Double points of free projective line arrangements

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Abstract

We prove Anzis and Tohaneanu conjecture, that is the Dirac-Motzkin conjecture for supersolvable line arrangements in the projective plane over an arbitrary field of characteristic zero. Moreover, we show that a divisionally free arrangements of lines contain at least one double point, that can be regarded as the Sylvester-Gallai theorem for some free arrangements. This is a corollary of a general result that if you add a line to a free projective line arrangement, then that line has to contain at least one double point. Also we prove some conjectures and one open problems related to supersolvable line arrangements and the number of double points.

1 Introduction

Let $\mathbb{K}$ be a field, $V = \mathbb{K}^3$, $S = \text{Sym}^*(V^*) = \mathbb{K}[x, y, z]$, $\mathbb{P}^2 := \text{Proj}(S)$ and $\text{Der } S := S \partial_x \oplus S \partial_y \oplus S \partial_z$. Then $\text{Der } S$ is an $S$-graded module and we say that $\theta \in \text{Der } S$ is homogeneous of degree $d$ if $\theta(x), \theta(y), \theta(z)$ belong to the homogeneous part $S_d$ of $S$ of degree $d$. Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ (called a projective line arrangement), equivalently, an arrangement of linear planes in $V$. For each $H \in \mathcal{A}$ let $\alpha_H$ be the defining linear form. Let $L(\mathcal{A})$ be the intersection lattice of $\mathcal{A}$ defined by

$$L(\mathcal{A}) := \{ \cap_{H \in B} H \mid B \subset \mathcal{A} \}$$

where we consider everything in $V = \cap_{H \in \emptyset} H$. So $V$ is always contained in $L(\mathcal{A})$. Also we always assume that the origin is in $L(\mathcal{A})$ (equivalently,
A is essential. Let $L_2(\mathcal{A})$ be the set $X \in L(\mathcal{A})$ such that codim$_V X = 2$. So they are points in $\mathbb{P}^2$, and we call $q \in L_2(\mathcal{A})$ a point. Define the Möbius function $\mu : L_2(\mathcal{A}) \to \mathbb{Z}$ by $\mu(p) := |\{H \in \mathcal{A} \mid p \in H\}| - 1$ for $p \in L_2(\mathcal{A})$. We say that $p \in L_2(\mathcal{A})$ is a double point if $\mu(p) = 1$. Let

$$n_2(\mathcal{A}) := \{|p \in L_2(\mathcal{A}) \mid \mu(p) = 1\}|,$$

$$n_2(H) := \{|p \in L_2(\mathcal{A}) \mid \mu(p) = 1, p \in H\}| (H \in \mathcal{A}).$$

Then we can define a characteristic polynomial $\chi_0(\mathcal{A}; t)$ of a projective line arrangement $\mathcal{A}$ by

$$\chi_0(\mathcal{A}; t) := t^2 - (|\mathcal{A}| - 1)t + \left( \sum_{q \in L_2(\mathcal{A})} \mu(q) \right) - |\mathcal{A}| + 1.$$

Now we can define the logarithmic vector field $D(\mathcal{A})$ of $\mathcal{A}$ as follows:

$$D(\mathcal{A}) = \{\theta \in \text{Der} S \mid \theta(\alpha_H) \in S\alpha_H (\forall H \in \mathcal{A})\}.$$

$D(\mathcal{A})$ is a reflexive $S$-graded module of rank 3, and not free in general. We say that $\mathcal{A}$ is free with exponents $\exp(\mathcal{A}) = (1, d_2, d_3)$ if there are $\theta_2, \theta_3 \in D(\mathcal{A})$ of degrees $d_2, d_3$ such that $\theta_2, \theta_3$ and the Euler derivation $\theta_E = x \partial_x + wy \partial_y + z \partial_z$ form a free basis for $D(\mathcal{A})$. By Terao’s factorization theorem in [18], in this case

$$\chi_0(\mathcal{A}; t) = (t - d_2)(t - d_3).$$

Hence in this case $|\mathcal{A}| = 1 + d_2 + d_3$. Free arrangements have been intensively studied in the theory of hyperplane arrangements. Moreover, in the free arrangements, a very important class of arrangements, so called the supersolvable arrangements exists. We say that $\mathcal{A}$ is supersolvable if there is a point $p \in L_2(\mathcal{A})$ (called the modular point) such that for $\mathcal{A}_p := \{H \in \mathcal{A} \mid p \in H\}$ and for all distinct $H, L \in \mathcal{A} \setminus \mathcal{A}_p$, there is the unique $K \in \mathcal{A}_p$ such that $H \cap L \subset K$. If $\mu(p) = m - 1$, then $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, m - 1, |\mathcal{A}| - m)$ (see [15], Theorem 4.58 for example). For general reference on the above results, please refer [15] and [21].

Supersolvable line arrangements have been intensively studied from viewpoints of algebra, combinatorics, geometry, and singularity. Among them, one of the most interesting and important conjectures is the following.

