Starlikeness of Analytic Functions with Subordinate Ratios

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Abstract. Let \( h \) be a non-vanishing analytic function in the open unit disc with \( h(0) = 1 \). Consider the class consisting of normalized analytic functions \( f \) whose ratios \( f(z)/g(z) \), \( g(z)/zp(z) \), and \( p(z) \) are each subordinate to \( h \) for some analytic functions \( g \) and \( p \). The radius of starlikeness is obtained for this class when \( h \) is chosen to be either \( h(z) = \sqrt{1+z} \) or \( h(z) = e^z \). Further \( G \)-radius is also obtained for each of these two classes when \( G \) is a particular widely studied subclass of starlike functions. These include \( G \) consisting of the Janowski starlike functions, and functions which are parabolic starlike.

1. Classes of Analytic Functions

Let \( \mathcal{A} \) denote the class of normalized analytic functions \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \) in the unit disc \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). A prominent subclass of \( \mathcal{A} \) is the class \( \mathcal{S}^* \) consisting of functions \( f \in \mathcal{A} \) such that \( f(\mathbb{D}) \) is a starlike domain with respect to the origin. Geometrically, this means the linear segment joining the origin to every other point \( w \in f(\mathbb{D}) \) lies entirely in \( f(\mathbb{D}) \). Every starlike function in \( \mathcal{A} \) is necessarily univalent.

Since \( f'(0) \) does not vanish, every function \( f \in \mathcal{A} \) is locally univalent at \( z = 0 \). Further, each function \( f \in \mathcal{A} \) mirrors the identity mapping near the origin, and thus in particular, maps small circles \( |z| = r \) onto curves which bound starlike domains. If \( f \in \mathcal{A} \) is also required to be univalent in \( \mathbb{D} \), then it is known that \( f \) maps the disc \( |z| < r \) onto a domain starlike with respect to the origin for every \( r \leq r_0 := \tanh(\pi/4) \) (see [4, Corollary, p. 98]). The constant \( r_0 \) cannot be improved. Denoting by \( \mathcal{S} \) the class of univalent functions \( f \in \mathcal{A} \), the number \( r_0 = \tanh(\pi/4) \) is commonly referred to as the radius of starlikeness for the class \( \mathcal{S} \).

Another informative description of the class \( \mathcal{S} \) is its radius of convexity. Here it is known that every \( f \in \mathcal{S} \) maps the disc \( |z| < r \) onto a convex domain for every \( r \leq r_0 := 2 - \sqrt{3} \) [4, Corollary, p. 44]. Thus the radius of convexity for \( \mathcal{S} \) is \( r_0 = 2 - \sqrt{3} \).

To formulate a radius description for other entities besides starlikeness and convexity, consider in general two families \( \mathcal{G} \) and \( \mathcal{M} \) of \( \mathcal{A} \). The \( \mathcal{G} \)-radius for the class \( \mathcal{M} \), denoted by \( R_\mathcal{G}(\mathcal{M}) \), is the largest number \( R \) such that \( r^{-1}f(rz) \in \mathcal{G} \) for every \( 0 < r \leq R \) and \( f \in \mathcal{M} \). Thus, for example, an equivalent description of the radius of starlikeness for \( \mathcal{S} \) is that the \( \mathcal{S}^* \)-radius for the class \( \mathcal{S} \) is \( R_{\mathcal{S}^*}(\mathcal{S}) = \tanh(\pi/4) \).

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In this paper, we seek to determine the radius of starlikeness, and certain other \( G \)-radius, for particular subclasses \( G \) of \( A \). Several widely-studied subclasses of \( A \) have simple geometric descriptions; these functions are often expressed as a ratio between two functions. Among the very early studies in this direction is the class of close-to-convex functions introduced by Kaplan \cite{9}, and Reade’s class \cite{22} of close-to-starlike functions. Close-to-convex functions are necessarily univalent, but not so for close-to-starlike functions. Several works, for example those in \cite{2}\cite{11}\cite{14}\cite{25}\cite{27}, have advanced studies in classes of functions characterized by the ratio between functions \( f \) and \( g \) belonging to given subclasses of \( A \).

In this paper, we examine two different subclasses of functions in \( A \) satisfying a certain subordination link of ratios. Interestingly, these classes contain non-univalent functions. An analytic function \( f \) is subordinated to an analytic function \( g \), written \( f \prec g \), if

\[
f(z) = g(w(z)), \quad z \in \mathbb{D},
\]

for some analytic self-map \( w \) in \( \mathbb{D} \) with \( |w(z)| \leq |z| \). The function \( w \) is often referred to as a Schwarz function.

Now let \( h \) be a non-vanishing analytic function in \( \mathbb{D} \) with \( h(0) = 1 \). The classes treated in this paper consist of functions \( f \in A \) whose ratios \( f(z)/g(z), g(z)/zp(z), \) and \( p(z), \) are each subordinate to \( h \) for some analytic functions \( g \) and \( p \):

\[
\frac{f(z)}{g(z)} \prec h(z), \quad \frac{g(z)}{zp(z)} \prec h(z), \quad p(z) \prec h(z).
\]

When \( p \) is the constant one function, then the class contains functions \( f \in A \) satisfying the subordination of ratios

\[
\frac{f(z)}{g(z)} \prec h(z), \quad \frac{g(z)}{z} \prec h(z).
\]

For \( h(z) = (1 + z)/(1 - z) \), and other appropriate choices of \( h \), these functions have earlier been studied, notably by MacGregor in \cite{11}\cite{14}, and Ratti in \cite{19}\cite{20}. Recent investigations include those in \cite{2}\cite{25}\cite{27}.

