Disordered XY models and Coulomb gases: renormalization via traveling waves

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We present a novel RG approach to 2D random XY models using direct and replicated Coulomb gas methods. By including fusion of environments (charge fusion in the replicated CG) it follows the distribution of local disorder, found to obey a Kolmogorov non linear equation (KPP) with traveling wave solutions. At low T and weak disorder it yields a glassy XY phase with broad distributions and precise connections to Derrida’s GREM. Finding marginal operators at the disorder-induced transition is related to the front velocity selection problem in KPP equations yielding new critical behaviour. The method is applied to critical random Dirac problems.

Two dimensional random systems have attracted considerable recent interest in domains ranging from localization in quantum Hall systems to vortices in superconductors. In the context of localization, progress was made to characterize the multifractal statistics of 2D wavefunctions using random Dirac models [1,2], extending previous studies in 1D [3]. On the other hand the glassy properties of vortex phases with disorder was investigated using random XY models. While these lines of studies have developed in an apparently disconnected way, they led to similar proposals [4,1] that remarkable connections exist between the large fluctuation properties of these systems and Derrida’s random energy models GREM [5]. To study these connections further, consistent RG techniques are needed. Our aim is to develop such an approach, which, as in other glassy systems, e.g. in 1D [3], requires a proper treatment of broad distributions.

Here we focus on random gauge XY models and discuss at the end related random Dirac problems. Recently the phase diagram predicted long ago in [3] was reexamed [6] using energy arguments. It was proposed that defects (vortices) are not thermalized at low T, leading to a non-reentrant transition line between the XY and the defective phases, and to a failure [7] of the conventional Coulomb Gas perturbative expansion of [6]. This motivated several interesting proposals for RG procedures [8,9]. However, these approaches, while giving the correct topology of the phase diagram, are not fully consistent, since they do not take into account the renormalization of local disorder. As shown here, this changes quantitatively the results of [6,7] for the XY phase and becomes crucial at the disorder driven transition.

In this Letter we reexamine the RG procedure for the disordered Coulomb gas (CG) and for the random gauge XY model and propose an approach which allows to treat the probability distribution of the local disorder. Technically this amounts to introduce composite charge fugacities and to study their fusion (in the CG sense) upon coarse graining, while only dipole fugacities were considered previously [8,9]. A precise connection between the fugacity distribution and the GREM free energy distribution is found via the Kolmogorov (KPP) equation [10] which arises as its RG flow equation. Universality in the corresponding non linear front selection problem unexpectedly translates into the RG universality around the disorder driven transition. This allows to describe the phase dominated - and the transition driven - by rare configurations of frozen defects, where correlations are broadly distributed. Restriction to the single charge sector yields a RG derivation of the multifractal properties of the critical Dirac wavefunction.

The 2D square lattice XY model with random phases $\theta$ is defined by its partition sum $Z[A] = \prod_i \int_{-\pi}^\pi d\theta_i e^{-\beta H[\theta,A]}$ with:

$$\beta H[\theta, A] = \sum_{\langle i,j \rangle} V(\theta_i - \theta_j - A_{ij})$$  \hspace{1cm} (1)

and $V(\theta) = -\frac{K}{\gamma} \cos(\theta)$, $K = \beta J$, $\beta = 1/T$. The $A_{ij}$ are independent gaussian random gauge fields, with $
abla \cdot A = \pi \sigma$. This model can be transformed exactly [14,15] into a CG with integer charges defined on the sites $r$ of the dual lattice with $Z[V] = \sum_{\{n_r\}} e^{-\beta H}$ and:

$$\beta H = -K \sum_{r \neq r'} n_r G_{rr'} n_{r'} + \sum_r n_r V_r$$  \hspace{1cm} (2)

where $G_k^{-1} = \frac{1}{2} (2 - \cos(k_x a) - \cos(k_y a))$ is the lattice Laplacian. The bare disorder potential, $V_r = G_{rr'} (\nabla \times A_{rr'})$ is gaussian with logarithmic long range correlations $V_r V_{-k} = 2\pi K^2 G_k$. The usual continuum approximation with (integer) charges of hard core $a$ and fugacities $y = e^{-\gamma K}$ of this lattice model is obtained by using the asymptotic form $G_{rr'} \approx (\ln |r-r'|/a + \gamma)(1-\delta_{rr'})$. Here, the perturbative expansion of $Z[V]$ in $y$, valid in the dilute limit uniformly over the system, fails.

