THE CONE OF EFFECTIVE ONE–CYCLES OF CERTAIN G–VARIETIES

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Abstract. Let $X$ be a normal projective variety admitting an action of a semisimple group with a unique closed orbit. We construct finitely many rational curves in $X$, all having a common point, such that every effective one–cycle on $X$ is rationally equivalent to a unique linear combination of these curves with non–negative rational coefficients. When $X$ is nonsingular, these curves are projective lines, and they generate the integral Chow group of one–cycles.

To C. S. Seshadri for his 70th birthday

Introduction

One associates to any complete algebraic variety $X$ the group $N_1(X)$ of one–cycles on $X$ modulo numerical equivalence; this is a free abelian group of finite rank [6, Example 19.1.4]. In the corresponding real vector space $N_1(X)_\mathbb{R}$, the convex cone generated by the classes of (closed irreducible) curves in $X$ is the cone of effective one–cycles, denoted by $NE(X)$.

Dually, one has the group $N^1(X)$ of Cartier divisors on $X$ modulo numerical equivalence, endowed with a non–degenerate pairing $N^1(X) \times N_1(X) \to \mathbb{Z}$ via intersection numbers of Cartier divisors with curves. Again, $N^1(X)$ is a free abelian group of finite rank; in the corresponding real vector space $N^1(X)_\mathbb{R}$, the dual cone of $NE(X)$ is the cone of numerically effective divisors (called nef for brevity). The interior of the nef cone is the ample cone, by Seshadri’s criterion [4, Theorem I.7.1].

The cone $NE(X)$ encodes much information about morphisms with source $X$. Specifically, any surjective separable morphism $f : X \to Y$ with connected fibers, where $Y$ is a normal projective variety, is uniquely determined by the subcone $NE(f) \subseteq NE(X)$ generated by the classes of curves lying in fibers of $f$. Moreover, $NE(f)$ is a face of the convex cone $NE(X)$.

However, the structure of $NE(X)$ may be quite complicated: this cone may not be closed, its closure may not be polyhedral, and certain faces may not arise from morphisms with source $X$, due to the existence of nef divisors having no globally generated positive multiple. Only the part of $NE(X)$ where the canonical divisor is
negative is well–understood, if \( X \) is a nonsingular projective variety in characteristic zero (see e.g. [3]).

In this note, we consider certain varieties where the structure of the whole cone of effective one–cycles turns out to be very simple. These are the normal projective varieties where a semisimple group acts with a unique closed orbit. For such a variety \( X \), we show that the cone \( NE(X) \) is generated by the closures of positive strata of dimension one, for an appropriate Bialynicki–Birula decomposition of \( X \) (recalled in Section 1). These closures are rational curves passing through a common point, the sink of the decomposition; their classes form a basis of the rational vector space \( N_1(X)_{\mathbb{Q}} \), and the latter is isomorphic to the rational Chow group of one–cycles. If, in addition, \( X \) is nonsingular, then so are our rational curves, and their classes form a basis of the group \( N_1(X) \) (Theorem 2 and Corollary 2).

Moreover, the cone of nef divisors is generated by the closures of negative strata of codimension one. These are globally generated Cartier divisors, and their classes in \( N^1(X)_{\mathbb{Q}} \) form the dual basis to our basis of \( N_1(X)_{\mathbb{Q}} \); finally, \( N^1(X) \) is isomorphic to the Picard group. As a consequence, every nef divisor on \( X \) is globally generated (Theorem 1).

The simplest examples of projective varieties where a semisimple group acts with a unique closed orbit are of course flag varieties. For these, our results are well–known: the cone of effective one–cycles is freely generated by the classes of Schubert varieties of dimension one, while the classes of opposite Schubert varieties of codimension one form the dual basis of the nef cone. Moreover, numerical and rational equivalence coincide.

These results were generalized in [3] and [4] to all projective simple spherical varieties (i.e., normal projective varieties where a connected reductive group acts with a unique closed orbit, and where a Borel subgroup acts with a dense orbit), in characteristic zero. There the main tool was the classification of embeddings of spherical homogeneous spaces. In the present note, the use of the Bialynicki–Birula decomposition (for possibly singular varieties) yields more general results.

Another interesting class of examples consists of orbit closures of a finite set of ordered points of the projective line, under the diagonal action of PGL(2). Clearly, these orbit closures are projective varieties where PGL(2) acts with a unique closed orbit; and they turn out to be normal, as shown by Iozzi and Poritz (in characteristic zero; see [10]). In the final section of this note, we show that their cone of nef divisors is generated by the pull–backs of points under the various projections to the projective line, while the cone of effective curves is generated by the projective lines mapped to points under all but one projection (Proposition 1). We also obtain another proof of the normality of these orbit closures, and we show that they yield all simple complete embeddings of PGL(2), in arbitrary characteristics.
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Note finally that the duality between closures of positive strata of dimension one
and of negative strata of codimension one, is very seldom satisfied by (say) nonsingular
projective varieties where a torus acts with only finitely many fixed points. One
checks, for example, that this duality fails for all toric surfaces having at least 5 fixed
points.

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1. THE CONE OF NUMERICALLY EFFECTIVE DIVISORS

We begin by fixing notation and recalling some general results on group actions,
to be used throughout this work.

All algebraic varieties and groups are defined over an algebraically closed field $K$
of arbitrary characteristic.

Let $G$ be a connected reductive algebraic group. Let $B, B^\sim \subseteq G$ be opposite Borel
subgroups with common torus $T$ and unipotent radicals $U = R_u(B), U^\sim = R_u(B^\sim)$;
then $B = TU$ and $B^\sim = TU^\sim$. Let $X^*(T)$ be the character group of $T$ (the weight
lattice of $G$), and $X_*(T)$ the group of one–parameter subgroups of $T$ (the dual lattice
of $X^*(T)$).

A $G$–module is a the space of a rational, finite–dimensional $G$–representation. Im-
portant examples of $G$–modules are the induced modules $H^0(\omega)$, where $\omega \in X^*(T)$:
these consist of all regular functions $f$ on $G$ such that $f(gtu) = \omega(t^{-1})f(g)$ for all
$g \in G$, $t \in T$ and $u \in U$. The $G$–action is defined by $(g \cdot f)(h) = f(g^{-1}h)$. Recall
that $H^0(\omega)$ is non–zero if and only if $\omega$ is dominant; then the set of $U$–fixed points
in $H^0(\omega)$ is a line where $T$ acts with weight $\omega$ (see [11, II.2] for these results, and for
more on induced modules).

