SEMICLASSICAL STATES FOR FRACTIONAL LOGARITHMIC
SCHRÖDINGER EQUATIONS

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ABSTRACT. In this paper, we consider the following fractional logarithmic Schrödinger equation
\[ \varepsilon^{2s}(-\Delta)^s u + V(x)u = u \log |u|^2 \text{ in } \mathbb{R}^N, \]
where \( \varepsilon > 0, \ N \geq 1, \ V(x) \in C(\mathbb{R}^N, [-1, +\infty)). \) By introducing an interesting penalized function, we show that the problem has a positive solution \( u_\varepsilon \) concentrating at a local minimum of \( V \) as \( \varepsilon \to 0. \) There is no restriction on decay rates of \( V, \) especially it can be compactly supported.

Key words: Fractional Logarithmic Schrödinger; penalized; concentration; compactly supported.

AMS Subject Classifications: 35J05, 35J20, 35J10.

1. Introduction

We study the following fractional Schrödinger equation with logarithmic nonlinear term:
\[ \varepsilon^{2s}(-\Delta)^s u + V(x)u = u \log |u|^2, \ x \in \mathbb{R}^N, \quad (1.1) \]
where \( \varepsilon > 0, \ N \in \mathbb{N}, \ V(x) \in C(\mathbb{R}^N, \mathbb{R}) \) is a continuous potential. This type of problem comes from the study of standing waves \( \psi(x, t) = e^{iEt/\varepsilon}u(x) \) of the following fractional nonlinear Schrödinger equation:
\[ i \frac{\partial \psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s \psi + (V(x) + E)\psi - f(\psi), \quad (1.2) \]
where \( f : \mathbb{C} \to \mathbb{R} \) is a function with \( f(z) = g(|z|)z \) and \( g : \mathbb{R}^+ \to \mathbb{R} \) is a real function (which is \( \log |\cdot|^2 \) in (1.1)). For power type nonlinearities, the fractional Schrödinger equation was introduced by Laskin (\[11\], \[12\]) as an extension of the classical nonlinear Schrödinger equations \( s = 1 \) in which the Brownian motion of the quantum paths is replaced by a Lévy flight.

Equation (1.1) is a generalization of the classical Nonlinear Schrödinger Equation with logarithmic nonlinearity:
\[ -\varepsilon^2 \Delta u + V(x)u = u \log |u|^2, \ x \in \mathbb{R}^N \quad (1.3) \]
which admits applications related to quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity and Bose-Einstein condensation (see \[18, 20\] and the references...
therein for more details). For this problem, one can check easily that there always exists $u \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |u|^2 \log |u|^2 = -\infty,$$

which makes the natural functional corresponding to (1.3) is not well defined in $H^1(\mathbb{R}^N)$. To overcome this difficulty, considering the case that $\varepsilon > 0$ is fixed, M. Squassini et al. in [15], decompose the functional $I$ into the sum of a $C^1$ functional and a convex l.s.c (short for lower semicontinuous hereafter) functional and then use the Mountain Pass Theorem 3.2 in [16] to find a critical point. W. Shuai in [14] added some growth conditions such as $\liminf_{|x| \to +\infty} V(x)|x|^{2\sigma} > 0$ (which makes $\int_{\mathbb{R}^N} |u|^2 \log |u|^2$ is $C^1$ in $H^1_V(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) : V(x)|u|^2 \in L^1(\mathbb{R}^N)\}$) and then proved that the problem has a least energy sign-changing solution and a positive ground state. Considering that $\varepsilon > 0$ is fixed and $V(x) \equiv \lambda > 0$, William C. Troy in [17] developed a new comparison method to show that the positive solution of (1.3) is unique up to translation when $1 \leq N \leq 9$; Using the result of Serrin-Tang [13], D’Avenia et al. pointed out in [5] that the positive solution of (1.3) is also unique up to translation. Following, letting $u_p$ be the unique positive solution of (1.3), i.e.,

$$-\Delta u + \lambda u = |u|^{p-2} u,$$

where $p > 2$, Wang et al. in [18] proved that $u_p$ will converge to the unique solution of (1.3) in the sense of $C^{2,\alpha}(\mathbb{R}^N)$ if $p \to 2$. Considering the case that $\varepsilon \to 0$, by penalized idea, C. Zhang et al. in [21] showed that (1.3) has solutions $u_\varepsilon$ concentrating at various types of topological critical points of $V$ as $\varepsilon \to 0$ provided that $\lim_{|x| \to +\infty} V(x)|x|^{-2} > -\infty$ ($V$ can be unbounded below).

To our best knowledge, there are few results about (1.1) in the nonlocal case $0 < s < 1$. When $\varepsilon > 0$ is fixed, D’Avenia et al. in [1] obtained existence of infinitely weak solutions. In [3], it was proved by compactness method that (1.1) has ground states which are stable.

In this paper, we are interesting in semiclassical analysis of (1.1). From a mathematical point of view, the transition from quantum to classical mechanics can be formally performed by letting $\varepsilon \to 0$. For small $\varepsilon > 0$, solutions $u_\varepsilon$ are usually referred to as semiclassical bound states.

In order to state our main result, we need to give some notations and assumptions. For $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined as

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : \frac{u(x) - u(y)}{|x - y|^{N/2 + s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)\},$$

endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} u \right|^2 + u^2 \, dx \right)^{1/2},$$

where

$$\int_{\mathbb{R}^N} \left| (-\Delta)^{s/2} u \right|^2 \, dx = \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy.$$
Like the classical case, we define the space \( \dot{H}^s(\mathbb{R}^N) \) as the completion of \( C_c^\infty(\mathbb{R}^N) \) under the norm
\[
\|u\|^2 = \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2 \, dx = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx \, dy.
\]

We will use the following local fractional Sobolev space
\[
W^{s,2}(\Omega) = \left\{ u \in L^2(\Omega) : \frac{|u(x) - u(y)|}{|x - y|^{N+2s}} \in L^2(\Omega \times \Omega) \right\}.
\]

It is easy to check that \( W^{s,2}(\Omega) \) is a Hilbert space under the following inner product
\[
(u, v) = \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} \, dx \, dy + \int_{\Omega} uv \, dx \quad \forall u, v \in W^{s,2}(\Omega),
\]
see [8] for more details. Also from [8], the fractional Laplacian is defined as
\[
(-\Delta)^s u(x) = C(N,s) \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy
\]
\[
= C(N,s) \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B\varepsilon(x)} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy.
\]

For the sake of simplicity, we define for every \( u \in \dot{H}^s(\mathbb{R}^N) \) the fractional \((-\Delta)^s\) as
\[
(-\Delta)^s u(x) = 2 \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} \, dy.
\]

Note that for every \( \varphi \in C^\infty_c(\mathbb{R}^N) \), it holds
\[
\int_{\mathbb{R}^N} (-\Delta)^s u(x) \varphi(x) = \frac{d}{dt} \int_{\mathbb{R}^N} |(-\Delta)^{s/2}(u + t \varphi)|^2 |_t=0.
\]

Our solutions will be found in the following weighted fractional Sobolev space:
\[
\mathcal{D}^{s,V}_{\varepsilon,\varepsilon}(\mathbb{R}^N) = \left\{ u \in \dot{H}^s(\mathbb{R}^N) : u \in L^2(\mathbb{R}^N, (V(x) + 1) \, dx) \right\},
\]
endowed with the norm
\[
\|u\|_{\mathcal{D}^{s,V}_{\varepsilon,\varepsilon}(\mathbb{R}^N)} = \left( \int_{\mathbb{R}^N} \varepsilon^{2s} |(-\Delta)^{s/2}u|^2 + (V(x) + 1)u^2 \, dx \right)^{\frac{1}{2}}.
\]

For the potential term \( V \), we assume that \( V \) is continuous and
\begin{itemize}
  \item \((V_1)\) \( V(x) + 1 \geq 0 \);
  \item \((V_2)\) there exist open bounded sets \( \Lambda \subset \subset U \) with smooth boundaries \( \partial \Lambda, \partial U \), such that
\end{itemize}
\[
0 < \lambda = \inf_{\Lambda} (V + 1) < \inf_{U \setminus \Lambda} (V + 1).
\]

Without loss of generality, we assume that \( 0 \in \Lambda \).

