Stable Bound Orbits in Six-dimensional Myers-Perry Black Holes

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The existence of stable bound orbits of test particles is one of the most characteristic properties in black hole spacetimes. In higher-dimensional black holes, due to the dimensionality of gravity, there is no stable bound orbit balanced by Newtonian gravitational monopole force and centrifugal force as in the same mechanism of the four-dimensional Kerr black hole case. In this paper, however, the existence of stable bound orbits of massive and massless particles is shown in six-dimensional singly spinning Myers-Perry black holes with a value of the spin parameter larger than a critical value. The innermost stable circular orbits and the outermost stable bound orbit are found on the rotational axis.

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I. INTRODUCTION

In the last decade, higher-dimensional black holes have been studied in a broad range of research, highly motivated by superstring theories (see, for example, [1]). One of the most important black hole solutions in higher-dimensions is the Myers-Perry black hole solutions [2], which are generalization of the four-dimensional Kerr solutions to higher dimensions. There also exist the black hole solutions that have ring horizon topology in five dimensions, called the black ring solutions [3]. Many efforts are devoted to understanding physics of these black hole solutions.

One of the most important step to comprehend black holes is analysis of geodesics. Geodesics have been extensively studied in various black holes [4], which provide new insights into not only background geometries but physical phenomena related to test particles in such backgrounds. For example, the geodesic system in the Kerr geometry was studied by Carter [5], and the Hamilton-Jacobi equation of its system was shown to occur the separation of variables by finding the non-trivial constant called Carter’s constant. The discovery of Carter’s constant led to the understanding of non-trivial hidden symmetries in the Kerr geometry such as Killing tensors [6].

In the case that the geodesic equation occurs the separation of variables in a black hole space-
time, it is possible to study various particle orbits around the black hole analytically. Among several special orbits in the Kerr black hole, stable bound orbits of massive particles are one of the most important characteristics \[7\]. The stable bound orbits mean that a particle ranges over a finite interval of radius, neither being captured by the black hole nor escaping to infinity. Basically, such orbits as Earth goes around the sun are formed by a balance between Newtonian gravitational monopole force and centrifugal force in four dimensions. Stable bound orbits play important role in the astrophysical context. For example, in a black hole and disk system, the innermost stable circular orbit are thought to be the inner edge of the accretion disk.

In \(D\)-dimensional black holes in the case \(D \geq 5\), there exist no stable bound orbit balanced by Newtonian gravitational monopole force and centrifugal force. More precisely, Newtonian gravitational monopole potential term and centrifugal potential term in radial motion cannot form a potential well because the centrifugal potential proportional to \(\propto r^{-2}\) in any dimensions is dominant in the region far from a black hole than the Newtonian gravitational monopole potential proportional to \(-r^{3-D}\) in general. In five-dimensional black rings, however, there exist potential wells, which are formed in different mechanism \[8-10\], by contrast to the case in four dimensions, and then a particle can be bounded in such a potential well stably. This physics is led by having ring horizon topology completely different from round sphere.

In higher-dimensional Myers-Perry black holes, horizon shape is deformed by the effect of black hole spin angular momenta. Although a horizon takes the shape of almost round sphere for small values of spin parameters, in general, its shape becomes non-trivial for large values of spin parameters. One of the most interesting limit for Myers-Perry black holes is the ultra-spinning limit in more than six dimensions \[11\]. Since the horizon is flattened in this limit, gravitational field that particles feel changes drastically, and the existence of stable bound orbits is expected to be non-trivial as in the case of five-dimensional black rings.

In \(D\)-dimensional singly rotating Myers-Perry black holes, unstable circular particle orbits was studied in \[16\]. Althought particle motion has been extensively studied in five-dimensional Myers-Perry black holes \[12-15\], no stable bound orbit has found yet. The purpose of this paper is to show that there exist stable bound orbits of particles even in higher-dimensional Myers-Perry black holes. Note that the mechanism of taking stable bound orbits is physically non-trivial as mentioned above. In this paper, the existence of stable bound orbits is demonstrated in six dimensional singly spinning Myers-Perry black holes.

