A Quadratic Convex Approximation of Optimal Power Flow in Distribution System with Application in Loss Allocation

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Abstract—Price is the key to resource allocation. In the electricity market, the price is settled by two steps: i) determine the optimal dispatches for generators; ii) calculate the prices for consumers at different locations. In this paper, a novel quadratic optimal power flow model, namely MDOPF, is proposed to decide the optimal dispatches for distribution systems. The model is proved to be convex if the summation of generation marginal costs is over zero. According to the result of MDOPF, the electricity price can be calculated by two methods, namely marginal loss method (MLM) and loss allocation method (LAM), respectively. The MLM can yields very accurate distribution locational marginal prices (DLMP) if compared with the DLMP solved by ACOPF. While the LAM is aimed to eliminate the over-collected losses caused by DLMP. These two methods are proposed in explicit forms which can release the computational burden. Numerical tests show that the proposed MDOPF model generates very close optimal dispatches if compared with the benchmarks provided by ACOPF.

Index Terms—Convex, locational marginal price, loss allocation, optimal power flow

 NOMENCLATURE

Indices and sets
\( \Phi_B \) Set of all buses
\( \Phi_G \) Set of distributed generation buses
\( \Phi_L \) Set of all branches
\( N_i \) Neighboring buses of bus \( i \)
\( \Psi_i \) Sets of the branches from bus \( k \) to the root
\( \Omega_{ij} \) Sets of the buses with branch \( ij \) in its path to root
\( l_{ij} \) Branch \( ij \)

Parameters
\( R_{ij}, X_{ij} \) Resistance and reactance of branch \( ij \)
\( P_i^D, P_i^G \) Maximum and minimum active power provided by DG at bus \( i \)
\( Q_i^D, Q_i^G \) Maximum and minimum reactive power provided by DG at bus \( i \)
\( P_i^D, Q_i^D \) Active and reactive power demand at bus \( i \)
\( V_i, \bar{V}_i, \underline{V}_i \) Upper and lower bound of voltage magnitude at bus \( i \)
\( P_{ij}, Q_{ij} \) Upper bound of active and reactive branch flow
\( \bar{I}_i \) Upper bound of current of branch \( ij \)
\( C_i^P, C_i^Q \) Active and reactive power cost of DG at bus \( i \)
\( Qloss \) Reactive power loss of the network
\( Ploss^A \) Active power loss caused by active power injection
\( Ploss^R \) Active power loss caused by reactive power injection
\( Qloss^A \) Reactive power loss caused by active power injection
\( Qloss^R \) Reactive power loss caused by reactive power injection

Decision Variables
\( V_i \) Voltage magnitude at bus \( i \)
\( W_i \) Auxiliary variable (approximate \( 1/V_i \))
\( \hat{P}_{ij}, \hat{Q}_{ij} \) Modified active and reactive power flow on branch \( ij \)
\( \hat{P}_i, \hat{Q}_i \) Modified active and reactive power injection at bus \( i \)
\( \hat{P}_i^G, \hat{Q}_i^G \) Modified active and reactive power injection provided by DG

Vector and Matrix
\( V_R \) Voltage magnitude vector of branch receiving ends
\( W_R \) Vector of auxiliary variable \( W_i \)
\( V_0 \) Vector with all values equal to the voltage magnitude of PSP (the reference bus)
\( \hat{P}^{BR}, \hat{Q}^{BR} \) Vector of modified active and reactive power flow on branches
\( P_R^D, Q_R^D \) Vector of active and reactive power load of branch receiving ends
\( \hat{P}_R^G, \hat{Q}_R^G \) Vector of modified active and reactive power generation of branch receiving ends
\( \hat{P}_R^{IN}, \hat{Q}_R^{IN} \) Vector of bus modified active and reactive injected power (\( \hat{P}_R^{IN} = -P_R^D \cdot W_R + \hat{P}_R^G, \hat{Q}_R^{IN} = -Q_R^D \cdot W_R + \hat{Q}_R^G \))
\( \text{Price}^e_R \) Vector of active power price of branch receiving ends
\( \text{Price}^e_Q \) Vector of reactive power price of branch receiving ends
\( T \) Path-branch incidence matrix for a radial network
\( V_N \) The Diagonal matrix of \( V_R \)
\( P_N, Q_N \) Diagonal matrix of active and reactive power load of branch receiving ends
\( R_N, X_N \) Diagonal matrix of branch resistance and reactance

I. INTRODUCTION

The dramatically increasing penetration of distributed generations (DGs) leads to the development of active distribution networks (ADNs). On one hand, DGs can reduce the cost of electricity supply and establish a diverse energy
ecosystem among end-users. On the other hand, they may also bring security hazards without proper dispatch and control. To that end, optimal power flow (OPF) is of crucial importance. However, alternating current OPF (ACOPF) is non-convex [1] and the standard direct current OPF (DCOPF) does not apply to distribution systems. Thus efforts on ACOPF convexification are indispensable for the operation of ADNs.

There have been several essential works on ACOPF convexification. The works in [2-5] set forth the semidefinite relaxation and prove it will be exact with allowance of load over-satisfaction and utilization of virtual phase shifters. However, solutions produced by the semidefinite relaxation become physically meaningless when the duality gap is nonzero [6, 7]. To extend the applications, the second-order cone relaxation method based branch flow model [6-8] was proposed and proved to be exact in conditions, where the objective function is convex, strictly increasing in branch losses, non-increasing in loads, and independent of complex branch flows. Though it can be applied to more situations, second-order cone relaxation may produce inexact solutions when the optimal power injection so that voltage bounds are valid. Therefore, there is hitherto no convexification approach that can always provide feasible solutions, while OPF based on linear power flow (LPF) model can always obtain approximately feasible and optimal solutions.

