Universe’s memory and spontaneous coherence in loop quantum cosmology

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The quantum bounce a priori connects several (semi)classical epochs of Universe evolution, however determining if and how well the semiclassicality is preserved in this transition is highly nontrivial. We review the present state of knowledge in that regards in the isotropic sector of loop quantum cosmology. This knowledge is next extended by studies of an isotropic universe admitting positive cosmological constant (featuring an infinite chain of large Universe epochs). It is also shown, that such universe always admits a semiclassical epoch thanks to spontaneous coherence, provided it is semiclassical in certain constant of motion playing the role of energy.

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I. INTRODUCTION

Over one and a half decade since its birth the field of loop quantum cosmology (LQC) has experienced tremendous progress [1–3]. One of the flag features of the models developed within the area is the so called quantum bounce – a high energy (Planck order) epoch of an universe evolution providing a deterministic connection between two low energy epochs (when given universe adheres to the rules of classical General Relativity) [4].

Since the process of the bounce is of pure quantum nature, bearing in fact a lot of similarities with the scattering process, it is a-priori not obvious whether the universe which is semiclassical before the bounce will evolve into a semiclassical one after it. Conversely, the same hold in regards of the question whether the expanding post-bounce semiclassical universe (like the observed one) had necessarily a semiclassical past. The numerical studies which originally established the bounce have shown strong indication that the answer to both these questions is in the affirmative, however by their very nature these studies could only cover a tiny (non-generic) portion of the space of physical states. As a consequence, the problem of semiclassicality preservation across the bounce is far from trivial even in the simplest isotropic sector of the theory and for a time was an arena of disagreement between researchers [5–7].

Over time, several groups addressed this issue using both analytical and numerical methods. At present it is established, that the semiclassicality is indeed preserved at least in the isotropic sector of the theory for the models featuring single bounce. There are also strong indications that this feature carries to the anisotropic homogeneous sector. Here we present a short review of the results which finally led to this conclusion.

It it important to note though, that the vast majority of these studies features a particular model (Friedman-Robertson-Walker (FRW) universe admitting massless scalar field). Furthermore, up to date there are no substantial studies (in the considered aspect) of models featuring infinite chains of low energy epochs connected by a sequence of bounces. Since the universe admitting positive cosmological constant (which is a feature of our Universe established with quite strong observational evidence) falls within this category, it is critical that the results are extended to it. The second part of this article is dedicated to this issue. There, the preservation result is extended to the case of (again isotropic) FRW universe with massless scalar field and positive cosmological constant.

The presence of an infinite chain of low energy epochs with generically varying (between epochs) semiclassicality properties leads to an interesting question: how generic the semiclassical sector is within the whole physical Hilbert space? Will an arbitrarily quantum universe eventually admit a semiclassical epoch? This question is addressed in the last part of the article, again in the context of FRW universe with positive cosmological constant and massless scalar field. We indeed show, that, due to a process known in quantum mechanics as spontaneous coherence, for a universe to admit a semiclassical epoch it is enough that the quantum state representing it is semiclassical with respect to an observable (scalar field momentum) representing a constant of motion.

Since the vast majority of the presented material regards the specific model of isotropic universe with massless scalar field with the notion of physical (time) evolution tied to this field we start with briefly introducing the details of this model.

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II. ISOTROPIC SECTOR OF LQC

The particular model we will focus on is the flat (isotropic) FRW universe with massless scalar field and cosmological constant. We will follow the specific Hamiltonian formulation and quantization procedure as specified in [8, 9].

A. The Hamiltonian formulation

Our starting point is the standard Einstein-Hilbert action of gravity coupled to matter (in our case massless scalar field) with partial gauge fixing using the (physically distinguished) foliation by homogeneity surfaces. In the chosen gauge the spacetime metric takes the well known form

\[ g = -N^2(t)dt^2 + a^2(t)q^\alpha, \]  

(2.1)

where \( N \) is the lapse function, \( a \) is a scale factor and \( q^\alpha \) is the fixed, positive definite, flat metric (constant in the co-moving coordinates) known as fiducial metric. The natural 3 + 1 splitting is next employed in the transition to Hamiltonian formalism, however due to homogeneity and noncompactness of the slices the integrals representing the symplectic structure and Hamiltonian diverge. To build meaningful theory one thus introduces an infrared regulator—a cell \( V \) taken to be cubical with sides along co-moving coordinates—and restricts all integrals to it. The actual physical theory is then expected to emerge in the regulator removal limit defined by expanding \( V \) to fill entire slice.

The treatment follows that of LQG where one uses triads instead of 3-metric directly. In the case considered here one can again fix the gauge (triad orientation) such that the triad is determined by a single configuration variable \( v \) which encodes both the volume \( V \) of the cell \( V \) (with respect to physical spatial metric \( q := a^2(t)q^\alpha \)) and the orientation of the triad

\[ (\text{sgn } v) v = \frac{V}{2\pi\gamma\sqrt{\Delta}\ell_{Pl}^2} \equiv \frac{a^3V_o}{2\pi\gamma\sqrt{\Delta}\ell_{Pl}^2}, \]  

(2.2)

where \( \gamma \) is the Barbero-Immirzi parameter of LQG, \( \Delta = 4\pi\sqrt{3}\gamma\ell_{Pl}^2 \) is the so-called LQC area gap (see the next subsection) and \( V_o \) is the volume of \( V \) with respect to \( q^\alpha \). The canonically conjugate momentum (denoted by \( b \)) is on classical solutions given by

\[ b = \gamma\sqrt{\Delta}H \equiv \gamma\sqrt{\Delta}\frac{1}{a} \frac{da}{dt}. \]  

(2.3)

where \( H \) is the Hubble parameter and \( t \) is the proper (or cosmological) time.

