Magnetic reconnection in 3D fusion devices: non-linear reduced equations and linear current-driven instabilities

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Abstract
Magnetic reconnection in 3D fusion devices is investigated. With the use of Boozer co-ordinates, we reduce the non-linear resistive magnetohydrodynamic equations in the limit of large aspect ratio and finite pressure fluctuations, to obtain a set of non-linear equations suitable for magnetic reconnection studies in stellarators. Magnetic flux unfreezing due to a finite electron mass is also considered. Equations that govern the linear regime and some of their general properties are given. We emphasise the role of magnetic geometry and identify how some aspects of stellarator optimisation could have an impact on reconnecting instabilities, in particular by exacerbating those enabled by electron inertia. The effect of 3D coupling on the linear reconnection rates and the mode structure is quantitatively addressed in the case in which the equilibrium rotational transform has one specific resonant location for which one mode can reconnect while coupled to an arbitrary number of non-resonant harmonics. The full problem is rigorously reduced to an equivalent cylindrical one, by introducing some geometrically modified plasma inertial and dissipative scales. The 3D scalings for the growth rates of reconnection instabilities and their destabilisation criteria are given.

Keywords: magnetic reconnection, stellarators, reduced magnetohydrodynamics

1. Introduction
The high level performance in the recent operational phase of Wendelstein 7-X (W7-X) [1–4] has been a key step in the development of the international stellarator research programme [5–10]. While, on the one hand, W7-X has set new standards in stellarators and magnetic confinement fusion devices, on the other it has provided a wealth of stimulating physics results. One of the most intriguing is the presence of current driven instabilities, externally driven by electron cyclotron resonance heating, which results in sawtooth-like plasma oscillations, and even electron temperature collapse [11]. Since the physics of sawtooth oscillations in tokamaks [12] is understood on the basis of magnetic reconnection theory [13–15], the explanation of such phenomena in W7-X requires a new look at the theory of magnetic reconnection in stellarators. Some fundamental aspects are, of course, expected to remain unaffected by the absence of axisymmetry. One of these is the boundary layer character of reconnecting instabilities, which occur as a local response acting to regularise current singularities otherwise driven by some special types of perturbed ideal magnetohydrodynamics (MHD) plasma displacements. Some others are, as we shall see, different. In the literature, we can already find studies of magnetic reconnection in stellarators [16, 17]. In most of them, one sees that the rotational transform (poloidal/toroidal winding number) plays an important role. The reason lies in the fact that the singular currents regularised by the instability are radially located where the rotational transform is rational. Some authors (see Matsuoka et al [18], for instance) have produced stability diagrams for reconnecting instabilities, highlighting the role of...
the rotational transform of the helical field, $\psi^2$, and that generated by the plasma current $\psi^0$. It is established that the value of the derivative of the rotational transform at rational surfaces is also a critical parameter for the destabilisation of sawteeth in tokamaks [13, 15], as well as for neoclassical tearing modes in quasihelically symmetric stellarators [19].

The effect of resistivity in hydromagnetic linear instabilities in general geometry was first studied by Johnson et al [20]. Here the authors used the so-called stellarator expansion [21], i.e., their analysis is limited to axial equilibria. A more general approach was presented by Glasser et al [22], who used Hamada [23] co-ordinates but, for the stellarator case, only considered high mode-numbers, thus not fully addressing general reconnection instabilities in stellarators (which are mostly low mode-number instabilities). In these works, an ordering, specific to the resistive interchange instability, is implemented for the linearised equations. In our work, we propose a more general ordering in large aspect ratio and small $\beta = \mu_0 \rho_0 / B^2$, (the ratio of kinetic to magnetic pressure) which is independent of the linear instability we consider and generates a non-linear reduced model for magnetic reconnection in stellarators.

The pervasiveness of magnetic reconnection in many phenomena occurring in fusion plasmas makes its understanding in 3D devices indispensable, not only for stellarators, but also for externally perturbed tokamaks and conceptual devices that combine features of stellarators and tokamaks.

A pictorial understanding of 3D magnetic reconnection is a notoriously hard exercise, as fascinating as arid of quantitative results. For this work, we find it useful to exploit the concepts of ‘small’ and ‘large’ solutions introduced by Newcomb [24]. That is, one first solves the equation for the reconnecting magnetic flux, $\chi$, in two distinct, asymptotically separated regions: one at the resonant surface ($\psi \sim \psi_{res}$, inner region), where radial derivatives are relatively large, and one far from it (outer region), where reconnection is negligible. Here $\psi$ is a radial co-ordinate. Then, the study of the asymptotic behaviour of inner and outer solutions, in the overlapping domain, generates an indicial equation that gives the leading order terms of a power series in $x = \psi - \psi_{res}$. For small enough pressure gradients, from both regions, two solutions with different asymptotic behaviours are generated: $\chi_L \sim A[1 + \log x O(x)]$, the ‘large’ solution, and $\chi_S = B[x + O(x^2)]$, the ‘small’ solution. In the external region, the complete solution is a linear combination of the small and large ones, and for $x \to 0$ is $\chi_{ext} = A_2 + B_2 x$. The ratio $2B_2/A_2$ is related to the so-called tearing mode instability parameter, $\Delta'$. In the simplest case, $2B_2/A_2 = \Delta'$. For $\Delta'$ of order unity, the relative perturbation is nearly constant, while for large $\Delta'$ it is not. The first type of solutions are called ‘tearing modes’, the second are dissipative kink modes. We generally refer to ‘reconnecting modes’ encompassing both types. For the external region solutions, first in axisymmetry, Glasser Green and Johnson discovered a stabilising effect due to toroidal geometry [25], which sets a critical $\Delta'$ for tearing mode destabilisation. Other authors have stressed the relation between $\Delta'$ coefficients at different resonant locations in axisymmetric devices, for low [26, 27] and moderate high [28] mode numbers. One of the main consequences of the work of Connor et al [26] is that the usual distinction between constant-$\chi$ and non-constant-$\chi$ fails, since the final eigenfunction is the result of a mixing of these two ‘flavours’. In general geometry, resonant modes contribute to magnetic reconnection at their respective rational surfaces, but they do interact. In this article we address some aspects of this problem. We show that mode coupling is important not only in the external regions, but also right at the locations where magnetic perturbations can be resonant and reconnection occurs. In general geometry, for low $\beta$, the scaling of reconnection rates with key geometric factors is derived. We find that collisionless instabilities scale more unfavourably with the metric element that regulates how close magnetic surfaces are. In a subsidiary low-$\beta$ limit, for a rotational transform that allows for one resonant reconnecting mode and an arbitrary number of non-resonant ones, it is shown that non-resonant modes have an order $O(1)$ effect on the growth rate of the resonant one by re-defining inertial and dissipative scales. The full external region solution is also given in a special case, and the singular behaviour of the non-resonant mode, induced by 3D coupling to the resonant one, is quantified.

This article is organised as follows. In section 2, the non-linear reduced magnetohydrodynamics equations for large aspect ratio and finite-$\beta$ 3D devices are derived. In section 3, the linear equations governing reconnecting modes in 3D are presented, and some of their general properties are discussed. In a subsidiary low-$\beta$ expansion, explicit growth rates for magnetic reconnection instabilities in 3D are derived. Conclusions are in section 4.

2. Reduced magnetohydrodynamics equations for stellarators

Magnetic reconnection in plasmas occurs only if some physical mechanism allows for magnetic flux unfreezing. In this work we consider resistive MHD, and briefly discuss electron inertial effects. The full MHD equations, however, are not always ideal for numerical simulation, in particular when the aim is to simulate processes that are slower than magneto-sonic waves propagating across the magnetic field. ‘Reduced’ MHD equations, where these waves are eliminated by a suitable ordering, are then more appropriate. Since these equations were first derived by Strauss [29], many derivations have been published, but their application to stellarator physics is very limited [30]. Most derivations are specific to tokamaks, and even those that are not, such as the one by Hazeltine and Meiss [31], tend to make assumptions that are not appropriate for stellarators. The aim of this section is to present a derivation that overcomes this problem.

