On $R$-matrix representations of Birman-Murakami-Wenzl algebras

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Abstract. We show that to every local representation of the Birman-Murakami-Wenzl algebra defined by a skew-invertible $R$-matrix $\hat{R} \in \text{Aut}(V \otimes V)$ one can associate pairings $V \otimes V \to \mathbb{C}$ and $V^* \otimes V^* \to \mathbb{C}$, where $V$ is the representation space. Further, we investigate conditions under which the corresponding quantum group is of $SO$ or $Sp$ type.

To the memory of Andrei Nikolayevich Tyurin

Let $G$ be either orthogonal or symplectic Lie group, $\mathfrak{g}$ its Lie algebra and $U_q(\mathfrak{g})$ the corresponding quantum group (i.e., the quantized universal enveloping algebra $[1, 2]$). Denote by $V$ the space of the vector representation of $G$ or $U_q(\mathfrak{g})$.

In [3] R. Brauer constructed centralizers $\text{End}_G(V \otimes V^n)$ of the action of $G$ on tensor powers of the vector representation. He introduced a one-parametric family of algebras $Br_n(x)$; for certain values of the parameter $x = x_G$ $^2$, the algebras $Br_n(x_G)$ possess representations $Br_n(x_G) \to \text{End}(V \otimes V^n)$ commuting with the action of $G$. These representations are generated by the permutation $P \in \text{Aut}(V \otimes V^n)$:

$$P(u \otimes v) = v \otimes u, \forall u, v \in V$$

and the operation related to the $G$-invariant pairing $g : V \otimes V \to \mathbb{C}$.

In case of $U_q(\mathfrak{g})$ the role of the Brauer centralizer algebras is played by a two-parametric family of algebras $W_n(q, \nu)$ introduced independently by J. Murakami [4] and by J. Birman and H. Wenzl [5]. Centralizers $\text{End}_{U_q(\mathfrak{g})}(V \otimes V^n)$ are then realized by specific representations of the Birman-Murakami-Wenzl algebras $W_n(q, \nu_{G,q}) \to \text{End}(V \otimes V^n)$ which are generated by $q$-analogs of permutations called the R-matrices.

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$^2$Values $x_G$ and $\nu_{G,q}$ (introduced in a paragraph below) depend essentially on a particular choice of the group $G$. 
In the present paper we study an inverse problem. Given an $R$-matrix representation of the Birman-Murakami-Wenzl algebra we find conditions under which the associated quantum group can be called an orthogonal or symplectic one. More precisely, we prove that for any $R$-matrix which generates representations of algebras $W_n(q, \nu)$ on spaces $V \otimes \mathbb{C}^n$ one can construct a unique, up to a multiplicative constant, nondegenerate pairing on the space $V$ (see Theorem and Proposition 2). We further describe conditions under which this pairing is invariant (see Proposition 3 and comments after it). So in the $q$-case the information on both the permutation and the pairing is advantageously encoded in a single $R$-matrix.

An algebra $W_n(q, \nu)$ is a $(2n-1)!!$ dimensional quotient of the group algebra of the braid group $B_n$. It is given in terms of generators $\{e_i, \kappa_i\}_{i=1}^{n-1}$ and relations [7]

1. $e_i e_{i+1} e_i = e_{i+1} e_i e_{i+1}$, $e_i e_j = e_j e_i$, $|i-j| > 1$,

2. $e_i^2 = 1 + \lambda (e_i - \nu \kappa_i)$, $\lambda := q - q^{-1}$,

3. $e_i \kappa_i = \kappa_i e_i = \nu \kappa_i$,

4. $\kappa_{i+1} e_i \kappa_{i+1} = \nu^{-1} \kappa_{i+1}$, $\kappa_{i+1} e_i^{-1} \kappa_{i+1} = \nu \kappa_{i+1}$.

