ASPECTS OF CHIRAL SYMMETRY

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We give a pedagogical review of implications of chiral symmetry in QCD. First, we briefly discuss classical textbook subjects such as the axial anomaly, spontaneous breaking of the flavor-nonsinglet chiral symmetry, formation of light pseudo-Goldstone particles, and their effective interactions. Then we proceed to other issues. We explain in some detail a recent discovery how to circumvent the Nielsen–Ninomiya’s theorem and implement chirally symmetric fermions on the lattice. We touch upon such classical issues as the Vafa-Witten’s theorem and ’t Hooft’s anomaly matching conditions. We derive a set of exact theorems concerning the dynamics of the theory in a finite Euclidean volume and the behavior of the Dirac spectral density. Finally, we discuss an imaginary world with a nonzero value of the vacuum angle $\theta$.

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1 Classic Sagas

1.1 Basics

When a field theory enjoys a symmetry, it is always a fortunate circumstance because the presence of the symmetry always simplifies analysis and, more often than not, allows one to obtain exact or semi-exact results. The richer is the symmetry, the stronger are the results thus obtained. Sometimes, the presence of a rich enough symmetry allows one to solve the theory (the best example is the exact results for the spectrum of the $\mathcal{N} = 2$ supersymmetric Yang–Mills theory due to Seiberg and Witten\cite{Seiberg-Witten}). The symmetry of QCD is not so rich and the theory is not exactly solvable. Still, the presence of light quarks such that the theory enjoys chiral symmetry in the chiral limit, when the quarks become massless, allows one to extract a lot of consequences concerning QCD dynamics, with some of them having the status of exact theorems. In this mini-review we will concentrate on some less known and/or comparatively recent results obtained in this direction. To make the discussion coherent, we will review some well-known facts as well.

The Lagrangian of QCD reads

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{2g^2} \text{Tr} \{ F_{\mu\nu} F^{\mu\nu} \} + \sum_{f=1}^{6} \bar{\psi}_f (i\slashed{D} - m_f) \psi_f + \frac{\theta}{32\pi^2} F^{a\mu\nu} \tilde{F}_{a\mu\nu},$$

(1)

where $\tilde{F}_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}$, the physical value of the parameter $\theta$ is very small $\theta \lesssim 10^{-9}$, and $m_f$ are the quark masses. The $u$, $d$, and $s$ quarks are relatively light, with masses $m_u \approx 4$ MeV, $m_d \approx 7$ MeV, and $m_s \approx 150$ MeV. $m_u,d$ are especially small. It makes sense to consider the chiral limit when $N_f = 2$ or $N_f = 3$ quarks become massless. In this limit, the Lagrangian (1) is invariant under the transformations

$$\delta \psi_f = i\alpha_A [t^A \psi]_f$$

(2)

and

$$\delta \psi_f = i\beta_A \gamma^5 [t^A \psi]_f$$

(3)

where $t^A$ ($A = 0, 1, \ldots, N_f^2 - 1$) are the generators of the flavor $U(N_f)$ group. The symmetry (2) is the ordinary isotopic symmetry. It is still present even if the quarks are endowed with a mass (of the same magnitude for all flavors). The symmetry (3) holds only in the massless theory. The corresponding Noether currents are

$$(j^\mu)^A = \bar{\psi} t^A \gamma^\mu \psi, \quad (j^{\mu5})^A = \bar{\psi} t^A \gamma^{\mu5} \psi.$$  

(4)

They are conserved upon applying the classical equations of motion.
1.2 Singlet Axial Anomaly

Let us discuss first the singlet axial symmetry (with $t^A = 1$). An important fact is that this symmetry exists only in the classical case. The full quantum path integral is not invariant under the transformations

$$\delta \psi = i \alpha \gamma^5 \psi, \quad \delta \bar{\psi} = i \alpha \bar{\psi} \gamma^5,$$

where the flavor index is now omitted. This explicit symmetry breaking due to quantum effects can be presented as an operator identity involving an anomalous divergence,

$$\partial_\mu j^{\mu 5} = -\frac{1}{8\pi^2} \epsilon^{\alpha \beta \mu \nu} \text{Tr} \{ F_{\alpha \beta} F_{\mu \nu} \}.$$

There are many ways to derive and understand this relation. Historically, it was first derived by purely diagrammatic methods. The amplitude

$$\delta^{ab} T_{\mu \nu \lambda}(k, q) = \int d^4 x d^4 y e^{i k x} e^{i q y} \langle T\{ j^a_\mu(x) j^b_\nu(0) j^5_\lambda(y) \} \rangle_0,$$

(\(j^{a, b}_\mu = \bar{\psi} t^{a, b} \gamma_\mu \psi\), where \(a, b\) are color indices) cannot be rendered transverse with respect to both vector and axial indices such that \(k_\mu T_{\mu \nu \lambda} = 0\) and \(q_\lambda T_{\mu \nu \lambda} = 0\) due to ultraviolet divergences in the anomalous triangle graph. We will comment more on this later when we will be discussing 't Hooft’s self-consistency conditions.

The anomaly relation (6) can also be derived as an operator identity. To this end, one should carefully define the operator of the axial current as

$$j^{\mu 5} = \lim_{\epsilon \to 0} \bar{\psi}(x + \epsilon) \gamma^\mu \gamma^5 \exp \left\{ i \int_x^{x+\epsilon} A_\alpha(y) dy^\alpha \right\} \psi(x)$$

and perform the limit, averaging over the directions of \(\epsilon_\mu\) and making use of the operator equations of motion (see e.g. Ref. 2 for more details.).

Let us dwell here on the third and the most elegant way to derive the anomaly relation (6) which uses path integral methods. The anomaly appears due to the necessity to regularize the theory in the ultraviolet. The most politically correct approach would be to study a path integral regularized by a lattice, and we will do so in Sec. 2. Let us, first, describe a more standard and habitual finite mode regularization.

Consider an Euclidean path integral for the partition function in QCD with one massless quark flavor. The fermionic part of the integral is

$$\int \prod_x d\bar{\psi}(x) d\psi(x) \exp \left\{ i \int d^4 x \bar{\psi} D^E \psi \right\},$$
which formally coincides with the determinant of the Euclidean Dirac operator

$-\mathcal{D}^E = -i\gamma^E_\mu \mathcal{D}_\mu$, where $\gamma^E_\mu$ are Euclidean $\gamma$ matrices. The latter are anti-Hermitean and satisfy

$$\gamma^E_\mu \gamma^E_\nu + \gamma^E_\nu \gamma^E_\mu = -2\delta_{\mu\nu}.$$ 

From now on, we will stay in the Euclidean space and will suppress the superscript “$\text{E}$” for $\mathcal{D}$ and $\gamma^\mu$.

Let us assume that the theory is somehow regularized in the infrared so that the spectrum of the operator $\mathcal{D}$ is discrete. Note now that the spectrum of the massless Euclidean Dirac operator enjoys the following symmetry: for any eigenfunction $u_k$ of the operator $\mathcal{D}$ with an eigenvalue $\lambda_k$, the function $u'_k = \gamma^5 u_k$ is also an eigenfunction with the eigenvalue $-\lambda_k$ ($\mathcal{D}$ and $\gamma^5$ anticommute).

Therefore,

$$\det \| -i\mathcal{D} + m \| = \prod_k (-i\lambda_k + m) = m^q \prod_k (\lambda_k^2 + m^2), \quad (9)$$

where $q$ is the number of possible exact zero modes and the product $\prod_k$ runs over the nonvanishing eigenvalues. We see that the expression (9) is real, and the Euclidean partition function is real, too.

Let us expand $\psi(x)$, $\bar{\psi}(x)$ as

$$\begin{cases}
    \psi(x) = \sum_k c_k u_k(x) \\
    \bar{\psi}(x) = \sum_k \bar{c}_k \bar{u}_k(x)
\end{cases}, \quad (10)$$

where $\{u_k(x)\}$ is a a set of eigenfunctions of $\mathcal{D}$ forming a complete basis in the corresponding Hilbert space. Then

$$\prod_x d\bar{\psi}(x) d\psi(x) \equiv \prod_k d\bar{c}_k dc_k. \quad (11)$$

Suppose the field variables are transformed by an infinitesimal global chiral transformation $\delta \psi$. Now, $\psi' = \psi + \delta \psi$ and $\bar{\psi}' = \bar{\psi} + \delta \bar{\psi}$ can be again expanded in the series (10). The new expansion coefficients are related to the old ones,

$$c'_k = c_k + i\alpha \sum_m c_m \int d^4x ~ u^{\dagger}_k(x) \gamma^5 u_m(x) \equiv \sum_m (\delta_{km} + i\alpha A_{km}) c_m$$

$$\bar{c}'_k = \bar{c}_k + i\alpha \sum_m \bar{c}_m \int d^4x ~ \bar{u}^{\dagger}_m(x) \gamma^5 \bar{u}_k(x) \equiv \sum_m \bar{c}_m (\delta_{km} + i\alpha A_{mk}). \quad (12)$$

\[^a\text{We keep the Euclidean } \psi \text{ and } \bar{\psi} \text{ independent and are not bothered by the fact that } \delta \bar{\psi} \neq (\delta \psi)^\dagger.\]
The point is that the transformation (12) has a nonzero Jacobian. We have

\[ \prod_k dc'_k dc_k' = J^{-2} \prod_k dc_k dc_k , \]  

(13)

where

\[ J = \det(1 + i\alpha A) \approx \exp \left\{ i\alpha \sum_k A_{kk} \right\} . \]  

(14)

Or, in other words,

\[ \ln J = i\alpha \int d^4x \sum_k u'^\dagger_k(x) \gamma^5 u_k(x) + o(\alpha) . \]  

(15)

One’s first (wrong!) impression might be that \( \int d^4x \sum_k u'^\dagger_k(x) \gamma^5 u_k(x) \) is just zero. Indeed, as was already mentioned above, the function \( u'_k = \gamma^5 u_k \) is also an eigenfunction of the Dirac operator with the eigenvalue \(-\lambda_k\). If \( \lambda_k \neq 0 \), \( u_k(x) \) and \( \gamma^5 u_k(x) \) thereby represent different eigenfunctions and the integral \( \int d^4x \ u'_k(x) \gamma^5 u_k(x) \) vanishes.

A nonzero value of (13) is due to the fact that, for intricate enough topologically nontrivial gauge fields, the spectrum of the Dirac operator involves some number of exact zero modes, for which \( \gamma^5 u_0(x) = \mp u_0(x) \) (depending on whether the modes are left-handed or right-handed), and their contribution to the sum (15) is responsible for the whole effect. A famous theorem of Atiyah and Singer dictates

\[ \int d^4x \sum_k u'^\dagger_k(x) \gamma^5 u_k(x) = n_R^{(0)} - n_L^{(0)} = q , \]  

(16)

where \( n_{L,R}^{(0)} \) is the number of the left–handed (right–handed) zero modes and \( q \) is the topological charge of the gauge field configuration,

\[ q = \frac{1}{16\pi^2} \int d^4x \text{ Tr} \{ F_{\mu\nu} \tilde{F}_{\mu\nu} \} . \]  

(17)

Substituting (16) in Eqs. (13, 13), we now see that the change of the measure under the chiral transformation can be presented as a shift of the effective action

\[ \delta S = \frac{i\alpha}{16\pi^2} \epsilon_{\alpha\beta\mu\nu} \int d^4x \text{ Tr} \{ F_{\alpha\beta} F_{\mu\nu} \} . \]  

(18)
In other words, a global singlet chiral transformation is equivalent to leaving
the fermionic fields intact, but shifting instead the parameter $\theta$ in the original

theory $[\text{I}]$,

$$\theta \rightarrow \theta + 2\alpha.$$ 

1.3 Spontaneous Breaking of Nonsinglet Chiral Symmetry

Consider now the whole set of symmetries $[\text{II}], [\text{III}]$. It is convenient to introduce

$$\psi_{L,R} = \frac{1}{2} (1 \mp \gamma^5) \psi, \quad \bar{\psi}_{L,R} = \frac{1}{2} \bar{\psi} (1 \pm \gamma^5)$$ 

and rewrite $[\text{II}], [\text{III}]$ as

$$\psi_L \rightarrow V_L \psi_L, \quad \psi_R \rightarrow V_R \psi_R ,$$ 

where $V_L$ and $V_R$ are two different $U(N_f)$ matrices. The singlet axial transfor-
mations with $V_L = V_R^* = e^{i\phi}$ are anomalous by the same token as in the theory
with a single quark flavor. Therefore, the true fermionic symmetry group of
massless QCD is

$$G = SU_L(N_f) \times SU_R(N_f) \times U_V(1).$$ 

A fundamental experimental fact is that the symmetry $[\text{II}]$ is actually
spontaneously broken, which means that the vacuum state is not invariant under the
action of the group $G$. The symmetry $G$ is, however, not broken completely.
The vacuum is still invariant under transformations with $V_L = V_R$, generated by
the vector isotopic current.

