The naturality of natural deduction (II)  
Some remarks on atomic polymorphism

Paolo Pistone¹, Luca Tranchini², and Mattia Petrolo³

¹ Dipartimento di Informatica-Scienza e Ingegneria, Università di Bologna  
² Wilhelm-Schickard-Institut, Universität Tübingen  
³ Centro de Ciências Naturais e Humanas, Universidade Federal do ABC

Abstract

In a previous paper (of which this is a prosecution) we investigated the extraction of proof-theoretic properties of natural deduction derivations from their impredicative translation into System F. Our key idea was to introduce an extended equational theory for System F codifying at a syntactic level some properties found in parametric models.

In a recent series of papers a different approach to extract proof-theoretic properties of natural deduction derivations was proposed by defining predicative variants of the usual translation, embedding intuitionistic propositional logic into the atomic fragment of System F.

In this paper we show that this approach finds a general explanation within our equational study of second-order natural deduction, and a clear semantic justification provided by parametricity.

Keywords  second-order logic, propositional quantification, identity of proofs, Russell-Prawitz translation, atomic polymorphism, naturality condition, instantiation overflow.

1 Introduction

Russell was the first to observe that propositional connectives like disjunction and conjunction can be defined using only implication and propositional quantification and in his monograph on natural deduction, Prawitz showed how the natural deduction system for intuitionistic propositional logic (henceforth $\text{NI}$) can be embedded into the implicational fragment of second-order propositional intuitionistic logic (also known as $\text{System F}$, and here referred to as $\text{NI}^2$). We will refer to this embedding as the Russell-Prawitz translation (shortly RP-translation).

Taking inspiration from this embedding, in recent work in proof-theoretic semantics (see for instance in [16] and [8]) $\text{NI}^2$ has been suggested as a suitable setting to investigate the proof theory of propositional connectives. This way of looking at $\text{NI}^2$ faces however two kinds of difficulties.
The equivalence-preservation problem  In proof-theoretic semantics, one is not only concerned with provability—i.e. with whether there is a derivation of a given formula in a certain system—but also with identity of proofs—i.e. with whether two distinct derivations of the same formula can be viewed as different syntactic representations of the same proof (understood as an abstract object).

A common way to characterize identity of proofs is by declaring two derivations equivalent when they converge, under the usual conversions used for normalization, to the same normal derivation. Equivalent derivations are then taken to represent the same proof. This intuition is made precise by the categorical semantics for natural deduction systems. For instance \( \text{NI} \) can be interpreted in any bi-cartesian closed category, with equivalent derivations being mapped onto the same morphism.

If not only provability but also identity of proofs is considered, then the RP-translation might not seem entirely satisfactory as equivalent derivations in \( \text{NI} \) need not translate into equivalent derivations in \( \text{NI}^2 \). Although the translation works for the equivalence induced by \( \beta \)-conversions only, it fails for the one induced by \( \eta \)-conversions and permutations, here referred to as \( \gamma \)-conversions. In categorical terms, the RP-translation of, say, a disjunction, is not interpreted as a co-product in every categorical model of \( \text{NI}^2 \), but only as a “weak” variant of it. We will refer to this fact as the equivalence-preservation problem of the RP-translation.

In a previous paper, of which the present one is a follow-up, we explored a solution to the equivalence-preservation problem based on the fact that the RP-translation of conjunctions and disjunctions does yield categorical products/co-products in the class of parametric models of \( \text{NI}^2 \) [22, 1, 13]. With the goal of making this result accessible to the proof theory community at large, in [30] we provided a purely syntactic reconstruction of it: we described an equational theory extending the one arising from the usual \( \beta \)- and \( \eta \)-conversions for \( \text{NI}^2 \)-derivations using a new class of conversions—that we called \( \varepsilon \)-conversions—expressing a naturality condition for \( \text{NI}^2 \)-derivations that holds in all parametric models of \( \text{NI}^2 \), and we showed that the RP-translation does preserve the full equivalence of \( \text{NI} \)-derivations as soon as \( \text{NI}^2 \)-derivations are considered under this stronger equivalence.

Impredicative vs predicative translations  A second difficulty is of a foundational nature and stems from the fact that the RP-translation and, more generally, the second-order encoding of inductive types (e.g. the types of natural numbers and well-founded trees) inside \( \text{NI}^2 \) are impredicative. In fact, the embedding of \( \text{NI} \) into \( \text{NI}^2 \) requires the full power of second-order quantification: in the elimination rule for the second-order quantifier \( \forall E \):

\[
\forall X.A \quad \text{A}[B/X] \quad \forall E
\]

\[\frac{\forall X.A}{A[B/X]} \quad \forall E\]

\[\text{In the functorial semantics of System F [1, 13], } \text{NI}^2 \text{-derivations are actually interpreted as dinatural transformation. The reason to focus on naturality, rather than on the more general notion of dinaturality, is briefly discussed in Section 6.}\]
no restriction can be imposed on the choice of the formula $B$ (called the *witness* of the rule application).

A solution to this problem can be found in a recent series of papers by Fernando Ferreira and Gilda Ferreira, who proposed a variant of the RP-translation (to which we will refer to as FF-translation) which encodes $\text{NI}$ in *atomic* System F (here referred to as $\text{NI}^2_{at}$), a weak predicative fragment of $\text{NI}^2$ in which the witnesses of $\forall \exists$ are required to be atomic formulas. A further refinement of the FF-translation was later proposed by José Espírito Santo and Gilda Ferreira in [4] (we will refer to it as the ESF-translation).

Besides being predicative, the FF- and ESF-translations have another significant advantage over the RP-translation: they do preserve the equivalence arising not only from $\beta$-conversions, but also from $\eta$- and $\gamma$-conversions for disjunction and $\bot$ [7, 10, 4]. For these reasons the predicative translations were advocated in [8] as evidence in favor of taking $\text{NI}^2$ and its fragments as a convenient framework to investigate propositional connectives.

**From impredicative to atomic polymorphism through $\varepsilon$-conversions** In this paper we investigate the predicative translations into System $F_{at}$ using the equational framework we developed in our previous paper, and we show that the syntactic results on atomic polymorphism can be given a semantic explanation ultimately relying on parametricity, a well-investigated semantics of (full) polymorphism.

Our first observation is that the results of Ferreira and co-authors do not hold only for $\lor$ and $\bot$, but for the class of connectives that are definable in $\text{NI}$ by arbitrarily composing $\land$, $\lor$, $\top$, and $\bot$ (to describe such connectives we borrow another concept from the toolbox of category theory, that of a *finite polynomial functor* [12]).

By extending the RP-translation to a natural deduction system for this class of propositional connectives (called $\text{NI}^p$), we are led to consider another fragment of $\text{NI}^2$, that we call the *Russell-Prawitz fragment* ($\text{NI}^2_{RP}$). Unlike the atomic fragment $\text{NI}^2_{at}$, the fragment $\text{NI}^2_{RP}$ is impredicative, since no restriction is imposed on the witnesses of the applications of $\forall \exists$.

Nonetheless, we show that every derivation in $\text{NI}^2_{RP}$ can be “atomized”, i.e. it can be mapped onto a derivation in $\text{NI}^2_{at}$ with the same conclusion and the same assumptions, by applying instances of $\eta$-expansion and the $\varepsilon$-conversion. By composing the RP-translation from $\text{NI}^p$ to $\text{NI}^2_{RP}$ with the atomization from $\text{NI}^2_{RP}$ to $\text{NI}^2_{at}$ one thereby obtains another predicative translation from $\text{NI}^p$ to $\text{NI}^2_{at}$ (we call it the $\varepsilon$-*translation*) which only differs from the FF- and ESF-translation by some $\beta$-reduction steps.

An immediate consequence of this fact is that the RP-translation and its three predicative variants are all equivalent modulo $\beta$, $\eta$- and $\varepsilon$-conversions. Thus, under the notion of identity of proofs induced by $\varepsilon$-conversions, all translations of a given propositional derivation are different syntactic descriptions of the same second-order proofs (that is, all these translations interpret an $\text{NI}^p$-derivation as the same morphism in all parametric models of $\text{NI}^2$).

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2 Actually, preservation of $\eta$-conversions fails for conjunction.
Comparing predicative translations and $\varepsilon$-conversions  On the one hand, we highlight two limitations of the approach based on atomic polymorphism: first, the predicative translations do not preserve the full $\eta$-rule needed to interpret disjunction as a categorical co-product, and thus fail to provide a full solution to the equivalence-preservation problem. Moreover, we show that once $\forall E$ is restricted to atomic witnesses it is not possible to prove the logical equivalence between a propositional formula and its second-order translation (with the terminology of [19], this means that connectives are not strongly definable, but only weakly definable in $\text{NI}_2^{\text{st}}$).

On the other hand, we observe that while the predicative translations are well-suited for the study of proof reductions, as shown for instance by the results in [5], the use of $\varepsilon$-conversions comes at the price of a rather involved and still not well-understood reduction behavior.

Goals and plan of the paper  One of the motivations for the previous and present papers is that of making some ideas underlying the categorical semantics of System $F$ accessible to the proof-theoretic community at large, and to show that these ideas can be fruitfully connected with strands of research arisen within more philosophically-oriented areas of proof theory. This was the reason for reformulating in the first paper categorical notions such as functors and natural transformations in the language of natural deduction, at the expenses of typographic conciseness.

To keep the presentation compact and readable for the largest audience, we chose to present the main results of the paper using the natural deduction notation and restricting the attention only to the case of a particular ternary connective $\bullet(A, B, C)$. Full proofs for the whole class of connectives we consider are postponed to a (large) technical appendix written using the drastically more economical $\lambda$-calculus notation.

In Section 2, we introduce the natural deduction calculus $\text{NI}^p$ for the class of propositional connectives we intend to investigate and a fragment of $\text{NI}^2$, that we call the Russell-Prawitz fragment (noted $\text{NI}_2^{\text{RP}}$). Both $\text{NI}^p$ and $\text{NI}_2^{\text{RP}}$ are inspired by the notion of finite polynomial functor from category theory, and we introduce a generalization of the usual RP-translation as a derivability-preserving embedding between these two systems. In Section 3 we recall the framework introduced in our previous paper to describe functors and natural transformations within natural deduction, based on the $\varepsilon$-conversion. In Section 4 we generalize the FF- and the ESF-translations to $\text{NI}^p$ and we investigate their relationship to the RP-translation. To do this, we first show how the FF-translation can be analyzed as the composition of the RP-translation and of an embedding from the fragment $\text{NI}_2^{\text{RP}}$ into $\text{NI}_2^{\text{st}}$ that we call FF-atomization, and then defining an alternative embedding from $\text{NI}_2^{\text{RP}}$ into $\text{NI}_2^{\text{st}}$ atomization using the $\varepsilon$-conversions, the $\varepsilon$-atomization. In Section 5 we discuss some limitations as well as some advantages of predicative translations for the study of identity of proofs and proof reduction. To do this, we first show how the FF-translation can be analyzed as the composition of the RP-translation and of an embedding from the fragment $\text{NI}_2^{\text{RP}}$ into $\text{NI}_2^{\text{st}}$ that we call FF-atomization, and then defining an alternative embedding from $\text{NI}_2^{\text{RP}}$ into $\text{NI}_2^{\text{st}}$ atomization using the $\varepsilon$-conversions, the $\varepsilon$-atomization. In Section 6 we briefly summarize the results of the paper, we draw some connections with related work, and we suggest further directions of investigation. Finally, the rich appendix provides full proofs (in $\lambda$-calculus notation) of the results discussed or simply sketched in the main text.
2 Polynomial connectives and their RP-translation

In this section we introduce a formal framework for natural deduction which extends the one from [30] to a more general class of propositional connectives.

2.1 Polynomial connectives

As suggested in the previous paper (cf. [30] Section 4.3), the results we are concerned with are not limited to the standard intuitionistic connectives, but scale smoothly to a wider class of connectives investigated in proof-theoretic semantics (see e.g. [21, 23]). These are those connectives that can be defined by composing \( \land, \lor, \top, \bot \), such as the ternary connective \( (A_1, A_2, A_3) \) definable as \( (A_1 \land A_2) \lor A_3 \) whose introduction and elimination rules are as follows:

\[
\begin{align*}
A_1 & \quad A_2 & \quad (A_1, A_2, A_3) & \quad I_1 \\
(\to) & & \to (A_1, A_2, A_3) & \to I_2 \\
& & \forall y \in \{1, 2, 3\} \quad A_{I_y} & \quad E
\end{align*}
\]

In general, each such connective, is definable in \( \mathbb{NI} \) as \( \bigvee_{p=1}^n \bigwedge_{q=1}^{m_p} A_{pq} \) for some choice of \( n, m_p \), and \( A_{pq} \), but here it will be treated as primitive.

Borrowing ideas from the theory of finite polynomial functors [12], each such connective can be described as determined by three finite lists \( \mathcal{I}, \mathcal{J}, \mathcal{K} \) (to be thought of as lists of indices) and two functions \( f : \mathcal{J} \rightarrow \mathcal{I} \) and \( g : \mathcal{J} \rightarrow \mathcal{K} \) that we depict in a diagram as follows:

\[
\mathcal{I} \xleftarrow{f} \mathcal{J} \xrightarrow{g} \mathcal{K}
\]

Any such diagram determines what we will call a polynomial connective to be indicated with \( \uparrow \mathcal{J}(f,g) \), or simply \( \uparrow \mathcal{J} \) when \( f, g \) are clear from the context, in the following way:

- The length \( |\mathcal{I}| \) of \( \mathcal{I} \) measures the arity of \( \uparrow \mathcal{J} \), so that when \( \uparrow \mathcal{J} \) is applied to an \( \mathcal{I} \)-indexed list of formulas \( \langle A_i \rangle_{i \in \mathcal{I}} \) one obtains a new formula \( \uparrow \langle A_i \rangle_{i \in \mathcal{I}} \).

- The length \( |\mathcal{K}| \) of \( \mathcal{K} \) is the number of distinct introduction rules of \( \uparrow \mathcal{J} \).

- Any element \( k \) in \( \mathcal{K} \) determines a sublist of \( \mathcal{J} \), namely the list of all \( j \in \mathcal{J} \) such that \( g(j) = k \), that we indicate with \( g^{-1}(k) \) and whose length \( |g^{-1}(k)| \) is the number of premises of the \( k \)-th introduction rule of \( \uparrow \mathcal{J} \).

Using the functions \( f \) and \( g \) we can describe the introduction and elimination rules for \( \uparrow \mathcal{J} \) as follows. Given an \( \mathcal{I} \)-indexed list of formulas \( \langle A_i \rangle_{i \in \mathcal{I}} \), the \( k \)-th introduction rule \( \uparrow \mathcal{I} \) for \( \uparrow \mathcal{J} \), allows us to infer \( \uparrow \langle A_i \rangle_{i \in \mathcal{I}} \) from the list of premises \( \langle A_{f(j)} \rangle_{j \in g^{-1}(k)} \). Given \( \uparrow \langle A_i \rangle_{i \in \mathcal{I}} \) and

\footnote{In the language of category theory this configuration describes a unary finite polynomial functor, which is the reason for our terminological choice.}

\footnote{\( \langle A_i \rangle_{i \in \mathcal{I}} \) abbreviates the sequence of formulas \( A_{i_1} \ldots A_{i_n} \) when \( \mathcal{I} = \langle i_1, \ldots, i_n \rangle \).

\footnote{\( \mathcal{J} \) can be seen as a family of lists indexed by the elements of \( \mathcal{K} \).}
a $K$-indexed list of derivations of an arbitrary formula $C$ from (respectively) the premises of the $k$-th introduction rule $\langle A_{f(j)} \rangle_{j \in g^{-1}(k)}$, we can infer $C$ thereby discharging in the $k$-th derivation of $C$ the assumptions $\langle A_{f(j)} \rangle_{j \in g^{-1}(k)}$. We depict the rules as follows:

$$
\frac{\langle A_{f(j)} \rangle_{j \in g^{-1}(k)} \vdash \langle A_{i} \rangle_{i \in I}}{\langle \langle A_{f(j)} \rangle_{j \in g^{-1}(k)} \rangle_{k \in K}} \quad \vdash \langle A_{i} \rangle_{i \in I} \quad \frac{\langle \langle A_{f(j)} \rangle_{j \in g^{-1}(k)} \rangle_{k \in K} \vdash C}{\vdash C}
$$

**Remark 2.1.** Let $0$ be the empty list, and $1, 2, 3, 4, 5$ be the lists $\langle 1 \rangle$, $\langle 1, 2 \rangle$, $\langle 1, 2, 3 \rangle$, $\langle 1, 2, 3, 4 \rangle$, $\langle 1, 2, 3, 4, 5 \rangle$ respectively. The connective $\bullet(A_1, A_2, A_3) = (A_1 \land A_2) \lor A_3$ is shorthand for $\vdash^{(f,g)}\langle A_1, A_2, A_3 \rangle$, given by $3 \vdash_{f_0} 3 \vdash 2$, where $f_0$ indicates the identity function on the list $A$ and $g : \{1, 2 \mapsto 1; 3 \mapsto 2\}$. Similarly, the usual connectives $\lor, \land, \top, \bot$ are obtained through the configurations $2 \vdash_{f_0} 1 \vdash_{g_0} 2, 2 \vdash_{f_0} 1 \vdash 1, 0 \vdash_{g_0} 0 \vdash 1$ and $0 \vdash_{g_0} 0$, respectively (where $\vdash$ indicates the empty function and $1$ the constant function with value $1$). We observe that some arguments of a connective may play a "dummy" role or may be used more than once, such as in the connective $\land(A_1, A_2, A_3, A_4, A_5) = (A_2 \land A_3) \lor (A_4 \land A_5)$ given by $5 \vdash_{f} 4 \vdash_{g} 2$, where $f : \{1 \mapsto 2; 2, 4 \mapsto 3; 3 \mapsto 4\}$ and $g : \{1, 2 \mapsto 1; 3, 4 \mapsto 2\}$.