In addition, there are also researches on the linearization of distribution power flow model, which naturally will make the OPF model convex. Existing works in this category include warm-start models and cold-start models. [9] proposed [10] and [9] calculated the initial operating point of the system with CSM, and then improved the accuracy of solutions by using WSM based OPF model. One of the problems is that if the CSM lacks accuracy and thus cannot provide an accurate operating point, then the WSM-based OPF model may achieve suboptimal solutions. Imprecision is the common drawback of CSM, so there are few works focusing on CSM-based OPF model. The most classical CSM-based OPF model is direct current OPF (DCOPF). Although the DCOPF provides a good approximation of active load flow under certain assumptions, it may fail to yield acceptable solutions for distribution networks with large R/X ratios [11] or with insufficiently flat voltage profiles [12]. To bridge this gap, modified DistFlow, a cold-start linear branch flow model, is adopted in this paper to construct a convex OPF model. Numerical test shows that the model can achieve more accurate optimal solutions than the state-of-art linearized OPF model.

CSM-based OPF can guarantee computational robustness, optimality and approximately optimal solutions. But it is difficult to obtain accurate locational marginal price (LMP), which is an important economic signal in electricity market, by Lagrange multiplier. [10] presents a method that calculates distribution LMP (DLMP) with “acceptable” error if compared with benchmarks provided by ACOPF. However, on the one hand, the method is computationally complex, since it requires to find the sink bus, and calculate the marginal loss using backward/forward sweep algorithm. On the other hand, there would be large errors if the systems work under heavy load or high impedance, as the effect of bus injected power on voltage magnitude is not considered when calculating marginal loss. To remedy the deficiencies, we derive the explicit expression of network loss by path-branch matrix to reduce computational complexity, and consider the influence of bus injected power on voltage magnitude, when calculating the marginal loss, to obtain more accurate DLMP.

Charging with LMP is considered as a reasonable method to collect the cost of electricity consumption. However, the loss component in LMP is determined by marginal loss method (MLM), which was confirmed that it will cause over-collection of losses (OCL) [22]. Naturally, ISOs were required to develop a method to eliminate the OCL [23]. To solve this problem, many loss allocation method (LAM) were developed, including average cost, Z-bus [13], power flow tracing [14], Shapley value [15], etc [16, 17]. These works, either based on physical rules or considering game theory, failed to propose a systematic pricing approach that combines MLM and LAM. In this paper, based on the characteristics of MLM, the loss allocation matrices are introduced, which can accurately allocate the network loss.

The main contributions of the paper are two-fold:

(i) A quadratic optimal power flow model, namely MDOPF, is proposed. The MDOPF is proved to be convex if the summation of generation cost in the objective function is over zero. This model can always find a near-optimal solution, which is more accurate than the state-of-art linearized OPF model in distribution systems, if compare with ACOPF.

(ii) Two pricing methods, namely MLM and LAM, for network losses are systematically combined and proposed. Based on MLM, DLMP can be calculated in explicit form and more accurate than current methods. Based on LAM, network losses can be explicitly allocated to each bus and decoupled into active power contributions and reactive power contributions as MLM can. Moreover, the LAM can effectively eliminate the OCL caused by MLM.

The remainder of this paper is organized as follows. Section II presents the MDOPF model. Section III describes the pricing method (i.e., MLM and LAM). Section IV outlines the test results of the proposed MDOPF model, MLM and LAM using modified IEEE test systems and several larger distribution systems. Section V concludes the paper.

II. OPTIMAL POWER FLOW

A. Modified DistFlow

Consider a single radial network structure shown in Fig. 1:

![Fig. 1. One-line diagram of a main distribution feeder.](image)

Modified DistFlow is a cold-start linear branch flow model for distribution system [18], which can be represented as follows:

\[ \hat{P}_j = \hat{P}_q + \hat{P}_i. \]  
\[ \hat{Q}_j = \hat{Q}_q + \hat{Q}_i. \]  
\[ W_j - W_i = R_j \hat{P}_q + X_j \hat{Q}_q. \]
\[ \hat{P}_i = PW_i. \]  
\[ \hat{Q}_i = QW_i. \]  
\[ W_i = 2 - V_i. \]  

For both PQ and PV nodes, (1)-(6) are linear equations. We can solve the recursions of branch flow equations and write as:  
\[ \hat{P}_{br} = -T \left( \hat{P}_R^G - \hat{P}_R^D \right). \]  
\[ \hat{Q}_{br} = -T \left( \hat{Q}_R^G - \hat{Q}_R^D \right). \]  

Then, the voltage profile is an affine mapping of modified power injections.  
\[ \hat{V}_r = V_0 + T^T R_n T \hat{P}_r + T^T X_n \hat{Q}_r. \]  

Moreover, according to (4)-(6), the voltage profile can be solved in an explicit matrix form:  
\[ \hat{V}_r = 2 - \left( I + T^T R_n T \hat{P}_r + T^T X_n \hat{Q}_r \right)^{-1} (2 - \hat{V}_0). \]  

**B. Model**

The MDOPF model is:  
\[
\begin{align*}
\text{Min} : & \quad \sum_{i \in \Phi_B} C_i^p \cdot \hat{P}_i^G + C_i^q \cdot \hat{Q}_i^G, \\
\text{S.T.}: & \quad \sum_{i \in N_j} \hat{P}_i + \hat{P}_j = 0, \forall j \in \Phi_B, \\
& \quad \sum_{i \in N_j} \hat{Q}_i + \hat{Q}_j = 0, \forall j \in \Phi_B, \\
& \quad W_i - W_j = R_i \hat{P}_i + X_i \hat{Q}_i, \forall i \in \Phi_L, \\
& \quad V_i = 2 - W_i, \forall i \in \Phi_B, \\
& \quad \hat{P}_i = -P_i^W W_i + \hat{P}_i^G, \forall i \in \Phi_B, \\
& \quad \hat{Q}_i = -Q_i^W W_i + \hat{Q}_i^G, \forall i \in \Phi_B, \\
& \quad P_i^W W_i \leq \hat{P}_i^G \leq P_i^G W_i, \forall i \in \Phi_G, \\
& \quad Q_i^W W_i \leq \hat{Q}_i^G \leq Q_i^G W_i, \forall i \in \Phi_G, \\
& \quad 2 - \hat{V}_i \leq W_i \leq 2 - \hat{V}_i, \forall i \in \Phi_B, \\
& \quad -P_i W_i \leq \hat{P}_i \leq P_i W_i, \forall i \in \Phi_L, \\
& \quad -Q_i W_i \leq \hat{Q}_i \leq Q_i W_i, \forall i \in \Phi_L, \\
& \quad \hat{P}_j^G + \hat{Q}_j^G \leq \hat{T}_j, \forall j \in \Phi_B.
\end{align*}
\]