Thus, the pattern of breaking is

$$SU_L(N_f) \times SU_R(N_f) \rightarrow SU_V(N_f).$$ 

The nonvanishing vacuum expectation values

$$\Sigma^{fg} = \langle \psi_L^f \bar{\psi}_R^g \rangle_0$$ 

are the order parameters of the spontaneously broken axial symmetry. The
matrix $\Sigma^{fg}$ is referred to as the quark condensate matrix.

Non-breaking of the vector symmetry implies that the matrix order pa-

rameter $[\text{II}]$ can be cast in the form

$$\Sigma^{fg} = \frac{1}{2} \Sigma \delta^{fg}$$ 

$^b$An exact theorem $^b$ that the vector symmetry cannot be broken spontaneously in QCD will
be proven in Sec. 3.
by group transformations \((20)\). This means that the general condensate matrix \(\Sigma^{fg}\) is a unitary \(SU(N_f)\) matrix multiplied by \(\Sigma\).

In general, \(\Sigma\) could be any complex number. It can be made real by a global \(U_A(1)\) rotation which, according to Eq. \((18)\) (with the factor \(N_f\)), amounts to a shift of the vacuum angle \(\theta\). For massless quarks, physics does not depend on the phase of \(\Sigma\) and, thereby, on \(\theta\). It is convenient then to choose \(\Sigma\) real and positive and \(\theta = 0\). From experiment, we know that \(\Sigma \approx (240 \text{ MeV})^3/2\) with about 30\% uncertainty (this value refers to some particular normalization point \(\mu \sim 0.5 \text{ GeV}\) on which the operator \(\bar{\psi}\psi\) and its vacuum expectation value depend).

By Goldstone’s theorem, spontaneous breaking of a global continuous symmetry leads to the appearance of purely massless Goldstone bosons. Their number coincides with the number of broken generators, which is \(N_f^2 - 1\) in our case. As it is the axial symmetry which is broken, the Goldstone particles are pseudoscalars. They are nothing but pions for \(N_f = 2\) or the octet \((\pi, K, \bar{K}, \eta)\) for \(N_f = 3\).

It is a fundamental and important fact that spontaneous breaking of continuous symmetries not only creates massless Goldstone particles, but also fixes the interactions of the latter at low energies. To see this, let us first recall that the Goldstone field describes fluctuations of the order parameter,

\[
\Sigma^{fg} \to \Sigma^{fg}(x) = \frac{1}{2} \Sigma U(x)
\]

with \(U(x) \in SU(N_f)\). It is convenient to express \(U(x)\) as an exponential

\[
U(x) = \exp \left\{ \frac{2i\phi^a(x) t^a}{F_\pi} \right\},
\]

where \(\phi^a(x)\) are the physical meson fields [so that \(\phi^a(x) = 0\) corresponds to the vacuum \((2)\)] and \(F_\pi\) is a constant of dimension of mass.

The Goldstone particles are massless whereas all other states in the physical spectrum have nonzero mass. Therefore, we are in the Born-Oppenheimer situation: there are two distinct energy scales and one can write down an effective Lagrangian depending only on slow Goldstone fields with the fast degrees of freedom corresponding to all other particles being integrated out.

To fix the exact form of this Lagrangian, note that the transformations \((20)\) are realized at the level of the effective Lagrangian as \(U \to V_L U V_R^\dagger\). Any scalar function depending on \(U\) and invariant under this symmetry is also a function of \(U^\dagger U = 1\), i.e. it is just a constant. There is only one invariant structure involving two derivatives, namely

\[
\mathcal{L}_{\text{eff}}^{(2)} = \frac{F_\pi^2}{4} \text{Tr} \left\{ \partial_\mu U \partial^\mu U^\dagger \right\}. \tag{26}
\]
Take $N_f = 2$. The perturbative expansion of Eq. (26) in powers of $\phi$ reads

$$L^{(2)}_{\text{eff}} = \frac{1}{2} (\partial_\mu \phi^a)^2 + \frac{1}{6F^2_\pi} \left[ (\phi^a \partial_\mu \phi^a)^2 - (\phi^a \phi^a)(\partial_\mu \phi^b)^2 \right] + \ldots$$

Also for $N_f \geq 3$, we have, on top of the standard kinetic term, a quartic term involving two derivatives, with somewhat more complicated group structure. We see that the symmetry dictates rather specific interactions between the pions. They do not interact at the $s$-wave level, which means that all amplitudes vanish at zero momenta, but the strength of interaction grows rapidly with energy.

Equation (26) describes the effective chiral Lagrangian of massless Goldstone bosons at leading order. The first corrections involve 4 derivatives and there are 3 different invariant functions

$$L^{(4)}_{\text{eff}} = L_1 \text{Tr} \left\{ \partial_\mu U \partial^\nu U^\dagger \right\}^2 + L_2 \text{Tr} \left\{ \partial_\mu U \partial_\nu U^\dagger \right\} \text{Tr} \left\{ \partial^\mu U \partial^\nu U^\dagger \right\}
+ L_3 \text{Tr} \left\{ \partial_\mu U \partial^\nu U^\dagger \partial_\nu U \partial^\mu U^\dagger \right\}$$

(only 2 linearly independent structures are left for $N_f = 2$).

The relevant Born-Oppenheimer expansion parameter is $\kappa_{\text{chir}} \sim p_{\text{chir}}/F_\pi$. When $\kappa_{\text{chir}} \sim 1$ (in practice, one should rather take $\kappa_{\text{chir}} \sim 2\pi$), the Born-Oppenheimer approach as well as the whole effective Lagrangian approach breaks down, and non-Goldstone degrees of freedom become important. The physical meaning of $F_\pi$ is thus clarified. It characterizes the gap in the spectrum and sets a scale below which massive degrees of freedom can be disregarded.

Let us discuss the actual world now. The Lagrangian of real QCD (1) is not invariant under the axial symmetry transformations just because quarks have nonzero masses. The symmetry (21) is still very much relevant to QCD because some of the quarks happen to be very light.

For $N_f = 2$, spontaneous breaking of an exact $\text{SU}_L(2) \times \text{SU}_R(2)$ symmetry would lead to the existence of 3 strictly massless pions. As the symmetry is not quite exact, the pions have a small mass. However, their mass $M$ goes to zero in the chiral limit $m_u, d \to 0$. Indeed, trading the mass term

$$- m_u \bar{u}u - m_d \bar{d}d = m_u (u_L \bar{u}_R + u_R \bar{u}_L) + m_d (d_L \bar{d}_R + d_R \bar{d}_L)$$

in the QCD Lagrangian for the contribution

$$L^{(m)}_{\text{eff}} = \Sigma \text{Re} \left\{ \text{Tr} \{ MU^\dagger \} \right\}$$
(\mathcal{M} is the quark mass matrix which is chosen here in the form \mathcal{M} = \text{diag}(m_u, m_d) with real \( m_u, m_d \) in the effective chiral Lagrangian\(^{30}\) and expanding \( \phi^a \) in \( \phi \), we obtain the Gell-Mann-Oakes-Renner relation

\begin{equation}
F_\pi^2 M^2 = (m_u + m_d)\Sigma + O(m_q^2). \tag{31}
\end{equation}

The constant \( F_\pi \) appears also in the matrix element

\[ \langle \text{vac} | A_\mu^+ | \pi \rangle_p = i\sqrt{2}F_\pi p_\mu \]

of the axial current \( A_\mu^+ = \bar{d} \gamma_\mu \gamma^5 u \) and determines the charged pion decay rate. Experimentally, \( F_\pi \approx 93 \text{ MeV} \).

The effective chiral Lagrangian involving the lowest \( (26), (30) \), and higher order terms (in addition to the terms of higher order in derivatives as in Eq. \( (28) \), there are also terms of higher order in mass) can be and was used to calculate the amplitudes of different processes involving pseudoscalar mesons at low energies. The corresponding technique is called chiral perturbation theory\(^{44}\).

2 Quarks on Euclidean Lattice

The material of the previous section is well known and can be found in many textbooks. We are in a position now to discuss less known issues and will start with the question of how to put quarks on the lattice. This issue looked very confusing for a long time, and has been clarified only recently.

2.1 The Nielsen–Ninomiya’s No-go Theorem

The only way to define what quantum field theory in nonperturbative regime really means is to introduce a lattice ultraviolet regularization so that the symbol of path integral is defined as a limit of finite-dimensional lattice integrals when the lattice spacing goes to zero. We want the regularized theory to preserve as many symmetries which the original continuous theory has as possible. Actually, some symmetries can be broken on the lattice (as rotational and Lorentz symmetries are) and be restored in the continuum limit, but it is not always straightforward and not always true that the symmetries are restored, indeed. For example, breaking gauge symmetry on the lattice would probably lead to a nonsensical, not gauge-invariant continuum theory. Gauge-invariant lattice action for a pure gauge theory was written by Wilson long time ago. But the task to put fermions on the lattice so that not only

\(^{44}\)Equation \( (30) \) is the leading chiral-noninvariant contribution in \( \mathcal{L}_{\text{eff}} \). Also terms of higher order in \( \mathcal{M} \), as well as terms of first order in \( \mathcal{M} \) but involving derivatives of \( U \), are allowed.
the gauge symmetry, but also chiral symmetry would be preserved turned out to be much more difficult.

Let us define to this end Grassmann variables $\bar{\psi}_n$, $\psi_n$ in the nodes of the lattice for each quark flavor (color and Lorentz indices are not displayed). As a first and natural guess, let us write the lattice counterpart of the Dirac action as follows

$$S_{\text{ferm.lat.}} = -\frac{ia^3}{2} \sum_{n,\mu} \left[ \bar{\psi}_n U_{n,n+e_\mu} \gamma_\mu \psi_{n+e_\mu} - \bar{\psi}_n U_{n,n-e_\mu} \gamma_\mu \psi_{n-e_\mu} \right] + ma^4 \sum_n \bar{\psi}_n \psi_n . \quad (32)$$

We see that the action (32) reproduces the Euclidean action

$$S_{\text{ferm}} = -\int d^4x [i\bar{\psi} \gamma_\mu (\partial_\mu - i A_\mu) \psi - m\bar{\psi}\psi] \quad (33)$$

in the continuum limit. Indeed, for free fermions

$$-\frac{ia^3}{2} \sum_{n,\mu} \bar{\psi}_n \gamma_\mu [\psi_{n+e_\mu} - \psi_{n-e_\mu}] \to -i \int d^4x \bar{\psi} \gamma_\mu \partial_\mu \psi .$$

Expanding $U_{n,n+e_\mu} \equiv 1 - ia A_\mu + O(a^2)$, we also restore the interaction term, and the last term in Eq. (32) turns into the continuum mass term. The action (32) is invariant under the gauge transformations when the $U$ and $\psi$ are transformed according to

$$U_{n+e_\mu} \rightarrow \Omega_{n+e_\mu} U_{n+e_\mu,n} \Omega_n^\dagger ,$$

$$\psi_n \rightarrow \Omega_n \psi_n , \quad (34)$$

where $\{\Omega_n\}$ is a set of unitary matrices defined on the nodes of the lattice.

Equation (32) is called the “naive lattice fermion action”, and I have to say that, if the reader was convinced by the above reasoning that, in the continuum limit, it goes over to Eq. (33), he/she was naive, too. Our implicit assumption was that the fermion fields $\psi_n$ depend on the lattice node $n$ in a smooth manner, so that the finite difference $\psi_{n+e_\mu} - \psi_{n-e_\mu}$ goes over to the continuum derivative. It turns out, however, that fermion field configurations which behave as $\psi_n \sim (-1)^{n_1}$ or $\psi_n \sim (-1)^{n_2+n_4}$ and change significantly at the microscopic lattice scale, are equally important. After carefully performing the continuum limit, these wildly oscillating modes give rise to 15 extra light fermion species with the same mass, the so-called doublers.
To understand it, consider first free massless fermions. Let

$$\left( \partial^+_\mu \psi \right)_n = \frac{1}{a} [\psi_{n+e_\mu} - \psi_n], \quad \left( \partial^-_\mu \psi \right)_n = \frac{1}{a} [\psi_n - \psi_{n-e_\mu}]$$ \tag{35}

be the forward and backward lattice derivative operators. The naive free massless Dirac operator is

$$D^0_{\text{free}} = -\frac{i}{2} \gamma_\mu (\partial^+_\mu + \partial^-_\mu).$$ \tag{36}

The eigenfunctions of $D^0_{\text{free}}$ are characterized by the Euclidean 4-momentum $p_\mu$,

$$u_p(n) = C_p e^{iap_{\mu} n_\mu},$$

where $C_p$ is a constant Grassmann bispinor. The eigenvalue equation

$$D^0_{\text{free}} u_p(n) = -i\lambda_p u_p(n)$$

implies

$$\left[ -\frac{1}{a} \gamma_\mu \sin(ap_\mu) \right] C_p = -i\lambda_p C_p$$ \tag{37}

with

$$\lambda_p = \pm \frac{1}{a} \sqrt{\sum_\mu \sin^2(ap_\mu)}. \tag{38}$$

(The operator (36) is anti-Hermitean and its eigenvalues are purely imaginary.)

