BRAIDED QUANTUM SU(2) GROUPS

PAWEŁ KASPRZAK, RALF MEYER, SUTANU ROY,
AND STANISŁAW LECH WORONOWICZ

Abstract. We construct a family of $q$-deformations of SU(2) for complex parameters $q \neq 0$. For real $q$, the deformation coincides with Woronowicz’ compact quantum SU$_q$(2) group. For $q \not\in \mathbb{R}$, SU$_q$(2) is only a braided compact quantum group with respect to a certain tensor product functor for C$^*$-algebras with an action of the circle group.

1. Introduction

The $q$-deformations of SU(2) for real deformation parameters $0 < q < 1$ discovered in [9] are among the first and most important examples of compact quantum groups. Here we construct a family of $q$-deformations of SU(2) for complex parameters $q \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$. For $q \not\in \mathbb{R}$, SU$_q$(2) is only a braided compact quantum group in a suitable tensor category.

A compact quantum group $G$ as defined in [10] is a pair $G = (A, \Delta)$ where $\Delta : A \to A \otimes A$ is a coassociative morphism satisfying the cancellation law (1.4) below. The C$^*$-algebra $A$ is viewed as the algebra of continuous functions on $G$.

The theory of compact quantum groups is formulated within the category C$^*$ of C$^*$-algebras. This category with the minimal tensor functor $\otimes$ is a monoidal category (see [2]). A more general theory may be formulated within a monoidal category $(D^*, \circledast)$, where $D^*$ is a suitable category of C$^*$-algebras with additional structure and $\bigotimes : D^* \times D^* \to D^*$ is a monoidal bifunctor on $D^*$. Braided Hopf algebras may be defined in braided monoidal categories (see [4, Definition 9.4.5]). The braiding becomes unnecessary when we work in categories of C$^*$-algebras.

Let $A$ and $B$ be C$^*$-algebras. The multiplier algebra of $B$ is denoted by $M(B)$. A morphism $\pi \in \text{Mor}(A, B)$ is a $^*$-homomorphism $\pi : A \to M(B)$ with $\pi(A)B = B$. If $A$ and $B$ are unital, a morphism is simply a unital $^*$-homomorphism.

Let $T$ be the group of complex numbers of modulus 1 and let $C^*_T$ be the category of $T$-C$^*$-algebras; its objects are C$^*$-algebras with an action of $T$, arrows are $T$-equivariant morphisms. We shall use a family of monoidal structures $\circledast_\zeta$ on $C^*_T$ parametrised by $\zeta \in T$, which is defined as in [5].

The C$^*$-algebra $A$ of SU$_q$(2) is defined as the universal unital C$^*$-algebra generated by two elements $\alpha, \gamma$ subject to the relations

\[
\begin{align*}
\alpha^*\alpha + \gamma^*\gamma &= 1, \\
\alpha\alpha^* + q^2 \gamma^*\gamma &= 1, \\
\gamma\gamma^* &= \gamma^*\gamma, \\
\alpha\gamma &= q\gamma\alpha, \\
\alpha\gamma^* &= q\gamma^*\alpha.
\end{align*}
\]

For real $q$, the algebra $A$ coincides with the algebra of continuous functions on the quantum SU$_q$(2) group described in [9]: $A = C(SU_q(2))$. Then there is a unique

2010 Mathematics Subject Classification. 81R50 (46L55, 46L06).
Key words and phrases. braided compact quantum group; SU$_q$(2); U$_q$(2).
S. Roy was supported by a Fields–Ontario Postdoctoral fellowship. St.L. Woronowicz was supported by the Alexander von Humboldt-Stiftung.
morphism $\Delta: A \to A \otimes A$ with
\begin{equation}
\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma, \\
\Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.
\end{equation}

It is coassociative, that is,
\begin{equation}
(\Delta \otimes \text{id}_A) \circ \Delta = (\text{id}_A \otimes \Delta) \circ \Delta,
\end{equation}
and has the following cancellation property:
\begin{equation}
A \otimes A = \Delta(A)(A \otimes I), \\
A \otimes A = \Delta(A)(I \otimes A);
\end{equation}

here and below, $EF$ for two closed subspaces $E$ and $F$ of a $C^*$-algebra denotes the norm-closed linear span of the set of products $ef$ for $e \in E$, $f \in F$.

If $q$ is not real, then the operators on the right hand sides of (1.2) do not satisfy the relations (1.1), so there is no morphism $\Delta$ satisfying (1.2). Instead, (1.2)
defines a $\mathbb{T}$-equivariant morphism $A \to A \otimes \zeta A$ for the monoidal functor $\otimes \zeta$ with $\zeta = q/7$. This morphism in $C^*_\mathbb{T}$ satisfies appropriate analogues of the coassociativity and cancellation laws (1.3) and (1.4), so we get a braided compact quantum group. Here the action of $\mathbb{T}$ on $A$ is defined by $\rho_\mathbb{T}(\alpha) = \alpha$ and $\rho_\mathbb{T}(\gamma) = z\gamma$ for all $z \in \mathbb{T}$.

For $X, Y \in \text{Obj}(C^*)$, $X \otimes Y$ contains commuting copies $X \otimes I_Y$ of $X$ and $I_X \otimes Y$ of $Y$ with $X \otimes Y = (X \otimes I_Y)(I_X \otimes Y)$. Similarly, $X \otimes \zeta Y$ for $X, Y \in C^*_\mathbb{T}$ is a $C^*$-algebra with injective morphisms $j_1 \in \text{Mor}(X, X \otimes \zeta Y)$ and $j_2 \in \text{Mor}(Y, X \otimes \zeta Y)$ such that $X \otimes \zeta Y = j_1(X)j_2(Y)$. For $\mathbb{T}$-homogeneous elements $x \in X_k$ and $y \in Y_l$ (as defined in (2.1)), we have the commutation relation
\begin{equation}
j_1(x)j_2(y) = \zeta^{kl} j_2(y)j_1(x)
\end{equation}
The following theorem contains the main result of this paper:

**Theorem 1.1.** Let $q \in \mathbb{C} \setminus \{0\}$ and $\zeta = q/7$. Then

1. there is a unique $\mathbb{T}$-equivariant morphism $\Delta \in \text{Mor}(A, A \otimes \zeta A)$ with
   \begin{equation}
   \Delta(\alpha) = j_1(\alpha)j_2(\alpha) - qj_1(\gamma)^*j_2(\gamma), \\
   \Delta(\gamma) = j_1(\gamma)j_2(\alpha) + j_1(\alpha)^*j_2(\gamma);
   \end{equation}
2. $\Delta$ is coassociative, that is,
   \begin{equation}
   (\Delta \otimes \text{id}_A) \circ \Delta = (\text{id}_A \otimes \Delta) \circ \Delta;
   \end{equation}
3. $\Delta$ obeys the cancellation law
   \begin{equation}
   j_1(A)\Delta(A) = \Delta(A)j_2(A) = A \otimes \zeta A.
   \end{equation}

We also describe some important features of the representation theory of $SU_q(2)$ to explain the definition of $SU_q(2)$, and we relate $SU_q(2)$ to the quantum $U(2)$ groups defined by Zhang and Zhao in [11].

