Harmonic-Coupled Riccati Equation and Its Applications in Distributed Filtering

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Abstract—The coupled Riccati equations (CREs) are a set of multiple Riccati-like equations whose solutions are coupled with each other through matrix means. They are a fundamental mathematical tool to depict the inherent dynamics of many complex systems, including Markovian systems or multiagent systems. This article investigates a new kind of CREs called harmonic-CREs (HCREs), whose solutions are coupled using harmonic means. We first introduce the specific form of HCREs and then analyze the existence and uniqueness of its solutions under the conditions of collective observability and primitiveness of coupling matrices. In addition, we manage to find an iterative law with low computation-complexity to obtain the solutions to HCREs. Based on this newly established theory, we greatly simplify the steady-state estimation error covariance of consensus-on-information-based distributed filtering (CIDF) into the solutions to a discrete-time Lyapunov equation (DLE). This leads to a significant conservativeness reduction of traditional performance evaluation techniques for CIDF. The obtained results are remarkable since they not only enrich the theory of CREs but also provide a novel insight into the synthesis and analysis of CIDF algorithms. We finally validate our theoretical findings through several numerical experiments.

Index Terms—Coupled Riccati equations (CRE), distributed filtering, matrix harmonic mean.

I. INTRODUCTION

A. Background

The Riccati equation, particularly the algebraic Riccati equation, provides a solid theoretical foundation for control and filtering technologies. For instance, it can be used for performance evaluation of Kalman filter [1], design of linear quadratic regulator (LQR) [2], and behavior assessment in dynamic noncooperative games [3]. Hence, a comprehensive investigation of the Riccati equation can greatly advance the development of related fields.

With the emergence of networked control systems, research object has shifted from single linear time-invariant systems to more complex systems, such as Markovian systems and multiagent systems. In these scenarios, traditional theories about a single Riccati equation are insufficient to capture system properties, such as correlations induced by the random jumps of the system and information flow within the network. As a result, there is an urgent need to develop more effective and general mathematical tools, while coupled Riccati equations (CREs) are one of the most critical techniques among them.

B. Coupled Riccati Equations

The CREs consist of multiple Riccati-like equations, with their solutions coupled with each other in the form of matrix means, such as algebraic mean. These equations were first formulated from the Markovian-jump LQR problem [4], [5] and optimal controllers can be designed using solutions to the corresponding algebraic coupled Riccati-like equations (ACREs), i.e.,

\[
P_i = A \left( \hat{P}_i^{-1} + \sum_{j=1}^{N} l_{ij} C_j^T R_j^{-1} C_j \right)^{-1} A^T + Q
\]

\[
\hat{P}_i \triangleq \sum_{j=1}^{N} l_{ij} P_j
\]

where \( i, j = 1, 2, \ldots, N \), \( A \) and \( C \triangleq [C_1^T, C_2^T, \ldots, C_N^T]^T \) are system matrices, \( l_{ij} \) is the \((i, j)\)th element of any stochastic matrix \( L \), \( Q \), and \( R \triangleq \text{diag}\{R_1, \ldots, R_N\} \) are positive definite, and \( P_j \) is the solution to each Riccati-like equation coupled with each other in the form of algebraic means \( \hat{P}_j \). The basic theories of ACREs, including the existence of solutions to ACREs [5], [6], [7], [8], upper and lower matrix bounds of the solutions [9], [10], [11], and numerical algorithms to obtain the solutions [6], [12], have been extensively investigated in literature.

However, in some scenarios, such as information-weighted distributed state estimation [13], [14], [15], [16], [17], the harmonic matrix mean rather than the algebraic one is primarily utilized for node interaction. Therefore, the theories of ACREs are infeasible here. As a remedy, another kind of CREs that involve harmonic means should be formulated to demonstrate the
properties of these situations. Compared with ACREs proposed in [5], [6], and [7], the coupled term \( \tilde{P}_i \) in HCREs is converted from algebraic mean to harmonic mean, i.e.,

\[
\tilde{P}_i = \sum_{j=1}^{N} l_{ij} P_j \Rightarrow \tilde{P}_i = \left( \sum_{j=1}^{N} l_{ij} P_j^{-1} \right)^{-1}
\]

which adopts a nonlinear form and embodies stronger couplings between solutions \( P_i, i = 1, \ldots, N \).

The existing results on HCREs primarily focus on two control areas: 1) consensus-on-information-based distributed filtering (CIDF) [14], [15], [16], [18], [19], [20] and 2) LQR cooperative regulator [21]. In the CIDF framework, recent remarkable studies have shown that harmonic means, also referred to as “covariance intersection fusion” in [14], can guarantee the stability of the distributed filter with mild requirements on network connectivity, system observability, and fusion steps [14], [15], [16], [20]. Specifically, Battistelli et al. [14] formulated a CIDF framework by embedding the harmonic mean into the traditional Kalman filter. Specifically, the matrix iteration of CIDF, i.e.,

\[
P_{i,k+1} = A \left( \sum_{j=1}^{N} l_{ij} P_{j,k-1}^{-1} + l_{ij} C_j^T R_j^{-1} C_j \right)^{-1} A^T + Q
\]

is shown to be bounded under a mild collective observability of \((A, C)\) and primitivity of the weighting matrix \(\mathcal{L}\). In this setting, HCREs can be interpreted as the steady-state form of the matrix iterative law (2). Some studies have investigated the properties of the iteration (2) or its variants to evaluate the performance of CIDF frameworks. For instance, Battistelli et al. [15] proposed a hybrid information fusion framework, which adopted two weighting matrices for the information terms \( P_{j,k-1} \) and \( C_j^T R_j^{-1} C_j \) to improve the performance of CIDF. Wang et al. [17] studied the iteration (2) with time-varying weighting matrix \(\mathcal{L}_k\) and relaxed the requirement of the weighting matrix to joint connectivity. He et al. [16] discussed the stability of CIDF algorithms with a more general system model [16], where state matrix \(A\) was noninvertible for some time steps. Duan et al. [20] generalized the matrix iteration (2) for the perturbation of matrix \(A\) and demonstrated the convergence of \( P_{i,k+1} \) with some elaborated but strict initial conditions.

However, all above results on CIDF related to HCREs are subjected to one fundamental problem, i.e., the stability of the iteration (2) can only be guaranteed by proving the boundedness of \( P_{i,k+1} \) with huge conservativeness [14], [15], [16], [17]. Even though it was shown in [20] that \( P_{i,k+1} \) (2) converges to a fixed point for some specific initial conditions \( P_{i,0}\), the hypothesis was too strict to satisfy in practice. Generally speaking, the upper bound of \( P_{i,k+1} \) derived in literature is much restrictive, and more precise properties of CIDF, such as the convergence of the matrix iterative law and the steady-state performance, remain not fully found. The in-depth discussion on the conservatism of the previous studies on CIDF is lacking.