Here eqs. (1) define the Artin presentation of the braid group $B_n$ and relations (2)-(4) extract an appropriate quotient. The domains of the algebra parameters $q \in \mathbb{C}\{0, \pm 1\}$ and $\nu \in \mathbb{C}\{0, q, -q^{-1}\}$ are chosen in such a way that the elements $\kappa_i$ can be expressed in terms of $e_i$

$$\kappa_i = \lambda^{-1} \nu^{-1} (q - e_i) (q^{-1} + e_i) = \lambda^{-1} (e_i^{-1} - e_i + \lambda),$$

and satisfy relations

$$\kappa_i^2 = \mu \kappa_i$$

with a nonzero coefficient $\mu := \lambda^{-1} \nu^{-1} (q - \nu) (q^{-1} + \nu)$.  

Note that for generic values of $\nu$ and $q$ the set of defining relations (1)–(4) is not a minimal one. To show this, start with the first one of eqs.(4) and multiply both sides by $e_i^{-1} e_{i+1}^{-1}$ from the right

$$\kappa_{i+1} \kappa_i = \kappa_{i+1} e_i^{-1} e_{i+1}^{-1}.$$  

Then, multiplying by $\lambda \kappa_{i+1}$ from the right and performing straightforward transformations we get

$$\lambda \nu^{-1} \kappa_{i+1} e_i^{-1} \kappa_{i+1} = \lambda \kappa_{i+1} \kappa_i \kappa_{i+1} =$$

$$= \kappa_{i+1} (e_i^{-1} - e_i + \lambda) \kappa_{i+1} = \kappa_{i+1} (e_i^{-1} \kappa_{i+1} - \nu^{-1} + \lambda \mu),$$

3Defining relations for Brauer centralizer algebra $Br_n(x)$ follow from relations (1)-(6) in a limiting case $\nu = q^{1-x}$, $q \to 1$; note that in this limit the generators $\kappa_i$ and $e_i$ become independent.
wherefrom it follows that \((\lambda \nu^{-1} - 1)(\kappa_{i+1}e_i^{-1}\kappa_{i+1} - \nu\kappa_{i+1}) = 0\). Thus, in case \(\nu \neq \lambda\) the two relations in (4) are algebraically dependent.

In the sequel we shall use a few more relations for the generators \(e_i\) and \(\kappa_i\):

\[
\begin{align*}
\kappa_i\kappa_{i \pm 1} &= \kappa_i e_{i \pm 1} e_i, \\
\kappa_i\kappa_{i \pm 1}\kappa_i &= \kappa_i, \\
\k_i^{\pm 1}\kappa_i &= \nu^{\mp 1}\k_i.
\end{align*}
\]
(8) (9) (10)

All these equalities follow easily from the defining relations (1)–(4).

We are aiming to study a specific family of representations of algebras \(W_n(q, \nu)\), the so-called local (or R-matrix) representations. We are using a compact matrix notation of [2]. Necessary explanations are given below.

Let \(V\) be a finite dimensional vector space. We label the component space \(V\) of the tensor power \(V \otimes V \otimes \ldots \otimes V\) from left to right in the ascending order starting from 1. For any element \(X \in \text{End}(V \otimes 2)\) and for all \(1 \leq k \neq l \leq n\) the symbol \(X_{kl}\) stands for an element of \(\text{End}(V \otimes n)\) whose action differs from the identity only on the tensor product of the \(k\)-th and \(l\)-th component spaces where it coincides with \(X\). In case \(l = k + 1\) a concise notation \(X_k := X_{k,k+1}\) is often applied. The symbols \(I\) and \(P\) are reserved for the identity and the permutation operators respectively.

An element \(\hat{R} \in \text{Aut}(V \otimes 2)\) is called an R-matrix if it satisfies the so called Yang-Baxter equation

\[
\hat{R}_1 \hat{R}_2 \hat{R}_1 = \hat{R}_2 \hat{R}_1 \hat{R}_2.
\]
(11)

With any R-matrix \(\hat{R}\) one associates a family of representations of the braid groups \(B_n, \rho^R_n : B_n \rightarrow \text{Aut}(V \otimes n), n = 1, 2, \ldots\), defined on the generators by \(\rho^R_n(e_i) := \hat{R}_i, i = 1, 2, \ldots, n - 1\).

An R-matrix \(\hat{R}\) whose minimal polynomial is cubic and which induces representations \(\rho^R_n\) of the quotient algebras \(W_n(q, \nu)\) (for some values of \(q\) and \(\nu\)) is called an R-matrix of BMW type.