When $ap_\mu \ll 1$, we reproduce the continuum massless fermions with the spectrum $\lambda_p = \pm \sqrt{p^2_\mu}$. Each eigenvalue (38) is doubly degenerate due to 2 possible polarizations. The eigenfunctionss with negative $\lambda_p$ are obtained from the ones with positive $\lambda_p$ by multiplication by $\gamma^5$.

Note, however, that the lattice Dirac equation (37) has an additional discrete symmetry ($Z_2|^4$: for any eigenfunction $u_p$, the function $\hat{Q}_\mu u_p = \gamma_\mu \gamma^5 u_p + (\pi/a)e_\mu$ (no summation over $\mu$) is also the eigenfunction of $D^0_{\text{free}}$ with the same eigenvalue $\lambda_p$. The operators $\hat{Q}_\mu$ commute with $D^0_{\text{free}}$ and anticommute with each other

$$\hat{Q}_\mu \hat{Q}_\nu + \hat{Q}_\nu \hat{Q}_\mu = 2\delta_{\mu\nu}. \tag{39}$$

The functions

$$u_p, \quad \hat{Q}_\mu u_p, \quad \hat{Q}_\mu \hat{Q}_\nu u_p, \quad \hat{Q}_\mu \hat{Q}_\nu \hat{Q}_\lambda u_p, \quad \hat{Q}_\mu \hat{Q}_\nu \hat{Q}_\lambda \hat{Q}_\rho u_p$$

form a degenerate 16-plet.
In the free case, each eigenstate of the naive Dirac operator is not just 16-fold, but 32-fold degenerate due to polarizations. In the interacting case (on a generic gauge field background), polarization is not a good quantum number, but the 16-fold degeneracy still holds. The naive lattice Dirac operator in Eq. (32) which can be written in the form

\[ D_0 = -\frac{i}{2} \gamma_\mu (D_\mu^+ + D^-_\mu), \]

are the covariant lattice forward and backward derivatives, still enjoys the

\[ \hat{Q}_\mu : \psi_n \longrightarrow (-1)^{n_\mu} \gamma_\mu \gamma_5 \psi_n \]  

(no summation over \( \mu \)). We see that if \( \psi_n \) changes smoothly from node to node, its 15 doublers wildly oscillate on the microscopic lattice spacing scale. We might call these modes “unphysical” but they would not listen to us and contaminate with a vengeance any numerical lattice calculation we might wish to do. Some way to get rid of them should be suggested, otherwise QCD, the theory involving only 6 quarks with different masses, would not be operationally defined.

The problem is that it is not so simple. Let us look for some other lattice Dirac operator \( \mathcal{D} \neq \mathcal{D}^0 \) satisfying the following 4 natural conditions:

1. At distances much larger that the lattice spacing \( a \), \( \mathcal{D} \rightarrow -i \mathcal{D} \) giving rise to a massless fermion in the continuum limit\(^d\).

2. All the modes of \( \mathcal{D} \) not associated with the latter are of order \( 1/a \) (no doublers!).

3. \( \mathcal{D} \) is local. In other words, the matrix elements \( \mathcal{D}_{nn'} \) decay exponentially fast at large distances \( |n - n'| \gg 1 \).

4. Chiral symmetries (5), (3) of the massless fermionic action are not broken by the regularization explicitly [the singlet axial symmetry (5) is eventually going to be broken due to noninvariance of the fermionic measure, but we require the absence of explicit breaking in the regularized Lagrangian]. This seems to imply the condition \( \mathcal{D} \gamma^5 + \gamma^5 \mathcal{D} = 0 \).

\(^d\)Adding a finite mass term to \( \mathcal{D} \) presents no difficulties [see Eq. (32)].
The no-go theorem due to Nielsen and Ninomiya tells us, however, that such $D$ does not exist. To understand it, consider first the free fermion case. The momentum $p_\mu$ is then a good quantum number, and the Dirac operator in the momentum representation has the form

$$D(p) = \gamma_\mu F_\mu(p) + G(p).$$

(42)

The condition 4 tells us that $G(p) = 0$. The condition 1 implies that $F_\mu(p) = p_\mu + O(ap^2)$ for $ap_\mu \ll 1$. Now, $F_\mu(p)$ is a periodic function of its four arguments $p_\mu$ with the period $2\pi/a$. It realizes thus a smooth map $T^4 \to R^4$, where $R^4$ is the tangent space. A look at Fig. 1 can convince the reader that the point on the tangent space where it touches our torus has at least one more pre-image. And his/her intuition would not betray him/her: this statement can be proven in a rigorous mathematical manner. Basically, it follows from the fact that the degree of the map $T^d \to R^d$ is zero which means that

$$\sum_{\text{pre-images}} \text{sign} \left[ \det |\partial_\nu F_\mu(p)| \right] = 0.$$

(43)

As the Jacobian of the mapping $p_\mu \to F_\mu(p)$ is equal to 1 at the point $p_\mu = 0$, Eq. (43) implies that some other pre-images of zero, i.e. some other solutions of the equation system $F_\mu(p) = 0$ should be present (one can have just one extra solution as in Fig. 1 or more: for the “round upright torus” $F_\mu(p) = \sin(ap_\mu)/a$, there are $2^d - 1$ extra solutions). And that means the presence of doublers in contradiction with condition 2.

The only remaining possibility is that $F_\mu(p)$ are not continuous. Besides being ugly, it contradicts also condition 3: the matrix elements $D_{nn'} = D(n - n') = \gamma_\mu F_\mu(n - n')$ present actually the Fourier coefficients of the periodic function $\gamma_\mu F_\mu(p)$. If the latter is discontinuous, the Fourier coefficients cannot decay faster than $1/|n - n'|$ (otherwise, the Fourier series would converge uniformly on the torus $p_\mu \in [0, 2\pi/a)$ and the sum of such a series would be continuous).

We have proven that a lattice Dirac operator satisfying the above four conditions cannot be found for free fermions, but that also means that it cannot be found in QCD: any $D$ with this property should also enjoy it for any smooth set of the link variables and, in particular, for the set $U_{\text{each link}} = 1$ corresponding to free theory.

*We believe that, in the continuum limit $a \to 0$, the characteristic fields $\{U\}$ contributing to the path integral can be gauge transformed into the form where $U = 1 + O(a)$ for all links. Note that the statement that characteristic fields are always smooth is just wrong: there are
2.2 Ways to Go. The Ginsparg-Wilson Way.

If we still want to build up a lattice version of QCD, we have to relax at least one of our four conditions. Conditions 1 and 2 are, however, indispensable: a lattice theory where they do not hold just has nothing to do with QCD. Therefore, either locality or chiral invariance of the lattice action should be abandoned.

One of the possible procedures is that only one mode of each degenerate 16-plet of $D^0$ is taken into account in the fermionic determinant and in the spectral decomposition of fermion Green’s functions

$$\langle \psi_n \bar{\psi}_{n'} \rangle = \sum_k \frac{u_k(n)u_k^+(n')}{m - i\lambda_k},$$

instantons which, in the singular gauge, involve singularities of the gauge potential $\sim x_\mu / x^2$ at the instanton center. Note also that a gauge transformation removing the singularity at $x = 0$ moves it to some other point or to infinity. For the Euclidean torus that means that we cannot simultaneously require $U_{\text{all links}} = 1 + O(a)$ and periodicity of $U$.

To the best of our knowledge, a rigorous proof of this crucial assumption is absent, but it can be justified by arguing that the action of field configurations which are “essentially singular” (so that the singularity cannot be removed by a gauge transformation) would be infinite in the continuum limit.
etc, where $u_k(n)$ describe the $k$-th eigenmode of $\mathcal{D}^0$ as a function of the node. This amounts to choosing the lattice Dirac operator in the form $(\mathcal{D}^0)^{1/16}$, which is not local. A similar method is sometimes used in the practical lattice calculations, but besides purely technical inconveniences it is unsatisfactory from a philosophical viewpoint: we would like to have a local lattice approximation for a local field theory.

But then the chiral invariance is necessarily lost. Though renouncing chiral invariance is also not desirable — when regularizing the theory, we should try to preserve as much of its symmetries as possible — it is still considered as the least of evils.

Two ways of chiral noninvariant lattice regularization have been known for some time and used in practical calculations: (i) Wilson fermions and (ii) Kogut-Susskind or staggered fermions. We will describe here the first method which consists in adding to $\mathcal{D}^0$ the term $\sim aD_\mu^+D_\mu^- = aD_\mu^+D_\mu^-$, where the covariant lattice derivatives $D_\mu^\pm$ are defined in Eq. (40). Thus, the Wilson-Dirac operator is defined as

$$D^W = -\frac{i}{2} \gamma_\mu (D_\mu^+ + D_\mu^-) - \frac{r a}{2} D_\mu^+ D_\mu^-$$

(45)

where $r$ is an arbitrary nonzero real constant. For free fermions, $D_\mu^\pm \to \partial_\mu^\pm$ and

$$(\partial_\mu^+ \partial_\mu^- \psi)_n = \frac{1}{a^2} \sum_\mu (\psi_{n+c_\mu} + \psi_{n-c_\mu} - 2\psi_n)$$

which is the lattice laplacian. Passing to the momentum representation, we obtain

$$\mathcal{D}^W_{\text{free}}(p) = \frac{1}{a} \gamma_\mu \sin(ap_\mu) + \frac{2r}{a} \sum_\mu \sin^2 \left(\frac{ap_\mu}{2}\right).$$

(46)

The second term has the form of momentum-dependent mass. For small $p_\mu \ll 1/a$, it can be neglected and the continuum massless Dirac operator is reproduced. In contrast to $\mathcal{D}^0$, the operator $\mathcal{D}^W$ is not anti-Hermitean, and its eigenvalues are complex. What is important is that, for not small $p_\mu$, the absolute values of the eigenvalues of $\mathcal{D}^W$,

$$-i\lambda^W_p = \pm \frac{i}{a} \sqrt{\sum_\mu \sin^2(ap_\mu) + \frac{2r}{a} \sum_\mu \sin^2 \left(\frac{ap_\mu}{2}\right)}$$

(47)

are of order $1/a$. The doublers disappear. At $p_\mu = (\frac{2\pi}{a}, 0, 0, 0)$, the eigenvalue is $\frac{2r}{a}$; at $p_\mu = (0, \frac{2\pi}{a}, 0, \frac{2\pi}{a})$, it is $\frac{4r}{a}$, etc.
The chiral symmetry is broken, however, and it is messy. In principle, when the continuum limit $a \to 0$ is taken, the effects due to the breaking of $\gamma^5$ invariance must be suppressed, but in this particular problem the continuum limit with restoration of chiral symmetry is rather slow to reach, and it is even not shown quite conclusively that it is reached at all. In particular, it is difficult to render pions light. In practical calculations, it is achieved by introducing a large bare quark mass of order $g^2(a)/a$ and fine-tuning it so that the effects due to two chiral noninvariant terms – the Wilson term and the bare quark term – would cancel each other. Needless to say, this is a rather artificial and unaesthetic procedure.

As we see, this Nielsen-Ninomiya puzzle defies attempts to solve it. A recent remarkable observation is that the best strategy here is to follow the example of the Alexandre the Great and just cut it through! The adequate sword was forged back in 1982 by Ginsparg and Wilson. They suggested to consider the lattice Dirac operators satisfying the relation

$$\gamma^5 \mathcal{D} + \mathcal{D} \gamma^5 = a \mathcal{D} \gamma^5 \mathcal{D}. \quad (48)$$

The anticommutator $\{\mathcal{D}, \gamma^5\}$ does not vanish which means that the lattice Lagrangian is not invariant with respect to chiral transformation

$$\delta \psi_n = i\alpha \gamma^5 \psi_n, \quad \delta \bar{\psi}_n = i\alpha \bar{\psi}_n \gamma^5, \quad (49)$$

a lattice Euclidean counterpart of Eq. (5).

