ON THE LOGARITHMIC PROBABILITY THAT A RANDOM INTEGRAL IDEAL IS \( \mathscr{A} \)-FREE

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Abstract. This extends a theorem of Davenport and Erdős [7] on sequences of rational integers to sequences of integral ideals in arbitrary number fields \( K \). More precisely, we introduce a logarithmic density for sets of integral ideals in \( K \) and provide a formula for the logarithmic density of the set of so-called \( \mathscr{A} \)-free ideals, i.e. integral ideals that are not multiples of any ideal from a fixed set \( \mathscr{A} \).

1. Introduction

Recently, the dynamical and spectral properties of so-called \( \mathscr{A} \)-free systems as given by the orbit closure of the square-free integers, visible lattice points and various number-theoretic generalisations have received increased attention; see [1, 2, 5, 6] and references therein. One reason is the connection of one-dimensional examples such as the square-free integers with Sarnak’s conjecture [12] on the ‘randomness’ of the Möbius function, another the explicit computability of correlation functions as well as eigenfunctions for these systems together with intrinsic ergodicity properties. Here, we provide a very first step towards the study of a rather general notion of freeness for sets of integral ideals in an algebraic number field \( K \).

A well known result by Benkoski [3] states that the probability that a randomly chosen \( m \)-tuple of integers is relatively \( l \)-free (the integers are not divisible by a common nontrivial \( l \)th power) is \( 1/\zeta(lm) \), where \( \zeta \) is the Riemann zeta function. In a recent paper Sittinger [13] reproved that formula and gave an extension to arbitrary rings of algebraic integers in number fields \( K \). Due to a lack of unique prime factorisation of integers in this general situation, one certainly passes to counting integral ideals as a whole and, with a natural notion of asymptotic density, the outcome is \( 1/\zeta_K(lm) \), where

\[
\zeta_K(s) = \sum_{0 \neq a \subset \mathcal{O}_K} \frac{1}{N(a)^s}
\]

is the Dedekind zeta function of \( K \). This immediately leads to the question if the result allows for a further generalisation to more general notions of freeness, where one forbids common divisors from an arbitrary set \( \mathscr{A} \) of non-zero integral ideals instead of considering merely the set consisting of all prime-powers of the form \( p^l \) with \( p \subset \mathcal{O}_K \) prime. In the special case \( K = \mathbb{Q} \) and \( m = 1 \), this was successfully done in a paper by Davenport and Erdős [7] from 1951. The goal of this short note is to provide a full generalisation of their result to arbitrary rings of algebraic integers. It turns out that, building on old and new results from analytic
number theory, one can easily adjust their argument to the more general situation. In this
generality, the case \( m \geq 2 \) remains open.

2. Preliminaries

Let \( K \) be a fixed algebraic number field of degree \( d = [K : \mathbb{Q}] \in \mathbb{N} \). Let \( \mathcal{O}_K \) denote the
ring of integers of \( K \) and recall that \( \mathcal{O}_K \) is a Dedekind domain \([10]\). Hence we have unique
factorisation of non-zero ideals into prime ideals at our disposal, i.e. any non-zero integral
ideal \( a \subset \mathcal{O}_K \) has a (up to rearrangement) unique representation of the form
\[
a = p_1 \cdot \ldots \cdot p_l,
\]
where the \( p_i \) are prime ideals. Recall that the (absolute) norm \( N(a) = [\mathcal{O}_K : a] \) of a non-zero
integral ideal \( a \subset \mathcal{O}_K \) is always finite. Moreover, the norm is completely multiplicative, i.e.
one always has \( N(ab) = N(a)N(b) \). A proof of the following fundamental result can be found
in \([9]\).

**Proposition 2.1.** Let \( H(x) \) be the number of non-zero integral ideals with norm less than or
equal to \( x \). Then
\[
H(x) = cx + O(x^{1 - \frac{1}{d}})
\]
for some positive constant \( c \).

**Corollary 2.2.** As \( x \to \infty \), one has
\[
\sum_{N(a) \leq x} \frac{1}{N(a)} \sim c \log x,
\]
where \( c \) is the constant from Proposition 2.1.

