THE POPESCU-GABRIEL THEOREM FOR TRIANGULATED CATEGORIES

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ABSTRACT. The Popescu-Gabriel theorem states that each Grothendieck abelian category is a localization of a module category. In this paper, we prove an analogue where Grothendieck abelian categories are replaced by triangulated categories which are well generated (in the sense of Neeman) and algebraic (in the sense of Keller). The role of module categories is played by derived categories of small differential graded categories. An analogous result for topological triangulated categories has recently been obtained by A. Heider.

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1. INTRODUCTION

One of the aims of the present article is to try and answer the question: what is the analogue, in the realm of triangulated categories, of the notion of Grothendieck category in the realm of abelian categories? The best way to proceed seemed to us that of lifting perhaps the most important theorem involving these notions from the abelian to the triangulated world. The theorem we are speaking of is due to Popescu-Gabriel:
**Theorem 1.1** (Popescu-Gabriel [28]). Let $\mathcal{T}$ be a Grothendieck category. Then the following statements are equivalent:

1. $G \in \mathcal{T}$ is a generator of $\mathcal{T}$;
2. the functor $\text{Hom}(G, -) : \mathcal{T} \to \text{Mod}(A)$, where $A = \text{Hom}(G, G)$, is a localization.

We refer to [27], [33], [20], [18] for complete proofs of the theorem.

In his book [24, Def. 1.15, p. 15], A. Neeman defined the class of *well generated triangulated categories*. It turns out that this class is a very good generalization to higher cardinals of the concept of *compactly generated triangulated category*. In fact, it preserves the most interesting properties, e.g. the validity of the Brown representability theorem [3] and of the Thomason localization theorem [34, Key Proposition 5.2.2, p. 338], and at the same time introduces new good features, such as the stability of the new class under localizations (assuming the quite weak hypothesis that the kernel of the localization functor is generated by a set of objects). H. Krause characterized the class of categories introduced by Neeman as follows [15]. Let $\mathcal{T}$ be a triangulated category with suspension functor $\Sigma$ admitting arbitrary set-indexed coproducts. $\mathcal{T}$ is well generated in the sense of Krause [15] if and only if there exists a set $G_0$ of objects with $\Sigma G_0 = G_0$ satisfying the conditions:

1. (G1) an object $X \in \mathcal{T}$ is zero provided that $T(G, X) = 0$ for all $G \in G_0$;
2. (G2) for each family of morphisms $f_i : X_i \to Y_i$, $i \in I$, the induced map
   $$T(G, \coprod_{i \in I} X_i) \to T(G, \coprod_{i \in I} Y_i)$$
   is surjective for all $G \in G_0$ provided that the maps
   $$T(G, X_i) \to T(G, Y_i)$$
   are surjective for all $i \in I$ and all $G \in G_0$;
3. (G3) there is some regular cardinal $\alpha$ such that the objects $G \in G_0$ are $\alpha$-small, i.e. for each family of objects $X_i$, $i \in I$, of $\mathcal{T}$, each morphism
   $$G \to \coprod_{i \in I} X_i$$
   factors through a subsum $\coprod_{i \in J} X_i$ for some subset $J$ of $I$ of cardinality strictly smaller than $\alpha$.

In the case $\alpha = \aleph_0$, the $\aleph_0$-compact objects are the compact objects of the classical literature [21], [22] and the definition of well generated category reduces to that of *compactly generated* one. Well generated triangulated categories arise very naturally when one localizes compactly generated ones, as it will be shown in detail in section [3]. For example, the unbounded derived category $\mathcal{D}(\text{Sh}(X))$ of sheaves of abelian groups over a topological space $X$ is well generated since it is a localization of the derived category of presheaves $\mathcal{D}(\text{Presh}(X))$, which is compactly generated. However, Neeman shows in [25] that not all derived categories of sheaves are compactly generated. An example is the category $\mathcal{D}(\text{Sh}(X))$ where $X$ is a connected, non compact real manifold of dimension at least one; in this case, there do not exist non zero compact objects. In the same article, it is shown that the derived categories of Grothendieck categories are always well generated. Another large class of examples arises when one localizes the derived category $\mathcal{D}A$ of a small DG category $A$ at the localizing subcategory generated by a set of objects. Indeed, since $\mathcal{D}A$ is a compactly generated triangulated category, such a localization is always well generated. Now we can state the main result of this paper. It also gives a positive answer to Drinfeld’s question [26] whether all well generated categories arise as localizations of module
categories over DG categories, for the class of algebraic triangulated categories. Here algebraic means triangle equivalent to the stable category of a Frobenius category. One can show that each algebraic triangulated category is triangle equivalent to a full triangulated subcategory of the category up to homotopy of complexes over some additive category.

**Theorem 1.2.** Let $\mathcal{T}$ be an algebraic triangulated category. Then the following statements are equivalent:

(i) $\mathcal{T}$ is well generated;

(ii) there is a small DG category $\mathcal{A}$ such that $\mathcal{T}$ is triangle equivalent to a localization of $\mathcal{D}\mathcal{A}$ with respect to a localizing subcategory generated by a set of objects.

Moreover, if (i) holds and $\mathcal{G} \subseteq \mathcal{T}$ is a full triangulated subcategory stable under coproducts of strictly less than $\alpha$ factors and satisfying (G1), (G2) and (G3) for some regular cardinal $\alpha$, the functor

$$\mathcal{T} \rightarrow \text{Mod}\mathcal{G}, \ X \mapsto \text{Hom}_\mathcal{T}(-, X)|_\mathcal{G}$$

lifts to a localization $\mathcal{T} \rightarrow \mathcal{D}(\tilde{\mathcal{G}})$, where $\tilde{\mathcal{G}}$ is a small DG category such that $H^0(\tilde{\mathcal{G}})$ is equivalent to $\mathcal{G}$.

If $\mathcal{T}$ is compactly generated, the theorem yields a triangle equivalence $\mathcal{T} \rightarrow \mathcal{D}\mathcal{A}$, and we recover Theorem 4.3 of [12]. Note the structural similarity with the abelian case. One notable difference is that in the abelian case, one can work with a single generator whereas in the triangulated case, in general, it seems unavoidable to use a (small but usually infinite) triangulated subcategory.

An analogous result for topological triangulated categories has recently been proved by A. Heider [7].

1.1. **Organization of the paper.** In section 2 we present some auxiliary results about well generated triangulated categories. After recalling the definition given by Krause (subsection 2.1), we establish a small set of conditions which allows us to show that two well generated triangulated categories are triangle equivalent (subsection 2.2).

In section 3, we recall some basic results about localizations of triangulated categories and about their thick subcategories. In subsection 3.3 we state a theorem concerning particular localizations of well generated triangulated categories, those which are triangle quotients by a subcategory generated by a set.

Section 4 is the heart of the paper. We construct the $\alpha$-continuous derived category $\mathcal{D}_\alpha\mathcal{A}$ of a homotopically $\alpha$-cocomplete small DG category $\mathcal{A}$ (section 4). This construction enjoys a useful and beautiful property which is the key technical result for proving the main theorem of the paper: Given a homotopically $\alpha$-cocomplete pretriangulated DG category $\mathcal{A}$, we show that its $\alpha$-continuous derived category $\mathcal{D}_\alpha\mathcal{A}$ is $\alpha$-compactly generated by the images of the free DG modules. The proof heavily uses theorem 3.10 of subsection 3.3 about localizations of well generated triangulated categories.

The categories $\mathcal{D}_\alpha\mathcal{A}$ turn out to be the prototypes of the $\alpha$-compactly generated algebraic DG categories. This characterization is what we have called the Popescu-Gabriel theorem for triangulated categories. This is the main result of the paper. We present it in section 5. As an application, we also give a result about compactifying subcategories of an algebraic well generated triangulated category. The notion of compactifying subcategory generalizes that of compactifying generator introduced by Lowen-Vanden Bergh [19, Ch. 5] in the case of a Grothendieck abelian category.

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2. Well generated triangulated categories

2.1. Definitions of Krause and Neeman. The notion of well generated triangulated category is due to A. Neeman [24, Def. 1.15, p. 15]. Instead of his original definition, we will use a characterisation due to H. Krause [15] which is closer in spirit to the definition of Grothendieck abelian categories. We recall that a regular cardinal \( \alpha \) is a cardinal which is not the sum of fewer than \( \alpha \) cardinals, all smaller than \( \alpha \) (see any standard reference about set theory for definitions and properties of ordinals and cardinals, a very readable one is [17]). In this article, we will usually assume that the cardinals we use are infinite and regular.

Definition 2.1. Let \( \mathcal{T} \) be a triangulated category with arbitrary coproducts and suspension functor \( \Sigma \). Let \( \alpha \) be an infinite regular cardinal. Then the category \( \mathcal{T} \) is \( \alpha \)-compactly generated if there exists a set of \( \alpha \)-good generators, i.e. a set of objects \( \mathcal{G}_0 \) such that \( \Sigma \mathcal{G}_0 = \mathcal{G}_0 \), satisfying the conditions:

(G1) an object \( X \in \mathcal{T} \) is zero if \( \mathcal{T}(G,X) = 0 \) for all \( G \in \mathcal{G}_0 \);
(G2) for each family of morphisms \( f_i : X_i \to Y_i \), \( i \in I \), the induced map

\[
\mathcal{T}(G, \bigoplus_{i \in I} X_i) \to \mathcal{T}(G, \bigoplus_{i \in I} Y_i)
\]

is surjective for all \( G \in \mathcal{G}_0 \) if the maps

\[
\mathcal{T}(G, X_i) \to \mathcal{T}(G, Y_i)
\]

are surjective for all \( i \in I \) and all \( G \in \mathcal{G}_0 \);
(G3) all the objects \( G \in \mathcal{G}_0 \) are \( \alpha \)-small, i.e. for each family of objects \( X_i, i \in I \), of \( \mathcal{T} \), each morphism

\[
G \to \bigoplus_{i \in I} X_i
\]

factors through a subsum \( \bigoplus_{i \in J} X_i \) for some subset \( J \) of \( I \) of cardinality strictly smaller than \( \alpha \).

A triangulated category is well generated [15], if there exists a regular cardinal \( \delta \) such that it is \( \delta \)-compactly generated.

Let \( \mathcal{T} \) be a triangulated category with arbitrary coproducts. We will say that condition (G4) holds for a class of objects \( \mathcal{G} \) of \( \mathcal{T} \) if the following holds:

(G4) for each family of objects \( X_i, i \in I \), of \( \mathcal{T} \), and each object \( G \in \mathcal{G} \), each morphism

\[
G \to \bigoplus_{i \in I} X_i
\]

factors through a morphism \( \bigoplus_{i \in I} \phi_i : \bigoplus_{i \in I} G_i \to \bigoplus_{i \in I} X_i, \) with \( G_i \) in \( \mathcal{G} \) for all \( i \in I \).

Clearly, condition (G4) holds for the empty class and, if it holds for a family of classes, then it holds for their union. Thus, for a given regular cardinal \( \alpha \), there exists a unique maximal class satisfying (G4) and formed by \( \alpha \)-small objects. Following Krause [15], we denote this class, and the triangulated subcategory on its objects, by \( \mathcal{T}^\alpha \). Its objects are called the \( \alpha \)-compact objects of \( \mathcal{T} \).
Remark 2.2. This definition of $T^\alpha$ is not identical to the one of Neeman [24] Def. 1.15, p. 15. However, as shown in [15] Lemma 6), the two definitions are equivalent if the isomorphism classes of $T^\alpha$ form a set. This always holds when $T$ is well generated, cf. [15].

In the case $\alpha = \aleph_0$, the $\aleph_0$-compact objects are the objects usually called compact (also called small). We recall that an object $K$ of $T$ is called compact if the following isomorphism holds

$$\bigoplus_{i \in I} T(K, X_i) \rightarrow T(\bigcoprod_{i \in I} X_i),$$

where the objects $X_i$ lie in $T$ for all $i \in I$, and $I$ is an arbitrary set. The triangulated category with coproducts $T$ is usually called compactly generated if condition (G1) holds for a set $G_0$ contained in the subcategory of compact objects $T^c = T^{\aleph_0}$. In the case $\alpha = \aleph_0$, the definition of well generated category specializes to that of compactly generated category.

Let $T$ be a triangulated category with arbitrary coproducts and $G_0$ a small full subcategory of $T$. Let $G = \text{Add}(G_0)$ be the closure of $G_0$ under arbitrary coproducts and direct factors. A functor $F : G \to \text{Ab}$ is coherent [3, 1] if it admits a presentation

$$G(\cdot, G_1) \to G(\cdot, G_0) \to F \to 0$$

for some objects $G_0$ and $G_1$ of $G$. Let $\text{coh}(G)$ be the category of coherent functors on $G$. It is a full subcategory of the category $\text{Mod} G$ of all additive functors $F : G^{\text{op}} \to \text{Ab}$. Part c) of the following lemma appears in [16] Lemma 3], in a version with countable coproducts instead of arbitrary coproducts. We give a new, more direct proof.

Lemma 2.3.

a) For each object $X$ of $T$, the functor $h(X)$ obtained by restricting $T(\cdot, X)$ to $G$ is coherent.

b) The functor $G \to \text{coh}(G)$ taking $G$ to $h(G)$ commutes with arbitrary coproducts.

c) Condition (G2) holds for $G_0$ iff $h : T \to \text{coh}(G)$ commutes with arbitrary coproducts.