From above discussion, it is beneficial for understanding the performance of CIDF, particularly its steady-state behavior, if we have a good understanding of HCREs. However, this is not a handy task. First, the harmonic mean \( \tilde{P}_i \) in each Riccati-like equation makes the correlation between \( P_i \) extremely nonlinear and nonconvex. Second, instead of each local \((A, C)\), the collective \((A, C)\) is assumed to be observable, which means that traditional mathematical techniques used for analyzing algebraic Riccati equation [22] and ACREs [5] are no longer applicable. Therefore, new mathematical techniques are urgently needed to excavate more properties of HCREs. The ongoing research is likely to enhance our understanding of these equations and their solutions.

C. Contributions

Motivated by the previous discussions, this article is aimed to uncover more valuable properties of HCREs. Specifically, we attempt to establish some sufficient conditions for the existence and uniqueness of solutions to HCREs and develop novel mathematical techniques for obtaining these solutions. By leveraging these outcomes, we aim to derive a closed-form expression for the steady-state performance of CIDF and construct a systematic performance evaluation framework. The contributions of this article are summarized as follows.

1) Only two mild requirements, namely, the collective observability of \((A, C)\) and the primitivity of the information weighting matrix \(\mathcal{L}\), are required to ensure the existence and uniqueness of solutions to HCREs (Theorem 1). Fundamental mathematical techniques for the analysis of HCREs are developed. These new findings greatly enrich the HCREs theory.

2) In addition to the basic theory of HCREs, we manage to find an iterative law with low computation-complexity to obtain the solutions to HCREs. It is demonstrated that the matrix iterative law of CIDF guarantees convergence of \( P_{i,k+1} \) to the solution \( P_i \) of the HCREs, regardless of initial value (Theorem 2). This result provides a constructive insight into the steady-state behavior of the CIDF matrix iterative law and further reveals the essence of the stability of CIDF.

3) By applying the obtained novel theories of HCREs, it has been demonstrated that the estimation error covariance matrix of CIDF also converges. The closed-form of the steady-state covariance matrix can be simplified as the solution to a discrete-time Lyapunov equation (DLE) (Section IV-A). Furthermore, the proposed HCREs framework unifies all classical CIDF algorithms [13], [14], [15], only differing in the parameter matrix \(\mathcal{L}\). These precise results are established for the CIDF for the first time.

The rest of this article is organized as follows. Some preliminaries, including the background of HCREs and the problem formulation, are presented in Section II. The main results, including the analysis of the solution to HCREs and the application of HCREs to CIDF, are presented in Sections III and IV. Some illustrative numerical experiments are presented in Section V. Finally, Section VI concludes this article.
For two symmetric matrices \( X_1 \) and \( X_2 \), \( X_1 > X_2 \) (\( X_1 \geq X_2 \)) means that \( X_1 - X_2 \) is positive definite (positive semidefinite). \( \exp(\cdot) \) denotes the exponential function. \( \{a\} \) denotes the absolute value of real number \( a \) or the norm of complex number \( a \). \( \mathcal{L} \) denotes that all the elements of matrix \( \mathcal{L} \) are positive (nonnegative). \( \mathbb{E}\{x\} \) denotes the expectation of a random variable \( x \). \( \lambda(A) \) denotes the eigenvalue of matrix \( A \). \( \rho(A) \) denotes the spectral radius of \( A \). \( \|A\|_2 \) denotes the 2-norm (the largest singular value) of matrix \( A \). \( X \otimes Y \) denotes the Kronecker product of matrix \( X \) and \( Y \). \( I_n \) denotes the identity matrix with dimension \( n \). \( O \) denotes the zero matrix with corresponding dimension.

II. PRELIMINARIES AND PROBLEM FORMULATION

In this section, the system model and the CIDF algorithm are provided. In addition, a comprehensive literature review of CIDF and the problem formulation of HCREs are also given.

A. System Model

Consider a network of \( N \) sensors that measure and estimate the state of a linear time-invariant system, described by

\[
\begin{align*}
x_{k+1} &= Ax_k + \omega_k, \quad k = 0, 1, 2, \ldots, \\
y_{i,k} &= C_i x_k + v_{i,k}, \quad i = 1, 2, \ldots, N
\end{align*}
\]

(3)

where \( x_k \in \mathbb{R}^n \) is the state vector of the system, \( y_{i,k} \in \mathbb{R}^{m_i} \) is the measurement vector of sensor \( i \), \( \omega_k \in \mathbb{R}^n \) is the process noise with covariance \( Q > 0 \in \mathbb{R}^{n \times n} \), and \( v_{i,k} \in \mathbb{R}^{m_i} \) is the observation noise with covariance \( R_i > 0 \in \mathbb{R}^{m_i \times m_i} \). The sequences \( \\{\omega_k\}_{k=0}^\infty \) and \( \{v_{i,k}\}_{k=0, i=1}^{\infty,N} \) are assumed to be mutually uncorrelated white Gaussian noise. Besides, \( A \) is the state-transition matrix and \( C_i \) is the observation matrix of sensor \( i \). Moreover, let \( C = [C_1^T, C_2^T, \ldots, C_N^T]^T \) and \( R = \text{diag}\{R_1, \ldots, R_N\} \).

The communication topology of the sensor network is denoted by \( \mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L}) \), where \( \mathcal{V} = \{1, 2, \ldots, N\} \) is the node set, \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is the edge set, and \( \mathcal{L} = [l_{ij}] \) is the adjacency matrix. The adjacency matrix reflects the interactions among the nodes, e.g., \( l_{ij} > 0 \iff (i, j) \in \mathcal{E} \), which means that sensor \( i \) can receive information from sensor \( j \). In this case, sensor \( j \) is called an in-neighbor of sensor \( i \), and sensor \( i \) is called an out-neighbor of sensor \( j \). For simplicity, let \( i \in \mathcal{V} \) represent the \( i \)th sensor of the network. \( N_i \) denotes the in-neighbor set of sensor \( i \), and \( l_i \) denotes the \( i \)th row of \( \mathcal{L} \). \( l_{i,\cdot}^{(L)} \) is the \( (i, j) \)th element of matrix \( \mathcal{L}^L \). In this article, let \( \mathcal{L} \) denote a special kind of adjacency matrix, where the \( (i, j) \)th element of \( \mathcal{L} \) is \( 1 \iff (i, j) \in \mathcal{E} \). The diameter \( d \) of graph \( \mathcal{G} \) is the length of the longest path between any two nodes in the graph.

B. Information Weighted Distributed Filtering

In this subsection, some typical CIDF algorithms are introduced, which is the background of HCREs and one important motivation.

CIDF is one of the most important branches of the well-established distributed state estimation techniques [23], [24], [25], [26], [27], [28]. The distributed state estimation problem mainly considers that multiple sensor nodes cooperatively observe a target system. Basic theories, such as algorithm design [14], [23], [24], [25], and performance analysis [26], [27], are extensively studied. Meanwhile, some applications of distributed state estimation techniques, such as distributed localization [28] and cooperative SLAM [29] are also deeply investigated. Among these theories and applications, the CIDF algorithms can guarantee the stability of the filter with low communication cost and mild observability requirement, hence attract considerable research interests. Different from the Kalman filter [30], the framework of CIDF consists of two essential parts. The first part is a local Kalman filter, where each sensor performs the Kalman iterative law to obtain the a posteriori state estimates using local observations. The second part is the covariance intersection-based information fusion step, where each sensor combines the a priori information received from neighboring nodes to compute a more precise estimate.

