COLLISION-AVOIDANCE AND FLOCKING IN THE CUCKER–SMALE-TYPE MODEL WITH A DISCONTINUOUS CONTROLLER

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ABSTRACT. The collision-avoidance and flocking of the Cucker–Smale-type model with a discontinuous controller are studied. The controller considered in this paper provides a force between agents that switches between the attractive force and the repulsive force according to the movement tendency between agents. The results of collision-avoidance are closely related to the weight function $f(r) = (r - d_0)^{-\theta}$. For $\theta \geq 1$, collision will not appear in the system if agents' initial positions are different. For the case $\theta \in [0, 1)$ that was not considered in previous work, the limits of initial configurations to guarantee collision-avoidance are given. Moreover, on the basis of collision-avoidance, we point out the impacts of $\psi(r) = (1 + r^2)^{-\beta}$ and $f(r)$ on the flocking behaviour and give the decay rate of relative velocity. We also estimate the lower and upper bound of distance between agents. Finally, for the special case that agents moving on the 1-D space, we give sufficient conditions for the finite-time flocking.

1. Introduction. The complex and orderly group behaviour of multi-agent systems that widely exist in nature have been a hot topic in many research fields. Such as starling flocks [2, 4] in nature, investment behaviour [1, 12] in economics and the evolution of languages [10, 16] in human society, the dynamic behaviour that appears in multi-agent systems presents many excellent properties. Moreover, multi-agent systems provide theoretical frameworks to many practical problems in science and engineering, such as the control law to space flight formations [21]. Therefore, in recent years, the theoretical researches on the dynamic behaviour of multi-agent systems continue to appear, which play a significant role in promoting the development of multi-agent systems. In 1987, using the method of algorithm modelling, Reynolds simulated the group behaviour of birds flock in [24]. By observing the behaviour characteristics of birds flock, he designed three rules in the simulation algorithm: Collision-avoidance, Aggregation and Velocity-matching. These three rules are of great practical significance in the process of group behaviour evolution of multi-agent systems. In the field of mathematics, mathematical modelling, as an effective tool to deal with practical problems, is also applied to research of multi-agent systems. To provide some justification of their observation of birds...
flock, Cucker and Smale presented the Cucker–Smale model in work [9] to describe the evolution of a flock. The Cucker–Smale model can be written as

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t), \quad i \in \{1, 2, ..., N\}, \\
\dot{v}_i(t) &= \lambda \sum_{j=1}^{N} \psi(r_{ij})(v_j(t) - v_i(t)),
\end{align*}
\]

where \(x_i \in \mathbb{R}^d\), \(v_i \in \mathbb{R}^d\) indicate the position and velocity of agent \(i\) and \(d \in \mathbb{N}^+\) is the dimension of space. The intensity of the influence between particles is indicated by \(\lambda > 0\) and \(r_{ij}(t) = ||x_i(t) - x_j(t)||\). The communication \(\psi\) decreases with distance between agents and satisfies that

\[
\psi(r) = (1 + r^2)^{-\beta}, \quad \beta > 0.
\]

Once Cucker–Smale model was put forward, it received extensive attention, and many properties of flocking behaviour were studied, see references [5, 6, 13, 19, 26, 27].

From the work in [29, 30], it can be seen that the coordination mechanism of Cucker–Smale model has the function of velocity-matching. When the relative velocity between agents decays rapidly, the aggregation will appear in Cucker–Smale model, which means that Cucker–Smale model does not specifically consider collision-avoidance and aggregation. Thus, considering the great practical significance of collision-avoidance and aggregation in multi-agent systems, a goal of this paper is to design a controller to ensure that collision-avoidance and aggregation appear in the Cucker–Smale-type model.

The structure of this paper is as follows. In Section 2, we introduce the Cucker–Smale-type model with a discontinuous controller and do some necessary preliminaries. The main results about collision-avoidance and flocking behaviour are given in Section 3. Section 4 provides several numerical simulations to validate the theoretical results. Finally, the conclusions are drawn in Section 5.

2. Model statement and some preliminaries. Inspired by the Cucker–Smale model with inter-particle bonding forces presented in work [20], we design the following Cucker–Smale-type model to ensure collision-avoidance and aggregation by using a discontinuous controller.

\[
\begin{align*}
\dot{x}_i(t) &= v_i(t), \quad i \in \{1, 2, ..., N\}, \\
\dot{v}_i(t) &= \lambda \sum_{j=1}^{N} \psi(r_{ij})(v_j(t) - v_i(t)) + \frac{\gamma}{N} \sum_{j \neq i} \chi_{ij}(x, v) f(r_{ij})(x_j(t) - x_i(t)),
\end{align*}
\]

where \(\lambda\) and \(\gamma\) are positive parameters to weight the influence of different parts of the model. Similar to paper [9], the communication function \(\psi(r)\) is taken as (2). Considering works about the relationship between the singular function and collision-avoidance in references [3, 7, 8, 13, 22, 23], we take the weight function \(f(r)\) of the discontinuous controller as

\[
f(r) = (r - d_0)^{-\theta}, \quad d_0 > 0, \quad \theta > 0,
\]

where \(d_0\) is the safe distance between agents and we want to show that the distance between any two agents is greater than \(d_0\) at any time, i.e., collision-avoidance. It can be seen that the weight function \(f(r)\) is singular at \(r = d_0\) only.

