Recent progress in Liouville field theory

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Abstract

An explicit construction for the monodromy of the Liouville conformal blocks in terms of Racah-Wigner coefficients of the quantum group $U_q(su(2,R))$ is proposed. As a consequence, crossing-symmetry for four point functions is analytically proven, and the expression for the correlator of three boundary operators is obtained.

1 Introduction

Liouville field theory has permanently attracted attention in the last two decades in the context of strings in non critical space-time dimensions; early works on the subject can be found in [1, 2, 3, 4, 5, 6]. It is also an alternative approach to matrix models in the study of 2D gravity [7]. In the context of the $AdS/CFT$ correspondence (see [8] and references therein), it is also very closely related to the $SL(2,C)/SU(2)$ and $SL(2,R)$ WZNW models that describe strings propagating on Euclidean and Lorentzian $AdS_3$.

Liouville field theory is the simplest case of non compact conformal field theories, i.e. with a continuous spectrum of primary fields, and serves as a prototype to develop techniques that can also be useful in the study of more complicated CFT that share similar features such as non compactness.

Given knowledge of conformal symmetry, spectrum and three point function, one has an unambiguous construction for any genus zero correlation function by summing over intermediate states. But the decomposition of an n-point function as a sum over three point functions can be performed in different ways. Equivalence of such decompositions (crossing symmetry, also called bootstrap condition) can be seen as the most difficult sufficient condition to verify in showing consistency of the CFT as characterized by the spectrum and the three point function. In this lecture, we will first review the articles [9,10]. In [9], the crossing-symmetry condition for four-point functions is proven analytically in the weak coupling regime where the

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1This article gathers results obtained in collaboration with J. Teschner [9,10,21] between 1997 and 2001.
Liouville central charge $c_L \geq 25$, thanks to the explicit construction of the monodromy (or fusion matrix) of the conformal blocks\(^2\). The expressions can be continued into the strong coupling regime $1 < c_L < 25$ by analytical continuation, providing thus a solution of the bootstrap in this region. The fusion matrix is constructed in terms of Racah-Wigner coefficients (or 6j symbol) for an appropriate continuous series of representations of the quantum group $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$, where the deformation parameter $q = e^{\pi b^2}$ ($b$ real) is related to the Liouville central charge by $c_L = 1 + 6(b + b^{-1})^2$, $c_L \geq 25$. We refer the reader to \cite{10} for the mathematical details of this construction.

Although the relation between CFTs and quantum groups is still rather mysterious, there is a long story of connections between Liouville field theory and quantum groups \cite{11, 12, 13, 14}, and more recently in \cite{15, 16, 17}. Our approach is somewhat an extension of the one developed in the latter papers. We would like to mention that the representations of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ whose study is relevant for our purposes (we also refer the reader to \cite{18} for discussions on the closely related concept of modular double, first introduced by L. Faddeev), are continuous, autodual in the sense that they remain unchanged when $b$ is replaced by $b^{-1}$, and do not have any classical counterpart.

Once the monodromy of the conformal blocks is constructed, it is not so difficult to deduce the explicit expression for the boundary three point function: an ansatz for it in terms of the fusion matrix leads to a solution of the consistency condition that expresses the associativity of the product of boundary operators \cite{20}. The last task is to fix the remaining freedom by imposing certain normalization conditions. This result is published in \cite{21} and is derived in the last section of the lecture.

\section{Preliminaries}

We consider Liouville field theory defined in the bulk by the following action:

$$\mathcal{A}_L = \frac{1}{4\pi} \int_{\Gamma} \left[ g^{ab} \partial_a \phi \partial_b \phi + QR\phi + 4\pi \mu e^{2b\phi} \right] \sqrt{g} d^2 x ,$$

where $R$ is the scalar curvature associated with the background metric $g$ and $Q = b + 1/b$ is called the background charge that determines the central charge of the theory

$$c_L = 1 + 6Q^2 .$$

We will consider in the following the so-called weak coupling regime, where the central charge $c_L \geq 25$. Let us note that the latter remains unchanged under the transformation $b \rightarrow 1/b$; so we take here the opportunity to mention that at the quantum level (\textit{only}), Liouville theory enjoys the property

\(^2\)Another method obtained after the one presented here can be found in \cite{19}.
of autoduality, i.e. the observables of the theory are invariant when $b$ and $1/b$ are exchanged.

In what follows, we will consider Liouville theory on the flat plane with trivial background metric $g_{ab} = \delta_{ab}$; in this case the action reads

$$\mathcal{A}_L = \int \left( \frac{1}{4\pi} (\partial_a \phi)^2 + \mu e^{2b\phi} \right) d^2 x,$$

with the curvature sent at spatial infinity, which means for the Liouville field the following boundary condition

$$\phi(z, \bar{z}) = -Q \log(z\bar{z}) + O(1) \quad \text{at } |z| \to \infty.$$

We note the conformal primary fields $V_\alpha(z, \bar{z})$ (classically one has $V_\alpha(z, \bar{z}) = e^{\alpha/2b\varphi_z}$, where $\varphi = 2b\phi$ is the classical Liouville field). These fields are primaries with respect to the energy momentum tensor

$$T(z) = - (\partial \phi)^2 + Q \partial^2 \phi,$$
$$\bar{T}(\bar{z}) = - (\bar{\partial} \phi)^2 + Q \bar{\partial}^2 \phi,$$

and have conformal weight $\Delta_\alpha = \bar{\Delta}_\alpha = \alpha(Q - \alpha)$. Because of the invariance $\alpha \to Q - \alpha$, one identifies the operator $V_\alpha$ with its reflected image $V_{Q - \alpha}$. In this lecture we will consider the following two important sets of values for $\alpha$.

- **Space of states:** $\alpha = \frac{Q}{2} + iP$, $P > 0$

  Because of conformal invariance, the space of states $\mathcal{H}$ decomposes as a direct sum over tensor products $V_\alpha \otimes \overline{V}_\alpha$ of highest weight representations of the left/right Virasoro algebra. Arguments based on canonical quantization \[2\] suggest the following spectrum:

$$\mathcal{H} = \int_{\mathbb{S}} V_\alpha \otimes \overline{V}_\alpha, \quad \mathbb{S} = \frac{Q}{2} + i\mathbb{R}^+$$

where each $V_\alpha$ contains a primary state $v_\alpha$ satisfying

$$L_nv_\alpha = \bar{L}_n v_\alpha = 0, \quad n > 0$$
$$L_0 v_\alpha = \bar{L}_0 v_\alpha = (Q^2/4 + P^2) v_\alpha.$$

- **Degenerate Virasoro representations:**

  The fields $V_{-nb/2}$, $n \in \mathbb{N}$, are degenerate with respect to the conformal symmetry algebra and satisfy linear differential equations \[22\]. For example, the first non trivial case consists of $\alpha = -b/2$, and the corresponding operator satisfies

$$\left( \frac{1}{b^2} \partial^2 + T(z) \right) V_{-b/2} = 0,$$
as well as the complex conjugate equation.

For $n = 2$, the operator $V_{-b}$ satisfies two complex conjugate differential equations of the third order

$$
\left( \frac{1}{2b^2} \partial^3 + 2T(z)\partial + (1 + 2b^2)\partial T(z) \right) V_{-b} = 0.
$$

It follows from these equations that when one performs the operator product expansion of one of these degenerate operators with a generic operator, then the OPE is truncated \[22\]. For example:

$$
V_{-b/2} V_{\alpha} = c_+ V_{\alpha - b/2} + c_- V_{\alpha + b/2}.
$$

So $\alpha = -b/2$ is nothing but the usual spin 1/2 (multiplied by $-b$); according to the fusion rules, its tensor product with a generic representation of spin $\alpha_1$ gives a sum of representations with spin $\alpha_2 = \alpha_1 \pm b/2$.

The structure constants $c_\pm$ are special cases of the bulk three point function, and can be computed perturbatively\[3\] as Coulomb gas (or screening) integrals \[24,25\]. One can take $c_+ = 1$, as in this case there is no need of insertion of interaction, whereas $c_-$ requires one insertion of the Liouville interaction $-\mu \int e^{2b\phi} d^2z$, and

$$
c_- = -\mu \int d^2z \left\langle V_{\alpha}(0)V_{-b/2}(1)e^{2b\phi(z,\bar{z})}V_{Q-\alpha-b/2}(\infty) \right\rangle
$$

$$
= -\mu \int d^2z |z|^{2b\alpha} |1 - z|^{-b^2}
$$

$$
= -\mu \frac{\pi \gamma(2b\alpha - 1 - b^2)}{\gamma(-b^2)\gamma(2b\alpha)}.
$$

where $\gamma(x) = \Gamma(x)/\Gamma(1 - x)$; in the first line, we used the property of invariance under global transformations to set $z_1 = 0$, $z_3 = 1$, $z_4 = \infty$, and in the second line $\langle \phi(x)\phi(y) \rangle = -\log|x - y|$.

There exists also a dual series of degenerate operators $V_{-m/2b}$ which have truncated operator expansion. If $\alpha_2$ is a degenerate Virasoro representation ($\alpha_2 = -nb/2 - m/2b$), then the fusion rules are such that $\alpha_{21} - \alpha_1 - \alpha_2 = sb + tb^{-1}$, $s, t \in \mathbb{N}$ \[22\]. In this case, the structure constants can be computed perturbatively with $s$ insertions of the Liouville interaction $-\mu \int e^{2b\phi(z,\bar{z})} d^2z$ and $t$ insertions of the dual interaction $-\tilde{\mu} \int e^{2b^{-1}\phi(z,\bar{z})} d^2z$ where the dual cosmological constant $\tilde{\mu}$ is related to the cosmological constant by the formula

$$
\pi \tilde{\mu} \gamma(1/b^2) = (\pi \mu \gamma(b^2))^{1/b^2}.
$$

This means that from the point of view of the path integral, everything occurs as if we had added a dual interaction term in the action. This

\[3\] One of the first step in this direction was developed in \[23\].
term is compatible with conformal invariance, as $e^{2b^{-1}}\phi$ has conformal weight $(1,1)$. Finally, if $\alpha_{21} - \alpha_1 - \alpha_2 = sb + tb^{-1}$, then the three point function can be evaluated by computing the integral
\[
\frac{(-\mu)^s(-\tilde{\mu})^t}{s!t!} \int d^2u_1 \ldots d^2u_s \int d^2v_1 \ldots d^2v_t \left( V_{\alpha_1}(0)V_{\alpha_2}(1) \prod_{i=1}^{s} e^{2b\phi(u_i,u_i)} \prod_{j=1}^{t} e^{2b^{-1}\phi(v_j,v_j)} V_{Q-\alpha_{21}}(\infty) \right).
\]
whose explicit expression can be found in [25].