A variety is a separated integral scheme of finite type over $K$. For a connected linear
algebraic group $H$, a $H$–variety is a variety $X$ endowed with an algebraic action
of $H$. The fixed point subset of $H$ in $X$ is denoted by $X^H$ and regarded as a reduced
closed subscheme of $X$. An example of a $H$–variety is the projectivization $\mathbb{P}(M)$ of
a $H$–module $M$; an $H$–variety $X$ is $H$–quasi–projective if it admits an equivariant
embedding into some $\mathbb{P}(M)$. Equivalently, $X$ admits an ample $H$–linearizable invert-
ible sheaf. Any normal $H$–variety $X$ admits a covering by $H$–quasi–projective open
subsets; as a consequence, $X$ is $H$–quasi–projective if it is quasi–projective, or if it
contains a unique closed $H$–orbit [12, Theorem 1].

Let $L$ be an invertible sheaf on a normal $H$–variety. If the Picard group of $H$
is trivial, then $L$ admits a $H$–linearization; in the general case, there exists a finite
covering $\tilde{H} \to H$ of algebraic groups such that $\tilde{H}$ has a trivial Picard group. Thus,
$L$ is $\tilde{H}$–linearizable, and some positive power $L^n$ is $H$–linearizable (for all this, see
[13, I.3] and also [12, 13]).
Next we obtain a criterion for a $G$–projective variety to contain a unique closed orbit.

Lemma 1. The following conditions are equivalent for a $G$–projective variety $X$:

1. $X$ contains a unique closed $G$–orbit.
2. $X^U$ consists of a unique point.
3. There exists a finite equivariant morphism $f : X \to \mathbb{P}(H^0(\omega))$, for some dominant weight $\omega$.

Then $X$ is fixed pointwise by the connected centre $Z(G)^0$, so that $G$ acts on $X$ via its semisimple quotient $G/Z(G)^0$.

Proof. (1)⇒(2) By assumption, $X^B$ consists of a unique point $x$. In other words, $T$ acts on $X^U$ with $x$ as its unique fixed point. Since $X^U$ is a $T$–projective variety, it follows that $X^U = \{x\}$.

(2)⇒(1) follows from Borel’s fixed point theorem.

(1)⇒(3) Let $L$ be a $G$–linearized, very ample invertible sheaf on $X$; let $Y \subseteq X$ be the closed $G$–orbit. Then the $G$–module $H^0(Y, L)$ contains a $B$–eigenvector $\sigma$. By [14, Korollar 2.3], there exist a positive integer $n$ such that $\sigma^n \in H^0(Y, L^n)$ lifts to a $B$–eigenvector $\tau \in H^0(X, L^n)$. Let $N \subseteq H^0(X, L^n)$ be the $G$–submodule generated by $\tau$. Then the set of common zeroes in $X$ of all sections in $N$ is closed, $G$–stable, and does not contain $Y$. Hence this set is empty, since $Y$ is the unique closed orbit.

So the canonical rational map $X \dashrightarrow \mathbb{P}(N^*)$ is in fact a finite equivariant morphism. Moreover, the $G$–module $N$ being generated by a $B$–eigenvector, it is a quotient of a universal highest weight module; thus, its dual $N^*$ is a submodule of some $H^0(\omega)$, by [11, II.2.13].

(3)⇒(2) Since $U$ is unipotent connected and $H^0(\omega)^U$ is a line, $\mathbb{P}(H^0(\omega))^U$ consists of a unique point; thus, the set $X^U$ is finite. On the other hand, $X^U$ is connected by [11, II.2.8], hence it consists of a unique point.

Finally, the centre $Z(G)$ acts on $H^0(\omega)$ by scalars [11, II.2.8], and hence fixes $\mathbb{P}(H^0(\omega))$ pointwise. Since $f$ is finite, it follows that $Z(G)^0$ fixes $X$ pointwise. □

Definition. A $G$–projective variety satisfying one of the conditions of Lemma 1 is called simple.

We will construct generators of the nef cone of a simple $G$–projective variety, by using the Bialynicki–Birula decomposition that we now review, after [4] (in the nonsingular case) and [14] (in the general case).

Let $X$ be a $T$–projective variety and let $\lambda \in \mathcal{X}_*(T)$. The multiplicative group $\mathbb{G}_m$ acts on $X$ via $\lambda : \mathbb{G}_m \to T$, and we denote by $X^\lambda$ the corresponding fixed point set. Given $x \in X$, the map $\mathbb{G}_m \to X$, $t \mapsto \lambda(t) \cdot x$ extends uniquely to a morphism $\mathbb{P}^1 \to X$; this defines $\lim_{t \to 0} \lambda(t) \cdot x$ and $\lim_{t \to \infty} \lambda(t) \cdot x$. Both are $\lambda$–fixed points, distinct unless $x \in X^\lambda$. 


Recall also that $X^\lambda$ equals $X^T$ for any $\lambda \in \mathcal{X}_*(T)$ outside a finite union of proper subgroups; such a $\lambda$ is called regular.

For any $\lambda \in \mathcal{X}_*(T)$ and any closed subset $Y \subseteq X^\lambda$, let

$$X^+(Y) = \{ x \in X \mid \lim_{t \to 0} \lambda(t) \cdot x \in Y \}.$$  

Then $X^+(Y)$ is a locally closed $T$–invariant subset of $X$, and the map

$$p^+: X^+(Y) \to Y, \ x \mapsto \lim_{t \to 0} \lambda(t) \cdot x$$  

is a surjective affine $T$–invariant morphism (these facts are easily checked in the ambient projectivization of a $T$-module). Moreover, $X$ is the disjoint union of the subsets $X^+(Y)$, where $Y$ runs over all connected components of $X^T$. These subsets are called the positive strata.

As a consequence, there exists a unique open positive stratum $X^+(Y)$. Then $Y = Y^+$ is irreducible; it is called the source of $X$ for the $\lambda$–action.

