UNIQUELY LABELLED GEODESICS OF COXETER GROUPS

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Abstract. Studying geodesics in Cayley graphs of groups has been a very active area of research over the last decades. We introduce the notion of a uniquely labelled geodesic, abbreviated with u.l.g. These will be studied first in finite Coxeter groups of type $A_n$. Here we introduce a generating function, and hence are able to precisely describe how many u.l.g.’s we have of a certain length and with which label combination. These results generalize several results about unique geodesics in Coxeter groups. In the second part of the paper, we expand our investigation to infinite Coxeter groups described by simply laced trees. We show that any u.l.g. of finite branching index has finite length. We use the example of the group $\tilde{D}_6$ to show the existence of infinite u.l.g.’s in groups which do not have any infinite unique geodesics. We conclude by exhibiting a detailed description of the geometry of such u.l.g.’s and their relation to each other in the group $\tilde{D}_6$.

1. Introduction

Coxeter groups are historically very important and occur naturally as reflection groups (see e.g. [Hu90]). Over the decades they have sparked immense interest from various sides of mathematics and physics.

In particular, geodesics on their Cayley graphs, shortest connections between points which represent reduced words, have been of special interest [St84], [BH93], [He94], [Ed95], [St97], [EE10], [LP13], [HNW], [Ha17]. In [AC13] and [CK16] the authors introduce a formal power series with coefficients the number of geodesics for right-angled and even Coxeter groups based on trees. The paper [MT13] relates geodesics and quasi-geodesics for Coxeter groups. Related to the formal power series of geodesic growth is the growth series of a group introduced in [P90]. This series has been studied for different types of Coxeter groups in for example [CD91], [M03], [A03] and been generalized in [GN97].

In this paper we introduce the notion of a uniquely labelled geodesic. Instead of limiting our investigation to the existence of geodesics and their uniqueness, we reach out to geodesics which are unique with respect to their total label combination seen along the path going to a fixed point in the graph. For example, in abelian groups these consist only of powers of the generators, as any word with more than one letter can be written in any other order of the generators and would still yield the same element. On the other hand, any unique geodesic is also the unique geodesic with that label combination reaching to the element it represents.

2010 Mathematics Subject Classification. 20F55, 20F65, 05E15.
Key words and phrases. Coxeter group, unique geodesic, generating function, reduced word.
Geometrically speaking, it can be shown that a word in the generators of a group can only be a u.l.g. if it is a connected path on the Coxeter diagram which is a graph describing the group. Hence asking for a certain u.l.g. is the equivalent of the graph theoretical problem of finding a connected path in a graph with visiting each vertex a given number of times.

In the first part of the paper, we study the finite Coxeter groups of type $A$ with $n$ generators. We introduce a generating function, a power series in $n$ variables, where each monomial represents a certain label combination. We then give a precise formula for each coefficient depending on the monomial (Corollary 3.6). Based on this, we give exact formulas for the number of non-zero coefficients (Corollary 3.7) as well as the total number of u.l.g.’s in these groups (Theorem 3.8), both dependent on $n$.

In the second part of the paper, we expand our study to infinite groups. We introduce the notion of a branching index (Definition 5.1) which roughly speaking describes the turning behaviour of a connected path in the Coxeter-Dynkin diagram. We show that any u.l.g. with finite branching index has finite length (Theorem 5.9). We then study the affine group $\tilde{D}_6$ [BB, Appendix A1, Table 2]. In this group, we exhibit an infinite periodic u.l.g. (Theorem 6.6). We show that there are indeed two more different u.l.g.’s and relate these with each other geometrically (Theorem 6.8).

Acknowledgements. Both authors were partially supported by the NSERC Discovery Grant RGPIN-2015-04469.

2. Coxeter groups

Let $W$ be a Coxeter group of rank $n$ that is given by generators and relations

$$W = \langle s_1, \ldots, s_n \mid s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where $m_{ij} \geq 2$ are the Coxeter exponents. Consider its Cayley graph $C$ with respect to the chosen generators $s_1, \ldots, s_n$: vertices of $C$ correspond to elements $w \in W$ and two vertices $w$ and $w'$ are connected by an edge and labelled by $i$ iff

$$w' = ws_i \text{ and } l(w') = l(w) + 1,$$

where $l: W \to \mathbb{Z}_{\geq 0}$ is the length function on $W$. Then the shortest path $\rho(w)$ connecting 1 and $w \in W$ corresponds to a reduced expression for $w = s_{i_1} \ldots s_{i_l}$ and its length coincides with the length $l$ of $w$; such a path will be called a geodesic.

We will use a slightly different labeling of the Cayley graph: instead of an integer $i \in \{1, \ldots, n\}$ we put the standard vector $\{(0, \ldots, 1, \ldots, 0)\}$ where 1 is at the $i$th position. By a (total) label of a geodesic $\rho(w)$ denoted by $\tilde{\rho}(w)$ we call the sum of labels (considered as vectors in $\mathbb{Z}^n$) of all edges of $\rho$ or, equivalently, it is an $n$-tuple $(i_1, \ldots, i_n)$, where $i_k$ is the number of generators $s_k$ used to express $w$.

We say $\rho(w)$ is a uniquely labelled geodesic (u.l.g.) on the Cayley graph of $W$, if there is only one geodesic connecting 1 and $w$ with label $\tilde{\rho}(w)$. Observe that u.l.g.’s correspond to elements $w \in W$ that have a unique reduced expression for a given label.

We define the generating function

$$U_C(t_1, \ldots, t_n) = \sum_{(i_1, \ldots, i_n)} a_{i_1, \ldots, i_n} t_1^{i_1} \cdots t_n^{i_n},$$

where $a_{i_1, \ldots, i_n}$ is the coefficient of the monomial $(i_1, \ldots, i_n)$.
where \( a_{i_1, \ldots, i_n} \) is the number of u.l.g.'s with label \((i_1, \ldots, i_n)\), i.e.,
\[
a_{i_1, \ldots, i_n} = |\{w \in W \mid \rho(w) \text{ is a u.l.g. with } \vec{\rho}(w) = (i_1, \ldots, i_n)\}|.
\]

2.1. Example. The Cayley graph of the symmetric group
\[
S_4 = \langle s_1, s_2, s_3 \mid s_1^2 = 1, (s_1s_2)^3 = 1, (s_2s_3)^3 = 1, [s_1, s_3] = 1 \rangle
\]
has the form

![Cayley graph of S_4](image)

which gives a polynomial generating function
\[
U_C(t_1, t_2, t_3) = 1 + (t_1 + t_2 + t_3) + (2t_1t_2 + 2t_2t_3)
+ (t_1^2t_2 + t_1t_3 + t_2t_3 + t_2t_3^2)
+ 2t_1t_2^2t_3 + (t_1^2t_2^2t_3 + t_1t_2^2t_3 + 2t_1t_2t_3^2).
\]

2.2. Example. Suppose \( \Gamma \) is a finite graph where each two vertices are connected by at most one edge. Let \( W \) be a right-angled Coxeter group associated to \( \Gamma \), i.e. adjacent vertices \( \alpha \) and \( \beta \) of \( \Gamma \) correspond to generators \( s_{\alpha} \) and \( s_{\beta} \) in \( W \) with \((s_{\alpha}s_{\beta})^\infty = 1\) (there are no relations between \( s_{\alpha} \) and \( s_{\beta} \)), nonadjacent vertices correspond to commuting generators and \( s_{\alpha}^2 = 1 \) for all \( \alpha \).

Since there are no relations between adjacent generators, u.l.g.'s in \( W \) are in 1-1 correspondence to (connected) paths on \( \Gamma \). Hence, if we index vertices of \( \Gamma \) (generators of \( W \)) as \( \alpha_1, \ldots, \alpha_n \), then the coefficient \( a_{i_1, \ldots, i_n} \) of the generating function \( U_C \) counts

the number of connected paths in \( \Gamma \) that pass through the vertex \( \alpha_j \) exactly \( i_j \) times.

