EXACT, E = 0, SOLUTIONS FOR GENERAL
POWER-LAW POTENTIALS.
I. CLASSICAL ORBITS

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ABSTRACT

For zero energy, \( E = 0 \), we derive exact, classical solutions for all power-law potentials, \( V(r) = -\gamma/r^\nu \), with \( \gamma > 0 \) and \( -\infty < \nu < \infty \). When the angular momentum is non-zero, these solutions lead to the orbits \( \rho(t) = [\cos \mu(\varphi(t) - \varphi_0(t))]^{1/\mu} \), for all \( \mu \equiv \nu/2 - 1 \neq 0 \). When \( \nu > 2 \), the orbits are bound and go through the origin. This leads to discrete discontinuities in the functional dependence of \( \varphi(t) \) and \( \varphi_0(t) \), as functions of \( t \), as the orbits pass through the origin. We describe a procedure to connect different analytic solutions for successive orbits at the origin. We calculate the periods and precessions of these bound orbits, and graph a number of specific examples. Also, we explain why they all must violate the virial theorem. The unbound orbits are also discussed in detail. This includes the unusual orbits which have finite travel times to infinity and also the special \( \nu = 2 \) case.

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1 Introduction

Since the birth of quantum mechanics, studying the connections between classical and quantum physics has been an enormous industry [1, 2, 3]. Both the intuitive and also the analytic aspects have been studied. In particular, one can observe a kind of “Folk Theorem.” Problems that are amenable to well-defined, exact, analytic solutions tend to be solvable in both the classical and the quantum regimes.

For example, the fundamental three-dimensional problems of quantum mechanics, the harmonic oscillator and the hydrogen atom, are those classical problems which, by Bertrand’s theorem [4], have exact, closed orbits for all energies. The standard, one-dimensional, quantum-mechanical potential problems that are exactly solvable, such as the Morse [5], Rosen-Morse [6], and Pöschl-Teller [7] potentials, are also solvable classically [8, 9, 10]. Further, it has even been shown that quantum problems which are solvable, but have certain “mathematical diseases,” manifest these diseases already in the classical problem [11].

Usually, when one solves a potential system, one takes a particular potential from a family and solves it for all values of the energy, \( E \). Here, we are going to do the opposite. We will consider the entire class of power-law potentials, parametrized as

\[
V(r) = -\frac{\gamma}{r^\nu} = -\frac{\gamma}{r^{2\mu+2}}, \quad \gamma > 0, \quad -\infty < \nu < \infty, \tag{1}
\]

and exactly solve it for all \( \nu \) with the particular energy \( E = 0 \). Note that it will be useful to switch back and forth between the variables \( \nu \) and \( \mu \), related by

\[
\mu = (\nu - 2)/2, \quad \nu = 2(\mu + 1). \tag{2}
\]

The potentials (1) are attractive for \( \nu > 0 \) and repulsive for \( \nu < 0 \). For \( \nu = 0 \), the potential is a constant, \( V(r) = -\gamma \), so the particle is force-free.

The above system is exactly solvable in both the classical and quantum-mechanical cases, and there are similarities in the properties of the solutions. In the present paper, hereafter called I, we shall discuss the very unusual and enlightening classical solutions for the orbits. In the following paper [12], called II, we will solve the quantum problem and discuss the wave functions.

In Appendix A we give, for completeness, the simple solutions for zero angular momentum. The main body of the paper concentrates on the more interesting general solutions, with non-zero, angular momentum.

We begin, in Sec. 2, by defining our “dynamical” units, which allow us to simplify the equations and their solutions. Then we show that for \( \nu \neq 2 \), the classical solutions have only one finite, non-zero turning radius, which we will denote by \( a \). This \( a \) has opposite interpretations, depending on the value of the power index \( \nu \). For \( \nu > 2 \) the solutions yield bound orbits with \( 0 \leq r \leq a \), but for \( \nu < 2 \) they yield unbound orbits, with \( r \geq a \). In Sec. 3 we derive the general \( E = 0 \) solutions, using two methods. In Sec. 4 we investigate the general properties of the bound orbits, \( \nu > 2 \) or \( \mu > 0 \), such as their precessions and periods.

Continuing, in Sec. 5 we graphically depict examples of the bound orbits, thereby demonstrating the general properties derived previously. Beginning with the case
ν = 4, which is a boundary case that does not precess, we find that the orbit describes a circle which goes through the origin and continually repeats itself. For ν > 4, the typical solution looks like a flower. A single orbit starts at the origin, goes out to a and returns to the origin. This first orbit describes what we will call a “petal.” The following orbits then go on to describe further petals which in general are precessed from each other. After a number of petals, the trajectory closes if ν is a rational fraction. The petals become narrower and narrower as ν → ∞. Contrariwise, for 4 > ν > 2, the orbits become tighter and tighter spirals in and out of the origin as ν → 2. Once again, for ν being a rational fraction, the orbits eventually close on themselves.

In Secs. 6 and 7 we discuss the classically unbound orbits. These are given by all ν < 2 as well as the special case ν = 2. We close with comments on the classical problem and defer comparisons between the classical and quantum problems to paper II. Appendix B discusses the scaling properties of the solutions. Appendix C investigates violations of the virial theorem.

2 Classical Units and E = 0 Orbits

Power-law potentials do not have a built-in length scale. To specify a power potential completely, two units are needed: a unit of energy, $\mathcal{E}_0$, and a unit of length, $a$. Instead of the unit of energy, one can use a unit of angular momentum, $L_0$, which can be related to $\mathcal{E}_0$, for example, by

$$\mathcal{E}_0 = L_0^2 / 2ma^2.$$  \hfill (3)

Thus, we can parametrize the power potentials as follows:

$$V(r) \equiv -\frac{\gamma}{r^\nu} \equiv -\mathcal{E}_0 \frac{g^2}{\rho^{\nu}} = -\frac{L_0^2}{2ma^2} \frac{g^2}{\rho^\nu}, \quad \rho \equiv r/a.$$  \hfill (4)

Therefore, the dimensional coupling constant, $\gamma$, is related to the dimensionless coupling constant, $g^2$, by

$$\gamma = \frac{L_0^2 a^{\nu-2}}{2m} g^2.$$  \hfill (5)

Dimensionless units, such as $g^2$, are convenient in quantum mechanics. There one has other physical constants, such as Planck’s constant, $\hbar$, to make a convenient choice of $L_0$ and $\mathcal{E}_0$. For example, in Sec. 2 of II we choose $L_0 = \hbar$ to obtain $\mathcal{E}_0 = \hbar^2 / 2ma^2$ as the unit of energy, but we leave $a$ arbitrary. (We also could have chosen $a$ to be equal to $\hbar/mc = \lambda_c$, the (angular) Compton wave-length, where $c$ is the velocity of light. Then we would have obtained $\mathcal{E}_0 = mc^2 / 2$.)

However, for classical orbits, it is often convenient to use dynamical units. By these we mean units which depend on the orbit itself and are determined by the initial conditions. For our purposes it is convenient to choose the dynamical units such that $a$ is the turning point and $L$ is the corresponding angular momentum for the $E = 0$ solution. The possibility of such a choice is based on the following result:
(a) Let “a” be a prescribed, non-zero length scale. Then every power-law potential, 
\[ V = -\frac{\gamma}{r^\nu} \] 
with \( \gamma > 0 \) and \( \nu \neq 2 \), will have an \( E = 0 \) classical orbit with \( r = a \) as a turning radius. We call this orbit the “standard orbit.” 

(b) This turning radius corresponds to the maximum (minimum) distance from the center of the force for \( \nu > 2 \) (\( \nu < 2 \)), respectively. 

(For \( \nu = 2 \) the orbit has no turning point, except for the origin.)

