1. Introduction

The quantization of gravity is currently an unresolved problem in theoretical physics. A main
obstacle to its consistent quantization lies in the observation that the theory of Einstein’s gravity,
unlike the standard model and quantum chromodynamics, is perturbatively nonrenormalizable
in metric variables. This impasse has led to two main alternative approaches to its quantization,
namely string theory and loop quantum gravity. String theory is based upon the idea that
Einstein’s theory of gravity is the low energy limit of a more fundamental theory which should
rather be quantized instead and leads to 26 (or 10, in the case of the superstring) dimensional
spacetime. Loop quantum gravity attempts to nonperturbatively quantize gravity in four-dimensional spacetime in the loop representation and has led to many insights at the kinematic level of the gravitational phase space.

One of the ingredients needed for a finite theory of quantum gravity, by the interpretation we adopt in this and in subsequent works, is that given a model coupled to Einstein’s relativity in four dimensions, one may be able to explicitly construct the physical quantum states devoid of ultraviolet infinities for the full theory as well as for minisuperspace. A physical quantum state is defined, by this criterion, as a wavefunctional satisfying the quantum version of the constraints of Einstein’s general relativity in Ashtekar variables. To explicitly construct such states one may quantize the theory in the Schrödinger representation utilizing Dirac’s method for quantizing constrained systems [1]. A good review of the background behind the Ashtekar variables and their resulting simplifications of general relativity can be found in [2–6].

There is one special state in the full theory of quantum gravity, known as the Kodama state, known to exactly solve the quantum constraints to all orders for a particular operator ordering [7]. The fact that this state as well solves the classical constraints exactly [8] leads to a new conjecture: the principle of the semiclassical-quantum correspondence (SQC). For the pure Kodama state, the SQC amounts to the imposition of a self-duality condition constraining the Ashtekar electric and magnetic fields to be proportional to each other by a factor of the cosmological constant $\Lambda$. The expansion of the Hamiltonian constraint reveals its division into a semiclassical part and singular quantum terms. In the case of pure gravity with $\Lambda$ term, the quantum terms cancel out and the semiclassical part leads directly, via the SQC, to the pure Kodama state. When matter fields are present in addition to gravity, the SQC is broken due to the existence of induced singular quantum terms. These singular quantum terms constitute an obstacle to the construction of a finite state of quantum gravity. Once these singularities are dealt with, then the resulting state once constructed should be finite. One can then focus on other issues such as normalizability, expectation values, probability currents and reality conditions. We save treatment of these aspects for future work.

The outline of this paper is as follows. In section 2, we outline the relevant attributes of the pure Kodama state and introduce the concept of the semiclassical-quantum correspondence, with a view to generalizing these attributes. In section 3, we generalize the Kodama state by introducing some degrees of freedom (via a CDJ matrix) necessary to incorporate the presence of matter fields quantized with gravity on the same footing. In section 4, we revisit the kinematic constraints within the context of matter coupling and quantize the constraints subject to the CDJ Ansatz, showing that the SQC remains unbroken. In section 5 we quantize the Hamiltonian constraint coupled to matter, subject to the CDJ Ansatz. By requiring that the SQC remain unbroken in spite of the composite nature of the quantum operators involved, we delineate the criteria for finiteness of the generalized Kodama state via a system of nine equations in nine unknowns, shown in section 6.1 In section 7 we then delineate the construction of the generalized Kodama state for the full theory from the CDJ matrix elements, explicitly showing its restriction to the final spatial hypersurface $\Sigma_f$ and independence of any field velocities or evolution within the interior of the spacetime manifold $M = \Sigma \times R$. We show the equivalence of the generalized Kodama states $\Psi_{\text{GKod}}$ with respect to a variety of representations, including a direct generalization of topological field theory to accommodate the presence of the matter fields as well as via techniques of geometric quantization applied

1 The general solution for the CDJ matrix elements in the full theory is beyond the scope of this paper. The interested reader is directed to [11], where our method of solution is developed in some detail.

2
to the cohomology of field theory\(^2\). It is hoped that this work should constitute a first step in the construction of a finite theory in that we provide a prescription for eliminating ultraviolet infinities usually present in the canonical approach to a quantum theory containing composite momentum operators by way of the SQC. Also, we demonstrate the equivalence of at least three quantization procedures, namely reduced phase space, Dirac and geometric quantization for the generalized Kodama states.

We must make a few notes on conventions. First, our use of the term ‘generalized Kodama states’ is not to be confused with the use in [9, 10], where Andrew Randono constructs Kodama states for pure gravity using different values of the Immirzi parameter \(\gamma\) to label states. The use of the term in this publication will signify the generalization from pure gravity to the analogous state when additional fields besides gravity are present for \(\gamma = \sqrt{-1}\). Secondly, a quick note on the Ashtekar variables: the basic dynamical variables are a left-handed \(SU(2)\) connection, \(A^a_i\) and its conjugate momentum, a densitized triad \(\tilde{\sigma}^a_i\) living in a four-dimensional spacetime manifold \(M = \Sigma \times R\). Our convention for index labelling is that letters from the beginning of the Latin alphabet \(a,b,c,...\) signify internal \(SU(2)\) indices and that letters from the middle of the alphabet \(i,j,k,...\) signify spatial indices. Spacetime position in \(M\) will be written as \(x\), however we will on occasion highlight the significance of spatial position in \(\Sigma\) by use of the boldface \(\bf{x}\), where \(x = (\bf{x}, t)\).

2. The pure Kodama state in perspective

The Einstein–Hilbert action in Ashtekar variables can be written in terms of its 3+1 ADM-type decomposition [2–4]

\[
I = \int dt \int_\Sigma d^3x \left( \frac{i}{G} \tilde{\sigma}^a_i \dot{A}^a_i - iN H_{\text{grav}} - N^i (H_i)_{\text{grav}} - \theta^a G_a \right),
\]

which is a canonical 1-form minus a linear combination of first-class constraints. The constraints are given by the classical equations of motion for the corresponding Lagrange multipliers, which are nondynamical fields since their time derivatives do not appear in the action. These are the lapse density \(N = N/\sqrt{\det h_{ij}}\) (where \(h_{ij}\) is the 3-metric), the shift vector \(N^i\) and the time component of the self-dual Ashtekar connection \(A^0_a = \theta^a\). The classical constraints read

\[
G_a(x) = \frac{\delta I}{\delta \theta^a(x)} = D_i \tilde{\sigma}^a_i(x) = \partial_i \tilde{\sigma}^a_i + f^{abc} A^b_i \tilde{\sigma}^c_i = 0 \quad \forall x \in M,
\]

which is the \(SU(2)\) Gauss’ law constraint with structure constants \(f^{abc}\). Then there is the diffeomorphism constraint

\[
(H_i)_{\text{grav}}(x) = \frac{\delta I}{\delta N^i(x)} = \epsilon_{ijk} \tilde{\sigma}^j_i(x) B^k_j(x) = 0 \quad \forall x \in M,
\]

and the Hamiltonian constraint

\[
H_{\text{grav}} = \frac{\delta I}{\delta \Lambda(x)} = e^{abc} \epsilon_{ijk} \tilde{\sigma}^j_i \tilde{\sigma}^k_j B^k_j + \frac{1}{6} e^{abc} \epsilon_{ijk} \tilde{\sigma}^j_i \tilde{\sigma}^k_j \Lambda = 0 \quad \forall x \in M.
\]

These constraints must hold, classically, at all points \(x = (\bf{x}, t)\) in the four-dimensional manifold \(M\). Here, \(x\) is the spatial position on a spatial hypersurface \(\Sigma\) labelled by time \(t\). We

\(\text{We relegate the demonstration of the equivalence of the aforementioned representations to the path-integral representation to [12], in which we formulate an analogue to the Hartle Hawking no boundary proposal [13, 14] in Ashtekar variables.}\)
are interested in the reduced phase space for this system, which corresponds to the physical degrees of freedom. The Hamiltonian constraint admits a nontrivial classical solution,

\[
\epsilon^{abc} \epsilon_{ijk} \tilde{\sigma}^i_a \tilde{\sigma}^j_b \frac{\Lambda}{6} \tilde{\sigma}^k_c + B^k_c = 0 \quad \forall \, k, \, c, \tag{5}
\]

which is the self-duality relation between the Ashtekar electric and magnetic fields, somewhat analogous to the self-duality relation for the electromagnetic field propagating in a vacuum in which \(\Lambda^{-1}\) plays the role of \(c\), the constant and finite speed of light. Consistency must be checked with the remaining constraints,

\[
G_a = D_i \tilde{\sigma}^i_a = -\frac{6}{\Lambda} D_i B^i_c = 0, \quad (H_i)_{\text{grav}} = \epsilon_{ijk} \tilde{\sigma}^j_a B^k_c = -\frac{6}{\Lambda} \epsilon_{ijk} B^j_a B^k_c = 0, \tag{6}
\]
due to the Bianchi identity and to antisymmetry, respectively. To evaluate the action on the reduced phase space one substitutes this classical solution back into the starting Lagrangian (1) yielding

\[
I = \int dt \int_{\Sigma} \left( \frac{i}{G} \tilde{\sigma}^i_a \dot{A}^i_a - i N H - N^i H_i + \theta^a G_a \right) \bigg|_{\tilde{\sigma}^i_a = -\frac{6}{\Lambda} B^i_c} \tag{7}
\]
on account of the constraints. Using the identification for the Ashtekar curvature

\[
B^i_a = \epsilon^{ijk} \delta_{ar} F^r_{jk} \quad \text{where} \quad F^r_{jk} = \partial_r A^r - \partial_j A^r + f^r_{js} A^s_j A^k_s,
\]
we can extend this to include the analogous four-dimensional connection by defining \(F^a_0 = \partial_0 A^a - \partial_i A^a_0 + f^a_{j0} A^i_j A^k_s\). Solving for \(\dot{A}^i_a\) and substituting into (7) one has

\[
I = -6i(GA)^{-1} \int dt \int_{\Sigma} B^i_a \dot{A}^i_a \tag{8}
\]
Integrating by parts and dropping boundary terms leads to

\[
I = -6i(GA)^{-1} \int d^4x \epsilon^{ijk} e_{ar} F^a_{jk} \left( F^a_0 + \partial_0 A^a_0 - f^a_{j0} A^i_j A^k_s \right) \tag{9}
\]
which can be written in covariant notation by defining \(\epsilon^{ijk} = \epsilon_{0ijk}\), noting that the second term in brackets in (9) vanishes due to the Bianchi identity, in the form

\[
I = -6i(GA)^{-1} \int d^4x \epsilon^{\mu\nu\rho\sigma} F^a_{\mu\nu} F^a_{\rho\sigma} = -6i(GA)^{-1} \int_M \text{tr}(F \wedge F) \tag{10}
\]
where the trace in (10) is taken over left-handed \(SU(2)_-\) indices. Let us write the state corresponding to (10) in a more recognizable form. Applying Stokes’ theorem

\[
\int_M \text{tr}(F \wedge F) = \int_{\partial M} L_{CS}[A] = I_{CS}|_{\partial M}, \tag{11}
\]
where \(L_{CS}\) is the Chern–Simons action for the left-handed \(SU(2)_-\) Ashtekar connection living on the boundary \((\Sigma_T, \Sigma_0) \equiv \partial_M\) of \(M\), we have

\[
I = I_{CS}[A(\Sigma_T)] - I_{CS}[A(\Sigma_0)] \tag{12}
\]
where \(I_{CS}\) is the Chern–Simons functional of the \(SU(2)_-\)-valued Ashtekar connection, given by

\[
I_{CS} = \int_{\Sigma_T} \left( \text{Ad} A + \frac{2}{3} A \wedge A \wedge A \right). \tag{13}
\]
A semiclassical wavefunction can be constructed from this functional by exponentiating $I$, in units of $i/\hbar$, evaluated on the reduced phase space. The exponential of the Chern–Simons functional for quantum gravity is known as the Kodama state, discovered by Hideo Kodama [7, 8], and corresponds to the Hamilton function for the system in Hamilton–Jacobi theory:

$$\Psi_{\text{Kod}}[A] \propto \exp\left[ \frac{-i}{\hbar} I \right] = \exp[-6(\hbar G A)^{-1} I_{\text{CS}}[A]]. \quad (14)$$

Note how the requirement that the classical constraints be satisfied at all $x$ within $M$ leads to a wavefunctional defined on the three-dimensional boundary $\Sigma_T = \partial M$. This holographic effect has been demonstrated by Horowitz in [15] and is typical of topological field theories.

2.1. Quantization of the constraints and the semiclassical-quantum correspondence (SQC)

In order to determine the physical states of quantum gravity the procedure for canonical quantization of constrained systems, developed by Dirac [1], can in some sense be used as an alternative to the reduced phase space method introduced below. In this procedure, one promotes the canonically conjugate variables $(A^a_i, \tilde{\sigma}^a_i)$ to quantum operators $(\hat{A}^a_i, \hat{\tilde{\sigma}}^a_i)$ and Poisson brackets to commutators via the equal-time commutation relations

$$\left[ A^a_i(x, t), \tilde{\sigma}^j_b(y, t) \right] \rightarrow \left[ \hat{A}^a_i(x, t), \frac{i}{G} \tilde{\sigma}^j_b(y, t) \right] = \hbar \delta^a_i \delta^j_b \delta^{(3)}(x, y), \quad (15)$$

with remaining trivial commutation relations

$$\left[ A^a_i(x, t), A^b_j(y, t) \right] = \left[ \tilde{\sigma}^a_i(x, t), \tilde{\sigma}^j_b(y, t) \right] = 0 \quad (16)$$

and defines a Hilbert space for the quantum operators to act on. To transform the relations (15) into the Schrödinger representation one chooses the basis vectors $|A^a_i\rangle$ of the quantum states to be eigenstates of the quantum operator $\hat{A}^a_i(x)$ for a given point $x$. The state $|A^a_i\rangle$ satisfies the orthogonality and completeness relations

$$\langle A_{a_1}^{(1)}(x) | A_{a_2}^{(2)}(x) \rangle = \prod_x w[A_{a_1}^{(1)}(x)]^{-1} \delta(A_{a_1}^{(1)}(x) - A_{a_2}^{(2)}(x)) \int D\mu[A] \langle A | \sim \prod_{x, a, i} \int dA^a_i(x) w[A(x)] |A^a_i(x)\rangle |A^a_i(x)\rangle = I. \quad (17)$$

We have not used a gauge invariant, diffeomorphism invariant, measure as in loop quantum gravity [2, 16] since our method will be to allow gauge and diffeomorphism invariance of the state to be imposed by the explicit solution of the quantum constraints. The ‘weighting’ functional $w[A]$ can be chosen judiciously. A good example for judicious choices of weighting functions in quantum theories was given by Ashtekar and Rovelli in [17] in Maxwell theory, in which the weighting function for the Bargmann representation was chosen as a measure for the normalization such that certain operators become Hermitian. We will for the time being leave the weighting function $w[A]$ unspecified, but reserve the freedom to choose it appropriately when the opportunity presents itself.

