KNOT EXTERIORS WITH ADDITIVE HEEGAARD GENUS AND MORIMOTO’S CONJECTURE

TSUYOSHI KOBAYASHI AND YO’AV RIECK

Abstract. Given integers \(\{g_i \geq 2\}_{i=1}^n\) we prove that there exists infinitely many knots \(K_i \subset S^3\) so that \(g(E(K_i)) = g_i\) and \(g(E(\#_{i=1}^n K_i)) = \Sigma_{i=1}^n g(E(K_i))\). (Here, \(E(\cdot)\) denotes the exterior and \(g(\cdot)\) the Heegaard genus.) Together with [8, Theorem 1.5], this proves the existence of counterexamples to Morimoto’s Conjecture [14].

1. Introduction and statements of results

Let \(K_i\) (\(i = 1, 2\)) be knots in the 3-sphere \(S^3\), and let \(K_1 \# K_2\) be their connected sum. We use the notation \(t(\cdot), E(\cdot), \) and \(g(\cdot)\) to denote tunnel number, exterior, and Heegaard genus respectively (we follow the definitions and notations given in [9]). It is well known that the union of a tunnel system for \(K_1\), a tunnel system for \(K_2\), and a tunnel on a decomposing annulus for \(K_1 \# K_2\) forms a tunnel system for \(K_1 \# K_2\). Therefore:

\[
t(K_1 \# K_2) \leq t(K_1) + t(K_2) + 1.
\]

Since (for any knot \(K\)) \(t(K) = g(E(K)) - 1\), this gives:

\[
(1) \quad g(E(K_1 \# K_2)) \leq g(E(K_1)) + g(E(K_2)).
\]

We say that a knot \(K\) in a closed orientable manifold \(M\) admits a \((g, n)\) position if there exists a genus \(g\) Heegaard surface \(\Sigma \subset M\), separating \(M\) into the handlebodies \(H_1\) and \(H_2\), so that \(H_i \cap K\) (\(i = 1, 2\)) consists of \(n\) arcs that are simultaneously parallel into \(\partial H_i\). We say that \(K\) admits a \((g, 0)\) position if \(g(E(\cdot)) \leq g\). Note that if \(K\) admits a \((g, n)\) position then \(K\) admits both a \((g, n+1)\) position and a \((g+1, n)\) position.

Remark 1.1. The definition given in [5] for \((g, n)\) position with \(n \geq 1\) is identical to our definition. However, in [5] \(K\) is said to admit a \((g, 0)\) position if \(K\) is isotopic into a genus \(g\) Heegaard surface for \(M\). Thus,
if $K$ admit a $(g,0)$ position in the sense of [3] and $g(X) > g$ then $K$ admits a $(g,1)$ position in our sense. For example, a non-trivial torus knot in $S^3$ is called $(1,0)$ in [5] and $(1,1)$ here. (Cf. [10, Remark 2.4].)

It is known [14, Proposition 1.3] that if $K_i (i = 1 \text{ or } 2)$ admits a $(t(K_i), 1)$ position then equality does not hold in Inequality (1):

$$g(E(K_1 \# K_2)) < g(E(K_1)) + g(E(K_2)).$$

Morimoto proved that if $K_1$ and $K_2$ are m-small knots in $S^3$ then the converse holds [14, Theorem 1.6]. This result was generalized to arbitrarily many m-small knots in general manifolds by the authors [9]. Morimoto conjectured that the converse holds in general [14, Conjecture 1.5]:

**Conjecture 1.2** (Morimoto’s Conjecture). Given knots $K_1, K_2 \subset S^3$, $g(E(K_1 \# K_2)) < g(E(K_1)) + g(E(K_2))$ if and only if $K_i$ admits a $(t(K_i), 1)$ position (for $i = 1 \text{ or } i = 2$).

**Remark 1.3.** We note that Morimoto stated the above conjecture in terms of 1-bridge genus $g_1(K)$. It is easy to see that the Conjecture 1.5 of [14] is equivalent to the statement above.

In [8] the authors showed that certain conditions imply existence of counterexamples to Morimoto’s Conjecture. One such condition is the existence of an m-small knot $K$ that does not admit a $(t(K), 2)$ position. We asked [8, Question 1.9] if there exists a knot $K$ with $g(E(K)) = 2$ that does not admit a $(1,2)$ position; this question was answered affirmatively by Johnson and Thompson [5, Corollary 2], who showed that for any $n$ there exist infinitely many knots with $g(E(K)) = 2$ not admitting a $(1,n)$ position. At about the same time Minsky, Moriah and Schleimer [10, Theorem 4.2] proved a more general result, showing that for any integers $g \geq 2, n \geq 1$ there exist infinitely many knots with $g(E(K)) = g$ that do not admit a $(g-1,n)$ position (more precisely, this follows from [10, Theorem 3.1] and Proposition 2.6 below). Although it is not known if any of these examples are m-small, in this paper we show that some of these examples have the property described in the theorem below, that also implies existence of counterexamples to Morimoto’s Conjecture.

