THE LOCAL TRACE FUNCTION FOR SUPER-WAVELETS

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Abstract. We define an affine structure on $L^2(\mathbb{R}) \oplus \cdots \oplus L^2(\mathbb{R})$ and, following some ideas developed in [Dut1], we construct a local trace function for this situation. This trace function is a complete invariant for a shift invariant subspace and it has a variety of properties which make it easily computable. The local trace is then used to give a characterization of super-wavelets and to analyze their multiplicity function, dimension function and spectral function. The ”n×” oversampling result of Chui and Shi [CS] is refined to produce super-wavelets.

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1. Introduction

The wavelet theory involves the study of an affine structure existent on a Hilbert space $H$, given by two unitary operators, $T$ the translation, and $U$ the dilation that satisfy the commutation relation:

$$UTU^{-1} = T^N,$$

where $N \geq 2$ is an integer called the scale. For the classical wavelet theory, the Hilbert space is $L^2(\mathbb{R})$, the translation is

$$T_0f(x) = f(x−1), \quad (x \in \mathbb{R}, f \in L^2(\mathbb{R})),$$

and the dilation operator is

$$U_0f(x) = \frac{1}{\sqrt{N}}f\left(\frac{x}{N}\right), \quad (x \in \mathbb{R}, f \in L^2(\mathbb{R})),$$

where $N \geq 2$ is an integer.

Consider an affine structure $U$, $T$ on a Hilbert space $H$. A wavelet is a set $\Psi = \{\psi_1, ..., \psi_L\}$ such that the affine system

$$\{U^mT^k\psi \mid m, k \in \mathbb{Z}, \psi \in \Psi\}$$
is an orthonormal basis for $H$. Often the requirement is weaker, and one looks for the affine system to be only a frame. We recall that a family $(e_i)_{i \in I}$ of vectors in a Hilbert space $H$ is called a frame with constants $A > 0$ and $B > 0$ if

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, e_i \rangle|^2 \leq B\|f\|^2, \quad (f \in H).$$

When only the second inequality is satisfied, the family is called Bessel. When $A = B = 1$, the frame is called a normalized tight frame.

Each orthonormal wavelet has a generalized multiresolution attached to it. A generalized multiresolution analysis (GMRA) is a collection $(V_j)_{j \in \mathbb{Z}}$ of subspaces of $H$ that satisfy:

(i) $V_j \subset V_{j+1}$ for all $j$.
(ii) $UV_j = V_{j-1}$ for all $j$.
(iii) $\cup V_j$ is dense in $H$ and $\cap V_j = \{0\}$.
(iv) $V_0$ is invariant under $T_k$ for all $k \in \mathbb{Z}$.

If, in addition, there is a vector $\varphi \in V_0$ such that $\{T^k\varphi | k \in \mathbb{Z}\}$ is an orthonormal basis for $V_0$, then $(V_j)_{j}$ is called a multiresolution analysis (MRA) and $\varphi$ is called a scaling vector for this MRA. For more information on GMRA’s we refer to [BMM].

The GMRA associated to a wavelet $\Psi$ is defined by

$$V_j := \{U^m T^k \Psi | m > -j, k \in \mathbb{Z}\}, \quad (j \in \mathbb{Z}).$$

If the GMRA associated to $\Psi$ is actually a MRA, then $\Psi$ is called a MRA wavelet.

As it is shown in [HL] and [Dut2], it is possible to construct an affine structure and wavelets on the larger Hilbert space $L^2(\mathbb{R}) \oplus \ldots \oplus L^2(\mathbb{R})$, i.e. ”super-wavelets”.

One way to do this is by considering the operators:

$$T := T_0 \oplus \ldots \oplus T_0, \quad U := U_0 \oplus \ldots \oplus U_0,$$

and as shown in [HL], wavelets can be constructed for this affine structure, but none of them are MRA wavelets.

The affine structure described in [Dut2] takes into consideration the cycles of the map $z \mapsto z^N$. A set $\{z_1, \ldots, z_p\}$ of points on the unit circle $T := \{z \in \mathbb{C} | |z| = 1\}$ is called a cycle if the points are distinct and $z_1^N = z_2, \ldots, z_{p-1}^N = z_p, z_p^N = z_1$. Then the translation is defined by

$$T(f_1, \ldots, f_p) = (z_1 T_0 f_1, \ldots, z_p T_0 f_p), \quad (f_1, \ldots, f_p \in L^2(\mathbb{R}));$$

and the dilation is defined by

$$U(f_1, \ldots, f_p) = (U_0 f_2, U_0 f_3, \ldots, U_0 f_p, U_0 f_1), \quad (f_1, \ldots, f_p \in L^2(\mathbb{R})).$$

Note that the dilation operator permutes cyclically the components $f_1, \ldots, f_p$. Then one can do finite direct sums of these representations for different cycles to obtain new affine structures.

The advantage of the second type of affine structure is that it possesses MRA-wavelets (see [Dut2]).

The objective of this paper is to analyze these affine structures (see section 2 for the definition) and their ”super-wavelets”, their characterization, their dimension function, multiplicity function and spectral function, the relation between super-wavelets and super-scaling functions. We follow the ideas introduced in [Dut1], and the main tool will be the local trace function.

The subspaces of $L^2(\mathbb{R}) \oplus \ldots \oplus L^2(\mathbb{R})$ which are invariant under all integer translations $T^k$ are called shift invariant. The local trace function is a complex valued
map defined on $\mathbb{R}$ which is associated to a shift invariant space and an operator on $l^2(\mathbb{Z}) \oplus \ldots \oplus l^2(\mathbb{Z})$ (definition \textit{4.1}). Varying the operator, we obtain a lot of information about the shift invariant subspace. Taking the operator to be the identity yields the dimension function (definition \textit{6.7}), the spectral function is obtained by taking the operator to be a certain canonical one-dimensional projection (definition \textit{6.3}).

The central results are theorem \textit{4.4} and \textit{4.5}. Their power is twofold: they give a formula for computing the local trace in terms of normalized tight frame generators and also they show that this formula is independent of the choice of the normalized tight frame generator. Particular instances of these theorems give fundamental results about wavelets and shift invariant subspaces:

(i) Take the shift invariant subspace to be $L^2(\mathbb{R}) \oplus \ldots \oplus L^2(\mathbb{R})$ and the consequence is the theorem that characterizes normalized tight frames generated by translations: theorem \textit{4.9}. This in turn gives a characterization result for super-wavelets, theorem \textit{5.4}.

(ii) Take the shift invariant subspace to be $V_0$, the core space of the GMRA of a wavelet and take the operator to be $I$, then the result is the equality between the dimension function and the multiplicity function (proposition \textit{6.11}).

(iii) For $V_0$, take the operator to be a canonical one-dimensional projection and the result is the Gripenberg-Weiss formula of corollary \textit{6.6}.

Also, using the spectral function we are able to give a lower bound estimate on the dimension function (corollary \textit{6.13}), which shows not only that some of our affine structures do not have MRA wavelets, but also we can give a lower bound on the number of scaling functions any such wavelet needs.

Another interesting application, which follows from the characterization theorem \textit{5.4}, is a refinement of an oversampling result of Chui and Shi \textit{[CS]}, which states that when $\psi$ is a NTF wavelet for $N = 2$, then $\eta := 1/\rho \psi(\cdot/\rho)$ is also a NTF wavelet, $p$ being an odd number. We prove in theorem \textit{5.8} that much more is true: $\eta$ is part of a super-wavelet $\vec{\eta}:=(\eta, \ldots, \eta)$ in $L^2(\mathbb{R})^p$ and there is also a converse: if $\vec{\eta}$ is a NTF super-wavelet then $\psi$ is a wavelet. Moreover, going from the wavelet $\psi$ to the super-wavelet $\vec{\eta}$ preserves the orthogonality.

Some notations:

$\mathbb{T}$ is the unit circle in $\mathbb{C}$. We will often identify it with the interval $[-\pi, \pi)$ and the functions on $\mathbb{T}$ with functions on $[-\pi, \pi)$ and with $2\pi$-periodic functions on $\mathbb{R}$. The identification is done via $z = e^{-i\theta}$.

The Fourier transform is given by

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x} \, dx,$$

$(\xi \in \mathbb{R})$.

If $T$ in an operator on $L^2(\mathbb{R})$ we denote by $\hat{T}$ its conjugate by the Fourier transform $\hat{T} \hat{f} = \hat{T}f$, for $f \in L^2(\mathbb{R})$. On $L^2(\mathbb{R}) \oplus \ldots \oplus L^2(\mathbb{R})$ the Fourier transform is done componentwise.

For $f \in L^\infty(\mathbb{T})$, denote by $\pi_0(f)$ the operator on $L^2(\mathbb{R})$ defined by

$$\pi_0(f)\varphi = f\varphi,$$

$(f \in L^\infty(\mathbb{T}))$;

note that $T_0 = \pi_0(z)$ (here $z$ indicates the identity function on $\mathbb{T}$).
2. Permutative wavelet representations

We define here the affine structure on $L^2(\mathbb{R}) \oplus \cdots \oplus L^2(\mathbb{R})$ that we will work with. This structure contains the ones used in [HL] and [Dut2] which we mentioned in the introduction.

Let $\sigma$ be a permutation of the set $\{1, \ldots, n\}$ and $z_k = e^{-i\theta_k} \in \mathbb{T}$, $\theta_k \in [-\pi, \pi)$, with the property that $z_k^N = z_{\sigma(k)}$ for all $k \in \{1, \ldots, n\}$. We denote by $Z := (z_1, \ldots, z_n)$.

