ON FACTORS ASSOCIATED WITH QUANTUM MARKOV STATES CORRESPONDING TO NEAREST NEIGHBOR MODELS ON A CAYLEY TREE

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Abstract. In this paper we consider nearest neighbour models where the spin takes values in the set $\Phi = \{\eta_1, \eta_2, \ldots, \eta_q\}$ and is assigned to the vertices of the Cayley tree $\Gamma^k$. The Hamiltonian is defined by some given $\lambda$-function. We find a condition for the function $\lambda$ to determine the type of the von Neumann algebra generated by the GNS - construction associated with the quantum Markov state corresponding to the unordered phase of the $\lambda$-model. Also we give some physical applications of the obtained result.

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1. Introduction

It is known that in the quantum statistical mechanics concrete systems are identified with states on corresponding algebras. In many cases the algebra can be chosen to be a quasi-local algebra of observables. The states on this algebra satisfying the KMS condition, describe equilibrium states of the quantum system. Basically, limiting Gibbs measures of classical systems with the finite radius of interactions are Markov random fields (see e.g. [8],[22]). In connection with this, there arises a problem to construct analogues of non-commutative Markov chains. In [1] Accardi explored this problem, he introduced and studied noncommutative Markov states on the algebra of quasi-local observables which agreed with the classical Markov chains. In [3],[15],[2], modular properties of the non-commutative Markov states were studied. In [10] Fannes, Nachtergale and Werner showed that ground states of the valence- bond- solid modles on a Cayley tree were quantum Markov chains on the quasi-local algebra. In the present paper we will consider Markov states associated with nearest neighbour models on a Cayley tree. Note the investigation of the type of quasi-free factors (i.e. factors generated by quasi-free representations) has been
an interesting problem since the appearance of the pioneering work of Araki and Wyss [5]. In [21] a family of representations of uniformly hyperfinite algebras was constructed, which can be treated as a free quantum lattice system. In that case factors corresponding to those representations are of type \( \text{III}_\lambda \), \( \lambda \in (0, 1) \). More general constructions of product states were considered in [4].

Observe that the product states can be viewed as the Gibbs states of Hamiltonian system in which interactions between particles of the system are absent, i.e. the system is a free lattice quantum spin system. So it is interesting to consider quantum lattice systems with non-trivial interactions, which leads us, as mentioned above, to consider the Markov states. Simple examples of such systems are the Ising and Potts models, which have been studied in many papers (see, for example, [24]). We note that all Gibbs states corresponding to these models are Markov random fields. The full analysis of the type of von Neumann algebras associated with the quantum Markov states is still an open problem. Some particular cases of the Markov states were considered in [11],[14],[17],[18],[19].

The present paper is devoted to the type analysis of some class of diagonal quantum Markov states, which correspond to a \( \lambda \)-model on the Cayley tree, in which spin variables take their values in a set \( \Phi = \{\eta_1, \ldots, \eta_q\} \), where \( \eta_k \in \mathbb{R}^{q-1} \), \( k = 1, \ldots, q \). Observe that the considered model generalizes a notion of \( \lambda \)-model introduced in [23], where the spin variables take their values \( \pm 1 \).

2. Definitions and preliminary results

The Cayley tree \( \Gamma^k \) of order \( k \geq 1 \) is an infinite tree, i.e., a graph without cycles, such that each vertex of which lies on \( k + 1 \) edges. Let \( \Gamma^k = (V, \Lambda) \), where \( V \) is the set of vertices of \( \Gamma^k \), \( \Lambda \) is the set of edges of \( \Gamma^k \). The vertices \( x \) and \( y \) are called nearest neighbor, which is denoted by \( l = < x, y > \) if there exists an edge connecting them. A collection of the pairs \( < x, x_1 >, \ldots, < x_{d-1}, y > \) is called path from the point \( x \) to the point \( y \). The distance \( d(x, y), x, y \in V \), on the Cayley tree, is the length of the shortest path from \( x \) to \( y \).

We set

\[
W_n = \{x \in V | d(x, x^0) = n\},
\]

\[
V_n = \bigcup_{m=1}^{n} W_m = \{x \in V | d(x, x^0) \leq n\},
\]

\[
L_n = \{l = < x, y > \in L | x, y \in V_n\},
\]

for an arbitrary point \( x^0 \in V \). Denote \( |x| = d(x, x^0), x \in V \).
Denote 

\[ S(x) = \{ y \in W_{n+1} : d(x, y) = 1 \}, \quad x \in W_n. \]

This set is made of the direct successors of \( x \). Observe that any vertex \( x \neq x^0 \) has \( k \) direct successors and \( x^0 \) has \( k + 1 \).

**Theorem 2.1.** There exists a one-to-one correspondence between the set \( V \) of vertices of the Cayley tree of order \( k \geq 1 \) and the group \( G_{k+1} \) of the free products of \( k + 1 \) cyclic groups of the second order with generators \( a_1, a_2, ..., a_{k+1} \).

