The conformal Killing spinor initial data equations

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Abstract

We obtain necessary and sufficient conditions for an initial data set for the vacuum conformal Einstein field equations to give rise to a spacetime development in possession of a Killing spinor. The fact that the conformal Einstein field equations are used in our derivation allows for the possibility of the initial hypersurface \(S\) intersecting non-trivially with (or even being a subset of) null infinity \(\mathcal{I}\). For conciseness, these conditions are derived assuming that the initial hypersurface is spacelike. Hence, in particular, these conformal Killing spinor initial data equations encode necessary and sufficient conditions for the existence of a Killing spinor in the development of asymptotic initial data on spacelike components of \(\mathcal{I}\).

1 Introduction

The discussion of symmetries in General Relativity is ubiquitous. From the question of the integrability of the geodesic equations to the existence of explicit solutions to the Einstein field equations and the black hole uniqueness problem, symmetries play an important role. Symmetry assumptions are usually incorporated into the Einstein field equations—which in vacuum read

\[ \tilde{R}_{ab} = \lambda \tilde{g}_{ab} \]  

(1)

—through the use of Killing vectors. From the spacetime point of view, the existence of Killing vectors allows one to perform symmetry reductions of the Einstein field equations—see \([13]\) for instance. This approach has been exploited in classical uniqueness results such as \([38]\). Closely related to the black hole uniqueness problem, characterisations and classifications of solutions to the Einstein field equations usually exploit the symmetries of the spacetime in one way or another, e.g. in the characterisations of the Kerr spacetime via the Mars–Simon tensor—see \([27, 28, 37, 29, 30]\). On the other hand, from the point of view of the Cauchy problem, symmetry assumptions should be imposed only at the level of initial data. In this regard, symmetry assumptions can be phrased in terms of the Killing vector initial data. The Killing vector initial data (KID) equations are a system of PDEs, defined over a spacelike hypersurface \(\tilde{S}\) and with coefficients computable in terms of the first and second fundamental forms \(\tilde{h}\) and \(\tilde{K}\), whose solutions (whenever they exist) correspond to initial data for Killing vectors on the ensuing spacetime development \((\tilde{M}, \tilde{g})\), in the form of lapse-shift pairs—see \([5]\). While Killing vectors play a central role in the discussion of the symmetries, their existence is sometimes not enough to encode all the symmetries and conserved quantities enjoyed by a spacetime e.g. the Carter constant in the Kerr spacetime. One approach to unraveling some of these hidden symmetries is to consider a more fundamental type

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of object, namely Killing spinors, denoted here by $\tilde{\kappa}_{AB}$. For vacuum spacetimes, the existence of a Killing spinor directly implies the existence of a Killing vector. The Killing spinor initial data equations have been derived in the physical framework —i.e. where the manifold on interest is a solution to the Einstein field equations— in [19]. These equations have been used in the construction of a geometric invariant which detects whether or not an initial data set corresponds to initial data for the Kerr spacetime —see [11] and [31] for a comprehensive discussion. The CFEs have found application in the stability analysis of spacetimes —see for instance [15, 14] for the proof of the global and semi-global non-linear stability of the de Sitter and Minkowski spacetimes, respectively. From the point of view of this article, the main advantage of the conformal (unphysical) approach to the Einstein field equations is that null infinity $\mathcal{I}$, being a submanifold of $(\tilde{M}, \tilde{g})$, is a bonafide hypersurface on which to prescribe data, to be evolved using (regular) evolution equations. This set up is particularly attractive in cases where $\lambda > 0$, in which, given the appropriate conditions (sufficient decay of matter fields at infinity), null infinity is a spacelike hypersurface, allowing for one to pose an asymptotic initial value problem: an initial value problem where the initial hypersurface is $\mathcal{I}$ and the initial data on the conformal boundary satisfy the prescribed data equations.

In [18], the authors generalise the KID equations to the broader class of conformal Killing vectors and in [16] these equations are used to characterise initial data for PP-wave spacetimes. It should be noted that, as the analysis is carried out in the physical framework, these conditions only apply to solutions of the Einstein field equations; in particular, the initial hypersurface does not extend to $\mathcal{I}$. On the other hand, the conformally-regular counterpart of the KID equations was derived in [32]; see also [30] for a generalisation to higher spacetime dimensions. That is to say, intrinsic conditions on an initial hypersurface $S \subset \mathcal{M}$ of the unphysical spacetime $(\tilde{M}, \tilde{g})$ —a solution of the CFEs— are found such that the development of the data gives rise to a conformal Killing vector on $(\mathcal{M}, g)$, which moreover corresponds to a Killing vector of the physical spacetime $(\tilde{M}, \tilde{g})$. This construction, in contrast to the previous work, allows for the possibility of $S$ intersecting non-trivially with, or even being a subset of, $\mathcal{I}$. 

\[ \varphi^* g = \Xi^2 \tilde{g}. \]
As previously mentioned, in the case of Petrov type D spacetimes such as the Kerr-de Sitter spacetime, the symmetries of the spacetime are closely related to the existence of Killing spinors. Hence, a natural question is whether there exists a conformal counterpart —i.e. in the unphysical framework— of the Killing spinor initial data equations introduced in [19]. In other words, what are the extra conditions that one has to impose on an initial data set for the CFEs so that the arising development contains a Killing spinor? This question is answered in this article by deriving such conditions, these being termed the conformal Killing spinor initial data equations.

The main result of this article, the more precise statement of which can be found in Theorem 3, is summarised informally in the following:

**Theorem.** If the conformal Killing spinor initial data equations (71) admit a solution on an open set $\mathcal{U} \subset \mathcal{S}$, where $\mathcal{S}$ is a spacelike hypersurface on which initial data for the conformal Einstein field equations has been prescribed, then there exists a Killing spinor on some open (spacetime) neighbourhood $\mathcal{W}$ of $\mathcal{U}$ contained in the domain of dependence, $\mathcal{W} \subseteq D^+(\mathcal{U})$.

The core of the proof of this theorem is obtaining a closed system of homogeneous wave equations for certain fields (so called zero-quantities) encoding the existence of a Killing spinor. Although these wave equations hold regardless of the causal character of $\mathcal{S}$, when obtaining conditions intrinsic to $\mathcal{S}$ we assume for conciseness that it is spacelike. A similar computation could be performed on a hypersurface $\mathcal{S}$ with a different causal character, allowing for applications to the black hole uniqueness problem in general. In the present set up, of a spacelike $\mathcal{S}$, the equations derived here have applicability in the asymptotic characterisation of the Kerr-de Sitter spacetime —which would comprise a spinorial analogue of [29]— in terms of the existence of a Killing spinor at $\mathcal{I}$. Although the main objective of the present paper is the valence-2 Killing spinor case, the analogous conditions encoding the existence of a valence-1 Killing spinor —the conformal twistor initial data equations— are also derived. The latter serves as a warm-up exercise for the valence-2 case in which one can already see some of the essential features of the analysis.

**Overview of the article**

Section 2 summarises relevant background material: subsection 2.1 fixes the conventions and notation and gives an abridged discussion of the main spinorial identities to be used and the space spinor formalism; subsection 2.2 gives an overview of Killing spinors and their conformal properties; subsection 2.3 introduces the conformal Einstein field equations (CFEs). In section 3 the conformal twistor (i.e. valence-1 Killing spinor) initial data equations are derived. In section 4 the conformal (valence-2) Killing initial data equations are derived and discussed.

Many of the more involved computations in this article were facilitated through the xAct suite in Mathematica.

**Notations and conventions**

Throughout this article, $(\mathcal{M}, g)$ will denote a 4-dimensional manifold equipped with a Lorentzian metric $g$ of signature $(+, -, -, -)$, with associated Levi-Civita connection $\nabla$. Moreover, $(\mathcal{M}, g)$ is assumed to be globally-hyperbolic. The Upper case Latin indices $^{ABC} \cdots \cdots_{A'B'C'}$ will be used as abstract indices of the spacetime spinor algebra and $\epsilon_{AB}$ will denote the skew-symmetric spinor metric. The bold numerals $012\cdots$ denote components with respect to a fixed spin dyad $o^A := \epsilon^A_0, r^A := \epsilon^A_1$—see Penrose & Rindler [34] for further details. Lower case Latin indices $a, b, c, \cdots$ will be used as abstract tensor indices. Our curvature conventions are fixed by

$$\nabla_a \nabla_b \kappa^c - \nabla_b \nabla_a \kappa^c = R^c_{\ qab} \kappa^q.$$

The future domain of dependence of an achronal set $\mathcal{A}$ will be denoted by $D^+(\mathcal{A})$. 

3
2 Background

In this section, we give a recap of spacetime and space spinor calculus, in addition to giving a brief introduction to Killing spinors and the conformal Einstein field equations.

2.1 Spinorial formalism in a nutshell

Since \((M, g)\) is, by assumption, globally-hyperbolic and of signature\(^{1}\), it admits a spinor structure —see Proposition 4 in [11].

For spinors, the curvature conventions are fixed via the spinorial Ricci identities which will be written in accordance with the above convention for tensors. Recall that the commutator of covariant derivatives \([\nabla_{AA'}, \nabla_{BB'}]\) can be expressed in terms of the symmetric operator \(\square_{AB}\) as

\[
[\nabla_{AA'}, \nabla_{BB'}] = \epsilon_{AB} \square_{A'B'} + \epsilon_{A'B'} \square_{AB},
\]

where

\[
\square_{AB} := \nabla_{Q'A'}(A \nabla_{B}) Q',
\]

The action of the symmetric operator \(\square_{AB}\) on valence-1 spinors is encoded in the spinorial Ricci identities

\[
\square_{AB} \xi_C = -\Psi_{ABCD} \xi_D + 2\Lambda \xi_{(A} \epsilon_{B)C}, \tag{3a}
\]

\[
\square_{A'B'} \xi_C = -\Phi_{A'A'B'} \xi_A', \tag{3b}
\]

where \(\Psi_{ABCD}, \Phi_{A'A'B'}\) and \(\Lambda\) are the standard curvature spinors of the standard Newmann–Penrose (NP) formalism, namely the Weyl spinor, tracefree Ricci spinor and the scalar curvature\(^{1}\), respectively. The above identities can be extended to higher valence spinors in the obvious way; further discussion (albeit using slightly different conventions) can be found in [39]. A related identity which will be used in the following discussion is

\[
\nabla_{AQ'} \nabla_{BQ'} = \square_{AB} + \frac{1}{2} \epsilon_{AB} \square,
\]

where \(\square_{AB}\) is the symmetric operator defined above and \(\square := \nabla_{AA'} \nabla_{AA'}\).

To keep the discussion self-contained, we briefly recall the space spinor formalism, originally introduced in [38]; see also [11] [2] [41]. Let \(\tau^{AA'}\) denote the spinorial counterpart of a timelike vector \(\tau^a\), normal to a spacelike hypersurface \(S\) and normalised so that \(\tau_a \tau_a = 2\). Then, it follows that \(\tau^{AA'} \tau_{A'A'} = 2\) and, consequently,

\[
\tau_{AA'} \tau_{B'A'} = \epsilon_{AB}.
\]

Given a spacetime spinor \(u_{AA'}\), its space spinor decomposition reads

\[
u_{AA'} = \frac{1}{2} \tau^{AA'} u - \tau^{A'A'} u_{AB},
\]

where \(u := \tau^{AA'} u_{AA'}\) and \(u_{AB} := \tau_{(A'} u_{B)B}\). This split extends to higher valence spinors in an analogous way —see [10] [2] [11]. Similarly, the covariant derivative \(\nabla_{AA'}\) is then decomposed into the normal and Sen derivatives:

\[
\nabla_{\tau} := \tau^{AA'} \nabla_{AA'}, \quad \nabla_{AB} := (\tau^{A'A'} \nabla_{B}) A'.
\]

But we will not need them here, for completeness we note that the Weingarten spinor and the acceleration of the congruence are then defined by

\[
K_{ABCD} := \tau_{D}^{C'} \nabla_{AB} \tau_{CC'}, \quad K_{AB} := \tau_{B}^{C'} \nabla_{\tau} \tau_{AC'}.\n\]

The distribution induced by \(\tau_{AA'}\) is integrable if and only \(K_{D(A'B)D} = 0\), in which case \(K_{ABCD}\) describes the extrinsic curvature of the resulting foliation. The Sen connection is related to the intrinsic Levi-Civita connection, \(\nabla\), as follows

\[
\nabla_{AB} \eta_{C} = D_{AB} \eta_{C} + \frac{1}{2} K_{ABC}^{D} \eta_{D}.
\]

\(^{1}\)More precisely, \(\Lambda = R/24\), with \(R\) the Ricci scalar curvature.
2.2 Killing spinors

Let \((\tilde{M}, \tilde{g})\) also be a 4-dimensional manifold equipped with a Lorentzian metric \(\tilde{g}\) and denote by \(\tilde{\nabla}\) its associated Levi-Civita connection. Later, we will reserve the \(\tilde{\cdot}\) notation for a vacuum spacetime --- that is to say, a solution of the vacuum Einstein field equations (1). For much of the present section, however, no such restriction is necessary.

