Quantum theory for the Standard Model

M. Talon *

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Abstract

These lectures present some basic facts in field theory necessary to understand the quantum theory of the Standard Model of weak and electromagnetic interactions.
Contents

1 Introduction 2

2 The path integral 3
   2.1 Path integral representation of the evolution operator . . . . . . . 3
   2.2 The ordering problem . . . . . . . . . . . . . . . . . . . . . . 6
   2.3 The time ordered products . . . . . . . . . . . . . . . . . . . 7
   2.4 The forced harmonic oscillator . . . . . . . . . . . . . . . . 9
   2.5 Vacuum expectation values . . . . . . . . . . . . . . . . . . . 11
   2.6 Generalization to quantum field theory . . . . . . . . . . . . . 13
   2.7 The perturbation theory . . . . . . . . . . . . . . . . . . . . 15

3 The effective action 17
   3.1 Generating function of connected Green's functions . . . . . . . 17
   3.2 One-particle-irreducible graphs . . . . . . . . . . . . . . . . 20

4 Goldstone and Higgs mechanisms 25
   4.1 The Goldstone theorem . . . . . . . . . . . . . . . . . . . . . 26
   4.2 The Higgs mechanism . . . . . . . . . . . . . . . . . . . . . . 27
   4.3 The Standard model . . . . . . . . . . . . . . . . . . . . . . . 29
1 Introduction

The aim of these lectures is to introduce the reader to the theoretical setup of the Standard Model of weak and electromagnetic interactions. This entails a presentation of the perturbative expansion of quantum field theory, which is also a prerequisite to the lectures of R. Stora on renormalization of gauge field theory. Finally this standard view on the Standard Model is an introduction to the lectures of D. Kastler which will present a modern viewpoint on this theory based on the ideas on non-commutative geometry developed by A. Connes.

The quickest way (hence the clearest) to the perturbative expansion is the quantification by path integration, historically introduced by R. Feynman in this context. This approach leads to some problems with respect to a rigorous definition of the path integral. These problems have been overcome, notably by an analytic continuation to the Euclidean region; in turn this allows powerful developments in Constructive Quantum Field Theory, that we shall not be concerned about. More pragmatically path integration does the magical job of dissolving the ordering problems of quantum mechanics; we show that this is of course an illusion and that these problems are simply transformed into discretization ones.

Another important issue is the question of causality in Field Theory. It is frequently unclear how the correct $i\epsilon$ prescription appears in the propagators, and why the path integral of product of operators yields a T–product. We have chosen to show, following the original work of R. Feynman [1, 2], that a sufficiently thorough study of a simple example, the forced harmonic oscillator, gives the correct answers.

We then introduce the important generating functionals of connected diagrams and one–particle–irreducible diagrams. They are combinatorial objects of much use in the discussion of the symmetries of the considered theories and their renormalization. Since their properties are relegated to exercises in the usual textbooks we present proofs of the main facts. These combinatorial objects are also useful in the applications of field theory to Statistical Mechanics, see for example [3, 4].

Finally we briefly discuss an essential ingredient of the Standard Model, i.e., the Higgs mechanism, but we do not offer a discussion of the quantification of gauge theories [5] since this would duplicate R. Stora’s lectures. Lack of space prevents us to introduce the subject of anomalies, that is breaking
of gauge invariance by quantum effects due to the divergences of quantum field theory, which lead to an important constraint on the construction of the Standard Model, namely that the families must be complete.

All these subjects are treated fully in the standard textbooks on Quantum Field Theory, either the classical ones \[6, 7\], or the modern ones \[8, 9, 10\] which moreover offer a complete discussion of renormalization group theory, which is essential to understand QCD, i.e., the present theory of strong interactions. Streamlined accounts can be found in \[11, 12\] and notably in the famous report of E. Abers and B. Lee \[13\]. Finally we must mention the beautiful lectures of S. Coleman \[14\] which cover roughly the same subjects as presented below.

2 The path integral

2.1 Path integral representation of the evolution operator

We work in the Heisenberg representation, for a one dimensional system, and denote \(Q(t)\) the position operator at time \(t\).

\[
Q(t) = e^{iHt}Qe^{-iHt}
\]

Let us recall that the quantum states have no time evolution in this representation. We denote \(|q, t>\) the fixed eigenstate of \(Q(t)\) for the eigenvalue \(q\), where \(t\) is given some fixed value. If \(|q>\) is such that \(Q|q >= q|q>\), then \(|q, t> = e^{iHt}|q>\). Finally, for any state \(|\psi>\), \(<q, t|\psi> = <q|e^{-iHt}\psi>\) is the Schrödinger representation of this state at the time \(t\).

The evolution operator is the amplitude to go from to \(|q, t>\) to \(|q', t'>\), i.e., knowing that the particle is sitting at \(q\) at time \(t\), the amplitude to find it at \(q'\) at time \(t'\).

\[
<q', t'|q, t> = <q'|e^{-iH(t'-t)}|q>
\]

For any decomposition of the time interval \((t, t')\) we have:

\[
t = t_0 < t_1 < \cdots < t_n < t_{n+1} = t'
\]

\[
<q', t'|q, t> = \int dq_1 \cdots dq_n <q', t'|q_n, t_n><q_n, t_n| \cdots |q, t>
\]
since for any $t_j$ the states $|q_j, t_j>$ are a complete set.

For $\epsilon$ small one traditionally performs the following approximations, hardly justified in general:

$$<q^{'}, t + \epsilon q, t > = <q^{'}|e^{-i\epsilon H(P,Q)}|q> \simeq <q^{'}|1 - i\epsilon H(P, Q)|q>$$

$$= \int \frac{dp}{2\pi} e^{ip(q^{'} - q)} \left(1 - i\epsilon H(p, \frac{q + q^{'}}{2})\right)$$

(under appropriate ordering of the operators $P$ and $Q$ in $H$, as explained later on)

$$\simeq \int \frac{dp}{2\pi} e^{ip(q^{'} - q) - i\epsilon H(p, \frac{q + q^{'}}{2})}$$

Notice that all operators have disappeared in the final expression. This approximation has first been noted by Dirac.

Finally one gets with $\epsilon = \frac{t^{'} - t}{n+1}$ the expression of the evolution operator $<q^{'}, t^{'}|q, t>$ in the form:

$$\int dq_1 \cdot \cdot \cdot dq_n \frac{dp_0}{2\pi} \cdot \cdot \cdot \frac{dp_n}{2\pi} \exp i\epsilon \sum_{k=0}^{n} \left\{ p_k \frac{q_{k+1} - q_k}{\epsilon} - H(p_k, \frac{q_k + q_{k+1}}{2}) \right\}$$

Of course, one defines:

$$\dot{q}_k = \frac{q_{k+1} - q_k}{\epsilon}, \quad \ddot{q}_k = \frac{q_k + q_{k+1}}{2}$$

so that in the limit $n \to \infty$ one gets the path integral expression of the evolution operator:

$$<q^{'}, t^{'}|q, t> = \int \prod_{\tau} \frac{dp(\tau)}{2\pi} \frac{dq(\tau)}{2\pi} e^{i\int_{t^{'}}^{t} (p\dot{q} - H(p, q))d\tau}$$  \hspace{1cm} (2.1)

Notice that in this expression $H$ may well be explicitly time dependent. Also notice that this path integral takes the form of an integral over partially constrained paths in phase space of the exponential of the classical action, averaged with the invariant measure of phase space. So it seems naively well behaved under canonical transformations when ordering and discretization problems are not taken into account.
In the particular case in which the phase space is polarized by a globally defined choice of a configuration variable $Q$ and of the momentum $P$ and the Hamiltonian is of the form:

$$H(P, Q) = \frac{1}{2}P^2 + V(Q)$$

notice that there are no ordering problems in the above approximation steps, and moreover one can explicitly integrate over all variables, using the Gaussian integration formula:

$$\int \frac{dp}{2\pi} e^{ie[pq-q^2/2]} = \frac{1}{\sqrt{2\pi i\epsilon}} e^{ieq^2/2}$$

One ends up with:

$$\langle q', t' | q, t \rangle = \int dq_1 \cdots dq_n \frac{1}{(2\pi i\epsilon)^{n+1}} e^{ie\sum_{k=0}^{n}(\frac{1}{2}q'^2_k - V(q_k))}$$

In this situation one introduces the Lagrangian

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^2 - V(q)$$

so that we get in the limit $n \to \infty$:

$$\langle q', t' | q, t \rangle = \int \prod_{\tau} \frac{dq(\tau)}{\sqrt{2\pi i\epsilon}} e^{i \int_{t}^{t'} L(q, \dot{q}) d\tau}$$

(2.2)

This is the form of the Path Integral first written by Feynman. Notice that $S = \int_{t}^{t'} L(q, \dot{q}) d\tau$ is the classical action along the path $q(\tau)$ in configuration space leading from $q$ at $t$ to $q'$ at $t'$ and one averages over all such paths.

