CONTROL OF PORT-HAMILTONIAN SYSTEMS WITH MINIMAL ENERGY SUPPLY

MANUEL SCHALLER\textsuperscript{3}, FRIEDRICH PHILIPP\textsuperscript{3}, TIMM FAULWASSER\textsuperscript{1}, KARL WORTHMANN\textsuperscript{3} AND BERNHARD MASCHKE\textsuperscript{2}

Abstract. We investigate optimal control of linear port-Hamiltonian systems with control constraints, in which one aims to perform a state transition with minimal energy supply. Decomposing the state space into dissipative and non-dissipative (i.e. conservative) subspaces, we show that the set of reachable states is bounded in dissipative directions. We analyze the corresponding steady-state optimization problem and prove that all optimal steady states lie in a non-dissipative subspace, which is induced by the kernel of the dissipation operator. We prove that all solutions of the optimal control problem exhibit the turnpike phenomenon with respect to this subspace, i.e., for varying initial conditions and varying horizon length the optimal state trajectories evolve close to the conservative subspace for the majority of the time. We conclude this paper by illustrating our finding by a numerical example.

1. INTRODUCTION

It stands to reason that the impact of the port-Hamiltonian (pH) framework for modelling, simulation, and analysis of interconnected physical systems is substantial, see the monographs \textsuperscript{[12, 28, 3]}. Indeed the port-Hamiltonian framework extends Hamiltonian structures, which arise naturally in dynamic models of physical systems due to energy conservation and dissipation, to input and output ports. The later point is of crucial interest for control, where inputs and outputs are fundamental for feedback design.

Actually, Hamiltonian structures arise in two distinct contexts in systems and control: (a) via energy-based modelling, where the (energy) Hamiltonian represents the total energy, which is the avenue towards pH systems, and (b) in context of optimal control, where the application of a variational principle leads to a Hamiltonian structure composed of the state dynamics and the adjoint/co-state/dual dynamics. In terms of (b), the (optimality) Hamiltonian is fundamental in stating Pontryagin’s Maximum Principle (PMP). Moreover, in case of time-invariant Optimal Control Problems (OCPs) the optimality Hamiltonian is known to be invariant along optimal trajectory lifts. The classical
link between both domains is given by variational modelling approaches in mechanics—
i.e., the Euler-Lagrange formalism and the Hamilton formalism—which in turn can be
considered as precursors of variational calculus and optimal control [24].

However, given the common historical origins of port-Hamiltonian systems and the
optimality Hamiltonian, it is surprising that little research has investigated the exploita-
tion of pH structures for optimal control. Indeed as pH systems are passive w.r.t. the
usual \( y^\top u \) supply rate [28, Chapter 7], one may apply to them the classical results on
inverse optimality of passive feedbacks, see [23, 20] and references therein. Moreover,
recently the preprint [13] has suggested to combine inverse optimality with learning con-
cepts, see also the references therein. Besides these works, a few results exist on the
LQG control using the structure of PH systems [32, 14]. In total, little appears to have
been done on exploiting pH structures in optimal control.

The present paper attempts first steps to close this gap. In Section 2, after a concise
analysis of the spectral properties of linear pH systems, we analyze the reachable set of
linear pH systems. In Section 3 we pose a natural OCP for pH systems, i.e., the transition
between given states under minimum supply of energy subject to input constraints.
While this OCP is natural in terms of the considered objective functional, it is also
singular as the energy supplied to the pH system is given by the supply rate \( y^\top u \).
Subsequently, we use the Hamiltonian structure of the optimality system arising from
the application of the PMP in combination with the underlying pH structure of the
dynamics to analyze this OCP. We show that the specific structure allows the statement
of an equivalent OCP, in which the terminal state constraint is replaced by a linear
Mayer term. Then, in Section 4, we present our main results on the presence of turnpike
phenomena in the considered class of OCPs. To this end, we leverage recent results
on dissipativity notions for OCPs [10, 7]. We consider (controlled) linear port-Hamiltonian systems given by

\[
\begin{align*}
\dot{x}(t) &= (J - R)Qx + Bu, \quad x(0) = x^0, \\
y(t) &= B^\top Qx(t),
\end{align*}
\]

1 Turnpike properties of OCPs are a phenomenon which dates back to [30, 6]. They refer to the
situation wherein, for varying initial conditions and different horizon lengths, the optimal solutions stay
close to the optimal steady state during the middle part of the optimization horizon and the time spend
far from the optimal steady state is bounded independent of the horizon length. See [4, 19] for classical
treatments and [26, 5, 10, 11] for recent results.

2. DISSIPATIVE AND CONSERVATIVE SUBSPACES

We consider (controlled) linear port-Hamiltonian systems given by

\[
\begin{align*}
\dot{x}(t) &= (J - R)Qx + Bu, \quad x(0) = x^0, \\
y(t) &= B^\top Qx(t),
\end{align*}
\]
where $J \in \mathbb{R}^{n \times n}$ is skew-symmetric, $R \in \mathbb{R}^{n \times n}$ is symmetric positive semidefinite, $Q \in \mathbb{R}^{r \times n}$ is symmetric positive definite, $B \in \mathbb{R}^{r \times m}$ has full rank $m \leq n$ and $u \in L^1(0,T;\mathbb{U})$ is the control function, where either $\mathbb{U}$ is an $m$-dimensional cuboid, i.e.,$\mathbb{U} = [u_1, \pi_1] \times \ldots \times [u_m, \pi_m]$ or $\mathbb{U} = \mathbb{R}^m$. If not otherwise stated we will always assume that $0 \in \text{int}(\mathbb{U})$. The system (1) is to be understood in an a.e.-sense in time with solution $x \in W^{1,1}(0,T;\mathbb{R}^n)$.

It is well-known that pH systems of the form (1) are passive with respect to the usual supply rate $w = y^T u$ [3] Section 6.3 (see also [28] Chapter 7) and the relation of dissipative linear time-invariant systems and port-Hamiltonian systems. Moreover note that $y^T u$ can be understood as the energy per time unit supplied to the system.