Moreover, for any initial position \(x(0) = (x_1(0), x_2(0), ..., x_N(0))^T\) satisfying

\[
\|x_i(0) - x_j(0)\| > d_0, \quad i \neq j,
\]
due to the continuity of trajectories \( x_i, \ i \in \{1, 2, ..., N\} \), one can define the maximum collision-free interval similar to our previous work \([6]\) as \([0, T_0]\), where \( T_0 > 0 \) meets

\[
T_0 \triangleq \sup \{ s \in \mathbb{R}_+ : \text{the trajectory } x(t) \text{ corresponding to } (5) \text{ for system (3) meets } \|x_i(t) - x_j(t)\| > d_0, \ t \in [0, s]\}.
\]

(6)

Then, we will analyse the dynamic behaviour of system (3) on the interval \([0, T_0]\) and point out that, when \( f(r) \) satisfies a certain relationship with the initial data \((x(0), v(0))\), there is no collision at any time, i.e., \( T_0 = +\infty \).

Inspired by the event-triggered impulsive control in \([17, 18]\), we take the discontinuous controller in (3) that switches between the attractive force and the repulsive force according to the following rule

\[
\chi_{ij}(x, v) = \begin{cases} 
1, & (x_i - x_j, v_i - v_j) > 0; \\
-1, & (x_i - x_j, v_i - v_j) \leq 0.
\end{cases}
\]

(7)

The basic idea of the discontinuous controller is that, when the movement tendency of the agent \( i \) is far away from agent \( j \), i.e., \((x_i - x_j, v_i - v_j) > 0 \) and \( \chi_{ij}(x, v) = 1 \), there is an attractive force between agents \( i \) and \( j \); otherwise, there is a repulsive force between agents \( i \) and \( j \). Different from the repulsive force used only to avoid collision in the references \([7, 8]\), the discontinuous controller can lead to both collision-avoidance and aggregation of system (3) since it can switch between the attractive force and the repulsive force, which will be shown in the results of the next section.

Because of the existence of the discontinuous controller, there may be no solution to system (3). Actually, considering the continuity of \((x_i, v_i), \ i \in \{1, 2, ..., N\}\), one can know that there exist at most countable many switches in \([0, T_0]\) and there is a non empty time interval between two adjacent switches. Thus, before giving the main results, it is worth giving the definition of the solution similar to that in \([14, 15]\).

**Definition 2.1** \((14, 15)\). Let the curve \((x_i(t), v_i(t)) \in \mathbb{R}^d \times \mathbb{R}^d\) be continuous in \([0, T_0]\) and \(+\infty \bigcup_{k=0}^{+\infty} [t_k, t_{k+1}) = [0, T_0]\), where \( T_0 \) is given in (6). We call the continuous curve \((x_i(t), v_i(t))\) a solution to system (3) if it solves the equation on every \([t_k, t_{k+1})\), \( k \in \mathbb{N}^+ \cup \{0\}\).

**Remark 2.2.** Since the solution of system (3) in the sense of Definition 2.1 may be not in general \(C^1\) smooth, we will use upper-right Dini derivative in the following discussion like references \([11, 25]\). For a given function \( w \) that is continuous at \( t \), its upper-right Dini derivative is defined by

\[
D^+ w(t) = \lim_{h \to 0^+} \frac{w(t + h) - w(t)}{h}.
\]

Because swarming and flocking are ubiquitous phenomena in group behaviour and have been widely studied, this paper will consider the two interesting phenomena of system (3) on the basis of collision-avoidance. We give some useful preliminaries that will be significantly used for the proof of the main results. Taking the center of positions and velocities of system (3) as

\[
\bar{x}(t) = \frac{1}{N} \sum_{i=1}^{N} x_i(t), \quad \bar{v}(t) = \frac{1}{N} \sum_{i=1}^{N} v_i(t),
\]
we can obtain that
\[
\Lambda^2(x(t)) = \sum_{i,j=1}^{N} \|x_i(t) - x_j(t)\|^2 = 2N \sum_{i=1}^{N} \|x_i(t) - \bar{x}(t)\|^2,
\]
\[
\Gamma^2(v(t)) = \sum_{i,j=1}^{N} \|v_i(t) - v_j(t)\|^2 = 2N \sum_{i=1}^{N} \|v_i(t) - \bar{v}(t)\|^2,
\]
where \(\|\cdot\|\) is the Euclidean norm. Then, similarly to [13, 28], we list the definitions of swarming and flocking as follows.