Before turning to the more systematic replica approach, let us first sketch the direct RG method suited to the present case where disorder favors some regions, resulting in a site dependent local fugacity $y_r$. Our expansion captures the limit where the fugacity is negligible almost everywhere except in a few rare favorable regions. This is achieved by following the local disorder distribution which is not gaussian, a novel feature from all previous approaches. We find that the disorder $V_r = V_r ^\gamma + v_r$ naturally splits into two parts, a long range correlated gaussian part $V_r ^\gamma$ with logarithmic
correlator $(V_{r} - V_{r'})^2 = 4\sigma K^2 \ln(|r - r'|/a)$ and a local non gaussian part $\varphi$, which defines the local fugacity variables $z_{\pm} = y_{e} \exp(\pm \varphi_{a})$ for $\pm 1$ charges \cite{1} which have only short range correlations. The RG equation for the distribution $P(z_{+}, z_{-})$ of local environments is obtained from two contributions \( \text{(i) "rescaling";} \) upon coarse graining $a \rightarrow \tilde{a} = ae^{dl}$, $V(r)$ produces a gaussian additive contribution to $v$: \( \delta \left( \frac{V(r) - V_{r'}}{\sqrt{2a}} \right)^2 = 4\sigma K^2 \ln(|r - r'|/\tilde{a}) + dl \equiv \left( \frac{V_{r} - V_{r'}}{\sqrt{2\alpha}} \right)^2 + (dv_{r} - dv_{r'})^2 \)

\[
\partial_{t}P(z_{+}, z_{-}) = OP - 2P(z_{+}, z_{-}) + 2 \left\{ \delta(z_{+} - \frac{z'_{+} + z''_{+}}{1 + z'_{+}z''_{+} + z'_+z''_+}) \delta(z_{-} - \frac{z'_- + z''_{-}}{1 + z'_-z''_{-} + z'_-z''_{-}}) \right\}
\]

where \( \langle A \rangle_{pp''} \) denotes $\int d_{r} z_{r}^{(k)} A P(z_{r}^{(k)})$, and $O = K(2 + z_{+} \partial_{z_{-}} + z_{-} \partial_{z_{+}}) + \sigma K^2 (z_{+} \partial_{z_{-}} - z_{-} \partial_{z_{+}})^2$ is the diffusion operator \cite{2}.

To put this derivation on a firmer footing and to capture broad distributions of local fugacities we introduce \cite{3} an expansion of physical quantities in the number of points \cite{4}, which for the free energy $F[V] = -T \ln Z[V]$ as a functional of the disorder reads:

\[
F[V] = \sum_{r_{1} \neq r_{2}} f_{r_{1}, r_{2}}^{(2)} [V] + \sum_{r_{1} \neq r_{2} \neq r_{3}} f_{r_{1}, r_{2}, r_{3}}^{(3)} [V] + \ldots
\]

where by definition $f_{r_{1}, r_{2}, r_{3}}^{(k)}$ depends only \cite{5} on $V(r)$ at points $r_{i}$, $i = 1, ..., k$. The continuum limit of \cite{4} consists in replacing the sums by integrals with hard core constraints around each $r_{i}$. The first two terms read:

\[
-\beta f_{r, r'}^{(2)} = \ln(1 + W_{r, r'}), \quad W_{r, r'} = w_{r, r'} + w_{r, r'}\]

\[
-\beta f_{r_{1}, r_{2}, r_{3}}^{(3)} = \ln(1 + W_{r_{1}, r_{2}, r_{3}})(1 + W_{r_{2}, r_{3}, r_{1}})(1 + W_{r_{3}, r_{1}, r_{2}})
\]

