Instantons for 4-manifolds with periodic ends and an obstruction to embeddings of 3-manifolds

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Abstract

We construct an obstruction to the existence of embeddings of a homology 3-sphere into a homology \( S^3 \times S^1 \) under some cohomological condition. The obstruction are defined as an element in the filtered version of the instanton Floer cohomology due to [6]. We make use of the \( \mathbb{Z} \)-fold covering space of homology \( S^3 \times S^1 \) and the instantons on it.

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1 Introduction

There are two typical studies of gauge theory for 4-manifolds with periodic end by C.H.Taubes [13] and J.Lin [11]. They gave a sufficient condition to exist a natural compactification of the instanton and the Seiberg-Witten moduli spaces
for such non-compact 4-manifolds. The condition of Taubes is the non-existence of non-trivial SU(2) flat connection of some segment of the end. The condition of Lin is the existence of a positive scalar curvature metric on the segment of the end. As an application of the existence of the compactification, Taubes showed the existence of uncountable family of exotic $\mathbb{R}^4$ and Lin constructed an obstruction to the existence of a positive scalar curvature metric.

In this paper, we also give a similar sufficient condition for the instanton moduli spaces. The condition is an uniform bound on the $L^2$-norm of curvature of instantons. When we bound the $L^2$-norm of curvature, we use an invariant which is a generalization of the Chern-Simons functional. Under this condition, we prove a compactness theorem (Theorem 5.1).

As the main theorem of this paper, we construct an obstruction of the existence of embeddings of a homology 3-sphere into a homology $S^3 \times S^1$ with some cohomological condition (Theorem 2.4). To formulate the obstruction, we need a variant of the instanton Floer cohomology $HF^*_r$ whose filtration was essentially considered by R.Fintushel-R.Stern in [6]. The obstruction is an element of the filtered instanton cohomology. We denote the element by $[\theta^r]$. The element $[\theta^r] \in HF^*_r$ is a filtered version of $[\theta]$ which was already defined by S.K.Donaldson [3] and K.Froyshov [9]. The class $[\theta]$ is defined by counting the gradient lines of the Chern-Simons functional which converge to the trivial flat connection. In order to show $[\theta^r]$ is actually an obstruction of embeddings, we count the number of the end of the 1-dimensional instanton moduli space for 4-manifold which has both of cylindrical and periodic end. For the counting, we use the compactness theorem (Theorem 5.1).

This paper is organized as follows. In Section 2, we give a precise formulation of our main theorem (Theorem 2.4). In Section 3, we prepare several notations and constructions which are used in the rest of this paper. In particular, we introduce the filtered instanton Floer homology $HF^*_r$ and the obstruction class $[\theta^r]$. We also review Fredholm, moduli theory for 4-manifolds with periodic end. In Section 4, we generalize the Chern-Simons functional and introduce the invariants $Q^r_X$. In Section 5, we prove the compactness theorem (Theorem 5.1). We use $Q^r_X$ to control the $L^2$-norm of curvature. In Section 6, we deal with technical arguments about the transversality and the orientation for the instanton moduli spaces for 4-manifolds with periodic end. In Section 7, we prove Theorem 2.4.

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2 Main theorem

Let $X$ be a homology $S^3 \times S^1$, i.e., $X$ is a closed 4-manifold equipped with an isomorphism $\phi : H_*(X,\mathbb{Z}) \to H_*(S^3 \times S^1,\mathbb{Z})$ in this paper. Then $X$ has an orientation induced by the standard orientation of $S^3 \times S^1$ and $\phi$. Let $Y$ be an oriented homology $S^3$. We construct an obstruction of embeddings $f$ of $Y$ into $X$ satisfying $f_*[Y] = 1 \in H_3(X,\mathbb{Z})$ as an element in the filtered instanton Floer cohomology. We use information of the compactness of the instanton moduli spaces for periodic-end 4-manifold in a crucial step of our construction. In order to formulate our main theorem, we need to prepare several notations.
For any manifold $Z$, we denote by $P_Z$ the product $SU(2)$ bundle. The product connection on $P_Z$ is written by $\theta$.

$$A(Z) := \{SU(2)\text{-connections on } P_Z\},$$

$$A^\text{flat}(Z) := \{SU(2)\text{-flat connections on } P_Z\} \subset A(Y),$$

$$\tilde{B}(Z) := A(Z)/\text{Map}_0(Z, SU(2)),$$

$$\tilde{R}(Z) := A^\text{flat}/\text{Map}_0(Z, SU(2)) \subset \tilde{B}(Z),$$

and

$$R(Z) := A^\text{flat}(Z)/\text{Map}(Z, SU(2)),$$

where $\text{Map}_0(Z, SU(2))$ is a set of smooth functions with mapping degree 0.

When $Z$ is equal to $Y$, the Chern-Simons functional $cs_Y : A(Y) \to \mathbb{R}$ is defined by

$$cs_Y(a) := \frac{1}{8\pi^2} \int_Y \text{Tr}(a \wedge da + \frac{2}{3} a \wedge a \wedge a).$$

It is known that $cs_Y$ descends to a map $\tilde{B}(Y) \to \mathbb{R}$, which we denote by the same notation $cs_Y$.

**Notation 2.1.** We denote the number of elements in $R(Y)$ by $l_Y$. If $R(Y)$ is not a finite set, we set $l_Y = \infty$.

We will use the following assumption on $Y$ in our main theorem (Theorem 2.4).

**Assumption 2.2.** All $SU(2)$ flat connections on $Y$ are non-degenerate, i.e. the first cohomology group of the next twisted de Rham complex:

$$0 \to \Omega^0(Y) \otimes su(2) \xrightarrow{d_a} \Omega^1(Y) \otimes su(2) \xrightarrow{d_a} \Omega^2(Y) \otimes su(2) \xrightarrow{d_a} \Omega^3 \otimes su(2) \to 0$$

vanishes for $[a] \in R(Y)$.

**Example 2.3.** All flat connections on the Brieskorn homology 3-sphere $\Sigma(p, q, r)$ are non-degenerate. ([5])

Under Assumption 2.2, $l_Y$ is finite ([14]).

In this paper without the use of Assumption 2.2, we will introduce the following invariants:

- $HF_r^i(Y)$ for $Y$ and $r \in (\mathbb{R} \setminus cs_Y(\tilde{R}(Y))) \cup \{\infty\}$ in Definition 3.2 satisfying $HF_{\infty}^i(Y) = HF^i(Y)$,

- $[\theta^r] \in HF_r^i(Y)$ for $Y$ and $r \in (\mathbb{R} \setminus cs_Y(\tilde{R}(Y))) \cup \{\infty\}$ in Definition 3.3 satisfying $[\theta^\infty] = [\theta] \in HF^i(Y)$, and

- $Q_X^i \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ for $i \in \mathbb{N}$ and $X$ in Definition 4.6 (When $X$ is homotopy equivalent to $S^3 \times S^1$, $Q_X^i = \infty$ for all $i \in \mathbb{N}$).

Our main theorem is:

**Theorem 2.4.** Under Assumption 2.2, if there exists an embedding $f$ of $Y$ into $X$ with $f_\ast[Y] = 1 \in H_3(X, \mathbb{Z})$ then $[\theta^r]$ vanishes for any $r \in [0, \min\{Q_X^{2i+3}, 1\}] \cap (\mathbb{R} \setminus cs_Y(\tilde{R}(Y))) \cup \{\infty\}$.
Example 2.5. Let $s$ satisfying $0 \neq X$ with $S$ to obtain a section $\text{Hol}(f \{7,1\})$ to define the instanton Floer homology. In this subsection we review classes of perturbations which were considered in Subsection 3.1 Holonomy perturbation(1)

In this section, we review the (filtered) instanton Floer theory and moduli theory on the periodic end 4-manifolds.

3 Preliminaries

In this section, we review the (filtered) instanton Floer theory and moduli theory on the periodic end 4-manifolds.

3.1 Holonomy perturbation(1)

In this subsection we review classes of perturbations which were considered in [7, 1] to define the instanton Floer homology.

Let $Y$ and $P_Y$ be as in Section 2. We fix a Riemannian metric $g_Y$ on $Y$. We define the set of embeddings from solid tori to $Y$ by

$$\mathcal{F}_d := \{(f_i)_{1 \leq i \leq d} : S^1 \times D^2 \to Y \mid f_i : \text{orientation preserving embedding}\}.$$ Fix a two form $dS$ on $D^2$ supported in the interior of $D^2$ with $\int_{D^2} dS = 1$. We denote by $C^1(SU(2)^d, \mathbb{R})_{\text{ad}}$ the adjoint invariant $C^1$-class functions from $SU(2)^d$ to $\mathbb{R}$ and define

$$\prod(Y) := \bigcup_{d \in \mathbb{N}} \mathcal{F}_d \times C^1(SU(2)^d, \mathbb{R})_{\text{ad}}.$$ We use the following notation,

$$\tilde{\mathcal{B}}^\pi(Y) := \left\{[a] \in \tilde{\mathcal{B}}(Y) \mid a \text{ is an irreducible connection}\right\},$$

where $\tilde{\mathcal{B}}(Y)$ is defined in Section 2. For $\pi = (f, h) \in \prod(Y)$, the perturbed Chern-Simons functional $cs_{Y, \pi} : \tilde{\mathcal{B}}^\pi(Y) \to \mathbb{R}$ is defined by

$$cs_{Y, \pi}(a) = cs_Y(a) + \int_{x \in D^2} h(\text{Hol}(a)_{f_1(-, x)}, \ldots, \text{Hol}(a)_{f_d(-, x)})dS,$$

where $\text{Hol}(a)_{f_i(-, x)}$ is the holonomy around the loop $t \mapsto f_i(t, x)$ for each $i \in \{1, \ldots, d\}$. If we identify $su(2)$ with its dual by the Killing form, the derivative of $h_i = pr^*h$ is a Lie algebra valued 1-form over $SU(2)$ for $h \in C^1(SU(2)^d, \mathbb{R})_{\text{ad}}$. Using the value of holonomy for the loops $\{f_i(x, t) \mid t \in S^1\}$, we obtain a section $\text{Hol}_{f_i(t, x)}(a)$ of the bundle $\text{Aut}P$ over $\text{Im}f_i$. Sending the section $\text{Hol}_{f_i(t, x)}(a)$ by the bundle map induced by $h_i' : \text{Aut}P \to \text{ad}P$, we obtain

In particular, if there exists an element

$$r \in [0, \min\{Q_X^{2Y+3}, 1\}] \cap (\mathbb{R} \setminus cs_Y(\tilde{\mathcal{R}}(Y)) \cup \{\infty\})$$
satisfying $0 \neq [\theta]$. Theorem 2.4 implies that there is no embedding from $Y$ to $X$ with $f_*[Y] = 1 \in H_3(X, \mathbb{Z})$.

Example 2.5. Let $X$ be a closed 4-manifold which is homotopy equivalent to $S^3 \times S^1$. There is no embedding $f$ of $\Sigma(2, 3, 6k - 1)$ into $X$ satisfying $f_*[\Sigma(2, 3, 6k - 1)] = 1 \in H_3(X, \mathbb{Z})$ for a positive integer $k$ satisfying $1 \leq k \leq 12$.

The proof of Example 2.5 is given in the end of Subsection 3.2.
a section \( h'_t(\text{Hol}_{f_i(t,x)}(a)) \) of \( \text{ad} P \over \text{Im} f_i \). We now describe the gradient-line equation of \( c s_{Y,\pi} \) with respect to \( L^2 \) metric:

\[
\frac{\partial}{\partial t} a_t = \text{grad}_{a_{cs_{Y,\pi}}} \ast g_Y(F(a_t) + \sum_{1 \leq i \leq d} h'_t(\text{Hol}(a_t) f_{i(t,x)})(f_i)_\ast \text{pr}_2^* dS), \quad (1)
\]

where \( \text{pr}_2 \) is the second projection \( \text{pr}_2 : S^1 \times D^2 \to D^2 \) and \( \ast g_Y \) is the Hodge star operator. We denote \( \text{pr}_2^* dS \) by \( \eta \). We set

\[
\tilde{R}(Y)_{\pi} := \left\{ a \in \tilde{B}(Y) \mid F(a) + \sum_{1 \leq i \leq d} h'_t(\text{Hol}(a_t) f_{i(t,x)})(f_i)_\ast \eta = 0 \right\},
\]

and

\[
\tilde{R}^*(Y)_{\pi} := \tilde{R}(Y)_{\pi} \cap \tilde{B}^*(Y).
\]

The solutions of (1) correspond to connections \( A \) over \( Y \times \mathbb{R} \) which satisfy an equation:

\[
F^+(A) + \pi(A)^+ = 0,
\]

where

- The two form \( \pi(A) \) is given by
  \[
  \sum_{1 \leq i \leq d} h'_t(\text{Hol}(A) f_{i(t,x,s)})(f_i)_\ast (\text{pr}_1^* \eta).
  \]
- The map \( \text{pr}_1 \) is a projection map from \( (S^1 \times D^2) \times \mathbb{R} \) to \( S^1 \times D^2 \).
- The notation \( + \) is the self-dual component with respect to the product metric on \( Y \times \mathbb{R} \).
- The map \( \tilde{f} : S^1 \times D^2 \times \mathbb{R} \to Y \times \mathbb{R} \) is \( f_i \times \text{id} \).

We also use several classes of the perturbations.

**Definition 3.1.** A class of perturbation \( \prod(Y)^{\text{flat}} \) is defined by a subset of \( \prod(Y) \) with the conditions:

- \( c s_Y \) coincides with \( c s_{Y,\pi} \) on a small neighborhood of critical points of \( c s_Y \)
- \( \tilde{R}(Y) = \tilde{R}(Y)_{\pi} \),

for all element in \( \prod(Y)^{\text{flat}} \).

If the cohomology groups defined by the complex (12) in [12] satisfies \( H^i_{\pi,a} = 0 \) for all \([a] \in \tilde{R}(Y)_{\pi} \setminus \{[\theta]\}\) for \( \pi \), we call \( \pi \) non-degenerate perturbation. If \( \pi \) satisfies the following conditions, we call \( \pi \) regular perturbation.

- The linearization of (2)
  \[
  d_A^+ + d\pi^+ : \Omega^1(Y \times \mathbb{R}) \otimes \text{su}(2)_{L^2_q} \to \Omega^+(Y \times \mathbb{R}) \otimes \text{su}(2)_{L^2_q}
  \]
  is surjective for \([A] \in M(a,b)_{\pi} \) and all irreducible critical point \( a,b \) of \( c s_{Y,\pi} \).
• The linearization of (2)

\[ d_A^+ + d\pi_A^+ : \Omega^1(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2_{q,\delta}} \rightarrow \Omega^+(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2_{q,\delta}} \]

is surjective for \([A] \in M(a, \theta)\) and all irreducible critical point \(a\) of \(c_{SY,\pi}\).

