Global well-posedness of the 2D Boussinesq equations with fractional Laplacian dissipation

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Abstract: As a continuation of the previous work [47], in this paper we focus on the Cauchy problem of the two-dimensional (2D) incompressible Boussinesq equations with fractional Laplacian dissipation. We give an elementary proof of the global regularity of the smooth solutions of the 2D Boussinesq equations with a new range of fractional powers of the Laplacian. The argument is based on the nonlinear lower bounds for the fractional Laplacian established in [13]. Consequently, this result significantly improves the recent works [13, 45, 47].

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1. Introduction

In this paper, we are interested in studying the following 2D incompressible Boussinesq equations with fractional Laplacian dissipation

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u + \nu \Lambda^\alpha u + \nabla p &= \theta e_2, \quad x \in \mathbb{R}^2, \quad t > 0, \\
\partial_t \theta + (u \cdot \nabla) \theta + \kappa \Lambda^\beta \theta &= 0, \quad x \in \mathbb{R}^2, \quad t > 0, \\
\nabla \cdot u &= 0, \quad x \in \mathbb{R}^2, \quad t > 0, \\
u(x, 0) &= u_0(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in \mathbb{R}^2,
\end{aligned}
\]  

(1.1)

where the numbers \(\nu \geq 0, \kappa \geq 0, \alpha \in [0, 2]\) and \(\beta \in [0, 2]\) are real parameters. Here \(u(x, t) = (u_1(x, t), u_2(x, t))\) is a vector field denoting the velocity, \(\theta = \theta(x, t)\) is a scalar function denoting the temperature, \(p\) is the scalar pressure and \(e_2 = (0, 1)\). The fractional Laplacian operator \(\Lambda^\alpha\), \(\Lambda := (-\Delta)^{\alpha/2}\) denotes the Zygmund operator which is defined through the Fourier transform, namely

\[
\hat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \hat{f}(\xi),
\]

where

\[
\hat{f}(\xi) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) \, dx.
\]

The fractional dissipation operator severs to model many physical phenomena (see [17]) in hydrodynamics and molecular biology such as anomalous diffusion in semiconductor growth (see [36]). We remark the convention that by \(\alpha = 0\) we mean that there is no dissipation in \((1.1)_1\), and similarly \(\beta = 0\) represents that there is no dissipation in \((1.1)_2\).
The standard Boussinesq equations (namely $\alpha = \beta = 2$) are of relevance to study a number of models coming from atmospheric or oceanographic turbulence where rotation and stratification play an important role (see for example [32, 35]). Moreover, as point out in [32], the 2D inviscid Boussinesq equations, namely (1.1) with $\alpha = \beta = 0$ are identical to the incompressible axi-symmetric (away from the $z$-axis) swirling 3D Euler equations. There are geophysical circumstances in which the Boussinesq equations with fractional Laplacian may arise. The effect of kinematic and thermal diffusion is attenuated by the thinning of atmosphere. This anomalous attenuation can be modeled by using the space fractional Laplacian (see [9, 18]).

The global well-posedness of the 2D Boussinesq equations has recently drawn a lot of attention and many important results have been established. It is well-known that the system (1.1) with full Laplacian dissipation (namely, $\alpha = \beta = 2$) is global well-posed, see, e.g., [7]. In the case of inviscid Boussinesq equations, the global regularity problem turns out to be extremely difficult and remains outstandingly open. Therefore, it is natural to consider the intermediate cases. Actually, many important progress has recently been made on this direction. Almost at the same time, Chae [10] and Hou and Li [23] proved the global regularity for the system (1.1) when $\alpha = 2$ and $\beta = 0$ or $\alpha = 0$ and $\beta = 2$ independently. Since then, much efforts are devoted to the global regularity of (1.1) with the smallest possible $\alpha \in (0, 2)$ and $\beta \in (0, 2)$. As pointed out in [24], we can classify $\alpha$ and $\beta$ into three categories: the subcritical case when $\alpha + \beta > 1$, the critical case when $\alpha + \beta = 1$ and the supercritical case when $\alpha + \beta < 1$. As a rule of thumb, with current methods it seems impossible to obtain the global regularity for the 2D Boussinesq equations with supercritical dissipation. Recently, Jiu, Wu and Yang [25] established the eventual regularity of weak solutions of the system (1.1) when $\alpha$ and $\beta$ are in the suitable supercritical range. For the critical case, there are several works are available. In the two elegant papers, Hmidi, Keraani and Rousset [21, 22] established the global well-posedness result to the system (1.1) with two special critical cases, namely $\alpha = 1$ and $\beta = 0$ or $\alpha = 0$ and $\beta = 1$. The more general critical case, that is $\alpha + \beta = 1$ with $0 < \alpha, \beta < 1$ is extremely difficult. The standard energy estimates do not yield the global bounds in any Sobolev spaces when $\alpha$ and $\beta$ in the critical case. Very recently, the global regularity of the general critical case $\alpha + \beta = 1$ with $\alpha > 23 - \sqrt{\frac{1777}{24}} \approx 0.9132$ and $0 < \beta < 1$ was recently examined by Jiu, Miao, Wu and Zhang [24]. This result was further improved by Stefanov and Wu [38] by further enlarging the range of $\alpha$ with $\alpha + \beta = 1$ and $1 > \alpha > 23 - \sqrt{\frac{1777}{24}} \approx 0.7981$ and $0 < \beta < 1$. Here we want to state that even in the subcritical ranges, namely $\alpha + \beta > 1$ with $0 < \alpha < 1$ and $0 < \beta < 1$, the global regularity of (1.1) is also definitely nontrivial and quite difficult. In fact, to the best of our knowledge there are only several works concerning the subcritical cases. More precisely, Miao and Xue [34] obtained the global regularity for system (1.1) for the case $\nu > 0$, $\kappa > 0$ and

$$\frac{6 - \sqrt{6}}{4} < \alpha < 1, \quad 1 - \alpha < \beta < \min \left\{ \frac{7 + 2\sqrt{6}}{5} \alpha - 2, \frac{\alpha(1 - \alpha)}{\sqrt{6} - 2\alpha}, \frac{2}{2 - 2\alpha} \right\}.$$  

In addition, Constantin and Vicol [13] verified the global regularity of the system (1.1) on the case when the thermal diffusion dominates, namely

$$\nu > 0, \ \kappa > 0, \ 0 < \alpha < 2, \ 0 < \beta < 2, \ \beta > \frac{2}{2 + \alpha}.$$
Recently, Yang, Jiu and Wu [45] proved the global well-posedness of the system (1.1) with
\[ \nu > 0, \ \kappa > 0, \ 0 < \alpha < 1, \ 0 < \beta < 1, \ \beta > 1 - \frac{\alpha}{2}, \ \beta \geq \frac{2 + \alpha}{3}, \ \beta > \frac{10 - 5\alpha}{10 - 4\alpha}. \]

Very recently, the authors [48] established the global regularity for the 2D Boussinesq equations with a new range of fractional powers, namely \( \nu > 0, \ \kappa > 0 \) and
\[ 0.783 \approx \frac{21 - \sqrt{217}}{8} < \alpha < 1, \ \ 1 - \alpha < \beta < \min \left\{ \frac{\alpha}{2}, \ \frac{(3\alpha - 2)(\alpha + 2)}{10 - 7\alpha}, \ \frac{2 - 2\alpha}{4\alpha - 3} \right\}. \]

Here we also want to mention that the two works [13, 45] have been improved by the recent manuscript [47]. More precisely, the authors in [47] established the global regularity result for the 2D Boussinesq equations with
\[ \nu > 0, \ \kappa > 0, \ 0 < \alpha < 1, \ 0 < \beta < 1, \ \beta > 1 - \frac{\alpha}{2}, \ \beta \geq \frac{2 + \alpha}{3}. \]

The case of partial anisotropic dissipation has been considered in several settings (see for instance [1, 8, 16, 29, 28]). For the global smooth solutions to the damped Boussinesq equations with small initial datum, we refer the readers to the recent works [2, 42]. Moreover, the global unique solution of the Boussinesq equations for the Yudovich type data has been established by many works, and we refer the readers to the interesting works [15, 40, 44, 41]. It is worth remarking that there are several works concerning the global regularity for the 2D Boussinesq equations with logarithmical dissipation (see, e.g., [20, 12, 26]). Many other interesting recent results on the Boussinesq equations can be found, with no intention to be complete (see, e.g., [11, 14, 24, 27, 28, 31, 43, 46] and the references therein).

The goal of this paper is to establish the global regularity of solutions to the system (1.1) with a new range of fractional powers of the Laplacian. Since the concrete values of the constant \( \nu, \kappa \) play no role in our discussion, for this reason, we shall assume \( \nu = \kappa = 1 \) throughout this paper. Now let us state our main result as follows

**Theorem 1.1.** Let \( 0 < \alpha < 1 \) and \( 0 < \beta < 1 \) satisfy
\[ \beta > \beta^* := \begin{cases} \max \left\{ \frac{2}{3}, \ \frac{4 - \alpha^2}{4 + 3\alpha} \right\}, & 0 < \alpha \leq \frac{2}{3}, \\ \frac{2 - \alpha}{2}, & \frac{2}{3} \leq \alpha < 1. \end{cases} \] (1.2)

Assume that \((u_0, \theta_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)\) for any \( s > 2 \) and satisfies \( \nabla \cdot u_0 = 0 \). Then the system (1.1) admits a unique global solution such that for any given \( T > 0 \)
\[ u \in C([0, T]; H^s(\mathbb{R}^2)) \cap L^2([0, T]; H^{s+\frac{\alpha}{2}}(\mathbb{R}^2)), \]
\[ \theta \in C([0, T]; H^s(\mathbb{R}^2)) \cap L^2([0, T]; H^{s+\frac{\beta}{2}}(\mathbb{R}^2)). \]

Let us give some remarks about our result.

**Remark 1.2.** On the one hand, one can easily check that \( \frac{2}{2+\alpha} > \beta^* \), thus Theorem 1.1 significantly improves Theorem 6.1 of [13]. On the other hand, our theorem also
significantly improves Theorem 1.1 of [45], which obtained the global regularity result under the assumption
\[
\beta > \max \left\{ \frac{2 - \alpha}{2}, \frac{2 + \alpha}{3}, \frac{10 - 5\alpha}{10 - 4\alpha} \right\} > \beta^*.
\]
Finally, we [47] have proved that the system (1.1) admits a unique global solution provided
\[
\beta > \frac{2 - \alpha}{2} \quad \text{and} \quad \beta \geq \frac{2 + \alpha}{3}.
\]
Obviously, Theorem 1.1 significantly improves the result of [47].

Remark 1.3. Through the proof, we find that Theorem 1.1 is always true for \(\beta > \frac{2 - \alpha}{2}\) with any \(0 < \alpha < 1\). However, it is easy to check that when \(0 < \alpha < \frac{2}{3}\), it holds
\[
\max \left\{ \frac{2}{3}, \frac{4 - \alpha^2}{4 + 3\alpha} \right\} < \frac{2 - \alpha}{2}.
\]
In fact, the proof of Case 1 (below) is much complicated than the proof of the case \(\beta > \frac{2 - \alpha}{2}\) with any \(0 < \alpha < 1\).