We define a wavelet representation on the Hilbert space

$$L^2(\mathbb{R})^n = L^2(\mathbb{R}) \oplus \cdots \oplus L^2(\mathbb{R}) \text{, n times}.$$

For $f \in L^\infty(\mathbb{T})$ define the operator $\pi_{\sigma,Z}(f)$ on $L^2(\mathbb{R})^n$ by:

$$\pi_{\sigma,Z}(f) (\varphi_1, \ldots, \varphi_n) = (\pi_0(f(z_1))\varphi_1, \ldots, \pi_0(f(z_n))\varphi_n),$$

for $\varphi_i \in L^2(\mathbb{R})$, $(i \in \{1, \ldots, n\})$, which means that the Fourier transform of this operator is

$$\hat{\pi}_{\sigma,Z}(f)(\varphi_1, \ldots, \varphi_n) = (f(\cdot + \theta_1)\varphi_1, \ldots, f(\cdot + \theta_n)\varphi_n).$$

Define the dilation $U_{\sigma,Z}$ on $L^2(\mathbb{R})^n$ by

$$U_{\sigma,Z}(\varphi_1, \ldots, \varphi_n) = (U_0\varphi_{\sigma(1)}, \ldots, U_0\varphi_{\sigma(n)}).$$

Then $(L^2(\mathbb{R})^n, \pi_{\sigma,Z}, U_{\sigma,Z})$ is a normal wavelet representation, i.e. it satisfies the following conditions:

(i) $U_{\sigma,Z}$ is unitary;
(ii) $\pi_{\sigma,Z}$ is a unital representation of the $C^*$-algebra $L^\infty(\mathbb{T})$;
(iii) $U_{\sigma,Z} \pi_{\sigma,Z}(f) U_{\sigma,Z}^{-1} = \pi_{\sigma,Z}(f(z^N))$ for all $f \in L^\infty(\mathbb{T})$;
(iv) For every uniformly bounded sequence $(f_n)_{n \in \mathbb{N}}$ in $L^\infty(\mathbb{T})$ which converges a.e to a function $f \in L^\infty(\mathbb{T})$, one has that $\pi_{\sigma,Z}(f_n)$ converges to $\pi_{\sigma,Z}(f)$ in the strong operator topology.

Checking these properties requires just some simple computations. The translation (or shift) of this wavelet representation is

$$T_{\sigma,Z} := \pi_{\sigma,Z}(z),$$

where $z$ indicates the identity function on $\mathbb{T}$, $z \mapsto z$. Also note that $T_{\sigma,Z}^n = \pi_{\sigma,Z}(z^n)$.

Remark 2.1. Each permutation $\sigma$ can be decomposed into a finite product of disjoint cycles. Since $z_i^N = z_{\sigma(i)}$ for all $i$, this implies that all $z_i$ have a finite orbit under the map $z \mapsto z^N$, i.e. they are cycles, as defined in the introduction.

It may be that the length of the cycle given by $z_i$ is shorter than the length of the corresponding cycle of $\sigma$ (for example, when all $z_i = 1$ and $\sigma$ is a product of disjoint transpositions, then the length of the cycle of $z_i$ is 1, but the cycles of $\sigma$ have length 2). However when these lengths coincide we obtain the representations defined in [Dut2].

When $z_i = 1$ for all $i$ and $\sigma$ is the identity we have the amplification of the standard representation on $L^2(\mathbb{R})$, amplification which was studied in [HL].

For the rest of the paper we will consider a fixed permutation $\sigma$ and the points $z_1, z_2, \ldots, z_n$, and we will omit the subscripts.
3. Shift invariant subspaces

In this section we present some structural theorem for subspaces of $L^2(\mathbb{R})^n$ which are invariant under the integer translations $T^k$.

**Definition 3.1.** A subspace $V$ of $L^2(\mathbb{R})^n$ is called shift invariant (SI) if $T^kV \subseteq V$ for all $k \in \mathbb{Z}$.

If $\mathcal{A}$ is a subset of $L^2(\mathbb{R})^n$, we denote by $S(\mathcal{A})$ the shift invariant subspace generated by $\mathcal{A}$.

**Definition 3.2.** Let $V$ be a shift invariant subspace of $L^2(\mathbb{R})^n$. A subset $\Phi$ of $V$ is called a normalized tight frame generator (or NTF generator) for $V$ if

$$\{T_k\varphi \mid k \in \mathbb{Z}, \varphi \in \Phi\}$$

is a NTF for $V$.

We use also the notation $S(\varphi) := S(\{\varphi\})$. $\varphi$ is called a quasi-orthogonal generator for $S(\varphi)$ if

$$\{T_k\varphi \mid k \in \mathbb{Z}\}$$

is a NTF for $S(\varphi)$ and for all $\xi \in \mathbb{R}$,

$$\text{Per} |\hat{\varphi}|^2(\xi) := \sum_{k \in \mathbb{Z}} |\hat{\varphi}(\xi + 2k\pi)| \in \{0, 1\}.$$

(actually, the second condition is a consequence of the first but we include it anyway).

We will need the "fiberization" techniques based on the range function of Helson $\mathbb{R}$, which were used in the classical case in the work of de Boor, DeVore and Ron [BDR] and further developed by Ron and Shen [RS1]-[RS4], Bownik [Bo1] and others.

For $\varphi = (\varphi_1, \ldots, \varphi_n) \in L^2(\mathbb{R})^n$, the map $T_{\text{per}} \varphi$ assigns to each point $\xi \in \mathbb{R}$ a vector in $l^2(\mathbb{Z})^n$, called the fiber of $\varphi$ at $\xi$:

$$T_{\text{per}}(\varphi_1, \ldots, \varphi_n)(\xi) = (\hat{\varphi}_1(\xi - \theta_1 + 2k\pi) + \mathbb{Z}, \ldots, (\hat{\varphi}_n(\xi - \theta_n + 2k\pi))_{k \in \mathbb{Z}}), \; (\xi \in \mathbb{R}),$$

for all $(\varphi_1, \ldots, \varphi_n) \in L^2(\mathbb{R})^n$.

This map transforms the translations into multiplications by scalars in each fiber:

**Proposition 3.3.** For all $f \in L^\infty(\mathbb{T})$ and $(\varphi_1, \ldots, \varphi_n) \in \mathbb{R}$,

$$T_{\text{per}}(\pi(f)(\varphi_1, \ldots, \varphi_n))(\xi) = f(\xi) T_{\text{per}}(\varphi_1, \ldots, \varphi_n)(\xi), \; (\xi \in \mathbb{R}).$$

In particular, for all $k \in \mathbb{Z}$,

$$T_{\text{per}}(T^k(\varphi_1, \ldots, \varphi_n))(\xi) = e^{-ik\xi} T_{\text{per}}(\varphi_1, \ldots, \varphi_n)(\xi), \; (\xi \in \mathbb{R}).$$

Note also the periodicity property of $T_{\text{per}}$ (which justifies the subscript "per"): $T_{\text{per}} f(\xi) = T_{\text{per}} f(\xi + 2s\pi)$, $\forall s \in \mathbb{Z}$,

where, for $s \in \mathbb{Z}$, $\lambda(s)$ is the shift on $l^2(\mathbb{Z})^n$:

$$(\lambda(s)\alpha)(k,i) = \alpha(k-s,i), \; (k \in \mathbb{Z}, i \in \{1, \ldots, N\}).$$

For a shift invariant subspace the fibers $T_{\text{per}} \varphi(\xi)$ at a fixed point $\xi$ form a subspace of $l^2(\mathbb{Z})^n$, which will be denoted by $J_{\text{per}}(\xi)$. Thus $J_{\text{per}}$ will be a bundle map which assigns to each point $\xi \in \mathbb{R}$ a subspace of $l^2(\mathbb{Z})^n$. This is the idea behind the range function.
Definition 3.4. A range function is a measurable mapping

\[ J : [-\pi, \pi) \to \{ \text{closed subspaces of } l^2(\mathbb{Z})^n \} \].

Measurable means weakly operator measurable, i.e., \( \xi \mapsto \langle P_J(\xi) a \mid b \rangle \) is measurable for any choice of vectors \( a, b \in l^2(\mathbb{Z})^n \).

A periodic range function is a measurable function \( J_{per} : \mathbb{R} \to \{ \text{closed subspaces of } l^2(\mathbb{Z})^n \} \), with the periodicity property:

\[ J_{per}(\xi + 2k\pi) = \lambda(k)^* J_{per}(\xi) \lambda(k), \quad (k \in \mathbb{Z}, \xi \in \mathbb{R}). \]

Sometimes we will use the same letter to denote the subspace \( J_{per}(\xi) \) and the projection onto \( J_{per}(\xi) \). In terms of projections, the periodicity can be written as:

\[ J_{per}(\xi + 2k\pi) = \lambda(k)^* J_{per}(\xi) \lambda(k), \quad (k \in \mathbb{Z}, \xi \in \mathbb{R}). \]

There is a one-to-one correspondence between shift invariant subspaces and range functions. This correspondence is made precise in the next theorem due to Helson [H]. The proof given in [Bo1], proposition 1.5, carries over here, without any significant modification.

Theorem 3.5. A closed subspace \( V \) of \( L^2(\mathbb{R}^n) \) is shift invariant if and only if

\[ V = \{ f \in L^2(\mathbb{R}^n) \mid T_{per} f(\xi) \in J_{per}(\xi) \text{ for a.e. } \xi \in \mathbb{R}^n \}, \]

for some measurable periodic range function \( J_{per} \). The correspondence between \( V \) and \( J_{per} \) is bijective under the convention that range functions are identified if they are equal a.e. Furthermore, if \( V = S(A) \) for some countable \( A \subset L^2(\mathbb{R}^n) \), then

\[ J_{per}(\xi) = \text{span}\{ T_{per} \varphi(\xi) \mid \varphi \in A \}, \quad \text{for a.e. } \xi \in \mathbb{R}. \]

Another fundamental fact about the range function is that it transforms the frame property into a local property: more precisely, a family of vectors generate a normalized tight frame by translations if and only if the fibers form normalized tight frames at each point in \( \mathbb{R} \). This is given in the next theorem, for a proof look in [Bo1], theorem 2.3.

Theorem 3.6. Let \( V \) be a SI subspace of \( L^2(\mathbb{R}^n) \), \( J_{per} \) its periodic range function and \( \Phi \) a countable subset of \( V \). \( \{ T_k \varphi \mid k \in \mathbb{Z}, \varphi \in \Phi \} \) is a frame with constants \( A \) and \( B \) for \( V \) (Bessel family with constant \( B \)) if and only if \( \{ T_{per} \varphi(\xi) \mid \varphi \in \Phi \} \) is a frame with constants \( A \) and \( B \) for \( J_{per}(\xi) \) (Bessel sequence with constant \( B \)) for almost every \( \xi \in \mathbb{R} \).