A totally symmetric \(\tilde{\kappa}_{A_1...A_p} = \tilde{\kappa}_{(A_1...A_p)}\) valence \(-p\) spinor is said to be a (valence \(-p\)) Killing spinor if it satisfies the following equation

\[
\tilde{\nabla}_{Q'}(Q\tilde{\kappa}_{A_1...A_p}) = 0. \tag{5}
\]

An important property of the Killing spinor equation is that it is conformally invariant, in other words if \(g\) is conformally related to \(\tilde{g}\) — namely \(g = \Xi^2 \tilde{g}\) — then \(\kappa_{A_1...A_p} = \Xi^2 \tilde{\kappa}_{A_1...A_p}\) satisfies

\[
\nabla_{Q'}(Q\kappa_{A_1...A_p}) = 0. \tag{6}
\]

In this paper we will only focus only the cases \(p = 1\) and \(p = 2\). If \(p = 1\), the equation

\[
\tilde{\nabla}_{A'}(Q\tilde{\kappa}_A) = 0. \tag{7}
\]

The Killing spinor equation and twistor equations are, in general, overdetermined; in particular, they imply the so-called Buchdahl constraints. In the twistor case \((p = 1)\), the Buchdahl constraint takes the form

\[
\tilde{\kappa}^D \Psi_{ABCD} = 0,
\]

while in the Killing spinor case \((p = 2)\) it takes the form

\[
\tilde{\kappa}_{(A'}Q_{BCD)Q} = 0,
\]

where \(\Psi_{ABCD}\) denotes the Weyl spinor, which is conformally invariant. This constraint restricts \(\Psi_{ABCD}\) to be algebraically special, in particular of Petrov type D, N or O. In the twistor case, which can be considered a degenerate case in which \(\kappa_{AB}\kappa^{AB} = 0\) (implying that \(\kappa_{AB} = \kappa_{A'B'}\) for some twistor \(\kappa_A\)), the spacetime is necessarily of Petrov type N or O, hence restricting its utility in black hole characterisation. On the other hand, given a vacuum spacetime \((\tilde{M}, \tilde{g})\) of Petrov type D, there is an explicit formula for a Killing spinor: choosing an adapted dyad \(\{o, \iota\}\)

\[
\kappa_{AB} = \tilde{\psi}^{-1/3}\psi^{oA}\psi^{\iota B} \tag{8}
\]

yields a Killing spinor. Indeed, the fact that the Killing spinor equation is satisfied follows from the vacuum Bianchi identity \(\tilde{\nabla}_A\Psi_{ABCD} = 0\) by a short calculation — see [34, 42] for more details.

Although the Killing spinor equation is conformally invariant, note that one cannot simply transcribe the analysis of [19, 2] into the conformal setting since we would like to allow for the possibility that \(S \cap \mathcal{I} \neq \emptyset\); points in \(\mathcal{I}\) do not have corresponding points in the physical spacetime. Moreover, we shall see that the method given here departs substantially from that of [19, 2], owing in part to the fact that the Einstein field equations are not conformally invariant, an important difference being that the set of zero-quantities used to encode the existence of a Killing spinor are different. The results of [19] can however be recovered from the analysis presented here by setting \(\Xi = 1\). The need for a different set of Killing spinor zero-quantities in the conformal case can be traced back to the observation that in \((\tilde{M}, \tilde{g})\) the vector \(\xi_{AA'} = \nabla_{A'}Q\kappa_{QA}\) does not
correspond to a Killing (or even a conformal Killing) vector. Although for a general Lorentzian manifold this vector appears not to have any clear geometric significance, a by-product of the present analysis is that, for conformally Einstein manifolds (i.e. solutions to the CFEs) the vector \( \xi_A \) represents a collineation of the rescaled Weyl curvature —see [24] for definitions of curvature collineations. Once the existence of a Killing spinor is established one can use the conformal factor \( \Xi \), the Killing spinor \( \kappa_{AB} \) and \( \xi_A \) to construct a conformal Killing vector \( X_a \) associated to a Killing vector \( X_a \) of the physical spacetime \((\tilde{M}, \tilde{g})\) —see Remark 1 later. In the analysis of [19], the fact that \( \xi_{AA} = \nabla_{AA} K_{QA} \) is a Killing vector is crucial; indeed, it motivates the introduction of \( \tilde{S}_{ab} := \tilde{\nabla}(\tilde{\xi})_{ab} \) as a zero-quantity. Similarly, in the work of [9], where the results of [19] are generalised to the case where \((\tilde{M}, \tilde{g})\) satisfies the Einstein-Maxwell equations, the condition \( \tilde{S}_{ab} = 0 \) is satisfied by virtue of an assumed matter alignment condition. In the conformal setting of interest in this article, the analogous quantity \( S_{ab} \) is not as geometrically motivated as in the physical cases and its usage as a variable in the system does not lead to a closed system of explicitly regular homogeneous wave equations. Again, the adjective “regular” refers to the absence of formally singular terms, such as \( \Xi^{-1} \), in the equations. Instead, the quantity that is central for the present analysis turns out to be the so-called Buchdahl zero-quantity (and derivatives thereof), the vanishing of which relates the existence of Killing spinors with the Petrov type of \((M, g)\), in line with the above discussion.

**Remark 1.** The notion of Killing spinors is related to that of Killing–Yano tensors. Given a Killing spinor \( \kappa_{AB} \), the Killing spinor is the spinorial counterpart of a Killing–Yano tensor \( \tilde{\Upsilon}_{ab} \) —i.e. an antisymmetric 2–tensor satisfying \( \tilde{\nabla}_{(a} \tilde{\Upsilon}_{b)} = 0 \)— as follows

\[
\tilde{\Upsilon}_{AA'B'B'} = i(\kappa_{AB}\epsilon_{A'B'} - \tilde{\kappa}_{A'B'}\epsilon_{AB}).
\]

Conversely, given a Killing–Yano tensor, one can construct a Killing spinor —see [9, 31, 35]. The (Killing–Yano) tensor counterpart of the Killing spinor initial data result of [19] has been recently derived in [17].

From now on, \((\tilde{M}, \tilde{g})\) will be reserved for the physical spacetime, while \((M, g)\) will refer to the unphysical spacetime, related to \((\tilde{M}, \tilde{g})\) via \( g = \Xi^2 \tilde{g} \) —as is customary, in a slight abuse of notation, the pullback \( \varphi^* \) of the embedding \( \varphi : \tilde{M} \to M \) will be omitted.

### 2.3 The conformal Einstein field equations

The conformal Einstein field equations (CFEs), first given in [11], are a conformal reformulation of the Einstein field equations. In other words, given a spacetime \((\tilde{M}, \tilde{g})\) satisfying the Einstein field equations, the CFEs encode a system of implied differential conditions for the curvature and concomitants of the conformal factor associated to a Killing vector \( \tilde{X}_a \) that is central for the present analysis turns out to be the so-called Buchdahl zero-quantity —see [24] for definitions of curvature collineations.

The so-called “metric” version of the standard vacuum conformal Einstein field equations are encoded in the following zero-quantities —see [11] [10] [12] [13]:

\[
Z_{ab} := \nabla_a \nabla_b \Xi + \Xi L_{ab} - s g_{ab} = 0, \tag{8a}
\]

\[
Z_a := \nabla_a s + L_{ac} \nabla^c \Xi = 0, \tag{8b}
\]

\[
\delta_{bac} := \nabla_b L_{ac} - \nabla_a L_{bc} - d_{abcd} \nabla^d \Xi = 0, \tag{8c}
\]

\[
\lambda_{abc} := \nabla_a d_{abc} = 0, \tag{8d}
\]

\[
Z := \lambda - 6 \Xi s + 3(\nabla_a \Xi) \nabla^a \Xi, \tag{8e}
\]

where \( \Xi \) is the conformal factor, \( L_{ab} \) is the Schouten tensor, defined in terms of the Ricci tensor \( R_{ab} \) and the Ricci scalar \( R \) via

\[
L_{ab} = \frac{2}{3} R_{ab} - \frac{1}{12} R g_{ab}, \tag{9}
\]

\( s \) is the so-called Friedrich scalar defined as

\[
s := \frac{1}{3} \nabla_a \nabla^a \Xi + \frac{1}{4} R \Xi, \tag{10}
\]
and \(d^a_{\text{b}cd}\) denotes the \textit{rescaled Weyl tensor}, defined as

\[
d^a_{\text{b}cd} = \Xi^{-1}C^a_{\text{b}cd},
\]

where \(C^a_{\text{b}cd}\) denotes the Weyl tensor. The geometric meaning of these zero-quantities is as follows. The equation \(Z_{ab} = 0\) encodes the conformal transformation law between \(R_{ab}\) and \(\tilde{R}_{ab}\). The equation \(Z_a = 0\) is obtained considering \(\nabla^a Z_{ab}\) and commuting covariant derivatives. Equations \(\delta_{abc} = 0\) and \(\lambda_{abc} = 0\) encode the contracted second Bianchi identity. Finally, \(Z = 0\) is a constraint in the sense that if it is verified at one point \(p \in M\) then \(Z = 0\) holds in \(M\) by virtue of the previous equations. A solution to the metric conformal Einstein field equations consists of a collection of fields

\[
\{g_{ab}, \Xi, s, \Lambda, d_{abcd}\}
\]

satisfying

\[
Z_{ab} = 0, \quad Z_a = 0, \quad \delta_{abc} = 0, \quad \lambda_{abc} = 0, \quad Z = 0. \quad (11)
\]

**Remark 2.** If one opts to use the Ricci tensor \(R_{ab}\) instead of the Schouten tensor \(L_{ab}\) then the Ricci scalar \(R\) appears in the right-hand side of equations but no equation for it has been provided. In the CFEs the Ricci scalar encodes the \textit{conformal gauge source function}, hence there is no equation to fix that variable as it represents a gauge quantity.

Since we are concerned here with spinor fields, we will need the spinorial transcription of the CFEs (see \(\text{(11)}\)), which reads

\[
\begin{align*}
&Z_{ABBB'} = \nabla_{AAV} \nabla_{BB'} \Xi - \Xi \Phi_{ABAB'B'} - s \epsilon_{AB} \epsilon_{A'B'} + \Xi \Lambda \epsilon_{A'B'}, \\
&Z_{AA'} = \nabla_{AA'} s + \Lambda \nabla_{AA} \Xi - \Phi_{ABAB'B'} \nabla_{BB'} \Xi, \\
&\delta_{ABCC'} = \nabla_{A(\Phi_{BCC'}) - iC_{(A} \nabla_{B)C'} \Lambda + \Phi_{ABCD} \nabla^D \nabla_{CC'} \Xi, \\
&\Lambda_{C'ABC} = \nabla_{DC'} \Phi_{ABCD}, \\
&Z = \lambda - 6 \Xi s + 3(\nabla_{AA} \Xi) \nabla_{AA'} \Xi,
\end{align*}
\]

where \(\Phi_{ABAB'B'}\) and \(\Lambda\) are as in section \(\text{(2.1)}\) and the Weyl spinor enters via the \textit{rescaled Weyl spinor}, \(\phi_{ABCD}\), defined as

\[
\phi_{ABCD} := \Xi^{-1} \Psi_{ABCD} \quad (13)
\]

—see \(\text{[39, 34]}\) for more details. As in the tensorial case, one can choose the Schouten (tensor) spinor or the Ricci (tensor) spinor as a variable.

**Remark 3.** In the initial value problem for the CFEs, \(\phi_{ABCD}\) is determined by the initial data on a spacelike hypersurface \(\mathcal{S}\). First decompose the rescaled Weyl spinor as

\[
\phi_{ABCD} = E_{ABCD} + iB_{ABCD}
\]

where \(E\) and \(B\) are the \textit{electric} and \textit{magnetic} parts

\[
E_{ABCD} := \frac{i}{2}(\phi_{ABCD} + \tilde{\phi}_{ABCD}), \quad B_{ABCD} := \frac{i}{2}(\phi_{ABCD} - \tilde{\phi}_{ABCD}),
\]

with \(\tilde{\phi}_{ABCD} := \tau_{A'B'} \tau_{B'C'} \tau_{D'D'} \phi_{A'B'C'D'}\). The fields \(E, B\) are the spinorial counterparts of the electric and magnetic parts of the rescaled Weyl tensor and comprise (part of) the initial data: away from \(\mathcal{J}\) they are determined by a conformal analogue of the Gauss–Codazzi–Mainardi equations, while for the asymptotic initial value problem the constraint equations implied by the CFEs acquire a particularly simple form so that initial data for the magnetic part is determined algebraically by the Bach tensor of the induced metric \(h_{ij}\) and the electric part must be prescribed, the only constraint being that it satisfies the \(TT\) condition with respect to \(h_{ij}\)—see \(\text{[11, 24]}\). For the discussion of this paper we will assume such data \(E, B\) (and hence \(\phi\)) to be given. Note also that one can formally define the Petrov type of an initial data set for the CFEs by applying the Petrov classification to the initial datum \(\phi_{ABCD}|_{\mathcal{S}}\), rather than to \(\Psi_{ABCD}\) —see \(\text{[39]}\) for an introduction to the Petrov classification.
The CFEs as previously presented can be regarded as a set of covariant conditions for geometric fields on \((M, g)\) and, hence, they do not have a particular PDE character. However, there are various hyperbolic reduction strategies, depending on the choice of gauge fixing procedure, for extracting a set of evolution and constraint equations. For the subsequent discussion only the evolution and constraint equations implied by the \(\Lambda_{C'ABC} = 0\) equation, namely,
\[
\nabla^D_{C'}\phi_{ABCD} = 0,
\]
will play an important role. A direct calculation using the space spinor formalism shows that equation (13) can be recast as the following system of evolution and constraint equations
\[
\nabla_{\tau}\phi_{ABCD} = 2\mathcal{D}_{(A}F_{BCD)F}, \quad \mathcal{D}_{CD}\phi_{CDAB} = 0.
\]
The CFEs can also be recast as a second-order system of wave equations; see [32] for the tensorial formulation and [22] for the spinorial formulation. We shall only need one of these wave equations here, namely
\[
\Box_{ABCF} = 12\Lambda_{ABCF} - 6\Xi_{(AB}D_{FG)CF},
\]
which is derived by considering \(\nabla_{D'}\Lambda_{C'ABC} = 0\) and applying identity (4). It is worth noting here that one of the tools used in [22] to show the equivalence between the system (12a)–(12e) and their wave-equation counterpart is the uniqueness property of solutions to a certain class of homogeneous wave equations, a result which we shall also use repeatedly in this article and which is given below in Theorem 1.

