Electric $S$-brane solutions with parallel forms on Ricci-flat factor space

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Abstract: In this paper we generalize electric $S$-brane solutions with maximal number of branes. Previously for the action containing $D$-dimensional gravity, a scalar field and antisymmetric $(p + 2)$-form we found composite, electric $S$-brane solutions with all non-zero “charge” densities which obeyed self-duality or anti-self-duality relations. These solutions occurred when $D = 4m + 1 = 5, 9, 13, ...$ and $p = 2m − 1 = 1, 3, 5, ...$. Here we generalize these solutions to the case when the spatial $4m$-dimensional submanifold is Ricci-flat rather than simply Euclidean-flat and the charge density form is a parallel self-dual or anti-self-dual form of rank $2m$. Also generalizations are found for the case when there is an extra “internal” Ricci-flat manifold not covered by the $S$-branes. In the case when one allows a phantom scalar field a subset of these solutions lead to accelerated expansion of this extra spatial factor space not covered by the $S$-branes while the other spatial factor space of dimension $4m$ contracts. Some of these $S$-brane solutions also provide specific examples of solutions of type IIA supergravity.
1. Introduction

The important role played by the relationship between the charge densities of D-branes in string theory was made clear in Polchinski's work on D-branes [1]. K-theory (see, [2, 3, 4] and references therein) gives a general mathematical framework for describing such relationships between charge densities of branes.

Other types of relations between brane charge densities may follow from the fields equations when certain gravitational background solutions are considered. For example, in reference [5] composite electric S-brane solutions were studied, and it was found that constraints coming from the fields equations (in particular the requirement that the off-diagonal components of the Einstein tensor vanish) fixed the relationships between the charges densities of the various S-branes. For more on S-brane solutions see [5]-[15] and references, therein. Specifically in 5-dimensions with a 3-form field and a scalar field the non-zero charge densities of the six electric branes obeyed the following relations

\[ Q_{12} = \mp Q_{34}, \quad Q_{13} = \pm Q_{24}, \quad Q_{14} = \mp Q_{23}. \]  
(1.1)

or, equivalently,

\[ Q_{ij} = \pm \frac{1}{2} \epsilon_{ijkl} Q^{kl} = \pm (\ast Q)_{ij} \]  
(1.2)

and where \( Q_{ij} = -Q_{ji} \). Thus, the charge density form is self-dual or anti-self-dual. When all \( Q_{ij} \) are non-zero, the configuration from (1.2) is the only possible one that follows from the non-diagonal part of Hilbert-Einstein equations.

One interesting feature of the \( D = 5 \) electric S-brane solutions with the 3-form and charge densities satisfying (1.1) was that when the scalar field was absent one was able to avoid the
BKL type [16] oscillating behavior as one approached the initial singularity [6]. BKL type oscillations are asymptotical never ending oscillations of the scale factors and the Kasner parameters which characterize these solutions. This oscillatory behavior can be described graphically in terms of the billiard picture. The billiard approach for $D = 4$ Bianchi-IX model was introduced by Chitre [17]. The multidimensional case was studied (for block-diagonal metrics on product of Einstein spaces) in [18]-[21]. A review of the billiard approach can be found in [22]. In terms of the billiard description the solution from [6] corresponds to a non-moving ("frozen") points in the billiard.

In this article we obtain generalizations of the solutions investigated in [5]. The generalizations we consider are: (i) allowing the spatial factor space covered by the $S$-branes to be Ricci-flat rather than Euclidean flat; (ii) considering some of the extra Ricci-flat factor space not covered by the $S$-branes. We briefly mention possible cosmological applications of some subset of these solutions and show that some of these solutions give examples of type $IIA$ supergravity solutions.

2. S-brane solutions on product of flat spaces

Here as in [5] we consider the model governed by the action

$$S = \int_M d^Dz \sqrt{|g|} \left[ R[g] - g^{MN} \partial_M \varphi \partial_N \varphi - \frac{1}{q!} \exp(2\lambda \varphi) F^2 \right],$$

(2.1)

where $g = g_{MN} dz^M \otimes dz^N$ is the metric, $\varphi$ is a scalar field, $\lambda \in \mathbb{R}$ is a constant dilatonic coupling and

$$F = dA = \frac{1}{q!} F_{M_1...M_q} dz^{M_1} \wedge ... \wedge dz^{M_q},$$

is a $q$-form, $q = p + 2 \geq 1$, on a $D$-dimensional manifold $M$. In (2.1) we denote $|g| = |\det(g_{MN})|$, and $F^2 = F_{M_1...M_q} F_{N_1...N_q} g^{M_1N_1} ... g^{M_qN_q}$.

