Covariant evolution of perturbations during reheating in two-field inflation

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Abstract. We develop a covariant method for studying the effects of a reheating phase on the primordial adiabatic and isocurvature perturbations in two-field models of inflation. To model the decay of the scalar fields into radiation at the end of inflation, we introduce a prescription in which radiation is treated as an additional effective scalar field, requiring us to extend the two-field setup into a three-field system. In this prescription, the coupling between radiation and the scalars can be interpreted covariantly in terms of geometrical quantities that parametrize the evolution of a background trajectory in a three-field space. In order to obtain concrete results, we consider two scenarios characterized for having unsuppressed isocurvature fluctuations at the end of inflation: (1) canonical two-field inflation with the product exponential potential, which sources a large negative amount of non-gaussianity and, (2) two-field inflation with an ultra-light field, a model in which the isocurvature mode becomes approximately massless, and its interaction with the curvature perturbation persists during the entire period of inflation. In both cases we discuss how their predictions are modified by the coupling of the scalar fields to the radiation fluid.

Keywords: cosmological perturbation theory, inflation, physics of the early universe

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1 Introduction

Single field slow-roll inflation represents the most successful framework to describe the dynamics of the very early universe. It solves the classical shortcomings of the hot big-bang scenario: the horizon, flatness, and monopole problems [1–6]. It also gives us a mechanism to explain the origin of the large-scale structure (LSS) of the universe and the cosmic microwave background (CMB) anisotropies [7–9]. Within this paradigm, the primordial curvature perturbations are the result of quantum fluctuations of the inflaton scalar field during the inflationary expansion [10–15].

One of the salient points of single field slow-roll inflation is that the evolution of curvature perturbations happens adiabatically. While the inflationary background evolves slowly, curvature perturbations freeze, recording in their amplitude information about the background at the time of horizon crossing. The subsequent evolution of modes is insensitive to the details of the universe’s background evolution, and so the measurements of correlation functions in the CMB provides direct information about the epoch of inflation when the fluctuations where produced.

This simple picture might break down in more realistic versions of inflation. More fundamental theories, where the standard model interactions are unified with the gravitational interaction, such as supergravity and string theory, require the existence of a large number of scalar fields. In these models, many of the scalar fields have a geometrical role, and so they are not necessarily massive at high energies. It is therefore not well understood how these scalar fields (isocurvature fields) could have had a role during inflation, where the typical energy available to fluctuations during horizon crossing is of the order of the grand unification theory scale.
One possibility is that isocurvature degrees of freedom did not affect the evolution of curvature perturbations during inflation after all. In that case, regardless of how complicated the ultraviolet theory underlying inflation is, one can understand the evolution of curvature perturbations in terms of an effective field theory (EFT) describing the evolution of a canonical scalar field. Another possibility is that isocurvature fields did not have a relevant kinematic role during inflation, but their couplings to the inflaton still affected the evolution of curvature perturbations. In such cases, the evolution of curvature perturbations continues to be well described by an effective field theory of a single scalar degree of freedom, but this time the theory contains self-interactions incorporating departures from the canonical picture [16–19]. An example of this, is the appearance of the sound-speed $c_s$ of curvature perturbations that is able to generate potentially large values of equilateral and orthogonal non-Gaussianity [20, 21].

Yet another possibility is that isocurvature degrees of freedom were both coupled to curvature perturbations and kinematically relevant at the time of horizon crossing [22–50]. In this case, the dynamics of curvature perturbations cannot be reduced to a single field EFT, and the computation of correlation functions has to take into account the full multi-field nature of inflation during horizon crossing. The wide range of potentially observable effects in this third possibility (multi-field inflation) is far from having been studied exhaustively.\footnote{For a comprehensive review on multi-field inflation, see refs. [51, 52].} A crucial difference between single-field and multi-field inflation is the presence of isocurvature perturbations. Because of these modes, the slow-roll solution is no longer an attractor, leading to non-vanishing non-adiabatic pressure perturbations and thus possible super-horizon evolution of the primordial curvature perturbation [23, 25, 53].

Irrespective of the number of fields that drove inflation, at the end of the inflationary stage the universe is typically found to be in a highly non-thermal state. The key feature of inflation, allowing it to homogenize the universe, also means that it leaves the universe at an effectively zero temperature. Hence, a successful theory of inflation must also explain how the universe was heated to the high temperatures required by the standard hot big-bang picture. This is achieved at the end of inflation by reheating [54, 55], a process in which the inflaton field starts oscillating around the minimum of its potential. Reheating sets the post-inflationary conditions of the universe before it enters the radiation era. Once the universe enters the radiation era, its evolution is described by the physics of the hot big bang model.

Essentially, all elementary particles are produced during reheating. This makes reheating one of the most fundamental phases of the very early universe. At the end of inflation all the energy stored in the inflaton field is transferred to the thermal energy of the particles produced by the decay of the inflaton. These particles interact and eventually thermalize into equilibrium at a reheating temperature $T_{rh}$ which is rather model-dependent [56]. Once all the energy of the inflaton is transferred to the particles, reheating ends and the Universe starts its radiation era. The details of reheating are sensitive to the inflationary background, choice of parameters, and initial conditions. In many inflationary models, reheating consists of distinct stages with different features, starting with a stage called preheating [57], where parametric resonance of the inflaton field leads to particle production, and then moving to a second stage of thermalization. For a treatment of preheating after multi-field inflation, see refs. [58–62].

A fundamental issue related to reheating is whether it can affect the evolution of super-horizon primordial perturbations. Our models of inflation are tested with the help of CMB...
observations, hence, if reheating modifies how the spectra of perturbations depend on the background during horizon crossing, we might be unable to constrain the role of fundamental theories during inflation. In the case of single field inflation, the conditions for the conservation of curvature perturbations over superhorizon scales are well understood \cite{15, 63}. Satisfying these conditions allows one to constrain certain aspects of reheating with CMB observations \cite{64}.

On the other hand, if additional isocurvature fields (also known as entropy perturbations) remain unsuppressed during reheating the situation may change drastically. This is possible if the effective mass of the entropy modes remain light compared to Hubble rate $H$ during the whole duration of inflation \cite{23, 53}. For instance, it was shown in ref. \cite{65} that parametric resonance during reheating may lead to an exponential amplification of superhorizon fluctuations. This has been confirmed in refs. \cite{29, 66, 67}. However, these works did not take into account the possible coupling of the perturbations (curvature and isocurvature) to radiation and pressureless matter components. This issue was considered in ref. \cite{68}, for a model of multi-field chaotic inflation. There it was found that, as long as the isocurvature modes remain light, the total curvature perturbation may experience large variations due to the non-adiabatic components. For latter studies on reheating after multi-field inflation, see refs. \cite{69, 70}.

It has been pointed out \cite{71} that multi-field scenarios should a priori be considered as non-predictive unless one demands the so called adiabatic limit, in which isocurvature perturbations decay before the end of inflation. The models for which the adiabatic limit is not reached are characterized by having isocurvature modes that remain nearly massless during the whole period of inflation. However, there are well motivated examples of models where the adiabatic limit is not reached, even at the end of inflation. One example is given by the product exponential potential $V(\phi_1, \phi_2) = V_0 \phi_1^2 e^{-\lambda \phi_2^2}$ \cite{72, 73}. Another example is offered by models with ultra-light entropy modes, where the isocurvature mode remains nearly massless thanks to a non-trivial symmetry of the two-field system \cite{50}.

The purpose of this article is to study the behavior of perturbations during reheating in models of inflation where the curvature perturbation interacts continuously with a non-adiabatic mode that remains approximately massless from horizon crossing until reheating \cite{26, 28, 50, 74–76}. We are particularly interested in addressing the possibility of having the fluctuations coupled to an effective radiation component. This possibility was addressed, for instance, in ref. \cite{77}, for the particular case of inflation driven by a single-field coupled to radiation. Notice that in the single-field case, we do not have an isocurvature contribution to modify the evolution of the curvature component during reheating. Here we extend the treatment of ref. \cite{77} to the case of two field models, where an isocurvature field alters the evolution of the curvature fluctuation during reheating. In doing so, we will show that a two-field system plus a radiation component may be effectively treated as a system with three scalar fields. In this way, we may use the covariant formalism in field-space, introduced in \cite{25}, to derive the complete set of background and perturbed equations which describe the universe starting from two-field inflation up to reheating phase. An alternative approach to the one considered here consists of the $\delta N$ formalism \cite{78, 79}, in which one evaluates the evolution of super-horizon perturbations with the help of the homogeneous background solutions under the slow-roll approximation. In particular, this formalism has been used in \cite{69, 80, 81} to study the evolution of large-scale fluctuations through the reheating phase in two-field inflation. However, in order to study the possible effects of deviations from slow-roll inflation as well as the effects on fluctuations at several scales due to the modified dynamics at the end
of inflation, in this work we opted to directly solve the full set of linear equations of motion for the perturbations.

We have organized this article as follows. In section 2, we derive the background equations of motion for a two-field system coupled to a radiation fluid. There we show that the equations of motion may be effectively understood in terms of a three-field system. Then, in section 3 we deduce the equations of motion for gauge invariant perturbations, such as the total curvature perturbation, the curvature perturbation associated with each individual component, and also the equations governing the evolution of the isocurvature modes. In section 4, we present the numerical results based on two inflation models: the product exponential potential and a model of inflation with ultra-light fields. We summarize our findings and present our conclusions in section 5.

2 Two scalar fields plus radiation

Here we introduce a covariant prescription to study two-field systems coupled to a radiation fluid. We will first consider the background equations of motion, and then move on to consider the dynamics for the perturbations (next section).

2.1 Basic background equations

Let us start by considering the most general action describing a two-field system with two space-time derivatives [52] coupled to a radiation field (setting the reduced Planck mass $M_{Pl}^2 = 1$ for simplicity):

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} R - \frac{1}{2} g^{\mu\nu} \gamma_{ab} \partial_\mu \phi^a \partial_\nu \phi^b - V(\phi) + L_{rad} + L_{int} \right].$$

(2.1)

Here $g_{\mu\nu}$ and $R$ are the usual metric and Ricci scalar respectively describing the gravitational sector. On the other hand $\gamma_{ab}$ is the $\sigma$-model metric characterizing the two-dimensional target space spanned by the fields $\phi^1$ and $\phi^2$ (with an inverse given by $\gamma^{ab}$), and $V$ is the scalar field potential of the model. The term $L_{rad}$ describes a radiation component that could be given by any class of field (Gauge bosons, scalars, fermions, etc) as long as their quanta are in a relativistic regime. Since we focus on a cosmological setup, in the present work we represent such a sector with a perfect fluid to simplify the equations of motion. We further assume that radiation interacts with the fields $\phi^a$ through an interaction term $L_{int}$. These contributions will be parameterized through effective terms that will be introduced in what follows. The equations of motion for the scalar fields are found to be:

$$\Box \phi^a + \Gamma^a_{bc} g^{\mu\nu} \partial_\mu \phi^b \partial_\nu \phi^c = V^a + J^a,$$

(2.2)

where $\Box \phi^a = g^{\alpha\beta} \nabla_\alpha \nabla_\beta \phi^a = g^{\alpha\beta} (\partial_\alpha \partial_\beta \phi^a - \Gamma^\gamma_{\alpha\beta} \partial_\gamma \phi^a)$, and $V^a \equiv \gamma^{ab} V_b$ with $V_b = \partial_b V$. In the previous expressions, both $\Gamma^a_{\alpha\beta}$ and $\Gamma^a_{bc}$ correspond to the usual Christoffel symbols for the gravitational and scalar sectors respectively. For instance, in the case of the target space one has $\Gamma^a_{bc} = \frac{1}{2} \gamma^{ad} (\partial_c \gamma_{db} + \partial_b \gamma_{cd} - \partial_d \gamma_{cb})$. Notice that we have introduced a source term $J^a$ that appears as a consequence of $L_{int}$. Hence, the effects of particle production into the radiation fluid are dictated by $J^a$. Now, if we take eq. (2.2) at background level, the Klein-Gordon equation is reduced to:

$$\frac{D \dot{\phi}_0^a}{dt} + 3 H \dot{\phi}_0^a + V^a = -J_0^a,$$

(2.3)
with \( \frac{dX^a}{dt} = \dot{X}^a + \Gamma^a_{bc} \dot{\phi}_b^0 X^c \). The 0-label reminds us that we are dealing with background quantities. Note that while \( V \) depends only on \( \phi_0^0 \), the source term \( J^a_0 \) may depend on both \( \phi_0^0 \) and the background radiation component.

Given that the source \( J^a_0 \) is due to the interaction between the scalar fields and radiation, we need to introduce a background equation of motion describing the evolution of the radiation energy density \( \rho_R^0 \) with a source term. This is given by:

\[
\dot{\rho}_R^0 + 4H\rho_R^0 = Q. \tag{2.4}
\]

Here, \( Q \) describes the energy transfer between the scalar fields and \( \rho_R \). Such a transfer of energy describes the perturbative decay of the fields into the relativistic particles constituting the radiation fluid, hence \( \dot{\rho}_R^0 = \frac{1}{3} \rho_R^0 \). In the single field case, to phenomenologically describe the damping of scalar oscillations, an extra friction term is added to the classical equation of motion of the field during the rapid oscillations regime \([54, 55]\):

\[
\ddot{\phi} + 3H\dot{\phi} + V = -\Gamma \dot{\phi}, \tag{2.5}
\]

where \( V_\phi \equiv \frac{\partial V}{\partial \phi} \). This requires that \( Q = \Gamma \dot{\phi}^2 \), where \( \Gamma \) is the decay rate of the inflaton into particles which can be computed using perturbation theory in quantum field theory \([82]\). During the slow-roll regime, the friction term becomes negligible as compared to the Hubble rate, i.e. \( \Gamma \ll 3H \). When \( \Gamma > 3H \) the energy density of the inflaton field decays quickly, becoming subdominant as compared to that of the decay product \( \rho_R^0 \) and reheating is said to be completed. Now, in view of the covariant formalism in our two-dimensional field space, the energy transfer between the scalar fields and the radiation fluid may be constructed from the source term \( J^a \). In order to find the dependence between \( J^a_0 \) and \( Q \) one may contract eq. (2.3) by \( \dot{\phi}_0^0 \) to obtain:

\[
\dot{\phi}_0^0 \ddot{\phi}_0^0 + 3H\dot{\phi}_0^0 + \dot{\phi}_0^a \partial_a V = -J^a_0 \dot{\phi}_0^a, \tag{2.6}
\]

where we have defined \( \dot{\phi}_0^2 \equiv \gamma_{ab} \dot{\phi}_0^a \dot{\phi}_0^b \). Now, in order to have energy-momentum conservation \( \nabla_\mu T_{\mu \nu} = 0 \), we require:

\[
Q = J^a_0 \dot{\phi}_0^a. \tag{2.7}
\]

One way to think about \( Q \) is that it corresponds to the zeroth component of a vector \( Q^\nu = (Q, 0) \) that defines the source for the radiation and scalar field contributions to the energy momentum tensor as \([83]\):

\[
\nabla_\mu T^{\mu \nu} = Q^\nu, \tag{2.8}
\]

\[
\nabla_\mu T^{\phi \nu} = -Q^\nu. \tag{2.9}
\]

Next, it is direct to verify that the Friedmann equation is given by:

\[
3H^2 = \rho^0 = \frac{1}{2} \dot{\phi}_0^2 + V + \rho_R^0. \tag{2.10}
\]

This equation, together with eqs. (2.4) and (2.6) implies the following additional equation for \( \dot{H} \):

\[
\dot{H} = -\frac{\rho^0 + \rho_R^0}{2} = -\frac{1}{2} \left( \dot{\phi}_0^2 + \frac{4}{3} \rho_R^0 \right). \tag{2.11}
\]
Eqs. (2.2), (2.4) and (2.10) are the background equations of motion describing the system. The dependence of $J_\alpha^0$ on the scalar fields and the radiation energy density is model dependent, and needs to be supplied. In this sense, it turns convenient to parametrize the source term in order to obtain the energy transfer $Q$, and then derive an expression for an effective decay rate $\Gamma_{\text{eff}}$, which enters directly in the equations of motion. We will describe this in more detail in the following subsection.

2.2 Standard parametrization of multi-field background trajectories

The impact of the inflationary background on the evolution of perturbations may be studied by the introduction of dimensionless parameters with clear geometrical interpretations. In particular, given that we are dealing with a multi-field trajectory, we will have two classes of useful dimensionless parameters: the first class corresponds to the set of slow-roll parameters, that give us information about the gradual evolution of the quasi-de Sitter background. The second class consists of quantities that parametrize the multi-field nature of the trajectory. In ref. [25] the following set of parameters were introduced in order to deal with general multi-field inflationary models:

$$\varepsilon \equiv -\frac{\dot{H}}{H^2},$$  \hspace{1cm} (2.12)

$$\eta^a \equiv -\frac{1}{H}\frac{D\dot{\phi}_0^a}{dt}.$$  \hspace{1cm} (2.13)

The function $\varepsilon$ corresponds to the usual first slow-roll parameter that describes the rate at which the quasi-de Sitter spacetime change in time. The quantity $\eta^a$, on the other hand, contains mixed information about the evolution of the quasi-de Sitter spacetime and the multi-field inflationary trajectory. To clarify the meaning of $\eta^a$, it is convenient to introduce a basis of vectors aligned with the background inflationary trajectory. In the case of the present two-field system described by eq. (2.3) one may introduce the following two unit vectors, $T^a$ and $N^a$, defined as:

$$T^a = \frac{\dot{\phi}_0^a}{\dot{\phi}_0},$$ \hspace{1cm} (2.14)

$$N^a = -\frac{1}{\Omega} \frac{D T^a}{dt},$$ \hspace{1cm} (2.15)

where $\Omega$ is a factor that keeps $N^a$ normalized, and that may be interpreted as the rate of turn of the inflationary path whenever it bends. Now, these two vectors may be used to project eq. (2.3) along the inflationary trajectory, and orthogonal to it. One way of doing this is to directly compute a covariant time derivative of $T^a$ with the help of the equations of motion (2.3). One finds:

$$\frac{D T^a}{dt} = -\left(\frac{\ddot{\phi}_0}{\dot{\phi}_0} + 3H\right) T^a - \frac{1}{\dot{\phi}_0} \left(V^a + J_0^a\right).$$ \hspace{1cm} (2.16)

Then, projecting this equation along $T^a$, and using the fact that $T^a \frac{D T^a}{dt} = 0$, one finds:

$$\ddot{\phi}_0 + 3H\dot{\phi}_0 + V_T = -\frac{Q}{\dot{\phi}_0},$$ \hspace{1cm} (2.17)

where $V_T = V^a T_a$. Then, by direct comparison with the equation for a single scalar field with a friction term (2.5) we find that the interaction term $Q$ may be written as

$$Q = \Gamma_{\text{eff}}\dot{\phi}_0^2,$$ \hspace{1cm} (2.18)
where $\Gamma_{\text{eff}}$ represents an effective decay rate. On the other hand, the projection of eq. (2.16) along $N^a$ gives:

$$\Omega = \frac{1}{\dot{\varphi}_0} (V_N + J_0^a N_a),$$

(2.19)

where we have defined $V_N = V^a N_a$. Coming back to eq. (2.13), we may decompose $\eta^a$ along the two directions in the following way:

$$\eta^a = T^a \eta_{||} + N^a \eta_{\perp}.$$

(2.20)

Then, it is direct to find that the components $\eta_{||}$ and $\eta_{\perp}$ are given by:

$$\eta_{||} = -\frac{\ddot{\varphi}_0}{H \dot{\varphi}_0} = 3H + \frac{V_T}{\dot{\varphi}_0} + \frac{Q}{\dot{\varphi}_0^2},$$

(2.21)

$$\eta_{\perp} = \frac{\Omega}{H} = \frac{1}{H \dot{\varphi}_0} (V_N + J_0^a N_a).$$

(2.22)

Now, it may be appreciated explicitly that this parametrization, based on eq. (2.13), becomes singular when $\dot{\varphi}_0 \to 0$, which can happen at the end of inflation. In the following subsection we offer a simple extension of this parametrization allowing the essentially the same geometrical interpretation, but that has the benefit of avoiding any singular behavior at $\dot{\phi}_0 \to 0$.

## 2.3 Radiation as an effective scalar

We will now reconsider the previous set of background equations of motion by accommodating the radiation energy density (hereby described as a perfect fluid) as an extra scalar. Given that we are not adopting a particular form for the Lagrangian contributions $L_{\text{rad}}$ and $L_{\text{int}}$, this will be done at the level of equations of motion. The idea is to extend our previous two-dimensional field space, given by eq. (2.3), to a three-dimensional one, where eq. (2.4) is included by introducing a new field $\phi_0^{(3)}$. This means that the real physics is always given by the equations of motion presented in section 2.1, given by the Lagrangian density (2.1), but we will rewrite them like a three scalar fields equations system, without assuming a Lagrangian determining the dynamics of the new field.

