Tractable Profit Maximization over Multiple Attributes under Discrete Choice Models

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Abstract

A fundamental problem in revenue management is to optimally choose the attributes of products, such that the total profit or revenue or market share is maximized. Usually, these attributes can affect both a product’s market share (probability to be chosen) and its profit margin. For example, if a smart phone has a better battery, then it is more costly to be produced, but is more likely to be purchased by a customer. The decision maker then needs to choose an optimal vector of attributes for each product that balances this trade-off. In spite of the importance of such problems, there is not yet a method to solve it efficiently in general. Past literature in revenue management and discrete choice models focus on pricing problems, where price is the only attribute to be chosen for each product. Existing approaches to solve pricing problems tractably cannot be generalized to the optimization problem with multiple product attributes as decision variables. On the other hand, papers studying product line design with multiple attributes all result in intractable optimization problems. Then we found a way to reformulate the static multi-attribute optimization problem, as well as the multi-stage fluid optimization problem with both resource constraints and upper and lower bounds of attributes, as a tractable convex conic optimization problem. Our result applies to optimization problems under the multinomial logit (MNL) model, the Markov chain (MC) choice model, and with certain conditions, the nested logit (NL) model.
1 Introduction

Consider an assortment with multiple products. A classical optimization problem in revenue management is to find the optimal prices of these products to be provided to customers. In general, when the price of a product increases, its market share shrinks, while its profit margin becomes larger. The goal of price optimization is to find a balance between market shares and profit margins, such that the overall profit or revenue is maximized.

In many applications, however, price is not the only variable that affects both market share and profit margin. Here we list some common scenarios, under which there are multiple attributes need to be determined, in addition to price:

- A ride-sharing service operator wants customers to walk to a pick-up location and to wait for some time before being picked up. With more waiting time and walking distance, the operator can make better matchings, and reduce the operational cost. However, customers are less likely to accept rides if waiting times and walking distances are long.

- A health insurance company needs to design both the price and the coverage of a plan. A insurance plan that covers more health care providers is more attractive to customers, but is also more costly to the company.

- An airline company sets (1) price, (2) cancelation fee, (3) baggage allowance, and (4) frequent flyer credits for each type of airline ticket (fare class). Lower fees and better services can attract more customers, but reduce the profit margin.

- A hotel decides (1) price, (2) cancellation policy, (2) earliest check-in time, and (4) latest check-out time for its rooms. Stricter cancelation policy and check-in / check-out time hurt customer satisfaction (and thus total sales), but also reduce the operational cost.

- An e-commerce platform chooses a combination of (1) price, (2) delivery time, and (3) return policy for each product. By offering better delivery time and return policy, the platform is able to charge a higher price without losing sales.

Literature that characterize products as a bundle of multiple attributes usually focus on the product line design problem (e.g. Bertsimas and Misic (2017) and Akcakucu and Misic (2021)).
The objective of a product line design problem is usually to maximize the market share, but papers like Akcakucs and Misic (2021) extend the discussion to profit maximization as well. Under the profit maximization scenario, the attributes also play a role in the profit margin. Papers along this line assumed product attributes to be discrete (usually binary). The solution method presented in these works are usually mixed-integer approaches, which is intractable.

In this paper, we present a tractable approach to solve the profit (or revenue) maximization problem, where for each product, there are multiple attributes that need to be determined simultaneously. We formulate both the static optimization problem $SP$ and the multi-stage fluid problem $FP$ and consider both resource constraints and upper and lower bounds of attributes. More concretely, we discuss problems $SP$ and $FP$ under the multinomial logit (MNL) model, the Markov chain (MC) choice model, and the nested logit (NL) model. We show that under these choice models, $SP$ and $FP$ can be solved by solving a convex conic program (for the NL model, additional conditions are needed to establish the results). We also give the necessary and sufficient conditions that an optimal solution exists for each of the problems. The formulations, results and discussions are provided in Section 3.

Our work is built upon existing literature on pricing problems under discrete choice models. Papers on pricing problems consider price as the only attribute for each product. Thus, our formulation of $SP$ and $FP$ is a generalization to the pricing problems discussed in these papers. Existing methods to solve the pricing problem cannot be applied to $SP$ and $FP$. First, the solution approaches in Song and Xue (2007), Dong et al. (2009), Li and Huh (2011), Keller et al. (2014) and Dong et al. (2019) relies on the ability of writing prices as inverse functions of market shares (the expected overall profit is concave in market shares). However, when each product has multiple attributes instead of price alone, it is no longer possible to replace the attributes by close-form functions of the market shares. Second, the optimality condition approach in Wang (2012), Gallego and Wang (2014) and Zhang et al. (2018) only applies to unconstrained optimization problems. However, having constrains in $SP$ and $FP$ is not only important but also necessary. When a product have multiple attributes that can control both the profit margin and the choice probability, it is very likely that we can "scarifies" one attribute for another. That is, we may be able to increase one attribute and decrease another attribute at the same time, such that the choice probability of the product remain unchanged, while the profit margin increases. If there is no constraint in attributes, the problem is certainly unbounded. Third, although discretizing attributes (like in Davis et al.
may be a theoretically valid approach, the number of candidate attribute vector grows exponentially with the number of attributes. As we embed more attributes into the problem, such a method would easily be impractical. A review of these papers are provided in Section 2.

Most results in the papers above can be reproduced with our method. Thus, besides solving the generalized optimization problem $\text{SP}$ and $\text{FP}$ our method also provides a more unified approach to solve the price (single attribute) optimization problems. We discuss these results as well as some potential extension directions in Section 5.

2 Related Literature

Multi-product price optimization problems under discrete choice models have been the subject of active research in the last two decades. Comparing to the past literature, these research use choice models to capture the price-driven substitution behavior from customers. Pricing problems under discrete choice models are usually not easy to solve in its original form. Hanson and Martin (1996) study this problem of maximizing the expected revenue under the multinomial logit (MNL) model, and observe that the expected profit is not concave in price. (They then proposed a path-following approach to perturb the objective function into a concave one.) Since then, many efficient solution methods have been introduced. We give a review to some of the most important works here:

**Market Share Reformulation:** An important approach to solve the pricing problems efficiently is to use market shares as decision variables. Song and Xue (2007) and Dong et al. (2009) considered the same problem as in Hanson and Martin (1996), and showed that the expected profit is concave in the market shares. In order words, the pricing problem under the MNL model can be solved efficiently by using the market shares of the products as decision variables. (Both works assumed that the price sensitivity parameters are constant for all products.) Li and Huh (2011) extended the concavity results to the nested logit (NL) model (which subsumes the MNL model), and showed that the expected profit is concave in the market shares if (1) the price-sensitivity parameters are identical for all the products within a nest, and (2) the nest coefficients are restricted to be in the unit interval. As a special case, the expected profit under a general MNL model (with asymmetric price-sensitivity parameters) is concave in the market shares. Keller et al. (2014) discussed simi-
lar concavity results under the general attraction demand model (which also subsumes the MNL model), and showed that constraints such as price bounds and joint price constraints can be added to the pricing problem as linear constraints in the market shares. Zhang et al. (2018) discussed the multi-product pricing problem under the generalized extreme value models (which subsumes the NL model). They showed that with homogeneous price sensitivity parameters, the problem can formulation as a convex program. Recently, Dong et al. (2019) formulate the pricing problem under the Markov chain (MC) choice model. They showed that the pricing problem can be solved as a dynamic program. They also presented a market-share based approach to solve the dynamic pricing problem with a single resource.

**First-order Conditions:** Another approach to solve the problems is to study the first-order condition. (Results that are built upon first-order conditions usually only apply to unconstrained optimization problems.) Anderson and De Palma (1992), Besanko et al. (1998) and Aydin and Ryan (2000) observex that under a MNL model with homogeneous price sensitivity parameters, the profit margin (price minus cost) is constant across all the products at the optimality of the expected profit. Aydin and Porteus (2008) and Akcay et al. (2010) then pointed out that under such scenarios, the profit function is uni-modal with respect to the markup, and its unique optimal solution can be found by solving the first-order conditions. Gallego and Wang (2014) extended this result and showed that under the general NL model, the adjusted markup (price minus cost minus the reciprocal of price sensitivity) is constant for all the products within a nest at optimality. They then defined an adjusted nest-level markup and showed it to be invariant for all nests, which reduces the pricing problem to a single dimensional optimization problem. This one-dimensional problem has a single local maxima when either (1) the difference between price sensitivity parameters are bounded by a certain value, or (2) the dissimilarity parameter is greater than 1. These conditions are more general than Li and Huh (2011). Wang (2012) also showed the constant adjusted markup property for pricing problem under the MNL model.) Huh and Li (2015) further extended this result to the pricing problem under the multistage nested attraction model. They showed that the problem can be reduced to single dimensional, and the optimal solution is unique under generalized conditions. Zhang et al. (2018) discussed the multi-product pricing problem under the generalized extreme value models with homogeneous price sensitivity parameters, and provided similar results.

**Discretization of Price:** A third approach is to limit price candidates to a small, finite set. Such methods are usually very similar to approaches used for solving assortment planning. For instance,
Davis et al. (2013) studied assortment planning problems under the MNL model with totally unimodular constraints. They showed that such problems can be solved efficiently as linear programs. As a consequence, pricing problems under the MNL model with finite possible prices can also be formulated as a linear program. Gallego and Topaloglu (2014) followed a similar approach and extend the results to pricing problems under the nested logit model. Researchers also used this approach to model constraints in the pricing problem in a tractable way. As an example, Davis et al. (2017) show that under the NL model, quality consistency constraints (pairwise inequalities in prices) can be added to the pricing problem, and the problem can be solved as a linear program.

Research on discrete choice models dates back to McFadden and Others (1973). The work integrates the random utility framework from Thurstone (1927) and the choice axioms introduced by Luce (1959), and provides a general econometric procedure to the analysis of individual choice behavior. We point the readers to Ben-Akiva et al. (1985) and Train (2009) for more in-depth discussions on the (parameterized) random utility choice models, such as the MNL model and the NL model mentioned above. Our paper also covers the Markov chain (MC) choice model. The MC model was recently analyzed in Blanchet et al. (2016). A discussion on different choice-modeling approaches and their mathematical equivalence can be found in Feng et al. (2017).

3 Optimization of Product Attributes

A seller offers an assortment of \( J \) products, indexed by \( j \in J \). Each product has a number of features, such as price and service characteristics, the values of which can be chosen by the seller. Different products may have different numbers of these features. In this paper, attribute refers to a feature for which the seller has to choose values for one or more products, and that has unique parameter values, as specified below. The value of a feature for a product affects the seller’s profit in two ways: (1) The feature’s value affects the demand for the product (and the demand for other products). (2) The feature’s value affects the seller’s revenue and/or cost, and thus profit margin, per unit product sold. Both of these effects are specified by a model with parameters that may depend on the product and the feature. For example, in a discrete choice model, the systematic utility associated with a product \( j \) and feature \( \ell \) may be given by \( \beta_{j\ell} y_{j\ell} \), where \( \beta_{j\ell} \) is a parameter and \( y_{j\ell} \) is the value of feature \( \ell \) for product \( j \). Similarly, the profit margin associated with a product \( j \) and feature \( \ell \) may be given by \( \tilde{\phi}_{j\ell} y_{j\ell} \), where \( \tilde{\phi}_{j\ell} \) is a parameter. We will simplify
the notation by defining an attribute to be a feature with unique parameter values. Thus, if two
products \( j \) and \( j' \) have different parameter values associated with a feature \( \ell \), for example \( \beta_{j\ell} \neq \beta_{j'\ell} \)
or \( \tilde{\phi}_{j\ell} \neq \tilde{\phi}_{j'\ell} \), then there are separate attributes \( k \) and \( k' \) in the model for feature \( \ell \) when used in
combination with product \( j \), and for feature \( \ell \) when used in combination with product \( j' \). Also,
if two products have the same parameter values associated with a feature, then the two products
have the same attribute, but they may have different values for the attribute. That way, parameter
values are uniquely specified by the attribute. Let \( K \) denote the resulting index set of attributes,
and for each \( j \in J \), let \( K_j \subset K \) denote the index set of the attributes for product \( j \) for which the
seller has to choose a value. Thus, \( K \neq \emptyset \) (\( K := |K_j| \geq 1 \)) for each \( j \in J \), \( K = \bigcup_{j \in J} K_j \), and
\( K_j \cap K_{j'} \) may be nonempty for two products \( j \) and \( j' \).

For each \( j \in J \) and \( k \in K_j \), the value chosen for attribute \( k \) for product \( j \) is denoted with \( y_{jk} \).
Let \( y := (y_{jk}, j \in J, k \in K_j) \) denote the list of attribute values, and let \( \hat{P}_j(y) \) denote the probability
that a customer chooses product \( j \). We assume that the profit margin per unit of product \( j \) sold
depends on \( y_{jk}, k \in K_j \), and that this dependence is specified by an affine function \( \sum_{k \in K_j} \hat{\phi}_k y_{jk} - \tilde{\psi}_j \),
where \( \hat{\phi}_k, k \in K \) and \( \tilde{\psi}_j, j \in J \) are parameters. For example, if attribute \( k \) represents product price,
then \( \hat{\phi}_k = 1 \). In general, \( \sum_{k \in K_j} \hat{\phi}_k y_{jk} - \tilde{\psi}_j \) can be regarded as an affine approximation of the relation
between the attribute values of product \( j \) and the profit margin per unit of product \( j \) sold.

It is important to make provision for bounds on attribute values, for the following reasons:

1. Usually discrete choice models and profit margin models are calibrated with a limited range
   of attribute value data, and it would be unwise to allow selection of attribute values much
   outside this range.

2. As explained in Remark \[3.1\] below, if multiple attributes are unbounded, then one attribute
can be traded off for another attribute to obtain unbounded profit.

Let \( K_j \) denote the set of attributes \( k \in K_j \) such that \( y_{jk} \) has a lower bound \( \underline{y}_{jk} \), and let \( K_j \) denote
the set of attributes \( k \in K_j \) such that \( y_{jk} \) has an upper bound \( \overline{y}_{jk} \). It is assumed that \( \underline{y}_{jk} \leq y_{jk} \leq \overline{y}_{jk} \)
for all \( j \in J \) and \( k \in K_j \cap K_j \).

The resulting static attribute optimization problem is

\[
\max_y \sum_{j \in J} \left( \sum_{k \in K_j} \hat{\phi}_k y_{jk} - \tilde{\psi}_j \right) \hat{P}_j(y) \quad (SP)
\]
We will also consider the following revenue management problem. Products require the use of resources. Let $\mathcal{R}$ denote the set of resources, and for each product $j \in \mathcal{J}$ and each resource $r \in \mathcal{R}$, let $a_{rj}$ denote the amount of resource $r$ needed per unit of product $j$. For each resource $r \in \mathcal{R}$, there are $b_r$ units of the resource available for use over a selling horizon. The selling horizon is discretized into time periods indexed by $t = 0, 1, \ldots, T$. In each time period $t$, the expected number of customer arrivals is denoted by $\lambda_t$. Parameters and decisions may depend on $t$, which is reflected in the notation. We will consider the following fluid optimization problem:

$$\max_y \sum_{t=0}^{T} \lambda_t \sum_{j \in \mathcal{J}} \left( \sum_{k \in \mathcal{K}_{jt}} \phi_{kt} y_{jkt} - \psi_{jt} \right) \bar{P}_{jt}(y) \tag{FP}$$

s.t. $\sum_{t=0}^{T} \lambda_t \sum_{j \in \mathcal{J}} a_{rj} \bar{P}_{jt}(y) \leq b_r \quad \forall r \in \mathcal{R}$

$$y_{jkt} \geq y_{jkt} \quad \forall j \in \mathcal{J}, k \in \mathcal{K}_{jt}, t = 0, 1, \ldots, T$$

$$y_{jkt} \leq \bar{y}_{jkt} \quad \forall j \in \mathcal{J}, k \in \mathcal{K}_{jt}, t = 0, 1, \ldots, T$$

In the remainder of the section, we consider problems $\text{SP}$ and $\text{FP}$ for various widely used discrete choice models.

### 3.1 $\text{SP}$ under the MNL Model

First we consider problem $\text{SP}$ in which $\bar{P}_j(y)$ is given by a multinomial logit model:

$$\bar{P}_j(y) = \frac{\exp \left( \alpha_j - \sum_{k \in \mathcal{K}_j} \beta_k y_{jk} \right)}{1 + \sum_{j' \in \mathcal{J}} \exp \left( \alpha_{j'} - \sum_{k \in \mathcal{K}_{jt}} \beta_k y_{jkt} \right)}$$

where $\alpha_j, j \in \mathcal{J}$ are the product “baseline attractiveness” parameters, and $\beta_k, k \in \mathcal{K}$ are the attribute “sensitivity” parameters.

Without loss of generality, we assume that $\beta_k > 0$ (because the sign of each $y_{jk}$ can be chosen such that $\beta_k > 0$). Therefore we also assume that $\phi_k > 0$, that is, there is a trade-off between the effect of attribute $k$ on the demand for products and the effect of attribute $k$ on the profit.
margin of products — as \( y_{jk} \) increases the demand for product \( j \) decreases and the profit margin per unit of product \( j \) increases. To further simplify notation, consider the scaled attributes \( x_{jk} := \beta_k y_{jk} - \alpha_j / K_j \), the scaled attribute lower bounds \( \underline{x}_{jk} := \beta_k \bar{y}_{jk} - \alpha_j / K_j \), the scaled attribute upper bounds \( \overline{x}_{jk} := \beta_k \bar{y}_{jk} - \alpha_j / K_j \), the scaled profit margins \( \phi_k := \beta_k \hat{\phi}_k > 0 \), the scaled item fixed costs \( \psi_j := \hat{\psi}_j + (\alpha_j / K_j) \sum_{k \in K_j} \hat{\phi}_k \), for every \( j \in \mathcal{J} \) and \( k \in K_j \). Let \( x := (x_{jk}, j \in \mathcal{J}, k \in K_j) \). Then the choice probabilities are given by

\[
P_j(x) = \frac{\exp\left(-\sum_{k \in K_j} x_{jk}\right)}{1 + \sum_{j' \in \mathcal{J}} \exp\left(-\sum_{k \in K_{j'}} x_{j'k}\right)}
\]

and the expected profit of product \( j \) becomes \( \left( \sum_{k \in K_j} \phi_k x_{jk} - \psi_j \right) P_j(x) \). Thus, we consider the static attribute optimization problem

\[
\max_{d, x} \sum_{j \in \mathcal{J}} \left( \sum_{k \in K_j} \phi_k x_{jk} - \psi_j \right) d_j \quad \text{(SPMNL)}
\]

s.t. \[
d_j = \frac{\exp\left(-\sum_{k \in K_j} x_{jk}\right)}{1 + \sum_{j' \in \mathcal{J}} \exp\left(-\sum_{k \in K_{j'}} x_{j'k}\right)} \quad \forall j \in \mathcal{J} \quad (1)
\]

\[
x_{jk} \geq \underline{x}_{jk} \quad \forall j \in \mathcal{J}, k \in K_j \quad (2)
\]

\[
x_{jk} \leq \overline{x}_{jk} \quad \forall j \in \mathcal{J}, k \in K_j \quad (3)
\]

In \( \text{SPMNL}_1 \) \( d := (d_j, j \in \mathcal{J}) \) denotes the vector of market shares.

**Remark 3.1.** Bounds on attribute values are important when multiple attributes are chosen for a product. Suppose that \( \phi_k > \phi_{k'} \) and that \( k, k' \in K_j \) for some \( j \in \mathcal{J} \). If attribute \( k \) is not bounded above and attribute \( k' \) is not bounded below, then \( x_{jk} \) can be increased and \( x_{j'k} \) can be decreased by the same amount, while keeping all other attribute values constant. Thereby all demands remain constant, while the profit contribution of product \( j \) grows without bound. In an application, it makes sense that attributes can be traded off to some extent, but arbitrarily large or small attribute values are not sensible.

Problem \( \text{SPMNL}_1 \) is not a convex optimization problem. One of the reasons is that nonlinear constraint \( (1) \) is an equality constraint. Also, if some products have multiple attributes \( (K_j > 1 \) for some \( j \)), then the approach in Song and Xue (2007), Dong et al. (2009), Li and Huh (2011), Keller et al. (2014), and Dong et al. (2019) of inverting the constraint to write \( x \) as a function of \( d \) does not apply here. We will relax the equality constraints to inequality constraints, show that
an optimal solution of the relaxation provides an optimal solution of $SP_{MNL}^1$, then reformulate the relaxation as a convex optimization problem, and then provide necessary and sufficient conditions for this convex optimization problem to have an optimal solution.