This Feynman formula can be proven rigorously assuming $H$ to be self-adjoint, by an application of the Trotter product formula, as first shown by Nelson [13]. The above more "general" one (2.1) has no such solid foundation. In turn the Feynman formula can be used to prove rigorous results. In this context one generally works in the Euclidean formulation in which time is rotated to pure imaginary values. Then

$$\prod_{\tau} \frac{dq(\tau)}{\sqrt{2\pi i\epsilon}} e^{-\int_{t}^{t'} \dot{q}'^2/2 d\tau}$$
can be shown to be a well-defined measure on the space of paths, first introduced by N. Wiener, with which one averages the potential-dependent functional of the path. An extended account of these applications can be found in the book of B. Simon [16]. At a more “physical” level Feynman has discovered a nice application of the path integral in his famous study of the polaron problem, discussed as well as other interesting applications to Statistical Mechanics in his book [3].

2.2 The ordering problem

As we have seen before, the operators $P$ and $Q$ present in the evolution operator disappear in the computation of the path integral and one ends up with purely numerical quantities. The crucial steps in the computation is reached when we write:

$$<q'|H(P,Q)|q> = \int \frac{dp}{2\pi} e^{ip(q'-q)} H(p, \frac{q + q'}{2})$$

under “appropriate” ordering hypothesis.

Notice that if one writes $H(P,Q)$ with all the $P$’s at the left and all the $Q$’s at the right, as in Faddeev [17], then trivially:

$$<q'|P^n Q^m |q> = \int dp <q'|P^n|p><p|Q^m|q>$$

$$= \int \frac{dp}{2\pi} e^{ip(q'-q)} p^n q^m$$

so that one ends up with $H(p,q)$ instead of $H(p, \frac{q + q'}{2})$. The problem is that $H(P,Q)$ is then not obviously self-adjoint. We see that ordering ambiguities translate into discretization problems in the Path Integral formulation.

Let us now define the so-called Weyl ordering which leads to the above “mid-point splitting”. We shall only concern ourselves with functions of the form $f(q)p^r$ with $r = 0, 1, 2$ the only ones of physical interest. A more general discussion can be found in the book of Berezin [18]. For $r = 0$ there is no problem. Then we define following T.D. Lee [19]:

$$[Q^n P]_W = \frac{1}{n+1} [Q^n P + Q^{n-1} PQ + \cdots + PQ^n]$$

$$[Q^n P^2]_W = \frac{2}{(n+1)(n+2)} \sum_{l,m} Q^{n-l-m} PQ^l PQ^m$$

(2.3)
i.e., one averages over all possible orders.

Then by using the commutation relations:

\[ PQ^l = -i l Q^{l-1} + Q^l P \]

one can show by brute force computation that for any polynomial \( f \):

\[
[f(Q)P]_W = \frac{1}{2} [f(Q)P + Pf(Q)]
\]

\[
[f(Q)P^2]_W = \frac{1}{4} [f(Q)P^2 + 2Pf(Q)P + P^2f(Q)]
\]

\[
[Q^n P^r]_W = \frac{1}{2^n} \sum_l C^n_l Q^{n-l} P^r Q^l \tag{2.4}
\]

As a matter of fact, each of the expressions in (2.3, 2.4) for \( r = 1 \) can be reduced to \( Q^n P - in/2 Q^{n-1} P \), and each of the expressions for \( r = 2 \) boils down to \( Q^n P^2 - inQ^{n-1} P - 1/4 n(n-1)Q^{n-2} \), but notice that Weyl ordered expressions are obviously self-adjoint.

Now we can easily compute for \( r = 0, 1, 2 \) the required kernel:

\[
\langle q' \mid [Q^n P^r]_W \mid q \rangle = 
\]

\[
= \frac{1}{2^m} \sum_l \int \frac{dp}{2\pi} e^{ip(q'-q)} q^{m-l} p^r q^l
\]

\[
= \int \frac{dp}{2\pi} e^{ip(q'-q)} p^r \left( \frac{q + q'}{2} \right)^m
\]

yielding the mid-point splitting as promised earlier.

### 2.3 The time ordered products

One can give path integral formulations of more general expectation values such as:

\[
\langle q', t' \mid F_n(t_n) \cdots F_1(t_1) \mid q, t \rangle
\]

where \( F_j(t_j) \) are Heisenberg operators, i.e., functions of \( P(t_j) \) and \( Q(t_j) \) assumed for example Weyl ordered. We also assume

\[ t < t_1 < \cdots t_n < t' \]
Then one can choose time splittings of the interval \((t, t')\) such that all the \(t_j\)'s occur in the decomposition. Proceeding as above we shall express the above expectation value in the form:

\[
\int \prod_l dq_l < q', t' | \cdots | q_{k+1}, t_k | F_j | q_k, t_k > < q_k, t_k | q_{k-1}, t_{k-1} > \cdots | q, t >
\]

where \(t_k\) is the same time occurring in \(F_j\). Since \(F_j\) is Weyl ordered one has:

\[
< q_{k+1}, t_k | F_j | q_k, t_k > = \int \frac{dp_k}{2\pi} e^{ip_k(q_{k+1}-q_k)} F_j(p_k, \frac{q_k + q_{k+1}}{2})
\]

and in the limit we get:

\[
\int \prod \frac{dp(\tau)}{2\pi} \prod F_j(p(t_j), q(t_j)) e^{i \int_{t}^{t'} (p\dot{q} - H(p, q)) d\tau}
\]

Notice that for the given path \((p(\tau), q(\tau))\) one has to evaluate the function \(F_j\) at the position corresponding to the time \(t_j\).

In the case in which \(H = 1/2 P^2 + V(Q)\) and \(F_j = [A_j(Q) + B_j(Q)P]_W\) one can easily eliminate the integrations over variables \(p_k\) and obtain a Feynman type formula for the expectation value:

\[
\int \prod dq(\tau) \frac{1}{\sqrt{2\pi i\epsilon}} \prod F_j(\dot{q}(t_j), q(t_j)) e^{iS}
\]

Of course the product \(\prod_j F_j\) is now a product of numbers, i.e., does not depend on the order of the times \(t_j\). Hence the preceding derivation shows that the above path integral is in general the expression of

\[
< q', t' | F_{\sigma(n)} \cdots F_{\sigma(1)} | q, t >
\]

where the permutation \(\sigma\) is such that:

\[
t < t_{\sigma(1)} < \cdots t_{\sigma(n)} < t'
\]

Such a permuted product is called a time-ordered product, and we have shown that:

\[
< q', t' | T(F_n(t_n) \cdots F_1(t_1)) | q, t > = \int \mathcal{D}q F_1 \cdots F_n e^{iS} \quad (2.5)
\]
where $Dq$ denotes the appropriate product of the $dq(\tau)$.

Remark that the appearance of the $T$–product in this expression ultimately comes from the assumption that the dissection of the time interval $(t, t')$ is time–ordered, which can be taken as a definition of the Path Integral, while this is compulsory in the Euclidean formulation. Moreover the appearance of the mid–point splitting in this formula leads to a more refined version of the ordinary $T$–product, called the $T^*$–product. The subtlety lies in the definition of $T$–product for coincident points. The implications are developed for example in Adler’s lectures 20.

### 2.4 The forced harmonic oscillator

We shall now follow Feynman and indicate how one can compute the path integral for a forced harmonic oscillator, a situation that directly generalizes to Field Theory. Let us take:

$$L = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2) + J(t)q$$

where $J(t)$ is some external, time–dependent source (exemplifying a time-dependent Hamiltonian).

