Stückelberg-Modified Massive Abelian 3-Form Theory: Constraint Analysis, Conserved Charges and BRST Algebra

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Abstract: For the Stückelberg-modified massive Abelian 3-form theory in any arbitrary D-dimension of spacetime, we show that its classical gauge symmetry transformations are generated by the first-class constraints. We establish that the Noether conserved charge (corresponding to the local gauge symmetry transformations) is same as the standard form of the generator for the underlying local gauge symmetry transformations (expressed in terms of the first-class constraints). We promote these classical local, continuous and infinitesimal gauge symmetry transformations to their quantum counterparts Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST symmetry transformations which are respected by the coupled (but equivalent) Lagrangian densities. We derive the conserved (anti-)BRST charges by exploiting the theoretical potential of Noether’s theorem. However, these charges turn out to be non-nilpotent. Some of the highlights of our present investigation are (i) the derivation of the off-shell nilpotent versions of the (anti-)BRST charges from the standard non-nilpotent Noether conserved (anti-)BRST charges, (ii) the appearance of the operator forms of the first-class constraints at the quantum level through the physicality criteria w.r.t. the nilpotent versions of the (anti-)BRST charges, and (iii) the deduction of the CF-type restrictions from the straightforward equality of the coupled (anti-)BRST invariant Lagrangian densities as well as from the requirement of the absolute anticommutativity of the off-shell nilpotent versions of the conserved (anti-)BRST charges.

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1 Introduction

The modern developments in the domain of (super)string theories have brought together, in an unprecedented manner, the top-class mathematicians and theoretical physicists of the highest calibre on a single platform which has led to the confluence of thoughts as well as the pollination of ideas that have been specifically responsible for many interesting developments in the realm of theoretical high energy physics. In this context, mention can be made of the AdS/CFT correspondence, gauge-gravity duality, classification of the spacetime manifolds, higher-spin gauge theories, higher $p$-form ($p = 2, 3, 4, ...$) gauge theories, etc. The central objective of our present endeavor is connected with the study of the Stückelberg-modified massive higher $p$-form (i.e. $p = 3$) Abelian gauge theory within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism where the symmetry considerations (i.e. continuous symmetry transformations, conserved Noether charges, algebra obeyed by the conserved charges, etc.) are given utmost importance. Our present endeavor is essential in view of the fact that it is connected with the BRST-quantization of a D-dimensional modified massive Abelian 3-form theory which is beyond the purview of the theoretical potential and power of the standard model of particle physics which is based on the non-Abelian 1-form (i.e. $p = 1$) interacting gauge theory that incorporates into its folds the matter as well as the gauge fields together with a well-defined coupling on the basis of the principle of local gauge invariance and ensuing interaction.

A deeper understanding of the quantum field theory of the massive and massless higher $p$-form ($p = 2, 3, 4, ...$) gauge theories has become essential and important because the higher $p$-form gauge fields appear in the excitations of the (super)strings which are one of the most promising candidates (i) to provide a consistent theory of quantum gravity, (ii) to address the question of the complete unification of all the four fundamental interactions of nature, and (iii) to go beyond the realm of the standard model of particle physics which is based on the non-Abelian interacting 1-form gauge theory (see, e.g. [1-5] for details). The latter issue is important because the experimental evidence of the masses of the neutrinos has led to the conclusion that, even though very successful, the standard model of particle physics is not a complete theory. The purpose of our present investigation is to study some field-theoretic aspects of the Stückelberg-modified massive Abelian 3-form gauge theory in any arbitrary D-dimension of spacetime. In our earlier works (see, e.g. [6-8]), we have studied the BRST quantization of the $p$-form ($p = 1, 2, 3, ...$) gauge theories in $D = 2p$ dimensions of spacetime and established that such massive and massless gauge theories are the tractable field-theoretic examples of Hodge theory where the discrete and continuous symmetry transformations (and corresponding conserved Noether charges) provide the physical realizations of the de Rham cohomological operators (see, e.g. [9-12]) of differential geometry. One of the highlights of such studies has been the observation that a set of fields, with the negative kinetic terms, appear in the theory on the ground of symmetry considerations alone. These “exotic” fields are important in the context of the cyclic, bouncing and self-accelerated cosmological models of the Universe where they have been christened as the “ghost” and/or “phantom” fields (see, e.g. [13-15] and references therein). One of the possible candidates of the dark matter/dark energy is also intimately related with the existence of the fields/particles with negative kinetic terms with well-defined mass and without mass (see, e.g. [16, 17] for details).
In a very recent work [18], we have exploited the theoretical beauty of the augmented version of superfield approach (AVSA) to BRST formalism to derive (i) the complete set of (anti-)BRST symmetry transformations, and (ii) the (anti-)BRST invariant Curci-Ferrari (CF) type restrictions [19] for the Stuckelberg-modified massive Abelian 3-form gauge theory in any arbitrary D-dimension of spacetime. The existence of the latter (i.e., CF-type restrictions) is one of the key signatures of a BRST-quantized theory as it is connected with the idea of the geometrical objects called gerbes (see, e.g., [20, 21]). We would like to lay emphasis on the fact that for the BRST-quantized theory, the existence of the Curci-Ferrari type restrictions is as fundamental as the existence of the first-class constraints in the definition of a classical gauge theory. The central theme of our present endeavor is to perform a thorough constraint analysis of the D-dimensional Stuckelberg-modified Abelian 3-form massive gauge theory (cf. Sec. 3) and show that its first-class constraints generate the continuous and infinitesimal classical gauge symmetry transformations that can be easily exploited for the quantum versions of the off-shell nilpotent (anti-)BRST symmetry transformations. It is pertinent to point out, at this stage, that in our earlier works (see, e.g., [6-8,18]), the constraint analysis of the modified massive Abelian 3-form theory has not been performed. In our present investigation, we derive the conserved Noether currents and corresponding conserved charges for our system and show that the underlying nilpotent (anti-)BRST symmetry transformations and the infinitesimal ghost-scale symmetry transformations lead to the derivation of the conserved (anti-)BRST and ghost charges. It turns out that the standard Noether conserved (anti-)BRST charges \([Q_{\alpha}]\) are not nilpotent of order two (i.e. \([Q_{\alpha}]^2 \neq 0\)). A systematic method has been proposed in [22] to obtain the off-shell nilpotent versions of the (anti-)BRST charges. These latter nilpotent charges and ghost charge obey the standard BRST algebra (cf. Appendix C). The physicality criteria w.r.t. the above nilpotent versions of the (anti-)BRST charges are crucial as far as the appearance of the first-class constraints at the quantum level is concerned. One of the highlights of our present investigation is the observation that the proof of the absolute anticommutativity property of the nilpotent (anti-)BRST charges (cf. Appendix B) leads to the derivation of the (anti-)BRST invariant CF-type restrictions. The latter are also derived by the requirement of the straightforward equality of the coupled (anti-)BRST invariant Lagrangian densities (cf. Appendix A). It is worthwhile to point out that, in our present endeavor, we have confined ourselves to the theoretical techniques of the BRST formalism because our system is simple and endowed with the first-class constraint only. Mathematically more sophisticated and more general BV formalism has not been adopted in our present endeavor because of the simplicity of our system.

Even though, our main results have been mentioned in a somewhat haphazard manner throughout our earlier paragraphs, we would like to assimilate systematically our key results in a single paragraph for the readers’ convenience. We have been able to show the existence of the first-class constraints at the classical level in the Lagrangian formulation (cf. Sec. 3) as well as at the quantum level within the framework of BRST formalism (cf. Sec. 6). In our present endeavor, we have computed all the conserved charges corresponding to the continuous symmetry transformations of our theory and explicitly shown that the Noether conserved (anti-)BRST charges are not nilpotent of order two (cf. Sec. 5). The off-shell nilpotent versions of these charges have been derived and their usefulness have been shown in the context of the physicality criteria (cf. Sec. 6). Two of the sacrosanct properties of
the BRST formalism are the nilpotency property and absolute anticommutativity property that are associated with the (anti-)BRST symmetries. We have devoted an Appendix (cf. Appendix E) of our present endeavor to establish the connections between these properties and the CF-type restrictions at the level of the (anti-)BRST transformed fields (that are present in our theory). Two appendices (i.e. Appendices A and B) deal with the derivations of these CF-type restrictions from different theoretical angles. These derivations are completely different from such derivations in our previous work [18]. We have derived the standard BRST algebra amongst the appropriate conserved charges of our theory.

Our present endeavor is essential and important on the following counts. First, we have already derived the coupled (but equivalent) Lagrangian densities for the St"uckelberg-modified massive Abelian 3-form theory that respect the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformations. The latter and associated CF-type restrictions have been deduced by exploiting the theoretical strength of the AVSA to BRST formalism (see, e.g. [18]). However, we have not derived the expressions for the conserved charges corresponding to the underlying continuous symmetry transformations of our theory. In our present investigation, we accomplish this goal and deduce the explicit expressions for the (anti-)BRST and ghost charges. Second, we have been able to show the existence of the CF-type restrictions (on our theory) by symmetry considerations alone in our earlier work [18]. In our present endeavor, we show the presence of the above CF-type restrictions by proving the absolute anticommutativity of the nilpotent (anti-)BRST charges and straightforward equality of the coupled Lagrangian densities. Third, we have established that the operator forms of the first-class constraints annihilate the physical quantum states of our theory through the physicality criteria w.r.t. the nilpotent versions of the (anti-)BRST charges (cf. Sec. 6). This observation is one of the key results of our present investigation. Finally, our present investigation is a modest step in the direction to prove that the 6D massive Abelian 3-form theory is a field-theoretic example of Hodge theory where the fields with negative kinetic terms are expected to appear on the basis of symmetry considerations alone. Such “exotic” fields are one of the possible candidates for dark matter/dark energy [16, 17] and these fields (popularly christened as the “ghost” and/or “phantom” fields) are found to play a crucial role in the cyclic, bouncing and self-accelerated cosmological models of the Universe [13-15].

The theoretical contents of our present investigation are organized as follows. First of all, in Sec. 2, we discuss the infinitesimal and local gauge symmetry transformations for the St"uckelberg-modified Lagrangian density and (anti-)BRST symmetries for the coupled (but equivalent) Lagrangian density. In Sec. 3, we write down the standard generator for the infinitesimal and local classical gauge symmetry transformations in terms of the first-class constraints and establish a connection between this generator and the Noether conserved charge corresponding to the infinitesimal local gauge symmetry transformations (at the classical level). Our Sec. 4 deals with the derivation of the Euler-Lagrange (EL) equations of motion (EoM) from the coupled (but equivalent) Lagrangian densities where we comment on the existence of the CF-type restrictions [cf. Eq. (29) below]. The subject matter of our Sec. 5 is connected with the derivations of the Noether conserved currents, corresponding conserved charges and a few comments on the non-nilpotency of the standard Noether (anti-) BRST charges. Our Sec. 6 is devoted to the discussion on the physicality criteria w.r.t. the off-shell nilpotent versions of the (anti-)BRST charges where we show the
existence of the first-class constraints of the theory at the quantum level in their operator form. Finally, in Sec. 7, we summarize our key results and point out a few future directions of further investigation(s) in the light of our present investigation.

In our Appendices A and B, we demonstrate the existence of a set of (anti-)BRST invariant CF-type restrictions by (i) requiring the straightforward equality of the coupled Lagrangian densities, and (ii) demanding the absolute anticommutativity of the off-shell nilpotent versions of the conserved (anti-)BRST charges. Our Appendix C is devoted to a brief description of the standard BRST algebra which exists amongst the off-shell nilpotent versions of the (anti-)BRST charges and the conserved ghost charge of our BRST-quantized theory. In Appendix D, we provide a glossary of the key properties and nature of all the tower of fields that are present in our BRST-quantized theory (cf. Sec. 2). Our Appendix E deals with the proof that the sanctity and usefulness of the (anti-)BRST invariant CF-type restrictions (that are present on our theory) are true even at the level of (anti-)BRST transformed fields.

General Convention and Notations: We adopt the convention of the left derivative w.r.t. all the fermionic fields of our theory in the computations of the canonical conjugate momenta, equations of motion, Noether’s conserved currents/charges, etc. The background flat D-dimensional Minkowskian spacetime manifold is endowed with a metric tensor $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, \ldots)$ so that the dot product between two non-null vectors $P_\mu$ and $Q_\mu$ is defined as: $P \cdot Q = \eta_{\mu\nu} P^\mu Q^\nu \equiv P_0 Q_0 - P_i Q_i$, where the Latin indices $i, j, k, \ldots = 1, 2, \ldots, D - 1$ stand for the space directions only and the Greek indices $\mu, \nu, \lambda, \ldots = 0, 1, 2, \ldots, D - 1$ correspond to the time and space directions together. Einstein’s summation convention has been taken into account in the whole body of the text. We denote the (anti-)BRST symmetry transformations by the symbols $s^{(a)} b$ and corresponding conserved charges have been represented by the notations $Q^{(a)} b$. Due to their fermionic nature, the (anti-)BRST symmetry transformation operators $s^{(a)} b$ commute with the bosonic fields of our theory and they anticommute with their counterparts fermionic fields.

2 Preliminaries: Local and Infinitesimal Gauge and Off-shell Nilpotent (Anti-)BRST Symmetries

Within the framework of Lagrangian formulation for the D-dimensional massive Abelian 3-form theory (see e.g. [18] for details), we discuss the infinitesimal, continuous and local gauge symmetry transformations for the following St"uckelberg-modified Lagrangian density $[\mathcal{L}^{\text{A}}_S]$

$$\mathcal{L}^{\text{A}}_S = \frac{1}{24} H^{\mu\nu\lambda\zeta} H_{\mu\nu\lambda\zeta} - \frac{m^2}{6} A^{\mu\nu\lambda} A_{\mu\nu\lambda} \pm \frac{m}{3} A^{\mu\nu\lambda} \Sigma_{\mu\nu\lambda} - \frac{1}{6} \Sigma^{\mu\nu\lambda} \Sigma_{\mu\nu\lambda}, \quad (1)$$

where the totally antisymmetric field-strength (curvature) tensor $H_{\mu\nu\lambda\zeta}$ has been derived from the 4-form $H^{(4)} = d A^{(3)}$ [18] and superscript (A) denotes the modified Lagrangian density $[\mathcal{L}^{\text{A}}_S]$ for the massive Abelian gauge field $A_{\mu\nu\lambda}$. The Abelian 3-form $A^{(3)} = \frac{1}{3!} A_{\mu\nu\lambda} (dx^\mu \wedge dx^\nu \wedge dx^\lambda)$ defines the totally antisymmetric tensor gauge field $A_{\mu\nu\lambda}$.
and $d = \partial_{\mu} d x^\mu$ (with $d^2 = \frac{1}{3!} (\partial_{\mu} \partial_{\nu} - \partial_{\nu} \partial_{\mu})(d x^\mu \wedge d x^\nu) = 0$) is the exterior derivative of differential geometry [9-12]. The explicit form of $H_{\mu\nu\lambda\zeta}$, in terms of the 3-form Abelian basic gauge field $A_{\mu\nu\lambda}$, is as follows:

$$H_{\mu\nu\lambda\zeta} = \partial_{\mu} A_{\nu\lambda\zeta} - \partial_{\nu} A_{\mu\lambda\zeta} + \partial_{\lambda} A_{\mu\nu\zeta} - \partial_{\zeta} A_{\mu\nu\lambda}. \quad (2)$$

In the above equation (1), the symbol $m$ is the rest mass of the 3-form field $A_{\mu\nu\lambda}$ and the notation $\Sigma_{\mu\nu\lambda}$ stands for the totally antisymmetric field-strength tensor

$$\Sigma_{\mu\nu\lambda} = \partial_{\mu} \Phi_{\nu\lambda} + \partial_{\nu} \Phi_{\lambda\mu} + \partial_{\lambda} \Phi_{\mu\nu}, \quad (3)$$

for the St"{u}ckelberg 2-form field $\Phi_{\mu\nu}$ defined through the 2-form: $\Phi^{(2)} = \frac{1}{3!} \Phi_{\mu\nu\rho} (d x^\mu \wedge d x^\nu \wedge d x^\rho)$. It is self-evident that the 3-form $\Sigma^{(3)} = d \Phi^{(2)}$ defines the field-strength tensor (3) with the help of the exterior derivative $d$ and the St"{u}ckelberg 2-form field $\Phi_{\mu\nu}$ because $\Sigma^{(3)} = \frac{1}{3!} \Sigma_{\mu\nu\lambda} (d x^\mu \wedge d x^\nu \wedge d x^\lambda)$. The infinitesimal, continuous and local gauge symmetry transformations (\(\delta_g\)) for the fields $A_{\mu\nu\lambda}$, $\Phi_{\mu\nu}$, $\Sigma_{\mu\nu\lambda}$ and $H_{\mu\nu\lambda\zeta}$ are as follows [18]

\[
\begin{align*}
\delta_g A_{\mu\nu\lambda} &= \partial_{\mu} \Lambda_{\nu\lambda} + \partial_{\nu} \Lambda_{\lambda\mu} + \partial_{\lambda} \Lambda_{\mu\nu}, \\
\delta_g \Sigma_{\mu\nu\lambda} &= \pm m (\partial_{\mu} \Lambda_{\nu\lambda} + \partial_{\nu} \Lambda_{\lambda\mu} + \partial_{\lambda} \Lambda_{\mu\nu}), \\
\delta_g \Phi_{\mu\nu} &= \pm m \Lambda_{\mu\nu} - (\partial_{\mu} \Lambda_{\nu\lambda} - \partial_{\nu} \Lambda_{\mu\lambda}), \\
\delta_g H_{\mu\nu\lambda\zeta} &= 0, \quad (4)
\end{align*}
\]

where the antisymmetric $[\Lambda_{\mu\nu}(x) = -\Lambda_{\nu\mu}(x)]$ tensor and Lorentz vector $\Lambda_{\mu}(x)$ are the local gauge symmetry transformation parameters. It is straightforward to note that $\delta_g L^{(A)}_S = 0$. The above transformations are generated by the first-class constraints that exist for the Lagrangian density $L^{(A)}_S$ (cf. Sec. 3 below for details).

The St"{u}ckelberg-modified Lagrangian density (1) has been generalized to the coupled (but equivalent) (anti-)BRST invariant Lagrangian densities in our earlier work [18]. The perfectly BRST invariant Lagrangian density ($L_B$), incorporating the gauge-fixing as well as the Faddeev-Popov (FP) ghost terms, is as follows

$$L_B = L^{(A)}_S + (\partial_{\mu} A^{(\mu\lambda\nu\xi)}) B_{\mu\lambda\nu\xi} - \frac{1}{2} B_{\mu\nu} B^{\mu\nu} + \frac{1}{2} B^{\mu\nu} \left[ \partial_{\mu} \phi_{\nu} - \partial_{\nu} \phi_{\mu} \mp m \Phi_{\mu\nu} \right]$$

$$- \left( \partial_{\mu} \Phi^{\mu\nu} \right) B_{\nu} - \frac{1}{2} B^{\mu} B_{\mu} + \frac{1}{2} B^{\mu} \left[ \pm m \phi_{\mu} - \partial_{\mu} \phi \right] + \frac{m^2}{2} \bar{C}_{\mu\nu} C^{\mu\nu}$$

$$+ (\partial_{\mu} \bar{C}_{\nu} + \partial_{\nu} \bar{C}_{\lambda} + \partial_{\lambda} \bar{C}_{\mu})(\partial^{\mu} C^{\nu\lambda}) \pm m (\partial_{\mu} \bar{C}^{\mu\nu}) C_{\nu} \pm m \bar{C}^{\mu}(\partial^{\mu} C_{\mu\nu})$$

$$+ (\partial_{\mu} \bar{C}_{\nu} - \partial_{\nu} \bar{C}_{\mu})(\partial^{\mu} C^{\nu}) - \frac{1}{2} \left[ \pm m \bar{\beta}^{\nu} - \partial_{\mu} \bar{\beta} \right] \left[ \pm m \beta_{\mu} - \partial_{\mu} \beta \right]$$

$$- (\partial_{\mu} \bar{\beta}_{\nu} - \partial_{\nu} \bar{\beta}_{\mu})(\partial^{\mu} \beta_{\nu}) - \partial_{\mu} \bar{C}_{2}\partial^{\mu} C_{2} - m^2 \bar{C}_{2} C_{2} + [(\partial \cdot \bar{\beta}) \mp m \bar{\beta}] B$$

$$- [(\partial \cdot \phi) \mp m \phi] B_{1} - [(\partial \cdot \beta) \mp m \beta] B_{2} + \left[ \partial_{\mu} \bar{C}^{\mu\nu} + \partial^{\mu} \bar{C}^{\nu} \mp \frac{m}{2} \bar{C}^{\nu} \right] f_{\mu}$$

$$- 2 F^{\mu} f_{\mu} - 2 F_{-} f_{-} - \left[ \partial_{\nu} C^{\mu\nu} + \partial^{\mu} C_{1} \mp \frac{m}{2} C_{1} \right] F_{\mu} + \left[ \frac{1}{2} (\partial \cdot C) \mp m C_{1} \right] F$$

$$- \left[ \frac{1}{2} (\partial \cdot \bar{C}) \mp m \bar{C}_{1} \right] f - B B_{2} - \frac{1}{2} B_{2}^{2}, \quad (5)$$

where the antisymmetric ($B_{\mu\nu} = -B_{\nu\mu}$) tensor ($B_{\mu\nu}$), bosonic and fermionic sets of the Lorentz vector ($B_{\mu}, F_{\mu}, f_{\mu}$) and scalar ($B, B_{1}, B_{2}, F, f$) auxiliary fields, respectively,
appear in the above Lagrangian density along with the fermionic (anti-)ghost fields \((C_2)C_2, (C_{\mu\nu})C_{\mu\nu}, (C_{\mu})C_{\mu}, (C_1)C_1\) as well as the bosonic (anti-)ghost fields \((\beta_\mu)\beta_\mu\) and \((\bar{\beta})\bar{\beta}\). In addition to the antisymmetric \((\Phi_{\mu\nu} = -\Phi_{\nu\mu})\) Stuckelberg bosonic field \(\Phi_{\mu\nu}\), we also have the vector bosonic field \(\phi_\mu\) and scalar bosonic field \(\phi\). We have a set of special kinds of auxiliary fields (\(B_1, B, B_2\)) which carry the ghost numbers \((0, +2, -2)\), respectively. Similarly, we have perfectly anti-BRST invariant Lagrangian density \((L_B)\) as

\[
L_B = L_{\text{S}}^{(A)} - \left(\partial_\mu A^{\mu\lambda}\right)B_{\nu\lambda} - \frac{1}{2}\bar{B}_{\mu\nu}B^{\mu\nu} + \frac{1}{2}B^{\mu\nu}\left[\partial_\mu \phi_\nu - \partial_\nu \phi_\mu \pm m \Phi_{\mu\nu}\right] + \left(\partial_\mu \Phi_{\mu\nu}\right)B_{\nu\lambda} - \frac{1}{2}B^{\mu\nu}B_{\mu\nu} + \left(\partial_\mu C_{\nu\lambda} + \partial_\nu C_{\mu\lambda} + \partial_\lambda C_{\mu\nu}\right)\left(\partial^\mu C^{\nu\lambda}\right) \pm m \left(\partial_\mu C^{\nu\lambda}\right)C_\nu \pm m C^{\nu\lambda}\left(\partial^\mu C_{\mu\nu}\right) + \left(\partial_\mu \bar{\beta}_\nu - \partial_\nu \bar{\beta}_\mu\right)\left(\partial^\mu C^{\nu\lambda}\right) - \frac{1}{2}\left[\pm m \bar{\beta}_\mu - \partial^\mu \bar{\beta}\right]\left[\pm m \beta_\mu - \partial^\mu \beta\right] - \left(\partial_\mu \bar{\beta}_\nu - \partial_\nu \bar{\beta}_\mu\right)\left(\partial^\mu \beta^{\nu}\right) - \partial^\mu C_2 \partial^\nu C_2 - m^2 C_2 C_2 + \left[(\partial \cdot \bar{\beta}) \mp m \bar{\beta}\right]B - \left[(\partial \cdot \phi) \mp m \phi\right]B_1 - \left[(\partial \cdot \beta) \mp m \beta\right]B_2 + \left[\partial_\nu C^{\mu\nu} - \partial^\mu C_2 \mp \frac{m}{2} C^{\mu\nu}\right]\bar{f}_\mu + 2\bar{F}^\mu\bar{f}_\mu + 2\bar{F}\bar{f} - \left[\partial_\nu C^{\mu\nu} - \partial^\mu C_1 \mp \frac{m}{2} C^{\mu\nu}\right]\bar{F}_\mu + \left[\frac{1}{2} (\partial \cdot \bar{C}) \pm m C_1\right]\bar{F} - \left[\frac{1}{2} (\partial \cdot C) \pm m C_1\right]\bar{f} - B B_2 - \frac{1}{2} B^2,
\]

which also incorporates the gauge-fixing and FP-ghost terms. The Lagrangian density \(L_B\) is characterized by the two bosonic as well as the four fermionic auxiliary fields \(B_{\mu\nu}, B_\mu, \bar{F}_\mu, \bar{f}_\mu, \bar{F}, \bar{f}\), respectively, and the rest of the symbols are same as in the Lagrangian density \((5)\). For the sake of readers’ convenience, we have provided a glossary of all the fields that are present in \((5)\) and \((6)\), various aspects that describe their true nature, their ghost numbers, etc., in a tabulated form in our Appendix D.