This paper is organized as follows. In the following section, the geometry of the singly rotating Myers-Perry solution in six dimensions is reviewed. In Sec. \[11\], particle motion orbiting around
a six-dimensional singly spinning Myers-Perry black hole is analyzed, and the existence of stable bound orbits is demonstrated by the analysis of an effective potential. In Sec. IV, the effective potential restricted on the rotational axis of a singly spinning Myers-Perry black hole is analyzed, and the critical value of the spin parameter classifying whether or not stable bound orbits exist is determined. In Sec. V, stable bound orbits of massless particles are considered. Finally, the implication of our results is discussed in Sec. VI.

II. SINGLY ROTATING MYERS-PERRY BLACK HOLES IN SIX DIMENSIONS

In this section, the geometry of singly rotating Myers-Perry black holes in six dimensions is briefly reviewed. Its metric in the Boyer-Lindquist coordinates is given by

\[ ds^2 = -dt^2 + \mu \frac{r}{\Sigma} (dt - a \mu_1^2 d\phi - b \mu_2^2 d\psi)^2 + \frac{\Sigma}{\Delta} dr^2 + r^2 d\alpha^2 + \left( r^2 + a^2 \right) (d\mu_1^2 + \mu_1^2 d\phi^2) + \left( r^2 + b^2 \right) (d\mu_2^2 + \mu_2^2 d\psi^2), \]

where

\[ \Delta = \frac{(r^2 + a^2) (r^2 + b^2)}{r^2}, \]
\[ \Sigma = \frac{(r^2 + a^2) (r^2 + b^2)}{r^2} \left( 1 - \frac{a^2 \mu_1^2}{r^2 + a^2} - \frac{b^2 \mu_2^2}{r^2 + b^2} \right), \]
\[ \mu_1^2 + \mu_2^2 + \alpha^2 = 1, \]

and \( \mu, a, \) and \( b \) are the mass and two spin parameters of the family. The event horizon is located at the largest real \( r \)-coordinate value that \( \Delta = 0 \). In the case \( a = b = 0 \), this reduces to the Schwarzschild-Tangherlini geometry.

Figure 1 shows the allowed region of spin parameters that \( \Delta = 0 \) has the real positive roots. Note that, in the case of a single rotation, i.e., \( a = 0 \) or \( b = 0 \), the allowed region is not bounded along the axes in the figure. Indeed, as will be discussed below, the spin parameter in the singly rotating case may take arbitrarily large value, which is called the ultra-spinning regime.

In order to focus on singly spinning Myers-Perry black holes, one of the spin parameters is eliminated, \( b = 0 \), in what follows. In terms of the following angular coordinates satisfying (4),

\[ \mu_1 = \sin \theta, \]
\[ \mu_2 = \cos \theta \sin \chi, \]
\[ \alpha = \cos \theta \cos \chi, \]

Figure 1 shows the allowed region of spin parameters that \( \Delta = 0 \) has the real positive roots.
the metric of the singly rotating Myers-Perry geometry in six dimensions is given by
\[ ds^2 = -dt^2 + \frac{\mu}{r} \left( dt - a \sin^2 \theta \, d\phi \right)^2 + \frac{\Sigma}{\Delta} \left( dr^2 + \sum \, d\theta^2 + \left( r^2 + a^2 \right) \sin^2 \theta \, d\phi^2 + r^2 \cos^2 \theta \, d\Omega_2^2 \right), \] (8)
where
\[ d\Omega_2^2 = d\chi^2 + \sin^2 \chi \, d\psi^2. \] (9)
Note that (2) and (3) reduce to
\[ \Delta = r^2 + a^2 - \frac{\mu}{r}, \] (10)
\[ \Sigma = r^2 + a^2 \cos^2 \theta. \] (11)
Hence, for arbitrary value of \( a \), there exists at least one real positive root to \( \Delta = 0 \).