We want to compute $\int Dq e^{iS}$. When $\omega = 0$, the free–particle situation, the answer is immediate using Fourier expansion. For example for $J = 0$ one can write:

$$<q', t'|e^{-iH(t'-t)}|q, t> = \int \frac{dk}{2\pi} e^{ik(q'-q) - \frac{i}{2}k^2(t'-t)}$$

$$= \frac{1}{\sqrt{2\pi T}} e^{\frac{(q'-q)^2}{2T}} \quad \text{where} \quad T = t' - t$$

When $\omega \neq 0$ the standard method is to expand around the classical solution. So let us write $q = q_0 + y$ with $q_0(t) = q$, $q_0(t') = q'$, $y(t) = y(t') = 0$, where $q_0$ extremizes the action, i.e., $S(q_0 + y)$ has no linear term in $y$. Here the expansion is exact:

$$S(q) = S(q_0) + \int_t^{t'} (\dot{q}_0 \tilde{y} - \omega^2 q_0 y + Jy) d\tau + \int_t^{t'} \frac{1}{2} (\dot{y}^2 - \omega^2 y^2) d\tau$$

Since $y$ vanishes at end points one can write $\int_t^{t'} \dot{q}_0 \tilde{y} d\tau = -\int_t^{t'} y \ddot{q}_0 d\tau$ so that $q_0$ obeys the classical equation of motion:

$$\ddot{q}_0 + \omega^2 q_0 = J$$
In order to find the classical solution it is convenient to use the method of Green’s functions, i.e., to first consider the case in which $q = q' = 0$ and $J(\tau) = \delta(\tau - \sigma)$. Then, setting $T = t' - t$, the solution, an appropriate combination of sinusoids, is:

$$G(\tau, \sigma) = -\frac{1}{\omega \sin \omega T} \begin{cases} \sin \omega(t' - \sigma) \sin \omega(\tau - t) & \tau < \sigma \\
\sin \omega(\sigma - t) \sin \omega(t' - \tau) & \tau > \sigma \end{cases}$$

For general $J$ the solution is obviously $\int_{t'}^t G(\tau, \sigma)J(\sigma)d\sigma$ and finally one fulfills the correct boundary conditions by adding a pure sinusoid. One gets:

$$q_0(\tau) = -\frac{1}{\omega \sin \omega T} \left\{ \sin \omega(t' - \tau) \int_{t'}^\tau \sin \omega(\sigma - \tau)J(\sigma)d\sigma + \sin \omega(\tau - t) \int_t^{t'} \sin \omega(t' - \sigma)J(\sigma)d\sigma \right\}$$

$$+ \frac{q}{\sin \omega T} \sin \omega(t' - \tau) + \frac{q'}{\sin \omega T} \sin \omega(\tau - t)$$

Then the path integral reads:

$$<q', t'|q, t> = e^{iS_J(q_0)} \int \mathcal{D}y e^{iS_0(y)}$$

The classical action is computed according to:

$$S_J(q_0) = \int_t^{t'} \left[ \frac{1}{2}(\dot{q}_0^2 - \omega^2 q_0^2) + J q_0 \right] d\tau$$

$$= \frac{1}{2} [q_0 \ddot{q}_0]_t^{t'} + \frac{1}{2} \int_t^{t'} J q_0 d\tau$$

by integrating by parts (the boundary terms do not cancel). Substituting the above expression of $q_0$ one notices that the two terms have the same value so the $1/2$ disappears.

It remains to compute the path integral for the free harmonic oscillator. This can be done by following the definition as a limiting procedure, see Schulman [21], but we shall content ourselves with a quick computation. Setting $t = 0$ $t' = T$ and, due to the boundary conditions:

$$y(\tau) = \sum_{n=1}^{\infty} y_n \sin \frac{n \pi \tau}{T}$$

so that

$$S_0(y) = \frac{T}{4} \sum_{n=1}^{\infty} \left( \frac{\pi^2 n^2}{T^2} - \omega^2 \right) y_n^2$$

10
one writes \( D y = \prod_n dy_n \) up to some normalizing factor, so that the path integral reduces to independent Gaussian integrations leading to:

\[
\prod_{n=1}^{\infty} \frac{1}{\sqrt{1 - \frac{\omega^2 T^2}{\pi^2 n^2}}} \quad \text{or} \quad \sqrt{\frac{\omega T}{\sin \omega T}}
\]

up to some constants. One adjusts the constant by comparing with the case \( \omega = 0 \) in which the computation is trivial, and one finally gets:

\[
<q', t'|q, t> = \sqrt{\frac{\omega}{2\pi i \sin \omega T}} e^{i\Omega}
\] (2.6)

\[
\Omega = \frac{\omega}{2\sin \omega T} [(q^2 + q'^2) \cos \omega T - 2qq'] + \frac{q'}{\sin \omega T} \int_t^{t'} J(\tau) \sin \omega(\tau - t) d\tau
\]

\[
+ \frac{q}{\sin \omega T} \int_t^{t'} J(\tau) \sin \omega(t' - \tau) d\tau
\]

\[
- \frac{1}{\omega \sin \omega T} \int_t^{t'} d\sigma \int_t^{\sigma} d\tau J(\sigma) J(\tau) \sin \omega(t' - \tau) \sin \omega(\sigma - t)
\] (2.7)

It is interesting to consider the limiting form of this result when \( \omega \to 0 \), and to compare with the direct computation for \( \omega = 0 \).

### 2.5 Vacuum expectation values

We shall introduce now the objects of interest in the generalization to Quantum Field Theory, i.e., vacuum expectation values. In general, let us consider a time-independent system coupled to a source \( J(\tau) \) such that \( J(\tau) \neq 0 \) only for \( t < \tau < t' \) and finally take \( T \ll t \) and \( T' \gg t' \). Then:

\[
< Q', T'|Q, T > = \int dq dq' < Q', T'|q', t'> < q', t'|q, t > < q, t|Q, T >
\]

But outside of \((t, t')\) we have free motion under the Hamiltonian \( H \) with spectrum \((E_n, \phi_n)\). Hence:

\[
< q, t|Q, T > = < q| e^{-iH_0(t-T)}|Q >
\]
\[ \sum_n \phi_n(q) \phi_n^*(Q) e^{-iE_n(t-T)} \]

Let us now formally assume some analytic continuation into Euclidean space, i.e.:

\[ T \to -(1-i\epsilon)\infty \quad \text{and} \quad T' \to +(1-i\epsilon)\infty \]

This has the effect of projecting on the ground state:

\[ \langle q,t|Q,T \rangle \simeq \phi_0(q) \phi_0^*(Q) e^{-iE_0(t-T)} \]

We see that:

\[ W[J] \equiv \frac{\langle Q',T'|Q,T \rangle}{e^{-iE_0(T'-T)} \phi_0^*(Q) \phi_0(Q')} = \int dq dq' \phi_0(q,t) \phi_0^*(q',t') \langle q',t'|q,t \rangle J(t' - t) \quad (2.8) \]

in which we set \( \phi_0(q,t) = e^{-iE_0t} \phi_0(q) = \langle q,t|\phi_0 \rangle \) i.e., the Schrödinger wavefunction at time \( t \) corresponding to the ground state \( |\phi_0 \rangle \). In particular this shows that this vacuum expectation value does not depend on \( t \) and \( t' \).

We apply this idea to the forced harmonic oscillator, where:

\[ \phi_0(q) = \left( \frac{\omega}{\pi} \right)^{\frac{1}{4}} e^{-\omega q^2/2} \]

We need to compute:

\[ W[J] = \frac{\omega}{\pi} \frac{e^{i\omega T/2}}{\sqrt{2i \sin \omega T}} \int dq dq' e^{-\omega(q^2+q'^2)/2+i\Omega} \]

where \( \Omega \) is the previously defined expression. This is a Gaussian integration which separates when setting \( q = u + v, \ q' = u - v \). After a lengthy computation, one observes that all factors like \( \sin \omega T \) (here \( T = t' - t \) cancel, and one gets the remarkably simple result:

\[ W[J] = \exp\{-i \int_t^{t'} d\sigma \int_\sigma^\sigma d\tau J(\sigma) e^{-i\omega(\sigma-\tau)}/2i\omega J(\tau)\} \]

The point of this computation, due to Feynman is that one ends up with the correct Feynman propagator between \( J(\sigma) \) and \( J(\tau) \) as a result of having correctly taken into account the boundary conditions, and without
any appeal to continuations into the Euclidean region. This is to be compared to the analysis in Ramond’s book [12], in which a convergence factor is used.

Due to the vanishing of $J$ outside $(t, t')$ one can write:

$$ W[J] = \exp\left\{ -\frac{i}{2} \int_{\infty}^{\infty} \int_{\infty}^{\infty} d\sigma d\tau J(\sigma) D_F(\sigma - \tau) J(\tau) \right\} $$

$$ D_F(\sigma) = \frac{1}{2i\omega} [\theta(\sigma)e^{-i\omega\sigma} + \theta(-\sigma)e^{i\omega\sigma}] $$

This means that the harmonic oscillator induces an effective interaction between $J(\sigma)$ and $J(\tau)$ such that positive frequencies are propagated forward in time, and negative ones backwards.

The same result is obtained straightforwardly by setting $\omega^2 \to \omega^2 - i\epsilon$ in the Lagrangian, i.e., introducing a convergence factor in the Path Integral, which in effect destroys the time-reversal symmetry of Quantum Mechanics.