The property of infinite-dimensional spaces, which is indigenous to all quantum field theories including quantum gravity, is a direct consequence of consideration of the full theory as opposed to minisuperspace. Any state $|\Psi\rangle$ can be expressed in this basis by projecting it onto

3 Units of $i/\hbar$ correspond to the Euclidean version of the Kodama state, while units of $1/\hbar$ correspond to the Lorentzian version.
the complete set of states (17) defined on a particular spatial hypersurface $\Sigma_t$ corresponding to time $t$, as presented by Misner in [18]:

$$|\Psi(t)\rangle = \int D\mu[A(t)]|A(t)\rangle\langle A(t)|$$

(18)

with $\langle A|\Psi\rangle = \Psi[A]$. Taking a basis of quantum states in the holomorphic representation of the (three-dimensional) Ashtekar connection $\Psi_{Kod}[A] = \langle A|\Psi\rangle$, the action of $\left(\hat{A}^a_i, \hat{\sigma}^j_b\right)$ is represented respectively by multiplication and functional differentiation

$$\hat{A}^a_i(x)\Psi_{Kod}[A] = A^a_i(x)\Psi_{Kod}[A]$$

$$\hat{\sigma}^j_b(x)\Psi_{Kod}[A] = \bar{h}G \frac{\delta}{\delta A^a_i(x)} \Psi_{Kod}[A].$$

(19)

According to Dirac, the physical Hilbert space $\Psi_{phys}$ forms the subset of the full Hilbert space satisfying the quantum version of the constraints, with operator ordering taken into account. We will attempt to find physical states in the simultaneous kernel of the quantum constraints for an operator ordering with the ‘momenta’ to the left of the ‘coordinate’ variables:

$$\hat{G}_a(x)\Psi_{Kod}[A] = \hat{H}_i(x)\Psi_{Kod}[A] = \hat{H}(x)\Psi_{Kod}[A] = 0 \quad \forall x.$$  

(20)

Using an Ansatz $\Psi_{Kod}[A] = \exp[(\bar{h}G)^{-1}I[A]]$, the Gauss’ law constraint, which is a statement of the invariance of the quantum state under $SU(2)$ rotations of the connection, reads

$$\frac{\delta\Psi_{Kod}[A]}{\delta \theta^a(x)} = \hat{G}_a\Psi = \hbar G D_{ij} \frac{\delta\Psi_{Kod}[A]}{\delta A^a_i(x)} = \Psi_{Kod}[A]D_i \left( \frac{\delta I}{\delta A^a_i(x)} \right) = 0.$$  

(21)

Note that the quantum condition on the quantum wavefunction $\Psi_{Kod}[A]$ implies an identical condition on its ‘phase’ $I$, which can be viewed as a semiclassical condition. This constitutes a semiclassical-quantum correspondence for the Gauss’ law constraint due to the constraint’s being linear in momenta. The diffeomorphism constraint is a statement of the invariance of the wavefunction under spatial coordinate transformations of its argument $A^a_i(x)$ and reads

$$\frac{\delta\Psi_{Kod}[A]}{\delta N^i(x)} = \hat{H}_i\Psi_{Kod}[A] = \left[ \epsilon_{ijk}hG \frac{\delta}{\delta A^a_j(x)} B^k_a(x) \right] \Psi_{Kod}[A]$$

$$= \left[ \hbar G\epsilon_{ijk} D^{jk}_{ab}(x) + \hbar G \frac{\delta I}{\delta A^a_j(x)} B^k_a(x) \right] \Psi_{Kod}[A] = 0.$$  

(22)

where we have used the definition

$$D^{ij}_{ab}(x) = \frac{\delta B^j_a(x)}{\delta A^i_b(x)} = \frac{\delta}{\delta A^i_b(x)} \left[ \delta_{ac} \partial_j A^c_k(x) + \frac{1}{2} f_{abc} A^j_b(x) A^c_k(x) \right]$$

$$= \epsilon^{ijk} \left( \delta_{ab} \partial_k (\delta^{(3)}(0) + \delta^{(3)}(0) f_{abc} A^c_k(x)) \right).$$  

(23)

The application of (23) to (22) implies that $D^{jk}_{ab} = 0$ due to antisymmetry of the structure constants and that $\delta^{(3)}(0) = 0$.

The Gauss’ law and diffeomorphism constraints, kinematic constraints, do not correspond to physical transformations. The Hamiltonian constraint is the dynamical constraint, which does encode nontrivial dynamics of the theory in this case since it is at least quadratic in momenta:

$$\frac{\delta\Psi_{Kod}[A]}{\delta N(x)} = \hat{H}\Psi_{Kod}[A] = 0.$$  

(24)

4 This latter condition, which differs from conventional functional calculus, is based on the requirement that spatial and functional differentiation must commute [25].
Expanded out this reads
\[ \hbar^2 G^2 e^{abc} \delta \frac{\delta}{\delta A^a_i(x)} \left[ B^b_i(x) + \frac{\hbar G A}{6} \frac{\delta}{\delta A^b_i(x)} \right] \exp[ \frac{1}{\hbar} I[A] ] = 0. \] (25)

On the one hand, one can see from (25) that due to the operator ordering chosen there exists a nontrivial solution in which the operator in square brackets annihilates the state, given by
\[ \frac{\delta I}{\delta A^a_i(x)} = -6(G\Lambda \hbar)^{-1} B^b_i(x) \quad \forall x \in M, \] (26)
from which, if one could ‘functionally integrate’, one could explicitly determine \( I \) and construct a wavefunction. The condition is defined on a particular 3-surface \( \Sigma_1 \) on which the constraint is evaluated. Let us contract the left-hand side of (26) by the time derivative \( \dot{A}_a^i(x) \) and integrate over all 3-space of the manifold \( \Sigma \):
\[ \int \Sigma d^3 x \frac{\delta I}{\delta A^a_i(x)} \dot{A}_a^i(x, t) = \frac{dI}{dt} = -6(G\Lambda \hbar)^{-1} \int \Sigma d^3 x B^b_i(x, t) \dot{A}_a^b(x, t) \] (27)
which is nothing but the definition of the time derivative of a functional of an independent variable \( A_a^i \) defined on 3-space in terms of the evolution of the variable. Recall that for functional variation on the infinite-dimensional functional spaces of the type encountered in field theory,
\[ \frac{\delta I}{\delta A^a_i(x)} = \int \Sigma d^3 x \frac{\delta I}{\delta A^a_i(x)} \delta A^a_i(x). \] (28)
If so happens, then, that \( dI/dt \) is a total time derivative. Integrating from \( t = t_0 \) to \( t = T \), one has that the functional \( I \) evolves from the initial 3-surface \( \Sigma_0 \) to the final 3-surface \( \Sigma_T \),
\[ I(T) - I(t_0) = \int_{t_0}^{T} \frac{dI}{dt} dt = \int_M \text{tr } F \wedge F = I_{CS}[A(\Sigma_T)] - I_{CS}[A(\Sigma_0)]. \] (29)
where we have used the results from (11). One can obtain the same result by applying (26) as evaluated on the final spatial hypersurface \( \Sigma_T \) to (28), unsuppressing the time label to yield
\[ \delta I_T = \int \Sigma d^3 x B^b_i(x, T) \delta A^a_i(x, T) = \delta I_{CS}[A(T)] \] (30)
and consequently
\[ I_T = I_{CS}[A(T)]. \] One can now write the solution to the quantum Hamiltonian constraint as
\[ \Psi_{Kod}[A] = \exp[-6(G\Lambda \hbar)^{-1} I_{CS}[A]], \] (31)
where in (31) we have suppressed the label \( T \) of the spatial hypersurface \( \Sigma_T \) forming the boundary \( \partial M \). So, the quantum state (31) and the semiclassically determined state (14) coincide to all orders with no quantum corrections. We will define this property, the ‘semiclassical-quantum correspondence’ (SQC). The usual prescription by which a classical theory gets promoted to its quantum counterpart is a rough rule of thumb which leads to an ambiguity in quantum theories to choose from of order \( \hbar \). The correct quantum theory is fixed by comparison with experiment. However, we have demonstrated that from the infinite set of possibilities to choose from there is a unique quantum state which coincides with the classical state exactly to all orders, namely the Kodama state.\(^5\)

A reasonable question to ask is what physical theories admit a pure Kodama state. The construction of such states by alternate methods for \( N = 1 \) and \( N = 2 \) supergravities in four dimensions has been demonstrated \([3, 19, 20]\). In [21], a canonical analysis was performed for

\(^5\) We rename this the ‘pure’ Kodama state since it exists for pure gravity with \( \Lambda \) term, devoid of any matter fields.
\( N = 3 \) supergravity. The reason that no such Kodama state seems to have been constructed, to the present author’s knowledge, is due to the fact that for \( N \geq 3 \) there automatically exist additional lower spin fields which ruin the topological nature of the models that exhibit the SQC. We hope to ultimately demonstrate, in this series of publications, a new way to extend the SQC to such models.

2.2. Semiclassical-quantum correspondence for the pure Kodama state

The pure Kodama state is the exact solution to the constraints of the full theory with \( \Lambda \) term when there are no matter fields present in addition to gravity, given by \( \Psi_{\text{Kod}} = \Psi_{\text{Kod}}[A] \). This can be represented in terms of the self-duality Ansatz

\[
\tilde{\sigma}_a^i = -6\Lambda^{-1}\delta_{ae}B_i^e. \tag{32}
\]

The pure Kodama state arguably, issues of normalizability aside, can be said to represent a canonical quantization of four-dimensional general relativity in the full superspace theory, exactly to all orders with no quantum corrections for pure gravity with \( \Lambda \) term. Of course \( \Psi_{\text{Kod}} \) is a special state, which satisfies the self-duality condition and as well the semiclassical-quantum correspondence (SQC). Equation (32) satisfies the SQC since its quantized counterpart yields the same condition to all orders. The quantized version of (32) is given by

\[
\hbar G \frac{\delta}{\delta A_i^a(x)} \Psi_{\text{Kod}}[A] = -\frac{6}{\Lambda}B_i^e(x)\Psi_{\text{Kod}}[A]. \tag{33}
\]

3. Generalized Kodama state and the CDJ Ansatz

We have reviewed the method for the quantization of gravity for pure gravity with cosmological term \( \Lambda \) as it relates to the pure Kodama state \( \Psi_{\text{Kod}} \). We now outline a method to extend this to the more general case for a nondegenerate magnetic field \( B_i^e \), namely gravity coupled to quantized matter fields. There are a few complications relative to the pure Kodama state which will arise due to the presence of quantized matter fields in the full theory. The generalized Kodama state contains functional dependence upon the gravitational \( A_i^a \) and matter variables \( \Psi_{\text{GKod}} = \Psi_{\text{GKod}}[A, \phi] \), where \( \phi = \phi(x) \) represents the matter fields of the model. First let us write the starting action for gravity coupled to matter, the analogue of (1), for completeness,

\[
I = \int dt \int_S d^3x \left[ \frac{i}{G} \tilde{\sigma}_a^i \dot{A}_i^a + \pi \dot{\phi} - iN(H_{\text{grav}} + G\Omega) - \chi_a(G_{a} + GQ_a) \right], \tag{34}
\]

where \( \Omega \) is the matter contribution to the Hamiltonian constraint, \( H_i \) is the matter contribution to the diffeomorphism constraint, \( \chi_a \) is the matter contribution to the Gauss' law constraint and \( \pi = \pi(x) \) is the conjugate momentum to the matter field \( \phi(x) \). One can then go through the analogous manipulations with respect to (34) as in section 2.

It has been shown by Thiemann in [22] that when matter fields are present in addition to gravity, the constraints can still be solved at the classical level by use of the CDJ Ansatz

\[
\tilde{\sigma}_a^i(x) = \Psi_{ae}(x)B_i^e(x). \tag{35}
\]

In (35), the CDJ matrix \( \Psi_{ae} = \Psi_{ae}[A, \phi] \) in general contains nontrivial functional dependence upon the gravitational and the matter variables at each point \( x \). The CDJ Ansatz (35) can be viewed as a generalization of the self-duality condition (32) to accommodate the presence of the matter fields and contains sufficient degrees of freedom to allow solution of the constraints.
at the classical level [22]. Our claim in this paper is that since the CDJ Ansatz is linear in gravitational momentum $\tilde{\sigma}_a$, it as well satisfies the SQC. Therefore, one should be able to promote (35) to its quantized version without breaking this correspondence. This is the analogue of (33) for the generalized Kodama state $\Psi_{\text{GKod}}$:

$$\frac{\hbar G}{\delta A_i^a(x)} \Psi_{\text{GKod}}[A, \phi] = (\Psi_{\text{or}}(x) B_i^a(x)) \Psi_{\text{GKod}}[A, \phi].$$

(36)

In (36) the self-duality condition is generalized from the case of a homogeneous isotropic CDJ matrix, and the CDJ matrix in this context now plays the role of a tensor-valued ‘generalized’ inverse cosmological constant.

The equal-time commutation relations can be read off from the phase space structure of (34). In the case of a Klein–Gordon scalar field, they read

$$[\hat{\phi}(x, t), \hat{\pi}(y, t)] = i\hbar \delta^{(3)}(x - y)$$

(37)

along with the ‘trivial’ relations

$$[\hat{\phi}(x, t), \hat{\phi}(y, t)] = [\hat{\pi}(x, t), \hat{\pi}(y, t)] = 0$$

(38)

amongst the matter variables, and

$$[\hat{\phi}(x, t), \hat{\Lambda}_a^i(y, t)] = [\hat{\pi}(x, t), \hat{\Lambda}_a^i(y, t)] = 0$$

(39)

for ‘cross’ commutation relations, as well as

$$[\hat{\phi}(x, t), \hat{\sigma}_a^i(y, t)] = [\hat{\pi}(x, t), \hat{\sigma}_a^i(y, t)] = 0.$$  

(40)

Equation (39) and (40) signify the requirement that $\phi(x)$ and $A_i^a(x)$ as well as their conjugate momenta be dynamically independent variables, which will be important in the construction of generalized Kodama states. Hence while these particular commutation relations appear trivial, we will find that they do indeed contain nontrivial physical content.