For the scalar field, the basic canonical pair is the standard one \( \phi, p_\phi \). The total phase space is thus topologically \( \mathbb{R}^4 \) and equipped with basic Poisson brackets:

\[ \{b, v\} = \frac{2}{\hbar}, \quad \{\phi, p_\phi\} = 1. \]  

(2.4)

Following the standard procedure of building the Hamiltonian applied in LQG [10] one arrives to reduced algebra of constraints, however due to gauge choice specified earlier the only nontrivial generator of this algebra is the Hamiltonian constraint:

\[ C = p_\phi^2 - 3\pi\hbar^2G\ell_{Pl}^2 + \pi\gamma^2\Delta\hbar^2G \Lambda \phi^2 \approx 0, \]  

(2.5)

Since there is no explicit dependence on \( \phi \) there, the momentum \( p_\phi \) is a constant of motion. This in turn implies that on the dynamical trajectory the field \( \phi(t) \) is monotoneous function of cosmic time, thus is a good choice for an internal clock parametrising the evolution. In order to directly tie the Hamiltonian time to this field (which can be thought of as a kind of deparameterization procedure) we select the lapse function to be \( N = a^3 \) (thus following the so called SLQC prescription [8]).

B. LQC quantization

The loop quantization procedure of the specified model is presented in detail in [9] (which in turn follows the techniques of [8, 11] and [12]) and is an adaptation of the procedure used for full LQG, implementing the Dirac program of quantizing the theories with constraints. This program involves the following steps:
(i) **Kinematic level quantization:** Building a suitable representation of the (reduced) holonomy-flux algebra through GNS technique (with GNS spectrum providing the Hilbert space).

(ii) **Building a quantum constraint operator:** In this case the Hamiltonian constraint \( \hat{H} \) is expressed in terms of holonomies and fluxes via procedure of Thiemann regularization \[10\]. In particular, following the heuristic argument of consistency of LQC with full LQG the value of area gap in \[22\] is fixed as the lowest nonzero eigenvalue of LQG area operator (although more intricate examination of LQC-LQG connection may lead to different choices \[13\]).

(iii) **Physical level quantization:** In this final step one constructs the physical Hilbert space as kernel of quantum constraint operator and defines suitable algebra of Dirac observables through partial observable formalism (essentially a family of constants of motion parametrized by value of internal clock).

As a result of (i) one ends up with a kinematical Hilbert space being a tensor product \( \mathcal{H}_{\text{kin}} = \mathcal{H}_{\text{gr}} \otimes \mathcal{H}_\phi \), where \( \mathcal{H}_\phi = L^2(\mathbb{R}, \phi) \) and \( \mathcal{H}_{\text{gr}} = L^2(\mathbb{R}, d\mu_{\text{Bohr}}) \) (with \( \mathbb{R} \) being a Bohr compactification of the real line and \( d\mu_{\text{Bohr}} \) the Haar measure thereon). A convenient basis on \( \mathcal{H}_{\text{gr}} \) is provided by the eigenvectors of the operator \( \hat{v} \):

\[
\hat{v}|v\rangle = v|v\rangle, \quad \text{so that} \quad \hat{V}|v\rangle = (2\pi \gamma \sqrt{\Delta t_{\text{Pl}}} |v\rangle |v\rangle. \tag{2.6}
\]

In the volume representation states in \( \mathcal{H}_{\text{gr}} \) become wave functions \( \psi(v) \). To incorporate the fact that \( v \to -v \) is a large gauge transformation corresponding to the flip of the orientation of the physical triad they are taken to be symmetric \( \psi(v) = \psi(-v) \). Unlike in standard quantum mechanics (Schrödinger representation) the \( \psi(v) \) have support only on a countable set of points along the \( v \)-axis and their inner product is given by a sum

\[
\langle \psi|\psi' \rangle = \sum_{v \in \mathbb{R}} \bar{\psi}(v)\psi'(v). \tag{2.7}
\]

The regularization of step (ii) yields (after promoting the holonomies and fluxes involved to operators) the quantum Hamiltonian constraint operator of the form

\[
\hat{C} = 1 \otimes \partial^2_o + \Theta_A \otimes 1, \quad \Theta_A := \Theta_o - \pi G \gamma^2 \lambda^2 \Lambda v^2, \tag{2.8}
\]

where

\[
-[\Theta_o|v\rangle = f_+(v)\psi(v - 4) - f_o(v)\psi(v) + f_-(v)\psi(v + 4), \tag{2.9}
\]

with the coefficients \( f_{o, \pm} \) given by

\[
f_{\pm}(v) = (3\pi G/4) \sqrt{v(v \pm 4)} (v \pm 2), \quad f_o(v) = (3\pi G/2)v^2. \tag{2.10a}
\]

The operator \( \Theta_o \) is a second order difference operator with uniform steps of size \( v = \pm 4 \) well defined on the domain \( \mathcal{D} \) of finite linear combinations of \(|v\rangle\). Therefore, there is super-selection: one can investigate dynamics separately on uniform lattices in the \( v \)-space and each sector consisting of wave functions with support on any one of these lattices is preserved by the complete set of Dirac observables of interest. In this paper, we will restrict ourselves to the lattice \( \mathcal{L} = \{v = 4n, n \in \mathbb{Z}\} \) for simplicity as in LQC physical results are largely insensitive to the choice of the sector \[14\].