It took 16 years to realize that the lattice fermion action

$$S_F = a^4 \sum_{nn'} \bar{\psi}_n \mathcal{D}_{nn'} \psi_{n'}, \quad (50)$$

(color and spinor indices being suppressed), with $\mathcal{D}$ satisfying the relation (48), is invariant with respect to the following transformations:

$$\delta \psi = i\alpha \gamma^5 \left[ 1 - \frac{1}{2} a \mathcal{D} \right] \psi$$

$$\delta \bar{\psi} = i\alpha \bar{\psi} \left[ 1 - \frac{1}{2} a \mathcal{D} \right] \gamma^5. \quad (51)$$

If $\mathcal{D}$ is local (in the sense of condition 3 in the Nielsen-Ninomiya list), Eq. (51) is as good a lattice approximation of the continuous chiral symmetry (5) as the trivial (49). In particular, the pions would automatically be light (massless in the chiral limit), and no fine tuning is required. But condition 4 above is no more satisfied and one can hope now to find a local $\mathcal{D}$ not involving doublers.
The problem is still not trivial: as we will see a bit later, many solutions of the Ginsparg-Wilson relation (48) can be found and most of them are not what we are looking for. The simplest good solution was suggested by Neuberger. It has the form

$$D = \frac{1}{a} \left[ 1 - \frac{A}{\sqrt{A^\dagger A}} \right]$$

(52)

where $A = 1 - aD^W$ and $D^W$ is the Wilson-Dirac operator (45) with $r > 1/2$. In particular, for $r = 1$ and for the free fermions, we have

$$aD(p) = 1 - \frac{1 - 2 \sum_{\mu} \sin^2 \left( \frac{ap_{\mu}}{a} \right) - \gamma_{\mu} \sin(ap_{\mu})}{\left[ 1 + 8 \sum_{\mu<\nu} \sin^2 \left( \frac{ap_{\mu}}{a} \right) \sin^2 \left( \frac{ap_{\nu}}{a} \right) \right]^{1/2}} \ .$$

(53)

The eigenvalues of (53) are different from zero provided $p_{\mu} \neq 0$. In particular, for

$$p_{\mu} = \left( \frac{\pi}{a}, 0, 0, 0 \right), \quad p_{\mu} = \left( \frac{\pi}{a}, \frac{\pi}{a}, 0, 0 \right), \quad p_{\mu} = \left( \frac{\pi}{a}, \frac{\pi}{a}, \frac{\pi}{a}, 0 \right),$$

and

$$p_{\mu} = \left( \frac{\pi}{a}, \frac{\pi}{a}, \frac{\pi}{a}, \frac{\pi}{a} \right)$$

the eigenvalues $-i\lambda$ of $D$ are all equal to $2/a$. The doublers are absent. A second look at Eq. (53) reveals a beautiful feature displayed in Fig. 2: the eigenvalues of $D$ lie on the circle

$$(\text{Re} \ \lambda)^2 + \left( \text{Im} \ \lambda - \frac{1}{a} \right)^2 = \frac{1}{a^2} \ .$$

(54)

This property holds also in the interacting case. Using the property $D^\dagger = \gamma^5 D^5$ and the Ginsparg-Wilson relation (48), it is not difficult to see that the operator $V = 1 - aD$ is unitary, i.e. its eigenvalues lie on the circle $\{ e^{i\phi} \}$. The eigenvalues of $D = (1 - V)/a$ lie on the circle in Fig. 2.

The function (53) is analytic on the torus $p_{\mu} \in \left[ 0, \frac{2\pi}{a} \right]$ which means that its Fourier image decays exponentially at large distances. The Dirac operator thus constructed is local. In the interacting case, it stays local if the gauge field is smooth enough, i.e. the link variables $U_{n,n+e_{\mu}}$ are sufficiently close to 1.

As was mentioned, the singlet axial symmetry (5) is anomalous which shows up in the noninvariance of the fermionic measure. The measure $\prod_n \text{d}\psi_n \text{d}\bar{\psi}_n$ is obviously invariant, however, with respect to the ultralocal transformations (49). This follows from the fact that $\text{Tr}\{\gamma^5\} = 0$. On the
other hand, Eq. (13) relates the modification of the measure to the operator trace of $\gamma^5$,

$$\text{Tr}\{\gamma^5\} = \int d^4x \sum_k u_k^\dagger(x) \gamma^5 u_k(x)$$

which is thus zero too. For the naive Dirac operator, only the zero modes contribute to the sum (55) which means that the number of the right-handed and left-handed zero modes of $D^0$ should be equal. That also follows from the fact that $D^0$ commutes with $\hat{Q}_\mu$ defined in Eq. (11). It is not difficult to see that if $u_k$ is, say, right-handed, $\hat{Q}_{[\mu} \hat{Q}_{\nu]} u_k$ and $\hat{Q}_{[\mu} \hat{Q}_{\nu} \hat{Q}_{\lambda} \hat{Q}_{\rho]} u_k$ are right-handed too, but $\hat{Q}_\mu u_k$ and $\hat{Q}_{[\mu} \hat{Q}_{\nu} \hat{Q}_{\lambda} \hat{Q}_{\rho]} u_k$ are left-handed. Thus, instead of a single right-handed zero mode in (the lattice approximation for) the instanton background, we have a degenerate 16-plet with 8 right-handed and 8 left-handed modes, which are no more necessarily zero modes. The vanishing of the index of $D^0$ is closely related to the identity (43). Indeed, the sign of the Jacobian $\det \|\partial_\mu F_{\mu}(p)\|$ describes the orientation of a neighborhood of the preimage $P_i \in T^4$ with respect to the orientation of the tangent space $R^4$ onto which it is mapped. This orientation is obviously related to chirality.

The absence of the anomaly is one of the diseases of the naive lattice

Figure 2: The circle of eigenvalues for Neuberger’s operator. The eigenmodes with the eigenvalues marked by crosses are related by a $\gamma^5$ transformation.
Dirac operator. For the operators of the Ginsparg-Wilson type, the situation is different. Chiral symmetry is now implemented as in Eq. (51) and, generically, the measure is not invariant with respect to these transformations. We have, instead of Eq. (15)

\[ \ln J = i\alpha \text{Tr} \left\{ \gamma^5 \left( 1 - \frac{1}{2} aD \right) \right\}. \] (56)

Even though \( \text{Tr} \{\gamma^5\} = 0 \) (we derived this in the basis involving the eigenvalues of \( D \)), but it is true in any basis), \( \text{Tr} \{\gamma^5 aD\} \) need not vanish and the anomaly is there.

It is instructive to see how a nonzero operator trace in the right-hand side of Eq. (56) is obtained in the basis involving the eigenvalues of \( D \). First, it is still true that for any eigenfunction \( u_k \) of \( D \), \( \gamma^5 u_k \) is also an eigenfunction with the eigenvalue \( \lambda_k' = -\lambda_k^* \) (complex conjugation appears and this is what distinguishes the Ginsparg-Wilson case from the continuum or naive lattice Dirac operators). Thus, for almost all eigenstates on the circle in Fig. 2, \( u_k \) and \( \gamma^5 u_k \) have different eigenvalues, are orthogonal to each other, and the corresponding contribution vanishes. The only exception are two points on the circle: \( \lambda = 0 \) and \( \lambda = 2i/a \) where \( \lambda_k = \lambda'_k \), and the eigenstates can have a definite chirality. But the doublers \(-i\lambda = \frac{3}{2} \) obviously give zero contribution in Eq. (56). Only the zero modes of \( D \) are relevant. We have derived the lattice index theorem

\[ n^0_R(D) - n^0_L(D) = -\frac{1}{2} \text{Tr} \{\gamma^5 aD\}. \] (57)

The right-hand side of Eq. (57) presents a functional depending on the link variables \( \{U\} \). By definition, it presents a sum over all lattice nodes of some local expression. After some work, one can be convinced that it goes over to the topological charge in the continuum limit.

3 Continuum Theory: Some Exact Results

We abandon the lattice now and will discuss in this section various aspects of chiral symmetry in the continuum limit. Sometimes, we will think in terms of quarks, of symmetries (3) and (5), and of the order parameter (23) associated with the spontaneous breaking of the flavor-nonsinglet symmetry. Sometimes, we will describe the system in terms of the pseudo-Goldstone degrees of freedom and the effective Lagrangians (26), (28), and (30). Sometimes we will confront the two languages using the philosophy of the quark-hadron duality — that is how most of the results discussed in this section, a bunch of beautiful exact theorems of QCD, will be obtained.
3.1 QCD Inequalities. The Vafa-Witten Theorem

As was discussed above, the octet of pseudoscalar mesons ($\pi, K, \eta$) can be interpreted as that of the pseudo-Goldstone particles appearing due to the spontaneous chiral symmetry breaking according to the pattern (22) in the massless limit. That is the reason why the pseudoscalar mesons are lighter than those with other quantum numbers. It is interesting that the latter statement can be formulated as an exact theorem of QCD without any reference to the experimental! fact that the chiral symmetry is broken.

Consider a QCD-like theory with at least 2 quark flavors and assume that these quarks (denote them by $u$ and $d$) have equal masses $m_u = m_d = m$. Consider a set of Euclidean correlators

$$C_\Gamma(x, y) = \langle \bar{J}^{\bar{u}d}(x) J^{da}(y) \rangle_{\text{vac}} , \quad (58)$$

where $J^{\bar{u}d}(x)$ are flavor-changing bilinear quark currents $J^{\bar{u}d} = \bar{\Gamma} d$ with Hermitian

$$\Gamma = 1, \gamma^5, i\gamma_\mu, \gamma_\mu\gamma^5, i\sigma_{\mu\nu} .$$

At large distances, the correlators (58) decay exponentially

$$C_\Gamma(x, y) \propto \exp\{-M_\Gamma |x - y|\} \quad (59)$$

where $M_\Gamma$ is the mass of the lowest meson state in the corresponding channel.\footnote{We assume here that the quarks are confined, otherwise the whole discussion is pointless.}

On the other hand, the correlators (58) of the quark currents can be presented in the form

$$C_\Gamma(x, y) = -Z^{-1} \int d\mu_A \text{Tr} \{ \Gamma G_A(x, y) \Gamma G_A(y, x) \} , \quad (60)$$

where

$$d\mu_A = \prod_{x, \mu} dA_\mu(x) \prod_f \det ||m_f - i\not{D}|| \exp \left\{ -\frac{1}{4g^2} \int F_{\mu\nu}^a F_{\mu\nu}^a d^4x \right\} \quad (61)$$

is the standard QCD measure and $G_A(x, y)$ is the Euclidean Green function of the $u-$ and $d-$ quarks in a given gauge-field background. Note that, when writing down Eq. (60), we used the fact that $J^{\bar{u}d}$ is not a singlet in flavor [otherwise, the disconnected contribution $\propto \text{Tr} \{ \Gamma G_A(x, x) \} \text{Tr} \{ \Gamma G_A(y, y) \}$ would appear in the right-hand side]. In addition the assumption $m_u = m_d$ was made [otherwise, we would have two different Green’s functions $G^u_A(x, y) \neq G^d_A(x, y)$].
An important nontrivial relation
\[
\gamma^5 G_A(x, y) \gamma^5 = G_A^\dagger(y, x) \tag{62}
\]
holds. To understand it, write the spectral decomposition for \(G_A(x, y)\),
\[
G_A(x, y) = \langle \psi(x) | \bar{\psi}(y) \rangle^A = \sum_k \frac{u_k(x) u_\dagger_k(y)}{m - i\lambda_k}, \tag{63}
\]
(cf. Eq. (44)). Using the symmetry \(u_k \to \gamma^5 u_k, \quad \lambda_k \to -\lambda_k\), we obtain
\[
\gamma^5 G_A(x, y) \gamma^5 = \sum_k \left[ \gamma^5 u_k(x) | \gamma^5 u_k(y) \rangle \right] \frac{1}{m - i\lambda_k}
\]
\[
= \sum_p \frac{u_p(x) u_\dagger_p(y)}{m + i\lambda_p} \left[ \sum_p \frac{u_p(y) u_\dagger_p(x)}{m - i\lambda_p} \right]^\dagger = G_A^\dagger(y, x), \tag{64}
\]
as annonced. We see that the pseudoscalar correlator
\[
\sim \langle \text{Tr} \{ \gamma^5 G_A(x, y) \gamma^5 G_A(y, x) \} \rangle = \langle \text{Tr} \{|G_A(x, y)|^2 \} \rangle
\]
plays a distinguished role – it presents an absolute upper bound for any other such correlator. The fastest way to show this is to expand the \(4 \times 4\) matrix \(G_A(x, y)\) over the full basis
\[
G_A(x, y) = s(x, y) + \gamma^5 p(x, y) + i\gamma_\mu v_\mu(x, y) + \gamma_\mu \gamma^5 a_\mu(x, y)
\]
\[
+ \frac{1}{2} i\sigma_{\mu\nu} t_{\mu\nu}(x, y). \tag{65}
\]
Then
\[
-\frac{1}{4} C_{\gamma^5}^A(x, y) = \frac{1}{4} \text{Tr} \{ \gamma^5 G_A(x, y) \gamma^5 G_A(y, x) \}
\]
\[
= \frac{1}{4} \text{Tr} \{|G_A(x, y)|^2 \} = |s|^2 + |p|^2 + |v_\mu|^2 + |a_\mu|^2 + \frac{1}{2} |t_{\mu\nu}|^2, \tag{66}
\]
but, say
\[
-\frac{1}{4} C_1^A(x, y) = \frac{1}{4} \text{Tr} \{G_A(x, y) G_A(y, x)\} = \frac{1}{4} \text{Tr} \{G_A(x, y) \gamma^5 G_A^\dagger(x, y) \gamma^5 \}
\]
\[
= |s|^2 + |p|^2 - |v_\mu|^2 - |a_\mu|^2 + \frac{1}{2} |t_{\mu\nu}|^2. \tag{67}
\]
The inequalities