**Definition 2.3** ([28]). System (3) forms a swarming if and only if its solution \((x,v)\), which is in the sense of Definition 2.1, satisfies
\[
\sup_{0 \leq t < +\infty} \Lambda(x(t)) < +\infty \text{ and } \sup_{0 \leq t < +\infty} \Gamma(v(t)) < +\infty.
\]

**Definition 2.4** ([13]). System (3) forms a flocking if and only if its solution \((x,v)\), which is in the sense of Definition 2.1, satisfies
\[
\sup_{0 \leq t < +\infty} \Lambda(x(t)) < +\infty \text{ and } \lim_{t \to +\infty} \Gamma(v(t)) = 0.
\]

Inspired by our previous work in [6], we take the following Lyapunov function \(E(x,v)\) to analyse the dynamic behaviour of system (3)
\[
E(x(t),v(t)) = \sum_{i=1}^{N} \|v_i(t) - \bar{v}(t)\|^2 + \frac{\gamma}{N} \sum_{i=1}^{N} \sum_{j \neq i} \int_{r_{ij}(0)}^{\min r_{ij}(t)} f(r)rdr.
\]

### 3. Main results

In this section, we focus on the dynamic behaviour of the Cucker–Smale-type model with a discontinuous controller given in (3). We first give the result of collision-avoidance in system (3) and ensure the solution in the sense of Definition 2.1 exists globally. Then, for the flocking behaviour in system (3), we study the impacts of \(\psi(r)\) and \(f(r)\) separately. Finally, for the case that agents moving on the real line, we give sufficient conditions for the finite-time flocking behaviour in system (3).

#### 3.1. Collision-avoidance

For the Cucker–Smale-type model with a discontinuous controller in (3), we give the following sufficient condition to ensure that there is no collision between any two agents at any time.

**Theorem 3.1.** Suppose that one of the following hypotheses holds:

(i) \(\theta \in [0, 1)\) and \(E(x(0),v(0)) < \frac{\gamma}{N} \int_{\min r_{ij}(0)}^{\min r_{ij}(0)} f(r)rdr\);

(ii) \(\theta \in [1, +\infty)\) and \(\min_{i \neq j} \|x_i(0) - x_j(0)\| > d_0\).

Then, system (3) admits a global solution \((x(t),v(t))\) in the sense of Definition 2.1 such that
\[
\min_{i \neq j} \|x_i(t) - x_j(t)\| \geq d > d_0, \quad t \geq 0,
\]
where \(d \equiv d(x(0),v(0))\) satisfies
\[
E(x(0),v(0)) = \frac{\gamma}{N} \int_{d}^{\min r_{ij}(0)} f(r)rdr.
\]
Remark 3.2. It can be found from Theorem 3.1 that, when $\theta \in [1, +\infty)$, the discontinuous controller ensures collision-avoidance with no-collisional initial configurations, which is similar to the results in [3, 7, 8, 6] and implies that the discontinuous controller plays its role in avoiding collision. In addition, for the case $\theta \in [0, 1)$ where references [3, 7, 8, 6] did not give result of collision-avoidance, Theorem 3.1 gives the limits of initial configurations to guarantee collision-avoidance.

Proof. It is a fact that (5) can be deduced from either initial configurations (i) or (ii), which means that we can define the maximum collision-free interval as $[0, T_0]$ and $T_0 > 0$ is given in (6). First, we will prove that the maximum collision-free interval is $[0, +\infty)$ if the initial configuration meets (i) or (ii), i.e., there is no collision between any two agents at any time. If $T_0 = +\infty$, we complete the proof directly. Otherwise, if $T_0 < +\infty$, one can know that there exists at least one collision at $T_0$, that is

$$\lim inf_{t \to T_0} r_{ij}(t) = d_0.$$  

Without loss of generality, we can take $i, j \in \{1, 2, ..., N\}$ such that $\lim_{t \to T_0} r_{ij}(t) = d_0$. Thus, combining with the definition of $E(x, v)$ in (9), we have the following estimation

$$\lim_{t \to T_0} E(x(t), v(t)) \geq \lim_{t \to T_0} \frac{\gamma}{N} \left| \int_{r_{ij}(0)}^{r_{ij}(t)} f(r) r dr \right|$$  

$$= \frac{\gamma}{N} \int_{d_0}^{\min_{j \neq i} r_{ij}(0)} f(r) r dr$$  

$$\geq \frac{\gamma}{N} \int_{d_0}^{\min_{j \neq i} r_{ij}(0)} f(r) r dr. \quad (13)$$

