Abstract. We prove that every \( \mathbb{Q} \)-factorial complete toric variety is a finite quotient of a poly weighted space (PWS), as defined in our previous work \cite{10}. This generalizes the Batyrev–Cox and Conrads description of a \( \mathbb{Q} \)-factorial complete toric variety of Picard number 1, as a finite quotient of a weighted projective space (WPS) \cite{2} Lemma 2.11 and \cite{5} Prop. 4.7, to every possible Picard number, by replacing the covering WPS with a PWS. As a consequence we describe the bases of the subgroup of Cartier divisors inside the free group of Weil divisors and the bases of the Picard subgroup inside the class group, respectively, generalizing to every \( \mathbb{Q} \)-factorial complete toric variety the description given in \cite{10} Thm. 3.13 for a PWS.

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Introduction

The present paper is the second part of a longstanding tripartite study aimed at realizing, for \( \mathbb{Q} \)-factorial projective toric varieties, a classification inspired by what V. Batyrev did in 1991 for smooth complete toric varieties \cite{1}. The first part of this study is given by \cite{10}, in which we studied Gale duality from the \( \mathbb{Z} \)-linear point of view and defined poly weighted spaces (PWS, for short: see the following Definition 1.4). The reader will often be referred to this work for notation, preliminaries and results.

In this paper, as a first application of \( \mathbb{Z} \)-linear Gale duality, we prove that every \( \mathbb{Q} \)-factorial complete toric variety is a finite quotient of a PWS. This generalizes, to every \( \mathbb{Q} \)-factorial complete toric variety, results \cite{2} Lemma 2.11 and \cite{5}.
Prop. 4.7] by V. Batyrev–D. Cox and H. Conrads, respectively, in terms of PWS. Roughly speaking, by adopting the usual terminology for toric varieties of Picard number (in the following called rank, for short) equal to 1 as suitable finite quotients of a weighted projective space (WPS, for short), such a generalization can be stated by saying that every \( \mathbb{Q} \)-factorial complete toric variety is a fake PWS. Section 2 is entirely devoted to explaining this fact. The main result is given by Theorem 2.2: the group action realizing the quotient is determined by the torsion subgroup \( \text{Tors}(\text{Cl}(X)) \) of the classes group \( \text{Cl}(X) \) of the given \( \mathbb{Q} \)-factorial complete toric variety \( X \), and it is represented by a suitable torsion matrix \( \Gamma \). In other words the codimension 1 structure of the \( \mathbb{Q} \)-factorial complete toric variety \( X \) can be completely assigned by a weight matrix \( Q \), in the sense of Definition 1.3 describing the covering PWS as a Cox’s quotient and the torsion matrix \( \Gamma \). In this section we make a significant use of W. Buczynska’s results presented in [3] and briefly summarized in [1.2], since a PWS can be characterized as a 1-connected in codimension 1 \( \mathbb{Q} \)-factorial complete toric variety.

A similar description of a \( \mathbb{Q} \)-factorial complete toric variety as a fake PWS allows us to extend the description of the bases of the subgroup of Cartier divisors inside the free group of Weil divisors and of the Picard subgroup inside the class group, given in [10 Thm. 3.13] for a PWS and presented for every \( \mathbb{Q} \)-factorial complete toric variety. Section 3 is entirely devoted to this purpose. The main result in this context is given by Theorem 3.2, whose proof is essentially the same as that of [10 Thm. 3.13], although the torsion in \( \text{Cl}(X) \) makes the situation significantly more intricate.

For the most part, results stated in Theorems 2.2 and 3.2 are based on the algorithms giving Hermite and Smith normal forms of a matrix (HNF and SNF, respectively), and associated switching matrices, which are well known algorithms (see e.g. [4] Algorithms 2.4.4 and 2.4.14) implemented in many computer algebra procedures. Theorem 3.2 gives quite effective and constructive methods for producing large amounts of interesting information characterizing a given \( \mathbb{Q} \)-factorial complete toric variety \( X \) (see Remark 3.3 for further details).

Finally section 4 is dedicated to reconstructing a fan matrix \( V \), in the sense of Definition 1.2 of a \( \mathbb{Q} \)-factorial complete toric variety \( X \), assigned by means of a weight matrix \( Q \) and a torsion matrix \( \Gamma \). Here, the main result is given by Theorem 4.2 in which the fan matrix is obtained by Gale duality and by reconstructing the matrix \( \beta \) characterizing the universal property (2) and linking a fan matrix of \( X \) with a fan matrix of the covering PWS.

The last section 5 is devoted to giving several examples and applications of all the techniques described in the previous sections: examples are presented following the lines stated in Remark 3.3. Here it is rather important for the reader to be equipped with some computer algebra package which has the ability to produce Hermite and Smith normal forms of matrices and their switching matrices. For example, using Maple, similar procedures are given by HermiteForm and SmithForm with their output options.

Acknowledgements. We would like to thank Antonella Grassi for helpful conversations and suggestions.
A Q–factorial complete toric variety is a quotient of a PWS

1. Preliminaries and notation

The present paper gives a first application of the $\mathbb{Z}$–linear Gale Duality developed in the previous paper [10], to which the reader is referred for notation and all the necessary preliminary results. In particular concerning toric varieties, cones and fans, the reader is referred to [10 § 1.1], for linear algebraic preliminaries about normal forms of matrices (Hermite and Smith normal forms - HNF and SNF for short) to [10 § 1.2]. $\mathbb{Z}$–linear Gale Duality and what is concerning fan matrices ($F$–matrices) and weight matrices ($W$–matrices) is developed in [10 § 2].

Every time the needed nomenclature will be recalled either directly by giving the necessary definition or by reporting the precise reference. Here is a list of main notation and relative references:

1.1. List of notation.
Let $X(\Sigma)$ be a $n$–dimensional toric variety and $T \cong (\mathbb{C}^*)^n$ the acting torus, then

- $M, N, M_\mathbb{R}, N_\mathbb{R}$ denote the group of characters of $T$, its dual group and their tensor products with $\mathbb{R}$, respectively;
- $\Sigma \subseteq N_\mathbb{R}$ is the fan defining $X$;
- $\Sigma(i)$ is the $i$–skeleton of $\Sigma$, which is the collection of all the $i$–dimensional cones in $\Sigma$;
- $\langle \mathbf{v}_1, \ldots, \mathbf{v}_s \rangle \subseteq N_\mathbb{R}$ denotes the cone generated by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_s \in N_\mathbb{R}$; if $s = 1$ then this cone is also called the ray generated by $\mathbf{v}_1$;
- $\mathcal{L}(\mathbf{v}_1, \ldots, \mathbf{v}_s) \subseteq \mathbb{N}$ denotes the sublattice spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_s \in \mathbb{N}$.

Let $A \in \mathbf{M}(d, m; \mathbb{Z})$ be a $d \times m$ integer matrix, then

- $\mathcal{L}_r(A) \subseteq \mathbb{Z}^m$ denotes the sublattice spanned by the rows of $A$;
- $\mathcal{L}_c(A) \subseteq \mathbb{Z}^d$ denotes the sublattice spanned by the columns of $A$;
- $A_I, A^I$ $\forall I \subseteq \{1, \ldots, m\}$ the former is the submatrix of $A$ given by the columns indexed by $I$ and the latter is the submatrix of $A$ whose columns are indexed by the complementary subset $\{1, \ldots, m\}\setminus I$;
- $sA, s^A$ $\forall 1 \leq s \leq d$ the former is the submatrix of $A$ given by the lower $s$ rows and the latter is the submatrix of $A$ given by the upper $s$ rows of $A$;
- HNF($A$), SNF($A$) denote the Hermite and the Smith normal forms of $A$, respectively.
- REF Row Echelon Form of a matrix;
- positive a matrix (vector) whose entries are non-negative.
Given a $F$–matrix $V = (v_1, \ldots, v_{n+r}) \in M(n, n + r; \mathbb{Z})$ (see Definition 1.2 below), then
\[
\langle V \rangle = \langle v_1, \ldots, v_{n+r} \rangle \subseteq \mathbb{N}^R
\]
denotes the cone generated by the columns of $V$;
\[
\mathcal{SF}(V) = \mathcal{SF}(v_1, \ldots, v_{n+r})
\]
is the set of all rational simplicial fans $\Sigma$ such that
\[
\Sigma(1) = \{\langle v_1 \rangle, \ldots, \langle v_{n+r} \rangle\} \subseteq \mathbb{N}^R
\]
[10, Def. 1.3].

Let us start by recalling four fundamental definitions:

**Definition 1.1.** A $n$–dimensional $\mathbb{Q}$–factorial complete toric variety $X = X(\Sigma)$ of rank $r$ is the toric variety defined by a $n$–dimensional simplicial and complete fan $\Sigma$ such that $|\Sigma(1)| = n + r$ [10, § 1.1.2]. In particular the rank $r$ coincides with the Picard number i.e. $r = \text{rk}(\text{Pic}(X))$.

**Definition 1.2** ([10], Def. 2.10). An $F$–matrix is a $n \times (n + r)$ matrix $V$ with integer entries, satisfying the conditions:

- a) $\text{rk}(V) = n$;
- b) $V$ is $F$–complete i.e. $\langle V \rangle = N_R \cong \mathbb{R}^n$ [10] Def. 2.4;
- c) all the columns of $V$ are non zero;
- d) if $v$ is a column of $V$, then $V$ does not contain another column of the form $\lambda v$ where $\lambda > 0$ is real number.

A $CF$–matrix is a $F$-matrix satisfying the further requirement

- e) the sublattice $L_c(V) \subset \mathbb{Z}^n$ is cotorsion free, which is $L_c(V) = \mathbb{Z}^n$ or, equivalently, $L_r(V) \subset \mathbb{Z}^{n+r}$ is cotorsion free.

A $F$–matrix $V$ is called reduced if every column of $V$ is composed by coprime entries [10, Def. 2.13].

**Definition 1.3.** [10] Def. 2.9] A $W$–matrix is an $r \times (n + r)$ matrix $Q$ with integer entries, satisfying the following conditions:

- a) $\text{rk}(Q) = r$;
- b) $L_r(Q)$ does not have cotorsion in $\mathbb{Z}^{n+r}$;
- c) $Q$ is $W$–positive, which is $L_c(Q)$ admits a basis consisting of positive vectors [10] Def. 2.4],
- d) Every column of $Q$ is non-zero.
- e) $L_r(Q)$ does not contain vectors of the form $(0, \ldots, 0, 1, 0, \ldots, 0)$.
- f) $L_r(Q)$ does not contain vectors of the form $(0, a, 0, \ldots, 0, b, 0, \ldots, 0)$, with $ab < 0$.