Remark 1.4. Through the proof of Theorem 1.1, we strongly believe that if one may establish Lemma 2.7 under somewhat weaker conditions than \(\beta > \beta^*\), then Theorem 1.1 can also be improved. As suggested by Jiu, Miao, Wu and Zhang in [24], the expected subcritical result is \(\beta > 1 - \alpha\) (\(1 - \alpha < \beta^*\)). However, at the moment we are not able to weaken the conditions \(\beta > \beta^*\). We will investigate this issue further in our future work.

Remark 1.5. The nonlinear lower bounds for the fractional Laplacian [13] or the Hölder estimates for advection fractional-diffusion equations [37] entails us that if one can show that for any given \(T > 0\)
\[
\sup_{0 \leq t \leq T} \|u(t)\|_{C^\alpha} < \infty \quad \text{or} \quad \sup_{0 \leq t \leq T} \|\omega(t)\|_{L^\infty} < \infty,
\]
under the assumption \(\beta > 1 - \alpha\), then the equations are well-posed in the smooth category up to time \(T\). Here \(u := \nabla \times u = \partial_{x_1} u_2 - \partial_{x_2} u_1\) is the vorticity and \(C^\alpha\) stands for the classical Hölder space.

We outline the main idea in the proof of this theorem. A large portion of the efforts are devoted to obtaining global \textit{a priori} bounds for \(u\) and \(\theta\) on the interval \([0, T]\). According to the definition of \(\beta^*\), the proof of Theorem 1.1 is divided into two cases, that is,
\[
\text{Case } 1 : \quad 0 < \alpha \leq \frac{2}{3}, \quad \max \left\{ \frac{2}{3}, \frac{4 - \alpha^2}{4 + 3\alpha} \right\} < \beta < 1,
\]
\[
\text{Case } 2 : \quad \frac{2}{3} \leq \alpha < 1, \quad \frac{2 - \alpha}{2} < \beta < 1.
\]
To start, let us say some words about the proof of the work [45], where the main idea of the work [45] is to consider the combined quantity \(G\) (see (2.10) for more details)
\[
\partial_t G + (u \cdot \nabla)G + \Lambda^{\alpha} G = -[\mathcal{R}_\beta, u \cdot \nabla] \theta + \Lambda^{\alpha - \beta} \partial_{x_1} \theta.
\]
Here and in sequel, we have used the standard commutator notation
\[
[\mathcal{R}_\beta, u \cdot \nabla] \theta := \mathcal{R}_\beta (u \cdot \nabla \theta) - u \cdot \nabla \mathcal{R}_\beta \theta.
\]
Invoking some commutator estimates and some computations, the combined quantity $G$ satisfies
\[ \sup_{0 \leq t \leq T} \| G(t) \|_{L^2}^2 + \int_0^T \| \Lambda^\frac{\beta}{2} G(\tau) \|_{L^2}^2 d\tau \ < \infty, \]
which is true for $\beta > \frac{2-\alpha}{2}$ and $\beta \geq \frac{\alpha+2}{3}$. Then they show the estimate
\[ \sup_{0 \leq t \leq T} \| G(t) \|_{L^{p_0}} < \infty \]
for some $p_0 > 2$. This estimate together with the iterative process leads to
\[ \sup_{0 \leq t \leq T} \| G(t) \|_{L^p} < \infty, \quad \text{for any } p_0 \leq p < \infty, \]
which is valid for $\beta > 1 - \frac{\alpha}{2}$, $\beta \geq \frac{2+\alpha}{3}$ and $\beta > \frac{10-5\alpha}{10-4\alpha}$.

However, the main argument used here is completely different from the work [45]. For **Case 1**, in view of several commutator estimates, we can show by combining $L^2$-norm of the combined quantity $G$ and the temperature $\theta$
\[ \sup_{0 \leq t \leq T} \left( \| G \|_{L^2}^2 + \| \Lambda^\beta \theta \|_{L^2}^2 \right) (t) + \int_0^T \left( \| \Lambda^\frac{\beta}{2} G \|_{L^2}^2 + \| \Lambda^{\frac{\beta}{2}} \theta \|_{L^2}^2 \right) (\tau) d\tau \ < \infty. \] (1.4)
whenever $0 \leq \beta < \frac{\beta}{2}$. For **Case 2**, by combining $L^2$-norm of the vorticity $\omega$ and the temperature $\theta$, one can conclude that
\[ \sup_{0 \leq t \leq T} \left( \| \omega \|_{L^2}^2 + \| \Lambda^\delta \theta \|_{L^2}^2 \right) (t) + \int_0^T \left( \| \Lambda^\frac{\beta}{2} \omega \|_{L^2}^2 + \| \Lambda^{\frac{\beta}{2}} \theta \|_{L^2}^2 \right) (\tau) d\tau \ < \infty. \] (1.5)
whenever $0 \leq \delta < \frac{\beta}{2}$.

The above two bounds (1.4) and (1.5) are the key component of this paper. With the help of the two bounds (1.4) and (1.5), we will establish the following key global bound
\[ \max_{0 \leq t \leq T} \| u(t) \|_{L^r} \ < \infty \]
for any $2 \leq r < \infty$. Thanks to the nonlinear lower bounds for the fractional Laplacian established in [13], the following key estimate holds
\[ \max_{0 \leq t \leq T} \| \nabla \theta(t) \|_{L^\infty} \ < \infty. \]

Finally, with the above estimate at our disposal, the global regularity of $u$ and $\theta$ following a standard approach (see for instance [11, 14, 32]).

The rest of the paper is organized as follows. In Section 2, we obtain the *a priori* estimates for sufficiently smooth solutions of the system (1.1). Section 3 is devoted to the proof of Theorem 1.1. Finally, in the Appendix, we give the proof of Lemmas 2.1 and 2.3 for the sake of completeness.
2. A priori estimates

This section is devoted to the a priori estimates which can be viewed as a preparation for the proof of Theorem 1.1. To simplify the notations, we shall use the letter $C$ to denote a generic constant which may vary from line to line. The dependence of $C$ on other parameters is usually clear from the context and we shall explicitly specify it whenever necessary.

The first lemma concerns the following commutator estimate, which plays a key role in proving our main result. The proof can be performed by making use of the Littlewood-Paley technique. To facilitate the reader, we will sketch the proof in the Appendix.

**Lemma 2.1.** Let $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ with $p \in [2, \infty)$ and $p_1, p_2 \in [2, \infty]$. Assume $r \in [1, \infty)$, $\delta \in (0, 1)$, $s \in (0, 1)$ such that $r + \delta < 1$, then it holds

$$\|[\Lambda^\delta, f]g\|_{B^s_{p,r}} \leq C(p, r, \delta, s)(\|\nabla f\|_{L^{p_1}}\|g\|_{B^{s+\delta-1}_{p_2,r}} + \|f\|_{L^2}\|g\|_{L^2}). \quad (2.1)$$

In particular,

$$\|[\Lambda^\delta, f]g\|_{B^s_{p,r}} \leq C(p, r, \delta, s)(\|\nabla f\|_{L^p}\|g\|_{B^{s+\delta-1}_{p,p}} + \|f\|_{L^2}\|g\|_{L^2}). \quad (2.2)$$

Here and in what follows, $B^s_{p,r}$ stands for the classical Besov space (see appendix for its precise definition).

In order to prove Case 2, we shall use the next two commutator estimates involving $R_\beta := \partial_x \Lambda^{-\beta}$.

**Lemma 2.2** (see [38]). Assume that $\frac{1}{p} < \beta < 1$ and $1 < p_2 < \infty$, $1 < p_1, p_3 \leq \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$. Then for $0 \leq s_1 < 1 - \beta$ and $s_1 + s_2 > 1 - \beta$, the following holds true

$$\left| \int_{\mathbb{R}^2} F[R_\beta, u_G \cdot \nabla] \theta \, dx \right| \leq C\|\Lambda^{s_1}\theta\|_{L^{p_1}}\|F\|_{W^{s_2, p_2}}\|G\|_{L^{p_3}}, \quad (2.3)$$

where $u_G := \nabla \Delta^{-1} G$ and $W^{s,p}$ denotes the standard Sobolev space.

**Lemma 2.3** (see [30]). Let $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$ for any $2 \leq p_1, p_2 \leq \infty$ and $2 \leq p < \infty$. Assume that $0 < \beta < 2$ and $\nabla \cdot u = 0$, then we have for any $r \in [1, \infty]$

$$\|[R_\beta, u \cdot \nabla] \theta\|_{L^p} \leq C(\|\nabla u\|_{L^{p_1}}\|\theta\|_{B^{1-\beta}_{p_2,1}} + \|u\|_{L^r}\|\theta\|_{L^2}). \quad (2.4)$$

For the sake of completeness, we will give the proof of Lemma 2.3 in the Appendix.

Finally, let us recall the following fractional type Gagliardo-Nirenberg inequality which is due to Hajaiej-Molinet-Ozawa-Wang [19].

**Lemma 2.4.** Let $0 < p, p_0, p_1, q, q_0, q_1 \leq \infty$, $s, s_0, s_1 \in \mathbb{R}$ and $0 \leq \vartheta \leq 1$. Then the following fractional type Gagliardo-Nirenberg inequality

$$\|v\|_{\dot{B}^{s_0}_{p_0, q_0}(\mathbb{R}^n)} \leq C\|v\|_{\dot{B}^{s_1}_{p_1, q_1}(\mathbb{R}^n)}^{\vartheta}\|v\|_{\dot{B}^{s_0}_{p_0, q_0}(\mathbb{R}^n)}^{1-\vartheta} \quad (2.5)$$

holds for all $v \in \dot{B}^{s_0}_{p_0, q_0} \cap \dot{B}^{s_1}_{p_1, q_1}$ if and only if

$$\frac{n}{p} - s = (1 - \vartheta)\left(\frac{n}{p_0} - s_0\right) + \vartheta\left(\frac{n}{p_1} - s_1\right), \quad s \leq (1 - \vartheta)s_0 + \vartheta s_1,$$

$$\frac{1}{q} \leq \frac{1 - \vartheta}{q_0} + \frac{\vartheta}{q_1}, \quad \text{if } p_0 \neq p_1 \text{ and } s = (1 - \vartheta)s_0 + \vartheta s_1,$$
Remark 2.5.

Lemma 2.4 is also true in the nonhomogeneous framework.

Lemma 2.6.

the system (1.1) obeys the following global bounds.

\[ \frac{1}{q} \leq \frac{1 - \vartheta}{q_0} + \frac{\vartheta}{q_1}, \quad \text{if } p_0 = p_1 \text{ and } s = (1 - \vartheta)s_0 + \vartheta s_1, \]

\[ \frac{n}{p_0} \neq s - \frac{n}{p} \text{ or } \frac{1}{q} \leq \frac{1 - \vartheta}{q_0} + \frac{\vartheta}{q_1}, \quad \text{if } s < (1 - \vartheta)s_0 + \vartheta s_1. \]

A special consequence of (2.5) is the following bound

\[ \|v\|_{B^{1,\beta}_2} \leq C\|v\|_{B^{\alpha}_2} \|v\|_{L^{\infty}}^{1-\lambda}, \quad \lambda = \frac{2\beta - 1}{2 - 2s}, \tag{2.6} \]

where \(2 - 2\beta < s < \frac{3-2\beta}{2}\) with \(\frac{1}{2} < \beta < 1\).