Braided Hopf algebras that deform $SL(2, \mathbb{C})$ are already described in [3]. We could, however, find no precise relationship between Majid’s braided Hopf algebra $BSL(2)$ and our braided compact quantum group $SU_q(2)$.

2. The algebra of $SU_q(2)$

The following elementary lemma explains what the defining relations (1.1) mean:

**Lemma 2.1.** Two elements $\alpha$ and $\gamma$ of a $C^*$-algebra satisfy the relations (1.1) if and only if the following matrix is unitary:
\begin{equation}
\begin{pmatrix}
\alpha & -q\gamma^* \\
\gamma & \alpha^*
\end{pmatrix}
\end{equation}
There are at least two ways to introduce a C*-algebra with given generators and relations. One may consider the algebra $\mathcal{A}$ of all non-commutative polynomials in the generators and their adjoints and take the largest C*-seminorm on $\mathcal{A}$ vanishing on the given relations. The set $\mathcal{N}$ of elements with vanishing seminorm is an ideal in $\mathcal{A}$. The seminorm becomes a norm on $\mathcal{A}/\mathcal{N}$. Completing $\mathcal{A}/\mathcal{N}$ with respect to this norm gives the desired C*-algebra $\mathcal{A}$. Another way is to consider the operator domain consisting of all families of operators satisfying the relations. Then $\mathcal{A}$ is the algebra of all continuous operator functions on that domain (see [1]). Applying one of these procedures to the relations (1.1) gives a C*-algebra $\mathcal{A}$ with two distinguished elements $\alpha, \gamma \in \mathcal{A}$ that is universal in the following sense:

**Theorem 2.2.** Let $\tilde{\mathcal{A}}$ be a C*-algebra with two elements $\tilde{\alpha}, \tilde{\gamma} \in \tilde{\mathcal{A}}$ satisfying

\[
\begin{align*}
\tilde{\alpha}^* \tilde{\alpha} + \tilde{\gamma}^* \tilde{\gamma} &= I, \\
\tilde{\alpha} \tilde{\alpha}^* + |q|^2 \tilde{\gamma}^* \tilde{\gamma} &= I, \\
\tilde{\gamma} \tilde{\gamma}^* &= \tilde{\gamma}^* \tilde{\gamma}, \\
\tilde{\alpha} \tilde{\gamma} &= \overline{\tilde{q}} \tilde{\gamma} \tilde{\alpha}, \\
\tilde{\alpha} \tilde{\gamma}^* &= q \tilde{\gamma}^* \tilde{\alpha}.
\end{align*}
\]

(2.1)

Then there is a unique morphism $\rho \in \text{Mor}(\mathcal{A}, \tilde{\mathcal{A}})$ with $\rho(\alpha) = \tilde{\alpha}$ and $\rho(\gamma) = \tilde{\gamma}$. □

The elements $\tilde{\alpha} = 1_{\mathcal{C}(\mathbb{T})} \otimes \alpha$ and $\tilde{\gamma} = z \otimes \gamma$ of $\mathcal{C}(\mathbb{T}) \otimes \mathcal{A}$ satisfy (2.1). Here $z \in \mathcal{C}(\mathbb{T})$ denotes the coordinate function on $\mathbb{T}$. (Later, we also denote elements of $\mathbb{T}$ by $z$.) Theorem 2.2 gives a unique morphism $\rho^A \in \text{Mor}(\mathcal{A}, \mathcal{C}(\mathbb{T}) \otimes \mathcal{A})$ with

\[
\begin{align*}
\rho(\alpha) &= 1_{\mathcal{C}(\mathbb{T})} \otimes \alpha, \\
\rho(\gamma) &= z \otimes \gamma.
\end{align*}
\]

This is a continuous $\mathbb{T}$-action, so we may view $(\mathcal{A}, \rho^A)$ as an object in the category $\mathcal{C}_T$ described in detail in the next section.

**Theorem 2.3.** The C*-algebras $\mathcal{A}$ for different $q$ with $|q| \neq 0, 1$ are isomorphic.

**Proof.** During this proof, we write $A_q$ for our C*-algebra with parameter $q$.

First, $A_q \cong A_{q'}$ for $q' = q^{-1}$ by mapping $A_q \ni \alpha \mapsto \alpha' = \alpha^* \in A_{q'}$ and $A_q \ni \gamma \mapsto \gamma' = q^{-1} \gamma \in A_{q'}$. Routine computations show that $\alpha'$ and $\gamma'$ satisfy the relations (1.1), so that Theorem 2.2 gives a unique morphism $A_q \rightarrow A_{q'}$ mapping $\alpha \mapsto \alpha'$ and $\gamma \mapsto \gamma'$. Doing this twice gives $q'' = q, \alpha'' = \alpha$ and $\gamma'' = \gamma$, so we get an inverse for the morphism $A_q \rightarrow A_{q'}$. This completes the first step. It reduces to the case $0 < |q| < 1$, which we assume from now on.

Secondly, we claim that $A_q \cong A_{|q|}$ if $0 < |q| < 1$. Equation (1.1) implies that $\gamma$ is normal with $||\gamma|| \leq 1$. So we may use the functional calculus for continuous functions on the closed unit disc $\mathbb{D}^1 = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$.

We claim that

\[
(2.3)
\]

for all $f \in C(\mathbb{D}^1)$. Indeed, the set $B \subseteq C(\mathbb{D}^1)$ of functions satisfying (2.3) is a norm-closed, unital subalgebra of $C(\mathbb{D}^1)$. The last two equations in (1.1) say that $B$ contains the functions $f(\lambda) = \lambda$ and $f^*(\lambda) = \overline{\lambda}$. Since these separate the points of $\mathbb{D}^1$, the Stone–Weierstrass Theorem gives $B = C(\mathbb{D}^1)$.