An operator \(\hat{R} \in \text{Aut}(V \otimes 2)\) (not necessarily an R-matrix) is called skew invertible iff there exists some \(\hat{\Psi} \in \text{End}(V \otimes 2)\), called the skew inverse of \(\hat{R}\), such that relations

\[
\text{Tr}(1)\hat{R}_{12}\hat{\Psi}_{23} = \text{Tr}(2)\hat{\Psi}_{12}\hat{R}_{23} = P_{13}
\]
(12)

are satisfied. Here the subscript \(i\) in the notation of trace \(\text{Tr}_{(i)}\) indicates the label of the space where the trace is evaluated.

Denote

\[
C_2 := \text{Tr}(1)\hat{\Psi}_{12}, \quad D_1 := \text{Tr}(2)\hat{\Psi}_{12}.
\]

As a direct consequence of the definitions one has

\[
\text{Tr}(1)C_i\hat{R}_{12} = I_2, \quad \text{Tr}(2)D_2\hat{R}_{12} = I_1.
\]
(13)

In what follows while referring to relations (1)–(10) we always imply their images in the R-matrix representations, that is

\[
e_i \mapsto \hat{R}_i, \quad \kappa_i \mapsto K_i := \lambda^{-1}\nu^{-1}(qI - \hat{R}_i)(q^{-1}I + \hat{R}_i).
\]
(14)
Proposition 1 [8, 9]. For a skew invertible R-matrix $\hat{R}$ the following relations hold

\begin{align}
C_1 \hat{\Psi}_{12} &= \hat{R}_{21}^{-1} C_2, \quad \hat{\Psi}_{12} C_1 = C_2 \hat{R}_{21}^{-1}, \quad (15) \\
D_2 \hat{\Psi}_{12} &= \hat{R}_{21}^{-1} D_1, \quad \hat{\Psi}_{12} D_2 = D_1 \hat{R}_{21}^{-1}. \quad (16)
\end{align}

**Proof.** First, we rewrite conditions (1) for the R-matrix $\hat{R}$ in the form $\hat{R}_{12}^{\pm 1} \hat{R}_{23} \hat{R}_{12}^{\pm 1} = \hat{R}_{23}^{\pm 1} \hat{R}_{12} \hat{R}_{23}^{\pm 1}$. Multiplying by $\hat{\Psi}_{01} \hat{\Psi}_{34}$ and taking traces in spaces with labels 1 and 3 we get

\[
\text{Tr}(1)(\hat{\Psi}_{01} \hat{R}_{12}^{\pm 1} P_{24} \hat{R}_{12}^{\mp 1}) = \text{Tr}(3)(\hat{\Psi}_{34} \hat{R}_{23}^{\mp 1} P_{02} \hat{R}_{23}^{\pm 1}).
\]

Next, evaluating trace in space 0 or 4 we get four equalities

\[
\text{Tr}(1)(C_1 \hat{R}_{12}^{\pm 1} P_{24} \hat{R}_{12}^{\mp 1}) = C_4 I_2, \quad \text{Tr}(3)(D_3 \hat{R}_{23}^{\mp 1} P_{02} \hat{R}_{23}^{\pm 1}) = D_0 I_2,
\]

which can be further transformed to

\[
\text{Tr}(1)(C_1 \hat{R}_{12}^{\pm 1} \hat{R}_{14}^{\mp 1}) = C_4 P_{24}, \quad \text{Tr}(3)(D_3 \hat{R}_{23}^{\mp 1} \hat{R}_{03}^{\pm 1}) = D_0 P_{02}.
\]

Consider the left one of equalities in (18) with upper/lower signs. Multiply both its sides by $\hat{\Psi}_{23}/\hat{\Psi}_{43}$ and take trace in the space with label 2/4. Then, apply definition (12) and use the relation $\text{Tr}(2)(U_2 P_{12} W_2) = W_1 U_1$ which holds for any $U, W \in \text{End}(V)$ and follows from the properties of the trace and the permutation. The resulting equality is just the left/right formula in (15).