The equations of motion corresponding to (2.1) are

$$R^M_N - \frac{1}{2} g^M_N R = T^M_N,$$

(2.2)

$$\triangle[g] \varphi - \frac{\lambda}{q!} e^{2\lambda \varphi} F^2 = 0,$$

(2.3)

$$\nabla_{M_1}[g](e^{2\lambda \varphi} F^{M_1...M_q}) = 0.$$  

(2.4)

In (2.3) and (2.4), $\triangle[g]$ and $\nabla[g]$ are Laplace-Beltrami and covariant derivative operators corresponding to $g$. Equations (2.2), (2.3) and (2.4) are, respectively, the multidimensional Einstein-Hilbert equations, the “Klein-Fock-Gordon” equation for the scalar field and the “Maxwell” equations for the $q$-form.

The source terms in (2.2) can be split up as

$$T^M_N = T^M_N[\varphi, g] + e^{2\lambda \varphi} T^M_N[F, g],$$

(2.5)
with
\[ T^M_N[\varphi, g] = \partial^M \varphi \partial_N \varphi - \frac{1}{2} \delta^M_N \partial_P \varphi \partial^P \varphi, \] (2.6)
\[ T^M_N[F, g] = \frac{1}{q!} \left[ -\frac{1}{2} \delta^M_N F^2 + q F_{NM2...Mq} F^{MM2...Mq} \right], \] (2.7)
which are the stress-energy tensor of the scalar field and \( q \)-form, respectively.

In [5] composite, electric \( S \)-brane solutions were found to the equations (2.2), (2.3), (2.4) which were maximal in the sense that all the charge densities of the \( S \)-branes were non-zero. These solutions occurred in spacetimes of dimension \( D = n + 1 = 4m + 1 = 5, 9, 13, \ldots \) and with form field having \( p = 2m - 1 = 1, 3, 5, \ldots \). In addition these solutions had non-exceptional dilatonic coupling
\[ \lambda^2 \neq \frac{n}{4(n-1)} \equiv \lambda_0^2, \] (2.8)
and were defined on the manifold
\[ M = (t_-, t_+) \times \mathbb{R}^n \] (2.9)
where \( \mathbb{R}^n \) is a flat Euclidean space. From [5] these \( S \)-brane solutions had the explicit form
\[ ds^2 = -e^{2n\phi(t)} dt^2 + e^{2\phi(t)} \sum_{i=1}^{n} (dy^i)^2 \] (2.10)
\[ \varphi = \frac{n}{4(1-n)K} (C_2 t + C_1) - \frac{\lambda f(t)}{K}, \] (2.11)
\[ F = e^{2f(t)} dt \wedge Q, \] (2.12)
\[ Q = \frac{1}{(p+1)!} Q_{i_0...i_p} dy^{i_0} \wedge \ldots \wedge dy^{i_p}. \] (2.13)
The functions \( f(t) \) and \( \phi(t) \) are given by
\[ f(t) = -\ln \left[ |z(t)||KQ^2|^{1/2} \right] \quad \text{and} \quad \phi(t) = \frac{1}{2(1-n)K} \left[ \lambda(C_2 + C_1) + f(t) \right], \] (2.14)
where \( K \) and \( Q^2 \) are given by
\[ K \equiv \lambda^2 - \frac{n}{4(n-1)} \quad \text{and} \quad Q^2 \equiv \frac{1}{(p+1)!} \sum_{i_0,...,i_p} Q_{i_0...i_p}^2 > 0, \] (2.15)
and \( C_2, C_1 \) are integration constants. The function \( z(t) \), takes one of the four following forms depending on the value of \( K \) and another integration constant, \( C \)
\[ z(t) = \frac{1}{\sqrt{-C}} \sinh \left[ (t-t_0)\sqrt{C} \right], \quad K < 0, \ C > 0; \] (2.16)
\[ z(t) = \frac{1}{\sqrt{-C}} \sin \left[ (t-t_0)\sqrt{-C} \right], \quad K < 0, \ C < 0; \] (2.17)
\[ z(t) = t - t_0, \quad K < 0, \ C = 0; \] (2.18)
\[ z(t) = \frac{1}{\sqrt{C}} \cosh \left[ (t-t_0)\sqrt{C} \right], \quad K > 0, \ C > 0, \] (2.19)
Here due to the zero energy constraint [5]
\[ C \equiv \frac{n}{4(n-1)}(C_2)^2 \geq 0. \]
Under this condition we exclude the solution (2.17) above. Later when we discuss the general case we will consider this solution.

