Deepening the (Parameterized) Complexity Analysis of Incremental Stable Matching Problems

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November 9, 2022

Abstract

When computing stable matchings, it is usually assumed that the preferences of the agents in the matching market are fixed. However, in many realistic scenarios, preferences change over time. Consequently, an initially stable matching may become unstable. Then, a natural goal is to find a matching which is stable with respect to the modified preferences and as close as possible to the initial one. For Stable Marriage/Roommates, this problem was formally defined as Incremental Stable Marriage/Roommates by Bredereck et al. [AAAI ’20]. As they showed that Incremental Stable Roommates and Incremental Stable Marriage with Ties are NP-hard, we focus on the parameterized complexity of these problems. We answer two open questions of Bredereck et al. [AAAI ’20]: We show that Incremental Stable Roommates is W[1]-hard parameterized by the number of changes in the preferences, yet admits an intricate XP-algorithm, and we show that Incremental Stable Marriage with Ties is W[1]-hard parameterized by the number of ties. Furthermore, we analyze the influence of the degree of “similarity” between the agents’ preference lists, identifying several polynomial-time solvable and fixed-parameter tractable cases, but also proving that Incremental Stable Roommates and Incremental Stable Marriage with Ties parameterized by the number of different preference lists are W[1]-hard.

1 Introduction

Efficiently adapting solutions to changing inputs is a core issue in modern algorithmics [3, 6, 11, 10, 27]. In particular, in incremental combinatorial problems, roughly speaking, the goal is to build new solutions incrementally while adapting to changes in the input. Typically, one wants to avoid (if possible) too radical changes in the solution relative to perhaps moderate changes in the input. The corresponding study of incremental algorithms attracted research on numerous problems and scenarios [25], including among many others shortest path computation [41], flow computation [32], clustering problems [10, 36], and graph coloring [28].

In this paper, we study the problem of adapting stable matchings under preferences to change. Consider for instance the following two scenarios: First, as reported by Feigenbaum et al. [19], school seats in public schools are centrally assigned in New York. Ahead of the start of the new year, all interested students are asked to submit their preferences over public schools. Then, a stable matching of students to public schools is computed and transmitted. However, in the past, shortly before the start of the new year typically around 10% of students changed their preferences and decided to attend a private school instead, leaving the initially
implemented matching unstable and triggering lengthy decentralized ad hoc updates. Second, consider the assignment of freshmen to double bedrooms in college accommodation. After the orientation weeks, it is quite likely that students got to know each other (and in particular their roommates) better and thus their initially uninformed preferences changed, making the matching unstable.

In our work, we focus on the problem of finding a stable matching after the “change” that is as close as possible to a given initially stable matching. The closeness condition here is due to the fact that in most applications reassignments come at some cost which we want to minimize (e.g., in the above New York example, reassigning students might make it necessary for the family to reallocate within the city). We build upon the work of Bredereck et al. [7], who performed a first systematic study of incremental versions of stable matching problems, and the recent (partially empirical) follow-up work by Boehmer et al. [5], who proved among others that different types of changes are “equivalent” to each other. The central focus of our studies lies on the Stable Roommates (SR) problem: given a set of agents with each agent having preferences over other agents, the task is to find a stable matching, i.e., a matching so that there are no two agents preferring each other to their assigned partner. We also consider a famous special case of SR, called Stable Marriage (SM), where the set of agents is partitioned into two sets, and each agent may only be matched to an agent from the other set. Formally, in the incremental versions of SR and SM, called Incremental Stable Roommates (ISR) and Incremental Stable Marriage (ISM), we are given two preference profiles containing the preferences of each agent before and after the “change” and a matching that is stable in the preference profile before the change. Then, the task is to find a matching that is stable after the change and as close as possible to the given matching, i.e., has a minimum symmetric difference to it.

**Related Work.** Bredereck et al. [7] formally introduced Incremental Stable Marriage [with ties] (ISM/[ISM-T]) and Incremental Stable Roommates [with ties] (ISR/[ISR-T]). They showed that ISM without ties (in the preference lists) is solvable in polynomial time by a simple reduction to finding a stable matching maximizing the weight of the included agent pairs (which is solvable in polynomial time [17]). In contrast to this, ISR is NP-complete [12, Theorem 4.2], yet admits an FPT-algorithm for the parameter $k$, that is, the maximum allowed size of the symmetric difference between the two matchings [7]. With ties, Bredereck et al. [7] showed that ISM-T and ISR-T are NP-complete and W[1]-hard for $k$ even if the two preference profiles only differ in a single swap in some agent’s preference list. As ISR-T can be considered as a generalization of ISM-T, their results motivate us to focus on the NP-hard ISR and ISM-T problems (which are somewhat incomparable problems). Recently, Boehmer et al. [5] followed up on the work of Bredereck et al. [7], proving that different types of changes such as deleting agents or performing swaps of adjacent agents in some preference list are “equivalent”. Moreover, they introduced incremental variants of further stable matching problems and performed empirical studies.

More broadly considered, matching under preferences problems in the presence of change are of high current interest in several application domains. Many such works fall into the category of “dynamic matchings” [1 2 14 15 35]. However, different from our work, there the focus is typically on adapting classic stability notions to dynamic settings while we rather aim for “reestablishing” (classic) stability at minimal change cost.

Closer to our work, there are several papers on adapting a given matching to change (while minimizing the number of reassignments): First, Gajulapalli et al. [21] designed a polynomial-time (and incentive-compatible) algorithm for an incremental variant of the many-to-one version of Stable Marriage (known as Hospital Residents) where new agents are added. Sec-
| Parameter | ISR | ISM-T |
|-----------|-----|-------|
| \(|P_1 \oplus P_2|\) | W[1]-h. (Th. 1) & XP (Th. 2) | NP-h. for \(|P_1 \oplus P_2| = 1\) (Th. 3) |
| \(#\text{ties+}k\) | FPT wrt. \(k\) (Th. 1 in [7]) | W[1]-h. even if \(|P_1 \oplus P_2| = 1\) (Th. 3) |
| \(#\text{outliers}\) | FPT (Th. 4) | XP (even for parameter \#agents with ties) (Pr. 2) |
| \(#\text{master lists}\) | W[1]-h. even for complete preferences (Th. 5/6) | \(\) |

Table 1: Overview of our main results where each row contains results for one parameterization. Note that ISM is polynomial-time solvable as proven by Bredereck et al. [7].

Feigenbaum et al. [19] considered an incremental variant of Hospital Residents where some agents may leave the system. They designed a “fair”, Pareto-efficient, and strategy-proof algorithm for finding a matching before and after the change. The problems studied in both works are closest related to the polynomial-time solvable ISM problem, which we do not study.

Bhattacharya et al. [3] studied one-to-one matching markets where agents are added and deleted over time and for some agents the set of acceptable partners may change. Their focus is on updating the matching in each step such that the number of reassignments is small while maintaining a small unpopularity factor. So in contrast to our work, they do not maintain that the matching is stable but (close to) popular.

Also motivated by temporally evolving preferences, several papers study the robustness of stable matchings subject to changing preferences [4,11,22,23,24,37]. By selecting a robust initial stable matching, one can increase the odds that it remains stable after some changes.

**Our Contributions.** Focusing on the two NP-hard problems ISR and ISM-T, we significantly extend the work of Bredereck et al. [7] on incremental stable matchings. In particular, we answer their two main open questions. Moreover, we strengthen several of their results. In addition, we analyze the impact of the degree of “similarity” between the agents’ preference lists. Doing so, from a conceptual perspective, we complement work of Meeks and Rastegari [39]. They studied the influence of the number of agent types on the computational complexity of stable matching problems (two agents are of the same type if they have the same preferences and all other agents are indifferent between them). By way of contrast, we consider the smaller so far unstudied parameter “number of different preference lists”.

Next, we present a brief summary of the structure of the paper (for each section marking the main studied problem(s)) and our main contributions (see Table 1 for an overview):

**Section 3 (ISR)** Motivated by the observation that ISM-T is NP-hard even if just one swap has been performed, Bredereck et al. [7] asked for the parameterized complexity of ISR with respect to the difference \(|P_1 \oplus P_2|\) between the two given preference profiles \(P_1\) and \(P_2\). We design and analyze an involved algorithm solving ISR in polynomial time if \(|P_1 \oplus P_2|\) is constant (in other words, this is an XP-algorithm). Our algorithm relies on the observation that if we know how certain agents are matched in the matching to be constructed and we adapt the given matching accordingly, then we can find an optimal solution by propagating these changes through the matching until a new stable matching is reached; a general approach that might be of independent interest. We complement this result by proving that ISR parameterized by \(|P_1 \oplus P_2|\) is W[1]-hard.

**Section 4 (ISM-T)** Bredereck et al. [7] considered the total number of ties to be a promising parameter to potentially achieve fixed-parameter tractability results. We prove that this is not the case as ISM-T is W[1]-hard with respect to \(k\) plus the total number of ties.
even if $|\mathcal{P}_1 \oplus \mathcal{P}_2| = 1$. Notably, this result strengthens the W[1]-hardness with respect to $k$ for $|\mathcal{P}_1 \oplus \mathcal{P}_2| = 1$ of Bredereck et al. \cite{Bredereck2020} for ISM-T, while presenting a fundamentally different yet less technical proof. On the positive side, we devise an XP-algorithm for the number of agents with at least one tie in their preferences.

**Section 5 (ISR; ISM-T)** We study different cases where the agents have “similar” preferences. For instance, we consider the case where all but $s$ agents have the same preference list (we call these $s$ agents outliers), or the case where each agent has one out of only $p$ different master preference lists. We devise an algorithm that enumerates all stable matchings in an SR instance in FPT time with respect to $s$, implying an FPT algorithm for ISR parameterized by $s$. In contrast to this and to a simple FPT algorithm for the number of agent types \cite{Bredereck2021}, we prove that ISR and ISM-T are W[1]-hard with respect to the number $p$ of different preference lists.

## 2 Preliminaries

The input of **Stable Roommates with Ties** (SR-T) is a set $A = \{a_1, \ldots, a_{2n}\}$ of agents. Each agent $a \in A$ has a subset $Ac(a) \subseteq A \setminus \{a\}$ of agents it accepts and a preference relation $\succeq_a$ which is a weak order over the agents $Ac(a)$. We assume that acceptance is symmetric, i.e., for two agents $a, a' \in A$, we have $a' \in Ac(a)$ if and only if $a \in Ac(a')$. The collection $\mathcal{P} = (\succeq_a)_{a \in A}$ of the preferences of all agents is called a preference profile. For two agents $a', a'' \in Ac(a)$, agent $a$ weakly prefers $a'$ to $a''$ if $a' \succeq_a a''$. If $a$ weakly prefers $a'$ to $a''$ but does not weakly prefer $a''$ to $a'$, then $a$ strictly prefers $a'$ to $a''$, and we write $a' \succ_a a''$. If $a$ weakly but not strictly prefers $a'$ to $a''$, then $a$ is indifferent between $a'$ and $a''$ and we write $a' \sim_a a''$; in other words, $a'$ and $a''$ are tied. If $a$ strictly prefers $a'$ to $a''$ or $a' = a''$ holds, then we write $a' \succeq_a a''$.

We say that an agent $a$ has strict preferences, which we denote as $\succ_a$, if $\succeq_a$ is a strict order, and, in this case, we use the terms “strictly prefer” and “prefer” interchangeably. The **Stable Roommates** (SR) problem is the special case of SR-T where the preference relation of each agent is strict.

For two preference relations $\succeq$ and $\succeq'$ defined over the same set, the swap distance between $\succeq$ and $\succeq'$ is the number of agent pairs that are ordered differently by the two relations, i.e., $|\{(a, b) : a \succ b \land b \succeq' a\}| + |\{(a, b) : a \sim b \land \neg(a \sim' b)\}|$; for two preference relations over different sets, we define the swap distance to be infinite. For two preference profiles $\mathcal{P}_1$ and $\mathcal{P}_2$ containing the preferences of the same agents, $|\mathcal{P}_1 \oplus \mathcal{P}_2|$ denotes the sum over all agents $a \in A$ of the swap distance of the two preference of $a$ in $\mathcal{P}_1$ and $\mathcal{P}_2$.

A matching $M$ is a set of pairs $\{a, a'\}$ with $a \neq a' \in A$, $a \in Ac(a')$, and $a' \in Ac(a)$, where each agent appears in at most one pair. In a matching $M$, an agent $a$ is matched if $a$ is part of one pair from $M$; otherwise, $a$ is unmatched. A perfect matching is a matching in which all agents are matched. For a matching $M$ and an agent $a \in A$, we denote by $M(a)$ the partner of $a$ in $M$, i.e., $M(a) = a'$ if $\{a, a'\} \in M$ and $M(a) := \varnothing$ if $a$ is unmatched in $M$. All agents $a \in A$ strictly prefer any agent from $Ac(a)$ to being unmatched, i.e., $a' \succ_a \varnothing$ for all $a \in A$ and $a' \in Ac(a)$.

Two agents $a \neq a' \in A$ block a matching $M$ if $a$ and $a'$ accept each other and strictly prefer each other to their partners in $M$, i.e., $a \in Ac(a')$, $a' \in Ac(a)$, $a' \succ_a M(a)$, and $a \succ_a M(a')$. A matching $M$ is stable if it is not blocked by any agent pair. An agent pair $\{a, a'\}$ is called a

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1. Notably, by the equivalence theorem of Boehmer et al. \cite{Bredereck2021} Theorem 1], all our results (except for Theorem 2) where the constant $|\mathcal{P}_1 \oplus \mathcal{P}_2|$ increases slightly) still hold if $|\mathcal{P}_1 \oplus \mathcal{P}_2|$ instead denotes the number of agents whose preferences changed, the number of deleted agents, or the number of added agents.

2. This definition of stability in the presence of ties is the by far most frequently studied variant known as
stable pair if there is a stable matching \( M \) with \( \{a, a'\} \in M \). For two matchings \( M \) and \( M' \), we denote by \( M \triangle M' \) the set of pairs that appear only in \( M \) or only in \( M' \), i.e., \( M \triangle M' = \{\{a, a'\} \mid (\{a, a'\} \in M \land \{a, a'\} \notin M') \lor (\{a, a'\} \notin M \land \{a, a'\} \in M')\} \).

The incremental variant of Stable Roommates [with Ties] is defined as follows.

**Incremental Stable Roommates [with Ties] (ISR/[ISR-T])**

**Input:** A set \( A \) of agents, two preference profiles \( P_1 \) and \( P_2 \) containing the strict [weak] preferences of all agents, a stable matching \( M_1 \) in \( P_1 \), and an integer \( k \).

**Question:** Is there a matching \( M_2 \) that is stable in \( P_2 \) such that \( |M_1 \triangle M_2| \leq k \)?

We also consider the incremental variant of Stable Marriage (SM). Instances of Stable Marriage are instances of Stable Roommates where the set of agents is partitioned into two sets \( U \) and \( W \) such that agents from one of the sets only accept agents from the other set, i.e., \( A(m) \subseteq W \) for all \( m \in U \) and \( A(w) \subseteq U \) for all \( w \in W \). Following traditional conventions, we refer to the agents from \( U \) as men and to the agents from \( W \) as women. All definitions from above still analogously apply to Stable Marriage. Thus, in **Incremental Stable Marriage [with Ties] (ISM/[ISM-T])**, we are given a set \( A = U \cup W \) of agents and two preference profiles \( P_1 \) and \( P_2 \) containing the strict [weak] preferences of all agents, where each \( m \in U \) accepts only agents from \( W \) and the other way round. The preferences of an agent \( a \in A \) are complete if \( A(a) = A \setminus \{a\} \). In Stable Marriage, the preferences of an agent \( a \in U \cup W \) are complete if \( A(a) = W \) for \( a \in U \) or if \( A(a) = U \) for \( a \in W \). If the preferences of an agent are not complete, then they are incomplete.

Note that for SM and SM-T, we know that a stable matching always exists and that we can check whether an SR instance admits a stable matching in \( O(|A|^2) \) time. Thus, in our algorithms for the incremental variants of these problems we always assume without loss of generality that there exists a stable matching in \( P_2 \). Moreover, the stable matchings in SM and SR instances have some structure: If the preferences of agents in an SM or SR instance are complete, then all stable matchings are perfect. If the preferences are incomplete, then in both variants by the Rural Hospitals Theorem \([20]\), all stable matchings match the same set of agents. This in particular implies that for ISR and ISM instead of minimizing \( |M_1 \triangle M_2| \), we can alternatively minimize the number of agents that are matched differently in \( M_1 \) and \( M_2 \).

**Parameterized Complexity Theory.** A parameterized problem \( L \) consists of a problem instance \( I \) and a parameter value \( k \in \mathbb{N} \). An **XP-algorithm** for \( L \) with respect to \( k \) is an algorithm deciding \( L \) in \( |I|^{f(k)} \) time for some computable function \( f \). A **fixed-parameter tractable algorithm** for \( L \) with respect to \( k \) is an algorithm deciding \( L \) in \( f(k) \cdot |I|^{O(1)} \) time for some computable function. The corresponding complexity classes are called XP, resp., FPT. There is also a theory of hardness for parameterized problems. For our purposes, the central class here is W[1], where it holds FPT \( \subseteq \text{W[1]} \subseteq \text{XP} \) and it is commonly believed that both inclusions are strict. Thus, W[1]-hard problems are commonly believed not to be in FPT. To show that a problem is W[1]-hard, one typically constructs a parameterized reduction from a known W[1]-hard problem \( L' \). Such a reduction from a parameterized problem \( L' \) to another parameterized problem \( L \) is a function that maps instances of \( L' = (I', k') \) to equivalent instances of \( L = (I, k) \) with \( k \leq f(k') \) running in \( f(k') \cdot |I|^{O(1)} \) time for some computable function. Finally, a parameterized problem \( L \) is para-NP-hard if it is NP-hard even when the parameter is bounded by a constant.

**weak stability. Strong stability and super stability** are the two most popular alternatives. Notably, ISM-T (as defined later) becomes polynomial-time solvable for both strong and super stability, as for these two stability notions a stable matching maximizing a given weight function on all pairs of agents can be found in polynomial time \([20, 33, 34]\).
3 Incremental Stable Roommates Parameterized by $|\mathcal{P}_1 \oplus \mathcal{P}_2|$ 

Bredereck et al. [7] showed that ISR-T and ISM-T are NP-hard even if $\mathcal{P}_1$ and $\mathcal{P}_2$ differ only by a single swap. While Bredereck et al. showed that ISR (without ties) is NP-hard, they asked whether it is fixed-parameter tractable parameterized by $|\mathcal{P}_1 \oplus \mathcal{P}_2|$. We show that ISR is W[1]-hard with respect to $|\mathcal{P}_1 \oplus \mathcal{P}_2|$ (Section 3.1), yet admits an intricate polynomial-time algorithm for constant $|\mathcal{P}_1 \oplus \mathcal{P}_2|$ (Section 3.2), thus still clearly distinguishing it from the case with ties.

3.1 W[1]-Hardness

This section is devoted to proving that ISR with respect to $|\mathcal{P}_1 \oplus \mathcal{P}_2|$ is W[1]-hard:

**Theorem 1.** ISR parameterized by $|\mathcal{P}_1 \oplus \mathcal{P}_2|$ is W[1]-hard.

To prove Theorem 1, we reduce from the W[1]-hard Multicolored Clique problem parameterized by the solution size $\ell$ [40]. In Multicolored Clique, we are given an $\ell$-partite graph $G = (V^1 \cup V^2 \cup \cdots \cup V^\ell, E)$ and the question is whether there is a clique $X$ of size $\ell$ in $G$ with $X \cap V^c \neq \emptyset$ for all $c \in [\ell]$. To simplify notation, we assume that $V^c = \{v^c_1, \ldots, v^c_\ell\}$ for all $c \in [\ell]$ and that the given graph $G$ is $r$-regular for some $r \in \mathbb{N}$. We refer to the elements of $[\ell]$ as colors and say that a vertex $v$ has color $c \in [\ell]$ if $v \in V^c$. The structure of the reduction is as follows. For each color $c \in [\ell]$, there is a vertex-selection gadget, encoding which vertex from $V^c$ is part of the multicolored clique. Furthermore, there is one edge gadget for each edge. Unless both endpoints of an edge are selected by the corresponding vertex-selection gadgets, the matching $M_2$ selected in the edge gadget contributes to the difference $M_1 \triangle M_2$ between $M_1$ and $M_2$. We set $k$ (that is, the upper bound on $|M_1 \triangle M_2|$) such that at least $\binom{k}{2}$ edges need to have both endpoints in the selected set of vertices, implying that the selected set of vertices forms a clique.

**Vertex-Selection Gadget.** For each color $c \in [\ell]$, we add a vertex selection gadget. For each vertex $v^c_i \in V^c$, we add one 4-cycle consisting of agents $a^c_{i,1}$, $a^c_{i,2}$, $a^c_{i,3}$, and $a^c_{i,4}$. Further, in $\mathcal{P}_2$, two agents $s^c$ and $\bar{s}^c$ are “added” to the gadget (more specifically, $s^c$ and $\bar{s}^c$ are matched to dummy agents $t^e$ and $\bar{t}^e$ in all stable matching in $\mathcal{P}_1$ but cannot be matched to $t^e$ and $\bar{t}^e$ in a stable matching in $\mathcal{P}_2$). We construct the vertex-selection gadget such that the agents $s^c$ and $\bar{s}^c$ have to be matched to agents from the same 4-cycle in a stable matching in $\mathcal{P}_2$. This encodes the selection of the vertex corresponding to this 4-cycle to be part of the multicolored clique. Lastly, we add a second 4-cycle consisting of agents $\bar{a}^c_{i,1}$, $\bar{a}^c_{i,2}$, $\bar{a}^c_{i,3}$, and $\bar{a}^c_{i,4}$ for each vertex $v^c_i \in V^c$ to achieve that $M_1 \triangle M_2$ contains the same number of pairs from the vertex-selection gadget, independent of which vertex is selected to be part of the clique. See Figure 1 for an example.

Apart from agents $s^c$ and $\bar{s}^c$, all agents from the vertex-selection gadget only find agents from the gadget acceptable, while $s^c$ and $\bar{s}^c$ also find agent $a_{e,1}$ (this agent will be introduced in the next paragraph “Edge Gadget”) for each edge $e$ incident to a vertex from $V^c$ acceptable. For $v^c_i \in V^c$, let $A_{\delta(v^c_i), 1}$ denote the set of agents $a_{e,1}$ such that $e$ is an edge incident to $v^c_i$, i.e., $A_{\delta(v^c_i), 1} := \{a_{e,1} \mid e \in E \land e \cap v^c_i \neq \emptyset\}$, and let $[A_{\delta(v^c_i), 1}]$ denote an arbitrary strict order of $A_{\delta(v^c_i), 1}$. For all $c \in [\ell]$ and $i \in [n]$, each vertex-selection gadget contains the following agents with the indicated preferences in $\mathcal{P}_1$:

$s^c : t^c \succ a^c_{i,1} \succ \bar{a}^c_{i,1} \succ [A_{\delta(v^c_i), 1}] \succ a^c_{2,1} \succ \bar{a}^c_{2,1} \succ [A_{\delta(v^c_i), 1}] \succ \cdots \succ a^c_{n,1} \succ \bar{a}^c_{n,1} \succ [A_{\delta(v^c_i), 1}]$

$s^c : \bar{t}^c \succ a^c_{i,4} \succ \bar{a}^c_{i,4} \succ [A_{\delta(v^c_i), 1}] \succ a^c_{n-1,4} \succ \bar{a}^c_{n-1,4} \succ [A_{\delta(v^c_i), 1}] \succ \cdots \succ a^c_{1,4} \succ \bar{a}^c_{1,4} \succ [A_{\delta(v^c_i), 1}]$
The matching preferences are within the cycle in both $\mathcal{P}_1$ and $\mathcal{P}_2$.

The depicted preferences are those in $\mathcal{P}_1$. The preferences in $\mathcal{P}_2$ arise from swapping the red numbers. The matching $M_1$ is marked in bold.

Figure 1: An example for the vertex-selection gadget from Theorem 1. For an edge between two agents $a$ and $a'$, the number $x$ closer to agent $a$ means that $a$ ranks $a'$ at position $x$, i.e., there are $x - 1$ agents which $a$ prefers to $a'$. For example, the preferences of $a_{1,1}^c$ are $a_{1,1}^c > a_{1,3}^c$.

In $\mathcal{P}_2$, only the preferences of agents $t^c$ and $t^e$ change to $u^c > s^c$, respectively, $\bar{u}^c > \bar{s}^c$. Notably, in each of the added 4-cycles, there exist two matchings of the four agents that are stable within the cycle in both $\mathcal{P}_1$ and $\mathcal{P}_2$ (i.e., $\{a_{1,1}^e, a_{1,2}^e\}, \{a_{1,3}^e, a_{1,4}^e\}$ or $\{a_{1,1}^e, a_{1,4}^e\}, \{a_{1,2}^e, a_{1,3}^e\}$) and $\{\{\bar{a}_{1,1}^c, \bar{a}_{1,2}^c\}, \{\bar{a}_{1,3}^c, \bar{a}_{1,4}^c\}\}$ or $\{\bar{a}_{1,1}^c, \bar{a}_{1,4}^c\}, \{\bar{a}_{1,2}^c, \bar{a}_{1,3}^c\}$) for $c \in [\ell]$ and $i \in [\nu]$. From each 4-cycle $a_{1,1}^e-a_{1,2}^e-a_{1,3}^e-a_{1,4}^e$, $\bar{a}_{1,1}^c-a_{1,2}^c-a_{1,3}^c-a_{1,4}^c$, matching $M_1$ contains edges $\{a_{1,1}^e, a_{1,2}^e\}$ and $\{a_{1,3}^e, a_{1,4}^e\}$, and from each 4-cycle $\bar{a}_{1,1}^c-a_{1,2}^c-a_{1,3}^c-a_{1,4}^c$, $\bar{a}_{1,1}^c-a_{1,2}^c-a_{1,3}^c-a_{1,4}^c$, matching $M_1$ contains edges $\{\bar{a}_{1,1}^c, \bar{a}_{1,4}^c\}$ and $\{\bar{a}_{1,2}^c, \bar{a}_{1,3}^c\}$.