### 2.6 Generalization to quantum field theory

The preceding discussion immediately generalizes to the case of a free field $\phi(\vec{x}, t)$ coupled to external sources. As a matter of fact, such a field (with mass term $\mu$) can be considered under Fourier transformation on $\vec{x}$, as a collection of harmonic oscillators $\phi_k(t)$ of frequency $\omega_k = \sqrt{\vec{k}^2 + \mu^2}$. The corresponding Lagrangian density is:

$$ L_J = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{\mu^2}{2} \phi^2 + J\phi $$

Without further ado we write the vacuum expectation value as:

$$ W[J] = \exp \left\{ -\frac{i}{2} \int d^4x d^4y J(x) \Delta_F(x - y) J(y) \right\} $$

with the Feynman propagator:

$$ \Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik.x}}{k^2 - \mu^2 + i\epsilon} \quad (2.10) $$
Let us remark that time–ordered products of fields are simply obtained by functional differentiation of $W[J]$:  

$$<0|T(\phi(x_1)\cdots \phi(x_n))|0>_J = \int \mathcal{D}\phi \phi(x_1)\cdots \phi(x_n) e^{i \int dx L_J} = \left(\frac{1}{i \delta J(x_1)}\right) \cdots \left(\frac{1}{i \delta J(x_n)}\right) W[J]$$  

(2.11)

Since $W[J]$ is known we can compute this expression, and then go to the limit $J \to 0$. Obviously to get something non zero in this limit we have to consider terms like:

$$\left(\frac{1}{i \delta J(x_1)}\right) e^{-\frac{1}{2} \int J \Delta_F J} = -\int dy \Delta_F(x_1 - y) J(y) e^{-\frac{1}{2} \int J \Delta_F J}$$

$$\left(\frac{1}{i \delta J(x_1)}\right) \left(\frac{1}{i \delta J(x_2)}\right) e^{-\frac{1}{2} \int J \Delta_F J} = i \Delta_F(x_1 - x_2)$$

plus terms vanishing in the limit $J \to 0$.

This means that the result is obtained by connecting all pair of points by propagators in all possible ways (a propagator is a line connecting $x_k$ to $x_l$, to which is associated the value $i \Delta_F(x_k - x_l)$), and adding all such expressions. One such product is called a Feynman graph, of tree type. We have expressed the Green function:

$$G(x_1, \cdots, x_n) = <0|T(\phi(x_1)\cdots \phi(x_n))|0>$$  

(2.12)

as functional derivative of $W[J]$ when $J \to 0$. This means that $W[J]$ is the generating function of Green’s functions:

$$W[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod dx_i G(x_1, \cdots, x_n) J(x_1) \cdots J(x_n)$$  

(2.13)

In the above trivial (free) situation, we have also shown that one can write:

$$W[J] = \exp(iZ[J])$$

$$Z[J] = -\frac{1}{2} \int dx dx dy J(x) \Delta_F(x - y) J(y)$$

We shall elaborate in the following on these important generating functions of Field Theory.
2.7 The perturbation theory

When self-interaction terms are present in the Lagrangian it is impossible to express $W[J]$ in closed form as above, and Green’s functions are computed as perturbation series in the coupling constants.

One writes

$$L = L_0 + L_I$$

where all terms quadratic or linear in the fields are collected in $L_0$ and all terms cubic and higher in $L_I$. Finally one expands

$$e^{i \int d^4xL} = e^{i \int d^4xL_0} \sum_{n=0}^{\infty} \frac{1}{n!} \left( i \int d^4xL_I \right)^n$$

This brings us back to the computation, order by order, of Green’s functions for the free field.

Taking for example $L = L_0 + \frac{\lambda}{4!} \phi^4$ where $L_0$ is the previously defined quadratic Lagrangian density we see that

$$G(x_1, \ldots, x_n) = \int D\phi \phi(x_1) \cdots \phi(x_n) e^{i \int d^4xL} = e^{i \int d^4xL_0} \sum_{m=0}^{\infty} \frac{1}{m!} \left( i \int d^4y \phi^4(y) \right)^m e^{iS_0} \cdot L_I$$

At each order $m$ in the coupling constant we have to compute the Green function corresponding to the product of fields

$$\frac{1}{m!} \left( \frac{i\lambda}{4!} \right)^m \int dy_1 \cdots dy_m \phi^4(y_1) \cdots \phi^4(y_m) \phi(x_1) \cdots \phi(x_n)$$

Using the method of generating functions each $\phi^4(y_k)$ is converted to

$$\left[ \frac{1}{i} \frac{\delta}{\delta J(y_k)} \right]^4$$

which, encountering a product of four $J$’s in the expansion of $\exp(iZ[J])$ gives

$$\prod_{l=1}^{4} J(z_l) \to 4! \prod_{l=1}^{4} \delta(z_l - y_k)$$

Notice that the $i$’s cancel correctly as explained above, leaving $i\Delta_F$ for each propagator.

In other words we get a Feynman diagram with $n$ terminals $x_1, \ldots, x_n$ and $m$ vertices $y_1, \ldots, y_n$ (which is then integrated other the $y_k$’s). To each vertex $y_k$ are attached four propagators. Each propagator has value $i\Delta_F(z_i - z_j)$ and ends either in some $x_i$ or some $y_k$ and each vertex carries a factor $(i\lambda)$, since the $4!$ cancels. Notice that many different terms in the expansion
of equation (2.14) may give rise to the same Feynman diagram, through relabelling of lines or vertices. Let $N$ be the number of such equivalent terms. Assuming the diagram contains $m$ vertices as above, and $q$ lines there is an explicit $\frac{1}{m!}$ in the perturbative expansion and an explicit $\frac{1}{q!}$ in the expansion of $\exp(iZ_0[J])$. Then the combinatorial factor $\frac{N}{m!q!}$ is called the symmetry factor of the Feynman graph. In simple situations it is equal to 1 but may be smaller, when permutation of lines or vertices do not correspond to different terms.

Finally the perturbative expansion of $G(x_1, \cdots, x_n)$ at order $m$ is obtained by summing over all Feynman graphs with $m$ vertices, each with its symmetry factor, and integrating over the variables $y_k$. Notice that such graphs in general contain loops, i.e., are not of tree type. In fact it is easy to see that there are exactly $m$ loops at order $m$.

Unfortunately some diagrams with loops produce divergent results when integrated over variables $y_k$. This happens because $\Delta_F(x)$ is singular for $x = 0$. This is cured by renormalization, as will be discussed in R. Stora’s lectures. One then gets a renormalized perturbation expansion which is generally a divergent power series. In some particularly simple situations this series can be resummed by appropriate methods. In practice the first few terms of the perturbative expansion are considered as an accurate representation of the “exact” quantum result. The classical limit $\hbar \to 0$ reduces to the sum of tree graphs, i.e., graphs with no loop.

Finally we must mention that the above perturbative scheme breaks down in the case of Gauge theories (which is relevant for the Standard model) because the quadratic part of the Lagrangian happens to be non invertible, precisely because of the symmetry of the theory. This means that propagators do not exist, and the solution is to break the symmetry by the addition of a “gauge–breaking term”. In turn we then have to add new fields in the theory called “Faddeev–Popov” fields in order to recover essentially the gauge symmetry in the form of a new symmetry, mixing bosonic and fermionic fields called BRS symmetry. This symmetry allows to show that the physical content of the theory is independent of the gauge breaking term, so that a solid foundation for the perturbative expansion is regained. This will be discussed in R. Stora’s lecture.
3 The effective action

3.1 Generating function of connected Green’s functions

We have already introduced (formally) the generating function $W[J] = \int \mathcal{D}\phi \exp(iS_J)$ which produces all Green’s functions under functional differentiation with respect to the external source $J$. We have also seen that in the free case $W[J] = \exp iZ[J]$. We shall see that such a formula holds true in general and that $Z[J]$ is the generating function of connected Green’s function. By a connected Green’s function we mean the sum of connected Feynman diagrams, i.e., diagrams that cannot be written as a product of disjoint parts (in particular, such diagrams do not contain pure vacuum parts, i.e., each line is connected to at least one of the terminals $x_k$). For example, a disconnected four point function is of the form:

![Disconnected Four Point Function Diagram]

Obviously in the free situation the only connected diagram is the two point diagram:

![Two Point Diagram]

and this corresponds to the fact that, in this case, $Z[J] = -\frac{1}{2} \int J \Delta F J$.

In the interacting case one can consider the following connected four point functions:

![Connected Four Point Functions Diagrams]

and so on.

