A set of exactly solvable Ising models with half-odd-integer spin

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Abstract

We present a set of exactly solvable Ising models, with half-odd-integer spin-S on a square-type lattice including a quartic interaction term in the Hamiltonian. The particular properties of the mixed lattice, associated with mixed half-odd-integer spin-(S,1/2) and only nearest-neighbor interaction, allow us to map this system either onto a purely spin-1/2 lattice or onto a purely spin-S lattice. By imposing the condition that the mixed half-odd-integer spin-(S,1/2) lattice must have an exact solution, we found a set of exact solutions that satisfy the free fermion condition of the eight vertex model. The number of solutions for a general half-odd-integer spin-S is given by \( S + \frac{1}{2} \). Therefore we conclude that this transformation is equivalent to a simple spin transformation which is independent of the coordination number.

Keywords: Mathematical physics, two dimensional Ising model, exact results.

Two-dimensional lattice models are one of the most interesting subjects of statistical mechanics, both experimentally[1,2] and theoretically. Several approximation methods are used to investigate these models on the lattice, such as mean-field theory[1,3], the Bethe approximation[4], the correlated effective field theory[5], the renormalization group[6], series expansion methods[7], Monte Carlo methods[8] and cluster variation methods. Following Onsager’s[9] solution for the square two dimensional Ising lattice, other solutions for regular

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two-dimensional lattices have been considered, such as triangular[10][11], honeycomb[12][13], kagome[14] lattices and others were explored in several works and their importance in statistical physics has waked up the search for a set of completely solvable models. Some other exact results have been obtained with restricted parameters, as investigated by Mi and Yang[16] using a non-one-to-one transformation[15].

Some half-odd-integer spin-S Ising models were already discussed in the literature[17]. Using the method proposed by Wu[18], Izmailian[19] obtained an exact solution for a spin-3/2 square lattice with only nearest-neighbor and two-body spin interactions. Izmailian and Ananikian[20] have also obtained an exact solution for a honeycomb lattice with spin-3/2. A particular case of solutions of these models were obtained by the method proposed by Joseph[21], where any spin-S could be decomposed in terms of spin-1/2. Another interesting method for mapping the spin-S lattice into a spin-1/2 lattice has been proposed by Horiguchi[22].

Mapping between models is an important tool for the study of exactly solvable models. The aim of this letter is to present the transformation of higher half-odd-integer spin-S systems into a simple spin-1/2 system.

To demonstrate this transformation we use a two dimensional mixed spin-(S,1/2) on a square Ising lattice with a quartic interaction $L_b$ as displayed in fig.1. First this mixed model can be mapped onto an exactly solvable two dimensional spin-1/2 Ising model with a quartic interaction $L_a$, such as presented in the literature[19]. Second it is also possible to transform a mixed spin lattice $L_o$ onto an effective spin-S lattice when we consider the spin-1/2 as a decorated Ising model of the lattice $L_b$. Then the classical Hamiltonian for a mixed spin-(3/2,1/2) lattice is given by

$$H_{1/2,3/2} = \sum_{<i,j>} (K_r^{(1)} S_i \sigma_j + K_r^{(3)} S_i^3 \sigma_j) + \sum_i D S_i^2,$$

with $<i,j>$ meaning summation over nearest interacting neighbors on the square lattice, and the last summation is performed over all spin-3/2 sites. The coefficient $K_r^{(1)}$ is the nearest-neighbor interaction parameter of the bilinear term; $K_r^{(3)}$ corresponds to the parameter of the non-bilinear interaction, where $r$ runs from 1 up to the coordination number (in our case up to 4); $D$ is the single ion-anisotropy parameter acting on spin-3/2; $S_i$ represents the spin-3/2 particle; whereas $\sigma_i$ corresponds the spin-1/2 particle, with two possible values $\pm 1$ (we use these values conveniently instead of $\pm 1/2$ for all spin $\sigma_i$).