**Edge Gadget.** For each edge $e = \{v_i^c, v_j^c\} \in E$, we add an edge gadget. This gadget consists of a 4-cycle with agents $a_{e,1}, a_{e,2}, a_{e,3}$, and $a_{e,4}$, admitting two different matchings that are stable within the gadget in both $\mathcal{P}_1$ and $\mathcal{P}_2$. The matching $M_1$ contains $\{a_{e,1}, a_{e,4}\}$ and $\{a_{e,3}, a_{e,2}\}$ in this 4-cycle and remains stable in $M_2$ if all of $s^c, \bar{s}^c, s^e,$ and $\bar{s}^e$ are matched at least as good as the respective agents corresponding to $v_i^e$ and $v_j^e$. This notably only happens if the vertex-selection gadgets of $V^c$ and $V^e$ “select” the endpoints of $e$. Otherwise, the agents in this component need to be matched as $\{a_{e,1}, a_{e,2}\}$ and $\{a_{e,3}, a_{e,4}\}$ in $M_2$, thereby contributing four pairs to $M_1 \triangle M_2$. For each $e = \{v_i^c, v_j^c\} \in E$, the agent’s preferences are as follows:

$$
\begin{align*}
a_{e,1} : a_{e,2} > s^c > s^e > \bar{s}^e > s^c > a_{e,4}, \\
a_{e,3} : a_{e,4} > a_{e,2}, \\
a_{e,2} : a_{e,3} > a_{e,1}, \\
a_{e,4} : a_{e,1} > a_{e,3}.
\end{align*}
$$

**The Reduction.** To complete the description of the parameterized reduction, we set $M_1 := \{\{s^c, t^c\}, \{s^e, t^e\} \mid c \in [\ell]\} \cup \{\{a_{1,1}^e, a_{1,2}^e\}, \{a_{1,3}^e, a_{1,4}^e\}, \{a_{1,1}^c, a_{1,2}^c\}, \{a_{1,3}^c, a_{1,4}^c\} \mid c \in [\ell], i \in [\nu]\} \cup \{\{a_{e,1}, a_{e,4}\}, \{a_{e,3}, a_{e,2}\} \mid e \in E\}$ and $k := \ell \cdot (4\nu + 5) + 4(|E| - \binom{\nu}{2})$. 

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For the correctness of the reduction one can show that in $M_2$ for each $c \in [\ell]$ there is some $i^* \in [\nu]$ such that the matching $M_2$ contains pairs \{s^c, a_{i^*}^c\}, \{s^c, a_{i^*}^c\}$ (this corresponds to selecting vertex $v_i^c$ for color $c$). Then, the only agents $a_{e_1}$ for an edge $e \in E$ incident to some vertex from $V_c$ that both $s^c$ and $s^c$ do not prefer to their partner in $M_2$ are those in $A_3(v_{i^*}^c)$. This implies that for all edges $e = \{v_i^c, v_j^c\} \in E$ with both endpoints selected we can match $a_{e_1}$ worse than $s^c$, $s^c$, $s^c$, and $s^c$. Thus, we can select \{a_{e_1}, a_{e_1}, a_{e_2}, a_{e_2}\} as in $M_1$ in the respective edge gadget. In contrast, for all other edges we have to select the other matching in the edge gadget. To upper-bound the overall symmetric difference, one needs to further prove that for all $j < i^*$, matching $M_2$ contains \{(a_{j,1}^c, a_{j,3}^c), (a_{j,3}^c, a_{j,4}^c)\}, and that for all $j > i^*$, matching $M_2$ contains \{(a_{j,1}^c, a_{j,4}^c), (a_{j,2}^c, a_{j,3}^c)\}. Thus, independent of the selected vertex, each vertex-selection gadget contributes $4\nu + 5$ pairs to $M_1 \Delta M_2$.

### Proof of Correctness
We start the proof of correctness by showing that the constructed instance is indeed a feasible instance of \textsc{Incremental Stable Roommates}, i.e., that $M_1$ is stable in $P_1$.

**Lemma 1.** $M_1$ is stable in $P_1$.

**Proof.** Since $s^c$, $s^c$, $t^c$, and $\bar{t}^c$ are matched to their first choice, they are not part of a blocking pair. Thus, the only possible remaining blocking pairs are inside a 4-cycle $a_{i,1}^c-a_{i,2}^c-a_{i,3}^c-a_{i,4}^c-a_{i,1}^c$ or $\bar{a}_{i,1}^c-\bar{a}_{i,2}^c-\bar{a}_{i,3}^c-\bar{a}_{i,4}^c-\bar{a}_{i,1}^c$ for some $i \in [\nu]$ and $c \in [\ell]$, or $a_{e_1}^c-a_{e_2}^c-a_{e_3}^c-a_{e_4}^c-a_{e_1}^c$ for some $e \in E$. It is easy to verify that there is no blocking pair inside such a 4-cycle. Thus, $M_1$ is stable.

Next, we classify the stable matchings in $P_2$. In order to simplify notation, we define for every vertex $v_i^c$ with $i \in [\nu]$ and $c \in [\ell]$, matchings $M_i^c := \{(a_{i,1}^c, a_{i,2}^c), (a_{i,3}^c, a_{i,4}^c), (a_{i,1}^c, \bar{a}_{i,2}^c), (\bar{a}_{i,3}^c, \bar{a}_{i,4}^c)\}$ and $\bar{M}_i^c := \{(a_{i,1}^c, \bar{a}_{i,4}^c), (a_{i,3}^c, \bar{a}_{i,2}^c), (\bar{a}_{i,1}^c, a_{i,4}^c), (\bar{a}_{i,3}^c, a_{i,2}^c)\}$.

**Lemma 2.** A matching $M$ is stable in $P_2$ if and only if

1. for every $c \in [\ell]$, matching $M$ contains $\{t^c, u^c\}$ and $\{\bar{t}^c, \bar{u}^c\}$,
2. for every $c \in [\ell]$, there exists some $i \in [\nu]$ such that matching $M$ contains pairs \{s^c, a_{i,1}^c\}, \{s^c, a_{i,4}^c\}, \{a_{i,2}^c, a_{i,3}^c\}$, matching $M_j^c$ for all $j < i$, matching $M_j^c$ for $j > i$, and \{(a_{i,1}^c, \bar{a}_{i,2}^c), (\bar{a}_{i,3}^c, \bar{a}_{i,4}^c)\} or \{(a_{i,1}^c, \bar{a}_{i,4}^c), (\bar{a}_{i,3}^c, \bar{a}_{i,2}^c)\},
3. for every edge $e \in E$, matching $M$ contains pairs $\{a_{e_1,1}, a_{e_2,2}\}$ and $\{a_{e_3,3}, a_{e_4,4}\}$ or $\{a_{e_1,1}, a_{e_4,4}\}$ and $\{a_{e_3,3}, a_{e_2,2}\}$, and
4. for every edge $e = \{v_i^c, v_j^c\} \in E$ with $\{s^c, a_{i,1}^c\} \notin M$ or $\{s^c, a_{j,1}^c\} \notin M$, we have that $\{a_{e_1,1}, a_{e_2,2}\}$ and $\{a_{e_3,3}, a_{e_4,4}\}$ are contained in $M$.

**Proof.** We first show that every matching $M$ fulfilling the above conditions is stable. So assume for a contradiction that matching $M$ fulfills the above conditions but there exists a blocking pair $\{a, b\}$. By Item 1 for every $c \in [\ell]$, agents $t^c$, $\bar{t}^c$, $u^c$, and $\bar{u}^c$ are matched to their first choice and thus not part of a blocking pair. It is easy to check that for each $c \in [\ell]$ and $i \in [\nu]$, none of $\{a_{i,1}^c, a_{i,2}^c\}, \{a_{i,1}^c, a_{i,4}^c\}, \{a_{i,3}^c, a_{i,2}^c\}$, and $\{a_{i,3}^c, a_{i,4}^c\}$ is blocking, and the same holds for $\{a_{i,1}^c, \bar{a}_{i,2}^c\}, \{a_{i,1}^c, \bar{a}_{i,4}^c\}, \{\bar{a}_{i,3}^c, a_{i,2}^c\}$, and $\{\bar{a}_{i,3}^c, a_{i,4}^c\}$.

We now check whether for some $c \in [\ell]$, agent $s^c$ or $s^c$ forms a blocking pair together with an agent from the same vertex-selection gadget. By Item 2 there exists some $i \in [\nu]$ such that $\{s^c, a_{i,1}^c\} \in M$ and $\{s^c, a_{i,4}^c\} \in M$. For $j < i$, agent $a_{j,1}^c$ prefers $M(a_{j,1}^c) = a_{j,2}^c$ to $s^c$, while for $j > i$, agent $s^c$ prefers $M(s^c) = a_{i,1}^c$ to $a_{i,1}^c$. Thus, $\{s^c, a_{i,1}^c\}$ is not blocking. Similarly, for each $c \in [\ell]$ and $j \in [\nu]$, $\{s^c, a_{j,1}^c\}$ is not blocking. For $j < i$, agent $s^c$ prefers $M(s^c) = a_{i,1}^c$, while for $j > i$, agent $s^c$ prefers $M(s^c) = a_{i,1}^c$ to $a_{i,1}^c$. Thus, $\{s^c, a_{i,1}^c\}$ is not blocking. For $j < i$, agent $s^c$ prefers $M(s^c) = a_{i,1}^c$, while for $j > i$, agent $s^c$ prefers $M(s^c) = a_{i,1}^c$ to $a_{i,1}^c$.
to $a_{j,4}^c$, while for $j > i$, agent $a_{j,4}^c$ prefers $M(a_{j,4}^c) = a_{i,1}^c$ to $s^c$. Thus, $\{s^c, a_{j,4}^c\}$ is not blocking. Similarly, for each $c \in [\ell]$ and $j \in [\nu]$, pair $\{s^c, a_{j,4}^c\}$ is not blocking.

It is easy to verify that there is no blocking pair inside an edge gadget. Lastly, consider a pair $\{s^c, a_{e,1}\}$ for an edge $e = \{v_i^c, v_j^c\}$. If $\{a_{e,1}, a_{e,2}\} \in M$, then $a_{e,1}$ prefers $M(a_{e,1}) = a_{e,2}$ to $s^c$ and $\bar{s}^c$. Otherwise, Item 4 implies that $\{s^c, a_{i,1}^c\} \in M$ and by Item 2 $\{\bar{s}^c, a_{i,1}^c\} \in M$. Thus, $s^c$ prefers $M(s^c) = a_{e,1}^c$ to $a_{e,1}$ and $\bar{s}^c$ prefers $M(\bar{s}^c) = a_{i,1}^c$ to $a_{e,1}$. Consequently, $\{s^c, a_{e,1}\}$ and $\{s^c, a_{i,1}\}$ are not blocking. Therefore, $M$ is stable.

We now show that every stable matching $M$ fulfills the above conditions. Since for all $c \in [\ell]$, agents $t^c$ and $u^c$ as well as $\bar{t}^c$ and $\bar{u}^c$ are their mutual top choice, it follows that $M$ fulfills Item 1. Note that there clearly always exists a matching $M^*$ fulfilling Items 1 to 4 and that $M^*$ is perfect. As proven above, $M^*$ is stable, and by the Rural-Hospitals-Theorem for SR \cite{18}, every stable matching in $\mathcal{P}_2$ is perfect. It follows that no stable matching contains $\{s^c, a_{e,1}\}$ or $\{\bar{s}^c, a_{e,1}\}$ for some $c \in [\ell]$ and $e \in E$, as otherwise $a_{e,2}$ or $a_{e,4}$ remains unmatched. As $M$ is perfect, it follows that $M$ fulfills Item 3. We now turn to Item 2. Fix some $c \in [\ell]$. Since $s^c$ is matched in $M$, matching $M$ contains pair $\{s^c, a_{i,1}^c\}$ or $\{s^c, a_{i,1}^c\}$ for some $i \in [\nu]$. We first assume for a contradiction that $M$ contains $\{s^c, a_{i,1}^c\}$ for some $i \in [\nu]$. As $M$ is perfect, this implies that $M$ also contains $\{\bar{s}^c, a_{i,1}^c\}$. We now distinguish two cases. Assuming that $M$ contains $\{a_{i,1}^c, a_{i,2}^c\}$, then it also contains $\{a_{i,3}^c, a_{i,4}^c\}$, and it follows that $\{s^c, a_{i,4}^c\}$ blocks $M$, a contradiction. Otherwise $M$ contains $\{a_{i,1}^c, a_{i,4}^c\}$, and $\{s^c, a_{i,2}^c\}$ blocks $M$. Therefore, $\{s^c, a_{i,1}^c\}$ is part of $M$ for some $i \in [\nu]$. As $M$ is perfect, $M$ also contains $\{a_{e,2}^c, a_{e,4}^c\}$ and $\{\bar{s}^c, a_{i,4}^c\}$. Since, for every $j \in [\nu] \setminus \{i\}$, pairs $\{s^c, a_{j,4}^c\}$ and $\{\bar{s}^c, a_{j,4}^c\}$ are not blocking, it follows that $M$ contains $M_j$ for $j < i$. Since $\{s^c, a_{e,4}^c\}$ and $\{\bar{s}^c, a_{e,4}^c\}$ are not blocking, it follows that $M$ contains $M_j$ for $j < i$. Therefore, $M$ fulfills Item 2.

Assume for a contradiction that $M$ does not fulfill Item 1. Then there exists some $c \in [\ell]$ and $i \in [\nu]$ such that $\{s^c, a_{i,1}^c\} \in M$ and $\{\bar{s}^c, a_{i,4}^c\} \in M$, and an edge $e = \{v_j^c, v_j^c\}$ with $i \neq j$ such that $\{a_{e,1}, a_{e,4}\} \in M$. If $j < i$, then $\{s^c, a_{e,1}\}$ blocks $M$. If $j > i$, then $\{\bar{s}^c, a_{e,1}\}$ blocks $M$. Therefore, $M$ is not stable, a contradiction.

We are now ready to prove the correctness of our reduction.

**Theorem 1.** ISR parameterized by $|\mathcal{P}_1 \oplus \mathcal{P}_2|$ is W[1]-hard.

**Proof.** In this section, we have described a reduction from MULTICOLORED CLIQUE parameterized by solution size $\ell$ to ISR parameterized by $|\mathcal{P}_1 \oplus \mathcal{P}_2|$. This reduction clearly runs in polynomial time. Further, as the preference profiles $\mathcal{P}_1$ and $\mathcal{P}_2$ only differ by a single swap in the preference profiles of $t^c$ and $\bar{t}^c$ for every $c \in [\ell]$, we have $|\mathcal{P}_1 \oplus \mathcal{P}_2| = 2\ell$. It remains to show the correctness of the reduction.

$(\Rightarrow)$ : Given a multicolored clique $\{v_1^c, \ldots, v_\ell^c\}$, we construct a stable matching as follows. Set

$$M_2 := \left\{ \{t^c, u^c\}, \{\bar{t}^c, \bar{u}^c\}, \{s^c, a_{h,c,1}^c\}, \{\bar{s}^c, a_{h,c,4}^c\}, \{a_{h,c,2}^c, a_{h,c,3}^c\}, \{\bar{a}_{h,c,1}^c, \bar{a}_{h,c,4}^c\}, \right. \left. \{a_{h,c,2}^c, a_{h,c,3}^c\} \mid c \in [\ell] \right\}$$

$$\bigcup_{c \in [\ell], i \in [\nu] : i < h_c} M_i^c \bigcup_{c \in [\ell], i > h_c} \bar{M}_i^c \bigcup_{e = \{v_j^c, v_j^c\} \in E} \{a_{e,1}, a_{e,4}\}, \{a_{e,3}, a_{e,2}\} \bigcup_{e \neq \{v_j^c, v_j^c\} \in E} \{a_{e,1}, a_{e,2}\}, \{a_{e,3}, a_{e,4}\}$$

By Lemma 2, $M_2$ is stable. Note that

$$M_1 \Delta M_2 = \left\{ \{a_{i,1}^c, a_{i,2}^c\}, \{a_{i,1}^c, a_{i,4}^c\}, \{a_{i,2}^c, a_{i,3}^c\}, \{a_{i,3}^c, a_{i,4}^c\} \mid c \in [\ell], i > h_c \right\}$$

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follows that the symmetric difference of $c$ edges $M$ at least $\text{ISR}$

Complementing the above $\text{W}[1]$-hardness result, we now present an intricate XP-algorithm for $a$ pair the main ingredients of our algorithm. We say that we update a matching also of independent interest and might find applications elsewhere. In the following, we sketch stable matching. We believe that our technique to propagate changes through a matching is reached. This matching is then guaranteed to be as close as possible to the original by the initial changes by “propagating” these constraints through the instance until a new stable

These changes and guesses then impose certain constraints how good/bad some agents must be

Thus, we have $|M_1 \triangle M_2| = \ell \cdot (4v + 5) + 4(|E| - (\frac{\ell}{2})) = k$.

$(\Leftarrow)$: Let $M_2$ be a stable matching in $P_2$ with $|M_1 \triangle M_2| \leq k$. By Lemma 2, it follows that the symmetric difference of $M_1$ and $M_2$ in each vertex gadget is at least $4v + 5$. Furthermore, for each color $c \in [\ell]$, there exists some $h_c$ such that $\{s^c, a^c_{h_c,1}\} \in M$. Lemma 2 implies that the symmetric difference of $M_1$ and $M_2$ in an edge gadget can only be smaller than four for edges $e = \{v^c_i, v^c_j\}$ such that $\{s^c, a^c_{i,1}\} \in M_2$ and $\{s^c, a^c_{j,1}\} \in M_2$. Since $|M_1 \triangle M_2| \leq k$, it follows that the symmetric difference of $M_1$ and $M_2$ in an edge gadget is smaller than four for at least $\binom{\ell}{2}$ edge gadgets. Thus, $\{v^c_{h_c} | c \in [\ell]\}$ is a multicolored clique.

**3.2 XP-Algorithm**

Complementing the above $\text{W}[1]$-hardness result, we now present an intricate XP-algorithm for $\text{ISR}$ parameterized by $|P_1 \oplus P_2|$, resulting in the following theorem:

**Theorem 2.** $\text{ISR}$ can be solved in $O(2^{|P_1 \oplus P_2|} \cdot n^5|P_1 \oplus P_2| + 3)$ time.

As our algorithm and its proof of correctness are quite complicated and uses some novel ideas, we split the reminder of this subsection into two parts. In Section 3.2.1 we give an high-level overview of the algorithm and some intuitive explanations how and why it works. Subsequently, in Section 3.2.2 we present the algorithm in all its details and formally prove its correctness. Throughout this section, we assume that $M_1$ and $M_2$ are both perfect matchings (we will argue why we can assume this without loss of generality in Lemma 5).

**3.2.1 High-Level Description**

Our algorithm works in two phases: In the initialization phase, we make some guesses how the stable matching $M_2$ looks like and accordingly change the original stable matching $M_1$. These changes and guesses then impose certain constraints how good/bad some agents must be matched in $M_2$. Subsequently, in the propagation phase, we locally resolve blocking pairs caused by the initial changes by “propagating” these constraints through the instance until a new stable matching is reached. This matching is then guaranteed to be as close as possible to the original stable matching. We believe that our technique to propagate changes through a matching is also of independent interest and might find applications elsewhere. In the following, we sketch the main ingredients of our algorithm. We say that we update a matching $M$ to contain a pair $e = \{a, b\}$ if we delete all pairs containing $a$ or $b$ from $M$ and add pair $e$ to $M$.

**Initialization Phase (First Part of Description).** Our algorithm maintains a matching $M$. At the beginning, we set $M := M_1$. Before we change $M$, we make some guesses on how the output matching $M_2$ shall look like. These guesses are responsible for the exponential part of the running time (the rest of our algorithm runs in polynomial time). The guesses result in some changes to $M$ and, for some agents $a \in A$, in a “best case” and “worst case” to which partner $a$ can be matched in $M_2$. Consequently, we will store in $bc(a)$ the best case how $a$ may
be matched in $M_2$, i.e., the most-preferred (by $a$ in $P_2$) agent $b$ for which we cannot exclude that $a$ is matched to $b$ in a stable matching in $P_2$ respecting the guesses (in other words, this means that $a$ cannot be matched to a better partner that $bc(a)$ in $M_2$). Similarly, $wc(a)$ stores the worst case to which $a$ can be matched. We initialize $bc(a) = wc(a) = \bot$ for all $a \in A$, encoding that we do not know a best or worst case yet. We will say that agent $a$ has a trivial best case if $bc(a) = \bot$ and a trivial worst case if $wc(a) = \bot$.

To be more specific, among others, in the initialization phase we guess for each agent $a \in A$ with modified preferences as well as for $M_1(a)$ how they are matched in $M_2$ and update $M$ to include the guessed pairs. Moreover, as an unmatched agent $a$ shall always have $bc(a) \neq \bot$ or $wc(a) \neq \bot$, we guess for all agents $a$ that became unmatched by this whether they prefer $M_1(a)$ to $M_2(a)$ (in which case we set $bc(a) := M_1(a)$) or $M_2(a)$ to $M_1(a)$ (in which case we set $wc(a) := M_1(a)$). Our algorithm also makes further guesses in the initialization phase. However, in order to understand the purpose of these additional guesses, it is helpful to first understand the subsequent propagation phase in some detail. Thus, we postpone the description of the additional guesses to the end of this section.

**Propagation Phase.** After the initialization phase, blocking pairs for the current matching $M$ force the algorithm to further change $M$ and force a propagation of best and worst cases through the instance until a stable matching is reached. As our updates to $M$ are in some sense “minimally invasive” and exhaustive, once $M$ is stable in $P_2$, it is guaranteed to be the stable matching in $P_2$ which is closest to $M_1$ among all matchings respecting the initial guesses. At the core of the technique lies the simple observation that in an SR instance for each stable pair $\{c, d\}$ and each stable matching $N$ not including $\{c, d\}$ exactly one of $c$ and $d$ prefers the other to its partner in $N$:

**Lemma 3 ([26], Lemma 4.3.9).** Let $N$ be a stable matching and $e = \{c, d\} \notin N$ be a stable pair in an SR instance. Then either $N(c) \succ_e d$ and $c \succ_d N(d)$ or $d \succ_e N(c)$ and $N(d) \succ_d c$.

From this we can draw conclusions in the following spirit: Assuming that for a stable pair $\{c, d\}$ in $P_2$ we have that $wc(c) \succ_e d$, i.e., $c$ is matched better than $d$ in $M_2$, it follows from Lemma 3 that $d$ is matched worse than $c$ in $M_2$, implying that we can safely set $bc(d) = c$.

In the following, we will now explain simplified versions of some parts of the propagation phase, while leaving out others.

Assume for a moment that the current matching $M$ is perfect and that there is a blocking pair for $M$ in $P_2$ (see Algorithm 1 for a pseudocode-description of the procedure described in the following). Because $M_1$ is stable in $P_1$, all pairs that currently block $M_1$ either involve an agent with changed preferences or resulted from previous changes made to $M$. Using this, one can show that at least one of the two agents from a blocking pair $\{a, b\}$, say $b$, will have

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**Algorithm 1** Simplified propagation step performed for a pair $\{a, b\}$ of two matched agents blocking $M$ with $bc(b) = M(b)$ or for an unmatched agent $a$ with $wc(a) \neq \bot$.

1. **if** $a$ is unmatched **then** Let $e = \{a, c\}$ be the stable pair in $P_2$ such that $c \succ^P_2 wc(a)$ and $bc(c) \succ^P_2 a$ (or $bc(c) = \bot$) and $c$ is worst-ranked by $a$ among all such pairs.
2. **else** Let $e = \{a, c\}$ be the stable pair in $P_2$ such that $c \succ^P_2 M(a)$ and $bc(c) \succ^P_2 a$ (or $bc(c) = \bot$) and $c$ is worst-ranked by $a$ among all such pairs.
3. **if** no such pair exists **then** Reject this guess.
4. **else** Update $M$ such that it contains $e$, set $wc(a) := c$ and $bc(c) := a$.
5. **if** $M(a) \neq \square$ **then** $bc(M(a)) := a$.
6. **if** $M(c) \neq \square$ **then** $wc(M(c)) := c$.
a \succ b \ bc(b) = M(b). Thus, we know that b is matched worse than a in any stable matching in \( P_2 \) respecting our current guesses. Accordingly, for \( \{a, b\} \) not to block \( M_2 \) agent a has to be matched to b or better and, in particular, better than \( M(a) \) in the solution. As a consequence, we update the worst case of a to be the next agent c which a prefers to \( M(a) \) such that \( \{a, c\} \) is a stable pair in \( P_2 \), i.e., we set \( wc(a) := c \) (see Line 4). Subsequently, we propagate this change further through the instance: Note that from Lemma 3 it follows that if \( \{a', a''\} \) is a stable pair in \( P_2 \) and agent \( a'' \) is the worst possible partner of \( a' \) in a stable matching in \( P_2 \) (or \( a' \) prefers its worst possible partner to \( a'' \)), then agent \( a'' \) cannot be matched better than agent \( a' \) in a stable matching in \( P_2 \). Thus, by setting \( wc(a') := a'' \) we also get \( bc(a'') := a' \). Consequently, applying this to our previous update \( wc(a) := c \), we can also set \( bc(c) := a \) (see Line 4). Moreover, recall that by the definition of c, agent a prefers c to a’s current partner \( M(a) \) in \( M \). Thus, assuming that in a stable matching \( M^* \) in \( P_2 \) one of a and \( M(a) \) prefers the other to its partner in \( M^* \), whereas the other prefers its partner in \( M^* \), as \( wc(a) = c \succ_a M(a) \), we can conclude \( bc(M(a)) := a \) (see Line 6 we will discuss in the next part “Initialization Phase (Second Part of Description)” why this assumption can be made). A similar reasoning applies to the update in Line 6.

So far, we assumed that all agents are matched. If there is an unmatched agent a involved in a blocking pair, then one can show that it cannot be matched to \( bc(a) \) or \( wc(a) \). Thus, as each stable matching for \( P_2 \) is perfect, if \( wc(a) \neq \bot \), then we match a to the next-better agent c before \( wc(a) \) in its preferences such that \( \{a, c\} \) is a stable pair in \( P_2 \) and set \( wc(a) = c \). Subsequently, we propagate this change as in the above described case of a blocking pair (see Algorithm 1). Otherwise, we have \( bc(a) \neq \bot \) and we match a to the next-worse agent c after \( bc(a) \) in the preferences of a such that \( \{a, c\} \) is a stable pair in \( P_2 \). Here, a slightly more complicated subsequent propagation step is needed (as described in the next section).

