CONTINUOUS-TIME IMPORTANCE SAMPLING: MONTE CARLO METHODS WHICH AVOID TIME-DISCRETISATION ERROR

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CONTENTS

1 Introduction ......................................................... 2
1.1 Acknowledgements .............................................. 4
2 Intuitive derivation: CIS as a limiting algorithm ............. 4
3 Setup and the CIS algorithm .................................... 8
3.1 Notation ......................................................... 8
3.2 Setup ............................................................ 9
3.3 The CIS algorithm ........................................... 9
4 Unbiasedness and stability of generic CIS .................... 11
4.1 Stability of Weights .......................................... 11
4.2 Unbiasedness ................................................... 14
5 The copycat CIS for diffusions ............................... 18
6 Unbiasedness and stability of the copycat CIS for diffusions 20
6.1 A cautionary example ........................................ 20
6.2 Assumptions .................................................... 22
6.3 Stability of weights ......................................... 23
7 Extensions .......................................................... 30
7.1 Proposal densities and optimal implementation ........... 31
7.2 Transition density estimation and guided CIS ............ 31
7.3 Resampling ....................................................... 32
8 Numerical illustrations ......................................... 34
8.1 Overview of Wagner’s approach ............................. 34
8.2 Stochastic volatility model ................................... 35
8.2.1 Implementation considerations .......................... 35
8.2.2 Estimation over large time intervals ..................... 36
8.3 Bivariate CIR model ........................................... 37
9 Discussion .......................................................... 40

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1. Introduction. Continuous-time Markov processes (CTMPs) are used extensively for modelling random processes which evolve continuously in time. They have found numerous applications in many diverse scientific areas, such as physics [26, 23], chemistry [26], finance [24, 11, 6], insurance [24] and systems biology [28]. The goal of this paper is to introduce a novel algorithm for simulating from CTMPs and develop Monte Carlo (MC) estimators of expectations of functionals of CTMPs which arise frequently in many practical applications.

A specific class of CTMPs which has motivated the work in this paper is diffusion processes. A diffusion process \( X \in \mathbb{R}^d \) is defined as the solution to a stochastic differential equation (SDE),

\[
dX_t = b(X_t)dt + \sigma(X_t)dB_t, \quad X_0 = x_0, \quad t \in [0,T],
\]

where \( B \) is a standard \( d \)-dimensional Brownian motion. The functionals \( b : \mathbb{R}^d \to \mathbb{R}^d \) and \( \sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d} \) are known as the drift and diffusion coefficient respectively and model the mean and variance of the infinitesimal increments of the process. We assume that the functionals satisfy growth and Lipschitz conditions, so that (1.1) admits a unique weak non-explosive solution [19].

Simulation and inference for diffusion processes is a very challenging task due to the intractability of the transitional dynamics of the process. Most contemporary approaches are based on approximating the diffusion by a discrete-time Markov process. If we choose \( M \in \mathbb{Z}^+ \) then we can partition a time interval \([0,T]\) into \( M \) subintervals of size \( h := T/M \).

The dynamics of the SDE can then be approximated by those of a discrete-time Markov process, with

\[
X_{t_{i+1}} = X_{t_i} + hb(X_{t_i}) + \sqrt{h}\sigma(X_{t_i})Z_{i+1},
\]

where \( t_i := ih, \ i = 0, 1, \ldots, M \), and \( Z_i \)s are independent standard Gaussian random variables. The above scheme, known as the Euler scheme, provides a straightforward way for simulating the target process and allows us to design MC estimators using independent draws from (1.2). However, the discrete approximation of the continuous-time model introduces a systematic error which is eliminated only when \( M \to \infty \). The value of \( M \) that guarantees a sufficiently accurate approximation is typically unknown, and is found empirically by multiple runs of the MC estimators over increasing values of \( M \) until no significant change is shown. Alternatively, \( M \) is selected as large as the computational resources allow, which may result in discretisations finer than necessary. Both selection procedures induce a significant computational burden and waste of resources. Furthermore, the approximation...
error of such MC estimators is hard to quantify; available results only provide upper bounds for the bias and these are typically known only up to a constant.

More recently, an unbiased approach was introduced with the development of the exact algorithm (EA)\cite{4,3}. The EA is a rejection sampling algorithm on the diffusion path space which returns, in principle, a continuous trajectory from (1.1). In practice, the output is an exact draw from the finite-dimensional distribution of a “skeleton” for the diffusion path. This skeleton can then be used to evaluate the diffusion path at any time instance with no further reference to the target process. The EA allows the design of MC estimators whose only source of error is due to MC, and thus easier to quantify. In particular, for a fixed computational cost $K$, the approximation error of the unbiased methods is of order $O(K^{-1/2})$, in contrast to the rate $O(K^{-1/3})$ which is usually attained Euler based approaches (though see \cite{15} for some interesting improvements on this).

Unfortunately, the range of applicability of the EA is limited by two conditions. First, the existence of a variance-stabilising transformation, known as the Lamperti transform, after which the process has unit diffusion matrix, and second, the drift of the transformed process has to be of gradient form. The reason that these conditions are needed for the EA is that the EA is based upon rejection and importance sampling methods that work on the path-space of an SDE. These require tractable proposal processes whose law for the diffusion path is absolutely continuous with the SDE of interest. In general such tractable proposal processes only exist if the SDE of interest can, for example, be transformed to unit diffusion matrix by the Lamperti transform. Both conditions for the EA are usually satisfied in a univariate setting, but pose serious limitations for general multivariate processes, as illustrated in \cite{1}.

In this paper we develop a continuous-time sequential importance sampling (CIS) algorithm which eliminates time-discretisation errors and provides online unbiased estimation for diffusions. Our work removes the strong conditions imposed by the EA and thus extends significantly the class of discretisation error-free MC methods for diffusions. The reason that CIS can be applied more generally than EA is that it no longer works on the path space of the SDE. Instead it uses proposal distributions for the transition density of the diffusion, and proposal distributions that are absolutely continuous with respect to the true transition density exist for general SDEs.

The CIS algorithm is built by modifying a standard discrete-time sequential importance sampling (SIS) algorithm by introducing updates at random times according to a Bernoulli process and replacing intractable importance weights by unbiased estimators, similarly to \cite{14}. The CIS algorithm is then obtained by taking the continuous-time limit of the discrete SIS. The key is that in the limit the Bernoulli process converges to a Poisson process implying that updates occur only at a finite number of points. Therefore, in principle, CIS evolves continuously in time but is implemented using only finite computation. Furthermore, the sequential nature of our approach allows us to boost the algorithmic performance of CIS. By using batch implementations and standard resampling techniques we develop a
sequential importance resampling algorithm and demonstrate empirically that it can lead to significantly more accurate estimates than the plain version of CIS when the time horizon $T$ is large.

Whilst motivated for sampling from SDEs, the CIS can be applied to other CTMPs. One important application is to use CIS to simulate from CTMPs that have been designed to have a particular stationary distribution of interest. The results in this paper provide the underpinning theoretical justification for some of the extensions of the recently proposed SCALE algorithm of [21] that are suggested in [13].

The paper is structured as follows. The next section gives an intuitive derivation of the CIS algorithm in terms of a continuous-time limit of standard discrete-time sequential importance sampling algorithms. Section 3 then formally defines the generic version of the CIS algorithm along with the resulting MC estimators. Unbiasedness and stability of the generic CIS is addressed in Section 4. Sections 5 and 6 provide the CIS version for diffusions and detailed treatment of its stability respectively. Section 7 deals with boosting the algorithms performance and presents two modifications based on optimal proposals and batch implementation with resampling. Section 8 assesses the performance of our MC estimators and compares it to several existing methods. Section 9 concludes with a discussion.

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2. Intuitive derivation: CIS as a limiting algorithm. To help understand the idea behind CIS, we first give an intuitive derivation of the CIS algorithm as the limiting algorithm of a standard discrete-time sequential importance sampler. Our target process is a CTMP with transition density

$$p(x, y, t) = \mathbb{P}(X_{s+t} \in dy | X_s = x) / dy$$

for $s, t > 0$, with $X_0 = x_0$. For the importance sampler we will use a family of proposal distributions that approximate this transition density. Denote these by $q_\theta(x, y, t)$, where $\theta$ is the parameter of the family of distributions.

Standard Sequential Importance Sampling. We first describe a standard discrete-time sequential importance sampler [17]. Firstly discretise the time interval $[0, T]$ into intervals of length $h$. The sequential importance sampler will simulate values of the process and importance weights at times $h, 2h, \ldots$. If at time $(i-1)h$ we have a value and weight, $x_{(i-1)h}, w_{(i-1)h}$, then we first generate $x_{ih}$ from $q_{\theta_{i-1}}(x_{(i-1)h}, \cdot, h)$, for some suitably chosen $\theta_{i-1}$, and set

$$w_{ih} = w_{(i-1)h} p(x_{(i-1)h}, x_{ih}, h) / q_{\theta_{i-1}}(x_{(i-1)h}, x_{ih}, h).$$
Full details are given in Algorithm 1.

**Algorithm 1:** Discrete-time Sequential Importance Sampler (SIS).

**Input:** $T$ a time to stop, $M$ a discretisation level and $x_0$ the initial value.

**Initialise:** Set $w_0 = 1$ and $h = T/M$.

**Iterate:** For $1 \leq i \leq M$

1. choose $\theta_{i-1}$ and draw $x_{ih} \sim q_{\theta_{i-1}}(x_{(i-1)h}, \cdot, h)$,
2. update the weight
   \[ w_{ih} = w_{(i-1)h} \frac{p(x_{(i-1)h}, x_{ih}, h)}{q_{\theta_{i-1}}(x_{(i-1)h}, x_{ih}, h)}. \]

Let $X_{ih}, W_{ih}$ denote the random variables of the value and weight at time $ih$, respectively. Standard importance sampling arguments show that whenever the left-hand side is defined, then

\[ \mathbb{E}_p[f(X_{ih})] = \mathbb{E}_q[W_{ih}f(X_{ih})], \tag{2.1} \]

where $\mathbb{E}_q$ denotes expectation of $X_{ih}, W_{ih}$ with respect to the sequential importance sampling process.

Unfortunately Algorithm 1 cannot be implemented since $p(x, y, h)$ is unknown, hence the weights cannot be updated at each iteration. To overcome this, the aim is to consider the limit as $h \to 0$. This cannot be considered directly as such a limit would not make sense. So first we adapt Algorithm 1 using in turn ideas of random weight sequential importance sampling \[14\], retrospective sampling \[20\] and Rao-Blackwellisation \[18\].

**Random Weight sequential Importance Sampling.** Our first idea is to change how the weights are updated, and to make this update random. This will still produce a valid importance sampler, for which (2.1) is satisfied, provided that the mean of the random weight update is equal to the true incremental weight.

We will implement our random weight update by introducing a series of Bernoulli random variables, $U_{ih}$, for $i = 1, 2, \ldots$. We call successes of the Bernoulli random variables, events. We will allow the distribution of $U_{ih}$ to depend on the time $ih$ and also the history of the Bernoulli process, but for concreteness and notational simplicity we will consider only the case where it depends on the time since the most recent event has occurred. That is, if the most recent event has happened at time $jh$, then we assume there exists a function $\lambda(\cdot) > 0$ such that with probability $h\lambda(ih - jh)$, $U_{ih} = 1$, otherwise $U_{ih} = 0$. It is trivial to see that if

\[ r_{\theta}(x, y, h, u) := 1 + \frac{1}{h\lambda(u)} \left\{ \frac{p(x, y, h)}{q_{\theta}(x, y, h)} - 1 \right\}, \]
then
\[
p(x_{(i-1)h}, x_{ih}, h) / q_{\theta_{i-1}}(x_{(i-1)h}, x_{ih}, h) = \mathbb{E} \left[ (1 - U_{ih}) + U_{ih} r_{\theta_{i-1}}(x_{(i-1)h}, x_{ih}, h, (i - j)h) \right].
\]

Thus, we can define an appropriate random incremental weight that takes the value 1 if \( U_{ih} = 0 \) and \( r_{\theta_{i-1}}(x_{(i-1)h}, x_{ih}, h, (i - j)h) \) if \( U_{ih} = 1 \). The resulting algorithm is shown in Algorithm 2.

**Algorithm 2: Random weight SIS.**

**Input:** \( T \) a time to stop, \( M \) a discretisation level and \( x_0 \) the initial value.

**Initialise:** Set \( w_0 = 1 \), \( h = T/M \) and \( j = 0 \).

**Iterate:** For \( 1 \leq i \leq M \)

1. choose \( \theta_{i-1} \) and draw \( x_{ih} \sim q_{\theta_{i-1}}(x_{(i-1)h}, \cdot, h) \),
2. draw \( U_{ih} \), a Bernoulli random variable with success probability \( h\lambda(ih - jh) \),
3. if \( U_{ih} = 1 \) set
   \[
   w_{ih} = w_{(i-1)h} r_{\theta_{i-1}}(x_{(i-1)h}, x_{ih}, h, (i - j)h),
   \]
   and \( j = i \); else set \( w_{ih} = w_{(i-1)h} \).

**Retrospective Sampling and Rao-Blackwellisation.** Algorithm 2 cannot be implemented in practice, as the random incremental weights still depend on the unknown transition density of the proposal process. Furthermore, even if it could be implemented, it would be less efficient than Algorithm 1 because of the extra randomness introduced by the random variables \( U_{ih}, i = 1, 2, \ldots \). However its use is that it now enables us to use the ideas of retrospective sampling and Rao-Blackwellisation.

The idea behind retrospective sampling is to interchange the order of simulating \( x_{ih} \) and \( U_{ih} \), and to note that if \( U_{ih} = 0 \) then we do not need to simulate \( x_{ih} \) at all. So we will only simulate \( x_{ih} \) at time points when there is an event. If \( jh \) is the time of the most recent event prior to \( ih \), then \( x_{ih} \) is simulated from the proposal transition density \( q_{\theta_j}(x_{jh}, \cdot, (i - j)h) \). To update the weight at time \( ih \) we also need to know the value of \( x_{(i-1)h} \) and so this has to be simulated, conditionally on \( x_{jh} \) and \( x_{ih} \), from the density

\[
q(x_{(i-1)h} \mid x_{jh}, x_{ih}) := \frac{q_{\theta_j}(x_{jh}, x_{(i-1)h}, (i - j - 1)h) q_{\theta_j}(x_{(i-1)h}, x_{ih}, h)}{q_{\theta_j}(x_{jh}, x_{ih}, (i - j)h)}.
\]

This can be viewed as introducing a random incremental weight. The weight depends on the value of \( x_{(i-1)h} \) that is simulated. The idea behind Rao-Blackwellisation is that a more efficient algorithm can be produced by replacing this random incremental weight by a deterministic one that is equal to the mean of the random incremental weight. We denote this mean by

\[
\bar{r}_{\theta_j}(x_{jh}, x_{ih}, h, (i - j)h) = \int r_{\theta_j}(x_{(i-1)h}, x_{ih}, h, (i - j)h) q(x_{(i-1)h} \mid x_{jh}, x_{ih}) dx_{(i-1)h}.
\]
The resulting algorithm is given in detail in Algorithm 3. If we stop the algorithm at a time point at which there is no event, then the weight at that time is equal to the value of the weight at the last event time, and the value of the process can be simulated from the appropriate transition density of the proposal process given the time and value at the last event time.