The stability of CIDF under weak observability is essentially ensured by the covariance intersection-based fusion technique. This technique was first proposed in [31] and [32], where the matrix intersection technique was used to guarantee the consistency of the estimator when there are unknown information correlations. Hu et al. [33] applied the covariance intersection technique to solve the distributed estimation problem and formulated the diffusion Kalman filter. However, this approach required a large number of fusion steps among sensors to achieve the local observability to further guarantee the filter stability. On considering these problems, Battistelli et al. [14] formulated the CIDF framework that achieved the filter stability with a weak observability assumption and single fusion step between two sampling instants. Since then, extensive efforts have been devoted to improving the performance of CIDF [15], [18] or applying the CIDF algorithm to more complex scenarios [16], [20].

While the theory of CIDF becomes sophisticated, several significant challenges remain unsolved. In literature regarding CIDF algorithms [14], [15], [16], the stability of the designed filters was ensured through the proof of the uniform boundedness of the iterative term \( \hat{P}_{i,k} \) for all \( i \in \mathcal{V} \) and \( k \in \mathbb{N} \), regardless of the initial value \( P_{i,0} \). The key techniques involve using the observation Gramian matrix to obtain a lower bound of \( P_{i,k}^{-1} \), as demonstrated in [14] and [16]. Afterward, a Lyapunov function can be constructed using \( \hat{P}_{i,k} \) to prove the stability of the noise-free dynamics of estimation error. Although these bounds of \( P_{i,k}^{-1} \) can be further used to prove the boundedness of the estimation error covariance matrix, it may be restrictive and much larger than the actual value of \( P_{i,k} \). Moreover, the boundedness of \( P_{i,k} \) does not reveal a direct connection between the filtering parameter \( \mathcal{L} \) and the filtering performance. Consequently, it is impossible to optimize the filter’s performance through the parameter tuning.

Based on the above analysis, it becomes apparent that the current theoretical basis for CIDF greatly hampers its further development. Hence, it is necessary to develop new theories to address this issue.
Algorithm 1: CIDF Algorithm.

Input: \[ \hat{x}_{i,0}, P_{i,0}, ~ i \in V, \]

Prediction: \[ \hat{x}_{i,k+1} = A \hat{x}_{i,k} - 1, \]
\[ P_{i,k+1} = AP_{i,k} - 1 A^T + Q, \]

Correction: \[ P_{i,k,k} = \left( P_{i,k-1} - 1 + C_i^T R_i^{-1} C_i \right)^{-1}, \]
\[ \hat{x}_{i,k} = P_{i,k,k} \left( \hat{x}_{i,k-1} + C_i R_i^{-1} y_{i,k} \right), \]

Information Fusion:
\[ P_{i,k} = \left( \sum_{j=1}^N l_{ij} P_{j,k}^{-1} \right)^{-1}, \]
\[ \hat{x}_{i,k} = P_{i,k} \left( \sum_{j=1}^N l_{ij} \hat{x}_{j,k} \right). \]

C. Problem Formulation

Generally, the CIDF algorithms take the form demonstrated in Algorithm 1, where the matrix terms \( P_{i,k-1} \) play an important part in the design of feedback gain \( K_{i,k} \) and information fusion gain \( l_{ij} P_{i,k} P_{j,k}^{-1} \). Meanwhile, the iterative law of \( P_{i,k+1} \) is equivalently rewritten as (2).

It is evident that HCREs represent the steady-state form of the iterative law (2). By studying HCREs, researchers can have a deeper understanding of the iterative law and develop the theory of CIDF. This, in turn, can provide valuable insights for the design of CIDF algorithms and parameters. In this article, we formulate two fundamental problems related to the HCREs as follows.

1) How to establish a quantitative relation between matrices \( A, C, L \) and properties of HCREs, particularly leveraging the connection between the iterative law (2) and the solution to HCREs?
2) Based on the theory of HCREs, how to develop a unified closed-form performance evaluation technique of all CIDF algorithms?

III. HARMONIC-COUPLED RICCATI EQUATIONS

In this section, the basic properties of the solution to HCREs will be revealed, especially the existence and uniqueness of the solution, together with the iterative law to obtain the unique solution. To do so, two preliminary assumptions are presented here.

Assumption 1: The matrix \( A \) is invertible and \( (A, C) \) is observable.

Assumption 2: The matrix \( L \) is primitive and row stochastic, i.e., \( \sum_{j=1}^N l_{ij} = 1 \) \( \forall i \in V \).

As proposed in [13], [14], and [15], both of the above two assumptions are mild for distributed filtering problems. Generally speaking, the invertibility of \( A \) in Assumption 1 is automatically satisfied in sampled-data systems as the matrix \( A \) is obtained through discretization of continuous-time systems. Meanwhile, the observability of \( (A, C) \) is essential for the stability of the filtering algorithm. As for Assumption 2, note that if the corresponding communication graph of \( L \) is strongly connected and the diagonal elements of \( L \) is positive, then the matrix \( L \) is primitive [34]. In this section, to simplify the notations, we replace the term \( P_{i,k+1|k} \) by \( P_{i,k} \) with slight abuse of notations. Under Assumptions 1 and 2, we mainly aim to prove the following facts.

1) (Uniqueness) The HCREs
\[ P_i = A \left( \sum_{j=1}^N l_{ij} P_{j,k}^{-1} + l_{ij} C_i^T R_i^{-1} C_i \right)^{-1} A^T + Q \]

have one unique group of solutions \( \{P_i\} \approx \{P_1, P_2, \ldots, P_N\} \).
2) (Convergence) The iterative law
\[ P_{i,k+1} = A \left( \sum_{j=1}^N l_{ij} P_{j,k}^{-1} + l_{ij} C_j^T R_j^{-1} C_j \right)^{-1} A^T + Q \]
converges to the unique solution to the HCREs with \( k \to \infty \), regardless of the initial value \( P_{i,0} \).

A. Uniqueness of the Solution to HCREs

In this subsection, the proof of the existence and uniqueness of the solution to the HCREs (4) is first proposed. To do so, the following two Lemmas are needed.

Lemma 1 ([14, Th. 4]): For any given matrices \( A, C, Q, R, L \) that satisfy Assumptions 1 and 2, there exist a number \( \hat{k} \) and a matrix \( P \) such that \( P_{i,k} \leq P \) \( \forall i \in V, k \geq \hat{k} \).

Similar to the idea proposed in [20, Th. 2], one has the following Lemma.

Lemma 2: Let Assumptions 1 and 2 hold, and suppose the initial value of the iteration (5) satisfies \( P_{i,0} \leq \epsilon I \) \( \forall i \in V \). Then, for sufficient small \( \epsilon \), there is \( P_{i,k+1} \geq P_{i,k} \) and \( P_{i,k} \) converges with the increase of \( k \).

Proof: The proof of this lemma takes a similar argument to the proof of [20, Th. 2].