The generalizations of the classical gauge symmetry transformations \((4)\) at the quantum level has been obtained in our earlier work \([18]\) where the AVSA to BRST formalism has been exploited in the derivations of the proper (anti-)BRST transformations. It is a bit involved but straightforward to note that the following off-shell nilpotent \((s_b^2 = 0)\) BRST transformations \((s_b)\)

\[
s_b A^{\mu\nu\lambda} = \partial_\mu C^{\mu\lambda} + \partial_\nu C^{\nu\lambda} + \partial_\lambda C_{\mu\nu}, \quad s_b C^{\mu\nu\lambda} = \partial_\mu \beta_\nu - \partial_\nu \beta_\mu, \\
s_b B_{\mu\nu} = \left(\partial_\mu f_\nu - \partial_\nu f_\mu\right), \quad s_b C_{\mu\nu} = B_{\mu\nu}, \\
s_b \Phi_{\mu\nu} = \pm m C_{\mu\nu} - \left(\partial_\mu C_\nu - \partial_\nu C_\mu\right), \quad s_b \bar{F}_\mu = -\partial_\mu B, \\
s_b C_\mu = \pm m \beta_\mu - \partial_\mu \beta, \quad s_b \phi_\mu = f_\mu, \quad s_b \bar{\beta}_\mu = \partial_\mu C_2, \\
s_b \bar{B}_\mu = \pm m f_\mu - \partial_\mu \bar{f}, \quad s_b \bar{f}_\mu = -\partial_\mu B_1, \quad s_b \bar{\beta}_\mu = F_\mu, \\
s_b \bar{C}_\mu = B_\mu, \quad s_b \bar{C}_2 = B_2, \quad s_b \bar{C}_1 = -B_1, \\
s_b \phi = f, \quad s_b \beta = \pm m C_2, \quad s_b \bar{\beta} = F, \\
s_b \bar{F} = \mp m B, \quad s_b \bar{f} = \mp m B_1, \quad s_b C_1 = -B, \\
s_b \left[H_{\mu\nu\lambda\zeta}, B_{\mu\nu}, B_\mu, f_\mu, F_\mu, F, \bar{f}, B, B_1, B_2, C_2\right] = 0,
\]

are the symmetry transformations for the action integral \(S_1 = \int d^D x L_B\) because the La-
The action integral remains invariant under the anti-BRST symmetry transformations (sab) as

\[
\begin{align*}
    s_{ab} A_{\mu\lambda} &= \partial_\mu \bar{C}_{\nu\lambda} + \partial_\nu \bar{C}_{\lambda\mu} + \partial_\lambda \bar{C}_{\mu\nu}, \\
    s_{ab} B_{\mu\nu} &= (\partial_\mu \bar{f}_\nu - \partial_\nu \bar{f}_\mu), \\
    s_{ab} C_{\mu\nu} &= \bar{B}_{\mu\nu}, \\
    s_{ab} \Phi_{\mu\nu} &= \pm m \bar{C}_{\mu\nu} - (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu), \\
    s_{ab} F_{\mu} &= -\partial_\mu B_2, \\
    s_{ab} \bar{C}_\mu &= \pm m \bar{\beta}_\mu - \partial_\mu \bar{\beta}_\mu, \\
    s_{ab} \phi_{\mu} &= \bar{f}_\mu, \\
    s_{ab} \bar{\beta}_\mu &= \bar{B}_2, \\
    s_{ab} F &= \mp m B_2, \\
    s_{ab} \bar{C}_1 &= -B_2, \\
    s_{ab} \bar{C}_1 &= B_1, \\
    s_{ab} [H_{\mu\nu\lambda}, \bar{B}_{\mu\nu}, \bar{B}_\mu, \bar{F}_\mu, \bar{f}_\mu, \bar{f}, B, B_1, B_2, \bar{C}_2] &= 0,
\end{align*}
\]

which transform the perfectly anti-BRST invariant Lagrangian density (L_B) to the following total spacetime derivative, namely;

\[
\begin{align*}
    s_{ab} L_B &= \partial_\mu \left[(\partial^\mu \bar{C}^{\nu\lambda} + \partial^\nu \bar{C}^{\lambda\mu} + \partial^\lambda \bar{C}^{\mu\nu}) \bar{B}_{\nu\lambda} + B \partial^\mu \bar{C}_2 \right] \\
    &= - B_2 \bar{F}_\mu - B_1 \bar{f}_\mu - (\partial^\mu \bar{\beta}^{\nu} - \partial^\nu \bar{\beta}^{\mu}) \bar{F}_\nu + \frac{1}{2} (\pm m \bar{\beta}^{\mu} - \partial^\mu \bar{\beta}) \bar{F} \\
    &= (\partial^\mu \bar{C}^{\nu} - \partial^\nu \bar{C}^{\mu}) \bar{B}_\nu \mp m \bar{B}_{\mu\nu} \bar{C}_\nu - \frac{1}{2} B^\mu \bar{f} \\
    &= \pm m C^{\mu\nu} (\pm m \bar{\beta}_\nu - \partial_\nu \bar{\beta}) \pm m (\partial^\mu \bar{\beta}^{\nu} - \partial^\nu \bar{\beta}^{\mu}) C_\nu.
\end{align*}
\]

This observation, once again, establishes the fact that the action integral \( S_2 = \int d^D x \, L_B \) remains invariant under the anti-BRST symmetry transformations (sab) which have been listed in (9). This happens due to the application of Gauss’s divergence theorem which implies that all the physical fields vanish off as \( x \to \pm \infty \).

We end this section with a few final remarks. First, the (anti-)BRST invariant coupled (but equivalent) Lagrangian densities (6) and (5) respect both the BRST and anti-BRST symmetry transformations together on the submanifold of the quantum Hilbert space of fields where the Curci-Ferrari (CF) type restrictions \((B_{\mu\nu} + \bar{B}_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu, B_\mu + \bar{B}_\mu = 0)\).
$\pm m \phi_{\mu} - \partial_{\mu} \phi$, $f_{\mu} + \tilde{F}_{\mu} = \partial_{\mu} C_{1}$, $\tilde{f}_{\mu} + F_{\mu} = \partial_{\mu} \tilde{C}_{1}$, $f + \tilde{F} = \pm m C_{1}$ and $\tilde{f} + F = \pm m \tilde{C}_{1}$ of our St"{u}ckelberg-modified theory are satisfied (see, e.g. [18] for details). Second, it is quite straightforward to note that the original St"{u}ckelberg-modified Lagrangian density $L_{S}^{(A)}$ [cf. Eq. (1)] remains invariant $[s_{(a)b}] L_{S}^{(A)} = 0$ under the infinitesimal, off-shell nilpotent and continuous (anti-)BRST symmetry transformations $s_{(a)b}$ [cf. Eqs (9), (7)]. Finally, it is worthwhile to mention here that the absolute anticommutativity property $\{s_{b}, s_{ab}\} = 0$ is also satisfied when we invoke the validity of the above CF-type restrictions. For, instance, we note the following explicit computations, namely;

$$\begin{align*}
\{s_{b}, s_{ab}\} A_{\mu\nu\lambda} &= \partial_{\mu} (B_{\nu\lambda} + \tilde{B}_{\nu\lambda}) + \partial_{\nu} (B_{\lambda\mu} + \tilde{B}_{\lambda\mu}) + \partial_{\lambda} (B_{\mu\nu} + \tilde{B}_{\mu\nu}), \\
\{s_{b}, s_{ab}\} C_{\mu\nu} &= \partial_{\mu} (f_{\nu} + F_{\nu}) - \partial_{\nu} (f_{\mu} + F_{\mu}), \\
\{s_{b}, s_{ab}\} \tilde{C}_{\mu\nu} &= \partial_{\mu} (\tilde{f}_{\nu} + F_{\nu}) - \partial_{\nu} (\tilde{f}_{\mu} + F_{\mu}), \\
\{s_{b}, s_{ab}\} \phi_{\mu\nu} &= \pm m (B_{\mu\nu} + \tilde{B}_{\mu\nu}) - \{\partial_{\mu} (B_{\nu\lambda} + \tilde{B}_{\nu\lambda}) - \partial_{\nu} (B_{\lambda\mu} + \tilde{B}_{\lambda\mu})\}, \\
\{s_{b}, s_{ab}\} C_{\mu} &= \pm m (f_{\mu} + \tilde{F}_{\mu}) - \partial_{\mu} (f + \tilde{F}), \\
\{s_{b}, s_{ab}\} \tilde{C}_{\mu} &= \pm m (\tilde{f}_{\mu} + F_{\mu}) - \partial_{\mu} (\tilde{f} + F),
\end{align*}$$

which demonstrate that $\{s_{b}, s_{ab}\} = 0$ for the above fields provided we invoke the validity of: $B_{\mu\nu} + \tilde{B}_{\mu\nu} = \partial_{\mu} \phi_{\nu} - \partial_{\nu} \phi_{\mu}$, $B_{\mu} + \tilde{B}_{\mu} = \pm m \phi_{\mu} - \partial_{\mu} \phi$, $f_{\mu} + \tilde{F}_{\mu} = \partial_{\mu} C_{1}$, $\tilde{f}_{\mu} + F_{\mu} = \partial_{\mu} \tilde{C}_{1}$, $f + \tilde{F} = \pm m C_{1}$ and $\tilde{f} + F = \pm m \tilde{C}_{1}$. It is straightforward to check that the absolute anticommutativity property (i.e. $\{s_{b}, s_{ab}\} = 0$) is automatically satisfied for the rest of all the fields of our theory which is described by the Lagrangian densities $L_{B}$ and $L_{\tilde{B}}$. We also point out that only a single CF-type restriction is required in the proof of the absolute anticommutativity property of the (anti-)BRST symmetry transformations for the fields $A_{\mu\nu\lambda}$, $C_{\mu\nu}$, $\tilde{C}_{\mu\nu}$. On the other hand, for the fields $\Phi_{\mu\nu}$, $C_{\mu}$, $\tilde{C}_{\mu}$, the absolute anticommutativity property of the (anti-)BRST symmetry transformations is satisfied when a set of two CF-type restrictions are invoked together. Finally, we establish that the sanctity of the CF-type restrictions is maintained even at the level of the transformed fields under the (anti-)BRST symmetry transformations where we find that the off-shell nilpotency, absolute anticommutativity and (anti-)BRST invariant CF-type restrictions are all intertwined together in an elegant manner (cf. Appendix E for details).

### 3 Generator for the Gauge Symmetry Transformations: First-Class Constraints and Conserved Charge of our theory

In this section, we perform the constraint analysis for the St"{u}ckelberg-modified Lagrangian density ($L_{S}^{(A)}$) and establish that it is endowed with the first-class constraints in the terminology of Dirac's prescription for the classification of constraints [23-26]. These first-class constraints are present in the generator for the infinitesimal, continuous and local classical gauge symmetry transformations (4). In this context, we note that we have the following
canonical conjugate momenta for the St"uckelberg-modified Lagrangian density (1), namely;

\[ \Pi_{\mu\nu\lambda}^{(A)} = \frac{\partial L_s}{\partial (\partial_0 A_{\mu\nu\lambda})} \equiv \frac{1}{3} H^{0\mu\nu\lambda}, \]

\[ \Pi_{\mu\nu}^{(\phi)} = \frac{\partial L_s}{\partial (\partial_0 \Phi_{\mu\nu})} \equiv -\Sigma^{0\mu\nu} \pm m A^{0\mu\nu}, \tag{12} \]

w.r.t. the basic 3-form field \( A_{\mu\nu\lambda} \) and St"uckelberg 2-form field \( \Phi_{\mu\nu} \). From (12), it is straightforward to check that we have the following primary constraints, respectively, namely;

\[ \Pi_{0ij}^{(A)} = \frac{1}{3} H^{00ij} \approx 0, \quad \Pi_{0i}^{(\phi)} = -\Sigma^{00i} \pm m A^{00i} \approx 0. \tag{13} \]

These constraints are weakly zero in the terminology of Dirac. As a consequence, the first-order time derivative on them can be performed to derive the secondary constraints on the theory. The Hamiltonian formalism is the best approach to obtain the successive constraints. However, for our simple case, the Lagrangian formalism (see, e.g. [27]) is simple, beautiful and straightforward. The Euler-Lagrange (EL) equation of motion (EoM) w.r.t. the basic fields \( A_{\mu\nu\lambda} \) and \( \Phi_{\mu\nu} \), obtained from \( L_s^{(A)} \) [cf. Eq. (1)], are as follows:

\[ \partial_\rho H^{\rho\mu\nu\lambda} + m^2 A^{\mu\nu\lambda} = 0, \quad \partial_\rho \Sigma^{\rho\mu\nu} \pm m \partial_\rho A^{\rho\mu\nu} = 0. \tag{14} \]

It is clear that, for the choices: \( \mu = 0, \nu = j, \lambda = k \), we have the following:

\[ \partial_0 H^{00jk} + \partial_i H^{i0jk} + m^2 A^{0ijk} = 0, \]

\[ \partial_0 \Sigma^{00j} + \partial_i \Sigma^{0ij} \pm m \partial_0 A^{00j} \pm m \partial_i A^{i0j} = 0. \tag{15} \]

The above equations lead to the derivations of the secondary constraints on our theory as these are nothing but the outcome of setting the first-order “time” derivative of the primary constraints (13) equal to zero. In other words, we have the following secondary constraints emerging out from the time-evolution invariance [27] of the primary constraints:

\[ 3 \frac{\partial \Pi_{0ij}^{(A)}}{\partial t} = 3 \partial_k \Pi_{kij}^{(A)} \pm m \Pi_{ij}^{(\phi)} \approx 0 \implies \frac{\partial \Pi_{0ij}^{(A)}}{\partial t} = \partial_k \Pi_{kij}^{(A)} \mp m \Pi_{ij}^{(\phi)} \approx 0, \]

\[ \frac{\partial \Pi_{0i}^{(\phi)}}{\partial t} = \partial_j \Pi_{ji}^{(\phi)} \approx 0. \tag{16} \]

In the above, we have used, primarily, the definitions of the space components of the canonical conjugate momenta w.r.t. \( A_{\mu\nu\lambda} \) and \( \Phi_{\mu\nu} \) [cf. Eq. (12)], namely;

\[ \Pi_{0ij}^{(A)} = \frac{1}{3} H^{00ij}, \quad \Pi_{0i}^{(\phi)} = -\Sigma^{00i} \pm m A^{00i}, \tag{17} \]

which are deduced from the original definitions of the canonical conjugate momenta in (12). There are no further constraints on the theory. Since the primary as well as secondary constraints are expressed in terms of the components of the canonical conjugate momenta only, it is pretty obvious that their commutator(s) will be zero. Hence, our theory is endowed with the first-class constraints in the terminology of Dirac’s prescription for the
classification scheme of the constraints [23-26]. In other words, our theory is an example of a massive gauge theory where the mass and gauge invariance co-exist together. This is, primarily, due to the existence of the first-class constraints on our theory.

Before we write down the final expression for the generator of the gauge transformations (4) in terms of the constraints (13) and (16), we define the non-trivial equal-time canonical commutators for the St"{u}ckelberg-modified Lagrangian density \( \mathcal{L}_S^{(A)} \) as:

\[
\begin{align*}
[A_{0ij}(\vec{x}, t), \Pi_{0k}^{(A)}(\vec{y}, t)] &= \frac{i}{2!} (\delta_k^i \delta_j^l - \delta_k^j \delta_l^i) \delta^{(D-1)}(\vec{x} - \vec{y}), \\
[A_{ijk}(\vec{x}, t), \Pi_{lmn}^{(A)}(\vec{y}, t)] &= \frac{i}{3!} \left[ \delta_k^l \left( \delta_j^m \delta_l^k - \delta_j^k \delta_l^m \right) + \delta_i^m \left( \delta_j^k \delta_l^i - \delta_j^i \delta_l^k \right) + \delta_i^m \left( \delta_j^k \delta_l^m - \delta_j^m \delta_l^k \right) \right] \delta^{(D-1)}(\vec{x} - \vec{y}), \\
[\Phi_{0i}(\vec{x}, t), \Pi_{0j}^{(A)}(\vec{y}, t)] &= i \delta_l^j \delta^{(D-1)}(\vec{x} - \vec{y}), \\
[\Phi_{ij}(\vec{x}, t), \Pi_{kl}^{(A)}(\vec{y}, t)] &= \frac{i}{2!} (\delta_k^i \delta_j^l - \delta_l^i \delta_j^k) \delta^{(D-1)}(\vec{x} - \vec{y}).
\end{align*}
\]

The rest of all the brackets are trivially zero. In terms of all the four first-class constraints, the generator \( G^{(sm)} \), for our modified massive Abelian 3-form theory, is \([28,29]\)

\[
G^{(sm)} = \int d^{D-1}x \left[ \Pi^{0ij} \left( \partial_0 \Lambda_{ij} + \partial_j \Lambda_{0i} \right) + \Pi^{ij0} \left( \partial_i \Lambda_{j0} + \partial_j \Lambda_{0i} \right) + \Pi^{0ij} \left( \partial_i \Lambda_{0j} + \partial_j \Lambda_{0i} \right) \right]
- \left[ \partial_i \Pi^{ijk} + \frac{m}{3} \Pi_{(A)}^{ijk} \right] \Lambda_{jk} - \left[ \partial_j \Pi^{jki} + \frac{m}{3} \Pi_{(A)}^{jki} \right] \Lambda_{ki}
- \left[ \partial_k \Pi^{kij} + \frac{m}{3} \Pi_{(A)}^{kij} \right] \Lambda_{ij} + \Pi_{(A)}^{ij} \left[ \pm m \Lambda_{0i} - (\partial_0 \Lambda_i - \partial_i \Lambda_0) \right]
+ \partial_i \Pi_{(A)}^{ij} \Lambda_j + \partial_j \Pi_{(A)}^{ij} \Lambda_i,
\]

where the superscript \((sm)\) on the generator denotes that it is explicitly written for the St"{u}ckelberg-modified massive Abelian 3-form theory. In equation (19), the totally antisymmetric nature of \( \Pi^{0ij}, \Pi^{ijk}, \Lambda_{ij}, \Lambda_{0i}, \) etc., has been taken into account. Using the Gauss divergence theorem (where the gauge transformation parameters and fields have been treated as the physically well-defined objects which vanish off as \( x \rightarrow \pm \infty \), the above gauge symmetry generator \( G^{(sm)} \) can be re-expressed, in a more transparent, beautiful and useful fashion, as follows

\[
G^{(sm)} = \int d^{D-1}x \left[ \Pi^{0ij} \left( \partial_0 \Lambda_{ij} + \partial_i \Lambda_{j0} + \partial_j \Lambda_{0i} \right) \right]
+ \Pi^{ijk} \left( \partial_i \Lambda_{jk} + \partial_j \Lambda_{ki} + \partial_k \Lambda_{ij} \right)
+ \Pi_{(A)}^{0i} \left\{ \pm m \Lambda_{0i} - (\partial_0 \Lambda_i - \partial_i \Lambda_0) \right\}
+ \Pi_{(A)}^{ij} \left\{ \pm m \Lambda_{ij} - (\partial_i \Lambda_j - \partial_j \Lambda_i) \right\},
\]

where we have exploited the potential of a simple (but cute) algebraic trick: \( m \Pi^{ij} \Lambda_{ij} = (m/3) \Pi^{ij} \Lambda_{ij} + (m/3) \Pi^{jk} \Lambda_{jk} + (m/3) \Pi^{ki} \Lambda_{ki} \) where three Latin indices have been used. It is straightforward, using the non-trivial existing canonical brackets of Eq. (18), to check that
we have the following expressions for the infinitesimal gauge symmetry transformations for the basic fields \( (A_{\mu \nu \lambda}, \Phi_{\mu \nu}) \) in their independent component form, namely:

\[
\begin{align*}
\delta_g A_{0ij}(\vec{x}, t) &= -i \left[ A_{0ij}(\vec{x}, t), G \right] \\
\equiv &\left( \partial_0 \Lambda_{ij} + \partial_i \Lambda_{j0} + \partial_j \Lambda_{0i} \right), \\
\delta_g A_{ijk}(\vec{x}, t) &= -i \left[ A_{ijk}(\vec{x}, t), G \right] \\
\equiv &\left( \partial_i \Lambda_{jk} + \partial_j \Lambda_{ki} + \partial_k \Lambda_{ij} \right), \\
\delta_g \Phi_{0i}(\vec{x}, t) &= -i \left[ \Phi_{0i}(\vec{x}, t), G \right] \\
\equiv &\left( \pm m \Lambda_{0i} - (\partial_0 \Lambda_i - \partial_i \Lambda_0) \right), \\
\delta_g \Phi_{ij}(\vec{x}, t) &= -i \left[ \Phi_{ij}(\vec{x}, t), G \right] \\
\equiv &\left( \pm m \Lambda_{ij} - (\partial_i \Lambda_j - \partial_j \Lambda_i) \right),
\end{align*}
\]

(21)

which establish that the first-class constraints are indeed the generator for the classical local, continuous and infinitesimal gauge symmetry transformations (4). A close look at (21) shows that we have already obtained: \( \delta_g A_{\mu \nu \lambda} = \partial_\mu \Lambda_{\nu \lambda} + \partial_\nu \Lambda_{\lambda \mu} + \partial_\lambda \Lambda_{\mu \nu} \) and \( \delta_g \Phi_{\mu \nu} = \pm m \Lambda_{\mu \nu} - (\partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu) \). These results automatically imply that we also have: \( \delta_g H_{\mu \nu \lambda \zeta} = 0 \) and \( \delta_g \Sigma_{\mu \nu} = \pm m (\partial_\mu \Lambda_{\nu \lambda} + \partial_\nu \Lambda_{\lambda \mu} + \partial_\lambda \Lambda_{\mu \nu}) \) due to the local and infinitesimal gauge symmetry transformations (i.e. \( \delta_g A_{\mu \nu \lambda}, \delta_g \Phi_{\mu \nu} \)) on the basic fields \( A_{\mu \nu \lambda} \) and \( \Phi_{\mu \nu} \). Thus, we have derived all the infinitesimal gauge transformations that are listed in (4).