**Proof.** For \( k \in \mathbb{N} \), let \( h(k) \) denote the number of non-zero integral ideals with norm equal to
\( k \). Summation by parts yields
\[
\sum_{N(a) \leq x} \frac{1}{N(a)} = \sum_{k=1}^{\lfloor x \rfloor} \frac{h(k)}{k}
\]
\[
= \frac{H(\lfloor x \rfloor)}{\lfloor x \rfloor} + \sum_{k=1}^{\lfloor x \rfloor - 1} \frac{H(k)}{k(k + 1)}
\]
\[
= c + O(x^{1 - \frac{1}{d}}) + c \sum_{k=1}^{\lfloor x \rfloor - 1} \frac{1}{k + 1} + O\left( \sum_{k=1}^{\lfloor x \rfloor - 1} \frac{k^{-\frac{1}{d}}}{k + 1} \right)
\]
\[
= c \sum_{k=1}^{\lfloor x \rfloor - 1} \frac{1}{k + 1} + O(1)
\]
\[
\sim c \log x,
\]
since \( \sum_{k=1}^{\lfloor x \rfloor} \frac{1}{k} \sim \log x \) as \( x \to \infty \). \( \Box \)
The following generalisation of Mertens’ third theorem to partial Euler products of the Dedekind zeta function \( \zeta_K(s) \) of \( K \) at \( s = 1 \) was shown by Rosen. It will turn out to be crucial for our main result.

**Theorem 2.3.** There is a positive constant \( C \) such that
\[
\prod_{N(p) \leq x} \left( 1 - \frac{1}{N(p)} \right)^{-1} = C \log x + O(1),
\]
where \( p \) ranges over the prime ideals of \( \mathcal{O}_K \). In particular, \( \prod_{N(p) \leq x} \left( 1 - \frac{1}{N(p)} \right)^{-1} \sim C \log x \) as \( x \to \infty \).

**Remark 1.** In fact, Rosen shows that the constant \( C \) above is given by \( C = \alpha_K e^{-\gamma} \), where \( \alpha_K \) is the residue of \( \zeta_K(s) \) at \( s = 1 \) and \( \gamma \) is the Euler-Mascheroni constant.

Let \( \mathcal{A} = \{ a_1, a_2, \ldots \} \) be a fixed set of non-zero integral ideals \( a_i \subset \mathcal{O}_K \). We are interested in the set
\[
\mathcal{M}_\mathcal{A} := \{ b \neq 0 \mid \exists i \ b \subset a_i \}
\]
of non-zero integral ideals that are multiples of some \( a_i \) respectively its complement in the set of all non-zero integral ideals
\[
\mathcal{V}_\mathcal{A} := \{ b \mid \forall i \ b \not\subset a_i \}
\]
of so-called \( \mathcal{A} \)-free (or \( \mathcal{A} \)-prime) integral ideals. More precisely, we ask if the natural asymptotic densities of these sets exist. In general, one defines densities of sets of non-zero integral ideals as follows.

**Definition 1.** Let \( S \) be a set of non-zero integral ideals \( b \subset \mathcal{O}_K \). and let \( S(x) \) be the subset of those \( b \) with \( N(b) \leq x \).

(1) The upper/lower (asymptotic) density \( D(S)/d(S) \) of \( S \) is defined as
\[
\limsup_{x \to \infty} / \liminf_{x \to \infty} \frac{S(x)}{H(x)}.
\]
If these numbers coincide, the common value is called the (asymptotic) density of \( S \), denoted by \( \text{dens}(S) \).

(2) The upper/lower (asymptotic) logarithmic density \( \Delta(S)/\delta(S) \) of \( S \) is defined as
\[
\limsup_{x \to \infty} / \liminf_{x \to \infty} \frac{\sum_{b \in S, N(b) \leq x} \frac{1}{N(b)}}{\sum_{0 \neq b \subset \mathcal{O}_K, N(b) \leq x} \frac{1}{N(b)}},
\]
where one might substitute the denominator by \( c \log x \) due to Corollary 2.2. Again, if these numbers coincide, the common value is called the (asymptotic) logarithmic density of \( S \), denoted by \( \text{dens}_{\log}(S) \).

