Index hypergeometric integral transform

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This is a brief overview of the index hypergeometric transform (other terms for this integral operator are: Olevskii transform, Jacobi transform, generalized Mehler–Fock transform). We discuss applications of this transform to special functions and harmonic analysis. The text is an addendum to the Russian edition of the book by G. E. Andrews, R. Askey, and R. Roy, Special Functions, Encycl. of Math. Appl. 71, Cambridge Univ. Press, 1999.

As is well-known, the continuous analog of Fourier series is the Fourier transform. It turns out that expansions in the Jacobi polynomials also have a continuous analog, namely the integral ‘Jacobi transform’ (the terminology is discussed below). The theory of this transform is rich, but there is no detailed modern exposition of this topic in the existing literature, this text also has not such a purpose. Numerous additional facts are contained in works [1], [3], [6], [13]–[15], [22]–[23], [35].

There is a well-known Askey–Wilson hierarchy of hypergeometric orthogonal polynomials (see [11]). There is a parallel hierarchy of hypergeometric integral transforms, see [2], [8], [3], [12], [20], [24]–[25], [34], on multidimensional analogs, see [9], [18], [1]. Our topic is neither the simplest object of this hierarchy (certainly the Hankel transform and the Kontorovich–Lebedev transform are simpler), nor the most complicated (see, for instance [8]). It is sufficiently simple to be versatile tool for special functions (see below Section 2), on the other hand it controls harmonic analysis on hyperbolic symmetric spaces (i.e., Lobachevsky spaces and their complex and quaternionian analogs), we briefly discuss this in Section 4, for more details see [12], [22].

1 The index hypergeometric transform

1.1. The Jacobi polynomials. Consider the Jacobi orthogonal system on the segment [0, 1],

\[ P_n^{\alpha,\beta}(x) = \frac{(-1)^n \Gamma(n + \beta + 1)}{\Gamma(\beta + 1)n!} \, _2F_1\left[ -n, n + \alpha + \beta + 1; \beta + 1 \right]. \]

We have

\[ \gamma_n := \|P_n^{\alpha,\beta}\|^2 = \]

\[ = \int_0^1 P_n^{\alpha,\beta}(x)^2 x^\beta (1 - x)^\alpha \, dx = \frac{\Gamma(n + \alpha + 1)\Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1)n!\Gamma(n + \alpha + \beta + 1)}. \quad (1.1) \]

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For a function \( f(x) \) consider the numbers ('the Fourier coefficients') defined by the formula
\[
c_n(f) := \int_0^1 f(x) P_n^{\alpha,\beta}(x) x^\alpha (1-x)^\beta \, dx.
\]
Then the function \( f(x) \) can be restored by the formula
\[
f(x) = \sum_{n=0}^{\infty} \frac{c_n(f)}{\gamma_n} P_n^{\alpha,\beta}(x).
\] (1.2)
Moreover, the following 'Plancherel formula' holds:
\[
\int_0^1 f(x) g(x) x^\beta (1-x)^\alpha \, dx = \sum_{n=0}^{\infty} \frac{1}{\gamma_n} c_n(f) c_n(g).
\]
The expansion in the Jacobi polynomials has a continuous analog, in which series are replaced by integrals.

1.2. The index hypergeometric transform. Let \( b, c > 0 \). For a function \( f \) on the half-line \([0, \infty)\) we define a function of the variable \( s \geq 0 \) by
\[
J_{b,c} f(s) = [\hat{f}]_{b,c}(s) = \frac{1}{\Gamma(b+c)} \int_0^\infty \Gamma(b+c-1+y) \Gamma(b+c+y) \times \Gamma(b+c+y) \times \Gamma(2y) \times ds.
\] (1.3)

**Theorem 1.1**
a) The operator \( J_{b,c} \) is a unitary operator
\[
J_{b,c} : L^2([0, \infty), x^{b+c-1}(1+x)^{b-c} \, dx) \to L^2([0, \infty), \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \, ds).
\]
In other words, the following Plancherel formula holds
\[
\int_0^\infty f_1(x) f_2(x) x^{b+c-1}(1+x)^{b-c} \, dx = \int_0^\infty [\hat{f}_1]_{b,c}(s) [\hat{f}_2]_{b,c}(s) \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \, ds.
\]
b) The inverse operator is given by the formula
\[
f(x) = \frac{1}{\Gamma(a+b)} \int_0^{\infty} [\hat{f}]_{b,c}(s) 2F_1 \left[ b+is, b-is; -x \right] \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)} \, ds.
\] (1.4)
Notice that the statement b) follows from a), because for a unitary operator \( U \) we have \( U^{-1} = U^* \).

As in the case of the Fourier transform, we have a question about a precise definition. For instance, we can say that the integral transform \( J_{b,c} \) is defined on functions with compact support, next we extend it by continuity to a unitary operator defined in the space \( L^2 \).
1.3. Holomorphic extension to a strip.

**Lemma 1.2** Let $f$ be integrable on $\mathbb{R}_+$ and

$$f(x) = o(x^{-\alpha - \epsilon}), \quad x \to +\infty,$$

where $\epsilon > 0$. Then $[\hat{f}(s)]_{b,c}$ is holomorphic in the strip

$$|\text{Im } s| < \alpha - b$$

and satisfy the condition $\hat{f}(-s) = \hat{f}(s)$ in the strip.

**Proof.** This follows from the asymptotics for the hypergeometric function, (see [4], V. 1, (2.3.2.9)) as $x \to +\infty$:

$$2F_1(b + is, b - is; b + c; -x) = \lambda_1 x^{-b + is} + \lambda_2 x^{-b - is} + O(x^{-b + is - 1}) + O(x^{-b - is - 1})$$

where $2is \notin \mathbb{Z}$, and $\lambda_1, \lambda_2$ are certain constants (for $2is \in \mathbb{Z}$ there arises an additional factor $\ln x$ at the leading term). □

1.4. The operator calculus. Denote by $D$ the hypergeometric differential operator

$$D := -x(x + 1) \frac{d^2}{dx^2} - \left[ (c + b) + (2b + 1)x \right] \frac{d}{dx} + b^2$$

(1.5)

(in comparision with the common notation we replaced $x$ by $-x$). The hypergeometric functions in (1.3) are (generalized) eigenfunctions of the operator $D$:

$$D \ 2F_1 \left[ b + is, b - is; b + c; -x \right] = -s^2 \ 2F_1 \left[ b + is, b - is; b + c; -x \right].$$

(1.6)

It is easy to see that $D$ is formally self-adjoint in the following sense

$$\int_0^\infty Df_1(x) \cdot f_2(x) x^{b+c-1}(1 + x)^{b-c} dx = \int_0^\infty f_1(x) \cdot Df_2(x) x^{b+c-1}(1 + x)^{b-c} dx$$

where $f_1, f_2$ are smooth compactly supported functions on $(0, \infty)$ (in fact $D$ is essentially self-adjoint, see below) and Theorem 1.1 is a theorem about expansion of $D$ in eigenfunctions.