Proof. a) We have to show that, for each $X \in T$, the functor $T(\cdot, X)|_G$, which, $a\ priori$, is in $\text{Mod} G$, is in fact coherent. We choose a morphism $\prod_{i \in I} G_i \to X$, $G_i \in G_0$, such that each $G \to X$, $G \in G_0$, factors through a morphism $G_i \to X$. Then

$$T(\cdot, \prod_{i \in I} G_i)|_G \to T(\cdot, X)|_G$$

is an epimorphism in $\text{Mod} G$. We form a distinguished triangle

$$X' \to \prod_{i \in I} G_i \to X \to \Sigma X'$$

in $T$. We can continue the construction and choose a morphism $\prod_{i \in I'} G'_i \to X'$, $G'_i \in G_0$, such that each $G \to X'$, $G \in G_0$, factors through a morphism $G'_i \to X'$. Then the sequence

$$T(\cdot, \prod_{i \in I'} G'_i)|_G \to T(\cdot, \prod_{i \in I} G_i)|_G \to T(\cdot, X)|_G \to 0$$

is a presentation of $T(\cdot, X)|_G$.

b) Let $(G_i)_{i \in I}$ be a family of objects of $G$. We have to show that the canonical morphism

$$\text{coh}(G)(h(\prod_{i \in I} G_i), F) \to \prod_{i \in I} \text{coh}(G)(h(G_i), F)$$

is invertible for each coherent functor $F$. Since $h(G)$ is projective for each $G$ in $G$, it is enough to check this for representable functors $F$. For these, it follows from Yoneda’s lemma and the definition of $\prod_{i \in I} G_i$. 

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c) We suppose that (G2) holds for \( \mathcal{G}_0 \).

**First step.** For each family \((X_i)_{i \in I}\) of \( T \), the canonical morphism \( \prod_{i \in I} h(X_i) \rightarrow h(\prod_{i \in I} X_i) \) is an epimorphism. Indeed, for each \( i \in I \), let \( G_i \rightarrow X_i \) be a morphism such that

\[
h(G_i) \rightarrow h(X_i)
\]

is an epimorphism, where \( G_i \) belongs to \( \mathcal{G} \). By b), the functor \( h : \mathcal{G} \rightarrow \text{coh}(\mathcal{G}) \) commutes with coproducts. Thus, we obtain a commutative square

\[
\begin{array}{ccc}
\prod_{i \in I} h(G_i) & \rightarrow & \prod_{i \in I} h(X_i) \\
\downarrow & & \downarrow \varphi \\
\pi(\prod_{i \in I} G_i) & \rightarrow & \pi(\prod_{i \in I} X_i)
\end{array}
\]

By condition (G2), \( \pi \) is an epimorphism. Thus, \( \varphi \) is an epimorphism.

**Second step.** For each family \((X_i)_{i \in I}\) of \( T \), the canonical morphism \( \prod_{i \in I} h(X_i) \rightarrow h(\prod_{i \in I} X_i) \) is an isomorphism. Indeed, for each \( i \in I \), we choose distinguished triangles

\[X'_i \rightarrow G_i \rightarrow X_i \rightarrow \Sigma X'_i,\]

and morphisms \( G'_i \rightarrow X'_i \), where \( G_i \rightarrow X_i \) is as in the first step and \( G'_i \) belongs to \( \mathcal{G} \), such that

\[
h(G'_i) \rightarrow h(X'_i)
\]

is an epimorphism. Then the sequence

\[0 \rightarrow h(\prod_{i \in I} X'_i) \rightarrow h(\prod_{i \in I} G_i) \rightarrow h(\prod_{i \in I} X_i) \rightarrow 0\]

is exact. Indeed, coproducts preserve distinguished triangles and \( h \) is cohomological since it is the composition of the Yoneda functor with the restriction functor \( F \rightarrow F|\mathcal{G} \), which is clearly exact. In particular, \( i \) is a monomorphism. Since the coproduct functor \( \prod_{i \in I} \) is right exact, the top morphism of the square

\[
\begin{array}{ccc}
\prod_{i \in I} h(G'_i) & \rightarrow & \prod_{i \in I} h(X'_i) \\
\downarrow & & \downarrow \varphi \\
h(\prod_{i \in I} G'_i) & \rightarrow & h(\prod_{i \in I} X'_i)
\end{array}
\]

is an epimorphism. By the first step, it follows that the morphism \( \varphi \) is an epimorphism. By b), the morphism \( \prod_{i \in I} h(G'_i) \rightarrow h(\prod_{i \in I} G'_i) \) is an isomorphism. Therefore, the morphism

\[
h(\prod_{i \in I} G'_i) \rightarrow h(\prod_{i \in I} X'_i)
\]

is an epimorphism and the sequence

\[
h(\prod_{i \in I} G'_i) \rightarrow h(\prod_{i \in I} G_i) \rightarrow h(\prod_{i \in I} X_i) \rightarrow 0
\]

is exact. The claim now follows from b) since, \( \prod_{i \in I} \) being a right exact functor, we have a diagram with exact rows

\[
\begin{array}{ccc}
\prod_{i \in I} h(G'_i) & \rightarrow & \prod_{i \in I} h(G_i) \\
\downarrow & & \downarrow \varphi \\
h(\prod_{i \in I} G'_i) & \rightarrow & h(\prod_{i \in I} G_i)
\end{array}
\]

\[
\begin{array}{ccc}
\prod_{i \in I} h(G_i) & \rightarrow & \prod_{i \in I} h(X_i) \\
\downarrow & & \downarrow \varphi \\
h(\prod_{i \in I} G_i) & \rightarrow & h(\prod_{i \in I} X_i)
\end{array}
\]
We suppose now that $h$ commutes with coproducts. We will show that condition (G2) holds for $\mathcal{G}$. Let $(f_i : X_i \to Y_i)_{i \in I}$ be a family of morphisms in $\mathcal{T}$ such that $T(G, f_i) : T(G, X_i) \to T(G, Y_i)$ is surjective for all $i \in I$ and all $G \in \mathcal{G}_0$. Then $T(\prod_{i \in I} G_i, f_i) : T(\prod_{i \in I} G_i, X_i) \to T(\prod_{i \in I} G_i, Y_i)$ is surjective for all the families $(G_i)_{i \in I}$ of $\mathcal{G}_0$ and all $i \in I$, thanks to the isomorphisms $T(\prod_{i \in I} G_i, X_i) \to \prod_{i \in I} T(G_i, X_i)$. Moreover, it is trivial to verify that $T(A, f_i) : T(A, X_i) \to T(A, Y_i)$ is surjective, for all $i \in I$, for each direct factor $A$ of any object $G \in \mathcal{G}_0$. Therefore, $T(G, f_i)$ is surjective for all $i \in I$ and all $G \in \mathcal{G}$. Thus, $T(-, X_i)|_G \to T(-, Y_i)|_G$ is an epimorphism for all $i \in I$. The coproduct $\prod_{i \in I} T(-, X_i)|_G \to \prod_{i \in I} T(-, Y_i)|_G$ is still an epimorphism. Since $h$ commutes with coproducts, it follows that $T(G, \prod_{i \in I} X_i) \to T(G, \prod_{i \in I} Y_i)$ is surjective for all $G \in \mathcal{G}$, in particular for all $G \in \mathcal{G}_0$. 

Consider a triangulated category $\mathcal{T}$ and a class of its objects $\mathcal{G}_0$, satisfying some or all the conditions of definition 2.1. It will be important for us to know if these conditions continue to hold for different closures of $\mathcal{G}_0$.

**Proposition 2.4.** Let $\mathcal{T}$ be a cocomplete triangulated category, i.e. $\mathcal{T}$ admits all small coproducts. Let $\mathcal{G}_0$ be a class of objects in $\mathcal{T}$, stable under $\Sigma$ and $\Sigma^{-1}$, satisfying conditions (G2) and (G3) of the definition 2.1. Let $\alpha$ be an infinite cardinal. Let $\mathcal{G}$ be the closure of $\mathcal{G}_0$ under $\Sigma$ and $\Sigma^{-1}$, extensions and $\alpha$-small coproducts. Then, conditions (G3) and (G4) hold for $\mathcal{G}$.

**Proof.** (G3) We directly show that condition (G3) holds for shifts, $\alpha$-coproducts and extensions of objects in $\mathcal{G}_0$.

Since the functor $\Sigma : \mathcal{T} \to \mathcal{T}$ is an equivalence, an object $X$ of $\mathcal{T}$ is $\alpha$-small iff $\Sigma X$ is $\alpha$-small. Thus, condition (G3) holds for all objects $\Sigma^n G, G \in \mathcal{G}_0, n \in \mathbb{Z}$.

Since $\alpha$-small coproducts commute with $\alpha$-filtered colimits, condition (G3) holds for $\alpha$-small coproducts of objects of $\mathcal{G}_0$. Indeed, let $(G_j)_{j \in J}, |J| < \alpha$, be a family of $\alpha$-small objects of $\mathcal{G}_0$ and let $(X_i)_{i \in I}$ be an arbitrary family of objects of $\mathcal{T}$. We have the following sequence of isomorphisms:

$$\text{Hom}_\mathcal{T}(\prod_{j \in J} G_j, \prod_{i \in I} X_i) = \prod_{j \in J} \text{Hom}_\mathcal{T}(G_j, \prod_{i \in I} X_i)$$

$$= \prod_{j \in J} \text{colim}_{I' \subset I} \text{Hom}_\mathcal{T}(G_j, \prod_{i \in I'} X_i)$$

$$\text{colim}_{I' \subset I} \text{Hom}_\mathcal{T}(\prod_{j \in J} G_j, \prod_{i \in I'} X_i) = \text{colim}_{I' \subset I} \prod_{j \in J} \text{Hom}_\mathcal{T}(G_j, \prod_{i \in I'} X_i),$$

where the cardinality of the subset $I'$ is strictly smaller than $\alpha$. The only non trivial isomorphism is the vertical third which holds since the cardinal $\alpha$ is supposed regular, hence the colimit is taken over an $\alpha$-filtered set $I$.

Let us consider the (mapping) cone of an arbitrary morphism $G \to G'$ of $\mathcal{G}_0$

$$G \longrightarrow G' \longrightarrow C \longrightarrow \Sigma G.$$

We can form two long exact sequences by applying the cohomological functors

$$\text{Hom}_\mathcal{T}(-, \prod_{i \in I} X_i) \text{ and } \text{Hom}_\mathcal{T}(-, \prod_{i \in J} X_i)$$

to the last distinguished triangle. Now we consider the colimit over the subsets $J \subset I$ of cardinality strictly smaller than $\alpha$ of the long exact sequence induced by $\text{Hom}_\mathcal{T}(-, \prod_{i \in J} X_i)$. We obtain a long sequence which is still exact since we are using filtered colimits. There
is a natural map of the two long exact sequences just formed. Let us represent a part of it in the following diagram, where we write $\text{colim}_J$ for $\text{colim}_{J \subseteq I}$

\[
\begin{array}{cccccc}
\text{col}_J & T(G, \coprod_J X_i) & \xrightarrow{=} & \text{col}_J & T(G', \coprod_J X_i) & \xrightarrow{=} & \text{col}_J & T(C, \coprod_J X_i) & \xrightarrow{=} & \text{col}_J & T(\Sigma G, \coprod_J X_i)
\end{array}
\]

\[
\begin{array}{ccccc}
T(G, \coprod_I X_i) & \xrightarrow{=} & T(G', \coprod_I X_i) & \xrightarrow{=} & T(C, \coprod_I X_i) & \xrightarrow{=} & T(\Sigma G, \coprod_I X_i)
\end{array}
\]

The vertical arrows are isomorphisms since $G, G'$ are in $G_0$, and we have seen that $\Sigma G$ is $\alpha$-small. Thus, the third vertical arrow is an isomorphism by the Five-Lemma and $C$ is $\alpha$-small, too.

(G4) We call $\mathcal{U}$ the full subcategory of $T$ formed by the objects $X \in \mathcal{G}$ which satisfy the following condition. Given a morphism

\[
f : X \to \prod_{i \in I} Y_i,
\]

where $(Y_i)_{i \in I}$ is a family of objects in $T$, there exists a family $(X_i)_{i \in I}$ of objects of $\mathcal{G}$ and some morphisms $\varphi_i : X_i \to Y_i$ such that $f$ factors as in the diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & \prod_{i \in I} Y_i \\
& \searrow & \downarrow \\
& & \prod_{i \in I} X_i.
\end{array}
\]

We shall show:

a) the subcategory $\mathcal{U}$ contains $\mathcal{G}_0$;

b) the subcategory $\mathcal{U}$ is stable under formation of $\alpha$-coproducts;

c) the subcategory $\mathcal{U}$ is closed under $\Sigma, \Sigma^{-1}$ and under extensions.