The \( Q_{i_0...i_p} \) are constant components of the charge density form \( Q \) which obey self-duality or anti-self-duality relations [5], i.e.
\[ Q_{i_0...i_p} = \pm \frac{1}{(p+1)!} \varepsilon_{i_0...i_p j_0...j_p} Q^{j_0...j_p} = \pm (\ast Q)_{i_0...i_p}. \]
This solution describes a collection of \( k \leq \frac{(4m)!}{(2m)!^2} \) electric \( Sp \)-branes with non-zero charge densities \( Q_{i_0...i_p} \neq 0, i_0 < \ldots < i_p \).

The above solutions, can also be enlarged by allowing multiple scalar fields [23].

3. Generalization to Ricci-flat factor space

We now show that one can generalize the solution from the previous section to the case when the manifold (2.9) is replaced by the manifold
\[ M = (t_-, t_+) \times N, \quad (3.1) \]
where \( N \) is \( n \)-dimensional oriented manifold with a Ricci-flat metric \( h = h_{ij}(y)dy^i \otimes dy^j \) of Euclidean signature. As before we will find \( S \)-brane solutions when \( D = n + 1 = 4m + 1, m = 1, 2, \ldots \). The charge density form (2.13) generalizes to
\[ Q = \frac{1}{(p+1)!} Q_{i_0...i_p}(y)dy^{i_0} \wedge \ldots \wedge dy^{i_p}, \]
where the components are now \( y \)-dependent. The solutions will again occur when the rank of \( Q \) is \( 2m \) and the rank of the form field is \( 2m + 1 \).

The form \( Q \) is required to be parallel, i.e. covariantly constant, with respect to \( h \)
\[ (i) \quad \nabla [h]Q = 0, \quad (3.2) \]
and also self-dual or anti-self-dual
\[ (ii) \quad Q = \pm * Q, \quad (3.3) \]
Here \( * = *[h] \) is Hodge operator for the metric \( h \). It follows from (i) that
\[ Q^2 \equiv \frac{1}{(p+1)!} h^{i_0 j_0} \ldots h^{i_p j_p} Q_{i_0...i_p} Q_{j_0...j_p}, \quad (3.4) \]
is constant. This in turn implies that \( Q^2 > 0 \), since \( Q \) is non-zero and the metric \( h \) has Euclidean signature.
For non-exceptional value of dilatonic coupling (2.8) we will show that the form of the solution given in (2.10)-(2.19) carries over to the present case simply by replacing Euclidean flat metric by the Ricci-flat one, i.e.

$$ds^2 = -e^{2n\phi(t)} dt^2 + e^{2\phi(t)} h_{ij}(y) dy^i dy^j,$$

(3.5)

$$\varphi = \frac{n}{4(1-n)K} \left( C_2 t + C_1 - \frac{\lambda f(t)}{K} \right),$$

(3.6)

$$F = e^{2f(t)} dt \wedge Q,$$

(3.7)

provides a solution to the field equations (2.2) -(2.4) for the manifold (3.1). As before the functions $f(t)$ and $\phi(t)$ are given by (2.14), $K$ is defined in (2.15), $C_2, C_1$ are integration constants, $z(t)$ is given by (2.16)-(2.19), and $C = \frac{n}{4(n-1)} (C_2)^2$.

The scalar field in (3.6) still solves (2.3) since $\varphi$ only depends on $t$ and, for the block diagonal metric considered here, going from Euclidean flat to Ricci-flat metric, $h_{ij}$, only modifies the spatial parts of the Laplace-Beltrami operator. The form field (3.7) also still solves the “Maxwell” equation, (2.4). The time dependent part of $F$ still works in (2.4) for the same reason as for the scalar field. The one possible deviation could come from the covariant derivative operator acting on $Q$, but from (3.2) one sees this does not give any additional contribution.

To conclude we need to show that the metric given by the line element (3.5) still satisfies the field equation (2.2). To verify the Hilbert-Einstein eqs. (2.2), we first show that

$$T[F, g]_{ij}^j = 0,$$

(3.8)

for all $i, j = 1, \ldots, n$. Physically this means that when $\lambda = 0$ the form field contributes as dust matter.