Repeating these steps, i.e., matching so far unmatched agents and resolving blocking pairs, eventually either results in a conflict (i.e., an agent preferring its worst case to its best case, or changing a pair which we guessed to be part of \( M_2 \)) or in a stable matching. In the first case, we conclude that no stable matching obeying our guesses exists, while in the latter case, we found a stable matching obeying the guesses and minimizing the symmetric difference to \( M_1 \) among all such matchings. The reason for the optimality of this matching is that every matching obeying our initial guesses has to obey the best and worst cases at the termination of the algorithm, and the computed matching \( M \) contains all pairs from \( M_1 \) which comply with the best and worst cases.

Initialization Phase (Second Part of Description). In addition to the guesses described above, the algorithm guesses the set \( F \) of pairs from \( M_1 \) for which both endpoints prefer \( M_2 \) to \( M_1 \). Similarly, the algorithm guesses the set \( H \) of pairs from \( M_2 \) for which both endpoints prefer \( M_1 \) to \( M_2 \). Notably, one can prove that the cardinality of both \( F \) and \( H \) can be upper-bounded by \( |P_1 \oplus P_2| \), ensuring XP-running time. The reason why we need to guess the set \( F \) is that pairs \( \{a, b\} \) from \( M_1 \) may not be stable pairs in \( P_2 \). In this case, if \( \{a, b\} \in M \) (which initially holds) and we know that a prefers \( M_2(a) \) to b it does not follow that b prefers a to \( M_2(b) \). Thus, if we would treat the pairs from \( F \) as “normal” pairs, we would propagate an incorrect best case in Line 5. Note that all pairs from \( M \setminus M_1 \) are stable pairs in \( P_2 \), as throughout the algorithm, we only add pairs that are stable in \( P_2 \) (see Line 4). The reason why we need to guess the set \( H \) is more subtle but also due to fact that pairs from \( H \) might cause problems for our propagation step. To incorporate our guesses, in the initialization phase, for each \( \{a, b\} \in F \), we delete \( \{a, b\} \) from \( M \) and set \( wc(a) = b \) and \( wc(b) = a \), while for each \( \{a, b\} \in H \) we update \( M \) to include \( H \). We remark that from the proof of Theorem 1 it follows that ISR is NP-complete even if we know for each agent a whose preferences changed.
as well as $M_1(a)$ how they are matched in $M_2$ and the set of pairs $F \subseteq M_1$ for which both endpoints prefer $M_2$ to $M_1$. This indicates that guessing the set $H$ might be necessary for an XP-algorithm.

### 3.2.2 Formal Description and Proof of Correctness

In the following, we start by giving a formal description of the XP algorithm in Section 3.2.2.1. Subsequently, in Section 3.2.2.2, we show that we can indeed assume that both $M_1$ and $M_2$ are perfect matchings. Next, in Section 3.2.2.3, we prove some useful observations on the best and worst cases of agents, how they relate to the current matching $M$ and to the matching $M_1$, and how they change during the execution of the algorithm. These observations will be crucial when proving the correctness of the algorithm. In Section 3.2.2.4, we continue by establishing that the algorithm is well-defined. Moreover, in Section 3.2.2.5 we prove that we propagate the best and worst cases of agents correctly, i.e., a matching that obeys our initial guesses also obeys the best and worst cases of all agents throughout the execution of the algorithm. We conclude by proving in Section 3.2.2.6 that the number of our initial guesses that we need to make is bounded in $n^{O(|P_1 \oplus P_2|)}$ and finally combine all parts of the proof in Section 3.2.2.7.

#### 3.2.2.1 Formal Description of Algorithm

We assume that in the given ISR instance matchings $M_1$ and $M_2$ are perfect (we later argue in Section 3.2.2.2 how we can preprocess instances to achieve this).

A pseudocode description of our XP-algorithm for ISR parameterized by $|P_1 \oplus P_2|$ can be found in Algorithms 2 to 4. Algorithm 2 contains the main part of the algorithm while Algorithm 3 concerns the initialization and Algorithm 4 the propagation step.

We start the algorithm with a call of the `Initialization(·)` function described in Algorithm 3 in which we initialize the matching $M$, the best and worst cases of agents, and make some guesses as described in Section 3.2.1. Next, as long as the current matching $M$ is blocked by some pair, we further update the matching and agent’s best and worst cases. For this, we make a case distinction based on whether there is a blocking pair involving an unmatched agent $a$. In this case we call the `Propagate(·)` function on $a$. In the case that all agents that are part of a blocking pair are matched, we pick an arbitrary blocking pair $\{a, b\}$, do some preprocessing steps and call the `Propagate(·)` function on the agent $x \in \{a, b\}$ with $bc(x) = \bot$ and $bc(y) \neq \bot$ with $y \in \{a, b\} \setminus \{x\}$ (we show in Section 3.2.2.4 that such an agent exists in every blocking pair). Lastly, in Algorithm 4, we describe the `Propagate(·)` function called on some agent $a$. Here we make a case distinction based on whether $bc(a) = \bot$ or $bc(a) \neq \bot$. In both cases, we search for a stable pair $\{a, b\}$ that respects the current best and worst cases, initial guesses, and the constraints imposed by the blocking pair and that is worst ranked among these pairs. Subsequently, we update the best or worst cases of $a$ and $b$ and under certain constraints also of $M(a)$ and $M(b)$. Finally, we update $M$ to include $\{a, b\}$.

#### 3.2.2.2 Initial Assumptions on $M_1$ and $M_2$

Note that our algorithm assumes that in the given ISR instance $(A, P_1, P_2, M_1, k)$ matching $M_1$ is a perfect matching and there is a perfect stable matching in $P_2$. We now show that we may indeed assume this by showing in two steps that each ISR instance can be reduced in linear time to an ISR instance satisfying these properties.

We first establish that we can assume that there is a perfect stable matching in $P_2$. 

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Algorithm 2 Algorithm for INCREMENTAL SR

Input: An ISR instance $\mathcal{I} = (A, \mathcal{P}_1, \mathcal{P}_2, M_1, k)$ where $M_1$ is a perfect stable matching in $\mathcal{P}_1$ and there is a perfect stable matching in $\mathcal{P}_2$.

Output: A stable matching $M_2$ in $\mathcal{P}_2$ with $|M_1 \triangle M_2| \leq k$ if one exists.

1: Initialization()
2: while there exists a blocking pair for $M$ in $\mathcal{P}_2$ do
3: if there is an agent $a$ not matched by $M$ that is part of a blocking pair for $M$ then
4: Propagate($a$).
5: else
6: Let $\{a, b\}$ be a blocking pair for $M$ in $\mathcal{P}_2$.
7: if $bc(a) \neq \bot$ and $bc(b) \neq \bot$ then Reject this guess.
8: Let $\{x, y\} = \{a, b\}$ such that $bc(x) = \bot$ and $bc(y) \neq \bot$.
9: if $wc(x) = \bot$ then Set $wc(x) := M(x)$.
10: Propagate($x$).
11: if there exists an agent $a$ with $wc(a) \succ_{\mathcal{P}_2} bc(a)$ then Reject this guess.
12: if $|M \triangle M_1| \leq k$ then Return $M$. else Reject this guess.

Algorithm 3 Function Initialization used in Algorithm 2

12: Set $M := M_1$.
13: Set $bc(a) := \bot$ and $wc(a) := \bot$ for every agent $a$.
14: Let $B$ be the set of agents with modified preferences and their partners in $M_1$. Guess for each agent $a \in B$ how $a$ is matched in $M_2$, and update $M$ such that it contains the guessed pairs.
15: Guess a set $H \subseteq A \times A$ of up to $|\mathcal{P}_1 \oplus \mathcal{P}_2|$ pairs and update $M$ to include $H$.
16: $(\triangleright)$ pairs from $M_2$ where both endpoints prefer $M_1$ to $M_2$ in $\mathcal{P}_2$.
17: Let $X$ be the set of all agents from $B$ and their guessed partners and all agents that are part of a pair from $H$.
18: Set $bc(a) := M(a)$ and $wc(a) := M(a)$ for every agent $a \in X$.
19: if we guessed that two agents from $X$ have the same partner then Reject this guess.
20: for every $a \in X$ with $M_1(a) \notin X$ do
21: Guess whether $M_1(a)$ prefers $M_2(a)$ to $a$ or prefers $a$ to $M_2(a)$ in $\mathcal{P}_2$.
22: In the former case, set $wc(M_1(a)) := a$, while in the latter case, set $bc(M_1(a)) := a$.
23: Guess a set $F$ of up to $|\mathcal{P}_1 \oplus \mathcal{P}_2|$ pairs from $M_1$.
24: $(\triangleright)$ pairs for which both endpoints prefer $M_2$ to $M_1$ in $\mathcal{P}_2$.
25: if there is a pair $\{a, M_1(a)\} \in F$ with $a \in X$ with $M_1(a) \succ_{\mathcal{P}_2} bc(a)$ then Reject this guess.
26: for every $e = \{a, b\} \in F$ do Set $wc(a) = b$, set $wc(b) = a$ and delete $e$ from $M$.

Lemma 4. An instance $\mathcal{I} = (A, \mathcal{P}_1, \mathcal{P}_2, M_1, k)$ of ISR can be transformed in linear time into an equivalent instance $\mathcal{I}' = (A', \mathcal{P}_1', \mathcal{P}_2', M_1', k')$ of ISR with a perfect stable matching in $\mathcal{P}_1'$ and $|\mathcal{P}_1 \oplus \mathcal{P}_2| = |\mathcal{P}_1' \oplus \mathcal{P}_2'|$.

Proof. Recall that in each stable matching in an SR instance the same set of agents is matched by the Rural Hospitals Theorem for SR [29]. Note further that we may assume without loss of generality that there is a stable matching in $\mathcal{P}_2$, as we can check this in linear time [29] and otherwise can map $\mathcal{I}$ to a trivial no-instance. Let $A_1$ be the set of agents that are unmatched
in a stable matching in $P_1$ and $A_2$ be the set of agents that are unmatched in a stable matching in $P_2$. We modify $I$ to arrive at $I'$ as follows. For each agent $a \in A_2$, we add an agent $a'$ which in both preference profiles only finds $a$ acceptable and append $a'$ at the end of the preferences of $a$ in both preference profiles. Moreover, for each $a \in A_1 \cap A_2$, we add to matching $M_1$ the pair $\{a, a'\}$, i.e., $M'_1 := M_1 \cup \{\{a, a'\} : a \in A_1 \cap A_2\}$. Lastly, we set $k'$ to $k + |A_2 \setminus A_1|$.

First note that $M'_1$ is a stable matching in $P'_1$, as each agent $a \in A_2$ is either matched to $a'$ or prefers its partner $M_1(a)$ to $a'$. Moreover, there exists a perfect stable matching in $P'_2$. Let $M$ be a stable matching in $P_2$ (we know that such a matching needs to exist). Then $M \cup \{\{a, a'\} : a \in A_2\}$ is a perfect stable matching in $P'_2$.

It remains to argue that $I$ is a yes-instance if and only if $I'$ is a yes-instance. Assume that $M_2$ is a solution for $I$. Then $M'_2 = M_2 \cup \{\{a, a'\} : a \in A_2\}$ is a stable matching in $P'_2$. Moreover we have that $M'_1 \Delta M'_2 = M_1 \Delta M_2 \cup \{\{a, a'\} : a \in A_2 \setminus A_1\}$ and thus $|M'_1 \Delta M'_2| = k + |A_2 \setminus A_1|$.

For the other direction, assume that $M'_2$ is a solution for $I'$. First of all note that the matching $M_2$ resulting from $M'_2$ after removing all agent pairs including agents from $\{a' \mid a \in A_2\}$ is stable in $P_2$, as all agents $a \in A$ prefer the same set of agents to $M_2(a)$ in $P_2$ as they prefer to $M'_2(a)$ in $P'_2$. To bound the symmetric difference note that no agent $a \in A_2$ can be unmatched in $M'_2$, as otherwise it forms a blocking pair with agent $a'$. As no agent from $A_2$ is matched in $M_2$ from this it follows that $\{\{a, a'\} : a \in A_2\} \subseteq M'_2$. Consequently, we have that $M'_1 \Delta M'_2 = M_1 \Delta M_2 \cup \{\{a, a'\} : a \in A_2 \setminus A_1\}$ and thus $|M_1 \Delta M_2| = |M'_1 \Delta M'_2| - |A_2 \setminus A_1| \leq k$. \hfill $\square$

Using similar arguments as in the previous lemma, we now show that we can additionally assume that the given stable matching $M_1$ in $P_1$ is perfect:

**Lemma 5.** An instance $I = (A, P_1, P_2, M_1, k)$ of ISR with a perfect stable matching in $P_2$ can be reduced transformed linear time into an equivalent instance $I' = (A', P'_1, P'_2, M'_1, k')$ of ISR with $|P_1 \cup P_2| = |P'_1 \cup P'_2|$ where $M'_1$ is a perfect stable matching in $P'_1$ and there is a perfect stable matching in $P'_2$.

**Proof.** Recall that in each stable matching in an SR instance the same set of agents is matched by the Rural Hospitals Theorem for SR \cite{26}. Let $A_1 = \{a_1, \ldots, a_x\}$ be the set of agents that

\begin{algorithm}
\caption{Function PROPAGATE used in Algorithm 2}
\begin{algorithmic}[1]
\State \textbf{Input:} An agent $a$
\State \textbf{if} $bc(a) \neq \perp$ \textbf{then} $\triangleright$ $a$ prefers $M_1(a)$ to $M_2(a)$ in $P_2$.\State \textbf{if} $bc(a) \neq \perp$ \textbf{then} Let $e = \{a, b\}$ be the stable pair in $P_2$ such that $bc(a) \succ'_{P_2} a, a \succ'_{P_2} b$ and $b$ is best-ranked by $a$ among all such pairs.\State \textbf{if} no such pair exists or $a \in X$ or $b \in X$ \textbf{then} Reject this guess.\State \textbf{if} no such pair exists or $a \in X$ or $b \in X$ \textbf{then} \textbf{else}\State \textbf{if} $bc(M(b)) = \perp$ and $\{b, M(b)\} \in M_1$ \textbf{then} $\triangleright$ $a$ prefers $M_2(a)$ to $M_1(a)$ in $P_2$.\State \textbf{if} $bc(M(b)) = \perp$ and $\{b, M(b)\} \in M_1$ \textbf{then} $\triangleright$ $a$ prefers $M_2(a)$ to $M_1(a)$ in $P_2$.\State \textbf{else} $\triangleright$ $a$ prefers $M_2(a)$ to $M_1(a)$ in $P_2$.\State \textbf{else} $\triangleright$ $a$ prefers $M_2(a)$ to $M_1(a)$ in $P_2$.\State \textbf{else} $\triangleright$ $a$ prefers $M_2(a)$ to $M_1(a)$ in $P_2$.\State Let $e = \{a, b\}$ be the stable pair in $P_2$ such that $b \succ'_{P_2} a$, as each agent $b$ prefers the same set of agents to $M_2(a)$ as they prefer to $M_1(a)$ in $P_2$. To bound the symmetric difference note that no agent $a \in A_2$ can be unmatched in $M'_2$, as otherwise it forms a blocking pair with agent $a'$. As no agent from $A_2$ is matched in $M_2$ from this it follows that $\{\{a, a'\} : a \in A_2\} \subseteq M'_2$. Consequently, we have that $M'_1 \Delta M'_2 = M_1 \Delta M_2 \cup \{\{a, a'\} : a \in A_2 \setminus A_1\}$ and thus $|M_1 \Delta M_2| = |M'_1 \Delta M'_2| - |A_2 \setminus A_1| \leq k$. \hfill $\square$
are unmatched in a stable matching in \( P_1 \). Note that \( x \) is even, as \(|A|\) needs to be even for a perfect matching to exist in \( P_2 \).

We modify \( T \) to arrive at \( T' \) as follows. For each \( i \in [x] \), we add a dummy agent \( a'_i \) that in both preference profiles ranks \( a_i \) in the first position and ranks in the second position \( a'_{i+1} \) if \( i \) is odd and \( a'_{i-1} \) if \( i \) is even. Moreover, we append \( a'_i \) at the end of the preferences of \( a_i \) in both preference profiles. Moreover, for each \( i \in [x] \), we add to matching \( M_1 \) the pair \( \{a_i, a'_i\} \). Lastly, we set \( k' \) to \( k + \frac{3x}{2} \).

First note that \( M'_1 \) is clearly perfect and also a stable matching in \( P'_1 \), as each dummy agent is matched to its top-choice and blocking pairs for \( M_1 \) where both agents are from \( A \) would also block \( M_1 \) in \( P_1 \). Let \( M \) be a perfect stable matching in \( P_2 \). Then, \( M \cup \{\{a'_{2i-1}, a'_{2i}\} \mid i \in [\frac{x}{2}] \} \) is clearly perfect and also a stable matching in \( P'_2 \), as each agent from \( A \) is matched to an agent it prefers to the newly added dummy agents and each dummy agent is matched to the only other dummy agent it finds acceptable.

It remains to argue that \( T \) is a yes instance if and only if \( T' \) is a yes instance. Assume that \( M_2 \) is a solution for \( T \). Let \( M'_2 = M_2 \cup \{\{a'_{2i-1}, a'_{2i}\} \mid i \in [\frac{x}{2}]\} \). As argued above \( M'_2 \) is stable in \( P'_2 \). Moreover note that as \( M'_1 = M_1 \cup \{\{a_i, a'_i\} \mid i \in [x] \} \) and \( M'_2 = M_2 \cup \{\{a'_{2i-1}, a'_{2i}\} \mid i \in [\frac{x}{2}]\} \), we have that \( M'_1 \triangle M'_2 = M_1 \triangle M_2 \cup \{\{a_i, a'_i\} \mid i \in [x]\} \cup \{\{a'_{2i-1}, a'_{2i}\} \mid i \in [\frac{x}{2}]\} \) and thus \(|M'_1 \triangle M'_2| \leq k + \frac{3x}{2} \).

For the other direction, assume that \( M'_2 \) is a solution for \( T' \). First of all note that the matching \( M_2 \) resulting from \( M'_2 \) after removing all agent pairs including a dummy agent is stable in \( P_2 \), as all agents \( a \in A \) prefer the same set of agents to \( M_2(a) \) in \( P_2 \) as they prefer to \( M'_2(a) \) in \( P'_2 \). To bound the symmetric difference note that all agents from \( A \) must be matched in \( M_2 \), as \( M_2 \) is a stable matching in \( P_2 \) and there is a perfect stable matching in \( P_2 \). Thus in \( M'_2 \) no agent from \( A \) can be matched to a dummy agent and thus \( \{\{a'_{2i-1}, a'_{2i}\} \mid i \in [\frac{x}{2}]\} \subseteq M'_2 \). Consequently, as above, we get that \(|M'_1 \triangle M'_2| = M_1 \triangle M_2 \cup \{\{a_i, a'_i\} \mid i \in [x]\} \cup \{\{a'_{2i-1}, a'_{2i}\} \mid i \in [\frac{x}{2}]\} \) and thus \(|M_1 \triangle M_2| = |M'_1 \triangle M'_2| - \frac{3x}{2} \leq k \).

3.2.2.3 Initial observations on \( bc \) and \( wc \)

We continue by making some observations on \( bc \) and \( wc \), how they relate to one another and to \( M_1 \), and when they are set to a non-trivial value. These observations will be vital in the reminder of the proof of correctness.

We start by showing that as soon as an agent is matched differently in \( M \) than in \( M_1 \), the agent has a non-trivial best or worst case.

**Lemma 6.** After calling the Initialization() function, for each agent \( c \in A \), it always holds that if \( M(c) \neq M_1(c) \), then \( bc(c) \neq \perp \) or \( wc(c) \neq \perp \).

**Proof.** For some agent \( c \in A \), we only change \( M(c) \) in Lines 14 and 15 and Line 26 of Algorithm 3 and Line 34 and Line 41 of Algorithm 4. In each of these cases we either set \( wc(c) \neq \perp \) or \( bc(c) \neq \perp \). For Lines 14 and 15 this happens in Lines 18 and 22 of Algorithm 3. For Line 26 this happens in Line 26 of Algorithm 3. Moreover for Line 34 this happens in Lines 32 and 34 and for Line 41 this happens in Lines 39 to 41 from Algorithm 4.

From this it directly follows that for every agent for which the propagation function is called the agent has a non-trivial best case or worst case:

**Lemma 7.** Whenever Propagate is called for an agent \( c \in A \), it follows that it holds that \( bc(c) \neq \perp \) or \( wc(c) \neq \perp \).

**Proof.** If \( c \) is currently unmatched by \( M \), then Lemma 6 (together with our assumption that \( M_1 \) is perfect) implies that \( bc(c) \neq \perp \) or \( wc(c) \neq \perp \). If \( c \) is currently matched, then by Line 8 of Algorithm 2 it follows that \( wc(c) \neq \perp \). □
We next show that over the course of the algorithm from the perspective of an agent \( c \in A \), \( \text{bc}(c) \) becomes worse and worse, while \( \text{wc}(c) \) becomes better and better.

**Lemma 8.** Consider an agent \( c \in A \) and two timesteps \( \mathcal{A} \) and \( \mathcal{B} \) of the execution of Algorithm 3 where \( \mathcal{A} \) appears before \( \mathcal{B} \). If \( \text{bc}(c) \neq \bot \) at point \( \mathcal{A} \), then either \( \text{bc}(c) \) is the same at points \( \mathcal{A} \) and \( \mathcal{B} \) or \( c \) prefers \( \text{bc}(c) \) at point \( \mathcal{A} \) to \( \text{bc}(c) \) at point \( \mathcal{B} \). If \( \text{wc}(c) \neq \bot \) at point \( \mathcal{A} \), then either \( \text{wc}(c) \) is the same at points \( \mathcal{A} \) and \( \mathcal{B} \) or \( c \) prefers \( \text{wc}(c) \) at point \( \mathcal{B} \) to \( \text{wc}(c) \) at point \( \mathcal{A} \).

**Proof.** The only points during the algorithm where \( \text{bc}(c) \neq \bot \) and \( \text{bc}(c) \) is changed or \( \text{wc}(c) \neq \bot \) and \( \text{wc}(c) \) is changed are in Line 23 or 11 of Algorithm 4. By the definition of the pair \( \{a, b\} \) from Line 29 or Line 30 of Algorithm 4 the lemma follows.

We now continue by making several general observation concerning the values of \( \text{bc}(\cdot) \) and \( \text{wc}(\cdot) \) of the different agents. We start with the crucial observation that if an agent \( c \in A \setminus X \) has a non-trivial best case \( \text{bc}(c) \), then \( c \) prefers \( M_1(c) \) to \( \text{bc}(c) \) or \( M_1(c) = \text{bc}(c) \) and if an agent \( c \in A \) has a non-trivial worst case \( \text{wc}(c) \), then \( c \) prefers \( \text{wc}(c) \) to \( M_1(c) \) or \( M_1(c) = \text{wc}(c) \):

**Lemma 9.** For each agent \( c \in A \setminus X \) at the beginning of each execution of the while loop in Line 2 it holds that if \( \text{bc}(c) \neq \bot \) then \( M_1(c) \geq^P_a \text{bc}(c) \) and if \( \text{wc}(c) \neq \bot \) then \( \text{wc}(c) \geq^P_a M_1(c) \).

**Proof.** Fix some \( c \in A \setminus X \). At the beginning of the algorithm, the statement is trivially fulfilled, as \( \text{bc}(c) = \text{wc}(c) = \bot \). If \( \text{wc}(c) \) or \( \text{bc}(c) \) is changed in Line 22 or Line 26 of Algorithm 3 then it is set to \( M_1(c) \) so again the statement is fulfilled.

It remains to consider changes made within the while-loop from Line 2. We prove that the statement holds at the beginning of the \( i \)th execution of the while-loop by induction over \( i \).

For \( i = 0 \), we have already argued that the statement holds. So assume that the statement holds for an arbitrary but fixed \( i \geq 0 \). We will argue that from this we can conclude that it also holds for \( i + 1 \). We denote by \( \text{bc}_{\text{before}} \) and \( \text{wc}_{\text{before}} \) the best and worst case after the \( i \)th iteration, but before the \( i + 1 \)th iteration.

The only point outside of the PROPAGATE function where the best or worst case of an agent might be changed in the \( i + 1 \)th iteration of the while loop is in Line 8 of Algorithm 2. However, note that here it holds that \( \text{wc}_{\text{before}}(c) = \bot \) and \( \text{bc}_{\text{before}}(c) = \bot \) and thus by Lemma 6 that \( \text{M}(c) = M_1(c) \). As we set \( \text{wc}(c) = M(c) = M_1(c) \), the statement clearly holds after the update.

It remains to consider a call of PROPAGATE and let \( \{a, b\} \) be the examined pair (i.e., the pair defined in Line 29 or Line 30 of Algorithm 4).

First assume that \( \text{bc}_{\text{before}}(a) \neq \bot \). If \( c = a \), then, using the induction hypothesis, \( M_1(a) \geq^P_a \text{bc}_{\text{before}}(a) \) follows. Lemma 8 then implies that the statement also holds after the update. If \( c = b \), then we know by the definition of \( b \) that \( a \geq^P_b M_1(b) \). As we set \( \text{wc}(b) \) to \( a \), after the update we have \( \text{wc}(b) \geq^P_b M_1(b) \). If \( c = M(a) \) or \( c = M(b) \), then \( \text{bc}(c) \) and \( \text{wc}(c) \) either remains unchanged or is set to \( M_1(c) \).

Now we turn to the remaining case where \( \text{bc}_{\text{before}}(a) = \bot \). Lemma 7 implies that \( \text{wc}_{\text{before}}(a) \neq \bot \). If \( c = a \), then by the induction hypothesis \( \text{wc}_{\text{before}}(a) \geq^P_a M_1(a) \) follows. Lemma 8 then implies that \( \text{wc}(a) \geq^P_a M_1(a) \) also holds after the update. If \( c = b \), then by the induction hypothesis for agent \( a \) and the definition of \( b \) it follows that \( b \geq^P_a \text{wc}_{\text{before}}(a) \geq^P_a M_1(a) \).

