Abstract. We generalize well-known Catalan-type integrals for Euler’s constant to values of the generalized-Euler-constant function and its derivatives. Using generating functions appeared in these integral representations we give new Vaccum and Ramanujan-type series for values of the generalized-Euler-constant function and its derivative. As a consequence, we get base $B$ rational series for $\log \frac{\pi}{4}$, $\pi G$ (where $G$ is Catalan’s constant), $\zeta'(2)\frac{\pi}{2}$ and also for logarithms of Somos’s and Glaisher-Kinkelin’s constants.

1. Introduction

In [11], J. Sondow proved the following two formulas:

$$\gamma = \sum_{n=1}^{\infty} \frac{N_{1,2}(n) + N_{0,2}(n)}{2n(2n + 1)},$$

(1)

$$\log \frac{4}{\pi} = \sum_{n=1}^{\infty} \frac{N_{1,2}(n) - N_{0,2}(n)}{2n(2n + 1)},$$

(2)

where $\gamma$ is Euler’s constant and $N_{i,2}(n)$ is the number of $i$’s in the binary expansion of $n$. The series (1) is equivalent to the well-known Vaccum series [13]

$$\gamma = \sum_{n=1}^{\infty} (-1)^n \frac{\lfloor \log_2 n \rfloor}{n} = \sum_{n=1}^{\infty} (-1)^n \frac{N_{1,2}(\lfloor \frac{n}{2} \rfloor) + N_{0,2}(\lfloor \frac{n}{2} \rfloor)}{n},$$

(3)

and both series (1) and (3) may be derived from Catalan’s integral [6]

$$\gamma = \int_{0}^{1} \frac{1}{1 + x} \sum_{n=1}^{\infty} x^{2n-1} \, dx.$$

To see this it suffices to note that

$$G(x) = \frac{1}{1 - x} \sum_{n=0}^{\infty} x^n = \sum_{n=1}^{\infty} (N_{1,2}(n) + N_{0,2}(n)) x^n$$

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is a generating function of the sequence $N_{1,2}(n) + N_{0,2}(n)$, (see [10 sequence A070939]), which is the binary length of $n$, rewrite (4) as

$$\gamma = \int_0^1 (1 - x) G(x^2) \frac{dx}{x}$$

and integrate the power series termwise. In view of the equality

$$1 = \int_0^1 \sum_{n=1}^{\infty} x^{2n-1} dx,$$

which is easily verified by termwise integration, (6) is equivalent to the formula

$$\gamma = 1 - \int_0^1 \frac{1}{1+x} \sum_{n=1}^{\infty} x^{2n} dx$$

obtained independently by Ramanujan (see [4 Cor. 2.3]). Catalan’s integral (5) gives the following rational series for $\gamma$:

$$\gamma = 1 - \int_0^1 (1 - x)G(x^2) \frac{dx}{x} = 1 - \sum_{n=1}^{\infty} \frac{N_{1,2}(n) + N_{0,2}(n)}{(2n+1)(2n+2)}.$$

Averaging (1), (6) and (4), (5), respectively, we get Addison’s series for $\gamma$:

$$\gamma = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{N_{1,2}(n) + N_{0,2}(n)}{2n(2n+1)(2n+2)}$$

and its corresponding integral

$$\gamma = \frac{1}{2} + \frac{1}{2} \int_0^1 \frac{1-x}{1+x} \sum_{n=1}^{\infty} x^{2n-1} dx$$

respectively. Integrals (5), (4) were generalized to an arbitrary integer base $B > 1$ by S. Ramanujan and B. C. Berndt and D. C. Bowman (see [4])

$$\gamma = 1 - \int_0^1 \left( \frac{1}{1-x} - \frac{Bx^{B-1}}{1-x^B} \right) \sum_{n=1}^{\infty} x^{Bn} dx \quad \text{(Ramanujan)},$$

$$\gamma = \int_0^1 \left( \frac{B}{1-x^B} - \frac{1}{1-x} \right) \sum_{n=1}^{\infty} x^{B^n-1} \quad \text{(Berndt-Bowman)}.$$}

Formula (9) implies the generalized Vaccar series for $\gamma$ (see [4 Th. 2.6]) proposed by L. Carlitz

$$\gamma = \sum_{n=1}^{\infty} \frac{\varepsilon(n)}{n} \lfloor \log_B n \rfloor,$$

where

$$\varepsilon(n) = \begin{cases} B - 1 & \text{if } B \text{ divides } n \\ -1 & \text{otherwise}, \end{cases}$$
and the averaging integral of (8) and (9) produces the generalized Addison series for \( \gamma \) found by Sondow in [11]

\[
\gamma = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{|\log_B Bn| P_B(n)}{Bn(Bn+1) \cdots (Bn+B)},
\]

where \( P_B(x) \) is a polynomial of degree \( B - 2 \) denoted by

\[
P_B(x) = (Bx + 1)(Bx + 2) \cdots (Bx + B - 1) \sum_{m=1}^{B-1} \frac{m(B-m)}{Bx + m}.
\]

In this short note, we generalize Catalan-type integrals (8), (9) to values of the generalized-Euler-constant function

\[
\gamma_{a,b}(z) = \sum_{n=0}^{\infty} \left( \frac{1}{an+b} - \log \left( \frac{an+b+1}{an+b} \right) \right) z^n, \quad a, b \in \mathbb{N},
\]

and its derivatives, which is related to constants [11], [2] as \( \gamma_{1,1}(1) = \gamma, \gamma_{1,1}(-1) = \log \frac{4}{x} \).

