Probabilistic properties of the elliptic motion

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Abstract

In this paper we consider the plane elliptic motion which occurs if the moving centrode is a circle of radius $r$ and the fixed centrode a circle of radius $2r$. Every point of the moving plane generates an ellipse in the fixed plane. Let a disk of radius $R, 0 \leq R < \infty$, concentric to the moving centrode be attached to the moving plane. If a point $P$ is chosen at random from this disk, then the area and the perimeter of the ellipse generated by $P$ are random variables. We determine the moments and the distributions of these random variables for the case that $P$ is uniformly distributed over the area of the disk.

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1 Introduction

We consider a fixed Euclidean plane $S$ with a Cartesian frame of origin $O$ and $x, y$ axes, and a moving Euclidean plane $\Sigma$ with a Cartesian frame of origin $\Omega$ and $\xi, \eta$ axes. To $\Sigma$ we attach a circle $C_1$ with centre $\Omega$ and radius $r$; to $S$ we attach a circle $C_0$ with centre $O$ and radius $2r$ (see Fig. 1).

If $C_1$ rolls inside $C_0$, then every point $P$ fixed in $\Sigma$ generates an ellipse in $S$. Therefore, this motion is called elliptic motion. If $P = \Omega$, then the ellipse is a circle with centre $O$ and radius $r$; if $P \in C_1$, then the ellipse degenerates to a (double) line segment of length $4r$ which is one diameter of $C_0$. [4, pp. 2-3, 8-9], [1, pp. 14-15]

We denote by $\varphi$ the angle between the $x$-axis and the line segment $O\Omega$. W.l.o.g. we assume that for $\varphi = 0$ the $\xi$-axis lies on the $x$-axis and both axes have equal direction. Then the equation in complex form of the ellipse generated by $P$ is given by

$$X = X(\varphi) = re^{i\varphi} + \Xi e^{-i\varphi}, \quad 0 \leq \varphi < 2\pi,$$

with

1) $\Xi := \rho e^{i\alpha}$, where $\rho, \alpha$ are the polar coordinates of $P$ with respect to the $\xi, \eta$-frame, or
2) $\Xi := \xi + i\eta$, where $\xi, \eta$ are the Cartesian coordinates of $P$ with respect to the $\xi, \eta$-frame.

In the first case, from (1.1) we get

$$\begin{cases} x = r \cos \varphi + \rho \cos(\varphi - \alpha), \\ y = r \sin \varphi - \rho \sin(\varphi - \alpha) \end{cases}$$

(1.2)

as parametric representation of the ellipse, and in the second case,

$$\begin{cases} x = r \cos \varphi + \xi \cos \varphi + \eta \sin \varphi, \\ y = r \sin \varphi + \eta \cos \varphi - \xi \sin \varphi \end{cases}$$

(1.3)
Fig. 1: Elliptic motion

Fig. 2: 100 random ellipses and their generating points; $r = 1$, $R = 1.5$
From (1.2) one finds that the length of the semi-major axis is equal to \( r + \varrho \), and the length of the semi-minor axis equal to \( |r - \varrho | \). Hence all points \( P \in \Sigma \) with equal distance \( \varrho \) from \( \Omega \) generate congruent ellipses. The centres of all ellipses lie in \( O \). The angle between the \( x \)-axis and the major-axis of an ellipse is equal to \( \alpha / 2 \).

\( C_0 \) is the fixed centrode and \( C_1 \) the moving centrode of the elliptic motion. It is possible to get the equations of \( C_0 \) and \( C_1 \) backwards from the equation(s) of the motion (1.1), (1.2) or (1.3). \[4, \text{p.} 8-9\], \[1, \text{p.} 14-15\] (For the notions of the fixed and the moving centrode see e.g \[2, \text{pp.} 257-259\].)

Now we consider a disk

\[
D_R = \left\{ (\xi, \eta) \in \Sigma : 0 \leq \xi^2 + \eta^2 \leq R^2 \right\}, \quad 0 \leq R < \infty.
\]

If \( P \) is chosen at random from \( D_R \), then the area and the perimeter of the ellipse generated by \( P \) are random variables which we denote by \( A_R \) and \( U_R \), respectively. In this paper we determine the moments and the distributions of these random variables for the case that \( P \) is uniformly distributed over the area of \( D_R \). Since \( r \) is no essential parameter, we identify \( A_x \equiv A_R \) and \( U_x \equiv U_R \), where \( x = R/r \).

Fig. 2 shows the result of a simulation with 100 random points and ellipses.

