A New Bound for the Uniform Admissibility Theorem

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Let $G$ be a reductive group defined over a $p$-adic field $F$, that is $G = F$-points of a reductive group defined over $F$. A representation $(\pi, V)$ of $G$ is called smooth if for every $v \in V$ there exists an open compact subgroup $K < G$ such that $\pi(g)v = v$ for every $g \in K$. Let us denote by $V^K$ the subspace of all $K$-fixed vectors in $V$. A representation is called admissible if $V^K$ is finite-dimensional for every open compact subgroup $K < G$. In this work we consider only smooth admissible representations of $G$. Also, algebras are associative and with unit. Let $K$ be a fixed open compact subgroup of $G$. In [Ber], Bernstein proved the following

Theorem 1 (Uniform Admissibility Theorem). There exists a uniform constant $N(G, K)$ such that for every irreducible representation $(\pi, V)$ of $G$ there is an inequality $\dim V^K \leq N(G, K)$.

In this work we reprove one of the two lemmas (Lemma 3 in Bernstein’s original proof. As a result we obtain a sharper estimate for the bound $N(G, K)$ in the theorem. For the convenience of the reader we sketch here (almost without proofs) the main steps in Bernstein’s proof. We will demonstrate the new bound on $\dim V^K$ only in the case $G = \text{GL}_n(F)$. The case of a general reductive group is similar.

Let us reformulate the theorem in terms of Hecke algebras. Denote by $H(G, K)$ the Hecke algebra consisting of all compactly supported functions $f : G \to \mathbb{C}$ such that $f(k_1 g k_2) = f(g)$ for all $g \in G$ and $k \in K$. Denote by $H(G)$ the Hecke algebra consisting of all locally constant and compactly supported functions $f : G \to \mathbb{C}$. Let $(\pi, V)$ be an irreducible representation of $G$ such that $V^K \neq 0$. The Hecke algebra $H(G)$ acts on $(\pi, V)$. Note that the space $V^K$ is an irreducible representation of $H(G, K)$. Indeed, let $0 \neq v \in V^K$, $0 \neq w \in V^K$. The representation $(\pi, V)$ is irreducible, hence there exists a function $f \in H(G)$ such that $\pi(f)v = w$. Obviously $\pi(1_K * f * 1_K)v = w$ and the convolution $1_K * f * 1_K \in H(G, K)$. Thus we can reformulate Theorem 1 as the following

Theorem 2. There exists a uniform constant $N(G, K)$ such that for every irreducible finite-dimensional $H(G, K)$-module $W$ there is an inequality $\dim W \leq N(G, K)$.

The theorem follows from the following lemmas.

Lemma 3. [Ber], Proposition 2] Let $A_1, A_2, ..., A_l$ be a commuting family of matrices in $M_{n \times n}(\mathbb{C})$. Let $\mathcal{F}$ be the algebra of matrices in $M_{n \times n}(\mathbb{C})$ generated by $A_1, A_2, ..., A_l$ and the identity. Then

$$\dim \mathcal{F} \leq (l + 1)n^2 - \frac{1}{4}.$$

We will prove this lemma later.

Lemma 4. [Ber], Proposition 1] Let $\mathcal{L}$ be an algebra over $\mathbb{C}$, $\mathcal{A}$, $\mathcal{Z}$ commutative subalgebras in $\mathcal{L}$, $A_1, A_2, ..., A_l \in \mathcal{A}$, $X_1, ..., X_p, Y_1, ..., Y_q \in \mathcal{L}$. Let us assume that $\mathcal{Z}$ lies in the center of the algebra $\mathcal{L}$, $\mathcal{Z} \subset \mathcal{A}$, $\mathcal{A}$ is the commutative algebra generated by $A_1, ..., A_l$ and $\mathcal{Z}$, and that any element $X \in \mathcal{L}$ can be written in the form $X = \sum X_i P_{ij}$, where $P_{ij} \in \mathcal{A}$, $(i = 1, ..., p; \; j = 1, ..., q)$. Then any irreducible finite dimensional representation of the algebra $\mathcal{L}$ has dimension at most $(pq)^{(l+1)/2}(l + 1)^{(l+1)/2}$.

Proof. If $\rho : \mathcal{L} \to \text{End}(V)$ is an irreducible representation, $\dim V = n$, then $\rho(\mathcal{Z}) = \mathbb{C} \cdot 1$ (by Schur’s Lemma), $\dim \rho(\mathcal{L}) = n^2$ (by Burnside’s theorem). By the conditions of Lemma 4

$$\dim \rho(\mathcal{L}) \leq pq \dim \rho(\mathcal{A}) \leq pq (l + 1)n^2 - 2^{2/(l+1)}.$$

Hence $n^2 \leq pq(l + 1)n^2 - 2^{2/(l+1)}$, that is $n \leq (pq)^{(l+1)/2}(l + 1)^{(l+1)/2}$. $\square$

Lemma 5. The Hecke algebra $H(G, K)$ satisfies the conditions of Lemma 4.