Derivation of relations (16) from the right equality in (18) proceeds similarly. □

**Corollary.** Evaluating traces of relations (15)/(16) in spaces with labels 2/1 one finds

\[
\text{Tr}(2)C_2 \hat{R}_{21}^{-1} = \text{Tr}(2)D_2 \hat{R}_{12}^{-1} = C_1 D_1 = D_1 C_1.
\]

**Theorem.** Let $\hat{R}$ be a skew invertible BMW type R-matrix. Then the rank of the operator $\hat{K} \in \text{End}(V^{\otimes 2})$ (see eq.(14)) equals 1.

**Proof.** Consider an R-matrix version of the left equation in (4)

\[
\hat{K}_{23} \hat{R}_{12} \hat{K}_{23} = \nu^{-1} \hat{K}_{23}.
\]

Multiplying by $\hat{\Psi}_{01}$ and taking trace in space 1 we obtain

\[
\hat{K}_{23} P_{02} \hat{K}_{23} = \nu^{-1} D_0 \hat{K}_{23}.
\]

Evaluating traces in spaces 2 and 3 in the left hand side of relation (20) one finds

\[
\text{Tr}(23)(\hat{K}_{23} P_{02} \hat{K}_{23}) = \mu \text{Tr}(23)(\hat{K}_{23} P_{02}) = \mu \text{Tr}(23)(P_{02} \hat{K}_{03}) = \mu \text{Tr}(3) \hat{K}_{03},
\]

where (6) and the properties of the permutation were used. On the other hand, (6) implies

\[
\text{Tr}(12) \hat{K}_{12} = \mu \text{rank}(\hat{K}),
\]

4Here it is suitable to label component spaces in $V^{\otimes n}$ starting from 0.
and so, applying $\text{Tr}(\mathit{23})$ to the right hand side of relation (20), one gets

$$\nu^{-1} D_0 \text{Tr}(\mathit{23}) \hat{K}_{23} = \frac{\mu}{\nu} \text{rank}(\hat{K}) D_0.$$  \hspace{1cm} (23)

Equating the results of calculations in (21) and (23) we obtain

$$\text{Tr}(\mathit{2}) \hat{K}_{12} = \nu^{-1} \text{rank}(\hat{K}) D_1.$$ \hspace{1cm} (24)

In the same way the equality

$$\text{Tr}(\mathit{1}) \hat{K}_{12} = \nu^{-1} \text{rank}(\hat{K}) C_2.$$ \hspace{1cm} (25)

follows from relation (10) with the upper choice of signs.

Consider now an R-matrix version of the right formula in (4)

$$\hat{K}_{23} \hat{R}_{12}^{-1} \hat{K}_{23} = \nu \hat{K}_{23}.$$ \hspace{1cm} (26)

Taking traces of this equality in spaces 2 and 3 and using eq.(24) one obtains

$$\text{Tr}(\mathit{2}) D_2 \hat{R}_{12}^{-1} = \nu^2 I_1,$$ \hspace{1cm} (27)

which in view of (16) is equivalent to

$$CD = DC = \nu^2 I.$$ \hspace{1cm} (28)

Thus, in the conditions of the theorem, the matrices $C$ and $D$ are invertible and we can write relation (25) in a form

$$\text{Tr}(\mathit{1}) D_2 \hat{K}_{12} = \nu \text{rank}(\hat{K}) I_2.$$ \hspace{1cm} (29)

On the other hand, applying $\text{Tr}(\mathit{23})$ to an R-matrix version of eq.(9), that is $\hat{K}_{23} \hat{K}_{12} \hat{K}_{23} = \hat{K}_{23}$, and taking into account relations (22) and (24) we obtain

$$\text{Tr}(\mathit{2}) D_2 \hat{K}_{12} = \nu I_1.$$ \hspace{1cm} (30)

Finally, evaluating $\text{Tr}(\mathit{2})$ of the equality (28) and $\text{Tr}(\mathit{1})$ of the equality (29) and comparing the results we conclude that rank $\hat{K} = 1$. \hspace{1cm} \square