where $w_{r, r'} = e^{-V_{r} + V_{r'} - G_{r, r'}} = z_{r}^{+}z_{r'}^{+}e^{-V_{r} + V_{r'}} - G_{r, r'}$ is the Boltzmann weight of a dipole. The first term \cite{6} corresponds to the independent dipole approximation \cite{7} while \cite{8} takes into account contributions from triplets of sites. Though there are no actual configuration with three charges in a given environment (neutrality), this term (and higher orders), overlooked in previous approaches, is crucial for the renormalization as it leads to fusion of environments when coarse graining. Upon increase of the cutoff $a \rightarrow ae^{dl}$ on the continuum version of \cite{4}, the $k$ point integral gives a correction to order $dl$ to the $k - 1$ point integral, e.g. $\int d_{r_{1}} d_{r_{2}} d_{r_{3}} f_{r_{1}, r_{2}, r_{3}}^{(3)},$ \cite{4} corrects $f_{r, r'}^{(2)}$ (with $\tilde{r} = \frac{1}{2}(r + r')$) as $\delta \ln(1 + W_{r, r'}) = dl \ln(1 + W_{r, r'}) - \ln(1 + W_{r', r}) - \ln(1 + W_{r, r'})$. First $W_{r, r'} = z_{r}^{+}z_{r'}^{+} + z_{r}^{+}z_{r'}^{-}$ since $V_{r}^{+} = V_{r'}^{+} + O(dl)$ (we denote $z_{r}^{+} = z_{r}^{+}$ etc.), thus the combination $(w_{r, r'} + w_{r, r'})/(1 + W_{r, r'})$ in \cite{8} can be rewritten as $\tilde{w}_{r, r'} = z_{r}^{+}z_{r}^{-}e^{-V_{r} + V_{r'}} - G_{r, r'}$ in terms of the new fugacity $z_{\pm}$ defined above. Second, $w_{r, r'} = z_{+}z_{-}e^{-V_{r} + V_{r'}} - G_{r, r'}$ (similarly $w_{r, r'}$ using $z_{\pm}$).

One gets $z_{r}^{+} \rightarrow z_{r}^{+} e^{Kdl + \delta v_{r}}$ with $\delta v_{r} = 2\sigma K^{2} d l_{r, r'}$. (ii) “fusion of charges” (fusion of environments) upon the change of cutoff, two regions with fugacities $z_{r}^{+}, z_{r}^{+}$ are replaced by a single region at $\tilde{r} = \frac{1}{2}(r + r')$ of effective fugacities $z_{\pm} = (z_{r}^{+} + z_{r}^{-})/(1 + z_{r}^{+}z_{r}^{-} + z_{r}^{+}z_{r}^{-})$ obtained from the relative weight $W_{r} / W_{0}$ of a charge 1 configuration (either in $r'$ or $r''$) versus a neutral one (either no charge or a dipole). (i) and (ii) yield:

\[
\text{Averaging this correction over disorder yields a result corresponding to the RG equation for } P \langle 1 \rangle. \text{ A complete derivation of (3) involves a similar procedure for all moments of } F[V] \text{, provided one adds to (3) the free energy sum of all degrees of freedom eliminated up to scale } l \text{ with } \partial_{t} F_{0} = -T \ln (1 + z_{r}^{+}z_{r}^{-} + z_{r}^{+}z_{r}^{-})/P_{r, r'}.
\]

Finally, the renormalization of $K$ and $\sigma$ is obtained from screening \cite{9, 11, 12}.

\[
\frac{dK^{-1}}{dl} = 4dl^{2} \left\{ \frac{z_{r}^{+}z_{r}^{-} + z_{r}^{+}z_{r}^{-}}{(1 + z_{r}^{+}z_{r}^{-} + z_{r}^{+}z_{r}^{-})^{2}} \right\}
\]

Expanding in the number of sites using \cite{4}, we obtain \cite{3} the following, which together with \cite{8} forms our complete set of RG equations \cite{2, 23}:

\[
\frac{d\sigma}{dl} = 4dl^{2} \left\{ \frac{(z_{r}^{+}z_{r}^{-} - z_{r}^{+}z_{r}^{-})^{2}}{(1 + z_{r}^{+}z_{r}^{-} + z_{r}^{+}z_{r}^{-})^{2}} \right\}
\]