As \( Q \) is a positive definite matrix, one can choose a sufficient small \( \epsilon \), such that \( P_{i,1} \geq P_{i,0} \) \( \forall i \in V \). Then, for all \( k \geq 1 \), one has
\[ (P_{i,k+1} - Q)^{-1} - (P_{i,k} - Q)^{-1} = (A^{-1})^T \left( \sum_{j=1}^N l_{ij} \left( P_{j,k}^{-1} - P_{j,k-1}^{-1} \right) \right) A^{-1}. \]

With mathematical induction, one has \( P_{i,k+1} \geq P_{i,k} \) \( \forall i \in V \). Together with the boundedness of \( P_{i,k} \) proved in Lemma 1, one can obtain that the iterative law (5) converges with specific initial value \( P_{i,0} \).

Lemma 3: For any given matrices \( A, C, Q, R, L \) that satisfy Assumptions 1 and 2, there exists at least one group of solution \( \{P_i\} \) to the harmonic-coupled Riccati (4).

Proof: Denote the convergent value of \( P_{i,k} \) in Lemma 2 as \( P_i \), then the result follows.

Before we prove the uniqueness of the solution to HCREs (4), the following notations are defined to simplify the proof.

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For any \( i \in \mathcal{V} \), the following notations are denoted as:
\[
\tilde{C}_i = [\text{sign}(l_{i1}) C_{i1}^T, \ldots, \text{sign}(l_{iN}) C_{iN}^T]^T
\]
\[
\tilde{R}_i = \text{diag}\left(\frac{1}{l_{i1}} R_1, \ldots, \frac{1}{l_{iN}} R_N\right)
\]
\[
\tilde{P}_i = \left(\sum_{j=1}^N l_{ij} P_{j-1}\right)^{-1}
\]
where the function \( \text{sign}(\cdot) \) is signum function for indication and the term \( 1/l_{ij} \) is set to be 0 if \( l_{ij} = 0 \). Then, the HCREs (4) can be rewritten as
\[
P_i = A \left(\tilde{P}_i^{-1} + \tilde{C}_i^T \tilde{R}_i^{-1} \tilde{C}_i\right)^{-1} A^T + Q
\]
\[
\tilde{P}_i = \left(\sum_{j=1}^N l_{ij} P_{j-1}\right)^{-1}.
\] (6)

With the existence of the solution to HCREs (6), for arbitrary one group of solution \( \{P_i\} \), one can obtain
\[
P_i = A \tilde{P}_i \tilde{P}_i A^T + Q + K_{\tilde{P}_i} \tilde{R}_i K_{\tilde{P}_i}^T
\]
\[
A_{\tilde{P}_i} = A - K_{\tilde{P}_i} \tilde{C}_i
\]
\[
K_{\tilde{P}_i} = A \tilde{P}_i \tilde{C}_i^T \left(\tilde{C}_i \tilde{P}_i \tilde{C}_i^T + \tilde{R}_i\right)^{-1}.
\]

Moreover, there holds
\[
\tilde{P}_i = \tilde{P}_i^{-1} \tilde{P}_i = \sum_{j=1}^N l_{ij} \tilde{P}_j^{-1} P_{j-1} \tilde{P}_i
\]
\[
= \sum_{j=1}^N l_{ij} \tilde{P}_j^{-1} P_{j-1} \tilde{P}_i
\]
\[
= \sum_{j=1}^N l_{ij} \tilde{P}_j^{-1} \left( A \tilde{P}_j \tilde{P}_j A^T + Q + K_{\tilde{P}_j} \tilde{R}_j K_{\tilde{P}_j}^T \right) P_{j-1} \tilde{P}_i.
\]

Let
\[
\tilde{A}_{ij} = \sqrt{l_{ij}} \tilde{P}_j^{-1} A_{\tilde{P}_j}
\]
\[
Q_i = \sum_{j=1}^N l_{ij} \tilde{P}_j^{-1} Q P_{j-1} \tilde{P}_i
\]
\[
\tilde{R}_i = \sum_{j=1}^N l_{ij} \tilde{P}_j^{-1} K_{\tilde{P}_j} \tilde{R}_j K_{\tilde{P}_j}^T P_{j-1} \tilde{P}_i.
\]

The expression of \( \tilde{P}_i \) can be rewritten as
\[
\tilde{P}_i = \sum_{j=1}^N \tilde{A}_{ij} \tilde{P}_j \tilde{A}_{ij}^T + Q_i + \tilde{R}_i
\]
\[
= \sum_{j_1=1}^N \sum_{j_2=1}^N \tilde{A}_{ij_1} \tilde{A}_{j_1 j_2} \tilde{P}_{j_2} \tilde{A}_{j_1 j_2}^T + Q_i + \sum_{j=1}^N \tilde{A}_{ij} Q_j \tilde{A}_{ij}^T
\]
\[
+ \tilde{R}_i + \sum_{j=1}^N \tilde{A}_{ij} \tilde{R}_j \tilde{A}_{ij}^T.
\]

In order to formulate the infinite series form of \( \tilde{P}_i \) in a neat and compact form, let
\[
\Phi_{i,j}^{(m)}(P) = \sum_{j_1=1}^N \cdots \sum_{j_m-1=1}^N \tilde{A}_{ij_1} \tilde{A}_{j_1 j_2} \cdots \tilde{A}_{j_{m-1} j_m} \times P \times \tilde{A}_{j_m j_1}^T \cdots \tilde{A}_{j_2 j_1}^T \tilde{A}_{j_1 j_2}^T, \quad m > 0
\]
\[
\Phi_{i,j}^{(0)}(P) = \tilde{A}_{ij} P \tilde{A}_{ij}^T.
\]

Then, the series form of \( \tilde{P}_i \) can be formulated as
\[
\tilde{P}_i = \sum_{j=1}^N \Phi_{i,j}^{(0)}(\tilde{P}_j) + Q_i + \tilde{R}_i
\]
\[
= \sum_{j=1}^N \Phi_{i,j}^{(0)}(\tilde{P}_j) + Q_i + \sum_{j=1}^N \Phi_{i,j}^{(0)}(Q_j)
\]
\[
+ \tilde{R}_i + \sum_{j=1}^N \Phi_{i,j}^{(0)}(\tilde{R}_j)
\]
\[
= \sum_{j=1}^N \Phi_{i,j}^{(m)}(\tilde{P}_j) + Q_i + \sum_{k=0}^{m-1} \sum_{j=1}^N \Phi_{i,j}^{(k)}(Q_j)
\]
\[
+ \tilde{R}_i + \sum_{k=0}^{m-1} \sum_{j=1}^N \Phi_{i,j}^{(k)}(\tilde{R}_j).
\] (7)

The following Lemma depicts the property of the operator \( \Phi_{i,j}^{(m)} \).