Likewise, $X$ is the disjoint union of the

$$X^-(Y) = \{ x \in X \mid \lim_{t \to \infty} \lambda(t) \cdot x \in Y \},$$  

where $Y$ runs over the connected components of $X^\lambda$. The unique component $Y = Y^-$ such that $X^-(Y)$ is open in $X$ is called the sink.

Note that $X^+(Y) \cap X^-(Y) = Y$ for any $Y \subseteq X^\lambda$, so that $X^-(x) = \{ x \}$ for all $x \in Y^+$. Moreover, by [16], one has for any $x \in X^\lambda$:

$$(*) \quad \dim X - \dim_x X^\lambda \leq \dim_x X^+(x) + \dim_x X^-(x).$$  

As a consequence, the source of $X$ consists of those $x \in X^\lambda$ such that $X^-(x) = \{ x \}$ (indeed, if $X^-(x) = \{ x \}$, then $X^-(x') = (p^-)^{-1}(x')$ equals $\{ x' \}$ for all $x'$ in a neighborhood of $x$ in $X^\lambda$, since $\{ x' \}$ is the unique closed $\mathbb{G}_m$–orbit in its fiber under $p^-$. Now $(*)$ implies that $\dim X = \dim X^+(Y)$ for any irreducible component $Y \subseteq X^\lambda$ containing $x$.)

If, in addition, $X$ is nonsingular, then so are $X^\lambda$ and $X^+(Y), X^-(Y)$ for all components $Y$ of $X^\lambda$. Moreover, $X^+(Y)$ and $X^-(Y)$ intersect transversally along $Y$, and the morphisms $p^+, p^-$ are affine bundles, by [1].

Next we recall the definition of certain subgroups of $G$ associated with a given $\lambda \in \mathcal{X}_*(T)$; see [13, II.2.6] for details. Let

$$G(\lambda) = \{ g \in G \mid \lambda(t) g \lambda(t^{-1}) \text{ has a limit in } G \text{ as } t \to 0 \},$$  

so that

$$G(-\lambda) = \{ g \in G \mid \lambda(t) g \lambda(t^{-1}) \text{ has a limit in } G \text{ as } t \to \infty \}.$$  

Then $G(\lambda)$ and $G(-\lambda)$ are opposite parabolic subgroups of $G$, with common Levi subgroup the centralizer of $\lambda$. Moreover, the unipotent radical of $G(\lambda)$ is

$$R_u G(\lambda) = \{ g \in G \mid \lim_{t \to 0} \lambda(t) g \lambda(t^{-1}) = 1 \}.$$
Finally, $G(\lambda)$ equals $B$ if and only if $\lambda$ lies in the interior of the Weyl chamber associated with $B$; then $G(-\lambda) = B^-$.

Consider now a $G$–projective variety $X$. Then we may choose a regular $\lambda \in X_*(T)$ such that $G(\lambda) = B$. Hence $X^+(Y)$ is $B$–invariant for any closed subset $Y \subseteq X^\lambda = X^T$ (since $\lambda(t)g \cdot x = \lambda(t)g(\lambda(t^{-1})\lambda(t) \cdot x)$). Likewise, $X^-(Y)$ is $B^-$–invariant. This implies at once that the sink $Y^-$ is fixed pointwise by $B$. In particular, we obtain

**Lemma 2.** Let $X$ be a simple $G$–projective variety. Then the sink consists of the unique $B$–fixed point $x^-$, and the source consists of the unique $B^-$–fixed point $x^+$.

Put for simplicity $X^- = X^-(x^-)$. Then $X^-$ is an open affine $B^-$–invariant neighborhood of $x^-$; one easily checks that such a neighborhood is unique. Moreover, $\mathbb{G}_m$ acts on the algebra of regular functions $R = H^0(X^-, \mathcal{O}_{X^-})$ via $(t \cdot f)(x) = f(\lambda(t^{-1})x)$, and this yields a positive grading of $R$. By Nakayama’s Lemma, any graded invertible $R$–module is generated by a homogeneous element, unique up to scalar. This implies

**Lemma 3.** With the preceding notation, the group of isomorphism classes of $T$–linearized invertible sheaves on $X^-$ is isomorphic to $X^*(-T)$, via pull–back to the fixed point $x^-$.  

We will also need the following variant of a result of Knop [15, Lemma 2.2].

**Lemma 4.** Let $H$ be a connected linear algebraic group, $X$ a normal $H$–variety, and $D$ an effective Weil divisor on $X$ whose support contains no $H$–orbit. Then $D$ is a globally generated Cartier divisor.

**Proof.** Let $i : X^{\text{reg}} \to X$ be the inclusion of the nonsingular locus. Then the sheaf $i^*\mathcal{O}_X(D)$ is invertible. Replacing $H$ by a finite cover, we may assume that this sheaf is $H$–linearized. Since $\mathcal{O}_X(D) \cong i_*i^*\mathcal{O}_X(D)$ (by normality of $X$), it follows that $\mathcal{O}_X(D)$ is $H$–linearized as well.

Let $Y \subseteq X$ be the subset of all points where $\mathcal{O}_X(D)$ is not invertible. Then $Y$ is closed, $H$–invariant, and contained in the support of $D$; thus, $Y$ contains no orbit. Hence $Y$ is empty, and $D$ is Cartier. Likewise, the base locus of $D$ is empty, so that $D$ is globally generated. \hfill \Box

We now come to our first main result.

**Theorem 1.** Let $X$ be a normal, simple $G$–projective variety. Let $x^- \in X$ be the $B$–fixed point, $X^- \subseteq X$ its unique $B^-$–invariant open affine neighborhood, and $D_1, \ldots, D_r$ the irreducible components of $X \setminus X^-$. Then the following hold:

1. $D_1, \ldots, D_r$ are globally generated Cartier divisors. Their linear equivalence classes form a basis of the Picard group of $X$.
2. Every ample divisor on $X$ is linearly equivalent to a unique linear combination of $D_1, \ldots, D_r$ with positive integer coefficients.
(3) Every nef divisor on $X$ is linearly equivalent to a unique linear combination of $D_1, \ldots, D_r$ with non-negative integer coefficients.