From this one can as well derive a matter-momentum analogue on the spatial hypersurface $\Sigma$, labelled by $t$, using the Schrödinger representation, to the CDJ Ansatz,

$$\hat{\pi}(x) \Psi_{\text{GKod}}[A, \phi] = -i\hbar \frac{\delta}{\delta \phi(x)} \Psi_{\text{GKod}}[A, \phi] = \pi(x) \Psi_{\text{GKod}}[A, \phi],$$

(41)

where $\pi(x)$ is by definition an Ansatz for the action of the operator $\hat{\pi}$ on the generalized Kodama state $\Psi_{\text{GKod}}$.

We have generalized the basis states in the Schrödinger representation to accommodate the presence of the matter fields via the identifications

$$\langle A_i^{a_1}(x), \phi(x)|A_i^{a_2}(x), \phi'(x)\rangle = \prod_x W[A_i^{a_1}(x), \phi(x)]^{-1} \delta(A_i^{a_1}(x) - A_i^{a_2}(x)) \delta(\phi(x) - \phi'(x))$$

$$\int D\mu[A, \phi]|A, \phi\rangle|A, \phi\rangle \sim \prod_{x,a,i} \int dA_i^{a}(x) d\phi(x) W[A(x), \phi(x)]$$

$$\times |A_i^{a_1}(x), \phi(x)|A_i^{a_2}(x), \phi(x)\rangle = I,$$

(42)

for some appropriately chosen weighting functional $W = W[A_i, \phi]$. Any state $|\Psi\rangle$ can now be expressed in this basis by projecting it onto the complete set of states (42) defined on a particular spatial hypersurface $\Sigma$, corresponding to time $t$,

$$|\Psi(t)\rangle = \int D\mu[A(t), \phi(t)]A(t), \phi(t)\rangle|A(t), \phi(t)\rangle |\Psi\rangle$$

(43)

such that $\Psi_{\text{GKod}}[A_i^a, \phi] = \langle A_i^a, \phi|\Psi\rangle$.

6 This is the case irrespective of their individual time histories within $M$.

7 Not all states are ‘momentum eigenstates’ satisfying equation (41).

8 We demonstrate in [12] how $W[A, \phi]$ can be chosen such as to establish formal equivalence of the path-integral representation of $\Psi_{\text{GKod}}$ to its canonically determined version.
The CDJ Ansatz applies only when the Ashtekar magnetic field $B^a_i$ is nondegenerate. A degenerate magnetic field implies a degenerate densitized triad $\tilde{\sigma}^i_a$, which implies a degenerate 3-metric. We are not considering in this work states that have support on degenerate metrics. Note that the existence of matter fields in the theory, which is what distinguishes $\Psi_{GKod}$ from $\Psi_{Kod}$ in this paper, implies the nondegeneracy of $B^a_i$ [22]. This can also be argued from the standpoint that the standard matter Lagrangians require an inverse metric $g^{\mu\nu}$ [23].

3.1. Mixed partials condition

Since the gravitational and matter fields $A^i_a(x)$ and $\phi(x)$ respectively are independent dynamical variables, then they must have trivial commutation relations with each other. So the commutation relations from (40) read

$$\left[ \hat{\Lambda}^i_a(x, t), \hat{\phi}(y, t) \right] = \left[ \hat{\tilde{\sigma}}^i_a(x, t), \hat{\pi}(y, t) \right] = 0 \quad \forall \, x, y.$$  \hspace{1cm} (44)

The right-hand side of (44) in its action on the generalized Kodama state

$$\left[ \hat{\tilde{\sigma}}^i_a(x, t), \hat{\pi}(y, t) \right] \Psi_{GKod}[A^i_a, \phi] = 0$$  \hspace{1cm} (45)
leads to a condition known as the mixed partials condition. Let us proceed into the Schrödinger representation taking $x = y$:

$$-i\hbar^2 G \left[ \frac{\delta}{\delta A^i_a(x)} \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta A^i_a(x)} \right] \Psi_{GKod} = 0.$$  \hspace{1cm} (46)

Taking the first functional derivatives in (46),

$$\left[ \hbar G \frac{\delta}{\delta A^i_a(x)} (\pi(x)) + \hbar \frac{\delta}{\delta \phi(x)} (\Psi_{ac}(x) B^i_e(x)) \right] \Psi_{GKod}$$

$$= \left[ \pi \Psi_{ac} B^i_e - \Psi_{ac} B^i_e \pi + \delta^{(3)}(0) \left( \hbar G \frac{\partial \pi}{\partial A^i_a} + \hbar B^i_e \frac{\partial \Psi_{ac}}{\partial \phi} \right) \right] \Psi_{GKod},$$  \hspace{1cm} (47)

the semiclassical part cancels out and does not lead to anything new. In order for (47) to be valid, the coefficient of $\delta^{(3)}(0)$ must vanish as well\(^9\). This implies

$$\frac{\partial \pi}{\partial A^i_a} = -\frac{i}{G} B^i_e \frac{\partial \Psi_{ac}}{\partial \phi} \quad \forall \, x.$$  \hspace{1cm} (48)

Note that (48) is a condition which holds separately at each point $x$ on the hypersurface $\Sigma_t$ for each time $t$. The mixed partials condition will be useful in the elimination of matter momentum in the construction of the generalized Kodama states and is a key input for the definition of $\Psi_{GKod}$.

The general solution of (48) can be written, by integration over the functional space of Ashtekar connections $A \in \Gamma$ at fixed position $x$, for the specified spatial hypersurface $\Sigma_t$, as\(^10\)

$$\pi = \pi[A, \phi] = f(\phi) - \frac{i}{G} \frac{\partial}{\partial \phi} \left( \int_{\Gamma} \delta X^{ae} \right) \Psi_{ae}[A, \phi],$$  \hspace{1cm} (49)

where $f(\phi)$ is an arbitrary function of the matter field $\phi(x)$ acting as a ‘constant’ of functional integration with respect to the gravitational configuration variables $A^i_a(x)$ for each spatial point $x$, and where the ‘functional’ 1-form $\delta X^{ae}$ is given by

$$\delta X^{ae}(x) = B^i_e(x) \delta A^i_a(x).$$  \hspace{1cm} (50)

\(^9\) Also, it is assumed that $\Psi_{GKod}$ is an eigenstate of the momentum operator.

\(^10\) The brackets around $\int \delta X^{ae}$ are meant to signify that the integration constitutes a linear operator, which must be separated from the vector space $\Psi_{ae}$ which it acts on.
There are four things to note concerning (49). First, it is a linear relation between the matter momentum $\pi(x)$ and the CDJ matrix $\Psi_{1a\beta}(x)$ arising as a consistency condition of the quantization procedure. Secondly, it has the same functional form for each position $x$ in $\Sigma$, and therefore resembles a minisuperspace equation but is in actuality the full theory. Third, one can consider the allowable semiclassical matter momenta to be labelled by the arbitrary function $f$. Fourth, even though $f$ is freely specifiable, it can be judiciously chosen for instance such as to produce a ‘boundary condition’ on the generalized Kodama state such that the proper wavefunction is obtained in the limit when there is no gravity. The function $f$ then can serve to reproduce the wavefunction which would result from solving the functional Schrödinger equation for the Klein–Gordon Hamiltonian in Minkowski spacetime\(^{11}\).

4. Quantization of the generalized kinematic constraints

4.1. Diffeomorphism constraint

The classical diffeomorphism constraint in the presence of matter reads

$$\epsilon_{ijk} \tilde{\sigma}_j^a B_k^a = G H_i = G \pi \partial_i \phi,$$  \hfill (51)

where $H_i$ is the matter contribution. In (51), we have introduced a factor of $G$ to balance the dimensions in accordance with the time-space part of the Einstein equations $G_{0i} = G T_{0i}$. Substitution of the CDJ Ansatz (35) into (51) yields the condition

$$\epsilon_{ijk} (\Psi_{1a\beta} B_j^\beta ) B_k^a = G H_i.$$  \hfill (52)

Using the relation $\epsilon_{ijk} B_j^i B_k^j = (\det B)(B^{-1})^i_{\beta} \delta_{\alpha\beta}$ and assuming the nondegeneracy of the Ashtekar magnetic field $B_{\alpha\beta}$, we have

$$\epsilon_{\alpha\beta} \Psi_{1\alpha\beta} = G \frac{B_j^i H_i}{\det B} = G \tilde{\tau}_{0\alpha},$$  \hfill (53)

where $\tilde{\tau}_{0\alpha}$ is the projection of the time-space component of the matter energy–momentum tensor $T_{0\alpha}$ into $SU(2)_{\gamma}$. The tilde signifies that it is rescaled, or densitized, by a factor of $\det^{-1} B$, signifying again the nondegeneracy requirement alluded to earlier. Equation (53) is a statement that the antisymmetric part of the CDJ matrix is uniquely fixed by the matter contribution to the quantum diffeomorphism constraint, as noted in [22].

There are two main points of interest regarding (53): first, the constraint is locally satisfied as a linear relation. The antisymmetric components of $\Psi_{\alpha\beta}$ here and now depend upon the local matter momentum here and now. Secondly, a nontrivial right-hand side to (52) signifies one of the differences between the full superspace and minisuperspace theories since it contains spatial gradients of the matter fields which would otherwise be zero due to spatial homogeneity. In the minisuperspace theory one would have $H_i = 0$ for a Klein–Gordon scalar field, corresponding to a symmetric CDJ matrix.

The quantized diffeomorphism constraint reads, taking $H_i$ as the matter contribution, as

$$\left(\epsilon_{ijk} \tilde{\sigma}_j^i (x) B_k^i (x) + G \pi (x) \partial_i \phi (x) \right) |\Psi_{\text{GKod}}\rangle = 0.$$  \hfill (54)

\(^{11}\) Or any appropriate semiclassical limit even which corresponds to observable effects below the Planck scale, if desired.
In (54), we have used the quantized CDJ Ansatz as well as the definition $D_{ij}^{ab} = \delta B_i^a / \delta A_j^b$. The quantum terms in (54) vanish due to antisymmetry of $D_{ij}^{ab}$ and the assumption that functional differentiation and spatial differentiation commute as in (23). The requirement that the quantized diffeomorphism constraint be satisfied leads to the condition

$$
\epsilon_{ijk} \Psi_{be} B_i^b B_j^e + G\pi \partial_i \phi = 0.
$$

Equation (55) is precisely the same condition that would arise from the classical part of the constraint, and therefore satisfies a semiclassical-quantum correspondence. In fact, this result holds independently of the chosen operator ordering. It is a property of constraints which are linear in conjugate momenta that the operator ordering for the quantized version is immaterial [15, 24].

4.2. Gauss’ law constraint

The classical Gauss’ law constraint in the presence of the matter fields reads

$$
D_i \tilde{\sigma}_i^a(x) = -G Q_a(x). \quad (56)
$$

Again, note that the matter contribution $Q_a$ contains a factor of $G$ relative to the gravitational contribution in (56) in order to balance the dimensions. Implicit in (56) is a dimensionless constant $\lambda$, which represents the numerical value of the matter $SU(2)_-$ charge, in the definition of $Q_a$. This would be the analogue of the electric charge $e$ in Maxwell theory and is expected to be very small. Substitution of the CDJ Ansatz into (56) leads to the condition

$$
D_i \left( \Psi_{ae} B_i^e \right) = \Psi_{ae} D_i B_i^e + B_i^e D_i \Psi_{ae} = -G Q_a. \quad (57)
$$

Using the Bianchi identity this yields

$$
B_i^e D_i \Psi_{ae} = -G Q_a. \quad (58)
$$

The quantum Gauss’ law constraint reads

$$
\hat{G}_a|\Psi_{GKod}\rangle = \left[ D_i \left( \hbar G \frac{\delta}{\delta A_\alpha^a(x)} \right) + G Q_a(x) \right]|\Psi_{GKod}\rangle = 0, \quad (59)
$$

where $Q_a$ is the $SU(2)_-$ charge for a general matter field, given by

$$
Q_a(x) = \lambda \pi_a(x) (T_a)^{\alpha\beta} \phi^\beta(x). \quad (60)
$$

Incorporating the matter contribution and applying the CDJ Ansatz, the matter constraint reads

$$
\left[ D_i \left( \Psi_{ae}(x) B_i^e(x) \right) - i\hbar \lambda G \frac{\delta \phi^\beta(x)}{\delta \phi^\alpha(x)} (T_a)^{\alpha\beta}(x) - i\hbar \lambda G \phi^\beta (T_a)^{\alpha\beta}(x) \right]|\Psi_{GKod}\rangle
$$

$$
= \left( D_i \left( \Psi_{ae} B_i^e \right) - i\hbar \lambda G \delta^{ij}(0) \delta_i^a (T_a)^{\alpha\beta} + G\lambda \pi_a(x) (T_a)^{\alpha\beta} \delta \phi^\beta(x) \right)|\Psi_{GKod}\rangle
$$

$$
= \left( D_i \left( \Psi_{ae} B_i^e \right) + G Q_a \right)|\Psi_{GKod}\rangle \quad (61)
$$

where we have used in the last line of (61) that the $SU(2)_-$ charge generators $(T_a)^{\alpha\beta}$ are traceless in order to get rid of the singular $\delta^{(3)}(0)$ term in the second line of (61). So we see from (61) that the quantum condition precisely implies the semiclassical condition. The Gauss’ law constraint satisfies the SQC due to being linear in conjugate momenta.