2.1. Background and orderings

Reduced MHD relies fundamentally on a disparity between length scales parallel and perpendicular to the magnetic field $B$. 

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\[ \frac{L_\perp}{L_\parallel} = \epsilon \ll 1. \]

Most quantities of interest are thus taken to vary on the length scale \( L_\parallel \) in the direction of \( \mathbf{B} \) and a factor \( \epsilon^{-1} \) faster in the other two directions. An exception pertains to the equilibrium field \( \mathbf{B}_0 \), which varies on the length scale \( L_\parallel \) in all directions. The magnetic field is written as a sum of this ‘guide field’ and a ‘perturbation’,

\[ \mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B}, \]

where \( |\delta \mathbf{B}|/B_0 \sim \epsilon \), so that \( |\nabla \cdot \mathbf{B}_0| \sim |\nabla \cdot \delta \mathbf{B}| \sim B_0/L_\parallel \). The currents \( \mathbf{J}_0 = \mu_0^{-1} \nabla \times \mathbf{B}_0 \) and \( \delta \mathbf{J} = \mu_0^{-1} \nabla \times \delta \mathbf{B} \) are then comparable,

\[ \mathbf{J}_0 \sim \delta \mathbf{J}, \]

and so are the gradients in the direction of \( \mathbf{B}_0 \) and \( \delta \mathbf{B} \),

\[ \mathbf{B}_0 \cdot \nabla \sim \delta \mathbf{B} \cdot \nabla, \]

thus allowing for a corresponding non-linearity in the equations.

For the case of a tokamak with inverse aspect ratio \( \epsilon \), Hazeltine and Meiss [31] choose \( \mathbf{B}_0 \) to denote the equilibrium toroidal field and \( \delta \mathbf{B} \) thus to represent both the equilibrium poloidal field and the perturbation due to instabilities. This choice is necessary for the study of magnetic reconnection because otherwise the gradient of the parallel equilibrium current would be smaller than that of the perturbed current,

\[ \nabla \mathbf{J}_0 \sim \nabla \delta \mathbf{J}, \]

and the primary linear drive for the instability, \( d \mathbf{J}_0/dr \), would not appear in the appropriate order.

In a stellarator, it is not immediately obvious how best to choose \( \mathbf{B}_0 \). Indeed, several choices are possible, and since there is no unique way of identifying the toroidal component of the magnetic field, we shall identify \( \mathbf{B}_0 \) with the full equilibrium field, which we assume satisfies a scalar-pressure equilibrium relation, \( \mathbf{J}_0 \times \mathbf{B}_0 = \nabla p_0 \), with nested magnetic surfaces. It is then possible to write \( \mathbf{B}_0 \) in Boozer coordinates [32, 9],

\[ \mathbf{B}_0 = \nabla \psi \times \nabla \theta + \iota \nabla \varphi \times \nabla \psi = G(\psi) \nabla \varphi + \iota(\psi) \nabla \theta + K(\psi, \theta, \varphi) \nabla \psi. \tag{1} \]

In the first of these expressions, the first term is assumed to be dominant,

\[ \mathbf{B}_0 \sim \nabla \psi \times \nabla \theta + O(\epsilon B_0), \]

if the inverse aspect ratio is \( \epsilon \ll 1 \). If we thus take the major radius to be \( L_\parallel \) and minor radius to be \( L_\perp \), we have

\[ |\nabla \varphi| \sim |\nabla \theta| \sim \epsilon |\nabla \perp \theta| \sim \frac{1}{L_\parallel}, \]

\[ G \sim \epsilon^{-2} I \sim B L_\parallel. \]

In order to eliminate the fast magnetosonic wave but retain the shear Alfvén wave, we only consider perturbations that vary slowly with respect to the former wave and scale in the same way as the magnetic-field perturbation \( |\delta \mathbf{B}|/B \sim \epsilon \), i.e.

\[ \frac{\partial/\partial t}{v_A/L_\parallel} \sim \frac{V}{v_A} \sim \epsilon, \]

where \( v_A = B/(\mu_0 \rho)^{1/2} \) denotes the Alfvén speed, \( \rho \) the plasma density, and \( V \) the plasma flow velocity. It follows that the plasma displacement is then comparable to the minor radius,

\[ |\xi| \sim \frac{V}{\partial/\partial t} \sim L_\perp, \]

so that the plasma pressure perturbation is of order unity,

\[ \delta p \sim \epsilon \nabla \psi \sim p_0. \]

Our equations will therefore be appropriate for describing non-linear instabilities.

Finally, we need to choose an ordering for the normalised plasma pressure, \( \beta = 2 \mu_0 p_0 / B^2 \). The natural choice follows from the equation of motion,

\[ \frac{d \mathbf{V}}{dt} = - \nabla \left( \frac{B^2}{2 \mu_0} + p \right) + \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B}, \tag{2} \]

which for the equilibrium (\( d \mathbf{V}/dt = 0 \)) suggests that, since the pressure varies on the length scale of the minor radius but \( \mathbf{B}_0 \) only on the length scale \( L_\parallel \), we should choose

\[ \frac{p_0}{L_\perp} \sim \frac{B_0^2}{\mu_0 L_\parallel}, \]

i.e. \( \beta = O(\epsilon) \). This ordering has consequences for the last term in equation (1), because of the equilibrium relation [9]

\[ \mu_0 \mathbf{J}_0 \times \mathbf{B}_0 = \frac{1}{\sqrt{\epsilon}} \left( \iota I' + G' - \iota \frac{\partial K}{\partial \theta} - \iota \frac{\partial K}{\partial \varphi} \right) \nabla \psi = \mu_0 \nabla p_0, \]

where \( 1/\sqrt{\epsilon} = B_0^2/(G + \iota I) \approx B^2/G \) so that

\[ \iota I' + G' - \iota \frac{\partial K}{\partial \theta} - \iota \frac{\partial K}{\partial \varphi} = - \frac{\mu_0 G}{B^2} \frac{dp_0}{d\psi}. \tag{3} \]

In this equation, the only quantities that vary over a flux surface are \( K \) and \( B \), so it follows that \( K \) must be of the order of

\[ K \sim \frac{\mu_0 GM}{B^2} \frac{dp_0}{d\psi}, \]

where \( M = \Delta B / B \) denotes the ‘mirror ratio’, i.e. the relative variation of \( B \) over the surface. Interestingly, it follows that the final term in equation (1) is in general not negligible,

\[ \frac{|K \nabla \psi|}{|G \nabla \varphi|} \sim \frac{M \beta}{\epsilon}. \tag{4} \]

This ratio is not small unless \( M \ll 1 \), since we have taken \( \beta = O(\epsilon) \). We further note that the general expression for
the parallel equilibrium current, obtained from the curl of equation (1),
\[
\frac{\mu_0 J_0 \cdot B_0}{B_0^2} = \frac{1}{\sqrt{8}} \left( G \mathbf{I} - IG - G \frac{\partial K}{\partial \theta} + I \frac{\partial K}{\partial \varphi} \right),
\]
becomes
\[
\frac{\mu_0 J_0 \cdot B_0}{B_0^2} \approx I' - \frac{\partial K}{\partial \theta},
\]
to lowest order in \( \varepsilon \ll 1 \), because \( G \sim \varepsilon^{-2} I \), and \( G' I \sim \varepsilon^2 G \), since \( \nabla \cdot B_0 \sim B_0 / L_\perp \).