First we would like to obtain an exactly solvable spin-(3/2,1/2) lattice $L_b$ (see fig.1); to this end, we write...
the first term of the Hamiltonian (1) as \((K^{(1)}_r S_i + K^{(2)}_r S^3_i)\sigma_j\), then for spin-3/2 we have four possible values ±1/2 and ±3/2, thus the term \(K^{(1)}_r S_i + K^{(2)}_r S^3_i\) can be written

\[
A^{(1)}_r = \pm \frac{1}{2} K^{(1)}_r \pm \frac{1}{8} K^{(3)}_r, \quad S_i = \pm 1/2,
\]

\[
A^{(2)}_r = \pm \frac{3}{2} K^{(1)}_r \pm \frac{27}{8} K^{(3)}_r, \quad S_i = \pm 3/2,
\]

with \(r = 1, \ldots, 4\).

In order to project the spin-3/2 onto spin \(\sigma\) with only two possible values ±1, we impose the condition \(|A^{(1)}_r| = |A^{(2)}_r| = A_r\). Therefore we are able to find the parameters \(K^{(1)}_r\) and \(K^{(3)}_r\) as a function of \(A_r\), by solving the system (2) and (3), we obtain following

\[
\begin{align*}
K^{(3)}_r &= -\frac{4}{3} A_r; & K^{(1)}_r &= \frac{7}{3} A_r \\
\text{or} & \quad & \text{in which} & \quad r = 1, \ldots, 4. \\
K^{(3)}_r &= -\frac{2}{3} A_r; & K^{(1)}_r &= -\frac{13}{6} A_r
\end{align*}
\]

The associated Boltzmann weight for a mixed spin-(3/2,1/2) lattice has a similar structure to that of the model discussed by Wu and Lin[24], the associated Boltzmann weights are given by \(W(\{\sigma_r\}) = \sum_3 \exp(H_{1/2,3/2})\). For simplicity we consider our calculation in units of \(-\beta\). Using the solution given by (4) the associated Boltzmann weights are simplified, which read as

\[
\begin{align*}
w_1 &= W(++,+,+) = \alpha \cosh(A_1 + A_2 + A_3 + A_4), \\
w_2 &= W(++,+,--) = \alpha \cosh(A_1 - A_2 + A_3 - A_4), \\
w_3 &= W(++,--,+) = \alpha \cosh(A_1 - A_2 - A_3 + A_4), \\
w_4 &= W(++,--,--) = \alpha \cosh(A_1 + A_2 - A_3 - A_4), \\
w_5 &= W(+-,+,+) = \alpha \cosh(A_1 - A_2 + A_3 + A_4), \\
w_6 &= W(+-,+,--)= \alpha \cosh(A_1 + A_2 + A_3 - A_4), \\
w_7 &= W(+-,--,+) = \alpha \cosh(A_1 + A_2 - A_3 + A_4), \\
w_8 &= W(-,++,+) = \alpha \cosh(-A_1 + A_2 + A_3 + A_4),
\end{align*}
\]
where $\alpha = 2(e^{i\Phi} + e^{i\Phi'})$.

The lattice $\mathcal{L}_b$ can be transformed into an effective spin-1/2 lattice $\mathcal{L}_a$, as shown in fig. 1. Then at each site of $\mathcal{L}_a$ there can be eight lines; we are thus led to consider an eight-vertex model $\mathcal{L}_a$ with weights given by (5) and which satisfies the free fermion condition [25],

\[
\omega_1 \omega_2 + \omega_3 \omega_4 = \omega_5 \omega_6 + \omega_7 \omega_8.
\]  

Consider the following Hamiltonian,

\[
H_{1/2} = J_0 N + \sum_{(i,j)} J_{i,j} \sigma_i \sigma_j + \sum_{\text{all tetrahedron}} J_{1,2,3,4} \sigma_{i_1} \sigma_{i_2} \sigma_{i_3} \sigma_{i_4}
\]

for the effective spin-1/2 Ising model $\mathcal{L}_a$, which is exactly solvable using the eight-vertex model [20]. Here, $(i, j)$ means summation over all pairs of sites over the tetrahedron (the boldface tetrahedron in Fig. 1), and $J_{i,j}$ are their corresponding interacting parameters, whereas $J_{1,2,3,4}$ is the quartic interaction parameter; $N$ represents the number of spin-3/2 sites on the lattice.