**Definition.** An operator \(h\) is said to be homogeneous in \(u\) and \(\partial u\), if \(h(\mu u, \mu \partial u) = \mu h(u, \partial u)\) for all \(\mu \in \mathbb{C}\).

**Theorem 1.** Let \(M\) be a smooth manifold equipped with a Lorentzian metric \(g\) and consider the wave equation
\[
\Box u = h(u, \partial u)
\]
where \(u \in \mathbb{C}^m\) is a complex vector-valued function on \(M\), \(h : \mathbb{C}^2m \to \mathbb{C}^m\) is a smooth homogeneous function of its arguments and \(\Box = g^{ab}\nabla_a \nabla_b\). Let \(U \subseteq S\) be an open set and \(S \subseteq M\) be a spacelike hypersurface with normal \(\tau^{a}\) respect to \(g\). Then the Cauchy problem
\[
\Box u = h(u, \partial u),
\]
\(u|_U = u_0\), \(\nabla_{\tau}u|_U = u_1\),
where \(u_0\) and \(u_1\) are smooth on \(U\) and \(\nabla_{\tau} := \tau^{\mu}\nabla_{\mu}\), has a unique solution \(u\) in an open neighbourhood \(W\) of \(U\), with \(W \subseteq D^+(U)\).

We refer the reader to Proposition 3.2 of [40] for a proof.

**Remark 4.** Analogous to the physical case, given a Petrov type D \(\phi_{ABCD}\), one can give an explicit construction of a Killing spinor, namely
\[
\kappa_{AB} = \phi^{-1/3}D_{(A'}\kappa_{B)'}
\]
in terms of the relevant adapted spin dyad \(\{\alpha, \xi\}\). This can be seen directly by noting that the equation (13) satisfied by the rescaled Weyl spinor \(\phi_{ABCD}\) is formally identical to the physical vacuum Bianchi constraint and so the same computations as in [42] follow through. An alternative approach to the construction of Killing spinor initial data equations would be to attempt to determine under what conditions the Petrov type of the (rescaled) Weyl tensor restricted to \(S\) is propagated into the spacetime development. Later, we shall see that such a result follows, rather, as a product of our analysis —see Corollary [41].

### 3 Conformal twistor initial data

In this section, the conformal twistor initial data equations are derived. Although the main result of this article is the valence-2 (Killing spinor) case, the twistor case illustrates the main features of the calculation for the Killing spinor case (given in section [4] in a simpler setting.
3.1 Twistor zero-quantities

For the following discussion is convenient to define the following zero-quantities

\[ H_{A'B} := 2\nabla_{A'}(A'B), \]  
\[ B_{ABC} := \phi_{ABCD}^D. \]  

(17a)

(17b)

The spinors \( H_{A'B} \) and \( B_{ABC} \) will be denoted in index free notation as \( H \) and \( B \) and will be called the twistor zero-quantity and the Buchdahl zero-quantity, respectively. The Buchdahl zero-quantity arises as an integrability condition of the twistor equation. To see this, notice that, taking the following derivative of \( H \) and substituting definition (17a), one obtains

\[ \nabla_{AA'} H_{A'B} = 2\nabla_{AA'} \nabla_{(B'A')} \nabla_{(C'A')} = \frac{1}{2} \epsilon_{AB} \Box_{AC} + \frac{1}{2} \epsilon_{AC} \Box_{BC} + \Box_{AB} \nabla_{AC} + \Box_{AC} \nabla_{AB}. \]  

(18)

Symmetrising and using equation (3a) gives

\[ \nabla_{(A'A')} H_{A'B} = -2\Psi_{ABCD}^D. \]  

The vanishing of the right-hand side of latter equation encodes the Buchdahl constraint, namely the fact that if \((\mathcal{M}, g)\) admits a twistor then it is necessarily of Petrov type N or O. To write this in terms of the variables appearing in the CFEs, we substitute the definition of the rescaled Weyl spinor to obtain

\[ \nabla_{(A'A')} H_{A'B} = 2\Xi_{ABC}. \]  

(19)

It is clear that if the unphysical spacetime \((\mathcal{M}, g)\) admits a twistor then \( H_{A'B} = B_{ABC} = 0 \).

3.2 Auxiliary quantities and the twistor candidate equation

The following auxiliary quantities

\[ Q_A := \nabla^{QA'} H_{A'QA}, \]  
\[ \xi_A := \nabla^{B'A'} k_B. \]  

(20a)

(20b)

will prove to be a useful bookkeeping device for the subsequent calculations. The spinor \( \xi_A \) is merely a convenient placeholder for making irreducible decompositions of derivatives of \( k_A \):

\[ \nabla_{AA'} k_B = \frac{1}{2} \epsilon_{AB} \nabla_{CA'} k_C + \nabla_{(A'A')} k_B = \frac{1}{2} H_{A'B} - \frac{1}{2} \epsilon_A \epsilon_{AB}. \]  

(21)

It is illustrative to introduce this shorthand since the analogous quantity in the Killing spinor case will play an important role in the calculation. On the other hand, the auxiliary quantity \( Q_A \) will be central for the following discussion since it encodes a wave equation for \( k_A \). To see this, observe that tracing the identity (18) and substituting definition (20a) gives

\[ Q_A = \frac{3}{2} \Box k_A + 3\Lambda k_A. \]  

(22)

Hence, \( Q_A = 0 \) encodes the following wave equation for \( k_A \):

\[ \Box k_A + 2\Lambda k_A = 0. \]  

(23)

A valence-1 spinor \( k_A \) satisfying (23) will be called a twistor candidate. To understand the motivation for this definition and its name, notice that in general any twistor trivially satisfies the twistor candidate equation but the converse is not necessarily true:

\[ H = 0 \Rightarrow Q = 0, \quad \text{but in general} \quad Q = 0 \nRightarrow H = 0. \]

However, the initial data \( (\kappa_A, \nabla_{\tau} k_A)|_{S} \) for the wave equation (23) have not yet been fixed. The present aim is to determine conditions on the twistor candidate initial data, on an initial hypersurface \( S \), which if propagated off \( S \) using equation (23) ensure that the corresponding twistor candidate \( \kappa_A \) is indeed a twistor —i.e. such that

\[ Q = 0 \quad \& \quad \text{twistor initial data} \implies H = 0. \]  

(24)
The strategy for obtaining such conditions on the initial data \((\kappa_A, \nabla \tau \kappa_A)|_{\mathcal{S}}\) will be to derive a closed system of homogeneous wave equations for the zero-quantities \(H\) and \(B\) such that, if trivial initial data is given:

\[
H_{A'AB} = 0, \quad \nabla_{A'} H_{A'AB} = 0, \quad B_{ABC} = 0, \quad \nabla_{A'} B_{ABC} = 0 \quad \text{on} \quad \mathcal{U} \subset \mathcal{S} \tag{25}
\]

then Theorem \[\text{I}\] will guarantee that \(H = 0\) and \(B = 0\) on some open neighbourhood \(\mathcal{W} \subset \mathcal{D}^+\mathcal{(S)}\). Conditions \[(25)\] will imply the desired restrictions, the \textit{conformal twistor initial data equations}, that must be satisfied by \(\kappa_A\).

### 3.3 Wave equations for the zero-quantities

To derive a wave equation for the zero-quantity \(H\), we start with the irreducible decomposition

\[
\nabla_D A' H_{A'AB} = \frac{1}{3} \epsilon_{BD} \nabla_{CA'} H_{A'B} + \frac{1}{3} \epsilon_{AD} \nabla_{CA'} H_{A'B} + \nabla_{(A} A' H_{A'|BD)}.
\]

Substituting the definition \(\text{(20a)}\) and equation \(\text{(19)}\), it follows that

\[
\nabla_D A' H_{A'AB} = 2\Xi_{ABD} + \frac{1}{3} Q_{B} \epsilon_{AD} + \frac{1}{3} Q_{A} \epsilon_{BD}. \tag{26}
\]

Applying \(\nabla_D \kappa^D\) to the last expression, and using identity \([4]\) along with the spinorial Ricci identities \([35] - [31]\), renders

\[
\Box H_{B'|AB} = 6\Lambda H_{B'|AB} + 4\Xi_{DB'} B_{AB} - 4B_{ABD} \nabla_{\mathcal{D'}} B_{A} \Xi - 4\Phi_{(A} B'|A' H_{A'|BD} + \frac{4}{3} \nabla_{(A} |B'|Q_{B)}. \tag{27}
\]

To derive a wave equation for \(B\), on the other hand, one begins by applying the D’Alembertian operator \(\Box\) to the definition \(\text{(17b)}\) to obtain

\[
\Box B_{ABC} = \kappa^D \Box \phi_{ABC} + \phi_{ABC} \Box \kappa^D + 2(\nabla_{FA'} \phi_{ABC}) \nabla_{F'A'} \kappa^D. \tag{28}
\]

Substituting the definition \(\text{(17b)}\), the identity \(\text{(22)}\), and the wave equation satisfied by the rescaled Weyl spinor \(\text{(16)}\) into the last expression gives

\[
\Box B_{ABC} = 10B_{ABC} + H_{A'DF} \nabla_{FA'} \phi_{ABC} - 6\Xi_{B_{ABC} (A} D_{F} \phi_{BC} ) D_F - \frac{1}{2} \phi_{ABC} Q_{B}. \tag{29}
\]

Observe that if \(Q_A = 0\), namely if the twistor candidate wave equation is imposed, then \(H\) and \(B\) satisfy the following set of wave equations

\[
\Box H_{B'|AB} = 6\Lambda H_{B'|AB} + 4\Xi_{DB'} B_{AB} - 4B_{ABD} \nabla_{\mathcal{D'}} B_{A} \Xi - 4\Phi_{(A} B'|A' H_{A'|BD} \tag{30a}
\]

\[
\Box B_{ABC} = 10B_{ABC} + H_{A'DF} \nabla_{FA'} \phi_{ABC} - 6\Xi_{B_{ABC} (A} D_{F} \phi_{BC} ) D_F. \tag{30b}
\]

Notice that the only place where the CFEs (in their wave equation form) were used is in substituting for the \(\Box \phi_{ABC}\) term in equation \(\text{(25)}\).

The important observation about equations \(\text{(30a)} - \text{(30b)}\) is that they comprise a closed system of \textit{regular and homogeneous} wave equations for \(H\) and \(B\). Hence, we have the following:

**Proposition 1.** Given initial data for the conformal Einstein field equations on \(\mathcal{U} \subset \mathcal{S}\) where \(\mathcal{S}\) is a spacelike hypersurface \(\mathcal{S}\) with normal vector \(\tau^{A'}\), then a twistor candidate on \(\mathcal{D}^+(\mathcal{U})\), is a true twistor (valence-1 Killing spinor) on an open neighbourhood \(\mathcal{W}\) of \(\mathcal{U}\), with \(\mathcal{W} \subset \mathcal{D}^+(\mathcal{U})\), if and only if

\[
H_{A'AB} = B_{ABC} = 0, \quad \nabla_{A'} H_{A'AB} = \nabla_{A'} B_{ABC} = 0, \tag{31a}
\]

\[
\text{hold on } \mathcal{U}. \tag{31b}
\]
Proof. The only if direction is immediate. Suppose, on the other hand, that $\kappa_A$ is a twistor candidate satisfying (31a)–(31b) on $U \subset S$. As the zero-quantities $H_{A'AB}$, $B_{ABC}$ satisfy the homogeneous wave equations (30a)–(30b) then the uniqueness result for homogeneous wave equations, given in Theorem 1, ensures that

$$H_{A'AB} = 0, \quad B_{ABC} = 0,$$

in an open neighbourhood $\mathcal{W}$ of $\mathcal{U}$, with $\mathcal{W} \subseteq \mathcal{D}^+(\mathcal{U})$. In other words, $\kappa_A$ solves the twistor equation in $\mathcal{W}$.

Remark 5. One important difference with [19], in which the twistor equations are derived on a vacuum spacetime $(\mathcal{M}, \tilde{g})$, is that there the wave system closes with $\tilde{H}_{A'AB}$ alone and there is no need to introduce the analogous physical Buchdahl zero-quantity $\tilde{B}_{ABC}$. Therefore it is interesting to check if in the conformal case one can also close the system with $H_{A'AB}$ alone. Observe that if one substitutes the expression for the Buchdahl constraint into equation (30a) and uses the CFEs, then one obtains

$$\Box H_{A'AB} = -2\Xi^{-1}\nabla^C A \Xi \nabla_{(A} B'|_{BC)} + 6\Lambda H_{A'AB} - 2\Xi \phi_{ABCD} H_{A'C'D} - 4\Phi_{[A} C |A' B'|_{B'C}.$$ (32)

Hence, $H$ satisfies a closed and homogeneous, though singular, equation due to the $\Xi^{-1}$ coefficient. Theorem 1 does not apply in this case. From equation (32) one can recover the analogous wave equation in the physical case discussed in [19] simply by adding a tilde to the fields and setting $\Xi = 1$. Arguably, one could try to use the theory of Fuchsian systems, as used in [7, 33], to see if the analogue of Theorem 1 applies for the singular equation (32). However, one of the advantages of the conformal approach of the CFEs is that one deals with manifestly regular equations. Therefore, from this perspective, it is preferable to deal with manifestly regular equations by introducing $B_{ABC}$ as a further zero-quantity to be propagated. The same observation holds for the conformal valence-2 Killing spinor initial data discussion of the following sections, where, to close the system in a regular way, one needs to introduce not only a “Buchdahl” zero-quantity but also a further derivative thereof.