Part (a) of the above result follows from energy conservation:

\[ T + V(r) = \frac{m}{2} \left( \frac{dr}{dt} \right)^2 + \frac{L^2}{2mr^2} - \frac{\gamma}{r^\nu} = \frac{m}{2} \left( \frac{dr}{dt} \right)^2 + U(L, r) = E, \tag{6} \]

where \( L^2/2m^2 \) is the centripetal-barrier potential and

\[ U(L, r) = \frac{L^2}{2ma^2} - \frac{\gamma}{r^\nu} \tag{7} \]

is the effective potential. Since a turning radius corresponds to \( dr/dt = 0 \), the turning radius \( a \) must be a zero of \( U(r) \) for \( E = 0 \). Thus, by combining Eqs. (4) and (7), we obtain

\[ U(L, r = a) = \frac{L^2}{2ma^2} - \frac{\gamma}{a^\nu} = \frac{L_0^2}{2ma^2} \left( \frac{L^2}{L_0^2} - g^2 \right) = 0. \tag{8} \]

Therefore, by choosing the angular momentum to be

\[ L = gL_0, \tag{9} \]

we obtain the desired “standard orbit.”

By using the dynamical unit \( L \), instead of \( L_0 \), the potential \( U \) is simplified to

\[ U(L, r) = \frac{L^2}{2ma^2} \left( \frac{1}{\rho^2} - \frac{1}{\rho'^2} \right), \quad \rho = \frac{r}{a}, \tag{10} \]

so that

\[ V(r) = -\frac{\gamma}{r^\nu} = -\frac{L^2}{2ma^2} \frac{1}{\rho^\nu} = -\frac{L^2 a^{\nu-2}}{2m} \frac{1}{r^\nu}. \tag{11} \]
Finally, from Eq. (11) we can read off the relation between $L$ and the turning radius $a$:

$$a = \left(\frac{2m\gamma}{L^2}\right)^\frac{1}{\nu - 2} = \left(\frac{2m\gamma}{L^2}\right)^\frac{1}{\nu - 2}, \quad \nu \neq 2, \quad \mu \neq 0.$$ \hfill (12)

In particular, for $L = 0$ there is no turning point, except for $a = \infty$. This special case is treated in Appendix A.

Part (b) of the result follows by substituting $E = 0$ into Eq. (6) to yield

$$U(L, r) = -\frac{m}{2} \left(\frac{dr}{dt}\right)^2 \leq 0.$$ \hfill (13)

Combining this with Eq. (10) means that

$$\rho^{\nu - 2} \leq 1.$$ \hfill (14)

This condition determines the physical domain of the distance $r$:

$$r \leq a, \quad \nu > 2,$$

$$r \geq a, \quad \nu < 2.$$ \hfill (15)

Therefore, $a$ is the apogee length for bound orbits ($\nu > 2$) and the distance of closest approach for unbound orbits ($\nu < 2$).

For $\nu = 2$ the above arguments break down, since the effective potential $U = (L^2/2m - \gamma)r^{-2}$ does not allow a finite, non-zero turning point when $E = 0$. This special case will be discussed further in Sec. 6.1.

Finally, we note that, having established the existence of a “standard orbit,” it follows by scaling arguments that there are an infinity of similar orbits whose turning radii and angular momenta are related to those of the standard orbit as follows:

$$a_\lambda = \lambda a, \quad L_\lambda = \lambda^{1-\nu/2}L = \lambda^{-\mu}L.$$ \hfill (16)

These scaling relations are, of course, just special cases of the more general $E \neq 0$ scaling law. (See Refs. [13, 14] and Appendix B.)

### 3 Classical Solutions

#### 3.1 From the orbit equation

A direct, mathematically-inspired solution can be obtained by integrating the orbit equation [15, 16]

$$\phi - \phi_0 = \int_{r_0}^{r} \frac{r^{-1} dr}{\sqrt{2m/\ell^2(E - V)r^2 - 1}}.$$ \hfill (17)
The general solution has the constant $\varphi_0$ in it. We presciently set $\varphi_0 = 0$ when $\rho = 1$ (the turning-point condition), so that the turning point is along the positive $x$-axis.

Then, setting $E = 0$ makes the integral (17) doable for all $\nu$. Changing variables successively to $\rho = r/a$, $y = \rho^\mu$, and $x = \cos^{-1} y$ means that Eq. (17) can be written as

$$\varphi = \int_1^\rho \frac{\rho^{(\mu - 1)}}{\sqrt{1 - \rho^{2\mu}}} \, d\rho = \mu^{-1} \int_1^y \frac{dy}{\sqrt{1 - y^2}} = -\mu^{-1} \int_0^\cos^{-1} y \, dx .$$

\hspace{1in} (18)

Therefore, the solution is

$$\rho^\mu = y = \cos (-\mu \varphi) = \cos (\mu \varphi) ,$$

\hspace{1in} (19)

or

$$\rho = \left[ \cos \left( \frac{\nu - 2}{2} \varphi \right) \right]^{\frac{2}{\nu - 2}} = [\cos (\mu \varphi)]^{1/\mu} .$$

\hspace{1in} (20)

We will discuss the allowed angular variations of $\varphi$ in the separate sections on bound and unbound orbits.

3.2 From the energy conservation equation

We now give a more intuitive derivation of the solution (20). By substituting the angular-momentum conservation condition

$$\dot{\varphi} = L/(mr^2)$$

\hspace{1in} (21)

into the energy conservation condition

$$E - V = \frac{m}{2} \dot{\varphi}^2 \left[ \left( \frac{dr}{d\varphi} \right)^2 + r^2 \right] ,$$

\hspace{1in} (22)

one obtains [17, 18]

$$\left( \frac{dr}{d\varphi} \right)^2 + r^2 = \frac{2m(E - V)r^4}{L^2} .$$

\hspace{1in} (23)

This is essentially a first-order differential equation, which can be formally integrated to yield the angular equation (17) of the last subsection.

However, for $E = 0$, it is much more efficient to solve Eq. (23) directly. Converting to the dimensionless variable $\rho = r/a$ and substituting $V$ from Eq. (11) into Eq. (23), we obtain

$$\left( \frac{d\rho}{d\varphi} \right)^2 + \rho^2 = \rho^{(4 - \nu)} = \rho^{(2 - 2\mu)} .$$

\hspace{1in} (24)
For \( \nu = 4 \) the right-hand side of this equation is unity, so the solution is a cosine. This is the circular orbit \( \rho = \cos \varphi \) which we will discuss in detail in the next section. Guided by this and the substitution \( y = \rho^\mu \) of the last subsection, we multiply Eq. (24) by \( \rho^{2\mu - 2} \) to yield

\[
\left( \rho^{\mu - 1} \frac{d\rho}{d\varphi} \right)^2 + \rho^{2\mu} = \left( \frac{d\rho^\mu}{\mu d\varphi} \right)^2 + (\rho^\mu)^2 = 1 .
\]  

Now \( \rho^\mu \) satisfies the differential equation for the trigonometric functions. Therefore, the general solution of Eq. (25) is given by

\[
\rho^\mu = \cos \mu (\varphi - \varphi_0) = \cos \left( \frac{\nu - 2}{2} (\varphi - \varphi_0) \right),
\]  

or

\[
\rho = \left[ \cos \mu (\varphi - \varphi_0) \right]^{1/\mu} = \left[ \cos \left( \frac{\nu - 2}{2} (\varphi - \varphi_0) \right) \right]^{2/\nu - 2}.
\]

The phase, \( \varphi_0 \), is the integration constant. It depends on the initial conditions and, as before, we will set \( \varphi_0 \) to zero for our first-orbit turning-point condition.

### 4 Properties of the Bound Trajectories, \( 2 < \nu \) or \( 1 < \mu \)

In Fig. 1 we show a typical effective potential \( U(r) \) in this regime. One sees that the value of \( U(r) \) starts from \(-\infty\) at \( r = 0 \), rises through zero at \( r = a \), reaches a maximum, and then decreases to zero at \( r \to \infty \). Therefore, the \( E = 0 \) solution for \( r = r(t) \) can only vary between \( r = 0 \) and \( r = a \). We shall now discuss the dependence of \( r \) on the azimuthal angle, \( \varphi \).