Now, with the notations and assumptions above at hand, we are in a position to state our main result:
Theorem 1.1. Let $V$ satisfy $(V_1)$ and $(V_2)$. Then there exists an $\varepsilon_0 > 0$ such that (1.1) has a positive solution $u_\varepsilon$ if $\varepsilon \in (0, \varepsilon_0)$. Moreover, $u_\varepsilon$ has a global maximum point $x_\varepsilon$ which satisfies

$$\lim_{\varepsilon \to 0} V(x_\varepsilon) = \inf_{x \in \Lambda} V(x)$$

and

$$u_\varepsilon(x) \leq \frac{C \varepsilon^{N+2s}}{\varepsilon^{N+2s} + |x - x_\varepsilon|^{N+2s}}, \; x \in \mathbb{R}^N,$$

where $C$ is a positive constant.

The difficulties in the proof of Theorem 1.1 are stated as follows. Firstly, there exists $u \in D_{\lambda V,\varepsilon}^s(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} |u|^2 \log |u|^2 = -\infty,$$

which makes the natural Euler-Lagrange functional corresponding to (1.1), i.e.,

$$I_\varepsilon(u) := \frac{1}{2} \int_{\mathbb{R}^N} \left( \varepsilon^{2s} |(-\Delta)^{s/2}u|^2 + (V(x) + 1)|u|^2 \right) - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 \, dx,$$

is not well defined in $D_{\lambda V,\varepsilon}^s(\mathbb{R}^N)$. Then all the methods developed for (1.1) in the case that the logarithmic term replaced by power type nonlinear term failed (see [1, 2]). Moreover, the expectation that the concentration should occur at a local minimum of $V$ in $\Lambda$ makes us have to truncate the nonlinear term outside $\Lambda$. Noting that since the ground states of limiting equation of (1.1) decay like $\frac{1}{|x|^{N+2s}}$, we can not use the penalized idea in [21] which deal with the case $s = 1$ to truncate the nonlinear term or the potential term.

Creatively, to overcome the two difficulties above, we use the characteristic function $\chi_{\mathbb{R}^N \setminus \Lambda}$ to truncate the nonlinear term (see (2.2) below), i.e., we firstly study the following penalized problem

$$\varepsilon^{2s}(-\Delta)^s u_\varepsilon + (V(x) + 1)u_\varepsilon = \chi_A (1 + \log |u|^2)u_\varepsilon - \chi_{\mathbb{R}^N \setminus \Lambda} \max\{2u, -u(1 + \log |u|^2)\}. \; (1.5)$$

The corresponding functional $J_\varepsilon$, all through is not well defined in $D_{\lambda V,\varepsilon}^s(\mathbb{R}^N)$, is the sum of a $C^1$ functional and a convex l.s.c. functional (see (2.2) and Remarks 2.3 below), whose critical points can be found by the Mountain Pass Theorem 3.2 in [16]. Then, a penalized solution $u$ of (1.5) is indeed a solution of the origin problem (1.1) if

$$-(1 + \log |u|^2) \geq 2 \; \text{on} \; \mathbb{R}^N \setminus \Lambda.$$

This, after the proof of concentration of $u$, can be checked easily by the well-known fact that a sub-solution of $(-\Delta)^s u + u \leq 0$ decays like $\frac{1}{|x|^{N+2s}}$ (see the last part of Section 3 for more details).

We need to emphasize that the nonlocal term $(-\Delta)^s$ makes the estimates more difficult than the classical case. In fact, for a smooth function $f \in C_c^\infty(\mathbb{R}^N)$, one can not compute $(-\Delta)^s f$ as precisely as $-\Delta f$ (see Appendix A below for example).

The nonlocal effect makes us have to know the global $L^2$-norm information of penalized solution, which in [2] was given by the assumption $\lim_{|x| \to \infty} \inf (V(x) + 1)|x|^{2s} > 0$. But, in
the present paper, the global $L^2$-norm information of penalized solution will be obtained
by the fractional Hardy inequality: There exists a positive constant $C_{N,s}$ such that
\begin{equation}
\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} \, dx \leq C_{N,s}|(-\Delta)^{s/2}|^2
\end{equation}
for all $u \in \dot{H}^s(\mathbb{R}^N)$ (see the proof of (A.3) in Appendix A for example).

**Plan of the paper.** In Section 2, we obtain the penalized problem by truncating the
logarithmic term in (1.1) outside by $4\chi_{\mathbb{R}^N \setminus \Lambda}u$, then we use the Mountain Pass Theorem 3.2
in [16] to obtain a penalized solution $u_{\varepsilon}$. In Section 3, we study the concentration of
$u_{\varepsilon}$ and then linearize the penalized equation in Section 2. At the last part of Section 3, we use
the well-known decay estimates of fractional Schrödinger equations to prove the asymptotic
behaviour of $u_{\varepsilon}$, which implies that $u_{\varepsilon}$ solves the origin problem.

2. THE PENALIZED PROBLEM

The following inequality exposes the relationship between $H^s(\mathbb{R}^N)$ and the Banach space
$L^q(\mathbb{R}^N)$.

**Proposition 2.1.** *(Fractional version of the Gagliardo–Nirenberg inequality [8])* For every
$u \in H^s(\mathbb{R}^N)$,
\begin{equation}
\|u\|_q \leq C \|(-\Delta)^{s/2} u\|_2 \|u\|_2^{1-\theta},
\end{equation}
where $q \in [2, 2_s^*)$ and $\theta$ satisfies $\frac{a}{2s} + \frac{(1-\theta)}{2} = \frac{1}{q}$.

By the proposition above, we know that $H^s(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$
for $q \in [2, 2_s^*)$. Moreover, on bounded set, the embedding is compact (see [8]), i.e.,
\[H^s(\mathbb{R}^N) \subset \subset L^q_{\text{loc}}(\mathbb{R}^N) \text{ compactly, if } q \in [1, 2_s^*].\]

Our proof will rely on the following fractional logarithmic Sobolev inequality (see [4]):

**Proposition 2.2.** For any $u \in H^s(\mathbb{R}^N)$, it holds
\begin{equation}
\int_{\mathbb{R}^N} |u|^2 \log \left( \frac{|u|^2}{\|u\|_2^2} \right) + \left( N + \frac{N}{s} \log a + \log \frac{s\Gamma(\frac{N}{s})}{\Gamma(\frac{N}{2s})} \right) \|u\|_2^2 \leq \frac{a^2}{n^s} \|(-\Delta)^s\|_2^2, \quad a > 0.
\end{equation}

Now we are going to modify the origin problem (1.1). In order to use the Mountain Pass
Theorem 3.2 in [16], considering the vanishing of $V$ and the concentration should occur in
$\Lambda$, we modified the nonlinear term as follows. Define
\begin{align*}
G_1(x, s) &= \frac{1}{2} \chi_{\Lambda}(x)s_+^2 \log s_+^2, \\
G_2(x, s) &= \frac{1}{2} \chi_{\mathbb{R}^N \setminus \Lambda}(x) \int_0^s \max\{4t_+, -2t_+(1 + \log t_+^2)\} \, dt.
\end{align*}
Remark 2.3. Importantly, since $G'_2(x, s)$ is nondecreasing on $s$, $G_2(x, s) \geq 0$ is convex on $s$ for all $x \in \mathbb{R}^N \setminus \Lambda$ and the functional $G^2 : \mathcal{D}^s_{V, \epsilon}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$G^2(u) = \int_{\mathbb{R}^N} G_2(x, u)$$

is convex and l.s.c by Fatou’s Lemma.

Proposition 2.1 and the boundedness of $\Lambda$ imply that the functional $G^1 : \mathcal{D}^s_{V, \epsilon}(\mathbb{R}^N) \to \mathbb{R}$ given by

$$G^1(u) = \int_{\mathbb{R}^N} G_1(x, u)$$

is $C^1$. So the functional

$$J_\epsilon(u) = \Phi_\epsilon(u) + \Psi(u)$$

with

$$\Phi_\epsilon(u) = \frac{1}{2}\|u\|^2_{V, \epsilon} - G^1(u) \quad \text{and} \quad \Psi = G^2(u)$$

has the form stated in [16].