In what follows we analyze the spectral properties of the system matrix $(J-R)Q$. If $\lambda \in \mathbb{C}$ is a complex eigenvalue of a real matrix $A \in \mathbb{R}^{n \times n}$, then so is $\overline{\lambda}$ with eigenspace $\ker(A - \overline{\lambda}I_n) = \{ \overline{x} : x \in \ker(A - \lambda I_n) \}$, where $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)^T$. If $\text{Im} \lambda \neq 0$, from the linear independence of $x$ and $\overline{x}$ it follows that also $\text{Re}(x)$ and $\text{Im}(x)$ are linearly independent in $\mathbb{R}^n$. We set

$$N_\lambda(A) = \text{span}\{ \text{Re}(x), \text{Im}(x) : x \in \ker(A - \lambda I_N) \} \subseteq \mathbb{R}^n.$$  

This space has even dimension if $\text{Im} \lambda \neq 0$. We say that a matrix $A$ is $Q$-symmetric ($Q$-skew-symmetric, $Q$-positive (semi-)definite) if it has the respective property with respect to the inner product $(\cdot, \cdot)_Q$.

**Lemma 2.1.** The matrix $A = (J-R)Q$ has the following spectral properties:

(i) Each eigenvalue of $A$ has non-positive real part.

(ii) For all $\alpha \in \mathbb{R}$, we have $\ker(A - i\alpha I_n) = \ker((A - i\alpha I_n)^2)$, i.e., the corresponding Jordan block is diagonal if $\alpha \in \sigma(A)$, and it holds that

$$N_{i\alpha}(A) \subseteq N_{i\alpha}(JQ) \cap \ker(RQ).$$

(iii) There is a $Q$-orthogonal subspace decomposition $\mathbb{R}^n = M_1 \oplus_Q M_2$ with respect to which

$$JQ = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \quad \text{and} \quad RQ = \begin{bmatrix} 0 & 0 \\ 0 & R_2 \end{bmatrix},$$

such that $J_1$ and $J_2$ are $Q$-skew-symmetric (on $M_1$ and $M_2$, respectively), $R_2$ is $Q$-positive semidefinite, and $J_2 - R_2$ is Hurwitz.

**Proof.** First note that $Q^{1/2}[(J-R)Q]Q^{-1/2} = \tilde{J} - \tilde{R}$, where $\tilde{J} = Q^{1/2}JQ^{1/2}$ is skew-symmetric and $\tilde{R} = Q^{1/2}RQ^{1/2}$ is positive semi-definite. Hence, we may assume WLOG that $Q = I$.

(i). Let $(J-R)x = \lambda x$ for some $\lambda \in \mathbb{C}$ and $x \in \mathbb{C}^n$, $\|x\| = 1$. Then $\lambda = \langle \lambda x, x \rangle = \langle Jx, x \rangle - \langle Rx, x \rangle$. Since $J$ is skew-symmetric, we have $\text{Re}\langle Jx, x \rangle = 0$ and thus $\text{Re} \lambda \leq 0$.

(ii). Let $\lambda = i\alpha$, $\alpha \in \mathbb{R}$, and assume that $(J-R)x = i\alpha x$. Then, by the same calculation as before, $Rx = 0$ and thus $Jx = i\alpha x$. This proves the inclusion (2). Let $y \in \mathbb{C}^n$ such that $(J-R+\alpha)x = \tilde{x}$. Then $\|x\|^2 = \langle x, (J-R+\alpha)x \rangle = \langle (J-R+\alpha)x, x \rangle = 0$, so $x = 0$.

(iii). Define $M_1 = \text{span}\{N_{\alpha} : \alpha \in \sigma(A)\}$. By [2], this space is both $J$- and $R$-invariant. Hence, so is $M_2 \supseteq M_1^\perp$. Since $R$ vanishes on $M_1$, it is clear that the
representations of $J$ and $R$ with respect to the decomposition $\mathbb{R}^n = M_1 \oplus M_2$ take the form \((3)\). Since $N_{\alpha}(J_2 - R_2) \subset N_{\alpha}(J - R) \subset M_1$, and thus $N_{\alpha}(J_2 - R_2) = \{0\}$, it is also clear that $J_2 - R_2$ is Hurwitz. \hfill \square

**Remark 2.2.** Note that $R_2$ might still have a non-trivial kernel.

Now, with respect to the decomposition $\mathbb{R}^n = M_1 \oplus M_2$ from Lemma 2.1, the control system \((1a)\) takes the form

\begin{align}
(4a) \quad & \dot{x}_1 = J_1 x_1 + B_1 u \\
& x_1(0) = x_1^0 \\
(4b) \quad & \dot{x}_2 = (J_2 - R_2)x_2 + B_2 u \\
& x_2(0) = x_2^0.
\end{align}

**Assumption 2.3.** System \((1a)\) with $U = \mathbb{R}^n$ is controllable, i.e., $\text{rank}[B, AB, \ldots, A^{n-1}B] = n$, where $A = (J - R)Q$.

The following lemma deals with the reachable sets of the systems \((4a)\) and \((4b)\).

**Lemma 2.4.** Let Assumption 2.3 hold and suppose that $U$ is compact. Then the following statements hold:

1. For every state $x_1^* \in M_1$ there exist a time $T > 0$ and a control $u \in L^1(0, T; U)$ which steers $x_1^*$ to $x_1^*$ under the dynamics in \((4a)\).

2. The set of states in $M_2$ that can be reached from $x_2^0$ in arbitrary time under the dynamics \((4b)\) is bounded in $M_2$.