For technical reasons it is more convenient to work in the dual representation in which states are wave functions \( \psi(b) \) of the conjugate variable \( b \). While operators corresponding to \( b \) do not exist in the theory, mathematically one can define the transformation via Fourier series

\[
|F\psi\rangle = \frac{1}{2\sqrt{d}} \sum_{\xi_o \setminus \{0\}} |v\rangle^{-\xi_o/2} \psi(v) e^{\pm ivb}, \tag{2.11}
\]

where the point \( v = 0 \) was removed from the transform because the state with support just at \( v = 0 \) is dynamically decoupled from the orthogonal sub-space spanned by states which vanish at \( v = 0 \). Since \( \psi \) are supported on \( \mathcal{L}_0 \), their images \( F\psi \) are periodic in \( b \) with the period \( \pi \). Therefore one can restrict the support of the wave functions \( |F\psi\rangle \) just to the circle \( b \in [0, \pi] \), with the identification \( |F\psi\rangle(0) = |F\psi\rangle(\pi) \).

By inspection, the elementary operators \( \hat{v} \) and \( \hat{N}_\mu \) defined by

\[
\hat{v}|v\rangle = v|v\rangle, \quad \text{and} \quad \hat{N}_\mu|v\rangle = |v + \mu\rangle, \tag{2.12}
\]

in the \( v \) representation are transformed to

\[
\hat{v} = 2i\partial_b, \quad \text{and} \quad \hat{N}_\mu = e^{-i\mu b/2}. \tag{2.13}
\]
in the $b$ representation. As a consequence, the operator $\Theta_\Lambda$ assumes the form

$$\Theta_\Lambda = -12\pi G \left[ (\sin(b)\partial_b)^2 - \text{sgn}(\Lambda)b_o^2\partial_b^2 \right],$$

(2.14)

in the $b$ representation, where $b_o := \gamma \sqrt{\Lambda\Delta/3}$.

The properties of the operator $\Theta_\Lambda$ depend on the value (sign) of the cosmological constant, thus the last step in quantization program has to be performed for each case $\Lambda = 0$, $\Lambda > 0$, $\Lambda < 0$ separately. Here we focus only on former two cases.

1. $\Lambda = 0$

The operator $\Theta_o$ is positive definite and essentially self-adjoint (thus generating a unique unitary evolution). Its spectrum $\text{Sp}(\Theta_o) = \mathbb{R}^+$ is continuous and nondegenerate (on symmetric sector). Per analogy to Klein-Gordon equation we restrict the physical Hilbert space to positive frequency states – satisfying

$$-i\partial_\phi \Psi(v, \phi) = \sqrt{\Theta_o}\Psi(v, \phi).$$

(2.15)

Thus, the relevant physical states are described by wave functions of the form

$$\Psi(v, \phi) = \int_0^{+\infty} dk \tilde{\Psi}(k)e_k(v)e^{i\omega(k)\phi},$$

(2.16)

where $\tilde{\Psi}$ is a spectral profile of the wave function, the dispersion relation is $\omega(k) = \sqrt{12\pi Gk}$ and $e_k$ are eigenbasis elements satisfying $\langle e_k|e_{k'}\rangle = \delta(k - k')$.

The physical inner product is

$$\langle \Psi|\Phi \rangle = \int_0^{+\infty} dk \tilde{\Psi}(k)\Phi(k).$$

(2.17)

As the physical observables it is convenient to select

(i) The scalar field momentum (analog of energy in KG equation) operator

$$\hat{p}_\phi = \sqrt{\Theta_o},$$

(2.18)

(ii) The family of ‘volume at given $\phi$’ operators defined by action

$$[\hat{V}_{\phi_o}\Psi](v, \phi) = 2\pi\gamma \sqrt{\Delta}\epsilon_{\text{Pl}}^{2}e^{i\Theta_o(\phi - \phi_o)}|v|\Psi(v, \phi_o).$$

(2.19)

2. $\Lambda > 0$

The case of positive cosmological constant is a bit more complicated. Since the observations indicate the cosmological constant order of magnitude $\Lambda \sim 10^{-120}\ell^{-2}_{\text{Pl}}$ it is safe to assume $b_o < 1$ in (2.14). In this case $\Theta_\Lambda$ is no longer positive definite (although Hamiltonian constraint still selects out its positive part). Furthermore, it is no longer essentially self-adjoint admitting instead a 1-parameter family of selfadjoint extensions. Each of these extensions has purely discrete spectrum consisting of isolated points selected out by condition (with dispersion relation $\omega(k) = C_\omega k$, see (2.22a) for the value of $C_\omega$)

$$\tan(k_n y_o) + \tanh(k_n(\pi - y_o))\tan(\beta) = 0,$$

(2.20)

where $\beta \in [0, \pi)$ labels the extensions, the constant $y_o$ is expressed in terms of the elliptic integral of the first kind

$$\pi y_o^{-1} := 1 + \sqrt{1 - b_o^2} \frac{F(\arcsin(b_o), 1/b_o)}{F(\pi/2 - \arcsin(b_o), 1/(1 - b_o^2))},$$

(2.21)

and the proportionality constant in the dispersion relation is

$$C_\omega := \sqrt{12\pi G\pi/x_M},$$

(2.22a)

$$x_M := \frac{1}{\sqrt{1 - b_o^2}}F(\pi/2 - \arcsin(b_o), 1/(1 - b_o^2)) + \frac{1}{b_o}F(\arcsin(b_o), 1/b_o^2).$$

(2.22b)
An important relation is the asymptotic behavior of $k_n$

$$k_n = \frac{(n\pi - \beta)}{\gamma} + O(e^{-2\pi n(\pi - \gamma)/\nu}).$$

(2.23)

This relation will play a crucial role in obtaining the results of sections IV and V.

The physical states have the form

$$\Psi(v, \phi) = \sum_{n=0}^{+\infty} \tilde{\Psi}_n e^{\beta_n(v)} e^{i\omega(k_n)\phi},$$

(2.24)

where again $\tilde{\Psi}_n$ is (this time discrete) spectral wave function profile and $e^{\beta_n}$ are (explicitly) normalized eigenfunctions of the (positive part of the) extension of $\Theta_L$ corresponding to given value of $\beta$.