3.4 Intrinsic conformal twistor initial data conditions

Proposition 1 of the previous section gives necessary and sufficient conditions for a twistor candidate to correspond to a true twistor. We would like now to reduce the conditions (31a)–(31b), which contain not only derivatives tangential to $S$ but also normal to it, to conditions on the field $\kappa_A|_U$ that are computable at the level of initial data for the CFEs. More precisely, our twistor candidate will be constructed as the solution to the following initial value problem:

$$\begin{cases}
\Box \kappa_A + 2\Lambda \kappa_A = 0 & \text{on } \mathcal{D}^+(\mathcal{U}), \\
\kappa_A = \tilde{\kappa}_A & \text{on } \mathcal{U}, \\
\nabla_\tau \kappa_A + \frac{2}{3} D_A \tilde{\kappa}_B = 0 & \text{on } \mathcal{U},
\end{cases}$$

(33)

where the initial data $\tilde{\kappa}_A$ will be appropriately restricted in order to ensure that the conditions (31a) and (31b) hold for the solution $\kappa_A$. Note that the bulk equation here is precisely $Q_A = 0$, the twistor candidate equation.

We begin by introducing the following definitions:

$$\mathcal{H}_{ABC} := \tau_{(A'} A' H_{A'|BC)}, \quad \mathcal{H}_A := \tau^A H_{A'AQ},$$

(34)

in terms of which the space spinor split of $H_{A'AB}$ reads

$$H_{A'AB} = \frac{1}{2} \tau^C A \mathcal{H}_{ABC} + \frac{1}{6} \tau_{AA'} \mathcal{H}_{B} + \frac{1}{6} \tau_{BA'} \mathcal{H}_{A}.$$ (35)

Note the space spinors $\mathcal{H}_{ABC}$ and $\mathcal{H}_A$ contain all the information of $H_{A'AB}$; in particular,

$$H_{A'AB} = 0 \iff \mathcal{H}_A = 0 \land \mathcal{H}_{ABC} = 0.$$
Substituting the definition of $H$, equation (17a), one obtains
\[ H_A \equiv \frac{1}{2} \nabla \tau \kappa_A + D_A \kappa_B, \quad H_{ABC} \equiv 2D_{(AB}\kappa_{C)}, \quad (36) \]
Notice that $H_{A}|_{U} = 0$ is precisely the second initial condition of (33). The condition $H_{ABC} = 0$, on the other hand, contains only quantities intrinsic to $S$, which is also true of the condition $B_{ABC} = 0$ from equation (31a). This motivates the following definition:

**Definition.** A spinor field $\bar{\kappa}_A$ defined on some $U \subset S$ and satisfying
\[ H(\bar{\kappa})_{ABC} \equiv 2D_{(AB}\bar{\kappa}_{C)} = 0, \quad B(\bar{\kappa})_{ABC} \equiv \phi_{ABCD}\bar{\kappa}^D = 0 \quad (37) \]
will be called a conformal twistor initial data set on $U$. We will show that a conformal twistor initial dataset, $\bar{\kappa}_A$, indeed comprises initial data for a spacetime twistor, in that the resulting solution of the initial value problem (33) necessarily satisfies the twistor equation on an open neighbourhood of $\bar{U}$, with $\bar{W} \subseteq D^+(U)$. First we establish the following:

**Lemma 1.** Given initial data for the conformal Einstein field equations on $U \subset S$, where $S$ is a spacelike hypersurface with normal $\tau$, if $\bar{\kappa}_A$ is a conformal twistor initial data set on $U$, then the solution $\kappa_A$ of the initial value problem (33) satisfies
\[ H(\kappa)_{A'AB} = B(\kappa)_{ABC} = \nabla \tau H(\kappa)_{A'AB} = \nabla \tau B(\kappa)_{ABC} = 0 \]
on $U$.

**Proof.** The assumption that $\bar{\kappa}_A$ satisfies $H(\bar{\kappa})_{ABC} = 0$ implies that the solution of (33) also satisfies
\[ H_{ABC}|_{U} = 0, \quad H_{A}|_{U} = 0, \quad (38) \]
the latter following from the second initial condition of the initial value problem (33). Hence, as remarked above, we have $H_{A'ABC}|_{U} = 0$. Clearly $B_{ABC}|_{U} = 0$ also, since $\kappa_A|_{U} = \bar{\kappa}_A$. Now, substituting the space spinor split of $\nabla$ in the identity (26), it follows that
\[ \tau_{A'}\nabla \tau H_{A''AB} = 2\tau_{A'}\nabla_{DC}H_{A''AB} = 4\Xi B_{ABD} + \frac{4}{3}Q(\kappa_{ABD}). \quad (39) \]
Transvecting with $\tau^B B^*$ and rearranging gives
\[ \nabla \tau H_{B''AB} = -4\Xi B_{ABD}\tau^D B^* - 2\tau_{A'}\nabla_{DC}H_{A''AB} - \frac{4}{3}\tau^D B^*Q(\kappa_{ABD}). \quad (40) \]
Now, since $\kappa_A$ satisfies (33), we have in particular that $Q_A = 0$ and hence the conditions $H_{A'ABC}|_{U} = B_{ABC}|_{U} = 0$ also imply that
\[ \nabla \tau H_{A''AB}|_{U} = 0. \quad (41) \]
All that remains, then, is to establish that $\nabla \tau B_{ABC}|_{U} = 0$. Using the definition of $B$, (17b), we find
\[ \nabla \tau B_{ABC} = \phi_{ABCD}\nabla \tau \kappa^D + \kappa^D \nabla \tau \phi_{ABCD}. \quad (42) \]
At this point one can exploit the evolution equation for $\phi_{ABCD}$, namely (15), along with the initial conditions of (33), to obtain
\[ \nabla \tau B_{ABC}|_{S} = -2\kappa^D D_{DF} \phi_{ABC} F + \frac{2}{3} \phi_{ABCD} D^D F K F + \frac{2}{3} \phi_{ABCD} H^D. \quad (43) \]
In fact, the right-hand-side can be completely rewritten in terms of $H_{A}|_{S}$, $H_{ABC}|_{S}$ and $B_{ABC}|_{U}$. To do so, first swap indices $D$ and $A$ in the first term on the right-hand-side of equation (33), which is made possible by the constraint equation for the rescaled Weyl spinor, equation (15), to obtain
\[ \nabla \tau B_{ABC}|_{U} = -2\kappa^D D_{AF} \phi_{DBC} F + \frac{2}{3} \phi_{ABCD} D^D F K F + \frac{2}{3} \phi_{ABCD} H^D. \quad (44) \]
Substituting the definition of $B_{ABC}$ into the expression $\mathcal{D}_{AD}B_{BC}^D$ and using the Leibnitz rule, one sees that the above as equivalent to

$$\nabla_x B_{ABC}|_U = -2\mathcal{D}_{AD}B_{BC}^D - 2\phi_{BCDF}\mathcal{D}_A^F\kappa^D + 2\phi_{ABCF}\mathcal{D}_D^F\kappa^D + \frac{2}{3}\phi_{ABCD}\mathcal{H}^D. \quad (41)$$

Now, performing the irreducible decomposition of $\mathcal{D}_{ABC}$ and using the expression for $\mathcal{H}_{ABC}$ in (30), one has

$$\mathcal{D}_{ABC} = \frac{1}{2}\mathcal{H}_{ABC} + \frac{1}{4}\epsilon_{BCD}\mathcal{D}_A^D\kappa^D + \frac{1}{8}\epsilon_{ACD}\mathcal{D}_B^D\kappa^D.$$ \quad (42)

Finally, substituting decomposition (42) into equation (41) gives

$$\nabla_x B_{ABC}|_U = -\phi_{BCDF}\mathcal{H}_A^{DF} - 2\mathcal{D}_{AD}B_{BC}^D + \frac{2}{3}\phi_{ABCD}\mathcal{H}^D.$$ \quad (43)

Hence it follows that if $\mathcal{H}_{ABC} = \mathcal{H}_A = B_{ABC} = 0$ on $U$, then

$$\nabla_x B_{ABC}|_U = 0,$$

and the conclusion follows.

**Remark 6.** Observe that although the CFEs were not needed in deriving equation (38), they were however used when deriving (43), so we have made crucial use of the assumption of having initial data satisfying the CFE constraints in Lemma 1.

We then obtain the main theorem of this section as a simple application of Lemma 1:

**Theorem 2.** Consider an initial data set for the vacuum conformal Einstein field equations, as encoded in the CFE zero-quantities (12a–12e), on a spacelike hypersurface $S$ and let $U \subset S$ be an open set. The development of the initial data set will have a twistor (valence-1 Killing spinor) in an open neighbourhood $W$ of $U$, with $W \subseteq \mathcal{D}^+(U)$, if and only if there exists a conformal twistor initial data set $\bar{\kappa}_A$ on $U$. Given the existence of such a $\bar{\kappa}_A$, the twistor $\kappa_A$ is obtained as the solution of the initial value problem (33).

**Proof.** Lemma 1 implies that if $\bar{\kappa}_A$ is a conformal twistor initial data set, then $\kappa_A$ satisfies (31a–31b), then, by virtue of Proposition 1 one has that $H_{A'ABC} = 0$ in an open neighbourhood $W$ of $U$, with $W \subseteq \mathcal{D}^+(U)$. Hence, $\kappa_A$ is a twistor on $W$. \hfill $\square$

### 4 Conformal Killing spinor initial data

In this section we perform the valence-2 counterpart of the analysis in previous section: we derive necessary and sufficient conditions for a spinor field defined on (a subset of) an initial hypersurface $S$ to give rise to a Killing spinor on the spacetime development. These conditions will be given by the so-called *conformal Killing spinor initial data equations*, or *CKSIDs* for short. Although the calculations will be more involved, in general terms we follow the same strategy as in the twistor case.

#### 4.1 Killing spinor zero-quantities

Analogous to the twistor case, we define the zero-quantities:

$$H_{A'ABC} := 3\nabla_{A'(A}\kappa_{BC)}, \quad B_{ABCD} := \kappa_{(A}^Q\phi_{BCD)Q}. \quad (44)$$

A short computation shows that

$$\nabla_{(A'}^A'H_{A'|BCD)} = 6\Xi B_{ABCD}. \quad (45)$$

Despite the formal resemblance with the equations of section 3.1, the following discussion will show that, unlike the twistor case, one cannot obtain a closed homogeneous wave system in terms of these variables alone; we shall need the following additional zero-quantity

$$F_{A'BCD} := \nabla_{A'}^Q B_{QBCD}. \quad (46)$$

We shall show that it is however possible to derive a closed homogeneous wave system for the fields $(H, B, F)$. 

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13
4.2 Auxiliary quantities and the Killing spinor candidate equation

By analogy with the twistor case, it will prove useful to define the following auxiliary quantity

\[ Q_{BC} := \frac{1}{2} \nabla^{A'} H_{A'ABC}. \] (47)

A direct calculation shows that the auxiliary quantity \( Q_{AB} = 0 \) encodes a wave equation for \( \kappa_{AB} \):

\[ Q_{BC} \equiv \Box k_{BC} + 4 \lambda k_{BC} - \Xi \phi_{BCAD} k^{AD}. \] (48)

A solution of \( Q_{AB} = 0 \) will be called a Killing spinor candidate. It will also prove convenient to define the auxiliary spinor \( \xi_{AA'} := \nabla^{B'} \kappa_{AB} \) in terms of which one can perform the following decomposition

\[ \nabla_{AA'} \kappa_{BC} = \frac{1}{2} H_{A'ABC} - \frac{1}{3} \xi_{CA'B} \xi_{AB} - \frac{1}{3} \xi_{BA' \epsilon AC}. \] (49)

Using the latter expression, one can show by a straightforward computation that

\[ \nabla^{Q_{A'} H_{B'ABC}} + Q_{AB} \xi_{A'B'} = \nabla_{AA'} \xi_{BB'} + \nabla_{BB'} \xi_{AA'} + 6 \kappa_{(A' \Phi_{B')QA'B'}}, \] (50)

which in turn implies the following identity

\[ \nabla_{AA'} \xi_{BB'} = -\frac{1}{2} \epsilon_{AB} \nabla^{C} (A' C_{B'}) - 3 \kappa_{(A' \Phi_{B')C'A'B'}} - 3 \lambda \kappa_{AB} \xi_{A'B'} + \frac{1}{2} \Xi \phi_{CD} \xi_{B'D'B'} \xi_{AB} + \frac{1}{4} \Xi \phi_{AB} \xi_{B'D'B'} + \frac{1}{4} \nabla^{Q_{A' H_{B')ABC}}, \] (51)

which will prove useful later. On the other hand, it is straightforward to show using equations (45) and (47) that

\[ \nabla_{B'ABC} = 6 \Xi B_{ABCD} + \frac{1}{4} Q_{(AB \epsilon C')D}. \] (52)