It follows by the properties a), b), c) that $\mathcal{U} = \mathcal{G}$, which shows that the condition (G4) holds for $\mathcal{G}$.

a) Let $G_0$ be an object in $\mathcal{G}_0$ and $f : G_0 \to \prod_{i \in I} Y_i$ a morphism in $T$, where $(Y_i)_{i \in I}$ is a family of objects in $T$. For every $i \in I$, let $(G_{ij} \to Y_i)_{j \in J_i}$, where $G_{ij} \in \mathcal{G}$, be a family of morphisms such that every morphism $G_0 \to Y_i$ factors through one of the morphisms $G_{ij} \to Y_i$. Then, the morphism

\[
\varphi_i : \prod_{j \in J_i} G_{ij} \to Y_i
\]

induces a surjection

\[
\text{Hom}_T(G_0, \prod_{j \in J_i} G_{ij}) \to \text{Hom}_T(G_0, Y_i),
\]

for every $i \in I$. By (G2), the map

\[
\text{Hom}(G_0, \prod_{i \in I} \prod_{j \in J_i} G_{ij}) \to \text{Hom}(G_0, \prod_{i \in I} Y_i)
\]

is a surjection. Therefore, there exists a morphism

\[
\tilde{f} : G_0 \to \prod_{i \in I} \prod_{j \in J_i} G_{ij}
\]
such that the composition

\[
G_0 \xrightarrow{\tilde{f}} \prod_{i \in I} \prod_{j \in J_i} G_{ij} \xrightarrow{\prod_{i \in I} \varphi_i} \prod_{i \in I} Y_i
\]

is equal to \( f \). We have supposed that \( G \) is \( \alpha \)-small (condition (G3) holds for \( G_0 \)). Therefore the morphism

\[
G_0 \xrightarrow{\tilde{f}} \prod_{i \in I} \prod_{j \in J_i} G_{ij} = \prod_{(i,j) \in \mathcal{L}} G_{ij},
\]

where \( \mathcal{L} \) is the set of pairs \((i, j)\) with \( i \in I \) and \( j \in J_i \), factors through the sub-sum

\[
\prod_{(i,j) \in \Lambda} G_{ij} = \prod_{j \in \tilde{I}} \prod_{j \in \tilde{J}_i} G_{ij},
\]

where \( \Lambda \subseteq \mathcal{L} \) is a subset of cardinality strictly smaller than \( \alpha \). Let \( \tilde{I} \) be the set of indices \( i \in I \) such that \( \Lambda \) contains a pair of the form \((i, j)\). Then \( \tilde{I} \) is of cardinality strictly smaller than \( \alpha \). Now for each \( i \in \tilde{I} \), let \( \tilde{J}_i \) be the set of indices \( j \in J_i \) such that \( \Lambda \) contains the pair \((i, j)\). Then each \( \tilde{J}_i \) is of cardinality strictly smaller than \( \alpha \). Now for \( i \notin \tilde{I} \), put \( \tilde{J}_i = \emptyset \). Then we have

\[
\prod_{(i,j) \in \Lambda} G_{ij} = \prod_{i \in I} \prod_{j \in \tilde{J}_i} G_{ij}.
\]

Let \( Y_i = \prod_{j \in \tilde{J}_i} G_{ij} \). Then \( f \) factors as

\[
G_0 \xrightarrow{\tilde{f}} \prod_{i \in I} Y_i \xrightarrow{\prod_{i \in I} (\varphi_i | Y_i)} \prod_{i \in I} X_i.
\]

As \( |\tilde{J}| < \alpha \), \( Y_i \) lies in \( \mathcal{G} \) for all \( i \in I \).

b) Let \((U_j)_{j \in J}\) be a family of \( \mathcal{U} \) where \( |J| < \alpha \). Let

\[
f : \prod_{j \in J} U_j \longrightarrow \prod_{i \in I} X_i,
\]

be a morphism in \( \mathcal{T} \), where \((X_i)_{i \in I}\) is a family of \( \mathcal{T} \). Let

\[
f_j : U_j \longrightarrow \prod_{i \in I} X_i
\]

be the component of \( f \) associated to \( j \in J \). For each \( j \in J \), since \( U_j \) lies in \( \mathcal{U} \), there exists a factorization

\[
U_j \xrightarrow{f_j} \prod_{i \in I} X_i \xrightarrow{\prod_{i \in I} \varphi_i} \prod_{i \in I} Y_{ji},
\]

where \((Y_{ji})_{i \in I}\) is a family of \( \mathcal{G} \). Then, we have the factorization

\[
\prod_{j \in J} U_j \xrightarrow{\varphi} \prod_{j \in J} \prod_{i \in I} Y_{ji} \xrightarrow{\prod_{i \in I}} \prod_{i \in I} X_i,
\]
which we can write as

\[
\prod_{j \in J} U_j \longrightarrow \prod_{i \in I} \prod_{j \in J} Y_{ji} \longrightarrow \prod_{i \in I} X_i,
\]

where \( \prod_{j \in J} Y_{ji} \) belongs to \( \mathcal{G} \) since \(|J| < \alpha\). Therefore, \( \prod_{j \in J} U_j \) lies in \( \mathcal{U} \).

c) Clearly, \( \mathcal{U} \) is stable under the action of \( \Sigma \) and \( \Sigma^{-1} \). Let

\[
X \longrightarrow X' \longrightarrow X'' \longrightarrow \Sigma X
\]

be a distinguished triangle of \( \mathcal{T} \) such that \( X, X' \) are in \( \mathcal{U} \). Let

\[
X'' \longrightarrow \prod_{i \in I} Y_i
\]

be a morphism of \( \mathcal{T} \) where \((Y_i)_{i \in I}\) is a family of \( \mathcal{T} \). We have the factorization

| \[
\begin{array}{c}
X' \\
\downarrow f' \\
\prod_{i \in I} X'_i
\end{array}
\longrightarrow
\begin{array}{c}
X'' \\
\downarrow f'' \\
\prod_{i \in I} Y_i
\end{array}
\longrightarrow
\begin{array}{c}
\Sigma X \\
\downarrow \Sigma f \\
\prod_{i \in I} Z_i
\end{array}
\]

where \( X' \in \mathcal{G} \) and \( \varphi_i : X'_i \to Y_i \) are morphisms in \( \mathcal{T} \) for all \( i \in I \). We can extend each \( \varphi_i \) to a distinguished triangle, take coproducts over \( I \) and then complete the square above to a morphism of distinguished triangles (using axiom TR3 of triangulated categories):

\[
\begin{array}{c}
X \\
\downarrow f \\
\prod_{i \in I} X_i
\end{array}
\longrightarrow
\begin{array}{c}
X' \\
\downarrow f' \\
\prod_{i \in I} X'_i
\end{array}
\longrightarrow
\begin{array}{c}
X'' \\
\downarrow f'' \\
\prod_{i \in I} Y_i
\end{array}
\longrightarrow
\begin{array}{c}
\Sigma X \\
\downarrow \Sigma f \\
\prod_{i \in I} Z_i
\end{array}
\]

The objects \( X, X' \) and \( \Sigma X \) belong to \( \mathcal{U} \). Thus, the morphisms \( f, f' \) and \( \Sigma f \) above factor through a coproduct taken over \( I \) of objects in \( \mathcal{G} \). We have the commutative diagram

\[
\begin{array}{c}
X \\
\downarrow f \\
\prod_{i \in I} X_i
\end{array}
\longrightarrow
\begin{array}{c}
X' \\
\downarrow f' \\
\prod_{i \in I} X'_i
\end{array}
\longrightarrow
\begin{array}{c}
X'' \\
\downarrow f'' \\
\prod_{i \in I} Y_i
\end{array}
\longrightarrow
\begin{array}{c}
\Sigma X \\
\downarrow \Sigma f \\
\prod_{i \in I} Z_i
\end{array}
\]

where the morphisms \( u_i : X_i \to X'_i, i \in I \), are in \( \mathcal{G} \). Now, we extend the morphisms \( u_i \) to distinguished triangles and then form the distinguished triangle of coproducts over \( I \). Successively, we form morphisms of distinguished triangles using axiom TR3 of triangulated
Note that the subdiagram between $X''$ and $\prod_{i \in I} Y_i$ does not commute, i.e., the composition $(\prod_I \psi_i) \circ g$ is in general not equal to $f''$. Anyway, by composing with $\prod_I \varepsilon_i$, we obtain

$$\left(\prod_I \varepsilon_i\right) \circ (\prod_I \psi_i) \circ g = (\prod_I \varepsilon_i) \circ f''.$$ 

Therefore, by applying $\text{Hom}_T(X'', \_)$ to the distinguished triangle in the third row of the last diagram, it is immediate that

$$\left(\prod_I \psi_i\right) \circ g - f'' = \left(\prod_I \varphi_i\right) \circ h,$$

for some morphism $h : X'' \to \prod_{i \in I} X'_i$, as in the diagram. Then, the correct expression of $f''$ is

$$f'' = \left(\prod_I \psi_i\right) \circ g + \left(\prod_I \varphi_i\right) \circ (-h)$$

which shows that $f''$ factors as

$$X'' \xrightarrow{[g\ -h]} \prod_{i \in I} X''_i \oplus \prod_{i \in I} X'_i \xrightarrow{\left[\prod_I \psi_i\right], \left[\prod_I \varphi_i\right]} \prod_{i \in I} Y_i.$$

The previous factorization of $f''$ is trivially equivalent to the following

$$X'' \xrightarrow{[g\ -h]} \prod_{i \in I} (X''_i \oplus X'_i) \xrightarrow{\left[\prod_I \psi_i\right], \left[\prod_I \varphi_i\right]} \prod_{i \in I} Y_i.$$

Now, $X''_i \oplus X'_i$ is in $\mathcal{G}$ for all $i \in I$ by construction.

There are two immediate and useful corollaries.

**Corollary 2.5.** Let $\mathcal{T}$ be a cocomplete triangulated category. Let $\mathcal{G}_0$ be a class of objects in $\mathcal{T}$ satisfying all the conditions of the last proposition. Let $\alpha$ be an infinite cardinal. Let $\mathcal{G}$ be the closure of $\mathcal{G}_0$ under $\Sigma$ and $\Sigma^{-1}$, extensions, $\alpha$-small coproducts and direct factors, i.e., $\mathcal{G} = (\mathcal{G}_0)_\alpha$ in the notation of [12] below. Then, conditions (G3) and (G4) hold for $\mathcal{G}$.

**Proof.** The proof of the preceding proposition works for (G4) if we verify that the subcategory $\mathcal{U}$ is also closed under direct factors, i.e., that it is thick [12].

Let $U$ be an object in $\mathcal{U}$ and $U = U' \oplus U''$. Then, there is a section $i$ of the projection $p : U \to U'$. Let $f : U' \to \prod_{i \in I} W_i$ be a morphism in $\mathcal{T}$. The composition $f \circ p$ factors as

$$U \xrightarrow{p} U' \xrightarrow{f} \prod_{i \in I} W_i.$$
where the objects $V_i$ are in $G$ and the morphisms $\phi : V_i \to W_i$ in $T$, for all $i \in I$. Then $f$ also factors over $\coprod_{i \in I} V_i$, through the morphism $g \circ i$. Indeed, $f \circ p = (\coprod_i \phi_i) \circ g$, and $f \circ p \circ i = (\coprod_i \phi_i) \circ g \circ i$, but $p \circ i$ is the identity morphism of $U'$.

The proof of the preceding proposition works for (G3) if we verify that the direct factors of the objects in $G_0$ are $\alpha$-small, too. This requires the construction of a diagram structurally identical to the one above. Therefore, we omit it. \hfill $\square$

**Corollary 2.6.** Let $\alpha$ be an infinite regular cardinal. Let $T$ be a triangulated category $\alpha$-compactly generated by a set $G_0$. Let $G$ be the closure of $G_0$ under $\Sigma$ and $\Sigma^{-1}$, extensions and $\alpha$-small coproducts. Let $(G_0)_\alpha$ be the closure of $G$ under direct factors. Then, $T$ is $\alpha$-compactly generated by both $G$ and $(G_0)_\alpha$.

**Proof.** The condition (G1) clearly holds for both $G$ and $(G_0)_\alpha$, since they contain $G_0$. The conditions (G3) and (G4) hold for $G$ by proposition 2.4 and for $(G_0)_\alpha$ by corollary 2.5. Moreover, condition (G4) easily implies (G2). \hfill $\square$

### 2.2. Equivalences of well generated triangulated categories

This subsection is devoted to establishing a small set of conditions which allows us to show that two well generated triangulated categories are triangle equivalent.

**Proposition 2.7.** Let $T$ and $T'$ be two triangulated categories admitting arbitrary set-indexed coproducts. Let $\alpha$ be a regular cardinal and $G \subset T$ and $G' \subset T'$ two $\alpha$-localizing subcategories, i.e. thick and closed under formation of $\alpha$-small coproducts [3.2]. Suppose that $G$ and $G'$ satisfy conditions (G1), (G2), (G3) for the cardinal $\alpha$. Let $F : T \to T'$ be a triangle functor which commutes with all coproducts and induces an equivalence $G \to G'$. Then $F$ is an equivalence of triangulated categories.

