Abstract. Marc Rieffel had introduced the notion of the quantum Gromov-Hausdorff distance on compact quantum metric spaces and found a sequence of matrix algebras that converges to the space of continuous functions on 2-sphere in this distance. One finds applications of similar approximations in many places in the theoretical physics literature. In this paper, we have defined a compact quantum metric space structure on the sequence of Toeplitz algebras on generalized Bergman spaces and have proved that the sequence converges to the space of continuous function on odd spheres in the quantum Gromov-Hausdorff distance.

1. Introduction

Rieffel introduced the notion of a compact quantum metric space.

Definition 1.1 ([18], Definition 1.1). Let $A$ be an order-unit space with identity element $e_A$, and let $L$ be a seminorm on $A$ taking finite values. $L$ is known as a Lip-norm if

1. $L(e_A) = 0$,
2. The topology on the state space $S(A)$ of $A$ from the metric

$$\rho_L(\mu, \nu) = \sup \{|\mu(a) - \nu(a)| : L(a) \leq 1\},$$

coincides with the weak$^*$ topology on $S(A)$.

A compact quantum metric space is a pair $(A, L)$, where $A$ is an order-unit space and $L$ is a Lip-norm $A$.

The justification for the name ‘compact quantum metric space’ comes from the fact that if $(X, d)$ is a compact metric space, then the Lipschitz seminorm $L_d$ is a Lip-norm on the space of Lipschitz functions (a dense subset of the space of continuous functions $C(X)$ on $X$). Since points of $X$ are extreme points of $S(C(X))$ and the restriction of the metric $\rho_{L_d}$ to $X$ coincide with $d$, the data $(C(X), L_d)$ is equivalent to data

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Thus, the notion of a compact metric space \((X, d)\) motivates a compact quantum metric space \((C(X), L_d)\).

Generalizing these ideas from \(C(X)\) to a non-commutative \(C^*\)-algebra, an important class of compact quantum metric spaces has been considered in literature (see [7, 22]) by virtue of a Lip-norm on a dense subset of the order-unit space of self adjoint elements of a \(C^*\)-algebra. Note that the state space of the order-unit space of self adjoint elements of a \(C^*\)-algebra and the state space of the \(C^*\)-algebra coincide. So, we can start with a seminorm \(L\) on a \(C^*\)-algebra \(\mathcal{A}\), taking finite values on a dense subset of \(\mathcal{A}\), which satisfies the properties of Definition 1.1 and \(L(a^*) = L(a)\) for all \(a \in \mathcal{A}\). A simple argument (as mentioned in section 2 of [20]) shows that \(L\), and the restriction of \(L\) to the order-unit space of self adjoint elements of \(\mathcal{A}\), determine the same metric on \(\mathcal{S}(\mathcal{A})\). In this paper, we shall be using \((\mathcal{A}, L)\) as a notation for the corresponding compact quantum metric space. Although an order-unit space is a vector space over \(\mathbb{R}\) and a \(C^*\)-algebra is a vector space over the complex field \(\mathbb{C}\), nevertheless the justification for the notation \((\mathcal{A}, L)\) for the compact quantum metric space comes from the fact above. Most of the classical examples of the subjects arise from \(C^*\)-algebras. For example, see [7] for the compact quantum metric spaces arising from a spectral triple and [21] for compact quantum metric spaces arising from ergodic strongly continuous action of a compact group on \(\mathcal{A}\) by automorphism. For more details about compact quantum metric spaces, see [18, 23].

Taking motivation from the notion of Gromov-Hausdorff distance between two compact metric spaces [9], Rieffel also introduced the notion of the quantum Gromov-Hausdorff distance between two compact quantum metric spaces in [20]. Let \((\mathcal{A}, L_A)\) and \((\mathcal{B}, L_B)\) be two compact quantum metric spaces. Let \(\mathcal{M}(L_A, L_B)\) denote the set of Lip-norms on \(\mathcal{A} \oplus \mathcal{B}\) that induce \(L_A\) and \(L_B\) on \(\mathcal{A}\) and \(\mathcal{B}\) respectively.

**Definition 1.2** ([20], Definition 4.2). The quantum Gromov-Hausdorff distance \(\text{dist}_q(\mathcal{A}, \mathcal{B})\) between \((\mathcal{A}, L_A)\) and \((\mathcal{B}, L_B)\), is defined as

\[
\text{dist}_q(\mathcal{A}, \mathcal{B}) = \inf \{ \text{dist}_{\rho_L}(\mathcal{S}(\mathcal{A}), \mathcal{S}(\mathcal{B})) : L \in \mathcal{M}(L_A, L_B) \},
\]

where \(\text{dist}_{\rho_L}(\mathcal{S}(\mathcal{A}))\) denotes the classical Gromov-Hausdorff distance between (compact subsets) \(\mathcal{S}(\mathcal{A})\) and \(\mathcal{S}(\mathcal{B})\) in the metric space \((\mathcal{S}(\mathcal{A} \oplus \mathcal{B}), \rho_L)\).