**Conjecture 1.1 ([8], Conjecture 3.2)**

Let $K = \mathbb{C}$, and let $\mathcal{A}$ be supersolvable. Then $n_2(\mathcal{A}) \geq \frac{|\mathcal{A}|}{2}.$
Let us review a history of Conjecture 1.1. The origin of Conjecture 1.1 is the *Sylvester’s problem* asking all real line arrangements that is not a pencil in $\mathbb{P}_2^R$ have at least one double points in $[17]$. Here we say that a projective line arrangement $\mathcal{A}$ is pencil if all the lines in $\mathcal{A}$ intersects at the same one point. This problem is solved by Gallai in [11] in 1944, thus it is now called the *Sylvester-Gallai theorem*. So it is natural to ask a lower bound of the cardinality of double points for such a line arrangement. When $\mathcal{A}$ is an arbitrary arrangement and $\mathbb{K} = \mathbb{R}$, Conjecture 1.1 is called the *Dirac-Motzkin conjecture*, and it is proved to be true when $|\mathcal{A}|$ is sufficiently large by Green and Tao in [12]. In [6], when $\mathcal{A}$ is supersolvable and $\mathbb{K} = \mathbb{R}$, this is proved without the assumption on $|\mathcal{A}|$. However, it is known that there is a line arrangement in the complex projective plane which has no double points like the dual Hesse arrangement, see [8], [11], and [14]. So it is natural to ask the supersolvable version of the Dirac-Motzkin conjecture over the complex number field as in Conjecture 1.1. Also, a supersolvable version of the Sylvester-Gallai theorem is reformulated as follows, which is a weaker version of Conjecture 1.1.

**Conjecture 1.2 ([13], Conjecture 11)**

*Let $\mathbb{K} = \mathbb{C}$, and let $\mathcal{A}$ be supersolvable. Then $n_2(\mathcal{A}) > 0$.***

These conjectures motivated several related researches on supersolvable line arrangements form several points of view, e.g., algebra, algebraic geometry, topology and combinatorics aiming at the classification of such arrangements. A part of such studies are in [5], [6], [7], [9], [10], [13], and [19]. On Conjectures 1.1 and 1.2, there have been several progresses and partial answers in [5], [6], [13] (e.g., see Theorem 2.7). The main result in this article is to prove (a) Conjecture 1.1 over an arbitrary field $\mathbb{K}$ of characteristic zero in full generality, and (b) Conjecture 1.2 in a wider category of supersolvable arrangements, so called the divisionally free arrangements. Let us state the first main result in this article.

**Theorem 1.3**

*Let $\mathbb{K}$ be a field of characteristic zero, and let $\mathcal{A}$ be a supersolvable line arrangement in $\mathbb{P}^2$. Then $n_2(\mathcal{A}) \geq \frac{|\mathcal{A}|}{2}$.***

Actually, we can give a new lower bound for $n_2(\mathcal{A})$ when $\mathcal{A}$ is supersolvable as follows.

**Corollary 1.4**

*Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ over a field $\mathbb{K}$ of characteristic zero. Assume that $\mathcal{A}$ is supersolvable with $|\mathcal{A}| = m + k, \quad k \geq 1, \quad p \in L_2(\mathcal{A})$ a*
modular point with $\mu(p) = m - 1 \geq 1$. Note that $\exp(A) = (1, m - 1, k)$. Then we have the following:

(1) If $k \leq m$, then $n_2(A) \geq k(m - k + 1)$.

(2) If $k \geq m$, then $n_2(A) \geq k$.

Remark 1.5

If $k \leq m$, then a lower bound

$$n_2(A) \geq k(m - k + 1) \geq \frac{|A|}{2}$$

is proved in Theorem 1.7 in [5] when $K = \mathbb{C}$. In this case Conjecture 1.1 holds true. We give another proof of this inequality over an arbitrary field $K$ of characteristic zero in the proof of Theorem 1.3.

Next let us state the second main result. For that, let us introduce a divisionally free arrangements of lines. We say that $A$ is divisionally free if there is $H \in A$ such that $\chi_0(A; |A^H| - 1) = 0$. Here $A^H := \{q \in L_2(A) \mid q \in H\}$. Note that divisional freeness is a combinatorial property, and divisionally free arrangements contain the famous inductively free arrangements. Thus in this class, Terao’s conjecture is true, that asserts that the freeness is combinatorial. Supersolvable arrangements are divisionally free, see [1] and [2] for details of divisional freeness. In this class we can show the Sylvester-Gallai theorem over an arbitrary field of characteristic zero.

Theorem 1.6

If $A$ is a divisionally free line arrangement in $\mathbb{P}_K^2$ over an arbitrary field of characteristic zero, then $n_2(A) > 0$.

Actually, Theorem 1.6 is a corollary of the following general result, that plays an important role in the proof of Theorem 1.3.