Here the spaces \(M(a, b)\) and \(M(a, \theta)\) are given in (3) in Subsection 3.2, \(L^2_{q,\delta}\) is the Sobolev norm and \(L^2_{\delta,\delta}\) is the weighted Sobolev norm which is same one as in Subsection 3.3.1 in [3].

### 3.2 Filtered instanton Floer (co)homology

In this subsection, we give the definition of the filtration of the instanton Floer (co)homology by using the technique in [6]. First, we give the definition of usual instanton Floer homology.

Let \(Y\) be a homology \(S^3\) and fix a Riemannian metric \(g_Y\) on \(Y\). Fix a non-degenerate regular perturbation \(\pi \in \prod(Y)\). Roughly speaking, the instanton Floer homology is infinite dimensional Morse homology with respect to

\[ c_{SY,\pi} : \tilde{R}^*(Y) \rightarrow \mathbb{R}. \]

Floer defined \(\text{ind} : \tilde{R}^*(Y)_{\pi} \rightarrow \mathbb{Z}\), called the Floer index. The (co)chains of the instanton Floer homology are defined by

\[ CF_i := \mathbb{Z}\{[a] \in \tilde{R}^*(Y)_{\pi} | \text{ind}(a) = i\} \quad (CF^i := \text{Hom}(CF_i, \mathbb{Z})). \]

The boundary maps \(\partial : CF_i \rightarrow CF_{i-1} (\delta : CF^i \rightarrow CF^{i+1})\) are given by

\[ \partial([a]) := \sum_{b \in \tilde{R}^*(Y)_{\pi} \text{ with ind}(b) = i-1} \#(M(a, b) / \mathbb{R})[b] \] (\(\delta := \partial^*\),

where \(M(a, b)\) is the space of trajectories of \(c_{SY,\pi}\) from \(a\) to \(b\). We now write the explicit definition of \(M(a, b)\). Fix a positive integer \(q \geq 3\). Let \(A_{a,b}\) be an SU(2) connection on \(Y \times \mathbb{R}\) satisfying \(A_{a,b}|_{Y \times (-\infty,1]} = p^*a\) and \(A_{a,b}|_{Y \times [1,\infty)} = p^*b\) where \(p\) is projection \(Y \times \mathbb{R} \rightarrow Y\).

\[ M(a, b)_{\pi} := \left\{ A_{a,b} + c \mid c \in \Omega^1(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2} \text{ with } (2) \right\} / \mathcal{G}(a, b) \quad (3) \]

where \(\mathcal{G}(a, b)\) is given by

\[ \mathcal{G}(a, b) := \left\{ g \in \text{Aut}(P_Y \times \mathbb{R}) \subset \text{End}(C^2)_{L^2_{q+1,\text{loc}}} \mid \nabla A_{a,b}(g) \in L^2_q \right\}. \]

The action of \(\mathcal{G}(a, b)\) on \(A_{a,b} + c \mid c \in \Omega^1(Y \times \mathbb{R}) \otimes \mathfrak{su}(2)_{L^2_q}\) given by the pull-backs of connections. The space \(\mathbb{R}\) has an action on \(M(a, b)\) by the translation. Floer show that \(M(a, b) / \mathbb{R}\) has structure of a compact oriented 0-manifold whose orientation is induced by the orientation of some determinant line bundles and \(\partial^2 = 0\) holds. The instanton Floer (co)homology \(HF_*(Y) (HF^*(Y))\) is defined by

\[ HF_* (Y) := \text{Ker} \partial / \text{Im} \partial \quad (HF^*(Y) := \text{Ker} \delta / \text{Im} \delta). \]
Second, we introduce the filtration in the instanton Floer homology. This filtration is essentially considered by Fintushel-Stern in [6]. We follow Fintushel-Stern and use the class of perturbations which they call $\epsilon$-perturbation defined in Section 3 of [6]. They constructed $\mathbb{Z}$-graded Floer homology whose chains are generated by the critical points of $cs_{Y,\pi}$ with $cs_{Y,\pi}(a) \in (m, m+1)$. We now consider Floer homology whose chains generated by the critical points of $cs_{Y,\pi}$ with $cs_{Y,\pi}(a) \in (-\infty, r)$.

Let $\tilde{R}(Y)$ be as in Section 2 and $\Lambda_Y$ be $\mathbb{R}\setminus \text{Im } cs_{Y}|_{\tilde{R}(Y)}$. For $r \in \Lambda_Y$, we define the filtered instanton homology $HF_r^*(Y)(HF_r^*(Y))$ by using $\epsilon$-perturbation. For $r \in \Lambda_Y$, we set $\epsilon = \inf_{a \in \tilde{R}(Y)} |cs_Y(a) - r|$ and choose such a $\epsilon$-perturbation $\pi$.

**Definition 3.2 (Filtered version of the instanton Floer homology).** The chains of the filtered instanton Floer (co)homology are defined by

$$CF_r^i := \mathbb{Z} \left\{ [a] \in \tilde{R}^*(Y)_{\pi} \mid \text{ind}(a) = i, \ cs_{Y,\pi}(a) < r \right\} (CF_r^i := \text{Hom}(CF_r^i, \mathbb{Z})).$$

The boundary maps $\partial^r : CF_r^i \to CF_r^{i-1}$ (resp. $\delta^r : CF_r^i \to CF_r^{i+1}$) are given by the restriction of $\partial$ to $CF_r^i$ (resp. $\delta := (\partial^r)^*$). This maps are well-defined and $(\partial^r)^2 = 0$ holds as in Section 4 of [6]. The filtered instanton Floer (co)homology $HF_r^*(Y)(HF_r^*(Y))$ is defined by

$$HF_r^*(Y) := \text{Ker}(\partial^r)/\text{Im}(\partial^r) (\text{resp. } HF_r^*(Y) := \text{Ker}(\delta^r)/\text{Im}(\delta^r)).$$

We can also show $HF_r^*(Y)$ and $HF_r^i(Y)$ are independent of the choices of the perturbation and the metric by similar discussion in [6]. For $r \in \Lambda_Y$, we now introduce obstruction classes in $HF_r^*(Y)$. These invariants are generalizations of $[\theta] \in HF^*(Y)$ considered in Subsection 7.1 of [3] and Subsection 2.1 of [9].

**Definition 3.3 (Obstruction class).** For $r \in \Lambda_Y$, we set homomorphism

$$\theta^r : CF_1^r \to \mathbb{Z}$$

by

$$\theta^r(a) := \#(M(a, \theta)_{\pi}/\mathbb{R}). \quad (4)$$

As in [3] and [9], we use the weighted norm on $M(a, \theta)_{\pi}$ to use Fredholm theory. From the same discussion for the proof of $(\delta^r)^2 = 0$, we can show $\delta^r(\theta^r) = 0$. Therefore it defines the class $[\theta^r] \in HF^1_r(Y)$. We call the class $[\theta^r]$ an obstruction class.

The class $[\theta^r]$ does not depend on the small perturbation and the metric. The proof is similar to the proof for original one $[\theta]$. Now we give the proof of Example 2.5.

*Proof.* Because $X$ is homotopy equivalent to $S^3 \times S^1$, $Q_X = \infty$ for $i \in \mathbb{N}$. If the element $[\theta^1] \in HF_1^1(\Sigma(2, 3, 6k - 1))$ does not vanish for $r = 1$, we can apply Theorem 2.4. Frøyshov showed $0 \neq [\theta] \in HF^1(\Sigma(2, 3, 6k - 1))$ by using the property of $h$-invariant of Proposition 4 in [9]. Then we get nonzero homomorphism $\theta : CF_1^r(Y) \to \mathbb{Z}$ for $r = \infty$. (If $r = \infty$, $HF_1^r$ is the usual instanton Floer cohomology by the definition.) By using calculation about the value of the Chern-Simons functional of Section 7 in [6], we can see $\theta^1 : CF_1^r \to \mathbb{Z}$ is nonzero for $r = 1$ and $CF_1^r$ is zero for $r = 1$ and $i \in 2\mathbb{Z}$. This implies

$$0 \neq [\theta^1] \in HF_1^1(\Sigma(2, 3, 6k - 1)) \text{ for } r = 1.$$

$\square$

7
3.3 Fredholm theory and moduli theory on 4-manifolds with periodic end

In [13], Taubes constructed the Fredholm theory of some class of elliptic operators on 4-manifolds with periodic ends. He also extends moduli theory of SU(2) gauge theory on such non-compact 4-manifolds. In this subsection, we review Fredholm theory of a certain elliptic operator on 4-manifolds with the periodic ends as in [13] and define the Fredholm index of the class of operators, which gives the formal dimension of a suitable instanton moduli space on such non-compact 4-manifolds. First we formulate the 4-manifolds with periodic ends.

Let \( Y \) be an oriented homology \( S^3 \) as in Section 2. Let \( W_0 \) be an oriented homology cobordism from \( Y \) to \( -Y \). We get a compact oriented 4-manifold \( X \) by pasting \( W_0 \) with itself along \( Y \) and \( -Y \). We give several notations in our argument.

1. The manifold \( W_i \) is a copy of \( W_0 \) for \( i \in \mathbb{Z} \).
2. We denote \( \partial(W_i) \) by \( Y_i^+ \cup Y_i^- \) where \( Y_i^+ \) (resp. \( Y_i^- \)) is equal to \( Y \) (resp. \( -Y \)) as oriented manifolds.
3. For \((m, n) \in (\mathbb{Z} \cup \{-\infty\}) \times (\mathbb{Z} \cup \{\infty\})\) with \( m < n \), we set
   \[
   W_{[m, n]} := \bigcup_{m \leq i \leq n} W_i / \{Y_j^+ \sim Y_j^+ + 1 \mid j \in \{m, \ldots, n\}\}.
   \]
   We denote by \( W \) the following non-compact 4-manifold
   \[
   W := Y \times (-\infty, 0] \cup W[0, \infty]/\{\partial(Y \times (-\infty, 0]) \sim Y_0^+\}.
   \]

We denote by \( g_Y \) the Riemannian metric on \( Y \) and choose a Riemannian metric \( g_W \) on \( W \) which satisfies

1. \( g_W|_{Y \times (-\infty, -1]} = g_Y \times g_{\tan} \).
2. The restriction \( g_W|_{W[0, \infty]} \) is a periodic metric.

There is a natural orientation on \( W[0, \infty] \) and \( W \) induced by the orientations of \( W_0 \). The infinite cyclic covering space of \( X \) can be written by
   \[
   \tilde{X} \cong W[-\infty, \infty].
   \]

Let \( T \) be the deck transformation of \( \tilde{X} \) which maps each \( W_i \) to \( W_{i+1} \). By restriction, \( T \) has an action on \( W[0, \infty] \). We use the following smooth functions \( \tau \) and \( \tau' \) on \( W \)
   \[
   \tau, \, \tau': W \to \mathbb{R}
   \]
satisfying

1. \( \tau(T|_{W[0, \infty]}(x)) = \tau(x) + 1, \quad \tau'(T|_{W[0, \infty]}(x)) = \tau'(x) + 1 \) for \( x \in W[0, \infty] \).
2. \( \tau|_{Y \times (-\infty, -2]} = 0, \quad \tau'(y, t) = -t \) for \( (y, t) \in Y \times (-\infty, -2] \).

By the restriction of \( \tau \), we have a function on \( W[0, \infty] \) which we denote by same notation \( \tau \).

In this subsection, we review the setting of the configuration space of fields on \( W \) and define the Fredholm index of a kind of operator on \( W \). We fix \( \pi \in \prod(Y) \) in Subsection 3.1 and assume that \( \pi \) is a non-degenerate perturbation. Let \( PW \) be the product SU(2) bundle.
Definition 3.4. For each element \([a] \in \tilde{R}(Y)\), we fix an \(SU(2)\) connection \(A_a\) on \(P_W\) which satisfying \(A_a|_{Y \times (-\infty, -1]} = \text{pr}^\pi a\) and \(A_a|_{W[0, \infty]} = \theta\). If \(a\) is an irreducible (resp. reducible) connection, we define the space of connections on \(P_W\) by
\[
A^W(a)_\delta := \left\{ A_a + c \mid c \in \Omega^1(W) \otimes \mathfrak{su}(2)_{L^2_{q, \delta}} \right\},
\]
where \(\Omega^1(W) \otimes \mathfrak{su}(2)_{L^2_{q, \delta}}\) (resp. \(\Omega^1(W) \otimes \mathfrak{su}(2)_{L^2_{q, \delta}}\)) is the completion of \(\Omega^1(W) \otimes \mathfrak{su}(2)\) with \(L^2_{q, \delta}\)-norm (resp. \(L^2_{q, \delta}\)-norm), \(q\) is a natural number greater than 3, and \(\delta\) is a positive real number. For \(f \in \Omega^1(W) \otimes \mathfrak{su}(2)\) with compact support, we define \(L^2_{q, \delta}\)-norm (resp. \(L^2_{q, \delta}\)-norm) by
\[
||f||^2_{L^2_{q, \delta}} := \sum_{0 \leq j \leq q} \int_W e^{\delta\tau} \left| \nabla^j f \right|^2 \text{dvol},
\]
where \(\nabla^j\) is covariant derivative with respect to the product connection \(\theta\). We use a periodic metric \(|| - ||\) on the bundle. Its completion is denoted by \(\Omega^1(W) \otimes \mathfrak{su}(2)_{L^2_{q, \delta}}\). We define the gauge group
\[
G^W(a)_{\delta} := \left\{ g \in \text{Aut}(P_W)_{L^2_{q + 1, \text{loc}}} \mid \nabla_{A_a}(g) \in L^2_{q, \delta} \right\},
\]
where \(\nabla_{A_a}(g) \in L^2_{q, \delta}\) has the action on \(A^W(a)_{\delta}\) induced by the pull-backs of connections. The space \(G^W(a)_{\delta}\) has structure of Banach Lie group and the action of \(G^W(a)_{\delta}\) on \(A^W(a)_{\delta}\) is smooth. The configuration space for \(W\) is defined by
\[
B^W(a)_{\delta} := A^W(a)_{\delta}/G^W(a)_{\delta}, \quad B^W(a)_{(\delta, \delta)} := A^W(a)_{(\delta, \delta)}/G^W(a)_{(\delta, \delta)}.
\]
Let \(s\) be a smooth function from \(W\) to \([0, 1]\) with
\[
s|_{Y \times (-\infty, -2]} = 1, \quad s|_{Y \times [-1, 0] \cup W[0, \infty]} = 0.
\]
We define the instanton moduli space for \(W\) by
\[
M^W(a)_{\pi, \delta} := \{ [A] \in B^W(a)_{\delta} \mid F_{\pi}(A) = 0 \} \quad (5)
\]
where \(F_{\pi}\) is the perturbed ASD-map
\[
F_{\pi}(A) := F^+(A) + s\pi(A).
\]
For each \(A \in A^W(a)_{\delta}\), we have the bounded linear operator:
\[
d^*_{A} + d_{A} : \Omega^1(W) \otimes \mathfrak{su}(2)_{L^2_{q, \delta}} \to (\Omega^0(W) \oplus \Omega^1(W)) \otimes \mathfrak{su}(2)_{L^2_{q, \delta}} \quad (6)
\]
Proposition 3.7. (resp. $d_A^+ + d_{A^\delta} : \Omega W \otimes \text{su}(2) \to (\Omega W \otimes \Omega^+ W) \otimes \text{su}(2)._Z^A_{,(\delta,\delta)}$.)