Proof. (1) The first assertion follows from Lemma \ref{lem:linear-equivalence} (applied to $H = B$). For the second assertion, let $D$ be a Cartier divisor on $X$. Then the invertible sheaf $\mathcal{O}_X(D)$ admits a $T$–linearization; by Lemma \ref{lem:linearization}, it follows that the pull–back of $\mathcal{O}_X(D)$ to $X^-$ is trivial (as an invertible sheaf). Thus, $D$ is linearly equivalent to a linear combination of $D_1, \ldots, D_r$ with integer coefficients. These coefficients are unique, since every regular invertible function on $X^-$ is constant.

(2) Let $D$ be an ample divisor on $X$. There exists a positive integer $n$ such that the invertible sheaf $\mathcal{O}_X(nD)$ is very ample and $G$–linearized. Then the $G$–module $H^0(G \cdot x^-, \mathcal{O}_X(nD))$ contains a $B^-$–eigenvector $\sigma$. Note that $\sigma(x^-) \neq 0$, since $\sigma \neq 0$ and $B^- \cdot x^- = B^- B \cdot x^-$ is dense in $G \cdot x^-$. Replacing $n$ by a positive multiple, we may also assume that $\sigma$ lifts to $\tau \in H^0(X, \mathcal{O}_X(nD))$; then we may further assume that $\tau$ is a $T$–eigenvector. Since $\tau(x^-) \neq 0$, it follows that $\tau$ has no zero on $X^-$, so that $nD$ is linearly equivalent to a linear combination of $D_1, \ldots, D_r$ with non–negative integer coefficients.

Since the divisor $mD - D_1 - \cdots - D_r$ is ample for large $m$, this shows that $D$ is linearly equivalent to a linear combination of $D_1, \ldots, D_r$ with positive rational coefficients. Together with (1), this implies our statement.

(3) Let $E$ be a nef divisor on $X$. Since $D_1 + \cdots + D_r$ is ample (by (2)), the divisor $E + D_1 + \cdots + D_r$ is ample as well. Applying again (2) completes the proof. \hfill \Box

In particular, the Picard group $\text{Pic}(X)$ is a free abelian group of finite rank. Together with \cite[Examples 19.1.2, 19.3.3]{example}, this implies

**Corollary 1.** For any normal, simple $G$–projective variety $X$, the natural map $\text{Pic}(X) \to N^1(X)$ is an isomorphism. In other words, rational and numerical equivalence coincide for Cartier divisors.

**2.** THE CONE OF EFFECTIVE ONE–CYCLES

We will construct generators of the cone of effective one–cycles of a simple $G$–projective variety $X$, that are dual to the generators of the nef cone obtained in Theorem \ref{thm:cone-effective}. For this, recall that every effective cycle on $X$ is rationally equivalent to a linear combination of $B$–invariant subvarieties with positive coefficients, by \cite. Thus, we will study the $B$–invariant curves in $X$; we begin with the following easy result, generalizing \cite[Proposition 2.1]{generalization} to arbitrary characteristics.

**Lemma 5.** Let $C$ be a $B$–invariant curve in a simple $G$–projective variety. Then $C$ is fixed pointwise by the radical of a unique minimal parabolic subgroup $P(C) \supset B$, and $C = \overline{B \cdot x}$ for a unique $x \in C^T$. Moreover, the normalization of $C$ is isomorphic to $\mathbb{P}^1$, and the normalization map is bijective.
Proof. By assumption, $C^B$ consists of a unique point. And since $C$ is $T$–invariant, $C \setminus C^B$ contains a unique $T$–fixed point $x$. The reduced isotropy group $B_x$ equals $U_x T$, where $U_x$ is a closed reduced subgroup of codimension one in $U$, normalized by $T$. It follows easily that $U_x = R_u(P)$ for a unique minimal parabolic subgroup $P \supset B$. Then the radical $R(P)$ is a normal subgroup of $B$ contained in $B_x$. Thus, $R(P)$ fixes pointwise $B \cdot x = C$, and $R_u(P)$ is the kernel of the $U$–action in $C$.

Since $C$ consists of a dense $B$–orbit and a unique fixed point, it is rational, and the normalization map is bijective. □

Remark 1. In characteristic zero, $C$ is isomorphic to $\mathbb{P}^1$ by [3, 2.2, 2.6]. This does not generalize to arbitrary characteristics: let indeed $K$ be a field of characteristic $p \geq 3$ and let $G = \text{SL}(2)$, with its Borel subgroup $B$ of upper triangular matrices. Let $M$ be the space of homogeneous polynomials of degree $p + 2$ in two variables $x, y$, where $G$ acts by linear substitutions. The line $K y^{p+2} \subset M$ is $B$–invariant; let $N = M / K y^{p+2}$ be the quotient $B$–module, and let $[x^{p+1}y]$ be the image of $x^{p+1}y$ in $\mathbb{P}(N)$. Then

$$C = B \cdot [x^{p+1}y]$$

is a $B$–invariant curve in $\mathbb{P}(N)$. Since

$$(x + ty)^{p+1}y = x^{p+1}y + tx^p y^2 + t^p xy^{p+1} + t^{p+1} y^{p+2},$$

the curve $C$ is singular at $[xy^{p+1}]$, the unique $B$–fixed point in $\mathbb{P}(N)$. On the other hand, $C$ is a $B$–invariant curve in

$$X = G \times^B \mathbb{P}(N),$$

and the latter is a simple, nonsingular $G$–projective variety.

Another useful observation is the following

Lemma 6. Any simple $G$–projective variety contains only finitely many $B$–invariant curves.

Proof. Let $X$ be a simple $G$–projective variety. By Lemma 3, it suffices to show that $X^{R(P)}$ contains only finitely many $B$–invariant curves, where $R(P)$ is the radical of a minimal parabolic subgroup $P \supset B$. Then the quotient group $P/R(P)$ is isomorphic to $\text{SL}(2)$ or $\text{PGL}(2)$; it acts on $X^{R(P)}$ with a unique closed orbit (since $R_u(P) \subset U$, and $X^U$ consists of a unique point). Thus, we may assume that $G = \text{SL}(2)$. Then every $H^0(\omega)$ is multiplicity–free as a $T$–module, so that the set $\mathbb{P}(H^0(\omega))^T$ is finite. Thus, $X^T$ is finite by Lemma 1. This implies our statement, by Lemma 3 again. □

We may now state our second main result.
Theorem 2. \( \text{(1)} \) With the notation of Theorem 4, the sink of each \( D_i \) is a unique point \( x_i^- \), isolated in \( X^T \). Moreover, \( D_i \) is the closure of \( X^-(x_i^-) \), whereas \( X^+(x_i^-) = B \cdot x_i^- \); the \( B \)-invariant curve \( C_i = B \cdot x_i^- \) intersects \( D_i \) at the unique point \( x_i^- \), and intersects no other \( D_j \). As a consequence, \( (D_i \cdot C_i) \) is a positive integer, and \( (D_i \cdot C_j) = 0 \) for \( j \neq i \).