**Proof.** 1st step: The functor $F$ induces an equivalence

$$\text{Add } G \to \text{Add } G'.$$

As $F$ commutes with coproducts and induces a functor $G \to G'$, $F$ induces a functor $\text{Add } G \to \text{Add } G'$. Clearly, the induced functor is essentially surjective. Let us show that it is fully faithful. For any objects $G$ and $G'$ in $\text{Add } G$ we consider the map

$$F(G, G') : T(G, G') \to T'(FG, FG').$$

By hypothesis, it is bijective if $G$ and $G'$ are in $G$. Let $G$ be in $G$ and $G' = \coprod_{i \in I} G'_i$, where $(G'_i)_{i \in I}$ is a family in $G$. Then, $F(G, G')$ is still bijective since we have the following sequence of isomorphisms

$$T(G, G') = T(G, \coprod_{i \in I} G'_i)$$

$$= \text{colim}_{J \subset I} T(G, \coprod_{i \in J} G'_i)$$

$$\cong \text{colim}_{J \subset I} T'(F(G), F(\coprod_{i \in J} G'_i))$$

$$\cong \text{colim}_{J \subset I} T'(F(G), \coprod_{i \in J} F(G'_i))$$

$$= T'(F(G), \coprod_{i \in I} F(G'_i))$$

$$T'(F(G), F(G')) \cong T'(F(G), F(\coprod_{i \in I} G'_i)).$$
where $J$ runs through the subsets of cardinality strictly smaller than $\alpha$ of $I$. Here, we have used: (1) $G$ is $\alpha$-small; (2) $G$ contains $\prod_{i \in J} G_i$ since $G$ is $\alpha$-localizing; (3) $F$ commutes with coproducts; (4) $F(G)$ is $\alpha$-small; (5) $F$ commutes with coproducts. If $G'$ is in Add $\mathcal{G}$ and $G = \prod_{i \in I} G_i$, where $(G_i)_{i \in I}$ is a family in $\mathcal{G}$, we have

$$T(G, G') = T(\prod_{i \in I} G_i, G')$$

$$\cong \prod_{i \in I} T(G_i, G')$$

$$\cong \prod_{i \in I} T'(F(G_i), F(G'))$$

$$\cong T'(\prod_{i \in I} F(G_i), F(G'))$$

$$T'(F(G), F(G')) = T'(F(\prod_{i \in I} G_i), F(G')).$$

2nd step: For each object $G$ in $\mathcal{G}$ and each object $X$ in $T$, $F$ induces a bijection

$$T(G, X) \longrightarrow T'(FG, FX).$$

Let $\mathcal{U}$ be the full subcategory of $T$ formed by the objects $X$ such that $F$ induces a bijection

$$T(G, X) \longrightarrow T'(FG, FX),$$

for each $G$ in $\mathcal{G}$. Clearly, $\mathcal{U}$ is a triangulated subcategory. Let us show that $\mathcal{U}$ is stable under formation of coproducts. Let $(X_i)_{i \in I}$ be a family of objects in $\mathcal{U}$. We will show that the map

$$T(G, \prod_{i \in I} X_i) \longrightarrow T'(FG, F(\prod_{i \in I} X_i)) = T'(FG, \prod_{i \in I} F(X_i))$$

is bijective. Let us show that it is surjective. Let

$$f : FG \longrightarrow \prod_{i \in I} F(X_i)$$

a morphism in $T'$. The condition (G4) holds for the subcategory $\mathcal{G}'$ by corollary 2.5. Therefore, as $F$ is an equivalence $\mathcal{G} \rightarrow \mathcal{G}'$, there exists a family of objects $(G_i)_{i \in I}$ in $\mathcal{G}$ and a factorization of $f$

$$FG \xrightarrow{g} \prod_{i \in I} F(G_i) \xrightarrow{\prod_{i \in I} h_i} \prod_{i \in I} F(X_i),$$

for a family of morphisms $h_i : F(G_i) \rightarrow F(X_i)$. As each $X_i$ is in $\mathcal{U}$, we have $h_i = F(k_i)$ for some morphisms $k_i : G_i \rightarrow X_i$. Since the object

$$\prod_{i \in I} F(G_i) = F(\prod_{i \in I} G_i)$$

is in Add $\mathcal{G}'$ and $F$ induces an equivalence

$$\text{Add } \mathcal{G} \xrightarrow{\sim} \text{Add } \mathcal{G}',$$

there exists a morphism $l : G \rightarrow \prod_{i \in I} G_i$ such that $F(l)$ gives $g$. Thus, $f$ is the image of the composition

$$G \xrightarrow{l} \prod_{i \in I} G_i \xrightarrow{\prod_{i \in I} k_i} \prod_{i \in I} X_i.$$
under $F$. Let us show that it is injective. Let

$$f : G \rightarrow \coprod_{i \in I} X_i$$

be a morphism such that $F(f) = 0$. As $\mathcal{G}$ has property (G4) by corollary 2.5, we have a factorization

$$G \xrightarrow{g} \coprod_{i \in I} G_i \xrightarrow{\coprod_{i \in I} h_i} \coprod_{i \in I} X_i,$$

for a family of objects $G$ in $\mathcal{G}$ and a family of morphisms $h_i : G_i \rightarrow X_i$. We have

$$F(\coprod_{i \in I} h_i) \circ F(g) = 0.$$

Let us extend the morphism $\coprod_{i \in I} h_i$ and form a distinguished triangle

$$\coprod_{i \in I} Y_i \xrightarrow{\coprod_{i \in I} k_i} \coprod_{i \in I} G_i \xrightarrow{\coprod_{i \in I} h_i} \coprod_{i \in I} X_i \rightarrow \Sigma \coprod_{i \in I} Y_i.$$

There exists a morphism $m : FG \rightarrow \coprod_{i \in I} Y_i$ such that

$$F(\coprod_{i \in I} k_i) \circ m = F(g).$$

Note that each $Y_i$ is in $\mathcal{U}$ since $G_i$ and $X_i$ are in $\mathcal{U}$. By the surjectivity already shown, we have

$$m = F(l)$$

for a morphism $l : G \rightarrow \coprod_{i \in I} Y_i$. Thus,

$$F(\coprod_{i \in I} k_i \circ l) = F(g).$$

As $G$ and $\coprod_{i \in I} G_i$ are in Add $\mathcal{G}$, it follows that

$$\prod_{i \in I} k_i \circ l = g.$$

Thus,

$$f = (\prod_{i \in I} h_i) \circ g = (\prod_{i \in I} h_i) \circ (\prod_{i \in I} k_i) \circ l = 0.$$

3\textsuperscript{rd} step: The functor $F$ is fully faithful.

Let $Y$ be an object in $\mathcal{T}$. Let $\mathcal{U}$ be the full subcategory of $\mathcal{T}$ formed by the objects $X$ such that $F$ induces a bijection

$$\mathcal{T}(X, Y) \rightarrow \mathcal{T}'(FX, FY).$$

By the second step, $\mathcal{U}$ contains $\mathcal{G}$. Clearly, $\mathcal{U}$ is a triangulated subcategory. Let us show that $\mathcal{U}$ is stable under formation of coproducts. Let $(X_i)_{i \in I}$ be a family of objects in $\mathcal{U}$. Then we have

$$\mathcal{T}(\coprod_{i \in I} X_i, Y) \cong \prod_{i \in I} \mathcal{T}(X_i, Y) \cong \prod_{i \in I} \mathcal{T}'(FX_i, F(Y)) = \mathcal{T}'(\coprod_{i \in I} F(X_i), F(Y)).$$
Thus, $\coprod_{i \in I} X_i$ is indeed in $\mathcal{U}$. It is easy to see that $\mathcal{U}$ contains the direct factors of its objects. So, we have checked that $\mathcal{U}$ is an $\alpha$-localizing subcategory of $\mathcal{T}$ and contains $\mathcal{G}$. By proposition \ref{prop:localizing} below, the smallest localizing subcategory containing $\mathcal{G}$ is the whole category $\mathcal{T}$. It follows that $\mathcal{U} = \mathcal{T}$.

4th step: The functor $F$ is essentially surjective.

The functor $F$ induces an equivalence from $\mathcal{T}$ onto a localizing subcategory $\mathcal{V}$ of $\mathcal{T}'$ by the third step. Indeed, $\mathcal{V}$ is triangulated, stable under coproducts and thick since $F$ is a triangle functor commuting with coproducts. It follows that $\mathcal{V} = \mathcal{T}'$, as $\mathcal{V}$ contains $\mathcal{G}'$, which generates $\mathcal{T}'$. $\square$

It remains to find conditions such that the functor $F$ of the preceding proposition commutes with coproducts. This is made in the following

**Theorem 2.8.** Let $\alpha$ be a regular cardinal. Let $\mathcal{T}$ and $\mathcal{T}'$ be two cocomplete triangulated categories. Let $\mathcal{G} \subset \mathcal{T}$ and $\mathcal{G}' \subset \mathcal{T}'$ be two $\alpha$-localizing subcategories, both of them satisfying conditions (G1), (G2), (G3) for $\alpha$. Let $F: \mathcal{T} \to \mathcal{T}'$ be a triangle functor. Suppose that $F$ induces a functor

$$\mathcal{G} \to \mathcal{G}'$$

which is essentially surjective and induces bijections

$$T(G, X) \xrightarrow{\sim} T'(FG, FX)$$

for all $G$ in $\mathcal{G}$ and $X$ in $\mathcal{T}$. Then $F$ is an equivalence of triangulated categories.

**Remark 2.9.** We do not suppose that $F$ commutes with coproducts.

**Proof.** 1st step: For each family $(G_i)_{i \in I}$ in $\mathcal{G}$, the morphism

$$\coprod_{i \in I} F(G_i) \to F(\coprod_{i \in I} G_i)$$

is invertible.

It is sufficient to show that, for all $G'$ in $\mathcal{G}'$, the map

$$T'(G', \coprod_{i \in I} F(G_i)) \to T'(G', F(\coprod_{i \in I} G_i))$$

is bijective since $\mathcal{G}'$ verifies (G1). As $F : \mathcal{G} \to \mathcal{G}'$ is essentially surjective, it is sufficient to verify this for $G' = FG$ for all $G$ in $\mathcal{G}$. Let $G \in \mathcal{G}$. We have

\begin{align*}
T'(FG, \coprod_{i \in I} FG_i) &= \colim_{J \subseteq I} T'(FG, \coprod_{i \in J} FG_i) \\
&= \colim_{J \subseteq I} T'(FG, \coprod_{i \in J} G_i) \\
&= \colim_{J \subseteq I} T(G, \coprod_{i \in J} G_i) \\
&= T(G, \coprod_{i \in I} G_i),
\end{align*}

where $J$ are subsets of $I$ of cardinality strictly smaller than $\alpha$. Here, we have used: \(\mathcal{G} = \mathcal{G}'\) is $\alpha$-small; \(\mathcal{G}'\) has $\alpha$-small coproducts; \(F\) induces an equivalence $\mathcal{G} \to \mathcal{G}'$; \(G\)
is $\alpha$-small. On the other hand, we have
\[ T'(FG, F \prod_{i \in I} G_i) \Rightarrow T(G, \prod_{i \in I} G_i), \]
by the hypothesis, with $X = \prod_{i \in I} G_i$.

2nd step: The functor $F$ induces an equivalence $\text{Add}\, \mathcal{G} \to \text{Add}\, \mathcal{G}'$.

By the first step and the essential surjectivity of $F: \mathcal{G} \to \mathcal{G}'$, $F$ induces an essentially surjective functor from $\text{Add}\, \mathcal{G} \to \text{Add}\, \mathcal{G}'$. By hypothesis, for $G$ in $\mathcal{G}$ and $X$ in $\text{Add}\, \mathcal{G}$, $F$ induces a bijection
\[ T(G, X) \to T'(FG, FX). \]

Let $(G_i)_{i \in I}$ be a family in $\mathcal{G}$ and $X$ an object in $\text{Add}\, \mathcal{G}$. Then,
\[
\begin{align*}
T(\prod_{i \in I} G_i, X) & \Rightarrow \prod_{i \in I} T(G_i, X) \\
& \Rightarrow \prod_{i \in I} T'(F(G_i), F(X)) \\
T'(F(\prod_{i \in I} G_i), F(X)) & = T'(\prod_{i \in I} F(G_i), F(X)).
\end{align*}
\]
Thus, $F$ restricted to $\text{Add}\, \mathcal{G}$ is fully faithful.

3rd step: The functor $F$ commutes with coproducts.

Let us consider the diagram
\[
\begin{array}{ccc}
T & \xrightarrow{F} & T' \\
\downarrow^{h_T} & & \downarrow^{h'_T} \\
\text{coh}(\text{Add}\, \mathcal{G}) & \xrightarrow{\sim\, F^*} & \text{coh}(\text{Add}\, \mathcal{G}').
\end{array}
\]
We will show that it commutes up to isomorphism. Let $X$ be an object in $T$ and let
\[ h(G_1) \to h(G_0) \to h(X) \to 0 \]
be a projective presentation, where $G_1$, $G_0$ are in $\text{Add}\, \mathcal{G}$. Then, for all $G$ in $\mathcal{G}$, we obtain an exact sequence
\[ T(G, G_1) \to T(G, G_0) \to T(G, X) \to 0. \]
Therefore, the sequence
\[ T'(FG, FG_1) \to T'(FG, FG_0) \to T'(FG, FX) \to 0 \]
is exact (since isomorphic to the first). It follows that the sequence
\[ h(FG_1) \to h(FG_0) \to h(FX) \to 0 \]
is exact (since the objects $h(FG)$, $G \in \mathcal{G}$, form a family of projective generators of $\text{coh}(\text{Add}\, \mathcal{G}')$). Thus,
\[ F^*(h(X)) = \text{cok}(h(FG_1) \to h(FG_0)) \]
is indeed canonically isomorphic to $h(FX)$. To conclude, note that $F^*$ (which is an equivalence!) and $h_T$ commute with coproducts and that $h'_T$ detects the isomorphisms.

4th step: The claim follows thanks to the preceding proposition 2.7.\[ \square \]
3. Thick subcategories and localization of triangulated categories

We recall now some known results about the localizations of triangulated categories and about their thick subcategories, before stating the most important theorem of this section concerning the localization of well generated triangulated categories. For complete proofs of the cited results, we refer to Neeman's book [24, Ch. 2, p. 73] and the classical [35, Ch. 2.2, p. 111-133].

3.1. Localization of triangulated categories. We begin with a collection of properties of the triangle quotient [35, Ch. 2.2, p. 111-133] of triangulated categories.

Proposition 3.1. Let \( T \) be a triangulated category with arbitrary coproducts, let \( \Phi \) be a set of morphisms in \( T \) and \( N \) the smallest triangulated subcategory of \( T \) containing the cone(s), with \( s \in \Phi \), stable under arbitrary coproducts. Then the following assertions hold:

a) \( T/N \) is a triangulated category and admits arbitrary coproducts;

b) the canonical functor \( Q : T \to T/N \) commutes with all coproducts;

c) the morphisms \( Q(s) \) are invertible for all \( s \in \Phi \);

d) if \( F : T \to S \) is a triangle functor, where \( S \) is a triangulated category which admits all coproducts and the functor \( F \) commutes with all coproducts and makes every \( s \in \Phi \) invertible in \( S \), then \( F = \overline{F} \circ Q \) for a unique coproduct preserving triangle functor \( \overline{F} : T/N \to S \);

e) more precisely, if \( S \) is a triangulated category with arbitrary coproducts, there is an isomorphism of categories

\[
\mathcal{F}un_{cont}(T/N, S) \cong \mathcal{F}un_{cont, \Phi}(T, S),
\]

where \( \mathcal{F}un_{cont} \) is the category of triangulated functors commuting with arbitrary coproducts, and \( \mathcal{F}un_{cont, \Phi} \) is the category of the functors in \( \mathcal{F}un_{cont} \) which have the additional property of making all \( s \in \Phi \) invertible.