**Theorem 1.4.** Given integers $g \geq 2$ and $n \geq 1$, there exists a family of knots in $S^3$ (denoted $K_{g,n}$) with the following properties:

1. For each $h$ with $2 \leq h \leq g$, there exists infinitely many knots $K \in K_{g,n}$ with $g(E(K)) = h$. 
For any collection of knots $K_1, \ldots, K_m \in \mathcal{K}_{g,n}$ (possibly, $K_i = K_j$ for $i \neq j$) with $m \leq n$,
$$g(E(\#_{i=1}^m K_i)) = \sum_{i=1}^m g(E(K_i)).$$
Moreover, for each $g$, we have:
$$\bigcap_{n=1}^\infty \mathcal{K}_{g,n} = \emptyset.$$

Remarks 1.5.  (1) The knots in $\mathcal{K}_{g,n}$ need not be prime. In fact, it is clear from the definition of $\mathcal{K}_{g,n}$ that if $K \in \mathcal{K}_{g,pn}$ then $pK \in \mathcal{K}_{pg,n}$ ($pK$ is defined in Definitions 2.1). We do not know if $\mathcal{K}_{g,n}$ contains a knot of the form $pK$ (for $p > 1$) when $g$ is prime.

(2) Existence of knots $K_1$, $K_2$ with $g(E(K_1 \# K_2)) = g(E(K_1)) + g(E(K_2))$ is known from [13] and [15]. Theorem 1.4 is new in the following ways:
(a) It is the first time that the connected sum of more than two knots are shown to have additive Heegaard genus.
(b) The proof in [13] uses minimal surfaces in hyperbolic manifolds and in [15] quantum invariants. Our proof is purely topological.

(3) The sets $\mathcal{K}_{g,n}$ are not uniquely defined; for example, we can remove any finite set from $\mathcal{K}_{g,n}$. However, for any sets $\mathcal{K}_{g,n}$ fulfilling Theorem 1.4 (1) and (2), we have that $\bigcap_{n=1}^\infty \mathcal{K}_{g,n} = \emptyset$.

A knot $K \subset M$ is called admissible (see [8]) if $g(E(K)) > g(M)$. Thus any knot $K \subset S^3$ is admissible. By [8, Theorem 1.2] for any admissible knot $K$ there exists $N$ so that if $n \geq N$ then $g(E(nK)) < ng(E(K))$. In contrast to that we have:

Corollary 1.6. Given integers $g \geq 2$ and $n \geq 1$, there exist an infinitely many knots $K \subset S^3$ so that $g(E(K)) = g$ and for any $m \leq n$, $g(E(mK)) = mg$.

Proof. For $K \in \mathcal{K}_{g,n}$ with $g(E(K)) = g$ we have $g(E(nK)) = ng$. □

Remark 1.7. By [8, Proposition 1.7], a knot $K$ with $g(E(K)) = g$ and $g(E(nK)) = ng$ cannot admit a $(g(X) - 1, n - 1)$ position.

Another consequence of Corollary 1.6 is:

Corollary 1.8. There exists a counterexample to Morimoto’s Conjecture, specifically, there exist knots $K_1$, $K_2 \subset S^3$ so that $K_i$ does not admit a $(t(K_i), 1)$ position ($i = 1, 2$), and (for some integer $m$) $g(E(K_1)) = 4$, $g(E(K_2)) = 2(m - 2)$, and $g(E(K_1 \# K_2)) < 2m$. 
Proof of Corollary 1.8. This argument was originally given in [8, Theorem 1.4]. We outline it here for completeness. Let $K$ be a knot as in Corollary 1.6, for $g = 2$ and $n = 3$. By [8, Theorem 1.2], for some $m > 1$, $g(E(mK)) < mg(E(K)) = 2m$. Let $m$ be the minimal number with that property. By Corollary 1.6, $m \geq 4$. Hence $g(E(2K)) = 2g(E(K)) = 4$. By the minimality of $m$, $g(E((m - 2)K)) = (m - 2)g(E(K)) = 2(m - 2)$.

Let $K_1 = 2K$ and $K_2 = (m - 2)K$. Note that $K_1\#K_2 = mK$. We have seen:

1. $g(E(K_1)) = 4$.
2. $g(E(K_2)) = 2(m - 2)$.
3. $g(E(K_1\#K_2)) < 2m$.