**Remark 2.2.** Treating $\lor$ and $\land$ as polynomial connectives (see Remark 2.1) yields their usual introduction and elimination rules. This is not the case for $\land E_1$ and $\land E_2$ (see also Remark 2.3 below):

$$
\frac{[A][B]}{C} \quad \frac{A \land B}{C} \quad \frac{A \land B}{A} \quad \frac{A \land B}{B} \quad \frac{A \land B}{A \land E_1} \quad \frac{A \land B}{A \land E_2}
$$

**Remark 2.3.** We will adopt the convention of using $i$ for indices in $I$, $k$ for indices in $K$ and $j$ for indices in $g^{-1}(k)$. Hence, to enhance readability, we will often omit the indication of the index set, so that for instance $\langle A_{i} \rangle_{i \in I}$ will be abbreviated as $\langle A_{i} \rangle$.

Given a set of propositional variables $V$, indicated as $X, Y, Z, \ldots$, the formulas of the language $L^{2p}$ will be constructed using implication, the universal quantifier and the family of polynomial connectives $\vdash^{f,g}$. Besides $L^{2p}$, we will mainly be concerned with two restrictions thereof, $L^p$ and $L^2$:

$$
L^{2p} := X \mid A \supset B \mid \vdash^{(f,g)}\langle A_{i} \rangle_{i \in I} \mid \forall X.A
$$

$$
L^p := X \mid A \supset B \mid \vdash^{(f,g)}\langle A_{i} \rangle_{i \in I}
$$

$$
L^2 := X \mid A \supset B \mid \forall X.A
$$

We moreover indicate with $L^\lor$ (respectively $L^\land$) the restriction of $L^p$ in which $\lor$ (resp. $\land$) is the only connective besides $\supset$, and similarly for $L^{2\lor}$ and $L^{2\land}$.

The natural deduction system $\mathfrak{N}^{2p}$ over the language $L^{2p}$ is obtained by adding to the rules of the standard system $\mathfrak{N}^{2}$ for $L^{2}$ (recalled in Definition 1.1 and Table 1 on
\[(A \land B)^* = \forall X.(A^* \supset B^* \supset X) \supset X \tag{2.1}\]
\[(A \lor B)^* = \forall X.(A^* \supset X) \supset (B^* \supset X) \supset X \tag{2.2}\]
\[
\bot^* = \forall X.X \\
\top^* = \forall X.X \supset X \tag{2.3}
\]

Table 1: RP-translation of standard connectives.

pages 197–198 of \cite{30}, all introduction and elimination rules for the connectives \((f, g)\). The system \(\mathcal{NI}^{2^p}\), along with its equational theory, is described in detail in \(\lambda\)-notation in Appendix A. Finally, by restricting \(\mathcal{NI}^{2^p}\) to the languages \(L^p, L^\bullet, \ldots\) we obtain the subsystems \(\mathcal{NI}^p, \mathcal{NI}^\bullet, \ldots\)

Remark 2.4. Observe that the standard intuitionistic natural deduction system \(\mathcal{NI}\) is *not* a fragment of \(\mathcal{NI}^p\) since the elimination rule for conjunction in the latter system is in general form (see Remark 2.2). Although replacing \(\land E_1\) and \(\land E_2\) with \(\land E_p\) does not alter derivability, the two forms of elimination rules behave differently with respect to identity of proofs. More on this below in Section 5.

The system \(\mathcal{NI}^2_{at}\) (as well as \(\mathcal{NI}^{2^p}_{at}, \mathcal{NI}^{2^\bullet}_{at}, \mathcal{NI}^{2^\top}_{at}\)) is obtained by replacing the rule \(\forall E\) with the following:

\[
\frac{\forall X.A}{A[Y/X]} \forall E_{at}
\]
in which the witness must be an atomic formula.

We indicate derivability in \(\mathcal{NI}^p, \mathcal{NI}^2, \ldots\) with \(\vdash_{\mathcal{NI}^p}, \vdash_{\mathcal{NI}^2}, \ldots\).

In the following we will often indicate an arbitrary formula \(A \in \mathcal{L}^2\) as:

\[
\forall(Y_1)(F_1 \supset \forall(Y_2)(F_2 \supset \cdots \supset \forall(Y_n)(F_n \supset \forall(Y_{n+1})(X)) \ldots)) \tag{\ast}
\]
where \(\forall(Y_i)\) (for \(1 \leq i \leq n + 1\)) indicates a list of consecutive quantifications. Moreover, we let \(at(A) = X\) indicate the rightmost atom of \(A\). Finally, we will often abbreviate \(A_1 \supset (\cdots \supset (A_{n-1} \supset A_n) \cdots)\) with \(A_1 \supset \cdots \supset A_{n-1} \supset A_n\).

2.2 Polynomial formulas and the RP-fragment of \(\mathcal{NI}^2\)

The RP-translation of standard connectives \(\land, \lor, \bot, \top\) is recalled in Table 1. Although the connective \(\bullet\) can be translated by composing the translations of \(\land\) and \(\lor\), a natural and more economical way to encode \(\bullet\) in \(\mathcal{NI}^2\) is given by

\[
\bullet(A, B, C)^* = \forall X.(A^* \supset (B^* \supset X)) \supset (C^* \supset X) \supset X
\]

The universal formula above shares a common structure with those in Table 1: all such formulas are of the form \(\forall X.A_1 \supset \cdots \supset A_n \supset X\), where the \(A_i\) have a unique occurrence of \(X\) in rightmost position. This suggests the following definition:
Definition 2.5. A formula $A \in L^2$ is strongly positive in $X$ (short sp-$X$) when it is of the form $A_1 \supset \cdots \supset A_n \supset X$ for some $n \in \mathbb{N}$ and $X$ does not occur in any of the $A_i$.

A formula $A \in L^2$ is polynomial in $X$ when it is of the form $A_1 \supset \cdots \supset A_n \supset X$ for some $n \in \mathbb{N}$ and all $A_i$ are sp-$X$.

A formula $\forall X.A$ is called universal polynomial if $A$ is polynomial in $X$.

Remark 2.6. We introduce the following compact notation for $\mathbb{N}^2$-formulas. Given a finite list $\mathcal{A} = \langle a_1, \ldots, a_k \rangle$, an $\mathcal{A}$-indexed list of formulas $A_{a_1}, \ldots, A_{a_k}$ and a formula $B$, we let

$$\langle A_a \rangle_{a \in \mathcal{A}} \supset B$$

be shorthand for the formula $A_{a_1} \supset \cdots \supset A_{a_k} \supset B$ if $\mathcal{A}$ is non-empty, otherwise for $B$. When $\mathcal{A}$ is clear from the context the index $a \in \mathcal{A}$ will be omitted so we simply write $\langle A_a \rangle \supset B$.

A universal polynomial formula $A$ can thus be written as $\forall X. \langle B_b \rangle_{b \in \mathcal{B}} \supset X$, where $\mathcal{B}$ is some list and the $B_b$s are sp-$X$. Moreover, any sp-$X$ formula $B$ can in turn be written as $\langle A_a \rangle_{a \in \mathcal{A}} \supset X$, for some list $\mathcal{A}$ such that each $A_a$ has no occurrence of $X$. Hence, a universal polynomial formula can be written as $\forall X. \langle \langle A_a \rangle_{a \in \mathcal{A}} \supset X \rangle_{b \in \mathcal{B}} \supset X$ for some list $\mathcal{B}$ and family of lists $\mathcal{A}_b$ indexed by $\mathcal{B}$. Since a diagram $\mathcal{I} \xrightarrow{f} \mathcal{J} \xrightarrow{g} \mathcal{K}$ describes a family of lists indexed by the elements of a list (see footnote above), it can be used to associate to each $\mathcal{I}$-indexed family of formulas $\langle A_i \rangle_{i \in \mathcal{I}}$ the universal polynomial formula $\forall X. \langle \langle A^*_{(j)} \rangle_{j \notin \{k\}} \supset X \rangle_{k \in \mathcal{K}} \supset X$, which in turn we propose to take as the RP-translation of the propositional formula $\uparrow f \cdot g \langle A_i \rangle_{i \in \mathcal{I}}$.

The Russell-Prawitz translation can thus be generalized to the whole of $L^{2p}$ by translating the formulas of $L^{2p}$ whose outermost connective is polynomial with a universal polynomial formula of $L^2$.

Definition 2.7 (RP-translation of formulas). We define a translation $^*$ from formulas of $L^{2p}$ to formulas of $L^2$ as follows:

$$X^* = X \quad (A \supset B)^* = A^* \supset B^* \quad (\forall X.A)^* = \forall X.A^*$$

$$(\uparrow \langle A_i \rangle_{i \in \mathcal{I}})^* = \forall X. \langle \langle A^*_{(j)} \rangle_{j \notin \{k\}} \supset X \rangle_{k \in \mathcal{K}} \supset X \quad (\text{for } X \text{ not free in any } A_i)$$

where $\uparrow$ is determined by $\mathcal{I} \xrightarrow{f} \mathcal{J} \xrightarrow{g} \mathcal{K}$.

Remark 2.8. For readability, we will abbreviate $(A \supset X) \supset ((B \supset X) \supset X)$ as $A \supset B$ (and thus $(A \lor B)^*$ as $\forall X. (A^* \lor B^*)$), and similarly $(A \supset (B \supset X)) \supset (C \supset X) \supset X$ as $\circ(A, B, C)$ (and thus $\bullet(A, B, C)^*$ as $\forall X. \circ(A^*, B^*, C^*)$).

The RP-translation scales well from formulas to derivations, yielding an embedding $^* : \mathbb{N}^{2p} \hookrightarrow \mathbb{N}^2$. The embedding for usual connectives is recalled in (30) Def. 2.2. Its extension to $\mathbb{N}^{2p}$ is defined in detail in Appendix D in $\lambda$-calculus notation.
Remark 2.9. If $\mathcal{D}$ is an $\mathcal{N}^{2*}$-derivation of $A$ from undischarged assumptions $A_1, \ldots, A_n$, then $\mathcal{D}^*$ is an $\mathcal{N}^{2*}$-derivation of $A^*$ from undischarged assumptions $A_1^*, \ldots, A_n^*$. (For a proof-sketch for the whole $\mathcal{N}^{2p}$, see Appendix D).

If we restrict the language of $\mathcal{N}^2$ by requiring every universal formula to be polynomial, then we obtain a fragment of $\mathcal{N}^2$ that we will call the Russell-Prawitz fragment.

Definition 2.10 (Russell-Prawitz fragment). We let $\mathcal{L}_{RP}^2$ be the subset of $\mathcal{L}^2$ in which all universal formulas are polynomial, and we let $\mathcal{N}_{RP}^2$, called the Russell-Prawitz fragment of $\mathcal{N}^2$, be the fragment of $\mathcal{N}^2$ obtained by restricting to $\mathcal{L}_{RP}^2$-formulas. The system $\mathcal{N}_{RP}^2$ is the subsystem of $\mathcal{N}_{2}^{2}$ in which universal formulas $\forall X.A$ are all of the form $\forall X.(A_1, A_2, A_3)$.

Remark 2.11. The RP-fragment of $\mathcal{L}^2$ can be equivalently defined inductively as follows:

$$
\mathcal{L}_{RP}^2 := X \mid A \supset B \mid \forall X.\langle A_{f_{(j)}}\rangle_{j \in g^{-1}(k)} \supset X \rangle_{k \in K} \supset X
$$

where in the last clause we assume $\langle A_i \rangle$ to be an $I$-indexed family of $\mathcal{L}_{RP}^2$-formulas and $I \xrightarrow{f} J \xrightarrow{g} K$ to be the diagram determining a polynomial connective. A quantified formula of $\mathcal{L}_{RP}^2$ may therefore have quantified formulas as proper subformulas, provided they are in turn universal polynomial formulas.

Remark 2.12. It is clear that the restriction of the RP-translation to $\mathcal{L}^p$ is into $\mathcal{L}_{RP}^2$, and that if $\mathcal{D}$ is an $\mathcal{N}^{*}$-derivation then $\mathcal{D}^*$ is an $\mathcal{N}_{RP}^2$-derivation.

Remark 2.13. It is easy to check that the equational theory of $\mathcal{N}_{RP}^2$ is well-defined, since $\mathcal{L}_{RP}^2$ is closed under substitution.

Remark 2.14. The system $\mathcal{N}_{RP}^2$ is the smallest fragment of $\mathcal{N}^2$ closed under $\supset$ and containing the RP-translation of all polynomial connectives.

Observe that not only the translation of a formula $\mathcal{N}^{f,g}(A_i)$ is a universal polynomial formula, but for every universal polynomial formula $A$ there is at least (but in general more than) one configuration $I \xrightarrow{f} J \xrightarrow{g} K$ such that $A = \langle \mathcal{N}^{f,g}(A_i) \rangle^*$. To wit, let $A = \forall X.A_1 \supset \cdots \supset A_n \supset X$, where $A_i = A_{i_1} \supset \cdots \supset A_{i_k} \supset X$. It is enough to take $I = \langle 1, 1 \rangle, \ldots, \langle 1, k_1 \rangle, \ldots, \langle n, 1 \rangle, \ldots, \langle n, k_n \rangle$, so that $A_{(i,j)} = A_{ij}$, $K = \langle 1, \ldots, n \rangle$, $f$ to be the identity function and $g = \pi^2_f : \langle i, j \rangle \mapsto i$.

Given this it is easily seen by induction that for any $A \in \mathcal{L}_{RP}^2$, there is at least (but in general more than) one $B \in \mathcal{L}^p$ such that $B^* = A$. Note however that not every $\mathcal{N}_{RP}^2$-derivation is the image of some $\mathcal{N}^p$-derivation under the RP-translation.

Remark 2.15. In [30] we alluded to the fact that the translation of polynomial connectives can be described through a class of formulas called nested sp-$X$. There we used ‘sp-$X$’ as short for ‘strictly positive’ rather than ‘strongly positive’, where strictly positive formulas are a slight generalization of strongly positive formulas in which $X$ may not occur at all, and nested sp-$X$ formulas roughly stand to universal polynomial as strictly positive formulas stand to strongly positive formulas.
The class of polynomial formulas we consider here is thus a proper subset of the class of nested sp-X formulas and it is already sufficient for the goal of RP-translating polynomial connectives. However, most of the results we prove for $\Pi^2_{RP}$ can be extended to a similar (and slightly larger) system defined using nested sp-X (more generally these systems are fragments of a more general system $\Lambda^{n \leq 2}$ we are currently investigating, see [18]).

3 The $\varepsilon$-conversions

In this section we introduce some notational conventions and shortly recall some notions and results from the previous paper. These are based on the introduction of a notation for functors in natural deduction and of a class of conversions, called the $\varepsilon$-conversions, that express a naturality condition for natural deduction derivations.

3.1 Weak expansion

Definition 3.1. Given a second formula $A$ (whose structure can be describe as in (7), see page 7 above), the weak expansion of $A$ is the derivation that one obtains by repeatedly applying $\eta$-expansions to the derivation consisting only of the assumption of $A$ until the minimal formula of the main branch is atomic, i.e.:

$$
\begin{array}{c}
A \\
\vdash \forall (Y_2)(F_2 \supset \cdots \supset \forall (Y_n)(F_n \supset \forall (Y_{n+1})(X)) \ldots) \\
\forall (Y_2)(F_2 \supset \cdots \supset \forall (Y_n)(F_n \supset \forall (Y_{n+1})(X)) \ldots)
\end{array}
$$

$\Rightarrow \! E$

where $\forall G \vdash E$ and $\forall I \vdash G$ indicate (possibly empty) sequences of applications of $\forall E$ and $\forall I$. We will indicate by $El_{\vec{m}} \vdash \forall (Y_1)G \vdash G$ and $\forall (Y_1)G \vdash G$ indicate (possibly empty) sequences of applications of $\forall E$ and $\forall I$. We will indicate by $El_{\vec{m}} \vdash$ the “first half” of the derivation above, consisting of a chain of elimination rules depending on indices $\vec{m} = m_1, \ldots, m_n$ for the undischarged assumptions $F_1, \ldots, F_n$. Similarly, we will indicate by $In_{\vec{m}}$ the “second half” of the derivation above,
Remark 3.2. The weak expansion of $A$ might differ from the expanded normal form (also known as $\eta$-long normal form) of the derivation consisting only of the assumption of $A$, since in the latter not only the minimal formula in the main branch, but the minimal formulas of all branches are atomic, see Prawitz [20, §II.3.2.2.].

