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Complexity and algorithms for injective edge-coloring in graphs

Florent Foucaud†‡§ Hervé Hocquard† Dimitri Lajou†

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Abstract

An injective k-edge-coloring of a graph G is an assignment of colors, i.e. integers in \{1, \ldots, k\}, to the edges of G such that any two edges each incident with one distinct endpoint of a third edge, receive distinct colors. The problem of determining whether such a k-coloring exists is called \textsc{Injective k-Edge-Coloring}. We show that \textsc{Injective 3-Edge-Coloring} is NP-complete, even for triangle-free cubic graphs, planar subcubic graphs of arbitrarily large girth, and planar bipartite subcubic graphs of girth 6. \textsc{Injective 4-Edge-Coloring} remains NP-complete for cubic graphs. For any \( k \geq 45 \), we show that \textsc{Injective k-Edge-Coloring} remains NP-complete even for graphs of maximum degree at most \( 5\sqrt{3}k \). In contrast with these negative results, we show that \textsc{Injective k-Edge-Coloring} is linear-time solvable on graphs of bounded treewidth. Moreover, we show that all planar bipartite subcubic graphs of girth at least 16 are injectively 3-edge-colorable. In addition, any graph of maximum degree at most \( \sqrt{k/2} \) is injectively k-edge-colorable.

1 Introduction

We study the algorithmic complexity of the injective edge-coloring problem. Our aim is to determine restricted graph classes where the problem is NP-hard, while in contrast, designing algorithms for other graph classes. An injective k-edge-coloring of a graph \( G = (V(G), E(G)) \) is an assignment of colors, i.e. integers in \{1, \ldots, k\}, to the edges of G in such a way that two edges that are each incident with one distinct endpoint of a third edge, receive distinct colors. In other words, for any 3-edge path of G (possibly forming a triangle), the first and last edge of the path receive distinct colors. The injective chromatic index of G, denoted \( \chi_s(G) \), is the least integer \( k \) for which G admits an injective k-edge-coloring.

This concept was recently introduced in [4], where it is studied for some classes of graphs, and proved to be NP-complete. Bounds on the injective chromatic index of planar graphs, graphs of given maximum degree, and other important graph classes, have been recently determined in [1, 3, 7, 14, 16]. In particular, as mentioned in [7], it follows from [1] that all planar graphs are injectively 30-edge-colorable, while outerplanar graphs are injectively 9-edge-colorable [7]. It is also proved in [14] that subcubic graphs are injectively 7-edge-colorable, while subcubic bipartite graphs [7] and subcubic planar graphs [14] are injectively 6-edge-colorable. Moreover all subcubic planar bipartite graphs are injectively 4-edge-colorable [14].

Note that in [1], this notion is studied as the induced star arboricity of a graph, that is, the smallest number of star forests into which the edges of the graph can be partitioned: this is an equivalent way to interpret injective edge-coloring (see [2]). The concept of an injective edge-coloring is the natural edge-version of the notion of an injective vertex-coloring, introduced in [10] and well-studied since then.

Another closely related notion is the one of strong edge-coloring of a graph G, introduced in [8] and well-studied since then, especially in view of a celebrated conjecture by Erdős and Nešetřil [6]. In this type of coloring, edges that are the endpoints of a same 3-edge path or 2-edge path must receive distinct colors. The strong chromatic index \( \chi'_s(G) \) of a graph G is the least integer k for which G admits a strong edge-coloring with k colors. It follows from the definitions that for any graph G, \( \chi'_s(G) \leq \chi_s(G) \) holds.

The algorithmic complexity of determining the strong chromatic index of a graph is well-studied, see for example [12] for a classic reference, and [3, 11] for more recent ones. In this paper, we wish to undertake similar types of studies for the injective chromatic index. The problem at hand is formally defined as follows.

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Injective $k$-Edge-Coloring
Instance: A graph $G$.
Question: Does $G$ admit an injective $k$-edge-coloring?

Injective $k$-Edge-Coloring was proved NP-complete (for every fixed $k \geq 3$) in [1], with no particular restriction on the inputs. We strengthen this as follows.

Theorem 1 The two following are NP-Complete:

1. Injective 3-Edge-Coloring, even for triangle-free cubic graphs, and
2. Injective 4-Edge-Coloring, even for cubic graphs.

Answering a question from [3] about the complexity of Injective $k$-Edge-Coloring for planar graphs, we also study restricted subclasses of planar graphs.

Theorem 2 Let $g \geq 3$. Injective 3-Edge-Coloring is NP-Complete even for:

1. planar subcubic graphs with girth at least $g$,
2. planar bipartite subcubic graphs of girth 6.

The two items in Theorem 2 cannot be combined, because we can prove the following (note that all planar bipartite subcubic graphs are injectively 4-edge-colorable [4]).

Theorem 3 Every planar bipartite subcubic graph of girth at least 16 is injectively 3-edge-colorable.

We also obtain the following positive result ($tw(G)$ denotes the treewidth of $G$).

Theorem 4 For every graph $G$ of order $n$ and every positive integer $k$, there exists a $2^{O(k \cdot tw(G)^2)}n$-time algorithm that solves Injective $k$-Edge-Coloring.

It is proved in [1] that $\chi'_i(G) \leq 3\left(\frac{tw(G)}{2}\right)^2$, and so using the above algorithm, one can determine the injective chromatic index of a graph of order $n$ in time $2^{O(tw(G)^3)}n$.

Contrasting with our hardness results for planar graphs, Theorem 4 implies that Injective $k$-Edge-Coloring can be solved in polynomial-time on subclasses of planar graphs: $K_4$-minor-free graphs (i.e. graphs of treewidth 2), and thus, on outerplanar graphs.

In [4], Cardoso et al. use a reduction on graphs having their maximum degree linear in the number of colors. We improve it with the following result.

Theorem 5 For every integer $k \geq 45$, Injective $k$-Edge-Coloring is NP-Complete even for graphs with maximum degree at most $5\sqrt{k}$.

The bound of Theorem 5 is tight up to a constant factor: by a standard maximum degree argument of a conflict graph, every graph with maximum degree at most $\sqrt{k}/2$ is injectively $k$-edge-colorable. (Indeed, for every edge $e$ of a graph $G$, there are at most $2(\Delta(G) - 1)^2$ edges which cannot have the same color as $e$, where $\Delta(G)$ is the maximum degree of $G$.)