Decision variables herein are the modified active and reactive power output of distributed generators \( \hat{P}_i^G \) and \( \hat{Q}_i^G \), modified power injections \( \hat{P}_i \) and \( \hat{Q}_i \), voltage magnitudes \( V_i \) and its auxiliary variables \( W_i \), and modified branch flows \( \hat{P}_j \) and \( \hat{Q}_j \). The first two terms are control variables and the other variables change accordingly.

According to modified DistFlow, (12) and (13) are the branch flow equations; (14) and (15) are the voltage equation; (16) and (17) are the active and reactive power injection constraints; (18) and (19) are the DG operation constraints; (20) is the bus voltage limits; (21), (22) and (23) are the branch flow limits. Notice that all the constraints are either linear or convex, while the objective function is non-convex.

**C. Convexification**

Considering that the voltage magnitude \( V_i \) close to 1 p.u., define \( \delta_i = 1 - \Delta V_i \). Then \( W_i = 1 + \Delta V_i \), and we have:  
\[ V_i \cdot W_i = 1 - \Delta V_i^2 \approx 1 \]  

Combining (24), the objective function can be estimated by:  
\[
\begin{align*}
\text{min} & \quad \sum_{i \in \Phi_B} V_i \cdot C_i^p \cdot \hat{P}_i^G + V_i \cdot C_i^q \cdot \hat{Q}_i^G.
\end{align*}
\]

However, all the terms in (25) are bilinear, which is still non-convex. To transform (25) to a convex objective function, \( V_r \) should be replaced by \( \hat{P} \) and \( \hat{Q} \). According to (9), \( V_r \) can be written as follows:
\[
\begin{align*}
V_r = V_0^r + T^T R_n T \hat{P}_r^G + T^T X_n \hat{Q}_r^G,
\end{align*}
\]

where \( V_r^0 \) is obtained by:
\[
\begin{align*}
V_r^0 = V_0 - T^T R_n T \hat{P}_r^D - T^T X_n \hat{Q}_r^D.
\end{align*}
\]

According to (9) and (10), \( V_r^0 \) can be solved:
\[
\begin{align*}
V_r^{0^2} = 2 - (1 - T^T R_n T \hat{P}_r^D - T^T X_n \hat{Q}_r^D)^{-1} (2 - V_0).
\end{align*}
\]

(26)-(28) shows that if \( \hat{P}_r^D \) and \( \hat{Q}_r^D \) are given, \( V_r^0 \) is fixed, and \( V_r \) is an affine mapping of \( \hat{P}_r^G \) and \( \hat{Q}_r^G \).

Now, the \( V_r \) is replaced by \( \hat{P}_r^G \) and \( \hat{Q}_r^G \), and the objective function is transformed to a quadratic function, so we need to prove that the quadratic function is convex.

Before the proof, (25) should be written in rectangular representation:
\[
\begin{align*}
\text{min} & \quad V^T C^p \hat{P}^G + V^T C^q \hat{Q}^G.
\end{align*}
\]

where \( V = [V_0; V_r] \) are the voltage magnitudes of all buses. \( \hat{P}^G = [\hat{P}_r^G; \hat{P}_r^D] \) and \( \hat{Q}^G = [\hat{Q}_r^G; \hat{Q}_r^D] \) are the active and reactive power output of generators. \( C^p \) and \( C^q \) are the active and reactive power price of generators:
\[
\begin{align*}
C^p = \begin{bmatrix} C_{1}^p & 0 \\ 0 & C_{2}^p \end{bmatrix}, \quad C^q = \begin{bmatrix} C_{1}^q & 0 \\ 0 & C_{2}^q \end{bmatrix}
\end{align*}
\]

To clarify, the objectives are split into three parts:
\[
\begin{align*}
\text{min} & \quad C_1(\hat{P}_r^G, \hat{Q}_r^G) + C_2(\hat{P}_r^G, \hat{Q}_r^G) + C_3(\hat{P}_r^G, \hat{Q}_r^G),
\end{align*}
\]

where \( C_1(\hat{P}_r^G, \hat{Q}_r^G) \) is the cost of the generator at the PSP:
\[
\begin{align*}
C_1(\hat{P}_r^G, \hat{Q}_r^G) = V_r^T C_r^p \hat{P}_r^G + V_r^T C_r^q \hat{Q}_r^G
\end{align*}
\]

Since the \( V_0 \) is the voltage magnitude of PSP, which is fixed in the optimization process, \( C_1(\hat{P}_r^G, \hat{Q}_r^G) \) is linear.

\( C_2(\hat{P}_r^G, \hat{Q}_r^G) \) and \( C_3(\hat{P}_r^G, \hat{Q}_r^G) \) represent the generation costs of DG at other locations. According to (26), we define \( C_2(\hat{P}_r^G, \hat{Q}_r^G) \) and \( C_3(\hat{P}_r^G, \hat{Q}_r^G) \) as follows:
\[
\begin{align*}
C_2(\hat{P}_r^G, \hat{Q}_r^G) = (V_r^D)^T C_r^p \hat{P}_r^G + (V_r^D)^T C_r^q \hat{Q}_r^G,
\end{align*}
\]

\[
\begin{align*}
C_3(\hat{P}_r^G, \hat{Q}_r^G) = (T^T R_n T \hat{P}_r^G)^T C_r^p \hat{P}_r^G + (T^T X_n \hat{Q}_r^G)^T C_r^q \hat{Q}_r^G + (T^T R_n T \hat{P}_r^D)^T C_r^p \hat{P}_r^D + (T^T X_n \hat{Q}_r^D)^T C_r^q \hat{Q}_r^D.
\end{align*}
\]

According to (28), \( V_r^0 \) is fixed. So \( C_1(\hat{P}_r^G, \hat{Q}_r^G) \) is linear. Therefore, we need to prove that \( C_2(\hat{P}_r^G, \hat{Q}_r^G) \) is convex.