### 3.2 $C$-expansion

**Definition 3.3.** If $C = C_1 \supset \ldots \supset C_n \supset X$ is sp-$X$ and $D$ is a derivation of $B$ from undischarged assumptions $A, \Delta$ (where $A$ is freely chosen among the undischarged assumptions of $D$), the $C$-expansion of $D$ relative to $X$ on main assumption $A$, notation $C^X \begin{array}{c} A \\ \Delta \end{array} \begin{array}{c} D \\ B \end{array}$, is the derivation of $C[B/X]$ from $C[A/X], \Delta$ defined by induction on $n$ as follows:

- If $n = 0$, then $C = X$ and $C^X \begin{array}{c} A \\ \Delta \end{array} \begin{array}{c} D \\ B \end{array}$ is just $D$.

- If $n \geq 1$ then $C = C_1 \supset D$ where $X \notin FV(C_1)$ and $D$ is sp-$X$. We define:

$$
\begin{array}{c}
C^X \begin{array}{c} A \\ \Delta \end{array} \begin{array}{c} D \\ B \end{array} = C_1 \supset D[A/X] & \frac{C_1 \supset D[B/X]}{D^X} \\
\end{array}
$$

Remark 3.4. In [30, sec. 3.2] we defined the notion of $C$-expansion for a broader class of formulas called pn-$X$ formulas. The restriction of the definition to the class of sp-$X$ formulas allows a straightforward reformulation of the definition using the notion of weak expansion that we give in Appendix C.

Remark 3.5. In the functorial semantics of $\mathcal{L}^2$, what we call the $C$-expansion of a derivation $D$ is just the result of applying the functor interpreting $C$ to the morphism interpreting $D$. As any $\mathcal{L}^2$-formula can be interpreted as a functor, the notion of $C$-expansion can be extended to any $C \in \mathcal{L}^2$.

Remark 3.6. Whenever $X$ is clear from the context, with the notation for substitution introduced in [30, sec. 3.2], we write $C[B/X]$ and $C[A/X]$ as $C \begin{array}{c} B \\ D \end{array}$ and $C \begin{array}{c} A \\ D \end{array}$, and leaving the main assumption $A$ implicit we indicate the $C$-expansion of $D$ as $C \begin{array}{c} D \\ \Delta \end{array}$.

Remark 3.7. In the present paper, we will only be concerned with the $C$-expansion of derivations of the form $E_{\vec{m}} \begin{array}{c} A \end{array}$. Observe that, by the definition of $C$-expansion, the
undischarged assumptions of $C A$ besides $C A$ are the same as the undischarged assumptions of $E^m_{r A}$ besides $A$, namely (see Definition 3.1 above) $m_1, \ldots, m_n$.

### 3.3 The $\varepsilon$-conversions

As recalled in the introduction, it is common to characterize identity of proofs using an equivalence induced by (the symmetric closure of) some reduction relation over derivations. The equivalence $\simeq_{\beta\eta}^\gamma$ generated by the $\beta$- and $\eta$-conversions for $\text{NI}^2$ is standard (and recalled in Appendix B). Propositional connectives like $\lor$ require, in addition to $\beta$- and $\eta$-conversion rules, the so-called permutative conversions (which allow to permute an elimination rule upwards a $\lor$-elimination rule), that we call here $\gamma$-conversions, generating an equivalence relation $\simeq_{\beta\eta\gamma}$. Analogous conversions can be defined for all polynomial connectives, see Appendix B.

**Remark 3.8.** The equivalence relation $\simeq_{\beta\eta\gamma}$ can be further extended in two equivalent ways: either by replacing $\gamma$ by a stronger permutation $\gamma^+$ which allows the permutation of an arbitrary derivation upwards across an application of $\top E$, or by replacing $\eta$ by a stronger $\eta^+$ which expresses in categorical terms the universality of the connective ($\eta^+$ in fact subsumes both $\gamma$ and $\gamma^+$).

The equivalence-preservation problem of the RP-translation can be formulated as the failure of the implication below (see [30]):

$$\mathcal{D}_1 \simeq_{\beta\eta\gamma} \mathcal{D}_2 \quad \Rightarrow \quad \mathcal{D}_1^* \simeq_{\beta\eta} \mathcal{D}_2^*$$

In [30] (see Section 4.1) we showed that the implication above does hold when the equivalence we consider for $\text{NI}^2$ is the one induced by adding to $\beta$- and $\eta$-conversions a new class of conversions, called $\varepsilon$-conversions (for the case of $\circ$ these are shown in Table 2, for arbitrary universal polynomial formulas see Appendix E).

**Proposition 1 ([30]).** For all $\text{NI}^\gamma$-derivations $\mathcal{D}_1$ and $\mathcal{D}_2$, if $\mathcal{D}_1 \simeq_{\beta\eta\gamma} \mathcal{D}_2$, then $\mathcal{D}_1^* \simeq_{\beta\varepsilon} \mathcal{D}_2^*$.

Semantically, the $\varepsilon$-conversions express a naturality condition for $\text{NI}^2$-derivations. Thus, the results of our previous paper were a syntactic reformulation of some well-known properties which hold in parametric models of $\text{NI}^2$ (see [1] and [13]).

**Remark 3.9.** The proof of Proposition 1 scales straightforwardly to the whole of $\text{NI}^p$ (see Appendix F), and it actually holds if one replaces $\eta$ (resp. $\gamma$) with the more general $\eta^+$ (resp. $\gamma^+$) (see Remark 3.8 above, Remark B.1 in Appendix B and Section 5.1 below). The class of $\varepsilon$-conversions needed to establish the analog of Proposition 1 for $\text{NI}^\bullet$ are depicted in Table 2.

### 4 Predicative translations via atomization

In this section we show that the RP-translation can be related to the predicative translations by embedding the $\text{NI}^z_{\text{RP}}$ fragment of $\text{NI}^2$ into the atomic fragment $\text{NI}^z_{\text{at}}$. 


4.1 The FF- and ESF-translations

As we recalled in the introduction, Ferreira and Ferreira [6, 7, 8, 10], proposed an alternative translation of \( \text{NI} \)-derivations into \( \text{NI}^2 \) that we call the FF-translation. The FF-translation agrees with the RP-translation on how to translate formulas, but not on how to translate derivations. In particular, the FF-translation does not use the full power of the \( \forall E \) rule of \( \text{NI}^2 \), but rather it maps derivations in \( \text{NI} \) into derivations in \( \text{NI}^2_{\text{at}} \).

The FF-translation exploits the property of instantiation overflow:

**Definition 4.1.** A formula \( \forall X.A \in L^2 \) enjoys the instantiation overflow property iff for all formulas \( B \in L^2 \)

\[
\forall X.A \vdash_{\text{NI}^2_{\text{at}}} A[B/X]
\]

While Ferreira and Ferreira only address standard intuitionistic connectives, it is easily seen that the instantiation overflow property holds for all universal polynomial formulas \( \forall X.A \in L^2_{\text{RP}} \) (and actually for many more, see [17]). The results of Ferreira and Ferreira thus scale to the whole system \( \text{NI}^p \). For simplicity we will focus here on the connective \( \bullet \) and on the fragments \( \text{NI}^* \) and \( \text{NI}^2_{\text{at}} \) of \( \text{NI}^p \) and \( \text{NI}^2_{\text{RP}} \) respectively. However, all definitions and results here presented scale to all polynomial connectives, as shown in Appendix.

By reformulating Ferreira and Ferreira’s insight, we define an embedding of \( \text{NI}^2_{\text{at}} \) into \( \text{NI}^2_{\text{at}} \) that we call FF-atomization, and using it we define the FF-translation from \( \text{NI}^* \) into \( \text{NI}^2_{\text{at}} \) as the composition of the RP-translation and FF-atomization.

**Definition 4.2 (FF-atomization, FF-translation).** If \( \mathcal{D} \) is an \( \text{NI}^2_{\text{at}} \)-derivation, the FF-atomization of \( \mathcal{D} \), which we indicate as \( \mathcal{D}^\dagger \), is the \( \text{NI}^2_{\text{at}} \)-derivation defined by induction on \( \mathcal{D} \) as follows. We only consider the case in which the last rule \( \mathcal{D} \) is \( \forall E \) with a non-atomic witness, since all other rules are translated in a trivial way. In this case observe
We define $D^1$ by a sub-induction on $F$.

- If $F = F_1 \supset F_2$ then

$$D^1 = \frac{\forall X.\, \phi(A, B, C)}{\phi(A, B, C)[F / X]} \forall E$$

- If $F = \exists Z. F'$, the clause is analogous to the previous one (for a fully detailed definition, see Definition 12 in Appendix G).

Remark 4.3. Ferreira and Ferreira present their result in a different way, by using the inductive clauses of Definition 4.2 to give a direct proof of instantiation overflow for the universal formulas of the form $(A \lor B)^*$, and they refer to what we here called the FF-translation of $D$ as to “the canonical translation of $D$ in $F_{at}$ provided by instantiation overflow” [10]. This difference in presentation will allow a more straightforward formulation of our results.

Whereas the FF-translation is defined by combining the RP-translation from $\mathbf{NI}^*$ into $\mathbf{NI}^*_{RP}$ with the FF-atomization embedding $\mathbf{NI}^*_{RP}$ into $\mathbf{NI}^*_{at}$, more recently Espirito Santo and Ferreira [4] introduced an alternative translation by “directly” defining an embedding $\sharp$ from $\mathbf{NI}$ into $\mathbf{NI}^*_{at}$. We will refer to this translation as the ESF-translation. Using the notation we introduced, the definition of $\sharp$ can be adapted to $\mathbf{NI}^*$ (the definition actually scales to the whole of $\mathbf{NI}^P$, see Def. G.3 in Appendix G), and the crucial case of the definition, that of an $\mathbf{NI}^*$-derivation ending with an application of $\bullet E$ runs as follows (assuming $at(C^*) = Z$):
What is remarkable about these two translations is not only that they show that one can translate $\mathbf{NI}^*$ using only a very weak predicative fragment of $\mathbf{NI}^2$, but also the fact that, unlike the RP-translation, they preserve the equivalence induced by $\gamma$- and $\eta$-conversion. In fact, both translations map not only $\beta$-, but also $\eta$- and $\gamma$-reduction steps in $\mathbf{NI}$ onto chains of $\beta\eta$-reduction steps (respectively $\beta\eta$-equivalences) in $\mathbf{NI}^2$:

**Proposition 2.** For all $\mathbf{NI}^*$-derivations $\mathcal{D}_1, \mathcal{D}_2$, if $\mathcal{D}_1 \Rightarrow^\beta\eta\gamma \mathcal{D}_2$, then

1. $\mathcal{D}_1^\downarrow \Rightarrow^\beta\eta \mathcal{D}_2^\downarrow$.
2. $\mathcal{D}_1^\downarrow \approx_{\beta\eta} \mathcal{D}_2^\downarrow$.

**Proof.** Point 1 of the proposition was proved for the whole of $\mathbf{NI}$ by Ferreira and Ferreira [7] (for $\beta\gamma$) and by Ferreira [10] for $\eta$ (although in this case by adding a primitive conjunction to $\mathbf{NI}^2$, see below Section 5.1), and we claim that, at the cost of tedious computations, those proofs can be scaled to the whole of $\mathbf{NI}^p$. In Appendix H we give a proof scaled to the whole of $\mathbf{NI}^p$ of point 2, that generalizes the analogous result for $\mathbf{NI}$ of Espírito Santo and Ferreira in [4].

### 4.2 The $\varepsilon$-translation

We now show that the $\varepsilon$-rule can be used to clarify the relationship between the RP-translation and the FF- and ESF-translations. To see how, we define an alternative atomization procedure yielding yet another translation of $\mathbf{NI}^*$ into $\mathbf{NI}^2_{\varepsilon}$ (also this atomization scales to the whole of $\mathbf{NI}^p$, see Definition G.2 in Appendix G):

**Definition 4.4** ($\varepsilon$-atomization, $\varepsilon$-translation). The definition differs from Definition 4.2 in the following respect: In case

$$\mathcal{D} = \forall X. \circ(A, B, C) \quad \circ(A, B, C)[F/X] \quad \forall E$$

where $F$ is not atomic and $Z$ is the variable in rightmost position in $F$, using the notation introduced in Sections 3.1–3.1 (see in particular Remark 3.7), we define:
If $D$ is an $\mathbf{NI}^\ast$-derivation, we call the $\mathbf{NI}_2^\ast$-derivation $D^{\ast\epsilon}$ the $\epsilon$-translation of $D$.

The name "$\epsilon$-atomization" is justified by the fact that, as can be easily verified in the case of $\circ$, the $\epsilon$-atomization $D^{\ast\epsilon}$ of an $\mathbf{NI}_2^\ast$-derivation $D$ is the result of applying $\eta$-expansions and $\epsilon$-conversions to $D$:

**Proposition 3.** If $D$ is an $\mathbf{NI}_2^\ast$-derivation, then $D \simeq_{\eta\epsilon} D^{\ast\epsilon}$.

**Proof.** See Appendix I. 

The relationship between the three predicative translations is very close. In fact, they yield $\beta$-equivalent derivations, with the $\epsilon$-translation lying between the FF-translation and the ESF-translation, in the sense below:

**Proposition 4.** For all $\mathbf{NI}^\ast$-derivations $D$, $D^{\ast\epsilon} \leadsto_{\beta} D^{\ast\epsilon} \leadsto_{\beta} D^\dagger$.

**Proof.** See Appendix J. 

**Remark 4.5.** That $D^{\ast\epsilon} \leadsto_{\beta} D^\dagger$ was already known from [4].

By putting together Proposition 2 and 4 we can deduce that also the $\epsilon$-translation preserves permutative conversions and $\eta$-conversions:

**Corollary 4.6.** For all $\mathbf{NI}^\ast$-derivations $D_1, D_2$, if $D_1 \simeq_{\beta\gamma\eta} D_2$, then $D_1^{\ast\epsilon} \simeq_{\beta\eta} D_2^{\ast\epsilon}$.

**Remark 4.7.** The statement of Corollary 16 cannot be expressed in terms of reduction, but only in terms of equivalence, due to the fact that neither $D_1 \leadsto_{\eta} D_2$ nor $D_1 \leadsto_{\gamma} D_2$ imply $D_1^{\ast\epsilon} \leadsto_{\beta\eta} D_2^{\ast\epsilon}$, but only $D_1^{\ast\epsilon} \simeq_{\beta\eta} D_2^{\ast\epsilon}$.

By combining Proposition 3 with Proposition 4 we deduce that the FF- and ESF-translations are $\beta\eta\epsilon$-equivalent to the RP-translation:

**Corollary 4.8.** For all $\mathbf{NI}^\ast$-derivations $D$, $D^\ast \simeq_{\beta\eta\epsilon} D^{\ast\epsilon}$ and $D^\ast \simeq_{\beta\eta\epsilon} D^\dagger$.

The relationship between the four different translations is illustrated in Table 3.

Summing up, all three predicative translations are equivalent, modulo $\beta\eta\epsilon$-equivalence to the RP-translation and thus, semantically, they all interpret an $\mathbf{NI}^\ast$-derivation using the same second-order proof (provided one understands second-order proofs as satisfying the naturality conditions expressed by $\epsilon$-conversions).
For all $\text{NI}^*$ derivations $D_1$ and $D_2$ such that $D_1 \rightsquigarrow_{\beta\eta}\eta\varepsilon D_2$:

$D_1^* \rightsquigarrow_{\eta\varepsilon} D_1$

$D_1^* \rightsquigarrow_{\beta} D_1^* \rightsquigarrow_{\beta} D_1^* \rightsquigarrow_{\eta\varepsilon} D_1$

$D_2^* \rightsquigarrow_{\eta\varepsilon} D_2^* \rightsquigarrow_{\beta} D_2^* \rightsquigarrow_{\beta} D_2^* \rightsquigarrow_{\eta\varepsilon} D_2$

Table 3: Relationship among the RP-, FF-, ESF- and $\varepsilon$-translations.