2 Proof of Theorem [1]

For these two problems, we reduce from 3-Edge-Coloring, which is NP-Complete even for cubic graphs [12]. (Recall that a proper edge-coloring is an edge-coloring for which edges that are incident to a same vertex receive different colors.)

3-Edge-Coloring
Instance: A cubic graph $G$.
Question: Does $G$ admit a proper 3-edge-coloring?
2.1 Proof of Theorem 1.1

Proof. Let $G$ be the input cubic graph. We will proceed in two steps: first, we create a triangle-free subcubic graph $G'$ which has an injective 3-edge-coloring if and only if $G$ is properly 3-edge-colorable. Then we describe how to make the graph cubic.

We create the graph $G'$ from $G$ by removing all the edges of $G$. For each edge $uv$ of $G$, we create a copy of a gadget $E_{uv}$ (see Figure 1(a) for an illustration) and connect it to $u$ and $v$ as follows. We add eight new vertices $w_{uv}, z_{uv}, a_{uv}, b_{uv}, c_{uv}, d_{uv}, e_{uv}$ and $f_{uv}$. We create the following edges $uw_{uv}, vw_{uv}, wu_{uv}, wv_{uv}, zw_{uv}, zb_{uv}, ac_{uv}, ad_{uv}, au_{uv}, bu_{uv}, cu_{uv}, e_{uv}f_{uv}$ and $f_{uv}w_{uv}$.

Claim 6: $E_{uv}$ is injectively 3-edge-colorable, and for every valid edge-coloring $\gamma$ of $E_{uv}$, $\gamma(w_{uv}) = \gamma(vw_{uv}) = \gamma(wu_{uv})$. Moreover, for any choice of the same color for these three edges, we can extend the coloring to an injective 3-edge-coloring of $E_{uv}$.

Proof. Let us injectively 3-edge-color $E_{uv}$. W.l.o.g., we can assume that $d_{uv}f_{uv}$ is colored 1, $b_{uv}e_{uv}$ is colored 2 and $a_{uv}z_{uv}$ is colored 3. We deduce that $b_{uv}e_{uv}$ is colored 2, $c_{uv}f_{uv}$ is colored 1, $a_{uv}d_{uv}$ and $a_{uv}z_{uv}$ are colored 3, $b_{uv}z_{uv}$ is colored 2 and $e_{uv}f_{uv}$ is colored 1. Hence $uw_{uv}, vw_{uv}$ and $wu_{uv}$ must all be colored 1.

Now, given one same color for these three edges, one can color the rest of the gadget, for example using the previously constructed coloring.

If $G$ has a proper 3-edge-coloring $\gamma$, we injectively 3-edge-color $G'$ by assigning to $uw_{uv}, vw_{uv}$ and $wu_{uv}$ in $G'$ the color $\gamma(uv)$; then we extend the coloring to each $E_{uv}$ using Claim 6.

Conversely, if $G'$ has an injective 3-edge-coloring, then we color an edge $uv$ of $G$ with the color of the edge $uw_{uv}$ (or $vw_{uv}$) of $G'$. This coloring is proper since Claim 5 ensures that $uw_{uv}$ and $vw_{uv}$ have the same color. Indeed if $uv$ is an edge adjacent to $w$, then $uw_{uv}$ and $xw_{uv}$ have different colors.

We now show how to make the construction cubic. We create the cubic graph $G''$ as follows. First, take three disjoint copies $G_1, G_2$ and $G_3$ of $G'$. To differentiate the vertices of each copy, we add an exponent to the name of the vertex corresponding to the number of the copy. For example, vertex $w_{uv}$ of $G_1$ will be noted $w_{uv}^1$. For each edge $uv$ of $G$, connect $G_1, G_2$ and $G_3$ via $K_{1,3}$ with vertex classes $\{r_u\}$ and $\{s_{uv}, p_{uv}, q_{uv}\}$ as follows. The vertex $s_{uv}$ (resp. $p_{uv}$, resp. $q_{uv}$) is adjacent to $d_{uv}^1$ (resp. $d_{uv}^2$, resp. $d_{uv}^3$), $c_{uv}^1$ (resp. $c_{uv}^2$, resp. $c_{uv}^3$) and $r_u$ (see Figure 1(b)). The graph $G''$ is simply the graph where the edge gadget is represented in Figure 1 and for each $u \in V(G)$, the three copies of $u'$ for $i \in \{1, 2, 3\}$ are identified.

As $G$ is cubic, $G''$ is triangle-free and cubic. Note that if $G'$ admits an injective 3-edge-coloring, then in particular $G'$ also admits an injective 3-edge-coloring and thus by our previous arguments, $G$ is properly 3-edge-colorable.

If $G$ is properly 3-edge-colorable, then we fix such a coloring $\gamma : E(G) \rightarrow \{1, 2, 3\}$. For $i \in \{1, 2, 3\}$, we color the edges incident with $w_{uv}^i$ with the color $\gamma(uv) + i$, where the colors are considered to be taken modulo 3 (considering 0 = 3). Then it suffices to extend the obtained coloring to each edge gadget (see Figure 1).

2.2 Proof of Theorem 1.2

Proof. Let $G$ be the input graph. For each vertex $u$ of $G$, we replace it by the following vertex gadget $S_u$ (see Figure 2). The gadget $S_u$ is made of a 9-cycle $x_0u, x_1u, \ldots, x_8u$ and three other vertices $y_iu$ ($i \in \{0, 3, 6\}$) that will be connected to the rest of the graph. We add the edges $x_1u, x_2u, x_3u, x_4u, x_5u, x_6u, x_7u, x_8u, x_9u$. For any edge-coloring $\gamma$ of $S_u$, we note $C^u_i(\gamma) = \{\gamma(x_iu, x_{i+1}u) \mid i \in \{0, 3, 6\}\}$ and where the indices are taken modulo 9.

Claim 7: For every injective 4-edge-coloring $\gamma$ of $S_u$ and for every $i \in \{0, 3, 6\}$, the color $\gamma(x_iu, y_iu)$ belongs to the set $C^u_i(\gamma)$. Moreover, $C^u_0(\gamma) \cup C^u_3(\gamma) \cup C^u_6(\gamma) = \{1, 2, 3, 4\}$ and there exists a color $a \in \{1, 2, 3, 4\}$ such that for all $i \in \{0, 3, 6\}$, $a \in C^u_i(\gamma)$.