Theorem 1.7

Let $A$ be a projective line arrangement over an arbitrary field of characteristic zero $K$, and $H \in A$. If $A' := A \setminus \{H\}$ is free, then $n_2(A) > n_2(H) > 0$. Equivalently, if you add a line to a free projective line arrangement, then that line has to contain at least one double points.

In Theorem 1.7, $A$ could be not free. Also note that Theorem 1.7 fails if $K$ has a positive characteristic. See Remark 3.1 for details.

The organization of this article is as follows. In §2 we introduce several results and definitions for the proof. In §3 we prove main results posed in
this section, and also prove several conjectures and open problems by using them. In §4 we pose several conjectures related to the cardinality of double points of non-free projective line arrangements.

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2 Preliminaries

From now on we assume that \( \mathbb{K} \) is a field of characteristic zero unless otherwise specified. Our approach to prove them is purely algebraic, and different from the previous approaches. To prove Theorem 1.3 we need a few results and definitions. First let us recall the multiarrangement on \( \mathbb{K}^2 \).

Let \( A \) be a central arrangement of lines in \( V' := \mathbb{K}^2 \) and let \( m : A \rightarrow \mathbb{Z}_{>0} \) be a multiplicity. Then a pair \((A, m)\) is called a multiarrangement. Let \( S' := \mathbb{K}[x, y] \) be the coordinate ring of \( V' \). Then we can define the logarithmic derivation module \( D(A, m) \) as

\[
D(A, m) := \{ \theta \in \text{Der} S' \mid \theta(\alpha_H) \in S' \alpha_H^{m(H)} \ (\forall H \in A) \}.
\]

It is known (see [22] for example) that \( D(A, m) \) is always free, so we have its exponents \( \exp(A, m) = (d_1, d_2) \). It is known that \( d_1 + d_2 = |m| := \sum_{H \in A} m(H) \). By definition, we have the following easy but important lemma.

Lemma 2.1

Let \( A_1 \subset A_2 \) be arrangements in \( \mathbb{K}^2 \) and \( m_i : A_i \rightarrow \mathbb{Z} \) be multiplicities. Assume that \( m_2(H) \geq m_1(H) \) for all \( H \in A_1 \). Then \( D(A_2, m_2) \subset D(A_1, m_1) \).

In particular, for \( \exp(A_1, m_1) = (d_1, d_2) \) and \( \exp(A_2, m_2) = (e_1, e_2) \) with \( d_1 \leq d_2, \ e_1 \leq e_2 \), it holds that \( d_1 \leq e_1 \) and \( d_2 \leq e_2 \).

It is not easy to determine exponents of multiarrangements, but in some cases we can determine them completely. The following is one of them and it plays the key role in our proof.

Proposition 2.2 ([16], Proposition 5.4. See [20], Example 2.2 too)

Let \( A \) be a line arrangement in \( \mathbb{K}^2 \) and \( 2 \) be the constant multiplicity 2 on \( A \), i.e., \( 2(H) = 2 \) for any \( H \in A \). Then \( \exp(A, 2) = (|A|, |A|) \).

Remark 2.3

Note that Proposition 2.2 holds true only over the field of characteristic zero. For example, consider the multiarrangement \((A, m)\) in \( \mathbb{F}_2^2 \) defined by
\[ x^2 y^2 (x - y)^2 = 0. \]

Then \( D(\mathcal{A}, \mathfrak{m}) \) has a basis \( x^2 \partial_x + y^2 \partial_y, x^4 \partial_x + y^4 \partial_y \). Thus \( \exp(\mathcal{A}, \mathfrak{m}) = (2, 4) \).

From a line arrangement \( \mathcal{A} \) in \( \mathbb{P}^2 \) and \( H \in \mathcal{A} \), we can canonically construct a multiarrangement \( (\mathcal{A}^H, \mathfrak{m}^H) \) in \( H \simeq \mathbb{K}^2 \) as follows: \( \mathcal{A}^H := \{ H \cap L \mid L \in \mathcal{A} \setminus \{ H \} \} \), and

\[
\mathfrak{m}^H(X) := \left| \{ L \in \mathcal{A} \setminus \{ H \} \mid L \cap H = X \} \right|
\]

for \( X \in \mathcal{A}^H \). Here \( \mathcal{A}^H \) is an arrangement in \( H \simeq \mathbb{K}^2 \). To relate \( D(\mathcal{A}) \) and \( D(\mathcal{A}^H, \mathfrak{m}^H) \), we need an \( S \)-graded submodule \( D_H(\mathcal{A}) \subset D(\mathcal{A}) \). Namely, for \( H \in \mathcal{A} \), the \( S \)-graded submodule \( D_H(\mathcal{A}) \) is defined by

\[
D_H(\mathcal{A}) := \{ \theta \in D(\mathcal{A}) \mid \theta(\alpha_H) = 0 \}.
\]

It is known that \( D(\mathcal{A}) = S\theta_E \oplus D_H(\mathcal{A}) \) for any \( H \in \mathcal{A} \) (e.g., see Lemma 1.33, [21]). So \( D_H(\mathcal{A}) \simeq D_L(\mathcal{A}) \) for any lines \( H, L \) in \( \mathcal{A} \). Then by [22], there is the **Ziegler restriction map** \( \pi_H : D_H(\mathcal{A}) \rightarrow D(\mathcal{A}^H, \mathfrak{m}^H) \) defined by

\[
\pi_H(\theta)(\overline{f}) := \overline{\theta(f)},
\]

where for \( f \in S \), \( \overline{f} \) indicates the image of \( f \) in \( S/\alpha_H S \). Then Ziegler proved the following.