The combinatorics necessary to this method is much easier performed using replicas. We start again from \cite{4} and represent $Z^{m}$ as the partition sum of a CG with $m$-vector charges $n_{r}^{m}$ living on the dual lattice sites. Averaging over disorder, and taking the continuum approximation we obtain the $m$-vector (hard core) CG of partition sum expanded in power of the vector fugacity $Y_{n}$:

\[
Z^{m} = 1 + \sum_{r_{1} \neq r_{2}} \sum_{n_{r_{1}} \neq n_{r_{2}}} \int d_{r_{1}, r_{2}} Y_{n_{r_{1}}} Y_{n_{r_{2}}} \prod_{i \neq j} \frac{|r_{i} - r_{j}|}{a} n_{r_{i}^{m}} n_{r_{j}^{m}}
\]

with $K_{bc} = K \delta_{bc} - \sigma K^{2}$, all integrals being restricted to $|r_{i} - r_{j}| > a$, and the sum is over all distinct neutral configurations $\sum_{r} n_{r}^{m} = 0$. $Y_{n}$ is a function of the replicated charge $n = (n_{1}^{1}, ..., n_{m}^{m})$ with bare value $Y_{n} \approx e^{-\gamma_{n} K^{2} n_{m}^{m}}$. Since $K_{bc} \neq 0$, one cannot restrict to single non zero component charges \cite{1}, as it leads to the erroneous results of \cite{8} at low temperature. However, we stress that this quadratic form for $Y_{n}$, which results from the Gaussian nature of the bare disorder, is not preserved by the RG as shown below. We now perform the RG analysis of the m-vector CG, extending the scalar case \cite{29}, leaving the above form unchanged with \cite{24}:
\begin{align}
\partial_t K_{bc}^{-1} &= d' \sum_{n \neq 0} n^b n^c Y_n Y_{-n} \\
\partial_t Y_{n\neq0} &= (2 - n^b K_{bc} n^c) Y_n + d \sum_{n' \neq 0, n} Y_{n-n'} Y_{n'}.
\end{align}

Rescaling and annihilation of opposite replica charges separated by $a \leq |r_1 - r_j| \leq a e^{d_1}$ gives the first term of (8) and (9). The second term of (8) which comes from fusion of two replica charges as usual in vector CG, was absent in (9) but is necessary for consistency of RG to order $Y_{n-n'}$.

Why should one consider the expansion in $Y_n$? Technically, it is valid, together with (9), in the limit of a small density of vector charges [20], which here corresponds to a small density of favorable local regions. Indeed we checked that this expansion is identical term by term, for $m \to 0$, to the expansion in number of points of the free energy [1]. Thus the set of $Y_n$ should encode the full scale dependent distribution $P(z_{\pm})$ of local disorder, the perturbative parameter being $P(z_+ \sim 1)$. Remarkably, the correspondence between $P(z_+, z_-)$ and $Y_n$ emerges when performing the analytical continuation $m \to 0$ of (8) which we now present. To capture the most relevant operators it is sufficient to consider $Y_n$ with $n^b = 0, \pm 1$ in each replica [16], which, using replica permutation symmetry, depends only on the numbers $n_\pm$ of $\pm 1$ components of $n$. This leads to the general parametrization in term of a function $\Phi(z_+, z_-)$:

$$Y_n = \langle z_+^{n_+} z_-^{n_-} \rangle \Phi = \prod_b \left[ \delta_{n_0,0} + z_+ \delta_{n_+,+1} + z_- \delta_{n_-,1} \right] \Phi,$$

where $\langle \rangle \Phi = \int_{z_+} \Phi(z_+, z_-)$. After some combinatorics [21] the limit $m \to 0$ of (8) can be rewritten equivalently as an equation for $\Phi$, detailed in [17]. The first term in (8) gives a diffusion contribution $(2 + O) \Phi$ and the second term in (9) yields a term of fusion of environments, analogous to the one in (8). From this equation $N = \int_{z_-<z_+>0} \Phi(z_\pm)$ is found to satisfy $\partial_t N = 2N - dN^2$, and thus converges quickly [22] towards $N^* = 2/d$. We thus define the normalized $P = \Phi/N$ which satisfies (3) and is naturally interpreted as a probability distribution [24]. Finally, with the same definitions, (3) yields (5).