For a Klein–Gordon scalar field $\phi$ the source term $Q_a$ is zero since the field is a Lorentz scalar and therefore does not transform under $SU(2)$. The Gauss’ law constraint for the

12 This assumption differs from conventional calculus. A brief development of some basic techniques for dealing with field theoretical singularities can be found in [25].
general case of matter coupled to gravity is given, by the coefficient of $\Psi_{G\text{Kod}}$ in the last line of (61), by
\begin{equation}
D_i (\Psi_{ae} B^i_{ae}) + G Q_a = 0.
\end{equation}
Let us focus first on the first term on the left-hand side of (62). This reads
\begin{equation}
D_i (\Psi_{ae} B^i_{ae}) = (D_i B^i_{ae}) \Psi_{ae} + B^i_{ae} D_i \Psi_{ae} = B^i_{ae} D_i \Psi_{ae},
\end{equation}
where we have used the Bianchi identity. The last term of (63) forms the covariant derivative of a $SU(2)_- \otimes SU(2)_-$ valued tensor $\Psi_{ae}$. Expanding this, applying the tensor representation of the covariant derivative, we have
\begin{equation}
B^i_{ae} D_i \Psi_{ae} = B^i_{ae} \left( \partial_i \Psi_{ae} + f_{afg} A^j_{ae} (\Psi_{ge} + f_{efg} A^i_{ag}) \right)
= B^i_{ae} \partial_i \Psi_{ae} + B^i_{ae} A^j_{ae} (f_{afg} \Psi_{ge} + f_{efg} \Psi_{ag}).
\end{equation}
Note that for the case of a homogeneous and isotropic CDJ matrix $\Psi_{ab} = 6 \Lambda^{-1} \delta_{ab}$ (e.g. the pure Kodama state), (64) would be zero. Hence factoring out $\Lambda$ which is a numerical constant, we would have
\begin{equation}
B^i_{ae} \partial_i \Psi_{ae} + B^i_{ae} A^j_{ae} (f_{afg} \Psi_{ge} + f_{efg} \Psi_{ag}) = B^i_{ae} \partial_i \delta_{ae} + B^i_{ae} A^j_{ae} (f_{afg} \delta_{ge} + f_{efg} \delta_{ag})
= 0 + B^i_{ae} A^j_{ae} (f_{afe} + f_{efa}) = 0
\end{equation}
due to antisymmetry of the structure constants.$^{13}$

4.3. Recapitulation of the kinematic constraints

There are a few items of note regarding the kinematic constraints:

(i) Both sets of constraints are linear in conjugate momenta and their solution depends linearly upon the matter source, namely the Noether charges corresponding to the respective kinematic symmetries. The kinematic constraints by definition satisfy the SQC since the operator ordering for the quantized version is immaterial.

(ii) As a corollary to (i), the processes of Dirac quantization and phase space reduction are the same for the kinematic constraints.$^{15}$

(iii) The solutions to the six kinematic constraints eliminate, modulo boundary conditions on the Gauss’ law constraints, six out of nine degrees of freedom of the CDJ matrix $\Psi_{ab}$, leaving three degrees of freedom remaining for the Hamiltonian constraint.

(iv) The diffeomorphism constraint determines the antisymmetric part of the CDJ matrix and depends locally upon the spatial gradients of the matter field, which distinguishes one aspect of the full theory from minisuperspace. The Gauss’ law constraint in general should determine three alternate CDJ matrix elements, reducing the CDJ matrix $\Psi_{ae}$ by an additional three degrees of freedom. The matter contribution to this constraint is of exactly the same form in the minisuperspace and in the full theory, but does distinguish $SU(2)_-$ scalars from fields transforming nontrivially under $SU(2)_-$. The diffeomorphism constraint makes this distinction as well, via the difference between a spatial gradient $\partial_i$ and a $SU(2)_-$ covariant derivative $D_i = \partial_i + A_i$.

(v) Lastly, the quantized versions of the kinematic constraints do not produce any information not already contained in their classical counterparts due to the SQC. The form of these constraints is model independent, since they are expressed entirely as a representation of the kinematic gauge algebra, namely a semidirect product of $SU(2)_-$ gauge transformations with diffeomorphisms.$^{14}$

13 We provide an explicit general solution to the Gauss’ law constraint, in the full theory coupled to matter fields, in $^{[26]}$.

14 Some interesting relationships between gauge transformations and diffeomorphisms can be found by the interested reader in $^{[27]}$. 
5. Quantization of the Hamiltonian constraint

The classical Hamiltonian constraint is given by

\[ \frac{1}{6} \epsilon_{ijk} \epsilon_{abc} \tilde{\sigma}_i^a \tilde{\sigma}_j^b \tilde{\sigma}_k^c + \epsilon_{ijk} \epsilon_{abc} \tilde{\sigma}_i^a \tilde{\sigma}_j^b B^c + G \Omega = 0, \]  

(66)

where \( \Omega \) is the matter contribution. Substitution of the classical CDJ Ansatz results in one condition relating the invariants of the CDJ matrix, namely

\[ \det B (\text{Var} \Psi + \Lambda \det \Psi) + G \Omega = 0, \]  

(67)

where we have defined \( \text{Var} \Psi \), the variance of the CDJ matrix, as

\[ \text{Var} \Psi = (\text{tr} \Psi^2) - \text{tr} \Psi^2. \]  

(68)

The matter contribution in general contains dependence upon the CDJ matrix elements in addition to the matter fields \( \Omega = \Omega[\phi, \pi, \Psi_{ab}] \), due to the gravity–matter coupling \( g^{\mu\nu} T_{\mu\nu} \) stemming from the Einstein equations \( G_{\mu\nu} \propto T_{\mu\nu} \). Unlike for the kinematic constraints, the form of the matter contribution to the Hamiltonian constraint is model specific.

The solution of (67) would eliminate one degree of freedom in the CDJ matrix. Combined with the six kinematic constraint solutions this leaves, modulo boundary conditions due to the Gauss' law constraint, two degrees of freedom remaining in the CDJ matrix \( \Psi_{ab} \). This results in a two-parameter ambiguity in the solution at the classical level.

Since we are interested in generalized quantum Kodama states, we must solve the quantized version of the Hamiltonian constraint. The quantized version of (66) is given, with respect to the spatial hypersurface \( \Sigma_T \) labelled by \( T \), by

\[ \hat{H}_{\Psi_{\text{GKod}}}[(A, \phi)] = \left[ \frac{\Lambda}{6} \hbar^2 G^3 \epsilon_{abc} \epsilon_{ijk} \frac{\delta}{\delta A_i^a(x)} \frac{\delta}{\delta A_j^b(x)} \frac{\delta}{\delta A_k^c(x)} + \hbar^2 G^2 \epsilon_{abc} \epsilon_{ijk} \frac{\delta}{\delta A_i^a(x)} \frac{\delta}{\delta A_j^b(x)} \frac{\delta}{\delta A_k^c(x)} + \hat{\Omega} \left[ \phi^a, \delta/\delta \phi^a, \delta/\delta A_i^a \right] \Psi_{\text{GKod}}[(A, \phi)] = 0 \quad \forall x. \]  

(69)

A question arises as to whether the SQC is broken due to (69) being cubic in conjugate momenta unlike for the kinematic constraints. Substitution of the quantized CDJ Ansatz (36) into (69) leads to a condition of the form (suppressing the implicit \( T \) dependence)

\[ \hat{H}(x)[\Psi_{\text{GKod}}] = (q_0(x) + \hbar G \delta^{(3)}(x) q_1(x) + (\hbar G \delta^{(3)}(x))^2 q_2(x)) \Psi_{\text{GKod}} = 0 \]  

(70)

for all \( x \) in the spatial hypersurface \( \Sigma_T \), whereupon the nonlinear action of the Hamiltonian constraint upon the state leads to the presence of singular quantum gravitational terms.

In a usual field-theoretical treatment, such terms would be regularized by some prescription in order to yield a finite result. However, there is no guarantee that the result obtained by solving the regularized constraint is independent of the regularization prescription [16]. Therefore, we shall dispense with any regularization procedures altogether in the canonical part of our quantum treatment of gravity. Instead, we will show that, due to the choice of the generalized Kodama state, singular terms in the constraint equations are cancelled out.

5.1. Cosmological contribution to the expansion of the quantized Hamiltonian constraint

There are a total of three contributions to the quantized Hamiltonian constraint, namely the cosmological term, the curvature and the matter terms. Starting with the cosmological term
\( H_\Lambda \), suppressing the implicit \( x \) dependence,

\[
H_\Lambda |_{\Psi_{\text{GKod}}} = \frac{\Lambda}{6} (\hbar G)^3 \epsilon_{ijk} \epsilon^{abc} \frac{\delta}{\delta A_i^a} \frac{\delta}{\delta A_j^b} \frac{\delta}{\delta A_j^c} |_{\Psi_{\text{GKod}}}
\]

\[= \frac{\Lambda}{6} (\hbar G)^2 \epsilon_{ijk} \epsilon^{abc} \left( \frac{\delta}{\delta A_i^a} \frac{\delta}{\delta A_j^b} \frac{\delta}{\delta A_j^c} \right) (\Psi_{ce} B_k^e) |_{\Psi_{\text{GKod}}}. \tag{71} \]

In (71), we have made the CDJ Ansatz \( \tilde{\sigma}_e^k = \Psi_{ce} B_k^e \). Note from (71) that the CDJ matrix \( \Psi_{ce} \) now plays the role of an inverse ‘generalized’ cosmological constant. Although it is in general a field-dependent \( SU(2) \otimes SU(2) \)-valued tensor, it is analogous to \( \Lambda^{-1} \) for the pure Kodama state and has mass dimensions \( [\Psi_{ce}] = -2 \). Since the Ashtekar magnetic field \( B_e^k \) has mass dimensions \( [B_e^k] = 2 \) it provides a check on dimensional consistency to note that the densitized triad \( \tilde{\sigma}_e^k \) must be dimensionless \( ([\tilde{\sigma}_e^k] = 0) \). Continuing along from (71),

\[
\frac{\Lambda}{6} (\hbar G)^2 \epsilon_{ijk} \epsilon^{abc} \left( \frac{\delta}{\delta A_i^a} \frac{\delta}{\delta A_j^b} \frac{\delta}{\delta A_j^c} \right) (\Psi_{ce} B_k^e) \bigg|_{\Psi_{\text{GKod}}}
\]

\[= \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \left( \frac{\hbar G}{\delta A_i^a} \frac{\hbar G}{\delta A_j^b} \frac{\hbar G}{\delta A_j^c} \right) \left[ (\Psi_{ce} \psi_{bf} B^0_k B^j_0) + \Psi_{ce} \psi_{bf} B^k_e B^j_0 \right] \bigg|_{\Psi_{\text{GKod}}}
\]

\[= \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \left( \frac{\hbar G}{\delta A_i^a} \frac{\hbar G}{\delta A_j^b} \frac{\hbar G}{\delta A_j^c} \right) \left[ (\Psi_{ce} \psi_{bf} B^0_k B^j_0) + \Psi_{ce} \psi_{bf} B^k_e B^j_0 \right] \bigg|_{\Psi_{\text{GKod}}}. \tag{72} \]

The observation that the second and third terms in the last two lines of (72) are proportional to each other enables a simplification of the ‘eigenvalue’ of the cosmological term. This can be seen by expanding out the coefficient of the quantum gravitational singularity and reshuffling indices:

\[
\frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \left( \frac{\partial}{\partial A_i^a} \right) (\Psi_{ce} \psi_{bf} B^k_e B^j_0) + \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \left( \frac{\partial}{\partial A_j^b} \right) (\Psi_{ce} B_k^e)
\]

\[= \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \left( \frac{\partial}{\partial A_i^a} \right) (\Psi_{ce} \psi_{bf} B^k_e B^j_0) + \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \left( \frac{\partial}{\partial A_j^b} \right) (\Psi_{ce} B_k^e)
\]

\[+ \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \left( \frac{\partial}{\partial A_j^c} \right) (\Psi_{ce} B_k^e) \tag{73} \]

in (73) we have used the Liebnitz rule. To show all three terms on the right-hand side of (73) are equal, relabel \( b \leftrightarrow c, f \leftrightarrow e \) and \( j \leftrightarrow k \) on the second term and \( a \leftrightarrow c, f \leftrightarrow e \) and \( i \leftrightarrow k \) on the third term. This leads to

\[
\frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \left( \frac{\partial}{\partial A_i^a} \right) (\Psi_{ce} \psi_{bf} B^k_e B^j_0) + \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \left( \frac{\partial}{\partial A_j^b} \right) (\Psi_{ce} B_k^e)
\]

\[+ \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \left( \frac{\partial}{\partial A_j^c} \right) (\Psi_{ce} B_k^e) = \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \left( \frac{\partial}{\partial A_i^a} \right) (\Psi_{ce} \psi_{bf} B^k_e B^j_0)
\]

\[+ \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \left( \frac{\partial}{\partial A_j^b} \right) (\Psi_{ce} B_k^e) + \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \left( \frac{\partial}{\partial A_j^c} \right) (\Psi_{ce} B_k^e). \tag{74} \]

Relabelling \( j \leftrightarrow i \) and \( a \leftrightarrow b \) on the last term on the right-hand side of (74), we obtain a final result for the second and third terms on the right-hand side on the bottom two lines of (72)
which constitute the first-derivative terms of the constraint, of
\[ \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} (\psi_{ce} B^k_i) \frac{\partial}{\partial A^c_j} (\psi_{bf} B^j_f) + \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} (\psi_{ce} B^k_i) \frac{\partial}{\partial A^c_j} (\psi_{bf} B^j_f) + \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} (\psi_{ce} B^k_i) \frac{\partial}{\partial A^c_j} (\psi_{bf} B^j_f) \]
\[ = \frac{1}{2} \left( \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \frac{\partial}{\partial A^c_i} (\psi_{ce} \psi_{bf} B^k_i B^j_f) = \frac{\Lambda}{4} \frac{\partial}{\partial A^c_i} \left( \epsilon_{ijk} \epsilon^{abc} \psi_{ce} \psi_{bf} B^k_i B^j_f \right). \right. \] (75)

The semiclassical part of (72) is given by
\[ \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} B^k_i B^j_f \psi_{ce} \psi_{bf} \psi_{af} = \frac{1}{6} (\text{det} B) \psi_{ce} \psi_{bf} \psi_{af} = \Lambda (\text{det} B) \psi. \]

The factor of 6 due to the definition of the determinant has cancelled out. So the total contribution due to the eigenvalue of the cosmological term is given by
\[ \hat{H}_\Lambda |\Psi_{\text{GKod}}\rangle = \frac{\Lambda}{6} (hG)^3 \epsilon_{ijk} \epsilon^{abc} \delta_{A^c_i} \delta_{A^c_j} |\Psi_{\text{GKod}}\rangle \]
\[ = \left[ \Lambda (\text{det} B) \psi + \hbar G \delta^{(3)}(x) \left( \frac{\Lambda}{4} \frac{\partial}{\partial A^c_i} \left( \epsilon_{ijk} \epsilon^{abc} \psi_{ce} \psi_{bf} B^k_i B^j_f \right) \right) \right. \]
\[ + \left. (\hbar G \delta^{(3)}(x))^2 \left( \frac{\Lambda}{6} \epsilon_{ijk} \epsilon^{abc} \frac{\partial}{\partial A^c_i} \frac{\partial}{\partial A^c_j} (\psi_{ce} \psi_{bf} B^k_i B^j_f) \right) \right] |\Psi_{\text{GKod}}\rangle. \] (76)

We shall now move on to the curvature contribution.