Taubes gave a criterion for the operator $d_A^+ + d_{A^\delta} = d_A^+ + d_{A}^+ + s\pi^+_{a,A}$ in (6)(resp. (7)) to be Fredholm in Theorem 3.1 of [13].

Theorem 3.5. (Taubes,[13]). There exists a discrete set $D$ in $\mathbb{R}$ with no accumulation points such that (6)(resp. (7)) is Fredholm for each $\delta$ in $\mathbb{R}\setminus D$.

The discrete set $D$ is defined by

$$D := \{ \delta \in \mathbb{R} | \text{the cohomology groups } H^i_{\delta} \text{ are acyclic for all } z \text{ with } |z| = e^\frac{4}{7} \}.$$ 

The cohomology groups $H^i_{\delta}$ are given by the complex:

$$0 \to \Omega^0(X) \otimes \text{su}(2) \xrightarrow{d_\theta,z} \Omega^1(X) \otimes \text{su}(2) \xrightarrow{d^+_\theta,z} \Omega^+(X) \otimes \text{su}(2) \to 0, \quad (8)$$

where

$$d_\theta,z : \Omega^0(X) \otimes \text{su}(2) \to \Omega^1(X) \otimes \text{su}(2)$$

and

$$d^+_\theta,z : \Omega^1(X) \otimes \text{su}(2) \to \Omega^+(X) \otimes \text{su}(2)$$

are given by

$$d_\theta,z(f) = z^\tau d_{p^\theta}(z^{-\tau}(p^\theta f)), \quad d^+_\theta,z(f) = z^\tau d^+_{p^\theta,a}(z^{-\tau}(p^\theta f)),$$

where $p$ is the covering map $\tilde{X} \to X$. (We fix a branch of $\ln z$ to define $z^\tau = e^{\tau \ln z}$.)

In above definition, $d_\theta,z(f)$ and $d^+_\theta,z(f)$ are sections of $p^\ast P_X$, however these are invariant under the deck transformation, we regard $d_\theta,z(f)$ and $d^+_\theta,z(f)$ as sections on $P_X$.

The operators $d_\theta,z$ and $d^+_\theta,z$ in (8) depend on the metric on $X$ and $\tau$ however the cohomology groups $H^i_{\delta}$ are independent of the choice of them. We now introduce the formal dimension of the instanton moduli spaces:

Definition 3.6. Suppose $a$ is an irreducible critical point of $cs_{Y,a}$. From Theorem 3.5, there exists $\delta_0 > 0$ such that (6) is Fredholm for any $\delta \in (0,\delta_0)$ and $A \in \mathcal{A}W(a).$ We define the formal dimension $\text{ind}_W(a)$ of the instanton moduli spaces for $W$ by the Fredholm index of (6). For the case of $a = \theta$, we also set $\text{ind}_W(a)$ as the Fredholm index of (7).

The formal dimension $\text{ind}_W(a)$ is calculated by using the following proposition.

Proposition 3.7. Suppose $a$ is an irreducible critical point of $cs_{Y,a}$. The formal dimension $\text{ind}_W(a)$ is equal to the Floer index $\text{ind}(a)$ of $a$. If $a$ is equal to $\theta$, $\text{ind}_W(a) = \text{ind}(a) = -3$.

Proof. First we take a compact oriented 4-manifold $Z$ with $\partial Z = -Y$. It is easy to show there is an isomorphism $H^+(W_{[0,\infty]}) \cong H^+(S^3).$ We define $Z^+ := Z \cup Y W_{[0,\infty]}$ and fix a periodic Riemannian metric $g_{Z^+}$ satisfying $g_{Z^+}|_{W_{[0,\infty]}} = g_W$. In Proposition 5.1 of [13], Taubes computed the Fredholm index of $d_0^+ + d_{a^\delta}^+.$
as a operator on $Z^+$ in the situation that $H_*(W[0, \infty], \mathbb{Z}) \cong H_*(S^3, \mathbb{Z})$. (The proof is given by using the admissibility of each segment $W_0$, however Taubes just use the condition $H_*(W[0, \infty], \mathbb{Z}) \cong H_*(S^3, \mathbb{Z})$.)

$$\text{ind}(d^+_a + d^+_b) = -3(1 - b_1(Z) + b^+(Z))$$

for a small $\delta$. Fix an $SU(2)$-connection $A_{a, \theta}$ on $W$ with $A|_{W \times \mathbb{R}} = a$, $A|_{W[0, \infty)} = \theta$ and an $SU(2)$-connection $B_a$ on $X \cup Y \times [0, \infty)$ By the similar discussion about gluing of the operators on cylindrical end in Proposition 3.9 of [3], we have

$$\text{ind}(d^+_a + d^+_b) = \text{ind}(d^+_a + d^+_b) = \text{ind}(d^+_a + d^+_b).$$

Donaldson show that $\text{ind}(d^+_a + d^+_b)$ is equal to $-\text{ind}(a) - 3(1 - b_1(Z) + b^+(Z))$ in Proposition 13.17 of [3]. The second statement is similar to the first one. 

\[4\] Chern-Simons functional for homology $S^3 \times S^1$

For a pair $(X, \phi)$ consisting of an oriented 4-manifold and non-zero element $0 \neq \phi \in H^1(X, \mathbb{Z})$, we generalize the Chern-Simons functional to a functional $cs_{(X, \phi)}$ on the flat connections on $X$. We define the invariants $Q_i(X) \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ for $i \in \mathbb{N}$ by using the value of $cs_{(X, \phi)}$. In our construction, $cs_{(X, \phi)}$ cannot be extended to a functional for arbitrary $SU(2)$ connections on $X$.

Let $X$ be an oriented closed 4-manifolds equipped with $0 \neq \phi \in H^1(X, \mathbb{Z})$ and $p : \tilde{X} \to X$ be the $\mathbb{Z}$-hold covering of $X$ corresponding to $\phi \in H^1(X, \mathbb{Z}) \cong [X, \mathbb{Z}]$. Recall that the bundle $P_X$ and the set $\tilde{R}(X)$ as in Section 2. Let $f$ be a smooth map representing the class $\phi \in H^1(X, \mathbb{Z}) \cong [X, S^1]$, and $i$ is a lift of $f$.

**Definition 4.1.** [Chern-Simons functional for a homology $S^3 \times S^1$] We define the Chern-Simons functional for $X$ as the following map

$$cs_{(X, \phi)} : \tilde{R}(X) \times \tilde{R}(X) \to \mathbb{R},$$

$$cs_{(X, \phi)}([a], [b]) := \frac{1}{8\pi^2} \int_{\tilde{X}} \text{Tr}(F(A_{a, b}) \wedge F(A_{a, b})),$$

where $a, b$ are flat connections on $P_X$ and $A_{a, b}$ is an $SU(2)$-connection on $P_{\tilde{X}} := \tilde{X} \times SU(2)$ which satisfies $A_{a, b}|_{\tilde{X} \times \mathbb{R}} = p^*a$ and $A_{a, b}|_{\tilde{X} \times \mathbb{R}} = p^*b$ for some $r > 0$.

We have an alternative description of $cs_{(X, \phi)}([a], [b])$ when a closed oriented 3-manifold $Y$ is given as a sub-manifold of $X$ satisfying $i_*[Y] = PD(\phi) \in H_3(X, \mathbb{Z})$, where $i$ is the inclusion $Y \to X$. Such $Y$ is given as an inverse image of a regular value of $f$. We can take $Y$ to be connected, and we assume this. We denote by $W_0$ the cobordism from $Y$ to itself obtained from cutting $X$ open along $Y$. Since $Y$ is connected and $\phi \neq 0$, $W_0$ is also connected. Then we choose the identification of $\tilde{X} \times 0$ and $\cdots \cup Y W_0 \cup Y W_1 \cup Y \cdots$ We have the following formula.

**Lemma 4.2.**

$$cs_{(X, \phi)}([a], [b]) = cs_Y([i^*a]) - cs_Y([i^*b])$$
Lemma 4.7.

Proposition 4.3. For \([a] \in \tilde{R}(Y \times S^1)\), the restriction \([i^*a] \in \tilde{R}(Y)\) satisfies
\[
\text{cs}_Y([i^*a]) = \text{cs}_{(Y \times S^1, PD(Y))}([a], [\theta]),
\]
where \(i\) is a inclusion \(Y = Y \times 1 \rightarrow Y \times S^1\) and \(PD\) is the Poincaré duality.

This is a corollary of Lemma 4.2. We have the following well-definedness.

Lemma 4.4. \(\text{cs}_{(X, \phi)}\) does not depend on the choices of \(f\), representatives \(a\) and \(b\), and \(A_{a,b}\).

This is also a consequence of Lemma 4.2

Definition 4.5. Let \(X\) be a closed oriented 4-manifold equipped with \(\phi \in H^1(X, \mathbb{Z})\). The invariant \(Q_{(X, \phi)}\) is defined by
\[
\begin{cases}
\infty & \text{if } \tilde{R}^*(X) = \emptyset, \\
\inf \left\{ \text{cs}_{(X, \phi)}([a], [\theta]) + m \left| \begin{array}{l}
m \in \mathbb{Z}, [a] \in \tilde{R}^*(X) \end{array} \right. \right\} & \text{if } \tilde{R}^*(X) \neq \emptyset,
\end{cases}
\]
where \(\tilde{R}^*(X)\) is the subset of the classes of the irreducible connections in \(\tilde{R}(X)\).

We now give a definition of \(Q_X^\ell \in \mathbb{R}_{\geq 0} \cup \{\infty\}\).

Definition 4.6. Suppose that \(X\) is a homology \(S^3 \times S^1\) and \(i\) is a positive integer. Let \(\tilde{X}\) be \(Z\)-fold covering space over \(X\) corresponding to the \(1 \in H^1(X, \mathbb{Z}) \cong_{PD} H_3(X, \mathbb{Z})\). We set \(\tilde{X}^i := \tilde{X}/i\mathbb{Z}\). Since the quotient map \(p^i : \tilde{X} \rightarrow \tilde{X}^i\) is a \(Z\)-fold covering, this determine a class \(\phi^i \in H^1(\tilde{X}^i, \mathbb{Z})\). We define \(Q_X^i \in \mathbb{R}_{\geq 0} \cup \{\infty\}\) by
\[
Q_X^i := \min_{0 \leq \ell \leq l} \tilde{Q}_{(\tilde{X}^i, \phi^i)}.
\]

We show the following lemma which is used in the proof of Key lemma(Lemma 5.3).

Lemma 4.7. Suppose that \(\gamma\) is a flat connection on \(W[m, n]\) satisfying the following conditions.

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• $\gamma|_{Y^m} \cong \gamma|_{Y^n}$.

• There exists $u \in \mathbb{Z}$ satisfying $|cs_{(W[m,n], PD[Y^m])}(r(\gamma), \theta) + u| < Q_X^{n-m+1}$, where $r(\gamma)$ is a flat connection on $W[0, k]$ given by pasting $\gamma$ with itself along $Y^0_0 \cup Y^k_k$.

Then $\gamma$ is gauge equivalent to $\theta$.

Proof. Suppose $\gamma$ is not gauge equivalent to $\theta$. The calculation

$H_4(W[m, n]) \cong 0$

and holonomy correspondence

$R(W[m, n]) \cong \text{Hom}(\pi_1(W[m, n]), SU(2))/\text{conjugate}$

imply that there is no reducible $SU(2)$ connection on $W[m, n]$ except $\theta$. Therefore $\gamma$ is an irreducible connection on $W[m, n]$. Because $W[m, n] \to X$ is the $(n - m + 1)$-fold covering space of $X$,

$Q_X^{n-m+1} \leq |cs_{(W[m,n], PD[Y^m])}(r(\gamma), \theta) + u|

holds by the definition of $Q_X$. This is a contradiction.

5 Compactness

The compactness of the instanton moduli spaces for non-compact 4-manifolds is treated in [7], [10], [3] for cylindrical end case and in [13] for periodic end case. In [10] and [13], they consider the instanton moduli spaces with the connections asymptotically convergent to the trivial connection on the end. We also follow their strategy by using $Q_X^{n+3}$ defined in the previous section. More explicitly, in this section we explain a compactness result for the instanton moduli spaces for a non-compact manifold $W[0, \infty]$ with periodic end.

5.1 Key lemma

Let $W_0$, $W[0, \infty]$ be the oriented Riemannian 4-manifolds introduced in the beginning of Subsection 3.3. By pasting $W_0$ with itself along its boundary $Y$ and $-Y$, we obtain a homology $S^3 \times S^1$ which we denote by $X$. We consider the product $SU(2)$-bundle $P_{W[0, \infty]}$ on $W[0, \infty]$. For $q \geq 3$ and $\delta > 0$, we define the instanton moduli space $M_\delta^{W[0, \infty]}$ by

$M_\delta^{W[0, \infty]} := \left\{ \theta + c \in \Omega^1(W[0, \infty]) \otimes \mathfrak{su}(2)_{L_{\delta}} | F^+(\theta + c) = 0 \right\} / \mathcal{G}$,

where $\mathcal{G}$ is the gauge group

$\mathcal{G} := \left\{ g \in \text{Aut}(P_{W[0, \infty]}) \subset \text{End}(\mathbb{C}^2)_{L_{\delta+1, \infty}} | dg \in L_{q, \delta}^2 \right\}$,

and the action of $\mathcal{G}$ is given by the pull-backs of connections. For $f \in \Omega^1(W[0, \infty]) \otimes \mathfrak{su}(2)$ with compact support, we define $L_{q, \delta}^2$ norm by the following formula

$||f||_{L_{q, \delta}^2}^2 := \sum_{0 \leq j \leq q} \int_{W[0, \infty]} e^{\delta r} \left| \nabla_{\theta} f \right|^2 \text{dvol},$
where $\nabla_\theta$ is the covariant derivative with respect to the product connection. We use the periodic metric $| - |$ which is induced from the Riemannian metric $g_W$. Its completion is denoted by $\Omega'(W[0,\infty]) \otimes \mathfrak{su}(2)_{L^2_{\delta,\delta}}$.