2.2. Magnetic-field perturbation

We now consider perturbations to this equilibrium. The equation of motion (2) simply becomes (to lowest order)
\[
\nabla \cdot (B_0 \cdot \mathbf{\delta B} + \mu_0 \mathbf{\delta p}) = 0,
\]
and thus implies that \( \mathbf{\delta B} = \mathbf{\delta B}_\perp + B_0 \mathbf{\delta B}_\parallel \), with \( \mathbf{B} = B_0 / B_0 \) and
\[
\mathbf{\delta B}_\parallel = -\frac{\mu_0 \varepsilon \mathbf{p}}{B_0},
\]
ensuring that fast magnetoacoustic waves are absent. Furthermore, the relation \( 0 = \nabla \cdot \mathbf{\delta B} \approx \nabla \cdot \mathbf{\delta B}_\perp \) implies that we can write
\[
\mathbf{\delta B}_\perp = \mathbf{B} \times \nabla \chi + O(\varepsilon^2 B_0).
\]
It follows that, for any function \( f(\psi, \theta, \varphi) \) varying more quickly across the field than along it,
\[
\mathbf{B}_0 \cdot \nabla f = \frac{1}{\sqrt{8}} \left( \frac{\partial f}{\partial \varphi} + \frac{\partial f}{\partial \theta} \right) \equiv B_0 \partial f / \partial s,
\]
\[
\mathbf{\delta B} \cdot \nabla f = \mathbf{B}_0 \cdot \left( \nabla \chi \times \nabla f \right) \equiv B_0 [\chi, f],
\]
where
\[
B_0 \sqrt{8} [f, g] = \sum \left( \frac{\partial g}{\partial \psi} \frac{\partial f}{\partial \theta} - \frac{\partial g}{\partial \theta} \frac{\partial f}{\partial \psi} + \frac{1}{4} \left( \frac{\partial^2 g}{\partial \varphi^2} - \frac{\partial^2 g}{\partial \theta^2} - \frac{\partial^2 g}{\partial \psi^2} + \frac{\partial^2 g}{\partial \varphi \partial \theta} \right) \right).
\]
In the last expression, the term proportional to \( I \) is negligible, being of order \( O(\varepsilon^2) \) compared with the first term. The Jacobian can similarly be approximated by \( 1/\sqrt{8} \approx B_0^2 / (G + \varepsilon I) \approx B_0^2 / G \).

Since \( \nabla \times \mathbf{\delta B} \approx B_0 \nabla^2 \chi + O(\varepsilon) \), the total current parallel to \( \mathbf{B} \) becomes
\[
\frac{\mu_0 \mathbf{J} \cdot \mathbf{B}}{B_0^2} \approx I' - \frac{\partial K}{\partial \theta} + \nabla^2 \chi,
\]
with a relative error of order \( \varepsilon \). The two first terms on the right represent the equilibrium current from equation (5), which in contrast to \( B_0 \) varies perpendicularly to the field on the length scale \( L_\perp \). The gradients of the equilibrium current and its perturbation are thus comparable.

2.3. The shear-Alfvén law and vorticity equation

We are now ready to consider the shear-Alfvén law [31]
\[
\mathbf{B} \cdot \nabla \left( \frac{\mathbf{J} \cdot \mathbf{B}}{B_0^2} \right) + \frac{B_0}{B^2} \left( 2 \kappa \times \nabla p - \nabla \times (\rho \mathbf{V}) + 2 \rho \mathbf{\kappa} \times \mathbf{V} \right) = 0,
\]
where \( \mathbf{V} = \partial \mathbf{V} / \partial t + \mathbf{V} \cdot \nabla \mathbf{V} \), and the curvature vector, defined by \( \mathbf{\kappa} = \mathbf{b} \cdot \nabla \mathbf{b} \), is almost equal to its equilibrium value. The last term in the square bracket is relatively small since \( \kappa \sim 1/L_\parallel \), and in the previous term we may make the approximation
\[
\mathbf{B} \cdot \nabla \times (\rho \mathbf{V}) \approx \nabla \cdot (\rho \mathbf{V} \times \mathbf{B}).
\]
From Ohm’s law,
\[
E + \mathbf{V} \times \mathbf{B} = \eta \mathbf{J},
\]
we conclude that \( \mathbf{V} \times \mathbf{B} \approx d\nabla \phi / dt \), where \( \phi \) denotes the electrostatic potential and the resistivity has assumed to be small. If we subtract the equilibrium version of the shear-Alfvén law,
\[
\mathbf{B}_0 \cdot \nabla \left( \frac{\mathbf{J} \cdot \mathbf{B}_0}{B_0^2} \right) + \frac{2B_0}{B_0} \left( \kappa \times \nabla p_0 \right) = 0,
\]
from equation (7), the remainder becomes
\[
\partial_\parallel \nabla^2 \chi + \left[ \chi \cdot \nabla^2 \chi + I' - \frac{\partial K}{\partial \theta} \right] + \frac{2\mu_0}{B_0^2} \mathbf{b} \times \kappa \cdot \nabla \mathbf{p} \approx B_0 \nabla \cdot \left( \rho \frac{d}{dt} \nabla \phi \right),
\]
where we have written \( B \) instead of \( B_0 \) since they are equal to the requisite accuracy. This is an appropriate form for the flute-reduced shear-Alfvén law in stellarator geometry. It is very similar to that of Hazeltine and Meiss [31] but has been derived under slightly different assumptions. In particular, the current associated with \( \mathbf{B}_0 \) is allowed to vary rapidly with the minor radius and the pressure perturbation can be as large as the equilibrium pressure.

2.4. Ohm’s law

It is not difficult to express Ohm’s law,
\[
\mathbf{B} \cdot \left( \frac{\partial \mathbf{A}}{\partial t} + \nabla \phi + \eta \mathbf{J} \right) = B \mathbf{E}_{\text{ext}},
\]
in this notation. The right-hand side expresses the effect of any non-inductive current drive, which is assumed to drive the equilibrium current. To proceed, we need only recall equation (6) and note that \( \mathbf{A} \approx -\chi \mathbf{B} \), so that
\[
\mathbf{B} \cdot \frac{\partial \mathbf{A}}{\partial t} \approx -B_0^2 \frac{\partial \chi}{\partial t},
\]
and
\[
\mathbf{B} \cdot \nabla \phi \approx B \left( \partial_\parallel \phi + [\chi, \phi] \right),
\]
to find
\[ \frac{\partial \chi}{\partial t} + \frac{1}{B} [\phi, \chi] = \frac{1}{B} \partial_t \phi + \frac{\eta}{\mu_0} \nabla^2 \chi. \] (9)

2.5. Equations for density and pressure

The density is governed by the continuity equation
\[ \frac{d \rho}{dt} = -\varrho \nabla \cdot \mathbf{V}, \]
where the compressibility is relatively small,
\[ \varrho \nabla \cdot \mathbf{V} = \tilde{\varrho} \nabla \left( \frac{\mathbf{B} \times \nabla \phi}{B} + \mathbf{V} \frac{\varrho V}{E} \right) \ll \mathbf{V} \cdot \nabla \varrho \]
and can therefore be neglected. The continuity equation therefore reduces to
\[ \frac{d \rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{1}{B} [\phi, \rho] = 0. \] (10)

The pressure obeys the entropy production law
\[ \frac{d}{dt} \left( \frac{p}{\varrho^{5/3}} \right) = \frac{2\eta \mu^2}{3\varrho^{2/3}} + \text{heating-radiation}, \]
where the right-hand side is usually neglected on the grounds that only phenomena occurring on time scales shorter than the energy confinement time are considered. The equation thus becomes
\[ \frac{\partial}{\partial t} \left( \frac{p}{\varrho^{5/3}} \right) + \frac{1}{B} [\phi, \left( \frac{p}{\varrho^{5/3}} \right)] = 0. \] (11)
where we have earlier written \( p = p_0 + \delta p \).

Equations (8)–(11) form a closed system of four non-linear equations for the four unknowns \( (\chi, \phi, \rho, \delta p) \). The density profile is however usually not of any great importance for the study of MHD instabilities. If it is assumed to be flat \( (\nabla \rho = 0) \), the pressure equation reduces to
\[ \frac{\partial \delta p}{\partial t} + \frac{1}{B} [\phi, p_0 + \delta p] = 0. \]

Note that all nonlinearities now appear in the form of Poisson brackets.

3. Linear theory of low-\( \beta \) reconnecting instabilities

We now consider \( \beta \ll \epsilon \), neglect pressure perturbations, and linearise equations (8) and (9), to obtain
\[ \frac{\mu_0}{B^2} \nabla \cdot \left( \frac{\partial}{\partial t} \nabla \phi \right) = \partial_t \| \nabla^2 \chi + [\chi, I] \|, \] (12)
and
\[ \frac{\partial \chi}{\partial t} = \frac{1}{B} \partial_t \phi + \frac{\eta}{\mu_0} \nabla^2 \chi. \] (13)

This is our starting point for the investigation of linear reconnecting instabilities in 3D devices.