Proof. See Chapter 2 in [24] or [35, Ch. 2.2, p. 111-133]. We give only an argument for the commutativity of \( Q \) with coproducts because it is a general one, useful in other contexts. Let \( \prod_I T \) be the product category of copies of \( T \) indexed by \( I \). Using the universality of coproducts it is easy to check that the functor \( \coprod_{i \in I} : \prod_I T \to T \) which takes a family \( (X_i)_{i \in I} \) to the coproduct \( \coprod_{i \in I} X_i \) is left adjoint to the diagonal functor \( \Delta \). It is clear that \( \Delta(\Phi) \subseteq \prod_I (\Phi) \) and that \( \coprod_{i \in I} (\prod_I (\Phi)) \subseteq \Phi \). Therefore, the pair \( \coprod_{i \in I} \dashv \Delta \) induces the following commutative diagram

\[
\begin{array}{ccc}
\prod_I T & \xrightarrow{can} & (\prod_I T)[(\prod_I \Phi)^{-1}] \\
\coprod_{i \in I} \Delta \downarrow & & \downarrow \coprod_{i \in I} \Delta \\
T & \xrightarrow{can} & T[\Phi^{-1}]
\end{array}
\]

which entails the required commutativity of \( Q \) with all the coproducts over the set \( I \). Of course, this construction is possible for every set \( I \). \( \square \)

The functor \( Q \) is usually called (canonical) quotient functor, even if \( T \) does not have coproducts. In general, the morphisms between two objects in a triangle quotient do not form a set. However, this is the case if the quotient functor \( Q \) admits a right adjoint \( Q_\rho \), because \( Q_\rho \) is automatically fully faithful.

Definition 3.2. Let \( T \) and \( T' \) be triangulated categories. A triangle functor \( F : T' \to T \) is a localization functor if it is fully faithful and admits a left adjoint functor.
If $F : \mathcal{T}' \to \mathcal{T}$ is a localization functor and $F_\lambda$ is left adjoint, then $F_\lambda$ induces an equivalence from the triangle quotient $\mathcal{T}/\ker(F_\lambda)$ to $\mathcal{T}'$. Via this equivalence, $F_\lambda$ identifies with the quotient functor $\mathcal{T} \to \mathcal{T}/\ker(F_\lambda)$ and $F$ with its right adjoint.

### 3.2. Some thick subcategories of triangulated categories.

Let us recall that a full triangulated subcategory of a triangulated category is called **thick**, (épaisse in the French literature, saturée in the original definition in Verdier’s thesis [35, 2.2.6, p. 114]) if it contains the direct factors of its objects. We remark that this property is automatically verified if the triangulated category has countable coproducts, since in this case idempotents split (see [2], [24], Prop. 1.6.8, p. 65) for definitions and properties of idempotents in triangulated categories.

Now we give definitions and notations about some important subcategories of a triangulated category $\mathcal{T}$ with arbitrary coproducts and suspension functor $\Sigma$. The best reference for this material is [24, Ch. 3-4]. We recall that a full triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ is called **localizing** if it is closed under arbitrary coproducts. It is called **$\alpha$-localizing**, for a given regular cardinal $\alpha$, if it is thick and closed under $\alpha$-coproducts of its objects, i.e., coproducts of objects formed over a set of cardinality strictly smaller than $\alpha$. We write $\langle S \rangle_\alpha$ for the smallest $\alpha$-localizing subcategory of $\mathcal{T}$ containing $S$, where $S$ is a set or a class of objects in $\mathcal{T}$ and $\alpha$ a regular cardinal. Note that in the above definitions the requirement that the subcategories are thick is necessary only for the case $\alpha = \aleph_0$, since for $\alpha > \aleph_0$ these subcategories are automatically thick as we underlined at the beginning of the section. In his book [24, Ch. 3-4] Neeman shows the very important properties of the previous subcategories and of the subcategories of the $\alpha$-small objects $\mathcal{T}^{(\alpha)}$ and that of the $\alpha$-compact objects $\mathcal{T}^\alpha$. They are triangulated, $\alpha$-localizing and thick subcategories of $\mathcal{T}$ for $\alpha > \aleph_0$. There is the following filtration: if $\alpha \leq \beta$ then $\mathcal{T}^\alpha \subset \mathcal{T}^\beta$. If $S$ contains a good set of generators for $\mathcal{T}$, then $\mathcal{T} = \bigcup_\alpha \langle S \rangle_\alpha$, where $\alpha$ runs over all regular cardinals. If $\mathcal{T}$ is a well generated triangulated category, then $\mathcal{T} = \bigcup_\alpha \mathcal{T}^\alpha$, where $\alpha$ runs over all regular cardinals.

We have the

**Theorem 3.3** ([24, Lemma 3.2.10, p. 107]). Let $\beta$ be an infinite cardinal. Let $\mathcal{T}$ be a triangulated category closed under the formation of coproducts of fewer than $\beta$ of its objects. Let $\mathcal{N}$ be a $\beta$-localising subcategory of $\mathcal{T}$. Then $\mathcal{T}/\mathcal{N}$ is closed with respect to the formation of coproducts of fewer than $\beta$ of its objects, and the universal functor $F : \mathcal{T} \to \mathcal{T}/\mathcal{N}$ preserves coproducts.

and its

**Corollary 3.4** ([24, Cor. 3.2.11, p. 110]). If $\mathcal{T}$ is a triangulated category with all coproducts and $\mathcal{N}$ is a localizing subcategory of $\mathcal{T}$, then $\mathcal{T}/\mathcal{N}$ is a triangulated category which admits all coproducts and the universal functor $\mathcal{T} \to \mathcal{T}/\mathcal{N}$ preserves coproducts.

Now we state one of the major results in the theory of triangulated categories. This result has a long story (see for example [3], [23], [10, Ch. 10]), which comes from algebraic topology. We state it in the modern and general form Neeman gives it in his book, see [24, Thm. 1.17, p. 16] and [24, Thm. 8.3.3, p. 282] for a more general statement and the proof.

**Theorem 3.5.** (Brown representability). Let $\mathcal{T}$ be a well-generated triangulated category. Let $H$ be a contravariant functor $H : \mathcal{T}^{op} \to \mathsf{Ab}$. The functor $H$ is representable if and only if it is cohomological and takes coproducts in $\mathcal{T}$ to products of abelian groups.

Let us now clarify the meaning of two similar but different notions. Let $\mathcal{T}$ be a triangulated category and $\mathcal{G}$ a set of objects in $\mathcal{T}$. We say that $\mathcal{T}$ is **generated by $\mathcal{G}$** or, equivalently, that $\mathcal{G}$ **generates $\mathcal{T}$** if $\mathcal{T} = \langle \mathcal{G} \rangle$. In contradistinction, we say that $\mathcal{G}$ is a **generating set** for
Proof. Let us call \( \mathcal{N} \) the subcategory \( \langle \mathcal{G} \rangle \). Since \( \mathcal{N} \) is a localizing subcategory generated by the set \( \mathcal{G} \), it is well generated by corollary 3.12 below. Then, the Brown representability theorem \( (3.11) \) holds for \( \mathcal{N} \). Therefore, for each object \( X \in \mathcal{T} \), the functor \( \text{Hom}_{\mathcal{T}}(-,X) \mid_{\mathcal{G}} : \langle \mathcal{G} \rangle^{\text{op}} \to \text{Ab} \), which is cohomological and sends coproducts into products, is representable. For each object \( X \in \mathcal{T} \) there exists an object \( X_{\mathcal{N}} \) in \( \mathcal{N} \) such that \( \text{Hom}_{\mathcal{T}}(-,X) \mid_{\mathcal{G}} \cong \text{Hom}_{\mathcal{N}}(-,X_{\mathcal{N}}) \). Thus, we have obtained a functor \( i_{\mathcal{G}} \) right adjoint to the fully faithful inclusion: \( i : \mathcal{N} \to \mathcal{T} \). Consider now, for every \( X \in \mathcal{T} \), the distinguished triangle in \( \mathcal{T} \)

\[
i_{\mathcal{G}}X \longrightarrow X \longrightarrow Y \longrightarrow \Sigma i_{\mathcal{G}}X.
\]

Applying to the triangle the covariant functor \( \text{Hom}_{\mathcal{T}}(iN, -) \), \( N \) an object of \( \mathcal{N} \), we obtain a long exact sequence of abelian groups. Consider the part corresponding to the input triangle: The map from the first term \( \text{Hom}_{\mathcal{T}}(iN, i_{\mathcal{G}}X) \) to the second term \( \text{Hom}_{\mathcal{T}}(iN, X) \) is easily seen to be an isomorphism, since \( i \) is fully faithful and \( i_{\mathcal{G}} \) is its right adjoint. Similarly, the map from the fourth to the fifth term is an isomorphism. Therefore, the third group \( \text{Hom}_{\mathcal{T}}(iN, Y) \) must be zero for all \( N \in \mathcal{N} \). This forces the object \( Y \) to lie in \( \mathcal{N}^\perp \). But \( \mathcal{N}^\perp \) is zero. Indeed, condition (G1) holds for \( \mathcal{G} \), i.e. \( \mathcal{G}^\perp = 0 \). Thus, the inclusion \( \mathcal{G} \subseteq \mathcal{N} \) gives \( \mathcal{N}^\perp \subseteq \mathcal{G}^\perp = 0 \), i.e. \( \mathcal{N}^\perp = 0 \). Therefore, we have \( Y = 0 \). By the triangle above, this means \( i_{\mathcal{G}}X \cong X \), for all \( X \) in \( \mathcal{T} \). It follows that \( i \) is an equivalence of categories, which gives \( \mathcal{T} = \langle \mathcal{G} \rangle \).  

3.3. Localization of well generated triangulated categories. In this section, we will state a Theorem about particular localizations of well generated triangulated categories, those which are triangle quotients by a subcategory generated by a set. One could obtain this result using Thomason’s powerful Theorem [34] Key Proposition 5.2.2, p. 338] in its generalized form given by Neeman in [24] Thm. 4.4.9, p. 143]. Neeman himself does this in [25] in proving that the derived category of a Grothendieck category is always a well generated triangulated category. We will give a more detailed and slightly different proof in order to make clear the machinery behind Thomason-Neeman’s Theorem. Before doing this task we recall the key ingredient of the proof.

Theorem 3.7 ([24] Ch. 4, Thm. 4.3.3, p. 131]). Let \( \mathcal{T} \) be a triangulated category with small coproducts. Let \( \beta \) be a regular cardinal. Let \( \mathcal{S} \) be some class of objects in \( T^\beta \). Let \( X \) be a \( \beta \)-compact object of \( \mathcal{T} \), i.e. \( X \in T^\beta \), and let \( Z \) be an object of \( \langle \mathcal{S} \rangle \). Suppose that \( f : X \to Z \) is a morphism in \( \mathcal{T} \). Then there exists an object \( Y \in \langle \mathcal{S} \rangle^\beta \) so that \( f \) factors as \( X \to Y \to Z \).

Proof. As in Neeman’s book, since that proof uses only the facts that \( T^\beta \) is a \( \beta \)-localizing triangulated subcategory of \( \mathcal{T} \), that all the objects in \( T^\beta \) are \( \beta \)-small, and that condition (G4) holds for \( T^\beta \). These properties are also valid for Krause’s definition of \( T^\beta \).  

The power of this property is seen at once, since it is the key to obtain the following results.
Corollary 3.8 ([24] Ch. 4, Lemma 4.4.5, p. 140 for item a) and Lemma 4.4.8, p. 142 for item b)). Let $T$ be a triangulated category with small coproducts. Let $S$ be some class of objects in $T^\alpha$ for some infinite cardinal $\alpha$. Let $\beta \geq \alpha$ be a regular cardinal. Then:

a) if $\langle S \rangle = T$, then the inclusion $\langle S \rangle_\beta \subseteq T^\beta$ is an equality;

b) let $N = \langle S \rangle$. Then there is an inclusion $\mathcal{N} \cap T^\beta \subseteq N^\beta$.

Proof. a) Let $X$ be an object of $T^\beta$ and consider the identity map $1_X : X \to X$. As $X$ is at the same time in $T^\beta$ and in $\langle S \rangle$ we can apply the theorem 3.7 and factor $1_X$ through some object $Y \in \langle S \rangle_\beta$. Thus the object $X$ is a direct factor of $Y$. Since $\langle S \rangle_\beta$ is thick, we have $X \in \langle S \rangle_\beta$.

b) Let $K$ be an object of $\mathcal{N} \cap T^\beta$. Then $K$ is $\beta$-small as an object of $\mathcal{N}$ since the inclusion $\mathcal{N} \subseteq T$ commutes with coproducts. Now, let $K \to \coprod_{i \in I} X_i$ be a morphism, where the objects $X_i$ belong to $\mathcal{N}$. It factors through a morphism $\coprod_{i \in I} f_i : \coprod_{i \in I} K_i \to \coprod_{i \in I} X_i$, where the objects $K_i$ belong to $T^\beta$. By the theorem above, each morphism $K_i \to X_i$ factors through an object $K_i'$ belonging to $\langle S \rangle_\beta \subseteq \mathcal{N} \cap T^\beta$. Therefore the class $\mathcal{N} \cap T^\beta$ satisfies (G4) in $\mathcal{N}$ and we obtain the required inclusion. □

The next proposition states some useful properties of the images in the quotient category $T/\mathcal{N}$ of the maps of the subcategories $\langle G \rangle_\beta$ of $T$ under the canonical quotient functor $Q$.