In first place, we need to define $\phi_0^{(3)}$ in such a way that it satisfies a Klein Gordon Equation as (2.3). Therefore:

$$\frac{D}{dt} \dot{\phi}_0^{(3)} + 3H \dot{\phi}_0^{(3)} + \mathcal{V}(3) = -J_0^{(3)},$$

(2.23)

where $\mathcal{V}(3)$ represents the derivative with respect to $\phi_0^{(3)}$ of an extended potential $\mathcal{V}$ that will be defined below (see eq. (2.27)). In order to achieve this, an effective field metric and an effective potential with a radiation component must be defined in such a way that they are consistent with eqs. (2.3)–(2.11). The aim is to accommodate this equation as part of the

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It has been shown [84], that a perfect fluid can be modelled with the help of general Lagrangian density, but the scalar field that we will introduce here does not correspond to a fundamental scalar field derived from a variational principle.
set of field equations describing multi-field inflation. With this in mind, we extend the set of two fields \( \phi^a \) to the following set of scalar fields:

\[
\phi_0^A \equiv \begin{pmatrix} \phi_0^a \\ \phi_0^{(3)} \end{pmatrix},
\]

(2.24)

where \( \phi_0^a \) stands for the original two fields. In addition, we extend the \( \sigma \)-model metric \( \gamma_{ab} \) of the two-field target space to:

\[
q_{AB} = \begin{pmatrix} \gamma_{ab} & 0 \\ 0 & q_{(33)} \end{pmatrix},
\]

(2.25)

where \( q_{(33)} \) is an arbitrary function of \( \phi^{(3)} \), such that the covariant time derivative must satisfy \( \frac{Dq_{AB}}{dt} = 0 \). A complete description about the 3D field space is presented in appendix A. Having defined these extended objects, we may achieve eq. (2.23), by identifying the radiation density with the kinetic energy of the new scalar field \( \phi^{(3)} \) in the following way:

\[
\frac{2}{3} \rho_0^R = \frac{1}{2} q_{(33)} \left( \dot{\phi}_0^{(3)} \right)^2.
\]

(2.26)

Then, the continuity equation (2.4) is equivalent to (2.23) as long as the effective potential \( V \) and the source term \( J_0^{(3)} \) are given by

\[
V = V + \frac{1}{3} \rho_0^R, \tag{2.27}
\]

\[
J_0^{(3)} = -\frac{Q}{q_{(33)} \dot{\phi}_0^{(3)}}. \tag{2.28}
\]

In the previous expressions, \( \rho_0^R \) may be thought of as an explicit function of \( \phi_0^{(3)} \), then \( V_0^{(3)} = \frac{1}{3} \frac{\partial \rho_0^R}{\partial \phi_0^{(3)}} \). However, given that \( \phi_0^{(3)} \) is an auxiliary field, the specific dependence of \( \rho_0^R \) on \( \phi_0^{(3)} \) is in fact irrelevant. Instead, what matters is that the radiation density is determined by the scalar field \( \phi_0^{(3)} \) in such a way that eq. (2.26) is satisfied. Then, the auxiliary field \( \phi_0^{(3)} \) satisfies eq. (2.23), where multiplied by \( q_{(33)} \dot{\phi}_0^{(3)} \) is equivalent to (2.4), and (2.26) is a consistency equation.

Now, the full system consisting of two scalar fields plus radiation may be formally described through a covariant Klein-Gordon equation for three scalar fields given by:

\[
\frac{D\phi_0^A}{dt} + 3H\dot{\phi}_0^A + V^A = -J_0^A, \tag{2.29}
\]

\[
3H^2 = \frac{1}{2} \ddot{\phi}_0^2 + V, \tag{2.30}
\]

\[
\dot{H} = -\frac{1}{2} \ddot{\phi}_0^2, \tag{2.31}
\]

where we have defined:

\[
\ddot{\phi}_0^2 \equiv q_{AB} \dot{\phi}_0^A \dot{\phi}_0^B = \dot{\phi}_0^2 + \frac{4}{3} \rho_0^R. \tag{2.32}
\]
This time, the covariant time derivative acts on a given vector $X^A$ as:

$$\frac{DX^A}{dt} = \dot{X}^A + \Gamma^A_{BC} \phi^B \dot{X}^C,$$

(2.33)

where the Christoffel symbols are now represented by $\Gamma^A_{BC} = \frac{1}{2} q^{AD} (\partial_C q_{DB} + \partial_B q_{CD} - \partial_D q_{BC})$.

In addition, the source term $J^A_0$ of eq. (2.29) is:

$$J^A_0 = \left( J^a_0, - J^b_0 \dot{\phi}^b_0 \right),$$

(2.34)

where the condition $J^A_0 \dot{\phi}^A_0 = 0$ is directly satisfied. This term represents the complete interaction at the background level. In principle, the interaction is completely arbitrary, however we need to impose some restrictions to study the period between the beginning of inflation, where the slow roll approximation are satisfied, and the reheating era, where the inflationary parameters present the problem introduced in section 2.2. A detailed analysis of interaction term at the background level as well as the perturbative level can be found in appendix B. Particularly, using $Q$ introduced in eq. (2.18) and (B.6), we may infer that:

$$\Gamma_{\text{eff}} = H q_1 \sin^2(\alpha).$$

(2.35)

This is the background expression for the effective decay rate that we will use along this work. Note that we have derived a decay rate varying in time, which has interesting features as we will see later. In all of the previous expressions, capital latin indices go from 1 to 3. Then, eq. (2.29) with $A = 1, 2$ is equivalent to eq. (2.3) whereas $A = 3$ represents eq. (2.4) after introducing equation (2.23) with $J^3_0$ as given by (2.28). Additionally, we have that the Friedmann equations are given by (2.30) and (2.31).

Following ref. [25], we introduce a set of orthogonal unit vectors to parameterize our new three-dimensional field-space. First, we write the Tangent vector to the inflationary trajectory as $T^A = \dot{\phi}_0 / \dot{\phi}_0$. Then, by introducing it in eq. (2.29), we obtain:

$$\frac{DT^A}{dt} = - \left( \frac{\ddot{\phi}_0}{\phi_0} + 3H \right) T^A - \frac{1}{\phi_0} \left( \mathcal{V}^A + J^A_0 \right).$$

(2.36)

Now, by projecting this equation along $T^A$, we obtain the following Klein-Gordon equation for the field $\phi_0$:

$$\ddot{\phi}_0 + 3H \dot{\phi}_0 + \mathcal{V}_T = 0,$$

(2.37)

where $\mathcal{V}_T \equiv T^A \partial_A \mathcal{V}$. Notice that, in order to derive the previous equation, we used that $J_T = T_A J^A_0 = 0$ and $T_A \frac{DT^A}{dt} = 0$. Next, we define two additional vectors, $N^A$ and $B^A$, to complete an orthogonal basis around the inflationary trajectory, such that $T^A N_A = T^A B_A = N^A B_A = 0$ and $T^A T_A = N_A N_A = B_A B_A = 1$ (for instance, see ref. [85]). The Normal vector $N^A$ is defined in such a way that it remains parallel to the time variation of $T^A$ (that is $N^A \propto D_t T^A$). This definition implies:

$$\frac{DT^A}{dt} = - \frac{\left( \mathcal{V}_N + J_N \right)}{\phi_0} N^A.$$

(2.38)
Then, it is direct to verify that the combination $V^A + J_A^0$ decomposes only along the subspace spanned by $T^A$ and $N^A$:

$$V^A + J_A^0 = V_T T^A + (V_N + J_N) N^A, \quad (2.39)$$

where $V_N = V^A N_A$ and $J_N = J_A^0 N_A$. This, in turn, implies that the Bi-Normal vector $B^A$ satisfies $V_B + J_B = 0$. The time variations of $N^A$ and $B^A$ are found to satisfy:

$$\frac{dN^A}{dt} = (V_N + J_N) T^A - \frac{DB^B}{dt} N_B^A, \quad (2.40)$$

$$\frac{dB^A}{dt} = DB^B \frac{dN_B}{dt} N^A. \quad (2.41)$$

We present more details about these orthogonal vectors like a base of the space fields in appendix A, where we introduce a special parametrization to solve the equations of motion. Then, we can now introduce the slow-roll parameters $\varepsilon$ and $\eta^A$, defined as:

$$\varepsilon \equiv -\frac{\dot{H}}{H^2} = \frac{\dot{\phi}_0^2}{2H^2}, \quad (2.42)$$

$$\eta^A \equiv -\frac{1}{H\phi_0} \frac{D\phi_0^A}{dt} = (3 + \frac{V_T}{H\phi_0}) T^A + (\frac{V_N + J_N}{H\phi_0}) N^A, \quad (2.43)$$

where we have used eq. (2.29). It is interesting to mention that eq. (2.43) corresponds to the three-dimensional extension of $\eta^a$ of ref. [86]. We can decompose $\eta^A$ along the Normal and Tangent directions by introducing two independent parameters $\eta^A = \eta|| T^A + \eta\perp N^A$, with:

$$\eta|| = -\frac{\dot{\phi}_0}{H\phi_0} = \varepsilon - \frac{\dot{\varepsilon}}{2H\varepsilon}, \quad (2.44)$$

$$\eta\perp = \frac{V_N + J_N}{H\phi_0}. \quad (2.45)$$

We note that $\eta||$ may be recognized as the usual $\eta$ slow-roll parameter in single field inflation. On the other hand $\eta\perp$ informs us about the rate with which $T^A$ rotates, and therefore it parameterizes the rate of turn of the trajectory followed by the scalar fields. Finally, we introduce the parameters:

$$\xi|| \equiv -\frac{\dot{\eta||}}{H\eta||}, \quad (2.46)$$

$$\xi\perp \equiv -\frac{\dot{\eta}\perp}{H\eta\perp}. \quad (2.47)$$

in order to deal with the perturbative equations in the next section.

Let us briefly emphasize an important aspect of this new parametrization. Notice that $\dot{\phi}_0^2$ and $\phi_0^2$ are related by eq. (2.32). This implies that the new geometrical parameters appearing in (2.43) will not be singular as $\dot{\phi}_0 \rightarrow 0$. As we shall see, after inflation ends, and we enter into the radiation era, one has $\phi_0 \rightarrow 2H$. In conclusion, this parametrization allows us to deal with well-defined inflationary parameters using an effective potential $V$ in addition to multi-field interactions. However, we should not forget that the system presented in section 2.1 is the real system, and that the method introduced in this section is just a tool.
to solve the equations of motion and avoid the problem discussed in section 2.2. Although in our formalism, both fields are coupled to a radiation fluid (which is parametrized as a third effective scalar field) during and after inflation, the effective decay must be such that $\Gamma_{\text{eff}} \ll 3H$ during slow-roll inflation, and it becomes important only when the scalar fields are oscillating about the minimum of the potential, satisfying $\Gamma_{\text{eff}} > 3H$ at the end of reheating.

So, the relevant equations relating geometrical dimensionless parameters and the potential $V$, are given by

$$V = H^2 (3 - \varepsilon),$$  \hspace{1cm} (2.48)

$$V_T = -H \dot{\phi}_0 (3 - \eta),$$ \hspace{1cm} (2.49)

$$V_N + J_N = H \dot{\phi}_0 \eta, $$ \hspace{1cm} (2.50)

$$V_B + J_B = 0,$$ \hspace{1cm} (2.51)

and the equations of motion:

$$\dot{\phi}_0^2 = 2H^2 \varepsilon,$$ \hspace{1cm} (2.52)

$$\frac{D T^A}{dt} = -H \eta \eta^A,$$ \hspace{1cm} (2.53)

$$\frac{D N^A}{dt} = H \eta \eta^A - HCB^A,$$ \hspace{1cm} (2.54)

$$\frac{DB^A}{dt} = HCN^A,$$ \hspace{1cm} (2.55)

with $C \equiv H^{-1} \frac{DB^B}{dt}$. Additionally, the initial conditions will be fixed in such a way that the radiation density is null in the beginning of inflation. This means that $T^{(3)}, N^{(3)} \to 0$ and we can choose:

$$B^A = \left(0, 0, \frac{1}{\sqrt{q_{(33)}}} \right).$$ \hspace{1cm} (2.56)

More details about this are provided in appendix A. Also, we describe the interaction contribution, given by $J_N$ and $J_B$, in appendix B.

### 3 Perturbations

In this section we consider the dynamics of scalar perturbations, parameterizing departures from the homogeneous and isotropic background. This may be done by defining perturbations $\delta \phi^a, \delta g_{\mu\nu}, \delta \Gamma_{\gamma}^{\mu
u}$ and $\delta \Gamma_{\alpha}^{a bc}$ as

$$\phi^a (t, x) = \phi^a_0 (t) + \delta \phi^a (t, x),$$ \hspace{1cm} (3.1)

$$g_{\mu\nu} (t, x) = g_{\mu\nu}^0 (t) + \delta g_{\mu\nu} (t, x),$$ \hspace{1cm} (3.2)

$$\Gamma_{\mu\nu}^a (t, x) = \Gamma_{\mu\nu}^{a0} (t) + \delta \Gamma_{\mu\nu}^a (t, x),$$ \hspace{1cm} (3.3)

$$\gamma_{ab} (\phi) = \gamma_{ab}^0 (\phi_0) + \delta \gamma_{ab} (\phi_0, \phi),$$ \hspace{1cm} (3.4)

$$\Gamma_{\alpha}^{bc} (\phi) = \Gamma_{\alpha}^{bc} (\phi_0) + \delta \Gamma_{\alpha}^{bc} (\phi_0, \phi),$$ \hspace{1cm} (3.5)

where the 0-label denotes background quantities. The Greek and Latin indices correspond to the coordinate space and field space respectively. So, $\Gamma_{\mu\nu}^a$ are the Christoffel symbols associated to $g_{\mu\nu}$ and $\Gamma_{\alpha}^{bc}$ is related to $\gamma_{ab}$. 


We must consider perturbations to the homogeneous background space-time and the energy-momentum tensor of the universe. The most general first-order perturbation to a spatially flat FLRW metric is \[83\]:

\[ ds^2 = -(1 + 2\Phi) dt^2 + 2aB_i dx^i dt + a^2 ((1 - 2\Psi) \delta_{ij} + 2E_{ij}) dx^i dx^j, \]  

(3.6)

where \( \Phi \) is the lapse, \( B_i \) is the shift, \( \Psi \) is the spatial curvature perturbation, and \( E_{ij} \) is the symmetric shear tensor. The symmetries of the FLRW background space-time allow linear perturbations to be decomposed into independent scalar, vector and tensor components. This reduces the linearized Einstein equations to a set of decoupled ordinary differential equations. So, the metric in eq. (3.6) can be rewritten as:

\[ ds^2 = -(1 + 2\Phi) dt^2 + 2a\partial_i B_i dx^i dt + a^2((1 - 2\Psi) \delta_{ij} + 2\partial_i \partial_j E_{ij}) dx^i dx^j + 2aG_i dx^i dt + 2a^2(\partial_i C_j + \partial_j C_i) dx^i dx^j + h_{ij} dx^i dx^j, \]  

(3.7)

where we have introduced the scalars \( B \) and \( E \), the vectors \( G_i \) and \( C_i \) and the pure tensor field \( h_{ij} \) to decompose the shift and the shear. For this decomposition to be unique, the fields \( G_i \), \( C_i \) and \( h_{ij} \) must satisfy the constraints \( \partial_i G_i = \partial_i C_i = \partial_i h_{ij} = \partial_j h_{ij} = h_{ii} = 0 \). Note that two of the four scalar perturbations \( (\Phi, B, \Psi \text{ and } E) \) can be eliminated by the specific gauge choice. In the same way, one of the vector perturbations \( (G_i \text{ and } C_i) \) can be removed, but the tensor perturbation \( (h_{ij}) \) is a gauge invariant. In this paper, we will rewrite the perturbation equations in a gauge-invariant form, because these become particularly simple as we will see soon.

Then focusing our attention to the scalar degrees of freedom, the metric is found to be given as:

\[ g_{\mu\nu} = \begin{pmatrix} - (1 + 2\Phi) & a \partial_i B \\ a \partial_i B & a^2((1 - 2\Psi) \delta_{ij} + 2\partial_i \partial_j E) \end{pmatrix}, \]  

(3.8)

whereas the inverse metric, up to linear order, is given by

\[ g^{\mu\nu} = \begin{pmatrix} - (1 - 2\Phi) & a^{-1} \partial^i B \\ a^{-1} \partial^i B & a^{-2}((1 + 2\Psi) \delta_{ij} - 2\partial^i \partial^j E) \end{pmatrix}. \]  

(3.9)

On the other hand the Christoffel symbols become:

\[ \Gamma^0_{00} = \dot{\Phi}, \quad \Gamma^0_{0i} = \Gamma^0_{i0} = \partial_i (\Phi + aHB), \]  

(3.10)

\[ \Gamma^0_{ij} = \delta_{ij}a^2 \left(H - 2H (\Psi + \dot{\Phi}) - \dot{\Psi}\right) + a^2 \partial_i \partial_j \left(2HE + \left(\dot{E} - \frac{B}{a}\right)\right), \]  

(3.11)

\[ \Gamma^i_{00} = a^{-2}\partial_i \left(\Phi + (aB)\right), \]  

(3.12)

\[ \Gamma^i_{0j} = \Gamma^j_{00} = \delta_{ij}(H - \Psi) + \partial_i \partial_j \dot{E}, \]  

(3.13)

\[ \Gamma^j_{ik} = -aH\delta_{jk}\partial_i B - \delta_{ij}\partial_k \Psi - \delta_{ik}\partial_j \Psi + \delta_{jk}\partial_i \dot{\Phi} + \partial_i \partial_j \partial_k E. \]  

(3.14)
From these, we may compute the perturbed Einstein tensor, whose components are given by:

\[
\delta G_{00} = 6H \left( \dot{\Psi} + H \Phi \right) - 2\frac{\partial^2}{a^2} \left[ H(a^2 \dot{E} - aB) + \Psi \right], \\
\delta G_{ij} = \left[ 2 \left( 3H^2 + 2H \right) \Phi + 2H \left( \Phi + 3\Psi \right) + 2\dot{\Psi} - \partial^2 \left( \frac{d}{dt} \left( \dot{E} - \frac{B}{a} \right) \right) + 3H \left( \dot{E} - \frac{B}{a} \right) \right] \delta_{ij} + \delta^{ik} \partial_k \partial_j \left[ \frac{d}{dt} \left( \dot{E} - \frac{B}{a} \right) \right] + 3H \left( \dot{E} - \frac{B}{a} \right) + \frac{(\Psi - \Phi)}{a^2}, \\
\delta G_{0i} = \frac{2}{a^2} \partial^i \left( \dot{\Psi} + H \Phi - \dot{H}B \right), \\
\delta G_{i0} = -\frac{2}{a^2} \partial_i \left( \dot{\Psi} + H \Phi \right). 
\]

In order to construct the perturbed field equations, we need to consider, besides the gravitational fields, the effects that induce the matter perturbations. The components of the perturbed total energy-momentum tensor can be expressed as [83]:

\[
\delta T_{00}^0 = -\rho^1, \\
\delta T_{00}^0 = q_i^1, \quad \delta T_{00}^i = \frac{\delta^i}{a^2} \left[ (\rho^0 + p^0) \partial_j B - q_j^1 \right], \quad \delta T_{00}^j = p^1 \delta^i + \Sigma_{ij}, \\
\delta T_{ij}^i = \rho^1 \delta_{ij} + \Sigma_{ij}.
\]

where \( q_i^1 = - (\rho^0 + p^0) u_i^1 \) is the 3-momentum density, and \( u_i^1 \) is the velocity perturbation of the matter content. \( \Sigma_{ij} \) is an anisotropic stress tensor. To relate the metric and energy-momentum perturbations, we must use the perturbed Einstein field equations at linear order given by:

\[
2\frac{\partial^2}{a^2} \left[ H(a^2 \dot{E} - aB) + \Psi \right] - 6H \left( \dot{\Psi} + H \Phi \right) = \rho^1, \\
\left[ 2 \left( 3H^2 + 2H \right) \Phi + 2H \left( \Phi + 3\Psi \right) + 2\dot{\Psi} - \partial^2 \left( \frac{d}{dt} \left( \dot{E} - \frac{B}{a} \right) \right) + 3H \left( \dot{E} - \frac{B}{a} \right) \right] \delta_{ij} + \delta^{ik} \partial_k \partial_j \left[ \frac{d}{dt} \left( \dot{E} - \frac{B}{a} \right) \right] + 3H \left( \dot{E} - \frac{B}{a} \right) + \frac{(\Psi - \Phi)}{a^2} = \rho^1 \delta_{ij} + \Sigma_{ij}, \\
-2\partial_i \left( H \Phi + \dot{\Psi} \right) = q_i^1.
\]