2.3. Example. Suppose \( W \) is an affine Weyl group \( \tilde{A}_2 \), i.e.
\[
W = \{s_1, s_2, s_3 \mid s_1^2 = 1, (s_is_j)^3 = 1, i \neq j\}.
\]
Its Coxeter-Dynkin diagram \( \Gamma \) is a triangle

![Coxeter-Dynkin diagram](image)
where edges correspond to the braid relations \((s_is_j)^3 = 1\) (there are no commuting generators). Then the infinite word \((s_1s_2s_3)\) has the property that any finite connected subword is a unique geodesic, in particular, it is a u.l.g. So the generating function \(U_C\) is a formal power series (not a polynomial) in \(\mathbb{Z}[[t_1, t_2, t_3]]\). This has been extensively studied in [LP13] and is closely related to the famous problem of constructing infinite reduced words.

In the present paper we construct infinite u.l.g.’s (hence, infinite reduce words) for some Coxeter groups whose Coxeter-Dynkin diagrams are simply laced tree.

3. Coxeter groups of type A

Consider the case of a finite Coxeter group \(W\) of type A of rank \(n\), i.e., \(m_{ij} = 2\) if \(|i - j| > 1\) and \(m_{ij} = 3\) if \(|i - j| = 1\) or, equivalently, the Coxeter-Dynkin diagram of \(W\) is a chain:

\[
\circ_1 \rightarrow \circ_2 \rightarrow \circ_3 \rightarrow \ldots \rightarrow \circ_{n-1} \rightarrow \circ_n
\]

We have the following observations that hold for any group of type A

3.1. Lemma. All non-zero monomials of \(U_C\) are of the form

\[
t_1^{i_1}t_2^{i_2} \cdots t_m^{i_m}, \quad 1 \leq l \leq m \leq n,
\]

where all the exponents \(i_l, i_{l+1}, \ldots, i_m\) are non-zero.

Proof. Assume that a geodesic (reduced word) \(\rho(w)\) contains no generator \(s_k\), where \(2 \leq k \leq n - 1\), but it contains generators from both subsets \(S = \{s_1, \ldots, s_{k-1}\}\) and \(T = \{s_{k+1}, \ldots, s_n\}\). Then without loss of generality, \(\rho(w)\) must contain a subword \(xy\) where \(x \in S, y \in T\), i.e., \(w = u \cdot xy \cdot v\). Since \(xy = yx\), \(w\) can also be written as \(w = u \cdot yx \cdot v\). Both ways of writing \(w\) result in two different geodesics with the same labels. Hence, \(\rho(w)\) can not be a u.l.g. \(\Box\)

From the proof it follows

3.2. Corollary. In a u.l.g. \(\rho(w)\) with \(l(w) \geq 2\), any two adjacent generators \(w = \ldots s_is_j\ldots\) must satisfy the condition \(|i - j| = 1\).

3.3. Lemma. Suppose \(\rho(w)\) is a u.l.g. Then it can not contain subwords of the form

\[
\begin{align*}
&s_1s_{m-1} \ldots s_{l+1}s_ls_{l+1} \ldots s_{m-1}s_m s_{m-1}, \\
&s_1s_{m-1} \ldots s_{l+1}s_ls_{l+1} \ldots s_{m-1}s_m, \\
&s_1s_{m-1} \ldots s_{m-1}s_m s_{m-1} \ldots s_{l+1}s_l s_{l+1}, \\
&s_1s_{m-1} \ldots s_{m-1}s_m s_{m-1} \ldots s_{l+1}s_l, \text{ where } l + 2 \leq m.
\end{align*}
\]

Proof. It is enough to prove it for the first word only (other words follow by symmetry). Suppose \(\rho(w) \in W\) contains such a subword. Then applying relations in the Coxeter group we obtain

\[
\begin{align*}
&s_m s_{m-1} \ldots s_{l+1}s_l s_{l+1} \ldots s_{m-1}s_m s_{m-1} = \\
&s_m s_{m-1} \ldots s_{l+1}s_l s_{l+1} \ldots s_{m-1}s_m = \\
&s_m s_{m-1} \ldots s_{l+1}s_l s_{l+1} \ldots s_m s_m = \\
&s_m s_{m-1} \ldots s_{l+1}s_l s_{l+1} \ldots s_m = \\
&s_m s_{m-1} \ldots s_{l+1}s_l s_{l+1} \ldots s_{m-1}s_m.
\end{align*}
\]
Since the first and the last subwords are different but have the same number of occurrences of each generator, i.e., the same label, \( \rho(w) \) can not be a u.l.g.

Any word \( \rho(w) \) that does not contain subwords of the lemma, must have one of the following forms (up to inversing the indices of generators \( s_k \mapsto s_{n+1-k} \)):

I. Suppose \( n \geq 1 \) and \( l \leq m \). It decreases from \( m \) to \( l \), that is \( \rho(w) = s_ms_{m-1} \ldots s_l \).

II. Suppose \( n \geq 2 \) and \( l < i, j \). First, it decreases from \( i \) till \( l \) and then it increases till \( j \).

III. Suppose \( n \geq 3 \) and \( l < i, j < m \). First, it decreases from \( i \) till the absolute minimum index \( l \), then it increases till the absolute maximum index \( m \) and, finally, decreases again till some index \( j \). This can be depicted as follows:

\[
\begin{array}{ccccc}
    & & s_i & \rightarrow & s_j \\
    s_i & \rightarrow & s_j & \rightarrow & s_l
\end{array}
\]

3.4. Corollary. For \( n \geq 3 \) the maximal length of a u.l.g. is \( 3n - 4 \).

Proof. The longest such word is of the form III (\( i = n-1, l = 1, m = n, j = 2 \))

\[ s_{n-1}s_{n-2} \ldots s_2s_1s_2 \ldots s_{n-1}s_{n-1} \ldots s_2, \]

its length is \( 3n - 4 \) and it is a u.l.g. with label \( (1, 3, 3, \ldots, 3, 3, 1) \).

The following theorem describes monomials of \( U_C \)

3.5. Theorem. A non-zero monomial of \( U_C \) has to be of the following type

I. \( t_lt_{l+1} \ldots t_m \) for \( l \leq m \),

II. \( t_l(t_{l+1} \ldots t_i)^2t_{i+1} \ldots t_m \) (and the inverse by \( t_k \mapsto t_{n+1-k} \)) for \( l < i \leq m \),

III. (a) \( t_l(t_{l+1} \ldots t_i)^2(t_{i+1} \ldots t_{j-1})(t_j \ldots t_{m-1})^2t_m \) for \( l < i < j < m \),

(b) \( t_l(t_{l+1} \ldots t_{j-1})^2(t_j \ldots t_i)^3(t_{i+1} \ldots t_{m-1})^2t_m \) for \( l < j \leq i < m \).

Observe that Type III(a) for \( |i - j| = 1 \) overlaps with Type II for \( m = i + 1 \).

Proof. All words of forms I, II and III are u.l.g.’s that have labels and, hence, monomials of the respective types I, II and III.

3.6. Corollary. We have the following formula for the generating function

\[
U_C(t_1, \ldots, t_n) = \sum_{i=(t_1, \ldots, t_n)} a_i t_1^{i_1} \ldots t_n^{i_n},
\]
where the coefficients \( a_i \) depend on the type of the label (monomial) \( i = (i_1, \ldots, i_n) \) as follows

\[
a_i = \begin{cases} 
1 & \text{Type I with } l = m \text{ or Type II with } m = i \\
2 & \text{Type I with } l < m \text{ or Type II with } m > i + 1 \text{ or Type III(b)}, \\
4 & \text{Type III(a) with } |i - j| > 1, \\
2(m - l) & \text{Type III(a) with } |i - j| = 1 \text{ or Type II with } m = i + 1.
\end{cases}
\]

**Proof.** We prove the last case only (previous cases follow similarly). Given a minimum index \( l \) and a maximum index \( m \) \((m \geq l + 3)\) as the initial index \( i \) of a generator of the word we can choose any \( i \in \{l + 1, \ldots, m - 2\} \) which gives \((m - 2) - (l + 1) + 1 = m - l - 2\) different options. Inversing the indices gives the same number of options. Hence, we obtain \(2(m - l - 2)\) options for Type III(a).