First, note that a decision variable $d_0$ and the constraints $d_0 + \sum_{j \in J} d_j = 1$ and $d_0 > 0$ can be added to problem $SP_{MNL}^1$ without changing the set of feasible $(d, x)$-values or the objective value for any feasible $(d, x)$-value, because the constraints will force

$$d_0 = \frac{1}{1 + \sum_{j \in J} \exp \left( - \sum_{k \in K_j} x_{jk} \right)}$$

without affecting the objective value. Next, we rewrite constraints (1) to obtain the following static attribute optimization problem

$$\max_{d, d_0, x} \sum_{j \in J} \left( \sum_{k \in K_j} \phi_k x_{jk} - \psi_j \right) d_j \quad (SP_{MNL}^2)$$

s.t.

$$\ln \left( \frac{d_j}{d_0} \right) = - \sum_{k \in K_j} x_{jk} \quad \forall j \in J$$

$$x_{jk} \geq \underline{x}_{jk} \quad \forall j \in J, k \in \underline{K}_j$$

$$x_{jk} \leq \overline{x}_{jk} \quad \forall j \in J, k \in \overline{K}_j$$

$$d_0 + \sum_{j \in J} d_j = 1$$

$$d > 0, \quad d_0 > 0$$

Note that for any $(d, x)$ feasible for $SP_{MNL}^1$ it holds that $(d, d_0, x)$ with $d_0$ given by (4) is feasible for $SP_{MNL}^2$ and has the same objective value. Conversely, consider any $(d, d_0, x)$ feasible for $SP_{MNL}^2$. It follows from (5) that $d_j = d_0 \exp \left( - \sum_{k \in K_j} x_{jk} \right)$, then it follows from (8) that $d_0$ satisfies (4), and finally it follows that $d_j$ satisfies (1). Thus, for any $(d, d_0, x)$ feasible for $SP_{MNL}^2$ it holds that $(d, x)$ is feasible for $SP_{MNL}^1$ and has the same objective value.

Note that constraint (5) is equivalent to $- \sum_{k \in K_j} x_{jk} \leq \ln (d_j/d_0) \leq - \sum_{k \in K_j} x_{jk}$. Next, we relax constraint (5) as follows. Let $\overline{J} := \{ j \in J : \overline{K}_j = K_j \}$ denote the set of products for which all attributes are upper bounded. For each $j \in \overline{J}$, constraint (5) is relaxed to $- \sum_{k \in K_j} \overline{x}_{jk} \leq \ln (d_j/d_0) \leq - \sum_{k \in K_j} x_{jk}$, and for each $j \in J \setminus \overline{J}$, constraint (5) is relaxed to $\ln (d_j/d_0) \leq
Thus we consider the following relaxation of $\text{SP}_M^{\text{MNL}}_2$:

$$
\begin{align*}
\max_{d, d_0, x} & \quad \sum_{j \in J} \left( \sum_{k \in K_j} \phi_k x_{jk} - \psi_j \right) d_j \\
\text{s.t.} & \quad \ln \left( \frac{d_j}{d_0} \right) \leq - \sum_{k \in K_j} x_{jk} \quad \forall j \in J \\
& \quad \ln \left( \frac{d_0}{d_j} \right) \leq \sum_{k \in K_j} \pi_{jk} \quad \forall j \in J \\
& \quad x_{jk} \geq \pi_{jk} \quad \forall j \in J, k \in K_j \\
& \quad x_{jk} \leq \pi_{jk} \quad \forall j \in J, k \in K_j \\
& \quad d_0 + \sum_{j \in J} d_j = 1 \\
& \quad d > 0, \quad d_0 > 0
\end{align*}
$$

Next we show that an optimal solution of the relaxation gives an optimal solution of $\text{SP}_M^{\text{MNL}}_2$. Consider any $(d, d_0, x)$ feasible for $\text{SP}_M^{\text{MNL}}_3$. If \( \ln \left( \frac{d_j}{d_0} \right) = - \sum_{k \in \mathcal{K}_j} x_{jk} \) for all \( j \in \mathcal{J} \), then \( (d, d_0, x) \) is feasible for $\text{SP}_M^{\text{MNL}}_2$ and has the same objective value. Otherwise, \( \ln \left( \frac{d_j}{d_0} \right) < - \sum_{k \in \mathcal{K}_j} x_{jk} \) for some \( j \). Then consider 2 cases: (1) \( j \in \mathcal{J} \setminus \mathcal{J}_0 \) or \( x_{jk} < \pi_{jk} \) for some \( k \in \mathcal{K}_j \), or (2) \( j \in \mathcal{J}_0 \) and \( x_{jk} = \pi_{jk} \) for all \( k \in \mathcal{K}_j \). In case (1), \( x_{jk} \) can be increased in $\text{SP}_M^{\text{MNL}}_3$ while keeping \( (d, d_0) \) unchanged. Since \( \phi_k > 0 \), the objective value will improve, and thus such \( (d, d_0, x) \) cannot be optimal for $\text{SP}_M^{\text{MNL}}_3$. In case (2), it follows that \( \ln \left( \frac{d_j}{d_0} \right) < - \sum_{k \in \mathcal{K}_j} \pi_{jk} \), that is, \( \ln \left( \frac{d_0}{d_j} \right) > \sum_{k \in \mathcal{K}_j} \pi_{jk} \), which violates constraint (10). Thus, an optimal solution for $\text{SP}_M^{\text{MNL}}_3$ satisfies \( \ln \left( \frac{d_j}{d_0} \right) = - \sum_{k \in \mathcal{K}_j} x_{jk} \) for all \( j \), and hence is feasible and therefore optimal for $\text{SP}_M^{\text{MNL}}_2$.

Problem $\text{SP}_M^{\text{MNL}}_3$ can be reformulated as a convex optimization problem as follows. Note that the functions \( (d_j, d_0) \mapsto d_j \ln \left( \frac{d_j}{d_0} \right) \) and \( (d_j, d_0) \mapsto d_0 \ln \left( \frac{d_0}{d_j} \right) \) are convex on \((0, \infty)^2\). Also, introduce a new variable \( u_{jk} = d_j x_{jk} \), and let \( u := (u_{jk}, j \in \mathcal{J}, k \in \mathcal{K}_j) \). Thus we consider the following convex reformulation of $\text{SP}_M^{\text{MNL}}_3$:

$$
\begin{align*}
\max_{d, d_0, u} & \quad \sum_{j \in \mathcal{J}} \left( \sum_{k \in \mathcal{K}_j} \phi_k u_{jk} - \psi_j d_j \right) \\
\text{s.t.} & \quad d_j \ln \left( \frac{d_j}{d_0} \right) \leq - \sum_{k \in \mathcal{K}_j} u_{jk} \quad \forall j \in \mathcal{J} \\
& \quad d_0 \ln \left( \frac{d_0}{d_j} \right) \leq \sum_{k \in \mathcal{K}_j} \pi_{jk} d_0 \quad \forall j \in \mathcal{J}_0
\end{align*}
$$

11
\[
\begin{align*}
  u_{jk} & \geq x_{jk}d_j & \forall j \in J, \ k \in K_j \\
  u_{jk} & \leq \varphi_{jk}d_j & \forall j \in J, \ k \in \bar{K}_j \\
  d_0 + \sum_{j \in \mathcal{J}} d_j & = 1 \\
  d & > 0, \ d_0 & > 0
\end{align*}
\]

Note that for any \((d, d_0, x)\) feasible for \(\text{SP}_3^{\text{MNL}}\) with \(u_{jk} = d_j x_{jk}\) for all \(j \in J, \ k \in K_j\) is feasible for \(\text{SP}_4^{\text{MNL}}\) and has the same objective value. Conversely, for any \((d, d_0, u)\) feasible for \(\text{SP}_4^{\text{MNL}}\) with \(u_{jk} = d_j x_{jk}/d_j\) for all \(j \in J, \ k \in K_j\) is feasible for \(\text{SP}_3^{\text{MNL}}\) and has the same objective value.

The feasible set of \(\text{SP}_4^{\text{MNL}}\) may not be closed. For example, suppose that \(j \in J \setminus \mathcal{J}\) and that \(u_{jk} \leq 0\) for all \(k \in K_j\). Consider a sequence \(\{(d^n, d_0^n, u^n)\}_{n=0}^\infty\) of feasible solutions with \(d_0^n \leq d_0^0\), \(d_j^n = \frac{d_j^0}{n}\), \(d_0^n = d_0^0 + d_j^0 (1 - 1/n)\), \(u_{jk} = 0\) for all \(k \in K_j\), and \((d^n_j, u^n_j)\) remain constant for all \(j' \neq j\). Then \((d^n, d_0^n, u^n) \to (\tilde{d}, \tilde{d}_0, \tilde{u})\) with \(\tilde{d}_j = 0\), and thus the limit \((\tilde{d}, \tilde{d}_0, \tilde{u})\) is not in the feasible set of \(\text{SP}_4^{\text{MNL}}\).

Next we relax \(\text{SP}_4^{\text{MNL}}\) to make the feasible set closed, and then we show that optimal solutions are not affected by the relaxation. Let

\[
K_{\exp} := \text{closure}\{(a_1, a_2, a_3) : a_3 \leq a_2 \ln (a_1/a_2), \ a_1 > 0, \ a_2 > 0\} = \{(a_1, a_2, a_3) : a_3 \leq a_2 \ln (a_1/a_2), \ a_1 > 0, \ a_2 > 0\} \cup \{(a_1, 0, a_3) : a_1 \geq 0, \ a_3 \leq 0\}
\]

denote the exponential cone. Then consider the following convex conic relaxation of \(\text{SP}_4^{\text{MNL}}\):

\[
\begin{align*}
\max_{d, d_0, u} & \quad \sum_{j \in \mathcal{J}} \left( \sum_{k \in K_j} \phi_k u_{jk} - \psi_j d_j \right) \\
\text{s.t.} & \quad \left( d_0, d_j, \sum_{k \in K_j} u_{jk} \right) \in K_{\exp} & \forall j \in J \\
& \quad \left( d_j, d_0, -\sum_{k \in K_j} \varphi_{jk} d_0 \right) \in K_{\exp} & \forall j \in \mathcal{J} \\
& \quad u_{jk} \geq \varphi_{jk} d_j & \forall j \in J, \ k \in K_j \\
& \quad u_{jk} \leq \varphi_{jk} d_j & \forall j \in J, \ k \in \bar{K}_j \\
& \quad d_0 + \sum_{j \in \mathcal{J}} d_j = 1
\end{align*}
\]
The following results hold for $\text{SP}^{\text{MNL}}_5$.

**Lemma 3.1.** $\text{SP}^{\text{MNL}}_5$ has an optimal solution if and only if for each $j \in J$, and for all $k_1 \in K_j \setminus K_j$, $k_2 \in K_j \setminus K_j$, it holds that $\phi_{k_1} \leq \phi_{k_2}$; otherwise, $\text{SP}^{\text{MNL}}_5$ is unbounded.

**Lemma 3.2.** Every optimal solution for $\text{SP}^{\text{MNL}}_5$ is also optimal for $\text{SP}^{\text{MNL}}_4$.

Next we summarize the results for $\text{SP}$ under the MNL choice model.

**Theorem 3.3.** $\text{SP}^{\text{MNL}}_1$ has an optimal solution if and only if for each $j \in J$, and for all $k_1 \in K_j \setminus K_j$, $k_2 \in K_j \setminus K_j$, it holds that $\phi_{k_1} \leq \phi_{k_2}$, in which case an optimal solution can be found by solving the convex conic program $\text{SP}^{\text{MNL}}_5$. Otherwise, $\text{SP}^{\text{MNL}}_1$ is unbounded.

### 3.2 $\text{FP}$ under the MNL Model

Next, we consider the fluid revenue management problem $\text{FP}$ under the following MNL model:

$$
\hat{P}_{jt}(y) = \frac{\exp \left( \alpha_{jt} - \sum_{k \in K_{jt}} \beta_{kt} y_{jkt} \right)}{1 + \sum_{j' \in J} \exp \left( \alpha_{j't} - \sum_{k \in K_{j't}} \beta_{kt} y_{j'kt} \right)}
$$

Let $x_{jkt} := \beta_{kt} y_{jkt} - \alpha_{jt} / K_{jt}$, $\bar{x}_{jkt} := \beta_{kt} \bar{y}_{jkt} - \alpha_{jt} / K_{jt}$, $\bar{x}_{jkt} := \beta_{kt} \bar{y}_{jkt} - \alpha_{jt} / K_{jt}$, $\phi_{kt} := \beta_{kt} \phi_{kt} > 0$, $\psi_{jt} := \tilde{\psi}_{jt} + (\alpha_{jt} / K_{jt}) \sum_{k \in K_{jt}} \phi_{kt}$, for every $j \in J$, $t = 0, 1, \ldots, T$, and $k \in K_{jt}$. Let $x := (x_{jkt}, j \in J, k \in K_{jt}, t = 0, 1, \ldots, T)$, $d := (d_{jt}, j \in J, t = 0, 1, \ldots, T)$, $d_0 := (d_{0t}, t = 0, 1, \ldots, T)$, and $u := (u_{jkt}, j \in J, k \in K_{jt}, t = 0, 1, \ldots, T)$. Thus, we consider the optimization problem

$$\max_{d, x} \quad \sum_{t=0}^T \lambda_t \sum_{j \in J} \left( \sum_{k \in K_{jt}} \phi_{kt} x_{jkt} - \psi_{jt} \right) d_{jt}$$

s.t. $d_{jt} = \frac{\exp \left( - \sum_{k \in K_{jt}} x_{jkt} \right)}{1 + \sum_{j' \in J} \exp \left( - \sum_{k \in K_{j't}} x_{j'kt} \right)}$ \quad $\forall j \in J, t = 0, 1, \ldots, T$

$$\sum_{t=0}^T \lambda_t \sum_{j \in J} a_{rj} d_{jt} \leq b_r \quad \forall r \in R$$

$$x_{jkt} \geq \bar{x}_{jkt} \quad \forall j \in J, k \in K_{jt}, t = 0, 1, \ldots, T$$

$$x_{jkt} \leq \bar{x}_{jkt} \quad \forall j \in J, k \in K_{jt}, t = 0, 1, \ldots, T$$
Consider the following convex conic relaxation of $FP_{MNL}^1$:

$$\begin{align*}
\max_{d, d_0, u} & \quad \sum_{t=0}^{T} \lambda_t \sum_{j \in J} \left( \sum_{k \in K_{jt}} \phi_{kt} u_{jkt} - \psi_{jt} d_{jt} \right) \\
\text{s.t.} & \quad \sum_{t=0}^{T} \lambda_t \sum_{j \in J} a_{rj} d_{jt} \leq b_r, \quad \forall r \in R \\
& \quad \left( d_{0t}, d_{jt}, \sum_{k \in K_{jt}} u_{jkt} \right) \in \mathcal{K}_\exp, \quad \forall j \in J, t = 0, 1, \ldots, T \\
& \quad \left( d_{jt}, d_{0t}, - \sum_{k \in K_{jt}} x_{jkt} d_{0t} \right) \in \mathcal{K}_\exp, \quad \forall j \in \overline{J}_t, t = 0, 1, \ldots, T \\
& \quad u_{jkt} \geq x_{jkt} d_{jt}, \quad \forall j \in J, k \in K_{jt}, t = 0, 1, \ldots, T \\
& \quad u_{jkt} \leq x_{jkt} d_{jt}, \quad \forall j \in J, k \in \overline{K}_{jt}, t = 0, 1, \ldots, T \\
& \quad d_{0t} + \sum_{j \in J} d_{jt} = 1, \quad \forall t = 0, 1, \ldots, T 
\end{align*}$$

(\text{FP}_{2\text{MNL}})

Unlike $SP_{5\text{MNL}}$, problem $FP_{2\text{MNL}}$ may not be feasible. For example, suppose that $J = \{j\}$ and $\overline{K}_{jt} = K_{jt}$ for all $t$. Let

$$d_{jt} := \frac{\exp \left( - \sum_{k \in K_{jt}} x_{jkt} \right)}{1 + \exp \left( - \sum_{k \in \overline{K}_{jt}} x_{jkt} \right)}$$

denote the minimum feasible demand. If $\sum_{t=0}^{T} \lambda_t a_{rj} d_{jt} > b_r$ for some $r$, then problem $FP_{2\text{MNL}}$ is not feasible, and hence problem $FP_{1\text{MNL}}$ is not feasible.

Examples in Section 4 show that it is possible for problem $FP_{1\text{MNL}}$ to be infeasible while problem $FP_{2\text{MNL}}$ is feasible, either with $FP_{2\text{MNL}}$ having an optimal solution or being unbounded. Next we show that the status of problems $FP_{1\text{MNL}}$ and $FP_{2\text{MNL}}$ is easy to determine by solving only $FP_{2\text{MNL}}$.

**Lemma 3.4.** If $FP_{2\text{MNL}}$ is feasible, then $FP_{1\text{MNL}}$ has an optimal solution if and only if for each $j \in J, t \in \{0, 1, \ldots, T\}$, $k_1 \in K_{jt} \setminus \overline{K}_{jt}$, and $k_2 \in \overline{K}_{jt} \setminus K_{jt}$, it holds that $\phi_{k_1, t} \leq \phi_{k_2, t}$.

**Lemma 3.5.** Suppose that $FP_{2\text{MNL}}$ has a feasible solution $(d', d_0', u')$ with $d' > 0$. Then, any optimal solution $(d^*, d_0^*, u^*)$ for $FP_{2\text{MNL}}$ satisfies $d^* > 0$, $d_0^* > 0$.

Next we summarize the results for $FP$ under the MNL choice model.

**Theorem 3.6.** $FP_{1\text{MNL}}$ can be solved by solving $FP_{2\text{MNL}}$ and taking into account the following possibilities:
1. If \( \text{FP}_{2}^{\text{MNL}} \) is infeasible, then \( \text{FP}_{1}^{\text{MNL}} \) is infeasible.

2. If \( \text{FP}_{2}^{\text{MNL}} \) is feasible, then the following possibilities hold:
   (a) If \( \text{FP}_{2}^{\text{MNL}} \) is unbounded (which happens if and only if for some \( j \in J, t \in \{0,1,\ldots,T\} \), \( k_1 \in K_{jt} \setminus K_{jt}^t \), and \( k_2 \in K_{jt} \setminus K_{jt}^t \), it holds that \( \phi_{k_1,t} > \phi_{k_2,t} \)), then the following possibilities hold:
      i. If \( \text{FP}_{2}^{\text{MNL}} \) does not have a feasible solution \( (d',d'_{0},u') \) with \( d' > 0 \), then \( \text{FP}_{1}^{\text{MNL}} \) is infeasible.
      ii. If \( \text{FP}_{2}^{\text{MNL}} \) has a feasible solution \( (d',d'_{0},u') \) with \( d' > 0 \), then \( \text{FP}_{1}^{\text{MNL}} \) is unbounded.
   (b) If \( \text{FP}_{2}^{\text{MNL}} \) has an optimal solution \( (d^*,d^*_{0},u^*) \) (which happens if and only if for each \( j \in J, t \in \{0,1,\ldots,T\} \), \( k_1 \in K_{jt} \setminus K_{jt}^t \), and \( k_2 \in K_{jt} \setminus K_{jt}^t \), it holds that \( \phi_{k_1,t} \leq \phi_{k_2,t} \)), then the following possibilities hold:
      i. If \( d^* > 0 \), then \( (d^*,x^*) \) with \( x^* \) given by \( x^*_{jkt} = u^*_{jkt}/d^*_{jt} \) for every \( j \in J, t \in \{0,1,\ldots,T\} \), and \( k \in K_{jt} \), is an optimal solution for \( \text{FP}_{1}^{\text{MNL}} \).
      ii. If \( d^*_{jt} = 0 \) for some \( j \in J \) and \( t \in \{0,1,\ldots,T\} \), then \( \text{FP}_{1}^{\text{MNL}} \) is infeasible.

### 3.3 \( \text{SP} \) and \( \text{FP} \) under the MC Model

Next we consider problem \( \text{FP} \) (and as a special case, \( \text{SP} \)) in which the choice probabilities \( \bar{P}_{jt}(y_t) \) are given by a Markov chain choice model. The Markov chain choice model here is similar to the model considered in Dong et al. (2019), adjusted to make provision for multiple attributes. The choice probabilities are given by \( \bar{P}_{jt}(y_t) = \bar{Q}_{jt}(y_t)\bar{V}_{jt}(y_t) \), where \( \bar{V}_{t}(y_t) = (\bar{V}_{jt}(y_t) : j \in J) \) is the unique solution of the linear system

\[
\bar{V}_{jt}(y_t) = \theta_{jt} + \sum_{i \in J} [1 - \bar{Q}_{jt}(y_t)] \rho_{ijt}\bar{V}_{it}(y_t), \quad j \in J
\]

for every \( t = 0,1,\ldots,T \). Here, the model parameters \( \theta_t := (\theta_{jt} : j \in J) \), \( \rho_t := (\rho_{ijt} : i,j \in J) \) and functions \( (\bar{Q}_{jt}(y_t) : j \in J) \) should satisfy the following assumptions:

1. \( \theta_t \geq 0 \) and \( \sum_{j \in J} \theta_{jt} > 0 \) (otherwise the problem is trivial);
2. \( \rho_t \geq 0 \) and \( I - \rho_t \) is nonsingular (\( I \) is an identity matrix with proper dimensionality);
3. \( \bar{Q}_{jt}(y_t) \in [0,1] \) for all feasible \( y_t \).
Note that $(I - \rho_t)^{-1} = I + \sum_{n=1}^{\infty} \rho_t^n$. Thus, $I - \rho_t$ is non-singular if and only if $\sum_{n=1}^{\infty} \rho_t^n < \infty$. Let \( \rho'_{ijt}(y_t) := [1 - \tilde{Q}_{jt}(y_t)] \rho_{ijt} \) for every $i, j \in J$, and let $\rho'_t(y_t) := (\rho'_{ijt}(y_t) : i, j \in J)$. Since $\tilde{Q}_{jt}(y_t) \in [0, 1]$, we have $\sum_{n=1}^{\infty} (\rho'_t(y_t))^n \leq \sum_{n=1}^{\infty} \rho_t^n < \infty$, which implies that [MC] has a unique solution $\tilde{V}_t(y_t) = (I - (\rho'_t(y_t))^T)^{-1} \theta_t$ for any feasible $y_t$.