As the set of physical observables one can use analogs of the ones specified in section II B, however, as even classically the trajectory $V(\phi)$ reaches infinity for finite $\phi$ the operators $\hat{V}_\phi$ would not preserve the Hilbert space. Therefore one is forced to use their compactified versions. Thus, finally we end up with

(i) The “energy” operator

$$\hat{\rho}_\phi = \sqrt{|\Theta_L|},$$

(2.25)

(ii) The family of 'compactified volume at given $\phi'$ operators defined by action

$$[\hat{\theta}_{\phi_0}\Psi](v, \phi) = \theta_K(v) e^{i\Theta_0(\phi - \phi_0)} \Psi(v, \phi_0),$$

(2.26)

where $\theta_K(v) := \arctan(|v|/K)$, with $K$ being positive constant of dimension of the volume (of which particular value can be selected arbitrarily).

III. PRESERVATION OF SEMICLASSICALITY: STATE OF THE ART SO FAR

As mentioned in the introduction the studies of semiclassicality focus almost entirely on the case of vanishing cosmological constant. There are essentially three lines of approach explored in the literature: (i) analytical studies in manageable prescriptions, (ii) numerical studies of selected classes of states, and (iii) estimates following from employing the interpretation of bounce as scattering. Let us start with the analytic approach.

A. Solvable prescription of LQC: analytical results

In general the loop quantization procedure presented in section II B features a series of ambiguities. Various ways of fixing them lead to many prescriptions of LQC, several of which have been explored in the literature (see [14] for their detailed comparative analysis). Most of them however require numerical methods to probe the state properties. The first prescription permitting reliable analytical treatment of the simplest case of universe with $\Lambda = 0$ is known as Solvable LQC [8] (this prescription is the one specified in section II B and further used in studies of sections IV and V). That prescription allowed to provide a strong estimate on the dispersion growth across the bounce for quite large class of states [5].

The key feature of the analysis was the fact, that (for $\Lambda = 0$) upon switching to $b$ representation and further reparametrizing the “momentum” coordinate $b$ to a new one

$$x := \ln(\tan(\sqrt{\Delta b}/2))$$

(3.1)

one reduces the Hamiltonian constraint (2.8) to explicit Klein-Gordon equation, which in turn yields the following form of physical states

$$\Psi(x, \phi) = \int_0^{+\infty} dk \tilde{\Psi}(k) \cos(kx) e^{i\omega(k)\phi},$$

(3.2)

with the dispersion relation $\omega(k)$ same as the one in sec. II B. Moreover, upon defining a simple transformation from the physical Hilbert space to certain auxiliary one, the relevant observables also take a quite simple analytic form. This allows to parametrize the quantum trajectories (evolution of expectation values of the observables selected in
sec. [11,12] and their dispersions) by a set of just 5 parameters – expectation values of a set of operators corresponding to constants of motion [8]. This set of parameters captures in particular the information on how the dispersion grows across the bounce.

The analysis of [2] focuses of the states which at some moment (value of \( \phi \)) had a support at \( x \in [x_o - \epsilon, x_o + \epsilon] \) for certain large value \( x_o \) (moment of evolution featuring low energy density) with \( \epsilon \ll x_o \). For this class of states it was shown, that

\[
\left| \lim_{\phi \to -\infty} - \lim_{\phi \to +\infty} \langle \Delta \hat{V}_\phi / \langle \hat{V} \rangle_\phi \rangle \right| \leq (1 + \delta)(e^{8\epsilon} - 1) \sim 8\epsilon(1 + \delta),
\]

where \( \delta := \lim_{\phi \to +\infty} (\Delta \hat{V}_\phi / \langle \hat{V} \rangle_\phi) \).

Found inequality is exact for selected class of states, however the requirement of compactness of the support is quite restrictive and in general is believed to be too restrictive to admit large semiclassical sector (with respect to sufficiently large family of physically relevant observables). The results of [2] can be however extended beyond that family at the cost of becoming estimates rather than exact inequalities.

This method, although strong and precise, strongly relies on the ability to cast the studied model as a very simple one (Klein-Gordon system). This is possible only for a very narrow family of scenarios in isotropic LQC like flat FRW universe with dust [13] (for any value of \( \Lambda \)) or with radiation [14] (for \( \Lambda = 0 \)) but so far has been impossible to extend even to the case of universe with massless scalar field and \( \Lambda \neq 0 \). Such scenarios require numerical analysis.

**B. Generalized Gaussian states: numerical studies**

In the pioneering work in which the bounce has been discovered as a feature of the model [11] the dynamics of quantum universe has been studied by purely numerical methods. The direct inspection of the large population of states have shown that its relative dispersions always satisfied the inequality

\[
\left| \frac{\Delta \hat{V}_\phi - \Delta \hat{V}_\pm}{\langle \hat{V} \rangle_\phi - \langle \hat{V} \rangle_\pm} \right| < \frac{\Delta \hat{p}_\phi}{\langle \hat{p} \rangle_\phi},
\]

throughout the evolution (that is for all probed values of \( \phi \)), where \( \langle \hat{V} \rangle_\pm := \lim_{\phi \to \pm\infty} (\hat{V})_\phi \). In all the studied cases the states which started as semiclassical (sharply peaked in selected observables) at given initial \( \phi = \phi_o \) remained so during the whole evolution. The feature of semiclassicality preservation has been subsequently confirmed (by direct inspection) for all cases of LQC dynamical evolution studied on the genuine quantum level: universe with spherical topology [17], with nonvanishing cosmological constant [3,18] and with Maxwell field as matter content [16]. However due to technical limitations in most of these cases the studies have been restricted to a finite number of examples corresponding to the Hamiltonian Gaussians, that is the states of spectral profile [4]

\[
\hat{\Psi} = \frac{1}{\sqrt{\pi \sigma}} e^{-(k-k_o)^2/2\sigma^2}.\]