In general, \( \Sigma_{ij} \) for \( i \neq j \) contains a non diagonal component called viscous pressure. On the other hand, the velocity perturbation can be decomposed into a scalar and vector components, where the first one can be written as \( u_i^1 = \partial_i u^1 \). Then, the perturbed field equations, in the
absence of anisotropic stress, reduce to:

\[
\frac{d}{dt} \left( \dot{E} - \frac{B}{a} \right) + 3H \left( \dot{E} - \frac{B}{a} \right) + \frac{\left( \Psi - \Phi \right)}{a^2} = 0,
\]

(3.26)

\[
2 \frac{\partial^2}{a^2} \left[ a^2 H \left( \dot{E} - \frac{B}{a} \right) + \Psi \right] - 6H \left( \dot{\Psi} + H \Phi \right) - \rho^1 = 0,
\]

(3.27)

\[
2 \left( 3H^2 + 2 \dot{H} \right) \Phi + 2H \left( \dot{\Phi} + 3 \dot{\Psi} \right) + 2 \ddot{\Psi} - \rho^1 = 0,
\]

(3.28)

\[
\left( \dot{\Psi} + H \Phi \right) + \dot{H} u^1 = 0,
\]

(3.29)

where we used that \( \rho^0 + p^0 = -2 \dot{H} \). To extract physical results it is useful to define
gauge-invariant combinations of the scalar metric perturbations. Two relevant quantities are known
as the Bardeen potentials, defined as [83]:

\[
\Phi_B \equiv \Phi - \frac{d}{dt} \left[ a^2 \left( \dot{E} - \frac{B}{a} \right) \right],
\]

(3.30)

\[
\Psi_B \equiv \Psi + a^2 H \left( \dot{E} - \frac{B}{a} \right).
\]

(3.31)

On the other hand, the matter perturbations are also gauge-dependent. However, it is possible
to construct a set of gauge-invariant quantities given by:

\[
\rho^{1I} = \rho^1 + \frac{\rho^0}{H} \Psi,
\]

(3.32)

\[
p^{1I} = p^1 + \frac{\rho^0}{H} \Psi,
\]

(3.33)

\[
\mathcal{R} = \Psi + Hu^1.
\]

(3.34)

These correspond to gauge invariant expressions for the energy density, pressure, and the
total curvature perturbation, respectively. In terms of these gauge-invariant quantities, the
perturbed Einstein equations may be rewritten as:

\[
\Phi_B - \Psi_B = 0,
\]

(3.35)

\[
2 \frac{\partial^2}{a^2} \Phi_B - 6H^2 \varepsilon \mathcal{R} - \rho^{1I} = 0,
\]

(3.36)

\[
2H \varepsilon \mathcal{R} + 2H^2 \varepsilon (3 - 2 \eta) \mathcal{R} - p^{1I} = 0,
\]

(3.37)

\[
\dot{\Phi}_B - H \varepsilon \mathcal{R} + H (1 + \varepsilon) \Phi_B = 0,
\]

(3.38)

for which we used the inflationary parameters given by eqs. (2.42)–(2.45). In this work, we
will consider that the total matter content of the universe is given by a two-fields system
interacting with radiation. However, in the last section, radiation was represented by an
additional scalar field. This fact has to be taken into account to define the total density,
pressure and scalar velocity, presented in appendix C where \((\delta \phi^{(3)})\) is introduced in the
system. We will give more detail about this later.

Now, we will perturb the Klein-Gordon equation (2.2) at first order. This yields:

\[
(g_0^{\alpha \beta} + \delta g^{\alpha \beta}) \left[ \partial_{\alpha} \partial_{\beta} (\phi_0^\alpha + \delta \phi^\alpha) - \left( \Gamma_\alpha^{\gamma \beta} + \delta \Gamma_\alpha^{\gamma \beta} \right) \partial_{\gamma} (\phi_0^\beta + \delta \phi^\beta) \right] + \left( \Gamma_\alpha^{(3) \gamma} + \delta \Gamma_\alpha^{(3) \gamma} \right) (g_0^{\mu \nu} + \delta g^{\mu \nu}) \partial_{\mu} (\phi_0^\beta + \delta \phi^\beta) \partial_{\nu} (\phi_0^\alpha + \delta \phi^\alpha) = V^\alpha + \partial_0 V^a \delta \phi^b + J_0^{(a)} + (\delta J^a).
\]

(3.39)
Expanding, we note that the zeroth order term corresponds to the Klein Gordon equation. Now, the first order term is:

\[
\frac{\partial^2}{\partial t^2} (\delta \phi^a) - \frac{\partial^2}{\partial a^2} (\delta \phi^a) + 3H(\delta \phi^a) + 2\Gamma^a_{bc} \dot{\phi}^b \dot{\phi}_c + \dot{\phi}^b \phi^c (\delta \Gamma^a_{bc}) + \partial_b (\gamma^{ac} \partial_c V) (\delta \phi^b) + 2(V^a + J^a_0) \Phi - \dot{\phi}_0^a \left( \dot{\Phi} + 3\dot{\Psi} - \partial^2 \left( \dot{E} - \frac{B}{a} \right) \right) + (\delta J^a) = 0, \tag{3.40}
\]

for which we used eqs. (3.8)–(3.10). To write this equation in a covariant form, we have:

\[
\partial_b (\gamma^{ac} \partial_c V) = V^a_b - \Gamma^a_{bc} V^c, \quad (\delta \Gamma^a_{bc}) = \partial_c \Gamma^a_{bc}(\delta \phi^d), \quad (\delta \phi^a) = \frac{D(\delta \phi^a)}{dt} - \Gamma^a_{bc} \dot{\phi}^b (\delta \phi^c), \quad (3.41)
\]

\[
(\delta \phi^a) = (\Delta J^a) - \Gamma_{bc}^{a0} \dot{\phi}_0^c \left( \delta \phi^b \right), \quad (\delta \phi^a) = \frac{D(\delta \phi^a)}{dt} - \Gamma^a_{bc} \dot{\phi}^b (\delta \phi^c) \tag{3.42}
\]

\[
(\dot{\phi}_0^c) = \frac{D^2(\delta \phi^a)}{dt^2} - 2\Gamma^a_{bc} \Phi \frac{D(\delta \phi^a)}{dt} - \Gamma^a_{bc} \dot{\phi}_0^b (\delta \dot{\phi}_c) - \dot{\phi}_0^c \dot{\phi}_0^b \left( \partial_c \Gamma^a_{bd} - \Gamma^a_{bd} \partial_c - \Gamma^a_{bc} \Gamma^c_{bd} - \Gamma^a_{de} \Gamma^c_{de} \right) (\delta \phi^d). \tag{3.43}
\]

Here, \( V^a_b = D_b D^a V \), where \( D_a \) represents the covariant derivative in the field space, and \((\Delta J^a)\) is a vector in the field space. So, by using eq. (2.3) and the expressions from above, the perturbative Klein-Gordon’s equation becomes:

\[
\frac{D^2}{dt^2} (\delta \phi^a) - \frac{\partial^2}{\partial a^2} (\delta \phi^a) + 3H \frac{D}{dt} (\delta \phi^a) + V^a_b \left( \delta \phi^b \right) - \dot{\phi}_0^c \phi^a \mathcal{R}^{a}_{bcd} \left( \delta \phi^d \right) + 2(V^a + J^a_0) \Phi - \dot{\phi}_0^a \left( \dot{\Phi} + 3\dot{\Psi} - \partial^2 \left( \dot{E} - \frac{B}{a} \right) \right) + (\Delta J^a) = 0, \tag{3.44}
\]

where \( \mathcal{R}^{a}_{bcd} = \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^a_{ce} \Gamma^e_{bd} - \Gamma^a_{de} \Gamma^e_{bc} \) is the Riemann tensor in the 2D-field space. Now, just as we did with the background, we now extend eq. (3.44) to a three-field version, in which one of the perturbations \((\delta \phi^3)\) corresponds to an auxiliary field identified with the radiation perturbation. In other words, we define a fluctuation \( \delta \phi^3 \) in such a way that the following set of equations are valid:

\[
\frac{D^2}{dt^2} (\delta \phi^A) - \frac{\partial^2}{\partial a^2} (\delta \phi^A) + 3H \frac{D}{dt} (\delta \phi^A) + \left( V^A_B - \dot{\phi}_0^B \mathcal{R}^{B}_{A,TB} \right) (\delta \phi^B) + 2(V^A + J^A_0) \Phi - \dot{\phi}_0^A \mathcal{T}^A \left( \dot{\Phi} + 3\dot{\Psi} - \partial^2 \left( \dot{E} - \frac{B}{a} \right) \right) + (\Delta J^A) = 0, \tag{3.45}
\]

where \( \mathcal{R}^{A}_{T,TB} = \mathcal{T}^C \mathcal{T}^D \mathcal{R}^{A}_{CDB} \) corresponds to the 3D Riemann tensor with the metric \( \mathcal{q}_{AB} \), mentioned in section 2.1, and \((\Delta J^A)\) is the effective perturbative interaction. Recall that \( V = V(\phi^0_1, \phi^0_2) + \frac{1}{4} \rho^0_B(\phi^0_3) \), and so \( V^{A=1,2} = 0 \). In addition, the only non-zero component of the Riemann tensor is \( \mathcal{R}_{(123)} \). Therefore, it is easy to see that eq. (3.45) with \( A = 1, 2 \) reduces to (3.44). In appendix C, we show that the third component of eq. (3.45) is equivalent to a perturbative equation of a perfect radiation fluid.

To continue, we will rewrite (3.45) in terms of gauge invariant quantities. For this, we define:

\[
(\delta \phi^A) = \frac{\dot{\phi}_0^A}{H} \left[ (R - \Psi) \mathcal{T}^A + \mathcal{S}_N \mathcal{N}^A + \mathcal{S}_B \mathcal{B}^A \right], \tag{3.46}
\]

\[
(\Delta J^A) = (\Delta J^A)^I - \frac{1}{H} \frac{D}{dt} (J^A_0) \Psi, \tag{3.47}
\]
where the total curvature $\mathcal{R}$, the isocurvature components $S_N$ and $S_B$ and $(\Delta J^A)^I$ are gauge invariant. Now, from eq. (3.46), we have:

$$
\frac{D}{dt} (\delta \phi^A) = f_T T^A + f_N N^A + f_B B^A,
$$

\[ (3.48) \]

$$
\frac{D^2}{dt^2} (\delta \phi^A) = \left( \dot{J}_T + H \eta_{\perp} f_N \right) T^A + \left( \dot{J}_N - H \eta_{\perp} f_T + H C f_B \right) N^A + \left( \dot{J}_B - H C f_N \right) B^A,
$$

\[ (3.49) \]

with $C = \frac{N_B}{H} \frac{D}{dt} (B^B)$, and:

$$
f_T \equiv T_A \frac{D}{dt} (\delta \phi^A) = \frac{\dot{\phi}_0}{H} \left( \mathcal{R} - \dot{\Psi} \right) + \dot{\phi}_0 \left( \epsilon - \eta_{\parallel} \right) \left( \mathcal{R} - \Psi \right) + \dot{\phi}_0 \eta_{\perp} S_N,
$$

\[ (3.50) \]

$$
f_N \equiv N_A \frac{D}{dt} (\delta \phi^A) = \frac{\dot{\phi}_0}{H} \dot{S}_N + \dot{\phi}_0 \left( \epsilon - \eta_{\parallel} \right) S_N - \dot{\phi}_0 \eta_{\perp} \left( \mathcal{R} - \Psi \right) + \dot{\phi}_0 C S_B,
$$

\[ (3.51) \]

$$
f_B \equiv B_A \frac{D}{dt} (\delta \phi^A) = \frac{\dot{\phi}_0}{H} \dot{S}_B + \dot{\phi}_0 \left( \epsilon - \eta_{\parallel} \right) S_B - \dot{\phi}_0 C S_N.
$$

\[ (3.52) \]

In addition, from appendix B, we have:

$$
(\Delta J_T)^I = \eta_{\perp} J_N \mathcal{R} + \dot{\phi}_0 \tau_N \dot{S}_N + \left( \frac{\dot{J}_N}{H} + C J_B + \dot{\phi}_0 \left( \tau_N + H \left( 3 - 2 \eta_{\parallel} \right) \tau_N \right) \right) S_N
$$

$$
+ \dot{\phi}_0 \tau_B \dot{S}_B + \left( \frac{\dot{J}_B}{H} - C J_N + \dot{\phi}_0 \left( \tau_B + H \left( 3 - 2 \eta_{\parallel} \right) \tau_B \right) \right) S_B,
$$

\[ (3.53) \]

$$
(\Delta J_N)^I = -\dot{\phi}_0 \tau_N \mathcal{R} + \dot{\phi}_0 \eta_{\parallel} \dot{S}_N + \left( \frac{\dot{J}_N}{H} + C J_B + \dot{\phi}_0 \left( \tau_N + H \left( 3 - 2 \epsilon \right) \tau_N \right) + H \dot{\phi}_0 C \tau_B \right) \mathcal{R}
$$

$$
+ H \dot{\phi}_0 \Lambda_{NB} S_N + H \dot{\phi}_0 \left( \Lambda_{NB} + \left( \epsilon - \eta_{\parallel} \right) j_0 \right) S_B + H \dot{\phi}_0 \kappa_N \Phi_B,
$$

\[ (3.54) \]

$$
(\Delta J_B)^I = -\dot{\phi}_0 \tau_B \mathcal{R} - \dot{\phi}_0 \eta_{\parallel} \dot{S}_N
$$

$$
+ \left( \frac{\dot{J}_B}{H} - C J_N + \dot{\phi}_0 \left( \tau_B + H \left( 3 - 2 \epsilon \right) \tau_B \right) + H \dot{\phi}_0 \left( \eta_{\parallel} j_0 - C \tau_N \right) \right) \mathcal{R}
$$

$$
+ H \dot{\phi}_0 \left( \Lambda_{NB} - \left( \epsilon - \eta_{\parallel} \right) j_0 - \eta_{\perp} \tau_B \right) S_N + H \dot{\phi}_0 \Lambda_{BB} S_B + H \dot{\phi}_0 \kappa_B \Phi_B.
$$

\[ (3.55) \]

On the other hand

$$
V^A_B = V_{TT} T^A T_B + V_{NN} N^A N_B + V_{BB} B^A B_B + V_{TN} \left( T^A N_B + N^A T_B \right)
$$

$$
+ V_{TB} \left( T^A B_B + B^A T_B \right) + V_{NB} \left( N^A B_B + B^A N_B \right),
$$

\[ (3.56) \]

where the projections are defined as:

$$
V_{TT} = T^A T_B D_A D_B V, \quad V_{TN} = N^A T_B D_A D_B V, \quad V_{TB} = B^A T_B D_A D_B V,
$$

\[ (3.57) \]

$$
V_{NN} = N^A N_B D_A D_B V, \quad V_{NB} = N^A B_B D_A D_B V, \quad V_{BB} = B^A B_B D_A D_B V,
$$

\[ (3.58) \]
where $D_A$ is the covariant derivative in the 3D field space. From eqs. (2.42)–(2.46), we know that:

$$
\mathcal{V}_{TT} = \frac{\dot{\mathcal{V}}_T + H\eta_\perp \dot{\mathcal{V}}_N}{\phi_0} = H^2 \left( (3 - \eta_\parallel) (\varepsilon + \eta_\parallel) - \eta_\parallel \xi_\parallel + \eta_\perp^2 \right) - \frac{H\eta_\perp \mathcal{J}_N}{\phi_0}, \quad (3.59)
$$

$$
\mathcal{V}_{TN} = \frac{\dot{\mathcal{V}}_N - H\eta_\perp \dot{\mathcal{V}}_T + HC\mathcal{V}_B}{\phi_0} = H^2 \eta_\perp (3 - \varepsilon - 2\eta_\parallel - \xi_\parallel) - \frac{\mathcal{J}_N + HC\mathcal{J}_B}{\phi_0}, \quad (3.60)
$$

$$
\mathcal{V}_{TB} = \frac{\dot{\mathcal{V}}_B - HC\mathcal{V}_N}{\phi_0} = -H^2 C\eta_\perp - \frac{\mathcal{J}_B - HC\mathcal{J}_N}{\phi_0}. \quad (3.61)
$$

Finally, we can rewrite the perturbative Einstein equations (3.35)–(3.38) using appendix C:

$$
\Psi_B = \Phi_B, \quad (3.62)
$$

$$
\frac{\partial^2}{a^2} \Phi_B = H^2 \varepsilon \left( \frac{\dot{\mathcal{R}}}{H} + (2\eta_\perp + \tau_N) S_N + \tau_B S_B \right), \quad (3.63)
$$

$$
\dot{\Phi}_B + H (1 + \varepsilon) \Phi_B = H \varepsilon \mathcal{R}. \quad (3.64)
$$

Then, with all these expressions and using eqs. (3.30)–(3.31), the components of the perturbative Klein-Gordon equation (3.45), become:

$$
\ddot{\mathcal{R}} - \frac{\partial^2}{a^2} \mathcal{R} + H (3 + 2\varepsilon - 2\eta_\parallel) \dot{\mathcal{R}} \quad (3.65)
$$

$$
+ H \left( 2\eta_\perp + \tau_N \right) \dot{S}_N + H^2 \left( 2\eta_\perp (3 - \varepsilon - 2\eta_\parallel - \xi_\parallel) + \frac{\dot{\mathcal{J}}_N}{H} + (3 + \varepsilon - 2\eta_\parallel) \tau_N \right) S_N
$$

$$
+ H \tau_B S_B + H^2 \left( \frac{\dot{\mathcal{J}}_B}{H} + (3 + \varepsilon - 2\eta_\parallel) \tau_B \right) S_B = 0, \quad (3.66)
$$

$$
+ H^2 \left( \frac{\mathcal{V}_{NN}}{H^2} + \varepsilon (3 + 2\varepsilon - 2\Re_{NTT}) - \eta_\perp^2 - C^2 - \eta_\parallel (3 + 3\varepsilon - \eta_\parallel - \xi_\parallel) + \Lambda_{NN} \right) S_N
$$

$$
+ H^2 \left( \frac{\mathcal{V}_{NB}}{H^2} - 2\varepsilon \Re_{NTTB} + C (3 + \varepsilon - 2\eta_\parallel) \frac{\dot{\mathcal{C}}}{H} + \Lambda_{NB} + (\varepsilon - \eta_\parallel) j_0 \right) S_B
$$

$$
- H (2\eta_\perp + \tau_N) \dot{\mathcal{R}} + H^2 \left( \frac{\dot{\mathcal{J}}_N}{H} + (3 - 2\varepsilon) \tau_N + C \tau_B \right) \mathcal{R} + H^2 \kappa_N \Phi_B = 0, \quad (3.67)
$$

$$
\ddot{S}_B - \frac{\partial^2}{a^2} S_B + H \left( 3 + 2\varepsilon - 2\eta_\parallel \right) \dot{S}_B - H \left( 2C + j_0 \right) \dot{S}_N \quad (3.68)
$$

$$
+ H^2 \left( \frac{\mathcal{V}_{BB}}{H^2} + \varepsilon (3 + 2\varepsilon - 2\Re_{BBTT}) - C^2 - \eta_\parallel (3 + 3\varepsilon - \eta_\parallel - \xi_\parallel) + \Lambda_{BB} \right) S_B
$$

$$
+ H^2 \left( \frac{\mathcal{V}_{NB}}{H^2} - 2\varepsilon \Re_{NTTB} - C (3 + \varepsilon - 2\eta_\parallel) \frac{\dot{\mathcal{C}}}{H} + \Lambda_{NB} - (\varepsilon - \eta_\parallel) j_0 - \eta_\perp \tau_B \right) S_N
$$

$$
- H \tau_B \dot{\mathcal{R}} + H^2 \left( \frac{\dot{\mathcal{J}}_B}{H} + (3 - 2\varepsilon) \tau_B + \eta_\perp j_0 - C \tau_N \right) \mathcal{R} + H^2 \kappa_B \Phi_B = 0.
$$

We note that the background interactions, given by $\mathcal{J}_N$ and $\mathcal{J}_B$, do not appear explicitly in the perturbative equations. They are given by the behaviour of the inflationary parameters,
presented in the coefficients of eqs. (3.62)–(3.67). However, we have other parameters to represent the perturbative interaction. In appendix C, we show that only two parameters, $\tau_N$ and $\tau_B$, represent the interaction on the total fluid system. We can see this in eq. (3.65), where they are the only contribution to the interaction. From this equation, we note that $\tau_N$ can be interpreted as a perturbative correction to $\eta_\perp$, defining $\hat{\eta}_\perp = \eta_\perp + \frac{\tau_N}{2}$. This inflation parameter represents the rate of turn of $T^A$ on $N^A$ in the fields space, see eq. (2.53). For this reason, $\tau_N$ produces an additional effect by the isocurvature in the Normal direction in (3.65). In the same way, $\tau_B$ gives us a contribution in the Bi-Normal direction. As an additional effect, they insert new curvature terms in eqs. (3.66)–(3.67).