A $W$–matrix is called reduced if $V = G(Q)$ is a reduced $F$–matrix [10] Def. 2.14, Thm. 2.15]

**Definition 1.4** ([10] Def. 3.11). A poly weighted space (PWS) is a $n$–dimensional $\mathbb{Q}$–factorial complete toric variety $X(\Sigma)$ of rank $r$, whose reduced fan matrix $V$ (see [10] Def. 1.3)) is a $CF$–matrix i.e. if

- $V = (v_1, \ldots, v_{n+r})$ is a $n \times (n + r)$ $CF$–matrix,
- $\Sigma \in \mathcal{SF}(V)$. 


1.2. The fundamental group in codimension 1. This subsection is devoted to recall notation and results introduced in [3 § 3] to which the interested reader is referred for any further detail.

Definition 1.5. The fundamental group in codimension 1 of an irreducible, complex, algebraic variety $X$ is the inverse limit

$$\pi_1^1(X) := \lim_{\longleftarrow} \pi_1(U)$$

where $U$ is an open, non-empty, algebraic subset of $X$ such that $\text{codim}_X (X \setminus U) \geq 2$. $X$ is called 1–connected in codimension 1 if $\pi_1^1(X)$ is trivial.

Theorem 1.6 ([3], Thm. 3.4). Assume $X$ be a smooth variety and $V \subset X$ be a closed subset with $\text{codim}_X V \geq 2$. Then $\pi_1^1(X) = \pi_1^1(X \setminus V)$.

Corollary 1.7 ([3], Cor.3.9 and Cor.3.10). If $X$ is a normal variety then $\pi_1^1(X) = \pi_1^1(X \setminus \text{Sing}(X))$. In particular if $X$ is smooth then $\pi_1^1(X) = \pi_1^1(X)$.

In fact, by Theorem 1.6, the inverse limit in the Definition 1.5 has a realization on the smooth subset $X \setminus \text{Sing}(X)$.

Definition 1.8. A finite surjective morphism $\varphi : Y \to X$ is called a covering in codimension 1 (or simply a 1–covering) if it is unramified in codimesion 1, which is exists a subvariety $V \subset X$ such that $\text{codim}_X V \geq 2$ and $\varphi_{|Y_V}$ is a topological covering, where $Y_V := \varphi^{-1}(X \setminus V)$. Moreover a universal covering in codimension 1 is a 1-covering $\varphi : Y \to X$ such that for any 1–covering $\phi : X' \to X$ of $X$ there exists a 1-covering $f : Y \to X'$ such that $\varphi = \phi \circ f$.

Proposition 1.9 ([3], Rem. 3.14). A 1–covering $\varphi : Y \to X$ is universal if and only if $Y$ is 1-connected in codimension 1 i.e. $\pi_1^1(Y) = 0$.

Theorem 1.10 (π_1^1 for toric varieties [3], Thm. 4.8). Let $X = X(\Sigma)$ with $\Sigma$ be a fan in $N_{\mathbb{R}}$. For any ray $\rho \in \Sigma(1)$ let $v_\rho$ be the generator of the monoid $\rho \cap N$ and consider the $\mathbb{Z}$-module

$$N_{\Sigma(1)} := \mathcal{L}(v_\rho \mid \rho \in \Sigma(1))$$

as a subgroup of the lattice $N$. Then $\pi_1^1(X) \cong N/N_{\Sigma(1)}$.

This result is an application of Van Kampen Theorem to the open covering $\{X_\rho\}_{\rho \in \Sigma(1)}$ of $X$, where $X_\rho = X(\rho) \cong \mathbb{C} \times (\mathbb{C}^*)^{n-1}$ is the toric variety associated with the fan given by the single ray $\rho \in \Sigma$.

2. Universal 1–coverings

Jointly with [10] Thm. 3.6, Prop. 3.8], the previous Theorem 1.10 gives the following equivalence between a 1–connected $\mathbb{Q}$–factorial complete variety and a PWS:

Theorem 2.1. Let $X = X(\Sigma)$ be a $\mathbb{Q}$–factorial complete $n$–dimensional toric variety of rank $r$. Then the following are equivalent:

1. $X$ is 1-connected in codimension 1,
2. $\pi_1^1(X) \cong N/N_{\Sigma(1)} \cong \text{Tors}(\text{Cl}(X)) = 0$,
3. the reduced fan matrix $V$ of $X$ has coprime $n \times n$ minors,
The HNF of the transposed matrix $V^T$ is given by $\begin{pmatrix} I_n & 0_{r,n} \end{pmatrix}$.

(5) $X$ is a PWS.

The previous results allows us to sketch a nice geometric picture associated with any $\mathbb{Q}$–factorial, complete toric variety: the following Theorem 2.2 is a generalization on the rank $r$ of a well known result holding for $r = \text{rk}(\text{Pic}(X)) = 1$ [2, Lemma 2.11], [5, Prop. 4.7]. In fact a 1-connected in codimension 1, $\mathbb{Q}$–factorial, complete $n$–dimensional toric variety of rank 1 is necessarily a weighted projective space (WPS) whose weights are given by the $1 \times (n + 1)$ weight matrix $Q$.

**Theorem 2.2.** A $\mathbb{Q}$–factorial, complete toric variety $X$ admits a canonic universal 1-covering, $Y$ which is a PWS and such that the 1-covering morphism $\varphi : Y \to X$ is equivariant with respect to the torus actions. In particular every $\mathbb{Q}$–factorial, complete toric variety $X$ can be canonically described as a finite geometric quotient $X \cong Y/\pi_1^1(X)$ of a PWS $Y$ by the torus–equivariant action of $\pi_1^1(X)$.

**Proof.** Let

- $\Sigma$ be a fan defining $X$;
- $V$ be an associated reduced fan matrix, whose transpose matrix represents the morphism

$$\text{div} : M \to \mathcal{P}(X) \cap \mathcal{W}_T(X) =: \mathcal{P}_T(X)$$

$$\mathbf{u} \mapsto \text{div}(\mathbf{u}) := \sum_{\rho \in \Sigma(1)} (\mathbf{u}, \mathbf{v}_\rho) D_\rho$$

over suitable fixed bases;
- $Q = \mathcal{G}(V)$ be a Gale dual matrix of $V$ [10 § 2.1], which is a $W$–matrix [10 Def. 2.9], also called a weight matrix of $X$, whose transposed matrix represents, over suitable fixed bases, the dual morphism

$$d^! : \text{Hom}(\text{Cl}(X), \mathbb{Z}) \to \text{Hom}(\mathcal{W}_T(X), \mathbb{Z})$$

of the morphism $d$ giving the linear equivalence class of a Weil divisor [10 Def. 3.2];
- $\mathcal{N} = \mathcal{G}(Q)$; it is a $CF$–matrix by [10 Prop. 2.11]. It is important to recall here that

$$(2) \quad \exists \beta \in \text{GL}_n(\mathbb{Q}) \cap \text{M}_n(\mathbb{Z}) : V = \beta \cdot \mathcal{N}$$

[10 Prop. 2.1(3)]

The fan $\Sigma \in \mathcal{SF}(V)$ corresponds to a suitable choice of subsets $I \subset \{1, \ldots, n + r\}$. Let $\mathcal{N} \in \mathcal{SF}(\mathcal{N})$ be the fan constructed as follows: for every cone $\sigma_I = \langle \mathbf{v}_i \mid i \in I \subset \{1, \ldots, n + r\} \rangle \in \Sigma$ consider the cone $\mathcal{N}_I \in \mathcal{N}$ generated by the columns of the matrix $\mathcal{V}_I$, under the notation introduced in [11]. By [10 Cor. 2.3, Cor. 2.7] we can conclude that:

(a) $\hat{\Sigma}$ is simplicial if and only if $\Sigma$ is simplicial,
(b) $\hat{\Sigma}$ is complete if and only if $\Sigma$ is complete.
Define \( Y = Y(\hat{\Sigma}) \) as the toric variety associated with the fan \( \hat{\Sigma} \). Then \( Y \) is \( \mathbb{Q} \)-factorial and complete. Moreover the following diagram holds

\[
\begin{array}{cccccc}
0 & \rightarrow & M & \overset{\text{div}_Y}{\rightarrow} & \mathcal{W}_T(Y) & = \mathbb{Z}^{\hat{\Sigma}(1)}_{\alpha} \\
\downarrow & & \downarrow \alpha & & \downarrow d_Y \\
0 & \rightarrow & M & \overset{\text{div}_X}{\rightarrow} & \mathcal{W}_T(X) & = \mathbb{Z}^{\Sigma(1)}_{\text{div}_X} \\
\downarrow & & \downarrow \alpha & & \downarrow d_X \\
0 & \rightarrow & \mathcal{W}_T(X)/\mathcal{W}_T(Y) & \overset{\cong}{\rightarrow} & \text{Tors}(\text{Cl}(X)) & \overset{\pi_1(X)}{\rightarrow} 0 \\
\end{array}
\]

where, fixing, once for all, all the bases:

- morphisms \( \text{div}_Y \) and \( d_Y \) are the \( \mathbb{Z} \)-linear ones represented by matrices \( \hat{V}^T \) and \( Q \), respectively;
- then \( \text{Cl}(Y) \) turns out to be a free abelian group: in fact \( \text{Im}(\text{div}_Y) \) is generated by the rows of \( \hat{V} \) and it is a co-torsion free subgroup of \( \mathbb{Z}^{\hat{\Sigma}(1)}_{\alpha} \) since \( \hat{V} \) is a \( CF \)-matrix and \( \text{Tors}\left( \mathbb{Z}^{n+r}/\mathcal{L}_r(\hat{V}) \right) \cong \text{Tors}\left( \mathbb{Z}^n/\mathcal{L}_e(\hat{V}) \right) \);
- then Theorem 2.1 shows that \( Y \) is a 1-connected in codimension 1 toric variety;
- the vertical morphism \( \alpha : \mathbb{Z}^{\hat{\Sigma}(1)}_{\alpha} \rightarrow \mathbb{Z}^{\Sigma(1)}_{\text{div}_X} \) can be thought of embedding the co-torsion free subgroup \( \mathcal{L}_r(\hat{V}) \subseteq \mathbb{Z}^{\hat{\Sigma}(1)}_{\alpha} = \mathbb{Z}^{n+r} \), generated by the rows of \( \hat{V} \), as the subgroup \( \mathcal{L}_r(V) \subseteq \mathbb{Z}^{\Sigma(1)}_{\text{div}_X} = \mathbb{Z}^{n+r+1} \), generated by the rows of \( V \), and leaving invariant a complementary part of \( \mathbb{Z}^{n+r} \); in fact one can easily check by a diagram chase that \( \alpha(\mathcal{L}_r(\hat{V})) = \mathcal{L}_r(V) \); concretely \( \alpha \) is the \( \mathbb{Z} \)-linear morphism represented by a matrix \( A \) obtained as follows:
  - fix, first of all, the standard bases of torus invariant Weil divisors \( \{ \hat{D}_j \}_{j=1}^{n+r} \) and \( \{ D_j \}_{j=1}^{n+r} \), given by the orbit closure of the rays in \( \hat{\Sigma}(1) \) and \( \Sigma(1) \), respectively;
  - consider \( r \) vectors \( w_1, \ldots, w_r \in \mathbb{Z}^{n+r} \) completing the \( n \) rows of \( \hat{V} \) to a basis of \( \mathcal{W}_T(Y) \cong \mathbb{Z}^{n+r} \);
  - define

\[
\hat{V}_c := \begin{pmatrix} w_1 \\
\vdots \\
w_r \\
V 
\end{pmatrix}, \quad V_c = \begin{pmatrix} w_1 \\
\vdots \\
w_r \\
V 
\end{pmatrix};
\]