#### 4.1 The first orbit

The radius, \( \rho \), is nonnegative. This means the range of the angle variable for the first orbit is restricted. Since, by convention, the first orbit has \( \varphi_0 = \varphi_0^1 = 0 \), this means \( \varphi^1 = \varphi \) satisfies

\[
-\frac{\pi}{2\mu} = -\frac{\pi}{\nu - 2} \leq \varphi^1 \leq \frac{\pi}{\nu - 2} = \frac{\pi}{2\mu}.
\]

The corresponding curve \( \rho = \rho(\varphi^1) \) in the x-y plane, begins at \( \rho = 0 \) for \( \varphi^1 = -\pi / 2\mu = -\pi / (\nu - 2) \), evolves counterclockwise to \( \rho = 1 \) at \( \varphi^1 = 0 \), and then continues on back to \( \rho = 0 \) at \( \varphi^1 = \pi / (2\mu) = \pi / (\nu - 2) \). The curve evolves counterclockwise
because we have chosen, by convention, that the angular momentum is in the positive z direction. (Of course, there is also a mathematically-rotated clockwise solution.) Every such closed circuit, beginning and ending at the origin, we will call an orbit. Because of the shapes of the orbits (which we will present in the next Section), when \( \nu > 4 \) we will also call an orbit a petal (of a flower) and when \( \nu < 4 \) we will also call an orbit a spiral, which actually is a double spiral since it spirals out and then spirals in. These two classes of orbits meet at \( \nu = 4 \), which is the circle that goes through the origin. As we will see below, the entire physical trajectory will consist of a pattern of either (i) a finite integral number of orbits which then repeat over themselves, when \( \mu \) is a rational number, or else (ii) an infinite number of nonrepeating orbits, if \( \mu \) is an irrational number. (In Sec. 5 we will discuss the classically unbound orbits described by \( \nu \leq 2 \) or \( \mu \leq 1 \).)

### 4.2 Solutions for later bound orbits

#### 4.2.1 The angles and the phase shifts

To obtain the entire physical solution, we must connect different orbits. That is, the first orbit evolves into the second orbit...evolves into the \( k \)th orbit. Because the orbits go through the origin, both \( \varphi \) and \( \varphi_0 \) are discontinuous, as we will demonstrate below. Therefore, we will label \( \varphi \) and \( \varphi_0 \) for the \( k \)th orbit as \( \varphi^k \) and \( \varphi_0^k \), respectively.

The successive orbits must be connected at the origin, \( r = 0 \), in such a way that the directions of the linear momentum \( \hat{p} \) and the angular momentum \( L \) are continuous functions of time. We defined the first orbit to have its apogee, \( r = a \), at \( \varphi^1 = 0 \), which corresponded to the choice \( \varphi_0^1 = 0 \). In order to obtain a physical solution, we must connect the first orbit to the second orbit in such a way that the tangent vector to the combined trajectory is continuous at the origin.

Before the particle starts the second orbit, the trajectory goes through the origin. This creates a singular transition in the polar coordinates. (Since the trajectory passes through the origin and because \( \hat{p} \) must be continuous, the direction \( \hat{r} \) of the position vector must change sign at \( r = 0 \).) The angle advances by \( \pi \) (instead of \( -\pi \)) in going through the origin because, in order to conserve angular momentum, the position vector must continue rotating in the counterclockwise direction. Then the particle begins its second orbit, travelling another angular distances \( \pi/\mu \). Before beginning its third orbit, the particle goes through the origin again, advancing another \( \pi \) radians. Therefore, the angular variation of the \( k \)th orbit, \( \varphi^k \), is given by

\[
\begin{align*}
\varphi^k_{\min} &\equiv \left( \frac{(k - 3/2)}{\mu} + (k - 1) \right) \pi \\
\varphi^k &\leq \left( \frac{(k - 1/2)}{\mu} + (k - 1) \right) \pi \\n&\equiv \varphi^k_{\max}
\end{align*}
\]

and one has the condition

\[
\varphi^k_{\min} + \pi = \varphi^k_{\max} + \pi .
\]

However, the phase shift, \( \varphi^k_0 \), must also change with each orbit. Recall, in summary, that the \( k \)-th orbit is described in polar coordinates by

\[
\rho_k = [\cos \mu (\varphi^k - \varphi^k_0)]^{1/\mu} ,
\]

and one has the condition

\[
\varphi^k_{\min} + \pi = \varphi^k_{\max} + \pi .
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\]

and one has the condition

\[
\varphi^k_{\min} + \pi = \varphi^k_{\max} + \pi .
\]
with
\[ |\varphi^k - \varphi_0^k| \leq \pi/2\mu \] (32)

or, equivalently,
\[ \varphi^k_{\min} \equiv \varphi_0^k - \frac{\pi}{2\mu} \leq \varphi^k \leq \varphi_0^k + \frac{\pi}{2\mu} \equiv \varphi^k_{\max} . \] (33)

Then, from Eqs. (29) and (33), the phase shift, \( \varphi_0^k \), is
\[ \varphi_0^k = \frac{\varphi_{\min}^k + \varphi_{\max}^k}{2} = (k - 1) \left( \frac{\nu}{\nu - 2} \right) \pi = (k - 1) \left( 1 + \frac{1}{\mu} \right) \pi . \] (34)

Mathematically this change of phase shift, \( \varphi_0^k \), with each orbit is because the orbit must be symmetric about any apsidal vector. This is true even though the vector is zero length from the origin. (See the interesting application of these concepts in the \( \nu = 6 \) orbit discussion of the next section.) Further, the other apsidal vectors, \( r_k (\varphi^k = \varphi_0^k) \), are the symmetry axes of the \( k \)th orbits.

### 4.2.2 Angle vs. phase-shift differences

It is instructive to observe that, instead of using the label, \( k \), for each orbit, formally we can write the general solution for all times \( t \geq 0 \) in the form
\[ \rho(t) = \left[ \cos \mu (\varphi(t) - \varphi_0(t)) \right]^{1/\mu} \equiv \left[ \cos \mu \chi(t) \right]^{1/\mu} , \] (35)

with
\[ \chi(t) \equiv \varphi(t) - \varphi_0(t) . \] (36)

Here, both the azimuthal angle, \( \varphi \), and also the phase shift, \( \varphi_0 \), are regarded as functions of time. These angles have discrete jumps at the crossing times
\[ t_k \equiv (k - 1/2)\tau , \] (37)

where the \( t_k \) are the times at which the particle passes through the origin and \( \tau \) is the period of one orbit. Using our same physical convention, which here is \( r(0) = a \) at \( t = 0 \) with \( \varphi(0) = \varphi_0(0) = 0 \), we have that the jumps are
\[ \varphi(t_k + \epsilon) = \varphi(t_k - \epsilon) + \pi \] (38)

and
\[ \varphi_0(t) = \pi \left( 1 + \frac{1}{\mu} \right) \sum_{k=1}^{\infty} \Theta(t - t_k) , \] (39)

where \( \Theta(t) \) is the Heaviside step function.
We see that both $\varphi$ and $\varphi_0$ are monotonically increasing functions of time. In contrast, their difference, $\chi(t)$, is a periodic function of $t$. For every new orbit, $\chi$ starts with the value $-\pi/(2\mu)$ and then increases monotonically to $\pi/(2\mu)$, due to the continuous increase in $\varphi(t)$ as the particle goes through one orbit (petal or spiral). Next, $\chi$ decreases stepwise by $\pi/\mu$ at the crossing time. Therefore, as in our previous representation, $\cos \mu \chi$ starts and ends each orbit at $\rho = 0$.