\[ |C^A_{\gamma}(x,y)| \geq |C^A_{\Pi}(x,y)| \]  \hspace{1cm} (68)

in any given gauge background, the positivity of the measure (61) in Eq. (60), and the asymptotics (63) imply that the mass \( M_{PS} \) of the (lightest) pseudoscalar meson in the \( ud \) channel is less or may be equal to the masses \( M_S, M_V, M_A, M_T \) of the lightest scalar, vector, axial, and tensor states.

Let us emphasize again that this statement can be justified only in the theory with the positive measure (61) [e.g. in the theory with nonzero vacuum angle \( \theta \neq 0 \), the measure is not positive and pseudoscalar states need not to be the lightest], with equal quark masses, and only for those states that are not flavor-singlet. For flavor-singlet states it need not be true. Consider, for example, the theory with just one quark of a large mass. Then the lowest meson states would be made of gluons and would know nothing about quarks. The lowest glueball state is believed to be scalar rather than pseudoscalar.

In Sec. 1, we have mentioned already the Vafa-Witten theorem saying that the vector isotopic symmetry is not broken spontaneously in QCD. Now we are ready to prove it. Indeed, if such a breaking occurred, the massless Goldstone scalar particles would appear in the spectrum. The inequality \( M_{PS} \leq M_S \) implies that a massless pseudoscalar particle would also exist. But the theory with \( m_u = m_d \neq 0 \) (which duly enjoys the exact isovector symmetry, a possible spontaneous breaking of which is under discussion now) has no exact axial isotopic symmetry, and there are no reasons for the massless pseudoscalar state to exist. So, it does not exist, hence the massless scalar does not exist either, and the isovector symmetry is not broken.

Many more inequalities of this kind (e.g. \( M_N \geq M_\pi \) or \( M_{\pi^+} \geq M_{\pi^0} \)) can be formulated, but their proof relies on some extra assumptions. We address the reader to Ref. 10 for a nice recent review (see also S. Nussinov’s contribution in this Volume).

### 3.2 Euclidean Dirac Spectral Density

Consider the Euclidean Dirac operator \( \mathcal{D} \) in a given gauge field background \( A_\mu(x) \). We assume that the system is placed in a finite 4-volume so that the spectrum of \( \mathcal{D} \) is discrete. Let \( \{\lambda_k\} \) be the background-dependent set of eigenvalues of \( \mathcal{D} \).

\footnote{We were a little bit sloppy here. The inequalities (68) hold strictly speaking only for tree correlators, not for renormalized ones. However, renormalization only brings about multiplicative factors, which do not depend on distance. Thus, taking \( |x-y| \) to infinity before the limit \( \Lambda_{UV} \to \infty \) is done, we can ensure that the inequalities (68) for renormalized correlators at large distances are fulfilled.}
eigenvalues of $\mathcal{D}$. The spectral density is defined as follows:

$$\rho(\lambda) = \frac{1}{V} \left( \sum_k \delta \left( \lambda - \lambda_k[A_{\mu}(x)] \right) \right),$$  \hspace{1cm} (69)$$

where the average is done with the weight function \((61)\). The $\gamma^5$ symmetry of the spectrum implies that $\rho(\lambda)$ is an even function of $\lambda$.

In contrast to solids or nuclei, the spectral density \((69)\) is defined in the Euclidean space and seems to have no direct physical meaning. There are, however, a set of remarkable identities which relate the spectral density of the Euclidean Dirac operator to physical observables. The simplest such identity relates the spectral density at “zero virtuality” $\lambda = 0$ to the quark condensate.

To derive it, set $x = y$ in the spectral decomposition \((63)\), integrate it over $\frac{1}{V} d^4x$, and perform the averaging over the gauge fields with weight \((61)\). In view of the definitions \((23)\), \((24)\), and \((69)\), using the symmetry $\rho(-\lambda) = \rho(\lambda)$, and assuming the reality of $\Sigma$, we obtain

$$\Sigma = \langle \sum_k \frac{1}{m - i\lambda_k} \rangle = \int_{-\infty}^{\infty} \frac{\rho(\lambda) d\lambda}{m - i\lambda} = 2m \int_{0}^{\infty} \frac{\rho(\lambda) d\lambda}{\lambda^2 + m^2}. \hspace{1cm} (70)$$

To understand better this formula, let us look first what happens for free fermions. As there is no physical dimensionfull scale in this case [remember that $\lambda_k$ in Eq. \((69)\) are eigenvalues of the massless Dirac operator], $\rho(\lambda) = C\lambda^3$ on dimensional grounds. By counting the eigenvalues of the free Dirac operator

$$\lambda(n_\mu) = \frac{2\pi}{L} \sqrt{\sum_\mu \left( n_\mu + \frac{1}{2} \right)^2} \hspace{1cm} (71)$$

[antiperiodic boundary conditions for the fermions in all 4 directions are chosen, $n_\mu$ are integer, and each level \((71)\) involves an extra $2N_c$-fold degeneracy] in the 4D ball $1/L \ll \lambda < \Lambda$, it is not difficult to determine $C = N_c/(4\pi^2)$. Thus,

$$\rho_{\text{free}}(\lambda) = \frac{N_c}{4\pi^2} \lambda^3. \hspace{1cm} (72)$$

The notion of spectral density and the definition \((69)\) are also widely used in condensed matter physics and nuclear physics. It is especially useful if a system is disordered or involves elements of disorder like it is the case for electron spectra in most solids or for energy levels in complicated nuclei. It makes sense also for ordered systems (such as metals). In this case, rather than averaging over stochastic external field, one averages over some interval of eigenvalues $\Delta\lambda$ much larger than characteristic level spacing, but much less than a characteristic scale of $\lambda$ on which $\rho(\lambda)$ is essentially changed.
In the interacting theory, the spectral density behaves as \( \rho(\lambda) \propto \lambda^3 \) for \( \lambda \) much greater than the characteristic hadron scale \( \mu_{\text{hadr}} \) so that interaction is weak. To be more precise, the power \( \lambda^3 \) can be multiplied by an anomalous dimension factor

\[
\sim \left( \ln \frac{\lambda}{\mu_{\text{hadr}}} \right)^\alpha.
\]

A recent one-loop calculation\(^1\) implies that \( \alpha \) is nonzero and negative.

We see that the integral in Eq. (70) diverges quadratically in the ultraviolet. The same result can be obtained directly by calculating the fermion bubble graph in the momentum representation

\[
\langle \bar{\psi}(0) \psi(0) \rangle = \int d^4p_E \left( \frac{p_E + m}{p_E^2 + m^2} \right) \propto m \Lambda^2_{\text{UV}}.
\]

Thus, strictly speaking, the formula (70) does not make much sense as it stands. Note, however, that even though the (purely perturbative) contribution (73) diverges in the ultraviolet, it vanishes in the chiral limit \( m \to 0 \). The whole point is that in QCD, the integral (70) acquires an additional nonperturbative contribution coming from the region of small \( \lambda \) which survives in the “continuum chiral thermodynamic limit” (first \( V \to \infty \), then \( m \to 0 \), and only then the ultraviolet cutoff is lifted \( \Lambda_{\text{UV}} \to \infty \)). The fact that chiral symmetry is broken spontaneously means that the vacuum expectation value \( \langle \bar{\psi}(0) \psi(0) \rangle \) is nonzero in this particular limit.

Obviously, the necessary condition for the condensate to develop is \( \rho(0) \neq 0 \). Neglecting all terms which vanish in the continuum chiral thermodynamic limit defined above, we obtain finally the famous Banks-Casher relation\(^2\)

\[
\langle \bar{\psi}(0) \psi(0) \rangle_{\text{vac}} \equiv \Sigma = \pi \rho(0).
\]

Note that the result does not depend on flavor which tells us again that the flavor vector symmetry is not broken.

Not only \( \rho(0) \), but also the form of \( \rho(\lambda) \) at small \( \lambda \ll \mu_{\text{hadr}} \) can be determined. Consider the theory with \( N_f \geq 2 \) light quarks of common mass \( m \). Let us study an integrated correlator in this theory,

\[
\int d^4x \langle S^a(x) S^b(0) \rangle = \frac{1}{V} \int d^4x d^4y \langle S^a(x) S^b(y) \rangle,
\]

where \( S^a(x) = \bar{\psi}(x) t^a \psi(x) \) and \( t^a \) is the generator of the SU\((N_f)\) flavor group. Fix a particular gluon background and define

\[
C^{ab} \big|_A = -\frac{1}{V} \int d^4x d^4y \text{Tr} \left\{ t^a G_A(x,y) t^b G_A(y,x) \right\}.
\]

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Substitute here the spectral decomposition (63) for $G_A(x, y)$, do the integration and perform averaging over the gluon fields trading the sum over eigenvalues for the integral over the spectral density (69). We obtain

$$C_{ab} = -\frac{\delta_{ab}}{2V} \left( \sum_k \frac{1}{(m - i\lambda_k)^2} \right) = -\frac{\delta_{ab}}{2} \int_{-\infty}^{\infty} \frac{\rho(\lambda) d\lambda}{(m - i\lambda)^2} =$$

$$- \frac{\delta_{ab}}{2} \int_0^{\infty} \frac{\rho(\lambda)(m^2 - \lambda^2)}{(m^2 + \lambda^2)^2} d\lambda,$$

where the property $\rho(-\lambda) = \rho(\lambda)$ was used.

On the other hand, the same correlator can be saturated by physical states among which the pseudo-Goldstone states play a distinguished role. Consider the 1-loop graph in Fig. 3 describing the contribution of the 2-Goldstone intermediate state $\sim \int \langle 0 | S^a \phi^c \phi^d \rangle \langle \phi^c \phi^d | S^b | 0 \rangle$ (obviously, the one-particle state does not contribute because the pseudo-Goldston mesons are pseudoscalars while $S^a(x)$ is scalar). To calculate it, we need to know the vertex $\langle 0 | S^a \phi^c \phi^d \rangle$ which can be determined via the generating functional of QCD involving scalar sources $u^a$ coupled to the current $S^a$. Adding the source term $u^a S^a$ to the Lagrangian amounts to adding $u^a t^a$ to the quark mass matrix $M$. The latter enters also the mass term (30) in the effective Lagrangian. Expanding $U$ up to the second order in $\phi^a$ and varying it with respect to $u^a$, we obtain

$$\langle 0 | S^a \phi^c \phi^d \rangle = -\frac{\Sigma}{F_\pi^2} d^{abc}.$$
The vertex is nonzero only for three or more flavors. Now we can calculate the graph in Fig. 3. Actually, we cannot because the integral diverges logarithmically in the ultraviolet, but anyway the effective theory is not valid at high momenta (technically, the divergence is absorbed into local counterterms of higher order in $\rho^{\text{chir}}$ and $m$). Only the infrared-sensitive part of the integral is relevant. A simple calculation gives

\[ (C^{ab})_{\text{infrared}} = -\frac{N_f^2 - 4}{32\pi^2 N_f} \left( \frac{\sum |F|^2}{F^2} \right)^2 \delta^{ab} \ln \frac{M^2}{\mu_{\text{hadr}}} \].