## 2 Moments of the area

The moments of the random area \( A_R \) enclosed by the ellipse generated by a random point \( \in D_R \) (see (1.4)) are given by

\[
\mathbb{E}[A_R^k] = \left( \int_{P \in D_R} A^k \, dP \right) / \left( \int_{P \in D_R} dP \right),
\]

where \( A \) is the area of the ellipse generated by the point \( P \in D_R \), and \( dP = d\xi \wedge d\eta \) is the density for sets of points in the plane. Up to a constant factor this density is the only one that is invariant under motions \[5, \text{p.} 13\]. Due to the point symmetry we use the polar coordinates \( \varrho, \alpha \). With \( \xi = \varrho \cos \alpha, \eta = \varrho \sin \alpha \) we get

\[
\begin{align*}
    d\xi &= \frac{\partial \xi}{\partial \varrho} \, d\varrho + \frac{\partial \xi}{\partial \alpha} \, d\alpha = \cos \alpha \, d\varrho - \varrho \sin \alpha \, d\alpha, \\
    d\eta &= \frac{\partial \eta}{\partial \varrho} \, d\varrho + \frac{\partial \eta}{\partial \alpha} \, d\alpha = \sin \alpha \, d\varrho + \varrho \cos \alpha \, d\alpha
\end{align*}
\]

and

\[
\begin{align*}
    d\xi \wedge d\eta &= \cos \alpha \sin \alpha \, d\varrho \wedge d\varrho + \varrho \cos^2 \alpha \, d\varrho \wedge d\alpha - \varrho \sin^2 \alpha \, d\alpha \wedge d\varrho - \varrho^2 \sin \alpha \cos \alpha \, d\alpha \wedge d\alpha \\
    &= \varrho \, d\varrho \wedge d\alpha,
\end{align*}
\]

hence

\[
\mathbb{E}[A_R^k] = \frac{1}{\pi R^2} \int_{\varrho=0}^{2\pi} \int_{\varrho=0}^{R} A^k(\varrho) \, \varrho \, d\varrho \, d\alpha = \frac{2}{R^2} \int_{\varrho=0}^{R} A^k(\varrho) \, \varrho \, d\varrho.
\]

The area enclosed by an ellipse generated by a point \( P \in D_R \) with distance \( \varrho \) from \( \Omega \) is given by

\[
A(\varrho) = \begin{cases} 
    \pi \left( r^2 - \varrho^2 \right) & \text{if } \ 0 \leq \varrho \leq r, \\
    \pi \left( \varrho^2 - r^2 \right) & \text{if } \ r < \varrho < \infty.
\end{cases}
\]

With \( w = \varrho/r \), the area (2.2) function may be written as

\[
\tilde{A}(w) = \begin{cases} 
    \pi r^2 \left( 1 - w^2 \right) & \text{if } \ 0 \leq w \leq 1, \\
    \pi r^2 \left( w^2 - 1 \right) & \text{if } \ 1 < w < \infty.
\end{cases}
\]
Theorem 2.1. The k-th moment, \( k = 1, 2, \ldots \), of the random area \( A_\kappa \equiv A_R, \kappa = R/r \), of an ellipse generated by a random point \( P \in D_R \) (\( P \) uniformly distributed over the area of the disk \( D_R \)) is given by

\[
E[A_\kappa^k] = \begin{cases} 
\pi k r 2k & \text{if } \kappa = 0, \\
\pi k r 2k \frac{1 - (1 - \kappa^2)^{k+1}}{k+1} & \text{if } 0 < \kappa \leq 1, \\
\pi k r 2k \frac{1 + (\kappa^2 - 1)^{k+1}}{\kappa^2} & \text{if } 1 < \kappa < \infty.
\end{cases}
\]

Proof. First, we consider the case \( 0 < \kappa \leq 1 \) (\( 0 < R \leq r \)). Eq. (2.1) becomes

\[
E[A_R^k] = \frac{2\pi k}{R^2} \int_0^R (r^2 - \varrho^2)^k \varrho \, d\varrho.
\]

The substitution \( \varrho = rw \) gives

\[
E[A_R^k] = \frac{2\pi k r 2k+2}{R^2} \int_{w=0}^{R/r} (1 - w^2)^k w \, dw,
\]

which, with \( A_\kappa \equiv A_R \) may also be written as

\[
E[A_\kappa^k] = \frac{2\pi k r 2k}{\kappa^2} \int_{w=0}^{\kappa} (1 - w^2)^k w \, dw.
\]

The substitution

\[
y = 1 - w^2, \quad dy = -2w \, dw, \quad dw = -\frac{dy}{2w}
\]

yields

\[
E[A_\kappa^k] = -\frac{\pi k r 2k}{\kappa^2} \int y^k \, dy = \frac{\pi k r 2k}{\kappa^2} \int_{y=1}^{1-\kappa^2} y^k \, dy = \frac{\pi k r 2k}{\kappa^2 (k+1)} \left( \frac{y^{k+1}}{k+1} \right)_{1-\kappa^2} = \frac{\pi k r 2k}{k+1} \frac{1 - (1 - \kappa^2)^{k+1}}{\kappa^2}.
\]