### 5.2. Curvature contribution to the expansion of the quantized Hamiltonian constraint

The curvature contribution \( \hat{H}_{\text{curv}} \) to the quantized Hamiltonian constraint is given by
\[ \hat{H}_{\text{curv}} |\Psi_{\text{GKod}}\rangle = (hG)^2 \epsilon_{ijk} \epsilon^{abc} \frac{\delta}{\delta A^c_i} \frac{\delta}{\delta A^c_j} B^k_i |\Psi_{\text{GKod}}\rangle \]
\[ = \epsilon_{ijk} \epsilon^{abc} hG \frac{\delta}{\delta A^c_i} \left[ hG \delta^{(3)}(x) D_{cb}^{kj} + B^k_i (\psi_{ce} B^j_f) \right] |\Psi_{\text{GKod}}\rangle. \] (77)

Note that we have maintained an operator ordering with momenta to the left of the coordinates in analogy to that determining the pure Kodama state. It has been demonstrated that for this ordering the quantum algebra of constraints formally closes \([5, 6]\). Continuing along,
\[ \epsilon_{ijk} \epsilon^{abc} hG \frac{\delta}{\delta A^c_i} \left[ hG \delta^{(3)}(x) D_{cb}^{kj} + B^k_i (\psi_{ce} B^j_f) \right] |\Psi_{\text{GKod}}\rangle \]
\[ = \epsilon_{ijk} \epsilon^{abc} hG \frac{\delta}{\delta A^c_i} \left[ hG \delta^{(3)}(x) \epsilon_{cba} + B^k_i (\psi_{ce} B^j_f) \right] |\Psi_{\text{GKod}}\rangle. \] (78)

The semiclassical part of (78) is given by
\[ \epsilon_{ijk} \epsilon^{abc} B^k_i (\psi_{ce} B^j_f) (\psi_{af} B^j_f) = \left( \text{det} B \right) \epsilon_{fca} \epsilon^{abc} \psi_{af} \psi_{be} \]
\[ = \left( \text{det} B \right) (\delta^e_d \delta^f_c - \delta^e_c \delta^f_d) \psi_{af} \psi_{be} = (\text{det} B) \text{Var} \psi. \] (79)
The coefficient of the highest degree of singularity \((\hbar G\delta^{(3)}(x))^2\) in (78) is a nonzero numerical constant equal to 36, as can be seen from the manipulation
\[
\epsilon_{ijk} \epsilon^{abc} \epsilon^{kij}_{\ cba} = (\epsilon_{ijk} \epsilon^{ijk})^2 = 36.
\] Therefore, in order to satisfy the quantum Hamiltonian constraint by canonical methods without complications, it is necessary to have a contribution to \(\hat{H}\) that cancels this numerical constant. So, the total contribution due to the curvature is given by
\[
\hat{H}_{\text{curv}} |\Psi_{G}\rangle = \left(\det B\right) \text{Var} |\Psi_{G}\rangle + \hbar G\delta^{(3)}(x)\left(\epsilon_{ijk} \epsilon^{abc} \frac{\partial}{\partial A^b_j} (B^k_{\ i} \Psi_{ba} B^i_j) + D^k_{cb} \Psi_{ac} B^i_c \right) + 36(\hbar G\delta^{(3)}(x))^2.
\]

Next we move on to the matter contribution.

5.3. Matter contribution to the expansion of the quantized Hamiltonian constraint

The contributions calculated thus far to the quantized Hamiltonian constraint are of the same form regardless of the model, as in the case of the kinematic constraints. It is the matter contribution that distinguishes one model from another. The quantized matter contribution to the Hamiltonian constraint for a general matter field will be of the general form
\[
G \hat{\Omega}_{\text{GKod}} |\Psi_{G}\rangle = \left(\sum_{n=-\infty}^{\infty} (\hbar G\delta^{(3)}(0))^n \Omega_n\right) |\Psi_{G}\rangle
\]
where, in (82), \(\Omega_n\) are the model-specific coefficients for a given degree of singularity and we have made the replacement, in an abuse of notation, of \(\delta(3)(x) \rightarrow \delta(3)(0)\). Before attempting to solve the constraints, we must take into account the contributions due to the matter fields for the full theory.

Let us illustrate a Klein–Gordon scalar \(\phi\) with conjugate momentum \(\pi\). We will assume that the scalar potential \(V(\phi)\) can be included as a contribution to the cosmological term \(\Lambda\), hence it suffices to consider the kinetic and spatial gradient terms. Starting with the classical form of this contribution,
\[
H_{\text{KG}} = \frac{\pi^2}{2} + \frac{1}{2} \partial_i \phi \partial_j \phi \bar{\sigma}_i^{\alpha} \bar{\sigma}_j^{\alpha},
\] we have, upon quantization and making the identification \((1/2)\partial_i \phi \partial_j \phi = T_{ij}\),
\[
(\hat{H})_{\text{KG}} |A^\nu, \phi\rangle = \left[ -\frac{\hbar^2}{2} \frac{\delta^2}{\delta \phi(x) \delta \phi(x)} + \hbar^2 G^2 T_{ij} \frac{\delta}{\delta A^\eta_i(x)} \frac{\delta}{\delta A^\eta_j(x)} \right] |A^\nu, \phi\rangle.
\] Continuing on from (84) and making use of the CDJ Ansatz,
\[
(\hat{H})_{\text{KG}} |A^\nu, \phi\rangle = \left[ -\frac{\hbar^2}{2} \frac{\delta^2}{\delta \phi(x) \delta \phi(x)} \pi(x) + \hbar G T_{ij} \frac{\delta}{\delta A^\eta_i(x)} \left( \Psi_{ae}(x) B^j_e(x) \right) \right] |A^\nu, \phi\rangle
\] where \(T_{ij} D_{ij}^\eta\) in (85) vanishes due to symmetry of \(T_{ij}\) and antisymmetry of \(D_{ij}^\eta\). One can thus read off from (85) the contributions to \(\Omega_0\) and \(\Omega_1\) in (82) due to the Klein–Gordon field.
These are given by
\[
\Omega_0 = \frac{\pi^2}{2} + T_{ij} \psi_{ae} \psi_{bf} B^i_j = \frac{\pi^2}{2} + \delta_{ab} \tau_{ef} \psi_{ae} \psi_{bf};
\]
\[
\Omega_1 = -\frac{i}{2G} \frac{\partial \pi}{\partial \phi} + T_{ij} B^i_e \frac{\partial \psi_{ae}}{\partial A^e_a} = -\frac{i}{2G} \frac{\partial \pi}{\partial \phi} + \tau_{ef} \frac{\partial \psi_{ae}}{\partial A^e_a}
\]
and \( \Omega_N = 0 \) for \( N \geq 2 \) and \( N < 0 \), where we have defined \( \tau_{ef} = T_{ij} \psi_{ae} \psi_{bf} B^i_j \) as the undensitized projection of the space–space components of the energy–momentum tensor from \( \Sigma \) into \( SU(2)_+ \), with \( \tau_{ef} = T_{ij} B^i_e \) corresponding to the projection of just one spatial index into \( SU(2)_+ \).

5.4. Putting it all together

The full expansion of the quantum Hamiltonian constraint can be written in the form, combining all terms,
\[
\hat{H} |\psi_{\text{GKod}}\rangle = \langle H_\Lambda + H_{\text{curv}} + H_{\text{matter}} | \psi_{\text{GKod}} \rangle = \left[ \det B \left( \text{Var} \psi + \langle \Lambda + G V \rangle \text{det} \psi \right) + G \Omega_0 
+ hG\delta^{(3)}(x) \left( \epsilon_{ijk} \epsilon^{abc} \left[ D_{ij}^{k} B^i_e \psi_{ae} + \frac{\partial}{\partial A^a_i} \right] + \left( B^i_e B^j_e \psi_{be} + \frac{\langle \Lambda + G V \rangle}{4} B^i_e B^j_e \psi_{bf} \psi_{bf} \right) + G \Omega_1 \right) + \langle hG\delta^{(3)}(x) \rangle^2 \left( \frac{\langle \Lambda + G V \rangle}{6} \epsilon_{ijk} \epsilon^{abc} \frac{\partial}{\partial A^a_i} \frac{\partial}{\partial A^b_j} \left( \psi_{ce} B^c_e \right) + 36 \right) \right] |\psi_{\text{GKod}}\rangle = 0.
\]

(87)

We see from (87) that a third-order functional differential condition on the generalized Kodama wavefunction \( \psi_{\text{GKod}} \) is equivalent to a second-order partial differential condition on the CDJ matrix elements. The tradeoff is that whereas the former functional differential condition is linear, the latter partial differential condition is nonlinear. The expansion (87) can be written in compact form as
\[
\hat{H}(x) \psi_{\text{GKod}} = [q_0(x) + hG\delta^{(3)}(x)q_1(x) + \langle hG\delta^{(3)}(x) \rangle^2 q_2(x)] \psi_{\text{GKod}} = 0 \quad \forall x.
\]

(88)

In order for (88) to be satisfied for all \( x \) in \( \Sigma_T \), which is equivalent to the condition that the quantum Hamiltonian constraint, upon direct promotion from its classical counterpart (which stems from the requirement \( \delta I_{\text{LH}}/\delta N(x) = 0 \quad \forall x \in M \)), be identically satisfied \( \forall x \), we must impose that \( q_0(x) = 0 \quad \forall x \) on \( \Sigma_T \). This is the classical part of the Hamiltonian constraint and also forms the semiclassical part of the SQC.

Note for \( x \neq 0 \) that the quantized Hamiltonian constraint is identically zero due to the delta functions, which have support only at the origin \( (x, t) = (0, t) \) for all times \( t \) in \( M \). So there is an automatically manifest semiclassical-quantum correspondence for all points not including the spatial origin. But we require that the quantized Hamiltonian constraint be satisfied everywhere, including the origin, as a necessary condition for a finite state. This dictates, and is often put in an abuse of notation, that
\[
\hat{H} |\psi_{\text{GKod}}\rangle = [q_0 + hG\delta^{(3)}(0)q_1 + \langle hG\delta^{(3)}(0) \rangle^2 q_2] |\psi_{\text{GKod}}\rangle = 0.
\]

(89)

The continuity of the SQC imposes conditions on the coefficients of the singular delta functions in (89), namely that \( q_0 = q_1 = q_2 = 0 \) for all \( x \) on the hypersurface \( \Sigma_t \) for each \( t \). Since the origin of \( \Sigma \) can be arbitrarily chosen, then these conditions must be satisfied at all points \( x \) in
This implies certain functional relationships in the coefficients \( q_0, q_1 \) and \( q_2 \), amongst the fields \( A^a_i = A^a_i(x) \) and \( \phi^a = \phi^a(x) \) which must be true for all \( x \) independently of position in \( \Sigma \). The explicit \( x \) dependence of the fields themselves can then be suppressed, since \( x \) is merely a dummy label.

### 6. Constraints corresponding to the finite states of quantum gravity

The existence of \( \Psi_{\text{GKod}} \) in the full theory (including minisuperspace as a subset) will depend upon the existence of solutions to the quantum constraints for the CDJ matrix elements for an arbitrary model coupled to gravity with cosmological term. As long as (55), (58) and (67) are satisfied, quantum states obeying (36) and (41) are semiclassical in the precise sense that these states will be exponentials (of the Hamiltonian function) and the Hamilton–Jacobi equations will also hold.

The resulting condition upon the CDJ matrix elements \( \Psi_{ae} \) is a system of nine equations in nine unknowns. There are a total of three equations from the quantized Gauss’ law constraints, three equations from the quantized diffeomorphism constraint and three equations from the quantized Hamiltonian constraint. It will be convenient to think of these nine equations as a map from the nine CDJ matrix elements \( \Psi_{ae} \) to the nine equations, thought of as the components of a 9-vector \( C_{ab} = 0 \):

\[
\Psi_{ab}(x) \mapsto C_{ab}[(\Psi_{ef}[A^a_i(x)])].
\]

First let us write the system corresponding to the pure Kodama state:

\[
\epsilon_{aed} \Psi_{ae} = 0;
\]

\[
\left( \delta_{df} \frac{\partial}{\partial f^g} + C_{af}^f \right) \Psi_{fg} = 0;
\]

\[
q_0 = \det B(\Lambda \det \Psi + \text{Var}\Psi) = 0;
\]

\[
q_1 = \epsilon_{ijk} \epsilon^{abc} D^j_{ik} \Psi_{ae} B^a_e + \epsilon_{ijk} \epsilon^{abc} \frac{\partial}{\partial A^a_i} \left[ B^b_i B^c_j \Psi_{be} + \frac{\Lambda}{4} B^b_i B^c_j \Psi_{ce} \Psi_{bf} \right] = 0;
\]

\[
q_2 = \frac{\Lambda}{6} \frac{\partial}{\partial A^a_i} \frac{\partial}{\partial A^b_j} \left( \epsilon_{ijk} \epsilon^{abc} B^a_e \Psi_{ce} \right) + 36 = 0.
\]

The system (91) is a nonlinear system with solution \( \Psi_{ae} = -6\Lambda^{-1} \delta_{ae} \), which corresponds to the pure Kodama state \( \Psi_{\text{Kod}} \) in a quantum theory of gravity free of field-theoretical singularities at the level of the state. Note that all nine degrees of freedom in the CDJ matrix for \( \Psi_{\text{Kod}} \) are exhausted in order to produce a unique solution.