**Theorem 1.3** Let $f, Df \in L^2$, then

$$[\tilde{D} \hat{f}(s)]_{b,c} = -s^2 \hat{f}(s).$$

(1.7)

**Proof.** This is a rephrasing of formula (1.6).

**Theorem 1.4** Let a function $f$ be continuous on $\mathbb{R}^+$ and satisfies the condition

$$f(x) = o(x^{-b - 1 - \epsilon}); \quad x \to +\infty.$$  

Then

$$[\hat{xf}(x)]_{b,c} = P[f(x)]_{b,c},$$

(1.9)
where the difference operator $Pg$ is given by

$$Pg(s) = \frac{(b-is)(c-is)}{(-2is)(1-2is)}(g(s+i)-g(s)) + \frac{(b+is)(c+is)}{(2is)(1+2is)}(g(s-i)-g(s)).$$

(1.10)

Remark. Emphasize amusing characteristics of this theorem.

1. The operator $P$ is a difference operator, but a shift $s \mapsto s+i$ is in the imaginary direction, and integration is along the real axis.

2. The transformation $J_{b,c}^{-1}$ send the operator $P$ to the operator of multiplication by a function $x$, i.e., our operator $J_{b,c}^{-1}$ determines a spectral decomposition of the difference operator $P$. For several examples of spectral decompositions of difference operators in imaginary direction, see [26], [8].

3. The operator $P$ is similar to difference operators, related to Wilson, continuous Hahn, continuous dual Hahn, Meixner–Pollachek orthogonal polynomials, see [AAR], formulas (6.10.6), (6.10.9), (6.10.12) and Problem 6.37.c (see also [11]). The rational coefficients of the operator $P$ are 'catenated' with the $\Gamma$-factors in the formula (1.4).

Proof. This is reduced to a verification of the identity

$$P 2F_1 \left[ \begin{array}{c} b+is, b-is \\ b+c \end{array}; x \right] = x 2F_1 \left[ \begin{array}{c} b+is, b-is \\ b+c \end{array}; -x \right].$$

Theorem 1.5 Let $f$ and $f'$ be continuous and decrease as (1.8). Then

$$[x(x+1) \frac{d}{dx} f]_{b,c} = H[\hat{f}]_{b,c},$$

(1.11)

where the difference operator $H$ is given by

$$Hg(s) = \frac{(b-is)(b+1-is)(c-is)}{(-2is)(1-2is)}(g(s+i)-g(s)) + \frac{(b+is)(b+1+is)(c+is)}{(+2is)(1+2is)}(g(s-i)-g(s)) - (b+c)g(s).$$

(1.12)

Proof. We evaluate $J_{b,c}$-image of the commutator $[x, D]$.

1.5. Historical remarks. The transformation $J_{1/2,1/2}$ was introduced by F. G. Mehler [19] in 1881. He presented the inversion formula without proof (it has to be said that the formula is not obvious at all). A proof was published by V. A. Fock [7] in 1943. As a result, the transformation $J_{1/2,1/2}$ is called the Mehler–Fock transform. The general transformation $J_{b,c}$ was introduced by H. Weyl in 1910 the work [33] on the spectral theory of differential operators. It seems that this result have not met the eye. Again this transform had appeared in the book of Titchmarsh [31] in 1946. In 1949 the transform was rediscovered by M. N. Olevsky [27], apparently this was related to his works on multi-dimensional Lobachevsky spaces.

The most common terms for $J_{b,c}$ are the Olevsky transform and the Jacobi transform (introduced by T. Koornwinder).
2 Applications to special functions

Our first aim is to evaluate index hypergeometric transforms of some functions. We do this in Subsection 2.2 by the Mellin transform. Next, in 2.3-2.4 we demonstrate effectivity of the index transform as a tool of theory of special functions.

2.1. The Mellin transform. Let \( f(x) \) be a function defined on the half-line \( x > 0 \). Its Mellin transform is defined by the formula

\[
F(s) = \mathcal{M}f(s) := \int_0^\infty f(x) x^s \frac{dx}{x}.
\]

The domain of absolute convergence of this integral is a certain vertical strip of the form \( u < \Re s < v \), the function \( F(s) \) is holomorphic in this strip, the boundaries of the strip can belong or do not belong the domain of convergence; a strip can be degenerated to a vertical line of the form \( \Re s = u \). Certainly, it can be empty.

The Mellin transform is a unitary operator from \( L^2(\mathbb{R}, dx/x) \) to \( L^2 \) on vertical line \( \Re s = 1/2 \). In particular,

\[
\int_0^\infty f(x)g(x) \frac{dx}{x} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(1/2 + is)G(1/2 + is) ds.
\]

Recall the theorem about convolution. If the domains of definition of \( F(s) = \mathcal{M}f(s) \) and \( G(s) = \mathcal{M}g(s) \) have an intersection (a strip or a line), then \( \mathcal{M} \) send the multiplicative convolution

\[
f \ast g(x) := \int_0^\infty f(y)g(x/y) \frac{dy}{y}
\]
to \( F(s)G(s) \) (on the common domain of definition).

Certainly, the Mellin transform is reduced to the Fourier transform by a substitution \( x = e^y \). From the point of view of abstract theory there is no difference between the Mellin transform and the Fourier transform. But their role in theory of special functions is different.

2.2. A game in the Mellin transform. A short table of index transforms. Since we will meet long products of \( \Gamma \)-functions, we will use the following notation

\[
\Gamma \left[ \begin{array}{c} a_1, \ldots, a_k \\ b_1, \ldots, b_l \end{array} \right] := \frac{\Gamma(a_1) \cdots \Gamma(a_k)}{\Gamma(b_1) \cdots \Gamma(b_l)}.
\]

Consider arbitrary convergent Barnes-type integral

\[
\frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \Gamma \left[ \begin{array}{c} a_1 + s, \ldots, a_k + s, b_1 - s, \ldots, b_l - s \\ c_1 - s, \ldots, c_m - s, d_1 + s, \ldots, d_n + s \end{array} \right] x^s \frac{ds}{s}.
\]

It can be represented as a linear combination of hypergeometric functions \( pF_q \) with \( \Gamma \)-coefficients. The idea is explained in the book [AAR], Section 2.4. A
calculation requires a watching of some asymptotics, but it can be done once and forever in ‘general case’. The final rules can be found in [30], [28].

On the other hand there are unexpectedly many cases when the integral admits a simpler expression than the result of the general algorithm, see the tables of Prudnikov, Brychkov, Marichev, v.3, [28], Chapter 8 (I do not know rational explanations of this phenomenon).

Now we evaluate two auxiliary integrals.