In this paper, two specific choices of the function \( h \) are made: \( h(z) = \sqrt{1+z} \), and \( h(z) = e^z \).

**The Class \( T_1 \).** This is the class given by

\[
T_1 := \left\{ f \in A : \frac{f(z)}{g(z)} \prec \sqrt{1+z}, \frac{g(z)}{zp(z)} \prec \sqrt{1+z} \text{ for some } g \in A, p(z) \prec \sqrt{1+z} \right\}.
\]

This class is non-empty: let \( f_1, g_1, p_1 : \mathbb{D} \to \mathbb{C} \) be given by

\[
f_1(z) = z(1 + z)^{3/2}, \quad g_1(z) = z(1 + z) \quad \text{and} \quad p_1(z) = \sqrt{1+z}.
\]

Then \( f_1(z)/g_1(z) \prec \sqrt{1+z} \) and \( g_1(z)/zp_1(z) \prec \sqrt{1+z} \), so that \( f_1 \in T_1 \). The function \( f_1 \) will be shown to play the role of an extremal function for the class \( T_1 \). Since \( f_1' \) vanishes at \( z = -2/5 \), the function \( f_1 \) is non-univalent, and thus, the class \( T_1 \) contains non-univalent functions. Incidentally, \( f_1 \) demonstrates the radius of univalence for \( T_1 \) is at most \( 2/5 \). In Theorem \cite{27}, the radius of starlikeness for \( T_1 \) is shown to be \( 2/5 \), whence \( T_1 \) has radius of univalence \( 2/5 \).

The following is a useful result in investigating the starlikeness of the class \( T_1 \).
Lemma 1.1. Let \( p(z) \prec \sqrt{1 + z} \). Then \( p \) satisfies the sharp inequalities
\[
\sqrt{1 - r} \leq |p(z)| \leq \sqrt{1 + r}, \quad |z| \leq r,
\]
and
\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \frac{r}{2(1 - r)}, \quad |z| \leq r.
\]

Proof. If \( p(z) \prec \sqrt{1 + z} \), then \( p^2(z) = 1 + w(z) \) for some Schwarz function \( w \). The well-known Schwarz lemma shows that \( |w(z)| \leq |z| \) and
\[
|w'(z)| \leq \frac{1 - |w(z)|^2}{1 - |z|^2}.
\]
Therefore,
\[
|p(z)|^2 = |1 + w(z)| \leq 1 + |w(z)| \leq 1 + |z| \leq 1 + r
\]
for \( |z| \leq r \), that is, \( |p(z)| \leq \sqrt{1 + r} \) for \( |z| \leq r \). Similarly, \( |p(z)| \geq \sqrt{1 - r} \) for \( |z| \leq r \).

Since \( 2zp'(z)/p(z) = zw'(z)/(1 + w(z)) \), the inequality (1.3) readily shows
\[
2 \left| \frac{zp'(z)}{p(z)} \right| \leq \frac{|z||w'(z)|}{1 - |w(z)|} \leq \frac{|z|(|1 + w(z)|)}{1 - |z|^2} \leq \frac{|z|(|1 + |z|)}{1 - |z|^2} = \frac{|z|}{1 - |z|} \leq \frac{r}{1 - r}
\]
for \( |z| \leq r \). This proves (1.2). The inequalities are sharp for the function \( p(z) = \sqrt{1 + z} \).

For \( f \in \mathcal{T}_1 \), let \( p_1(z) = f(z)/g(z) \) and \( p_2(z) = g(z)/zp(z) \). Then \( f(z) = zp(z)p_1(z)p_2(z) \) and
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \frac{zp'(z)}{p(z)} \right| + \left| \frac{zp_1'(z)}{p_1(z)} \right| + \left| \frac{zp_2'(z)}{p_2(z)} \right|.
\]
Since \( p, p_1, p_2 \prec \sqrt{1 + z} \), we deduce from (1.2) and (1.4) that
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{3r}{2(1 - r)}, \quad |z| \leq r,
\]
for each function \( f \in \mathcal{T}_1 \). Sharp growth inequalities also follow from (1.1):
\[
r(1 - r)^{3/2} \leq |f(z)| \leq r(1 + r)^{3/2}
\]
for each \( f \in \mathcal{T}_1 \). Crude distortion inequalities can readily be obtained from (1.5) and the growth inequality; however, finding sharp estimates remain an open problem.

The Class \( \mathcal{T}_2 \). This class is defined by
\[
\mathcal{T}_2 := \left\{ f \in \mathcal{A} : \frac{f(z)}{g(z)} \prec e^z, \frac{g(z)}{zp(z)} \prec e^z \quad \text{for some} \quad g \in \mathcal{A}, \; p(z) \prec e^z \right\}.
\]
Let \( f_2, g_2, p_2 : \mathbb{D} \rightarrow \mathbb{C} \) be given by
\[
f_2(z) = ze^{3z}, \quad g_2(z) = ze^{2z} \quad \text{and} \quad p_2(z) = e^z.
\]
Evidently, \( f_2(z)/g_2(z) \prec e^z, \; g_2(z)/zp_2(z) \prec e^z \) so that \( f_2 \in \mathcal{T}_2 \), and the class \( \mathcal{T}_2 \) is non-empty. Similar to \( f_1 \in \mathcal{T}_1 \), the function \( f_2 \) plays the role of an extremal function for the class \( \mathcal{T}_2 \). The Taylor series expansion for \( f_2 \) is
\[
f_2(z) = z + 3z^2 + \frac{9z^3}{2} + \frac{9z^4}{2} + \frac{27z^5}{8} + \cdots.
\]
Comparing the second coefficient, it is clear that $f_2$ is non-univalent. Hence the class $\mathcal{T}_2$ contains non-univalent functions. The derivative $f_2'$ vanishes at $z = -1/3$, which shows the radius of univalence for $\mathcal{T}_2$ is at most $1/3$. From Theorem 2.1 the radius of starlikeness is shown to be $1/3$, and so the radius of univalence for $\mathcal{T}_2$ is $1/3$.