Subsequently, in the case of flat FRW universe with massless scalar field and \( \Lambda = 0 \) the preservation of the semiclassicality has been confirmed (heuristically, without explicit test of inequality [8,4] however with the observed increase of dispersion remaining within the same level of magnitude) for profiles different than Gaussian [14,15] but still for technical reasons the (necessarily numerical) studies have been restricted to few specific shapes only (i.e. Gaussians in \( \ln(k) \), triangle “sawtooth” profiles). Note, that for specifically tailored states the increase in the spread can be substantial [20], however this can happen only for the states which were never semiclassical during their whole evolution (usually very quantum in “energy” \( p_\phi \)).

The abovementioned results have been strengthened (again just in case of flat FRW universe with massless scalar field and vanishing cosmological constant) in [21] where authors performed systematic numerical analyses of the parameter space of solutions corresponding to generalized Gaussian spectral profile, that is

\[
\hat{\Psi}(k) = k^n e^{-\eta(k-\beta)^2}, \quad \eta, \beta \in \mathbb{C}, \, n \in \mathbb{N}
\]

Within this class of states it was shown, that once the physically reasonable dispersion of a state is selected in its asymptotic past (relative dispersions of the order of \( 10^{-6} \)) the inequality (3.4) is strongly undersaturated.

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1 In cases when the spectrum of the evolution generator was discrete the cutoff to the values of \( k \) such that \( \omega(k) \) was in its spectrum was taken.
The above results, while being strong, have been obtained just for specific nongeneric classes of states, thus can be taken only as the indication of the general property of semiclassality preservation rather than its solid proof. To get a truly firm confirmation one needs to resort to methods which on the one hand are not restricted to just solvable prescriptions in LQC and on the other hand allow to reliably probe the properties of general physical states. One of such methods comes from treating the loop quantum cosmology evolution as a process of scattering of large semiclassical universe.

C. The scattering picture of the bounce

Loop quantization is not a unique route of quantizing the cosmological models. One can start with the reduced classical phase space and the Hamiltonian constraint (2.13) and apply to it the standard methods of quantum mechanics (also following Dirac program parallel to LQC). This is known as Wheeler-DeWitt (WDW) quantization. The description it provides is usually much simpler than that of LQC, however (unless the modification due to exotic matter field is introduced) it fails to resolve the singularity. Indeed, the wave packets representing semiclassical universe hit the boundary representing classical singularity (\( v = 0 \)) which in turn generically introduces a nonuniqueness in the unitary evolution of a given universe.

The principal case considered here: flat FRW universe with massless scalar field and vanishing cosmological constant has a very simple description in WDW approach: The kinematical Hilbert space is the product \( L^2(\mathbb{R}, d\phi) \otimes L^2(\mathbb{R}, dv) \), the physical states are representing by wave function (decomposed in basis of generalized eigenvectors of kinematical volume and scalar field operator)

\[
\Psi(v, \phi) = \int_{-\infty}^{+\infty} \Psi(k)e^{ik\phi}e^{i\omega(k)v}, \quad \xi_k(v) = e^{ik\ln|v|}, (3.7)
\]

(with \( \omega(k) = \sqrt{12\pi G}|k| \) and all the symbols in the equation being defined analogously to (2.10) and the observables can be defined analogously to LQC, as the analog of Schrödinger equation (in \( \phi \) reparametrization) takes the form

\[
-\imath \partial_\phi \Psi(v, \phi) = 12\pi G \sqrt{|v\partial_v|^2}\Psi(v, \phi).
\]

An important feature of those LQC models where the universe can expand to infinite size is the existence of their well defined WDW limit. In particular the model considered in sec. [11B.1] features the following large \( v \) behavior of the basis functions

\[
e_k(v) = e^{i\alpha(k)}\xi_k(v) + e^{-i\alpha(k)}\xi_{-k}(v) + O(|v|^{-3}). \quad (3.8)
\]

That allows to associate with each LQC state a contracting (\( k > 0 \)) and expanding (\( k < 0 \)) WDW “limit” state. For the states satisfying \( \Delta \hat{p}_\phi < \infty \) the expectation values and dispersions of observable \( \ln |\hat{v}|_{\phi} \) (defined analogously to (2.6) of LQC state approach in the limit of \( \phi \to \pm \infty \)) the values of analogous observables of the corresponding WDW (expanding/contracting respectively) limit states [22]. Thus, as long as we are interested in distant future/past only the global evolution of LQC state can be considered as a scattering process with a very simple scattering matrix

\[
\langle k|\hat{p}|k'\rangle = e^{-2i\alpha(k)}\delta(k+k'). \quad (3.9)
\]

The behavior of the phase rotation \( \alpha(k) \) could be systematically analyzed via numerical methods, which in turn allows to establish the following triange inequality true for every physical state with finite dispersion in \( \hat{p}_\phi \)

\[
|\sigma_+ - \sigma_-| < 2\Delta \ln |\hat{v}|_{\phi}/\hbar, \quad \sigma_\pm = \lim_{\phi \to \pm \infty} \Delta \ln |\hat{v}|_{\phi}. \quad (3.10)
\]

Preservation of semiclassality is then a straightforward consequence of this inequality.

The scattering process provides not only general and exact result in considered case but also can be easily generalized to more complicated systems not treatable analytically. Its downside is that it only allows to probe the asymptotic values, in particular not teling anything about the properties of a universe in high curvature region near the bounce.