\( \text{(2)} \) The cone \( NE(X) \) is generated by the classes of \( C_1, \ldots, C_r \), and these form a basis of the rational vector space \( N_1(X) \).

\( \text{(3)} \) Any negative stratum having an irreducible component of codimension one is actually irreducible, with closure some \( D_i \). Likewise, any positive stratum having an irreducible component of dimension one is open in some \( C_i \).

\( \text{(4)} \) If, in addition, \( X \) is nonsingular, then every \( D_i \) intersects transversally \( C_i \) at \( x_i^- \). In particular, \( D_i \) is nonsingular at \( x_i^- \), and \( (D_i \cdot C_i) = 1 \). Moreover, all \( C_i \) are isomorphic to \( \mathbb{P}^1 \), and their classes form a basis of the group \( N_1(X) \).

Proof. (1) Let \( Y_i \) be the sink of \( D_i \) and let \( x \in Y_i \). Then \( X^+(x) \) is positive–dimensional (since \( x \neq x^- \)), and \( X^+(x) \cap D_i = D_i^+(x) = \{x\} \). Since \( D_i \) is a Cartier divisor, it follows that \( X^+(x) \) is one–dimensional. As a consequence, \( B \cdot x \) is a \( B \)-invariant curve with \( T \)-fixed points \( x \) and \( x^- \). Now Lemma 3 implies that \( Y_i = \{x\} \). Since \( X^+(Y_i) \) is \( B \)-invariant, we must have \( X^+(Y_i) = B \cdot x \).

Let \( Y \) be an irreducible component of \( X^T \) through \( x \). Then \( X^+(Y) \cap D_i = D_i^+(Y \cap D_i) \) contains \( x \) as an isolated point, so that \( \dim X^+(Y) \leq 1 \). On the other hand, \( X^+(Y) \) contains the curve \( B \cdot y \) for any \( y \in Y \); it follows that \( Y = \{x\} \). In other words, \( x \) is isolated in \( X^T \); hence \( X^+(x) = B \cdot x \).

To show that \( X^-(x) = D_i \), we choose an open affine \( T \)-invariant neighborhood \( X(x) \subset X(x) \), where \( D_i \) is a principal divisor associated to a regular function \( f \), eigenvector of \( T \). Then \( X(x) \) contains both \( X^-(x) \) and \( X^+(x) = B \cdot x \); moreover, the weight of \( f \) is negative on \( \lambda \) (since \( f \) restricts to a non–zero function on \( B \cdot x \)). It follows that \( f \) vanishes at any point of \( X^-(x) \), so that \( X^-(x) = D_i^-(x) \).

Since \( x^- \notin D_i \), the curve \( C_i = B \cdot x \) intersects \( D_i \) at \( x \) only. We show that \( C_i \) intersects no \( D_j \) for \( j \neq i \). Otherwise, \( x \in D_j \). Moreover, \( x \) is not the sink of \( D_j \) by the preceding step, so that \( D_j^+(x) \neq \{x\} \). But \( D_j^+(x) \subseteq X^+(x) = B \cdot x \), and hence \( B \cdot x \subseteq D_j \). Thus, \( x^- \in D_j \), a contradiction.

(2) follows from (1) together with Theorem 4.

(3) Let \( X^-(Y) \) be a negative stratum having an irreducible component \( D \) of codimension one in \( X \). Then \( D = D_i \) for some index \( i \). Together with (1), it follows that \( x_i^- \in D_i \), whence \( Y = \{x_i^-\} \) and \( X^-(Y) = D_i^- \).

Next let \( X^+(Y) \) be a positive stratum having an irreducible component \( C \) of dimension one. Then \( C \) is \( B \)-invariant, so that \( C = B \cdot x \) for some \( x \in Y \). Let \( Z \) be an irreducible component of \( Y \) containing \( x \), then \( C \subseteq B \cdot Z \subseteq X^+(Y) \), and \( B \cdot Z \) is irreducible. Hence \( C = B \cdot Z \), that is, \( Y = \{x\} \). Now every irreducible component
of $X^+(x)$ is $B$–invariant and contains $x$; thus, $X^+(x) = B \cdot x = C$. By $(*)$, we have \[ \dim X \leq \dim_x X^+(x) + \dim_x X^-(x) \] Thus, $X^-(x)$ is a divisor at $x$. Since $X^-(x)$ is disjoint from $X^-$, it follows that $x = x_i^-$ for some index $i$, whence $C = C_i$.

(4) Since $X$ is nonsingular and $x_i^-$ is an isolated fixed point, $X^-(x_i^-)$ and $X^+(x_i^-)$ intersect transversally at $x_i^-$. Thus, the same holds for their closures $D_i$ and $C_i$. Together with (1) and Theorem \[ \text{[1]}, \] it follows that $D_i$ restricts to a globally generated Cartier divisor of degree 1 on $C_i$. But the normalization of $C_i$ is isomorphic to $\mathbb{P}^1$, so that $C_i \cong \mathbb{P}^1$. Finally, by (2), any $\gamma \in N_1(X)$ decomposes as $\gamma = \sum_{i=1}^r c_i C_i$, and every $c_i = (D_i \cdot \gamma)$ is an integer. \[ \square \]

**Remark 2.** In particular, any stratum of codimension at most one in a simple $G$–projective variety contains a unique $T$–fixed point. But the whole fixed point set may be infinite; in fact, it may have arbitrary irreducible components, as shown by the following construction.