2.1 Three point function [26, 27]

The spatial dependence of the three point correlation function is completely determined by conformal symmetry,
\[
\mathcal{G}_{\alpha_1,\alpha_2,\alpha_3}(z_1, z_2, z_3) = |z_{12}|^{2\gamma_1} |z_{23}|^{2\gamma_2} |z_{31}|^{2\gamma_3} \mathcal{C}(\alpha_3, \alpha_2, \alpha_1)
\]
where $\gamma_1 = \Delta_{\alpha_1} - \Delta_{\alpha_2} - \Delta_{\alpha_3}$, $\gamma_2 = \Delta_{\alpha_2} - \Delta_{\alpha_3} - \Delta_{\alpha_1}$, $\gamma_3 = \Delta_{\alpha_3} - \Delta_{\alpha_1} - \Delta_{\alpha_2}$. It remains only to determine the pure number $\mathcal{C}(\alpha_3, \alpha_2, \alpha_1)$. For generic values of $\alpha_i, i = 1,2,3$, the following expression for this structure constant was proposed by [26, 27]
\[
\mathcal{C}(\alpha_3, \alpha_2, \alpha_1) = \frac{\pi \mu \gamma(b^2) b^{2-2b^2}}{\Theta_0(2\alpha_1) \Theta_0(2\alpha_2) \Theta_0(2\alpha_3)} 
\mathcal{Y}_b(\alpha_1 + \alpha_2 + \alpha_3 - Q) \mathcal{Y}_b(\alpha_1 + \alpha_2 - \alpha_3) \mathcal{Y}_b(\alpha_1 + \alpha_3 - \alpha_2) \mathcal{Y}_b(\alpha_2 + \alpha_3 - \alpha_1)
\]
where the special function\footnote{Definitions and properties of the special functions $\Gamma_b, \mathcal{Y}_b, S_b$ are collected in the Appendix B.} $\mathcal{Y}_b$ is such defined: $\mathcal{Y}_b^{-1}(x) \equiv \Gamma_b(x)\Gamma_b(Q - x)$, with $\Gamma_b(x) \equiv \Gamma_2(x|x|b^{-1})$, and $\Gamma_2(x|\omega_1, \omega_2)$ is the Double Gamma function introduced by Barnes [28], which definition is
\[
\log \Gamma_2(x|\omega_1, \omega_2) = \left( \frac{\partial}{\partial t} \sum_{n_1,n_2=0}^{\infty} (x + n_1 \omega_1 + n_2 \omega_2)^{-t} \right)_{t=0}.
\]
The function $\Gamma_b(x)$ defined above is such that $\Gamma_b(x) \equiv \Gamma_{b^{-1}}(x)$, and satisfies the following functional relation
\[
\Gamma_b(x + b) = \frac{\sqrt{2\pi} b^{bx - \frac{1}{2}}}{\Gamma(bx)} \Gamma_b(x),
\]
as well as the dual functional relation with $b$ replaced by $1/b$. $\mathcal{Y}_0 = \text{res}_{x=0} \frac{d \mathcal{Y}_b(x)}{dx}$.

The three point function enjoys remarkable properties:
• It has poles located at \( \alpha_{21} - \alpha_1 - \alpha_2 = sb + tb^{-1} \), \( s, t \in \mathbb{N} \) (other poles are obtained by reflection \( \alpha_i \rightarrow Q - \alpha_i \)), whose residues exactly coincide with the perturbative computation in terms of screening integrals explained above.

• It satisfies the reflection property, for each \( \alpha_i \)

\[
C(\alpha_3, \alpha_2, \alpha_1) = C(\alpha_3, \alpha_2, Q - \alpha_1)S(\alpha_1),
\]

where \( S(\alpha) \) is the reflection amplitude that enters the two point function

\[
\langle V_{\alpha_2}(z_2, \bar{z}_2)V_{\alpha_1}(z_1, \bar{z}_1) \rangle = [\delta(\alpha_2 + \alpha_1 - Q) + S(\alpha_1)\delta(\alpha_2 - \alpha_1)] |z_1 - z_2|^{4\Delta_{\alpha_1}}.
\]

Explicitly:

\[
S(\alpha) = \left( \frac{\pi \mu \gamma(b^2)}{b^2} \right)^{(Q-2\alpha)/b} \frac{\gamma(2\alpha b - b^2)}{\gamma(2 - 2\alpha/b + 1/b^2)}.
\]

The reflection amplitude satisfies the unitarity property

\[ S(\alpha)S(Q - \alpha) = 1. \]

• It is invariant when \( b \) is replaced by \( 1/b \), and \( \mu \) replaced by \( \tilde{\mu} \) defined in \([11]\).

2.2 Crossing symmetry

With the three point function and the space of states at hand, we are in a position to write down a four point function by summing over intermediate states. Let us denote the four point function

\[
\langle V_{\alpha_4}(\infty)V_{\alpha_3}(1)V_{\alpha_2}(z, \bar{z})V_{\alpha_1}(0) \rangle \equiv V_{\alpha_4,\alpha_3,\alpha_2,\alpha_1}(z, \bar{z}).
\]

It can be written first in the \( s \)-channel

\[
V_{\alpha_4,\alpha_3,\alpha_2,\alpha_1}(z, \bar{z}) = \int d\alpha_{21} C(\alpha_4, \alpha_3, \alpha_{21})C(Q - \alpha_{21}, \alpha_2, \alpha_1)|F^s(\Delta_{\alpha_i}, \Delta_{\alpha_{21}} | z)\|^2
\]

\( F^s(\Delta_{\alpha_i}, \Delta_{\alpha_{21}} | z) \) is the \( s \)-channel Liouville conformal block \([22]\), represented by power series of the form

\[
F^s(\Delta_{\alpha_i}, \Delta_{\alpha_{21}} | z) = z^{\Delta_{\alpha_i}+\Delta_{\alpha_{21}}-\Delta_3-\Delta_4} \sum_{n=0}^{\infty} z^n \mathcal{F}^s_n(\Delta_{\alpha_i}, \Delta_{\alpha_{21}})
\]
where \( z_{ij} = z_i - z_j, \ i, j = 1, \ldots, 4 \) and the cross ratio \( z = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}. \) It is a \textit{chiral} object, completely determined by Virasoro symmetry (although there is no closed form known for it in general). It depends on conformal weights only. We can use invariance under global transformations to set \( z_1 = 0, z_3 = 1, z_4 = \infty, \) so \( z \equiv z_2. \) The conformal block is normalized such that

\[
\mathcal{F}^s(\Delta_{\alpha_i}, \Delta_{\alpha_{21}}, z) \sim z \to 0 \ z^{\Delta_{\alpha_{21}} - \Delta_{\alpha_1} - \Delta_{\alpha_2} (1 + O(z))}.
\]

- We would like to stress that in general, the set \( \mathcal{D} \) does not coincide with the spectrum \( \mathcal{S}. \) This is the case however if the external \( \alpha_i \) are in the range

\[
2|\text{Re}(\alpha_1 + \alpha_2 - Q)| < Q, \quad 2|\text{Re}(\alpha_1 - \alpha_2)| < Q,
\]

\[
2|\text{Re}(\alpha_3 + \alpha_4 - Q)| < Q, \quad 2|\text{Re}(\alpha_3 - \alpha_4)| < Q.
\]

as the three point functions are analytic in this range. Outside of this range, meromorphic continuation should be understood. Some poles of the three point functions will cross the real \( P \) axis and their residues will give additional contributions besides the integral over the spectrum \( [20, 27]. \) For our discussion, it will be enough to consider the external spins in the range above, so we will integrate over the spectrum only.

- The four point function can be written equivalently in the \( t \)-channel

\[
\mathcal{V}_{\alpha_4, \alpha_3, \alpha_2, \alpha_1}(z, \bar{z}) = \int_\mathcal{S} d\alpha_{32} C(\alpha_4, \alpha_{32}, \alpha_1) C(Q - \alpha_{32}, \alpha_3, \alpha_2) |\mathcal{F}^t(\Delta_{\alpha_i}, \Delta_{\alpha_{32}} | 1 - z)|^2
\]

By comparison with the \( s \)-channel, \( (\alpha_1, z_1) \) and \( (\alpha_3, z_3) \) have been exchanged, so the cross ratio \( z \) is replaced by \( 1 - z \) in the \( t \)-channel.

- There exist\(^5\) invertible fusion transformations between \( s \)- and \( t \)-channel conformal blocks, defining the Liouville fusion matrix (or monodromy of conformal blocks):

\[
\mathcal{F}^s(\Delta_{\alpha_i}, \Delta_{\alpha_{21}} | z) = \int_\mathcal{S} d\alpha_{32} F_{\alpha_{21}, \alpha_{32}} [ \begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{array} ] \mathcal{F}^t(\Delta_{\alpha_i}, \Delta_{\alpha_{32}} | 1 - z).
\]

Then the crossing symmetry condition becomes

\[
\int_\mathcal{S} d\alpha_{21} C(\alpha_1, \alpha_3, \alpha_{21}) C(Q - \alpha_{21}, \alpha_2, \alpha_1) F_{\alpha_{21}, \alpha_{32}} [ \begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{array} ] \left( F_{\alpha_{21}, \beta_{32}} [ \begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{array} ] \right)^* = \delta(\alpha_{32} - \beta_{32}) C(\alpha_4, \alpha_{32}, \alpha_1) C(Q - \alpha_{32}, \alpha_3, \alpha_2)
\]

\(^5\)This is supported by an explicit computation in a particular case treated in [30], and was subsequently proven in [19].
2.2.1 Properties of the fusion matrix

- It is quite obvious that the fusion matrix should be invariant when exchanging columns and rows:

\[
F_{\alpha_2 \alpha_3} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} = F_{\alpha_2 \alpha_3} \begin{bmatrix} \alpha_4 & \alpha_1 \\ \alpha_3 & \alpha_2 \end{bmatrix} = F_{\alpha_2 \alpha_3} \begin{bmatrix} \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_4 \end{bmatrix}
\]

- As the conformal blocks depend on conformal weights only, so does the fusion matrix, i.e. it remains unchanged when any of the six \(\alpha_i, i = 1, \ldots, 4\), is changed into \(Q - \alpha_i\).