**Algorithm 3: SIS with Retrospective Sampling and Rao-Blackwellisation.**

*Input:* $T$ a time to stop, $M$ a discretisation level and $x_0$ the initial value.

*Initialise:* Set $w_0 = 1$, $h = T/M$, $j = 0$ and choose $\theta_j$.

*Iterate:* For $1 \leq i \leq M$

1. draw $U_{ih}$, a Bernoulli random variable with success probability $h\lambda(ih - jh)$,
2. if $U_{ih} = 1$, then
   (a) draw $x_{ih} \sim q_{\theta_j}(x_{jh}, \cdot, h)$,
   (b) update the weight $w_{ih} = w_{(i-1)h} \bar{r}_{\theta_j}(x_{jh}, x_{ih}, h, (i-j)h)$,
   (c) set $j = i$ and choose $\theta_j$.

*CIS as a limiting algorithm.* Whilst Algorithm 3 cannot be implemented, we can now take the limit as $h \to 0$. In doing this the Bernoulli random variables will converge to a renewal process with the rate of an event at a time $\tau$ since the last event being $\lambda(\tau)$. We will thus obtain a limiting algorithm where events are simulated from a renewal process, the value of the process at these events are drawn from the proposal transition density, and the weights are updated based on the simulated value. The key thing is that in this limit the incremental weight is tractable. If we fix $s = jh$ and $t = kh$ and let $h \to 0$, then

$$
\lim_{h \to 0} \bar{r}_{\theta_s}(x_s, x_t, h, t - s) = 1 + \frac{1}{\lambda(t - s)} \lim_{h \to 0} \frac{1}{h} \int \left\{ \frac{q_{\theta_s}(x_s, y, t - s - h)p(y, x_t, h) - q_{\theta_s}(y, x_t, h)}{q_{\theta_s}(x_s, x_t, t - s)} \right\} dy.
$$

However, by integrating the Kolmogorov’s forward equation (c.f. (3.5)) over a time interval of length $h$, with an initial distribution for the process being $q_{\theta_s}(x_s, y, t - s - h)$ we get:

$$
\int q_{\theta_s}(x_s, y, t - s - h)p(y, x_t, h)dy = q_{\theta_s}(x_s, x_t, t - s - h) + hKq_{\theta_s}(x_s, x_t, t - s - h) + o(h),
$$

where $\mathcal{K}$ is the forward operator of the target CTMP. A similar argument applies for the term which involves $q_{\theta_s}(y, x_t, h)$. Thus

$$
\lim_{h \to 0} \bar{r}_{\theta_s}(x_s, x_t, h, t - s) = 1 + \frac{[K - \mathcal{K}_{\theta_s}]q_{\theta_s}(x_s, y, t - s)}{\lambda(t - s)q_{\theta_s}(x_s, x_t, t - s)}|_{y=x_t},
$$

where $\mathcal{K}_{\theta_s}$ is the forward operator associated with the proposal transition density $q_{\theta_s}(y, x_t, h)$. 

3. Setup and the CIS algorithm. This section introduces the basic notation and presents the CIS algorithm.

3.1. Notation. The $i$th element of a vector $x$ is denoted by $x_i$ or $[x]_i$. The Euclidean inner product of two vectors $x$ and $y$ is denoted by

$$x \cdot y = \sum_i x_i y_i.$$ 

For a matrix $x$, its $(i,j)$th element is denoted by $x_{ij}$ or $[x]_{ij}$, its transpose by $x^T$, its inverse by $x^{-1}$, its inverse transpose by $x^{-T}$, and its complex conjugate by $\bar{x}$. An $m \times n$ matrix whose elements are equal to one is denoted by $\mathbf{1}_{m \times n}$. The Frobenius inner product of two matrices $x$ and $y$ of the same dimension, i.e., the sum of all entries of their element-wise product, is denoted by

$$x : y = \sum_{i} \sum_{j} x_{ij} y_{ij}.$$ 

Unless stated otherwise, the Frobenius (Euclidean) matrix norm

$$\|x\|^2 = \sum_{i} \sum_{j} |x_{ij}|^2 = \text{tr}(x \bar{x}^T)$$

is used throughout the paper and when applying the norm we treat vectors as matrices. The first and second derivative of a scalar function $f : \mathbb{R} \to \mathbb{R}$ are denoted by $f'$ and $f''$ respectively. The gradient of a scalar function $f : \mathbb{R}^d \to \mathbb{R}$ and the Hessian matrix are written respectively as

$$[\nabla_x f(x)]_i := \frac{\partial f(x)}{\partial x_i}$$

$$[H_x f(x)]_{ij} := \frac{\partial^2 f(x)}{\partial x_i \partial x_j}.$$ 

Furthermore, for the drift $b$ and the diffusion coefficient $\sigma$ in (1.1) we define

$$\gamma : \mathbb{R}^d \to \mathbb{R}^{d \times d} \quad \text{as} \quad \gamma = \sigma \sigma^T,$$

$$b_{(1)} : \mathbb{R}^d \to \mathbb{R}^d \quad \text{as} \quad [b_{(1)}(x)]_i = \frac{\partial b_i(x)}{\partial x_i},$$

$$\gamma_{(1)} : \mathbb{R}^d \to \mathbb{R}^{d \times d} \quad \text{as} \quad [\gamma_{(1)}(x)]_{ij} = \frac{\partial \gamma_{ij}(x)}{\partial x_j},$$

$$\gamma_{(2)} : \mathbb{R}^d \to \mathbb{R}^{d \times d} \quad \text{as} \quad [\gamma_{(2)}(x)]_{ij} = \frac{\partial^2 \gamma_{ij}(x)}{\partial x_i \partial x_j}.$$ 

For a finite sequence $\{\tau_k\}_{k \in \mathbb{N}}, 0 < \tau_k < \tau_{k+1}$, we define

$$\tau_t := \max\{\tau_k : \tau_k \leq t\},$$

$$n_t := \max\{k : \tau_k \leq t\},$$

$$\Delta \tau_k := \tau_{k+1} - \tau_k.$$
Additionally, \( N(x, \mu, \Sigma) \) denotes the density of a Gaussian random variable with mean \( \mu \in \mathbb{R}^d \) and covariance matrix \( \Sigma \in \mathbb{R}^{d \times d} \) evaluated at \( x \in \mathbb{R}^d \). Finally, to avoid overloading the notation, any constant that appears in all subsequent assumptions and proofs is denoted by \( C \).

3.2. Setup. We consider a general setup, where our target process is a CTMP evolving on state space \( \mathcal{X} \) with unknown transition density with respect to reference measure \( dy \):

\[
 p(x, y, t) = \mathbb{P}(X_{s+t} \in dy | X_s = x) / dy \quad \text{for} \quad s, t > 0.
\]

For notational simplicity we have assumed that the target process is time-homogeneous and the initial state value is known \( X_0 = x_0 \). Extensions to cases where the initial value is from a known distribution are straightforward, while applicability to time-inhomogeneous processes would follow by extending the current setup under appropriate smoothness conditions in the time variable.

While the transition density is assumed to be intractable, under mild regularity conditions it will satisfy Kolmogorov’s forward equation

\[
 \frac{\partial}{\partial t} p(x, y, t) = \mathcal{K} p(x, y, t),
\]

where \( \mathcal{K} \) is the forward operator of the process, acting on \( y \). For Markov jump processes this is just the rate matrix, while for the diffusion (1.1) this is the second order differential operator defined via

\[
 \mathcal{K} f(y) = - \sum_{i=1}^{d} \frac{\partial}{\partial y_i} [b_i(y)f(y)] + \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial}{\partial y_i \partial y_j} [ \gamma_{ij}(y)f(y)].
\]

To implement importance sampling we need a proposal process. We consider a family of proposal processes, parameterised by \( \theta \), and assume these have a known transition density, denoted by \( q_{\theta}(x, y, t) \). The proposal process will also have a family of forward operators, which we denote by \( \mathcal{K}_{\theta} \). As an example, when analysing diffusions, a natural choice of proposal processes will be diffusions with constant drift and diffusion coefficient. In this case \( \theta \) would be the value of the drift and diffusion coefficient, and the transition density would be Gaussian. Throughout the following we assume that for any \( \theta \), any \( t > 0 \) and any \( x \), the support of \( q_{\theta}(x, \cdot, t) \) contains the support of \( p(x, \cdot, t) \).

3.3. The CIS algorithm. We are now in a position to give a formal description of the algorithm outlined in Section 2. The CIS algorithm simulates values of the process and importance weights at times indicated by a renewal process. If \( \tau_k \), for some \( k \in \mathbb{N} \), is the most recent renewal time and \( x_{\tau_k}, w_{\tau_k} \) denote the value of the process and importance weight at that time, then the algorithm iterates two steps. First, the waiting time for the
next renewal $\tau_{k+1}$ is simulated as the time to the event of a renewal process with time-dependent intensity $\lambda(s), s > 0$. If $\tau_{k+1} < T$ then $x_{\tau_{k+1}}$ is generated from the proposal density $q_\theta_k(x_{\tau_k}, \cdot, \Delta \tau_k)$ and the weight is updated to

$$w_{\tau_{k+1}} = w_{\tau_k} \rho_\theta(x_{\tau_k}, x_{\tau_{k+1}}, \Delta \tau_k),$$

where $\rho_\theta(x, y, u)$ is referred to as the *incremental weight* and is defined by

$$\rho_\theta(x, y, u) = 1 + \frac{[K - K_\theta] q_\theta(x, Y, u)}{\lambda(u)q_\theta(x, y, u)}_{Y=y}. \quad (3.6)$$

If $\tau_{k+1} > T$ then the algorithm stops and outputs a distribution for the value of the process at time $T$, namely $q_\theta_k(x_{\tau_k}, \cdot, T - \tau_k)$, along with the importance weight $w_T = w_{\tau_k}$. Full details are given in Algorithm 4 below.

**Algorithm 4: Continuous-time Importance Sampling algorithm (CIS).**

*Input:* $T$ a time to stop and $x_0$ the initial value.

*Initialise:* Set $k = 0, \tau_0 = 0, x_{\tau_0} = x_0, w_{\tau_0} = 1$ and choose $\theta_0$.

*Iterate:* Repeat

1. draw $u$, the interarrival time of a renewal process of rate $\lambda(s)$, and set $\tau_{k+1} = \tau_k + u$,

2. if $\tau_{k+1} > T$, set $w_T = w_{\tau_k}$ and stop.

3. Otherwise
   (a) draw $x_{\tau_{k+1}} \sim q_\theta_k(x_{\tau_k}, \cdot, \Delta \tau_k)$,
   (b) compute the incremental weight using (3.6)
      $$H_{k+1} = \rho_\theta(x_{\tau_k}, x_{\tau_{k+1}}, \Delta \tau_k)$$
      and update the weight $w_{\tau_{k+1}} = w_{\tau_k}H_{k+1}$,
   (c) choose $\theta_{k+1} = \theta(K, x_{\tau_{k+1}})$ and set $k = k + 1$.

*Output:* A distribution for the value of the process at time $T$, namely $q_\theta_k(x_{\tau_k}, \cdot, T - \tau_k)$, and the corresponding importance sampling weight $w_T$.

The output of the CIS algorithm should be interpreted as a (signed) measure valued estimate of the transition measure, i.e. $w_T q_\theta_k(x_{\tau_k}, \cdot, T - \tau_k)$ is an estimate of $p(x_{\tau_k}, \cdot, T)$ and it can be readily used to estimate functional expectations, $E_p(f(X_T))$, where $E_p$ denotes expectation with respect to the target process. Specifically, if we repeat Algorithm 4 $N$ times and denote the output of the $i$th run to be weight $w_T^{(i)}$ and distribution

$$q_T^{(i)}(\cdot) := q_\theta^{(i)}(x_T^{(i)}, T - \tau_T^{(i)}),$$
then our estimate of $\mathbb{E}_p(f(X_T))$ is

$$
\frac{1}{N} \sum_{i=1}^{N} w_T^{(i)} \int f(x_T)q^{(i)}(x_T)dx_T.
$$

If we cannot calculate the integral in the above expression analytically, we can approximate it by a standard Monte Carlo estimate as we can draw samples from the proposal process density $q^{(i)}(\cdot)$. In particular, we can simulate $x_T^{(i)}$ from $q^{(i)}(\cdot)$, and approximate the expectation by $\sum_{i=1}^{N} w_T^{(i)} f(x_T^{(i)})/N$.

We should refer to Algorithm 4 as generic CIS, and a more detailed version will be considered in Section 5 to specifically deal with SDEs. This will allow for precise and rigorous stability and validity analysis in Section 6. However, even at this level of generality we can identify convenient ingredients for ensuring stability and unbiasedness, as outlined in the next section.

4. Unbiasedness and stability of generic CIS. In this section we provide some general results about stability and unbiasedness of the generic CIS algorithm. These are then used in the particular case of SDEs in Sections 5 and 6, and may be similarly applied in other settings of interest.

More specifically, recall that the output of Algorithm 4 amounts to the weight $w_T$ and the distribution $q_{\theta_k}(x_{\tau_k}, \cdot, T - \tau_k)$, thus the expectation of the resulting density estimate is

$$
\tilde{p}(x_0, y, T) := \mathbb{E}_\text{CIS} \left[ W_T q_{\theta_{\tau_T}}(X_{\tau_T}, y, T - \tau_T) \right],
$$

where $\mathbb{E}_\text{CIS}$ denotes the expectation with respect to the probability space generated by the CIS algorithm until target time horizon $T$, i.e. with respect to the renewal, the proposal process and its parameters. Consequently, unbiasedness means $p(x_0, y, T) = \tilde{p}(x_0, y, T)$ for every $T > 0$, and requires suitable assumptions.

Furthermore, the tails of $W_T$, the random variable of the weight $w_T$ returned by the CIS algorithm, determine the stability of the CIS based estimators, e.g. (3.7), and in particular we need to establish that $W_T \in L^1$, otherwise (4.1) makes no sense. Thus stability of the algorithm is studied by examining conditions that guarantee that $W_T$ is in $L^p$. Stability and unbiasedness are studied in the following subsections.