By the remark above, although $J_\epsilon$ is not $C^1$, we can still use the Mountain Pass Theorem 3.2 in [16] to find a critical point for $J_\epsilon$. We first state some necessary definitions corresponding to those functionals has the form of $J_\epsilon$.

Definition 2.4. Let $E$ be a Banach space, $E'$ be the dual space of $E$ and $\langle \cdot, \cdot \rangle$ be the duality paring between $E'$ and $E$. Let $J : E \to \mathbb{R}$ be a functional of the form $J(u) = \Phi(u) + \Psi(u)$, where $\Phi \in C^1(E, \mathbb{R})$ and $\Psi$ is convex and l.s.c.. We have the following definitions:

(i) A critical point of $J$ is a point $u \in E$ such that $J(u) < +\infty$ and $0 \in \partial J(u)$, i.e.

$$\langle \Phi'(u), v - u \rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in E.$$

(ii) A Palais-Smale sequence at level $c$ for $J$ is a sequence $(u_n) \subset E$ such that $J(u_n) \to c$ and there is a numerical sequence $\sigma_n \to 0^+$ with

$$\langle \Phi'(u_n), v - u_n \rangle + \Psi(v) - \Psi(u_n) \geq -\sigma_n\|v - u_n\|, \quad \forall v \in E.$$

(iii) The functional $J$ satisfies the Palais-Smale condition at level $c$ $(PS)_c$ condition if all Palais-Smale sequence at level $c$ has a convergent subsequence.

(iv) The set $D(J) := \{u \in E : J(u) < +\infty\}$ is called the effective domain of $J$.

According to the Corollary 2.6 in [15], we have

Proposition 2.5. Let $\sup_n J_\epsilon(u_n) < +\infty$. Then $(u_n)$ is a Palais-Smale sequence if and only if $J'(u_n) \to 0$ in $(\mathcal{D}^s_{V, \epsilon}(\mathbb{R}^N))'$.

To use Theorem 3.2 in [16], we need to prove that $J_\epsilon$ satisfies the $(PS)_c$ condition (iii) above.

Proposition 2.6. $J_\epsilon$ satisfies $(PS)_c$ condition, i.e., each sequence $(u_n) \subset \mathcal{D}^s_{V, \epsilon}(\mathbb{R}^N)$ with $\lim_{n \to \infty} J_\epsilon(u_n) \to c$ has a convergent subsequence in $\mathcal{D}^s_{V, \epsilon}(\mathbb{R}^N)$. 

Proof. We first show that \((u_n)\) is bounded in \(D_{V,\varepsilon}^s(\mathbb{R}^N)\). Observing firstly that \(\Psi \geq 0 \ \forall u \in D_{V,\varepsilon}^s(\mathbb{R}^N)\), hence we have

\[
J_\varepsilon(u_n) \geq \Phi_\varepsilon(u_n) = \frac{1}{2}\|u_n\|_{V,\varepsilon}^2 - \frac{1}{2} \int_{\Lambda} |u_n|^2 \log |u_n|^2.
\]  

(2.3)

Moreover, by (2.2), Proposition 2.5 and the fact that \(\int \max\{f, g\} \geq \max\{\int f, \int g\}\), it holds

\[
\int_{\mathbb{R}^N} |u_n|^2 dx \leq 2 J_\varepsilon(u_n) - \langle J'_\varepsilon(u_n), u_n \rangle \leq C + o_n(1)\|u_n\|_{V,\varepsilon}.
\]  

(2.4)

For a set \(A \subset \mathbb{R}^N\), we define \(A^d := \{x \in \mathbb{R}^N : \text{dist}(A, x) < d\}\). Let \(\eta \in C_c^\infty(\Lambda^\delta)\) be a function satisfying \(0 \leq \eta \leq 1\) and \(\eta \equiv 1\) on \(\Lambda\), where the \(\delta\) is a small parameter such that \(\Lambda^{2\delta} \subset \subset U\). Defining \(v_n(x) = u_n(\varepsilon x)\eta(\varepsilon x)\), we have \(v_n(x) \in H^s(\mathbb{R}^N)\setminus\{0\}\). Then, by Proposition 2.2 we have

\[
\int_{\mathbb{R}^N} |v_n|^2 \log |v_n|^2 \leq \frac{a^s}{\pi} \|(-\Delta)^{s/2} v_n\|^2 + (C_{s,N} + \frac{N}{s} \log a)\|v_n\|^2_2 + \|v_n\|^2_2 \log \|v_n\|^2_2.
\]

Rescaling back, we find

\[
\int_{\mathbb{R}^N} |\eta u_n|^2 \log \|\eta u_n\|^2
\]

\[
\leq \frac{a^s}{2} \|(-\Delta)^{s/2}(\eta u_n)\|^2_2 + (C_{s,N} + \frac{N}{s} \log a)\|\eta u_n\|^2_2 + \|\eta u_n\|^2_2 \log \left(\frac{1}{\varepsilon^N}\|\eta u_n\|^2_2\right).
\]  

(2.5)

By fractional Hardy inequality (1.6) and some delicate nonlocal estimates, we will prove in Appendix A that

\[
T_1(\eta) := \varepsilon^s\|(-\Delta)^{s/2}(\eta u_n)\|^2_2 \leq C\|u_n\|^2_{V,\varepsilon},
\]  

(2.6)

where \(C\) is a positive constant. Hence, returning back to (2.5), letting \(a > 0\) be small enough, by (2.4), we get

\[
\int_{\mathbb{R}^N} |\eta u_n|^2 \log \|\eta u_n\|^2 \leq \frac{1}{2}\|u_n\|_{V,\varepsilon}^2 + C(1 + \|\eta u_n\|_{2(1+\delta)}^2) - N\|u_n\|^2_2 \log \varepsilon
\]

\[
\leq \frac{1}{2}\|u_n\|_{V,\varepsilon}^2 + C(1 + \|u_n\|_{V,\varepsilon}^{1+\delta}) - N(C + \|u_n\|_{V,\varepsilon}) \log \varepsilon,
\]  

(2.7)

where \(\delta \in (0, 1)\) is a parameter. Note that for every \(f : \mathbb{R}^N \to \mathbb{R}\), it holds

\[
\int_{U \setminus \Lambda} |f|^2 \log |f|^2 \geq -\frac{1}{\varepsilon}|U \setminus \Lambda|.
\]

Then by (2.7) and (2.3), we have

\[
\tilde{C} \geq \frac{1}{2}\|u_n\|_{V,\varepsilon}^2 - \|u_n\|_{V,\varepsilon}^{1+\delta} + N(C + \|u_n\|_{V,\varepsilon}) \log \varepsilon,
\]

where \(\tilde{C}\) is a positive constant.
where $\tilde{C}$ is a positive constant. Consequently, since $0 < \delta < 1$, we can conclude that $(u_n)$ is bounded in $\mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)$. Going if necessary to a subsequence, we assume that

$$u_n \rightharpoonup u \text{ weakly in } \mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N).$$

Next we show that $(u_n)$ has a convergent subsequence. Since $u_n \rightharpoonup u \in \mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)$, by the boundedness of $\Lambda$ and the fact that $|G_1(x,s)| \leq C_1|t|^\frac{s}{2} + C_2|t|\frac{s}{2}$, where $C_1, C_2 > 0$ are two positive constants, we have

$$G_1'(x,u_n)u_n \rightarrow G_1'(x,u)u,$$

Noting that since $J'_\varepsilon(u_n)\varphi = o_n(1)\|\varphi\|_{V,\varepsilon}$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, we deduce that $J'_\varepsilon(u)\varphi = 0$ for all $\varphi \in C_0^\infty(\mathbb{R}^N)$, and so, $J'_\varepsilon(u)u = 0$. Combining with $\|J'_\varepsilon(u_n)\|_{\mathcal{D}'_{V,\varepsilon}} \leq \|J'_\varepsilon(u_n)\|_{\mathcal{D}'_{V,\varepsilon}} \|u_n\|_{V,\varepsilon} = o_n(1)\|u_n\|$, we have

$$\|u_n\|^2_{V,\varepsilon} + \int_{\mathbb{R}^N} \chi_{\mathbb{R}^N \setminus \Lambda} G_2'(x,u_n)u_n = \|u\|^2_{V,\varepsilon} + \int_{\mathbb{R}^N} \chi_{\mathbb{R}^N \setminus \Lambda} G_2'(x,u)u + o_n(1). \quad (2.8)$$

Finally, by $\|u\| \leq \liminf_{n \to \infty} \|u_n\|^2_{V,\varepsilon}$ and the l.s.c property (by Fatou’s Lemma), we get that $u_n \rightarrow u$ strongly in $\mathcal{D}^s_{V,\varepsilon}(\mathbb{R}^N)$. This completes the proof. \hfill \Box

**Remark 2.7.** The proof of (2.8) is trivial if $s = 1$, but delicate if $0 < s < 1$. It will need the global $L^2$ information of $u_n$, which is given by the nonlocal operator $(-\Delta)^s$ and the fractional Hardy inequality (1.6) (see (A.2) for example).