Note that the preferences of \( a \) and \( b \) are the same in \( P_1 \) and \( P_2 \) (otherwise the guess would have been rejected in Line 37 of Algorithm 4). Thus, from \( b \geq^P_a M_1(a) \), it follows that \( M_1(b) \geq^P_b a \), as otherwise \( \{a, b\} \) is a blocking pair for \( M_1 \). As we set \( \text{bc}(b) \) to \( a \) it follows that after the update of \( \text{bc}(b) \), agent \( b \) prefers \( M_1(b) \) to \( \text{bc}(b) \). If \( c = M(a) \) or \( c = M(b) \), then \( \text{bc}(c) \) and \( \text{wc}(c) \) either remain unchanged or are set to \( M_1(c) \).

From the previous lemma, we can easily conclude that if an agent has a non-trivial best case and worst case, they need to be identical:
Lemma 10. For each agent \(c \in A\) at the beginning of each execution of the while-loop in Line 2 it holds that \(bc(b) = wc(b)\) or \(bc(b) = \perp\) or \(wc(c) = \perp\).

Proof. For every agent from \(X\), we set \(bc\) and \(wc\) accordingly in Line 18 of Algorithm 2 (and never change their best and worst cases afterwards again).

For every other agent \(c \in A \setminus X\), the statement clearly holds before the first execution of the while-loop. For all other executions of the while-loop, Lemma 9 implies that as soon as \(bc(c) \neq wc(c)\) and \(bc(c) \neq \perp\) and \(wc(c) \neq \perp\), we have \(wc(c) \succ _c P^* bc(c)\), which leads to a rejection in Line 10 of Algorithm 2

Using this, we conclude that if one of \(bc(c)\) or \(wc(c)\) is set to some agent, then \(c\) is matched to this agent in \(M\) or \(M\) is currently unmatched and this agent was \(c\)'s most recent partner:

Lemma 11. For each agent \(c \in A\) at the beginning of each execution of the while-loop in Line 2 the following holds:

1. If \(c\) is matched by \(M\) and \(bc(c) \neq \perp\), then \(M(c) = bc(c)\).
2. If \(c\) is matched by \(M\) and \(wc(c) \neq \perp\), then \(M(c) = wc(c)\).
3. If \(c\) is unmatched by \(M\) and \(bc(c) \neq \perp\), then \(bc(c)\) is the last agent \(c\) was matched to by \(M\).
4. If \(c\) is unmatched by \(M\) and \(wc(c) \neq \perp\), then \(wc(c)\) is the last agent \(c\) was matched to by \(M\).

Proof. At the beginning of the algorithm, all four items trivially hold, as all agents have trivial worst and best cases. We start by proving Item 1 and Item 2. Observe that each time the partner of \(c\) is changed and \(c\) is matched by \(M\) after this change (this can happen in Lines 14 or 15 of Algorithm 3 or Lines 34 or 41 of Algorithm 4), \(bc(c)\) or \(wc(c)\) is set to \(M(c)\) after the change. The only other changes of \(bc(c)\) or \(wc(c)\) appear in Lines 22 and 24 of Algorithm 3 and in Lines 32, 33, 39 and 40 of Algorithm 4 and after these changes, \(c\) is unmatched. Applying Lemma 11, Item 1 and Item 2 follow. We now turn to Item 3 and Item 4. Note that before the first execution of the while-loop, if an agent \(c\) becomes unmatched, then its best or worst case is set to \(M_1(c)\), which is the last agent \(c\) was matched to. To see that Item 3 and Item 4 also hold before the ith iteration of the while-loop for \(i > 1\), we distinguish two cases.

First, observe that we never change the best and worst case of an agent that is unmatched before and after the change. If agent \(c\) becomes unmatched and \(bc(c)\) and \(wc(c)\) are not modified in this execution of the while-loop, then by Item 1 and Item 2 we know that before \(c\) becomes unmatched, agent \(c\) is matched to \(bc(c)\) if \(bc(c) \neq \perp\) and to \(wc(c)\) if \(wc(c) \neq \perp\). Thus, Item 3 and Item 4 follow.

If \(c\) becomes unmatched and \(bc(c)\) or \(wc(c)\) is modified (Lines 32 and 33 and Lines 39 and 40 of Algorithm 4), then \(bc(c)\) or \(wc(c)\) is set to the agent to which \(c\) is matched by \(M\) before the last modification of \(M\). Thus, Item 3 and Item 4 follow.

We conclude this part by proving two further useful lemmas:

Lemma 12. For each agent \(c \in A \setminus X\) at the beginning of each execution of the while-loop in Line 2 with \(M(c) \neq M_1(c)\) and \(M(c) \neq \emptyset\), we have \(M(c) = bc(c)\) or \(M(c) = wc(c)\).

Proof. From \(M(c) \neq M_1(c)\), by Lemma 7 it follows that \(bc(c) \neq \perp\) or \(wc(c) \neq \perp\). As \(c\) is matched by \(M\), the lemma now follows from Item 1 of Lemma 11 or Item 2 of Lemma 11

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Lemma 13. If \textsc{Propagate}(\cdot) is called for an agent \(a \in A \setminus X\) and we have that the selected stable pair is \(e = \{a, b\}\) with \(b \in A \setminus X\), then \(\{a, b\} \notin M_1\).

Proof. If \(bc(a) \neq \perp\), then from Lemma 9 we get that \(M_1(a) \succeq_{\mathcal{P}_2} bc(a)\). As we require that \(bc(a) \succeq_{\mathcal{P}_2} b\), we get that \(M_1(a) \succeq_{\mathcal{P}_2} b\).

Otherwise, by Lemma 4 we have \(wc(a) \neq \perp\). Then from Lemma 9 we get that \(wc(a) \succeq_{\mathcal{P}_2} M_1(a)\). As we require that \(b \succeq_{\mathcal{P}_2} wc(a)\), we get that \(b \succeq_{\mathcal{P}_2} M_1(a)\).

\(\Box\)

3.2.2.4 Well-Definedness

We continue by showing that Algorithm 2 is well-defined. To do so, we need to show that in Line 7 of Algorithm 2 in every blocking pair one agent has a trivial best case and the other a non-trivial best case. As otherwise the guess would have been rejected in Line 6 of Algorithm 2, we know that at least one agent from the blocking pair has a trivial best case. Thus, it suffices to show the following:

Lemma 14. At any point of the execution of the algorithm, for every agent pair \(\{c, d\}\) currently blocking \(M\) where both \(c\) and \(d\) are currently matched by \(M\), we have \(bc(c) \neq \perp\) or \(bc(d) \neq \perp\).

Proof. If the preferences of \(c\) or \(d\) differ in \(\mathcal{P}_1\) and \(\mathcal{P}_2\), then a non-trivial best case for \(c\) or \(d\) was set in Line 18 of Algorithm 2. So assume that \(c\) and \(d\) have the same preferences in \(\mathcal{P}_1\) and \(\mathcal{P}_2\). Then, as \(\{c, d\}\) does not block \(M_1\), agent \(c\) or \(d\) is matched worse in \(M\) than in \(M_1\). Assume that \(c\) is matched worse to an agent \(c'\), i.e., we have \(M(c) = c'\) with \(M_1(c) \succ_{c} c'\). By Lemma 12 it follows that \(bc(c) = c'\) or \(wc(c) = c'\). By Lemma 9 and as \(c\) prefers \(M_1(c)\) to \(c'\), it follows that \(bc(c) = c'\).

\(\Box\)

3.2.2.5 Propagation of best and worst cases

Now we show that we correctly propagate \(bc(\cdot)\) and \(wc(\cdot)\), i.e., every stable matching obeying a given guess must for every agent \(c \in A\) obey \(bc(c)\) and \(wc(c)\) as set at each point during the execution of the algorithm: For \(c \in A\), we say that a matching \(M\) obeys \(bc(c)\) if \(bc(c) = \perp\) or \(bc(c) \succeq_{\mathcal{P}_2} M(c)\), and we say that \(M\) obeys \(wc(c)\) if \(wc(c) = \perp\) or \(M(c) \succeq_{\mathcal{P}_2} wc(c)\). Further, we say that a matching obeys the best and worst cases if it obeys \(bc(c)\) and \(wc(c)\) for every agent \(c \in A\). Lastly, we say that a matching \(M\) obeys our guesses if \(M\) obeys all best and worst cases after the call of the \textsc{Initialization}(\cdot) function and the guessed set \(F\) contains exactly the set of agent pairs from \(M_1\) where both endpoints prefer \(M\) to \(M_1\) in \(\mathcal{P}_2\) and the guessed set \(H\) contains exactly the agent pairs from \(M\) where both endpoints prefer \(M_1\) to \(M\) in \(\mathcal{P}_2\).

As the final piece before proving that we propagate the best and worst cases correctly, we show the following:

Lemma 15. If \textsc{Propagate}(\cdot) is called for an agent \(c \in A\), then it holds that

1. If \(bc(c) \neq \perp\), then \(c\) cannot be matched to \(bc(c)\) in any stable matching obeying the current guess and the current best and worst cases.

2. If \(wc(c) \neq \perp\), then \(c\) cannot be matched to \(wc(c)\) in any stable matching obeying the current guess and the current best and worst cases.

Proof. We distinguish two cases based on whether \(c\) is matched in \(M\) when \textsc{Propagate}(\cdot) is called or not. If \(c\) is currently matched, then there exists a blocking pair \(\{c, d\}\) for \(M\) selected in Line 5 of Algorithm 2 where both \(c\) and \(d\) are matched. Moreover, by Lines 7 and 8 of Algorithm 2 it needs to hold that \(bc(d) \neq \perp\) and \(wc(c) \neq \perp\). By Lemma 14 from this it follows
Lemma 16. Every stable matching \( M^* \) in \( P_2 \) obeying a guess obeys the best and worst cases at every point during the execution of the algorithm for this guess.

If a stable matching \( M^* \) obeying a guess exists, then we do not reject in Line 33 and Line 27 of Algorithm 2 during the execution of the algorithm for this guess.

Proof. Let \( M^* \) be a stable matching in \( P_2 \) obeying our guesses. We now show that \( M^* \) obeys the best and worst cases at every point during the execution of the algorithm for this guess. Before the first execution of the while-loop in Line 2 of Algorithm 2 the current best and worst cases are obeyed since \( c, d \) and \( c \) got unmatched it holds that \( bc(d) = c \) if \( bc(d) \neq \bot \) and \( wc(d) = c \) if \( wc(d) \neq \bot \). Using this, as in the call of Propagate(·) examining the pair \( \{a, b\} \) the best or the worst cases of both \( a \) and \( b \) are changed, after the call of the Propagate(·) function in which \( c \) got unmatched, by Lemma 8 it holds that \( c \succ_d bc(d) \) if \( bc(d) \neq \bot \) or \( wc(d) \succ_d c \) if \( wc(d) \neq \bot \). Thus, a matching respecting the current worst and best cases cannot contain \( \{c, d\} \).

We are now ready to prove that we correctly propagate agent’s best and worst cases, the cornerstone of our proof of correctness:

**Lemma 16.** Every stable matching \( M^* \) in \( P_2 \) obeying a guess obeys the best and worst cases at every point during the execution of the algorithm for this guess.

If a stable matching \( M^* \) obeying a guess exists, then we do not reject in Line 33 and Line 27 of Algorithm 2 during the execution of the algorithm for this guess.

**Proof.** Let \( M^* \) be a stable matching in \( P_2 \) obeying our guesses. We now show that \( M^* \) obeys the best and worst cases at every point during the execution of the algorithm for this guess. Before the first execution of the while-loop in Line 2 of Algorithm 2 the current best and worst cases are obeyed since \( M^* \) obeys the guess. We now argue that the matching \( M^* \) continues to obey the best and worst cases. For this we examine all points during the execution of the algorithm where best and worst cases may change.

**Changes outside of Propagate(·).** The only point where an update may happen inside of the while-loop but outside of the Propagate(·) function is in Line 8 of Algorithm 2. Let \( M_{\text{before}}, bc_{\text{before}} \) and \( wc_{\text{before}} \) be the matching \( M \), best and worst case before the update, and \( M_{\text{after}}, bc_{\text{after}} \) and \( wc_{\text{after}} \) the matching \( M \), best and worst case after the update. We set \( wc_{\text{after}}(x) \to M_{\text{before}}(x) \) in Line 8 if \( \{x, y\} \) forms a blocking pair for \( M_{\text{before}} \), both \( x \) and \( y \) are matched in \( M_{\text{before}} \), \( bc_{\text{before}}(y) \neq \bot \) and \( wc(x) = \bot = bc(x) \). By Lemma 11 and as \( bc_{\text{before}}(y) \neq \bot \), it follows that \( bc_{\text{before}}(y) = M_{\text{before}}(y) \). Thus, as \( \{x, y\} \) does not block \( M^* \) and \( M^* \) obeys \( bc_{\text{before}}(y) \) (i.e., \( x \succ_{P_2} M_{\text{before}}(y) = bc_{\text{before}}(y) \succ_{P_2} M^*(y) \)) it needs to hold that \( M^*(x) \succ_{P_2} y \) and thus that \( M^*(x) \) respects \( wc_{\text{after}}(x) = M_{\text{before}}(x) \prec_{P_2} y \), where \( M_{\text{before}}(x) \succ_{P_2} y \) holds as \( \{x, y\} \) blocks \( M \).

**Propagate(·).** Let us now consider a call of the function Propagate(·) and let \( a \in A \) be the agent on which the function was called and \( \{a, b\} \) the considered stable pair from Line 29 or 36 of Algorithm 3. We prove that \( M^* \) respects the best and worst cases by induction
over the number of iterations of the while loop. Assume that the statements holds after iteration \( i \geq 0 \). And let us examine the \( i + 1 \)th iteration of the while-loop. For this let \( M^\text{before} \), \( b^\text{before} \) and \( wc^\text{before} \) be the matching \( M \), best and worst case at the beginning of this call of \( \text{Propagate}(\cdot) \), and \( M^\text{after} \), \( b^\text{after} \) and \( wc^\text{after} \) the matching \( M \), best and worst case after this call of \( \text{Propagate}(\cdot) \). By our induction hypothesis, we know that \( M^* \) respects \( b^\text{before} \) and \( wc^\text{before} \) (for \( i = 0 \) this follows from the paragraph above).

We first show that \( a \not\in X \). Suppose towards a contradiction that \( a \in X \). As \( M^* \) respects our guesses, we have guessed in the \text{INITIALIZATION}(\cdot) function the partner of all agents from \( X \) in \( M^* \) and in particular set \( M(a) = M^*(a) \) and \( bc(a) = wc(a) = M^*(a) \). Moreover, by Lemma 8 and as \( M^* \) still respects the current guesses, we have \( bc^\text{before}(a) = wc^\text{before}(a) = M^*(c) \) and by Lemma 13 thus also \( M^\text{before} = M^*(a) \). However, from Lemma 15 we get that if \( \text{Propagate}(\cdot) \) is called on \( a \), then \( a \) cannot be matched to \( bc^\text{before}(a) = M^*(a) \) in a stable matching obeying the current guess and \( b^\text{before} \) and \( wc^\text{before} \), a contradiction to the existence of \( M^* \).

We now argue chronologically one change of best and worst cases after each other why \( M^* \) still obeys the best and worst cases \( b^\text{after} \) and \( wc^\text{after} \).

\text{Propagate}(\cdot)—Changing best and worst cases of \( a \). First, we consider the case \( bc^\text{before}(a) \neq \perp \). We now argue that \( b \succeq_{\mathcal{P}_2} M^*(a) \) and thus that \( M^* \) respects \( b^\text{after}(a) = b \). First of all note that \( \{a, M^*(a)\} \) clearly needs to be a stable pair in \( \mathcal{P}_2 \). As \( M^* \) respects \( bc^\text{before} \) and \( wc^\text{before} \), it needs to hold that \( bc(a) \neq_{\mathcal{P}_2} M^*(a) \) (where we additionally apply Lemma 15) and that \( a \succeq_{\mathcal{M}_2} wc^\text{before}(M^*(a)) \) or \( wc^\text{before}(M^*(a)) = \perp \). Moreover, by Lemma 9 it holds that \( M_1(a) \succeq_{\mathcal{P}_2} bc^\text{before}(a) \succ_{\mathcal{P}_2} M^*(a) \). We now make a case distinction based on the preferences of \( M^*(a) \).

If \( M^*(a) \) prefers \( M_1(M^*(a)) \) to \( a \), then for the pair \( \{a, M^*(a)\} \in M^* \) both endpoints prefer \( M_1 \) to \( M^* \). This means that \( \{a, M^*(a)\} \in H \), as \( M^* \) respects the current guess. Thus, we have \( a \in X \), a contradiction to \( a \not\in X \). Consequently, we have \( a \succeq_{\mathcal{P}_2} M^*(a) \). Thus, we have proven that \( \{a, M^*(a)\} \) is a stable pair, that \( a \succeq_{\mathcal{P}_2} wc^\text{before}(M^*(a)) \) or \( wc^\text{before}(M^*(a)) = \perp \), and \( bc(a) \neq_{\mathcal{P}_2} M^*(a) \). These are exactly the constraints \( b \) has to fulfill in Line 29. As \( b \) is the agent best ranked by \( a \) fulfilling these constraints, we clearly get \( b \succeq_{\mathcal{P}_2} M^*(a) \). Thus, \( M^* \) also respects \( b^\text{after}(a) = b \).

We now turn to the case that \( bc^\text{before}(a) = \perp \). Then, very similar to the previous case, Lemma 7 implies \( wc^\text{before}(a) \neq \perp \). First of all note that in \( M^* \), the pair containing a clearly needs to be a stable pair in \( \mathcal{P}_2 \). By Lemma 15 and as \( M^* \) respects the current best and worst cases, it needs to hold that \( M^*(a) \succeq_{\mathcal{P}_2} wc^\text{before}(a) \) and that \( bc^\text{before}(M^*(a)) \succeq_{\mathcal{P}_2} M_1(a) \) or \( bc^\text{before}(M^*(a)) = \perp \). By the definition of \( b \) it follows that \( M^*(a) \succeq_{\mathcal{P}_2} b \). Thus, \( M^* \) also respects \( wc^\text{after}(a) = b \).

\text{Propagate}(\cdot)—Changing best and worst cases of \( b \). \text{Propagate}(\cdot) also updates \( bc(b) \) and \( wc(b) \). First assume that we modified \( wc(b) \) (which happens if and only if \( bc^\text{before}(a) \neq \perp \)). Note that in this case we set \( b^\text{after}(a) = b \). Since we have shown above that \( M^* \) respects \( b^\text{after}(a) = b \), we have \( b \succeq_{\mathcal{P}_2} M^*(a) \). As \( \{a, b\} \) does not block \( M^* \) and \( b \succeq_{\mathcal{P}_2} M^*(a) \), it follows that \( M^*(b) \succeq_{\mathcal{P}_2} a \), and thus, \( M^* \) respects \( b^\text{after}(a) = b \).

Now assume that we modified \( bc(b) \) by setting \( b^\text{after}(a) = a \) (which happens if and only if \( bc^\text{before}(a) = \perp \)). In this case, we set \( wc^\text{after}(a) = b \). As \( M^* \) respects \( wc^\text{after}(a) = b \), it needs to hold that \( M^*(a) \succeq_{\mathcal{P}_2} b \). As \( \{a, b\} \) is a stable pair in \( \mathcal{P}_2 \) and \( M^*(a) \succeq_{\mathcal{P}_2} b \) it follows from Lemma 5 that \( a \succeq_{\mathcal{P}_2} M^*(b) \). This proves that \( M^* \) respects \( b^\text{after}(a) = a \).

Note that we have not shown (or assumed) so far that \( b \notin X \). To establish that we do not reject this guess in Line 29 assume for the sake of contradiction that \( b \in X \). As \( M^* \) respects
our guesses, we have guessed in the \textsc{Initialization}($\cdot$) function the partner of all agents from $X$ in $M^*$ and in particular set $M(b) = M^*(b)$ and $bc(b) = wc(b) = M^*(b)$. Moreover, by Lemma\textsuperscript{8} and as $M^*$ respects $bc_{\text{before}}(b)$ and $wc_{\text{before}}(b)$, we have $bc_{\text{before}}(b) = wc_{\text{before}}(b) = M^*(b)$ and by Lemma\textsuperscript{11} thus also $M^*_{\text{before}} = M^*(b)$. However, as clearly $\{a, b\} \notin M^*_{\text{before}}$, we either have $bc_{\text{before}}(b) \succ P_2^c bc_{\text{after}}(b)$ or $wc_{\text{after}}(b) \succ P_2^c wc_{\text{before}}(b)$. In both cases, we afterwards have $wc_{\text{after}}(b) \succ P_2^c bc_{\text{after}}(b)$. However, this cannot be the case as we have shown that $M^*$ respects both $wc_{\text{after}}(b)$ and $bc_{\text{after}}(b)$.

\textbf{Propagate($\cdot$)—Changing best and worst cases of $M_1(a)$}. \textsc{Propagate($\cdot$)} may also update $bc(M_1(a))$ or $wc(M_1(a))$ if $M(a) = M_1(a)$. First, we consider the case that the algorithm modified $wc(M_1(a))$. As $bc_{\text{before}}(a) \neq \perp$ in this case, by Lemma\textsuperscript{9} we get that $M_1(a) \geq P_2^a bc_{\text{before}}(a)$. Moreover, as we have set $bc_{\text{after}}(a) = b$ with $bc_{\text{before}}(a) \succ P_2^a b$, it follows that $M_1(a) \succ P_2^a bc_{\text{after}}(a)$ and as we have established above that $M^*$ respects $bc_{\text{after}}(a)$ we get $M_1(a) \succ P_2^a M^*(a)$. As $\{a, M_1(a)\}$ does not block $M^*$, it follows that $M^*(M_1(a)) \succ P_2^a M_1(a) = wc_{\text{after}}(M_1(a))$ and thus that $M^*$ respects $wc_{\text{after}}(M_1(a))$.

Now we consider the case that the algorithm modified $bc(M_1(a))$. In this case we have $wc_{\text{before}}(a) \neq \perp$ by Lemma\textsuperscript{7}. By Lemma\textsuperscript{9} we get that $wc_{\text{before}}(a) \geq P_2^a M_1(a)$. Moreover, as we have set $wc_{\text{after}}(a) = b$ with $wc_{\text{before}}(a) \succ P_2^a b$, it follows that $wc_{\text{after}}(a) \succ P_2^a M_1(a)$ and as we have established above that $M^*$ respects $wc_{\text{after}}(a)$ we get $M^*(a) \succ P_2^a M_1(a)$. Assume for the sake of contradiction that $M^*$ does not respect $bc_{\text{after}}(M_1(a)) = a$, i.e., $M^*(M_1(a)) \succ P_2^a M_1(a)$. However, as $\{a, M_1(a)\} \in M_1$ this implies that $\{a, M_1(a)\}$ is a pair from $M_1$ where both endpoints prefer $M^*$ to $M_1$. Thus, as $M^*$ respects the current guess it needs to hold that $\{a, M_1(a)\} \in F$ and that the pair $\{a, M_1(a)\}$ was deleted from $M$ already before the first call of the \textsc{Propagate($\cdot$)} function (and was clearly never inserted again as we only insert the selected stable pairs $\{a, b\}$ in a call of the \textsc{Propagate($\cdot$)} function and by Lemma\textsuperscript{13} $\{a, b\} \notin M_1$), a contradiction to $M(a) = M_1(a)$.

\textbf{Propagate($\cdot$)—Changing best and worst cases of $M_1(b)$}. Similar arguments as for the previous case apply here. If $wc(M_1(b))$ was modified, then we have $bc_{\text{after}}(b) = a$ and as we have established above that $M^*$ respects $bc_{\text{after}}(b)$ that $a \geq P_2^b M^*(b)$. From Lemma\textsuperscript{9} (applied to the next iteration of the while loop) it further follows that $M_1(b) \geq P_2^b bc_{\text{after}}(b) = a$. It follows that $M_1(b) \geq P_2^b M^*(b)$. Thus, for $\{b, M_1(b)\}$ not to form a blocking pair for $M^*$, it needs to hold that $M^*(M_1(b)) \geq P_2^b M_1(b) = wc_{\text{after}}(M_1(b))$. Consequently, $M^*$ respects $wc_{\text{after}}(M_1(b))$.

If $bc(M_1(b))$ was modified, then we have by the definition of $b$ that $wc_{\text{after}}(b) = a \geq P_2^b M_1(b)$. Moreover, as we know that $\{a, b\} \notin M_1$ by Lemma\textsuperscript{13} it even holds that $wc_{\text{after}}(b) = a \geq P_2^b M_1(b)$. As we have established above that $M^*$ respects $wc_{\text{after}}(b)$ it follows that $M^*(b) \geq P_2^b M_1(b)$. Assume for the sake of contradiction that $M^*$ violates $bc_{\text{after}}(M_1(b)) = b$, i.e., $M^*(M_1(b)) \succ P_2^b M_1(b)$. However, as $\{b, M_1(b)\} \in M_1$ this implies that $\{b, M_1(b)\}$ is a pair from $M_1$ where both endpoints prefer $M^*$ to $M_1$. Thus, as $M^*$ respects the current guess it needs to hold that $\{b, M_1(b)\} \in F$ and that the pair $\{b, M_1(b)\}$ was deleted from $M$ already before the first call of the \textsc{Propagate($\cdot$)} function (and was clearly never inserted again as we only insert the selected stable pairs $\{a, b\}$ in a call of the \textsc{Propagate($\cdot$)} function and by Lemma\textsuperscript{13} $\{a, b\} \notin M_1$), a contradiction to $M_1(b) = M(b)$.

We now turn to the second part of the lemma. First of all note that from the existence of $M^*$ it follows that a stable pair as defined in Line\textsuperscript{29} or Line\textsuperscript{30} of Algorithm\textsuperscript{1} always exists, as we have shown that the partner of $a$ in $M^*$ fulfills all the required properties. Moreover, we have argued above that we always have $a \notin X$ and $b \notin X$ implying that we do not reject in
3.2.2.6 Bounding the Number of Guesses

It remains to argue that our guesses are exhaustive and indeed cover all cases. In particular, we now show that the set $F$ guessed in Line 23 of Algorithm 3 is large enough, i.e., for every stable matching $M_2$ in $P_2$ there are at most $|P_1 \oplus P_2|$ pairs of $M_1$ for which both endpoints strictly prefer $M_2$ to $M_1$, and that the set $H$ guessed in Line 15 of Algorithm 3 is large enough, i.e., for every stable matching $M_2$ in $P_2$ there are only $|P_1 \oplus P_2|$ pairs of $M_2$ for which both endpoints strictly prefer $M_1$ to $M_2$.