Remark. Although in this paper the matrices $C$ and $D$ are auxiliary, they play a conceptual role in the theory of quantum groups and are used, in particular, for the definition of quantum traces (more details on that can be found in [10, 8, 9, 12]). While proving the theorem we have derived a number of formulas — (29), (27), and (24), (25) (where one has to put rank$\hat{K} = 1$) — which are characteristic for matrices $C$ and $D$ corresponding to BMW type R-matrices. One more relation can be added

$$\text{Tr} D = \text{Tr} C = \nu \mu.$$ \hspace{1cm} (31)

It follows by evaluation of traces of relations (24) and (25).
From now on we shall fix some basis \( \{v^i\}_{i=1}^N \) in space \( V \) \( (N := \dim V) \). Let

\[
\hat{K}_{ij}^{kl} = \bar{g}_{ij} g^{kl},
\]

be the matrix of the rank one operator \( \hat{K} \) in this basis. Define operators \( X, Y \in \text{End}(V) \) whose matrices in the chosen basis are

\[
X^j_i := \sum_k g^{ik} \bar{g}_{kj}, \quad Y^j_i := \sum_k g^{kj} \bar{g}_{ik}.
\]

**Proposition 2.** Let \( \hat{R} \) be a skew invertible BMW type R-matrix. Bivectors \( g^{kl} \) and \( g_{ij} \) (31) define nondegenerate bilinear pairings \( g : V \otimes V \to \mathbb{C} \) and \( \bar{g} : V^* \otimes V^* \to \mathbb{C} \)

\[
g(x, y) := \sum_{i,j=1}^N x_i y_j g^{ij}, \quad \bar{g}(z, t) := \sum_{i,j=1}^N z^i t^j \bar{g}_{ij}, \quad \forall \ x, y \in V, \ z, t \in V^*,
\]

where \( x_i, y_j \), and \( z^i, t^j \) stand for coordinates of vectors \( x, y \), and \( z, t \) in the basis \( \{v^i\} \) and the dual basis \( \{v^*_i\} \), respectively.

Operators \( X \) and \( Y \) are inverse to each other; the coefficients of the characteristic polynomial of \( X \):

\[
\det(xI - X) = \sum_{k=0}^N (-1)^k C_k x^{N-k}
\]

satisfy reciprocity relations

\[
C_k = \epsilon C_{N-k} \quad (\forall \ 0 \leq k \leq N), \quad \epsilon = \pm 1.
\]

**Proof.** Consider an R-matrix version of relation (9), \( \hat{K}_{12} \hat{K}_{23} \hat{K}_{12} = \hat{K}_{12} \). By a substitution of eq.(31) and by evaluation of traces in spaces 1 and 2 (note that \( \sum_{i,j}(g^{ij}\bar{g}_{ij}) = \text{Tr}(12)\hat{K}_{12} = \mu \neq 0 \)) the above equality acquires a form\(^5\)

\[
XY = I,
\]

wherefrom it also follows that pairings (33) are nondegenerate.

The definition of matrices \( X \) and \( Y \) implies that \( \text{Tr}(X^k) = \text{Tr}(Y^k), \forall k = 1, 2, \ldots \), and, hence, matrices \( X \) and \( Y = X^{-1} \) obey the same characteristic polynomial. Taking into account the identities \( C_N(X) C_k(X^{-1}) = C_{N-k}(X) \), we then conclude

\[
C_N(X) C_k(X) = C_{N-k}(X) \quad \forall \ 1 \leq k \leq N.
\]

For \( k = N \) this gives \( C_N(X) = \epsilon = \pm 1 \); substituting the expression for \( C_N(X) \) back to (36) one obtains (34). \( \square \)

Following [2] for any R-matrix \( \hat{R} \) we define an associative unital bialgebra \( \mathcal{F}(\hat{R}) \) generated by components of the matrix \( T := ||T^j_i||_{i,j=1}^N \) subject to relations

\[
\hat{R}_{12} T_1 T_2 = T_1 T_2 \hat{R}_{12}.
\]

The coproduct and the counit in the bialgebra are defined by

\[
\Delta(T^j_i) = \sum_{k=1}^N T^j_k \otimes T^j_k, \quad \varepsilon(T^j_i) = \delta^j_i.
\]

\(^5\)

It is this equation which was used in a classification of quantum groups in dimension 2 [11].
For the skew invertible BMW type R-matrix $\hat{R}$ the eqs. (14), (37) and the rank one property of the matrix $\hat{K}$ together imply
\[
\hat{K}_{12}T_1T_2 = \mu^{-1}\hat{K}_{12}T_1T_2\hat{K}_{12} = \tau\hat{K}_{12}
\]
for some $\tau \in \mathcal{F}(\hat{R})$.