Using generating functions appeared in these integral representations we give new Vacca and Ramanujan-type series for values of \( \gamma_{a,b}(z) \) and Addison-type series for values of \( \gamma_{a,b}(z) \) and its derivative. As a consequence, we get base \( B \) rational series for \( \log \frac{4}{x}, \frac{G}{x} \), (where \( G \) is Catalan’s constant), \( \frac{G(2)}{x^2} \) and also for logarithms of Somos’s and Glaisher-Kinkelin’s constants. We also mention on connection of our approach to summation of series of the form

\[
\sum_{n=1}^{\infty} N_{\omega,B}(n)Q(n, B) \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{N_{\omega,B}(n)P_B(n)}{Bn(Bn+1) \cdots (Bn+B)},
\]

where \( Q(n, B) \) is a rational function of \( B \) and \( n \)

\[
Q(n, B) = \frac{1}{Bn(Bn+1)} + \frac{2}{Bn(Bn+2)} + \cdots + \frac{B-1}{Bn(Bn+B-1)},
\]

and \( N_{\omega,B}(n) \) is the number of occurrences of a word \( \omega \) over the alphabet \( \{0, 1, \ldots, B-1\} \) in the \( B \)-ary expansion of \( n \), considered in [2]. In this notation, the generalized Vacca series (11) can be written as follows:

\[
\gamma = \sum_{k=1}^{\infty} L_B(k)Q(k, B),
\]

where \( L_B(k) := |\log_B Bk| = \sum_{a=0}^{B-1} N_{a,B}(k) \) is the \( B \)-ary length of \( k \). Indeed, representing \( n = Bk + r, 0 \leq r \leq B - 1 \) and summing in (10) over \( k \geq 1 \) and \( 0 \leq r \leq B - 1 \) we get

\[
\gamma = \sum_{k=1}^{\infty} |\log_B Bk| \left( \frac{B-1}{Bk} - \frac{1}{Bk+1} - \cdots - \frac{1}{Bk+B-1} \right) = \sum_{k=1}^{\infty} |\log_B Bk| Q(k, B).
\]
By the same notation, the generalized Addison series \( \text{(12)} \) gives another base \( B \) expansion of Euler’s constant
\( \gamma \)
\[
\frac{1}{2} + \sum_{n=1}^{\infty} \frac{L_B(n)P_B(n)}{Bn(Bn+1)\cdots(Bn+B)} = \frac{1}{2} + \sum_{n=1}^{\infty} L_B(n) \left( Q(n,B) - \frac{B-1}{2Bn(n+1)} \right)
\]
which converges faster than \( \text{(16)} \) to \( \gamma \). Here we used the fact that
\[
\sum_{n=1}^{\infty} \sum_{\alpha=0}^{B-1} N_{\alpha,B}(n) = \frac{B}{B-1},
\]
which can be easily checked by \([3, \text{Section 3}]\). On the other hand,
\[
Q(n,B) - \frac{B-1}{2Bn(n+1)} = \frac{1}{2} \sum_{m=1}^{B-1} \left( \frac{1}{Bn} - \frac{2}{Bn+m} + \frac{1}{Bn+B} \right)
\]
\[
= \frac{1}{Bn(Bn+B)} \sum_{m=1}^{B-1} \left( 2m - B + \frac{2m(B-m)}{Bn+m} \right) = \frac{P_B(n)}{Bn(Bn+1)\cdots(Bn+B)}.
\]

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2. **Analytic continuation**

We consider the generalized-Euler-constant function \( \gamma_{a,b}(z) \) defined in \([12]\), where \( a, b \) are positive real numbers, \( z \in \mathbb{C} \), and the series converges when \( |z| \leq 1 \). We show that \( \gamma_{a,b}(z) \) admits an analytic continuation to the domain \( \mathbb{C} \setminus [1, +\infty) \). The following theorem is a slight modification of \([12, \text{Th.3}]\).

**Theorem 1.** Let \( a, b \) be positive real numbers, \( z \in \mathbb{C}, |z| \leq 1 \). Then
\[
\gamma_{a,b}(z) = \int_{0}^{1} \int_{0}^{1} \frac{(xy)^{b-1}(1-x)}{1-xy^a} \left( -\log xy \right) dxdy = \int_{0}^{1} \frac{x^{b-1}(1-x)}{1-zx^a} \left( \frac{1}{1-x} + \frac{1}{\log x} \right) dx.
\]
The integrals converge for all \( z \in \mathbb{C} \setminus (1, +\infty) \) and give the analytic continuation of the generalized-Euler-constant function \( \gamma_{a,b}(z) \) for \( z \in \mathbb{C} \setminus [1, +\infty) \).

**Proof.** Denoting the double integral in \((18)\) by \( I(z) \) and for \( |z| \leq 1 \), expanding \((1-zx^a)^{-1}\) in a geometric series we have
\[
I(z) = \sum_{k=0}^{\infty} z^k \int_{0}^{1} \int_{0}^{1} \frac{(xy)^{ak+b-1}(1-x)}{(-\log xy)} dxdy
\]
\[
= \sum_{k=0}^{\infty} z^k \int_{0}^{1} \int_{0}^{+\infty} (xy)^{t+ak+b-1}(1-x) dxdy dt
\]
\[
= \sum_{k=0}^{\infty} z^k \int_{0}^{+\infty} \left( \frac{1}{(t+ak+b)^2} - \left( \frac{1}{t+ak+b} - \frac{1}{t+ak+b+1} \right) \right) dt = \gamma_{a,b}(z).
\]
On the other hand, making the change of variables $u = x^a$, $v = y^a$ in the double integral we get

$$I(z) = \frac{1}{a} \int_0^1 \int_0^1 \frac{(uv)^{\frac{b}{a}} - 1 - u^\frac{b}{a}}{(1 - zuv)(-\log uv)} dudv.$$ 

Now by [8, Corollary 3.3], for $z \in \mathbb{C} \setminus [1, +\infty)$ we have

$$I(z) = \frac{1}{a} \Phi(z, 1, \frac{b}{a}) - \frac{\partial \Phi}{\partial s}(z, 0, \frac{b}{a}) + \frac{\partial \Phi}{\partial s}(z, 0, \frac{b + 1}{a}),$$

where $\Phi(z, s, u)$ is the Lerch transcendent, a holomorphic function in $z$ and $s$, for $z \in \mathbb{C} \setminus [1, +\infty)$ and all complex $s$ (see [8, Lemma 2.2]), which is the analytic continuation of the series

$$\Phi(z, s, u) = \sum_{n=0}^{\infty} \frac{z^n}{(n + u)^s}, \quad u > 0.$$

To prove the second equality in (18), make the change of variables $X = xy$, $Y = y$ and integrate with respect to $Y$. □