After transforming the Hamiltonian (1) into (7), we relate their parameters using the Boltzmann weights, thus to obtain

\[
J_0 = \ln \left(2\alpha\right) + \frac{1}{8} \ln \left(\frac{w_1 w_2 w_3 w_4 w_5 w_6 w_7 w_8}{w_6 w_7 w_2 w_3}\right)
\]

\[
J_{1,2} = \frac{1}{8} \ln \left(\frac{w_1 w_4 w_5 w_8}{w_6 w_7 w_2 w_3}\right)
\]

\[
J_{2,3} = \frac{1}{8} \ln \left(\frac{w_1 w_8 w_3 w_6}{w_7 w_4 w_5 w_2}\right)
\]

\[
J_{3,4} = \frac{1}{8} \ln \left(\frac{w_1 w_6 w_7 w_4}{w_5 w_2 w_9 w_8}\right)
\]

\[
J_{4,1} = \frac{1}{8} \ln \left(\frac{w_1 w_7 w_5 w_3}{w_8 w_4 w_2 w_6}\right)
\]

\[
J_{1,3} = \frac{1}{8} \ln \left(\frac{w_1 w_7 w_2 w_8}{w_6 w_4 w_5 w_3}\right)
\]

\[
J_{2,4} = \frac{1}{8} \ln \left(\frac{w_1 w_6 w_5 w_2}{w_7 w_4 w_3 w_8}\right)
\]

\[
J_{1,2,3,4} = \frac{1}{8} \ln \left(\frac{w_1 w_4 w_2 w_3}{w_8 w_7 w_5 w_6}\right).
\]

The partition function $Z$ of the decorated (or mixed) model in the thermodynamic limit is related to the partition function $Z_{8v}$ of the effective eight-vertex spin-1/2 model (7) by the expression $Z = e^{-\beta J_0} Z_{8v}$. An
analytical expression for the free energy of the free fermion model is well known[25] and after some manipulation, was expressed in the thermodynamic limit by

$$f = J_0 + \frac{1}{16\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\theta d\phi \ln[2a + 2b \cos(\theta) + 2c \cos(\phi) + 2d \cos(\theta - \phi) + 2e \cos(\theta + \phi)],$$

(16)

where

$$a = \frac{1}{2}(w_1^2 + w_2^2 + w_3^2 + w_4^2), \quad b = w_1 w_3 - w_2 w_4,$$

$$c = w_1 w_4 - w_2 w_3, \quad d = w_3 w_4 - w_7 w_8, \quad e = w_3 w_4 - w_5 w_6.$$  

(17)

The system exhibits an Ising transition at the critical points

$$w_1 + w_2 + w_3 + w_4 = 2 \max\{w_1, w_2, w_3, w_4\}.$$  

(18)

At low temperature the system exhibits ordered states such as the ferromagnetic state (for \(w_1 > w_2, w_3, w_4\)), the antiferromagnetic state (for \(w_2 > w_1, w_3, w_4\)) and the metamagnetic state (for \(w_3 \text{ or } w_4 > w_1, w_2\)).

Thus our goal is to transform an exactly solvable Ising model with spin-1/2 via the eight-vertex model[26] into an equivalent Ising model on the lattice with spin-3/2. For this purpose we introduce an auxiliary lattice \(\mathcal{L}_b\), with mixed spin-1/2 and spin-3/2, the schematic transformation of which is displayed in fig[4]. If we consider the spin-1/2 model as a decoration of the mixed spin model and transform it into an equivalent spin-3/2 Ising model on lattice, then we can conclude that there exist a transformation from the spin-1/2 Ising model onto the spin-3/2 Ising model, with a non-bilinear interaction and four-body interactions terms over the tetrahedron or quartic interaction.