The following proposition demonstrates the role of the matrix $X$ for the algebra $\mathcal{F}(\hat{R})$.

**Proposition 3.** Under the assumptions of the theorem the element $\tau$ is group-like, i.e., $\Delta(\tau) = \tau \otimes \tau$, $\varepsilon(\tau) = 1$. It also satisfies relations
\[
\tau T_i^j = (XTX^{-1})^j_i \tau .
\]

**Proof.** The group-like properties of the element $\tau$ are directly checked by application of the coproduct and the counit operations to relation (38). Relation (39) is justified by a calculation
\[
\tau \hat{K}_{12}T_3 = \hat{K}_{12}T_1T_2T_3 = \hat{K}_{12}\hat{R}_{23}\hat{R}_{12}T_1T_2T_3\hat{R}_{12}^{-1}\hat{R}_{23}^{-1} =
\]
\[
\hat{K}_{12}\hat{K}_{23}T_1T_3\hat{R}_{12}^{-1}\hat{R}_{23}^{-1} = \hat{K}_{12}\hat{K}_{23}\hat{R}_{12}^{-1}\hat{R}_{23}^{-1} \tau =
\]
\[
= \hat{K}_{12}\hat{K}_{23}\hat{K}_{12} \tau = \hat{K}_{12}\hat{K}_{23}\hat{K}_{12}(XTX^{-1})_3 \tau = \hat{K}_{12}(XTX^{-1})_3 \tau .
\]
Here we have used relations (8), (7), (9), as well as formula
\[
T_1\hat{K}_{23}\hat{K}_{12} = \hat{K}_{23}\hat{K}_{12}(XTX^{-1})_3,
\]
which is checked by a substitution of expressions (31), (32) for $\hat{K}$ and $X$ with a subsequent use of (35).

Given a quantum group $U_q(g)$ (recall: $g$ is an orthogonal or symplectic Lie algebra), its dual Hopf algebra $G_q$ can be constructed as a quotient of the bialgebra $\mathcal{F}(\hat{R})$ (here the R-matrix $\hat{R}$ is defined by a canonical element of $U_q(g)$) by an ideal $\tau = 1$. By duality, the left $U_q(g)$-module $V$ admits the right coaction of $G_q$
\[
\delta(v^i) = \sum_{j=1}^{N} v^j \otimes T_j^i .
\]

As one can see from eq.(38) it is the condition $\tau = 1$ that makes the pairings $g$ and $\bar{g}$ (33) invariant with respect to the coaction (40).

One can start with an algebra $\mathcal{F}(\hat{R})$ defined by some skew invertible BMW type R-matrix $\hat{R}$. Then the factorization of $\mathcal{F}(\hat{R})$ by the relation $\tau = 1$ would imply linear dependencies among generators $T_j^i$ (c.f., eq.(39)) unless the matrix $X$ (32) is scalar. As it is seen from an example below this is not always the case.

The standard $so_N$ and $sp_N$ series of BMW type R-matrices (see [2])\(^6\) are
\[
\hat{R} = \sum_{i,j=1}^{N} q^{(\delta_{ij}-\delta_{ij'})} e_{ij} \otimes e_{ji} + \lambda \sum_{1 \leq j < i}^{N} e_{jj} \otimes e_{ii} - \lambda \sum_{1 \leq j < i}^{N} q^{(\rho_j-\rho_i)} e_i e_j e_{ij} \otimes e_{ij'} .
\]