Probably, the other six parameters are not completely independent of $\tau_N$ and $\tau_B$, but we do not know how they are related without previous information about the radiation component and interaction terms. On one side, $\Lambda_{NN}$, $\Lambda_{NB}$ and $\Lambda_{BB}$ are a kind of frequency terms in eqs. (3.66)–(3.67), so we expect that they will produce an oscillatory contribution in the isocurvature components. On the other side, $j_0$ is more like a friction term in the isocurvature equations, then it should produce a decrease (increase) in $S_N$ ($S_B$). In any case, these parameters represent the internal effects of the interactions, modifying the behaviour of the isocurvature components, therefore they could be important, for instance, to understand how a particular interaction affects the isocurvature evolution during reheating. Finally, $\kappa_N$ and $\kappa_B$ are related to a possible dependence on the space-time metric in the Lagrangian interaction. These kind of terms could be uncommon, but they can not be discarded.

All the natural restrictions were introduced to fix the interactions (See appendix B and C), surviving all these parameters. We know that they are not completely independent of one another, but their nature is model dependent, so they are completely unknown. This means that we could fix these parameters if the particular characteristics of radiation component are exhibited. However, the goal of this work is to develop a new method to study reheating and analyze particular models where the isocurvature term could produce some modifications in this era, introducing the radiation component in a natural way but not assuming anything about it.

Therefore, in spite of the importance of the interaction components, we will set all the perturbative parameters to zero in order to prove the method and simplify the calculation, preserving just the background components, $J_N$ and $J_B$. This fixing is not a special choice, but we could say that it is a model where the interactions are completely given by the background components. Naturally, the final results are sensitive to the interaction parameters, but understand these effects is not our actual goal, but they will be considered in future works.

Finally, it will be useful to express the cosmic time derivatives as derivatives with respect to the number of $e$-folds, $N$, according to the following relations:

$$\frac{d}{dt} = H \frac{d}{dN}, \quad (') \equiv \frac{d}{dN}. \quad (3.68)$$

Additionally, we introduce the following useful dimensionless parameters:

$$v_{NN} = \frac{V_{NN}}{H^2}, \quad v_{NB} = \frac{V_{NB}}{H^2}, \quad v_{BB} = \frac{V_{BB}}{H^2} \quad (3.69)$$

$$q_N = \frac{J_N}{H^2 \sqrt{2\varepsilon}}, \quad q_B = \frac{J_B}{H^2 \sqrt{2\varepsilon}}. \quad (3.70)$$

$$Q = \frac{k}{aH}. \quad (3.71)$$
Then, the background equations of motion acquire the following forms:

\[ V = H^2 (3 - \varepsilon), \]  \tag{3.72}
\[ \phi_0'' = 2\varepsilon, \]  \tag{3.73}
\[ \frac{DT^A}{dN} = -\eta_L N^A, \]  \tag{3.74}
\[ \frac{DN^A}{dN} = \eta_L T^A - CB^A, \]  \tag{3.75}
\[ \frac{DB^A}{dN} = CN^A, \]  \tag{3.76}

and the projections of the covariant derivative of \( V \) are found to be given by:

\[ V_T = H^2 \sqrt{2\varepsilon} (\eta_\parallel - 3), \quad V_N = H^2 \sqrt{2\varepsilon} (\eta_\perp - q_N), \quad V_B = -H^2 \sqrt{2\varepsilon} q_B, \]  \tag{3.77}

where we have introduced the covariant derivative with respect to \( N \) as:

\[ \frac{DX^A}{dN} = X^A + \Gamma^A_{BC} \phi_0^B X_C, \]  \tag{3.78}
\[ \Gamma^A_{BC} = \frac{1}{2} q^{AD} (\partial_C q^{DB} + \partial_B q^{CD} - \partial_D q^{BC}). \]  \tag{3.79}

In addition, the perturbative equations (3.62)–(3.67) become:

\[ R'' + Q^2 R + (3 + \varepsilon - 2\eta_\parallel) R' + 2\eta_L S_N' + \eta_\perp (3 + \varepsilon - 2\eta_\parallel - \xi_\parallel) S_N' = 0, \]  \tag{3.80}
\[ S_N'' + Q^2 S_N' + (3 + \varepsilon - 2\eta_\parallel) S_N' 
+ (v_{NN} - 2\varepsilon R_{NTTN} - \eta_\perp^2 - C^2 - \eta_\parallel (3 + 3\varepsilon - \eta_\parallel - \xi_\parallel) + 3\varepsilon + 2\varepsilon^2) S_N' 
+ 2CS_B' + (v_{NB} - 2\varepsilon R_{NTTB} + C (3 + \varepsilon - 2\eta_\parallel) + C') S_B - 2\eta_\perp R' = 0, \]  \tag{3.81}
\[ S_B'' + Q^2 S_B' + (3 + \varepsilon - 2\eta_\parallel) S_B' 
+ (v_{BB} - 2\varepsilon R_{BTTB} - C^2 - \eta_\parallel (3 + 3\varepsilon - \eta_\parallel - \xi_\parallel) + 3\varepsilon + 2\varepsilon^2) S_B' 
- 2CS_B' + (v_{NB} - 2\varepsilon R_{NTTB} - C (3 + \varepsilon - 2\eta_\parallel) - C') S_N = 0, \]  \tag{3.82}
\[ Q^2 \Phi_B + \varepsilon (R' + 2\eta_\perp S_N) = 0, \]  \tag{3.83}
\[ \Phi_B' - \varepsilon R + (1 + \varepsilon) \Phi_B = 0, \]  \tag{3.84}

(recall that \( \Psi_B = \Phi_B \)). On the other side, each single matter component has an intrinsic entropy perturbation associated. A single component perfect fluid, by definition, does not have any intrinsic entropy perturbation, whereas for a multi-scalar field system, the intrinsic entropy perturbation is defined as (see ref. [83]):

\[ S_\phi = \frac{P_{\text{had}}^1}{2H^2 (3 - \eta_\parallel)}, \]  \tag{3.85}
\[ P_{\text{had}}^1 \equiv p^1 - \frac{\rho_1}{\rho^1} \rho^1 = p^{1I} - \left( \frac{2}{3} \eta_\parallel - 1 \right), \]  \tag{3.86}
where $p_{\text{nad}}^1$ is the non-adiabatic pressure perturbation of our composite system. Using eqs. (3.32)–(3.33), it is easy to see that the intrinsic entropy is a gauge-invariant quantity. Using the density and pressure components presented in appendix C, the non-adiabatic pressure perturbation becomes:

$$p_{\text{nad}}^1 = 2H^2 \varepsilon (\mathcal{R}' + (3 - 2\eta_\parallel) \mathcal{R}) - \left(\frac{2}{3} \eta_\parallel - 1\right) 2H^2 \varepsilon (\mathcal{R}' - 3\mathcal{R} + 2\eta_\perp \mathcal{S}_\mathcal{N}).$$

(3.87)

In appendix A, a new basis $(T_A^A, N_{A0}^A, B_{A0}^A)$ is introduced, where $N_{A0}^A$ is such that $N_{A0}^{(3)} = 0$. The equations, in this case, are given by eqs. (A.11)–(A.13). In that new basis, similarly to eq. (3.46), we can write $(\delta \phi^A)$ as:

$$(\delta \phi^A) = \frac{\dot{\phi}_0}{H} \left[ (\mathcal{R} - \Psi) T^A + S_{N0} N^A_0 + S_{B0} B^A_0 \right],$$

(3.88)

where $S_{N0}$ and $S_{B0}$ represent the isocurvature elements in this basis. On the other side, we can introduce the inflaton curvature, $\mathcal{R}_\phi$, which is defined on hypersurfaces orthogonal to comoving world-lines [83], such that:

$$\left(\delta \phi^a\right) = \frac{\dot{\phi}_0}{H} \left[ (\mathcal{R}_\phi - \Psi) T^a_{2D} + F_{\phi} N^a_{2D} \right].$$

(3.89)

Here, $(T^a_{2D}, N^a_{2D})$ is the 2D basis, with $a = 1, 2$ considering just the two inflatons, $\phi_0^{(1)}$ and $\phi_0^{(2)}$, and $F_{\phi}$ is the inflaton isocurvature component. In this case, we have $T^a_{2D} = \frac{\dot{\phi}_0}{\phi_0}$, so:

$$T^a = T^a_{2D} \cos(\alpha), \quad N^a_0 = N^a_{2D}, \quad B^a_0 = -T^a_{2D} \sin(\alpha),$$

(3.90)

with $\cos(\alpha) = \frac{\dot{\phi}_0}{\phi_0}$, $T^a_{2D} T^b_{2D} = N^a_{2D} N^b_{2D} = 1$, $T^a_{2D} N^b_{2D} = 0$ and we used eq. (A.15) from appendix A. Therefore, if we compare eq. (3.88) with (3.89), we obtain:

$$\zeta_\phi \equiv \left(\frac{\dot{\phi}_0}{\phi_0}\right)^2 \mathcal{R}_\phi = \cos^2(\alpha) \mathcal{R} - \sin(\alpha) \cos(\alpha) S_{B0},$$

(3.91)

$$F_{\phi} = \frac{S_{N0}}{\cos(\alpha)}.$$  

(3.92)

By convenience, we use a normalized curvature, $\zeta_\phi$, given by:

$$\mathcal{R} = \sum_i \zeta_i,$$

(3.93)

$$\zeta_i = \frac{\rho^i_0 + p^i_0}{\phi_0^2} \mathcal{R}_i,$$

(3.94)

where we used that $\dot{\phi}_0^2 = \rho^0 + p^0$ and $i$ represents all the component in the universe, in our case: Inflatons and Radiation. In this way, we avoid the divergence problem, presented in section 2.2, in the curvature definitions. From (3.91), we conclude that $S_{B0}$ represents the change of the contribution to the total curvature from inflaton perturbations to radiation. On the other side, (3.92) says that $S_{N0}$ represents the inflaton’s isocurvature. In fact, we know
that \( \cos(\alpha) = 0 \) in the radiation epoch, so \( S_{\phi 0} = \zeta_\phi = 0 \) as we expected it. Additionally, using eq. (3.93), we can define the radiation curvature as:

\[
\zeta_R \equiv R - \zeta_\phi = \sin^2(\alpha) R + \sin(\alpha) \cos(\alpha) S_{\phi 0}.
\]

which amounts to the contribution of the total curvature perturbation \( R \) from the radiation fluid. Although our formalism and numerical methods for solving of the full linear equations require that the inflaton fields are coupled to radiation during and after inflation, our choice for the parameters characterizing the interaction yields an amount of radiation such that \( \zeta_R \) becomes negligible in comparison to the curvature associated to the fields, i.e. \( R \approx \zeta_\phi \).

On the other hand, after the decay of the inflaton is completed and reheating ends, the curvature perturbation is sourced by the radiation fluid, i.e. \( R \approx \zeta_R \). This ensures that the coupling between curvature and isocurvature perturbations is sourced only by the dynamics of the multi-field model itself, and not determined by the production of radiation. Similar arguments can be found in the analysis done in [77] for a single-field coupled to radiation. In the next section, we will use all these definitions to describe the evolution of perturbations in different cases.

4 Analysis and results

In this section we set ourselves to study a few concrete examples of multi-field models coupled to radiation. Our analysis will benefit from the covariant formalism offered in the previous two sections to parametrize the evolution of both, background fields and perturbations. All of the models that we consider in the next subsections are characterized for having a diagonal metric in the extended field space:

\[
q_{AB} = \begin{pmatrix}
q_{(11)} & 0 & 0 \\
0 & q_{(22)} & 0 \\
0 & 0 & q_{(33)}
\end{pmatrix}.
\]

With this form of the field-metric, we may work with the basis of unit vectors given in eq. (A.14), parametrized by the angles \( \alpha, \beta \) and \( \gamma \). Alternatively, we may work with the basis shown in (A.15) parametrized by the same angles. Using this parametrization back in eqs. (3.72)–(3.77), the \( \beta \)-angle is found to be given by (see appendix A)

\[
\tan(\beta) = \frac{\sin(\alpha) \left( g(\theta) + \left( 1 + q_1 \left( 1 - \frac{\cos^2(\alpha)}{3} \right) \right) \cos(\alpha) \right)}{f(\theta) + q_2},
\]

where \( f(\theta) \) and \( g(\theta) \) are defined in eqs. (A.18)–(A.19). On the other hand, the parameters \( q_1 \) and \( q_2 \) are defined to respect eq. (B.5). That is, they parametrize the projection of the source \( \mathcal{J}^A_0 \) on the directions \( \mathcal{N}^A_0 \) and \( \mathcal{B}^A_0 \) respectively. It may be seen that \( q_1 \) parameterizes the coupling of the two fields system and radiation, whereas the self-interaction of the two fields system is parameterized by \( q_2 \). To continue, the inflationary parameters describing the background dynamics acquire the form:

\[
\varepsilon = \frac{\dot{\phi}_0^2}{2} = \frac{3 - \frac{V}{\Pi^2}}{1 + \frac{\sin^2(\alpha)}{2}},
\]

\[
\eta_|| = 2 + \cos^2(\alpha) \left( 1 + \frac{q_1 \sin^2(\alpha)}{3} \right) + g(\theta) \cos(\alpha),
\]

\[
\eta_\perp = \sqrt{(f(\theta) + q_2)^2 + \sin^2(\alpha) \left( g(\theta) + \left( 1 + q_1 \left( 1 - \frac{\cos^2(\alpha)}{3} \right) \right) \cos(\alpha) \right)^2}.
\]
In this way, the equations of motion for the background are reduced to:

\[ \alpha' - \sin(\alpha) \left( g(\theta) + \left( 1 + q_1 \left( 1 - \frac{\cos^2(\alpha)}{3} \right) \right) \cos(\alpha) \right) = 0, \]  

\[ (4.6) \]

\[ \theta' - \frac{(f(\theta) + q_2)}{\cos(\alpha)} + \sqrt{\frac{\varepsilon}{2}} \cos(\alpha) \left( \frac{\cos(\theta) \partial_{(2)}q_{(11)}}{\sqrt{q_{(22)}q_{(11)}}} + \frac{\sin(\theta) \partial_{(1)}q_{(22)}}{\sqrt{q_{(11)}q_{(22)}}} \right) = 0, \]  

\[ (4.7) \]

\[ \beta' - C + (f(\theta) + q_2) \tan(\alpha) = 0. \]  

\[ (4.8) \]

Equation (4.2) comes from \( \mathcal{V}_B + J_B = 0 \), hence (4.8) gives us an expression of \( C \). In addition, from eq. (B.5) in appendix B, we deduce:

\[ q_N = q_2 \cos(\beta) + q_1 \sin(\alpha) \cos(\alpha) \sin(\beta), \]  

\[ (4.9) \]

\[ q_B = q_2 \sin(\beta) - q_1 \sin(\alpha) \cos(\alpha) \cos(\beta). \]  

\[ (4.10) \]

In order to solve the perturbative equations, we will need \( R_{NTTN}, R_{NTTB} \) and \( R_{BTTB} \). Given that the field metric is given by (4.1), the only non-zero component of the Riemann tensor is given by:

\[ R_{(1212)} = \frac{q_{(11)}q_{(22)}}{2} R, \]  

\[ (4.11) \]

with \( R \) the Ricci scalar given by:

\[ R = \frac{1}{2q_{(11)}q_{(22)}} \left( -2q_{(11)}q_{(22)} \partial_{(2)}q_{(11)} - 2q_{(11)}q_{(22)} \partial_{(1)}q_{(22)} + q_{(22)} \left( \partial_{(2)}q_{(11)} \right)^2 \right. \]

\[ + q_{(22)} \left( \partial_{(1)}q_{(11)} \right) \left( \partial_{(1)}q_{(22)} \right) + q_{(11)} \left( \partial_{(2)}q_{(11)} \right) \left( \partial_{(2)}q_{(22)} \right) + q_{(11)} \left( \partial_{(1)}q_{(22)} \right)^2 \right). \]  

\[ (4.12) \]

Then, it follows that

\[ R_{NTTN} = \left( N^{(1)}T^{(2)}T^{(1)}N^{(2)} - N^{(1)}T^{(1)}T^{(2)}N^{(1)} - N^{(2)}T^{(1)}T^{(1)}N^{(2)} \right. \]

\[ + \left. N^{(2)}T^{(1)}T^{(2)}N^{(1)} \right) R_{(1212)} \]

\[ = - \cos^2(\alpha) \cos^2(\beta) \frac{R}{2}, \]  

\[ (4.13) \]

\[ R_{NTTB} = \left( N^{(1)}T^{(2)}T^{(1)}B^{(2)} - N^{(1)}T^{(1)}T^{(2)}B^{(1)} - N^{(2)}T^{(1)}T^{(1)}B^{(2)} \right. \]

\[ + \left. N^{(2)}T^{(1)}T^{(2)}B^{(1)} \right) R_{(1212)} \]

\[ = - \cos^2(\alpha) \sin(\beta) \cos(\beta) \frac{R}{2}, \]  

\[ (4.14) \]

\[ R_{BTTB} = \left( N^{(1)}T^{(2)}T^{(1)}N^{(2)} - N^{(1)}T^{(1)}T^{(2)}N^{(1)} - N^{(2)}T^{(1)}T^{(1)}N^{(2)} \right. \]

\[ + \left. N^{(2)}T^{(1)}T^{(2)}N^{(1)} \right) R_{(1212)} \]

\[ = - \cos^2(\alpha) \sin^2(\beta) \frac{R}{2}. \]  

\[ (4.15) \]

It is important to recall that the main motivation for developing the covariant formalism in a three-dimensional field-space is that it provides us well-defined slow-roll parameters \( \varepsilon, \eta_\parallel, \) and \( \eta_\perp \) during inflation as well as the transition from inflation up to radiation-dominated epoch.
In addition, we stress that the present prescription necessarily introduces a coupling between the two-field system and a radiation fluid, parameterized by \( q_1 \), making the smooth transition into reheating possible. On the other hand, \( q_2 \) parameterizes the self-interaction of the two-field system. Thus, in order to analyze the consequences of the interaction between the field fluctuations and the thermal bath (and to simplify our analysis) we will set \( q_2 = 0 \) from the beginning. As we shall see, the range of acceptable values for \( q_1 \) is somewhat restricted. If \( q_1 \) is too small, the transition from inflation to the radiation-dominated epoch becomes extremely oscillatory, and the thermalization of the universe is not efficient. On the other hand, if \( q_1 \) is too large, radiation will start to be produced too early during inflation, drastically modifying the usual analytical predictions describing the epoch at which the fluctuations crossed the horizon (we will come back to this issue later on).

In what follows, we apply the angular parametrization specified by \((\alpha, \beta, \theta)\) and solve the equations of motion for these angles, given by eqs. (4.6)–(4.8). In this way, the slow-roll parameters \( \varepsilon, \eta_\parallel, \) and \( \eta_\perp \) which describe the background evolution may be completely determined. Once the background evolution of our system is known, we solve the set of eqs. (3.80)–(3.84) for the variables \( R, \Phi_B, S_N, \) and \( S_B \) numerically. By imposing the standard Bunch-Davies initial conditions, we shall evolve these perturbations from the sub-horizon to the super-horizon scales, and analyze the effects of the transition from inflation to the radiation dominated epoch on the evolution of the large-scale perturbations.

4.1 Product-exponential (PE) potential

The first model we consider is the product-exponential (PE) potential. This is a canonical two-field model with the product separable form:

\[
V(\phi_1, \phi_2) = V_0 \phi_1^2 e^{-\lambda \phi_2^2},
\]

where \( V_0 \) sets the energy scale of the potential and is of mass dimension two. Its value sets the scale of inflation and determines the amplitude of the primordial power spectrum. Inflation takes place at super-Planckian values for the \( \phi_1 \) field and the exponential factor is very much suppressed, i.e. \( \lambda \phi_2^2 \ll 1 \). For this model \( \phi_1 \) is identified as the inflaton and \( \phi_2 \) as the subdominant field which sources the isocurvature perturbations. Additionally, the field-space metric is given by eq. (4.1) with \( q_{(11)} = q_{(22)} = 1 \). This potential was first introduced in [73] to study the generation of non-Gaussianity.