Consider a left (resp. right) transformation shift on \( G_{k+1} \) defined as follows. For \( g_0 \in G_{k+1} \) we put

\[ T_{g_0} h = g_0 h \quad (\text{resp. } T_{g_0} h = hg_0), \quad h \in G_{k+1}. \]

It is easy to see that the set of all left (resp. right) shifts on \( G_{k+1} \) is isomorphic to the group \( G_{k+1} \).

Let \( \Phi = \{ \eta_1, \eta_2, ..., \eta_q \} \), where \( \eta_1, \eta_2, ..., \eta_q \) are vectors in \( \mathbb{R}^{q-1} \), such that

\[ \eta_i \eta_j = \begin{cases} 1, & \text{if } i = j \\ \frac{1}{q-1}, & \text{if } i \neq j \end{cases}. \tag{1} \]

We consider models where the spin takes values in the set \( \Phi = \{ \eta_1, \eta_2, ..., \eta_q \} \) and is assigned to the vertices of the tree. A configuration \( \sigma \) on \( V \) is then defined as a function \( x \in V \rightarrow \sigma(x) \in \Phi \); the set of all configurations coincides with \( \Omega = \Phi^{V_n} \). The Hamiltonian is of an \( \lambda \)-model form:

\[ H_\lambda(\sigma) = \sum_{<x,y>} \lambda(\sigma(x), \sigma(y); J), \tag{2} \]

where \( J \in \mathbb{R}^n \) is a coupling constant and the sum is taken over all pairs of neighboring vertices \( < x, y >, \sigma \in \Omega \). Here and below \( \lambda : \Phi \times \Phi \times \mathbb{R}^n \rightarrow \mathbb{R} \) is some given function.

We note that \( \lambda \)-model of this type can be considered as a generalization of the Ising model. The Ising model corresponds to the case \( q = 2 \) and \( \lambda(x, y; J) = -J_{xy} \).

We consider a standard \( \sigma \)-algebra \( \mathcal{F} \) of subsets of \( \Omega \) generated by cylinder subsets, all probability measures are considered on \((\Omega, \mathcal{F})\). A probability measure \( \mu \) is called a Gibbs measure (with Hamiltonian \( H_\lambda \)) if it satisfies the DLR equation: for \( n = 1, 2, ... \) and \( \sigma_n \in \Phi^{V_n} \):

\[ \mu\left( \{ \sigma \in \Omega : \sigma|_{V_n} = \sigma_n \} \right) = \int_{\Omega} \mu(d\omega)\nu^{V_n}_{\omega|_{W_{n+1}}}(\sigma_n) \]
where $\nu_{\omega|_{W_{n+1}}}^V$ is the conditional probability
\[ \nu_{\omega|_{W_{n+1}}}^V(\sigma_n) = Z^{-1}(\omega|_{W_{n+1}}) \exp(-\beta H(\sigma_n||\omega|_{W_{n+1}})). \]

where $\beta > 0$. Here $s_n|_{V_n}$ and $\omega|_{W_{n+1}}$ denote the restriction of $\sigma, \omega \in \Omega$ to $V_n$ and $W_{n+1}$ respectively. Next, $\sigma_n : x \in V_n \rightarrow \sigma_n(x)$ is a configuration in $V_n$ and $H(\sigma_n||\omega|_{W_{n+1}})$ is defined as the sum $H(\sigma_n) + U(\sigma_n, \omega|_{W_{n+1}})$ where
\[ H(\sigma_n) = \sum_{<x,y> \in L_n} \lambda(\sigma_n(x), \sigma_n(y); J), \]
\[ U(\sigma_n, \omega|_{W_{n+1}}) = \sum_{<x,y> : x \in V_n, y \in W_{n+1}} \lambda(\sigma_n(x), \omega(y); J). \]

Finally, $Z(\omega|_{W_{n+1}})$ stands for the partition function in $V_n$ with the boundary condition $\omega|_{W_{n+1}}$:
\[ Z(\omega|_{W_{n+1}}) = \sum_{\tilde{\sigma}_n \in \Phi_{V_n}} \exp(-\beta H(\tilde{\sigma}_n||\omega|_{W_{n+1}})). \]

Since we consider nearest neighbour interactions, the Gibbs measures of the $\lambda$-model possess a Markov property: given a configuration $\omega_n$ on $W_n$, random configurations in $V_{n-1}$ and in $V \setminus V_{n+1}$ are conditionally independent. It is known (see [26]) that for any sequence $\omega(n) \in \Omega$, any limiting point of measures $\tilde{\nu}_{\omega(n)|_{W_{n+1}}}^V$ is a Gibbs measure. Here $\tilde{\nu}_{\omega(n)|_{W_{n+1}}}^V$ is a measure on $\Omega$ such that $\forall n' > n$:
\[ \tilde{\nu}_{\omega(n)|_{W_{n+1}}}^V \left( \{ \sigma \in \Omega : \sigma|_{V_{n'}} = \sigma_{n'} \right) = \begin{cases} \nu_{\omega(n)|_{W_{n+1}}}^V(\sigma_{n'}|_{V_{n'}}) & \text{if } \sigma_{n'}|_{V_{n'}\setminus V_n} = \omega(n)|_{V_{n'}\setminus V_n} \\ 0 & \text{otherwise}. \end{cases} \]