- then set \( A := V_c^T \cdot (\hat{V}_c^T)^{-1} \);

it is quite convenient to further specialize the choice of the matrix \( A \) by taking as \( w_1, \ldots, w_r \) the upper \( r \) rows of a matrix \( U_Q \in \text{GL}_{n+r}(\mathbb{Z}) \) such
that
\[ U_Q \cdot Q^T = \text{HNF}(Q^T) = \begin{pmatrix} I_r \\ 0_{n,r} \end{pmatrix} \]
whose existence is a well known result, see e.g. [4, Thm. 2.4.3]; since the lower \( n \) rows of \( U_Q \) give a fan matrix of \( Y \), by [10, Prop. 3.5], then \( \tilde{V}_c = U_Q \); recalling [2], let \( \beta \in \text{GL}_n(Q) \cap M_n(Z) \) be such that \( \beta \cdot \tilde{V} = V \), then

\[ V_c = \begin{pmatrix} I_r \\ 0_{n,r} \end{pmatrix} \cdot U_Q \]

hence giving
\[ A = U_Q^T \cdot \begin{pmatrix} I_r \\ 0_{n,r} \end{pmatrix} \cdot \left( \begin{array}{cc} 0_{r,n} \\ \beta \end{array} \right) \cdot (U_Q^T)^{-1} \]

• while the morphism \( \text{div}_X \) is the \( Z \)-linear morphism represented by the matrix \( V^T \), the morphism \( d_X \) has the following description: by the fundamental theorem of finitely generated abelian groups,

\[ \text{Cl}(X) = F \oplus \text{Tors}(\text{Cl}(X)) \cong \mathbb{Z}^r \oplus \bigoplus_{k=1}^s \mathbb{Z}/\tau_k \mathbb{Z} \]

where \( F \cong \mathbb{Z}^r \) is a free abelian group of rank \( r \) and \( \tau_1, \ldots, \tau_s \) are \( s \geq 0 \) positive integers such that \( \tau_1 \cdots \tau_s \); then \( d_X = f_X \oplus \tau_X \) where the free part \( f_X \) is still represented by the weight matrix \( Q \), while the torsion part \( \tau_X \) is represented by a \( s \times (n + r) \) “torsion matrix” \( \Gamma = (\gamma_{kj}) \) whose \( k \)-th row has entries in \( \mathbb{Z}/\tau_k \mathbb{Z} \), for all \( 1 \leq k \leq s \); by [4] one can easily check that \( Q \cdot A = Q \), in fact

\[ Q \cdot A = (U_Q \cdot Q^T)^T \cdot \begin{pmatrix} I_r \\ 0_{n,r} \end{pmatrix} \cdot \left( \begin{array}{cc} 0_{r,n} \\ \beta \end{array} \right) \cdot (U_Q^T)^{-1} = \begin{pmatrix} I_r \\ 0_{n,r} \end{pmatrix} \cdot \left( \begin{array}{cc} I_r \\ 0_{n,r} \end{array} \right) \cdot (U_Q^T)^{-1} = \begin{pmatrix} I_r \\ 0_{n,r} \end{pmatrix} \cdot (U_Q^T)^{-1} = Q \]

this means that the vertical morphism \( \text{Cl}(Y) \to \text{Cl}(X) \) is actually represented, over the fixed bases, by the identity matrix \( I_n \), isomorphically mapping \( \text{Cl}(Y) \) onto the free part \( F \) of \( \text{Cl}(X) \); then

\[ \text{Cl}(X) \cong \text{Cl}(Y) \oplus \pi_1^1(X) \]

by Theorem [2, 2(2)], giving the splitting of the vertical exact sequence between divisor class groups on the right of diagram [3].
Apply the functor $\text{Hom}(\cdot, \mathbb{C}^*)$ to the previous diagram (8). Since $\text{Hom}$ is left exact and $\mathbb{C}^*$ is a divisible group, we get the following diagram

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & \mu & \mu & 1 \\
G(X) & d_X^* (\mathbb{C}^*)_{\Sigma(1)} & d_{\text{div}} X & T_N & 1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
G(Y) & d_Y^* (\mathbb{C}^*)_{\Sigma(1)} & d_{\text{div}} Y & T_N & 1 \\
1 & 1 & 1 & 1
\end{array}
\]

where

\[
\mu := \text{Hom}(\pi_1(X), \mathbb{C}^*)
\]
\[
G(X) := \text{Hom}(\text{Cl}(X), \mathbb{C}^*)
\]
\[
G(Y) := \text{Hom}(\text{Cl}(Y), \mathbb{C}^*) \cong (\mathbb{C}^*)^r
\]
\[
T_N := \text{Hom}(M, \mathbb{C}^*) \cong N \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^n
\]

and, in particular, (8) gives that $G(X) \cong (\mathbb{C}^*)^r \oplus \mu$. Finally apply to $X$ and $Y$ the Cox’s presentation of toric varieties as geometric quotients, [6, Thm. 2.1]: notice that $\mathbb{C}^{[\Sigma(1)]} = \mathbb{C}^{[\Sigma(1)]} = \mathbb{C}^{n+r}$, hence

\[
Y \cong (\mathbb{C}^{n+r} \setminus Z)/(\mathbb{C}^*)^r, \quad X \cong (\mathbb{C}^{n+r} \setminus Z)/(((\mathbb{C}^*)^r \oplus \mu)
\]

where the locus $Z \subset \mathbb{C}^{n+r}$ is the same both for $Y$ and $X$, since it is completely defined by choice of maximal cones assigned by $\Sigma \in SF(V)$ and $\Sigma \in SF(V)$, respectively. The actions of $(\mathbb{C}^*)^r$ and $(\mathbb{C}^*)^r \oplus \mu$ are induced by the natural multiplicative action of $(\mathbb{C}^*)^{n+r}$ on $\mathbb{C}^{n+r}$, via inclusions $d_Y^*$ and $d_X^*$ in diagram (9), respectively. By the explicit description of $d_Y^*$ and $d_X^*$ given above we get that

\[
(9) \quad \forall \ t \in G(Y) \cong (\mathbb{C}^*)^r \quad d_Y^*(t) = \left( \prod_{i=1}^r q_{i,1}^{t_i}, \ldots, \prod_{i=1}^r q_{i,n+r}^{t_i} \right) \in (\mathbb{C}^*)^{n+r}
\]

where exponents $q_{ij}$ are the entries of the weight matrix $Q$; moreover, since

\[
\mu \cong \bigoplus_{i=1}^s \text{Hom}(\mathbb{Z}/\tau_i \mathbb{Z}, \mathbb{C}^*) \cong \bigoplus_{i=1}^s \mu_{\tau_i}
\]

with $\mu_{\tau_i}$ being the cyclic group of $\tau_i$-th roots of unity,

\[
\forall \ (t; \varepsilon_1^a, \ldots, \varepsilon_s^a) \in G(X) \cong (\mathbb{C}^*)^r \oplus \bigoplus_{i=1}^s \mu_{\tau_i}
\]

\[
(10) \quad d_X^*(t; \varepsilon^a) = \left( \prod_{i=1}^r q_{i,1}^{t_i}, \prod_{k=1}^s \varepsilon_k^{a_{k,1}}, \ldots, \prod_{i=1}^r q_{i,n+r}^{t_i}, \prod_{k=1}^s \varepsilon_k^{a_{k,n+r}} \right)
\]
where:
- $\varepsilon_k$ is a primitive $\tau_k$–th root of unity, for $1 \leq k \leq s$,
- $\alpha_k$ is any representative of a class $\alpha_k \in \mathbb{Z}/\tau_k\mathbb{Z}$, for $1 \leq k \leq s$,
- $c_{kj}$ is any representative of entry $\gamma_{kj} \in \mathbb{Z}/\tau_k\mathbb{Z}$ in the torsion matrix $\Gamma$.

Comparing (9) and (10), the quotients in (8) can then be composed to give the following commutative diagram

\[
\begin{array}{c}
\mathbb{C}^{n+r} \setminus \mathbb{Z} \\
p_Y \downarrow \quad \downarrow p_X \\
Y \quad \varphi \quad X
\end{array}
\]

where $p_Y$ and $p_X$ are the quotient maps induced by the actions of $G(Y)$ and $G(X)$, respectively, and $\varphi$ is the finite quotient map induced by the action of $\mu$ on $Y$. The equivariance of $\varphi$ with respect to the toric actions on $Y$ and $X$ follows by observing that the torus acting is the same for both $Y$ and $X$ and is given by $T_N$ in diagram (7). □

Remark 2.3. Let us notice that the given proof of Theorem 2.2 shows that a $\mathbb{Q}$–factorial complete toric variety $X$ uniquely determines its universal 1-covering $Y$, which is the PWS defined by the following data:

- a weight matrix $Q = G(V)$, where $V$ is a reduced fan matrix of $X$: in fact a fan matrix of $Y$ is given by $G(Q) = \hat{V}$,
- the choice of a fan $\hat{\Sigma} \in \mathcal{SF}(\hat{V})$ uniquely determined by the given choice of the fan $\Sigma \in \mathcal{SF}(V)$ defining $X$.