**Remark 3.2.** [Dong et al. (2019)](https://example.com) provided the following interpretation to the MC model: Each arriving customer first visits product $j$ with probability $\theta_{jt}$. A customer visiting product $j$ purchases the product with probability $\tilde{Q}_{jt}(y_t)$, or transition from the product with probability $1 - \tilde{Q}_{jt}(y_t)$. When a customer choose to transition from product $j$, she either transition to another product $i$ with probability $\rho_{ijt}$, or transition to the no purchase option (and leaves the system) with probability $1 - \sum_{i \in J} \rho_{ijt}$. In this way, each arriving customer transitions between different products until purchasing one of the products, or deciding to leave without making a purchase. Thus, $\tilde{V}_t(y_t)$ gives the expected number of times that a customer visits product $j$.

However, as [Dong et al. (2019)](https://example.com) stated, we should not view the MC model as a faithful model of the mental thought process of the customers when they purchase a product. The value of the MC model comes from the facts that (1) this model is compatible with the random utility maximization principle (see Theorem 1 in [Dong et al. (2019)](https://example.com)), (2) its parameter values can be set so that the purchase probabilities under it become identical to the generalized attraction model, which subsumes the MNL model (see Lemma 2 in [Dong et al. (2019)](https://example.com)), and (3) it yields tractable optimization problems (see [Blanchet et al. (2016)], [Feldman and Topaloglu (2017)](https://example.com) and [Dong et al. (2019)](https://example.com)).

**Remark 3.3.** [Dong et al. (2019)](https://example.com) showed that parameter values of the MC model can be set so that the purchase probabilities under it become identical to the MNL model. The result is established assuming that price attribute is lower bounded for all products. We can extend this result to the case with multiple attributes, assuming that all attributes are lower bounded. See Section 4.

In this section, we let

$$
\tilde{Q}_{jt}(y_t) = \exp \left( \alpha_{jt} - \sum_{k \in K_{jt}} \beta_{kt} y_{jkt} \right), \quad j \in J, \quad t = 0, 1, \ldots, T \tag{2}
$$

Similar to the previous discussion, we let $x_{jkt} := \beta_{kt} y_{jkt} - \alpha_{jt}/K_{jt}$, $\underline{x}_{jkt} := \beta_{kt} \underline{y}_{jkt} - \alpha_{jt}/K_{jt}$, $\overline{x}_{jkt} := \beta_{kt} \overline{y}_{jkt} - \alpha_{jt}/K_{jt}$, $\phi_{kt} := \beta_{kt} \phi_{kt} > 0$, $\psi_{jt} := \psi_{jt} + (\alpha_{jt}/K_{jt}) \sum_{k \in K_{jt}} \phi_{kt}$, for every $j \in J$.
\[ t = 0, 1, \ldots, T, \text{ and } k \in \mathcal{K}_{jt}. \] In addition, to make Assumption 3 above hold, we need
\[ \sum_{k \in \mathcal{K}_j} x_{jkt} \geq 0, \quad j \in \mathcal{J}, \; t = 0, 1, \ldots, T \]
and \( \mathcal{K}_j = \mathcal{K}_j \) (all attributes are lower bounded).

Let \( x := (x_{jkt}, j \in \mathcal{J}, k \in \mathcal{K}_{jt}, t = 0, 1, \ldots, T), d := (d_{jt}, j \in \mathcal{J}, t = 0, 1, \ldots, T), v := (v_{jt}, t = 0, 1, \ldots, T), \) and \( u := (u_{jkt}, j \in \mathcal{J}, k \in \mathcal{K}_{jt}, t = 0, 1, \ldots, T). \) We consider the problem

\[
\begin{align*}
\max_{v, d, x} & \quad \sum_{t=0}^{T} \lambda_t \left( \sum_{j \in \mathcal{J}} \left( \sum_{k \in \mathcal{K}_j} \phi_{kt} x_{jkt} - \psi_{jt} \right) \right) d_{jt}^r
\end{align*}
\]

\( (\text{FP}_1^{MC}) \)

s.t.
\[
\begin{align*}
d_{jt} &= \exp \left( - \sum_{k \in \mathcal{K}_j} x_{ik} \right) v_{jt} \quad \forall j \in \mathcal{J}, \; t = 0, 1, \ldots, T \\
v_{jt} &= \theta_{jt} + \sum_{i \in \mathcal{J}} \rho_{ij} (v_{it} - d_{it}) \quad \forall j \in \mathcal{J}, \; t = 0, 1, \ldots, T \\
\sum_{t=0}^{T} \lambda_t \sum_{j \in \mathcal{J}} a_{rj} d_{jt} &\leq b_r \quad \forall r \in \mathcal{R} \\
x_{jkt} &\geq \overline{x}_{jkt} \quad \forall j \in \mathcal{J}, \; k \in \mathcal{K}_{jt}, \; t = 0, 1, \ldots, T \\
x_{jkt} &\leq \underline{x}_{jkt} \quad \forall j \in \mathcal{J}, \; k \in \mathcal{K}_{jt}, \; t = 0, 1, \ldots, T
\end{align*}
\]

and its following convex conic relaxation:

\[
\begin{align*}
\max_{v, d, u} & \quad \sum_{t=0}^{T} \lambda_t \sum_{j \in \mathcal{J}} \left( \sum_{k \in \mathcal{K}_j} \phi_{kt} u_{jkt} - \psi_{jt} d_{jt} \right) \\
\text{s.t.} & \quad v_{jt} = \theta_{jt} + \sum_{i \in \mathcal{J}} \rho_{ij} (v_{it} - d_{it}) \quad \forall j \in \mathcal{J}, \; t = 0, 1, \ldots, T \\
\sum_{t=0}^{T} \lambda_t \sum_{j \in \mathcal{J}} a_{rj} d_{jt} &\leq b_r \quad \forall r \in \mathcal{R} \\
\left( v_{jt}, d_{jt}, \sum_{k \in \mathcal{K}_j} u_{jkt} \right) &\in \mathcal{K}_{\exp} \quad \forall j \in \mathcal{J}, \; t = 0, 1, \ldots, T \\
\left( d_{jt}, v_{jt}, \sum_{k \in \mathcal{K}_j} \overline{x}_{jkt} v_{jt} \right) &\in \mathcal{K}_{\exp} \quad \forall j \in \mathcal{J}, \; t = 0, 1, \ldots, T \\
u_{jkt} &\geq \overline{x}_{jkt} d_{jt} \quad \forall j \in \mathcal{J}, \; k \in \mathcal{K}_{jt}, \; t = 0, 1, \ldots, T \\
u_{jkt} &\leq \underline{x}_{jkt} d_{jt} \quad \forall j \in \mathcal{J}, \; k \in \mathcal{K}_{jt}, \; t = 0, 1, \ldots, T
\end{align*}
\]

Next we summarize the results for \( \text{FP} \) under the MC choice model.
Theorem 3.7. \( \text{FP}^{MC}_1 \) can be solved by solving \( \text{FP}^{MC}_2 \) and taking into account the following:

1. If \( \text{FP}^{MC}_2 \) is infeasible, then \( \text{FP}^{MC}_1 \) is infeasible.

2. If \( \text{FP}^{MC}_2 \) is feasible, then:
   
   (a) \( \text{FP}^{MC}_2 \) is bounded, and has an optimal solution.
   
   (b) \( \text{FP}^{MC}_1 \) is bounded, and has an optimal solution.
   
   (c) Let \((v^*, d^*, u^*)\) be an optimal solution of \( \text{FP}^{MC}_2 \). Let \( x^*_{jkt} = \frac{u^*_{jkt}}{d^*_jt} \) for every \( j \in J \), \( t \in \{0, 1, \ldots, T\} \), and \( k \in K_{jt} \) such that \( d^*_jt > 0 \), and let \( x^*_{jkt} = 0 \) otherwise. Then \((v^*, d^*, x^*)\) is an optimal solution of \( \text{FP}^{MC}_1 \).

Remark 3.4. Recall that, with the three assumptions hold, any feasible \( y \) corresponds to a unique solution of the system \( \text{MC} \). Thus, if there is no resource constraint in \( \text{FP}^{MC}_1 \), then it must be feasible (assuming that the upper and lower bounds of attributes are proper, such that a feasible \( y \) exists). Therefore, while \( \text{SP} \) is a special case of \( \text{FP} \), the problem \( \text{SP} \) under the MC model is always feasible, and has an optimal solution.

3.4 \text{SP} and \text{FP} under the NL Model

Next we consider problem \( \text{FP} \) (and as a special case, \( \text{SP} \) in which the choice probabilities are given by a nested logit model with non-overlapping nests, and one single null alternative. Let \( I \) denotes the set of all nests. For each \( i \in I \), let \( J_i \in J \) denote the set of products in nest \( i \). In this model:

(1) \( J_i \neq \emptyset \) for each \( i \in I \), (2) \( J = \cup_{i \in I} J_i \), and (3) \( J_i \cap J_i' = \emptyset \) for each pair of two different nests \( i, i' \in I \). Then the choice probabilities \( \tilde{P}_{ji}(y_i) \) are:

\[
\tilde{P}_{ji}(y) = \frac{\exp \left( \alpha_{jt} - \sum_{k \in K_{jt}} \beta_{kt} y_{jkt} \right) \left( \sum_{j' \in J_i} \exp \left( \alpha_{j't} - \sum_{k \in K_{j't}} \beta_{kt} y_{j'kt} \right) \right)^{\gamma_{jt} - 1}}{1 + \sum_{i' \in I} \left( \sum_{j' \in J_{i'}} \exp \left( \alpha_{j'jt} - \sum_{k \in K_{j't}} \beta_{kt} y_{j'kt} \right) \right)^{\gamma_{j't}}}
\]

for each \( i \) and \( j \) such that \( j \in J_i \). Same as in the MNL model, \( \alpha_{jt} \) are the product “baseline attractiveness” parameters, and \( \beta_{kt} > 0 \) are the attribute “sensitivity” parameters. In the nested logit model, \( \gamma_{jt} > 0 \) denotes the “nest dissimilarity” parameters. Note that, when \( \gamma_i > 1 \) for some \( i \in I \), the model can be inconsistent with the random utility theory. See the following quote from Train (2009):
The value of \( \gamma_i \) must be within a particular range for the model to be consistent with utility-maximizing behavior. If \( \gamma_i \) is between zero and one for all \( i \in \mathcal{I} \), the model is consistent with utility maximization for all possible values of the explanatory variables.

For \( \gamma_i \) greater than one, the model is consistent with utility-maximizing behavior for some range of the explanatory variables but not for all values. Kling and Herriges (1993) and Herriges and Kling (1996) provide tests of consistency of nested logit with utility maximization when \( \gamma_i > 1 \); and Train et al. (1987) and Lee (1999) provide examples of models for which \( \gamma_i > 1 \). A negative value of \( \gamma_i \) is inconsistent with utility maximization and implies that improving the attributes of an alternative (such as lowering its price) can decrease the probability of the alternative being chosen. With positive \( \gamma_i \), the nested logit approaches the “elimination by aspects” model of Tversky (1972) as \( \gamma_i \to 0 \).

In this part, we let \( \mathcal{I}_< := \{ i \in \mathcal{I} : \gamma_i \leq 1 \} \), let \( \mathcal{I}_> := \{ i \in \mathcal{I} : \gamma_i > 1 \} \), let \( \mathcal{J}_< := \cup_{i \in \mathcal{I}_<} \mathcal{J}_i \), and let \( \mathcal{J}_> := \cup_{i \in \mathcal{I}_>} \mathcal{J}_i \). Different types of nests will be discussed separately. In addition, similar to the previous parts, we let \( x_{jkt} := \beta_{kt}y_{jkt} - \alpha_{jt}/K_{jt}, \quad \bar{x}_{jkt} := \beta_{kt}\tilde{y}_{jkt} - \alpha_{jt}/K_{jt}, \quad \phi_{kt} := \beta_{kt}\tilde{\phi}_{kt} > 0, \quad \psi_{jt} := \tilde{\psi}_{jt} + (\alpha_{jt}/K_{jt})\sum_{k \in K_{jt}} \tilde{\phi}_{kt}, \) for every \( j \in \mathcal{J}, t = 0, 1, \ldots, T, \) and \( k \in K_{jt} \).

Let \( x := (x_{jkt}, j \in \mathcal{J}, k \in K_{jt}, t = 0, 1, \ldots, T), \) \( d := (d_{jt}, j \in \mathcal{J}, t = 0, 1, \ldots, T), \) \( p := (p_{it}, i \in \mathcal{I}, t = 0, 1, \ldots, T), \) \( u := (u_{jkt}, j \in \mathcal{J}, k \in K_{jt}, t = 0, 1, \ldots, T), \) \( u_0 := (u_{jkt}, j \in \mathcal{J}, k \in K_{jt}, t = 0, 1, \ldots, T), \) \( u_{1/2} := (u_{jkt}, j \in \mathcal{J}_1, k \in K_{jt}, t = 0, 1, \ldots, T), \) \( u_{1/2} := (u_{jkt}, j \in \mathcal{J}_2, k \in K_{jt}, t = 0, 1, \ldots, T), \) \( v_{1/2} := (v_{ikt}, i \in \mathcal{I}, k \in K_{it}, t = 0, 1, \ldots, T). \) We consider the optimization problem

\[
\begin{align*}
\max_{d, p, x} & \sum_{t=0}^T \lambda_t \sum_{j \in \mathcal{J}_t} \left( \sum_{k \in K_{jt}} \phi_{kt} x_{jkt} - \psi_{jt} \right) d_{jt} \\
\text{s.t.} & \quad d_{jt} = \frac{p_{it}}{\sum_{j' \in \mathcal{J}_t} \exp \left( \alpha_{j't} - \sum_{k \in K_{j't}} \beta_{kt'y_{jkt}} \right)} \\
& \quad \sum_{j' \in \mathcal{J}_t} \exp \left( \alpha_{j't} - \sum_{k \in K_{j't}} \beta_{kt'y_{jkt}} \right) = 1 \\
& \quad p_{it} = \frac{\sum_{j' \in \mathcal{J}_t} \exp \left( \alpha_{j't} - \sum_{k \in K_{j't}} \beta_{kt'y_{jkt}} \right)^{\gamma_t}}{\sum_{i \in \mathcal{I}_t} \sum_{j' \in \mathcal{J}_t} \exp \left( \alpha_{j't} - \sum_{k \in K_{j't}} \beta_{kt'y_{jkt}} \right)^{\gamma_t}} \\
& \quad \sum_{t=0}^T \lambda_t a_{jt} d_{jt} \leq b_r \\
& \quad x_{jkt} \geq x_{jkt} \quad \forall j \in \mathcal{J}, k \in K_{jt}, t = 0, 1, \ldots, T \\
& \quad x_{jkt} \leq x_{jkt} \quad \forall j \in \mathcal{J}, k \in K_{jt}, t = 0, 1, \ldots, T
\end{align*}
\]
and the convex program:

$$\max_{d, p, p_0, \psi, u_o} \sum_{t=0}^{T} \sum_{i \in I} \sum_{k \in K'_t} \phi_{kt} v_{ikt}$$

$$+ \sum_{t=0}^{T} \sum_{j \in J} \sum_{k \in K_{jt}} \phi_{kt} u_{jkt}$$

$$- \sum_{t=0}^{T} \lambda_t \sum_{j \in J} \psi_{jt} d_{jt}$$

subject to:

$$\sum_{t=0}^{T} \lambda_t \sum_{j \in J} \alpha_{rj} d_{jt} \leq b_r \quad \forall r \in R$$

$$\left(\frac{1}{\gamma_{it}} - 1\right) p_{it} \ln \left(\frac{p_{0t}}{p_{it}}\right) + \sum_{j \in J} d_{jt} \ln \left(\frac{p_{0t}}{d_{jt}}\right) \geq \sum_{k \in K'_{it}} v_{ikt} \quad \forall i \in I, t = 0, \ldots, T$$

$$\left(\frac{1}{\gamma_{it}} - 1\right) p_{0t} \ln \left(\frac{p_{0t}}{p_{it}}\right) + p_{0t} \ln \left(\frac{p_{0t}}{d_{jt}}\right) \leq p_{0t} \sum_{k \in K_{jt}} \phi_{jkt} \quad \forall i \in I, j \in J, t = 0, \ldots, T$$

$$\left(1 - \frac{1}{\gamma_{it}}\right) d_{jt} \ln \left(\frac{p_{0t}}{d_{jt}}\right) + \left(\frac{1}{\gamma_{it}}\right) d_{jt} \ln \left(\frac{p_{0t}}{d_{jt}}\right) \geq \sum_{k \in K_{jt}} u_{jkt} \quad \forall i \in I, j \in J, t = 0, \ldots, T$$

$$p_{0t} + \sum_{i \in I} d_{jt} = 1 \quad \forall t = 0, \ldots, T$$

$$d \geq 0$$

$$v_{ikt} \geq \sum_{j \in J} \phi_{jkt} d_{jt} \quad \forall i \in I, k \in K'_{it}, t = 0, \ldots, T$$

$$v_{ikt} \leq \sum_{j \in J} \phi_{jkt} d_{jt} \quad \forall i \in I, k \in K'_{it}, t = 0, \ldots, T$$

$$u_{jkt} \geq \phi_{jkt} d_{jt} \quad \forall j \in J, k \in K_{jt}, t = 0, \ldots, T$$

$$u_{jkt} \leq \phi_{jkt} d_{jt} \quad \forall j \in J, k \in K_{jt}, t = 0, \ldots, T$$

where $K_{it}' := \cap_{j \in J} K_{jt}', K_{lt}' := \cap_{j \in J} K_{jt}$ and $K_{lt} := \cup_{j \in J} K_{jt}$. Then the following result holds:

**Theorem 3.8.** If the following conditions hold:

Condition (i): for any feasible solution $(d, p, p_0, v, u_o)$ to \(\text{FP}_{NL}^{2}\) at which

$$\left(\frac{1}{\gamma_{it}} - 1\right) p_{it} \ln \left(\frac{p_{0t}}{p_{it}}\right) + \sum_{j \in J} d_{jt} \ln \left(\frac{p_{0t}}{d_{jt}}\right) = \sum_{k \in K_{it}'} v_{ikt} \quad \forall i \in I, t = 0, \ldots, T$$

the system

$$\left(\frac{1}{\gamma_{it}} - 1\right) d_{jt} \ln \left(\frac{p_{0t}}{d_{jt}}\right) + d_{jt} \ln \left(\frac{p_{0t}}{d_{jt}}\right) = \sum_{k \in K_{jt}} u_{jkt} \quad \forall i \in I, j \in J, t = 0, \ldots, T$$
A simple sufficient condition for Condition (i) to hold in Theorem 3.6, is that for any \( i \in I_\leq \), \( k \in K'_{it} \), \( t = 0,1,\ldots,T \)

\[
v_{ikt} = \sum_{j \in J'_{ikt}} u_{jkt} \quad \forall i \in I_\leq, \ k \in K'_{it}, \ t = 0,1,\ldots,T
\]

\[
u_{jkt} \geq \overline{x}_{jkt}d_{jt} \quad \forall j \in J_\leq, \ k \in K_{jt}, \ t = 0,1,\ldots,T
\]

\[
u_{jkt} \leq \overline{x}_{jkt}d_{jt} \quad \forall j \in J_\leq, \ k \in \overline{K}_{jt}, \ t = 0,1,\ldots,T
\]

has a solution \( u_\leq \). Here \( J'_{ikt} := \{ j \in J_i : k \in K_{jt} \} \).

Condition (ii): \( \overline{K}_{jt} \subsetneq K_{jt} \) for every \( j \in J_\geq \), i.e. \( \sum_{k \in K_{jt}} u_{jkt} \) is not upper bounded for any \( j \in J_\geq \). then \( FP_{NL1} \) can be solved by solving \( FP_{NL2} \) and taking into account the following possibilities:

1. If \( FP_{NL2} \) is infeasible, then \( FP_{NL1} \) is infeasible.

2. If \( FP_{NL2} \) is feasible and unbounded, then \( FP_{NL1} \) is feasible and unbounded.

3. If \( FP_{NL2} \) is feasible and has an optimal solution \((d^*, p^*, p_0^*, v_\leq^*, u_\leq^*)\), then \((d^*, p^*, x^*)\) with \( x^* \) given by \( x^*_{jkt} = u^*_{jkt}/d^*_{jt} \) for every \( j \in J, t \in \{0,1,\ldots,T\} \), and \( k \in K_{jt} \), is an optimal solution for \( FP_{NL1} \). (Here \( u_\leq^* \) is part of the optimal solution \((d^*, p^*, p_0^*, v_\leq^*, u_\leq^*)\) to \( FP_{NL2} \) while \( u_\leq^* \) is obtained by solving the system in Condition (i), using \((d^*, p^*, p_0^*, v_\leq^*, u_\leq^*)\) as an input.)