Let $q = e^{i\theta} |q|$ be the polar decomposition of $q$. For $\lambda \in \mathbb{D}^1$, let

\[
g(\lambda) = \begin{cases} 
\lambda e^{i \theta \log |q|} |\lambda| & \text{for } \lambda \neq 0, \\
0 & \text{for } \lambda = 0.
\end{cases}
\]

This is a homeomorphism of $\mathbb{D}^1$ because we get the map $g^{-1}$ if we replace $\theta$ by $-\theta$. Thus $\gamma$ and $\gamma' = g(\gamma)$ generate the same C*-algebra. We also get $g(\overline{q} \lambda) = |q|^{-1} g(\lambda)$,
so inserting \( f = g \) and \( f = \overline{g} \) in (2.3) gives
\[
a\gamma' = |q|\gamma\alpha, \quad a(\gamma')^* = |q| (\gamma')^*\alpha.
\]
Moreover, \(|g(\lambda)| = |\lambda|\) and hence \(|\gamma'| = |\gamma|\). Thus we may replace \( \gamma \) by \( \gamma' \) in the first three equations of (1.1). As a result, \( \alpha \) and \( \gamma' \) satisfy the relations (1.1) with \(|q|\) instead of \( q \). Since \( g \) is a homeomorphism, an argument as in the first step now shows that \( A_q \cong A_{|q|} \). Finally, [9, Theorem A2.2, page 180] shows that the C*-algebras \( A_q \) for \( 0 < q < 1 \) are isomorphic. □

3. Monoidal structure on \( \mathbb{T} \)-C*-algebras

We are going to describe the monoidal category \((\mathcal{C}_\mathbb{T}^\ast, \boxtimes)\) for \( \zeta \in \mathbb{T} \) that is the framework for our braided quantum groups. Monoidal categories are defined in [2].

The C*-algebra \( C(\mathbb{T}) \) is a compact quantum group with comultiplication
\[
\delta: C(\mathbb{T}) \to C(\mathbb{T}) \otimes C(\mathbb{T}), \quad z \mapsto z \otimes z.
\]

An object of \( \mathcal{C}_\mathbb{T}^\ast \) is, by definition, a pair \((X, \rho^X)\) where \( X \) is a C*-algebra and \( \rho^X \in \text{Mor}(X, C(\mathbb{T}) \otimes X) \) makes the diagram
\[
\begin{array}{ccc}
X & \xrightarrow{\rho^X} & C(\mathbb{T}) \otimes X \\
\rho^X \downarrow & & \downarrow \delta \otimes \text{id} \\
C(\mathbb{T}) \otimes X & \xrightarrow{\text{id}(C(\mathbb{T})) \otimes \rho^X} & C(\mathbb{T}) \otimes C(\mathbb{T}) \otimes X
\end{array}
\]
commute and satisfies the Podleś condition
\[
\rho^X(X)(C(\mathbb{T}) \otimes I_X) = C(\mathbb{T}) \otimes X.
\]
This is equivalent to a continuous \( \mathbb{T} \)-action on \( X \) by [5, Proposition 2.3].

Let \( X, Y \) be \( \mathbb{T} \)-C*-algebras. The set of morphisms from \( X \) to \( Y \) in \( \mathcal{C}_\mathbb{T}^\ast \) is the set \( \text{Mor}_\mathbb{T}(X, Y) \) of \( \mathbb{T} \)-equivariant morphisms \( X \to Y \). By definition, \( \varphi \in \text{Mor}(X, Y) \) is \( \mathbb{T} \)-equivariant if and only if the following diagram commutes:
\[
\begin{array}{ccc}
X & \xrightarrow{\rho^X} & C(\mathbb{T}) \otimes X \\
\varphi \downarrow & & \downarrow \text{id}(C(\mathbb{T})) \otimes \varphi \\
Y & \xrightarrow{\rho^Y} & C(\mathbb{T}) \otimes Y
\end{array}
\]

Let \( X \in \mathcal{C}_\mathbb{T}^\ast \). An element \( x \in X \) is homogeneous of degree \( n \in \mathbb{Z} \) if
\[
\rho^X(x) = z^n \otimes x.
\]
The degree of a homogeneous element \( x \) is denoted by \( \text{deg}(x) \). Let \( X_n \) be the set of homogeneous elements of \( X \) of degree \( n \). This is a closed linear subspace of \( X \), and \( X_n X_m \subseteq X_{n+m} \) and \( X_n^* = X_{-n} \) for \( n, m \in \mathbb{Z} \). Moreover, finite sums of homogeneous elements are dense in \( X \).

Let \( \zeta \in \mathbb{T} \). The monoidal functor \( \boxtimes_\zeta: \mathcal{C}_\mathbb{T}^\ast \times \mathcal{C}_\mathbb{T}^\ast \to \mathcal{C}_\mathbb{T}^\ast \) is introduced as in [5]. We describe \( X \boxtimes_\zeta Y \) using quantum tori. By definition, the C*-algebra \( C(\mathbb{T}_\zeta^\mathbb{Z}) \) of the quantum torus is the C*-algebra generated by two unitary elements \( U, V \) subject to the relation \( UV = \zeta VU \).

There are unique injective morphisms \( \iota_1, \iota_2 \in \text{Mor}(C(\mathbb{T}), C(\mathbb{T}_\zeta^\mathbb{Z})) \) with \( \iota_1(z) = U \) and \( \iota_2(z) = V \). Define \( j_1 \in \text{Mor}(X, C(\mathbb{T}_\zeta^\mathbb{Z}) \otimes X \otimes Y) \) and \( j_2 \in \text{Mor}(Y, C(\mathbb{T}_\zeta^\mathbb{Z}) \otimes X \otimes Y) \).
We have deg\((q\jmath = (q_1 \otimes \text{id}_X) \circ \rho^X(y) \quad \text{for all } y \in Y.

Let \(x \in X_k\) and \(y \in Y_l\). Then \(j_1(x) = u^k \otimes x \otimes 1\) and \(j_2(y) = V^l \otimes 1 \otimes y\), so that we get the commutation relation \((1.5)\). This implies \(j_1(X)j_2(Y) = j_2(Y)j_1(X)\), so that \(j_1(X)j_2(Y)\) is a C*-algebra. We define

\[ X \mathbb{E}_\zeta Y = j_1(X)j_2(Y). \]

This construction agrees with the one in \[1\] because \(C(\mathbb{T}_\zeta^2) \cong C(\mathbb{T}) \mathbb{E}_\zeta C(\mathbb{T})\), see also the end of \[5\] Section 5.2.

There is a unique continuous \(\mathbb{T}\)-action \(\rho_X^{X \mathbb{E}_\zeta Y}\) on \(X \mathbb{E}_\zeta Y\) for which \(j_1\) and \(j_2\) are \(\mathbb{T}\)-equivariant, that is, \(j_1 \in \text{Mor}_\mathbb{T}(X, X \mathbb{E}_\zeta Y)\) and \(j_2 \in \text{Mor}_\mathbb{T}(Y, X \mathbb{E}_\zeta Y)\). This action is constructed in a more general context in \[6\]. We always equip \(X \mathbb{E}_\zeta Y\) with this \(\mathbb{T}\)-action and thus view it as an object of \(C^*_\mathbb{T}\).