Another approach, with a more representation-theoretic flavor, is proposed in [BMM], [BM], [B]. The group \( \mathbb{Z} \) has a unitary representation on a shift invariant space and the abstract harmonic analysis arguments given in the papers by Baggett et al work here with minor changes and yield the following results.

Theorem 3.7. If \( V_0 \) is a shift invariant subspace of \( L^2(\mathbb{R})^n \) then there exists a multiplicity function \( m : \mathbb{T} \to \{0, 1, \ldots, \infty\} \) such that:

(i) If

\[ S_j := \{ \xi \in \mathbb{T} \mid m(\xi) \geq j \}, \]
then there is a unitary
\[ \mathcal{J} : V_0 \to \bigoplus_{j=1}^{\infty} L^2(S_j) \]
which intertwines the representations
\[ (\mathcal{J} T^k) = \rho_k \mathcal{J}, \quad (k \in \mathbb{Z}), \]
where \( \rho_k \) is multiplication by the function \( \xi \mapsto e^{-ik\xi} \) on each component:
\[ \rho_k(f_1, f_2, \ldots) = (e^{-ikf_1}, e^{-ikf_2}, \ldots), \quad (k \in \mathbb{Z}). \]

(ii) If
\[ \phi_i := \mathcal{J}^{-1}(\chi_{S_i}), \quad (i \in \{1, 2, \ldots\}), \]
(where \( \chi_{S_i} \) stands for the element of \( \bigoplus L^2(S_j) \) whose only nonzero component is \( \chi_{S_i} \) in \( L^2(S_i) \)), then
\[ \{\phi_i \mid i \in \{1, 2, \ldots\}\} \text{ is a NTF generator for } V_0, \]

(iii) \( V_0 \) can be decomposed as the orthogonal sum of the shift invariant subspaces generated by the quasi-orthogonal generators \( \phi_i \):
\[ V_0 = \bigoplus_{i=1}^{\infty} S(\phi_i). \]

Proof. We follow the arguments given in [BMM] which use the spectral theory of Stone and Mackey for the commutative group \( \mathbb{Z} \) that has as dual \( \hat{\mathbb{Z}} = \mathbb{T} \).

For the representation of \( \mathbb{Z} \) by translations \( T^k \) on \( V_0 \), there is a unique projection-valued measure \( p \) on \( \mathbb{T} \) for which
\[ T^k|_{V_0} = \int_{\mathbb{T}} e^{-ik\xi} dp(\xi), \quad (k \in \mathbb{Z}). \]

The projection-valued measure \( p \) is completely determined by a measure class \([\mu]\) on \( \mathbb{T} \) and a measurable multiplicity function \( m : \mathbb{T} \to \{0, 1, \ldots, \infty\} \).

As in [BMM] proposition 1.2, we will show that the measure class \([\mu]\) is absolutely continuous with respect to the Haar measure on \( \mathbb{T} \).

Indeed, the Fourier transform establishes an equivalence between the translation \( T \) on \( L^2(\mathbb{R})^n \) and its Fourier version
\[ \hat{T}^k(f_1, \ldots, f_n) = (e^{-ik(\cdot + \theta_i)}f_i)_{i \in \{1, \ldots, n\}}, \quad (k \in \mathbb{Z}). \]
Define \( W : L^2(\mathbb{R})^n \to L^2(\mathbb{R})^n \) by
\[ W(f_1, \ldots, f_n) = (f_i(\cdot - \theta_i))_{i \in \{1, \ldots, n\}}. \]
This intertwines the given representation with the one defined by
\[ \hat{T}^k(f_1, \ldots, f_n) = (e^{-ikf_i})_{i \in \{1, \ldots, n\}}. \]
But $\mathbb{R}$ can be decomposed as
\[ \mathbb{R} = \bigcup_{k \in \mathbb{Z}} [(2k - 1)\pi, (2k + 1)\pi), \]
thus, the representation of $\mathbb{Z}$ on the entire space $L^2(\mathbb{R})^n$ is equivalent to $n$ times an infinite multiple of the regular representation of $\mathbb{Z}$ (which is multiplication by $e^{-ik\xi}$ on $L^2(\mathbb{T})$).

Therefore, the projection-valued measure associated to the representation of $\mathbb{Z}$ on the entire space $L^2(\mathbb{R})^n$ is equivalent to the Haar measure on $\mathbb{T}$, so the projection-valued measure associated to any subrepresentation of this representation must have a measure which is absolutely continuous w.r.t. the Haar measure.

For (3.1), (3.2) and (3.3) the argument is the one used on [BM] lemma 1.1. (3.4) can be proved as in lemma 1.2 from [BM]. It follows from the fact that $J$ is unitary and intertwining so, for $k \in \mathbb{Z}, i, j \in \{1, 2, \ldots\}$,
\[ \langle T^k \phi_i | \phi_j \rangle = \int_{\mathbb{T}} e^{-ik\xi} \chi_S \delta_{ij} d\xi, \]
which implies
\[ \int_{\mathbb{T}} e^{-ik\xi} \langle T_{\text{per}} \phi_i(\xi) | T_{\text{per}} \phi_j(\xi) \rangle d\xi = \int_{\mathbb{T}} e^{-ik\xi} \chi_S \delta_{ij} d\xi. \]
(3.5) is just a rephrasing of the previous statements. □

4. The local trace function

We are able now to define the local trace function. The proofs from [Dut1] work here in most of the instances and, when a result is presented without a proof, the reader can look in [Dut1].

Each shift invariant subspace $V$, can be seen as a bundle of vector subspaces of $l^2(\mathbb{Z})^n$ by means of a range function $J_{\text{per}}$. The local trace function associated to $V$ and an operator $T$ on $l^2(\mathbb{Z})^n$ is the function which associates to each point $\xi$ the value of the canonical trace of the restriction of $T$ to the subspace $J_{\text{per}}(\xi)$. We recall that the trace of a positive or trace-class operator $T$ on some Hilbert space $H$, can be computed by
\[ \text{Trace}(T) = \sum_{i \in I} \langle Tf_i | f_i \rangle, \]
where $(f_i)_{i \in I}$ is any normalized tight frame of the Hilbert space $H$ (see [Dut1]).

**Definition 4.1.** Let $V$ be a SI subspace of $L^2(\mathbb{R}^n)$, $T$ a positive (or trace-class) operator on $l^2(\mathbb{Z})^n$ and let $J_{\text{per}}$ be the range function associated to $V$. We define the local trace function associated to $V$ and $T$ as the map from $\mathbb{R}$ to $[0, \infty]$ (or $\mathbb{C}$) given by the formula
\[ \tau_{V,T}(\xi) = \text{Trace} \left( T J_{\text{per}}(\xi) \right), \quad (\xi \in \mathbb{R}). \]
We define the restricted local trace function associated to $V$ and a vector $f$ in $l^2(\mathbb{Z})^n$ by
\[ \tau_{V,f}(\xi) = \text{Trace} \left( P_f J_{\text{per}}(\xi) \right) = \tau_{V,P_f}(\xi), \quad (\xi \in \mathbb{R}^n), \]
where $P_f$ is the operator on $l^2(\mathbb{Z})^n$ defined by $P_f(v) = \langle v | f \rangle f$. 

Remark 4.2. Even though, $T J_{\text{per}}(\xi)$ is not necessarily positive when $T$ is, when computing the trace, we can consider instead of $T J_{\text{per}}(\xi)$ the operator $J_{\text{per}}(\xi) T J_{\text{per}}(\xi)$, or we can use normalized tight frames just for the subspace $J_{\text{per}}(\xi)$ (not for the entire $l^2(\mathbb{Z})^n$), in any case the expression $\text{Trace}(T J_{\text{per}}(\xi))$ makes sense and is a positive number.

When $T$ is trace-class, since the trace class operators form an ideal, $T J_{\text{per}}(\xi)$ is also trace class so $\text{Trace}(T J_{\text{per}}(\xi))$ makes sense and is a positive number.

Here are some ways to compute the local trace function.

Proposition 4.3. For all $f \in l^2(\mathbb{Z})^n$, 

$$\tau_{V,f}(\xi) = \| J_{\text{per}}(\xi)(f) \|^2,$$

(4.1) for a.e. $\xi \in \mathbb{R}$.

The next result is central for the paper. As we have mentioned in the introduction, this theorem can be used for two purposes: it gives a method to compute the local trace function and secondly, and maybe even more important, it shows that the formula on the right does not depend on the choice of the normalized tight frame. So one can look at the local trace function as an invariant for the SI subspace, and the theorem is an index theorem.

Theorem 4.4. Let $V$ be a SI subspace of $L^2(\mathbb{R})^n$ and $\Phi \subset V$ a NTF generator for $V$. Then for every positive (or trace-class) operator $T$ on $l^2(\mathbb{Z})^n$ and any $f \in l^2(\mathbb{Z})^n$,

$$\tau_{V,T}(\xi) = \sum_{\varphi \in \Phi} \langle TT_{\text{per}}\varphi(\xi) \mid T_{\text{per}}\varphi(\xi) \rangle,$$

(4.1) for a.e. $\xi \in \mathbb{R}$;

$$\tau_{V,f}(\xi) = \sum_{\varphi \in \Phi} | \langle f \mid T_{\text{per}}\varphi(\xi) \rangle |^2,$$

(4.2) for a.e. $\xi \in \mathbb{R}$.

The converse is also true: if a family of vectors can be used to compute the local trace then they form a NTF generator for the SI space. Moreover it is sufficient that this is satisfied only for some very particular vectors (see (iii) in the next result):

Theorem 4.5. Let $V$ be a SI subspace of $L^2(\mathbb{R})^n$, $J_{\text{per}}$ its periodic range function and $\Phi$ a countable subset of $L^2(\mathbb{R})^n$. Then following affirmations are equivalent:

(i) $\phi \subset V$ and $\phi$ is a NTF generator for $V$;

(ii) For every $f \in l^2(\mathbb{Z})^n$

$$\sum_{\varphi \in \Phi} | \langle f \mid T_{\text{per}}\varphi(\xi) \rangle |^2 = \| J_{\text{per}}(\xi)(f) \|^2,$$

(4.3) for a.e. $\xi \in \mathbb{R}$

(iii) For every $0 \neq k \in \mathbb{Z}$, $i,j \in \{1,\ldots,n\}$ and $\alpha \in \{0,1,i = \sqrt{-1}\}$,

$$\sum_{\varphi \in \Phi} | \hat{\varphi}_i(\xi) + \alpha \hat{\varphi}_j(\xi + 2k\pi) |^2 = \| J_{\text{per}}(\xi)(\delta_{il} + \alpha\delta_{kj}) \|^2,$$

(4.4) for a.e. $\xi \in \mathbb{R}$.