Remark 3.5. A simple sufficient condition for Condition (i) to hold in Theorem 3.6, is that for any \( i \in I_\leq \), there is one unbounded attribute \( k \) that is shared by all products in \( J_i \) (for example, the price attribute). That is, for any \( i \in I_\leq, j \in J_i \), \( t = 0,1,\ldots,T \), we need \( k \in K_{jt} \) (which implies \( J_i = J'_{ikt} \)), \( k \notin \overline{K}_{jt} \), and \( k \notin K_{jt} \). Then we can split all other attribute \( k' \in J_i \) for each \( i \in I_\leq \), \( j \in J_i \) and \( t = 0,1,\ldots,T \), such that each of those attributes only apply to one product (i.e. \( J'_{ikt} \) are all singletons). Now, given any feasible solution \((d,p,p_0,v_\leq,u_\leq)\) to \( FP_{NL2} \), at which

\[
\left( \frac{1}{\gamma_{it}} - 1 \right) p_{it} \ln \left( \frac{p_{ot}}{p_{it}} \right) + \sum_{j \in J_i} d_{jt} \ln \left( \frac{p_{ot}}{d_{jt}} \right) = \sum_{k \in K_{it}} v_{ikt} \quad \forall i \in I_\leq, \ t = 0,1,\ldots,T
\]

we can find a feasible solution to the system in Condition (i) as follow: Consider any \( i \in I_\leq \) and \( j \in J_i \). Let

\[
u_{ilk'} = v_{ik'} \quad \forall t = 0,1,\ldots,T
\]

for any \( k' \neq k \), and let

\[
u_{ikt} = \left( \frac{1}{\gamma_{it}} - 1 \right) d_{jt} \ln \left( \frac{p_{ot}}{p_{it}} \right) + d_{jt} \ln \left( \frac{p_{ot}}{d_{jt}} \right) - \sum_{k' \in K_{jt}\{k\}} u_{ik'} \quad \forall t = 0,1,\ldots,T
\]
Note that we automatically have

\[ v_{ikt} = \sum_{j \in J_i} u_{jkt} \quad \forall \ t = 0, 1, \ldots, T \]

which implies that the \( u \)\( \le \) constructed is a feasible solution to the system in Condition (i).

Similar to the case under the MNL model, the feasible set of \( \text{FP}_2^{\text{NL}} \) may not be closed. Next we relax \( \text{FP}_2^{\text{NL}} \) to make the feasible set closed, and then we show that optimal solutions are not affected by the relaxation. Consider the following convex conic program:

\[
\max_{d, p, p_0, v_{\le}, u_{\le}} \sum_{t=0}^{T} \lambda_t \sum_{i \in I_{\le}} \sum_{k \in K'_{it}} \phi_{ikt} v_{ikt} \\
+ \sum_{t=0}^{T} \lambda_t \sum_{j \in J_{\le}} \sum_{k \in K_{jt}} \phi_{jkt} u_{jkt} \\
- \sum_{t=0}^{T} \lambda_t \sum_{j \in J} v_{jkt} d_{jt} \\
\sum_{t=0}^{T} \lambda_t \sum_{j \in J} a_{rj} d_{jt} \leq b_r \quad \forall \ r \in \mathcal{R} \tag{32a}
\]

s.t. \[
\left( \frac{1}{\gamma_{it}} - 1 \right) r_{it} + \sum_{j \in J_i} e_{jt} = \sum_{k \in K'_{it}} v_{ikt} \quad \forall \ i \in I_{\le}, \ t = 0, 1, \ldots, T \tag{32b}
\]
\[
\left( \frac{1}{\gamma_{it}} - 1 \right) s_{it} + f_{jt} = -p_{0t} \sum_{k \in K_{jt}} \pi_{jkt}^{||i} \quad \forall \ i \in I_{\le}, \ j \in J, \ t = 0, 1, \ldots, T \tag{32c}
\]
\[
\left( 1 - \frac{1}{\gamma_{it}} \right) g_{jt} + \left( \frac{1}{\gamma_{it}} \right) e_{jt} = \sum_{k \in K_{jt}} u_{jkt} \quad \forall \ i \in I_{\ge}, \ j \in J, \ t = 0, 1, \ldots, T \tag{32d}
\]
\[(p_{0t}, d_{jt}, e_{jt}) \in K_{\text{exp}} \quad \forall \ i \in I_{\le}, \ j \in J, \ t = 0, 1, \ldots, T \tag{32e}
\]
\[(d_{jt}, p_{0t}, f_{jt}) \in K_{\text{exp}} \quad \forall \ i \in I_{\le}, \ j \in J, \ t = 0, 1, \ldots, T \tag{32f}
\]
\[(p_{it}, d_{jt}, g_{jt}) \in K_{\text{exp}} \quad \forall \ i \in I_{\ge}, \ j \in J, \ t = 0, 1, \ldots, T \tag{32g}
\]
\[(p_{it}, p_{0t}, s_{it}) \in K_{\text{exp}} \quad \forall \ i \in I_{\le}, \ t = 0, 1, \ldots, T \tag{32h}
\]
\[(p_{it}, p_{it}, r_{it}) \in K_{\text{exp}} \quad \forall \ i \in I_{\le}, \ t = 0, 1, \ldots, T \tag{32i}
\]
\[p_{0t} + \sum_{i \in I} p_{it} = 1 \quad \forall \ t = 0, 1, \ldots, T \tag{32j}
\]
\[p_{it} = \sum_{j \in J_i} d_{jt} \quad \forall \ i \in I, \ t = 0, 1, \ldots, T \tag{32k}
\]
\[v_{ikt} \geq \sum_{j \in J_i} x_{jkt} d_{jt} \quad \forall \ i \in I_{\le}, \ k \in K'_{it}, \ t = 0, 1, \ldots, T \tag{32l}
\]
\[v_{ikt} \leq \sum_{j \in J_i} x_{jkt} d_{jt} \quad \forall \ i \in I_{\le}, \ k \in K'_{it}, \ t = 0, 1, \ldots, T \tag{32m}
\]
\[u_{jkt} \geq x_{jkt} d_{jt} \quad \forall \ j \in J, \ k \in K_{jt}, \ t = 0, 1, \ldots, T \tag{32n}
\]
Then the following result holds:

**Theorem 3.9.** Assume that Condition (i) in Theorem 3.8 holds.

Then, $FP_{NL}^2$ can be solved by solving $FP_{NL}^3$ and taking into account the following:

1. If $FP_{NL}^3$ is infeasible, then $FP_{NL}^2$ is infeasible.

2. If $FP_{NL}^3$ is feasible, then the following possibilities hold:

   (a) If $FP_{NL}^3$ is unbounded (which happens if and only if for some $j \in J$, $t \in \{0,1,\ldots,T\}$, $k_1 \in K_{jt} \setminus K_{jt}$, and $k_2 \in K_{jt} \setminus K_{jt}$, it holds that $\phi_{k_1,t} > \phi_{k_2,t}$), then the following possibilities hold:

      i. If $FP_{NL}^3$ does not have a feasible solution $(d',p',p_0',v'_\leq,u'_>,e',f'_\leq,g'_>,r'_\leq,s'_>)$ with $d' > 0$, then $FP_{NL}^2$ is infeasible.

      ii. If $FP_{NL}^3$ has a feasible solution $(d',p',p_0',v'_\leq,u'_>,e',f'_\leq,g'_>,r'_\leq,s'_>)$ with $d' > 0$, then $FP_{NL}^2$ is unbounded.

   (b) If $FP_{NL}^3$ has an optimal solution $(d^*,p^*,p_0^*,v^*_\leq,u^*_>,e^*,f^*_\leq,g^*_>,r^*_\leq,s^*_>)$ (which happens if and only if for each $j \in J$, $t \in \{0,1,\ldots,T\}$, $k_1 \in K_{jt} \setminus K_{jt}$, and $k_2 \in K_{jt} \setminus K_{jt}$, it holds that $\phi_{k_1,t} \leq \phi_{k_2,t}$), then the following possibilities hold:

      i. If $d^* > 0$, then $(d^*,p^*,p_0^*,v^*_\leq,u^*_>)$ is an optimal solution for $FP_{NL}^2$.

      ii. If $d^*_{jt} = 0$ for some $j \in J$ and $t \in \{0,1,\ldots,T\}$, then $FP_{NL}^2$ is infeasible.

### 4 Remarks

**Examples where $FP_{MNL}^1$ is infeasible, while $FP_{MNL}^2$ is feasible:**

**Example 4.1.** In this example $FP_{MNL}^1$ is infeasible, while $FP_{MNL}^2$ has an optimal solution. Consider the following single-period instance of $FP_{MNL}^1$ with two products indexed $\{1,2\}$. Each product $i$ has

\[
0 \leq u_{jkt} \leq \pi_{jkt} d_{jt} \quad \forall j \in J, \ k \in K_{jt}, \ t = 0,1,\ldots,T
\]
one attribute also indexed $i$. The time index is omitted.

$$\max_{d,x} \left\{ x_{1,1}d_1 + x_{2,2}d_2 \right\} \quad \text{(FP}_{\text{Example}} \text{)}$$

s.t.  
\[ d_1 + d_2 \leq \frac{1}{2} \]
\[ d_1 = \frac{\exp(-x_{1,1})}{1 + \exp(-x_{1,1}) + \exp(-x_{2,2})} \]
\[ d_2 = \frac{\exp(-x_{2,2})}{1 + \exp(-x_{1,1}) + \exp(-x_{2,2})} \]
\[ x_{1,1} \leq 0 \]

For any $x$ feasible for $\text{FP}_{\text{Example}}$, it holds that

\[ d_1 + d_2 = \frac{\exp(-x_{1,1}) + \exp(-x_{2,2})}{1 + \exp(-x_{1,1}) + \exp(-x_{2,2})} \geq \frac{\exp(0)}{1 + \exp(0)} = \frac{1}{2} \]

Thus, $\text{FP}_{\text{Example}}$ is infeasible.

The corresponding instance of $\text{FP}_{\text{MNL}}$ is

$$\max_{d_0, d_1, u} \left\{ u_{1,1} + u_{2,2} \right\} \quad \text{(FP}_{\text{Example}} \text{)}$$

s.t.  
\[ d_1 + d_2 \leq \frac{1}{2} \]
\[ (d_0, d_1, u_{1,1}) \in K_{\text{exp}} \]
\[ (d_0, d_2, u_{2,2}) \in K_{\text{exp}} \]
\[ (d_1, d_0, 0) \in K_{\text{exp}} \]
\[ d_0 + d_1 + d_2 = 1 \]
\[ u_{1,1} \leq 0 \]

The only feasible solution of $\text{FP}_{\text{Example}}$ is $d_0 = 1/2, d_1 = 1/2, d_2 = 0, u_{1,1} = 0, u_{2,2} = 0$. Thus, $\text{FP}_{\text{Example}}$ has an optimal solution.

**Example 4.2.** In this example $\text{FP}_{\text{MNL}}$ is infeasible, while $\text{FP}_{\text{MNL}}$ is unbounded. Consider the following single-period instance of $\text{FP}_{\text{MNL}}$ with two products indexed $\{1, 2\}$. Product 1 has one
attribute indexed 1, and product 2 has two attributes indexed 1, 2. The time index is omitted.

\[
\begin{align*}
\max_{d, x} & \quad x_{1,1}d_1 + (x_{2,1} + 2x_{2,2})d_2 \\
\text{s.t.} & \quad d_1 + d_2 \leq \frac{1}{2} \\
& \quad d_1 = \frac{\exp(-x_{1,1})}{1 + \exp(-x_{1,1}) + \exp(-x_{2,1} - x_{2,2})} \\
& \quad d_2 = \frac{\exp(-x_{2,1} - x_{2,2})}{1 + \exp(-x_{1,1}) + \exp(-x_{2,1} - x_{2,2})} \\
& \quad x_{1,1} \leq 0
\end{align*}
\]

For any \( x \) feasible for \( \text{FP}_3 \), it holds that

\[
\begin{align*}
d_1 + d_2 & = \frac{\exp(-x_{1,1}) + \exp(-x_{2,1} - x_{2,2})}{1 + \exp(-x_{1,1}) + \exp(-x_{2,1} - x_{2,2})} \\
& > \frac{\exp(-x_{1,1})}{1 + \exp(-x_{1,1})} \geq \frac{\exp(0)}{1 + \exp(0)} = \frac{1}{2}
\end{align*}
\]

Thus, \( \text{FP}_3 \) is infeasible.

The corresponding instance of \( \text{FP}_2^{\text{MNL}} \) is

\[
\begin{align*}
\max_{d, d_0, u} & \quad \{ u_{1,1} + u_{2,1} + 2u_{2,2} \} \\
\text{s.t.} & \quad d_1 + d_2 \leq \frac{1}{2} \\
& \quad (d_0, d_1, u_{1,1}) \in K_{\text{exp}} \\
& \quad (d_0, d_2, u_{2,1} + u_{2,2}) \in K_{\text{exp}} \\
& \quad (d_1, d_0, 0) \in K_{\text{exp}} \\
& \quad d_0 + d_1 + d_2 = 1 \\
& \quad u_{1,1} \leq 0
\end{align*}
\]

Consider the sequence of solutions \( \{(d_{1,1}^n, d_{2,2}^n, d_0^0, u_{1,1}^n, u_{2,1}^n, u_{2,2}^n) = (1/2, 0, 1/2, 0, -n, n)\}_{n=1}^{\infty} \). The solutions are feasible for \( \text{FP}_4 \) for all \( n \), and the objective value of \( (d_{1,1}^n, d_{2,2}^n, d_0^0, u_{1,1}^n, u_{2,1}^n, u_{2,2}^n) \) is \( n \). Thus, \( \text{FP}_4 \) is unbounded.

**The MC model subsumes the MNL model when attributes are lower bounded**: Assume that all attributes are lower bounded. Consider a generic MNL model

\[
\hat{P}_{jt}(y) = \frac{\exp \left( \alpha_{jt} - \sum_{k \in K_{jt}} \beta_{kjt}y_{kjt} \right)}{1 + \sum_{j' \in \mathcal{J}} \exp \left( \alpha_{j't} - \sum_{k \in K_{j't}} \beta_{kjt}y_{kjt} \right)}
\]
where
\[ y_{jkt} \leq y_{jkt}, \quad \forall \; j \in J, \; k \in K_{jt} \]

We let
\[ \theta_{jt} = \frac{1}{\exp \left( \sum_{k \in K_{jt}} \beta_{kt}y_{jkt} - \alpha_{jt} \right) + |J|}, \quad j \in J \]
\[ \rho_{ijt} = \frac{1}{\exp \left( \sum_{k \in K_{jt}} \beta_{kt}y_{jkt} - \alpha_{jt} \right) + |J|}, \quad j \in J \]

let
\[ \hat{Q}_{jt}(y_t) = \exp \left( \sum_{k \in K_{jt}} \beta_{kt} \left( y_{jkt} - y_{jk} \right) \right), \quad j \in J \]

and let
\[ \hat{V}_{jt}(y_t) = \frac{1}{\exp \left( \sum_{k \in K_{jt}} \beta_{kt}y_{jkt} - \alpha_{jt} \right) + \sum_{j' \in J} \exp \left( \sum_{k \in K_{j'jt}} \beta_{kt} \left( y_{j'kt} - y_{j'k} \right) \right)}, \quad j \in J \]

Now, we have
\[ \hat{P}_{jt}(y) = \hat{Q}_{jt}(y_t)\hat{V}_{jt}(y_t), \quad j \in J \]

Meanwhile, since
\[ 1 + \sum_{i \in J_t} \left[ 1 - \hat{Q}_{ii}(y_t) \right] \hat{V}_{ii}(y_t) \]
\[ = 1 + \sum_{i \in J_t} \frac{1 - \exp \left( \sum_{k \in K_{it}} \beta_{kt} \left( y_{ikt} - y_{ik} \right) \right)}{\exp \left( \sum_{k \in K_{it}} \beta_{kt}y_{ikt} - \alpha_{it} \right) + \sum_{j' \in J} \exp \left( \sum_{k \in K_{j'it}} \beta_{kt} \left( y_{j'kt} - y_{j'k} \right) \right)} \]
\[ = 1 + \sum_{i \in J_t} \frac{\exp \left( \alpha_{it} - \sum_{k \in K_{it}} \beta_{kt}y_{ikt} - \sum_{k \in K_{it}} \beta_{kt}y_{ikt} \right) - \exp \left( \alpha_{it} - \sum_{k \in K_{it}} \beta_{kt}y_{ikt} \right)}{1 + \sum_{j' \in J} \exp \left( \alpha_{it} - \sum_{k \in K_{j'it}} \beta_{kt}y_{j'k} \right) + \sum_{j' \in J} \exp \left( \sum_{k \in K_{j'it}} \beta_{kt} \left( y_{j'kt} - y_{j'k} \right) \right)} \]
\[ = \frac{1 + \sum_{i \in J_t} \exp \left( \alpha_{it} - \sum_{k \in K_{it}} \beta_{kt}y_{ikt} \right) + \sum_{j' \in J} \exp \left( \sum_{k \in K_{j'it}} \beta_{kt} \left( y_{j'kt} - y_{j'k} \right) \right)}{1 + \sum_{j' \in J} \exp \left( \sum_{k \in K_{j'it}} \beta_{kt}y_{j'k} - \alpha_{jt} \right) + |J|} \]

we know that \( \hat{V}(y) \) is a solution to the system \( \text{MC} \).

It left to prove that the \( \hat{V}(y) \) is the unique solution to \( \text{MC} \). Indeed, it is clear that \( \theta_t \geq 0, \sum_{j \in J} \theta_{jt} > 0, \rho_t \geq 0 \) and \( \sum_{n=1}^{\infty} \rho_t^n < \infty \). In addition, since \( y_{jkt} \leq y_{jkt} \) for any \( j \in J \) and \( k \in K_{jt} \), we have \( \hat{Q}_j(y) \in [0, 1] \). Thus, all necessary assumptions hold, and the solution is unique.
5 Conclusion

In this paper, we formulate the profit maximization problems $\mathbf{SP}$ and $\mathbf{FP}$ with multiple product attributes, and provide their convex conic reformulations under three choice models. When there is only one attribute associating with each product, $\mathbf{SP}$ and $\mathbf{FP}$ reduces to the classical pricing problems and revenue management problems with resource constraints. Thus, our paper reproduces the tractability results of pricing problems under the MNL model (Song and Xue (2007), Dong et al. (2009)), the MC model (Dong et al. (2019)), and the NL model (Li and Huh (2011)). For the NL model, our approach also covers the case when the nest dissimilarity parameters $\gamma$ is larger than 1. Tractability result of this scenario was given in Gallego and Topaloglu (2014). However, their method only works for unconstrained problems, while our approach allows different type of constraints, such as resource constraints and upper and lower bound of attributes.

While we kept our discussion minimum, our approach can be extended to work with multiple types of objective functions and constraints. For example, revenue maximization can be viewed as a special case of profit maximization with zero costs, while market share maximization can be viewed as a special case of profit maximization with zero profit margins. In general, any objective function is allowed as long as (1) it can be transformed into a concave function in the convex conic reformulation, and (2) the nonlinear inequality constraints hold tight at optimality. Similarly, we can work with different types of constraints, as long as there is a good convex transformation. Another possible direction of extension is allowing utility and / or profit margin to be nonlinear functions of attributes. One example is to replace an attribute $x_j$ by $x_j^+ - x_j^-$, where $x_j^+, x_j^- \geq 0$. Then $x_j^+$ and $x_j^-$ can have different sensitivity parameters $\beta_j^+$ and $\beta_j^-$. (Note that $\beta_j^+$ and $\beta_j^-$ need to satisfy certain conditions, such that one cannot benefit from increasing $x_j^+$ and $x_j^-$ simultaneously.) Thus, the utility is a piecewise linear function of $x_j$. 

27
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A Proofs

A.1 Proof of Lemma 3.1

Lemma 3.1. $\text{SP}^{\text{MNL}}_5$ has an optimal solution if and only if for each $j \in J$, and for all $k_1 \in K_j \setminus K_j$, $k_2 \in K_j \setminus K_j$, it holds that $\phi_{k_1} \leq \phi_{k_2}$; otherwise, $\text{SP}^{\text{MNL}}_5$ is unbounded.

Proof. First, note that $\text{SP}^{\text{MNL}}_5$ is feasible. For example, consider $(d_0, d_0^0, u^0)$ given as follows: For all $j \in J$ and $k \in K_j$, let $x_{jk}^0 = \pi_{jk}$. For all $j \in J$ and $k \in K_j \setminus (K_j \cup K_j)$, let $x_{jk}^0 = 0$. Next, let $d_0^0$ be given by (1), let $d_j^0$ be given by (1) for all $j \in J$, and let $u_{jk}^0 = d_{jk}^0$ for all $j \in J$ and $k \in K_j$. Then $(d_0, d_0^0, u^0)$ is feasible for $\text{SP}^{\text{MNL}}_5$.

Suppose that there is a $j \in J$, a $k_1 \in K_j \setminus K_j$, and a $k_2 \in K_j \setminus K_j$, such that $\phi_{k_1} > \phi_{k_2}$. Then consider a sequence $\{(d_n, d_n^0, u^n)\}_{n=0}^\infty$ of feasible solutions with $u_{jk}^n = u_{jk}^0 + n$, $u_{jk_2}^n = u_{jk_2}^0 - n$, $u_{jk}^n = u_{jk}^0$ for all $k \notin \{k_1, k_2\}$, $u_{jk}^n = u_{jk}^0$ for all $j' \neq j$ and $k \in K_j$, $d_j^n = d_j^0$ for all $j' \in J$, and $d_0^n = d_0^0$. Then $(d_n, d_n^0, u^n)$ is feasible for $\text{SP}^{\text{MNL}}_5$ for all $n$, and the objective value of $(d_n, d_n^0, u^n)$ exceeds the objective value of $(d_n, d_n^0, u^n)$ by $\phi_{k_1} - \phi_{k_2}$ for all $n$, and thus the objective value of $(d_n, d_n^0, u^n)$ increases without bound as $n \to \infty$.