Proposition 3.9. Let $\alpha$ be a regular cardinal. Let $T$ be a triangulated category with small coproducts, generated by a class of objects $G \subseteq T^\alpha$. Let $S$ be an arbitrary class of objects in $T^\alpha$ and $Q$ the canonical quotient functor $Q : T \to T/\langle S \rangle$.

Let $\beta \geq \alpha$ be a regular cardinal. Then:

a) each morphism $u : Q(G) \to Q(X)$, where $G$ is an object of $\langle G \rangle_\beta$ and $Q(X)$ an arbitrary object of $T/\mathcal{N}$, is the equivalence class of a diagram in $T$

\[
\begin{array}{ccc}
G' & \sim & X \\
\downarrow & & \downarrow \\
G & \sim & X,
\end{array}
\]

where the object $G'$ belongs to $T^\beta = \langle G \rangle_\beta$ and the arrow $\sim$ means a morphism whose image under $Q$ is invertible; in particular, the morphisms from $Q(G)$ to $Q(X)$ in $T/\langle S \rangle$ form a set if $G$ is a set;

b) the image of $\langle G \rangle_\beta$ under the (restriction of the) functor $Q$ is a full triangulated subcategory of $T/\mathcal{N}$;

c) if $\beta$ is uncountable, then $\langle Q(G) \rangle_\beta$ equals $Q(\langle G \rangle_\beta)$. If $\beta$ is countable, then $\langle Q(G) \rangle_\beta$ equals the closure of $Q(\langle G \rangle_\beta)$ under taking direct factors.

Proof. a) Let $u : Q(G) \to Q(X)$ be a morphism in $T/\mathcal{N}$. It is the equivalence class of a ‘roof’ diagram in $T$

\[
\begin{array}{ccc}
T & \sim & X \\
\downarrow & & \downarrow \\
G & \sim & X,
\end{array}
\]
where the object $T$ belongs to $\mathcal{T}$. We can form the distinguished triangle
\[
N \xleftarrow{\sim} G \xrightarrow{\sim} T \xleftarrow{\sim} \Sigma^{-1}N,
\]
where $N$ and $\Sigma^{-1}N$ lie in $\langle \mathcal{S} \rangle$. The object $G$ is $\beta$-compact in $\mathcal{T}$. Therefore, we can apply theorem 3.7 to the morphism $N \leftarrow G$ and factor it as
\[
N \xleftarrow{\sim} N' \xleftarrow{\sim} G,
\]
where $N'$ belongs to $\langle \mathcal{S} \rangle_\beta$. The class $\langle \mathcal{S} \rangle_\beta$ is contained in $\mathcal{T}^\beta$, since $S$ is contained in $\mathcal{T}^\alpha$ by the hypothesis. Therefore, we can complete the morphism $N' \leftarrow G$ to a distinguished triangle in $\mathcal{T}^\beta$
\[
N' \xleftarrow{\sim} G' \xleftarrow{\sim} \Sigma^{-1}N'
\]
and deduce a map of distinguished triangles
\[
\begin{array}{c}
\Sigma^{-1}N' \\
\downarrow \\
G' \\
\downarrow \\
\Sigma^{-1}N
\end{array}
\begin{array}{c}
N' \\
\downarrow \\
G \\
\downarrow \\
X
\end{array}
\begin{array}{c}
\sim \\
\sim
\end{array}
\begin{array}{c}
T \\
\sim
\end{array}
\]

adding the morphism $G' \to T$. The wavy arrow stands for the given morphism $Q(G) \to Q(X)$ in $\mathcal{T}/N'$, whereas the dotted arrow is the composition $G' \to T \xrightarrow{\sim} X$. The roof diagrams $G \xrightarrow{\sim} G' \to X$ and $G \xrightarrow{\sim} T \to X$ are clearly equivalent. We have supposed that $\mathcal{T}$ has small coproducts and that $\langle \mathcal{G} \rangle = \mathcal{T}$, with $\mathcal{G}$ contained in $\mathcal{T}^\alpha$, hence in $\mathcal{T}^\beta$. Therefore, $\langle \mathcal{G} \rangle_\beta = \mathcal{T}^\beta$ by point a) of corollary 3.8. This shows that $G'$ also lies in $\langle \mathcal{G} \rangle_\beta$.

b) Clearly, the image of $\langle \mathcal{G} \rangle_\beta$ under $Q$ is stable under $\Sigma$ and $\Sigma^{-1}$. We have to show that it is stable under forming cones. Let $G_1$ and $G_2$ be two objects of $\langle \mathcal{G} \rangle_\beta$ and $u$ a morphism from $QG_1$ to $QG_2$. By part a), the morphism $u$ equals the equivalence class of a diagram
\[
\begin{array}{c}
G_1' \\
\sim
\end{array}
\begin{array}{c}
G_1 \\
\sim
\end{array}
\begin{array}{c}
\downarrow \\
\sim
\end{array}
\begin{array}{c}
G_2 \\
\sim
\end{array}
\]
where $G_1'$ belongs to $\langle \mathcal{G} \rangle_\beta$. Therefore, the cone $C$ on $v$ still belongs to $\langle \mathcal{G} \rangle_\beta$. Clearly, the cone on $u$ is isomorphic to $Q(C)$, which still belongs to the image under $Q$ of $\langle \mathcal{G} \rangle_\beta$.

c) Let $\mathcal{U}$ be the closure of $Q(\langle \mathcal{G} \rangle_\beta)$ under taking direct factors. We claim that $\mathcal{U}$ equals $\langle Q\mathcal{G} \rangle_\beta$ for all $\beta \geq \alpha$. Indeed, we have $Q(\langle \mathcal{G} \rangle_\beta) \subseteq \langle Q\mathcal{G} \rangle_\beta$ since $Q$ is a triangle functor and commutes with arbitrary coproducts. It follows that $\mathcal{U} \subseteq \langle Q\mathcal{G} \rangle_\beta$ since $\langle Q\mathcal{G} \rangle_\beta$ is thick. For the reverse inclusion, we notice that $\mathcal{U}$ contains $Q\mathcal{G}$, that it is a triangulated subcategory since $Q(\langle \mathcal{G} \rangle_\beta)$ is a triangulated subcategory (by b), and that it is thick (by definition). We have thus proved the claim for countable $\beta$. Now suppose $\beta$ is uncountable. Then $Q(\langle \mathcal{G} \rangle_\beta)$
is a triangulated subcategory stable under forming countable coproducts. Therefore, it is stable under taking direct factors (cf. [32]) and thus equals $\mathcal{U} = \langle Q\mathcal{G} \rangle_\beta$. □

Now we can state the most important theorem of this section. This theorem has been inspired by Neeman’s generalization to well generated categories [24, Thm. 4.4.9, p. 143] of Thomason-Trobaugh’s theorem [34, Key Proposition 5.2.2, p. 338].

**Theorem 3.10.** Let $\mathcal{T}$ be an $\alpha$-compactly generated triangulated category and $\mathcal{G}$ a set of good generators for $\mathcal{T}$, contained in $\mathcal{T}^\alpha$. Let $\mathcal{S}$ be a set of objects contained in $\mathcal{T}^\gamma$, for some fixed regular cardinal $\gamma$. Let $\mathcal{N} = \langle \mathcal{S} \rangle$ and $Q$ the canonical quotient functor $Q : \mathcal{T} \to \mathcal{T}/\mathcal{N}$. 

a) The localizing triangulated subcategory $\mathcal{N}$ is the union 
$$
\mathcal{N} = \bigcup_{\delta \geq \gamma} \mathcal{N}_\delta,
$$
where $\delta$ runs through the regular cardinals. Equivalently, $\mathcal{N}$ is given by the same union as above, formed over all regular cardinals; 

b) the subcategory $\mathcal{N}$ is $\delta$-compactly generated for all regular cardinals $\delta \geq \gamma$ by the set $\langle \mathcal{S} \rangle_\gamma$; 

c) the subcategory $Q(\langle \mathcal{G} \rangle_\beta)$ equals $\langle Q\mathcal{G} \rangle_\beta$ for $\beta > \aleph_0$ and its closure under taking direct factors equals $\langle Q\mathcal{G} \rangle_\beta$ for $\beta = \aleph_0$; 

d) the quotient category $\mathcal{T}/\mathcal{N}$ is a $\delta$-compactly generated triangulated category for all regular cardinals $\delta \geq \beta$, where $\beta = \sup(\alpha, \gamma)$, with set of good generators $Q(\langle \mathcal{G} \rangle_\beta)$.

**Proof.** It is clearly sufficient to prove b) for $\delta = \gamma$ and c) for $\delta = \beta$.

a) The triangulated category $\mathcal{T}$ is well generated. Therefore, it is the union over all the regular cardinals $\sigma$ of its subcategories $\mathcal{T}^\sigma$ [15, Corollary of Thm. A]. We know from the hypothesis that $\mathcal{S} \subseteq \mathcal{T}^\gamma$, hence $\mathcal{S} \subseteq \mathcal{N} \cap \mathcal{T}^\gamma$. Clearly, $\langle \mathcal{S} \rangle_\gamma \subseteq \mathcal{N} \cap \mathcal{T}^\gamma$, since $\langle \mathcal{S} \rangle_\gamma$ is the smallest $\gamma$-localizing subcategory of $\mathcal{T}$ containing the set $\mathcal{S}$. Moreover, $\mathcal{N} \cap \mathcal{T}^\gamma \subseteq \mathcal{N}^\gamma$ by point b) of corollary 3.8. Thus, we have the following sequence of inclusions: 

$$
\mathcal{S} \subseteq \langle \mathcal{S} \rangle_\gamma \subseteq \mathcal{N} \cap \mathcal{T}^\gamma \subseteq \mathcal{N}^\gamma.
$$

Therefore, for each regular cardinal $\delta \geq \gamma$, we obtain $\langle \mathcal{S} \rangle_\delta = \mathcal{N} \cap \mathcal{T}^\delta = \mathcal{N}^\delta$, by point a) of corollary 3.8. The claim now follows by the equalities 

$$
\mathcal{N} = \mathcal{N} \cap \mathcal{T} = \mathcal{N} \cap (\bigcup_{\lambda} \mathcal{T}^\lambda) = \bigcup_{\lambda} (\mathcal{N} \cap \mathcal{T}^\lambda) = \bigcup_{\delta \geq \gamma} \mathcal{N}^\delta = \bigcup_{\lambda} \mathcal{N}^\lambda.
$$

The two last equalities hold since the set of the subcategories $\mathcal{N}^\lambda$ is filtered over regular cardinals. This means that $\mathcal{N}^\alpha \subseteq \mathcal{N}^\beta$ if $\alpha \leq \beta$, for all regular cardinals $\alpha$ and $\beta$.

b) The isomorphism classes of the objects of the subcategory $\langle \mathcal{S} \rangle_\gamma$ form a set, since it is explicitly constructed from the objects in $\mathcal{S}$, which is also a set. Moreover, $\langle \mathcal{S} \rangle_\gamma$ is stable under shifts because it is triangulated. Let us show condition (G1). Let $Y$ be an object of $\mathcal{N}$ such that $\text{Hom}_\mathcal{N}(X, Y) = 0$ for all $X$ in $\langle \mathcal{S} \rangle_\gamma$. Then, it is easy to check that this equality holds for $X$ in $\langle \mathcal{S} \rangle_\gamma$. In particular it holds for $X = Y$. Hence $Y$ vanishes. Therefore, condition (G1) holds for $\langle \mathcal{S} \rangle_\gamma$. By the proof of point a), $\langle \mathcal{S} \rangle_\gamma = \mathcal{N}^\gamma$. Therefore, conditions (G2) and (G3) trivially hold by the definition of $\mathcal{N}^\gamma$.

c) All the conditions of proposition 3.9 hold. Thus, this point results from part c) of proposition 3.9.

d) The subcategory $\mathcal{N}$ is well generated by point b). Thus, the Brown representability theorem (3.5) holds for $\mathcal{N}$ and we conclude that the inclusion $i$ of $\mathcal{N}$ into $\mathcal{T}$ admits a right
adjoint $i_\rho$ as in the proof of proposition 3.6. Now this implies that the quotient functor $Q : T \to T/\mathcal{N}$ admits a right adjoint $Q_\rho$ (which takes an object $X$ to the cone of the adjunction morphism $ii_\rho X \to X$). The functor $Q_\rho$ is a localization functor (3.2). Thus, it is fully faithful. Let us sum up the situation in the following diagram,

We have to show that the conditions (G1), (G2) and (G3) of definition 2.1 hold for $Q(\langle G \rangle_\beta)$. We begin by observing that the sets $G$ and $S$ are both contained in $T^\beta$, since we have chosen $\beta = \sup(\alpha, \gamma)$. The condition (G1) holds even for the smaller set $QG$, hence for $Q(\langle G \rangle_\beta)$. Indeed, suppose Hom$_{T/\mathcal{N}}(QG, X) = 0$, for an arbitrary object $X$ in $T/\mathcal{N}$. By the adjunction, this is equivalent to Hom$_T(G, Q_\rho(X)) = 0$. The condition (G1) holds for the set $G$ in $T$ and implies $Q_\rho(X) = 0$. Therefore, $X = QQ_\rho X = 0$, since $QQ_\rho$ is naturally equivalent to the identity endofunctor of $T/\mathcal{N}$. Thus, condition (G1) holds for the set $QG$. The subcategory $Q(\langle G \rangle_\beta)$ contains its $\beta$-coproducts because $Q$ commutes with all coproducts and its objects form a set. Therefore, conditions (G2) and (G4) are equivalent for $Q(\langle G \rangle_\beta)$ (cf. [15, Lemma 4]). Let us now simultaneously show that conditions (G4) and (G3) hold for $Q(\langle G \rangle_\beta)$. Consider a morphism $u : QG \to \coprod_{i \in I} X_i$, where $G$ is an arbitrary object in $\langle G \rangle_\beta$. We know from point a) of proposition 3.9 that $u$ is the equivalence class of a diagram in $T$,

where the object $G'$ belongs to $T^\beta = \langle G \rangle_\beta$. The conditions (G3) and (G4) also hold for $\langle G \rangle_\beta$, by corollary 2.6. Therefore, there exists a set $J \subset I$ of cardinality strictly smaller than $\beta$ and a set of morphisms $(f_i : G_i \to X_i)_{i \in I}$, where $G_i$ lies in $\langle G \rangle_\beta$ for all $i \in J$, so that the morphism $f : G' \to \coprod_{i \in I} X_i$ factors through $\coprod_{i \in J} X_i$ (G3)

and through the morphism $\coprod_{i \in I} f_i$ (G4)

The image under $Q$ of the last two diagrams shows that the morphism $u$ factors in $T/\mathcal{N}$ in the same way. Therefore, conditions (G3) and (G4) hold for $Q(\langle G \rangle_\beta)$.
Remark 3.11. The construction of the cardinal $\beta$ in the preceding proof is not optimized at all. In spite of the constructive proof, this result will be useful mainly for existence problems.