The local trace function is related in a very simple way to another important invariant, the dual Gramian. This was introduced by A. Ron and Z. Shen and successfully used for the analysis of the structure of SI spaces and of frames generated by translations in [RS1]-[RS4].
Definition 4.6. Let $V$ be a shift invariant subspace of $L^2(\mathbb{R})^n$ and $\Phi$, a NTF generator for $V$. The dual Gramian of $\Phi$ is the function $\tilde{G}_\Phi$ which assigns to each point $\xi \in \mathbb{R}$ the matrix

$$
\tilde{G}_{ki,lj}^\Phi(\xi) = \sum_{\varphi \in \Phi} \hat{\varphi}_i(\xi - \theta_i + 2k\pi) \hat{\varphi}_j(\xi - \theta_j + 2l\pi), \quad (k, l \in \mathbb{Z}, i, j \in \{1, ..., n\}).
$$

We can recuperate the Gramian from the local trace function just by computing it at some rank-one operators.

Proposition 4.7. If $V$ is a SI space, $J_{\text{per}}$ its range function, and $\Phi$ a NTF generator for it then, for all $k, l \in \mathbb{Z}, i, j \in \{1, ..., n\}$,

$$
\tilde{G}_{ki,lj}^\Phi(\xi) = \tau_{V, P_{ki,lj}}(\xi) = \langle J_{\text{per}}(\xi) \delta_{ki} | J_{\text{per}}(\xi) \delta_{lj} \rangle, \quad (\xi \in \mathbb{R}),
$$

where $P_{ki,lj}$ is the rank-one operator defined by

$$
P_{ki,lj}v = \langle v | \delta_{ki} \rangle \delta_{lj}, \quad (v \in l^2(\mathbb{Z})^n).
$$

Proof. The first part is a consequence of theorem 4.4. For the last equality we use the following argument: if $(f_q)_{q \in Q}$ is an orthonormal basis for $J_{\text{per}}(\xi)$ then

$$
\text{Trace}(P_{ki,lj} J_{\text{per}}(\xi)) = \sum_{q \in Q} \langle P_{ki,lj} f_q | f_q \rangle = \sum_{q \in Q} \langle f_q | \delta_{ki} \rangle \langle \delta_{lj} | f_q \rangle = \sum_{q \in Q} \langle f_q | J_{\text{per}}(\xi) \delta_{ki} \rangle \langle J_{\text{per}}(\xi) \delta_{lj} | f_q \rangle = \langle J_{\text{per}}(\xi) \delta_{lj} | J_{\text{per}}(\xi) \delta_{ki} \rangle.
$$

□

Having these it is clear that the dual Gramian is an invariant and in fact it is a complete one (see [Dut1, theorem 4.6]) for the proof:

Theorem 4.8. Let $V$ be a SI subspace of $L^2(\mathbb{R})^n$, $\Phi_1$ a NTF generator for $V$ and $\Phi_2$ a countable family of vectors from $L^2(\mathbb{R})^n$. The following affirmations are equivalent:

(i) $\Phi_2 \subset V$ and $\Phi_2$ is a NTF generator for $V$;

(ii) The dual Gramians $\tilde{G}_\Phi^1$ and $\tilde{G}_\Phi^2$ are equal almost everywhere.

If we particularize this result to the case when the shift invariant subspace is $L^2(\mathbb{R})^n$, then since the range function for it is constant $l^2(\mathbb{Z})^n$ at each point, we obtain the following characterization of families that generate normalized tight frames by translations:

Theorem 4.9. Let $\Phi$ be a countable subset of $L^2(\mathbb{R})^n$. The following affirmations are equivalent:

(i) $\Phi$ is a NTF generator for $L^2(\mathbb{R})^n$;

(ii)

$$
\sum_{\varphi \in \Phi} \hat{\varphi}_i(\xi - \theta_i) \hat{\varphi}_j(\xi - \theta_j + 2k\pi) = \delta_{ij} \delta_k, \quad (\xi \in \mathbb{R}, i \in \{1, ..., n\}, k \in \mathbb{Z}).
$$

Next, we present some elementary properties of the local trace function. The proofs of these results are simple and can be found in [Dut1] for the classical case.
Proposition 4.10. [Periodicity] Let $V$ be a SI subspace, $T$ a positive or trace-class operator on $l^2(\mathbb{Z})^n$, $f \in l^2(\mathbb{Z})^n$. Then, for $k \in \mathbb{Z}$

$$(4.5) \quad \tau_{V,T}(\xi + 2k\pi) = \tau_{V,\lambda(k)T\lambda(k)^*}(\xi), \quad (\text{for a.e. } \xi \in \mathbb{R});$$

$$(4.6) \quad \tau_{V,f}(\xi + 2k\pi) = \tau_{V,\lambda(k)f}(\xi), \quad (\text{for a.e. } \xi \in \mathbb{R}),$$

where $\lambda(k)$ is the shift on $l^2(\mathbb{Z})^n$, i.e. the unitary operator on $l^2(\mathbb{Z})^n$ defined by $(\lambda(k)\alpha)(l, i) = \alpha(l - k, i)$ for all $l \in \mathbb{Z}, i \in \{1, ..., n\}, \alpha \in l^2(\mathbb{Z})^n$.

Proposition 4.11. [Additivity] Suppose $(V_i)_{i \in I}$ are mutually orthogonal SI subspaces (I countable) and let $V = \oplus_{i \in I} V_i$. Then, for every positive or trace-class operator $T$ on $l^2(\mathbb{Z})^n$ and every $f \in l^2(\mathbb{Z})^n$:

$$(4.7) \quad \tau_{V,T} = \sum_{i \in I} \tau_{V_i,T}, \quad \text{a.e. on } \mathbb{R};$$

$$(4.8) \quad \tau_{V,f} = \sum_{i \in I} \tau_{V_i,f}, \quad \text{a.e. on } \mathbb{R}.$$

Proposition 4.12. [Monotony and injectivity] Let $V, W$ be SI subspaces.

(i) $V \subset W$ iff $\tau_{V,T} \leq \tau_{W,T}$ a.e. for all positive operators $T$ iff $\tau_{V,f} \leq \tau_{W,f}$ a.e. for all $f \in l^2(\mathbb{Z})^n$.

(ii) $V = W$ iff $\tau_{V,f} = \tau_{W,f}$ a.e. for all $f \in l^2(\mathbb{Z})^n$.

Theorem 4.13. Let $(V_j)_{j \in \mathbb{N}}$ be an increasing sequence of SI subspaces of $L^2(\mathbb{R}^n)$,

$$ V := \bigcup_{j \in \mathbb{N}} V_j, $$

$T$ a positive (trace-class) operator on $l^2(\mathbb{Z})^n$ and $f \in l^2(\mathbb{Z})^n$. Then, for a.e. $\xi \in \mathbb{R}$, $\tau_{V_j,T}(\xi)$ increases (converges) to $\tau_{V,T}(\xi)$ and $\tau_{V_j,f}(\xi)$ increases to $\tau_{V,f}(\xi)$.

We will use the local trace function for the analysis of wavelets. For this reason we have to see how it behaves under dilation.

Theorem 4.14. Let $V$ be a shift invariant subspace of $L^2(\mathbb{R})^n$ and $T$ a positive or trace-class operator on $l^2(\mathbb{Z})^n$. For $l \in \{0, ..., N-1\}$ define the operators $D_l$ on $l^2(\mathbb{Z})^n$ by

$$D_l v(k, i) = \begin{cases} v(s, i), & \text{if } 2l\pi = 2l\pi + \theta_i - N\theta_{\sigma^{-1}(i)} + 2Ns\pi, \text{ with } s \in \mathbb{Z}, \\ 0, & \text{otherwise}, \end{cases}$$

and the unitary operator $S$ on $l^2(\mathbb{Z})^n$, by

$$S(v_1, ..., v_n) = (v_{\sigma^{-1}(1)}, ..., v_{\sigma^{-1}(n)}), \quad (v_1, ..., v_n) \in l^2(\mathbb{Z})^n).$$

Then $U^{-1}V$ is shift invariant and its local trace function is given by

$$\tau_{U^{-1}V,T}(\xi) = \frac{1}{N} \sum_{i=0}^{N-1} \tau_{V,S \cdot D_l T D_l S}(\frac{\xi + 2l\pi}{N}), \quad (\xi \in \mathbb{R}).$$

Proof. By theorem 3.7, $V$ can be decomposed into an orthogonal sum of singly-generated SI spaces. Also the local trace is additive and $U$ is unitary so we may assume that $V = S(\varphi)$ with $\varphi$ a quasi-orthogonal generator.
Due to the commutation relation $UTU^{-1} = T^N$, the vectors
\[ \{ \varphi_{-1,r} := U^{-1}T^r \varphi, | r \in \{0, ..., N-1\} \} \]
form a NTF generator for $U^{-1}V$. Their Fourier transforms are
\[ \hat{\varphi}_{-1,r}(\xi) = \left( \frac{1}{\sqrt{N}} e^{-ir(\xi/2N + \theta_{\sigma^{-1}(i)})} \hat{\varphi}_{\sigma^{-1}(i)}(\xi/N) \right)_{i \in \{1, ..., n\}}, \quad (\xi \in \mathbb{R}). \]

Then a short computation shows that, if we consider the vectors \( v_l(\xi), l \in \{0, ..., N-1\} \) in $l^2(\mathbb{Z})^n$,
\[ v_l(\xi)(k,i) = \left\{ \begin{array}{ll}
\hat{\varphi}_{\sigma^{-1}(i)} \left( \frac{\xi - \theta + 2k\pi}{N} \right), & \text{if } -\frac{\theta + 2k\pi}{N} \in \frac{2\pi l}{N} - \theta_{\sigma^{-1}(i)} + 2\pi \mathbb{Z}, \\
0, & \text{otherwise},
\end{array} \right. \]
then
\[ T_{\text{per}} \varphi_{-1,r}(\xi) = \frac{1}{\sqrt{N}} e^{-ir\frac{2\pi}{N}} \sum_{l=0}^{N-1} e^{-ir\frac{2\pi l}{N}} v_l(\xi), \quad (r \in \{0, ..., N-1\}). \]

The matrix
\[ \left( \frac{1}{\sqrt{N}} e^{-ir\frac{2\pi l}{N}} \right)_{l,r \in \{0, ..., N-1\}} \]
is unitary, so the vectors \( \{ v_l(\xi) \mid l \in \{0, ..., N-1\} \} \) span the same subspace of $l^2(\mathbb{Z})^n$ as the vectors \( \{ T_{\text{per}} \varphi_{-1,r}(\xi) \mid r \in \{0, ..., N-1\} \} \), namely $J_{U^{-1}V}(\xi)$.