This is not just an extension of concepts of classical compact metric spaces, but these gave mathematical justification for the assertions found in theoretical physics literature which deal with string theory and related parts of quantum field theory, that the complex matrix algebras converge to two-sphere \(S^2\) (or to related spaces). This has been explored in detail by Rieffel in [22]. More examples of convergence in
the quantum Gromov-Hausdorff distance can be found in [11, 3, 11]. To understand this convergence, it is important to understand elements of $\mathcal{M}(L_A, L_B)$. In [20], Rieffel introduced the notion of bridges that we recall below.

**Definition 1.3** ([20], Definition 5.1). A bridge between $(\mathcal{A}, L_A)$ and $(\mathcal{B}, L_B)$ is a seminorm, $N$ on $\mathcal{A} \oplus \mathcal{B}$ such that

1. $N$ is continuous for the norm on $\mathcal{A} \oplus \mathcal{B}$,
2. $N(e_A, e_B) = 0$ but $N(e_A, 0) \neq 0$,
3. For any $a \in \mathcal{A}$ and $\delta > 0$, there is an element $b \in \mathcal{B}$ such that
   \[ \max \{ L_B(b), N(a, b) \} \leq L_A(a) + \delta, \]
   and similarly for $\mathcal{A}$ and $\mathcal{B}$ interchanged.

It was proved in Theorem 5.2 of [20] that if $N$ is a bridge between $(\mathcal{A}, L_A)$ and $(\mathcal{B}, L_B)$ and $L$ is defined as $L(a, b) = \max \{ L_A(a), L_B(b), N(a, b) \}$, then $L \in \mathcal{M}(L_A, L_B)$. In [22], Rieffel found a sequence of complex matrix algebras (arising from finite-dimensional representations of $SU(2)$) converging to the space of continuous complex valued functions on the two-sphere $S^2$. In this paper, we have found a compact quantum metric space structure on the Toeplitz algebras on generalized Bergman spaces on the closed unit ball in $\mathbb{C}^d$ such that a sequence of these algebras converges to the space of continuous functions on the sphere in $\mathbb{R}^{2d}$, denoted by $C(S^{2d-1})$, in the quantum Gromov-Hausdorff distance.

Motivated by the quantum Gromov-Hausdorff distance, various notions of convergence have been introduced in the literature of noncommutative geometry. Order-unit quantum Gromov–Hausdorff distance was introduced by Li in [16]. The notion of Gromov-Hausdorff propinquity was introduced by Latrémolière, see [2, 12, 15]. For the notion of the quantum Gromov-Hausdorff propinquity, see [14]. The notion of Gromov-Hausdorff propinquity for metric spectral triples can be found in [13]. The notion of Gromov-Hausdorff convergence of state spaces for spectral truncations is of interest to mathematicians, see [19, 24]. For a general setup of spectral truncations in noncommutative geometry and operator systems, see [8]. For operator systems with Lip-norm, even a matricial version of the quantum Gromov-Hasudorff distance can be defined, see [10]. The convergence of the particular sequence obtained in this paper can also be described in terms of the matricial quantum Gromov-Hausdorff distance which we shall remark on in the last section.

In Section 2, we explain the space of Toeplitz algebras on the generalized Bergman space and describe the compact quantum metric space
structure on these spaces. In Section 3, we prove our main theorem that the sequence of Toeplitz algebras converges to odd spheres. In Section 4, a few remarks are mentioned.

2. Toeplitz algebras on generalized Bergman spaces

Let \( d \in \mathbb{N} \) be fixed. Let \( B^{2d} \) denote the open unit ball in \( \mathbb{C}^d \). For all \( n \geq d \), let \( dV_n \) denote the volume measure on \( B^{2d} \) given by

\[
dV_n = c_n (1 - |z|^2)^{n-d} dV,
\]

where \( dV \) denotes the Lebesgue measure and \( c_n = \frac{n!}{(n-d)! \pi^d} \) is a normalizing constant so that \( dV_n \) is a probability measure on \( B^{2d} \).