We establish now a deep connection between the Noether conserved current and charge (corresponding to the infinitesimal gauge transformations) and the first-class constraints of the theory under consideration. In this context, we can compute the Noether conserved current for the infinitesimal gauge symmetry transformations (4) for the Stückelberg-modified massive Abelian 3-form theory. This current \( [J^\mu_{(sm)}] \) turns out to be the following

\[
J^\mu_{(sm)} = \frac{1}{3} H'^{\mu \nu \lambda \zeta} (\partial_\nu \Lambda_{\lambda \zeta} + \partial_\lambda \Lambda_{\nu \zeta} + \partial_\zeta \Lambda_{\nu \lambda})
\]

\[
+ \left( \pm m A^{\mu \nu \lambda} - \Sigma^{\mu \nu \lambda} \right) \left[ \pm m \Lambda_{\mu \lambda} - (\partial_\mu \Lambda_\lambda - \partial_\lambda \Lambda_\mu) \right],
\]

(22)

which can be shown to be conserved \( (\partial_\mu J^\mu_{(sm)} = 0) \) due to the EoMs (15) that have already been derived from the Stückelberg-modified Lagrangian density \( (\mathcal{L}_S^{(A)}) \) w.r.t. the basic fields \( A_{\mu \nu \lambda} \) and \( \Phi_{\mu \nu} \) of our theory. Here the subscript \( (sm) \) on the Noether current denotes that it is written for the Stückelberg-modified massive Abelian 3-form theory. This Noether conserved current leads to the definition of the conserved charge \( Q_{(sm)} \) as

\[
Q_{(sm)} = \int d^{D-1}x \ J^0_{(sm)} = \int d^{D-1}x \ \left[ \frac{1}{3} H'^{0 \nu \lambda \zeta} (\partial_\nu \Lambda_{\lambda \zeta} + \partial_\lambda \Lambda_{\nu \zeta} + \partial_\zeta \Lambda_{\nu \lambda}) \right]
\]

\[
+ \left( \pm m A^{0 \nu \lambda} - \Sigma^{0 \nu \lambda} \right) \left[ \pm m \Lambda_{\nu \lambda} - (\partial_\nu \Lambda_\lambda - \partial_\lambda \Lambda_\nu) \right]
\]

(23)

In view of the fact that our theory is endowed with a set of four first-class constraints, we have to expand the r.h.s. of the above equation carefully. In other words, we do not have to set the constraints strongly equal to zero which appear in the expression for the zeroth component of the conserved Noether current (i.e. \( J^0_{(sm)} \)). To be precise, we have the following expression for the conserved charge, namely:

\[
Q_{(sm)} = \int d^{D-1}x \ \left[ \frac{1}{3} H'^{0 \nu \lambda} (\partial_\nu \Lambda_{ij} + \partial_i \Lambda_{j0} + \partial_j \Lambda_{0i}) \right]
\]

\[
+ \frac{1}{3} H'^{0ijk} (\partial_i \Lambda_{jk} + \partial_j \Lambda_{ki} + \partial_k \Lambda_{ij})
\]

\[
+ \left( \pm m A^{0ij} - \Sigma^{0ij} \right) \left[ \pm m \Lambda_{0i} - (\partial_0 \Lambda_i - \partial_i \Lambda_0) \right]
\]

\[
+ \left( \pm m A^{0ij} - \Sigma^{0ij} \right) \left[ \pm m \Lambda_{ij} - (\partial_i \Lambda_j - \partial_j \Lambda_i) \right]
\]

(24)
Taking into account, the definition of the constraints in the equations (13) and (16), we have the following charge in terms of the first-class constraints, namely;

\[
Q_{(sm)} = \int d^{D-1}x \left[ \Pi^{0ij} (\partial_0 \Lambda_{ij} + \partial_i \Lambda_{j0} + \partial_j \Lambda_{0i}) 
+ \Pi^{ijk} (\partial_i \Lambda_{jk} + \partial_j \Lambda_{ki} + \partial_k \Lambda_{ij}) 
+ \Pi_i^0 \{ \pm m \Lambda_{0i} - (\partial_0 \Lambda_i - \partial_i \Lambda_0) \} 
+ \Pi_j^i \{ \pm m \Lambda_{ij} - (\partial_i \Lambda_j - \partial_j \Lambda_i) \} \right].
\]

(25)

which is nothing but the final form of the generator \(G^{(sm)}\) that is quoted in (20). Taking into account the Gauss divergence theorem and the following simple algebraic trick

\[
\pm m \Pi_{(\phi)}^{ij} \Lambda_{ij} = \pm \frac{m}{3} \Pi_{(\phi)}^{jk} \Lambda_{jk} \pm \frac{m}{3} \Pi_{(\phi)}^{ki} \Lambda_{ki},
\]

(26)

which involves three Latin indices, we can express the above conserved charge as follows

\[
Q_{(sm)} = \int d^{D-1}x \left[ \Pi^{0ij} (\partial_0 \Lambda_{ij} + \partial_i \Lambda_{j0} + \partial_j \Lambda_{0i}) 
+ \Pi_i^0 \{ \pm m \Lambda_{0i} - (\partial_0 \Lambda_i - \partial_i \Lambda_0) \} 
- (\partial_i \Pi^{ijk} + \frac{m}{3} \Pi_{(\phi)}^{jk} \Lambda_{jk} - (\partial_j \Pi^{kji} + \frac{m}{3} \Pi_{(\phi)}^{ki} \Lambda_{ki} 
- (\partial_k \Pi^{kij} + \frac{m}{3} \Pi_{(\phi)}^{ij} \Lambda_{ij} + \partial_i \Pi_{(\phi)}^{ij} \Lambda_j + \partial_j \Pi_{(\phi)}^{ji} \Lambda_i \right],
\]

(27)

which is nothing but the expression for the generator \(G^{(sm)}\) that has been written in the equation (19) in terms of the primary and secondary constraints. Hence, there is a deep connection between the classical generator and the Noether conserved charge for the D-dimensional St"uckelberg-modified massive Abelian 3-form gauge theory.

We conclude this section with the following crucial and clinching remarks. First, the St"uckelberg-technique of compensating field(s) is responsible for the conversion of the second-class constraints of the original massive Abelian 3-form theory into their counterparts first-class constraints. Second, the resulting first-class constraints generate the classical local, continuous and infinitesimal gauge symmetry transformations for the St"uckelberg-modified Lagrangian density \((L_{s(A)})\). Third, appropriate power(s) of the rest-mass \(m\) has to be taken into account so that, in the gauge symmetry transformations, the different gauge symmetry transformation parameters appear with the appropriate mass dimension (in the natural units: \(\hbar = c = 1\)) as is the case for \(\delta_g \Phi_{\mu \nu}\) [cf. Eqs. (4), (21)]. Finally, if the expressions for the first-class constraints appear in the zeroth component of the Noether conserved current [e.g. \(J^0_{(sm)}\)], they should not be set strongly equal to zero in the derivation of the final expression for the Noether conserved charge.

4 Equations of Motion: Coupled (but Equivalent) (Anti-)BRST Invariant Lagrangian Densities

It is very interesting to pinpoint the fact that the Euler-Lagrange (EL) equations of motion (EoM) from the coupled Lagrangian densities \(L_B\) and \(\bar{L}_B\) [cf. Eqs. (5), (6)] with respect
to the auxiliary fields \((B_{\mu\nu}, \bar{B}_{\mu\nu}, B_\mu, \bar{B}_\mu, f_\mu, \bar{f}_\mu, F_\mu, \bar{F}_\mu, f, \bar{f}, F)\) are:

\[
\begin{align*}
B_{\mu\nu} &= (\partial^\nu A_{\rho\mu}) + \frac{1}{2} (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu \mp m \Phi_{\mu\nu}), \\
\bar{B}_{\mu\nu} &= - (\partial^\nu A_{\rho\mu}) + \frac{1}{2} (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu \pm m \Phi_{\mu\nu}), \\
B_\mu &= - (\partial^\rho \Phi_{\rho\mu}) + \frac{1}{2} (\pm m \phi_\mu - \partial_\mu \phi), \quad \bar{B}_\mu = (\partial^\rho \Phi_{\rho\mu}) + \frac{1}{2} (\pm m \phi_\mu - \partial_\mu \phi), \\
F_\mu &= \frac{1}{2} (\partial^\rho \bar{C}_{\rho\mu}) + \frac{1}{2} \partial_\mu \bar{C}_1 \mp \frac{m}{4} \bar{C}_\mu, \quad \bar{F}_\mu = - \frac{1}{2} (\partial^\rho \bar{C}_{\rho\mu}) + \frac{1}{2} \partial_\mu \bar{C}_1 \pm \frac{m}{4} \bar{C}_\mu, \\
f_\mu &= \frac{1}{4} (\partial \cdot \bar{C}) \pm \frac{m}{2} \bar{C}_1, \quad \bar{f} = \frac{1}{4} (\partial \cdot \bar{C}) \pm \frac{m}{2} \bar{C}_1, \\
f &= \frac{1}{4} (\partial \cdot C) \pm \frac{m}{2} C_1, \quad \bar{f} = \frac{1}{4} (\partial \cdot C) \pm \frac{m}{2} C_1. 
\end{align*}
\]

It is evident that we obtain the following relationships using the straightforward algebra

\[
\begin{align*}
B_{\mu\nu} + \bar{B}_{\mu\nu} &= \pm \partial_\mu \phi_\nu - \partial_\nu \phi_\mu, \quad B_\mu + \bar{B}_\mu = \pm m \phi_\mu - \partial_\mu \phi, \\
\bar{f}_\mu + F_\mu &= \partial_\mu \bar{C}_1, \quad f_\mu + \bar{F}_\mu = \partial_\mu C_1, \\
f + \bar{f} &= \pm m C_1, \quad \bar{f} + F = \pm m \bar{C}_1, 
\end{align*}
\]

which are nothing but the \((anti-)BRST\) invariant CF-type restrictions that are responsible for the absolute anticommutativity (i.e. \(\{s_a, s_b\} = 0\)) of the \((anti-)BRST\) symmetry transformations \([s_a]b\). Furthermore, we obtain the following \emph{common} EL-EoMs from the coupled (but equivalent) Lagrangian densities \(\mathcal{L}_B\) and \(\mathcal{L}_{\bar{B}}\), namely;

\[
\begin{align*}
B_1 &= - (\partial \cdot \phi \mp m \phi), \quad B_2 = (\partial \cdot \bar{\beta} \mp m \bar{\beta}), \quad B = - (\partial \cdot \bar{\beta} \mp m \beta), \\
\Box \beta \mp m (\partial \cdot \beta) \mp 2m B &= 0, \quad \Box \bar{\beta} \mp m (\partial \cdot \bar{\beta}) \mp 2m \bar{B} &= 0, \\
\Box \bar{\beta}_\mu - \partial_\mu (\partial \cdot \bar{\beta}) + \partial_\mu B_2 - \frac{m^2}{2} \bar{\beta}_\mu \pm \frac{m}{2} \partial_\mu \bar{\beta} &= 0, \\
\Box \bar{\beta}_\mu - \partial_\mu (\partial \cdot \bar{\beta}) - \partial_\mu B - \frac{m^2}{2} \beta_\mu \pm \frac{m}{2} \partial_\mu \beta &= 0, \\
(\Box - m^2) C_2 &= 0, \quad (\Box - m^2) \bar{C}_2 = 0. 
\end{align*}
\]

w.r.t. the auxiliary fields \((B_1, B, B_2)\) and the \emph{bosonic} \((anti-)\)ghost vector and scalar fields \((\beta_\mu, \bar{\beta}_\mu, \beta, \bar{\beta})\) as well as the \emph{fermionic} \((anti-)\)ghost fields \((\bar{C}_2)C_2\). It is clear, from the above equations, that the auxiliary fields \((B_1, B, B_2)\) carry the ghost numbers \((0, +2, -2)\), respectively, because the basic fields \((\phi, \beta, \bar{\beta})\) also have the same ghost numbers. Before we proceed further, it is worthwhile to mention that the \((anti-)BRST\) invariant CF-type restrictions (29) are \textit{not} the EL-EoMs as they are not derived from a single Lagrangian density and/or the minimization of the corresponding action integral.

We now focus on the EL-EoM, derived from the Lagrangian densities \(\mathcal{L}_B\) and \(\mathcal{L}_{\bar{B}}\), that are \textit{different} in appearance but \textit{equivalent} on the submanifold of the Hilbert space of quantum fields where the CF-type restrictions (29) are satisfied/valid. At this stage, first
of all, we derive explicitly the EL-EoMs from the Lagrangian density $\mathcal{L}_B$ w.r.t. the fields: $A_{\mu\nu\lambda}, \Phi_{\mu\nu}, C_{\mu\nu}, \Phi_\mu, C_\mu, \phi, C_1, C_1$. These are as follows:

\[
\begin{align*}
\partial^\rho H^\rho_{\mu\nu\lambda} + m^2 A_{\mu\nu\lambda} &\equiv m (\partial_\mu \Phi_{\nu\lambda} + \partial_\nu \Phi_{\lambda\mu} + \partial_\lambda \Phi_{\mu\nu}) \\
+ (\partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}) &= 0, \\
\Box \Phi_{\mu\nu} - \partial_\mu (\partial^{\rho} \Phi_{\rho\mu}) + \partial_\nu (\partial^{\rho} \Phi_{\rho\nu}) &\equiv m (\partial^\rho A_{\rho\mu\nu}) \\
+ \frac{1}{2} (\partial_\mu B_\nu - \partial_\nu B_\mu) &\pm \frac{m}{2} B_{\mu\nu} = 0, \\
\Box C_{\mu\nu} - \partial_\mu (\partial^{\rho} C_{\rho\mu}) + \partial_\nu (\partial^{\rho} C_{\rho\nu}) &\equiv m (\partial^\rho C_{\rho\mu\nu}) \\
+ \frac{1}{2} (\partial_\mu f_\nu - \partial_\nu f_\mu) &\pm \frac{m}{2} C_{\mu\nu} = 0, \\
\partial_\mu B^{\mu\nu} - \partial^\nu B_1 &\equiv \frac{m}{2} B^\nu = 0, \quad \frac{1}{2} (\partial \cdot B) \equiv m B_1 = 0, \\
\Box \bar{C}_\mu - \partial_\mu (\partial^{\rho} \bar{C}_{\rho\mu}) &\equiv m (\partial^\rho \bar{C}_{\rho\mu}) \\
(\partial \cdot f) &\equiv m f = 0.
\end{align*}
\]

The above kinds of EL-EoMs can be derived from the Lagrangian density $\mathcal{L}_B$, too. These equations of motion are:

\[
\begin{align*}
\partial^\rho H^\rho_{\mu\nu\lambda} + m^2 A_{\mu\nu\lambda} &\equiv m (\partial_\mu \Phi_{\nu\lambda} + \partial_\nu \Phi_{\lambda\mu} + \partial_\lambda \Phi_{\mu\nu}) \\
- (\partial_\mu \bar{B}_{\nu\lambda} + \partial_\nu \bar{B}_{\lambda\mu} + \partial_\lambda \bar{B}_{\mu\nu}) &= 0, \\
\Box \Phi_{\mu\nu} - \partial_\mu (\partial^{\rho} \Phi_{\rho\mu}) + \partial_\nu (\partial^{\rho} \Phi_{\rho\nu}) &\equiv m (\partial^\rho A_{\rho\mu\nu}) \\
- \frac{1}{2} (\partial_\mu \bar{B}_\nu - \partial_\nu \bar{B}_\mu) &\pm \frac{m}{2} \bar{B}_{\mu\nu} = 0, \\
\Box C_{\mu\nu} - \partial_\mu (\partial^{\rho} C_{\rho\mu}) + \partial_\nu (\partial^{\rho} C_{\rho\nu}) &\equiv m (\partial^\rho C_{\rho\mu\nu}) \\
- \frac{1}{2} (\partial_\mu \bar{f}_\nu - \partial_\nu \bar{f}_\mu) &\pm \frac{m}{2} C_{\mu\nu} = 0, \\
\partial_\mu \bar{B}^{\mu\nu} - \partial^\nu B_1 &\equiv \frac{m}{2} \bar{B}^\nu = 0, \quad \frac{1}{2} (\partial \cdot \bar{B}) \equiv m B_1 = 0, \\
\Box \bar{C}_\mu - \partial_\mu (\partial^{\rho} \bar{C}_{\rho\mu}) &\equiv m (\partial^\rho \bar{C}_{\rho\mu}) \\
(\partial \cdot \bar{f}) &\equiv m \bar{f} = 0.
\end{align*}
\]

(31)
It is quite straightforward to note that the above EL-EoMs [cf. Eqs (31) and (32)] are equivalent on the submanifold of the Hilbert space of the quantum fields where the CF-type conditions (29) are satisfied. Let us take an example to corroborate this claim. The equation of motion w.r.t. $\Phi_{\mu\nu}$ fields from $\mathcal{L}_B$ and $\bar{\mathcal{L}}_B$ can be subtracted from each-other to yield the following explicit expression:

$$\frac{1}{2} \left[ \partial_\mu (B_\nu + \bar{B}_\nu) - \partial_\nu (B_\mu + \bar{B}_\mu) \right] \mp \frac{m}{2} (B_{\mu\nu} + \bar{B}_{\mu\nu}) = 0. \quad (33)$$

Now it is elementary to check that the substitution of the CF-type restrictions: $B_{\mu\nu} + \bar{B}_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu$, $B_\mu + \bar{B}_\mu = \pm m \phi_\mu - \partial_\mu \phi$ in the above equation establishes that the equations of motion, derived from $\mathcal{L}_B$ and $\bar{\mathcal{L}}_B$, are equivalent on the submanifold in the Hilbert space of quantum fields which is defined by the CF-type restrictions. Similarly, it can be seen that all the equations of motion (31) and (32) are equivalent.

We end this section with the following final remarks. First of all, one can deduce some very beautiful equations from the ones that have been obtained as the EL-EoMs from $\mathcal{L}_B$ and $\bar{\mathcal{L}}_B$. For instance, we observe that the following

\begin{align*}
(\Box - m^2) C_1 &= 0, \quad (\Box - m^2) \bar{C}_1 = 0, \quad (\Box - m^2) (\partial \cdot C) = 0, \\
(\Box - m^2) (\partial \cdot \bar{C}) &= 0, \quad (\Box - m^2) B_1 = 0, \quad (\Box - m^2) B_2 = 0, \\
(\Box - m^2) B &= 0, \quad \Box B_{\mu} - \partial_\mu (\partial \cdot B) \mp m (\partial^\rho B_{\rho\mu}) = 0, \\
\pm m [\Box \bar{C}_\mu - \partial_\mu (\partial \cdot \bar{C})] + [\Box F_\mu - \partial_\mu (\partial \cdot F)] - m^2 (\partial^\rho \bar{C}_{\rho\mu}) &= 0, \\
\pm m [\Box C_\mu - \partial_\mu (\partial \cdot C)] + [\Box f_\mu - \partial_\mu (\partial \cdot f)] - m^2 (\partial^\rho C_{\rho\mu}) &= 0, \quad (34)
\end{align*}

can be deduced from the EL-EoM [cf. Eq. (31)] that have been derived from the perfectly BRST invariant Lagrangian density $\mathcal{L}_B$. In exactly similar fashion, the concise and beautiful equations of motion can be derived from the Lagrangian density $\mathcal{L}_B$, too. Second, it can be seen, once again, that the above equations (that have been derived from $\mathcal{L}_B$) will be equivalent to the ones that are obtained from $\mathcal{L}_B$ on the submanifold in the quantum Hilbert space of the fields that are defined by the CF-type restrictions (29). Third, it is very interesting to observe that the CF-type restrictions can be derived from the EL-EoMs from the coupled (but equivalent) Lagrangian densities $\mathcal{L}_B$ and $\bar{\mathcal{L}}_B$, too, besides their derivations from the requirement of the absolute anticommutativity property of the (anti-)BRST symmetry transformations [cf. Eq. (11)]. Finally, it is worthwhile to point out that the CF-type restrictions (29) have been precisely derived from the augmented superfield approach to BRST formalism in our earlier work [18]. It turns out that they are physical restrictions on our theory because they are found to be (anti-)BRST invariant quantities. In other words, it can be checked that the following are true, namely;

\begin{align*}
&\quad \quad \quad \quad s_{(a)b}[B_{\mu\nu} + \bar{B}_{\mu\nu} - (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu)] = 0, \quad s_{(a)b}[\bar{f} + F \mp m \bar{C}_1] = 0, \\
&\quad \quad \quad \quad s_{(a)b}[\bar{f} + F - \partial_\mu \bar{C}_1] = 0, \quad s_{(a)b}[B_\mu + \bar{B}_\mu \mp m \phi_\mu + \partial_\mu \phi] = 0, \\
&\quad \quad \quad \quad s_{(a)b}[f + \bar{F} \mp m C_1] = 0, \quad s_{(a)b}[f_\mu + \bar{F}_\mu - \partial_\mu C_1] = 0, \quad (35)
\end{align*}

where $s_{(a)b}$ are the off-shell nilpotent (anti-)BRST symmetry transformations (9) and (7), respectively. We shall see later that the (anti-)BRST invariant CF-type restrictions will
be also derived from (i) the equality of the coupled Lagrangian densities \( \mathcal{L}_B \) and \( \mathcal{L}_B \) in our Appendix A, and (ii) the requirement of the absolute anticommutativity of the off-shell nilpotent versions of the (anti-)BRST charges \( [Q_{(a)b}] \) in our Appendix B. It is worthwhile to mention, in passing, that the CF-type restrictions have also been derived from the (anti-)BRST invariance of \( \mathcal{L}_B \) and \( \mathcal{L}_B \) (see, e.g. [18] for details).

5 Continuous Symmetries: Conservation Laws

This section is divided into two parts. In Subsec. A, we derive the Noether conserved currents and provide the proof of their conservation laws. Our Subsec. B is devoted to the derivation of the Noether conserved charges where their salient features have been pointed out in a quite elaborate manner.

5.1 Conserved Currents: Noether Theorem

Whenever the action integral (or the Lagrangian density) remains invariant under the continuous symmetry transformation, according to Noether’s theorem, there is always existence of the conserved current (as well as corresponding conserved charge). In this context, first of all, we focus on the following ghost-scale continuous symmetry transformations with a global (i.e. spacetime independent) bosonic scale transformation parameter \( \Omega \) as

\[
\begin{align*}
C_2 & \rightarrow e^{3\Omega} C_2, \quad \bar{C}_2 \rightarrow e^{-3\Omega} \bar{C}_2, \quad \beta_\mu \rightarrow e^{2\Omega} \beta_\mu, \quad \bar{\beta}_\mu \rightarrow e^{-2\Omega} \bar{\beta}_\mu, \\
\beta & \rightarrow e^{2\Omega} \beta, \quad \bar{\beta} \rightarrow e^{-2\Omega} \bar{\beta}, \quad B \rightarrow e^{2\Omega} B, \quad B_2 \rightarrow e^{-2\Omega} B_2, \\
C_1 & \rightarrow e^{\Omega} C_1, \quad \bar{C}_1 \rightarrow e^{-\Omega} \bar{C}_1, \quad C_{\mu\nu} \rightarrow e^{\Omega} C_{\mu\nu}, \quad \bar{C}_{\mu\nu} \rightarrow e^{-\Omega} \bar{C}_{\mu\nu}, \\
f & \rightarrow e^{\Omega} f, \quad \bar{f} \rightarrow e^{-\Omega} \bar{f}, \quad F \rightarrow e^{\Omega} F, \quad \bar{F} \rightarrow e^{\Omega} \bar{F}, \quad f_\mu \rightarrow e^{\Omega} f_\mu, \\
\bar{f}_\mu & \rightarrow e^{-\Omega} \bar{f}_\mu, \quad F_\mu \rightarrow e^{-\Omega} F_\mu, \quad \bar{F}_\mu \rightarrow e^{\Omega} \bar{F}_\mu, \quad \Sigma \rightarrow e^0 \Sigma,
\end{align*}
\]

(36)

where the numerical factors, in the exponents, denote the ghost numbers of the specific fields. The generic field \( \Sigma \) stands for all the fields: \( A_{\mu\nu\lambda}, \Phi_{\mu\nu}, \phi_{\mu}, \phi, B_1, B_{\mu\nu}, \bar{B}_{\mu\nu}, B_\mu, \bar{B}_\mu \) which carry the ghost number equal to zero. It can be readily checked that the coupled (but equivalent) Lagrangian densities \( \mathcal{L}_B \) and \( \mathcal{L}_B \) [cf. Eqs. (5), (6)] remain invariant under the ghost-scale continuous symmetry transformations (36).