**Lemma 4:** For any positive definite matrix \( P \), there holds
\[
\lim_{m \to \infty} \Phi_{i,j}^{(m)}(P) = \mathbf{O} \quad \forall i, j \in \mathcal{V}.
\]

**Proof:** Note that matrices \( \Phi_{i,j}^{(m)}(\tilde{P}_j) \), \( Q_i \), \( \tilde{R}_i \), \( \Phi_{i,j}^{(k)}(Q_j) \), and \( \Phi_{i,j}^{(k)}(\tilde{R}_j) \) are positive semidefinite. With the series form of \( \tilde{P} \) in (7), one can obtain
\[
\tilde{P}_i \geq Q_i + \sum_{k=0}^{m-1} \sum_{j=1}^N \Phi_{i,j}^{(k)}(Q_j) \quad \forall m \in \mathbb{N}.
\]

Consider the matrix sequence
\[
B_m = Q_i + \sum_{k=0}^{m-1} \sum_{j=1}^N \Phi_{i,j}^{(k)}(Q_j).
\]

One can verify that the sequence \( \{B_m\} \) has the following two properties:
\[
B_m \leq B_{m+1}, \quad B_m \leq \tilde{P}_i \quad \forall m \in \mathbb{N}.
\]

Namely, the sequence \( \{B_m\} \) is monotonically increasing and uniformly upper bounded. Hence, the sequence \( \{B_m\} \) is convergent. Denotes the supremum of \( \{B_m\} \) as \( B \), one has for any \( \epsilon > 0 \), there exists a number \( L \in \mathbb{N} \), for all \( m \leq L \), there is
\[
\mathbf{O} \leq B - B_m \leq \epsilon I
\]
and
\[ \Phi_{i,j}^{(m)}(Q_j) \leq \sum_{k=m}^{\infty} \sum_{j=1}^{N} \Phi_{i,j}^{(k)}(Q_j) = B - B_m \leq \epsilon I. \]
Together with the positive semidefiniteness of \( \Phi_{i,j}^{(k)}(Q_j) \), there is
\[ \lim_{m \to \infty} \Phi_{i,j}^{(m)}(Q_j) = O \quad \forall i, j \in \mathcal{V}. \]
It is easy to verify that the operator \( \Phi_{i,j}^{(m)}(\cdot) \) has the following two properties that:
\[ \Phi_{i,j}^{(m)}(\lambda P) = \lambda \Phi_{i,j}^{(m)}(P) \quad \forall i, j \in \mathcal{V} \]
and for any positive definite matrices \( P_1 \geq P_2 \), there is
\[ \Phi_{i,j}^{(m)}(P_1) \geq \Phi_{i,j}^{(m)}(P_2) \quad \forall i, j \in \mathcal{V}. \]
For a fixed positive definite matrix \( P \), one can choose a sufficient large number \( \lambda \), such that \( P \leq \lambda Q_j \), then one has
\[ \Phi_{i,j}^{(m)}(P) \leq \lambda \Phi_{i,j}^{(m)}(Q_j) \Rightarrow \lim_{m \to \infty} \Phi_{i,j}^{(m)}(P) = O \quad \forall i, j \in \mathcal{V}. \]

The above lemma shows that for any group of solution \( \{P_i\} \) to the HCREs (4), the operator \( \Phi_{i,j}^{(m)}(\cdot) \) converges to zero operator with the increase of \( m \), which is essential for the following deduction.

To fully excavate the property of the operator \( \Phi_{i,j}^{(m)}(P) \), consider the following matrix:
\[ \tilde{A}_{ij} = \left[ \tilde{A}_{ij_1} \times \tilde{A}_{ij_2} \cdots \times \tilde{A}_{ij_m} \right] \]
j_1, \ldots., j_m = 1, \ldots., N
which is an \( n \times nN^m \) matrix and contains all the matrices \( \tilde{A}_{ij_1}, \tilde{A}_{ij_2} \cdots \tilde{A}_{ij_m} \) as row blocks, with \( j_1, \ldots., j_m \) chosen from 1 to \( N \). Hence, the operator \( \Phi_{i,j}^{(m)}(P) \) can be rewritten as
\[ \Phi_{i,j}^{(m)}(P) = \left[ \tilde{A}_{ij} \right]^{(m)} (I_{N^m} \otimes P) \left[ \tilde{A}_{ij} \right]^{(m) T}. \]
Let \( P = I \), with the result in Lemma 4 that
\[ \lim_{m \to \infty} \Phi_{i,j}^{(m)}(I) = O \quad \forall i, j \in \mathcal{V} \], one has
\[ \lim_{m \to \infty} \left\| \left[ \tilde{A}_{ij} \right]^{(m)} \right\|_2 = 0 \quad \forall i, j \in \mathcal{V}. \]
The asymptotic property of \( [\tilde{A}_{ij}]^{(m)} \) with \( m \to \infty \) will be repeatedly used in the following proof of the main theorems.