5 RP-translation and $\approx_{\beta\eta\varepsilon}$ vs predicative translations and $\approx_{\beta\eta}$

Given the results of the previous sections, it might be tempting to say that the approach based on atomic polymorphism might provide a fully syntactic alternative to categorical semantics and related techniques for the study of identity of proofs for propositional connectives.

In this section we argue that this is not entirely the case, by stressing two important limitations of the predicative translations. First, they do not preserve the whole equational theory of propositional connectives, and in particular the predicative translations of $\land$, $\lor$ do not yield products and co-products in $\text{NI}_1^2$. Hence, strictu sensu the predicative translations do not fully solve the equivalence-preservation problem. Second, the $\forall$E rule restricted to atomic witnesses is too weak to prove the logical equivalence between a connective and its second-order translation. In Prawitz’ terminology [19], we show that the connective $\lor$ is not strongly definable in $\text{F}_1$ but only weakly definable.

Conversely, we highlight that the approach based on atomic polymorphism looks more apt to the syntactic study of proof reduction, since the rewriting theory induced by $\varepsilon$-conversions looks rather intricate.

5.1 Predicative translations and generalized permutations

As is well known, in order to obtain a perfect match between the syntax of $\text{NI}$ and the free bi-cartesian closed category, it is necessary to consider the following stronger permutative conversions in which every chunk of derivation (and not just those consisting of applications of elimination rules) can be permuted-up across an application of disjunction
Proposition 1 can be strengthened by replacing $\gamma$ with $\gamma^+$, so that the RP-translations of the left- and right-hand side of any instance of $\gamma^+$ are $\beta\varepsilon$-equivalent $\mathcal{NI}^2$-derivations (see Appendix F).

However, Proposition 2 ceases to hold as soon as one replaces $\gamma$ with $\gamma^+$ (even if one replaces reduction $\rightsquigarrow$ with equivalence $\simeq$ in point 1). That is, although the predicative translations preserve the equivalences induced by $\eta$- and $\gamma$-conversions, they do not preserve all equivalences needed to interpret $\mathcal{NI}$ as a bi-cartesian closed category.

To see this, it is enough to consider the instance of $\gamma^+$ in which $A$, $B$, $C$ and $D$ are atoms, $\mathcal{D}$ (respectively $\mathcal{D}_1$, $\mathcal{D}_2$) consists only of the assumptions of $A \lor B$ (resp. $C$), and $\mathcal{D}_3$ is the derivation $C \Rightarrow D \Rightarrow C \Rightarrow E$. As the reader can easily check, the predicative translations of the left and right-hand side of this instance of $\rightsquigarrow_{\gamma^+}$ are not $\beta\eta$-equivalent, just like their RP-translations.

That the predicative translations fail to preserve the stronger permutations provides an explanation for another puzzling aspect of the approach of Ferreira and co-authors. As observed in [10], Proposition 2 fails if $\mathcal{NI}^\gamma$ is extended to include the standard rules of conjunction. In particular, the predicative translations fail to preserve all instances of the $\eta$-equation for the standard conjunction rules. Nonetheless, when the two standard elimination rules $\wedge^E_1$ and $\wedge^E_2$ are replaced by the general elimination rule $\wedge^E_p$ as in $\mathcal{NI}^p$ (see above Remark 2.1 and Remark 2.4), Proposition 2 applies to the equivalence relation induced by the $\beta$-, $\eta$- and $\gamma$-conversions for “generalized” conjunction (see Appendix B for a formulation of these conversions in $\lambda$-calculus notation and Appendix [11] for a proof of Proposition 2 scaled to $\mathcal{NI}^p$).

The standard elimination rules can be defined using the general elimination rule as follows:

$$
\begin{array}{c}
A \land B \\
A
\end{array} \quad [A] \quad \wedge^E_p \quad \begin{array}{c}
A \land B \\
B
\end{array} \quad [B] \quad \wedge^E_p
$$

and given these definitions, the $\eta$-equations for the standard conjunction rules become the following:

These stronger permutations were first formulated in natural deduction format by Seely [26] and have been recently discussed in the context of proof-theoretic semantics by Tranchini [28, 29].

The general form of the $\gamma^+$-conversions for an arbitrary polynomial connective (of which the one given here are a simple instance) is given in Appendix B.

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6These stronger permutations were first formulated in natural deduction format by Seely [26] and have been recently discussed in the context of proof-theoretic semantics by Tranchini [28, 29].

7The general form of the $\gamma^+$-conversions for an arbitrary polynomial connective (of which the one given here are a simple instance) is given in Appendix B.
The different behavior of the standard and the general elimination rules is explained by the fact that, as in the case of disjunction, the predicative translations of two $\gamma^+$-equivalent $\text{NI}^2_{at}$-derivations need not be $\beta\eta$-equivalent $\text{NI}^2_{at}$-derivations, and that the $\gamma^+$-conversion for conjunction:

\[
\begin{array}{c}
\text{Proposition 5. For all } A \in \mathcal{L}^2: \\
A \vdash \text{NI}^2_{at}\ A^* \\
\end{array}
\]

Proof. See Appendix K

That is, as soon as one extends $\text{NI}^2$ with a primitive disjunction, one can show in the extended system $\text{NI}^2_{at}$ that every formula is interderivable with its RP-translation.\(^8\)

In contrast to what happens in $\text{NI}^2_{at}$, a propositional formula $A$ and its RP-translation $A^*$ may fail to be interderivable in the extension of $\text{NI}^2_{at}$ with a primitive disjunction. By inspecting one direction of the proof of Proposition 5 it is clear that $A \vdash_{\text{NI}^2_{at}} A^*$. However, the inspection of the other direction clearly suggests that, at least in some cases, in order to establish that $A^* \vdash_{\text{NI}^2_{at}} A$ it is essential to apply $\forall\text{E}$ with a non-atomic witness. In particular,

Proposition 6. $(Y \lor Z)^* \not\vdash_{\text{NI}^2_{at}} Y \lor Z$

Proof. See Appendix K

\(^8\)The same remarks of footnote \(^7\) apply.

\(^9\)Actually, $A$ and its RP-translation are not just interderivable, but can be shown to be isomorphic modulo $\sim_{\beta\eta\gamma^+\epsilon}$ (see for details [13]).
In $\text{NT}_{\text{al}}^{2\upnu}$ the RP-translation of $Y \lor Z$ is thus strictly weaker than $Y \lor Z$.

**Remark 5.1.** An immediate consequence of Proposition 6 concerns the faithfulness of the RP-translation. We recall that a translation $(\cdot)^{2}$ from $\mathcal{N}$ to $\mathcal{N'}$ is faithful if for all $\Gamma, A$, if $\Gamma \vdash A^{2}$ then $\Gamma \vdash_{\mathcal{N}} A$. Although the FF-translation is a faithful translation of $\text{NT}^{\nu}$ into $\text{NT}_{\text{al}}^{2\upnu}$ [9], it is *not* a faithful translation of $\text{NT}_{\text{al}}^{2\upnu}$ into $\text{NT}_{\text{al}}^{2\upnu}$ (since—keeping in mind that the FF- and the RP-translations of formulas coincide—$((Y \lor Z)^{\ast})^{\ast} = (Y \lor Z)^{\ast}$ and hence obviously $((Y \lor Z)^{\ast})^{\ast} \vdash_{\text{RP}}^{2\upnu} (Y \lor Z)^{\ast}$, but $(Y \lor Z)^{\ast} \not\vdash_{\text{RP}}^{2\upnu} Y \lor Z$ by the above proposition).

**Remark 5.2.** In addition to the failure of faithfulness, also the disjunction property fails for the translation of disjunction in $\text{NT}_{\text{al}}^{2\upnu}$ in the following sense: there exist formulas $A, B$ of $\text{NT}_{\text{al}}^{2\upnu}$ such that $\vdash_{\text{NT}_{\text{al}}^{2\upnu}} (A \lor B)^{\ast}$ holds but neither of $\vdash_{\text{NT}_{\text{al}}^{2\upnu}} A^{\ast}$ and $\vdash_{\text{NT}_{\text{al}}^{2\upnu}} B^{\ast}$ holds.

Let $A = \forall X Y. X \supset Y$ and $B = \exists X. A \supset X$, where $\exists X. A$ is shorthand for $\forall Y. (\forall X. A \supset Y) \supset Y$. One can easily check that $\not\vdash_{\text{NT}_{\text{al}}^{2\upnu}} A$ and that for all variable $Z$, $\not\vdash_{\text{NT}_{\text{al}}^{2\upnu}} A \supset Z$, from which one can deduce that $\not\vdash_{\text{NT}_{\text{al}}^{2\upnu}} B$, and notice that $A^{\ast} = A$ and $B^{\ast} = B$. On the other hand one can show that $\vdash_{\text{NT}_{\text{al}}^{2\upnu}} (A \lor B)^{\ast}$ as follows:

\[
\begin{array}{c}
\forall X. (A \supset X) \supset Y \\
(A \supset X) \supset Y \\
\forall E
\end{array}
\]

\[
\begin{array}{c}
\forall X. (A \supset X) \supset Y \\
\vdash_{\text{E}}
\end{array}
\]

\[
\begin{array}{c}
Y \\
\forall I
\end{array}
\]

\[
\begin{array}{c}
B \supset X \\
\vdash_{\text{I}(n)}
\end{array}
\]

\[
\begin{array}{c}
(B \supset X) \supset X \\
\forall I
\end{array}
\]

Another way of highlighting the difference between $(A \lor B)^{\ast}$ and $(A \lor B)$ in $\text{NT}_{\text{al}}^{2\upnu}$ is by observing that, whereas the disjunction elimination rule warrants that $A \lor B \vdash_{\text{NT}_{\text{al}}^{2\upnu}} (A \supset C) \supset (B \supset C) \supset C$ for all $A, B, C$ in $\mathcal{L}^{2\upnu}$, the same does not hold if one replaces $A \lor B$ with $\forall X. (A \lor B)$:

**Proposition 7.** There are $\mathcal{L}^{2\upnu}$-formulas $A, B, C$ such that

\[
\forall X. (A \lor B) \not\vdash_{\text{NT}_{\text{al}}^{2\upnu}} (A \lor B)[C/X]
\]

**Proof.** See Appendix [K].

That is, in contrast to what happens in $\text{NT}_{\text{al}}^{2\upnu}$, the instantiation overflow property fails for $\forall X. (A \lor B)$ in $\text{NT}_{\text{al}}^{2\upnu}$.

Given Proposition 6 the analog of Proposition 5 cannot hold for $\text{NT}_{\text{al}}^{2\upnu}$. One may therefore wonder in which sense, if at all, can one say that disjunction is definable in $\text{NT}_{\text{al}}^{2\upnu}$. Following Prawitz [19, p. 58], we can distinguish between *strong and weak definability* of a connective. In Prawitz’ terminology, disjunction is weakly definable in $\text{NT}_{\text{al}}^{2\upnu}$ if and only if there is a faithful translation of $\text{NT}^{\nu}$ into $\text{NT}_{\text{al}}^{2\upnu}$, which is the case, since the FF-translation
is faithful (see Remark 5.1 above); on the other hand, disjunction is strongly definable in \(\mathcal{L}^2_{\text{at}}\) iff for all \(A, B \in \mathcal{L}^\gamma\) there is an \(\mathcal{L}^2\)-formula \(C\), such that \(C \models_{\mathcal{L}^2_{\text{at}}} A \vee B\). \(^{10}\) Not only Proposition 6 shows that disjunction is not strongly defined by its RP-translation, but there is in fact no other formula strongly defining it in \(\mathcal{L}^2_{\text{at}}\):

**Proposition 8.** \(\vee\) is not strongly definable in \(\mathcal{L}^2_{\text{at}}\).

**Proof.** See Appendix K \(\square\)

### 5.3 Equivalence-preservation vs reduction-preservation

Although our discussion so far showed that the RP-translation coupled with \(\varepsilon\) yields stronger results concerning preservation of equivalence, when one restricts the attention to \(\beta, \eta\) and the weaker \(\gamma\) conversions the FF-translation yield a stronger result in that they preserve reduction and not merely equivalence. As observed, Proposition 1 cannot be strengthened by replacing equivalence with reduction, at least not in the case of \(\eta\) (as it is clear by inspecting the proof of Proposition 1 in Appendix F, see in particular Proposition 11).

At the same time, the fact that the predicative translations do preserve \(\eta\)- and \(\gamma\)-equivalences can be explained using Proposition 3 and Corollary 4.8, which essentially show that the translations into \(\mathcal{L}^2_{\text{at}}\) encapsulate those bits of \(\eta\)-expansions and of \(\varepsilon\)-conversions needed to translate \(\eta\)-reductions and \(\gamma\)-permutations of \(\mathcal{L}^\ast\).

Moreover, since \(\varepsilon\)-conversions allow to translate not only \(\gamma\) but also \(\gamma^+\)-permutation (see Remark 3.8), all disadvantages of the fully extensional reduction theory for disjunction (induced by \(\beta\) together with \(\eta^+\) or, equivalently with \(\eta\) and \(\gamma^+\)) are “imported” into \(\mathcal{L}^2\), in particular non-confluence and non-termination (see e.g. [15] for a discussion).

As in the case of disjunction, these problems do not exclude the possibility of considering restricted forms of \(\varepsilon\)-conversion (essentially, those in which only elimination rules are permuted across an application of \(\forall\mathcal{E}\)), and it is not implausible to thereby obtain a well-behaving reduction theory.

It is however remarkable that the \(\beta\eta\gamma^+\)-equational theory has been recently shown to be the maximum consistent equational theory in \(\mathcal{L}^2\) and this suggests the possibility of showing that the \(\beta\eta\gamma\)-equational theory is the maximum equational theory in the fragment \(\mathcal{L}^2_{\text{RP}}\) of \(\mathcal{L}^2\) (this line of research is currently being pursued by the authors [18]).

Summing up, we can say that \(\varepsilon\) is probably more useful for investigating proof-identity rather than proof reduction, but that its connections with category theory provide an explanation of how and why the predicative translations work, by embedding the syntactic results about them into a wider picture.

---

10Note that strong definability implies the existence of a faithful translation from \(\mathcal{L}^2_{\text{at}}\) to \(\mathcal{L}^2_{\text{at}}\).

11These remarks do not exclude the possibility of devising further reductions in \(\mathcal{L}^2\) that can allow to “simulate” those steps of \(\varepsilon\)-conversions and of \(\eta\)-expansions needed to formulate a version of Proposition 1 using reduction in place of equivalence, as done by Espírito Santo and Ferreira in [3].
6 Concluding remarks

6.1 Summary of the results

In this paper we have shown how the category-theory-inspired framework introduced in our previous paper can be used to clarify the relationship between the alternative translations proposed by Ferreira, Ferreira and Espírito Santo and the original Russell-Prawitz translation, and to provide semantic insights on the proof-theoretic properties of the former.

Our approach consisted in focusing on an atomizing translation from a suitable fragment of $\mathbb{NI}_2$ (in which universal formulas correspond to the translation of polynomial connectives) into $\mathbb{NI}_2^{at}$, and showing that such atomizations are obtained by using the $\varepsilon$-conversions.

This made it possible to show that the predicative translations produce derivations that are equivalent to the RP-translation modulo the $\varepsilon$-conversions (hence, in semantic terms, modulo parametricity), and that their proof-theoretic properties (the preservation of $\eta$- and permutative conversions) result from the fact that $\varepsilon$-conversions express a naturality condition for the proofs denoted by $\mathbb{NI}_2$-derivations.

Finally, we highlighted the trade-off between the approach based on atomic polymorphism and the more semantics-inspired approach based on the $\varepsilon$-conversions, and we argued that the former is better adapted to investigate proof reduction, while the second is more adapted for investigating identity of proofs.

6.2 Related and further work

Although a generalization of our $\varepsilon$-conversions (expressing a dinaturality condition, rather than just naturality) can be formulated for any formula $\forall X.A$ of $\mathcal{L}_2$, the tight connection between the $\varepsilon$-conversions and the phenomenon of instantiation overflow underlying the present paper seems to be limited to the class of universal polynomial formulas. In fact, in [30] the $\varepsilon$-conversions were defined for a broader class of formulas, that of nested $p\times X$ formulas, all enjoying the instantiation overflow property. However, it does not seem possible to define a uniform atomization procedure in the style of our $\varepsilon$-atomization for the $\mathbb{NI}_2$-derivations in which the premises of all applications of $\forall E$ are nested $p\times X$ (rather than universal polynomial) formulas. For an exact characterization of the class of $\mathcal{L}_2$-formulas enjoying instantiation overflow, see [2] and [17].