- Moore-Seiberg equations (or polynomial equations) \[31\]. Although the Moore-Seiberg equations are proven to hold in rational conformal field theory, we believe they continue to hold even in the non compact case for coherency of the operator algebra. Actually, we will prove in the next section that they do hold in the case of Liouville field theory. Let us remind that they consist of

  – one pentagonal equation

\[
\int_{\frac{Q}{2} + i\mathbb{R}^+} d\delta_1 F_{\beta_1 \delta_1} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \beta_2 & \alpha_1 \end{bmatrix} F_{\beta_2 \gamma_2} \begin{bmatrix} \alpha_4 & \delta_1 \\ \alpha_5 & \alpha_1 \end{bmatrix} F_{\delta_1 \gamma_1} \begin{bmatrix} \alpha_4 & \alpha_3 \\ \gamma_2 & \alpha_2 \end{bmatrix} = F_{\beta_2 \gamma_1} \begin{bmatrix} \alpha_4 & \alpha_3 \\ \beta_1 & \alpha_1 \end{bmatrix} F_{\beta_1 \gamma_2} \begin{bmatrix} \gamma_1 & \alpha_2 \\ \alpha_5 & \alpha_1 \end{bmatrix}
\]

  – two hexagonal equations

\[
F_{\alpha_2 \beta_1} \begin{bmatrix} \alpha_4 & \alpha_2 \\ \alpha_3 & \alpha_1 \end{bmatrix} e^{i\pi \epsilon(\Delta_{\alpha_2} + \Delta_{\alpha_4} + \Delta_{\alpha_3} + \Delta_{\alpha_1} - \Delta_{\alpha_2} - \Delta_{\beta})} = \\
\int_{\frac{Q}{2} + i\mathbb{R}^+} d\alpha_3 F_{\alpha_2 \alpha_3} \begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} F_{\alpha_3 \beta_2} \begin{bmatrix} \alpha_2 & \alpha_4 \\ \alpha_3 & \alpha_1 \end{bmatrix} e^{i\pi \epsilon \Delta_{\alpha_3}}
\]

where \(\epsilon = \pm\).

These equations generate in genus zero all other polynomial equations \[31\].

- If one of the \(\alpha_i, i = 1, \ldots, 4\), corresponds to a degenerate Virasoro representation, then the fusion transformations simplify: the conformal blocks then exist only for a finite number of values of \(\alpha_{21}\) and \(\alpha_{32}\), so that the fusion coefficients form a finite dimensional matrix \[22\]. It is enough to consider the case where \(\alpha_2 = -b/2\) or \(-1/2b\),...
higher cases are generated by the use of the pentagonal equation\(^6\).
For \(\alpha_2 = -b/2\), the conformal blocks are known explicitly in terms of hypergeometric functions \(2F_1\)\(^{[22]}\). They read in the \(s\)-channel:

\[
\mathcal{F}^s_+ = z^{b\alpha_1} (1-z)^{b\alpha_3} \times 2F_1(b(\alpha_1 + \alpha_3 + \alpha_4 - 3b/2) - 1, b(\alpha_1 + \alpha_3 - \alpha_4 + b/2); b(2\alpha_1 - b); z),
\]

\[
\mathcal{F}^s_- = z^{1-b(\alpha_1-b)} (1-z)^{b\alpha_3} \times 2F_1(b(-\alpha_1 + \alpha_3 + \alpha_4 - b/2), b(-\alpha_1 + \alpha_3 - \alpha_4 + b/2) + 1; 2 - b(2\alpha_1 - b); z).
\]

and in the \(t\)-channel:

\[
\mathcal{F}^t_+ = z^{b\alpha_1} (1-z)^{b\alpha_3} \times 2F_1(b(\alpha_1 + \alpha_3 + \alpha_4 - 3b/2) - 1, b(\alpha_1 + \alpha_3 - \alpha_4 + b/2); b(2\alpha_3 - b); 1 - z),
\]

\[
\mathcal{F}^t_- = z^{b\alpha_1} (1-z)^{1-b(\alpha_3-b)} \times 2F_1(b(\alpha_1 - \alpha_3 - \alpha_4 + b/2) + 1, b(\alpha_1 - \alpha_3 + \alpha_4 - b/2); 2 - b(2\alpha_3 - b); 1 - z).
\]

The computation of the monodromy of these quantities is then standard. Let us introduce

\[
F_{\alpha_1-sb/2,\alpha_3-s'b/2} \binom{\alpha_3}{\alpha_4} \frac{-b/2}{\alpha_1} \equiv F_{s,s'}^{L}, \ s, s' = \pm.
\]

Then the entries of this \(2 \times 2\) matrix are:

\[
F_{++} = \frac{\Gamma(b(2\alpha_1 - b))\Gamma(b(b - 2\alpha_3) + 1)}{\Gamma(b(\alpha_1 - \alpha_3 - \alpha_4 + b/2) + 1)\Gamma(b(\alpha_1 - \alpha_3 + \alpha_4 - b/2))}
\]

\[
F_{+-} = \frac{\Gamma(b(\alpha_1 + \alpha_3 + \alpha_4 - 3b/2) - 1)\Gamma(b(\alpha_1 + \alpha_3 - \alpha_4 - b/2))}{\Gamma(b(2\alpha_1 - b))\Gamma(b(b - 2\alpha_3 - b) - 1)}
\]

\[
F_{-+} = \frac{\Gamma(2 - b(2\alpha_1 - b))\Gamma(b(b - 2\alpha_3 - b) + 1)}{\Gamma(2 - b(\alpha_1 + \alpha_3 + \alpha_4 - 3b/2))\Gamma(1 - b(\alpha_1 + \alpha_3 - \alpha_4 - b/2))}
\]

\[
F_{--} = \frac{\Gamma(2 - b(\alpha_1 - b))\Gamma(b(2\alpha_3 - b) - 1)}{\Gamma(b(-\alpha_1 + \alpha_3 + \alpha_4 - b/2))\Gamma(b(-\alpha_1 + \alpha_3 - \alpha_4 + b/2) + 1)}
\]

The dual case with \(\alpha_2 = -b^{-1}/2\) is obtained by substituting \(b\) by \(b^{-1}\).

### 2.2.2 Construction of the fusion matrix: strategy \(^9\)

We now make the following change of normalization that preserves the Moore-Seiberg equations:

\[
F_{\alpha_2 \alpha_3} \binom{\alpha_3}{\alpha_4} \binom{\alpha_2}{\alpha_1} = \frac{N(\alpha_4, \alpha_3, \alpha_2)N(\alpha_21, \alpha_2, \alpha_1)}{N(\alpha_4, \alpha_32, \alpha_1)N(\alpha_32, \alpha_3, \alpha_2)} G_{\alpha_2 \alpha_3} \binom{\alpha_3}{\alpha_4} \binom{\alpha_2}{\alpha_1}
\]

\(^6\)For example, the fusion coefficients \(F_{\beta_2 \gamma_1} \binom{\alpha_4}{\alpha_5} \binom{\alpha_3}{-b}\) are computed by setting in the pentagonal equation \(\alpha_1 = \alpha_2 = -b/2\).
where the function \( N(\alpha_3, \alpha_2, \alpha_1) \) reads

\[
N(\alpha_3, \alpha_2, \alpha_1) = \frac{\Gamma_b(2\alpha_1)\Gamma_b(2\alpha_2)\Gamma_b(2Q - 2\alpha_3)}{\Gamma_b(2Q - \alpha_1 - \alpha_2 - \alpha_3)\Gamma_b(\alpha_1 + \alpha_2 + \alpha_3 - \alpha_2)\Gamma_b(\alpha_2 + \alpha_3 - \alpha_1)}.
\]

Now the crossing symmetry condition takes the form of an orthogonality relation for \( G_{\alpha_21\alpha_32} \):

\[
\int d\alpha_{21} M_b(\alpha_{21}) G_{\alpha_{21}\alpha_{32}} \left[ \begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{array} \right] \left( G_{\alpha_{21}\beta_{32}} \left[ \begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{array} \right] \right)^* = M_b(\alpha_{32})\delta(\alpha_{32} - \beta_{32})
\]

where

\[
M_b(\alpha) = -4\sin \pi b(2\alpha - Q)\sin \pi b^{-1}(2\alpha - Q).
\]

If \( \alpha_2 = -b/2 \), the matrix \( G_{s,s', s, s'} = \pm \)

\[
G_{s,s'} = \begin{pmatrix}
\frac{[\alpha_4 + \alpha_3 - \alpha_4 - \frac{b}{2}]}{[2\alpha_3 - b]} & \frac{[\alpha_4 + \alpha_3 + \alpha_1 - \frac{3b}{2}]}{[2\alpha_3 - b]}
\frac{[\alpha_4 + \alpha_3 - \frac{b}{2}]}{[2\alpha_3 - b]} & \frac{[\alpha_4 + \alpha_1 - \alpha_3 - \frac{b}{2}]}{[2\alpha_3 - b]}
\end{pmatrix}
\]

where \([x]_b = \frac{\sin(\pi bx)}{\sin \pi b^2}\). The entries of the matrix are thus purely trigonometrical.

To construct the fusion matrix explicitly, we propose to do the following:

- Choose an appropriate quantum group: \( \mathcal{U}_q(sl(2, \mathbb{R})) \) with deformation parameter \( q = e^{i\pi b^2}, |q| = 1 \), seems to be a reasonable candidate to describe the internal structure of Liouville field theory.

- Construct the Clebsch-Gordan coefficients, then the \( 6j \) symbol (or Racah-Wigner coefficients), for the continuous unitary representations \( P_\alpha, \alpha = \frac{Q}{2} + iP \) of \( \mathcal{U}_q(sl(2, \mathbb{R})) \).

- Find a suitable normalization for the Clebsch-Gordan coefficients such that the \( 6j \) symbol constructed coincides with the function \( G_{\alpha_{21}\alpha_{32}} \left[ \begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{array} \right] \) (this will be done by comparing the found expression for the \( 6j \) symbol for the special case \( \alpha_2 = -b/2 \) with (6)).

We would like to emphasize that:

- Such a construction by means of representation theory methods will ensure the validity of the Moore-Seiberg equations for our fusion matrix.