As for Type II, a minimum index \( l \) and a maximum index \( m = i + 1 \) \((m \geq l + 2)\) give two different words (up to an inverse), so we have exactly 4 options. Hence, \(a_i = 2(m - l - 2) + 4 = 2(m - l)\). \(\square\)

### 3.7. Corollary. There are exactly \( \frac{1}{12}n^4 - \frac{1}{6}n^3 + \frac{17}{12}n^2 - \frac{1}{3}n + 1 \) non-zero coefficients in \( U_C \) (incl. the constant term).

**Proof.** We sum the number of respective coefficients for each type of a u.l.g.

In type I there are following cases for each pair \((l, m)\)

1. Case \(|m - l| > 1\): the choices of \( m \) and \( l \) divide the list of \( n \) indices into three parts \(0 \ldots 01 \mid 1 \ldots 1 \mid 10 \ldots 0\) which gives \(\binom{n-1}{2}\) options.
2. Case \(|m - l| = 1\): there are \( n - 1 \) options to choose \((l, l + 1)\).
3. Case \(m = l\): we have exactly \( n \) options.

Hence, in total for type I, we obtain \(\binom{n-1}{2} + 2n - 1\) options.

In type II we have the following cases

1. Case \(m = i + 1\): this amounts to a partition into three parts \(0 \ldots 01 \mid 2 \ldots 2 \mid 10 \ldots 0\) which gives \(\binom{n-1}{2}\) options.
2. Case \(|m - i| > 1\): this amounts to a partition into four non-trivial parts \(0 \ldots 01 \mid 2 \ldots 2 \mid 1 \ldots 1 \mid 10 \ldots 0\), hence, giving us \(\binom{n-1}{3}\) options. Since a monomial is not symmetric, this number doubles to \(2 \cdot \binom{n-1}{3}\) by reversing the generators.
3. Case \(m = i\): if \( m < n \), we split the list of indices into three non-trivial parts \(0 \ldots 01 \mid 2 \ldots 2 \mid 0 \ldots 0\) which leads to \(\binom{n-1}{2}\) options; if \( m = n \), then we split it into two non-trivial parts and, hence, obtain \( n - 1 \) options. So, in total we get \(\binom{n-1}{2} + n - 1\) possibilities. Since a monomial is not symmetric, this number doubles to \((n - 1)(n - 2) + 2n - 2\) by reversing the generators.

Finally, in type III we have

1. Type III(a), case \(|j - i| > 1\): this is the same number as in Type III(b) with \(|i + 1 - (j - 1)| > 1\). Both need 5 partitions of \( n \)
   \[
   0 \ldots 01|2 \ldots 2|1 \ldots 1|2 \ldots 2|10 \ldots 0, \\
   0 \ldots 01|2 \ldots 2|3 \ldots 3|2 \ldots 2|10 \ldots 0,
   \]


hence, in both cases individually we have \( \binom{n-1}{5-1} \) options. This gives \( 2 \cdot \binom{n-1}{4} \) options in total.

(2) Type III(a), case \( |j - i| = 1 \): we have a partition into three parts \( 0 \ldots 01 \mid 2 \ldots 2 \mid 10 \ldots 0 \). However, we have already considered coefficients with these exponents in Type II, case 1. 

(3) Type III(b), \( j = l + 1 \): we have a partition into three parts \( 0 \ldots 01 \mid 3 \ldots 3 \mid 10 \ldots 0 \) giving again \( \binom{n-1}{2} \) options.

(4) Type III(b) with \( m = i + 1 \): we have a partition into four non-trivial parts \( 0 \ldots 01 \mid 2 \ldots 2 \mid 3 \ldots 3 \mid 10 \ldots 0 \) for which we obtain again \( \binom{n-1}{3} \) options.

Since a monomial is not symmetric, this number doubles to \( 2 \cdot \binom{n-1}{3} \) by reversing the generators. \( \square \)

A complete list of different monomials with non-zero coefficients is given for \( A_3, A_4, A_5 \) in the Appendix 1.1.

3.8. Theorem. In a Coxeter group of type \( A_n \) there are exactly

\[
U_C(1, 1, \ldots, 1) = \begin{cases} 
6 & \text{if } n = 2 \\
19 & \text{if } n = 3 \\
\frac{1}{5}n^4 - \frac{3}{2}n^3 + \frac{29}{3}n^2 - \frac{19}{2}n + 1 & \text{if } n \geq 4 
\end{cases}
\]

uniquely labelled geodesics.

Proof: We combine the Corollaries 3.6 and 3.7. The geodesics counted in the proof of 3.7 have the following coefficients and multiplicities:

| Case of the proof | multiplicity | number of different geodesics of this type |
|-------------------|-------------|------------------------------------------|
| Type I, 1         | 2           | \( \binom{n-1}{2} \)                      |
| Type I, 2         | 2           | \( n - 1 \)                               |
| Type I, 3         | 1           | \( n \)                                  |
| Type II, 1        | 4           | \( \binom{n-1}{2} \)                      |
| Type II, 2        | 2           | \( 2 \cdot \binom{n-1}{3} \)             |
| Type II, 3        | 1           | \((n - 1)(n - 2) + 2n - 2\)              |
| Type III, 1       | 4           | \( 2 \cdot \binom{n-1}{4} \)             |
| Type III, 2       | \( 2(m - l - 2) \) | \( 1 \leq l < m \leq n \)          |
| Type III, 3       | 2           | \( 2 \cdot \binom{n-1}{3} \)             |
| Type III, 4       | 2           | \( 2 \cdot \binom{n-1}{3} \)             |

The only non-obvious case is Type III, case 2, which only applies if \( n \geq 4 \). Here we obtain the sum:

\[
\sum_{l=1}^{n-2} \sum_{m=l+2}^{n} 2(m - l - 2).
\]
Since only the case \( k = m - l - 2 > 0 \) matters and each \( k \) occurs \( (n - k) \)-times, the sum transforms into

\[
(*) \quad \sum_{l=1}^{n-2} \sum_{k=1}^{n-l-2} 2k = \frac{1}{3}n^3 - 2n^2 + \frac{11}{3}n - 2.
\]

In view of the table above, it gives

\[
\frac{1}{3}n^4 - \frac{4}{3}n^3 + \frac{4}{9}n^2 - 5n
\]

for \( n < 4 \). If \( n \geq 4 \), then we add \((*)\) and the result becomes

\[
\frac{1}{3}n^4 - n^3 + \frac{8n^2}{3} - n + \frac{1}{3}.
\]

\[\square\]

Using the description of u.l.g.'s we can recover a result of Hart [Ha17].

3.9. **Corollary.** In a symmetric group \( S_{n+1} \) there are \( n^2 + 1 \) elements with a unique geodesic, i.e. a uniquely reduced expression.