### 4.3 Precession of the orbits

Because of angular-momentum conservation, the azimuthal angle, $\varphi$, is a monotonic function of time, $t$. If there is no precession, $\varphi$ increases exactly by $2\pi$ after one period, $\tau$. However, if after one period the axis of the orbit has rotated forwards or backwards (i.e. clockwise or counterclockwise) from $2\pi$ by an angle $P_{\nu}$, then this is the precession per period. Specifically,

$$\varphi(t + \tau) - \varphi(t) = 2\pi + P_{\nu},$$

where $\tau$ is the period of one orbit. If we choose $t$ to be the time at which the particle was at the apogee of the $k$th orbit, then $t + \tau$ will be the time at which the particle reaches the apogee of the $(k+1)$th orbit. To do this, the particle must first rotate by $\pi/2\mu$ to reach the origin, have its angle $\varphi$ jump by $\pi$ at the origin, and finally rotate by another $\pi/2\mu$ to reach the new apogee. Thus, from Eq. (40) we have

$$\varphi(t + \tau) - \varphi(t) = \frac{\pi}{2\mu} + \pi + \frac{\pi}{2\mu} = \left(\frac{1}{\mu} + 1\right)\pi = 2\pi + P_{\nu},$$

so that

$$P_{\nu} = \left(\frac{1}{\mu} - 1\right)\pi = \frac{4 - \nu}{(\nu - 2)}\pi.$$  \hspace{1cm} (42)

In Fig. 2 we plot $P_{\nu}$ as a function of $\nu$. The precession is infinite at $\nu = 2$, decreases through zero at $\nu = 4$, and asymptotes to $-\pi$ as $\nu$ goes to infinity. Therefore, the precession will be zero for $\nu = 4$, negative (counter-clockwise) for $\nu > 4$, and positive (clockwise) for $\nu < 4$.

This also demonstrates that, for $\nu$ equal to a rational fraction, the orbits will eventually close on themselves, and repeat. This happens when $kP_{\nu}$ is an integer times $2\pi$.

### 4.4 Classical period

The classical period, $\tau$, can be obtained by integrating the angular-velocity equation

$$\frac{d\varphi}{dt} = \frac{L}{mr^2} = \frac{1}{\tau_0 \rho^2},$$

where $L$ is the angular momentum, $m$ is the mass, $r$ is the radial coordinate, and $\rho$ is the polar coordinate. The precession $P_{\nu}$ is given by

$$P_{\nu} = \left(\frac{1}{\mu} - 1\right)\pi = \frac{4 - \nu}{(\nu - 2)}\pi.$$
where $\tau_0$ is a convenient unit of time,

$$\tau_0 = \frac{ma^2}{L}.$$  \hfill (44)

One then has that the classical period in units of $\tau_0$ is

$$T_\nu = \frac{\tau_\nu}{\tau_0} = \int_{-\pi}^{\pi} \rho^2(\varphi) \, d\varphi = \int_{-\pi/2}^{\pi/2} \left[ \cos(|\mu|\varphi) \right]^{(\frac{\pi}{2})} \, d\varphi.$$  \hfill (45)

(In Eq. (45) we have inserted absolute values around $\mu$ in the arguments of the integral and integrand. This allows us to include the negative $\mu$ case that we will return to at the end of this discussion.)

Changing variables to $x = |\mu|\varphi$ allows the integral to be evaluated as \hfill [20]

$$T_\nu = \frac{2}{|\mu|} \int_0^{\frac{\pi}{|\mu|}} \cos x \left[ \left( \frac{x}{|\mu|} \right) \right] \, dx = \frac{1}{|\mu|} B(1/2, b)$$

$$= \frac{\sqrt{\pi}}{|\mu|} \frac{\Gamma(b)}{\Gamma(b+1/2)}, \quad b > 0,$$  \hfill (47)

where

$$b \equiv \frac{1}{\mu} + \frac{1}{2} = \frac{\nu + 2}{2\nu - 4} > 0.$$  \hfill (48)

$B(b, c) = \Gamma(b)\Gamma(c)/\Gamma(b + c)$ is the beta function and we used $\Gamma(1/2) = \sqrt{\pi}$. We see that $T_\nu$ is finite and well-defined for $\mu > 0$ or $\nu > 2$, as would be expected for bound orbits.

In Fig. 3 we plot $T_\nu$ as a decreasing function of $\nu$. From Eq. (47) special cases are

$$\lim_{\epsilon \to 0} T_{2+\epsilon} = \left( \frac{2\pi}{\epsilon} \right)^{1/2},$$  \hfill (49)

$$T_3 = \frac{3\pi}{4},$$

$$T_4 = \frac{\pi}{2},$$

$$T_6 = 1,$$

$$\lim_{\nu \to \infty} T_\nu = \frac{2\pi}{\nu}.$$
The first equality in Eq. (49) is obtained by using the relation \( \Gamma(z + 1/2) \approx \Gamma(z) \sqrt{z} \), which holds for \( |z| \to \infty \).

In passing, note that Eqs. (47) and (48) also tell us that \( T_\nu \) is finite and well-defined for the unbound, infinite orbits corresponding to \( \mu < -2 \) or \( \nu < -2 \). This means that it takes a finite time for the particle to travel in from infinity, reach the turning point \( r = a \), and then go back to infinity.

4.5 Geometric approximation to the area of the orbit

Kepler’s 2nd Law states that equal areas are swept out in equal times. Although usually considered for the gravitational potential, it actually holds for any central potential. The proportionality constant follows from the angular-momentum conservation equation (21):

\[
dA = \frac{1}{2} r^2 d\varphi = \frac{1}{2} r^2 \dot{\varphi} dt = \frac{L}{2m} dt ,
\]

which is related to the angular-velocity equation (43) in an obvious way. Integrating this equation over a complete period, we get

\[
A_\nu = \frac{L}{2m} \tau_\nu ,
\]

where once again the subscript \( \nu \) labels the potential we are considering. We now introduce a dimensionless area, \( A_\nu \), as the area of an orbit in units of \( a^2 \):

\[
A_\nu \equiv \frac{A_\nu}{a^2} = \frac{L}{2ma^2} \tau_\nu = \frac{\tau_\nu}{2\tau_0} \equiv \frac{1}{2} T_\nu .
\]

Thus, the reduced area, \( A_\nu \), is equal to one-half the reduced period, \( T_\nu \).

For these \( \nu > 2 \) or \( \mu > 0 \) cases, we can give amusing, geometric, upper and lower bounds to the areas of the orbits. Recalling that the opening angle of a petal or spiral is

\[
\Phi_\nu = \frac{\pi}{\mu} = \frac{2\pi}{\nu - 2} ,
\]

an upper bound is defined by the area of section of a unit circle with opening angle \( \Phi_\nu \). This bounds the real area as

\[
A_\nu < \frac{1}{2} \Phi_\nu .
\]

12
Of course, this sectional area is equal to half the opening angle itself, measured in radians.

We can also obtain another bound, by considering the quantity

\[ \tilde{\rho}(\varphi) \equiv \cos \mu \varphi \]  

in comparison to the real orbital radius

\[ \rho = [\cos \mu \varphi]^{1/\mu} . \]  

By using \( \tilde{\rho}(\varphi) \) in place of \( \rho \) in the integral for \( A_\nu \),

\[ A_\nu = \frac{1}{2} \int_{\frac{\pi}{2\mu}}^{\frac{\pi}{2\mu}} \rho^2(\varphi) \, d\varphi = \frac{1}{2} \int_{\frac{\pi}{2\mu}}^{\frac{\pi}{2\mu}} [\cos \mu \varphi]^{2} \, d\varphi , \]  

one obtains the quantity

\[ \tilde{A}_\nu \equiv \frac{1}{2} \int_{\frac{\pi}{2\mu}}^{\frac{\pi}{2\mu}} \tilde{\rho}^2(\varphi) \, d\varphi = \frac{\Phi_\nu}{4} . \]  

\( \tilde{A}_\nu \) is an upper bound of the reduced area when \( 2 < \nu \leq 4 \) or \( 0 < \mu \leq 1 \), because then \( \tilde{\rho}(\varphi) \) will be outside the area of the orbit. However, for \( 4 \leq \nu \) or \( 1 \leq \mu \), \( \tilde{A}_\nu \) becomes a lower bound, because then \( \tilde{\rho}(\varphi) \) is inside the orbit.