An interesting feature of this model is that the adiabatic regime is not reached during inflation, and so the curvature perturbation \( R \) and its power spectrum will continue to evolve after inflation has ended. In addition, the scale dependence of the curvature perturbation is not close to the maximum likelihood value from current observational data. In particular, at the end of inflation the spectral index of the curvature perturbations is around \( n_R \simeq 0.794 \) [87, 88]. However, the effects of reheating can not be neglected for this model. Indeed, the coupling of this model to radiation makes it possible that the adiabatic limit be reached before the end of inflation. In addition, after reheating the value of \( n_R \) will be closer to the currently accepted value.

For the dimensionless parameter \( q_1 \), which parameterizes the coupling of the two-field system and radiation, we set the value \( q_1 = 3.4 \). Regarding the choice for the parameter values of the product exponential potential, firstly we consider \( \lambda = 0.03/M_p^2 \), which is close to the value used in refs. [87, 88]. Now, by regarding that this model may reach the adiabatic limit at the end of reheating, \( V_0 \) should be chosen to match the Planck 2015 maximum likelihood value \( P_R \simeq 2.2 \times 10^{-9} \) for the pivot scale \( k_0 = 0.002 \) Mpc\(^{-1} \) at that time. As
Figure 1. Transition from inflation to the radiation dominated era, through reheating, at background level for the product-exponential potential (4.16). Figures (a) and (b) show the evolution from the last stage of inflation to the radiation dominated epoch for the slow-roll parameters $\varepsilon$ and $\eta_\parallel$, respectively, which are plotted against the number of $e$-folds starting from the end of inflation. The fields $\phi_1$ (solid blue line) and $\phi_2$ (dashed red line) as function of the number of $e$-folds from the last $e$-folds of inflation to the end of reheating is depicted in figure (c). Figure (d) shows the evolution of fractional contributions of the energy density from the inflaton fields (solid blue line) and radiation (dashed red line).

It can be seen, the value $V_0 = 1.23 \times 10^{-11} M_p^2$ sets the correct normalization of the power spectra at the end of reheating. For the results shown here, the initial values taken for the scalar fields $\phi_1$ and $\phi_2$ are $17 M_p$ and $0.0025 M_p$, respectively. Finally, the initial value for the dimensionless density parameter for radiation is set to $3.46 \times 10^{-17}$. Figure 1 shows the transition from inflation to radiation-dominated epoch. In particular, figure 1(a) presents the evolution of the $\varepsilon$ slow-roll parameter from the last $e$-folds of inflation to a late evolution where $\varepsilon = 2$, consistent with a radiation-dominated universe, after oscillating during a few
e-folds. On the other hand, figure 1(b) presents the evolution of $\eta_{\parallel}$ during the last e-folds of inflation and the beginning of the radiation-dominated epoch. It is interesting to note that $\eta_{\parallel}$ has an oscillatory behavior with a larger amplitude than $\varepsilon$ about the value 2, which is the expected value that this takes when the universe becomes radiation-dominated. As it is depicted in figure 1(c), the $\phi_1$ field oscillates about its minimum and its kinetic energy is transferred to the radiation fluid. As radiation becomes the dominant component, Hubble damping slows the motion of $\phi_2$, which becomes a constant value. Finally, in figure 1(d), we have plotted the fractional contributions (with respect to the total energy density) of the energy densities of the inflaton fields (solid blue line) and radiation (dashed red line). Notice that the value of the fractional contribution of the energy density from radiation at the end of inflation becomes $\Omega_R \simeq 0.07$, being negligible in comparison to the contribution of the inflaton fields. On the other hand, we also confirm that the universe reheats properly in this model. We can see that as the model reheats, the proportion of energy density stored in the fields drops to zero and is converted into radiation. The numerical computation stops around 3 e-folds of reheating, when $\Omega_R \simeq 0.994$. Then we assume that reheating ends at that time.

In figure 2, the ratio $\Gamma_{\text{eff}}/3H$ is plotted against the number of e-folds during and after inflation. As it can be seen in figure 2(a), the effective decay rate $\Gamma_{\text{eff}}$ stays much smaller than Hubble rate during inflation, ensuring that the dynamics of inflation under slow-roll is not modified (figure 1). On the other hand, right after inflation, this ratio increases and when reheating is complete we have that $\Gamma_{\text{eff}} \gtrsim 3H$ (figure 2(b)).

Figure 3 shows the evolution of primordial perturbations during inflation as well as during reheating. In first place, figures 3(a) and 3(b) show the comparison of the power spectra $P_R$ (solid blue line), $P_{S_{N_0}}$ (dashed red line), and $P_{S_{B_0}}$ (dotted green line) at the pivot scale $k_0 = 0.002$ Mpc$^{-1}$ during inflation and reheating, respectively. At this point we recall that, in the basis $(T^A_0, N^A_0, B^A_0)$, $S_{B_0}$ represents the interaction between the inflaton fields and radiation at perturbative level, in the sense that it interchanges the contribution to the total curvature from one of them to another, as it can be inferred from eq. (3.95). In addition, eq. (3.92) shows that $S_{N_0}$ represents the isocurvature of the inflaton fields. What is clear from figure 3(a) is that, during inflation, the isocurvature perturbation measured by
$S_{N0}$ has a significantly larger amplitude than the curvature perturbation $R$ which slowly increases. In particular, when inflation ends, the amplitude of the isocurvature $S_{N0}$ becomes $10^{-8}$, being one order of magnitude larger than the curvature power spectra. In addition, an interesting feature produced by the small, but non-vanishing coupling between the two-field system and the radiation fluid during inflation, is that the scalar spectral index at the end of inflation becomes $n_R = 0.974$, which is larger than those obtained for the decoupled case [87, 88]. On the other hand, regarding the power spectra of the isocurvature $S_{B0}$, it has a lower value in comparison with the other ones during inflation. In addition, as it can be seen from figure 3(b), the amplitude of the curvature perturbation remains almost a constant value after the end of inflation and also presents an oscillatory behavior with tiny
amplitude. This behavior is more clearly shown in figure 3(b), which displays the evolution of the magnitude of $\mathcal{R}$ in comparison to the Bardeen potential $\Phi_B$. During reheating, the amplitude of $\mathcal{S}_{\mathcal{V}0}$ also presents oscillatory patterns, staying almost the same value reached at the end of inflation. Approximately at 2.5 $e$-folds after inflation ends, the power spectra of $\mathcal{S}_{\mathcal{V}0}$ begins to be suppressed and then, at the end of reheating, the curvature and isocurvature power spectra take the values $2.2 \times 10^{-9}$ and $3.2 \times 10^{-10}$, respectively. Our results suggest that the adiabatic limit is only reached at the end of reheating. This is due that $\phi_1$ asymptotes to a constant as we approach to $\Omega_R \simeq 1$, then the trajectory in the field space does not evolve any longer and $\mathcal{R}$ approaches to a constant value. However, the numerical computation needs to be improved in order to go more $e$-folds further and obtain a more definitive result.

Regarding the post-inflationary evolution of both the Bardeen potential $\Phi_B$ and the curvature perturbation $\mathcal{R}$, figure 3(c) shows that at the end of the reheating phase, $\Phi_B \simeq 0.669\mathcal{R}$, being in agreement with the condition $\Phi_B = \frac{3}{7}\mathcal{R}$, which holds for a radiation-dominated phase [89]. The comparison of the scale dependence of the power spectra at the end of reheating is displayed in figure 3(b) for the range $0.1 \leq k/k_0 \leq 10$. This plot illustrates the near scale invariance of the power spectra. In particular, for the curvature power spectra (solid blue line), the scalar spectral index becomes $n_R \simeq 0.973$, which deviates from the maximum likelihood value from Planck 2015.

Alternatively, we may split the total curvature perturbation $\mathcal{R}$ as $\mathcal{R} = \zeta_\phi + \zeta_R$, where $\zeta_\phi$ and $\zeta_R$ are the individual curvature perturbations of the two-field system and radiation defined on hypersurfaces orthogonal to comoving world lines, given by eqs. (3.91) and (3.95), respectively. The evolution of power spectra of $\zeta_\phi$ (dashed red line) and $\zeta_R$ (dotted green line) during inflation as well as during reheating, in comparison to the power spectra of $\mathcal{R}$ (solid blue line), is depicted in figure 4. Since we have introduced a coupling between the inflaton fields and radiation fluid from the beginning of the numerical computation, figure 4(a) confirms that during the last 20 $e$-folds of inflation, $\zeta_\phi$ deviates from $\mathcal{R}$, however the contribution from the radiation fluid to curvature perturbation may be regarded negligible.
Around 0.5 e-folds after the end of inflation, the power spectra of \( \zeta_\phi \) and \( \zeta_R \) become equal, but shortly after, the power spectra of \( \zeta_\phi \) begins to increase with oscillatory behavior and becomes equal to the power spectra of curvature perturbation, i.e. \( R \simeq \zeta_R \), and hence the power spectra of \( \zeta_\phi \) rapidly gets suppressed, as can be seen from figure 4(b). Our results show that during the first stages of reheating, although the energy stored in the fields has not been totally transferred to radiation fluid, the main contribution to total curvature comes from radiation fluid.

We have shown for this model that, at the end of reheating, the curvature perturbation becomes almost constant, achieving the adiabatic limit. Additionally, the initial conditions for the perturbations during radiation-dominated phase are set at the end of reheating phase. Finally, the tilt of the curvature power spectrum \( n_R \) is enhanced during reheating, reaching a value marginally consistent with the maximum likelihood from current Planck data. Notice that our result for the spectral index at the end of reheating differs from that obtained in [81], where the authors used the \( \delta N \) formalism to study the effects of a reheating phase on the statistics of large-scale fluctuations in two-field inflation with a product exponential potential. In particular, they considered a constant decay rate and found that \( n_R \simeq 0.761 - 0.763 \). In view of this, we infer that an energy transfer from the inflaton fields to the radiation fluid during and after inflation can affect the amplitude and the spectral index as well.

### 4.2 An ultra-light field (ULF) coupled to inflation

As it was mentioned in the introduction, the presence of light scalar field with masses much smaller than the Hubble expansion rate are known to produce potentially large levels of isocurvature perturbations, leading to super-horizon evolution of curvature perturbations and possibly observable features in the CMB. Motivated by this, in ref. [50], the authors studied the consequences of considering the extreme situation in which a non-adiabatic mode is approximately massless, and its interaction with the curvature perturbation persists during the whole period of inflation, from horizon crossing until reheating. The authors provided a concrete example in which an ultra-light field emerges, that appears within a well studied class of models consisting of a multi-field action with a non-canonical kinetic term [53], typical of supergravity and string theory compactifications. For this class of models, the field-space metric is given by eq. (4.1) with \( q_{(11)} = e^{2\phi_2/R_0} \) and \( q_{(22)} = 1 \), which describes a two-dimensional hyperbolic manifold of curvature \(-2/R_0^2\).

In order to obtain concrete results, the authors studied the dynamics of the system for a monomial potential of the form:

\[
V(\phi_1) = V_0 (\phi_1/\phi_0)^n, \tag{4.17}
\]

where \( \phi_0 \) is the value of the field \( \phi_1 \) at a given reference time \( t_0 \). For this non-canonical model with the monomial potential, the authors found that, in order to obtain 60 e-folds of inflation and a value for the scalar spectral index \( n_R \) close to the maximum likelihood value from Planck 2015, the power \( n \) must satisfy \( n < 4/5 \), implying that the potential \( V \) must be concave. For the particular case \( n = 1/2 \), it is required that \( R_0 = 2/3 \) which implies that \( n_s = 0.967 \). This ensures a huge enhancement to the curvature power spectrum at super-horizon scales. Then, the presence of such a light field implies non-vanishing isocurvature modes and a super-horizon evolution of the curvature perturbation \( R \). This makes it really interesting to study the post-inflationary evolution of this class of model to see how reheating modifies the evolution of the primordial observables. In that sense, the results regarding the evolution after inflation for this model are new.
A first point to note here is that if the power \( n \) of the monomial potential is an odd number, the potential of eq. (4.17) is not suitable to properly finish inflation. However, by introducing the modified potential given by:

\[
V_1(\phi_1) = V_0 \left(1 + \left(\frac{\phi_1}{\phi_0}\right)^2\right)^{\frac{n}{2}} - 1,
\]

it is possible to address the end of inflation. For this modified potential, \( \tilde{\phi}_0 \) denotes the value of the scalar field \( \phi_1 \) for which the potential changes its concavity. When \( \phi_1/\tilde{\phi}_0 \gg 1 \), this potential behaves as the monomial one and also presents a minimum at \( \phi_1 = 0 \). On the other hand, by adding an axion-like potential [90]:

\[
V_2(\phi_1) = \Lambda^4 \left[1 - \cos\left(\frac{\phi_2}{f}\right)\right],
\]

characterized by two mass scales \( f \) and \( \Lambda \), with \( f \gg \Lambda \), and provided that \( V_0 \gg \Lambda^4 \), the effects of this potential on the inflationary dynamics will be sub-dominant comparing to \( V(\phi_1) \), however, as it will be shown, the new term will be relevant in the post-inflationary dynamics, since the light field will acquire a small mass term, stabilizing its dynamics at the end of reheating and providing the suppression of the isocurvature perturbation during reheating. Then, the total potential for studying the inflationary dynamics together with the transition into reheating, is considered to be:

\[
V = V_1 + V_2 = V_0 \left(1 + \left(\frac{\phi_1}{\phi_0}\right)^2\right)^{\frac{n}{2}} - 1 + \Lambda^4 \left[1 - \cos\left(\frac{\phi_2}{f}\right)\right].
\]

In order to model reheating for this potential and determine the possible effects of the coupling between the inflaton fields and the radiation fluid, parameterized by \( q_1 \), and the value of the field \( \phi_1 \) for which the concavity of potential changes, in our numerical implementation, we will study three different cases separately, namely i) \( q_1 = 3.28 \) and \( \tilde{\phi}_0 = \phi_{10}/50 \), ii) \( q_1 = 3.28 \) and \( \tilde{\phi}_0 = \phi_{10}/90 \), and finally, iii) \( q_1 = 3.33 \) and \( \tilde{\phi}_0 = \phi_{10}/50 \). For all cases to be studied, we use the following set of values:

\[
\begin{align*}
\phi_{10} &= 11M_p, & \phi_{20} &= -0.2R_0M_p, \\
A &= 2 \times 10^{-2}V_0^{1/4}, & f &= 0.4M_p, \\
R_0 &= 2, & V_0 &= A \times 10^{-11} \left(\frac{\phi_0}{\phi_{10}}\right)^n, \\
\Lambda &= 2 \times 10^{-2}V_0^{1/4}, & \phi_{10} &= 11M_p,
\end{align*}
\]

where \( \phi_{10} \) and \( \phi_{20} \) denote the initial field values and \( A \) is a dimensionless parameter. These choices allow us to compare our results with the previous work [50]. In addition, for each case, the \( A \) parameter must be set in order to give the correct normalization of the curvature power spectra. Finally, the corresponding initial values for the dimensionless density parameter for radiation \( \Omega_R \) are set to \( 9.74 \times 10^{-15}, 1.21 \times 10^{-11}, \) and \( 1.28 \times 10^{-11} \), respectively.

i) \( q_1 = 3.28 \) and \( \tilde{\phi}_0 = \phi_{10}/50 \). The smooth transition from inflation to the radiation dominated-epoch is depicted in figure 5. Particularly, figure 5(a) shows the evolution of the \( \varepsilon \) slow-roll parameter from the last e-folds of inflation to the radiation dominated-epoch.
Figure 5. Transition from inflation to the radiation dominated era, through reheating, at background level for the case (i), corresponding to the potential (4.20). In particular, figures (a) and (b) show the evolution from the last stage of inflation to the radiation dominated epoch for the slow-roll parameters $\varepsilon$ and $\eta_\parallel$, respectively, which are plotted against the number of $e$-folds from the end of inflation. The fields $\phi_1$ (solid blue line) and $\phi_2$ (dashed red line) as function of the number of $e$-folds from the last $e$-folds of inflation to the end of reheating is depicted in figure (c). Figure (d) shows the evolution of fractional contributions of the energy density from the inflaton fields (solid blue line) and radiation (dashed red line).

After oscillating during a few $e$-folds after the end of inflation, $\varepsilon$ reaches the constant value 2. Interestingly, at around 3 $e$-folds after inflation ends, $\varepsilon$ decreases and reaches a minimum value, and later it starts to increase and oscillate about 2. On the other hand, figure 5(b) presents the evolution of $\eta_\parallel$ during the last $e$-folds of inflation and the beginning of the radiation-dominated epoch. As the in previous model, $\eta_\parallel$ also has an oscillatory behavior about the value 2, consistent with a radiation-dominated universe, however the amplitudes of oscillations of both $\varepsilon$ and $\eta_\parallel$ during reheating become smaller compared to the previous model. In addition, there is a feature between 3 and 4 $e$-folds after the end of inflation.
This feature in the behavior of both $\varepsilon$ and $\eta_{||}$ may be explained by studying the evolution of the fields during reheating, which is depicted in figure 5(c). We observe that both field $\phi_1$ (solid blue line) and $\phi_2$ (dashed red line) oscillate about their minimum and the kinetic energy stored in the fields is transferred to the radiation fluid. As radiation becomes the dominant component, Hubble damping slows down the motion of $\phi_1$ and $\phi_2$. In particular, $\phi_1$ decays to its minimum before than $\phi_2$. This result suggest that most of the kinetic energy is stored in $\phi_2$, then the reheating phase for this case is driven by $\phi_2$. This becomes clearer from figure 6(d), where we have plotted the fractional contributions of the energy densities of the inflaton fields and radiation. Before decaying at the minimum of its potential, $\phi_2$ reaches a maximum value during its oscillations to later invert its motion and finally decay. This behavior yields to a suddenly decrease of the amplitude of the oscillating $\varepsilon$ and $\eta_{||}$, and a sudden increase of the fractional contribution of the inflaton field during a short period. The fractional contribution of radiation at the end of inflation becomes $\Omega_R \simeq 0.007$, then our choice for the values of parameters ensures that this contribution becomes negligible in comparison to the contribution of the inflaton fields. We can see that at the moment when $\phi_1$ reaches its minimum, most of the proportion of energy density in the fields comes from $\phi_2$ and, as the model reheats, the proportion of energy density in $\phi_2$ drops to zero and is converted into radiation. For this case, the numerical computation stops around 6 e-folds of reheating, when $\Omega_R \simeq 0.993$ (dashed red line). Then, we set this time as the end of reheating.

Regarding the dynamics of perturbations, we can see from figure 6(a) that, during inflation, the curvature power spectrum at pivot scale $k_0 = 0.002$ Mpc$^{-1}$ (solid blue line) increases monotonically, whereas the spectrum of the isocurvature perturbation $S_{\phi_0}$ (red dashed line) is slowly decreasing and, as inflation ends, it becomes four order of magnitude smaller than the curvature perturbation. Since we are introducing a coupling between the two-field system and the radiation fluid from the beginning of the inflationary phase, the choice of parameters for the case $q_1 = 3.2$ and $\phi = \phi_{10}/50$ yields a value for the scalar spectral index of $n_R = 0.974$ for the pivot scale at the end of inflation, which deviates from the value obtained in [50], given by 0.967. An explanation for this discrepancy may be found in the value we set for $\phi_0 = \phi_{10}/50$. Since that $\phi$ represents the value for $\phi_1$ at which the potential changes its concavity in order to achieve the end of inflation, as $\phi_0$ is increased, $n_R$ becomes greater than the maximum likelihood of Planck.