The construction \(\mathbb{E}_\zeta\) is a bifunctor; that is, \(\mathbb{T}\)-equivariant morphisms \(\pi_1 \in \text{Mor}_\mathbb{T}(X_1, Y_1)\) and \(\pi_2 \in \text{Mor}_\mathbb{T}(X_2, Y_2)\) induce a unique \(\mathbb{T}\)-equivariant morphism \(\pi_1 \mathbb{E}_\zeta \pi_2 \in \text{Mor}_\mathbb{T}(X_1 \mathbb{E}_\zeta X_2, Y_1 \mathbb{E}_\zeta Y_2)\) with

\[ (\pi_1 \mathbb{E}_\zeta \pi_2)(j_1(x_1)j_2(x_2)) = j_Y(\pi_1(x_1))j_Y(\pi_2(x_2)) \]

for all \(x_1 \in X_1\) and \(x_2 \in X_2\).

**Proposition 3.1.** Let \(x \in X\) and \(y \in Y\) be homogeneous elements. Then

\[ j_1(x)j_2(Y) = j_2(Y)j_1(x), \]

\[ j_1(X)j_2(y) = j_2(y)j_1(X). \]

**Proof.** Equation \((1.3)\) shows that

\[ j_1(x)j_2(y) = j_2(y)j_1(\rho^{X}_{\deg(y)}(x)) \]

for any \(x \in X\) and any homogeneous \(y \in Y\). Since \(\rho^{X}_{\deg(y)}\) is an automorphism of \(X\), this implies \(j_1(X)j_2(y) = j_2(y)j_1(X)\). Similarly, \(j_1(x)j_2(y) = j_2(y)j_1(x)\) for homogeneous \(x \in X\) and any \(y \in Y\) implies \(j_1(x)j_2(Y) = j_2(Y)j_1(x)\). \(\square\)

### 4. Proof of the main theorem

Let \(\alpha\) and \(\gamma\) be the distinguished elements of \(A\). Let \(\tilde{\alpha}\) and \(\tilde{\gamma}\) be the elements of \(A \mathbb{E}_\zeta A\) appearing on the right hand side of \((1.6)\):

\[ \tilde{\alpha} = j_1(\alpha)j_2(\alpha) - qj_2(\gamma)j_2(\gamma), \]

\[ \tilde{\gamma} = j_1(\gamma)j_2(\alpha) + j_1(\alpha)j_2(\gamma). \]

We have \(\deg(\alpha) = \deg(\alpha^*) = 0\), \(\deg(\gamma) = 1\) and \(\deg(\gamma^*) = -1\) by \((2.2)\). Assume \(\overline{\gamma} = q\). Using \((1.5)\) we may rewrite \((4.1)\) in the following form:

\[ \tilde{\alpha} = j_2(\alpha)j_1(\alpha) - \overline{\gamma}j_2(\gamma)j_1(\gamma)^*, \]

\[ \tilde{\gamma} = j_2(\alpha)j_1(\gamma) + j_2(\gamma)j_1(\alpha)^*. \]

Therefore,

\[ \tilde{\alpha}^* = j_2(\alpha)^*j_1(\alpha) - qj_1(\gamma)j_2(\gamma)^*, \]

\[ \tilde{\gamma}^* = j_1(\gamma)^*j_2(\alpha)^* + j_1(\alpha)j_2(\gamma)^*. \]

The four equations \((4.1)\) and \((4.2)\) together are equivalent to

\[ \begin{pmatrix} \tilde{\alpha} & -q\gamma^* \\ \tilde{\gamma} & \tilde{\alpha}^* \end{pmatrix} = \begin{pmatrix} j_1(\alpha) & -qj_1(\gamma)^* \\ j_1(\gamma) & j_1(\alpha)^* \end{pmatrix} \begin{pmatrix} j_2(\alpha) & -qj_2(\gamma)^* \\ j_2(\gamma) & j_2(\alpha)^* \end{pmatrix}. \]
Lemma 2.1 shows that the matrix
\[
\begin{pmatrix}
\alpha & -q\gamma \\
\gamma & \alpha^*
\end{pmatrix}
\in M_2(A)
\]
is unitary. Hence so is the matrix \(j_1(u)j_2(u)\) on the right hand side of (4.3). Now Lemma 2.1 shows that \(\tilde{\alpha}, \tilde{\gamma} \in A \otimes_{\mathbb{C}} A\) satisfy (2.1). So the universal property of \(A\) in Theorem 2.2 gives a unique morphism \(\Delta\) with \(\Delta(\alpha) = \tilde{\alpha}\) and \(\Delta(\gamma) = \tilde{\gamma}\).

The elements \(\alpha\) and \(\gamma\) are homogeneous of degrees 0 and 1, respectively, by (2.2). Hence \(\tilde{\alpha}\) and \(\tilde{\gamma}\) are homogeneous of degree 0 and 1 as well. Since \(\alpha\) and \(\gamma\) generate \(A\), it follows that \(\Delta\) is \(\mathbb{T}\)-equivariant. This proves statement (1) in Theorem 1.1. Here we use the action \(\rho A \otimes_{\mathbb{C}} A\) with \(\rho^A_{A}(j_1(a_1)j_2(a_2)) = j_1(\rho^A_{A}(a_1))j_2(\rho^A_{A}(a_2))\).

We may rewrite (3.3) as
\[
\begin{pmatrix}
\Delta(\alpha) & -q\Delta(\gamma)^* \\
\Delta(\gamma) & \Delta(\alpha)^*
\end{pmatrix} = \begin{pmatrix} j_1(\alpha) & -qj_1(\gamma)^* \\
j_1(\gamma) & j_1(\alpha)^*
\end{pmatrix} \begin{pmatrix} j_2(\alpha) & -qj_2(\gamma)^* \\
j_2(\gamma) & j_2(\alpha)^*
\end{pmatrix}.
\]
Identifying \(M_2(A)\) with \(M_2(\mathbb{C}) \otimes A\), we may further rewrite this as
\[
(id \otimes \Delta)(u) = (id \otimes j_1)(u) (id \otimes j_2)(u),
\]
where \(id\) is the identity map on \(M_2(\mathbb{C})\).