We first study numerically the RG equations (3) and (5) and find at low $T$, $\sigma < \sigma_c$, an XY phase as in Fig.1 ($K$, $\sigma$ converge to $K_R$, $\sigma_R$). The typical $z$ goes to zero but $P$ develops a broad tail up to $z \sim O(1)$. While in this phase and at criticality the concentration of rare favorable regions $P(1)$ decreases, it eventually increases at large $l$ in the disordered phase $\sigma_c \lesssim \sigma$.

To go beyond numerics, we argue [25] that it is consistent to discard the $z_+^l z_-^l$ terms in the denominators in (3). This leads to a closed equation for a single fugacity distribution $P(z) = \int_{z_-} P(z, z_-)$ which, using the parametrization $G_l(x) = \int_{z_-} \langle \exp(-z e^{-\beta(z-E_l)}) \rangle P_l(z)$ where $E_l = \int_0^1 \int_0^1 \beta = 1/T$, can be rewritten as:

$$\frac{1}{2} \partial_t G = D_l \partial_z^2 G + (1 - G)G.$$

The diffusion coefficient is $D_l = \frac{1}{2} \sqrt{\sigma J}$ and by construction $G_l(-\infty) = 1$ and $G_l(+\infty) = 0$. Remarkably, for constant $D$ this is the much studied KPP equation, which describes diffusive invasion of an unstable state ($G = 0$) by a stable one ($G = 1$), also related to branching diffusions and glassy REM-like models [28]. It is known [14, 27] that $G_l(x)$ converges at large $l$ towards traveling waves solutions $h(x-m_l)$ selected by the behaviour at infinity of $G_l=0(x) \sim e^{-\beta x}$. This implies $P_l(z) \sim z^{-1} \phi((\ln z - \beta X_l))$ with $X_l = m_l - E_l$ ($X_l < 0$ in the XY phase, see Fig.1).

At low $T$ in the XY phase, $P_l(z)$ becomes very broad and one must distinguish two different tails. As shown in Fig.1, the bulk of the distribution (typical values) is located around $z_{typ} \sim e^{\beta X_l}$. It corresponds to the front region which has a tail of size $\sqrt{t}$ ahead of the front. There from the velocity selection studies [13, 27] for $T > T_g = J/\sqrt{\sigma/2}$ we find the front position $m_l \sim 2(\beta - 1 + D/\beta)l$. For $T < T_g$ the velocity freezes with $m_l = \sqrt{D} (l - \frac{3}{2} \ln l + O(1))$ and $h_l(y) \sim \frac{\sqrt{\pi}}{\sqrt{\sigma}} \exp(-\frac{y^2}{\sigma})$. This corresponds to $P_l(z) \sim z^{-(1+\mu)}$ within the tail of the front, with $\mu = T/T_g < 1$. Thus for $T < T_g$ the distribution function of $\ln z$ travels at the relative velocity $\partial_t X_l = J/(\sqrt{8\sigma} - 1)$, which determines the phase diagram: it is negative (decrease of $P(1)$) in the low $T$ XY regime, positive for $\sigma \geq \sigma_c$. Furthermore, at low $T$, there is also a far tail ahead $\sim l$ of the front which corresponds to rare events $z \sim 1$, of small probability $P_l(1)$, but which dominate average correlations (and thus $\partial_t K$ and $\partial_t \sigma$). The linearized KPP equation, valid in this region, leads to $P_l(z) \sim P_l(1) z^{1-\beta}$ with $\beta(T) = T/T^* < 1$ and to $P_l(1) \sim e^{[2 - \frac{1}{2\beta}]}$, for $T < T^* = 2\sigma J$. A more detailed study of the XY phase is given in [17].