Our goal of this section is to show the next theorem under the above setting.

**Theorem 5.1.** Under Assumption 2.2 the following statement holds. There exist $\delta' > 0$ satisfying the following property. Suppose that $\delta$ is a non-negative number less than $\delta'$ and $\{A_n\}$ is a sequence in $M^{W[0,\infty]}_\delta$ satisfying

$$\sup_{n \in \mathbb{N}} ||F(A_n)||^2_{L^2(W[0,\infty])} < \min\{8\pi^2, Q^{2\nu+3}\}.$$  

Then for some subsequence $\{A_{n_i}\}$, a positive integer $N_0$ and some gauge transformations $\{g_j\}$ on $W[N_0,\infty]$ the sequences $\{g_j^*A_{n_i}\}$ converges to some $A_\infty$ in $L^2_{\delta,\delta}(W[N_0,\infty])$.

The proof of Theorem 5.1 is given in the end of Subsection 5.3.

**Lemma 5.2.** For a positive number $c_1 > 0$, there exists a positive number $c_2 > 0$ satisfying the following statement.

For any $SU(2)$-connection $a$ on $Y^0_+$ and any flat connection $\gamma$ on $W[0, k]$ satisfying the following conditions

- $\sup_{x \in Y^0_+} \sum_{0 \leq j \leq 1} \left| \nabla_\gamma^{(j)}(a - (l^0_+)^*\gamma)(x) \right| < c_1$.
- $\gamma|_{Y^0_+} \cong \gamma|_{Y_0^k}$.

the inequality

$$\left| c_{SY}(a) - c_{SY}(W[0,k],PD(Y^0_+))(r(\gamma),\theta) \right| \leq c_2 \sup_{x \in Y^0_+} \sum_{0 \leq j \leq 1} \left| \nabla_\gamma^{(j)}(a - (l^0_+)^*\gamma)(x) \right|^2$$

holds, where $r(\gamma)$ is a flat connection on $W[0,k]$ given by pasting $\gamma$ with itself along $Y^0_+ \cup Y^k$ and $l^0_+: Y^0_+ \to W_0$ is the inclusion.

**Proof.** Lemma 4.2 imply

$$\left| c_{SY}(a) - c_{SY}(W[0,k],PD(Y^0_+))(\gamma,\theta) \right| = |c_{SY}(a) - c_{SY}((l^0_+)^*\gamma)|.$$

Since $(l^0_+)^*\gamma$ is a flat connection on $Y^0_+$, we have

$$\frac{1}{8\pi^2} \left| \int_{Y^0_+} Tr((a - (l^0_+)^*\gamma) \wedge d(l^0_+)^*\gamma)(a - (l^0_+)^*\gamma) + \frac{2}{3} (a - (l^0_+)^*\gamma)^3 \right|$$

$$\leq \frac{1}{8\pi^2} \text{vol}(\gamma) \sup_{x \in Y^0_+} \left| \nabla_\gamma^{(j)}(a - (l^0_+)^*\gamma)(x) \right| \left| (a - (l^0_+)^*\gamma) + \frac{2}{3} \sup_{x \in Y^0_+} |(a - (l^0_+)^*\gamma)|^3 \right|$$

$$\leq c_2 \sup_{x \in Y^0_+} \sum_{0 \leq j \leq 1} \left| \nabla_\gamma^{(j)}(a - (l^0_+)^*\gamma)(x) \right|^2.$$
Next lemma gives us a key estimate. We use $Q_X^{2l_Y+3}$ to obtain an estimate of the difference between an ASD-connection and the trivial flat connection on the end $W[0, \infty]$.

**Lemma 5.3.** Suppose that $Y$ satisfies Assumption 2.2. There exists a positive number $c_3$ satisfying the following statement.

For $A \in M^W_{\#}[0, \infty]$ satisfying $\frac{1}{8\pi^2}||F(A)||_{L^2(W[0, \infty])}^2 < \min\{1, Q_X^{2l_Y+3}\}$, there exists a positive number $\eta_0$ which depends only on the difference $\min\{1, Q_X^{2l_Y+3}\} - \frac{1}{8\pi^2}||F(A)||_{L^2}^2$ such that the following condition holds.

Note that if $K$ is sufficiently large, the inequality $||F(A)||_{L^2(W_k)}^2 < \eta_0$ is satisfied for every $k > K$. When $K$ satisfies this property, there exist gauge transformations $g_k$ over $W[k, k+2]$ such that

$$\sup_{x \in W[k, k+2]} \sum_{0 \leq j \leq q+1} ||\nabla^j g_k \cdot A||_{W[k, k+2]}^2$$

$$\leq c_3 ||F(A)||_{L^2(W[k-l_Y-2, k+l_Y+3])}^2$$

holds for $k > K + l_Y + 3$.

**Proof.** For $k > K + l_Y + 3$, we apply Lemma 10.4 in [13] to $A|_{W[k-l_Y-1, k+l_Y+2]}$. Then we obtain gauge transformations $g_k$ and flat connections $\gamma_k$ over $W[k-l_Y-1, k+l_Y+2]$ satisfying

$$\sup_{x \in W[k-l_Y-1, k+l_Y+2]} \sum_{0 \leq j \leq q+1} ||\nabla^j g_k \cdot A||_{W[k-l_Y-1, k+l_Y+2]}^2$$

$$\leq c_3 ||F(A)||_{L^2(W[k-l_Y-2, k+l_Y+3])}^2 \leq (2l + 5)c_3 \eta$$

for a small $\eta$. By using the pull-backs of $\gamma_k$ from $W[k-1, k-1]$ (resp. $W[k+2, k+l_Y+2]$) to $Y_k$, we get the flat connections over $Y$. Then we get $l_Y + 1$ flat connections by using this method. Under the assumption that $l_Y = \#R(Y)$, same flat connections appear by the pigeonhole principle. We choose two numbers $k(1)^\pm < k(2)^\pm$ which satisfy $\{l_+^{k(1)^\pm}\} \gamma_k \equiv \{l_+^{k(2)^\pm}\}$ as connections on $Y$, where $k(1)^+$ and $k(2)^+$ are elements in $\{k-l_Y-1, \ldots, k-1\}$ (resp. $k-1, k-2\} \in \{k+2, k+l_Y+2\}$). The map $l_+^k : Y_k \to W_k$ is the inclusion.

**Claim.** Suppose $||F(A)||_{L^2(W_k)}^2 < \eta$ holds for $k > K + l_Y + 3$. For sufficiently small $\eta$, the flat connection $\gamma_k$ is isomorphic to $\theta$.

The properties of $k(1)^\pm$ and $k(2)^\pm$, (9) and Lemma 5.2 imply

$$|c_Y((l_+^{k(1)^\pm})^* g_k \cdot A) - c_Y((l_+^{k(1)^\pm})^* A)|_{PD(Y_{k+1}^{(1)^\pm})} (r(\gamma_k), \theta) \leq (2l + 5)c_3 \eta c_2.$$  

(10)

We also have

$$c_Y((l_+^{k(1)^\pm})^* g_k \cdot A) = c_Y((l_+^{k(1)^\pm})^* A) + \deg(g_k|_{Y_{k+1}^{(1)^\pm}})$$

$$= ||F(A)||_{L^2(W[k(l_k)^+], \infty)}^2 + \deg(g_k|_{Y_{k+1}^{(1)^\pm}}).$$

(11)

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We choose \( \eta_0 \) satisfying the following condition:

\[
(2l + 5)c_3\eta_0 c_2 < Q_X^{2l+3} - \frac{1}{8\pi^2} \int_{W[k(1)\pm, \infty]} |F(A)|^2, \tag{12}
\]

where the right hand side is positive by the assumption of \( A \). We obtain

\[
\left| cs_{\left(W[k(1)\pm, k(2)\pm], PD[Y_k(1)\pm]\right)} (r(\gamma_k), \theta) + \deg(g_k|_{Y_k(1)\pm}) \right| < Q_X^{2l+3}
\]

from (10), (11) and (12). Then Lemma 4.7 imply \( \gamma|_{W[k(1)\pm, k(2)\pm]} \cong \theta \).

Similarly the inequality

\[
\sup_{x \in W[k(2)^-, k(1)^+]} \sum_{0 \leq j \leq q} |\nabla^j \gamma_k (g_k^* A|_W - \gamma_k|_{W[k(2)^-, k(1)^+]}) (x)| \leq (2l+5)c_3\eta_0
\]

holds over \( W[k(2)^-, k(1)^+] \). From above discussion, \( l_+^{k(2)^-} \gamma_k \) and \( \gamma_k^{(1)+} \) are gauge equivalent. By Lemma 5.2, we also get

\[
\left| cs_Y (l_+^{k(2)^-})^* g_k^* A - cs_{\left(W[k(2)^-, k(1)^+], PD[Y_k(2)^-]\right)} (r(\gamma_k), \theta) \right| \leq (2l+5)c_3\eta_0 c_2.
\]

By choice of \( \eta_0 \), we have

\[
\left| cs_{\left(W[k(2)^-, k(1)^+], PD[Y_k(2)^-]\right)} (r(\gamma_k), \theta) + \deg(g_k|_{Y_k(2)^-}) \right| < Q_X^{2l+3}
\]

and get \( \gamma|_{W[k(2)^-, k(1)^+]} \cong \theta \) by using Lemma 4.7 as above.

\[
\square
\]

### 5.2 Chain convergence

We introduce the following notion which is crucial for our proof of the compactness theorem (Theorem 5.1).

**Definition 5.4.** [Chain decomposition of a sequence of the instantons] For a fixed number \( \eta > 0 \) and a sequence \( \{A_n\} \subset M_8^{W[0, \infty]} \) satisfying

\[
\sup_{n \in \mathbb{N}} \|F(A_n)\|_{L^2(W[0, \infty])} < \infty,
\]

when a finitely many sequences \( \{s^j_n(\eta)\}_{1 \leq j \leq m} \) of non-negative numbers satisfies

\[
\|F(A_n)\|^2_{L^2(V_s)} > \eta \iff s = s^j_n \text{ for some } j,
\]

and

\[
s^1_n < \cdots < s^m_n,
\]

we call \( \{s^j_n\}_{1 \leq j \leq m} \text{ chain decomposition of } \{A_n\} \subset M_8^{W[0, \infty]} \) for \( \eta \).

**Remark 5.5.** For any sequence \( \{A_n\} \) satisfying \( \sup_{n \in \mathbb{N}} \|F(A_n)\|^2_{L^2(W[0, \infty])} < \infty \) and any \( \eta > 0 \), we can show the existence of a chain decomposition for \( \eta > 0 \) if we take a subsequence of \( \{A_n\} \).

First we give the next technical lemma.
Lemma 5.6. Let $A$ be a $L^2_q$ ASD-connection on $W[k,\infty]$ satisfying

$$\frac{1}{8\pi^2}\|F(A)\|_{L^2(W[k,\infty])}^2 < 1.$$ 

Then there exists a positive number $c_4$ which depends only on the difference $1 - \frac{1}{8\pi^2}\|F(A)\|_{L^2(W[k,\infty])}^2$ such that the following statement holds.

Suppose there exists a gauge transformation $g$ on $Y^k$ satisfying

$$\sum_{0 \leq j \leq 1} \sup_{x \in Y^k_+} \left| \nabla_\theta (g^*(t^k_i)^* A - \theta)(x) \right|^2 \leq c_4.$$ 

Then $g$ is homotopic to the identity gauge transformation.

Proof. By the property of $c_{SY}$, we have

$$|\text{deg}(g)| = |\text{cs}_{SY}(g^*(t^k_i)^* A) - \text{cs}_{SY}((t^k_i)^* A)|$$

$$\leq |\text{cs}_{SY}(g^*(t^k_i)^* A)| + \|F(A)\|_{L^2(W[k,\infty])}^2$$

$$\leq \sum_{0 \leq j \leq 1} \sup_{x \in Y^k_+} \left| \nabla_\theta (g^*(t^k_i)^* A - \theta)(x) \right|^2 + \|F(A)\|_{L^2(W[k,\infty])}^2.$$ 

We define $c_4 < \frac{1}{8}(1 - \|F(A)\|_{L^2(W[k,\infty])}^2)$, then

$$|\text{deg}(g)| < 1$$

holds. This implies the conclusion.

The next proposition is a consequence of Lemma 5.3 and 5.6.

Proposition 5.7. Let $\eta$ be a positive number and $\{s^k_i\}_{1 \leq j \leq m}$ be a chain decomposition for $\eta$ of a sequences $\{A_n\}$ in $M^W_{[0,\infty]}$ with

$$\frac{1}{8\pi^2} \sup_{n \in N} \|F(A_n)\|_{L^2(W[0,\infty])}^2 < \min\{1, Q^{2y+3}\}.$$ 

Then there exists a subsequence $\{A_{n_0}\}$ of $\{A_n\}$ such that $\sup_{n \in N} \|s^m_{n_0}\| < \infty$ holds.

Proof. Suppose there exists $n_0 > 0$ which does not satisfy the condition of Proposition 5.7. There exist a chain decomposition $\{s^k_i(\eta_0)\}_{0 \leq j \leq m}$ of $\{A_n\}$ which satisfies $s^m_n \to \infty$ as $n \to \infty$. We take a sufficiently small $\eta > 0$, we will specify $\eta$ later. Choose a subsequence of $\{A_n\}$ which allows a chain decomposition for $\eta$. For simplify, we denote the subsequence by same notation $\{A_n\}$.

We denote the chain decomposition of $\{A_n\}$ for $\eta$ by $\{t^k_i(\eta)\}_{1 \leq j \leq m'}$. There are two cases for $\{t^k_i(\eta)\}_{1 \leq j \leq m'}$: 

- There exists $j'' \in \{0, \cdots, m'-1\}$ satisfying $t^k_{n''} - t^k_{n'' + 1} \to -\infty$.
- There is no $j'' \in \{0, \cdots, m'-1\}$ satisfying $t^k_{n''} - t^k_{n'' + 1} \to -\infty$.