Obviously, there exists $\rho > 0$ such that

$$J_\varepsilon(u) \geq \Phi_\varepsilon(u) \geq \frac{1}{2}\|u\|^2_{V,\varepsilon} - C\|u\|^p_{V,\varepsilon} > 0 \text{ for all } u \text{ with } \|u\|^2_{V,\varepsilon} = \rho$$

and for each $u \in C_0^\infty(\Lambda) \setminus \{0\}$, it holds

$$J_\varepsilon(su) \rightarrow -\infty \text{ as } s \rightarrow +\infty,$$

i.e., $J_\varepsilon$ owns mountain pass geometry. Thus by Proposition 2.6 and Theorem 3.2 in [16], we immediately have:

**Lemma 2.8.** The mountain pass value

$$c_\varepsilon = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_\varepsilon(\gamma(t))$$

is positive and can be achieved by a positive function $u_\varepsilon$ which is a critical point of $J_\varepsilon$ and solves the following penalized problem

$$\varepsilon^{2s}(-\Delta)^s u_\varepsilon + (V(x) + 1)u_\varepsilon = G_1'(x,(u_\varepsilon)_+) - G_2'(x,(u_\varepsilon)_+). \quad (2.9)$$

**Proof.** By Theorem 3.2 in [16], $c_\varepsilon$ is a critical value, i.e., there exists $u_\varepsilon \in D(J_\varepsilon) = \{v \in \mathcal{D}^s_{V,\varepsilon} : J_\varepsilon(v) < +\infty\}$ with $J_\varepsilon(u_\varepsilon) = c_\varepsilon$ such that

$$\langle \Phi'_\varepsilon(u_\varepsilon), v - u_\varepsilon \rangle + \Psi(v) - \Psi(u_\varepsilon) \geq 0 \text{ for all } v \in \mathcal{D}^s_{V,\varepsilon}.$$
In particular, letting \( t > 0 \) and \( v = u_\varepsilon + t \varphi \) with \( \varphi \in C_c^\infty (\mathbb{R}^N) \), we have
\[
\langle \Phi'_\varepsilon (u_\varepsilon), \varphi \rangle + \frac{\Psi(u_\varepsilon + t \varphi) - \Psi(u_\varepsilon)}{t} \geq 0 \quad \forall t > 0
\]
and then
\[
\langle J'_\varepsilon (u_\varepsilon), \varphi \rangle = \langle \Phi'_\varepsilon (u_\varepsilon), \varphi \rangle + \int_{\mathbb{R}^N} G'_2(x, u_\varepsilon) \varphi \geq 0.
\]
Rearranging \( \varphi = -\psi \), we eventually have
\[
\langle J'_\varepsilon (u_\varepsilon), \varphi \rangle = 0 \quad \forall \varphi \in C_c^\infty (\mathbb{R}^N),
\]
which implies (2.9).

Finally, letting \( (u_\varepsilon)_- \) be a test function to (2.9), we find \( u_\varepsilon \geq 0 \). By the standard regularity assertion and maximum principle in [10, Appendix D], we conclude that \( u_\varepsilon \) is positive. □

3. Concentration and the origin problem

In this section we will prove the concentration phenomenon of \( u_\varepsilon \) via energy comparison, by which we will prove at the last of this section that
\[
- (1 + \log |u_\varepsilon|^2) \geq 2 \quad \forall x \in \mathbb{R}^N \setminus \Lambda,
\]
which and (2.9) indicate that \( u_\varepsilon \) solves the origin problem (1.1).

3.1. Concentration. In the first subsection, we prove the concentration of \( u_\varepsilon \) via comparing the energy \( c_\varepsilon \) with the least energy of the limiting problem of (1.1).

The limiting problem corresponding to (1.1) is
\[
(-\Delta)^s u + \lambda u = u \log |u|^2,
\]
where \( \lambda > -1 \). Its Euler-Lagrange functional is
\[
\mathcal{L}_\lambda (u) = \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u|^2 + (\lambda + 1)|u|^2 - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2.
\]
In [3], it was proved that the limiting problem (3.2) has a least energy solution \( U_\lambda \) with
\[
\mathcal{L}_\lambda (U_\lambda) = C_\lambda := \inf_{\varphi \in H^s (\mathbb{R}^N) \setminus \{0\}} \max_{t > 0} \mathcal{L}_\lambda (t \varphi) = \inf_{\varphi \in C_c^\infty (\mathbb{R}^N) \setminus \{0\}, \varphi \geq 0} \max_{t > 0} \mathcal{L}_\lambda (t \varphi).
\]
For \( C_\lambda \), we have

**Proposition 3.1.** the function \( C : (-1, +\infty) \rightarrow (0, +\infty) \) is continuous and increasing.

**Proof.** Let \(-1 < \lambda < \lambda' < +\infty\) and \( U_{\lambda'} \) be the ground solution of equation (3.2) with \( \lambda = \lambda' \). An easy analysis shows that the following function \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \)
\[
f(t) = \mathcal{L}_\lambda (t U_{\lambda'}) = \mathcal{L}_{\lambda'} (t U_{\lambda'}) + \frac{t^2}{2} (\lambda - \lambda') \int_{\mathbb{R}^N} |U_{\lambda'}|^2
\]
has a unique maximum point $t' \in (0, +\infty)$, from which we have
\[ C_\lambda \leq \max_{t > 0} f(t) = \mathcal{L}_\lambda'(t'U_{t'}) + \frac{(t')^2}{2}(\lambda - \lambda') \int_{\mathbb{R}^N} |U_{t'}|^2 \]
\[ \leq C_\lambda' + \frac{(t')^2}{2}(\lambda - \lambda') \int_{\mathbb{R}^N} |U_{t'}|^2 \]
\[ < C_\lambda'. \]
Similarly, it holds
\[ C_\lambda' \leq C_\lambda + \tilde{t}^2(\lambda' - \lambda) \int_{\mathbb{R}^N} |U'|^2 \]
for some unique $\tilde{t} \in (0, +\infty)$. Then $C_\lambda'$ is increasing and continuous.

By the analysis above, we have the following upper bound of $c_\varepsilon$.

**Proposition 3.2.** It holds
\[ \limsup_{\varepsilon \to 0} \frac{c_\varepsilon}{\varepsilon N} \leq \min_{x \in \Lambda} C_{V(x)}. \]

**Proof.** Let $\varphi \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}$, $\varphi \geq 0$ and define for each $x_0 \in \Lambda$
\[ \varphi_\varepsilon(x) = \varphi\left(\frac{x - x_0}{\varepsilon}\right). \]
Obviously, $\text{supp} \varphi_\varepsilon \subset \Lambda$ for small $\varepsilon$ and $\gamma_\varepsilon(t) = tT_0 \varphi_\varepsilon \in \Gamma_\varepsilon$ for some $T_0$ large enough. Then we have
\[ \frac{c_\varepsilon}{\varepsilon N} \leq \max_{t \in [0,1]} \mathcal{J}_\varepsilon(\gamma_\varepsilon(t)) \leq \mathcal{L}_{V(x_0)}(t\varphi) + o_\varepsilon(1) \]
and
\[ \limsup_{\varepsilon \to 0} \frac{c_\varepsilon}{\varepsilon N} \leq \inf_{\varphi \in C_c^\infty(\mathbb{R}^N) \setminus \{0\}} \mathcal{L}_{V(x_0)}(t\varphi) = C_{V(x_0)}, \]
which completes the proof. \boxdot

Next, we give the lower bounds of solutions of (2.9).