The infinitesimal version \( (s_g) \) of the above ghost-scale symmetry transformations (36), with the very simple choice \( \Omega = 1 \) for the sake of bravity, can be expressed as:

\[
\begin{align*}
& s_g C_2 = 3 C_2, \quad s_g \bar{C}_2 = -3 \bar{C}_2, \quad s_g \beta_\mu = 2 \beta_\mu, \quad s_g \bar{\beta}_\mu = -2 \bar{\beta}_\mu, \\
& s_g \beta = 2 \beta, \quad s_g \bar{\beta} = -2 \bar{\beta}, \quad s_g B = 2 B, \quad s_g B_2 = -2 B_2, \\
& s_g C_1 = C_1, \quad s_g \bar{C}_1 = -\bar{C}_1, \quad s_g C_{\mu\nu} = C_{\mu\nu}, \quad s_g \bar{C}_{\mu\nu} = -\bar{C}_{\mu\nu}, \\
& s_g f = f, \quad s_g \bar{f} = -\bar{f}, \quad s_g F = -F, \quad s_g \bar{F} = \bar{F}, \quad s_g f_\mu = f_\mu, \\
& s_g \bar{f}_\mu = -\bar{f}_\mu, \quad s_g F_\mu = -F_\mu, \quad s_g \bar{F}_\mu = \bar{F}_\mu, \quad s_g \Sigma = 0.
\end{align*}
\]

(37)
According to the basic concept behind the Noether theorem, the expression for the ghost conserved current \([J^\mu_{(g)}]\), derived from the Lagrangian density \(\mathcal{L}_B\), is as follows

\[
J^\mu_{(g)} = (s_g C_2) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu C_2)} \right] + (s_g \bar{C}_2) \left[ \frac{\partial \mathcal{L}_B}{\partial (\bar{C}_2 \partial_\mu)} \right] + (s_g C_1) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu C_1)} \right] \\
+ (s_g \bar{C}_1) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu C_1)} \right] + (s_g \beta) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \bar{C}_1)} \right] + (s_g \bar{\beta}) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \beta)} \right] \\
+ (s_b C_\nu) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu C_\nu)} \right] + (s_b \bar{C}_\nu) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \bar{C}_\nu)} \right] + (s_b \beta_\nu) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \bar{\beta}_\nu)} \right] \\
+ (s_b \bar{\beta}_\nu) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \beta_\nu)} \right] + (s_b C_{\nu\lambda}) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu C_{\nu\lambda})} \right] + (s_b \bar{C}_{\nu\lambda}) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \bar{C}_{\nu\lambda})} \right],
\]

(38)

where the convention of the left-derivative w.r.t. the fermionic (anti-)ghost fields has been adopted. The explicit computation of the above equation, with the help of (37) and the Lagrangian density \(\mathcal{L}_B\), leads to the following expression for the ghost current:

\[
J^\mu_{(g)} = (\partial^\mu \bar{C}^\nu + \partial^\nu \bar{C}^\mu + \partial^\lambda \bar{C}^{\mu\nu}) C_{\nu\lambda} \mp m \bar{C}^{\mu\nu} C_\nu - \bar{C}^{\mu\nu} f_\nu \\
+ (\partial^\mu \bar{C}^{\nu\lambda} + \partial^\nu \bar{C}^{\mu\lambda} + \partial^\lambda \bar{C}^{\mu\nu}) C_{\nu\lambda} \mp m \bar{C}^{\mu\nu} C_\nu - \bar{C}^{\mu\nu} F_\nu \\
+ (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) C_\nu + (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \bar{C}_\nu + \frac{1}{2} (\bar{C}^\mu f + C^\mu F) \\
- C_1 f^\mu - \bar{C}_1 f^\mu + 2 (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{\beta}_\nu - 2 (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) \beta_\nu \\
+ (\pm m \bar{\beta}^\mu - \partial^\mu \beta) \beta - (\pm m \beta^\mu - \partial^\mu \beta) \bar{\beta} - 2 \beta^\mu B_2 - 2 \bar{\beta}^\mu B \\
+ 3 \bar{C}_2 \partial^\mu C_2 + 3 C_2 \partial^\mu \bar{C}_2.
\]

(39)

Using the EL-EoMs (that have been derived in Sec. 4) from the Lagrangian density \(\mathcal{L}_B\), it is not very difficult to check that \(\partial_\mu J^\mu_{(g)} = 0\) which proves the conservation law for the ghost Noether current. It will be worthwhile to mention here that the ghost Noether current \([J^\mu_{(g)}]\) (that has been computed from \(\mathcal{L}_B\) using the same formula as (38) with the replacement: \(\mathcal{L}_B \longrightarrow \mathcal{L}_B\)) is:

\[
J^\mu_{(g)} = (\partial^\mu \bar{C}^{\nu\lambda} + \partial^\nu \bar{C}^{\mu\lambda} + \partial^\lambda \bar{C}^{\mu\nu}) C_{\nu\lambda} \mp m \bar{C}^{\mu\nu} C_\nu + \bar{C}^{\mu\nu} F_\nu \\
+ (\partial^\mu \bar{C}^{\nu\lambda} + \partial^\nu \bar{C}^{\mu\lambda} + \partial^\lambda \bar{C}^{\mu\nu}) C_{\nu\lambda} \mp m \bar{C}^{\mu\nu} C_\nu + \bar{C}^{\mu\nu} \bar{F}_\nu \\
+ (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) C_\nu + (\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu) \bar{C}_\nu - \frac{1}{2} (\bar{C}^\mu \bar{F} + C^\mu \bar{F}) \\
+ (\pm m \bar{\beta}^\mu - \partial^\mu \beta) \beta - (\pm m \beta^\mu - \partial^\mu \beta) \bar{\beta} - 2 \beta^\mu B_2 - 2 \bar{\beta}^\mu B \\
+ 3 \bar{C}_2 \partial^\mu C_2 + 3 C_2 \partial^\mu \bar{C}_2.
\]

(40)

Using the appropriate equations of motion from Sec. 4 (that have been derived from \(\mathcal{L}_B\)), it is not very hard to prove that \(\partial_\mu J^\mu_{(g)} = 0\) which establishes the validity of the conservation law.

Some of the key points to be noted, at this juncture, are as follows. First, we observe that a part of the ghost currents in (39) and (40), depending on the transformations of the fields
\(\beta, \bar{\beta}, \beta_\mu\) and \(\bar{\beta}_\mu\), remains conserved on its own. Second, in exactly similar fashion, a part of the ghost currents, generated by the transformations: \(s_y C_2 = 3 C_2\) and \(s_y \bar{C}_2 = -3 \bar{C}_2\), also remains conserved on its own. Third, we have to use the expressions for the auxiliary fields of (28) and \(B = -(\partial \cdot \beta \mp m \bar{\beta})\) as well as \(B_2 = (\partial \cdot \beta \mp m \bar{\beta})\) at appropriate places for the proof of the conservation laws. Finally, the part of the ghost currents, derived from the transformations for the fields \(C_1, \bar{C}_1, C_\mu, \bar{C}_\mu, C_{\mu\nu}, \bar{C}_{\mu\nu}\), etc., are conserved due to the equations of motion of all these fields (that have been derived from the Lagrangian densities \(\mathcal{L}_B\) and \(\mathcal{L}_{\bar{B}}\) in Sec. 4). To sum up, we observe that the part of the ghost currents, corresponding to the infinitesimal ghost-scale transformations for the (anti-)ghost fields with ghost numbers: \((-3, +3), (-2, +2)\) and \((-1, +1)\), are separately and independently conserved on their own due to the EL-EoMs.

We are in the position now to discuss the continuous and infinitesimal BRST symmetry transformations (7) and corresponding Noether current. Due to our observation in (8), it is clear that we shall have conserved BRST current (and corresponding charge) for the Lagrangian density \(\mathcal{L}_B\). In exactly similar fashion, our observation in (10) would lead to the derivation of the anti-BRST conserved current (and corresponding conserved anti-BRST charge). First of all, we concentrate on the derivation of Noether’s current corresponding to the BRST symmetry transformations. The expression for the Noether current \(J^\mu_{(b)}\) is

\[
J^\mu_{(b)} = (s_b A_{\alpha\beta\gamma}) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu A_{\alpha\beta\gamma})} \right] + (s_b \Phi_{\alpha\beta}) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \Phi_{\alpha\beta})} \right] + (s_b \bar{C}_{\alpha\beta}) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \bar{C}_{\alpha\beta})} \right] \\
+ (s_b C_{\alpha\beta}) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu C_{\alpha\beta})} \right] + (s_b \phi_\alpha) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \phi_\alpha)} \right] + (s_b C_\alpha) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu C_\alpha)} \right] \\
+ (s_b \bar{C}_\alpha) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \bar{C}_\alpha)} \right] + (s_b \beta_\alpha) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \beta_\alpha)} \right] + (s_b \bar{\beta}_\alpha) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \bar{\beta}_\alpha)} \right] \\
+ (s_b \bar{C}_2) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \bar{C}_2)} \right] + (s_b \bar{C}_1) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \bar{C}_1)} \right] + (s_b \bar{C}_1) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \bar{C}_1)} \right] \\
+ (s_b \phi) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \phi)} \right] + (s_b \beta) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \beta)} \right] + (s_b \bar{\beta}) \left[ \frac{\partial \mathcal{L}_B}{\partial (\partial_\mu \bar{\beta})} \right] - X^\mu, \tag{41}
\]

where \(\mathcal{L}_B\) is the perfectly BRST invariant Lagrangian density (5) and \(X^\mu\) is expression in the square bracket of Eq. (8). In other words, we have:

\[
X^\mu = (\partial^\mu C^{\nu\lambda} + \partial^\nu C^{\lambda\mu} + \partial^\lambda C^{\mu\nu}) B_{\nu\lambda} + B^{\mu\nu} f_\nu - B_2 \partial^\mu C_2 \\
- B_1 f^\mu + B F^\mu - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) F + \frac{1}{2} (\pm m \beta^\mu - \partial^\mu \beta) F \\
+ (\partial^\mu C^\nu - \partial^\nu C^\mu) B_\nu \pm m B^{\mu\nu} C_\nu - \frac{1}{2} B^\mu f \\
\mp m \bar{C}^{\mu\nu} (\mp m \beta_\nu - \partial_\nu \beta) \mp m (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \bar{C}_\nu. \tag{42}
\]

We call a Lagrangian density as the perfectly symmetry invariant Lagrangian density where no EL-EoMs and/or CF-type restrictions are invoked for its symmetry invariance (from outside). For instance, the Lagrangian density \(\mathcal{L}_B\) is a perfectly symmetry invariant Lagrangian density w.r.t. the BRST symmetry transformations \((s_b)\). It will be noted that, in (41), we have not taken the BRST symmetry transformations on the auxiliary fields
because they do not have their derivatives in the Lagrangian density $L_B$. Furthermore, due to our observation that $s_b C_2 = 0$, we have not taken its contribution to (41) even though the derivative on $C_2$ field exists. The substitutions of the BRST symmetry transformations (7) and the appropriate expressions for the derivatives w.r.t. the suitable fields of the Lagrangian density $L_B$ lead to the following:

$$J^\mu_{(b)} = H^{\mu\lambda\chi} (\partial_\nu C_{\lambda\chi}) \pm \frac{m}{2} [\pm m (\beta^\mu - \partial^\mu \beta)] C_2 \pm m \bar{C}^{\mu\nu} (\pm m \beta_\nu - \partial_\nu \beta)$$

$$+ (\partial^\mu C^{\nu\lambda} + \partial^\nu C^{\lambda\mu} + \partial^\lambda C^{\mu\nu}) B_{\nu\lambda} - (\partial^\mu \bar{C}^{\nu\lambda} + \partial^\nu \bar{C}^{\lambda\mu})$$

$$+ \partial^\lambda \bar{C}^{\mu\nu} (\partial_\nu \beta_\lambda - \partial_\lambda \beta_\nu) \mp m C^{\mu\nu} B_\nu - B_1 f^\mu + B F^\mu - \frac{1}{2} B^\mu f$$

$$+ \frac{1}{2} (\pm m \beta^\mu - \partial^\mu \beta) F - (\partial^\mu \bar{C}^{\nu\mu} - \partial^\nu \bar{C}^{\mu\nu}) (\pm m \beta_\nu - \partial_\nu \beta).$$

(43)

According to the basic tenets of Noether’s theorem, we know that the conservation law $\partial_\mu J^\mu_{(b)} = 0$ can be proven by using the appropriate EL-EoMs that have been derived in our Sec. 4 where some of the common EL-EoMs [cf. Eq. (30)] have been listed (that emerge out from both the Lagrangian densities $L_B$ and $\mathcal{L}_B$). In the proof of $\partial_\mu J^\mu_{(b)} = 0$, we have utilized the EL-EoMs (31) that have been derived from $L_B$. Furthermore, we have invoked the explicit expressions (28) for the auxiliary fields ($B_{\mu\nu}, B_\mu, F_\mu, f_\mu, F, f$) in the above proof. We lay emphasis on the fact that the expressions for these auxiliary fields in Eq. (28) are nothing but the EL-EoMs derived from $\mathcal{L}_B$ w.r.t. these auxiliary fields themselves. Ultimately, we note that the conservation law $[\partial_\mu J^\mu_{(b)} = 0]$ is true which proves the sanctity of Noether’s theorem.

We now concentrate on the proof of the conservation of the Noether current $[J^\mu_{(ab)}]$ corresponding to the infinitesimal, continuous and off-shell nilpotent ($s_{ab}^2 = 0$) anti-BRST symmetry transformations (9). The explicit expression for $J^\mu_{(ab)}$ can be computed in exactly similar fashion as $J^\mu_{(b)}$ [cf. Eq. (41)] where we have to replace: $s_b \rightarrow s_{ab}$ and $L_B \rightarrow \mathcal{L}_B$. In this context, we note that the anti-BRST symmetry transformations ($s_{ab}$) are listed in (9) and the explicit expression for $\mathcal{L}_B$ is given in (6). In addition, we here to replace $X^\mu$ of equation (41) by $Y^\mu$ which is as follows:

$$Y^\mu = \bar{B}^{\mu\nu} \tilde{f}_\nu - (\partial^\mu \bar{C}^{\nu\lambda} + \partial^\nu \bar{C}^{\lambda\mu} + \partial^\lambda \bar{C}^{\mu\nu}) \bar{B}_{\nu\lambda} + B \partial^\mu \bar{C}_2$$

$$- B_2 \tilde{F}^\mu - B_1 \tilde{f}^\mu - (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) \tilde{F}_\nu + \frac{1}{2} (\pm m \beta^\mu - \partial^\mu \beta) \tilde{F}$$

$$- (\partial^\mu \bar{C}^{\nu\mu} - \partial^\nu \bar{C}^{\mu\nu}) \bar{B}_\nu \mp m \bar{B}^{\mu\nu} \bar{C}_\nu - \frac{1}{2} B^\mu \tilde{f}$$

$$\pm m C^{\mu\nu} (\pm m \beta_\nu - \partial_\nu \beta) \pm m (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) C_\nu.$$  

(44)

The above expression in (44) is nothing but the terms that are present in the square bracket of equation (10) on the r.h.s. The exact mathematical form of the anti-BRST Noether current is as follows
\[ J_{(ab)}^\mu = H^{\mu\nu\lambda\chi} (\partial_\nu \bar{C}_{\lambda\chi}) \pm \frac{m}{2} [\pm m \beta^\mu - \partial^\mu \beta] \bar{C}_2 \mp m C^{\mu\nu} (\pm m \tilde{\beta}_\nu - \partial_\nu \tilde{\beta}) \]
\[ - (\partial^\mu \bar{C}^{\nu\lambda} + \partial^\rho \bar{C}^{\rho\lambda} + \partial^\lambda \bar{C}^{\mu\nu}) B_{\nu\lambda} + (\partial^\mu C^{\nu\lambda} + \partial^\nu C^{\mu\lambda}) \]
\[ + \partial^\lambda C^{\mu\nu} (\partial_\nu \bar{\beta}_\lambda - \partial_\lambda \bar{\beta}_\nu) \pm m \bar{C}^{\mu\nu} B_{\nu\mu} - \bar{f}^\mu B_1 - B_2 F^\mu - \frac{1}{2} \bar{B}^\mu \bar{f} \]
\[ + [\pm m A^{\mu\nu\lambda} - \Sigma^{\mu\nu\lambda}] [\pm m \bar{C}_{\nu\lambda} - (\partial_\nu \bar{C}_{\lambda\chi} - \partial_\lambda \bar{C}_{\nu\chi})] + B \partial^\mu \bar{C}_2 + \bar{B}^{\mu\nu} \bar{f}_\nu \]
\[ - (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) (\partial_\nu \bar{C}_2) - (\partial^\mu \bar{\beta}_\nu - \partial^\nu \bar{\beta}_\mu) \bar{F}_\nu - (\partial^\mu \bar{C}_\nu - \partial^\nu \bar{C}_\mu) B_{\nu\mu} \]
\[ + \frac{1}{2} \left( \pm m \bar{\beta}^\mu - \partial^\mu \bar{\beta} \right) \bar{F} + (\partial^\mu C^\nu - \partial^\nu C^\mu) (\pm m \bar{\beta}_\nu - \partial_\nu \bar{\beta}), \quad (45) \]

where it should be noted that \( s_b \bar{C}_2 \) will be replaced by \( s_{ab} \bar{C}_2 \) because we note that \( s_{ab} \bar{C}_2 = 0 \) but \( s_{ab} C_2 = B \). The conservation law (i.e. \( \partial_\mu J_{(ab)}^\mu = 0 \)) can be proven by using the appropriate equations of motion of Sec. 4. We would like to lay emphasis on the fact that the following points are pertinent in our present discussion. First, we have not used the CF-type restrictions (29) anywhere in the proof of \( \partial_\mu J_{(b)}^\mu = 0 \) and \( \partial_\mu J_{(ab)}^\mu = 0 \) as the (anti-)BRST invariant CF-type restrictions, as pointed out earlier, are not the EL-EoMs. Second, the explicit expressions for the auxiliary fields in (28) have been used in the proof of the conservation laws because the entries of equation (28) are primarily the EL-EoMs from the Lagrangian densities \( L_B \) and \( L_{\bar{B}} \). Finally, we have used quite frequently the common equations of motion (30) from \( L_B \) and \( L_{\bar{B}} \). On top of it, in the proof of \( \partial_\mu J_{(b)}^\mu = 0 \) and \( \partial_\mu J_{(ab)}^\mu = 0 \) we have invoked the validity of EL-EoMS (31) and (32), respectively.

### 5.2 Conserved Charges from Conserved Currents

The central purpose of our present subsection is to derive the conserved Noether charges \( Q_r \equiv \int d^{D-1} x \, j^{0}_{(r)} (r = a, ab, g) \) from the conserved Noether currents \( J_{(r)}^\mu (r = a, ab, g) \) of our previous subsection and comment on some salient features. For instance, we show that the standard Noether conserved charges \( [Q_{(a)\bar{b}}] \) are off-shell nilpotent (i.e. \( Q_{(a)\bar{b}}^2 \neq 0 \)) of order two (where we have taken the help of the celebrated relationship between the continuous symmetries and their generators as the conserved Noether charges). In this context, first of all, we concentrate on the anti-BRST charge \( Q_{ab} \) which is explicitly written, in any arbitrary D-dimension of spacetime, as follows

\[ Q_{ab} = \int d^{D-1} x \, j^{0}_{(ab)} = \int d^{D-1} x \left[ H^{0ijk} (\partial_i \bar{C}_{jk}) \pm \frac{m}{2} [\pm m \beta^0 - \partial^0 \beta] \bar{C}_2 \right. \]
\[ \mp m C^{0i} (\pm m \tilde{\beta}_i - \partial_i \tilde{\beta}) - (\partial^0 \bar{C}^{ij} + \partial^i \bar{C}^{\bar{0}j} + \partial^j \bar{C}^{\bar{0}i}) \bar{B}_{ij} - B_2 F^0 \]
\[ + (\partial^0 C^{ij} + \partial^i C^{\bar{0}j} + \partial^j C^{\bar{0}i}) (\partial_\nu \tilde{\beta}_{ij} - \partial_{\nu} \tilde{\beta}_{ji}) + B \partial^0 \bar{C}_2 + \bar{B}^{0ij} \tilde{f}_i - \frac{1}{2} \bar{B}^0 \bar{f} \]
\[ \pm m C^{0i} B_i - \bar{f}^0 B_1 + [\pm m A^{0ij} - \Sigma^{0ij}] [\pm m C_{ij} - (\partial_i \bar{C}_j - \partial_j \bar{C}_i)] \]
\[ - (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) (\partial_i \bar{C}_2) - (\partial^0 \bar{\beta}_i - \partial^i \bar{\beta}^0) \bar{F}_i - (\partial^0 \bar{C}_i - \partial^i \bar{C}^0) B_i \]
\[ + \frac{1}{2} \left( \pm m \bar{\beta}^0 - \partial^0 \bar{\beta} \right) F + (\partial^0 C^i - \partial^i C^0) (\pm m \bar{\beta}_i - \partial_i \bar{\beta}), \quad (46) \]

where the expression for \( j^{0}_{(ab)} \) has been derived from the Noether conserved anti-BRST current \( J_{(ab)}^\mu \) [cf. Eq. (45)] in a straightforward fashion. In exactly similar manner, we
write down the BRST charge \( Q_b \equiv \int d^{D-1}x \, J_0^{(b)} \) from the expression for conserved current \([J_0^{(b)}]_\mu\) that has been written in (43). To be precise, we have the following expression for \( Q_b \) in any arbitrary D-dimensions of spacetime for our present system, namely;

\[
Q_b = \int d^{D-1}x \, J_0^{(b)} = \int d^{D-1}x \left[ H^{\alpha ij} (\partial_\alpha C_{ij}) \pm \frac{m}{2} [\pm m \, \bar{\beta}^0 - \partial^0 \bar{\beta}] \, C_2 \right. \\
\left. \pm \, m \, C^0i (\pm m \, \bar{\beta}_i - \partial_i \beta) + (\partial^0 C^i + \partial^i C^0 + \partial^\alpha C^{0\alpha}) \, B_{ij} \right. \\
- \, (\partial^0 \bar{C}^0i + \partial^i \bar{C}^00 + \partial^\alpha \bar{C}^{0\alpha}) (\partial_i \bar{\beta}_j - \partial_j \bar{\beta}_i) - B_2 \, \partial^0 C_2 + B \, F^0 + B^0 \, f_i \\
- \, B_1 \, f^0 \mp m \, C^0i \, B_i + [\pm m \, A_0i - \Sigma_0ij] [\pm m \, C_{ij} - (\partial_i \beta_j - \partial_j \beta_i)] \\
- \, (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) (\partial_i C_2) - (\partial^0 \beta^i - \partial^i \beta^0) \, F_i + (\partial^0 C^i - \partial^i C^0) \, B_i \\
- \, \frac{1}{2} \, B^0 \, f + \frac{1}{2} \left( \pm m \, \bar{\beta}^0 - \partial^0 \beta \right) \, F - (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \left( \pm m \, \bar{\beta}_i - \partial_i \beta \right), \tag{47}
\]

where \( J_0^{(b)} \) is the zeroth component of \( J_0^\mu \) that has been quoted in (43). Finally, we write down the expression for the ghost charge \( Q_g = \int d^{D-1}x \, J_0^{(g)} \) as follows

\[
Q_g = \int d^{D-1}x \, J_0^{(g)} = \int d^{D-1}x \left[ (\partial^0 \bar{C}^0i - \partial^i \bar{C}^00 + \partial^\alpha \bar{C}^{0\alpha}) \, C_{ij} \mp m \, C^0i \, C_i \right. \\
- \, \bar{C}^0i \, F_i + (\partial^0 C^i + \partial^i C^0 + \partial^\alpha C^{0\alpha}) \, C_{ij} \mp m \, C^0i \, \bar{C}_i - C^0i \, F_i - C_1 \, F^0 \\
+ \, (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \, C_i + (\partial^0 C^i - \partial^i C^0) \, \bar{C}_i + \frac{1}{2} \, (\bar{C}^0 \, f + C^0 \, F) - \bar{C}_1 \, f^0 \\
+ \, 2 \, (\partial^0 \beta^i - \partial^i \beta^0) \, \bar{\beta} - 2 \, (\partial^0 \bar{\beta}_i - \partial^i \bar{\beta}_0) \, \beta_i + (\pm m \, \beta^0 - \partial^0 \beta) \, \beta \\
- \, (\pm m \, \beta^0 - \partial^0 \beta) \, \bar{\beta} - 2 \, \beta^0 \, B_2 - 2 \, \bar{\beta}^0 \, B + 3 \, \bar{C}_2 \, \partial^0 C_2 + 2 \, C_2 \, \partial^0 \bar{C}_2 \right], \tag{48}
\]

where the mathematical form of of the conserved ghost current \([J_0^\mu]_\mu\) has been quoted in Eq. (39) for our system. A similar kind of ghost charge \( \bar{Q}_g = \int d^{D-1}x \, J_0^{(g)} \) can be computed from (40). However, we shall not explicitly express it here and we shall only concentrate on \( Q_g = \int d^{D-1}x \, J_0^{(g)} \) [cf. the above Eq. (48)].