**Lemma 2.1**

\[
\int_0^\infty \frac{x^{\alpha-1}}{(x+z)^\rho} \, x^r F_1(p,q;r;-x) \, dx = \frac{z^{\alpha-\rho}}{2\pi i} \Gamma \left[ \frac{r}{p,q,\rho} \right] \int_{-i\infty}^{i\infty} \Gamma \left[ \frac{t + \alpha, \rho - t - \alpha, p + t, q + t - t}{r + t} \right] z^t \, dt. \tag{2.1}
\]

\[
\int_0^\infty x^{\alpha-1} \, \frac{2F_1\left[p,q;\omega x\right]}{x^{r-w}} \, \frac{2F_1\left[u,v;\tilde{\omega} x\right]}{x^{c-b}} \, dx = \frac{\omega^{-\alpha}}{2\pi i} \Gamma \left[ \frac{r, w}{u, v, p, q} \right] \int_{-i\infty}^{i\infty} \Gamma \left[ \frac{\alpha + t, u + t, v + t, p - \alpha - t, q - \alpha - t, t}{r - \alpha - t, w + t} \right] \left(\frac{\omega}{\tilde{\omega}}\right)^{-t} \, dt. \tag{2.2}
\]

**Proof.** To be definite, we evaluate the first integral. The Mellin transform of the function \( f(x) := x^{\alpha-1}/(x+z)^\rho \) is \( B(s + \alpha, \rho - s - \alpha)z^{s+\alpha-\rho} \). The Mellin transform of \( g(x) := 2F_1(p,q;r;-x) \) is evaluated in [AAR], Section 2.4, and is a product of \( \Gamma \)-functions. Our integral is a convolution of \( xf(1/x) \) and \( g(x) \). Next, we observe that the Mellin transform of the function \( xf(1/x) \) is \( F(1-s) \). It remains to apply the theorem about convolution. \( \blacksquare \)

Thus, our calculations was reduced to an rearrangement of \( \Gamma \)-functions.

In the right hand side there are Barnes integrals, which can be represented as linear combinations of functions \( 3F_2 \) and \( 4F_3 \) respectively. We will not write them, instead of this we notice that for some values of the parameters \( \Gamma \)-factors in the right hand sides can cancel.

**Lemma 2.2** The transform \( J_{b,c} \) send

\[
(1 + x)^{-a-c} \rightarrow \frac{\Gamma(c + is)\Gamma(c - is)}{\Gamma(c + a)\Gamma(c + b)}; \tag{2.3}
\]

\[
\frac{(1 + x)^{b-a}}{(x + z)^{c+b}} \rightarrow \Gamma \left[ \frac{c + is, c - is}{c + a, c + b} \right] 2F_1 \left[ \frac{c + is, c - is}{c + a}; 1 - z \right]; \tag{2.4}
\]

\[
x^{-u-a} \rightarrow \frac{\Gamma(-u + b)}{\Gamma(a + u)} \cdot \frac{\Gamma(u + is)\Gamma(u - is)}{\Gamma(b + is)\Gamma(b - is)} \tag{2.5}
\]
\[ 2F_1 \left[ \frac{p + b, q + b}{a + b}; -\frac{x}{y} \right] (1 + x)^{b-a} \rightarrow \]
\[ y^{b-q} \Gamma \left[ \frac{a + b}{p + q, p + b, q + b} \right] \Gamma \left[ \frac{p + i s, p - i s, q + i s, q - i s}{a + i s, a - i s} \right] 2F_1 \left[ \frac{p + i s, p - i s}{p + q}; 1 - y \right]; \]

(2.6)

\[ 2F_1 \left[ \frac{p + b, q + b}{a + b}; -x \right] (1 + x)^{b-a} \rightarrow \]
\[ \Gamma \left[ \frac{a + b}{p + q, p + b, q + b} \right] \Gamma \left[ \frac{p + i s, p - i s, q + i s, q - i s}{a + i s, a - i s} \right]; \]

(2.7)

\[ 2F_1 \left[ \frac{a + c, a + d}{a + b + c + d}; -x \right] \rightarrow \]
\[ \frac{\Gamma(a + b + c + d) \Gamma(c + i s) \Gamma(d + i s) \Gamma(d - i s)}{\Gamma(a + c) \Gamma(a + d) \Gamma(b + c) \Gamma(b + d) \Gamma(c + d)}. \]

(2.8)

**Proof.** We look to the right hand side of (2.2). If \( \alpha = r \), then two \( \Gamma \)-factors cancel. The rest is the integral representation of \( 2F_1 \). This gives the second formula. Substituting \( z = 1 \) we get the first formula.

Next, if \( z = 1, r = p + q + \rho \), then we get one of Barnes integrals in the right hand side (see Theorem 2.4.3 of [AAR]). This gives us (2.5).

Further, we watch possible simplifications in the right hand side of (2.2). After the substitution \( \alpha = w = r \) we get a cancelation of four \( \Gamma \)-factors. This gives formula (2.6). Substituting \( y = 1 \) to (2.6) we get (2.7).

To verify (2.8), we observe that two \( \Gamma \)-factors in the right hand side of (2.2) cancel, and we again apply Theorem 2.4.3 of [AAR].

**2.3. Game in the Plancherel formula.** Thus we wrote a short table with 6 row for the transform \( J_{b,c} \). Applying the Plancherel formula for \( J_{b,c} \) we can get an amusing collection of integrals. We present several examples.

a) The De Branges–Wilson integral. Applying the Plancherel formula for pair of functions \((1 + x)^{-a-c}\) and \((1 + x)^{-a-d}\), we after a trivial calculation we get the De Branges–Wilson integral (see Section 3.6 of [AAR]), this derivation is due to Koornwinder, [15]. Recall that its is given by

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\prod_{k=1}^{4} \Gamma(a_k + i s)}{\Gamma(2 i s)} \right|^2 ds = \frac{\prod_{1 \leq k < l \leq 4} \Gamma(a_k + a_l)}{\Gamma(a_1 + a_2 + a_3 + a_4)}. \]

(2.9)

b) Another beta-integral. Applying the Plancherel formula to \( x^{-u-a}, x^{-v-a} \), we get the integral

\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\prod_{k=1}^{3} \Gamma(a_k + i s)}{\Gamma(2 i s) \Gamma(b + i s)} \right|^2 ds = \frac{\Gamma(b - a_1 - a_2 - a_3) \prod_{1 \leq k < l \leq 3} \Gamma(a_k + a_l)}{\prod_{k=1}^{3} \Gamma(b - a_k)}. \]
c) An integral representation for $\, _3F_2(1)$. The pair of functions $(1 + x)^{-a - e}$ and $\,_2F_1 \begin{bmatrix} a + c, a + d \\ a + b + c + d \end{bmatrix} - x$ gives the following integral representation of $\, _3F_2(1)$,

$$\frac{1}{\pi} \int_{0}^{\infty} \frac{\left| \Gamma(a + is)\Gamma(b + is)\Gamma(c + is)\Gamma(d + is)\Gamma(e + is) \right|^2 ds}{\Gamma(2is)} = \frac{\Gamma(a + b + c + d)\Gamma(a + b + c + e)}{\Gamma(a + b + c + d)\Gamma(a + b + c + e)} \times \, _3F_2 \begin{bmatrix} a + c, b + c, a + b \\ a + b + c + d, a + b + c + e \end{bmatrix} 1 \right]. \quad (2.10)$$