We start by proving this bound for the guessed set $F$:

Lemma 17. Let $M_2$ be a perfect stable matching in $P_2$ and $M_1$ be a perfect stable matching in $P_1$. The number of pairs $e = \{b, c\} \in M_1$ such that $b$ prefers $M_2(b)$ to $c$ and $c$ prefers $M_2(c)$ to $b$ in $P_2$ is bounded by the number of agents whose preferences differ in $P_1$ and $P_2$.

Proof. $M_1 \triangle M_2$ is a disjoint union of cycles of even length. For each such cycle $C$, fix an orientation of $C$. Let $C = (v_1, \ldots, v_{2r}, v_{2r+1} = v_1)$ be a cycle containing a pair $e = \{b, c\} \in M_1$ such that $b$ prefers $M_2(b)$ to $c$ and $c$ prefers $M_2(c)$ to $b$ in $P_2$. We assume without loss of generality that $e = \{v_1, v_2\}$. Let $i$ be the smallest index in $[1, 2r]$ such that the preferences of $v_i$ differ in $P_1$ and $P_2$ (and $i = \infty$ if no such agent exists). We show by induction on $j$ that $v_j$ prefers $v_{j+1}$ to $v_{j-1}$ (in $P_1$ and $P_2$) for every $j \in \{2, 3, \ldots, i-1\}$. For $j = 2$, this follows by the definition of $e$. So consider $j \geq 3$, and let $\{v_j, v_{j+1}\} \in M_p$ for some $p \in \{1, 2\}$. By induction, $v_{j-1}$ prefers $v_j$ to $v_{j-2}$. Since $\{v_{j-1}, v_j\}$ does not block $M_p$, it follows that $v_j$ prefers $v_{j+1}$ to $v_{j-1}$. This implies that $i < \infty$ because if $i = \infty$, then $v_{2r+1} = v_1$ prefers $v_2$ to $v_{2r}$ by the above proven claim. This leads to a contradiction, as $v_1$ prefers $v_{2r}$ to $v_2$ by the definition of $e$. Note that from the above claim it also follows that apart from $\{b, c\}$ there cannot be a second pair from $M_1$ in $\{v_1, \ldots, v_i\}$ with both agents from this pair preferring $M_2$ to $M_1$, as we have proven above that there cannot exist a $j \in [2, i]$ such that $\{v_j, v_{j+1}\} \in M_1$ and $v_{j-1} \succ_v v_{j+1}$. Thus, mapping pair $e$ to agent $v_i$ yields an injection from the set of pairs $\{b, c\} \in M_1$ with both $b$ and $c$ preferring $M_2$ to $M_1$ to the set of agents with modified preferences.

By swapping the roles of $M_2$ and $M_1$ and the roles of $P_1$ and $P_2$, Lemma 17 also shows that the set $H$ of pairs guessed in Line 15 of Algorithm 3 is large enough, i.e., for every stable matching $M_2$ in $P_2$ there exist at most $|P_1 \oplus P_2|$ many pairs from $M_2$ where both endpoints prefer $M_1$ to $M_2$.

3.2.2.7 Proof of Correctness: Putting the Pieces Together

We are now ready to put the pieces together and prove the correctness of the algorithm.

Theorem 2. ISR can be solved in $O(2^{|P_1 \oplus P_2|} \cdot n^{|P_1 \oplus P_2|+3})$ time.

Proof. Lemma 14 and Line 6 of Algorithm 2 show that Line 7 of Algorithm 2 and thus the algorithm is well-defined.

Note that for each guess executing the algorithm takes $O(n^3)$ time: For each agent $a$, the best case and worst case can only be changed $n$ times by Lemma 5 Since in every iteration of the while-loop Propagate(·) for some agent $a$ is called and either $bc(a)$ or $wc(a)$ is modified, the while-loop is executed at most $n^2$ times. Moreover, given the set of stable pairs in $P_2$, which can be precomputed in $O(n^3)$ time [18], each iteration of the while-loop takes $O(n)$ time. Moreover, for our guesses made in the initialization phase in Algorithm 3 there exist overall $O(2^{|P_1 \oplus P_2|} \cdot n^{|P_1 \oplus P_2|})$ possibilities: There are $n^2|P_1 \oplus P_2|$ possibilities how the agents with changed preferences
and their partners in $M_1$ are matched (Line 13). There are $n^2 |\mathcal{P}_1 \oplus \mathcal{P}_2|$ possibilities for the set $H$ from Line 15. Further, there are $n |\mathcal{P}_1 \oplus \mathcal{P}_2|$ possibilities for the set $F$ from Line 23 (as we only need to iterate over $|\mathcal{P}_1 \oplus \mathcal{P}_2|$-subsets of $M_1$). Lastly, as $X$ contains at most $4 |\mathcal{P}_1 \oplus \mathcal{P}_2|$ agents, there exist at most $2^4 |\mathcal{P}_1 \oplus \mathcal{P}_2|$ possibilities for our guesses in Line 24. Thus, we get an overall running time of $O(2^4 |\mathcal{P}_1 \oplus \mathcal{P}_2| \cdot n^2 |\mathcal{P}_1 \oplus \mathcal{P}_2| \cdot n^3)$. It remains to show the correctness of Algorithm 2.

If Algorithm 2 returns a matching, then this matching $M$ is stable, as all blocking pairs get resolved in the while-loop and also satisfies $|M \triangle M_1| \leq k$ by Line 11 of Algorithm 2.

It remains to prove that if there exists a stable matching $M_2$ in $\mathcal{P}_2$ with $|M_1 \triangle M_2| \leq k$, then Algorithm 2 returns a matching for some guess. Let $F'$ be the set of agent pairs from $M_1$ where both endpoints prefer $M_2$ to $M_1$ in $\mathcal{P}_2$ and let $H'$ be the set of agent pairs from $M_2$ where both endpoints prefer $M_1$ to $M_2$ in $\mathcal{P}_2$. Then, as proven in Lemma 14, we have max|$|M_1 \triangle M_2|| \leq |\mathcal{P}_1 \oplus \mathcal{P}_2|$. So assume for the sake of contradiction that Algorithm 2 rejected the guess with $F = F'$, $H = H'$, and guesses in Lines 14 and 21 in Algorithm 3 made according to $M_2$. Lemma 16 implies that $M_2$ obeys $bc(c)$ and $wc(c)$ for every agent $c \in A$ at any point of the execution of the algorithm for this guess. The guess clearly cannot be rejected during the initialization in Algorithm 3 because there exists a stable matching obeying the guess.

Assume for the sake of contradiction that the guess is rejected in Line 6 of Algorithm 2 because there exists two agents $\{a, b\}$ with $bc(a) \neq \perp$ and $bc(b) \neq \perp$ that form a blocking pair for $M$. As Lemma 11 implies that $a$ is matched to $bc(a)$ and $b$ to $bc(b)$ in $M$ and as $M_2$ respects $bc(a)$ and $bc(b)$, it follows that $\{a, b\}$ also blocks $M_2$, a contradiction.

Moreover, the existence of $M_2$ implies that the current guess cannot be rejected in Line 30 or Line 67 of Algorithm 3 by Lemma 15. Lastly, as $M_2$ obeys the best and worst cases of all agents at any point during the execution of the algorithm, the guess cannot be rejected in Line 10 of Algorithm 2.

This means that the algorithm can only reject in Line 11 of Algorithm 2. Let $M$ be the matching in Line 11 of Algorithm 2. Matching $M$ contains every pair $\{a, b\} \in M_1$ with $bc(a) = \perp \lor bc(b) = M_1(a)$, $wc(a) = \perp \lor M_1(a) = wc(a)$, $bc(b) = \perp \lor bc(b) = M_1(a)$, and $wc(b) = \perp \lor M_1(b) = wc(b)$. As $M_2$ respects the best and worst cases, by Lemma 9 if a pair from $M_1$ does not satisfy the aforementioned criteria, then it cannot be part of $M_2$. Thus, it holds that $|M_1 \triangle M| \leq |M_1 \triangle M_2| \leq k$, a contradiction.

Finally note that Algorithm 2 (which requires that $M_1$ is a perfect matching and that there is a perfect stable matching in $\mathcal{P}_2$) also gives rise to an algorithm for general ISR instances, as we have shown in Lemmas 4 and 5 that each ISR instance can be reduced to an ISR instance fulfilling the above-described properties in linear time.

We remark that ISR is NP-complete even if we know for each agent $a$ whose preferences changed as well as $M_1(a)$ how they are matched in $M_2$ and the set of pairs $F \subseteq M_1$ for which both endpoints prefer $M_2$ to $M_1$. This indicates that guessing the set $H$ might be necessary for the XP-algorithm. To prove this we only need to slightly alter the construction from Theorem 1. Specifically, for each $c \in [\ell]$, we add agents $x^c$, $y^c$, $\bar{x}^c$, and $\bar{y}^c$ and replace pair $\{s^c, t^c\}$ by pairs $\{s^c, \bar{x}^c\}$, $\{x^c, y^c\}$, $\{y^c, t^c\}$ and pair $\{\bar{s}^c, \bar{t}^c\}$ by pairs $\{\bar{s}^c, \bar{x}^c\}$, $\{\bar{x}^c, \bar{y}^c\}$, and $\{\bar{y}^c, \bar{t}^c\}$. Here, $x^c$ prefers $y^c$ to $s^c$ and $y^c$ prefers $t^c$ to $x^c$ (and symmetrically, we have that $\bar{x}^c$ prefers $\bar{y}^c$ to $\bar{s}^c$ and $\bar{y}^c$ prefers $\bar{t}^c$ to $\bar{x}^c$) in $\mathcal{P}_1$ and $\mathcal{P}_2$. Matching $M_1$ contains edges $\{s^c, x^c\}$, $\{y^c, t^c\}$, $\{\bar{s}^c, \bar{x}^c\}$, and $\{\bar{y}^c, \bar{t}^c\}$ instead of $\{s^c, t^c\}$ and $\{\bar{s}^c, \bar{t}^c\}$. Note that every stable matching in $\mathcal{P}_2$ then contains pairs $\{u^c, t^c\}$, $\{x^c, y^c\}$, $\{u^e, \bar{t}^c\}$, and $\{\bar{x}^c, \bar{y}^c\}$ (so there is nothing to guess how the agents with changed preferences and their partners in $M_1$ are matched). Setting $F := \emptyset$ now results in an NP-complete problem by the same proof as the one of Theorem 1.
4 Incremental Stable Marriage with Ties Parameterized by the Number of Ties

Bredereck et al. [7] raised the question how the total number of ties influences the computational complexity of ISM-T. Note that the number of ties in a preference relation is the number of maximal sets of pairwise tied agents containing more than one agent. For instance the preference relation $a \sim b \sim c \succ d \sim e \succ f$ contains two ties, where the first tie $(a \sim b \sim c)$ has size three and the second tie $(d \sim e)$ has size two. In this section, following a fundamentally different and significantly simpler path than Bredereck et al., we show that their W[1]-hardness result for ISM-T parameterized by $k$ for $|P_1 \oplus P_2|=1$ still holds if we parameterize by $k$ plus the number of ties. To prove this, we introduce a natural extension of ISM-T called INCREMENTAL STABLE MARRIAGE WITH FORCED EDGES AND TIES (ISMFE-T). ISMFE-T differs from ISM-T in that as part of the input we are additionally given a subset $Q \subseteq M_1$ of the initial matching, and the question is whether there is a stable matching $M_2$ for the changed preferences with $|M_1 \Delta M_2| \leq k$ containing all pairs from $Q$, i.e., $Q \subseteq M_2$.

We first show that ISMFE-T is intractable even if $|Q|=1$ by reducing from a W[1]-hard local search problem related to finding a perfect stable matching with ties [38]:

**Proposition 1.** ISMFE-T parameterized by $k$ and the summed number of ties in $P_1$ and $P_2$ is W[1]-hard, even if $|P_1 \oplus P_2|=1$ and $|Q|=1$ and only women have ties in their preferences.

ISMFE-T parameterized by $k$ is W[1]-hard, even if $|P_1 \oplus P_2|=1$, $|Q|=1$, only women have ties in their preferences, and each tie has size at most two.

**Proof.** We show both parts by reducing from the following problem related to finding a perfect matching in an STABLE MARRIAGE WITH TIES instance: Given a STABLE MARRIAGE WITH TIES instance consisting of $n$ men and $n$ women, an integer $\ell$, and a stable matching $N$ of size $n-1$, decide whether there exists a perfect stable matching $N^*$ with $|N \Delta N^*| \leq \ell$. Marx and Schlotter showed that this problem is W[1]-hard parameterized by the number ties plus $\ell$, even if only the preferences of women contain ties [38, Theorem 2] and W[1]-hard parameterized by $\ell$, even if all ties have size two and are in the preferences of women [38, Theorem 3].

We now establish a reduction from the above defined problem to INCREMENTAL STABLE MARRIAGE WITH FORCED EDGES AND TIES. As, in the reduction, the number of ties and the length of each tie remain unchanged, we thereby establishing both statements at the same time. The reduction works as follows. Let $(U \cup W, \mathcal{P})$ be an instance of STABLE MARRIAGE with ties, let $N$ be a stable matching of size $n-1$, where $n=|U|=|W|$, and let $\ell \in \mathbb{N}$. Let $m_{\text{single}}$ and $w_{\text{single}}$ be the two agents unmatched by $N$. We add a man $m^*$ and a woman $w^*$ accepting each other and all agents from $U \cup W$ which prefer to be matched to any agent from $W$ respectively $U$ to being matched together, and set $Q := \{\{m^*, w^*\}\}$ to be the set of forced pairs. Moreover, we add $m^*$ at the end of the preferences of all women and $w^*$ at the end of the preferences of all men. We now modify the instance such that $m_{\text{single}}$ and $w_{\text{single}}$ are “bounded” in $P_1$ but become “free” and are thereby added to the relevant part of the instance in $P_2$. As $M_2$ needs to contain the pair $\{\{m^*, w^*\}\}$, we need to find a matching that matches all agents from $U \cup W$ and has a large intersection with $N$. To realize these constraints on $m_{\text{single}}$ and $w_{\text{single}}$, we additionally add a man $m_{\text{single}}$ and woman $w_{\text{single}}$, with preferences $m_{\text{single}} : w_{\text{single}} \succ w^*_{\text{single}}$ and $w_{\text{single}} : m^*_{\text{single}}$. In $P_1$, we modify the preferences of man $m_{\text{single}}$ such that he prefers $w^*_{\text{single}}$ to all other agents, and change the preferences of $w_{\text{single}}$ such that she prefers $m^*_{\text{single}}$ to all other agents. The preferences of $P_2$ now arise from $P_1$ by swapping the first two women in the preferences of all men and all women.

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4 Notably, Marx and Schlotter [38] use a different measure for the difference between two matchings, i.e., the number of agents that are matched differently. However, as we here know the number of pairs in $N$ and $N^*$ the two distance measures can be directly converted to each other and differ by at most a factor of two.
preferences of $m_{\text{single}}^*$, i.e., the preferences of $m_{\text{single}}^*$ in $\mathcal{P}_2$ are $w_{\text{single}}^* > w_{\text{single}}$. Matching $M_1$ is defined as $M_1 := N \cup \{(m^*, w^*) \mid (m_{\text{single}}^*, w_{\text{single}}^*) \}$, $\{m_{\text{single}}^*, w_{\text{single}}^*\}$, $\{m^*, w^*\}$. The stability of $M_1$ follows from the stability of $N$, as none of $w_{\text{single}}^*$, $m_{\text{single}}^*$, $w^*$, and $m^*$ is part of a blocking pair. We set $k := \ell + 3$. We now prove that there exists a perfect matching $N^*$ with $|N \triangle N^*| \leq \ell$ if and only if $|M_1 \triangle M_2| \leq k$.

\( \Rightarrow \): Given a perfect stable matching $N^*$ with $|N \triangle N^*| \leq \ell$, we claim that $M_2 := N^* \cup \{m^*, w^*\} \subset \{m_{\text{single}}^*, w_{\text{single}}^*\}$ is a stable matching in $\mathcal{P}_2$. As $N$ is stable, every blocking pair must contain $w_{\text{single}}^*$ and $m_{\text{single}}^*$. Agents $w_{\text{single}}^*$ and $m_{\text{single}}^*$ are not part of a blocking pair, as they are matched to their first choice. Since $N$ is a perfect matching, no agent prefers to be matched to $m^*$ or $w^*$. Therefore, $M_2$ is stable with respect to $\mathcal{P}_2$. Since $|N \triangle N^*| \leq \ell$, it follows that $|M_1 \triangle M_2| \leq \ell + 3 = k$. Thus, $M_2$ is a solution to the constructed ISM-T instance.

\( \Leftarrow \): Vice versa, let $M_2$ be a stable matching with respect to $\mathcal{P}_2$ such that $\{m^*, w^*\} \subset \{m_{\text{single}}^*, w_{\text{single}}^*\}$ and $|M_1 \triangle M_2| \leq k$. As $w_{\text{single}}^*$ and $m_{\text{single}}^*$ are their mutual unique first choice, it follows that $M_2$ contains $\{w_{\text{single}}^*, m_{\text{single}}^*\}$. As $M_2$ contains $\{m^*, w^*\}$, it follows that for each agent $a \in U \cup W$, agent $a$ has to be matched to an agent it prefers to $m^*$ and $w^*$, i.e., an agent from $U \cup W$. Therefore, $N := M_2 \setminus \{m^*, w^*\}$, $\{w_{\text{single}}^*, m_{\text{single}}^*\}$ is a perfect matching on $U \cup W$. It is also a stable one, as any blocking pair would also be a blocking pair for $M_2$. Furthermore, $|N \triangle N^*| = |M_1 \triangle M_2| - 3 \leq \ell$.

Second, we reduce ISMF-E-T to ISM-T. The general idea of this parameterized reduction is to replace a forced pair $\{m, w\} \in Q$ by a gadget consisting of $6(k + 1)$ agents. In $M_1$, we match the agents from the gadget in a way such that if “$m$ and $w$ are not matched to each other” in $M_2$, then, compared to $M_1$, the matching in the whole gadget needs to be changed, thereby exceeding the given budget $k$. This reduction implies:

**Theorem 3.** ISM-T parameterized by $k$ and the summed number of ties in $\mathcal{P}_1$ and $\mathcal{P}_2$ is $W[1]$-hard, even if $|\mathcal{P}_1 \oplus \mathcal{P}_2| = 1$ and only women have ties in their preferences.

Parameterized by $k$, ISM-T is $W[1]$-hard, even if $|\mathcal{P}_1 \oplus \mathcal{P}_2| = 1$ and each tie has size at most two and is in the preferences of some women.

**Proof.** We show the theorem by reducing Incremental Stable Marriage with Forced Edges and Ties to ISM-T without increasing $k$, $|\mathcal{P}_1 \oplus \mathcal{P}_2|$, and the number and length of ties (the theorem then follows from Proposition 1). Let $I = (A, \mathcal{P}_1, \mathcal{P}_2, M_1, k, Q)$ be an instance of Incremental Stable Marriage with Forced Edges and Ties. From $I$ we construct an instance $I' = (A', \mathcal{P}_1', \mathcal{P}_2', M_1', k')$ of ISM: We create preference profiles $\mathcal{P}_1'$ and $\mathcal{P}_2'$ by replacing every forced pair $e = \{v, w\} \in Q$ by a gadget inspired by a gadget designed by Cechlárová and Fleiner [9] which allows to replace a pair by a cycle of six agents (see Figure 2; note that the superscripts of each agent describes its position in the gadget). Our gadget replaces every forced pair by concatenating $k + 1$ copies of the gadget by Cechlárová and Fleiner [9] (see Figure 3). Formally, for each forced pair $e = \{v, w\} \in Q$, we add $6(k + 1)$ agents $a_{e,p}^{lb}$, $a_{e,p}^{lm}$, $a_{e,p}^{lt}$, $a_{e,p}^{rb}$, $a_{e,p}^{rm}$, and $a_{e,p}^{rt}$ for $p \in [k + 1]$. Agent $v$ replaces $w$ in its preferences by $a_{e,1}^{lm}$, and $w$ replaces $v$ by $a_{e,k+1}^{lm}$. The newly added agents have the following preferences:

\[
\begin{align*}
& a_{e,p}^{lt} > a_{e,p}^{lm}, & & p \in [k + 1]; \\
& a_{e,1}^{lt} > v > a_{e,1}^{lb}; \\
& a_{e,p}^{lm} > a_{e,p-1}^{lm} > a_{e,p}^{lb}, & & p \in \{2, 3, \ldots, k + 1\}; \\
& a_{e,p}^{lb} > a_{e,p}^{lm} > a_{e,p}^{rt}, & & p \in [k + 1]; \\
& a_{e,p}^{rt} > a_{e,p}^{lm} > a_{e,p}^{lt}, & & p \in [k + 1]; \\
& a_{e,p}^{rm} > a_{e,p+1}^{lm} > a_{e,p}^{rt}. & & p \in [k];
\end{align*}
\]
Figure 2: Gadget by Cechlárová and Fleiner [9] for pair $e = \{v, w\}$, where $v$ ranks $w$ at the $i$-th position and $w$ ranks $v$ at the $j$-th position.

Figure 3: The replacement for pair $e = \{v, w\}$, where $v$ ranks $w$ at the $i$-th position and $w$ ranks $v$ at the $j$-th position, for $k = 2$. The matching corresponding to matching $v$ and $w$ in the original instance is marked in bold.

$$a_{e,k+1}^m : a_{e,k+1}^b \succ w \succ a_{e,k+1}^a;$$

$$a_{e,p}^b \succ a_{e,p}^m, \quad p \in [k+1].$$

The preferences of the other agents remain unchanged. For the sake of readability, for $e = \{v, w\} \in Q$, we define $M_a := \{\{v, a_{e,1}^m\}, \{a_{e,k+1}^m, w\}\} \cup \{\{a_{e,p}^a, a_{e,p}^b\}, \{a_{e,p}^b, a_{e,p}^m\} \mid p \in [k + 1]\} \cup \{\{a_{e,q}^m, a_{e,q+1}^m\} \mid q \in [k]\}$ (this matching is marked by bold edges in Figure 3).

To finish the construction of the instance $I'$ of ISM, we set $M'_1 := (M_1 \setminus Q) \cup \{M_a \mid e \in Q\}$ and $k' := k$. Note that by definition we have $Q \subseteq M_1$ and thus $M'_1$ is a matching and is stable in $P_1$.

It remains to show that $I$ and $I'$ are equivalent. Given solution $M_2$ to $I$, we get a solution $M'_2$ to $I'$ by replacing every forced pair $e \in Q$ (note that $e$ needs to be part of $M_2$) by $M_a$. Clearly, it holds that $|M'_1 \triangle M'_2| = |M_1 \triangle M_2| \leq k = k'$. It is straightforward to verify that $M_2$ is indeed stable.

Given a solution $M'_2$ to $I'$, first observe that every stable matching $M$ in $P_2'$ which does not contain both $\{v, a_{e,1}^m\}$ and $\{a_{e,p}^m, w\}$ for some forced pair $e = \{v, w\} \in Q$ contains none of the pairs $\{v, a_{e,1}^m\}$, $\{a_{e,1}^m, w\}$, and $\{a_{e,p}^m, a_{e,p+1}^m\}$ for $p \in [k]$. As $|M_1 \triangle M'_2| \leq k$, it follows that $M'_2$ contains pairs $\{v, a_{e,1}^m\}$, $\{w, a_{e,k+1}^m\}$, and $\{a_{e,p}^m, a_{e,p+1}^m\}$ for every $p \in [k]$ for every forced pair $\{v, w\} \in Q$. Then $M_2 := (M'_2 \setminus \{M_a \mid e \in Q\}) \cup Q$ is a stable matching in $P_2$. We have $|M_1 \triangle M_2| = |M'_1 \triangle M'_2| \leq k' = k$, and thus $M_2$ is a solution to $I$.

On the algorithmic side, parameterized by the number of agents with at least one tie in their preferences in $P_2$, ISM-T lies in XP. The idea of our algorithm is to first guess the partners of
all agents in $M_2$ with a tie in their preferences in $P_2$ and subsequently reduce the problem to an instance of Weighted Stable Marriage, which is polynomial-time solvable [17]. Moreover, parameterizing by the summed size of all ties results in fixed-parameter tractability, as we can iterate over all possibilities of breaking the ties and subsequently apply the algorithm for ISM.

**Proposition 2.** ISM-T parameterized by the number of agents with at least one tie in their preferences in $P_2$ lies in XP. ISM-T parameterized by the summed size of all ties in $P_2$ is fixed-parameter tractable.

**Proof.** For the FPT-algorithm, we just enumerate all possibility of breaking the ties and subsequently apply the algorithm for ISM.

We now turn to the XP-algorithm. For each agent $a \in A$ with ties in their preferences in $P_2$, we guess its partner in $M_2$. Let $B \subseteq A$ be the set of agents who do not have an assigned partner in $M_2$ after this. To decide how these agents are matched, we now construct an instance of Weighted Stable Marriage with agent set $B$. For every $a \in A \setminus B$ and $b \in B$ with $b \succ_a M_2(a)$, we modify the preferences of $b$ by deleting each agent $b' \in B$ with $a \succ b'$. The reason for this is that $b$ needs to be matched better than $a$ in the stable matching $M_2$, as otherwise $\{a, b\}$ blocks $M_2$. Finally, we delete all agents from $A \setminus B$ from the preferences of agents from $B$, as we already know the partners of agents from $A \setminus B$ in $M_2$ and in particular that none of them is matched to an agent from $B$. Turning to the weights, all pairs from $M_1$ have weight one, while all other pairs have weight zero. Thereby, we maximize the overlap between $M_1$ and the computed matching. Let $M'$ be the computed maximum-weight stable matching in this instance. We set $M_2(b) := M'(b)$ for every $a \in B$. If $M_2$ is stable and fulfills $|M_1 \triangle M_2| \leq k$, then we return $M_2$; otherwise we reject the current guess.