In what follows we use the following notation

$$C_{ij}^j = \sum_{i_1, \ldots, i_p = 1}^n Q_{i_1 \ldots i_p} Q_{j i_1 \ldots i_p}.$$

(3.9)

For $i \neq j$, $T[F, g]_{ij}^j$ is proportional to $C_{ij}^j$.

First, prove the relation (3.8) for $i \neq j$. Due to the (anti-) self-duality of $Q$ we get

$$C_{ij}^j = \sqrt{|h|} \sum_{i_1, \ldots, i_p = 1}^n \sum_{j_0, \ldots, j_p = 1}^n \pm \frac{1}{(p+1)!} \varepsilon_{i_1 \ldots i_p j_0 \ldots j_p} Q_{j_0 \ldots j_p} Q_{j i_1 \ldots i_p}.$$

(3.10)

This can be further rewritten as

$$C_{ij}^j = \sqrt{|h|} \sum_{i_1, \ldots, i_p = 1}^n \sum_{j_1, \ldots, j_p = 1}^n \pm \frac{1}{p!} \varepsilon_{i_1 \ldots i_p j_1 \ldots j_p} Q_{j_1 \ldots j_p} Q_{j i_1 \ldots i_p}$$

$$= \sqrt{|h|} \sum_{i_1, \ldots, i_p = 1}^n \sum_{j_1, \ldots, j_p = 1}^n \pm (-1)^p \frac{1}{p!} \varepsilon_{i_1 \ldots i_p j_1 \ldots j_p} Q_{j i_1 \ldots i_p} Q_{j j_1 \ldots j_p}$$

$$= (-1)^p C_{ij}^j.$$

(3.11)
Note that \( j \) is not summed over in the two sums above, and we have explicitly written out the sums that are performed. In going from (3.9) to the first line of (3.11) we have carried out \( p + 1 \) identical sums with: \( j_0 = j, j_1 = j, \ldots, j_p = j \), respectively. From (3.11) one finds that for odd \( p = 2m - 1 \)

\[ C_i^j = -C_i^j \Rightarrow C_i^j = 0, \quad i \neq j, \]

and, hence, relation (3.8) is valid for \( i \neq j \).

Next we prove relation (3.8) for \( i = j \), i.e.

\[ T_i^i[F, g] = 0, \quad (3.12) \]

(no summation in \( i \)) for all \( i = 1, \ldots, n \).

It follows from (2.7) and (2.12)

\[ T_i^i[F, g] = B(t) \left[ -\frac{1}{2} \sum_{k=1}^{n} C_k^i + (q - 1)C^i_i \right], \quad (3.13) \]

for all \( i = 1, \ldots, n \) where \( B(t) \) is function of \( t \). The matrix \( (T_j^i[F, g]) \) is traceless

\[ \sum_{k=1}^{n} T_k^k[F, g] = 0, \]

since \( n = 2(p + 1) = 2(q - 1) \). To prove (3.12) it sufficient to verify that \( T_1^1 = \ldots = T_n^n \), or, equivalently, \( C_1^1 = \ldots = C_n^n \).

Next we show without restriction of generality that \( C_1^1 = C_2^2 \). Indeed, using (3.10) we get (the summation over repeated indices is understood)

\[ C_1^1 = \pm \frac{\sqrt{|h|}}{(p + 1)!} \varepsilon_{i_1 \ldots i_p j_0 \ldots j_p} Q^{j_0 j_1 \ldots j_p} Q^{i_1 \ldots i_p} \]

\[ = \pm \frac{\sqrt{|h|}}{(p + 1)!} \left[ p \varepsilon_{i_1 \ldots i_p j_0 j_1 \ldots j_p} Q^{j_0 j_1 \ldots j_p} Q^{i_1 \ldots i_p} + (p + 1) \varepsilon_{i_1 \ldots i_p j_0 j_1 \ldots j_p} Q^{j_0 j_1 \ldots j_p} Q^{i_1 \ldots i_p} \right] \]

\[ = \pm \frac{\sqrt{|h|}}{(p + 1)!} \left[ p \varepsilon_{i_1 \ldots i_p j_0 j_1 \ldots j_p} Q^{i_0 i_1 \ldots i_p} + (p + 1) \varepsilon_{i_1 \ldots i_p j_0 j_1 \ldots j_p} Q^{j_0 j_1 \ldots j_p} Q^{i_1 \ldots i_p} \right] \]

\[ = \pm \frac{\sqrt{|h|}}{(p + 1)!} \varepsilon_{i_1 \ldots i_p j_0 j_1 \ldots j_p} Q^{i_0 i_1 \ldots i_p} Q^{j_0 j_1 \ldots j_p} Q^{i_1 \ldots i_p} \]

\[ = C_2^2. \]

Here we used the fact that \( p + 1 = 2m \) is even.