We claim that $K_1$ does not admit a $(t(K_1), 1)$ position; assume for a contradiction it does. By Inequality (2) and the above (1), we would have that $g(E(3K)) = g(E(K_1\#K)) < g(E(K_1)) + g(E(K)) = 6$, contradicting our choice of $K$.

We claim that $K_2$ does not admit a $(t(K_2), 1)$ position; assume for a contradiction it does. Then by Inequality (2) and the above (2), $g(E((m - 1)K)) < g(E((m - 2)K)) + g(E(K)) = (m - 1)g(E(K))$, contradicting minimality of $m$.

We note that $K_1$ and $K_2$ are composite knots. This leads Moriah [12, Conjecture 7.14] to conjecture that if $K_1$ and $K_2$ are prime then Conjecture 1.2 holds.

Outline. Section 2 is devoted to three propositions necessary for the proof of Theorem 1.4: Proposition 2.2 that relates strongly irreducible Heegaard splittings and bridge position, Proposition 2.5 that relates essential surfaces and the distance of Heegaard splitting (Proposition 2.5 is exactly Theorem 3.1 of [19]), and Proposition 2.6 which relates bridge position and distance of Heegaard splittings (Proposition 2.6 is based on and extends Theorem 1 of [5]). In Section 3 we calculate the genera of certain manifolds that we denote by $X(m)$. In Section 4 we prove Theorem 1.4.

Remark 1.9. The reader may wish to read [6], where an easy argument is given for a special case of Corollary 1.6, namely, $g = 2$ and $n = 3$. Note that this special case is sufficient for Corollary 1.8; [6] can be used as an introduction to the ideas in the current paper.

2. Decomposing $X(m)$.

In this and the following sections, we adopt the following notations.
Definitions 2.1. Let $K$ be a knot in a closed orientable manifold and $X$ its exterior. Let $n \geq 1$ be an integer.

1. The connected sum of $n$ copies of $K$ is denoted by $nK$ and its exterior by $X(n)$.
2. For an integers $c \geq 0$ and $n \geq 1$ we denote by $X(n)(c)$ the manifold obtained by drilling $c$ curves out of $X(n)$ that are simultaneously parallel to meridians of $nK$. For convenience, we denote $X(1)(c)$ by $X(c)$. (Note that $X(0) = X$, and $X(n)(0) = X(n)$.)

Proposition 2.2. Let $X$, $X(c)$ be as above and $g \geq 0$ an integer. Suppose $c > 0$, and $X(c)$ admits a strongly irreducible Heegaard surface of genus $g$. Then one of the following holds:

1. $X$ admits an essential surface $S$ with $\chi(S) \geq 4 - 2g$.
2. For some $b$, $c \leq b \leq g$, $K$ admits a $(g - b, b)$ position.

Remarks 2.3. (1) If $c > g$ then conclusion (1) holds.
2. Compare the proof to [20, Theorem 3.8].

Proof of Proposition 2.2. Let $C_1 \cup_S C_2$ be a genus $g$ strongly irreducible Heegaard splitting of $X(c)$.

Since $c > 0$, $X(c)$ admits an essential torus $T$ that gives the decomposition $X(c) = X' \cup_T Q(c)$, where $X' \cong X$ and $Q(c)$ is a $c$-times punctured annulus cross $S^1$. Since $T$ is incompressible and $\Sigma$ is strongly irreducible, we may isotope $\Sigma$ so that every component of $\Sigma \cap T$ is essential in both surfaces. Isotope $\Sigma$ to minimize $|\Sigma \cap T|$ subject to that constraint. Denote $\Sigma \cap X'$ by $\Sigma_X$ and $\Sigma \cap Q(c)$ by $\Sigma_Q$. Note that (by essentiality of $T$) $\Sigma \cap T \neq \emptyset$ and (by minimality) no component of $\Sigma_X$ (resp. $\Sigma_Q$) is boundary parallel in $X'$ (resp. $Q(c)$). By the argument of [9, Claim 4.5] we may assume that $\Sigma_X$ is connected and compresses into both sides in $X'$ and $\Sigma_Q$ is incompressible in $Q(c)$, for otherwise Conclusion (1) holds.

Every component of $\Sigma_Q$ is a vertical annulus (see, for example, [3, VI.34]). Hence, $\partial \Sigma_X$ consists of meridians of $K$. For $i = 1, 2$, let $\Sigma_i$ be the surface obtained by simultaneously compressing $\Sigma_X$ maximally into $C_i \cap X$. Then the argument of Claim 6 (page 248) of [7] shows that every component of $\Sigma_i$ is incompressible. Hence, we may assume that every component is a boundary parallel annulus in $X'$ or a 2-sphere (for otherwise Conclusion (1) holds). Denote that number of annuli by $b$ (note that $b = \frac{1}{2} |\partial \Sigma_X|$ and is the same for $\Sigma_1$ and $\Sigma_2$). Denote the solid tori that define that boundary parallelism of $\Sigma_i$ by $N_{i,1}, \ldots, N_{i,b}$.