**Lemma 1.2.** Every $p(z) < e^z$ satisfies the sharp inequalities

\[
e^{-r} \leq |p(z)| \leq e^r, \quad |z| \leq r,
\]

and

\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \begin{cases} r, & |z| \leq \sqrt{2} - 1 \\ \frac{(1 + r^2)^2}{4(1 - r^2)}, & |z| = \sqrt{2} - 1. \end{cases}
\]

**Proof.** Let $p(z) < e^z$. Since $p(z) = e^{w(z)}$ for some Schwarz self-map $w$ satisfying $|w(z)| \leq |z|$, it follows that

\[
|p(z)| = e^{Re(w(z))} \leq e^{|w(z)|} \leq e^{|z|}.
\]

The function $w$ also satisfy the sharp inequality (see [41 Corollary, p. 199])

\[
|w'(z)| \leq \begin{cases} 1, & r = |z| \leq \sqrt{2} - 1 \\ \frac{(1 + r^2)^2}{4r(1 - r^2)}, & r \geq \sqrt{2} - 1. \end{cases}
\]

From $zp'(z)/p(z) = zw'(z)$, we conclude that

\[
\left| \frac{zp'(z)}{p(z)} \right| \leq \begin{cases} r, & r = |z| \leq \sqrt{2} - 1 \\ \frac{(1 + r^2)^2}{4(1 - r^2)}, & r \geq \sqrt{2} - 1. \end{cases}
\]

This inequality is sharp for $p(z) = e^z$ and $r = |z| \leq \sqrt{2} - 1$. It is also sharp in the remaining interval for the function $p(z) = e^{w(z)}$, where $w$ is the extremal function for which equality holds in (1.8).

For $f \in \mathcal{T}_2$, let $p_1(z) = f(z)/g(z)$ and $p_2(z) = g(z)/zp(z)$. Then $f(z) = zp(z)p_1(z)p_2(z)$ and

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \left| \frac{zp'(z)}{p(z)} \right| + \left| \frac{zp_1'(z)}{p_1(z)} \right| + \left| \frac{zp_2'(z)}{p_2(z)} \right|.
\]

Since $p, p_1, p_2 < e^z$, estimates (1.7) and (1.8) show that

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \begin{cases} 3r, & r = |z| \leq \sqrt{2} - 1 \\ \frac{3(1 + r^2)^2}{4(1 - r^2)}, & r \geq \sqrt{2} - 1. \end{cases}
\]

for each function $f \in \mathcal{T}_2$. It also follows from (1.6) that

\[re^{-3r} \leq |f(z)| \leq re^{3r}\]

holds for each function $f \in \mathcal{T}_2$, and that these estimates are sharp.
In this paper, we shall adopt the commonly used notations for subclasses of $\mathcal{A}$. First, for $0 \leq \alpha < 1$, let $S^\alpha(\alpha)$ denote the class of starlike functions of order $\alpha$ consisting of functions $f \in \mathcal{A}$ satisfying the subordination

$$zf'(z) \leq f(z) < \frac{1 + (1 - 2\alpha)z}{1 - z}.$$  

Thus

$$\Re \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{D}.$$  

The case $\alpha = 0$ corresponds to the classical functions whose image domains are starlike with respect to the origin. Various other starlike subclasses of $\mathcal{A}$ occurring in the literature can be expressed in terms of the subordination

$$zf'(z) < \varphi(z)$$  

for suitable choices of the superordinate function $\varphi$. When $\varphi : \mathbb{D} \rightarrow \mathbb{C}$ is chosen to be $\varphi(z) := (1 + A\alpha)/(1 + B\alpha)$, $-1 \leq B < A \leq 1$, the subclass derived is denoted by $S^\alpha[A, B]$. Functions $f \in S^\alpha[A, B]$ are known as Janowski starlike. When $\varphi(z) := 1 + (2/\pi^2)((\log((1 + \sqrt{z})/(1 - \sqrt{z})))^2)$, the subclass is denoted by $S^\alpha_\sigma$, and its functions are called parabolic starlike.

In Section 2 of this paper, the radius of starlikeness, Janowski starlikeness, and parabolic starlikeness are found for the classes $T_i$, with $i = 1, 2$. Section 3 deals with the determination of the $G$-radius for the class $T_i$ with $i = 1, 2$, for certain other subclasses $G$ occurring in the literature. These classes are associated with particular choices of the superordinate function $\varphi$ in (1.11). As mentioned earlier, the $G$-radius for a given class $\mathcal{M}$, denoted by $R_G(\mathcal{M})$, is the largest number $R$ such that $r^{-1}f(rz) \in G$ for every $0 < r \leq R$ and $f \in \mathcal{M}$. It will become apparent in the forthcoming proofs that there are common features in the methodology of finding the $G$-radius for each of these subclasses.