This completes the demonstration of (3.8). Now we can verify the Hilbert-Einstein equations (2.2) for the Ricci-flat case of (3.5)-(3.7). The Hilbert-Einstein equations are satisfied for the non-diagonal components \( (M \neq N) \), since the Einstein tensor on the left hand side of (2.2), for the metric (3.5), is diagonal (see the appendix in [9]) and the stress-energy tensor (2.5) \( T_M^M \) is also diagonal due to eqs. (3.8) and because the scalar field only has a temporal
dependence \( \varphi = \varphi(t) \). Next, because of the Ricci-flatness of \( h \) and (3.8) the diagonal part of the Hilbert-Einstein equations (2.2) give the same ordinary differential equations for the metric warp factor, \( \phi(t) \), (with constant parameter \( Q^2 \) from (3.4)) as in the Euclidean flat case. Thus the solution for \( \phi(t) \) is again given by (2.14) [5].

This shows that the solutions given in (3.5) - (3.7) satisfy the field equations (2.2) - (2.4) when the metric, \( h \), is generalized from Euclidean to Ricci-flat. As in the Euclidean flat case the solutions correspond to \( D = 4m + 1 \) and \( p = 2m - 1 \).

### 3.1 Generalization to extra Ricci-flat space not covered by \( S \)-brane

In this subsection we give a generalization of the solution from the previous section when the manifold of (3.1) is replaced by

\[
M = (t_-, t_+) \times N \times N_1,
\]

where \( N_1 \) is a Ricci-flat manifold with the metric \( h^1 \), of dimension \( d_1 \) which is not covered by the \( S \)-branes.

For manifold above we now find

\[
ds^2 = e^{\frac{m f(t)}{K}} \left[ -e^{2 \varphi + 2 \bar{\varphi}} dt^2 + e^{(f(t) + 2 \varphi)h_{ij}(y)dy^i dy^j} + e^{2 \varphi t + 2 \bar{\varphi}} ds_1^2 \right],
\]

\[
\varphi = -\frac{\lambda}{K} f(t) + c_\varphi t + \bar{c}_\varphi,
\]

\[
F = e^{2f(t)} dt \wedge Q,
\]

where \( ds_1^2 = h^1_{mn}(z_1)dz^m dz^n \) is line element corresponding to the metric \( h^1 \), \( f(t) \) is given by (2.14) and (2.16), (2.17), (2.18), (2.19). The constants \( c, \bar{c} \) and \( K \) are given by

\[
c = 4mc^0 + d_1 c^1 \quad \bar{c} = 4mc^0 + d_1 \bar{c}^1
\]

and

\[
K = \lambda^2 + m + \frac{4m^2}{2 - D} \neq 0
\]

where now \( D = 4m + 1 + d_1 \). The integration constants obey the following relations:

\[
CK^{-1} + (c_\varphi)^2 + 4m(c^0)^2 + (c^1)^2 d_1 - (4mc^0 + c^1 d_1)^2 = 0,
\]

\[
2mc^0 = \lambda c_\varphi, \quad 2mc^0 = \lambda \bar{c}_\varphi.
\]

When internal space \( N_1 \) is omitted we recover the solution from the previous subsection with the following identifications between constants:

\[
c_\varphi = \frac{C_{2n}}{4(1 - n)K}, \quad \bar{c}_\varphi = \frac{C_{1n}}{4(1 - n)K}.
\]

For a flat Euclidean space \( N = \mathbb{R}^{4m} \) – the solution presented above can be obtained as a special (one-block) case of the so-called block-orthogonal, composite \( S \)-brane solutions
given in [9, 12]. Block-orthogonal solutions first appeared in [24] for configurations having one factor space of non-zero curvature and were subsequently generalized in [9, 12, 25, 26].

For a Ricci-flat space $N$ the solution under consideration may be verified just along the lines of the previous section when the “internal” space $N_1$ was absent. Here we get the same ordinary differential equations for the ansatz functions which depend only on time as in the case of flat $N$.