**Proof** see appendix.
D. Remark:
We illustrate that based on the Modified DistFlow model, all constraints in the standard OPF model satisfy requirements for convex programming. However, the objective function may or may not be convex. There are two situations:

- **Voltage and loss:** If our objective function is to minimize the network loss, the voltage deviation, or both, as in the following, then the objective function is already convex and the OPF problem can be solved directly:

\[
\min \sum_{i \in \mathcal{O}} a_i R_i (\hat{P}_i + \hat{Q}_i) + \sum_{i \in \mathcal{Q}} \beta_i (V_i - 1)^2.
\]  
(35)

for some given constants \(a_i, \beta_i \geq 0\).

- **Power injection:** If the objective function contains the power terms \(P, Q\), based on modified DistFlow model, it should be replaced by \(\hat{P}/W\) and \(\hat{Q}/W\) respectively, which is non-convex. To convexify this model, we firstly estimate the non-convex function by (25), then replace the bilinear terms in (25) with the quadratic function (31), finally prove that the quadratic function is convex.

III. NETWORK LOSS PRICING
To meet the demand in the real power system, power transmission will cause network losses. Calculated in electricity markets, the loss can be charged by marginal loss method (MLM) and loss allocation method (LAM).

A. Network Loss
With the Path-branch matrix \([18]\), the total network loss can be obtained by:

\[
Ploss = \sum_{i \in \mathcal{O}} R_i \frac{P_i^2 + Q_i^2}{V_i^2} = \sum_{i \in \mathcal{O}} R_i \hat{P}_i^2 + \sum_{i \in \mathcal{Q}} R_i \hat{Q}_i^2. 
\]  
(36)

\[
Qloss = \sum_{i \in \mathcal{Q}} X_i \frac{P_i^2 + Q_i^2}{V_i^2} = \sum_{i \in \mathcal{Q}} X_i \hat{P}_i^2 + \sum_{i \in \mathcal{Q}} X_i \hat{Q}_i^2. 
\]  
(37)

According to (7) and (8), (36) and (37) can be written in rectangular form:

\[
Ploss = (T_{R}^R)^T R_n T_{R}^R + (T_{Q}^Q)^T R_n T_{Q}^R. 
\]  
(38)

\[
Qloss = (T_{R}^Q)^T X_n T_{R}^R + (T_{Q}^Q)^T X_n T_{Q}^R. 
\]  
(39)

In (38) and (39), the total network loss consists of two parts, and each part is a complete square form of the modified power injection with coefficient \(R\) or \(X\). Therefore, the network loss can be decomposed into active power contributions and reactive power contributions.

\[
Ploss^p = (T_{R}^R)^T R_n T_{R}^R. 
\]  
(40)

\[
Ploss^q = (T_{Q}^Q)^T R_n T_{Q}^R. 
\]  
(41)

\[
Qloss^p = (T_{R}^Q)^T X_n T_{R}^R. 
\]  
(42)

\[
Qloss^q = (T_{Q}^Q)^T X_n T_{Q}^R. 
\]  
(43)

B. Marginal Loss Pricing
In electricity market, the LMP of a certain location is defined as the marginal cost to supply an increment of load at this location. When there is no congestion, the LMP of a distribution network consists of two parts \([19]\): the energy price at the root node and the loss component for different locations. Therefore, DLMP can be seen as pricing network losses with the marginal loss method. Thus we discuss the calculation of DLMP here. After the MD-OPF is solved, the DLMP can be obtained by definition:

\[
DLMP^p_i = \lambda^p_i / V_i, DLMP^q_i = \lambda^q_i / V_i. 
\]  
(44)

where \(\lambda^p_i\) and \(\lambda^q_i\) are the shadow prices of constraints in (16) and (17). However, because that modified DistFlow is an approximation of power flow, it is not accurate enough if uses the dual variables as DLMP. Therefore, here we propose an accurate method to calculate the DLMP.

Assume that there is no congestion happened in system, and the generations at power supply point (PSP) are always larger than zero. According to the definition of LMP, which only contains energy component and marginal loss component, the DLMP can be obtained by:

\[
DLMP^p_i = C_0^p - C_0^p \cdot \partial Ploss / \partial P_i, DLMP^q_i = C_0^q - C_0^q \cdot \partial Qloss / \partial Q_i. 
\]  
(45)

\[
DLMP^q_i = C_0^q - C_0^q \cdot \partial Qloss / \partial Q_i. 
\]  
(46)

Then, the loss factors to w.r.t. the system total losses calculated as following four equations:

\[
\partial Ploss / \partial P_i = 2 \left( T_{R}^R \right)^T R_n T_{R}^R \hat{P}_R + 2 \left( T_{Q}^Q \right)^T R_n T_{Q}^R \hat{Q}_R. 
\]  
(47)

\[
\partial Qloss / \partial Q_i = 2 \left( T_{R}^Q \right)^T X_n T_{R}^R \hat{Q}_R + 2 \left( T_{Q}^Q \right)^T X_n T_{Q}^R \hat{Q}_R. 
\]  
(48)

\[
\partial Qloss / \partial P_i = 2 \left( T_{R}^Q \right)^T X_n T_{R}^R \hat{Q}_R + 2 \left( T_{Q}^Q \right)^T X_n T_{Q}^R \hat{Q}_R. 
\]  
(49)

\[
\partial Qloss / \partial Q_i = 2 \left( T_{R}^Q \right)^T X_n T_{R}^R \hat{Q}_R + 2 \left( T_{Q}^Q \right)^T X_n T_{Q}^R \hat{Q}_R. 
\]  
(50)