**Corollary 1.** Let $a, b$ be positive real numbers, $l \in \mathbb{N}$, $z \in \mathbb{C} \setminus [1, +\infty)$. Then for the $l$-th derivative we have

$$\gamma_{a, b}^{(l)}(z) = \int_0^1 \int_0^1 \frac{(xy)^{al+b-1}(x - 1)}{(1 - zx^a)^{l+1}} \log xy \ dxdy = \int_0^1 x^{al+b-1}(1 - x) \left( \frac{1}{1 - x} + \frac{1}{\log x} \right) dx.$$

From Corollary [8, Cor.3.3, 3.8, 3.9] and [2, Lemma 4] we get

**Corollary 2.** Let $a, b$ be positive real numbers, $z \in \mathbb{C} \setminus [1, +\infty)$. Then the following equalities are valid:

$$\gamma_{a, b}(1) = \log \Gamma\left(\frac{b + 1}{a}\right) - \log \Gamma\left(\frac{b}{a}\right) - \frac{1}{a} \psi\left(\frac{b}{a}\right),$$

$$\gamma_{a, b}(z) = \frac{1}{a} \Phi(z, 1, \frac{b}{a}) - \frac{\partial \Phi}{\partial s}(z, 0, \frac{b}{a}) + \frac{\partial \Phi}{\partial s}(z, 0, \frac{b + 1}{a}),$$

$$\gamma_{a, b}'(z) = -\frac{b}{a^2} \Phi\left(z, 1, \frac{b}{a} + 1\right) + \frac{1}{a(1 - z)} + \frac{b}{a} \frac{\partial \Phi}{\partial s}(z, 0, \frac{b}{a} + 1) - \frac{\partial \Phi}{\partial s}(z, -1, \frac{b}{a} + 1) - \frac{b + 1}{a} \frac{\partial \Phi}{\partial s}(z, 0, \frac{b + 1}{a} + 1) + \frac{\partial \Phi}{\partial s}(z, -1, \frac{b + 1}{a} + 1),$$

where $\Phi(z, s, u)$ is the Lerch transcendent and $\psi(x) = \frac{d}{dx} \log \Gamma(x)$ is the logarithmic derivative of the gamma function.
3. Catalan-type integrals for $\gamma_{a,b}^{(l)}(z)$.

In [4] it was demonstrated that for $x > 0$ and any integer $B > 1$, one has

$$
\frac{1}{1-x} + \frac{1}{x} \log x = \sum_{k=1}^{\infty} \frac{(B-1) + (B-2)x^{\frac{1}{B^k}} + (B-3)x^{\frac{2}{B^k}} + \cdots + x^{\frac{B-2}{B^k}}}{B^k(1 + x^{\frac{1}{B^k}} + x^{\frac{2}{B^k}} + \cdots + x^{\frac{B-1}{B^k}})}.
$$

The special cases $B = 2, 3$ of this equality can be found in Ramanujan’s third notebook [9, p.364]. Using this key formula we prove the following generalization of integral (9).

**Theorem 2.** Let $a, b, B > 1$ be positive integers, $l$ a non-negative integer. If either $z \in \mathbb{C} \setminus [1, +\infty)$ and $l \geq 1$, or $z \in \mathbb{C} \setminus (1, +\infty)$ and $l = 0$, then

$$
\gamma_{a,b}^{(l)}(z) = \int_{0}^{1} \left( \frac{B}{1-x^B} - \frac{1}{1-x} \right) F_l(z, x) \, dx
$$

where

$$
F_l(z, x) = \sum_{k=1}^{\infty} \frac{x^{(b+a)B^{k-1}}(1-x^{B^k})}{(1-zx^{aB^k})^{l+1}}.
$$

**Proof.** First we note that the series of variable $x$ on the right-hand side of (20) uniformly converges on $[0, 1]$, since the absolute value of its general term does not exceed $\frac{B-1}{2B^k}$. Then for $l \geq 0$, multiplying both sides of (19) by $\frac{x^{(a+b-1)(1-x)}}{(1-zx^{aB^k})^{l+1}}$ and integrating over $0 \leq x \leq 1$ we get

$$
\gamma_{a,b}^{(l)}(z) = \sum_{k=1}^{\infty} \int_{0}^{1} x^{(a+b-1)(1-x)} \cdot \frac{(B-1) + (B-2)x^{\frac{1}{B^k}} + \cdots + x^{\frac{B-2}{B^k}}}{B^k(1 + x^{\frac{1}{B^k}} + x^{\frac{2}{B^k}} + \cdots + x^{\frac{B-1}{B^k}})} \, dx.
$$

Replacing $x$ by $x^{B^k}$ in each integral we find

$$
\gamma_{a,b}^{(l)}(z) = \sum_{k=1}^{\infty} \int_{0}^{1} \frac{x^{(a+b)(B^{k-1})(1-x^{B^k})}}{(1-zx^{aB^k})^{l+1}} \cdot \frac{(B-1) + (B-2)x + \cdots + x^{B^2}}{1 + x + x^2 + \cdots + x^{B-1}} \, dx
$$

$$
= \int_{0}^{1} \left( \frac{B}{1-x^B} - \frac{1}{1-x} \right) F_l(z, x) \, dx,
$$

as required. □

From Theorem 2 we readily get a generalization of Ramanujan’s integral.