Next suppose that for each $j \in J$, and for all $k_1 \in K_j \setminus K_j$, $k_2 \in K_j \setminus K_j$, it holds that $\phi_{k_1} \leq \phi_{k_2}$. Let

$$
K^*_\exp := \text{closure}\{(y_1, y_2, y_3) : y_1 \geq -y_3 \exp(y_2/y_3 - 1), y_1 > 0, y_3 < 0\}
$$

$$
= \{(y_1, y_2, y_3) : y_1 \geq -y_3 \exp(y_2/y_3 - 1), y_1 > 0, y_3 < 0\} \cup \{(y_1, y_2, 0) : y_1 \geq 0, y_2 \geq 0\}
$$

denote the dual cone of $K_\exp$. Let $\pi_j \in K^*_\exp$ denote the dual variables corresponding to (20), let $\nu_j \in K^*_\exp$ denote the dual variables corresponding to (21), let $\nu_jk \geq 0$ denote the dual variables corresponding to (22), let $\nu_jk \geq 0$ denote the dual variables corresponding to (23), and let $\gamma$ denote the dual variable corresponding to (24). Then the dual of $\text{SP}^{\text{MNL}}_5$ is

$$
\min_{\pi, \nu, \gamma} \gamma \\
\text{s.t.} \quad \pi_j = -\phi_k \quad \forall j \in J, k \in K_j \setminus (K_j \cup K_j) \quad (33a) \\
\pi_j + \nu_jk = -\phi_k \quad \forall j \in J, k \in K_j \setminus K_j \quad (33b) \\
\pi_j - \nu_jk = -\phi_k \quad \forall j \in J, k \in K_j \setminus K_j \quad (33c) \\
\pi_j + \nu_jk - \nu_jk = -\phi_k \quad \forall j \in J, k \in K_j \setminus K_j \quad (33d)
$$

(33d)
\[
\pi_j + \omega_j - \sum_{k \in K_j} \nu_{jk} x_{jk} + \sum_{k \in K_j} \tau_{jk} x_{jk} - \gamma = \psi_j \quad \forall j \in J \tag{33c}
\]

\[
\pi_j - \sum_{k \in K_j} \nu_{jk} x_{jk} + \sum_{k \in K_j} \tau_{jk} x_{jk} - \gamma = \psi_j \quad \forall j \in J \setminus J \tag{33f}
\]

\[
\sum_{j \in J} \pi_j + \sum_{j \in J} \omega_j - \sum_{j \in J} \omega_j - \sum_{j \in J} \tau_{jk} x_{jk} - \gamma = 0 \tag{33g}
\]

\[
\pi_j \in K^*_\exp \quad \forall j \in J \tag{33h}
\]

\[
\omega_j \in K^*_\exp \quad \forall j \in J \tag{33i}
\]

\[
\nu_{jk} \geq 0 \quad \forall j \in J, k \in K_j \tag{33j}
\]

\[
\tau_{jk} \geq 0 \quad \forall j \in J, k \in K_j \tag{33k}
\]

Next we show that there is a feasible solution \((\pi, \omega, \nu, \tau, \gamma)\) of \(\text{SDMNL}_2\) such that \(\pi_j \in \text{int} (K^*_\exp)\) for all \(j \in J\) and \(\omega_j \in \text{int} (K^*_\exp)\) for all \(j \in J\). Choose \(\omega_{j2} = 0\), \(\omega_{j3} = -1\), and \(\omega_{j1} = \exp(-1) + 1\) for all \(j \in J\). Thus \(\omega_j \in \text{int} (K^*_\exp)\) for all \(j \in J\). For each \(j \in J\), choose \(\pi_{j3}, \nu_{jk}\) for all \(k \in K_j\), and \(\tau_{jk}\) for all \(k \in K_j\), by considering the following cases:

1. If \(K_j \setminus K_j \neq \emptyset\), then choose \(\pi_{j3} = -\min \{\phi_k : k \in K_j \setminus K_j\}\). Since \(\phi_k > 0\) for all \(k\), it follows that \(\pi_{j3} < 0\). Then, for each \(k \in K_j\), consider the following four cases.

   (a) Suppose \(k \in K_j \setminus (K_j \cup K_j)\). Since \(k \in K_j \setminus K_j\), it follows that \(-\pi_{j3} \leq \phi_k\). Since \(k \in K_j \setminus K_j\) and \(\phi_{k1} \leq \phi_{k2}\) for all \(k_1 \in K_j \setminus K_j, k_2 \in K_j \setminus K_j\), it follows that \(\phi_k \leq -\pi_{j3}\). Thus, \(\phi_k = -\pi_{j3}\), and hence constraint (33a) is satisfied.

   (b) Suppose \(k \in K_j \setminus K_j\). Then choose \(\nu_{jk} = -\pi_{j3} - \phi_k\). Hence constraint (33b) is satisfied.

      Since \(k \in K_j \setminus K_j\) and \(\phi_{k1} \leq \phi_{k2}\) for all \(k_1 \in K_j \setminus K_j, k_2 \in K_j \setminus K_j\), it follows that \(\phi_k \leq -\pi_{j3}\). Thus \(\nu_{jk} = -\pi_{j3} - \phi_k \geq 0\), and hence constraint (33c) is satisfied.

   (c) Suppose \(k \in K_j \setminus K_j\). Then choose \(\tau_{jk} = \pi_{j3} + \phi_k\). Hence constraint (33c) is satisfied.

      Since \(k \in K_j \setminus K_j\), it follows that \(-\pi_{j3} \leq \phi_k\). Thus \(\tau_{jk} = \pi_{j3} + \phi_k \geq 0\), and hence constraint (33c) is satisfied.

   (d) Suppose \(k \in K_j \cap K_j\). Then choose \(\nu_{jk} = \max \{0, -\pi_{j3} - \phi_k\}\) and \(\tau_{jk} = \max \{0, \pi_{j3} + \phi_k\}\).

      Hence constraints (33d), (33f) and (33k) are satisfied.

2. If \(K_j \setminus K_j = \emptyset\) (that is, \(K_j = K_j\)), then choose \(\pi_{j3} = -\max \{\phi_k : k \in K_j\}\). Since \(\phi_k > 0\) for all \(k\), it follows that \(\pi_{j3} < 0\). Then, for each \(k \in K_j\), consider the following two cases (only these two cases are possible).
(a) Suppose \( k \in K_j \setminus \overline{K}_j \). Then choose \( \nu_{jk} = -\pi_{j3} - \phi_k \). Hence constraint (33b) is satisfied.

Since \( \phi_k \leq -\pi_{j3} \), it follows that \( \nu_{jk} = -\pi_{j3} - \phi_k \geq 0 \), and hence constraint (33) is satisfied.

(b) Suppose \( k \in K_j \cap \overline{K}_j \). Then choose \( \nu_{jk} = \max \{0, -\pi_{j3} - \phi_k \} \) and \( \overline{\nu}_{jk} = \max \{0, \pi_{j3} + \phi_k \} \).

Hence constraints (33d), (33) and (33k) are satisfied.

Thus \( \pi_{j3}, \nu_{jk}, \) and \( \overline{\nu}_{jk} \) have been chosen for all \( j \) and \( k \).

Next we use (33a) and (33) to express \( \pi_{j2} \) as a function of \( \gamma \) for each \( j \in J \). Then we choose \( \pi_{j1} \) as a function of \( \gamma \) such that \( \pi_{j1} = -\pi_{j3} \exp (\pi_{j2}/\pi_{j3} - 1) + 1 \). Then it will follow that \( \pi_j \in \text{int}(K^*_\text{exp}) \) for all \( j \). Finally we will show that there exists a \( \gamma \) such that (33g) is satisfied, and thus all the constraints of \( \text{SD}^{\text{MNL}}_5 \) will be satisfied.

If \( j \in J \), then let \( e_j = \omega_{j1} + \sum_{k \in K_j} \nu_{jk} x_{jk} - \sum_{k \in K_j} \overline{\nu}_{jk} x_{jk} + \psi_j \), and if \( j \in J \setminus \overline{J} \), then let \( e_j = \sum_{k \in K_j} \nu_{jk} x_{jk} - \sum_{k \in K_j} \overline{\nu}_{jk} x_{jk} + \psi_j \). For all \( j \in J \), let \( \pi_{j2} = \pi_{j2}(\gamma) = \gamma + e_j \), and let \( \pi_{j1}(\gamma) = -\pi_{j3} \exp (\pi_{j2}(\gamma)/\pi_{j3} - 1) + 1 = -\pi_{j3} \exp ([\gamma + e_j]/\pi_{j3} - 1) + 1 \). It follows that for any \( \gamma \), (33e) is satisfied for all \( j \in J \) and (33f) is satisfied for all \( j \in J \setminus \overline{J} \). It also follows that for any \( \gamma \), \( \pi_j \in \text{int}(K^*_\text{exp}) \) for all \( j \). For constraint (33g) to be satisfied, \( \gamma \) has to satisfy

\[
\sum_{j \in J} \pi_{j1}(\gamma) + \sum_{j \in \overline{J}} \omega_{j2} - \sum_{j \in J} \omega_{j3} \sum_{k \in K_j} \overline{\nu}_{jk} - \gamma = 0,
\]

that is,

\[
h(\gamma) := \sum_{j \in J} \left(-\pi_{j3} \exp \left( \frac{\gamma + e_j}{\pi_{j3}} - 1 \right) + 1 \right) + \sum_{j \in \overline{J}} \omega_{j2} - \sum_{j \in J} \omega_{j3} \sum_{k \in K_j} \overline{\nu}_{jk} - \gamma = 0.
\]

Since \( \pi_{j3} < 0 \) for all \( j \), it follows that \( h(\gamma) \) is decreasing, \( h(\gamma) \to \infty \) as \( \gamma \to -\infty \), and \( h(\gamma) \to -\infty \) as \( \gamma \to \infty \). Thus it follows from the intermediate value theorem that there is a (unique) \( \gamma^* \) such that \( h(\gamma^*) = 0 \), and hence \( (\gamma^*, \pi_{j1}(\gamma^*), j \in J) \) satisfies (33g). The resulting solution \( (\pi(\gamma^*), \omega, \nu, \overline{\nu}, \gamma^*) \) is feasible for \( \text{SD}^{\text{MNL}}_5 \) and satisfies \( \pi_j(\gamma^*) \in \text{int}(K^*_\text{exp}) \) for all \( j \in J \) and \( \omega_j \in \text{int}(K^*_\text{exp}) \) for all \( j \in J \).

Therefore it follows from the conic duality theorem (Theorem 1.4.2 in [Ben-Tal and Nemirovski (2001)]) that \( \text{SD}^{\text{MNL}}_5 \) has an optimal solution. \( \square \)
A.2 Proof of Lemma 3.2

Lemma 3.2. Every optimal solution for $SP_{MNL}^{4}$ is also optimal for $SP_{MNL}^{5}$.

Proof. Consider any $(d,d_0,u)$ feasible for $SP_{MNL}^{4}$. Then $(d,d_0,u)$ is also feasible for $SP_{MNL}^{5}$ and has the same objective value. Next, consider any $(d,d_0,u)$ feasible for $SP_{MNL}^{5}$. Note that if $d > 0$ and $d_0 > 0$, then $(d,d_0,u)$ is feasible for $SP_{MNL}^{5}$ and has the same objective value. We show by contradiction that $d_0 > 0$, and that $d_j > 0$ for any $j \in J$. Suppose that $d_0 = 0$. Then it follows from constraint (20) that $d_j = 0$ for all $j$, which would violate constraint (21). Similarly, suppose that $d_j = 0$ for some $j \in J$, then it follows from constraint (21) that $d_0 = 0$, which as shown above cannot be feasible for $SP_{MNL}^{5}$. However, $d_j = 0$ for $j \in J \setminus J$ can be feasible for $SP_{MNL}^{5}$.

Next we show that if $(d^*, d_0^*, u^*)$ is an optimal solution for $SP_{MNL}^{5}$, then it is also optimal for $SP_{MNL}^{4}$. It remains to show that $d_j^* > 0$ for every $j \in J \setminus J$. Suppose that $d_j^* = 0$ for some $i \in J \setminus J$. Let $K_i^+ := \{k \in K_i : \xi_{ik} > 0\}$ and $K_i^- := \{k \in K_i : \xi_{ik} < 0\}$. Consider any $\delta > 0$, $\min \{1, \min\{u_{ik}/\xi_{ik} : k \in K_i^+, u_{ik} > 0\}, \min\{u_{ik}/\xi_{ik} : k \in K_i^-, u_{ik} > 0\}\}$. Consider the solution $(d', d_0', u')$ for $SP_{MNL}^{4}$ given by

\[
\begin{align*}
d_i' &= \delta \\
d_0' &= d_0^* (1 - \delta) \\
d_j' &= d_j^* (1 - \delta) \quad \text{for all } j \in J \setminus \{i\} \\
u_{il}' &= u_{il}^* - \sum_{\{k \in K_i^+ \setminus \{l\} : u_{ik}^* = 0\}} \xi_{ik} \delta - \sum_{\{k \in K_i^- : u_{ik}^* = 0\}} \xi_{ik} \delta - \delta \ln \left( \frac{\delta}{d_0^* (1 - \delta)} \right) \\
u_{ik}' &= u_{ik}^* + \xi_{ik} \delta \\
u_{ik}' &= u_{ik}^* + \xi_{ik} \delta \\
u_{ik}' &= u_{ik}^* \\
u_{jk}' &= u_{jk}^* (1 - \delta) \\
\end{align*}
\]

for all $k \in K_i^+ \setminus \{l\} : u_{ik}^* = 0$

for all $k \in K_i^- : u_{ik}^* = 0$

for all $k \in K_i \setminus (\{l\} \cup \{k \in K_i^+ : u_{ik}^* = 0\} \cup \{k \in K_i^- : u_{ik}^* = 0\})$

for all $j \in J \setminus \{i\}, k \in K_j$

Note that, since $i \in J \setminus J$, it holds that $K_i \setminus K_i \neq \emptyset$, and thus an attribute $l \in K_i \setminus K_i$ exists.

Next we show that $(d', d_0', u')$ is feasible for $SP_{MNL}^{4}$.

\begin{itemize}
  \item Note that for all $j \in J \setminus \{i\}$, it holds that $d_j' / d_0' = d_j^* / d_0^*$, and thus $(d', d_0', u')$ satisfies (21).
  \item Next we show that $(d', d_0', u')$ satisfies (20).
\end{itemize}
Next, we show that

\[ \sum_{k \in K_i} u_{ik}' = u_{ik}' - \sum_{\{k \in K_i^+ \setminus \{l\} : u_{ik}' = 0\}} \bar{x}_{ik}\delta - \sum_{\{k \in K_i^- : u_{ik}' = 0\}} \bar{\pi}_{ik}\delta - \delta \ln \left( \frac{\delta}{d_0'(1 - \delta)} \right) \]

\[ + \sum_{\{k \in K_i^+ \setminus \{l\} : u_{ik}' = 0\}} (u_{ik}' + \bar{x}_{ik}\delta) + \sum_{\{k \in K_i^- : u_{ik}' = 0\}} (u_{ik}' + \bar{\pi}_{ik}\delta) \]

\[ + \sum_{k \in K_i \setminus \{l\} \cup \{k \in K_i^+ : u_{ik}' = 0\} \cup \{k \in K_i^- : u_{ik}' = 0\}} u_{ik}' \]

\[ = \sum_{k \in K_i} u_{ik}' - \delta \ln \left( \frac{\delta}{d_0'(1 - \delta)} \right) \leq -d_j' \ln \left( \frac{d_j'}{d_0'} \right) \]

and hence \((d', d_0', u')\) satisfies \((20)\) for \(j = i\).

For \(j \in J \setminus \{i\}\), if \(d_j^* > 0\) then it holds that

\[ \sum_{k \in K_j} u_{jk}' = (1 - \delta) \sum_{k \in K_j} u_{jk}' \leq (1 - \delta) d_j^* \ln \left( \frac{d_j^*}{d_0'} \right) = -d_j' \ln \left( \frac{d_j'}{d_0'} \right) \]

and if \(d_j' = 0\) then it holds that \(d_j' = 0\) and

\[ \sum_{k \in K_j} u_{jk}' = (1 - \delta) \sum_{k \in K_j} u_{jk}' \leq 0 \]

and hence \((d', d_0', u')\) satisfies \((20)\) for all \(j \in J\).

Next, we show that \((d', d_0', u')\) satisfies \((22)\).

Suppose that \(l \in K_j\). Let \(f(\delta) := -\sum_{\{k \in K_i^+ \setminus \{l\} : u_{ik}' = 0\}} \bar{x}_{ik}\delta - \sum_{\{k \in K_i^- : u_{ik}' = 0\}} \bar{\pi}_{ik}\delta - \delta \ln \left( \frac{\delta}{d_0'(1 - \delta)} \right) - \frac{\delta}{d_0'}\). Note that \(f'(\delta) = -\sum_{\{k \in K_i^+ \setminus \{l\} : u_{ik}' = 0\}} \bar{x}_{ik} - \sum_{\{k \in K_i^- : u_{ik}' = 0\}} \bar{\pi}_{ik} - \ln \left( \frac{\delta}{d_0'} \right) - 1 - \frac{\delta}{(1 - \delta) - \frac{\delta}{d_0'}} \rightarrow \infty \text{ as } \delta \downarrow 0.\) Thus, \(f'(s) > 0\) for all \(s > 0\) sufficiently small, hence \[u_{ik}' - \bar{x}_{ik}d_i^l - [u_{ik}' - \bar{x}_{il}d_i^l] = \int_0^\delta f'(s) \, ds > 0,\] and therefore \[u_{ik}' - \bar{x}_{ik}d_i^l > u_{ik}' - \bar{x}_{il}d_i^l \geq 0\] for all \(\delta > 0\) sufficiently small.

Consider any \(k \in K_i^+ \setminus \{l\}\). If \(u_{ik}^* = 0\), then \(u_{ik}' = \bar{x}_{ik}\delta = \bar{x}_{ik}d_i^l\). Otherwise, \(u_{ik}^* > 0\), then

\[ u_{ik}' = u_{ik}' - \bar{x}_{ik}\delta = \bar{x}_{ik}d_i^l. \]

Consider any \(k \in K_j \setminus (K_i^+ \cup \{l\})\). Then \(u_{ik}' = u_{ik}' \geq 0 \bar{x}_{ik}\delta = \bar{x}_{ik}d_i^l.\)

Next, consider any \(j \in J \setminus \{i\}\) and any \(k \in K_j\). Then \(u_{jk}' = u_{jk}'(1 - \delta) \geq \bar{x}_{jk}d_j^* (1 - \delta) = \bar{x}_{jk}d_j'.\) Thus, \((d', d_0', u')\) satisfies \((22)\) for all \(j \in J\) and \(k \in K_j\).

Next, we show that \((d', d_0', u')\) satisfies \((23)\).
Consider any $k \in \overline{K}_i$. If $u_{ik}^* = 0$, then $u_{ik}' = \overline{x}_{ik}d_i'$. Otherwise, $u_{ik}^* < 0$, then $u_{ik}' = u_{ik}^* < \overline{x}_{ik}d_i'$.

Consider any $k \in \overline{K}_i \setminus \overline{K}_j$. Then $u_{ik}' = u_{ik}^* \leq 0 \leq \overline{x}_{ik}d_i'$.

Consider any $j \in J \setminus \{i\}$ and any $k \in \overline{K}_j$. Then $u_{jk}' = u_{jk}^*(1 - \delta) \leq \overline{x}_{jk}d_j'(1 - \delta) = \overline{x}_{jk}d_j'$.

Hence $(d', d_0, u')$ satisfies (23) for all $j \in J$ and $k \in \overline{K}_j$.

Since $d_i' = 0$, it follows from (24) that $d_0^* + \sum_{j \in J \setminus \{i\}} d_j^* = 1$, and thus

$$d_0 + \sum_{j \in J} d_j' = d_0^*(1 - \delta) + \sum_{j \in J \setminus \{i\}} d_j^*(1 - \delta) = \left(d_0^* + \sum_{j \in J \setminus \{i\}} d_j^*\right)(1 - \delta) + \delta = 1 - \delta + \delta = 1$$

Hence $(d', d_0^*, u')$ satisfies (24), and therefore $(d', d_0^*, u')$ is feasible for $\mathcal{SP}_{MNL}^5$.