IV. DISPERION AND SEMICLASSICALITY OF DESITTER UNIVERSE

The scattering picture presented in last sub-section can be generalized with reasonable effort to the case of positive cosmological constant. The LQC quantization of this model as well as its WDW limit have been studied extensively in [3]. In comparision to \( \Lambda = 0 \) the model exhibits two important differences:
(i) The volume of the universe reaches infinity for finite value of scalar field. That leads to nontrivial extensions past this point and to transition between expanding and contracting epoch of evolution through deSitter timelike future/past SCRI. This is a feature of both LQC and WDW approach, thus, unlike in $\Lambda = 0$ case, the eigenfunctions of the WDW evolution generator also take the form of standing waves.

(ii) Due to above, the bounce leads to infinite chain of large size low curvature epochs (universes) connected by quantum bounces and SCRI transitions.

Furthermore, the unitary evolution of the system is nonunique (in both LQC and WDW approach). The choice of self-adjoint extension of the “Hamiltonian” $\sqrt{\Theta\Lambda}$ corresponds to the choice of boundary conditions at the SCRI. From the point of view of semiclassicality preservation this nonuniqueness is not critical, as one can always work with a single extension (with discrete spectrum of the evolution generator).

Similarly to $\Lambda = 0$ one could start with establishing the asymptotics between the eigenfunctions of respectively LQC and WDW evolution generators. However, since WDW eigenfunctions themselves are standing waves they too would have to be decomposed onto simpler “expanding” (incoming to SCRI) and “contracting” (outgoing from SCRI) components.

In LQC approach the direct inspection (for detailed description of method used in the identification of the limit see [23]) shows that

$$e_n^\beta(v) = N_n[e^{i\alpha}e_n^+(v) + e^{-i\alpha}e_n^-(v)] + O(v^{-3}), \quad (4.1)$$

where

$$e_n^\pm = |v|^{-1} e^{\pm i(\Omega (\Lambda)) |v|} e^{\pm i\omega n,\Lambda,\beta}/|v| \quad (4.2a)$$
$$\cos(4\Omega (\Lambda)) = 1 - 2\Lambda/\Lambda_c =: 1 - 2\lambda, \quad (4.2b)$$
$$\kappa(n, \Lambda, \beta) = \frac{3\pi G(1 - 2\lambda) + \omega_n^2}{12\pi G\sqrt{\Lambda (1 - \lambda)}} =: A\omega_n^2 + B, \quad (4.2c)$$

and $N_n$ is a normalization constant. Note that the function $\theta$ used in (2.26) behaves at large $v$ as follows

$$1/|v| \approx (1/K)(\theta_0 - \pi/2). \quad (4.3)$$

Similarly, the WDW basis eigenfunctions (corresponding to possibly different value of cosmological constant here denoted as $\Lambda$) exhibit the limit

$$\lambda_n^\beta(v) = N(k)[e^{i\alpha}\lambda_n^+(v) + e^{-i\alpha}\lambda_n^-(v)] + O(v^{-3}), \quad (4.4)$$

where the phase rotation $\alpha$ depends on both $\beta$ and $k$ and

$$\lambda_n^\pm = |v|^{-1} e^{\pm i(\Omega (\Lambda)) |v|} e^{\pm i\omega n,\Lambda,\beta}/|v| \quad (4.5a)$$
$$\kappa = \kappa(\omega = \sqrt{12\pi G k}), \quad (4.5b)$$

and $\Omega$ the same as in (4.2b). We then observe that the functions $e_n^\pm$ and $\lambda_n^\pm$ do agree, provided we choose

$$\Lambda/\Lambda_c = \lambda = \arccos(1 - 2\lambda). \quad (4.6)$$

This allows us to associate with each LQC basis eigenfunction a WDW one. This in principle allows to define a WDW limit of LQC state (barring one caveat which we will discuss in detail below), however with WDW basis functions themselves being quite complicated exploiting this fact is not practical for the purpose of semiclassicality preservation analysis. Instead we will construct the auxiliary limit Hilbert spaces directly from $e_n^\pm$. Let us denote them by $H^\pm$ and $H^\pm_\Lambda$ for LQC and WDW respectively. We equip them with inner products selected in such a way that the norm of each (component) limit determined through (4.1), (4.4) agrees with the norm of the original (LQC/WDW) state. We note, that on each limit space the term $e^{\pm i(\Omega (\Lambda)) |v|}$ is a global rotation, thus can be dropped. After this modification the bases of the limit spaces become regular in $\theta$.

Let us return for a moment to relating the LQC and WDW states. We do this through the sequence of transformations

$$H_{\text{phy}} \to H^\pm \to H^\pm_\Lambda \to H_{\text{phy}}. \quad (4.7)$$

We note immediately however, that the direct association through basis functions (as specified earlier) would lead to associating with any formalizable element of $H_{\text{phy}}$ the WDW state(s) of zero norm. Thus, to be meaningful, the
transformation $\mathcal{H}_\pm \to \mathcal{H}_\pm$ has to be modified. In order to select the appropriate modification of it we recall, that we defined the auxiliary spaces and searched for the limit in order to determine the semiclassicity properties of LQC state. Thus, a natural requirement for the desired transformation is that the physical parameters (expectation values and dispersions of relevant observables) of the localized state near the SCRI are well reflected by those of the limit state. We can then define an instantiation of the state: require an agreement of the relevant observables at time $\phi = \phi_o$. A priori one could use for that the so-called Hamburger \cite{24} decomposition and require that all the Hamburger moments are preserved upon the transformation. This approach however, although precise is impractical as reproducing the state (wave function) out of its Hamburger moments is extremely difficult and till now remains an open problem.