We now recall some basic facts from the theory of von Neumann algebras. Let $B(H)$ be the algebra of all bounded linear operators on the Hilbert space $H$ (over the field of complex numbers $\mathbb{C}$). A weak (operator) closed $*$-subalgebra $\mathcal{N}$ in $B(H)$ is called von Neumann algebra if it contains the identity operator $\mathbf{1}$. By $\text{Proj}(\mathcal{N})$ we denote the set of all projections in $\mathcal{N}$. A von Neumann algebra is a factor if its center
\[ Z(\mathcal{N}) := \{ x \in \mathcal{N} : xy = yx, \forall y \in \mathcal{N} \} \]
is trivial, i.e., $Z(\mathcal{N}) = \{ \lambda \mathbf{1} : \lambda \in \mathbb{C} \}$. The von Neumann algebras are direct sum of the classes I ($I_n, n < \infty, I_\infty$), II ($II_1, II_\infty$) and III. Further, a factor is of only one type among these listed above, see e.g. [30]. An element $x \in \mathcal{N}$ is called positive if there is an element $y \in \mathcal{N}$.
such that $x = y^* y$. A linear functional $\omega$ on $\mathcal{N}$ is called a state if $\omega(x^* x) \geq 0$ for all $x \in \mathcal{N}$ and $\omega(1) = 1$. A state $\omega$ is said to be normal if $\omega(\sup x_\alpha) = \sup \omega(x_\alpha)$ for any bounded increasing net $\{x_\alpha\}$ of positive elements of $\mathcal{N}$. A state $\omega$ is called trace (resp. faithful) if the condition $\omega(xy) = \omega(yx)$ holds for all $x, y \in \mathcal{N}$ (resp. if the equality $\omega(x^* x) = 0$ implies $x = 0$).

Let $\mathcal{N}$ be a factor, $\omega$ be a faithful normal state on $\mathcal{N}$ and $\sigma^{\omega}_t$ be the modular group associated with $\omega$ (see Definition 2.5.15 in [6]). We let $\Gamma(\sigma^{\omega}_t)$ denote the Connes spectrum of the modular group $\sigma^{\omega}_t$ (see Definition 2.2.1 in [6]).

For the type III factors, there is a finer classification.

**Definition** ([9]). The type III factor $\mathcal{N}$ is of type
(i) III$_1$, if $\Gamma(\sigma^{\omega}) = \mathbb{R}$;
(ii) III$_\lambda$, if $\Gamma(\sigma^{\omega}) = \{n \log \lambda, n \in \mathbb{Z}\}$, $\lambda \in (0, 1)$;
(iii) III$_0$, if $\Gamma(\sigma^{\omega}) = \{0\}$;
see, e.g. [6], [28] for details of von Neumann algebras and the modular theory of operator algebras.)

3. CONSTRUCTION OF GIBBS STATES FOR THE $\lambda$-MODEL

In this section we give a construction of a special class of limiting Gibbs measures for the $\lambda$-model on the Cayley tree.

Let $h : x \rightarrow h_x = (h_{1,x}, h_{2,x}, ..., h_{q-1,x}) \in \mathbb{R}^{q-1}$ be a real vector-valued function of $x \in V$. Given $n = 1, 2, ...$ consider the probability measure $\mu^{(n)}$ on $\Phi^{V_n}$ defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp\{-\beta H(\sigma_n) + \sum_{x \in W_n} h_x \sigma(x)\},$$

(3)

Here, as before, $\sigma_n : x \in V_n \rightarrow \sigma_n(x)$ and $Z_n$ is the corresponding partition function:

$$Z_n = \sum_{\sigma_n \in \Omega_{V_n}} \exp\{\beta H(\tilde{\sigma}_n) + \sum_{x \in W_n} h_x \tilde{\sigma}(x)\}.$$

The consistency conditions for $\mu^{(n)}(\sigma_n), n \geq 1$ are

$$\sum_{\sigma^{(n)}_{n-1}} \mu^{(n)}(\sigma_{n-1}, \sigma^{(n)}) = \mu^{(n-1)}(\sigma_{n-1}),$$

(4)

where $\sigma^{(n)}_{n-1} = \{\sigma(x), x \in W_n\}$.

Let $V_1 \subset V_2 \subset ... \cup_{n=1}^{\infty} V_n = V$ and $\mu_1, \mu_2, ...$ be a sequence of probability measures on $\Phi^{V_1}, \Phi^{V_2}, ...$ satisfying the consistency condition (4). Then, according to the Kolmogorov theorem (see, e.g. [25]), there exists
a unique limit Gibbs measure $\mu_h$ on $\Omega$ such that for every $n = 1, 2, \ldots$ and $\sigma_n \in \Phi V_n$ the equality holds

$$\mu \left( \{ \sigma | V_n = \sigma_n \} \right) = \mu^{(n)}(\sigma_n).$$

(5)

Further we set the basis in $\mathbb{R}^{q-1}$ to be $\eta_1, \eta_2, \ldots, \eta_{q-1}$.