The next corollary is a result about the localization of well generated categories obtained by inverting a set of arrows, implicitly contained in Neeman’s book [24].

Corollary 3.12. Let $T$ be a well generated triangulated category and $N$ a localizing triangulated subcategory of $T$ generated by a set of objects $S$. Then $N$ and $T/N$ are well generated triangulated categories.

Proof. Take the coproduct of all the objects in $S$. Since $S$ is a set, the coproduct will be in $T^\gamma$ for some regular cardinal $\gamma$. Therefore, we have $S \subseteq T^\gamma$, because $T^\gamma$ is thick in $T$ and so contains the direct factors of its objects. Now apply theorem 3.10. □

4. THE $\alpha$-CONTINUOUS DERIVED CATEGORY

In this section, we construct the $\alpha$-continuous derived category of a homotopically $\alpha$-cocomplete DG category. This construction enjoys a useful and beautiful property. Given a homotopically $\alpha$-cocomplete (cf. below) pretriangulated DG category $A$, we will show that its $\alpha$-continuous derived category $D_\alpha A$ is $\alpha$-compactly generated by the free DG modules. The categories $D_\alpha A$ will be the prototypes of the $\alpha$-compactly generated algebraic DG categories. We use the notations of [14].

Definition 4.1. Let $\alpha$ be a regular cardinal and $A$ a small DG $k$-category. We assume that $A$ is homotopically $\alpha$-cocomplete, i.e. that the category $H^0(A)$ admits all $\alpha$-small coproducts. For each $\alpha$-small family $(A_i)_{i \in I}$ of objects of $A$, we write

$$\prod_{i \in I} A_i$$

for their coproduct in $H^0(A)$. Each DG functor $M : A^{op} \to C_{dg}(k)$ induces a functor $H^0 M : (H^0(A))^{op} \to H(k)$ and so we have a canonical morphism

$$(H^* M) \left( \prod_{i \in I} A_i \right) \longrightarrow \prod_{i \in I} (H^* M)(A_i).$$

Let $DA$ be the derived category [12] of $A$. It is a triangulated category (cf. [12], [14]). The $\alpha$-continuous derived category $D_\alpha A$ is defined as the full subcategory of $DA$ whose objects are the DG functors $M$ such that, for each $\alpha$-small family of objects $(A_i)_{i \in I}$ of $A$, the canonical morphism above is invertible.

Remark 4.2. All the small $k$-linear DG categories which are $\alpha$-cocomplete, i.e. admit all $\alpha$-small coproducts, are homotopically $\alpha$-cocomplete. A partial converse is given in remark 4.4 below.

This definition describes $D_\alpha A$ as a subcategory of $DA$. One can give an equivalent definition in terms of a localization of $DA$, which yields a category $DA/N$ triangle equivalent to $D_\alpha A$. For this, let us define some sets of morphisms in the category of DG modules $CA$ (cf. [12], [14]). We recall that the notation $A^\wedge$ means $\text{Hom}_A(\cdot, A)$. Let $\Sigma_0$ be the set of all morphisms of $CA$

$$\sigma_{\lambda} : \prod_{i \in I} A_i^\wedge \longrightarrow \left( \prod_{i \in I} A_i \right)^\wedge,$$
where \( \lambda \) ranges over the set \( \Lambda \) of all families \((A_i)_{i \in I}\) in \( \mathcal{A} \) of cardinality strictly smaller than \( \alpha \). We define \( \Sigma \) to be the set of cofibrations \([8], [9], [10]\) between cofibrant DG modules \([14]\)

\[
\left[ \sigma_\lambda \right] : \prod_{i \in I} A_i^\wedge \longrightarrow \left( \prod_{i \in I} A_i \right)^\wedge \oplus I\left( \prod_{i \in I} A_i^\wedge \right),
\]

where \( \lambda \in \Lambda \) and, for each object \( X \), the morphism \( inc : X \rightarrow IX \) is the inclusion of \( X \) into the cone over its identity morphism. We can also consider the set

\[
\mathcal{M} = \{ N_\lambda \rightarrow I N_\lambda \mid N_\lambda = \text{cone}(\sigma_\lambda), \lambda \in \Lambda \}.
\]

The cones over the morphisms in \( \Sigma_0 \), \( \Sigma \) or \( \mathcal{M} \) generate the same localizing subcategory \( \mathcal{N} \) of \( \mathcal{D}A \) because the objects \( IX \) are contractible and thus become zero objects in \( \mathcal{D}A \). The quotient functor

\[
\mathcal{D}A \longrightarrow \mathcal{D}A/\mathcal{N}
\]

induces an equivalence

\[
\mathcal{D}_\alpha A \sim \mathcal{D}A/\mathcal{N}.
\]

Following [5], we say that a DG category \( \mathcal{A} \) is pretriangulated if the essential image of the Yoneda functor is a triangulated subcategory of the derived category \( \mathcal{D}A \). In the case of pretriangulated DG categories, the definition of quasi-equivalence of DG categories of [14] specializes to the following.

**Definition 4.3.** Let \( \mathcal{A} \) and \( \mathcal{A}' \) be pretriangulated DG categories. A DG functor \( F : \mathcal{A} \rightarrow \mathcal{A}' \) is a quasi-equivalence of pretriangulated DG categories if the induced triangle functor

\[
H^0(F) : H^0(\mathcal{A}) \rightarrow H^0(\mathcal{A}')
\]

is an equivalence of triangulated categories.

**Remark 4.4.** We will show in [29] that if \( \mathcal{A} \) is a homotopically \( \alpha \)-cocomplete pretriangulated DG category, then there exists a quasi-equivalence \( \mathcal{A} \rightarrow \mathcal{A}' \), where \( \mathcal{A}' \) is a pretriangulated DG category which is \( \alpha \)-cocomplete. This establishes the link between this article and [32].

We now come to the result which motivated the definition of the \( \alpha \)-continuous derived category.

**Theorem 4.5.** Let \( \mathcal{A} \) be a homotopically \( \alpha \)-cocomplete pretriangulated DG category. The \( \alpha \)-continuous derived category of \( \mathcal{A} \) is \( \alpha \)-compactly generated by the images of the free DG modules \( A^\wedge, A \in \mathcal{A} \). More precisely, the full subcategory \( \mathcal{G} \) of \( \mathcal{D}_\alpha A \) formed by the images of the free DG modules \( A^\wedge \), \( A \in \mathcal{A} \), is a triangulated subcategory satisfying conditions (G1), (G2) and (G3) of definition 2.1.

**Remark 4.6.** We prove the theorem in the case where \( \alpha \) is strictly greater than \( \aleph_0 \), the case \( \alpha = \aleph_0 \) being trivial. In fact, \( \aleph_0 \)-coproducts are finite coproducts. Thus, the morphisms \( \sigma_\lambda \) above are isomorphisms already in \( \mathcal{D}A \), and \( \mathcal{D}_{\aleph_0} A \) equals \( \mathcal{D}A \).

**Proof.** This proof depends heavily on theorem 3.10. Therefore, let us explain how the notations correspond. The triangulated category \( \mathcal{T} \) is \( \mathcal{D}A \). The set \( \mathcal{S} \) is formed by the cones on the following morphisms

\[
\sigma_\lambda : \prod_{i \in I} (A_i^\wedge) \longrightarrow \left( \prod_{i \in I} A_i \right)^\wedge,
\]

where \( \lambda \) ranges over the set \( \Lambda \) of all families \((A_i)_{i \in I}\) in \( \mathcal{A} \) of cardinality strictly smaller than \( \alpha \). The set \( \mathcal{G} \) is formed by the free DG modules \( A^\wedge, A \in \mathcal{A} \). It is contained in \( \mathcal{T}^{\aleph_0} \),
whereas $S$ is contained in $T^\alpha$. We have $\beta = \sup(\aleph_0, \alpha) = \alpha$. Let $\mathcal{N}$ be $\langle S \rangle$ and $Q$ the projection functor

$$Q : \mathcal{D}A \longrightarrow \mathcal{D}_\alpha A \cong T/\mathcal{N}.$$  

Then, according to theorem 3.10, the $\alpha$-continuous derived category $\mathcal{D}_\alpha A$ is $\alpha$-compactly generated by $\langle QG \rangle_\alpha$. Hence, the claim of the theorem is equivalent to the following claim: $\langle QG \rangle_\alpha = QG$ and the functor $Q$ induces an equivalence $G \cong QG$. We begin with the equivalence $G \cong QG$. It amounts to the same as to show that the functor $Q|_G$ is fully faithful. We know from the proof of the point d) of theorem 3.10 that $Q$ admits a right adjoint $Q\rho$ (3.2). From the general theory of Bousfield localizations [24, Ch. 9], we have that $Q|_N$ is an equivalence of triangulated categories. In particular $Q|_N$ is fully faithful. Therefore, it is sufficient to show that $G$ is contained in $\mathcal{N}$. By definition, $\mathcal{N}$ is the localizing subcategory generated by the cones cone($\sigma\lambda$), which we call $C(A\lambda)$. We have to show that each $A^\wedge \in G$ is right orthogonal to the objects $C(A\lambda)$. By applying the cohomological functor $Hom_{\mathcal{D}A}(\bigoplus_i A_i^\wedge, \Sigma^n A^\wedge)$, $n \in \mathbb{Z}$, to the distinguished triangle

$$\prod_i (A_i^\wedge) \longrightarrow (\bigoplus_i A_i)^\wedge \longrightarrow C(A\lambda) \longrightarrow \Sigma \prod_i (A_i^\wedge),$$

it is clear that it is sufficient to show that the natural morphism

$$Hom_{\mathcal{D}A}(\prod_i (A_i^\wedge), \Sigma^n A^\wedge) \leftarrow Hom_{\mathcal{D}A}(\bigoplus_i A_i)^\wedge, \Sigma^n A^\wedge)$$

is an isomorphism for all $n \in \mathbb{Z}$. This follows from the following sequence of isomorphisms

$$Hom_{\mathcal{D}A}(\prod_i (A_i^\wedge), \Sigma^n A^\wedge) \cong \prod_i Hom_{\mathcal{D}A}(A_i^\wedge, \Sigma^n A^\wedge)$$

$$= \prod_i H^n Hom_A(A_i, A)$$

$$\cong H^n(\prod_i Hom_A(A_i, A))$$

$$Hom_{\mathcal{D}A}(\bigoplus_i A_i^\wedge, \Sigma^n A^\wedge) \cong H^n(Hom_{\mathcal{D}A}(\bigoplus_i A_i^\wedge, A), A)).$$

The second and the last isomorphisms are justified by the following one

$$Hom_{\mathcal{D}A}(A^\wedge, \Sigma^n B^\wedge) = H^n Hom_A(A, B),$$

valid for all $A$ and $B$ in $\mathcal{A}$. The third isomorphism is the fact that cohomology commutes with formation of products. For the fourth, we observe that the natural homomorphism

$$Hom_A(\prod_i A_i, A) \longrightarrow \prod_i Hom_A(A_i, A)$$

is a homotopy equivalence, by the definition of $\prod^H$. Therefore, it becomes invertible in cohomology.