It remains to argue that if there is a solution to the problem, then our algorithm finds one. Assume that $M^*$ is a stable matching in $P_2$ with $|M_1 \triangle M^*| \leq k$. Consider the guess in which we guess for each agent $a$ with ties in its preferences that it is matched to $M^*(a)$. Let $B$ be the set of agents who are unmatched after this guess and let $I'$ be the Weighted Stable Marriage instance constructed for this guess. Moreover, let $N^*$ be the stable matching $M^*$ restricted to the agents from $B$. Then, $N^*$ is a stable matching in $I'$ (as argued above, none of the pairs deleted in the process of creating $I'$ can be included in $M^*$). Note that $N^*$ has weight $|N^* \cap M_1|$ in $I'$. Moreover, let $M'$ be the computed maximum-weight stable matching in $I'$ and $M_2$ the final matching constructed by the algorithm for this guess.

We continue by showing that that $M_2$ is stable and $|M_1 \triangle M_2| \leq k$. We start with the latter. As $N^*$ is of weight $|N^* \cap M_1|$, we get that $M'$ overlaps with $M_1$ in at least $|N^* \cap M_1|$ pairs. As in $I'$ all agents have strict preferences, by the Rural Hospitals Theorem, $N^*$ and $M'$ match the same set of agents and thus we have that $|N^* \triangle M_1| \geq |M' \triangle M_1|$ and thus $k \geq |M^* \triangle M_1| \geq |M_2 \triangle M_1|$. To see that $M_2$ is stable, note first of all that there cannot be a blocking pair involving two agents from $A \setminus B$, as such a pair would also block $M^*$. Next, we prove that there cannot be a blocking pair involving two agents from $B$. Assume towards a contradiction that $\{u, w\}$ with $u, w \in B$ is a blocking pair for $M_2$. We will show that both $u$ and $w$ find each other acceptable in $I'$, directly implying that $\{u, w\}$ blocks $M'$ in $I'$, a contradiction to the stability of $M'$. Note that when creating $I'$ for each agent $a \in B$ which is matched in $M'$ it holds that if an agent $b$ from $B$ is deleted from $a$'s preferences, then $a$ prefers $M'(a)$ to $b$. Thus, if $u$ is matched in $M'$, then as $u$ prefers $w$ to $M'(u)$, agent $u$ still finds $w$ acceptable. So consider the case that $u$ is unmatched in $M'$. Then $u$ is also unmatched in $N^*$ and $M^*$ (due to the Rural Hospitals Theorem for $I'$). Recall that we only delete a woman $b$ from $B$ from the preferences of $u$ if there exists a woman $w' \in A \setminus B$ for which we have that $u \succ w' M^*(w')$ and $w' \succ_u b$; however if such a women $w'$ exists, then $\{u, w'\}$ blocks $M^*$, as $u$ is unmatched in $M^*$. Thus, if $u$ is unmatched, then he finds the same women from $B$ acceptable in $I$ and $I'$. Symmetric
arguments apply for $w$, proving that $u$ and $w$ still find each other acceptable in $I'$ and thus block $M'$.

It remains to consider pairs $\{u, w\}$ where one of them is contained in $A \setminus B$ and the other in $B$. Without loss of generality assume that $u \in A \setminus B$ and $w \in B$. Then, as $\{u, w\}$ blocks $M_2$, we have that $w \succ_u M_2(u) = M^*(u)$. Consequently, all agents that come after $u$ in the preferences of $w$ are deleted from the preferences of $w$ in $I'$. Thus, if $w$ is matched in $M'$, then $w$ prefers $M_2(w) = M'(w)$ to $u$, implying that $\{u, w\}$ does not block $M_2$. If $w$ is unmatched in $M'$, then it is also unmatched in $N^*$ and $M^*$ (due to the Rural Hospitals Theorem for $I'$), implying that $\{u, w\}$ blocks $M^*$, a contradiction. \hfill $\Box$

5 Master Lists

After having shown in the previous section that ISM-T and ISR mostly remain intractable even if we restrict several problem-specific parameters, in this section we analyze the influence of the structure of the preference profiles by considering what happens if the agents’ preferences are similar to each other. The arguably most popular approach in this direction is to assume that there exists a single central order (called master list) and that all agents derive their preferences from this order. This approach has already been applied to different stable matching problems in the quest for making them tractable [8, 13, 30, 31]. Specifically, we analyze in Section 5.1 the case where the preferences of all agents follow a single master list, in Section 5.2 the case where all but few agents have the same preference list, and in Section 5.3 the case where each agent has one of few different preference lists (which generalizes the setting considered in Section 5.2).

5.1 One Master List

In an instance of Stable Marriage/Roommates with agent set $A$, we say that the preferences of agent $a \in A$ can be derived from some preference list $\succ^*_a$ over agents $A$ if the preferences of $a$ are $\succ^*_a$ restricted to $Ac(a)$. If the preferences of all agents in $P_2$ can be derived from the same strict preference list (which is typically called master list), then there is a unique stable matching in $P_2$ which iteratively matches the so-far unmatched top-ranked agent in the master list to the highest ranked agent it accepts:

**Observation 1.** If all preferences in $P_2$ can be derived from the same strict preference list, then ISR can be solved in linear time.

This raises the question what happens when the master list is not a strict but a weak order. If the preferences of the agents may be incomplete, then reducing from the NP-hard Weakly Stable Pair problem (the question is whether there is a stable matching in an SM-T/SR-T instance containing a given pair [30, Lemma 3.4]), one can show that even assuming that all preferences are derived from the same weak master list is not sufficient to make ISM-T or ISR-T polynomial-time solvable.

**Proposition 3.** ISM-T and ISR-T are NP-hard even if all preferences in $P_1$ and $P_2$ can be derived from the same weak preference list.

**Proof.** As ISR-T generalizes ISM-T it suffices to prove NP-hardness of the latter problem. We do so by reducing from the Weakly Stable Pair problem where we are given a set $U$ of men and a set $W$ of women with preference profile $P$ and a man-woman pair $\{u, w\} \in U \times W$ and the question is to decide whether there exists a stable matching $M$ with $\{u, w\} \in M$. Irving et al. [30, Lemma 3.4] proved that Weakly Stable Pair is NP-hard even if all preferences from $P$ are derived from the same weak master list. Given an instance of Weakly Stable
PAIR \((U \cup W, \mathcal{P}, \{u, w\})\), we construct an instance of ISM with ties as follows. For each agent \(a \in U \cup W\), we add two agents \(c_a\) and \(d_a\). Agent \(d_a\) has empty preferences in \(\mathcal{P}_1\) and only accepts \(d_a\) in \(\mathcal{P}_2\), thereby ensuring that \(c_a\) is matched in every stable matching in \(\mathcal{P}_2\). If \(a \in (U \cup W) \setminus \{u, w\}\), then agent \(c_a\) has empty preferences in \(\mathcal{P}_1\) and the preferences of \(a\) in \(\mathcal{P}_2\), appended by \(d_a\), i.e., in \(\mathcal{P}_2\) we have \(Ac(c_a) = \{c_b \mid b \in Ac(a)\} \cup \{d_a\}\) and \(c_b \succ_{c_a} c_y\) if and only if \(b \succ_a b'\), and \(c_b \succ_{c_a} d_a\) for every \(b \in Ac(a)\). For each agent \(a \in \{u, w\}\), we add an agent which has the preferences of \(a\) (where each agent \(b \in Ac(a)\) is replaced by \(c_b\)) in \(\mathcal{P}_1\) and \(\mathcal{P}_2\).

We set \(M_1 := \{(u, w)\}\) and \(k := |U| + |W| - 1\). Note that every stable matching in \(\mathcal{P}_2\) has size \(|U| + |W|\). This implies that there is a solution to the constructed ISM with ties instance if and only if there exists a stable matching in \(\mathcal{P}_2\) containing the pair \(\{u, w\}\).

In contrast to this, if we assume that the preferences of agents in \(\mathcal{P}_2\) are complete and derived from a weak master list, then we can solve ISM-T and ISR-T in polynomial time. While for ISM-T this follows from a characterization of stable matchings in such instances as the perfect matchings in a bipartite graph due to Irving et al. [30] Lemma 4.3, for ISR-T this characterization does not directly carry over. Thus, we need a new algorithm that we present below. For this, we define an indifference class of a master list to be a maximal set of tied agents (note that an indifference class may consist of only one agent). Assume that the master list consists of \(q\) indifference classes and let \(A_i \subseteq A\) be the set of agents from the\(i\)th indifference class (where we order the indifference class according to the master list from most preferred to least preferred). For instance, for the master list \(a \succ b \sim c \sim d \succ e \sim f\), we have \(p = 3\), \(A_1 = \{a\}\), \(A_2 = \{b, c, d\}\), and \(A_3 = \{e, f\}\). Distinguishing between several cases, we build the matching \(M_2\) by dealing for increasing \(i \in [q]\) with each tie separately while greedily maximizing the overlap of the so-far constructed matching with \(M_1\). Our algorithm exploits the observation that in a stable matching, for \(i \in [q]\), all agents from \(A_i\) are matched to agents from \(A_i\) except if (i) \(|\bigcup_{j \in [i-1]} A_j|\) is odd in which case one agent from \(A_i\) is matched to an agent from \(A_{i-1}\), or (ii) if \(|\bigcup_{j \in [i]} A_j|\) is odd in which case one agent from \(A_i\) is matched to an agent from \(A_{i+1}\).

**Proposition 4.** If the preferences of agents in \(\mathcal{P}_2\) are complete and derived from a weak master list, then ISM-T/ISR-T can be solved in polynomial time.

**Proof.** We first give an algorithm for ISM-T. Consider an instance of ISM-T consisting of agents \(U \cup W\), preference profiles \(\mathcal{P}_1\) and \(\mathcal{P}_2\), and a stable matching \(M_1\) for \(\mathcal{P}_1\), where the preferences of agents in \(\mathcal{P}_2\) are complete and derived from a weak master list. We can find a stable matching as close as possible to \(M_1\) in polynomial-time as follows: Irving et al. [30] Lemma 4.3] proved that if \(\mathcal{P}_2\) satisfies these constraints, then it is possible to construct in polynomial-time a bipartite graph \(G\) on \(U \cup W\) such that the perfect matchings in \(G\) one-to-one correspond to stable matching of agents \(U \cup W\) in \(\mathcal{P}_2\). By assigning each edge in \(G\) that is contained in \(M_1\) weight one and all other edges weight zero and computing a perfect maximum-weight matching, we get the stable matching in \(\mathcal{P}_2\) with maximum overlap and thus minimum symmetric difference with \(M_1\).

For ISR-T, assume that the master list consists of \(q\) indifference classes and let \(A_i \subseteq A\) be the set of agents from the\(i\)th indifference class for \(i \in [q]\). Given a matching \(M\) of agents \(A\), for \(A' \subseteq A\), let \(M|_{A'}\) be the matching \(M\) restricted to agents \(A'\).

First of all we observe that, in a stable matching, for \(i \in [q]\) all agents from \(A_i\) are matched to agents from \(A_i\) except if (i) \(|\bigcup_{j \in [i-1]} A_j|\) is odd in which case one agent from \(A_i\) is matched to an agent from \(A_{i-1}\) and/or, (ii) if \(|\bigcup_{j \in [i]} A_j|\) is odd in which one agent from \(A_i\) is matched to an agent from \(A_{i+1}\). It now remains to find a matching fulfilling this constraint maximizing the intersection with the given matching \(M_1\).

To achieve this, we build the matching \(M_2\) by iterating over the master list and dealing with each indifference class \(A_i\) separately. To deal with situations where agents are matched outside
Algorithm 5 Algorithm for ISR-T with complete preferences derived from a weak master list

**Input:** A matching \( M_1 \) and sets of agents \( A_1, \ldots, A_q \) constituting the indifference classes of the master list (with all agents from \( A_i \) being preferred to all agents from \( A_{i+1} \)).

**Output:** A stable matching \( M_2 \) with \( |M_1 \triangle M_2| \leq k \) if one exists.

1. \( M_2 := \{\} \); \( a := \emptyset \)
2. \( \text{for } i = 1 \text{ to } q \) do
3. \( \text{if } \left| A_i \cup a \right| \text{ is even then } \triangleright \text{ Case 1} 
\)
4. \( \text{Add all pairs } M_1|_{A_i \cup a} \text{ to } M_2 \) and match remaining agents from \( A_i \cup a \) arbitrarily
5. \( a := \emptyset \)
6. else
7. \( \text{if } \text{there are agents } b'' \in A_{i+1} \text{ and } b' \in A_i \text{ with } \{b', b''\} \in M_1 \text{ then } \triangleright \text{ Case 2a} 
\)
8. \( \text{Add all pairs } M_1|_{(A_i \setminus \{b'\}) \cup a} \text{ to } M_2 \) and match remaining agents from \((A_i \setminus \{b'\}) \cup a\) arbitrarily
9. \( a := \{b'\} \)
10. else
11. \( \text{if } M_1|_{A_i} \text{ is a perfect matching for } A_i \text{ then } \triangleright \text{ Case 2b} 
\)
12. \( \text{Pick an arbitrary pair } \{a', a''\} \in M_1|_{A_i} \)
13. \( \text{Add } M_1|_{A_i \setminus \{a', a''\}} \text{ and } \{a, a'\} \text{ to } M_2 \)
14. \( a := \{a''\} \)
15. else \( \triangleright \text{ Case 2c} 
\)
16. \( \text{Add } M_1|_{A_i \cup a} \text{ to } M_2 \) and match remaining agents from \( A_i \cup a \) arbitrarily such that \( a \) gets assigned a partner
17. Set \( a := \{b\} \) with \( b \) being the agent from \( A_i \) that is not matched by \( M_2 \)
18. \( \text{if } |M_1 \triangle M_2| \leq k \) then Return \( M_1 \)
19. else Return NO

Their indifference class, we introduce a variable \( a \). After the processing of the \( i \)th indifference class, this variable is set to \( \{b\} \) for an agent \( b \in A_i \) if \( \left| \bigcup_{j \in [i]} A_j \right| \) is odd, which implies that \( b \) needs to be matched to an agent from \( A_{i+1} \), and to \( \emptyset \) otherwise. For each indifference class \( i \in [q] \), we distinguish several cases. First of all, we distinguish based on whether \( \left| \bigcup_{j \in [i]} A_j \right| \) is even (Case 1) or odd (Case 2). In Case 1, all agents from \( A_i \cup a \) need to be matched among themselves and we simply match them in a way maximizing the overlap with \( M_1 \). In Case 2, in which one agent \( a \) from \( A_i \) needs to be selected to be matched to an agent from \( A_{i+1} \), we again distinguish different cases: If there are agents \( b'' \in A_{i+1} \) and \( b' \in A_i \) with \( \{b', b''\} \in M_1 \) (Case 2a), we match agents from \( A_i \setminus \{b'\} \) as to maximize the overlap with \( M_1 \) and set \( a \) to \( \{b'\} \). Otherwise, we match the agents from \( A_i \cup a \) in a way to maximize the overlap with \( M_1 \), while leaving one agent from \( A_i \) unmatched. This reasoning gives rise to Algorithm 5.

It is easy to verify that the matching returned by Algorithm 5 satisfies the above stated conditions for being a stable matching so it remains to show that it maximizes the intersection between \( M_1 \) and a stable matching in \( \mathcal{P}_2 \). Note that maximizing the intersection is equivalent to minimizing the symmetric difference because all stable matchings for \( \mathcal{P}_2 \) have the same size as the preferences are complete and thus every stable matching matches every or all but one agents (depending on whether the number of agents is even). Let \( M_2 \) be the returned matching.

We claim that for all \( i \in [q] \), matching \( M_2 \) simultaneously maximizes \( |M_1 \cap M_2 \cap \{\{a, b\} : a \in A_i, b \in A_i\}| \) and \( |M_1 \cap M_2 \cap \{\{a, b\} : a \in A_i, b \in A_{i+1}\}| \) among all stable matchings in \( \mathcal{P}_2 \). As for all pairs \( \{a, b\} \) in a stable matching it either holds that \( a, b \in A_i \) or \( a \in A_i \) and \( b \in A_{i+1} \), from this the maximality of \( |M_1 \cap M_2| \) follows.

The maximality of \( |M_1 \cap M_2 \cap \{\{a, b\} : a \in A_i, b \in A_i\}| \) is clearly ensured in Cases 1, 2a,
and 2c, as in each case we add to $M_2$ the matching $M_1|_{A_i}$. For Case 2b, we add all but one pair from $M_1|_{A_i}$ to $M_2$. Note however that no stable matching can add all pairs from $M_1|_{A_i}$ to $M_2$, as in this case one agent from $\bigcup_{j=1}^{i-1} A_j$ is matched to an agent from $\bigcup_{j=i+1}^{n} A_j$, resulting in a blocking pair.

The maximality of $|M_1 \cap M_2 \cap \{(a, b) : a \in A_i, b \in A_{i+1}\}|$ is ensured by always adding a pair $(a, b) \in M_1$ with $a \in A_i$ and $b \in A_{i+1}$ to $M_2$ in Case 2a if this is possible in a matching fulfilling the necessary criteria for stability.

### 5.2 Few Outliers

Next, we consider the case that almost all agents derive their complete preferences from a single strict preference list (we will call these agents followers), while the remaining agents (we will call those agents outliers) have arbitrary preferences. We will show that ISR is fixed-parameter tractable with respect to the number of outliers by showing that all stable matchings in a Stable Roommates instance can be enumerated in FPT time with respect to this parameter:

**Theorem 4.** Given a Stable Roommates instance $(A, \mathcal{P})$ and a partitioning $F \cup S$ of the agents $A$ such that all agents from $F$ have complete preferences that can be derived from the same strict preference list in $\mathcal{P}_2$, one can enumerate all stable matchings in $(A, \mathcal{P})$ in $O(n^2 \cdot |S|^{5}|S|^{5}+1)$ time. Consequently, ISR is solvable in $O(n^2 \cdot |S|^{5}|S|^{5}+1)$ time.

**Proof.** We start by guessing the set $S^* \subseteq S$ of outliers which are matched to another outlier. For every $a \in S^*$, we additionally guess to which agent from $S^*$ it is matched. We denote this guess by $M^*(a)$.

We say that a stable matching $M$ respects a guess $(S^*, M^*)$ if $S^* = \{s \in S \mid M(s) \in S\}$ and $M^*(a) = M(a)$ for each $a \in S^*$. The basic idea of our algorithm now is that we can pass through the master list and greedily find pairs contained in the stable matching.

We claim that there is at most one stable matching respecting the guess $(S^*, M^*)$, and that the respective stable matching $M$ can be found, if it exists, as follows. We start with $M := \{(a, M^*(a)) : a \in S^*\}$ and delete all agents from $S^*$ from the instance including the master list. Subsequently, we construct the matching for the agents from $A \setminus S^*$ iteratively: As long as there are at least two agents unmatched by $M$, we consider the first unmatched agent $a$ from the master list. If $a \in S$, then we match $a$ to the unmatched (by $M$) agent $a'$ from $A \setminus S$ which $a$ likes most (i.e., we add $(a, a')$ to $M$). If $a \in F$, then let $a'$ be the next follower in the master list and let $b_1, \ldots, b_j$ be the outliers which appear (in that order) between $a$ and $a'$ in the master list. Intuitively, as $a$ is the so-far unmatched agent best-ranked in the master list, $a$ cannot be matched worse than $a'$ in any stable matching. We now compute a temporary matching $M^\text{temp}$ by starting with $M^\text{temp} := M$ and for $i = 1$ to $j$, if $b_i$ is unmatched in $M^\text{temp}$, we iteratively add the pair $(b_i, b^*_i)$ to $M^\text{temp}$, where $b^*_i$ is the follower which $b_i$ likes most among all agents which are currently unmatched in $M^\text{temp}$. If after this for-loop agent $a$ is matched in $M^\text{temp}$, then we set $M := M^\text{temp}$. Otherwise, we discard $M^\text{temp}$ and add $(a, a')$ to $M$. When all but at most one agent are matched, we check whether the resulting matching is stable. If this is the case, then we output the matching and afterwards proceed with the next guess.

We now prove the correctness of the described algorithm. Assume for the sake of contradiction that there exists a stable matching $M'$ respecting a guess $(S^*, M^*)$ that differs from the matching $M$ constructed for this guess. As $M$ and $M'$ both respect the same guess $(S^*, M^*)$, we have $M(a) = M'(a)$ for each $a \in S^*$.

**Claim 1.** Let $\hat{a} \in A \setminus S^*$ be the first agent in the master list that is matched differently in $M$ and $M'$ and let $b := M(\hat{a})$. Agent $b$ prefers $\hat{a}$ to $M'(b)$. 

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Proof (Claim). First we consider the case that $b$ is a follower. As $b$ is not matched to $\hat{a}$ in $M'$ and all agents appearing before $\hat{a}$ in the master list are matched the same in $M$ and $M'$, agent $b$ prefers $\hat{a}$ to $M'(b)$.

It remains to consider the case that $b$ is an outlier. Because $\hat{a} \notin S^*$, it follows that $\hat{a}$ is a follower. Therefore, by the construction of $M$, between $\hat{a}$ and $b$, the master list contains only followers which are matched to outliers appearing before $\hat{a}$ in the master list (and none of these agents can be $M'(b)$ by the definition of $\hat{a}$) and some outliers $b_1, \ldots, b_k$. We assume that for every $i \in [k-1]$, outlier $b_i$ appears before $b_{i+1}$ in the master list. Note that the pair \{\hat{a}, b\} must have been added to $M$ by setting $M$ to a matching $M^{\text{temp}}$. Let $X$ be the set of agents which are matched in $M$ before the algorithm set $M$ to $M^{\text{temp}}$ (i.e., all agents appearing before $\hat{a}$ in the master list and their partners in $M$).

Note that all agents from $X$ are matched the same in $M$ and $M'$ by the definition of $\hat{a}$. Moreover, we later show that $M(b_i) = M'(b_i)$ for all $i \in [k]$. From this, the statement of the claim follows as follows: Further note that by the construction of $M'$, $\hat{a}$ is the most preferred follower by $b$ among the agents $F \setminus (X \cup \{M(b_1), \ldots, M(b_k)\})$. As $b$ is matched to a follower in $M'$ (due to $b \notin S^*$ and $M'$ respecting our guess) and all agents from $(X \cup \{M(b_1), \ldots, M(b_k)\})$ are matched the same in $M$ and $M'$, it follows that $b$ prefers $\hat{a}$ to $M'(b)$.

Thus, it suffices to show that $M(b_i) = M'(b_i)$ for every $i \in [k]$. Assume towards a contradiction that $M(b_i) \neq M'(b_i)$ for some $i \in [k]$, and let $i$ be the minimal index fulfilling $M(b_i) \neq M'(b_i)$. By the definition of $i$, all agents from $X \cup \{M(b_1), \ldots, M(b_{i-1})\}$ are matched the same in $M$ and $M'$. Thus, as $M$ matches $b_i$ to its favorite follower from $F \setminus (X \cup \{M(b_1), \ldots, M(b_{i-1})\})$ and $b_i$ is also matched to a follower in $M'$ (as $b_i \notin S^*$), it follows that $b_i$ prefers $M(b_i)$ to $M'(b_i)$. Because $M(b_i)$ is a follower and $M'$ can match $M(b_i)$ only to $\hat{a}$ or an agent after $b_i$ in the master list, we have that either \{\hat{a}, $M(b_i)$\} is matched in $M'$ or $M'(b_i)$ prefers $b_i$ to $M'(M(b_i))$. In the latter case, we have that \{b_i, $M(b_i)$\} blocks $M'$, a contradiction to the stability of $M'$. Thus, we focus on the former case. We claim that in this case $M(b_j) \in \{M'(b_i), M'(b_{i+1}), \ldots, M'(b_j)\}$ for every $i < j \leq k$. To see this, assume towards a contradiction that this is not the case and let $j$ be the minimal such index. Then $M(b_j)$ prefers $b_j$ to $M'(M(b_j))$ because $M(b_j)$ is a follower, matched neither to $\hat{a}$ nor to one of $b_1, \ldots, b_j$ in $M'$, and all agents from $X \cup \{b_1, \ldots, b_{j-1}\}$ are matched the same in $M$ and $M'$ (and thus in particular not to $M'(b_j)$). Moreover, outlier $b_j$ prefers $M(b_j)$ to $M'(b_j)$ because all agents from $X \cup \{M(b_1), \ldots, M(b_{j-1})\}$ are matched the same in $M$ and $M'$, and $b_j$ is not matched to $M(b_{\ell})$ for $\ell < j$, and $M(b_j)$ is its first choice among the remaining agents. Consequently, \{b_j, $M(b_j)$\} blocks $M'$, a contradiction to the stability of $M'$. Thus, $b$ prefers $\hat{a}$ to $M'(b)$.

Having shown that $b$ prefers $\hat{a}$ to $M'(b)$, it is enough to show that $\hat{a}$ prefers $b$ to $M'(\hat{a})$, which implies that \{\hat{a}, $b$\} blocks $M'$, contradicting the stability of $M'$:

Claim 2. Let $\hat{a} \in A \setminus S^*$ be the first agent appearing in the master list that is matched differently in $M$ and $M'$ and let $b := M'(\hat{a})$. Agent $\hat{a}$ prefers $b$ to $M'(\hat{a})$.

Proof (Claim). If $\hat{a}$ is an outlier, then $\hat{a}$ prefers $b$ to $M'(\hat{a})$, as the algorithm matches $\hat{a}$ to the most preferred follower that is not matched to an agent which comes before $\hat{a}$ in the master list. Since all agents before $\hat{a}$ in the master list are matched the same in $M$ and $M'$, agent $\hat{a}$ cannot be matched better than $M'(\hat{a})$.

Otherwise, $\hat{a}$ is a follower. Let $X$ contain the agents appearing before $\hat{a}$ in the master list and their partners in $M$. For the sake of contradiction, assume that $\hat{a}$ prefers $M'(\hat{a})$ to $b$ (we will later distinguish two cases and in each cases establish a contradiction, thereby ultimately proving that $\hat{a}$ prefers $b$ to $M'(\hat{a})$). Let $a'$ be the next follower in the master list that is not matched to an agent appearing before $\hat{a}$ in the master list in $M$ (and thereby also in $M'$).

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Further, let $b_1, \ldots, b_k$ be the outliers that appear between $a$ and $a'$ in the master list. We assume that $b_i$ is before $b_{i+1}$ in the master list for $i \in [k-1]$. By the construction of $M$, agent $\hat{a}$ is matched to $b_1, \ldots, b_k$ or $a'$ in $M$. Thus, for $\hat{a}$ to be matched better in $M'$ than in $M$, it needs to hold that $M'(\hat{a}) = b_i$ for some $i \in [k]$.