The following statement describes conditions on $h_x$ guaranteeing the consistency condition of measures $\mu^{(n)}(\sigma_n)$.

**Theorem 3.1.** The measures $\mu^{(n)}(\sigma_n), \ n = 1, 2, \ldots$ satisfy the consistency condition (4) if and only if for any $x \in V$ the following equation holds:

$$h'_x = \sum_{y \in S(x)} F(h'_y, \lambda),$$

(6)

Here, and below $h'_x$ stands for the vector $\frac{q-1}{q} h_x$ and $F : \mathbb{R}^{q-1} \to \mathbb{R}^{q-1}$ function is $F(h; \lambda) = \left( F_1(h; \lambda), \ldots, F_{q-1}(h; \lambda) \right)$, with

$$F_i(h_1, h_2, \ldots, h_{q-1}; \lambda)$$

$$= \log \frac{\sum_{j=1}^{q-1} \exp\{-\beta \lambda(\eta_i, \eta_j; J)\} \exp h_j + \exp\{-\beta \lambda(\eta_i, \eta_q; J)\}}{\sum_{j=1}^{q-1} \exp\{-\beta \lambda(\eta_q, \eta_j; J)\} \exp h_j + \exp\{-\beta \lambda(\sigma_q, \eta_j; J)\}},$$

$i = 1, 2, \ldots, q-1, h = (h_1, \ldots, h_{q-1})$.

The proof uses the same argument as in [18],[19]. Denote

$$\mathcal{D} = \{ h = (h_x \in \mathbb{R}^{q-1} : x \in V) : h_x = \sum_{y \in S(x)} F(h_y, \lambda), \ \forall x \in V \}.$$ 

According to Theorem 3.1 for any $h = (h_x, x \in V) \in \mathcal{D}$ there exists a unique Gibbs measure $\mu_h$ which satisfies the equality (5).

If the vector-valued function $h^0 = (h_x = (0, \ldots, 0), x \in V)$ is a solution, i.e. $h^0 \in \mathcal{D}$ then the corresponding Gibbs measure $\mu_0^{(\lambda)}$ is called the unordered phase of the $\lambda$-model. Since we deal with this unordered phase, we have to make an assumption which guarantees us the existence of the unordered phase.

**Assumption A.** For the considered model the vector-valued function $h^0 = (h_x = (0, 0, \ldots, 0), x \in V)$ belongs to $\mathcal{D}$.

This means that the equation (6) has a solution $h_x = h_0 = 0, x \in V$.

According to Theorem 2.1 any transformation $S$ of the group $G_{k+1}$ induces an automorphism $\hat{S}$ on $\Omega$. By $G_{k+1}$ we denote the left group of shifts of $G_{k+1}$. Any $T \in G_{k+1}$ induces a shift automorphism $\tilde{T} : \Omega \to \Omega$ by

$$ (\tilde{T} \sigma)(h) = \sigma(Th), \ h \in G_{k+1}, \ \sigma \in \Omega.$$
It is easy to see that $\mu_0^{(\lambda)} \circ \tilde{T} = \mu_0^{(\lambda)}$ for every $\tilde{T} \in \mathcal{G}_{k+1}$. As mentioned above, the measure $\mu_0^{(\lambda)}$ has a Markov property (see, [27]).

**Assumption B.** We suppose that the measure $\mu_0^{(\lambda)}$ enables a mixing property, i.e. for any $A, B \in \mathcal{F}$ the following holds

$$\lim_{|g| \to \infty} \mu_0^{(\lambda)}(\tilde{T}_g(A) \cap B) = \mu_0^{(\lambda)}(A) \mu_0^{(\lambda)}(B).$$

(8)

Note that the last condition is satisfied, for example, if the phase transition does not occur for the model under consideration.

4. Diagonal states and corresponding von Neumann algebras

Consider $C^*$-algebra $A = \otimes_{\Gamma^k} M_q(\mathbb{C})$, where $M_q(\mathbb{C})$ is the algebra of $q \times q$ matrices over the field $\mathbb{C}$ of complex numbers. By $e_{ij}$, $i, j \in \{1, 2, \ldots, q\}$ we denote the basis matrices of the algebra $M_q(\mathbb{C})$. We let $CM_q(\mathbb{C})$ denote the commutative subalgebra of $M_q(\mathbb{C})$ generated by the elements $e_{ii}$ $i = \{1, 2, \ldots, q\}$. We set $CA = \otimes_{\Gamma^k} CM_q(\mathbb{C})$. Elements of commutative algebra $CA$ are functions on the space $\Omega = \{e_{11}, \ldots, e_{qq}\}^{\Gamma^k}$. Fix a measure $\mu$ on the measurable space $(\Omega, B)$, where $B$ is the $\sigma$-algebra generated by cylindrical subsets of $\Omega$. We construct a state $\omega_\mu$ on $A$ as follows. Let $P : A \to CA$ be the conditional expectation, then the state $\omega_\mu$ is be defined by $\omega_\mu(x) = \mu(P(x))$, $x \in A$, here $\mu(P(x))$ means an integral of a function $P(x)$ under measure $\mu$, i.e. $\mu(P(x)) = \int_\Omega P(x)(s) d\mu(s)$ (see [29]). The state is called diagonal.