It is trivial that $QG$ is stable under shifts. Moreover, it is automatically thick for $\alpha > \aleph_0$ (3.2). To prove that $QG$ equals $\langle QG \rangle_\alpha$ it is then sufficient to show that $QG$ is closed under
α-coproducts and extensions. We have
\[
\prod_{i \in I} (QA_i^\wedge) \xrightarrow{\sim} Q(\prod_{i \in I} A_i^\wedge) \xrightarrow{H^0} Q((\prod_{i \in I} A_i)^\wedge),
\]
where the cardinality of \( I \) is strictly smaller than \( \alpha \) and the last isomorphism holds by the construction of \( \mathcal{N} \). This shows that \( Q\mathcal{G} \) is closed under formation of \( \alpha \)-coproducts. Finally, \( Q\mathcal{G} \) is stable under extensions in \( D\alpha A \). Indeed, \( \mathcal{G} \) is stable under extensions in \( DA \) and hence in \( \mathcal{N}^\perp \), since we have shown that \( \mathcal{G} \) is contained in \( \mathcal{N}^\perp \). We have also seen that the restriction \( Q|_{\mathcal{N}^\perp} \) is an equivalence of the categories \( \mathcal{N}^\perp \) and \( D\alpha A \). It follows that \( Q\mathcal{G} \) is stable under extensions in \( D\alpha A \). □

5. The Popescu-Gabriel theorem for triangulated categories

5.1. Algebraic triangulated categories. Let us recall that an exact category \( \mathcal{E} \) \cite{11, 30} is a Frobenius category \cite{6} if it has enough injectives, enough projectives, and the two classes of the injectives and projectives coincide. For all pairs of objects \( X, Y \) of \( \mathcal{E} \), let \( I_{\mathcal{E}}(X,Y) \) be the subgroup of the abelian group \( \text{Hom}_{\mathcal{E}}(X,Y) \) formed by the morphisms which factor over an injective-projective object of \( \mathcal{E} \). The stable category of \( \mathcal{E} \) \cite{6}, written \( \mathcal{E}_s \), is the category which has the same objects as \( \mathcal{E} \) and the morphisms

\[
\text{Hom}_{\mathcal{E}_s}(X,Y) = \text{Hom}_{\mathcal{E}}(X,Y)/I_{\mathcal{E}}(X,Y).
\]

Definition 5.1. \cite{14} An algebraic triangulated category is a \( k \)-linear triangulated category which is triangle equivalent to the stable category \( \mathcal{E}_s \) of some \( k \)-linear Frobenius category \( \mathcal{E} \).

The class of algebraic triangulated categories is stable under taking triangulated subcategories and forming triangulated localizations (up to a set-theoretic problem). Examples abound since categories of complexes up to homotopy are algebraic. Therefore, the categories arising in homological contexts in algebra and geometry are algebraic. The area where one often encounters non algebraic triangulated categories is topology. In particular the stable homotopy category of spectra is not algebraic. More examples can be found in section 3.6 of \cite{14}.

5.2. The main theorem. We recall \cite{12} that a graded category over a commutative ring \( k \) is a \( k \)-linear category \( \mathcal{B} \) whose morphism spaces are \( \mathbb{Z} \)-graded \( k \)-modules

\[
\text{Hom}_{\mathcal{B}}(X,Y) = \prod_{p \in \mathbb{Z}} \text{Hom}_{\mathcal{B}}(X,Y)^p
\]
such that the composition maps

\[
\text{Hom}_{\mathcal{B}}(X,Y) \otimes_k \text{Hom}_{\mathcal{B}}(Y,Z) \longrightarrow \text{Hom}_{\mathcal{B}}(Y,Z)
\]
are homogeneous of degree 0, for all \( X, Y, Z \) in \( \mathcal{B} \). Now we can state and prove the main theorem of this paper.

Theorem 5.2. Let \( T \) be a triangulated category. Then the following statements are equivalent:

(i) \( T \) is algebraic and well generated;

(ii) there is a small DG category \( \mathcal{A} \) such that \( T \) is triangle equivalent to a localization of \( DA \) with respect to a localizing subcategory generated by a set of objects.
Moreover, if $T$ is algebraic and $\alpha$-compactly generated, and $U \subset T$ is a full triangulated subcategory stable under $\alpha$-small coproducts and such that conditions (G1), (G2) and (G3) of definition 2.1 hold for $U$, then there is an associated localization functor \[ T \to DA \]
for some small DG category $A$ such that $H^*(A)$ is equivalent to the graded category $U_{gr}$ whose objects are those of $U$ and whose morphisms are given by

\[ U_{gr}(U_1, U_2) = \bigoplus_{n \in \mathbb{Z}} T(U_1, \Sigma^n U_2). \]

Proof. (ii) \implies (i) : $T$ is a localization of $DA$, i.e. there is a fully faithful functor

\[ T \xrightarrow{\sim} F : DA, \]

admitting a left adjoint functor. The category $DA$ is algebraic. Triangulated subcategories of algebraic categories are algebraic, implying that $T$ is algebraic, too. Moreover, $DA$ is compactly generated by the set $(A$ is small)

\[ \{ X^\wedge[n] \mid n \in \mathbb{Z}, X \in A \} \]

thanks to the isomorphism

\[ \text{Hom}_{DA}(X^\wedge[n], M) \xrightarrow{\sim} H^{-n}(M(X)) , \]

where $M$ is a DG module and $X$ is an object of $A$ (cf. [12], [13]). Therefore, $T$ is well generated by corollary 3.12 since it is assumed to be a localization generated by a set of the $\aleph_0$-compactly generated category $DA$.

(i) \implies (ii) : for the sake of clarity, we will give the proof of this implication in several steps, after making the main construction.

Let $T$ be an algebraic, well generated triangulated category, i.e. $T$ is equivalent to $\mathcal{E}$ for some Frobenius category $\mathcal{E}$. By the definition of well generated triangulated category (in the sense of Krause), there are a regular cardinal $\alpha$ and a set of $\alpha$-good generators $G_0 \subset T$ such that $\Sigma G_0 = G_0$ and the conditions (G1), (G2) and (G3) of definition 2.1 hold. Let $G$ be the closure of the set $G_0$ under extensions and $\alpha$-coproducts. The set $G$ is stable under the suspension functor $\Sigma$ of $T$ and under its inverse. Therefore, it is a small triangulated subcategory of $T$. Let us recall and summarize the properties which hold for $G$.

(G0) The set $G$ is a small full triangulated subcategory of $T$, stable under the formation of all $\alpha$-small coproducts;

(G1) the set $G$ is a generating set for $T$: An object $X \in T$ is zero if $\text{Hom}_T(G, X) = 0$ for all $G$ in $G$;

(G3) all the objects $G \in G$ are $\alpha$-small: For each family of objects $X_i, i \in I, of T$, we have $\text{Hom}_T(G, \bigsqcup_i X_i) = \text{colim}_{J \subset I} \text{Hom}_T(G, \bigsqcup_J X_i)$, where the sets $J$ have cardinality strictly smaller than $\alpha$;

(G4) for each family of objects $X_i, i \in I, of T$, and each object $G \in G$, each morphism $G \to \bigsqcup_i X_i$

factors through a morphism $\bigsqcup_{i \in I} \phi_i: \bigsqcup_{i \in I} G_i \to \bigsqcup_{i \in I} X_i$, with $G_i$ in $G$ for all $i \in I$.

Condition (G0) clearly holds for $G$. Condition (G3) of definition 2.1 has just been rewritten using colimits. Condition (G4) holds for $G$ by proposition 2.4. Note that conditions (G2) and (G4) are equivalent for $G$. Indeed, we can apply [15, Lemma 4], since the set $G$ has $\alpha$-coproducts and its objects are $\alpha$-small.

We may assume that the category $\mathcal{E}$ is of the form $Z^0(\mathcal{E})$ for an exact DG category $\mathcal{E}$ by the argument of the proof of theorem 4.4 of [12]. Let us recall that a DG category $A$ is
an exact DG category [13] if the full subcategory $Z^0(A)$ of $CA$ formed by the image of the Yoneda functor is closed under shifts and extensions (in the sense of the exact structure of [12]). Then, $H^0(A)$ becomes a triangulated subcategory of $H(A)$ and the subcategory of the representable functors becomes a triangulated subcategory of $DA$. Thus, an exact DG category is also a pretriangulated DG category (cf. section 1). Let us now define a small full DG subcategory $A \subset \tilde{E}$ as follows. For each isomorphism class of objects of $G$, we choose a representative $G$ and we denote by $A_G$ the same object considered in the category $E$. By definition, these objects $A_G$ are objects of $A$. Then, clearly, the category $H^0(A)$ is a full subcategory of $H^0(\tilde{E}) = \tilde{E} = T$ and it is equivalent to $G$ by the functor sending $A_G$ to $G$. In particular, $H^0(A)$ is a triangulated category and it admits all $\alpha$-small coproducts. Thus, $A$ is a homotopically $\alpha$-cocomplete pretriangulated DG category. We define the functor

$$F : T \longrightarrow DA$$

by sending an object $X$ of $T = H^0(\tilde{E})$ to the DG module $FX$ taking $A_G \in A$ to $\text{Hom}_E(G, X)$. A priori, $FX$ lies in $DA$. Let us show that it belongs in fact to the full subcategory $DA \subset DA$. Let $A_{G_i}$, $i \in I$, be an $\alpha$-small family in $A$. Then the coproduct $\bigoplus_{i \in I} A_{G_i}$ of the $A_{G_i}$ in $H^0(A)$ is isomorphic to $A_{\bigoplus_{i \in I} G_i}$. Thus, we have a quasi-isomorphism

$$(FX)(\prod_{i \in I} A_{G_i}) = \text{Hom}_{\tilde{E}}(\prod_{i \in I} G_i, X) \longrightarrow \prod_{i \in I} \text{Hom}_{\tilde{E}}(G_i, X) = \prod_{i \in I} (FX)(A_{G_i}),$$

induced by liftings to $E = Z^0(\tilde{E})$ of the canonical morphisms $G_j \to \prod_{i \in I} G_i$ in $T$, respectively by representatives in $Z^0(A)$ of the canonical morphisms $A_{G_j} \to \prod_{i \in I} H^0(A_{G_i})$ in $H^0(A)$. For $A_G \in A$, we have

$$FG = \text{Hom}_{\tilde{E}}(-, G) = \text{Hom}_A(-, A_G) = A_G^\alpha,$$

which shows that $F$ induces an essentially surjective functor from $G$ to the full subcategory of the $A_G^\alpha$ in $DA$. For $A_G$ in $A$ and $X$ in $T$, we have

$$\text{Hom}_{DA}(FG, FX) = \text{Hom}_{DA}(A_G^\alpha, FX) = \text{Hom}_{DA}(A_G^\alpha, FX) = H^0(FX(A_G)) \quad \text{Hom}_T(G, X) = H^0(\tilde{E}(G, X)).$$

We would like to apply theorem 2.8 to conclude that $F$ is a triangle equivalence: In the notations of theorem 2.8, we take $T = T$, $G = G$, $T' = DA$, $A = A_G^\alpha$ to be the full subcategory on the objects $A_G^\alpha$ in $DA$. By theorem 4.5, $T'$ and $A_G^\alpha$ do satisfy the hypothesis of theorem 2.8 and so $F$ is indeed a triangle equivalence.

Now suppose that $T$ is an algebraic well generated triangulated category. Let $U \subset T$ be a full small subcategory as in the last assertion of the theorem. Then the conditions (G0), (G1) and (G3) above hold for $G = U$. Moreover, condition (G4) holds for $G = U$ by [15, Lemma 4]. Therefore, we can construct a DG category $A$ and an equivalence $F : T \sim DA$ as above in the proof of the implication from i) to ii). Moreover, $H^*(A)$ equals $U_{gr}$. Indeed,
both have the same objects and we have
\[H^n(A)(A_{G_1}, A_{G_2}) = \mathcal{H}A(A^n_{G_1}, (\Sigma^n A_{G_2})^\wedge) = H^0(A)(A_{G_1}, (\Sigma^n A_{G_2}))\]
\[U_{gr}(G_1, G_2)^n = U(G_1, \Sigma^n G_2).\]

If \(T\) is compactly generated we recover a result obtained by B. Keller in [12, Thm. 4.3]:

**Corollary 5.3.** Let \(T\) be an algebraic triangulated category. Then the following statements are equivalent:

(i) \(T\) is compactly generated;

(ii) \(T\) is equivalent to the derived category \(DA\) for some small DG category \(A\).

**Proof.** See remark [4.6] \(\Box\)

5.3. **Application.** We apply theorem 5.2 to a certain class of subcategories of algebraic triangulated categories we are going to define.

**Definition 5.4.** Let \(T\) be an algebraic triangulated category which is triangle equivalent to the stable category of the Frobenius category \(E\) and admits arbitrary coproducts. Let \(\mathcal{E}\) be a DG category (not necessarily small) such that \(H^0(\mathcal{E})\) is triangle equivalent to \(T\). Given a subcategory \(G\) of \(T\), let \(\tilde{G}\) be the DG subcategory of \(A\) with the same objects as \(G\). Thus, the category \(H^0(\tilde{G})\) is isomorphic to \(G\). We will say that \(G\) is a compactifying subcategory of \(T\) if it is small and the functor
\[T \longrightarrow D\tilde{G}, \ X \mapsto \text{Hom}_\mathcal{E}(-, X)|_{\tilde{G}}\]
is fully faithful.

For example, W. T. Lowen and M. Van den Bergh proved in [19, Ch. 5] that, given a Grothendieck category \(A\) with a generator \(G\), the one-object subcategory \(G = \{G\}\) of the derived category \(DA\) is a compactifying subcategory. For this reason we call such a generator \(G\) compactifying.

**Theorem 5.5.** Let \(T\) be a well generated algebraic triangulated category. Then there is a regular cardinal \(\alpha\) such that the subcategory \(sk(T^\beta)\) formed by a system of representatives of the isomorphism classes of \(T^\beta\) is compactifying for each regular cardinal \(\beta \geq \alpha\).

**Proof.** Suppose that \(\alpha\) is the first regular cardinal such that \(T = (T^\alpha)\). This cardinal exists because the category \(T\) is well generated. For each \(\beta \geq \alpha\), the subcategory \(sk(T^\beta)\) is small and satisfies conditions (G1), (G2) and (G3) of definition 2.1 by definition of the subcategory \(T^\beta\) and the filtration by increasing regular cardinals. Now the claim follows from the last part of theorem 5.2 \(\Box\)

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