Instead, we propose a simpler (although non-unique) construction motivated by properties of 1-dimensional Klein-Gordon equation. We note that the same problem occurs if for the system of free particle between two parallel walls we want to associate with a given particle (say at the moment of reflection from the wall) a wave packet of a particle moving freely on $\mathbb{R}$. To define an instantiation at $\phi = \phi_o$ we thus follow the construction natural for that scenario.

1. First by transformation of the spectral profile $\hat{\Psi}(k) \to \hat{\Psi}(k)e^{\imath \omega(k)\phi_o}$ we reduce the problem to constructing the instantiation at $\phi = 0$.

2. We extend the spectral profile from the discrete set of $k_n$ by linear interpolation of the modulus and phase of $\hat{\Psi}$ separately.

3. We transform the new wave function back using the inverse of the first step.

Have the spectrum of $\Theta_\Lambda$ been uniform, this procedure would lead to definition of the WDW state of the same expectation value and dispersion of $\hat{p}_\phi$ as those of the original LQC state. However, it is almost regular for large $k$. Furthermore to be semiclassical the states have to be peaked at large $k$ where the estimate \cite{22,23} is extremely accurate (the deviation from uniformity can be bounded by $C \exp(-2\pi n(\pi - y_o)/y_o)$ where $C$ is of the order of 1). As a consequence, for a state of $\hat{p}_\phi$ the deviation between expectation values and dispersions of $\ln |\hat{p}_\phi/\hbar|$ will be of the same order: $C \exp(-2(1 - \epsilon)\pi n(\pi - y_o)/y_o)$.

In order to determine how the state disperses through the bounce we need to compare the dispersions of observable $\hat{\theta}$ at two consecutive moments of reflection from SCRI. Due to complicated form of the basis functions $e_n^\beta(v)$ doing so directly on $\mathcal{H}$ is extremely difficult. Therefore, we cast the problem as comparing the relevant dispersions of the analog of $\hat{\theta}$ between two consecutive instantiations (corresponding to the points where the expectation value of $\hat{\theta}$ reaches maximum) on the auxiliary Hilbert space $\mathcal{H}^\pm$. This analog observable takes at $\phi = 0$ a very simple form

$$\hat{\theta}^\pm = \hat{x} + \frac{\pi}{2} = \frac{iK}{2\lambda \omega} \hat{\theta} + \frac{\pi}{2},$$

where $\Lambda$ has been defined in \eqref{4.2c}.

The main limitation of this step is a direct consequence of the fact, that even at the point where universe reaches SCRI $\Delta \hat{\theta}$ remains finite, thus even for sharply peaked states there would be finite differences between the dispersion of the original observable and its analog on the auxiliary space — there is no exact convergence as observed in the case of $\Lambda = 0$. The form of asymptotics \cite{11} and the numerical observations of the behavior of $e_n^\beta$ allow to conclude, that for as long as $\Delta \hat{\theta}$ remains small the difference is of higher order, thus one can provide estimate

$$1/C_\theta < \Delta \hat{\theta}/\Delta \hat{x} < C_\theta,$$

where $C_\theta$ is of the order of 1. The sufficiently optimal value of $C_\theta$ and the domain of validity of the above estimate (corresponding to such value) can be determined precisely via numerical analysis. For the purpose of studies of this article we take it as a conjecture.

To estimate the difference of $\hat{x}$ between specified instantiations we note, that, have the spectrum of $\Theta_\Lambda$ been uniform, the values of interest would have agreed exactly. By the same argument as used in case of $\hat{p}_\phi$ we conclude that the relative change of dispersion of $\hat{x}_\phi$ is of the order $C \exp(-2(1 - \epsilon)\pi n(\pi - y_o)/y_o)$. Thus for physically relevant semiclassical states they remain extremely small. As a consequence the semiclassicity is preserved at least in the sense of limiting states and by conjecture \cite{4.9} we can extend it to physical states.

An important property of the studied system follows from the fact that the spectrum of $\Theta_\Lambda$ does deviate from uniformity, resulting in a nontrivial spread of the semiclassical wave packet which does occur across the bounce. In consequence, after sufficiently large (although enormous) number of cycles the originally semiclassical state will eventually loose the semiclassicity. As a consequence even the universe semiclassical at some moment of evolution will preserve this property only for finite time (although spanning many evolution cycles) which in turn may render semiclassicality a non-generic feature even for a single (dynamical trajectory of a) universe. To check whether this is indeed the case we can ask a converse question: given a generic quantum state, under what condition it will ever admit a semiclassical epoch? To answering that question we dedicate the next section.

V. SPONTANEOUS COHERENCE IN LQC

The process of the (originally dispersed) quantum state attaining in the process of dynamical evolution semiclassical properties is a feature of several quantum mechanical systems and known as spontaneous coherence. The simple and regular structure of isotropic quantum cosmological models allows to expect that such process will occur also in isotropic sector of LQC. In this section we investigate this process in context of (again) flat FRW universe with massless scalar field and positive cosmological constant.

To start with, we note that, since \( p_\phi \) is a constant of motion, the semiclassicality properties tied to observable \( \hat{p}_\phi \) cannot change, thus small relative dispersion with respect to this observable is a necessary condition for the state to be able to ever feature semiclassical epoch.

To probe the spread in volume we employ the tools developed in previous section, casting the problem as the issue of coherence of observable \( \hat{\delta}_\phi \) on the instantiation of LQC state.

Since the limiting states bear some similarity to plane Klein-Gordon waves the question of coherence can be posed as question about the existence of epoch when the Heisenberg uncertainty principle is close to be saturated for the properties is a feature of several quantum mechanical systems and known as by differences in phases of (instantiated) \( \hat{\Psi}(k) \).