Remark 2.4. In a sense, the universality property of the 1-covering $Y(\hat{\Sigma})$ is summarized by the linear algebraic property linking a $F$–matrix $V$ and a $CF$–matrix $\hat{V}$ such that $\hat{V} = G(G(V))$. In fact, the given proof of Theorem 2.2 shows that any further 1-covering of the $\mathbb{Q}$–factorial complete toric variety $X(\Sigma)$ admitting $V$ as a fan matrix is given by the choice of an integer matrix $\gamma \in \text{GL}_n(\mathbb{Q}) \cap \text{M}_n(\mathbb{Z})$ dividing the matrix $\beta$ defined in (2), i.e. such that $\beta \cdot \gamma^{-1} \in \text{GL}_n(\mathbb{Q}) \cap \text{M}_n(\mathbb{Z})$. Then setting $V' := \gamma \cdot \hat{V}$ and constructing a fan $\Sigma' \in \mathcal{SF}(V')$ as done for $\hat{\Sigma} \in \mathcal{SF}(\hat{V})$, it turns out the natural factorization of 1-coverings

\[
\begin{array}{c}
Y(\hat{\Sigma}) \\
\varphi \\
X(\Sigma)
\end{array}
\]

as in Definition 1.8. In particular Theorems 1.10 and 2.1 give that

\[
\pi_1^1(X') \cong N/N_{\Sigma'(1)}
\]

where $N_{\Sigma'(1)}$ is defined like in (11) starting from the columns of $V'$ instead of $V$. 
3. Weil versus Cartier

The aim of the present section is that of extending [10, Thm. 3.13] to every \( \mathbb{Q} \)-factorial complete toric variety. By Theorem 2.2 this means extending this result, currently holding for PWS, to their quotients.

Let us start by the following preliminary

**Proposition 3.1.** Let \( X(\Sigma) \) be a \( \mathbb{Q} \)-factorial complete toric variety and \( Y(\hat{\Sigma}) \) be its universal 1-covering. Let \( \{ D_\rho \}_{\rho \in \Sigma(1)} \) and \( \{ \hat{D}_\rho \}_{\rho \in \hat{\Sigma}(1)} \) be the standard bases of \( W_T(X) \) and \( W_T(Y) \), respectively, given by the torus orbit closures of the rays. Then

\[
D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \in C_T(X) \iff \hat{D} = \sum_{\rho \in \hat{\Sigma}(1)} a_\rho \hat{D}_\rho \in C_T(Y).
\]

Therefore \( C_T(X) \cong C_T(Y) \) is a maximal rank subgroup of \( \mathbb{Z}^{\Sigma(1)} = \mathbb{Z}^{\hat{\Sigma}(1)} \) and \( \text{Pic}(X) \cong \text{Pic}(Y) \) is a maximal rank subgroup of the free part \( \text{Cl}(Y) \) of \( \text{Cl}(X) \), under the decomposition (6).

**Proof.** The result follows by writing down the condition of being locally principal for a Weil divisor and observing that

\[
I_{\Sigma} = \{ I \subset \{ 1, \ldots, n+r \} : \langle V^I \rangle \in \Sigma(n) \} = I_{\hat{\Sigma}}
\]

by the construction of \( \hat{\Sigma} \in SF(\hat{V}) \), given the choice of \( \Sigma \in SF(V) \). Then the thesis follows by the same argument proving part (2) of [10, Thm 3.13]. \( \square \)

We are then in a position of stating and proving the main result of this section

**Theorem 3.2.** Let \( X = X(\Sigma) \) be a \( n \)-dimensional \( \mathbb{Q} \)-factorial complete toric variety of rank \( r \) and \( Y = Y(\Sigma) \) be its universal 1-covering. Let \( V \) be a reduced fan matrix of \( X \), \( Q = G(V) \) a weight matrix of \( X \) and \( \hat{V} = G(Q) \) be a CF–matrix giving a fan matrix of \( Y \).

1. Consider the matrix

\[
U_Q = (u_{ij}) \in \text{GL}_{n+r}(\mathbb{Z}) : U_Q \cdot Q^T = \text{HNF}(Q^T)
\]

then the rows of \( ^t U_Q \) (recall notation in [14]) describe the following set of generators of the free part \( F(\cong \text{Cl}(Y)) \) of \( \text{Cl}(X) \)

\[
\forall 1 \leq i \leq r \quad L_i := \sum_{j=1}^{n+r} u_{ij} D_j \in W_T(X) \quad \text{and} \quad F = \bigoplus_{i=1}^{r} \mathbb{Z}[d_X(L_i)]
\]

where \( d_X : W_T(X) \to \text{Cl}(X) \) is the morphism giving to a Weil divisor its linear equivalence class and appearing in the central horizontal exact sequence of diagram (3).

2. Define \( I_{\Sigma} \) like in (12). Through the identification \( F \cong \mathbb{Z}^r \), just fixed in (13), one gets

\[
\text{Pic}(X) = \bigcap_{I \in I_{\Sigma}} L_{\Sigma}(Q_I).
\]

Therefore a basis of \( \text{Pic}(X) \subseteq F \cong \mathbb{Z}^r \) can be computed by applying the procedure described in [10] § 1.2.3.
(3) Let $b_1, \ldots, b_r$ be a basis of $\text{Pic}(X)$ in $\mathbb{Z}^{r} \cong F$, and let $B$ be the $r \times r$ matrix having $b_1, \ldots, b_r$ on the rows. Then a basis of $C_T(X) \subseteq \mathbb{Z}^{n+r} \cong W_T(X)$ is given by the rows of the matrix

$$C_X = \begin{pmatrix} B & 0_{r,n} \\ 0_{n,r} & \beta \end{pmatrix}. U_Q = \begin{pmatrix} B \cdot rU_Q \\ V \end{pmatrix},$$

where $\beta \in \text{GL}_n(\mathbb{Q}) \cap M_n(\mathbb{Z})$ is such that $\beta \cdot \hat{V} = V$ (recall (3)).

(4) There exists a choice of the fan matrices $V$ and $\hat{V} = G(G(V))$ and a diagonal matrix $\Delta = \text{diag}(c_1, \ldots, c_n) \in \text{GL}_n(\mathbb{Q}) \cap M_n(\mathbb{Z})$ such that:

(a) $1 = c_1 \mid \ldots \mid c_n$,

(b) $V = \Delta \cdot \hat{V}$,

(c) $\text{Tors}(\text{Cl}(X)) \cong \bigoplus_{i=1}^{n} \mathbb{Z}/c_i \mathbb{Z} = \bigoplus_{k \leq n} \mathbb{Z}/\tau_k \mathbb{Z},$

according with the decomposition (5) of $\text{Cl}(X)$, which is $s < n$, $\tau_k = c_{n-s+k} > 1$, $c_1 = \cdots = c_{n-s} = 1$.

(5) Given the choice of $\hat{V} = (\hat{v}_{ij})$ and $V$ as in the previous part (4), then the rows of the submatrix $\hat{V}$ describe the following set of generators of $\text{Tors}(\text{Cl}(X))$:

(14) \forall 1 \leq k \leq s \ T_k := \sum_{j=1}^{n+r} \hat{v}_{n-s+k,j} D_j \in W_T(X)$ and $\text{Tors}(\text{Cl}(X)) = \mathcal{L}(d_X(T_1), \ldots, d_X(T_s))$.

(6) Given the choice of $\hat{V}$ and $V$ as in the previous parts (4) and (5), consider

$$W \in \text{GL}_{n+r}(\mathbb{Z}) : W \cdot (n-s)V^T = \text{HNF} ((n-s)V^T)$$

$$G := \hat{V} \cdot (r+s)W^T$$

$$U_G \in \text{GL}_{r+s}(\mathbb{Z}) : U_G \cdot G^T = \text{HNF}(G^T).$$

Then a “torsion matrix”, like $\Gamma$ in diagram (5) and representing the “torsion part” of the morphism $d_X$, which is $\tau_X : W_T(X) \rightarrow \text{Tors}(\text{Cl}(X))$, is given by

(15) $\Gamma = U_G \cdot r+s W \mod \tau$

where this notation means that the $(k, j)$-entry of $\Gamma$ is given by the class in $\mathbb{Z}/\tau_k \mathbb{Z}$ represented by the corresponding $(k, j)$-entry of $U_G \cdot r+s W$, for every $1 \leq k \leq s$, $1 \leq j \leq n+s$.

(7) Setting $\delta_\Sigma := \text{lcm} (\text{det}(Q_I) : I \in I_\Sigma)$ then

$$\delta_\Sigma W_T(X) \subseteq C_T(X)$$

and

$$\delta_\Sigma W_T(Y) \subseteq C_T(Y)$$

and there are the following divisibility relations

$$\delta_\Sigma | [F : \text{Pic}(X)] = [\text{Cl}(Y) : \text{Pic}(Y)] = [W_T(Y) : C_T(Y)] = [W_T(X) : C_T(X)].$$

Proof. (1): Recall that $Q$ is a representative matrix of the “free part” of the morphism $d_X : W_T(X) \rightarrow \text{Cl}(X)$, which is the morphism $f_X : W_T(X) \rightarrow F \cong \text{Cl}(Y)$, where the last isomorphism comes from (10). Then the proof goes on as proving part (1) of (10) Thm. 3.13: namely

$$U_Q \cdot Q^T = \text{HNF} (Q^T) = \begin{pmatrix} I_r \\ 0_{n,r} \end{pmatrix} \implies Q \cdot rU_Q = I_r.$$

(2): Recalling Proposition 3.1, the proof proceeds as for part (2) in [10] Thm. 3.13.

(3): As in the proof of part (3) in [10] Thm. 3.13, observe that

\[ Q \cdot C^T_X = Q \cdot \left( (U_Q)^T \cdot B^T \mid V^T \right) = (B^T \mid 0_{r,n}), \]

so that \( \mathcal{L}_r(C_X) \subseteq \mathcal{C}_T(X) \), where \( \mathcal{L}_r(C_X) \) is the subgroup spanned by the rows of \( C_X \); moreover

\[
[\mathcal{W}_T(X) : \mathcal{L}_r(C_X)] = \det(B) \det(\beta) = [F : \text{Pic}(X)] \cdot [\mathcal{C}(X) : F] = [\mathcal{C}(X) : \text{Pic}(X)] = [\mathcal{W}_T(X) : \mathcal{C}_T(X)];
\]

therefore \( \mathcal{L}_r(C_X) = \mathcal{C}_T(X) \).

(4): By (2) there exists a matrix \( \beta \in \text{GL}_n(Q) \cap M_n(Z) \) is such that \( \beta \cdot \hat{V} = V \). Define \( \Delta := \text{SNF}(\beta) \). It is then a well known fact (see e.g. [4, Algorithm 2.4.14]) the existence of matrices \( \mu, \nu \in \text{GL}_n(Z) \) such that

\[
(16) \quad \Delta = \mu \cdot \beta \cdot \nu \quad \Rightarrow \quad \Delta \cdot (\nu^{-1} \cdot \hat{V}) = \mu \cdot V.
\]

Notice that \( \nu^{-1} \cdot \hat{V} \sim \hat{V} \) are equivalent \( CF \)-matrices and \( \mu \cdot V \sim V \) are equivalent \( F \)-matrices giving a choice of the fan matrices of \( Y \) and \( X \), respectively, satisfying conditions (a) and (b); notice that \( c_1 = \gcd(c_1, \ldots, c_n) = 1 \) since \( V \) is a reduced \( F \)-matrix.