Furthermore, for any choice of color for $x_0u, y_0u, x_3u, y_3u, x_6u, y_6u$ and sets of colors $C^u_i(\gamma), i \in \{0, 3, 6\}$ verifying the previous necessary conditions, there exists an injective 4-edge-coloring $\gamma$ of $S_u$ matching those choices.

Proof. Let us try to construct an injective 4-edge-coloring $\gamma$ of $S_u$. Up to permuting the colors, we assume that $\gamma(x_0u, x_1u) = 1, \gamma(x_3u, x_4u) = 2$ and $\gamma(x_6u, x_7u) = 3$. Note that $x_0u, x_1u, x_3u, x_4u$ cannot both be colored 4, w.l.o.g. assume that $\gamma(x_0u, x_1u) \neq 4$. Hence $\gamma(x_3u, x_4u) = 2$ and $\gamma(x_6u, x_7u) = 4$. Remark that $\gamma(x_0u, x_1u) \neq 2$. Moreover $x_3u, x_4u$ can only receive colors 1 or 4 and they must receive different colors. Hence
$\gamma(x_1^u x_2^u) = 3$, $\gamma(x_3^u x_4^u) = 1$, $\gamma(x_5^u x_6^u) = 4$ and $\gamma(x_7^u x_8^u) = 1$. Now there are two ways to complete the coloring of $S_u$, either $\gamma(x_1^u x_2^u) = 4$, $\gamma(x_3^u x_4^u) = 3$ and $\gamma(x_5^u x_6^u) = 2$ or, $\gamma(x_1^u x_2^u) = 3$, $\gamma(x_3^u x_4^u) = 2$ and $\gamma(x_5^u x_6^u) = 4$. In both cases all properties of the first part of the claim hold (with $a = 1$).

Finally, note that the second of the two previous coloring options allows us to color $x_i^u y_i^u$, $i \in \{0, 3, 6\}$ with any color among those of $x_i^u x_{i+1}^u$ and $x_i^u x_{i-1}^u$, and to complete the coloring.

For every edge $uv$ of $G$, we construct the following edge gadget $E_{uv}$ (see Figure 2). First, choose $y_i^u$ (resp. $y_j^v$) of degree 1 among the vertices of $S_u$ (resp. $S_v$). Create two new adjacent vertices $w_{uv}$ and $z_{uv}$ such that $y_i^u w_{uv} y_j^v z_{uv}$ is a 4-cycle.

**Claim 8** For every injective 4-edge-coloring $\gamma$ of $G$ and every edge gadget $E_{uv}$ connecting $y_i^u$ and $y_j^v$ ($i, j \in \{0, 3, 6\}$), we have $C_i^u(\gamma) = C_j^v(\gamma)$.

Furthermore, any injective 4-edge-coloring $\gamma$ of $S_u$ and $S_v$ such that $C_i^u(\gamma) = C_j^v(\gamma)$ and $\gamma(x_i^u y_i^u) = \gamma(x_j^v y_j^v)$ can be extended to an injective 4-edge-coloring of $S_u \cup E_{uv} \cup S_v$.

**Proof.** Suppose, w.l.o.g. by Claim 7 that $x_i^u x_{i+1}^u$ is colored 1, $x_i^u x_{i-1}^u$ is colored 2 and $x_i^u y_i^u$ is colored 1. Now w.l.o.g., $y_i^u w_{uv}$ is colored 3 and $y_i^u z_{uv}$ is colored 4. This implies that $w_{uv} z_{uv}$ is colored 2, $y_j^v w_{uv}$ is colored 3, $y_j^v z_{uv}$ is colored 4, $y_j^v x_j^v$ is colored 1 and $C_i^u(\gamma) = \{1, 2\}$.

The second part of the claim is proved by taking the previous coloring and extending it using the second part of Claim 7.

Let $G'$ be the cubic graph constructed from $G$ by the above process. By Claim 8 if $uv$ is an edge connecting $y_i^u$ and $y_j^v$ then for any injective coloring $\gamma$ of $G'$, $C_i^u(\gamma) = C_j^v(\gamma) = \{a, b\}$ for some and $b$. Hence this set somehow characterizes the edge gadget $E_{uv}$, we say that $E_{uv}$ is colored by $\{a, b\}$.

Suppose that there exists an injective 4-edge-coloring $\gamma$ of $G'$. For each edge $uv$ of $G$, we color $uv$ depending on the coloring of $E_{uv}$. When $E_{uv}$ is colored $\{1, 2\}$ or $\{3, 4\}$ (resp. $\{1, 3\}$ or $\{2, 4\}$, resp. $\{1, 4\}$ or $\{2, 3\}$) then we color $uv$ by color 1 (resp. 2, resp. 3). We argue that this edge-coloring, noted $\gamma$, is proper. Indeed suppose it is not, then for some vertex $u$, w.l.o.g., $uv$ and $uw$ are both colored 1. This means that the coloring of $G$ is such that $C_i^u(\gamma) = C_j^u(\gamma)$ or $C_i^u(\gamma) \cap C_j^u(\gamma) = \emptyset$ for $i \neq j$ and $i, j \in \{0, 3, 6\}$. This contradicts Claim 7. Hence we get a proper 3-edge-coloring of $G$.

Conversely, suppose that there exists a proper 3-edge-coloring of $G$. In $G'$, we color each edge of the form $x_i^u y_i^u$ by 1. If an edge $uv$ of $G$ is colored 1 (resp. 2, resp. 3) then we assign the color $\{1, 2\}$ (resp. 2, resp. 3) to $uv$. This is well-defined, and we get the corresponding edge-coloring of $E_{uv}$.
For any injective 4-edge-coloring of each $S_u$, $u \in V(G)$. By Claim 8, this injective 4-edge-coloring can be extended to each edge gadget to color the whole graph.

\section{Proof of Theorem 2}

We will reduce from the following problem:

\begin{itemize}
  \item [Planar 3-Vertex-Coloring]
  \item [Instance:] A planar graph $G$ with maximum degree 4.
  \item [Question:] Does $G$ admit a proper 3-vertex-coloring?
\end{itemize}

This problem was proven to be NP-Complete in \cite{planar3vertex}. Let $G$ be a planar graph with maximum degree 4.

\subsection{Proof of Theorem 2}

\textbf{Proof.} Recall that we want to construct a graph $G'$ with girth at least $g$.