(79)

Compare it now with Eq. (77). Note first of all that the constant part $\rho(0)$ does not contribute here,

\[ \int_0^\infty \frac{m^2 - \lambda^2}{(m^2 + \lambda^2)^2} d\lambda = 0 \].

Thus, only the difference $\rho(\lambda) - \rho(0)$ is relevant. It is easy to see that, in order to reproduce the singularity $\sim \ln M^2 \sim \ln m$, we should have $\rho(\lambda) - \rho(0) = C|\lambda|$ at small $|\lambda|$. Substituting it in Eq. (77) and comparing the coefficient of $\ln m$ with the coefficient of $\ln M^2$ in Eq. (79), we finally obtain

\[ \rho(\lambda) = \frac{\Sigma}{\pi} + \frac{N_f^2 - 4}{32\pi^2 N_f} \left( \frac{\Sigma \sum |F|^2}{F^2} \right)^2 |\lambda| + o(\lambda) \].

(80)

Thus, for $N_f \geq 3$, the spectral density has a nonanalytic “dip” at $\lambda = 0$. The behavior is smooth in the theory with two light flavors. Physically, it is rather natural that the more is number of flavors, the stronger is the suppression of $\rho(0)$. The determinant factor in the measure (61) punishes small eigenvalues, and the larger is $N_f$, the more important is this factor. By “analytic continuation” of this argument, one should expect a nonanalytic bump rather than a nonanalytic dip at $\lambda = 0$ in the case $N_f = 1$. Indeed, Eq. (80) displays such a bump. One should not forget, of course, that the whole derivation was based on the effective chiral Lagrangian approach and does not directly apply to the case $N_f = 1$.

3.3 Chiral Symmetry Breaking and Confinement

Look again at Eq. (8). The axial current entering the left-hand side is an external current in a sense that no dynamical field is coupled directly to $j_{\mu 5}$. It is also confirmed by a numerical calculation in the instanton liquid model.
But the fields entering on the right-hand side are dynamical gluon fields present in the QCD Lagrangian.

In the chiral (left-right asymmetric) gauge theories like the standard electroweak model, both vector and axial currents are coupled directly to the physical gauge fields. Anomalous divergence of such current would mean explicit breaking of the gauge invariance which is not nice. Therefore, in chiral theories, one should always take care that such purely internal anomalies would cancel out at the end of the day. In the Standard Model, they do.

Let us discuss, however, purely external anomalies in QCD which are not related to breaking of any symmetry but just mean that certain correlators involving external currents are singular.

As a simplest nontrivial example, consider the theory with two massless flavors and look at the correlator

$$K^{AB}_{\mu \nu} \mathcal{H}(q) = i \int \langle T\{ j^A_{\mu}(x) j^B_\nu(0) \} \rangle \mathcal{H} \ e^{i q \cdot x} d^4 x ,$$  \hspace{1cm} (81)

where $A, B$ are flavor indices and $\mathcal{H}$ is the external flavor-singlet “magnetic field.” The correlator (81) is nothing but a three-point vacuum expectation value (7) in kinematics in which one of the external momenta associated with the vector current is set to zero.

The one-loop calculation of the corresponding graph displays a singularity,

$$K^{AB}_{\mu \nu} \mathcal{H}(q) = - \frac{\mathcal{H}}{2\pi^2} \frac{q_\mu \varepsilon_{\nu\alpha} q^\alpha}{q^2} \cdot N_c \cdot \frac{1}{2} \delta^{AB} .$$  \hspace{1cm} (82)

The imaginary part of this amplitude is also singular, $\sim \delta(q^2)$, which can be related to the masslessness of quarks. However, the quarks (in contrast to electrons in QED) do not exist as physical particles due to confinement and one can ask where does the singularity in the imaginary part $\text{Im} \ K^{AB}_{\mu \nu} \mathcal{H}(q) \propto \delta(q^2)$ come from? This is a good question, the answer is better still: the singularity $\sim 1/q^2$ comes from the propagator of a massless Goldstone boson, which appears due to the spontaneous chiral symmetry breaking and is directly coupled to the axial current $j_\mu 5$ (see the middle graph in Fig. 4).

Let us ask now: can one reproduce the singularity in Eq. (82) without Goldstone bosons and without spontaneous chiral symmetry breaking, but in some other way?  

---

1. The conventional anomaly (6) is related to the correlator (5) involving both internal ($j^a_\mu$) and external ($j^5_\mu$) currents and can be called mixed in this setting.

---

2. The quotation marks distinguish $\mathcal{H}$ which couples to the baryon charge from the physical magnetic field which has the matrix structure $\text{diag}(2/3, -1/3)$ and is a mixture of isotriplet and isosinglet. But we are not interested in dynamics of electromagnetic or weak currents here. In QCD proper, all color-singlet currents are external. $\mathcal{H}$ is just a source of such vector flavor-singlet current.
As far as the theory with two light quarks is concerned, the answer is positive: the singularity of the correlator above can be reproduced, in principle, if massless baryons are present. Proton and neutron represent, like quarks, a flavor SU(2) doublet. There are $N_c = 3$ quark doublets and only one baryon doublet $|P⟩ = |uud⟩$ and $|N⟩ = |udd⟩$. The absence of the overall $N_c$ factor is compensated, however, by the fact that the baryon charge of nucleon is 3 times larger than that of the quark, and the vertex involving the “magnetic field” $H$ is 3 times larger for baryons.

Thus, this purely algebraic anomaly matching argument due to 't Hooft does not rule out a dynamical scenario where the physical spectrum in the theory with just two massless quark flavors would not involve massless pions, but, instead, the massless proton and neutron. It is rather remarkable that, in the theories with $N_f \geq 3$, the scenario with massless baryons is ruled out. Suppose that, instead of the octet of massless Goldstone fields, we have an octet of massless baryons. The contribution of the corresponding triangle graph in (81) would have the same structure as in Eq. (82), but with the factor

$$\text{Tr} \{T^a T^b\} = C_8 \delta^{ab}$$

(83)

instead of $\delta^{ab}/2$, where $T^a$ are the flavor generators in the octet representation. To find the Dynkin index of the octet representation $C_8$, it is sufficient to assume that $a = b = 1$ and decompose the octet with respect to the SU(2) flavor subgroup: $8 = 3 + 2 + 2 + 1$. The contribution of each doublet to $C_8$ is 1/2 and the contribution of the triplet is 2. Adding it together, we obtain $C_8 = 3 \neq 1/2$ and the required result (82) is not reproduced. Also a massless decuplet and all other possible color-singlet baryon representations would give the coefficient in front of the singularity much larger than that in Eq. (81), and the anomaly matching condition would not be fulfilled. Therefore, massless
baryons do not exist.\footnote{To be quite precise, one could, in principle, saturate the anomaly with several baryon multiplets with positive and negative baryon charges. This possibility is so unaesthetic, however, that it can be rejected by that reason.}

We have arrived at a remarkable result. In QCD with three massless quarks, the assumption of confinement allowing the existence of only colorless states in the physical spectrum and the anomaly matching condition lead to the conclusion that massless Goldstone states must appear and chiral symmetry should be spontaneously broken. If the latter is not true, the only possibility to saturate the anomaly is to assume that massless quarks still exist as physical states in the spectrum and there is no confinement!

In the real world, confinement and spontaneous chiral symmetry breaking in the limit of massless $u$, $d$, and $s$ quarks are experimental facts. Whether or not these phenomena take place in hypothetic theories with $N_f \geq 4$ is an open question. It is quite possible that starting, say, from $N_f = 6$, the small eigenvalues in the Dirac operator spectrum are punished by the determinant factor so strongly that $\rho(0) = 0$ and, in view of the Banks-Casher relation (74), the quark condensate vanishes, and the symmetry is not broken. The anomaly matching argument tells then that there is no confinement in this case.\footnote{We know, of course, that if the number of the quark flavors is very high $N_f > 16$, the asymptotic freedom is lost and we cannot expect confinement. We are almost sure that quarks and gluons are not confined at $N_f = 16$ or $N_f = 15$, in which case the theory is asymptotically free, but has an infrared fixed point at a small value of $\alpha_s$, and the coupling constant never grows large. There are some reasons to believe, however, that confinement is lost at a smaller value of $N_f$, not just $N_f = 15.$}

We have called this result – that the chiral symmetry breaking and confinement go together – remarkable. It is also somewhat mysterious. Even though we do not understand well dynamical reasons for confinement to occur, we still expect that the same mechanism which works for the theory with $N_f = 3$ works also for the theory with $N_f = 2$. But ’t Hooft’s argument works only for $N_f \geq 3$...

4 Mesoscopic QCD

To make the path integral finite-dimensional and the spectrum of the Dirac operator discrete, not only an ultraviolet lattice regularization should be introduced, but one also has to consider the theory in a finite volume. If the volume is very large, much larger that the pion Compton wavelength, the dynamics of the theory is basically the same as that in the continuum limit. However, in practical calculations characteristic lattice sizes are not too large, and finite volume effects are essential. It is therefore important to estimate theoretically
these effects. In this section, we derive a set of exact relations for QCD placed in a finite Euclidean box. They rely heavily on the presence of the chiral symmetry and on the notion of quark-hadron duality used in the previous section. We will always assume that the size of the box $L$ is large $L \gg \mu_{\text{hadr}}^{-1}$, but the requirement $L \gg M_{\pi}^{-1}$ is not imposed.

4.1 Partition Function: $N_f = 1$

Consider first the theory with just one light quark flavor. We assume that the quark mass $m$ is complex so that the mass term in the Lagrangian has the form

$$L_{N_f=1} = m \bar{\psi}_R \psi_L + \bar{m} \bar{\psi}_L \psi_R ,$$

and the vacuum angle $\theta$ is nonzero. The partition function of the theory can be written in the form

$$Z(\theta) = \exp\{-V_\epsilon(\theta, m, L)\},$$

where $V = L^4$ and $\epsilon(\theta, m, L)$ tends to the vacuum energy density $\epsilon_{\text{vac}}(\theta, m)$ in the thermodynamic limit $L \to \infty$. It is important that the finite volume corrections $\epsilon(\theta, m, L) - \epsilon_{\text{vac}}(\theta, m)$ are exponentially suppressed $\propto \exp\{-\mu_{\text{hadr}} L\}$. Indeed, the corrections appear due to modification of the spectrum of excitations in the finite box by the Casimir mechanism. At the technical level, they are due to modification of the corresponding Green functions by finite volume effects. But in the theory with $N_f = 1$, all physical (meson and baryon) excitations are massive and their Green functions decay as $\exp\{-\mu_{\text{hadr}} R\}$ at large distances. Also the modification of the Green functions (at coinciding points) due to boundary effects has the order $\exp\{-\mu_{\text{hadr}} L\}$.

The vacuum energy depends on the quark mass $m$ and vacuum angle $\theta$. It turns out that $\epsilon_{\text{vac}}$ is actually a function of just one complex parameter which is $m e^{i \theta}$. To see that, express $Z(\theta)$ as a series

$$Z(\theta) \sim \sum_{q=-\infty}^{\infty} e^{-i q \theta} \int dA_{\mu}^a(\tau, x) \det \left[ -i D + m \frac{1 - \gamma^5}{2} + \bar{m} \frac{1 + \gamma^5}{2} \right]$$

$$\times \exp \left\{ -\frac{1}{4g^2} \int_0^{\beta} d\tau \int d\chi \left[ (F_{\mu\nu}^a)^2 \right] \right\} \equiv \sum_{q=-\infty}^{\infty} e^{-i q \theta} Z_q ,$$

where $Z_q$ is given by the functional integral over the quark and gluon fields in the sector of definite topological charge $q$. Consider some particular $Z_q$ and
suppose for definiteness that \( q > 0 \). One of the factors entering the integrand of \( Z_q \) is the quark determinant,

\[
\det \left( -i\gamma^5 - \frac{1}{2}\bar{\psi} + \frac{1}{2}\gamma^5 \psi \right) = \tilde{m}^q \prod_k \det \left( \begin{pmatrix} \bar{\psi} & \lambda_k \end{pmatrix} \right)
\]

where the product \( \prod_k \) runs over nonzero \( \lambda_k \)'s, and the factor \( \tilde{m}^q \) reflects the presence of \( q \) right-handed fermion zero modes as the index theorem (16) requires [cf. Eq. (17) written earlier for the case of real masses]. If \( q < 0 \), the factor \( \tilde{m}^q \) should be substituted by \( \tilde{m}^{-q} \). Averaging (87) over the gluon field configurations with a given \( q \) and substituting it in Eq. (86), we obtain that each term in the sum depends, indeed, only on the combinations \( z = \frac{m}{e^{-\theta}} \) and \( \bar{z} \) (being real, the partition function and \( \epsilon_{\text{vac}} \propto \ln Z \) cannot, of course, be holomorphic in \( z \)).