As in the well known special case of rational integers, the above lower and upper densities are related as follows.
Lemma 2.4 (Density inequality). For any set $S$ of non-zero integral ideals of $K$, one has
\[ d(S) \leq \delta(S) \leq \Delta(S) \leq D(S). \]
In particular, the existence of the density of $S$ implies the existence of the logarithmic density of $S$.

Proof. The assertion follows from summation by parts as follows. Let us first show that $\Delta(S) \leq D(S)$. To this end, let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $\frac{S(n)}{H(n)} \leq D(S) + \varepsilon$ for all $n \geq N$. For $k \in \mathbb{N}$, let $s(k)$ denote the number of non-zero integral ideals $a \in S$ with norm equal to $k$. Summation by parts yields for $n \geq N$
\[ n \sum_{k=1}^{n} \frac{s(k)}{k} = \frac{S(n)}{n} + \sum_{k=1}^{n-1} \frac{S(k)}{k(k+1)} \]
\[ \leq \frac{H(n)}{n} + \sum_{k=1}^{N} \frac{S(k)}{k(k+1)} + \sum_{k=N}^{n-1} \frac{S(k)}{k(k+1)} \]
\[ \leq \frac{H(n)}{n} + \sum_{k=1}^{N} \frac{S(k)}{k(k+1)} + (D(S) + \varepsilon) \sum_{k=N}^{n-1} \frac{H(k)}{k(k+1)}. \]
Since $\frac{H(n)}{n} \to c$ and $\sum_{k=N}^{n-1} \frac{H(k)}{k(k+1)} \sim c \log n$ as $n \to \infty$ (see the proof of Corollary 2.2), one obtains $\Delta(S) \leq D(S) + \varepsilon$. The assertion follows.

For the left inequality $d(S) \leq \delta(S)$, let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $\frac{S(n)}{H(n)} \geq d(S) - \varepsilon$ for all $n \geq N$. Again, summation by parts yields for $n \geq N$
\[ n \sum_{k=1}^{n} \frac{s(k)}{k} \geq \frac{S(n)}{n} + \sum_{k=N}^{n-1} \frac{S(k)}{k(k+1)} \]
\[ \geq (d(S) - \varepsilon) \sum_{k=N}^{n-1} \frac{H(k)}{k(k+1)}, \]
which as above implies $\delta(S) \geq d(S) - \varepsilon$ and thus the assertion. □

3. The Davenport-Erdős theorem for number fields

Next, we shall study the densities of the set $\mathcal{M}_\mathcal{A}$. Let us start with the finite case. Note that, for a finite set $\mathcal{J}$ of integral ideals, their least common multiple is just the intersection $\bigcap \mathcal{J}$.

Proposition 3.1. If $\mathcal{A}$ is finite, then the density of $\mathcal{M}_\mathcal{A}$ exists and is given by
\[ \text{dens}(\mathcal{M}_\mathcal{A}) = \sum_{\emptyset \neq \mathcal{J} \subset \mathcal{A}} (-1)^{|\mathcal{J}|+1} \frac{1}{N(\bigcap \mathcal{J})}. \]

Proof. If $b$ is a non-zero integral ideal of norm $N(b)$ and divisible by $a$, then there is a unique non-zero integral ideal $a'$ such that $b = aa'$. In particular, $N(a') = N(b)/N(a)$ by the multiplicativity of the norm. This provides a bijection from the set of multiples of $a$ of norm
n to the set of non-zero integral ideals of norm $n/N(a)$. Hence, by the inclusion-exclusion principle, one has

$$\frac{S(x)}{H(x)} = \sum_{\emptyset \neq J \subset A} (-1)^{|J|+1} H\left(\frac{x}{N(A \setminus J)}\right) - H(x).$$

Application of Proposition 2.1 now yields the assertion. $\square$

Now let $A = \{a_1, a_2, \ldots\}$ be (countably) infinite. Since $\text{dens}(M_{\{a_1, \ldots, a_r\}})$ is an increasing sequence with upper bound 1, we may define

$$A := \lim_{r \to \infty} \text{dens}(M_{\{a_1, \ldots, a_r\}}).$$

It is then natural to ask if, in general, $A$ is the density of $M_A$. Already in the special case $K = \mathbb{Q}$ the answer is negative in the sense that the natural lower and upper densities may differ; cf. [4].