For each vertex $u \in V(G)$, we construct a vertex gadget $S_u$ as follows (see Figure 3). First create a cycle $x_1^u, x_2^u, \ldots, x_{\ell}^u$ where $\ell \geq g$ and $\ell$ is an odd multiple of 3. To each $x_i^u$ add a single pendant neighbor $y_i^u$ of degree 1. To the vertex $y_i^u$, add two non-adjacent neighbors $w^u$ and $z^u$. Create four more vertices $a_1^u, b_1^u, c_1^u$ and $d_1^u$. The vertex $w^u$ is adjacent to $a_1^u$ and $b_1^u$ while $z^u$ is adjacent to $c_1^u$ and $d_1^u$. Now construct a path $a_1^u, a_2^u, \ldots, a_{\ell}^u$ of length $\ell$ and add to each $a_i^u$ for $i \leq g - 1$ a pendant vertex of degree 1 called $a_{i+1}^u$. Similarly we create the vertices $b_1^u, b_2^u, \ldots, b_{\ell-u}^u$, $c_1^u, \ldots, c_{\ell-u}^u$, $d_1^u, \ldots, d_{\ell-u}^u$ and $d_1^u, \ldots, d_{\ell-u}^u$. Finally add a vertex $\alpha^u$ (resp. $\beta^u$, $\gamma^u$, $\delta^u$) adjacent to $a_2^u$ (resp. $b_2^u$, resp. $c_2^u$, resp. $d_2^u$).

\textbf{Claim 9} For any injective 3-edge-coloring $\rho$ of $S_u$, $\rho(a_1^u, \alpha^u) = \rho(b_1^u, \beta^u) = \rho(c_1^u, \gamma^u) = \rho(\delta^u)$. We call this color $\rho(S_u)$. Moreover, for any choice of a color $\rho(S_u)$, there exists an injective 3-edge-coloring $\rho$ with these properties.

\textbf{Proof.} Suppose that there exists $i \in \{1, \ldots, \ell\}$ such that the property $P(i) := \rho(x_i^u, x_{i+1}^u) = \rho(x_{i+1}^u, y_{i+1}^u) \neq \rho(x_{i-1}^u, x_i^u)^{-1}$ holds (the indices are taken modulo $\ell$, considering $0 = \ell$). Then $P(i)$ holds for all $i \in \{1, \ldots, \ell\}$. Indeed, take such an $i$, then $\rho(x_{i+1}^u, x_{i+2}^u) = \rho(x_{i+1}^u, y_{i+1}^u) = \rho(x_{i-1}^u, x_i^u)$. Hence the property holds for $i + 1$, by induction it holds for every $i$. Note that the same can be said for the property $P(i) := \rho(x_{i-1}^u, x_i^u) = \rho(x_i^u, y_i^u) \neq \rho(x_{i+1}^u, x_{i-1}^u)^{-1}$. Also note that if $\rho(x_{i+1}^u, x_{i-1}^u) = \rho(x_i^u, x_{i+1}^u) = \rho(x_i^u, y_i^u)$ then we have $P(i+1)$ which is a contradiction because we do not have $P(i)$.

Suppose now that for all $i$, neither $P(i)$ nor $P(i)^{-1}$ holds. This means that the edges incident to a vertex $x_i^u$ are either of the same color, or of three distinct colors. If they have the same color, then the edges incident with $x_{i-1}^u$ have three distinct colors, the ones incident to $x_{i+2}^u$ have the same color, and so on. This would imply that the cycle $x_1^u, x_2^u, \ldots, x_{\ell}^u$ is even, which is a contradiction. Moreover, if the edges incident to $x_i^u$ have three distinct colors, then the edges incident to $x_{i+1}^u$ (or $x_{i-1}^u$) would all have the same color, and therefore no injective 3-edge-coloring would be possible.
Thus, w.l.o.g. we can suppose that $\rho(x^1_1 x^2_1) = \rho(x^1_1 y^1_1) = 1$ and $\rho(x^1_1 x^3_1) = 3$. By extending the coloring to the rest of $S_u$, we can infer that $\rho(y^2_1 w^1_1) = \rho(y^2_1 z^1_1) = 2$, $\rho(w^1_1 b^1_1) = \rho(w^1_1 d^1_1) = 3$ and $\rho(z^1_1 c^1_1) = \rho(z^1_1 d^1_1) = 3$. By the same reasoning, we can see that all the edges of $S_u$ have only one possible color which depends only on their distance to $y^1$ and in particular $\rho(a^u_1 a^u_2) = \rho(b^u_1 b^u_2) = \rho(c^u_1 c^u_2) = \rho(d^u_1 d^u_2)$.

Conversely, $S_u$ admits a coloring (see Figure 3 for an example). To choose a coloring of $S_u$ having the desired color $\rho(S_u)$, it suffices to permute the colors in the previous coloring.

To finish the construction, for any edge $uv \in E(G)$, we add an edge $e^{uv}$ to $G'$ between a vertex among $\{u^v, \beta^v, \gamma^v, \delta^v\}$ and a vertex among $\{\alpha^v, \beta^v, \gamma^v, \delta^v\}$ such that the planarity of $G'$ is preserved. This can be done by cyclically ordering the vertices of $\{\alpha^v, \beta^v, \gamma^v, \delta^v\}$ according to a planar embedding of $G$, and adding the edge $e^{uv}$ between the right pair of vertices.

Note that $G'$ is planar, subcubic with girth at least $g$.

Suppose that $G'$ admits an injective 3-edge-coloring $\rho$. Assign to the vertex $u$ of $G$ the color $\rho(S_u)$. Take two adjacent vertices $u$ and $v$ of $G$. The edge $e^{uv}$ in $G'$ is an edge between two vertices, one of $S_u$ and one of $S_v$: w.l.o.g. say $e^{uv} = \alpha^v \alpha^v$. This implies that $a^v_1 \alpha^u$ and $a^v_1 \alpha^v$ receive different colors and thus $\rho(S_u) \neq \rho(S_v)$. Hence this coloring of $G$ is a proper 3-vertex-coloring.