Remark 7. As stressed in section 1 in general the auxiliary spinor \( \xi_{AA'} \) is not the spinorial counterpart of a Killing vector. Contrast this with the case of the physical framework—namely \((\mathcal{M}, \tilde{g})\) satisfying the vacuum Einstein field equations—where the last term in equation (50) vanishes and hence \( \tilde{\xi} \) is a Killing vector. This point is subtle even in the physical framework if one departs from the vacuum case: for instance, if one considers matter models such as Einstein-Maxwell then it is necessary to make further assumptions such as the matter alignment condition to ensure that \( \tilde{\xi} \) is a Killing vector—see [9] for details. This property of \( \tilde{\xi} \) is crucial for the derivation of the physical Killing spinor data equations presented in [10] and [9], which involves the Killing vector zero-quantity \( S_{ab} := \nabla_{(a} \tilde{\xi}_{b)} \). In the unphysical framework one cannot appeal to this strategy since the unphysical Ricci spinor \( \Phi \) is non-vanishing and does not satisfy any useful algebraic relation. As we will see in the following, the key to solving this problem in the unphysical framework is not to introduce the analogous quantity \( S \) but, instead, to focus on the Buchdahl constraint \( B \) and its derivative \( F \). Note however that the quantity

\[ X_{AA'} \equiv \Xi \xi_{AA'} - 3 \kappa_{AQ} \nabla_{A'} Q_{\Xi}, \] (53)

which we do not make explicit use of, does have geometric significance in that it is a conformal Killing vector on \((\mathcal{M}, \bar{g})\). Indeed, a short computation verifies that

\[ \nabla_{AA'} \xi_{BB'} + \nabla_{BB'} X_{AA'} - \frac{1}{2} \epsilon_{AB} \epsilon_{A'B'} \nabla^{C C'} X_{C C'} = -\Xi \nabla^{Q_{A' H_{B')}ABC} - 2 \nabla^{Q_{A' H_{B')}ABC} \] (54)

and moreover that \( X_{AA'} = \Xi^{2} \tilde{\xi}_{AA'} \), where \( \tilde{\xi}_{AA'} \) is the Killing vector associated to \( \tilde{\kappa}_{AB} = \Xi^{-2} \kappa_{AB} \) in the physical spacetime (with \( \bar{g} = \Xi^{-2} \bar{g} \)).

4.3 Wave equations for the zero-quantities

Applying \( \nabla^{A'} B' \) to equation (52), one obtains the following wave equation:

\[ \Box H_{B'ABC} = 6 \lambda H_{B'ABC} - 12 \Xi F_{B'ABC} - 12 (\nabla^{D} B' \Xi) B_{ABCD} + 3 \nabla_{A(B'} \xi_{BC)} - 6 \Phi_{A(D} B' A' H_{A'BC)} D. \] (55)
Similarly, applying $\nabla A^A$ to equation (16), it is straightforward to verify the following wave equation for $B_{ABCD}$:

$$\Box B_{ABCD} = 12 \Lambda B_{ABCD} - 6 \Xi \phi_{ABCD} + 2 \nabla A^A F^A_{BCD}. \quad (56)$$

The task remaining is to derive a wave equation for $F_{A'B''C'D'}$. To do so, we will need some ancillary identities. Firstly, a direct calculation shows that

$$2 \phi_{(AB} G_{CD)} F_{G} = \kappa_A^D \phi_{(BC} G_{D)F_G} + \kappa_B^D \phi_{(AC} G_{DF_G}} + \kappa_C^D \phi_{(AB} G_{DF}) + \kappa_F^D \phi_{(BC)G_A}G_{HF}.$$ \quad (57)

This identity, along with the irreducible decomposition

$$\phi_{ABCD} \phi_{FGH} = \frac{1}{6} \phi_{DLM} \phi_{G}^{DLM} \epsilon_{AHF} \epsilon_{CG} + \frac{1}{6} \phi_{DLM} \phi_{G}^{DLM} \epsilon_{AFH} \epsilon_{CG} + \frac{1}{6} \phi_{DLM} \phi_{G}^{DLM} \epsilon_{AGH} \epsilon_{CF} + \frac{1}{6} \phi_{DLM} \phi_{G}^{DLM} \epsilon_{AGC} \epsilon_{BF}$$

allows one to derive the following identity:

$$\kappa^D G \phi_{(ABC} G^H \phi_{F)H}G = 2 \phi_{(AB} G_{CF)FG}. \quad (58)$$

Now, using the definition of the Buchdahl zero-quantity (11), the CFEs for $\phi$ in its first and second order form, namely equations (14) and (15), and using the decomposition (19), we get

$$\Box B_{ABCD} = \frac{1}{2} \nabla \phi_{ABCD} + 8 \Lambda B_{ABCD} - \phi_{(AB} G_{Q)D} + \frac{1}{2} \nabla \phi_{(A} G_{F)B} G_{CD)}G$$

$$- 6 \Xi \phi_{(A} G_{FGH)} G_{D)FG} - 2 \kappa^F G \phi_{(ABC} G^H \phi_{F)D}.$$ \quad (59)

where $\nabla \phi := \xi^A \nabla A^A$. Substituting the above identities (57)–(58), we can derive the following alternative (non-homogeneous) wave equation for $B_{ABCD}$:

$$\Box B_{ABCD} = \frac{1}{2} \nabla \phi_{ABCD} + 8 \Lambda B_{ABCD} - 14 \Xi \phi_{ABCD} G_{Q)D}$$

$$+ \frac{1}{2} (\nabla \phi_{(A} G_{F)B} G_{Q)D}) G - \phi_{(ABC} G_{D} G_{Q)D}. \quad (60)$$

**Remark 8.** Equation (60) actually encodes the fact that, given a Killing spinor $\kappa_{AB}$, the field $\xi_{AA'}$ is a collineation for the rescaled Weyl tensor —i.e. that

$$\mathcal{L}_{\xi} \phi_{ABCD} := \nabla \phi_{ABCD} + \phi_{(ABC} \nabla D) A^A' \xi_{F} = 0,$$

the definition of the “Lie derivative” being derived from the spinorialised counterpart of $\mathcal{L}_{\xi} d_{abcd}$. Indeed, this fact follows from a straightforward calculation

$$\mathcal{L}_{\xi} \phi_{ABCD} := \nabla \phi_{ABCD} + \phi_{(ABC} \nabla D) A^A' \xi_{F}$$

$$= \nabla \phi_{ABCD} - 6 \Lambda \kappa_{D} F \phi_{ABC)F} - \frac{1}{2} \kappa^F \phi_{ABC} G_{D} G - \frac{1}{2} \phi_{(ABC} G_{Q)D} F,$$

where we are using (51) along with the identity (58). Given a Killing spinor $\kappa_{AB}$, the resulting zero-quantities vanish: $B_{ABCD} = H_{A'B''C'D'} = Q_{AB} = 0$, and hence it follows from (60) that

$$\mathcal{L}_{\xi} \phi_{ABCD} = 0. \quad (61)$$

Thus, a sub-product of our analysis is that if $(\mathcal{M}, g)$ admits a Killing spinor, then the Weyl-collineation condition (61) is satisfied. This is not trivial since, as remarked above, the vector $\xi_{AA'}$ is not in general a conformal Killing vector. In contrast, for the physical spacetime it is clear that this condition holds since in that case $\xi_{AA'}$ is a Killing vector and thus $\mathcal{L}_{\xi} C_{abcd} = 0$, trivially.

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2The Lie derivative does not extend to spinor fields, in general. See [33] for a discussion.
The key observation to close the system is that there are no derivatives of zero-quantities appearing on the right-hand-side of equation (60). This, combined with the CFE $\Lambda A A B C = 0$, equation (14), suggests that by applying $\nabla A A'$ to equation (60) we may be able to derive a wave equation for $F A B C D$ with the desired properties, namely being homogeneous in $(H, B, F)$ and their first derivatives (apart from the $D'$Alembertian term). We will see that this strategy does indeed work; the difficult part is in deriving a suitable expression for $\nabla A A' \nabla_\xi \phi_{A B C D}$ in terms of the zero-quantities. To do so, first, we first note some further useful identities: from the definition of the zero-quantities $H$, $B$ and the auxiliary spinor $\xi$, we obtain

$$\kappa_A F \nabla_{F F'} \phi_{B C D G} = \kappa_A F \nabla_{D F'} \phi_{B C G F}$$

$$= \frac{1}{4} \xi_A F \phi_{B C D G} - \frac{1}{4} \xi_{F'} \phi_{B C G F} \epsilon_{A D} - \frac{1}{4} \xi_F \phi_{B C (D F') A}$$

$$- \frac{1}{2} \xi_{F'} \phi_{B [D | C | F] A} + \epsilon_{A D} F F' B C G + \nabla_{(B F')} B [C | D] + \frac{1}{2} \phi_{B C (D F') A} F F' A$$

$$+ 2 \epsilon_{A B F} F F' H_{(F'|G)} F H_{A (F|G)} - \frac{1}{2} \phi_{B C F H} F F' H_{(F'|G)} F A (F|G)$$

$$- \frac{1}{2} \phi_{B [D | C | F] A} F F' H_{F A (F|G)} - \frac{1}{6} \phi_{D (B F') H F A (F|G)}$$

from which it follows that

$$\kappa^{A D} \phi_{A D} G H \nabla_{H A'} \phi_{B C D G} = 4 \epsilon_{A D} F \phi_{B C (D G') A} + \frac{1}{2} \phi_{A D G H} \phi^{A D G H} H_{A' B C F}$$

$$- 4 \epsilon_{B} F \phi_{A D G H} H_{A' F A C D} - 8 \epsilon_{A D} \phi_{A D G H} H_{A' F A C D} + 4 \epsilon_{B} F \phi_{A D G H} H_{A' F A C D}$$

$$- \frac{1}{2} \phi_{A D G H} H_{A' F A C D}$$

$$= \frac{1}{2} \phi^{A D} \phi_{A D G H} H_{A' F A C D}$$

(63)

—see Appendix A for details.

Now, commuting derivatives and using equation (14) gives

$$\nabla A A' \nabla_\xi \phi_{A B C D} = \nabla_\xi (\nabla A A' \phi_{A B C D}) + (\nabla A' A' \phi_{A B C D}) \nabla_\xi F F' \phi_{A B C D} + \phi^{A F} \left[\nabla A A', \nabla_{F F'}\right] \phi_{A B C D}$$

$$= (\nabla A A' \phi^{F F'}) \nabla_{F F'} \phi_{A B C D} + \phi^{A F} \left[\nabla A A', \nabla_{F F'}\right] \phi_{A B C D}$$

(64)

Then, using equations (51), (14) and expanding the commutator gives

$$\nabla A A' \nabla_\xi \phi_{A B C D} = 6 \epsilon A D F \phi_{A B C D} - \xi^{D F} \Phi^{D F}_{A' A'} \phi_{A B C D} - 3 \epsilon^{D F} \Phi^{D F}_{A' A'} \phi_{A B C D}$$

$$- 2 \epsilon D F \phi_{B} G A' F \nabla_{F F'} \phi_{A B C D} - 2 \epsilon^{D F} \phi_{B} G A' F \nabla_{F F'} \phi_{A B C D}$$

$$+ 2 \epsilon D F \phi_{B} G A' F \nabla_{F F'} \phi_{A B C D} - \frac{1}{2} (\nabla_{F F'} \phi_{A B C D}) \nabla Q_A H_{A' F} F D Q.$$ 

Finally, using equations (62)—(65), one obtains

$$\nabla A A' \nabla_\xi \phi_{A B C D} = -3 \epsilon A D F \phi_{A B C D} - \frac{1}{2} \epsilon D F \phi_{A B C D} H_{A' A' B C D} + \epsilon D F \phi_{A B C D} H_{A' A' B C D}$$

$$+ \frac{1}{2} \epsilon^{D F} \phi_{(B A) D} H_{A' A' B C D} + \frac{1}{2} \epsilon D F \phi_{(B A) D} H_{A' A' B C D}$$

$$+ \frac{1}{2} \epsilon^{D F} \phi_{(B A) D} H_{A' A' B C D} + \frac{1}{2} \epsilon D F \phi_{(B A) D} H_{A' A' B C D}$$

Note that the final expression is homogeneous in the zero-quantities $(H, B, F)$ and their first derivatives, as required.
Collecting together the above, we derive the required wave equation for $F_{A'B'CD}$:

$$\square F_{A'B'CD} = \square (\nabla^A_{A'} B_{ABCD})$$

$$= [\square, \nabla^A_{A'}] B_{ABCD} + \nabla^A_{A'} \square B_{ABCD}$$

$$= [\square, \nabla^A_{A'}] B_{ABCD} + \frac{2}{3} \nabla^A_{A'} \nabla \phi_{ABCD}$$

$$+ \nabla^A_{A'} \left( 8AB_{ABCD} - 14\Phi_{(AB} F_{CD)FG} + \frac{2}{3} (\nabla F_A \phi_{(ABC)H^{A'}D^{FG}}) \right)$$

$$= \frac{2}{3} \nabla^A_{A'} \nabla \phi_{ABCD} - 6A_{ABCD} - 6\Phi_{(A'} B_{|A'B'|F_B|CD)A} - 9(\nabla^A_{A'} A) B_{BCDA}$$

$$+ 3B_{(BC} F_{D)A} F_{G} A^{A'} \Xi + 3B_{AFCB} \nabla \phi_{FB'B'} \phi^{A'} \phi_{D} - 6\Phi_{(A'B'} F_{DC|A'B'} \phi_{G|} B_{CD|A}$$

$$+ \nabla^A_{A'} \left( 8AB_{ABCD} - 14\Phi_{(AB} F_{CD)FG} + \frac{2}{3} (\nabla F_A \phi_{(ABC)H^{A'}D^{FG}}) \right),$$

where we are using equation (66) in the third line and in the fourth we are expanding out the commutator and using the Bianchi identities. Substituting equation (65) and setting $Q_{AB} = 0$, we obtain a homogeneous expression in $(H, B, F)$ and their first derivatives, as required.