Let us show that $Z[J]$ indeed generates connected Green’s functions.
First, vacuum diagrams are removed by considering the quotient:

\[
\int \mathcal{D}\phi \phi_1 \cdots \phi_n \ e^{iS_J} \quad \frac{\int \mathcal{D}\phi \ e^{iS_J}}{\int \mathcal{D}\phi \ e^{iS_J}}
\]

But setting \( W[J] = \exp iZ[J] \), a variation \( \delta J_i \) gives rise to:

\[
\delta W = i\delta Z W = \int \mathcal{D}\phi \ i\delta J_i \phi_i e^{iS_J}
\]

so that the above quotient is exactly for \( n=1 \) (\( i \) collects all indices, including space–time position):

\[
\frac{\delta Z}{\delta J_i} \equiv \Phi_i = \frac{1}{W} \int \mathcal{D}\phi \phi_i e^{iS_J} \quad (3.1)
\]

Hence, in the limit \( J \to 0 \) is the vacuum expectation value of the field \( \phi_i \), which may very well be equal to some constant (by translation invariance) value \( v_i \) (the situation with \( v_i \neq 0 \) happens to be important in the Standard model). We then define for any \( J \) the field \( \bar{\phi}_i = \phi_i - v_i \) such that \( <\phi_i> = 0 \) when \( J = 0 \), i.e., \( \bar{\phi}_i \) has no tadpole. Then we have:

**Theorem.** **Functional differentiation of** \( Z[J] \)

\[
\frac{\delta^n Z[J]}{\delta J_1(x_1) \cdots \delta J_n(x_n)} \bigg|_{J=0} = (i)^{n-1} < T \left( \bar{\phi}_1(x_1) \cdots \bar{\phi}_n(x_n) \right) >^c
\]

yields connected expectation values of time ordered product of fields (denoted by the subscript \( c \)). In particular the connected one point function vanishes.

**Proof.** For any \( J \) let us denote:

\[
<\bar{\phi}_k>^c_j = \frac{1}{W[J]} \int \mathcal{D}\phi \bar{\phi}_k e^{iS_J} = \Phi_k - v_k
\]

\[
<\bar{\phi}_k \bar{\phi}_l>_J = \frac{1}{W[J]} \int \mathcal{D}\phi \bar{\phi}_k \bar{\phi}_l e^{iS_J}
\]

Then we have:

\[
\frac{\delta}{\delta J_i} <\bar{\phi}_k>^c_j = -i\Phi_l <\bar{\phi}_k>^c_j + <\bar{\phi}_k i (\bar{\phi}_l + v_l)>_J = i <\bar{\phi}_k \bar{\phi}_l>_J - i <\bar{\phi}_k>^c_j <\bar{\phi}_l>^c_j
\]
We shall denote (skipping the $T$ symbol for brevity):

$$<\bar{\phi}_k \bar{\phi}_l>^c_J = <\bar{\phi}_k \bar{\phi}_l>^c_J - <\bar{\phi}_k>^c_J <\bar{\phi}_l>^c_J$$  \hspace{1cm} (3.2)$$

and we shall show that this two point function is indeed connected, so that we get:

$$\frac{\delta}{\delta J_j} <\bar{\phi}_k>^c_J = i <\bar{\phi}_k \bar{\phi}_l>^c_J$$  \hspace{1cm} (3.3)$$

Notice that $<\bar{\phi}_k \bar{\phi}_l>^c_J$ is obtained by removing from $<\bar{\phi}_k \bar{\phi}_l>_J$ the possible disconnected parts. One then has to worry about the combinatorial factors, i.e., the symmetry factors.

Let us consider a disconnected graph that can be separated into two parts with respectively $m_1$ loops, $q_1$ lines, weight $N_1$ (number of identical terms in the perturbative expansion), and $m_2$, $q_2$, $N_2$. Then each component occurs with combinatorial factor $\frac{N_i}{m_i!q_i!}$ while the disconnected graph will receive a factor $\frac{N}{(m_1+m_2)!(q_1+q_2)!}$. Each permutation of lines or vertices between the two components corresponds to a different term in the perturbative expansion, precisely because the parts are disjoint. The number of such moves is \(\frac{(m_1+m_2)!}{m_1!m_2!} \cdot \frac{q_1+q_2)!}{q_1!q_2!}\). Multiplying by the number $N_1 N_2$ of identical terms in each connected part we get the total weight $N$, and we see that the combinatorial factor of the disconnected graph is just the product of the combinatorial factors of its parts. Obviously a similar reasoning works for any number of parts. Finally the connected two point function is obtained by removing from the ordinary two point function the disconnected graphs, correctly counted, as it should be.

We can now proceed to the next inductive step.

$$\frac{\delta}{\delta J_j} <\bar{\phi}_k \bar{\phi}_l>^c_J = -i \Phi_j <\bar{\phi}_k \bar{\phi}_l>^c_J$$

$$+ <\bar{\phi}_k \bar{\phi}_l (\bar{\phi}_j + v_j)>^c_J$$

$$- i <\bar{\phi}_k \bar{\phi}_j>^c_J <\bar{\phi}_l>^c_J$$

$$- i <\bar{\phi}_l \bar{\phi}_j>^c_J <\bar{\phi}_k>^c_J$$

Since we have by equation (3.2)

$$(-i \Phi_j + iv_j) <\bar{\phi}_k \bar{\phi}_l>_J = -i <\phi_j>^c_J \left( <\bar{\phi}_k \bar{\phi}_l>_J + <\bar{\phi}_k>^c_J <\bar{\phi}_l>^c_J \right)$$
we get the equation similar to equation [3.3]:
\[
\frac{\delta}{\delta J_j} < \bar{\phi}_j \bar{\phi}_l >_J = i < \bar{\phi}_j \bar{\phi}_k \bar{\phi}_l >_J
\]
with the definition similar to equation [3.2]:
\[
< \bar{\phi}_j \bar{\phi}_k \bar{\phi}_l >_J = < \bar{\phi}_j \bar{\phi}_k \bar{\phi}_l >_J - < \bar{\phi}_j >_J < \bar{\phi}_k \bar{\phi}_l >_J + < \bar{\phi}_k >_J < \bar{\phi}_j \bar{\phi}_l >_J - < \bar{\phi}_l >_J < \bar{\phi}_j \bar{\phi}_k >_J
\]
By the preceding arguments it is obvious that this is the connected three
point function, and that this computation extends similarly to the \(n\)-point
function, thereby completing the proof of the theorem.

3.2 One–particle–irreducible graphs

We now proceed to define the generating function of one particle irreducible
graphs which turns out to be the effective action, that is the generalization
of the classical action at the quantum level. It is simply the Legendre transfor
of the generating function \(Z[J]\) of connected graphs, in the following way. We
have defined \(\frac{\delta Z}{\delta J_i} = \Phi_i[J]\) as the mean value of \(\phi_i\) under path integration.
Let us invert this relation to express \(J_i = J_i[\Phi]\) and form:
\[
\Gamma[\Phi] = Z[J] - \sum_i J_i \Phi_i
\]  
(more precisely the summation on \(i\) means \(\sum_i \int d^4x J_i(x) \Phi_i(x)\)).

Under variations \(\delta J_i\) producing some \(\delta \Phi_j\) we get:
\[
\delta \Gamma = \sum_i \frac{\delta Z}{\delta J_i} \delta J_i - \sum_i \Phi_i \delta J_i - \sum_i J_i \delta \Phi_i = - \sum_i J_i \delta \Phi_i
\]
so that:
\[
J_i = - \frac{\delta \Gamma}{\delta \Phi_i}
\]
In particular, for \(J \rightarrow 0\) all \(\Phi_i\)'s go to their vacuum expectation values \(v_i\) and
we see that:
\[
\frac{\delta \Gamma}{\delta \Phi_i}\bigg|_{\Phi=v} = -J = 0
\]
Hence \( v \) is the value of \( \Phi \) which extremizes \( \Gamma \) as in the classical case with respect to the classical action.

Moreover consider the limit \( \hbar \to 0 \) in:

\[
W[J] = e^{\frac{i}{\hbar}Z[J]} = \int \mathcal{D}\phi e^{\frac{i}{\hbar}S_J}
\]

In this classical limit, everything reduces to the action evaluated on the classical solution so we have \( Z[J] = S_J[\phi_{cl}] + O(\hbar) \) (where \( \phi_{cl} \) extremizes \( S_J \)) hence \( Z[J] = S[\phi_{cl}] + J\phi_{cl} \). By definition of \( \phi_{cl} \) this quantity is stationary when \( \phi_{cl} \) is varied with \( J \) fixed, so that only the explicit \( J \) term contributes in:

\[
\Phi = \frac{\delta Z}{\delta J} = \phi_{cl}
\]

hence we get finally:

\[
\Gamma[\Phi] = Z[J] - J\Phi = S[\Phi]
\]

We have shown that \( \Gamma[\Phi] = S[\Phi] \) at order 0 in \( \hbar \), so that \( \Gamma[\Phi] \) is the quantum generalization of the classical action. In general the symmetries present in \( S \) will remain as such in \( \Gamma \), this being notably the case for the BRS symmetry, when the path integration measure \( \mathcal{D}\phi \) is formally invariant under the considered symmetry. Nevertheless problems associated with the divergences of quantum field theory may break this nice scheme. In such cases the lack of symmetry of the effective action is called an “anomaly”. Of course even the first (one loop) correction \( \Gamma^1[\Phi] \) is completely non local, and moreover divergent, but fortunately the divergent part is local and can be absorbed in a redefinition of the classical \( \Gamma^0 \). This can be done at all orders consistently in a renormalizable theory. An example of a direct computation of \( \Gamma^1 \) can be found in Coleman–Weinberg [22], in a simple situation, and more extensive computations along the same way have been performed by S. Weinberg for the Standard model [23].