As discussed in [11], the boundlessness of the spin parameter allows us to take the limit of large \( a \) compared to \( \mu \), which is called ultra-spinning limit. Then (8) near the axis, \( \theta \approx 0 \), goes to
\[ ds^2 = -f \, dt^2 + f^{-1} \, dr^2 + r^2 \, d\Omega_2^2 + d\sigma^2 + \sigma^2 \, d\phi^2, \] (12)
\[ f = 1 - \frac{\hat{\mu}}{r}, \] (13)
where \( \hat{\mu} = \mu/a^2 \) kept finite, and the new coordinates \( \sigma \) is introduced by
\[ \sigma = a \sin \theta. \] (14)
Note that \( \sigma \) also is kept finite to obtain the regular metric. Hence, the six-dimensional ultra-spinning Myers-Perry black hole geometry near the rotational axis can be regarded as a direct product space of a four-dimensional Schwarzschild black hole and a two-dimensional flat space, i.e., a Schwarzschild black brane.
III. STABLE BOUND ORBITS FOR MASSIVE PARTICLES

In this section, massive particle motion around a six-dimensional singly spinning Myers-Perry black hole is analyzed. Our attention is focused on stable bound orbits, which mean that a particle ranges over a finite interval of radius outside the horizon, neither being captured by the black hole nor escaping to infinity. In the method of analyzing an effective potential, the existence of stable bound orbits is demonstrated by showing explicit shape of effective potentials including a local minimum point. Furthermore, the region that stable bound orbits can exist in is also found by the analysis of the extremal potential problem.

To reduce the problem of finding stable bound orbits of a particle to the extremal problem of an effective potential in six-dimensional singly rotating Myers-Perry black holes, let us define an effective potential in the Hamiltonian formalism. Let $g^{ab}$ be the inverse metric of (8). The Hamiltonian of particle system in this black holes is given by

$$H = \frac{1}{2} \left( g^{ab} p_a p_b + \kappa \right)$$

$$= \frac{1}{2} \left[ \frac{\Delta}{\Sigma} p_r^2 + \frac{1}{\Sigma} p_\theta^2 + \frac{1}{r^2 \cos^2 \theta} \left( p_\lambda^2 + \frac{p_\psi^2}{\sin^2 \chi} \right) \right. - \left. \left( 1 + \frac{\mu (r^2 + a^2)}{r \Delta \Sigma} \right) p_t^2 - \frac{2 \mu a}{r \Delta \Sigma} p_t p_\phi + \frac{1}{\Sigma} \left( \frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right) p_\phi^2 + \kappa \right],$$

where $p_a$ are the canonical momenta conjugate to the coordinates, and $\kappa$ takes the value of 1 for massive particles and 0 for massless particles. Note that $p_t$, $p_\phi$, and $p_\psi$ are constants of motion because $t$, $\phi$, and $\psi$ are the cyclic coordinates due to the spacetime symmetries. Since the relativistic particle system has the reparameterization invariance, there exist a constraint, $H = 0$, or explicitly in this case,

$$\frac{1}{\Sigma} \left( \Delta p_r^2 + p_\theta^2 \right) + E^2 \left( U + \frac{\kappa}{E^2} \right) = 0,$$

where $E = -p_t$, and $U$ is defined as

$$U = \frac{1}{E^2} \left( g^{tt} E^2 - 2 g^{t \phi} p_\phi E + g^{\phi \phi} p_\phi^2 + g^{\lambda \lambda} p_\lambda^2 + g^{\psi \psi} p_\psi^2 \right)$$

$$= -1 + \frac{\mu (r^2 + a^2)}{r \Delta \Sigma} + \frac{2 \mu a}{r \Delta \Sigma} \lambda + \frac{1}{\Sigma} \left( \frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right) \lambda^2 + \frac{L^2}{r^2 \cos^2 \theta},$$

$$L^2 = \frac{1}{E^2} \left( p_\lambda^2 + \frac{p_\psi^2}{\sin^2 \chi} \right),$$

and $\lambda = p_\phi / E$. Note that $L^2$ is a constant of motion, which is Poisson commutable with $p_t$, $p_\phi$, and $p_\psi$, and is associated with a reducible Killing tensor. The function $U$ depends on two variables,
\( r \) and \( \theta \), and is called the effective potential in what follows. The allowed range of \( U \) must be
\[
-\infty < U \leq -\frac{\kappa}{E^2} \leq 0
\]
outside the horizon because the kinetic term, the first term in [17], is non-negative. The region that satisfies this inequality shows the allowed region for particle motion.