**Proof.** (i). Since $\sigma(J_1) \subset i\mathbb{R}$, by \cite{18} Theorem 5, p. 45 there exist $T_1 \geq 0$ and a control $u \in L^1(0, T_1; U)$ that steers $x_1^0$ into $0 \in \mathbb{R}^n$ in time $T_1$. Let $x_1 \in W^{1,1}(0, T_1; \mathbb{R}^n)$ denote the corresponding internal state. By the same reason there exist a time $T_2$ and a control $v \in L^1(0, T_2; U)$, which steers $x_1^*$ to $0$ in time $T_2$ under the dynamics

$$
\dot{z}_1(t) = -J_1z_1(t) - B_1 v(t).
$$

By $z_1$ denote the corresponding state solution. Set $T = T_1 + T_2$ and define $x_1(t) \equiv z_1(T - t)$ as well as $u(t) \equiv v(T - t)$, $t \in (T_1, T]$. Then $u \in L^1(0, T; U)$, $x_1$ is absolutely continuous on $[0, T]$, and

$$
\dot{x}_1(t) = -\dot{z}_1(T - t) = J_1 z_1(T - t) + B_1 v(T - t) = J_1 x(t) + B_1 u(t)
$$

for $t \in (T_1, T]$. Also, $x_1(0) = x_1^0$ and $x_1(T) = z(0) = x_1^*$.

(ii). This can be easily seen from the variation of constants formula. Indeed, for any control $u \in L^1(0, T; U)$ the solution of \((4b)\) can be represented as

$$
x_2(t) = e^{tA_2} x_2^0 + \int_0^t e^{(t-s)A_2} B_2 u(s) \, ds,
$$

where $A_2 = J_2 - R_2$. As $A_2$ is Hurwitz, there exists $\mu > 0$, $M \geq 1$ such that $\|e^{tA_2}\| \leq Me^{-\mu t}$. Hence,

$$
\|x(t)\| \leq Me^{-\mu t}\|x_0^0\| + \int_0^t Me^{-\mu(t-s)}\|B_2\|\|u(s)\| \, ds \leq M\|x_2^0\| + \frac{M}{\mu}\|B_2\|\left(\max_{v \in U} \|v\|\right)
$$
Corollary 2.5. Under the assumptions of Lemma 2.4, the reachable states of the system (1a) are contained in $M \oplus Q K$, where $M \subset \mathbb{R}^n$ is a linear subspace and $K \subset M^\perp Q$ is compact in $M^\perp Q$.

3. Minimum Energy Supply OCPs

We turn to the following OCP

$$\min_{u \in L^1(0,T;\mathbb{U})} C(u) = \int_0^T u(t)^\top y(t) \, dt$$

$$\dot{x}(t) = (J - R)Qx(t) + Bu(t),$$

$$x(0) = x^0, \quad x(T) = x_T,$$

$$y(t) = B^\top Qx(t),$$

which models the task of transferring the system state from $x^0$ to $x_T$ by minimizing the amount of supplied energy.

As is well known, the task of steering $x^0$ to $x_T$ at (any) time $T$ is surely feasible in the case where $\mathbb{U} = \mathbb{R}^n$ in view of Assumption 2.3. However, as Lemma 2.4 shows, this is much more delicate if $\mathbb{U}$ is compact. To make OCP (5) feasible, we shall make the following assumption.

Assumption 3.1. There exists a control $u \in L^1(0,T;\mathbb{U})$ which steers $x^0$ to $x_T$ at time $T$ under the dynamics in (1a).

Then the next proposition follows immediately from [18, Theorem 2, p. 91] or [15, Theorem 4, p. 259].

Proposition 3.2. Under Assumption 3.1, the OCP (5) has an optimal solution.

Considering the energy Hamiltonian $H(x) \doteq \frac{1}{2} x^\top Qx$, straightforward computations yield the energy balance

$$\frac{d}{dt} H(x(t)) = u(t)^\top y(t) - \|R \frac{1}{2} Qx(t)\|_2^2,$$

and hence we have that

$$C(u) = H(x(T)) - H(x(0)) + \int_0^T \|R \frac{1}{2} Qx(t)\|_2^2 \, dt.$$  

Moreover, we consider the alternative problem:

$$\min_{u \in L^1(0,T;\mathbb{U})} \int_0^T u(t)^\top y(t) \, dt + \varphi(x(T))$$

$$\dot{x}(t) = (J - R)Qx(t) + Bu(t),$$

$$x(0) = x^0,$$

$$y(t) = B^\top Qx(t),$$

where $\varphi \in C^1(\mathbb{R}^n, \mathbb{R})$ is a terminal cost (or Mayer term). In contrast to OCP (5), in this OCP the existence of optimal solutions does not depend on the horizon length as it
does not include a terminal constraint. In particular, for all $T > 0$, the above problem is (trivially) feasible, i.e., there is a control-state pair that satisfies the dynamics.

**Remark 3.3** (Spherical energy coordinates). Setting $\tilde{x} = Q^{1/2}x$, $\tilde{J} = Q^{1/2}JQ^{1/2}$, $\tilde{R} = Q^{1/2}RQ^{1/2}$, and $\tilde{B} = Q^{1/2}B$, the control system in (5) and (7) transfers into

$$\dot{\tilde{x}} = (\tilde{J} - \tilde{R})\tilde{x} + \tilde{B}u, \quad y = \tilde{B}^\top \tilde{x}$$

and the cost functional remains the same (with $\varphi$ replaced by $\varphi \circ Q^{-1/2}$ in (7)). The energy becomes $H(\tilde{x}) = \frac{1}{2}||\tilde{x}||^2$. Hence, it is no restriction to set $Q = I$ and we will do this occasionally to simplify proofs.