**Corollary 3.** Let $a, b, B > 1$ be positive integers, $l$ a non-negative integer. If either $z \in \mathbb{C} \setminus [1, +\infty)$ and $l \geq 1$, or $z \in \mathbb{C} \setminus (1, +\infty)$ and $l = 0$, then

$$
\gamma_{a,b}^{(l)}(z) = \int_{0}^{1} \frac{x^{b+a-1}(1-x)}{(1-zx^{a})^{l+1}} \, dx + \int_{0}^{1} \left( \frac{Bx^B}{1-x^B} - \frac{x}{1-x} \right) F_l(z, x) \, dx.
$$

**Proof.** First we note that the series (21) considered as a sum of functions of variable $x$ uniformly converges on $[0, 1-\varepsilon]$ for any $\varepsilon > 0$. Then integrating termwise we have

$$
\int_{0}^{1-\varepsilon} F_l(z, x) \, dx = \sum_{k=1}^{\infty} \int_{0}^{1-\varepsilon} \frac{x^{(b+a)B^{k-1}}(1-x^{B^k})}{(1-zx^{aB^k})^{l+1}} \, dx.
$$
Making the change of variable \( y = x^B \) in each integral we get

\[
\int_0^{1-\varepsilon} F_1(z, x) \, dx = \sum_{k=1}^{\infty} \frac{1}{B^k} \int_0^{(1-\varepsilon)B^k} \frac{y^{b+al-1}(1-y)}{(1-zy^a)^{l+1}} \, dy.
\]

Since the last series of variable \( \varepsilon \) uniformly converges on \([0, 1]\), letting \( \varepsilon \) tend to zero we get

(23)

\[
\int_0^1 F_1(z, x) \, dx = \frac{1}{B-1} \int_0^1 \frac{y^{b+al-1}(1-y)}{(1-zy^a)^{l+1}} \, dy.
\]

Now from (20) and (23) it follows that

\[
\gamma^{(l)}_{a,b}(z) = \frac{1}{2} \int_0^1 \frac{x^{b+al-1}(1-x)}{(1-zx^a)^{l+1}} \, dx + \frac{1}{2} \int_0^1 \left( \frac{B\left(1+x^B\right) - 1+x}{1-x} \right) F_1(z, x) \, dx,
\]

and the proof is complete. □

Averaging both formulas (20), (22) we get the following generalization of integral (7).

Corollary 4. Let \( a, b, B > 1 \) be positive integers, \( l \) a non-negative integer. If either \( z \in \mathbb{C} \setminus [1, +\infty) \) and \( l \geq 1 \), or \( z \in \mathbb{C} \setminus (1, +\infty) \) and \( l = 0 \), then

\[
\gamma^{(l)}_{a,b}(z) = 1 - \frac{1}{2} \int_0^1 \frac{x^{b+al-1}(1-x)}{(1-zx^a)^{l+1}} \, dx + \frac{1}{2} \int_0^1 \left( \frac{B\left(1+x^B\right) - 1+x}{1-x} \right) F_1(z, x) \, dx.
\]

4. VACCÀ-TYPE SERIES FOR \( \gamma_{a,b}(z) \) AND \( \gamma'_{a,b}(z) \).

Theorem 3. Let \( a, b, B > 1 \) be positive integers, \( z \in \mathbb{C}, |z| \leq 1 \). Then for the generalized-Euler-constant function \( \gamma_{a,b}(z) \), the following expansion is valid:

\[
\gamma_{a,b}(z) = \sum_{k=1}^{\infty} a_k Q(k, B) = \sum_{k=1}^{\infty} a_k \frac{\varepsilon(k)}{k},
\]

where \( Q(k, B) \) is a rational function given by (15), \( \{a_k\}_{k=0}^{\infty} \) is a sequence defined by the generating function

(24)

\[
G(z, x) = \frac{1}{1-x} \sum_{k=0}^{\infty} \frac{x^{B^k}(1-x^{B^k})}{1-zx^{B^k}} = \sum_{k=0}^{\infty} a_k x^k
\]

and \( \varepsilon(k) \) is denoted in (11).

Proof. For \( l = 0 \), rewrite (20) in the form

\[
\gamma_{a,b}(z) = \int_0^1 \frac{1-x^B}{x} \left( \frac{B}{1-x^B} - \frac{1}{1-x} \right) G(z, x^B) \, dx
\]

where \( G(z, x) \) is defined in (24). Then, since \( a_0 = 0 \), we have

(25)

\[
\gamma_{a,b}(z) = \int_0^1 (B - 1 - x - x^2 - \cdots - x^{B-1}) \sum_{k=1}^{\infty} a_k x^{Bk-1} \, dx.
\]
Expanding $G(z, x)$ in a power series of $x$

$$G(z, x) = \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} z^m x^{(am+b)B^k} (1 + x + \ldots + x^{B^k-1})$$

we see that $a_k = O(\ln B k)$. Therefore, by termwise integration in (25), which can be easily justified by the same way as in the proof of Corollary 3 we get

$$\gamma_{a,b}(z) = \sum_{k=1}^{\infty} a_k \int_0^1 \left[ (x^{Bk-1} - x^{Bk}) + (x^{Bk-1} - x^{Bk+1}) + \ldots + (x^{Bk-1} - x^{Bk+B-2}) \right] dx$$

$$= \sum_{k=1}^{\infty} a_k Q(k, B). \quad \square$$

**Theorem 4.** Let $a, b, B > 1$ be positive integers, $z \in \mathbb{C}$, $|z| \leq 1$. Then for the generalized-Euler-constant function, the following expansion is valid:

$$\gamma_{a,b}(z) = \int_0^1 \frac{x^{B-1}(1-x)}{1-zx^a} \, dx - \sum_{k=1}^{\infty} a_k \tilde{Q}(k, B),$$

where

$$\tilde{Q}(k, B) = \frac{B - 1}{Bk(k+1)} - Q(k, B)$$

$$= \frac{B - 1}{(Bk+B)(Bk+1)} + \frac{B - 2}{(Bk+B)(Bk+2)} + \ldots + \frac{1}{(Bk+B)(Bk+B-1)}$$

and the sequence $\{a_k\}_{k=1}^{\infty}$ is defined in Theorem 3.