Note that
\[ v_l(\xi) = D_l S T_{\text{per}} \varphi \left( \frac{\xi + 2l\pi}{N} \right), \quad (l \in \{0, ..., N-1\}), \]
\( D_l \) are isometries and \( S \) is unitary. In addition \( \varphi \) is quasi-orthogonal so \( \| T_{\text{per}} \varphi(\xi) \| \in \{0,1\} \) for almost every $\xi \in \mathbb{R}$, hence \( \| v_l(\xi) \| \in \{0,1\} \) for all $l$. It is also clear that the vectors $v_l(\xi)$ are mutually orthogonal for a fixed $\xi$. Having all these, it follows that \( \{ v_l(\xi) \mid l \in \{0, ..., N-1\} \} \) is a NTF for $J_{U^{-1}V}(\xi)$, therefore we can compute the trace with them.
\[ \tau_{U^{-1}V,T}(\xi) = \sum_{l=0}^{N-1} (T v_l(\xi) \mid v_l(\xi)) \]
\[ = \sum_{l=0}^{N-1} \left\langle T D_l S T_{\text{per}} \varphi \left( \frac{\xi + 2l\pi}{N} \right) \mid D_l S T_{\text{per}} \varphi \left( \frac{\xi + 2l\pi}{N} \right) \right\rangle \]
\[ = \sum_{l=0}^{N-1} \left\langle S^* D_l^* T D_l S T_{\text{per}} \varphi \left( \frac{\xi + 2l\pi}{N} \right) \mid T_{\text{per}} \varphi \left( \frac{\xi + 2l\pi}{N} \right) \right\rangle, \]
which proves the formula. \( \square \)

5. A CHARACTERIZATION OF SUPER-WAVELETS

We apply now the local trace function to super-wavelets. Our first goal is to obtain a characterization of NTF super-wavelets. This is done in theorem 5.2.

**Definition 5.1.** A finite subset $\Psi = \{ \psi^1, ..., \psi^L \}$ of $L^2(\mathbb{R})^n$ is a NTF wavelet if the affine system
\[ X(\Psi) := \{ \psi_{j,k} := U^{-j}T^k \psi \mid j \in \mathbb{Z}, k \in \mathbb{Z}, \psi \in \Psi \} \]
is a NTF for $L^2(\mathbb{R})^n$. 
The quasi-affine system $X^q(\Psi)$ is
$$X^q(\Psi) := \{ \tilde{\psi}_{j,k} \mid j \in \mathbb{Z}, k \in \mathbb{Z}, \psi \in \Psi \},$$
with the convention
$$\tilde{\psi}_{j,k} := \begin{cases} U^{-j}T^k\psi, & \text{if } j \geq 0, k \in \mathbb{Z} \\ N^{j/2}T^kU^{-j}\psi & \text{if } j < 0, k \in \mathbb{Z}. \end{cases}$$

There is an equivalence between affine and quasi-affine frames. This was proved in full generality for the $L^2(\mathbb{R})$-case in CSS. Again the result will work here with some slight modifications in the proof.

**Theorem 5.2.** Let $\Psi = \{\psi^1, ..., \psi^L\}$ be a finite subset of $L^2(\mathbb{R})^n$.

(i) $X(\Psi)$ is a Bessel family if and only if $X^q(\Psi)$ is a Bessel family. Furthermore, their exact frame bounds are equal.

(ii) $X(\Psi)$ is an affine frame with constants $A$ and $B$ if and only if $X^q(\Psi)$ is a quasi-affine frame with constants $A$ and $B$.

**Proof.** The proof of theorem 2 in CSS works here word for word once we have the following lemma, which is the analogue of lemma 4 in CSS.

**Lemma 5.3.** Let $\Psi = \{\psi_1, ..., \psi_L\} \subset L^2(\mathbb{R})^n$ and let $f \in L^2(\mathbb{R})^n$ be a vector such that $f_1, ..., f_n$ have all compact supports. Then
$$\lim_{M \to \infty} \sum_{j < 0} \sum_{\psi \in \Psi} \sum_{k \in \mathbb{Z}} |\langle U^{-M}f, \tilde{\psi}_{j,k} \rangle|^2 = 0,$$
and
$$\lim_{M \to \infty} N^{-M} \sum_{j \leq -M} \sum_{\psi \in \Psi} \sum_{k \in \mathbb{Z}} |\langle T^j f, \tilde{\psi}_{j,k} \rangle|^2 = 0.$$

Lemma 4 from CSS shows that this is true for the classical wavelet representation on $L^2(\mathbb{R})$ (i.e. $n = 1$, $\sigma =$ identity, $\theta_1 = 0$). We will show that the general case follows from this particular one.

For $M \geq 0$, $j < 0$, $\psi \in \Psi$, $k \in \mathbb{Z}$ we evaluate
$$|\langle U^{-M}f, \tilde{\psi}_{j,k} \rangle|^2 \leq \sum_{i=1}^n \int_{\mathbb{R}} |U_0^{-M}f_{\sigma^{-M}(i)}(x)N^{j/2}T^kU_0^{-j}\psi_{\sigma^{-j}(i)}(x) dx|^2 \leq |\sum_{i=1}^n \int_{\mathbb{R}} |U_0^{-M}f_{\sigma^{-M}(i)}(x)N^{j/2}T^kU_0^{-j}\psi_{\sigma^{-j}(i)}(x) dx|^2$$
where
$$F_f := \max\{|f_i| \mid i \in \{1, ..., n\}\}, P_\psi := \max\{|\psi_i| \mid i \in \{1, ..., n\}\},$$
and the last term in the previous inequality is actually
$$= n^2 |\langle U_0^{-M}F_f, (P_\psi)^{j,k} \rangle|^2.$$

Apply now lemma 4 of CSS for $f \leftrightarrow F_f$ and $\Psi \leftrightarrow \{P_\psi \mid \psi \in \Psi\}$ and with the previous inequality we obtain the first limit.

The second one can be obtain by a similar argument. $\Box$
Theorem \ref{thm:NTF} is one of the crucial steps in obtaining a characterization for wavelets. The only thing that remains to do is to use theorem \ref{thm:NTF} for the quasi-affine system and carefully compute the Gramians to obtain the next result which gives the characterizing equation for NTF wavelets in $L^2(\mathbb{R})^n$.

**Theorem 5.4.** Let $\Psi = \{\psi_1, ..., \psi_p\}$ be a finite subset of $L^2(\mathbb{R})^n$. The following affirmations are equivalent:

(i) The affine system $X(\Psi)$ is a NTF for $L^2(\mathbb{R})^n$;

(ii) The following equation hold for almost every $\xi \in \mathbb{R}$, $i, j \in \{1, ..., n\}$:

\begin{equation}
\sum_{\psi \in \Psi} \sum_{m=-\infty}^{\infty} \hat{\psi}_{\sigma^m(i)}(N^m \xi) \overline{\psi}_{\sigma^m(j)}(N^m \xi) = \delta_{ij}.
\end{equation}

For all $i, j$ and for all $s \in \mathbb{Z} \setminus N\mathbb{Z}$:

\begin{equation}
\sum_{\psi \in \Psi} \sum_{m \geq 0} \hat{\psi}_{\sigma^m(i)}(N^m \xi) \overline{\psi}_{\sigma^m(j)}(N^m (\xi + N(\theta_{\sigma^{-1}(i)} - \theta_{\sigma^{-1}(j)}) + 2s\pi)) = 0.
\end{equation}

**Proof.** According to theorem \ref{thm:NTF}, the affine system $X(\Psi)$ is a normalized tight frame if and only if the quasi-affine system $X(\Psi)$ is. This can be reformulated as

$$\{\hat{\psi}_{m,0} \mid m \leq 0, \psi \in \Psi\} \cup \{\hat{\psi}_{m,r} \mid m > 0, r \in \{0, ..., N^m - 1\}, \psi \in \Psi\}$$

is a NTF generator for $L^2(\mathbb{R})^n$. We will use theorem \ref{thm:NTF} for this family.

We have to compute the Fourier transforms:

$$\hat{\psi}_{m,0}(\xi) = \left(\hat{\psi}_{\sigma^{-m}(i)}(N^{-m}\xi)\right)_{i \in \{1, ..., n\}}, \quad (m \leq 0),$$

and

$$\hat{\psi}_{m,r}(\xi) = N^{-m/2} \left(e^{-i(N^{-m}\xi + \theta_{\sigma^{-m}(i)})r}\hat{\psi}_{\sigma^{-m}(i)}(N^{-m}\xi)\right)_{i \in \{1, ..., n\}},$$

when $m > 0, r \in \{0, ..., N^m - 1\}$. Change $m$ into $-m$ and the equations of theorem \ref{thm:NTF} are

\begin{equation}
\sum_{\psi \in \Psi} \sum_{m \geq 0} \hat{\psi}_{\sigma^m(i)}(N^m(\xi - \theta_i)) \overline{\psi}_{\sigma^m(j)}(N^m(\xi - \theta_j + 2k\pi)) + \sum_{\psi \in \Psi} \sum_{m < 0} N^m \left(\sum_{r=0}^{N^m-1} e^{-ir(\theta_{\sigma^m(i)} + N^m(\xi - \theta_i) - (\theta_{\sigma^m(j)} + N^m(\xi - \theta_j + 2k\pi)))}\right) \hat{\psi}_{\sigma^m(i)}(N^m(\xi - \theta_j + 2k\pi)) \overline{\psi}_{\sigma^m(j)}(N^m(\xi - \theta_j + 2k\pi)) = \delta_{ij}\delta_k,
\end{equation}

$i, j \in \{1, ..., n\}, k \in \mathbb{Z}$ for almost all $\xi \in \mathbb{R}$.