In the high $T$ regime of the XY phase, $P_l(z)$ is not so broad, and one recovers from (10) the usual RG result [1] $\partial_t y_l = (2 - K + \sigma K^2) y_l$ for the average fugacity $y_l = \langle z_l \rangle \sim +\infty$ ($\sim z_{typ}$ for $T > T_g$), using $G_l(x) \sim e^{-\beta(x-E_l)} \phi(z)$ at large $x$.  

![Diagram](image.png)

FIG. 1. Scale dependent distribution $P(z)$ and its two tail regions for $T < T_g$. Inset: phase diagram
The critical behaviour at the transition from the XY to the disordered phases is determined by the front region, since the velocity $\partial_t X_i$ vanishes. Here we sketch the analysis at $T = 0$: defining $z = e^{\beta u}$ and using $\lim_{\beta \to \infty} \beta p_{l}(\beta (u - X_i)) = -h'_{l}(u - X_i)$, (10) yields:

$$\partial_t J^{-1} = -\frac{8d'}{d''} h'_{l}(-2X_i) ; \quad \partial_t \sigma = \frac{8d'}{d''} h(-2X_i) \quad (11)$$

(which is of order $P(1)^2$ perturbative for $X_i \gg 1$). From the universal corrections (12) to the velocity:

$$\partial_t X_i = J(\sqrt{2\alpha} - 1) - 3\sqrt{D}/2I + O(l^{-2}) \quad (12)$$

we recover a projection of the RG flow on the plane $\sigma \sim \sigma_c = 1/3$ and $g \sim P(1) \sim h(-X_i)$ which reads (13):

$$\partial_t g = (16(\sigma - \sigma_c) - \frac{3}{2I})g ; \quad \partial_t (\sigma - \sigma_c) = g^2 \quad (12)$$

yielding $g \sim l^{-\frac{1}{2}}$ at criticality and a correlation length $\xi \sim e^{\frac{3}{2I} - \sigma_c}$. This new universality class (14) is different from KT and from the prediction of (10). Note that although most details of $P(z_{+\pm})$ e.g. its bulk, depend on the cutoff procedure (and fusion rule...), here the universality appears in a remarkable way. It comes from the independence of the velocity and the front tail (which also determine the relevant operators) on the precise form $F[G]$ of the non linear term in (10) (see (13)).

Finally, our RG also applies to the problem of a single charge $Z = \sum_{r} e^{-V_{r}}$ in a random potential with logarithmic correlations, related to diffusion in random media (15) or wavefunction of 2D Dirac fermions in a random magnetic field (16). The same decomposition of disorder, and fusion of environments ($z = z' + z''$, fugacities being local partition sums) yields (10). We recover for $P(z)$ the mapping to directed polymers (DP) on Cayley trees, conjectured in (17), with the same universal intensive free energy and wavefunction multifractal spectrum (18).

To conclude, we developed a RG approach to random XY models, disordered CG and random Dirac problems. By following the whole fugacity distribution, it appears perturbative in the concentration of rare favorable regions, which corresponds to the vector fugacity in the replica method. This expansion is highly non perturbative in the original fugacity $y$. A precise connection to the free energy distribution of DP on Cayley trees and GREM arises from the RG (19) and turns out to be crucial to describe the disorder driven transition.

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[1] C. de C. Chamon, C. Mudry and X. G. Wen. Phys. Rev. Lett. 77 4194 (1996).