So it is nice to be able to compute \( \Gamma[\Phi] \) by summing over one–particle irreducible graphs. By this we mean graphs that cannot be decomposed into disjoint parts by cutting one line. We begin by considering propagators. In the free situation we have seen that \( Z[J] = -1/2 \sum J_i \Delta_{ij} J_j \) so that:

\[
\frac{\delta^2 Z}{\delta J_i \delta J_j} = -\Delta_{ij} = i < \bar{\phi}_i \phi_j >^c
\]
and $\Delta_{ij}$ is the “bare” propagator. In the interacting situation we similarly define the “dressed” propagator $\Delta'_{ij}$ by:

$$\frac{\delta^2 Z}{\delta J_i \delta J_j} = -\Delta'_{ij} \quad (3.5)$$

As a matter of fact $\Delta'_{ij}$ is the sum of connected graphs leading from $i$ to $j$ (the bubbles below mean two point diagrams which cannot be separated into two parts by cutting a line):

This is a geometric series which sums up to $1/(\Delta^{-1} + \Sigma)$. When $\Delta = 1/(k^2 - \mu^2)$ we get $\Delta' = 1/(k^2 - \mu^2 + \Sigma(p^2))$ and $\Sigma(p^2)$ describes the (one-particle irreducible) quantum corrections to the propagator, which are called the self-energy corrections. Essentially this shifts the position of the pole of the propagator, i.e., the mass of the particle, and introduces a broadening of this pole, through the imaginary part of $\Sigma(\mu^2)$.

Similarly to equation (3.5) we define:

$$\frac{\delta^2 \Gamma}{\delta \Phi_i \delta \Phi_j} = X_{ij}$$

and we have since $\delta Z/\delta J_i = \Phi_i$:

$$\sum_j \frac{\delta^2 Z}{\delta J_j} \frac{\delta J_i}{\delta \Phi_j} \frac{\delta \Phi_j}{\delta \Phi_k} = \frac{\delta \Phi_i}{\delta \Phi_k} = \delta_{ik}$$

Similarly $J_j = -\delta \Gamma/\delta \Phi_j$ hence $\delta J_i/\delta \Phi_k = -\delta^2 \Gamma/\delta \Phi_j \delta \Phi_k$ so that:

$$\sum_j \Delta'_{ij} X_{jk} = \delta_{ik} \quad (3.6)$$

Recalling that $\Delta'_{ij}$ is the dressed propagator while

$$\Gamma[\Phi] = \sum_{jk} \frac{1}{2} \Phi_j X_{jk} \Phi_k + O(\Phi^3)$$
we see that the dressed propagator is the inverse of the coefficient of the quadratic term in the effective action. In particular in the classical limit the bare propagator is the inverse of the quadratic part of the Lagrangian, which may be seen as the lowest order 1PI contribution to $\Gamma$. When interactions are taken into account, $X_{jk}$ is obtained by adding to this $\Sigma_{jk}$ i.e., all 1PI two point functions.

Then one derives the relation (3.6) with respect to $J_k$:

$$\frac{\delta^3 Z}{\delta J_i \delta J_j \delta J_k} X_{jl} + \left(-\Delta'_{ij}\right) \frac{\delta^3 \Gamma}{\delta \Phi_j \delta \Phi_l \delta \Phi_m} \frac{\delta \Phi_m}{\delta J_k} = 0$$

Noticing that $\delta \Phi_m / \delta J_k = -\Delta'_{km}$ and inverting $X_{jl}$ with a $\Delta'$ we get:

$$\frac{\delta^3 Z}{\delta J_i \delta J_j \delta J_k} = -\Delta'_{il} \Delta'_{jm} \Delta'_{kn} \frac{\delta^3 \Gamma}{\delta \Phi_l \delta \Phi_m \delta \Phi_n}$$

Defining the 3–vertex $\Gamma^{(3)}_{kmn}$ as $\delta^3 \Gamma / \delta \Phi_l \delta \Phi_m \delta \Phi_n$ we see that:

$$< T(\bar{\phi}_i \bar{\phi}_j \bar{\phi}_k) > = \left( i \Delta'_{il} \right) \left( i \Delta'_{jm} \right) \left( i \Delta'_{kn} \right) \left( i \Gamma^{(3)}_{kmn} \right)$$ (3.7)

This means that the connected three point function is obtained by attaching dressed propagators to the dressed vertex $\Gamma^{(3)}_{kmn}$ which is therefore the so-called amputated three point function. Obviously there is no way to separate such a function into parts by cutting a line, so the amputated vertex is 1PI.

![Diagram](https://via.placeholder.com/150)

Taking as an example a theory with a coupling $\frac{1}{3!} \phi^3$ the lowest order value of $\Gamma^{(3)}$ is precisely $\lambda$ and at the next order we have the 1PI diagram:

![Diagram](https://via.placeholder.com/150)
We can now state the:

**Theorem.** Functional differentiation of the effective action $\Gamma[\Phi]$ yields the one–particle–irreducible graphs.

**Proof.** Inductively assume that:

$$\frac{1}{i^{n-1}} \delta^n Z[J] = (i\Delta'_{ul})(i\Delta'_{jm}) \cdots i \frac{\delta^n \Gamma}{\delta \Phi_i \delta \Phi_m} \cdots + \text{1–part. red. graphs}$$

and differentiate with:

$$\frac{1}{i} \delta J_k = (i\Delta'_{kr}) \frac{\delta}{\delta \Phi_r}$$

Either this derivation acts on some $\delta^m \Gamma / \delta \Phi_\alpha \cdots$ and produces $(i\Delta'_{kr}) \delta^{m+1} \Gamma / \delta \Phi_r \delta \Phi_\alpha \cdots$ i.e., the external leg $k$ is attached to an $m$–vertex producing a $(m + 1)$–vertex (the corresponding graph will be one–particle–irreducible except if $m = n$) or the derivation acts on some $(i\Delta'_{m\alpha})$. Then as we have seen:

$$\frac{1}{i} \frac{\delta}{\delta J_k} (i\Delta'_{ma}) = -\frac{\delta^3 Z}{\delta J_k \delta J_m \delta J_\alpha} = (i\Delta'_{ka})(i\Delta'_{mb})(i\Delta'_{ac})\Gamma^{(3)}_{abc}$$

This means that the leg $k$ gets attached in the following way:

This always give a one–particle–irreducible graph. For example, starting from the above three point function we get the four point function as:

The five point function is obtained similarly by attaching a new leg to the four external legs, moreover we get new diagrams by attaching the new leg to the internal dressed propagator as in:
It is now clear that the recursion is verified at order \((n+1)\) and moreover that the remaining one–particle–reducible graphs form all the tree graphs that can be constructed with the irreducible vertices \(\Gamma^{(n)}\).

This means that the full quantum theory of the fields \(\phi_i\) with the Lagrangian \(L[\phi]\) is the same as the classical limit of the theory of the fields \(\Phi_i\) with the effective action \(\Gamma[\Phi]\), a fact which is easily verified by noting that for \(a \to 0\) the path integral \(\int D\Phi \exp \frac{i}{a} \{\Gamma[\Phi] + J\Phi\}\) becomes equal to \(\exp \frac{i}{a} \{\Gamma[\Phi_S] + J\Phi_S\}\) where \(\Phi_S\) is the stationary value, i.e., such that \(J = -\delta \Gamma / \delta \Phi\) hence equal to \(\exp \frac{i}{a} Z[J]\).