Claim. For $i = 1, 2$, $N_{i,1}, \ldots, N_{i,b}$ are mutually disjoint.
**Proof of claim.** Assume for a contradiction that two components (say $N_{i,1}$ and $N_{i,2}$) intersect, say $N_{i,2} \subset N_{i,1}$. By construction $\Sigma_X$ is a connected surface obtained by tubing the annuli $\Sigma_i$ and (possibly empty) collection of 2-spheres into one side, therefore the tubes are all contained in $\text{cl}(N_{i,1} \setminus N_{i,2})$, and we see that (for all $j \geq 2$) $N_{i,j} \subset N_{i,1}$. This shows that $\Sigma$ is isotopic into $Q^{(c)}$, hence $T \subset C_1$ or $T \subset C_2$. Since $T$ is essential, this is impossible. This proves the claim. \[\Box\]

By the claim, $C_i \cap X'$ is obtained from $N_{i,1}, \ldots, N_{i,b}$ and a (possibly empty) collection of 3-balls by attaching 1-handles. This implies $C_i \cap X'$ is obtained from a handlebody $H_i$ (say of genus $h_i$) by removing a regular neighborhood of $b$ trivial arcs, say $\gamma_{i,1}, \ldots, \gamma_{i,b}$, where $N_{i,j} \cap T$ corresponds to the frontier of the regular neighborhood of $\gamma_{i,j}$ ($j = 1, \ldots, b$). Since every component of $\Sigma_{Q}$ is an annulus, $\chi(\Sigma_{X}) = \chi(\Sigma)$. $\partial H_i$ is obtained by capping $\Sigma_X$ off with $2b$ disks, hence $\chi(\partial H_i) = \chi(\Sigma) + 2b$; this shows that $h_i = g - b$.

We obtained a $(g - b, b)$ position for $K$, and to complete the proof we need to show that $b \geq c$. Suppose for a contradiction that $b < c$. Note that $\Sigma_Q$ consists of $b$ vertical annuli. Since $\partial \Sigma_Q \subset T$, we see that $\Sigma_Q$ separates $Q^{(c)}$ into $b + 1$ components. Note that $\partial X^{(c)}$ consists of $c + 1$ tori; thus if $b < c$ then two components of $\partial Q^{(c)}$ are in the same component of $Q^{(c)}$ cut open along $\Sigma_Q$. It is easy to see that there is a vertical annulus connecting these tori which is disjoint from $\Sigma$. Hence this annulus is contained in the compression body $C_i$ and connects components of $\partial - C_i$ for $i = 1$ or 2, a contradiction (for the notation $\partial - C_i$, see, for example, [9]). This contradiction completes the proof of Proposition 2.2. \[\Box\]

**Definition 2.4** (Hempel [2]). Let $H_1 \cup_{\Sigma} H_2$ be a Heegaard splitting. The **distance** of $\Sigma$, denoted $d(\Sigma)$, is the least integer $d$ so that there exist meridian disks $D_i \subset H_i$ $(i = 1, 2)$ and essential curves $\gamma_0, \ldots, \gamma_d \subset \Sigma$ so that $\gamma_0 = \partial D_1$, $\gamma_d = D_2$, and $\gamma_{i-1} \cap \gamma_i = \emptyset$ $(i = 1, \ldots, d)$. If $\Sigma$ is the trivial Heegaard splitting of a compression body (that is, $\Sigma$ is boundary parallel) this definition does not apply, since on one side of $\Sigma$ there are no meridional disks. In that case, we define $d(\Sigma)$ to be zero.

The properties of knots with exteriors of high distance that we need are given in the next two propositions:

**Proposition 2.5.** Let $K$ be a knot and $d \geq 0$ an integer. Suppose $X$ admits a Heegaard splitting with distance greater than $d$. Then $X$ does not admit a connected essential surface $S$ with $\chi(S) \geq 2 - d$.

**Proof.** This is Theorem 3.1 of [19]. \[\Box\]
Our next proposition is a combination of Theorem 1 of [5] and Corollary 3.5 of [21]:

**Proposition 2.6.** Let $K \subset M$ be a knot and $p$, $q$ integers so that $K$ admits a $(p, q)$ position.

If $p < g(X)$ then any Heegaard splitting for $X$ has distance at most $2(p + q)$.