We consider \( H_n = \left\{ f : B^{2d} \to \mathbb{C} : f \text{ is analytic and } \int_{B^{2d}} |f|^2 dV_n < \infty \right\} \). Then \( H_n \) with the inner product \( \langle f | g \rangle_{H_n} = \int_{B^{2d}} f \overline{g} dV_n \) is the reproducing kernel Hilbert space with the kernel function

\[
K_n(z, w) = \frac{1}{(1 - \langle z | w \rangle)^{n+1}} \quad \text{for all } z, w \in \mathbb{C}^d.
\]

For \( n = d \), the space \( H_n \) is known as the Bergman space and for all \( n > d \), we call it a generalized Bergman space. For more details, see Chapter 2 of [25]. For each \( n \geq d \), an orthonormal basis for \( H_n \) is given by \( (e_{k,n}) \), where

\[
e_{k,n} = \left( \frac{(|k| + n)!}{k! \ n!} \right)^{1/2} z^k,
\]

where \( k = (k_1, \ldots, k_d) \) is an \( d \)-tuple of non-negative integers and we take \( |k| = k_1 + \cdots + k_d \), \( k! = k_1! \cdots k_d! \), \( z^k = z_1^{k_1} \cdots z_d^{k_d} \).

Let \( \bar{B}^{2d} \) denote the closed unit ball in \( \mathbb{C}^d \). For \( \phi \in C(\bar{B}^{2d}) \), we define the Toeplitz operator \( T_{\phi,n} : H_n \to H_n \) as

\[
T_{\phi,n}(f) = P_n(\phi f),
\]

where \( P_n : L^2(B^{2d}, dV_n) \to H_n \) is the orthogonal projection.

Let \( T_n \) be the \( C^* \)-subalgebra of operators on \( H_n \) generated by \( \{ T_{\phi,n} : \phi \in C(\bar{B}^{2d}) \} \). In Theorem 1 of [6], it was proved that the Toeplitz algebra \( T_d \) contains the space of compact operators \( \mathcal{K}(H_d) \)

\[
T_d = \{ T_{\phi,d} + K : \phi \in C(\bar{B}^{2d}) \text{ and } K \in \mathcal{K}(H_d) \}.
\]

The quotient algebra \( T_d / \mathcal{K}(H_d) \) is \( C^* \)-isomorphic to \( C(S^{2d-1}) \) with the isomorphism given by

\[
T_{\phi,d} + K \mapsto \phi|_{S^{2d-1}} \quad \text{for all } \phi \in C(\bar{B}^{2d}), K \in \mathcal{K}(H_d).
\]

Using techniques used in [6], we prove that the same holds for \( T_n \).
Theorem 2.1. For all $n \geq d$, we have
\[ T_n = \{ T_{\phi,n} + K : \phi \in C(\tilde{B}^{2d}) \text{ and } K \in \mathcal{K}(\mathcal{H}_n) \}, \]
and the quotient $C^*$-algebra $T_n/\mathcal{K}(\mathcal{H}_n)$ is $C^*$-isomorphic to $C(S^{2d-1})$ with the isomorphism given by
\[ T_{\phi,n} + K \overset{\sim}{\rightarrow} \phi|_{S^{2d-1}} \text{ for all } f \in C(\tilde{B}^{2d}), K \in \mathcal{K}(\mathcal{H}_n). \]
So, we have the existence of the following short exact sequence:
\[ 0 \rightarrow \mathcal{K}(\mathcal{H}_n) \overset{i}{\rightarrow} T_n \overset{\pi}{\rightarrow} C(S^{2d-1}) \rightarrow 0, \]
where $i$ denotes the inclusion map.

Proof. For $n = d$, the theorem was proved in Theorem 1 of [6]. Let $n > d$ be fixed and let $U : \mathcal{H}_n \rightarrow \mathcal{H}_d$ be the unitary transformation given by $U(e_{k,n}) = e_{k,d}$. Along the lines of the proof of Lemma 3 of [6], we have that $U^*T_{\phi,d}U - T_{\phi,n}$ is a compact operator. It follows that the commutator ideal $C_n$ of $T_n$ is contained in $\mathcal{K}(\mathcal{H}_n)$. Along the lines of the proof of Lemma 1 of [6], $T_n$ is an irreducible $C^*$-algebra.