**Proposition 3.3.** Let $(u_\varepsilon^n)$ with $\varepsilon_n > 0$, $\varepsilon_n \to 0$ as $n \to \infty$ be a family of solutions of (2.9). If for each $k \in \mathbb{N}$, there exists $k$ families of points $\{(x^i_{\varepsilon_n}) : 1 \leq i \leq k\}$ with $\lim_{n \to \infty} x^i_{\varepsilon_n} = x^i_*$ such that
\[ \liminf_{n \to \infty} \|u_\varepsilon^n\|_{L^\infty(B_{\varepsilon_n}(x^i_{\varepsilon_n}))} > 0, \quad V(x^i_*) + 1 > 0, \quad 1 \leq i \leq k, \]
\[ \liminf_{n \to \infty} \frac{|x^i_{\varepsilon_n} - x^j_{\varepsilon_n}|}{\varepsilon_n} = +\infty, \quad 1 \leq i \neq j \leq k \]
and
\[ \limsup_{n \to \infty} \frac{J_{\varepsilon_n}(u_\varepsilon^n)}{\varepsilon_n^N} < +\infty, \]
then
\[ \lim_{n \to \infty} \inf_{\varepsilon N} \frac{J_{\varepsilon N}(u_{\varepsilon n})}{\varepsilon} \geq \sum_{i=1}^{k} C_{V(x_i)}. \]

**Proof.** Fixing a \( 1 \leq i \leq k \) and rescaling the function \( u_{\varepsilon n} \) as \( v^i_n(x) = u(\varepsilon_n x + x^i_{\varepsilon n}), \ x \in \mathbb{R}^N \), we have by the estimate in Proposition 2.6 that
\[ \sup_n \int_{\mathbb{R}^N} \left( |(-\Delta)^{s/2} v^i_n|^2 + (V^i_n(x) + 1)|v^i_n|^2 \right) < +\infty, \]
where \( V^i_n(\cdot) = V(\varepsilon_n \cdot + x^i_{\varepsilon n}) \). Obviously, \( v^i_n \) satisfies
\[ (-\Delta)^s v^i_n + (V^i_n(x) + 1)v^i_n = G^i_1(\varepsilon_n x + x^i_{\varepsilon n}, v^i_n) - G^i_2(\varepsilon_n x + x^i_{\varepsilon n}, v^i_n) \text{ in } \mathbb{R}^N. \]
(3.3)

Fixing \( R > 0 \), we have by continuity that
\[ \lim sup_{n \to \infty} \|v^i_n\|^2_{H^s(B_R)} \leq \lim sup_{n \to \infty} \left( \inf_{\Lambda} \int_{\mathbb{R}^N} \left( |(-\Delta)^{s/2} v^i_n|^2 + (V^i_n(x) + 1)|v^i_n|^2 \right) < +\infty, \]
which says that \( (v^i_n) \) is bounded in \( H^s_{\text{loc}}(\mathbb{R}^N) \) and then by diagonal argument, we can assume without loss of generality that \( v^i_n \to v^i_* \) weakly in \( H^s_{\text{loc}} \) as \( n \to \infty \). By
\[ \|v^i_*\|^2_{H^s(B_R)} \leq \lim inf_{n \to \infty} \|v^i_n\|^2_{H^s(B_R)} < +\infty, \]
we have \( v^i_* \in H^s(\mathbb{R}^N) \).

The smoothness of \( \Lambda \) implies that the set \( \Lambda_n^i = \{ x : \varepsilon_n x + x^i_{\varepsilon n} \in \Lambda \} \) converges to a set \( \Lambda_*^i \in \{ 0, H, \mathbb{R}^N \} \) as \( n \to \infty \), where \( H \) is a half plane, by which we have
\[
\int_{\mathbb{R}^N} \left( G^i_1(\varepsilon x + x^i_{\varepsilon n}, v^i_n)\varphi - G^i_2(\varepsilon x + x^i_{\varepsilon n}, v^i_n)\varphi \right)
\to \int_{\mathbb{R}^N} \chi_{\Lambda_*^i} (1 + \log |v^i_*|^2) v^i_* \varphi - \int_{\mathbb{R}^N} \chi_{\mathbb{R}^N \setminus \Lambda_*^i} \max\{2v^i_* - (1 + \log |v^i_*|^2) v^i_* \} \varphi
\]
for all \( \varphi \in C_c^\infty(\mathbb{R}^N) \) as \( n \to \infty \). Then we conclude that \( v^i_* \) satisfies the following equation:
\[
(-\Delta)^s v^i_* + (V(x^*_*) + 1)v^i_* = \chi_{\Lambda_*^i} (1 + \log |v^i_*|^2) - \chi_{\mathbb{R}^N \setminus \Lambda_*^i} \max\{2v^i_* - (1 + \log |v^i_*|^2) v^i_* \} \text{ in } R^N. \]
(3.4)

By the similar regularity argument in [10, Appendix D], we have
\[ \|v^i_*\|_{L^\infty(B_r(x_1))} = \lim_{n \to \infty} \|v^i_n\|_{L^\infty(B_r(x^i_n))} = \lim_{n \to \infty} \|u^i_n\|_{L^\infty(B_{\varepsilon_n r}(x^i_{\varepsilon n}))} > 0, \]
which implies that \( v^i_* \) is nontrivial.

The Euler-Lagrange functional corresponding to (3.4) is
\[ J^i_*(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left( |(-\Delta)^{s/2} u|^2 + (V(x^*_*) + 1)|u|^2 \right) - \frac{1}{2} \int_{\mathbb{R}^N} \chi_{\Lambda_*^i} u^2 \log |u|^2. \]
which implies

\[
J_i^*(v_i^*) = \max_{t > 0} J_i^*(tv_i^*) \geq \max_{t > 0} \mathcal{L}_{V(x_i^*)}(tv_i^*) \geq C_{V(x_i^*)}.
\]

Then, after rescaling, we have

\[
\liminf_{n \to \infty} \frac{1}{2 \varepsilon_n^2} \int_{B_{2\varepsilon_n}(x_i^*)} \left( ((\varepsilon_n^2 |(-\Delta)^{s/2} u_n|^2 + (V(x) + 1)|u_n|^2) \right. \\
- \chi_\Lambda |u_n|^2 \log |u_n|^2 + \chi_{\mathbb{R}^N \setminus \Lambda} G_2(x, u_n) \\
\left. \geq J_i^*(v_i^*) + o_R(1) \geq C_{V(x_i^*)} + o_R(1). \right.
\]

Now let us estimate the energy outside \( \bigcup_{i=1}^k B_{\varepsilon_n}(x_i^*) \). Choose \( \eta \) as another cut-off function with \( \eta_R \equiv 0 \) in \( B_R \) and \( \eta_R \equiv 1 \) on \( B_{2R}^c \) and define

\[
\eta_{n,R}(\cdot) = \prod_{i=1}^k \eta_R \left( \frac{\cdot - x_i^n}{\varepsilon_n} \right).
\]