**Proof.** Recall from Remark 1.1 that our definition of $(p, q)$ position is not quite the same as [5]. As explained in Remark 1.1 either $K$ admits a $(p, q)$ position in the sense of [5] or $q = 1$ and $K$ admits a $(p, 0)$ position in the sense of [5]. In the former case, Proposition 2.6 is exactly Theorem 1 of [5]. Thus we may assume:

1. $q = 1$.
2. $K$ admits a $(p, 0)$ position in the sense of [5], that is, $M$ admits a Heegaard splitting of genus $p$ (say $H_1 \cup_\Sigma H_2$) so that $K$ is isotopic into $\Sigma$.

We base our analysis on [16][18][17]. After isotoping $K$ into $\Sigma$, let $N = N_{H_1}(K)$ be a neighborhood of $K$ in $H_1$. Then $N$ is a solid torus and $K \subset \partial N$ a longitude. Let $\Delta$ be a meridian disk of $N$ that intersects $K$ in one point. Let $\alpha \subset \Delta$ be a properly embedded arc with $\partial \alpha \subset (\Delta \cap \Sigma)$, so that $K \cap (\Delta \cap \Sigma)$ separates the points of $\partial \alpha$. Let $K'$ be a copy of $K$ pushed slightly into $H_1$, so that $K' \cap \Delta$ is a single point and $\alpha$ does not separate $K \cap \Delta$ from $K' \cap \Delta$ in $\Delta$.

We stabilize $\Sigma$ by tubing it along $\alpha$; denote the tube by $t$, the surface obtained after tubing by $S(\Sigma)$, and the complementary handlebodies by $H'_1$ and $H'_2$ (with $K' \subset H'_1$).

Let $X'$ be the exterior of $K'$. Since $K'$ is isotopic to $K$, $X' \cong X$. Note that $H'_1$ admits an obvious meridian disk that intersects $K'$ once (a component of $\Delta \cap H'_1$). Note also that $K'$ is isotopic into $S(\Sigma)$ in $H'_1$. Therefore, $S(\Sigma)$ is a Heegaard surface for $X'$. Since $g(S(\Sigma)) = g(\Sigma) + 1 = p + 1$ and by assumption $p < g(X) = g(X')$, we have that $S(\Sigma)$ is a minimal genus Heegaard surface for $X'$.

We claim that $d(S(\Sigma)) \leq 2$. To prove this we will show that $H'_1$ and $H'_2$ admit meridian disks that are disjoint from $K \subset S(\Sigma)$. In $H'_2$ we take the compressing disk for the tube $t$. For $H'_1$, let $D \subset H_1$ be any meridian disk. We will use $D$ to construct $D'$, a meridian disk for $H'_1$, so that $D' \cap K = \emptyset$. (Intuitively, we construct $D'$ by pushing $D$ over $t$.) Via isotopy we may assume that $D$ intersects $N$ (if at all) in disks $D_1, \ldots, D_l$ (for some integer $l$) that are parallel to $\Delta$, and close enough to $\Delta$ so that $t$ intersects $D_i$ in the same pattern as it intersects $\Delta$ ($i = 1, \ldots, l$). Note that $D$ cut open along $S(\Sigma)$ has $2l + 1$
components: $l$ components inside $t$, $l$ components that intersect $K$, and exactly one other component, denoted $D'$. It is easy to see that $D'$ is a meridian disk for $H'_1$ disjoint from $K$. Thus $d(S(\Sigma)) \leq 2$.

Let $\Sigma'$ be any Heegaard surface for $X'$. To estimate $d(\Sigma')$ we apply [21, Corollary 3.5] (with $S(\Sigma)$ corresponding to $Q$ and $\Sigma'$ to $P$). Then by [21, Corollary 3.5] one of the following holds:

1. $S(\Sigma)$ is isotopic to a stabilization of $\Sigma'$.
2. $d(\Sigma') \leq 2g(S(\Sigma))$.

In case (1), since $S(\Sigma)$ is a minimal genus Heegaard splitting, $S(\Sigma)$ is isotopic to $\Sigma'$ (with no stabilizations). Therefore $d(\Sigma') = d(S(\Sigma)) \leq 2 < 2(p+q)$. On case (2), $d(\Sigma') \leq 2g(S(\Sigma)) = 2(p+1) = 2(p+q)$. As $X' \cong X$, any Heegaard surface for $X$ has distance at most $2(p+q)$. □

3. Calculating $g(X(m)^{(c)})$.

Recall that we follow the notations in Definition 2.1.

Lemma 3.1. Let $K \subset M$ be a knot, $X$ the exterior of $K$ and $c \geq 0$ an integer. Denote $g(X)$ by $g$.

Then $g(X^{(c)}) \leq g + c$.