Next we assume that \( |m| \ll \mu_{\text{hadr}} \), that the dependence of \( \epsilon_{\text{vac}} \) on mass is analytic, and expand

\[
\epsilon_{\text{vac}} = \epsilon_0 - \Sigma \Re(e^{i\theta}) + o(m),
\]

where \( \Sigma \) is a real constant which, for \( \theta = 0 \) and real \( m \), coincides with the variation of the vacuum energy with respect to mass, i.e. with the quark condensate \( \langle \bar{\psi}\psi \rangle_{\text{vac}} \). Substituting (88) in Eq. (87) and comparing it with the Fourier expansion (86), we can derive

\[
Z_q \propto \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{i\theta} e^{V \Sigma m \cos \theta + o(m)} = I_q(m\Sigma V)
\]

\( [I_q(x) = I_{-q}(x) \) stands for the Bessel functions which grow exponentially at large real \( x \).

It is very instructive now to consider two limits: (i) \( m\Sigma V \gg 1 \) and (ii) \( m\Sigma V \ll 1 \) (keeping the assumption \( L\mu_{\text{hadr}} \gg 1 \)). The limit (i) is a genuine thermodynamic limit. The sum (86) is saturated by many terms with large values of \( |q| \). Actually, one easily derives (for any \( m\Sigma V \))

\[
\langle q^2 \rangle_{\text{vac}} = m\Sigma V.
\]

\( m \) A finite nonzero value of the quark condensate was exactly our assumption when we wrote the expansion (88).
The quantity $\chi = \langle q^2 \rangle_{\text{vac}} / V$ can also be expressed via the correlator of topological charge densities,

$$\chi = \int d^4 x \left\langle \frac{1}{16\pi^2} \text{Tr}\{F_{\mu\nu} \tilde{F}_{\mu\nu}\}(x) \right\rangle \frac{1}{16\pi^2} \text{Tr}\{F_{\mu\nu} \tilde{F}_{\mu\nu}\}(0), \quad (91)$$

and is called the topological susceptibility of the vacuum. Taking a double variation of Eq.(86) with respect to $\theta$, we relate it to the curvature of the function $\epsilon_{\text{vac}}(\theta)$ at $\theta = 0$:

$$\chi = - \lim_{V \to \infty} \frac{1}{V} \frac{\partial^2 Z(\theta)}{\partial \theta^2} \bigg|_{\theta=0} = \frac{\partial^2 \epsilon_{\text{vac}}(\theta)}{\partial \theta^2} \bigg|_{\theta=0}. \quad (92)$$

The region

$$\mu_{\text{hadr}}^{-1} \ll L \lesssim (m\Sigma)^{-1/4} \quad (93)$$

can be called mesoscopic. The word is borrowed from the condensed matter physics where mesoscopic system is defined as a system involving a large enough number of particles for the statistical description to be adequate, but which is not sufficiently large for the boundary effects to be irrelevant. The same is true in our case. In particular, when $m\Sigma V \ll 1$, the partition function is well approximated by just the first term $q = 0$ and, if we are interested in the mass dependence, — by the terms $q = \pm 1$ corresponding to the instanton and anti-instanton sectors.

If $m\Sigma V \ll 1$, the fermionic condensate is provided by instanton-like configurations supporting a single fermion zero mode. If $m\Sigma V \gg 1$, the condensate appears due to finite average density of small but nonzero eigenvalues of the Dirac operator, according to the Banks-Casher formula (74). Quite remarkably, the value of the condensate is the same, however, and does not depend on the parameter $m\Sigma V$.

Note that, if $m$ is real and $\theta = 0$ (more generally, if the combination $\theta_{\text{phys}} = \theta + \arg(m)$ on which everything depends vanishes), all terms in the partition function (86) are positive and the path integral for $Z(\theta)$ has a probabilistic interpretation. And if not – then not.

4.2 Partition Function: $N_f \geq 2$

The spectrum of the theory with several light flavors involves light pseudo-Goldstone states. If the length of our box is comparable with the Compton wavelength of pseudo–Goldstone particles, finite volume corrections in
\(\epsilon(\theta, m, L)\) in Eq. (85) are not exponentially small and should be taken into account. We will be interested here in the region

\[
\mu^{-1}_{\text{hadr}} \ll L \ll \frac{1}{M_\pi} \sim \frac{1}{\sqrt{m\mu_{\text{hadr}}}}.
\]

(94)

The scale \(\sim m^{-1/4}\mu^{-3/4}_{\text{hadr}}\) entering Eq. (93) and lying in the middle of the newly defined mesoscopic interval (94) is relevant in the multiflavor case too. The formulae we are going to derive in this section will be universally valid, however, for any value of \(m\Sigma V\) as long as the condition (94) is satisfied.

The main idea is to present the partition function as the path integral in the effective theory (26), (30) and notice that, when the condition \(LM_\pi \ll 1\) is fulfilled, only the zero Fourier harmonics of the pseudo–Goldstone fields are relevant. Thus, we can disregard the kinetic term, and the path integral is reduced to an ordinary one. We want to calculate the partition function at arbitrary \(\theta\) and need to know the form of the effective potential at arbitrary \(\theta\).

In the case \(N_f = 1\) all physical quantities depended on the combination \(m_\epsilon\theta\), but not on \(m\) and \(\theta\) separately. By the same token, for \(N_f \geq 2\), they depend on the combination \(M e^{i\theta/N_f}\) (where \(M\) is a complex mass matrix). The proof is quite analogous and is based on the multiflavor version of Eq. (87),

\[
\det \left\{ -i\gamma^\nu + \mathcal{M} \frac{1 - \gamma^5}{2} + \mathcal{M}^\dagger \frac{1 + \gamma^5}{2} \right\} = \left[ \det \|M\| \right]^q \times \prod_k \left\| M^\dagger \frac{-i\lambda_k}{\mathcal{M}} \right\| = \left[ \det \|M\| \right]^q \prod_k \det \|\lambda_k^2 + M\mathcal{M}\|.(95)\]

Similar to the case \(N_f = 1\), we see that the physical vacuum angle on which all physical quantities depend is

\[
\theta_{\text{phys}} = \theta + \arg(\det \|M\|),
\]

(96)

rather than just \(\theta\). In particular, the statement made earlier that the vacuum angle in QCD is either zero or very small refers to the combination (96).

Thus, we can write

\[
Z_{N_f}(\theta) = \int dU \exp \left\{ V \Sigma \text{Re} \left[ \text{Tr} \{ M e^{i\theta/N_f} U^\dagger \} \right] \right\},
\]

(97)

One can observe it by looking at the pseudo-Goldstone Green function in finite volume,

\[
G_\pi(x) = \frac{1}{V} \sum_{(n_\mu)} \frac{e^{ipx}}{M_\pi^2 + \left( \frac{2\pi n_\mu}{L} \right)^2}.
\]
where the integral is done over the group SU($N_f$) with the proper Haar measure. Integrating it further over $d\theta$ as in Eq. (89), we find the partition function in the sector with a definite topological charge $Z_q$. In the simple case $M = m\mathbf{1}$, a beautiful analytic expression for $Z_q$ can be derived,

$$Z_q = \text{det} \begin{pmatrix} I_q(\kappa) & I_{q+1}(\kappa) & \ldots & I_{q+N_f-1}(\kappa) \\ I_{q-1}(\kappa) & I_q(\kappa) & \ldots & I_{q+N_f-2}(\kappa) \\ \vdots & \vdots & \ddots & \vdots \\ I_{q-N_f+1}(\kappa) & \ldots & \ldots & I_q(\kappa) \end{pmatrix}, \quad (98)$$

where $\kappa = |m\Sigma V|$. Let us assume $\theta = 0$, $m$ real, and let us study the quantity

$$\langle \bar{\psi} \psi \rangle_{m,V} = \frac{1}{N_fV} \frac{\partial}{\partial m} \ln Z, \quad (99)$$

the quark condensate at finite mass and finite volume. In the limit $\kappa \to \infty$, the integral in Eq. (97) has the main support in the region $\Upsilon \sim 1$, and we obtain

$$Z_{N_f} \propto e^{\frac{N_f\kappa}{\kappa(N_f^2-1)/2}}, \quad (100)$$

so that the condensate (99) tends to a constant $\Sigma$ as it should.

In the opposite limit $\kappa \ll 1$, $Z_{q \neq 0}$ are suppressed compared to $Z_0$ as $\kappa |q| N_f$ reflecting the presence of $N_f |q|$ fermion zero modes in the sector with the topological charge $q$. The condensate (99) is also suppressed $\propto \kappa$. This is due to the fact that, in the multiflavor case, the condensate plays the role of the order parameter signaling the spontaneous chiral symmetry breaking. Strictly speaking, spontaneous symmetry breaking never occurs in quantum systems at finite volume in the sense that the true ground state wave function is always symmetric even though the gap separating it from excited asymmetric states may be very (exponentially) small. So, for $N_f > 1$, the condensate should vanish in the chiral limit $m \to 0$ no matter how large the fixed volume $V$ is. And it does.

\footnote{Note the presence of the power-like preexponential factor in Eq. (100). It reflects the fact that, while the condition $L M_p \ll 1$ is satisfied and higher Fourier harmonics of the pion fields are decoupled, we are not in the true thermodynamic limit yet. In the latter, the partition function is given by a simple exponential as in Eq. (85), and the finite-volume corrections to $\epsilon(\theta, m, L)$ are suppressed as $\exp(-M_p L)$.}

\footnote{Never say never: the statement above is not correct for supersymmetric systems. But QCD is not supersymmetric and we may forget this subtlety.}
4.3 Spectral Sum Rules

Let us return to a simpler case, $N_f = 1$. Assume that $m$ is real and $q$ is positive, and rewrite the result (89) as

$$
\left\langle m^q \prod_k (\lambda_k^2 + m^2) \right\rangle_q = CI_q(\kappa), \quad (101)
$$

where the averaging is done over the gluon field configurations with a given topological charge $q$. We can expand now the two sides of Eq. (101) in $m$ and compare the coefficients of the expansion. The first nontrivial relation is obtained when comparing the terms $\propto m^q$ and $\propto m^{q+2}$. We obtain

$$
\left\langle \sum_k 1 \lambda_k^2 \right\rangle_q = \frac{(\Sigma V)^2}{4(q+1)}, \quad (102)
$$

where the sum runs over the nonzero eigenvalues and the symbol $\langle \cdots \rangle_q$ means averaging with weight $\exp\{-S_g\} \prod_k \lambda_k^2$ (the second factor is the fermionic determinant in the limit $m \to 0$, with the discarded zero modes).

Expanding Eq. (101) up to the order $\sim m^q + 4$, we obtain the following sum rule:

$$
\left\langle \left( \sum_{k \neq \ell} \frac{1}{\lambda_k^2} \right)^2 \right\rangle_q = \frac{(\Sigma V)^4}{16(q+1)(q+2)}, \quad (103)
$$

Similar relations can be derived in the multiflavor case. Expanding Eq. (88) in mass and comparing the terms $\sim m^{N_f q}$, $\sim m^{N_f q+2}$, and $\sim m^{N_f q+4}$, we obtain the same relations (102), (103), with $q$ being substituted by $q + N_f - 1$. We can extract more information by considering the expression for the partition function with different quark masses. Two different sum rules can be derived by considering the terms $\sim m^{N_f q+4}$ of the expansion,

$$
\left\langle \left( \sum_k \frac{1}{\lambda_k^2} \right)^2 \right\rangle_q = \frac{(\Sigma V)^4}{16[(q + N_f)^2 - 1]},
$$

$$
\left\langle \sum_k 1 \lambda_k^2 \right\rangle_q = \frac{(\Sigma V)^4}{16(q + N_f)[(q + N_f)^2 - 1]}, \quad (104)
$$

Subtracting one sum rule in Eq. (104) from the other and setting $N_f = 1$, we reproduce Eq. (103), but the relations (104) are valid only for $N_f \geq 2$. 
Actually, the average $\langle \sum_k (1/\lambda_k^4) \rangle_q$ diverges at $N_f = 1$. For $N_f = 2$, there are two sum rules at each level. For $N_f = 3$, three different sum rules can be derived starting from the terms $\sim m_{N_f+6}$ of the expansion, etc.