Conversely, suppose that $G$ admits a proper 3-vertex-coloring. Let $\rho$ be a partial edge-coloring of $G'$ with no colored edges. We choose the color $\rho(S_u)$ to be the color of $u$ in $G$ (and we color the appropriate edges of $G'$). By Claim 9 we can extend $\rho$ to each gadget $S_u$. Note that by the choice of $\rho(S_u)$, there is no conflict between edges of $S_u$ and $S_v$ when $u$ and $v$ are adjacent in $G$. It is left to color the edges of the form $e^{uv}$. By construction, there are only two edges at distance 2 of $e^{uv}$ (and this edge does not belong to a triangle). Hence there is at least one remaining color for $e^{uv}$. After coloring these edges, $\rho$ is an injective 3-edge-coloring of $G'$.

\[\square\]

### 3.2 Proof of Theorem 2.2

Proof. In order to prove this result, we will modify the previous construction to make it bipartite (the girth condition will be lost).

First we modify $S_u$ (see Figure 4). Create the following gadget $H$. Start with a complete graph on four vertices $x_1, \ldots, x_4$. For each edge $x_i x_j$, create a vertex $x_{ij}$ adjacent to both $x_i$ and $x_j$, and remove the edge $x_i x_j$. To each of these vertices of degree 2, add a pendant edge, with $y_{ij}$ the vertex of degree 1 adjacent to $x_{ij}$.

![Figure 3: Vertex gadget $S_u$ for planar subcubic graphs with girth at least $g$ (in this example $g = 4$ and $\ell = 9$).](image-url)
We claim that in every injective 3-edge-coloring $\gamma$ of $H$, for any $i \neq j$, the vertex $x_{ij}$ is incident to only one color. Suppose it is not the case, then there must exist an injective 3-edge-coloring $\gamma$ for which we have one of $x_{12}x_2$ and $x_{12}x_1$ colored differently from $x_{12}y_12$, say w.l.o.g. $\gamma(x_{12}x_1) = 1$ and $\gamma(x_{12}y_12) = 2$. We deduce that $\gamma(x_{22}x_{23}) = \gamma(x_{22}x_{24}) = 3$, $\gamma(x_{14}x_{14}) = \gamma(x_{3}x_{13}) = 2$, $\gamma(x_{3}x_{34}) = 1$, and there is no color available for $x_{23}y_{23}$, a contradiction.

Now, take two disjoint copies of $H$ named $H_u^1$ and $H_u^2$. Add an edge between the two vertices $y_{12,1}^u$ and $y_{12,2}^u$ and add the edge $y_{12,1}^u y_{12,2}^u$ where $y_{12,1}^u$ is a new vertex. Now repeat the construction process of $S_u$, for $g = 6$ for example, as described in the previous section by starting at the step where the vertices $w^u$ and $z^u$ are added. As we observed, the edges incident to vertex $x_{12,2}^u$ of $H_u^2$ (resp. $x_{12,2}^u$ of $H_u^2$) have the same color in any injective 3-edge-coloring $\rho$. Hence, $\rho(y_{12,1}^u y_{12,2}^v) = \rho(y_{12,1}^u y_{12,2}^v) \neq \rho(x_{12,1}^u y_{12,2}^v)$. Note that this graph also admits an injective 3-edge-coloring (see Figure 4). We are in the same configuration as in the proof of Theorem 2.1. Thus Claim 9 also holds for this gadget $S_u$. Note that this gadget is bipartite.

The edge gadget does not change, it is still the edge $e^u$. We need to be careful with the bipartiteness of the constructed graph. To ensure that the constructed graph is bipartite, it suffices that all vertices $y_{1}^u$, $u \in V(G)$, belong to the same part of the bipartition. To that end, if there is a path of odd length between $y_{1}^u$ and $y_{1}^v$, then w.l.o.g. this path is $y_{1}^u a_{1}^u \ldots a_{1}^u a_{2}^u \ldots a_{1}^v y_{1}^v$. If we increase the length of a sequence $a_{1}^u \ldots a_{1}^v$ in $S_u$ by 3 (and also adding $a_{g+1}^u$, $a_{g+1}^v$ and $a_{g+2}^u$, then this path now has even length. With this trick, we can ensure the bipartiteness of the constructed graph $G'$ as well as keeping Claim 9 true in this new setting.

Hence, as before, $G$ admits a proper vertex-3-coloring if and only if $G'$ admits an injective 3-edge-coloring.

\[\square\]

4 Proof of Theorem 3

**Proof.** Let $G$ be a planar bipartite subcubic graph with girth at least 16. Let $A$ and $B$ be the two parts of the bipartition of $G$. We construct the graph $G_A$ as follows: for each $u \in A$, we create a vertex $u$ in $G_A$. For each pair of vertices $u, v$ of $A$ which are at distance 2, we add an edge between $u$ and $v$ in $G_A$. As $G$ is subcubic, a planar embedding of $G$ also serves as a planar embedding of $G_A$, where the edges of $G_A$ follow their corresponding path of length 2 in $G$. Hence, $G_A$ is a planar graph with maximum degree at most 6. Note that, by the girth condition on $G$, $G_A$ does not have any $k$-cycle, for all $k$ with $4 \leq k \leq 7$. Then, by the main result from [2], the graph $G_A$ admits a vertex-3-coloring $\gamma$.

We now color $G$ as follows: each edge $uv$ of $G$, where $u \in A$ and $v \in B$, is colored by the color $\gamma(u)$ in $G_A$. We claim that this is an injective 3-edge-coloring of $G$. Indeed, take any path $uvwz$ of $G$. W.l.o.g., assume $u, w \in A$ and $v, z \in B$. By construction, $uw \in E(G_A)$ and thus $uw$ and $wz$ receive different colors.

\[\square\]
5 Proof of Theorem 5

Proof. We give an fixed-parameter tractable (FPT) algorithm parameterized by the treewidth \( \text{tw}(G) \) of our input graph \( G \). We use a *nice tree decomposition* (see [3]) of the input graph for our dynamic programming algorithm. Nice tree decompositions are a well-known tool for designing algorithms on graphs of bounded treewidth using dynamic programming. In our notation, the set of vertices of the graph associated to a node \( v \) of the tree, its *bag*, is denoted \( X_v \).

A nice tree decomposition of a graph is a tree decomposition, rooted at a node *Root*, with the following types of nodes. A *join node* has exactly two children, with the same bags as their parent join node. An *introduce node* has a unique child and contains exactly one more vertex in its bag than its child's bag. A *forget node* also has a unique child, but the forget node's bag has exactly one less vertex than its child's bag. A *leaf node* is a leaf of the tree and contains no vertices. We call \( G_{\leq v} \) the subgraph of \( G \) induced by the subtree of the decomposition rooted at \( v \) and \( G_v \) the subgraph of \( G \) induced by \( X_v \). We note \( N_H(u) \) for the neighborhood of a vertex \( u \) in a subgraph \( H \) of \( G \).