**Proof.** From Corollary 3 with $l = 0$, by the same way as in the proof of Theorem 3 we get

$$\int_0^1 \left( \frac{Bx^B}{1-x^B} - \frac{x}{1-x} \right) F_0(z, x) = \int_0^1 \frac{1-x^B}{x} \left( \frac{Bx^B}{1-x^B} - \frac{x}{1-x} \right) G(z, x^B) \, dx$$

$$= \int_0^1 (Bx^{B-1} - (1 + x + \ldots + x^{B-1})) \sum_{k=1}^{\infty} a_k x^{Bk} \, dx$$

$$= \sum_{k=1}^{\infty} a_k \int_0^1 \left[ (x^{Bk+B-1} - x^{Bk+B-2}) + \ldots + (x^{Bk+B-1} - x^{Bk+1}) + (x^{Bk+B-1} - x^{Bk}) \right] dx$$

$$= - \sum_{k=1}^{\infty} a_k \tilde{Q}(k, B). \quad \square$$

**Theorem 5.** Let $a, b, B > 1$ be positive integers, $z \in \mathbb{C}$, $|z| \leq 1$. Then for the generalized-Euler-constant function $\gamma_{a,b}(z)$ and its derivative, the following expansion is valid:

$$\gamma_{a,b}^{(l)}(z) = \frac{1}{2} \int_0^1 \frac{x^{B+n-1}(1-x)}{(1-zx^a)^{l+1}} \, dx + \sum_{k=1}^{\infty} a_k \frac{P_B(k)}{Bk(Bk+1)\ldots(Bk+B)^l}, \quad l = 0, 1,$$
where \( P_B(k) \) is a polynomial of degree \( B - 2 \) given by (13), \((z - 1)^2 + (l - 1)^2 \neq 0 \) and the sequence \( \{a_{k,l}\}_{k=0}^{\infty} \) is defined by the generating function

\[
G_l(z, x) = \frac{1}{1 - x} \sum_{k=0}^{\infty} \frac{x^{(b+al)B^k}(1 - x^{B^k})}{(1 - z x^{a B^k})^{l+1}} = \sum_{k=0}^{\infty} a_{k,l} x^k, \quad l = 0, 1.
\]

**Proof.** Expanding \( G_l(z, x) \) in a power series of \( x \)

\[
G_l(z, x) = \sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \binom{m+l}{l} z^m x^{(b+al+am)B^k}(1 + x + x^2 + \cdots + x^{B^k-1})
\]

we see that \( a_{k,l} = O(k^l \ln B k) \). Therefore, for \( l = 0, 1 \), by termwise integration we get

\[
\int_0^1 \left( \frac{B(1 + x^B)}{1 - x^B} - \frac{1 + x}{1 - x} \right) F_l(z, x) dx = \int_0^1 \frac{1 - x^B}{x} \left( \frac{B(1 + x^B)}{1 - x^B} - \frac{1 + x}{1 - x} \right) G_l(z, x) dx
\]

\[
= \int_0^1 [(B - 1) - 2x - 2x^2 - \cdots - 2x^{B-1} + (B - 1)x^B] \sum_{k=1}^{\infty} a_{k,l} x^{Bk-1} dx
\]

\[
= \sum_{k=1}^{\infty} a_{k,l} \left( \frac{B - 1}{Bk} - \frac{2}{Bk + 1} - \frac{2}{Bk + 2} - \cdots - \frac{2}{Bk + B - 1} + \frac{B - 1}{Bk + B} \right)
\]

\[
= 2 \sum_{k=1}^{\infty} a_{k,l} \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},
\]

where \( P_B(k) \) is defined in (13) and the last series converges since \( \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)} = O(k^{-3}) \). Now our theorem easily follows from Corollary 4. \( \square \)

5. **Summation of series in terms of the Lerch transcendent**

It is easily seen that the generating function (26) satisfies the following functional equation:

\[
G_l(z, x) - \frac{1 - x^B}{1 - x} G_l(z, x^B) = \frac{x^{b+al}}{(1 - z x^a)^{l+1}},
\]

which is equivalent to the identity for series:

\[
\sum_{k=0}^{\infty} a_{k,l} x^k - (1 + x + \cdots + x^{B-1}) \sum_{k=0}^{\infty} a_{k,l} x^{Bk} = \sum_{k=1}^{\infty} \binom{k}{l} z^{-k-l} x^{ak+b}.
\]

Comparing coefficients of powers of \( x \) we get an alternative definition of the sequence \( \{a_{k,l}\}_{k=0}^{\infty} \) by means of the recursion

\[
a_{0,l} = a_{1,l} = \cdots = a_{al+b-1,l} = 0
\]

and for \( k \geq al + b \),

\[
a_{k,l} = \begin{cases} 
   a_{\left\lfloor \frac{k}{b} \right\rfloor,l} & \text{if } k \equiv b \pmod{a}, \\
   a_{\left\lfloor \frac{k}{b} \right\rfloor,l} + \binom{(k-b)/a}{l} \cdot \binom{k-b}{a-1} & \text{if } k \not\equiv b \pmod{a}.
\end{cases}
\]
On the other hand, in view of Corollary \[2\] \(\gamma_{a,b}(z)\) and \(\gamma'_{a,b}(z)\) can be explicitly expressed in terms of the Lerch transcendent, \(\psi\)-function and logarithm of the gamma function. This allows us to sum the series figured in Theorems \[3,5\] in terms of these functions.

6. Examples of rational series

**Example 1.** Suppose that \(\omega\) is a non-empty word over the alphabet \(\{0, 1, \ldots, B-1\}\). Then obviously \(\omega\) is uniquely defined by its length \(|\omega|\) and its size \(v_B(\omega)\) which is the value of \(\omega\) when interpreted as an integer in base \(B\). Let \(N_{\omega,B}(k)\) be the number of (possibly overlapping) occurrences of the block \(\omega\) in the \(B\)-ary expansion of \(k\). Note that for every \(B\) and \(\omega\), \(N_{\omega,B}(0) = 0\), since the \(B\)-ary expansion of zero is the empty word. If the word \(\omega\) begins with 0, but \(v_B(\omega) \neq 0\), then in computing \(N_{\omega,B}(k)\) we assume that the \(B\)-ary expansion of \(k\) starts with an arbitrary long prefix of 0's. If \(v_B(\omega) = 0\) we take for \(k\) the usual shortest \(B\)-ary expansion of \(k\).

Now we consider equation \[(27)\] with \(l = 0, z = 1\)

\[
G(1, x) - \frac{1 - xB}{1 - x}G(1, x^B) = \frac{x^b}{1 - x^a}
\]

and for a given non-empty word \(\omega\), set in \[(29)\] \(a = B^{|\omega|}\) and

\[
b = \begin{cases} 
B^{|\omega|} & \text{if } v_B(\omega) = 0 \\
v_B(\omega) & \text{if } v_B(\omega) \neq 0.
\end{cases}
\]

Then by \[(28)\], it is easily seen that \(a_k := a_{k,0} = N_{\omega,B}(k), k = 1, 2, \ldots,\) and by Theorem \[3\] we get one more proof of the following statement (see [2, Sections 3, 4.2]).