Of course to fully justify these considerations one still has to show that the combinatorial weights of the diagrams match correctly. The argument is similar as in the case of connected diagrams. Assume that a one–particle–reducible graph is obtained by joining two parts with respectively \(m_i\) lines \(q_i\) vertices and weight \(N_i\) with a dressed propagator. The resulting graph has \((m_1 + m_2 + 1)\) lines, \((q_1 + q_2)\) vertices and weight \(N\). Moves that contribute to \(N\) are permutation of vertices between the two parts whose number is \(\frac{(q_1 + q_2)!}{q_1!q_2!}\). All permutation of lines between the two parts, and exchange of the line which connects the two parts with one of the other correspond to different terms of the perturbative expansion due to the topology of the graph. The number of these moves is given by dividing the number \((m_1 + m_2 + 1)!\) of all permutation of lines by the number \(m_1!m_2!\) of permutations inside each part. Hence we get:

\[
N = N_1N_2\frac{(q_1 + q_2)!}{q_1!q_2!} \frac{(m_1 + m_2 + 1)!}{m_1!m_2!}
\]

so that the symmetry number of the whole is the product of the symmetry number of its parts. This completes the proof.

4 Goldstone and Higgs mechanisms
4.1 The Goldstone theorem

The Goldstone theorem concerns a theory in which a global symmetry is spontaneously broken. Let us take the simplest example of two scalar fields $\sigma$ and $\pi$ with a $U(1)$ symmetry. The corresponding Lagrangian density is:

$$L = \frac{1}{2} \left( (\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2 \right) - \frac{\mu^2}{2} (\sigma^2 + \pi^2) - \frac{\lambda}{4} (\sigma^2 + \pi^2)^2$$

and the global symmetry corresponds to the rotation of fields:

$$\begin{pmatrix} \sigma \\ \pi \end{pmatrix} \rightarrow \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \cdot \begin{pmatrix} \sigma \\ \pi \end{pmatrix}$$

Let us now assume that the mass $\mu$ has the “unphysical” value such that $\mu^2 < 0$. Then the minimal value of the potential is given by:

$$\sigma^2 + \pi^2 = -\frac{\mu^2}{\lambda}$$

In other words there is a whole circle of minima (due to the symmetry) and one has to choose one of them as the starting point of the perturbative expansion. This is called spontaneous symmetry breakdown. On the contrary for physical values of the mass there is only one minimum at $\sigma = \pi = 0$. So let us take the minimum at $\sigma = \sigma_0$ and $\pi = 0$ hence $\sigma_0 = \sqrt{-\mu^2/\lambda}$. One analyzes the theory by simply shifting the fields according to $\sigma \rightarrow \sigma_0 + \sigma$, $\pi \rightarrow \pi$, so that the vacuum now corresponds to $\sigma = \pi = 0$.

The Lagrangian density is immediately written using these variables:

$$L = \frac{1}{2} \left( (\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2 \right) + \mu^2 \sigma^2 - \lambda \sigma_0 \sigma (\sigma^2 + \pi^2) - \frac{\lambda}{4} (\sigma^2 + \pi^2)^2$$

This means that at the classical level, the field $\sigma$ has a (now physical) mass term $-2\mu^2$ while the field $\pi$ has mass 0. This is the content of the Goldstone theorem. In the situation in which a global symmetry is spontaneously broken there appear particles with no mass called Goldstone particles. Of course in realistic models of particle physics this may be useful in some circumstances (for example the pion may be seen as a Goldstone particle in the symmetric limit) but is more frequently considered catastrophic since light particles cannot be concealed and relegated to some high energy region. So it is very
important that the Goldstone theorem remains true at the quantum level. This has been proven by Goldstone [24] by using direct quantum mechanical arguments, and also by Jona–Lasinio [25] using the formalism of the effective action.

The proof runs as follows, using the notations of the previous subsection. Let us consider an infinitesimal transformation \( \delta \phi_i(x) = t_{ij} \phi_j(x) \) under which \( S[\phi] \) is invariant. If one compensates with \( \delta J_i(x) = -t_{ji} J_j(x) \) the product \( J_i \phi_i \) is invariant, hence \( Z[J] \) and \( \Gamma[\Phi] \) are also invariant. This yields the equality:

\[
-t_{ji} J_j = \delta J_i = \sum_k \int d^4x \frac{\delta J_i}{\delta \Phi_k(x)} \delta \Phi_k(x)
\]

\[
= -\sum_k \int d^4x \frac{\delta^2 \Gamma}{\delta \Phi_i \delta \Phi_k(x)} t_{kl} \Phi_l(x)
\]

So in the limit \( J \to 0 \) we get (notice that \( \int d^4x \) yields the Fourier coefficient at zero momentum):

\[
\sum_k X_{ik}(\text{momentum} = 0).t_{kl}v_l = 0
\]

Hence all transformations that effectively move the vacuum expectation value \( v \) (or that break the vacuum) also produce eigenvectors of the inverse propagator for the eigenvalue 0 at zero momentum. This means that there are as many modes developing a pole of the propagator at zero mass, i.e., Goldstone particles. In general there will be an isotropy subgroup of the vacuum \( v \) and the number of the Goldstone particles will be equal to the dimension of the quotient of the symmetry group by this isotropy subgroup.

Fortunately there is a way out of this situation when the symmetry is realized as a gauge symmetry, so that the Goldstone modes can be gauged away. This is the Higgs mechanism.

### 4.2 The Higgs mechanism

We shall explain this mechanism using the above simple model and denoting \( \phi = \sigma + i\pi \) so that the \( U(1) \) symmetry reads \( \phi \to e^{i\theta} \phi \). This symmetry can be localized, i.e., one can take for \( \theta \) a function of the space–time point \( x \) by adding a gauge field \( A_\mu(x) \) and introducing the covariant derivative \( D_\mu \phi = \partial_\mu \phi - ieA_\mu \phi \). Then under the above symmetry, \( D_\mu \phi \to e^{i\theta} D_\mu \phi \).
if $A_\mu$ transforms according to $A_\mu \rightarrow A_\mu + \frac{1}{e} \partial_\mu \theta$ (here $e$ is the “charge” or the coupling constant between $A_\mu$ and $\phi$, hence is the same for different multiplets). Using the “field–strength” $F_{\mu\nu}$ which is invariant under gauge transformation the Lagrangian density reads:

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\ast (D^\mu \phi) - \mu^2 \phi^\ast \phi - \lambda (\phi^\ast \phi)^2$$

All this situation can be generalized readily to any semisimple symmetry group (then $F_{\mu\nu}$ is covariant) following Yang and Mills [26].

Still assuming unphysical mass one chooses $<\phi>_0 = v/\sqrt{2}$ with a real $v$ such that $v = \sqrt{-\mu^2/\lambda}$. Shifting as above and expanding there seems to appear a Goldstone particle. In fact it can be gauged away as follows. One can always parametrize $\phi$ as:

$$\phi = e^{i\xi/v} \frac{v + \eta}{\sqrt{2}}$$

and perform the gauge transformation:

$$\phi \rightarrow e^{-i\xi/v} \phi = \frac{v + \eta}{\sqrt{2}} \quad A_\mu \rightarrow A_\mu - \frac{1}{ev} \partial_\mu \xi$$

Since $L$ is gauge invariant it now reads:

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu (v + \eta) D^\mu (v + \eta) - \frac{\mu^2}{2} (v + \eta)^2 - \frac{\lambda}{4} (v + \eta)^4$$

where of course the new $A_\mu$ has been used. Notice that $\xi$ has completely disappeared from the theory and one recovers a coupling:

$$\frac{1}{2} (D_\mu (v + \eta))^2 = \frac{1}{2} (\partial_\mu \eta)^2 + \frac{1}{2} e^2 A_\mu A^\mu (v^2 + 2v\eta + \eta^2)$$

Replacing $v$ by its value we see that $\eta$ has acquired a mass term $-2\mu^2$ as in the Goldstone situation but moreover the vector field $A_\mu$ which seemed to be massless has also acquired a mass $ev$. Notice that this mass can be adjusted by varying $v$ or equivalently $\lambda$, but then other particles in the theory similarly coupled to $\phi$ will acquire a mass proportional to their coupling constant. Hence the Higgs mechanism succeeds in both giving a mass to the vector field and gauging away the Goldstone particle. This is the main ingredient in the construction of the Standard model.
4.3 The Standard model

We finally describe briefly the “unified” model of weak and electromagnetic interactions due to Glashow Salam Weinberg [27]. The weak interactions are notably responsible of the decay of the neutron or the muon, and were as such described traditionally by a four fermions interaction. Such a theory presents numerous problems; so T.D. Lee and C.N. Yang long ago postulated the existence of a heavy charged vector meson $W$ mediating the weak interaction.

Unfortunately the theory of a massive vector field is also non renormalizable so it was necessary to wait till the introduction of the Higgs mechanism so as to get a consistent theory along these lines. The simplest gauge theory involving gauge mesons $W^+$ and $W^-$ is based on the group $SU(2)$ hence will also involve a neutral meson. As a matter of fact, the above authors found that it was necessary to go to the next simple situation, with a gauge group $SU(2) \times U(1)$ hence involving two neutral mesons, one of them being the photon. This produced a consistent framework for studying weak and electromagnetic interactions.