**Theorem 2.4 ([22])**

Let \( H \in \mathcal{A} \). Then

1. there is an exact sequence

\[
0 \rightarrow \alpha_H D_H(\mathcal{A}) \rightarrow D_H(\mathcal{A}) \xrightarrow{\pi_H} D(\mathcal{A}^H, \mathfrak{m}^H).
\]

2. Assume that \( \mathcal{A} \) is free with \( \exp(\mathcal{A}) = (1, d_2, d_3) \) and let \( H \in \mathcal{A} \). Then \( \exp(\mathcal{A}^H, \mathfrak{m}^H) = (d_2, d_3) \).

Next let us recall some fundamental results on supersolvable line arrangements.

**Lemma 2.5**

Let \( \mathcal{A} \) be a supersolvable line arrangement with a modular point \( p \in L_2(\mathcal{A}) \) such that \( \mu(\mathcal{A}) = m - 1 \) and \( |\mathcal{A}| = m + k \). Thus \( \exp(\mathcal{A}) = (1, m - 1, k) \).

1. Let \( H \in \mathcal{A} \setminus \mathcal{A}_p \). Then \( |\mathcal{A}^H| = m \).
2. Let \( \mathcal{A} \setminus \mathcal{A}_p = \{ H_1, \ldots, H_k \} \). Then \( n_2(\mathcal{A}) \geq \sum_{i=1}^{k} n_2(H_i) \).

**Proof.** (1) Since \( |\mathcal{A}_p| = m \) and \( H \not\subseteq \mathcal{A}_p \), it is clear that \( |\mathcal{A}^H| \geq m \). Let \( q \in L_2(\mathcal{A}) \) be a point on \( H \) that is not on a line in \( \mathcal{A}_p \). Then \( q = H \cap L \)
for $L \in \mathcal{A} \setminus (\mathcal{A}_p \cup \{H\})$. Since $\mathcal{A}$ is supersolvable and $p$ is a modular point, there is $K \in \mathcal{A}_p$ such that $H \cap L = q \in K$, a contradiction. So $|\mathcal{A}^H| = m$.

(2) Let $q_1 \in H_1$ and $q_2 \in H_2$ be double points belonging to $L_2(\mathcal{A})$. It suffices to show that $q_1 \neq q_2$. Assume that $q_1 = q_2$. Then $q_1 = q_2 = H_1 \cap H_2$. Since $\mathcal{A}$ is supersolvable and $p$ is a modular point, there is $K \in \mathcal{A}_p$ such that $H_1 \cap H_2 = q_1 = q_2 \in K$. So $q_1 = q_2 = H_1 \cap H_2 \cap K$. As a consequence, $q_1 = q_2$ is not a double point, a contradiction. \hfill \Box

To count $|\mathcal{A}^H|$, the following is the key.

**Theorem 2.6 ([1], Theorem 1.1, [2], Theorem 1.1, Corollary 1.2)**
Let a line arrangement $\mathcal{A}$ be free with $\exp(\mathcal{A}) = (1, d_2, d_3)$ with $d_2 \leq d_3$.

Then

(1) $|\mathcal{A}^H| \leq d_2 + 1$ or $|\mathcal{A}^H| = d_3 + 1$.

(2) For $L \notin \mathcal{A}$, let $\mathcal{B} := \mathcal{A} \cup \{L\}$. Then $|\mathcal{B}^L| = 1 + d_2$ or $|\mathcal{B}^L| \geq d_2 + d_3 + 1$.

(3) $\mathcal{A}$ is (divisionally) free if $|\mathcal{A}^H| = d_2 + 1$ or $|\mathcal{A}^H| = d_3 + 1$.

Finally recall a partial result on Conjecture 1.1.

**Theorem 2.7 ([5], Theorem 1.7)**
Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ over $\mathbb{C}$. Assume that $\mathcal{A}$ is supersolvable with $|\mathcal{A}| = m + k$, $k \geq 1$, $p \in L_2(\mathcal{A})$ a modular point with $\mu(p) = m - 1 \geq 1$. If $k \leq m$, then $n_2(\mathcal{A}) \geq k(m - k + 1)$.

3 Proof of main results

First we prove Theorem 1.7.