Applying L’Hôpital’s rule we get

\[
\lim_{\kappa \to 0} \frac{1 - (1 - \kappa^2)^{k+1}}{\kappa^2} = \lim_{\kappa \to 0} \frac{(k+1) (1 - \kappa^2)^k 2\kappa}{2\kappa} = (k+1) \lim_{\kappa \to 0} (1 - \kappa^2)^k = k+1,
\]

hence

\[
E[A_0^k] = \pi k r 2k.
\]

Now we consider the case \( 1 < \kappa < \infty \) (\( r < R < \infty \)). Here we have

\[
E[A_R^k] = \frac{2}{R^2} \int_0^R A^k(\varrho) \varrho \, d\varrho = \frac{2}{R^2} \left( \int_0^r A^k(\varrho) \varrho \, d\varrho + \int_r^R A^k(\varrho) \varrho \, d\varrho \right)
\]

\[
= \frac{2}{R^2} \left( \int_0^r \pi k (r^2 - \varrho^2)^k \varrho \, d\varrho + \int_r^R \pi k (\varrho^2 - r^2)^k \varrho \, d\varrho \right)
\]

\[
= \frac{2\pi k r 2k+2}{R^2} \left( \int_0^1 (1 - w^2)^k w \, dw + \int_1^{R/r} (w^2 - 1)^k w \, dw \right),
\]

hence

\[
E[A_\kappa^k] = \frac{2\pi k r 2k}{\kappa^2} \left( \frac{1}{2(k+1)} + \int_1^\kappa (w^2 - 1)^k w \, dw \right) = \frac{2\pi k r 2k}{\kappa^2} \left( \frac{1}{2(k+1)} + \frac{(\kappa^2 - 1)^{k+1}}{2(k+1)} \right)
\]

\[
= \frac{\pi k r 2k}{k+1} \frac{1 + (\kappa^2 - 1)^{k+1}}{\kappa^2}.
\]

\( \Box \)
The graph of $E[A_{\kappa}/r^2]$ is shown in Fig. 3. For $\kappa > 1$ we have

$$\frac{d}{d\kappa} E[A_{\kappa}] = \frac{\pi r^2 (\kappa^4 - 2)}{\kappa^4}.$$ 

It follows that the expectation $E[A_{\kappa}]$ has its global minimum at point $\kappa = \sqrt{2} \approx 1.18921$ with value

$$E\left[A_{\sqrt{2}}\right] = (\sqrt{2} - 1)\pi r^2 \approx 1.30129 r^2.$$

Furthermore one finds that

$$E[A_1] = E\left[A_{\sqrt{2}}\right] = \frac{1}{2} \pi r^2 = \frac{1}{2} \tilde{A}(0) = \frac{1}{2} \tilde{A}(\sqrt{2}) = \frac{1}{2} E[A_0].$$

For the variance of $A_{\kappa}$, $\text{Var}[A_{\kappa}] = E[A_{\kappa}^2] - E[A_{\kappa}]^2$, we get

$$\text{Var}[A_{\kappa}] = \frac{\pi^2 r^4}{3} \frac{1 - (1 - \kappa^2)^3}{\kappa^2} - \frac{\pi^2 r^4}{4} \frac{1 - (1 - \kappa^2)^2}{\kappa^4} = \frac{\pi^2 r^4 \kappa^4}{12}$$

if $0 \leq \kappa \leq 1$, and

$$\text{Var}[A_{\kappa}] = \frac{\pi^2 r^4}{3} \frac{1 + (\kappa^2 - 1)^3}{\kappa^2} - \frac{\pi^2 r^4}{4} \frac{1 + (\kappa^2 - 1)^2}{\kappa^4} = \pi^2 r^4 \left[ -1 - \frac{1}{\kappa^4} + \frac{2}{\kappa^2} + \frac{\kappa^4}{12} \right]$$

if $1 < \kappa < \infty$. One finds that $\text{Var}[A_{\kappa}]$ has local extrema at

$$\kappa_1 \approx 1.06840 \quad \text{and} \quad \kappa_2 \approx 1.30621$$

with values

$$\text{Var}[\kappa_1] \approx 0.920036 \quad \text{and} \quad \text{Var}[\kappa_2] \approx 0.703487,$$

respectively (see Fig. 4).

![Fig. 3: a) $\tilde{A}(\kappa)/r^2$ (see (2.3)), b) $E[A_{\kappa}/r^2$](image)

![Fig. 4: $\text{Var}[A_{\kappa}/r^4$](image)