Next we show that the objective value of $\mathcal{SP}_{MNL}^5$ at $(d', d_0^*, u')$ is greater than the objective value at $(d^*, d_0^*, u^*)$. The change in objective value from $(d^*, d_0^*, u^*)$ to $(d', d_0^*, u')$ is

$$\sum_{j \in J} \left(\sum_{k \in K_j} \phi_k u_{jk}^* - \psi_{j} d_j^*\right) - \sum_{j \in J} \left(\sum_{k \in K_j} \phi_k u_{jk}^* - \psi_{j} d_j^*\right)$$

$$= \phi_l \left[u_{il}' - \sum_{\{k \in K_i^+ \setminus \{l\} : u_{ik}^* = 0\}} \overline{x}_{ik}d_i' - \sum_{\{k \in K_i^- : u_{ik}^* = 0\}} \overline{x}_{ik}d_i' - \delta \ln \left(\frac{\delta}{d_0^*(1 - \delta)}\right)\right]$$

$$+ \sum_{\{k \in K_i^+ \setminus \{l\} : u_{ik}^* = 0\}} \phi_k \left[u_{ik}^* + \overline{x}_{ik}d_i\right] + \sum_{\{k \in K_i^- : u_{ik}^* = 0\}} \phi_k \left[u_{ik}^* + \overline{x}_{ik}d_i\right]$$

$$+ \sum_{k \in K_i \setminus \{l\} : \{k \in K_i^+ : u_{ik}^* = 0\} \cup \{k \in K_i^- : u_{ik}^* = 0\}} \phi_k u_{ik}^* - \psi_{j} d_j^*$$

$$+ \sum_{j \in J \setminus \{i\}} \left(\sum_{k \in K_j} \phi_k u_{jk}^*(1 - \delta) - \psi_{j} d_j^*(1 - \delta)\right) - \sum_{j \in J} \left(\sum_{k \in K_j} \phi_k u_{jk}^* - \psi_{j} d_j^*\right)$$

$$= \int_0^1 g'(s) \ ds$$

where $g : (0, 1) \mapsto \mathbb{R}$ is given by

$$g(s) := \phi_l \left[u_{il}' - \sum_{\{k \in K_i^+ \setminus \{l\} : u_{ik}^* = 0\}} \overline{x}_{ik}s - \sum_{\{k \in K_i^- : u_{ik}^* = 0\}} \overline{x}_{ik}s - s \ln \left(\frac{s}{d_0^*(1 - s)}\right)\right]$$

$$+ \sum_{\{k \in K_i^+ \setminus \{l\} : u_{ik}^* = 0\}} \phi_k \left[u_{ik}^* + \overline{x}_{ik}s\right] + \sum_{\{k \in K_i^- : u_{ik}^* = 0\}} \phi_k \left[u_{ik}^* + \overline{x}_{ik}s\right]$$
\[
\begin{align*}
&+ \sum_{k \in K \setminus \{i\} : u_{ik}^* = 0} \sum_{k \in K \setminus \{i\} : u_{ik}^* = 0} \phi_k u_{ik}^* - \psi_i s \\
&+ \sum_{j \in J \setminus \{i\}} \left( \sum_{k \in K} \phi_k u_{jk}^* (1 - s) - \psi_j d_j^* (1 - s) \right)
\end{align*}
\]

Note that
\[
g'(s) = \phi_i \left[ - \sum_{k \in K \setminus \{i\} : u_{ik}^* = 0} x_{ik} - \sum_{k \in K \setminus \{i\} : u_{ik}^* = 0} x_{ik} - \ln \left( \frac{s}{d_j^* (1 - s)} \right) \right] - 1 - \frac{s}{1 - s}
\]
\[
+ \sum_{k \in K \setminus \{i\} : u_{ik}^* = 0} \phi_k x_{ik} + \sum_{k \in K \setminus \{i\} : u_{ik}^* = 0} \phi_k x_{ik} - \psi_i - \sum_{j \in J \setminus \{i\}} \left( \sum_{k \in K} \phi_k u_{jk}^* - \psi_j d_j^x \right) \to \infty \text{ as } s \downarrow 0
\]

Thus \(g'(s) > 0\) for all \(s > 0\) sufficiently small, and therefore for \(\delta > 0\) sufficiently small the objective value of \(\text{SP}_5^{\text{MNL}}\) at \((d', d_0', u')\) is greater than the objective value at \((d^*, d_0^*, u^*)\). This contradiction establishes that \(d_j^* > 0\) for all \(j \in J \setminus J\). In conclusion, it has been shown that \((d^*, d_0^*, u^*)\) is feasible for and therefore also optimal for \(\text{SP}_4^{\text{MNL}}\). \(\square\)
A.3 Proof of Lemma 3.4

Lemma 3.4. If $\text{FP}^{\text{MNL}}_M$ is feasible, then $\text{FP}^{\text{MNL}}_2$ has an optimal solution if and only if for each $j \in J$, $t \in \{0, 1, \ldots, T\}$, $k_1 \in K_{jt} \setminus K_{jt}$, and $k_2 \in K_{jt} \setminus K_{jt}$, it holds that $\phi_{k_1,t} \leq \phi_{k_2,t}$.

Proof. Suppose that there is a $j \in J$, a $t \in \{0, 1, \ldots, T\}$, a $k_1 \in K_{jt} \setminus K_{jt}$, and a $k_2 \in K_{jt} \setminus K_{jt}$, such that $\phi_{k_1,t} > \phi_{k_2,t}$. Consider any feasible solution $(d^0, d^0_0, u^0)$ and a sequence $\{(d^n, d^n_0, u^n)\}_{n=0}^{\infty}$ of feasible solutions with $u^n_{j,k_1,t} = u^n_{j,k_1,t} + n$, $u^n_{j,k_2,t} = u^n_{j,k_2,t} - n$, $u^n_{k,t} = u^n_{j,k,t}$ for all $k \notin \{k_1, k_2\}$, $u^n_{j',kt'} = u^n_{j',kt'}$ for all $j' \neq j$ or $t' \neq t$, and $k \in K_{j't'}$, $d^n_{j,t'} = d^n_{j,t'}$ for all $j' \in J$ and $t' \in \{0, 1, \ldots, T\}$, and $d^n_{0,t'} = d^n_{0,t'}$ for all $t' \in \{0, 1, \ldots, T\}$. Then $(d^n, d^n_0, u^n)$ is feasible for $\text{FP}^{\text{MNL}}_2$ for all $n$, and the objective value of $(d^{n+1}, d^{n+1}_0, u^{n+1})$ exceeds the objective value of $(d^n, d^n_0, u^n)$ by $\phi_{k_1,t} - \phi_{k_2,t}$ for all $n$, and thus the objective value of $(d^n, d^n_0, u^n)$ increases without bound as $n \to \infty$.

Suppose that, for each $j \in J$, $t \in \{0, 1, \ldots, T\}$, and $k_1 \in K_{jt} \setminus K_{jt}$, $k_2 \in K_{jt} \setminus K_{jt}$, it holds that $\phi_{k_1,t} \leq \phi_{k_2,t}$. Let problem $\text{FP}^{\text{MNL}}_3$ be the same as problem $\text{FP}^{\text{MNL}}_2$ with the resource constraint (25) removed. Thus, $\text{FP}^{\text{MNL}}_3$ is a collection of separate problems, one problem for each $t \in \{0, 1, \ldots, T\}$, similar to $\text{SP}^{\text{MNL}}_5$. By the proof of Lemma 3.1, $\text{FP}^{\text{MNL}}_3$ is feasible, the dual of $\text{FP}^{\text{MNL}}_3$ is bounded, and has a feasible solution in the interior of the dual cones.

Since $\text{FP}^{\text{MNL}}_2$ is feasible, the dual of $\text{FP}^{\text{MNL}}_2$ is bounded. Also, since $\text{FP}^{\text{MNL}}_2$ is the same as $\text{FP}^{\text{MNL}}_3$ with additional linear inequality constraints, the dual of $\text{FP}^{\text{MNL}}_2$ has additional signed variables beyond the dual of $\text{FP}^{\text{MNL}}_3$. Then a feasible solution of the dual of $\text{FP}^{\text{MNL}}_3$ in the interior of the dual cones, combined with these additional signed variables set to 0, gives a feasible solution of the dual of $\text{FP}^{\text{MNL}}_2$ in the interior of the dual cones. Then it follows from the conic duality theorem (Theorem 1.4.2 in Ben-Tal and Nemirovski (2001)) that $\text{FP}^{\text{MNL}}_2$ has an optimal solution. \qed
A.4 Proof of Lemma 3.5

**Lemma 3.5.** Suppose that \( \text{FP}^\text{MNL}_2 \) has a feasible solution \((d', d'_0, u')\) with \(d' > 0\). Then, any optimal solution \((d^*, d'_0, u^*)\) for \( \text{FP}^\text{MNL}_2 \) satisfies \(d^* > 0, d'_0 > 0\).

Proof. First, suppose that \(d^*_0 = 0\) for some \(t \in \{0, 1, \ldots, T\}\). Then it follows from (26) that \(d^*_j = 0\) for all \(j \in J\), which violates (30). Thus \(d^*_0 > 0\) for all \(t \in \{0, 1, \ldots, T\}\).

Next, suppose that \(d^*_j = 0\) for some \(t \in \{0, 1, \ldots, T\}\) and some \(j \in \tilde{J}_t\). Then it follows from (27) that \(d^*_0 = 0\). It was shown above that \(d^*_0 > 0\) for all \(t \in \{0, 1, \ldots, T\}\), and thus \(d^*_j > 0\) for all \(t \in \{0, 1, \ldots, T\}\) and \(j \in \tilde{J}_t\).

Next, suppose that \(d^*_i = 0\) for some \(s \in \{0, 1, \ldots, T\}\) and some \(i \in J \setminus \tilde{J}_s\). Note that it follows from \(\phi_{ks} > 0\) for all \(k \in K_{is}, i \in J \setminus \tilde{J}_s\), and (26), that \(\sum_{k \in K_{is}} u^*_{iks} = 0\). Let \((d', d'_0, u')\) be a feasible solution of \( \text{FP}^\text{MNL}_2 \) with \(d' > 0\). By the same argument used above, it holds that \(d'_0 > 0\). Consider any \(\delta \in (0, 1)\). Since the feasible set of \( \text{FP}^\text{MNL}_2 \) is convex, it follows that \((1-\delta)(d^*, d'_0, u^*) + \delta(d', d'_0, u')\) is feasible for \( \text{FP}^\text{MNL}_2 \). Consider the feasible solution \((\hat{d}, \hat{d}_0, \hat{u})\) for \( \text{FP}^\text{MNL}_2 \) constructed to be the same as \((1-\delta)(d^*, d'_0, u^*) + \delta(d', d'_0, u')\), except for a single variable \(u_{ils}\), as follows: Since \(i \in J \setminus \tilde{J}_s\), it holds that \(K_{is} \setminus \tilde{K}_{is} \neq \emptyset\). Choose any \(l \in K_{is} \setminus \tilde{K}_{is}\). Let \(\hat{d}_{jt} = (1-\delta)d^*_j + \delta d'_j\) for all \(j \in J\) and \(t \in \{0, 1, \ldots, T\}\), \(\hat{d}_{0t} = (1-\delta)d^*_0 + \delta d'_0 > 0\) for all \(t \in \{0, 1, \ldots, T\}\), and \(\hat{u}_{jkt} = (1-\delta)u^*_j + \delta u'_{jkt}\) for all \(j \in J, t \in \{0, 1, \ldots, T\}\), and \(k \in K_{jt}\) such that \((j, k, t) \neq (i, l, s)\). Note that \(\hat{d}_i = (1-\delta)d^*_i + \delta d'_i = \delta d'_i > 0\). Finally, let

\[
\hat{u}_{ils} = \hat{d}_i \ln \left( \frac{\hat{d}_0}{\hat{d}_i} \right) - \sum_{k \in K_{is} \setminus \{l\}} \hat{u}_{iks} = \delta d'_i \ln \left( \frac{(1-\delta)d'^*_0 + \delta d'_0}{\delta d'^*_i} \right) - \sum_{k \in K_{is} \setminus \{l\}} [(1-\delta)u^*_ks + \delta u'_{iks}]
\]

Note that, by choice of \(\hat{u}_{ils}\), (26) is satisfied for \(i \in J \) and \(s \in \{0, 1, \ldots, T\} \). If \(l \in \tilde{K}_{is}\), then note that, since \((1-\delta)(d^*, d'_0, u^*) + \delta(d', d'_0, u')\) is feasible for \( \text{FP}^\text{MNL}_2 \), it follows from (28) and (26) that

\[
\begin{align*}
\bar{w}_{ils} \hat{d}_i &= \bar{w}_{ils} [(1-\delta)d^*_i + \delta d'^*_i] \\
&\leq (1-\delta)u^*_ils + \delta u'_{ils} \\
&\leq [(1-\delta)d^*_i + \delta d'^*_i] \ln \left( \frac{(1-\delta)d^*_0 + \delta d'_0}{(1-\delta)d^*_i + \delta d'^*_i} \right) - \sum_{k \in K_{is} \setminus \{l\}} [(1-\delta)u^*_iks + \delta u'_{iks}] \\
&= \hat{d}_i \ln \left( \frac{\hat{d}_0}{\hat{d}_i} \right) - \sum_{k \in K_{is} \setminus \{l\}} \hat{u}_{iks} = \hat{u}_{ils}
\end{align*}
\]

and thus \(\hat{u}_{ils}\) satisfies (28). In addition, since \(l \in K_{is} \setminus \tilde{K}_{is}\), the variable \(u_{ils}\) is not upper bounded. Hence \((\hat{d}, \hat{d}_0, \hat{u})\) is feasible for \( \text{FP}^\text{MNL}_2 \).
The difference between the objective values of $\text{FP}_2^{\text{MNL}}$ at $(\hat{d}, \hat{d}_0, \hat{u})$ and $(d^*, d_0^*, u^*)$ is

$$
\begin{align*}
&\sum_{t=0}^{T} \lambda_t \sum_{j \in J} \left( \sum_{k \in K_{jt}} \phi_{kt} u_{jkt} - \psi_{jt} d_{jt} \right) - \sum_{t=0}^{T} \lambda_t \sum_{j \in J} \left( \sum_{k \in K_{jt}} \phi_{kt} u_{jkt}^* - \psi_{jt} d_{jt}^* \right) \\
&= \sum_{t=0}^{T} \lambda_t \sum_{j \in J} \left( \sum_{k \in K_{jt}} \phi_{kt} \left[ (1 - \delta)u_{jkt} + \delta u_{jkt}' \right] - \psi_{jt} \left[ (1 - \delta)d_{jt}^* + \delta d_{jt}' \right] \right) \\
&\quad + \lambda_s \phi_{\text{ls}} \left\{ \delta d_{\text{ls}}' \ln \left( \frac{(1 - \delta)d_{0s}' + \delta d_{0s}}{d_{\text{ls}}'} \right) - \sum_{k \in K_{\text{ls}}} \left[ (1 - \delta)u_{\text{iks}}' + \delta u_{\text{iks}} \right] - \left[ (1 - \delta)u_{\text{ils}}' + \delta u_{\text{ils}} \right] \right\} \\
&\quad - \sum_{t=0}^{T} \lambda_t \sum_{j \in J} \left( \sum_{k \in K_{jt}} \phi_{kt} u_{jkt} - \psi_{jt} d_{jt} \right) \\
&= \sum_{t=0}^{T} \lambda_t \sum_{j \in J} \left( \sum_{k \in K_{jt}} \phi_{kt} u_{jkt}^* - \psi_{jt} d_{jt}^* \right) + \delta \sum_{t=0}^{T} \lambda_t \sum_{j \in J} \left( \sum_{k \in K_{jt}} \phi_{kt} \left[ u_{jkt}^* - u_{jkt} \right] - \psi_{jt} \left[ d_{jt}^* - d_{jt} \right] \right) \\
&\quad + \lambda_s \phi_{\text{ls}} \left\{ \delta d_{\text{ls}}' \ln \left( \frac{(1 - \delta)d_{0s}' + \delta d_{0s}}{d_{\text{ls}}'} \right) - \delta \sum_{k \in K_{\text{ls}}} u_{\text{iks}}' - u_{\text{iks}} \right\} \\
&\quad - \sum_{t=0}^{T} \lambda_t \sum_{j \in J} \left( \sum_{k \in K_{jt}} \phi_{kt} u_{jkt} - \psi_{jt} d_{jt} \right) \\
&= \delta \sum_{t=0}^{T} \lambda_t \sum_{j \in J} \left( \sum_{k \in K_{jt}} \phi_{kt} \left[ u_{jkt}^* - u_{jkt} \right] - \psi_{jt} \left[ d_{jt}^* - d_{jt} \right] \right) \\
&\quad + \lambda_s \phi_{\text{ls}} \left\{ \delta d_{\text{ls}}' \ln \left( \frac{(1 - \delta)d_{0s}' + \delta d_{0s}}{d_{\text{ls}}'} \right) - \delta \sum_{k \in K_{\text{ls}}} u_{\text{iks}}' - u_{\text{iks}} \right\} \\
&= \int_{0}^{\delta} h'(v) \, dv
\end{align*}
$$

where the third equality follows from the result that $\sum_{k \in K_{\text{ls}}} u_{\text{iks}}' = 0$, and $h : (0, 1) \mapsto \mathbb{R}$ is given by

$$
\begin{align*}
&h(v) = v \sum_{t=0}^{T} \lambda_t \sum_{j \in J} \left( \sum_{k \in K_{jt}} \phi_{kt} \left[ u_{jkt}^* - u_{jkt} \right] - \psi_{jt} \left[ d_{jt}^* - d_{jt} \right] \right) \\
&\quad + \lambda_s \phi_{\text{ls}} \left\{ v d_{\text{ls}}' \ln \left( \frac{(1 - v)d_{0s}' + v d_{0s}}{vd_{\text{ls}}'} \right) - v \sum_{k \in K_{\text{ls}}} u_{\text{iks}}' - u_{\text{iks}} \right\}
\end{align*}
$$

Note that

$$
\begin{align*}
&h'(v) = \sum_{t=0}^{T} \lambda_t \sum_{j \in J} \left( \sum_{k \in K_{jt}} \phi_{kt} \left[ u_{jkt}^* - u_{jkt} \right] - \psi_{jt} \left[ d_{jt}^* - d_{jt} \right] \right)
\end{align*}
$$
+ \lambda_s \phi_{is} \left\{ d'_{is} \ln \left( \frac{(1-v)d^*_0 + vd^*_s}{vd'_{is}} \right) - \frac{d'_{is}d^*_0}{(1-v)d^*_0 + vd^*_s} - \sum_{k \in \mathcal{K}_{is}} [u'_{iks} - u^*_{iks}] \right\} \to \infty \text{ as } v \downarrow 0

Thus $h'(v) > 0$ for all $v > 0$ sufficiently small, and therefore for $\delta > 0$ sufficiently small the objective value of $\text{FP}_2^{\text{MNL}}$ at $(\hat{d}, \hat{d}_0, \hat{u})$ is greater than the objective value at $(d^*, d^*_0, u^*)$. This contradiction establishes that $d^*_{jt} > 0$ for all $t \in \{0, 1, \ldots, T\}$ and $j \in \mathcal{J} \setminus \mathcal{J}_t$. In conclusion, it has been shown that $d^* > 0, d^*_0 > 0$ for any optimal solution $(d^*, d^*_0, u^*)$ of $\text{FP}_2^{\text{MNL}}$. \(\square\)
A.5 Proof of Theorem 3.7

Theorem 3.7. \textit{FP} can be solved by solving \textit{FP} and taking into account the following:

1. If \textit{FP} is infeasible, then \textit{FP} is infeasible.

2. If \textit{FP} is feasible, then:
   
   (a) \textit{FP} is bounded, and has an optimal solution.
   
   (b) \textit{FP} is bounded, and has an optimal solution.
   
   (c) Let \((v^*,d^*,u^*)\) be an optimal solution of \textit{FP}. Let \(x_{jkt}^* = u_{jkt}^*/d_{jkt}^*\) for every \(j \in J\), \(t \in \{0,1,\ldots,T\}\), and \(k \in K_{jt}\) such that \(d_{jkt}^* > 0\), and let \(x_{jkt}^* = 0\) otherwise. Then \((v^*,d^*,x^*)\) is an optimal solution of \textit{FP}.

Proof. First, consider any \((v,d,u)\) that is feasible to \textit{FP}. Consider any \(j \in J\) and \(t \in \{0,1,\ldots,T\}\) such that \(d_{jkt} > 0\) (which implies \(v_{jkt} > 0\)). By constraint (31c), we know

\[ d_{jkt} \ln \left( \frac{v_{jkt}}{d_{jkt}} \right) \geq \sum_{k \in K_{jt}} u_{jkt} \]

Suppose that

\[ d_{jkt} \ln \left( \frac{v_{jkt}}{d_{jkt}} \right) > \sum_{k \in K_{jt}} u_{jkt} \]

If \(j \in J_t\), then by constraint (31d), we know

\[ \sum_{k \in K_{jt}} x_{jkt} d_{jkt} \geq d_{jkt} \ln \left( \frac{v_{jkt}}{d_{jkt}} \right) \geq \sum_{k \in K_{jt}} u_{jkt} \]

otherwise, \(\sum_{k \in K_{jt}} u_{jkt}\) is not upper bounded by any constraint other than (31c). In either case, we can increase \(u_{jkt}\) for some \(k\) while let \(v_{jkt}\) and \(d_{jkt}\) remain unchanged, and get a feasible solution with a better objective value than \((v,d,u)\). Thus, an optimal solution for \textit{FP} satisfies

\[ d_{jkt} \ln \left( \frac{v_{jkt}}{d_{jkt}} \right) = \sum_{k \in K_{jt}} u_{jkt} \]

for any \(j \in J\) and \(t \in \{0,1,\ldots,T\}\) such that \(d_{jkt} > 0\).