Let us introduce new coordinates \( \zeta \) and \( \rho \) instead of \( r \) and \( \theta \) as
\[
r = \sqrt{\left(\zeta^2 + \rho^2 - a^2\right) + \Sigma},
\]
\[
\sin \theta = \frac{\zeta}{\sqrt{r^2 + a^2}},
\]
\[
\Sigma = \sqrt{(\zeta - a)^2 + \rho^2}\sqrt{(\zeta + a)^2 + \rho^2},
\]
to grasp effective potential figures intuitively.\(^1\) Thus \( U \) is transformed into the function of the two variables, \( \zeta \) and \( \rho \), as
\[
U = U(\zeta, \rho; \lambda, \mathcal{L}^2).
\]

The conditions for \( U \) to exist stable bound orbits of massive particles in six-dimensional singly rotating Myers-Perry black holes are as follows: (i) \( U \) includes local minimum points; (ii) \( U \) takes a non-positive value at the local minimum points, which comes from [21]. Therefore let us analyze the extremal problem of \( U \) in what follows. The extrema of \( U \) are given by
\[
\partial_\zeta U = 0,
\]
\[
\partial_\rho U = 0.
\]
These equations have a solution \((\zeta, \rho) = (\zeta_0, \rho_0)\), where \( \zeta_0 \) and \( \rho_0 \) are written by \( \lambda \) and \( \mathcal{L}^2 \). The expressions for \( \zeta_0 \) and \( \rho_0 \) are solved in terms of \( \lambda \) and \( \mathcal{L}^2 \) as
\[
\lambda = \lambda(\zeta_0, \rho_0) \equiv \lambda_0,
\]
\[
\mathcal{L}^2 = \mathcal{L}^2(\zeta_0, \rho_0) \equiv \mathcal{L}^2_0,
\]
where \( \lambda_0 \) is one of the two branches for given \((\zeta_0, \rho_0)\). The value of \( U \) at an extremum \((\zeta_0, \rho_0)\) is given by
\[
U_0 = U(\zeta_0, \rho_0; \lambda_0, \mathcal{L}^2_0).
\]

\(^1\) The new coordinates, \( \zeta \) and \( \rho \), show radial labels of the polar coordinates on each two-dimensional flat planes because in the flat limit the metric goes to
\[
ds^2 = -dt^2 + d\zeta^2 + \zeta^2 d\psi^2 + d\rho^2 + \rho^2 d\phi^2.
\]
The next step is to classify whether an extremum of $U$ at $(\zeta_0, \rho_0)$ is local minimum or not. The second partial derivative test is useful to classify an extremal point as local maximum or local minimum, and therefore let us evaluate the determinant of the Hessian matrix of $U$

$$\mathcal{H} (\zeta, \rho; \lambda, \mathcal{L}^2) = \partial_\zeta^2 U \partial_\rho^2 U - (\partial_\zeta \partial_\rho U)^2$$

at each extremal point. Let $\mathcal{H}_0$ be the value of $\mathcal{H}$ at $(\zeta_0, \rho_0)$, i.e.,

$$\mathcal{H}_0 = \mathcal{H} (\zeta_0, \rho_0; \lambda_0, \mathcal{L}_0^2),$$

which is the function of $\zeta_0$ and $\rho_0$ through (29) and (30). The second partial derivative test asserts that if both $\text{det}\mathcal{H}_0$ and $\text{tr}\mathcal{H}_0$ are positive, then $(\zeta_0, \rho_0)$ is a local minimum point of $U$. 