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[2] H. Castillo et al. Phys. Rev. B. 56 10668 (1997).
[3] For review see A. Comtet, C. Texier cond-mat/9707313, Lect. notes in Phys. 502 (1998).
[4] L. H. Tang, Phys. Rev. B 54, 3350 (1996).
[5] B. Derrida Phys. Rev. B 24 2613 (1981), J. Phys. Lett. 46 401 (1985).
[6] D. S. Fisher, Phys. Rev. B 50 (1994), D. S. Fisher, P. Le Doussal, C. Monthus cond-mat/9710270.
[7] M. Rubinstein et al., Phys. Rev. B 27, 1800 (1983).
[8] T. Nattermann et al., J. Phys. I (France) 5, 565 (1995).
[9] M. Cha and H.A. Fertig, Phys. Rev. Lett. 74, 4867 (1995).
[10] S. E. Korshunov, Phys. Rev. B 48, 1124 (1993).
[11] S. Scheidl, Phys. Rev. B 55, 457 (1997).
[12] S.E. Korshunov and T. Nattermann, Phys. Rev. B 53, 2746 (1996).
[13] M. Bramson Mem. Am. Math Soc. 44 No 285 (1983).
[14] It is exact for the Villain form $e^{-V(\theta)} = \sum_{e} e^{-\frac{d}{2}((\theta - 2e \pi p)^2}$ which should be in the same universality class.
[15] As will appear here (13) is equivalent to the random Sine Gordon model $H = \int \frac{1}{2} \left( \nabla \phi \right)^2 + i a \cdot \nabla \phi + z_{+} e^{i \phi} + z_{-} e^{-i \phi}$ with a natural splitting of disorder $a, z_{\pm}$. The replica OPE of $Y_{a} e^{i \phi}$ also yields (13).
[16] Higher charges, e.g. $\pm 2$, are less relevant since the diffusion operator for $\int \sum_{\mu} P(z_{\pm}, z_{\mp}, z_{\pm\mp})$ is as in (13) with $K \to 4K$ and $\sigma \to 2\sigma$ and fusion leads to $P(z_{\pm} \sim 1) \sim P(z_{\pm} \sim 1)^2$.
[17] D. Carpentier, P. Le Doussal in preparation.
[18] i.e. in the number of favorable regions, $P(1)$.
[19] A Taylor expansion shows (13) that $f^{(n)}(r_{1}, \ldots r_{k}) = \sum_{l=0}^{\infty} (-1)^{k-l} \sum_{r_{1},\ldots,r_{k},l\in[1,k]} F_{r_{1},\ldots,r_{k}}$ in terms of the site free energy $F(r_{1},\ldots,r_{k})$.
[20] B. Nienhuis, in Phase transitions and critical phenomena, Domb Green Ed, Vol. 11 (1987).
[21] (13) corresponds to a given branching process, associated to a particular cutoff, which even at $\sigma = 0$ contains disorder in the positions of the branching nodes. The most appropriate cutoff under $\sum_{n} Y_{n} |n| = \sum_{n} e^{\mu n}$ corresponding to a non linear term $F[G] = (G - 1) \ln(1 - G)$ in (14).
[22] Without fusion our diffusion formalism yields $z_{\pm} = \gamma e^{\pm v_{t}}$ where $v_{t}$ is gaussian. Redefining dipole fugacity via $v = v' + v''$ we recover (26) of (13) using $N \sim e^{z}$.
[23] $d^{3} = 2\pi^{2}, d = \pi$ for our cutoff choice with $d'/d^{2}$ universal. The cubic term in $\partial_{V} Y$ in (26) drops out for $m \to 0$.
[24] The fraction of fused environments yields the universal factor $\partial_{V} Y = 2$.
[25] This amounts to neglect terms of order $P(1)^{3}$ in the RG equation for $P(1)$ and in (13) since $P(z_{\pm} \sim 1, z_{\mp} \sim 1)$ consistently remains of order $P(1)^{2}$.
[26] B. Derrida, H. Spohn J. Stat. Phys. 51 817 (1988).
[27] E. Brunet and B. Derrida Phys. Rev. E 56 2597 (1997).
[28] U. Ebert and W. Van Sarloos, Phys. Rev. Lett. Feb. 1998.
[29] In (13) subdominant contributions are neglected, e.g. the variations of $D_{i}$.
[30] Direct replica solution (13) of GREM models requires replica symmetry breaking (RSB) for $T < T_{c}$. It yields exponential free energy distributions (Boltzman weights $z_{r} / \sum_{r} z_{r}$ being dominated by a few states, a characteristic of RSB). Within the RG it translates into generation of broad distributions $P(z) \sim z^{-(1+\mu)} (\mu < 1, no
first moment) which have similar properties. This suggests to follow broad distributions alternatively in a more conventional RG (with gaussian distributions) but with RSB [17].