We note, that all the information about the instantiated state is still contained in the cutoff of the profile \( \hat{\Psi}(k) \) to the original \( k_o \) corresponding to spectrum of \( \Theta_\Lambda \). We then can encode the dynamical evolution as rotation of the spectral profile by phases \( e^{i\omega(k_n,\phi)} \) (with again linear interpolation between the discrete points as defined for the instantiations). The observable \( x \) can then be always evaluated at \( \phi = 0 \).

Since the state is localized in \( p_\phi \) (that is \( \Delta \hat{p}_\phi/\langle \hat{p}_\phi \rangle \leq \epsilon_p \ll 1 \) we can restrict the spectral profile to the finite number of points within the interval \( \omega_n \in \omega_o[1-3\epsilon_o,1+3\epsilon_o] \), where \( \epsilon_p \ll \epsilon_o \ll 1 \). Indeed, the neglected part of the wave function has a norm necessarily smaller than \( \epsilon_p/(3\epsilon_o) \), which allows in turn to estimate the correction to \( \Delta \hat{x} \) via \( 1.8 \text{ as } K/(2A\omega_o\Delta \omega) \cdot \epsilon_p/(3\epsilon_o) \). Upon this restriction the evolution of the phases of the state is a smooth trajectory on \( N \)-dimensional torus (where \( N \) is the number of eigenvalues in the selected interval).

Since now the system is reduced to a finite one, we can use number theory to estimate how big \( \epsilon_o \) has to be in order for the trajectory to eventually hit a cell centered at \( 0 \) and of size \( 2\epsilon_o \). This is an exctbook ergodicity problem for a linear dynamical systems on \( n \)-torus. We observe, that the density of trajectory on the torus depends on whether pairs of frequencies are rationally related. If none of frequencies \( \omega_n \) are rationally related then the trajectory is (truly) dense on \( N \)-torus, thus \( \epsilon_n \) can be arbitrarily small. Therefore we can divide the selected set of frequencies (eigenvalues) onto classes of rationally related ones and consider each class separately. Within each class, in order to ensure that the trajectory intersects the distinguished cell one needs to select the size of the cell such that

\[
2\pi/\epsilon_o \leq \min_{(m,n)\in P_k} q_{m,n},
\]

where \( P_k \) is a set of all possible pairs of eigenvalues such that \( \omega_m < \omega_n \) and the ratios of eigenvalues within each pair have the reduced form \( \omega_m/\omega_n = p_{m,n}/q_{m,n} \) with \( p_{m,n}, q_{m,n} \in \mathbb{N} \).

While we cannot determine the relevant minimum, the fact, that the spectrum approaches uniformity extremely fast (while not being exactly uniform) provides us with the lower bound for it. Indeed \( 2.23 \) immediately implies that the (reduced) denominator is higher than

\[
e^{2(1-3\epsilon_o)\pi n(\pi-y_o)/y_o}.
\]

As a consequence one can safely take

\[
\epsilon_o \sim 4\pi e^{-2(1-3\epsilon_o)\pi n(\pi-y_o)/y_o}. \tag{5.5}
\]

Together with \( 5.2 \) this estimate ensures existence of an epoch where the product \( \delta \hat{p}_\phi \Delta \hat{x}_\phi \) is of the order of its minimal value allowed by uncertainty principle (see \( 5.1 \)). Thus by the conjecture \( 4.3 \) we conclude that every state
sharply peaked in \( \hat{p}_\phi \) admits semiclassical epoch in its dynamical evolution. Furthermore, finite size of \( \epsilon_{\phi} \) ensures that the semiclassical epoch will be always hit after final time, thus excluding the possibility that semiclassical epoch is nongeneric along a single universe dynamical trajectory.

VI. PERSPECTIVES

To summarize the original research reported in this article, we extended the known results regarding semiclassicality preservation of isotropic universe within the LQC framework to the case of DeSitter FRW universe admitting massless scalar field as the sole matter content. In such case:

1. The semiclassical universe remains sharply peaked through many cycles of the evolution (separated by quantum bounce), although it very slowly looses its semiclassicality properties, and

2. it is enough that the universe is semiclassical with respect to the constant of motion \( p_\phi \) to admit a semiclassical epoch somewhere along its dynamical evolution.

These results, together with existing ones reviewed in sec. III, while promising, are restricted to just several models within the isotropic sector of LQC. In order to be considered reliable they have to be extended not only to wider class of isotropic systems (different topology and matter content) but also beyond the class of isotropic models: to homogeneous non-isotropic (like for example Bianchi I [25]) and ultimately inhomogeneous scenarios.

Within the homogeneous sector of LQC most of the methods presented in this article: scattering picture, numerical analysis of generalized Gaussians as well as methods applied in sec IV and V can be applied to wide class of the models [3]. Beyond the homogeneity the situation complicated significantly. There, two particular approaches give a hope of success:

1. The so called Abelianization procedure [26] when applied to cosmological models (see in particular an application to Gowdy cosmology [27]) allows to bring the (otherwise unbearably complicated) evolution generator to the form quite similar to the one known from isotropic sector of LQC. Provided, that the method is improved to unambiguously reproduce general relativity as its low energy limit, that property in principle allows to apply the methods discussed in this article either directly or after an extension.

2. While the present studies via either dressed [28] or rainbow [29] metric start to include the effects of quantum dispersion of the states, up to now they do not allow for precise control of the dispersion’s behavior. One may however hope that the synthesis of above methods with the semiclassical approach using the Hamburger decomposition of the state [24] will give birth to a methodology of probing the universe dynamics sufficiently robust to address the semiclassicality loss or spontaneous coherence problems in realistic cosmological scenarios.

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