From the above, we can make special case comparisons of these bounds, similar to the results of Eq.(49). Differing by a factor of 2 because of the reduced area definition, and written in the form “lower bound” \( \leq \) “exact reduced area” \( \leq \) “upper bound,” one has

\[ \lim_{\epsilon \to 0} A_{2+\epsilon} = \left(\frac{\pi}{2\epsilon}\right)^{1/2} < \frac{\pi}{2\epsilon} = 0 \neq \Phi_{2+\epsilon} , \]  

\[ A_3 = \frac{3\pi}{8} < \frac{\pi}{2} = \frac{1}{4} \Phi_3 , \]  

\[ \frac{1}{4} \Phi_4 = \frac{\pi}{4} = A_4 = \frac{\pi}{4} = \frac{1}{4} \Phi_4 < \frac{\pi}{2} = \frac{1}{2} \Phi_4 , \]  

\[ \frac{1}{4} \Phi_6 = \frac{\pi}{8} < A_6 = \frac{1}{2} < \frac{\pi}{4} = \frac{1}{2} \Phi_6 , \]  

\[ \frac{1}{4} \Phi_\nu = \frac{\pi}{2(\nu-2)} < \lim_{\nu \to \infty} A_\nu = \frac{\pi}{\nu} < \frac{\pi}{\nu-2} = \frac{1}{2} \Phi_\nu . \]  

The upper bound \( \frac{1}{2} \Phi_\nu \) is very good as \( \nu \to \infty \), because the area of the orbit is a very thin petal. As \( \nu \) becomes smaller, the approximation becomes worse. By the
time \( \nu = 4 \) or \( \mu = 1 \) this upper bound is off by a factor of 2. This is because the reduced area is that of a circle of radius 1/2, whereas this upper bound is equal to half the area of a unit circle. Both upper bounds become useless as \( \nu \to 2 \). This is because the upper bounds \( \Phi_\nu/4 \) and \( \Phi_\nu/2 \) are describing areas with angular widths approaching infinity and radius 1, whereas the true area has a radius that is spiraling in to zero as the angular position becomes large.

Finally, note that the lower bound goes from being off by a factor of 2 as \( \nu \to \infty \) to equality at \( \nu = 4 \).

5 Examples of Classical Bound Trajectories

5.1 Petals: \( \nu > 4 \) or \( \mu > 1 \)

From the previous section we see that, as \( \nu \) approaches infinity, the orbit is an ever-narrowing petal of width

\[
\Phi_\nu = \frac{\pi}{\mu} = \frac{2\pi}{\nu - 2}.
\]

Similarly, the counterclockwise precession reaches \( -\pi \) per orbit as \( \nu \to \infty \). As \( \nu \) becomes smaller, the petals increase in width. We illustrate this with several illuminating examples.

\( \nu = 8 \), three petals: We begin with the case \( \nu = 8 \). Here a petal is \( \pi/3 \) wide and the precession per orbit is \( -2\pi/3 \). Thus, there are three orbits before the trajectory closes. Note that the three petals in a closed trajectory cover only half of the opening angle from the origin. We show this in Figure 4.

\( \nu = 7 \), ten petals: For \( \nu = 7 \) a petal is \( 2\pi/5 \) wide and the precession per orbit is \( -3\pi/5 \). This means that, before the trajectory closes, there must be ten orbits and the precession goes around three times.

\( \nu = 6 \), perpendicular lemniscates: The case \( \nu = 6 \) is very interesting. The width of a petal is \( \pi/2 \) and the precession is \( -\pi/2 \) per orbit. Here, the width of a petal and the precession are exactly such that there is no overlap and also no "empty angles." It takes four orbits to close a trajectory. This is shown in Figure 5. We see that the physical solution consists of two perpendicular lemniscates (figure-eight curves composed of two opposite petals). This conclusion is in contradiction to one given in the literature [22]. There, a single lemniscate was predicted. From the physical arguments we have given above, this is not valid. This incorrect solution resulted from studying the equation for the square of the orbit.

\( \nu = 5 \), six overlapping petals: The petals are \( 2\pi/3 \) wide and they precess by \( -\pi/3 \) per orbit. This means there are six petals in a closed trajectory (flower). Note that here a petal is still so wide and the precession is so small, that the successive petals overlap.
5.2 Circle through the origin: $\nu = 4$ or $\mu = 1$

For $\nu = 4$, the solution is well known \[23\]. It is a circle that starts at the origin, travels symmetrically about the positive $x$-axis, and returns to the origin. The precession is zero, so the orbit continually repeats itself. In Fig. 6 we show the orbit.

5.3 Spirals: $2 < \nu < 4$ or $0 < \mu < 1$

As $\nu$ becomes less than 4, we can think of a petal obtaining a width greater than $\pi$, i.e., an orbit consists of two spirals, one out and one in, at opposite ends of the orbit. As $\nu$ approaches 2, the spirals become tighter and tighter and the precession (now clockwise) becomes larger. In fact, the spirals’ angular variation as well as the orbit’s precession both become infinite in magnitude as $\nu$ approaches 2.

Consider the special case $\nu = 3$. The width of the double-spiral orbit is still given by the formula for $\Phi_\nu$, and is $2\pi$. Therefore, the first orbit begins and ends towards the negative $x$-axis. The precession is $\pi$, so the trajectory closes after two orbits. We show this case in Figure 7.

When $\nu = 7/3$, the width of the double-spiral is $6\pi$. This means the first orbit starts towards the negative $x$-axis, winds around one and a half times before reaching the positive axis, and then winds one and a half times more to reach the origin again. The precession is $5\pi$, so the trajectory closes after two orbits. We show this case in Figure 8. However, on this scale it is impossible to see the tight winding of the spirals near the origin. Therefore, we demonstrate it with expanded views in Fig. 9.

As $\nu \to 2$ from above, the spirals get tighter and tighter. Their angular widths, $\Phi_\nu = 2\pi/|\nu - 2|$, become larger and larger as $\nu \to 2$.

6 Unbound Classical Trajectories, $\nu \leq 2$ or $\mu \leq 0$

6.1 The infinite spiral: $\nu = 2$ or $\mu = 0$

When $\nu$ reaches 2, there is a singular change. First, the double spiral becomes infinite in angular width. But also, the joining of the two sides of the double spiral at $\rho = 1$ and $\varphi = 0$ breaks down. It is as if a tightly-wound double spring broke. The ends spiral out to infinity, as we discuss below.

The $\nu = 2$ is a special case in other ways. For us, with our $E = 0$ condition, it is the demarcation between the bound orbits, $\nu > 2$, and the unbound orbits, $\nu < 2$. But also, for $\nu \equiv 2$ and $E \neq 0$, it is the demarcation between the $E < 0$ bound solutions and $E > 0$ unbound solutions. These latter are Cotes’ spirals \[24\]. The $E \neq 0$ solutions are given in Ref. \[25\].

Returning to $E = 0$, Eq. (7) shows that, for $\mu = 0$ or $\nu = 2$, there is no natural length scale. This is because both the centripetal and the external potentials have the same power dependence on $r$:

$$U(r) = \left(\frac{L^2}{2m} - \gamma\right)\frac{1}{r^2}.$$  

(61)
Thus, for all $r$, the effective potential $U(r)$ is either repulsive (and positive-valued) or else it is attractive (and negative-valued).

For $U(r) > 0$ there is no solution for $E = 0$. However, for $U < 0$ the energy-conservation condition (22) with $E = 0$ gives

$$
\left( \frac{dr}{d\varphi} \right)^2 = \left( 2m\gamma/L^2 - 1 \right) r^2 \equiv \lambda^2 r^2 ,
$$

so that

$$
dr/d\varphi = \mp \lambda r .
$$

Therefore, the two possible solutions are

$$
r = r_0 \exp \left[ \mp \lambda (\varphi - \varphi_0) \right] , \quad -\infty < \varphi < \infty .
$$

These correspond to orbits which pass through the point $(r_0, \varphi_0)$ and spiral inwards or outwards for the minus or plus signs, respectively. For example, a complete, counterclockwise orbit, starting from the initial value $r = r_{in} = r(\varphi_{in})$, going into the origin, and then going out of the origin towards the final value $r = r_f = r(\varphi_f)$, is given by

$$
\frac{r}{r_{in}} = \exp \left[ -\lambda (\varphi - \varphi_{in}) \right] , \quad \varphi_{in} \leq \varphi < \infty ,
$$

$$
\frac{r}{r_f} = \exp \left[ +\lambda (\varphi - \varphi_f) \right] , \quad -\infty < \varphi \leq \varphi_f .
$$

However, it is important to note that Eqs. (65) and (66) are only special cases of Eq. (64). For example, Eq. (64), with the plus sign, can also describe an $r$ starting at $r(\varphi_0) = r_0$ which then goes out to infinity as $\varphi \to \infty$.