**Theorem 1:** If Assumption 1 and 2 hold, the solution to (4) is unique.

**Proof:** Based on Lemma 3, suppose there exist two different groups of solutions \( \{P_{1}\} \) and \( \{P_{2}\} \) to the HCREs (4). The corresponding notations are modified to be
\[ \tilde{P}_1^{(\tau)} = \left( \sum_{j=1}^{N} l_{ij} \left( P_{1}^{(\tau)} \right)^{-1} \right)^{-1}, \quad \tau = 1, 2. \]
The corresponding modifications of \( A_{P_{1}^{(\tau)}}, K_{P_{1}^{(\tau)}}, \tilde{A}_{ij}, Q_{i}^{(\tau)} \), and \( \tilde{R}_{ij}^{(\tau)} \) are made through the replacement of the matrix \( P_{1}, \tilde{P}_{1}, P_{2}, \tilde{P}_{2} \). The operator \( \Phi_{i,j}^{(m)}(P) \) is also modified to \( \Phi_{i,j}^{(m)\tau}(P) \), with the expression as
\[ \Phi_{i,j}^{(m)\tau}(P) = \left[ \tilde{A}_{ij}^{(m)} \right] (I_{N^m} \otimes P) \left[ \tilde{A}_{ij}^{(m)\tau} \right]^{T}. \]
where
\[ \left[ \tilde{A}_{ij}^{(m)} \right] \triangleq \left[ \tilde{A}_{ij_1}^{(\tau)} \times \tilde{A}_{ij_2}^{(\tau)} \cdots \times \tilde{A}_{ij_m}^{(\tau)} \right] \]
j_1, \ldots., j_m = 1, \ldots., N.
A new operator is defined as
\[ \Psi_{i,j}^{(m)}(P) = \left[ \tilde{A}_{ij}^{(1)} \right]^{(m)} (I_{N^m} \otimes P) \left[ \tilde{A}_{ij}^{(2)} \right]^{(m) T}. \]
particularly with
\[ \Psi_{i,j}^{(0)}(P) = \tilde{A}_{ij}^{(1)} P \left( \tilde{A}_{ij}^{(2)} \right)^{T}. \]
Consider the difference between two groups of solutions, for any \( i \in \mathcal{V} \), one has
\[ \tilde{P}_{1}^{(1)} - \tilde{P}_{2}^{(1)} = \tilde{P}_{1}^{(1)} \left( \left( \tilde{P}_{1}^{(2)} \right)^{-1} - \left( \tilde{P}_{2}^{(1)} \right)^{-1} \right) \tilde{P}_{2}^{(2)} = \sum_{j=1}^{N} l_{ij} \tilde{P}_{1}^{(1)} \left( P_{1}^{(1)} \right)^{-1} \left( P_{1}^{(1)} - P_{2}^{(2)} \right) \left( P_{2}^{(2)} \right)^{-1} \tilde{P}_{2}^{(1)}. \]
With some calculations, one has
\[ \tilde{P}_{1}^{(1)} - \tilde{P}_{2}^{(2)} = A \left( \left( \tilde{P}_{1}^{(1)} \right)^{-1} + \tilde{C}^{T} \tilde{R}^{-1} \tilde{C} \right)^{-1} A^{T} \]
\[ = A \tilde{P}_{1}^{(1)} \left( \tilde{P}_{1}^{(1)} - \tilde{P}_{2}^{(2)} \right) A^{T} \tilde{P}_{2}^{(1)}. \]
where
\[ A_{\tilde{P}_{j}^{(1)}} = A - A \tilde{P}_{j}^{(1)} \tilde{C}^{T} \left( \tilde{C} \tilde{P}_{j}^{(1)} \tilde{C}^{T} + \tilde{R} \right)^{-1} \tilde{C}, \]
\[ A_{\tilde{P}_{j}^{(2)}} = A - A \tilde{P}_{j}^{(2)} \tilde{C}^{T} \left( \tilde{C} \tilde{P}_{j}^{(2)} \tilde{C}^{T} + \tilde{R} \right)^{-1} \tilde{C}. \]
The difference of \( \tilde{P}_{j}^{(1)} \) and \( \tilde{P}_{j}^{(2)} \) can be reformulated as
\[ \tilde{P}_{j}^{(1)} - \tilde{P}_{j}^{(2)} = \sum_{j=1}^{N} \Psi_{i,j}^{(0)} \left( \tilde{P}_{j}^{(1)} - \tilde{P}_{j}^{(2)} \right). \]
With the property of \( \Psi_{i,j}^{(m)}(P) \), one has
\[ \tilde{P}_{j}^{(1)} - \tilde{P}_{j}^{(2)} = \sum_{j=1}^{N} \Psi_{i,j}^{(m)} \left( \tilde{P}_{j}^{(1)} - \tilde{P}_{j}^{(2)} \right), \quad m \in \mathbb{N}. \]
Hence, to prove Theorem 1, one only need to prove that for any matrix \( P \), there is \( \lim_{m \to \infty} \Psi_{i,j}^{(m)}(P) = O \).
For any matrix $P$, there is
\[
\left\| \Psi_{i,j}^{(m)}(P) \right\|_2 = \left\| \begin{bmatrix} A_{ij}^{(1)} \\ A_{ij}^{(2)} \end{bmatrix} (I_{N \times N} \otimes P) \begin{bmatrix} A_{ij}^{(3)} \end{bmatrix}^T \right\|_2 \\
\leq \left\| \begin{bmatrix} A_{ij}^{(1)} \\ A_{ij}^{(2)} \end{bmatrix} \right\|_2 \left\| (I_{N \times N} \otimes P) \right\|_2
\]
where the inequality holds due to the fact that $\left\| (I_{N \times N} \otimes P) \right\|_2 = \left\| P \right\|_2$. Hence, with the discussion above Theorem 1, for any matrix $P$, one has
\[
\lim_{m \to \infty} \left\| \Psi_{i,j}^{(m)}(P) \right\|_2 = 0 \quad \forall i, j \in V.
\]
Thus the uniqueness of the solution $\{P_i\}$ is proved.

Remark 1: The above analysis implies that the matrix iterative law of the CIDF algorithm has a single fixed-point at the network level with Assumptions 1 and 2 satisfied. In order to prove the uniqueness of this fixed-point, some novel mathematical tools are provided, including the formulation of the infinite series form of $\hat{P}_i$ and the asymptotic property analysis of the novel matrix operator $\Phi_{i,j}^{(m)}$ and $\Psi_{i,j}^{(m)}$. The matrix $A_{ij}$ in the operator $\Phi_{i,j}^{(m)}$ can be interpreted as the one-step coupling strength of solution $\hat{P}_i$ and $\hat{P}_j$. Hence, the operator $\Phi_{i,j}^{(m)}$ is related to the $m$-step coupling strength of solution $\hat{P}_i$ and $\hat{P}_j$. It is also illustrative that the matrix $[A_{ij}]^{(m)}$ contains all the possible $m$-step coupling between the solutions $\hat{P}_i$ and $\hat{P}_j$. With the increase of $m$, although the dimension of the matrix $[A_{ij}]^{(m)}$ will become infinity with the rate to be exponentially fast, the norm of $[A_{ij}]^{(m)}$ will still decay to zero. This phenomenon indicates that the decay rate of the product $A_{ij}, A_{j_1j_2}, \ldots A_{j_mj}$ to zero with the increase of $m$ is not slower than exponential convergence.

### B. Iterative Law for Solving HCREs

In this subsection, the convergence of the iterative law (5) with respect to $k$ will be analyzed.

To simplify the proof, let $P_{i,k+1}$ represent the term $P_{i,k+1}$ in the iterative law of the CIDF Algorithm 1.

The iterative law (5) can be reformulated as
\[
P_{i,k+1} = A \left( \hat{P}_{i,k}^{-1} + C_i^T R_{i,k}^{-1} C_i \right)^{-1} A^T + Q
\]
\[
\hat{P}_{i,k+1} = \left( \sum_{j=1}^N l_{ij} P_{j,k+1}^{-1} \right)^{-1}.
\]

With the similar technique proposed before, one can rewrite the iteration of $P_{i,k}$ as
\[
P_{i,k+1} = A \hat{P}_{i,k} \hat{P}_{i,k} A_{i,k}^T + Q + K_{i,k} R_{i,k}^{-1} K_{i,k}^T
\]
\[
A \hat{P}_{i,k} = A - K_{i,k} C_i
\]
\[
K_{i,k} = A \hat{P}_{i,k} C_i^T \left( \hat{C_i} \hat{P}_{i,k} \hat{C_i}^T + R_i \right)^{-1}.
\]

Similarly, one can also rewrite the iteration of $\hat{P}_{i,k+1}$ as
\[
\hat{P}_{i,k+1} = \hat{P}_{i,k+1} \hat{P}_{i,k+1}^{-1} \hat{P}_{i,k+1} + \tilde{P}_{i,k+1} + \tilde{R}_{i,k+1}
\]
\[
\tilde{P}_{i,k+1} = \sum_{j=1}^N l_{ij} P_{j,k+1}^{-1} \tilde{P}_{j,k+1} \tilde{P}_{j,k} A_{i,j,k+1} + Q_{i,k+1} + \tilde{R}_{i,k+1}
\]
\[
\tilde{R}_{i,k+1} = \sum_{j=1}^N l_{ij} \tilde{P}_{i,k+1}^{-1} \tilde{P}_{j,k+1} K_{i,j,k}^{-1} K_{j,k}^{-1} \tilde{R}_{j,k+1} + \tilde{R}_{i,k+1}.
\]