We conclude this subsection with the following crucial remarks. First of all, we note that the standard expressions for the Noether conserved charges \( Q_r \) (with \( r = b, ab, g \)) lead the derivation of the infinitesimal continuous symmetry transformations (7), (9) and (37) which are fermionic (i.e. off-shell nilpotent) as well as bosonic in nature, respectively. In other words, these standard Noether charges are the generators for the above infinitesimal continuous symmetry transformations. Mathematically, this statement can be captured in the following (anti)commutators between the generic fermionic/bosonic field \( \Phi \) of our theory (described by \( L_B \) and \( L_B \)) and the Noether conserved charges \( Q_r \) (with \( r = b, ab, g \), namely;

\[
s_r \, \Phi = -i \left[ \Phi, Q_r \right]_{(\pm)}, \quad r = b, ab, \\
s_g \, \Phi = -i \left[ \Phi, Q_g \right], \tag{49}
\]

where the subscripts \( (\pm) \) on the square bracket, in the first entry of the above equation, stand for the (anti)commutator for the generic field \( \Phi \) being fermionic/bosonic in nature. Since the infinitesimal ghost-sacle symmetry transformations (37) are bosonic in nature,
we have a commutator in the second entry of the above equation between the generic fermionic/bosonic field $\Phi$ and the conserved Noether ghost charge $Q_g$. Second, the relationship between the continuous symmetry transformation and its generator (as the standard conserved Noether charge) is general in nature. Thus, it can be readily checked that we have

$$s_b Q_b = -i \{Q_b, Q_b\} = -2i Q_b^2, \quad s_{ab} Q_{ab} = -i \{Q_{ab}, Q_{ab}\} = -2i Q_{ab}^2,$$  

(50)

where the l.h.s. can be computed by the direct applications of the (anti-) BRST symmetry transformations [cf. Eqs. (9), (7)] on the precise expressions for the Noether conserved (anti-)BRST charges [cf. Eqs. (46), (47)]. Ultimately, we observe that the following are transformations [cf. Eqs. (9), (7)] on the precise expressions for the Noether conserved (anti-)BRST charges [cf. Eqs. (46), (47)]. Ultimately, we observe that the following are true, namely:

$$s_b Q_b = \int d^{D-1}x \left[ \pm \frac{m}{2} \left( \pm m F^0 - \partial^0 F \right) C_2 - (\partial^0 F^i - \partial^i F^0) (\partial_i C_2) \right. - (\partial^0 B^{ij} + \partial^i B^{0j} + \partial^j B^{0i}) (\partial_i \beta_j - \partial_j \beta_i) + \left. \left[ \pm m B^{0i} - (\partial^0 B^i - \partial^i B^0) \right] (\pm m \beta_i - \partial_i \beta) \right] \neq 0,$$

$$s_{ab} Q_{ab} = \int d^{D-1}x \left[ \pm \frac{m}{2} \left( \pm m \bar{F}^0 - \partial^0 \bar{F} \right) \bar{C}_2 - (\partial^0 \bar{F}^i - \partial^i \bar{F}^0) (\partial_i \bar{C}_2) \right. + (\partial^0 \bar{B}^{ij} + \partial^i \bar{B}^{0j} + \partial^j \bar{B}^{0i}) (\partial_i \bar{\beta}_j - \partial_j \bar{\beta}_i) - \left. \left[ \pm m \bar{B}^{0i} - (\partial^0 \bar{B}^i - \partial^i \bar{B}^0) \right] (\pm m \bar{\beta}_i - \partial_i \bar{\beta}) \right] \neq 0,$$  

(51)

which demonstrate that the standard Noether conserved (anti-)BRST charges are not nilpotent of order two (i.e. $Q_{(a)b}^2 \neq 0$). As a consequence, one can not obtain the standard BRST algebra with the Noether conserved charges $Q_{(a)b}$ and $Q_g$. Furthermore, the BRST cohomology w.r.t. the Noether conserved (anti-)BRST charges can not be performed.

### 6 Physicality Criteria: Nilpotent Charges

Our starting classical theory is described by the Lagrangian density (1) for the modified massive Abelian 3-form theory which is endowed with a set of four first-class constraints [cf. Eqs. (13), (16)] in the terminology of Dirac’s prescription for the classification scheme of constraints (see, e.g. [23-27] for details). According to the celebrated Dirac quantization scheme, for the quantization of a classical system endowed with any kind of constraints, the Dirac quantization conditions require that the operator form of these constraints must annihilate the physical states (i.e. $|phys >$) existing in the total quantum Hilbert space of states. In other words, the Dirac quantization conditions are as follows [23,24]:

$$|\Pi_{(A)}^{0ij}|phys >= 0, \quad |\Pi_{(\phi)}^{0i}|phys >= 0,$$

$$\left[ \partial_i \Pi_{(A)}^{jkl} + \frac{m}{3} \Pi_{(\phi)}^{jkl} \right] |phys >= 0, \quad \left[ \partial_i \Pi_{(\phi)}^{jkl} \right] |phys >= 0.$$  

(52)

Within the framework of BRST-quantization scheme, the conditions (52) must be realized through the physicality criteria w.r.t. the conserved and nilpotent (anti-)BRST charges.
where we demand that the physical states (i.e. $|\text{phys} \rangle$), in the total quantum Hilbert space of states, are those that are annihilated by the above conserved and nilpotent charges [25-27, 30, 31]. However, we have already noted (at the end of our previous subsection) that the standard Noether conserved (anti-)BRST charges are not nilpotent (i.e. $Q^2_{(a)b} \neq 0$) of order two. In other words, they are conserved but non-nilpotent.

It has been shown in our earlier work (see, e.g. [22] for details) that the Noether theorem does not lead to the derivations of the nilpotent (anti-)BRST charges for the BRST-quantized theories where the CF-type restrictions are non-trivial. A systematic theoretical method has been developed in [22] where we have discussed the precise theoretical methodology to obtain the nilpotent (i.e. $[Q^2_{(1)}] = 0$) version of the BRST charge $Q_{(1)}^b$ from the standard Noether conserved charge $Q_b$ which is found to be non-nilpotent (i.e. $Q^2_b \neq 0$). It is obvious here that we have chosen the notation for the non-nilpotent Noether conserved charge as $Q_b$ (which is derived by using the standard Noether’s theorem). The nilpotent version of the conserved BRST charge $Q_{(1)}^b$, for our modified D-dimensional massive Abelian 3-form theory, is as follows [22]:

$$
Q_{(1)}^b = \int d^{D-1}x \left[ (\partial^0 C^{ij} + \partial^i C^{0j} + \partial^j C^{0i}) B_{ij} - [\pm m C_{0i} - (\partial^0 C^i - \partial^i C^0)] B_i 
- (\partial^0 B^{ij} + \partial^i B^{0j} + \partial^j B^{0i}) C_{ij} + [\pm m B_{0i} - (\partial^0 B^i - \partial^i B^0)] C_i 
- [\pm m C_{0i} - (\partial^0 C^i - \partial^i C^0) (\pm m \beta_i - \partial_i \beta)] + 2 (\partial^0 \beta_i - \partial^i \beta^0) \partial_i C_2 
\mp m (\pm m \beta^0 - \partial^0 \beta) C_2 + (\partial^0 C^{ij} + \partial^i C^{0j} + \partial^j C^{0i}) (\partial_i \beta_j - \partial_j \beta_i)
+ 2 (\partial^0 F^i - \partial^i F^0) \beta_i - (\pm m F^0 - \partial^0 F) \beta + 3 B_2 C_2 - B_2 C_2 + B F^0
- B_1 F^0 - \frac{1}{2} B^0 f + B^{0a} f_i + \frac{1}{2} (\pm m \beta^0 - \partial^0 \beta) F - (\partial^0 \beta^i - \partial^i \beta^0) F_i \right].
$$

(53)

At this juncture, it is pertinent to point out that in our earlier work [22], we have exploited the interplay of (i) the appropriate El-EoMs from the Lagrangian density $\mathcal{L}_B$, (ii) the Gauss divergence theorem, and (iii) the off-shell nilpotent BRST symmetry transformations $s_b$ together to obtain the above off-shell nilpotent version of the BRST charge $Q_{(1)}^b$. It is interesting to point out that, out of the whole tower of the EL-EoMs (cf. Sec. 4 for details), our proposal has utilized only the following three relevant and useful EL-EoMs (see, e.g. [22])

$$
\partial_{\mu} H^{\mu\nu\lambda\xi} + m^2 A^{\nu\lambda\xi} + m \Sigma^{\nu\lambda\xi} + (\partial^\nu B^{\lambda\xi} + \partial^\lambda B^{\xi\nu} + \partial^\xi B^{\nu\lambda}) = 0,
$$

$$
\partial_{\mu} (\partial^\nu C^{\mu\nu} - \partial^\nu C^{\mu\nu}) \mp m (\partial_{\mu} C^{\mu\nu}) \pm \frac{m}{2} F^\nu - \frac{1}{2} \partial^\nu F = 0,
$$

$$
\partial_{\mu} (\partial^\nu \beta^{\mu\nu} - \partial^\nu \beta^{\mu\nu}) + \partial^\nu B_2 \pm \frac{m}{2} \beta^\nu - \frac{1}{2} m^2 \beta^\nu = 0,
$$

(54)

to obtain the off-shell nilpotent version of the BRST charge $Q_{(1)}^b$ from the non-nilpotent standard conserved Noether BRST charge $Q_b$ (which has been derived using the Noether theorem).

The coupled (but equivalent) (anti-)BRST invariant Lagrangian densities $\mathcal{L}_B$ and $\mathcal{L}\bar{B}$ do not support any kind of constraints unlike the starting classical Lagrangian density $\mathcal{L}_c^{(3)}$ [cf. Eq. (1)] for the modified massive Abelian 3-form theory where we have a set of four
first-class constraints [cf. Eqs. (13), (16)] in the terminology of Dirac’s prescription for the classification scheme of constraints. However, these constraints have been traded with the Nakanishi-Lautrup auxiliary fields and the specific combination of derivatives on them for the Lagrangian densities \( \mathcal{L}_B \) and \( \mathcal{L}_B \) at the quantum level. To corroborate this statement, first of all, let us focus on the following expressions for the canonical conjugate momenta w.r.t. the basic fields \( A_{\mu\nu\lambda} \) and \( \Phi_{\mu\nu} \) for the BRST invariant Lagrangian density \( \mathcal{L}_B \), namely:

\[
\Pi^{\mu\nu\lambda}_{(A)} = \frac{\partial \mathcal{L}_B}{\partial (\partial_\nu A_{\mu\nu\lambda})} = \frac{1}{3} H^{0\mu\nu\lambda} + \frac{1}{3} (\eta^{0\mu} B^{\nu\lambda} + \eta^{0\nu} B^{\lambda\mu} + \eta^{0\lambda} B^{\mu\nu}),
\]
\[
\Pi^{\mu\nu}_{(\phi)} = \frac{\partial \mathcal{L}_B}{\partial (\partial_\nu \Phi_{\mu\nu})} = -\Sigma^{0\mu\nu} \pm m A^{0\mu\nu} - \frac{1}{2} (\eta^{0\mu} B^\nu - \eta^{0\nu} B^\mu).
\]

(55)

It is clear, from the above expressions, that we have the following explicit components

\[
\Pi^{0ij}_{(A)} = \frac{1}{3} H^{00ij} + \frac{1}{3} B^{ij} \equiv \frac{1}{3} B_{ij} \quad \implies \quad B_{ij} = 3 \Pi^{0ij}_{(A)},
\]
\[
\Pi^{0i}_{(\phi)} = -\Sigma^{00i} \pm m A^{00i} - \frac{1}{2} B^i \equiv \frac{1}{2} B_i \quad \implies \quad B_i = 2 \Pi^{0i}_{(\phi)},
\]
\[
\Pi^{ijk}_{(A)} = \frac{1}{3} H^{0ijk} \quad \implies \quad H^{0ijk} = 3 \Pi^{ijk}_{(A)},
\]
\[
\Pi^{ij}_{(\phi)} = -\Sigma^{0ij} \pm m A^{0ij},
\]

(56)

which demonstrate that the original primary constraints (i.e. \( \Pi^{0ij}_{(A)} \approx 0 \), \( \Pi^{0i}_{(\phi)} \approx 0 \)) have been traded with the Nakanishi-Lautrup auxiliary fields \( B_{ij} \) and \( B_i \) of the BRST-invariant theory (which are present in the Lagrangian density \( \mathcal{L}_B \)). Now, we concentrate on the top equation of (54) and the following EL-EoM derived from \( \mathcal{L}_B \) w.r.t. \( \Phi_{\mu\nu} \):

\[
\partial_\nu \Sigma^{\mu\nu\lambda} \mp m (\partial_\mu A^{\nu\mu\lambda}) + \frac{1}{2} (\partial_\nu B^\lambda - \partial_\lambda B^\nu) \mp \frac{m}{2} B^{\nu\lambda} = 0.
\]

(57)

Making the choices: \( \nu = 0 \), \( \lambda = j \) and \( \xi = k \), we obtain the following from the top equation of (54) and (57), namely;

\[
(\partial^\theta B^{ij} + \partial^i B^{0j} + \partial^j B^{0i}) = 3 \left[ \partial_k \Pi^{kij}_{(A)} \mp \frac{m}{3} \Pi^{ij}_{(\phi)} \right],
\]
\[
(\partial^\theta B^i - \partial^i B^0) \mp m B^{0i} = 2 \left[ \partial_j \Pi^{jij}_{(\phi)} \right],
\]

(58)

where we have used the notations from (56) for the purely space components of the canonical conjugate momenta. Thus, we have observed here that the secondary constraints [cf. Eq. (16)] of our classical Stückelberg-modified massive Abelian 3-form theory have been traded with the specific combinations of derivatives (as well as the specific components of) on the Nakanishi-Lautrup auxiliary fields \( (B_{\mu\nu}, B_\mu) \) of the BRST-invariant Lagrangian density \( \mathcal{L}_B \) at the quantum level [cf. Eq. (58)].

In view of the definitions (56), it is straightforward to note that the physicality criteria [25-27, 30, 31] w.r.t. the conserved standard Noether charge (47) leads to the following

\[
Q_b |_{phys} > 0 \quad \implies \quad B_{ij} |_{phys} > 0, \quad B_i |_{phys} > 0,
\]

(59)
where we have taken into account the fact that the physical states are those that are annihilated by the conserved BRST charge \( Q_b \). Thus, we note that only the operator form of the primary constraints (i.e. \( \Pi_{(A)}^{bij} \approx 0 \), \( \Pi_{(\phi)}^{i} \approx 0 \)) of the original classical theory (which have been traded with the Nakanishi-Lautrup auxiliary fields: \( B_{ij} = 3 \Pi_{(A)}^{bij}, B_{i} = 2 \Pi_{(\phi)}^{i} \)) are able to annihilate the physical state (i.e. \( |\text{phys} \rangle \)). Hence, the quantization conditions (59) do not say anything about the secondary constraints [cf. Eq. (16)] of the original classical modified Abelian 3-form theory. We, ultimately, infer that the Noether conserved (but non-nilpotent) BRST charge \( Q_b \) [cf. Eq. (47)] fails in satisfying the Dirac quantization conditions for the precise quantization of our classical theory [cf. Eq. (1)] that is endowed with a set of four first-class constraints. We draw the conclusion that the standard conserved (but non-nilpotent) Noether charge is not suitable for the physicality criterion at the quantum level (for our present classical system that is endowed with four first-class constraints).

Against the backdrop of the above paragraph, we concentrate on the physicality criteria (i.e. \( Q_b^{(1)} |\text{phys} \rangle = 0 \)) w.r.t. the nilpotent version of the BRST charge [cf. Eq. (53)] which leads to the following quantization conditions [25-27, 30, 31] on the physical states (i.e. \( |\text{phys} \rangle = 0 \)) existing in the total quantum Hilbert space of states, namely;

\[
\begin{align*}
B_{ij} |\text{phys} \rangle = 0 & \implies \Pi_{(A)}^{bij} |\text{phys} \rangle = 0, \\
B_{i} |\text{phys} \rangle = 0 & \implies \Pi_{(\phi)}^{i} |\text{phys} \rangle = 0,
\end{align*}
\]

\[
(\partial^0 B^{ij} + \partial^i B^{0j} + \partial^j B^{0i}) |\text{phys} \rangle = 0 \implies \left( \partial_k \Pi_{(A)}^{kij} + \frac{m}{3} \Pi_{(\phi)}^{ij} \right) |\text{phys} \rangle = 0,
\]

\[
\left[ (\partial^0 B^i - \partial^i B^0) \mp m B^0 \right] |\text{phys} \rangle = 0 \implies (\partial_i \Pi_{(\phi)}^{ij}) |\text{phys} \rangle = 0,
\]

(60)

where we have taken into consideration the theoretical strength of the EL-EoMs (58). Thus, we have obtained the conditions on the physical states from the coefficients of the basic ghost fields (i.e. \( C_{\mu\nu}, C_{\mu} \)) in (the expression for \( Q_b^{(1)} \)) where the first four terms have played the crucial role. In other words, the physical states are annihilated by only the field operators which are endowed with the ghost numbers equal to zero. To be precise, the physical states (i.e. \( |\text{phys} \rangle \)), existing in the total quantum Hilbert space of states, are unaffected by (i) the presence of the basic ghost fields, (ii) the auxiliary fields with specific non-zero ghost numbers, and (iii) the ordinary derivatives acting on both of them. This is why, the first four terms of \( Q_b^{(1)} \) have contributed to the physicality criterion w.r.t. it in the equation (60). It will be worthwhile to point out that we have not taken \( B_{i} |\text{phys} \rangle = 0, B^{0i} |\text{phys} \rangle = 0 \) and \( B^{0} |\text{phys} \rangle = 0 \) because these bosonic auxiliary fields are associated with the components of the auxiliary ghost fields \( f_{\mu} \) and \( f \) which are not independent as they have to obey: \( f_{\mu} + \tilde{F}_{\mu} = \partial_{\mu} C_1, f + \tilde{F} = \pm m C_1 \).

For the completion of our arguments, we now say a few words about the anti-BRST charge. As discussed in the context of the conserved BRST charge, in exactly similar fashion, we find that the standard Noether conserved anti-BRST charge (46) leads to the following conditions [25-27, 30, 31] on the physical states, namely;

\[
Q_{ab} |\text{phys} \rangle = 0 \implies \bar{B}_{ij} |\text{phys} \rangle = 0, \quad \bar{B}_{i} |\text{phys} \rangle = 0,
\]

(61)

which is not different from the conditions (59) that have been derived from the conserved Noether charge \( Q_b \) (except the change of notations for the Nakanishi-Lautrup fields). We
also observe that the anti-BRST charge (46) does not say anything about the secondary constraints [cf. Eq. (16)] of the original classical modified massive Abelian 3-form theory [described by the Lagrangian density (1)]. Thus, we focus on the physicality criteria (i.e. \( Q_{ab}^{(1) |\text{phys} = 0} \)) w.r.t. the nilpotent version of the anti-BRST charge \( Q_{ab}^{(1)} \) which has been explicitly derived in our earlier work [22] as follows

\[
Q_{ab}^{(1)} = \int d^{D-1}x \left[ \left( \partial^0 B^{ij} + \partial^i B^{0j} + \partial^j B^{0i} \right) \bar{C}_{ij} - \left[ \pm m B^{0i} - \left( \partial^0 B^i - \partial^i B^0 \right) \right] \bar{C}_i \right.
\]

\[
- \left( \partial^0 \bar{C}^{ij} + \partial^i \bar{C}^{0j} + \partial^j \bar{C}^{0i} \right) \bar{B}_{ij} + \left[ \pm m \bar{C}^{0i} - \left( \partial^0 \bar{C}^i - \partial^i \bar{C}^0 \right) \right] \bar{B}_i \\
+ \left[ \pm m \bar{C}^{0i} - \left( \partial^0 C^{i} - \partial^i C^0 \right) \right] \left( \pm m \bar{\beta}_i - \partial_i \bar{\beta} \right) + 2 \left( \partial^0 \beta^i - \partial^i \beta^0 \right) \partial_i \bar{C}_2
\]

\[
\mp \left( \pm m \beta^0 - \partial^0 \beta \right) \bar{C}_2 - \left( \partial^0 C^{ij} + \partial^i C^{0j} + \partial^j C^{0i} \right) \left( \partial_i \bar{\beta}_j - \partial_j \bar{\beta}_i \right)
\]

\[
+ 2 \left( \partial^0 \bar{F}^i - \partial^i \bar{F}^0 \right) \bar{\beta}_i - \left( \pm m F^0 - \partial^0 F \right) \beta - 3 \bar{B} \bar{C}_2 + B \bar{C}_2 - B_2 F^0
\]

\[
- B_1 \bar{f}^0 - \frac{1}{2} \bar{B}^0 f + B^0 \bar{f}_i + \frac{1}{2} \left( \pm m \beta^0 - \partial^0 \beta \right) \bar{F} - \left( \partial^0 \beta^i - \partial^i \beta^0 \right) \bar{F}_i \right].
\]

(62)

where the above form of the nilpotent anti-BRST charge has been derived using the interplay of (i) the appropriate EL-EoMs derived from \( \mathcal{L}_B \), (ii) the Gauss divergence theorem, and (iii) the off-shell nilpotent anti-BRST symmetry transformations \( s_{ab} \) [cf. Eq. (9)] at appropriate places.

We observe that the physicality criteria (i.e. \( Q_{ab}^{(1) |\text{phys} = 0} \)) w.r.t. \( Q_{ab}^{(1)} \), leads to the following conditions on the physical state (i.e. \( |\text{phys} = 0 \)) with the help of the first four terms, namely:

\[
\bar{B}_{ij} |\text{phys} = 0 \implies \Pi_0^{ij}_{(A)} |\text{phys} = 0,
\]

\[
\bar{B}_i |\text{phys} = 0 \implies \Pi_0^{ji}_{(A)} |\text{phys} = 0,
\]

\[
\left( \partial^0 \bar{B}^{ij} + \partial^i \bar{B}^{0j} + \partial^j \bar{B}^{0i} \right) |\text{phys} = 0 \implies \left( \partial_k \Pi_k^{ij}_{(A)} \mp \frac{m}{3} \Pi_0^{ij}_{(A)} \right) |\text{phys} = 0,
\]

\[
\left[ \pm m \bar{B}^{0i} - \left( \partial^0 \bar{B}^i - \partial^i \bar{B}^0 \right) \right] |\text{phys} = 0 \implies \left( \partial_i \Pi_0^{ij}_{(A)} \right) |\text{phys} = 0.
\]

(63)

where we have used the following definitions of the canonical conjugate momenta w.r.t. the basic fields \( A_{\mu \lambda} \) and \( \Phi_{\mu \nu} \) for our theory described by the Lagrangian density \( \mathcal{L}_B \), namely:

\[
\Pi^{\mu \nu \lambda}_{(A)} = \frac{\partial \mathcal{L}_B}{\partial (\partial_0 A_{\mu \lambda})} = \frac{1}{3} H_0^{0 \mu \lambda} - \frac{1}{3} (\eta^0 \bar{B}^\nu \lambda + \eta^\nu \bar{B}^\lambda \mu + \eta^{\lambda \mu} \bar{B}^0),
\]

\[
\Pi^{\mu \nu}_{(B)} = \frac{\partial \mathcal{L}_B}{\partial (\partial_0 \Phi_{\mu \nu})} = - \Sigma_0^{\mu \nu} \pm m A_0^{\mu \nu} + \frac{1}{2} (\eta^0 \bar{B}^\nu - \eta^\nu \bar{B}^0).
\]

(64)

From the above, it is clear that we have the following explicit independent components of the canonical conjugate momenta (derived from the Lagrangian density \( \mathcal{L}_B \)):

\[
\Pi_0^{ij}_{(A)} = \frac{1}{3} \bar{B}_{ij}, \quad \Pi_0^{0i}_{(A)} = - \frac{1}{2} \bar{B}_i,
\]

\[
\Pi_0^{ijk}_{(A)} = \frac{1}{3} H_0^{0ijk}, \quad \Pi_0^{ij}_{(B)} = - \Sigma_0^{ij} \pm m A_0^{ij}.
\]

(65)
In obtaining the result (63), we have also used the following EL-EoMs (derived from the Lagrangian density $\mathcal{L}_B$), namely:

$$\partial_\mu H^{\mu\nu\lambda} - (\partial^\nu B^{\lambda\xi} + \partial^\xi B^{\nu\lambda} + \partial^\xi B^{\nu\lambda}) = 0,$$
$$\partial_\mu \Sigma^{\mu\nu\lambda} + m (\partial_\mu A^{\mu\nu}) - \frac{1}{2} (\partial^\nu B^{\lambda\xi} - \partial^\xi B^{\nu\lambda}) \pm \frac{m}{2} \bar{B}^{\mu\lambda} = 0,$$  \hspace{1cm} (66)

with the specific choices: $\nu = 0, \lambda = j$ and $\xi = k$ which lead to the relationships between the specific combinations of the derivatives on the Nakanishi-Lautrup type auxiliary fields and the secondary constraints [cf. Eq. (63)]. In fact, the last two entries of (63) are due to the EL-EoMs (66) that are derived from the Lagrangian density $\mathcal{L}_B$. We would like to add here that the physical states, in the physicality criterion (63) w.r.t. the nilpotent version of the anti-BRST charge (i.e. $Q^{(1)}_{ab}|_{phys \gg 0}$), remains unaffected by the presence of (i) the basic anti-ghost fields, (ii) the auxiliary fields with specific non-zero ghost numbers, (iii) the ordinary derivatives acting on both of them. This happens because all such terms, present in the expression for the nilpotent anti-BRST charge $Q^{(1)}_{ab}$, do not act on the physical states. To be precise, the field operator with ghost number equal to zero act on the physical states. Such field operators turn out to be the coefficient of the basic (anti-)ghost fields and/or the derivatives on them. The field operators with ghost numbers equal to zero do not contribute to the physicality criteria if they are associated with the auxiliary (anti-)ghost fields. For instance, the terms $(B^{0i} f_i, B_1 f^0, B^0 f)$ in (62) do not contribute. This is why, we do not have: $\bar{B}^{0i}|_{phys \gg 0}, B_1|_{phys \gg 0}$, and $B^0|_{phys \gg 0}$ in (63).