We define the following set associated with a node \( v \):

\[
\mathcal{T}_v = \{ t_1 : X_v \to \mathcal{P}([1, 2, \ldots, k])^2 \} \times \{ t_2 : E(G_v) \to \{ [1, 2, \ldots, k] \} \},
\]

where \( \mathcal{P}(X) \) is the power set of \( X \). For \( T \in \mathcal{T}_v \) with \( T = (t_1, t_2) \), to simplify notation, we note \( T[u] \) for \( t_1(u) \) when \( u \in X_v \) and \( T[e] = t_2(e) \) when \( e \in E(G_v) \). For a vertex \( u \in X_v \), we also note \( A_u \) and \( B_u \) the two sets such that \( T[u] = t_1(u) = (A_u, B_u) \).

The set \( \mathcal{V}_{al}(v) \) is the subset of \( \mathcal{T}_v \) such that \( T \in \mathcal{V}_{al}(v) \) if and only if there exists an injective \( k \)-edge-coloring \( \gamma \) of \( G_{\leq v} \) such that:

1. for all \( u \in X_v \), \( A_u = \{ \gamma(uw), w \in V(G_{\leq v}) \setminus X_v \} \), i.e., \( A_u \) is the set of colors of the edges of \( G_{\leq v} \) (not in \( G_v \)) incident with \( u \),
2. for all \( u \in X_v \), \( B_u = \{ \gamma(zw), zw \in E(G_{\leq v}) \setminus E(G_v) \text{ and } z \in N_{G_{\leq v}}(u) \} \), i.e., \( B_u \) is the set of colors of the edges of \( G_{\leq v} \) (not in \( G_v \)) at distance 2 of \( u \) (or contained in a triangle containing \( u \)),
3. for all \( e \in E(G_v) \), \( T[e] \) is the color \( \gamma(e) \).

In this case we say that \( \gamma \) is *associated with* \( T \). Note that for each injective \( k \)-edge-coloring of \( G_{\leq v} \), there exists an associated \( T \in \mathcal{T}_v \) and hence, \( T \in \mathcal{V}_{al}(v) \). The set \( \mathcal{V}_{al}(v) \) is thus the set of \( T \in \mathcal{T}_v \) associated with an injective \( k \)-edge-coloring of \( G_{\leq v} \).

Note that \( \mathcal{V}_{al}(\text{Root}) \neq \emptyset \) if and only if there exists an injective \( k \)-edge-coloring of \( G \). We will compute \( \mathcal{V}_{al}(\text{Root}) \) with a dynamic programming algorithm. Also note that \( |\mathcal{T}_v| \leq 2^{O(ktw(G)^2)} \).

First suppose that \( v \) is a leaf node. Then \( \mathcal{V}_{al}(v) = \mathcal{T}_v = \{ (\emptyset, \emptyset) \} \).

Suppose that \( v \) is a forget node where \( v' \) is its child node such that \( X_v \cup \{ a \} = X_v' \). Let \( T \in \mathcal{T}_v \), \( T \in \mathcal{V}_{al}(v) \) if and only if there exists an associated coloring \( \gamma \) of \( G_{\leq v} \). This coloring \( \gamma \) is also a coloring of \( G_{\leq v'} \) and thus is associated to a \( T' \in \mathcal{V}_{al}(v') \). In this case, since \( T \) and \( T' \) share the same coloring \( \gamma \), we have the following constraints on \( T \) and \( T' \):

- for all \( e \in E(G_v) \), \( T[e] = T'[e] = \gamma(e) \),
- for all \( u \in X_v \) such that \( au \in E(G_v) \), \( A_u = A'_u \cup \{ T[au] \} \) and \( B_u = B'_u \cup \{ T[au], w \in X_v \cap N_G(a), w \neq u \} \) where \( T[u] = (A_u, B_u) \) and \( T'[u] = (A'_u, B'_u) \),
- for all \( u \in X_v \) such that \( au \notin E(G_v) \), \( A_u = A'_u \) and \( B_u = B'_u \cup \{ T[au], w \in X_v \cap N_G(u) \cap N_G(a) \} \) where \( T[u] = (A_u, B_u) \) and \( T'[u] = (A'_u, B'_u) \).

The last two constraints reflect the fact that \( A_u \) and \( B_u \) must be updated after the removal of \( a \). The only new colors that can be added to these sets come from edges incident with \( a \). There are multiple cases, depending on whether \( u \) and \( a \) are adjacent or not, determining which colors of edges need to be added to these sets.

Hence, for all \( T \in \mathcal{V}_{al}(v) \), it suffices to check whether there exists a \( T' \in \mathcal{V}_{al}(v') \) for which the previous conditions are verified. This can be done in time \( 2^{O(ktw(G)^2)} \), as \( T \) is uniquely determined by \( T' \) in the above constraints.
Suppose that \( v \) is an introduce node where \( v' \) is its child node such that \( X_v = X_{v'} \cup \{ a \} \). Let \( T \in \mathcal{T}_v \), \( T' \in \mathcal{V}_d(v') \) if and only if there exists an associated coloring \( \gamma \) of \( G_{\leq v} \). This coloring \( \gamma \) is also a coloring of \( G_{\leq v'} \) and thus is associated to a \( T' \in \mathcal{V}_d(v') \). In other words, \( T \) is associated to a coloring \( \gamma \) obtained by extending a coloring \( \gamma' \) associated to some \( T' \in \mathcal{V}_d(v') \). Thus \( T' \in \mathcal{V}_d(v') \), we have the following constraints on \( T \) and \( T' \), in order to ensure that \( \gamma \) is the extension of \( \gamma' \):

- for all \( e \in E(G_v), T[e] = T'[e] \),
- for all \( u \in X_v, T[u] = T'[u] \),
- for all \( r \in \gamma, A_r = \{ a \} \) and \( B_r = \bigcup_{u \in X_v, u \in E(G_v)} A_u \),
- the coloring of \( X_v \) is an injective k-edge-coloring,
- for all \( uu \in E(G_v), T[ua] \notin B_u \bigcup \bigcup_{\ell \in X_v, \ell \notin u, a \in E(G_v)} A_u \).