We also have

\[ \|\Lambda^{\gamma\beta}v\|_{L^2} \leq C\|\Lambda^{\beta/2}v\|_{L^2} \|v\|_{L^\infty}^{1-2\gamma}, \quad \beta > 0, \quad 0 < \gamma < \frac{1}{2}. \tag{2.7} \]

Remark 2.5. Lemma 2.4 is also true in the nonhomogeneous framework.

It follows from the basic energy estimates that the corresponding solution \((u, \theta)\) of the system (1.1) obeys the following global bounds.

Lemma 2.6. Assume \((u_0, \theta_0)\) satisfies the assumptions stated in Theorem 1.7. Then the corresponding solution \((u, \theta)\) of (1.1) admits the following bounds for any \(t > 0\)

\[ \|\theta(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\alpha}{2}}\theta(\tau)\|_{L^2}^2 d\tau \leq \|\theta_0\|_{L^2}^2, \]

\[ \|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}, \quad \forall p \in [2, \infty], \]

\[ \|u(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{\alpha}{2}}u(\tau)\|_{L^2}^2 d\tau \leq (\|u_0\|_{L^2}^2 + t\|\theta_0\|_{L^2}^2)^2. \tag{2.8} \]

2.1. Case 1. As well-known, when \(0 < \alpha, \beta < 1\), it is impossible to obtain the global \(H^1\) bound of \((u, \theta)\) by direct energy estimate method. Actually, applying operator \(\text{curl}\) to the first equation in (1.1), we have the following vorticity \(w = \nabla \times u\)

\[ \partial_t w + (u \cdot \nabla)w + \Lambda \omega = \partial_{x_1} \theta. \tag{2.9} \]

However, the "vortex stretching" term \(\partial_{x_1} \theta\) appears to prevent us from proving any global bound for \(w\). To circumvent this difficulty, a natural idea would be to eliminate \(\partial_{x_1} \theta\) from the vorticity equation. To this end, we generalize the idea of Hmidi, Keraani and Rousset [21] [22] to introduce a new quantity. More precisely, we set the combined quantity

\[ G = \omega - \mathcal{R}_\beta \theta, \quad \mathcal{R}_\beta := \partial_{x_1} \Lambda^{-\beta}, \]

which obeys the following equation

\[ \partial_t G + (u \cdot \nabla)G + \Lambda \omega = -[\mathcal{R}_\beta, u \cdot \nabla] \theta + \Lambda^{\alpha-\beta} \partial_{x_1} \theta. \tag{2.10} \]

Since \(u\) is determined by \(\omega\) through the Biot-Savart law, we have

\[ u = \nabla^\perp \Delta^{-1} \omega = \nabla^\perp \Delta^{-1} (G + \mathcal{R}_\beta \theta) = \nabla^\perp \Delta^{-1} G + \nabla^\perp \Delta^{-1} \mathcal{R}_\beta \theta := u_G + u_\theta. \tag{2.11} \]

We are now in the position to derive the following estimates concerning \(G\) and \(\theta\).
Lemma 2.7. Assume \((u_0, \theta_0)\) satisfies the assumptions stated in Theorem 1.1. Let \((u, \theta)\) be the corresponding solution of the system \((L_1)\). If \(\beta > \max \left\{ \frac{4-5\beta}{2}, \frac{2\alpha-3\beta}{2} \right\}\), then the following estimate holds for any max \(\left\{ \frac{4-5\beta}{2}, \frac{2\alpha-3\beta}{2} \right\} < \varrho < \frac{\beta}{2} \) and \(t \in [0, T]\)

\[
\|G(t)\|_{L^2}^2 + \|\Lambda^\varrho \theta(t)\|_{L^2}^2 + \int_0^t (\|\Lambda^\varrho G\|_{L^2}^2 + \|\Lambda^{\varrho + \frac{\beta}{2}} \theta\|_{L^2}^2)(\tau) \, d\tau \leq C(T, u_0, \theta_0),
\]

where \(C(T, u_0, \theta_0)\) is a constant depending on \(T\) and the initial data.

Remark 2.8. Although the above estimate \((2.12)\) holds for max \(\left\{ \frac{4-5\beta}{2}, \frac{2\alpha-3\beta}{2} \right\} < \varrho < \frac{\beta}{2}\), yet by energy estimate \((2.8)\) and the classical interpolation, we find that \((2.12)\) is true for any \(0 \leq \varrho < \frac{\beta}{2}\).

Proof of Lemma 2.7. Applying \(\Lambda^\varrho\) to \((1.1)_2\) and taking the inner product with \(\Lambda^\varrho \theta\), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\Lambda^\varrho \theta\|_{L^2}^2 + \|\Lambda^{\varrho + \frac{\beta}{2}} \theta\|_{L^2}^2 = - \int_{\mathbb{R}^2} \Lambda^\varrho (u \cdot \nabla) \Lambda^\varrho \theta \, dx.
\]

Hence, an application of the divergence-free condition and commutator estimate \((2.2)\) directly yields

\[
\left| \int_{\mathbb{R}^2} \Lambda^\varrho (u \cdot \nabla) \Lambda^\varrho \theta \, dx \right| = \left| \int_{\mathbb{R}^2} \left[ \Lambda^\varrho, u \cdot \nabla \right] \varrho \theta \, dx \right|
= \left| \int_{\mathbb{R}^2} \nabla \left[ \Lambda^\varrho, u \right] \varrho \theta \, dx \right|
\leq C \|\Lambda^{1-\frac{\beta}{2}} [\Lambda^\varrho, u] \varrho \theta\|_{L^2} \|\Lambda^{\varrho + \frac{\beta}{2}} \theta\|_{L^2}
\leq C \|\varrho \theta\|_{H^{1-\frac{\beta}{2}}} \|\Lambda^{\varrho + \frac{\beta}{2}} \theta\|_{L^2}
\leq C \left( \|u\|_{L^2} \|\theta\|_{H^{1-\frac{\beta}{2}}} + \|u\|_{L^2} \|\theta\|_{L^2} \right) \|\Lambda^{\varrho + \frac{\beta}{2}} \theta\|_{L^2}
\leq C \left( \|G\|_{L^2} \|\theta\|_{H^{1-\frac{\beta}{2}}} + \|u\|_{L^2} \|\theta\|_{L^2} \right) \|\Lambda^{\varrho + \frac{\beta}{2}} \theta\|_{L^2}
\leq \frac{1}{4} \|\Lambda^{\varrho + \frac{\beta}{2}} \theta\|_{L^2}^2 + C(1 + \|G\|_{L^2}^2 + \|\theta\|_{H^{\frac{\beta}{2}}}^2),
\]

where we have used the following facts

\[
\|f\|_{H^{1-\frac{\beta}{2}}} \approx \|f\|_{B^{1-\frac{\beta}{2}}} \quad \text{and} \quad \|R_\beta \theta\|_{L^2} \leq C \|\theta\|_{H^{\frac{\beta}{2}}}, \quad \beta \geq \frac{2}{3}.
\]

Inserting the above estimate in \((2.13)\), we thus obtain

\[
\frac{1}{2} \frac{d}{dt} \|\Lambda^\varrho \theta\|_{L^2}^2 + \frac{3}{4} \|\Lambda^{\varrho + \frac{\beta}{2}} \theta\|_{L^2}^2 \leq C(1 + \|G\|_{L^2}^2 + \|\theta\|_{H^{\frac{\beta}{2}}}^2).
\]
In order to close the above inequality, we need to consider the equation (2.10). To this end, we multiply the equation (2.10) by $G$ to obtain

$$\frac{1}{2} \frac{d}{dt} \|G\|_{L^2}^2 + \|\Lambda^\frac{\beta}{2} G\|_{L^2}^2 = \int_{\mathbb{R}^2} \Lambda^{\alpha-\beta} \partial_{x_1} \theta \|G\|_{L^2}^2 - \int_{\mathbb{R}^2} [\mathcal{R}_\beta, u \cdot \nabla] \theta \|G\|_{L^2}^2$$

$$= \int_{\mathbb{R}^2} \Lambda^{\alpha-\beta} \partial_{x_1} \theta \|G\|_{L^2}^2 - \int_{\mathbb{R}^2} [\mathcal{R}_\beta, u_G \cdot \nabla] \theta \|G\|_{L^2}^2$$

$$- \int_{\mathbb{R}^2} [\mathcal{R}_\beta, u_\theta \cdot \nabla] \theta \|G\|_{L^2}^2. \quad (2.16)$$

Bounding the first term at the R-H-S of (2.16) according to the Hölder inequality and the interpolation inequality, we thus get

$$\int_{\mathbb{R}^2} \Lambda^{\alpha-\beta} \partial_{x_1} \theta \|G\|_{L^2}^2 \leq C \|\Lambda^{1+\frac{\alpha-\beta}{2}} \theta\|_{L^2} \|\Lambda^\frac{\beta}{2} G\|_{L^2}$$

$$\leq C \|\theta\|_{L^2} \|\Lambda^{\alpha+\frac{\beta}{2}} \theta\|_{L^2} \|\Lambda^\frac{\beta}{2} G\|_{L^2}$$ ( $\theta > \frac{2 + \alpha - 3\beta}{2} \Rightarrow \tau = \frac{2 + \alpha - 2\beta}{2\theta + \beta} \in (0, 1)$

$$\leq \frac{1}{8} \|\Lambda^\frac{\beta}{2} G\|_{L^2}^2 + \frac{1}{8} \|\Lambda^{\alpha+\frac{\beta}{2}} \theta\|_{L^2}^2 + C \|\theta\|_{L^2}^2. \quad (2.17)$$

Next we appeal to the commutator estimate (2.4) and the fractional type Gagliardo-Nirenberg inequality (2.6) to bound the third term at the R-H-S of (2.16)

$$\int_{\mathbb{R}^2} [\mathcal{R}_\beta, u_\theta \cdot \nabla] \theta \|G\|_{L^2}^2 \leq \|\mathcal{R}_\beta, u_\theta \cdot \nabla\|_{L^2} \|\theta\|_{L^2} \|G\|_{L^2}$$

$$\leq C \|\nabla u_\theta\|_{L^1} \|\|\theta\|_{B^1_{4,1}} \|\theta\|_{B^1_{4,1}} \|\theta\|_{L^2} \|G\|_{L^2}$$

$$\leq C \|\Lambda^{1-\beta} \theta\|_{L^1} \|\theta\|_{B^1_{4,1}} \|\theta\|_{L^2}^2 + \|\theta\|_{L^2} \|G\|_{L^2}$$

$$\leq C \|\theta\|_{B^1_{4,1}} \|\theta\|_{L^2}^2 + \|\theta\|_{L^2} \|G\|_{L^2}, \quad (2.18)$$

where we have used the classical embedding $B^1_{4,1} \hookrightarrow W^{1-\beta, 4}$. Thanks to (2.6), we have that for $2 - 2\beta < s < \frac{3-2\beta}{2}$ with $\frac{1}{2} < \beta < 1$

$$\|\theta\|_{B^1_{4,1}(\mathbb{R}^2)} \leq C \|\theta\|_{H^s_{2,2}(\mathbb{R}^2)} \|\theta\|_{L^{\frac{2}{s-2\beta}}(\mathbb{R}^2)}, \quad \lambda = \frac{2\beta - 1}{2 - 2s} \in (0, 1). \quad (2.19)$$

