Polygonal Derivation of the Neutrino Mass Matrix

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Abstract

Representations of the symmetry group $D_n$ of the $n$–sided regular polygon have generic multiplication rules if $n$ is prime. Using $D_n$ with $n = 5$ or greater, a particular well-known form of the Majorana neutrino mass matrix is derived.

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The form of the $3 \times 3$ Majorana neutrino mass matrix $\mathcal{M}_\nu$ has been the topic of theoretical study for some time. If $\mathcal{M}_\nu$ has less than the full 6 parameters, then there exists at least one relationship among masses and mixing angles, which may be tested against the increasingly more precise experimental data from neutrino oscillations. However, even if such a comparison is successful, the question still remains as to why it has such a form. A possible answer is that it comes from an underlying symmetry. In this paper, it is shown how

$$\mathcal{M}^{(e, \mu, \tau)}_\nu = \begin{pmatrix} a & c & d \\ c & 0 & b \\ d & b & 0 \end{pmatrix}$$

(1)

may be derived from $D_n$, the symmetry group of the $n$–sided regular polygon, where $n$ is a prime number, equal to or greater than 5.

Consider $D_5$, the symmetry group of the regular pentagon. It has 10 elements, 4 equivalence classes, and 4 irreducible representations. Its character table is given by

Table 1: Character Table of $D_5$. 

| class | $n$ | $h$ | $\chi_1$ | $\chi_2$ | $\chi_3$ | $\chi_4$ |
|-------|----|----|--------|--------|--------|--------|
| $C_1$ | 1  | 1  | 1      | 1      | 2      | 2      |
| $C_2$ | 5  | 2  | 1      | $-1$   | 0      | 0      |
| $C_3$ | 2  | 5  | 1      | 1      | $\phi - 1$ | $-\phi$ |
| $C_4$ | 2  | 5  | 1      | 1      | $-\phi$ | $\phi - 1$ |

Here $n$ is the number of elements and $h$ is the order of each element. The number $\phi$ is the Golden Ratio (or Divine Proportion) known to the ancient Greeks:

$$\phi = \frac{\sqrt{5} + 1}{2} \approx 1.618,$$

(2)

and satisfies the equation

$$\phi^2 = \phi + 1,$$

(3)
which implies that
\[ \phi^{k+1} = \phi F_{k+1} + F_k, \] (4)
where \( F_k \) are the Fibonacci numbers. [Zadar on the Dalmatian coast in Croatia is an ancient
city with a rich history and a university whose origin dates back to 1396. One person who
taught there was Luca Pacioli, whose famous work *Divina Proportione* (1509) was illustrated
by Leonardo da Vinci.]

The character of each representation is its trace and must satisfy the following two orthogonality conditions:
\[ \sum_{C_i} n_i \chi_{ai} \chi_{bi}^* = n \delta_{ab}, \quad \sum_{\chi_a} n_i \chi_{ai} \chi_{aj}^* = n \delta_{ij}, \] (5)
where \( n \) is the total number of elements. The number of irreducible representations must be
equal to the number of equivalence classes.

The two irreducible two-dimensional representations of \( D_5 \) may be chosen as follows. For 2, let
\[
\begin{aligned}
C_1 &: \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & C_2 &: \begin{pmatrix} 0 & \omega^k \\ \omega^{5-k} & 0 \end{pmatrix}, & (k &= 0, 1, 2, 3, 4); \\
C_3 &: \begin{pmatrix} \omega & 0 \\ 0 & \omega^4 \end{pmatrix}, & C_4 &: \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^3 \end{pmatrix}, & (\omega &= \exp(2\pi i/5), \text{then}\ 2' \text{is simply obtained by interchanging } C_3 \text{ and } C_4. \text{ Note that}
\end{aligned}
\] (6)
\[
2 \cos(2\pi/5) = \phi - 1, \quad 2 \cos(4\pi/5) = -\phi,
\] (7)
as expected.

For \( D_n \) with \( n \) prime, there are \( 2n \) elements divided into \((n+3)/2\) equivalence classes: \( C_1 \) contains just the identity, \( C_2 \) has the \( n \) reflections, \( C_k \) from \( k = 3 \) to \((n+3)/2\) has 2 elements
each of order \( n \). There are 2 one-dimensional representations and \((n-1)/2\) two-dimensional
ones. For \( D_3 = S_3 \), the above reduces to the “complex” representation with \( \omega = \exp(2\pi i/3) \)
discussed in a recent review [1].
The group multiplication rules of $D_5$ are:

$$1' \times 1' = 1, \quad 1' \times 2 = 2, \quad 1' \times 2' = 2', \quad (8)$$

$$2 \times 2 = 1 + 1' + 2', \quad 2' \times 2' = 1 + 1' + 2, \quad 2 \times 2' = 2 + 2'. \quad (9)$$

In particular, let $(a_1, a_2), (b_1, b_2) \sim 2$, then

$$a_1 b_2 + a_2 b_1 \sim 1, \quad a_1 b_2 - a_2 b_1 \sim 1', \quad (a_1 b_1, a_2 b_2) \sim 2'. \quad (10)$$

Similarly, in the decomposition of $2' \times 2'$, $(a'_2 b'_2, a'_1 b'_1) \sim 2$, and in the decomposition of $2 \times 2'$, $(a_2 a'_1, a_1 a'_2) \sim 2$, and $(a_2 a'_2, a_1 a'_1) \sim 2'$.