4.1. Stability of Weights. To allow for precise statements let $(F_t^{X,\lambda})_{t \geq 0}$ be the filtration of the proposal process $X$ together with the Poisson event process $\tau$, i.e. the filtration generated by the CIS algorithm. The following two assumptions imply a generic result regarding stability of weights.
Assumption 1 (Uniform boundedness of moments of incremental weights). For a constant $p^* > 1$ consider the family of random variables

$$
\mathcal{H}_k := \mathbb{E}_{\text{CIS}} \left[ |H_k|^{p^*} \left| \mathcal{F}_{X_{\tau_k-1}} \right| \right],
$$

and assume that for some $p^* > 1$ there exists a constant $C(p^*)$, s.t. for every $k = 1, 2, \ldots$

(4.2) \hspace{1cm} \mathcal{H}_k \leq C(p^*) \ a.s.

Remark 4.1. If the choice of $\theta_{k+1}$ in step 3(c) of Algorithm 4 depends on $x_{\tau_{k+1}}$ only and the intensity function of the Poisson process used in step 1 of Algorithm 4 is fixed, then condition (4.2) is implied by the more tractable

(4.3) \hspace{1cm} \sup_x \mathbb{E}_{\text{CIS}} \left[ |H_1|^{p^*} \left| X_0 = x \right| \right] < C(p^*),

which will be verified for the diffusion case in Section 6.

Assumption 2 (Finitness of the generating function of $n_T$). Consider the sequence $\tau_k$ of renewal times generated by Algorithm 4 and the resulting $n_T$ as defined in equation (3.2). Assume that the generating function of $n_T$ is everywhere finite, i.e. $\mathbb{E}_{\text{CIS}}[\exp\{cn_T\}] < \infty$ for every $c \in \mathbb{R}$.

The stability of weights is then guaranteed by the following result.

Theorem 4.2. Under Assumptions 1 and 2 the weight $W_T$ returned by Algorithm 4 is in $L^p$ for all $p < p^*$.

Proof of Theorem 4.2. Fix arbitrary $p < p^*$ and let $\varepsilon := p^* - p$. Following Assumption 1, define

$$
K := C(p^*)^{\frac{p}{p^*}} = C(p + \varepsilon)^{\frac{p}{p + \varepsilon}} < \infty.
$$

We first show that

(4.4) \hspace{1cm} \mathbb{E}_{\text{CIS}}|W_T|^p \leq \sum_{j=0}^{\infty} K^j \left[ \mathbb{P}(n_T = j) \right]^{\varepsilon/p^*}.

To this end we use the Hölder inequality (with $\frac{p}{p^*} + \frac{p}{p} = 1$) term by term in the sum below
as follows.

\[
E_{\text{CIS}}|W_T|^p = E_{\text{CIS}} \left( \prod_{i=0}^{n_T} |H_i|^p \right) = E_{\text{CIS}} \left[ \sum_{j=0}^{\infty} \mathbb{I}(n_T = j) \prod_{i=0}^{j} |H_i|^p \right] \\
= \sum_{j=0}^{\infty} E_{\text{CIS}} \left[ \mathbb{I}(n_T = j) \prod_{i=0}^{j} |H_i|^p \right] \\
\leq \sum_{j=0}^{\infty} \left[ P(n_T = j) \right] \frac{C(p^*)}{p^*} \left( \sum_{j=0}^{\infty} \mathbb{I}(n_T = j) \prod_{i=0}^{j} |H_i|^p \right)^{\frac{p^*}{p}} \\
= \sum_{j=0}^{\infty} \left[ P(n_T = j) \right] \frac{C(p^*)}{p^*} \left( \mathbb{E}_{\text{CIS}} \left[ \prod_{i=0}^{j} |H_i|^p \mathbb{E}_{\text{CIS}} \left[ |H_j|^p | \mathcal{F}_{X_\lambda,\tau_j-1} \right] \right] \right)^{\frac{p^*}{p}} \\
\leq \sum_{j=0}^{\infty} \left[ P(n_T = j) \right] \frac{C(p^*)}{p^*} \left( P(n_T = j) \right)^{\frac{p^*}{p}} \leq \cdots \\
\leq \sum_{j=0}^{\infty} C(p^*)^{\frac{p^*}{p}} \left[ P(n_T = j) \right] \left( P(n_T = j) \right)^{\frac{p^*}{p}} = \sum_{j=0}^{\infty} K^j \left[ P(n_T = j) \right]^{\frac{p^*}{p}} \\
= \sum_{j=0}^{\infty} K^j \left[ P(n_T = j) \right]^{\frac{p^*}{p}} < \infty,
\]

where the last conditioning is repeated recursively.

Now by Assumption 2, the moment generating function of \( n_T \) is finite for

\[
c := \frac{p^*}{\epsilon} \log(2K).
\]

This implies that for some constant \( M < \infty \),

\[
P(n_T = j) \leq M (2K)^{-\frac{p^*}{\epsilon} j}
\]

and consequently by (4.4), we obtain

\[
E_{\text{CIS}}|W_T|^p \leq \sum_{j=0}^{\infty} K^j \left[ P(n_T = j) \right]^{\epsilon/p^*} \leq M^{\epsilon/p^*} \sum_{j=0}^{\infty} K^j (2K)^{-j} < \infty.
\]

Thus \( W_T \in L^p \). \( \square \)
4.2. Unbiasedness. The generic CIS relies on the Kolmogorov forward equation, hence the fundamental assumption that underpins the approach is:

**Assumption 3 (Validity of forward equations).**

(a) The transition density of the target process \( p(x, y, t) \), defined in (3.4), is the unique solution of the Kolmogorov forward equation (3.5).

(b) Similarly, for every \( \theta \), the transition density of the proposal process \( q_\theta(x, y, t) \), is the unique solution of the Kolmogorov forward equation

\[
\frac{\partial}{\partial t} q_\theta(x, y, t) = K_\theta q_\theta(x, y, t).
\]

Unbiasedness of the CIS algorithm will follow from verifying that \( \tilde{p}(x_0, y, T) \) defined by (4.1) also satisfies the Kolmogorov forward equation (3.5), and hence by uniqueness is the same as \( p(x, y, T) \).

To this end we relate the CIS algorithm to Piecewise Deterministic Markov Processes [8] (see also [9, 24]). These are processes that have stochastic jumps at event times of a point process, but where the state evolves deterministically between event times.

We can restrict our attention to the case where the statespace of a Piecewise Deterministic Markov Process (PDP) can be written as a single open set (rather than a sum of indexed open sets, c.f. [8]), which simplifies notation. The following objects define a PDP:

- The state space \((E, \mathcal{B}(E))\) of such a PDP process \( Z_t \in E \);
- \( \lambda^{\text{PDP}}(z) : E \rightarrow \mathbb{R}_+ \), a state dependent rate function of the event times;
- \( \psi : E \rightarrow \mathbb{R}^d \), a vector field that specifies the deterministic dynamics between jump times. If the vector field has a unique integral curve solution \( \Psi(z, t) : E \times \mathbb{R}_+ \rightarrow E \) and there are no events in \([t, t + s]\), then \( Z_{t+s} = \Psi(Z_t, s) \);
- \( Q(z, \cdot) : E \times \mathcal{B}(E) \rightarrow [0, 1] \), a transition kernel for the process at jump times, i.e. if there is an event at \( t \), then \( Z_t \sim Q(z_{t-}, \cdot) \).

In the following definition we write the process generated by the CIS (i.e. Algorithm 4) as a PDP.

**Definition 4.3 (CIS as PDP).**

- Let \( z = (u, x, w) \in \mathbb{R}_+ \times X \times \mathbb{R} =: E \), where \( X \) is the state space of the target CTMP. Here \( u \) is the time since the most recent jump (or the current time if there has been no jump); \( x \) is the value of the process simulated at the last event time (or the initial value of the process if there has been no jump), and \( w \) is the current value of the weight, that is \( z_t := (t - \tau_t, x_{\tau_t}, w_{\tau_t}) \).

- Take \( \lambda(u) \), the rate of the CIS algorithm interarrival renewal process, and let \( \lambda^{\text{PDP}}(z) := \lambda(u) \).
• Define the deterministic dynamics between events as
  \[ \Psi(u, x, w, s) := (u + s, x, w), \]
  or equivalently
  \[ \psi(z_t) := \frac{\partial z_t}{\partial t} = (1, 0, 0). \]

• Given \( z_{t^-} = (u, x, w) \), the transition kernel \( Q(z_{t^-}, \cdot) \) is defined through sampling
  \[ y \sim q_{\theta(x)}(x, \cdot, u), \]
  and setting
  \[ z_t := (0, y, w_{\rho_{\theta(x)}}(x, y, u)). \]

Algorithm 4 generates the process \( Z_t \) defined as above with initial value \( Z_0 = (0, x_0, 1) \).

Denote by \( M(E) \) the family of real valued measurable functions on \( E \), and by \( M_b(E) \subset M(E) \), the subfamily of all bounded functions. Let \( \|g\|_{\infty} \) be the sup norm of \( g \in M(E) \). Let \( T(s) \) be the semigroup on \( M_b(E) \) associated with the transition kernels of our PDP \( Z_t \) and let \( A \) be its full generator.

In the CIS setting, where in particular there is no active boundary of \( E \) (i.e. the set of boundary points of \( E \) that can be reached from \( E \) via integral curves is empty), the full generator \( A \) and its domain \( D(A) \) are characterised by the following result, which is a straightforward simplification of Theorem 11.2.2 of [24] and the preceding discussion of the Dynkin formula.

**Theorem 4.4.** Let \( Z_t \) be the PDP constructed in Definition 4.3 and let \( g \in M(E) \) satisfy the following conditions:

1. for each \( z \in E \) the function \( h(t) := g(\Psi(z, t)) \) is absolutely continuous on \( (0, \infty) \);
2. for each \( t \geq 0 \),
   \[ \mathbb{E}\left( \sum_{i=1}^{n_t} \left| g(Z_{\tau_i}) - g(Z_{\tau_i^-}) \right| \right) < \infty. \]

Then \( g \in D(A) \). Moreover, at \( z = (u, x, w) \),

\[ Ag(z) = \psi(z) \cdot \nabla g(z) + \lambda^{\text{PDP}}(z) \int_E (g(z') - g(z)) Q(z, dz') \]

\[ = \frac{\partial}{\partial u} g(u, x, w) + \lambda(u) \int_x \left( g(0, y, w_{\rho_{\theta(x)}}(x, y, u)) - g(u, x, w) \right) q_{\theta(x)}(x, y, u) dy. \]

(4.6)

Furthermore, the Dynkin formula holds for \( g \), hence if \( Z_0 = z_0 \), then:

\[ \mathbb{E}[g(Z_T)] = g(z_0) + \mathbb{E} \left[ \int_0^T Ag(Z_s) ds \right]. \]

(4.7)
To prove unbiasedness we will relate the generator $A$ of the PDP to the forward operator $K$ of the target for a suitable class of functions. This will require some further assumptions. Denote by $L$ the generator of the target process (i.e. the adjoint of $K$) and its domain by $D(L)$.

**Assumption 4 (Regularity conditions).** Assume there exists a family of functions $S \subseteq D(L) \cap M_b(X)$ satisfying the following conditions:

(a) For each $f \in S$ the function $g_f : E \to \mathbb{R}$ defined as

$$g_f(u, x, w) := w \int_X f(y)q_{\theta(x)}(x, y, u)dy,$$

satisfies conditions 1. and 2. of Theorem 4.4 and hence $g_f \in D(A)$.

(b) For each $f \in S$, $x \in X$ and $u \in (0, T)$, time differentiation can be moved under the integral sign, i.e.

$$\frac{\partial}{\partial u} \left\{ \int_X f(y)q_{\theta(x)}(x, y, u)dy \right\} = \int_X f(y)\frac{\partial}{\partial u} \left\{ q_{\theta(x)}(x, y, u) \right\} dy;$$

$$\frac{\partial}{\partial u} \left\{ \int_X f(y)\tilde{p}(x, y, u)dy \right\} = \int_X f(y)\frac{\partial}{\partial u} \left\{ \tilde{p}(x, y, u) \right\} dy.$$

(c) For each $f \in S$, $x \in X$ and $t > 0$, the following adjoint equations hold:

$$\int_X q_{\theta(x)}(x, y, t)Lf(y)dy = \int_X f(y)Kq_{\theta(x)}(x, y, t)dy;$$

$$\int_X \tilde{p}(x, y, t)Lf(y)dy = \int_X f(y)K\tilde{p}(x, y, t)dy.$$

(d) $S$ is separating, i.e:

$$\left\{ \forall f \in S \int_X f(y)h_1(y)dy = \int_X f(y)h_2(y)dy \right\} \implies h_1 = h_2 \text{ a.s.}$$

The following lemma helps to establish part (a) of Assumption 4, and does not rely on further details of the setting.

**Lemma 4.5.** If Assumptions 1 and 2 hold, then for any $f \in S$ the function $g_f$ defined in (4.8) satisfies Condition 2 of Theorem 4.4.

**Proof.** Recall that $S \subseteq D(L) \cap M_b(X)$. Fix $f \in S$ and let $C := \max\{1, \|f\|\}$. Recall the definition of $k$–th incremental weight $H_k$ and define $H^*_k := C(|H_k| + 1) \geq 1$. Let $W^*_{\tau_i}$ and $W^*_T$ be the analogues of $W_{\tau_i}$ and $W_T$, respectively, built from $H^*_k$ instead of $H_k$. Note that
if \( \{H_k\} \) satisfies Assumption 1, so does \( \{H_k^*\} \). Hence Theorem 4.2 yields \( W_t^* \in L^p \) for all \( p < p^* \) and \( t \leq T \). Now compute

\[
\begin{align*}
\mathbb{E} \left( \sum_{i=1}^{n_t} \left| g_f(Z_{\tau_i}) - g_f(Z_{\tau_i}^-) \right| \right) &= \mathbb{E} \left( \sum_{i=1}^{n_t} \left| g_f(0, X_{\tau_i}, W_{\tau_i}) - g_f(\tau_i - \tau_{i-1}, X_{\tau_{i-1}}, W_{\tau_{i-1}}) \right| \right) \\
&= \mathbb{E} \left( \sum_{i=1}^{n_t} W_{\tau_{i-1}} \left| H_i \int_X f(y)q_\theta(X_{\tau_i}, y, 0)dy - \int_X f(y)q_\theta_{t-1}(X_{\tau_{i-1}}, y, \tau_i - \tau_{i-1})dy \right| \right) \\
&\leq \mathbb{E} \left( \sum_{i=1}^{n_t} |W_{\tau_{i-1}}| (|H_i| C + C) \right) \leq \mathbb{E} \left( \sum_{i=1}^{n_t} |W^*_i| \right) \leq \mathbb{E} \left( \sum_{i=1}^{n_t} |W^*_i| \right) \\
&= \mathbb{E} (n_t |W^*_t|) \leq \left( \mathbb{E} \left( n_t^{p-1} \right) \right)^{\frac{p-1}{p}} (\mathbb{E} (|W^*_t|^p))^{\frac{1}{p}} < \infty,
\end{align*}
\]

where \( 1 < p < p^* \) and we used that \( n_t \) has all moments by Assumption 2. □

A key step to verifying unbiasedness of CIS is to compute how the generator acts on functions of the form (4.8).