The left hand side is symmetric with respect to the parameters, therefore the right hand side also is symmetric. This symmetry is the Kummer identity (see [AAR], Corollary 3.3.5).

d) Adding a $\Gamma$-factor to the numerator. Applying the Plancherel formula to the pair

$$\,_2F_1 \begin{bmatrix} a + c, a + d \\ a + b + c + d \end{bmatrix} - x$$ and $\,_2F_1 \begin{bmatrix} a + e, a + f \\ a + b + e + f \end{bmatrix} - x$,

we get the identity

$$\frac{1}{\pi} \int_{0}^{\infty} \frac{\left| \Gamma(a + is)\Gamma(b + is)\Gamma(c + is)\Gamma(d + is)\Gamma(e + is) \right|^2 ds}{\Gamma(2is)} = \frac{1}{2\pi i} \Gamma(a + c)\Gamma(a + d)\Gamma(c + d)\Gamma(b + e)\Gamma(b + f)\Gamma(e + f) \times \int_{-\infty}^{+\infty} \Gamma \begin{bmatrix} a + b + s, a + e + s, a + f + s, d - a - s, c - a - s, -s \\ c + d - s, a + b + e + f + s \end{bmatrix} ds. \quad (2.11)$$

The right hand side is a linear combination of three functions $\, _4F_3(1)$ with $\Gamma$-coefficients. By the way a Barnes integral can be regarded as a final answer.

e) Adding a $\Gamma$-factor to the denominator. Now we apply the Plancherel formula to the pair

$$\, _2F_1 \begin{bmatrix} p + b, q + b \\ a + b \end{bmatrix} - x (1 + x)^{b-a}$$ and $\, _2F_1 \begin{bmatrix} u + b, v + b \\ a + b \end{bmatrix} - x (1 + x)^{b-a}$. 

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We omit intermediate calculations and present the final result

$$\frac{1}{\pi} \int_0^\infty \left| \frac{\Gamma(b + is)\Gamma(p + is)\Gamma(q + is)\Gamma(u + is)\Gamma(v + is)}{\Gamma(2is)\Gamma(a + is)} \right|^2 ds =$$

$$= \frac{1}{2\pi i} \Gamma \left[ u + v, p + q, p + b, q + b \atop a - v, u - v \right] \times$$

$$\times \int_{-i\infty}^{i\infty} \Gamma \left[ u + p + s, u + q + s, b + u + s, a - v + s, v - u - s, -s \atop u + a + s, u + b + p + q + s \right].$$

(2.12)

Two last identities are not as aesthetic as previous. However, consider two special cases of the last integral.

f) **The Nassrallah–Rahman integral.** In the last integral we set $a = b + u + v + p + q$ (this leads to cancellation of $\Gamma$-factors) and applying Theorem 2.4.3 of \text{[AAR]} (after changing notation), we obtain the Nassrallah–Rahman integral (its $q$-version is present in the book \text{[AAR]}, Theorem 10.8.2)

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \prod_{j=1}^5 \Gamma(a_j + is) \right|^2 ds = 2 \prod_{k=1}^5 \Gamma(a_k + a_l) \prod_{1 \leq k < l \leq 5} \Gamma(a_k + a_l).$$

(2.13)

The left hand side of (2.12) is symmetric with respect to the parameters $b, p, q, u, v$, but the right hand side in the form (2.13) does not looks as symmetric. This gives a symmetry relation for $4F_3$ in the form "a linear combination of four summands equals 0". This is the "nonterminating Whipple identity". Its "terminating version" (Theorem 3.3.3 of \text{[AAR]}) can obtained by the substitution

\[W_n(a, b, c, d; s^2) = (a+b)n(a+c)n(a+d)/a+b+a+c+a+d\] of the weight

\[w(s) = \frac{1}{\pi} \left| \frac{\Gamma(a + is)\Gamma(b + is)\Gamma(c + is)\Gamma(d + is)}{\Gamma(2is)} \right|^2.\]
\[ a = v - m \] with integer \( m \), then two summands disappear due factors \( \Gamma(-m) \)
in the denominators (this identity is the symmetry of the Wilson polynomials
with respect to the parameters).

h) **One again extension of the De Branges–Wilson integral.** Applying the
Plancherel formula to \((1 + x)^{b-a}(1 + x + y)^{-b-c}\) and \((1 + x)^{b-a}(1 + x + y)^{-b-d}\),
we get

\[
\frac{1}{\pi} \int_0^\infty \left| \frac{\Gamma(a + is)\Gamma(b + is)\Gamma(c + is)\Gamma(d + is)}{\Gamma(2is)} \right|^2 \times \int_0^\infty \frac{2F_1 \left[ \begin{array}{c} c - is, c + is \\ a + c \end{array} ; -y \right]}{2F_1 \left[ \begin{array}{c} d - is, d + is \\ a + d \end{array} ; -y \right]} ds = (2.14)
\]

\[
= \frac{\pi \Gamma(a + b)\Gamma(a + c)\Gamma(b + c)\Gamma(b + d)\Gamma(c + d)}{\Gamma(a + b + c + d)} \times \int_0^\infty \Gamma(a + is)\Gamma(b + is)\Gamma(c + is) \Gamma(2is) \int_0^\infty \frac{2F_1 \left[ \begin{array}{c} c - is, c + is \\ a + c \end{array} ; -y \right]}{2F_1 \left[ \begin{array}{c} d - is, d + is \\ a + d \end{array} ; -y \right]} ds.
\]

**2.4. A derivation of the orthogonality relations for Wilson polynomials.**

Now we derive \( J^{-1}_n \)-image of the function

\[ |\Gamma(a + is)|^2 W_n(s^2), \quad \text{where } W_n(s^2) = W_n(a, b, c, d; s^2) \text{ is a Wilson polynomial.} \]

We must evaluate the integral

\[
\frac{1}{\Gamma(b + c)} \int_0^\infty \frac{\Gamma(a + is)\Gamma(b + is)\Gamma(c + is)\Gamma(d + is)}{\Gamma(2is)} \Gamma(a + c) \Gamma(b + c) \Gamma(b + d) \Gamma(c + d) \Gamma(2is) \int_0^\infty \Gamma(a + is)\Gamma(b + is)\Gamma(c + is) \Gamma(2is) \left( \begin{array}{c} b + is, b - is \\ b + c \end{array} ; -x \right) W_n(s^2) ds =
\]

\[
= \frac{(a + b)_n(a + c)_n(a + d)_n}{\Gamma(b + c)} \sum_{k=0}^n \frac{\left( -n \right)_k (n + a + b + c + d - 1)_k (a + is)_k (a - is)_k}{k!(a + b)_k(a + c)_k(a + d)_k} ds.
\]

We get a linear combination of known for us (in virtue of the inversion formula
and \( 2.3 \)) integrals of the type

\[
\int_0^\infty \frac{\Gamma(a + k + is)\Gamma(b + is)\Gamma(c + is)\Gamma(d + is)}{\Gamma(2is)} \Gamma(2is) \left( \begin{array}{c} b + is, b - is \\ b + c \end{array} ; -x \right) ds = \Gamma(a + b + k)\Gamma(a + c + k) \frac{\Gamma(a + k + is)\Gamma(b + is)\Gamma(c + is)}{(1 + x)^{a+b}}.
\]

As a result we get

\[
\Gamma(a + b)\Gamma(a + c)(a + b)_n (1 + x)^{-a-b} \frac{\Gamma(c + d - 1)}{(a + d)_n} \int_0^\infty \frac{\Gamma(a + k + is)\Gamma(b + is)\Gamma(c + is)\Gamma(d + is)}{\Gamma(2is)} \left( \begin{array}{c} b + is, b - is \\ b + c \end{array} ; -x \right) ds = \Gamma(a + b + k)\Gamma(a + c + k) \frac{\Gamma(a + k + is)\Gamma(b + is)\Gamma(c + is)}{(1 + x)^{a+b}}.
\]

This is a Jacobi polynomial of the variable \( 1/(1 + x) \).