Let $Y \subseteq \mathbb{P}^n$ be a projective variety. Let $G = \text{SL}(3)$ with standard opposite Borel subgroups $B, B^-$; let $\omega$ be the highest root, i.e., the highest weight of the adjoint representation. Then the zero–weight space $H^0(\omega)^T$ has dimension 2, so that $H^0(\omega)^T$ has dimension $\geq n + 1$. Thus, we may regard $Y$ as a subvariety of $\mathbb{P}(H^0(\omega)^T)$. Now let $X = \overline{G \cdot Y}$ (closure in $\mathbb{P}(H^0(\omega))$. Clearly, $X$ is a simple $G$–projective variety, and $Y$ is an irreducible component of $X^T$. In fact, both $X^+(Y)$ and $X^-(Y)$ have codimension 3 in $X$, since $U \cdot Y$ (resp. $U^- \cdot Y$) is dense in $X^+(Y)$ (resp. $X^-(Y)$).

Finally, we compare the group $N_1(X)$ with the Chow group $A_1(X)$ of one–cycles modulo rational equivalence.

**Corollary 2.** For any normal, simple $G$–projective variety $X$, the map $A_1(X) \to N_1(X)$ is an isomorphism over the rationals. If, in addition, $X$ is nonsingular, then this map is an isomorphism.

**Proof.** We begin with the case where $X$ is nonsingular. Then it suffices to show that the classes of $C_1, \ldots, C_r$ generate the group $A_1(X)$. Decomposing $X$ into the positive strata $X^+(Y)$ and using the long exact sequence for Chow groups \[ \text{[1], Proposition 1.8], \] this reduces to checking the vanishing of $A_1(X^+(Y))$ whenever $\dim X^+(Y) \geq 2$.

Recall that $p^+ : X^+(Y) \to Y$ is an affine bundle. By Lemma \[ \text{[2]} \] and \[ \text{[3], Proposition 1.9], \] it follows that $A_1(X^+(Y)) = 0$ unless $\dim X^+(x) = 1$ for all $x \in Y$. In the latter case, since $B \cdot x \subseteq X^+(x)$ for all $x \in Y$, the fibers of $p^+$ are $B$–invariant curves in $X^+(Y)$. By Lemma \[ \text{[3]}, \] it follows that $Y$ is a unique point, so that $X^+(Y)$ is one–dimensional.

For arbitrary $X$, recall that the group $A_1(X)$ is generated by classes of $B$–invariant curves. Together with Theorem \[ \text{[2] (3),} \] this reduces to checking the following assertion:
let $X^+(Y)$ be a positive stratum and let $C \subseteq X^+(Y)$ be a $B$–invariant curve which is not an irreducible component. Then the class of $C$ is zero in $A_1(X^+(Y))_Q$.

Since $C$ is contained in an irreducible component of $X^+(Y)$ of dimension $\geq 2$, we may find a closed irreducible $B$–stable surface $S \subseteq X^+(Y)$ containing $C$. Let $Z = p^+(S)$, then $Z \neq S$ (otherwise $S \subseteq X^T$, whence $S \subseteq X^B$, a contradiction). For any $x \in Z$, the fiber at $x$ of $p^+: S \to Z$ contains $B \cdot x$. By Lemma 3, it follows that $Z$ consists of a unique point $x$, the source of $S$. In particular, $S$ is affine, and its algebra of regular functions is negatively graded. Consider now the normalization $\tilde{S}$ of $S$. The group $B$ acts on $\tilde{S}$ without fixed points, so that $\tilde{S}$ is nonsingular. On the other hand, the algebra of regular functions of $\tilde{S}$ is negatively graded as well; as a consequence, $\tilde{S}$ is isomorphic to the affine plane. In particular, $A_1(\tilde{S}) = 0$, whence $A_1(S)_Q = 0$. This proves our assertion. □

3. The simple complete $\text{PGL}(2)$–embeddings

As an illustration of our results, we describe the cones of effective one–cycles and nef divisors for all simple complete embeddings of $\text{PGL}(2)$, that is, for the normal complete $\text{PGL}(2)$–varieties containing an open orbit isomorphic to $\text{PGL}(2)$ and a unique closed orbit. Along the way, we obtain a realization of these embeddings as $\text{PGL}(2)$–orbit closures in a product of copies of the projective line, suggested by work of Iozzi and Poritz [10].

Note that there is a combinatorial classification of all embeddings of $\text{PGL}(2)$, presented e.g. in [17], as part of the Luna–Vust theory of embeddings of homogeneous spaces; the geometric realization of simple complete $\text{PGL}(2)$–embeddings can be deduced from this classification. The Chow rings of smooth complete $\text{SL}(2)$–embeddings, and their cones of effective one–cycles, are described in [18].

We introduce some notation. Let $G = \text{PGL}(2)$ and let $T$ be the image in $G$ of the torus of diagonal matrices in $\text{SL}(2)$. The map $t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ defines a one–parameter subgroup $\lambda$ of $T$, which generates the group $X_*(T)$. The opposite Borel subgroups $B = G(\lambda)$, $B^- = G(-\lambda)$ are the images in $G$ of the subgroups of upper (resp. lower) triangular matrices in $\text{SL}(2)$.

For the standard action of $G$ on $\mathbb{P}^1$, the isotropy group of the sink $\infty$ (resp. the source 0) is $B$ (resp. $B^-$). We will consider the diagonal action of $G$ on the product $(\mathbb{P}^1)^r$ of $r \geq 3$ copies of $\mathbb{P}^1$. This is a (very) simple $G$–projective variety, with closed orbit being the small diagonal, $\text{diag} \mathbb{P}^1$; the sink is $\infty^r$, with $B^-$–invariant open affine neighborhood $(\mathbb{P}^1 \setminus \{0\})^r \cong \mathbb{A}^r$ where $T$ acts by scalar multiplication. The irreducible divisors of Theorem 4 are the

$$D_i = (\mathbb{P}^1)^{i-1} \times \{0\} \times (\mathbb{P}^1)^{r-i},$$
and the curves of Theorem 2 are the lines
\[ C_i = \infty^{i-1} \times \mathbb{P}^1 \times \infty^{r-i}, \]
where \(1 \leq i \leq r\).

**Proposition 1.** Let \(p_1, \ldots, p_r \in \mathbb{P}^1\) be pairwise distinct points and let
\[ X = X(p_1, \ldots, p_r) \subseteq (\mathbb{P}^1)^r \]
be the closure of the orbit \(G \cdot (p_1, \ldots, p_r)\). Then the following statements hold.