Despite the infinite spirals in the example of Eqs. (65) and (66), the journey in and out takes only a finite time. To see this, consider the time dependence of $r$. From the energy-conservation conditions (3) and (62), we have

$$
\dot{r}^2 = \frac{2m}{m} \frac{\gamma}{m^2 r^2} - \frac{L^2}{m^2} = \frac{L^2}{m^2} \left( \frac{2m\gamma}{L^2} - 1 \right) \frac{1}{r^2} = \left( \frac{L\lambda}{m} \right)^2 \frac{1}{r^2} .
$$

Therefore,

$$
\frac{dr^2}{dt} = 2r\dot{r} = \mp \frac{2L\lambda}{m}
$$

and

$$
r^2 = r_0^2 + \frac{2L\lambda}{m}(t - t_0) .
$$
This tells us that, in spite of the infinite spiralling, a particle moving \textit{inwards} reaches the origin, \( r = 0 \), from any point \( r_0 \) in a finite time, given by

\[
\Delta t = t - t_0 = \frac{m}{2L\lambda} r_0^2. \tag{70}
\]

Finally, substituting Eq. (64) into Eq. (69) gives the general time dependence of \( \varphi \) as

\[
t - t_0 = \pm \frac{m}{2L\lambda} (r^2 - r_0^2) = \pm \frac{mr_0^2}{2L\lambda} \{ \exp[\mp 2\lambda(\varphi - \varphi_0)] - 1 \}. \tag{71}
\]

Because of our choice \( \hat{L} = \hat{z} \), in Eq. (71) we are assuming that \( \varphi \) increases monotonically with time.

\section*{6.2 General unbound orbits: \( \nu < 2 \) or \( \mu < 0 \)}

When the potential parameter \( \nu \) just leaves that of the infinite spiral, that is, when one barely has \( \nu < 2 \) or \( \mu < 0 \), there is another change. Although the two ends of the entire trajectory still reach to infinity and the spirals in and out almost have infinite angular widths, the distance of closest approach jumps from \( \rho = 0 \) to \( \rho = 1 \).

As the value of \( \nu \) decreases, the value of the angular width of the trajectory, now given by \( \Phi_{\nu} = \pi/|\mu| \), also decreases accordingly. By the time \( \nu = 1 \), the angular width has decreased to \( 2\pi \). Eventually it becomes less than \( \pi \), meaning the orbit comes in and out in the same half plane. That is, when \( \nu < 0 \), the force is repulsive. In the next section we give some illustrative examples.

\section*{7 Examples of Classical Unbound Trajectories}

\subsection*{7.1 Kepler-like potentials: \( 0 < \nu < 2 \) or \( -1 < \mu < 0 \)}

When \( 0 < \nu < 2 \), the repulsive centripetal barrier dominates at small \( r \) whereas the attractive potential \( V = -\gamma / r^\nu \) dominates at large \( r \). A typical shape of the effective potential is sketched in Fig. 10. It is familiar from the Kepler problem. Therefore, for \( 0 < \nu < 2 \), the \( E = 0 \) classical orbits are all unbounded. The distance, \( a \), defined in Eq. (12) now has a completely different interpretation. It is now the distance of closest approach. Even so, the formal solution (26) remains valid for negative values of \( \mu \).

As a first example consider the case \( \nu = 3/2 \) or \( \mu = -1/4 \). This orbit has a total angular width of \( 4\pi \). It is shown in the two drawings of Fig 11. The large-scale first drawing shows the trajectory coming in from the top, performing some gyration, and going out at the bottom. The small-scale second drawing shows the trajectory winding around twice near the origin, with the distance of closest approach being one.
A second example is the exact Kepler potential, $\nu = 1$ or $\mu = -1/2$. Eq. (26) gives
\[ \rho^{-1/2} = \cos \varphi / 2 , \] (72)
so that
\[ \frac{1}{\rho} = (\cos \varphi / 2)^2 = \frac{1 + \cos \varphi}{2} . \] (73)
This is the famous parabolic orbit for the Kepler problem with $E = 0$. This orbit is shown in the first drawing of Figure 12. The parabola yields an angular width of $2\pi$, as it should.

### 7.2 The straight line: $\nu = 0$ or $\mu = -1$

If we formally set $\nu = 0$ in the expression (4), we get a negative constant potential $V(r) = -\gamma$. Therefore, in this case the force vanishes and we have a free particle. Its orbit must be a straight line. However, Eq. (27) shows that one still has the same type of solution, Eq. (27). Here it is
\[ \rho = [\cos \varphi]^{-1} , \quad x = r \cos \varphi = a . \] (74)
This is the equation for a vertical straight line that crosses the $x$-axis at $x = a$, as required by the initial conditions. This orbit is shown in the second drawing Fig. 12, it subtending an angular width of $\pi$ from the origin.

### 7.3 Repulsive potentials: $-\infty < \nu < 0$ or $-\infty < \mu < -1$

For $\nu < 0$ the potentials $V(r)$ in Eq. (1) are repulsive and negative-valued for all $r > 0$, with $V(r)$ going to $-\infty$ at large distances. Since both the potential, $V(r)$, and the centripetal potential decrease monotonically, the effective potential has no minima or maxima. Even so, for $E = 0$ these unbounded orbits behave qualitatively like those for $0 \leq \nu < 2$. The distance of closest approach again obeys the formula (12) and the solutions are given by the same expression (26), which is valid for all $\mu \neq 0$.

The solution (27) for the first orbit ($\varphi_0^1 = 0$) gives
\[ \rho = [\cos \mu \varphi]^{1/\mu} = [\cos |\mu|\varphi]^{-1/|\mu|} . \] (75)
This shows that the particle is at $r = a$ when $\varphi = 0$ and it is at infinity when $\varphi = \pm \pi / (2|\mu|)$. Thus, these orbits become narrower as $|\mu|$ becomes larger. This is similar to the case of the width of the petals of the bound orbits.

**Hyperbolic orbits:** The most famous special case of these potentials is the “inverted” harmonic-oscillator potential, with $\nu = \mu = -2$. The orbit is given by $\rho = [\cos 2\varphi]^{-1/2}$, so that
\[ 1 = \frac{r^2}{a^2} \cos 2\varphi = \frac{r^2}{a^2} (\cos^2 \varphi - \sin^2 \varphi) = \frac{x^2}{a^2} - \frac{y^2}{a^2} . \] (76)
Thus, the trajectory is a special hyperbolic orbit, whose minor and major axes are equal, \( b^2 = a^2 \). (In fact, every \( \nu = -2 \) solution, i.e. for arbitrary \( E \neq 0 \), also yields a hyperbolic orbit, but with \( b \neq a \).) We show this orbit as the third drawing in Fig. 12. Now the angular width has decreased to \( \pi/2 \).

As the last case, we consider the orbit for \( \nu = -4 \) or \( \mu = -3 \). This orbit is shown in the last drawing of Fig. 12. The orbit subtends an angle of \( \pi/3 \), again as it should. One sees that as \( \nu \) becomes more and more negative, the orbits will become narrower and narrower. This is just as in the bound case, where the petals became narrower and narrower as \( \nu \) became more and more positive.

8 Comments on the Classical Problem

It is well-known that classical orbits precess for general central potentials. There are two exceptions, the Kepler and the isotropic harmonic oscillator potentials [4]. Further, these central potentials are known to have additional conserved quantities: the Runge-Lenz vector, \( \mathbf{A} \), in the Kepler problem and the quadrupole moment tensor, \( Q_{ij} \), in the harmonic oscillator problem [26, 27]. These two examples show that there is a close connection between the absence of precession and the existence of dynamically conserved quantities.