We note that previously (when $N_1$ was absent) solution (2.17) was excluded, by the zero energy constraint. In the present case the new zero energy constraint (3.18) allows solution (2.17).

The preceding analysis can simply be generalized to the case when there are several Ricci-flat spaces not covered by $S$-branes, i.e. when $M = (t_-, t_+) \times N \times N_1 \times N_2 \times \ldots$. This was done in [27] for the case when the scalar field was absent.

### 3.2 Solution with acceleration

If one considers the above solutions in the simple case when the integration constants vanish (i.e. $C = c\varphi = c^0 = c^1 = 0$) one finds the physically interesting solution with accelerated expansion for $N_1$. Under these conditions the solution takes the form given by (2.18) and the explicit form of $f(t)$ is $f(t) = -\ln |t - t_0| [K|Q|^2]^{1/2}$. For this solution $K < 0$ or $\lambda^2 < m^{-\frac{d_1}{D-2}}$. This is possible when $\lambda$ is pure imaginary. This implies that the scalar field, after the redefinition $\varphi \rightarrow i\varphi$, is a phantom field [28], i.e. a scalar field with a negative kinetic energy term.

The metric (3.15) for this case reads

$$ds^2 = -d\tau^2 + B_0\tau^{2\nu_0}h_{ij}(y)dy^i dy^j + \tau^{2\nu_1}B_1 ds_1^2,$$  \hspace{1cm} (3.20)

where $\tau > 0$ is “synchronous” time variable given by

$$d\tau^2 = e^{\left(\frac{4m f(t)}{K(D-2)}\right)} dt^2$$

Solving for $\tau$ gives

$$\tau \propto |t - t_0| \left(\frac{2m + K(D-2)}{K(D-2)}\right)^{1/2}$$

In (3.20) $B_0, B_1$ are positive constants and exponents $\nu_0, \nu_1$ are

$$\nu_0 = \frac{4m + 2 - D}{2\Delta}, \quad \nu_1 = \frac{2m}{\Delta}.$$  \hspace{1cm} (3.21)

Here $\Delta = (D - 2)K + 2m$. When $\nu_1 > 1$, or $-2m/(D - 2) < K < 0$ we get an accelerated expansion of factor space $N_1$. This takes place when

$$-(d_1 + 1)m < \lambda^2(D - 2) < -(d_1 - 1)m.$$  \hspace{1cm} (3.22)

For this range of $K, \lambda$ one also finds that $\nu_0 < 0$ if $d_1 > 1$. Thus the Ricci-flat factor space covered by the $S$-branes contracts for $d_1 > 1$, while the other factor space with the line element $ds_1^2$, expands with accelerated expansion.

As a final comment we note that the solutions of this section for $\lambda = 0$ are in agreement with the perfect fluid solutions from [29].
3.3 \(IIA\) supergravity solutions

In this subsection we show how the solutions of this section provide specific examples of supergravity solutions. In \(D = 10\) \(IIA\) supergravity the bosonic part of the action is given by

\[
S = \int d^{10}z \sqrt{|g|} \left[ R[g] - (\partial \phi)^2 - \sum_{a=2}^{4} e^{2\lambda_a \phi} F^2_a \right] - \frac{1}{2} \int F_4 \wedge F_4 \wedge A_2, \quad (3.21)
\]

where \(F_a = dA_{a-1} + \delta_a A_1 \wedge F_3\) is an \(a\)-form and

\[
\lambda_3 = -2\lambda_4, \quad \lambda_2 = 3\lambda_4, \quad \lambda_4^2 = \frac{1}{8}. \quad (3.22)
\]

The example we consider here corresponds to zero forms \(A_1, A_3\) (and hence \(F_2\) and \(F_4\)) in (3.21). This is the so-called NS-NS (Neveu-Schwarz) sector of the model, and the solution from this section gives a solution to this sector of the supergravity model. It can be seen that in this case the solution describes a collection electric \(S_1\) -branes, i.e. \(S\)-fundamental strings (SFS). In this case we have \(m = 1\), and \(K = 1\), and the solution reads

\[
\begin{align*}
\text{ds}^2 &= e^{-f(t)} \left[ -e^{2c_0^2 + \bar{c}_0^2} dt^2 + e^{f(t) + 2c_0^2 + \bar{c}_0^2} h_{ij} dy^i dy^j + e^{2c_1^2 + \bar{c}_1^2} ds_1^2 \right], \\
\phi &= -\lambda_3 f(t) + c_\phi t + \bar{c}_\phi, \\
F_3 &= e^{2f(t)} dt \wedge Q. \quad (3.23)
\end{align*}
\]