On the other hand, along the solution $(x, v)$ of system (3) in the sense of Definition 2.1, we calculate the derivative of the Lyapunov function $E(x(t), v(t))$ on interval $[0, T_0]$ as

$$D^+ E(x(t), v(t)) = 2 \sum_{i=1}^{N} (\dot{v}_i - \ddot{v}_i - \ddot{v}) + D^+ \frac{\gamma}{N} \sum_{i=1}^{N} \sum_{j \neq i} \left| \int_{r_{ij}(0)}^{r_{ij}(t)} f(r) r dr \right|$$

$$= -\frac{\lambda}{N} \sum_{i,j=1}^{N} \psi(r_{ij}) \|v_j - v_i\|^2 - \frac{\gamma}{N} \sum_{i=1}^{N} \sum_{j \neq i} f(r_{ij}) \langle x_i - x_j, v_i - v_j \rangle$$

$$+ \frac{\gamma}{N} \sum_{i=1}^{N} \sum_{j \neq i} \left( \int_{r_{ij}(0)}^{r_{ij}(t)} \frac{f(r) r dr}{f(r_{ij}(0))} \right) f(r_{ij}) \langle x_i - x_j, v_i - v_j \rangle,$$ 

where we use the fact that

$$\ddot{v} = 0 \text{ and } \sum_{i=1}^{N} \langle \dddot{v}_i, \dddot{v}_i \rangle = N \langle \dddot{v}, \dddot{v} \rangle = 0.$$  

Thus, from (14), one can obtain that

$$D^+ E(x(t), v(t)) \leq -\frac{\lambda}{N} \sum_{i,j=1}^{N} \psi(r_{ij}) \|v_j - v_i\|^2 \text{ a.e. } t \in [0, T_0), \quad (15)$$
which tells us that, on the interval \([0, T_0]\), the Lyapunov function \(E(x(t), v(t))\) is non increasing along the solution \((x(t), v(t))\) of system (3), i.e., \(E(x(t), v(t)) \leq E(x(0), v(0))\). Then, we show that both the initial configurations \((i)\) and \((ii)\) can lead out

\[
E(x(0), v(0)) < \frac{\gamma}{N} \int_{d_0}^{\min r_{ij}(0)} f(r) r dr. \tag{16}
\]

It is obvious that (16) can be obtained from \((i)\) directly and we consider \((ii)\). For the case \(\theta \geq 1\), one can estimate \(\frac{\gamma}{N} \int_{d_0}^{\min r_{ij}(0)} f(r) r dr\) as

\[
\frac{\gamma}{N} \int_{d_0}^{\min r_{ij}(0)} f(r) r dr = \frac{\gamma}{N} \int_{d_0}^{\min r_{ij}(0)} (r - d_0)^{-\theta} dr \\
\geq \frac{\gamma d_0}{N} \int_{d_0}^{\min r_{ij}(0)} (r - d_0)^{-\theta} dr \\
= +\infty,
\]

which yields (16) directly since \(E(x(0), v(0)) < +\infty\). Furthermore, applying (16) and the truth that \(E(x, v)\) is non increasing along \((x, v)\), we obtain the following formula

\[
E(x(t), v(t)) \leq E(x(0), v(0)) < \frac{\gamma}{N} \int_{d_0}^{\min r_{ij}(0)} f(r) r dr, \quad t \in [0, T_0). \tag{17}
\]

Thus, we obtain a contradiction between the estimation (17) and (13), which means the assumption of \(T_0 < +\infty\) is false and there is no collision between any two agents at any time, i.e.,

\[
\min_{i \neq j} \|x_i(t) - x_j(t)\| > d_0, \quad t \geq 0.
\]

Now, we estimate the lower bound of distance between agents in (11). Following from (17), one can find \(d_0 < \bar{d} \leq \min_{i \neq j} r_{ij}(0)\) such that

\[
E(x(0), v(0)) = \frac{\gamma}{N} \int_{\bar{d}}^{\min r_{ij}(0)} f(r) r dr. \tag{18}
\]

This together with the structure of \(E(x, v)\) in (9) gives us the following inequality

\[
\frac{\gamma}{N} \int_{r_{ij}(0)}^{r_{ij}(t)} f(r) r dr \leq E(x, v) \leq E(x(0), v(0)) = \frac{\gamma}{N} \int_{\bar{d}}^{\min r_{ij}(0)} f(r) r dr.
\]

If \(r_{ij}(t) \geq r_{ij}(0)\), then \(r_{ij}(t) \geq r_{ij}(0) \geq \bar{d}\). Otherwise, if \(r_{ij}(t) < r_{ij}(0)\), we have

\[
\int_{r_{ij}(0)}^{r_{ij}(t)} f(r) r dr \leq \int_{\bar{d}}^{\min r_{ij}(0)} f(r) r dr,
\]

which means \(r_{ij}(t) \geq \bar{d}\) directly. Thus, (11) holds globally and the lower bound of distance between agents is given in (12). Hence we complete the proof.

\[
\boxed{\square}
\]

3.2. **Impacts of** \(f(r)\) **on flocking behaviour.** In this part, we study the impacts of the weight function \(f(r)\) on flocking behaviour. Sufficient conditions related to \(f(r) = (r - d_0)^\theta\) and initial configurations are given in this subsection.
Theorem 3.3. Suppose that the condition in Theorem 3.1 holds. Moreover, if one of the following hypotheses holds:

(I) \( \theta \in [0, 2] \);

(II) \( \theta \in (2, +\infty) \) and \( E(x(0), v(0)) < \frac{\gamma}{N} \int_{\max r_{ij}(0)}^{+\infty} f(r)rdr \).