Using the first of solution of Eq.(4), we express the Hamiltonian of the transformed spin-3/2 lattice \(\mathcal{L}_c\) in terms of four constrained parameters and one arbitrary parameter \(D\), thus obtaining

$$\mathcal{H}_{3/2} = J_0 N + \sum_i DS_i^2 + \sum_{(i,j)} J_{i,j} \left( \frac{49}{9} S_i S_j - \frac{28}{9} (S_i^3 S_j^3 + S_i^3 S_j^3) + \frac{16}{9} S_i^3 S_j^3 \right) + \sum_{\text{all tetrahedron}} J_{1,2,3,4} \left( \frac{2401}{81} S_i S_j S_k S_l + \frac{1372}{81} S_i^3 S_j S_k S_l + \frac{784}{81} S_i^3 S_j^3 S_k S_l - \frac{448}{81} S_i S_j S_k^3 S_l^3 + \frac{256}{81} S_i^3 S_j S_k S_l^3 S_l^3 \right),$$

(19)

where the parameters \(J_{i,j}\) were already defined by Eqs.(8)-(15), and these parameters are constrained. Whereas the parameters \(A_1, A_2, A_3\) and \(A_4\) are free, as well as the single ion anisotropy parameter \(D\). Summations are performed as indicated in [4].
By factorizing the terms under the summation sign, we obtain

\[ H_{3/2} = J_0 N + \sum_i DS_i^2 + \sum_{(i,j)} \left( J_{i,j} \frac{S_i S_j}{9} (7 - 4S_i^2)(7 - 4S_j^2) \right) + \sum_{\text{all tetrahedron}} J_{1,2,3,4} \frac{S_{i_1} S_{i_2} S_{i_3} S_{i_4}}{81} (7 - 4S_{i_1}^2)(7 - 4S_{i_2}^2)(7 - 4S_{i_3}^2)(7 - 4S_{i_4}^2). \] (20)

We can reduce the Hamiltonian (20) further for both solutions in Eq. (4), using the spin transformation \( \sigma^{(k)}(S) \), where this function only take two possible values \( \pm 1 \), for all values of \( S = \{-3/2, -1/2, 1/2, 3/2\} \). These spin transformations read as

\[ \sigma^{(k)}(S) = \begin{cases} \frac{S}{3}(7 - 4S^2); & k = 1 \\ \frac{S}{3}(13 - 4S^2); & k = 2 \end{cases} \] (21)

We remark that the transformation given by (21) are the same as those obtained by Izmailian[19] using the method proposed by Wu[18] (but for another model, the spin-3/2 Ising model on a square lattice). As we can see, the previous transformation is independent of the coordination number. Therefore it may be used to yield a particular case of solution, for the exactly solvable spin-3/2 Ising model in a honeycomb lattice as was obtained by Izmailian and Ananikian[20].

We are now able to write the Hamiltonian (20) as

\[ H_{3/2}^{(k)} = J_0 N + \sum_i DS_i^2 + \sum_{(i,j)} J_{i,j} \sigma_i^{(k)} \sigma_j^{(k)} + \sum_{\text{all tetrahedron}} J_{1,2,3,4} \sigma_{i_1}^{(k)} \sigma_{i_2}^{(k)} \sigma_{i_3}^{(k)} \sigma_{i_4}^{(k)}. \] (22)

The second term of the Hamiltonian (22) may be written using a new spin variable transformation, similar to that performed by Izmailian[19]. Note that there the transformation was performed only for spin-3/2 case and for higher spin that method could become a more complex task.

We also remark that the model discussed by Izmailian[19] can be completely re-obtained, by using our simple transformation instead of Wu’s[18] method.