2. Starlikeness of order $\alpha$, Janowski and parabolic starlikeness

The first result deals with the $S^\alpha(\alpha)$-radius (radius of starlikeness of order $\alpha$) for the classes $T_1$ and $T_2$. This radius is shown to equal the $S^\alpha_\alpha$-radius, where $S^\alpha_\alpha$ is the subclass containing functions $f \in \mathcal{A}$ satisfying $|zf'(z)/f(z) - 1| < 1 - \alpha$. The latter condition also implies that $S^\alpha_\alpha \subset S^\alpha(\alpha)$.

**Theorem 2.1.** Let $0 \leq \alpha < 1$. The radius of starlikeness of order $\alpha$ for $T_1$ and $T_2$ are

(i) $R_{S^\alpha(\alpha)}(T_1) = R_{S^\alpha_\alpha}(T_1) = 2(1 - \alpha)/(5 - 2\alpha)$,

(ii) $R_{S^\alpha(\alpha)}(T_2) = R_{S^\alpha_\alpha}(T_2) = (1 - \alpha)/3$.

**Proof.** (i) The function $\sigma(r) = (2 - 5r)/(2 - 2r)$ is a decreasing function on $[0, 1)$. Further, the number $R_1 := (2(1 - \alpha)/(5 - 2\alpha))$ is the root of the equation $\sigma(r) = \alpha$. For $f \in T_1$ and $0 < r \leq R_1$, the inequality (1.5) readily yields

$$\Re \frac{zf'(z)}{f(z)} \geq 1 - \frac{3r}{2(1 - r)} = 2 - 5r = \sigma(r) \geq \sigma(R_1) = \alpha$$

and

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{3r}{2(1 - r)} = 1 - \sigma(r) \leq 1 - \sigma(R_1) = 1 - \alpha.$$
At \( z = -R_1 \), the function \( f_1 \in T_1 \) given by \( f_1(z) = z(1 + z)^{3/2} \) yields
\[
\frac{zf_1'(z)}{f_1(z)} = \frac{2 + 5z}{2 + 2z} = \frac{2 - 5R_1}{2 - 2R_1} = \alpha.
\]
Thus
\[
\Re \frac{zf_1'(z)}{f_1(z)} = \alpha \quad \text{and} \quad \left| \frac{zf_1'(z)}{f_1(z)} - 1 \right| = 1 - \alpha.
\]
This proves that the \( S^*(\alpha) \) and \( S^*_\alpha \) radii for \( T_1 \) are the same number \( R_1 \).

(ii) Consider \( \omega(r) = 1 - 3r, \quad 0 \leq r < 1 \). The number \( R_2 = (1 - \alpha)/3 < 1/3 \) is clearly the root of the equation \( \omega(r) = \alpha \). Since \( \omega \) is decreasing, then \( \omega(r) \geq \omega(R_2) = \alpha \) for each \( f \in T_2 \) and \( 0 < r \leq R_2 \). It follows from (1.10) that
\[
\Re \frac{zf'(z)}{f(z)} \geq 1 - 3r = \omega(r) \geq \alpha,
\]
and
\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 3r = 1 - \omega(r) \leq 1 - \alpha.
\]
Evaluating the function \( f_2(z) = ze^{3z} \) at \( z = -R_2 \) yields
\[
\frac{zf_2'(z)}{f_2(z)} = 1 - 3R = \alpha.
\]
Hence
\[
\Re \frac{zf_2'(z)}{f_2(z)} = \alpha \quad \text{and} \quad \left| \frac{zf_2'(z)}{f_2(z)} - 1 \right| = 1 - \alpha.
\]
This proves that the \( S^*(\alpha) \) and \( S^*_\alpha \) radii for the class \( T_2 \) are the same number \( R_2 \).

Next we find the \( S^*[A,B]\)-radius (Janowski starlikeness) for \( T_1 \) and \( T_2 \). Recall that \( S^*[A,B] \) consists of analytic functions \( f \in A \) satisfying the subordination \( zf'(z)/f(z) < (1 + Az)/(1 + Bz), -1 \leq B < A \leq 1 \).

**Theorem 2.2.** (i) Every \( f \in T_1 \) is Janowski starlike in the disc \( D_r = \{ z : |z| < r \} \) for \( r \leq 2(A - B)/(3(1 + |B|) + 2(A - B)). \) If \( B < 0 \), then \( R_{S^*[A,B]}(T_1) = 2(A - B)/(3 + 2A - 5B) \).
(ii) The radius of Janowski starlikeness for \( T_2 \) is \( R_{S^*[A,B]}(T_2) = (A - B)/(3(1 - B)). \)

**Proof.** Since \( S^*[A, -1] = S^*((1 - A)/2, \) the results in the case \( B = -1 \) follow from Theorem 2.1. We now prove the results when \( -1 < B < A \leq 1 \).

(i) Let \( f \in T_1 \) and write \( w = zf'(z)/f(z) \). Then (1.23) shows that \( |w - 1| \leq 3r/(2(1 - r)) \) for \( |z| \leq r \). For \( 0 \leq r \leq R_1 := 2(A - B)/(3(1 + |B|) + 2(A - B)) \), then \( 3R_1/(2(1 - R_1)) = (A - B)/(1 + |B|) \).