For the proof see [Ber, Lemma]. We only note that $l$ is the $F$-semi-simple rank of the group $G$ and the numbers $p, q$ depend linearly on $[K_0 : K]$ where $K_0$ is a maximal compact subgroup of $G$ such that $K \subset K_0$. Let us demonstrate the choices of $p, q, x_i$, and $y_j$ in the case $G = \text{GL}_n(F)$. Let $\mathcal{O} = \{x \in F \mid |x| \leq 1\}$ and let $\varpi$ be a
generator of the maximal ideal in $\mathcal{O}$. Then $K_0 = GL_n(\mathcal{O})$ is a maximal compact subgroup of $G$. Let $K$ be a “good enough” compact open subgroup of $GL_n(\mathcal{O}_F)$, for example a congruence subgroup of $GL_n(\mathcal{O}_F)$,

$$K = K_m := \{ x \in G \mid ||1 - x|| \leq ||x||^m \},$$

where $m \geq 1$ and $||x|| = \max |x_{ij}|$. In this case one can take $a_j$ ($j = 1, \ldots, n - 1$) to be a diagonal matrix, $(a_j)_{ii} = 1$ for $i \leq j$ and $(a_j)_{ii} = \infty$ for $i > j$ and define $A_j = 1_{K_{a_j}K}$. Let $Z$ be the algebra generated by $1_{Kg}$ for $g \in Z(G)$. Let $\mathcal{A}$ be the algebra generated by $Z$ and $A_j$, $j = 1, \ldots, n - 1$. Let $p = q = [K : K]$, decompose $K_0 = \cup Kg_i$, $i = 1, \ldots, p$ and choose $x_i = y_i = 1_{Kg_iK}$. The elements $x_i, y_j, i, j = 1, \ldots, p$ and the algebras $\mathcal{A}$, $Z$, and $H(G, K)$ satisfy the conditions of Lemma $4$. As a consequence, we obtain

$$\dim V^K \leq [GL_n(\mathcal{O}_F) : K]^n \cdot n^{n/2}$$

for every irreducible representation $(\pi, V)$ of $GL_n(F)$. This improves Bernstein’s bound

$$\dim V^K \leq [GL_n(\mathcal{O}_F) : K]^{2n-1}.$$

See [Ber] for more details. In the proof of Lemma $3$ we need the following

**Lemma 6.** Let $A \in M_{n \times n}(\mathbb{C})$ be a nilpotent matrix and let $m \geq 1$. Then

$$\dim \left( \text{Span} \{ A^mB \mid AB = BA \} \right) \leq \frac{n^2}{m}.$$

**Proof.** Suppose $A = \text{diag}(J_{i_1}, J_{i_2}, \ldots, J_{i_l})$ where $J_i$ is a Jordan block of dimension $i \times i$. The assertion $AC = CA$ for $C_{n \times n} = (C_{ij})$ with $C_{ij}$ a block of dimension $l_i \times l_j$ means $J_{i_1}C_{ij} = C_{ij}J_{i_j}$. The dimension spanned by such blocks is $\min(l_i, l_j)$. Let us call this dimension $d_{ij}$ and note that

$$d_{ij} \leq \frac{l_i l_j}{\max(l_i, l_j)}.$$

The matrix $A^m$ kills every Jordan cell of size $\leq m$. Thus, the dimension of the vector space spanned by the matrices of the form $A^mB$ less than

$$\sum_{\max(l_i, l_j) \geq m} d_{ij} \leq \sum_{\max(l_i, l_j) \geq m} \frac{l_i l_j}{m} = \frac{n^2}{m}. \quad \Box$$

**Proof of Lemma $3$.** By a standard argument, we can assume that the matrices $A_1, \ldots, A_l$ are nilpotent. Namely, let us view the matrices $A_1, \ldots, A_l$ as operators on $V = \mathbb{C}^n$. Since the algebra $\mathcal{F}$ is commutative, we can decompose the space $V$ into the direct sum of $\mathcal{F}$-invariant subspaces $V_j$ such that for every $A \in \mathcal{F}$ and every $j$, the eigenvalues of $A|_{V_j}$ coincide. We can restrict ourselves to the case $V = V_j$, and substracting suitable constants from the operators $A_i$, we may assume that all the $A_i$ are nilpotent.

Let $x < n^2$, we will choose it later. Divide matrices of the form $A_1^{j_1}A_2^{j_2} \ldots A_l^{j_l}$ into two families. One with $j_i < x$ for every $1 \leq i \leq l$ and the other with at least one of the powers $j_i \geq x$. The first family consists of $x^l$ matrices. Let us estimate the dimension of the subspace generated by the second family. Suppose $j_1 \geq x$. The number of linearly independent matrices of the form $A_1^{j_1}B$ where $A_1B = BA_1$ is at most $\frac{x^2}{2}$ by Lemma $6$. Thus $\dim \mathcal{F}$ is bounded by

$$f(x) := \frac{ln^2}{x} + x^l.$$  

The minimum is achieved for $f'(x_0) = 0$, $x_0 = n^{2/(l+1)}$. We obtain

$$\dim \mathcal{F} \leq ln^2 -2^{2/(l+1)} + 2^{2/(l+1)} = (l + 1)n^2 - \frac{x^2}{2}. \quad \Box$$

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**References**

[Ber] I. N. Bernstein, All reductive $p$-adic groups are tame, Functional Analysis and its Applications 8, No.2, 3-6 (1974).