The rest of the proof will be just the attempt to rewrite the equation \ref{eq:NTF} in the nicer form given in the statement of the theorem.

We evaluate the inner sum, for fixed $\xi, i, j, k$. Denote by

$$\alpha(m) := \theta_{\sigma^m(i)} - \theta_{\sigma^m(j)} - N^m(\theta_i - \theta_j + 2k\pi), \quad (m \leq 0).$$
\[ S_m := N^m \sum_{r=0}^{N^m-1} e^{-ir(\theta_{\sigma^m(i)}+N^m(\xi-\theta_i)-(\theta_{\sigma^m(j)}+N^m(\xi-\theta_j+2k\pi))} \]
\[
= \begin{cases} 
N^m \frac{1-e^{-i(N^m(\theta_{\sigma^m(i)}-\theta_{\sigma^m(j)})-(\theta_i-\theta_j))}}{1-e^{-i\alpha(m)}} & \text{if } e^{-i\alpha(m)} \neq 1 \\
1 & \text{if } e^{-i\alpha(m)} = 1.
\end{cases}
\]

But \( m \leq 0 \) so \( N^{-m}\theta_{\sigma^m(i)} \equiv \theta_i \mod 2\pi \) and similarly for \( j \) so that
\[
S_m = \begin{cases} 
0, & \text{if } \alpha(m) \not\equiv 0 \mod 2\pi \\
1, & \text{if } \alpha(m) \equiv 0 \mod 2\pi.
\end{cases}
\]

Also note that, if for some \( m < 0 \), \( \alpha(m) \in 2\pi\mathbb{Z} \) then \( \alpha(m+1) \in 2\pi\mathbb{Z} \).

Indeed,
\[ 2N\pi\mathbb{Z} \ni N\alpha(m) = N\theta_{\sigma^m(i)} - N\theta_{\sigma^m(j)} - N^{m+1}(\theta_i - \theta_j + 2k\pi) \]

But
\[ N\theta_{\sigma^m(i)} \equiv \theta_{\sigma^{m+1}(i)} \mod 2\pi \]
and similarly for \( j \) therefore we see that \( \alpha(m+1) \) is in \( 2\pi\mathbb{Z} \) also.

Thus, if for some \( m_0 \leq 0 \) we have \( \alpha(m_0) \in 2\pi\mathbb{Z} \) then all \( \alpha(m) \) are in \( 2\pi\mathbb{Z} \) for \( 0 \geq m \geq m_0 \), and so \( S_m = 1 \) for \( m \geq m_0 \).

Define
\[ m_0 := \min\{m \leq 0 | \alpha(m) \in 2\pi\mathbb{Z}\} \in \{-\infty, \ldots, -1, 0\}. \]

Rewrite (5.4):
\[
\sum_{\psi \in \Psi} \sum_{m \geq m_0} \hat{\psi}_{\sigma^m(i)}(N^m(\xi-\theta_i)) \overline{\hat{\psi}_{\sigma^m(j)}(N^m(\xi-\theta_j + 2k\pi))} = \delta_{ij} \delta_k.
\]

Of course, \( m_0 \) depends on \( i, j, k \) and we have to make this dependence more precise.

If \( z_i = z_j \) and \( k = 0 \) then \( \theta_{\sigma^m(i)} = \theta_{\sigma^m(j)} \) for all \( m \in \mathbb{Z} \) so \( \alpha(m) = 0 \) for all \( m \leq 0 \) and therefore \( m_0 = -\infty \). In this case the equation (5.4) is
\[
\sum_{m=-\infty}^{\infty} \hat{\psi}_{\sigma^m(i)}(N^m(\xi-\theta_i)) \overline{\hat{\psi}_{\sigma^m(j)}(N^m(\xi-\theta_j))} = \delta_{ij},
\]
which after the change of variable \( \xi - \theta_i \rightarrow \xi \) becomes (5.1).

If \( z_i \neq z_j \) or \( k \neq 0 \) then \( m_0 = 0 \) is finite. When \( z_i \neq z_j \) we have \( \theta_{\sigma^m(i)} \neq \theta_{\sigma^m(j)} \) for all \( m \), by cyclicity \( \theta_{\sigma^m(i)} - \theta_{\sigma^m(j)} \) will assume only a finite number of nonzero values in \((-2\pi, 2\pi)\) as \( m \) varies. For \( m \) close to \(-\infty\) the quantity \(-N^m(\theta_i - \theta_j + 2k\pi)\) is too small to cover the distance from \( \theta_{\sigma^m(i)} - \theta_{\sigma^m(j)} \) to any point in \( 2\pi\mathbb{Z} \) so that \( \alpha(m) \notin 2\pi\mathbb{Z} \).

We change the variable in (5.4), \( \xi' - \theta_{\sigma^m_0(i)} = N^{m_0}(\xi - \theta_i) \). Then
\[ N^{m_0}(\xi' - \theta_j + 2k\pi) = \xi' - \theta_{\sigma^{m_0}(j)} - \alpha(m_0), \]
so (5.4) is equivalent to
\[
\sum_{m \geq 0} \hat{\psi}_{\sigma^{m_0}(i)}(N^m(\xi - \theta_{\sigma^{m_0}(i)})) \overline{\hat{\psi}(N^m(\xi - \theta_{\sigma^{m_0}(j)} - \alpha(m_0))))} = 0,
\]

We prove that (5.5) is equivalent to:
For all \( i, j \) and \( s \in \mathbb{Z} \backslash N\mathbb{Z} \):
\[
\sum_{\psi \in \Psi} \sum_{m \geq 0} \hat{\psi}_{\sigma^m(i)}(N^m(\xi - \theta_i)) \overline{\hat{\psi}_{\sigma^m(j)}(N^m(\xi - \theta_j + A(i, j) + 2s\pi))} = 0,
\]
where $A(i, j) = N(\theta_{\sigma^{-1}(i)} - \theta_{\sigma^{-1}(j)}) - (\theta_i - \theta_j)$.

First, assume (5.6) holds. Fix $i, j \in \{1, \ldots, n\}$, $k \in \mathbb{Z}$ with $z_i \neq z_j$ or $k \neq 0$. We prove that $-\alpha(m_0)$ is of the form $A(\sigma^{m_0}(i), \sigma^{m_0}(j)) - 2s\pi$ with $s \in \mathbb{Z} \setminus NZ$.

We have

$$\alpha(m_0) = \theta_{\sigma^{m_0}(i)} - \theta_{\sigma^{m_0}(j)} - N^{m_0}(\theta_i - \theta_j + 2k\pi).$$

By the definition of $m_0$, $\alpha(m_0 - 1) \notin 2\pi\mathbb{Z}$. But observe that

$$N\alpha(m_0 - 1) = N(\theta_{\sigma^{-1}m_0(i)} - \theta_{\sigma^{-1}m_0(j)}) - N^{m_0}(\theta_i - \theta_j + 2k\pi)
= \alpha(m_0) + A(\sigma^{m_0}(i), \sigma^{m_0}(j)).$$

Also $A(\sigma^{m_0}(i), \sigma^{m_0}(j)) \in 2\pi\mathbb{Z}$ and $\alpha(m_0) \in 2\pi\mathbb{Z}$ so $N\alpha(m_0 - 1) \in 2\pi\mathbb{Z}$ which means that $\alpha(m_0 - 1) \notin \frac{2\pi}{N}\mathbb{Z}$.

We also know that $\alpha(m_0 - 1) \not\in 2\pi\mathbb{Z}$, hence $\alpha(m_0 - 1) = 2\pi s\overline{N}$ with $s \in \mathbb{Z} \setminus NZ$.

Then

$$-\alpha(m_0) = -2s\pi + A(\sigma^{m_0}(i), \sigma^{m_0}(j)).$$

Now using (5.6) for $\sigma^{m_0}(i), \sigma^{m_0}(j)$ and $-s$, we obtain (5.5).

For the converse, suppose (5.5) holds and fix $i, j$ and $s \in \mathbb{Z} \setminus NZ$. Let $k := 2s\pi + A(i, j)$ and we show that the $m_0$ associated to $i, j$ and $k$ is 0.

Indeed, we compute: $\alpha(-1) = -2s\overline{N}\pi \not\in 2\pi\mathbb{Z}$. Since $\alpha(0) = -2s\pi - A(i, j)$, using (5.5), the relation (5.6) is obtained.

Equation (5.6) is clearly equivalent to (5.2) after a change of variable. □

**Remark 5.5.** Note that there is some redundancy in the equations of theorem 5.4. Indeed, the relations (5.1) and (5.2) for $i > j$ follow from the corresponding relations for $i \leq j$ after a conjugation and a change of variable.

Also, if $i_1, \ldots, i_l$ form a cycle for $\sigma$ then, once we have equation (5.1) for $i = j = i_1$, the equation (5.1) will be also true for $i_2, \ldots, i_l$, because one needs only to change the variable $\xi \leftrightarrow \overline{Np}\xi$ for some appropriate $p$.

We can apply theorem 5.4 to the particular case of the amplification of the classic representation on $L^2(\mathbb{R})$. Our result complements the equations obtained for orthogonal super-wavelets in [HL], theorem 5.13.