With this closed system of homogeneous wave equations —(55), (56) and (67)— at hand, a direct application of Theorem 1 gives the following:

**Proposition 2.** Given initial data for the conformal Einstein field equations on $U \subset S$ where $S$ is a spacelike hypersurface $S$ with normal vector $\tau^{AA'}$, then a (valence-2) Killing spinor candidate on $D^+(U)$, is a true Killing spinor on an open neighbourhood $W$ of $U$, with $W \subseteq D^+(U)$, if and only if

$$H_{A'B'ABC} = F_{A'B'ABC} = 0,$$

$$\nabla_\tau H_{A'B'ABC} = \nabla_\tau B_{ABCD} = \nabla_\tau F_{A'B'ABC} = 0,$$

hold on $U$.

**Proof.** The only if direction is immediate. Suppose, on the other hand, that $\kappa_{AB}$ is a Killing spinor candidate on $D^+(U)$ satisfying (68a)–(68b) on $U$. In particular, $Q_{AB} = 0$, and the identities (55), (56) and (67) reduce to a closed system of homogeneous wave equations for the zero-quantities $H$, $B$ and $F$. The uniqueness result for homogeneous wave equations, given in Theorem 1 ensures then that

$$H_{A'B'ABC} = 0, \quad B_{ABCD} = 0, \quad F_{A'B'ABC} = 0,$$

on an open neighbourhood $W$ of $U$, with $W \subseteq D^+(U)$. In particular, $\kappa_{AB}$ solves the Killing spinor equation on $W$.

To summarise: we have found necessary and sufficient conditions, namely (68a)–(68b), defined on $U \subset S$, for a given Killing spinor candidate —i.e. a field satisfying the wave equation encoded by $Q_{AB} = 0$, see (18)— to be a Killing spinor.

### 4.4 Conformal Killing spinor initial data conditions (CKSID)

By analogy with the twistor case in section 3.3, in this section we aim to reduce (68a)–(68b) to a set of intrinsic conditions —that is to say, conditions on $\kappa_{AB}$ that are computable at the level of an initial data set for the CFEs.

This time, the initial value problem of interest is the following:

$$\square \kappa_{AB} + 4\Lambda \kappa_{AB} - 3\Phi_{ABCD} \kappa^{CD} = 0$$

on $D^+(U)$,

$$\nabla_\tau \kappa_{AB} + \xi_{AB} = 0$$

on $U$.

where $\xi_{AB} := D_{A'B'} [\kappa_{BCD}]$ is used as a shorthand. We define

$$H_{ABCD} := \tau (A'B') H_{A'B'BCD}, \quad H_{BC} := \tau A'B' H_{A'B'ABC},$$

in which

$$\begin{equation}
\begin{aligned}
\square \kappa_{AB} + 4\Lambda \kappa_{AB} - 3\Phi_{ABCD} \kappa^{CD} &= 0 \quad \text{on } D^+(U), \\
\nabla_\tau \kappa_{AB} + \xi_{AB} &= 0 \quad \text{on } U.
\end{aligned}
\end{equation}$$
in terms of which

\[ H_{A'B'C'} = -\frac{1}{2} \tau^D A' \mathcal{H}_{A'B'C'D} + \frac{1}{2} \tau_{AB} A' \mathcal{H}_{AB} + \frac{1}{2} \tau_{BA} A' \mathcal{H}_{BA} + \frac{1}{2} \tau_{CA} A' \mathcal{H}_{AB}. \]

Analogous to the twistor case, \( \mathcal{H}_{AB} \) and \( \mathcal{H}_{ABCD} \) together contain all the information of \( H_{A'B'C'} \); in particular,

\[ H_{A'B'C'} = 0 \quad \iff \quad \mathcal{H}_{AB} = 0 \quad \& \quad \mathcal{H}_{ABCD} = 0. \]

Substituting the definition of \( H_{A'B'C'} \) from (44),

\[ \mathcal{H}_{AB} \equiv 3(\nabla_{\tau} \kappa_{AB} + \xi_{BC}) \quad \& \quad \mathcal{H}_{ABCD} \equiv 3D_{(AB} \kappa_{CD)}. \] (70)

Note that \( \mathcal{H}_{AB}|_{\mathcal{U}} = 0 \) is precisely the second initial condition of (59). Again, the conditions \( \mathcal{H}_{ABCD} = B_{ABCD} = 0 \) involve only quantities intrinsic to \( \mathcal{S} \). Following the twistor case, we make the following definition:

**Definition.** A symmetric spinor field \( \bar{\kappa}_{AB} \) defined on some \( \mathcal{U} \subset \mathcal{S} \) and satisfying

\[ \mathcal{H}(\kappa)_{ABCD} \equiv 3D_{(AB} \kappa_{CD)} = 0, \quad B(\kappa)_{ABCD} \equiv \kappa \phi_{(ABCD)} = 0 \] (71)

will be called a conformal Killing spinor initial data set (CKSID) on \( \mathcal{U} \).

While conditions (71) are clearly necessary for \( \kappa_{AB} \) to be a Killing spinor, we will only be able to prove their sufficiency under an additional, albeit minor, assumption; see the statement of Lemma 3 below. We begin with the following:

**Lemma 2.** Suppose we have an initial data for the conformal Einstein field equations on \( \mathcal{U} \subset \mathcal{S} \) where \( \mathcal{S} \) is a spacelike hypersurface. Let \( \kappa_{AB} \) be a CKSID on \( \mathcal{U} \) which moreover satisfies the equations

\[ \bar{\kappa}(A^F D_B \phi_{CD})_F G + \phi_{(ABC} F \bar{\xi}_{D)}_F = 0, \] (72a)

\[ \bar{\xi}^F G D_{FG} \phi_{ABCD} + \frac{1}{2} \bar{\xi} D_{(A^F \phi_{BCD})_F} = 0, \] (72b)

where \( \bar{\xi}_{AB} \equiv D(A^C \kappa_{BC}) \) and \( \bar{\xi} \equiv D_{AB} \kappa_{AB} \). Then the resulting solution \( \kappa_{AB} \) of system (59) satisfies

\[ H(\kappa)_{A'B'C'} = B(\kappa)_{A'B'C'} = F(\kappa)_{A'B'C'} = \nabla_{\tau} H(\kappa)_{A'B'C'} = \nabla_{\tau} B(\kappa)_{A'B'C'} = \nabla_{\tau} F(\kappa)_{A'B'C'} = 0 \text{ on } \mathcal{U}. \]

**Proof.** First we note that for \( \kappa_{AB} \) satisfying equation (59), the following holds

\[ \xi_{AB}|_{\mathcal{U}} = -\frac{1}{2} \tau^B B \xi_{AB} + \xi_{AB}^B, \] (73)

with \( \xi := D_{AB} \kappa_{AB} \) and where, recall, \( \xi_{AB} = D_{(A^C \kappa_{BC})} \). To see this, note that

\[ \xi_{AB} = -\tau_B A^A \tau^B B \xi_{AB} = -\tau_B A^A \tau^B B \nabla^C A^B \kappa_{AC} = \frac{1}{2} \tau_B^A \nabla_{\tau} \kappa_{AB} + \tau_B^B D_{BC} \kappa_{AC} = \frac{1}{2} \tau_B^B \nabla_{\tau} \kappa_{AB} + \tau_B^B \xi_{AB} + \xi_{AB} \] (74)

where we are decomposing the covariant derivative. Finally, substituting the initial condition from (59), we obtain equation (73). Note also that \( \xi = \xi, \xi_{AB} = \xi_{AB} \) on \( \mathcal{U} \) as a result of the initial condition \( \kappa_{AB} = \bar{\kappa}_{AB} \). Starting from equation (52), performing the decomposition of the covariant derivative and substituting the Killing spinor candidate equation \( Q_{AB} = 0 \), we get

\[ \nabla_{\tau} H_{B'A'B'C'} + 2 \tau^D B'A' \mathcal{H}_{DF} H_{A'B'C'D} = -12 \tau^D B'A' \mathcal{H}_{B'A'B'C'D}. \]

At this point we see that, since the CKSID conditions hold by assumption, then

\[ \nabla_{\tau} H_{B'A'B'C'}|_{\mathcal{U}} = 0. \]
also. Now, a similar computation to the twistor case yields
\[ \nabla_\tau B_{ABCD}|_U = 2\kappa(A^F D_B G^F \phi)_{FG} + \phi(ABC^F \xi D)_{F} \]  
where we are again making use of equations (15) and (73). Note that the quantity on the right-hand-side is intrinsic to \( S \). Hence, if we assume (72a), then we have
\[ \nabla_\tau B_{ABCD}|_U = 0 \]  
(76)

Consider now the quantity \( F_{A'ABC} \). Recall that, by definition, \( F_{A'ABC} = \nabla D_{A'} B_{ABCD} \), and so decomposing the covariant derivative one obtains
\[ F_{A'BCD} = \frac{1}{2} \tau_{A'} \nabla_\tau B_{ABCD} - \tau_{A'} \nabla_{A'} B_{ABCD} \]

(77)

Decomposing the covariant derivative and imposing the Killing spinor candidate equation, \( Q_{AB} = 0 \), we get
\[ \nabla_\tau F_{B'BCD} + 2\tau_{A'F^A' D_{A'} F_{A'BCD}} = \tau_{A'} \nabla_{A'}(\frac{1}{4} \nabla_\xi \phi_{BCD}) - 4\Omega_{BCD} + 6\Omega_{(BC)(DF)AG} - 14\Omega_{(BC)(DF)AG} + \frac{2}{3} H_{A'}(B^F G \nabla_{AF')\phi_{CDG})}. \]  
(78)

Hence, if \( F_{A'ABC}|_U = H_{A'ABC}|_U = B_{ABCD}|_U = 0 \), then \( \nabla_\tau F_{A'ABC}|_U = 0 \) if and only if
\[ \nabla_\tau \phi_{ABC}|_U = 0 \]

Decomposing the covariant derivative, and using the evolution equation (15) for \( \phi_{ABCD} \) again, along with equation (73), we have
\[ \nabla_\tau \phi_{ABC}|_U = \xi_{D} (A^F \phi_{BCD})_{F} + \frac{2}{3} \xi_{FG} \phi_{FG} \phi_{ABCD} \]

Hence, if we assume (72b) to hold, then
\[ \nabla_\tau F_{B'BCD}|_U = 0, \]
and the result of Lemma 2 follows.

Condition (72a) is in fact the “unphysical” counterpart of the condition appearing in [19], which was later shown to be redundant in [3], modulo a minor algebraic assumption on the Killing spinor initial data—see (i) and (ii) in the lemma below. Although no such counterpart of (72b) appears in the physical case, this same algebraic assumption ensures redundancy of both (72a) and (72b):

Lemma 3. Suppose that \( \kappa_{AB} \) is CKSID set on \( U \) satisfying one of the following two conditions on \( U \):

(i) \( \kappa_{AB} \kappa^{AB} = 0 \) but \( \kappa_{AB} \neq 0 \), or

(ii) \( \kappa_{AB} \kappa^{AB} = 0 \).

Then conditions (72a) and (72b) are redundant in the sense that they are automatically satisfied by \( \kappa_{AB} \) by virtue of the CKSID conditions (71).
The proof of this lemma requires decomposing the fields respect to a spin dyad and considering the cases where $\phi$ is of different Petrov types. This is a long but direct calculation that is given in Appendix A.2.

**Remark 9.** Note that if it is not the case that $\bar{\kappa}_{AB}\bar{\kappa}^{AB} \equiv 0$ on $\mathcal{U}$, then there must exist an open subset $\emptyset \neq \mathcal{U} \subset \mathcal{U}$ on which (ii) holds, since the vanishing of $\bar{\kappa}_{AB}\bar{\kappa}^{AB}$ is a closed condition by the assumed continuity (in fact, differentiability) of the Killing spinor candidate. Hence, the conditions of Lemma 4 imply, at worst, a restriction the of domain of applicability.

Finally, putting together Proposition 2, Lemmas 2 and 3 gives the following valence-2 analogue of Theorem 2.

**Theorem 3.** Consider an initial data set for the vacuum conformal Einstein field equations, as encoded in the CFE zero-quantities (12a)–(12d), on a spacelike hypersurface $\mathcal{S}$ and let $\mathcal{U} \subset \mathcal{S}$ be an open set. If there exists a CKSID set $\bar{\kappa}_{AB}$ on $\mathcal{U}$ satisfying either of conditions (i) or (ii) from Lemma 3, then the development of the initial data set admits a Killing spinor on an open neighbourhood $\mathcal{W}$ of $\mathcal{U}$, with $\mathcal{W} \subseteq \mathcal{D}^{+}(\mathcal{U})$, given by the solution $\kappa_{AB}$ of the initial value problem (29). If condition (i) holds, then in fact there exists a twistor on $\mathcal{W}$.