Now we prove statement (2) in Theorem 1.1. Let \(j_1, j_2, j_3\) be the natural embeddings of \(A\) into \(A \otimes_{\mathbb{C}} A \otimes_{\mathbb{C}} A\). Since \(\Delta\) is \(\mathbb{T}\)-equivariant, we may form \(\Delta \otimes_{\mathbb{C}} id\) and \(id \otimes \Delta\). The values of \(id \otimes (\Delta \otimes_{\mathbb{C}} id)\) and \(id \otimes (id \otimes \Delta)\) on the right hand side of (4.3) are equal:
\[
(id \otimes (\Delta \otimes_{\mathbb{C}} id)) \circ \Delta(u) = (id \otimes j_1)(u) (id \otimes j_2)(u) (id \otimes j_3)(u),
\]
\[
(id \otimes (id \otimes \Delta)) \circ \Delta(u) = (id \otimes j_1)(u) (id \otimes j_2)(u) (id \otimes j_3)(u).
\]
Thus \((\Delta \otimes_{\mathbb{C}} id) \circ \Delta\) and \((id \otimes \Delta) \circ \Delta\) coincide on \(\alpha, \gamma, \alpha^*, \gamma^*\). Since the latter generate \(A\), this proves statement (2) of Theorem 1.1.

Now we prove statement (3). Let \(S = \{x \in A : j_1(x) \in \Delta(A)j_2(A)\}\).

This is a closed subspace of \(A\). We may also rewrite (4.3) as
\[
\begin{pmatrix} j_1(\alpha) & -qj_1(\gamma)^* \\
j_1(\gamma) & j_1(\alpha)^*
\end{pmatrix} = \begin{pmatrix} \Delta(\alpha) & -q\Delta(\gamma)^* \\
\Delta(\gamma) & \Delta(\alpha)^*
\end{pmatrix} \begin{pmatrix} j_2(\alpha) & -qj_2(\gamma)^* \\
j_2(\gamma) & j_2(\alpha)^*
\end{pmatrix}.
\]
Thus \(\alpha, \gamma, \alpha^*, \gamma^* \in S\). Let \(x, y \in S\) with homogeneous \(y\). Proposition 3.1 gives
\[
j_1(xy) = j_1(x)j_1(y) \in \Delta(A)j_2(A)j_1(y) = \Delta(A)j_1(y)j_2(A)
\subseteq \Delta(A)\Delta(A)j_2(A)j_2(A) = \Delta(A)j_2(A).
\]
That is, \(xy \in S\). Therefore, all monomials in \(\alpha, \gamma, \alpha^*, \gamma^*\) belong to \(S\), so that \(S = A\). Hence \(j_1(A) \subseteq \Delta(A)j_2(A)\). Now \(A \otimes_{\mathbb{C}} A = j_1(A)j_2(A) \subseteq \Delta(A)j_2(A)j_2(A) = \Delta(A)j_2(A)\), which is one of the Podleś conditions. Similarly, let
\[
R = \{x \in A : j_2(x) \in j_1(\Delta(A))\}.
\]
Then \(R\) is a closed subspace of \(A\). We may also rewrite (4.3) as
\[
\begin{pmatrix} j_2(\alpha) & -qj_2(\gamma)^* \\
j_2(\gamma) & j_2(\alpha)^*
\end{pmatrix} = \begin{pmatrix} j_1(\alpha) & -qj_1(\gamma)^* \\
j_1(\gamma) & j_1(\alpha)^*
\end{pmatrix} \begin{pmatrix} \Delta(\alpha) & -q\Delta(\gamma)^* \\
\Delta(\gamma) & \Delta(\alpha)^*
\end{pmatrix}.
\]
Thus \(\alpha, \gamma, \alpha^*, \gamma^* \in R\). Let \(x, y \in R\) with homogeneous \(x\). Proposition 3.1 gives
\[
j_2(xy) = j_2(x)j_2(y) \in j_2(x)j_1(A)\Delta(A) = j_1(A)j_2(x)\Delta(A)
\subseteq j_1(A)j_1(A)\Delta(A)\Delta(A) = j_1(A)\Delta(A).
\]
Thus $xy \in R$. Therefore, all monomials in $\alpha, \gamma, \alpha^*, \gamma^*$ belong to $R$, so that $R = A$, that is, $j_2(A) \subseteq j_1(A)\Delta(A)$. This implies $A \otimes \gamma A = j_1(A)j_2(A) \subseteq j_1(A)j_1(A)\Delta(A) = j_1(A)\Delta(A)$ and finishes the proof of Theorem 4.4.

5. The representation theory of $SU_q$

For real $q$, the relations defining the compact quantum group $SU_q(2)$ are dictated if we stipulate that the unitary matrix in Lemma 4.11 is a representation and that a certain vector in the tensor square of this representation is invariant. Here we generalise this to the complex case. This is how we found $SU_q(2)$.

Let $H$ be a $\mathbb{T}$-Hilbert space, that is, a Hilbert space with a unitary representation $U : \mathbb{T} \to U(H)$. For $x \in \mathbb{T}$ and $x \in K(H)$ we define

$$\rho^K_{x}(x) = U_x U_x^*.$$ 

Thus $(K(H), \rho^K_H)$ is a $\mathbb{T}$-C*-algebra. Let $(X, \rho^X) \in \text{Obj}(C^*_\mathbb{T})$. Since $\rho^K_H$ is inner, the braided tensor product $K(H) \otimes K(\mathbb{T})$ may (and will) be identified with $K(H) \otimes X$ – see [5] Corollary 5.18 and [5] Example 5.19.

**Definition 5.1.** Let $H$ be a $\mathbb{T}$-Hilbert space and let $v \in M(K(H) \otimes A)$ be a unitary element which is $\mathbb{T}$-invariant, that is, $(\rho^K_H \otimes \rho^X)(v) = v$. We call $v$ a representation of $SU_q(2)$ on $H$ if

$$(\text{id}_H \otimes \Delta)(v) = (\text{id}_H \otimes j_1)(v) (\text{id}_H \otimes j_2)(v).$$

Theorem 5.1 below will show that representations of $SU_q(2)$ are equivalent to representations of a certain compact quantum group. This allows us to carry over all the usual structural results about representations of compact quantum groups to $SU_q(2)$. In particular, we may tensor representations. To describe this directly, we need the following result:

**Proposition 5.2.** Let $X, Y, U, T$ be $\mathbb{T}$-C*-algebras. Let $v \in X \otimes T$ and $w \in Y \otimes U$ be homogeneous elements of degree 0. Denote the natural embeddings by

$$i_1 : X \to X \otimes \mathbb{C} Y, \quad i_2 : Y \to X \otimes \mathbb{C} Y,$$

$$j_1 : U \to U \otimes \mathbb{C} T, \quad j_2 : T \to U \otimes \mathbb{C} T.$$ 

Then $(i_1 \otimes j_2)(v)$ and $(i_2 \otimes j_1)(w)$ commute in $(X \otimes \mathbb{C} Y) \otimes (U \otimes \mathbb{C} T)$.