The Hamiltonian (1) can be extended to arbitrary half-odd-integer spin-(S,1/2); the general Hamiltonian has the following form

\[ H_{1/2,S} = \sum_{<i,j>} \left( K_i^{1}(S_i S_j^3 + K_i^{3}(S_i^5) \cdots + K_i^{(S+\frac{1}{2})}(S_i^{S+\frac{1}{2}}) \sigma_j + \sum_i DS_i^2, \right. \] (23)
where $K_r^{(1)}$ is the bilinear interaction parameter, and $K_r^{(u)}$ are non-bilinear interaction parameters of $S_u\sigma_j$, with $u = 2, ..., S/2 + 1$ and $r = 1, ..., 4$, thus the set of transformations for each half-odd-integer spin-S have $S + 1/2$ unknown parameters to be determined. We extend eq.(4) to the general case (23) by constructing a Vandermonde-like matrix, explicitly

$$
\begin{pmatrix}
\frac{1}{2} & \left(\frac{1}{2}\right)^3 & \left(\frac{1}{2}\right)^5 & \ldots & \left(\frac{1}{2}\right)^{2S} \\
\frac{1}{2} & \left(\frac{1}{2}\right)^3 & \left(\frac{1}{2}\right)^5 & \ldots & \left(\frac{1}{2}\right)^{2S} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
S & S^3 & S^5 & \ldots & S^{2S}
\end{pmatrix}
\begin{pmatrix}
K_r^{(1)} \\
K_r^{(3)} \\
\vdots \\
K_r^{(S+\frac{1}{2})}
\end{pmatrix}
= 
\begin{pmatrix}
A_r^{(1)} \\
A_r^{(2)} \\
\vdots \\
A_r^{(S+\frac{1}{2})}
\end{pmatrix},
$$

(24)

where the parameters $A_r^{(u)}$ must satisfy the following identities $|A_r^{(1)}| = |A_r^{(2)}| = \cdots = |A_r^{(S+\frac{1}{2})}| = A_r$, in order to project the spin-S onto spin-1/2. By inverting the Vandermonde-like matrix, we are able to obtain the solution of the algebraic system. In what follows, we show the solutions of these system of linear equations for some higher spin-S values.

For $S = 5/2$, there are three solutions:

$$
\sigma^{(k)}(S) = \begin{cases}
\frac{1067}{480} S - \frac{11}{12} S^3 + \frac{1}{10} S^5; & k = 1 \\
\frac{529}{240} S - \frac{5}{6} S^3 + \frac{1}{15} S^5; & k = 2 \\
\frac{1183}{480} S - \frac{23}{12} S^3 + \frac{7}{30} S^5; & k = 3.
\end{cases}
$$

(25)

For $S = 7/2$, there are four solutions:

$$
\sigma^{(k)}(S) = \begin{cases}
\frac{30251}{13710} S - \frac{301}{258} S^3 + \frac{61}{969} S^5 - \frac{1}{126} S^7; & k = 1 \\
\frac{60577}{20880} S - \frac{3047}{2580} S^3 + \frac{127}{1260} S^5 - \frac{11}{1260} S^7; & k = 2 \\
\frac{14887}{16720} S - \frac{637}{720} S^3 + \frac{37}{1350} S^5 - \frac{1}{315} S^7; & k = 3 \\
\frac{68123}{26880} S - \frac{1289}{376} S^3 + \frac{293}{720} S^5 - \frac{5}{252} S^7; & k = 4.
\end{cases}
$$

(26)
For $S = 9/2$, we obtain five solutions:

$$\sigma^{(k)}(S) = \begin{cases} 
5851067 & 2580480 \ S^{-46573} S^3 + 7501_{41956} S^5 - 97_{6048} S^7 + \frac{1}{2592} S^9; & k = 1 \\
2924921 & 1290240 \ S^{-813413} S^3 + 3727_{17280} S^5 - 393_{6048} S^7 + \frac{97}{45360} S^9; & k = 2 \\
5868067 & 230400 \ S^{-334907} S^3 + 8117_{34560} S^5 - 113_{6048} S^7 + \frac{43}{90720} S^9; & k = 3 \\
5725183 & 230400 \ S^{-47337} S^3 + 3593_{34560} S^5 - \frac{29}{6048} S^7 + \frac{1}{12960} S^9; & k = 4 \\
6651283 & 230400 \ S^{-506017} S^3 + 18341_{34560} S^5 - \frac{1237}{30240} S^7 + \frac{13}{12960} S^9; & k = 5.
\end{cases}$$

(27)

Using the spin transformations given by eqs. (21), (25), (26) and (27), we are able to recover the transformation obtained by Joseph\cite{21} by considering $k = 1$, whereas the remaining transformation are new.