The first two constraints correspond to the fact that \( \gamma \) is an extension of \( \gamma' \). As \( a \) is a new vertex, \( A_a = \emptyset \) and the only colors in \( B_a \) can be obtained by edges incident with some vertex \( u \in X_v \) itself adjacent to \( a \), hence the third constraint. The last two constraints correspond to the fact that the coloring of the new edges around \( u \) cannot be in conflict with edges already colored. The fourth constraint checks that no such conflict arises in \( X_v \) and the fifth constraint ensures that for each new edge \( uu \) the color \( T[ua] \) does not appear around an edge at distance 2 from \( u \) or \( u' \). For each \( T' \), there are at most \( 2^{|E(G_v)|} \) possible candidates to be added to \( \mathcal{V}_d(v) \). Hence \( 2^{O(k \cdot |E(G_v)|^{2})} \) time is sufficient to compute \( \mathcal{V}_d(v) \) from \( \mathcal{V}_d(v') \).

Suppose that \( v \) is a join node where \( v_1 \) and \( v_2 \) are its children nodes such that \( X_v = X_{v_1} = X_{v_2} \). Let \( T \in \mathcal{T}_v, T' \in \mathcal{V}_d(v) \) if and only if there exists an associated coloring \( \gamma \) of \( G_{\leq v} \). As both \( G_{\leq v_1} \) and \( G_{\leq v_2} \) are subgraphs of \( G_{\leq v} \), \( \gamma \) is also a coloring of \( G_{\leq v} \) (\( i \in \{ 1, 2 \} \)) and is associated to \( T_1 \in \mathcal{V}_d(v_1) \). In this case, since \( T \) and \( T_1 \) share the same coloring \( \gamma \), we have the following constraints on \( T, T_1 \) and \( T_2 \):

- for all \( e \in E(G_v), T[e] = T_1[e] = T_2[e] \),
- for all \( u \in X_v, A_u = A_{u_1} \cup A_{u_2} \) and \( B_u = B_{u_1} \cup B_{u_2} \) where \( T_1[u] = (A_{u_1}, B_{u_1}) \) for \( i \in \{ 1, 2 \} \),
- for all \( uu \in E(G_v), A_u \cap A_w = \emptyset \).

The last constraint corresponds to the fact that the coloring is an injective k-edge-coloring (i.e. with no conflicts between the two subtrees). Given \( T_1 \in \mathcal{V}_d(v_1) \) and \( T_2 \in \mathcal{V}_d(v_2) \), \( T \) is uniquely determined by the above constraints. Hence it suffices to try all the pairs of \( T_1, T_2 \) and when the obtained set \( T \) verifies all conditions, we can add it to \( \mathcal{V}_d(v) \). This can be done in time \( (2^{O(k \cdot |E(G)|^{2})})^2 = 2^{O(k \cdot |E(G)|^{2})} \).

### 6 Proof of Theorem 5

**Proof.** We reduce from k-Edge-Coloring, proven to be NP-Complete even for k-regular graphs in [13].

#### k-Edge-Coloring

**Instance:** A k-regular graph \( G \).

**Question:** Does \( G \) admit a proper k-edge-coloring?

We choose \( p \) to be the largest integer such that \( k = \binom{p}{2} + r \) (and thus \( r < p \)) and recall that \( k \geq 45 \).

Moreover we set \( \ell = 2p \). Let \( G \) be the input k-regular graph. For \( uv \in E(G) \), we define the edge gadget \( E_{uv} \) as follows (see Figure 5). First create the following vertices \( a_{uv}, b_{uv}, x_{uv}, ..., x_{uv}^{p-3}, c_{uv}, d_{uv}, e_{uv}, y_{uv}, ..., y_{uv}^{p-3}, s_{1uv}, ..., s_{2uv} \). The vertices \( x_{uv}^{p-1} \) have degree 1 in \( E_{uv} \) and will be connected to the rest of the graph. The vertices \( \{ x_{1uv}, ..., x_{uv}^{p-3}, a_{uv}, b_{uv}, c_{uv} \} \) form a clique; this is also the case for \( \{ y_{1uv}, ..., y_{uv}^{p-3}, a_{uv}, b_{uv}, d_{uv} \} \) and \( \{ s_{1uv}, ..., s_{2uv}^{p-3}, s_{1uv}^{p-1}, s_{2uv}^{p-1} \} \). The vertex \( e_{uv} \) is adjacent to \( e_{uv}, d_{uv}, x_{uv}, ..., x_{uv}^{p-3}, s_{1uv}, ..., s_{2uv}^{p-3} \). In the case where \( r = 0 \), i.e. \( k = \binom{p}{2} \), we delete \( d_{uv} \).

Let \( u \) be a vertex of \( G \) with \( v_1, ..., v_k \) its neighbors. We construct the vertex gadget \( S_u \) from \( k \times \ell \) vertices \( v_1, ..., v_{k+1}, v_1, ..., v_{k+1}, ..., v_k, v_k \) and successively consider pairs \( v_i, v_j \) of neighbors. For each pair, we add an edge between one of \( v_{i+1}, ..., v_{i+\ell} \) of minimum degree and one of \( v_{j+1}, ..., v_{j+\ell} \) with minimum degree. By adding edges one by one in this way, we ensure that the maximum degree of the vertices of \( S_u \) is at most 5 + \frac{\ell}{r} + 1.
Finally, for each edge $uv$ of $G$, we identify the $2\ell$ vertices $s_1^{uv}, \ldots, s_2^{uv}$ with the $\ell$ vertices of $S_u$ corresponding to $v$ (since $v$ is a neighbour of $u$, by the construction of $S_u$ in the previous paragraph, there are $\ell$ such vertices in $S_u$) and with the $\ell$ vertices of $S_v$ corresponding to $u$. This creates the graph $G'$. Note that its maximum degree is $max(2\ell + p - 1, \frac{p}{2} + 2) \leq 5\sqrt{3k}$.

**Claim 10** For any injective $k$-edge-coloring $\gamma$ of $E_{uv}$, we have $\gamma(e_{uv} s_1^{uv}) = \gamma(e_{uv} s_2^{uv}) = \cdots = \gamma(e_{uv} s_{2\ell}^{uv})$. Moreover if $\gamma$ is a partial injective $k$-edge-coloring of $E_{uv}$ where $\gamma(e_{uv} s_1^{uv}) = \gamma(e_{uv} s_2^{uv}) = \cdots = \gamma(e_{uv} s_{2\ell}^{uv})$ and there are no other colored edges, we call $\gamma$ to $E_{uv}$.