**Corollary 5.** Let \(\omega\) be a non-empty word over the alphabet \(\{0, 1, \ldots, B-1\}\). Then

\[
\sum_{k=1}^{\infty} N_{\omega,B}(k)Q(k, B) = \begin{cases} 
\gamma_{B^{|\omega|}, v_B(\omega)}(1) & \text{if } v_B(\omega) \neq 0 \\
\gamma_{B^{|\omega|}, B^{|\omega|}}(1) & \text{if } v_B(\omega) = 0.
\end{cases}
\]

By Corollary \[2\] the right-hand side of the last equality can be calculated explicitly and we have

\[
\sum_{k=1}^{\infty} N_{\omega,B}(k)Q(k, B) = \begin{cases} 
\log \Gamma \left( \frac{v_B(\omega) + 1}{B^{|\omega|}} \right) - \log \Gamma \left( \frac{v_B(\omega)}{B^{|\omega|}} \right) - \frac{1}{B^{|\omega|}} \psi \left( \frac{v_B(\omega)}{B^{|\omega|}} \right) & \text{if } v_B(\omega) \neq 0 \\
\log \Gamma \left( \frac{1}{B^{|\omega|}} \right) + \frac{\gamma}{B^{|\omega|}} - |\omega| \log B & \text{if } v_B(\omega) = 0.
\end{cases}
\]

**Corollary 6.** Let \(\omega\) be a non-empty word over the alphabet \(\{0, 1, \ldots, B-1\}\). Then

\[
\sum_{k=1}^{\infty} \frac{N_{\omega,B}(k)P_B(k)}{Bk(Bk + 1) \cdots (Bk + B)}
\]

\[
= \begin{cases} 
\gamma_{B^{|\omega|}, v_B(\omega)}(1) - \frac{1}{2B^{|\omega|}} \left( \psi \left( \frac{v_B(\omega) + 1}{B^{|\omega|}} \right) - \psi \left( \frac{v_B(\omega)}{B^{|\omega|}} \right) \right) & \text{if } v_B(\omega) \neq 0 \\
\gamma_{B^{|\omega|}, B^{|\omega|}}(1) - \frac{1}{2B^{|\omega|}} \psi \left( \frac{1}{B^{|\omega|}} \right) - \frac{\gamma}{2B^{|\omega|}} - \frac{1}{2} & \text{if } v_B(\omega) = 0.
\end{cases}
\]

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Proof. The required statement easily follows from Theorem 5, Corollary 5 and the equality
\[
\int_0^1 \frac{x^{b-1}(1-x)}{1-x^a} \, dx = \sum_{k=0}^{\infty} \left( \frac{1}{ak+b} - \frac{1}{ak+b+1} \right) = \frac{1}{a} \left( \psi \left( \frac{b+1}{a} \right) - \psi \left( \frac{b}{a} \right) \right).
\]
\[\square\]
From Theorem 3, (27) and (28) with \(a = 1, l = 0\) we have

Corollary 7. Let \(b, B > 1\) be positive integers, \(z \in \mathbb{C}, |z| \leq 1\). Then
\[
\gamma_{1,b}(z) = \sum_{k=1}^{\infty} a_k Q(k, B) = \sum_{k=1}^{\infty} a_{\lfloor \frac{k}{B} \rfloor} \varepsilon(k) \frac{k}{k},
\]
where \(a_0 = a_1 = \ldots = a_{b-1} = 0, a_k = a_{\lfloor \frac{k}{B} \rfloor} + z^{k-b}, k \geq b\).

Similarly, from Theorem 5 we have

Corollary 8. Let \(b, B > 1\) be positive integers, \(z \in \mathbb{C}, |z| \leq 1\). Then
\[
\gamma_{1,b}(z) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{z^k}{(k+b)(k+b+1)} + \sum_{k=1}^{\infty} a_k \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},
\]
where \(a_0 = a_1 = \ldots = a_{b-1} = 0, a_k = a_{\lfloor \frac{k}{B} \rfloor} + z^{k-b}, k \geq b\).

Example 2. If in Corollary 7 we take \(z = 1\), then we get that \(a_k\) is equal to the \(B\)-ary length of \(\lfloor \frac{k}{B} \rfloor\), i.e.,
\[
a_k = \sum_{\alpha=0}^{B-1} N_{\alpha,B} \left( \left\lfloor \frac{k}{B} \right\rfloor \right) = L_B \left( \left\lfloor \frac{k}{B} \right\rfloor \right).
\]
On the other hand,
\[
\gamma_{1,b}(1) = \log b - \psi(b) = \log b - \sum_{k=1}^{b-1} \frac{1}{k} + \gamma
\]
and hence we get
\[
(31) \quad \log b - \psi(b) = \sum_{k=1}^{\infty} L_B \left( \left\lfloor \frac{k}{b} \right\rfloor \right) Q(k, B).
\]
If \(b = 1\), formula (31) gives (16). If \(b > 1\), then from (31) and (16) we get
\[
(32) \quad \log b = \sum_{k=1}^{b-1} \frac{1}{k} + \sum_{k=1}^{\infty} \left( L_B \left( \left\lfloor \frac{k}{B} \right\rfloor \right) - L_B(k) \right) Q(k, B),
\]
which is equivalent to [4, Theorem 2.8]. Similarly, from Corollary 5 we obtain (17) and
\[
(33) \quad \log b = \sum_{k=1}^{b-1} \frac{1}{k} - \frac{b-1}{2b} + \sum_{k=1}^{\infty} \frac{L_B \left( \left\lfloor \frac{k}{B} \right\rfloor \right) - L_B(k)}{Bk(Bk+1)\cdots(Bk+B)} P_B(k).
\]
Example 3. Using the fact that for any integer $B > 1$

$$L_B \left( \left\lfloor \frac{k}{B} \right\rfloor \right) - L_B(k) = -1,$$

from (30), (16) and (32) we get the following rational series for $\log \Gamma(1/B)$:

$$\log \Gamma \left( \frac{1}{B} \right) = \sum_{k=1}^{B-1} \frac{1}{k} + \sum_{k=1}^{\infty} \left( N_{0,B}(k) - \frac{1}{B} L_B(k) - 1 \right) Q(k,B).$$

Example 4. Substituting $b=1$, $z=-1$ in Corollary 7 we get the generalized Vácca series for $\log \frac{4}{\pi}$.