3.1. General properties of the external region

In the external region, as already discussed in the Introduction, the plasma is considered to be marginally stable and resistive effects are negligible. Thus, after taking \( \partial_t \equiv \gamma \to 0 \), and \( \eta \to 0 \) in equation (12), we obtain
\[ \partial_t \| \nabla^2 \chi_{\text{ext}} = -\left[ \chi_{\text{ext}}, I' \right], \] (14)
and
\[ \partial_t [\phi_{\text{ext}} = \gamma B \chi_{\text{ext}}. \]

Equation (14) is the fundamental external region equation for arbitrary 3D geometry in the limit of negligible equilibrium pressure gradients. As opposed to the cylindrical case, considerable complications come from the differential operator on the LHS, since, in arbitrary geometries, the Laplacian is
\[ \nabla^2 = \frac{1}{\sqrt{g}} \partial_{ij} g^{ij} \partial_i, \]
where each element of the metric tensor, \( g^{ij} \), as well as the Jacobian \( 1/\sqrt{g} \) are functions of the Boozer angles, \( \theta \) and \( \varphi \). This introduces a coupling of the Fourier harmonics of the magnetic perturbation, \( \chi_{\text{ext}} \). Such coupling has been studied in axisymmetric geometry by several authors. Obviously, in this case, only poloidal harmonics are coupled. However, in non-axisymmetric geometry, toroidal coupling is also expected. Indeed, it is not difficult to show that the Fourier expansion of equation (14) is
\[ (n - m) \sum_{m' \neq m} \left\{ h_{m' n'}^m \left( \partial_j \chi \right)_{m - m', n - n'} + g_{m' n'}^{ij} \left( \partial_i \partial_j \chi \right)_{m - m', n - n'} \right\} \]
\[ = -G^{t'\theta} m \chi_{\text{ext}}, \] (15)
where
\[ h_{m' n'}^m \left( \partial_j \chi \right)_{m - m', n - n'} = h_{m' n'}^j \left( \partial_j \chi \right)_{m - m', n - n'} + \frac{1}{\mu_0} \frac{d}{d\psi} \chi_{m - m', n - n'}, \]
\[ + i \left( (m - m') h_{m' n'}^0 \chi_{m - m', n - n'} + \frac{d}{d\psi} \left[ \chi_{m - m', n - n'} \right] \right) \chi_{m - m', n - n'}, \]
\[ g_{m' n'}^{ij} \left( \partial_i \partial_j \chi \right)_{m - m', n - n'} \]
\[ = \left\{ g_{m' n'} \frac{d^2}{d\psi^2} + 2i (m - m') g_{m' n'}^{00} \frac{d}{d\psi} - 2i (n - n') g_{m' n'}^{\psi \varphi} \frac{d}{d\psi} \right\} \chi_{m - m', n - n'}. \] (16)

and subscripts refer to Fourier indices, with perturbed quantities written as \( \delta Q(\psi, \theta, \varphi, t) = \exp \left[ -i(n \varphi - m \theta) + \gamma t \right] \delta Q(\psi) \), and we suppressed the subscript ‘ext’. By using these results, equation (15) is then written in the following useful form

\[ h' = \frac{1}{\sqrt{g}} \partial_{ij} g^{ij} \delta \varphi, \]
Due to the biguously. One remarkable feature of these equations is the Fourier series of \( e^\psi \), easily be seen by using equation (18). The system of equations (18) perturbed quantities are coupled. This issue did not arise in previous works on coupled reconnecting modes in tokamaks. As usual, in the reconnection layer, the electrostatic potential cannot be neglected. The analysis is further complicated by the second order derivatives of any other, non-resonant, mode can be infinite! We will study this phenomenon in detail for the simplest case of one resonant and one non-resonant mode, but we first introduce the inner layer equations for an arbitrary number of modes.

3.2. General properties of the inner region

As usual, in the reconnection layer, the electrostatic potential cannot be neglected. The analysis is further complicated by the inclusion of a finite growth rate and resistivity. The only simplification allowed pertains to high order radial derivatives. We shall assume

\[
\sum_{m,n} a_{mn} e^{4\bar{\Theta}^2} = G' \frac{\partial \chi_{mn}}{\epsilon - c}, \tag{18}
\]

where each coefficient can be inferred from the expressions above. With equation (18), we are aiming at describing a mode \( \chi_{mn} \) that is resonant at the radial location \( \psi_{mn} \). Then, if we introduce \( \psi = \psi_{mn} + \eta \psi \) and \( \chi = \chi_{mn} + \eta \chi \), at the RHS of equation (18) we obtain the familiar singular term also found in cylindrical geometry. It is interesting to note that the Fourier coefficients of the metric element \( \psi^\psi \) play a key role in the classification of solutions of equations (18), since they multiply the highest order derivative. In fact, when the second order derivatives dominate, for \( \psi \rightarrow \psi_{mn} \), equations (18) are a coupled system of equations first studied by Giovanni Plana [33, 34] (see appendix A). A particular property of this system of equations is that, if for one given resonant mode \( \chi_{mn} \neq 0 \), the second order derivatives of any other, non-resonant, mode can be infinite! We will study this phenomenon in detail for the simplest case of one resonant and one non-resonant mode, but we first introduce the inner layer equations for an arbitrary number of modes.

The inner layer equations (18)–(21) for all types of modes (resonant and non-resonant) are

\[
\gamma \chi_{mn} = \frac{\eta}{\mu_0} \left( \frac{g^\psi}{\sqrt{B^2}} \frac{d^2}{d\psi^2} \chi \right)_{mn}, \tag{20}
\]

and

\[
\gamma \phi_{mn} \left( \frac{g^\psi}{\sqrt{B^2}} \frac{d^2}{d\psi^2} \phi \right)_{mn} = \eta \left( \frac{g^\psi}{\sqrt{B^2}} \frac{d^2}{d\psi^2} \chi \right)_{mn}, \tag{21}
\]

where we took care to combine the fields in such a way that the Fourier series of \( \delta J_i / B \) in \( \theta \) and \( \varphi \) can be introduced unambiguously. One remarkable feature of these equations is the fact that, due to the \( \theta \) and \( \varphi \) dependence of \( B \), \( g^\psi \), and \( 1 / \sqrt{B} \), perturbed quantities are coupled. This issue did not arise in previous works on coupled reconnecting modes in tokamaks. The system of equations (20)–(21) is of fourth order. This can easily be seen by writing \( \phi \) as a function of \( \chi \) and \( \chi'' \), and inserting the result in equation (21). We neglect the coupling for simplicity, and find

\[
\hat{\eta} \chi'' - 2x^2 \chi' = \left( \frac{2}{x} + 1 + \eta \right) \chi'' + \frac{2}{x} \chi' - \frac{2}{x} \chi = 0, \tag{22}
\]

where \( \eta = \eta g^{\psi} / (\mu_0 \gamma) \), \( \zeta = \gamma^2 \mu_0 g^{\psi} / (\mu_0 \gamma)^2 \), and \( x = \psi - \psi_{mn} \). Thus, we have two exponential solutions that, far from the reconnection region, \( x \gg 1 \) with \( \zeta = \zeta \sim O(1) \), behave exponentially [35] \( \chi \sim \exp \pm x^2 / 2 \sqrt{\eta} \). We also find an exact solution

\[
\phi_{mn} = C, \tag{23}
\]

\[
\chi_{mn} = - \frac{\eta m l}{\sqrt{B^2}} \chi, \tag{24}
\]

where \( C \) is a constant. This solution has the so-called ‘twisting parity’, that is \( \phi'(0) = 0 \), and does not provide magnetic reconnection. The correction to this solution, as we approach the ideal region, is evaluated from the full equation (12), which we re-write, again neglecting geometric couplings and taking constant coefficients for illustrative purposes, in the following form

\[
\hat{\gamma} (\chi'' - \nu^2 \chi) = \mu x \chi'' + \nu \chi, \tag{25}
\]

with \( \hat{\gamma} = G \mu_0 g^{\psi} / \sqrt{B^2} \), \( \nu = m^2 g^{\psi} / g^{\psi} \), \( \mu = \mu ' g^{\psi} \) and \( \nu = -m l \) constants. We remind the reader that, far from the reconnection region, the second order derivatives are no longer dominant, thus the \( m^2 \) terms must be retained. We then find

\[
\chi = -\frac{C}{\mu} (\log x - x) + C \frac{\eta m l}{\sqrt{B^2}} \gamma \left( \frac{x^2}{6} + \frac{x^2}{2} \right) + C' x + C'', \tag{26}
\]

where \( C' \) and \( C'' \) are constants of integration. An \( \log x \) term of the external region solution is therefore driven by the layer twisting parity solution. The role of the \( \log x \) term in the matching of the tearing parity solution is discussed in appendix A.