**Proof of Theorem 1.7** Let $\exp(\mathcal{A}) = (1, a, b)$ with $a \leq b$. By Theorem 2.6 (2), $|\mathcal{A}^H| = 1 + a$ or at least $b + 1$. First assume the latter. Since $|\mathcal{m}^H| = a + b + 1 \leq 2(b + 1) = 2|\mathcal{A}^H|$, it holds that $n_2(H) > 0$. Thus we may assume that $|\mathcal{A}^H| = a + 1$, thus $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1, a, b + 1)$ by Theorem 2.6 (3). Assume that $n_2(H) = 0$. Then by definition, $m^H(X) \geq 2$ for all $X \in \mathcal{A}^H$. So Lemma 2.1 shows that $D(\mathcal{A}^H, \mathcal{m}^H) \subset D(\mathcal{A}^H, 2)$. By Theorem 2.4 (2), $\exp(\mathcal{A}^H, \mathcal{m}^H) = (a + 1, a + 1)$ and by Proposition 2.2 $\exp(\mathcal{A}^H, 2) = (a + 1, a + 1)$, which contradicts Lemma 2.1. \hfill \Box

**Proof of Theorem 1.6** Combining Theorems 2.6 and 1.7 we have $n_2(\mathcal{A}) > 0$. \hfill \Box
Remark 3.1
Theorem 1.7 is not true if \( \text{ch}(K) \neq 0 \). Let \( K = \mathbb{F}_2 \) and let \( A \) be a line arrangement in \( \mathbb{P}^2_{\mathbb{F}_2} \) consisting of all lines in \( \mathbb{P}^2_{\mathbb{F}_2} \). Thus \( |A| = 7 \). Let \( A \ni H : x = 0 \). Then it is easy to show that \( D(A \setminus \{H\}) \) is free with basis
\[
\theta_E, \ x^2\partial_x + y^2\partial_y + z^2\partial_z, \ (x + y)(x + z)(x + y + z)\partial_x.
\]
However, it is also clear that \( n_2(H) = 0 \).

Of course even if \( \text{ch}(K) = p > 0 \), it is easy to show that Theorem 1.7 is true if (1) \( A \) is not free in terms of Theorem 1.7, or (2) \( a \) is not divisible by \( p \). See the proof of Proposition 1.53 in [21].

Now we can prove Theorem 1.3.

Proof of Theorem 1.3. Let \( p \in L_2(A) \) be the modular point with \( \mu(p) = m - 1 \) and let \( |A| = k + m \). Thus \( \exp(A) = (1, m - 1, k) \). If \( m = 1 \), then the statement is clear. So we may assume that \( m \geq 2 \). First we prove when \( k \leq m \). If \( k = 0 \), then it is not essential. So we may assume that \( k \geq 1 \). If \( k = 1 \), then it is the coning of \( A_p \), so \( n_2 = m \), which satisfies the inequality.

Let \( k \geq 2 \), and let \( H \in A \setminus A_p \), and count the number of double points \( n_2(H) \) on \( H \). By Lemma 2.5 (1), \( |A^H| = m \). Let \( q \in L_2(A) \cap A^H \) be a non-double point. Since \( p \) is a modular point and \( H \notin A_p \), there is \( L \in A \setminus (A_p \cup \{H\}) \) such that \( H \cap L = q \). Since \( A \) is supersolvable again, there is \( K \in A_p \) such that \( K \ni q = H \cap L \cap K \). Thus \( |A^H| - n_2(H) \leq |A \setminus (A_p \cup \{H\})| = k - 1 \). Thus \( n_2(H) \geq m - (k - 1) > 0 \) since \( m \geq k \). Now Lemma 2.5 (2) implies that \( n_2(A) \geq (m - k + 1)k \). Now compare
\[
2(n_2(A) - \frac{|A|}{2}) \geq 2k(m - k + 1) - (m + k) = (m - k)(2k - 1) \geq 0.
\]
So the proof is completed when \( k \leq m \).

Next consider when \( k \geq m \). The statement is clear if \( |A| - |A_p| = 1 \). So \( |A| - |A_p| \geq 2 \). Let \( H \in A \setminus A_p \). By definition of the supersolvable arrangement, \( A \setminus \{H\} \) is supersolvable, thus free. Hence \( n_2(H) > 0 \) by Theorem 1.7. Now let \( A \setminus A_p = \{H_1, \ldots, H_k\} \). Since \( n_2(H_i) > 0 \) for \( i = 1, \ldots, k \), Lemma 2.5 (2) shows that
\[
n_2(A) \geq \sum_{i=1}^{k} n_2(H_i) \geq k.
\]
Now compute
\[
2(n_2(A) - \frac{|A|}{2}) \geq 2k - (m + k) = k - m \geq 0,
\]
which completes the proof. □

Proof of Corollary 1.4. Immediate by the proof of Theorem 1.3. □

Based on the proof of Theorem 1.3 we can show the following general relation between $n_2(H)$ with $H \in \mathcal{A}$ and the minimal degree relation $r(\mathcal{A}) = mdr(\mathcal{A})$ defined by

$$r(\mathcal{A}) = mdr(\mathcal{A}) := \min\{d \mid D_H(\mathcal{A})_d \neq (0)\}$$

Here $M_d$ for an $S$-graded module $M$ indicates the homogeneous degree $d$-part of $M$. Then the most general version of Theorem 1.7 is given as follows:

**Theorem 3.2**

Let $\mathcal{A}$ be a projective line arrangement with $H \in \mathcal{A}$ such that $r(\mathcal{A}) < |\mathcal{A}^H|$. Then $n_2(\mathcal{A}) > n_2(H) > 0$.