### 3.1. Necessary Optimality Conditions

We deduce the first-order optimality conditions using Pontryagin’s maximum principle (PMP) for the OCPs (5) and (7). By PMP there is $(\lambda_0, \lambda) \in \mathbb{R}_{\geq 0} \times W^{1,1}(0,T;\mathbb{R}^n)$, $(\lambda_0, \lambda(t)) \neq 0$ for all $t \in [0,T]$, such that, defining the optimality Hamiltonian

$$H(x,u,\lambda_0,\lambda) = \lambda_0 \cdot u^\top y + \lambda^\top ((J - R)Qx + Bu)$$

on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_{\geq 0} \times \mathbb{R}^n$, optimal solutions $(x^*, u^*) \in W^{1,1}(0,T;\mathbb{R}^n) \times L^1(0,T;\mathbb{U})$ of (5) and (7) satisfy

\begin{align}
(8a) \quad & \dot{x}^*(t) = H_x(x^*(t), u^*(t), \lambda_0, \lambda(t)) \\
(8b) \quad & \dot{\lambda}(t) = -H_x(x^*(t), u^*(t), \lambda_0, \lambda(t)) \\
(8c) \quad & u^*(t) \in \arg\min_{\tilde{u} \in \mathbb{U}} H(x^*(t), \tilde{u}, \lambda_0, \lambda(t)).
\end{align}

For the OCP (7) with terminal costs, the (additional) terminal condition $\lambda(T) = \varphi(x(x^*(T)))$ has to hold.

### 3.2. Normality of extremal lifts

If $\varphi(x) = v^\top x$ holds for some $v \in \mathbb{R}^n$ in OCP (7), the objective function can be written as

$$\int_0^T \|R^{1/2}Qx(t)\|^2 \, dt + (Qx(T) + v)^\top x(T) - H(x(0))$$

using $H(x(T)) = x(T)^\top Qx(T)$. Then, every optimal solution of the OCP (5) satisfies the PMP for (7) with terminal cost $\varphi(x(T)) = (\lambda(T) - Qx_T)^\top x(T)$. Furthermore, invoking Assumptions 2.3 and 3.1 and using the structural assumption $\varphi(x(T)) = v^\top x(T)$, we obtain the converse statement provided that the desired terminal state is given by $x_T = Q^{-1}(\lambda(T) - v)$. In other words, under mild technical assumptions, one can choose the Mayer term in (7) such that the necessary optimality conditions are equivalent.

In general, the stage cost multiplier $\lambda_0$ in the Hamiltonian $H$ may be zero. This case is called abnormal, see [16] [17]. However, for OCP without terminal constraints, one can show that w.l.o.g. a normalization, i.e. setting $\lambda_0 = 1$, can be applied, cf. [17] Rem. 6.9, p. 168. Hence, we set $\lambda_0 = 1$ in OCP (7). Given the shown equivalence of the OCPs (5) and (7), we also have that, for sufficiently long horizons $T$ and provided the state $x_T$ is reachable under the given input constraints, $\lambda_0 = 1$ holds in OCP (5).
3.3. **Singular Arcs.** We begin the analysis with a discussion of the Hamiltonian minimization \( (8c) \). Using the output equation \( y(t) = B^T Q x(t) \) allows to write the optimality system

\[
\mathcal{H}(x, u, \lambda) = u^T B^T (Q x + \lambda) + \lambda^T (J - R) Q x.
\]

As the second term does not depend on \( u \), it has no influence on the minimization w.r.t. \( u \) in \( (8c) \). Hence, rewriting \( (8) \), thereby using skew-symmetry and symmetry, we obtain the optimality system

\[
\begin{align*}
\dot{x}(t) &= (J - R) Q x(t) + B u(t) \\
\dot{\lambda}(t) &= -Q B u(t) + Q (J + R) \lambda(t) \\
u(t) &\in \arg\min_{u \in U} \tilde{u}^T B^T (Q x(t) + \lambda(t)).
\end{align*}
\]

Due to the fact that the Hamiltonian \( \mathcal{H} \) is affine in \( u \), i.e. the OCP is singular, the \( i \)-th switching function \( s_i \) is given by

\[
s_i(t) = b_i^T (Q x(t) + \lambda(t)) = (B^T (Q x(t) + \lambda(t)))_i,
\]

where the \( b_i \) denote the columns of the matrix \( B \). Since we have \( \sum_i \tilde{u}_i s_i(t) = \tilde{u}^T B^T (Q x(t) + \lambda(t)) \) in \( (9) \), it follows that \( s_i(t) > 0 \) implies \( u_i(t) = u_i \). Similarly, \( s_i(t) < 0 \) implies \( u_i(t) = -u_i \). If \( s_i(t) = 0 \) on an interval \( I \), the OCP is said to exhibit a singular arc \( [16] \). It is well understood that the presence of singular arcs complicates the analysis of OCPs, cf. the classical example of Fuller \( [8] \), see also \( [16] \). Here, however, we completely characterize the optimal control in dependence of the optimal state trajectory and the corresponding adjoint on such singular arcs under a certain structural assumption.

**Theorem 3.4.** Assume that \( \text{im}(B) \cap \ker(RQ) = \{0\} \) and that \( (x, u, \lambda) \) satisfies the optimality system \( (9) \) of OCP \( (5) \) or \( (7) \). Then \( u \) is completely determined by \( x \) and \( \lambda \).

Furthermore, for a given interval \( I \), define a partition of the index set \( \{1, \ldots, m\} \) into (in-) active switching indices by

\[
\mathcal{I} \doteq \{ i : s_i \equiv 0 \text{ on } I \} \quad \text{and} \quad \mathcal{A} \doteq \{ i : s_i \not\equiv 0 \text{ a.e. on } I \}.
\]

Set \( u_{\mathcal{I}} = (u_i)_{i \in \mathcal{I}}, u_{\mathcal{A}} = (u_i)_{i \in \mathcal{A}}, B_{\mathcal{I}} = (b_i)_{i \in \mathcal{I}}, \) and \( u_{\mathcal{A}} = (b_i)_{i \in \mathcal{A}} \). Then, on a singular interval \( I \), i.e. \( \mathcal{I} \neq \emptyset \), we have

\[
u_{\mathcal{I}} = M^{-1} B_{\mathcal{I}}^T \left[ \frac{1}{2} (Q A^2 x + (A^2)^T \lambda) - Q R Q B A u_{\mathcal{A}} \right],
\]

where \( M = B_{\mathcal{I}}^T Q R Q B_{\mathcal{I}} \) and \( A = (J - R) Q \).