As to further directions of investigations, we observe that, in the extension of System $F$ with primitive propositional connectives, not only every proposition and its Russell-Prawitz translation are interderivable, but they are actually isomorphic (modulo the equivalence relation induced by $\beta$-, $\eta$-, $\gamma^+$-, and $\varepsilon$-equations). From a categorical perspective, such isomorphisms belong to a more general class of isomorphisms induced by a proof-theoretic formulation of the Yoneda lemma. This and further categorical aspects underlying the present work are the object of current ongoing work by the authors [18].
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24
A The system $\mathbf{NI}^{2p}$

We indicate by $\mathcal{TV}$ a countably infinite set of term variables $x, y, z, \ldots$. The terms of the system $\mathbf{NI}^{2p}$ are given by the following grammar (where, given a list $A = \langle a_1, \ldots, a_n \rangle$, we indicate an $A$-indexed list of terms $t_{a_1}, \ldots, t_{a_n}$ as $\langle t_{a_i} \rangle_{a \in A}$ and, below, simply with $\langle t_a \rangle$ whenever $A$ is clear from the context, see Remark 2.3):

$$t, u := x \in \mathcal{TV} \mid \lambda x. t \mid t u \mid \Lambda X.t \mid tB \mid \delta k(t_j)_{j \in g^{-1}(k)} \mid \delta k(t, \langle x_j \rangle_{j \in g^{-1}(k)} \cdot s_k)_{k \in K}$$

That is, for every connective $\dagger$ generated by $I \overset{\mathcal{I}}{\rightarrow} \mathcal{J} \overset{\mathcal{S}}{\rightarrow} \mathcal{K}$, we have a family of generalized injections $\delta_k^I$ indexed by elements of $\mathcal{K}$, and a generalized “case” constructor $\delta_\dagger$ with $|\mathcal{K}| + 1$ arguments.

Let $A = \langle a_1, \ldots, a_k \rangle$ be a finite list. We will use the following abbreviated notation to indicate $A$-indexed sequences of $\lambda$-abstractions and applications: if $\langle x_a \rangle_{a \in A}$ is an $A$-indexed list of variables, for any term $t$, we let $\lambda(x_a)_{a \in A} t$ (often abbreviated as $\lambda(x_a) t$) indicate the term $\lambda x_{a_1} \ldots \lambda x_{a_k} t$. Similarly, if $\langle t_a \rangle$ is an $A$-indexed list of terms, for any term $u$, we let $u(x_a)_{a \in A}$ (often abbreviated as $u(x_a)$) indicate the term $u t_{a_1} \ldots t_{a_k}$.

As usual, by a typing context we indicate a finite set of type declarations $x : A$ where all declared variables are distinct. We indicate typing contexts with $\Gamma, \Delta, \Sigma$.

The typing rules for $\mathbf{NI}^{2p}$ are the following:

$$\frac{}{\Gamma, x : A \vdash x : A} \text{Ax} \quad \frac{}{\Gamma, x : A \vdash t : B} \quad \frac{\Gamma \vdash t : A}{\Gamma \vdash \lambda x. t : \forall X. A} \quad \frac{\Gamma \vdash \forall X. A}{\Gamma \vdash \forall X. A} \quad \frac{\Gamma \vdash \forall X. A}{\Gamma \vdash t : A \vdash B} \quad \frac{\Gamma \vdash u : A}{\Gamma \vdash u : A}$$

$$\frac{\Gamma \vdash t : A \vdash B}{\Gamma \vdash \lambda x. t : A \vdash B} \quad \frac{\Gamma \vdash t : \forall X. A}{\Gamma \vdash t : A \vdash B} \quad \frac{\Gamma \vdash \forall X. A}{\Gamma \vdash \forall X. A}$$

We write $\Gamma \vdash_{\mathbf{NI}^{2p}} t : A$ iff there is a derivation of $\Gamma \vdash t : A$ using the above rules. Similarly for $\Gamma \vdash_{\mathbf{NI}^{2p}} t : A$ and all other systems introduced in Sections 2.1 and 2.2.

A term context (or simply context when no ambiguity with typing contexts arise, indicated as $\mathbb{T}, \mathbb{U}, \ldots$) is a term with a distinguished variable that we indicate as the “hole” $[\ ]$. Observe that this is a generalization of the standard notion of context, since we allow the hole $[\ ]$ to occur zero, one or (finitely) many times in a context $\mathbb{T}$. For
any context \( T \), we let \( T[t] \) (resp. \( T[\mathcal{U}] \)) indicate usual—i.e. non variable-capturing—substitution (short n.v.c.-substitution) of the term \( t \) (resp. context \( \mathcal{U} \)) for \([ \ ] \) in \( T \), and \( T[t] \) (respectively \( T[\mathcal{U}] \)) indicate variable-capturing substitution (short v.c.-substitution) of the term \( t \) (resp. context \( \mathcal{U} \)) for \([ \ ] \) in \( T \). Moreover, we let \( T : A \vdash B \) be a shorthand for \( \Gamma, x : A \vdash T[x] : B \). Note that if \( T : A \vdash B \) and \( \Gamma \vdash t : A \), not only \( \Gamma \vdash T[t] : B \) but also \( \Gamma \vdash T[t] : B \) hold.

We will use the following special families of contexts:

- the **principal contexts** (indicated as \( C, \mathcal{D}, \ldots \)) are defined by the grammar below:

\[
C := [ ] \mid Cu \mid CB \mid \delta_1(C,\langle y_j \rangle .s_k) \mid \lambda x.C \mid \Lambda Y.C \mid \lambda x.<t_1, \ldots, t_{i-1} , C, t_{i+1}, \ldots, t_{|g^{-1}(k)|}> 
\]

- the **elimination contexts** are defined by dropping the cases \( \lambda x.C, \Lambda Y.C \) and \( \langle t_1, \ldots, t_{i-1} , C, t_{i+1}, \ldots, t_{|g^{-1}(k)|} \rangle \) from the grammar above;

- the **introduction contexts** are defined by dropping the cases \( Cu, CB \) and \( \delta_1(E,\langle y_j \rangle .s_k) \) from the grammar above.

Observe that all principal contexts are contexts in the standard sense (i.e. they all contain exactly one occurrence of the hole). Note that if \( C, \mathcal{D} \) are principal contexts, then \( C[\mathcal{D}] \) and \( C[\mathcal{D}] \) are principal contexts as well.

It is easily checked that if \( E \) is an elimination context, then for all term \( t \), \( E[t] \) cannot capture variables of \( t \), whence \( E[t] = E[t] \).

## B The standard equivalence on derivations

The rules of equivalence are the following:

\[
\begin{align*}
\Gamma, x : A \vdash t : B \quad & \quad \Gamma \vdash s : A \\
\Gamma \vdash (\lambda x.t)s = t[s/x] : B & \quad \Rightarrow \beta \\
\Gamma \vdash t : A \supset B & \quad \Rightarrow \eta \quad (x \notin \text{FV}(t)) \\
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash t : A \quad & \quad \forall \beta \quad (X \notin \text{FV}(\Gamma)) \\
\Gamma \vdash t : \forall X. A & \quad \Rightarrow \forall \eta \quad (X \notin \text{FV}(\Gamma)) \\
\end{align*}
\]

\[
\begin{align*}
\langle \Gamma \vdash t_j : A_{f(j)} \rangle_{j \in g^{-1}(k)} \quad & \quad \langle \Gamma, \langle y_j : A_{f(j)} \rangle_{j \in g^{-1}(k)} \vdash s_k : C \rangle_{k \in \mathcal{K}} \\
\Gamma \vdash \delta_1(t_j, \langle y_j \rangle_{j \in g^{-1}(k)}, \langle \langle y_j \rangle_{j \in g^{-1}(k)} . s_k \rangle_{k \in \mathcal{K}}) & \Rightarrow s_k \langle t_j \rangle_{\langle y_j \rangle : \langle \langle y_j \rangle : C \rangle_{k \in \mathcal{K}} \rangle_{k \in \mathcal{K}} \\
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash t : \hat{\langle A_i \rangle} & \quad \Upsilon : \hat{\langle A_i \rangle} = \Gamma \quad \Upsilon \quad \eta^+ \\
\Gamma \vdash \delta_1(t, \langle y_j \rangle . \Lambda[\langle y_j \rangle]_{k \in \mathcal{K}}) & \Rightarrow \Upsilon \langle t \rangle : C \\
\end{align*}
\]

together with reflexivity, transitivity and symmetry and congruence rules.

**Remark B.1.** As in the case of disjunction (see, e.g. [26] and [15]) the “generalized” rule \( \eta^+ \) for \( \hat{-} \) connectives can be decomposed into a simple form of \( \eta \)-rule and a “generalized” permutation rule:

\[
\begin{align*}
\Gamma \vdash t : \hat{\langle A_i \rangle} & \quad \Upsilon : \hat{\langle A_i \rangle} = \Gamma \quad \Upsilon \quad \eta^+ \\
\Gamma \vdash \delta_1(t, \langle y_j \rangle . \Lambda[\langle y_j \rangle]_{k \in \mathcal{K}}) & \Rightarrow \Upsilon \langle t \rangle : C \\
\end{align*}
\]

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\[
\Gamma \vdash t : \dagger(A_i) \quad \langle \Gamma, \langle y_j : A_{f(j)} \rangle \vdash s_k : F' \rangle_{k \in K} \quad U : F' \vdash \Gamma \iff 
\Gamma \vdash U[\delta_1(t, \langle \langle y_j \rangle \cdot s_k \rangle_{k \in K})] \simeq \delta_1(t, \langle \langle y_j \rangle \cdot U[s_k] \rangle_{k \in K}) : G \quad \dagger \gamma^+
\]

The rule \(\dagger \gamma\) expressing the standard permuting conversions used in establishing the subformula of normal derivations in \(NI\) is the special case of \(\dagger \gamma^+\) in which the context \(U\) is an elimination context.

We write

\[
\begin{array}{lcl}
\Gamma \vdash \eta_{2^p} t \equiv \beta s : A & \text{iff there is a derivation of} & \exists \beta, \forall \beta, \dagger \beta_k \\
\Gamma \vdash \eta_{2^p} t \equiv \eta^+ s : A & \text{if there is a derivation using the rules underlying both} & \exists \eta, \forall \eta, \dagger \eta^+
\end{array}
\]

as well as reflexivity, transitivity, symmetry and the congruence rules. With e.g. \(\Gamma \vdash \eta_{2^p} t \equiv \eta s : A\) we indicate that there is a derivation using the rules underlying both \(\Gamma \vdash \eta_{2^p} t \equiv \beta s : A\) and \(\Gamma \vdash \eta_{2^p} t \equiv \eta s : A\), and similarly for other “combined” labels.

Similar notation will be used for sub-systems of \(\eta_{2^p}\). If the derivations do not use symmetry we write \(\rightsquigarrow\) in place of \(\equiv\).

**Remark B.2.** The rules \(\dagger \eta^+\) and \(\dagger \gamma^+\) above are formulated using n.v.c-substitution of a term in the context \(U\). Observe that the variants \(\dagger \eta^+_{v.c.}\) and \(\dagger \gamma^+_{v.c.}\) of these rules with v.c.-substitution in place of n.v.c-substitution are derivable from \(\dagger \eta^+\) and \(\dagger \gamma^+\) (together with the other conversion rules and symmetry). The proof of the derivability of the v.c.-variants is by induction on the context \(U\), and the only critical cases are those in which \(U = \lambda x. U'\), \(U = \lambda X. U'\) or \(U\) is a \(\eta_1\)-term in which one or more of its immediate subterms in \(\langle s_k \rangle\) are a context \(U'\). We give the details of the inductive case for \(\dagger \eta^+_v{e.}\) in which \(U = \lambda x. U'\) by informally showing how to construct a derivation of \(\Gamma \vdash U[\delta_1(t, \langle \langle y_j \rangle \cdot s_k \rangle)] \simeq \delta_1(t, \langle \langle y_j \rangle \cdot U[s_k] \rangle) : G\) as follows:

\[
\begin{align*}
\lambda x. U'[\delta_1(t, \langle \langle y_j \rangle \cdot s_k \rangle)] & \quad \text{I.H.} \\
\simeq_{\eta_1} & \quad \lambda x. \delta_1(t, \langle \langle y_j \rangle \cdot U'[s_k] \rangle) \\
\simeq_{\eta_1} & \quad \lambda x. \delta_1(t, \langle \langle y_j \rangle \cdot \lambda x. U'[s_k] \rangle) \quad \simeq_{\eta_1} \\
\simeq_{\gamma^+} & \quad \lambda x. \delta_1(t, \langle \langle y_j \rangle \cdot \lambda x. U'[s_k] \rangle) \ rightsquigarrow_{\eta_1} \delta_1(t, \langle \langle y_j \rangle \cdot \lambda x. U'[s_k] \rangle)
\end{align*}
\]

The other critical cases for \(\dagger \gamma^+\) and those for \(\dagger \eta^+\) are similar.

**Remark B.3.** Assuming the standard elimination rules for \(\land\) are defined as \(\pi_1(t) := \delta_0(t, y_1 y_2 y_1)\) and \(\pi_2(t) := \delta_0(t, y_1 y_2 y_2)\) (and taking \(\iota_\alpha(t_1, t_2)\) as prefix notation for the standard pairing), we can derive the standard \(\eta\) rule for conjunction \(\iota_\alpha(\pi_1(t), \pi_2(t)) \simeq t\) as follows:

\[
\iota_\alpha(\delta_0(t, y_1 y_2 y_1), \delta_0(t, y_3 y_4 y_4)) \rightsquigarrow \dagger \gamma^+
\]

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\[\vdash_{\gamma^+} \delta_\gamma(t, y_1 y_2, \underline{\underline{x}}(y_1, \delta_{\lambda_\gamma}(t, y_1 y_4, y_4))) \quad \vdash_{\gamma^+} \delta_\gamma(t, y_1 y_2, \underline{\underline{x}}(y_1, y_4), y_3 y_4, \underline{\underline{x}}(t, y_1, y_4)) \]
\[\cong_{\eta} \delta_\gamma(t, z_1 z_2, \underline{\underline{x}}(z_1, z_2)), y_1 y_2, \delta_{\lambda_\gamma}(t, z_1 z_2, \underline{\underline{x}}(z_1, z_2)), y_3 y_4, \underline{\underline{x}}(y_1, y_4)) \quad \vdash_{\eta} \]
\[\vdash_{\beta} \delta_\beta(t, z_1 z_2, \underline{\underline{x}}(t, z_1, z_2)), y_1 y_2, \underline{\underline{x}}(t, z_1, z_2)) \quad \vdash_{\beta} \delta_\beta(t, z_1 z_2, \underline{\underline{x}}(t, z_1, z_2)) \]
\[\vdash_{\eta} t \]

where the three applications of \(\uparrow_{\gamma^+}\) can be described as:

- \(\mathcal{U}[\delta_\gamma(t, y_1 y_2, \underline{\underline{x}}(y_1, \delta_{\lambda_\gamma}(t, y_1 y_4, y_4))) \approx_{\gamma^+} \delta_\gamma(t, y_1 y_2, \mathcal{U}[y_1]) \text{ with } \mathcal{U} := \varepsilon_{\lambda}(\mathcal{U}[\mathcal{U}[y_1]]);\)
- \(\mathcal{U}[\delta_\gamma(t, y_3 y_4, y_4)] \vdash_{\gamma^+} \delta_\gamma(t, y_3 y_4, \mathcal{U}[y_4]) \text{ with } \mathcal{U} := \varepsilon_{\lambda}(y_1, y_4];\)
- \(\mathcal{U}[\delta_\gamma(t, z_1 z_2, \underline{\underline{x}}(z_1, z_2)) \vdash_{\gamma^+} \delta_\gamma(t, z_1 z_2, \mathcal{U}[\underline{\underline{x}}(z_1, z_2)]) \text{ with } \mathcal{U} := \cdot_{\lambda}(\cdot_{\lambda}(\cdot_{\lambda}(t, z_1, z_2), \cdot_{\lambda}(y_1, y_4)).\]

### C Weak expansion and A-expansion

**Definition C.1.** For any \(A \in \mathcal{L}^2\) we define a family of atomic elimination contexts \(\text{Elim}(A)\) and a family of introduction contexts \(\text{Intro}(A)\) by induction as follows:

- If \(A = Z\), then \(\text{Elim}(A) = \text{Intro}(A) = \{[\mathcal{U}]\};\)
- If \(A = B \rightarrow C\), then \(\text{Elim}(A) = \{\mathcal{E}[\mathcal{U}[x]] \mid x \text{ not free in } \mathcal{E} \text{ and } \mathcal{E} \in \text{Elim}(C)\}\) and \(\text{Intro}(A) = \{\lambda x.\mathcal{I} \mid \mathcal{I} \in \text{Intro}(C)\}\)
- If \(A = \forall X.C\), then \(\text{Elim}(A) = \{\mathcal{E}[\mathcal{U}[X]] \mid X \text{ not free in } \mathcal{E} \text{ and } \mathcal{E} \in \text{Elim}(C)\}\) and \(\text{Intro}(A) = \{\lambda X.\mathcal{I} \mid \mathcal{I} \in \text{Intro}(C)\}\)

An expansion pair for \(A\) is a pair of contexts \((\mathcal{E}_1, \mathcal{I}_n)\), where \(\mathcal{E}_1 \in \text{Elim}(A), \mathcal{I}_n \in \text{Intro}(A)\) and \(\mathcal{I}_n[\mathcal{E}_1] \cong_{\eta} [\mathcal{U}]\).