- The orthogonality condition for the Racah-Wigner coefficients is equivalent to proving crossing symmetry in Liouville field theory.
3 Clebsch-Gordan coefficients and Racah-Wigner symbol for a continuous series of representations of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ \cite{9, 10}

3.1 A class of representations of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$

$\mathcal{U}_q(\mathfrak{sl}(2))$ is a Hopf algebra with generators $E$, $F$, $K$, $K^{-1}$ that satisfy the $q$-commutation relations

$$KE = qEK, \quadKF = q^{-1}FK, \quad[E, F] = -\frac{K^2 - K^{-2}}{q - q^{-1}}.$$ 

The coproduct is

$$\Delta(K) = K \otimes K, \quad\Delta(E) = E \otimes K + K^{-1} \otimes E, \quad\Delta(F) = F \otimes K + K^{-1} \otimes F.$$ 

The center of $\mathcal{U}_q(\mathfrak{sl}(2))$ is generated by the $b$-Casimir

$$C = FE - \frac{qK^2 + q^{-1}K^{-2} - 2}{(q - q^{-1})^2}.$$ 

The real form $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$, $q = e^{i\pi b^2}$, $|q| = 1$, has the following star structure for its unitary representations

$$K^* = K, \quad F^* = F, \quad E^* = E.$$ 

We realize the set of unitary continuous representations $\mathcal{P}_\alpha$, $\alpha = \frac{Q}{2} + iP$ on the Hilbert space $L^2(\mathbb{R})$ in terms of the Weyl algebra generated by $U = e^{2\pi bx}$ and $V = e^{-\frac{2\pi}{b}p}$ where $[x, p] = i$:

$$E = U^{1+\frac{1}{e^{i\pi b(Q-\alpha)}V - e^{-i\pi b(Q-\alpha)}V^{-1}}}}{e^{i\pi b^2} - e^{-i\pi b^2}}},$$
$$F = U^{-1}\frac{e^{-i\pi b(Q-\alpha)}V - e^{i\pi b(Q-\alpha)}V^{-1}}{e^{i\pi b^2} - e^{-i\pi b^2}}},$$
$$K = V.$$ 

where $U$ is the operator of multiplication by $e^{2\pi bx}$ and $V$ is the shift operator acting on a function $f(x)$ of the real variable $x$ as $Vf(x) = f(x + \frac{ib}{2})$. Note that we are dealing with unbounded operators, as the quantum group is non-compact.

Remark:
In the limit where $b \to 0$, $\alpha \sim -bj$, $z = e^{2\pi bx}$, one has

$$E \sim z^2 \partial_z + (j + 1)z, \quad F \sim \partial_z - (j + 1)z^{-1}, \quad \frac{K^2 - K^{-2}}{q - q^{-1}} \sim 2\partial_z.$$
The representation will be realized on the space $\mathcal{P}_\alpha$ of entire analytic functions $f(x)$ that have a Fourier transform $f(\omega)$ which is meromorphic in $\mathbb{C}$ with possible poles at
\[
\begin{align*}
\omega &= i(\alpha - Q - nb - mb^{-1}), \\
\omega &= i(Q - \alpha + nb + mb^{-1}).
\end{align*}
\]
They are unitarily equivalent to a subset of the integrable representations of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$ that appear in the classification of \[32\]. In particular, the generators have self-adjoint extensions in $L^2(\mathbb{R})$. The $b$-Casimir acts as a scalar in this representation
\[
C = FE - \frac{qR^2 + q^{-1}K^{-2} - 2}{(q - q^{-1})^2} \equiv [\alpha - \frac{Q}{2}]_b, \quad [x]_b \equiv \frac{\sin \pi bx}{\sin \pi b}. 
\]

**Remark:**
The representations we consider here have the remarkable property that if one introduces $\tilde{E}, \tilde{F}, \tilde{K}$ by replacing $b \to 1/b$ in the expressions for $E, F, K$ given above, one obtains a representation of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$, $\tilde{q} = e^{i\pi b^{-2}}$ on the same space $\mathcal{P}_\alpha$. On this space, the generators of the two dual quantum groups commute. This autoduality of our representations $\mathcal{P}_\alpha$ is related to the fact that they do not have a classical limit when $b \to 0$. This is after all not so surprising, because autoduality in Liouville field theory is a characteristic of the quantum theory only. Lastly, we note that representations $\mathcal{P}_\alpha$ and $\mathcal{P}_{Q-\alpha}$ are equivalent.

### 3.2 Clebsch-Gordan decomposition of tensor products

The Clebsh-Gordan map $C(\alpha_3|\alpha_2, \alpha_1) : \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1} \to \mathcal{P}_{\alpha_3}$ can be explicitly represented by an integral transform
\[
f(x_2, x_1) \to F[f](\alpha_3, x_3) \equiv \int dx_2 dx_1 \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} f(x_2, x_1)
\]

The distributional kernel $\begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}$ is the Clebsh-Gordan coefficients (3j symbol). It is an invariant with respect to the action of the quantum group and satisfies
\[
\begin{align*}
K_3 & \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} = \Delta_{21}^t(K) \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} \\
E_3 & \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} = \Delta_{21}^t(E) \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} \\
F_3 & \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix} = \Delta_{21}^t(F) \begin{bmatrix} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{bmatrix}
\end{align*}
\]
where $\Delta^t$ means transposed action. These equations can be solved with the following result [2]:

$$
\begin{bmatrix}
\alpha_3 & \alpha_2 & \alpha_1 \\
x_3 & x_2 & x_1
\end{bmatrix} = e^{-2\pi(Q-\alpha_3)x_3-2\pi(\alpha_1+\alpha_3-Q)x_2+2\pi\alpha_1x_1} M(\alpha_3, \alpha_2, \alpha_1)
$$

We introduced the function $G_b(x) = e^{i\pi(x-Q)}S_b(x)$ (see Appendix B for the definition of $S_b$) that satisfies the functional relation $G_b(x+b) = (1 - e^{2\pi bx}) G_b(x)$, as well as the dual equation with $b$ replaced by $1/b$. The coefficient $M(\alpha_3, \alpha_2, \alpha_1)$ is a constant that normalizes the 3j symbol. We choose, for reasons that will be explained later,

$$
M(\alpha_3, \alpha_2, \alpha_1) = e^{i\pi(\alpha_1^2+\alpha_1\alpha_3-Q\alpha_1)}S_b(\alpha_1 + \alpha_2 - \alpha_3). \quad (7)
$$

It follows from the construction of a common spectral decomposition [10] for the Casimir operator $C_{21}$ and the third generator $K_{21}$ that the Clebsch-Gordan coefficients satisfy the orthogonality and completeness relations:

$$
\int_{\mathbb{R}} dx_1 dx_2 \left[ \begin{array}{ccc} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{array} \right]^* \left[ \begin{array}{ccc} \beta_3 & \alpha_2 & \alpha_1 \\ y_3 & x_2 & x_1 \end{array} \right] = |S_b(2\alpha_3)|^{-2} \delta(\alpha_3 - \beta_3) \delta(x_3 - y_3)
$$

Note that $|S_b(2\alpha)|^2 \equiv M_b(\alpha)$ (see equation (5)).

$$
\int_{\mathbb{R}} d\alpha_3 |S_b(2\alpha_3)|^2 \int_{\mathbb{R}} dx_3 \left[ \begin{array}{ccc} \alpha_3 & \alpha_2 & \alpha_1 \\ x_3 & x_2 & x_1 \end{array} \right]^* \left[ \begin{array}{ccc} \alpha_3 & \alpha_2 & \alpha_1 \\ y_3 & x_2 & x_1 \end{array} \right] = \delta(x_2 - y_2) \delta(x_1 - y_1)
$$

From the completeness relation it follows that our autodual representations are closed under tensor product, which is a priori non trivial if there are other unitary representations of $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$.

### 3.3 Racah-Wigner coefficients for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{R}))$

It is possible to construct two canonical bases of $\mathcal{P}_{\alpha_3} \otimes \mathcal{P}_{\alpha_2} \otimes \mathcal{P}_{\alpha_1}$. The decomposition of this representation into irreducibles can be done in two different ways, by iteration of the Clebsch-Gordan mapping

- either by using $(id \otimes \Delta) \circ \Delta$
- or $(\Delta \otimes id) \circ \Delta$

The expression for the first base in the $s$-channel is given by

$$
\Phi_{\alpha_21}^s \left[ \begin{array}{ccc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{array} \right](x) = \int_{\mathbb{R}} dx_{21} \left[ \begin{array}{ccc} \alpha_4 & \alpha_3 & \alpha_{21} \\ x_4 & x_3 & x_{21} \end{array} \right] \left[ \begin{array}{ccc} \alpha_{21} & \alpha_2 & \alpha_1 \\ x_{21} & x_2 & x_1 \end{array} \right] \quad (8)
$$
and similarly for the $t$-channel base

$$\Phi_{α32}^t \left[ \begin{array}{ccc} α_3 & α_2 \\ α_4 & α_1 \end{array} \right] (x) = \int dx_{32} \left[ \begin{array}{ccc} α_4 & α_3 & α_1 \\ x_4 & x_{32} & x_1 \end{array} \right] \left[ \begin{array}{ccc} α_{32} & α_3 & α_2 \\ x_{32} & x_3 & x_2 \end{array} \right]$$  \hspace{1cm} (9)

From an argument of completeness and orthogonality it follows that the bases $Φ^s$ and $Φ^t$ are related by a transformation

$$Φ_{α21}^s \left[ \begin{array}{ccc} α_3 & α_2 \\ α_4 & α_1 \end{array} \right] (x) = \int dα_{32} \left\{ \begin{array}{ccc} α_1 & α_3 & α_{21} \\ α_2 & α_4 & α_{32} \end{array} \right\}_b \Phi_{α32}^t \left[ \begin{array}{ccc} α_3 & α_2 \\ α_4 & α_1 \end{array} \right] (x)$$

defining thus the Racah-Wigner coefficients. Their construction implies the orthogonality condition which is equivalent to proving crossing-symmetry in Liouville field theory

$$\int S_b(α_{21}) \left[ \begin{array}{ccc} α_1 & α_3 & α_{21} \\ α_2 & α_4 & α_{32} \end{array} \right]_b \left[ \begin{array}{ccc} α_1 & α_3 & α_{21} \\ α_2 & α_4 & α_{32} \end{array} \right]_b^* \left[ \begin{array}{ccc} α_1 & α_3 & α_{21} \\ α_2 & α_4 & α_{32} \end{array} \right]_b^* = |S_b(α_{32})|^2 δ(α_{32} - β_{32})$$

### 3.4 Explicit computation of the Racah-Wigner coefficients

It seems a priori difficult to compute the integrals and δ functions, as $Φ^s, Φ^t$ are functions of the four variables $x_4, x_3, x_2, x_1$; however, one simplifies the problem in the limit where $x_4 \to +∞$ and $x_2 \to −∞$ by making use of the asymptotics of the $S_b$ function

$$S_b(x) \sim e^{∓\frac{2π}{x}(x^2-Q)}, \quad \text{Im} x \to ±∞.$$  

The result is then expressed in terms of $x_1$ and $x_3$. It turns out that it is convenient to set $x_3 = \frac{1}{2}(α_2 + α_4)$. Let $x \equiv x_1$.