**Proof.** Apply Theorem 2.2 and the same argument in the proof of Theorem 1.7. □

Thus in the terminology of $mdr(\mathcal{A}) = r(\mathcal{A})$, we can give a lower bound of $n_2(\mathcal{A})$.

**Theorem 3.3**

Let $\mathcal{A}$ be a line arrangement in $\mathbb{P}^2$ with $r = mdr(\mathcal{A})$. If there is $H \in \mathcal{A}$ with $|\mathcal{A}^H| = k > r$, then $n_2(H) \geq k - r$. Moreover, if $\mathcal{A}_{>r} := \{L \in \mathcal{A} \mid |\mathcal{A}^L| > r\}$, then

$$n_2(\mathcal{A}) \geq \frac{1}{2} \sum_{L \in \mathcal{A}_{>r}} (|\mathcal{A}^L| - r).$$

In particular,

$$n_2(\mathcal{A}) \geq \frac{|\mathcal{A}_{>r}|}{2},$$

and $n_2(\mathcal{A}) > 0$ if $\mathcal{A}_{>r} \neq \emptyset$.

**Proof.** Assume that $n_2(H) < k - r$. Let $(\mathcal{B}, m)$ be a multiarrangement of lines in $\mathbb{K}^2$ obtained by removing all double points on $\mathcal{A}^H$, and putting $m := m^H|_{\mathcal{B}}$. Thus $|\mathcal{B}| = k - n_2(H) > r$, and $m(X) \geq 2$ for all $X \in \mathcal{B}$. By definition of $r$, there is $0 \neq \theta \in D_H(\mathcal{A})_r$. Assume that $\theta$ is divisible by $\alpha_H$. Then by definition of $D_H(\mathcal{A})$, it holds that $\theta/\alpha_H \in D_H(\mathcal{A})_{r-1} = (0)$, a contradiction. Thus $\theta \not\in \alpha_H D_H(\mathcal{A})_r$. Since $\ker(\pi_H) = \alpha_H D_H(\mathcal{A})$ by Theorem 2.4 (1), it holds that $0 \neq \pi_H(\theta) \in D(\mathcal{A}^H, m^H)_r \subset D(\mathcal{B}, m)_r \subset D(\mathcal{B}, 2)_r = (0)$ because $\exp(\mathcal{B}, 2) = (k - n_2(H), k - n_2(H))$ with $k - n_2(H) > r$ by Lemma 2.1 and Proposition 2.2, a contradiction. The rest statements are clear. □
Remark 3.4
Theorem 3.3 is weaker than the argument used in the proof of Theorem 1.3 since all double points in the proof of Theorem 1.3 are distinct, but those in Theorem 3.3 are not. The difference comes from whether there is a modular point or not.

In [6], two equivalent Conjectures 3.3 and 3.6 are posed over $\mathbb{C}$, which are stronger than Conjecture 1.1. By Theorem 3.3 or the same proof as in Theorem 1.3, we can prove them (explicitly, we can show [6], Conjecture 3.3 as Theorem 3.5, and Conjecture 3.6 follows by the equivalence) over any arbitrary field of characteristic zero.

Theorem 3.5 ([6], Conjectures 3.3)
There are no supersolvable line arrangement $A$ in $\mathbb{P}^2$ with a modular point $p$ of $\mu(p) \geq 2$ such that $n_2(H) = 0$ for some $H \in A \setminus A_p$.

Theorem 3.6 ([6], Conjectures 3.6)
Let $K$ be an arbitrary field of characteristic zero, $f_1, \ldots, f_n \in K[x, y]$ be $m$-linear forms with $\gcd(f_i, f_j) = 1$ for any $i \neq j$. Then

$$\prod_{i=1}^{n} f_i \notin \bigcap_{1 \leq i < j \leq m} \langle f_i + z, f_j + z \rangle,$$

where $\langle f_i + z, f_j + z \rangle$ is the ideal in $K[x, y, z]$ generated by $f_i + z$ and $f_j + z$.

Moreover, Question 16 posed in [13] can be settled affirmatively as follows:

Theorem 3.7
Let $A$ be a supersolvable line arrangement in $\mathbb{P}^2$ over a field of characteristic zero. Let $p$ be a modular point of $A$ such that $\mu(p) \geq 1$. Assume that $|A \setminus A_p| \geq 1$. Then $n_2(A) \geq \max\{|A| - m, m\}$.