By $\omega_0^{(\lambda)}$ we denote the diagonal state generated by the unordered phase $\mu_0^{(\lambda)}$. The Markov property implies that the state $\omega_0^{(\lambda)}$ is a quantum Markov state (see [3]). On a finite dimensional $C^*$-subalgebra $A_{V_n} = \otimes_{V_n} M_q(\mathbb{C}) \subset A$ we rewrite the state $\omega_0^{(\lambda)}$ as follows

$$\omega_0^{(\lambda)}(x) = \frac{tr(e^{\hat{H}(V_n)}x)}{tr(e^{\hat{H}(V_n)})}, \quad x \in A_{V_n},$$

(9)

where $tr$ is the canonical trace on $A_{V_n}$. The term $\lambda(\sigma(x)\sigma(y); J)$ in (4) is given by a diagonal element of $M_q(\mathbb{C}) \otimes \mathbb{M}_n(\mathbb{C})$ in the standard basis.
as follows

\[
\beta \lambda(\sigma(x), \sigma(y); J) = \begin{pmatrix}
B^{(1)} & 0 & \cdots & 0 \\
0 & B^{(2)} & 0 & \cdots \\
& \ddots & \ddots & \ddots \\
0 & 0 & \cdots & B^{(q)}
\end{pmatrix} .
\] (10)

Here, \( B^{(k)} = (b_{ij,k})_{i,j=1}^q \), \( k = 1, \ldots, q \) are \( q \times q \) matrices, and

\[
b_{ij,k} = \begin{cases} 
-\beta \lambda(\eta_k, \eta_i; J), & i = j, i = 1, \ldots, q \\
0, & i \neq j
\end{cases} .
\] (11)

Consequently, using (9) and from (10),(11) (cp. [26], Ch.1, §1) the form of Hamiltonian \( \tilde{H}(V_n) \) in the standard basis of \( A_{V_n} \) (i.e. under the basis matrices) is regarded as

\[
\tilde{H}(V_n) = \sum_{<x,y> \in L_n} \Phi_{x,y}, \quad \Phi_{x,y} = \begin{pmatrix}
B^{(1)} & 0 & \cdots & 0 \\
0 & B^{(2)} & 0 & \cdots \\
& \ddots & \ddots & \ddots \\
0 & 0 & \cdots & B^{(q)}
\end{pmatrix} .
\]

Denote \( \mathcal{M} = \pi_{\omega_0^{(\lambda)}}(A)^\sigma \), where \( \pi_{\omega_0^{(\lambda)}} \) is the GNS - representation associated with the state \( \omega_0^{(\lambda)} \) (see Definition 2.3.18 in [6]). Our goal in this section is to determine a type of \( \mathcal{M} \).

Remark. In [29] general properties of a representation associated with diagonal state were studied, but there concrete constructions of states were not considered. In [20] a deep classification of types of factors generated by quasi-free states has been obtained. For translation-invariant Markov states the corresponding type analysis has been made in [13]. The investigation of the type of factor arising from translation invariant or periodic quantum Markov states on the one dimensional chains is contained in [11].

Now we define translations of the \( C^* \)-algebra \( A \). Every \( T \in \mathcal{G}_{k+1} \) induces a translation automorphism \( \tau_T : A \to A \) defined by

\[
\tau_T(\bigotimes_{x \in V_n} a_x) = \bigotimes_{x \in V_n} a_{T(x)} .
\]

Since measure \( \mu_0^{(\lambda)} \) satisfies a mixing property (see (8)), then we can easily obtain that \( \omega_0^{(\lambda)} \) also satisfies the mixing property under the
translations \( \{ \tau_T \}_{T \in G_{k+1}} \), i.e. for all \( a, b \in A \) the equality holds
\[
\lim_{|g| \to \infty} \omega_0(\lambda)(\tau_g(a)b) = \omega_0(\lambda)(a)\omega_0(\lambda)(b).
\]

According to Theorem 2.6.10 in [6], the algebra \( \mathcal{M} \) is a factor. We note that the modular group of \( \mathcal{M} \) associated with \( \omega_0(\lambda) \) is defined by
\[
\sigma_{t}(\omega_0(\lambda))(x) = \lim_{\Lambda \uparrow \Gamma_k} \exp \{ it\widetilde{H}(\Lambda) \} x \exp \{ -it\widetilde{H}(\Lambda) \}, \quad x \in \mathcal{M},
\]
where \( \widetilde{H}(\Lambda) = \sum_{<x,y> \in \Lambda} \Phi_{x,y} \). It is well known that the last limit exists if a suitable norm of the potential \( \widetilde{H} \) is finite (see Theorem 6.2.4. [7]). First of all, we recall the definition of the norm of a potential \( \Psi = \sum_{X \subset \Gamma_k} \Psi(X) \).
\[
\| \Psi \|_d = \sum_{n \geq 0} e^{dn} \left( \sup_{x \in \Gamma_k} \sum_{X, |X| = n+1} \| \Psi(X) \| \right),
\]
where \( d > 0 \). Here \( \Psi(X) \in A_X = \otimes_X M_q(C) \).