We define a sequence by

$$u_n(\eta) := \left\lfloor \frac{t^k_{n''}(\eta) + t^k_{n'' + 1}(\eta)}{2} \right\rfloor \in N$$
for the first case and by
\[ u_n(\eta) := 0 \]
for the second case, where \([-\cdot]\) is the floor function. Applying Lemma 5.3 to \(A_n|_{W[u_n(\eta), u_n(\eta)+2]}\), we get the gauge transformation \(g_n\) on \(W[u_n(\eta) - l_Y - 2, u_n(\eta) + l_Y + 3]\) satisfying
\[
\sup_{x \in W[u_n(\eta), u_n(\eta)+2]} \sum_{1 \leq j \leq q+1} |\nabla_j^\theta(g_n^*A_n|_{W[u_n(\eta), u_n(\eta)+2]} - \theta)|^2 \leq c_3(2l_Y + 5)\eta,
\]
for small \(\eta\) and large \(n\). Because \(A_n\) is the ASD-connection for each \(n\), we have
\[
\frac{1}{8\pi^2} \int_{W[u_n, \infty]} Tr(F(A_n) \wedge F(A_n)) = \frac{1}{8\pi^2} \int_{W[u_n, \infty]} |F(A_n)|^2 > \eta_0
\]
for large \(n\). On the other hand, by the Stokes theorem
\[
\frac{1}{8\pi^2} \int_{W[u_n, \infty]} Tr(F(A_n) \wedge F(A_n)) = cs_Y(l_n^*A_n)
\]
holds. By Lemma 5.6, we have
\[
cs_Y(l_n^*A_n) = cs_Y(l_n^*g_n^*A_n)
\]
for small \(\eta\). Therefore (13) and Lemma 5.2 gives
\[
|cs_Y(l_n^*A_n)| \leq c'c_3(2l_Y + 5)\eta.
\]
We choose \(\eta\) satisfying
\[
c'c_3(2l_Y + 5)\eta < \frac{1}{8\pi^2}\eta_0.
\]
For such \(\eta\), we have
\[
|cs_Y(l_n^*A_n)| < \frac{1}{8\pi^2}\eta_0.
\]
On the other hand, \(\eta_0\) satisfies
\[
\frac{1}{8\pi^2}\eta_0 < \frac{1}{8\pi^2} \int_{W[u_n, \infty]} |F(A_n)|^2 = |cs_Y(l_n^*A_n)|
\]
which is a contradiction.

5.3 Exponential decay

In the instanton Floer theory, there is an estimate called exponential decay about the \(L^2\)-norm of curvature of the instanton over cylindrical end. We give a generalization of the exponential decay estimate over \(W[0, \infty]\). In the end of this subsection, we also give a proof of Theorem 5.1.

**Lemma 5.8.** There exists a constant \(c_5\) satisfying the following statement. For \(A \in M^n_{W[0, \infty]}\) satisfying \(\frac{1}{8\pi^2}||F(A)||^2 < \min\{1, Q_X^{2l_Y+3}\}\), there exists \(\eta_1 > 0\) which depends only on the difference \(\min\{Q_X^{2l_Y+3}, 1\} - \frac{1}{8\pi^2}||F(A)||^2\) such that the following condition holds.
Let $K > 0$ be a positive number satisfying $\|F(A)\|_{L^2(W_k)}^2 < \eta_1$ for any $k > K$, the inequality
\[
\|F(A)\|_{L^2(W[k,k+m])}^2 \leq c_3(\|F(A)\|_{L^2(W[k-k_l+1+\delta+3])}^2 + \|F(A)\|_{L^2(W[k+m-k_l+1+\delta+3])}^2)
\]
holds for $k > K + l_\gamma + 3$.

Proof. Let $\eta$ be the positive number in Lemma 5.3 which depends only the difference $\tilde{\eta}^{X,y+3} = \frac{1}{\delta} \|F(A)\|^2$. Then for $k > K + l_\gamma + 3$, we have the following inequalities
\[
\sup_{x \in W_k} \sum_{0 \leq j \leq q} \|\nabla^j g^A \theta \|_{L^2(W[k-k_l+1+\delta+3])}^2 \leq c_3(2l_\gamma + 5) \eta_1
\]
and
\[
\sup_{x \in W_{k+m}} \sum_{0 \leq j \leq q} \|\nabla^j g^{k+m} A \theta \|_{L^2(W[k+m-k_l+1+\delta+3])}^2 \leq c_3(2l_\gamma + 5) \eta_1.
\]
These inequality (15), (16) and Lemma 5.6 imply that for sufficiently small $\eta_1$, the gauge transformation $g_{k,\gamma+\delta} = g_k$ is homotopic to the constant gauge transformation. Hence, there exists a gauge transformation $\hat{g}$ on $W[k,k+m]$ satisfying $\hat{g}|W_k = g_k$ and $\hat{g}|W_{k+m} = g_{k+m}$, moreover, since $A$ is the ASD connection, we have
\[
\|F(A)\|^2 = \|F(\hat{g}^* A)\|_{L^2(W[k,k+m])}^2 = 8\pi^2 c_{s_Y}(l_{k+1}^* g_k A) - c_{s_Y}(l_{k+m}^* g_{k+m} A).
\]
Applying the inequalities (15) and (16) again, we get the conclusion. □

Proposition 5.9 (Exponential decay). There exists $\delta' > 0$ satisfying the following statement.
Suppose $A$ is an element in $M^{W[0,\infty]}_s$ satisfying the assumption of Lemma 5.8. Then there exists $c_5(K) > 0$ satisfying the following inequality.
\[
\|F(A)\|_{L^2(W[k-k_l+1+\delta+3])}^2 \leq c_5(K)e^{-k\delta'}.
\]
for $k > K + l_\gamma + 3$.

Proof. This is a consequence of Lemma 5.8 and Lemma 5.2 in [10] by applying $\eta_i = \|F(A)\|_{L^2(W[k-k_l+1+\delta+3])}^2$. □

By using a similar argument in Lemma 4.2 and Lemma 7.1 of [10] we have:

Lemma 5.10 (Patching argument). For a positive number $c_7$, there exists a constant $c_8$ satisfying the following statement holds.

\[
\|F(A)\|_{L^2(W[k-k_l+1+\delta+3])}^2 \leq c_8(K)e^{-k\delta'}.
\]
Suppose we have an $L^2_q$ connection $A$, the gauge transformations $g_k$ on $W[k−1, k + 1]$ satisfying
\[
\int_{W[k−1, k + 1]} \sum_{0 \leq j \leq q+1} |\nabla^j \theta (g_k^* A|_{W[k−1, k + 1]} - \theta)|^2 \\
\leq c_7 ||F(A)||^2 L^2(W[k−3, k + l+2]).
\]
for any non-negative integer $k$.

Then there exists the positive integer $n_0$ and a gauge transformation $g$ on $W[n_0, \infty]$ satisfying the following condition:
\[
\int_{W[k−1, k + 1]} \sum_{0 \leq j \leq q+1} |\nabla^j \theta (g^* A|_{W[k−1, k + 1]} - \theta)|^2 \\
\leq c_8 ||F(A)||^2 L^2(W[k−3, k + l+2])
\]
for $k > n_0$.

We use this lemma to prove the next proposition.

**Proposition 5.11.** There exists $\delta' > 0$ satisfying the following condition. Let $K > 0$ be a positive number and $\{A_n\}$ be a sequence in $M^W_{\delta}[0, \infty]$ satisfying the following properties:

1. $0 < \min\{Q^2 \lambda + 3, 1\} - \sup_{n \in \mathbb{N}} \frac{1}{8\pi^2} ||F(A_n)||^2$.
2. There exists a chain decomposition $\{s^n_j\}$ of $\{A_n\}$ for $\eta_2$ satisfying
   \[
   \sup_{n \in \mathbb{N}} |s^n_j(\eta_\ast)| < \infty
   \]
   for
   \[
   \eta_\ast := \inf_{n \in \mathbb{N}} \{\eta| \text{ constants which depend on } A_n \text{ in Lemma 5.3 and 5.8}\}.
   \]

Then there exist a positive integer $N_0$, gauge transformations $\{g_j\}$ on $W[N_0, \infty]$ and subsequence $\{A_{n_j}\}$ of $\{A_n\}$ such that $\{g_j^* A_{n_j}\}$ converge to some $A_\infty$ in $L^2_q(W[N_0, \infty])$ for any $0 \leq \delta < \delta'$.

**Proof.** If we apply the Lemma 5.3 to $A_n$, there exists gauge transformations $g_n^k$ on $W[k−1, k + 1]$ satisfying the following condition: for $k > \eta Y + K + 3$,
\[
\int_{W[k−1, k + 1]} \sum_{0 \leq j \leq q+1} |\nabla^j \theta (g_n^k A_n|_{W[k−1, k + 1]} - \theta)|^2 \\
\leq c_9 ||F(A_n)||^2 L^2(W[k−3, k + l+2]).
\]

On the other hand, we have
\[
||F(A_n)||^2 L^2(L^2(W[k−3, k + l+2])) \leq c_6(K)e^{-\delta'k}
\]
by using the exponential decay estimate(Proposition 5.9). Using (17), we can show that $n_0$ uniformly with respect to $n$ in Lemma 5.10. So there exist a
large natural number $N_0$ and a gauge transformation on $W[N_0, \infty]$ for each $n$ satisfying

$$
\int_{W[k-1,k+1]} \sum_{0 \leq j \leq q+1} |\nabla^j \theta(g_n^* A_n|_{W[k-1,k+1]} - \theta)|^2
\leq c_6\|F(A_n)|^2_{L^2(W[k-1-3,k+1+2])} \leq c_6(K)c_8 e^{-\delta' k},
$$

(18)

where the last inequality follows from (17).

We set $g_n^* A_n = \theta + a_n$. Then we have

$$
||a_n||_{L^2_q(W[N_0, \infty])} = \sum_{0 \leq j \leq q+1} \int_{W[N_0, \infty]} e^{\delta} |\nabla^j \theta(a_n)|^2.
$$

Putting this estimate and (18) together, we have

$$
||a_n||_{L^2_q(W[k, \infty])} \leq c_9 e^{(\delta - \delta') k}
$$

(19)

for $k > N_0$. We take a subsequence of $\{a_n\}$ which converges on any compact set in $L^2_q(W[k, \infty])$ by using the Rellich Lemma. We denote the limit in $L^2_q, \text{loc}$ by $a_\infty$. Then the exponential decay (18) and a standard argument implies that $\{a_n\}$ converges $a_\infty$ on $W[N_0, \infty]$ in $L^2_q, \delta$-norm.

We now give the proof of Theorem 5.1.

**Proof.** We choose $\eta_2$ in Proposition 5.11. After taking subsequence of $\{A_n\}$, we consider the chain decomposition $\{s^j_n\}_{1 \leq j \leq m}$ for $\eta_2$ of $\{A_n\}$. From Proposition 5.7, $\{s^j_n\}$ has upper bound by some $K > 0$ after taking a subsequence of $\{A_n\}$ again. So we can apply Proposition 5.11, we get the conclusion. \hfill \qed

### 6 Perturbation and Orientation

To prove the vanishing $[\theta^r] = 0$ in Theorem 2.4, we use the moduli spaces $M^W(a)_{\pi, \delta}$ and need the transversality for the equation $F^+(A) + s\pi(A) = 0$. We also need the orientability of $M^W(a)_{\pi, \delta}$.

#### 6.1 Holonomy perturbation(2)

In [2], Donaldson introduced the holonomy perturbation with compact support for irreducible ASD-connections. Combining the technique in [2] and the compactness theorem (Theorem 5.1), we get sufficient perturbations to achieve required transversality.

**Definition 6.1.** Let $\pi$ be an element in $\prod(Y)$ and $a$ be a critical point of $e_{SV, \pi}$. We use the following notations:

- $\Gamma(W) := \{l : S^1 \times D^3 \to W|l|: \text{orientation preserving embedding}\}$.

- $\Lambda^d(W) := \{ (l_1, \mu_1^+)_{1 \leq i \leq d} \in \Gamma(W)^d \times (\Omega^+(W) \otimes \mathfrak{su}(2))^d|\text{supp} \mu_1^+ \subset \text{Im} l_1\}$. 

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\[ \Lambda(W) := \bigcup_{d \in \mathbb{N}} \Lambda_d(W). \]

Let \( \chi : SU(2) \to \mathfrak{su}(2) \) be
\[ \chi(u) := u - \frac{1}{2} \text{tr}(u)\text{id} \]
and fix \( \mu^+_i \in \Omega^+(W) \otimes \mathfrak{su}(2) \) supported on \( l_i(S^1 \times D^3) \) for \( i \in \{1, \cdots, d\} \). For \( \epsilon \in \mathbb{R}^d \), we set
\[ \sigma_\Psi(A, \epsilon) := \sum_{1 \leq i \leq d} \epsilon_i \chi(\text{Hol}_{x \in l_i(S^1 \times D^3)}(A)) \mu^+_i, \]
where \( \text{Hol}_{x \in l_i(S^1 \times D^3)} \) is a holonomy around the loop \( t \mapsto l_i(t, y) \) satisfying \( x = l_i(t_2, y_2) \) for some \( t_2 \) and \( \epsilon = (\epsilon_i)_{1 \leq i \leq d} \). For \( \Psi = (l_i, \mu_i)_{1 \leq i \leq d} \in \Lambda \), Donaldson defined the holonomy perturbation of the ASD-equation:
\[ F_{\pi, \Psi}(A, \epsilon) := F^+(A) + s\pi(A) + \sigma_\Psi(A, \epsilon) = 0. \] (20)
The map \( \sigma_\Psi(-, \epsilon) \) is smoothly extended to the map \( A^W(\delta) \to \Omega^+(W) \otimes \mathfrak{su}(2)_{l_{q-1}, q} \) and the map \( A^W(\delta, \delta) \to \Omega^+(W) \otimes \mathfrak{su}(2)_{l_{q-1}, q} \). For \( \Psi \) and \( \epsilon \in \mathbb{R}^d \), the perturbed instanton moduli space are defined by
\[ M^W(a)_{\pi, \Psi, \delta} := \{ c \in B^W(\delta) : F_{\pi, \Psi}(c, \epsilon) = 0 \} \]
in the case of \( a \in \tilde{R}^+(Y)_{\pi} \) and
\[ M^W(a)_{\pi, \Psi, \rho, \delta} := \{ c \in B^W(\rho, \delta) : F_{\pi, \Psi}(c, \epsilon) = 0 \} \]
in the case of \( \text{Stab}(a) = SU(2) \). For a fixed \( \epsilon \in \mathbb{R}^d \), if the operator
\[ d(F_{\pi, \Psi})_{(A, 0)} : T_A A^W(\delta) \times \mathbb{R}^d \to \Omega^+(W) \otimes \mathfrak{su}(2)_{l_{q-1}, q}. \]
is surjective for all \( [A] \in M^W(a)_{\delta, \pi, \Psi, \epsilon} \), we call \( (\Psi, \epsilon) \) regular perturbation for \( a \in \tilde{R}^+(Y)_{\pi} \).