Testing (2.9) against with \( \eta_{n,R} \varepsilon_n u_{x_i} \), by the definition of penalized function in (2.2), we find

\[
\frac{1}{2} \int \left( \bigcup_{i=1}^k B_{\varepsilon_n R}(x_i^*) \right) e(\varepsilon_n^2 |(-\Delta)^{s/2} u_n|^2 + (V(x) + 1)|u_n|^2) \\
- \frac{1}{2} \int \left( \bigcup_{i=1}^k B_{\varepsilon_n R}(x_i^*) \right) e \cap \Lambda \ G_1(x, u_n) + \frac{1}{2} \int \left( \bigcup_{i=1}^k B_{\varepsilon_n R}(x_i^*) \right) e \cap \mathbb{R}^N \setminus \Lambda \ G_2(x, u_n) \\
:= T_{n,R}^2 + \frac{1}{2} \int \left( \bigcup_{i=1}^k B_{\varepsilon_n R}(x_i^*) \right) \left( 1 - \eta_{n,R}(\cdot) \right) \left( \varepsilon_n^2 |(-\Delta)^{s/2} u_n|^2 + (V(x) + 1)|u_n|^2 \right) \\
+ \frac{1}{2} \int_\Lambda \eta_{n,R} \left| u_n \right|^2 \left( 1 + \log \left| u_n \right|^2 \right) - \frac{1}{2} \int_{\mathbb{R}^N \setminus \Lambda} \chi \left( \bigcup_{i=1}^k B_{\varepsilon_n R}(x_i^*) \right) e \cap \Lambda \left| u_n \right|^2 \log \left| u_n \right|^2 \\
- \frac{1}{2} \int_{\mathbb{R}^N \setminus \Lambda} \eta_{n,R}(x) \max \left\{ 2 \left| u_n \right|^2, -\left| u_n \right|^2 \left( 1 + \log \left| u_n \right|^2 \right) \right\} \\
+ \frac{1}{2} \int_{\mathbb{R}^N \setminus \Lambda} \chi \left( \bigcup_{i=1}^k B_{\varepsilon_n R}(x_i^*) \right) e \left| u_n \right|^2 \left( 1 + \log \left| u_n \right|^2 \right) \\
\geq T_{n,R}^2 + \frac{1}{2} \int_\Lambda \eta_{n,R} \left| u_n \right|^2 \left( 1 + \log \left| u_n \right|^2 \right) - \frac{1}{2} \int_{\mathbb{R}^N \setminus \Lambda} \chi \left( \bigcup_{i=1}^k B_{\varepsilon_n R}(x_i^*) \right) e \left| u_n \right|^2 \log \left| u_n \right|^2 \\
- \frac{1}{2} \int_{\mathbb{R}^N \setminus \Lambda} \eta_{n,R}(x) \max \left\{ 2 \left| u_n \right|^2, -\left| u_n \right|^2 \left( 1 + \log \left| u_n \right|^2 \right) \right\} \\
\text{(3.5)}
\]
+ \frac{1}{2} \int_{\mathbb{R}^N \setminus \Lambda} \chi \left( \bigcup_{i=1}^k B_{\varepsilon_n R}(x_i) \right)^c \max \{ 2|u_n|^2, -|u_n|^2(1 + \log |u_n|^2) \} dx \\
\geq T_{n,R}^2 + \frac{1}{2} \int_{\mathbb{R}^N} \left( B_{2\varepsilon_n R}(x_i) \right) \left( \eta_{n,R} - 1 \right) |u_n|^2 \log |u_n|^2 \\
+ \frac{1}{2} \int_{\mathbb{R}^N} \left( B_{2\varepsilon_n R}(x_i) \right) \left( \eta_{n,R} - 1 \right) \max \{ 2|u_n|^2, -|u_n|^2(1 + \log |u_n|^2) \},
\]

where

\[
T_{n,R}^2 = \frac{\varepsilon_n^{2s}}{2} \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{(\eta_{n,R}(y) - \eta_{n,R}(x)) u_n(y) (u_n(x) - u_n(y))}{|x - y|^{N+2s}} dy.
\]

In Appendix A, we will prove by fractional Hardy inequality (1.6) that

\[
\limsup_{n \to \infty} T_{n,R}^2 \geq o_R(1).
\]

Hence, by the fact that \( v^i_n \to v^* \) strongly in \( L^q_{\text{loc}}(\mathbb{R}^N) \) with \( 1 < q < 2^*_s \), we conclude that

\[
\lim_{n \to \infty} \left( \frac{1}{2\varepsilon_n^N} \int_{\mathbb{R}^N} \left( \bigcup_{i=1}^k B_{\varepsilon_n R}(x_i) \right)^c \left( \varepsilon_n^{2s} (-\Delta)^{s/2} u_n \right)^2 + (V(x) + 1)|u_n|^2 \right) \\
- \frac{1}{2\varepsilon_n^N} \int_{\mathbb{R}^N} \left( \bigcup_{i=1}^k B_{\varepsilon_n R}(x_i) \right)^c G_1(x, u_n) + \frac{1}{2\varepsilon_n^N} \int_{\mathbb{R}^N} \left( \bigcup_{i=1}^k B_{\varepsilon_n R}(x_i) \right)^c G_2(x, u_n) \\
\geq o_R(1) - C \int_{B_{2R} \setminus B_R} (|v^i_n| + |v^*_n|^q) \\
= o_R(1).
\]

Finally, by the analysis above, we have

\[
\liminf_{n \to \infty} \frac{J_{\varepsilon_n}(u_{\varepsilon_n})}{\varepsilon_n^N} \geq \sum_{i=1}^k C_{V(x_i)} + o_R(1),
\]

the conclusion then follows by letting \( R \to \infty \).

\[\square\]

**Remark 3.4.** It is easy to check that

\[
\int \max \{ f, g \} \geq \max \{ \int f, \int g \};
\]

which and the skillful choice of the truncated function in (2.2) play a key role in the proof of (3.5). The estimates (3.6) is trivial in the classical case, but delicate under the nonlocal effect of \((-\Delta)^s(0 < s < 1)\), some skillful global estimates will be involved (e.g., the \( L^2 \) information of \( u_{\varepsilon_n} \) outside \( \Lambda \), see (A.3) in Appendix for example).

Now we prove the concentration of \( u_{\varepsilon_n} \).
Lemma 3.5. Let $\rho > 0$ and $u_\varepsilon$ be the penalized solution given by Lemma 2.8. There exists a family of points $(x_\varepsilon) \subset \Lambda$ such that

(i) $\liminf_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty(B_{\varepsilon \rho}(x_\varepsilon))} > 0$.

(ii) $\lim_{\varepsilon \to 0} V(x_\varepsilon) = \inf_{\Lambda} V$.

(iii) $\lim_{R \to \infty} \|u_\varepsilon\|_{L^\infty(U \setminus B_{\varepsilon R}(x_\varepsilon))} = 0$.

Proof. Easily, we have

$$0 < \int_{\mathbb{R}^N} (\varepsilon^{2s}|(-\Delta)^{s/2}u_\varepsilon|^2 + (V(x) + 1)|u_\varepsilon|^2) \leq \int_{\Lambda \cap \{x:u_\varepsilon(x) > e^{-1/2}\}} |u_\varepsilon|^2(1 + \log |u_\varepsilon|^2)$$

which and the similar regularity assertion in [10] imply that there exist $x_\varepsilon \in \overline{\Lambda}$ such that

$$u_\varepsilon(x_\varepsilon) = \sup_{x \in \Lambda} u_\varepsilon(x) \quad \text{and} \quad \liminf_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty(B_{\varepsilon \rho}(x_\varepsilon))} > 0.$$ 

This proves $(i)$.

For $(ii)$, assuming without loss of generality that $\lim_{\varepsilon \to 0} x_\varepsilon = x_*$, by the lower and upper bounds of $u_\varepsilon$ in Propositions 3.2 and 3.3 we have

$$\min_{x \in \Lambda} C_V(x) \geq \liminf_{\varepsilon \to 0} \frac{J_\varepsilon(u_\varepsilon)}{\varepsilon^N} \geq C_V(x_*),$$

which implies $V(x_*) = \min_{x \in \Lambda} V(x)$.

For $(iii)$, if it is not true, then one will get a contradiction like

$$\min_{x \in \Lambda} C_V(x) \geq \liminf_{\varepsilon \to 0} \frac{J_\varepsilon(u_\varepsilon)}{\varepsilon^N} \geq C_V(x_*) + C_V(y_*)$$

for some $y_* \in \overline{U}$ by Proposition 3.3.$\square$

3.2. Back to the origin problem. In this subsection, we use Lemma 3.5 to linearize equation (2.9) and then use the well-known decay estimates of positive solutions of fractional Schrödinger equations to show that (3.1) is true.

Noting that by the regular assertion in Appendix D of [10], we can assume that

$$\sup_{\Lambda} u_\varepsilon(x) \leq C < \infty,$$  \hspace{1cm} (3.7)

where $C$ is a positive constant. Hence by Lemma 3.5, we can linearize the penalized equation (2.9) as follows.