1. The irreducible components of the boundary \(X \setminus G(p_1, \ldots, p_r)\) are the divisors
   \[ \partial_i X = \{(x^{i-1}, y, x^{r-i}) \mid x, y \in \mathbb{P}^1\} = G \cdot C_i \ (1 \leq i \leq r). \]
   These are isomorphic to \(\mathbb{P}^1 \times \mathbb{P}^1\), and they intersect along \(\text{diag} \mathbb{P}^1\).
2. The open subset \(X^-\) is isomorphic to \(A^1 \times \Sigma\), where \(\Sigma\) is the affine cone over the rational normal curve in \(\mathbb{P}^{r-2}\). As a consequence, \(X\) is a normal, Cohen–Macaulay, simple \(G\)-projective variety; it is singular along \(\text{diag} \mathbb{P}^1\), if \(r \geq 4\).
3. The irreducible divisors constructed in Theorem 2 are the pull-backs of \(0\) under the \(r\) projections \(X \to \mathbb{P}^1\); they are normal and Cohen–Macaulay. The curves constructed in Theorem 2 are the lines \(C_1, \ldots, C_r\).
4. The boundary divisors \(\partial_1 X, \ldots, \partial_r X\) form a basis of the divisor class group of \(X\) with rational coefficients; the latter is isomorphic to the Picard group with rational coefficients.
5. A canonical divisor for \(X\) is
   \[ -K_X = \partial_1 X + \cdots + \partial_r X = \frac{2}{r-2}(D_1 + \cdots + D_r). \]
   As a consequence, \(- (r-2)K_X\) is very ample, so that \(X\) is \(\mathbb{Q}\)-Fano.

Conversely, any simple complete embedding of \(G\) is isomorphic to \(X(p_1, \ldots, p_r)\), where \(r \geq 3\) and \(p_1, \ldots, p_r \in \mathbb{P}^1\) are uniquely determined up to permutation and diagonal action of \(G\).

**Proof.** (1) If \(r = 3\), then \(X = (\mathbb{P}^1)^3\) and our assertions are evident; so we may assume that \(r \geq 4\). We may also assume that \(p_1 = \infty\) and \(p_2 = 0\); then \(p_3, \ldots, p_r\) are distinct non-zero scalars.

The first projection \(z_1 : X \to \mathbb{P}^1 = G \cdot p_1 \cong G/B\) is a locally trivial fibration. It yields an isomorphism \(X \cong G \times^B S\), where \(S = \overline{B \cdot (p_1, \ldots, p_r)} \subset X\) is an irreducible projective \(B\)-invariant surface with a unique fixed point \(\infty^r\). Moreover, the map
\[ U^- \times S^- \to X^-, \ (g, s) \mapsto g \cdot s \]
is an isomorphism, where \( S^– = S \cap (\mathbb{P}^1 \setminus \{0\})^r \) is an irreducible surface in \( \mathbb{A}^r \), invariant under scalar multiplication. Since \( S^– \subseteq \{\infty\} \times \mathbb{A}^{r-1} \) (as \( p_1 = \infty \)), we will regard \( S^– \) in \( \mathbb{A}^{r-1} \) via projection.

Since \( \begin{pmatrix} t & u \\ 0 & t^{-1} \end{pmatrix} \cdot p_i = t^2 p_i + tu \) in \( \mathbb{P}^1 \setminus \{\infty\} \), we see that \( S^– \) is the closure of the image of the rational map

\[
\mathbb{A}^2 \rightarrow \mathbb{A}^{r-1}, \quad (t, u) \mapsto \left( \frac{1}{tu}, \frac{1}{t^2p_3 + tu}, \ldots, \frac{1}{t^2p_r + tu} \right).
\]

For \( 2 \leq i \leq r \), let \( x_i = (t^2p_i + tu)^{-1} \). Equivalently, \( tux_2 = 1 \), and \( tp_ix_i + u(x_i - x_2) = 0 \) for \( 3 \leq i \leq r \). Hence \( S^– \) is defined (as a closed subset of \( \mathbb{A}^{r-1} \)) by the vanishing of all \( 2 \times 2 \) minors of the \( 2 \times (r - 2) \) matrix

\[
A = \begin{pmatrix}
p_3x_3 & p_4x_4 & \cdots & p_rx_r \\
x_3 - x_2 & x_4 - x_2 & \cdots & x_r - x_2
\end{pmatrix}.
\]

It follows that the boundary \( S^– \setminus B \cdot (p_1, \ldots, p_r) \) consists of the diagonal

\[
(x_2 = x_3 = \cdots = x_r) = S^– \cap G \cdot C_1,
\]

together with the \( r - 1 \) coordinate lines

\[
(x_2 = \cdots = x_{i-1} = x_{i+1} = \cdots = x_r = 0) = C_i^–.
\]

On the other hand, the open subset \( X^– \) equals \( U^– \cdot S^– \), and intersects all \( G \)-orbits in \( X \) (since \( X^– \) intersects the unique closed orbit). Thus, the boundary \( X \setminus G \cdot (p_1, \ldots, p_r) \) is the union of the irreducible divisors \( \partial_i X = G \cdot C_1, \ldots, \partial_r X = G \cdot C_r \).

(2) Clearly, \( X \) is a simple \( G \)-projective variety. We claim that its regular locus intersects all irreducible components of the boundary. To see this, consider the projection \( (z_i, z_j, z_k) : (\mathbb{P}^1)^r \rightarrow (\mathbb{P}^1)^3 \) where \( 1 \leq i < j < k \leq r \). Then the restriction of \( (z_i, z_j, z_k) \) to \( X \) is birational, and bijective over \( G \cdot (C_j \setminus \{\infty\}) \). Now our claim follows from Zariski’s main theorem.