Proof. Let $|A| = m + k$, so $\exp(A) = (1, m - 1, k)$. By the assumption, $k \geq 2$. First assume that $m \geq k$. Then Corollary 1.4 shows that $n_2(A) \geq k(m - k + 1)$, and

$$k(m - k + 1) - m = k(m - k) - (m - k) \geq 0$$

since $k \geq 1$. Thus $n_2(A) \geq m = \max\{m, |A| - m = k\}$. Assume that $m \leq k$. Then $n_2(A) \geq k$ by Corollary 1.4. Thus $n_2(A) \geq k = \max\{m, |A| - m = k\}$. □

As mentioned in [13], Theorem 3.7 implies Conjecture 1.1 too.
4 Conjectures

Based on the previous results, the following problem is natural to ask.

**Problem 4.1**

Consider the Sylvester’s problem depending on $D(A)$. For example, consider it when $A$ is free.

For a generic line arrangement $A$ in $\mathbb{P}_K^2$, the Dirac-Motzkin conjecture is true. Also, when we construct a free line arrangement $A$, experimentally, we know that $n_2(A)$ decreases. Based on these very rough observation with analysis of the known Sylvester-Gallai configurations, we pose the following conjecture.

**Conjecture 4.2**

Let $A$ be a projective line arrangement in $\mathbb{P}_K^2$, where $K$ is a field of characteristic zero. If $A$ is not free, then $n_2(A) > 0$.

A weaker version is as follows:

**Conjecture 4.3**

Let $A$ be a free projective line arrangement in $\mathbb{P}_K^2$, where $K$ is a field of characteristic zero. Let $H \in A$. Assume that $A \setminus \{H\}$ is not free. Then $n_2(A \setminus \{H\}) > 0$.

Conjecture 4.3 is a deletion version of Theorem 1.7, i.e., Theorem 1.7 asserts that any addition to a free arrangement has a double point. Then how about the deletion? Since the dual Hesse arrangement, that consists of $9$-lines with $12$-triple points and no other intersection points, can be obtained by deleting a line from a free arrangement (see [4], Theorem 1.1), Conjecture 4.3 is not true if $A \setminus \{H\}$ is free. Even in that case, $D(A \setminus \{H\})$ has a good algebraic structure so called the plus-one generated property (see [3]). Thus we may have a chance to approach Conjecture 4.3 by using algebraic technique. Actually, the supersolvable version of Conjecture 4.3 is true as follows:

**Theorem 4.4**

Let $A$ be a supersolvable projective line arrangement with a modular point $p \in L_2(A)$, $\mu(p) = m - 1 \geq 1$. Let $A' := A \setminus \{H\}$. Assume that $A'$ is not a pencil. Then $n_2(A') > 0$.

**Proof.** Assume that $H \in A \setminus A_p$. Since $A'$ is not a pencil, $A'$ is still supersolvable. Thus Theorem 1.3 shows that $n_2(A') \geq |A'|/2 > 0$. Next assume that $H \in A_p$. By Theorem 1.7 there is at least one double point on any $L \in A \setminus A_p$. Assume that $n_2(A') = 0$, that occurs only when all such
double points are on $H$. So $|A^H| = 1 + |A| - m$. Hence it is clear that $A'$ is still supersolvable with a modular point $p$, $\mu(p) = m - 2$, contradicting Theorem 1.3. Thus $n_2(A') > 0$. □

**Remark 4.5**
Note that such an $A'$ in Theorem 4.4 is called **nearly supersolvable** in [10] if $A'$ is not supersolvable. Thus the proof of Theorem 4.4 asserts that if $A$ is nearly supersolvable, then $n_2(A) > 0$.

So far Conjecture 4.2 is true in the following class:

**Theorem 4.6**
For a projective line arrangement $A$, $n_2(A) > 0$ if

1. $A$ has a generic line (thus not free),
2. $A$ is (nearly) supersolvable,
3. there is $H \in A$ such that $A \setminus \{H\}$ is free (in particular, inductively and divisionally free arrangements),
4. there is $H \in A$ and $p \in A^H$ such that $|m^H(p)| \geq |A|/2$, or
5. there is $H \in A$ such that $|A^H| > mdr(A) = r(A)$.

**Proof.** Immediate by Theorems 1.3, 1.7, 3.2, 4.4, and the proof of Theorem 1.7. □

It is natural to ask whether the Dirac-Motzkin conjecture holds for non-free arrangements. However, it is pointed out by Hiraku Kawanoue that this is not true even if $A$ can be obtained by deleting a line from a free arrangement in general.

**Proposition 4.7 (Kawanoue)**
There is a free line arrangement $A$ in $\mathbf{P}^2_C$ such that $A \setminus \{H\}$ is not free, and $n_2(A \setminus \{H\}) < \frac{|A| - 1}{2}$ for any $H \in A$.