Proposition 3.6. Let $\varepsilon > 0$ be small enough, $x_\varepsilon$ be the point given by Lemma 3.5. Then there exists $R > 0$ such that

$$\begin{cases}
\varepsilon^{2s}(-\Delta)^s u_\varepsilon + \min\{\lambda, 2\} u_\varepsilon \leq 0, & \text{in } \mathbb{R}^N \setminus B_{\varepsilon R}(x_\varepsilon) \\
u_\varepsilon \leq C, & \text{in } B_{\varepsilon R}(x_\varepsilon)
\end{cases}$$
Proof. For \( \varepsilon > 0 \) small enough, by Lemma 3.5, there exists \( R > 0 \) such that \((1 + \log |u_\varepsilon|^2) \leq 0 \) for all \( x \in U \setminus B_{\varepsilon R}(x_\varepsilon) \), the conclusion then follows by the penalized function in (2.2) and inserting (3.7) into (2.9). □

At last, we prove Theorem 1.1.

Completes the proof of Theorem 1.1

By Proposition 3.6, the rescaling function \( v_\varepsilon = u_\varepsilon(\varepsilon x + x_\varepsilon) \) satisfies

\[
\begin{align*}
\left\{ \begin{array}{l}
(-\Delta)^s v_\varepsilon + \lambda v_\varepsilon \leq 0, \\
v_\varepsilon \leq C,
\end{array} \right. \quad \text{in } \mathbb{R}^N \setminus B_R \\
v_\varepsilon \leq C, \quad \text{in } B_R.
\end{align*}
\]

Then by the well-known decay estimates of fractional Schrödinger equation (see [10, Appendix D] for example), we have

\[
v_\varepsilon(x) \leq \frac{C}{1 + |x|^{N+2s}}, \quad x \in \mathbb{R}^N.
\]

Consequently, it holds

\[
u_\varepsilon(x) \leq \frac{C\varepsilon^{N+2s}}{\varepsilon^{N+2s} + |x - x_\varepsilon|^{N+2s}},
\]

from which, we can conclude that

\[
\max\{2u_\varepsilon, -(1 + \log(u_\varepsilon)^2)\} = -(1 + \log(u_\varepsilon)^2) \quad \text{for all } x \in \mathbb{R}^N \setminus \Lambda
\]

if \( \varepsilon > 0 \) is small. This proves that \( u_\varepsilon \) solves the origin problem (1.1) and complete the proof of this paper.

Appendix A.

In this section, we are going to verify (2.6) and (3.6). The fractional Hardy inequality (1.6) will be involved in the proof. We first give the proof of (2.6). By Cauchy inequality, we have

\[
T_1(\eta) \leq 2\varepsilon^{2s} \int_{\mathbb{R}^N} dx \int_{\mathbb{R}^N} \frac{|u_n(x)|^2(\eta(x) - \eta(y))^2}{|x - y|^{N+2s}} dy + C\|u_n\|^2_{V,\varepsilon, N}
\]

:= \frac{1}{2} T_{11}(\eta) + C\|u_n\|^2_{V,\varepsilon, N}.

By decomposition, denoting \( f(x, y) = \frac{|u_n(x)|^2(\eta(x) - \eta(y))^2}{|x - y|^{N+2s}} \), we have

\[
\frac{1}{2} T_{11}(\eta) = \varepsilon^{2s} \int_{\Lambda^s} dx \int_{\mathbb{R}^N} f(x, y) dy + \varepsilon^{2s} \int_{\mathbb{R}^N \setminus \Lambda^s} dx \int_{\Lambda^s} f(x, y) dy
\]

:= \sum_{i=1}^{2} T_{11i}(\eta).

Since

\[
\int_{\mathbb{R}^N} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{N+2s}} dy = \int_{B_1(x)} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{N+2s}} dy + \int_{\mathbb{R}^N \setminus B_1(x)} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{N+2s}} dy
\]
where the functions $\beta_{16}$ are defined skillfully as $\alpha_{n}^l(x) = \prod_{s=l+1}^{k} \frac{x - x_{n}^s}{\varepsilon_{n} R}$, $\beta_{n}^0(y) \equiv 1$. Following, we have

$$
\varepsilon_{n}^{2s - NT_{n,R}^2} = \sum_{l=1}^{k} \int_{B_{2R}} v_{n}^{l}(y) \beta_{n}^{l}(y) dy \int_{B_{2R}^{c}} \frac{\alpha_{n}^{l}(x)(v_{n}^{l}(x) - v_{n}^{l}(y))(\eta_R(x) - \eta_R(y))}{|x - y|^{N+2s}} dx
$$

For $T_{122}(\eta)$, similar to the estimates above, by fractional Hardy inequality (1.6), we have

$$
T_{122}(\eta) \leq C_{\Delta}^{2s} \int_{\mathbb{R}^N \setminus \Lambda^{2s}} |u_{n}(x)|^{2} \leq C \|u_{n}\|^2_{V_{\varepsilon}^{r}}. \quad (A.1)
$$

By $(A.1)$ and $(A.2)$, we conclude that

$$
T_1(\eta) \leq C \|u_{n}\|^2_{V_{\varepsilon}^{r}}.
$$

Next, we give the proof of (3.6). A change of variable tells us

$$
T_{2,1} = \int_{\mathbb{R}^N} u_{n}(y) dy \int_{\mathbb{R}^N} \frac{(u_{n}(x) - u_{n}(y))(\psi_{n,R}(x) - \psi_{n,R}(y))}{|x - y|^{N+2s}} dx
$$

$$
= \varepsilon_{n}^{N-2s} \sum_{l=1}^{k} \int_{\mathbb{R}^N} v_{n}^{l}(y) \beta_{n}^{l}(y) dy \int_{\mathbb{R}^N} \frac{\alpha_{n}^{l}(x)(v_{n}^{l}(x) - v_{n}^{l}(y))(\eta_R(x) - \eta_R(y))}{|x - y|^{N+2s}} dx,
$$

where the functions $\beta_{n}^{l}$ and $\alpha_{n}^{l}$ are defined skillfully as

$$
\beta_{n}^{l}(y) = \prod_{s=0}^{l-1} \frac{y - x_{n}^s}{\varepsilon_{n} R}, \quad \beta_{n}^{0}(y) \equiv 1
$$

and

$$
\alpha_{n}^{l}(x) = \prod_{s=l+1}^{k} \frac{x - x_{n}^s}{\varepsilon_{n} R}, \quad \alpha_{n}^{k}(x) \equiv 1.
$$

Following, we have

$$
\varepsilon_{n}^{2s - NT_{n,R}^2} = \sum_{l=1}^{k} \int_{B_{2R}} v_{n}^{l}(y) \beta_{n}^{l}(y) dy \int_{B_{2R}^{c}} \frac{\alpha_{n}^{l}(x)(v_{n}^{l}(x) - v_{n}^{l}(y))(\eta_R(x) - \eta_R(y))}{|x - y|^{N+2s}} dx
$$

$$
+ \sum_{l=1}^{k} \int_{B_{2R}^{c}} v_{n}^{l}(y) \beta_{n}^{l}(y) dy \int_{B_{2R}} \frac{\alpha_{n}^{l}(x)(v_{n}^{l}(x) - v_{n}^{l}(y))(\eta_R(x) - \eta_R(y))}{|x - y|^{N+2s}} dx
$$
\[ + \sum_{l=1}^{k} \int_{B_{2R}} v_{n}^l(y) \beta_{n}^l(y) \, dy \int_{B_{2R}} \frac{\alpha_{n}^l(x)(v_{n}^l(x) - v_{n}^l(y))(\eta_{R}(x) - \eta_{R}(y))}{|x - y|^{N + 2s}} \, dx \]

\[ := T_{n,R}^{21} + T_{n,R}^{22} + T_{n,R}^{23}. \]