On the other hand, it is also well-known that for a specific energy to angular-momentum combination, general central potentials can have closed orbits. But these combinations have different unique values, depending upon the potential.

Therefore, it is interesting that the class of \( E = 0 \) solutions for the potentials (1) with \( \nu > 2 \), have a countable number of examples with well-defined, closed trajectories. They are all the potentials with rational values of \( \nu \). One of them, the case \( \nu = 4 \), has zero precession.

All the \( E = 0 \) with \( \nu > 2 \) solutions are bound and go through the origin. The \( \nu < 2 \) solutions are unbound, and go to infinity. (The \( \nu = 2 \) case is special, and does both.)

We have seen that, with the \( \nu > 2 \) singular potentials, the physical solutions differ from more familiar solutions in that, at the origin, we must paste together single-orbit solutions which have different phase-shifts, \( \varphi^k_0 \), at \( r = 0 \). The discontinuity at the origin of the phase shifts, given by Eq. (39), is fundamental [28]. It is necessary because the bound orbits pass through the origin, where the potential is singular. This causes the second order time-derivative of the position, i.e., the acceleration, to be infinite at \( r = 0 \) when \( \nu \geq 2 \):

\[
\ddot{x}_i(t) = -\frac{\nu \gamma}{m r^{\nu+2}} x_i, \quad i = 1, 2, 3.
\]

Therefore, irrespective of what coordinate system we choose, at the origin we must still paste together different solutions of Newton’s equation which depend on different initial conditions, as exhibited in Eq. (31).

As we have just observed, when crossing the center of singular potentials, a particle will have infinite momentum and kinetic energy. This leads to a violation of the virial
theorem, even for these bound trajectories. Formally applying the virial theorem \cite{29, 30} to power potentials leads to the following well-known equation:

\[ \langle T \rangle = -\frac{\nu}{2} \langle V \rangle, \tag{78} \]

where \( \langle T \rangle \) and \( \langle V \rangle \) are the time averages of the kinetic and potential energies. The above equation leads to two famous relations: \( \langle T \rangle = -\langle V \rangle/2 \), for Kepler elliptical orbits \( (E < 0 \text{ and } \nu = 1) \), and \( \langle T \rangle = \langle V \rangle \) for the harmonic oscillator \( (E > 0 \text{ and } \nu = -2) \).

From Eq. (78) and energy conservation, we have

\[ E = T + V = \langle T \rangle + \langle V \rangle = \left(1 - \frac{2}{\nu}\right) \langle T \rangle. \tag{79} \]

Since \( \langle T \rangle > 0 \), Eq. (78) immediately shows that the virial theorem (78) is violated by all the \( E = 0 \) solutions of power potentials with \( \nu \neq 2 \). Such a violation of (78) is expected for infinite orbits, such as the parabolic orbit of the Kepler problem and all other \( E = 0, \, \nu < 2 \) orbits. What is of more interest is that the bound solutions, \( \nu > 2 \), also violate the theorem. In Appendix C we will go into more detail on this point.

In summary, the classical \( E = 0 \) solutions for the power-law potentials exhibit a fascinating set of properties. In paper II we will show that this characteristic is also true for the quantum solutions.
Appendix A: Zero Angular-Momentum Solutions

Here we discuss the simpler $E = L = 0$ solutions, where the effective potential $U(r)$ is equal to the potential $V(r)$ itself. A particle moves radially, in a straight line, between zero and infinity, with no turning points at any finite radius, $r \neq 0$.

From the energy-conservation equation (6) one has that, for $E = L = 0$,

$$
\frac{dr}{dt} = \pm \left( \frac{2\gamma}{m} \right)^{1/2} \frac{1}{r^{\nu/2}} \equiv v(t) .
$$

(80)

Unless $\nu = -2$ this is equivalent to

$$
\frac{d(r)^{\nu/2+1}}{(\nu/2 + 1)dt} = \pm \left( \frac{2\gamma}{m} \right)^{1/2} ,
$$

(81)

which has the solutions

$$
r(t) = \left[ \frac{\nu+2}{r_0^{\nu/2}} \pm \left( \frac{2\gamma}{m} \right)^{1/2} \left( \frac{\nu}{2} + 1 \right) (t - t_0) \right]^{\frac{1}{\nu/2}} , \quad \nu \neq -2 .
$$

(82)

where $r_0 \equiv r(t_0)$.

The solution (82) does not depend explicitly on the initial velocity $v_0 \equiv v(t_0)$ because $r_0$ and $v_0$ are not independent. However, because of the condition $E = 0$, using Eq. (80) yields $\pm (2\gamma/m)^{1/2} = v_0 r_0^{\nu/2}$. Substituting this relation into Eq. (82) and expanding yields

$$
r(t) = r_0 + v_0(t - t_0) - \frac{\nu v_0^2}{4 r_0} (t - t_0)^2 + \frac{\nu(\nu + 1) v_0^3}{12 r_0^2} (t - t_0)^3 + \cdots .
$$

(83)

This expansion is absolutely convergent for $|v_0(t - t_0)| < r_0$. This power series terminates, if the index $\nu$ is equal to negative integers, $\nu = 0, -1, -2, \cdots$. In particular, for a constant force, $\nu = -1$, we get the familiar result, $r(t) = r_0 + v_0(t - t_0) + \frac{1}{2}a(t - t_0)^2$; where the constant acceleration is given by $a = \gamma/m = v_0^2/(2r_0)$.

Eq. (82) shows that for $\nu > -2$ it takes an infinite time to travel between $r_0 \neq 0$ and $r = \infty$, but only a finite amount of time to reach the origin $r = 0$. This finite arrival time is given by

$$
\Delta t \equiv t - t_0 = \frac{r_0}{|v_0|} .
$$

(84)
Note that $\Delta t$ is equal to the time required by a particle moving with an effective constant velocity,
\[ v_{\text{eff}} = |(\nu/2 + 1) v_0| , \] (85)
to reach the origin from $r = r_0$. In particular, for the free particle, $\nu = 0$, the above effective speed (83) is equal to the actual constant speed $v_0$, as it should.

However, for $\nu < -2$, the power of the large square brackets in Eq. (82) is negative. This yields the opposite situation: Now, it takes an infinite time to reach $r = 0$ from any $r_0 \neq 0$. But it takes a finite time, given by (84), to travel between any $r_0 \neq 0$ and infinity. The latter result is consistent with the result of Sec. 4.4, that this travel time is finite for $L \neq 0$.

Finally, the special case $\nu = -2$ yields the differential equation
\[ \frac{dr}{dt} = \pm \left( \frac{2\gamma}{m} \right)^{1/2} r . \] (86)
The solutions are therefore,
\[ r(t) = r_0 \exp \left[ \pm \left( \frac{2\gamma}{m} \right)^{1/2} (t - t_0) \right] = r_0 \exp \left[ \left( \frac{v_0}{r_0} \right) (t - t_0) \right] . \] (87)

The Taylor expansion agrees with Eq. (83) with $\nu = -2$. In fact, the exponential solution (87) follows from the general solution (82), by using \( \lim_{\epsilon \to \infty} (1 + \epsilon x)^{1/\epsilon} = \exp[x] \). Note that for $\nu = -2$ the travel time between $r_0 \neq 0$ and either $r = 0$ or $r = \infty$ becomes infinite. This is consistent with $\nu = -2$ being a transition between the regimes where the travel time is infinite to reach the origin ($\nu < -2$) or infinity ($\nu > -2$).

Therefore, for the $L = 0$ case, the complete journey from the origin to infinity takes an infinite amount of time for all indices $-\infty < \nu < \infty$.