According to Sobolev interpolation, we can get that for any $\frac{\beta}{2} < s < \frac{\beta}{2} + \varrho$

$$\|\theta\|_{B^l_{2,2}(\mathbb{R}^2)} \leq \|\theta\|_{H^l_{2,2}(\mathbb{R}^2)} \|\Lambda^{\alpha+\frac{\beta}{2}} \theta\|_{L^2(\mathbb{R}^2)}, \quad l = \frac{s - \frac{\beta}{2}}{\varrho} \in (0, 1). \quad (2.20)$$

Inserting (2.20) into (2.19) and considering (2.18), we can conclude

$$\int_{\mathbb{R}^2} [\mathcal{R}_\beta, u_\theta \cdot \nabla] \theta \|G\|_{L^2}^2 \leq C \|\theta\|_{B^l_{2,2}(\mathbb{R}^2)} \|\theta\|_{L^2(\mathbb{R}^2)} \|G\|_{L^2}$$

$$\leq C \|\theta\|_{H^l_{2,2}(\mathbb{R}^2)} \|\Lambda^{\alpha+\frac{\beta}{2}} \theta\|_{L^2} \|\theta\|_{L^2(\mathbb{R}^2)} \|G\|_{L^2}$$

$$\leq \frac{1}{8} \|\Lambda^{\alpha+\frac{\beta}{2}} \theta\|_{L^2}^2 + C \|\theta\|_{H^l_{2,2}(\mathbb{R}^2)}^2 \|G\|_{L^2}^2, \quad C \|\theta\|_{H^l_{2,2}(\mathbb{R}^2)}^2 \|G\|_{L^2}^2, \quad (2.21)$$
where in the last line we have used the following fact
\[
0 < s < \frac{\beta (2\beta - 1) + \varrho}{2\beta - 1 + \varrho} \Rightarrow \lambda l \leq \frac{1}{2} \Leftrightarrow \frac{2\beta - 1}{2 - 2s} \frac{s - \frac{\beta}{2}}{\varrho} \leq \frac{1}{2} \Rightarrow \frac{1}{1 - \lambda l} \leq 2.
\]
Combing all the restrictions on \( s \) yields
\[
\max \left\{ 0, 2 - 2\beta, \frac{\beta}{2} \right\} < s < \min \left\{ \frac{3 - 2\beta}{2}, \frac{\beta}{2} + \varrho, \frac{\beta (2\beta - 1) + \varrho}{2\beta - 1 + \varrho} \right\},
\]
which would work as long as
\[
\varrho > \frac{4 - 5\beta}{2}, \quad \beta > \frac{1}{2}.
\]
Now we focus on the second term at the R-H-S of (2.16). The estimate (2.7) as well as energy estimate (2.8) leads to
\[
\left\| \Lambda^{\gamma \beta} \theta \right\|_{L^1_t L^1_x} \leq C \left\| \Lambda^{\frac{\alpha}{2}} \theta \right\|_{L^2_t L^2_x} \left\| \theta \right\|_{L^{1 - 2\gamma}_t L^\infty_x} < \infty, \quad 0 < \gamma < \frac{1}{2}.
\]
(2.22)
The commutator estimate (2.3) with \( s_1 = \gamma \beta < 1 - \beta, 1 - \beta - \gamma \beta < s_2 < \frac{\alpha}{2}, p_1 = \frac{1}{\gamma}, p_2 = \frac{4}{2 + 2s_2 - \alpha} \) and \( p_3 = \frac{4}{2 + \alpha - 2s_2 - 4\gamma} \) allows us to show
\[
\int_{\mathbb{R}^2} [\mathcal{R}_\beta, u_G \cdot \nabla] \theta \ G \ dx 
\leq C \left\| \Lambda^{\gamma \beta} \theta \right\|_{L^1_t L^1_x} \left\| \mathcal{D} G \right\|_{W^{s_2, p_2}} \left\| G \right\|_{L^p_3} 
\leq C \left\| \Lambda^{\gamma \beta} \theta \right\|_{L^1_t L^1_x} \left\| \Lambda^{\frac{\alpha}{2}} G \right\|_{L^2} \left( \left\| \theta \right\|_{L^{1 - 2\gamma}_t L^\infty_x} \left\| \Lambda^{\frac{\alpha}{2}} G \right\|_{L^2} \left( \frac{\alpha - 4\gamma}{2} < s_2 < \alpha - (\alpha + 2)\gamma \Rightarrow \mu = \frac{2s_2 + 4\gamma - \alpha}{\alpha} \in (0, 1) \right) \right) 
\leq \frac{1}{8} \left\| \Lambda^{\frac{\alpha}{2}} G \right\|_{L^2}^2 + C \left\| \Lambda^{\gamma \beta} \theta \right\|_{L^1_t L^1_x}^2 \left\| G \right\|_{L^2}^2, 
\]
(2.23)
where in the last line the following fact has been applied
\[
\frac{2}{1 - \mu} \leq \frac{1}{\gamma} \Rightarrow \mu = \frac{2s_2 + 4\gamma - \alpha}{\alpha} \leq 1 - 2\gamma.
\]
Putting all the restrictions on \( s_2 \) together, we have
\[
1 - \beta - \gamma \beta < s_2 < \frac{\alpha}{2}, \quad \frac{\alpha - 4\gamma}{2} < s_2 < \alpha - (\alpha + 2)\gamma.
\]
Consequently, the above \( s_2 \) would work as long as
\[
\max \left\{ 0, \frac{2 - 2\beta - \alpha}{2\beta} \right\} < \gamma < \min \left\{ \frac{1}{2}, \frac{1 - \beta}{\beta}, \frac{\alpha + \beta - 1}{2 + \alpha - \beta} \right\},
\]
which leads to the key assumption
\[
\beta > \frac{4 - \alpha^2}{4 + 3\alpha}.
\]
It is worth noting that the fact $\frac{4-\alpha^2}{4+\alpha} > 1 - \alpha$ and this is the only place where we use the assumption $\beta > \frac{4-\alpha^2}{4+\alpha}$. Inserting the above aforementioned estimates (2.17), (2.21) and (2.23) into (2.16) yields

$$\frac{d}{dt} \|G\|^2_{L^2} + \frac{3}{4} \|\Lambda^{\frac{\alpha}{2}} G\|^2_{L^2} \leq \frac{1}{8} \|\Lambda^{\frac{\alpha}{2}} \theta\|^2_{L^2} + C \|\theta\|^2_{L^2} + C(1 + \|\theta\|^2_H)(1 + \|G\|^2_{L^2})$$

$$+ C(1 + \|\Lambda^\beta \theta\|^2_{L^4}) \|G\|^2_{L^2}. \quad (2.24)$$

Summing up (2.24) and (2.16), we thereby obtain

$$\frac{d}{dt}(\|G\|^2_{L^2} + \|\Lambda \theta\|^2_{L^2}) + \|\Lambda^{\frac{\alpha}{2}} G\|^2_{L^2} + \|\Lambda^{\frac{\alpha}{2}} \theta\|^2_{L^2} \leq C(1 + \|\theta\|^2_H) + \|\Lambda^\beta \theta\|^2_{L^4}) (1 + \|G\|^2_{L^2}),$$

which together with the classical Gronwall inequality and (2.19) lead to

$$\sup_{0 \leq t \leq T} (\|G(t)\|^2_{L^2} + \|\Lambda \theta(t)\|^2_{L^2}) + \int_0^T \left(\|\Lambda^{\frac{\alpha}{2}} G\|^2_{L^2} + \|\Lambda^{\frac{\alpha}{2}} \theta\|^2_{L^2}\right) d\tau \leq C(T, u_0, \theta_0).$$

This completes the proof of Lemma 2.7. □

2.2. Case 2. In this case, we consider the vorticity $\omega$ instead of the combined quantity $\theta$. Now we derive the following estimates concerning vorticity $\omega$ and the temperature $\theta$, which can be stated as follows.

**Lemma 2.9.** Assume $(u_0, \theta_0)$ satisfies the assumptions stated in Theorem 1.1. Let $(u, \theta)$ be the corresponding solution of the system (1.1). If $\beta > \frac{2-\alpha}{2}$, then the vorticity $\omega$ and the temperature $\theta$ admit the following bound for any $\frac{2-\alpha-\beta}{2} < \delta < \frac{\beta}{2}$ and $t \in [0, T]$

$$\|\omega(t)\|^2_{L^2} + \|\Lambda^{\delta} \theta(t)\|^2_{L^2} + \int_0^T \left(\|\Lambda^{\frac{\alpha}{2}} \omega\|^2_{L^2} + \|\Lambda^{\delta + \frac{\alpha}{2}} \theta\|^2_{L^2}\right) d\tau \leq C(T, u_0, \theta_0), \quad (2.25)$$

where $C(T, u_0, \theta_0)$ is a constant depending on $T$ and the initial data.

**Remark 2.10.** Similarly, by energy estimate (2.28) and the classical interpolation, we find that (2.26) is true for any $0 \leq \delta < \frac{\beta}{2}$.

**Proof of Lemma 2.9.** With the same argument used in obtaining (2.14), we find that

$$\frac{d}{dt} \|\Lambda^{\delta} \theta\|^2_{L^2} + \|\Lambda^{\delta + \frac{\alpha}{2}} \theta\|^2_{L^2} \leq C\|\Lambda^{1-\frac{\alpha}{2}} \theta\|_{L^2} \|\Lambda^{\delta + \frac{\alpha}{2}} \theta\|_{L^2}$$

$$\leq C\|\Lambda^\delta u\|_{H^{1-\frac{\alpha}{2}}} \|\Lambda^{\delta + \frac{\alpha}{2}} \theta\|_{L^2}$$

$$\leq C\|\Lambda^{\delta} u\|_{B^{1-\frac{\alpha}{2}}_{2, \infty}} \|\Lambda^{\delta + \frac{\alpha}{2}} \theta\|_{L^2}$$

$$\leq C(\|\nabla u\|_{L^2} \|\theta\|_{B^{1-\frac{\alpha}{2}}_{2, \infty}} + \|u\|_{L^2} \|\theta\|_{L^2}) \|\Lambda^{\delta + \frac{\alpha}{2}} \theta\|_{L^2} \quad (\delta < \frac{\beta}{2})$$

$$\leq C(\|\omega\|_{L^2} \|\theta\|_{L^\infty} + \|u\|_{L^2} \|\theta\|_{L^2}) \|\Lambda^{\delta + \frac{\alpha}{2}} \theta\|_{L^2}$$

$$\leq \frac{1}{4} \|\Lambda^{\delta + \frac{\alpha}{2}} \theta\|^2_{L^2} + C\|\theta_0\|^2_{L^2} \|\omega\|^2_{L^2} + C\|u\|^2_{L^2} \|\theta\|^2_{L^2}.$$
Substituting the above estimate into (2.26), we thus obtain
\[ \frac{1}{2} \frac{d}{dt} \| \Lambda^\delta \theta \|_{L^2}^2 + \frac{3}{4} \| \Lambda^{\delta + \frac{2}{3}} \theta \|_{L^2}^2 \leq C \| \theta_0 \|_{L^\infty}^2 \| \omega \|_{L^2}^2. \] (2.26)