**Proof.** Let $A$ be defined by

$$(x^4 - y^4)(y^4 - z^4)(x^4 - z^4) = 0$$

in $\mathbf{P}^2_C$. This is called the monomial arrangement with respect to the group $G(4, 4, 3)$, see [4] Theorem 1.1 and section 5. It is known that (e.g., see [4]) $A$ is free with $\exp(A) = (1, 5, 6)$, $|A^H| = 5$, $A^H$ consists of one quadruple points and four triple points for any $H \in A$. It is also shown in [4] that
$\mathcal{A} := \mathcal{A} \setminus \{H\}$ is not free. Also, it is clear that $n_2(\mathcal{A}') = 4 < |\mathcal{A}'|/2 = 11/2$. □

Though the Dirac-Motzkin conjecture is not true for non-free arrangements in $\mathbb{P}_2^k$, as in Proposition 4.7, Theorem 1.3 shows that it is true for supersolvable arrangements. So let us pose the following problem.

**Problem 4.8**

Let $\mathbb{K}$ be a field of characteristic zero. Then in which class of line arrangements in $\mathbb{P}_2^k$ the inequality $n_2(\mathcal{A}) \geq |\mathcal{A}|/2$ holds?

**References**

[1] T. Abe, Roots of characteristic polynomials and and intersection points of line arrangements. *J. Singularities*, 8 (2014), pp100–179.

[2] T. Abe, Divisionally free arrangements of hyperplanes. *Invent. Math.* **204** (2016), no. 1, 317–346.

[3] T. Abe, Plus-one generated and next to free arrangements of hyperplanes. *Int. Math. Res. Not.*, to appear. arXiv:1808.04697 (2018).

[4] T. Abe, M. Cuntz, H. Kawanoue and T. Nozawa, Non-recursive freeness and non-rigidity of plane arrangements. *Discrete Math.* **339** (2016), Issue 5, 1430–1449.

[5] T. Abe and A. Dimca, On complex supersolvable line arrangements, arXiv:1907.12497.

[6] B. Anzis and S.O. Tohaneanu, On the geometry of real and complex supersolvable line arrangements, *J. Combin. Theory, Ser. A* **140** (2016), 76–96.

[7] D. Cook II, B. Harbourne, J. Migliore and U. Nagel, Line arrangements and configurations of points with an unexpected geometric property, *Compos. Math.* **154** (2018), 2150–2194.

[8] L. Dickson, The points of inflexion of a plane cubic curve, *Ann. of Math.* **16** (1914–1915), 50–66.

[9] M. Di Marca, G. Malara and A. Oneto, Unexpected curves arising from special line arrangements, *J. Alg. Combin.*, https://doi.org/10.1007/s10801-019-00871-0
[10] A. Dimca and G. Sticlaru, On supersolvable and nearly supersolvable line arrangements, *J. Alg. Combin.*, https://doi.org/10.1007/s10801-018-0859-6.

[11] T. Gallai, Solution to problem number 4065, *Amer. Math. Monthly* **51** (1944), 169–171.

[12] B. Green and T. Tao, On sets defining few ordinary lines, *Discrete Comput. Geom.* **50** (2013), 409–468.

[13] K. Hanumanthu and B. Harbourne, Real and complex supersolvable line arrangements in the projective plane, arXiv: 1907.07712.

[14] L. Kelly, A resolution of the Sylvester-Gallai problem of J. -P. Serre, *Discrete Comput. Geom.* **1** (1986), 210–219.

[15] P. Orlik and H. Terao, *Arrangements of hyperplanes*. Grundlehren der Mathematischen Wissenschaften, **300**. Springer-Verlag, Berlin, 1992.

[16] L. Solomon and H. Terao, The double Coxeter arrangement, *Comm. Math. Helv.* **73** (1998), no. 2, 237–258.

[17] J. Sylvester, Mathematical question 11851, *Educational Times* **46** (March 1893), 156.

[18] H. Terao, Generalized exponents of a free arrangement of hyperplanes and Shephard-Todd-Brieskorn formula. *Invent. Math.* **63** (1981), 159–179.

[19] S. O. Tohăneanu, A computational criterion for supersolvability of line arrangements, *Ars Combin.* **117** (2014), 217–223.

[20] M. Wakefield and S. Yuzvinsky, Derivations of an Effective Divisor on the Complex Projective Line. *Trans. Amer. Math. Soc.* **359** (2007), 4389–4403.

[21] M. Yoshinaga, Freeness of hyperplane arrangements and related topics. *Annales de la Faculte des Sciences de Toulouse*, **23** (2014), no. 2, 483–512.

[22] G. M. Ziegler, Multiarrangements of hyperplanes and their freeness. Singularities (Iowa City, IA, 1986), 345–359, Contemp. Math., **90**, Amer. Math. Soc., Providence, RI, 1989.