Proof. Note that $X^{(c)}$ is obtained from $X^{(c-1)}$ by drilling out a curve parallel to $\partial X$. Equivalently, we obtain $X^{(c-1)}$ by Dehn filling a component of $\partial X^{(c)}$, and the core of the attached solid torus is isotopic into $\partial X$. This shows that the core of the solid torus is isotopic to any Heegaard surface of $X^{(c-1)}$, because one compression body of the Heegaard splitting is obtained from a regular neighborhood of $(\partial X \cup (\text{some components of } \partial X^{(c-1)} \setminus \partial X))$ by adding some 1-handles. In [16], this situation is called a good Dehn filling, and it is shown that one of the following holds:

1. $g(X^{(c)}) = g(X^{(c-1)})$.
2. $g(X^{(c)}) = g(X^{(c-1)}) + 1$.

Hence we have $g(X^{(c)}) \leq g(X^{(c-1)}) + 1$ in general. Since $g(X^{(0)}) = g(X) = g$, this implies $g(X^{(c)}) \leq g + c$. □

Proposition 3.2. Let $M$ be a compact orientable manifold that does not admit a non-separating surface, and $K \subset M$ a knot. Let $c \geq 0$ be an integer. Denote $g(X)$ by $g$. Suppose that $X$ does not admit an essential surface $S$ with $\chi(S) \geq 4 - 2(g+c)$, and that $K$ does not admit a $(g-1, c)$ position.

Then $g(X^{(c)}) = g + c$.

Proof. The proof is an induction on $c$. For $c = 0$ there is nothing to prove. Fix $c > 0$ as in the statement of the proposition and let
Σ ⊂ X^{(c)} be a minimal genus Heegaard surface. Suppose that X does not admit an essential surface S with χ(S) ≥ 4 − 2(g + c), and that K does not admit a (g − 1, c) position. By the inductive hypothesis we have g(X^{(c−1)}) = g + c − 1. By the inequalities in the proof of Lemma 3.1 we have either g(X^{(c)}) = g + c − 1, or g(X^{(c)}) = g + c. The proof is divided into the following two cases.

**Case 1.** Σ is strongly irreducible.

By Proposition 2.12, one of the following holds:

1. X admits an essential surface S with χ(S) ≥ 4 − 2g(X^{(c)}).
2. K admits a (g(X^{(c)}) − b, b) position for some b ≥ c.

By Lemma 3.1, we have 4 − 2g(X^{(c)}) ≥ 4 − 2(g + c). By assumption, X does not admit an essential surface S with χ(S) ≥ 4 − 2(g + c), so (1) above cannot happen and we may assume that K admits a (g(X^{(c)}) − b, b) position for some b ≥ c. Since b − c ≥ 0 we can tube the Heegaard surface giving the (g(X^{(c)}) − b, b) position b − c times to obtain a (g(X^{(c)}) − b + (b − c), b − (b − c)) = (g(X^{(c)}) − c, c) position. Since K does not admit a (g − 1, c) position, this implies that g(X^{(c)}) − c ≥ g and in particular g(X^{(c)}) ≠ g + c − 1. Hence we have g(X^{(c)}) = g + c.

**Case 2.** Σ is weakly reducible.

By Casson and Gordon [1], an appropriately chosen weak reduction yields an essential surface ̂F (see [23], Theorem 1.1) for a relative version of Casson and Gordon’s Theorem). Let ̂F be a connected component of ̂F. Since ̂F ⊂ X^{(c)} ⊂ M it separates and by [9], Proposition 2.13] Σ weakly reduces to ̂F. Note that χ( ̂F) ≥ χ(Σ) + 4.

Claim. ̂F can be isotoped into Q^{(c)} (recall the definition of T, X' and Q^{(c)} from the proof of Proposition 2.12).

**Proof of Claim.** Assume for a contradiction this is not the case. Since ̂F and T are essential, the intersection consists of a (possibly empty) collection of curves that are essential in both surfaces. Minimize | ̂F ∩ T| subject to this constraint. If ̂F ∩ T compresses, then (since the curves of ̂F ∩ T are essential in ̂F) so does ̂F, contradiction. Since T is a torus, boundary compression of ̂F ∩ X' implies a compression (see, for example, [7], Lemma 2.7]). Finally, minimality of | ̂F ∩ T| implies that no component of ̂F ∩ X' is boundary parallel. Thus, every component of ̂F ∩ X' is essential. This includes the case ̂F ⊂ X' (in that case ̂F is essential in X', else it would be parallel to T and isotopic into Q^{(c)}).

Since no component of ̂F ∩ Q^{(c)} is a disk or a sphere, χ( ̂F ∩ X') ≥ χ( ̂F) ≥
χ(Σ)+4. By Lemma 3.1 we have χ(Σ)+4 ≥ 2−2(g+c)+4 = 6−2(g+c), contradicting our assumption. □

By [3 VI.34] F is a vertical torus in Q(c). First, if F is not parallel to a component of ∂Q(c) then F decomposes X(c) as X(p+1) ∪ F D(c−p), where p ≥ 0 is an integer and D(c−p) is a disk with c−p holes cross S^1. Note that since F is not parallel to a component of ∂Q(c), c−p ≥ 2. Therefore p + 1 < c and by the inductive hypothesis g(X(p+1)) = g + p + 1. By Schultens [22], g(D(c−p)) = c−p.