Second, consider any \((v,d,x)\) that is feasible to \textit{FP}. Let \(u_{jkt} = x_{jkt} d_{jkt}\) for every \(j \in J\), \(t \in \{0,1,\ldots,T\}\), and \(k \in K_{jt}\). Then the solution \((v,d,x)\) is feasible to \textit{FP} with the same
objective value as \((v, d, x)\) in \(\text{FP}_1\). On the other hand, consider any \((v, d, u)\) that is feasible to \(\text{FP}_2\). Then (by the previous paragraph) there is a solution \((v, d, u')\) that satisfies
\[
d_{jt} \ln \left( \frac{v_{jt}}{d_{jt}} \right) = \sum_{k \in K_{jt}} u'_{jkt}
\]
for any \(j \in J\) and \(t \in \{0, 1, \ldots, T\}\) such that \(d_{jt} > 0\), whose objective value is at least as good as \((v, d, u)\). Let \(x_{jkt} = u'_{jkt} / d_{jt}\) for every \(j \in J\), \(t \in \{0, 1, \ldots, T\}\), and \(k \in K_{jt}\) such that \(d_{jt} > 0\), and let \(x_{jkt} = x_{jkt}\) otherwise. Then \((v, d, x)\) is a feasible solution of \(\text{FP}_1\) whose objective value is at least as good as \((v, d, u)\) in \(\text{FP}_2\).

Thus, we know that:

1. \(\text{FP}_2\) is infeasible if and only if \(\text{FP}_1\) is infeasible.
2. \(\text{FP}_2\) is feasible and unbounded if and only if \(\text{FP}_1\) is feasible and unbounded.
3. \(\text{FP}_2\) is feasible and bounded if and only if \(\text{FP}_1\) is feasible and bounded.
4. \(\text{FP}_2\) has an optimal solution if and only if \(\text{FP}_1\) has an optimal solution.
5. Let \((v^*, d^*, u^*)\) be an optimal solution of \(\text{FP}_2\). Then
\[
d^*_{jt} \ln \left( \frac{v^*_{jt}}{d^*_{jt}} \right) = \sum_{k \in K_{jt}} u^*_{jkt}
\]
for any \(j \in J\) and \(t \in \{0, 1, \ldots, T\}\) such that \(d^*_{jt} > 0\).
6. Let \((v^*, d^*, u^*)\) be an optimal solution of \(\text{FP}_2\). Let \(x^*_{jkt} = u^*_{jkt} / d^*_{jt}\) for every \(j \in J\), \(t \in \{0, 1, \ldots, T\}\), and \(k \in K_{jt}\) such that \(d^*_{jt} > 0\), and let \(x^*_{jkt} = x^*_{jkt}\) otherwise. Then \((v^*, d^*, x^*)\) is an optimal solution of \(\text{FP}_1\).

It left to show that an optimal solution exists for \(\text{FP}_2\) if \(\text{FP}_2\) is feasible. Indeed. Let problem \(\text{FP}_3\) be the same as problem \(\text{FP}_2\) with the resource constraints removed. Thus, \(\text{FP}_3\) is a collection of separate problems, one problem for each \(t \in \{0, 1, \ldots, T\}\). Each of the separate
problem have the following formulation:

$$\max_{v, d, u} \sum_{j \in J} \left( \sum_{k \in K_j} \phi_k u_{jk} - \psi_j d_j \right)$$ (SP_MC^1)

s.t. \quad v_j = \theta_j + \sum_{i \in J} \rho_{ij} (v_i - d_i) \quad \forall j \in J \quad (34a)

$$\left( v_j, d_j, \sum_{k \in K_j} u_{jk} \right) \in \mathcal{K}_{\exp} \quad \forall j \in J \quad (34b)$$

$$\left( d_j, v_j, -\sum_{k \in K_j} x_{jk} v_j \right) \in \mathcal{K}_{\exp} \quad \forall j \in J \quad (34c)$$

$$u_{jk} \geq x_{jk} d_j \quad \forall j \in J, k \in K_j \quad (34d)$$

$$u_{jk} \leq x_{jk} d_j \quad \forall j \in J, k \in K_j \quad (34e)$$

and is feasible. The dual problem of SP_MC^1 can be written as:

$$\min_{\pi, \varpi, \nu, \eta, \nu} \sum_{j \in J} \theta_j \eta_j$$ (SD_MC^1)

s.t. \quad \pi_{j1} + \varpi_{j2} - \varpi_{j3} \sum_{k \in K_j} \nu_{jk} - \eta_j + \sum_{i \in J} \rho_{ji} \eta_i = 0 \quad \forall j \in J \quad (35a)

$$\pi_{j1} - \eta_j + \sum_{i \in J} \rho_{ji} \eta_i = 0 \quad \forall j \in J \setminus J \quad (35b)$$

$$\pi_{j2} + \varpi_{j1} + \sum_{k \in K_j} \nu_{jk} x_{jk} - \sum_{k \in K_j} \nu_{jk} x_{jk} - \sum_{i \in J} \rho_{ji} \eta_i = \psi_j \quad \forall j \in J \quad (35c)$$

$$\pi_{j2} + \sum_{k \in K_j} \nu_{jk} x_{jk} - \sum_{k \in K_j} \nu_{jk} x_{jk} - \sum_{i \in J} \rho_{ji} \eta_i = \psi_j \quad \forall j \in J \setminus J \quad (35d)$$

$$\pi_{j3} + \nu_{jk} - \nu_{jk} = -\phi_k \quad \forall j \in J, k \in K_j \quad (35e)$$

$$\pi_{j3} + \nu_{jk} = -\phi_k \quad \forall j \in J, k \in K_j \setminus K_j \quad (35f)$$

$$\pi_j \in \mathcal{K}_{\exp}^* \quad \forall j \in J \quad (35g)$$

$$\varpi_j \in \mathcal{K}_{\exp}^* \quad \forall j \in J \quad (35h)$$

$$\nu_{jk} \geq 0 \quad \forall j \in J, k \in K_j \quad (35i)$$

$$\nu_{jk} \geq 0 \quad \forall j \in J, k \in K_j \quad (35j)$$

The refined conic duality theorem (see Ben-Tal and Nemirovski (2001)) states that SP_MC^1 has an optimal solution if its dual problem SD_MC^1 is bounded below and is “strictly feasible” (i.e. it has a feasible solution in the interior of the cones). Since SP_MC^1 is feasible, we know that SD_MC^1 is bounded below. Next we show that SD_MC^1 has a feasible solution such that the conic variables are all in the
interior of $K_{\exp}^*$. First, let

$$
\nu_{jk} = 0 \quad \forall j \in J, k \in K_j
$$

$$
\nu_{jk} = \max_k \{ \phi_k \} - \phi_k \geq 0 \quad \forall j \in J, k \in K_j
$$

$$
\pi_{j1} = 1 + \max_k \{ \phi_k \} \cdot \exp \left( - \frac{\psi_j}{\max_k \{ \phi_k \}} - 1 \right) > 0 \quad \forall j \in J
$$

$$
\pi_{j2} = \psi_j + \sum_{k \in K_j} \nu_{jk} x_{jk} + \sum_{i \in J} \rho_{ji} \eta_i \geq \psi_j \quad \forall j \in J
$$

$$
\pi_{j3} = - \max_k \{ \phi_k \} < 0 \quad \forall j \in J
$$

$$
\varpi = 0
$$

where $\eta = (1 - \rho)^{-1} \pi_{j1} \geq 0$ is a solution to constraints (33a) and (35b). (Recall that $I - \rho$ needs to be non-singular and have nonnegative entries for the MC model). Clearly, the solution we constructed is feasible. Meanwhile, for each $j \in J$, we have

$$
\pi_{j1} = 1 + \max_k \{ \phi_k \} \cdot \exp \left( - \frac{\psi_j}{\max_k \{ \phi_k \}} - 1 \right)
$$

$$
\geq 1 + \max_k \{ \phi_k \} \cdot \exp \left( - \frac{\pi_{j2}}{\max_k \{ \phi_k \}} - 1 \right)
$$

$$
= 1 - \pi_{j3} \cdot \exp \left( \frac{\pi_{j2}}{\pi_{j3}} - 1 \right)
$$

$$
> -\pi_{j3} \cdot \exp \left( \frac{\pi_{j2}}{\pi_{j3}} - 1 \right)
$$

Thus, $\pi_j > \kappa_{\exp}^*$ for all $j \in J$. Then consider each $j \in J$. From the current solution, we increase $\varpi_{j1}$ and decrease $\varpi_{j3}$ by some small enough $\epsilon > 0$. To stay feasible, we need to decrease $\pi_{j1}$ by $\epsilon \sum_{k \in K_j} \pi_{jk}$ and decrease $\pi_{j2}$ by $\epsilon$. Let $(\pi', \varpi', \nu', \nu', \eta')$ be the updated solution, which is also feasible. For any $\epsilon > 0$, we have $\varpi_j' > \kappa_{\exp}^*$. Meanwhile, if $\epsilon$ is small enough, then $\pi_j'$ will stay in the interior of $K_{\exp}^*$ (as it is close enough to $\pi_j$). Thus, $\text{SP}_{1}^{MC}$ has an optimal solution.

Now, since $\text{FP}_{2}^{MC}$ is feasible, the dual of $\text{FP}_{2}^{MC}$ is bounded. Also, since $\text{FP}_{3}^{MC}$ is the same as $\text{FP}_{2}^{MC}$ with additional linear inequality constraints, the dual of $\text{FP}_{3}^{MC}$ has additional signed variables beyond the dual of $\text{FP}_{2}^{MC}$. Then a feasible solution of the dual of $\text{FP}_{3}^{MC}$ in the interior of the dual cones, combined with these additional signed variables set to 0, gives a feasible solution of the dual of $\text{FP}_{2}^{MC}$ in the interior of the dual cones. Then it follows from the conic duality theorem (Theorem 1.4.2 in Ben-Tal and Nemirovski (2001)) that $\text{FP}_{2}^{MC}$ has an optimal solution. \qed
A.6 Proof of Theorem 3.8

Theorem 3.8. If the following conditions hold:

Condition (i): for any feasible solution \((d,p,p_0, v_\leq, u_>)\) to \(\text{FP}_{NL}^1\) at which
\[
\left(\frac{1}{\gamma_{it}} - 1\right) d_{it} \ln \left(\frac{p_{it}}{p_{it}}\right) + \sum_{j \in J_i} d_{jt} \ln \left(\frac{p_{jt}}{d_{jt}}\right) = \sum_{k \in K_{it}} v_{ikt} \quad \forall i \in I_\leq, \ t = 0,1,\ldots,T
\]
the system
\[
\left(\frac{1}{\gamma_{it}} - 1\right) d_{jt} \ln \left(\frac{p_{it}}{p_{jt}}\right) + d_{jt} \ln \left(\frac{p_{jt}}{d_{jt}}\right) = \sum_{k \in K_{jt}} u_{jkt} \quad \forall i \in I_\leq, \ j \in J_i, \ t = 0,1,\ldots,T
\]
\[
v_{ikt} = \sum_{j \in J_{ikt}} u_{jkt} \quad \forall i \in I_\leq, \ k \in K_{it}, \ t = 0,1,\ldots,T
\]
\[
u_{jkt} \geq x_{jkt} d_{jt} \quad \forall j \in J_>, \ k \in K_{jt}, \ t = 0,1,\ldots,T
\]
\[
u_{jkt} \leq x_{jkt} d_{jt} \quad \forall j \in J_>, \ k \in K_{jt}, \ t = 0,1,\ldots,T
\]
has a solution \(u_\leq\). Here \(J_{ikt} := \{j \in J_i : k \in K_{jt}\}\).

Condition (ii): \(K_{jt} \subsetneq K_{jt}\) for every \(j \in J_>, \ i.e. \sum_{k \in K_{jt}} u_{jkt}\) is not upper bounded for any \(j \in J_>\).

then \(\text{FP}_{NL}^1\) can be solved by solving \(\text{FP}_{NL}^2\) and taking into account the following possibilities:

1. If \(\text{FP}_{NL}^2\) is infeasible, then \(\text{FP}_{NL}^1\) is infeasible.
2. If \(\text{FP}_{NL}^2\) is feasible and unbounded, then \(\text{FP}_{NL}^1\) is feasible and unbounded.
3. If \(\text{FP}_{NL}^2\) is feasible and has an optimal solution \((d^*, p^*, p_0^*, v_\leq^*, u_>)\), then \((d^*, p^*, x^*)\) with \(x^*\) given by \(x_{jkt}^* = u_{jkt}^*/d_{jt}^*\) for every \(j \in J, \ t \in \{0,1,\ldots,T\}\), and \(k \in K_{jt}\), is an optimal solution for \(\text{FP}_{NL}^1\). (Here \(u_>^*\) is part of the optimal solution \((d^*, p^*, p_0^*, v_\leq^*, u_>)\) to \(\text{FP}_{NL}^2\) while \(u_\leq^*\) is obtained by solving the system in Condition (i), using \((d^*, p^*, p_0^*, v_\leq^*, u_>)\) as an input.)

Proof. First, consider any feasible solution \((d,p,x)\) to \(\text{FP}_{NL}^1\) Let
\[
p_0 = 1 - \sum_{i \in I} p_i
\]
\[
v_{ikt} = \sum_{j \in J_{ikt}} x_{jkt} d_{jt} \quad \forall i \in I_\leq, \ k \in K_{it}, \ t = 0,1,\ldots,T
\]
\[
u_{jkt} = x_{jkt} d_{jt} \quad \forall j \in J_>, \ k \in K_{jt}, \ t = 0,1,\ldots,T
\]
Then \((d, p, p_0, v, u)\) is a feasible solution to \(\text{FP}_{2\text{NL}}\) with the same objective value as \((d, p, x)\) in \(\text{FP}_{1\text{NL}}\). Second, consider any feasible solution \((d, p, p_0, v, u)\) to \(\text{FP}_{2\text{NL}}\). Then we can find a solution \((d, p, p_0, v', u')\) (by increased some components in \(v\) and \(u\)) whose objective value is at least as good as \((d, p, p_0, v, u)\), such that

\[
\begin{align*}
\left(\frac{1}{\gamma_{it}} - 1\right) p_{it} \ln \left(\frac{p_{ot}}{p_{it}}\right) + \sum_{j \in \mathcal{J}_i} d_{jt} \ln \left(\frac{p_{ot}}{d_{jt}}\right) &= \sum_{k \in \mathcal{K}_{it}} v'_{ikt} \quad \forall i \in \mathcal{I}_\leq, t = 0, 1, \ldots, T \\
\left(1 - \frac{1}{\gamma_{it}}\right) d_{jt} \ln \left(\frac{p_{it}}{d_{jt}}\right) + \left(\frac{1}{\gamma_{it}}\right) d_{jt} \ln \left(\frac{p_{ot}}{d_{jt}}\right) &= \sum_{k \in \mathcal{K}_{jt}} u'_{jkt} \quad \forall i \in \mathcal{I}_>, j \in \mathcal{J}_i, t = 0, 1, \ldots, T
\end{align*}
\]

(The argument above can be proved the same way as in the proof of Theorem 3.7.) Let \(u'_{\leq}\) be a solution of the system in Condition (i). Let

\[
x_{jkt} = \begin{cases} 
  u'_{jkt}/d_{jt} & \text{if } d_{jt} > 0 \\
  \text{anything that satisfies the upper and lower bound (if any)} & \text{if } d_{jt} = 0
\end{cases}
\]

for each \(j \in \mathcal{J}_>, k \in \mathcal{K}_{jt}\) and \(t = 0, 1, \ldots, T\). The \((d, p)\) is a feasible solution to \(\text{FP}_{1\text{NL}}\) with the same objective value as \((d, p, p_0, v, u)\) in \(\text{FP}_{2\text{NL}}\), which is at least as good as the objective value of \((d, p, p_0, v, u)\). Thus, we know that:

1. If \(\text{FP}_{2\text{NL}}\) is infeasible, then \(\text{FP}_{1\text{NL}}\) is infeasible.
2. If \(\text{FP}_{2\text{NL}}\) is feasible and unbounded, then \(\text{FP}_{1\text{NL}}\) is feasible and unbounded.
3. If \(\text{FP}_{2\text{NL}}\) is feasible and has an optimal solution \((d^*, p^*, p_0^*, v^*_\leq, u^*_>)\), then \((d^*, p^*, x^*)\) with \(x^*\) given by \(x^*_{jkt} = u^*_jkt/d^*_j\) for every \(j \in \mathcal{J}, t \in \{0, 1, \ldots, T\}\), and \(k \in \mathcal{K}_{jt}\), is an optimal solution for \(\text{FP}_{1\text{NL}}\) (Here \(u^*_>\) is part of the optimal solution \((d^*, p^*, p_0^*, v^*_\leq, u^*_>)\) to \(\text{FP}_{2\text{NL}}\) while \(u^*_\leq\) is obtained by solving the system in Condition (i), using \((d^*, p^*, p_0^*, v^*_\leq, u^*_>)\) as an input.)
A.7 Proof of Theorem 3.9

Theorem 3.9 Assume that Condition (i) in Theorem 3.5 holds.

Then, $\text{FP}_{NL}^3$ can be solved by solving $\text{FP}_{NL}^2$ and taking into account the following:

1. If $\text{FP}_{NL}^3$ is infeasible, then $\text{FP}_{NL}^2$ is infeasible.

2. If $\text{FP}_{NL}^3$ is feasible, then the following possibilities hold:
   
   (a) If $\text{FP}_{NL}^3$ is unbounded (which happens if and only if for some $j \in J$, $t \in \{0,1,\ldots,T\}$, $k_1 \in K_{jt} \setminus K_{jt}'$, and $k_2 \in K_{jt} \setminus K_{jt}'$, it holds that $\phi_{k_1,t} > \phi_{k_2,t}$), then the following possibilities hold:
      
      i. If $\text{FP}_{NL}^3$ does not have a feasible solution $(d', p', p_0', v_{\leq}', u_{>}', e', f_{\leq}', g_{>}', r_{\leq}', s_{>}')$ with $d' > 0$, then $\text{FP}_{NL}^2$ is infeasible.
     
      ii. If $\text{FP}_{NL}^3$ has a feasible solution $(d', p', p_0', v_{\leq}', u_{>}', e', f_{\leq}', g_{>}', r_{\leq}', s_{>}')$ with $d' > 0$, then $\text{FP}_{NL}^2$ is unbounded.
   
   (b) If $\text{FP}_{NL}^3$ has an optimal solution $(d^*, p^*, p_0^*, v_{\leq}^*, u_{>}^*, e^*, f_{\leq}^*, g_{>}^*, r_{\leq}^*, s_{>}^*)$ (which happens if and only if for each $j \in J$, $t \in \{0,1,\ldots,T\}$, $k_1 \in K_{jt} \setminus K_{jt}'$, and $k_2 \in K_{jt} \setminus K_{jt}'$, it holds that $\phi_{k_1,t} \leq \phi_{k_2,t}$), then the following possibilities hold:
      
      i. If $d^* > 0$, then $(d^*, p^*, p_0^*, v_{\leq}^*, u_{>}^*)$ is an optimal solution for $\text{FP}_{NL}^2$
     
      ii. If $d_{jt}^* = 0$ for some $j \in J$ and $t \in \{0,1,\ldots,T\}$, then $\text{FP}_{NL}^2$ is infeasible.

Proof. Same as Theorem 3.6. Theorem 3.9 can be proved by showing two facts:

1. If $\text{FP}_{NL}^3$ is feasible, then $\text{FP}_{NL}^3$ has an optimal solution if and only if for each $j \in J$, $t \in \{0,1,\ldots,T\}$, $k_1 \in K_{jt} \setminus K_{jt}'$, and $k_2 \in K_{jt} \setminus K_{jt}'$, it holds that $\phi_{k_1,t} \leq \phi_{k_2,t}$.

2. Suppose that $\text{FP}_{NL}^3$ has a feasible solution $(d', d_0', u')$ with $d' > 0$. Then, any optimal solution $(d^*, p^*, p_0^*, v_{\leq}^*, u_{>}^*, e^*, f_{\leq}^*, g_{>}^*, r_{\leq}^*, s_{>}^*)$ for $\text{FP}_{NL}^3$ satisfies $d^* > 0$, $d_0^* > 0$.

The second result can be shown in the same way as in the proof of Lemma 3.5. On the other hand, if there is a $j \in J$, a $t \in \{0,1,\ldots,T\}$, a $k_1 \in K_{jt} \setminus K_{jt}'$, and a $k_2 \in K_{jt} \setminus K_{jt}'$, such that
\( \phi_{k_1,t} > \phi_{k_2,t} \), then \( \text{FP}_{3}^{NL} \) is unbounded when feasible. This can be shown by constructing a sequence of feasible solutions with objective values going to infinity, same as in the proof of Lemma 3.4.