**Proof.** Let \( s_{\mathcal{I}} = B_{\mathcal{I}}^T (Q x + \lambda) \). Then \( s_{\mathcal{I}} \equiv 0 \) on \( I \). Hence,

\[
0 = \dot{s}_{\mathcal{I}} = B_{\mathcal{I}}^T (Q \dot{x} + \dot{\lambda}) = B_{\mathcal{I}}^T (Q A x - A^T \lambda).
\]

Taking the derivative w.r.t. time once again on \( I \) and setting \( v \doteq \frac{1}{2} (Q A^2 x + (A^2)^T \lambda) \) gives

\[
0 = \ddot{s}_{\mathcal{I}} = B_{\mathcal{I}}^T (Q A \dot{x} - A^T \dot{\lambda}) = B_{\mathcal{I}}^T [Q A (A x + B u) - A^T (-Q B u - A^T \lambda)] = B_{\mathcal{I}}^T [2 v + (Q A + A^T Q) B u] = 2 B_{\mathcal{I}}^T v - 2 B_{\mathcal{I}}^T Q R Q B u,
\]

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where we have used that $QA + A^TQ = -2QR$. Thus,

$$B^TQRQ_Bt = B^Tv - B^TQRQ_BAu_A,$$

which proves the theorem. The matrix $B^TQRQ_Bt$ is positive definite (and thus indeed invertible) since $\ker(B^TQRQ_Bt) = \ker(RQBt)$, so if $RQBt = 0$, then $Btzt \in \text{im}(B) \cap \ker(RQ) = \{0\}$ and hence $Btzt = 0$. As $B$ has full rank, we conclude $zt = 0$. □

3.4. The Lossless Case $R = 0$. In case that the considered pH system is lossless (or conservative), the computation of optimal solutions can be simplified. To this end consider the free end-time counterpart to OCP (5):

$$\min_{T \geq 0, u \in L^1(0, T; U)} C(u) = \int_0^T u(t)^Ty(t) \, dt$$

$$\dot{x}(t) = JQx(t) + Bu(t),$$

$$x(0) = x^0, \quad x(T) = x_T,$$

$$y(t) = B^TQx(t).$$

(10)

Lemma 3.5 (Feasibility implies optimality). Consider OCP (5) and let the underlying pH system (1) be controllable and lossless ($R = 0$). Then any feasible solution $u \in L^1(0, T; U)$ is optimal. Moreover, any feasible solution to the free end-time problem (10) delivers the same performance as optimal solutions in OCP (5) (provided they exist).

Proof. The assertion follows from (6) as for $R = 0$ the value of the objective functional $C(u)$ is completely determined by the initial condition $x^0$ and the terminal condition $x_T$. Hence, also the free end-time problem (10) delivers the same performance. □

This observation motivates to obtain an optimal solution to (5) via an auxiliary problem in which the analytical optimal solution is well known. Indeed, by the bang-bang principle, if there is a control that steers $x^0$ to $x_T$, then there is a bang-bang control that does so as well, cf. [18, Theorem 10, p.48].

Hence, a natural question that arises is to ask for what happens, if we seek for a time-optimal solution in this case, i.e., solve

$$\min_{T \geq 0, u \in L^1(0, T; U)} T$$

$$\dot{x}(t) = JQx(t) + Bu(t),$$

$$x(0) = x^0, \quad x(T) = x_T.$$  

(11)

For pH systems, $Q$ is s.p.d. and $J$ is skew-symmetric and hence $\sigma(JQ) = \sigma(Q^{1/2}JQ^{1/2}) \subset i\mathbb{R}$. The following lemma can be proved in the same way as Lemma 2.4 (i).

Lemma 3.6. Let Assumption 2.3 hold. Then for any $x^0, x_T \in \mathbb{R}^n$ there exist a time $T \geq 0$ and a control $u \in L^1(0, T; U)$ which steers $x^0$ to $x_T$ in time $T$.

Hence we arrive at the main insight of this section: under the assumption of losslessness, optimal solutions for OCP (10) can be obtained solving (11). Moreover, the performance of these solutions evaluated in the objective (5) is identical to solving OCP (5) directly.
4. Dissipativity, turnpike and steady states

We analyse the OCP (5) and (7) in a dissipativity framework for the general dissipative case \( R \neq 0 \). Indeed beginning with [1] there has been widespread interest in dissipativity notions of OCPs in context of model predictive control, see [5, 9, 7]. The driving force behind these investigations is the close relation between dissipativity and turnpike properties of OCPs [10, 7, 25].

4.1. Equilibria of the extremal dynamics. Consider the steady state problem corresponding to OCPs (5) and (7), i.e.,

\[
\min_{\hat{u} \in U} \hat{u}^\top \hat{y}
\]

\[
\text{s.t. } 0 = (J - R)Q\hat{x} + B\hat{u},
\]

\[
\hat{y} = B^\top Q\hat{x}.
\]

In this part, we will assume that \((J - R)Q\) is invertible and we will comment on this assumption in Lemma 4.1. The first-order necessary conditions are the optimality dynamics of (9) considered at steady state, i.e., if \((\hat{x}, \hat{u})\) solves (12) there is a Lagrange multiplier \( \hat{\lambda} \in \mathbb{R}^n \) such that

\[
0 = (J - R)Q\hat{x} + B\hat{u}
\]

\[
0 = -B\hat{u} + (J + R)\hat{\lambda}
\]

\[
\hat{u}^\top (B^\top (Q\hat{x} + \hat{\lambda})) \leq \hat{u}^\top (B^\top (Q\hat{x} + \hat{\lambda})) \quad \forall \hat{u} \in U.
\]

The above equations follow, e.g., by eliminating the control via \( u = ((J - R)Q)^{-1}Bu \), defining \( f(u) \equiv u^\top B^\top Q((J - R)Q)^{-1}Bu \) and applying [27] Theorem 1.2 or Lemma 2.21.

We first state a lemma that allows for a continuous map from the control to the state in (12).