Next, suppose F is boundary parallel in Q(c). Since Σ is minimal genus, F is not parallel to a component of ∂X(c) [23]. Hence F is isotopic to T. This gives the decomposition X(c) = X′ ∪ F Q(c). Since X′ ≅ X, g(X′) = g. By [22] g(Q(c)) = c + 1.

Since F was obtained by weakly reducing a minimal genus Heegaard surface, [3 Proposition 2.9] (see also [22 Remark 2.7]) gives, in the first case:

\[ g(X(c)) = g(X(p+1)) + g(D(c−p)) − g(F) \]
\[ = (g + p + 1) + (c - p) - 1 \]
\[ = g + c. \]

And in the second case:

\[ g(X(c)) = g(X′) + g(Q(c)) − g(F) \]
\[ = g + (c + 1) − 1 \]
\[ = g + c. \]

This completes the proof. □

**Proposition 3.3.** Let m ≥ 1 and c ≥ 0 be integers and \{K_i \subset M_i\}_{i=1}^m knots in closed orientable manifolds so that (for all i) M_i does not admit a non-separating surface. Denote E(K_i) by X_i and E(\#_{i=1}^m K_i) by X. Let g be an integer so that g(X_i) ≤ g for all i.

Suppose that no X_i admits an essential surface S with χ(S) ≥ 4−2(m+c)g and that no K_i admit a (g(X_i) − 1, m+c−1) position. Then we have:

\[ g(X(c)) = \sum_{i=1}^m g(X_i) + c. \]

**Remarks 3.4.** (1) The proof for m ≥ 2 is an induction of (m,c) ordered lexicographically. During the inductive step, (m,c) is replaced by (say) (m_1,c_1). Since the complexity is reduced, m_1 ≤ m. However, c_1 > c is possible. We will see that if c_1 > c, then c_1 = c+1 and m_1 < m. Thus m_1 + c_1 ≤ m + c and the condition “no X_i admits an essential surface S with χ(S) ≥ 4−2(m+c)g” holds when m+c is replaced by m_1+c_1. The same
holds for the condition “no $K_i$ admit a $(g(X_i) - 1, m + c - 1)$ position”.

(2) For $m \geq 2$, the proof is an application of the Swallow Follow Torus Theorem [9 Theorem 4.1]. In [9 Remark 4.2] it was shown by means of a counterexample that the Swallow Follow Torus Theorem does not apply to $X^{(c)}$ when $m = 1$. Hence argument of Proposition 3.3 cannot be used to simplify the proof of Proposition 3.2.

**Proof.** The assumptions of Proposition 3.2 hold and so that the proposition establishes the case $m = 1$ (note that $4 - 2(1 + c)g \leq 4 - 2(c + g)$ holds). Hence we assume from now on that $m \geq 2$.

We induct on $(m, c)$ ordered lexicographically, where $m$ is the number of summands and $c$ is the number of curves drilled. Note that by Miyazaki [11] $m$ is well defined (see [9 Claim 1]).

By Lemma 3.1 and Inequality (1) in section 1, we get: $g(X^{(c)}) \leq g(X) + c = g(E(#_{i=1}^{m}K_i)) + c \leq \Sigma_{i=1}^{m}g(E(K_i)) + c \leq mg + c$. Since $g \geq 2$ we have that $g(X(m)^{(c)}) \leq (m + c)g$.

By assumption, for all $i$, $X_i$ does not admit an essential surface $S$ with $\chi(S) \geq 4 - 2(m + c)g$. Hence by the Swallow Follow Torus Theorem [9 Theorem 4.1] any minimal genus Heegaard surface for $X^{(c)}$ weakly reduces to a swallow follow torus $F$ giving the decomposition $X = X_I^{(c_1)} \cup_F X_J^{(c_2)}$, with $I \subset \{1, \ldots, m\}$, $K_I = \#_{i \in I}K_i$, $K_J = \#_{i \notin I}K_i$, $X_I = E(K_I)$, $X_J = E(K_J)$, and $c_1 + c_2 = c + 1$ (for details see the first paragraph of Section 4 of [9]). Note that $I = \emptyset$ or $I = \{1, \ldots, m\}$ are possible. However, at least one of $I$, $\{1, \ldots, m\} \setminus I$ is not empty and by symmetry we may assume $I \neq \emptyset$.