Now we compute \( \| \widetilde{H} \|_d \):
\[
\| \widetilde{H} \|_d = e^{2d} \left( \sup_{x \in \Gamma_k} \sum_{x \in X, X = \{u,v\}} \| \Phi_{u,v} \| \right) = k e^{2d} \sup_{\{u,v\} \in L} \| \Phi_{u,v} \|
\]
\[
= k e^{2d} \max_{i,j,k} | \log p_{i,j,k} | < \infty.
\]

Hence the norm of \( \widetilde{H} \) is finite, therefore the limit in (12) exists.

By \( \mathcal{M}^\sigma \) one denotes the centralizer of \( \omega_0(\lambda) \), which is defined as
\[
\mathcal{M}^\sigma = \{ x \in \mathcal{M} : \sigma_{t}(\omega_0(\lambda))(x) = x, \quad t \in \mathbb{R} \}.
\]

Since \( \omega_0(\lambda) \) is Gibbs state, according to Proposition 5.3.28 [7], the centralizer \( \mathcal{M}^\sigma \) coincides with the set
\[
\mathcal{M}_{\omega_0(\lambda)} = \{ x \in \mathcal{M} : \omega_0(\lambda)(xy) = \omega_0(\lambda)(yx), \quad y \in \mathcal{M} \},
\]
where we denote by \( \omega_0(\lambda) \) also the normal extension of the state under consideration to all of \( \mathcal{M} \).

By \( \Pi[n] \) we denote the group of all permutations \( \gamma \) of the set \( V_n \) such that
\[
\gamma(x) = x, \quad x \in W_n.
\]
Every $\gamma \in \Pi[n]$ defines an automorphism $\alpha_\gamma : \mathcal{M} \to \mathcal{M}$ by
\[
\begin{align*}
\alpha_\gamma (\prod_{x \in V_n} a_x) &= \prod_{x \in V_n} a_{\gamma(x)} \\
\alpha_\gamma |_{\otimes_{x \in V_n} \mathcal{M}(\mathbb{C})} &= \text{id},
\end{align*}
\]
where $\text{id}$ is the identity mapping.

Denote
\[
S_0 = \bigcup \{\alpha_\gamma | \gamma \in \Pi[n]\}.
\]

Simply repeating the proof of a proposition in [16], we can prove the following

**Lemma 4.1.** The group
\[
G_0 = \{\alpha \in S_0 | \omega^{(\lambda)}_0(\alpha(x)) = \omega^{(\lambda)}_0(x), \ x \in \mathcal{M}\},
\]
acts ergodically on $\mathcal{M}$, i.e. the equality $\alpha(x) = x$, $\alpha \in G_0$ implies $x = \theta \mathbf{1}, \ \theta \in \mathbb{C}$.

**Lemma 4.2.** The centralizer $\mathcal{M}^\sigma$ is a factor of type $II_1$.

**Proof.** From the definition of the automorphism $\alpha_\gamma$ (see (14)), it is easy to see that every automorphism $\alpha \in G_0$ is inner, i.e. there exists a unitary $u_\alpha \in \mathcal{M}$ such that $\alpha(x) = u_\alpha x u_\alpha^*$, $x \in \mathcal{M}$. From the condition $\omega^{(\lambda)}_0 \circ \alpha = \omega^{(\lambda)}_0$ we find
\[
\omega^{(\lambda)}_0(u_\alpha x u_\alpha^*) = \omega^{(\lambda)}_0(x), \ x \in \mathcal{M}.
\]
It follows from (13) that $u_\alpha \in \mathcal{M}^{(\lambda)}$. According to Lemma 4.1 the group $G_0$ acts ergodically, this means that the equality $u_\alpha x = x u_\alpha$ for every $\alpha \in G_0$ implies $x = \theta \mathbf{1}, \ \theta \in \mathbb{C}$. Hence, we obtain $\{u_\alpha | \alpha \in G_0\}' = \mathbb{C} \mathbf{1}$. Since $\mathcal{M}^\sigma \subset \{u_\alpha\}'$ we then get
\[
\mathcal{M}^\sigma \cap \mathcal{M} = \mathbb{C} \mathbf{1}.
\]
In particular $\mathcal{M}^\sigma \cap \mathcal{M} = \mathbb{C} \mathbf{1}$. This means that $\mathcal{M}^\sigma$ is a factor. $\square$

Now we are able to prove main result of the paper (compare with the analogous result in [11]).