We list some useful and easily established facts about introduction and elimination contexts (recalling that for all elimination context \(E\) and term \(t, E[t] = E[t]\)):

**Fact C.2.** If \(\Gamma \vdash_{\mathcal{L}^2} t : A\) then for all expansion pairs \((\mathcal{E}_1, \mathcal{I}_n)\) for \(A, \Gamma \vdash t \cong_{\eta} \mathcal{I}_n[\mathcal{E}_1[t]] : A\)

**Fact C.3.** For all expansion pairs \((\mathcal{E}_1, \mathcal{I}_n)\) for \(A \supset B\) there exists \(x \in TV\) such that \(\mathcal{E}_1 = \mathcal{E}[\mathcal{U}[x]]\) and \(\mathcal{I}_n = \lambda x.\mathcal{I}_n\) where \((\mathcal{E}, \mathcal{I}_n)\) is an expansion pair for \(B\). Similarly, for all expansion pairs \((\mathcal{E}_1, \mathcal{I}_n)\) for \(\forall X.A\) there exists \(X \in V\) such that \(\mathcal{E}_1 = \mathcal{E}[\mathcal{U}[X]]\) and \(\mathcal{I}_n = \lambda X.\mathcal{I}_n\), where \((\mathcal{E}, \mathcal{I}_n)\) is an expansion pair for \(A\).

**Fact C.4.** If \((\mathcal{E}_1, \mathcal{I}_1)\) and \((\mathcal{E}_2, \mathcal{I}_2)\) are two expansion pairs for \(A\), then for any \(U : at(A) \vdash_{\Gamma} at(A)\) and term \(\Gamma \vdash t : A,\) if no variable free in \(U\) and \(t\) is bound in either \(\mathcal{I}_1\) or \(\mathcal{I}_2\), then \(\mathcal{I}_1[\mathcal{U}[\mathcal{E}_1[t]]] = \mathcal{I}_2[\mathcal{U}[\mathcal{E}_2[t]]].\)
In the following we will suppose fixed for any formula \( A \in \mathcal{L}^2 \) an expansion pair \((\text{El}_A, \text{In}_A)\).

**Definition C.5** (Weak expansion and A-expansion). For all \( A \in \mathcal{L}^2 \), the weak expansion of \( A \) is the context \( \text{In}_A(\text{El}_A) \). Moreover, if \( A \) is sp-X, for all \( U : B \vdash \Gamma \), we let the \( A \)-expansion of \( U \) be \( A^U \Gamma = \text{In}_A\{U[\text{El}_A[x]]\} \).

Observe that by Fact C.3 the context \( A^U \Gamma \) does not depend on the chosen expansion pair for \( A \). Moreover, it is easily checked that for all sp-X formula \( A \in \mathcal{L}^2 \), if \( U : B \vdash \Gamma \) then \( A^U \Gamma : A[B/X] \vdash \Gamma \ A[C/X] \).

**Remark C.6.** In [30] we defined the notion of \( A \)-expansion in a more direct manner for the broader class of strictly positive formulas (and not, as done here, strongly positive, see Remark 2.15). The interested reader can easily check that the two definitions coincide in the case of strongly positive formulas.

## D The RP-translation

**Definition D.1** (RP-translation). Given the RP-translation of formulas (Definition 2.7 in Section 2.2), for every term \( u \) such that \( \Gamma \vdash_{\mathcal{N}^2p} u : A \) we define a term \( u^* \) as follows:

\[
\begin{align*}
x^* &= x \\
(\lambda x.t)^* &= \lambda x.t^* \\
(t^s)^* &= t^s^s^* \\
(\Lambda X.t)^* &= \Lambda X.t^* \\
(1B)^* &= t^s B^*
\end{align*}
\]

The following fact is easily checked by induction on a derivation of \( \Gamma \vdash_{\mathcal{N}^2p} u : A \):

**Fact D.2.** If \( \Gamma \vdash_{\mathcal{N}^2p} u : A \), then \( \Gamma^* \vdash_{\mathcal{N}^2} u^* : A^* \) and if \( \Gamma \vdash_{\mathcal{N}^p} u : A \), then \( \Gamma^* \vdash_{\mathcal{N}^2p} u^* : A^* \).

**Remark D.3.** The clauses for \( \iota \)- and \( \delta \)-terms generalize the standard translation of disjunction and conjunction constructors in System F (where \( \iota_\lambda(t_1, t_2) \) is prefix notation for the standard pairing).

\[
\begin{align*}
(\iota^u_\nu t)^* &= \Lambda X.\lambda x_1.x_2.x_1t^* \quad (u = 1, 2) \\
(\iota_\nu(t_1, t_2))^* &= \Lambda X.\lambda x.xt^*_2
\end{align*}
\]

**Remark D.4.** It can be checked that if \( C : A \vdash \Gamma B \) is a context in the language of \( \mathcal{N}^2p \), then \( C^* : A^* \vdash_{\Gamma^*} B^* \) is a context in the language of \( \mathcal{N}^2 \). Moreover, if \( E \) is an elimination context then \( E^* \) is also an elimination context.

## E The \( \varepsilon \)-equation

We write \( \Gamma \vdash t \approx^e u : A \) iff there is a derivation of \( \Gamma \vdash t \approx u \) using the rules of congruence, symmetry, transitivity, reflexivity and any instance of the rule below:
\[ \Gamma \vdash t : (\uparrow \langle A \rangle)^* \quad \left( \Gamma \vdash u_k : (\langle A_{f(j)} \rangle \Rightarrow X)[C/X] \right)_{k \in K} \quad \mathbf{u} : C \Rightarrow \Gamma \mathbf{D} \quad \varepsilon \]

\[ \Gamma \vdash \mathbf{u} \Gamma C(\langle u_k \rangle_{k \in K}) \simeq \mathbf{t} \mathbf{D} \left( \left( \langle A_{f(j)} \rangle \Rightarrow X \right)^\varepsilon \left[ \mathbf{u} [u_k] \right] \right)_{k \in K} : D \]

For the use of e.g. \( \bowtie_\eta \) and \( \sim_\varepsilon \) we adopt the same conventions introduced at the end of Appendix \[E].

**Remark E.1.** As for the rules \( \uparrow \eta^+ \) and \( \uparrow \gamma^+ \), we formulated the rule \( \varepsilon \) using n.v.c.-substitution of a term for the hole of \( U \). Similarly to the other cases (see Remark \[B.2\]), a variant of \( \varepsilon \) with v.c.-substitution in place of n.v.c.-substitution can be derived from the rule \( \varepsilon \) along with \( \beta^* \) - and \( \eta^* \)-conversions and symmetry.

### F \quad Proof of Proposition \[1\]

We will establish the stronger statement below.

**Proposition 9.** For all \( u, v \) if \( \Gamma \vdash_{\Pi_2^p} u \bowtie_\beta \eta^+ v : A \) then \( \Gamma^* \vdash_{\Pi_2^p} u^* \bowtie_\beta \eta^* v^* : A^* \).

We need the following two lemmas:

**Lemma F.1.** For all \( \mathbf{u} : A \vdash_{\Pi_2^p} B \) and \( t \) such that \( \Gamma \vdash_{\Pi_2^p} t : A \), \( \langle \mathbf{u}[t] \rangle^* = \mathbf{u}[t]^* \).

*Proof.* The lemma is easily established by induction on \( \mathbf{u} \). \( \square \)

**Lemma F.2.** For all sp-\( X \) formulas \( A \in \mathcal{L}^2 \) such that \( A = \langle A_a \rangle_{a \in A} \Rightarrow X \), if \( \mathbf{u} : B \vdash_{\Gamma} C \) and \( \Gamma, \langle x_a : A_a \rangle_{a \in A} \vdash_{\Pi_2^p} t : B \), the following hold:

\[ \Gamma \vdash \langle A_a \rangle_{a \in A} \Rightarrow X \quad \mathbf{u} \left[ \lambda \langle x_a \rangle_{a \in A} \cdot t \right] \bowtie_\beta \lambda \langle x_a \rangle_{a \in A} \mathbf{u}[t] : \langle A_a \rangle_{a \in A} \Rightarrow C \]

Moreover, if no free variable of \( \mathcal{E} \langle A_a \rangle_{a \in A} \Rightarrow X \) is bound in \( \mathbf{u} \), then

\[ \Gamma \vdash \langle A_a \rangle_{a \in A} \Rightarrow X \quad \mathbf{u} \left[ \lambda \langle x_a \rangle_{a \in A} \cdot t \right] \bowtie_\beta \lambda \langle x_a \rangle_{a \in A} \mathbf{u}[t] : \langle A_a \rangle_{a \in A} \Rightarrow C \]

*Proof.* The lemma is easily established by induction on the length of the list \( A \). \( \square \)

**Proposition 10.** For all \( u, v \) if \( \Gamma \vdash_{\Pi_2^p} u \bowtie_\gamma \gamma^+ v : A \) then \( \Gamma^* \vdash_{\Pi_2^p} u^* \bowtie_\gamma \gamma^* v^* : A^* \).

*Proof.* By induction on the typing derivation \( \mathcal{D} \) of \( \Gamma \vdash_{\Pi_2^p} u \bowtie_\gamma \gamma^+ v : A \). If \( \mathcal{D} \) ends with an application of either reflexivity, transitivity or one of the congruence rules, it is enough to apply the induction hypothesis to the derivations of the premises, and then applying the reflexivity, transitivity or the congruence rules.

If \( \mathcal{D} \) ends with an application of \( \uparrow \gamma^+ \), then for some context \( \mathbf{u} : B \vdash_{\Gamma} A, u = \mathbf{u}[\delta_\gamma(t, \langle \langle y_j \rangle \cdot s_k \rangle)] \) and \( v = \delta_\gamma(t, \langle \langle y_j \rangle \cdot \mathbf{u}[s_k] \rangle) \), where \( \Gamma \vdash_{\Pi_2^p} \delta_\gamma(t, \langle \langle y_j \rangle \cdot s_k \rangle) : B \), \( \Gamma \vdash_{\Pi_2^p} t : \uparrow \langle A \rangle \) for some \( \uparrow \), and \( \Gamma, \langle y_j : A_{f(j)} \rangle \vdash_{\Pi_2^p} s_k : B \) for every \( k \in K \). Thus \( \Gamma^* \vdash_{\Pi_2^T} \mathbf{u} : A_{f(j)} \Rightarrow X, \mathbf{u}^* \Rightarrow \langle \langle y_j \rangle \cdot \mathbf{u}[s_k] \rangle^* \quad \forall X, \langle \langle A_{f(j)} \rangle \Rightarrow X \rangle, \mathbf{u}^* \Rightarrow \langle \langle y_j \rangle \cdot \mathbf{u}[s_k] \rangle \)

for every \( k \in K \) and we can construct a derivation \( \mathcal{D}' \) of \( \Gamma^* \vdash_{\Pi_2^T} u^* \bowtie_\gamma \gamma^* v^* \) as shown below:

\[ u^* \quad \text{Def} \[E.1\] \quad \text{Lemma} \[E.1\] \quad \mathbf{u}^* \left[ \langle \langle A_{f(j)} \rangle \Rightarrow X \rangle^* \left[ \mathbf{u}^* \left[ \lambda \langle y_j \rangle \cdot s_k \rangle \right] \right] \right] \bowtie_\varepsilon \]

\[ (t^* A^*) \left[ \langle \langle A_{f(j)} \rangle \Rightarrow X \rangle^* \left[ \mathbf{u}^* \left[ \lambda \langle y_j \rangle \cdot s_k \rangle \right] \right] \right] \quad \text{Lemma} \[E.2\] \quad \bowtie_\beta \quad (t^* A^*) \left[ \lambda \langle y_j \rangle \cdot \mathbf{u}^* [s_k] \right] \quad \text{Lemma} \[E.3\] \quad \text{Def} \[E.1\] \quad v^* \]

\[ \square \]
Proposition 11. For all $u, v$ if $\Gamma \vdash _{\mathbf{H}^2 p} u \approx_\eta v : A$ then $\Gamma^* \vdash _{\mathbf{H}^2} u^* \approx_{\beta \eta} v^* : A^*$.

Proof. As in the proof above we can argue by induction on the derivation $\mathcal{D}$ of $\Gamma \vdash _{\mathbf{H}^2 p} u \approx_\eta v : A$. The only non-trivial case is when $\mathcal{D}$ ends with an application of $\eta^*$. So suppose $u = \delta_1(v, \langle \langle y_j \rangle \rangle, t)$. Hence $\Gamma \vdash _{\mathbf{H}^2 p} v : \top \langle A_i \rangle$ for some $i$, and thus $\Gamma^* \vdash _{\mathbf{H}^2} v^* : \forall X. \langle \langle A^*_i \rangle \rangle \supset X \supset X \rangle$, and we can construct a derivation $\mathcal{D}'$ of $\Gamma^* \vdash _{\mathbf{H}^2} u^* \approx_{\beta \eta} v^* : A^*$ as shown below:

$$
\begin{align*}
v^* \approx_\eta & \Lambda X. \lambda(f_k)_{k \in K}. v^* X \langle \lambda(y_j) . f_k(y_j) \rangle_{k \in K} = \mathcal{C} \{ v^* X \langle \lambda(y_j) . f_k(y_j) \rangle_{k \in K} \} \quad \text{Def. G.1} \\
(v^* A^*) \langle \langle A^*_i \rangle \rangle & \supset X \supset X \rangle \supset X \rangle \quad \text{Lemma B.2} \\
(v^* A^*) \langle \lambda(y_j) . \mathcal{C} \{ f_k(y_j) \} \rangle & \supset X \rangle \quad \text{Def. D.1} \\
\end{align*}
$$

where $\mathcal{C} = \Lambda X. \lambda(f_k)_{k \in K}$ and the variables $X, f_k$ are chosen so as not to occur free in the contexts $\mathcal{E} \Lambda (A^*_i \rangle \rangle \supset X \rangle$.

Proof of Proposition 9. It suffices to check rule-by-rule that equivalences are preserved. The only non-trivial case is the one of $\eta^*$, which is treated using Remark B.1, Proposition 10 and Proposition 11.

G. The three translations into $\mathbf{H}^2_{at}$

Definition G.1 (Atomization). For every term $u$ such that $\Gamma \vdash _{\mathbf{H}^2 p} u : A$ we define a term $u^\dagger$ as follows (observe that when $u = tB$ then $\Gamma \vdash _{\mathbf{H}^2 p} t : \forall W. F$ and since $F$ is polynomial in $W$, by what we observed in Subsection 2.2, $F$ can be written as $\langle \langle A^*_i \rangle \rangle \supset W \rangle \supset W \rangle$, with the $A_i$ having no occurrence of $W$):

$$
\begin{align*}
x^\dagger & = x \\
(\lambda x. t)^\dagger & = \lambda x.t^\dagger \\
(t s)^\dagger & = t^\dagger s^\dagger \\
(\Lambda X.t)^\dagger & = \Lambda X.t^\dagger \\
\end{align*}
$$

Definition G.2 ($\varepsilon$-atomization). For every term $u$ such that $\Gamma \vdash _{\mathbf{H}^2 p} u : A$ we define a term $u^\varepsilon$ by replacing in Definition G.1 the last two clauses for terms of the form $tB$ (i.e. with $B$ non-atomic) with the following single clause (assuming $Z = \text{at}(B)$):

$$
\begin{align*}
(tB)^\varepsilon & = \lambda(y_k). \mathcal{I} \{ t^\varepsilon Z \langle \langle A^*_i \rangle \rangle \supset X \rangle \supset X \rangle \} \\
\end{align*}
$$

It easy to check by induction on the derivation of $\Gamma \vdash _{\mathbf{H}^2 p} u : A$ that the following holds:

Fact G.3. For $\varepsilon \in \{ \dagger, \varepsilon \}$, if $\Gamma \vdash _{\mathbf{H}^2 p} u : A$, then $\Gamma \vdash _{\mathbf{H}^2_{at}} u^\varepsilon : A$.