Since the gauge group is a direct product, there are two coupling constants in the model, associated to a triplet $\mathbf{A}_\mu$ and a singlet $B_\mu$. The corresponding covariant derivative is ($\mathbf{\sigma}$ are the Pauli matrices):

$$D_\mu = \partial_\mu - ig\mathbf{A}_\mu \cdot \mathbf{\sigma} - ig'B_\mu$$

while the Yang–Mills Lagrangian of the vector fields reads:

$$L_{YM} = -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}\mathbf{A}_{\mu\nu} \mathbf{A}^{\mu\nu}$$

where the field–strengths are defined as usual as:

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \quad \mathbf{A}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + g\mathbf{A}_\mu \times \mathbf{A}_\nu$$
In order to give a mass to the appropriate vectors one introduces a doublet of complex scalars, the Higgs field:

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$$

where $\phi^+$ has electric charge $+1$ and $\phi^0$ has no electric charge. One assumes that the potential is such that spontaneous symmetry breakdown occurs so that the neutral component $\phi^0$ gets a vacuum expectation value $v/\sqrt{2}$. Hence the Higgs field has to be shifted around:

$$\langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

when considering its Lagrangian:

$$L_{\text{Higgs}} = (D_\mu \phi)^* (D^\mu \phi) + L_I (\phi^* \phi)$$

where $L_I$ is some quadratic polynomial providing a mass term and a coupling term for the Higgs. The action of $SU(2) \times U(1)$ on $\phi$ has an isotropy subgroup of dimension 1 hence one can gauge away three out of the four real fields involved in $\phi$, ending with just one real electrically neutral field $h$ which is called the Higgs field (what is really physical in the Higgs field). Hence one can set

$$\phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h \end{pmatrix}$$

in the Higgs Lagrangian, which then reads:

$$L_{\text{Higgs}} = \frac{1}{2} (\partial_\mu h)^2 + \frac{1}{4} g^2 (v + h)^2 W^+ W^- + \frac{1}{8} (v + h)^2 (g' B_\mu - g A^3_\mu)^2$$

where we have introduced as usual $W^\pm_\mu = (A^1_\mu \mp i A^2_\mu)/\sqrt{2}$.

One finds the mass matrix of the vectors by keeping the quadratic terms $(v + h)^2 \to v^2$ in the form

$$\frac{1}{8} v^2 \left\{ (g' B_\mu - g A^3_\mu)^2 + 2 g^2 W^+ W^- \right\}$$

This means that the charged vector meson acquires a mass (or has a mass term $M_W^2 W^+_\mu W^-_{\mu'}$)

$$M_W = \frac{1}{2} g v = M_{W^+} = M_{W^-} \quad (4.1)$$
while the neutral ones are still mixed. One diagonalizes the mass matrix by performing a rotation of fields through the so–called Weinberg angle $\theta_W$ defining:

\[
A_\mu = \cos \theta_W B_\mu + \sin \theta_W A_\mu^3
\]

\[
Z_\mu = \sin \theta_W B_\mu - \cos \theta_W A_\mu^3
\]

(4.2)

Since this is a rotation it does not affect the form of the kinetic term (invariance of the quadratic form) and one chooses $\theta_W$ so that $(g' B_\mu - g A_\mu^3)$ be proportional to $Z_\mu$ hence:

\[
\cos \theta_W = \frac{g}{\sqrt{g^2 + g'^2}} \quad \sin \theta_W = \frac{g'}{\sqrt{g^2 + g'^2}}
\]

(4.3)

Then obviously $Z_\mu$ acquires a mass (or has a mass term $M_Z^2 Z_\mu Z_\mu / 2$)

\[
M_Z = \frac{1}{2} \sqrt{g^2 + g'^2} \nu
\]

(4.4)

while $A_\mu$ remains massless and is identified to the photon. Of course by developing the Yang–Mills Lagrangian $L_{YM}$ with these notations one immediately finds trilinear and quadrilinear couplings between all these vectors, including the correct coupling (with charge $e$: the electronic charge) of the photon field $A_\mu$ with the charged bosons $W_\mu^\pm$ while the neutral $Z_\mu$ remains uncoupled to $A_\mu$. Notice that the previous equations lead to a relation between $M_W$ and $M_Z$:

\[
\rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta_W} = 1
\]

As it happens $\rho$ can be measured as the ratio of low energy interactions of charged and neutral currents and the deviations of $\rho$ to one, due to radiative corrections, are a sensitive test of the theory.

It now remains to introduce the fermions in the model. We shall present the standard example of the electron and its neutrino. One first decomposes the fermionic fields into chirality components:

\[
L = \frac{1 - \gamma^5}{2} \quad R = \frac{1 + \gamma^5}{2} \quad e_L = L e \quad e_R = R e \quad \nu_L = L \nu = \nu \quad \nu_R = 0
\]
and one affects the \( L \) components to a doublet of \( SU(2) \) with “hypercharge” \(-1/2\) under the \( U(1) \) while \( e_R \) is a singlet of \( SU(2) \) with hypercharge \(-1\). So the left doublet, on which \( SU(2) \) acts is:

\[
\psi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}
\]

and considering the covariant derivatives one immediately sees that the electron is correctly coupled to the photon while the neutrino is neutral:

\[
L_F = \bar{\psi}_L i \gamma^\mu D_\mu \psi_L + \bar{e}_R i \gamma^\mu D_\mu e_R
\]

reads after substituting the above definitions:

\[
L_F = \bar{e}(i \Phi - e \mathcal{A})e + \bar{\nu} i \Phi \nu + L_{(W,Z,\psi)}
\]

where \( L_{(W,Z,\psi)} \) contains the interactions of the fermions with \( W \) and \( Z \). For example one gets

\[
\frac{g}{\sqrt{2}} \left\{ \bar{\nu}_L W^+ \gamma^\mu e_L + \bar{e}_L W^- \gamma^\mu \nu_L \right\}
\]

which precisely describes the weak interaction mediated by the \( W^\pm \) as described above. One sees that its coupling constant is \( g/\sqrt{2} \). Hence in the low energy limit one recovers the four fermions interaction with the Fermi coupling constant:

\[
\left( \frac{G_F}{\sqrt{2}} \right) = \frac{g^2}{8M_W^2}
\]

Moreover one identifies the electric charge in equation (4.5) as:

\[
e = \sin \theta_W \cos \theta_W \sqrt{g^2 + g'^2}
\]

so that all the parameters of the theory are related to experimental quantities. The computation in equation (4.5) also produces the couplings of the fermions to the \( Z \) field, which are slightly more complicated and we shall not write them.

The Higgs mechanism also gives masses to the fermions. It is only necessary to introduce a Yukawa coupling between the Higgs and the fermions in the form:

\[
L_{HF} = -c \left[ \bar{\psi}_L \phi \psi_R + \bar{\psi}_R \phi^* \psi_L \right]
\]
in which $c$ is some coupling constant and we recall that $\psi_L$ is a doublet contracted with the doublet of Higgses $\phi$ while $\psi_R$ is a scalar, hence the whole stuff is invariant under the gauge group. When gauging away the unphysical Higgses this Lagrangian boils down to $-c(v + h)\bar{e}e/\sqrt{2}$ hence the electron acquires a mass $m_e = cv/\sqrt{2}$ while the neutrino remains massless. One can always adjust the coupling $c$ so as to obtain the correct mass $m_e$ of the electron, and we see that the coupling of the Higgs to such a particle will always be proportional to its mass (in particular negligible for light particles). One can give arbitrary masses to the two components of the doublet by also considering the charge conjugate of the doublet $\psi_L$. This is the mechanism that is used to give masses to quarks. When Majorana neutrinos are considered the procedure becomes extremely messy, and a large number of more or less arbitrary constants enter the game.

At this point near all the ingredients of the Standard model Lagrangian have been introduced. What is missing is the gauge fixing term and the associated Faddeev–Popov Lagrangian. One then faces a model in which consistent radiative corrections can be computed. As a matter of fact the model is in very good agreement with experiments, and the question of whether these radiative (i.e., quantum) corrections can be seen within present experimental precision is still controversial [28]. Other points which are still under active consideration are the question of masslessness of neutrinos (are they Weyl or Majorana neutrinos), the quest for the top quark which is necessary to avoid anomalies, hence keep a consistent quantum theory, and for the Higgs which still eludes experimental evidence. From the theoretical viewpoint a more embracing and symmetrical theory of strong, weak and electromagnetic interactions is still lacking.

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