**Proof.** First note that the clique $\{x_1^{uv}, \ldots, x_{p-3}^{uv}, a^{uv}, b^{uv}, e^{uv}\}$ needs exactly $\binom{p}{2}$ distinct colors. W.l.o.g. $a^{uv} b^{uv}$ is colored 1 and the colors used for this clique are $1, 2, \ldots, \binom{p}{2}$. No one of these colors can be used to color the $r$ edges of the form $d^{uv} y_j^{uv}$ hence they must be colored with $\binom{p}{2} + 1, \ldots, \binom{p}{2} + r$. One can observe that an edge $e_{uv} s_j^{uv}$ cannot have a color among $\binom{p}{2} + 1, \ldots, \binom{p}{2} + r$ as it is at distance 2 from the edges of the form $d^{uv} y_j^{uv}$ ($j \in \{1, \ldots, r\}$). Moreover this edge cannot receive the same color as one of the edges of the clique $\{x_1^{uv}, \ldots, x_{p-3}^{uv}, a^{uv}, b^{uv}, e^{uv}\}$ except for the color 1 on the edge $a^{uv} b^{uv}$. Hence all edges of the form $e_{uv} s_j^{uv}$ have the same color.

Now suppose we have a coloring $\gamma$ such that the edges $e_{uv} s_i^{uv}$ ($i \in \{1, \ldots, 2\ell\}$) are all colored with the same color, say 1. We color $a^{uv} b^{uv}$ with color 1 and use the $\binom{p}{2} + r - 1$ other colors to color the rest of the edges of the clique $\{x_1^{uv}, \ldots, x_{p-3}^{uv}, a^{uv}, b^{uv}, e^{uv}\}$ and the edges of the form $d^{uv} y_j^{uv}$ ($j \in \{1, \ldots, r\}$). We color $e_{uv} z$ for $z \in \{x_1^{uv}, \ldots, x_{p-3}^{uv}, y_1^{uv}, y_2^{uv}, \ldots, y_r^{uv}\}$ with the color of $a^{uv} z$.

If $r = 0$, then $E_{uv}$ is colored and $\gamma$ is an injective $k$-edge-coloring.

If $r > 0$, we color $d^{uv} e^{uv}$ and $d^{uv} a^{uv}$ with the color of $d^{uv} y_1^{uv}$. We color $d^{uv} e^{uv}$ for $z \in \{x_1^{uv}, \ldots, x_{p-3}^{uv}, y_1^{uv}\}$ with the color of $e^{uv} z$. It is left to color the edges of the clique $\{y_1^{uv}, \ldots, y_r^{uv}\}$, for which we have available the $\binom{p}{2} - 1$ colors used to color the clique $\{x_1^{uv}, \ldots, x_{p-3}^{uv}, a^{uv}, b^{uv}\}$, which is enough as $r \leq p - 1$. This is an injective $k$-edge-coloring of $E_{uv}$.

Suppose there is an injective $k$-edge-coloring $\gamma$ of $G'$. For an edge $uv$ of $G$, we color it with the color $\gamma(e_{uv} s_j^{uv})$. Take two adjacent edges of $G$: $u v_1$ and $u v_2$. In $S_u$, there is an edge between $v_{1,i}$ and $v_{2,j}$ for some indices $i$ and $j$. Thus the edges $e_{uv_1} v_{1,i}$ and $e_{uv_2} v_{2,j}$ receive different colors. By Claim 10 $u v_1$ and $u v_2$ receive different colors. Hence $G$ admits a $k$-edge-coloring.

Suppose there is a $k$-edge coloring $\gamma$ of $G$. For each edge $uv$, we color $e_{uv} s_i^{uv}$ with the color $\gamma(uv)$. By Claim 10 we can extend this coloring to all $E_{uv}$. At this point there is no conflict between the colored edges. Indeed the only pairs of edges which are at distance 2 and not in the same edge gadget are of the form $e_{uv} s_i^{uv}$ and $e_{uv} s_j^{uv}$, and since $\gamma$ is proper, there is no conflict here. It is left to color the edges inside the vertex gadget. Let $e = v_{1,i} v_{j',j}$ be an uncolored edge. As the maximum degree of the vertices of $S_u$ is at most $\frac{p}{2} + 2$, there are at most $(\frac{p}{2} + 2)^2$ edges incident to a vertex of $S_u$ that can be in conflict with $e$. We must also consider the edges incident with $e_{uv}$ and $e_{uv'}$. For each of the two vertices there is
one forbidden color $\gamma(uv_i)$ which is common to $2\ell$ edges incident to $e_{uv_i}$ to which we need to add $p - 1$ colors for the other edges of $e_{uv_i}$. In the end, there are at most $2p + \left(\frac{k}{2} + 2\right)^2$ forbidden colors for $e$. As $2p + \left(\frac{k}{2} + 2\right)^2 \leq 2p + \left(\frac{k-1}{2}\right)^2 = \left(\frac{k-1}{2}\right)^2 + 3p + 3 \leq k$ when $k \geq 45$ and $p \geq 10$, $G'$ admits an injective $k$-edge-coloring. \hfill $\square$

7 Conclusion

We proved that Injective 3-Edge-Coloring and Injective 4-Edge-Coloring are NP-complete on some restricted classes of subcubic graphs. One can ask whether Injective 5-Edge-Coloring is NP-complete on subcubic graphs. A conjecture proposed by Ferdjallah et al. \cite{7} states that every subcubic graph admits an injective 6-edge-coloring (it is proved for planar graphs in \cite{14}). In fact, we only know of two connected subcubic graphs which require six colors: $K_4$ and the prism. Perhaps these are the only examples that are not 5-colorable, in which case Injective 5-Edge-Coloring would be polynomial-time solvable for this class.

We have also proved that for planar bipartite subcubic graphs, Injective 3-Edge-Coloring is polynomial-time solvable when the girth is at least 16 (because the answer is always YES), but NP-Complete when the girth is 6. It would be interesting to determine the values of the girth of planar bipartite subcubic graphs for which Injective 3-Edge-Coloring stays NP-Complete, becomes polynomial-time solvable, and always has YES as an answer.

We also do not know whether Injective 4-Edge-Coloring is NP-Complete for bipartite subcubic graphs.

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