Then, system (3) forms a flocking as Definition 2.4 and the solution satisfies

\[
\sup_{0 \leq t \leq +\infty} \max_{1 \leq i, j \leq N} r_{ij}(t) \leq \bar{M} < +\infty, \quad t \geq 0,
\]

and

\[
\Gamma(v(t)) \leq \Gamma(v(0)) e^{-\lambda \psi(\bar{M}) t}, \quad t \geq 0,
\]

where \( \Gamma(v) \) is given in (8) and \( \bar{M} \) satisfying

\[
E(x(0), v(0)) = \frac{\gamma}{N} \int_{\max r_{ij}(0)}^{\bar{M}} f(r)rdr.
\]

Proof. Following from the flocking behaviour in Definition 2.4, we find that the flocking behaviour consists of two parts, i.e., aggregation and velocity-matching.

First, we consider the aggregation behaviour of system (3) and give the uniform estimate of maximum inter-particle distance. We start by showing that both (I) and (II) can derive out

\[
E(x(0), v(0)) < \frac{\gamma}{N} \int_{\max r_{ij}(0)}^{+\infty} f(r)rdr.
\]

Actually, (II) can derive out (23) directly and we only need consider (I). Since \( \theta \in [0, 2] \), we can estimate \( \frac{\gamma}{N} \int_{\max r_{ij}(0)}^{+\infty} f(r)rdr \) as

\[
\frac{\gamma}{N} \int_{\max r_{ij}(0)}^{+\infty} f(r)rdr = \frac{\gamma}{N} \int_{\max r_{ij}(0)}^{+\infty} (r - d_0)^{-\theta} rdr
\]

\[
\geq \frac{\gamma}{N} \int_{\max r_{ij}(0)}^{+\infty} r^{1-\theta} dr
\]

\[
= +\infty
\]

and obtain (23). Thus, one can find \( \bar{M} \geq \max_{i \neq j} r_{ij}(0) \) such that

\[
E(x(0), v(0)) = \frac{\gamma}{N} \int_{\max r_{ij}(0)}^{\bar{M}} f(r)rdr.
\]

On the other hand, following from the fact that \( E(x, v) \) is non increasing on \([0, +\infty)\) and the structure of \( E(x, v) \) in (9), we have

\[
\frac{\gamma}{N} \left| \int_{r_{ij}(0)}^{r_{ij}(t)} f(r)rdr \right| \leq E(x, v) \leq E(x(0), v(0)) = \frac{\gamma}{N} \int_{\max r_{ij}(0)}^{\bar{M}} f(r)rdr.
\]

If \( r_{ij}(t) \leq r_{ij}(0) \), then \( r_{ij}(t) \leq r_{ij}(0) \leq \bar{M} \). Otherwise, if \( r_{ij}(t) > r_{ij}(0) \), we get

\[
\int_{r_{ij}(0)}^{r_{ij}(t)} f(r)rdr \leq \int_{\max r_{ij}(0)}^{\bar{M}} f(r)rdr,
\]
which means $r_{ij}(t) \leq \bar{M}$. Therefore, if one of hypotheses in (I) and (II) holds, the distance between agents is bounded (i.e. (20)) and the upper bound $\bar{M}$ is given in (22).

Then, we study the velocity-matching behaviour of system (3) and estimate the decay rate of the relative velocities between agents. Following from (3), one can obtain

$$D^+ \sum_{i=1}^{N} \|v_i(t) - \bar{v}(t)\|^2 = 2 \sum_{i=1}^{N} \langle v_i - \bar{v}, \dot{v}_i - \dot{\bar{v}} \rangle$$

$$= 2 \sum_{i=1}^{N} \langle v_i, \dot{v}_i \rangle$$

$$= -\frac{\lambda}{N} \sum_{i,j=1}^{N} \psi(r_{ij}) \|v_j - v_i\|^2$$

$$- \frac{\gamma}{N} \sum_{i=1}^{N} \sum_{j \neq i} f(r_{ij}) |\langle x_i - x_j, v_i - v_j \rangle|$$

$$\leq -\frac{\lambda}{N} \sum_{i,j=1}^{N} \psi(r_{ij}) \|v_j - v_i\|^2.$$  \hspace{1cm} (24)