Corollary 9. Let $B \in \mathbb{N}$, $B > 1$. Then

$$\log \frac{4}{\pi} = \sum_{k=1}^{\infty} a_k Q(k,B) = \sum_{k=1}^{\infty} a_{\lfloor \frac{k}{B} \rfloor} \varepsilon(k),$$

where

$$a_0 = 0, \quad a_k = a_{\lfloor \frac{k}{B} \rfloor} + (-1)^{k-1}, \quad k \geq 1.$$  

In particular, if $B$ is even, then

$$\log \frac{4}{\pi} = \sum_{k=1}^{\infty} (N_{odd,B}(k) - N_{even,B}(k)) Q(k,B) = \sum_{k=1}^{\infty} \frac{(N_{odd,B}(\lfloor \frac{k}{B} \rfloor) - N_{even,B}(\lfloor \frac{k}{B} \rfloor)) \varepsilon(k)}{k},$$

where $N_{odd,B}(k)$ (respectively $N_{even,B}(k)$) is the number of occurrences of the odd (respectively even) digits in the $B$-ary expansion of $k$.

Proof. To prove (35), we notice that if $B$ is even, then the sequence $\tilde{a}_k := N_{odd,B}(k) - N_{even,B}(k)$ satisfies recurrence (34). In particular, if $B$ is even, then

$$\log \frac{4}{\pi} = \sum_{k=1}^{\infty} \left( L_B(\lfloor \frac{k}{2} \rfloor) - 2N_{even,B}(k) \right) P_B(k) = \sum_{k=1}^{\infty} \frac{(L_B(\lfloor \frac{k}{2} \rfloor) - 2N_{even,B}(k)) P_B(k)}{Bk(Bk+1)\cdots(Bk+B)}.$$  

Example 5. For $t > 1$, the generalized Somos constant $\sigma_t$ is defined by

$$\sigma_t = \sqrt{1\sqrt{2\sqrt{3\cdots}}} = 1^{1/t}2^{1/t^2}3^{1/t^3}\cdots = \prod_{n=1}^{\infty} n^{1/t^n}.$$  

In view of the relation [12, Th.8]

$$\gamma_{1,1} \left( \frac{1}{t} \right) = t \log \frac{t}{(t-1)\sigma_t^{t-1}},$$

(36)
by Corollary 7 and formula (32) we get

**Corollary 11.** Let \( B \in \mathbb{N}, B > 1, t \in \mathbb{R}, t > 1. \) Then

\[
\log \sigma_t = \frac{1}{(t-1)^2} + \frac{1}{t-1} \sum_{k=1}^{\infty} \left( L_B \left( \left\lfloor \frac{k}{t} \right\rfloor \right) - L_B \left( \left\lfloor \frac{k}{t-1} \right\rfloor \right) - \frac{a_k}{t} \right) Q(k, B),
\]

where \( a_0 = 0, \) \( a_k = a_{\lfloor \frac{k}{t} \rfloor} + t^{1-k}, k \geq 1. \)

In particular, setting \( B = t = 2 \) we get the following rational series for Somos’s quadratic recurrence constant:

\[
\log \sigma_2 = 1 - \frac{1}{2} \sum_{k=1}^{\infty} \frac{a_k}{2k(2k+1)},
\]

where \( a_1 = 3, \) \( a_k = a_{\lfloor \frac{k}{2} \rfloor} + \frac{1}{2^{k-1}}, k \geq 2. \)

From (36), (33) and Theorem 5 we find

**Corollary 12.** Let \( B \in \mathbb{N}, B > 1, t \in \mathbb{R}, t > 1. \) Then

\[
\log \sigma_t = \frac{3t-1}{4t(t-1)^2} + \frac{t+1}{2(t-1)} \sum_{k=1}^{\infty} \left( L_B \left( \left\lfloor \frac{k}{t} \right\rfloor \right) - L_B \left( \left\lfloor \frac{k}{t-1} \right\rfloor \right) - \frac{2a_k}{t(t+1)} \right) \frac{P_B(k)}{Bk(Bk+1)\cdots(Bk+B)},
\]

where the sequence \( a_k \) is defined in Corollary 11.

In particular, if \( B = t = 2 \) we get

\[
\log \sigma_2 = \frac{5}{8} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{a_k}{2k(2k+1)(2k+2)},
\]

where \( a_1 = 4, \) \( a_k = a_{\lfloor \frac{k}{2} \rfloor} + \frac{1}{2^{k-1}}, k \geq 2. \)

**Example 6.** The Glaisher-Kinkelin constant is defined by the limit \([7, \text{p.135}]\)

\[
A := \lim_{n \to \infty} \frac{1 \cdot 2^2 \cdots n^n}{n^{n+1/2} \cdot \frac{1}{\sqrt{2\pi}} e^{-n}} = 1.28242712\ldots.
\]

Its connection to the generalized-Euler-constant function \( \gamma_{a,b}(z) \) is given by the formula \([12, \text{Cor.4}]\)

\[
(37) \quad \gamma'_{1,1}(-1) = \log \frac{2^{11/6} A^6}{\pi^{3/2} e}.
\]

By Theorem 5 since

\[
\int_0^1 \frac{x(1-x)}{(1+x)^2} \, dx = 3 \log 2 - 2,
\]

we have

\[
\log A = \frac{4}{9} \log 2 - \frac{1}{4} \log \pi + \frac{1}{6} \sum_{k=1}^{\infty} \frac{a_{k,1}}{Bk(Bk+1)\cdots(Bk+B)},
\]
where the sequence \( a_{k,1} \) is defined by the generating function (26) with \( a = b = l = 1 \), \( z = -1 \), or using (28) by the recursion

\[
a_{0,1} = a_{1,1} = 0, \quad a_{k,1} = a_{\left\lfloor \frac{k}{B} \right\rfloor,1} + (-1)^k (k - 1), \quad k \geq 2.
\]

Now by Corollary 10 and (33) we get

**Corollary 13.** Let \( B > 1 \) be a positive integer. Then

\[
\log A = \frac{13}{48} - \frac{1}{36} \sum_{k=1}^{\infty} \left( 7L_B(k) - 7L_B\left( \left\lfloor \frac{k}{2} \right\rfloor \right) + b_k \right) \frac{P_B(k)}{Bk(Bk+1) \cdots (Bk+B)},
\]

where \( b_0 = 0, \ b_k = b_{\left\lfloor \frac{k}{B} \right\rfloor} + (-1)^{k-1}(6k+3), \ k \geq 1 \).