We are then left with a tearing parity solution, which we investigate in greater detail.

3.3. Dispersion relation

The inner layer equations (20)–(21), for all types of modes (resonant and non-resonant) are

\[
\gamma \chi_{mn} + \frac{i}{G} [n - \iota (\psi) m] \phi_{mn} = \frac{\eta}{\mu_0} \sum_{kl} g^{\psi} \chi_{mn-k,n-l}, \tag{27}
\]

and

\[
\gamma \mu_0 g^{\psi} \sum_{kl} h_{kl} \chi_{mn-k,n-l} = -\iota [n - \iota (\psi) m] \sum_{kl} g^{\psi} \chi_{mn-k,n-l}, \tag{28}
\]

where \( h_{kl} \) are the Fourier coefficients of \( g^{\psi} / B^4 \). It is useful considering the partial Fourier sums. Then, \( -N \leq n \leq N, -M \leq m \leq M \), with \( M, N \gg 1 \), and we are dealing with 2
sets of \((2M+1)(2N+1)\) equations. Among all the couples of integers \((m,n)\), there is a special one, and multiples thereof (which we do not consider here) for which \(\iota(\psi_{m0n}) = n_0/m_0\). Thus, for \((m,n) = (m_0,n_0)\), equation (27) gives

\[
\gamma \chi_{m0n} - \frac{i}{\iota} = \frac{\eta}{\mu_0} N \sum_{kl} g_{kl}^{\psi} \chi_{m0-k,n0-l},
\]

where \(x = \psi - \psi_{m0n}\). For \((m,n) \neq (m_0,n_0)\), we have

\[
\gamma \chi_{mn} + \frac{i}{\iota}(n - n_0/m_0) \phi_{mn} = \frac{\eta}{\mu_0} \sum_{kl} g_{kl}^{\psi} \chi_{m-k,n-l}.
\]

In (29) we see that, if \(\chi_{m0n} \sim \chi_{mn} \sim \phi_{mn}\), then

\[
\phi_{mn} \sim \iota x \phi_{mn},
\]

and, for \(\iota x \ll 1\), the LHS of equation (28) becomes

\[
\gamma \mu_0 G \sum_{kl} h_{kl} \phi_{m-k,n-l}'' = \iota \mu_0 G \phi_{m0n}'',
\]

since all the non-resonant harmonics of the electrostatic potential can be neglected.

We now notice that, for \((m,n) = (m_0,n_0)\), equation (28) gives

\[
\gamma \mu_0 G h_{00} \phi_{m0n}''' = \iota \mu_0 G h_{00} \phi_{m0n}'',
\]

while, for \((m,n) \neq (m_0,n_0)\), we have

\[
\gamma \mu_0 G \sum_{kl} h_{kl} \phi_{m-k,n-l}''' = -\iota \sum_{kl} g_{kl}^{\psi} \chi_{m-k,n-l}.''
\]

Since in the sum on the LHS of equation (33) there is at least one non-null term, then

\[
\sum_{kl} g_{kl}^{\psi} \chi_{m-k,n-l}.' = \iota \sum_{kl} g_{kl}^{\psi} \chi_{m-k,n-l},
\]

for \((m,n) \neq (m_0,n_0)\), and, for \(x \to 0\),

\[
\sum_{kl} g_{kl}^{\psi} \chi_{m-k,n-l} = 0,
\]

that is, for \((m,n) \neq (m_0,n_0)\),

\[
g_{00}^{\psi} \chi_{m0n} + g_{01}^{\psi} \chi_{m,n-1} + g_{10}^{\psi} \chi_{m-1,n} + \cdots + g_{m-n0-n0}^{\psi} \chi_{m-n0,n0} + \cdots = 0,
\]

and the second order derivative of a non-resonant harmonic is a linear combination of second order derivatives of all other harmonics, including the resonant one.

Let us recall that \(-N \leq n \leq N\), \(-M \leq m \leq M\), so equation (35) is in fact a set of \((2M+1)(2N+1) - 1\) equations. Each of them involves \(\chi_{m0n}''\) once. We write all these equations in matrix form, emphasising such \(\chi_{m0n}''\) terms

\[
G \chi_{m0n} = -\chi_{m0n}''
\]

where one procedure to obtain the matrix \(G\) is given in appendix B. Hence, we are finally able to obtain, by inverting the matrix \(G\), all the non-resonant terms in the sum of the RHS of the resonant Ohm’s law (29), which now reads

\[
\gamma \chi_{m0n} - \frac{i}{\iota} \mu_0 G h_{00} \phi_{m0n}''' = \frac{\eta}{\mu_0} G^{\psi} \chi_{m0n}',
\]

where \(G_{k,l}^{\psi} \chi_{k,l}''', k,l, 1\) are the entries at the \(k\)th row of the column vector \(G^{\psi} \chi_{m0n}''\). Basically, the role of non-resonant harmonics is to modify resistive and inertial scales by a factor of order unity. Then, the eigenvalue equation of the full 3D problem is obtained by introducing the generalised Fourier transform, \(\hat{\chi}(p) = \int dx \exp(ipx) \chi(x)\), and combining the transformed equations for the resonant mode. This gives

\[
\frac{d}{dp} \frac{p^2}{1 + \delta_{\eta,e} p^2} \frac{d\phi_{m0n}}{dp} = \delta_{\eta,e} p^2 \phi_{m0n},
\]

with

\[
\delta_{\eta,e} = \frac{\eta}{\mu_0} G_{00}^{\psi},
\]

and

\[
\delta_{\eta,e} = \left(\frac{\gamma G}{\mu_0}\right) \frac{\mu_0 h_{00}}{G_{00}^{\psi}}.
\]

We notice that \(p\) should not be confused with the plasma pressure, since we are considering the ultra low-\(\beta\) limit \(\beta \ll \epsilon\). Equation (39) yields the dispersion relation [35–38]

\[
\frac{1}{(\delta_{\eta,e} G_{00}^{\psi})^{1/3}} \Delta m_{m0n} = \frac{8}{\left(\frac{r_s}{\lambda_{00}}\right)^{5/6}} \Gamma \left[\frac{1}{3} \left(\frac{\delta_{\eta,e}}{\lambda_{00}} + 5\right)\right],
\]

where

\[
\Delta m_{m0n} = \frac{1}{\chi_{m0n}''} \left(\frac{d\chi_{m0n}''}{d\psi}\right)_{\psi_{m0n}} - \frac{d\chi_{m0n}''}{d\psi}_{\psi_{m0n}}
\]

and

\[
\lambda_{00} = \left(\frac{d\chi_{m0n}''}{d\psi}\right)_{\psi_{m0n}}.
\]
and \( \chi_{mn}^{ext} \) is the \((m_0, n_0)\) Fourier harmonic of the solution of the external region equation equation (18).

We now have a dispersion relation that encapsulates, to leading order, the effect of 3D geometry, and yet yields some very familiar scalings, since the problem of reconnecting low mode-number instabilities in stellarators has been reduced to an equivalent cylindrical limit.