Here the function \(f(t)\) is given by relation

\[
f(t) = -\ln \left[ |z(t)|Q^2|^{1/2} \right]
\]

with

\[
z(t) = \frac{1}{\sqrt{C}} \cosh \left[ (t - t_0)\sqrt{C} \right],
\]

and the integration constants obey

\[
c = 4c_0 + d_1 c_1, \quad \bar{c} = 4\bar{c}_0 + d_1 \bar{c}_1,
\]

and

\[
C = -(c_\phi)^2 - 4(c_0^2)^2 - (c_1^2)^2 d_1 + (4c_0 + c_1 d_1)^2 > 0,
\]

\[
2e^{0} = \lambda_3 c_\phi, \quad \bar{c}_\phi = \lambda_3 \bar{c}_\phi.
\]

The inequality \(C > 0\) is an important restriction here. It implies that \(c_\phi\) and the “anisotropy” parameter \(\delta c = c_1 - c_0\), should be small enough. We note that the solution of the previous section does not provide a solution for \(IIB\) supergravity model. In \(IIB\) supergravity model the 5-form, \(F_5\), should be self-dual and in our solution it is not.
4. Conclusions and discussions

In this paper we generalized the composite electric $Sp$-brane solutions from [5] for $D = 4m + 1 = 5, 9, 13, ...$ and $p = 2m − 1 = 1, 3, 5, ...$ to the case when $Q$-form of rank $2m$ is defined on $4m$-dimensional oriented Ricci-flat factor space $N$ of Euclidean signature. Here the form $Q$ is an arbitrary parallel self-dual or anti-self-dual 2$m$-form on $N$ with $Q^2 > 0$. For flat $N = R^{4m}$ [5] the components of this form in canonical coordinates are proportional to the charge densities of the electric $p$-branes. In addition we generalized these solutions to the case when there was an extra Ricci-flat factor space not covered by the $S$-branes. These generalized solutions also provided examples of solutions to IIA supergravity models. One could also extend the solutions of this paper by allowing multiple scalar fields as in [23].

For the case with a phantom field a certain subclass of these solutions with an extra Ricci-flat factor space had the interesting feature of the extra factor space having accelerated expansion while the space covered by the $S$-branes contracted.

We note that such parallel forms exist when the $4m$-dimensional manifold, $N$, is a Kähler, Ricci-flat manifold of holonomy group $SU(2m)$. Indeed the $m$th wedge power of a Kähler 2-form, i.e. $\alpha = \Omega^m$, gives an example of non-zero parallel (i.e. covariantly constant) form of rank $2m$. Splitting this form into a sum of self-dual and anti-self-dual parallel forms: $\alpha = \alpha_+ + \alpha_-$, (here $\alpha_{\pm} = \frac{1}{2} (1 \pm *) \alpha$, where $* = *[h]$ is the Hodge operator on $N$) we get that either $\alpha_+$ or $\alpha_-$ is a non-zero parallel form. Thus, we get an example of either self-dual or anti-self-dual parallel 2$m$-form on a Kähler, Ricci-flat manifold of dimension $4m$. When $N$ is a hyper-Kähler, Ricci-flat manifold of dimension $4m$ with holonomy group $Sp(m)$ there are three Kähler 2-forms: $\Omega_1, \Omega_2, \Omega_3$. In this case we have more examples of parallel forms, since any wedge product $\alpha = \Omega_1^{m_1} \wedge \Omega_2^{m_2} \wedge \Omega_3^{m_3}$, with $m_1 + m_2 + m_3 = m$, is a parallel form. Finally we mention that there exists a parallel (self-dual) 4-form on a 8-dimensional, Ricci-flat manifold of $Spin(7)$ holonomy. See item 10.124 (Table 1) in [30].

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References

[1] J. Polchinski, *Phys. Rev. Lett.* 75, 4724 (1995) [arXiv: hep-th/9510017].

[2] R. Minasian and G. Moore, *JHEP* 9711, 002 (1997) [arXiv: hep-th/9710230].

[3] E. Witten, *JHEP* 9812, 019 (1998) [arXiv: hep-th/9810188].
[4] S. Gukov, *Commun. Math. Phys.* **210** (2000) 621 [arXiv: hep-th/9901042].