**Remark 2.** Due to $\text{dens}(M_{\{a_1, \ldots, a_r\}}) \leq d(M_A)$ for all $r \in \mathbb{N}$, one has $A \leq d(M_A)$.

**Proposition 3.2.** If the series $\sum_{a \in A} \frac{1}{N(a)}$ converges, then the density of $M_A$ exists and is equal to $A$.

**Proof.** For fixed $r \in \mathbb{N}$, the number of elements of $M_A$ up to norm $n$ not divisible by any of $a_1, \ldots, a_r$ is at most $\sum_{i=r+1}^{\infty} \frac{1}{N(a_i)}$. Hence, the corresponding upper density is at most $\sum_{i=r+1}^{\infty} \frac{1}{N(a_i)}$ and this converges to 0 as $r \to \infty$. It follows that the upper density of $M_A$ is

$$\text{dens}(M_{\{a_1, \ldots, a_r\}}) + O\left(\sum_{i=r+1}^{\infty} \frac{1}{N(a_i)}\right),$$

which converges to $A$ as $r \to \infty$. This yields $D(M_A) \leq A$ and thus the assertion by Remark 2. $\square$

**Example 1.** Recall that the Dedekind zeta function $\zeta_K(s)$ converges for all $s > 1$ and has the Euler product expansion

$$\zeta_K(s) = \sum_{a \neq 0} \frac{1}{N(a)^s} = \prod_p \left(1 - \frac{1}{N(p)^s}\right)^{-1}.$$

It follows that, for $l \geq 2$ fixed and $A = \{p^l \mid p \text{ prime}\}$, the density of $M_A$ exists and is equal to

$$1 - \prod_p \left(1 - \frac{1}{N(p)^l}\right) = 1 - \frac{1}{\zeta_K(l)},$$

In other words, the density of $V_A$ exists and is equal to $\frac{1}{\zeta_K(l)}$, in accordance with [13, Thm. 4.1].

As a preparation of the proof below, we next introduce the so-called *multiplicative density* of $M_A$. Let $\{p_1, p_2, \ldots\}$ be the set of all prime ideals of $\mathcal{O}_K$, with a numbering that corresponds to increasing order with respect to the norms, i.e. $i \leq j$ always implies $N(p_i) \leq N(p_j)$. For
$k \in \mathbb{N}$ fixed, denote by $n'$ the general non-zero integral ideal composed entirely of the prime ideals $p_1, \ldots, p_k$ (a so-called $p_k$-ideal). Then, one has the convergence

$$
\sum_{n'} \frac{1}{N(n')} = \prod_{i=1}^{k} \left(1 - \frac{1}{N(p_i)}\right)^{-1} =: \Pi_k.
$$

Further, denote by $b'$ those ideals from $\mathcal{M}_{a'}$ that are $p_1, \ldots, p_k$-ideals and let

$$B_k := \frac{\sum_{b'} \frac{1}{N(b')}}{\sum_{n'} \frac{1}{N(n')}} = \Pi_k^{-1} \sum_{b'} \frac{1}{N(b')}.
$$

If the sequence $B_k$ converges as $k \to \infty$, the limit is called the multiplicative density of $\mathcal{M}_{a'}$. Let $\mathcal{A}' := \{a'_1, a'_2, \ldots\}$ be the subset of $\mathcal{A}$ consisting of the $p_1, \ldots, p_k$-ideals only. Then the $b'$ from above are precisely those of the form $a'_i n'$. It follows from the inclusion-exclusion principle and Proposition 3.2 in conjunction with the convergence of $\sum_{a' \in \mathcal{A}'} \frac{1}{N(a')}$ that

$$
\sum_{b'} \frac{1}{N(b')} = \sum_{n'} \frac{1}{N(n')} \sum_{\varnothing \neq J \subset \mathcal{A}'} (-1)^{|J|+1} \frac{1}{N(\bigcap J)} = \Pi_k \ \text{dens}(\mathcal{M}_{a'})
$$