Regarding growth rates, in particular, for \( \delta \psi_{in} \Delta_{m_0n_0} > 0 \), with \( \delta \psi_{in} \sim \left( \frac{4}{g(m_0'n_0')} \right)^{1/2} \left( \gamma g^{\psi\psi}_{00} \right)^{1/4} \), one solves equation (42) for \( \delta_{e/\delta} \to 0 \), to obtain

\[
\gamma = \left( \frac{\eta}{g^{\psi\psi}_{00}} \right)^{3/5} \left( \frac{m_0'l'}{g\mu_0} \right)^{2/5} \left( \frac{g^{\psi\psi}_{00}}{g\mu_0h_0} \right)^{2/5} \left( \gamma g^{\psi\psi}_{00} \right)^{1/5} (c_0 \Delta_{m_0n_0})^{4/5},
\]

which is the classical scaling of Furth Killeen and Rosenbluth [39, 40].

For \( \Delta_{m_0n_0} \sim \left( \frac{4}{g(m_0'n_0')} \right)^{1/2} \gg \delta \psi_{in}^{-1} \), one has [41]

\[
\gamma = \left[ \frac{\eta}{\mu_0} \left( g^{\psi\psi}_{00} \right)^{2} \left( \frac{m_0'l'}{g\mu_0} \right)^{2} \left( \gamma g^{\psi\psi}_{00} \right)^{1/3} \right],
\]

which is the Coppi scaling for nearly ideally marginal dissipative reconnecting modes, with a geometric \( O(1) \) correction. Equation (46) is obtained from equation (42) by taking the \( \delta_{e/\delta} \to 1 \) limit. Notice that the dispersion relation equation (42) is valid for both tearing and dissipative kink instability, therefore for constant and non-constant \( \chi \)-perturbations across the resonant layer.

In the collisionless case, we simply replace resistivity by electron inertia \( \eta/\mu_0 \to \gamma d_e^2 \) [39, 42] (where \( d_e = c/\omega_{pe} \) is the electron skin depth, with \( \omega_{pe} = (me^2/m_e\epsilon_0)^{1/2} \) the electron plasma frequency) to obtain the scaling [43–46]

\[
\gamma \sim d_e g^{\psi\psi}_{00} \frac{m_0'l'}{G\sqrt{\mu_0h_0}g},
\]

with a boundary layer width

\[
\delta \psi_{in} \sim \sqrt{\frac{g^{\psi\psi}_{00}}{d_e}}.
\]

The collisionless equivalent of the Furth Killeen Rosenbluth [39, 40] result is obtained in a similar way.

The results are

\[
\gamma = d_e^2 \left( \frac{g^{\psi\psi}_{00}}{g} \right)^{3/2} \frac{m_0'l'}{G\sqrt{\mu_0h_0}} \left( \frac{g^{\psi\psi}_{00}}{g\mu_0h_0} \right)^{2} \left( c_0 \Delta_{m_0n_0} \right)^{2},
\]

for the growth rate, and

\[
\delta \psi_{in} \sim d_e^2 g^{\psi\psi}_{00} \Delta_{m_0n_0}
\]

for the boundary layer width.

Thus, we can say that, depending on the regime of interest, several scalings with the modified metric element \( g^{\psi\psi}_{00} \) (and the Jacobian) are possible. Collisionless modes feature growth rates with stronger scalings in \( g^{\psi\psi}_{00} \) than resistive ones. As a result, an increase in \( g^{\psi\psi}_{00} \) would exacerbate reconnection instabilities, just as it does for ion-temperature-gradient driven electrostatic turbulence [47, 48], since \( g^{\psi\psi}_{00} \propto g^{\psi\psi}_{00}, \) and \( g^{\psi\psi}_{00} = \nabla \psi \cdot \nabla \psi \) measures the strength of radial gradients. This result applies even in the absence of non-resonant modes coupling. Another result, which is stellarator-specific and new, is that all harmonics contribute to the actual growth of the instability! The quantity that measures the contribution of all harmonics to the growth rate is the ratio

\[
\zeta = \frac{\Delta_{m_0n_0}^{2}}{g^{\psi\psi}_{00}}.
\]

A quantitative assessment of the effect of non-resonant harmonics is a nontrivial task, but is possible through the evaluation of the matrix \( G \). We note that the necessary condition for these modes to be simulated numerically is that the quantity \( \delta \psi_{in} \), and the relevant physical scales that define it, must be resolved.

We conclude this section by stressing that \( \Delta_{m_0n_0} > 0 \) is the sufficient condition for instability in the presence of only one resonant location. The stabilising effect of the average magnetic curvature [25], which causes a threshold in \( \Delta' \), has been neglected owing to the low \( \beta \equiv \epsilon \) subsidiary limit used to derive equations (12)–(13). This would be just one effect, among a plethora of other kinetic effects, which are here neglected. For large Larmor radii compared with the layer width (which is the most relevant case is fusion devices) all such kinetic effects can be treated analytically [49–51]. They all generally cause some finite \( \Delta' \) threshold with which the potential geometric thresholds will compete. In any case, regardless of the layer model used in the analysis, the quantity \( \Delta_{m_0n_0} \) must be evaluated from an ideal MHD calculation treating the exterior region. In the case in which two modes are considered, of which one is resonant and the other is not, an exact external solution has been found, and this is reported in appendix C.

4. Conclusions

In this work, we have presented several features of the theory of magnetic reconnection in 3D magnetic fusion confinement devices putting emphasis on the mathematical nature of the problems encountered.

A set of non-linear equations for reduced resistive magnetohydrodynamics in stellarator geometry has been introduced. The equations are derived in the limit of small inverse aspect ratios \( \epsilon \ll 1 \), and finite \( \beta \sim \epsilon \), where \( \beta \) is the ratio of kinetic to magnetic plasma pressure. They describe: the plasma flow evolution, via a vorticity equation, equation (8); the evolution...
of the component of the vector potential parallel to the equilibrium magnetic field, equation (9); and the plasma pressure fluctuations through the entropy law, equation (11). All equations are derived by using Boozer co-ordinates and include the equilibrium magnetic field mirror term to leading order. They are suitable for general 3D devices: tokamaks, stellarators, as well as hybrid, as long as the aspect ratio is large.

We studied linear reconnecting instabilities supported by the 3D reduced MHD equations derived in the subsidiary limit of small $\beta \ll \epsilon$. Equations (12)–(13) are the governing equations for magnetic reconnection this regime. Far from the resonant locations where reconnection occurs, $\psi \gg \psi_{mn}$, the perturbed vector potential (external solution) is described by a set of coupled equations of the Bessel-Plana type. Here $\psi$ is the radial flux co-ordinate, and $\psi_{mn}$ is the radial location where the rotational transform is rational, $\epsilon = n/m$. A key role in the 3D coupling of modes is played by a matrix constructed with the coefficients of the Fourier series in Boozer angles of the metric element $\eta^{\psi \psi} = \nabla \psi \cdot \nabla \psi$. In the limit of $\psi$ approaching the resonant location $\psi_{mn}$, the Fourier components of the external solution show the same asymptotic radial behaviour as the solution of the cylindrical problem first studied by Newcomb. If one resonant mode is present, 3D coupling is responsible for logarithmic singularities in the radial derivatives of all other modes, even non-resonant ones. We identified two types of reconnecting modes that can be destabilised, and show how they relate to the standard tearing and dissipative kink mode. The influence of the metric elements and other important geometric factors have been established.

The effect of 3D mode-coupling has been investigated by rigorously reducing the 3D problem to an equivalent cylindrical one. This was done for an equilibrium rotational transform with one resonant location, thus allowing for one reconnecting mode and an arbitrary number of non-resonant harmonics. The effect of geometry is that of redefining plasma inertial and dissipative scales. Explicit formulae for the growth rate, in terms of the device geometric factors, and destabilisation criteria were given.