FIG. 2: Region where stable bound orbits of massive particles can exist in the six-dimensional singly spinning Myers-Perry black holes with $a = 1.6$ (upper left panel), $a = 1.7$ (upper right panel), $a = 2.0$ (lower left panel), and $a = 2.3$ (lower right panel), in the unit $\mu = 1$. The larger $\lambda_0$-branch is taken here. Each black solid line shows the location of the horizon, and each gray region shows the set satisfying $\lambda_0, \mathcal{L}_0^2 \in \mathbb{R}$, $\text{tr}\mathcal{H}_0 > 0$, and $U_0 < 0$. Inside of each red line shows the set $\text{det}\mathcal{H}_0 > 0$. Hence, local minimum points of $U$ can exist in the gray region surrounded by the red lines.
Then the conditions that stable bound orbits exist in a neighborhood of $(\zeta_0, \rho_0)$ are

\begin{align}
\lambda_0, \mathcal{L}_0^2 &\in \mathbb{R}, \\
\text{det} \mathbb{H}_0 &> 0, \\
\text{tr} \mathbb{H}_0 &> 0, \\
U_0 &< 0.
\end{align}

Under several fixed values of $a$, let us see the region satisfying these conditions. The unit is normalized by $\mu$ in the remaining this section. The first line in Fig. 2 shows that local minimum points with non-positive local minimum value do appear in the case $a = 1.6$ although there is no local minimum point of $U$ in the case $a = 1.0$. Hence, there exist stable bound orbits of massive particles in six-dimensional singly spinning Myers-Perry black holes with $a = 1.6$, and these figures imply that particle motion qualitatively changes at a critical value in the range $1.6 \leq a \leq 1.7$, which will be considered in detail in the following section. The second line of Fig. 2 shows the case $a = 2.0, 2.3$. The region that stable bound orbits can exist becomes larger as $a$ gets larger. It is worth noting that the boundary of the region that stable bound orbits can exist means marginally stable bound orbits, and they include the innermost stable circular orbit and the outermost stable circular orbit as spacial cases, which will be analyzed in the next section.

Since Fig. 2 provides the location of a local minimum point $(\zeta_0, \rho_0)$, let us plot $U$ with some local minimum points by using the following expression

\[ U = U(\zeta, \rho; \lambda_0, \mathcal{L}_0^2). \]
Figure 3 shows explicit shapes of (38) with a local minimum point in the case \( a = 2 \). There exist a potential well near the local minimum point.

### IV. CRITICAL SPIN PARAMETER

In this section, massive particle motion on the rotational axis in six-dimensional singly spinning Myers-Perry black holes is analyzed. It was indicated in the previous section that there was the minimum value of the black hole rotational parameter at which the region where the existence of stable bound orbits disappeared, which is called the critical parameter in this paper. Since this region is located near the rotational axis for rapidly rotating case, the value of the critical parameter is expected to be determined by analyzing the effective potential on the axis.

First let us define a one-dimensional effective potential for particle motion on the rotational axis by deriving a set of ordinary differential equations for each variables. The Hamilton-Jacobi method is useful for the separation of variables of the equation of motion in this case. Let \( S \) be Hamilton’s principal function, and then canonical momenta are related to the partial derivatives of \( S \) in terms of the conjugate coordinates, \( p_{\mu} = \partial S / \partial x^\mu \). From (16) the Hamilton-Jacobi equation is given in the form

\[
- \left( 1 + \mu \frac{r^2 + a^2}{r \Delta \Sigma} \right) \left( \frac{\partial S}{\partial t} \right)^2 - \frac{2a\mu}{r \Delta \Sigma} \left( \frac{\partial S}{\partial t} \right) \left( \frac{\partial S}{\partial \phi} \right) + \frac{1}{\Sigma} \left( \frac{1}{\sin^2 \theta} - \frac{a^2}{\Delta} \right) \left( \frac{\partial S}{\partial \phi} \right)^2 \\
+ \frac{\Delta}{\Sigma} \left( \frac{\partial S}{\partial r} \right)^2 + \frac{1}{\Sigma} \left( \frac{\partial S}{\partial \theta} \right)^2 \left[ \left( \frac{\partial S}{\partial \chi} \right)^2 + \frac{1}{\sin^2 \chi} \left( \frac{\partial S}{\partial \psi} \right)^2 \right] + 1 = 0,
\]