**Corollary 5.6.** Let $\sigma$ be the identity permutation and $z_1 = \ldots = z_n = 1$. Then $\Psi$ is a NTF wavelet if and only if the following equations are satisfied for almost all $\xi \in \mathbb{R}$:

$$\sum_{\psi \in \Psi} \sum_{m=-\infty}^{\infty} \hat{\psi}_i(N^m\xi)\overline{\hat{\psi}_j(N^m\xi)} = \delta_{i,j}, \quad (i, j \in \{1, \ldots, n\});$$

$$\sum_{\psi \in \Psi} \sum_{m=0}^{\infty} \hat{\psi}_i(N^m\xi)\overline{\hat{\psi}_j(N^m(\xi + 2s\pi))} = 0, \quad (i, j \in \{1, \ldots, n\}, s \in \mathbb{Z} \setminus NZ).$$

In [HL], two normalized tight frames $\{e_i\}_{i \in I}$ for $H$ and $\{f_i\}_{i \in I}$ for $K$ are called strongly disjoint if their direct sum $\{e_i \oplus f_i\}_{i \in I}$ is a normalized tight frame for $H \oplus K$. Two normalized tight frame wavelets $\Psi = \{\psi_1, \ldots, \psi_L\}$, $\Psi' = \{\psi'_1, \ldots, \psi'_L\}$ for $L^2(\mathbb{R})$ (note that the cardinality is the same) are strongly disjoint if their affine systems $X(\psi)$ and $X(\Psi')$ are strongly disjoint.

If we look at corollary 5.6 we see that the fact that each component $\{\psi_i | \psi \in \Psi\}$, $(i \in \{1, \ldots, n\})$ is a NTF wavelet for $L^2(\mathbb{R})$ is equivalent to the equations with $i = j$ (and we reobtain in fact the well known characterization of wavelets in $L^2(\mathbb{R})$, see [Bo2], [Ca] or [HW]). Therefore the disjointness part is covered by the equations
with \(i \neq j\), hence we can use the corollary \(5.6\) to produce a characterization of disjointness:

**Corollary 5.7.** Two NTF wavelets \(\Psi := \{\psi_1, ..., \psi_L\}\) for \(L^2(\mathbb{R})\) are strongly disjoint if and only if

\[
\sum_{l=1}^{L} \sum_{m=-\infty}^{\infty} \hat{\psi}^l(N^m \xi) \bar{\psi}^l(N^m \xi) = 0,
\]

\[
\sum_{l=1}^{L} \sum_{m=0}^{\infty} \hat{\psi}^l(N^m \xi) \bar{\psi}^l(N^m (\xi + 2s\pi)) = 0, \quad (s \in \mathbb{Z} \setminus N\mathbb{Z}).
\]

In the remainder of this section we present another application of the characterization theorem. The following oversampling result is proved in [CS] for the scaling factor \(N = 2\): if one has a NTF wavelet \(\psi\) (for \(L^2(\mathbb{R})\)), then also \(\eta := \frac{1}{p} \psi \left( \frac{\cdot}{p} \right)\) is a NTF wavelet, where \(p\) is any odd number.

We will refine this result here: not only that \(\eta\) is a NTF wavelet, but it is part of a super-wavelet \((\eta, ..., \eta)\) for \(L^2(\mathbb{R})^p\), and the starting wavelet is an orthogonal one if and only if the super-wavelet is also orthogonal. Here is the precise formulation of this result:

Let \(p\) be a positive integer which is prime with \(N\). For \(\psi \in L^2(\mathbb{R})\), define

\[
\eta(\psi)(x) = \frac{1}{p} \psi \left( \frac{x}{p} \right), \quad (x \in \mathbb{R}),
\]

\[
\hat{\eta}(\psi) = (\eta(\psi), ..., \eta(\psi)) \in L^2(\mathbb{R})^p, \quad (\psi \in \Psi).
\]

Let \(\rho := e^{-2\pi i/p}, z_k = \rho^k, k \in \{1, ..., p\}\) and \(\sigma\) the permutation of \(\{1, ..., p\}\) defined by \((\rho^i)^N = \rho^{\sigma(i)}\) for all \(i \in \{1, ..., p\}\), and let \(\mathcal{A}_{\sigma, Z}\) be the corresponding affine structure on \(L^2(\mathbb{R})^p\).

**Theorem 5.8.** Let \(\Psi := \{\psi_1, ..., \psi_p\}\) in \(L^2(\mathbb{R})\). \(\Psi\) is a NTF wavelet for \(L^2(\mathbb{R})\) if and only if

\[
\hat{\eta}(\Psi) := \{\hat{\eta}(\psi) \mid \psi \in \Psi\}
\]

is a NTF wavelet on \(L^2(\mathbb{R})^p\) for the affine structure \(\mathcal{A}_{\sigma, Z}\).

Moreover, \(\Psi\) is an orthogonal wavelet for \(L^2(\mathbb{R})\) if and only if \(\eta(\Psi)\) is an orthogonal wavelet for \(L^2(\mathbb{R})^p\).

**Proof.** We have to check that the equations of theorem \(5.4\) are satisfied:

The Fourier transform of \(\eta(\psi)\) is

\[
\hat{\eta}(\psi)(\xi) = \hat{\psi}(p\xi), \quad (\xi \in \mathbb{R}).
\]

Then, we can rewrite the equations \(5.1\) and \(5.2\) for \(\hat{\eta}(\psi)\):

\[
\sum_{\psi \in \Psi} \sum_{m=-\infty}^{\infty} |\hat{\psi}(N^m p\xi)|^2 = 1,
\]

\[
\sum_{\psi \in \Psi} \sum_{m \geq 0} \hat{\psi}(N^m p\xi) \bar{\psi}(N^m (p\xi + a)) = 0,
\]

for all \(a \in \{p(N\theta_{\sigma^{-1}(i)} - N\theta_{\sigma^{-1}(j)} + 2s\pi) \mid i, j \in \{1, ..., p\}, s \in \mathbb{Z} \setminus N\mathbb{Z}\} =: A\). We want to see what this set is and we will show that it is equal to \(\{2q\pi \mid q \in \mathbb{Z} \setminus N\mathbb{Z}\}\).
Note that \( \theta_i \) takes all the values of the form \( 2k\pi/p \) inside \([−\pi, \pi)\) with \( k \) integer. Then \( p\theta_i \) takes all values of the form \( 2k\pi \) inside \([−p\pi, p\pi)\) and therefore \( p\theta_{\sigma^{-1}(i)} − p\theta_{\sigma^{-1}(j)} \) will cover all values of the form \( 2k\pi \) within \((-2p\pi, 2p\pi)\). Therefore

\[
A = \{2\pi(Nl + ps) \mid l \in \{-p + 1 − p + 2, ..., −1, 0, 1, ..., p − 1\}, s \in \mathbb{Z} \setminus N\mathbb{Z}\}.
\]

Since \( p \) is prime with \( N \) it is clear that all numbers in \( A \) are of the form \( 2\pi q \) with \( q \in \mathbb{Z} \setminus N\mathbb{Z} \).

For the converse, take \( q \) not divisible by \( N \). Since \( p \) and \( N \) are prime we can write \( q = Nl + ps \) with \( l_1, s_1 \) integers. Clearly \( s_1 \) cannot be divisible by \( N \). Also we can write \( l_1 = pr_1 + l \) with \( r_1 \in \mathbb{Z} \) and \( l \in \{0, ..., p − 1\} \). So

\[
q = Nl + p(Nr_1 + s_1)
\]

with \( l \in \{0, ..., p − 1\} \), \( Nr_1 + s_1 \) not divisible by \( N \) and therefore \( 2\pi q \) is in \( A \).

Finally, if we change the variable \( p\xi \leftrightarrow \xi \), we see that the equations are equivalent to the fact that \( \Psi \) is a NTF wavelet for \( L^2(\mathbb{R}) \).

The orthogonal case follows by an inspection of the norms: a normalized tight frame is an orthonormal basis if and only if all the norms of the vectors are 1. But observe that

\[
\|\tilde{\eta}(\psi)\|^2 = p\|\eta(\psi)\|^2 = \|\psi\|^2,
\]

and therefore the last equivalence is clear. \( \square \)

6. The spectral function and the dimension function

**Definition 6.1.** A finite subset \( \Psi = \{\psi^1, ..., \psi^L\} \) of \( L^2(\mathbb{R})^n \) is called a semi-orthogonal wavelet if the affine system

\[
\{U^jT^k\psi \mid j \in \mathbb{Z}, k \in \mathbb{Z}, \psi \in \Psi\}
\]

is a NTF for \( L^2(\mathbb{R}^n) \) and \( W_i \perp W_j \) for \( i \neq j \), where

\[
W_j = V_{j+1} \ominus V_j = \text{span}\{U^jT^k\psi \mid k \in \mathbb{Z}, \psi \in \Psi\} = U^{-j}(S(\Psi)), \quad (j \in \mathbb{Z}).
\]

and \( (V_j)_1 \) is the GMRA associated to \( \Psi \).

If \( \Phi \) is a NTF generator for \( V_0 \) then \( \Phi \) is called a set of scaling functions for \( \Psi \).

For a semi-orthogonal wavelet, theorem 5.2 and the orthogonality conditions given in the definition imply the following:

**Proposition 6.2.** Let \( \Psi \) be a semi-orthogonal wavelet and \( (V_j)_j \) its GMRA. Then the vectors

\[
\{\tilde{\psi}_{m,0} = N^{-m/2}U^m\psi \mid m > 0, \psi \in \Psi\}
\]

form a NTF generator for \( V_0 \).

This proposition will be one of the essential ingredients for our results. In \( V_0 \) we have two NTF generators: one given by the wavelet \( \Psi \) as in proposition 6.2 and another given by a set of scaling functions for \( V_0 \) which always exists according to theorem 3.7. Computing the local trace for various operators in two ways will give rise to some fundamental equalities about wavelets.

The spectral function was introduced for the classical case on \( L^2(\mathbb{R}) \) by M. Bownik and Z. Rzeszotnik in [BoR2]. It contains a lot of information about the shift invariant subspace and it has several nice features. In some sense, the spectral function corresponds to the diagonal entries of the dual Gramian.
Definition 6.3. Let $V$ be a shift invariant subspace of $L^2(\mathbb{R})^n$. For $i \in \{1, \ldots, n\}$, the $i$-th spectral function of $V$ is
\[
\sigma_{V,i}(\xi) := \tau_{V,P_0,i}(\xi), \quad (\xi \in \mathbb{R}),
\]
where $P_{0,i}$ is the projection onto the 0-th component of the $i$-th vector
\[
(P_{0,i}v)(k,j) = v_{0,i}\delta_{0k}\delta_{ij}, \quad \text{for all } j \in \{1, \ldots, n\}, k \in \mathbb{Z}.
\]

The next proposition is just a consequence of theorem 4.4.