**Proof.** We may assume that $v = x \otimes t$ and $w = y \otimes u$ for homogeneous elements $x \in X$, $t \in T$, $y \in Y$ and $u \in U$. Since $\deg(v) = \deg(w) = 0$, we get $\deg(x) = -\deg(t)$ and $\deg(y) = -\deg(u)$. The following computation completes the proof:

$$(i_1 \otimes j_2)(v) (i_2 \otimes j_1)(w) = (i_1(x) \otimes j_2(t))(i_2(y) \otimes j_1(u))$$

$$= i_1(x)i_2(y) \otimes j_2(t)j_1(u) = \zeta^{\deg(x) - \deg(t)}i_2(y)i_1(x) \otimes j_1(u)j_2(t)$$

$$= (i_2(y) \otimes j_1(u))(i_1(x) \otimes j_2(t)) = (i_2 \otimes j_1)(w)(i_1 \otimes j_2)(v)$$

$$= (i_2 \otimes j_1)(w)(i_1 \otimes j_2)(v).$$

**Proposition 5.3.** Let $H_1$ and $H_2$ be $\mathbb{T}$-Hilbert spaces and let $v_i \in M(K(H_i) \otimes A)$ for $i = 1, 2$ be representations of $SU_q(2)$. Define

$$v = (i_1 \otimes \text{id}_A)(v_1)(i_2 \otimes \text{id}_A)(v_2) \in M(K(H_1) \otimes K(H_2) \otimes A)$$

and identify $K(H_1) \otimes K(H_2) \cong K(H_1 \otimes H_2)$. Then $v \in M(K(H_1 \otimes H_2) \otimes A)$ is a representation of $SU_q(2)$ on $H_1 \otimes H_2$. It is denoted $v_1 \otimes v_2$ and called the tensor product of $v_1$ and $v_2$. 


Proof. It is clear that $v$ is $T$-invariant. We compute
\[
(id_{\mathcal{H}_1 \otimes \mathcal{H}_2} \otimes \Delta)(v) = (id_{\mathcal{H}_1 \otimes \mathcal{H}_2} \otimes \Delta)((t_1 \otimes \text{id}_A)(v_1)(t_2 \otimes \text{id}_A)(v_2))
\]
\[
= (t_1 \otimes j_1)(v_1)(t_2 \otimes j_1)(v_2)(t_2 \otimes j_2)(v_2)
\]
\[
= (t_1 \otimes j_1)(v_1)(t_2 \otimes j_1)(v_2)(t_2 \otimes j_2)(v_1)(t_2 \otimes j_2)(v_2)
\]
where the third step uses Proposition 5.2. □

Now consider the Hilbert space $\mathbb{C}^2$, let $\{e_0, e_1\}$ be its canonical orthonormal basis. We equip it with the representation $U : T \to \mathcal{U}(\mathbb{C}^2)$ defined by $Uze_0 = ze_0$ and $Uze_1 = e_1$. Let $\rho_{\mathbb{M}_2(\mathbb{C})}$ be the action implemented by $U$:
\[
\rho_{\mathbb{M}_2(\mathbb{C})}(a_{11} \ a_{12} \ a_{21} \ a_{22}) = (z_{11} \ z_{12} \ z_{21} \ z_{22}) ,
\]
where $a_{ij} \in \mathbb{C}$. We claim that
\[
u = \left(\begin{array}{cc}
\alpha & -q \gamma^* \\
\gamma & \alpha^*
\end{array}\right) \in \mathbb{M}_2(\mathbb{C}) \otimes A
\]
is a representation of $SU_q(2)$ on $\mathbb{C}^2$. By Lemma 2.4, the relations defining $A$ are equivalent to $u$ being unitary. The $T$-action on $A$ is defined so that $u$ is $T$-invariant. The comultiplication is defined exactly so that $u$ is a corepresentation, see 4.3.

The particular shape of $u$ contains further assumptions, however. To explain these, we consider an arbitrary compact quantum group $G = (\mathbb{C}(G), \Delta_G)$ in $\mathbb{C}_T^*$ with a unitary representation
\[
u = \left(\begin{array}{cc}
a & b \\
c & d
\end{array}\right) \in \mathbb{M}_2(\mathbb{C}(G)),
\]
such that $a, b, c, d$ generate the $C^*$-algebra $\mathbb{C}(G)$. We assume that $u$ is $T$-invariant for the above $T$-action on $\mathbb{C}^2$. Thus $\deg(a) = \deg(d) = 0$, $\deg(b) = -1$, $\deg(c) = 1$.

Theorem 5.4. Let $G$ be a braided compact quantum group with a unitary representation $u$ as above. Assume $b \neq 0$ and that the vector $e_0 \otimes e_1 - qe_1 \otimes e_0 \in \mathbb{C}^2 \otimes \mathbb{C}^2$ for $q \in \mathbb{C}$ is invariant for the representation $u \oplus u$. Then $q \neq 0$, $\Psi_G = q$, $d = a^*$, $b = -qc^*$, and there is a unique morphism $\pi : \mathbb{C}(SU_q(2)) \to \mathbb{C}(G)$ with $\pi(a) = a$ and $\pi(\gamma) = c$. This is $T$-equivariant and satisfies $(\pi \otimes \pi) \circ \Delta_{SU_q(2)} = \Delta_G \circ \pi$.

Proof. The representation $u \oplus u$ is of degree $k, l$ and $x, y \in \mathbb{C}^2$ of degree $m, n$, we let $t_1(T)t_2(S)(x \otimes y) = \zeta^{-m}Txy \otimes Sy$. By construction, $u \oplus u$ is $(t_1 \otimes \text{id}_{\mathbb{C}(G)})(u) \cdot (t_2 \otimes \text{id}_{\mathbb{C}(G)})(u)$. So we may rewrite the invariance of $e_0 \otimes e_1 - qe_1 \otimes e_0$ as
\[
(t_1 \otimes \text{id}_{\mathbb{C}(G)})(u^*)(e_0 \otimes e_1 - qe_1 \otimes e_0) = (t_2 \otimes \text{id}_{\mathbb{C}(G)})(u)(e_0 \otimes e_1 - qe_1 \otimes e_0)
\]
in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}(G)$. The left and right hand sides of 5.1 are
\[
eq e_0 \otimes e_1 \otimes 1 + e_1 \otimes e_1 \otimes b^* - qe_0 \otimes e_0 \otimes 1 + e_0 \otimes e_1 \otimes 1 - qe_1 \otimes e_0 \otimes a - q\zeta e_1 \otimes e_1 \otimes 1,
\]
respectively. These are equal if and only if $b = -qc^*$, $d = a^*$, and $b^* = -q\zeta c$. Since $b \neq 0$, this implies $q \neq 0$ and $a$ has the form in Lemma 2.1. Since $u$ is a representation, it is unitary. So $a, c$ satisfy the relations defining $SU_q(2)$ and Theorem 2.2 gives the unique morphism $\pi$. The conditions on $u$ in Definition 5.4 imply that $\pi$ is $T$-equivariant and compatible with comultiplications. □
The proof also shows that $q$ is uniquely determined by the condition that $e_0 \otimes e_1 - q e_1 \otimes e_0$ should be $SU_q(2)$-invariant. Up to scaling, the basis $e_0, e_1$ is the unique one consisting of joint eigenvectors of the $T$-action with degrees 1 and 0. Hence the braided quantum group $(C(SU_q(2)), \Delta)$ determines $q$ uniquely.