\(^6\)To realize a representation of the Birman-Murakami-Wenzl algebra the R-matrices given in [2] are to be multiplied (in our case, from the left) by the permutation operator.
Here the following notation is used: \( i' := N + 1 - i; \| e_{ij} \|_k := \delta_{ik} \delta_{j}^{k} \) are matrix units; \( \epsilon_i = 1 (\forall i) \) for the \( so_N \) case and \( \epsilon_i = 1 = -\epsilon_i \) \( \forall i \leq n \) for the \( sp_{2n} \) case; the numbers \( (\rho_1, \rho_2, \ldots, \rho_N) \) are chosen as \( (n-1/2, n-3/2, \ldots, 1/2, 0, -1/2, \ldots, -n+1/2), (n-1, n-2, \ldots, 1, 0, 0, -1, \ldots, -n+1) \) and \( (n, n-1, \ldots, 1, -1, \ldots, -n) \) in cases \( so_{2n+1}, so_{2n}, \) or \( sp_{2n} \) correspondingly. In calculations below we will use the relation \( \rho_i = -\rho_i' \) rather then the explicit expressions for \( \rho_i \). The R-matrices (41) generate representations of the Birman-Murakami-Wenzl algebras \( W_k(q, \nu) \) with specific values of their parameter \( \nu \), namely, \( \nu = q^{1-N} \) for the \( so_N \) case and \( \nu = -q^{-1-2N} \) for the \( sp_N \) case.

For the R-matrices (41) one calculates \( \bar{g}_{ij} = \delta_{ij} \epsilon_i q^{-\rho_i}, \) \( g^{ij} = \delta_{ij} \epsilon_i q^{-\rho_i} \), wherefrom it follows that \( X = I \). To construct R-matrices whose corresponding matrices \( X \) are not scalars we apply the twist procedure suggested in [13] (see also [12, 14]). Remind briefly that given a pair of R-matrices \( \hat{R} \) and \( \hat{F} \) one can produce a new R-matrix \( \hat{R}_F := (PF) \hat{R} (\hat{F}^{-1}P) \), called the twisted \( \hat{R} \), provided that additional relations on \( \hat{R} \) and \( \hat{F} \) are satisfied

\[
\hat{R}_{12} \hat{F}_{23} \hat{F}_{12} = \hat{F}_{23} \hat{F}_{12} \hat{R}_{23}, \quad \hat{F}_{12} \hat{F}_{23} \hat{R}_{12} = \hat{R}_{23} \hat{F}_{12} \hat{F}_{23}.
\]  

(42)

By construction the twist procedure preserves not only the Yang-Baxter equation (11) but all the additional relations (2)–(10) which characterize BMW type R-matrices.

Now we twist R-matrices (41). As a trial twisting R-matrix we use \( \hat{F} \) such that \( PF = \sum_{i,j} d_{ij} e_{ii} \otimes e_{jj} \) where \( d_{ij} \in \mathbb{C} \setminus \{0\} \). An easy check gives the conditions

\[
d_{ij} d_{ij}' = u_j, \quad d_{ij} d_{ij'} = w_i, \quad \forall i, j,
\]

under which relations (42) are satisfied. The latter in turn are consistent if

\[
u_i w_i' = w_i w_i' = \text{const}, \quad \forall i.
\]

The twisting procedure results in a usual family of multiparametric R-matrices (some of parameters here are inessential and can be removed by a linear change of basis in the space \( V \))

\[
\hat{R}_F = \sum_{i,j=1}^{N} q^{(\delta_{ij}-\delta_{ij'})} \frac{d_{ij}}{d_{jj}} e_{ij} \otimes e_{ji} + \lambda \sum_{1 \leq j < i}^{N} e_{jj} \otimes e_{ii} - \lambda \sum_{1 \leq j < i}^{N} q^{(\rho_i - \rho_j)} \epsilon_i \epsilon_j \frac{d_{ij}}{d_{jj'}} e_{ij} \otimes e_{ij'}.
\]

(43)

For these twisted R-matrices we have \( \bar{g}_{ij} = \delta_{ij} \epsilon_i q^{-\rho_i} d_{ii}, \) \( g^{ij} = \delta_{ij} \epsilon_i q^{-\rho_i} d_{ii}^{-1} \) which gives \( X_i^j = \delta_i^j d_{ii} d_{ii}^{-1} \). Thus, element \( \tau \) (38) is not necessarily central in the algebra \( \mathcal{F}(\hat{R}) \).

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