By virtue of the short exact sequence (1) in Theorem 2.1, we define a compact quantum metric space structure on $T_n$ for all $n \geq d$ using Theorem 3.4 of [5]. Let $n \geq d$ be fixed. Let $(e_j)_{j \geq 1}$ be an enumeration of the orthonormal basis of $\mathcal{H}_n$ mentioned above. We equip $\mathcal{K}(\mathcal{H}_n)$ with the compact quantum metric space structure $(\text{Lip}(\mathcal{K}(\mathcal{H}_n)) \oplus \mathbb{R}I, \tilde{L}_n)$ given by
\[ \text{Lip}(\mathcal{K}(\mathcal{H}_n)) = \{ T \in \mathcal{K}(\mathcal{H}_n) : T^* = T, \langle Te_i | e_j \rangle_{\mathcal{H}_n} \in \mathbb{R} \text{ for all } i, j \geq 1 \}
\text{and } \sup_{i,j \geq 1} (i + j)^{n+2} |\langle Te_i | e_j \rangle_{\mathcal{H}_n}| < \infty, \]
with $\tilde{L}_n(T) = \sup_{i,j \geq 1} (i + j)^{n+2} |\langle Te_i | e_j \rangle_{\mathcal{H}_n}|$ for all $T \in \text{Lip}(\mathcal{K}(\mathcal{H}_n))$. By Theorem 3.4 of [5], $(\text{Lip}(\mathcal{K}(\mathcal{H}_n)) \oplus \mathbb{R}I, \tilde{L}_n)$ is a compact quantum metric space. Now, we consider a fixed positive linear splitting $\sigma : C(S^{2d-1}) \rightarrow T_n$ of the short exact sequence (1), for example, $f \in C(S^{2d-1})$ is mapped to $T_f$ where $f$ is the unique solution of the Dirichlet’s problem. Then using Theorem 3.6 of [5], we have the following compact quantum compact space structure on $T_n$. 

\[ \]
Theorem 2.2. The space of Toeplitz operators on the generalized Bergman space $T_n$ has the compact quantum space structure $(\text{Lip}(T_n), L_n)$ where $\text{Lip}(T_n) = \text{Lip}(\mathcal{K}(H_n)) \oplus \{T_{\sigma(f)} : f \text{ real valued Lipschitz function on } S^{2d-1}\}$, and $L_n(T_f + K) = \tilde{L}_n(K) + L(f|_{S^{2d-1}})$ for all $T_f + K \in \text{Lip}(T_n)$ and $L(f|_{S^{2d-1}})$ is the Lipschitz norm of $f|_{S^{2d-1}}$.

Now we claim that $T_n$ converges to $C(S^{2d-1})$ with the compact quantum structure of real valued Lipschitz functions with the Lipschitz norm in the quantum Gromov-Hausdorff distance.

3. MAIN THEOREM

Now, we prove our main theorem. The idea of the proof is the same as that used by Rieffel in [22].

Theorem 3.1. The sequence of compact quantum metric spaces $(\text{Lip}(T_n), L_n)$ converges to $C(S^{2d-1})$ in the quantum Gromov-Hausdorff distance.

The following lemma is useful.

Lemma 3.2. For all $n \geq d$ and for all $T \in T_n$, we have

$$\|T - \sigma(\pi(T))\| \leq \gamma_n L_n(T),$$

where $\gamma_n$ is a decreasing sequence converging to 0.

Proof. Let $T = T_{\sigma(f)} + K \in T_n$ for some $K \in \text{Lip}(\mathcal{K}(H_n))$ and a real valued Lipschitz function $f$ on $C(S^{2d-1})$. Then we have,

$$\|T - \sigma(\pi(T))\| = \|K\| \leq \sup_{i \geq 1} \sum_{i \geq 1} |\langle T e_i | e_j \rangle_{H_n}|,$$

$$\leq \tilde{L}_n(K) \sup_{i \geq 1} \sum_{i \geq 1} (i + j)^{-n-2},$$

$$\leq \tilde{L}_n(K) \sum_{i \geq 1} (i + 1)^{-n-2},$$

$$\leq L_n(T)(\zeta(n + 2) - 1),$$

where $\zeta$ denotes the Riemann-Zeta function. The proof is completed by taking $\gamma_n = \zeta(n + 2) - 1$. \hfill \Box

Proof of Theorem 3.1. Let $\varepsilon > 0$. Let $n_0 \in \mathbb{N}$ be such that $\gamma_n \leq \varepsilon / 2$ for $n \geq n_0$.

For all $T \in T_n$ and $f \in C(S^{2d-1})$, we define $N(T, f) = \gamma_n^{-1} \|\pi(T) - f\|$ and

$$\tilde{L}(T, f) = \max\{L_n(T), L(f), N(T, f)\}.$$
Since \( L_n(\sigma(f)) = L(f) \) and \( N(\sigma(f), f) = 0 \), we have
\[
\tilde{L}(\sigma(f), f) = L(f).
\]
Also, \( L(\pi(T)) \leq L_n(T) \) and \( N(T, \pi(T)) = 0 \). Hence,
\[
\tilde{L}(T, \pi(T)) = L_n(T).
\]
So, \( N \) is bridge between the compact quantum metric spaces \( T_n \) and \( C(S^{2d-1}) \). By Theorem 5.2 of [22], \( L \) is a Lip-norm on \( T_n \oplus C(S^{2d-1}) \) that induces the previously defined Lip-norms on the compact quantum metric spaces \( T_n \) and \( C(S^{2d-1}) \).