\(^3\)Another simple derivation is contained in \( 2.3 \).
Transposing $a\, c\, d$, we evaluate the inverse index transform of $|\Gamma(d + is)|^2 W_m(s^2)$. Next we evaluate integral (see formula (3.8.3) of [AAR])

$$\frac{1}{\Gamma(b + c)} \int_0^\infty \left| \frac{\Gamma(a + is)\Gamma(b + is)\Gamma(c + is)\Gamma(d + is)}{\Gamma(2is)} \right|^2 W_n(s^2) W_m(s^2) ds$$

applying the Plancherel formula. We get

$$\text{const} \int_0^\infty 2F_1 \left[ -n, n + a + b + c + d - 1; \frac{1}{1 + x} \right] \times 2F_1 \left[ -m, m + a + b + c + d - 1; \frac{1}{1 + x} \right] x^{b+c-1}(1 + x)^{-a-d-c} dx.$$ 

Passing to the variable $y = 1/(1 + x)$, we get the integral, which express the orthogonality relations for the Jacobi polynomials.

At the first glance this proof of orthogonality can be invented only if we know the final result. Really this calculation gives us the following observation:

*The Wilson orthogonal system with $a = d$ is the image of the Jacobi system under the index transform (11).*

Notice that the index transform was discovered in 1910 and became well-known up to 1950, therefore it seems strange that the Wilson polynomials were discovered so late (1980).

## 3 Derivation of the inversion formula. Jump of resolvent

Many ways of derivations are known, see [14]. In particular, we can decompose the index transform as a product of simpler integral transforms and apply inversion formulas for the factors. However, the original way of Weyl based on spectral theory seems the most natural up to now (see, e.g., [3], §13.8, or [31]). We present a version of derivation using a minimum of theory but requiring superfluous calculations. In details, the spectral theory of differential operators is exposed in Titchmarsh [31], Dunford, Schwartz [3], Chapters XII–XIII, and Naimark [21].

### 3.1. Jump of the resolvent

Recall the Spectral Theorem. Consider a finite or countable collection of measures $\mu_1, \mu_2, \ldots$ on $\mathbb{R}$, the Hilbert space

$$V[\mu] := \oplus_{j} L^2(\mathbb{R}, \mu_j),$$

and an operator $Z_\mu : V[\mu] \to V[\mu]$ given by the formula

$$[Z_\mu(f_1 \oplus f_2 \oplus \ldots)](x) = x f_1(x) \oplus x f_2(x) \oplus \ldots$$

**Theorem 3.1** For any self-adjoint (generally, unbounded) operator in a Hilbert space $H$ there exists a collection of $\mu_j$ and a unitary operator $U : H \to V[\mu]$ such that $A = U^{-1} Z_\mu U$
For any Borel subset $M \subset \mathbb{R}$ consider the subspace $W(M) \subset V[\hat{\mu}]$ of functions, which equals 0 outside the set $M$. Define the spectral subspace $\Omega(M) := U^{-1}W(M)$. Denote by $P[\Omega]$ the projection operator to this subspace.

**Proposition 3.2** For any finite interval $(a, b) \subset \mathbb{R},$

$$P[(a, b)] = \frac{1}{2\pi i} \lim_{\delta \to +0} \lim_{\epsilon \to +0} \int_{a+\delta}^{b-\delta} ((\lambda - i\epsilon - A)^{-1} - (\lambda + i\epsilon - A)^{-1}) \, d\lambda.$$  

The limit here is the limit in the strong operator topology, $T_n \to T$ if for any vector $v$ we have $\|T_n v - Tv\| \to 0$.

A verification of this statement is straightforward (and is a good exercise, in particular for finite-dimensional spaces), we can from outset assume that our operator acts in $V[\hat{\mu}]$.

For any vector $v$,

$$v = \frac{1}{2\pi i} \lim_{N \to \infty} \lim_{\epsilon \to +0} \int_{-N}^{N} ((\lambda - i\epsilon - A)^{-1} - (\lambda + i\epsilon - A)^{-1}) v \, d\lambda.$$  

An evaluation of the limit gives the spectral decomposition. Now we will perform this for the hypergeometric differential operator $D$ defined above \[1.5\].

**3.2. Solutions of the equation** $(D - \lambda)f = 0$. For each $\lambda$ this equation has two linear independent solutions. We choose two bases in the space of solutions (both bases consist of Kummer series, see \[4\], Section 2.9). The first basis consists of functions

$$\varphi(x, \lambda) = {}_2F_1[b + \sqrt{\lambda}, b - \sqrt{\lambda}; b + c; -x]; \quad (3.1)$$

$$\psi(x, \lambda) = (-x)^{1-b-c} {}_2F_1[1 + \sqrt{\lambda} - c, 1 - \sqrt{\lambda} - c; 2 - b - c; -x]. \quad (3.2)$$

The second basis $u_{\pm}(x)$ is given by formulas

$$u_{\pm}(x, \lambda) = (-x)^{-b+c} {}_2F_1[b \pm \sqrt{\lambda}, 1 \pm \sqrt{\lambda} - c; 1 \pm 2\sqrt{\lambda}; -x^{-1}]. \quad (3.3)$$

We assume that the complex plane $\lambda$ is cut along the negative semi-axis.