In order to obtain the global $H^1$ bound of the velocity $u$, we resort to the vorticity $w$ equation (2.9)
\[ \partial_t w + (u \cdot \nabla) w + \Lambda^\alpha w = \partial_{x_1} \theta. \] (2.27)
Testing it by $\omega$ yields
\[ \frac{1}{2} \frac{d}{dt} \| \omega \|_{L^2}^2 + \| \Lambda^\frac{3}{2} \omega \|_{L^2}^2 = \int_{\mathbb{R}^2} \partial_{x_1} \theta \omega \, dx \leq C \| \Lambda^{1-\frac{3}{2}} \theta \|_{L^2} \| \Lambda^\frac{3}{2} \omega \|_{L^2} \leq C \| \Lambda^\delta \theta \|_{L^2}^{-\tau} \| \Lambda^{\delta + \frac{3}{2}} \theta \|_{L^2} \leq \frac{1}{2} \| \Lambda^\frac{3}{2} \omega \|_{L^2}^2 + \frac{1}{4} \| \Lambda^{\delta + \frac{3}{2}} \theta \|_{L^2}^2 + C \| \Lambda^\delta \theta \|_{L^2}^2, \] (2.28)
where we have applied the following Sobolev interpolation
\[ \| \Lambda^{1-\frac{3}{2}} \theta \|_{L^2} \leq C \| \Lambda^\delta \theta \|_{L^2}^{-\tau} \| \Lambda^{\delta + \frac{3}{2}} \theta \|_{L^2} \tau = \frac{2 - \alpha - 2\delta}{\beta} \in (0, 1). \]
Note the fact
\[ \frac{2 - \alpha - \beta}{2} < \delta < \frac{2 - \alpha}{2} \Rightarrow 0 < \tau < 1. \]
Putting all the restrictions on $\delta$ together, we have
\[ \frac{2 - \alpha - \beta}{2} < \delta < \min \left\{ \frac{2 - \alpha}{2}, \frac{\beta}{2} \right\} = \frac{\beta}{2}. \]
Thus, this is the only place in the proof where we use the assumption of the theorem, namely $\beta > \frac{2 - \alpha}{2}$.

Summing up (2.28) and (2.26), we thereby obtain
\[ \frac{d}{dt} (\| \omega \|_{L^2}^2 + \| \Lambda^\delta \theta \|_{L^2}^2) + \| \Lambda^\frac{3}{2} \omega \|_{L^2}^2 + \| \Lambda^{\delta + \frac{3}{2}} \theta \|_{L^2}^2 \leq C (\| \omega \|_{L^2}^2 + \| \Lambda^\delta \theta \|_{L^2}^2) + C \| u \|_{L^2}^2 \| \theta \|_{L^2}, \]
which together with the classical Gronwall inequality leads to
\[ \sup_{0 \leq t \leq T} (\| \omega \|_{L^2}^2 + \| \Lambda^\delta \theta \|_{L^2}^2) (t) + \int_0^T \left( \| \Lambda^\frac{3}{2} \omega \|_{L^2}^2 + \| \Lambda^{\delta + \frac{3}{2}} \theta \|_{L^2}^2 \right) (\tau) \, d\tau < \infty. \]
This completes the proof of Lemma 2.9. \qed

Both in Case 1 and Case 2, we can establish the following global a priori bound $\| u(t) \|_{L^r}$ for any $2 \leq r < \infty$ and $0 \leq t \leq T$.

**Lemma 2.11.** Assume $\beta$ satisfies the assumptions stated in Lemmas 2.7 and 2.9, then the velocity field $u$ obeys the following key global a priori bound for any $2 \leq r < \infty$ and $0 \leq t \leq T$
\[ \sup_{0 \leq t \leq T} \| u(t) \|_{L^r} \leq C(r, T, u_0, \theta_0), \] (2.29)
where $C(r, T, u_0, \theta_0)$ is a constant depending on $r$, $T$ and the initial data.
Proof of Lemma 2.11 Let us notice that in Case 1, we have \( \beta > \frac{2}{3} \). As a result, we can select \( \rho \) satisfying \( 1 - \beta < \rho < \frac{\beta}{2} \) such that
\[
\| \mathcal{R}_\beta \theta \|_{L^2} \leq \| \Lambda^{1-\beta} \theta \|_{L^2} \leq \| \theta \|_{H^\rho} < \infty.
\]
Recalling \( G = \omega - \mathcal{R}_\beta \theta \) and the estimate (2.12), we get
\[
\| \omega \|_{L^2} \leq \| G \|_{L^2} + \| \mathcal{R}_\beta \theta \|_{L^2} < \infty,
\]
which together with (2.25) implies that we have both in Case 1 and Case 2
\[
\sup_{0 \leq t \leq T} \| \omega(t) \|_{L^2} < \infty.
\]
By the Sobolev interpolation inequality
\[
\sup_{0 \leq t \leq T} \| u(t) \|_{L^r} \leq C(r) \| u \|_{L^2}^{\frac{1}{r}} \| \nabla u \|_{L^2}^{\frac{1}{r'}}
\leq C(r) \| u \|_{L^2}^{\frac{1}{r}} \| \omega \|_{L^2}^{\frac{1}{r'}}
\leq C(r, T, u_0, \theta_0),
\]
(2.30)
Consequently, this immediately completes the proof of Lemma 2.11. □

With the help of the above estimate (2.29), we are able to the next lemma, which is concerned with the global a priori bounda \( \| \nabla \theta \|_{L^\infty} \) as well as \( \| \omega \|_{L^\infty} \).

Lemma 2.12. Assume \((u_0, \theta_0)\) satisfies the assumptions stated in Theorem 1.1. Assume \( \beta \) satisfies the assumptions stated in Lemmas 2.7 and 2.9 then the temperature \( \theta \) and the vorticity \( \omega \) admit the following key global a priori bound
\[
\sup_{0 \leq t \leq T} \| \nabla \theta(t) \|_{L^\infty} \leq C(T, u_0, \theta_0),
\]
(2.31)
\[
\sup_{0 \leq t \leq T} \| \omega(t) \|_{L^\infty} \leq C(T, u_0, \theta_0),
\]
(2.32)
where \( C(T, u_0, \theta_0) \) is a constant depending on \( T \) and the initial data.

Proof of Lemma 2.12. The idea of the proof is based on the argument of nonlinear lower bounds for the fractional Laplacian established in [13]. For convenience the reader, we present the details as follows. By the elementary calculations, it is not hard to check that
\[
\beta^* \geq \frac{1}{1 + \alpha}.
\]
Therefore, this fact further implies
\[
\beta > \frac{1}{1 + \alpha}.
\]
We start with the following pointwise bound
\[
\nabla f(x) \cdot \Lambda^\alpha \nabla f(x) \geq \frac{1}{2} \Lambda^\alpha (|\nabla f(x)|^2) + \frac{|\nabla f(x)|^{2+\frac{2\alpha}{p}}+2}{c\|f\|_{L^p}^{\frac{2\alpha}{p}}},
\]
(2.33)
which can be proved by combining the proofs of Theorems 2.2 and 2.5 of [13]. Applying $\nabla$ to the temperature equation of (1.1) and multiplying the resulting equation by $\nabla \theta$ lead to
\[ \frac{1}{2} (\partial_t + u \cdot \nabla) |\nabla \theta|^2 + \nabla \theta \cdot \Lambda \nabla \theta = -\nabla u : \nabla \theta \cdot \nabla \theta. \] (2.34)
Thus, by making use of (2.33) with $p = \infty$, we immediately arrive at
\[ \frac{1}{2} (\partial_t + u \cdot \nabla + \Lambda) |\nabla \theta|^2 + c_1 \frac{|\nabla \theta(x)|^{2+\beta}}{\|\theta_0\|_{L^\infty}^{\beta}} \leq -\nabla u : \nabla \theta \cdot \nabla \theta. \] (2.35)
Suppose that $|\nabla \theta(x, t)|$ achieves the maximum at the point $\tilde{x} = \tilde{x}(t)$, then we get
\[ \partial_t |\nabla \theta(\tilde{x}, t)|^2 + c_1 \frac{\Phi(t)^{2+\beta}}{\|\theta_0\|_{L^\infty}^{\beta}} \leq \Phi(t)^2 \|\nabla u\|_{L^\infty}, \] (2.36)
where
\[ \Phi(t) = \|\nabla \theta(., t)\|_{L^\infty}. \]
Similarly, let us assume that $|\omega(x, t)|$ achieves the maximum at the point $\hat{x} = \hat{x}(t)$ and denote
\[ \Omega(t) = \|\omega(., t)\|_{L^\infty}. \]
Recalling the vorticity equation
\[ \partial_t \omega + (u \cdot \nabla) \omega + \Lambda \omega = \partial \omega \theta, \]
and adapting the same argument used above, we can conclude that
\[ \partial_t |\omega(\hat{x}, t)|^2 + c_2 \frac{\Omega(t)^{2+\alpha r}}{\|u\|_{L^r}^{\alpha r}} \leq \Phi(t)\Omega(t), \] (2.37)
where the number $r \in (2, \infty)$ will be fixed hereafter.
To bound $\|\nabla u(., t)\|_{L^\infty}$, we need the following logarithmic inequality which was established in (13)
\[ \|\nabla u(., t)\|_{L^\infty} \leq C_0 + C_0 \Omega(t) + C_0 \Omega(t) \log_+ \left(1 + \int_0^t \left(1 + K(\tau) + \Omega(\tau) + \Phi(\tau)\right)^\Gamma d\tau\right) \] (2.38)
where $C_0 > 0$ is a constant depending on initial data, $K(\tau)$ is a bounded function on the interval $[0, T]$ and $\Gamma = \Gamma(\alpha, \beta)$.
Therefore, it follows from (2.36) and (2.37) that
\[ \partial_t |\nabla \theta(\tilde{x}, t)|^2 + C_1 \Phi(t)^{2+\beta} \leq C_0 \Phi(t)^2 \left\{1 + \Omega(t) + \Omega(t) \log_+ \left(1 + \int_0^t \left(1 + K(\tau) + \Omega(\tau) + \Phi(\tau)\right)^\Gamma d\tau\right)\right\}, \] (2.39)
\[ \partial_t |\omega(\hat{x}, t)|^2 + C_2 \Omega(t)^{2+\alpha r} \leq \Phi(t)\Omega(t), \] (2.40)
where $\Phi(t) = |\nabla \theta(\tilde{x}, t)|$ and $\Omega(t) = |\omega(\hat{x}, t)|$, and the constants $C_0, C_1, C_2$ depend on the initial data, $\alpha, \beta$ and $\|u(T)\|_{L^r}$.
Suppose that $M > 0$ is large enough to be fixed hereafter. Assuming the solutions blow...
up at time $T$, thus $\lim_{t \to T} \Phi(t) = \infty$. Then we can select $T_0 \in (0, T)$ as the first time such that $\Phi(T_0) = M \geq 4\Phi(0)$. Now one may deduce from (2.40) that for any $t \in [0, T_0]$

$$\Omega(t) \leq \max\left\{ \Omega(0), \left( \frac{M}{C_2} \right)^{\frac{2 + \alpha r}{2 + (1 + \alpha) r}} \right\} = \left( \frac{M}{C_2} \right)^{\frac{2 + \alpha r}{2 + (1 + \alpha) r}} := \tilde{M},$$

as long as $M$ is large enough in terms of $\Omega(0)$, $\alpha$, $r$ and $C_2$. Let us give details about how to get the above estimate. Actually, if $\Omega(t) \geq \tilde{M}$, then

$$C_2 \Omega(t)^{2 + \frac{\alpha r}{2 + \alpha r}} - \Phi(t) \Omega(t) \geq C_2 \Omega(t)^{2 + \frac{\alpha r}{2 + \alpha r}} - M \Omega(t) \geq (C_2 \Omega(t)^{1 + \frac{\alpha r}{2 + \alpha r}} - M) \Omega(t) \geq (C_2 \tilde{M}^{1 + \frac{\alpha r}{2 + \alpha r}} - M) \Omega(t) = 0.$$