Let \( FM^W(a)_{\delta, \pi, \Psi} \) be the family version of the perturbed instanton moduli spaces defined by
\[ FM^W(a)_{\delta, \pi, \Psi} := \{ (c, \epsilon) \in B^W(\delta) \times \mathbb{R}^d : F_{\pi, \Psi}(c, \epsilon) = 0 \}. \]

**Lemma 6.2.** Suppose that \( Y \) satisfies Assumption 2.2. There exists \( \delta' > 0 \) such that for a fixed \( \delta \in (0, \delta') \), the following statement holds. Suppose \( \pi \) is a holonomy perturbation which is non-degenerate and regular. Let \( a \) be an irreducible critical point of \( cs_{Y, \pi} \) with \( cs_{Y, \pi}(a) < \min\{Q_X^{2r+3}, 1\} \). We assume the next three hypotheses for \( (\pi, \delta) \).

1. For \( [A] \in M^W(b)_{\pi, \delta}, \)
\[ \frac{1}{8\pi^2} \sup_{n \in \mathbb{N}} \| F(A) + s\pi(A) \|_{L^2(W)}^2 < \min\{1, Q_X^{2r+3}\}, \]
where \( b \) is an element of \( \tilde{R}(Y)_{\pi} \) with \( cs_{Y, \pi}(b) \leq cs_{Y, \pi}(a) \).
2. The linear operator

\[ d^+_A + s \pi^+ = T_\theta A^W(\theta,\delta) \times \mathbb{R}^d \to \Omega^+(W) \otimes \mathfrak{su}(2)_{L^{2,-1}(\delta,\delta)} \]

is surjective.

3. \( M(c)_{\pi,\delta} \) is empty set for \( c \in \tilde{R}_\pi(Y) \) satisfying \( cs_{Y,\pi}(c) < 0 \).

Then there exist a small number \( \eta > 0 \) and a perturbation \( \Psi \) such that the map

\[ dF_{\pi,\psi} : T A^W(\delta) \times \mathbb{R}^d \to \Omega^+(W) \otimes \mathfrak{su}(2)_{L^{2,-1}(\delta,\delta)} \]

is surjective for all point in \( F_{\pi,\psi}^{-1}(0) \cap (A^W(\delta) \times B^2(\eta)) \).

Proof. First we show that the surjectivity of \( dF_{\pi,\psi} \) at the point in \( F_{\pi,\psi}^{-1}(0) \cap (A^W(\delta) \times \{0\}) \). Second, we show that there exists a positive number \( \eta > 0 \) such that \( dF_{\pi,\psi} \) is surjectivitie at the point in \( F_{\pi,\psi}^{-1}(0) \cap (A^W(\delta) \times B^2(\eta)) \).

We name the critical point of \( cs_{Y,\pi} \) by

\[ 0 = cs_{Y,\pi}(\theta = a_0) \leq cs_{Y,\pi}(a_1) \leq cs_{Y,\pi}(a_2) \cdots \leq cs_{Y,\pi}(a_w = a) \]

The proof is induction on \( w \) and there are four steps.

**Step 1.** For an irreducible element \( A \in A(a_w) \) with \( 0 \neq \text{Coker}(dF_{\pi,\psi}(A,0)) \), there exists \( \Psi \) such that \( dF_{\pi,\psi}(A,0)|_{\mathbb{R}^{d(A)}} \) generates the space \( \text{Coker}(d_A^+ + s \pi_A^+) \).

The proof is essentially the same discussion of Lemma 2.5 in [2]. We fix \( h \in \Omega^+(W) \otimes \mathfrak{su}(2)_{L^{2,-1}(\delta,\delta)} \) satisfying \( 0 \neq h \in \text{Coker}(d_A^+ + s \pi_A) \) with \( ||h||_{L^2} = 1 \).

The unique continuation theorems: Proposition 8.6 (ii) of [12] for the equation \((d_A^+(-) + d \pi_A)\) satisfying \( 0 = 0 \) on \( Y \times (-\infty, -1] \) and Section 3 of [8] for the equation \((d_A^+)^{\ast}(-) = 0 \) on \( W[0, \infty] \) imply \( h|_{Y \times [-2,0]} \neq 0 \). Then we choose \( x_h \) in \( Y \times [-2,0] \) so that \( h(x_h) \neq 0 \) holds. Since \( A \) is the irreducible connection,

\[ \text{Hol}(A, x_h) = \{ \text{Hol}(A) \in SU(2) | t: \text{loop based at } x_h \} \]

is a dense subset of \( SU(2) \). So we can choose the loops \( (e_i \text{ } \chi(\text{Hol}_i(A))) \) for \( \mathfrak{su}(2) \).

For a small neighborhood \( U_{x_h} \) of \( x_h \), we can write \( h \) by

\[ h |_{U_{x_h}} = \sum_{i \leq 3} h_i \otimes e_i. \]

By using a smoothing of \( \delta \), function, we have

\[ \langle h |_{U_{x_h}}, \sum_{i \leq 3} \mu_i^+(h) \otimes (\chi(\text{Hol}_i(A)))|_{L^2(U_{x_h})} \rangle \neq 0, \quad (21) \]

where \( \mu_i^+(h) \) are three self dual 2-forms supported on \( U_{x_h} \). For a fixed generator \( \{ h^1, \ldots, h^w \} \) of \( \text{Coker}(d_A^+ + s \pi_A) \), we get the points \( \{ x_{h^1} \} \subset Y \times [-2,0] \cup Y \) \( W_0 \), small neighborhoods \( \{ U_{x_{h^1}} \} \), loops \( \{ h^1 \} \) and self dual 2-forms \( \{ \mu_i^+(h^1) \} \)
satisfying (21) for all \( h^j \). We extend the maps \( l^i_{h^j} : S^1 \rightarrow W \) to embeddings \( S^1 \times D^3 \rightarrow W \). We can choose \( U_{h^j} \) satisfying \( U_{h^j} \subset \text{Im} \ l^i_{h^j} \). We set

\[
\Psi(A) := (l^i_{h^j}, \rho^+_i(h^j))_{i,j} \in \Lambda(W),
\]

which satisfies the statement of Step 1.

For \( j \geq 0 \) satisfying \( cs_{Y, \pi}(w_j) = 0 \), we show:

**Step 2.** For an element \( b \in \bar{R}^+(Y)_{\pi}(\text{resp. } b = \theta) \) satisfying \( cs_{Y, \pi}(b) = 0 \), there exists a perturbation \( \Psi^b \) such that the operator

\[
dF_{\pi, \Psi^b}|(A,0) : TA\mathcal{A}^W(b)_{\delta} \times \mathbb{R}^d \rightarrow \Omega^+(W) \otimes \mathfrak{su}(2)|_{L^2_{q-1,\delta}}
\]

and \( \text{resp. } dF_{\pi, \Psi}|(A,0) : TA\mathcal{A}^W(b)_{(\delta, \delta)} \times \mathbb{R}^d \rightarrow \Omega^+(W) \otimes \mathfrak{su}(2)|_{L^2_{q-1,(\delta, \delta)}} \)

is surjective for \( A \in (F_{\pi, \Psi})^{-1}(0) \cap (\mathcal{A}W(a)_{\delta} \times \{0\}) \) (resp. \( A \in (F_{\pi, \Psi})^{-1}(0) \cap (\mathcal{A}W(a)_{(\delta, \delta)} \times \{0\}) \)).

First we show that \( M^W(b)_{\pi, \delta} \) is compact. Let \( \{[A_n]\} \) be any sequence in \( M^W(b)_{\pi, \delta} \). By the second hypothesis, we have

\[
\frac{1}{8\pi^2} \sup_{n \in \mathbb{N}} \|F(A_n)\|_{L^2(W[0,\infty])}^2 \leq \frac{1}{8\pi^2} \sup_{n \in \mathbb{N}} \|F(A_n) + s\pi(A_n)\|_{L^2(W)}^2 < \min\{1, QX^{\delta+3}\}.
\]

By Theorem 5.1, there exist a large positive number \( N \) and the gauge transformations \( \{g_n\} \) over \( W[N, \infty] \) such that \( \{g_n^*A_n\} \) converges over \( W[N, \infty] \) for small \( \delta \) after taking a subsequence. Note that \( Y \times (-\infty, 0] \cup Y W[0, N+1] \) is a cylindrical end manifold and we can apply the general theory developed on Section 5 of [3]. In particular, there exist gauge transformations \( \{h_n\} \) on \( Y \times (-\infty, 0] \cup Y W[0, N+1] \) such that \( \{h_n^*A_n|_{Y \times (-\infty, 0] \cup Y W[0, N+1]}\} \) has a chain convergent subsequence in the sense in Section 5 in [3] because the bubble phenomenon does occur under the first hypothesis

\[
\frac{1}{8\pi^2} \sup_{n \in \mathbb{N}} \|F(A_n) + s\pi(A_n)\|_{L^2(W)}^2 < 1.
\]

By gluing \( \{g_n\} \) and \( \{h_n\} \), we obtain a chain convergent subsequence

\[
[A_n] \rightarrow ([C^1], \ldots, [C^N], [A^0]) \in M(b = c_1, c_2)_{\pi} \times \cdots \times M(c_v, c_{v+1})_{\pi} \times M^W(c_{v+1})_{\pi, \delta}
\]

with \( c_i \in \bar{R}(Y)_{\pi} \). Suppose that \( [A_n] \rightarrow [A^0] \in M(b)_{\pi, \delta} \) does not hold. We get \( cs_{Y, \pi}(c_{v+1}) < 0 \) because the moduli spaces

\[
M(b, c_1)_{\pi} \times \cdots \times M(c_v, c_{v+1})_{\pi}
\]

are non-empty sets. However this contradicts to the assumption of \( M(c)_{\pi, \delta} = \emptyset \) for \( c \in \bar{R}(Y)_{\pi} \) with \( cs_{Y, \pi}(c) < 0 \).

When \( b \) is an irreducible connection, the compactness of \( M(b)_{\pi, \delta} \), Step 1 and the openness of surjective operators imply Step 2. When \( b \) is equal to \( \theta \), the second hypothesis implies Step 2.

For the inductive step, we show:
Step 3. Suppose there is a perturbation
\[ \Psi^{w-1} = (l_i^{w-1}, \mu_i^{w-1}), i \in \Lambda(W) \]
such that the operators
\[ dF_{\pi, \Psi^{w-1}|(A, 0)} : TA^W(a_j)_{\delta} \to \Omega^+(W)_{L^2_{-1, \delta}} \]
is surjective for \((A, 0) \in (F_{\pi, \Psi^{w-1}})^{-1}(A^W(a_j) \times \{0\}) \) and \(j \in \{1, \cdots, w-1\}\). Then the space
\[ K_w := \left\{ A \in M^W(a_w)_{\pi, \delta} \big| 0 \neq \text{Coker}(dF_{\pi, \Psi^{w-1}|(A, 0)}) \subset \Omega^+(W)_{L^2_{-1, \delta}} \right\} \]
is compact.

Let \(\{[A_n]\}\) be a sequence in \(K_w\). By the similar estimate in Step 2 and Theorem 5.1, we get a chain convergent subsequence
\[ [A_n] \to ([B^1], \cdots, [B^{N}], [A^0]) \in M(a_w = b_1, b_2)_{\pi} \times \cdots \times M(b_v, b_{v+1})_{\pi} \times M^W(b_{v+1})_{\pi, \delta} \]
with \(b_i \in R(Y)\). Suppose that \([A_n] \to [A^0] \in M(a_w)_{\pi, \delta}\) does not hold. In this case, the operators \(d^+_B + d\pi_B\) on \(Y \times \mathbb{R}\) and the operators \(dF_{\pi, \Psi^{w-1}|(A^0, 0)}\) on \(W\) are surjective in the suitable functional spaces by the assumption of \(\pi\) and the induction. For large \(j\), the operator \(dF_{\pi, \Psi^{w-1}|(A^j, 0)}\) can be approximated by the gluing of the operators \(d^+_B + d\pi_B, dF_{\pi, \Psi^{w-1}|(A^j, 0)}\). By gluing the right inverses of them as in Theorem 7.7 of [12], \(dF_{\pi, \Psi^{w-1}|(A^j, 0)}\) also has a right inverse for sufficiently large \(j\). This is a contradiction and we have the conclusion of Step 3.

For induction, we need to show:

Step 4. There exists the perturbation \(\Psi^w\) satisfying the surjectivity of the operator
\[ dF_{\pi, \Psi^w} : A^W(a_w)_{\delta} \times \mathbb{R}^d \to \Omega^+(W) \otimes \text{su}(2)_{L^2_{-1, \delta}} \]
for any point in \((F_{\pi, \Psi^w})^{-1}(0) \cap (A(a_w)_{\delta} \times \{0\})\).

We take the perturbation \(\Psi_A = ((l_A^j), (\mu_A^j(A)))\) for each \(A \in K_w\) in Step 1. Because \(K_w\) is compact and surjectivity of the operators is open condition, there exist \(\{A_1, \cdots, A_k\} \subset K_w\) and a perturbation \(\Psi^w\) such that
\[ dF_{\pi, \Psi^w| (A, 0)} : A^W(a_w)_{\delta} \times \mathbb{R}^d \to \Omega^+(W) \otimes \text{su}(2)_{L^2_{-1, \delta}} \]
is surjective for all \((A, 0) \in (F_{\pi, \Psi^w})^{-1}(0) \cap (A^W(a)_{\delta} \times \{0\})\). Here \(\Psi^w\) is defined by
\[ \Psi^w := ((l^w_A, \cdots, l^{w-1}_A), (\mu_A^1(A), \cdots, \mu_A^w(A))) \]
which satisfies the property in Step 4.

Second, we show that the operator \(dF_{\pi, \Psi^w}\) is surjective for any point in \((F_{\pi, \Psi^w})^{-1}(0) \cap (A^W(a)_{\delta} \times D^d(\eta))\). Suppose there is no \(\eta\) such that the statement holds. Then there is a sequence \(\{([A, \epsilon_n])\} \in M^W(a)_{\delta, \pi, \Psi, \epsilon_n}\) which satisfies that \(\epsilon_n \to 0\) as \(n \to \infty\) and \(dF_{\pi, \Psi^w| ([A, \epsilon_n])}\) is not surjective for all \(n \in \mathbb{N}\). Because the bubble does occur, \(\{[A_n]\}\) has a chain convergent subsequence to
\[ ([B^1], \cdots, [B^{N}], [A^0]) \in M(b_0, b_1)_{\pi} \times \cdots \times M(b_v, b_{v+1})_{\pi} \times M^W(b_{v+1})_{\delta, \pi, \Psi, 0} \]
for some \( b_i \in \tilde{R}(Y) \). Since \( \pi \) is a regular perturbation and \( dF_{\pi, \Psi} |_{(A^0, 0)} \) is surjective, there exist the right inverses of \( d^+_{B_1}, d^+_{B_2}, \ldots, d^+_{B_N} \) and \( dF_{\pi, \Psi} |_{(A^0, 0)} \) for suitable functional spaces. By the gluing of the right inverses as in Step 3, \( dF_{\pi, \Psi} |_{(A_N, \epsilon N)} \) also has the right inverse for large \( N \). This is a contradiction and this completes the proof.