For the first pair of functions the behavior near zero is easily observable, for the second pair we see the behavior near infinity. Below we need a formula expressing $\varphi$ in terms of $u_+$ and $u_-:

$$\varphi(x, \lambda) = B_+(\lambda) u_+(x, \lambda) + B_- (\lambda) u_-(x, \lambda),$$

where

$$B_{\pm}(\lambda) = \frac{\Gamma(b + c)\Gamma(\mp \sqrt{\lambda})}{\Gamma(b \mp \sqrt{\lambda})\Gamma(c \mp \sqrt{\lambda})}. \quad (3.4)$$

**3.3. Self-adjointness.** Let $b > 0$, $c > 0$. We define the operator $D$ on the space $D(\mathbb{R}_+)$ of smooth compactly supported functions on $(0, \infty)$. The operator $D$ is formally symmetric with respect to the weight $x^{b+c-1}(1 + x)^{b-c} \, dx$, i.e.,

$$\int_{0}^{\infty} (Df)(x)g(x)x^{b+c-1}(1 + x)^{b-c} \, dx = \int_{0}^{\infty} f(x)Dg(x)x^{b+c-1}(1 + x)^{b-c} \, dx$$

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where \( f, g \in D(\mathbb{R}_+) \). Its adjoint operator \( D^* \) is determined from the condition 
\[ D^* g = h \quad \text{if} \quad g, h \in L^2(\mathbb{R}_+, x^{b+c-1}(1 + x)^{b-c}) \]
and
\[
\int_0^\infty (Df)(x)\overline{g(x)}x^{b+c-1}(1 + x)^{b-c} \, dx = \int_0^\infty f(x)\overline{h(x)}x^{b+c-1}(1 + x)^{b-c} \, dx
\]
for all \( f \in D(\mathbb{R}_+) \). As before, this operator is given by formula (1.5), but its domain of definition have increased.

Recall that for any formally symmetric operator \( A \) the numbers \( \dim \ker(A^* - \lambda) \) (deficiency indexes) are constant on the half-planes \( \text{Im} \lambda > 0 \) and \( \text{Im} \lambda < 0 \). The operator \( A \) is essentially self-adjoint if the both numbers are 0. Therefore we must verify existence/nonexistence of solutions of the differential equations \( Df = \lambda f \) with \( \text{Im} \lambda \neq 0 \) such that \( f \) is contained in \( L^2 \) with respect to our weight. To be definite consider the upper half-plane \( \text{Im} \lambda > 0 \).

It is easy to see that for \( b + c > 2 \) such solutions do not exist. Indeed, \( \psi \) is too large near 0, and \( u_- \) is too large near \( \infty \). Therefore an \( L^2 \)-solution must coincide with \( \varphi \) and \( u_+ \) simultaneously. But these two solutions are different.

Therefore \( D \) is essentially self-adjoint.

\textbf{Remark.} If \( b + c < 2 \), then \( \varphi \) and \( \psi \) are in \( L^2 \) near 0. Therefore \( u_+ \in L^2 \) and the operator \( D \) is not self-adjoint. We extend the operator \( D \) and define it on the space of functions smooth on the closed half-line \([0, \infty)\) and vanishing for large \( x \). Then the operator became self-adjoint. Below we do not watch this case. \( \Box \)

\textbf{3.4. The resolvent.}

\textbf{Lemma 3.3} The resolvent \( (D-\lambda)^{-1} \) of the operator \( D \) is defined in the domain \( \mathbb{C} \setminus [\infty, 0) \) and is given by
\[
L(\lambda)f(x) = \int_0^\infty K(x, y; \lambda)y^{b+c-1}(1 + y)^{b-c} \, dy,
\]
(3.5)
where the 'Green function' \( K \) is given by
\[
K(x, y; \lambda) = \begin{cases} 
2B_-(\lambda)^{-1}\lambda^{-1/2}\varphi(x, \lambda)u_+(y, \lambda), & \text{if } x \leq y \\
2B_-(\lambda)^{-1}\lambda^{-1/2}\varphi(y, \lambda)u_+(x, \lambda), & \text{if } x \geq y
\end{cases}
\]
(3.6)
\(B_-(\lambda)\) is defined by (3.4).

The jump of the resolvent appear on the semi-axis \( \lambda \leq 0 \) due discontinuity of \( \sqrt{\lambda} \) on the cut. Evaluating the jump of resolvent we get
\[
f(x) = \frac{1}{2\pi} \int_{-\infty}^0 \frac{d\lambda}{2\sqrt{\lambda}B_+(\lambda)B_-(\lambda)} \int_0^\infty \varphi(z, \lambda)f(z)z^{b+c-1}(1 + z)^{b-c} \, dz.
\]
(3.7)

\textbf{Remark.} Formula is so simple, because \( \varphi \) has no jump; jump of \( u_+ \) is proportional to \( \varphi \).

The last formula is the desired inversion formula.
Proof of Lemma. First, we formally check the identity \((D - \lambda)L(\lambda) = 1\).

We must verify that the function \(K\) satisfies the equation

\[
(D - \lambda)K(x, y; \lambda) = \delta(x - y).
\]

(3.8)

Obviously, outside the diagonal \(x = y\) the equality \((D - \lambda)K = 0\) holds. The kernel \(K\) is continuous, but the first derivative has a jump. Therefore,

\[
(D - \lambda)K(x, y; \lambda) = x(x + 1)\left\{ \left. \frac{\partial K(x, y, \lambda)}{\partial x} \right|_{y = x + 0} - \left. \frac{\partial K(x, y, \lambda)}{\partial x} \right|_{y = x - 0} \right\} \delta(x - y) =
\]

\[= 2B_{-}(\lambda)^{-1} \lambda^{-1/2} \left[ \varphi(y, \lambda)' u_{+}(y, \lambda) - \varphi(y, \lambda) u_{+}(y, \lambda)' \right] \delta(x - y).\]

In square brackets we have Wronski determinant of two solutions of \((D - \lambda)f\).

Upto a constant factor Wronskian is determined by a differential equation, in our case it equals const \(\cdot y^{b - c - 1}(1 + y)^{c - b - 1}\). To evaluate the constant factor we watch asymptotics of the Wronskian as \(y \to \infty\).

In fact this calculation is sufficient to a proof. But a priori boundedness of \(L(\lambda)\) in \(L^2\) is not evident. We overcome this difficulty in the following way.

Since \(D\) is essentially self-adjoint, for \(\lambda \notin \mathbb{R}\) the operator \((D - \lambda)^{-1}\) is unbounded. In virtue of L. Schwartz's Kernel Theorem (see, e.g., (10) \((D - \lambda)^{-1}\) is an integral operator, its kernel \(K(x, y; \lambda)\) is a distribution of two variables. It satisfies equation (3.8) and the symmetry condition \(K(y, x; \lambda) = K(x, y; \lambda)\). Therefore, outside the diagonal \(x = y\) the distribution satisfies the system of equations

\[
(D_{x} - \lambda)K = 0, \quad (D_{y} - \lambda)K = 0.
\]

It can be readily checked that our kernel \(K\) is a unique admissible candidate, all other solutions of the system increase too rapidly.

3.5. The Romanovski Polynomials. Now let \(b < 0, b + c > 0\). Let \(m = 0, 1, \ldots, [-b]\). Consider the polynomials \(p_{m}\) given by

\[
p_{m}(x) := 2F_{1}\left[ \begin{array}{c} -m, 2b + m \\ b + c \end{array} ; -x \right] =
\]

\[
= \frac{x^{m} \Gamma(b + c) \Gamma(-m - b)}{\Gamma(2b + m) \Gamma(c + b + m)} \cdot 2F_{1}\left[ \begin{array}{c} -m, 1 - m - b - c \\ 2 - b - c \end{array} ; \frac{-1}{x} \right].
\]

Theorem 3.4 a) The polynomials \(p_{m}\) are contained in \(L^{2}(\mathbb{R}_{+}, x^{b+c-1}(1+x)^{b-c})\).

b) \(Dp_{m} = (b + m)^{2}p_{m}\).

c) The polynomials \(p_{m}\) are pairwise orthogonal.