The basic principle of the nonperturbative quantization of gravity in the general case in the full theory is to introduce a driving force to the right-hand side of (91) corresponding to a particular matter model. The associated criterion of finiteness of the quantum state produces a system which would hopefully converge, in the functional sense, to the CDJ matrix elements \( \Psi_{ae} \) for the generalized Kodama state for the model with all of its degrees of freedom similarly exhausted. In the case of the Klein–Gordon field with self-interaction potential \( V = V(\phi(x)) \) coupled to gravity the associated system then becomes

\[
\epsilon_{aed} \Psi_{ae} = G \tau_{ab};
\]

\[
\left( \delta_{df} \frac{\partial}{\partial f^g} + C_{af}^f \right) \Psi_{fg} = -G Q_{ab} = 0;
\]

\[
\det B((\Lambda + G) \det \Psi + \text{Var}\Psi) + G \left( \frac{\pi^2}{2} + \delta_{ab} \tau_{ef} \Psi_{ae} \Psi_{bf} \right) = 0;
\]

\[
\Psi_{ab}(x) \mapsto C_{ab}[(\Psi_{ef}[A^a_i(x)])].
\]
\[ \epsilon_{ijk} \epsilon^{abc} D^j \Psi_{ae} B^i_c + \epsilon_{ijk} \epsilon^{abc} \frac{\partial}{\partial A^i_a} \left[ B^k_b B^j_c \Psi_{be} + \frac{(\Lambda + GV)}{4} B^k_b B^j_c \Psi_{ce} \Psi_{bf} \right] \]

\[ \frac{-1}{2} \frac{\partial \pi}{\partial \phi} + G \frac{\partial}{\partial A^i_a} \Psi_{ae} = 0; \]

\[ \frac{(\Lambda + GV)}{6} \frac{\partial}{\partial A^i_a} \frac{\partial}{\partial A^j_f} \left( \epsilon_{ijk} \epsilon^{abc} B^k_b \Psi_{ce} \right) + 36 = 0. \] 

(92)

The system (92) represents a system of nine equations in nine unknowns, corresponding to the model of the Klein–Gordon field coupled to quantum gravity in the full theory. Likewise, this system should have a unique solution for its CDJ matrix elements \( \Psi_{ae} \) corresponding to its associated generalized Kodama state \( \Psi_{GKod} \), when the mixed partials condition is taken into account. Note that the self-interaction potential \( V \) can be treated as a contribution to the cosmological constant \( \Lambda \). This procedure should in general be applicable to a wide class of models coupled to quantum gravity. In the case of the Klein–Gordon field the \( SU(2) \) charge \( Q_a \) is zero. The shorthand notation for the system (92) can be written as a map

\[ \Psi_{ab} \left[ A^i_a(x), \phi(x) \right] \rightarrow C_{ab} \left[ \Psi_{ef} \left[ A^i_a(x), \phi(x) \right] \right] \]

\[ = O_{ab} \Psi_{cd} + \Lambda I_{ab}^{cd} \Psi_{ef} + \Lambda^2 E_{ab}^{cd} \Psi_{cd} \Psi_{ef} = G Q_{ab} \] 

(93)

for suitably defined \( O_{ab} \), \( I_{ab}^{cd} \) and \( E_{ab}^{cd} \), and \( Q_{ab} \) where \( \Lambda' = \Lambda + GV \).

7. Theoretical arguments for the existence of generalized Kodama states

The following section is designed to examine the pure and generalized Kodama states from various perspectives relevant to the mathematical proof of their existence. Section 7.1 shows that the analogue of the topological \( F \wedge F \) Chern–Simons term for gravity when quantized on an equal footing in the presence of matter fields is simply another representation of \( \Psi_{GKod} \) upon solution to the quantum constraints. We derive and contrast this to work of previous authors, which view the CDJ matrix as a Lagrange multiplier necessary to enforce metricity as opposed to our use here. The presence of matter fields coupled to gravity does not alter the holographic nature of the state. They simply, in a sense, ‘rotate’ the holographic ground state \( \Psi_{Kod} \) into a new holographic ground state \( \Psi_{GKod} \). In section 7.2 we show, using geometric quantization, that the uniqueness of the solution \( C_{ab} = 0 \) to the quantum constraints guarantees the existence of \( \Psi_{Kod} \) and \( \Psi_{GKod} \) via the cohomology of the fibre bundle with base space consisting of the configuration space variables. The significance of the geometric quantization procedure outlined in section 7.2 is that it implies the equivalence of the reduced phase space and the Dirac canonical quantization procedures.

15 One could eliminate all components of the semiclassical matter momentum from (92) by using the integrated from of the mixed partials condition \( \pi_\alpha (A^i_a, \phi) = f_\alpha (\phi) - \frac{1}{N} \sum_{\alpha} (f_\alpha \delta X^{ae}) \Psi_{ae}(A^i_a, \phi) \), \( N \) conditions where \( N \) is the number of semiclassical matter conjugate momentum components \( \pi_\alpha \) labelled by \( \alpha \), and then look for solutions of the resulting nine by nine system or, alternatively, incorporate these \( N \) conditions into the constraints themselves dimensionally expanding them to a \( 9 + N \times 9 + N \) system which must be solved. Note that the generalized Kodama \( \Psi_{GKod} \) then acquires the label \( f_\alpha \), the functional boundary condition corresponding to the semiclassical matter momentum in the absence of gravity. It is this label which establishes the link from the quantum gravitational state \( \Psi_{GKod} \) to the semiclassical limit below the Planck scale, where the manifestation of quantum gravitational effects may possibly be predicted or ruled out.

16 For a demonstration of the technique for the general solution to the system (92), (93) the interested reader is directed to [11].

17 This refers to velocity independence as well as independence of the bulk configuration of the fields, a characteristic shared by topological field theories.
We will show in this section an important representation of the generalized Kodama state \( \Psi_{GKod} \) in analogy to the instanton \( F \wedge F \) term for the pure Kodama state \( \Psi_{Kod} \) generalized to the case when matter fields are present. We must first show, in analogy to the steps leading from (26) to (31), that the matter-coupled case does in fact correspond to a wavefunction defined only on the three-dimensional boundary \( \Sigma_T \) at time \( T \) and is in fact independent of the gravitational and matter fields in the bulk \( M \) of spacetime. Complementarily to the matter-free case previously discussed, we shall proceed in a more direct manner. We will start with the assumption that the wavefunction is a boundary term defined on the final spatial hypersurface \( \Sigma_T \) at time \( T \), and then explicitly show that this boundary term is the same as its four-dimensional version, the precise form of which we will deduce. The ability to do so without contradiction should suffice to prove our assertion.

Assume that there exists a functional \( \Psi_{GKod}(T) = \Psi_{GKod}[A(T), \vec{\phi}(T)] = e^{i(T)} \) of two dynamical variables \( (A_I^a(x, T), \phi^a(x, T)) \) as determined by the solution to the constraints (93) on the final spatial hypersurface \( \Sigma_T \):

\[
\Psi(T) = e^{i(T)} = e^{i(T) - i(t_0)} e^{i(t_0)} = e^{i(T) dT/\partial \sigma} \Psi(t_0) \tag{94}
\]

where \( \Psi(t_0) = \Psi_{GKod}[A(t_0), \vec{\phi}(t_0)] \) is the state as determined by the solution to the constraints (93) on the initial spatial hypersurface \( \Sigma_0 \). On the right-hand side in the last term of (94), the functional \( I(T) \) is extended from \( \Sigma_T \) into the interior of \( M \) to produce \( I(t) \) for \( t_0 < t < T \).

Note that the right-hand side of (94) is independent of this detailed time dependence, since the left-hand side is independent of any such dependence by the starting assumption. Expansion of the time derivative in (94) while making use of the field-theoretical time variation of \( I \) leads to

\[
\Psi(T) = \exp \left[ \int_{t_0}^{T} dt \int_{\Sigma} d^3 x \left( \frac{\delta I}{\delta A_i^a} \frac{\partial A_i^a}{\partial t} + \frac{\delta I}{\delta \phi} \frac{\partial \phi}{\partial t} \right) \right] \Psi(t_0). \tag{95}
\]

We now specify the functional form of the state \( \Psi(T) \) on the left-hand side of (95) by choosing it to correspond to the solution to the constraints (92) on the final hypersurface \( \Sigma_T \). Since the functional form of the constraints is the same for each \( 0 \leq t \leq T \), we can extend the functional form of \( I(T) = I[A(T), \vec{\phi}(T)] \) on the spacelike boundary \( \partial M = \Sigma_T \) to the same functional form \( I(t) = I[A(t), \vec{\phi}(t)] \) on the spacelike hypersurface \( \Sigma_t \) for any \( t_0 < t < T \).

That the constraints have the same form independently of position \( x \) in \( M \) is a result of the classical equations of motion for the Lagrange multipliers which implement the constraints

\[
\frac{\delta S}{\delta N^i(x)} = \frac{\delta S}{\delta N^a(x)} = \frac{\delta S}{\delta \phi^a(x)} = 0 \quad \forall \ x \in M. \tag{96}
\]

Assuming that the constraints when implemented on the quantum level allow for a unique solution of (92) corresponding to the CDJ matrix elements \( \Psi_{cdj} \), we identify the quantity \( I \) in (95) with the Hamilton–Jacobi functional for a WKB state. Since the semiclassical-quantum correspondence holds by construction, then the WKB state is as well the quantum state. We can then make the replacements

\[
\frac{\delta I}{\delta A_i^a(x)} = (\hbar G)^{-1} \Psi_{cdj}[A_i^a(x), \phi^a(x)] B_i^a[A(x)], \tag{97}
\]

\[\text{Note that this is a total time derivative and hence is still independent of the bulk configuration of the fields, even though the functional } I \text{ has been extended into the interior of the spacetime region } M.\]
which merely renames the semiclassical gravitational conjugate momentum based upon the solution to the constraints, and
\[
\frac{\delta I}{\delta \phi^\alpha(x)} = \frac{i}{\hbar} \left( -i\hbar \frac{\delta I}{\delta \phi^\alpha(x)} \right) = \frac{i}{\hbar} \pi^\alpha(A^\mu_i(x), \phi^\alpha(x)),
\]
which renames the semiclassical matter conjugate momentum. Note that (97) and (98) suffice to imply the mixed partials condition which in turn stems from the requirement that \( I(T) \) be the integral of a total time derivative, since by evaluation of cross derivatives we have that
\[
\frac{\delta^2 I}{\delta A^\mu_i \delta \phi^\alpha} = \frac{\delta^2 I}{\delta A^\mu_i \delta \phi^\alpha} = \frac{i}{\hbar} \pi^\alpha(A^\mu_i(x), \phi^\alpha(x)),
\]
\[
\frac{\delta^2 I}{\delta A^\mu_i \delta \phi^\alpha} = \frac{\delta^2 I}{\delta \phi^\alpha \delta \pi^\alpha} = (\hbar G)^{-1} B^\mu_i \frac{\delta \Psi_{\text{GKod}}}{\delta \phi^\alpha}.
\]

The mixed partials condition is necessary in order to eliminate the semiclassical matter conjugate momentum \( \pi^\alpha \) from the constraints (92), so that the wavefunctional \( \Psi_{\text{GKod}} = \Psi_{\text{GKod}}[A^\mu_i, \phi^\alpha] \) can be expressed completely in terms of the configuration variables \( \{A^\mu_i, \phi^\alpha\} \).

Equations (37) and (38) in conjunction with the CDJ Ansatz are necessary in order to obtain the quantized version of the constraints with associated degrees of singularity, which place conditions on the CDJ matrix (93) which are unique to the Hamiltonian of general relativity. Equation (43) guarantees that the wavefunctional has the proper polarization to enable these conditions to be determined and is the correct boundary term corresponding to these conditions. Equations (39) and (40) simply follow as a consistency condition that follows from the existence of the aforementioned boundary term and its polarization. It then follows from all these observations that the existence of the generalized Kodama state \( \Psi_{\text{GKod}} \) is a natural consequence of the exhaustive application of the canonical quantization relations of quantum field theory to gravity in Ashtekar variables in the Schrödinger representation, which follows from the principle of the SQC.

Proceeding along by substitution of (97) and (98) into (95) leads to
\[
\Psi(T) = \Psi_{\text{GKod}}(T)
\]
\[
= \exp \left[ \int_{t_0}^{T} dt \int_\Sigma d^3x \left( (\hbar G)^{-1} \Psi_{\text{GKod}} B^\mu_i A^\mu_i + \frac{i}{\hbar} \pi^\alpha \phi^\alpha \right) \right] \quad (100)
\]
where the notation \( \pi^\alpha_i(\phi) \) signifies that the mixed partials condition in integrated form has already been incorporated into the solution to the constraints.