We make a case distinction based on whether $b = a'$. If $b \neq a'$, then let $i' \in [i]$ be the smallest $i'$ such that $b_{i'}$ is matched differently in $M$ and $M'$. Note that $b \neq b_{i'}$, as we have assumed that $\hat{a}$ prefers $M'(\hat{a})$ to $b = M(\hat{a})$. Because $b_{i'}$ is matched to a follower in both $M$ and $M'$ (as $b_{i'} \notin S^*$) and all agents that appear before $b_{i'}$ in the master list expect of $\hat{a}$ are matched the same in $M$ and $M'$, it follows that $M(b_{i'})$ prefers $b_{i'}$ to $M'(M(b_{i'}))$. Because $M(b_{i'})$ is the most preferred follower of $b_{i'}$ among $F \setminus (X \cup \{M(b_1), \ldots, M(b_{i'-1})\})$ and all agents from $X \cup \{M(b_1), \ldots, M(b_{i'-1})\}$ are matched the same in $M$ and $M'$, it follows that $b_{i'}$ prefers $M(b_{i'})$ to $M'(b_{i'})$. Consequently, $\{b_{i'}, M(b_{i'})\}$ blocks $M'$, a contradiction to $M'$ being a stable matching.

Otherwise, we have $b = a'$. As all agents from $X$ are matched the same in $M$ and $M'$, matching $M'$ needs to match $b_1$ to its most preferred follower $a_1$ from $F \setminus X$, as otherwise $b_1$ would form a blocking pair together with $a_1$ (note that $M'(\hat{a})$ is an outlier and thus $M'(a_1) \neq \hat{a}$). By induction, one easily sees that $M'$ matches agent $b_j$ for $j \in [k]$ to its most preferred follower that is not matched to an agent appearing before $b_k$ in the master list. As all agents from $X$ are matched the same in $M$ and $M'$ and we know that $M'$ matches $\hat{a}$ to $b_i$ for some $i \in [k]$, it follows that by construction matching $M$ also contains $\{a, b_i\}$, a contradiction to $\hat{a}$ being matched differently in $M$ and $M'$.

Finally, we analyze the running time of the algorithm. First observe that the number of possible sets $S^*$ and matchings on $S^*$ can be upper-bounded by $|S|^{|S|+1}$. Passing through the master list can be done in $O(n)$ time. For each agent $\hat{a}$ which is the first agent of the master list which is still unmatched, we may construct a matching $M_{\text{temp}}$, which can be done in $O(n)$ (the remaining computations when $\hat{a}$ is the first agent in the master list can be done in constant time). Checking whether a matching is stable trivially runs in $O(n^2)$ time. Thus, the running time of $O(n^2 |S|^{|S|+1})$ follows.

If the master list may contains ties, then enumerating stable matchings becomes a lot more complicated, as here we have much more flexibility on how the agents are matched. Specifically, even if there are no outliers, there may be exponentially many stable matchings (if the master list ties every agent, then every (near)-perfect matching is stable). We leave it open whether there exists a similar fixed-parameter tractability result for a weak master list (both in the roommates and marriage setting).

### 5.3 Few Master Lists

Motivated by the positive result from Section 5.2, in this section we consider the smaller parameter “number of different preference lists”. Recall that Observation 1 states that if the preference lists of all agents are derived from a strict master list in a STABLE ROOMMATES instance, then there exists only one stable matching (even if the preferences of the agents may be incomplete). This raises the question what happens if there exist “few” master lists and each agent derives its preferences from one of the lists. To the best of our knowledge, the parameter “number of master lists” has not been considered before. However, it nicely complements (and lower-bounds) the parameter “number of agent types” as studied by Meeks and Rastegari [39]. Two agents are of the same type if they have the same preferences and all other agents are indifferent between them. Notably, Boehmer et al. [3] Proposition 5] proved that ISM-T is fixed-parameter tractable with respect to the number of agent types. Their algorithm also works for ISR-T.
If the preferences of agents are incomplete, then as proven in Proposition 3, ISM-T is already NP-hard for just one weak master list. Moreover, note that a reduction of Cseh and Manlove [12, Theorem 4.2] implies that ISR with incomplete preferences is NP-hard even if the preferences of each agent are derived from one of two strict preference lists. The preferences of all agents in this reduction can be derived from the preference list \([\{q_i : i \in [n]\}] > [\{p_i : i \in [n]\}] > [\{q_i : i \in [n]\}]\) or the preference list \([\{q_i : i \in [n]\}] > [\{p_i : i \in [n]\}] > [\{q_i : i \in [n]\}]\) where for a set \(A\) of agents, \([A]\) denotes an arbitrarily but fixed strict order of \(A\). Consequently, in this subsection we focus on the case with complete preferences.

In the following, we show that ISR (Section 5.3.1) and ISM-T (Section 5.3.2) are \(W[1]\)-hard parameterized by the number of master lists, even if agent’s preferences are complete. For the sake of readability, we sometimes only specify parts of the agents’ preference relation and end the preferences with “\(\triangleright \)(rest)”, which means that all remaining agents appear afterwards in some arbitrary strict ordering.

### 5.3.1 Incremental Stable Roommates

In contrast to the two fixed-parameter tractability results for the number of outliers (Theorem 4) and the number of agent types [5], we show that parameterized by the number \(p\) of master lists, ISR is \(W[1]\)-hard even if the preferences of agents are complete:

**Theorem 5.** ISR is \(W[1]\)-hard parameterized by the minimum number \(p\) such that in \(\mathcal{P}_2\) the preferences of each agent can be derived from one of \(p\) strict preference lists, even if in \(\mathcal{P}_1\) as well as in \(\mathcal{P}_2\) all agents have complete preferences.

**Proof.** The basic structure of the reduction is similar to the one in Theorem [11]. We again reduce from MULTICOLORED CLIQUE, and will construct a vertex-selection gadget for every color but this time only one edge gadget for all edges together. Each gadget will have a constant number of master list in \(\mathcal{P}_2\), resulting in \(O(\ell)\) many master lists. Let \((G = (V^1 \cup \cdots \cup V^\ell, E), \ell)\) be an instance of MULTICOLORED CLIQUE. Again, we will refer to elements from \([\ell]\) as colors or color classes. We assume that \(\ell\) is even, \(G\) is \(r\)-regular and that every color class contains \(\nu\) vertices. Let \(m := \frac{\ell^2}{2}\) be the number of edges in \(G\).

**Vertex-selection gadget**

Note that later in the edge gadget, we will add agents \(a_{e,1}, a_{e,2}, a_{e,3}\), and \(a_{e,4}\) for each edge \(e \in E\). For every color class, we add a vertex-selection gadget (where the preferences of each agent are derived from one of four master lists, depending on the color). Fix some color \(c \in [\ell]\) and an arbitrary order \(v_{i,1}^c, v_{i,2}^c, \ldots, v_{i,\nu}^c\) of the vertices of color class \(c\). The gadget contains agents \(a_{i,1}^c, a_{i,2}^c, \ldots, a_{i,\nu}^c\) for each \(i \in \{1, \ldots, \nu \}\) and every \(j \in \{1, 2, 3, 4\}\). Let \(e_{i,1}^c, e_{i,2}^c, \ldots, e_{i,\nu}^c\) be the edges with one endpoint of color \(c\). For \(t \in [\nu]\), let \(v_{t}^c\) be the endpoint of \(e_{t}^c\) of color \(c\). For every \(s \in [n]\), we define

\[ A_{s,1}^c := a_{(s-1)\nu+1,2}^c > a_{(s-1)\nu+2,2}^c > \cdots > a_{(s-1)\nu+r_{t},2}^c > a_{s,1}^c > a_{(s-1)\nu+r_{t}+1,2}^c > a_{c} > \cdots > a_{\nu}^c \]

if \(s = i_t\) for some \(t \in [\nu]\) and \(A_{s,1}^c := a_{(s-1)\nu+1,2}^c > a_{(s-1)\nu+r_{t}+1,2}^c > \cdots > a_{\nu}^c\) otherwise (this incomplete preference list will be part of the preferences of the master list for agents \(a_{i,1}^c\)). Similarly, for \(s \in [n]\), let \(A_{s,2}^c := a_{(s-1)\nu+1,1}^c > a_{(s-1)\nu+r_{t}+1,1}^c > \cdots > a_{\nu}^c\) if \(s = i_t\) for some \(t \in [\nu]\) and \(A_{s,1}^c := a_{(s-1)\nu+1,1}^c > a_{(s-1)\nu+r_{t}+1,1}^c > \cdots > a_{\nu}^c\) otherwise (this incomplete preference list will be part of the preferences of the master list for agents \(a_{i,2}^c\)). In \(\mathcal{P}_2\), the preferences of the agents in the vertex-selection gadget for color class \(c\) look as follows.

\[ a_{i,1}^c : A_{i,1} \triangleright A_{i,2}^c \triangleright A_{i,3}^c \triangleright \cdots \triangleright A_{\nu}^c \triangleright a_{1,4}^c > a_{2,4}^c \cdots a_{\nu,4}^c \triangleright \cdots \]
We say that a vertex-selection gadget selects vertex $v_i^c$ with $i \in [v]$ if $M_2$ contains pairs $\{a_{j,1}^c, a_{j+(i-1),2}^c\}$ and $\{a_{j,3}^c, a_{j+(i-1),4}^c\}$ for $j \leq \nu m -(i-1)$ together with $\{a_{j,1}^c, a_{j-(\nu m-i+1),4}^c\}$ and $\{a_{j,3}^c, a_{j-(\nu m-i+1),2}^c\}$ for $j > \nu m - i + 1$.

**Edge gadget**

Fix an arbitrary order $e_1, \ldots, e_m$ of the edges, where $e_i = \{v, w\} \in E$ with $v \in V^c$ and $w \in V^d$. For every edge $e \in E$, the gadget contains agents $a_{e,1}$, $a_{e,2}$, $a_{e,3}$, and $a_{e,4}$. The preferences of these agents are as follows.

\[
a_{e,1} : a_{e,1}^2 > a_{e,1}^1 > a_{e,1}^c > a_{e,2}^d > a_{e,1} \vdash a_{e,2} > a_{e,2} > a_{e,3} > a_{e,4} > a_{e,4} > a_{e,4} > a_{e,4} > a_{e,4} = (\text{rest})
\]

\[
a_{e,2} : a_{e,2} > a_{e,3} > a_{e,1} > a_{e,2} > a_{e,4} > a_{e,1} > a_{e,2} > a_{e,3} > a_{e,4} > a_{e,3} = (\text{rest})
\]

\[
a_{e,3} : a_{e,3} > a_{e,4} > a_{e,1} > a_{e,2} > a_{e,3} > a_{e,4} > a_{e,1} > a_{e,2} > a_{e,3} = (\text{rest})
\]

\[
a_{e,4} : a_{e,4} > a_{e,3} > a_{e,2} > a_{e,3} > a_{e,1} = (\text{rest})
\]

The basic idea of this gadget is that for every edge $e = \{v, w\} \in E$ with $v \in V^c$ and $w \in V^d$ for some $c, d \in [\ell]$, we may pick either pairs $\{a_{e,1}, a_{e,2}\}$ and $\{a_{e,3}, a_{e,4}\}$ or pairs $\{a_{e,1}, a_{e,4}\}$ and $\{a_{e,3}, a_{e,2}\}$ to be part of a stable matching. The first possibility will not have any intersection with $M_1$ (thus increasing the symmetric difference by 4), while the second possibility will intersect with $M_1$. However, it is only possible to include these pairs if the vertex-selection gadgets for the colors $c$ and $d$ select the endpoints $v$ and $w$ of $e$.

Set $M_1 := \{a_{i,j}^{2c-1}, a_{i,j}^{2c}\} : i \in [\nu m], j \in [4], c \in [\ell] \cup \{a_{e,1}, a_{e,4}\}, a_{e,2}, a_{e,3} : e \in E$. As the distance between $P_1$ and $P_2$ is allowed to be unbounded, it is easy to construct a preference profile $P_1$ such that $M_1$ is a stable matching for $P_1$. Note that no stable matching for $P_2$ contains any edge from $\{a_{i,j}^{2c-1}, a_{i,j}^{2c}\} : i \in [\nu m], j \in [4], c \in [\ell] \cup \{a_{e,1}, a_{e,4}\}, a_{e,2}, a_{e,3} : e \in E$.

Furthermore, we set $k := 2|\nu| - 4(\ell^2)$, as $|\nu| = |M_2|$ this enforces that the intersection of $M_1$ and $M_2$ contains at least $2(\ell^2)$ pairs. Since the constructed instance uses $4\ell + 4$ master lists and the reduction clearly runs in polynomial time, it remains to show the correctness of the reduction.

**Forward Direction**

Let $X$ be a multicolored clique. We construct a stable matching $M_2$ as follows. For every edge $e$ not corresponding to an edge inside the clique, we add pairs $\{a_{e,1}, a_{e,2}\}$ and $\{a_{e,3}, a_{e,4}\}$ to $M_2$, while for every other edge $e \subseteq X$, we add pairs $\{a_{e,1}, a_{e,4}\}$ and $\{a_{e,2}, a_{e,3}\}$. For every vertex-selection gadget, we add the matching corresponding to selecting the vertex of this color class that is part of $X$. Clearly, $M_1 \cap M_2 = \{a_{e,1}, a_{e,4}\}, a_{e,2}, a_{e,3} : e \subseteq X$, and this set has cardinality $2(\ell^2)$. Thus, $|M_1 \triangle M_2| = |M_1| + |M_2| - 2 \cdot 2(\ell^2) = k$ (note that $|M_1| = |M_2|$ as the preferences in $P_1$ as well as $P_2$ are complete). It remains to show that $M_2$ is stable.
First, we show by induction on $s$ that none of the agents $a_{e_{s,1}}, a_{e_{s,2}}, a_{e_{s,3}},$ and $a_{e_{s,4}}$ is contained in a blocking pair. For $s = 0$, there is nothing to show. So assume towards a contradiction that $a_{e_{s,j}}$ is contained in a blocking pair for some $s > 0$. Using the induction hypothesis that no agent $a_{e_{s',j'}}$ for some $s' < s$ and $j' \in [4]$ is part of a blocking pair, it is easy to see that $j = 1$ and that the other of agent the blocking pair needs to be from a vertex gadget. Let $a_{e_{s,q}}^c$ for some $c \in [\ell], p \in [\nu_m], q \in [4]$, be the other agent of the blocking pair, and let $v^c_q$ be the endpoint of $e_s$ of color $c$. Further, let $v^c_q$ be the vertex from the multicolored clique $X$ of color $c$. We consider whether $q = 1$ or $q = 2$.

First assume that $q = 1$. Note that agent $a_{e_{s,1}}^c$ prefers $a_{e_{s,1}}$ to $M_2(a_{e_{s,1}}^c)$ only if $M_2(a_{e_{s,1}}^c) = a_{p,q}^c$ for some $p' \in [\nu_m]$ or $M_2(a_{e_{s,1}}^c) = a_{p',2}^c$ for some $p' > (s - 1)\nu + t$. These two conditions are by the construction of $M_2$ equivalent to $p > (s - 1)\nu + t - i + 1$. If $\{a_{e_{s,1}}, a_{e_{s,2}}\} \in M_2$, then $a_{e_{s,1}}$ prefers $a_{p,1}^c$ to $M_2(a_{e_{s,1}})$ only if $p \leq (s - 2)\nu + 1$. This contradicts $p > (s - 1)\nu + t - i + 1$ as $i \leq \nu$. Otherwise we have $\{a_{e_{s,1}}, a_{e_{s,4}}\} \in M_2$. By the construction of $M_2$, it follows that $i = t$. In this case agent $a_{e_{s,1}}$ prefers $a_{p,1}^c$ only if $p \leq (s - 1)\nu + 1$. However, this contradicts $p > (s - 1)\nu + t - i + 1$.

Next assume $q = 2$. Agent $a_{p,q_2}^c$ prefers $a_{e_{s,1}}$ to $M(a_{p,q_2}^c)$ only if $M_2(a_{p,q_2}^c) = a_{p',1}^c$ for some $p' > sv + 1 - t$ which by the construction of $M_2$ implies that $p > sv + i - t$. If $\{a_{e_{s,1}}, a_{e_{s,2}}\} \in M_2$, then $a_{e_{s,1}}$ prefers $a_{p,q_2}^c$ to $M(a_{e_{s,1}})$ only if $p \leq (s - 1)\nu + t$. This contradicts $p > sv + i - t$ as $t \leq \nu$. Otherwise we have $\{a_{e_{s,1}}, a_{e_{s,4}}\} \in M_2$. By the construction of $M_2$, it follows that $i = t$. Agent $a_{e_{s,1}}$ prefers $a_{p,q_2}^c$ to $M(a_{e_{s,1}})$ only if $p \leq sv$. This contradicts $p > sv = sv + i - t$. It follows that there cannot be a blocking pair involving an agent from the edge gadget.

Finally, we show that there is no blocking pair involving an agent from a vertex-selection gadget. So assume for a contradiction that there is a blocking pair $\{a_{p,q}, b\}$ for some $p \in [\nu_m], q \in [4], c \in [\ell]$. As we have previously shown that $b$ cannot be contained in the edge gadget, $b$ has to be contained in the same vertex-selection gadget, i.e., $b = a_{p,q_2}^c$, for some $p' \in [\nu_m]$ and $\nu_2' \in [4]$. We may assume that $q \in \{1, 3\}$ and $\nu_2' \in \{2, 4\}$. We assume $q = 1$; the case $q = 3$ is symmetric. Let $v^c_q$ be the vertex of color $c$ which is contained in the multicolored clique $X$. We make a case distinction based on the value of $p$.

If $p \leq \nu_m - (i - 1)$, then the only agents from the vertex selection gadget which $a_{e_{s,1}}^c$ prefers to $M_2(a_{p,q}) = a_{p,(i-1),1}^c$ are agents $a_{p+1}^c$, $x < p + i - 1$. However, every such agent $a_{p+1}^c$ is matched to an agent $a_{c,3}^e$ for some $y \in [\nu_m]$ or an agent $a_{c,1}^e$ for some $z < p$. Agent $a_{c,2}$ prefers these agents to $a_{e_{s,1}}^c$, a contradiction to $\{a_{p+1,q}, a_{c,1}^c\}$ being blocking.

If $p > \nu_m - i + 1$, then the only agents which $a_{e_{s,1}}^c$ prefers to $M_2(a_{p,q}) = a_{p-(\nu_m-i+1),4}$ are agents $a_{p+x}^c$, $x \in [\nu_m]$ or agents $a_{c,4}^e$ with $y < p - (\nu_m - i + 1)$. Every agent $a_{c,2}^e$ is matched to an agent $a_{c,1}^e$ with $z \leq \nu_m - (i - 1) < p$ or $a_{c,3}^e$ for some $z \in [\nu_m]$ and thus does not prefer $a_{p+1}^c$ to $M_2(a_{x,q})$. Agent $a_{c,4}^e$ with $y < p - (\nu_m - i + 1)$ is matched to an agent $a_{c,1}^e$ with $z < p$ and thus prefers $M_2(a_{c,4}, a_{p+1})$. Therefore, $a_{p+1}$ is not part of a blocking pair, a contradiction. Thus, $M_2$ is stable.

**Backward Direction**

Let $M_2$ be a stable matching with $|M_1 \Delta M_2| \leq 2|M_1| - 4\binom{\ell}{4}$. Note that $M^* := \{\{a_{e_{s,1}}, a_{e_{s,2}}\}, \{a_{e_{s,3}}, a_{e_{s,4}}\} : i \in [\ell], c \in [\ell]\} \cup \{\{a_{e_{s,1}}, a_{e_{s,2}}\}, \{a_{e_{s,3}}, a_{e_{s,4}}\} : e \in E\}$ is a stable matching in $P_2$. By Lemma [3] no stable matching in $P_2$ can contain a pair $\{a, b\}$ with both $a$ preferring $M^*(a)$ to $b$ and $b$ preferring $M^*(b)$ to $a$. Thus, every stable matching $M_2$ in $P_2$ is also stable in the instance $\mathcal{I}^\ast \cdots$ which arises from $P_2$ by deleting all edges $\{a, b\}$ with $b$ appearing in the $\succ \cdots$ part of the preferences of $a$ and $a$ appearing in the $\succ \cdots$ part of the preferences of $b$. Further, as $M^*$ is perfect in $\mathcal{I}^\ast \cdots$, the Rural Hospitals Theorem for SR [20] implies that
\( M_2 \) is perfect in \( \mathcal{I}^\text{(rest)} \).

First, we show by induction on \( s \) that for every edge \( e_s \), matching \( M_2 \) contains pairs \( \{a_{e_s,1}, a_{e_s,2}\} \) and \( \{a_{e_s,3}, a_{e_s,4}\} \) or pairs \( \{a_{e_s,1}, a_{e_s,4}\} \) and \( \{a_{e_s,2}, a_{e_s,3}\} \). In \( \mathcal{I}^\text{(rest)} \), agents \( \{a_{e,2}, a_{e,4} : e \in E\} \) are only incident to \( \{a_{e,1}, a_{e,3} : e \in E\} \). As \( M_2 \) is perfect in \( \mathcal{I}^\text{(rest)} \), each agent from \( \{a_{e,2}, a_{e,4} : e \in E\} \) is matched to an agent from \( \{a_{e,1}, a_{e,3} : e \in E\} \). We now turn to the induction. For \( s = 0 \) there is nothing to show. Fix \( s > 0 \). If \( M_2 \) contains neither \( \{a_{e,1}, a_{e,2}\} \) nor \( \{a_{e,1}, a_{e,4}\} \), then \( \{a_{e,1}, a_{e,4}\} \) blocks \( M_2 \) (note that \( a_{e,1} \) cannot be matched to an agent it prefers to \( a_{e,4} \) by the induction hypothesis and as \( a_{e,1} \) must be matched to an agent \( a_{e,4} \) for some \( j \in [m] \) and \( q \in \{2, 4\} \); the same holds for \( a_{e,4} \)), a contradiction to \( M_2 \) being stable. If \( M_2 \) contains neither \( \{a_{e,3}, a_{e,2}\} \) nor \( \{a_{e,3}, a_{e,4}\} \), then \( \{a_{e,3}, a_{e,2}\} \) blocks \( M_2 \), a contradiction to \( M_2 \) being stable.

Let us fix some color \( c \in [\ell] \). Since \( M_2 \) matches no agent from a vertex-selection gadget to an agent from the edge gadget, it follows that \( M_2 \) matches every agent \( a_{c,1}^{p,1} \) or \( a_{c,3}^{p,3} \) to an agent \( a_{c,2}^{p,2} \) or \( a_{c,4}^{p,4} \). If there existed an agent \( a_{c,2}^{p,2} \) with \( q_1 \in \{1, 3\} \) which this does not hold, then there also exists an agent \( a_{c,4}^{q,4} \) with \( q_2 \in \{2, 4\} \) which is not matched to an agent \( a_{c,3}^{q',3} \) with \( q' \in \{1, 3\} \) and \( a_{c,2}^{p,2}, a_{c,4}^{q,4} \) forms a blocking pair. If \( M_2(a_{c,1}^{p,1}) = a_{c,4}^{q_1} \) for some \( i \in [\nu m] \), then \( \{a_{c,1}^{p,1}, a_{c,4}^{q_1}\} \) is incident to a vertex of color \( c \) blocks \( M_2 \), contradicting the stability of \( M_2 \). Thus we have \( M_2(a_{c,1}^{p,1}) = a_{c,2}^{q_2} \) for some \( i \in [\nu m] \). We now continue by proving a more detailed statement about \( M_2 \).

**Claim 3.** For each \( c \in [\ell] \), we have \( M_2(a_{c,1}^{p,1}) = a_{c,2}^{q_2} \) for some \( i \in [\nu m] \). Moreover, we have \( \{a_{c,1}^{p,1}, a_{c,2}^{q_2}\} \subseteq \{a_{c,3}^{q_3,3}, a_{c,4}^{q_4,4}\} : j \in [\nu m + 1 - i] \cup \{a_{c,1}^{p,1}, a_{c,2}^{q_2}\} \subseteq M_2. \)

**Proof.** We have already observed above that the first part holds and now argue that the second part follows from the first part.

Note that \( M_2 \) cannot contain pairs \( \{a_{c,1}^{p,1}, a_{c,2}^{q_2}\} \) and \( \{a_{c,3}^{q_3,3}, a_{c,4}^{q_4,4}\} \) with \( p \in \{1, 3\} \), \( q \in \{2, 4\} \), and both \( j > j' \) and \( s < s' \), since otherwise \( \{a_{c,1}^{p,1}, a_{c,2}^{q_2}\} \) would be blocking. Further note that for every \( j > i \), agent \( a_{c,2}^{j,2} \) needs to be matched to an agent \( a_{c,1}^{j',1} \) for some \( j' \in [\nu m] \), as otherwise it is matched to some \( a_{c,3}^{q_3,3} \) and \( M_2(a_{c,2}^{j,2}, a_{c,2}^{j',2}) \) would be blocking. Lastly, note that, for every \( j < i \), agent \( a_{c,2}^{j,2} \) needs to be matched to an agent \( a_{c,3}^{j',3} \) for some \( j' \in [\nu m] \), as otherwise \( \{a_{c,1}^{p,1}, a_{c,2}^{q_2}\} \) would be blocking. From these three observations it follows that \( M_2 \) contains pair \( \{a_{c,1}^{p,1}, a_{c,2}^{q_2}\} \) for every \( j \in [\nu m + 1 - i] \).

It follows that only \( i - 1 \) agents \( a_{c,2} \) can be matched to some \( a_{c,3} \). Thus, in case that \( j < i \) where agent \( a_{c,2}^{j,2} \) is matched to an agent \( a_{c,3}^{j',3} \) for some \( j' \in [\nu m] \) we can conclude that \( j' = \nu m + 1 - i \): Otherwise there exists some \( j'' > j' \) such that \( a_{c,3}^{j'',3} \) is not matched to some \( a_{c,2} \) and thereby \( \{a_{c,1}^{p,1}, a_{c,2}^{q_2}\} \in M_2 \) for some \( t \in [\nu m] \). In this case, \( \{a_{c,1}^{p,1}, a_{c,2}^{q_2}\} \) would be a blocking pair. Consequently, \( M_2 \) contains pair \( \{a_{c,3}^{j,3}, a_{c,2}^{j,2}\} \) for every \( j \in [\nu m - i + 2, \nu m] \).