**Lemma 4.1.** It holds that

\[
\ker(J \pm R) = \ker R \cap \ker J.
\]

In particular, if one of \( J \) and \( R \) is non-singular, then so is \( J \pm R \).

**Proof.** \( Jx = Rx \) implies \( \|R^{1/2}x\|^2 = x^\top Rx = x^\top Jx = 0 \) and thus \( Rx = Jx = 0 \). \( \square \)

**Assumption 4.2.** In the sequel we shall assume

\[
\ker J \cap \ker R = \{0\}
\]

This is equivalent by [14] that the structure matrix \((J - R)\) does not admit any left Casimir function [21] which means that the system does not admit any dynamical invariant associated with the structure matrix.

Under this assumption \( J \pm R \) is invertible by Lemma 4.1. Hence, setting \( A = J - R \), for each optimal \( \hat{u} \) the corresponding optimal stationary state and adjoint (Lagrange multiplier) are given by \( \hat{x} = -Q^{-1}A^{-1}B\hat{u} \) and \( \hat{\lambda} = -A^{-T}B\hat{u} \).

**Lemma 4.3.** If \((\hat{x}, \hat{u}, \hat{\lambda})\) is an optimal equilibrium, then

\[
\frac{1}{2}\hat{u}^\top B^\top (Q\hat{x} + \hat{\lambda}) = \|R^{1/2}\hat{\lambda}\|^2 = \|R^{1/2}Q\hat{x}\|^2 \geq 0
\]
and thus \( u^\top B^\top (Q\hat{x} + \hat{\lambda}) \geq 0 \) for each \( u \in \mathbb{U} \).

**Proof.** Set \( A = (J - R)Q \). We compute
\[
\langle R\hat{\lambda}, \hat{\lambda} \rangle = \langle (J + R)\hat{\lambda}, \hat{\lambda} \rangle = \langle B\hat{u}, \hat{\lambda} \rangle = -\langle A\hat{x}, \hat{\lambda} \rangle
\]
\[
\langle RQ\hat{x}, Q\hat{x} \rangle = \langle (R - J)Q\hat{x}, Q\hat{x} \rangle = \langle B\hat{u}, Q\hat{x} \rangle
\]
\[
= \langle Q(J + R)\hat{\lambda}, \hat{x} \rangle = -\langle A\hat{x}, \hat{\lambda} \rangle.
\]
This shows \( \|R^{1/2}\hat{\lambda}\|^2 = \|R^{1/2}Q\hat{x}\|^2 = -\langle A\hat{x}, \hat{\lambda} \rangle \). But on our way we also saw that
\[
-\langle A\hat{x}, \hat{\lambda} \rangle = \frac{1}{2}\langle (B\hat{u}, \hat{\lambda}) + \langle B\hat{u}, Q\hat{x} \rangle \rangle = \frac{1}{2}\langle B\hat{u}, Q\hat{x} + \hat{\lambda} \rangle,
\]
which proves the lemma. \( \square \)

**Corollary 4.4.** We have either \( u^\top B^\top (Q\hat{x} + \hat{\lambda}) > 0 \) for all \( u \in \mathbb{U} \) or \( Q\hat{x} + \hat{\lambda} = 0 \). In the first case, we have \( 0 \notin \mathbb{U} \). In the second case, \( \hat{\lambda} = -Q\hat{x} \in \ker R \).

**Proof.** Assume that there exists some \( u \in \mathbb{U} \) such that
\[
(15) \quad u^\top B^\top (Q\hat{x} + \hat{\lambda}) = 0.
\]
Then, in view of the minimality condition in (13), it follows that \( \hat{u}^\top B^\top (Q\hat{x} + \hat{\lambda}) = 0 \). Hence, \( R\hat{\lambda} = RQ\hat{x} = 0 \) by Lemma 4.3. Adding the equations in (13) gives \( 0 = (J - R)Q\hat{x} + (J + R)\hat{\lambda} = (J - R)(Q\hat{x} + \hat{\lambda}) \) and thus \( Q\hat{x} + \hat{\lambda} = 0 \). \( \square \)

We finish this section with the main result characterizing optimal steady states in the case of non-active control constraints or if \( 0 \notin \mathbb{U} \).

**Theorem 4.5.** Consider an optimal solution \((\hat{x}, \hat{u})\) of (12) and let \( \hat{\lambda} \) be the corresponding adjoint state in (13). Suppose that either \( 0 \notin \mathbb{U} \) or \( \hat{u} \in \text{int} \mathbb{U} \). Then \( Q\hat{x} = -\hat{\lambda} \in \ker R \).

**Proof.** First, assume that \( 0 \notin \mathbb{U} \). By Lemma 4.4 we get \( Q\hat{x} + \hat{\lambda} = 0 \) and \( \hat{\lambda} = -Q\hat{x} \in \ker R \). For the second claim, let \( \hat{u} \in \text{int} \mathbb{U} \). Then, the minimality condition of (13) can be replaced by
\[
B^\top (Q\hat{x} + \hat{\lambda}) = 0.
\]
The result then follows by Corollary 4.4. \( \square \)

### 4.2. Strict dissipativity and the turnpike property.

We first characterize the distance to the conservative subspace. In that context, we denote the distance to a set \( S \) via
\[
\text{dist}(x, S) = \inf\{\|x - s\| : s \in S\}.
\]
It is clear that if \( S \) is closed (i.e., some nullspace of a matrix), then the inf can be replaced with a min.