Notice that \(W_i\) approximates the \(1/V_i\). To obtain more accurate DLMP result, the partial derivative of \(\hat{P}_R^R\) and \(\hat{Q}_R^R\) to \(P_i^p\) and \(Q_i^p\) are:

\[
\partial \hat{P}_R^R / \partial P_i^p = \begin{cases} \frac{P_i^p}{V_i^2} \cdot \frac{\partial V_i}{\partial P_i^p}, & i \neq j \\ -1 \cdot \frac{P_i^p}{V_i^2} \cdot \frac{\partial V_i}{\partial P_i^p}, & i = j \end{cases} 
\]  
(51)

\[
\partial \hat{Q}_R^R / \partial P_i^p = \frac{Q_i^p}{V_i^2} \cdot \frac{\partial V_i}{\partial P_i^p}. 
\]  
(52)

\[
\partial \hat{Q}_R^R / \partial Q_i^p = \begin{cases} \frac{Q_i^p}{V_i^2} \cdot \frac{\partial V_i}{\partial Q_i^p}, & i \neq j \\ -1 \cdot \frac{Q_i^p}{V_i^2} \cdot \frac{\partial V_i}{\partial Q_i^p}, & i = j \end{cases} 
\]  
(53)

\[
\partial \hat{Q}_R^R / \partial Q_i^p = \begin{cases} \frac{Q_i^p}{V_i^2} \cdot \frac{\partial V_i}{\partial Q_i^p}, & i \neq j \\ -1 \cdot \frac{Q_i^p}{V_i^2} \cdot \frac{\partial V_i}{\partial Q_i^p}, & i = j \end{cases} 
\]  
(54)
The partial derivative of voltage to injected power can be obtained from Jacobi matrix. Let \( J \) denote the Jacobi matrix:

\[
J = \begin{bmatrix}
\frac{\partial P^N}{\partial \delta} & \frac{\partial P^N}{\partial V} \\
\frac{\partial Q^N}{\partial \delta} & \frac{\partial Q^N}{\partial V}
\end{bmatrix}
\]  

(55)

\[
\frac{\delta V}{\delta P^N} = \left( \frac{\partial Q^N}{\partial V} \right)^{-1} \frac{\partial P^N}{\partial \delta} \left( \frac{\partial V}{\partial \delta} - \frac{\partial P^N}{\partial V} \left( \frac{\partial Q^N}{\partial V} \right)^{-1} \frac{\partial Q^N}{\partial \delta} \right)^{-1}
\]  

(56)

\[
\frac{\delta V}{\delta Q^N} = -\left( \frac{\partial Q^N}{\partial V} \right)^{-1} \frac{\partial P^N}{\partial \delta} \left( \frac{\partial V}{\partial \delta} - \frac{\partial P^N}{\partial V} \left( \frac{\partial Q^N}{\partial V} \right)^{-1} \frac{\partial Q^N}{\partial \delta} \right)^{-1}
\]  

(57)

The DLMP can be obtained by substituting (47)-(57) into (45) and (46). To be more clear, the steps to calculate the marginal loss and DLMP are summarized as follows:

### Algorithm: Marginal loss pricing

1. procedure MD-OPF
2. \( V^0 \leftrightarrow V_b \) \( (T, R, X, P^{in}, Q^{in}) \)
3. Constructing MD-OPF: \( F(V_b, W, \bar{P}^p, \bar{Q}^p, \bar{P}_o, \bar{Q}_o) \)
4. \( \{V', W, \bar{P}^p, \bar{Q}^p, \bar{P}_o, \bar{Q}_o\} = \arg \min_f F(V_b, W, \bar{P}^p, \bar{Q}^p, \bar{P}_o, \bar{Q}_o) \)
5. \( \{\delta_i\} - \delta_i = (X_b \bar{P}_o - R_b \bar{Q}_o) V'_i \)
6. \( \{\delta_i\} - \delta_i = (X_b \bar{P}_o - R_b \bar{Q}_o) V'_i \)
7. Calculate \( \{\frac{\partial P^n_p}{\partial P^n_p}, \frac{\partial P^n_p}{\partial Q^n_p}, \frac{\partial Q^n_p}{\partial P^n_p}, \frac{\partial Q^n_p}{\partial Q^n_p}\} \) by (51)-(54)
8. Calculate \( \{\frac{\partial P_{loss}}{\partial P^n_p}, \frac{\partial Q_{loss}}{\partial Q^n_p}, \frac{\partial P_{loss}}{\partial P^n_p}, \frac{\partial Q_{loss}}{\partial Q^n_p}\} \) by (47)-(50)
9. Calculate \( \{\text{DLMP}^p, \text{DLMP}^q\} \) by (45) and (46)
10. end procedure

Although, MLM is widely adopted, it remains potential drawbacks. Fig.2 shows the difference between the network loss and the loss collected from MLM. It can be seen that the MLM will cause over-collection of losses (OCL) [22].

![Fig. 2. Marginal loss curve.](image)

In this paper, we assume there is no congestion. Then the OCL can be obtained by:

\[
\text{OCL} = \text{Revenue} - \text{Payment}
\]  

(58)

### C. Loss Allocation Pricing

Since the marginal loss method will cause OCL, the network loss allocation should be developed properly. Consequently, the price consists two part: energy cost and loss cost, then the distribution locational prices (DLP) are defined as follows:

\[
\text{DLP}^p = C_0^p - C_{\text{loss}}^p \cdot \frac{\text{Ploss}^p}{P} - C_0^q \cdot \frac{\text{Qloss}^p}{Q^0}.
\]  

(59)

\[
\text{DLP}^q = C_0^q - C_0^p \cdot \frac{\text{Ploss}^q}{P} - C_0^q \cdot \frac{\text{Qloss}^q}{Q^0}.
\]  

(60)