The statements a), b) are evident, c) follows from a) and b).

Thus we get a finite system of orthogonal polynomials. We can not enlarge it because the monomials \(x^{N}\) with larger powers are not in \(L^2\).
Remark. A lot of such finite systems of orthogonal polynomials is known. Romanovski [29] introduced another two systems: the polynomials on the line orthogonal with respect to the weight
\[ \frac{dx}{(1 + ix)^\alpha(1 - ix)^\beta}, \quad \text{where } \alpha \in \mathbb{C}, \]
(they also are analytic continuations of the Jacobi polynomials) and polynomials on the half-line \( x \geq 0 \) with respect to the weight
\[ x^{-\beta} \exp(-1/x) \, dx \]
(this is an analytic continuation of the Laguerre polynomials of \( 1/x \)). More complicated finite systems of orthogonal polynomials were enumerated by P. Lesky [16], [17], some additions are in [23].

Our considerations explain this phenomenon. For \( b < 0, b + c > 0 \) our operator \( D \) has finite number of discrete eigenvalues corresponding to the Romanovski polynomials, which are added to the continuous spectrum.

It is necessary to modify our calculation. The resolvent (see formula (3.6)) now has a finite number of poles at points \( \lambda = (b+m)^2 \), they arise from the poles of \( B_-(\lambda)^{-1} \). To write the jump of the resolvent we must additionally evaluate residues at these poles. In the inversion formula (3.7) we get additional terms
\[ \cdots + \sum_m \frac{\langle f, p_m \rangle_{L^2}}{\langle p_m, p_m \rangle_{L^2}} p_m(x). \]

The expression (1.3) for \( J_{b,c} \) does not change. But the function \( J_{b,c} f(s) \) now is defined on the following subset in \( \mathbb{C} \): the half-line \( s \geq 0 \) and a finite set \( s = i(b + m) \) on the imaginary axis (these points correspond to Romanovski polynomials).

Remark. Such orthogonal systems arise in non-commutative harmonic analysis and correspond to discrete part of spectra (for instance, for \( L^2 \) on pseudo-Riemannian symmetric spaces of rank 1, see also [24] about tensor products of unitary representations of the group \( \text{SL}(2, \mathbb{R}) \). Apparently (nobody verified this) the discrete Flensted-Jensen series [5] are controlled by some multivariable orthogonal systems of Romanovski type.

4 Applications to harmonic analysis

4.1. Pseudounitary groups of rank 1. Let \( \mathbb{K} \) be \( \mathbb{R}, \mathbb{C} \) or the quaternion algebra \( \mathbb{H} \). The case \( \mathbb{K} = \mathbb{R} \) is sufficiently interesting. We present several simple facts without proofs, the reader can believe or verify.

Denote by \( r \) the dimension of \( \mathbb{K} \). Let \( \mathbb{K}^n \) be the \( n \)-dimensional space over \( \mathbb{K} \) with the standard inner product,
\[ \langle z, u \rangle = \sum z_j \overline{u}_j. \]
By \( U(1, n; \mathbb{K}) \) we denote the pseudounitary group over \( \mathbb{K} \), i.e., the group of \((1 + n) \times (1 + n)\)-matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) over \( \mathbb{K} \) satisfying the condition

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

The standard notations for the groups \( U(1, n; K) \) for \( K = \mathbb{R}, \mathbb{C}, \mathbb{H} \) are respectively: \( O(1, n), U(1, n), \text{Sp}(1, n) \).

4.2. Homogeneous hyperbolic spaces. Denote by \( B_n(\mathbb{K}) \) the open unit ball \( \langle z, z \rangle < 1 \) in \( \mathbb{K}^n \). By \( S^{n-1} \) we denote the sphere \( \langle z, z \rangle = 1 \). The group \( U(1, n; \mathbb{K}) \) acts on \( B_n(\mathbb{K}) \) by linear-fractional transformations

\[
z \mapsto z^g := (a + zc)^{-1}(b + zd).
\]

(4.1)

The stabilizer \( K \) of the point \( 0 \in B_n(\mathbb{K}) \) consists of matrices of the form

\[
\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}, \quad |a| = 1, \quad d \in U(n; \mathbb{K}).
\]

(4.2)

Therefore \( B_n(\mathbb{K}) \) is the homogeneous space

\[
B_n(\mathbb{K}) = U(1, n; \mathbb{K})/(U(1; \mathbb{K}) \times U(n; \mathbb{K})).
\]

Remark. If \( \mathbb{K} = \mathbb{R} \), then our ball is the \( n \)-dimensional Lobachevsky space in the Beltrami–Klein model. Recall that in this case straight lines in the Lobachevsky sense are segments (chords), the sphere \( S^{n-1} \) is the absolute in the Lobachevsky sense. The group \( O(1, n) \) is the group of motions of the Lobachevsky space.

For \( \mathbb{K} = \mathbb{C} \) and \( \mathbb{H} \) we get complex and quaternionic hyperbolic spaces. \( \square \)

The Jacobian of the transformation (1.5) is

\[
J(g; z) = |a + zc|^{-r(1+n)}.
\]

Note the following simple formula

\[
1 - \langle z^g, u^g \rangle = (a + zc)^{-1}(1 - \langle z, u \rangle)(a + uc)^{-1}.
\]

This implies that the \( U(1, n; \mathbb{K}) \)-invariant measure on \( B_n(\mathbb{K}) \) has the form

\[
dm(z) = (1 - \langle z, z \rangle)^{-(n+1)r/2}dz,
\]

where \( dz \) denotes the Lebesgue measure on \( B_n(\mathbb{K}) \).

The group \( U(1, n; \mathbb{K}) \) acts in \( L^2(B_n(\mathbb{K}), dm(z)) \) by changes of variable

\[
\rho(g)f(z) = f((a + zc)^{-1}(b + zd)).
\]

(4.3)

Evidently these operators are unitary. In other words we get an infinite-dimensional unitary representation of the group \( U(1, n, \mathbb{K}) \).
Our next problem is to decompose this representation into irreducible representations.

4.3. The spherical principal series. Let \( s \in \mathbb{R} \). A representation \( T_s \) of spherical principal series of the group \( \text{U}(1,n;\mathbb{K}) \) is realized in \( L^2(S^{rn-1}) \) and is given by the formula

\[
T_s \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(h) = f((a + hc)^{-1}(b + hd))|a + hc|^{-(n+1)r/2+1+is}, \tag{4.4}
\]

where \( h \in S^{rn-1} \). A straightforward calculation shows that these representations are unitary for \( s \in \mathbb{R} \).

Remark. All representations \( T_s \) are irreducible, representations \( T_s \) and \( T_{-s} \) are equivalent (this is not completely obvious, see [32]).

Remark. The term ‘series’ is used because these groups have different types of unitary representations. The term ‘spherical’ means that any representation \( T_s \) contains a (unique) \( K \)-invariant vector. In our model this vector is the function \( f = 1 \).