To match the time-varying property of the iterative law, the original operator $\Phi_{i,j,k}$ is correspondingly modified as $\Phi_{i,j,k}^{(m)}$, where
\[
\Phi_{i,j,k}^{(m)}(P) = \left[ A_{i,j,k} (I_{N \times N} \otimes P) \right] \left( \Phi_{i,j,k}^{(m)} \right)^T
\]
with
\[
\left[ A_{i,j,k} \right]^{(m)} \triangleq \left[ A_{i,j,k} A_{j_1j_2,k-1} \cdots A_{j_{m-1},j_m,k-m} \right]
\]
\[
j_1, j_2, \ldots, j_m = 1, \ldots, N.
\]

Then, one can also rewrite the iteration of $\hat{P}_{i,k+1}$ as
\[
\hat{P}_{i,k+1} = \sum_{j=1}^N \Phi_{i,j,k}^{(0)} \left( \hat{P}_{i,k} \right) + Q_{i,k+1} + \tilde{R}_{i,k+1}
\]
\[
\tilde{P}_{i,k+1} = \sum_{j=1}^N \Phi_{i,j,k+1} \left( \hat{P}_{i,k} \right) + Q_{i,k+1} + \tilde{R}_{i,k+1}
\]
\[
\hat{R}_{i,k+1} = \sum_{j=1}^N \Phi_{i,j,k+1} \left( \hat{P}_{j,k} \right) + Q_{i,k+1} + \tilde{R}_{i,k+1}
\]
where $\hat{P}_{i,0} = \sum_{j=1}^N l_{ij} P_{j,0}^{-1}$ and $P_{j,0}$ is the initial value of iteration (5). With Lemma 1, the matrix $P_{i,k}$ is uniformly bounded for all $i \in V$ and $k \geq \tilde{k}$. Hence, $\hat{P}_{i,k}$ is also uniformly bounded for all $i \in V$ and $k \geq \tilde{k}$, and one has
\[
\hat{P}_{i,k+1} \geq \sum_{j=1}^N \Phi_{i,j,k+1} \left( \hat{P}_{j,0} \right).
\]

Similar to the deduction in the previous subsection, for any positive definite initial value $P_{i,0}$, there exists a positive number
only related to \( P_{i,0}, \) such that
\[
\left\| \left[ \hat{A}_{i,j} \right]^{(k-1)} \right\|_2 \leq M \quad \forall i, j \in \mathcal{V}, k > 0.
\]

With the proposed theoretical preparation, one can finally prove the convergence of the iterative law (5).

**Theorem 2.** For any given matrices \( A, C, Q, R, \mathcal{L} \) that satisfy Assumption 1 and 2, the term \( P_{i,k} \) of the iterative law (5) converges to the unique solution \( \{ P_i \} \) of the HCREs (4), regardless of the initial value \( P_{i,0}, \) i.e.,
\[
\lim_{k \to \infty} P_{i,k} = P_i \quad \forall i \in \mathcal{V}.
\]

**Proof:** One can first rewrite the iteration of the gap between \( \hat{P}_{i,k} \), and the unique solution \( \hat{P}_i \) as
\[
\begin{align*}
\hat{P}_{i,k} - \hat{P}_i &= \hat{P}_{i,k} \left( \hat{P}_i^{-1} - (\hat{P}_{i,k})^{-1} \right) \hat{P}_i \\
&= \sum_{j=1}^{N} l_{ij} \hat{P}_{i,k} (P_{j,k})^{-1} (P_{j,k} - P_j) (P_j)^{-1} \hat{P}_{i,k} \\
&= \sum_{j=1}^{N} \hat{A}_{i,j,k} (\hat{P}_{j,k-1} - \hat{P}_j) \hat{A}_{i,j}^T \\
&= \sum_{j=1}^{N} \left[ \hat{A}_{i,j} \right]^{(k-1)} \left( I_{N_{k-1}} \otimes (\hat{P}_{j,0} - \hat{P}_j) \right) \left( \left[ \hat{A}_{i,j} \right]^{(k-1)} \right)^T
\end{align*}
\]
where the definition of \( \left[ \hat{A}_{i,j} \right]^{(k-1)} \) is the same as that of Section III-A. Hence, one can obtain the norm of the gap as
\[
\left\| \hat{P}_{i,k} - \hat{P}_i \right\|_2 \leq \sum_{j=1}^{N} \left\| \left[ \hat{A}_{i,j} \right]^{(k-1)} \right\|_2 \left\| \hat{P}_{i,0} - \hat{P}_i \right\|_2
\]
\[
\times \left\| \left[ \hat{A}_{i,j} \right]^{(k-1)} \right\|_2.
\]

For any initial value \( \hat{P}_{i,0}, \) due to the uniform boundedness of \( \left\| \left[ \hat{A}_{i,j} \right]^{(k-1)} \right\|_2 \) and the fact that
\[
\lim_{k \to \infty} \left\| \left[ \hat{A}_{i,j} \right]^{(k-1)} \right\|_2 = 0 \quad \forall i, j \in \mathcal{V}.
\]

One can finally obtain that
\[
\lim_{k \to \infty} \left\| \hat{P}_{i,k} - \hat{P}_i \right\|_2 = 0
\]
for any initial value \( \hat{P}_{i,0}. \) The convergence of the iteration (5) is finally proved. \( \square \)

**Remark 2:** Theorem 2 demonstrates that the a priori covariance matrix term \( P_{i,k+1|k} \) of CIDF converges to the unique steady-state performance \( P_i \) as \( k \) tends to infinity, regardless of the initial value. Compared with previous literature [14], [15], [16], [17], [19] that only proves the boundedness of \( P_{i,k+1|k}, \) Theorem 2 further confirms the convergence of \( P_{i,k+1|k}. \) This result is significant as it establishes a concise relationship between filtering performance, i.e., the steady-state matrix iteration \( P_i \) and the filtering parameter \( \mathcal{L} \) with HCREs (4), reducing the conservativeness of the performance evaluation. The steady-state performance of \( P_{i,k+1|k} \) also motivates the formulation of the steady-state performance of the real estimation error covariance matrix, which will be proposed in the next section.

The numerical example presented below demonstrates that the solution of HCREs (4) is significantly smaller than the upper bound proposed in literature of CIDF, such as [14] and [15]. This result confirms that the performance evaluation of CIDF is much less conservative than traditional techniques.

Consider a 1-D system with \( A = 1, C_1 = 1, C_2 = C_3 = 0, Q = 1, R = 1, \) and
\[
\mathcal{L} = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}.
\]
Through performing the iterative law (5) for sufficient many times, one can obtain the solution of HCREs (4) as \( P_1 = 2.0492, P_2 = 2.3990, \) and \( P_3 = 3.9901. \) With the method proposed in [14], one has
\[
P_{3,k+1|k} = AP_{3,k}A^T + Q
\]
\[
= A \left( 0.5P_{2,k}^{-1} + 0.5P_{3,k}^{-1} \right)^{-1} A^T + Q
\]
\[
\leq A \left( 0.5\beta A^T P_{2,k}^{-1} A^{-1} \right)^{-1} A^T + Q
\]
\[
\leq A \left( 0.5\beta A^T \left( \frac{1}{3} C_1 R^{-1} C_1 \right)^{-1} A^{-1} \right)^{-1} A^T + Q
\]
where \( \beta < 1. \) In this case, one has the upper bound of \( P_{3,k+1|k} \) is greater than 7, which is much larger than the exact solution \( P_3 = 3.9901. \) This indicates that the classical performance evaluation technique is conservative and the estimated values are much higher than the actual values. On the other hand, HCREs provides a solution that is much smaller than the upper bound proposed in literature for CIDF, which means that it is less conservative and provides a more accurate evaluation.