**Proof.** A uniquely reduced expression corresponds to a word of type I. Hence, there are at most \( n \) words of length 1, \( n - 1 \) (decreasing) words of length 2 which are of the form \( s_is_{i-1} \). In general, there are \( n - l \) words of length \( l \), all of which must have the form \( s_js_{j-1} \ldots s_{j-l+1} \). In total this gives \( n(n+1)/2 \) words. Adding an inverse for each word of length \( \geq 2 \) gives \( n(n+1)/2 \cdot 2 - n + 1 = n^2 + 1 \) unique geodesics. \[\square\]

3.10. **Remark.** Observe that in type \( A \) the property of being a u.l.g. can be also interpreted using the language of rhombic tilings of Elnitsky [El97]. Following [El97] we say that two geodesics \( \rho_1(w) \) and \( \rho_2(w) \) are \( C_1 \)-equivalent if \( \rho_2(w) \) is obtained from \( \rho_1(w) \) by applying a finite number of commuting relations, i.e., by commuting subsequent generators \( s_is_j \) with \( |i - j| > 1 \) in the reduced expression for \( w \). A function \( \rho(w) \rightarrow \tilde{\rho}(w) \) (which assigns to a geodesic its label) factors through \( C_1 \)-equivalence, hence, if \( w \) has a u.l.g. \( \rho(w) \), then the \( C_1 \)-equivalence class of \( \rho(w) \) must contain only one element (the geodesic \( \rho(w) \) itself). The latter means that

A rhombic tiling of the \( 2(n-1) \)-polygon corresponding to the equivalence class of a u.l.g. \( \rho(w) \) must have a unique ordering.

We say that a tile touches a border strongly if it touches it with 2 sides and the border is on the left from the tile. Then a tiling has a unique ordering if it satisfies the following property:

Any border except the rightmost one has exactly one tile that touches it strongly (i.e. with two sides).

4. **Simply laced trees**

We will now investigate the case when the Coxeter-Dynkin diagram \( \Gamma \) describing the Coxeter group \( W \) is no longer a chain as in the type \( A \) case, but a finite graph where any two vertices are connected by at most one edge, i.e., \( W \) has the Coxeter exponents \( m_{ij} = 2 \) or 3 only. More precisely, a vertex \( \alpha \in \Pi \) corresponds to a generator \( s_\alpha \) of \( W \). If two vertices \( \alpha, \beta \) are adjacent (connected by an edge), then the generators satisfy the braid relation \( s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta \), otherwise the generators \( s_\alpha \) and \( s_\beta \) commute.
We index elements of \( \Pi \) from 1 to \( n = |\Pi| \), i.e., \( \Pi = \{\alpha_1, \ldots, \alpha_n\} \). Consider the Cayley graph \( C \) of \( W \) with respect to the generators \( s_{\alpha_1}, \ldots, s_{\alpha_n} \) and the generating function

\[
U_C(t_1, \ldots, t_n) = \sum_{i=(i_1, \ldots, i_n)} a_i t_1^{i_1} \cdots t_n^{i_n}
\]

which counts the number of u.l.g.’s in the Cayley graph of \( W \). Observe that by reindexing \( \Pi \) we reindex the variables \( t_i \) and the label coordinates of \( i \). Sometimes we will write the elements of \( \Pi \) as subscripts meaning the respective indices, i.e.,

\[
t_{\alpha} = t_{\alpha} = t_{\alpha_j}
\]

By definition, the subpath \([\nu_1, \nu_r]\) is a connected subset of vertices in \( \Gamma \). . .

4.2. Corollary. In a u.l.g. \( \rho(w) \) with \( l(w) \geq 2 \) any two adjacent generators \( w = \ldots s_\alpha s_\beta \ldots \) correspond to adjacent (connected by an edge) vertices \( \alpha, \beta \) in \( \Gamma \).

In other words, a u.l.g. is necessarily a path (with possible returns) on the Coxeter-Dynkin diagram \( \Gamma \).

4.3. Lemma. A u.l.g. can not contain a subword of the following form

\[
s_\alpha s_\beta \cdot v \cdot s_\beta s_\alpha s_\beta \text{ and } s_\beta s_\alpha s_\beta \cdot v \cdot s_\beta s_\alpha,
\]

where \( v \) does not contain generators adjacent to \( s_\alpha \).

By Corollary 4.2 we restrict to study paths on \( \Gamma \). A path in \( \Gamma \) is called a simple path or a path with no returns, if every vertex on it occurs exactly once. By a turning vertex \( \nu \) of a path \( \rho(w) \) we call a vertex corresponding to a generator \( s_\nu \) such that \( s_\mu s_\nu s_\mu \) is a subword of \( w \): we go from \( \mu \) to \( \nu \) and then back to \( \mu \). Let \((\nu_1, \ldots, \nu_r)\) be a list of subsequent (following the direction of the path) turning vertices of \( \rho(w) \) (\( r = 0 \) corresponds to the empty list). Let \( \nu_0 \) and \( \nu_{r+1} \) denote the starting and the ending vertex of the path \( \rho(w) \).

We now restrict to the case when \( \Gamma \) is a tree.

4.4. Lemma. Let \( \rho(w) \) be a path corresponding to a reduced word in \( W \). Let \((\nu_0, \ldots, \nu_{r+1})\) be the list of turning vertices (incl. the starting and the end point).

1. Then for all \( 0 \leq i \leq r \) the subpath \([\nu_i, \nu_{i+1}]\) passes through \( \nu_i \) and \( \nu_{i+1} \) exactly one time. In particular, \( \nu_i \neq \nu_{i+1} \) for all \( i \).

2. Moreover, if \( \rho(w) \) is a u.l.g., then for all \( 1 \leq i \leq r \) the subpath \([\nu_{i-2}, \nu_{i+2}]\) passes through \( \nu_i \) exactly one time (here to simplify the notation we set \( \nu_{-1} = \nu_0 \) and \( \nu_{r+2} = \nu_{r+1} \)).

Proof. By definition, the subpath \([\nu_i, \nu_{i+1}]\) passes through the intermediate turning vertex \( \nu_i \). Hence, it is enough to show that all other vertices in the subpath are
different from $\nu_i$. Since $\Gamma$ is a tree, all the subpaths $[\nu_i, \nu_{i+1}]$ are simple, so (1) follows.

Let $\rho(w)$ be a u.l.g. Suppose the subpath $[\nu_{i-2}, \nu_{i+2}]$ contains a second copy of $\nu_i$. Then by (1) it has to be either in $[\nu_{i-2}, \nu_{i-1})$ or in $(\nu_{i+1}, \nu_{i+2}]$. Suppose it is in $(\nu_{i+1}, \nu_{i+2}]$. Since $\Gamma$ is a tree, any path of length $\geq 2$ that starts and ends at $\nu_i$ and does not go through $\nu_i$ has to go through the same adjacent to $\nu_i$ vertex $\mu_i$. Hence, $\rho(w)$ contains a subword $s_{\mu_i}s_{\nu_i}s_{\mu_i}vs_{\mu_i}s_{\nu_i}$ of Lemma 4.3, a contradiction. □

4.5. **Corollary.** A uniquely labelled geodesic $\rho(w)$ has to be necessarily of the following form

Type I. A simple path, i.e. each generator in $\rho(w)$ occurs exactly once.

Type II. A path with a single turn, i.e. $r = 1$.

Type III. A path with $r \geq 2$ turns such that for all $1 \leq i \leq r$ the subpath $[\nu_{i-2}, \nu_{i+2}]$ passes through the turning vertex $\nu_i$ exactly one time.

Observe that if $\Gamma$ is a chain, then the types I, II and III become the respective types of the $A$-case, hence, they are also provide sufficient conditions for being a u.l.g. In general, there are paths of type III which are not u.l.g’s.

4.6. **Example.** Consider the Weyl group $E_6$ with the Coxeter-Dynkin diagram

```
α_2
α_1 -- α_3 -- α_4 -- α_5 -- α_6
```

Consider a reduced word $w = s_3s_1s_3s_4s_2s_4s_5s_4s_3s_1$. It corresponds to a path of type III with $r = 3$ and turning vertices $(\alpha_3, \alpha_1, \alpha_2, \alpha_5, \alpha_1)$, however, it contains a subword of Lemma 4.3, hence, it is not a u.l.g. So the condition that a reduced word $\rho(w)$ has type III is not sufficient for being a u.l.g.