[5] V.D. Ivashchuk and D. Singleton, *JHEP* **0410**, 061 (2004) [arXiv: hep-th/0407224].

[6] V.D. Ivashchuk, V.N. Melnikov and D. Singleton, *Phys. Rev.* **D 72** (2005) 103511 [arXiv: gr-qc/0509065].

[7] V.D. Ivashchuk and V.N. Melnikov, *J. Math. Phys.*, **39**, 2866 (1998) [arXiv: hep-th/9708157].

[8] V.D. Ivashchuk and S.-W. Kim, *J. Math. Phys.*, **41** 444 (2000) [arXiv: hep-th/9907019].

[9] V.D. Ivashchuk and V.N. Melnikov, *Class. Quantum Grav.* **18**, R82-R157 (2001) [arXiv: hep-th/0110274].

[10] M. Gutperle and A. Strominger, *JHEP* **0204**, 018 (2002) [arXiv: hep-th/0202210].

[11] C.M. Chen, D.M. Gal’tsov and M. Gutperle, *Phys. Rev.* **D 66**, 024043 (2002) [arXiv: hep-th/0204071].

[12] V.D. Ivashchuk, *Class. Quantum Grav.* **20**, 261-276 (2003) [arXiv: hep-th/0208101].

[13] N. Ohta, *Phys. Lett.* **B 558**, 213 (2003) [arXiv: hep-th/0301095].

[14] V. D. Ivashchuk, “On composite S-brane solutions with orthogonal intersection rules” [arXiv: hep-th/0309027].

[15] V.D. Ivashchuk, “On exact solutions in multidimensional gravity with antisymmetric forms”, In: Proceedings of the 18th Course of the School on Cosmology and Gravitation: The Gravitational Constant. Generalized Gravitational Theories and Experiments (30 April-10 May 2003, Erice). Ed. by G.T. Gillies, V.N. Melnikov and V. de Sabbata, (Kluwer Academic Publishers, Dordrecht, 2004), pp. 39-64; [arXiv:gr-qc/0310114].

[16] V.A. Belinskii, E.M. Lifshitz and I.M. Khalatnikov, *Usp. Fiz. Nauk* **102**, 463 (1970) (in Russian); *Adv. Phys.* **31**, 639 (1982).

[17] D.M. Chitre, Ph. D. Thesis (University of Maryland), 1972.

[18] V.D. Ivashchuk, A.A. Kirillov and V.N. Melnikov, *Izu. Vuzov (Fizika) No 11*, 107 (1994) [Russian Physics Journal **37**, 1102 (1994)].

[19] V.D. Ivashchuk, A.A. Kirillov and V.N. Melnikov, *Pis’ma ZhETF* **60**, No 4, 225 (1994) [JETP Lett., **60**, 235 (1994)].

[20] V.D. Ivashchuk and V.N. Melnikov, *Class. Quantum Grav.* **12**, 809 (1995) [arXiv: gr-qc/9407028]

[21] V.D. Ivashchuk and V.N. Melnikov, *J. Math. Phys.*, **41**, 6341 (2000) [arXiv: hep-th/9904077].

[22] T. Damour, M. Henneaux and H. Nicolai, *Class. Quantum Grav.* **20**, R145 (2003).

[23] H. Dehnen, V. D. Ivashchuk, V. N. Melnikov, *Grav. Cosmol.* **11**, 345 (2005) [arXiv: hep-th/0509147].

[24] K.A. Bronnikov, *Grav. Cosmol.* **4**, 49 (1998) [arXiv: hep-th/9710207].

[25] V.D. Ivashchuk and V.N.Melnikov, “Multidimensional cosmological and spherically symmetric solutions with intersecting p-branes”, [arXiv: gr-qc/9901001].
[26] V.D. Ivashchuk and V.N. Melnikov, J. Math. Phys., 40, 6558 (1999).

[27] I.S. Goncharenko, V.D. Ivashchuk, S. Rudamentkin-Aguilar and D. Singleton, Grav. Cosmol., 12, 169 (2006) [arXiv: gr-qc/0609114].

[28] K. Bronnikov, Acta. Phys. Pol., B4, 251 (1973); H. Ellis, J. Math. Phys., 14, 104 (1973).

[29] V.D. Ivashchuk and V.N. Melnikov, Grav. Cosmol. 1 (1995) 133 [arXiv: hep-th/9503223].

[30] A. Besse, Einstein Manifolds, Springer, 1987.