Regarding the analysis of post inflationary evolution of super-horizon fluctuations, we recall that in the basis $(T^A, N_0^A, B_0^A)$, $S_{\phi_0}$ gives us an account on how the inflationary fields interact with radiation. On the other hand, $S_{\phi_0}$ corresponds to the isocurvature fluctuation. In figure 6(b) we zoom into the evolution through the almost 6 e-folds of reheating. From figure 6(c) we can see that during the initial phase of reheating, the curvature perturbation $\mathcal{R}$ presents a small enhancement, however this feature cannot be seen clearly in figure 6(b). Figure 6(c) also shows the evolution of the Bardeen potential $\Phi_B$ during reheating, and we notice that at the end of reheating, the curvature perturbation and the Bardeen potential satisfy the relation, $\Phi_B \simeq 0.655 \mathcal{R}$, being close to the condition $\Phi_B = \frac{3}{2} \mathcal{R}$, which is satisfied during the radiation-dominated phase. Back to figure 6(b), the isocurvature perturbation $S_{\phi_0}$ begins to increase, presenting oscillatory patterns, but always being smaller than the curvature perturbation, which is still evolving. At around 3 e-folds after the end of inflation, it can be noticed that the isocurvature perturbation $S_{\phi_0}$ begins to be suppressed and the curvature perturbation reaches an almost constant value when reheating ends. At this time, the curvature power spectra has the value $2.28 \times 10^{-9}$, in agreement with Planck normalization of $P_R$, whereas the isocurvature perturbation $S_{\phi_0}$ becomes seven orders of magnitude smaller than the curvature perturbation.
The comparison of the scale dependence of the several power spectra at the end of reheating is displayed in figure 6(d), for scales within the range $0.1 \leq k/k_0 \leq 10$. This plot illustrates the near scale invariance of the power spectra. In particular for the curvature power spectra (solid blue line), the scalar spectral index becomes $n_R \simeq 0.989$, which is very close to scale invariance. Clearly this value presents a significant deviation from the maximum likelihood value from Planck 2015. On the other hand, for all the range $0.1 \leq k/k_0 \leq 10$, the power spectrum of $S_{\eta_0}$ is about 7 orders of magnitude smaller than the curvature power spectrum.

Finally, by splitting the total curvature perturbation $\mathcal{R}$ as $\mathcal{R} = \zeta_\phi + \zeta_R$, as it was done for the product exponential potential, figure 7(a) shows that, during inflation, the main contribution to total curvature perturbation (solid blue line) comes from the individual
curvature of inflaton fields $\zeta_\phi$ (dashed red line), and so its contribution due radiation fluid $\zeta_R$ (dotted green line) becomes negligible. However, as it can be noticed in figure 7(b), 2 e-folds after inflation ends, $\zeta_R$ rapidly becomes the main contributor to curvature perturbation whereas the individual curvature of the inflationary fields is rapidly suppressed although $\phi_2$ has not decayed totally in radiation. This result confirms our assumption that in order to have a smooth transition from inflation to the radiation-dominated epoch at background and perturbative level, the effects of radiation production must be taken into account in the computations.

Then, we conclude that at the end of reheating for this case, the curvature perturbation becomes almost constant and the isocurvature mode becomes suppressed, achieving the adiabatic limit. Additionally, the initial conditions for the perturbations during the radiation-dominated phase are set at the end of the reheating phase. In addition, the curvature power spectrum is close to scale invariance, but it deviates from the maximum likelihood from current Planck data.

ii) $q_1 = 3.28$ and $\tilde{\phi}_0 = \phi_{10}/90$. In order to see the effects of modifying $\tilde{\phi}_0$ on the dynamics, the second case to be studied corresponds to $q_1 = 3.28$ and $\tilde{\phi}_0 = \phi_{10}/90$, for which the background evolution from inflation to the radiation-dominated epoch is displayed in figure 8. In particular, figures 8(a) and 8(b) show the evolution of the slow-roll parameters $\varepsilon$ and $\eta_\parallel$ from the last e-folds of inflation up the radiation-dominated epoch, respectively. For this case, the behavior of both $\varepsilon$ and $\eta_\parallel$ is smoother than in the previous case, without any feature. As we can see, like the previous case $\phi_1$ decays before $\phi_2$ does, and so most of the kinetic energy is stored in $\phi_2$. After $\phi_2$ reaches the minimum of its potential, the friction term produces a damping oscillation and finally decays into radiation. Regarding the behavior of the fractional energy densities, the contribution coming from radiation at the end of inflation becomes $\Omega_R \simeq 0.021$, which is smaller than the previous case. When the numerical computation stops, 5.3 e-folds after inflation ends, we find that $\Omega_R \simeq 0.9991$, signaling the end of reheating.
Figure 8. Transition from inflation to the radiation dominated era, through reheating, at background level for the parameter choice (ii). In particular, figures (a) and (b) show the evolution from the last stage of inflation to the radiation dominated epoch for the slow-roll parameters $\epsilon$ and $\eta$, respectively, which are plotted against the number of e-folds from the end of inflation. The fields $\phi_1$ (solid blue line) and $\phi_2$ (dashed red line) as functions of the number of e-folds from the last e-folds of inflation to the end of reheating is depicted in figure (c). Figure (d) shows the evolution of fractional contributions of the energy density from the inflaton fields (solid blue line) and radiation (dashed red line).

Figures 9(a) and 9(b) show the comparison of the power spectra $P_R$ (solid blue line), $P_{S_N}$ (dashed red line), and $P_{S_B}$ (dotted green line) at the pivot scale $k_0$ during inflation and reheating, respectively. From figure 9(a), we observe that the behavior of the power spectra during inflation is practically indistinguishable from the previous case. However, the crucial difference between both cases comes from the tilt of the curvature power spectrum and the post inflationary evolution of $R$. Regarding the value of the scalar spectral index at the end of inflation, we found that $n_R = 0.966$, which is closer than that obtained in [50]. This value for the tilt of the curvature power spectrum at the end of inflation can be achieved by
Figure 9. Figures 9(a) and 9(b) show the comparison of the power spectra $P_R$ (solid blue line), $P_{SN0}$ (dashed red line), and $P_{SB0}$ (dotted green line) at the pivot scale $k_0 = 0.002 \text{ Mpc}^{-1}$ during inflation and reheating, respectively, for the case (ii). In figure 9(c) we present the evolution of the amplitudes of the Bardeen potential $\Phi_B$ and total curvature perturbation during reheating, which also has been plotted as a function of the number of $e$-folds $N$. The power spectra $P_R$ (solid blue line), $P_{SN0}$ (dashed red line), and $P_{SB0}$ (dotted green line) in terms of the ratio $k/k_0$ at the end of reheating are plotted in figure 9(d).

Decreasing the value of $\phi_1$ at which the potential changes its concavity. Figure 9(c) shows the post-inflationary evolution of the curvature perturbation $R$ and the Bardeen potential, $\Phi_B$, which are plotted against the number of $e$-folds. The curvature perturbation is still evolving during the first 2 $e$-folds after inflation ends, and then becomes constant. A similar behavior is observed in $\Phi_B$, which presents an enhancement during the first $e$-fold after the end of inflation, and then becomes constant until the end of reheating like as $R$. In particular, at the end of reheating, these satisfy $\Phi_B \simeq 0.666R$, being closer to the conditions which holds for radiation-dominated universe in comparison to previous case.

Back to figure 9(b), during the initial phases of reheating, the isocurvature $S_{N0}$ is enhanced displaying a structure of spikes, but always being smaller than the curvature per-
turbation. At around 3 e-folds after inflation ends, it can be noticed that the isocurvature perturbation $S_{N0}$ reaches a maximum value and, shortly after, it begins to be suppressed and by the end of reheating, $P_{S_{N0}}$ becomes five orders of magnitude smaller than $P_{R}$, whose value is $2.23 \times 10^{-9}$.

The comparison of the scale dependence of the several power spectra at the end of reheating is displayed in figure 9(d), for scales within the range $0.1 \leq k/k_0 \leq 10$. For the curvature power spectra (solid blue line), the scalar spectral index now becomes $n_R \simeq 0.965$, which corresponds to maximum likelihood value from Planck 2015. For all the scales within the range $0.1 \leq k/k_0 \leq 10$, the power spectrum of $S_{N0}$ is about 5 orders of magnitude smaller than the curvature power spectra.

By splitting the total curvature perturbation $R$ as $R = \zeta_\phi + \zeta_R$, figure 10(a) shows that, during inflation, the evolution of $R$, $\zeta_\phi$, and $\zeta_R$ becomes indistinguishable from the case (i). However, from figure 10(b), it can be noticed that $\zeta_R$ becomes the main contribution to curvature perturbation around one e-fold after inflation ends, before than case (i) (see figure 7(b)). In a similar way, the individual curvature of the inflaton fields $\zeta_\phi$ is rapidly suppressed although $\phi_2$ has not decayed totally in radiation.

For this case we conclude that, for a value of $\tilde{\phi}_0$ closer to zero, the curvature perturbation does not evolve during the initial stages of reheating and the isocurvature mode becomes suppressed, achieving the adiabatic limit before the previous case. Additionally, the initial conditions for the perturbations during the radiation-dominated phase are also set at the end of reheating phase. Moreover, the scale dependence of the curvature power spectrum becomes compatible with maximum likelihood from current Planck data, since its value is smaller than in the previous case.

iii) $q_1 = 3.33$ and $\tilde{\phi}_0 = \phi_{10}/50$. In order to see whether by increasing $q_1$ one modifies the results in comparison to the case (i), we set $q_1 = 3.33$ and $\tilde{\phi}_0 = \phi_{10}/50$. As it can be seen from figure 11, the transition from the last stage of inflation up to the radiation-dominated epoch at background level is smoother than in case (i). The reason for this behavior is that,
in the present case, there is a stronger dissipation than in case (ii). Regarding the evolution of the fractional energy densities, the contribution coming from radiation at the end of inflation becomes $\Omega_R \simeq 0.052$, which is larger than the contribution of radiation compared to case (i), since $q_1 = 3.3$ leads to a stronger dissipation compared to $q_1 = 3.28$. For this case, the numerical computation stops around 5.3 $e$-folds after inflation ends, and $\Omega_R \simeq 0.999$, signaling the end of reheating.

From figure 12(a), we observe that the behavior of the several power spectra during inflation is similar to that of case (i), being of the same order at the end of inflation. However, the tilt of the curvature power spectrum at the end of inflation becomes $n_R = 0.979$, which
Figure 12. Figures 12(a) and 12(b) show the comparison of the power spectra $P_R$ (solid blue line), $P_{SN0}$ (dashed red line), and $P_{SB0}$ (dotted green line) at the pivot scale $k_0 = 0.002 \text{ Mpc}^{-1}$ during inflation and reheating, respectively, for the case (ii). In figure 12(c) we present the evolution of the amplitudes of the Bardeen potential $\Phi_B$ and the total curvature perturbation during reheating, which also has been plotted as a function of the number of $e$-folds $N$. The power spectra $P_R$ (solid blue line), $P_{SN0}$ (dashed red line), and $P_{SB0}$ (dotted green line) in terms of the ratio $k/k_0$ at the end of reheating are plotted in figure 12(d).

is smaller than that of case (i). Then, an enhancement of $q_1$ produces a lowering on the value of the scalar spectral index at the end of inflation. As it can be noticed from figure 12(c), the curvature perturbation is still evolving during first 2 $e$-folds of reheating, and then becomes constant up to the end of reheating. This behavior is due the non-vanishing of the isocurvature mode, which after reaching a maximum value, displays a structure of spikes of decreasing amplitude, being suppressed until reheating ends, as it can be noticed from figure 12(b). By the end of reheating, the curvature power spectrum becomes $2.2 \times 10^{-9}$, in agreement with Planck normalization of $P_R$, being grater by four orders of magnitude than the isocurvature perturbation $S_{N0}$. Moreover, at that time the ratio between $R$ and $\Phi_B$ becomes 0.665.
Table 1. In this table we summarize the values for the scalar spectral index \( n_R \), the tensor-to-scalar ratio \( r \), and the running of the scalar spectral index \( \alpha_R \), at the end of inflation \( t_{\text{end inf}} \) and reheating \( t_{\text{end rh}} \), respectively.

| Model | \( n_R(t_{\text{end inf}}) \) | \( r(t_{\text{end inf}}) \) | \( \alpha_R(t_{\text{end inf}}) \) | \( n_R(t_{\text{end rh}}) \) | \( r(t_{\text{end rh}}) \) | \( \alpha_R(t_{\text{end rh}}) \) |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| PE    | 0.974           | 2.113\times10^{-2} | -4.272\times10^{-4} | 0.973           | 2.141\times10^{-2} | 2.116\times10^{-3} |
| ULF (i) | 0.974           | 6.548\times10^{-4} | 1.680\times10^{-3} | 0.988           | 4.524\times10^{-4} | 2.040\times10^{-3} |
| ULF (ii) | 0.964          | 6.853\times10^{-4} | 1.637\times10^{-3} | 0.965           | 6.717\times10^{-4} | 1.678\times10^{-3} |
| ULF (iii) | 0.978         | 6.147\times10^{-4} | 1.622\times10^{-3} | 0.984           | 5.243\times10^{-4} | 1.769\times10^{-3} |

Figure 12(d) displays the comparison of the scale dependence of the power spectrum at the end of reheating, for the range \( 0.1 \leq k/k_0 \leq 10 \). For the curvature power spectrum (solid blue line), the scalar spectral index now becomes \( n_R = 0.984 \), which also deviates from the maximum likelihood value from Planck 2015 and is slower than in the previous cases. However, it is interesting to mention that this value becomes smaller than that obtained in case (i) at the end of reheating, which confirms the fact that, by increasing \( q_1 \), it produces a smaller tilt of the curvature power spectrum.

Finally, regarding the evolution of the individual contributions of the total curvature coming from \( \zeta_\phi \) and \( \zeta_R \), we find that during and after inflation, we found that their behavior becomes indistinguishable than in the previous cases (figure not shown).

Table 1 summarizes the corresponding values of the scalar spectral index \( n_R \) at the end of inflation as well as at the end of reheating for the PE model and the three cases corresponding to the ULF model. In addition, we also have included the values of the tensor-to-scalar ratio \( r \) and the running of the scalar spectral index \( \alpha_R \) both at the end of inflation and reheating.

5 Conclusions

We have studied the evolution of perturbations in two-field models of inflation in which the scalar fields remain coupled to a radiation fluid. We have payed special emphasis on those cases where the isocurvature mode remain nearly massless all the way from horizon crossing up to the reheating phase. To do so, we have introduced a formalism that allows one to treat the radiation fluid as an effective scalar field. This scalar field can be treated as the third partner of a three-field model, and so one can treat the perturbation system covariantly, in terms of a three dimensional target space.

By itself, the third scalar field \( \phi^{(3)} \) has no physical meaning. On the other hand, the energy density \( \rho_R \) introduced in eq. (2.26) does have a well defined meaning. This is in part behind the power of the method introduced here: one may deal with the full scalar field system, including the third fictitious scalar, as if it was a closed system. Another advantage of the present approach is that it allows us to deal with the initial conditions for the perturbations in a simple way, by imposing the Bunch-Davies conditions for the three scalar perturbations un sub-horizon scales.

We have checked well known results in the literature, and have examined how some observable quantities, such as the spectral index of the power spectrum of primordial curvature perturbations may change due to the coupling between the scalar fields and the thermal bath. Such a coupling enables us to follow the post inflationary evolution, into reheating, for
multi-field models having a non-vanishing amount of isocurvature when inflation ends. For
the models examined here, we found that a non-vanishing coupling between the scalar fields
and a radiation fluid (parametrized by $q_1$) forces the isocurvature modes to decay rapidly as
the universe reheats. In general, when the energy stored in the scalar fields is completely
transferred to the radiation fluid we make contact with the radiation-dominated epoch of
the hot big-bang scenario, and the power spectra of the isocurvature becomes completely
negligible in comparison to the power spectrum of curvature. A more careful analysis of this
process should consider the details of how the isocurvature mode decays into different species.

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A Field space

In this appendix, we provide some useful expressions concerning the field space spanned by
the three scalar fields employed in the body of the article. In a field space with a metric $q_{AB}$,
the set $(T^A, N^A, B^A)$ offer a complete orthogonal basis and the metric may be written as:

$$q_{AB} = T_A T_B + N_A N_B + B_A B_B ,$$  \hspace{1cm} (A.1)

The equation $\frac{Dq_{AB}}{dt} = 0$ implies that

$$-H\eta_\perp N^A T^B - H\eta_\perp T^A N^B + \frac{DN_A}{dt} N^B + N_A \frac{DN_B}{dt} + \frac{DB_A}{dt} N_B + B_A \frac{DB_B}{dt} = 0. \hspace{1cm} (A.2)$$

where we used eq. (2.53) to identify $\eta_\perp$. Then:

$$T^B \frac{Dq_{AB}}{dt} = 0 \rightarrow -H\eta_\perp N^A T^B - H\eta_\perp T^A N^B + \frac{DN_A}{dt} N^B + N_A \frac{DN_B}{dt} + B_A \frac{DB_B}{dt} = 0, \hspace{1cm} (A.3)$$

$$N^B \frac{Dq_{AB}}{dt} = 0 \rightarrow -H\eta_\perp T^A + \frac{DN_A}{dt} + N_A N_B \frac{DN_B}{dt} + B_A N_B \frac{DB_B}{dt} = 0, \hspace{1cm} (A.4)$$

$$B^B \frac{Dq_{AB}}{dt} = 0 \rightarrow N^A B_B \frac{DN_B}{dt} + B_A B_B \frac{DB_B}{dt} = 0. \hspace{1cm} (A.5)$$

Additionally, from the normalization of $N^A$ and $B^A$, we have $\frac{DN_B}{dt} N_B = \frac{DB_B}{dt} B_B = 0$. Then, defining $HC = \frac{DB_B}{dt} N_B = -\frac{DN_B}{dt} B_B$, we obtain from these equations:

$$\frac{DN_A}{dt} = H\eta_\perp T^A - HCB^A, \hspace{1cm} (A.6)$$

$$\frac{DB^A}{dt} = HCN^A. \hspace{1cm} (A.7)$$

They correspond to eqs. (2.54) and (2.55) respectively. These vectors contain 9 parameters,
but they are constrained by $T^A T_A = N^A N_A = B^A B_A = 1$ and $T^A N_A = T^A B_A = N^A B_A = 0.$
So, just three degrees of freedom survive. During inflation, radiation must be negligible, so $T^{(3)} \to 0$, making contact with the two-field case, where $T^3_T = N^a N_a = 1$ and $T^3 N_a = 0$, which means that $N^{(3)} = 0$. Additionally, from $T^a B_a = N^a B_a = 0$, we obtain $B^{(1)} = B^{(2)} = 0$. Therefore, the normalization is reduced to $B^A B_A = q_{(33)} (B^{(3)})^2 = 1$. To summarize, the initial conditions are given by:

$$B^A = \begin{pmatrix} 0 \\ 0 \\ \pm \frac{1}{\sqrt{q_{(33)}}} \end{pmatrix}.$$  (A.8)

On the other side, we can define another basis $(T^A, N_0^A, B_0^A)$, given by:

$$N_0^A \equiv \cos(\beta) N^A + \sin(\beta) B^A,$$

$$B_0^A = -\sin(\beta) N^A + \cos(\beta) B^A,$$  (A.9)

for which the equations of motion are:

$$\frac{DT^A}{dt} = -H \eta_\perp (\cos(\beta) N_0^A - \sin(\beta) B_0^A),$$  (A.11)

$$\frac{DN_0^A}{dt} = H \eta_\perp \cos(\beta) T^A + \left(\dot{\beta} - HC\right) B_0^A,$$  (A.12)

$$\frac{DB_0^A}{dt} = -H \eta_\perp \sin(\beta) T^A - \left(\dot{\beta} - HC\right) B_0^A.$$  (A.13)

In this article, we will use both set of basis to study the perturbative components (See figure 13). Also, we will study cases where $q_{AB}$ is given by eq. (4.1). So, we can parameterize our vectors as:

$$T^{(1)} = \frac{1}{\sqrt{q_{(11)}}} \cos(\theta) \cos(\alpha),$$

$$T^{(2)} = -\frac{1}{\sqrt{q_{(22)}}} \sin(\theta) \cos(\alpha),$$

$$T^{(3)} = \frac{1}{\sqrt{q_{(33)}}} \sin(\alpha),$$

$$N^{(1)} = \frac{1}{\sqrt{q_{(11)}}} \left(\sin(\theta) \cos(\beta) + \cos(\theta) \sin(\alpha) \sin(\beta)\right),$$

$$N^{(2)} = \frac{1}{\sqrt{q_{(22)}}} \left(\cos(\theta) \cos(\beta) - \sin(\theta) \sin(\alpha) \sin(\beta)\right),$$

$$N^{(3)} = -\frac{1}{\sqrt{q_{(33)}}} \cos(\alpha) \sin(\beta),$$

$$B^{(1)} = \frac{1}{\sqrt{q_{(11)}}} \left(\sin(\theta) \sin(\beta) - \cos(\theta) \sin(\alpha) \cos(\beta)\right),$$

$$B^{(2)} = \frac{1}{\sqrt{q_{(22)}}} \left(\cos(\theta) \sin(\beta) + \sin(\theta) \sin(\alpha) \cos(\beta)\right),$$

$$B^{(3)} = \frac{1}{\sqrt{q_{(33)}}} \cos(\alpha) \cos(\beta)$$  (A.14)
Figure 13. Representation of the basis vectors in field space. In black, the Tangent vector $T^A$. In red, the Normal vectors in the standard basis ($N^A, B^A$), given by eq. (A.14). In blue, the Normal vectors in the zero basis ($N^A_0, B^A_0$), given by eq. (A.15). The $\beta$ angle corresponds to $\angle N^A_0N^A_0$. This angle does not have a physical meaning; it is used to relate different bases. In the standard basis, $\beta$ is fixed in order to obtain $V_B + J_B = 0$.