**Appendix B: Similar Orbits and Scaling**

The power potentials have the property that for each solution, $r_1 = a\rho(\varphi)$, there are infinitely many similar solutions, $r_\lambda = \lambda a \rho(\varphi)$. The periods, $\tau$, the angular momenta, $L$, and the energies, $E$, of these similar orbits are related by powers of the scaling parameter, $\lambda$, as follows:
\[ \frac{\tau_\lambda}{\tau_1} = \lambda^{1+\nu/2} , \quad \frac{L_\lambda}{L_1} = \lambda^{1-\nu/2} , \quad \frac{E_\lambda}{E_1} = \lambda^{-\nu} , \] (88)

This scaling law can be demonstrated in various ways. For example, Landau and Lifshitz use the Lagrangian to show it [14]. We shall use Newton’s equation directly.
For the forces corresponding to the power potentials of Eq. (4), we have

\[ m \frac{d^2 \mathbf{r}}{dt^2} = - \frac{dV(r)}{dr} \mathbf{r} = - \frac{\nu \gamma}{r^{\nu+1}} \mathbf{r}. \]  \hspace{1cm} (89)

If we choose an arbitrary scale for the distance,

\[ \rho = \frac{r}{a}, \]  \hspace{1cm} (90)

scale the time as

\[ T = \frac{t}{t_1}, \quad t_1 = a^{1+\nu/2} \sqrt{m/\gamma}, \]  \hspace{1cm} (91)

and multiply Eq. (89) by \( \frac{t_2}{t_1} \frac{1}{ma} = \frac{a^{\nu+1}}{\gamma} \), we obtain, instead of Eq. (89), a dimensionless equation of motion,

\[ \frac{d^2 \rho}{dT^2} = \frac{t_1^2}{a} \frac{d^2 \mathbf{r}}{dt^2} = - \frac{a^{\nu+1}}{\gamma} \frac{\nu \gamma}{r^{\nu+1}} \mathbf{r} = - \frac{\nu}{\rho^{\nu+1}} \hat{\rho}. \]  \hspace{1cm} (92)

Therefore, the solutions of Eq. (89) must have the following form:

\[ r = r(t) = a \rho(\varphi(T)) = a \rho(\varphi(t/t_1)), \]  \hspace{1cm} (93)

where \( \rho(\varphi) \) describes the same functional dependence of \( \rho \) on \( \varphi \) for all similar orbits.

Thus, if we change the length scale by \( a_\lambda = \lambda a \) and the time scale according to Eq. (91),

\[ t_\lambda = a_\lambda^{1+\nu/2} \sqrt{m/\gamma} = \lambda^{1+\nu/2} a^{1+\nu/2} \sqrt{m/\gamma} = \lambda^{1+\nu/2} t_1, \]  \hspace{1cm} (94)

we obtain similar solutions, which differ in size and period. The other scaling ratios in Eq. (89) follow simply by dimensional analysis:

\[ \frac{L_\lambda}{L_1} = \left( \frac{a_\lambda}{a} \right)^2 \frac{t_1}{t_\lambda} = (\lambda)^2 \lambda^{-(1+\nu/2)} = \lambda^{1-\nu/2}, \quad \text{etc.} \]  \hspace{1cm} (95)

The scaling property (93) is a powerful result. For example, it immediately leads to Kepler’s third law for the ratio of the periods, \( \tau_\lambda/\tau_1 = \lambda^{3/2} \), and to the independence of the harmonic oscillator period on the amplitude of the motion. It also predicts the \( a \)-dependence of the unit of time, \( \tau_0 \), that we used in Sec. 4.4.

Appendix C: Violation of the Virial Theorem

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Clearly, the violation of the virial theorem by $E = 0$, $\nu > 2$ bound orbits is connected to the infinite values of the potential and kinetic energies at $r = 0$. But to better understand the phenomenon, let us return to the derivation of Eq. (78) and investigate the relevant assumptions.

One often starts by formally differentiating the scalar product $G = p \cdot r$ with respect to time \[29\]:

$$\dot{G} = \dot{p} \cdot r + p \cdot \dot{r} = F \cdot r + \frac{1}{m} p \cdot p .$$

(96)

Applying this result to power potentials, where

$$F = -\nabla V(r) = -\gamma \nu r^{-\nu-1} \hat{r} ,$$

(97)

one obtains

$$\dot{G} = \nu V(r(t)) + 2T(t) .$$

(98)

Integrating Eq. (98) with respect to time, we find

$$G(t_2) - G(t_1) = \int_{t_1}^{t_2} \dot{G} = (t_2 - t_1)[\nu \langle V \rangle + 2 \langle T \rangle] ,$$

(99)

where, in Eq. (99), $\langle T \rangle$ and $\langle V \rangle$ again denote time averages of $T$ and $V$, here over the time interval $(t_1, t_2)$.

Eq. (99) is valid, as long as the particle stays within one orbit (i.e. one petal) during the time interval. But Eq. (99) is violated when the particle passes through the origin. Then $G(t) = p \cdot r$ changes its value from $-\infty$ to $+\infty$ at the origin, $r = 0$, meaning the derivation of the virial theorem breaks down.

To demonstrate that $p \cdot r$ becomes infinite at $r = 0$, we can show that $p$ goes to infinity stronger than $r$ goes to zero. From angular-momentum conservation, we have

$$G(t) = p \cdot r = rp \cos \alpha = L \cos \alpha / |\sin \alpha| = L \cot \alpha ,$$

(100)

where $\alpha$ is the angle between the vectors $p$ and $r$. The angle $\alpha$ changes quickly from $\pi$ to 0 at the $r = 0$ crossing, since there $r$ reverses its direction from being antiparallel to $p$ to being parallel to $p$. Accordingly, $\cot \alpha$ passes from $-\infty$ to $+\infty$.

In a similar way we can extend these results to $E < 0$ orbits when $\nu > 2$. For clarity, we summarize our observations:

The $E = 0$ orbits of the power potentials $V = -\gamma / r^\nu$ with $\nu \neq 2$ must either go to infinity or pass through the origin. In addition, for $\nu > 2$, every bounded orbit with $E < 0$ goes through $r = 0$. 


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The discontinuity at the origin of the azimuthal angle, $\varphi$, by $\pi$, is only a technical problem associated with the nature of polar coordinates. It disappears in cartesian coordinates because both $x(t)$ and $y(t)$ are continuous functions of time.

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Figure Captions

Figure 1: For \( \nu = 4 \) a plot of the effective potential, \( U \), as a function of \( r \). (See Eq. (7).) \( U(r) = [1/\rho^2 - 1/\rho^4] \).

Figure 2: A plot of, \( P_\nu \), the precession per orbit, as a function of \( \nu \).

Figure 3. A plot of the dimensionless orbital period \( T_\nu = \tau_\nu / \tau_0 \) as a function of \( \nu \).

Figure 4. The first three orbits for \( \nu = 8 \). Each orbit is precessed \(-2\pi/3\) from the previous one, so that by the end of the 3rd orbit, the trajectory closes. In this and later figures, we show cartesian coordinates for orientation.

Figure 5: The first four orbits for \( \nu = 6 \). Each orbit is precessed \(-\pi/2\) from the previous one, so that by the end of the 4th orbit, the trajectory closes.

Figure 6. The orbit for \( \nu = 4 \). It is a circle, and repeats itself continually.

Figure 7: The first two orbits for \( \nu = 3 \). Each orbit is precessed \( \pi \) from the previous one, so that by the end of the 2nd orbit, the trajectory closes.

Figure 8: The first two orbits for \( \nu = 7/3 \). Each orbit is precessed \( 5\pi \) from the previous one, so that by the end of the 2nd orbit, the trajectory closes.

Figure 9: Close-up details of the beginning of the first orbit for \( \nu = 7/3 \). The orbit starts at \( \varphi = -3\pi \). The four graphs show the evolution for (i) \( -3\pi \leq \varphi \leq -2.91\pi \), (ii) \( -3\pi \leq \varphi \leq -5\pi/2 \), (iii) \( -3\pi \leq \varphi \leq -2\pi \), and (iv) \( -3\pi \leq \varphi \leq -\pi \).

Figure 10: The effective potential, \( U(\rho) = 1/\rho^2 - 1/\rho \), as a function of \( \rho \).

Figure 11: A large-scale view, and a small-scale view near the origin, of the trajectory for \( \nu = 3/2 \).

Figure 12: From left to right, the trajectories for the cases i) \( \nu = 1 \), ii) \( \nu = 0 \), iii) \( \nu = -2 \), and iv) \( \nu = -4 \). The curves are labeled by the numbers \( \nu \).
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