4.7. **Example.** Consider the Weyl group $D_4$ that is

$$ W = \{s_0, s_1, s_2, s_3 \mid s_k^2 = 1, \ [s_i, s_j] = 1 \text{ and } (s_is_0)^3 = 1 \text{ for } i, j > 0\}. $$

Its Dynkin diagram $\Gamma$ is (here $\alpha_i$ correspond to $s_i$)

```
α_2
α_1 -- α_0 -- α_3
```

Consider the reduced word $\rho(w) = s_0s_1s_0s_2s_0s_3s_0s_1s_0$. It corresponds to the path of type III

$$ \rho(w) : \alpha_0 \rightarrow \alpha_1 \rightarrow \alpha_0 \rightarrow \alpha_2 \rightarrow \alpha_0 \rightarrow \alpha_3 \rightarrow \alpha_0 \rightarrow \alpha_1 \rightarrow \alpha_0 $$

with $r = 4$ and turning vertices $(\alpha_1, \alpha_2, \alpha_3, \alpha_1)$.

The reduced expression graph taken modulo $C_1$-equivalence classes (two representatives of $C_1$-equivalence classes are connected by a directed edge labelled by $i$ if the first class is obtained from the second by applying $s_0s_is_0 \rightarrow s_is_0s_i$) for $w$ is
Since there is only one reduced expression with 5 generators $s_0$, the reduced word $\rho(w)$ is a u.l.g. for the label $(5, 2, 1, 1)$. Observe that the word $w = (s_0 s_1 s_0 s_2 s_0 s_3)^2 s_0$ is not reduced. Indeed, we have

$$\begin{align*}
0102030102030 &\rightarrow (101)2(303)1(202)30 \\
&\rightarrow 1023(101)230230 \\
&\rightarrow 1023010203020 \\
&\rightarrow 1023012023202 \\
&\rightarrow 10230120302.
\end{align*}$$

5. Uniquely labelled geodesics with finite branching index

Let $\Gamma$ be a simply laced tree. By the valency of a vertex in $\Gamma$ we denote the number of vertices adjacent to it. A branching vertex is a vertex of valency at least 3. An end vertex is a vertex of valency 1. A branch of $\Gamma$ is a maximal connected subchain of $\Gamma$ where all vertices have valency less or equal than 2.

5.1. Definition. Let $\rho(w)$ be a u.l.g. and let $(\nu_1, \ldots, \nu_r)$ be the list of its turning vertices (without the starting and the end points) so that $\rho(w)$ contains subwords $s_{\mu_i} s_{\nu_i} s_{\mu_i}$, $1 \leq i \leq r$. If the adjacent vertex $\mu_i$ is a branching vertex, then the $\nu_i$ is called a short turning vertex, otherwise it is called a long turning vertex.

We define a branching index of $\rho(w)$ with respect to the tree $\Gamma$ as the number (repetitions are possible) of short turning vertices that is

$$i(\rho(w)) = |\{i \in \{1, \ldots, r\} | \mu_i \text{ is a branching vertex of } \Gamma\}|.$$

If $\rho(w)$ does not have turning vertices, we set $i(\rho(w)) = 0$.

As an immediate consequence of Corollary 4.5 we obtain

5.2. Lemma. Suppose a u.l.g. visits a branch on the tree $\Gamma$, i.e., goes in and out via the branching vertex $\gamma$ attached to the branch. Then it has exactly one turning vertex inside that branch. In other words, each visit of a branch corresponds to a turning vertex in that branch.

5.3. Lemma. Suppose a u.l.g. $\rho(w)$ visits the same branch more than once. Let $(\gamma_1, \ldots, \gamma_s)$, $s > 1$ be the corresponding list of turning vertices on that branch (observe that it is a sublist of the list of all turning vertices of $\rho(w)$). Let $d_i$ denote the distance between $\gamma_i$ and the branching vertex $\gamma$ (observe that a short turning vertex has distance 1 and a long one has distance $> 1$).
Then any subpath \([\gamma_i, \gamma_j]\), \(i \neq j\) must contain a turning vertex \(\gamma_k\) of distance 
\[d_k = \max(1, \min(d_i, d_j) - 1).
\]

**Proof.** Assume this is not the case. Then there are two long turning vertices \(a\) and \(b\) of distances \(d_a > 1\) and \(d_b > 1\) such that all turning vertices between \(a\) and \(b\) are of distance \(< (\min(d_a, d_b) - 1)\). Hence, the subpath \([a, b]\) is a subword of a word of Lemma 4.3 a contradiction. \(\square\)

5.4. **Corollary.** If \(\gamma_i\) is a long turning vertex, then either \(\gamma_{i-1}\) or \(\gamma_{i+1}\) has to be a short turning vertex.

Consider now a u.l.g. \(\rho(w)\) of maximal length with trivial branching index, i.e., 
\(\nu(\rho(w)) = 0\).

5.5. **Lemma.** The u.l.g. \(\rho(w)\) contains all vertices of the tree.

**Proof.** Assume \(\rho(w)\) does not contain a vertex \(v\) of the tree but it does contain a vertex \(x\) adjacent to \(v\). Because of the tree structure, removing \(v\) will divide the tree into two disconnected sets \(A\) and \(B\) and now \(\rho(w)\) can only use vertices in either of the two sets. Since \(\rho(w)\) goes through \(x\) at least once it has the form \(w_1xw_2\) and we can extend the path by replacing \(w_1xw_2\) with \(\ldots\). Hence \(\rho(w)\) is not of maximal length, a contradiction. \(\square\)

5.6. **Corollary.** All turning vertices of \(\rho(w)\) have valency 1. In particular, every end vertex can only be visited once.

5.7. **Corollary.** Let \((\nu_1, \ldots, \nu_r)\) be turning vertices of \(\rho(w)\) and let \(d(\nu_i, \nu_{i+1})\) denote the distance between \(\nu_i\) and \(\nu_{i+1}\). Then the sum \(S = \sum_{i=1}^{r-1} d(\nu_i, \nu_{i+1})\) is maximal.

5.8. **Proposition.** Suppose the Coxeter-Dynkin diagram of \(W\) is a simply-laced tree with \(n\) vertices. The maximal length of a u.l.g. with \(\nu(\rho(w)) = 0\) is bounded above by 
\[\frac{3}{2}n^2 + \frac{5}{2}n - 7.
\]

**Proof.** The maximal distance between end vertices is \(n - 1\) (which is achieved in the case of a chain). Let \(V_i\) denote the set of end vertices of the tree. Since \(|V_i| \leq n\) we obtain \(\frac{1}{2}(n - 1)n\) as a bound for the maximal value of \(S\). However we need to consider the beginning and the end of a word which can lie on a maximal subchain. For this we use Corollary 3.4 and add \(3(n - 1) - 4\) and, hence, obtain the desired bound. \(\square\)

5.9. **Theorem.** Let \(\Gamma\) be an arbitrary simply-laced tree with \(n\) vertices. Then a u.l.g. with finite branching index \(\nu(\rho(w)) = B\) has length at most \(n^2(B + 1) + n\cdot B\).

In other words, any u.l.g. in \(\Gamma\) with finite branching index has finite length.

**Proof.** Assume the branching index of \(\rho(w)\) is \(\nu(\rho(w)) = B\). We argue using the total number of turning vertices \(V\). If a vertex \(v\) appears in the list \(V\) twice, then its occurrences have to be separated by a short turning vertex on the same branch. Consecutive long turning vertices have to be on different branches. Hence, a long turning vertex \(x\) occurs at most \(B + 1\) times in the list \(V\). So the list \(V\) has length at most \(B + (\text{the number of short turning vertices})(B + 1) \leq B + n \cdot (B + 1)\).