By that claim, \( X \) is nonsingular in codimension one. Next we show that \( X \) is Cohen–Macaulay, or, equivalently, that \( S^– \) is Cohen–Macaulay. For this, consider the closed subscheme \( \Sigma \subseteq \mathbb{A}^{r-1} \) associated with the ideal generated by the \( 2 \times 2 \) minors of the matrix \( A \). Then \( S^– \) is the support of \( \Sigma \). On the other hand, we may regard \( \Sigma \) as a closed subscheme of the space \( M_{2, r-2} \) of \( 2 \times (r - 2) \) matrices, an affine space of dimension \( 2r - 4 \). Then \( \Sigma \) is the scheme–theoretic intersection of the closed subscheme of matrices of rank at most one (a Cohen–Macaulay variety of dimension \( r - 1 \)) with a linear space of dimension \( r - 1 \). Since \( \dim \Sigma = 2 \), it follows that \( \Sigma \) is Cohen–Macaulay. Moreover, one easily checks that the intersection \( \Sigma \cap (x_2 = 0) \subseteq \mathbb{A}^{r-1} \) is reduced, of dimension one. Thus, \( x_2 \) is a non–zero–divisor on \( \Sigma \), and the latter is reduced. It follows that \( \Sigma = S^– \).
Thus, \( X^- \cong U^- \times S^- \cong \mathbb{A}^1 \times \Sigma \), where \( \Sigma \) is the affine cone over a curve \( C \subset \mathbb{P}^{r-2} \). Clearly, \( C \) is rational and not contained in any hyperplane; moreover, \( C \) is projectively normal, since \( \Sigma \) is Cohen–Macaulay and nonsingular in codimension one. Thus, \( C \) is a rational normal curve.

(3) follows from the fact that \( X \) contains \( C_1, \ldots, C_r \), together with Theorem 2.

(4) For the divisor class group \( A_2(X) \), we have an exact sequence

\[
\bigoplus_{i=1}^{r} \mathbb{Z} \partial_i X \to A_2(X) \to A_2(G \cdot (p_1, \ldots, p_r)) \to 0.
\]

Moreover, the group \( A_2(G \cdot (p_1, \ldots, p_r)) = \text{Pic}(\text{PGL}(2)) \) has order 2, and the classes of the \( \partial_i X \) are linearly independent in \( A_2(X)_\mathbb{Q} \). Therefore, these classes form a basis of \( A_2(X)_\mathbb{Q} \). The latter contains \( \text{Pic}(X)_\mathbb{Q} \) as a subspace of dimension \( r \); thus, they are equal.

(5) By (4), we have

\[ -K_X = a_1 \partial_1 X + \cdots + a_r \partial_r X \]

for unique rational coefficients \( a_1, \ldots, a_r \). If \( r = 3 \), then \( X = (\mathbb{P}^1)^3 \) and one obtains easily \( a_1 = a_2 = a_3 = 1 \). In the general case, any \( \partial_i X \) is mapped isomorphically to its image under some projection to \( (\mathbb{P}^1)^3 \); it follows that \( a_i = 1 \).

To express \( -K_X \) in terms of \( D_1, \ldots, D_r \), consider the divisors of the rational functions \( z_i - z_j \), where \( z_1, \ldots, z_r : X \to \mathbb{P}^1 \) denote the projections. We obtain

\[ \text{div}(z_i - z_j) = -D_i - D_j + \sum_{k,k \notin \{i,j\}} \partial_k X, \]

whence

\[ \partial_1 X + \cdots + \partial_r X = \frac{2}{r-2} (D_1 + \cdots + D_r) \]

in \( A_2(X)_\mathbb{Q} \).

For the final assertion, let \( X \) be a simple complete embedding of \( G \). Choose \( p \in X \) in the open \( G \)-orbit. By Lemma 3, there exists a finite equivariant morphism

\[ f : X \to \mathbb{P}(H^0(\omega)). \]

Now \( H^0(\omega) \) is the space of homogeneous polynomials of degree \( d \) in two variables \( x, y \), where \( G \) acts by linear substitutions; here \( d \) is an even positive integer (see e.g. [4, II.2.16]). Let \( P(x, y) \in H^0(\omega) \) be a representative of \( f(p) \in \mathbb{P}(H^0(\omega)) \); let

\[ P(x, y) = \prod_{i=1}^{s} (a_i x + b_i y)^{m_i}, \]

be a decomposition into a product of pairwise distinct linear forms, with multiplicities. Since \( f \) is finite, then so is the isotropy group of the line \( KP \). Thus, \( s \geq 3 \).
The source of the $\lambda$–action on $\mathbb{P}(H^0(\omega))$ consists of the image of $y^d$, and the corresponding open subset $\mathbb{P}(H^0(\omega))^-\times$ consists of the images of those homogeneous polynomials where the coefficient of $y^d$ is non–zero. Since $f$ is finite and equivariant, $X^- = f^{-1}(\mathbb{P}(H^0(\omega)))^-$. It follows that $(X \setminus X^-) \cap G \cdot p$ consists of $s$ irreducible components. On the other hand, any irreducible component $D_i$ of $X \setminus X^-$ intersects $G \cdot p$ (otherwise, $D_i$ is $G$–invariant, and hence contains the closed $G$–orbit, a contradiction). Thus, $s = r$ with the notation of Theorem 1.

Next let

$$f_i : X \to Y_i = \text{Proj} \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(nD_i))$$

be the morphism associated with the globally generated divisor $D_i$. Then $Y_i$ is a normal simple $G$–projective variety, and $f_i$ is equivariant, separable, with connected fibers. Moreover, the Picard group of $Y_i$ is freely generated by a unique ample divisor, which pulls back to $D_i$. By considering a finite morphism $Y_i \to \mathbb{P}(H^0(\omega_i))$ and arguing as above, one obtains that $\dim Y_i = 1$. On the other hand, $G$ acts on $Y_i$ with a dense separable orbit and a unique closed orbit, whence $Y_i$ is isomorphic to the projective line with standard $G$–action.

Consider now the product morphism

$$f = \prod_{i=1}^{r} f_i : X \to \prod_{i=1}^{r} Y_i \cong (\mathbb{P}^1)^r.$$ 

Then $f$ is finite, since $D_1 + \cdots + D_r$ is ample. Let $f(p) = (p_1, \ldots, p_r)$, then the $p_i$ are pairwise distinct (since so are the $D_i$), and their number is at least 3. Thus, the restriction $G \cdot p \to G \cdot f(p)$ is an isomorphism. On the other hand, $f(X)$ is normal by (1), so that $f$ is a closed immersion by Zariski’s main theorem.

So we have proved that $X$ embeds equivariantly into $(\mathbb{P}^1)^r$, where $r$ is the rank of $\text{Pic}(X)$; moreover, the $r$ projections $z_i : X \to \mathbb{P}^1$ are uniquely determined. This implies the remaining uniqueness assertion. \hfill \Box

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