**Theorem 6.3.** For a given data \((\delta, \pi, a)\) in Lemma 6.2, there exist \( \eta > 0 \), a perturbation \( \Psi \) and a dense subset of \( R \subset B^d(\eta) \subset \mathbb{R}^d \) such that \((\Psi, b)\) is a regular perturbation for \( b \in R \).

**Proof.** This is a conclusion of Lemma 6.2, the argument in Section 3 of [8] and the Sard-Smale theorem.

Applying the implicit function theorem, we get a structure of manifold of \( M^W(a)_{\delta, \pi, \Psi, b} \). Its dimension coincides with the Floer index \( \text{ind}(a) \) of \( a \) by Proposition 3.7. Therefore we have:

**Corollary 6.4.** For given data \((\delta, \pi, a)\) in Lemma 6.2, there exist \( \eta > 0 \), a perturbation \( \Psi \) and a dense subset of \( R \subset B^d(\eta) \subset \mathbb{R}^d \) such that \( M^W(a)_{\delta, \pi, \Psi, b} \) has a structure of manifold of dimension \( \text{ind}(a) \).

### 6.2 Orientation

In [2], Donaldson showed the orientability of the instanton moduli spaces for closed oriented 4-manifolds. In this subsection, we deal with the case for non-compact 4-manifold \( W \) by generalizing Donaldson’s argument. More explicitly, we show that the moduli space \( M^W(a)_{\delta, \pi} \) is orientable. We also follow Fredholm and moduli theory in [13] to formulate the configuration space for \( SU(l) \)-connections for \( l \geq 2 \).

Let \( \tilde{Z} \) be a compact oriented 4-manifold which satisfies \( \partial \tilde{Z} = Y \) and \( H_1(Z) \cong \mathbb{Z} \). We set \( Z^+ := (-Z) \cup_Y W [0, \infty) \) and \( \tilde{Z} := (-Z) \cup_Y Y \times [0, \infty) \). Fix a Riemannian metric \( g_{Z^+} \) on \( Z^+ \) with \( g_{Z^+}|_{W[0, \infty)} = g_W|_{W[0, \infty)} \) and Riemannian metric \( g_{\tilde{Z}} \) with \( g_{\tilde{Z}}|_{Y \times [0, \infty)} = g_Y \times g_{\tilde{Z}}^{\text{tan}} \). First, we introduce the configuration spaces for \( SU(l) \)-connections on \( W \) and \( Z^+ \) for \( l \geq 2 \) and \( SU(2) \)-configuration space for \( \tilde{Z} \).

**Definition 6.5.** Fix a positive integer \( q \geq 3 \). For an irreducible \( SU(2) \)-connection \( a \) on \( Y \), we define

\[
A^W(a)_{(\delta, \pi, l)} := \left\{ A_a + c \big| c \in \Omega^1(W) \otimes su(l)_{L^2_{(\delta, \pi)}} \right\},
\]

\[
A^Z_{\pi, l} := \left\{ \theta + c \big| c \in \Omega^1(Z^+) \otimes su(l)_{L^2_{\pi, l}} \right\},
\]

and

\[
A^{\tilde{Z}}(a) := \left\{ B_a + c \big| c \in \Omega^1(\tilde{Z}) \otimes su(2)_{L^2_{\tilde{Z}}} \right\},
\]

where

- \( A_a \) is an \( SU(l) \)-connection on \( W \) with \( A_a|_{Y \times (-\infty, -1]} = \text{pr}^*(a \oplus \theta) \), \( A_a|_{W[0, \infty]} = \theta \)
- \( B_a \) is an \( SU(2) \)-connection on \( \tilde{Z} \) with \( B_a|_{Y \times [0, \infty)} = \text{pr}^*a \).

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• $L^2_{q,(\delta,\lambda)}(W)$-norm is defined by

$$||f||^2_{L^2_{q,(\delta,\lambda)}(W)} := \sum_{0 \leq i \leq q} \int_W e^{\tau^\prime | \nabla_{A_g} f |^2} dvol,$$

where $\tau'$ is defined in Definition 3.4 and $f$ is an element in $\Omega^1(W) \otimes \mathfrak{su}(2)$ with compact support.

• $L^2_{q,\lambda}(Z^+)$-norm is defined by

$$||f||^2_{L^2_{q,\lambda}(Z^+)} := \sum_{0 \leq i \leq q} \int_{Z^+} e^{\tau'' | \nabla_{g_{\lambda}} f |^2} dvol,$$

where $\tau'': Z^+ \to [0,1]$ is a smooth function satisfying $\tau''|_{W[0,\infty]} = \tau$ defined in Definition 3.4 and $f$ is an element in $\Omega^1(Z^+) \otimes \mathfrak{su}(2)$ with compact support.

• $L^2_{q}(\hat{Z})$-norm is defined by

$$||f||^2_{L^2_{q}(\hat{Z})} := \sum_{0 \leq i \leq q} \int_{\hat{Z}} | \nabla_{B_{\lambda}} f |^2 dvol,$$

where $f$ is an element in $\Omega^1(\hat{Z}) \otimes \mathfrak{su}(2)$ with compact support.

We also define the $SU(l)$ configuration spaces $B^{W}(a)_{\delta,\lambda}, B^{Z^+}_{\lambda}$ and $SU(2)$-configuration spaces $B^{\hat{Z}}(a)$ by

$$B^{W}(a)_{\delta,\lambda} := A^{W}(a)_{\delta,\lambda} / G^{W}(a), B^{Z^+}_{\lambda} := A^{Z^+}_{\lambda} / G^{Z^+}_{\lambda}$$

and

$$B^{\hat{Z}}(a) := A^{\hat{Z}}(a) / G^{\hat{Z}}(a),$$

where $G^{W}(a), G^{Z^+}$ and $G^{\hat{Z}}(a)$ are given by

$$G^{W}(a) := \left \{ g \in \text{Aut}(W \times SU(l)) \subset \text{End}(\mathbb{C}^l)_{L^2_{q+1,\text{loc}}} \mid \nabla_{A_g} (g) \in L^2_{q,(\delta,\lambda)}(W) \right \},$$

$$G^{Z^+}_{\lambda} := \left \{ g \in \text{Aut}(Z^+ \times SU(l)) \subset \text{End}(\mathbb{C}^l)_{L^2_{q+1,\text{loc}}} \mid d(g) \in L^2_{q,\lambda}(Z^+) \right \}$$

and

$$G^{\hat{Z}}(a) := \left \{ g \in \text{Aut}(\hat{Z} \times SU(l)) \subset \text{End}(\mathbb{C}^2)_{L^2_{q+1,\text{loc}}} \mid \nabla_{B_{\lambda}} (g) \in L^2_{q,\delta}(\hat{Z}) \right \}.$$
Let $W$ be a fibration since (22) has a local slice due to Fredholm and moduli theory in $H_\pi$ for $0 < \delta < \delta'$ such that for a positive real number $\delta$ less than $\delta'$ the following properties hold.

- $\mathcal{B}^{W}(a)_{(\delta,\delta),l}$ is simply connected.

- $\mathcal{B}_{\delta,l}^{Z^+}$ is simply connected.

**Proof.** We will show only the first property. The second one is shown in a similar way to the first case. We use the condition $H_1(Z,Z) \cong 0$ for the second property.

Since $\pi_i(SU(l)) = 0$ for $i = 0,1,$

$$\pi_1(\mathcal{B}^{W}(a)_{(\delta,\delta),l}) \text{ is isomorphic to } \pi_1(\mathcal{B}^{W,fr}(a)_{(\delta,\delta),l}).$$

Therefore, we will show $\pi_1(\mathcal{B}^{W,fr}(a)_{(\delta,\delta),l}) = 0.$ There exists $\delta' > 0$ such that for $0 < \delta < \delta'$,

$$\hat{\mathcal{G}}^{W}(a)_{(\delta,\delta)} \rightarrow A^{W}(a)_{(\delta,\delta),l} \rightarrow \mathcal{B}^{W,fr}(a)_{(\delta,\delta),l} \quad (22)$$

is a fibration since (22) has a local slice due to Fredholm and moduli theory in [13]. Let $W^*$ be the one point compactification of $W.$ Using (22), we obtain

$$\pi_1(\mathcal{B}^{W,fr}(a)_{(\delta,\delta),l}) \cong \pi_0(\hat{\mathcal{G}}^{W}(a)_{(\delta,\delta)}) \cong [W^*,SU(l)].$$

Since $\pi_i(SU(l))$ vanishes for $i = 0,1,2,4,$ the obstruction for an element of $[W^*,SU(l)]$ to be homotopic to the constant map lives in $H^3(W^*,\pi_3(SU(l))) \cong H_{\text{comp}}^3(W,\pi_3(SU(l))) \cong H_1(W,\pi_3(SU(l))) = 0$ where the second isomorphism is the Poincaré duality.

This implies

$$\pi_1(\mathcal{B}^{W,fr}(a)_{(\delta,\delta),l}) \cong 0.$$

□
We now define the determinant line bundles. For simplify, we impose Assumption 2.2 on \( Y \).

**Definition 6.7.** Let \( \pi \) be an element in \( \prod(Y)^{\text{flat}} \) and \((\Psi, \epsilon)\) be a perturbation in Subsection 6.1 and fix an element \( a \in \hat{R}(Y) \). For \( c \in \mathcal{B}^W(a)(\delta, \delta, l) \), we have the following bounded operator

\[
d(\mathcal{F}_{\pi, \Psi})_c + d_c^{s+1(\delta, \delta)} : \Omega^1(Z^+) \otimes \mathfrak{su}(l)_{L^2_q(\delta, \delta)} \to \Omega^0(Z^+) \otimes \mathfrak{su}(l) \oplus \Omega^0(Z^+) \otimes \mathfrak{su}(l))_{L^2_q-1, \delta}.
\]

The operators \( d(\mathcal{F}_{\pi, \Psi})_c + d_c^{s+1(\delta, \delta)} \) are the Fredholm operators for small \( \delta \). Fix such a \( \delta \). We set

\[
\lambda^W(a, l, c) := \Lambda^{\text{max}} \ker(d(\mathcal{F}_{\pi, \Psi})_c) \otimes \Lambda^{\text{max}} \text{Coker}(d(\mathcal{F}_{\pi, \Psi})_c)^*.
\]

The determinant line bundles are defined by

\[
\lambda^W(a, l) := \bigcup_{c \in \mathcal{B}^W(a)(\delta, \delta, l)} \lambda(a, l, c) \to \mathcal{B}^W(a)(\delta, \delta, l)
\]

and

\[
\hat{\lambda}^W(a, l) := \bigcup_{c \in \mathcal{B}^W(a)(\delta, \delta, l)} \lambda(a, l, c) \to \hat{B}^W(a)(\delta, \delta, l).
\]

We also define

\[
\lambda^{Z^+}(l) \to \hat{B}^{\mathbb{Z}^+}_{\delta, l} \quad \text{and} \quad \hat{\lambda}^Z(a) \to \hat{B}^Z(a)
\]

in a similar way with respect to the operators

\[
d_c^+ + d_c^{\mathbb{Z}^+} : \Omega^1(Z^+) \otimes \mathfrak{su}(l)_{L^2_q(\delta, \delta)} \to \Omega^0(Z^+) \otimes \mathfrak{su}(l) \oplus \Omega^0(Z^+) \otimes \mathfrak{su}(l))_{L^2_q-1, \delta}
\]

for \( c \in \mathcal{B}^{\mathbb{Z}^+}_{\delta, l} \) and

\[
d_c^+ + d_c^{\mathbb{Z}^+} : \Omega^1(Z) \otimes \mathfrak{su}(2)_{L^2_q(\delta, \delta)} \to \Omega^0(Z) \otimes \mathfrak{su}(2) \oplus \Omega^0(Z) \otimes \mathfrak{su}(2))_{L^2_q-1, \delta}
\]

for \( c \in \mathcal{B}^{\mathbb{Z}^+}(a) \).

**Lemma 6.8.** For a given data \((a, \delta, l)\) in Proposition 6.6, the bundles \( \lambda^{Z^+}(a, l) \to \mathcal{B}^{\mathbb{Z}^+}_{\delta, l} \) and \( \lambda^W(a, l) \to \mathcal{B}^W(a)(\delta, \delta, l) \) are trivial.

**Proof.** Since the determinant line bundle is a real line bundle, the triviality of \( \lambda^{Z^+}(a, l) \to \mathcal{B}^{\mathbb{Z}^+}_{\delta, l} \) is a consequence of Proposition 6.6. Therefore we show the triviality of \( \lambda^W(a, l) \to \mathcal{B}^W(a)(\delta, \delta, l) \). We have a fibration

\[
\text{Stab}(a \oplus \theta) \to \hat{B}^W(a)(\delta, \delta, l) \to \mathcal{B}^W(a)(\delta, \delta, l).
\]

We also have an isomorphism \( j^{*} \lambda^W(a, l) \cong \hat{\lambda}^W(a, l) \) for \( j \) in (23). \( \hat{\lambda}^W(a, l) \) is the trivial bundle for \( l \geq 2 \) from Proposition 6.6. So if the fiber \( \text{Stab}(a \oplus \theta) \) of (23) is connected, \( \lambda^W(a, l) \) is also trivial. The possibilities of \( \text{Stab}(a \oplus \theta) \) are 

\[
SU(l), U(1) \times U(l - 1), S(U(2) \times U(l - 2)) \text{ and }
\]

\[
\{(z, A) \in U(1) \times U(l - 2) | z^2 \det A = 1 \}.
\]

Since these groups are connected, \( \lambda^W(a, l) \) is the trivial bundle. 

\[ \square \]
Lemma 6.9. Suppose that $Y$ satisfies Assumption 2.2 and $a$ is an element in $\hat{R}^* (Y)$. Let $i_1 : B^W (a)_{(\delta, \lambda)} \to B^W (a)_{(\delta, \lambda)}$ and $i_2 : B^W_{Z^+} \to B^W_{Z^+}$ be the maps induced by the product with the product connection. There exists a positive number $\delta'$ such that for a positive real number $\delta$ less than $\delta'$, $i^*_2 \lambda^W (a, 3) \cong \lambda^W (a, 2)$ and $i^*_2 \lambda^W (a, 3) \cong \lambda^W (a, 2)$ hold.