**Lemma 4.6.** Let Assumptions 3 and 4 be satisfied and let \( g_f \) be given by (4.8). Then

\[
(4.13) \quad \mathcal{A}g_f(u, x, w) = w \int_X f(y)\mathcal{K}q_\theta(x)(x, y, u)dy.
\]

**Proof.** Since Assumption 4 (a) is satisfied, using expressions (4.6) and (3.6), we obtain

\[
\begin{align*}
\mathcal{A}g_f(u, x, w) &= w \frac{\partial}{\partial u} \left\{ \int_X f(y)q_\theta(x)(x, y, u)dy \right\} \\
&\quad + \lambda(u)w \int_X \left[ \rho_\theta(x)(x, y, u)f(y) - \int_X f(y)q_\theta(x)(x, y, u)dy \right] q_\theta(x)(x, y, u)dy \\
&= w \frac{\partial}{\partial u} \left\{ \int_X f(y)q_\theta(x)(x, y, u)dy \right\} \\
&\quad + \lambda(u)w \int_X \left[ f(y) - \int_X f(y)q_\theta(x)(x, y, u)dy \right] q_\theta(x)(x, y, u)dy \\
&\quad + w \int_X f(y) \left[ \mathcal{K} - \mathcal{K}_\theta(x) \right] q_\theta(x)(x, y, u)dy = A + 0 + B.
\end{align*}
\]

By Assumptions 4 (b) and 3 (b)

\[
A = w \int_X f(y)\mathcal{K}_\theta(x)q_\theta(x)(x, y, u)dy,
\]

which combined with (4.14) yields (4.13). □
Theorem 4.7. Under Assumptions 1, 2, 3 and 4, the CIS Algorithm 4 is unbiased, i.e.
\begin{equation}
  p(x_0, y, T) = \tilde{p}(x_0, y, T) \quad \text{for every} \quad T > 0,
\end{equation}
where \( \tilde{p}(x_0, y, T) \) is defined in equation (4.1).

Proof. Let \( f \in S \). Since \( f \) is bounded and by Theorem 4.2 \( W_T \in L^p \) for some \( p > 1 \), the RHS of (4.16) below, exists by Hölder inequality, and by Fubini-Tonelli also the LHS of (4.16) exists. Furthermore, by Assumption 4, Dynkin formula (4.7) holds and writing CIS as PDP (c.f. Definition 4.3), yields
\begin{equation}
  \int_X f(y) \tilde{p}(x_0, y, T) dy = \mathbb{E}_{\text{CIS}} \left[ f(y) W_T q_{\theta_{nT}}(X_{\tau_T}, y, T - \tau_T) \right]
  = \mathbb{E}[g_f(Z_T)] = g_f(z_0) + \mathbb{E} \left[ \int_0^T A g_f(Z_s) ds \right]
  = f(x_0) + \int_0^T \mathbb{E}[A g_f(Z_s)] ds.
\end{equation}
Now use Lemma 4.6, then Assumption 4 (c), then Fubini-Tonelli, then Assumption 4 (c) again, to compute
\begin{equation}
  \int_X f(y) \tilde{p}(x_0, y, T) dy = f(x_0) + \int_0^T \mathbb{E} \left[ W_s \int_X f(y) K q_{\theta_{ns}}(X_{\tau_s}, y, s - \tau_s) dy \right] ds
  = f(x_0) + \int_0^T \mathbb{E} \left[ W_s \int_X q_{\theta_{ns}}(X_{\tau_s}, y, s - \tau_s) \mathcal{L} f(y) dy \right] ds
  = f(x_0) + \int_0^T \left[ \int_X \tilde{p}(x_0, y, s) \mathcal{L} f(y) dy \right] ds
  = f(x_0) + \int_0^T \left[ \int_X f(y) K \tilde{p}(x_0, y, s) dy \right] ds.
\end{equation}
Differentiate the above equation in \( T \), use Assumption 4 (b) to interchange differentiation and integration of the LHS, and then Assumption 4 (d) to conclude that
\[ \frac{\partial}{\partial t} \tilde{p}(x_0, y, t) = K \tilde{p}(x_0, y, t), \]
which by Assumption 3 implies (4.15).

5. The copycat CIS for diffusions. We now shift our attention to the case where the target CTMP is the diffusion process (1.1), and present in detail how to choose the renewal intensity and proposal density for a CIS implementation in this context.
If $\delta > 0$ and $0 < \alpha < 1$ are two user-defined constants then the renewal intensity is chosen as

\begin{equation}
\lambda(s) = \delta s^{\alpha-1}.
\end{equation}

Simulating renewal times from such a process is straightforward using the inverse transform. As we shall see later, this choice of $\lambda(s)$ ensures that the incremental weight is well behaved.

The proposal density is constructed by ‘adapting’ it to the current state $(\tau_k, x_{\tau_k}, w_{\tau_k})$ of the CIS algorithm. Specifically, the process is propagated at the next renewal time $\tau_{k+1}$ according to

\[
dX_s = b(x_{\tau_k})ds + \sigma(x_{\tau_k})dB_s, \quad X_0 = x_{\tau_k}, s \geq 0.
\]

Notice that the drift and diffusion coefficient are constant and fixed at their value given $x_{\tau_k}$, the value of the diffusion at the most recent renewal time. Hence, the proposal process is Gaussian with tractable transition density parameterised by $\theta_k = (b(x_{\tau_k}), \sigma(x_{\tau_k}))$ and $q_{\theta_k}(x, y, t) = N\{y; x + tb(x_{\tau_k}), t\gamma(x_{\tau_k})\}$.

Hereafter, the above settings are assumed to be the default for a CIS implementation for diffusions, unless otherwise stated. We refer to the resulting algorithm as copycat CIS since the proposal process adapts at every renewal event, and denote it simply by CIS. The algorithm is presented in Algorithm 5.

**Algorithm 5: Copycat CIS for diffusions.**

*Input:* $T$ a time to stop and $x_0$ the initial value.

*Initialise:* Set $k = 0$, $\tau_0 = 0$, $H_0 = 1$, $x_{\tau_0} = x_0$, $w_{\tau_0} = 1$ and $\theta_0 = (b(x_{\tau_0}), \sigma(x_{\tau_0}))$.

*Iterate:* Repeat

1. draw $U \sim \text{Un}(0, 1)$, and set $\tau_{k+1} = \tau_k + \{-\alpha \log(U) / \delta\}^{1/\alpha}$,
2. if $\tau_{k+1} > T$, set $w_T = w_{\tau_k}$ and stop.
3. Otherwise
   
   (a) draw $x_{\tau_{k+1}} \sim q_{\theta_k}(x_{\tau_k}, \cdot, \Delta \tau_k)$,
   
   (b) compute the incremental weight
   
   \[
   H_{k+1} = \rho_{\theta_k}(x_{\tau_k}, x_{\tau_{k+1}}, \Delta \tau_k)
   \]
   
   and update the weight $w_{\tau_{k+1}} = w_{\tau_k} H_{k+1}$,
   
   (c) set $\theta_{k+1} = (b(x_{\tau_{k+1}}), \sigma(x_{\tau_{k+1}}))$ and $k = k + 1$.

*Output:* A distribution for the value of the process at time $T$, namely $q_{\theta_k}(x_{\tau_k}, \cdot, T - \tau_k)$, and a corresponding importance sampling weight $w_T$.

The formula for the incremental weight based on the above settings is derived in the Lemma below, which is proved in the Appendix.
Lemma 5.1. The incremental weight \( \rho_\theta(x, y, u) \) for the copycat CIS presented in Algorithm 5 is given by

\[
\rho_\theta(x, y, u) = 1 + \frac{1}{\lambda(u)} \left( \frac{1}{2} \left\{ \gamma(y) - \gamma(x) \right\} : K_\theta(x, y, u) + \gamma_2(y) : 1_{d \times d} \right) + \left[ \gamma_1(y) 1_{d \times 1} - b(y) + b(x) \right] : \Lambda_\theta(x, y, u) - b_1(y) \cdot 1_{d \times 1},
\]

where \( b_1, \gamma_1, \gamma_2 \) as in Section 3.1 and

\[
\Lambda_\theta(x, y, u) := \nabla_y q_\theta(x, y, u) q_\theta(x, y, u) = -\gamma^{-1}(x) \frac{y - x - ub(x)}{u},
\]

\[
K_\theta(x, y, u) := \frac{H_q \theta(x, y, u)}{q_\theta(x, y, u)} = \Lambda(x, y, u) \Lambda^T(x, y, u) - \gamma^{-1}(x) \frac{y - x - ub(x)}{u}.
\]

The expressions above are presented conveniently in a matrix form which is very useful from a programming and computational point of view. In the univariate case \( (d = 1) \), the formula for the incremental weight simplifies to

\[
\rho_\theta(x, y, u) = 1 + \frac{1}{\lambda(u)} \left( \frac{1}{\gamma(x) u} \left\{ \gamma(y) - \gamma(x) \right\} \frac{(y - x - ub(x))^2}{\gamma(x) u} - 1 \right) + \left[ b(y) - b(x) - \gamma'(y) \right] \frac{[y - x - ub(x)]}{2} + b'(y).
\]

6. Unbiasedness and stability of the copycat CIS for diffusions. In this section we deal with stability and unbiasedness of the copycat CIS for diffusions detailed in Algorithm 5 in Section 5. The settings for the copycat CIS, in particular the choice of the renewal intensity function along with assumptions on drift and diffusion coefficients will guarantee that the incremental weights \( \{H_k\}_k \) are uniformly bounded, thus ensuring the stability of the overall weight returned by the algorithm. We start with a cautionary example followed by results on stability of weights and unbiasedness.

6.1. A cautionary example. To understand the issues, consider the single incremental weight for the copycat scheme for a univariate diffusion (5.5). First condition on \( x \) and \( u \), and consider this weight as a function of the new value of the process \( Y \). We can write \( Y = x + ub(x) + \sqrt{u \gamma(x)} Z \) where \( Z \) is a standard normal random variable. Then we get

\[
\rho_\theta(x, y, u) = 1 + \frac{1}{\lambda(u)} \left( \frac{1}{\gamma(x) u} \left\{ \frac{\gamma(Y) - \gamma(x)}{2} [Z^2 - 1] \right\} + \frac{\gamma''(Y)}{2} - b'(Y) \right).
\]
The term that causes problems with stability of the incremental weight is
\[
\frac{1}{\lambda(u)} \frac{1}{\gamma(x)u} \left\{ \frac{\gamma(Y) - \gamma(x)}{2} [Z^2 - 1] \right\},
\]
due to the \( u \) in the denominator. Thus any instability in the weight is caused by small values of \( u \), i.e., when two renewal events occur very close to each other.

We have to be careful that the variability of this contribution to the incremental weight does not blow-up too quickly as \( u \to 0 \). However, we immediately see that if \( \gamma \) is for example a Lipschitz continuous function and \( Y \) is “close to” \( x \), then the term in the numerator of the curly bracket of (6.1) has the effect of slowing down this explosion. In fact, we can see that
\[
\gamma(Y) - \gamma(x) = \gamma(x + ub(x) + \sqrt{u\gamma(x)}Z) - \gamma(x) = O_p(\sqrt{u}).
\]

Now when considering whether the incremental weight is in \( L^1 \), we are required to integrate with respect to the renewal distribution density which includes a term \( \lambda(u) \) which cancels with the corresponding term in the denominator of (6.1) and thus the integrand will be \( O_p(u^{-1/2}) \) for small \( u \) and hence is integrable. This is the crux of the integrability stability provided by the copycat scheme.

We note that, were we to be using a different scheme from the copycat one, where the proposal process variance was not chosen to be close to \( \gamma(x) \), then the integrand for the \( L^1 \) integrability will typically be \( O_p(u^{-1}) \) and the incremental weight will thus fail to be in \( L^1 \). Therefore \( L^1 \) stability of the weights will usually fail if we do not adopt the copycat scheme.

Now we return to the copycat scheme and informally consider integrability in \( L^p \) for some \( p > 1 \). Note that the choice of \( \alpha \) made no difference when considering \( L^1 \) stability as the term involving \( \lambda \) cancelled. However for \( p > 1 \) this cancellation does not occur. By choosing \( \lambda(u) \propto u^{\alpha - 1} \) for some \( \alpha < 1 \) the integrand for \( L^p \) stability resembles
\[
\frac{\lambda(u)(\gamma(Y) - \gamma(x))^p}{u^p \lambda(u)^p} = O_p(u^{-p/2 + (1 - \alpha)(p - 1)})
\]
for small \( u \). The integrand is thus integrable for fixed \( p \) whenever
\[
\alpha < \frac{1}{2(1 - p^{-1})}
\]
and in particular this holds whenever \( \alpha \leq 1/2 \). In conclusion, for the simplified setting of this informal calculation, \( \alpha \leq 1/2 \) is sufficient to ensure finiteness of all polynomial moments of the incremental weights.

It is natural to ask whether the strategy of setting \( \lambda(u) \propto u^{\alpha - 1} \) for some \( \alpha < 1 \) can save non-copycat schemes where the proposal process variance was not chosen to be close
However a similar calculation yields that the $L^p$ integrand has order $u^{(1-\alpha)(p-1)-p}$ which is not integrable at 0 for any choice of $0 \leq \alpha < 1$, $p \geq 1$.

These informal calculations for the single incremental weight in the univariate Lipshitz case demonstrate that the copycat scheme is more or less the only possible scheme to ensure stability of the incremental weights, and this explains why we concentrate on this case from now on. Although the arguments for the rest of this section are quite technical, generalisation of the above informal calculations form their core.

6.2. Assumptions. The following regularity conditions will be used to establish the stability of weights and unbiasedness. They are stated in a rather weak form with respect to the geometry induced by the diffusion coefficient $\sigma$, c.f. the remarks below about simplifying and verifying them.