If $I = \{1, \ldots, m\}$ then $c_2 \geq 2$. Hence $c_1 < c$ and the inductive hypotheses applies to $X_I^{(c_1)} \cong X^{(c_1)}$, showing that $g(X_I^{(c_1)}) = \Sigma_{i=1}^{m}g(X_i) + c_1$. Since $X_J^{(c_2)}$ is homeomorphic to a disk with $c_2$ holes cross $S^1$, by [22] $g(X_J^{(c_2)}) = c_2$. Amalgamation along $F$ gives (recall that $c_1 + c_2 = c + 1$):

$$
g(X^{(c)}) = g(X_I^{(c_1)}) + g(X_J^{(c_2)}) - g(F) = \Sigma_{i=1}^{m}g(X_i) + c_1 + c_2 - 1 = \Sigma_{i=1}^{m}g(X_i) + c.
$$

If $I \neq \{1, \ldots, m\}$ then the number of summands in $K_I$ and $K_J$ are $|I|$ and $m - |I|$ (respectively) and are both less than $m$. By construction $c_1 \leq c + 1$ and $c_2 \leq c + 1$ hence $m_1 + c_1 \leq m + c$ and $m_2 + c_2 \leq m + c$. By the inductive hypothesis $g(X_I^{(c_1)}) = \Sigma_{i \in I}g(X_i) + c_1$ and $g(X_J^{(c_2)}) = \Sigma_{i \notin I}g(X_i) + c_2$. Hence

$$
g(X^{(c)}) = g(X_I^{(c_1)}) + g(X_J^{(c_2)}) - g(F) = \Sigma_{i=1}^{m}g(X_i) + c_1 + c_2 - 1 = \Sigma_{i=1}^{m}g(X_i) + c.
$$
$\Sigma_{i\in I} g(X_i) + c_2$. Amalgamation along $F$ gives:

$$g(X^{(c)}) = g(X^{(c_1)}_I) + g(X^{(c_2)}_J) - g(F) = (\Sigma_{i\in I} g(X_i) + c_1) + (\Sigma_{i\in I} g(X_i) + c_2) - 1 = \Sigma_{i=1}^m g(X_i) + (c_1 + c_2) - 1 = \Sigma_{i=1}^m g(X_i) + c.$$

This proves the proposition in both cases. ☐

4. Proof of Theorem 1.4

Fix $g \geq 2$ and $n \geq 1$. Let $\mathcal{K}_{g,n}$ be the collection of all knots $K \subset S^3$ so that:

1. $g(E(K)) \leq g$.
2. $X$ does not admits an essential surface $S$ with $\chi(S) \geq 4 - 2gn$.
3. $K$ does not admit a $(g(X) - 1, n)$ position.

Applying Proposition 3.3 with $m \leq n$ and $c = 0$, we see that the knots in $\mathcal{K}_{g,n}$ fulfill Condition (2) of Theorem 1.4. Fix $h$, $2 \leq h \leq g$. By [10, Theorem 3.1] there exist infinitely many knots $K$ with $g(X) = h$, admitting a Heegaard splitting of distance greater than $\max\{2gn - 2, 2(h + n - 1)\}$ (for $g = 2$ this was obtained independently by Johnson [4, Lemma 4]). Let $K$ be such a knot and $X$ its exterior. By Proposition 2.5, since $X$ admits a Heegaard splitting with distance greater than $2gn - 2$, $X$ does not admit an essential surface $S$ with $\chi(S) \geq 4 - 2gn$. By Proposition 2.6, since $X$ admits a Heegaard splitting with distance greater than $2(h + n - 1)$, $K$ does not admit a $(g(X) - 1, n)$ position. We see that $K \in \mathcal{K}_{g,n}$ and hence, $\mathcal{K}_{g,n}$ contains infinitely many knots $K$ with $g(X) = h$. This proves that $\mathcal{K}_{g,n}$ fulfills condition (1) as well.

Let $K \subset S^3$ be a knot with $g(E(K)) = h$. As noted in the introduction, any knot in $S^3$ is admissible (in the sense of [8]) and therefore by [8, Theorem 1.2] there exists $N$ so that if $n \geq N$ then $g(E(nK)) < ng(E(K))$. This shows that $K \notin \mathcal{K}_{g,n}$ for $n \geq N$. Hence $K \notin \cap_{i=1}^\infty \mathcal{K}_{g,n}$. As $K$ was arbitrary, $\cap_{i=1}^\infty \mathcal{K}_{g,n} = \emptyset$.

This completes the proof of Theorem 1.4.

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Department of Mathematics, Nara Women’s University Kitauoya
Nishimachi, Nara 630-8506, Japan

Department of mathematical Sciences, University of Arkansas, Fayetteville, AR 72701
E-mail address: tsuyoshi@cc.nara-wu.ac.jp
E-mail address: yoav@uark.edu