We end our present section with a couple of crucial and clenching remarks. First of all, we have not taken the conditions: $(B^{0i}|_{phys \gg 0}, B_1|_{phys \gg 0}, B^0|_{phys \gg 0})$ and $(\bar{B}^{0i}|_{phys \gg 0}, B_1|_{phys \gg 0}, B^0|_{phys \gg 0})$ from the physicality criteria w.r.t. the nilpotent and conserved BRST charge (i.e. $Q^{(1)}_b|_{phys \gg 0}$ and nilpotent anti-BRST charge $Q^{(1)}_{ab}|_{phys \gg 0}$) because $(B^{0i}, B_1, B^0)$ and $(\bar{B}^{0i}, B_1, \bar{B}^0)$ are not associated with the basic (anti-)ghost fields (e.g. $C_{\mu\nu}, C_\mu, C_{\mu\nu}, C_\mu, \beta_\mu, \bar{\beta}_\mu$, etc.). Rather, we observe that the terms $(B^{0i} f_i, B_1 f^0, B^0 f)$ and $(\bar{B}^{0i} \bar{f}_i, B_1 \bar{f}^0, \bar{B}^0 \bar{f})$ are present in the expression for $Q^{(1)}_b$ and $Q^{(1)}_{ab}$, respectively. However, the fermionic (anti-)ghost auxiliary fields $(\bar{f}_i) f_i$ and $(\bar{f}) f$ are not the basic (anti-)ghost fields of our theory and they are also not independent in the sense that they obey the CF-type restrictions: $\bar{f} + \bar{F} = m C_1$, $\bar{f} + F = m C_1$. Furthermore, we observe, from the expressions for the following canonical conjugate momenta, derived from the Lagrangian densities $\mathcal{L}_B$ and $\mathcal{L}_\bar{B}$ w.r.t. the fields $\phi_\mu$ and $\phi$, namely:

$$\Pi^{(\phi)}_\mu = \frac{\partial \mathcal{L}_B}{\partial (\partial_0 \phi_\mu)} \equiv B^{0\mu} - \eta^{0\mu} B_1,$$
$$\Pi^{(\bar{\phi})}_\mu = \frac{\partial \mathcal{L}_\bar{B}}{\partial (\partial_0 \phi_\mu)} \equiv \bar{B}^{0\mu} - \eta^{0\mu} B_1,$$
$$\Pi^{(\phi)}_\eta = \frac{\partial \mathcal{L}_B}{\partial (\partial_0 \phi_\eta)} \equiv -\frac{1}{2} B^0,$$
$$\Pi^{(\bar{\phi})}_\eta = \frac{\partial \mathcal{L}_\bar{B}}{\partial (\partial_0 \phi_\eta)} \equiv -\frac{1}{2} \bar{B}^0,$$  \hspace{1cm} (67)

that $\Pi^{(\phi)}_\phi = B^{0i}, \Pi^{(\bar{\phi})}_\phi = -B_1, \Pi^{(\phi)}_\phi = -\frac{1}{2} B^0$ for the Lagrangian density $\mathcal{L}_B$ and $\Pi^{(\phi)}_\phi = \bar{B}^{0i}, \Pi^{(\bar{\phi})}_\phi = -B_1, \Pi^{(\phi)}_\phi = -\frac{1}{2} \bar{B}^0$ for the Lagrangian density $\mathcal{L}_\bar{B}$. It is pertinent to point
out that the subscript ($\phi$) has been taken in the expressions for the canonical conjugate momenta w.r.t. both the fields $\Phi_\mu$ and $\phi$. However, the contravariant indices differentiate them clearly. It is pretty obvious that these components of momenta $[\Pi_\mu(\phi)]$, and $\Pi(\phi)$ are not any type of constraints on our theory. Hence, we can not take $B_0^0|\text{phys} >= 0$, $B_1|\text{phys} >= 0$, $B_0^0|\text{phys} >= 0$, w.r.t. the nilpotent charge $Q_{b}^{(1)}$ and, similarly, the physicality criterion w.r.t. the anti-BRST charge $Q_{ab}^{(1)}|\text{phys} >= 0$ can not produce $\tilde{B}^0_0|\text{phys} >= 0$, $B_1|\text{phys} >= 0$, $\tilde{B}^0_0|\text{phys} >= 0$, as one of the quantization conditions on the physical states (i.e. $|\text{phys} >$) existing in the total quantum Hilbert space of states of our modified massive Abelian 3-form theory in any arbitrary D-dimension of spacetime. The second very important point is the observation that the Dirac quantization conditions, for our D-dimensional modified massive Abelian 3-form theory, are precisely satisfied through the physicality criteria w.r.t. the conserved and off-shell nilpotent (anti-)BRST charges (i.e. $Q_{(a)b}^{(1)}|\text{phys} >= 0$) because we observe that the quantization conditions on the physical states are the same [cf. Eqs. (63), (60)] in terms of the primary and secondary first-class constraints (of our present theory which is endowed with a set of four first-class constraints).

7 Conclusions

In our present investigation, we have taken a D-dimensional St"{u}ckelberg-modified massive Abelian 3-form theory and proven it to be an example of a massive model of a gauge theory because of the presence of the first-class constraints on it (cf. Sec. 3). The latter generate the gauge symmetry transformations [cf. Eqs. (4), (21)] for the St"{u}ckelberg-modified Lagrangian density (1) where the mass and gauge symmetry co-exist. The classical infinitesimal, local and continuous gauge symmetry transformations have been elevated to their quantum counterparts as the (anti-)BRST symmetry transformations [cf. Eqs. (9), (7)] which are respected by a set of coupled (but equivalent) Lagrangian densities. These latter set of Lagrangian densities are nothing but the generalizations of the classical St"{u}ckelberg-modified Lagrangian density (1). The existence of the coupled (but equivalent) (anti-)BRST invariant Lagrangian densities [cf. Eqs. (5), (6)] owe their origin to the existence of the (anti-)BRST invariant Curci-Ferrari restrictions [cf. Eq. (29)] which are also responsible for the absolute anticommutativity property of the nilpotent (anti-)BRST transformations and corresponding conserved and nilpotent (anti-)BRST charges (cf. Appendix B).

The central objective of our present endeavor has been to derive the Noether conserved currents and corresponding charges which are the generators for the infinitesimal and continuous BRST, anti-BRST and ghost-scale symmetry transformations. We have derived the standard BRST algebra (cf. Appendix C) from the ideas and concepts behind the continuous symmetry transformations and their generators (according to Noether’s theorem in the context of quantum field theory). The BRST algebra physically implies (i) the off-shell nilpotency of the (anti-)BRST conserved charges which establishes their fermionic nature, (ii) the absolute anticommutativity of the (anti-)BRST conserved charges (and, hence, their independent identities), and (iii) the ghost number of a field increases by one by the BRST symmetry transformations generated by the BRST charge. On the contrary, the ghost number of a field decreases by one by the anti-BRST symmetry transformations that are generated by the conserved anti-BRST charge. These claims are true for any arbitrary
(anti-)BRST invariant theory in any arbitrary dimension of spacetime [cf. Eq. (C.2)].

One of the highlights of our present investigations is the derivation of the (anti-)BRST invariant CF-type restrictions from the proof of the absolute anticommutativity of the conserved and nilpotent (anti-)BRST charges \( Q^{(1)}_{(a)b} \). These restrictions automatically appear when we demand the absolute anticommutativity [cf. Eq. (11)] of the (anti-)BRST symmetry transformations that are listed in (9) and (7), respectively. It is worthwhile to point out that the CF-type restrictions (29) also appear when we demand the equivalence of the coupled Lagrangian densities (5) and (6) w.r.t. the (anti-)BRST symmetry transformations. This proof has been established in our earlier recent work [18]. In our Appendix A, we have shown the direct equality of the total Lagrangian densities \( L_B \) and \( \bar{L}_B \) in a straightforward manner without any consideration of symmetry transformations and/or principles where, once again, the CF-type restrictions (29) appear in a beautiful fashion. In addition, the (anti-)BRST invariant CF-type restrictions (29) have also been derived from the EL-EoMs (w.r.t. the fermionic/bosonic auxiliary fields) from the coupled (but equivalent) Lagrangian densities \( L_B \) and \( \bar{L}_B \), respectively (cf. Sec. 4 for details). It is worthwhile to mention here that the massless limit of the CF-type restrictions [cf. Eq. (29)] has been found in our earlier work [32] in the context of the BRST approach to the D-dimensional Abelian 3-form gauge theory. The existence of the (non-)trivial CF-type restrictions is one of the hallmarks of the BRST-quantized gauge/diffeomorphism invariant theories. The D-dimensional Abelian 1-form theory is endowed with a trivial CF-type restriction. However, the latter turns out to be the limiting case of the non-trivial celebrated CF-condition [19] that exists for the BRST-quantized D-dimensional non-Abelian 1-form theory [30, 31].

It is gratifying to state that we have established a deep connection between the first-class constraints and the Noether conserved charge at the classical level (cf. Sec. 3 for details) for the St"uckelberg-modified massive Abelian 3-form theory which is described by the Lagrangian density (1). On the contrary, at the quantum level, we find that the standard Noether conserved (anti-)BRST charges \( Q_{(a)b} \), derived from the infinitesimal, continuous and nilpotent (anti-)BRST symmetry transformations [cf. Eq. (9), (7)], are found to be non-nilpotent. Furthermore, the physicality criteria \( (Q_{(a)b}|_{phys} >= 0) \) w.r.t. these standard Noether (anti-)BRST charges produce specific conditions on the physical states (i.e. \( |phys > \)) that are not complete and consistent with Dirac’s quantization conditions [cf. Eqs. (59), (61)]. Thus, we have derived the off-shell nilpotent versions of the (anti-) BRST charges [cf. Eqs. (62), (53)] by exploiting the theoretical proposal in our earlier work [22]. The physicality criteria (i.e. \( Q^{(3)}_{(a)b}|_{phys} >= 0 \)) w.r.t. these nilpotent (i.e. \( [Q^{(1)}_{(a)b}]^2 = 0 \)) (anti-BRST) charges \( Q^{(1)}_{(a)b} \) produce the quantization conditions [cf. Eqs. (63), (60)] which are found to be consistent with Dirac’s quantization conditions [cf. Eq. (52)] where the operator forms of the first-class constraints annihilate the physical states (i.e. \( |phys > \)) of the quantum version of our theory.

It will be an interesting future project to capture the off-shell nilpotency and absolute anticommutativity of the conserved (anti-)BRST charges within the framework of superfield approach to BRST formalism. The central goal of our present investigation (and earlier one on the modified massive Abelian 3-form theory [18]) is to show that the St"uckelberg-modified massive 6D Abelian 3-form theory is a massive model of Hodge theory where
(i) the standard Stückelberg technique gets modified, (ii) the off-shell nilpotent (anti-)co-
BRST symmetries exist, too, in addition to the off-shell nilpotent (anti-)BRST symmetry
transformations, and (iii) there are new “exotic” fields that are introduced in the theory
which possess negative kinetic terms. These “exotic” fields, as mentioned earlier, are now-
a-days very popular in the context of the cyclic, bouncing and self-accelerated cosmological
models of the Universe where they have been christened as the “ghost” and/or “phantom”
particles/fields [13-15]. These “exotic” fields also provide a set of possible candidates of
dark matter [16, 17] as they obey standard Klein-Gordon equation with a well-defined rest
mass. It is obvious that the massless limit of the “exotic” fields would represent the idea
dark energy where there is existence of fields with only negative kinetic terms (with
their rest mass equal to zero). Such fields automatically lead to the generation of negative
pressure which is one of the key characteristic features of the dark energy. The Abelian
3-form theory has been studied within the framework of the generalized BRST and BV
formulation [33, 34] using the superspace technique. Furthermore, the BRST analysis of
the ABJM theory (connected with the M-theory) has also been performed [35]. It will be a
nice future endeavor to look at these works [33-35] in the light of our present investigation
in terms of the constraints and conserved charges.

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Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Appendix A: On the Equality of the Coupled Lagrangian Densities and the
(Anti-)BRST Invariant CF-Type Restrictions

As pointed out earlier, the root-cause behind the existence of the coupled Lagrangian den-
sities $\mathcal{L}_B$ and $\mathcal{L}_\bar{B}$ [cf. Eqs. (5), (6)] is the set of six (anti-) BRST invariant CF-type
restrictions (29) that incorporate into themselves a set of four fermionic and two bosonic
type relationships. The purpose of our present Appendix is to demonstrate the existence of
the six (anti-)BRST invariant CF-type restrictions directly from the equality of $\mathcal{L}_B$ and $\mathcal{L}_\bar{B}$.
In other words, we plan to show that $\mathcal{L}_B - \mathcal{L}_\bar{B} = 0$ provided all the six CF-type restrictions
are satisfied. Towards this goal in mind, first of all, we note that there is a common part $[\mathcal{L}^{(C)}]$ that is present in the (anti-)BRST invariant Lagrangian densities $\mathcal{L}_B$ and $\mathcal{L}_\bar{B}$ [cf. Eqs. (5), (6)] as follows

$$\mathcal{L}^{(C)} = \mathcal{L}_S^{(A)} + \frac{m^2}{2} C_{\mu\nu} C^{\mu\nu} + (\partial_\mu \bar{C}_\nu - \partial_\nu \bar{C}_\mu) (\partial^\mu C^{\nu}) - \frac{1}{2} \left[ \pm m \beta^\mu - \partial^\mu \beta \right]$$

$$\pm m \bar{C}^{\nu} (\partial^{\mu} C_{\mu\nu}) - (\partial_\mu \bar{B}_\nu - \partial_\nu \bar{B}_\mu) (\partial^\mu B^{\nu})$$

$$-\partial_\mu \bar{C}_\nu \partial^\mu C_\nu - m^2 \bar{C}_\nu \partial^\mu C_{\mu\nu} C_\nu - B \bar{B}_\nu - \frac{1}{2} B^2$$

$$+ [(\partial \cdot \beta) \mp m \bar{\beta}] B - [(\partial \cdot \phi) \mp m \phi] B_1 - [(\partial \cdot \beta) \mp m \beta] B_2.$$  \hspace{1cm} (A.1)

where the superscript $(C)$, on the l.h.s. of the Lagrangian density, denotes the common part of the (anti-)BRST invariant Lagrangian densities $\mathcal{L}_B$ and $\mathcal{L}_\bar{B}$. It is obvious that $\mathcal{L}^{(C)}$ will cancel out in the mathematical proof of $\mathcal{L}_B - \mathcal{L}_\bar{B} = 0$.

Against the backdrop of the above paragraph, we note that there are differences in $\mathcal{L}_B$ and $\mathcal{L}_\bar{B}$ (i) in the bosonic sector of the physical gauge field $A_{\mu\nu\lambda}$, St"{u}ckelberg field $\Phi_{\mu\nu}$ as well as the auxiliary fields associated with them, and (ii) in the fermionic ghost-sector where the auxiliary fields $F_\mu, \bar{F}_\mu, f_\mu, \bar{f}_\mu, F, \bar{F}, f$ and $\bar{f}$ are present. First of all, we focus on the physically important bosonic sector of the coupled Lagrangian densities where we have:

$$\mathcal{L}_B^{(b)} - \mathcal{L}_{\bar{B}}^{(b)} = (\partial_\mu A^{\mu\nu\lambda}) (B_{\nu\lambda} + \bar{B}_{\nu\lambda}) + \frac{1}{2} (\bar{B}_{\mu\nu} \bar{B}^{\mu\nu} - B_{\mu\nu} B^{\mu\nu})$$

$$= \frac{m}{2} \Phi_{\mu\nu} (B^{\mu\nu} + \bar{B}^{\mu\nu}) - (\partial_\mu \Phi_{\mu\nu}) (B_{\nu\lambda} + \bar{B}_{\nu\lambda}) + \frac{1}{2} (\bar{B}_{\mu} \bar{B}^{\mu} - B_{\mu} B^{\mu})$$

$$+ \frac{1}{2} \left[ \pm m \Phi^{\mu} - \partial^\mu \phi \right] (B_{\mu} - \bar{B}_{\mu}) + \frac{1}{2} (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) (B^{\mu\nu} - \bar{B}^{\mu\nu}).$$  \hspace{1cm} (A.2)

In the above, the superscript $(b)$ on the Lagrangian densities (on the l.h.s.) denote the bosonic sector of the coupled (but equivalent) Lagrangian densities (5) and (6). We would like to point out, at this juncture, that we have freedom to add/subtract the total space-time derivative term(s) to (A.2) because the latter is a part of the physical Lagrangian density. Thus, it is elementary to note that first and fifth terms of (A.2) can be written as $(\partial_\mu A^{\mu\nu\lambda}) [B_{\nu\lambda} + \bar{B}_{\nu\lambda} - (\partial_\nu \phi_\lambda - \partial_\lambda \phi_\nu) - (\partial_\mu \Phi^{\mu\nu}) [B_{\nu\lambda} + \bar{B}_{\nu\lambda} + \partial_\nu \phi] \mp m \phi]$ without changing the corresponding action integral due to the celebrated Gauss’s divergence theorem. Using the following simple and straightforward algebraic tricks, namely;

$$\frac{1}{2} (\bar{B}_{\mu\nu} \bar{B}^{\mu\nu} - B_{\mu\nu} B^{\mu\nu}) = \frac{1}{2} (\bar{B}^{\mu\nu} - B^{\mu\nu}) (\bar{B}_{\mu\nu} + B_{\mu\nu}),$$

$$\frac{1}{2} (\bar{B}_{\mu} \bar{B}^{\mu} - B_{\mu} B^{\mu}) = \frac{1}{2} (\bar{B}^{\mu} - B^{\mu}) (\bar{B}_{\mu} + B_{\mu}),$$  \hspace{1cm} (A.3)

the above expression for $\mathcal{L}_B^{(b)} - \mathcal{L}_{\bar{B}}^{(b)}$ [cf. Eq. (A.2)] can be re-expressed as

$$\mathcal{L}_B^{(b)} - \mathcal{L}_{\bar{B}}^{(b)} = (\partial_\lambda A^{\lambda\mu\nu}) [B_{\mu\nu} + \bar{B}_{\mu\nu} - (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu)]$$
\[ + \frac{1}{2} (B^{\mu \nu} - B^{\mu \nu}) \left[ B_{\mu \nu} + \bar{B}_{\mu \nu} - (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) \right] \]

\[ \pm \frac{m}{2} \Phi_{\mu \nu}(\partial^\mu \phi^\nu - \partial^\nu \phi^\mu) - (\partial_\mu \Phi^{\mu \nu}) [B_\mu + \bar{B}_\mu + \partial_\mu \phi] \pm \frac{m}{2} \Phi_{\mu \nu} (\bar{B}^{\mu \nu} + B^{\mu \nu}) \]

\[ \pm \frac{m}{2} \Phi_{\mu \nu} (\partial^\mu \phi^\nu - \partial^\nu \phi^\mu) + \frac{1}{2} (\bar{B}^{\mu} - B^{\mu}) [B_\mu + \bar{B}_\mu - (\pm m \phi_\mu - \partial_\mu \phi)], \tag{A.4} \]

where we have added and subtracted \[ \pm \frac{m}{2} (\Phi_{\mu \nu}) \{ (\partial^\mu \phi^\nu - \partial^\nu \phi^\mu) \} \] in the above for the algebraic convenience. Re-arranging the whole equation, we have the following form of the r.h.s. of the above equation, namely:

\[ (\partial_\lambda A^\lambda_{\mu \nu}) \left[ B_{\mu \nu} + \bar{B}_{\mu \nu} - (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) \right] \]

\[ + \frac{1}{2} (\bar{B}^{\mu \nu} - B^{\mu \nu}) \left[ B_{\mu \nu} + B_{\mu \nu} - (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) \right] \]

\[ \pm \frac{m}{2} \Phi^{\mu \nu} \left[ B_{\mu \nu} + B_{\mu \nu} - (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) \right] \]

\[ - (\partial_\nu \Phi^{\nu \mu}) \left[ B_\mu + \bar{B}_\mu - (\pm m \phi_\mu - \partial_\mu \phi) \right] \]

\[ + \frac{1}{2} (\bar{B}^{\mu} - B^{\mu}) \left[ B_\mu + \bar{B}_\mu - (\pm m \phi_\mu - \partial_\mu \phi) \right], \tag{A.5} \]

where the explicit expressions for the bosonic CF-type restrictions appear [cf. Eq. (29)].

The above equation can be re-written, in a more compact form, as:

\[ \left[ (\partial_\lambda A^\lambda_{\mu \nu}) + \frac{1}{2} (\bar{B}^{\mu \nu} - B^{\mu \nu}) \right] \pm \frac{m}{2} \Phi^{\mu \nu} \left[ B_{\mu \nu} + B_{\mu \nu} - (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) \right] \]

\[ + \left[ \frac{1}{2} (\bar{B}^{\mu} - B^{\mu}) - (\partial_\nu \Phi^{\nu \mu}) \right] \left[ B_\mu + \bar{B}_\mu - (\pm m \phi_\mu - \partial_\mu \phi) \right]. \tag{A.6} \]

Thus, we observe that \( \mathcal{L}_B^{(b)} - \mathcal{L}_\bar{B}^{(b)} = 0 \) provided we invoke the validity of the two bosonic type of CF-type restrictions that have been quoted in (29), namely:

\[ B_{\mu \nu} + \bar{B}_{\mu \nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu, \quad B_\mu + \bar{B}_\mu = \pm m \phi_\mu - \partial_\mu \phi. \tag{A.7} \]

We draw the conclusion that the bosonic (physical) sector of the Lagrangian densities \( \mathcal{L}_B \) and \( \mathcal{L}_\bar{B} \) are equal (and equivalent w.r.t. the (anti-)BRST symmetries [18]) on the submanifold of the Hilbert space where the specific quantum fields satisfy (A.7).