Differ from the loss factor, which considers the incremental change of network losses due to bus power injections, the total network losses allocated to bus \( k \), i.e., \( \text{Ploss}^p_k, \text{Qloss}^p_k \), are the summation of branch losses allocation results:

\[
\text{Ploss}^p_k = \sum_{j \in \Psi_k} \text{Ploss}^p_{ij_k},
\]  

(61)

\[
\text{Qloss}^p_k = \sum_{j \in \Psi_k} \text{Qloss}^p_{ij_k},
\]  

(62)

\[
\text{Ploss}^q_k = \sum_{j \in \Psi_k} \text{Ploss}^q_{ij_k},
\]  

(63)

\[
\text{Qloss}^q_k = \sum_{j \in \Psi_k} \text{Qloss}^q_{ij_k}.
\]  

(64)

The branch losses are allocated to the related buses. Specifically, for the branch \( ij \), the loss should be allocated to its downstream buses, noted by the corresponding row of path-incidence matrix \( T \), proportionally. For bus \( k \), it should share the network loss of all branches on its path, noted by the corresponding column of path-incidence matrix \( T \). Quantitatively, it follows the rules:

- Branch \( ij \) should be on the path from bus \( k \) to slack bus if the network loss on branch \( ij \) is to be allocated to bus \( k \).
- The amount of the branch loss allocated to bus \( k \) is linear to the ratio of branch power to the voltage magnitude and also linear to the branch equivalent impedance \( R \) and \( X \).
- The amount of the branch loss allocated to bus \( k \) is linear to the ratio of bus \( k \)’s power injection to its voltage magnitude.

Therefore, \( \text{Ploss}^p_{ij_k}, \text{Qloss}^p_{ij_k}, \text{Ploss}^q_{ij_k} \) and \( \text{Qloss}^q_{ij_k} \) are defined as:

\[
\text{Ploss}^p_{ij_k} = \begin{cases} \hat{P} R T \hat{P} & l_{ij} \in \Psi_k \\ 0 & l_{ij} \notin \Psi_k \end{cases},
\]  

(65)

\[
\text{Qloss}^p_{ij_k} = \begin{cases} \hat{Q} X T \hat{Q} & l_{ij} \in \Psi_k \\ 0 & l_{ij} \notin \Psi_k \end{cases},
\]  

(66)

\[
\text{Ploss}^q_{ij_k} = \begin{cases} \hat{Q} R T \hat{Q} & l_{ij} \in \Psi_k \\ 0 & l_{ij} \notin \Psi_k \end{cases},
\]  

(67)

\[
\text{Qloss}^q_{ij_k} = \begin{cases} \hat{Q} X T \hat{Q} & l_{ij} \in \Psi_k \\ 0 & l_{ij} \notin \Psi_k \end{cases}.
\]  

(68)

Substitute (65)-(68) into (61)-(64):

\[
\text{Ploss}^p = T^0 R T \hat{P}^N.
\]  

(69)

\[
\text{Qloss}^p = T^0 X T \hat{Q}^N.
\]  

(70)

\[
\text{Ploss}^q = T^0 R T \hat{Q}^N.
\]  

(71)

\[
\text{Qloss}^q = T^0 X T \hat{Q}^N.
\]  

(72)

Then, the DLP becomes:

\[
\text{DLP}^p = C_0^p - C_0^p \cdot T^0 V^{-1} R T \hat{P}^N - C_0^p \cdot T^0 V^{-1} X \hat{Q}^N.
\]  

(73)

\[
\text{DLP}^q = C_0^q - C_0^p \cdot T^0 V^{-1} R T \hat{Q}^N - C_0^p \cdot T^0 V^{-1} X \hat{Q}^N.
\]  

(74)

Also, the DLP can be obtained in non-iterative manner:

\[
\text{DLP}^p = C_0^p - C_0^p \cdot (T^0 V^{-1})^T X \hat{Q}^N.
\]  

(75)

\[
\text{DLP}^q = C_0^q - C_0^p \cdot (T^0 V^{-1})^T X \hat{Q}^N.
\]  

(76)
IV. NUMERICAL TESTS

In this section, lots of scenarios based on 33-bus system [8], 69-bus system [17] and 141-bus system are set to test the proposed method. In those scenarios, MDOPF is applied to determine the optimal dispatch, and then, the proposed pricing method is implemented. The voltage of power supply point (PSP) is set as 1.05 p.u. and the generation costs at the PSP are set as 30 $/MWh and 3 $/MVArh for active and reactive power, respectively.

The benchmarking ACOPF results are calculated with MATPOWER. The MDOPF is solved by an embedded IBM CPLEX 12.8 solver with the YALMIP interface. All the simulations are programmed in MATLAB on a laptop with an Intel Core i7-5600U 2.60GHz CPU and 8GB of RAM.

A. Dispatch result on 33-bus system

To compare the optimal dispatch of MDOPF and LOPF-D, 7 scenarios are set. In each scenario, there is one DG with a capacity of (1 MW, 0.5 MVar). Its reactive power costs are set as 2 $/MVArh, and its active power costs and location information are shown in Table I. (Note that, to keep DG’s active output as marginal unit, the active power cost of DG is slightly higher than that of PSP). Then, we use ACOPF, MDOPF and LOPF-D to calculate the optimal power flow results and show the generation cost and dispatches. Since all the reactive power dispatches are 0.5 MVar, they are not shown in the table.

| NO. | DG Location | Price ($/MWh) | Generation Cost ($) | PG of DG (MW) |
|-----|-------------|---------------|---------------------|----------------|
|     |             |               | ACOPF | MDOPF | LOPF-D | ACOPF | MDOPF | LOPF-D |
| 1   | 18th bus    | 31            | 122.16 | 122.16 | 122.45 | 0.625 | 0.624 | 1.000 |
| 2   | 25th bus    | 31            | 123.32 | 123.32 | 123.55 | 0.353 | 0.368 | 1.000 |
| 3   | 33rd bus    | 31            | 121.62 | 121.66 | 121.66 | 0.822 | 1.000 | 1.000 |
| 4   | 6th bus     | 32            | 122.96 | 123.00 | 123.23 | 0.233 | 0.513 | 1.000 |
| 5   | 12th bus    | 32            | 122.57 | 122.58 | 122.85 | 0.515 | 0.614 | 1.000 |
| 6   | 15th bus    | 32            | 122.53 | 122.53 | 122.99 | 0.478 | 0.502 | 1.000 |
| 7   | 31st bus    | 32            | 122.23 | 122.28 | 122.54 | 0.501 | 0.704 | 1.000 |

From the table, it can be seen that MDOPF generates more accurate optimal value and dispatch if compared with ACOPF. Among scenarios 1, 2, 5 and 6, the results of ACOPF and MDOPF are very close. Although there are some differences in the dispatch results of ACOPF and MDOPF in 3 and 7, their generation costs are only slightly different. Notice that the total generation cost determined by LOPF-D is always larger than MDOPF.