3 Distribution of the area

Now we determine the distribution function

$$F_{\kappa}(x) = P[A_{\kappa} \leq x]$$

of the random variable $A_{\kappa}$. We have to distinguish the following three cases.
1) \(0 \leq R \leq r\) \((0 \leq \kappa \leq 1)\): The smallest area of an ellipse is equal to \(\pi(r^2 - R^2)\) and the biggest area equal to \(\pi r^2\). We have \(A_R > x\) if \(P\) lies in an open disk with area \(\pi \varrho^2\). From the first equation in (2.2), with \(x = A\) we get
\[
\varrho^2 = r^2 \left( 1 - \frac{x}{\pi r^2} \right).
\]
It follows that
\[
P[A_R > x] = \frac{\pi \varrho^2}{\pi R^2} = \frac{r^2}{R^2} \left( 1 - \frac{x}{\pi r^2} \right)
\]
and
\[
P[A_R \leq x] = 1 - P[A_R > x] = 1 - \frac{r^2}{R^2} \left( 1 - \frac{x}{\pi r^2} \right)
\]
So we have
\[
F_{\kappa}(x) = \begin{cases} 
0 & \text{if } -\infty < x < \pi r^2 \left( 1 - \kappa^2 \right), \\
1 - \frac{1}{\kappa^2} \left( 1 - \frac{x}{\pi r^2} \right) & \text{if } \pi r^2 \left( 1 - \kappa^2 \right) \leq x < \pi r^2, \\
1 & \text{if } \pi r^2 \leq x < \infty.
\end{cases}
\]

2) \(r \leq R \leq \sqrt{2} r\) \((1 \leq \kappa \leq \sqrt{2})\): For \(0 \leq A_R \leq \pi (R^2 - r^2)\) we have \(A_R > x\) if \(P\) lies
a) in an open disk with area \(\pi \varrho^2\) or
b) in an open annulus with area \(\pi (R^2 - \varrho^2)\), where \(\varrho'\) is the \(\varrho\) in the second equation in (2.2).

It follows that
\[
P[A_R > x] = \frac{\pi \varrho^2 + \pi (R^2 - \varrho^2)}{\pi R^2} = 1 + \frac{\varrho^2 - \varrho'^2}{R^2} = 1 + \frac{r^2}{R^2} \left[ 1 - \frac{x}{\pi r^2} - \left( 1 + \frac{x}{\pi r^2} \right) \right]
\]
\[
= 1 - \frac{2x}{\pi R^2},
\]
hence
\[
P[A_R \leq x] = 1 - P[A_R > x] = \frac{2x}{\pi R^2}.
\]
For \(\pi (R^2 - r^2) \leq A_R \leq \pi r^2\) we have \(A_R > x\) if \(P\) lies in an open disk with area \(\pi \varrho^2\). Therefore, the distribution function is given by
\[
F_{\kappa}(x) = \begin{cases} 
0 & \text{if } -\infty < x < 0, \\
\frac{2x}{\pi r^2 \kappa^2} & \text{if } 0 \leq x < \pi r^2 \left( \kappa^2 - 1 \right), \\
1 - \frac{1}{\kappa^2} \left( 1 - \frac{x}{\pi r^2} \right) & \text{if } \pi r^2 \left( \kappa^2 - 1 \right) \leq x < \pi r^2, \\
1 & \text{if } \pi r^2 \leq x < \infty.
\end{cases}
\]

3) \(\sqrt{2} r \leq R < \infty\) \((\sqrt{2} \leq \kappa < \infty)\): One easily finds
\[
F_{\kappa}(x) = \begin{cases} 
0 & \text{if } -\infty < x < 0, \\
\frac{2x}{\pi r^2 \kappa^2} & \text{if } 0 \leq x < \pi r^2, \\
\frac{1}{\kappa^2} \left( 1 + \frac{x}{\pi r^2} \right) & \text{if } \pi r^2 \leq x < \pi r^2 \left( \kappa^2 - 1 \right), \\
1 & \text{if } \pi r^2 \left( \kappa^2 - 1 \right) \leq x < \infty.
\end{cases}
\]
Now we determine the moments of the random variable $A_\kappa$ in an alternative way:

1) $0 \leq \kappa \leq 1$: The density function is given by $f_\kappa(x) = 1/(\pi r^2 x^2)$ if $\pi r^2 (1 - \kappa^2) \leq x < \pi r^2$, and $f_\kappa(x) = 0$ if the area is outside this interval. So we have

$$E[A_\kappa^k] = \int_{\pi r^2(1-\kappa^2)}^{\pi r^2} x^k f_\kappa(x) \, dx = \frac{1}{\pi r^2 \kappa^2} \int_{\pi r^2(1-\kappa^2)}^{\pi r^2} x^k \, dx = \frac{\pi \kappa 2k}{k+1} \frac{1 - (1 - \kappa^2)^{k+1}}{\kappa^2}.$$

2) $1 \leq \kappa < \sqrt{2}$: The restriction of the density function $f_\kappa(x)$ to the interval $0 \leq x \leq \pi r^2$ is given by

$$f_\kappa(x) = \frac{2}{\pi r^2 \kappa^2} \text{ if } 0 \leq x \leq \pi r^2 (\kappa^2 - 1), \text{ and } f_\kappa(x) = \frac{1}{\pi r^2 \kappa^2} \text{ if } \pi r^2 (\kappa^2 - 1) \leq x \leq \pi r^2.$$