Thus it follows from (2.40) that $\partial_t |\omega(x, t)|^2 \leq 0$. This implies that $\Omega(t)$ cannot exceed the value $\tilde{M}$. Hence, the following inequality is an easy consequence of (2.39)

$$\partial_t |\nabla \theta(x, t)|^2 + C_1 \Phi(t)^{2 + \beta} \leq C_0 \Phi(t)^2 \left\{ 1 + \tilde{M} + \tilde{M} \log_+ \left( 1 + (1 + K(T) + \tilde{M} + M)^{\Gamma} \right) \right\}.$$

Repeating the same argument as above, we obtain

$$\Phi(t)^\beta \leq \max\left\{ \Phi(0)^\beta, \frac{C_0}{C_1} \left( 1 + \tilde{M} + \tilde{M} \log_+ \left( 1 + (1 + K(T) + \tilde{M} + M)^{\Gamma} \right) \right) \right\},$$

for any $t \in [0, T_0]$. Now notice that $\tilde{M} \approx M^{\frac{2 + \alpha r}{2 + (1 + \alpha) r}}$, we select $M$ large enough such that

$$\frac{C_0}{C_1} \left( 1 + \tilde{M} + \tilde{M} \log_+ \left( 1 + (1 + K(T) + \tilde{M} + M)^{\Gamma} \right) \right) \leq \left( \frac{M}{4} \right)^\beta,$$

which is equivalent to

$$1 + M^{\frac{2 + \alpha r}{2 + (1 + \alpha) r}} (1 + \log_+ M) \leq \frac{M^\beta}{C^\gamma}. \quad (2.41)$$

Thanks to the fact $\beta > \frac{1}{1 + \alpha}$, it is sufficient to choose $r$ as

$$r_0 < r < \infty, \quad r_0 = \max\left\{ \frac{2(1 - \beta)}{(1 + \alpha)\beta - 1}, 2 \right\},$$

then the above inequality (2.41) can be guaranteed due to the following fact

$$r_0 < r \Rightarrow \frac{2 + r}{2 + (1 + \alpha) r} < \beta.$$

Hence, it is not difficult to verify that $\Phi(T_0) \leq M^\frac{4}{4}$, which contradict the definition of $T_0$. We thus get the fact that $\Phi(t)$ never blows up as $t \to T$ when $T < \infty$. As a direct consequence of above fact, we infer that

$$\sup_{0 \leq t \leq T} \| \nabla \theta(t) \|_{L^\infty} \leq C(T, u_0, \theta_0) < \infty.$$

As a consequence of the above estimate, it follows from the vorticity equation (2.27) that for any $0 \leq t \leq T$

$$\| \omega(t) \|_{L^\infty} \leq \| \omega_0 \|_{L^\infty} + \int_0^t \| \nabla \theta(\tau) \|_{L^\infty} \ d\tau \leq C(T, u_0, \theta_0) < \infty.$$
This concludes the proof of Lemma 2.12.

3. The proof of Theorem 1.1

In this section we give the proof of Theorem 1.1. With the estimates (2.31) and (2.32) at hand, the proof can be performed as the classical approach.

**Proof of Theorem 1.1.** To begin with, we smooth the initial data to consider the following approximate system

\[
\begin{aligned}
\partial_t u^{(N)} + (u^{(N)} \cdot \nabla) u^{(N)} + \Lambda^\alpha u^{(N)} + \nabla p^{(N)} &= \theta^{(N)} e_2, \quad x \in \mathbb{R}^2, \ t > 0, \\
\partial_t \theta^{(N)} + (u^{(N)} \cdot \nabla) \theta^{(N)} + \Lambda^\beta \theta^{(N)} &= 0, \quad x \in \mathbb{R}^2, \ t > 0, \\
\nabla \cdot u^{(N)} &= 0, \quad x \in \mathbb{R}^2, \ t > 0, \\
u^{(N)}(x, 0) &= S_N u_0(x), \quad \theta^{(N)}(x, 0) = S_N \theta_0(x), \quad x \in \mathbb{R}^2,
\end{aligned}
\]  
\tag{3.1}

where \( S_N \) is the low-frequency cut-off operator (see Appendix for its definition).

Now we apply \((I + \Lambda)^s\) to system (3.1) and multiply the resulting equations by \((I + \Lambda)^s u^{(N)}\) and \((I + \Lambda)^s \theta^{(N)}\) respectively, add them up to conclude that

\[
\begin{aligned}
\frac{d}{dt}(\|u^{(N)}(t)\|_{H^s}^2 + \|\theta^{(N)}(t)\|_{H^s}^2) + \|u^{(N)}\|_{H^{s+\frac{1}{2}}}^2 + \|\theta^{(N)}\|_{H^{s+\frac{1}{2}}}^2 \leq C(1 + \|\nabla u^{(N)}\|_{L^\infty} + \|\nabla \theta^{(N)}\|_{L^\infty})(\|u^{(N)}\|_{H^s}^2 + \|\theta^{(N)}\|_{H^s}^2),
\end{aligned}
\]

where we have used the embedding \( H^s(\mathbb{R}^2) \hookrightarrow W^{1,\infty}(\mathbb{R}^2) \) for any \( s > 2 \).

Therefore, there exists a time

\[
T^* := C^* \left(1 + \|u_0\|_{H^s}^2 + \|\theta_0\|_{H^s}^2\right)^{-\frac{1}{2}}
\]

for some absolute constant \( C^* > 0 \) such that

\[
\begin{aligned}
u^{(N)} \in L^\infty([0, T^*); H^s(\mathbb{R}^2)) \cap L^2([0, T^*); H^{s+\frac{1}{2}}(\mathbb{R}^2)), \\
\theta^{(N)} \in L^\infty([0, T^*); H^s(\mathbb{R}^2)) \cap L^2([0, T^*); H^{s+\frac{1}{2}}(\mathbb{R}^2)).
\end{aligned}
\]

Note that

\[
\begin{aligned}
\partial_t u^{(N)} &= -(\mathcal{P}((u^{(N)} \cdot \nabla) u^{(N)} - \Lambda^\alpha u^{(N)}) + \mathcal{P} \theta^{(N)} e_2, \\
\partial_t \theta^{(N)} &= -(u^{(N)} \cdot \nabla) \theta^{(N)} - \Lambda^\beta \theta^{(N)},
\end{aligned}
\]

where \( \mathcal{P} \) denote the Leray projection onto divergence-free vector fields.

Thus, it is not hard to see that

\[
\begin{aligned}
\partial_t u^{(N)}, \quad \partial_t \theta^{(N)} \in L^\infty_t([0, T^*]; H^{s-1}_x(\mathbb{R}^2)).
\end{aligned}
\]

Consequently, we assume that

\[
\begin{aligned}
\partial_t u^{(N)}, \quad \partial_t \theta^{(N)} \in L^4_{t, loc}([0, T^*]; H^{s-1}_x(\mathbb{R}^2)).
\end{aligned}
\]

Since the embedding \( H^s \hookrightarrow H^{s-1} \) is locally compact, the well-known Aubin-Lions argument and Cantor’s diagonal process, we conclude that there exists a solution satisfying

\[
\begin{aligned}
u \in L^\infty([0, T^*); H^s(\mathbb{R}^2)) \cap L^2([0, T^*); H^{s+\frac{1}{2}}(\mathbb{R}^2)),
\end{aligned}
\]
\[ \theta \in L^\infty([0, T^*); H^s(\mathbb{R}^2)) \cap L^2([0, T^*); H^{s+\frac{2}{p}}(\mathbb{R}^2)). \]

The continuity of \( u \) and \( \theta \) in time, namely \( u, \theta \in C([0, T^*); H^s(\mathbb{R}^2)) \) can be obtained by a standard approach. It suffices to consider \( u \in C([0, T^*); H^s(\mathbb{R}^2)) \) as the same fashion can be applied to \( \theta \) to obtain the desired result.

By the equivalent norm, it yields

\[ \|u(t_1) - u(t_2)\|_{H^s} = \left\{ \left( \sum_{j < N} + \sum_{j \geq N} (2^{js}\|\Delta_j u(t_1) - \Delta_j u(t_2)\|_{L^2})^2 \right)^{\frac{1}{2}}, \quad (3.3) \right. \]

where \( \Delta_j \) is the non-homogeneous Littlewood-Paley operator (see Appendix for its definition). Let \( \varepsilon > 0 \) be arbitrarily small. Due to \( u \in L^\infty([0, T^*); H^s(\mathbb{R}^2)) \), there exists an integer \( N > 0 \) such that

\[ \left\{ \sum_{j \geq N} (2^{js}\|\Delta_j u(t_1) - \Delta_j u(t_2)\|_{L^2})^2 \right\}^{\frac{1}{2}} < \frac{\varepsilon}{2}. \quad (3.4) \]

Recalling the system (1.1), we obtain for \( 0 \leq t_1 < t_2 < T^* \) that

\[ \Delta_j u(t_1) - \Delta_j u(t_2) = \int_{t_1}^{t_2} \frac{d}{d\tau} \Delta_j u(\tau) \, d\tau \]

\[ = \int_{t_1}^{t_2} \Delta_j \mathcal{P}[\theta e_2 - (u \cdot \nabla)u - \Lambda^\alpha u](\tau) \, d\tau. \quad (3.5) \]

Therefore, we can get

\[ \sum_{j < N} 2^{2js}\|\Delta_j u(t_1) - \Delta_j u(t_2)\|_{L^2}^2 \]

\[ = \sum_{j < N} 2^{2js} \left( \| \int_{t_1}^{t_2} \Delta_j \mathcal{P}[\theta e_2 - (u \cdot \nabla)u - \Lambda^\alpha u](\tau) \, d\tau \|_{L^2} \right)^2 \]

\[ \leq \sum_{j < N} 2^{2js} \left( \int_{t_1}^{t_2} \left[ \| \Delta_j \theta \|_{L^2} + \| \Delta_j (u \cdot \nabla u) \|_{L^2} + \| \Delta_j \Lambda^\alpha u \|_{L^2} \right](\tau) \, d\tau \right)^2 \]

\[ = \sum_{j < N} 2^{2j} \left( \int_{t_1}^{t_2} [2^{j(s-1)} \| \Delta_j \theta \|_{L^2} + 2^{j(s-1)} \| \Delta_j (u \cdot \nabla u) \|_{L^2} + 2^{j(s-1)+\alpha} \| \Delta_j u \|_{L^2}) \, d\tau \right)^2 \]

\[ \leq C \sum_{j < N} 2^{2j} \left( \| \theta \|_{H^{s-1}}^2 |t_1 - t_2| + \| (u \cdot \nabla)u \|_{H^{s-1}}^2 |t_1 - t_2| + \| u \|_{H^{s-1+\alpha}}^2 |t_1 - t_2| \right) \]

\[ \leq C \sum_{j < N} 2^{2j} \left( \| \theta \|_{H^{s-1}}^2 + \| u \|_{L^\infty}^2 \| \nabla u \|_{H^{s-1}}^2 + \| \nabla u \|_{L^\infty}^2 \| u \|_{H^{s-1}}^2 + \| u \|_{H^{s}}^2 \right) \]

\[ \leq C2^{2N} |t_1 - t_2| \left( \| u \|_{H^s}^2 + \| u \|_{H^{s}}^2 + \| \theta \|_{H^s}^2 \right), \quad (3.6) \]

where the Sobolev imbeddings \( H^s(\mathbb{R}^2) \hookrightarrow H^{s-1}(\mathbb{R}^2) \) and \( H^{s-1}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2) \) with \( s > 2 \) are used several times in the last inequality.