where \( \kappa = 1 \). To show the separability of (39), \( S \) is assumed to be

\[
S = -Et + p_\phi \phi + p_\psi \psi + S_\chi(\chi) + S_r(r) + S_\theta(\theta),
\]

where \( S_\chi(\chi) \), \( S_r(r) \), and \( S_\theta(\theta) \) depend on \( \chi \), \( r \), and \( \theta \), respectively. Substitution of (40) into (39) leads to the separation of variables of the Hamilton-Jacobi equation in the form

\[
-P(r, \theta) = Q(\chi, \psi) = L^2,
\]

where \( L^2 \) is the constant separation equivalent to (20), and \( P(r, \theta) \), \( Q(\chi, \psi) \) are given by

\[
P(r, \theta) = \frac{r^2 \cos^2 \theta}{\Sigma} \left[ \left( \frac{dS_\theta}{dr} \right)^2 + \frac{p_\theta^2}{\sin^2 \theta} + E^2 a^2 \sin^2 \theta + a^2 \cos^2 \theta \right]
+ \Delta \left( \frac{dS_r}{dr} \right)^2 - (r^2 + a^2) \left( 1 + \mu \frac{r}{r \Delta} \right) E^2 - a^2 \frac{p_\theta^2}{\Delta} \frac{2 \mu a}{r \Delta} p_\phi E + r^2,
\]

\[
Q(\chi, \psi) = \left( \frac{dS_\chi}{d\chi} \right)^2 + \frac{p_\psi^2}{\sin^2 \chi},
\]
respectively. Although one of these equation $-P(r, \theta) = L^2$ still depends on the two variables, $r$ and $\theta$, it occurs the separation of variable further

$$G(\theta) = -F(r) = C^2,$$  \hspace{1cm} (44)

where $C^2$ is the constant separation, and $F(r)$, $G(\theta)$ are given by

$$F(r) = \Delta \left( \frac{dS_r}{dr} \right)^2 - (r^2 + a^2) \left(1 + \frac{\mu}{r \Delta} \right) E^2 + \frac{2 \mu a}{r \Delta} p_\phi E - \frac{a^2}{\Delta} p_\phi^2 + \frac{a^2}{r^2} L^2 + r^2, \hspace{1cm} (45)$$

$$G(\theta) = \left( \frac{dS_\theta}{d\theta} \right)^2 + \frac{p_\phi^2}{\sin^2 \theta} + \frac{L^2}{\cos^2 \theta} + E^2 a^2 \sin^2 \theta + a^2 \cos^2 \theta, \hspace{1cm} (46)$$

respectively. Note that $C^2$ is associated with the non-trivial Killing tensor in the six-dimensional singly rotating Myers-Perry geometry. The equations of $r$-motion and $\theta$-motion is written by

$$\dot{r}^2 + \tilde{V} = 0, \hspace{1cm} (47)$$

$$\dot{\theta}^2 + \tilde{W} = 0, \hspace{1cm} (48)$$

respectively, where

$$\tilde{V} = \frac{1}{\Sigma^2} \left( - (r^2 + a^2) E^2 + \frac{2 \mu a}{r} p_\phi E - a^2 p_\phi^2 + \Delta \left( \frac{a^2}{r^2} L^2 + r^2 \right) + \Delta C^2 \right), \hspace{1cm} (49)$$

$$\tilde{W} = \frac{1}{\Sigma^2} \left( \frac{p_\phi^2}{\sin^2 \theta} + \frac{L^2}{\cos^2 \theta} - C^2 + a^2 \cos^2 \theta + E^2 a^2 \sin^2 \theta \right). \hspace{1cm} (50)$$

In what follows, $\tilde{V}$ is called the one-dimensional effective potential in $r$-motion, and $\tilde{W}$ is called the one-dimensional effective potential in $\theta$-motion.