By Proposition 1.3 of [22], we know that the state space of \( C(S^{2d-1}) \), denoted by \( \mathcal{S}_{C(S^{2d-1})} \), is contained in the \( \gamma_{n_0} \) neighbourhood of \( \mathcal{S}_{T_n} \) for \( \rho_L \). So, \( \mathcal{S}_{C(S^{2d-1})} \) is in the \( \varepsilon/2 \) neighbourhood of \( \mathcal{S}_{T_n} \) for \( \rho_L \).

Let \( T \in T_n \) and \( f \in C(S^{2d-1}) \) such that \( \tilde{L}(T, f) \leq 1 \) and \( \nu \in \mathcal{S}_{T_n} \). We consider \( \mu = \nu \circ \sigma \in \mathcal{S}_{C(S^{2d-1})} \). For all \( n \geq n_0 \), we have
\[
|\mu(T, f) - \nu(T, f)| = |\nu(T - \sigma(f))| \\
\leq ||T - \sigma(\pi(T))|| + ||\sigma(\pi(T)) - \sigma(f)|| \\
\leq ||T - \sigma(\pi(T))|| + ||\pi(T) - f|| \\
\leq 2\gamma_{n_0}.
\]
The last inequality uses Lemma 3.2 and the fact that \( \tilde{L}(T, f) \leq 1 \) (this implies that \( ||\pi(T)) - f|| \leq \gamma_n \) and \( L_n(T) \leq 1 \)). Hence for all \( n \geq n_0 \), we have that \( \mathcal{S}_{T_n} \) is contained in the \( \varepsilon \) neighbourhood of \( \mathcal{S}_{C(S^{2d-1})} \) for \( \rho_L \).

Hence, we get that the quantum Gromov-Hausdorff distance between the compact quantum metric spaces on \( T_n \) and \( C(S^{2d-1}) \) is less than or equal to \( \varepsilon \) for all \( n \geq n_0 \), that is, \( T_n \) converges to \( C(S^{2d-1}) \) in the quantum Gromov-Hausdorff distance.

\[\square\]

4. Remarks

Remark 4.1. Since \( T_n \) and \( C(S^{2d-1}) \) are unital \( C^* \)-algebras, these are also examples of Lip-normed unital \( C^* \)-algebras with the compact quantum metric space structures given in Theorem 2.2. Using Theorem 3.9 and Theorem 3.11 of [17], the maps \( \pi \) and \( \sigma \) are completely positive maps. Using arguments similar to the ones in Example 3.12 of [10], Lemma 3.2 holds at each matrix level and we get that \( T_n \) converges to \( C(S^{2d-1}) \) in the complete distance also, as defined in [10].

Remark 4.2. With the same methods as in the proof of Theorem 3.1, we can get a more general result. Let \( \mathcal{A}_n \) be a sequence of separable Hilbert spaces. Let \( \mathcal{A} \) be a \( C^* \)-algebra with a compact quantum metric...
space structure. If $\mathcal{B}_n$ be a sequence of $C^*$-algebras such that there exists a split exact sequence of $C^*$-algebra isomorphisms

$$0 \to \mathcal{K}(\mathcal{H}_n) \to \mathcal{B}_n \to \mathcal{A},$$

then there exists a compact quantum metric space structure on $\mathcal{B}_n$ such that $\mathcal{B}_n$ converges to $\mathcal{A}$ in quantum Gromov-Hausdorff distance.

**Remark 4.3.** For each $\alpha \in [d, \infty)$, we can consider the volume measure $dV_\alpha$ on $B^{2d}$ given by

$$dV_\alpha = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha - d + 1)\pi^d} (1 - |z|)^{\alpha - d} dV.$$

Then the space $\mathcal{H}_\alpha$ defined as the set of all analytic functions on $B^{2d}$ which are square-integrable with respect to $dV_\alpha$ and the Toeplitz algebra $T_\alpha$ on $\mathcal{H}_\alpha$ are well defined. Since arguments in this paper and [5] work when natural numbers are replaced by a positive real number, we get the following extension of Theorem [3.1]: For $\alpha \in [d, \infty)$, the net of compact quantum metric spaces $(\text{Lip}(T_\alpha), L_\alpha)$ converges to $C(S^{2d-1})$ in the quantum Gromov-Hausdorff distance.

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