For \( 0 \leq r \leq R_1 \), we first show that the disc
\[
\left\{ w : |w - 1| \leq \frac{3R_1}{2(1 - R_1)} = \frac{A - B}{1 + |B|} \right\}
\]
is contained in the images of the unit disc under the mapping \( (1 + Az)/(1 + Bz) \). As \( B \neq -1 \), the image is the disc given by
\[
\left\{ w : \left| w - \frac{1 - AB}{1 - B^2} \right| < \frac{A - B}{1 - B^2} \right\}.
\]
Silverman [30, p. 50-51] has shown that the disc
\[ \{ w : |w - c| < d \} \subset \{ w : |w - a| < b \} \]
if and only if \( |a - c| \leq b - d \). With the choices \( c = 1, d = (A - B)/(1 + |B|), a = (1 - AB)/(1 - B^2) \) and \( b = (A - B)/(1 - B^2) \), then \( |a - c| = |B|(A - B)/(1 - B^2) = b - d \). This proves that \( S^*[A, B] \) radius is at least \( R_1 \).

To prove sharpness, consider the function \( f_1 \in T_1 \) given by \( f_1(z) = z(1 + z)^{3/2} \). Evidently, \( zf'_1(z)/f_1(z) = (2 + 5z)/(2 + 2z) \). For \( B < 0 \), evaluating at \( z = -R_1 \), then \( zf'_1(z)/f_1(z) = 1 + 3z/(2 + 2z) = 1 - (A - B)/(1 + |B|) = (1 - A)/(1 - B) \). This shows that
\[
\frac{|zf'_1(z)|}{|f_1(z)|} \leq \frac{|A - B|}{1 - B^2},
\]
proving sharpness in the case \( B < 0 \).

(ii) Let \( f \in T_2 \) and \( w := zf'(z)/f(z) \). It follows from (1.10) that \( |w - 1| \leq 3r \) for \( |z| \leq r \). For \( 0 \leq r \leq R_2 := (A - B)/(3(1 + |B|)) \), we see that the disc \( \{ w : |w - 1| \leq 3R_2 = (A - B)/(1 + |B|) \} \) is contained in the disc \( \{ w : |w - 1| \leq 3R_2 < (A - B)/(1 - B^2) \} \), as in the proof of (i). This proves that \( S^*[A, B] \) radius is at least \( R_2 \). The result is sharp for the function \( f_2 \in T_2 \) given by the function \( f_2(z) = ze^{3z} \).

The function \( \varphi_{PAR} : \mathbb{D} \to \mathbb{C} \) given by
\[
\varphi_{PAR}(z) := 1 + \frac{2}{\pi^2} \left( \log \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)^2, \quad \text{Im} \sqrt{z} \geq 0,
\]
maps \( \mathbb{D} \) into the parabolic region
\[
\varphi_{PAR}(\mathbb{D}) = \{ w = u + iv : v^2 < 2u - 1 \} = \{ w : \text{Re} \, w > |w - 1| \}.
\]
The class \( C(\varphi_{PAR}) = \{ f \in A : 1 + zf''(z)/f'(z) < \varphi_{PAR}(z) \} \) is the class of uniformly convex functions introduced by Goodman [7]. The corresponding class \( S^*_p := S^*(\varphi_{PAR}) = \{ f \in A : zf''(z)/f'(z) < \varphi_{PAR}(z) \} \) introduced by Rønning [24] is known as the class of parabolic starlike functions. The class \( S^*_p \) consists of functions \( f \in A \) satisfying
\[
\text{Re} \left( \frac{zf''(z)}{f'(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right|, \quad z \in \mathbb{D}.
\]
Evidently, every parabolic starlike function is also starlike of order 1/2. The radius of parabolic starlikeness for the class \( T_1 \) and \( T_2 \) is given in the next result.

**Corollary 2.3.** The radius of parabolic starlikeness for \( T_1 \) and \( T_2 \) is respectively equal to its radius of starlikeness of order 1/2. Thus,

(i) \( R_{S^*_p}(T_1) = 1/4 \).

(ii) \( R_{S^*_p}(T_2) = 1/6 \).

**Proof.** Shanmugam and Ravichandran [25, p. 321] proved that
\[
\{ w : |w - a| < a - 1/2 \} \subseteq \{ w : \text{Re} \, w > |w - 1| \}
\]
for \( 1/2 < a \leq 3/2 \). Choosing \( a = 1 \), this implies that \( S^*_{1/2} \subset S^*_p \). Every parabolic starlike function is also starlike of order 1/2, whence the inclusion \( S^*_{1/2} \subset S^*_p \subset S^* \). Therefore, for any class \( \mathcal{F} \), readily \( R_{S^*_{1/2}}(\mathcal{F}) \leq R_{S^*_p}(\mathcal{F}) \leq R_{S^*_{(1/2)}}(\mathcal{F}) \).
When $\mathcal{F} = \mathcal{T}_i$, $i = 1, 2$, Theorem 2.1 gives $R_{\mathcal{S}^*_1}(\mathcal{T}_i) = R_{\mathcal{S}_{1}^*}(\mathcal{T}_i)$. This shows that $R_{\mathcal{S}^*_1}(\mathcal{T}_i) = R_{\mathcal{S}^*_2}(\mathcal{T}_i) = R_{\mathcal{S}^*(1/2)}(\mathcal{T}_i)$. Since $R_{\mathcal{S}^*(1/2)}(\mathcal{T}_1) = 1/4$ and $R_{\mathcal{S}^*(1/2)}(\mathcal{T}_2) = 1/6$ from Theorem 2.1, it follows that $R_{\mathcal{S}_{1}^*}(\mathcal{T}_1) = 1/4$ and $R_{\mathcal{S}_{1}^*}(\mathcal{T}_2) = 1/6$.