Noting the relationship in (8) and the truth that the communication function $\psi(r) = (1 + r^2)^{-\beta}$ is non increasing, one can estimate the formula (24) as

$$D^+ \Gamma^2(v) = D^+ 2N \sum_{i=1}^{N} \|v_i(t) - \bar{v}(t)\|^2$$

$$\leq -2\lambda \sum_{i,j=1}^{N} \psi(r_{ij}) \|v_j - v_i\|^2$$

$$\leq -2\lambda \psi(\bar{M}) \sum_{i,j=1}^{N} \|v_j - v_i\|^2$$

$$= -2\lambda \psi(\bar{M}) \Gamma^2(v),$$

which means $\Gamma(v(t)) \leq \Gamma(v(0)) e^{-\lambda \psi(\bar{M}) t}$ and we finish the proof. \hfill \Box

**Remark 3.4.** Theorem 3.3 implies that, different from the results in [9, 13, 19] that the flocking behaviour is related to the communication function $\psi(r)$, the flocking behaviour of system (3) is also affected by the weight function $f(r)$ of the discontinuous controller. The reason is that the discontinuous controller provides not only the repulsive force when agents approach others, but also the attractive force when agents move away from others. Moreover, it can be found from the following proof that the uniform estimate of maximum inter-particle distance $\bar{M}$ still holds without the velocity-matching function in classical Cucker–Smale model (1). That means, if we make $\lambda = 0$, the swarming behaviour in Definition 2.3 appears in system (3).

### 3.3. Impacts of $\psi(r)$ on flocking behaviour.

This subsection studies the impacts of $\psi(r)$ on flocking behaviour. Similar to works in [9, 13, 19], we give the
sufficient conditions, which related to \( \psi(r) = (1 + r^2)^{-\beta} \) and initial configurations, to ensure that the flocking behaviour appears in system (3).

**Theorem 3.5.** Suppose that the condition in Theorem 3.1 holds. Moreover, if one of the following hypotheses holds:

(a) \( \beta \in [0, 1/2] \);

(b) \( \beta \in (1/2, +\infty) \) and \( \Gamma(v(0)) < \lambda \int_{\Lambda(x(0))}^{+\infty} \psi(r)dr \).

Then, system (3) forms a flocking as Definition 2.4 and the solution satisfies

\[
\Lambda(x(t)) \leq D < +\infty, \quad t \geq 0,
\]

and

\[
\Gamma(v(t)) \leq \Gamma(v(0))e^{-\lambda \psi(D)t}, \quad t \geq 0,
\]

where \( \Lambda(x) \) and \( \Gamma(v) \) is given in (8) and \( D \) satisfying

\[
\Gamma(v(0)) = \lambda \int_{\Lambda(x(0))}^{D} \psi(r)dr.
\]

**Proof.** We first construct a Lyapunov function \( W(x, v) \) that is significantly useful in the following proof

\[
W(x(t), v(t)) = \Gamma(v(t)) + \lambda \int_{\Lambda(x(0))}^{\Lambda(x(t))} \psi(r)dr.
\]

Then, following from the structure of \( \Lambda(x) \) and \( \Gamma(v) \) in (8) and using Cauchy-Schwartz inequality, one can find that

\[
D^+ \Lambda^2(x(t)) = 2 \sum_{i,j=1}^{N} \langle x_i(t) - x_j(t), v_i(t) - v_j(t) \rangle \\
\leq 2 \sum_{i,j=1}^{N} \|x_i(t) - x_j(t)\| \|v_i(t) - v_j(t)\| \\
\leq 2\Lambda(x(t))\Gamma(v(t)),
\]

which yields

\[
D^+ \Lambda(x(t)) \leq \Gamma(v(t)).
\]

It follows from system (3) and formula (25), (31) that

\[
D^+ W(x(t), v(t)) = D^+ \Gamma(v(t)) + \lambda D^+ \int_{\Lambda(x(0))}^{\Lambda(x(t))} \psi(r)dr \\
\leq -\lambda \psi(\Lambda(x(t)))\Gamma(v(t)) + \lambda \psi(\Lambda(x(t)))\Gamma(v(t)) \\
= 0.
\]

Thus, the Lyapunov function \( W(x, v) \) is non increasing along the solution \( (x, v) \) of system (3), i.e.,

\[
W(x(t), v(t)) \leq W(x(0), v(0)) = \Gamma(v(0)).
\]

On the other hand, following from the truth that \( \lambda \int_{\Lambda(x(0))}^{+\infty} \psi(r)dr = +\infty \) when \( \beta \in [0, 1/2] \), one can say that both condition (a) and (b) can derive out

\[
\Gamma(v(0)) < \lambda \int_{\Lambda(x(0))}^{+\infty} \psi(r)dr.
\]
This together with (33) and the structure of $W(x,v)$ in (30) gives the following inequality

$$\lambda \int_{\Lambda(x(0))} \psi(r) dr < \lambda \int_{\Lambda(x(0))}^{+\infty} \psi(r) dr,$$

which means there exists a $\bar{D} < +\infty$ such that

$$\Gamma(v(0)) = \lambda \int_{\Lambda(x(0))}^{\bar{D}} \psi(r) dr,$$

and the solution $(x,v)$ of system (3) meets

$$\Lambda(x(t)) \leq \bar{D}, \quad t \geq 0.$$

Therefore, using the derivative in (25) and the uniform estimation $\bar{D}$ of $\Lambda(x)$, we have

$$D^+ \Gamma(v(t)) \leq -\lambda \psi(\Lambda(x(t))) \Gamma(v(t)) \leq -\lambda \psi(\bar{D}) \Gamma(v(t)).$$

Thus, we can estimate $\Gamma(v(t))$ as

$$\Gamma(v(t)) \leq \Gamma(v(0)) e^{-\lambda \psi(\bar{D}) t},$$

and the proof is finished.