B. DLMP Result on 33-bus system with low DG penetration

In this subsection, the proposed MDOPF and DLMP are compared with the LOPF-D mentioned in [10]. Therefore, the following four scenarios are set the same as those in [10].

Scenario A1: There are 4 identical DGs installed at Bus 18, 22, 25, and 33, each with an output range of [0, 0.2] MW and [0, 0.1] MVar. The real power price at the DG is set at $31/MWh which is $1/MWh higher than the PSP, and reactive power price is set at $4/MVARh which is $1/MVARh higher than the PSP.

Scenario A2: (High DG penetration scenario) The prices at the DG are set at $25/MWh and $2/MVARh which are both lower than the PSP. The size of each DG is increased to [0, 1] MW and [0, 0.5] MVar. Besides, a load of 0.5 MW is added to the PSP representing the load from transmission level in order to create reverse flow.

Scenario A3: (Heavy load scenario) The load of the 33-bus system is scaled up to 150% of the baseload, and no DG connection is considered.

Scenario A4: (High impedance scenario) The impedance of each branch in the 33-bus system is increased by 190%, and no DG connection is considered.

The detailed results of the proposed DLMP and the errors w.r.t. ACOPF are shown in Fig. 3 and Fig. 4. The summary of the average errors of the proposed method compared with the method in [10] is shown in Table II. Because the optimal dispatch results of the two methods are identical, which can also be reflected in the accuracy of DLMP, the dispatch results are not demonstrated.

In Fig.3, it can be observed that the active power prices calculated with the proposed method, based on the results of MDOPF, are very close to the benchmarks by ACOPF. In scenario A1 and A2, the systems work at the baseload of 33-bus system with different DG penetration. The errors in A1 and A2 are almost the same, which shows that the DG output has little effect on accuracy. In scenario A3 and A4, as the load/impedance increases, the DLMP errors of the buses located at the end of the feeders become larger. However, the voltage magnitude of the buses decreases to 0.87 p.u. ~ 0.9 p.u., which means the extreme condition happens. Such low voltage magnitude shall not be allowed in operating condition but only to demonstrate the performance of the proposed approach here. Even in such conditions, the DLMP errors with MDOPF are negligible.

As shown in Fig. 4, the reactive power prices follow a similar pattern to the active power price, while its errors are larger than
those of active power price, mainly because the reactive power cost is 3 $/MVarh, much lower than that of active power cost.

### Table II

**SUMMARY OF AVERAGE ERRORS OF DLMP RESULT**

| Scenarios | A1   | A2   | A3   | A4   |
|-----------|------|------|------|------|
| Active    | MDOPF 0.02% | 0.04% | 0.09% | 0.25% |
| Price     | LOPF-D 0.18% | 0.08% | 0.97% | 1.96% |
| Reactive  | MDOPF 0.17% | 0.39% | 0.44% | 0.73% |
| Price     | LOPF-D 0.31% | 0.61% | 3.34% | 5.84% |

From the table, it can be seen that the proposed method yields more accurate DLMPs than LOPF-D, especially in scenarios A3 and A4, where MDOPF shows more merits. That is because LOPF-D is a warm start model that requires an accurate operating point, but LPF-D [10] can hardly provide the point as the loads/impedance of the system increase, LPF-D [10], resulting in a significant increase of DLMP errors. Instead, MDOPF can get more accurate power flow solutions, and thus the DLMPs are more reliable.

### C. Verify the Loss Allocation Method

To illustrate the proposed LAM, the base case of 33-bus system is performed. Based on the system, we firstly calculate DLMP with ACOPF, and then determine the price according to the LAM. According to the two groups of prices, we use (58) to calculate OCL, and finally verify that the proposed LAM can effectively eliminate the OCL.

![Fig. 5. Loss allocation results.](image)

The results: (1) the price by LAM shares the same tendency with DLMPs, which means the network loss price increases as the bus’s marginal loss increases; (2) the DLMP of a specific bus is always slightly higher than the price by LAM; (3) the reactive power should be also charged for network loss, due to the effect of reactive power on network losses. (4) the OCL introduced by MLM is 6.6015$ while the OCL caused by LAM is -0.013$, which shows that LAM can effectively eliminate the OCL.

### D. Large Active Distribution Networks

To verify the proposed MDOPF, MLM and LAM in different systems, three large ADNs are tested. 33 bus system, 69 bus system and 141 bus system are used as the basic systems, and we use the method mentioned in [20] to extend those systems to a large system. The details are described as follows:

#### Scenario C1: The 3201-bus system

The 3201-bus system is obtained by duplicating the 33-bus system 100 times. There are 400 DGs distributed at the end of the distribution feeders.

#### Scenario C2: The 6801-bus system

The 6801-bus system is obtained by duplicating the 69-bus system 100 times. There are 400 DGs distributed at the end of the distribution feeders.

#### Scenario C3: The 14001-bus system

The 14001-bus system is obtained by duplicating the 141-bus system 100 times. There are 600 DGs distributed at the end of the distribution feeders.