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Appendix A Asymptotic behaviour near the origin for a prototype external region equation

$$\frac{d^2 \chi_{mm}}{d\psi^2} - \frac{\kappa}{\psi - \psi_{mn}} \chi_{mn} = 0$$

The leading order terms of the $r \to 0$ expansion of the resonant mode, equation (C1), are consistent with the resonant mode equation, equation (C2), being dominated by the second order derivative. We consider the equation

$$\frac{d^2 \chi_{mn}}{d\psi^2} - \frac{\kappa}{\psi - \psi_{mn}} \chi_{mn} = 0,$$  \hspace{1cm} (A1)

where $\kappa > 0$, and we study its asymptotic behaviour for $\psi \to \psi_{mn}$. Equation (A1) is of the same type found for the study of the tearing mode in cylindrical geometry [52]. We are interested in the limit $\psi \to \psi_{mn}$. It was shown by Giovanni Plana [33, 34, 53] that the equation

$$\frac{d^2 u}{d\zeta^2} - c^2 u = \frac{p(p + 1)}{2} u,$$  \hspace{1cm} (A3)

by defining $\zeta = \zeta^0 q$, and $u = v e^{-\zeta}$, where $q = 1/(2p + 1)$. The general integral of equation (A3) is known, and so that of equation (A2), which is $v = \zeta^{1/2} C_1/(2q) (c i \zeta^q/q)$, where $C_1$ is a linear combination of the Bessel functions. In our case, care must be taken in the choice of which combination, since we need to evaluate $\lim_{\psi \to \psi_{mn}} \chi_{mn}$, and the two limits can result in spurious phase factors that must be excluded, since all coefficients of equation (A1) are real. Let us then introduce $\psi - \psi_{mn} = s$, and consider $\chi_{mn} \equiv \chi_R$, for $s > 0$. We then have

$$\frac{d^2 \chi_R}{ds^2} - \frac{\kappa}{s} \chi_R = 0,$$  \hspace{1cm} (A4)

general solution is $\chi_R = \sqrt{\kappa} C_1 (2i \sqrt{\kappa} s)$. We now choose

$$\chi_R = A_R \sqrt{s} J_1 (2i \sqrt{\kappa} s) + B_R \sqrt{s} Y_1 (2i \sqrt{\kappa} s)$$

$$\sim i \sqrt{s} \left[ A_R s^{1/2} + \frac{B_R}{\pi} \frac{1}{\sqrt{\kappa} \log s} \right],$$  \hspace{1cm} (A5)

where $A_R$ and $B_R$ are arbitrary constants. We recognise the ‘small’, $J_1$, and ‘large’, $Y_1$, solutions in Newcomb language. Equation (A5) implies that

$$\chi_R' \sim A_R + \frac{B_R}{\pi} (1 + \log s),$$  \hspace{1cm} (A6)

with a logarithmic divergence for $s \to 0^+$. For $s < 0$, we introduce $\chi_{mn} \equiv \chi_L$, $\rho \to -s$, with $\rho > 0$, and equation (A2) becomes

$$\frac{d^2 \chi_L}{d\rho^2} + \frac{\kappa}{s} \chi_L = 0.$$  \hspace{1cm} (A7)

The general solution is now $\chi_L = \sqrt{\kappa} C (2 \sqrt{\kappa} \rho)$, with $C$ a linear combination of Bessel functions. After choosing $\chi_L = A_L \sqrt{\rho} J_1 (2 \sqrt{\kappa} \rho) + B_L \sqrt{\rho} Y_1 (2 \sqrt{\kappa} \rho)$, we find [52, 54]

$$\chi_L \sim A_L \rho^{1/2} \left[ 1 + \frac{B_L}{\pi \kappa} \log (-s) \right],$$  \hspace{1cm} (A8)

and

$$\chi_L' \sim A_L + \frac{B_L}{\pi} \left[ 1 + \log (-s) \right].$$  \hspace{1cm} (A9)

The logarithmic singularities cancel when evaluating the difference $\chi_R - \chi_L$, but determine the constant on integration of the twisting parity solution derived in section 3.2.
Appendix B The geometric matrix $\mathcal{G}$

One procedure to construct the matrix $\mathcal{G}$ is the following one. For any $(m,n) \neq (m_0,n_0)$, in each of the $(2M + 1) \times (2N + 1) - 1$ sums

$$\sum_{kl} \delta^{\psi \psi}_{kl} \chi_{m-k,n-l}, \quad (m,n) \neq (m_0,n_0), \quad (B1)$$

one extracts the coefficients multiplying $\chi_{(M-m_0),-(N-n_0)}$. These coefficients are the entries of the first column of $\mathcal{G}$. The second column is constructed by extracting the coefficients of the $\chi_{(M-m_0),-(N-1-n_0)}$ term, and so on. Lengthy and tedious algebra produces

$$\mathcal{G} = \begin{pmatrix}
\delta^{\psi \psi}_{00} & \delta^{\psi \psi}_{01} & \delta^{\psi \psi}_{02} & \cdots \\
\delta^{\psi \psi}_{1,M+1} & \delta^{\psi \psi}_{1,M+2} & \cdots \\
\delta^{\psi \psi}_{2,M+1} & \delta^{\psi \psi}_{2,M+2} & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
\delta^{\psi \psi}_{N-1,M+1} & \delta^{\psi \psi}_{N-1,M+2} & \cdots \\
\delta^{\psi \psi}_{N,M+1} & \delta^{\psi \psi}_{N,M+2} & \cdots \\
0 & 0 & \cdots & 0
\end{pmatrix} \quad (B2)$$

Appendix C Exact external solution for two-modes model

The role of mode coupling is mathematically investigated in this section.

Let us now consider the matrix whose entries are the coefficients of the Fourier series of the metric element $g^{\psi \psi}_{mn}$

$$g^{\psi \psi}_{mn} = \begin{pmatrix}
g_{-1-1} & g_{-10} & g_{-1-1} & \cdots \\
g_{0-1} & g_{00} & g_{01} & \cdots \\
g_{10} & g_{11} & \cdots & \cdots
\end{pmatrix} \quad (C1)$$

which we take to be independent of $\psi$, for simplicity. Indeed, we focus on the effect of having one resonant and one non-resonant mode induced by only two non-zero entries different from $g_{00}$. The obvious choice is to consider $g_{01}$ and $g_{10}$ for $\chi_{m,n}$ and $\chi_{m,n-1}$. We set $\chi_{0,n} \equiv 0$ (an all other non-resonant harmonics, except $\chi_{m,n-1}$) to keep the problem still tractable. For the external region, equation (18) becomes

$$\nu' (\psi - \psi_{\text{mz}}) \left\{ \frac{1}{m} \left\{ \left( g_{00} \frac{d^2}{d\psi^2} + a_{00} \frac{d}{d\psi} + b_{00} \right) \chi_{m,n} + \left( g_{01} \frac{d^2}{d\psi^2} + a_{01} \frac{d}{d\psi} + b_{01} \right) \chi_{m,n-1} \right\} = G\nu' \chi_{m,n}, \quad (C2)$$

and

$$\frac{1}{m} \left\{ \left( g_{00} \frac{d^2}{d\psi^2} + a_{00} \frac{d}{d\psi} + b_{00} \right) \chi_{m,n} + \left( g_{01} \frac{d^2}{d\psi^2} + a_{01} \frac{d}{d\psi} + b_{01} \right) \chi_{m,n-1} \right\} = G\nu' \chi_{m,n-1}. \quad (C3)$$

If we set $\psi - \psi_{\text{mz}} \equiv x$, $\chi_{\text{m,n-1}} \equiv y$, after introducing $z = \exp\{a_{001}/(2g_{00})\} x$, and $q = \exp\{x(001 - a_{011}/g_{00})/[2(g_{00} - g_{01}/g_{01})]\} y$, then $\nu' (\chi_{\text{m,n-1}}) \equiv \chi_{\text{m,n-1}}$, and $G\nu' \chi_{\text{m,n-1}} \equiv \chi_{\text{m,n-1}}$. After introducing $z = \exp\{a_{001}/(2g_{00})\} x$, and $q = \exp\{x(001 - a_{011}/g_{00})/[2(g_{00} - g_{01}/g_{01})]\} y$, we obtain our model equations

$$\chi' (\nu + \nu q + e^{\nu q} (\alpha', \nu z')) = \nu \chi' + \frac{1}{m} \left\{ \left( g_{00} \frac{d^2}{d\psi^2} + a_{00} \frac{d}{d\psi} + b_{00} \right) \chi_{m,n} + \left( g_{01} \frac{d^2}{d\psi^2} + a_{01} \frac{d}{d\psi} + b_{01} \right) \chi_{m,n-1} \right\} = G\nu' \chi_{m,n-1}. \quad (C3)$$

where

$$\nu = [\nu - \nu^2/(4\alpha)]/\nu, \quad \alpha = g_{00} - g_{01}/g_{01}, \quad \beta = a_{00} - a_{01}/g_{01}, \quad \delta = b_{00} - b_{01}/g_{01}, \quad \nu = \frac{\alpha}{\beta}, \quad \lambda = G\nu' \chi_{m,n-1}. \quad (C4)$$

The integral at the exponent can be readily evaluated if one introduces the Laplace transform $\hat{q}(p) = \int_0^\infty \exp(-px)q(x)$, where $x = \frac{\nu q + e^{\nu q} (\alpha', \nu z')}{\nu}$.