Proposition 6.4. Let $V$ be a shift invariant subspace and $\Phi$ be a NTF generator for $V$. Then
\[
\sigma_{V,i}(\xi) = \sum_{\varphi \in \Phi} |\hat{\varphi}_i|^2(\xi - \theta_i), \quad (\xi \in \mathbb{R}, i \in \{1, \ldots, n\}).
\]

Using the NTF generator given in proposition 6.2 we obtain a formula for the spectral functions associated to wavelets:

Proposition 6.5. Let $\Psi$ be a semi-orthogonal wavelet and $(V_j)_j$ its GMRA. Then the spectral functions of the core space $V_0$ can be computed by:
\[
\sigma_{V_0,i}(\xi) = \sum_{\psi \in \Psi} \sum_{m \geq 1} |\hat{\psi}_{\sigma_m(i)}|^2(N^m(\xi - \theta_i)), \quad (\xi \in \mathbb{R}, i \in \{1, \ldots, n\}).
\]

Combining proposition 6.4 and 6.5 we obtain a Gripenberg-Weiss-type formula:

Corollary 6.6. If $\Psi$ is a semi-orthogonal wavelet and $\Phi$ is a set of scaling function for it, then
\[
\sum_{\varphi \in \Phi} |\hat{\varphi}_i|^2(\xi - \theta_i) = \sum_{\psi \in \Psi} \sum_{m \geq 1} |\hat{\psi}_{\sigma_m(i)}|^2(N^m(\xi - \theta_i)), \quad (\xi \in \mathbb{R}, i \in \{1, \ldots, n\}).
\]

The dimension function was extensively used on $L^2(\mathbb{R})$ to decide whether a wavelet is a MRA wavelet or not, see [HW]. It also gives information about the multiplicity of the wavelets ([B], [BM], [BMM], [Web]).

Definition 6.7. Let $V$ be a shift invariant subspace of $L^2(\mathbb{R})^n$ and $J_{\text{per}}$ its range function. The dimension function of $V$ is
\[
\dim_V(\xi) = \dim(J_{\text{per}}(\xi)), \quad (\xi \in \mathbb{R}).
\]

Since the trace of a projection is the dimension of its range, using the definition of the local trace we have the following equality:

Proposition 6.8. Let $V$ be a shift invariant subspace of $L^2(\mathbb{R})^n$. Then
\[
\dim_V(\xi) = \tau_{V,I}(\xi),
\]
where $I$ is the identity operator on $l^2(\mathbb{Z})^n$.

Then, with theorem 4.4 and proposition 6.4 we have

Proposition 6.9. If $\Phi$ is a NTF generator for a shift invariant subspace $V$ then
\[
\dim_V(\xi) = \sum_{\varphi \in \Phi} \sum_{i=1}^n \sum_{k \in \mathbb{Z}} |\hat{\varphi}_i|^2(\xi - \theta_i + 2k\pi) = \sum_{i=1}^n \text{Per}(\sigma_{V,i})(\xi), \quad (\xi \in \mathbb{R}).
\]

where
\[
\text{Per}(f)(\xi) = \sum_{k \in \mathbb{Z}} f(\xi + 2k\pi), \quad (\xi \in \mathbb{R}, f \in L^1(\mathbb{R})).
\]
We can choose the scaling functions to be as in theorem 4.10 and the consequence is that the dimension function and the multiplicity function are the same (a generalization of the result of E. Weber [Web]).

**Theorem 6.10.** If \( \Psi \) is a semi-orthogonal wavelet and \( V_0 \) the core space of the associated GMRA. Then the multiplicity function and the dimension function of \( V_0 \) are equal and they are given by

\[
m_{V_0}(\xi) = \dim m_{V_0}(\xi) = \sum_{m \geq 1} \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}} |\hat{\psi}_{m(i)}|^2 (N^m (\xi - \theta_i + 2k\pi)) =: D_\Psi(\xi),
\]

for almost all \( \xi \in [-\pi, \pi) \).

Next we want to establish some dilation formulas for the spectral function and for the dimension function. We use theorem 4.14.

**Proposition 6.11.** Let \( V \) be a SI subspace of \( L^2(\mathbb{R})^n \). Then for all \( j \in \{1, ..., n\} \),

\[
\sigma_{U^{-1}V,j}(\xi) = \sigma_{V,\sigma^{-1}(j)} \left( \frac{\xi - \theta_j + N\theta_{\sigma^{-1}(j)}}{N} \right), \quad (\xi \in \mathbb{R}),
\]

\[
\dim_{U^{-1}V}(\xi) = \sum_{l=0}^{N-1} \dim_{V} \left( \frac{\xi + 2l\pi}{N} \right), \quad (\xi \in \mathbb{R}).
\]

**Proof.** With theorem 4.14, we have to compute for \( l \in \{0, ..., N-1\} \), \( S^* D_l^* P_{0,j} D_l S \) is the rank-one operator given by the vector \( S^* D_l^* e_{0,j} \). We have for \( k \in \mathbb{Z}, i \in \{0, ..., N-1\} \):

\[
S^* D_l^* e_{0,j}(k, i) = (D_l^* e_{0,j})(k, \sigma(i)) = e_{0,j}(2l\pi + \theta_{\sigma(i)} - N\theta_i + 2kN\pi),
\]

and this shows that

\[
S^* D_l^* e_{0,j} = \begin{cases} 
  e_{k_j, \sigma^{-1}(j)} & \text{if } l = l_j \\
  0, & \text{otherwise,}
\end{cases}
\]

where \( l_j \in \{0, ..., N-1\} \) and \( k_j \in \mathbb{Z} \) are the only such numbers that verify the equation \( 2l_j \pi + \theta_j - N\theta_{\sigma^{-1}(j)} + 2k_j N\pi = 0 \). Therefore

\[
S^* D_l^* P_{0,j} D_l S = \begin{cases}
  P_{k_j, \sigma^{-1}(j)} = \lambda(k_j) P_{0,\sigma^{-1}(j)} \lambda(k_j)^* & \text{if } l = l_j, \\
  0, & \text{otherwise.}
\end{cases}
\]

Using theorem 4.14 and then the periodicity property given in proposition 4.10, we obtain:

\[
\sigma_{U^{-1}V,j}(\xi) = \tau_{V, P_{k_j, \sigma^{-1}(j)}} \left( \frac{\xi + 2l_j \pi}{N} \right) = \sigma_{V_0, \sigma^{-1}(j)} \left( \frac{\xi + 2l_j \pi}{N} + 2k_j \pi \right),
\]

and, with the definition of \( k_j \) and \( l_j \) the equation follows.

The equation for the dimension functions follows from the fact that \( S^* D_l^* I D_l S = I \) for all \( l \in \{0, ..., N-1\} \).

**Proposition 6.12.** Let \( (V_j) \) be a GMRA and \( \Phi \) a NTF generator for \( V_0 \). Then, for almost every \( \xi \in \mathbb{R} \) and all \( i \in \{1, ..., n\} \),

\[
\lim_{m \to \infty} \sum_{\psi \in \Phi} |\hat{\varphi}_i|^2 \left( \frac{\xi}{N^m} \right) = 1.
\]
Proof. With the monotone convergence theorem 4.13 we have that $\sigma_{U-mV,i}(\xi)$ converges pointwise a.e. to $\sigma_{L^2(\mathbb{R}^n),i}(\xi) = 1$. With proposition 6.11 by induction we get:

$$\sigma_{U-mV,i}(\xi) = \sigma_{V_0,\sigma^{-m}(i)} \left( \frac{\xi - \theta_i + N^m \theta_{\sigma^{-m}(i)}}{N^m} \right)$$

$$= \sum_{\varphi \in \Phi} |\hat{\varphi}_{\sigma^{-m}(i)}|^2 \left( \frac{\xi - \theta_i}{N^m} \right).$$

Changing $\xi - \theta_i$ to $\xi$ we obtain

$$\lim_{m \to \infty} \sum_{\varphi \in \Phi} |\hat{\varphi}_{\sigma^{-m}(i)}|^2 \left( \frac{\xi}{N^m} \right) = 1.$$ 

But $\sigma$ is a finite permutation so $\sigma^p = \text{id}$ for some $p$. Thus, for the subsequence $mp$, $\sigma^{-mp}(i) = i$ for all $m$, hence

$$\lim_{m \to \infty} \sum_{\varphi \in \Phi} |\hat{\varphi}_i|^2 \left( \frac{\xi}{N^{mp}} \right) = 1,$$

and if we apply this to $\xi, \xi/N, ..., \xi/N^{p-1}$ we obtain the desired limit. \hfill \Box

The limit given in proposition 6.12 enables us to give a lower bound for the dimension function of a wavelet. We can conclude that, if there are two distinct indices $i, j$ such that $z_i = z_j$ then the dimension function of any wavelet is strictly bigger then 1 at some points so there are no MRA wavelets, and for any wavelet one needs at least 2 scaling functions. If there are three indices for which the points $z_i$ are the same, then any wavelet needs at least three scaling functions, and so on. This result generalizes and refines proposition 5.16 in [HL].

Corollary 6.13. If $\Psi$ is a semi-orthogonal wavelet then, for all $\alpha_0 \in [-\pi, \pi)$,

$$\limsup_{\xi \to \alpha_0} D_\Psi(\xi) \geq \text{card}\{i \in \{1, ..., n\} | \theta_i = \alpha_0\}.$$ 

In particular, if there are two distinct indices $i \neq j$ with $\theta_i = \theta_j$ then there are no MRA wavelets.

Proof. Let $\Phi$ be a set of scaling functions associated to $\Psi$ and compute $D_\Psi$ at $\xi/N^m + \alpha_0$ using proposition 6.12. Then take the limsup and use proposition 6.12.

For a MRA wavelet $D_\Psi$ is constant 1, and the second statement follows by contradiction. \hfill \Box

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