**Remark 3.6.** On the basis of collision-avoidance, the sufficient conditions of the flocking behaviour in Theorem 3.5 are the same as sufficient conditions given by papers [9, 13, 19], which means the discontinuous controller in system (3) does not break the velocity-matching function in classical Cucker–Smale model (1).

3.4. The finite-flocking behaviour in system (3) on the real line. Due to the properties of the discontinuous controller, system (3) will present the finite-time flocking behaviour on the real line. In this subsection, we consider the dynamic behaviour of system (3) on the real line and list sufficient conditions of the finite-time flocking behaviour.

**Theorem 3.7.** Suppose that the condition in Theorem 3.1 holds. Moreover, if one of the following hypotheses holds:

(C1) $\theta \in [0, 1]$;

(C2) $\theta \in (1, +\infty)$ and $\Gamma^2(v(0)) < 2\gamma_d \int_{\Lambda(x(0))}^{+\infty} f(r) dr.$

Then, system (3) forms a flocking in finite time, i.e.

$$\sup_{0 \leq t \leq \infty} \max_{1 \leq i,j \leq N} r_{ij}(t) \leq M_1 < +\infty, \quad t \geq 0,$$

and

$$\begin{cases} 
\Gamma(v(t)) \leq \Gamma(v(0)) - \gamma_d f(M_1) t, \quad t \leq T_1 = \frac{\Gamma(v(0))}{\gamma_d f(M_1)}; \\
\Gamma(v(t)) = 0, \quad t > T_1,
\end{cases}$$

where $d$ is given in (12) and $M_1$ satisfying

$$\Gamma^2(v(0)) = 2\gamma_d \int_{\Lambda(x(0))}^{M_1} f(r) dr.$$  

(34)
Proof. For the convenience of proof, we estimate the Dini derivative of $\Gamma(v(t))$ on the real line as follows

$$
D^+ \Gamma^2(v) = D^+ 2N \sum_{i=1}^{N} |v_i(t) - \bar{v}(t)|^2 \\
= -2\lambda \sum_{i,j=1}^{N} \psi(r_{ij}) |v_j - v_i|^2 \\
- 2\gamma \sum_{i=1}^{N} \sum_{j \neq i} f(r_{ij}) \|x_i - x_j, v_i - v_j\| \\
= -2\lambda \sum_{i,j=1}^{N} \psi(r_{ij}) |v_j - v_i|^2 \\
- 2\gamma \sum_{i=1}^{N} \sum_{j \neq i} f(r_{ij}) |x_i - x_j| |v_i - v_j| \\
\leq -2\gamma \sum_{i=1}^{N} \sum_{j \neq i} f(r_{ij}) |x_i - x_j| |v_i - v_j|,
$$

(38)

Since the condition in Theorem 3.1 holds, we have $|x_i - x_j| > d$. This together with (38) and $\sum_{i=1}^{N} \sum_{j \neq i} |v_i - v_j| \geq \Gamma(v(t))$ gives the following estimation

$$
D^+ \Gamma^2(v) \leq -2\gamma \sum_{i=1}^{N} \sum_{j \neq i} f(r_{ij}) |x_i - x_j| |v_i - v_j| \\
\leq -2\gamma df(\Lambda(x(t))) \sum_{i=1}^{N} \sum_{j \neq i} |v_i - v_j| \\
\leq -2\gamma df(\Lambda(x(t))) \Gamma(v(t)).
$$

(39)

Then, we construct a Lyapunov function $G(x, v)$ to analyse the upper bound of the distance between agents

$$
G(x(t), v(t)) = \Gamma^2(v(t)) + 2\gamma d \int_{\Lambda(x(0))}^{\Lambda(x(t))} f(r)dr.
$$

(40)

By the discussion similar to that in Theorem 3.3 and Theorem 3.5, one can obtain the fact that $G(x, v)$ is non increasing along the solution of system (3) in the sense of Definition 2.1 and, when one of the hypotheses in (34) holds, there exists a upper bound $M_1 < +\infty$ such that $M_1 \geq \Lambda(x(t))$. Thus, following from (39), we have

$$
D^+ \Gamma(v) \leq -\gamma df(M_1),
$$

which can derive out (36) directly by using comparison theorem, i.e., the finite-time flocking behaviour appears in system (3).