One obtains that $B_k = \text{dens}(\mathcal{M}_{a'})$ which shows that the $B_k$ increase with $k$. Since the $B_k$ are bounded above by 1, this proves that the $B_k$ indeed converge, say $\lim_{k \to \infty} B_k =: B$.

Next, we shall show that $B = A$. Clearly, if $k$ is sufficiently large in relation to $r$, then $\{a_1, \ldots, a_r\} \subset \mathcal{A}'$. Hence, one has

$$B \geq B_k = \text{dens}(\mathcal{M}_{a'}) \geq \text{dens}(\mathcal{M}_{\{a_1, \ldots, a_r\}})
$$

and therefore $B \geq A$. For the reverse inequality $A \geq B$, let $k$ be fixed. The convergence of $\sum_{a' \in \mathcal{A}'} \frac{1}{N(a')}$ implies that the density of $\mathcal{M}_{a'}$ exists and satisfies (see the proof of Proposition 3.2)

$$\text{dens}(\mathcal{M}_{a'}) \leq \text{dens}(\mathcal{M}_{\{a'_1, \ldots, a'_s\}}) + \sum_{i=r+1}^{\infty} \frac{1}{N(a'_i)}.
$$

Now choose $s$ large enough such that $\{a'_1, \ldots, a'_s\} \subset \{a_1, \ldots, a_r\}$. It follows that

$$\text{dens}(\mathcal{M}_{\{a'_1, \ldots, a'_s\}}) \leq \text{dens}(\mathcal{M}_{\{a_1, \ldots, a_r\}}) \leq A
$$

and further, by letting $r \to \infty$, $\text{dens}(\mathcal{M}_{a'}) \leq A$, i.e. $B_k \leq A$. It follows that $B \leq A$. Altogether, this proves the claim $B = A$. We are now in a position to proof the main result of this short note.

**Theorem 3.3.** The logarithmic density of $\mathcal{M}_{a'}$ exists and is equal to $A$. The number $A$ also equals the lower density of $\mathcal{M}_{a'}$.

**Proof.** We have to show, for $S = \mathcal{M}_{a'}$, the equality $d(S) = \delta(S) = \Delta(S) = A$, i.e. $\Delta(S) \leq A$ or, equivalently, $\Delta(S) \leq B$ since we have already seen above that $A = B$. Let $k \in \mathbb{N}$ be fixed. Divide the $b'$ from above of norm $\leq x$ into two classes, placing in the first class those from $\mathcal{M}_{a'}$ and in the second class the remaining ones. The $b'$ in the first class have density
B_k (see above), hence the sum \( \beta_1(x) \) corresponding to the \( b' \) in the first class satisfies (the density inequality is an equality in this case)

\[
\lim_{x \to \infty} \frac{\beta_1(x)}{c \log x} = B_k.
\]

For the sum \( \beta_2(x) \) corresponding to the \( b' \) in the second class, let \( \{p_1, \ldots, p_h\} \) be the set of all prime ideals with norm up to \( x \). The \( b' \) in the second class are \( p_1, \ldots, p_h \)-ideals, but are not in \( M_{\mathfrak{A}'} \). Denoting by \( b^* \) the \( b' \) of this kind (whether of norm \( \leq x \) or not), one has

\[
\beta_2(x) \leq \sum_{b^*} \frac{1}{N(b^*)}.
\]

The \( b^* \) are obtained by taking all \( p_1, \ldots, p_h \)-ideals \( b'' \), and removing from them all \( b' \), where \( b' \) is a \( p_1, \ldots, p_k \)-ideal and \( c \) is any \( p_{k+1}, \ldots, p_h \)-ideal. Hence

\[
\sum_{b'} \frac{1}{N(b')} = \sum_{b''} \frac{1}{N(b'')} - \sum_{b'} \frac{1}{N(b')} \sum_c \frac{1}{N(c)} = \Pi_h B_h - \Pi_k B_k \sum_c \frac{1}{N(c)}.
\]