By the choice of \( \eta_{R} \) and \( \lim_{n \to \infty} \frac{\|v_{n}^l - v_{n}^l \|}{\varepsilon_{n}} = \infty \) if \( l \neq s \), for \( n \) large, we have

\[ T_{n,R}^{21} = \sum_{l=1}^{k} \int_{B_{2R}} v_{n}^l(y) \beta_{n}^l(y) \, dy \int_{B_{2R}} \frac{\alpha_{n}^l(x)(v_{n}^l(x) - v_{n}^l(y))(1 - \eta_{R}(y))}{|x - y|^{N + 2s}} \, dx \]

\[ + \sum_{l=1}^{k} \int_{B_{2R}} v_{n}^l(y) \beta_{n}^l(y) \, dy \int_{B_{2R}} \frac{\alpha_{n}^l(x)(v_{n}^l(x) - v_{n}^l(y))}{|x - y|^{N + 2s}} \, dx \]

\[ = \sum_{l=1}^{k} \int_{B_{2R}} v_{n}^l(y) \, dy \int_{B_{2R}} \frac{\alpha_{n}^l(x)(v_{n}^l(x) - v_{n}^l(y))(1 - \eta_{R}(y))}{|x - y|^{N + 2s}} \, dx \]

\[ + \sum_{l=1}^{k} \int_{B_{2R}} v_{n}^l(y) \, dy \int_{B_{2R}} \frac{\alpha_{n}^l(x)(v_{n}^l(x) - v_{n}^l(y))}{|x - y|^{N + 2s}} \, dx \]

\[ := T_{n,R}^{21} + T_{n,R}^{22} + T_{n,R}^{23}. \]

and

\[ T_{n,R}^{22} = \sum_{l=1}^{k} \int_{B_{2R}} v_{n}^l(y) \beta_{n}^l(y) \, dy \int_{B_{2R} \setminus B_{R}} \frac{\alpha_{n}^l(x)(v_{n}^l(x) - v_{n}^l(y))(\eta_{R}(x) - 1)}{|x - y|^{N + 2s}} \, dx \]

\[ - \sum_{l=1}^{k} \int_{B_{2R} \setminus B_{R}} v_{n}^l(y) \beta_{n}^l(y) \, dy \int_{B_{R}} \frac{\alpha_{n}^l(x)(v_{n}^l(x) - v_{n}^l(y))}{|x - y|^{N + 2s}} \, dx \]

\[ = \sum_{l=1}^{k} \int_{B_{2R}} v_{n}^l(y) \beta_{n}^l(y) \, dy \int_{B_{2R} \setminus B_{R}} \frac{(v_{n}^l(x) - v_{n}^l(y))(\eta_{R}(x) - 1)}{|x - y|^{N + 2s}} \, dx \]

\[ - \sum_{l=1}^{k} \int_{B_{2R} \setminus B_{R}} v_{n}^l(y) \beta_{n}^l(y) \, dy \int_{B_{R}} \frac{v_{n}^l(x) - v_{n}^l(y)}{|x - y|^{N + 2s}} \, dx \]

\[ := T_{n,R}^{21} + T_{n,R}^{22}. \]

Also, for large \( n \),

\[ T_{n,R}^{23} = \sum_{l=1}^{k} \int_{B_{2R} \setminus B_{R}} v_{n}^l(y) \, dy \int_{B_{2R}} \frac{(v_{n}^l(x) - v_{n}^l(y))(\eta_{R}(x) - \eta_{R}(y))}{|x - y|^{N + 2s}} \, dx \]

\[ + \sum_{l=1}^{k} \int_{B_{2R}} v_{n}^l(y) \, dy \int_{B_{2R} \setminus B_{R}} \frac{(v_{n}^l(x) - v_{n}^l(y))(\eta_{R}(x) - \eta_{R}(y))}{|x - y|^{N + 2s}} \, dx \]
+ \sum_{l=1}^{k} \int_{B_{2R} \setminus B_{R}} v_{n}^l(y) \, dy \int_{B_{2R} \setminus B_{R}} \frac{(v_{n}^l(x) - v_{n}^l(y))(\eta_{R}(x) - \eta_{R}(y))}{|x - y|^{N+2s}} \, dx
\end{align*}
\end{align*}

For $|T_{n,R}^{2i}|$, $i = 1, 2$, it holds

$$\limsup_{n \to \infty} |T_{n,R}^{2i}| \leq CR^{-2s} + \limsup_{n \to \infty} 2 \sum_{l=1}^{k} \int_{B_{2R}^c} \, dy \int_{B_{R}} \frac{(v_{n}^l(y))^2}{|x - y|^{N+2s}} \, dx.$$ 

By the fractional Hardy inequality (1.6) and letting $\tilde{R} = R^{N+1}$, we find

\begin{align*}
&\limsup_{n \to \infty} \int_{B_{2R}^c} (v_{n}^l(y))^2 \, dy \int_{B_{R}} \frac{1}{|x - y|^{N+2s}} \, dx \\
&\leq C \limsup_{n \to \infty} \int_{B_{2R}^c} (v_{n}^l(y))^2 \frac{R^N}{|y|^{N+2s}} \, dy \\
&\leq C \limsup_{n \to \infty} \int_{B_{R} \setminus B_{2R}} (v_{n}^l(y))^2 \frac{R^N}{|y|^{N+2s}} \, dy + C \limsup_{n \to \infty} \int_{B_{2R}^c} (v_{n}^l(y))^2 \frac{R^N}{|y|^{N+2s}} \, dy \\
&\leq C \limsup_{n \to \infty} \int_{B_{R} \setminus B_{2R}} (v_{n}^l(y))^2 \, dy + C \limsup_{n \to \infty} \int_{B_{2R}^c} \frac{(v_{n}^l(y))^2}{|y|^{2s}} \frac{R^N}{|y|^N} \, dy \\
&\leq C \int_{B_{R} \setminus B_{2R}} (v_{n}^l(y))^2 \, dy + \frac{C}{R} \\
&= o_R(1).
\end{align*}

Noting that for each $\Omega \subset \subset \mathbb{R}^N$ is smooth, it holds

$v_{n}^l \rightharpoonup v_{*}^l$ weakly in $W^{s,2}(\Omega)$.

Then, for $T_{n,R}^{2i}$, by the estimates of $T_{n,R}^{2i}(i = 1, 2)$, we have

\begin{align*}
&\limsup_{n \to \infty} |T_{n,R}^{2i1}| \\
&\leq \limsup_{n \to \infty} \sum_{l=1}^{k} \int_{B_{2R} \setminus B_{R}} \, dy \int_{B_{2R}^c} \frac{|v_{n}^l(x) - v_{n}^l(y)|^2}{|x - y|^{N+2s}} \, dx \\
&+ \limsup_{n \to \infty} \sum_{l=1}^{k} \int_{B_{2R} \setminus B_{R}} (v_{n}^l(y))^2 \, dy \int_{B_{2R}^c} \frac{(1 - \eta_{R}(y))^2}{|x - y|^{N+2s}} \, dx \\
&\leq \limsup_{n \to \infty} \sum_{l=1}^{k} \int_{B_{2R} \setminus B_{R}} \, dy \int_{B_{2R}^c \cap B_{4R}} \frac{|v_{n}^l(x) - v_{n}^l(y)|^2}{|x - y|^{N+2s}} \, dx
\end{align*}
\[
+ \limsup_{n \to \infty} \sum_{l=1}^{k} \int_{B_{2R} \setminus B_R} dy \int_{B_{2R} \setminus B_R} \left| v_{n}(x) - v_{l}(y) \right|^2 \frac{1}{|x-y|^{N+2s}} \, dx + C \limsup_{n \to \infty} \sum_{l=1}^{k} \int_{B_{2R} \setminus B_R} (v_{n}(y))^2 \, dy
\]
\[
\leq \limsup_{n \to \infty} \sum_{l=1}^{k} \int_{B_{2R} \setminus B_R} dy \int_{B_{2R} \setminus B_R} \left| v_{n}(x) - v_{l}(y) \right|^2 \frac{1}{|x-y|^{N+2s}} \, dx + C \sum_{l=1}^{k} \int_{\mathbb{R}^N} (v_{l}(y))^2 \, dy
\]
\[
\leq C \int_{B_{4R} \setminus B_R} dy \int_{\mathbb{R}^N} \frac{\left| v_{n}(x) - v_{l}(y) \right|^2}{|x-y|^{N+2s}} \, dx + o_R(1)
\]
\[
= o_R(1).
\]

Similarly, we get
\[
\limsup_{n \to \infty} |T^{221}_{n,R}| \leq o_R(1)
\]
and
\[
\limsup_{n \to \infty} |T^{231}_{n,R} + T^{232}_{n,R} + T^{233}_{n,R}| \leq o_R(1).
\]
Therefore
\[
\varepsilon^{2s-N} T^{2}_{n,R} \leq o_R(1).
\]

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