**Proof.** Given such a $\bar{\kappa}_{AB}$, Lemmas 2 and 3 together imply that the Killing spinor candidate $\kappa_{AB}$ constructed as a solution of (69) satisfies conditions (68a)–(68b). Proposition 2 then implies that $H_{\lambda\gamma\delta\epsilon} = 0$ on an open neighbourhood $\mathcal{W}$ of $\mathcal{U}$, with $\mathcal{W} \subseteq \mathcal{D}^{+}(\mathcal{U})$, and hence that $\kappa_{AB}$ is indeed a Killing spinor on $\mathcal{W}$. In particular, if condition (i) holds then $\bar{\kappa}_{AB} = \bar{\kappa}_{A}\bar{\kappa}_{B}$ for some $\bar{\kappa} \neq 0$ on $\mathcal{U}$. It is straightforward to verify that $\bar{\kappa}_{A}$ solves the conformal twistor initial data conditions, equations (37); see the discussion of Appendix A.2. Theorem 2 then implies the existence of a twistor on $\mathcal{W}$.

**Remark 10.** Note that we recover the conditions from [1], namely

$$\bar{D}_{(AB}\bar{\kappa}_{CD)} = 0, \quad \bar{\kappa}_{(A}F_{B}C)F_{D)} = 0,$$

when $\Xi \equiv 1$ on $\mathcal{U}$.

Given the close connection between the notion of algebraic special Petrov types and the existence of Killing spinors, discussed in section 2, it is not surprising that one can use the previous result to establish conditions under which the Petrov type of an initial data set is “propagated” to the resulting spacetime development; see Theorem 3 of [19] for a similar result in the physical framework case. This is the content of the following corollary:

**Corollary 1.** Given initial data for the CFEs, suppose that the initial data for the rescaled Weyl spinor $\phi_{ABCD}$ is of Petrov type D on $\mathcal{U} \subset \mathcal{S}$ and suppose further that

$$\sigma \equiv -\phi^{A}\phi^{B}\phi^{C}D_{ABC} = 0 \quad \text{and} \quad \lambda \equiv \psi^{A}\psi^{B}D_{ABC} = 0$$

(79)

hold on $\mathcal{U}$, where $\{o, t\}$ is an adapted spin dyad in terms of which $\phi_{ABCD} = \phi_{0}(\phi_{B}^{C}(\phi_{C}^{D}))$ for some field $\phi : \mathcal{U} \rightarrow \mathbb{C}$. Then, the rescaled Weyl spinor $\phi_{ABCD}$ of the corresponding development $(\mathcal{M}, g)$ is of Petrov type D on an an open neighbourhood $\mathcal{W}$ of $\mathcal{U}$, with $\mathcal{W} \subseteq \mathcal{D}^{+}(\mathcal{U})$, as evidenced by the existence of a non-trivial (valence-2) Killing spinor satisfying $\kappa_{AB} \neq 0$.

**Proof.** Suppose that the initial data for the rescaled Weyl spinor $\phi_{ABCD}$ is of Petrov type D on $\mathcal{U}$, with $\{o, t\}$ an adapted spin dyad in terms of which $\phi_{ABCD} = \phi_{0}(\phi_{B}^{C}(\phi_{C}^{D}))$ for some field $\phi$ on $\mathcal{U}$. Note that $\phi \neq 0$ everywhere on $\mathcal{U}$; if $\phi = 0$ at some $p \in \mathcal{U}$ then $\phi_{ABCD}|_{p}$ would be of type O at $p$. Define $\bar{\kappa}_{AB} = \phi^{-1/3}\phi_{0}(\phi_{A}^{B})$ and note that $\bar{\kappa}_{AB} \neq 0$ on $\mathcal{U}$. A short calculation verifies that the Buchdahl constraint $B(\bar{\kappa})_{ABCD} = 0$ is satisfied. The equation $D_{(AB}\bar{\kappa}_{CD)} = 0$, on the other hand, decomposes to give

$$\sigma = 0,$$

The fields $\sigma$ and $\lambda$ (not to be confused with the cosmological constant), along with $\kappa, \tau, \rho, \mu, \nu, \pi$ to follow, are Newmann–Penrose (NP) scalars [34] [35]; see the Appendix for further details.
\begin{align*}
\mathcal{D}_{00}\phi &= 3\phi(\kappa - \tau), \\
\mathcal{D}_{01}\phi &= \frac{3}{4}\phi(\rho + \mu), \\
\mathcal{D}_{11}\phi &= 3\phi(\nu - \pi), \\
\lambda &= 0,
\end{align*}

in terms of the NP scalars — to see this, substitute \( e^{A} = \phi^{-1/3} \) in equations (86a)–(86c) of the Appendix. The first and last components are guaranteed by the assumption (70). The remaining equations are precisely those implied by the constraint \( \mathcal{D}^{AB}\phi_{ABCD} = 0 \) — see (88) of the Appendix. Hence, the conformal Killing spinor initial data conditions (1) are met by \( \bar{\kappa}_{AB} \) and propagating this data using (69) we obtain a Killing spinor \( \bar{\kappa}_{AB} \) on \( \mathcal{D}^{+}(U) \), according to Theorem 11. Now, by continuous dependence on the data, it follows that there exists some open subset \( V \subset \mathcal{D}^{+}(U) \) on which \( \kappa_{AB} \neq 0 \). Since the Buchdahl constraint necessarily holds on \( V \) and since \( \kappa_{AB} \neq 0 \), it follows that either \( \phi_{ABCD} \) is of Petrov type D or O on \( V \). Now \( \phi_{ABCD} \propto \phi^2 \neq 0 \) on \( U \) and so by continuity \( \phi_{ABCD} \propto \phi^2 \neq 0 \) on some open \( \tilde{V} \subset V \). Therefore, \( \phi_{ABCD} \) cannot be of type O anywhere on \( V \) and so must be of type D.

**Remark 11.** While the constraint \( \mathcal{D}^{AB}\phi_{ABCD} = 0 \) implies that the 0001, 0011 and 1111-components of \( \mathcal{D}_{(AB}\bar{\kappa}_{CD)} = 0 \) are all satisfied, as shown in the above proof, the 0000 and 1111-spin components are not guaranteed simply as a consequence of the expression \( \bar{\kappa}_{AB} = \phi^{-1/3}o_{A(B)} \). Hence, the conditions of Theorem 11 are more restrictive than simply assuming that \( \phi_{ABCD} \) be of Petrov type D on \( U \). These remaining two components of \( \mathcal{D}_{(AB}\bar{\kappa}_{CD)} = 0 \), namely \( \sigma = \lambda = 0 \) are however implied by the 0000 and 1111-components of the evolution equation \( \nabla_F\phi_{ABCD} = 2\mathcal{D}_F\phi_{ABCD}F \) if one assumes that the Petrov type extends to the spacetime development, consistent with Remark 11.

**Conclusions**

In this article a conformal version of the Killing spinor initial data equations given in [19], namely equations (71), are derived. By conformal it is understood that \( (\mathcal{M}, g) \) is conformally related to a vacuum Einstein spacetime \( (\mathcal{M}', \bar{g}) \). It is shown that, modulo a minor technical assumption, the existence of a non-trivial solution of equations (71) is a necessary and sufficient condition for the existence of a Killing spinor on the development in the unphysical spacetime \( (\mathcal{M}, g) \), as given by a solution of Friedrich’s conformal Einstein field equations. The initial data equations (71) are comprised of one differential condition and one algebraic condition. The differential condition corresponds to the so-called spatial Killing spinor equation while the algebraic condition is a restriction imposed by the Buchdahl constraint on the initial hypersurface. This constraint can be interpreted as restricting the Petrov type of the initial data set for the conformal Einstein field equations. Although conditions (71) look formally identical to those derived for the physical spacetime \( (\mathcal{M}, \bar{g}) \), only with the Weyl spinor being replaced by the rescaled Weyl spinor, the derivation of these conditions in an unphysical spacetime \( (\mathcal{M}, g) \) is non-trivial, owing to the non-trivial behaviour of the Einstein field equations under conformal rescaling.

In the case where the conformal rescaling is trivial i.e. \( \Xi \equiv 1 \), we recover the results of [19]. However, even for this case the set of variables to be propagated in the physical and the unphysical spacetimes are different. This difference can be traced back to the observation that for a general Lorentzian manifold the vector \( \xi_{AA'} = \nabla^B\bar{A}_KAB \) is not a Killing vector. Furthermore, even in the case where \( (\mathcal{M}, g) \) satisfies the conformal Einstein field equations, this vector does not correspond to a conformal Killing vector as one could naively expect but rather to a collineation for the rescaled Weyl tensor/spinor, as shown here. Naturally, once the existence of a Killing spinor in the unphysical spacetime \( (\mathcal{M}, g) \) is established one can always construct, a posteriori, a conformal Killing vector \( \hat{X}_{AA'} \) on \( (\mathcal{M}, g) \) which corresponds to a Killing vector \( \xi_{AA'} \) of the physical spacetime \( (\mathcal{M}, \bar{g}) \). Notice that the conformal approach followed in this article i.e. the use of the conformal Einstein field equations, allows for the possibility of \( S \) intersecting non-trivially with (or even being a subset of) null infinity \( \mathcal{I} \). One possible application is the characterisation of asymptotic initial data — initial data for the conformal Einstein field equations given on \( \mathcal{I} \) — for
de Sitter-like spacetimes and, in particular, an asymptotic characterisation of the Kerr-de Sitter spacetime, via the existence of Killing spinors at the (spacelike) conformal boundary. Such a result would mirror the spinorial characterisations of Kerr given in [1, 2, 3, 4]. It would also be of interest to compare with [20, 21], in which the propagation of Petrov type is explored in the context of both the Einstein and the conformal Einstein field equations.

The applications of the core analysis of this article, however, are not restricted to the study of de Sitter-like spacetimes. Indeed, the most taxing part of the procedure, generally speaking, consists of finding a closed system of homogeneous wave equations for the relevant zero-quantities, these equations being irrespective of the causal nature of . For the spacelike case, the uniqueness result for solutions of homogeneous wave equations given by Theorem 1 ensures that if trivial initial data on is provided then vanish on the domain of dependence of the data. Analogous theorems for the characteristic problem or the initial boundary value problem could be used to obtain similar conditions on a null or timelike hypersurface, as has been done in the case of the Killing vector initial data equations and the physical Killing spinor initial data equations —see [33, 8] and [6]. In the case of the characteristic problem, the conformal approach of this article would facilitate the analysis of the existence of Killing spinors at the conformal boundary of an asymptotically flat spacetime. In the case of a timelike hypersurface, the analogous conformal Killing spinor initial data equations could potentially be used in the analysis and characterisation of anti-de Sitter like spacetimes.

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A Appendix

Here we give the derivation of some of the key identities used in section 4.3 and the proof of Lemma 3 from section 4.4.

A.1 Spinorial identities

Here we give the derivation of identities (62) and (63). First, by expanding out the definition of ,

\[
\nabla^D G^G B_{ABFD} = -\frac{1}{4} \phi_{ABFD} \nabla_G \kappa^{CD} - \frac{1}{4} \kappa (A \nabla_{D[G}] \phi_{BF]C})^D \\
- \frac{1}{4} \kappa^{CD} \nabla_G \phi_{ABFC} + \frac{1}{4} \kappa (G[A \nabla_{D]F} C)]^C \\
= \frac{1}{4} \kappa^{CD} \nabla_G \phi_{ABFC} + \frac{1}{4} \kappa (G[A \nabla_{D]F} C)]^C \\
- \frac{1}{4} \kappa^{CD} \nabla_G \phi_{ABFC}
\]

the second equality following from \( \nabla^A \phi_{ABCD} = 0 \). Using (99), the definition of \( F_{A'ABC} \) and rearranging,

\[
\kappa^{CD} \nabla_G \phi_{ABFC} = 2 \kappa^{CD} \phi_{ABFC} + \kappa (G[A \nabla_{D]F} C)]^C - \frac{1}{4} \kappa^{CD} \nabla_G \phi_{ABFC}
\]

one calculates

\[
\kappa_F^D \phi_{ABCD} = B_{ABCF} + \frac{1}{4} \kappa^{DG} \phi_{BCDF} + \frac{1}{4} \kappa^{DG} \phi_{ACDG} \epsilon_{BF} + \frac{1}{4} \kappa^{DG} \phi_{ABDG} \epsilon_{CF},
\]

Then, using the irreducible decomposition

\[
\kappa_F^D \phi_{ABCD} = B_{ABCF} + \frac{1}{4} \kappa^{DG} \phi_{BCDF} + \frac{1}{4} \kappa^{DG} \phi_{ACDG} \epsilon_{BF} + \frac{1}{4} \kappa^{DG} \phi_{ABDG} \epsilon_{CF},
\]

one calculates

\[
\kappa_A^F \nabla_G F \phi_{BCDF} = \nabla_G (\kappa_A^F \phi_{BCDF}) - \phi_{BCDF} \nabla_G \kappa_A^F \\
= \nabla_G B_{ABCD} - \frac{1}{4} \kappa^{FH} \epsilon_{A(B} \nabla_G [(\phi_{CD}] F_H)
\]
\[- \phi_{BCDF} \nabla_G F \kappa^F - \frac{1}{2} \epsilon(B \phi_{CD}) F H \nabla_G F \kappa^F \]
\[- \nabla_G F B_{ABC} + \frac{1}{2} \kappa^F \phi_{BCDF} - \frac{1}{2} \phi_{BCDF} F H \epsilon^A + \frac{1}{2} \kappa^F \epsilon(A \phi_{CD}) G F \]
\[- \frac{1}{2} \epsilon(A \phi_{CD}) F H \kappa^F \phi_{BCDF} \epsilon^A - \frac{1}{2} \kappa^F \epsilon(B \phi_{CD}) F H \kappa^F \phi_{BCDF} \epsilon^A \]

where the third equality follows from (49). Then swapping indices on the \( \nabla \phi \) terms and using \( \nabla^A \phi_{ABCD} = 0 \), one obtains (62):

\[ \kappa^A \nabla_F \phi_{BCD} = \frac{1}{4} \epsilon(A \phi_{BDF}) \phi_{C} \nabla^A \phi_{ABCD} - \frac{1}{4} \epsilon(A \phi_{BDF}) \phi_{C} \nabla^A \phi_{ABCD} - \frac{1}{4} \epsilon(A \phi_{BDF}) \phi_{C} \nabla^A \phi_{ABCD} \]