An invariant vector for $SU_q(2)$ should also be homogeneous for the $T$-action. There are three cases of homogeneous vectors in $\mathbb{C}^2 \otimes \mathbb{C}^2$: multiples of $e_0 \otimes e_0$, multiples of $e_1 \otimes e_1$, and linear combinations of $e_0 \otimes e_1$ and $e_1 \otimes e_0$. If a non-zero multiple of $e_i \otimes e_j$ for $i, j \in \{0, 1\}$ is invariant, then the representation $u$ is reducible. Ruling out such degenerate cases, we may normalise the invariant vector to have the form $e_0 \otimes e_1 - q e_1 \otimes e_0$ assumed in Theorem 5.4.

Roughly speaking, $SU_q(2)$ is the universal family of braided quantum groups generated by a 2-dimensional representation with an invariant vector in $u \otimes u$.

There is, however, one extra symmetry that changes the $T$-action on $C(SU_q(2))$ and that corresponds to the permutation of the basis $e_0, e_1$. Given a $T$-algebra $A$, let $S(A)$ be the same $C^*$-algebra with the $T$-action by $\rho^S = (\rho^q)^{-1}$. Since the commutation relation (1.5) is symmetric in $\alpha, \gamma$, we claim that there is an isomorphism $S(\mathbb{C} \otimes \mathbb{C}) \cong S(A) \otimes \mathbb{C}$, $j_1(a) \mapsto j_1(a)$, $j_2(b) \mapsto j_2(b)$.

Hence the comultiplication on $C(SU_q(2))$ is one on $S(C(SU_q(2)))$ as well.

**Proposition 5.5.** The braided quantum groups $S(C(SU_q(2)))$ and $C(SU_q(2))$ for $q = \overline{q}^{-1}$ are isomorphic as braided quantum groups.

**Proof.** Let $\alpha, \gamma$ be the standard generators of $A_q = C(SU_q(2))$ and let $\tilde{\alpha}, \tilde{\gamma}$ be the standard generators of $A_{\tilde{q}}$. We claim that there is an isomorphism $\varphi: A_q \to A_{\tilde{q}}$ that maps $\alpha \mapsto \tilde{\alpha}^*$ and $\gamma \mapsto \tilde{\gamma}^*$ and that is an isomorphism of braided quantum groups from $S(A_q)$ to $A_{\tilde{q}}$. Lemma 2.1 implies that the matrix

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{\alpha} & \tilde{\gamma}^* \\
\tilde{\gamma} & \tilde{\alpha}
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}^{-1}
= \begin{pmatrix}
\varphi(\alpha) & \varphi(-q\gamma^*) \\
\varphi(\gamma) & \varphi(\alpha^*)
\end{pmatrix}
\]

is unitary. Now Lemma 2.1 and Theorem 2.2 give the desired morphism $\varphi$. Since the inverse of $\varphi$ may be constructed in the same way, $\varphi$ is an isomorphism. On generators, it reverses the grading, so it is $T$-equivariant as a map $S(A_q) \to A_{\tilde{q}}$.

Let $\Delta$ and $\tilde{\Delta}$ denote the comultiplications on $S(A_q)$ and $A_{\tilde{q}}$. We compute

\[
(\varphi \otimes \varphi) \Delta(\alpha) = (\varphi \otimes \varphi)(j_1(\alpha)j_2(\alpha) - q j_1(\gamma^*)j_2(\gamma))
= j_1(\varphi(\alpha))j_2(\varphi(\alpha)) - q j_1(\varphi(\gamma^*))j_2(\varphi(\gamma))
= j_1(\tilde{\alpha}^*)j_2(\tilde{\alpha}^*) - \tilde{q} j_1(\tilde{\gamma}^*)j_2(\tilde{\gamma}^*),
\]

\[
\tilde{\Delta}(\alpha) = \tilde{\Delta}(\alpha^*) = j_2(\tilde{\alpha})^*j_1(\tilde{\alpha}) - q^{-1} j_2(\tilde{\gamma})^*j_1(\tilde{\gamma})
= j_1(\tilde{\alpha})^*j_2(\tilde{\alpha})^* - q^{-1} j_2(\tilde{\gamma})^*j_1(\tilde{\gamma}).
\]

These are equal because $\tilde{q} = \overline{q}^{-1} = q^{-1}\zeta$. Similarly, $(\varphi \otimes \varphi) \Delta(\gamma) = \tilde{\Delta}(\gamma)$. Thus $\varphi$ is an isomorphism of braided quantum groups.

6. The semidirect product quantum group

A quantum analogue of the semidirect product construction for groups turns the braided quantum group $SU_q(2)$ into a genuine compact quantum group $(B, \Delta_B)$; we will publish details of this construction separately. Here $B$ is the universal $C^*$-algebra with three generators $\alpha, \gamma, z$ with the $SU_q(2)$-relations for $\alpha$ and $\gamma$ and

\[
\begin{align*}
\alpha z^* &= \alpha, \\
z \gamma z^* &= \zeta^{-1}\gamma, \\
zz^* &= z^*z = I;
\end{align*}
\]
Theorem 6.1. Let \( H \) be a Hilbert space.