**Assumption 5 (Moments).**

(i) Polynomial growth: there exists $C < \infty$, and $0 < \kappa_b < \infty$, s.t. for all $x, y \in \mathbb{R}$,

\[ ||\sigma^{-1}(x)(b(y) - b(x))|| \leq C(1 + ||\sigma^{-1}(x)(y - x)||^{\kappa_b}) \].

(ii) $||\sigma^{-1}(x)b(x)|| \leq C$, for some $C < \infty$ and all $x \in \mathbb{R}$.

(iii) The derivatives $\partial b_i(x)/\partial x_i$ exist and are bounded above, i.e. $||b(1)(x)|| \leq C$ for all $x \in \mathbb{R}^d$ and some $C < \infty$.

(iv) The infinitesimal variance $\gamma(x) = \sigma(x)\sigma^T(x)$ is such that the derivatives

\[ \partial\gamma_{ij}(x)/\partial x_j \text{ and } \partial^2\gamma_{ij}(x)/\partial x_i\partial x_j \]

exist and are bounded above, i.e. for all $x \in \mathbb{R}^d$ and some $C < \infty$,

\[ ||\gamma(1)(x)|| \leq C \text{ and } ||\gamma(2)(x)|| \leq C. \]

(v) The infinitesimal variance is uniformly bounded from below, i.e. there exists $c > 0$, s.t. for all $x, z \in \mathbb{R}^d$,

\[ z \cdot \gamma(x)z \geq cz \cdot z. \]

(vi) The infinitesimal variance is locally spectrally Hölder continuous of order $0 < \kappa_\gamma \leq 1$ with polynomial growth, in the following sense. There exists $C < \infty$ and $\kappa_\gamma < m < \infty$, s.t. for all $x, y \in \mathbb{R}^d$,

\[ ||\sigma^{-1}(x)(\gamma(y) - \gamma(x))|| \leq C(||\sigma^{-1}(x)(y - x)||^{\kappa_\gamma} + ||\sigma^{-1}(x)(y - x)||^m). \]

(vii) The diffusion coefficient is locally spectrally Hölder continuous of order $0 < \kappa_\sigma \leq 1$ with polynomial growth, in the following sense. There exists $C < \infty$ and $\kappa_\sigma < m < \infty$, s.t. for all $x, y \in \mathbb{R}^d$,

\[ ||\sigma^{-1}(x)(\sigma(y) - \sigma(x))|| \leq C(||\sigma^{-1}(x)(y - x)||^{\kappa_\sigma} + ||\sigma^{-1}(x)(y - x)||^m). \]
Remark 6.1. As for verifying the above conditions, note that (i) is implied by (v) and any of

(i') The drift is bounded above, i.e. $|b(x)| \leq C$, for all $x \in \mathbb{R}^d$ and some $C < \infty$,

(i'') $|b(y) - b(x)| \leq C(1 + \|\sigma^{-1}(x)(y - x)\|^{\kappa_y})$ for all $x \in \mathbb{R}^d$, some $C < \infty$.

Similarly, if (v) holds, then (vi) is implied by any of

(vi') $\|\gamma(y) - \gamma(x)\| \leq C(\|\sigma^{-1}(x)(y - x)\|^{\kappa_y} + \|\sigma^{-1}(y - x)\|^{m})$, for all $x \in \mathbb{R}^d$ and some $C < \infty$.

(vi'') The infinitesimal variance is Hölder continous and bounded above, i.e. $\|\gamma(y) - \gamma(x)\| \leq C\|y - x\|^{\kappa_y}$ and $\|\gamma(x)\| \leq C$ for all $x \in \mathbb{R}^d$ and some $C < \infty$.

Moreover, if (v) holds, then (vii) is implied by any of

(vii') $\|\sigma(y) - \sigma(x)\| \leq C(\|\sigma^{-1}(x)(y - x)\|^{\kappa_y} + \|\sigma^{-1}(y - x)\|^{m})$, for all $x \in \mathbb{R}^d$ and some $C < \infty$,

(vii'') The infinitesimal variance is Hölder continous and bounded above, i.e. $\|\sigma(y) - \sigma(x)\| \leq C\|y - x\|^{\kappa_y}$ and $\|\sigma(x)\| \leq C$ for all $x \in \mathbb{R}^d$ and some $C < \infty$.

Remark 6.2. Assumption 5 is stated using the Euclidean matrix norm, however due to equivalency of norms in normed finite-dimensional vector spaces, it can be equivalently rephrased in terms of other matrix norms. In particular, in case of the $L^2$ operator norm $\| \cdot \|_2$, that we shall use in the sequel, the latter condition of (iv) translates into upper bounds on the largest eigenvalue of $\gamma(2)(x)$ and (v) into the smallest eigenvalue of $\gamma(x)$ being bounded away from 0.

Remark 6.3. In Assumption 5 the diffusion coefficient $\sigma(x)$ can be replaced by any other version $\sigma_\ast(x)$, s.t. $\sigma_\ast(x)\sigma_\ast^T(x) = \gamma(x)$.

6.3. Stability of weights. In view of generic stability results of Section 4, it is enough to verify Assumptions 1 and 2 and conclude from Theorem 4.2. We shall follow this path and prove the following result.

Theorem 6.4. Consider Copycat CIS as detailed in Algorithm 5. The CIS weights $W_T$ are in $L^p$ for every time horizon $T$ and for every $0 < p < p^*$ under the following conditions from Assumption 5,

(a) Conditions (i), (ii), (iii), (iv), (v) and (vi) with $(\alpha - \kappa_\gamma/2)p^* < \alpha$, or

(b) Conditions (i), (ii), (iii), (iv), (v) and (vii) with $(\alpha - \kappa_\sigma/2)p^* < \alpha$.

Remark 6.5. Remarks 6.1, 6.2 and 6.3 apply.

Remark 6.6. For given $p$ and $\kappa_\gamma$ or $\kappa_\sigma$, one can choose $\alpha$ small enough s.t. $W_T \in L^p$. In particular, if $\alpha \leq \kappa_\gamma$ or $\alpha \leq \kappa_\sigma$ respectively, then all moments exist.
Proof of Theorem 6.4. Follows from Theorem 4.2 by verifying Assumption 1 via Proposition 6.9 below and Assumption 2 via Lemma 6.7 and Corollary 6.8 below.

The next lemma deals with the tails of \( n_T \).

**Lemma 6.7.** Let \( \lambda(u) \) and \( n_t \) be as in (5.1) and (3.2) respectively. Then for every \( c < 1 \) and \( t \), there exists \( C < \infty \) such that

\[
\mathbb{P}(n_t \geq j) \leq C j^{-c} \quad \text{for every } j \in \mathbb{N}.
\]

**Proof.** Without loss of generality assume \( \delta = 1 \) in the definition of \( \lambda(u) \) and thus the density and cdf of \( \Delta \tau_1 \) writes

\[
\begin{align*}
g_{\Delta \tau_1}(u) &= \lambda(u) \exp\{-\int_0^u \lambda(v)dv\} = \frac{1}{u^{1-\alpha}} \exp\{-\frac{1}{\alpha} u^\alpha\} \quad \text{and} \\
G_{\Delta \tau_1}(u) &= 1 - \exp\{-\int_0^u \lambda(v)dv\} = 1 - \exp\{-\frac{1}{\alpha} u^\alpha\} \leq \alpha^{-1} u^\alpha,
\end{align*}
\]

respectively. For \( a < \alpha^{1/\alpha} \), define \( b := \alpha^{-1} a^\alpha \). By (6.4), the cdf of \( \Delta \tau_1 \) is smaller than the cdf of \( aB(1 - b) \), where \( B(1 - b) \) is a Bernoulli random variable with success probability \( 1 - b \). Let \( NB(r, b) \) denote the negative binomial distribution of the number of success in a sequence of Bernoulli trials (with success probability \( b \)) before \( r \) failures occur. We allow for \( 0 < r \in \mathbb{R} \), resulting in the generalized negative binomial distribution. Now note that for every choice of \( a < \alpha^{1/\alpha} \), the random variable \( n_t \) is stochastically smaller than \( [t/a] + 1 + NB([t/a] + 1, b) \), and consequently, by stochastic monotonicity in the first parameter between generalized negative binomials, \( n_t \) is also stochastically smaller than \( N_f(a) := t/a + 1 + NB(t/a + 1, b) \). The moment generating function of \( N_f(a) \), say \( M_{N_f(a)}(s) \), exists for \( s \leq -\log b \) and writes

\[
M_{N_f(a)}(s) = e^{s\frac{1-a}{a} \left( \frac{1-b}{1-be^s} \right)^{\frac{1+a}{a}}}.
\]

Thus by stochastic ordering and then Markov inequality we obtain for every \( s \)

\[
\mathbb{P}(n(t) \geq j) \leq \mathbb{P}(N_f(a) \geq j) \leq e^{-sj} e^{s\frac{1-a}{a} \left( \frac{1-b}{1-be^s} \right)^{\frac{1+a}{a}}}.
\]

We set \( z := e^s \), take the log of (6.6) and differentiate it with respect to \( z \), to obtain that given \( j > 1 + t/a \), the RHS of (6.6) is minimized at \( z = \frac{a(j-1)-t}{a(j-1)} \). Using this and recalling that \( b = \alpha^{-1} a^\alpha \), by elementary algebra we arrive at

\[
\begin{align*}
\mathbb{P}(N_f(a) \geq j) &\leq \alpha^{-j} \left( \frac{aj-a-t}{a^{1+\alpha} j} \right)^{-j} \left( \frac{(aj-a-t)\alpha}{a^{1+\alpha} j} \right)^{\frac{1+a}{a}} \left( \frac{(\alpha-a^\alpha)aj}{\alpha(a+t)} \right)^{\frac{1+a}{a}} \\
&= \left( \frac{\alpha}{a^\alpha} \right)^{-j} \left( 1 - \frac{1}{j} - \frac{t}{aj} \right)^{-j} \left( \frac{aj}{a+t} - 1 \right)^{\frac{1+a}{a}} \left( \frac{\alpha}{a^\alpha - 1} \right)^{\frac{1+a}{a}}.
\end{align*}
\]
The next step is to minimize the above with respect to \( a \) provided it is in the admissible range, i.e. \( j > 1 + t/a \), for \( j \) large enough. The solution is not in closed form, however by standard calculus one can see that for large \( j \) the optimal value of \( a \), say \( a^* \) satisfies e.g. 
\[
\log \log j < a^* < \log j.
\]
In particular, since the inequality in (6.7) holds for every \( a \) from the admissible range, we can set \( a = \frac{\log j}{j} < \alpha^{1/\alpha} \). Rearranging terms yields for \( j \) large enough 
\[
\mathbb{P}(N_J(a) \geq j) \leq \left( \frac{\alpha^{\frac{1}{\alpha} j}}{\log j} \right)^{-\alpha j} \left( 1 - \frac{1}{j} - \frac{t}{\log j} \right)^{-j} \times 
\left( j \log j + j^2 - 1 \right)^{\frac{j^2 + \log j}{\log j}} \left( \frac{j^2 + \log j}{\log j} - 1 \right)^{\frac{\log j}{\log j}} \leq j^{-c(1 - \frac{c}{\log j} + \frac{1}{j})\alpha j} \times (\log j)^{-\alpha j + \frac{j^2 + \log j}{\log j}(1 - \alpha)} \times 
\alpha^{-j} \left( 1 - \frac{1}{j} - \frac{t}{\log j} \right)^{-j} \left( \frac{1}{2t} \right)^{\frac{j^2 + \log j}{\log j}} 
=: A(j) \times B(j) \times C(j),
\]
\[
\leq j^{-c\alpha j}, \quad \text{for every } c < 1 \text{ and } j \text{ large enough,}
\]
where the last inequality results from verifying asymptotics with \( j \to \infty \), namely 
\[
\frac{C(j)}{M^j} \to 0, \quad \text{for large enough } M,
\]
\[
\frac{B(j)}{(\log j)^{2j}} \to 0, \quad \text{and}
\]
\[
\frac{A(j)}{j^{-c_0\alpha j}} \to 0, \quad \text{for every } c_0 < 1.
\]
Thus the proof is complete for arbitrary \( c < 1 \) by setting \( c_0 = 1 - (1 - c)/2 \). \qed

The following weaker result is an immediate corollary from Lemma 6.7 and verifies Assumption 2.

**Corollary 6.8.** The random variable \( n_T \) has everywhere finite moment generating function, i.e. \( \mathbb{E}_{\text{CIS}}[\exp\{cn_T\}] < \infty \) for every \( c \in \mathbb{R} \).

Now we turn to establishing Assumption 1 via verifying equation (4.3).

**Proposition 6.9.** Consider the Copycat version of the CIS algorithm for SDEs. Under the following conditions from Assumption 5,
(a) Conditions (i), (ii), (iii), (iv), (v) and (vi) with \( p(\alpha - \kappa/2) < \alpha \), or
(b) Conditions (i), (ii), (iii), (iv), (v) and (vii) with \( p(\alpha - \kappa/2) < \alpha \).

Equation (4.3) holds with \( p^* \) replaced by \( p \), i.e. the \( p \)-th moment of the incremental weight is uniformly bounded.