Between each pair of turning vertices we visit at most \(n\) vertices. Hence, in total we obtain the bound \(n \cdot (B + n \cdot (B + 1))\). \(\square\)
Observe that one such example is produced in Example 2.3 which is not a tree that any finite connected subword is a u.l.g. We do it for a simply laced tree.

Proof. Lemma. 6.1.

The word \( w \) is infinite and irreducible, then the word \( \prod_{i \in I} s_i \prod_{j \in J} s_j \) is reduced for all \( n \).

We define \( I = \{ a, 3, 4 \} \) and \( J = \{ b, 1, 2 \} \) and show that there exists an infinite word which is 'close' to the one that is reduced by Lemma 6.2.

We will then show that for \( w = a_1a_2b3a_2b4b \) the word \( w^n \) is a u.l.g.

6.3. Lemma. The word \( w^2 \) can be transformed into

\[ 1a3 \cdot 1b2 \cdot 3a4 \cdot 2b1 \cdot 4a3 \cdot 1b2 \cdot 3a4 \cdot 2b4, \]

where the subword \( 1b2 \cdot 3a4 \cdot 2b1 \cdot 4a3 \cdot 1b2 \cdot 3a4 \) is reduced.

These word reductions are depicted in Figure 1 and show a part of the Cayley graph of \( D_6 \).

Proof. The word \( w^2 = 1a1 \cdot 3b3 \cdot 2a2 \cdot 4b4 \cdot 1a1 \cdot 3b3 \cdot 2a2 \cdot 4b4 \) can be written as

\[ 1a \cdot 3b1 \cdot 23 \cdot a \cdot 42 \cdot b \cdot 14 \cdot a \cdot 31 \cdot b \cdot 23 \cdot a4 \cdot 2 \cdot b4 \rightarrow 1a3 \cdot 1b2 \cdot 3a4 \cdot 2b1 \cdot 4a3 \cdot 1b2 \cdot 3a4 \cdot 2b4. \]

The subword \( 1b2 \cdot 3a4 \cdot 2b1 \cdot 4a3 \cdot 1b2 \cdot 3a4 \) is now reduced by Corollary 6.2
Proof. Note that letters. Now assume \( n \) of the path in the Cayley graph labelled by \( w \) end of the path in the Cayley graph labelled by \( w \) shorter connection between \( w \) with 3 letters each, cancelling at most 12 letters. The word \( \frac{1}{2} \) has 12 letters. The part of \( \frac{1}{2} \) reduces to
\[
1a3 \cdot 1b2 \cdot 3a4 \cdot 2b1 \cdot 4a3 \cdot 1b2 \cdot 3a4 \cdot 2b4 \cdot a1ab3a2ab4b \rightarrow \\
1a3 \cdot 1b23a42b14a31b23a4 \cdot 2b4 \cdot a13b32a24b4 \rightarrow \\
1a3 \cdot 1b23a42b14a31b23a4 \cdot 2b1 \cdot 4a3 \cdot 1b2 \cdot 3a4 \cdot 2b4 \rightarrow 1a3 \cdot (1b23a4)^2b4.
\]
In general, we see that \( w^n \) reduces to
\[
1a3 \cdot 1b2 \cdot 3a4 \cdot 1b2 \cdot 3a4 \cdot 1b2 \cdot 3a4 \cdot 2b4 \rightarrow 1a3(1b23a4)^{2n-1}b4.
\]
The estimate of the length comes from having \( 6(2n - 1) + 6 \) letters. The part of length \( 6(2n - 1) \) is reduced and we allow for cancellation of the prefix and suffix, with 3 letters each, cancelling at most 12 letters.

6.4. Corollary. The word \( w^n \) has form
\[
w^n = 1a3(1b23a4)^{2n-1}b4
\]
and, hence, it has length at least \( 12n - 12 \).

Proof. We assume it is not, then there is a power \( p \) of \( w \) such that \( w^p \) is not reduced. The word \( w^p \) has 12p letters. Denote by \( w^p_- \) and \( w^p_+ \) the beginning and end of the path in the Cayley graph labelled by \( w^p \). We assume first, there is a shorter connection between \( w^p_- \) and \( w^p_+ \) in the Cayley graph with at most 12p - 1 letters. Now assume \( n \geq 13 \cdot p \) and denote by \( w^n_- \) and \( w^n_+ \) the beginning and end of the path in the Cayley graph labelled by \( w^n \). This is depicted in Figure 2.

There are \( n/p \) copies of \( w^p \) connecting \( w^n_- \) with \( w^n_+ \). Denote by \( s_- \) and \( s_+ \) the two vertices connecting the reduced middle part \( (1b23a4)^{2n-1} \) of length 12n - 6 of \( w^n \). Because each part \( w^p \) of \( w^n \) is connected by a path of length 12p - 1, we get a connection from \( s_- \) to \( s_+ \) of length at most
\[
\frac{n}{p} \cdot (12p - 1) + 6 = 12n - \frac{n}{p} + 6 \leq 12n - 13 + 6 = 12n - 7.
\]
However, by Corollary 6.2 the subword \( (1b23a4)^{2n-1} \) between \( s_- \) and \( s_+ \) has length 12n - 6.

6.5. Proposition. The word \( w^n = (a1ab3ba2ab4b)^n \) is reduced for all \( n \geq 2 \).

Proof. We assume it is not, then there is a power \( p \) of \( w \) such that \( w^p \) is not reduced. The word \( w^p \) has 12p letters. Denote by \( w^p_- \) and \( w^p_+ \) the beginning and end of the path in the Cayley graph labelled by \( w^p \). We assume first, there is a shorter connection between \( w^p_- \) and \( w^p_+ \) in the Cayley graph with at most 12p - 1 letters. Now assume \( n \geq 13 \cdot p \) and denote by \( w^n_- \) and \( w^n_+ \) the beginning and end of the path in the Cayley graph labelled by \( w^n \). This is depicted in Figure 2.

There are \( n/p \) copies of \( w^p \) connecting \( w^n_- \) with \( w^n_+ \). Denote by \( s_- \) and \( s_+ \) the two vertices connecting the reduced middle part \( (1b23a4)^{2n-1} \) of length 12n - 6 of \( w^n \). Because each part \( w^p \) of \( w^n \) is connected by a path of length 12p - 1, we get a connection from \( s_- \) to \( s_+ \) of length at most
\[
\frac{n}{p} \cdot (12p - 1) + 6 = 12n - \frac{n}{p} + 6 \leq 12n - 13 + 6 = 12n - 7.
\]
However, by Corollary 6.2 the subword \( (1b23a4)^{2n-1} \) between \( s_- \) and \( s_+ \) has length 12n - 6.

6.6. Theorem. The word \( w^n = (a1ab3ba2ab4b)^n \) is a u.l.g.

Proof. We observe that asymptotically there are each one third \( a \)'s, one third \( b \)'s and \( \frac{1}{12} \)th every number 1, 2, 3 and 4. We first show that the only way to have a u.l.g. with this distribution is a periodic word. Assume the infinite word is not periodic. Then the maximal distance between two occurrences of one of the letters 1, 2, 3, 4 must be bigger than 12, because it is 12 on average. Assume without loss of generality that it is the letter 1 which has two occurrences in the infinite word of distance more than 12. Because of the symmetries of the Coxeter tree defining

\[\text{Figure 2. The word } w^n, \text{ each block represents } w^p \text{ as in Figure 1 where the red line represents the reduced part.}\]
\[ \tilde{D}_6 \] the same arguments work for 2, 3 and 4. This can only happen in one of the following cases. (Recall that a u.l.g. has to be a connected path in the graph.)