Thus, the following holds true

\[ \left\{ \sum_{j < N} (2^{js}\|\Delta_j u(t_1) - \Delta_j u(t_2)\|_{L^2})^2 \right\}^{\frac{1}{2}} < \frac{\varepsilon}{2}. \quad (3.7) \]
provided $|t_1 - t_2|$ small enough.

Combining (3.4) with (3.7) implies $u \in C([0, T^*); H^s(\mathbb{R}^2))$. Moreover, the uniqueness is clear since the velocity and the temperature are both in Lipschitz spaces.

Now, it remains for us to show that the local smooth solutions may be extended to all positive time. It suffices to state that under the assumption of the theorem and any given $T > 0$, we have

$$\sup_{0 \leq t \leq T} (\|u(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2) + \int_0^T (\|u(t)\|_{H^{s+\frac{3}{2}}}^2 + \|\theta(t)\|_{H^{s+\frac{3}{2}}}^2) \, dt \leq C(T, u_0, \theta_0).$$

In consequence, the energy estimate (3.2) ensures that

$$\frac{d}{dt} (\|u(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2) + \|u\|_{H^{s+\frac{3}{2}}}^2 + \|\theta\|_{H^{s+\frac{3}{2}}}^2 \leq C(1 + \|\nabla u\|_{L^\infty} + \|\nabla \theta\|_{L^\infty})(\|u\|_{H^s}^2 + \|\theta\|_{H^s}^2).$$

To obtain the global existence of smooth solutions, the standard procedure is to bound the term $\|\nabla u\|_{L^\infty}$ with $\|\omega\|_{L^\infty}$ and a Sobolev extrapolation inequality with logarithmic correction (see e.g., [31, 6])

$$\|\nabla u\|_{L^\infty(\mathbb{R}^2)} \leq C \left(1 + \|u\|_{L^2(\mathbb{R}^2)} + \|\omega\|_{L^\infty(\mathbb{R}^2)} \ln(e + \|u\|_{H^s(\mathbb{R}^2)})\right), \quad s > 2.$$

Consequently, it enables us to get

$$\frac{d}{dt} (\|u(t)\|_{H^s}^2 + \|\theta(t)\|_{H^s}^2) + \|u\|_{H^{s+\frac{3}{2}}}^2 + \|\theta\|_{H^{s+\frac{3}{2}}}^2 \leq C(1 + \|\omega\|_{L^\infty} + \|\nabla \theta\|_{L^\infty}) \ln(e + \|u\|_{H^s} + \|\theta\|_{H^s})(\|u\|_{H^s}^2 + \|\theta\|_{H^s}^2). \quad (3.8)$$

Applying the log-Gronwall type inequality as well as the estimates (2.31) and (2.32), we eventually obtain the desired estimates. This concludes the proof of Theorem 1.1. \( \square \)

**Appendix A. The proof of Lemmas 2.1 and 2.3**

Before proving Lemmas 2.1 and 2.3, we first recall the so-called Littlewood-Paley operators and their elementary properties which allow us to define the Besov spaces (see for example [3, 11, 33, 39]). It will be also convenient to introduce some function spaces and review some well-known facts.

Let $(\chi, \varphi)$ be a couple of smooth functions with values in $[0, 1]$ such that $\chi \in C_0^\infty(\mathbb{R}^n)$ is supported in the ball $B := \{ \xi \in \mathbb{R}^n, |\xi| \leq \frac{4}{3} \}$, $\varphi \in C_0^\infty(\mathbb{R}^n)$ is supported in the annulus $C := \{ \xi \in \mathbb{R}^n, \frac{2}{3} \leq |\xi| \leq \frac{5}{3} \}$ and satisfy

$$\chi(\xi) + \sum_{j \in \mathbb{N}} \varphi(2^{-j} \xi) = 1, \quad \forall \xi \in \mathbb{R}^n, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

For every $u \in S'$ (tempered distributions) we define the non-homogeneous Littlewood-Paley operators as follows,

$$\Delta_j u = 0 \quad j \leq -2; \quad \Delta_{-1} u = \chi(D) u; \quad \forall j \in \mathbb{N}, \quad \Delta_j u = \varphi(2^{-j} D) u.$$

We shall also denote

$$S_j u := \sum_{-1 \leq k \leq j-1} \Delta_k u, \quad \tilde{\Delta}_j u := \Delta_{j-1} u + \Delta_j u + \Delta_{j+1} u.$$
We now point out several simple facts concerning the operators $\Delta_j$: By compactness of the supports of the series of Fourier transform, we have
\[
\Delta_j \Delta_l u \equiv 0, \quad |j - l| \geq 2 \quad \text{and} \quad \Delta_k(S_l u \Delta_l v) \equiv 0, \quad |k - l| \geq 5.
\]
for any $u$ and $v$. Moreover, it is easy to check that
\[
\supp \mathcal{F}(S_{j-1} u \Delta_j v) \approx \{ \xi : \frac{1}{12} 2^j \leq |\xi| \leq \frac{10}{3} 2^j \},
\]
\[
\supp \mathcal{F}(\Delta_j u \Delta_j v) \subset \{ \xi : |\xi| \leq 8 \times 2^j \},
\]
where $\mathcal{F}$ denotes the Fourier transform and $A \approx B$ to denote $C^{-1} B \leq A \leq C B$ for some positive constant $C$.

Let us recall the definition of homogeneous and inhomogeneous Besov spaces through the dyadic decomposition.

**Definition A.1.** Let $s \in \mathbb{R}, (p, r) \in [1, +\infty]^2$. The inhomogeneous Besov space $B^s_{p,r}$ are defined as a space of $f \in S'(\mathbb{R}^n)$ such that
\[
B^s_{p,r} = \{ f \in S'(\mathbb{R}^n), \| f \|_{B^s_{p,r}} < \infty \},
\]
where
\[
\| f \|_{B^s_{p,r}} = \left\{ \left( \sum_{j \geq -1} 2^{jr s} \| \Delta_j f \|_{L^p}^r \right)^{\frac{1}{r}}, \quad \forall \ r < \infty, \right. \]
\[
\left. \sup_{j \geq -1} 2^{js} \| \Delta_j f \|_{L^p}, \quad \forall \ r = \infty. \right.
\]

Many frequently used function spaces are special cases of Besov spaces. For $s \in \mathbb{R}, (p, r) \in [1, +\infty]^2$, we have the following fact
\[
\| f \|_{B^s_{2,2}} \approx \| f \|_{H^s}.
\]
For any $s \in \mathbb{R}$ and $1 < q < \infty$,
\[
B^s_{q,\min(q,2)} \hookrightarrow W^{s,q} \hookrightarrow B^s_{q,\max(q,2)}.
\]

Bernstein inequalities are fundamental in the analysis involving Besov spaces and these inequalities trade integrability for derivatives.

**Lemma A.2** (see [3]). Let $k \in \mathbb{N} \cup \{0\}, 1 \leq a \leq b \leq \infty$. Assume $k = |\alpha|$, then there exist positive constants $C_1$ and $C_2$ independent of $j$ and $f$ only such that
\[
\sup \hat{f} \subset \{ \xi \in \mathbb{R}^n : |\xi| \lesssim 2^j \} \Rightarrow \| \partial^\alpha f \|_{L^b} \leq C_1 2^{jk + jn\left(\frac{1}{2} - \frac{1}{b}\right)} \| f \|_{L^a};
\]
\[
\sup \hat{f} \subset \{ \xi \in \mathbb{R}^n : |\xi| \approx 2^j \} \Rightarrow C_1 2^{jk} \| f \|_{L^b} \leq \| \partial^\alpha f \|_{L^b} \leq C_2 2^{jk + jn\left(\frac{1}{2} - \frac{1}{b}\right)} \| f \|_{L^a}.
\]
Here we use $A \lesssim B$ to denote $A \leq CB$ for some positive constant $C$.

To prove Lemmas 2.1 and 2.3 the following lemma will be used extensively.

**Proposition A.3.** Given $(p_1, p_2) \in [2, \infty]^2$ and $p \in [2, \infty)$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$. Let $f, g$ and $h$ be three functions such that $\nabla f \in L^{p_1}, g \in L^{p_2}$ and $xh \in L^1$. Then it holds
\[
\| h \ast (fg) - f (h \ast g) \|_{L^p} \leq \| xh \|_{L^1} \| \nabla f \|_{L^{p_1}} \| g \|_{L^{p_2}},
\]
where $\ast$ stands for the convolution symbol.
Proof of Proposition A.3 We remark that Proposition A.3 with \( p_1 = p \) and \( p_2 = \infty \) has been proven in [22]. The interested reader may refer to [24] for general case. To facilitate the reader, we give the details.