It left to show that if \( \text{FP}_{3}^{NL} \) is feasible, then \( \text{FP}_{4}^{NL} \) has an optimal solution when for each \( j \in J \), \( t \in \{0,1,\ldots,T\} \), \( k_1 \in K_{jt} \setminus \bar{K}_{jt} \), and \( k_2 \in K_{jt} \setminus \bar{K}_{jt} \), it holds that \( \phi_{k_1,t} \leq \phi_{k_2,t} \). Let problem \( \text{FP}_{4}^{NL} \) be the same as problem \( \text{FP}_{3}^{NL} \) with the resource constraints removed. Thus, \( \text{FP}_{4}^{NL} \) is a collection of separate problems, one problem for each \( t \in \{0,1,\ldots,T\} \). Each of the separate problem have the following formulation:

\[
\begin{align*}
\max_{d, p, p_0, v_\leq, u_\geq} \sum_{i \in I_\leq} \sum_{k \in K_i^\prime} \phi_k v_{ik} + \sum_{j \in J_\geq} \sum_{k \in K_j} \phi_k u_{jk} - \sum_{j \in J} \psi_j d_j & \quad (SP_1^{NL}) \\
\text{s.t.} \quad \left( \frac{1}{\gamma_i} - 1 \right) r_i + \sum_{j \in J_i} e_j &= \sum_{k \in K_i^\prime} v_{ik} & \forall i \in I_\leq \, ((\sigma_\leq)_i) \quad (36a) \\
\left( \frac{1}{\gamma_i} - 1 \right) s_i + f_j &= -p_0 \sum_{k \in K_j} \pi_{jk} & \forall i \in I_\leq, j \in J_i \, ((\kappa_\leq)_j) \quad (36b) \\
\left( \frac{1 - 1}{\gamma_i} \right) g_j + \left( \frac{1}{\gamma_i} \right) e_j &= \sum_{k \in K_j} u_{jk} & \forall i \in I_>, j \in J_i \, ((\kappa_>)_j) \quad (36c) \\
(p_0, d_j, e_j) & \in K_{\exp} & \forall i \in I, j \in J_i \, (\pi_j) \quad (36d) \\
(p_i, d_j, g_j) & \in K_{\exp} & \forall i \in I_\geq, j \in J_i \, ((\omega_\geq)_j) \quad (36e) \\
(d_j, p_0, f_j) & \in K_{\exp} & \forall i \in I_\leq, j \in J_i \, ((\omega_\leq)_j) \quad (36f) \\
(p_i, p_0, s_i) & \in K_{\exp} & \forall i \in I_\leq \, ((\tau_\leq)_i) \quad (36g) \\
(p_0, p_i, r_i) & \in K_{\exp} & \forall i \in I_\leq \, ((\eta_\leq)_i) \quad (36h) \\
p_0 + \sum_{i \in I} p_i &= 1 \quad (\eta_0) \quad (36i) \\
p_i &= \sum_{j \in J_i} d_j & \forall i \in I \, (\eta_j) \quad (36j) \\
v_{ik} & \geq \sum_{j \in J_i} \pi_{jk} d_j & \forall i \in I_\leq, k \in K_i^\prime \, ((\mu_\leq)_ik) \quad (36k) \\
v_{ik} & \leq \sum_{j \in J_i} \pi_{jk} d_j & \forall i \in I_\leq, k \in K_i \, ((\mu_\leq)_ik) \quad (36l) \\
u_{jk} & \geq \pi_{jk} d_j & \forall j \in J_>, k \in K_j \, ((\nu_>)_jk) \quad (36m) \\
u_{jk} & \leq \pi_{jk} d_j & \forall j \in J_>, k \in K_j \, ((\nu_>)_jk) \quad (36n)
\end{align*}
\]

and is feasible. The dual problem of \( SP_1^{NL} \) can be written as:

\[
\begin{align*}
\min_{\eta_0} \quad \eta_0 \quad (SD_1^{NL}) \\
\text{s.t.} \quad \sum_{i \in I} \sum_{j \in J_i} \pi_{ij} + \sum_{i \in I_\leq} \sum_{j \in J_i} \omega_{ij} + \sum_{i \in I_\leq} \omega_{ij} \quad \sum_{i \in I} \sum_{j \in J_i} \pi_{ij} + \sum_{i \in I_\leq} \sum_{j \in J_i} \omega_{ij} + \sum_{i \in I_\leq} \omega_{ij}
\end{align*}
\]

49
\[
\begin{align*}
+ \sum_{i \in I_>} \tau_{i1} + \sum_{i \in I_<} \sum_{j \in J_i} \left( \sum_{k \in K_j} \pi_{jk} \right) \kappa_j &= \eta_0 \\
\sum_{j \in J_i} w_{j1} + \eta_i &= \eta_0 & i \in I_> \quad (p_i) \\
\omega_i + \tau_{i2} + \eta_i &= \eta_0 & i \in I_< \quad (p_i) \\
\pi_{j2} + \omega_{j2} - \eta_i - \sum_{k \in K_j} \mu_{jk} \omega_{jk} + \sum_{k \in K_j} \pi_{jk} \tau_{jk} &= \psi_j & i \in I_>, j \in J_i \quad (d_j) \\
\pi_{j2} + \omega_{j2} - \eta_i - \sum_{k \in K_j} \mu_{jk} \omega_{jk} + \sum_{k \in K_j} \pi_{jk} \tau_{jk} &= \psi_j & i \in I_<, j \in J_i \quad (d_j) \\
\pi_{j2} - \eta_i - \sum_{k \in K_i} \mu_{jk} \omega_{jk} + \sum_{k \in K_i} \pi_{jk} \tau_{jk} &= \psi_j & i \in I_<, j \in J_i \setminus J_i \quad (d_j) \\
- \kappa_j &= -\phi_k \\
- \kappa_j + \nu_{jk} &= -\phi_k \\
- \kappa_j - \pi_{jk} &= -\phi_k \\
- \kappa_j + \nu_{jk} - \pi_{jk} &= -\phi_k \\
- \sigma_i &= -\phi_k \\
- \sigma_i + \mu_{ik} &= -\phi_k \\
- \sigma_i - \mu_{ik} &= -\phi_k \\
- \sigma_i + \mu_{ik} - \pi_{ik} &= -\phi_k \\
\pi_{j3} + \left( \frac{1}{\gamma_i} \right) \kappa_j &= 0 \\
\pi_{j3} + \sigma_i &= 0 \\
\omega_{j3} + \kappa_j &= 0 \\
\omega_{j3} + \left( 1 - \frac{1}{\gamma_i} \right) \kappa_j &= 0 \\
\tau_{i3} + \left( \frac{1}{\gamma_i} - 1 \right) \sigma_i &= 0 \\
\omega_{i3} + \left( \frac{1}{\gamma_i} - 1 \right) \sum_{j \in J_i} \kappa_j &= 0 \\
\pi_j &\in K^*_{\text{exp}} \\
(w_>)_j &\in K^*_{\text{exp}} \\
(w_<)_j &\in K^*_{\text{exp}} \\
(\omega_<)_i &\in K^*_{\text{exp}} \\
(\tau_<)_i &\in K^*_{\text{exp}} \\
(\mu_<)_{ik} &\geq 0 \\
\forall i \in I, j \in J_i \\
\forall i \in I_>, j \in J_i \\
\forall i \in I_<, j \in J_i \\
\forall i \in I_< \\
\forall i \in I_>
\end{align*}
\]
The refined conic duality theorem (see [Ben-Tal and Nemirovski (2001)]) states that \( \text{SP}^{\text{NL}}_1 \) has an optimal solution if its dual problem \( \text{SD}^{\text{NL}}_1 \) is bounded below and is “strictly feasible” (i.e. it has a feasible solution in the interior of the cones). Since \( \text{SP}^{\text{NL}}_1 \) is feasible, we know that \( \text{SD}^{\text{NL}}_1 \) is bounded below. Next we show that \( \text{SD}^{\text{NL}}_1 \) has a feasible solution such that the conic variables are all in the interior of \( \mathcal{K}_{\text{exp}}^* \). As a preparation, note that given Condition (i) in Theorem 3.8, the following two arguments are equivalent (either one applies the other):

1. For any \( j \in \mathcal{J} \), \( \phi_{k_1} \leq \phi_{k_2} \) for all \( k_1 \in \mathcal{K}_j \setminus \mathcal{K}_j, k_2 \in \mathcal{K}_j \setminus \mathcal{K}_j \).

2. For any \( i \in \mathcal{I} \), \( \phi_{k_1} \leq \phi_{k_2} \) for all \( k_1 \in \mathcal{K}_i \setminus \mathcal{K}_i, k_2 \in \mathcal{K}_i \setminus \mathcal{K}_i \).

For each \( i \in \mathcal{I}_> \), \( j \in \mathcal{J} \), choose \( \kappa_j, \nu_{jk} \) for all \( k \in \mathcal{K}_j \), and \( \theta_{jk} \) for all \( k \in \mathcal{K}_j \), by considering the following cases:

1. If \( \mathcal{K}_j \setminus \mathcal{K}_j \neq \emptyset \), then choose \( \kappa_j = -\min \{ \phi_k : k \in \mathcal{K}_j \setminus \mathcal{K}_j \} \). Since \( \phi_k > 0 \) for all \( k \), it follows that \( \kappa_j < 0 \). Then, for each \( k \in \mathcal{K}_j \), consider the following four cases.

   (a) Suppose \( k \in \mathcal{K}_j \setminus (\mathcal{K}_j \cup \mathcal{K}_j) \). Since \( k \in \mathcal{K}_j \setminus \mathcal{K}_j \) and \( \phi_{k_1} \leq \phi_{k_2} \) for all \( k_1 \in \mathcal{K}_j \setminus \mathcal{K}_j, k_2 \in \mathcal{K}_j \setminus \mathcal{K}_j \), it follows that \( \phi_k \leq \kappa_j \). Then \( \phi_k = \kappa_j \).

   (b) Suppose \( k \in \mathcal{K}_j \setminus \mathcal{K}_j \). Then choose \( \nu_{jk} = \kappa_j - \phi_k \). Since \( k \in \mathcal{K}_j \setminus \mathcal{K}_j \) and \( \phi_{k_1} \leq \phi_{k_2} \) for all \( k_1 \in \mathcal{K}_j \setminus \mathcal{K}_j, k_2 \in \mathcal{K}_j \setminus \mathcal{K}_j \), it follows that \( \phi_k \leq \kappa_j \). Thus \( \nu_{jk} = \kappa_j - \phi_k \geq 0 \).

   (c) Suppose \( k \in \mathcal{K}_j \setminus \mathcal{K}_j \). Then choose \( \theta_{jk} = \kappa_j + \phi_k \). Since \( k \in \mathcal{K}_j \setminus \mathcal{K}_j \), it follows that \( \kappa_j \leq \phi_k \). Thus \( \theta_{jk} = \kappa_j + \phi_k \geq 0 \).

   (d) Suppose \( k \in \mathcal{K}_j \cap \mathcal{K}_j \). Then choose \( \nu_{jk} = \max \{ 0, \kappa_j - \phi_k \} \) and \( \theta_{jk} = \max \{ 0, \kappa_j + \phi_k \} \).

2. If \( \mathcal{K}_j \setminus \mathcal{K}_j = \emptyset \) (that is, \( \mathcal{K}_j = \mathcal{K}_j \)), then choose \( \kappa_j = -\max \{ \phi_k : k \in \mathcal{K}_j \} \). Since \( \phi_k > 0 \) for all \( k \), it follows that \( \kappa_j < 0 \). Then, for each \( k \in \mathcal{K}_j \), consider the following two cases (only these two cases are possible).

   (a) Suppose \( k \in \mathcal{K}_j \setminus \mathcal{K}_j \). Then choose \( \nu_{jk} = \kappa_j - \phi_k \). Since \( \phi_k \leq \kappa_j \), it follows that \( \nu_{jk} = \kappa_j - \phi_k \geq 0 \).

51
(b) Suppose \( k \in \mathcal{K}_i \cap \overline{\mathcal{K}}_j \). Then choose \( \mu_{jk} = \max \{ 0, \kappa_j - \phi_k \} \) and \( \nu_{jk} = \max \{ 0, \kappa_j + \phi_k \} \).

For each \( i \in \mathcal{I}_< \), choose \( \sigma_i, \mu_{ik} \) for all \( k \in \mathcal{K}'_i \), and \( \nu_{ik} \) for all \( k \in \overline{\mathcal{K}}'_i \), by considering the following cases:

1. If \( \mathcal{K}'_i \setminus \mathcal{K}'_j \neq \emptyset \), then choose \( \sigma_i = -\min \{ \phi_k : k \in \mathcal{K}'_i \setminus \mathcal{K}'_j \} \). Since \( \phi_k > 0 \) for all \( k \), it follows that \( \sigma_i < 0 \). Then, for each \( k \in \mathcal{K}'_i \), consider the following four cases.

   (a) Suppose \( k \in \mathcal{K}'_i \setminus \left( \mathcal{K}'_j \cup \overline{\mathcal{K}}'_i \right) \). Since \( k \in \mathcal{K}'_i \setminus \mathcal{K}'_j \), it follows that \( \sigma_i \leq \phi_k \). Since \( k \in \mathcal{K}'_i \setminus \overline{\mathcal{K}}'_i \) and \( \phi_{k_1} \leq \phi_{k_2} \) for all \( k_1 \in \mathcal{K}'_i \setminus \overline{\mathcal{K}}'_i \), \( k_2 \in \mathcal{K}'_i \setminus \mathcal{K}'_j \), it follows that \( \phi_k \leq \sigma_i \). Thus, \( \phi_k = \sigma_i \).

   (b) Suppose \( k \in \mathcal{K}'_i \setminus \overline{\mathcal{K}}'_i \). Then choose \( \mu_{ik} = \sigma_i - \phi_k \). Since \( k \in \mathcal{K}'_i \setminus \overline{\mathcal{K}}'_i \) and \( \phi_{k_1} \leq \phi_{k_2} \) for all \( k_1 \in \mathcal{K}'_i \setminus \overline{\mathcal{K}}'_i \), \( k_2 \in \mathcal{K}'_i \setminus \mathcal{K}'_j \), it follows that \( \phi_k \leq \sigma_i \). Thus \( \mu_{ik} = \sigma_i - \phi_k \geq 0 \).

   (c) Suppose \( k \in \overline{\mathcal{K}}'_i \setminus \mathcal{K}'_j \). Then choose \( \nu_{ik} = \sigma_i + \phi_k \). Since \( k \in \overline{\mathcal{K}}'_i \setminus \mathcal{K}'_j \), it follows that \( \sigma_i \leq \phi_k \). Thus \( \nu_{ik} = \sigma_i + \phi_k \geq 0 \).

   (d) Suppose \( k \in \mathcal{K}'_i \cap \overline{\mathcal{K}}'_i \). Then choose \( \mu_{ik} = \max \{ 0, \sigma_i - \phi_k \} \) and \( \nu_{ik} = \max \{ 0, \sigma_i + \phi_k \} \).

2. If \( \mathcal{K}'_i \setminus \mathcal{K}'_j = \emptyset \) (that is, \( \mathcal{K}'_i = \mathcal{K}'_j \)), then choose \( \sigma_i = -\max \{ \phi_k : k \in \mathcal{K}'_i \} \). Since \( \phi_k > 0 \) for all \( k \), it follows that \( \sigma_i < 0 \). Then, for each \( k \in \mathcal{K}'_i \), consider the following two cases (only these two cases are possible).

   (a) Suppose \( k \in \mathcal{K}'_i \setminus \mathcal{K}'_j \). Then choose \( \mu_{ik} = \sigma_i - \phi_k \). Since \( \phi_k \leq \sigma_i \), it follows that \( \mu_{ik} = \sigma_i - \phi_k \geq 0 \).

   (b) Suppose \( k \in \mathcal{K}'_i \cap \mathcal{K}'_j \). Then choose \( \mu_{ik} = \max \{ 0, \sigma_i - \phi_k \} \) and \( \nu_{ik} = \max \{ 0, \sigma_i + \phi_k \} \).

In addition, let

\[
\begin{align*}
\pi_{j3} &= - \left( \frac{1}{\gamma_i} \right) \kappa_j < 0 & i \in \mathcal{I}_>, \ j \in \mathcal{J}_i \\
\pi_{j3} &= - \sigma_i < 0 & i \in \mathcal{I}_<, \ j \in \mathcal{J}_i \\
\varpi_{j3} &= - \varpi_{j1} = - \left( 1 - \frac{1}{\gamma_i} \right) \kappa_j < 0 & i \in \mathcal{I}_>, \ j \in \mathcal{J}_i \\
\varpi_{j3} &= - \varpi_{j1} = - \kappa_j < 0 & i \in \mathcal{I}_<, \ j \in \mathcal{J}_i \\
\omega_{3j} &= - \omega_{11} = - \left( \frac{1}{\gamma_i} - 1 \right) \sum_{j \in \mathcal{J}_i} \kappa_j < 0 & i \in \mathcal{I}_< \\
\tau_{3i} &= - \left( \frac{1}{\gamma_i} - 1 \right) \sigma_i < 0 & i \in \mathcal{I}_<
\end{align*}
\]
and let

\[
\begin{align*}
\pi_j &= C \quad i \in \mathcal{I}, \ j \in \mathcal{J}_i \\
\tau_i &= C \quad i \in \mathcal{I}_< \\
\omega_j &= 0 \quad i \in \mathcal{I}_>, \ j \in \mathcal{J}_i \\
\omega_j &= 0 \quad i \in \mathcal{I}_<, \ j \in \mathcal{J}_i \\
\omega_i &= 0 \quad i \in \mathcal{I}_<
\end{align*}
\]

\[
\begin{align*}
\eta_0 &= \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}_i} \pi_{j1} + \sum_{i \in \mathcal{I}_<} \tau_{i1} + \sum_{i \in \mathcal{I}_<} \sum_{j \in \mathcal{J}_i} \left( \sum_{k \in \mathcal{K}_j} \pi_{jk} \right) \kappa_j \\
\eta_i &= \eta_0 - \sum_{j \in \mathcal{J}_i} \omega_{j1} \quad i \in \mathcal{I}_> \\
\tau_{i2} &= \eta_0 - \eta_i - \omega_{i1} = \sum_{j \in \mathcal{J}_i} \omega_{j1} \quad i \in \mathcal{I}_< (p_i) \\
\pi_{j2} &= \eta_i + \psi_j - \omega_{j2} + \sum_{k \in \mathcal{K}_j} \nu_{jk} x_{jk} - \sum_{k \in \mathcal{K}_j} \nu_{jk} x_{jk} \quad i \in \mathcal{I}_>, \ j \in \mathcal{J}_i \\
\pi_{j2} &= \eta_i + \psi_j - \omega_{j1} + \sum_{k \in \mathcal{K}_j} \mu_{jk} x_{jk} - \sum_{k \in \mathcal{K}_j} \mu_{jk} x_{jk} \quad i \in \mathcal{I}_<, \ j \in \mathcal{J}_i \\
\pi_{j2} &= \eta_i + \psi_j + \sum_{k \in \mathcal{K}_j} \nu_{jk} x_{jk} - \sum_{k \in \mathcal{K}_j} \nu_{jk} x_{jk} \quad i \in \mathcal{I}_<, \ j \in \mathcal{J} \setminus \mathcal{J}_i
\end{align*}
\]

where \( C \) is a positive number. Thus, all equality and inequality constraints in SDNL hold. Meanwhile, we can make \( C \) large enough, such that

\[
\begin{align*}
\eta_0 &> 0 \\
\eta_i &> 0 \quad i \in \mathcal{I}_> \\
\pi_{j1} &\geq -\pi_{j3} > 0 \quad \forall \ i \in \mathcal{I}, \ j \in \mathcal{J}_i \\
\pi_{j2} &\geq 0 \quad \forall \ i \in \mathcal{I}_>, \ j \in \mathcal{J}_i \\
\omega_{j1} &= -\omega_{j3} > 0 \quad \forall \ i \in \mathcal{I}_>, \ j \in \mathcal{J}_i \\
\omega_{j2} &= 0 \quad \forall \ i \in \mathcal{I}_<, \ j \in \mathcal{J}_i \\
\omega_{j1} &= -\omega_{j3} > 0 \quad \forall \ i \in \mathcal{I}_<, \ j \in \mathcal{J}_i \\
\omega_{j2} &= 0 \quad \forall \ i \in \mathcal{I}_< \tau_{j1} > -\tau_{j3} \exp (\tau_{j2}/\tau_{j3} - 1) \quad \forall \ i \in \mathcal{I}_<
\end{align*}
\]

which implies

\[
\begin{align*}
\pi_j &> \kappa_j^{\exp} 0 \quad \forall \ i \in \mathcal{I}, \ j \in \mathcal{J}_i \\
(\omega_j)_{i} &> \kappa_j^{\exp} 0 \quad \forall \ i \in \mathcal{I}_>, \ j \in \mathcal{J}_i
\end{align*}
\]
\begin{align*}
(\omega_{\leq})_j & > \kappa^*_\exp \quad 0 \quad \forall i \in I_\leq, \ j \in J_i \\
(\omega_{\leq})_i & > \kappa^*_\exp \quad 0 \quad \forall i \in I_\leq \\
(\tau_{\leq})_i & > \kappa^*_\exp \quad 0 \quad \forall i \in I_\leq
\end{align*}

Thus, \( \text{SD}^{\text{NL}}_1 \) has a feasible solution such that the conic variables are all in the interior of \( \mathcal{K}^*_\exp \).

Now, since \( \text{FP}^{\text{NL}}_2 \) is feasible, the dual of \( \text{FP}^{\text{NL}}_2 \) is bounded. Also, since \( \text{FP}^{\text{NL}}_2 \) is the same as \( \text{FP}^{\text{NL}}_3 \) with additional linear inequality constraints, the dual of \( \text{FP}^{\text{NL}}_2 \) has additional signed variables beyond the dual of \( \text{FP}^{\text{NL}}_3 \). Then a feasible solution of the dual of \( \text{FP}^{\text{NL}}_3 \) in the interior of the dual cones, combined with these additional signed variables set to 0, gives a feasible solution of the dual of \( \text{FP}^{\text{NL}}_2 \) in the interior of the dual cones. Then it follows from the conic duality theorem (Theorem 1.4.2 in Ben-Tal and Nemirovski (2001)) that \( \text{FP}^{\text{NL}}_2 \) has an optimal solution. \( \square \)