Then (c) follows by recalling that

\[
\text{Tors}({\mathcal{C}(X)}) \cong \mathbb{Z}^n / \mathcal{L}_r(T_n),
\]

where \( T_n \) is the upper \( n \times n \) submatrix of \( \text{HNF}(V^T) \) (see relations (10) in [10] Thm. 3.6).

(5): If \( \hat{V} \) and \( V \) are such that condition (b) in (4) holds, then the \( k \)-th row of \( \hat{V} \) is actually the \( (n - s + k) \)-th row of \( V \) divided by the gcd of its entries, i.e. by \( c_{n-s+k} = \tau_k > 1 \). Since the rows of \( V \) span \( \ker(d_X) \), the \( k \)-th row of \( \hat{V} \) gives then rise to the torsion Weil divisor \( T_k = \sum_{j=1}^{n+r} \tilde{v}_{n-s+k,j} D_j \). Recalling (4)(c) the classes \( d_X(T_1), \ldots, d_X(T_s) \) suffice to generate \( \text{Tors}({\mathcal{C}(X)}) \).

(6): A representative matrix of the torsion morphism \( \tau_X : \mathcal{W}_T(X) \rightarrow \mathcal{C}(X) \) is any matrix satisfying the following properties:

\[
\begin{align*}
(i) \quad & \Gamma = (\gamma_{kj}) \text{ with } \gamma_{kj} \in \mathbb{Z}/\tau_k \mathbb{Z}, \\
(ii) \quad & \Gamma \cdot V^T \equiv 0_{s,n} \mod \tau, \text{ where } V \text{ is a fan matrix satisfying condition (4)(b): this is due to the fact that the rows of } V \text{ span } \ker(d_X), \\
(iii) \quad & \Gamma \cdot (\hat{V})^T = I_s \mod \tau, \text{ since the rows of } \hat{V} \text{ give the generators of } \text{Tors}({\mathcal{C}(X)}), \text{ as in (14)}.
\end{align*}
\]

Therefore it suffices to show that the matrix \( \ast U_G \cdot r+s W \) in (15) satisfies the previous conditions (ii) and (iii) without any reduction mod \( \tau \), which is

\[
\ast U_G \cdot r+s W \cdot (n-s) V^T = 0_{s,n-s}, \quad \ast U_G \cdot r+s W \cdot (\hat{V})^T = I_s.
\]

The first equation follows by the definition of \( W \), in fact

\[
W \cdot (n-s) V^T = \text{HNF}((n-s) V^T)
\]

so that

\[
\text{HNF}((n-s) V^T) = \left( \begin{array}{c} I_{n-s} \\ 0_{r+s, n-s} \end{array} \right) \Rightarrow r+s W \cdot (n-s) V^T = 0_{r+s, n-s}
\]

The second equation follows by the definition of \( U_G \), in fact

\[
U_G \cdot r+s W \cdot (\hat{V})^T = U_G \cdot G^T = \text{HNF}(G^T) = \left( \begin{array}{c} I_s \\ 0_{r, s} \end{array} \right).
\]
Finally, recalling the central vertical exact sequence in (3), one clearly gets

\[ \delta \tag{17} \]

Part (4) of [10, Thm. 3.13] gives that

the last equality is still guaranteed by Proposition 3.1.

Remark 3.3. This is the generalization of [10, Rem. 3.14]. The most part of results stated in Theorems 2.2 and 3.2 are based on the algorithms giving the HNF and the SNF of a matrix, and associated switching matrices, which are well known algorithms (see e.g. [4] Algorithms 2.4.4 and 2.4.14) implemented in many computer algebra procedures. Then Theorem 3.2 gives quite effective and constructive methods to produce a lot of interesting information characterizing a given \( \mathbb{Q} \)-factorial complete toric variety \( X \). In particular

i. given a fan matrix \( V \) of \( X \), a switching matrix \( U \in \text{GL}_{n+r}(\mathbb{Z}) \), such that

\[ \text{HNF}(V^T) = U \cdot V^T, \]

encodes a weight matrix \( Q = G(V) \) of \( X \) in the last \( r \) rows [10, Prop. 3.5];

ii. given a weight matrix \( Q \) of \( X \), a switching matrix \( U \in \text{GL}_{n+r}(\mathbb{Z}) \), such that

\[ \text{HNF}(Q^T) = U \cdot Q^T, \]

encodes both a basis \( \{ d_X(L_i) \}_{i=1}^r \) of the free part \( F \cong \mathbb{Z}^r \) of the divisor class group \( \text{Cl}(X) \), given by the upper \( r \) rows by [13], and a fan matrix \( \widehat{V} \) of the universal 1-covering \( Y \) of \( X \), given by the lower \( n \) rows [10 Prop. 3.5];

iii. by [2] there exists \( \beta \in \text{GL}_n(\mathbb{Z}) \) such that \( \beta \cdot \widehat{V} = V \); an effective procedure producing \( \beta \) is the following:

- set

\[ H = (h_{i,j}) := \text{HNF}(V), \quad \widehat{H} = (\hat{h}_{i,j}) := \text{HNF}(\widehat{V}) \]

\[ U \in \text{GL}(n, \mathbb{Z}) : \quad U \cdot V = H \]

\[ \widehat{U} \in \text{GL}(n, \mathbb{Z}) : \quad \widehat{U} \cdot \widehat{V} = \widehat{H} \]

\[ \beta_H = (b_{i,j}) := U \cdot \beta \cdot \widehat{U}^{-1}, \]

- then \( V = \beta \cdot \widehat{V} \Rightarrow H = \beta_H \cdot \widehat{H} \): since, up to a simultaneous permutations of columns of \( V \) and \( \widehat{V} \), one can assume that both the submatrices \( H_{\{1,...,n\}} \) and \( \widehat{H}_{\{1,...,n\}} \) are upper triangular, then also \( \beta_H \) turns out to be upper triangular,

- one can then get \( \beta_H \) from \( H \) and \( \widehat{H} \), by the following recursive relations

\[ \forall 1 \leq i \leq n \quad b_{i,i} = \frac{h_{i,i}}{\hat{h}_{i,i}} \]

\[ \forall i+1 \leq j \leq n \quad b_{i,j} = \frac{1}{\hat{h}_{i,i}} \left( h_{i,j} - \sum_{k=i}^{j-1} b_{i,k} \hat{h}_{k,j} \right) \]

iv. let \( \Delta := \text{SNF}(\beta) \) and \( \mu, \nu \in \text{GL}_n(\mathbb{Z}) \) be matrices realizing relations (16); setting \( V' := \nu^{-1} \cdot \widehat{V} \), construct the submatrix \( s \cdot V' \) as in part (5) of Theorem 3.2; then the rows of \( s \cdot V' \) describe a set of generators \( \{ d_X(T_k) \}_{k=1}^s \) of \( \text{Tors}(\text{Cl}(X)) \), as in [14];
v. get matrices $W$, $G$ and $U_G$ as in part (6) of Theorem 3.2 and construct the torsion matrix $\Gamma$ as in (15), giving a representative matrix of the torsion part $\tau_X$ of the morphism $d_X$;
vi. apply the procedure [10 § 1.2.3], also based on the HNF algorithm, to get a $r \times r$ matrix $B$ whose rows give a basis of $\text{Pic}(X) \subseteq F = \bigoplus_{i=1}^{r} \mathbb{Z}[d_X(L_i)]$;
vii. construct the matrix $C_X$ as in part (3) of Theorem 3.2 whose rows give a basis of $C_T(X) \subseteq W_T(X) = \bigoplus_{j=1}^{n+r} \mathbb{Z}[D_j]$. 

Moreover:

- recall the commutative diagram of exact sequences [3],
- recall that, for the universal 1-covering $Y$ of $X$, once fixed the basis $\{\hat{D}_j\}_{j=1}^{n+r}$ of $W_T(Y) \cong \mathbb{Z}^{n+r}$ and the basis $\{d_Y(\hat{L}_i)\}_{i=1}^{r}$ of $\text{Cl}(Y) \cong \mathbb{Z}^r$, constructed as in [13] by replacing $D_j$ with $\hat{D}_j$ (see (11) in [10 Thm. 3.13]), one gets the following commutative diagram

\[
\begin{array}{cccccc}
0 & \xrightarrow{\text{div}} & M & \xrightarrow{d_X} & W_T(Y) = \bigoplus_{j=1}^{n+r} \mathbb{Z} \cdot D_j & \xrightarrow{d_Y} & \text{Cl}(Y) = 0 \\
& & 0 & & 0 & & 0 \\
& & \downarrow{\text{div}_Y} & & \downarrow{d_Y} & & \downarrow{B_T} \\
0 & \xrightarrow{0} & M & \xrightarrow{C_T(Y) \cong \text{Pic}(Y) \oplus M} & \text{Pic}(Y) = \bigoplus_{i=1}^{r} \mathbb{Z} \cdot L_i & \xrightarrow{d_Y} & \text{Pic}(Y) = 0 \\
& & \downarrow{C_T(Y)} & & \downarrow{B_T} & & \downarrow{B_T} \\
& & 0 & & 0 & & 0 \\
& & \downarrow{0} & & \downarrow{0} & & \downarrow{0} \\
& & 0 & & 0 & & 0
\end{array}
\]

where $B$ is the $r \times r$ matrix constructed in [10 Thm. 3.13(3)] and which is actually the same of the matrix $B$ constructed in part (3) of Theorem 3.2 due to Proposition 3.11 and

\[
C_Y = \begin{pmatrix} B & 0_{r,n} \\ 0_{n,r} & I_n \end{pmatrix} \cdot U_Q = \begin{pmatrix} B \cdot rU_Q \\ \hat{\nu} \end{pmatrix},
\]

- once fixed the basis $\{D_j\}_{j=1}^{n+r}$ for $W_T(X) \cong \mathbb{Z}^{n+r}$ and the basis $\{d_X(L_i)\}_{i=1}^{r}$ of the free part $F \cong \mathbb{Z}^r$ of $\text{Cl}(X)$, constructed in [13], one gets the following
then, putting all together, one gets the following 3–dimensional commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & M & \xrightarrow{(0_{n,r} | I_n)} & C_T(X) \cong \text{Pic}(X) \oplus M & \xrightarrow{(I_r | 0_{r,n})} & \text{Pic}(X) & \to & 0 \\
& & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & M & \xrightarrow{\text{div}_X} & \mathcal{W}_T(X) = \bigoplus_{j=1}^{n+r} \mathbb{Z} \cdot D_j & \xrightarrow{d_X} & \text{Cl}(X) & \to & 0 \\
& & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & \mathcal{T} \oplus (\mathcal{W}_T(X)/\mathcal{W}_T(Y)) & \cong & \mathcal{T} \oplus \text{Tors}(\text{Cl}(X)) & \to & 0 \\
& & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
0 & \to & 0 & \to & 0 & & & & \\
\end{array}
\]

where \( C_T(X) = C_T(Y) =: C_T \) and \( \text{Pic}(X) = \text{Pic}(Y) =: \text{Pic} \) have been identified, by Proposition 3.1. In fact, diagram (18) follows by checking that

\[
A \cdot C_Y^T = U_Q^T \left( \begin{array}{cc} I_r & 0_{r,n} \\ 0_{n,r} & \beta_T \end{array} \right) (U_Q)^{-1} \cdot U_Q^T \left( \begin{array}{cc} B^T & 0_{r,n} \\ 0_{n,r} & I_n \end{array} \right) = U_Q^T \left( \begin{array}{cc} B^T & 0_{r,n} \\ 0_{n,r} & \beta_T \end{array} \right) = C_X^T.
\]