We go through all other possibilities which do not yield the word \((a1\bar{b}3\bar{b}a2\bar{a}b4b)\). Most of them either contain either \(xyxy\), \(xyx\) \ldots \(xy\) \ldots \(yx\) with no \(x\) in the dots. The first two are not reduced words, the latter not a u.l.g. A complete list of cases can be found in the Appendix 7.2 We discuss only the cases which are not immediate:

\[
\begin{align*}
(5) & \quad a1a2ab4b \\
(17) & \quad a1ab4ba2ab3ba1 \\
(23) & \quad a1ab4b3ba2ab4ba1a \\
(27) & \quad a1ab4b3baab4b3b \\
(28) & \quad a1ab4ba2ab3b4ab3b \\
(31) & \quad a1ab4ba2ab3b4ba1a \\
(33) & \quad a1ab4ba2ab3b4ab3b
\end{align*}
\]

\(5\) Isomorphic to \(6\) and follows because of the isomorphism \(3 \leftrightarrow 4\).

\(17\) \quad a1ab4ba2ab3ba1 in this case the letter 1 occurs with distance 12.

\(23\) \quad a1ab4b3ba2ab4ba1a this is the complex case: We easily verify that neither \(b4b, b3b\) nor \(ab\) can be a prefix. Hence the only case to check is \(a2a1a\ldots\):

This can only occur after \(aba2a1a\ldots\), which can only occur after either

\[ a1 \cdot aba2a1a \ldots \text{ or } a2 \cdot aba2a1a \ldots. \]

The word \(a2aba2a1a = 2a2 \cdot ba2a1a = 2ab \cdot 2a2a \cdot 1a\) is not reduced. Hence we need to check \(a1aba2a \cdot a1a \ldots\)

\[
\begin{align*}
& a1aba2a1ab4b3ba2ab4ba1a = a1ab \cdot 2a2 \cdot 1ab4b3ba2ab4ba1a \\
& a1ab \cdot 2a1 \cdot 2a \cdot b4b3ba2ab4ba1a = a1ab \cdot 2a1 \cdot a2a2 \cdot b4b3ba2ab4ba1a \\
& a1ab2a1a2ab4b3b \cdot 2a2a \cdot b4ba1a = 1a1 \cdot b2a1a2ab4b3b2ab4ba1a \\
& 1a \cdot b2 \cdot 1a1a \cdot 2ab4b3ba2ab4ba1a = 1ab2 \cdot a1 \cdot 2ab4b3ba2ab4ba1a.
\end{align*}
\]

The first word has length 23 the last word has length 21, hence it is not reduced and cannot occur as part of a u.l.g.

\[
\begin{align*}
(27) & \quad a1ab4b3bab4b3b = a1ab4b3 \cdot aba \cdot 4b3b = a1a \cdot 4b4 \cdot 3aba4b3b \\
& a1ab4b3a \cdot 4b4 \cdot ab3b = a1a4b3a \cdot b4b \cdot ab3b \\
& a14aba3 \cdot b4b \cdot ab3b = a14 \cdot bab \cdot 3abab3b.
\end{align*}
\]

Both the first and last term have 3 times \(a\), six times \(b\), once 1 and twice each 3 and 4. Hence it cannot be part of a u.l.g.

\(28\) This word is not reduced:

\[
\begin{align*}
& a1ab4ba2ab3b4ab3b = a1ab4ba2a \cdot 3b3 \cdot 4ab3b \\
& a1ab4bx2a \cdot 3b \cdot 4 \cdot 3b \cdot ab3b = a1ab4ba2a \cdot b3b3 \cdot 4 \cdot 3b \cdot ab3b \\
& a1ab4ba2a \cdot 3b4 \cdot b3b3 \cdot ab3b = a1ab4ba2a \cdot 3b4 \cdot b3b \cdot ab3b \\
& a1ab4ba2a \cdot 3b4b3b \cdot a \cdot b3 = a1ab4ba2a3 \cdot 4b4 \cdot 3bab3b \\
& = a1a \cdot b4b4 \cdot a2a3b43bab3.
\end{align*}
\]
This is the inverse of \( \text{[24]} \). Hence it cannot be succeeded by anything valid.

This word is not reduced:
\[
\begin{align*}
a1ab4ba2ab3b4ba1ab3b &= a1ab4ba2a \cdot 3b3 \cdot 4b4ba1ab3b \\ a1ab4ba2a3b &= a1ab4ba2a3b4b3b \cdot a1a \cdot 3b3b \\ a1ab4ba2a3b &= a1ab4ba2a3b4b3b \cdot a1a \cdot 3b3b \\
\end{align*}
\]
\[
\begin{align*}
a1ab4ba2a3b &= 4b4 \cdot 3ba1a \cdot b3 = a1ab4b4 \cdot a2a3 \cdot b4 \cdot 3ba1ab3 \\
&= a1a \cdot 4b \cdot a2a3b43baab3b.
\end{align*}
\]

This word is not reduced:
\[
\begin{align*}
a1ab4ba2a3b &= a1ab4ba2a3b4b3b \cdot a1a \cdot 3b3b \\
a1ab4ba2a3b &= a1ab4ba2a3b4b3b \cdot a1a \cdot 3b3b \\
a1a \cdot b4b4 \cdot a2a3ab4ab3ba2ab3b &= a1a \cdot 4b \cdot a2a3ab4ab3ba2ab3b.
\end{align*}
\]

We study the end of the sequence \( a2ab3b4ba1ab2ab1a2a \) and show it is not reduced:
\[
\begin{align*}
a2ab3b4ba1ab2ab1a2a &= a2ab3b4ba1 \cdot 2a2 \cdot ba1ab2a \\
a2ab3b4ba1ab2ab1a2a &= a2ab3b4ba2a \cdot 21 \cdot a2ba1ab2a = a2ab3b4ba2a \cdot 2a2a \cdot 1a2b \cdot 1a1 \cdot 2a \\
&= a2a \cdot 2 \cdot b3b4ba2 \cdot 1a1 \cdot 2b \cdot 1a1 \cdot 2a = a2 \cdot b3b4ba2 \cdot 1a \cdot 2b \cdot 1a \cdot 2a.
\end{align*}
\]

This word is not reduced:
\[
\begin{align*}
a1ab4b3ba2ab4b3ba1a2ab4b &= a1ab4b3ba2ab4b \cdot 3b3 \cdot a1a2ab4b \\
a1ab4b3ba2a \cdot 3b3b \cdot 4b3ba1a2ab4b &= a1ab4b3ba2a \cdot 3b \cdot a2ab3 \cdot 4b4 \cdot 3a1a2ab4b \\
a1ab4b3ba2ab4b &= 3a1a2a \cdot 4b4b.
\end{align*}
\]

This word is not reduced:
\[
\begin{align*}
a1ab4b3ba2ab4b3ba1a2ab4b &= a1ab4b3ba2ab4b3ba1 \cdot 2a \cdot ba1a \\
a1ab4b3ba2ab4b3ba1a2ab4b &= a1ab4b3ba2ab4b3ba1 \cdot 2a \cdot ba1a \\
&= a1ab4b3ba2ab4b3ba1a2ab4b.
\end{align*}
\]

The same as in \( \text{[51]} \) since the extra \( a1a2 \) in \( \text{[51]} \) has no effect on the technique.

\( a1ab4b3ba2ab4b3ba1a2ab4b \) is not reduced by the same argument as in \( \text{[51]} \) as well.

It is left to show that the above word does not transform into the one under the isomorphisms \( 3 \leftrightarrow 4 \) or \( 1 \leftrightarrow 2 \).