We obtain in the $s$-channel

$$Ψ_{α21}^s \left[ \begin{array}{ccc} α_3 & α_2 \\ α_4 & α_1 \end{array} \right] (x) = N_{α21}^s \left[ \begin{array}{ccc} α_3 & α_2 \\ α_4 & α_1 \end{array} \right] β_{α21}^s \left[ \begin{array}{ccc} α_3 & α_2 \\ α_4 & α_1 \end{array} \right] (x)$$

with

$$N_{α21}^s \left[ \begin{array}{ccc} α_3 & α_2 \\ α_4 & α_1 \end{array} \right] = \frac{M(α_{21}, α_2, α_1)M(α_4, α_3, α_{21})}{S_b(2α_{21})} S_b(α_{21} + α_2 - α_1)S_b(α_{21} + α_1 - α_2)$$

$$\times e^{-iπ[α_4(α_2 + α_4 + α_{21} + Q) + α_{21}(α_1 + α_{21} - Q) - α_3 α_2]}$$

$$Θ_{α21}^s \left[ \begin{array}{ccc} α_3 & α_2 \\ α_4 & α_1 \end{array} \right] (x) = e^{2π(α_{21} - α_1 - α_2)x}$$

$$\times F_b(α_{21} + α_1 - α_2, α_{21} + α_3 + α_4 - Q; 2α_{21}; −ix)$$

where the function $F_b(a, b; c; ix)$ is the $b$-deformed hypergeometric function in the Euler representation defined in the Appendix C.
Similarly, we have in the $t$-channel
\[
\Psi_t^{\alpha_3 \alpha_2 \alpha_1 \alpha_4}(x) = N_{\alpha_3 \alpha_2 \alpha_1 \alpha_4}^t \Theta_t^{\alpha_3 \alpha_2 \alpha_1 \alpha_4}(x)
\]
with
\[
N_{\alpha_3 \alpha_2 \alpha_1 \alpha_4}^t = \frac{M(\alpha_3, \alpha_2, \alpha_1)M(\alpha_4, \alpha_2, \alpha_1)}{S_b(2\alpha_3)} S_b(\alpha_3 + \alpha_2 - \alpha_1)
\]
\[
\Theta_t^{\alpha_3 \alpha_2 \alpha_1 \alpha_4}(x) = e^{-2\pi(\alpha_3 + \alpha_1 - \alpha_2 - \alpha_1)} \times
F_b(\alpha_3 + \alpha_1 + \alpha_2 - \alpha_1)
\]

Now we use the normalization
\[
\int dxe^{2\pi Qx} \left( \Theta_{\alpha_3 \alpha_2 \alpha_1}^s \right)^*(x) \Theta_{\alpha_3 \alpha_2 \alpha_1}^s(x) = \delta(\alpha_2 - \alpha_1'),
\]
and the fact that $\{\Theta_{\alpha_3 \alpha_2 \alpha_1}^s \alpha_2 \in S\}$ and $\{\Theta_{\alpha_3 \alpha_2 \alpha_1}^s \alpha_3 \in S\}$ form complete sets of generalized eigenfunctions for the operators $C_{21}$ and $C_{32}$ respectively
\[
C_{21} = [\delta_x + \alpha_1 + \alpha_2 - \frac{Q}{2}b^2 - e^{2\pi bx} \delta_x + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - Q]_b \delta_x + 2\alpha_1)_b
\]
\[
C_{32} = [\delta_x + \alpha_1 + \alpha_4 - \frac{Q}{2}b^2 - e^{-2\pi bx} [\delta_x + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - Q]_b \delta_x)_b
\]
where $\delta_x = \frac{1}{2\pi} \partial_x$, $[x]_b = \frac{\sin(\pi bx)}{\sinh(\pi bx)}$.

We get the following expression for the Racah-Wigner symbol:
\[
\left\{ \begin{array}{c}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
\alpha_4
\end{array} \mid \begin{array}{c}
\alpha_2 \\
\alpha_4 \\
\alpha_1
\end{array} \right\}_b = N_{\alpha_2 \alpha_1}^s \left[ \begin{array}{c}
\alpha_2 \\
\alpha_4 \\
\alpha_1
\end{array} \right] \times \int dxe^{2\pi Qx} \left( \Theta_{\alpha_3 \alpha_2 \alpha_1}^t \right)^*(x) \Theta_{\alpha_3 \alpha_2 \alpha_1}^t(x).
\]

Using now the Barnes representation for the $b$-deformed hypergeometric function $\mathbf{3F}_1$ presented in the Appendix C, the Racah-Wigner coefficients can be expressed explicitly in terms of a $b$-deformed hypergeometric function $\mathbf{4F}_3$ in the Barnes representation:
Now we set \( \alpha \) not difficult to compute the residues, and one finds

\[
\frac{S_b(\alpha_2 + \alpha_1 - \alpha_3)S_b(\alpha_2 + \alpha_1 - \alpha_4)S_b(\alpha_3 + \alpha_4 - \alpha_1 - \alpha_4 - Q)}{S_b(\alpha_2 + \alpha_3 - \alpha_3)S_b(\alpha_3 + \alpha_2 - \alpha_3)S_b(\alpha_3 + \alpha_4 - \alpha_1 - \alpha_4 - Q)}.
\]

where:

\[
\begin{align*}
U_1 &= \alpha_2 + \alpha_1 - \alpha_2 & V_1 &= Q + \alpha_2 - \alpha_3 - \alpha_2 + \alpha_4 \\
U_2 &= Q + \alpha_2 - \alpha_3 - \alpha_2 & V_2 &= \alpha_2 + \alpha_3 + \alpha_4 - \alpha_2 \\
U_3 &= \alpha_2 + \alpha_3 + \alpha_4 - Q & V_3 &= 2\alpha_2 \\
U_4 &= \alpha_2 - \alpha_3 + \alpha_4 &
\end{align*}
\]

Now we set \( \alpha_2 = -\frac{b}{2} \): the fusion rules are

\[
\begin{align*}
\{ \alpha_{21} &= \alpha_1 - s\frac{b}{2}, \\
\alpha_{32} &= \alpha_3 - s'\frac{b}{2}, & s, s' &= \pm .
\end{align*}
\]

In this case some poles of the integrand will cross the imaginary axis. It is not difficult to compute the residues, and one finds

\[
\begin{pmatrix}
\alpha_1 & \alpha_3 & \alpha_1 - s\frac{b}{2} \\
-\frac{b}{2} & \alpha_4 & \alpha_3 - s'\frac{b}{2}
\end{pmatrix}_b = \begin{pmatrix}
\frac{|a_4 + a_3 - a_1 - \frac{b}{2}|_b}{|2a_3 - b|_b} & \frac{|a_4 + a_3 + a_1 - \frac{b}{2}|_b}{|2a_3 - b|_b} \\
\frac{|a_3 + a_1 - a_4 - \frac{b}{2}|_b}{|2a_3 - b|_b} & \frac{|a_4 + a_1 - a_3 - \frac{b}{2}|_b}{|2a_3 - b|_b}
\end{pmatrix}
\]

The normalization \( M(\alpha_3, \alpha_2, \alpha_1) \) introduced in equation (14) has been chosen such that the Racah-Wigner coefficient equals \( G_{s,s'} \) for \( \alpha_2 = -\frac{b}{2} \). It follows from a uniqueness argument for the fusion matrix (see [9] and Appendix C of [21]) that these two objects are equal for any values of \( \alpha_i \).

### 3.5 Expression of the fusion matrix

It is now shown that \( G_{\alpha_{21} \alpha_{32}} \begin{pmatrix}
\alpha_3 & \alpha_2 \\
\alpha_4 & \alpha_1
\end{pmatrix} = \begin{pmatrix}
\alpha_1 & \alpha_3 & \alpha_{21} \\
\alpha_2 & \alpha_4 & \alpha_{32}
\end{pmatrix}_b \), so we deduce thanks to equation (4) the expression for the fusion matrix [9]:

\[
\begin{align*}
|S_b(2\alpha_{32})|^2 &= \int_{-\infty}^{\infty} ds \frac{S_b(U_1 + s)S_b(U_2 + s)S_b(U_3 + s)S_b(U_4 + s)}{S_b(V_1 + s)S_b(V_2 + s)S_b(V_3 + s)S_b(Q + s)}
\end{align*}
\]
\[
F_{\alpha_1 \alpha_2} \begin{bmatrix}
\alpha_3 & \alpha_2 \\
\alpha_4 & \alpha_1
\end{bmatrix} = \\
\frac{\Gamma_b(2Q - \alpha_3 - \alpha_2 - \alpha_3 \alpha_2) \Gamma_b(\alpha_3 + \alpha_3 - \alpha_2) \Gamma_b(Q - \alpha_2 - \alpha_3 + \alpha_3) \Gamma_b(Q - \alpha_3 - \alpha_2 + \alpha_3)}{\Gamma_b(2Q - \alpha_1 - \alpha_2 - \alpha_3 \alpha_2) \Gamma_b(\alpha_1 + \alpha_2 - \alpha_2) \Gamma_b(Q - \alpha_2 - \alpha_1 + \alpha_3) \Gamma_b(Q - \alpha_2 - \alpha_1 + \alpha_3)}
\]
\[
\frac{\Gamma_b(\alpha_3 + \alpha_2 + \alpha_4) \Gamma_b(\alpha_2 + \alpha_3 + \alpha_4 - Q) \Gamma_b(\alpha_3 + \alpha_4 - \alpha_1) \Gamma_b(\alpha_3 + \alpha_4 - \alpha_1)}{\Gamma_b(\alpha_3 + \alpha_4 - \alpha_3) \Gamma_b(\alpha_3 + \alpha_4 - \alpha_3)}
\]
\[
\frac{\Gamma_b(2Q - 2\alpha_2) \Gamma_b(2\alpha_2 - Q)}{\Gamma_b(2Q - 2\alpha_2) \Gamma_b(2\alpha_2 - Q)} \cdot \frac{1}{i \int_{-i\infty}^{i\infty} ds} \frac{S_b(U_1 + s) S_b(U_2 + s) S_b(U_3 + s) S_b(U_4 + s)}{S_b(V_1 + s) S_b(V_2 + s) S_b(V_3 + s) S_b(Q + s)}
\]

where:
\[
\begin{align*}
U_1 &= \alpha_1 + \alpha_2 - \alpha_2 \\
U_2 &= Q + \alpha_1 - \alpha_2 - \alpha_2 \\
U_3 &= \alpha_2 + \alpha_3 + \alpha_4 - Q \\
U_4 &= \alpha_2 + \alpha_3 + \alpha_4
\end{align*}
\]
\[
\begin{align*}
V_1 &= Q + \alpha_2 - \alpha_2 + \alpha_4 \\
V_2 &= \alpha_2 + \alpha_3 + \alpha_4 - \alpha_2 \\
V_3 &= 2\alpha_2
\end{align*}
\]

One can check explicitly that the fusion matrix is indeed invariant with respect to the exchange of rows and columns, and that it depends on conformal weights only. Some of these properties are trivial to check, others require the use of transformation properties of \(b\)-deformed hypergeometric functions like the one presented in the Appendix C.