3. Further radius of starlikeness

In this section, we find the $\mathcal{G}$-radius for the class $\mathcal{T}_i$ with $i = 1, 2$, for certain other widely studied subclasses $\mathcal{G}$. These are associated with particular choices of the superordinate function $\varphi$ in (1.11).

Denote by $\mathcal{S}_{\exp}^* := \mathcal{S}^*(e^z)$ the class associated with $\varphi(z) := e^z$ in (1.11). This class was introduced by Mendiratta et al. [16], and it consists of functions $f \in \mathcal{A}$ satisfying the condition $|\log(zf'(z)/f(z))| < 1$. The following result gives the radius of exponential starlikeness for the classes $\mathcal{T}_1$ and $\mathcal{T}_2$.

**Corollary 3.1.** The $\mathcal{S}_{\exp}^*$-radius for the class $\mathcal{T}_1$ is

$$R_{\mathcal{S}_{\exp}^*}(\mathcal{T}_1) = (2 - 2e)/(2 - 5e) \approx 0.296475,$$

while that of $\mathcal{T}_2$ is

$$R_{\mathcal{S}_{\exp}^*}(\mathcal{T}_2) = (e - 1)/3e.$$

**Proof.** Mendiratta et al. [16], Lemma 2.2 proved that

$$\{w : |w - a| < a - 1/e\} \subseteq \{w : |\log w| < 1\}$$

for $e^{-1} \leq a \leq (e + e^{-1})/2$, and this inclusion with $a = 1$ gives $S_{1/e}^* \subseteq \mathcal{S}_{\exp}^*$. It was also shown in [16], Theorem 2.1 (i) that $\mathcal{S}_{\exp}^* \subset \mathcal{S}^*(1/e)$. Therefore, $S_{1/e}^* \subset \mathcal{S}_{\exp}^* \subset \mathcal{S}^*(1/e)$, which, as a consequence of Theorem 2.1 established the result.

**Corollary 3.2** investigates the radius of cardioid starlikeness for each class $\mathcal{T}_1$ and $\mathcal{T}_2$. The class $\mathcal{S}_{\mathcal{C}}^* := \mathcal{S}^*(\varphi_{\mathcal{CAR}})$, where $\varphi_{\mathcal{CAR}}(z) := 1 + 4z^3 + 2z^2/3$ in (1.11), was introduced and studied in [21, 26, 23]. Descriptively, $f \in \mathcal{S}_{\mathcal{C}}^*$ provided $zf'(z)/f(z)$ lies in the region bounded by the cardioid $\Omega_{\mathcal{C}} := \{w = u + iv : (9u^2 + 9v^2 - 18u + 5)^2 - 16(9u^2 + 9v^2 - 6u + 1) = 0\}$.

**Corollary 3.2.** The following are the $\mathcal{S}_{\mathcal{C}}^*$-radius for the classes $\mathcal{T}_1$ and $\mathcal{T}_2$:

(i) $R_{\mathcal{S}_{\mathcal{C}}^*}(\mathcal{T}_1) = 4/13$,

(ii) $R_{\mathcal{S}_{\mathcal{C}}^*}(\mathcal{T}_2) = 2/9$.

**Proof.** Sharma et al. [27] proved that $\{w : |w - a| < a - 1/3\} \subseteq \Omega_{\mathcal{C}}$ for $1/3 < a < 5/3$, and this inclusion with $a = 1$ gives $S_{1/3}^* \subseteq \mathcal{S}_{\mathcal{C}}^*$. Thus $R_{S_{1/3}^*}(\mathcal{T}_i) \leq R_{\mathcal{S}_{\mathcal{C}}^*}(\mathcal{T}_i)$ for $i = 1, 2$.

To complete the proof, we demonstrate $R_{\mathcal{S}_{\mathcal{C}}^*}(\mathcal{T}_i) \leq R_{S_{1/3}^*}(\mathcal{T}_i)$ for $i = 1, 2$.

(i) Evaluating the function $f_1(z) = z(1 + z)^{3/2}$ at $z = -R = -R_{S_{1/3}^*}(\mathcal{T}_i) = -4/13$ gives

$$\frac{zf_1'(z)}{f_1(z)} = \frac{2 + 5z}{2 + 2z} = \frac{2 - 5R}{2 - 2R} = \frac{1}{3} = \varphi_{\mathcal{CAR}}(-1).$$

Thus, $R_{\mathcal{S}_{\mathcal{C}}^*}(\mathcal{T}_1) \leq 4/13$.

(ii) Similarly, at $z = -R = -R_{S_{1/3}^*}(\mathcal{T}_2) = -2/9$, the function $f_2(z) = ze^{3z}$ yields

$$\frac{zf_2'(z)}{f_2(z)} = 1 + 3z = 1 - 3R = \frac{1}{3} = \varphi_{\mathcal{CAR}}(-1).$$
This proves that $R_{S^*}(T_2) \leq 2/9$.

In 2019, Cho et al. [3] studied the class $S^*_\sin := S^*(1 + \sin z)$ consisting of functions $f \in \mathcal{A}$ satisfying the condition $zf'(z)/f(z) < 1 + \sin z$. We find the $S^*_\sin$-radius for the classes $T_1$ and $T_2$.