From this and Fact D.2 it follows moreover the following:

Corollary G.4. For $\varepsilon \in \{ \dagger, \varepsilon \}$, if $\Gamma \vdash _{\mathbf{H}^2 p} u : A$, then $\Gamma^* \vdash _{\mathbf{H}^2_{at}} u^\varepsilon : A^*$.
Definition G.5 (ESF-translation). The ESF-translation is obtained by replacing in Definition $\mathcal{D}$ $\mathcal{N}^2_{\Delta t}$ with $\mathcal{N}_{\Delta t}$ and the last clause with the following (we assume here $\Gamma \vdash \mathcal{N}_{\Delta t} u: A$ and $Z = at(A)$):

$$(\delta_t(t, \langle\langle y_j \rangle_\Delta \rangle_\Delta))^{\dagger} = In_{A^{\ast}} \left\{ (t^2 Z) \left( \lambda \langle y_j \rangle_\Delta . El_A[s^2_k] \right) \right\}$$

Also in this case, it easily checked by induction on the derivation of $\Gamma \vdash \mathcal{N}_{\Delta t} u: A$ that the following holds:

Fact G.6. If $\Gamma \vdash \mathcal{N}_{\Delta t} u: A$, then $\Gamma^{\ast} \vdash \mathcal{N}_{\Delta t}^2 u^2: A^\ast$.

The interested reader can check that the refined interpretation proposed by Espírito Santo and Ferreira in [4] coincides essentially with ours in the case of $\mathcal{N}^\gamma$.

Remark G.7. All three translations $\ast^1, \ast^\Delta, \ast^T: \mathcal{N}_{\Delta t} \rightarrow \mathcal{N}_{\Delta t}^2$ extend straightforwardly to contexts, similarly to what happens for the RP-translation (see Remark D.3). Moreover, for each of the three translations $\ast^T$ it can be checked by induction that for all elimination context $E$ of $\mathcal{N}_{\Delta t}$, $E^T$ is a principal context and that $E^T \{t^T\} = E^T[t^T] = (E[t])^T$.

H Proof of Proposition [2]

We prove the following generalization to $\mathcal{N}_{\Delta t}^0$ of Point 2 of Proposition [2]

Proposition 12. For all $u, v$, if $\Gamma \vdash \mathcal{N}_{\Delta t} u \rightarrow_{\beta\eta} v: A$ then $\Gamma^{\ast} \vdash \mathcal{N}_{\Delta t}^2 u^2 \approx_{\beta\eta} v^2: A^\ast$.

Let $\mathcal{A}$ be a finite list. By a $\mathcal{A}$-multicontext $M$ we indicate a term containing exactly one occurrence of $|\mathcal{A}|$ distinct special variables noted as $[ ]_a$, for each $a \in \mathcal{A}$. Given a $\mathcal{A}$-multicontext $M$ and a finite list of terms $\langle u_a \rangle_{a \in \mathcal{A}}$, we let $M \langle \langle u_a \rangle_{a \in \mathcal{A}} \rangle$ be the term obtained by simultaneous variable-capturing substitution of $[ ]_a$ by $u_a$. We let $M: \langle \Delta_a \vdash A \rangle_{a \in \mathcal{A}}$ $(\Gamma \vdash B)$ indicate that for all $\mathcal{A}$-indexed list of terms $\langle u_a \rangle_{a \in \mathcal{A}}$ such that $\Gamma, \Delta_a \vdash u_a: A$, $\Gamma \vdash M \langle \langle u_a \rangle_{a \in \mathcal{A}} \rangle : B$ (meaning that the free variables of $u_a$ in $\Delta_a$ are captured by $M$).

Lemma H.1. Let $A, B$ be formulas of $\mathcal{N}_{\Delta t}^2$; let $u_1, \ldots, u_{|\mathcal{A}|}$ be terms such that $\Gamma, \Delta_a \vdash \mathcal{N}_{\Delta t}^2 u_a: A$, $\mathcal{C}: A \vdash B$ be a principal context of $\mathcal{N}_{\Delta t}^2$ and $M: \langle \Delta_a \vdash \mathcal{N}_{\Delta t}^2 X \rangle_{a \in \mathcal{A}} \Rightarrow (\Gamma \vdash \mathcal{N}_{\Delta t}^2 X)$ be a $\mathcal{A}$-multicontext such that $X$ does not occur free neither in $\Gamma$ nor in the $\Delta_a$. Moreover, suppose that $M$ binds no term or type variable of $El_A$ and that $In_A$ binds no term or type variable of $M$, $\mathcal{C}$ and of the $u_a$. Then

$$C \left\{ In_A \langle M_{at(A)} \langle El_A[u_a] \rangle_{a \in \mathcal{A}} \rangle \right\} \approx_{\beta} In_B \left\{ M_{at(B)} \langle El_B[C[u_a]] \rangle_{a \in \mathcal{A}} \right\}$$

where $M_{\mathcal{Y}}$ is shorthand for $M[Y/X]$.

Proof. We argue by induction on the principal context $C$:

• if $C = [ ]$ the claim is immediate;
• if $C = C'v$, then $C' : A \vdash \Gamma$ and $\Delta \vdash C \triangleright B$ and $\Gamma \vdash \vdash_{\text{a}_1} v : C$, and noticing that $\text{at}(B) = \text{at}(C \triangleright B)$ we then have

\[
\mathcal{C}[\text{In}_{A}\{\mathcal{M}_{\text{at}}(A)\{\langle \mathcal{E}_{\text{a}}[u_{a,e}] \rangle_{a,e,A}\}\}] = C'[\text{In}_{A}\{\mathcal{M}_{\text{at}}(A)\{\langle \mathcal{E}_{\text{a}}[u_{a,e}] \rangle_{a,e,A}\}\}] v
\]

\[
\vdash_{\beta}^{1H} \left( \text{In}_{C \triangleright B}\{\mathcal{M}_{\text{at}}(B)\{\langle \mathcal{E}_{\text{a}}[u_{a,e}] \rangle_{a,e,A}\}\}\right) v = (\lambda x. \text{In}_{B}\{\mathcal{M}_{\text{at}}(B)\{\langle \mathcal{E}_{\text{a}}[u_{a,e}] \rangle_{a,e,A}\}\}) v
\]

\[
\vdash_{\beta} \text{In}_{B}\{\mathcal{M}_{\text{at}}(B)\{\langle \mathcal{E}_{\text{a}}[u_{a,e}] \rangle_{a,e,A}\}\} = \text{In}_{B}\{\mathcal{M}_{\text{at}}(B)\{\langle \mathcal{E}_{\text{a}}[C[u_{a,e}] \rangle_{a,e,A}\}\}
\]

• if $C = C'W$, then $C' : A \vdash \forall Y.B'$, $B = B'[W/Y]$; by noticing that $\text{at}(\forall Y.B')[W/Y] = \text{at}(B)$ we then have

\[
\mathcal{C}[\text{In}_{A}\{\mathcal{M}_{\text{at}}(A)\{\langle \mathcal{E}_{\text{a}}[u_{a,e}] \rangle_{a,e,A}\}\}] = C'[\text{In}_{A}\{\mathcal{M}_{\text{at}}(A)\{\langle \mathcal{E}_{\text{a}}[u_{a,e}] \rangle_{a,e,A}\}\}] W
\]

\[
\vdash_{\beta}^{1H} \left( \text{In}_{Y.Y.B'}\{\mathcal{M}_{\text{at}}(Y.Y.B')\{\langle \mathcal{E}_{\text{a}}[u_{a,e}] \rangle_{a,e,A}\}\}\right) W = (\Lambda Y. \text{In}_{B}\{\mathcal{M}_{\text{at}}(Y.Y.B')\{\langle \mathcal{E}_{\text{a}}[u_{a,e}] \rangle_{a,e,A}\}\}) W \vdash_{\beta} \text{In}_{B}\{\mathcal{M}_{\text{at}}(B)\{\langle \mathcal{E}_{\text{a}}[C[u_{a,e}] \rangle_{a,e,A}\}\}
\]

\[
= \text{In}_{B}\{\mathcal{M}_{\text{at}}(B)\{\langle \mathcal{E}_{\text{a}}[C[u_{a,e}] \rangle_{a,e,A}\}\}
\]

• if $C = \lambda w.C'$, then $B = B_1 \triangleright B_2$ and $C' : A \vdash \Gamma$, $B_2$ so by noticing that $\text{at}(B) = \text{at}(B_2)$ we have

\[
\text{In}_{B}\{\mathcal{M}_{\text{at}}(B)\{\langle \mathcal{E}_{\text{a}}[u_{a,e}] \rangle_{a,e,A}\}\} = \lambda w'. \text{In}_{B}\{\mathcal{M}_{\text{at}}(B)\{\langle \mathcal{E}_{\text{a}}[\lambda w.C'[u_{a,e}] \rangle_{a,e,A}\}\}
\]

\[
\vdash_{\beta} \lambda w'. \text{In}_{B}\{\mathcal{M}_{\text{at}}(B)\{\langle \mathcal{E}_{\text{a}}[\lambda w.C'[u_{a,e}] \rangle_{a,e,A}\}\} \vdash_{\beta}^{1H} \lambda w'. \text{C}'[\rangle\langle u_{a,e} / w']\}
\]

\[
\vdash_{\beta} \lambda w'. \text{In}_{B}\{\mathcal{M}_{\text{at}}(B)\{\langle \mathcal{E}_{\text{a}}[\lambda w.C'[u_{a,e}] \rangle_{a,e,A}\}\}
\]

\[
\vdash_{\beta} \lambda w'. \text{In}_{B}\{\mathcal{M}_{\text{at}}(B)\{\langle \mathcal{E}_{\text{a}}[\lambda w.C'[u_{a,e}] \rangle_{a,e,A}\}\}
\]

Proposition 13. For all $u, v$, if $\Gamma \vdash_{\text{a}} u \vdash \gamma v : A$, then $\Gamma^* \vdash_{\text{a}_1} u^\sharp \vdash_{\beta} v^\sharp : A^*$. 

Proof. By induction on the typing derivation $\mathcal{D}$ of $\Gamma \vdash_{\text{a}} u \vdash \gamma v : A$. If $\mathcal{D}$ ends with an application of either reflexivity, transitivity or of one of the congruence rules, it is enough to apply the induction hypothesis to the derivations of the premises, and then applying the reflexivity, transitivity or the congruence rules.

If $\mathcal{D}$ ends with an application of $\downarrow \gamma ; u = E[\delta_{\gamma}(t, \langle y_{j} \rangle, s_k)]$ and $v = \delta_{\gamma}(t, \langle y_{j} \rangle, E[s_k])$ for some elimination context $E : B \vdash \Gamma A$ (hence $\Gamma \vdash_{\text{a}} \delta_{\gamma}(t, \langle y_{j} \rangle, s_k) : B$ and $\Gamma \vdash_{\text{a}} t : \downarrow \langle A \rangle$ for some $\Gamma$). We show how to construct a derivation $\mathcal{D}'$ of $\Gamma^* \vdash_{\text{a}_1} u^\sharp \vdash_{\beta} v^\sharp : A^*$. Let $Z = \text{at}(A^*)$ and $Z' = \text{at}(B^*)$. By Remark $[3,7]$ we know that $E^\sharp$ is a principal context and that $E^\sharp[t] = E[t]$. Moreover, consider the $K$-multicontext $M : \langle \Delta_k \vdash \chi \rangle_{\kappa \in \kappa} \Rightarrow (\Gamma \vdash X)$ given by $M = (u^\sharp X)^\langle \lambda y_{j} \rangle_{j \neq \gamma^{-1}(k)}[k]_{\kappa \in \kappa}$, where $\Delta_k = \langle y_{j} : A_{\kappa} \rangle_{j \neq \gamma^{-1}(k)}$ and $X$ is
a fresh variable. The reader can check that all conditions of Lemma [H.1] do hold with $E^p$ in place of $C$, $A^s$, $B^p$ in place of $A$, $B$ and the $s^p_k$ in place of the $u_n$. Then we have:

\[
\gamma \text{ Def. G.5 } E^p \left[ \text{In}_{B^p} \left\{ \left( u^p Z \right) \left( \lambda \langle y_j \rangle \cdot \text{El}_{B^p} \left[ s^p_k \right] \right) \right\} \right] \\
\text{Lemma G.1 \text{ in place of G.0} } \text{Def. G.5 } E^p \left[ \text{In}_{A^s} \left\{ \left( u^p Z \right) \left( \lambda \langle y_j \rangle \cdot \text{El}_{A^s} \left[ E^p \left[ s^p_k \right] \right] \right) \right\} \right]
\]

\[
\beta
\]

Proposition 14. For all $u, v$ if $\Gamma \vdash_{\Pi^{2}_{\text{RT}}} u \leadsto_{\eta} v : A$ then $\Gamma^* \vdash_{\Pi^{2}_{\text{RT}}} u^\eta \leadsto_{\beta\eta} v^\eta : A^\eta$.

Proof. By induction on the typing derivation $\mathcal{D}$ of $\Gamma \vdash_{\Pi^{2}_{\text{RT}}} u \leadsto_{\eta} v : A$. If $\mathcal{D}$ ends with an application of either reflexivity, transitivity, one of the congruence rules or $\Rightarrow_\eta$ it is enough to apply the induction hypothesis to the derivations of the premises, and then applying reflexivity, transitivity, the congruence rules and $\Rightarrow_\eta$ or $\forall_\eta$.

If $\mathcal{D}$ ends with an application of $\Rightarrow_\eta$, $u = \delta_1(v, \langle y_j \rangle \cdot \langle y_j \rangle)$ (where $\Gamma \vdash_{\Pi^{2}_{\text{RT}}} v : \uparrow A_i$ for some $\uparrow$), and we informally show how to construct a derivation $\mathcal{D}'$ of $\Gamma^* \vdash_{\Pi^{2}_{\text{RT}}} u^\eta \leadsto_{\beta\eta} v^\eta : A^\eta$. By letting $C = \Lambda X. \lambda (x_k)^{h \in \mathbb{K}}[.]$ and $\text{In}_{A^s}$ be $\Lambda Z. \lambda (z_h)^{h \in \mathbb{K}}[.]$, we have that:

\[
\gamma \text{ Def. G.5 } \text{In}_{A^s} \left\{ \left( u^p Z \right) \left( \lambda \langle y_j \rangle \cdot \text{El}_{A^s} \left[ C(x_k)^{h \in \mathbb{K}} \right]\right) \right\} \leadsto_{\beta}
\]

\[
\Rightarrow_\eta \text{ In}_{A^s} \left\{ \left( v^p Z \right) \left( \lambda \langle y_j \rangle \cdot z_h \langle y_j \rangle \right)^{h \in \mathbb{K}} \right\} \leadsto_{\eta} \text{ In}_{A^s} \left\{ \left( v^p Z \right) \left( z_h \right)^{h \in \mathbb{K}} \right\} \leadsto_{\eta} v^\eta
\]

Proof of Proposition 13. It suffices to check rule-by-rule that reductions are preserved. The only non-trivial cases are those of $\gamma$ and $\eta$, which can be treated using Proposition 13 and Proposition 14.

I Proof of Proposition 3

We prove the following generalization of Proposition 3 to $\Pi^{2}_{\text{RT}}$:

Proposition 15. For all $u$ such that $\Gamma \vdash_{\Pi^{2}_{\text{RT}}} u : A$ we have that $\Gamma \vdash_{\Pi^{2}_{\text{RT}}} u \equiv_{\eta} u^\eta : A$.

Proof. By induction on the typing derivation $\mathcal{D}$ of $\Gamma \vdash_{\Pi^{2}_{\text{RT}}} u : A$. If $\mathcal{D}$ ends with an application of either $\text{Ax}$, $\Rightarrow_\eta$, $\Rightarrow_\eta$, or $\forall_\eta$ with an atomic witness, (i.e. $u$ is either a variable or of the form $\lambda x. t$, $\forall x. t$, $\forall x. t$ or $t Y$), then it is enough to use reflexivity or to apply the induction hypothesis to the immediate sub-derivations of $\mathcal{D}$ and use the congruence rules.

Otherwise $u = t B$ with $B$ non atomic, where the same remarks as in Def. G.1 apply to $t$ (in particular $\Gamma \vdash_{\Pi^{2}_{\text{RT}}} t : \forall W. F$), and we informally show how to construct a derivation $\mathcal{D}'$ of $\Gamma \vdash_{\Pi^{2}_{\text{RT}}} u \equiv_{\eta} u^\eta : A$. Assuming $Z = \text{at}(B)$ (so that $\text{El}_B : B \vdash \Gamma Z$) we have that:

\[
t B \text{ Fact G.3 } \text{In}_{F[B]/W} \left[ \text{El}_{F[B]/W} \left[ t B \right] \right] \text{ Fact G.3 } \lambda \langle y_k \rangle \cdot \text{In}_B \left\{ t Z \left( \left( \left( \langle A \rangle^i \Rightarrow Z \right) \Rightarrow B \right) \right) \right\} \text{ Def G.1-1.H. } \left( t B \right)^{\eta}
\]

where in the penultimate step the rule $\varepsilon$ is applied to the term in braces by taking $\text{El}_B$ for $U$. 