The most natural assignment of the 3 lepton families under $D_5$ is

$$(\nu_i, l_i), \quad l_i^c \sim 1 + 2. \quad (11)$$

Assuming two Higgs doublets $\Phi_1 \sim 1$, $\Phi_2 \sim 1'$, the charged-lepton mass matrix is then of the form

$$\mathcal{M}_l = \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b - c \\ 0 & b + c & 0 \end{pmatrix}, \quad (12)$$

where $a, b$ come from $\langle \phi_1^0 \rangle$, and $c$ from $\langle \phi_2^0 \rangle$. Redefining $l^c_{2,3}$ as $l^c_{3,2}$, $\mathcal{M}_l$ becomes diagonal with $m_e = |a|$, $m_\mu = |b - c|$, $m_\tau = |b + c|$.

Assuming that neutrino masses are Majorana and that they come from the naturally small vacuum expectation values $[2]$ of heavy Higgs triplets $\xi_1 \sim 1, \xi_{2,3} \sim 2$, then

$$\mathcal{M}_\nu = \begin{pmatrix} a & c & d \\ c & 0 & b \\ d & b & 0 \end{pmatrix} \quad (13)$$

as advertised, where $a, b$ come from $\langle \xi_1^0 \rangle$, and $c = f \langle \xi_3^0 \rangle$, $d = f \langle \xi_2^0 \rangle$. The two texture zeros are the result of the absence of a Higgs triplet transforming as $2'$. In the case of $D_3 = S_3$, there is only one two-dimensional representation, hence these zeros cannot be maintained without also making $c = d = 0$. 4
The decomposition $2 \times 2 = 1 + 1' + 2'$ holds not only in $D_5$, but also in $D_n$ with $n$ prime and $n > 5$. For example in $D_7$, there are 3 two-dimensional irreducible representations, corresponding to the 3 cyclic permutations of

$$C_3 : \begin{pmatrix} \omega & 0 \\ 0 & \omega^6 \end{pmatrix}, \begin{pmatrix} \omega & 0 \\ 0 & \omega^6 \end{pmatrix}, \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega^5 \end{pmatrix}, \begin{pmatrix} \omega^5 & 0 \\ 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} \omega^3 & 0 \\ 0 & \omega^4 \end{pmatrix}, \begin{pmatrix} \omega^4 & 0 \\ 0 & \omega^3 \end{pmatrix},$$

(14)

(15)

(16)

where $\omega = \exp(2\pi i/7)$. It is clear that

$$2_1 \times 2_1 = 1 + 1' + 2_2, \quad 2_2 \times 2_2 = 1 + 1' + 2_3,$$

(17)

e tc. Hence Eq. (13) is valid in all these symmetries.

Phenomenologically, Eq. (13) has been studied [3] as an example of the class of neutrino mass matrices with two texture zeros. It was first derived from a symmetry ($Q_8$ or $D_4$) only recently [4]. Whereas $Q_8$ or $D_4$ allows other forms, $D_n$ with $n$ prime and $n \geq 5$ allows only Eq. (13). Models based on $D_4 \times Z_2$ have also been proposed [5]. The 4 parameters of Eq. (13) imply that $m_{1,2,3}$ are related to the mixing angles. Given the present global experimental constraints [6]:

$$\Delta m_{atm}^2 = (1.5 - 3.4) \times 10^{-3} \text{ eV}^2, \quad \sin^2 2\theta_{atm} > 0.92,$$

(18)

$$\Delta m_{sol}^2 = (7.7 - 8.8) \times 10^{-5} \text{ eV}^2, \quad \tan^2 \theta_{sol} = 0.33 - 0.49,$$

(19)

and $|\sin \theta_{13}| < 0.2$, the allowed region in the $m_3 - m_2$ plane has been obtained in Ref. [4]. That figure is reproduced here for the convenience of the reader. It shows that there are lower bounds on $m_2$ and $m_3$ and that $m_3 < m_2$ up to about 0.1 eV. The parameter $a$ in Eq. (13) measures neutrinoless double beta decay and has a lower bound of about 0.02 eV in this case.
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Figure 1: Allowed region in $m_2 - m_3$ plane for Eq. (13)