**Corollary 3.3.** The following are the $S^*_\sin$-radius for each class $T_1$ and $T_2$:

1. $R_{S^*_\sin}(T_1) = 2(\sin 1)/(3 + 2 \sin 1) \approx 0.35938$.
2. $R_{S^*_\sin}(T_2) = (\sin 1)/3$.

**Proof.** It was proved in [3] that $\{w : |w-a| < \sin 1-|a-1|\} \subseteq q(\mathbb{D})$ for $|a-1| \leq \sin 1$, where $q(z) := 1 + \sin z$. For $a = 1$, this implies that $S^*_{\sin 1} \subset S^*_\sin$. Thus $R_{S^*_{\sin 1}}(T_i) \leq R_{S^*_\sin}(T_i)$ for $i = 1, 2$. The proof is completed by demonstrating $R_{S^*_\sin}(T_i) \leq R_{S^*_{\sin 1}}(T_i)$ for $i = 1, 2$.

(i) Evaluating the function $f_1(z) = z(1 + z)^{3/2}$ at $z = -R = -R_{S^*_{\sin 1}}(T_1) = -2\sin 1/(3 + 2 \sin 1)$ gives

$$\frac{zf_1'(z)}{f_1(z)} = \frac{2 + 5z}{2 + 2z} = \frac{2 - 5R}{2 - 2R} = 1 - \sin 1 = q(-1).$$

Thus, $R_{S^*_\sin}(T_1) \leq 2\sin 1/(3 + 2 \sin 1)$.

(ii) Similarly, at $z = \pm R = \pm R_{S^*_{\sin 1}}(T_2) = \pm(\sin 1)/3$, the function $f_2(z) = ze^{3z}$ yields

$$\frac{zf_2'(z)}{f_2(z)} = 1 + 3z = 1 \pm 3R = 1 \pm \sin 1 = q(\pm 1).$$

This proves that $R_{S^*_\sin}(T_2) \leq (\sin 1)/3$.

Consider next the class $S^*_\ell := S^*(z + \sqrt{1 + z^2})$ introduced by Raina and Sokól in [18]. Functions $f \in S^*_\ell$ provided $zf'(z)/f(z)$ lies in the region bounded by the lune $\Omega_\ell := \{w : |w^2 - 1| < 2|w|\}$. The result below gives the radius of lune starlikeness for each class $T_1$ and $T_2$.

**Corollary 3.4.** The following are the $S^*_\ell$-radius for each class $T_1$ and $T_2$:

1. $R_{S^*_\ell}(T_1) = 2(\sqrt{2} - 2)/(2\sqrt{2} - 7) \approx 0.280847$.
2. $R_{S^*_\ell}(T_2) = (2 - \sqrt{2})/3$.

**Proof.** It was shown by Gandhi and Ravichandran [5] Lemma 2.1 that $\{w : |w-a| < 1 - |\sqrt{2} - a|\} \subseteq \Omega_\ell$ for $\sqrt{2} - 1 < a \leq \sqrt{2} + 1$. Choosing $a = 1$, the inclusion gives $S^*_{\sqrt{2} - 1} \subset S^*_\ell$. Thus $R_{S^*_{\sqrt{2} - 1}}(T_i) \leq R_{S^*_\ell}(T_i)$ for $i = 1, 2$. We complete the proof by demonstrating $R_{S^*_\ell}(T_i) \leq R_{S^*_{\sqrt{2} - 1}}(T_i)$ for $i = 1, 2$.

(i) Evaluating the function $f_1(z) = z(1 + z)^{3/2}$ at $z = -R = -R_{S^*_{\sqrt{2} - 1}}(T_1) = -2(\sqrt{2} - 2)/(2\sqrt{2} - 7)$ gives

$$\left|\frac{zf_1'(z)}{f_1(z)} - 1\right| = \left|\frac{2 + 5z}{2 + 2z} - 1\right| = \left|\frac{2 - 5R}{2 - 2R} - 1\right| = 0.828 = 2 \left|\frac{zf_1'(z)}{f_1(z)}\right|.$$ 

Thus, $R_{S^*_\ell}(T_1) \leq 2(\sqrt{2} - 2)/(2\sqrt{2} - 7)$. 


(ii) Similarly, at \( z = -R = -R_{S^*_{\sqrt{2}-1}}(T_2) = -(2 - \sqrt{2})/3 \), the function \( f_2(z) = ze^{3z} \) yields

\[
\left( \frac{zf'_2(z)}{f_2(z)} \right)^2 - 1 = |(1 + 3z)^2 - 1| = |(1 - 3R)^2 - 1| = 0.828 = 2 \frac{zf'_2(z)}{f_2(z)}. 
\]

This proves that \( R_{S^*_{\sqrt{2}}}^*(T_2) \leq (2 - \sqrt{2})/3 \).

As a further example, consider next the class \( S^*_R := S^*(\eta(z)) \), where \( \eta(z) = 1 + ((z + z^2)/(k^2 - kz)) \), \( k = \sqrt{2} + 1 \). This class associated with a rational function was introduced and studied by Kumar and Ravichandran in [10].

**Corollary 3.5.** The following are the \( S^*_R \)-radius for the classes \( T_1 \) and \( T_2 \):

(i) \( R_{S^*_2}(T_1) = 2(-3 + 2\sqrt{2})/(4\sqrt{2} - 9) \approx 0.102642 \),

(ii) \( R_{S^*_2}(T_2) = (3 - 2\sqrt{2})/3 \).