**Proof.** We start with some preliminary calculations useful for both cases. Recall that for Copycat CIS for diffusions, conditionally on \( X_0 = x, X_{\tau_1} = y \) and \( \tau_1 = u \), for the incremental weight \( H_1 = H_1(x, y, u) \), we have

\[
H_1 = 1 + \frac{(K - K_\theta)q_{\theta_0}(x, y, u)}{\lambda(u)q_{\theta_0}(x, y, u)}
\]

\[
= 1 + \frac{1}{\lambda(u)} \left( \frac{1}{2}[\gamma(y) - \gamma(x)] : K(x, y, u) + \frac{1}{2}[\gamma(1)(y)1_{d\times1} - b(y) + b(x)] : \Lambda(x, y, u) + \frac{1}{2} \gamma(2)(y) : 1_{d\times d} - b(1)(y) \cdot 1_{d\times1} \right)
\]

(6.8)

\[
D_2 = [\gamma(y) - \gamma(x)] : K(x, y, u),
\]

\[
D_1 = [\gamma(1)(y)1_{d\times1} - b(y) + b(x)] : \Lambda(x, y, u),
\]

\[
D_0 = \frac{1}{2} \gamma(2)(y) : 1_{d\times d} - b(1)(y) \cdot 1_{d\times1}.
\]

Recall that in the above : and \( \cdot \) stand for the Frobenius and Euclidean inner products respectively. To rewrite \( \Lambda(x, y, u) \) and \( K(x, y, u) \), in a suitable form, we now use \( z \), the realisation of the \( d \)-dimensional standard normal vector used to generate \( y \) in the Copycat scheme.

\[
\Lambda(x, y, u) = -\frac{\gamma^{-1}(x)}{u} [y - x - ub(x)] = -\frac{\sigma^{-T}(x)\sigma^{-1}(x)}{u} \sqrt{u}\sigma(x)z = -\frac{\sigma^{-T}(x)}{\sqrt{u}} z,
\]

\[
K(x, y, u) = \Lambda(x, y, u)\Lambda^T(x, y, u) - \frac{\gamma^{-1}(x)}{u} = \frac{1}{u} \sigma^{-T}(x)(zz^T - I)\sigma^{-1}(x).
\]

For \( A = A_1 + \cdots + A_k \), and arbitrary \( c > 0 \), we make repeated use of

\[
|A|^c = |A_1 + \cdots + A_k|^c \leq (k \max_i |A_i|)^c \leq k^c(|A_1|^c + \cdots + |A_k|^c).
\]

In particular, since by (6.8)

\[
|H_1|^p \leq 4^p \left( 1 + 2^p \frac{|D_2|^p}{\lambda p(u)} + \frac{|D_1|^p}{\lambda p(u)} + \frac{|D_0|^p}{\lambda p(u)} \right),
\]

(6.9)
we can deal with expectations $\mathbb{E} \left| \frac{D_i}{\lambda(x)} \right|$, $i = 2, 1, 0$, separately.

Now assume (a).

Compute $D_2$ using first the cyclic property of trace, then Cauchy-Schwartz for the Frobenius norm.

$$
|D_2| = \frac{1}{u} \left| \text{tr} \left( [\gamma(y) - \gamma(x)]\sigma^{-T}(x)(zz^T - I)\sigma^{-1}(x) \right) \right|
= \frac{1}{u} \left| \text{tr} \left( \sigma^{-1}(x)[\gamma(y) - \gamma(x)]\sigma^{-T}(x)(zz^T - I) \right) \right|
\leq \frac{1}{u} \left\| \sigma^{-1}(x)[\gamma(y) - \gamma(x)] \right\| \left\| \sigma^{-T}(x)(zz^T - I) \right\| =: \Diamond_1.
$$

Next, use submultiplicativity of the Frobenius norm, uniform boundedness of the $L^2$ operator norm of $\sigma^{-1}(x)$ resulting from Assumption 5 (v), and subsequently Assumption 5 (vi).

$$
\Diamond_1 \leq \frac{1}{C} \left\| \sigma^{-1}(x)[\gamma(y) - \gamma(x)] \right\| \left\| \sigma^{-T}(x) \right\| \left\| zz^T - I \right\|
\leq \frac{1}{C} \left\| \sigma^{-1}(x)[\gamma(y) - \gamma(x)] \right\| \left\| zz^T - I \right\|
\leq \frac{C}{u} \left( \left\| \sigma^{-1}(x)(y - x) \right\|^\kappa + \left\| \sigma^{-1}(x)(y - x) \right\|^m \right) \left\| zz^T - I \right\|
\leq \frac{C}{u} \left( \left\| \sigma^{-1}(x)(y - x - ub(x)) \right\|^\kappa + \left\| \sigma^{-1}(x)b(x) \right\|^\kappa
+ \left\| \sigma^{-1}(x)(y - x - ub(x)) \right\|^m + \left\| \sigma^{-1}(x)b(x) \right\|^m \right) \left\| zz^T - I \right\| =: \Diamond_2.
$$

Finally, use Assumption 5 (ii), and realisation $z$ of the generic standard normal vector used to generate $y$ to rewrite the above as

$$
\Diamond_2 \leq \frac{C}{u} \left( \left\| \sigma^{-1}(x)u^{1/2}\sigma(x)z \right\|^\kappa + \left\| \sigma^{-1}(x)u^{1/2}\sigma(x)z \right\|^m + \left\| u^m \right\|^m \right) \left\| zz^T - I \right\|
= \frac{C}{u} \left( u^{\kappa/2} \left\| z \right\|^\kappa + u^\kappa + u^{m/2} \left\| z \right\|^m + u^m \right) \left\| zz^T - I \right\|.
$$

(6.10)

If $Z$ is a $d-$dimensional standard normal, then for every $p$ there exists $C < \infty$, s.t. $\mathbb{E} \left\| ZZ^T - I \right\|^p \leq C$ and $\mathbb{E} \left( \left\| Z \right\|^{p\kappa} \left\| ZZ^T - I \right\|^p \right) \leq C$. Similarly with $\kappa$ replaced by $m$. Denote the $d-$dimensional standard normal density by $\varphi(\cdot)$. Now recall the density of the interarrival
time (6.3) and using (6.10), compute

$$
\mathbb{E}\frac{|D_2|^p}{\lambda^p(U)} \leq C \int_0^T \frac{u^{(p-1)(1-\alpha)}}{u^p} \exp \left\{ -\frac{1}{\alpha} u^\alpha \right\} \times
\left( \int_{\mathbb{R}^d} \left( u^{\kappa_{+}/2}\|z\|^\kappa_{+} + u^{\kappa_{-} + \cdot \gamma} + u^{m/2}\|z\|^m + u^m \right)^p \|z^T - I\|^p \varphi(z) dz \right) \ du
\leq C \int_0^T u^{(1-p)-1} \left( u^{p\kappa_{+}/2} + u^{p\kappa_{-}} + u^{pm/2} + u^{pm} \right) du
$$

since $m > \kappa_{-}$ and we assumed $p(\alpha - \kappa_{-}/2) < \alpha$.

To deal with the scalar product in $D_1$ use first that $\sigma^{-1}(x)$ is the adjoint of $\sigma^{-T}(x)$, then Cauchy-Schwarz for the Frobenius norm, and finally Assumptions 5 (iv) and (v), i.e. boundedness of $\|\gamma(1)(x)\|$ and $\|\sigma^{-1}(x)\|$.

$$
|D_1| \leq \left| \gamma(1)(y)1_{d \times 1} - b(y) + b(x) \right| \cdot \frac{\sigma^{-T}(x)}{u^{\alpha}} \|z\| \leq u^{-1/2} \left( \|\sigma^{-1}(x)\gamma(1)(y)1_{d \times 1} \cdot z\| + \|\sigma^{-1}(x)(b(y) - b(x)) \cdot z\| \right) \leq C u^{-1/2} \left( \|z\| + \|\sigma^{-1}(x)(b(y) - b(x))\| \right) =: \clubsuit_1
$$

Next, use Assumption 5 (i) and (ii) and again $z$, the realisation of the standard normal vector used to generate $y$.

$$
\clubsuit_1 \leq C u^{-1/2} \left( \|z\| + \left(1 + \|\sigma^{-1}(x)(y - x)\|^{\kappa_{+}}\|z\| \right) \right) \leq C u^{-1/2} \left( \|z\| + \left( \|\sigma^{-1}(x)(y - x - ub(x))\|^{\kappa_{+}} + u^{\kappa_{+}} \|\sigma^{-1}(x)b(x)\|^{\kappa_{+}}\|z\| \right) \right) \leq C u^{-1/2} \left( \|z\| + \left( u^{\kappa_{+}/2}\|z\| + u^{\kappa_{+}}\|z\| \right) \right) \leq C u^{-1/2} \left( \|z\| + u^{\kappa_{+}/2}\|z\| + u^{\kappa_{+}/2}\|z\| \right) \right)
$$

Similarly as in (6.11), recalling (6.3) and noting that $\mathbb{E}\|Z\|^p < \infty$ and $\mathbb{E}\|Z\|^{2p} < \infty$ for $Z \sim N(0, I)$, yields

$$
\mathbb{E}\frac{|D_1|^p}{\lambda^p(U)} \leq C \int_0^T \frac{u^{(p-1)(1-\alpha)}}{u^{p/2}} \exp \left\{ -\frac{1}{\alpha} u^\alpha \right\} \times
\left( \int_{\mathbb{R}^d} \left( \|z\| + u^{\kappa_{+}/2}\|z\| + u^{\kappa_{+}/2}\|z\| \right)^p \varphi(z) dz \right) \ du
$$
\[
C \int_0^T u^{p(1/2 - \alpha) - 1 + \alpha} (1 + u^{pc_0} + u^{pc_1}) \, du \\
(6.13) \leq C \int_0^T u^{p(1/2 - \alpha) - 1 + \alpha} \, du \leq C_1(p) < \infty,
\]

since \( p(\alpha - \kappa / 2) < \alpha \) implies \( p(\alpha - 1/2) < \alpha \).

Finally, by Assumptions 5 (iii) and (iv) and Cauchy-Schwarz, for some \( C < \infty \),
\[
|D_0| \leq \frac{1}{2} |\text{tr}(\gamma(2)(y)1_{d \times d})| + |b(1)(y) \cdot 1_{d \times 1}| \\
\leq \|\gamma(2)(y)\|\|1_{d \times d}\| + \|b(1)(y)\|\|1_{d \times 1}\| \leq C_0 < \infty.
\]

Hence
\[
\mathbb{E} \left| \frac{D_0}{\chi_{p}(U)} \right|^p = C \int_0^T u^{(p-1)(1-\alpha)} \exp \left\{ - \frac{1}{\alpha} u^\alpha \right\} \left( \int_{\mathbb{R}^d} C_0^\beta \varphi(z) \, dz \right) \, du \\
\leq C \int_0^T u^{(p-1)(1-\alpha)} \, du \leq C < \infty.
\]

Combining (6.11), (6.13) and (6.15) yields the claim under assumption (a).

Now assume (b).

The calculations involving \( |D_1|^p \) and \( |D_0|^p \) do not change. To deal with \( |D_2| \) note that
\[
\gamma(y) - \gamma(x) = (\sigma(y) - \sigma(x) + \sigma(x)) \left( \sigma^T(y) - \sigma^T(x) + \sigma^T(x) \right) = (\sigma(y) - \sigma(x)) (\sigma^T(y) - \sigma^T(x)) \\
+ (\sigma(y) - \sigma(x)) \sigma^T(y) - \sigma^T(x).
\]

Now use (6.16) to deal with \( |D_2| \) in a similar manner that led to (6.10). We thus start with utilising the decomposition (6.16), next use cyclic property of the trace, and submultiplicativity of the Frobenius norm to arrive at
\[
|D_2| = \frac{1}{u} |\text{tr} \left( (\gamma(y) - \gamma(x)) |\sigma^{-T}(x)(zz^T - I)\sigma^{-1}(x) \right) | \\
\leq \frac{1}{u} |\text{tr} \left( (\sigma(y) - \sigma(x)) (\sigma^T(y) - \sigma^T(x)) |\sigma^{-T}(x)(zz^T - I)\sigma^{-1}(x) \right) |
+ \frac{1}{u} |\text{tr} \left( (\sigma(y) - \sigma(x)) |\sigma^T(x)(zz^T - I)\sigma^{-1}(x) \right) |
+ \frac{1}{u} |\text{tr} \left( (\sigma(y) - \sigma(x)) |\sigma^T(x)(zz^T - I)\sigma^{-1}(x) \right) |
\leq \frac{1}{u} |\sigma^{-1}(x)(\sigma(y) - \sigma(x))| \| (\sigma^T(y) - \sigma^T(x)) \| \| zz^T - I \|
+ \frac{1}{u} |(\sigma^T(y) - \sigma^T(x)) \| \| zz^T - I \|
The result of \((\alpha - \kappa_\sigma) / 2 \leq \alpha\).

Combining (6.18), (6.13) and (6.15) yields the claim under assumption (b).

7. Extensions. This section deals with boosting the algorithmic performance of CIS and introduces two modifications. The first generalises the basic algorithm by allowing the use of proposal distributions other than \(q\). An optimal distribution, in the sense of weight variability minimisation, is identified when interest lies in simulation/estimation at
a fixed time $T$. This extension also allows us to improve transition density estimation by using proposal processes that are guided towards a specific value at the terminal time. The second modification deals with estimation as $T$ increases and enhances stability by using a batch implementation of CIS, which monitors the performance of the algorithm through time and controls the variability of the weights via resampling.

7.1. Proposal densities and optimal implementation. Recall that the CIS algorithm described in Algorithm 5 updates the process at each event time using proposals from $q$, as shown in Step 3a. We now modify this step so that $x_{\tau_{k+1}}$ at each event time is drawn from a generic density, denoted by $g_\xi(\cdot)$ where $\xi$ is an abstract parameter. To preserve the unbiasedness of the algorithm, an importance sampling correction term is added to the incremental weight so that Step 3b becomes

$$w_{\tau_{k+1}} = w_{\tau_k} \rho_{\theta_k}(x_{\tau_k}, x_{\tau_k+1}, \Delta \tau_k) \frac{q_{\theta_k}(x_{\tau_k}, x_{\tau_k+1}, \Delta \tau_k)}{g_\xi(x_{\tau_k+1})}.$$

The above expression allows us to identify a proposal distribution which minimises the variance of $w_{\tau_{k+1}}$. It is trivial to show that the optimal $g_\xi$ is given by

$$g_\xi(x_{\tau_{k+1}}) \propto |\rho_{\theta_k}(x_{\tau_k}, x_{\tau_k+1}, \Delta \tau_k)| q_{\theta_k}(x_{\tau_k}, x_{\tau_k+1}, \Delta \tau_k).$$

The optimal density will typically be intractable but it can guide the choice of a suitable approximation, as illustrated later in Section 8.2.

7.2. Transition density estimation and guided CIS. When interest lies in estimating the transition density $p(x_0, x_T, T)$, the accuracy of CIS can be improved by choosing $g$ such that the proposed process is guided towards $x_T$. A process with this property which can be easily simulated is the so-called modified Brownian bridge [12]. If $g_{bb}(x_s, x_t; x_T, \alpha)$ denotes the transition density of a scaled Brownian bridge, i.e., for $0 \leq s \leq t \leq T$,

$$g_{bb}(x_s, x_t; x_T, \alpha) = \mathcal{N}\left\{x_t; \frac{x_s(T - t) + x_T(t - s)}{T - s}, \alpha(T - t)(t - s)\right\},$$

then we update the proposal at event time of CIS using $g_{bb}\{x_{\tau_k}, x_{\tau_{k+1}}; x_T, \gamma(x_{\tau_k})\}$. Notice that the density incorporates information from the end point through the mean and drifts the process towards $x_t$. We refer to a CIS algorithm using the above proposal as guided CIS (GCIS).