The next step is the derivation of the one-dimensional effective potential in $r$-motion restricted on the rotational axis, $\theta = 0$. The particle with the initial conditions, $\theta(0) = 0$ and $\dot{\theta}(0) = 0$, must satisfy

$$p_\phi = 0, \hspace{1cm} (51)$$

$$C^2 = L^2 + a^2. \hspace{1cm} (52)$$

These relations lead to $\dot{\theta}(0) = 0$, which means that the particle must keep moving on the rotational axis. Substitution of (51) and (52) into (49) yields

$$\tilde{V} = - (E^2 - 1) + \frac{L^2}{r^2} - \frac{\mu (r^2 + L^2)}{r^3 (r^2 + a^2)} \equiv V(r; E, L). \hspace{1cm} (53)$$

This is the one-dimensional effective potential for the particle moving on the axis $\theta = 0$. 
Let us determine the critical parameter by analyzing the extremal value problem of $V$. If $r = r_m$ is the radii of marginally stable circular orbits, the following conditions must be satisfied

\begin{align}
V(r_m; E, L) &= 0, \quad (54) \\
V'(r_m; E, L) &= 0. \quad (55)
\end{align}

These equations are solved in terms of $E$ and $L$ as $E = E_m$ and $L = L_m$, where

\begin{align}
E_m &= \frac{2 (r_m^3 + a^2 r_m - \mu)^2}{r_m \left( 2 r_m (r_m^2 + a^2)^2 - \mu (5 r_m^2 + 3 a^2) \right)}, \quad (56) \\
L_m^2 &= \frac{\mu r_m^2 (3 r_m^2 + a^2)}{2 r_m (r_m^2 + a^2)^2 - \mu (5 r_m^2 + 3 a^2)}. \quad (57)
\end{align}

In addition to (54) and (55), the marginal condition requires

\begin{equation}
V''(r_m; E_m, L_m) = \frac{2 \mu a^4 r_m + 3 a^2 (2 r_m^3 - \mu) - 3 (r_m^5 + 5 \mu r_m^2)}{r_m^3 2 r_m (r_m^2 + a^2)^3 - \mu (r_m^2 + a^2)(5 r_m^2 + 3 a^2)} = 0. \quad (58)
\end{equation}

The solution to this equation for $a$ is $a = a_m$, where $a_m$ is of the form

\begin{equation}
a_m = \sqrt{3 \mu - 6 r_m^3 + \sqrt{3 (3 \mu^2 + 8 \mu r_m^3 + 16 r_m^6)}}. \quad (59)
\end{equation}

Figure 4 shows the radii of the marginally stable circular orbits for each value of $a$, which are the innermost stable circular orbits and the outermost stable circular orbits. The result shows that $a_m$ has the minimum value $a_*$, which corresponds to the critical value of the spinning parameter as mentioned in Sec. [III]. Finally, the value of $a_*$ is exactly determined as

\begin{equation}
a_* = \left( \frac{15 \sqrt{2} \sqrt{10 - 5 \mu}}{4 \mu} \right)^{1/3}, \quad (60)
\end{equation}
FIG. 5: Region where stable bound orbits of massive particles can exist in the case $a = 2.3$ in the unit $\mu = 1$. Another branch of $\lambda_0$ is chosen, as compared with Fig. 2. The black solid line shows the location of the horizon, the gray region shows the set satisfying $\lambda_0, \mathcal{L}_0^2 \in \mathbb{R}, \text{tr} \mathcal{H}_0 > 0, U_0 < 0$. Local minimum points of $U$ exist in the gray region surrounded by the red solid line.

for which $r_m$ takes the value $r_*$ such that

\[
    r_* = \left( \frac{5 + \sqrt{10}}{4} \mu \right)^{1/3}.
\]  

(61)

The value of $a_*$ is approximated as

\[
    a_*/\mu^{1/3} = 1.628 \cdots.
\]  

(62)

This result implies that stable bound orbits exist for $a \geq a_*$ and disappear at $a < a_*$ as expected from the analysis of $U$ in the previous section.

V. STABLE BOUND ORBITS FOR MASSLESS PARTICLES

In this section let us consider stable bound orbits of massless particles in six-dimensional singly spinning Myers-Perry black holes. Although the analysis is not performed systematically as demonstrated for massive particles in Sec. III and IV, the evidence for the existence of such orbits is presented by showing the suitable shape of $U$.