**Lemma 4.6.** There are constants \( c_1, c_2 > 0 \) such that
\[
c_1 \text{dist}(x, \ker R^\frac{1}{2}Q) \leq \|R^\frac{1}{2}Qx\| \leq c_2 \text{dist}(x, \ker R^\frac{1}{2}Q).
\]
Proof. As $QR\hat{z}Q$ is symmetric p.s.d. it has eigenvalues $\lambda_1, \ldots, \lambda_n \geq 0$ with corresponding pairwise orthonormal eigenvectors $v_1, \ldots, v_n$. Let $r \in \mathbb{N}^{\geq 0}$ such that w.l.o.g. $\ker QR\hat{z}Q = \text{span}\{v_1, \ldots, v_r\}$. Then, for the orthogonal projection $P$ onto $\ker QR\hat{z}Q$ we have

$$Px = \sum_{i=1}^{r} \langle x, v_i \rangle v_i, \quad (I - P)x = \sum_{i=r+1}^{n} \langle x, v_i \rangle v_i.$$  

Furthermore,

$$\|QR\hat{z}Qx\|^2 = \sum_{i=1}^{n} \lambda_i^2 |\langle x, v_i \rangle|^2 = \sum_{i=r+1}^{n} \lambda_i^2 |\langle x, v_i \rangle|^2.$$  

Using the identity $\|\hat{z}x\|^2 = \sum_{i=r+1}^{n} |\langle x, v_i \rangle|^2$, we get

$$\lambda_{\min}^2 \|\hat{z}x\|^2 \leq \|QR\hat{z}Qx\|^2 \leq \lambda_{\max}^2 \|\hat{z}x\|^2.$$  

where $\lambda_{\min}$ and $\lambda_{\max}$ is the smallest resp. largest non-zero eigenvalue of $QR\hat{z}Q$. Further, by positive definiteness of $Q$ the norm $\|Q\cdot\|$ is equivalent to the standard norm $\|\cdot\|$ in $\mathbb{R}^n$ and $\ker QR\hat{z}Q = \ker R\hat{z}Q$. Hence, together with

$$\text{dist}(x, \ker R\hat{z}Q) = \text{dist}(x, \ker QR\hat{z}Q) = \|\hat{z}x\|,$$

the result follows. □

Next we present a generalization of a notion of strict dissipativity for OCPs which has appeared first in a discrete-time context in [1]. Similar ideas of dissipativity with respect to a subspace have been considered in [29], where the authors consider dissipativity with respect to a compact set. Here, we introduce a novel notion that formulates dissipativity with respect to a subspace.

Definition 4.7 (Dissipativity with respect to subspaces). Consider $n : \mathbb{R}^n \to \mathbb{R}_+$. OCPs (5) and (7) are said to be strictly dissipative with respect to a subspace $\mathcal{V} \subset \mathbb{R}^n$ if there exists a storage function $S : \mathbb{R}^n \to \mathbb{R}_+$ and a function $\alpha \in \mathcal{K}$ such that for all optimal controls $u^* \in L^1(0, T; \mathbb{U})$ and associated states $x^* \in W^{1,1}(0, T; \mathbb{R}^n)$ solving (5) or (7) the inequality

$$S(x^*(T)) - S(x^*(0)) \leq \int_0^T u^*(t)^\top B^\top Q x^*(t) - \alpha(\text{dist}(x^*(t), \mathcal{V})) \, dt \quad (16)$$

holds.

If $\mathcal{V} = \{0\}$, the OCPs (5) and (7) are said to be strictly dissipative with respect to $(\hat{x}, \hat{u}) = (0, 0)$.

An immediate consequence of this strict dissipativity w.r.t. a subspace $\mathcal{V}$ with this particular supply rate is the following turnpike property stating that on a large portion of the interval, the optimal states of (7) or (5) reside close to the subspace $\mathcal{V}$.

Lemma 4.8. Let (7) or (5) be strictly dissipative with respect to a subspace $\mathcal{V} \subset \mathbb{R}^n$. Further, assume that $((J - R)Q, B)$ is stabilizable. Then, for all compact sets $K \subset \mathbb{R}^n$
and \( \varepsilon > 0 \) there is \( C_{K,\varepsilon} > 0 \) independent of \( T \) such that for all optimal solutions \( x^*(t) \) starting in \( K \),
\[
\mu[t \in [0, T] : \text{dist}(x^*(t), V) \geq \varepsilon] < C_{K,\varepsilon},
\]
where \( \mu \) is the standard Lebesgue measure on \( \mathbb{R} \).

**Proof.** The proof follows by straightforward modifications of the proofs in [9, Section 3] or [7, Theorem 2]. \( \square \)

**Remark 4.9 (Available storage).** In the foundational work of Jan Willems (cf. [31]) the available storage for a dissipative system with supply rate \( w : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R} \) is defined by
\[
\tilde{S}(x^0) = -\sup_{T \geq 0, u \in L^1(0,T;\mathbb{U})} \int_0^T w((x(t,u,x^0),u(t))) \, dt.
\]

It is well-known that boundedness of the available storage is a necessary and sufficient condition for dissipativity. Considering our particular supply rate \( w(x,u) = u^\top B^\top Q x + \|R^\frac{1}{2} Q x\|^2 \) suggested by Definition 4.7, we get for solutions of (1a) that
\[
\tilde{S}(x^0) = \sup_{T \geq 0, u \in L^1(0,T;\mathbb{U})} (H(x(0)) - H(x(T,u,x^0))).
\]

The next result summarizes the main insights.

**Theorem 4.10 (Dissipativity with respect to subspaces).** Assume that the control constraints are not active at the optimal steady states, i.e., \( \hat{u} \in \text{int}(\mathbb{U}) \) or that \( 0 \in \text{int}(\mathbb{U}) \). Then:

(i) The optimal steady state and the corresponding Lagrange multiplier satisfy \( Q \hat{x}, \hat{\lambda} \in \ker R \) and \( Q \hat{x} + \hat{\lambda} = 0 \).

(ii) All \( \hat{u} \in \mathbb{R}^m \) satisfying \( ((J - R)Q)^{-1} Bu \in \ker R \) are optimal controls for (12).

(iii) OCP (5) and (7) are strictly dissipative with respect to \( \ker R^\frac{1}{2} Q \) with storage function \( H(x) \).

(iv) If \( R \) is invertible, then the unique optimal steady state is \( \hat{x} = \hat{u} = 0 \) and OCP (5) and (7) are strictly dissipative with storage function \( H(x) \).