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J Proof of Proposition 4

We prove the following generalization of Proposition 4 to $\Pi^P$:

**Proposition 16.** For all $u$ such that $\Gamma \vdash_{\Pi^P} u : A$: (1) $\Gamma^* \vdash_{\Pi^P_{at}} u^! \rightsquigarrow_{\beta} u^{*!} : A^*$;
(2) $\Gamma^* \vdash_{\Pi^P_{at}} u^{*!} \rightsquigarrow_{\beta} u^* : A^*$.

The proposition follows immediately from the following two:

**Proposition 17.** For all $u$ such that $\Gamma \vdash_{\Pi^P_{at}} u : A$ we have that: $\Gamma \vdash_{\Pi^P_{at}} u^! \rightsquigarrow_{\beta} u^{*!} : A$.

*Proof.* By induction on the typing derivation $\mathcal{D}$ of $\Gamma \vdash_{\Pi^P_{at}} u : A$. If $\mathcal{D}$ ends with an application of either $\text{Ax}$, $\Rightarrow_{\text{I}}$, $\Rightarrow_{\text{E}}$, $\Rightarrow_{\text{I}}$ or one of $\forall E$ with an atomic witness, (i.e. $u$ is either a variable or of the form $\lambda x.t$, $ts$, $\Lambda X.t$ or $t\gamma$), then it is enough to use reflexivity or to apply the induction hypothesis to the immediate sub-derivations of $\mathcal{D}$ and use the congruence rules.

Otherwise $u = tB$ with $B$ non atomic, where the same remarks as in Def. 4.1 apply to $t$, and we informally show how to construct a derivation $\mathcal{D'}$ of $\Gamma \vdash_{\Pi^P_{at}} u^! \rightsquigarrow_{\beta} u^{*!} : A$ by induction on $B$.

If $B = C \supset D$, assuming $Z = \text{at}(B)$, we have:

$$(tB)^{\text{Def g.1+I.H.}} \lambda (v_k) \mu (u).((tD)^{i^*}((\lambda (x_j).v_k(x_j))u)$$

so we have:

$$(tB)^{\text{Def g.2}} \lambda (v_k) \mu (u).\lambda (y_k).\text{Ind}_D \left\{t^{i^*}Z\langle\langle A_{(i)}^\ast\supset Z\rangle^\ast\text{El}_D[y_k]\rangle\langle\lambda (x_j).v_k(x_j)u]\right\}$$

and hence (by Def. 4.1

One can argue in a similar way if $B = \forall U.D$ (just replace $C \supset D$ with $\forall U.D$ and $u$ with $U$).

**Proposition 18.** For all $u$ such that $\Gamma \vdash_{\Pi^P} u : A$: $\Gamma^* \vdash_{\Pi^P_{at}} u^{*!} \rightsquigarrow_{\beta} u^* : A^*$.

*Proof.* By induction on the typing derivation $\mathcal{D}$ of $\Gamma \vdash_{\Pi^P} u : A$. If $\mathcal{D}$ ends with an application of either $\text{Ax}$, $\Rightarrow_{\text{I}}$, $\Rightarrow_{\text{E}}$, $\Rightarrow_{\text{I}}$ or one of $\forall E$ with an atomic witness, (i.e. $u$ is either a variable or of the form $\lambda x.t$, $ts$ or $t\gamma(t_j)$), then it is enough to use reflexivity or to apply the induction hypothesis to the immediate sub-derivations of $\mathcal{D}$ and use the congruence rules.

Otherwise $u = \delta(t,\langle y_j \rangle.s_k)$ and hence (by Def. 4.1 $u^* = \langle t^* A^*\rangle\langle\lambda (y_j).s_k^*\rangle$, where the same remarks as in Def. 4.1 apply to $t^*$, and we informally show how to construct a derivation $\mathcal{D'}$ of $\Gamma \vdash_{\Pi^P_{at}} u^{*!} \rightsquigarrow_{\beta} u^* : A$.

If $A$ is atomic, i.e. $A = Y$ we have that:

$$(\delta(t,\langle y_j \rangle.s_k)^{\text{Def l.1}})^{\text{Def l.3}}(t^* Y)^{\text{Def l.2}}\langle\langle A_{(i)}^\ast\supset Z\rangle^\ast\text{El}_Y[y_k]\rangle\langle\lambda (y_j).s_k^{*!}\rangle^{\text{Def g.2}}(t^* Y)^{\text{Def l.3}}\langle\langle A_{(i)}^\ast\supset Z\rangle^\ast\text{El}_Y[s_k]\rangle^{\text{Def g.2}}u^*$$
Otherwise, assuming $Z = \text{at}(A^*)$ we have that:

\[
\begin{align*}
\delta(t, \langle \langle y_j \rangle, s_k \rangle))^{*_i} & \overset{\text{Def.} \ref{def:delta}}{=} (t^{*_i} A^*) \langle \lambda \langle y_j \rangle, s_k^{*_i} \rangle) \\
= \lambda \langle z_k \rangle & . \text{In}_{A^*} \begin{cases} 
(t^{*_i} A) \langle \langle A_{f(j)} \rangle \supset Z \rangle \text{El}_{A^*} \langle \langle z_k \rangle \rangle \rangle \langle \lambda \langle y_j \rangle, s_k^{*_i} \rangle & \beta \\
\end{cases} \\
\overset{\text{Def.} \ref{def:in}}{=} & \lambda \langle z_k \rangle . \text{In}_{A^*} \begin{cases} 
(t^{*_i} A) \langle \langle A_{f(j)} \rangle \supset Z \rangle \text{El}_{A^*} \langle \lambda \langle y_j \rangle, s_k^{*_i} \rangle \rangle & \beta \\
\end{cases} \\
\end{align*}
\]

\[
\overset{\text{I.H.}}{=} u^2
\]

\[\square\]

K Proofs of Proposition 5, 6, 7 and 8

We establish the following generalization of Proposition 5 to the whole of $\mathcal{L}_{\Pi^{2}}$.

**Proposition 19.** For all $A \in \mathcal{L}_{\Pi^{2}}$, $A \models_{\mathcal{L}_{\Pi^{2}}} A^*$.

**Proof.** We can construct by induction on $A$ term contexts $\mathcal{C} : A^* \models_{\mathcal{L}} A$ and $\mathcal{D} : A \models_{\mathcal{L}} A^*$.

The only non trivial case is $A = \downarrow \langle \alpha \rangle$. In this case we have that:

\[
\begin{align*}
\mathcal{C} &= \left[ \downarrow \langle \alpha \rangle \right. \\
\mathcal{D} &= \lambda \langle x \rangle . \delta(t, \langle \langle y_j \rangle, \lambda \langle x_k \rangle \rangle_{k \in K} \langle \langle y_j \rangle \rangle_{k \in K} \langle \langle x_k \rangle \rangle_{k \in K})
\end{align*}
\]

where by induction hypothesis $\mathcal{C}_j : A_{f(j)}^* \models_{\mathcal{L}} A_{f(j)}$ and $\mathcal{D}_j : A_{f(j)} \models_{\mathcal{L}} A_{f(j)}^*$

\[\square\]

To establish Proposition 6 we will exploit a sound and complete semantics for $\mathcal{L}_{\Pi^{2}}$ from \cite{27}, that we briefly recall.

For any partially ordered set $\langle W, \leq \rangle$, let $W^\uparrow = \{ a \subseteq W \mid \forall \alpha, \beta \in W (\alpha \in a \land \alpha \leq \beta \Rightarrow \beta \in a) \}$ the set of upward closed subsets of $W$.

A $\mathcal{L}_{\Pi^{2}}$-model is a tuple $\mathcal{M} = \langle W, \leq, \bot, D, g \rangle$ such that $\langle W, \leq, \bot \rangle$ is a partially ordered set with a bottom element $\bot$, $D$ is a monotone function from $W$ to $\varphi(W^\uparrow)$ (i.e. $\alpha \leq \beta \Rightarrow D(\alpha) \subseteq D(\beta)$) and $g : V \rightarrow \bigcup_{\alpha \in W} D(\alpha)$.

For any formula $A \in \mathcal{L}$ and model $\mathcal{M} = \langle W, \leq, \bot, D, g \rangle$, for all $\alpha \in W$ such that $g(V) \subseteq D(\alpha)$, we let the relation $\alpha \models_{\mathcal{M}} A$ be defined by:

- $\alpha \models_{\mathcal{M}} X$ iff $\alpha \in g(X)$
- $\alpha \models_{\mathcal{M}} A \supset B$ iff for all $\beta \geq \alpha$, if $\beta \models_{\mathcal{M}} A$ then $\beta \models_{\mathcal{M}} B$
- $\alpha \models_{\mathcal{M}} A \lor B$ if $\alpha \models_{\mathcal{M}} A$ or $\alpha \models_{\mathcal{M}} B$
- $\alpha \models_{\mathcal{M}} \forall X. A$ iff for all $\beta \geq \alpha$ and $a \in D(\beta)$, $\beta \models_{\mathcal{M}[X \rightarrow a]} A$

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where $M[X \rightarrow a]$ is $\langle W, \leq, \bot, D, g[X \rightarrow a] \rangle$, with $g[X \rightarrow a]$ being like $g$ except that it sends $X$ to $a$.

A model $M = \langle W, \leq, \bot, D, g \rangle$ is called regular if $g(V) \subseteq D(\bot)$. We let $\Gamma \models A$ if for any regular model $M$, $\bot \models_M \Gamma$ implies $\bot \models_M A$.

**Proposition 20** (Soundness and completeness [27]). $\Gamma \not\models_{\text{MI}_2^{\text{CF}}} A$ iff $\Gamma \not\models A$.

We will now exhibit a countermodel to $p@X.Y \vDash Z \supseteq Y \vee Z$.

**Proof of Proposition 6.** Let $M = \langle W, \leq, \bot, D, g \rangle$ be the regular model given by:

- $W = \{ \bot, \alpha, \beta \}$, $\leq$ is reflexive, $\bot \leq \alpha, \beta$;
- $D(\bot) = D(\alpha) = D(\beta) = \{ \{\alpha\}, \{\beta\}\}$;
- $g(Y) = \{\alpha\}$, $g(Z) = \{\beta\}$ and $g(W)$ is chosen arbitrarily in $D(\bot)$ for $W \neq Y, Z$.

We have $\alpha, \beta \models_M Y \vee Z$ but $\bot \not\models_M Y \vee Z$. Given $a \in D(\bot) = D(\alpha) = D(\beta)$, we have the following facts (where $\alpha \models_M A_1, \ldots, A_n$ is short for $\alpha \models_M A_1$ and, $\ldots,$ and $\alpha \models_M A_1$, and similarly for $\not\models$):

- $\alpha \not\models_M [X \rightarrow a] Y \supseteq X, Z \supseteq X, X$
- $\alpha \not\models_M [X \rightarrow a] Z \supseteq X,$
- if $a = \{\alpha\}$, then $\beta \not\models_M [X \rightarrow a] Y \supseteq X$
- $\beta \not\models_M [X \rightarrow a] Z \supseteq X, X$
- $\beta \not\models M [X \rightarrow a] Y \supseteq X, Z \supseteq X, X$
- if $a = \{\beta\}$, then $\alpha \not\models_M [X \rightarrow a] Y \supseteq X, X$
- $\alpha \not\models_M [X \rightarrow a] Z \supseteq X,$
- $\beta \not\models_M [X \rightarrow a] Y \supseteq X, Z \supseteq X, X$

From these facts we deduce in turn:

- if $a = \{\alpha\}$, then
  - $\bot \not\models_M [X \rightarrow a] Y \supseteq X$ (since $\bot \not\models_M [X \rightarrow a] Y, \beta \not\models_M [X \rightarrow a] Y$ and $\alpha \models_M [X \rightarrow a] Y, X$)
  - $\bot \not\models_M [X \rightarrow a] Z \supseteq X, X$ (since $\beta \not\models_M [X \rightarrow a] Z$ but $\beta \not\models M [X \rightarrow a] X$)

- if $a = \{\beta\}$, then
  - $\bot \not\models_M [X \rightarrow a] Z \supseteq X$ (since $\bot \not\models_M [X \rightarrow a] Z, \alpha \not\models_M [X \rightarrow a] Z$ and $\beta \not\models_M [X \rightarrow a] Z, X$)
  - $\bot \not\models_M [X \rightarrow a] Y \supseteq X, X$ (since $\alpha \not\models_M [X \rightarrow a] Y$ but $\alpha \not\models M [X \rightarrow a] X$)

We now deduce that
• if \( a = \{\alpha\} \), then \( \bot \vdash_{\mathcal{M}(X \mathbin{-} a)} Y \mathbin{\gamma} Z \), since the only \( \gamma \succeq \bot \) such that \( \gamma \vdash_{\mathcal{M}(X \mathbin{-} a)} Y \mathbin{\gamma} X \) and for all \( \gamma' \succeq \gamma \), \( \gamma' \vdash_{\mathcal{M}(X \mathbin{-} a)} Z \mathbin{\gamma} X \) is \( \alpha \) and \( \alpha \vdash_{\mathcal{M}(X \mathbin{-} a)} X \);

• if \( a = \{\beta\} \), then \( \bot \vdash_{\mathcal{M}(X \mathbin{-} a)} Y \mathbin{\gamma} Z \), since the only \( \gamma \succeq \bot \) such that \( \gamma \vdash_{\mathcal{M}(X \mathbin{-} a)} Y \mathbin{\gamma} X \) and for all \( \gamma_1 \succeq \gamma \), \( \gamma_1 \vdash_{\mathcal{M}(X \mathbin{-} a)} Z \mathbin{\gamma} X \) is \( \beta \) and \( \beta \vdash_{\mathcal{M}(X \mathbin{-} a)} X \).

We deduce then \( \bot \vdash_{\mathcal{M}} \forall X.Y \mathbin{\gamma} Z \), and since \( \bot \not\vdash_{\mathcal{M}} Y \mathbin{\gamma} Z \), we conclude \( \bot \not\vdash_{\mathcal{M}} (\forall X.Y \mathbin{\gamma} Z) \mathbin{\gamma} Y \mathbin{\gamma} Z \), hence by the definition of \( \vdash \) we have that \( \not\vdash (\forall X.Y \mathbin{\gamma} Z) \mathbin{\gamma} Y \mathbin{\gamma} Z \).

Proof of Proposition 7. Take \( A = Y \), \( B = Z \) and \( C = Y \mathbin{\gamma} Z \). If there were an \( \mathcal{NI}_2^\forall \)-context \( \mathcal{IO} : \forall X.Y \mathbin{\gamma} Z \vdash \hat{\varnothing} \) \( \hat{(Y \mathbin{\gamma} Z)}[Y \mathbin{\gamma} Z/X] \), then one could derive \( Y \mathbin{\gamma} Z \) from \((Y \mathbin{\gamma} Z)^*\) by replacing \([\ ]\dagger\langle A_i\rangle\) with \( \mathcal{IO} \) in the context \( C \) of the proof of proposition 5, thereby contradicting Proposition 6.

Proof of Proposition 8. For all \( \Gamma \subset \mathcal{L}^2 \) such that \( \Gamma \vdash_{\mathcal{NI}_2^\forall} A \mathbin{\gamma} B \), either \( \Gamma \vdash_{\mathcal{NI}_2^\forall} A \) or \( \Gamma \vdash_{\mathcal{NI}_2^\forall} B \). This can be established following Prawitz' proof of the generalized disjunction property of \( \mathcal{NI} \) (see Corollary 6 [19, p. 56]). We suppose (given the normalization theorem for \( \mathcal{NI}_2^\forall \)) the existence of a normal derivation of \( \Gamma \vdash_{\mathcal{NI}_2^\forall} A \mathbin{\gamma} B \) and we reason as Prawitz. The only additional case to consider is that in which \( A \mathbin{\gamma} B \) is the conclusion of a (sequence of) applications of \( \forall E \), but due to the atomic restriction this case can be treated as the other cases of \( \mathcal{NI} \).

Now suppose \( \mathbin{\gamma} \) is strongly definable in \( \mathcal{NI}_2^\forall \) and let \( C \vdash_{\mathcal{NI}_2^\forall} (Y \mathbin{\gamma} Z) \). Then \( Y \vdash_{\mathcal{NI}_2^\forall} C \), \( Z \vdash_{\mathcal{NI}_2^\forall} C \) and \( C \vdash_{\mathcal{NI}_2^\forall} Y \mathbin{\gamma} Z \) hold; by the above we deduce then that either \( C \vdash_{\mathcal{NI}_2^\forall} Y \) or \( C \vdash_{\mathcal{NI}_2^\forall} Z \) holds, and so we conclude that either \( Y \vdash_{\mathcal{NI}_2^\forall} Z \) or \( Z \vdash_{\mathcal{NI}_2^\forall} Y \), which is impossible.

\[ \square \]