It follows that

$$E[A_\kappa^k] = \frac{2}{\pi r^2 \kappa^2} \int_0^{\pi r^2(\kappa^2-1)} x^k \, dx + \frac{1}{\pi r^2 \kappa^2} \int_{\pi r^2(\kappa^2-1)}^{\pi r^2} x^k \, dx$$

$$= \frac{2\pi \kappa 2k (\kappa^2 - 1)^{k+1}}{(k+1) \kappa^2} + \frac{\pi \kappa 2k \left[1 - (\kappa^2 - 1)^{k+1}\right]}{(k+1) \kappa^2} = \frac{\pi \kappa 2k}{k+1} \frac{1 + (\kappa^2 - 1)^{k+1}}{\kappa^2}.$$

3) $\sqrt{2} \leq \kappa < \infty$: One finds

$$E[A_\kappa^k] = \frac{2}{\pi r^2 \kappa^2} \int_0^{\pi r^2(\kappa^2-1)} x^k \, dx + \frac{1}{\pi r^2 \kappa^2} \int_{\pi r^2(\kappa^2-1)}^{\pi r^2} x^k \, dx$$

$$= \frac{2\pi \kappa 2k}{(k+1) \kappa^2} \frac{(\kappa^2 - 1)^{k+1} - 1}{(k+1) \kappa^2} = \frac{\pi \kappa 2k}{k+1} \frac{1 + (\kappa^2 - 1)^{k+1}}{\kappa^2}.$$

4 Moments of the perimeter

By analogy to the determination of $E[A_R^k]$, we get the moments of the perimeter with

$$E[U_R^k] = \left(\int_{P \in D_R} u^k \, dP\right) / \left(\int_{P \in D_R} \, dP\right) = \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R u^k(\rho) \, \rho \, d\rho \, d\alpha$$

$$= \frac{2}{R^2} \int_0^R u^k(\rho) \, \rho \, d\rho,$$

where $u(\rho)$ is the perimeter of an ellipse generated by a point $P \in D_R$ with distance $\rho$ from $\Omega$. The length of the semi-major axis is given by $r + \rho$, and the length of the semi-minor axis by $r - \rho$ or $\rho - r$. In both cases we have

$$u(\rho) = 4(r + \rho) E\left(\frac{2 \sqrt{r\rho}}{r + \rho}\right), \quad (4.1)$$

where

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta$$

is the complete elliptic integral of the second kind with modulus $k$, $0 \leq k \leq 1$. So we have

$$E[U_R^k] = \frac{2^{2k+1}}{R^2} \int_0^R \rho (r + \rho)^k E^k\left(\frac{2 \sqrt{r\rho}}{r + \rho}\right) \, d\rho. \quad (4.2)$$
By substituting \( w = \varrho/r \), the perimeter (4.1) may be written as
\[
\tilde{u}(w) = rh(w), \quad \text{where} \quad h(w) := 4(w+1) E \left( \frac{2\sqrt{w}}{w+1} \right), \quad (4.3)
\]
and, with \( U_{\kappa} \equiv U_{R}, \kappa = R/r \), the integral (4.2) becomes
\[
\mathbb{E}[U_{\kappa}^k] = \frac{2^{2k+1} r^k}{\kappa^2} \int_0^\kappa (w+1)^k E^k \left( \frac{2\sqrt{w}}{w+1} \right) dw. \quad (4.4)
\]

**Theorem 4.1.** The expectation of the random perimeter \( U_{\kappa} = U_{R}, \kappa = R/r \), of an ellipse generated by a random point \( P \in D_R \) (\( P \) uniformly distributed over the area of the disk \( D_R \)) is given by
\[
\mathbb{E}[U_{\kappa}] = \begin{cases} 
2\pi r & \text{if } \kappa = 0, \\
\frac{8\pi}{9\kappa^2} (7\kappa^2 + 1) E(\kappa) + (3\kappa^4 - 2\kappa^2 - 1) K(\kappa) & \text{if } 0 < \kappa < 1, \\
64\pi/9 & \text{if } \kappa = 1, \\
\frac{8r}{9\kappa} (7\kappa^2 + 1) E(\kappa^{-1}) - 4(\kappa^2 - 1) K(\kappa^{-1}) & \text{if } \kappa > 1
\end{cases}
\]
where
\[
K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}
\]
is the complete elliptic integral of the first kind with modulus \( k \), \( 0 \leq k \leq 1 \). The expectations for the cases \( 0 < \kappa < 1 \) and \( \kappa > 1 \) can be subsumed in the formula
\[
\mathbb{E}[U_{\kappa}] = \frac{4(\kappa + 1)r}{9\kappa^2} \left[ (7\kappa^2 + 1) E \left( \frac{2\sqrt{\kappa}}{\kappa+1} \right) - (\kappa - 1)^2 K \left( \frac{2\sqrt{\kappa}}{\kappa+1} \right) \right].
\]