4 Application: Boundary three point function [21]

We now consider Liouville theory on a simply connected domain \(\Gamma\) with a non-trivial boundary \(\partial\Gamma\). \(\Gamma\) can be either the unit disk, the upper half-plane, or the infinite strip. A conformally invariant boundary condition in LFT can be introduced through the following boundary interaction
\[
A_{\text{bound}} = A_{\text{bulk}} + \int_{\partial\Gamma} \left( \frac{QK}{2\pi} \phi + \mu_B e^b \right) g^{1/4} dx,
\]
where the integration in \(x\) is along the boundary while \(K\) is the curvature of the boundary in the background geometry \(g\). We will consider the geometry of the Upper Half Plane (flat metric background)
\[
A_{\text{bound}} = \int_{\text{UHP}} \left( \frac{1}{4\pi} (\partial_x \phi)^2 + \mu e^{2b\phi(z, \bar{z})} d^2 z + \mu_B \int_{\mathbb{R}} e^{b \phi(x)} dx \right)
\]
with the boundary condition at \(|z| \to \infty\)
\[
\phi(z, \bar{z}) = -Q \log(z \bar{z}) + O(1),
\]
and the Neumann condition for the Liouville field \( \phi \) on the real axis. We call \( \mu_B \) the boundary cosmological constant, by analogy with the bulk case \(^7\). There is actually a one-parameter family of conformally invariant boundary conditions characterized by different values of the boundary cosmological constant \( \mu_B \). We shall denote the boundary operators \( B_{\beta \sigma_1}^{\sigma_2}(x) \) (classically they correspond to the boundary value of \( e^{\beta/2b}\phi \), where \( \phi \) is the classical Liouville field: \( \varphi = 2b\phi \)). The boundary operators have conformal weight \( \Delta_\beta = \beta(Q - \beta) \), and are labelled by two left and right boundary conditions \( \sigma_1 \) and \( \sigma_2 \) related to \( \mu_{B_1} \) and \( \mu_{B_2} \) by the relation \(^3\)

\[
\cos\left(2\pi b\left(\sigma - \frac{Q}{2}\right)\right) = \frac{\mu_B}{\sqrt{\mu}} \sqrt{\sin(\pi b^2)}.
\]

(10)

Let us note for real values of \( \mu_B \) the two following regimes for the parameter \( \sigma \):

1. if \( \frac{\mu_B}{\sqrt{\mu}} \sqrt{\sin(\pi b^2)} > 1 \), then \( \sigma \) is of the form \( \sigma = Q/2 + iP \);

2. if \( \frac{\mu_B}{\sqrt{\mu}} \sqrt{\sin(\pi b^2)} < 1 \), then \( \sigma \) is real.

Anticipating that all relevant objects will be found to possess meromorphic continuation with respect to the boundary parameter \( \sigma \), we shall discuss only the first regime explicitly in the following. The Hilbert space was found \(^{35}\) to decompose into irreducible representations of the Virasoro algebra:

\[
\mathcal{H}^B = \int_0^\infty V_\beta.
\]

Contrary to the pure bulk case situation where the cosmological constant enters only as a scale parameter, in the boundary case, the observables depend on the scale invariant ration \( \mu/\mu_B^2 \): for example, a correlation function with the bulk operators \( V_{\alpha_1}V_{\alpha_2}...V_{\alpha_n} \) and the boundary operators \( B_{\beta_1}^{\sigma_1}B_{\beta_2}^{\sigma_2}...B_{\beta_m}^{\sigma_m} \) scales as follow

\[
\mathcal{G}(\alpha_1,...,\alpha_n,\beta_1...\beta_m) \sim \mu^{(Q-2)\sum_i \alpha_i - \sum_j \beta_j}/2b F\left(\frac{\mu_B^2}{\mu}, \frac{\mu_{B_2}^2}{\mu}, ..., \frac{\mu_{B_m}^2}{\mu}\right),
\]

where \( F \) is some scaling function. The observables are autodual provided the dual cosmological constant \( \mu \) is related to \( \tilde{\mu} \) as in \(^{11}\), and the dual boundary cosmological constant is defined as follows

\[
\cos\left(\frac{2\pi}{b}\left(\sigma - \frac{Q}{2}\right)\right) = \frac{\tilde{\mu}_B}{\sqrt{\tilde{\mu}}} \sqrt{\sin(\pi b^2)}.
\]

In order to characterize LFT on the upper half plane, one needs to know additional structure constants beside the bulk three point function \( C(\alpha_1, \alpha_2, \alpha_3) \).

\(^7\)A different set of boundary conditions is considered in the work \(^{33}\), following early works of \(^{34,35}\).
1. bulk one point function 
\[ \langle V_\alpha(z, \bar{z}) \rangle = \frac{U(\alpha|\mu_B)}{|z - \bar{z}|^{2\Delta_\alpha}} \]

2. boundary two point function
\[ \langle B_{\beta_1}^{\sigma_1\sigma_2}(x)B_{\beta_2}^{\sigma_2\sigma_1}(0) \rangle = \frac{\delta(\beta_2 + \beta_1 - Q) + S(\beta_1, \sigma_2, \sigma_1)\delta(\beta_2 - \beta_1)}{|x|^{2\Delta_{\beta_1}}} \]

We give here the explicit expression for the boundary reflection amplitude:
\[ S(\beta, \sigma_2, \sigma_1) = \left( \frac{\pi \mu \gamma(b^2) b^2 - 2b^2}{\pi}(Q - 2\beta) \right) \frac{\Gamma_b(2\beta - Q) S_b(\sigma_2 + \sigma_1 - \beta)S_b(2Q - \beta - \sigma_1 - \sigma_2)}{\Gamma_b(Q - 2\beta) S_b(\beta + \sigma_2 - \sigma_1)S_b(\beta + \sigma_1 - \sigma_2)} \]

It satisfies the unitarity relation \( S(\beta, \sigma_2, \sigma_1)S(Q - \beta, \sigma_2, \sigma_1) = 1 \).

3. bulk-boundary two point function
\[ \langle V_\alpha(z, \bar{z})B_{\beta}^{\sigma_3}(x) \rangle = \frac{R(\alpha, \beta|\mu_B)}{|z - \bar{z}|^{2\Delta_\alpha - 2\Delta_\beta}|z - x|^{2\Delta_\beta}} \]

4. boundary three point function
\[ \langle B_{Q-\beta_3}^{\sigma_1\sigma_3}(x_3)B_{\beta_2}^{\sigma_2\sigma_2}(x_2)B_{\beta_1}^{\sigma_2\sigma_1}(x_1) \rangle = \frac{C^{(\sigma_3\sigma_2\sigma_1)\beta_3}_{\beta_3\beta_2\beta_1}(x_3| x_1)}{|x_{21}|^{\Delta_1 + \Delta_2 - \Delta_3} |x_{32}|^{\Delta_2 + \Delta_3 - \Delta_1} |x_{31}|^{\Delta_3 + \Delta_1 - \Delta_2}} \]

We now proceed to the determination of the latter quantity.

4.1 Associativity condition

Let us consider a four point function of boundary operators. This quantity can be equivalently written:

- in the s-channel
\[ \langle B_{\beta_4}^{\sigma_1\sigma_4}(x_4)B_{\beta_3}^{\sigma_4\sigma_3}(x_3)B_{\beta_2}^{\sigma_3\sigma_2}(x_2)B_{\beta_1}^{\sigma_2\sigma_1}(x_1) \rangle = \int d\beta_{21} C^{(\sigma_4\sigma_3\sigma_1)\beta_4}_{\beta_4\beta_3\beta_2} C^{(\sigma_3\sigma_2\sigma_1)\beta_2}_{\beta_2\beta_1} F^4(\Delta_{\beta_1}, \Delta_{\beta_2}, x_i) \]

\( ^8 \)The bulk one point function is a special case of the bulk-boundary coefficient with \( \beta = 0 \).

\( ^9 \)One restricts oneself to the case where \( \text{Re}(\beta_i), \text{Re}(\sigma_i), i = 1 \ldots 4 \) are close enough to \( Q/2 \). In this case, \( \beta_{21} \) is of the form \( Q/2 + iP \). Meromorphic continuation is understood otherwise.
• in the $t$-channel

$$\langle B_{\beta_4}^{\sigma_4}(x_4)B_{\beta_3}^{\sigma_3}(x_3)B_{\beta_2}^{\sigma_2}(x_2)B_{\beta_1}^{\sigma_1}(x_1)\rangle = \int \mathcal{S} d\beta \sigma_4 \sigma_3 \sigma_2 \sigma_1 F(\Delta \beta_1, \Delta \beta_2, x_1)$$

Using the fusion transformation (3), the equivalence of the factorisation in the two channels can be rewritten:

$$\int \mathcal{S} d\beta \sigma_4 \sigma_3 \sigma_2 \sigma_1 F(\Delta \beta_1, \Delta \beta_2, x_1) = \int \mathcal{S} d\beta \sigma_3 \sigma_2 \sigma_1 F(\Delta \beta_2, \Delta \beta_3, x_1)$$

The following ansatz

$$C(\sigma_3 \sigma_1) = \frac{g(\beta_3, \sigma_3, \sigma_1)g(\beta_1, \sigma_2, \sigma_1)}{g(\beta_3, \sigma_2, \sigma_1)} F(\Delta \beta_3, \Delta \beta_2, x_1)$$

yields a solution to (11), as was noticed in (20) (one recognizes the pentagonal equation (3)). The functions $g(\beta, \sigma_2, \sigma_1)$ appearing are unrestricted by (11), and will turn out to correspond to the normalization of the boundary operators $B_{\sigma_2}^{\sigma_1}$. We will show that it is possible to compute this normalization explicitly, and that the ansatz (12) is indeed consistent with the normalization required for the boundary two point function.