Case 1: We look at \( 3 \leftrightarrow 4 \) first:
\[
\begin{align*}
w^2 &= a1ab4b3ba2ab4ba1ab4b3ba2ab4b = 1a1 \cdot 3b3 \cdot 2a2 \cdot 4b4 \cdot 1a1 \cdot 3b3 \cdot 2a2 \cdot 4b4 \\
&= 1a31b42a21b34a3123a42b4 = 1a32b41a31b42a21b34a3123a42b4 \\
&= 1a32b41a31b42a21b34a3123a42b4 = 1a32b41a31b42a21b34a3123a42b4.
\end{align*}
\]

Even though the last transformation of \( w^2 \) contains the word \( w_2 = a1ab4ba2ab3b \), the letter 1 occurs more often than in \( w^2 \). Further, the last line is not a u.l.g. which implies that \( w^n \) is a u.l.g.
6.7. Corollary. The words \( w_2^n = (a1ab4ba2ab3b)^n \) and \( w_3^n = (a2ab3ba1ab4b)^n \) are u.l.g.'s for all \( n \geq 1 \).

We finish with an observation regarding the two bi-infinite geodesic rays of \( w^n \) and \( w_2^n \) in the Cayley graph of \( W \). We say two geodesics \( \gamma_1 \) and \( \gamma_2 \) are fellow travelling if there exists a constant \( D \) such that every point on \( \gamma_2 \) is at distance at most \( D \) from a point on \( \gamma_1 \).

6.8. Theorem. The bi-infinite u.l.g.'s with labels

1. \( w^n = (a1ab3ba2ab4b)^n \)
2. \( w_2^n = (a1ab4ba2ab3b)^n \)
3. and \( w_3^n = (a2ab3ba1ab4b)^n \)

are fellow travelling at distance at most 5 but at least 2 from each other on their entire lengths.

Proof. We can reduce \( w^2 = a1ab3ba2ab4b \cdot a1ab3ba2ab4b \) as follows:

\[
w^2 = a1a \cdot b3b \cdot 2a2 \cdot 4b4 \cdot 1a1 \cdot 3b3 \cdot 2a2 \cdot 4b4
= 1a31b23a42b14a31b23a42b4 = 1a3 \cdot 2b1 \cdot 3a4 \cdot 1b2 \cdot 3a4 \cdot 2b1 \cdot 3a42b4
= 1a3 \cdot b41 \cdot 2a2 \cdot 3b3 \cdot 1a1 \cdot 4b4 \cdot 2a2 \cdot 3b4 = 1a3b41 \cdot a2ab3ba1ab4b \cdot 2a2b34.
\]

We note that a cyclic permutation of the last line gives \( w_2^2 \) and we applied 6 relations to get to the last line. However, as it can be verified in Figure 3, the shortest distance actually remains uniformly bounded above by 5.

We note that this method is independent of how many copies of \( w \) we had in the beginning. In a similar fashion it can be seen that we can obtain \( w_3 = a2ab3ba1ab4b \). Hence we have a bundle of three bi-infinite geodesic rays which remain at distance at most 5 from each other.

7. Appendix

7.1. Type A examples. We list the u.l.g.'s with different labels for the groups of type \( A_n \) for \( n = 3, 4, 5 \) by enumerating all total labels which give a u.l.g. including
the word of length 0, which gives the constant term in the generating function. It can be verified that these numbers correspond to the formula in Corollary 3.7.

| Type | Labels of u.l.g.’s                                                                 | Total number |
|------|----------------------------------------------------------------------------------|--------------|
| $A_3$| 100 110 120 122 010 011 012 221 001 111 021 131 121 210 000                     | 15           |
| $A_4$| Type I: 1000 0100 0010 0001 1100 0110 0011 0111 1110 1111 0000                   | 33           |
|      | Type II and III: 1200 0120 0012 1210 0121 1220 0122 0021 2210 0221 2221 1321 1331 1310 0131 1211 1121 1222 1221 0210 2100 |              |
| $A_5$| Type I: 10000 01000 00100 00001 00100 00011 11000 01100 00110 00011 11100 01110 00111 11111 11111 00000 | 66           |
|      | Type II: 12000 01200 00120 00012 21000 02100 00210 00021 01220 00122 22100 02210 22221 12100 01210 00121 12110 01211 11210 01221 12221 12200 12222 11121 |              |
|      | Type III: 12121 12321 12310 13210 13100 01310 00131 01321 12231 13231 13321 12232 13232 13331 01331 13331 01321 |              |

7.2. **Infinite u.l.g.’s in the group $\tilde{D}_4$.** We list all cases that have to be considered in Theorem 6.3. Figure 4 depicts the first part of these words, and the corresponding end-vertices of the trees labelled with $T2$, $T3$ and $T4$ indicate that the tree continues with Figure 5, 6 or 7.

(1) $a_1a_2a_1a$                       (21) $a_1ab_3ba_2ab_4b_3b_3$
(2) $a_1a_2a_2$                        (22) $a_1ab_3ba_2ab_4b_2a$
(3) $a_1a_2a_3a_2a$                    (23) $a_1ab_3ba_2ab_4b_1a$  
(4) $a_1a_2a_5b_5b$                    (24) $a_1ab_3baba$
(5) $a_1a_2a_4b$                       (25) $a_1ab_3b_3b_3$  
(6) $a_1a_2a_5b_3b_3$                  (26) $a_1ab_4b_3b_3b_3$  
(7) $a_1a_2a_5b_4b_2a$                 (27) $a_1ab_3baba$
(8) $a_1a_2a_5b_4b_3b$                 (28) $a_1ab_3b_3b_3b_3b$  
(9) $a_1a_2a_5b_3ab$                   (29) $a_1ab_3b_4b_4b_4b$  
(10) $a_1a_2a_5b_3b_3a_2a$             (30) $a_1ab_3b_4b_4b_3b_3$  
(11) $a_1a_3a_2a$                      (31) $a_1ab_3b_4b_3b_4b_3b$  
(12) $a_1a_2a_2a_2$                    (32) $a_1ab_3b_3b_3b_2a$  
(13) $a_1ab_3b_3b$                     (33) $a_1ab_3b_3b_3b_4b_4b_3b$  
(14) $a_1ab_3b_4b$                     (34) $a_1ab_3b_3b_3b_4b_4b_3b$  
(15) $a_1ab_4b_3b_3b$                  (35) $a_1ab_3b_3b_3b_4b_3b_3b$  
(16) $a_1ab_4b_3b_2ab_4b$              (36) $a_1ab_3b_3b_3b_2a_2a_2b_3b$  
(17) $a_1ab_4b_3b_2ab_3b_3a_1$        (37) $a_1ab_3b_3b_3b_2a_2a_2b_3b$  
(18) $a_1ab_3b_4b$                     (38) $a_1ab_3b_3b_3b_2a_2a_2b_3b$  
(19) $a_1ab_4b_3b_2a_2aba$             (39) $a_1ab_3b_3b_3b_2a_2a_2b_3b$  
(20) $a_1ab_4b_3b_2a_2ab_3b$          (40) $a_1ab_3b_3b_3b_2a_2a_2b_3b$
Checking the cases of the tree:

- The following cases have been shown in the proof of Theorem 6.6:
  \[ \{ 13, 17, 25, 31, 33, 34, 35, 36, 59, 60, 62, 65 \} \]

- The following cases contain a sequence \( xyxy \) and are hence not reduced:
  \[ \{ 2, 4, 6, 12, 21, 23, 30, 38, 40, 41, 42, 43, 44, 45, 46, 47, 49, 53, 62, 63 \} \]

- The following cases contain \( xy \ldots xy \) where the dots do not contain \( x \) or \( y \). These cases are not reduced:
  \[ \{ 11, 13, 14, 16, 18, 20, 22, 25, 29, 32, 34, 39, 50, 51, 52, 54, 56, 61, 65, 67 \} \]

- The following cases contain \( xy \ldots yxy \) and are hence not a u.l.g.:
  \[ \{ 3, 11, 13, 15, 16, 26, 35, 37, 45, 51, 56, 60 \} \]
Figure 4. Tree 1

Figure 5. Tree 2
Figure 6. Tree 3
Figure 7. Tree 4

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