**Proof.** First, we consider the case \( 0 < \kappa < 1 \). Using the relation
\[
E \left( \frac{2\sqrt{k}}{1+k} \right) = \frac{1}{1+k} \left[ 2E(k) - (1-k^2) K(k) \right]
\]
([3, Vol. 2, p. 299], Eq. (8.126.4)), (4.4) becomes
\[
\mathbb{E}[U_{\kappa}] = \frac{8r}{9\kappa^2} \left[ 2 \int_0^\kappa wE(w) dw - \int_0^\kappa wK(w) dw + \int_0^\kappa w^3 K(w) dw \right].
\]
With
\[
\int E(k) k dk = \frac{1}{3} \left[ (1 + k^2) E(k) - (1 - k^2) K(k) \right], \\
\int K(k) k dk = E(k) - (1 - k^2) K(k), \\
\int K(k) k^3 dk = \frac{1}{9} \left[ (4 + k^2) E(k) - (1 - k^2) (4 + 3k^2) K(k) \right]
\]
(Equations (5.112.4), (5.112.3), (5.112.5) in [3, Vol. 2, p. 13]) we get
\[
\mathbb{E}[U_{\kappa}] = \frac{8r}{9\kappa^2} \left[ (7\kappa^2 + 1) E(\kappa) + (3\kappa^4 - 2\kappa^2 - 1) K(\kappa) \right], \quad 0 < \kappa < 1. \quad (4.6)
\]
For \( \varkappa = 0 \) we have \( E(\varkappa) = \pi/2 = K(\varkappa) \), hence

\[
(7\varkappa^2 + 1) E(\varkappa) + (3\varkappa^4 - 2\varkappa^2 - 1) K(\varkappa) = 0.
\]

So \( \mathbb{E}[U_{\varkappa}] \) has the indeterminate form 0/0. Taking

\[
\frac{dE(k)}{dk} = \frac{E(k) - K(k)}{k} \quad \text{and} \quad \frac{dK(k)}{dk} = \frac{E(k) - (1 - k^2)K(k)}{k(1 - k^2)}
\]

(Equations (8.123.4), (8.123.2) in [3, Vol. 2, p. 298]) into account, applying L'Hôpital’s rule twice gives

\[
\lim_{\varkappa \to 0} \mathbb{E}[U_{\varkappa}] = 4r \lim_{\varkappa \to 0} \left[ 3E(\varkappa) + 2(\varkappa^2 - 1)K(\varkappa) \right] = 4r \left[ 3 \frac{\pi}{2} - 2 \frac{\pi}{2} \right] = 2\pi r.
\]

Now we consider the limit of (4.6) for \( \varkappa \to 1 \) (and hence \( R \to r \)). We have \( E(1) = 1, K(1) = \infty \). For \( \varkappa = 1 \),

\[
(3\varkappa^4 - 2\varkappa^2 - 1) K(\varkappa)
\]

has the indeterminate form 0 \cdot \infty. Mathematica finds

\[
\lim_{\varkappa \to 1^-} (3\varkappa^4 - 2\varkappa^2 - 1)K(\varkappa) = 0.
\]

It follows that

\[
\lim_{\varkappa \to 1^-} \mathbb{E}[U_{\varkappa}] = \frac{8r}{9\cdot1} \left( (7 + 1) + 0 \right) = \frac{64r}{9}.
\]

Now we consider the case \( \varkappa = R/r > 1 \). Here we have

\[
\mathbb{E}[U_{\varkappa}] = \frac{64r}{9\varkappa^2} + \frac{8}{R^2} \int_{\varrho = r}^{R} \varrho (r + \varrho) E \left( 2 \frac{\sqrt{r\varrho}}{r + \varrho} \right) \, d\varrho
\]

\[
= \frac{64r}{9\varrho^2} + \frac{8}{R^2} \int_{\varrho = r}^{R} \varrho^2 \left( \frac{r}{\varrho} + 1 \right) E \left( 2 \frac{\sqrt{r\varrho}}{(r/\varrho) + 1} \right) \, d\varrho.
\]

The substitution

\[
v = \frac{r}{\varrho}, \quad dv = -\frac{r}{\varrho^2} \, d\varrho, \quad d\varrho = -\frac{r}{v^2} \, dv, \quad \varrho = r \implies v = 1, \quad \varrho = R \implies v = r/R = 1/\varkappa
\]

gives

\[
\mathbb{E}[U_{\varkappa}] = \frac{64r}{9\varkappa^2} + \frac{8r}{\varkappa^2} \int_{1/\varkappa}^{1} \frac{v + 1}{v^4} E \left( 2 \frac{\sqrt{v}}{1 + v} \right) \, dv.
\]

Applying (4.5), we get

\[
\mathbb{E}[U_{\varkappa}] = \frac{64r}{9\varkappa^2} + \frac{8r}{\varkappa^2} \left[ 2 \int_{1/\varkappa}^{1} \frac{E(v)}{v^4} \, dv - \int_{1/\varkappa}^{1} \frac{K(v)}{v^4} \, dv + \int_{1/\varkappa}^{1} \frac{K(v)}{v^2} \, dv \right]
\]