4. **Going back: from the geometric quotient to the fan**

In the present section we want to reverse our point of view. Namely assume that a \( \mathbb{Q} \)–factorial complete \( n \)–dimensional toric variety \( X \) of rank \( r \) is presented as a geometric quotient as follows:
A $\mathbb{Q}$–factorial complete toric variety is a quotient of a PWS

- let $Z$ be an algebraic subset of $\mathbb{C}^{n+r}$ defined by a suitable monomial ideal $B \subseteq \mathbb{C}[x_1, \ldots, x_{n+r}]$;
- consider the reductive subgroup $G = \text{Hom}(A, \mathbb{C}^*) \subseteq (\mathbb{C}^*)^{n+r}$ where $A$ is a finitely generated abelian group of rank $r$; this means that

$$G \cong (\mathbb{C}^*)^r \oplus \bigoplus_{k=1}^s \mu_{\tau_k}$$

where $\mu_{\tau_k}$ is the cyclic group of $\tau_k$-th roots of unity with $1 < \tau_1 \mid \cdots \mid \tau_s$;
- assume that the action of $G$ over $\mathbb{C}^{n+r}$ is equivariant with respect to the usual multiplication of $(\mathbb{C}^*)^{n+r}$, meaning that for every $g = (t, \varepsilon) \in G$, with $t = (t_1, \ldots, t_r) \in (\mathbb{C}^*)^r$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_s) \in \bigoplus_{k=1}^s \mu_{\tau_k}$, there exist matrices

$$(19) \quad Q = (q_{ij}) \in M(r, n+r; \mathbb{Z}) \quad , \quad C = (c_{kj}) \in M(s, n+r; \mathbb{Z})$$

such that the action is given by the usual multiplication as follows

$$(20) \quad \forall \mathbf{x} \in \mathbb{C}^{n+r} \quad (t, \varepsilon) \cdot \mathbf{x} = \left( \prod_{i=1}^r t_i^{q_{ij}} \cdot \prod_{k=1}^s \varepsilon_k^{c_k j} \cdot x_j \right)_{1 \leq j \leq n+r}$$

- assume that the given action $G \times \mathbb{C}^{n+r} \to \mathbb{C}^{n+r}$ defines a geometric quotient $(\mathbb{C}^{n+r}/\mathbb{Z})/G$ giving precisely the toric variety $X$; this is possible for every $\mathbb{Q}$–factorial complete toric variety by the well known Cox’ result [6].

**Remark 4.1.** By (19) and (20) $X$ is completely assigned by the following three data:

i. a reduced $W$–matrix $Q = (q_{ij})$ defining the action of the free part $(\mathbb{C}^*)^r$ of $G$: the fact that $Q$ can be assumed to be a $W$–matrix follows by the completeness of $X$ [10] Thm. 2.8; moreover a $W$–matrix and its reduction define isomorphic actions of the free part of $G$;

ii. the torsion matrix $\Gamma = (\gamma_{kj})$, with $\gamma_{kj} = [c_{k,j}]_{\tau_k} \in \mathbb{Z}/\tau_k \mathbb{Z}$, defining the action of the torsion part of $G$: in fact such an action is invariant with respect to the choice of different representatives of the class $\gamma_{k,j}$;

iii. the algebraic subset $Z \subseteq \mathbb{C}^{n+r}$.

**Wanted:** a fan matrix $V$ of $X$ and a fan $\Sigma \in SF(V)$ defining $X$.

Let us first of all notice that Theorem 4.2 gives immediately the PWS $Y$ which is the universal 1–covering of $X$: this is obtained by Gale duality since $\widehat{V} = \mathcal{G}(Q)$ is a CF–matrix (by [10] Prop. 2.12(1)), hence giving a fan matrix of $Y$. Moreover the irrelevant ideal $B$ defining $Z$ reconstructs the $n$–skeleton of a fan $\widehat{\Sigma} \in SF(\widehat{V})$ obtained as the fan of all the faces of every cone in $\widehat{\Sigma}(n)$.

Therefore to get $V$ and a fan $\Sigma$ of $X$ it suffices to recover a matrix

$$\beta \in \text{GL}_n(\mathbb{Q}) \cap M_n(\mathbb{Z}) : \beta \cdot \widehat{V} = V$$

as in (2). Then the fan $\Sigma \in SF(V)$ is obtained from $\widehat{\Sigma} \in SF(\widehat{V})$, by reversing the process described in the proof of Theorem 4.2.

**Theorem 4.2.** Given a reduced $W$–matrix $Q$ and a torsion matrix $\Gamma$, as in parts i and ii of Remark 4.1, describing a $\mathbb{Q}$–factorial complete toric variety $X$ as a geometric quotient, and setting $\widehat{V} = \mathcal{G}(Q)$, then a matrix $\beta \in \text{GL}_n(\mathbb{Q}) \cap M_n(\mathbb{Z})$ such that $V = \beta \cdot \widehat{V}$ is a fan matrix of $X$ can be reconstructed as follows:
(1) define a matrix \( C \in M(s, n + r; \mathbb{Z}) \) by choosing a representative \( c_{kj} \) for any entry \( \gamma_{kj} \) of \( \Gamma \),

(2) consider the matrix 

\[
K := 
\begin{pmatrix}
\tilde{V} \cdot C^T \\
\tau_1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \tau_s 
\end{pmatrix} 
\in M(n + s, s; \mathbb{Z})
\]

(3) let \( U \in GL_{n+s}(\mathbb{Z}) : HNF(K) = U \cdot K \)

then, recalling notation in 1.1,

\[ \beta = (nU)_{\{1,\ldots,n\}} \]

Proof. A fan matrix \( V = (v_{ij}) \) of \( X \) has to satisfy the relations

\[
V \cdot Q^T = 0, \quad V \cdot \Gamma^T = 0 \mod \tau
\]

where the latter means that \( \sum_{j=1}^{n+r} v_{ij} \gamma_{kj} = [0]_{\tau_k} \), for every \( 1 \leq i \leq n, 1 \leq k \leq s \).

Since by (2) there exists a matrix \( \beta \) such that \( V = \beta \cdot \tilde{V} \), the former relation is a consequence of the fact that \( \tilde{V} = G(Q) \). Setting \( \beta = (x_{l,i}) \) and \( \tilde{V} = (\tilde{v}_{ij}) \), the second relation in (21) can be rewritten as follows

\[
\forall 1 \leq l \leq n, 1 \leq k \leq s \quad \sum_{i=1}^{n} \left( \sum_{j=1}^{n+r} v_{ij} \gamma_{kj} \right) x_{l,i} = [0]_{\tau_k} \\
\iff \sum_{i=1}^{n} \left( \sum_{j=1}^{n+r} v_{ij} c_{kj} \right) x_{l,i} + \tau_k (-x_{l+k}) = 0
\]

where \( c_{kj} \) is a representative of \( \gamma_{kj} \). Then we are looking for \( n \) independent integer common solutions of a system of \( s \) linear homogeneous equations in \( n + s \) variables whose matrix is given by \( K^T \), where \( K \) is defined as in part (2) of the statement. These solutions are given by the lower \( n \) rows of a matrix

\[ U \in GL_{n+s}(\mathbb{Z}) : U \cdot K = HNF(K) = \begin{pmatrix} T_s \\ 0_{n,s} \end{pmatrix} \]

where \( T_s \) is an upper triangular matrix. In particular the first \( n \) columns of \( nU \) give a \( n \times n \) matrix \( \beta \) such that

\[ \beta \cdot \tilde{V} \cdot \Gamma^T = 0 \mod \tau \]

ending up the proof. \( \square \)

Remark 4.3. The previous Theorem 4.2 gives also a method to check if two geometric quotients giving \( \mathbb{Q} \)-factorial complete toric varieties are actually isomorphic. In fact by recovering the fan matrices this problem is turned into the problem of checking if the fan matrices are equivalent, up to a permutation on columns, in the sense of [10, §2], and if their respective fans are each other related by such a fan matrices equivalence. If only the latter fact is not true then the two given toric geometric quotients are actually birationally isomorphic (and isomorphic in codimension 1 [17, §15.3]).
5. Examples

In this section we are going to present concrete applications of Theorems 2.2, 3.2 and 4.2.

Example 5.1. The following example is the same given in [8, Ex. 1.3], which is here re-discussed to introduce the reader to the use of all the above illustrated techniques in the easier case of an already known example with $r = 1$.