**Theorem 4.3.**

(i) If the fraction
\[
\frac{\lambda(\eta_i, \eta_j; J) - \lambda(\eta_m, \eta_l; J)}{\lambda(\eta_k, \eta_p; J) - \lambda(\eta_u, \eta_v; J)},
\]
is rational for every $i, j, m, l, k, p, u, v \in 1, 2, \ldots, q$, whenever the denominator is different from 0, then the von Neumann algebra
\( \mathcal{M} \) associated with the quantum Markov state corresponding to the unordered phase of \( \lambda \)-model (2) on a Cayley tree is a factor of type III\(_0\).

(ii) If \( \mathcal{M} \) is a type of III\(_1\) factor, then all the fraction cannot be rational.

Proof. It is known (see Proposition 2.2.2 in [9]) that Connes’ spectrum \( \Gamma(\alpha) \) of group of automorphisms \( \alpha = \{ \alpha_g \}_{g \in G} \) of von Neumann algebra \( M \) has the following form

\[
\Gamma(\alpha) = \cap \{ \text{Sp}(\alpha^e) | e \in \text{Proj}(Z(M^\alpha)), e \neq 0 \},
\]

where \( \alpha^e(x) = \alpha(exe), x \in eMe \) and \( Z(M^\alpha) \) is the center of subalgebra \( M^\alpha = \{ x \in M : \alpha_g(x) = x, \ g \in G \} . \)

Here, \( \text{Sp}(\alpha) \) be the Arveson’s spectrum of group of automorphisms \( \alpha \) (see for more details [9],[28]).

By virtue of Lemma 4.2 we have \( Z(\mathcal{M}^\sigma) = \mathbb{C} \). The equality (15) implies \( \Gamma(\sigma^\omega) = \text{Sp}(\sigma^\omega) \).

We now consider the operator \( \tilde{H}(V_n) = \sum_{<x,y> \in L_n} \Phi_{x,y} \). We let \( \text{Sp}(\tilde{H}(V_n)) \) denote the spectrum of the operator \( \tilde{H}(V_n) \). Setting

\[
\sigma^\omega_{\epsilon^n}(x) = \exp\{it\tilde{H}(V_n)\}x\exp\{-it\tilde{H}(V_n)\}, \ x \in \mathcal{M},
\]

we obtain

\[
\text{Sp}(\sigma^\omega_{\epsilon^n}) = \text{Sp}(\tilde{H}(V_n)) - \text{Sp}(\tilde{H}(V_n)) = \{ \lambda - \mu : \lambda, \mu \in \text{Sp}(\tilde{H}(V_n)) \} .
\]

(16)

It is clear that \( \beta \lambda(\eta_i, \eta_k; J) \in \text{Sp}(\tilde{H}(V_n)), \ \forall i, k \in 1, \ldots, q \). Formula (16) then implies that \( \text{Sp}(\sigma^\omega_{\epsilon^n}) \) is generated by elements of the form

\[
\lambda(\eta_i, \eta_j; J) - \lambda(\eta_k, \eta_l; J), \ i, j, k, l \in 1, \ldots, q .
\]

Since \( \frac{\lambda(\eta_i, \eta_j; J) - \lambda(\eta_m, \eta_n; J)}{\lambda(\eta_k, \eta_l; J) - \lambda(\eta_m, \eta_n; J)} \) is a rational number, then there is a number \( \gamma \in (0, 1) \) and integers \( m_{i,j,k,l} \in Z, \ (i, j, k, l \in 1, 2, \ldots, q) \) such that

\[
\lambda(\eta_i, \eta_j; J) - \lambda(\eta_k, \eta_l; J) = m_{i,j,k,l} \log \gamma .
\]

(17)

Hence we find that an increasing sequence \( \{ E(n) \} \) of subsets \( Z \) such that \( E(-n) = -E(n) \) and \( \text{Sp}(H(V_n)) = \{ m \log \gamma \}_{m \in E(n)} \) is valid. It follows that

\[
\text{Sp}(\sigma^\omega_{\epsilon^n}) \subset \{ n \log \gamma \}_{n \in Z} .
\]

Hence there exists a positive integer \( d \in Z \) such that we have

\[
\Gamma(\sigma^\omega_{\epsilon^n}) = \{ n \log \gamma^d \}_{n \in Z} .
\]
This means that $\mathcal{M}$ is a factor of type $\text{III}_\vartheta$, $\vartheta = \gamma^d$. □

5. Applications and examples

5.1. Potts model. We consider the Potts model on the Cayley tree $\Gamma^k$ whose Hamiltonian is regarded as

$$H(\sigma) = - \sum_{<x,y>} J_\sigma(x)\sigma(y),$$

where $J \in \mathbb{R}$ is a coupling constant, as usual $<x,y>$ stands for the nearest neighbor vertices and as before $\sigma(x) \in \Phi = \{\eta_1, \eta_2, ..., \eta_q\}$. Here $\delta$ is the Kronecker symbol.