The remaining \( i - 1 \) agents \( a_{c,1}^{p,1} \) as well as the remaining \( \nu m - i + 1 \) agents \( a_{c,3}^{q_3,3} \) need to be matched to agents \( a_{c,2}^{q_2} \) by \( M_2 \). It is straightforward to see that \( M_2 \) contains pairs \( \{a_{c,3}^{j,3}, a_{c,2}^{j,2}\} \) for \( j \in [\nu m + 1 - i] \) and \( \{a_{c,1}^{p,1}, a_{c,2}^{q_2}\} \) for \( j \in [\nu m - i + 2, \nu m] \).

Let \( M_2 \) contain pair \( \{a_{c,1}^{p,1}, a_{c,2}^{q_2}\} \) for an edge \( e_x = \{v_x^c, v_x^d\} \) with \( y, z \in [\nu] \) and \( v_x^c \in V^c \) and \( v_x^d \in V^d \). Then both \( a_{c,(x-1)\nu+1}^{p,1} \) and \( a_{c,(x-1)\nu+2}^{p,2} \) prefer their partner in \( M_2 \) to \( a_{c,1}^{p,1} \), as otherwise they form a blocking pair for \( M_2 \) together with \( a_{c,1}^{p,1} \). From \( a_{c,(x-1)\nu+1}^{p,1} \) preferring its partner in \( M_2 \) to \( a_{c,1}^{p,1} \), it follows that \( a_{c,(x-1)\nu+1}^{p,1} \) is matched to some agent \( a_{2} \). By Claim 3 we get that \( M_2(a_{c,(x-1)\nu+1}^{p,1}) = a_{c,(x-1)\nu+1+(i-1)}^{c,2} \) for some \( i \in [\nu m] \) with \( i \leq (m - x + 1)\nu \). As \( a_{c,(x-1)\nu+1}^{p,1} \) prefers \( a_{c,(x-1)\nu+1+(i-1)}^{c,2} \) to \( a_{c,1}^{p,1} \) it needs to hold that \( (x - 1)\nu + 1 + (i - 1) \leq (x - 1)\nu + y \).
which implies \( i \leq y \). Moreover, by Claim 3 we get that \( M(a_{xv}^i) = a_{xv-i+1} \) (as clearly \( xv - i + 1 \leq mv + 1 - i \)). As \( a_{xv}^i \) prefers \( a_{xv-i+1} \) to \( a_{xv-i} \) which is directly after \( a_{xv-y+1} \) in the preferences of \( a_{xv}^i \) it needs to hold that \( xv - i + 1 \leq xv - y + 1 \) which is equivalent to \( y \leq i \). Thus, we have \( i = y \), i.e., the corresponding vertex gadget needs to select the vertex \( v_y^i \). Using symmetric arguments for \( a_{xv}^d \) and \( a_{xv}^{d^1} \), we have that for every edge \( e_x \) with \( \{e_x, e_{x,4}\} \in M_2 \), both endpoints have to be selected by vertex-selection gadgets.

The only edges which may contribute to \( M_1 \cap M_2 \) are \( \{a_{e,1}, a_{e,4}\} \) and \( \{a_{e,2}, a_{e,3}\} \) for some \( e \in E \). By the definition of \( k \), there must be at least \( \binom{k}{2} \) edges \( e \) such that \( \{a_{e,1}, a_{e,4}\}, \{a_{e,2}, a_{e,3}\} \in M_2 \). Using the above arguments it follows that the endpoint of these edges form a multicolored clique.

Containment of this problem in XP is an intriguing open question; in other words, is there a polynomial-time algorithm if the number of master lists is constant?

### 5.3.2 Incremental Stable Marriage with Ties

Recall that ISM-T is polynomial-time solvable if agents have complete preferences derived from one weak master list (Proposition 4). Motivated by this, we now ask whether ISM-T parameterized by the number of master lists is fixed-parameter tractable for agents with complete preferences (similar to Section 5.3.1 for ISR). Using a similar but slightly more involved reduction than for Theorem 5 for ISR, we show that this problem is \( W[1]\)-hard parameterized by the number of master lists.

**Theorem 6.** ISM-T is \( W[1]\)-hard parameterized by the minimum number \( p \) such that in \( P_2 \) the preferences of each agent can be derived from one of \( p \) weakly ordered preference lists, even if in \( P_1 \) as well as in \( P_2 \) all agents have complete preferences.

Similar to ISR, it remains open whether ISM-T for a constant number of master lists is polynomial-time solvable or NP-hard.

To show Theorem 6 we adapt the reduction from above for ISR. The underlying idea is that removing the pairs \( \{a_{p+2}, a_{e,1}\} \) for some \( c \in [\ell] \), \( p \in [\nu m] \), and \( e \in E \) (and some pairs which appear only in the \( \ldots \) part of the preferences) from the constructed ISR instance in Section 5.3.1 already results in an ISM instance. Moreover, adding pairs \( \{a_{e,2}, a_{e,4}\} \) to the resulting ISM instance still maintains bipartiteness. Thus, if we achieve that \( a_{e,4} \) is matched “badly” if and only if \( a_{e,1} \) is matched “badly” then \( a_{e,1} \) together with \( a_{e,4} \) can perform the role of \( a_{e,1} \) in our previous reduction. To achieve this, we change the preferences of the edge gadget such that \( a_{e,2} \) is indifferent between \( a_{e,1} \) and \( a_{e,3} \), agent \( a_{e,3} \) is indifferent between \( a_{e,2} \) and \( a_{e,4} \), and \( a_{e,4} \) prefers \( a_{e,3} \) to \( a_{e,1} \).

However, these changes lead to another problem: A stable matching could now match \( a_{e,1} \) or \( a_{e,4} \) to an agent from a vertex-selection gadget. In order to prevent this, we will use another gadget which we call Forbidden Pairs Gadget. The basic idea behind this gadget is the same as behind the forced pair gadget from Proposition 1. We can replace a pair \( \{a, a'\} \) by a long “path”, which makes adding \( \{a, a'\} \) to \( M_2 \) result in many pairs from \( M_1 \) not being contained in \( M_2 \), and thus, it is too “expensive” to add \( \{a, a'\} \) to \( M_2 \). The forbidden pairs gadget is very similar to the forced pair gadget from Proposition 1 however, it consists of \( s \) (where \( s \) is a sufficiently large number) instead of \( k + 1 \) many repetitions of the parallel pairs gadget by Cechlárová and Fleiner [9]. Matching \( M_1 \) will contain pairs \( \{a^{1b}, a^{m}\} \) and \( \{a^{1t}, a^{m}\} \) for every of the parallel pairs gadgets. Thus, taking the pair modeled by the forbidden pairs gadget into \( M_2 \) results in \( s \) less pairs shared with \( M_1 \).
We will later replace for each $e \in \ell$, for each $e \in E$, and $p \in [\nu m]$ the pairs $\{a_{e,1}, a_{p,1}\}$ and $\{a_{e,4}, a_{p,2}\}$ by such a forbidden pair gadget. We now show how to construct such a forbidden pairs gadget and that a constant number of master lists are enough to cover all the agents from the introduced forbidden pairs gadgets.

**Lemma 18.** Let $\mathcal{I} = (A, \mathcal{P}_1, \mathcal{P}_2, M_1, k)$ be an ISM-T instance where in $\mathcal{P}_1$ and $\mathcal{P}_2$ all agents have complete preferences. Let $F = \{(v_1, w_1), \ldots, (v_r, w_r)\}$ be contained in a set of pairs such that $F \cap M_1 = \emptyset$. Further assume that the preferences of $v_i$ for each $i \in [r]$ are derived from the same master list in $\mathcal{P}_2$, the preferences of $w_i$ for each $i \in [r]$ are derived from the same master list in $\mathcal{P}_2$, and for each $k \in [r]$, it holds that $v_i \succ_w v_j$ and $w_i \succ_w w_j$ for all $i < j \in [r]$. Then, by adding a so-called forbidden pairs gadget, one can construct an ISM-T instance $\mathcal{I}' = (A', \mathcal{P}_1', \mathcal{P}_2', M_1', k')$ using only $O(1)$ additional master lists such that the following holds:

There is a stable matching $M_2$ in $\mathcal{P}_2$ with $M_2 \cap F = \emptyset$ and $|M_1 \Delta M_2| \leq k$ if and only if there is a stable matching $M'_2$ in $\mathcal{P}'_2$ with $|M_1' \Delta M_2'| \leq k'$.

**Proof.** The forbidden pairs gadget consists of $s$ parts; for our purposes it suffices to set $s := |A| + 1$. The forbidden pairs gadget contains agents $a_{i,j}^{0}, a_{i,j}^{1}, a_{i,j}^{2}, a_{i,j}^{3}, a_{i,j}^{4}, a_{i,j}^{5}, a_{i,j}^{6}, a_{i,j}^{7}$ for $i \in [r]$ and $j \in [s]$. For each $i \in [r]$, the preferences of the agents from the forbidden pairs gadget in $\mathcal{P}_2'$ are as follows (see Figure 4 for an illustration).

For each $i \in [r]$, the preferences of the agents from the forbidden pairs gadget in $\mathcal{P}_2'$ are as follows (see Figure 4 for an illustration).

Furthermore, for each $i \in [r]$, agent $a_{i,1}^{0}$ replaces $w_i$ in the preferences of $v_i$, and $a_{i,s}$ replaces $v_i$ in the preferences of $w_i$. For each agent $a \in A$, its preferences are completed by adding all preferences which are not contained in the preferences of $a$ so far at the end of the preferences of $a$ (in an arbitrary order).

Concerning the matching $M_1'$, we add to $M_1$ the pairs $\{a_{i,j}^{0}, a_{i,j}^{1}\}, \{a_{i,j}^{2}, a_{i,j}^{3}\}, \{a_{i,j}^{4}, a_{i,j}^{5}\}, \{a_{i,j}^{6}, a_{i,j}^{7}\}$ for $i \in [s], j \in [r]$. Moreover, we extend the matching to be a perfect matching where each currently unmatched agent is matched to an agent that appears in the $\succ (\text{rest})$ part of its preferences in $\mathcal{P}_2'$. 

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For every $i$ to overlap in the construction from Theorem 5 to prove an analogous result for

$\quad$I

Figure 4: The replacement for the number of pairs contained in a blocking pair shows that

$\quad$\[ |w_1| ≤ |v_1| \]

We get from matching $\succ$ matched to an agent from the

$\quad$\[ \text{rest} \]

if the first case applies we directly get that

$\quad$\[ \text{rest} \]

Thus the first case never applies. Let

$\quad$\[ \text{rest} \]

Given a matching $M_1$ in $\mathcal{P}_2$ with $F ∩ M_2 = \emptyset$ and $|M_1 ∩ M_2| ≥ t$, we construct $M_2^\prime$ as follows. For every $i ∈ [r]$,

$\quad$\[ \text{rest} \]

if $v_i$ prefers $w_i$ to $M_2(v_i)$, then we add to $M_2$ pairs $\{a_{i,j}^l, a_{i,j}^m\}$, $\{a_{i,j}^r, a_{i,j}^m\}$, and $\{a_{i,j}^l, a_{i,j}^r\}$ for $j ∈ [s]$, and

$\quad$\[ \text{rest} \]

otherwise, we add pairs $\{a_{i,j}^l, a_{i,j}^r\}$, $\{a_{i,j}^m, a_{i,j}^r\}$, and $\{a_{i,j}^l, a_{i,j}^m\}$ for $j ∈ [s]$.

We get from $|M_1 ∩ M_2| ≥ t$ that $|M_2^\prime ∩ M_1^\prime| ≥ t'$. Showing by induction on $i ∈ [r]$ that $a_{i,j}$ is not contained in a blocking pair shows that $M_2^\prime$ is stable in $\mathcal{P}_2^\prime$.

Given a stable matching $M_2^\prime$ in $\mathcal{P}_2^\prime$, it is easy to verify (by induction on $i$) that for any $i ∈ [r]$, matching $M_2^\prime$ contains either

1. pairs $\{a_{i,j}^l, a_{i,j}^r\}$, $\{a_{i,j}^l, a_{i,j}^m\}$ for $j ∈ [s]$, pair $\{a_{i,j}^l, a_{i,j}^m\}$ for $j ∈ [s − 1]$ and

$\quad$\[ \text{rest} \]

pair $\{a_{i,j}^m, w_i\}$.

2. pairs $\{a_{i,j}^l, a_{i,j}^m\}$, $\{a_{i,j}^l, a_{i,j}^m\}$, and $\{a_{i,j}^l, a_{i,j}^m\}$ for $j ∈ [s]$, or

3. pairs $\{a_{i,j}^l, a_{i,j}^m\}$, $\{a_{i,j}^l, a_{i,j}^m\}$, and $\{a_{i,j}^l, a_{i,j}^m\}$ for $j ∈ [s]$.

Note that it is further easy to see that no agent from the forbidden pairs gadget can be matched to an agent from the $\succ$ (rest) part of its preferences in a stable matching in $\mathcal{P}_2^\prime$. Thus, if the first case applies we directly get that $|M_2^\prime ∩ M_2^\prime| ≤ |A| + (r − 1)s < t'$, a contradiction. Thus the first case never applies. Let $M_2$ be the matching $M_2^\prime$ restricted to the agents from $A$. It is easy to see that $M_2$ is stable in $\mathcal{P}_2$. Moreover, as edges from the forbidden edge gadget can contribute at most $rs$ to $|M_2^\prime ∩ M_2^\prime|$ from $|M_2^\prime ∩ M_2^\prime| ≥ t + rs$ it follows that $|M_1 ∩ M_2| ≥ t$.

We will call the gadget constructed in Lemma 15 “forbidden pairs gadget”.

Having described the forbidden pairs gadget, we now formally describe how we adapt the construction from Theorem 5 to prove an analogous result for ISM-T. For this, let $\mathcal{I}' = (A', \mathcal{P}_1', \mathcal{P}_2', M_1')$ be the ISR instance constructed in the proof of Theorem 5. We will now describe how we modify $\mathcal{I}'$ to arrive at an instance $\mathcal{I}$ of ISM-T. The bipartition in $\mathcal{I}$ will
be \( U = \{a^c_{i,1}, a^d_{i,3} : i \in [v m], c \in [\ell]\} \cup \{a_{c,2}, a_{c,4} : c \in E\} \) and \( W = \{a^c_{i,2}, a^d_{i,4} : i \in [v m], c \in [\ell]\} \cup \{a_{e,1}, a_{e,3} : e \in E\} \).

We start by describing how to construct the preferences of the agents in the second preference profile. For this, we start with their preferences in \( P'_2 \). For each \( i \in [m] \), let \( e_i = \{v, w\} \in E \) with \( v \in V^c \) and \( w \in V^d \). We replace the master lists of \( a_{e_i,1}, a_{e_i,2}, a_{e_i,3}, \) and \( a_{e_i,4} \) as follows:

\[
\begin{align*}
     a_{e,1} : a_{e,1,2} &> a_{e,1,3} > a_{e,1,4} > a_{e,2,2} > a_{e,2,3} > a_{e,2,4} > \cdots > a_{e,m,2} \\
                      &> a^c_{e,v(m-1)+1,1} > a^d_{e,v(m-1)+1,1} > a_{e,m,4} > (\text{rest}) \\
     a_{e,2} : a_{e,1,3} &> a_{e,1,4} > a_{e,2,3} > a_{e,2,2} > \cdots > a_{e,m,3} > a_{e,m,1} > (\text{rest}) \\
     a_{e,3} : a_{e,1,1} &> a_{e,1,2} > a_{e,2,1} > a_{e,2,2} > \cdots > a_{e,m,2} > a_{e,m,1} > (\text{rest}) \\
     a_{e,4} : a_{e,1,3} &> a^c_{e,v,2} > a^d_{e,v,2} > a_{e,1,1} > a_{e,2,3} > a_{e,2,2} > a_{e,2,1} > \cdots > a_{e,m,3} \\
                      &> a^c_{e,m,2} > a^d_{e,m,2} > a_{e,m,1} > (\text{rest}) \\
\end{align*}
\]

That is, compared to the construction from Theorem 5, we delete all agents \( a^c_{e,2} \) and \( a^d_{e,2} \) from the preferences of \( a_{e,1} \) and add them to the preferences of \( a_{e,4} \). Furthermore, \( a_{e,2} \) is indifferent between \( a_{e,1} \) and \( a_{e,3} \) and \( a_{e,3} \) is indifferent between \( a_{e,2} \) and \( a_{e,4} \). Finally, \( a_{e,4} \) now prefers \( a_{e,3} \) to \( a_{e,1} \).

Regarding the vertex-selection gadgets, for each \( c \in [\ell] \) and \( i \in [v m] \), the preferences of \( a^c_{i,1} \), \( a^c_{i,3} \) and \( a^d_{i,3} \) remain unchanged, while agent \( a^c_{i,2} \) replaces \( a_{e,1} \) by \( a_{e,4} \) in its preferences. Note that the modified instance is now indeed bipartite and that we denote the constructed preference profile as \( P^*_2 \) (in the following, we will use this preference profile to construct an intermediate ISM-T instance \( I^* \)). Moreover, we construct the initial matching \( M^*_1 \) in the modified instance such that it contains the pairs \( \{ (a_{e,1}, a_{e,4}) : e \in E \} \) and such that all other agents are matched to agents that appear in the \( (\text{rest}) \) part of their preferences. As the difference between the first and second preference profile can be unbounded, we construct \( P^*_1 \) such that \( M^*_1 \) is stable. As all preferences are complete, the second matching needs to be perfect. Thus, we can set the budget \( k^* \) in such a way that \( M^*_1 \) and the solution need to share at least \( \frac{k^*}{2} \) edges. Thereby, we arrive at an ISM-T instance \( I^* = (A, P^*_1, P^*_2, M^*_1, k^*) \). Note in particular that in \( P^*_2 \) the preferences of all agents can still be derived from \( O(\ell) \) master lists.

We now continue by modifying \( I^* \) by forbidding some edges. Specifically, for every \( c \in [\ell] \), \( e \in E \), and \( p \in [v m] \), we “forbid” the pairs \( \{a_{e,1}, a^c_{p,1}\} \) and \( \{a_{e,4}, a^d_{p,2}\} \). We do this by iteratively applying Lemma 13 to \( I^* \). Specifically, for increasing \( c \in [\ell] \), we transform the previously construct ISM-T instance by applying Lemma 18 with \( F = \{a_{e,1, c^p_{1}}, a_{e,4, c^p_{2}} \mid p \in [v m], e \in E\} \). We call the resulting ISM-T instance \( I = (A, P_1, P_2, M_1, k) \), which is now our final constructed ISM-T instance. Note that by applying Lemma 18 \( \lceil \ell \rceil \) times we have only added \( O(\ell) \) many master lists, thus still arriving at overall \( O(\ell) \) master lists. By Lemma 18 we can directly conclude the following:

**Lemma 19.** There is a stable matching \( M^*_2 \) in \( P^*_2 \) without a pair from \( \{a_{e,1, c^p_{1}}, a_{e,4, c^p_{2}} \mid p \in [v m], e \in E, c \in [\ell]\} \) for which \( |M^*_1 \setminus M^*_2| \leq k^* \) if and only if there is a stable matching \( M^*_2 \) in \( P^*_2 \) with \( |M_1 \setminus M_2| \leq k \).

We are now ready to prove the correctness of the reduction, thereby proving Theorem 6 stated in the beginning of this section:

**Theorem 6.** ISM-T is \( W[1] \)-hard parameterized by the minimum number \( p \) such that in \( P_2 \) the preferences of each agent can be derived from one of \( p \) weakly ordered preference lists, even if in \( P_1 \) as well as in \( P_2 \) all agents have complete preferences.
Proof. We already described how to adapt the reduction from the proof of Theorem \([5]\). It remains to show its correctness.

Forward Direction

Let \(X\) be a multicolored clique. We construct a stable matching \(M_2^*\) the same way as in the proof of Theorem \([5]\). Note that in fact \(M_2^*\) only contains pairs that occur in \(\mathcal{I}^*\) and thus in particular respects that the instance is bipartite. Further, using the same arguments as in Theorem \([5]\) we can prove that \(M_2^*\) is stable in \(\mathcal{P}_2^*\) and that it overlaps with \(M_1^*\) in at least \(\binom{\ell}{2}\) edges. Thus we have that \(|M_1^* \Delta M_2^*| \leq k^*\). As \(M_2^*\) contains no pairs from \(\{(a_{e,1}, a_{e,2}^c), (a_{e,4}, a_{p,2}^c)\} | p \in [\nu m], e \in E, c \in [\ell]\}\). using Lemma \([19]\) it follows that \(\mathcal{I}\) is a yes instance.

Backward Direction

Let \(M_2\) be a stable matching in \(\mathcal{P}_2\) with \(|M_1 \Delta M_2| \leq k\). Then by Lemma \([19]\) we get a stable matching in \(\mathcal{P}_2^*\) with \(|M_1^* \Delta M_2^*| \leq k^*\) which does not contain any edges from \(\{(a_{e,1}, a_{e,2}^c), (a_{e,4}, a_{p,2}^c)\} | p \in [\nu m], e \in E, c \in [\ell]\}\). Specifically, by the design of \(k^*\) we have that \(M_1^*\) and \(M_2^*\) overlap in \(\binom{\ell}{2}\) edges.

Before we proceed, we show that in \(M_2^*\) no agent is matched to an agent that appears in the \(>\) \((rest)\) part of its preferences in \(\mathcal{P}_2^*\).

Lemma 20. Let \(M_2^*\) be a stable matching in \(\mathcal{P}_2^*\) with \(|M_1^* \Delta M_2^*| \leq k^*\). Then any agent \(a \in A^*\) is matched to an agent appearing before \(\succ \) \((\text{rest})\) in its preferences. Moreover, for any edge \(e \in E\), matching \(M_2^*\) contains either \(\{a_{e,1}, a_{e,2}\}\) and \(\{a_{e,3}, a_{e,4}\}\) or \(\{a_{e,1}, a_{e,4}\}\) and \(\{a_{e,2}, a_{e,3}\}\).

Proof. We first show the second part of the statement. By Lemma \([18]\) for every \(r \in [m]\) and \(e' \in E\), \(a_{e',1}\) is matched to neither \(a_{e',r+1}\) nor \(a_{e',r+1}^d\). Similarly, for every \(r \in [m]\) and \(e' \in E\), \(a_{e',4}\) is matched to neither \(a_{e',2}\) nor \(a_{e',2}^d\). Assume towards a contradiction that there exists some \(a_{e,i}\) for \(j \in \{1, 3\}\) with \(M_2^*(a_{e,j}) \notin \{a_{e,1}, a_{e,3}\}\), where \(i\) is minimal. Then using our above observations, there also exists some \(j' \in \{2, 4\}\) with \(M_2^*(a_{e,j'}) \notin \{a_{e,1}, a_{e,3}\}\). Then \(\{a_{e,i,j}, a_{e,j'}\}\) blocks \(M_2^*,\) a contradiction to the stability of \(M^*\).

To prove the first part, it remains to consider agents \(a_{e,j'}^c\). The proof is analogous to the case of an agent \(a_{e,i,j}\): By Lemma \([18]\) \(a_{e,1}\) is not matched to \(a_{e,1}\) for every \(c \in [\ell]\), \(p \in [\nu m]\) and \(e \in E\). Similarly, \(a_{e,4}^c\) is not matched to \(a_{e,4}\) for every \(c \in [\ell]\), \(p \in [\nu m]\) and \(e \in E\). Assume towards a contradiction that there exists some \(a_{e,p}^c\) for \(j \in \{1, 3\}\) with \(M_2^*(a_{e,p}) \notin \{a_{e,2}, a_{e,4}\}\), where \(p\) is minimal. Then, using our above observations, there also exists some \(j' \in \{2, 4\}\) with \(M_2^*(a_{e,p}) \notin \{a_{e,2}, a_{e,4}\}\). Then \(\{a_{e,p,j}, a_{e,p,j'}\}\) blocks \(M_2^*,\) a contradiction to the stability of \(M^*\). \(\square\)

Using this and our above observation on forbidden edges it follows that \(M_2^*\) contains only edges inside vertex-selection and inside edge gadgets. Note that for each edge \(e_r = \{v_p^c, v_q^d\}\), matching \(M_2^*\) can contain \(\{a_{e,1}, a_{e,4}\}\) and \(\{a_{e,2}, a_{e,3}\}\) only if agent \(a_{vr-1}^{c(t-1)+1}\) is matched at least as good as \(a_{vr-1}^{c(t-1)+1}\) and agent \(a_{vr-2}^{d(t-1)+1}\) is matched at least as good as \(a_{vr-2}^{d(t-1)+1}\). Symmetrically, \(a_{vr-2}^{d(t-1)+1}\) must be matched at least as good as \(a_{vr-2}^{d(t-1)+1}\) and agent \(a_{vr-1}^{d(t-1)+1}\) would block \(M_2^*\). From here on, the proof is along the lines of the proof of Theorem \([5]\). \(\square\)
6 Conclusion

Among others, answering two open questions of Bredereck et al. [7], we have contributed to the study of the computational complexity of adapting stable matchings to changing preferences. From a broader algorithmic perspective, in particular, the “propagation” technique from our XP-algorithm for the number of swaps, and the study of the number of different preference lists/master lists as a new parameter together with the needed involved constructions for the two respective hardness proofs could be of interest.

There are several possibilities for future work. As direct open questions, for the parameterization by the number of outliers, we do not know whether ISM-T or ISR-T are fixed-parameter tractable. Moreover, it remains open whether ISR or ISM-T with complete preferences is polynomial-time solvable for a constant number of master lists.

While we have already considered some (new) parameters measuring the distance of a preference profile from the case of a master list (e.g., the number of outliers and the number of master lists), there also exist various other possibilities to measure the similarity (or, more generally, the structure) of the preferences. Possible additional parameters to measure the similarity to a master lists that might be worthwhile to explore further include, for instance, the maximum or average distance of an agent’s preference list from the master list.

Acknowledgments

NB was supported by the DFG project MaMu (NI 369/19) and by the DFG project ComSoc-MPMS (NI 369/22). KH was supported by the DFG Research Training Group 2434 “Facets of Complexity” and by the DFG project FPTinP (NI 369/16).
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