We now derive the instanton representation of \( \Psi_{\text{GKod}} \), which highlights its relation to topological field theory. The first thing to note is that the argument of the exponential in (100) is nothing other than starting action (34) evaluated on the reduced phase space upon solution to the constraints as the integral of a total time derivative. Writing the argument as a spacetime integral, we have
\[
I = \int_M \left( (\hbar G)^{-1} \Psi_{\text{GKod}} B^\mu_i A^\mu_i + \frac{i}{\hbar} \pi^\alpha \phi^\alpha \right). \quad (101)
\]
To see more clearly the relation between the two terms from a different perspective, it helps to express (101) in covariant form. We have, for the gravitational part,
\[
\Psi_{\text{GKod}} B^\mu_i A^\mu_i = \Psi_{\text{GKod}} B^\mu_i \left( F^\alpha_{ij} + D_i A^\alpha_j \right) = \epsilon^{ijk} \Psi_{\text{GKod}} F^\alpha_{ik} F^\beta_{kj} + \hat{\sigma}^\alpha_i D_i A^\alpha_j
\]
\[
(102)
\]
20 As indicated under footnote 16 in section 6, one may view the mixed partials condition as an extension of the quantum constraints to be solved in conjunction with (92) as a \( 9+N \) by \( 9+N \) system, where \( N \) is the number of mixed partials conditions in integrated form, \( N \) of them corresponding to the \( N \) components of the matter fields, labelled by the index \( \alpha \).
where we have made the identification in analogy to $SU(2)$ Yang–Mills theory, that

$$F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu.$$  

(103)

To show the steps for the zeroth component,

$$F^a_0 = \partial_0 A^a_\nu - \partial_\nu A^a_0 + f^{abc} A^b_\nu A^c_0 \rightarrow \partial_0 A^a_\nu = F^a_\nu + \partial_\nu A^a_0 + f^{abc} A^b_\nu A^c_0 = F^a_\nu + D_\nu A^a_0.$$  

(104)

Performing an analogous operation for the matter fields,

$$\pi_\alpha \dot{\phi}^\alpha = \pi_\alpha D^0 \phi^\alpha - A^a_0 Q_a.$$  

(105)

In (105) we used the definition of the time component of the covariant derivative

$$(D_\mu \phi)^\alpha = \partial_\mu \phi^\alpha + A^a_\mu (T_a)^\alpha_\beta \phi^\beta.$$  

(106)

with the $SU(2)$ gauge charge $Q_a$, given by

$$Q_a = \pi_\alpha (T_a)^\alpha_\beta \phi^\beta.$$  

(107)

Let us first rewrite the last term of (102):

$$\partial_\nu D_\mu A^a_0 = \partial_\nu \partial_\mu A^a_0 + \partial_\nu f^{abc} A^b_\mu A^c_0 = \partial_\nu \left( \partial_\mu A^a_0 \right) - A^a_0 \partial_\mu \partial_\nu A^a_0 + \partial_\nu f^{abc} A^b_\mu A^c_0.$$  

(108)

The first term on the right-hand side of (108) corresponds to a gauge transformation on the two-dimensional boundary $\partial \Sigma$ of 3-space $\Sigma$, and we are left with

$$\partial_\nu \left( \partial_\mu A^a_0 \right) - A^a_0 \partial_\mu \partial_\nu A^a_0 + \partial_\nu f^{abc} A^b_\mu A^c_0 = \partial_\nu \left( \partial_\mu A^a_0 \right) - A^a_0 D_\nu \partial_\mu A^a_0.$$  

(109)

Having expressed both the gravitational and the matter contributions in covariant notation, making the extension $\epsilon_{ijk} = \epsilon_{ij\rho} \rightarrow \epsilon_{\mu \nu \rho \sigma}$, we can upon substitution of (102) rewrite (101) in covariant notation as

$$I = \int_T^T dt \int \Sigma d^3 x \left( (hG)^{-1} \Psi_{ac} B^a_\mu \dot{A}^c_\mu + \frac{i}{\hbar} \pi_\alpha \dot{\phi}^\alpha \right)$$

$$= \int_M \left( (hG)^{-1} \Psi_{ac} F^a \wedge F^b + \frac{i}{\hbar} \pi_\alpha D_0 \phi^\alpha + \frac{i}{\hbar} A^a_0 \left( D_\mu \partial_\mu A^a_0 + Q_a \right) \right)$$

$$= \int_M \left( (hG)^{-1} \Psi_{ac} F^a \wedge F^b + \frac{i}{\hbar} \pi_\alpha D_0 \phi^\alpha \right)$$  

(110)

on account of Gauss’ law. Equation (110) signifies an interaction between the gravitational and the matter field based on $SU(2)$ gauge invariance. Without knowledge of the CDJ matrix solution, there would be no physical input at this stage of simplicity different to that from $SU(2)$ Yang–Mills theory coupled to matter. But there is input from gravity, since the gravitational phase space is smaller than its Yang–Mills counterpart owing to the remaining constraints which determine the CDJ matrix. Note that in the absence of matter, we have $\phi^a = \pi_a = 0$ and the interactions are no longer present. Then (110) would reduce to the second Chern class $\int \tr F \wedge F$, a topological invariant of $M$, and ultimately to the Chern–Simons action $I_{CS}$ on the spatial boundary $\Sigma$ in direct analogy to [15]. In the more general case, the CDJ matrix acts as a kind of matter-induced metric on $SU(2)$, noting that since (110) arose from the same starting functional (101) which was chosen to be velocity independent and independent of histories within $M$, it follows that the instanton-like term is as well another representation of the generalized Kodama state, which is as well velocity and history independent and furthermore constitutes the analogue of a topological invariant in the presence of matter fields coupled to gravity:

$$\Psi_{GKod}(T) = e^{\int_{t_0}^T (hG)^{-1} \Psi_{ac} F^a \wedge F^b + \frac{i}{\hbar} \pi_\alpha D_0 \phi^\alpha} \Psi_{GKod}(t_0).$$  

(111)
Note that one can expand the generalized Kodama instanton representation about its pure Kodama counterpart via the change in variables \( \Psi_{ab} = -(6\Lambda^{-1}\delta_{ab} + \epsilon_{ab}) \), where \( \epsilon_{ab} \) parametrizes the departure of \( \Psi_{\text{GKod}} \) from \( \Psi_{\text{Kod}} \), to obtain

\[
\Psi_{\text{GKod}}[A(T), \Phi(T)] = e^{iBGX^{-1}f_{ab} F^a F^b} e^{i(bG X^{-1})\epsilon_{ab} F^a F^b + \frac{1}{2} \frac{4}{\Lambda G} D_\Phi^2} = \Psi_{\text{Kod}}[A(t_0)] e^{i(bG X^{-1})\epsilon_{ab} F^a F^b + \frac{1}{2} \frac{4}{\Lambda G} D_\Phi^2}.
\]

To go one step further, the gravitational sector of the generalized Kodama state can be taken 'off-shell' by the introduction of a self-dual 2-form \( \Sigma^a = \Sigma_{\mu\nu}^a dx^\mu \wedge dx^\nu \). The 'phase' of this sector then becomes

\[
-\frac{1}{4} \int_M \Psi_{ab} F^a \wedge F^b = \int_M (\Sigma^a \wedge F^a + (\Psi^{-1})_{ab} \Sigma^a \wedge \Sigma^b).
\]

Comparison of (113) with [28, 29] shows a distinct difference in that when matter is quantized with gravity on the same footing, the degrees of freedom in the CDJ matrix \( \Psi_{\text{GKod}} \) from \( \Psi_{\text{Kod}} \) is none other than the equivalence of the reduced phase space and Dirac quantization of a quantum system, which exists only for special states including \( \Psi_{\text{GKod}} \) and \( \Psi_{\text{Kod}} \).

### 7.2. Existence of \( \Psi_{\text{GKod}} \) from the perspective of geometric quantization vis-a-vis the SQC

The basic notion for the time-parametrization invariance of the generalized Kodama states can also be seen from the application of geometric quantization to constrained systems. We will demonstrate the relationship to the semiclassical-quantum correspondence and how this can lead uniquely to such independence. In a more mathematically precise sense, the relationship between geometric quantization and the SQC is none other than the equivalence of the reduced phase space and Dirac quantization of a quantum system, which exists only for special states including \( \Psi_{\text{GKod}} \) and \( \Psi_{\text{Kod}} \).

Let us start with the determination of the pure Kodama state \( \Psi_{\text{Kod}} \). One first starts with a symplectic structure \( \Omega = (\omega, M) \) with symplectic (functional) 2-form \( \Omega \) on the classical phase space \( M \equiv (\Gamma_\mu^I(x), A_\mu^i(x)) \) on the configuration space of connections \( \Gamma_\mu^I \) living at the point \( x \in M \). The symplectic 2-form on \( \Omega \) associated with a spatial hypersurface \( \Sigma_t \) labelled by time \( t \) is given by

\[
\omega_M = (h G)^{-1} \int_\Sigma d^3 x \delta \Gamma_\mu^I(x) \wedge \delta A_\mu^i(x),
\]

22 Pointed out in [15] as a requirement for the quantization of a topological theory on a spacetime manifold \( M \) to lead to a unique quantum state living on the boundary \( \partial M \).

21 This does not in general preclude the existence of a metric with which \( A_\mu^i \) is compatible.
where the wedge product is taken with respect to the cotangent bundle of the fields at each point \( x \). At the level of (115) the phase space per point is 18 dimensional. Hence, the associated volume form corresponding to (115)

\[
d\mu(A) = w[A] \prod_{a,i} \delta \sigma_a^i(x) \wedge \delta A^i_a(x)
\]

is nonzero. Note that (116) signifies the 18-dimensional phase space per point, and \( w[A] \) is the weighting factor used in (16). The volume form in (116) is nonzero, which in the language of geometric quantization corresponds to a nondegenerate 2-form \( \omega_m \).

At the classical level there are seven constraints, which upon solution leave a two-parameter ambiguity in the CDJ matrix \( \Psi_{ur} \). This corresponds to a two-parameter ambiguity in the conjugate momentum \( \tilde{\sigma}_a^i = \Psi_{ur} B^i_r \) at the level of the reduced phase space \( m \). Denote the undetermined momenta, without loss of generality, by \( \tilde{\sigma}_1^i \) and \( \tilde{\sigma}_2^i \). The volume form on the original (18-dimensional) phase space \( M \) is zero, which means that the corresponding 2-form \( \omega_M \), though not zero, is degenerate on the space \( M \). However, it is nondegenerate on the reduced (four-dimensional) phase space \( m \), with a symplectic 2-form given by

\[
\omega_m = (\hbar G)^{-1} \int_{\Sigma} d^3 x \left[ \delta \tilde{\sigma}_1^i(x) \wedge \delta A^i_a(x) + \delta \tilde{\sigma}_2^i(x) \wedge \delta A^i_a(x) \right] \neq 0.
\]

The volume form on this reduced (eleven-dimensional) phase space per point is nonzero, and is given by

\[
d\mu_{red}(A) = w[A] \delta \tilde{\sigma}_1^i(x) \wedge \delta A^i_a(x) \wedge \delta \tilde{\sigma}_2^i(x) \wedge \delta A^i_a(x) \prod_{b,j \neq 1,2} \delta A^i_b.
\]

This ambiguity in the state of the system presents an obstruction to progress from the prequantization to the quantization stage of geometric quantization in that some momenta would still remain in the wavefunction, assuming that it can be constructed. Labelling this by \( (\Psi_m)_{Wib} \) to signify the classical level of solution, one has

\[
(\Psi_m)_{Wib} = \Psi_m[\tilde{\sigma}_1^i(x), \tilde{\sigma}_2^i(x), A^i_a(x)].
\]

Equation (119) as it stands is unsuitable for a wavefunction of the universe, since one requires a polarization for which the wavefunction depends completely on ‘coordinate’ variables \( A^i_a \). Indeed, the requirement of integrability is not met at the naive classical level and the manner in which to construct a state on the boundary \( \partial M \) of spacetime is not in general clear.

However, we will show that the full solution to (91) at the quantum level removes this ambiguity, enabling the corresponding quantum state to be explicitly constructed. Our argument proceeds as follows. Upon full solution to the quantum constraints, all momenta, including \( \tilde{\sigma}_1^i \) and \( \tilde{\sigma}_2^i \), are completely determined by the coordinate variables \( A^i_a \). The interpretation is that the symplectic 2-form \( \omega_m \) in (117) collapses to zero\(^{23} \). This means that the reduced phase space has decreased from \( 4 \otimes \infty \) degrees of freedom per point to \( 0 \otimes \infty \) degrees of freedom per point.

A way to see this is to compute the full original 2-form (115) subject to the solution of the quantum constraints (all nine \( C_{ab} = 0 \) equations). In this case, \( \omega_M \) on the full phase space collapses into \( \omega_M \) on the configuration space. Suppressing \( x \) dependence, we have

\[
\omega_M = \omega_M|_{C_{ab}=0} = (\hbar G)^{-1} \int_{\Sigma} d^3 x \left. \delta \tilde{\sigma}_1^i(x) \wedge \delta A^i_a(x) \right|_{C_{ab}=0}.
\]

\(^{23}\)This is the field-theoretical analogy to finite-dimensional systems. Take the harmonic oscillator with \( \omega = dp \wedge dq \). Upon solution to the constraint of constant energy \( 2E = p^2 + q^2 \) the symplectic 2-form reduces to \( \omega = (2E - q^2)^{-1/2} q \wedge dq = 0 \).
Expanding this out, noting that \( \tilde{\sigma}^i_a(x) = \tilde{\sigma}^i_a[A^b_j(x)] \) gives

\[
(hG)^{-1} \int_{\Sigma} d^3 x \delta \tilde{\sigma}^i_a(x) \wedge \delta A^i_a(x) = (hG)^{-1} \int_{\Sigma} d^3 x \frac{\partial \tilde{\sigma}^i_a}{\partial A^j_b} \delta A^b_j \wedge \delta A^i_a
\]

\[
= \int_{\Sigma} d^3 x \left( \frac{\partial^2 I_{\text{Kod}}}{\partial A^i_a \partial A^j_b} \right) \delta A^b_j \wedge \delta A^i_a = 0. \tag{121}
\]

Equation (121) vanishes due to the symmetric indices in the double derivative being contracted with antisymmetric indices in the wedge product.

To show explicitly the manner in which \( \Psi_{\text{Kod}} \) arises, we argue in analogy to the cohomology of manifolds. A closed 1-form \( \theta \) on a manifold \( M \), such that \( d\theta = 0 \), is locally exact by the Poincaré lemma [30]. Hence \( \theta = d\lambda \) for some 0-form \( \lambda \). So, the manifold \( M \) can then be divided into cohomology classes \( H^1(M) \) of closed-modulo exact forms. We now extend this to field theory in the following manner. As noted in [4] the symplectic 2-form \( \omega_M \) can be seen as the Abelian curvature of a connection 1-form, namely the symplectic (canonical) 1-form (the analogy to \( \theta = p \, dq \) for the harmonic oscillator) for a \( U(1) \) line bundle, given by

\[
\theta_M = (hG)^{-1} \int_{\Sigma} d^3 x \tilde{\sigma}^i_a(x) \delta A^i_a(x). \tag{122}
\]

Equation (122) is a 1-form from the perspective of the functional space \( \Gamma^1_A \) of fields \( A^i_a \), but is actually a 4-form on spacetime \( M \) integrated over all of 3-space \( \Sigma \). It also corresponds to the symplectic form on the original classical phase space, prior to implementation of the constraints.