**Proof.** Part (i) follows immediately from Corollary 4.4. For (ii) we compute with Theorem 4.5 for all optimal steady states that
\[
\hat{u}^\top \hat{y} = \|R^\frac{1}{2} Q \hat{x}\|_2^2 = 0.
\]
as they are particular solutions of the pH system with constant energy, i.e., choosing the (constant) control \( \hat{u} \) and the initial state \( \hat{x} \). For (iii) we use (6) and obtain
\[
H(x(T)) - H(x(0)) = \int_0^T u(t)^\top y(t) - \|R^\frac{1}{2} Q x(t)\|^2 \, dt.
\]
To show Part (iv), we insert (18) into (12). As \( R \) is positive definite, \( R^\frac{1}{2} Q \) is and we can estimate
\[
\|R^\frac{1}{2} Q v\| \geq \gamma \|v\|,
\]
for \( \gamma > 0 \) and all \( v \in \mathbb{R}^n \). Hence, by invertibility of \( (J - R)Q \), \( \hat{u} = \hat{x} = 0 \) is the unique optimal solution with objective value zero. Moreover, (19) yields strict dissipativity. \( \square \)
Remark 4.11 (Connection with optimal steady states). Theorem 4.10 (i) states that optimal steady states lie in the conservative subspace, i.e., \( \hat{x} \in \ker R^\perp Q \). By the dissipativity (iii), we can conclude a turnpike property in the sense of (17) towards this subspace. This means, that solutions of the dynamic problem are close to the solutions of the steady state problem up to directions that lie in the conservative subspace. If \( R \) is invertible, then this states a classical turnpike property towards the unique optimal steady state \( \hat{x} = 0 \) by (iv).

Remark 4.12 (Regularization of the OCP). If we augment the cost functional with an additional control cost of the form \( \int_0^T \varepsilon \|u(t)\|^2 \, dt, \varepsilon > 0 \) by (6) we obtain

\[
\int_0^T u(t)^\top y(t) + \varepsilon \|u(t)\|^2 \, dt = H(x(T)) - H(x(0)) + \int_0^T \|R^\perp Q x(t)\|^2 + \varepsilon \|u(t)\|^2 \, dt.
\]

Then, assuming we have no specified terminal state, the optimization reduces to

\[
\min_{u \in L^1(0,T;U)} \int_0^T \|R^\perp Q x(t)\|^2 + \varepsilon \|u(t)\|^2 \, dt + H(x(T))
\]

\[
\dot{x}(t) = (J - R)Q x + Bu
\]

\[
x(0) = x^0.
\]

In [22, Proposition 1] it was proven that if \((J - R)Q, B)\) is stabilizable one obtains the estimate

\[
\|D(x(t) - \hat{x})\| + \|u(t) - \hat{u}\| \leq C(e^{-\mu t} + e^{-\mu(T-t)}),
\]

where \( D \in \mathbb{R}^{n \times n} \) is a projection onto the detectable subspace, i.e., the observable subspace that corresponds to eigenvalues of \((J - R)Q x\) with nonnegative real part. If \((J - R)Q, R^\perp Q)\) is detectable, then \( D = I \). Note, that this differs to our setting as we do not have a control penalization and control constraints, which rules out the Riccati theory used in [22]. We briefly discuss a possible extension of Theorem 4.10 to this control-regularized case. One immediately sees that the unique optimal steady state is given by \( \hat{x} = \hat{u} = 0 \) with associated Lagrange multiplier \( \hat{\lambda} = 0 \). Hence claim (i) of Theorem 4.10 trivially holds. It is clear, that also (16) still holds as we only add the positive term \( \int_0^T \varepsilon \|u(t)\|^2 \, dt \) on the right-hand side. Hence the claims (iii) and (iv) of Theorem 4.10 also remain valid. The second claim (ii) does obviously not hold by uniqueness of the optimal control.

Remark 4.13. (Open problem) If neither 0 \( \in \text{int}(U) \) and control constraints are active at the steady state, we can not assure that \( Q \hat{x}, \hat{\lambda} \in \ker R \). In this case, the optimal steady states can lie outside of the conservative subspace.
5. Numerical example

In this part, we consider an example in $\mathbb{R}^3$ and set

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, \quad Q = I_3, \quad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and the initial resp. terminal value

$$x^0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad x_T = \begin{pmatrix} 0.1 \\ 0.1 \\ 0.2 \end{pmatrix}.$$

To avoid chattering solutions, we regularize the objective with a term $10^{-3}u^2$ in the objective functional. Moreover, we consider the equivalent quadratic objective formulation \( (6) \). The solutions are computed via ipopt using a direct multiple shooting with 200 shooting intervals and a fixed stepsize RK4 implemented in CasADi.

The numerical results are shown in Figures 1–3. As one can see, all three cases show the structural similarity of optimal solutions for varying horizon lengths that one expects in OCPs exhibiting the turnpike phenomenon. Observe that in Figure 1 for $R = \text{diag}(0, 0, 1)$ the states $x_1$ and $x_2$ show linear trends during the middle part of the solutions, while the state $x_3$ and the input $u$ approach their optimal steady state values at 0. This can be understood as the turnpike being with respect to $\ker R$. Moreover, in Figure 2 for $R = \text{diag}(0, 1, 0)$ the solutions show oscillations, while in Figure 3 for $R = \text{diag}(0, 1, 1)$ exhibit typical turnpike behaviour at the optimal steady state.

![Figure 1. $R = \text{diag}(0, 0, 1)$ for $T \in \{20, 40, 60\}$.](image-url)
This paper has studied optimal control problems for linear port-Hamiltonian systems. Specifically, we consider the problem of state transition while minimizing the intrinsic pH objective, i.e. the supplied energy. We have shown that under mild assumptions the considered OCPs are strictly dissipative w.r.t. the kernel of the energy-dissipation matrix $R$. This induces the turnpike phenomenon. Finally, we have drawn upon a numerical example to illustrate the interplay between the energy-dissipation matrix and the structure of the turnpike in the optimal solutions. Future work will extend the analysis to other system classes.
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