There are two embeddings \( \iota_1, \iota_2 : A \cong B \otimes B \) defined by

\[
\iota_1(\alpha) = \alpha \otimes 1 \quad \iota_2(\alpha) = 1 \otimes \alpha,
\]

\[
\iota_1(\gamma) = \gamma \otimes 1 \quad \iota_2(\gamma) = z \otimes \gamma.
\]

Homogeneous elements \( x, y \in A \) satisfy

\[ \iota_1(x)\iota_2(y) = \zeta^{\text{deg}(x)\text{deg}(y)} \iota_2(y)\iota_1(x). \]

Thus we may rewrite the comultiplication as

\[
\Delta_B(z) = z \otimes z, \\
\Delta_B(\alpha) = \iota_1(\alpha)\iota_2(\alpha) - q_1(\gamma)^*\iota_2(\gamma), \\
\Delta_B(\gamma) = \iota_1(\gamma)\iota_2(\alpha) + \iota_1(\alpha)^*\iota_2(\gamma).
\]

In particular, \( \Delta_B \) respects the commutation relations for \((\alpha, \gamma, z)\), so it is a well-defined \( * \)-homomorphism \( B \to B \otimes B \). It is routine to check the cancellation conditions \([14]\) for \( B \), so \((B, \Delta_B)\) is a compact quantum group.

This is a compact quantum group with a projection as in \([7]\). Here the projection \( \pi : B \to B \) is the unique \( * \)-homomorphism with \( \pi(\alpha) = 1_B, \pi(\gamma) = 0 \) and \( \pi(z) = z \); this is an idempotent bialgebra morphism. Its “image” is the copy of \( C(T) \) generated by \( z \), its “kernel” is the copy of \( A \) generated by \( \alpha \) and \( \gamma \).

For \( q = 1, B \cong C(T \times SU(2)) \) as a \( C^* \)-algebra, which is commutative. The representation on \( C^2 \) combines the standard embedding of \( SU(2) \) and the representation of \( T \) mapping \( z \) to the diagonal matrix with entries \( z, 1 \). This gives a homeomorphism \( T \times SU(2) \cong U(2) \). So \((B, \Delta_B)\) is the group \( U(2) \), written as a semidirect product of \( SU(2) \) and \( T \).

For \( q \neq 1 \), \((B, \Delta_B)\) is the co-opposite of the quantum \( U_q(2) \) group described previously by Zhang and Zhao in \([11]\): the substitutions \( a = \alpha^*, b = \gamma^* \) and \( D = z^* \) turn our generators and relations into those in \([14]\), and the comultiplications differ only by a coordinate flip.

**Theorem 6.1.** Let \( U \in M(K(H) \otimes C(T)) \) be a unitary representation of \( T \) on a Hilbert space \( H \). There is a bijection between representations of \( SU_q(2) \) on \( H \) and representations of \((B, \Delta_B)\) on \( H \) that restrict to the given representation on \( T \).

**Proof.** Let \( v \in M(K(H) \otimes A) \) be a unitary representation of \( SU_q(2) \) on \( H \). Since \( B \) contains copies of \( A \) and \( C(T) \), we may view \( u = vU^* \) as an element of \( M(K(H) \otimes B) \).

The \( T \)-invariance of \( v \),

\[ (\text{id} \otimes \rho^A)(v) = U_{12}v_{13}U_{12} \]

and the formula for \( \iota_2 \) (which is basically given by the action \( \rho^A \)) show that

\[ U_{12}(\text{id} \otimes \iota_2)(v)U_{12}^* = v_{13}. \]

Using \( (\text{id} \otimes \iota_2)(v) = v_{12} \), we conclude that \( u \) is a unitary representation of \((B, \Delta_B)\):

\[ (\text{id} \otimes \Delta_B)(u) = v_{12}(\text{id} \otimes \iota_2)(v)U_{12}^*U_{14}^* = v_{12}U_{12}^*v_{13}U_{13}^* = u_{12}v_{13}. \]

Going back and forth between \( u \) and \( v \) is the desired bijection. \( \square \)
References

[1] Paweł Kruszyński and Stanisław Lech Woronowicz, A noncommutative Gelfand–Naimark theorem, J. Operator Theory 8 (1982), no. 2, 361–389, available at http://www.theta.ro/jot/archive/1982-008-002/1982-008-002-009.html MR 677419
[2] Saunders MacLane, Categories for the working mathematician, Springer, New York, 1971. Graduate Texts in Mathematics, Vol. 5., doi: 10.1007/978-1-4757-1721-8 MR 0354798
[3] Shahin Majid, Examples of braided groups and braided matrices, J. Math. Phys. 32 (1991), no. 12, 3246–3253, doi: 10.1063/1.529485 MR 1137374
[4] , Foundations of quantum group theory, Cambridge University Press, Cambridge, 1995., doi: 10.1017/CBO9780511613104 MR 1381692
[5] Ralf Meyer, Sutanu Roy, and Stanisław Lech Woronowicz, Quantum group-twisted tensor products of C*-algebras, Internat. J. Math. 25 (2014), no. 2, 1450019, 37, doi: 10.1142/S0129167X14500190 MR 3189775
[6] , Quantum group-twisted tensor products of C*-algebras II, J. Noncommut. Geom. (2015), accepted. arXiv: 1501.04432
[7] Sutanu Roy, C*-Quantum groups with projection, Ph.D. Thesis, Georg-August Universität Göttingen, 2013. http://hdl.handle.net/11858/00-1735-0000-0022-5EF9-0
[8] Piotr Mikołaj Sołtan, Examples of non-compact quantum group actions, J. Math. Anal. Appl. 372 (2010), no. 1, 224–236, doi: 10.1016/j.jmaa.2010.06.045 MR 2672521
[9] Stanisław Lech Woronowicz, Twisted SU(2) group. An example of a noncommutative differential calculus, Publ. Res. Inst. Math. Sci. 23 (1987), no. 1, 117–181, doi: 10.2977/prims/1195176848 MR 890482
[10] , Compact quantum groups, Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, pp. 845–884. MR 1616348
[11] Xiao Xia Zhang and Ervin Yunwei Zhao, The compact quantum group U_q(2). I, Linear Algebra Appl. 408 (2005), 244–258, doi: 10.1016/j.laa.2005.06.004 MR 2166867.

E-mail address: pawel.kasprzak@fuw.edu.pl
Katedra Metod Matematycznych Fizyki, Wydział Fizyki, Uniwersytet Warszawski, Pasteura 5, 02-093 Warszawa, Poland
E-mail address: rmeier2@uni-goettingen.de
Mathematisches Institut, Georg-August Universität Göttingen, Bunsenstrasse 3–5, 37073 Göttingen, Germany
E-mail address: rmssutanu85@gmail.com
Department of Mathematics and Statistics, University of Ottawa, 585 King Edward Avenue, Ottawa, ON K1N 6N5, Canada
E-mail address: Stanislaw.Woronowicz@fuw.edu.pl
Instytut Matematyki, Uniwersytet w Białymstoku, and, Katedra Metod Matematycznych Fizyki, Wydział Fizyki, Uniwersytet Warszawski, Pasteura 5, 02-093 Warszawa, Poland