**Proof.** It was shown in [10] that \( \{ w : |w - a| < a - 2(\sqrt{2} - 1) \} \subseteq \eta(\mathbb{D}) \) for \( 2(\sqrt{2} - 1) < a \leq \sqrt{2} \). This inclusion with \( a = 1 \) gives \( S^*_2(\sqrt{2}-1) \subset S^*_R \). Thus \( R_{S^*_R}(T_i) \leq R_{S^*_2}(T_i) \) for \( i = 1, 2 \). We next show that \( R_{S^*_R}(T_1) \leq R_{S^*_2}(T_i) \) for \( i = 1, 2 \).

(i) At \( z = -R = -R_{S^*_2}(T_1) = -2(-3 + 2\sqrt{2})/(4\sqrt{2} - 9) \), the function \( f_1(z) = z(1 + z)^{3/2} \) yields

\[
\frac{zf'_1(z)}{f_1(z)} = \frac{2 - 5R}{2 - 2R} = 2(\sqrt{2} - 1) = \eta(1).
\]

Thus, \( R_{S^*_R}(T_1) \leq 2(-3 + 2\sqrt{2})/(4\sqrt{2} - 9) \).

(ii) Evaluating \( f_2(z) = ze^{3z} \) at \( z = -R = -R_{S^*_2}(T_2) = -(3 - 2\sqrt{2})/3 \) gives

\[
\frac{zf'_2(z)}{f_2(z)} = 1 - 3R = 2(\sqrt{2} - 1) = \eta(1).
\]

Thus \( R_{S^*_R}(T_2) \leq (3 - 2\sqrt{2})/3 \).

The class \( S^*_N_e := S^*(\psi(z)) \), where \( \psi(z) = 1 + z - z^3/3 \), was introduced and studied by Wani and Swaminathan in [31]. Geometrically, \( f \in S^*_N_e \) provided \( zf'(z)/f(z) \) lies in the region bounded by the nephroid: a 2-cusped kidney shaped curve \( \Omega_{N_e} := \{ w = u + iv : ((u - 1)^2 + v^2 - 4/9)^3 - 4v^2/3 = 0 \} \).

**Corollary 3.6.** The following are the \( S^*_N_e \)-radius for the classes \( T_1 \) and \( T_2 \):

(i) \( R_{S^*_N_e}(T_1) = 4/13 \),

(ii) \( R_{S^*_N_e}(T_2) = 2/9 \).

**Proof.** It was shown in [31] that \( \{ w : |w - a| < a - 1/3 \} \subseteq \Omega_{N_e} \) for \( 1/3 < a \leq 1 \). This inclusion with \( a = 1 \) gives \( S^*_{1/3} \subset S^*_N_e \). This shows that \( R_{S^*_1}(T_i) \leq R_{S^*_N_e}(T_i) \) for \( i = 1, 2 \). We next show that \( R_{S^*_N_e}(T_i) \leq R_{S^*_1}(T_i) \) for \( i = 1, 2 \).

(i) Evaluating the function \( f_1(z) = z(1 + z)^{3/2} \) at \( z = -R = -R_{S^*_1}(T_i) = -4/13 \) results in

\[
\frac{zf'_1(z)}{f_1(z)} = \frac{2 - 5R}{2 - 2R} = \frac{1}{3} = \psi(-1).
\]
Thus, \( R_{S_{N_e}}(T_1) \leq 4/13 \).

(ii) Similarly, evaluating \( f_2(z) = ze^{3z} \) at \( z = -R = -R_{S_1/3}(T_2) = -2/9 \) yields

\[
\frac{zf_2(z)}{f_2(z)} = 1 - 3R = \frac{1}{3} = \psi(-1).
\]

This proves that \( R_{S_{N_e}}(T_2) \leq 2/9 \).

Finally, we consider the class \( S_{SG}^* := S^*(2/(1 + e^{-z})) \) introduced by Goel and Kumar in [6]. Here \( 2/(1 + e^{-z}) \) is the modified sigmoid function that maps \( \mathbb{D} \) onto the region \( \Omega_{SG} := \{ w = u + iv : |\log(w/(2 - w))| < 1 \} \). Thus, \( f \in S_{SG}^* \) provided the function \( zf'(z)/f(z) \) maps \( \mathbb{D} \) onto the region lying inside the domain \( \Omega_{SG} \).

**Corollary 3.7.** The \( S_{SG}^* \)-radius for the class \( T_1 \) is

\[
R_{S_{SG}}(T_1) = (2e - 2)/(1 + 5e) \approx 0.23552,
\]

while that of \( T_2 \) is

\[
R_{S_{SG}}(T_2) = (e - 1)/(3(1 + e)).
\]

**Proof.** The inclusion \( \{ w : |w - a| < ((e - 1)/(e + 1)) - |a - 1| \} \subseteq \Omega_{SG} \) holds for \( 2/(1 + e) < a < 2e/(1 + e) \) (see [6]). At \( a = 1 \), the set inclusion shows that \( S_{2/(e+1)}^* \subseteq S_{SG}^* \).

It was also shown in [6] that \( S_{SG}^* \subseteq S^*(\alpha) \) for \( 0 \leq \alpha \leq 2/(e + 1) \). The desired result is now an immediate consequence of Theorem 2.1.

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