4.4. An intertwining operator. Consider the space of functions \( \varphi(h,s) \) on the semi-cylinder \( S^{rn-1} \times \mathbb{R}_+ \) (a precise description of this space is given below), let \( \text{U}(1,n;\mathbb{K}) \) act in this space by the formula

\[
\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} \varphi(h,s) = \varphi((a + hc)^{-1}(b + hd),s)|a + hc|^{-(n+1)r/2+1+is}.
\]

For a fixed \( s \) we get the representation \( T_s(g) \) in functions depending on \( h \). Thus we have some kind of a direct sum of all representations \( T_s \) with respect to a continuous parameter \( s \) (thus is called a ‘direct integral’).

We define the following operator \( A \) from the space \( L^2(Bn(\mathbb{K}), dm(z)) \) to the space of functions on \( S^{rn-1} \times \mathbb{R}_+ \):

\[
Af(h,s) = \int_{Bn(\mathbb{K})} f(z) \frac{|1 - \langle z,h \rangle|^{-(n+1)r/2+1+is}}{|1 - \langle z,z \rangle|^{(n+1)r/4+1/2+is/2}} dz. \tag{4.5}
\]

Lemma 4.1 The operator \( A \) is intertwining, i.e.,

\[
A \rho(g) = \tau(g) A, \quad \text{for all } g.
\]

This statement is a useful two-step exercise. First, it is worth to verify the lemma in a straightforward way. Secondly, it is interesting to find a way to invent the formula for the operator \( A \) if you do not know it before.

4.5. The Plancherel formula.

Theorem 4.2 The operator \( A \) is a unitary operator

\[
L^2(Bn(\mathbb{K}), dm(z)) \rightarrow L^2 \left( S^{rn-1} \times \mathbb{R}_+, \frac{\Gamma(b+is)\Gamma(c+is)}{\Gamma(2is)}^2 ds \, dh \right), \tag{4.6}
\]

where

\[
b = (n + 1)r/4 - 1/2; \quad c = (n - 1)r/4 + 1/2. \tag{4.7}
\]
Keeping in mind the previous lemma we get that the operator $A$ identifies the representation of the group $U(1, n; \mathbb{K})$ in $L^2$ on the ball with a continuous direct sum of representations of principal series.

**Beginning of proof.** First, we explain the appearance of the $\Gamma$-factor. For this purpose we restrict the operator $A$ to the space of functions depending only on radius. It is convenient to define the variable $x = \frac{|h|^2}{1-|h|^2}$ and set $f = f(x)$. Then the corresponding function $G(h, s)$ depends only on the variable $s$, an uncomplicated calculation gives the familiar formula

$$G(s) = \text{const} \cdot \int_0^\infty f(x) {}_2F_1(b + is, b - is; b + c; -x)x^b e^{-c-1}(1 + x)^{b-c}dx. \quad (4.8)$$

Thus, we observe that the operator $A$ is a unitary operator from the space of $L^2$-functions on the ball depending only on radius to the space of functions on the half-cylinder depending only on $s$.

This is the main argument, it remains to apply some standard representation-theoretic tricks.

**4.6. The end of the proof.** Denote $G := U(1, n; \mathbb{K})$, $K := U(1, \mathbb{K}) \times U(n, \mathbb{K})$. Denote the Hilbert spaces $L^2$ from row (4.6) by $V$ and $W$ respectively. By $V^K$ and $W^K$ we denote the spaces of $K$-fixed vectors in $V$ and $W$. By $P_V$ and $P_W$ we denote the projection operators to $V^K$ and $W^K$.

Recall the following standard statement.

**Lemma 4.3** Let $\rho(k)$ be a unitary representation of a compact group $K$. Then the projection operator to the space of $K$-fixed vectors is given by the formula

$$P = \int_K \rho(k) \, dk,$$

where $dk$ is the Haar measure on $K$, normed in such a way that the measure of the whole group is 1.

**Corollary 4.4** $P_W A = A P_V$.

**Lemma 4.5** Any closed $G$-invariant subspace in $V$ contains a smooth function.

**Proof.** Consider a sequence of smooth compactly supported positive functions $r_j$ on $G$ approximating the $\delta$-function at unit. For a vector $v \neq 0$ from the subspace we get a sequence of smooth functions $\int r_j(g)\rho(g)v \, dg$, convergent to $v$. \quad \square

\footnote{For $\mathbb{K} = \mathbb{C}$ there are also actions of the group $U(1, n)$ given by

$$\rho(g)f(z) = f((a+zc)^{-1}(b+zd))(a+zc)^k(a+zc)^{-k}$$

The problem of decomposition also is reduced to the index transform (with another parameters), the representation has finite discrete spectrum which is controlled by the Romanovsky polynomials.}

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Lemma 4.6 Any closed \( G \)-invariant subspace in \( V \) contains a \( K \)-invariant vector.

Proof. Consider a smooth function \( f \) from the subspace. Let \( f(a) \neq 0 \). Consider \( g \in G \) such that \( 0^{[g]} = a \). Next, average the function \( f(x^{[g]}) \) by \( K \). □

Corollary 4.7 The linear span of vectors \( \rho(g)v \), where \( g \) ranges in \( G \) and \( v \) ranges in \( V^K \), is dense in \( V \).

Proof. If not, we consider the orthogonal complement to this linear span. It contains a \( K \)-invariant vector. □

Lemma 4.8 The linear span of vectors \( \tau(g)w \), where \( g \) ranges in \( G \) and \( w \) ranges in \( W^K \), is dense in \( W \).

Proof. We use irreducibility of representations \( T_s \) of the principal series. As \( u \) we take functions \( f(x, s) = 1 \), if \( |s - s_0| < \varepsilon \) and 0 otherwise. It is easy to show that there are no functions orthogonal to all \( \tau_g(f) \). □

End of proof of Theorem 4.2. Let \( \rho(g)v, \rho(g')v' \) be as in the last lemma. Then

\[
\langle \rho(g)v, \rho(g')v' \rangle_V \quad \text{(representation } U \text{ is unitary)} \quad \langle v, \rho(g^{-1}g')v' \rangle_V \quad \text{(} P_V \text{ is projection operator)}
\]

\[
= \langle v, P_V \rho(g^{-1}g')v' \rangle_V \quad \text{(Plancherel formula)} \quad \langle Av, AP_W \rho(g^{-1}g')v' \rangle_W =
\]

\[
= \langle Av, P_W \tau(g^{-1}g')Av' \rangle_W \quad \text{(} P_W \text{ is a projection operator)}
\]

\[
= \langle Av, \tau(g^{-1}g)Av' \rangle_W \quad \text{(representation } \tau \text{ is unitary)} \quad \langle \tau(g)Av, \tau(g')Av' \rangle_W.
\]

Therefore \( A \) is an isometry. On the other hand, the image of \( A \) contains \( W^K \) (by Theorem [4.1]) and therefore contains the whole \( W \) (by Lemma [4.8]).

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