By direct calculation, one may easily show that

\[
(h \ast (fg))(x) - f(h \ast g)(x) = \int_{\mathbb{R}^2} h(x-y)g(y)(f(y) - f(x)) \, dy
\]

\[
= \int_{\mathbb{R}^2} \int_0^1 h(x-y)g(y)(y-x)(\nabla f)(x+(y-x)t) \, dy \, dt
\]

\[
= \int_{\mathbb{R}^2} \int_0^1 h\left(\frac{z}{t}\right)g\left(x - \frac{z}{t}\right)\frac{z}{t^2}(\nabla f)(x-z) \, dz \, dt.
\]

According to the Minkowski inequality and the Hölder inequality, one has

\[
\|h \ast (fg) - f(h \ast g)\|_{L^p} \leq \int_{\mathbb{R}^2} \int_0^1 h\left(\frac{z}{t}\right)\frac{z}{t^2}\|\nabla f\|_{L^p_1} \|g\|_{L^p_2} \, dz \, dt
\]

\[
\leq \|xh\|_{L^1} \|\nabla f\|_{L^p_1} \|g\|_{L^p_2}, \quad (A.1)
\]

which is nothing but the desired result. \(\square\)

Now let us proceed to prove Lemma 2.1. To start, we use Bony’s decomposition to present the commutator as

\[
\Delta_k [A^\delta, f]g = \sum_{|j-k| \leq 4} \Delta_k (\{A^\delta, S_{j-1}f\}\Delta_jg) + \sum_{|j-k| \leq 4} \Delta_k (\{A^\delta, \Delta_jf\}S_{j-1}g)
\]

\[
+ \sum_{j-k \geq 4} \Delta_k (\{A^\delta, \Delta_jf\} \tilde{\Delta}_jg)
\]

\[
:= N_1 + N_2 + N_3. \quad (A.2)
\]

Now we recall the following fact. Let \( A \) bet an annulus centered at the origin. Then for every \( F \) with spectrum supported on \( 2^j A \), there exists \( \eta \in \mathcal{S}(\mathbb{R}^n) \) whose Fourier transform supported away from the origin, such that

\[
A^\delta F = 2^j(\eta(\cdot)) \ast F.
\]

For fixed \( k \), the summation over \( |j-k| \leq 4 \) involves only a finite number of \( j \)'s. For the sake of brevity, we shall replace the summations by their representative term with \( j = k \) in \( N_1 \) and \( N_2 \). In view of the above fact, Berstein’s lemma and Proposition A.3, we thus get

\[
\|N_1\|_{L^p} \leq C\|x 2^{k(\delta - \frac{1}{2})} \eta(2^k x)\|_{L^1} \|\nabla S_{k-1}f\|_{L^p_1} \|\Delta_k g\|_{L^p_2}
\]

\[
\leq C 2^k(\delta - 1) \|\nabla f\|_{L^p_1} \|\Delta_k g\|_{L^p_2}. \quad (A.3)
\]

Similarly, one can also deduce that

\[
\|N_2\|_{L^p} \leq C\|x 2^{k(\delta - \frac{1}{2})} \eta(2^k x)\|_{L^1} \|\Delta_k \nabla f\|_{L^p_1} \|S_{k-1}g\|_{L^p_2}
\]

\[
\leq C 2^k(\delta - 1) \|\nabla f\|_{L^p_1} \sum_{l \leq k-2} \|\Delta_l g\|_{L^p_2}
\]

\[
\leq C \|\nabla f\|_{L^p_1} \sum_{l \leq k-2} 2^{(k-l)(\delta - 1)} \|\Delta_l g\|_{L^p_2}. \quad (A.4)
\]
Finally, the last term $N_3$ can be rewritten as

$$N_3 = \sum_{j-k \geq -4} \Delta_k \left( \Lambda^\delta (\Delta_j f \tilde{\Lambda} g) - \Delta_j f \Lambda^\delta \tilde{\Lambda} g \right)$$

$$= \sum_{j-k \geq -4, j \geq 0} \Delta_k \left( \Lambda^\delta (\Delta_j f \tilde{\Lambda} g) - \Delta_j f \Lambda^\delta \tilde{\Lambda} g \right)$$

$$+ \sum_{-1-k \geq -4} \Delta_k \left( \Lambda^\delta (\Delta_{-1} f \tilde{\Lambda}^{-1} g) - \Delta_{-1} f \Lambda^\delta \tilde{\Lambda}^{-1} g \right)$$

$$:= N_{31} + N_{32} \quad (A.5)$$

By Berstein’s lemma, the term $N_{31}$ can be bounded without using commutator structure

$$\|N_{31}\|_{L^p} \leq C \sum_{j-k \geq -4, j \geq 0} \left( \|\Delta_k (\Lambda^\delta (\Delta_j f \tilde{\Lambda} g))\|_{L^p} + \|\Delta_k (\Delta_j f \Lambda^\delta \tilde{\Lambda} g)\|_{L^p} \right)$$

$$\leq C \sum_{j-k \geq -4, j \geq 0} \left( \|\Lambda^\delta (\Delta_j f \tilde{\Lambda} g)\|_{L^p} + \|\Delta_j f \Lambda^\delta \tilde{\Lambda} g\|_{L^p} \right)$$

$$\leq C \sum_{j-k \geq -4, j \geq 0} 2^{j(\delta - 1)} \|\Delta_j \nabla f\|_{L^p} \|\Delta_j g\|_{L^p}$$

$$\leq C \sum_{j-k \geq -4} 2^{j(\delta - 1)} \|\nabla f\|_{L^p} \|\Delta_j g\|_{L^p} \quad (A.6)$$

Resorting Berstein’s lemma again, the term $N_{32}$ admits the following bound

$$\|N_{32}\|_{L^p} \leq \sum_{-1 \leq k \leq 3} \|\Delta_k (\Lambda^\delta (\Delta_{-1} f \tilde{\Lambda}^{-1} g))\|_{L^p} + \sum_{-1 \leq k \leq 3} \|\Delta_k (\Delta_{-1} f \Lambda^\delta \tilde{\Lambda}^{-1} g)\|_{L^p}$$

$$\leq C \sum_{-1 \leq k \leq 3} (2^{k\delta} + 1) \|\Delta_{-1} f\|_{L^{2p}} \|\tilde{\Lambda}^{-1} g\|_{L^{2p}}$$

$$\leq C \chi_{\{-1 \leq k \leq 3\}} \|f\|_{L^2} \|g\|_{L^2} \quad (A.7)$$

By the definition of $B_{p,r}^s$, we have

$$\|\Lambda^\delta, f\|_{B_{p,r}^s} \leq \|2^{ks} N_1\|_{L^p} \|_{L^r} + \|2^{ks} N_2\|_{L^p} \|_{L^r} + \|2^{ks} N_3\|_{L^p} \|_{L^r}$$

$$\leq C \|\nabla f\|_{L^p} \|2^{k(s+\delta - 1)} \|\Delta_k g\|_{L^{p_2}} \|_{L^r}$$

$$+ C \|\nabla f\|_{L^p} \||\sum_{l \leq k - 2} 2^{(k-l)(s+\delta - 1)} 2^{l(s+\delta - 1)} \|\Delta_l g\|_{L^{p_2}} \|_{L^r} \quad (s + \delta - 1 < 0)$$

$$+ C \|\nabla f\|_{L^p} \||\sum_{j-k \geq -4} 2^{(k-j)s} 2^{j(s+\delta - 1)} \|\Delta_j g\|_{L^{p_2}} \|_{L^r} \quad (s > 0)$$

$$+ C \|f\|_{L^2} \|g\|_{L^2} \quad (A.8)$$

Therefore, this concludes the proof of Lemma 2.1.

The proof of Lemma 2.3 is very similar to that of Lemma 2.1. Indeed, we can regard the operator $R_\beta$ as the operator $\Lambda^{1-\beta}$ without any difference. Now we just view $f = u,$
\[ g = \nabla \theta \text{ and } \delta = 1 - \beta. \text{ Therefore, we have} \]

\[
\Delta_k [R_\beta, u \cdot \nabla] \theta = \sum_{|j-k| \leq 4} \Delta_k ( [R_\beta, S_{j-1} u \cdot \nabla] \Delta_j \theta) + \sum_{|j-k| \leq 4} \Delta_k ( [R_\beta, \Delta_j u \cdot \nabla] S_{j-1} \theta) 
\]

\[
+ \sum_{j-k \geq -4} \Delta_k ( [R_\beta, \Delta_j u \cdot \nabla] \tilde{\Delta}_j \theta) 
\]

\[
:= \tilde{N}_1 + \tilde{N}_2 + \tilde{N}_3.
\]

The same as \( N_1 \) and \( N_2 \), we can conclude

\[
\| \tilde{N}_1 \|_{L^p} \leq C 2^{k(1-\beta)} \| \nabla u \|_{L^p} \| \Delta_k \theta \|_{L^p}
\]

and

\[
\| \tilde{N}_2 \|_{L^p} \leq C \| \nabla u \|_{L^p} \sum_{l \leq k-2} 2^{(l-k)\beta} 2^{l(1-\beta)} \| \Delta_l \theta \|_{L^p}.
\]

However, we need to deal with the term \( \tilde{N}_3 \) differently. We rewrite \( \tilde{N}_3 \) as

\[
\tilde{N}_3 = \sum_{j-k \geq -4} \Delta_k ( [R_\beta, \Delta_j u \cdot \nabla] \tilde{\Delta}_j \theta) 
\]

\[
= \sum_{j-k \geq -4, j \geq 0} \Delta_k ( R_\beta (\Delta_j u \cdot \nabla \tilde{\Delta}_j \theta) - \Delta_j u \cdot \nabla R_\beta \tilde{\Delta}_j \theta) 
\]

\[
- \sum_{j-k \geq -4, j \geq 0} \Delta_k ( R_\beta (\Delta_{j-1} u \cdot \nabla \tilde{\Delta}_{j-1} \theta) - \Delta_{j-1} u \cdot \nabla R_\beta \tilde{\Delta}_{j-1} \theta) 
\]

\[
:= \tilde{N}_{31} + \tilde{N}_{32}.
\]

By Berstein’s lemma and the divergence-free condition

\[
\| \tilde{N}_{31} \|_{L^p} \leq C \sum_{j-k \geq -4, j \geq 0} \left( \left\| \Delta_k ( R_\beta (\Delta_j u \cdot \nabla \tilde{\Delta}_j \theta)) \right\|_{L^p} + \left\| \Delta_k (\Delta_j u \cdot \nabla R_\beta \tilde{\Delta}_j \theta) \right\|_{L^p} \right) 
\]

\[
\leq C \sum_{j-k \geq -4, j \geq 0} \left( \left\| \Delta_k ( R_\beta \nabla \cdot (\Delta_j u \tilde{\Delta}_j \theta)) \right\|_{L^p} + \left\| \Delta_k \nabla \cdot (\Delta_j u R_\beta \tilde{\Delta}_j \theta) \right\|_{L^p} \right) 
\]

\[
\leq C \sum_{j-k \geq -4, j \geq 0} 2^{(k-j)(2-\beta)} \| \nabla u \|_{L^p} 2^{(1-\beta)j} \| \Delta_j \theta \|_{L^p} 
\]

\[
+ C \sum_{j-k \geq -4, j \geq 0} 2^{k-j} \| \nabla u \|_{L^p} 2^{(1-\beta)j} \| \Delta_j \theta \|_{L^p},
\]

and

\[
\| \tilde{N}_{32} \|_{L^p} \leq C_{\chi_{\{-1 \leq k \leq 3\}}} \| u \|_{L^p} \| \theta \|_{L^p}.
\]

Putting all the above estimates together and making use of the Young inequality for series convolution yield

\[
\| [R_\beta, u \cdot \nabla] \theta \|_{L^p} \leq \sum_{k \geq -1} \| \Delta_k [R_\beta, u \cdot \nabla] \theta \|_{L^p} 
\]

\[
\leq C \| \nabla u \|_{L^p} \sum_{k \geq -1} 2^{k(1-\beta)} \| \Delta_k \theta \|_{L^p}.
\]
This completes the proof of Lemma 2.3.

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