From [3, Vol. 2, pp. 13-14], Equations (5.112.12), (5.112.9), we know that

\[
\int \frac{E(k)}{k^4} \, dk = \frac{1}{9k^3} \left[ 2(k^2 - 2)E(k) + (1 - k^2)K(k) \right],
\]

\[
\int \frac{K(k)}{k^2} \, dk = -\frac{E(k)}{k}.
\]

Mathematica finds

\[
\int_{1/\varkappa}^{1} \frac{K(v)}{v^3} \, dv = \frac{1}{9} \left[ -5 + \varkappa (\varkappa^2 + 4) E(\varkappa^{-1}) + 2\varkappa (\varkappa^2 - 1) K(\varkappa^{-1}) \right].
\]
So we get
\[ \mathbb{E}[U_\kappa] = \frac{8r}{9\kappa} \left( (7\kappa^2 + 1) E(\kappa^{-1}) - 4 (\kappa^2 - 1) K(\kappa^{-1}) \right), \quad \kappa > 1. \]

For \( \kappa = 1 \),
\[ (\kappa^2 - 1) K(\kappa^{-1}) \]
has the indeterminate form \( 0 \cdot \infty \). Mathematica finds
\[ \lim_{\kappa \to 1^+} (\kappa^2 - 1) K(\kappa^{-1}) = 0. \]

It follows that
\[ \lim_{\kappa \to 1^+} \mathbb{E}[U_\kappa] = \frac{8r}{9} \left( (7 + 1) - 0 \right) = \frac{64r}{9}. \]

Using the relations (4.5) and
\[ K \left( \frac{2 \sqrt{\kappa}}{1 + \kappa} \right) = (1 + k) K(k) \]
[3, Vol. 2, p. 299], Eq. (8.126.3), for \( 0 < \kappa < 1 \) we get
\[
\begin{align*}
f(\kappa) := & \frac{4(\kappa + 1)r}{9\kappa^2} \left[ (7\kappa^2 + 1) E(\kappa) \left( \frac{2 \sqrt{\kappa}}{\kappa + 1} \right) - (\kappa - 1)^2 K \left( \frac{2 \sqrt{\kappa}}{\kappa + 1} \right) \right] \\
= & \frac{8r}{9\kappa^2} \left[ (7\kappa^2 + 1) E(\kappa) + (3\kappa^4 - 2\kappa^2 - 1) K(\kappa) \right] = \mathbb{E}[U_\kappa].
\end{align*}
\]

For \( \kappa > 1 \) we have
\[
\begin{align*}
E \left( \frac{2 \sqrt{\kappa}}{\kappa + 1} \right) = & E \left( \frac{2 \sqrt{\kappa^{-1}}}{1 + \kappa^{-1}} \right) = \frac{1}{1 + \kappa^{-1}} \left[ 2E(\kappa^{-1}) - (1 - \kappa^{-2}) K(\kappa^{-1}) \right] \\
= & \frac{2\kappa}{\kappa + 1} E(\kappa^{-1}) - \frac{\kappa - 1}{\kappa} K(\kappa^{-1})
\end{align*}
\]
and
\[
K \left( \frac{2 \sqrt{\kappa}}{\kappa + 1} \right) = K \left( \frac{2 \sqrt{\kappa^{-1}}}{1 + \kappa^{-1}} \right) = \frac{\kappa + 1}{\kappa} K(\kappa^{-1}).
\]

It follows that
\[
\begin{align*}
f(\kappa) = & \frac{4r}{9\kappa^2} \left[ 2\kappa (7\kappa^2 + 1) E(\kappa^{-1}) - 8\kappa (\kappa^2 - 1) K(\kappa^{-1}) \right] \\
= & \frac{8r}{9\kappa} \left[ (7\kappa^2 + 1) E(\kappa^{-1}) - 4 (\kappa^2 - 1) K(\kappa^{-1}) \right] = \mathbb{E}[U_\kappa].
\end{align*}
\]

For \( \kappa = 0 \),
\[
(7\kappa^2 + 1) E \left( \frac{2 \sqrt{\kappa}}{\kappa + 1} \right) - (\kappa - 1)^2 K \left( \frac{2 \sqrt{\kappa}}{\kappa + 1} \right) = 0,
\]
hence \( f(0) \) has the indeterminate form \( 0/0 \). For \( \kappa = 1 \),
\[
(\kappa - 1)^2 K \left( \frac{2 \sqrt{\kappa}}{\kappa + 1} \right)
\]
has the indeterminate form \( 0 \cdot \infty \). Mathematica finds
\[
\lim_{\kappa \to 0} f(\kappa) = 2\pi r \quad \text{and} \quad \lim_{\kappa \to 1} f(\kappa) = \frac{64r}{9}. \]
The graph of \( \mathbb{E}[U_{\kappa}]/r \) is shown in Fig. 5. Mathematica finds the series expansion

\[
\mathbb{E}[U_{\kappa}] = \pi r \left( \frac{4\kappa}{3} + \frac{1}{\kappa} - \frac{1}{16\kappa^3} \right) + O(\kappa^{-5})
\]

about the point \( \kappa = \infty \). For abbreviation we put

\[
s(\kappa) := \pi r \left( \frac{4\kappa}{3} + \frac{1}{\kappa} - \frac{1}{16\kappa^3} \right). 
\]