Proof. Under Assumption 2.2 on $Y$, the isomorphism class of these line bundles are independent of the choices of the perturbations $\pi$ and $(\Psi, \epsilon)$ by considering a 1-parameter family of perturbations $\pi_t := (f, t h)$ and $(\Psi, t e)$ for $t \in [0, 1]$. So Lemma (5.4.4) in [4] implies the conclusion.

Theorem 6.10. Suppose that $Y$ satisfies Assumption 2.2 and $a$ is an element in $\hat{R}^* (Y)$. Let $\pi$ be an element in $\prod (Y)^{\text{flat}}$ and $(\Psi, \epsilon)$ be a regular perturbation for $a \in \hat{R}^* (Y)$. For sufficiently small $\delta$, $M^W (a)_{\pi, \Psi, \epsilon, \delta}$ is orientable. Furthermore the orientation of $M^W (a)_{\pi, \Psi, \epsilon, \delta}$ is induced by the orientation of $\lambda^W (a, 2)$.

Proof. Using the exponential decay estimate in Proposition 4.3 of [3], we have a inclusion $i : M^W (a)_{\pi, \Psi, \epsilon, \delta} \to B^W (a)_{(\delta, \lambda)}$ for small $\delta$ as a set. From this inclusion $i$, we regard $M^W (a)_{\pi, \Psi, \epsilon, \delta}$ as a subset in $B^W (a)_{(\delta, \lambda)}$. Applying result of convergence in Corollary 5.2 of [3], we can show that the topology of $M^W (a)_{\pi, \Psi, \epsilon, \delta}$ in $B^W (a)_{\delta}$ coincides with the topology of $M^W (a)_{\pi, \Psi, \epsilon, \delta}$ in $B^W (a)_{(\delta, \lambda)}$. Also using exponential decay estimate for solutions to the linearized equation in Lemma 3.3 of [3], $\lambda^W (a, 2)|_{M^W (a)_{\pi, \Psi, \epsilon, \delta}} \to M^W (a)_{\pi, \Psi, \epsilon, \delta}$ is canonically isomorphic to $\Lambda^\text{max} M^W (a)_{\pi, \Psi, \epsilon, \delta}$.

From Theorem 6.10, an orientation of $M^W (a)_{\pi, \Psi, \epsilon, \delta}$ is characterized by the trivialization of $\lambda^W (a, 2)$. On the other hand, to formulate the instanton Floer homology of $Y$ with $\mathbb{Z}$ coefficient, Donaldson introduced the line bundle $\lambda (a) := \lambda^W (a) \otimes \lambda^W (\tilde{Z}) \to B^W (\tilde{Z}) (a)$, where $\lambda (\tilde{Z})$ is given by

$$\Lambda^\text{max} (H^0_{DR} (-\tilde{Z}) \oplus H^1_{DR} (-\tilde{Z}) \oplus H^1_{DR} (-\tilde{Z}))$$

in Subsection 5.4 of [3]. The orientation of $\lambda (a)$ is essentially independent of the choice of $Z$.

Definition 6.11. We set

$$\lambda^W := \Lambda^\text{max} (H^0_{DR} (W) \oplus H^1_{DR} (W) \oplus H^1_{DR} (W)),$$

and $\lambda^W (a) := \lambda^W (a, 2) \otimes \lambda^W \to B^W (a)_{(\delta, \lambda)}, 2$.

Lemma 6.12. Suppose that $Y$ satisfies Assumption 2.2. For an irreducible flat connection $a$, there is a canonical identification between the orientations of $\lambda^W (a)$ and the orientations of $\lambda (a)$.

Proof. It suffices to construct an isomorphism $\lambda^W (a) \cong \lambda (a)$ which is canonical up to homotopy. First we fix two elements $[A] \in B^W (a)_{(\delta, \lambda)}$ and $[B] \in B^Z (a)$ which have representative $A$ and $B$ satisfying $A_{Y \times (-\infty, -1]} = pr^* a$ and $B_{Y \times [1, \infty)} = pr^* a$. For such two connections, we obtain an element $A \# B \in B_{Z^+}^Z$ by gluing of connections. This map induces an isomorphism

$$
\# : \det (d_A^* + d_A^*) \otimes \det (d_B^* + d_B^*) \to \det (d_{A \# B}^* + d_{A \# B}^*),
$$

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from the similar argument of Proposition 3.9 in [3]. Therefore we have identification
\[ \# : \lambda^W(a)|_{[A]} \otimes \lambda^Z(a)|_{[B]} \rightarrow \lambda^W(a)|_{[A\#B]} \cdot \]
If we choose a path from \([\theta]\) to \([A\#B]\) in \(B^\mathbb{Z}_2\), then we have an identification between \(\lambda^W(a)|_{[\theta]}\) and \(\lambda^W(a)|_{[A\#B]}\). The line bundle \(\lambda^W(a)|_{[\theta]}\) is naturally isomorphic to
\[ A^{\text{max}}(H^0_{DR}(Z^+) \oplus H^1_{DR}(Z^+) \oplus H^1_{DR}(Z^+)) \]
by using Proposition 5.1 in [13]. This cohomology group is isomorphic to
\[ A^{\text{max}}(H^0_{DR}(Z) \oplus H^1_{DR}(Z) \oplus H^+_1(Z)) \oplus A^{\text{max}}(H^0_{DR}(W) \oplus H^+_1(W)) \]
by using the Mayer-Vietoris sequence.
Therefore
\[ \lambda^W(a)|_{[A]} \otimes \lambda^W(a)|_{[B]} \cong (\lambda^{(-Z)}(a)|_{[B]} \otimes \lambda^{(-Z)}(a))^{*} \]
holds. Because \(\lambda^{Z^+}(a,2) \rightarrow B^\mathbb{Z}_2\) is orientable by Lemma 6.9 and Proposition 6.6, the homotopy class of this identification does not depend on choices of the path, \(A, B\), the bump functions of the gluing map. We also have the following canonical isomorphism
\[ (\lambda^{(-Z)}(a)|_{[B]} \otimes \lambda^{(-Z)}(a))^{*} \cong \lambda^{Z^+}(a)|_{[B]} \otimes \lambda^{Z^+} \]
by the gluing \(Z\) and \(-Z\) as above discussion and the Mayer-Vietoris sequence. This completes the proof.

Combining Theorem 6.10 and Lemma 6.12, we have:

**Theorem 6.13.** Under the assumption of Theorem 6.10, an orientation \(\lambda(a)\) and an orientation of \(\lambda_W\) give an orientation of \(M^W(a)_{\pi,\Psi,\epsilon,\delta}\).

### 7 Proof of main theorem

**Proof.** Let \(Y\), \(l_Y\) and \(X\) be as in Section 2. Take a Riemannian metric \(g_Y\) on \(Y\). Fix a non-negative real number \(r \in \Lambda_Y\) smaller than \(Q_X^{2r+3}\). Suppose that there is an embedding \(f\) of \(Y\) into \(X\) satisfying \(f_*[Y] = 1 \in H_3(X,\mathbb{Z})\). Then we obtain the oriented homology cobordism from \(Y\) to \(-Y\) by cutting open along \(Y\). Recall that \(W\) is a non-compact oriented Riemann 4-manifold \(W\) with both of cylindrical end and periodic end which is formulated at the beginning of Subsection 3.5.

We fix a holonomy perturbation \(\pi \in \prod(Y)\) satisfying the following conditions.

1. \(\pi\) is an \(\epsilon\)-perturbation in Subsection 3.2.
2. \(\pi\) is a regular perturbation in the end of Subsection 3.1.
3. \(\pi\) is an element of \(\prod(Y)^{\text{flat}}\) in Definition 3.1.
4. For \(a \in \tilde{R}(Y)\) with \(0 \leq c_X(a) < \min\{1, Q_X^{2r+3}\}\) and \(A \in M^W(a)_{\pi,\delta},\)
\[ \frac{1}{8\pi^2} \sup_{n \in \mathbb{N}} \|F(A) + s\pi(A)\|_{L^2(W)}^2 < \min\{1, Q_X^{2r+3}\} \]
holds.
5. \( d^\pi_\theta + s d\pi : \mathcal{A}^W(\theta)_{(\delta, \delta)} \times \mathbb{R}^d \to \Omega^+(W) \otimes \mathfrak{su}(2)_{L^2_{\eta^{-1}}(\delta, \delta)} \) is surjective.

6. For \( c \in \tilde{R}(Y)_\pi \) satisfying \( \text{csy}(c) < 0 \), \( M^W(c)_{\pi, \delta} \) is the empty set.

Assumption 2.2 and the proof of Theorem 8.4 (ii) of [12] implies the existence of the perturbation satisfying the third condition. The first, forth, fifth and sixth conditions follow from choosing small \( h \in C^\infty(SU(2)^d, \mathbb{R})_{ad} \) of \( \pi = (f, h) \).

Next we also fix a holonomy perturbation \( (\Psi, \epsilon) \) satisfying the following conditions.

1. \( (\Psi, \epsilon) \) is a regular for \( [b] \in \tilde{R}(Y) \) with \( 0 \leq \text{csy}(b) \leq \text{csy}(a) \).

2. For \( a \in \tilde{R}(Y) \) with \( 0 \leq \text{csy}(a) < \min\{1, Q^2_{X^{l+3}}\} \),

\[
\frac{1}{8\pi^2} \sup_{n \in \mathbb{N}} \| F(A) + s\pi(A) + \sigma_\Psi(A, \epsilon) \|^2_{L^2(W)} < \min\{1, Q^2_{X^{l+3}}\}
\]

hold.

To get the first condition, we use Lemma 6.2. The second condition satisfied when we take \( \epsilon \) sufficiently small.

In order to formulate the instanton Floer homology of \( Y \) with \( \mathbb{Z} \) coefficient, we fix an orientation of \( \lambda(a) \) for each \( a \in R(Y) \). The orientation of \( Y \) induce an orientation of \( \lambda_W \). To determine the orientation of \( M^W(a)_{\pi, \Psi, \epsilon, \delta} \), we fix a compact oriented manifold \( Z \) with \( H_1(Z, \mathbb{Z}) \cong 0 \) as in Subsection 6.2. The relation between \( \lambda_{Z,a} \) and \( \lambda(a) \) is given by

\[
\lambda_{Z,a}(a) \otimes \lambda_Z \cong \lambda(a).
\]

Let \( a \) be a flat connection satisfying \( \text{csy}(a) < r \leq \min\{Q^2_{X^{l+3}}, 1\} \) and \( \text{ind}(a) = 1 \). We consider the moduli space \( M^W(a)_{\pi, \Psi, \epsilon, \delta} \). From the choice of these perturbation data and Corollary 6.4, \( M^W(a)_{\pi, \Psi, \epsilon, \delta} \) has a structure of 1-dimensional manifold for small \( \delta \). From Theorem 6.13, we obtain an orientation of \( M^W(a)_{\pi, \Psi, \epsilon, \delta} \) induced by the orientation of \( \lambda_{Z,a} \).

Let \( (A, B) \) be a limit point of \( M^W(a)_{\pi, \Psi, \epsilon, \delta} \). Using Theorem 5.1 and the standard dimension counting argument, the limit points of \( M^W(a)_{\pi, \Psi, \epsilon, \delta} \) correspond to two cases:

1. \( (A, B) \in \bigcup_{B \in \tilde{R}(Y), \text{csy}(b) < r, \text{ind}(b) = 0} M(a, b)_\pi \times M^W(b)_{\pi, \Psi, \epsilon, \delta} \)

2. \( (A, B) \in M(a, \theta)_\pi \times M^W(\theta)_{\pi, \Psi, \epsilon, (\delta, \delta)} \).

For the second case, we use the exponential decay estimate to show \( B \in M^W(\theta)_{\pi, \Psi, \epsilon, (\delta, \delta)} \). Here \( M(a, b)_\pi \) and \( M(a, \theta)_{\pi, \delta} \) have a structure of 1-dimensional manifold. The quotient spaces \( M(a, b)_\pi/\mathbb{R} \) and \( M(a, \theta)_{\pi, \delta}/\mathbb{R} \) have a structure of compact oriented 0-dimensional manifold whose orientation induced by the orientation of \( \lambda_Z \) and \( \lambda_W \) by the translation as in Subsection 5.4 of [3]. Corollary 6.4 and Theorem 6.13 imply that \( M(b)_{\pi, \Psi, \epsilon, \delta} \) has a structure of compact oriented 0-manifold whose orientation induced by the orientation of \( \lambda_\theta \) and \( \lambda_W \) for small \( \delta \). Since the formal dimension of \( M(\theta)_{\pi, \Psi, \epsilon, (\delta, \delta)} \) is \(-3\) from Proposition 3.7 and there is no reducible solution except \( \theta \) for a regular perturbation.
\((\Psi, \epsilon)\). \(M(\theta)_{x, \Psi, \epsilon, \delta}\) consists of just one point. By the gluing theory as in Theorem 4.17 and Subsection 4.4.1 of [3], there is the following diffeomorphism onto its image:

\[
J : \bigcup_{b \in \tilde{R}^*(Y), c_{SV}(b) < r, \text{ind}(b) = 0} (M(a, b)_{x, \Psi, \epsilon, \delta} \times M(b, \Psi, \epsilon, \delta) \cup M(a, \theta)_{x, \Psi, \epsilon, \delta}) \times [T, \infty) \\
\rightarrow M^W(a, \Psi, \epsilon, \delta).
\]

By the definition of the orientation of \(M(a, b)_{x, \Psi, \epsilon, \delta}\) and \(M(a, \theta)_{x, \Psi, \epsilon, \delta}\), we can construct \(J\) as an orientation preserving map. Furthermore, the complement of \(\text{Im} J\) is compact. Therefore we can construct the compactification of \(M^W(a, \Psi, \epsilon, \delta)\) by adding the finitely many points

\[
\bigcup_{b \in \tilde{R}^*(Y), c_{SV}(b) < r, \text{ind}(b) = 0} (M(a, b)_{x, \Psi, \epsilon, \delta} \times M(b, \Psi, \epsilon, \delta) \cup M(a, \theta)_{x, \Psi, \epsilon, \delta})
\]

which has a structure of compact oriented 1-manifold. By counting of boundary points of the compactification, we obtain the relation

\[
\delta^r(n)(a) + \theta^r(a) = 0,
\]

where \(n \in CF^r_0(Y)\) is defined by \(n(b) := \#M(b)_{x, \Psi, \epsilon, \delta}\). This implies \(\theta^r\) is a coboundary. Therefore we have \(0 = [\theta^r] \in HF^1_r(Y)\) for \(0 \leq r \leq \min\{Q_{X^{2Y+3}}, 1\}\).

\[\square\]

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