Let us consider the 3-dimensional toric variety $X = X(Σ), Σ = \text{fan}(v_1, v_2, v_3, v_4)$ and $v_i$ is the $i$-th column of the $3 \times 4$ fan matrix

$$V = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 5 & -5 \end{pmatrix}.$$  

Let us follows the list from i. to vii. in Remark 3.3.

i. A matrix $U \in \text{GL}_4(\mathbb{Z})$ such that $\text{HNF}(V^T) = U \cdot V^T$ is given by

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

whose last row gives the weight matrix $Q = (1 \ 1 \ 1 \ 1 \ 1)$ of $X$: then the universal 1–covering $Y$ of $X$ is the projective space $Y = \mathbb{P}^3$.

ii. A switching matrix $U_Q \in \text{GL}_4(\mathbb{Z})$ such that $\text{HNF}(Q^T) = U_Q \cdot Q^T$ is given by

$$U_Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

encoding a basis $\{d_X(L)\}$ of the free part $F \cong \mathbb{Z}$ of $\text{Cl}(X)$, given by the first row and meaning that $L = D_1$, and a fan matrix $\hat{V}$ of the universal 1–covering $Y = \mathbb{P}^3$, given by the further rows, which is

$$\hat{V} = \begin{pmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$  

iii. One can then immediately get a matrix $β$ such that $β \cdot \hat{V} = V$, which is $β = \text{diag}(1, 1, 5)$ and already in SNF.

iv. Therefore $s = 1$ and the last row of $\hat{V}$ gives a generator of $\pi_1(X) = \text{Tors}(\text{Cl}(X)) = \mathbb{Z}/5\mathbb{Z}$, namely $d_X(T)$ with $T = D_3 - D_4$. Hence

$$\text{Cl}(X) = \mathcal{L}(f_X(D_1), τ_X(D_3 - D_4)).$$

v. A matrix $W \in \text{GL}_4(\mathbb{Z})$ such that $\text{HNF}((2V)^T) = W \cdot (2V)^T$ is given by

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 3 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{pmatrix}$$
the torsion matrix \( \Gamma \)
the fan matrix \( \mathbf{V} \)
quotient (24) and check that, up to a permutation on columns, it is equivalent to isomorphic quotients, one can go back obtaining the fan matrix of the geometric data of the Reid’s quotient (24) are the weight matrix \( \mathbf{Q} \)
As M. Kasprzyk observes, this example has been firstly presented by M. Reid as
\( \mu (24) \)
determined by \( \Sigma \), i.e.
while the columns of the torsion matrix \( \Gamma \) describe the exponents of the action of \( \text{Hom}(F, C^*) \cong C^* \) on \( C^4 \setminus \{0\} \) clearly giving \( \mathbb{P}^3 \), while the columns of the torsion matrix \( \Gamma \) describe the exponents of the action of \( \text{Hom}(\pi_1(X), C^*) \cong \mu_5 \) on \( Y = \mathbb{P}^3 \) whose quotient gives \( X \):
\[
\mathbb{P}^3 \quad \longrightarrow \quad \mathbb{P}^3
\]
\[
(\varepsilon, [x_1 : \ldots : x_4]) \quad \mapsto \quad [\varepsilon x_1 : \varepsilon^2 x_2 : \varepsilon^3 x_3 : \varepsilon^4 x_4]
\]
As M. Kasprzyk observes, this example has been firstly presented by M. Reid as the quotient of \( \mathbb{P}^3 \) by the action
\[
\mathbb{P}^3 \quad \longrightarrow \quad \mathbb{P}^3
\]
\[
(\varepsilon, [x_1 : \ldots : x_4]) \quad \mapsto \quad [\varepsilon^4 x_1 : \varepsilon^3 x_2 : \varepsilon^2 x_3 : \varepsilon x_4]
\]
(see [9] Ex. (4.15)). Recalling Remark 4.3 to show that actions (23) and (24) give isomorphic quotients, one can go back obtaining the fan matrix of the geometric quotient (24) and check that, up to a permutation on columns, it is equivalent to the fan matrix \( V \). Then let us follow steps (1), (2) and (3) of Theorem 4.2. The data of the Reid’s quotient (24) are the weight matrix \( \mathbf{Q} = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \) and the torsion matrix \( \Gamma_R := \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \). Let \( C = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix} \) be a matrix of representative integers for \( \Gamma_R \), as in step (1) of Theorem 4.2. The the matrix \( K \) defined in step (2) of the same Theorem is given by
\[
K = \left( \begin{array}{c} \hat{V} \cdot C^T \end{array} \right) = \begin{pmatrix} -4 & 1 \\ 1 & -1 \\ 5 \end{pmatrix}
\]
Step (3) then gives

$$\text{HNF}(K) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = U \cdot K$$

with

$$U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{pmatrix}$$

Therefore

$$\beta_R = (3U)_{(1,2,3)} = \begin{pmatrix} 1 & 4 \\ 0 & 1 \\ 2 & 3 \end{pmatrix}$$

giving the fan matrix

$$V_R := \beta_R \cdot \hat{V} = \begin{pmatrix} 1 & 4 & -11 & 6 \\ 0 & 1 & -2 & 1 \\ 2 & 3 & -7 & 2 \end{pmatrix}$$

The latter \(F\)-matrix turns out to be equivalent to the fan matrix \(V\) in (22), up to a permutation on columns, since

$$V_R = R \cdot V \cdot S$$

with

$$R = \begin{pmatrix} 1 & -11 & -6 \\ 0 & -2 & -1 \\ 2 & -7 & -4 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Example 5.2.** This is a less trivial example for which the procedure of Remark 3.3 turns out to be quite useful to get all the necessary information.

Let us consider the following matrix

\[
\begin{pmatrix}
18 & -21 & -9 & 333 & -492 & 120 \\
-3 & 8 & 4 & -14 & 13 & -4 \\
-23 & 33 & 14 & -404 & 588 & -144 \\
-20 & 26 & 12 & -337 & 493 & -121
\end{pmatrix}
\]

i. A matrix \(U_V \in \text{GL}_6(\mathbb{Z})\) such that \(\text{HNF}(V^T) = U_V \cdot V^T\) is given by

\[
U_V = \begin{pmatrix}
20 & 18 & 11 & 24 & 16 & 0 \\
5 & 4 & 4 & 6 & 4 & 0 \\
20 & 19 & 9 & 24 & 16 & 0 \\
12 & 10 & 9 & 15 & 10 & 0 \\
2 & 4 & 1 & 5 & 4 & 3 \\
1 & 1 & 3 & 2 & 3 & 7
\end{pmatrix}
\]

whose last two rows give the \(W\)-matrix

\[
Q = \begin{pmatrix}
2 & 4 & 1 & 5 & 4 & 3 \\
1 & 1 & 3 & 2 & 3 & 7
\end{pmatrix}
\]

Therefore \(V\) is a \(F\)-matrix [10, Prop. 2.12(2)]. Consider \(X(\Sigma)\) with \(\Sigma \in \mathcal{SF}(V)\). Then \(X\) is a 4-dimensional \(Q\)-factorial complete toric variety of rank 2. One can check that \(|\mathcal{SF}(V)| = 3\), meaning that we have 3 choices for the fan \(\Sigma\) i.e. for the toric variety \(X(\Sigma)\).
ii. A switching matrix $U_Q \in \text{GL}_6(\mathbb{Z})$ such that $\text{HNF}(Q^T) = U_Q \cdot Q^T$ is given by

$$U_Q = \begin{pmatrix}
5 & -2 & -1 & 0 & 0 & 0 \\
2 & -1 & 0 & 0 & 0 & 0 \\
11 & -5 & -2 & 0 & 0 & 0 \\
4 & -3 & -1 & 1 & 0 & 0 \\
7 & -4 & -2 & 0 & 1 & 0 \\
15 & -7 & -5 & 0 & 0 & 1
\end{pmatrix}$$

encoding a basis $\{d_X(L_1), d_X(L_2)\}$ of the free part $F \cong \mathbb{Z}^2$ of $\text{Cl}(X)$, given by the upper two rows and meaning that

$L_1 = 5D_1 - 2D_2 - D_3$, $L_2 = 2D_1 - D_2$,

and a fan matrix $\hat{V}$ of the universal 1–covering $Y$, given by the further rows, which is

$$\hat{V} = \begin{pmatrix}
11 & -5 & -2 & 0 & 0 & 0 \\
4 & -3 & -1 & 1 & 0 & 0 \\
7 & -4 & -2 & 0 & 1 & 0 \\
15 & -7 & -5 & 0 & 0 & 1
\end{pmatrix}.$$ (26)

iii. To get a matrix $\beta$ such that $\beta \cdot \hat{V} = V$ we need the HNF of both $\hat{V}$ and $V$:

$$\hat{H} := \text{HNF}(\hat{V}) = \begin{pmatrix}
1 & 0 & 0 & 16 & -25 & 6 \\
0 & 1 & 0 & 7 & -12 & 3 \\
0 & 0 & 1 & 4 & -6 & 1 \\
0 & 0 & 0 & 19 & -29 & 7
\end{pmatrix}$$

$$\hat{U} = \begin{pmatrix}
2 & 16 & -25 & 6 \\
1 & 7 & -12 & 3 \\
1 & 4 & -6 & 1 \\
2 & 19 & -29 & 7
\end{pmatrix} \in \text{GL}_4(\mathbb{Z}) : \hat{U} \cdot \hat{V} = \hat{H}.$$ (26)

$$H := \text{HNF}(V) = \begin{pmatrix}
1 & 0 & 2 & 100 & -153 & 36 \\
0 & 1 & 2 & 53 & -82 & 19 \\
0 & 0 & 3 & 69 & -105 & 24 \\
0 & 0 & 0 & 285 & -435 & 105
\end{pmatrix}$$

$$U = \begin{pmatrix}
-2 & 3 & -2 & 0 \\
0 & 1 & -1 & 1 \\
3 & -1 & -1 & 4 \\
-40 & 34 & -14 & -25
\end{pmatrix} \in \text{GL}_4(\mathbb{Z}) : U \cdot V = H.$$ (17)

The given recursive relations (17) define

$$\beta_H = \begin{pmatrix}
1 & 0 & 2 & 4 \\
0 & 1 & 2 & 2 \\
0 & 0 & 3 & 3 \\
0 & 0 & 0 & 15
\end{pmatrix} \Rightarrow \beta = U^{-1} \beta_H \hat{U} = \begin{pmatrix}
30 & 333 & -492 & 120 \\
2 & -14 & 13 & -4 \\
-33 & -404 & 588 & -144 \\
-28 & -337 & 493 & -121
\end{pmatrix}.$$
iv. Therefore $\Delta = \text{SNF}(\beta)$ and $\mu, \nu \in \text{GL}_4(\mathbb{Z}) : \Delta = \mu \cdot \beta \cdot \nu$ are given by

$$\Delta = \text{diag}(1, 1, 3, 15)$$

$$\mu = \begin{pmatrix}
1410 & -1138 & 551 & 780 \\
1140 & -916 & 420 & 661 \\
-1623 & 1304 & -598 & -941 \\
8425 & -6769 & 3104 & 4885
\end{pmatrix}$$

$$\nu = \begin{pmatrix}
1 & 58 & 2224 & 2022 \\
0 & 1 & 27 & 24 \\
0 & 0 & 1 & 1 \\
0 & -2 & -78 & -71
\end{pmatrix}$$

Then $s = 2$ and the last two rows of

$$\tilde{V}' = \nu^{-1} \tilde{V} = \begin{pmatrix}
521 & -251 & -168 & -2 & 14 & 28 \\
388 & -222 & -112 & 7 & 45 & 3 \\
-184 & 105 & 53 & -2 & -23 & -1 \\
191 & -109 & -55 & 2 & 24 & 1
\end{pmatrix}$$

give the generators of

$$\pi_1^1(X) = \text{Tors}(\text{Cl}(X)) = \mathbb{Z}/90\mathbb{Z},$$

namely $d_X(T_1)$ and $d_X(T_2)$ with

$$T_1 = -184D_1 + 105D_2 + 53D_3 - 2D_4 - 23D_5 - D_6$$

$$T_2 = 191D_1 - 109D_2 - 55D_3 + 2D_4 + 24D_5 + D_6$$

Hence $\text{Cl}(X) = \mathcal{L}(f_X(L_1), f_X(L_2), \tau_X(T_1), \tau_X(T_2))$.