By recalling that $\tau_k, i = 0, 1, \ldots, n_t$ denote the simulated time points in a single run of GCIS up to time $t$, then the transition density estimator is given by

$$q_{\theta_{wt}}(x_{\tau_t}, x_T, t - \tau_t) \prod_{k=0}^{n_T-1} \frac{q_{\theta_k}(x_{\tau_k}, x_{\tau_{k+1}}, \Delta \tau_k)}{g_{bb}\{x_{\tau_k}, x_{\tau_{k+1}}; x_T, \gamma(x_{\tau_k})\}} \rho_{\theta_k}(x_{\tau_k}, x_{\tau_{k+1}}, \Delta \tau_k).$$
An interesting link to the transition density estimator of Durham and Gallant [12] (henceforth denoted by DG) arises here. In particular, notice that after removing the incremental weight terms from the above quantity the density estimator only differs from a standard DG density estimator which imputes \( n_t \) points in that the partition of the time interval is neither equidistant nor deterministic. This allows us to give an intuitive explanation to the incremental weights \( \rho_{\theta_k} \) as bias correction terms to the subtransition densities \( q_{\theta_k} \).

### 7.3. Resampling.

The variance of estimates obtained by sequential importance samplers typically increases with time, implying that the approximations can be arbitrarily poor when \( T \) is large, see for example [17]. An established approach which attempts to improve the quality of the estimates over large time horizons is based on resampling procedures, see [10] for a recent review. The idea is to monitor the quality of the approximation through time using a cloud of \( N \) particles and rejuvenate them, when necessary, by eliminating particles with low importance weights. In this section, we adapt these ideas to the CIS framework and develop two algorithms which incorporate a resampling step. We show numerically that both algorithms improve greatly upon the basic CIS algorithm presented in the previous sections when \( T \) is large. The first version has a \( \mathcal{O}(N) \) computational complexity but its performance deteriorates superlinearly with \( T \). The second is inspired by the proposed method of [22], has an \( \mathcal{O}(N^2) \) cost and its performance degrades linearly with \( T \).

Let \( M \in \mathbb{Z}^+ \) and denote by \( t_j = jT/M, \ j = 0, 1, \ldots, M \) a partition of the time interval \([0, T]\) into \( M \) subintervals. The algorithms that we propose use an initial cloud of \( N \) particles and propagate them through all times \( t_j \) using CIS. At every time \( t_j \) we have available the set of weights \( \{w_j^{(i)}\}_{i=1}^N \), whose variability is assessed using an effective sample size (ESS), defined in step 1 of Algorithm 6 below. If the ESS is small then the particles are resampled according to their weights and only the surviving ones are propagated to the next time \( t_{j+1} \). We employ resampling based on \( \{|w_j^{(i)}|\}_{i=1}^N \) since the plain weights returned by CIS are not necessarily positive. This may weaken the benefits of the resampling procedure, since it can occasionally lead to resampling in areas where the variability of the weights is large, rather than where the mean weight is large. In the algorithms presented below, we denote the output of a CIS algorithm with input \( x \) and terminal time \( t \) by the triplet \((s, y, w)\) where \( s \) is the last Poisson event before time \( t \), \( y \) the value of the process at that time and \( w \) the output weight, and write \((s, y, w) \sim \text{CIS}(x, t)\). The first version of CIS with resampling, termed CIS-R1, is presented in Algorithm 6.

**Algorithm 6: CIS algorithm with resampling (CIS-R1).**

**Input:** \( T \) a time to stop, \( M \) a discretisation level, \( C \) an ESS threshold and \( x_0 \) the initial value.

**Initialise:** Set \( x_0^{(i)} = x_0, s_0^{(i)} = 0, w_0^{(i)} = 1, 1 \leq i \leq N \) and \( t_j = jT/M, 1 \leq j \leq M - 1 \).

For \( 0 \leq j \leq M - 1, 1 \leq i \leq N \) perform the following steps:

**Step 1:** Calculate the effective sample size of \( \{|w_j^{(i)}|\} \), \( \text{ESS} = a_j^2 / \sum_{i=1}^N (w_j^{(i)})^2 \), where \( a_j = \ldots \)
\[ \sum_{i=1}^{N} |w_j^{(i)}|. \] If ESS < C then sample \( k_{i,j} \sim p(k) \propto |w_j^{(k)}|, k = 1, \ldots, N \) and set \( \phi_j^{(i)} = \text{sign}(w_j^{(k_{i,j})})a_j/N; \) else set \( k_{i,j} = j \) and \( \phi_j^{(i)} = w_j^{(i)}. \)

**Step 2:** Draw \((\tau_{j+1}^{(i)}, x_{j+1}^{(i)}, w_{j+1}^{(i)}) \sim \text{CIS}(x_j^{(k_{i,j})}, t_{j+1} - s_j^{(k_{i,j})})\), set \( s_{j+1}^{(i)} = s_j^{(k_{i,j})} + \tau_{j+1}^{(i)} \), and compute the weights \( w_{j+1}^{(i)} = \phi_j^{(i)}w_{j+1}^{(i)} \).

Each step requires \( \mathcal{O}(N) \) computations, therefore the total complexity of CIS-R1 is \( \mathcal{O}(N) \). When resampling occurs in Step 1, the weights themselves are updated to \( \phi_j^{(i)} = a_j \text{sign}(w_j^{(k_{i,j})})/N \) to ensure the unbiasedness of the algorithm. This also implies that surviving particles retain their weight sign after resampling and CIS-R1 keeps propagating particles with negative weights.

A more efficient approach, albeit more computationally expensive, is presented below. The idea is based on simulating the process at resampling times \( t_j \) and assigning a numerically more stable weight to each resampled particle. The algorithm, termed CIS-R2, is presented below; for simplicity, we suppress the dependence of \( q \) on \( \theta \).

**Algorithm 7:** *Extended CIS algorithm with resampling* (CIS-R2).

**Input:** \( T \) a time to stop, \( M \) a discretisation level, \( C \) an ESS threshold and \( x_0 \) the initial value.

**Initialise:** Set \( x_0^{(i)} = x_0, s_0^{(i)} = 0, w_0^{(i)} = 1, 1 \leq i \leq N \) and \( t_j = jT/M, 1 \leq j \leq M - 1 \).

For \( 0 \leq j \leq M - 1, 1 \leq i \leq N \) perform the following steps

**Step 1:** Calculate the effective sample size of \( \{ |w_j^{(i)}| \} \), \( \text{ESS} = a_j^2/\sum_{i=1}^{N} (w_j^{(i)})^2 \), where \( a_j = \sum_{i=1}^{N} |w_j^{(i)}| \). If ESS < C then sample \( \bar{x}_j^{(i)} \sim \bar{p}_j(\cdot), s_j^{(i)} = t_j \) and \( \phi_j^{(i)} = \bar{p}_j(\bar{x}_j^{(i)})/\bar{p}_j(\bar{x}_j^{(i)}) \),

where

\[ (7.2) \quad \bar{p}_j(y) = \frac{1}{N} \sum_{k=1}^{N} w_j^{(i)} q(x_j^{(i)}, y, t_j - s_j^{(i)}), \]

\[ (7.3) \quad \bar{p}_j(y) = a_j^{-1} \sum_{i=1}^{N} |w_j^{(i)}| q(x_j^{(i)}, y, t_j - s_j^{(i)}), \]

and set \( x_j^{(i)} = \bar{x}_j^{(i)} \); else set \( \phi_j^{(i)} = w_j^{(i)}. \)

**Step 2:** Draw \((\tau_{j+1}^{(i)}, x_{j+1}^{(i)}, w_{j+1}^{(i)}) \sim \text{CIS}(x_j^{(i)}, t_{j+1} - s_j^{(i)})\), set \( s_{j+1}^{(i)} = s_j^{(i)} + \tau_{j+1}^{(i)} \), and compute the weights \( w_{j+1}^{(i)} = \phi_j^{(i)}w_{j+1}^{(i)} \).

Resampling is still based on the absolute value of the weights and is embedded in sampling from \( \bar{p}_j \). Contrary to CIS-R1, when resampling occurs at time \( t_j \) the particles are also propagated to that time. The weight given to each resampled particle \( \phi_j^{(i)} \) is given by the ratio \( \tilde{p}_j/\bar{p}_j \) where \( \tilde{p}_j \) is the unbiased transition density estimator at time \( t_j \) and \( \bar{p}_j \) is a mixture of our proposal distributions. After resampling, the weighted particle set at time
\[ t_j, \{\bar{x}_j^{(i)}, \phi_j^{(i)}\}_{i=1}^N \text{ provides an approximation to the density of the target process at time } t_j \text{ via} \]
\[ \hat{\pi}_j(dy) = \sum_{i=1}^N \phi_j^{(i)} \delta_{\bar{x}_j^{(i)}}(dy) \]

where \(\delta_{\bar{x}}(dy)\) denotes the Dirac delta mass located at \(x\). By conditioning on \(\{s_j^{(i)}, x_j^{(i)}, w_j^{(i)}\}_{i=1}^N\), it is trivial to show that \(\hat{\pi}_j/N\) is an unbiased estimator of \(\hat{p}_j\), and thus CIS-R2 retains the desired unbiased property.

Evaluating each of \(\{\phi_j^{(i)}\}_{i=1}^N\) requires \(O(N)\) calculations, hence the overall complexity of CIS-R2 is \(O(N^2)\). However, these weights will typically be numerically more stable than that of CIS-R1 since they are calculated by marginalising out the auxiliary variables up to time \(t_j\); see also [22] for a similar implementation. Numerical evidence in Section 8.2.2 suggest that CIS-R2 degrades only linearly with \(T\) whereas CIS-R1 superlinearly.

8. Numerical illustrations. In this section, we investigate numerically the performance of the CIS algorithm under various implementation schemes and compare it to two existing approaches. The first is the well known transition density estimator of [12] which is based on discrete-time approximations of the diffusion process and returns biased estimates. The second is an unbiased approach introduced by Wagner [27], an overview of which is given below. All methods under consideration are based on a finite imputation (either random or deterministic) of points, thus allowing us to compare the approaches in a fair manner on the basis of a fixed average number \(K\) of simulated values; that is, the number of trajectories \(N\) multiplied by the average number of points \(M\) each trajectory is evaluated. We refer to \(K\) as the computational cost.

8.1. Overview of Wagner’s approach. The approach by Wagner (denoted by WGR hereafter) is a method which estimates unbiasedly any quantities of interest by using unbiased estimates of the transition density. The key of this method is the representation of the unknown transition density via the following integral equation
\[ p(x_0, x_t, t) = p^{(1)}(x_0, x_t, t) + \int_0^t \int_{\mathbb{R}^d} p(x_0, y, s)\mathcal{C}(y, x_t, t-s)dyds, \]
where \(p^{(1)}(x_0, x_t, t)\) is the transition density of a tractable process and \(\mathcal{C}(y, x_t, t-s)\) is an analytically available quantity which involves the Kolmogorov’s backward operator [for more details see 27]. By successive substitutions of the transition density in the integral equation we obtain an infinite expansion which is intractable but can be estimated unbiasedly using importance sampling. In particular, if \(t_0 = t\) and \(q_u(t, s), t > s\) is a user-specified transition kernel, then the quantity
\[ \frac{p^{(1)}(x_0, x_{t_n}, t_n)}{p_u(t_n)} \prod_{k=1}^n q_{bb}\{x_0, x_{t_k}; x_{t_{k-1}}, \gamma(x_0)\}q_u(t_{k-1}, t_k)\{1 - p_u(t_{k-1})\} \]
is an unbiased estimator of the transition density. The algorithm creates a random partition of the time interval by repeatedly simulating time points \( t_k \mid t_{k-1} \sim q_u(t_{k-1}, t_k) \) and drawing \( x_{t_k} \mid x_0, x_{t_{k-1}} \) according to Brownian bridge dynamics. The function \( p_u(t) \) is an absorption probability which terminates the simulation and \( n \) is the total number of simulated time points. Wagner proposes using

\[
p_u(t) = \left\{ \sum_{m=0}^{\infty} \frac{\delta^m \Gamma(m \alpha)^m}{\Gamma(m \alpha + 1)} \right\}^{-1}, \quad q_u(t, s) = \frac{\delta \mathbf{1}_{(0,t)}(s)(t-s)^{\alpha-1}p_u(t)}{p_u(s)[1 - p_u(t)]},
\]

where \( \delta, \alpha > 0 \) are specified by the user.

Notice that when \( \alpha = 1 \) the above quantities can be calculated analytically and the time points \( \{t_k\} \) correspond to arrival times of a simple homogeneous Poisson process of rate \( \delta \) in \([0, t]\). We refer to this particular implementation as WGR1. However, when \( \alpha \neq 1 \), calculating these quantities without sacrificing the unbiasedness is not straightforward.

For an objective comparison with CIS, we propose simulating the time points \( \{t_k\} \) using a process similar to that of CIS. In particular, choosing

\[
p_u(t) = \exp\left\{ -\frac{\gamma t^\alpha}{\alpha} \right\}, \quad q_u(t, s) = \frac{\gamma \mathbf{1}_{(0,t)}(s)(t-s)^{\alpha-1}p_u(t-s)}{1 - p_u(t)},
\]

results in \( \{t_k\} \) being a simple transformation of \( \{\tau_k\} \) as \( t_k = t - \tau_k \), where \( \{\tau_k\} \) are the time points simulated in a CIS algorithm. We term such an implementation by WGR2. Empirical results in the subsequent section suggest that WGR2 can perform substantially better than WGR1.

8.2. Stochastic volatility model. We consider a simple bivariate diffusion process, solution to

\[
\begin{align*}
\mathrm{d}X_{1,t} &= -\frac{\sigma_1^2}{2} \tanh(X_{1,t}) \mathrm{d}t + \sigma_1 \mathrm{d}B_{1,t}, \\
\mathrm{d}X_{2,t} &= \sigma_2 [2 + \tanh(X_{1,t})] \mathrm{d}B_{2,t},
\end{align*}
\]

(8.1)

where \( B_1 \) and \( B_2 \) are two independent standard Brownian motions and \( \sigma_1, \sigma_2 > 0 \). We refer to the model as SV. The first coordinate of the system is an ergodic process with invariant mean 0 while \( \sigma_1 \) governs the speed at which the process returns to the invariant mean. It is straightforward to check that the drift and diffusion matrix of this model satisfy conditions for a valid CIS implementation. Despite its fairly simple nature, (8.1) cannot be transformed to a diffusion with constant volatility, and therefore unbiased estimation cannot be performed using any variant of the Exact Algorithm.