Let us focus on the fermionic (anti-)ghost part of the Lagrangian densities \( \mathcal{L}_B \) and \( \mathcal{L}_\bar{B} \). It turns out that we have the following for \( \mathcal{L}_B^{(f)} - \mathcal{L}_\bar{B}^{(f)} \), namely:

\[ -2 F f - 2 \bar{F} \bar{f} + \frac{1}{2} (\partial \cdot C) [\bar{f} + F] \mp m C_1 [F - \bar{f}] - \frac{1}{2} (\partial \cdot \bar{C}) [\bar{f} + \bar{F}] \]

\[ \pm m C_1 [f - \bar{F}] - 2 F^\mu f_\mu - 2 \bar{F}^\mu \bar{f}_\mu + [\partial_\nu \bar{C}^{\nu \mu}] \mp \frac{m}{2} \bar{C}^\mu (f_\mu + \bar{F}_\mu) \]

\[ - [\partial_\nu C^{\nu \mu}] \pm \frac{m}{2} C^\mu (\bar{f}_\mu + F_\mu) + (\partial^\mu C_1) [f_\mu - \bar{F}_\mu] + (\partial^\mu \bar{C}_1) [\bar{f}_\mu - F_\mu], \tag{A.8} \]
where the superscript \((f)\) on the Lagrangian densities \(\mathcal{L}_B^{(f)}\) and \(\mathcal{L}_B^{(f)}\) denotes a part of the fermionic ghost sector of the coupled Lagrangian densities. At this stage, let us, first of all, concentrate on the following part of the above equation, namely;

\[-2 F f - 2 \tilde{F} \tilde{f} + \frac{1}{2} (\partial \cdot C) [\tilde{f} + F] - \frac{1}{2} (\partial \cdot \tilde{C}) [f + \tilde{F}]\]

\[\pm m C_1 [F - \tilde{f}] \pm m \tilde{C}_1 [f - \tilde{F}] . \tag{A.9}\]

It can be seen that the substitutions of the CF-type restrictions

\[f + \tilde{F} = \pm m C_1, \quad \tilde{f} + F = \pm m \tilde{C}_1, \tag{A.10}\]

into (A.9) leads to the following:

\[\pm \frac{m}{2} (\partial \cdot C) \tilde{C}_1 \mp \frac{m}{2} (\partial \cdot \tilde{C}) C_1 . \tag{A.11}\]

We would like to point out that the substitutions of (A.10) into (A.9) results in the cancellation of the first two terms with the last two terms of the latter.

We are now in the position to concentrate on the remaining part of (A.8) which is:

\[-2 F^\mu f_\mu - 2 \tilde{F}^\mu \tilde{f}_\mu + [(\partial_\nu C^{\mu \nu}) \mp \frac{m}{2} \tilde{C}^\nu \tilde{C}^\mu] (f_\mu + \tilde{F}_\mu) - [(\partial_\nu \tilde{C}^{\mu \nu}) \pm \frac{m}{2} C^\nu \tilde{C}^\mu] (\tilde{f}_\mu + F_\mu)
\]

\[\mp \frac{m}{2} C^\mu (\tilde{f}_\mu + F_\mu) + (\partial_\mu \tilde{C}_1) [f_\mu - \tilde{F}_\mu] + (\partial_\mu C_1) [\tilde{f}_\mu - F_\mu] . \tag{A.12}\]

The substitutions of the following CF-type restrictions

\[f_\mu + \tilde{F}_\mu = \partial_\mu C_1, \quad \tilde{f}_\mu + F_\mu = \partial_\mu \tilde{C}_1, \tag{A.13}\]

into the above equation lead to (i) the cancellation of the first two terms with the last two terms, and (ii) the observations that \((\partial_\nu C^{\mu \nu}) \partial_\mu C_1 \equiv \partial_\mu [(\partial_\nu \tilde{C}^{\mu \nu}) \tilde{C}_1]\) and \((\partial_\nu \tilde{C}^{\mu \nu}) \partial_\mu \tilde{C}_1 \equiv \partial_\mu [(\partial_\nu \tilde{C}^{\mu \nu}) \tilde{C}_1]\) are the total spacetime derivatives that do not play any significant role in the dynamics of our theory. Hence, they can be ignored in the action integral due to the Gauss divergence theorem. Ultimately, we obtain [from (A.12)] the following

\[\mp \frac{m}{2} \tilde{C}^\mu \partial_\mu \tilde{C}_1 \pm \frac{m}{2} C^\mu \partial_\mu C_1 , \tag{A.14}\]

which cancels with (A.11) modulo a couple of total spacetime derivatives.

We end this Appendix with the concluding remarks that the straightforward requirement of the equality between \(\mathcal{L}_B\) and \(\mathcal{L}_B\) demonstrates that these Lagrangian densities are equivalent on the submanifold of the quantum Hilbert space of fields where, modulo some total spacetime derivatives, the CF-type restrictions (29) are satisfied. The proof of the two bosonic CF-type restrictions: \(B_\mu + \tilde{B}_\mu = \partial_\mu \phi_\mu - \partial_\mu \phi_\mu\) and \(B_\mu + \tilde{B}_\mu = \pm m \phi_\mu - \partial_\mu \phi\) has been demonstrated quite explicitly in the equality \(\mathcal{L}_B^{(b)} - \mathcal{L}_B^{(b)} = 0\) [cf. Eqs. (A.6), (A.7)] which is valid for the bosonic sector. However, the fermionic four CF-type restrictions: \(f + \tilde{F} = \pm m C_1, \tilde{f} + F = \pm m \tilde{C}_1, f_\mu + \tilde{F}_\mu = \partial_\mu C_1\) and \(\tilde{f}_\mu + F_\mu = \partial_\mu \tilde{C}_1\) are hidden in a subtle manner when we prove: \(\mathcal{L}_B^{(f)} - \mathcal{L}_B^{(f)} = 0\) for the coupled Lagrangian densities in their
fermionic sector. This is why there is cancellation between (A.11) and (A.14). However, there is also an explicit way to demonstrate the existence of the above four fermionic CF-type restrictions by re-arranging terms in (A.8) and throwing away all the total spacetime derivative terms. Ultimately, we can show that $\mathcal{L}_{B}^{(j)} - \mathcal{L}_{B}^{(j)}$, modulo some total spacetime derivative terms, is equal to the following explicit expression:

$$\left[\frac{1}{2} (\partial \cdot C) \pm m C_1 - 2 \bar{F}\right] (\bar{f} + F \equiv m \bar{C}_1)$$

$$- \left[\frac{1}{2} (\partial \cdot \bar{C}) \mp m \bar{C}_1 + 2 F\right] (f + \bar{F} \equiv m C_1)$$

$$+ [\bar{C}_\mu^\nu \mp \left(\frac{m}{2}\right) C_\mu \mp \partial_\mu \bar{C}_1 - 2 F_\mu] (f_\mu + \bar{F}_\mu - \partial_\mu C_1)$$

$$- [\partial_\nu C_\mu^\nu \mp \left(\frac{m}{2}\right) C_\mu - \partial_\mu C_1 + 2 F_\mu] (\bar{f}_\mu + F_\mu - \partial_\mu \bar{C}_1).$$ (A.15)

The above equation demonstrates that we have the equality of the fermionic sector $(\mathcal{L}_{B}^{(j)} - \mathcal{L}_{B}^{(j)} = 0)$ of the coupled Lagrangian densities if and only if all the four fermionic CF-type restrictions of (29) are satisfied.

**Appendix B: On the Absolute Anticommutativity of the Nilpotent (Anti-)BRST Charges: CF-Type Restrictions**

The purpose of this Appendix is to establish the existence of a set of six CF-type restrictions on our theory from the requirement of the absolute anticommutativity property of the off-shell nilpotent versions of the (anti-) BRST charges [cf. Eqs. (62), (53)]. In other words, the theoretical tricks we apply for this purpose are as follows

$$s_b Q^{(1)}_{ab} = -i \{ Q^{(1)}_a, Q^{(1)}_b \} = 0 \implies Q^{(1)}_{ab} Q^{(1)}_{ab} + Q^{(1)}_{ba} Q^{(1)}_{ba} = 0,$$

$$s_{ab} Q^{(1)}_a = -i \{ Q^{(1)}_b, Q^{(1)}_{ab} \} = 0 \implies Q^{(1)}_{ba} Q^{(1)}_{ba} + Q^{(1)}_{ab} Q^{(1)}_{ab} = 0,$$ (B.1)

where one can find the proper conditions under which the absolute anticommutativity property between the nilpotent versions of the BRST and anti-BRST charges (B.1) are precisely satisfied. Towards, this goal in mind, we shall focus on the explicit computation of the l.h.s. of the above equations by the direct applications of the (anti-)BRST symmetry transformations [cf. Eqs. (9), (7)] on the explicit expressions for the nilpotent versions of the BRST charge [cf. (53)] and the anti-BRST charge [cf. Eq. (62)], respectively.

Now we concentrate on the application of the BRST symmetry transformations on the explicit expression for the nilpotent anti-BRST charge $Q^{(1)}_{ab}$. This leads to the following:

$$s_b Q^{(1)}_{ab} = \int d^{D-1} x \left[ (\partial^0 \bar{B}^{ij} + \partial^i \bar{B}^j + \partial^j \bar{B}^i) B_{ij} - (\partial^0 B^{ij} + \partial^i B^j) \right]$$

$$+ \partial^j \bar{B}^0 \bar{B}_{ij} - 2 (\partial^0 \bar{C}^{ij} + \partial^i \bar{C}^j + \partial^j \bar{C}^i) (\partial_{ij} \bar{F}) + (\pm m F^0 - \partial^0 F) F$$

$$+ 2 (\partial^0 C^{ij} + \partial^i C^j + \partial^j C^i) (\partial_{ij} F) - (\partial^0 F^0 - \partial^i \bar{F}^i) \bar{F}_{ij}.$$
\(-2 \left( \partial^i \bar{F}^i - \partial^i \bar{F}^0 \right) F_i + 2 \left( \partial^0 \beta^i - \partial^i \beta^0 \right) \left( \partial_i B_2 \right) \)
\(\pm m \left[ \pm m C^{0i} - (\partial^0 C^i - \partial^i C^0) \right] F_i + \left[ \pm m C^{0i} - (\partial^0 C^i - \partial^i C^0) \right] \left( \partial_i F \right) \)
\(\pm m \left( \partial^0 \bar{F}^i - \partial^i \bar{F}^0 \right) \bar{C}_i \mp m \bar{B}^{0i} \bar{B}_i \pm m \left( \partial^0 \bar{f}^i - \partial^i \bar{f}^0 \right) \bar{C}_i \)
\(+ \left( \partial^0 \bar{B}^i - \partial^i \bar{B}^0 \right) \bar{B}_i \pm m \bar{B}^{0i} \bar{B}_i - (\partial^0 B^i - \partial^i B^0) \bar{B}_i \)
\(+ \left[ \pm m \bar{C}^{0i} - (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \right] \left( \partial_i f \right) \mp m \left[ \pm m \bar{C}^{0i} - (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \right] f_i \)
\(\equiv m \left( \pm m \beta^0 - \partial^0 \beta \right) B_2 + B \dot{B}_2 - 2 \dot{B} B_2 + B_1 \dot{B}_1 - \frac{1}{2} \left( \pm m f^0 - \partial^0 f \right) \bar{f} \)
\(- \bar{B}^{0i} \left( \partial_i B_1 \right) \pm \frac{m}{2} \bar{B}^0 B_1 - (\partial^0 F^i - \partial^i F^0) \bar{F}_i + (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) \partial_i B \)
\(+ \frac{1}{2} \left[ \pm m F^0 - \partial^0 F \right] \bar{F} \mp \frac{m}{2} \left( \pm m \bar{\beta}^0 - \partial^0 \bar{\beta} \right) B \). (B.2)

The stage is now set to apply the Gauss divergence theorem and the appropriate EL-EoMs from the Lagrangian densities \(L_B \) and \(L_{\bar{B}} \), respectively, at appropriate places so that the above computation of the operation of the BRST symmetry transformations \((s_B)\) on the nilpotent version of the anti-BRST charge \([Q^{(1)}_{ab}]\) can be completed and the necessary conditions can be found so that the absolute anticommutativity property is satisfied.

It is very interesting to note that all the terms containing the auxiliary fields \((B, B_1, B_2)\) and the derivatives on them can be explicitly expressed as follows

\[ \int d^{D-1} x \left[ (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0) (\partial_i B) + 2 \left( \partial^0 \beta^i - \partial^i \beta^0 \right) (\partial_i B_2) \right. \]
\(- \bar{B}^{0i} (\partial_i B_1) + B \dot{B}_2 - 2 \dot{B} B_2 \pm \frac{m}{2} \bar{B}^0 B_1 + B_1 \dot{B}_1 \)
\[ \left. \mp m \left( \pm m \beta^0 - \partial^0 \beta \right) B_2 \mp \frac{m}{2} \left( \pm m \bar{\beta}^0 - \partial^0 \bar{\beta} \right) B \right], \quad (B.3) \]

where we can apply the Gauss divergence theorem on the first three terms of (B.3) and drop the total space derivative terms. This mathematical operation leads to the following form of the above terms that are present in the integral, namely;

\[ \int d^{D-1} x \left[ - \partial_i \left( \partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0 \right) B - 2 \partial_i \left( \partial^0 \beta^i - \partial^i \beta^0 \right) B_2 \right. \]
\(+ (\partial_i \bar{B}^{0i}) B_1 + B \dot{B}_2 - 2 \dot{B} B_2 \pm \frac{m}{2} \bar{B}^0 B_1 + B_1 \dot{B}_1 \)
\[ \left. \mp m \left( \pm m \beta^0 - \partial^0 \beta \right) B_2 \mp \frac{m}{2} \left( \pm m \bar{\beta}^0 - \partial^0 \bar{\beta} \right) B \right]. \quad (B.4) \]

Using the following EL-EoMs from \(L_B \), namely;

\[ \partial_\mu \left( \partial^\mu \beta^\nu - \partial^\nu \beta^\mu \right) - \partial^\nu B - \frac{m^2}{2} \beta^\nu \pm \frac{m}{2} \partial^\nu \beta = 0, \]
\[ \partial_\mu \left( \partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu \right) + \partial^\nu B_2 - \frac{m^2}{2} \bar{\beta}^\nu \pm \frac{m}{2} \partial^\nu \bar{\beta} = 0, \]
\[ \partial_\mu \bar{B}^{\mu\nu} - \partial^\nu B_1 \mp \frac{m}{2} \bar{B}^\nu = 0, \]  

(B.5)

it is straightforward to prove that all the terms inside the integral (B.3) produce zero result. Hence, we note that, in (B.2), all the terms containing the auxiliary fields \((B, B_1, B_2)\) and/or derivatives on them contribute to zero as far as the explicit computation of \(s_b Q^{(1)}_{ab}\) (i.e. \(s_b Q^{(1)}_{ab} = -i \{Q^{(1)}_{ab}, Q^{(1)}_b\} = 0\)) is concerned.

We focus now on all the terms that do not contain any kind (i.e. bosonic/fermionic) of the (anti-)ghost fields. These terms of the non-ghost sector of (B.2) are:

\[ \int d^{D-1} x \left[ (\partial^0 \bar{B}^{ij} + \partial^i \bar{B}^{j0} + \partial^j \bar{B}^{0i}) B_{ij} - (\partial^0 B^{ij} + \partial^i B^{j0} + \partial^j B^{0i}) \bar{B}_{ij} \right. \]
\[ + m B^{0i} B_i - (\partial^0 B^i - \partial^i B^0) B_i \mp m B^{0i} B_i + (\partial^0 \bar{B}^i - \partial^i \bar{B}^0) B_i \right]. \]  

(B.6)

The above terms can be re-written as follows

\[ \int d^{D-1} x \left[ \partial^0 [B^{ij} + \bar{B}^{ij} - (\partial^i \phi^j - \partial^j \phi^i)] B_{ij} \right. \]
\[ + \partial^i \left[ B^{0i} + \bar{B}^{0i} - (\partial^0 \phi^i - \partial^i \phi^0) \right] B_{ij} \]
\[ + \partial^j \left[ \partial^0 \bar{B}^{0i} + \partial^i \bar{B}^{0i} - (\partial^0 \phi^i - \partial^i \phi^0) \right] B_{ij} \]
\[ - (\partial^0 B^{ij} + \partial^i B^{j0} + \partial^j B^{0i}) \left[ B_{ij} + \bar{B}_{ij} - (\partial_i \phi_j - \partial_j \phi_i) \right] \]
\[ + \partial^i \left[ B^i + \bar{B}^i \mp m \phi^i + \partial_i \phi^i \right] B_i - \partial^i \left[ B^0 + \bar{B}^0 \mp m \phi^0 + \partial^0 \phi \right] B_i \]
\[ \mp m \left[ B^{0i} + \bar{B}^{0i} - (\partial^0 \phi^i - \partial^i \phi^0) \right] B_i \]
\[ - \left( \partial^0 B^i - \partial^i B^0 \right) \left[ B_i + \bar{B}_i \mp m \phi_i + \partial_i \phi \right] \]
\[ \left. \pm m B^{0i} \left[ B_i + \bar{B}_i \mp m \phi_i + \partial_i \phi \right] \right], \]  

(B.7)

where we have exploited the Gauss divergence theorem at appropriate places and used the following EL-EoMs that are derived from \(\mathcal{L}_B\) namely:

\[ \partial_\mu H^{\mu\nu\lambda} + m^2 A^{\nu\lambda} = m \Sigma^{\nu\lambda} + (\partial^\nu B^{\lambda} + \partial^\lambda B^{\nu} + \partial^\epsilon B^{\nu\lambda}) = 0, \]

\[ \partial_\mu \Sigma^{\nu\lambda} + m \left( \partial_\mu A^{\nu\lambda} \right) - \frac{1}{2} (\partial^\nu B^{\lambda} - \partial^\lambda B^{\nu}) \mp \frac{m}{2} B^{\nu\lambda} = 0, \]  

(B.8)

where \(\Sigma^{\mu\nu\lambda} = \partial_\mu \Phi_{\nu\lambda} + \partial_\nu \Phi_{\lambda\mu} + \partial_\lambda \Phi_{\mu\nu}\) is a totally antisymmetric tensor defined in terms of the sum of the cyclic derivatives [cf. Eq. (3)] on the St"uckelberg field \(\Phi_{\mu\nu}\) (which is also antisymmetric: \(\Phi_{\mu\nu} = -\Phi_{\nu\mu}\)). Ultimately, we note that the non-ghost sector of the terms in (B.2) (i.e. \(s_b Q^{(1)}_{ab}\)) lead to the existence of the bosonic CF-type restrictions: \(B_{\mu\nu} + \bar{B}_{\mu\nu} - (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) = 0\) and \(B_\mu + \bar{B}_\mu \mp m \phi_\mu + \partial_\mu \phi = 0\) in equation (B.7) if we demand that the non-ghost sector of (B.2) must be equal to zero on its own so that the first entry of (B.1) could be satisfied (i.e. \(s_b Q^{(1)}_{ab} = -i \{Q^{(1)}_{ab}, Q^{(1)}_b\} = 0\).

We now concentrate on terms in (B.2) which contain bosonic/fermionic (anti-)ghost fields. These terms in the integral (B.2) are as follows:

\[ \int d^{D-1} x \left[ (\partial^0 \hat{C}^{ij} + \partial^i \hat{C}^{j0} + \partial^j \hat{C}^{0i}) (\partial_i \hat{F}_j - \partial_j \hat{F}_i) \right. \]
We can apply the Gauss divergence theorem on the first four terms and drop the total space derivative terms. This theoretical operation leads to the following explicit form (for the first four terms), namely:

\[
\int d^{D-1}x \left[ -2 \partial_i (\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i}) F_j + 2 \partial_i \left( \partial^0 \bar{C}^{ij} + \partial^i \bar{C}^{j0} + \partial^j \bar{C}^{0i} \right) \bar{F}_j - \partial_i \left[ \pm m \bar{C}^{0i} - (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \right] f - \partial_i \left[ \pm m C^{0i} - (\partial^0 C^i - \partial^i C^0) \right] F_i \right].
\]

(B.9)

Using the following EL-EoMs from \( \mathcal{L}_B \), namely:

\[
\partial_\mu \left[ \partial^\mu \bar{C}^{\nu \lambda} + \partial^\nu \bar{C}^{\lambda \mu} + \partial^\lambda \bar{C}^{\mu \nu} \right] \pm \frac{m}{2} (\partial^\nu \bar{C}^\lambda - \partial^\lambda \bar{C}^\nu)
\]

\[
+ \frac{1}{2} (\partial^\nu F^\lambda - \partial^\lambda F^\nu) - \frac{m^2}{2} \bar{C}^{\nu \lambda} = 0,
\]

\[
\partial_\mu \left[ \partial^\mu C^{\nu \lambda} + \partial^\nu C^{\lambda \mu} + \partial^\lambda C^{\mu \nu} \right] \pm \frac{m}{2} (\partial^\nu C^\lambda - \partial^\lambda C^\nu)
\]

\[
+ \frac{1}{2} (\partial^\nu f^\lambda - \partial^\lambda f^\nu) - \frac{m^2}{2} C^{\nu \lambda} = 0,
\]

\[
\partial_\mu \left[ \partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu \right] - \frac{1}{2} \partial^\nu F \mp m \left( \partial_\mu \bar{C}^{\mu \nu} \right) \pm \frac{m}{2} F^\nu = 0,
\]

\[
\partial_\mu \left[ \partial^\mu C^\nu - \partial^\nu C^\mu \right] - \frac{1}{2} \partial^\nu f \mp m \left( \partial_\mu C^{\mu \nu} \right) \pm \frac{m}{2} f^\nu = 0,
\]

we observe that (B.10) reduces to the following:

\[
\int d^{D-1}x \left[ \pm m \left[ \pm m C^{0i} - (\partial^0 C^i - \partial^i C^0) \right] F_i - \frac{1}{2} (\pm m \bar{f}^0 - \partial^0 \bar{f}) \right].
\]
\[ + \frac{1}{2} (\pm m f^0 - \partial^0 f) F \mp m \left[ \pm m \bar{C}^{0i} - (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \right] \bar{F}_i \]
\[ + (\partial^0 F^i - \partial^i F^0) \bar{F}_i + (\partial^0 \bar{F}^i - \partial^i \bar{F}^0) F_i. \]  

(B.12)

It is interesting to point out that the first term of the above equation cancels out with the first term in the fourth row of (B.9). A careful observation of all the terms of (B.12) and (B.9) as well as the use of the Gauss divergence theorem at appropriate places along with the utility of EL-EoMS (B.11), ultimately, lead to the following explicit expression for the ghost-sector of (B.2), namely:

\[
\int d^{D-1}x \left[ \pm m \left[ \partial^0 (f^i + \bar{F}^i - \partial^i C_1) - \partial^i (f^0 + \bar{F}^0 - \partial^0 C_1) \right] f_i \right.
\]
\[ \mp m \left[ \pm m \bar{C}^{0i} - (\partial^0 \bar{C}^i - \partial^i \bar{C}^0) \right] (f_i + \bar{F}_i - \partial_i C_1)
\]
\[ - (\partial^0 \bar{F}^i - \partial^i \bar{F}^0) (\bar{f}_i + F_i - \partial_i C_1) - \frac{1}{2} \left( \pm m f^0 - \partial^0 f \right) (\bar{f} + F \mp m C_1)
\]
\[ + \left[ \pm m (f^0 + \bar{F}^0 - \partial^0 C_1) - \partial^0 (\bar{f} + F \mp m C_1) \right] F
\]
\[ + \frac{1}{2} \left[ \pm m (\bar{f}^0 + F^0 - \partial^0 \bar{C}_1) - \partial^0 (\bar{f} + F \mp m \bar{C}_1) \right] \bar{F}
\]
\[ - \frac{1}{2} (\pm m \bar{f}^0 - \partial^0 \bar{f}) (f + \bar{F} \mp m C_1) \right]. \]

(B.13)

A close look at the above equation demonstrates that it can be zero if we exploit the validity of the four fermionic CF-type restrictions: \( f_\mu + \bar{F}_\mu - \partial_\mu C_1 = 0 \), \( \bar{f}_\mu + F_\mu - \partial_\mu C_1 = 0 \), \( f + \bar{F} \mp m C_1 = 0 \), and \( \bar{f} + F \mp m \bar{C}_1 = 0 \). In other words, if we demand the absolute anticommutativity of the nilpotent (anti-)BRST charges \( Q_{(a)b}^{(1)} \) from the condition: \( s_b Q_{ab}^{(1)} = 0 \), we observe that all the four set of fermionic CF-type restrictions emerge out.