The capacity of all DGs is set as [0.2mw 0.1mvar], and the biddings of those DGs are 25 $/MWh and 2 $/MVarh for active and reactive power respectively. In addition, the above three systems are modified by randomly scaling individual branch impedances ($R_i$, $X_i$) and loads ($P_i$, $Q_i$) in the range of (0.7, 1.3), respectively.

![Fig. 6. 3201-bus system.](image)

![Fig. 7. 6801-bus system.](image)

![Fig. 8. 14001-bus system.](image)

For all the scenarios, the optimal solutions are successfully solved by the proposed model, the active power prices for different locations are also calculated by proposed MLM and LAM. For comparison, the benchmarking ACOPF results are calculated with MATPOWER. The active power prices calculated by ACOPF, proposed MLM and LAM are sorted and illustrated in Fig. 6-8. The errors w.r.t. the ACOPF results are shown in Table III. The over-collected loss by ACOPF and the proposed LAM are shown in Table IV. The solving time of MDOPF is shown in Table V.

As seen in the three figures, the results of MDOPF and the proposed MLM follow the results of ACOPF closely. The red price curve always covers the blue curve, even if at the end of distribution feeders, where the prices are relatively higher due to higher losses.

From the errors in Table III, we can find that more errors are introduced in the 3201-bus system and 6801-bus system than 14001-bus system due to higher load/impedance. In the three scenarios, the average errors of active power prices are within a very small range, less than 0.03%, and the largest errors are less than 0.1%. The errors of reactive power prices are slightly larger than active power prices because of low energy cost (3 $/MVar).
Based on the same network loss formulas (38) and (39), the accuracy of MLM results reveals the rationality of LAM to allocate the total network loss to each bus. It can be seen from Table IV that in the three systems, MLM (DLMP) will cause OCL, while LAM can eliminate OCL. Although LAM cannot guarantee that OCL is 0, it can be finally achieved in practice, where the allocation proportion of losses can be determined by LAM under the condition that network losses are known.

TABLE IV

| System     | MLM Loss (bus system) | LAM Loss (bus system) |
|------------|-----------------------|-----------------------|
| 3201       | 405.55 $              | -0.82 $               |
| 6801       | 503.30 $              | -0.65 $               |
| 14001      | 1528.87 $             | -0.32 $               |

It can be seen from Table V that the proposed MDOPF is convex and thus with fast computation. It verified the proposed model works consistently on different large systems.

V. CONCLUSION

In this paper, a convex power flow model namely MDOPF is proposed. MDOPF can achieve very close optimal dispatches of active and reactive power. In order to provide the price information for energy consumptions, two methods namely MLM and LAM are presented and discussed. Based on the solutions of MDOPF, the proposed MLM formulation yields accurate DLMP for both active and reactive power if compared with ACOPF. Based on the same network loss formulas, the LAM can effectively eliminate the OCL caused by DLMP.

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APPENDIX

Proof:

\( C_s(\hat{P}_R^G, \hat{Q}_R^G) \) can be written in matrix form:

\[
\begin{bmatrix}
\hat{P}_R^G \cdot (\hat{Q}_R^G)^T
\end{bmatrix}
\begin{bmatrix}
T^T R_T \quad T^T R_T \\
T^T X_N T \quad T^T X_N T \\
\end{bmatrix}
\begin{bmatrix}
C^p_R \quad 0 \\
0 \quad C^o_R \\
\end{bmatrix}
\begin{bmatrix}
\hat{P}_R^G \\
\hat{Q}_R^G \\
\end{bmatrix}
\]

(77)

Let \( 2\nabla^2 f \) denote the hessian matrix. \( \nabla^2 f \) can be expressed as the Hadamard product of two matrices, i.e., \( \nabla^2 f = \nabla^2 f_1 \circ \nabla^2 f_2 \), where \( \nabla^2 f_1 \) and \( \nabla^2 f_2 \) are:

\[
\nabla^2 f_1 = 
\begin{bmatrix}
C^p_1 \quad \ldots \quad C^p_N \\
\vdots \quad \vdots \quad \vdots \quad \vdots \\
C^o_1 \quad \ldots \quad C^o_N \\
\end{bmatrix}
\]

(78)

\[
\nabla^2 f_2 = 
\begin{bmatrix}
C^p_1 \quad \ldots \quad C^p_N \\
\vdots \quad \vdots \quad \vdots \quad \vdots \\
C^o_1 \quad \ldots \quad C^o_N \\
\end{bmatrix}
\]

(79)

First, we prove that \( \nabla^2 f_1 \) is positive semi-definite (PSD):

\[
\nabla^2 f_1 = 
\begin{bmatrix}
T^T R_T \quad T^T R_T \\
T^T X_N T \quad T^T X_N T \\
\end{bmatrix}
\begin{bmatrix}
R_N \quad R_N \\
X_N \quad X_N \\
\end{bmatrix}
\begin{bmatrix}
T \quad 0 \\
0 \quad T \\
\end{bmatrix}
\]

(80)

Because that \( T \) is an upper triangular matrix and its diagonal elements are 1, \( T \) is invertible. Therefore, \( \nabla^2 f_1 \) is congruent with the matrix:

\[
\begin{bmatrix}
R_N \quad R_N \\
X_N \quad X_N \\
\end{bmatrix}
\]

(81)

It is apparent that the right-hand side is PSD, so \( \nabla^2 f_1 \) is PSD.

Meanwhile, since the elements of each column of \( \nabla^2 f_2 \) are equal, the rank of \( \nabla^2 f_2 \) is 1, and the non-zero eigenvalue of \( \nabla^2 f_2 \) is the trace of \( \nabla^2 f_2 \). If the trace tr(\( \nabla^2 f_2 \)) is positive, the \( \nabla^2 f_2 \) is PSD.

In OPF problems, the summation of generation marginal costs is usually larger than zero.

In linear algebra, the Schur product theorem states that the Hadamard product of two PSD matrices is also PSD. Therefore, \( \nabla^2 f \) is PSD and \( C_s(\hat{P}_R^G, \hat{Q}_R^G) \) is convex. ■