8.2.1. Implementation considerations. The purpose of this section is to illustrate the effect that the choice of the Poisson rate and proposal process can have on the efficiency of
the CIS algorithm. We employ CIS to estimate the mean vector of (8.1) at terminal time $T = 1$ using four different implementations. The first is the copycat CIS as described in Algorithm 5 and estimates the integral in (3.7) by Monte Carlo. The second is implemented as the first but sets a constant Poisson rate $\lambda(\tau) = \delta$; we refer to this estimator as CIS$_{con}$. The third attempts to optimise the first by sampling from the optimal density in (7.1) and evaluating the integral in (3.7) analytically; we refer to this estimator as CIS$_{opt}$. Simulating directly from the optimal density is not straightforward; instead, we use an approximation where $x_{1,t}$ is sampled from an Euler density as implied by (8.1), while the conditional distribution of $x_{2,t}$ given $x_{1,t}$ is analytically tractable and is sampled using rejection sampling. Finally, the fourth is implemented as the first but adapts only the drift functional at every Poisson event; we refer to this estimator as CIS$_{nc}$. To ensure a fair comparison, the value of $\delta$ for each estimator is selected such that their computational costs are comparable.

Figure 1 shows the kernel density estimates of the estimators, obtained by 1000 independent realisations. Notice that CIS$_{con}$ produces extreme estimates and its distribution is heavy tailed. Both of these issues are remedied by using a nonconstant Poisson rate, as illustrated by the distribution of the CIS. CIS$_{opt}$ outperforms substantially the other methods while the estimates obtained from CIS$_{nc}$ were highly unstable and exhibited very heavy tails (not shown here).

8.2.2. Estimation over large time intervals. We now assess the performance of CIS over large time intervals, illustrate the effect of resampling using the two approaches developed in Section 7.3 and compare CIS to the WGR approach. We consider various values for the time horizon from $t = 1/2$ to $t = 10$ with stepsize $1/2$.

We begin by comparing the plain CIS, WGR1 and WGR2. The CIS algorithm is employed with $N = 1000$ whilst the Monte Carlo size of the other two methods is tuned such that their
computational cost are comparable to that of CIS. Figure 2a illustrates the performance of these three approaches. Notice that the approximation error of CIS is substantially smaller than that of WGR1, and comparable to that of WGR2. The WGR1 approach was highly unstable and generated many extreme estimates. For this reason, and for this comparison only, we have assessed the performance of the estimators using the median absolute deviation (MAD) which is robust to outliers.

Figures 2b-2d present the root mean square (RMSE) of the plain CIS along with its two resampling modifications, CIS-R1 and CIS-R2. For an empirical illustration of the rate at which the error increases, a linear, quadratic and cubic fit overlay the plots. The plain CIS algorithm exhibits an approximation error which increases cubically with time, and is outperformed significantly by CIS-R1, whose error grows quadratically. Finally, at the expense of a $O(N^2)$ computational cost, the performance of CIS-R2 seems to deteriorate only linearly with time, implying that for sufficiently large $T$, CIS-R2 will always outperform the other two methods.

8.3. Bivariate CIR model. In this section we deal with a bivariate Cox-Ingersoll-Ross process (hereafter denoted by CIR), solution to

$$
\begin{align*}
    dX_{1,t} &= -\rho_1 (X_{1,t} - \mu_1) \, dt + \sigma_1 \sqrt{X_{1,t}} \, dW_t, \\
    dX_{2,t} &= -\rho_2 (X_{2,t} - \mu_2) \, dt + \sigma_2 \sqrt{X_{2,t}} \left( \rho dW_t + \sqrt{1 - \rho^2} dB_t \right).
\end{align*}
$$

The instantaneous correlation between the coordinates of the system is $\rho$, whilst each marginal is a univariate Cox-Ingersoll-Ross process [7]. Even though the model can be reduced into one with unit diffusion matrix, the transformed drift cannot be written in a gradient form. Therefore, existing exact methods are not applicable. The CIR process is an example for which the conditions of Theorem (toDo: REF THRM HERE) do not hold. It is therefore interesting to examine empirically the behaviour of the CIS algorithm. We perform two sets of simulations.

The first examines the stability of the weights at different terminal times for a given set of parameters and fixed initial point. The distribution of the weights is shown in Figure 3. Notice that, despite the deviation of the process from Theorem’s (toDo: ref here) assumptions, the weights behave in a stable manner.

The second assesses the performance of the GCIS method for estimating the transition density of the process and compare it to the conventional DG estimator [12] under two transformation schemes: log transformation and transformation to unit diffusion matrix (also known as Lamperti transform). The DG method is employed using $N = O(M^2)$, where $N$ and $M$ are the number of trajectories and imputed points per trajectory respectively.

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$^1$Implemented with $\lambda(\tau) = \tau^{-1/2}$

$^2$Implemented with $\lambda(\tau) = 1$

$^3$Implemented with $\lambda(\tau) = 1/2\tau^{-1/2}$
This setting follows from the results of [25] on optimal allocation of the computing resources. As recommended by the authors, we set $N = M^2$ as a crude choice. The performance of the estimators is examined under an increasing sequence of computational costs $K_i$, $i = 1, 2, \ldots, 7$. The $K_i$s are chosen by selecting various values for the level of discretisation for the DG approach, $M_i = 2^i$. Thus $K_i = N_i M_i = M_i^3$, $i = 1, 2, \ldots, 7$.

Table 1 summarises the performance of the transition density estimators for a given initial and terminal point. The GCIS algorithm performs substantially better than CIS, clearly illustrating the effect of using a guided proposal process. For lower values of the computational cost, the DG approach outperforms CIS but this behaviour changes as the cost increases. This is not surprising since DG tries to account for two sources of error, bias arising from the discretisation and variance due to Monte Carlo error, whereas CIS allocates all computational effort to the Monte Carlo error. The Lamperti transform is
Fig 3: The CIR model with \((\rho_1, \mu_1, \sigma_1, \rho_2, \mu_2, \sigma_2, \rho) = (0.6, 2.5, 0.45, 0.3, 3.0, 0.35, 0.5)\), and \(X_0 = (2.5, 3)'\). Kernel density estimates of the weight distribution using CIS at different terminal times \(T\).

particularly beneficial to the accuracy of the CIS and approximately halves its RMSE. Further comparisons were also performed on different terminal points which yielded similar results (not shown here).
Table 1

The CIR model with $(\rho_1, \mu_1, \sigma_1, \rho_2, \mu_2, \sigma_2, \rho) = (0.6, 2.5, 0.45, 0.3, 3.0, 0.35, 0.5)$, $X_0 = X_1 = (2.5, 3)'$. Mean, Monte Carlo error and RMSE of transition density estimators under increasing computational cost $K$. The results are averages of 1000 replications.

| Method       | Mean   | $K_1$ | $K_2$ | $K_3$ | $K_4$ | $K_5$ | $K_6$ | $K_7$ |
|--------------|--------|-------|-------|-------|-------|-------|-------|-------|
| CIS $^4$     | Mean   | 0.6692| 0.6200| 0.6369| 0.6380| 0.6391| 0.6387| 0.6386 |
| (log-transform)| MC Err.| 0.8901| 0.2904| 0.1261| 0.0640| 0.0227| 0.0082| 0.0032 |
|              | RMSE   | 0.8907| 0.2910| 0.1261| 0.0640| 0.0227| 0.0082| 0.0032 |
| GCIS $^5$    | Mean   | 0.6573| 0.6324| 0.6402| 0.6386| 0.6389| 0.6386| 0.6387 |
| (log-transform)| MC Err.| 0.4425| 0.1491| 0.0550| 0.0201| 0.0073| 0.0025| 0.0009 |
|              | RMSE   | 0.4429| 0.1492| 0.0550| 0.0201| 0.0073| 0.0025| 0.0009 |
| GCIS $^5$    | Mean   | 0.6361| 0.6357| 0.6388| 0.6386| 0.6387| 0.6386| 0.6386 |
| (lam-transform)| MC Err.| 0.2017| 0.0704| 0.0250| 0.0093| 0.0033| 0.0012| 0.0004 |
|              | RMSE   | 0.2017| 0.0704| 0.0250| 0.0093| 0.0033| 0.0012| 0.0004 |
| DG           | Mean   | 0.5231| 0.5786| 0.6084| 0.6235| 0.6310| 0.6348| 0.6367 |
| (log-transform)| MC Err.| 0.0997| 0.0431| 0.0167| 0.0067| 0.0028| 0.0013| 0.0006 |
|              | RMSE   | 0.1525| 0.0739| 0.0344| 0.0165| 0.0081| 0.0040| 0.0020 |
| DG           | Mean   | 0.5315| 0.5853| 0.6110| 0.6248| 0.6316| 0.6351| 0.6368 |
| (lam-transform)| MC Err.| 0.0873| 0.0324| 0.0124| 0.0049| 0.0018| 0.0008| 0.0004 |
|              | RMSE   | 0.1381| 0.0623| 0.0303| 0.0146| 0.0072| 0.0036| 0.0018 |

9. Discussion. This paper has introduced the CIS sampler for unbiased simulation of quite general classes of stochastic processes, and demonstrated that the method performs well in practice in a range of scenarios. In doing this we have circumvented the two major limitations of the exact algorithm framework for simulating diffusions, while also providing a framework which can be applicable well beyond the diffusion case.

We have shown that classical ideas due to Wagner can be used to provide alternative approaches, although it appears that CIS has distinct computational advantages, deriving in particular from its sequential construction. There are also other, more recent ideas, for unbiasedly sampling from a multivariate diffusions. Using a methodology based on Malliavin weights, [16] construct unbiased estimators for diffusion functionals at a fixed time, but it can only be applied when diffusion coefficients are constant. There is also theoretical work on importance sampling for diffusions by Bally and Kohatsu-Higa [2], from which it may be able to construct sequential Monte Carlo schemes. This latter approach is derived from the parametrix method for solving parabolic differential equations and is based on an iterated integral representation of the transition density. It is interesting that interpreting CIS as an

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$^4\lambda(\tau) = 1/2\tau^{-1/2}$; each trajectory was evaluated on average at 3.3 time points.

$^5\lambda(\tau) = \tau^{-1/2}$; each trajectory was evaluated on average at 6.1 time points.
iterated integral is not equivalent to the parametrix and thus, albeit related, CIS appears to be fundamentally different to this approach. Finally [5] suggest an approach for unbiasedly simulating multivariate diffusions using rough path techniques, but currently these methods have an expected computational cost which is infinite.

The theoretical stability of the method provides quite delicate and subtle questions, and without carefully constructed proposal stochastic processes, the Monte Carlo weight can easily fail to be stable (and may even fail to be in $L^1$ in some situations). However we give a collection of positive results which provide stability guarantees under quite reasonable and checkable conditions.

In Section 8 we have investigated resampling schemes which are required to provide some robustness for CIS as the time interval grows. There are many issues requiring further investigation here (both theoretically and empirically) including how best to deal with negative weights, and how to optimally choose times to update weights in order to minimise the need for resampling.

APPENDIX A: APPENDIX

Proof of Lemma 5.1. By definition,

$$\rho_\theta(x, y, u) = 1 + \frac{1}{\lambda(u)q_\theta(x, y, u)} [K - K_\theta] q_\theta(x, y, u).$$

We start by calculating the forward operator of the target process applied to the proposal density. Specifically, we have that

$$K_{q_\theta}(x, y, u) = \sum_{i,j} \frac{\partial^2}{\partial y_i \partial y_j} \left( \gamma_{ij}(y) q_\theta(x, y, u) \right) - \sum_i \frac{\partial}{\partial y_i} \left( b_i(y) q_\theta(x, y, u) \right)$$

$$= \frac{1}{2} \sum_{i,j} \left\{ \frac{\partial^2}{\partial y_i \partial y_j} \gamma_{ij}(y) + \gamma_{ij}(y) \left( K_{q_\theta}(x, y, u) \right)_{ij} + \frac{\partial \gamma_{ij}(y)}{\partial y_i} \left[ \Lambda_{q_\theta}(x, y, u) \right]_j \right\}$$

$$+ \frac{\partial \gamma_{ij}(y)}{\partial y_j} \left[ \Lambda_{q_\theta}(x, y, u) \right]_i - \sum_i \left\{ \frac{\partial b_i(y)}{\partial y_i} + b_i(y) \left[ \Lambda_{q_\theta}(x, y, u) \right]_i \right\}$$

$$= \frac{1}{2} \sum_{i,j} \left\{ \frac{\partial^2 \gamma_{ij}(y)}{\partial y_i \partial y_j} + \gamma_{ij}(y) \left[ K_{q_\theta}(x, y, u) \right]_{ij} \right\}$$

$$+ \sum_i \left\{ \sum_j \frac{\partial \gamma_{ij}(y)}{\partial y_j} - b_i(y) \right\} \left[ \Lambda_{q_\theta}(x, y, u) \right]_i - \sum_i \frac{\partial b_i(y)}{\partial y_i}.$$
Writing the above expression in terms of vectors and matrices, we obtain that

$$\frac{Kq_\theta(x,y,u)}{q_\theta(x,y,u)} = \frac{1}{2} \left[ \gamma(y) : K_\theta(x,y,u) + \gamma(2)(y) : 1_{d \times d} \right]$$

$$+ \sum_i \left\{ \left[ \gamma(1)(y) 1_{d \times 1} \right]_i - b_i(y) \right\} \left[ \Lambda_\theta(x,y,u) \right]_i - b(1)(y) \cdot 1_{d \times 1}$$

$$= \frac{1}{2} \left[ \gamma(y) : K_\theta(x,y,u) + \gamma(2)(y) : 1_{d \times d} \right]$$

$$+ \left[ \gamma(1)(y) 1_{d \times 1} - b(y) \right] \cdot \Lambda_\theta(x,y,u) - b(1)(y) \cdot 1_{d \times 1}.$$  \hfill (A.2)

The analogous calculation for the operator of the proposed process is simpler since the drift and diffusion matrix are constant. In particular, we obtain that

$$\frac{Kq_{\theta_0}(x,y,u)}{q_{\theta_0}(x,y,u)} = \frac{1}{2} \gamma(x) : K_{\theta_0}(x,y,u) - b(x) \cdot \Lambda_{\theta_0}(x,y,u).$$  \hfill (A.3)

Combining (A.1), (A.2) and (A.3), we obtain the formula for the incremental weight as

$$\rho_\theta(x,y,u) = 1 + \frac{1}{\lambda(u)} \left( \frac{1}{2} \left\{ \left[ \gamma(y) - \gamma(x) \right] : K_{\theta_0}(x,y,u) + \gamma(2)(y) : 1_{d \times d} \right\} 

+ \left[ \gamma(1)(y) 1_{d \times 1} - b(y) + b(x) \right] \cdot \Lambda_{\theta_0}(x,y,u) - b(1)(y) \cdot 1_{d \times 1} \right).$$

\[\square\]

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