Before we end this Appendix, we briefly comment on the appearance of the CF-type restrictions in the application of \( s_{ab} Q_b^{(1)} \). For this purpose, we now apply the anti-BRST symmetry transformations (9) on the off-shell nilpotent version of the BRST charge [cf. Eq. (53)] so that the explicit form of the l.h.s. of the second entry of (B.1) is as follows:

\[
\begin{align*}
\int d^{D-1} x \left[ ( & \partial^0 \bar{B}^{ij} + \partial^i \bar{B}^{j0} + \partial^j \bar{B}^{0i} ) B_{ij} - ( \partial^0 B^{ij} + \partial^i B^{j0} \\
+ & \partial^j \bar{B}^{0i} ) B_{ij} - ( \partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i} ) \right] ( \partial_i \bar{F}_j - \partial_j \bar{F}_i ) \\
+ & ( \partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i} ) \left( \partial_i F_j - \partial_j F_i \right) \\
+ & 2 \left[ \bar{B}_2 B - \bar{B} \bar{B}_2 \mp m \left[ \partial_0 \{ \bar{f}^i + F^i \} - \partial^i \{ \bar{f}^0 + F^0 \} \right] C_i \pm m B^{0i} \bar{B}_i \\
+ & \left[ \pm m \bar{C}^{0i} - ( \partial^0 \bar{C}^i - \partial^i \bar{C}^0 ) \right] ( \pm m \bar{F}_i - \partial_i \bar{F} ) - 2 \left( \partial^0 \bar{F}^i - \partial^i \bar{F}^0 \right) \bar{F}_i \\
+ & 2 \left( \partial^0 \bar{f}^i - \partial^i \bar{f}^0 \right) \partial_i B - B_1 \bar{B}_1 - \left[ \pm m B^{0i} - ( \partial^0 B^i - \partial^i B^0 ) \right] B_i \\
+ & \left( \partial^0 \bar{f}^i - \partial^i \bar{f}^0 \right) \partial_i B_2 \mp \left[ \pm m \bar{C}^{0i} - ( \partial^0 \bar{C}^i - \partial^i \bar{C}^0 ) \right] ( \pm m \bar{f}_i - \partial_i \bar{f} ) \\
\mp & \frac{m}{2} B^0 B_1 + \frac{1}{2} \left( \pm m \bar{F}^0 - \partial^0 \bar{F} \right) F \pm \frac{m}{2} \left( \pm m \beta^0 - \partial^0 \beta \right) B_2 + B^{0i} \partial_i B_1 \right].
\end{align*}
\]
\[-(\partial^0 F^i - \partial^i F^0) f_i + (\pm m F^0 - \partial^0 F) \bar{F} - \frac{1}{2} (\pm m \bar{f}^0 - \partial^0 \bar{f}) f \]

\[\mp m \left[ \pm m \bar{\beta}^0 - \partial^0 \bar{\beta} \right] B - \left( \partial^0 \bar{F}^i - \partial^i \bar{F}^0 \right) F_i - \left( \partial^0 B^i - \partial^i B^0 \right) \bar{B}_i \]. \hspace{1cm} (B.14)

To simplify the above expression, first of all, we select all the terms that contain the auxiliary fields \(B, B_1\) and \(B_2\). These are as follows:

\[\int d^{D-1} x \left[ 2 \dot{B}_2 B - \dot{B} B_2 - B_1 \dot{B}_1 \mp m \left( \pm m \beta^0 - \partial^0 \beta \right) B_2 \right. \]

\[\left. \mp m \left[ \pm m \bar{\beta}^0 - \partial^0 \bar{\beta} \right] B \mp \frac{m}{2} B^0 B_1 + B^{0i} \partial_i B_1 \right. \]

\[+ \left( \partial^0 \beta^i - \partial^i \beta^0 \right) \partial_i B_2 + 2 \left( \partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0 \right) \partial_i B \right]. \hspace{1cm} (B.15)\]

In the last three terms of the above equation, we can apply the Gauss divergence theorem and drop the total space derivative terms to obtain the following [from (B.15)], namely:

\[\int d^{D-1} x \left[ 2 \dot{B}_2 B - \dot{B} B_2 - B_1 \dot{B}_1 \mp m \left( \pm m \beta^0 - \partial^0 \beta \right) B_2 \right. \]

\[\left. \mp m \left[ \pm m \bar{\beta}^0 - \partial^0 \bar{\beta} \right] B - (\partial_i B^{0i}) B_1 - \partial_i \left( \partial^0 \beta^i - \partial^i \beta^0 \right) B_2 \right. \]

\[-2 \partial_i \left( \partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0 \right) B \right]. \hspace{1cm} (B.16)\]

At this stage, we use the following EL-EoMs, from \(\mathcal{L}_B\):

\[\partial_\mu (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu) + \partial^\nu B_2 - \frac{m^2}{2} \bar{\beta}^\nu \mp \frac{m}{2} \partial^\nu \beta = 0,\]

\[\partial_\mu B^{\mu\nu} - \partial^\nu B_1 \mp \frac{m}{2} B^\nu = 0,\]

\[\partial_\mu (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) - \partial^\nu B - \frac{m^2}{2} B^\nu \mp \frac{m}{2} \partial^\nu \beta = 0. \hspace{1cm} (B.17)\]

The substitutions of these values into (B.16) shows that all the terms, containing auxiliary fields \(B, B_1, B_2\), cancel out and the whole integral (B.16) turns out to be zero.

At this juncture, we now concentrate on the non-ghost sector of the terms that are present in (B.14). A close look at these terms demonstrate that they are same as (B.6). Hence, the integral with these terms can be expressed as (B.7) provided we use the EL-EoMs (B.8). Thus, we shall have validity of the two bosonic CF-type restrictions provided we assume that non-ghost sector contribute to zero in \(s_{ab} Q_b^{(1)}\) [cf. Eq. (B.18)]. We concentrate now on the ghost-sector of the terms that are present in (B.14) which necessarily contain bosonic/fermionic (anti-)ghost fields. These are:

\[\int d^{D-1} x \left[ (\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i}) \left( \partial_i F_j - \partial_j F_i \right) \right. \]

\[\left. - (\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i}) \left( \partial_i F_j - \partial_j F_i \right) \right. \]

\[\left. \mp m \left[ \partial^i \left\{ \bar{f} + F^i \right\} - \partial^i \left\{ \bar{f}^0 + F^{0i} \right\} \right] C_i - (\partial^0 \bar{F}^i - \partial^i \bar{F}^0) F_i \right]. \]
Hence, the contributions of the terms in the ghost-sector provided all the EL-EoMs from \( L \). We apply (i) the Gauss divergence theorem at appropriate places, and (ii) the appropriate integral \( \text{(B.18)} \) appears as follows:

\[
\begin{align*}
+ & \left[ \pm m \tilde{C}^0_i - (\partial^0 \tilde{C}^i - \partial^i \tilde{C}^0) \right] \left( \pm m \tilde{F}_i - \partial_i \tilde{F} \right) - 2 \left( \partial^0 \tilde{F}^i - \partial^i \tilde{F}^0 \right) F_i \\
- & \left[ \pm m C^0_i - (\partial^0 C^i - \partial^i C^0) \right] \left( \pm m f_i - \partial_i f \right) + \frac{1}{2} \left( \pm m \tilde{F}^0 - \partial^0 \tilde{F} \right) F \\
- & (\partial^0 F^i - \partial^i F^0) f_i + \left( \pm m F^0 - \partial^0 F \right) \tilde{F} - \frac{1}{2} \left( \pm m \tilde{F}^0 - \partial^0 \tilde{f} \right) f \].
\end{align*}
\]

We apply (i) the Gauss divergence theorem at appropriate places, and (ii) the appropriate EL-EoMs from \( L_B \) and \( \bar{L}_B \), respectively. It turns out that the final form of the above integral \( \text{(B.18)} \) appears as follows:

\[
\int d^{D-1} x \left[ \pm m \left[ \pm m C^0_i - (\partial^0 C^i - \partial^i C^0) \right] \left( \bar{f}_i + F_i - \partial_i C_1 \right) \right.
\]

\[
\mp m \left[ \partial^0 \left( \bar{f}^i + F^i - \partial^i C_1 \right) - \partial^i \left( f^0 + \tilde{F}^0 - \partial^0 C_1 \right) \right] C_i
\]

\[
- (\partial^0 F^i - \partial^i F^0) \left( \bar{f}_i + F_i - \partial_i C_1 \right) + \frac{1}{2} \left( \pm m F^0 - \partial^0 F \right) \left( f + \tilde{F} \mp m C_1 \right)
\]

\[
+ \frac{1}{2} \left[ \pm m \left( f^0 + \tilde{F}^0 - \partial^0 C_1 \right) - \partial^0 \left( f + \tilde{F} \mp m C_1 \right) \right] F
\]

\[
- \frac{1}{2} \left( \pm m f^0 - \partial^0 f \right) \left( \bar{f} + F \mp m \tilde{C}_1 \right)
\]

\[
- \frac{1}{2} \left[ \pm m \left( \bar{f}^0 + F^0 - \partial^0 \tilde{C}_1 \right) - \partial^0 \left( \bar{f} + F \mp m \tilde{C}_1 \right) \right] f
\].

Hence, the contributions of the terms in the ghost-sector of \( \text{(B.14)} \) will be equal to zero provided all the four fermionic CF-type restrictions: \( f_\mu + \tilde{F}_\mu - \partial_\mu C_1 = 0, \bar{f}_\mu + F_\mu - \partial_\mu \bar{C}_1 = 0, f + \tilde{F} \mp m C_1 = 0, \bar{f} + F \mp m \tilde{C}_1 = 0 \) are satisfied.

We end this Appendix with the final remark that the existence of the CF-type restrictions is the hallmark of a BRST-quantized theory. We have shown, in our present endeavor, the existence of a set of six CF-type restrictions (i) from the EL-EoMs [cf. Eq. (29)], (ii) from the equality of the Lagrangian density \( L_B \) and \( \bar{L}_B \) [cf. Appendix A], and (iii) from the requirement of absolute anticommutativity of the nilpotent (anti-)BRST charges.

**Appendix C: On the Standard BRST Algebra**

We have seen that the standard Noether conserved (anti-)BRST charges [i.e. \( Q_{(a)b} \)] are not nilpotent of order two (i.e. \( Q_{(a)b}^2 \neq 0 \)). Hence, they do not participate in the construction of the standard BRST algebra where the nilpotency of the (anti-)BRST charges is one of the key ingredients. In fact, the nilpotency property plays a decisive role in the discussion of the BRST cohomology (see, e.g. [27] for details) where the original state and the gauge transformed state both are cohomologically equivalent w.r.t. the nilpotent (anti-)BRST charges. Thus, in the construction and derivations of the standard BRST algebra, the off-shell nilpotent versions of the (anti-) BRST charges \( [Q_{(a)b}^{(1)}] \) (that have been derived
in an explicit forms in (62) and (53), respectively) play a crucial role. In the process of the derivation of the standard BRST algebra, we take into consideration the celebrated relationship between the continuous symmetry transformations and their generators as the conserved charges [cf. Eq. (49)]. In other words, we note the following explicit relationships, namely:

\[ s_b Q_b^{(1)} = -i \{ Q_b^{(1)}, Q_b^{(1)} \} = 0 \implies [Q_b^{(1)}]^2 = 0, \]

\[ s_{ab} Q_{ab}^{(1)} = -i \{ Q_{ab}^{(1)}, Q_{ab}^{(1)} \} = 0 \implies [Q_{ab}^{(1)}]^2 = 0, \]

\[ s_b Q_b^{(1)} = -i \{ Q_b^{(1)}, Q_b^{(1)} \} = 0 \implies Q_b^{(1)} Q_b^{(1)} + Q_b^{(1)} Q_b^{(1)} = 0, \]

\[ s_{ab} Q_{ab}^{(1)} = -i \{ Q_{ab}^{(1)}, Q_{ab}^{(1)} \} = 0 \implies Q_{ab}^{(1)} Q_{ab}^{(1)} + Q_{ab}^{(1)} Q_{ab}^{(1)} = 0, \]

\[ s_g Q_b = -i [Q_b, Q_g] = + Q_b, \quad s_g Q_g = -i [Q_g, Q_g] = 0, \]

\[ s_g Q_{ab} = -i [Q_{ab}, Q_g] = - Q_{ab}, \]  

where we have taken into account the general mathematical relationship (49).

We conclude this Appendix with the following remarks. First of all, we note that the absolute anticommutativity property between the (anti-)BRST charges is satisfied on a submanifold of the quantum Hilbert space of fields where the full set of all the six CF-type restrictions are satisfied [cf. Appendix B]. Second, the (anti-)BRST charges \( Q_{(a)b}^{(1)} \) are off-shell nilpotent of order two and, hence, they are fermionic in nature and they generate continuous symmetry transformations where the fermionic fields transform into bosonic fields and vice-versa. Third, the off-shell nilpotent (anti-)BRST charges are not like the \( \mathcal{N} = 2 \) SUSY off-shell nilpotent supercharges because the latter set of charges do not absolutely anticommutate. Fourth, the nilpotency property plays a crucial role in the discussion on the BRST cohomology (see, e.g. [27] for details). Fifth, we note that the ghost number of the BRST charge is (+1) and that of the anti-BRST charge is (−1), respectively. The ghost number of the ghost charge is zero as is clear from the algebra (C.1). Finally, we obtain, in a nutshell, the standard BRST algebra with the help of the nilpotent versions of the (anti-)BRST charges \( [Q_{(a)b}^{(1)}] \) and the ghost charge \( Q_g \) as follows:

\[ [Q_{(a)b}^{(1)}]^2 = 0, \]

\[ i [Q_g, Q_b^{(1)}] = + Q_b^{(1)}, \]

\[ i [Q_g, Q_{ab}^{(1)}] = - Q_{ab}^{(1)}, \]

\[ \{ Q_b^{(1)}, Q_{ab}^{(1)} \} \equiv \{ Q_{ab}^{(1)}, Q_b^{(1)} \} = 0, \]  

where the last entry (i.e. absolute anticommutativity property) is satisfied only on the submanifold of the Hilbert space of quantum fields where all the six CF-type restrictions are valid (see, e.g. Appendix B for details).

**Appendix D: On the Glossary of Fields**

In this Appendix, we provide the list of all the fields that are present in the coupled (but
equivalent) BRST and anti-BRST invariant Lagrangian densities $\mathcal{L}_B$ and $\mathcal{L}_{\bar{B}}$, respectively [cf. Eqs. (5),(6)]. We lay emphasis on their bosonic/fermionic, basic/auxiliary, independent/restricted, tensor/vector/scalar, etc., nature and mention their ghost number(s), too. We list them in the following tabulated form:

| Fields $A_{\mu\nu\lambda}$ | Bosonic | Basic | Tensorial Nature | Ghost Number | Independent/Restricted |
|-----------------------------|---------|-------|------------------|--------------|------------------------|
| $C_{\mu\nu}$ | Fermionic | Basic | Antisymmetric | $+1$ | Independent |
| $\bar{C}_{\mu\nu}$ | Fermionic | Basic | Antisymmetric | $-1$ | Independent |
| $B_{\mu\nu}$ | Bosonic | Auxiliary | Antisymmetric | $0$ | Restricted (D.1) |
| $\bar{B}_{\mu\nu}$ | Bosonic | Auxiliary | Antisymmetric | $0$ | Restricted (D.1) |

$\phi_{\mu\nu}$ Bosonic Basic Antisymmetric $0$ Independent
$\beta_{\mu}$ Bosonic Basic Vector $+2$ Independent
$\bar{\beta}_{\mu}$ Bosonic Basic Vector $-2$ Independent
$\beta$ Bosonic Basic Scalar $+2$ Independent
$\bar{\beta}$ Bosonic Basic Scalar $-2$ Independent
$C_{\mu}$ Fermionic Basic Vector $+1$ Independent
$\bar{C}_{\mu}$ Fermionic Basic Vector $-1$ Independent
$C_1$ Fermionic Basic Scalar $+1$ Restricted (D.2), (D.5)
$\bar{C}_1$ Fermionic Basic Scalar $-1$ Restricted (D.3), (D.6)
$C_2$ Fermionic Basic Scalar $+3$ Independent
$\bar{C}_2$ Fermionic Basic Scalar $-3$ Independent
$\phi_{\mu}$ Bosonic Basic Vector $0$ Restricted (D.4), (D.1)
$\phi$ Bosonic Basic Scalar $0$ Restricted (D.4)
$B_{\mu}$ Bosonic Auxiliary Vector $0$ Restricted (D.4)
$\bar{B}_{\mu}$ Bosonic Auxiliary Vector $0$ Restricted (D.4)
$B$ Bosonic Auxiliary Scalar $+2$ Independent
$B_1$ Bosonic Auxiliary Scalar $0$ Independent
$B_2$ Bosonic Auxiliary Scalar $-2$ Independent
$f_{\mu}$ Fermionic Auxiliary Vector $+1$ Restricted (D.2)
$\bar{f}_{\mu}$ Fermionic Auxiliary Vector $-1$ Restricted (D.3)
$F_{\mu}$ Fermionic Auxiliary Vector $-1$ Restricted (D.3)
$\bar{F}_{\mu}$ Fermionic Auxiliary Vector $+1$ Restricted (D.2)
$f$ Fermionic Auxiliary Scalar $+1$ Restricted (D.5)
$\bar{f}$ Fermionic Auxiliary Scalar $-1$ Restricted (D.6)
$F$ Fermionic Auxiliary Scalar $-1$ Restricted (D.6)
$\bar{F}$ Fermionic Auxiliary Scalar $+1$ Restricted (D.5)

Table 1: Tower of fields and their specifications

In the above, the equation numbers from (D.1) to (D.6) correspond to the celebrated (anti-)BRST invariant CF-type restrictions [cf. Eq. (29)] that are present on our theory.
These restrictions, in their explicit form, are as follows:

\[ B_{\mu\nu} + \bar{B}_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu, \quad (D.1) \]
\[ f_\mu + \bar{F}_\mu = \partial_\mu C_1, \quad (D.2) \]
\[ \bar{f}_\mu + F_\mu = \partial_\mu \bar{C}_1, \quad (D.3) \]
\[ B_\mu + \bar{B}_\mu = \pm m \phi_\mu - \partial_\mu \phi, \quad (D.4) \]
\[ f + \bar{F} = \pm m C_1, \quad (D.5) \]
\[ \bar{f} + F = \pm m \bar{C}_1. \quad (D.6) \]

The above equations demonstrate that there are many fields in the theory which are not independent because they are restricted by the CF-type restrictions from (D.1) to (D.6).

Appendix E: On the Off-Shell Nilpotency and Absolute Anticommutativity of the (Anti-)BRST Symmetries and Importance of the CF-Type Restrictions

We have already noted that the absolute anticommutativity property of the (anti-)BRST symmetry transformations [cf. Eq. (11)] crucially depends on the validity of the CF-type restrictions [cf. Eq. (29)]. This anticommutativity property between the BRST and anti-BRST transformations is very important because it distinguishes the nilpotent (anti-)BRST symmetry transformations from the nilpotent \( N = 2 \) SUSY transformations which do not anticommute (with each-other). The purpose of our present Appendix is to show that, even at the level of the (anti-)BRST transformed fields, the CF-type restrictions are valid in the sense that the off-shell nilpotency property of the (anti-)BRST symmetry transformations, at this stage, crucially depends upon them. We take here an example from the BRST symmetry transformations [cf. Eq. (7)] to corroborate our claim. Let us focus on the BRST transformation: \( s_b \bar{C}_{\mu\nu} = B_{\mu\nu} \) which is nilpotent of order two (i.e. \( s_b^2 = 0 \)) because we observe that \( s_b^2 \bar{C}_{\mu\nu} = s_b B_{\mu\nu} = 0 \) due to the fact that \( s_b B_{\mu\nu} = 0 \) [cf. Eq. (7)]. We would like to point out that the r.h.s. of the transformation: \( s_b \bar{C}_{\mu\nu} = B_{\mu\nu} \) is not independent in the sense that it is restricted by the CF-type restriction: \( B_{\mu\nu} + \bar{B}_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \). Taking recourse to this CF-type restriction, we can re-express the above BRST transformation as:

\[ s_b \bar{C}_{\mu\nu} = - \bar{B}_{\mu\nu} + (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu). \quad (E.1) \]

The question we address is whether the off-shell nilpotency is satisfied at this stage or not. Interestingly, we find that the off-shell nilpotency is maintained because we observe that

\[ s_b^2 \bar{C}_{\mu\nu} = s_b \left[ - \bar{B}_{\mu\nu} + (\partial_\mu \phi_\nu - \partial_\nu \phi_\mu) \right] = 0, \quad (E.2) \]

because of the fact that we have: \( s_b \bar{B}_{\mu\nu} = \partial_\mu f_\nu - \partial_\nu f_\mu \) and \( s_b \phi_\mu = f_\mu \) [cf. Eq. (7)]. Thus, it is very clear that the off-shell nilpotency (i.e. \( s_{(a)b}^2 = 0 \)) and absolute anticommutativity [cf.
Eq. (11)] properties of the (anti-)BRST symmetry transformations and the (anti-)BRST invariant CF-type restrictions are intertwined together in a very meaningful fashion.

To complete our discussion, let us take an example from the anti-BRST symmetry transformations (9) and focus on the application of the anti-BRST symmetry transformation on a bosonic field. For instance, let us choose \( s_{ab} \phi_\mu = \bar{f}_\mu \). It is clear that the off-shell nilpotency (i.e. \( s_{ab}^2 = 0 \)) is true, at this stage, because we observe that \( s_{ab} \bar{f}_\mu = 0 \) [cf. Eq. (9)]. However, we note that the r.h.s. of this transformation is not independent because \( \bar{f}_\mu \) is restricted by the CF-type restriction: \( \bar{f}_\mu + F_\mu = \partial_\mu \bar{C}_1 \). As a consequence, we can replace the r.h.s. of \( s_{ab} \phi_\mu = \bar{f}_\mu \) (due to the above CF-type restriction) as follows:

\[
s_{ab} \phi_\mu = - F_\mu + \partial_\mu \bar{C}_1. \quad (E.3)
\]

At this stage, we would like to check whether the off-shell nilpotency \( s_{ab}^2 = 0 \) is maintained or not after the application of the CF-type restriction. It turns out, interestingly, that the off-shell nilpotency (i.e. \( s_{ab}^2 \phi_\mu = 0 \)) is maintained due to the fact that we have the following anti-BRST transformations [cf. Eq. (9)]:

\[
s_{ab} F_\mu = - \partial_\mu B_2, \quad s_{ab} \bar{C}_1 = - B_2 \quad \implies \quad s_{ab}^2 \phi_\mu = s_{ab}[- F_\mu + \partial_\mu \bar{C}_1] = 0. \quad (E.4)
\]

Thus, we conclude that the nilpotency is maintained even at the level of the anti-BRST transformed fields (provided we exploit the appropriate variety of the CF-type restriction).

We end this Appendix with the following concluding remarks. First of all, we note that the absolute anticommutativity property (i.e. \( \{s_b, s_{ab}\} = 0 \)) of the (anti-)BRST symmetry transformations [cf. Eq. (11)] crucially depends on the existence of the CF-type restrictions [cf. Eq. (29)] on our theory. Second, we note that the straightforward equality of the BRST and anti-BRST invariant Lagrangian densities \( \mathcal{L}_B \) and \( \mathcal{L}_{\bar{B}} \) leads to the derivation of the CF-type restrictions (cf. Appendix A). In other words, Lagrangian densities \( \mathcal{L}_B \) and \( \mathcal{L}_{\bar{B}} \) are equivalent on the submanifold of the fields where the CF-type restrictions are satisfied. Third, at the level of the (anti-)BRST transformed fields, it turns out that the uses of the appropriate CF-type restrictions are responsible for the off-shell nilpotency of the (anti-)BRST symmetry transformations. Finally, it is worthwhile to point out that we have chosen only two examples to show the sanctity of the CF-type restrictions at the level of the (anti-)BRST transformed fields. However, this observation is true for other similar examples that are present in the (anti-)BRST transformations [cf. Eqs. (7),(9)], too.

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