Remark 3.8. Following from the proof of Theorem 3.7, we find the finite-time flocking behaviour can appear in system (3) even if $\lambda = 0$. This implies that the finite-time flocking behaviour on the real line is caused by the discontinuous controllers in system (3)2. The reason why the finite-time flocking behaviour can appear in system (3) on the real line is that the discontinuous controller switches
with $\chi_{ij}$, which can cause velocity-matching in finite time. However, when agents move on the space that higher than 1-D space, the angle between the relative position and relative velocity may not be 0 or $\pi$ (see Figure 3), i.e., the treatment in (38) will no longer be correct.

4. **Numerical simulations.** In this section, by three numerical simulations of different dynamic behaviours in system (3), we show the effectiveness of the theoretical analysis in Section 3.

**Example 1.** Flocking behaviour in system (3).

First, we show the flocking behaviour discussed in the Theorem 3.3 and Theorem 3.5. Considering $N = 4$ agents moving on a 2-D space, we take $\lambda = 1$, $\gamma = 4$, $\beta = 1/3$, $\theta = 2$ and the safe distance between agents $d_0 = 0.1$. The initial configurations are take as Table 1.

| $i$ | 1       | 2       | 3       | 4       |
|-----|---------|---------|---------|---------|
| $(x_i(0))$ | (1,1)   | (2,2)   | (3,3)   | (4,4)   |
| $(v_i(0))$ | (-3.25,-1.25) | (0.75,3.75) | (-1.25,0.75) | (3.75,-3.25) |

Figure 1. Flocking: the evolution of agents’ position (left) and velocity (right).

It can be seen from Figure 1 that the flocking behaviour appears in system (3). Figure 2 shows the evolution of the distance and related velocity between agents. The lower and upper bound of the distance between agents correspond to collision-avoidance and aggregation respectively and the decay of the relative velocity corresponds to velocity-matching.

**Example 2.** Swarming behaviour in system (3).

As stated in Remark 3.4, the swarming behaviour will appear in system (3) if we take $\lambda = 0$, i.e., without the velocity-matching function in classical Cucker–Smale model (1). Keeping the parameters and initial configurations the same as in Example 1, we change $\lambda$ from 1 to 0 and obtain Figure 3, 4 with the initial data in Table 2.
Figure 2. Flocking: the lower and upper bound of the distance between agents (left) and the logarithm of the maximum velocity difference (right).

Table 2. Initial configurations of system (3)

| i | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| \(x_i(0)\) | (1,1) | (2,2) | (3,3) | (4,4) |
| \(v_i(0)\) | (-0.325,-0.125) | (0.075,0.375) | (-0.125,0.075) | (0.375,-0.325) |

Figure 3. Swarming: the evolution of agents’ position (left) and velocity (right).

Under the impact of the discontinuous controller, system (3) presents the interesting swarming behaviour in Figure 3, which is similar to the circling behaviour of birds flock in nature. The reason for this phenomenon may be related to the switching rule \(\chi_{ij}\) in (7).

Example 3. Finite-time flocking behaviour in system (3) on the real line.

Taking the initial configurations of system (3) on the real line as Table 3, we keep the parameters the same as in Example 2 and obtain Figure 5, 6. Comparing Figure 2 and Figure 6, one can find that the difference between the asymptotic flocking (Figure 2) and the finite-time flocking behaviour (Figure 6).
Figure 4. Swarming: the lower and upper bound of the distance between agents (left) and the maximum velocity difference (right).

Table 3. Initial configurations of system (3) on the real line

| i  | 1   | 2     | 3     | 4     |
|----|-----|-------|-------|-------|
| \((x_i(0), v_i(0))\) | (1,-3.25) | (2,0.75) | (3,-1.25) | (4,3.75) |

Figure 5. Finite-time flocking: the evolution of agents’ position (left) and velocity (right).

Figure 6. Finite-time flocking: the lower and upper bound of the distance between agents (left) and the logarithm of the maximum velocity difference (right).
5. Conclusion. In order to avoid collisions in multi-agent system, this paper introduces a discontinuous controller into Cucker–Smale model and studies the impacts on the dynamic behaviour of the Cucker–Smale-type model. The controller considered in this paper provides a force between agents that switching between the attractive force and the repulsive force according to the movement tendency between agents. The results show that the weight function \( f(r) = (r - d_0)^\theta \) of the controller has a significant influence on the dynamic behaviour of the system. For \( \theta \geq 1 \), collision will not appear in the system if agents’ initial positions are different. For the case \( \theta \in [0, 1) \) that not considered in previous work \([3, 7, 8, 6]\), the limits of initial configurations to guarantee collision-avoidance are given. On the basis of collision-avoidance, we also present sufficient conditions for the flocking behaviour, which are related to \( f(r), \psi(r) \) and the initial configuration. We also estimate the lower and upper bounds of distance and the decay rate of the relative velocity between agents. Finally, considering the case that agents moving on the real line, our result shows that the finite-time flocking will appear in the Cucker–Smale-type model.

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