and:

$$
N_0^{(1)} = \frac{1}{\sqrt{q_{11}}} \sin(\theta), \quad B_0^{(1)} = -\frac{1}{\sqrt{q_{11}}} \cos(\theta) \sin(\alpha),
$$

$$
N_0^{(2)} = \frac{1}{\sqrt{q_{22}}} \cos(\theta), \quad B_0^{(2)} = \frac{1}{\sqrt{q_{22}}} \sin(\theta) \sin(\alpha),
$$

$$
N_0^{(3)} = 0, \quad B_0^{(3)} = \frac{1}{\sqrt{q_{33}}} \cos(\alpha).
$$

(A.15)

Applying eq. (A.8) in (A.14), we note that the initial conditions are given by $\sin(\alpha) = \sin(\beta) = 0$. Actually, we can see that $\alpha$ defines the magnitude of radiation density as:

$$
\sin(\alpha) = \sqrt{\frac{2\Omega_R}{\varepsilon}},
$$

(A.16)

where we used eq. (2.26), $\rho_R^0 = 3H^2\Omega_R$ and $\dot{\phi}_0^0 = 2H^2\varepsilon$. Then, $\alpha$ goes from 0, in the beginning of inflation, to $\frac{\pi}{2}$ in the radiation era. On the other side, $\beta$ is an angle that defines the Normal plane, such as $V_B + J_B = 0$ (See figure 13). This means:

$$
\tan(\beta) = 
\frac{\sin(\alpha) \left( g(\theta) + \frac{1}{H^2 \sqrt{2} \varepsilon} \left( \frac{J_0^{(1)}}{\sqrt{q_{11}}} \cos(\theta) - \frac{J_0^{(2)}}{\sqrt{q_{22}}} \sin(\theta) \right) + \left( 1 + \frac{Q}{3H^2 \varepsilon \sin^2(\alpha)} \right) \cos(\alpha) \right)}{
\left( f(\theta) + \frac{1}{H^2 \sqrt{2} \varepsilon} \left( \frac{J_0^{(1)}}{\sqrt{q_{11}}} \sin(\theta) + \frac{J_0^{(2)}}{\sqrt{q_{22}}} \cos(\theta) \right) \right)},
$$

(A.17)
with \( Q = J^0_a \dot{\phi}^a_0 \) and:

\[
    f(\theta) = \frac{1}{H^2 \sqrt{2 \varepsilon}} \left( \frac{V_{(1)}}{\sqrt{q_{(1)}}} \sin(\theta) + \frac{V_{(2)}}{\sqrt{q_{(2)}}} \cos(\theta) \right), \quad (A.18)
\]

\[
    g(\theta) = \frac{1}{H^2 \sqrt{2 \varepsilon}} \left( \frac{V_{(1)}}{\sqrt{q_{(1)}}} \cos(\theta) - \frac{V_{(2)}}{\sqrt{q_{(2)}}} \sin(\theta) \right). \quad (A.19)
\]

Finally, from eq. (B.5) in appendix B, we obtain that the \( \beta \) angle is given by:

\[
    \tan(\beta) = \sin(\alpha) \left( g(\theta) + (1 + q_1 \left( 1 - \frac{\cos^2(\alpha)}{3} \right)) \cos(\alpha) \right) f(\theta) + q_2, \quad (A.20)
\]

where \( Q = 2H^3 \varepsilon q_1 \sin^2(\alpha) \cos^2(\alpha) \).

### B Interaction analysis

In this appendix we offer a brief analysis about interactions, showing how we obtain the final expressions used in this paper to describe the background and perturbative equations of motion. In eq. (2.1), the interaction is given by \( \mathcal{L}_{\text{int}} \) and, in our prescription where we use a third scalar field to represent the radiation, it depend on \( \phi^A \). Particularly, we expect that the interaction Lagrangian is a function of \( \phi^A \) and \( \partial \mu \phi^A \). Then, the interaction at the background level is given by:

\[
    \mathcal{J}^0_A = - \frac{\delta \mathcal{L}_{\text{int}}(\phi_0, \dot{\phi}_0)}{\delta \phi^A_0} + D_t \left( \frac{\delta \mathcal{L}_{\text{int}}(\phi_0, \dot{\phi}_0)}{\delta \dot{\phi}^A_0} \right), \quad (B.1)
\]

where \( \frac{\delta}{\delta \phi_0} \) represents a covariant variation on the field space and \( \frac{D}{dt} \) is the covariant time derivative. On the other side, we have that:

\[
    \mathcal{L}_{\text{int}}(\phi_0, \dot{\phi}_0) = \dot{\phi}^A_0 \left( \frac{\delta \mathcal{L}_{\text{int}}}{\delta \phi^A_0} \right) + D_t \left( \dot{\phi}^A_0 \left( \frac{\delta \mathcal{L}_{\text{int}}}{\delta \dot{\phi}^A_0} \right) \right) = \frac{d}{dt} \left( \dot{\phi}^A_0 \left( \frac{\delta \mathcal{L}_{\text{int}}}{\delta \dot{\phi}^A_0} \right) \right), \quad (B.2)
\]

where we used that \( \dot{\phi}^A_0 = \dot{\phi}_0 \mathcal{T}^A \) and \( \mathcal{J}_T \equiv \mathcal{T}^A \mathcal{J}^0_A = 0 \). A quick analysis of the last expression tells us that \( \mathcal{L}_{\text{int}} \) is a first order expression in terms of \( \dot{\phi}^A \). Actually in general, \( \mathcal{L}_{\text{int}} \) must be at most linear on \( \partial \mu \phi^A \) to obtain up to first order contributions in the equations of motion.

We can see that the Friedman equations (2.10)–(2.11) do not depend on the interactions. That is because they take into account the complete system, producing \( \mathcal{J}_T = 0 \). This fact is related to (B.2). Therefore, the interaction Lagrangian and the source can be written as:

\[
    \mathcal{L}_{\text{int}}(\phi_0, \dot{\phi}_0) = \lambda^0_A(\phi_0) \dot{\phi}^A_0, \quad (B.3)
\]

\[
    \mathcal{J}^0_A = (D_B \lambda^0_A(\phi_0) - D_A \lambda^0_B(\phi_0)) \dot{\phi}^B_0. \quad (B.4)
\]
where the covariant derivative $D_A$ appears because $J^0_A$ must be define like a vector in the field space. Now, eq. (B.4) means $J^T = 0$, however a more convenient expression to the interaction source can be defined using eq. (A.15) from appendix A. That is:

$$J^0_A = H^2 \sqrt{2\varepsilon} \left( q_2 \lambda^{(3)}_0 - q_1 \sin(\alpha) \cos(\alpha) B^0_0 \right),$$  \hspace{1cm} (B.5)

where $q_1$ and $q_2$ are arbitrary and dimensionless functions that represent different contributions from the interaction. They are related to $\lambda^0_A(\phi_0)$ in a particular but irrelevant way, because the form of $L_{\text{int}}$ at this moment is not important. From (A.16), we know that $\alpha$ defines the magnitude of radiation density. To avoid possible divergences, it is necessary to include the trigonometric functions in the second term of (B.5). In this way, $q_1$ and $q_2$ are defined in such a way that $J^0_A$ is well-behaved for all values of $\alpha$. To understand the particular role of these parameters, we need to observe how they affect the equations of motion. Using eq. (B.5) on (2.28), we obtain:

$$Q = -q_1(33)\phi_0 T^{(3)} H^2 \sqrt{2\varepsilon} \left( q_2 \lambda^{(3)}_0 - q_1 \sin(\alpha) \cos(\alpha) B^0_0 \right)$$

$$= q_1(33)\phi_0 \sin(\alpha) \sqrt{q_1^{(3)}} H^2 \sqrt{2\varepsilon} q_1 \sin(\alpha) \cos(\alpha) \frac{\cos(\alpha)}{\sqrt{q_1^{(3)}}}$$

$$= 2H^3 \varepsilon q_1 \sin^2(\alpha) \cos^2(\alpha),$$

where we used (A.14) and (A.15). Then, we can see that (2.3)–(2.4) are:

$$\frac{D\phi_0^{(1)\prime}}{dN} + (3 - \varepsilon) \phi_0^{(2)\prime} + \frac{V_1}{H^2 q_{(11)}} = -\sqrt{2\varepsilon} \left( q_2 \sin(\theta) + q_1 \sin^2(\alpha) \cos(\alpha) \cos(\theta) \right),$$  \hspace{1cm} (B.7)

$$\frac{D\phi_0^{(2)\prime}}{dN} + (3 - \varepsilon) \phi_0^{(2)\prime} + \frac{V_2}{H^2 q_{(22)}} = -\sqrt{2\varepsilon} \left( q_2 \cos(\theta) - q_1 \sin^2(\alpha) \cos(\alpha) \sin(\theta) \right),$$  \hspace{1cm} (B.8)

$$\Omega'_R + 2 (2 - \varepsilon) \Omega_R = \frac{2q_1}{3} \varepsilon \sin^2(\alpha) \cos^2(\alpha).$$  \hspace{1cm} (B.9)

From these equations, we know that $q_1 > 0$ is related to the decay of the fields $\phi_0^{(1)}$, $\phi_0^{(2)}$ into radiation ($\Omega_R$) and $q_2$ gives us the interaction between $\phi_0^{(1)}$ and $\phi_0^{(2)}$, depending on $\theta$. On the other side, the third degree of freedom of $\lambda^0_A(\phi_0)$ is at first sight irrelevant to the equations of motion, but in appendix C we will see that it is necessary to guarantee the equivalence between the radiation fluid and the additional scalar field.

Now, we need an expression for the perturbative interaction. We do not know enough about interactions during reheating in this level, but we can obtain a general expression to understand the behavior of our fields. Previously, we said that $L_{\text{int}}$ must be a first order function of $\partial_\mu \phi^A$, so the perturbative interaction, $(\Delta J^A) = J^A - J^A_{0}$, must be linear too. Additionally, $L_{\text{int}}$ can be dependent on the metric, so some perturbative element in eq. (3.8) must be taken into account. Hence, if we use (3.46) and (3.47) to define the invariant component as:

$$\left( \delta \phi^A \right) = \left( \delta \phi^A \right)^I - \frac{\dot{\phi}_A^0}{H} \Psi,$$

$$\left( \Delta J^A \right) = \left( \Delta J^A \right)^I - \frac{1}{H} \frac{D}{dt} \left( J^A_0 \right) \Psi,$$

we can define:

$$J^0_A = H^2 \sqrt{2\varepsilon} \left( q_2 \lambda^{(3)}_0 - q_1 \sin(\alpha) \cos(\alpha) B^0_0 \right),$$

$$Q = -q_1(33)\phi_0 T^{(3)} H^2 \sqrt{2\varepsilon} \left( q_2 \lambda^{(3)}_0 - q_1 \sin(\alpha) \cos(\alpha) B^0_0 \right)$$

$$= q_1(33)\phi_0 \sin(\alpha) \sqrt{q_1^{(3)}} H^2 \sqrt{2\varepsilon} q_1 \sin(\alpha) \cos(\alpha) \frac{\cos(\alpha)}{\sqrt{q_1^{(3)}}}$$

$$= 2H^3 \varepsilon q_1 \sin^2(\alpha) \cos^2(\alpha),$$
and eqs. (3.35) and (3.38) to reduce the metric contribution, then the most general expression for the invariant perturbative interaction can be written as:

\[
(\Delta \mathcal{J}_A)^I = \mathcal{I}_{AB} \left( \delta \phi^B \right)^I + \mathcal{J}_{AB} \frac{D}{dt} \left( \delta \phi^B \right)^I + K_A \Phi_B,
\]

(B.12)

where \( \mathcal{I}_{AB} \) and \( \mathcal{J}_{AB} \) are symmetric and antisymmetric matrices respectively. In appendix C, some components of this parameters are fixed to guarantee that the radiation fluid can be represented by an additional scalar field. It is proved that the \textit{Tangent} component of eq. (B.12) is given by (C.56). Using eq. (C.27), it is reduced to:

\[
(\Delta \mathcal{J}_T)^I = \eta_\perp J_N R + \hat{\phi}_0 \tau_N \dot{S}_N + \left( \frac{\dot{J}_N}{H} + C J_B + \dot{\phi}_0 \left( \dot{\tau}_N + H (3 - 2\eta_\parallel) \tau_N \right) \right) S_N
\]

\[
+ \hat{\phi}_0 \tau_B \dot{S}_B + \left( \frac{\dot{J}_B}{H} - C J_N + \dot{\phi}_0 \left( \dot{\tau}_B + H (3 - 2\eta_\parallel) \tau_B \right) \right) S_B,
\]

(B.13)

With this result, we can obtain the other components. They are:

\[
(\Delta \mathcal{J}_N)^I = -\dot{\phi}_0 \tau_N \dot{R} + \dot{\phi}_0 j_0 \dot{S}_B + \left( \frac{\dot{J}_N}{H} + C J_B + \dot{\phi}_0 \left( \dot{\tau}_N + H (3 - 2\varepsilon) \tau_N \right) + H \dot{\phi}_0 C \tau_B \right) \mathcal{R}
\]

\[
+ H \dot{\phi}_0 \Lambda_{NN} \dot{S}_N + H \dot{\phi}_0 \left( \Lambda_{NB} + \left( \varepsilon - \eta_\parallel \right) j_0 \right) S_B + H \dot{\phi}_0 \kappa_N \Phi_B,
\]

(B.14)

\[
(\Delta \mathcal{J}_B)^I = -\dot{\phi}_0 \tau_B \dot{R} - \dot{\phi}_0 j_0 \dot{S}_N
\]

\[
+ \left( \frac{\dot{J}_B}{H} - C J_N + \dot{\phi}_0 \left( \dot{\tau}_B + H (3 - 2\varepsilon) \tau_B \right) + H \dot{\phi}_0 \left( \eta_\perp j_0 - C \tau_B \right) \right) \mathcal{R}
\]

\[
+ H \dot{\phi}_0 \left( \Lambda_{NB} - \left( \varepsilon - \eta_\parallel \right) j_0 - \eta_\perp \tau_B \right) \dot{S}_N + H \dot{\phi}_0 \Lambda_{BB} \dot{S}_B + H \dot{\phi}_0 \kappa_B \Phi_B,
\]

(B.15)

where \( \tau_N \), \( \tau_B \) and:

\[
\dot{j}_0 = \frac{J_{NB}}{H}, \quad \Lambda_{NN} = \frac{I_{NN}}{H^2} - \eta_\perp \tau_N - C j_0,
\]

\[
\Lambda_{NB} = \frac{I_{NB}}{H^2}, \quad \Lambda_{BB} = \frac{I_{BB}}{H^2} - C j_0,
\]

\[
\kappa_N = \frac{\kappa_N}{H \dot{\phi}_0}, \quad \kappa_B = \frac{\kappa_B}{H \dot{\phi}_0},
\]

(B.16)

(B.17)

(B.18)

are the perturbative interaction parameters. These expressions will be used to obtain the perturbative equations in section 3. From eq. (B.5), we can see that the interaction becomes completely determined by two parameters at the background level. They are given by \( q_1 \) and \( q_2 \), represented by \( \mathcal{J}_N \) and \( \mathcal{J}_B \) in eqs. (B.13)–(B.15). In the perturbative equations (3.65)–(3.67), both parameters do not appear explicitly, so they are not relevant, however we have additional parameters to represent the interaction at the perturbative level.

In first place, we have \( \tau_N \) and \( \tau_B \) as the only not-fixed parameters in appendix C to represent the interaction. This means that they are not completely independent of the other six parameters. Additionally, they define the perturbative interaction in the \textit{Tangent} component or, in the same way, the effect of the interaction on the total fluid, affecting the curvature \( \mathcal{R} \) directly. Then, the other parameters represent the internal effect. On one side, we have \( j_0 \), \( \Lambda_{NN} \), \( \Lambda_{NB} \) and \( \Lambda_{BB} \) as the interaction parameters related to the isocurvature...
components. On the other side, we have $\kappa_N$ and $\kappa_B$, related to a geometric contribution in the interaction. At the beginning, we do not have information about these parameters, so they are completely arbitrary. For that, we need to know the interaction Lagrangian. Each one of them could give us interesting properties, but in this paper we will fix them to zero in order to simplify the calculations. We present more details in section 3.

C Fluid components

In this appendix we describe the different components of the total fluid, particularly the perturbative contribution, in order to restrict the interaction contribution in our prescription where the radiation component is represented by an additional scalar field. In first place, we will find a general expression of density and pressure, and then fix the parameters. Finally, we will connect these results with $(\Delta J^A)$ defined in appendix B.

The Energy-Momentum tensor is given by:

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left( \sqrt{-g} L_M - g_{\mu\nu} L_M \right),$$

(C.1)

where $L_M$ is the matter Lagrangian. In our case, we have two inflatons and a radiation fluid, given by eq. (2.1). However, in this paper, a third scalar field is used to represent the radiation fluid, moreover we introduce the coupling between inflatons and this additional fluid, given by eq. (2.1). However, in this paper, a third scalar field is used to represent the radiation fluid, moreover we introduce the coupling between inflatons and this additional field at the Lagrangian level following the procedure of refs. [91, 92]. This means that the Lagrangian and the energy-momentum tensor must be given respectively by:

$$L_M = -\frac{1}{2} g^{\mu\nu} \tilde{q}_{AB} \partial_\mu \phi^A \partial_\nu \phi^B - V + L_{\text{int}},$$

(C.2)

$$T_{\mu\nu} = \tilde{q}_{AB} \partial_\mu \phi^A \partial_\nu \phi^B + 2 \delta g_{\mu\nu} \left( \frac{1}{2} g^{\alpha\beta} \tilde{q}_{AB} \partial_\alpha \phi^A \partial_\beta \phi^B + V - L_{\text{int}} \right),$$

(C.3)

where $\tilde{q}_{AB}$ is a non-perturbative version of $q_{AB}$ and $\delta g_{\mu\nu} = \frac{\delta L_{\text{int}}}{\delta g^{\mu\nu}}$. So, using $T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$, we can deduce the total density, pressure and the 4-velocity from eq. (C.3). They are:

$$\rho = \frac{(2 u^\alpha u^\beta + g^{\alpha\beta})}{2} \tilde{q}_{AB} \partial_\alpha \phi^A \partial_\beta \phi^B + V - L_{\text{int}} + 2 u^0 u^0 \delta \hat{H}_{0\beta},$$

(C.4)

$$p = \frac{(2 u^\alpha u^\beta - g^{\alpha\beta})}{6} \tilde{q}_{AB} \partial_\alpha \phi^A \partial_\beta \phi^B - V + L_{\text{int}} + \frac{2}{3} (u^0 u^0 + g^{\alpha\beta}) \delta \hat{H}_{0\beta},$$

(C.5)

$$u_\mu u_\nu = \frac{\tilde{q}_{AB} \partial_\mu \phi^A \partial_\nu \phi^B + 2 \delta g_{\mu\nu} \left( u^\alpha u^\beta + g^{\alpha\beta} \right) \left( \tilde{q}_{AB} \partial_\alpha \phi^A \partial_\beta \phi^B + 2 \delta \hat{H}_{0\beta} \right)}{\rho + p}.$$

(C.6)

From these equations, we can obtain the background components:

$$\rho^0 = \frac{1}{2} \dot{\phi}_0^2 + V(\phi_0) - \lambda_A^0 (\phi_0) \dot{\phi}_0^A + 2 \delta H_{00},$$

(C.7)

$$p^0 = \frac{1}{2} \dot{\phi}_0^2 - V(\phi_0) + \lambda^0_A (\phi_0) \dot{\phi}_0^A,$$

(C.8)

where eq. (B.3) from appendix B was used to represent $L_{\text{int}}$ in the background approximation and we will assume that $\delta \hat{H}_{0\nu} = H_{0\nu} + \delta H_{0\nu}$. These expressions are the components of the total fluid in the system, so eqs. (C.7) and (C.8) have to satisfy (2.10)–(2.11), then:

$$L_{\text{int}}(\phi_0) \equiv \lambda^0_A(\phi_0) \dot{\phi}_0^A \xrightarrow{\text{On-Shell}} 0,$$

(C.9)

$$H_{00} \xrightarrow{\text{On-Shell}} 0.$$
Basically, (C.9) means that $\lambda^0_A(\phi_0)$ has two effective degrees of freedom and they are represented by $q_1$ and $q_2$ in eq. (B.5) from appendix B. Additionally, we can use the fluid reference frame where:

$$u^0_\mu = (1 \ 0 \ 0 \ 0) .$$

(C.11)

In this case, (C.6) tells us that $\mathcal{H}_{ij} = \mathcal{H}_{i0} = \mathcal{H}_{0i} = 0$. Therefore, the complete condition on $\mathcal{H}_{\mu\nu}$ is:

$$\mathcal{H}_{\mu\nu} \xrightarrow{\text{On-Shell}} 0 .$$

(C.12)

In any case, we will evaluate this condition at the end.