4.2 Normalization of the boundary operator

The normalization is easily computed using the special value $\beta = -b/2$. The degenerate operator $B_{-b/2}^{\sigma_2 \sigma_1}$ is a non vanishing primary field in general: it is shown in (33) that already at the classical level we have

$$\left(\frac{d^2}{dx^2} + T_{cl}\right)e^{-\varphi/4} = \pi b^2 (\pi \mu b^2 - \mu) e^{3\varphi/4},$$

where $T_{cl}$ is the boundary value of the classical stress-energy tensor:

$$T_{cl} = \frac{1}{16} \varphi_x^2 + \frac{1}{4} \varphi_{xx} + \pi b^2 (\pi \mu b^2 - \mu) e^\varphi,$$

and $\varphi = 2b\phi$ is the classical Liouville field. This relation means that generically, the second order differential equation has some non zero terms in the right hand side, unless $\pi \mu b^2 / \mu = 1$ This effect holds at the quantum level too: if the boundary and bulk cosmological constant are related as

$$1 = \frac{2\mu b^2}{\mu} \tan \frac{\tau b^2}{2},$$

In (21), it is the value $\beta = -b$ that is used to determine to normalization of the boundary operators.
then the second order differential equations holds for the boundary operator $B_{\beta_{2}/b}^{\sigma_{1}/b}$. It is argued in [33] that this remark with the structure of singularities of the boundary two point function suggests that the level 2 degenerate boundary operator $B_{\beta_{2}/b}^{\sigma_{2}/b}$ has a vanishing null vector if and only if the right and left boundary conditions are related by $\sigma_2 = \sigma_1 \pm b/2$. Under this suggestion, the operator product expansion of this degenerate operator with any primary field $B_{\beta_{2}/b}^{\sigma_{1}/b}$ is truncated

$$B_{\beta_{2}/b}^{\sigma_{3}/b}B_{-b/2}^{\sigma_{2}/b} = c_{\pm}^+ B_{\beta_{2}/b}^{\sigma_{3}/b} + c_{\pm}^- B_{\beta_{2}/b}^{\sigma_{3}/b} + c_{\pm}^+ B_{\beta_{2}+b/2}^{\sigma_{3}/b}, \quad \sigma_2 = \sigma_1 \pm b/2$$

(13)

where as in the bulk situation, $c_{\pm}^\pm$ are structure constants and are obtained as certain screening integrals. In the first term of (13) there is no need of screening insertion and therefore $c_{\pm}^+\pm$ can be set to 1. Let us insert in (13) the operator $B_{Q/2-b}\sigma_{1/3}$. Together with the two point function

$$\left< B_{\beta_{2}/b}^{\sigma_{3}/b} B_{Q/2-b}^{\sigma_{3}/b} \right> = 1,$$

one gets the three point function

$$\left< B_{\beta_{2}/b}^{\sigma_{3}/b} B_{Q/2-b}^{\sigma_{3}/b} B_{Q/2-b}^{\sigma_{3}/b} \right> = 1, \quad \sigma_2 = \sigma_1 \pm b/2.$$

Plugging these special values in (12), one finds that the normalization satisfies the following first order difference equation

$$1 = \frac{g(\beta_2 - b/2, \sigma_3, \sigma_1)}{g(\beta_2, \sigma_3, \sigma_1 \pm b/2)g(-b/2, \sigma_3 \pm b/2, \sigma_1)F_{\sigma_1 \pm b/2, \beta_2 - b/2}^{\beta_2 - b/2}} \left[ \begin{array}{c} \beta_2 \\ \sigma_3 \\ \sigma_1 \end{array} \right]$$

where the fusion coefficient on the right hand side $F_{\pm, \pm}$ is explicitly written in the Appendix A. One finds for the normalization

$$g(\beta, \sigma_3, \sigma_1) = \frac{f(\sigma_3, \sigma_1)^{b-1/2} \Gamma_b(Q) \Gamma_b(Q - 2\beta) \Gamma_b(2\sigma_1) \Gamma_b(2Q - 2\sigma_3)}{\Gamma_b(2Q - \beta - \sigma_1 - \beta - \sigma_3) \Gamma_b(\sigma_1 + \sigma_3 - \beta) \Gamma_b(Q - \beta + \sigma_1 - \sigma_3) \Gamma_b(Q - \beta + \sigma_3 - \sigma_1)}$$

the remaining freedom being parameterized by the function $f(\sigma_3, \sigma_1)$. Let us furthermore note that one may derive a second finite difference equation for the normalization $g(\beta, \sigma_2, \sigma_1)$ obtained by replacing $b$ by $b^{-1}$, if one considers the dual operator $B_{Q/2-b}^{\sigma_{1/3}}$. Taken together, these functional relations allow one to conclude that our solution is unique at least for irrational values of $b$. Let us now turn to the determination of the function $f(\sigma_3, \sigma_1)$: we now insert the operator $B_{Q/2-b}^{\sigma_1/2}$ and take the expectation value

$$\left< B_{\beta_{2}/b}^{\sigma_{3}/b} B_{Q/2-b}^{\sigma_{3}/b} B_{Q/2-b}^{\sigma_{3}/b} \right> = c_{\pm}^+ \pm.$$
On the other hand, \( c^\pm \) requires one insertion of the boundary interaction 

\[-\mu_B \int e^{\phi(x)} dx\]

and was explicitly computed in [21] with the following result

\[
c^\pm = 2 \left( -\frac{\mu}{\pi \gamma(-b^2)} \right)^{1/2} \Gamma(2b\beta_2 - b^2 - 1) \Gamma(1 - 2b\beta_2) \times \sin \pi b(\beta_2 \pm (\sigma_1 - \sigma_3) - b/2) \sin \pi b(\beta_2 \pm (\sigma_3 + \sigma_1 - Q) - b/2),
\]

on the other hand, [12] reads

\[
c^\pm = \frac{g(\beta_2 + b/2, \sigma_3, \sigma_1)}{g(\beta_2, \sigma_3, \sigma_1) g(-b/2, \sigma_1 \pm b/2, \sigma_1)} F_{\sigma_1 \pm b/2, \beta_2 + b/2} \left[ \begin{array}{c} \beta_2 \\ \beta_3 \\
\sigma_3 \\ \sigma_1 \end{array} \right] ,
\]

where the explicit expression for \( F_{\pm,-} \) can be found in the Appendix A. By identifying the expressions obtained, one finds

\[
f(\sigma_3, \sigma_1) = \pi \mu \gamma(b^2) b^{2 - 2b^2}.
\]

One now collects the pieces together and the expression for the structure constant follows [21]

\[
C_{\beta_2 \beta_1}^{(\sigma_3 \sigma_2 \sigma_1) \beta_3} = \left( \frac{\pi \mu \gamma(b^2) b^{2 - 2b^2}}{\Gamma(3 - 2b^2)} \right)^{1/2} \frac{\Gamma(2b\beta_3 - Q) \Gamma(2b\beta_2 - Q) \Gamma_b(Q + \beta_2 - \beta_1 - \beta_3) \Gamma_b(Q + \beta_3 - \beta_1 - \beta_2)}{\Gamma_b(2b\beta_3 - Q) \Gamma_b(Q - 2b\beta_2) \Gamma_b(Q - 2b\beta_1) \Gamma_b(Q)} \times \frac{S_b(\beta_3 + \sigma_1 - \sigma_3) S_b(Q + \beta_3 - \sigma_3 - \sigma_1)}{S_b(\beta_2 + \sigma_2 - \sigma_3) S_b(Q + \beta_2 - \sigma_3 - \sigma_2)} \times \frac{1}{i} \int_{-\infty}^{\infty} ds \frac{S_b(U_1 + s) S_b(U_2 + s) S_b(U_3 + s) S_b(U_4 + s)}{S_b(V_1 + s) S_b(V_2 + s) S_b(V_3 + s) S_b(Q + s)}
\]

the coefficients \( U_i, V_i \) and \( i = 1, \ldots, 4 \) read

\[
\begin{align*}
U_1 &= \sigma_1 + \sigma_2 - \beta_1 & V_1 &= Q + \sigma_2 - \sigma_3 - \beta_1 + \beta_3 \\
U_2 &= Q - \sigma_1 + \sigma_2 - \beta_1 & V_2 &= 2Q + \sigma_2 - \sigma_3 - \beta_1 - \beta_3 \\
U_3 &= \beta_2 + \sigma_2 - \sigma_3 & V_3 &= 2\sigma_2 \\
U_4 &= Q - \beta_2 + \sigma_2 - \sigma_3
\end{align*}
\]

4.3 Remarks

1. One may explicitly check [21] that

\[
\lim_{\beta_1 \to 0} C_{\beta_2 \beta_1}^{(\sigma_3 \sigma_2 \sigma_1) \beta_3} = \delta(\beta_3 - \beta_2) + S(\beta_3, \sigma_3, \sigma_1) \delta(\beta_3 + \beta_2 - Q)
\]

2. It can be proven that the boundary three point function is cyclic invariant with the help of symmetry properties of the fusion coefficients, see Appendix B.2 of [21].
3. One recovers the expression for the boundary reflection amplitude from
the boundary three point function the same way as in [27] where the
bulk reflection amplitude is recovered from the bulk three point func-
tion (see equation (2)). Using the fact that the fusion matrix depends
on conformal weights only, and is thus invariant when
\[ \beta_i \rightarrow Q - \beta_i, \]

\[ C_{\beta_2 \beta_1}^{(\sigma_3 \sigma_2 \sigma_1)Q-\beta_3} = \frac{g(Q - \beta_3, \sigma_3, \sigma_1)}{g(\beta_3, \sigma_3, \sigma_1)} C_{\beta_2 \beta_1}^{(\sigma_3 \sigma_2 \sigma_1)\beta_3} \]

From the expression of the normalization of the boundary operator,
one recovers the boundary reflection amplitude computed in [34]

\[ \frac{g(Q - \beta_3, \sigma_3, \sigma_1)}{g(\beta_3, \sigma_3, \sigma_1)} = S(\beta_3, \sigma_3, \sigma_1) = (\pi \mu \gamma (b^2) b^{2 - 2b} \frac{1}{2b}(Q - 2\beta_3) \times \]

\[ \times \frac{\Gamma_b(2\beta_3 - Q) S_b(\sigma_3 + 1 - \beta_3) S_b(2Q - \beta_3 - \sigma_3 - 1)}{\Gamma_b(Q - 2\beta_3) S_b(\beta_3 + \sigma_3 - 1) S_b(\beta_3 + \sigma_1 - \sigma_3)} \]

4. The expression of the boundary three point function in terms of the
fusion matrix confirms the remark made in [34] that any degenerate
field \( B^{\sigma_2 \sigma_1}_{-nb/2} \), \( n \in \mathbb{N} \) has truncated operator product expansion if \( \sigma_2 = \sigma_1 - \frac{sb}{b} \), \( s = -n, -n + 2, \ldots, n - 2, n \), and are therefore in analogy with
the fusion rules for the degenerate bulk fields (and respectively with \( b \)
replaced by \( 1/b \)).