In this report we present a transformation which maps a spin-3/2 lattice, with quartic and non-linear interactions terms, onto an effective spin-1/2 Ising model on lattice. First this transformation was carried out using an auxiliary mixed spin-(3/2,1/2) square lattice with only nearest neighbor interaction term, mapped onto an effective spin-1/2 or spin-3/2 lattice model, depending on which spin is considered to be the decoration spin. Finally, a systematic way of transformation for higher half-odd-integer spin-S is considered, inverting a Vandermonde like matrix, to obtain a families of mapping between spin-S models and spin-1/2 models, using the one-to-one transformation. Therefore we conclude that there exist a spin transformation, which can be applied to lattice models with arbitrary coordination number, including non-exactly solvable half-odd-integer spin-S models. We also recovered some results previously obtained in the literature\cite{19,20}, as a particular case of solution.

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References

[1] M. Blume, V. J. Emery, R. B. Griffiths, Phys. Rev. A 3, 1071 (1971).

[2] J. Bernasconi and F. Rys, Phys. Rev. B 4, 3045 (1982).

[3] H. W. Capel, Physica 32, 966 (1966); D. Mukamel and M. Blume, Phys. Rev. A 10 61, (1974).

[4] K. G. Chakraborty and T. Morita, Physica A 129, 415 (1985).
[5] T. Kaneyoshi, Physica A 164, 730 (1976).

[6] S. Krinsky and D. Furman, Phys. Rev. B 11, 2602 (1975); A. N. Berker and M. Wortis, Phys. Rev. 14, 4946 (1976).

[7] D. M. Soul, M. Wortis and D. Stauffer, Phys. Rev. B 9, 4964 (1974)

[8] A. K. Jain and D. P. Landau, Bull. Am. Phys. Soc. 21, 231 (1976).

[9] Onsager, Phys. Rev. 65,117 (1944).

[10] G.F. Newell, Phys. Rev. 79, 876 (1950).

[11] K. Husimi and I. Syozi, Prog. Theor. Phys. 5, 117 (1950).

[12] T. Horiguchi, Phys. Lett. A 113, 425 (1986).

[13] K. Husimi(1) and I. Syozi, Prog. Theor. Phys. 5, 341 (1950).

[14] I. Syozi, Prog. Theor. Phys. 6, 306 (1951).

[15] M. Kolesík and L. Samaj, Int. J. Mod. Phys. B 6, 1529 (1992).

[16] X. D. Mi and Z. R. Yang, J. Phys. A: Math. Gen. 28, 4883 (1995); Phys. Rev E 49, 3636 (1994).

[17] K. Tang, J. Phys. A: Math. Gen. 21, L1097 (1988).

[18] F. Y. Wu, Phys. Lett. A 117, 365 (1986).

[19] N. Sh. Izmailian, Pis’ma Zh. Éksp. Teor. Fiz. 63, 270-275 (1996)

[20] N. Sh. Izmailian and N. S. Ananikian, Phys. Rev. B 50, 6829 (1994).

[21] R. I. Joseph, J. Phys. A: Math. Gen. 9, L31 (1976).

[22] T. Horiguchi, Physica A 214, 452 (1995).

[23] J. Strečka, L. Čanová and Ján Dely, Phys. Stat. Solids (B) 243, 1946 (2006).
[24] F. Y. Wu and K. Y. Lin, J. Phys. A: Math. Gen. 20 5737, (1987).

[25] C. Fan and F. Y. Wu, Phys. Rev. B 2, 723 (1970).

[26] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, N.Y. 1982.
Figure 1: Schematic representation of mixed spin-(S,1/2) on a square lattice ($\mathcal{L}_b$), a square-type spin-1/2 Ising model ($\mathcal{L}_a$) and a square-type spin-S Ising model ($\mathcal{L}_c$)