Figure 5 shows the region where stable bound orbits of massive particles can exist in the case $a/\mu^{1/3} = 2.3$. Note that the other branch for $\lambda_0$ is chosen, as compared with Sec. III. It is found that the set of marginally stable bound orbits, the red line, intersects the boundary of the region satisfying $\lambda_0, \mathcal{L}_0^2 \in \mathbb{R}, \text{tr} \mathcal{H}_0 > 0$, and $U_0 < 0$, the gray region. It is worth noting that the condition $U_0 = 0$ determines this part of the boundary. Hence, massless particles can be marginally stably bounded at this point. Furthermore, near the marginally stable bound orbit of massless particles,
there can be found massless particles stably bounded inside a potential well. Indeed, Fig. 6 shows that an explicit form of $U$ in the case that a massless particle is stably bounded in a finite region. The allowed region for massless particles is restricted in the red solid closed line, and this result shows the existence of stable bound orbits of massless particles.

**VI. CONCLUSION**

In this paper, particle motion in higher-dimensional black holes has been investigated to understand fundamental properties of higher-dimensional gravity. Since higher-dimensional black holes in more than six dimensions are allowed to take arbitrarily large spin parameter, its characteristic feature of gravitational field leads to non-trivial particle dynamics. The main result of this paper is to show the existence of stable bound orbits of massive and massless particles in six-dimensional singly spinning Myers-Perry black holes. In the case that the spin parameter is larger than the critical value, massive particles can be stably bounded near rotational axis, and the region for the existence of stable bound orbits becomes finite. As special cases, there exist the innermost stable circular orbit and the outermost stable circular orbit on the rotational axis. On the other hand, there is no stable bound orbits for small value of the spin parameter. Therefore, behavior of particle motion around a six-dimensional singly rotating Myers-Perry black hole qualitatively changes at the critical value of the spin parameter $a_*$, which is determined as $a_*/\mu^{1/3} = 1.628 \cdots$. It has been
also shown that massless particles can be also stably bounded by six-dimensional singly spinning Myers-Perry black holes. Since massless particles are not stably bounded outside the horizon in four-dimensional Kerr black holes, this phenomenon in higher-dimensions is unusual.

As mentioned in Sec. II, the boundlessness of the spin parameter allows to take ultraspinning limit of a singly rotating Myers-Perry black hole in six dimensions, and then the geometry goes to a direct product spacetime of a four-dimensional Schwarzschild black hole and a two-dimensional flat space at least near rotational axis. Therefore, a particle moving near the rotational axis effectively feels gravity of the Schwarzschild black brane geometry. The appearance of stable bound orbits of massive particles is naively understood as in the same mechanism of the four-dimensional Schwarzschild black hole case. Furthermore, the appearance of stable bound orbits of massless particles is also understood in the same way because non-zero particle momentum in the residual two-dimensional flat direction leads to an effective particle mass.

In the sense of the ultraspinning limit of Myers-Perry black hole in \( D (\geq 6) \)-dimensions, limiting horizon topology is \( S^{D-2} \rightarrow \mathbb{R}^{2n} \times S^{D-2(n+1)} \), where \( n \) is the number of larger spin parameters than \( N - n \) residual spin parameters and \( N = \lfloor (D - 1)/2 \rfloor \) is the total number of spin parameters \([1]\).

In the case \( n = (D - 4)/2 \), which is satisfied only in even dimensions, the limiting horizon topology includes \( S^2 \). Hence, it is conjectured that there exist stable bound orbits of massive and massless particles at least in even-dimensional Myers-Perry black holes in more than six dimensions.

The existence of stable bound orbits for massless particles means that photons and gravitons can be stably bounded near a singly spinning Myers-Perry black hole. This leads to new physical phenomena or implies a sort of instability of the geometry. In this sense, the relation between stable bound orbits of massless particles and the Gregory-Laflamme instability \([17]\) is interesting issue. Not only particle dynamics in six-dimensional singly spinning Myers-Perry black holes but field dynamics is also interesting issue for future work.

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