The solution of equation (C5) is

$$\hat{q}(p) = Q_0 e^{-\gamma p} \exp \left\{ \frac{1}{A(p)} \right\} e^{-\lambda p} \exp \left\{ \frac{1}{\mu} \right\} \exp \left\{ \frac{1}{\mu} \right\} \exp \left\{ \frac{1}{\mu} \right\}. \quad (C6)$$

The integral at the exponent can be readily evaluated if one writes

$$\frac{1}{A(p)} = \sum_{i=1}^{4} \frac{c_i}{(p - p_i)}, \quad (C7)$$

where

$$c_i = \frac{(p_i - p_{p_i})^2 - \mu}{\Pi_{j=1}^{4} (p - p_j)}, \quad i = 1, 2, 3, 4,$$
with $\sum_i c_i = 0$, and $p_i$ are the four zeros of $A(p)$. Then, the homogeneous part of solution (C6) is

$$Q_0 e^{-\lambda \int_0^t \frac{\tau}{\tau_i} dt} = Q_0 \frac{(p-p_0)^2 - \mu}{\Pi_{i=1}^4 (p-p_i)^{1+\lambda\chi_i}},$$

(C8)

and its inverse Laplace transform can be written in terms of the general hypergeometric series of Lauricella [55]. Thus [56] (page 238)

$$Q_0^{-1} q_{hom}(x) = x \Phi_2 (1 + \lambda c_1, \ldots, 1 + \lambda c_4; 2; p_1 x, \ldots, p_4 x)$$

$$+ O(x^2) \Phi_2 (1 + \lambda c_1, \ldots, 1 + \lambda c_4; 3; p_1 x, \ldots, p_4 x)$$

$$+ O(x^3) \Phi_2 (1 + \lambda c_1, \ldots, 1 + \lambda c_4; 4; p_1 x, \ldots, p_4 x)$$

(C9)

where

$$\Phi_2(a_1, a_2, a_3, a_4; k; p_1 x, p_2 x, p_3 x, p_4 x)$$

$$= \sum_{m_1, m_2, m_3, m_4} \left( a_1 \right)_{m_1} \left( a_2 \right)_{m_2} \left( a_3 \right)_{m_3} \left( a_4 \right)_{m_4} \left( p_1 x \right)^{m_1} \left( p_2 x \right)^{m_2}$$

$$\times \left( m_1 + m_2 + m_3 + m_4 \right) \left( m_1 m_2 m_3 m_4 \right)$$

$$\left( m_1 + m_2 + m_3 + m_4 \right) \left( m_1 m_2 m_3 m_4 \right)$$

$$\left( m_1 + m_2 + m_3 + m_4 \right) \left( m_1 m_2 m_3 m_4 \right)$$

where $(a)_m = \Gamma(a + m)/\Gamma(a)$ is the Pochhammer symbol. Since $\Phi_2(a_1, \ldots, k; 0) = const$, we see that

$$\lim_{x \to 0} q_{hom}(x) = Q_0 \left( x + O(x^2) \right) = 0,$$

and the homogeneous solution cannot account for a non-zero resonant mode $\chi_{m,n}(\hat{\chi}_{mn}) \neq 0$. On the other hand, the inhomogeneous correction to $\hat{q}(p)$, for $p \to \infty$, is

$$\hat{q}_{inh}(p) = \frac{q(0)}{A(p)} e^{-\lambda \int_0^t \frac{\tau}{\tau_i} dt} \int_0^p dt e^{\lambda \int_0^t \frac{\tau}{\tau_i} dt}$$

$$= q(0) \frac{(p-p_0)^2 - \mu}{\Pi_{i=1}^4 (p-p_i)^{1+\lambda\chi_i}} \int_0^p dt \Pi_{i=1}^4 (t-p_i)^{\lambda\chi_i}$$

$$\sim q(0) \frac{(p-p_0)^2 - \mu}{\Pi_{i=1}^4 (p-p_i)^{1+\lambda\chi_i}} \int_0^p dt \sum_{i=1}^4 \left( 1 - \frac{1}{i} p_i \lambda \chi_i \right)$$

$$\sim q(0) - q(0) \sum_{i=1}^4 p_i \lambda \chi_i \frac{\log p}{p^2},$$

(C10)

Then

$$\chi_{m,n}(x) \sim q(0) \left[ 1 + \sum_{i=1}^4 p_i \lambda \chi_i x \log x \right] + Q_0 x, \text{ for } x \to 0.$$  

(C11)

Notice that, since $\chi_{m,n}(x) = \exp[-\beta/(2\alpha)x] q(x)$, then $\chi_{m,n}(p) = \hat{q}(p + \beta/(2\alpha))$, where $q(p)$ is the solution of equation (C5), and for large arguments $\chi_{m,n}(p) \approx \hat{q}(p)$. Thus, we back transformed to real space the leading order powers of $\chi_{m,n}(p)$. Taking $\lim_{x \to 0} \exp[-\beta/(2\alpha)x] L^{-1} \left[ \hat{q}(p) \right]$, where $L^{-1}$ is the inverse Laplace transform, would have given a different (and incorrect) answer. Since, $q(0) = \chi_{m,n}(0)$, we see that the constant $Q_0$ relates to the familiar tearing mode instability parameter. Indeed, equation (C11) reduces to the ideal MHD boundary condition, $\chi_{m,n}^{MHD} = 1 + \Delta_{m,n}/x$, for $Q_0/\chi_{m,n}(0) = \Delta_{m,n}/2$. We now show how the $\Delta_{m,n}$ of the resonant mode enters the non-resonant mode equation.

From

$$\dot{z}(p) \sim -\beta_2 \hat{q}(p) + \frac{z(0) + \beta_2 q(0)}{p} + \mathcal{O} \left( \frac{1}{p^2} \right)$$

$$\sim \frac{z(0)}{p} + \beta_2 q(0) \sum_{i=1}^4 p_i \lambda \chi_i \log p + \mathcal{O} \left( \frac{1}{p^2} \right), \text{ for } p \to \infty,$$

(C12)

we find, for $x \to 0$,

$$\chi_{m,n-1}(x) \sim \chi_{m,n-1}(0) - \beta_2 \sum_{i=1}^4 p_i \lambda \chi_i x \log x$$

$$+ \left( z'(0) - \beta_2 \left( q(0) + Q_0 \right) \right) x$$

$$\sim \chi_{m,n-1}(0) \left\{ 1 + \left( \frac{\chi_{m,n-1}(0)}{\chi_{m,n-1}(0)} \right) \frac{\sum_{i=1}^4 p_i \lambda \chi_i}{x} \log x$$

$$+ O(x) + \cdots \right\} + \chi_{m,n-1}(0) x \{ 1 + \cdots \}.$$  

(C13)

Notice the exact cancellation of the leading order resonant value, $\beta_2 q(0)$, so that, as it should be, the leading order term is $\chi_{m,n-1}(0)$. We must stress, at this point, that the only condition for instability at the resonant location is $\Delta_{m,n} > 0$. The new aspect introduced by the 3D coupling is that, the first order derivative of the non-resonant mode features the same logarithmic singularity of the resonant one. This is represented by the second term on the first line of equation (C13), which goes to zero in the absence of 3D coupling, since in that case $g_{0-1} \equiv 0$. 

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