v. Set

$$V' = \mu \cdot V = \begin{pmatrix}
521 & -251 & -168 & -2 & 14 & 28 \\
388 & -222 & -112 & 7 & 45 & 3 \\
-552 & 315 & 159 & -6 & -69 & -3 \\
2865 & -1635 & -825 & 30 & 360 & 15
\end{pmatrix}$$

A matrix $W \in \text{GL}_6(\mathbb{Z})$ such that $\text{HNF}((^T V')^T) = W \cdot ((^T V')^T)$ is given by

$$W = \begin{pmatrix}
5343 & 5282 & 8576 & 8576 & 0 & 0 \\
2626 & 2596 & 4215 & 4215 & 0 & 0 \\
11385 & 11255 & 18274 & 18274 & 0 & 0 \\
3689 & 3647 & 5921 & 5922 & 0 & 0 \\
573 & 567 & 919 & 919 & 1 & 0 \\
1908 & 1886 & 3063 & 3063 & 0 & 1
\end{pmatrix}$$

then

$$G = 2 \tilde{V}' \cdot (4W)^T = \begin{pmatrix}
18909 & 6128 & 949 & 3170 \\
-20782 & -6735 & -1043 & -3484
\end{pmatrix}$$

A matrix $U_G \in \text{GL}_4(\mathbb{Z})$ such that $\text{HNF}(G^T) = U_G \cdot G^T$ is given by

$$U_G = \begin{pmatrix}
351 & -1084 & 6 & 0 \\
315 & -974 & 13 & 0 \\
-11 & 31 & 19 & 0 \\
-8 & 22 & 14 & 1
\end{pmatrix}$$
hence giving
\[
\Gamma = {}^2U_G \cdot {}^4W \mod \tau
= \begin{pmatrix}
697 & 559 & 1324 & 240 & 6 & 0 \\
638 & 518 & 1203 & 229 & 13 & 0
\end{pmatrix} \mod \begin{pmatrix} 3 \\ 15 \end{pmatrix}
= \begin{pmatrix}
[1]_3 & [1]_3 & [1]_3 & [0]_3 & [0]_3 & [0]_3 \\
[8]_{15} & [8]_{15} & [3]_{15} & [4]_{15} & [13]_{15} & [0]_{15}
\end{pmatrix}
\]

vi. Applying the procedure [10 § 1.2.3] one gets the following three possibilities
\[
B_1 = \begin{pmatrix} 5909475 & 0 \\ -238040 & 165 \end{pmatrix},
B_2 = \begin{pmatrix} 5805800 & 0 \\ -4648596 & 1 \end{pmatrix},
B_3 = \begin{pmatrix} 5805800 & 0 \\ -217580 & 55 \end{pmatrix}
\]
depending on the choice of the fan \( \Sigma \in SF(V) \).

vii. A basis of \( C_T(X) \subseteq W_T(X) \) is then given, depending on the choice of the fan \( \Sigma \in SF(V) \), by the rows of the matrices
\[
\forall 1 \leq i \leq 3 \quad \Omega_{X,i} = \begin{pmatrix} B_i & \Omega_{2,4} \\ 0 & \beta \end{pmatrix} \cdot U_Q
\]
which are
\[
C_{X,1} = \begin{pmatrix}
29547375 & -11818950 & -5909475 & 0 & 0 & 0 \\
-1189870 & 475915 & 238040 & 0 & 0 & 0 \\
18 & -21 & -9 & 333 & -492 & 120 \\
-3 & 8 & 4 & -14 & 13 & -4 \\
-23 & 33 & 14 & -404 & 588 & -144 \\
-20 & 26 & 12 & -337 & 493 & -121
\end{pmatrix}
\]
\[
C_{X,2} = \begin{pmatrix}
29029000 & -11611600 & -5805800 & 0 & 0 & 0 \\
-23242978 & 9297191 & 4648596 & 0 & 0 & 0 \\
18 & -21 & -9 & 333 & -492 & 120 \\
-3 & 8 & 4 & -14 & 13 & -4 \\
-23 & 33 & 14 & -404 & 588 & -144 \\
-20 & 26 & 12 & -337 & 493 & -121
\end{pmatrix}
\]
\[
C_{X,3} = \begin{pmatrix}
29029000 & -11611600 & -5805800 & 0 & 0 & 0 \\
-1087790 & 435105 & 217580 & 0 & 0 & 0 \\
18 & -21 & -9 & 333 & -492 & 120 \\
-3 & 8 & 4 & -14 & 13 & -4 \\
-23 & 33 & 14 & -404 & 588 & -144 \\
-20 & 26 & 12 & -337 & 493 & -121
\end{pmatrix}
\]

On the other hand, a basis of \( C_T(Y) \subseteq W_T(Y) \) is given, depending on the choice of the fan \( \tilde{\Sigma} \in SF(\tilde{V}) \), by the rows of the matrices
\[
\forall 1 \leq i \leq 3 \quad \Omega_{Y,i} = \begin{pmatrix} B_i & \Omega_{2,4} \\ 0 & I_4 \end{pmatrix} \cdot U_Q
\]
which are
\[
C_{Y,1} = \begin{pmatrix}
29547375 & -11818950 & -5909475 & 0 & 0 & 0 \\
-1189870 & 475915 & 238040 & 0 & 0 & 0 \\
11 & -5 & -2 & 0 & 0 & 0 \\
4 & -3 & -1 & 1 & 0 & 0 \\
7 & -4 & -2 & 0 & 1 & 0 \\
15 & -7 & -5 & 0 & 0 & 1
\end{pmatrix}
\]
A Q-factorial complete toric variety is a quotient of a PWS

\[ g : ((C^*)^2 \oplus \mu_3 \oplus \mu_{15}) \times (C^6 \setminus Z) \longrightarrow C^6 \setminus Z \]

This is obtained by recovering a matrix \( \beta \) such that \( V = \beta \cdot \hat{V} \), as in \([2]\). At this purpose let us apply Theorem \([4,2]\).

\[
C = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
8 & 8 & 3 & 4 & 13 & 0
\end{pmatrix}
\]

**Example 5.3.** This example is devoted to give an application of Theorem \([4,2]\) by reversing the previous Example \([5,2]\). Let \( X \) be the \( \mathbb{Q} \)-factorial complete toric variety defined by the toric geometric quotient

\[
X = (C^6 \setminus Z) / g ((C^*)^2 \oplus \mu_3 \oplus \mu_{15})
\]

where \( g \) is the action defined in \([27]\). Our aim is recovering a fan matrix of \( X \).

From the exponents of the action \( g \) we get the two matrices:

\[
\begin{align*}
 Q & = \begin{pmatrix}
2 & 4 & 1 & 5 & 4 & 3 \\
1 & 1 & 3 & 2 & 3 & 7
\end{pmatrix} \\
\end{align*}
\]

\[
\begin{align*}
 \Gamma & = \begin{pmatrix}
1 & 3 \\
8 & 15
\end{pmatrix} \begin{pmatrix}
1 & 3 \\
8 & 15
\end{pmatrix} \begin{pmatrix}
1 & 3 \\
8 & 15
\end{pmatrix} \begin{pmatrix}
1 & 3 \\
8 & 15
\end{pmatrix} \begin{pmatrix}
1 & 3 \\
8 & 15
\end{pmatrix} \\
\end{align*}
\]

Finally, matrices \( Q \) and \( \Gamma \) give the geometric description of \( X(\Sigma) \) as a Cox geometric quotient. Namely, saying \( Z = Z_2 = Z_3 \) the exceptional subset of \( C^{n+r} \) determined by \( \Sigma \in SF(V) \) (or equivalently by \( \hat{\Sigma} \in SF(\hat{V}) \)), whose explicit determination is left to the reader, the columns of the weight matrix \( Q \) describe the exponents of the action of \( \text{Hom}(F, C^*) \cong (C^*)^2 \) on \( C^6 \setminus Z \), whose quotient gives \( Y \); on the other hand the columns of the torsion matrix \( \Gamma \) describe the exponents of the action of \( \text{Hom}(\pi_1(X), C^*) \cong \mu_3 \oplus \mu_{15} \) on \( Y \), whose quotient gives \( X \). Putting all together one gets the following action of \( G(X) = \text{Hom}(\text{Cl}(X), C^*) \cong (C^*)^2 \oplus \mu_3 \oplus \mu_{15} \)

\[
\begin{align*}
 C_Y : \mathbb{C}^6 & \longrightarrow X(\Sigma) \\
\end{align*}
\]
be a matrix of representative integers of $\Gamma$ and construct the matrix

$$K := \left( \hat{\mathbf{V}} \cdot C^T \right) = \begin{pmatrix} 4 & 42 \\ 0 & 9 \\ 1 & 31 \\ 3 & 49 \\ 3 & 0 \\ 0 & 15 \end{pmatrix}$$

Then

$$\text{HNF}(K) = U \cdot K$$

with

$$U = \begin{pmatrix} -5 & -49 & 21 & 0 & 0 & 0 \\ -1 & -9 & 4 & 0 & 0 & 0 \\ -9 & -82 & 36 & 0 & 0 & 0 \\ -8 & -68 & 29 & 1 & 0 & 0 \\ -3 & -17 & 9 & 0 & 1 & 0 \\ -3 & -29 & 12 & 0 & 0 & 1 \end{pmatrix}$$

giving

$$\beta = (4U)_{\{1,2,3,4\}} = \begin{pmatrix} -9 & -82 & 36 & 0 \\ -8 & -68 & 29 & 1 \\ -3 & -17 & 9 & 0 \\ -3 & -29 & 12 & 0 \end{pmatrix}$$

Therefore a fan matrix of $X$ is given by

$$V'' := \beta \cdot \hat{\mathbf{V}} = \begin{pmatrix} -175 & 147 & 28 & -82 & 36 & 0 \\ -142 & 121 & 21 & -68 & 29 & 1 \\ -38 & 30 & 5 & -17 & 9 & 0 \\ -65 & 54 & 11 & -29 & 12 & 0 \end{pmatrix}$$

As a final remark, let us notice that the so obtained matrix $V''$ is actually equivalent to the matrix $V$ in [25]: in fact $R \cdot V = V''$ for

$$R = \begin{pmatrix} -646 & 512 & -203 & -416 \\ -533 & 422 & -166 & -345 \\ -143 & 113 & -45 & -92 \\ -237 & 188 & -75 & -152 \end{pmatrix} \in \text{GL}_4(\mathbb{Z})$$

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