But we have shown that \( \omega_M \) vanishes, which corresponds to the unique solution to the quantum constraints being selected from the two-parameter family of semiclassical solutions or the self-duality condition \( \tilde{\sigma}^i_a = -6 \Lambda^{-1} B^i_a \) at each point \( x \). The canonical 1-form on the reduced phase space then becomes

\[
(\theta_m)_{\text{Kod}} = -6(hG\Lambda)^{-1} \int_{\Sigma} d^3 x B^i_a(x) \delta A^i_a(x). \tag{123}
\]

This enables the integrability condition to be satisfied, in analogy to the cohomology of finite-dimensional spaces. Hence, since \( \theta_m \) is closed (\( \omega_m = d\theta_m = 0 \)), then it must be ‘locally’ exact\(^{24}\). Hence it must be the case that \( \theta_m = d\lambda \) for some ‘functional’ 0-form \( \lambda \).\(^{25}\) We argue that the determination of this form \( \lambda \) amounts to finding the wavefunction of the universe and is the direct analogue of finding \( \lambda \in \Lambda^0(M) \) in the finite-dimensional case. The latter can be found directly via the integration \( \lambda = \int \theta \). The infinite-dimensional analogue is to integrate (123) over the functional space of fields, which requires that this functional integration commutes with the spatial integration \( d^3 x \) over \( \Sigma \). Hence,

\[
\lambda_{\text{Kod}} = \int_{\Gamma_A} (\theta_m)_{\text{Kod}} = -6(hG\Lambda)^{-1} \int_{\Sigma} \int_{\Gamma} d^3 x B^i_a(x) \delta A^i_a(x)
\]

\[
= -6(hG\Lambda)^{-1} \int_{\Sigma} \int_{\Gamma} B^i_a(x) \delta A^i_a(x)
\]

\[
= -6(hG\Lambda)^{-1} \int_{\Sigma} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)
\]

\[
= -6(hG\Lambda)^{-1} I_{\text{CS}}[A]. \tag{124}
\]

\(^{24}\) Locally in the functional sense on the space of functions per point \( x \).

\(^{25}\) A 0-form in the sense of a 3-form integrated over all 3-space \( \Sigma \).
There are a few points to note concerning (124). (i) First, it corresponds to an indefinite functional integral over $\Gamma_A$. Limits of functional integration can be associated with an initial time $t_0$ and a final time $T$, determining the ‘evolution’ of the state between these two times irrespective of histories. (ii) Secondly, since the self-duality condition $\delta\sigma^i_a(x) = -6(A^i) A^i (x)$ arose from the Dirac canonical quantization procedure, the reduced phase space quantization and Dirac quantization procedures are equivalent for the state, which can be written as $\Psi_{\text{Kod}} = e^{i\lambda_{\text{Kod}}}$, where $\lambda_{\text{Kod}} \in \Lambda^0(M)$ is an element of the functional 0-forms on the configuration space of connections. The pure Kodama state then has the interpretation of the section of the line bundle for which the configuration space of connections and matter fields $\Gamma \equiv (\Gamma_A, \Gamma_\phi)$ living at the point $x \in M'$. The symplectic 2-form on $\Sigma'$ associated with a spatial hypersurface $\Sigma_t$ labelled by time $t$ is given by
\[
\omega_{\Sigma_t} = \int_{\Sigma_t} d^3x \left[ (h_G)^{-1} \delta \tilde{\sigma}^a(x) \wedge \delta A^i_a(x) + \frac{i}{\hbar} \delta \pi_a(x) \wedge \delta \phi^a(x) \right]
\]
where the wedge product is taken with respect to the cotangent space of the fields at each point $x$. At the level of (125) the configuration space per point is $18 + 2N$ dimensional, where $N$ is the total number of fields (components labelled by the index $\alpha$) due to the presence of matter.

At the classical level there are seven constraints, which upon solution leave a two-parameter ambiguity in the CDJ matrix $\Psi_{\text{cdj}}$. This corresponds to a two-parameter ambiguity in the symplectic $2$-form $\sigma^i = \sigma^i_{\text{cdj}}(A^i_a, f_a, \phi^a)$ at the level of the reduced phase space $m$ for each value of $f_a(\phi^b)$ determined by the mixed partials condition. In analogy to $\Psi_{\text{Kod}}$ this constitutes an obstruction to proceeding from the prequantization into the quantization stage, in that the symplectic $2$-form does not in general vanish.

By solution of the system (92) one eliminates the two-parameter ambiguity to obtain the CDJ matrix $\Psi_{\text{cdj}} = \Psi_{\text{cdj}}[A^i_a, f_a, \phi^a]$, labelled by $f_a$, as a function of the configuration space variables $(A^i_a, \phi^a)$. Hence, one bypasses the semiclassical level to arrive directly at the conditions delineating the unique quantum state $\Psi_{\text{GKod}}$.

In order to be able to construct $\Psi_{\text{GKod}}$, we must show that the symplectic $2$-form $\omega_{\Sigma_t}'$ vanishes on the reduced phase space $m'$. It is true that there is an ambiguity due to the choice of $f_a$, however this does not affect the determination of the symplectic $2$-form since $f_a = f_a(\phi)$ is a function purely of the matter fields $\phi^a$ and not of any momenta. The full solution to the constraints (92) should result in the determination
\[
\tilde{\sigma}^i_a = \tilde{\sigma}^i_a(A^i_a, f_a, \phi^a); \quad \pi_a = \pi_a(A^i_a, f_a, \phi^a),
\]
or the conjugate momenta, labelled by $f_a$, in terms of the configuration space variables. Taking this dependence into account, suppressing the label $f_a$, we compute (125) on the reduced phase

26 We will call this the infinite-dimensional functional analogue of the Poisson bracket, for lack of a more descriptive term.

27 One possible criteria for the elimination of this freedom in $\tilde{f}$ is to demand that it corresponds to $\delta f_a(\phi)/h\phi^a$ for the effective action $\Gamma_{\text{eff}}[\phi]$ of the theory of matter quantized in the absence of gravity $(\lim_{\phi \to 0})$, such as to produce the correct semiclassical limit. We motivate this concept briefly in [11, 31].
The terms of (127) quadratic in the fields \( \delta A \wedge \delta A + \delta \phi \wedge \delta \phi \) vanish for the same reason that \( \delta A \wedge \delta A \) vanished for the pure Kodama state. This is the infinite-dimensional analogue to \( dx \wedge dx + dy \wedge dy = 0 \) for independent variables \( x \) and \( y \). Expanding out the quadratic terms in direct analogy to (121) leads to

\[
(\hbar G)^{-1} \int \Sigma d^3 x \left\{ \frac{-\partial \tilde{\sigma}^i_a}{\partial A^b_j} \delta A^b_j \wedge \delta A^a_i + \frac{i}{\hbar} \frac{\partial \pi_\alpha}{\partial A^a_i} \delta A^a_i \wedge \delta \phi^\alpha + \frac{i}{\hbar} \frac{\partial \pi_\alpha}{\partial \phi^\beta} \delta \phi^\alpha \wedge \delta \phi^\beta \right\} = 0
\]  

(128)

due to symmetric indices on the double partial derivatives being contracted with antisymmetric indices on due to the wedge product of the two forms. Hence all that remain are the cross terms \( \delta A \wedge \delta \phi \). We argue that these terms vanish due to the mixed partials consistency condition. Expanding these terms from (127) leads to

\[
\int \Sigma d^3 x \left\{ \frac{-\partial^2 I_{\text{GKod}}}{\partial A^a_i \partial \phi^\alpha} \right\} \delta A^a_i \wedge \delta \phi^\alpha = 0
\]  

(129)

The end result is that the symplectic 2-form \( \omega'_{m'} = 0 \). It then follows that the corresponding canonical 1-form is an exact functional differential \( \theta'_{m'} = \delta \lambda_{\text{GKod}} \) for some functional 0-form \( \lambda_{\text{GKod}}[A, \phi] \). The generalized Kodama state is then given by the exponential of this 0-form:

\[
\Psi_{\text{GKod}} = e^{-\theta'_{m'}} = e^{\lambda_{\text{GKod}}}
\]  

(130)

Hence, the generalized Kodama state is determined by the integral of the canonical 1-form corresponding to the matter-coupled theory. The arguments of this section support the concept that the generalized Kodama state is indeed the generalization of the pure Kodama state from another perspective.

8. Discussion and future directions

We have delineated a criterion for finiteness of the states of the quantum theory of gravity in the full theory by the introduction of the semiclassical-quantum correspondence. This can be thought of as the search for quantum states which due to cancellation of field-theoretical infinities are also WKB states. The significance of this work is to show that the generalized Kodama states \( \Psi_{\text{GKod}} \) are a special class of states of the full theory of quantum gravity, the
direct analogue of the pure Kodama state $\Psi_{Kod}$ in the presence of matter fields. The existence of the generalized Kodama states is a direct consequence of the consistent and exhaustive application of the canonical commutation relations to quantum gravity in Ashtekar variables in the functional Schrödinger representation.

We have also shown how $\Psi_{GKod}$ is restricted to any chosen spatial hypersurface $\Sigma_T$ of spacetime in the presence of matter fields, in analogy to a similar holographic effect for the pure Kodama states $\Psi_{Kod}$. The ability to construct the wavefunction $\Psi_{GKod}$ hinges crucially upon the ability to obtain solutions to the set of nine equations $C_{ab} = 0$ for the CDJ matrix elements $\Psi_{ae}$ for the full theory stemming from the quantum constraints. The integrated form of the mixed partials condition, a consistency condition arising from the quantization procedure, must be solved in conjunction with this system in order to obtain a wavefunction solely of configuration space variables defined on the final spatial hypersurface $\Sigma_T$ of spacetime $M = \Sigma \times \mathbb{R}$. The boundary condition $f_\alpha = f_\alpha(\vec{\phi})$ on this mixed partials condition provides a possible link from quantized gravity to its associated semiclassical limit in the case of the generalized Kodama states $\Psi_{GKod}$.

We have also provided a variety of representations for the generalized Kodama states $\Psi_{GKod}$ including arguments from geometric quantization, the natural generalization of the Chern–Simons instanton term to include matter fields, as well as arguments for independence of field bulk configurations. The principle of the SQC establishes that the reduced phase space, Dirac and geometric quantization procedures unambiguously produce the same quantum state for the generalized Kodama states by construction. This is significant in that the usual ambiguities inherent in the quantization of a given theory via different quantization schemes are absent in the case of the generalized Kodama states. It is anticipated that this feature should facilitate examination of the semiclassical limit of the quantum theory for these states.

A next step is to establish equivalence of the three quantization procedures considered in this paper to the path-integral quantization procedure introduced in [12]. In this way, we hope to show how several issues in quantum gravity can be resolved using these states. Also, we hope to illustrate in greater detail the specific algorithm to construct solutions to the constraints by inspection, both for minisuperspace and for the full theory, and from these solutions demonstrate the explicit construction of the generalized Kodama states for a wide class of models. Additional future directions will include but are not limited to those mentioned in the introduction.

Acknowledgments

I would like to thank Ed Anderson for his open-mindedness, guidance, advice and support during various lengthy discussions regarding the initial drafts of my work, as well as advice in various aspects of paper writing, as well as his patience and recommendations in reading over the final version. I would also like to thank various other members of DAMTP.

References

[1] Dirac P 1964 Lectures on Quantum Mechanics (New York: Yeshiva University Press)
[2] Rovelli C 1991 Ashtekar formulation of general relativity and loop-space non-perturbative quantum gravity: a report Class. Quantum Grav. 8 1613–75
[3] Ezawa K 1997 Nonperturbative solutions for canonical quantum gravity: an overview Phys. Rep. 286 271–348
[4] Ashtekar A 1988 New Perspectives in Canonical Gravity (Napoli: Bibliopolis)
[5] Ashtekar A 1987 New Hamiltonian formulation of general relativity Phys. Rev. D 36 1587
[6] Ashtekar A 1986 New variables for classical and quantum gravity Phys. Rev. Lett. 57
[7] Kodama H 1990 Holomorphic wavefunction of the universe Phys. Rev. D 42 2548
[8] Smolin L 2002 Quantum gravity with a positive cosmological constant Preprint hep-th/0207079
[9] Randono A 2006 Generalizing the Kodama state: I. Construction Preprint gr-qc/0611073
[10] Randono A 2006 Generalizing the Kodama state: II. Properties and physical interpretation Preprint gr-qc/0611074
[11] Ita E 2007 A systematic approach to the solution of the constraints of quantum gravity: the full theory Preprint 0710.2364
[12] Ita E 2008 Finite states in 4 dimensional quantized gravity. A brief introduction into the path integration approach in Ashtekar variables Preprint 0804.0793
[13] Hartle J B and Hawking S W 1983 Wave function of the universe Phys. Rev. D 28 2960
[14] Hawking S W 1978 Quantum gravity and path integrals Phys. Rev. D 18 1747
[15] Horowitz G T 1989 Exactly soluble diffeomorphism invariant theories Commun. Math. Phys. 125 417
[16] Gambini R and Pullin J 1996 Loops, Knots, Gauge Theories and Quantum Gravity (Cambridge: Cambridge University Press)
[17] Ashtekar A and Rovelli C 1992 A loop representation for the quantum Maxwell field Class. Quantum Grav. 9 1121–50
[18] Misner C 1957 Feynman quantization of general relativity Rev. Mod. Phys. 29 497
[19] Sano T 1992 The Ashtekar formalism and WKB wave functions of $N = 1, 2$ supergravities Preprint hep-th/9211103
[20] Tsuda M and Shirafuji T 2000 Phys. Rev. D 62 064020
[21] Tsuda M 2004 Graded algebraic structure in the canonical formulation of $N = 3$ chiral supergravity Preprint gr-qc/0402043
[22] Thieman T 1993 On the solution of the initial value constraints for general relativity coupled to matter in terms of Ashtekar variables Class. Quantum Grav. 10 1907–21
[23] Horowitz G T 1991 Topology change in classical and quantum gravity Class. Quantum Grav. 8 587–601
[24] Tsamis N C and Woodward R P 1987 The factor-ordering problem must be regulated Phys. Rev. D 36 3641
[25] Ita E 2007 Nonconventional functional calculus techniques in quantum field theories and in quantum gravity Preprint 0710.2337
[26] Finite states of four dimensional quantized gravity. General solution to the Gauss’ law constraint Preprint 0704.0367
[27] Ita E 2007 An analysis of the kinematic constraints of the general relativity in Ashtekar variables Preprint 0707.0452
[28] Capovilla R, Dell J and Jacobsen T 1991 A pure spin-connection formulation of gravity Class. Quantum Grav. 8 59–73
[29] Capovilla R, Dell J, Jacobsen T and Mason L 1991 Self-dual 2-forms and gravity Class. Quantum Grav. 8 41–57
[30] Nakahara M 2003 Differential Geometry, Topology, and Physics (Bristol: Institute of Physics Publishing)
[31] Ita E 2008 Finite states in four dimensional quantum gravity: the isotropic mini-superspace Ashtekar–Klein–Gordon model Class. Quantum Grav. 25 125002 (Preprint gr-qc/0703056)