In particular, if \( B = 2 \) we get

\[
\log A = \frac{13}{48} - \frac{1}{36} \sum_{k=1}^{\infty} \frac{c_k}{2k(2k+1)(2k+2)},
\]

where \( c_1 = 16, \ c_k = c_{\left\lfloor \frac{k}{2} \right\rfloor} + (-1)^{k-1}(6k+3), \ k \geq 1 \).

Using the formula expressing \( \zeta'(2) / \pi^2 \) in terms of Glaisher-Kinkelin's constant [7, p.135]

\[
\log A = -\frac{\zeta'(2)}{\pi^2} + \frac{\log 2\pi + \gamma}{12}
\]

by Corollaries 8, 10 and 13 we get

**Corollary 14.** Let \( B > 1 \) be a positive integer. Then

\[
\frac{\zeta'(2)}{\pi^2} = -\frac{1}{16} + \frac{1}{36} \sum_{k=1}^{\infty} \left( 4L_B(k) - L_B\left( \left\lfloor \frac{k}{2} \right\rfloor \right) + c_k \right) \frac{P_B(k)}{Bk(Bk+1) \cdots (Bk+B)},
\]

where \( c_0 = 0, \ c_k = c_{\left\lfloor \frac{k}{B} \right\rfloor} + (-1)^{k-1}6k, \ k \geq 1 \).

**Example 7.** First we evaluate \( \gamma_{2,1}(-1) \) for \( l = 0, 1 \). From Corollaries 1, 2 and [12, Ex.3.12, 3.13] we have

\[
\gamma_{2,1}(-1) = \int_0^1 \int_0^1 \frac{(x-1) \, dx \, dy}{(1 + x^2 y^2)} \log xy = \frac{\pi}{4} - 2 \log \Gamma\left( \frac{1}{4} \right) + \log \sqrt{2\pi^3}
\]

and

\[
\gamma'_{2,1}(-1) = -\frac{1}{4} \Phi(-1,1,3/2) + \frac{1}{2} \Phi(-1,0,3/2) + \frac{1}{2} \frac{\partial \Phi}{\partial s}(-1,0,3/2) - \frac{\partial \Phi}{\partial s}(-1,-1,3/2) - \frac{\partial \Phi}{\partial s}(-1,0,2) + \frac{\partial \Phi}{\partial s}(-1,-1,2).
\]

The last expression can be evaluated explicitly (see [12, Section 2]) and we get

\[
\gamma'_{2,1}(-1) = \frac{G}{\pi} + \frac{\pi}{8} - \log \Gamma\left( \frac{1}{4} \right) - 3 \log A + \log \pi + \frac{1}{3} \log 2,
\]

or

\[
\frac{G}{\pi} = \gamma'_{2,1}(-1) - \frac{1}{2} \gamma_{2,1}(-1) + \frac{1}{4} \log \frac{4}{\pi} + 3 \log A - \frac{7}{12} \log 2.
\]
On the other hand, by Theorem 3 and (28) we have

\begin{equation}
\gamma_{2,1}(-1) = \frac{\pi}{8} - \frac{1}{4} \log 2 + \sum_{k=1}^{\infty} \frac{a_{k,0}}{Bk(Bk + 1) \cdots (Bk + B)}.
\end{equation}

where \(a_{0,0} = 0, a_{2k,0} = a_{\lfloor \frac{2k}{B} \rfloor}, k \geq 1, a_{2k+1,0} = a_{\lfloor \frac{2k+1}{B} \rfloor,0} + (-1)^k, k \geq 0, \)

and

\begin{equation}
\gamma_{2,1}'(-1) = \frac{\pi}{16} - \frac{1}{4} \log 2 + \sum_{k=1}^{\infty} \frac{a_{k,1}}{Bk(Bk + 1) \cdots (Bk + B)},
\end{equation}

where \(a_{0,1} = 0, a_{2k,1} = a_{\lfloor \frac{2k}{B} \rfloor,1}, k \geq 1, a_{2k+1,1} = a_{\lfloor \frac{2k+1}{B} \rfloor,1} + (-1)^{k-1}k, k \geq 0. \)

Now from (38) - (40), (33) and Corollary 10 we get the following expansion for \(G/\pi.\)

**Corollary 15.** Let \(B > 1\) be a positive integer. Then

\[
\frac{G}{\pi} = \frac{11}{32} + \sum_{k=1}^{\infty} \left( \frac{1}{8} L_B \left( \left\lfloor \frac{k}{2} \right\rfloor \right) - \frac{1}{8} L_B(k) + c_k \right) \frac{P_B(k)}{Bk(Bk + 1) \cdots (Bk + B)},
\]

where \(c_0 = 0, c_{2k} = c_{\lfloor \frac{2k}{B} \rfloor} + k, k \geq 1, c_{2k+1} = c_{\lfloor \frac{2k+1}{B} \rfloor} + \frac{(-1)^{k-1}-1}{2}(2k+1), k \geq 0. \)

In particular, if \(B = 2\) we get

\[
\frac{G}{\pi} = \frac{11}{32} + \sum_{k=1}^{\infty} \frac{c_k}{2k(2k+1)(2k+2)},
\]

where \(c_1 = -\frac{9}{8}, c_{2k} = c_k + k, c_{2k+1} = c_k + \frac{(-1)^{k-1}-1}{2}(2k+1), k \geq 1. \)

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