To show (63), we first apply \( \nabla^A \phi \) to identity (65), using (49) and \( \nabla^A \phi_{ABCD} = 0 \), one obtains

\[ \frac{1}{2} \kappa^A \phi_{(BF) \phi_{DG} \phi_{C} \phi_{ADG}} + \frac{1}{12} \phi_{ADGH} \phi_{BCFG}^H \nabla_{H} \phi_{BCF}^A + \frac{1}{4} \phi_{(BF) \phi_{DG} \phi_{C} \phi_{ADG}} \]

Then, using (62), along with the irreducible decomposition

\[ \phi_{BCDF} G^D = \frac{1}{2} \phi_{CDHL} \phi_{CDHL} \epsilon_{AG} \epsilon_{BF} + \frac{1}{2} \phi_{CDHL} \phi_{CDHL} \epsilon_{AF} \epsilon_{BG} + \phi_{(AB) \phi_{CD} \phi_{FG}} \]

one finally obtains (63):

\[ \kappa^A \phi_{ADG} \nabla_{H} \phi_{BCF}^A = 4 \kappa^A \phi_{(BF) \phi_{DG} \phi_{C} \phi_{ADG}} + \frac{1}{2} \phi_{ADGH} \phi_{CDHL} \epsilon_{AG} \epsilon_{BF} + \frac{1}{2} \phi_{CDHL} \phi_{CDHL} \epsilon_{AF} \epsilon_{BG} + \phi_{(AB) \phi_{CD} \phi_{FG}} \]

A.2 Proof of Lemma 3

Assuming \( \kappa_{AB} \neq 0 \), the Buchdahl constraint restricts the Petrov type of \( \phi \) to be type D, N, or O. We follow the same strategy as in Lemma 3, expanding out conditions (72a)–(72b) in an adapted spin dyad, considering separately the cases (i): \( \kappa_{AB} \kappa^{AB} \equiv 0 \), \( \kappa_{AB} \neq 0 \) and (ii): \( \kappa_{AB} \kappa^{AB} \neq 0 \) on \( U_1 \), from Lemma 3, corresponding to Petrov types N and D, respectively. Observe that for Type O, for which \( \phi_{BCD} = 0 \), the proof of Lemma 3 trivialises, so only the types N and D are needed.

Recalling that \( \mathcal{D}_{AB} := \tau(A^A \nabla_B) \), a straightforward computation yields

\[ o^{A} o^{B} o^{C} \mathcal{D}_{ABQ} = - \sigma, \]
\[ o^{A} o^{B} o^{C} \mathcal{D}_{ABQ} = o^{A} o^{B} o^{C} \mathcal{D}_{ABQ} = - \beta, \]
\[ o^{A} o^{B} o^{C} \mathcal{D}_{ABQ} = \frac{1}{2} (\kappa - \tau), \]
\[ o^{A} o^{B} o^{C} \mathcal{D}_{ABQ} = o^{A} o^{B} o^{C} \mathcal{D}_{ABQ} = \frac{1}{2} (\epsilon - \gamma), \]
\[ o^{A} o^{B} o^{C} \mathcal{D}_{ABQ} = o^{A} o^{B} o^{C} \mathcal{D}_{ABQ} = \rho, \]
\[ o^{A} o^{B} o^{C} \mathcal{D}_{ABQ} = o^{A} o^{B} o^{C} \mathcal{D}_{ABQ} = \alpha, \]
\[ o^{A} o^{B} o^{C} \mathcal{D}_{ABQ} = o^{A} o^{B} o^{C} \mathcal{D}_{ABQ} = - \mu, \]
\[ o^{A} o^{B} o^{C} \mathcal{D}_{ABQ} = o^{A} o^{B} o^{C} \mathcal{D}_{ABQ} = \lambda, \]

where we are following the conventions of [34] in the definition of the Newmann–Penrose (NP) scalars \( \alpha, \beta, \epsilon, \gamma, \kappa, \mu, \lambda, \rho, \tau, \sigma, \nu, \pi. \)
The Buchdahl constraint then implies that
\[ Dφ = φo(φB)C, \]  
and follow easily from \( ε_{AB} = o_{AB} - o_{B,A} \).

A.2.1 Case I: \( \kappa_{AB} = 0, \kappa_{AB} \neq 0 \) on \( \mathcal{U} \)

The assumption on \( \kappa_{AB} \) imply that there exists a spin dyad \( \{o, \ell\} \) on \( \mathcal{U} \) such that \( \kappa_{AB} = o_{AB} \).

The Buchdahl constraint then implies that
\[ φ_{ABCD} = φo_{AB}o_{CD}D, \]  
for some scalar field \( φ : \mathcal{U} \to \mathbb{C} \) and hence that the curvature is of Petrov type \( N \) on \( \mathcal{U} \). Note that \( φ_{ABCD}D = 0 \), the equation \( D(φ_{ABCD}) = 0 \) implies
\[ o^A^B^C^Dφ_{ABCD} = o^A^B^C^Dφ_{ABCD} = A^B^C^Dφ_{ABCD} - o^A^B^C^Dφ_{ABCD} = 0, \]

implying that \( D(φ_{ABCD}) = 0 \) — that is to say that \( o_A \) is a twistor candidate. In terms of the NP scalars, the above read as follows
\[ \sigma = −β + κ - τ = ε - γ + ρ = α = 0. \]  
(83)

Using these relations, we obtain
\[ \xi = -3β, \quad ξ_{AB} = 2ρo_{AB} - 2βo_{(AB)} \]  
(84)

The non-trivial component of the constraint \( D(φ_{ABCD}) = 0 \) reduces to
\[ D_{00} = \frac{5}{2}(2β + κ - τ) = 5β. \]  
(85)

Then, substituting (84), condition (72a) reduces to
\[ o^{A^B^C^D}(2\kappa_{[F^G^H^I]F^G^H^I} + φ_{(ABCD)}F) = \frac{1}{2}φσ = 0, \]
\[ ε_{C^D}^{A^B^C^D}(2\kappa_{[F^G^H^I]F^G^H^I} + φ_{(ABCD)}F) = \frac{1}{2}φ(β - κ + τ) = 0, \]
with the second equality following from (83) and with all other components vanishing trivially. These are essentially the same computations as in [3]. On the other hand, substituting (84), condition (72b) reduces to
\[ o^{A^B^C^D}(2\kappa_{[F^G^H^I]F^G^H^I} + φ_{(ABCD)}F) = \frac{1}{2}φσ = 0, \]
\[ ε_{C^D}^{A^B^C^D}(2\kappa_{[F^G^H^I]F^G^H^I} + φ_{(ABCD)}F) = \frac{1}{2}φ(β - κ + τ) = 0, \]
\[ ε_{C^D}^{A^B^C^D}(2\kappa_{[F^G^H^I]F^G^H^I} + φ_{(ABCD)}F) = 2ρ(φ) - 10βρφ = 0, \]
where we are again using (83) and (84). All other components vanish trivially. Hence, in this case, both conditions (72a) and (72b) trivialise.

A.2.2 Case II: \( \kappa_{AB} \neq 0 \) on \( \mathcal{U} \)

There exists a spin dyad \( \{o, \ell\} \) such that \( \kappa_{AB} = εφ_{o(AB)} \) for some \( x : \mathcal{U} \to \mathbb{C} \). The Buchdahl constraint then implies that the rescaled Weyl spinor takes the form
\[ φ_{ABCD} = φo_{AB}o_{CD}, \]  
for some scalar field \( φ : \mathcal{U} \to \mathbb{C} \) and hence that the curvature is of Petrov type \( D \) on \( \mathcal{U} \). The equation \( D(φ_{ABCD}) = 0 \) is equivalent to
\[ \sigma = 0, \]  
(86a)
\[ D_{00} \pi = \tau - \kappa, \quad (86b) \]
\[ D_{01} \pi = -\frac{1}{2}(\rho + \mu), \quad (86c) \]
\[ D_{11} \pi = \pi - \nu, \quad (86d) \]
\[ \lambda = 0. \quad (86e) \]

Using the above, the auxiliary spinors can be written as
\[ \bar{\xi} = -\frac{1}{2}e^{\pi}(\rho + \mu), \quad (87a) \]
\[ \xi_{AB} = e^{\pi}(\nu - \pi)\delta_{AB} + e^{\pi}(\rho - \mu)\delta_{(A)}\delta_{(B)} + e^{\pi}(\tau - \kappa)\pi_{(A)}\pi_{(B)}. \quad (87b) \]

The constraint \[ D^{CD}\phi_{CDAB} = 0 \] is equivalent to
\[ D_{00}\phi = 3\phi(\kappa - \tau), \quad D_{01}\phi = \frac{3}{2}\phi(\rho + \mu), \quad D_{11}\phi = 3\phi(\nu - \pi). \quad (88) \]

Then, substituting (87a) and (87b), condition (72a) decomposes as follows
\[ o^{A}o^{B}o^{C}o^{D}(2\bar{R}(A)\bar{D}^{G}\phi_{CD}FG + \phi_{(ABC)}\bar{\xi}_{DF}) = -\frac{1}{2}e^{z}\phi\sigma = 0, \]
\[ o^{A}o^{B}o^{C}o^{D}(\bar{\xi}^{FG}D_{FG}\phi_{ABCD} + \frac{3}{2}\bar{D}(A)\phi_{BCDG}) = \frac{1}{2}e^{\pi}D_{00}\phi + \frac{1}{3}e^{\pi}\phi(\tau - \kappa) = 0, \]
\[ o^{A}o^{B}o^{C}o^{D}(2\bar{R}(A)\bar{D}^{G}\phi_{CD}FG + \phi_{(ABC)}\bar{\xi}_{DF}) = \frac{1}{2}e^{\pi}D_{01}\phi - \frac{1}{2}e^{\pi}\phi(\rho + \mu) = 0, \]
\[ o^{A}l^{B}l^{C}l^{D}(2\bar{R}(A)\bar{D}^{G}\phi_{CD}FG + \phi_{(ABC)}\bar{\xi}_{DF}) = \frac{1}{2}e^{\pi}D_{11}\phi + \frac{1}{3}e^{\pi}\phi(\pi - \nu) = 0, \]
\[ l^{A}l^{B}l^{C}l^{D}(2\bar{R}(A)\bar{D}^{G}\phi_{CD}FG + \phi_{(ABC)}\bar{\xi}_{DF}) = -\frac{1}{2}e^{\pi}\phi\lambda = 0, \]

the equality with zero following from (88). Again, these are essentially the same computations as in [3]. On the other hand, substituting (87a) and (87b) into (72a),
\[ o^{A}o^{B}o^{C}o^{D}(\bar{\xi}^{FG}D_{FG}\phi_{ABCD} + \frac{3}{2}\bar{D}(A)\phi_{BCDG}) = \frac{1}{2}e^{\pi}(\rho + \mu)\phi_{00} + \frac{1}{3}e^{\pi}\phi(\tau - \kappa)(\rho + \mu) + \frac{1}{2}e^{\pi}\phi(\phi - \nu)\sigma = 0, \]
\[ o^{A}o^{B}o^{C}o^{D}(\bar{\xi}^{FG}D_{FG}\phi_{ABCD} + \frac{3}{2}\bar{D}(A)\phi_{BCDG}) = \frac{1}{2}e^{\pi}\phi_{00} + \frac{1}{3}e^{\pi}\phi(\tau - \kappa)(\rho + \mu) + \frac{1}{2}e^{\pi}\phi(\phi - \nu)\sigma = 0, \]
\[ o^{A}o^{B}o^{C}o^{D}(\bar{\xi}^{FG}D_{FG}\phi_{ABCD} + \frac{3}{2}\bar{D}(A)\phi_{BCDG}) = \frac{1}{2}e^{\pi}\phi_{00} + \frac{1}{3}e^{\pi}\phi(\tau - \kappa)(\rho + \mu) + \frac{1}{2}e^{\pi}\phi(\phi - \nu)\sigma = 0, \]
\[ o^{A}o^{B}o^{C}o^{D}(\bar{\xi}^{FG}D_{FG}\phi_{ABCD} + \frac{3}{2}\bar{D}(A)\phi_{BCDG}) = \frac{1}{2}e^{\pi}\phi_{00} + \frac{1}{3}e^{\pi}\phi(\tau - \kappa)(\rho + \mu) + \frac{1}{2}e^{\pi}\phi(\phi - \nu)\sigma = 0, \]
\[ o^{A}o^{B}o^{C}o^{D}(\bar{\xi}^{FG}D_{FG}\phi_{ABCD} + \frac{3}{2}\bar{D}(A)\phi_{BCDG}) = \frac{1}{2}e^{\pi}\phi_{00} + \frac{1}{3}e^{\pi}\phi(\tau - \kappa)(\rho + \mu) + \frac{1}{2}e^{\pi}\phi(\phi - \nu)\sigma = 0, \]

Hence, again, (72a) and (72b) trivialise. Combining the result of this section with the previous, Lemma [3] follows immediately.

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