(4.7)

\( s(\kappa) \) provides a very good approximation for \( \mathbb{E}[U_{\kappa}] \) even for relatively small values of \( \kappa \) (see Fig. 5). One finds that

\[
\begin{align*}
\mathbb{E}[U_1] - s(1) &\approx -2.29222 \cdot 10^{-2} r, \\
\mathbb{E}[U_2] - s(2) &\approx -5.44238 \cdot 10^{-4} r, \\
\mathbb{E}[U_{10}] - s(10) &\approx -1.64009 \cdot 10^{-7} r.
\end{align*}
\]

5 Distribution of the perimeter

With (4.3) the maximum perimeter in the disk of radius \( R \) is given by

\[
\tilde{u}(\kappa) = rh(\kappa) \quad \text{with} \quad \kappa = R/r.
\]

Let \( G_{\kappa}(u) = P[U_{\kappa} \leq u] \) be the distribution function of the random variable \( U_{\kappa} \). One easily finds from geometrical considerations

\[
G_{\kappa}(u) = \begin{cases} 
0 & \text{if } -\infty < u < 2\pi r, \\
\frac{w^2(u)}{\kappa^2} & \text{if } 2\pi r \leq u < rh(\kappa), \\
1 & \text{if } rh(\kappa) \leq u < \infty,
\end{cases}
\]

(5.1)

where \( w = w(u) \) is the solution of

\[
2\pi r = rh(w).
\]

The Figures 6-11 show examples for graphs of distribution functions \( G_{\kappa} \) and corresponding density functions \( g_{\kappa} \) (multiplied with \( r \)). The density functions are obtained by numerical differentiation of the distribution functions. For comparison the distribution function and the density function (multiplied with \( r \)) of the uniform distribution with support interval \( 2\pi r \leq u \leq rh(\kappa) \) are shown (dashed lines).
For the $k$-th moment of the perimeter $U_\kappa$ we have the Stieltjes integral

$$E[U_\kappa^k] = \int_{-\infty}^{\infty} u^k \, dG_\kappa(u) = \int_{2\pi r}^{r h(\kappa)} u^k \, dG_\kappa(u).$$

Integration by parts yields

$$E[U_\kappa^k] = u^k G_\kappa(u) \bigg|_{2\pi r}^{r h(\kappa)} - k \int_{2\pi r}^{r h(\kappa)} u^{k-1} G_\kappa(u) \, du = r^k h^k(\kappa) - k \int_{2\pi r}^{r h(\kappa)} u^{k-1} G_\kappa(u) \, du.$$ 

From (5.1) it follows that

$$E[U_\kappa^k] = r^k h^k(\kappa) - \frac{k}{\kappa^2} \int_{2\pi r}^{r h(\kappa)} u^{k-1} w^2(u) \, du.$$ 

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The substitution $u = r x$ gives

$$E[U_k^k] = r^k \left[ h^k(x) - \frac{k}{x^2} \int_{2\pi}^{h(x)} x^{k-1} \tilde{w}^2(x) \, dx \right],$$

(5.2)

where $\tilde{w} = \tilde{w}(x)$ is the solution of the equation

$$r h(\tilde{w}) = r x \quad \Rightarrow \quad h(\tilde{w}) = x$$

for given value of $x \in [2\pi, h(x)]$.

Table 1 shows examples for numerical values of $E[U_k^k]/r^k$ which are obtained by numerical integration of (4.4) and (5.2) using Mathematica. The values for $k = 1$ also directly follow from Theorem 4.1.

| $k$ | $E[U_2^k]/r^k$ | $E[U_3^k]/r^k$ |
|-----|----------------|----------------|
| 1   | 9.9232888058187711084 | 13.606226799878091189 |
| 2   | 102.9664851991466206  | 199.6369601685873413  |
| 3   | 1110.1715673108248830 | 3094.81067941943393   |
| 4   | 12355.45526029594611  | 49903.06116964320575  |
| 5   | 141074.96324382298144 | 827860.92687516817    |
| 6   | 1.6440752317143806830 - 10^6 | 1.4025517639333515857 - 10^7 |
| 7   | 1.94749698992456378 - 10^7 | 2.414607953755357859 - 10^6 |
| 8   | 2.3373182556510529688 - 10^8 | 4.2096527656599463840 - 10^9 |
| 9   | 2.8351332586002858086 - 10^9 | 7.4140462419039734138 - 10^{10} |
| 10  | 3.4691601000674766672 - 10^{10} | 1.3167229871972133743 - 10^{12} |

Table 1: Numerical values of $E[U_k^k]/r^k$ for $x = 2, 3$

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