The sum rules (102) can be presented as the integrals

$$V \int \frac{\rho(\lambda)}{\lambda^2} \, d\lambda,$$

where $\rho(\lambda)$ is the microscopic spectral density in the sector with a given topological charge $q$. The sum rules (103) and the first sum rule in Eq. (104) are not expressed in terms of $\rho(\lambda)$ only, but also via a certain integral involving the correlation function $\rho(\lambda_1, \lambda_2)$. The sum rule for $\langle \sum_k (1/\lambda_k^2 \lambda_k^2 \lambda_k^2) \rangle_q$ is expressed in terms of an integral of the correlator $\rho(\lambda_1, \lambda_2, \lambda_3)$, etc. Using ingenious arguments which are beyond the scope of our discussion here, J. Verbaarschot and collaborators managed to determine the functional form of $\rho(\lambda)$ and of all relevant correlators. For example,

$$\rho_{N_f,q}(\lambda) = \frac{\Sigma V |\lambda|}{2} \left[ J_{N_f+|q|}^2 (\Sigma V \lambda) - J_{N_f+|q|+1} (\Sigma V \lambda) \right]. \quad (105)$$

The expression is valid for $\lambda \ll \mu_{\text{hadr}}$, and the function is “alive” in the region of very small eigenvalues $\lambda \sim 1/\Sigma V$. In the thermodynamic limit $\lambda \Sigma V \to \infty$, the function (105) tends to the constant $\Sigma/\pi$ in agreement with the theorem (74).

Spectral sum rules are well adapted to be confronted with the numerical lattice calculations. The main interest here is not so much to “confirm” these exact theoretical results by computer, but, rather, to test lattice methods. This was a challenge for lattice people for some time. By now reasonably good numerical data were obtained, they agree well with the theory. One can expect that the accuracy will grow substantially if the algorithms based on the lattice fermion action with exact chiral symmetry (as was discussed in detail in Sec. 2) are implemented.

5 QCD at finite $\theta$

We know that in the real world the gauge group is $SU(3)$ and we have 6 fundamental quarks with some particular mass values. We also know that $\theta_{\text{phys}}$ defined in (10) is very close to zero. It presents a considerable theoretical interest, however, to study what happens in the same theory, but with other
values of the parameters. Though these “fairy”\footnote{This word was coined by chess problemists. A fairy chess problem is a problem referring to a game with modified rules or on a nonstandard board.} variants of the theory do not have a direct relation to reality, in order to understand well the physics of our world, it is important to understand also how it varies if the rules of the game are changed.

One of the ways to modify the theory is to assume a nonzero $\theta_{\text{phys}}$. We will not assume that the quark masses coincide with their experimental values, but will keep them (or rather some of them) small enough for the chirality considerations to be relevant. The expressions for the partition function in finite volume as a function of $\theta$ were written in Sec. 3. In this section, we will concentrate, however, not on the mesoscopic regime $m\Sigma V \sim 1$, but on dynamics of the theory in the thermodynamic limit $m\Sigma V \gg 1$.

The vacuum energy at finite $\theta$ (its mass-dependent part) is obtained by minimizing the effective chiral potential over $U$,

$$
\epsilon_{\text{vac}}(\theta) = -\Sigma \min_U \left[ \text{Re} \text{ Tr} \left\{ \mathcal{M} e^{i\theta/N} U^\dagger \right\} \right].
$$

(106)

In the two-flavor case, a nice analytic expression for $\epsilon_{\text{vac}}(\theta)$ can be written for a generic mass matrix $\mathcal{M}$,

$$
\epsilon_{\text{vac}}(\theta) = -\Sigma \sqrt{\text{Tr} \{ \mathcal{M}^\dagger \mathcal{M} \} + e^{i\theta} \det \| \mathcal{M} \| + e^{-i\theta} \det \| \mathcal{M} \|}.
$$

(107)

In the general case it is a continuous smooth function of $\theta$. It is, of course, also periodic: $\epsilon_{\text{vac}}(\theta + 2\pi) = \epsilon_{\text{vac}}(\theta)$. The case of degenerate masses is more subtle. Substituting $\mathcal{M} = m1$ (with real $m$) in Eq. (107) we obtain

$$
\epsilon_{\text{vac}}(\theta) = -2m\Sigma \left| \cos \frac{\theta}{2} \right|.
$$

(108)

This function is still periodic in $\theta$, but is obviously singular at $\theta = \pi$. A mathematical reason for this is rather simple. The function (107) has two stationary points: a minimum and a maximum. At $\theta = \pi$, they fuse together. The maximum and minimum of energy are plotted in Fig. 5. We see that two smooth curves $\propto \pm \cos(\theta/2)$ cross at the point $\theta = \pi$ and, after passing this point, the maximum and minimum are interchanged. While $\epsilon_{\text{vac}}(\theta)$ is determined by the curve $-2m\Sigma \cos(\theta/2)$ at $\theta < \pi$, it is determined by the curve $2m\Sigma \cos(\theta/2)$ for $\theta > \pi$.

Physically, this is the situation of the second order phase transition: the energy is a continuous function of the parameter $\theta$, but its derivative is not.
A more detailed analysis shows, however, that the conclusion of the existence of the second order phase transition is an artifact of the approximation used. Indeed, for $N_f = 2$, the potential in Eq. (106) just vanishes identically at the point $\theta = \pi$. That means that the pion excitations become exactly massless and chiral symmetry is restored.

It is clear from physical considerations that, no matter what the value of $\theta$ is, the symmetry (21) is broken explicitly if the quarks are massive. For $\theta = \pi$ the leading term vanishes, but the breaking is still there being induced by the terms of higher order in mass in the chiral effective potential. The relevant term has the structure $\propto m^2 \sin^2(\theta/2)\text{Tr}\{U^2\}$.

One can show that taking this into account modifies the curves in Fig. 5 in the vicinity of $\theta = \pi$ so that, at the point $\theta = \pi$, one has two exactly degenerate vacuum states separated by a low but nonvanishing barrier. If $\theta$ is slightly less than $\pi$, the degeneracy is lifted, and we have a metastable vacuum state on top of the true vacuum. When $\theta$ slightly exceeds $\pi$, the roles of these

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$^7$Another way to understand it is to look at the Gell-Mann-Oakes-Renner relation (31). Equation (31) was originally written for $\theta = 0$, but a generalization for arbitrary $\theta$ is rather straightforward. In our case, we even do not need it: the point $\theta = \pi$, $m_u = m_d$ is equivalent to the point $\theta = 0$, $m_u = -m_d$ by virtue of (30), and we see that the pion mass vanishes.
two states is reversed: the one which was stable before becomes metastable, and the other way around. And this is exactly the physical picture of the first order phase transition with superheated water, supercooled vapor and all other familiar paraphernalia.

Consider now the theory with three degenerate light flavors. By a conjugation $U ightarrow RUR^\dagger$, any unitary matrix $U$ can be rendered diagonal,

$$U = \text{diag}(e^{i\alpha}, e^{i\beta}, e^{-i(\alpha+\beta)}).$$

When $\mathcal{M} = m\mathbf{1}$, the conjugation does not change the potential in Eq. (106). For diagonal $U$, the latter acquires the form

$$V(\alpha, \beta) = -m\Sigma \left[ \cos\left(\alpha - \frac{\theta}{3}\right) + \cos\left(\beta - \frac{\theta}{3}\right) + \cos\left(\alpha + \beta + \frac{\theta}{3}\right) \right]. \quad (109)$$

The function $U$ has six stationary points,

$$\begin{align*}
I & : \alpha = \beta = 0, & II & : \alpha = \beta = -\frac{2\pi}{3}, & III & : \alpha = \beta = \frac{2\pi}{3} \\
IV & : \alpha = \beta = -\frac{2\theta}{3} + \pi, & IVa & : \alpha = -\alpha - \beta = -\frac{2\theta}{3} + \pi, \\
IVb & : \beta = -\alpha - \beta = -\frac{2\theta}{3} + \pi. \quad (110)
\end{align*}$$

The points IVa and IVb are obtained from IV by the Weyl permutations $\alpha \leftrightarrow \beta, \alpha \leftrightarrow -\alpha - \beta$, etc, and their physical properties are the same. Actually, we have here not 3 distinct stationary points, but, rather, a four-dimensional manifold $\text{SU}(3)/[\text{SU}(2) \otimes \text{U}(1)]$ of the physically equivalent stationary points related to each other by conjugation. The values of the potential at the stationary points are

$$\begin{align*}
\epsilon_I &= -3m\Sigma \cos\frac{\theta}{3}, & \epsilon_{II} &= -3m\Sigma \cos\frac{\theta + 2\pi}{3}, \\
\epsilon_{III} &= -3m\Sigma \cos\frac{\theta - 2\pi}{3}, & \epsilon_{IV} &= m\Sigma \cos \theta. \quad (111)
\end{align*}$$

Studying the expressions (111), and the matrix of the second derivatives of the potential (109) at $\theta = \pi$, one can readily see that i) the points I and III are degenerate minima separated by a barrier; ii) the point II is a maximum, and iii) the points IV are saddle points. The physical picture we arrive at is that of the first order phase transition. The situation is even simpler than in the
Figure 6: $N_f = 3$: Stationary points of $V(\alpha, \beta)$ for different $\theta$. The solid lines are minima, the dotted lines are maxima, and the dashed line are saddle points.

two-flavor case since the barrier between the minima appears in the leading order and the subleading $m^2$ terms in the potential is of no concern to us in the case at hand.

Figure 6 illustrates how the stationary points of the potential move when the vacuum angle is changed. At $\theta = \pi/2$, $\theta = 3\pi/2$, etc, the metastable minima coalesce with the saddle points and disappear. No trace of them is left at $\theta = 0$. One can show that also for the physical values of the quark masses the metastable vacua are absent at $\theta = 0$.

If the quark masses are not degenerate but their values are close, nice curves in Fig. 6 are distorted a little bit, but physics remains largely the same: first order phase transition is robust and cannot be destroyed by small variations of parameters. If the masses are essentially different, the phase transition may disappear. This is true, in particular, for physical values of the quark masses. In this case, the vacuum energy depends on $\theta$ analytically, as in Eq. (107). Unfortunately, no such simple formula for $\epsilon_{\text{vac}}(\theta)$ can be written for a generic mass matrix $M$ for three or more flavors.

What we can do easily and quite generally is to determine the behavior of the function $\epsilon_{\text{vac}}(\theta)$ at small $\theta$ and the topological susceptibility (92). Let us do it for arbitrary $N_f$ still assuming, for simplicity, that $M$ is real and diagonal.
Then $U$ can be taken to be diagonal too,

$$U = \text{diag} \left( e^{i\alpha_1}, \ldots, e^{i\alpha_{N_f}} \right), \quad \sum_{j=1}^{N_f} \alpha_j = 0.$$  

For $\theta \ll 1$, we anticipate that the vacuum values of $\alpha_j$ will also be small. We can write the effective potential as

$$V_{\text{eff}}(U) = -\sum_{j=1}^{N_f} m_j \cos \left( \alpha_j - \frac{\theta}{N_f} \right)$$

$$= \text{const} + \frac{\sum_{j=1}^{N_f} m_j}{2} \left( \alpha_j - \frac{\theta}{N_f} \right)^2 + \cdots$$  \hspace{1cm} (112)

Adding here a Lagrange multiplier term $\lambda \sum_{j=1}^{N_f} \alpha_j$ and minimizing the expression thus obtained over $\alpha_j$ and $\lambda$, we find

$$\alpha_j = \theta \left[ \frac{1}{N_f} - \frac{1}{m_j \sum_{j=1}^{N_f} m_j} \right].$$  \hspace{1cm} (113)

Substituting this result for $\alpha_j$ in Eq. (112), we finally obtain

$$\chi = m_{\text{eff}}^{-1} = \frac{\partial^2 E_{\text{vac}}(\theta)}{\partial \theta^2} \bigg|_{\theta=0} = \frac{1}{V} \langle q^2 \rangle_{N_f} = \sum \left[ \frac{1}{m_1} + \cdots + \frac{1}{m_{N_f}} \right]^{-1}.$$  \hspace{1cm} (114)

Roughly speaking, $\chi$ is proportional to the lightest quark mass (lightest quark masses). In actual QCD

$$\lambda_{\text{QCD}} = \frac{m_u m_d}{m_u + m_d} \Sigma;$$  \hspace{1cm} (115)

the strange and other quarks are too heavy to be relevant. If $M = m_{\Sigma}$, Eq. (114) is reduced to $\chi = m_{\Sigma}/N_f$. The latter result can be easily reproduced by substituting the asymptotics for $Z_q$ in the definition

$$\langle q^2 \rangle = \frac{\sum_q q^2 Z_q}{\sum_q Z_q}.$$
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