Equality (1) implies that

$$\delta_{\sigma(x)\sigma(y)} = \frac{q-1}{q} \left( \sigma(x)\sigma(y) + \frac{1}{q-1} \right),$$

for all $x, y \in V$. The Hamiltonian $H(\sigma)$ is therefore

$$H(\sigma) = - \sum_{<x,y> \in L} J'\sigma(x)\sigma(y), \quad (18)$$

where $J' = \frac{q-1}{q} J$.

Hence the $\lambda$-model is a generalization of the Potts model, that is in this case the function $\lambda : \Phi \times \Phi \times \mathbb{R} \rightarrow \mathbb{R}$ is defined by $\lambda(x, y; J') = -J'(x, y)$. Here, $x, y \in \mathbb{R}^{q-1}$ and $(x, y)$ stands for the scalar product in $\mathbb{R}^{q-1}$. From (1) it is easy to see that

$$\lambda(\eta_i, \eta_j; J) = \begin{cases} J, & \text{if } i = j \\ -\frac{1}{q-1}, & \text{if } i \neq j \end{cases}. \quad (19)$$

From (19) and (6) we can check that the assumption A (see section 3) is valid for the Potts model. So there exists the unordered phase.

From (19) we can find that the fraction $\frac{\lambda(\eta_i, \eta_j; J')}\lambda(\eta_k, \eta_p; J') = \frac{\lambda(\eta_k, \eta_p; J') - \lambda(\eta_m, \eta_n; J')}{\lambda(\eta_k, \eta_p; J')}$, takes values $\pm 1$ and $0$. So by Theorem 4.3 a von Neumann algebra $\mathcal{M}$ is a $\text{III}_\varphi$-factor. From (17) we may obtain that $\varphi = \exp \left\{ \frac{-J'q}{(q-1)} \right\}$.

Hence we obtain the following

**Theorem 5.1.** The von Neumann algebra $\mathcal{M}$ corresponding to the unordered phase of the Potts model (18) on a Cayley tree is a factor of type $\text{III}_{\varphi^k}$, for some $k \in \mathbb{Z}$, $k > 0$, where $\varphi = \exp \left\{ \frac{-J'q}{(q-1)} \right\}$.
Remark. If $q = 2$ the considered Potts model reduces to the Ising model, for this model analogous results were obtained in [17]. For a class of inhomogeneous Potts model similar result has been also obtained in [18].

5.2. Markov random fields. In this subsection we consider a case when $\lambda(x, y)$ function is not symmetric and the corresponding Gibbs measure is a Markov random field (see [27]).

Let $P = (p_{ij})_{i,j=1}^d$ be a stochastic matrix such that $p_{ij} > 0$ for all $i, j \in \{1, \ldots, d\}$. Define a function $\lambda(x, y)$ as follows:

$$\lambda(\eta_i, \eta_j) = -\log p_{ij}, \quad (20)$$

for all $i, j \in \{1, \ldots, d\}$. From now on, we will consider the case $\beta = 1$ and $q = d$. It is easy to verify that Assumptions A and B, for the defined function $\lambda$, are satisfied. By $\mu$ we denote the corresponding unordered phase of the $\lambda$-model. Observe that if the order of the Cayley tree is $k = 1$ then the measure $\mu$ is a Markov measure, associated with the stochastic matrix $P$ (see [27]).

By $\omega_\mu$ one denotes the diagonal state corresponding to the measure $\mu$ on $C^*$-algebra $A = \otimes_{1}M_d(\mathbb{C})$.

**Theorem 5.2.** Let $P = (p_{ij})_{i,j=1}^d$ be a stochastic matrix such that $p_{ij} > 0$ for all $i, j = 1, \ldots, d$ and at least one element of this matrix is different from $1/2$, and $\omega_\mu$ be the corresponding Markov state. If there exist integers $m_{ij}$, $i, j \in \{1, \ldots, d\}$, and some number $\alpha \in (0, 1)$ such that

$$\frac{p_{11}}{p_{i,j}} = \alpha^{m_{ij}}, \quad (21)$$

then $\pi_{\omega_\mu}(A)^\prime\prime$ is a factor of type III_\theta for some $\theta \in (0, 1)$.

In order to prove this theorem it should be used the rationality condition of Theorem 4.3. Namely, if the condition (21) is satisfied then using (20) one can see that the rationality condition holds, so we get the assertion.

Remark. If all elements of the stochastic matrix $P$ equal to $1/2$ then the corresponding Markov state $\omega_\mu$ is a trace and consequently $\pi_{\omega_\mu}(A)^\prime\prime$ is the unique hyperfinite factor of type II_1.

Remark. There is a conjecture (see for example, [31]) that every factor associated with GNS representation of a Gibbs state of a Hamiltonian system having a non-trivial interaction is of type III_1. Theorems 4.3 and 5.2 show that the conjecture is not true even if the Hamiltonian has nearest neighbour interactions.
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