Since

\[
\sum_c \frac{1}{N(c)} = \prod_{i=k+1}^h \left(1 - \frac{1}{N(p_i)}\right)^{-1} = \Pi_h \Pi_k^{-1},
\]

this shows that

\[
\sum_{b^*} \frac{1}{N(b^*)} = \Pi_h (B_h - B_k).
\]

Finally, it follows from the Mertens type Theorem 2.3 by Rosen that

\[
\beta_2(x) \leq \sum_{b^*} \frac{1}{N(b^*)} = \Pi_h (B_h - B_k) \leq C \log x (B_h - B_k)
\]

and thus, with \( \beta(x) := \beta_1(x) + \beta_2(x) \),

\[
\limsup_{x \to \infty} \frac{\beta(x)}{c \log x} \leq B_k + \frac{C}{c} (B - B_k),
\]

since \( x \to \infty \) implies \( h \to \infty \) which in turn implies \( B_h \to B \). Letting \( k \to \infty \) and thus \( B_k \to B \), one obtains that \( \Delta(M_{\mathfrak{A}'}) \leq B \). \( \square \)

**Corollary 3.4.** The logarithmic density of \( \mathcal{V}_{\mathfrak{A}'} \) exists and is equal to \( 1 - A \). This number also equals the upper density of \( \mathcal{V}_{\mathfrak{A}'} \).

**Proof.** In general, one has \( d(S) = 1 - D(S^c) \) and \( \delta(S) = 1 - \Delta(S^c) \). \( \square \)

**Remark 3.** It is natural to ask for an extension of the above results to the case of \( m \)-tuples \( (b_1, \ldots, b_m) \) of non-zero integral ideals, where one studies the set of those tuples that consist of simultaneous multiples of ideals from \( \mathfrak{A} \) respectively its complement consisting of the relatively \( \mathfrak{A} \)-free tuples. This is work in progress.
Remark 4. There is a non-canonical possibility of defining upper and lower (asymptotic) densities of sets $S$ of non-zero integral ideals $b \subset \mathcal{O}_K$ by passing from $S$ to the subset
\[
\tilde{S} := \{ a \in \mathcal{O}_K \mid (a) \in S \}
\]
of $\mathcal{O}_K$ and considering the image $\alpha(\tilde{S}) \subset \mathbb{Z}^d$ under any isomorphism $\alpha : \mathcal{O}_K \to \mathbb{Z}^d$ of Abelian groups (recall that $d = [K : \mathbb{Q}]$). The set $\alpha(\tilde{S})$ then has natural upper and lower densities defined by counting points e.g. in centred balls (or cubes) of radius $R$ in $\mathbb{R}^d$ divided by the volume and then considering the lim sup resp. lim inf as $R \to \infty$. Note that this also extends componentwise to the case of $m$-tuples mentioned in the last remark. In general, it is not clear if the outcome is independent of the embedding $\alpha$ or coincides with the corresponding densities introduced above. However, for the set of coprime $m$-tuples $(b_1, \ldots, b_m)$ of non-zero integral ideals (i.e. $b_1 + \ldots + b_m = \mathcal{O}_K$) resp. the set of $m$-tuples $(a_1, \ldots, a_m) \in \mathcal{O}_K^m$ with $(a_1) + \ldots + (a_m) = \mathcal{O}_K$, even the (suitably defined) densities exist and all answers are affirmative (with both densities equal to $1/\zeta_K(m)$) as follows from [8, 13]. Another coincidence of the two ways of computing densities shows up (with both densities equal to $1/\zeta_K(l)$) in the case of $l$-free non-zero integral ideals (non-divisibility by any nontrivial $l$th power) resp. integers in $\mathcal{O}_K$, [5, 13]. Proving such a coincidence in our setting above for the lower density of $M_{\text{df}}$ remains open, even for the case $m = 1$.

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