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Appendix A. Some residues of the Liouville fusion
matrix

It is well known that in the case where one of \( \alpha_1, \ldots, \alpha_4 \), say \( \alpha_i \) equals \(-\frac{n}{2b} - \frac{m}{2} b^{-1} \) where \( n, m \in \mathbb{N} \) and where a triple \( (\Delta_{\alpha_4}, \Delta_{\alpha_3}, \Delta_{\alpha_2}) \), \( (\Delta_{\alpha_2}, \Delta_{\alpha_3}, \Delta_{\alpha_1}) \)
which contains \( \Delta_{\alpha_i} \) satisfies the fusion rules of [22][24], one will find that
the fusion coefficients that multiply the conformal blocks are residues of the
general fusion coefficient.

In the case where \( \alpha_2 = -\frac{1}{2} b \), the fusion rules are:

\[ \begin{cases} 
\alpha_{21} = \alpha_1 - s \frac{b}{2}, \\
\alpha_{32} = \alpha_3 - s' \frac{b}{2}, \\
\end{cases} \quad s, s' = \pm . \]
There are four entries for the fusion matrix in this special case

\[
F_{\alpha -sb/2,\alpha 3-s'b/2} \left[ \begin{array}{cc} \alpha 3 & -b/2 \\ \alpha 4 & \alpha 1 \end{array} \right] \equiv F^L_{s,s'}
\]

which expressions are well known to be:

\[
F_{++} = \frac{\Gamma(b(2\alpha 1 - b)) \Gamma(b(b - 2\alpha 3) + 1)}{\Gamma(b(\alpha 1 - \alpha 3 - \alpha 4 + b/2) + 1) \Gamma(b(\alpha 1 - \alpha 3 + \alpha 4 - b/2))}
\]

\[
F_{+-} = \frac{\Gamma(b(\alpha 1 + \alpha 3 + \alpha 4 - 3b/2) - 1) \Gamma(b(\alpha 1 + \alpha 3 - \alpha 4 - b/2))}{\Gamma(2 - b(2\alpha 1 - b)) \Gamma(b(2\alpha 3 - b)) + 1)}
\]

\[
F_{-+} = \frac{\Gamma(2 - b(\alpha 1 + \alpha 3 + \alpha 4 - 3b/2)) \Gamma(1 - b(\alpha 1 + \alpha 3 - \alpha 4 - b/2))}{\Gamma(2 - b(2\alpha 1 - b)) \Gamma(b(2\alpha 3 - b) - 1)}
\]

\[
F_{--} = \frac{\Gamma(-\alpha 1 + \alpha 3 + \alpha 4 - b/2)) \Gamma(b(-\alpha 1 + \alpha 3 - \alpha 4 + b/2) + 1)}{\Gamma(2 - b(2\alpha 1 - b)) \Gamma(b(-2\alpha 3 - b))}
\]

The dual case where \(\alpha 2 = -b^{-1}/2\) is obtained by substituting \(b\) by \(b^{-1}\).

Appendix B. Special functions

- \(\Gamma_b(x)\) function
  The Double Gamma function introduced by Barnes is defined by:

\[
\log \Gamma_2(s|\omega 1, \omega 2) = \left( \frac{\partial}{\partial t} \sum_{n_1, n_2=0}^{\infty} \left( s + n_1 \omega 1 + n_2 \omega 2 \right)^{-t} \right)_{t=0}
\]

Definition: \(\Gamma_b(x) \equiv \frac{\Gamma_2(x|b, b^{-1})}{\Gamma_2(Q/2|b, b^{-1})}\).

Functional relations:

\[
\Gamma_b(x + b) = \frac{\sqrt{2\pi b^{bx - \frac{1}{2}}}}{\Gamma(bx)} \Gamma_b(x),
\]

\[
\Gamma_b(x + 1/b) = \frac{\sqrt{2\pi b^{-x + \frac{1}{2}}}}{\Gamma(x/b)} \Gamma_b(x).
\]

\(\Gamma_b(x)\) is a meromorphic function of \(x\), whose poles are located at \(x = -nb - mb^{-1}, n, m \in \mathbb{N}\).

Integral representation convergent for \(0 < \text{Re} x\)

\[
\log \Gamma_b(x) = \int_0^\infty \frac{dt}{t} \left[ \frac{e^{-xt} - e^{-Qt/2}}{(1 - e^{-bt})(1 - e^{-t/b})} - \frac{(Q/2 - x)^2}{2} e^{-t} - \frac{Q/2 - x}{t} \right]
\]

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- $S_b(x)$ function
  Definition: $S_b(x) \equiv \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}$
  Functional relations:
  
  $S_b(x + b) = 2\sin(\pi bx)S_b(x)$,

  $S_b(x + 1/b) = 2\sin(\pi x/b)S_b(x)$.

  $S_b(x)$ is a meromorphic function of $x$, whose poles are located at $x = -nb - mb^{-1}, n, m \in \mathbb{N}$, and whose zeros are located at $x = Q + nb + mb^{-1}, n, m \in \mathbb{N}$.

  Integral representation convergent in the strip $0 < \text{Re}x < Q$

  $$
  \log S_b(x) = \int_0^\infty \frac{dt}{t} \left[ \frac{\sinh\left(\frac{Q}{2} - x\right)t}{2\sinh\left(\frac{Q}{2}\right)\sinh\left(\frac{t}{2b}\right)} - \frac{(Q - 2x)}{t} \right]
  $$

- $\Upsilon_b(x)$ function
  Definition: $\Upsilon_b(x)^{-1} \equiv \Gamma_b(x)\Gamma_b(Q - x)$
  Functional relations:

  $\Upsilon_b(x + b) = \frac{\Gamma(bx)}{\Gamma(1 - bx)} b^{1 - 2bx} \Upsilon_b(x)$,

  $\Upsilon_b(x + 1/b) = \frac{\Gamma(x/b)}{\Gamma(1 - x/b)} b^{2x/b - 1} \Upsilon_b(x)$.

  $\Upsilon_b(x)$ is an entire function of $x$ whose zeros are located at $x = -nb - mb^{-1}$ and $x = Q + nb + mb^{-1}, n, m \in \mathbb{N}$.

  Integral representation convergent in the strip $0 < \text{Re}x < Q$

  $$
  \log \Upsilon_b(x) = \int_0^\infty \frac{dt}{t} \left[ \left(\frac{Q}{2} - x\right)^2 e^{-t} - \frac{\sinh^2\left(\frac{Q}{2} - x\right)t}{\sinh^2\left(\frac{t}{2b}\right)} \right]
  $$

**Appendix C. $b$-deformed hypergeometric function**

The results presented here come mainly from [38]. The $b$-hypergeometric function $F_b(\alpha, \beta; \gamma; -ix)$ is solution of the second order finite difference equation:

$$
\left( [\delta_x + \alpha][\delta_x + \beta] - e^{-2\pi bx} [\delta_x][\delta_x + \gamma - Q] \right) F_b(\alpha, \beta; \gamma; -ix) = 0
$$

where $\delta_x = \frac{1}{i\pi} \partial_x$, $[x]_b = \frac{\sin(\pi bx)}{\sinh\pi b}$.

Remark: In the following limits, the equation becomes:

$$
\begin{cases}
  b \to 0, \\
  z = -e^{2\pi bx}, \\
  \alpha = bA, \beta = bB, \gamma = bC.
\end{cases}
$$
\[ z(1 - z) \frac{d^2 u}{dz^2} + \left[ C - (A + B + 1)z \right] \frac{du}{dz} - ABu = 0. \]

which is the usual hypergeometric equation. Note that in the deformed case, there is no singularity at \( z = 1 \) anymore.

Integral representations:

- **Analogue of the Barnes representation:**
  \[
  F_b(\alpha, \beta; \gamma; -ix) = \frac{1}{i} \frac{S_b(\gamma)}{S_b(\alpha)S_b(\beta)} \int_{-i\infty}^{+i\infty} dse^{2\pi isb} \frac{S_b(\alpha + s)S_b(\beta + s)}{S_b(\gamma + s)S_b(Q + s)}
  \]
  The integration contour is located to the right of the poles:
  \[
  \left\{ \begin{array}{l}
  s = -\alpha - nb - mb^{-1}, \\
  s = -\beta - nb - mb^{-1}.
  \end{array} \right.
  \]
  and to the left of the poles:
  \[
  \left\{ \begin{array}{l}
  s = Q - \gamma + nb + mb^{-1}, \\
  s = nb + mb^{-1}.
  \end{array} \right.
  \]
  where \( n, m \in \mathbb{N} \).
  The integral is uniformly convergent in the set of \( x \in \mathbb{C} \) such that \(|\text{Im} x| < \frac{1}{2} \text{Re}(Q + \gamma - \beta - \alpha)\).

- **Analogue of the Euler representation:**
  Let \( G_b(x) = e^{\frac{\pi x}{2}(x-Q)}S_b(x) \).
  \[
  F_b(\alpha, \beta; \gamma; -ix) = \frac{1}{i} \frac{G_b(\gamma)}{G_b(\alpha)G_b(\beta)} \int_{-i\infty}^{+i\infty} dse^{2\pi isb} \frac{G_b(s - ix')G_b(s + \gamma - \beta)}{G_b(s - ix' + \alpha)G_b(s + Q)}
  \]
  where \( ix' = ix + \frac{1}{2}(\alpha + \beta - \gamma - Q) \).
  The integration contour is located to the right of the poles:
  \[
  \left\{ \begin{array}{l}
  s = ix' - nb - mb^{-1}, \\
  s = \beta - \gamma - nb - mb^{-1}.
  \end{array} \right.
  \]
  and to the left of the poles:
  \[
  \left\{ \begin{array}{l}
  s = nb + mb^{-1} \\
  s = ix' - \alpha + Q + nb + mb^{-1}
  \end{array} \right.
  \]
  with \( n, m \in \mathbb{N} \).

A useful transformation:

\[
F_b(\alpha, \beta; \gamma; -ix) = e^{\pi x(\gamma - \alpha - \beta)} \frac{S_b(-ix + \frac{\gamma - \alpha - \beta + Q}{2})}{S_b(-ix + \frac{-\gamma + \alpha + \beta + Q}{2})} F_b(\gamma - \alpha, \gamma - \beta; \gamma; -ix)
\]
This is the equivalent in the deformed case of the classical transformation:

\[ 2F_1(a, b; c; z) = (1 - z)^{c-a-b} 2F_1(c - a, c - b; c, z). \]

Two identities:

\[ F_b(0, \beta; \gamma; -ix) = 1, \]
\[ F_b(-b, \beta; \gamma; -ix) = 1 + e^{2\pi bx} \frac{\sin \pi b \beta}{\sin \pi b \gamma}. \]

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