On the other side, we need the perturbative component of the fluid. At first order, the interaction Lagrangian can be written as:

$$\delta L_{\text{int}} = \left( \frac{\delta L_{\text{int}}(\phi_0, \dot{\phi}_0)}{\delta \phi^A_0} \right) (\delta \phi^A) + \left( \frac{\delta L_{\text{int}}(\phi_0, \dot{\phi}_0)}{\delta \dot{\phi}^A_0} \right) \frac{D}{dt} (\delta \phi^A) + O_A(\phi_0, \dot{\phi}_0) (\delta \phi^A)$$

$$+ \omega^0_A(\phi_0) \frac{D}{dt} (\delta \phi^A) ,$$

(C.13)

where $\delta L_{\text{int}} = L_{\text{int}}(\phi, \partial_\mu \phi) - L_{\text{int}}(\phi_0, \dot{\phi}_0)$ and we used the same variation presented in eq. (B.1) to make this expression covariant, and the last two terms are additional contributions in the Lagrangian interaction in a perturbative level, given by $O_A(\phi_0, \dot{\phi}_0)$ and $\omega^0_A(\phi_0)$. Using eq. (B.3) in (C.13), we obtain:

$$\delta L_{\text{int}} = \left( \frac{\dot{\phi}^B_0}{\phi^0_0} (D_A \lambda^0_B) + O_A \right) (\delta \phi^A) + \left( \lambda^A_0 + \omega^A_0 \right) \frac{D}{dt} (\delta \phi^A) .$$

(C.14)

With all these, the perturbative components are given by:

$$\rho^i = \left( \phi^A_0 - \lambda^0_A - \omega^A_0 \right) \frac{D}{dt} (\delta \phi^A) - \dot{\phi}^A_0 \Phi + \left( \mathcal{V}_A - \dot{\phi}^B_0 (D_A \lambda^0_B) - O_A \right) (\delta \phi^A)$$

$$+ 2 \delta \mathcal{H}_{00} + 4 u^0_\alpha u^1_1 \mathcal{H}_{\alpha\beta},$$

(C.15)

$$p^i = \left( \phi^0_A + \lambda^0_A + \omega^A_0 \right) \frac{D}{dt} (\delta \phi^A) - \dot{\phi}^A_0 \Phi - \left( \mathcal{V}_A - \dot{\phi}^B_0 (D_A \lambda^0_B) - O_A \right) (\delta \phi^A)$$

$$+ \frac{2}{3} \left( 2 u^0_\alpha u^1_1 + \delta g^{\alpha\beta} \right) \mathcal{H}_{\alpha\beta},$$

(C.16)

$$u^1_{(0)} = \Phi$$

(C.17)

$$u^1_{(i)} = \frac{\left( \phi_0 T_A \partial_i (\delta \phi^A) \right) + \delta \mathcal{H}_{0i}}{\phi^2_0 + 2 \mathcal{H}_{00}},$$

(C.18)

where we used that eq. (3.9) and $q_{AB}$ depend on $\phi^A$. In particular, the expression in (C.18) is completely inconvenient. $u^1_{(i)}$ is usually used to define the curvature, presented in eq. (3.34), so it must be directly related to the Tangent component of $(\delta \phi^A)$, without interaction terms. So, we need the On-Shell condition:

$$\delta \mathcal{H}_{0i} \xrightarrow{\text{On-Shell}} 0 .$$

(C.19)
Additionally, we know that \( \hat{\cal H}_{00} \) is linear in \( \dot{\phi}^A \), then \( \delta H_{00} \) depends on \( \frac{d}{dt}(\delta \phi^A) \) and \( (\delta \phi^A) \). Besides, any term proportional to \( \Phi \) in \( \delta H_{00} \) will be zero when we impose eqs. (C.23) and (C.24). With all these, the fluid components in our prescription using (C.9) and (C.12) are finally given by:

\[
\begin{align*}
\rho^0 &= \frac{1}{2} \dot{\phi}_0^2 + \mathcal{V}(\phi_0), \\
p^0 &= \frac{1}{2} \dot{\phi}_0^2 - \mathcal{V}(\phi_0), \\
u^0_i &= (1 \ 0 \ 0 \ 0), \\
r^1 &= \left( \phi_A^0 + \chi_A^0 - \lambda_A^0 \right) \frac{d}{dt}(\delta \phi^A) - \dot{\phi}_0^2 \Phi + \left( \mathcal{V}_A - \dot{\phi}_0^B (D_A \lambda_B^0) - \mathcal{U}_A \right) (\delta \phi^A), \\
p^1 &= \left( \phi_A^0 + \omega_A^0 + \lambda_A^0 \right) \frac{d}{dt}(\delta \phi^A) - \dot{\phi}_0^2 \Phi - \left( \mathcal{V}_A - \dot{\phi}_0^B (D_A \lambda_B^0) - \mathcal{O}_A \right) (\delta \phi^A), \\
u^1_{(0)} &= \Phi \\
u^1_i &\equiv \partial_i u^1 \to u^1 = \frac{T_A (\delta \phi^A)}{\phi_0},
\end{align*}
\]

These expressions give us the most general relation between the scalar fields and the fluid. Now, we can fix some of these parameters to restrict the fluid components using particular rules. In first place, we can write eqs. (C.23)–(C.26) in terms of invariant components using (B.10)–(B.11). Now, eq. (C.25) is just the Newtonian Potential, obeying eq. (3.30), and eq. (C.26) is given by (3.34), where:

\[
(\delta \phi^A)^I = \frac{\dot{\phi}_0}{H} \left( R T^A + S_{N^A} N^A + S_{B^A} B^A \right).
\]

On the other side, \( r^1 \) and \( p^1 \) are respectively given by eqs. (3.32)–(3.33) with:

\[
\begin{align*}
\rho^I &= \left( \phi_A^0 + \chi_A^0 - \lambda_A^0 \right) \frac{d}{dt}(\delta \phi^A) \\
&\quad + \left( \mathcal{V}_A - \dot{\phi}_0^B (D_A \lambda_B^0) - \mathcal{U}_A - \frac{\left( \dot{\phi}_0^A + \chi_A^0 \dot{\phi}_0^B \right)}{2H} \phi_A^0 \right) (\delta \phi^A) \\
&\quad + \left( 3H \lambda_A^0 \dot{\phi}_0^A + \chi_A^0 (\mathcal{V}^A + \mathcal{J}_0^A) + \dot{\phi}_0^A \mathcal{U}_A \right) \left( \frac{\Phi_B}{H} - f \right) + \chi_A^0 \dot{\phi}_0^A \left( \Phi_B + \dot{f} \right), \\
p^I &= \left( \phi_A^0 + \omega_A^0 + \lambda_A^0 \right) \frac{d}{dt}(\delta \phi^A) \\
&\quad - \left( \mathcal{V}_A - \dot{\phi}_0^B (D_A \lambda_B^0) - \mathcal{O}_A + \frac{\left( \dot{\phi}_0^A + \omega_A^0 \dot{\phi}_0^B \right)}{2H} \phi_A^0 \right) (\delta \phi^A) \\
&\quad + \left( 3H \omega_A^0 \dot{\phi}_0^A + \omega_A^0 (\mathcal{V}^A + \mathcal{J}_0^A) - \dot{\phi}_0^A \mathcal{O}_A \right) \left( \frac{\Phi_B}{H} - f \right) + \omega_A^0 \dot{\phi}_0^A \left( \Phi_B + \dot{f} \right),
\end{align*}
\]

where \( f = a^2 \left( \dot{E} - \frac{B}{a} \right) \) and we used the background equations in section 2.3, eqs. (3.30), (3.31), (3.35) and (3.38). \( f \) is a non-invariant element, so we need additional
conditions on the interactions given by:

\begin{align}
\chi^0_A \dot{\phi}^0_A &= 0, \\
\omega^0_A \dot{\phi}^0_A &= 0, \\
\dot{\phi}^0_A \mathcal{A}_A &= -\chi^0_A (\mathcal{V}^A + \mathcal{J}_0^A) = -\chi^0_N H \dot{\phi}_N \eta_L, \\
\dot{\phi}^0_A \mathcal{O}_A &= \omega^0_A (\mathcal{V}^A + \mathcal{J}_0^A) = \omega^0_N H \dot{\phi}_N \eta_L, \\
\end{align}

and eqs. (C.28)–(C.29) are reduced to:

\begin{align}
p^{1I} &= \left( \dot{\phi}^0_A + \chi^0_A - \chi^0_A \right) \frac{D}{dt} (\delta \phi^A) + \left( \mathcal{V} - \dot{\phi}^B (D_A \lambda^0_B) - \mathcal{U}_A - \frac{\dot{\phi}^2}{2H} \phi^0_A \right) (\delta \phi^A) \\
&= \frac{\dot{\phi}^2}{H} - 3 \phi^0_R + \frac{(\chi^0_N - \lambda^0_N)}{H} \dot{\phi}_N + \dot{\phi} \left( 2 \phi_0 \eta_L + (\chi^0_N - \lambda^0_N) (\epsilon - \eta) - \chi^0_B C - \frac{\lambda^0_N}{H} + \mathcal{U}_N \right) S_N \\
&+ \dot{\phi} \left( \frac{\lambda^0_B}{H} \dot{\phi} - (\chi^0_B - \lambda^0_B) (\epsilon - \eta) + \chi^0_C C + \frac{\dot{\lambda}^0_B + \mathcal{U}_B}{H} \right) S_B, \\
p^{1I} &= \left( \dot{\phi}^0_A + \omega^0_A + \lambda^0_A \right) \frac{D}{dt} (\delta \phi^A) - \left( \mathcal{V} - \dot{\phi}^B (D_A \lambda^0_B) - \mathcal{O}_A - \frac{\dot{\phi}^2}{2H} \phi^0_A \right) (\delta \phi^A) \\
&= \frac{\dot{\phi}^2}{H} - 3 \phi^0_R + \frac{(\omega^0_N + \lambda^0_N)}{H} \dot{\phi}_N + \dot{\phi} \left( (\omega^0_N + \lambda^0_N) (\epsilon - \eta) - \omega^0_B C + \frac{\dot{\lambda}^0_N + \mathcal{O}_N}{H} \right) S_N \\
&+ \dot{\phi} \left( \frac{\omega^0_B}{H} \dot{\phi} + (\omega^0_B + \lambda^0_B) (\epsilon - \eta) + \omega^0_C C + \frac{\dot{\lambda}^0_B + \mathcal{O}_B}{H} \right) S_B,
\end{align}

where we used the background equations of motion and eq. (C.27) in the last steps. On the other side, the perturbative Einstein equations say that \( p^{1I} \) satisfy eq. (3.37), so we need that:

\begin{align}
\omega^0_A &= -\lambda^0_A, \\
\mathcal{O}_A &= -\lambda^0_N H \eta_L T_A - \left( \lambda^0_N + H C \lambda^0_B \right) N_A - \left( \lambda^0_B - H C \lambda^0_N \right) B_A \\
&= -\frac{D \lambda^0_A}{dt}.
\end{align}

These conditions satisfy eqs. (C.31) and (C.33), and we can see from (C.14) that the interaction Lagrangian must be represented by:

\begin{equation}
\mathcal{L}_{\text{int}}(\phi, \dot{\phi}) = \dot{\phi}^0_A \chi^0_A (\phi_0) - \mathcal{J}^A_A (\phi_0, \dot{\phi}_0) (\delta \phi^A) + O \left( (\delta \phi)^2 \right)
\end{equation}

and \( \dot{\mathcal{H}}_{\mu \nu} = \frac{\delta \mathcal{L}_{\text{int}}}{\delta g_{\mu \nu}} \) gives us a first order contribution, all these in the on-shell approximation.
Therefore, we can rewrite eqs. (C.34)–(C.35) as:
\[
\rho^{11} = \left( \phi_0^0 + H \vartheta_A \right) \frac{D}{dt} (\delta \phi^A)^I + \left( V_A + J_A^0 + H \dot{\phi}_0 \tau_A - \frac{\dot{\phi}_0^2}{2H} \phi_A^0 \right) (\delta \phi^A)^I \\
= \frac{\dddot{\phi}_0^2}{H} \mathcal{R} - 3 \dddot{\phi}_0^2 \mathcal{R} + \vartheta_N \dot{\phi}_0 \dot{S}_N + \dot{\phi}_0 \left( \phi_0 \left( 2 \eta_\perp + \tau_N \right) + H \left( \left( \epsilon - \eta_\parallel \right) \partial_N - C \partial_B \right) \right) S_N \\
+ \vartheta_B \dot{\phi}_0 \dot{S}_B + \dot{\phi}_0 \left( \phi_0 \tau_B + H \left( \left( \epsilon - \eta_\parallel \right) \partial_B + C \partial_N \right) \right) S_B, \\
(C.39)
\]
\[
p^{11} = \dot{\phi}_0^0 \frac{D}{dt} (\delta \phi^A)^I - \left( V_A + J_A^0 + \frac{\dot{\phi}_0^2}{2H} \phi_A^0 \right) (\delta \phi^A)^I = \frac{\dddot{\phi}_0^2}{H} \mathcal{R} + (3 - 2 \eta_\parallel) \dot{\phi}_0^2 \mathcal{R}, \\
(C.40)
\]
with:
\[
\lambda_A^0 = \lambda_0^0 + H \vartheta_A, \\
U_A = - \frac{D\lambda_A^0}{dt} - H \dot{\phi}_0 \tau_A, \\
(C.41, 42)
\]
and the remaining parameters of the interaction satisfy:
\[
J_T = 0, \\
\vartheta_T = 0, \\
\tau_T = \frac{H \eta_\perp \vartheta_N}{\dot{\phi}_0}, \\
(C.43, 44, 45)
\]
Finally, we need to verify that eqs. (C.23)–(C.26) obey the perturbative equation of motion of a fluid. From [83], we know that it is given by:
\[
\dot{\rho} + 3H (\rho + p) + \frac{\partial^2}{a^2} q^1 - (\rho^0 + p^0) \left( 3 \Psi - \partial^2 \left( E - \frac{B}{a} \right) \right) = 0, \\
(C.46)
\]
where \( q^1 = -(\rho^0 + p^0) u^1 \). Besides, from eqs. (C.23)–(C.26) and considering (C.36)–(C.42), we have:
\[
\left( \phi_0^0 + H \vartheta_A \right) \frac{D}{dt} (\delta \phi^A)^I = \rho^1 + \dddot{\phi}_0^2 \Phi - \left( V_A + J_A^0 + H \dot{\phi}_0 \tau_A \right) (\delta \phi^A)^I, \\
(2 \left( V_A + J_A^0 \right) + H \dot{\phi}_0 \tau_A) (\delta \phi^A)^I = \rho^1 - p^1 - H \partial_A \frac{D}{dt} (\delta \phi^A), \\
\dot{\phi}_0^0 (\delta \phi^A) = -q^1. \\
(C.47)
\]
Now, in this paper is proposed that the perturbative equation of the scalar fields in our formalism is given by eq. (3.45). If we Project it on \( (\phi_0^0 + H \vartheta_A) \), we obtain:
\[
\dot{\rho} + 3H (\rho + p) + \frac{\partial^2}{a^2} q^1 - \dddot{\phi}_0^2 \left( 3 \Psi - \partial^2 \left( E - \frac{B}{a} \right) \right) + \left( \phi_0^0 + H \vartheta_A \right) (\Delta J_A) \\
- \frac{\partial^2}{a^2} H \vartheta_A (\delta \phi^A) + 2H \vartheta_A (V_A + J_A^0) \Phi - H \left( \frac{D\vartheta_A}{dt} - H \vartheta_A + \dot{\phi}_0 \tau_A \right) \frac{D}{dt} (\delta \phi^A) \\
- \left( \frac{D}{dt} (J_A^0 + H \dot{\phi}_0 \tau_A) + 3H^2 \dot{\phi}_0 \tau_A - H \vartheta_B \left( V_{AB} - \dddot{\phi}_0 \mathcal{R} A T B \right) \right) (\delta \phi^A) = 0. \\
(C.48)
\]
This means that eqs. (3.45) and (C.46) are equivalent if:

$$\left( \frac{\partial^2}{a^2} H \partial A (\delta \phi^A)^I + H \left( \frac{D \partial A}{dt} - H \varepsilon \partial A + \phi_0 \tau_A \right) \right) \frac{D}{dt} (\delta \phi^A)^I \quad \text{(C.49)}$$

$$+ \left( \frac{D}{dt} J^A_A \right) + H \phi_0 D \frac{dt}{dt} (\tau_A) + H^2 \phi_0 (3 - \varepsilon - \eta) \tau_A$$

$$- H \partial B \left( V_{AB} - \frac{\phi_0^2 \Re \alpha_{TTB}}{H} + \frac{\phi_0^0 (V_B + J^0_B)}{H} \right),$$

where we used (3.30), (3.31), (3.35), (3.38) and (B.10). From eq. (B.12) in appendix B, we know that (C.49) implies:

$$\left( \dot{\phi}_B^B + H \partial B \right) \mathcal{I}_{BA} = H \partial A \frac{\partial^2}{a^2} + H \frac{\partial A}{dt} (\tau_A) + H^2 \phi_0 (3 - \varepsilon - \eta) \tau_A$$

$$H \partial B \left( V_{AB} - \frac{\phi_0^2 \Re \alpha_{TTB}}{H} + \frac{\phi_0^0 (V_B + J^0_B)}{H} \right), \quad \text{(C.50)}$$

$$\left( \dot{\phi}_B^B + H \partial B \right) \mathcal{J}_{BA} = H \left( \frac{D \partial A}{dt} - H \varepsilon \partial A + \phi_0 \tau_A \right), \quad \text{(C.51)}$$

$$\left( \dot{\phi}_B^B + H \partial B \right) \mathcal{K}_B = 0. \quad \text{(C.52)}$$

In this paper, we will discard the Laplacian term in eq. (C.50), because we do not expect that second order terms appear in the interaction contributions, however it could be considered in a future work. Taking all of this into consideration, we will use $\partial_A = 0$, so:

$$\mathcal{I}_{\tau A} = \frac{1}{\phi_0} \frac{D}{dt} (\mathcal{J}_A^0) + H \frac{D}{dt} (\tau_A) + H^2 (3 - \varepsilon - \eta) \tau_A, \quad \text{(C.53)}$$

$$\mathcal{J}_{\tau A} = H \tau_A, \quad \text{(C.54)}$$

$$\mathcal{K}_\tau = 0, \quad \text{(C.55)}$$

and:

$$\left( \Delta \mathcal{J}_\tau \right)^I = H \tau_A \frac{D}{dt} (\delta \phi^A)^I + \left( \frac{1}{\phi_0} \frac{D}{dt} (\mathcal{J}_A^0) + H \frac{D}{dt} (\tau_A) + H^2 (3 - \varepsilon - \eta) \tau_A \right) (\delta \phi^A)^I, \quad \text{(C.56)}$$

with $\mathcal{J}_\tau = 0$ and $\tau_\tau = 0$. The final expression for $\left( \Delta \mathcal{J}_\tau \right)^I$ and other components of the perturbative interaction used in this paper are presented in eqs. (B.13)–(B.15) from appendix B. Finally, in relation to the fluid components, they are given by:

$$\rho^0 = \frac{1}{2} \phi_0^2 + \nu, \quad p^0 = \frac{1}{2} \phi_0^2 - \nu, \quad \text{(C.57)}$$

$$p^{\perp I} = \frac{\phi_0^2}{H} R - 3 \phi_0^2 R + \phi_0^2 (2 \eta_1 + \tau_\nu) \mathcal{S}_{\nu} + \phi_0^2 \tau_B S_B, \quad \text{(C.58)}$$

$$p^{\perp I} = \frac{\phi_0^2}{H} \dot{\mathcal{R}} + (3 - 2 \eta) \phi_0^2 \mathcal{R}. \quad \text{(C.59)}$$

This final result prove that a system with two inflatons and a radiation fluid can be represented by our prescription with three scalar fields, where the perturbative Klein-Gordon equation is given by eq. (3.45), and the interaction for the total fluid is given by $\tau_\nu$ and $\tau_B$. In fact, if we use eqs. (C.58)–(C.59) in (C.46), we obtain (3.65). Besides, (C.58) is used to obtain (3.63) from (3.36) to complete the perturbative system equations.
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