### IV. APPLICATION OF HCRES IN CIDF

In this section, some new perspectives of the CIDF algorithm are proposed on the basis of the HCREs theory obtained in the previous section.

#### A. Steady-State Performance of the Real Covariance Matrix

It is mentioned that the parameter matrix \( P_{i,k+1|k} \) converges to the steady-state form with the increase of \( k \) regardless of the initial value. In this subsection, the corresponding steady-state performance of the real estimation error covariance matrix will be formulated. Consider the LTI system proposed in Section II-A and the CIDF algorithm proposed in Algorithm 1. The estimation error is defined as
\[
e_{i,k+1|k-1} = x_k - \hat{x}_{i,k|k-1}
\]
\[
e_{i,k} = x_k - \hat{x}_{i,k}
\]
and iteration of the estimation error can be reformulated as
\[
e_{i,k+1|k} = \sum_{j=1}^{N} l_{ij} AP_{i,k} P_{j,k|k-1}^{-1} e_{j,k|k-1} + \omega_k
\]
\[
+ \sum_{j=1}^{N} l_{ij} AP_{i,k} C_j^T R_j^{-1} v_{j,k}.
\]

Denote the following notations as:
\[
A_k \triangleq \begin{bmatrix}
l_{11} AP_{1,k} P_{1,k|k-1}^{-1} & \cdots & l_{1N} AP_{1,k} P_{N,k|k-1}^{-1} \\
\vdots & \ddots & \vdots \\
l_{N1} AP_{N,k} P_{1,k|k-1}^{-1} & \cdots & l_{NN} AP_{N,k} P_{N,k|k-1}^{-1}
\end{bmatrix}
\]
\[
\Gamma_k \triangleq \begin{bmatrix}
l_{11} AP_{i,k} C_1 R_1^{-1} & \cdots & l_{1N} AP_{i,k} C_N R_N^{-1} \\
\vdots & \ddots & \vdots \\
l_{N1} AP_{N,k} C_1 R_1^{-1} & \cdots & l_{NN} AP_{N,k} C_N R_N^{-1}
\end{bmatrix}
\]
and
\[
e_{k+1|k} \triangleq \begin{bmatrix}
e^T_{1,k+1|k} + \cdots + e^T_{N,k+1|k}
\end{bmatrix}^T
\]
\[
v_k \triangleq \begin{bmatrix}
v_{1,k}^T + \cdots + v_{N,k}^T
\end{bmatrix}^T.
\]

Then, the iterative law of \( e_{k+1|k} \) can be reformulated as a compact form
\[
e_{k+1|k} = A_k e_{k|k-1} + \Gamma_k v_k + 1_N \otimes \omega_k.
\]
As the matrix \( P_{i,k} \) and \( P_{j,k|k-1} \) converge to the steady-state form with \( k \) tending to infinity, i.e.,
\[
\lim_{k \to \infty} P_{i,k} = \bar{P}_i, \quad \lim_{k \to \infty} P_{i,k|k-1} = P_i
\]
where
\[
\bar{P}_i = \left( \tilde{C}_i^T R_i^{-1} \tilde{C}_i + \sum_{j=1}^{N} l_{ij} P_j^{-1} \right)^{-1}
\]
and the steady-state matrices satisfy \( A \bar{P}_i A^T = P_i - Q \leq \beta \bar{P}_i \) \( \forall i \in \mathcal{V} \), where \( 0 < \beta < 1 \) due to the positive definiteness of \( Q \).

With Theorem 2, the matrices \( A_k \) and \( \Gamma_k \) will also converge to the steady-state form with the increase of \( k \), i.e.,
\[
\lim_{k \to \infty} A_k = A \triangleq \begin{bmatrix}
l_{11} AP_{1} P_{1}^{-1} & \cdots & l_{1N} AP_{1} P_{N}^{-1} \\
\vdots & \ddots & \vdots \\
l_{N1} AP_{N} P_{1}^{-1} & \cdots & l_{NN} AP_{N} P_{N}^{-1}
\end{bmatrix}
\]
\[
\lim_{k \to \infty} \Gamma_k = \Gamma \triangleq \begin{bmatrix}
l_{11} AP_{i} C_1 R_1^{-1} & \cdots & l_{1N} AP_{i} C_N R_N^{-1} \\
\vdots & \ddots & \vdots \\
l_{N1} AP_{N} C_1 R_1^{-1} & \cdots & l_{NN} AP_{N} C_N R_N^{-1}
\end{bmatrix}
\]
Consider the Perron–Frobenius left eigenvector \( q \) of the stochastic matrix \( L \), which satisfies \( q^T L = q^T \), and the matrix
\[
Q = \text{diag}( q_1 P_1^{-1}, \ldots, q_N P_N^{-1} )
\]
which is a block diagonal matrix. Note that
\[
\begin{bmatrix}
q_1 A^T P_1^{-1} A & \cdots & q_N A^T P_N^{-1} A
\end{bmatrix}
\leq \beta \tilde{Q}.
\]
where
\[
\tilde{Q} = \begin{bmatrix}
q_1 \bar{P}_1^{-1} \\
\vdots \\
q_N \bar{P}_N^{-1}
\end{bmatrix}.
\]
Then, one has
\[
A^T Q A = \begin{bmatrix}
l_{11} AP_{1} P_{1}^{-1} & \cdots & l_{1N} AP_{1} P_{N}^{-1} \\
\vdots & \ddots & \vdots \\
l_{N1} AP_{N} P_{1}^{-1} & \cdots & l_{NN} AP_{N} P_{N}^{-1}
\end{bmatrix}^T \times Q^* \geq \beta \begin{bmatrix}
l_{11} \bar{P}_1^{-1} & \cdots & l_{1N} \bar{P}_1^{-1} \\
\vdots & \ddots & \vdots \\
l_{N1} \bar{P}_N^{-1} & \cdots & l_{NN} \bar{P}_N^{-1}
\end{bmatrix}^T \times \tilde{Q}^* \geq \beta \sum_{i=1}^{N} q_i \bar{P}_i \begin{bmatrix}
l_{i1} P_1^{-1} & \cdots & l_{iN} P_N^{-1}
\end{bmatrix}^T
\]
where the term \( \star \) denotes the transpose of corresponding matrices. With the fact that \( \bar{P}_i \leq P_i \triangleq (\sum_{j=1}^{N} l_{ij} P_j^{-1})^{-1} \) and the lemma that (equivalent to [14, Lemma 2])
\[
\begin{bmatrix}
P_1 \\
\vdots \\
P_N
\end{bmatrix} \left( \sum_{j=1}^{N} P_j \right)^{-1} \begin{bmatrix}
P_1 & \cdots & P_N
\end{bmatrix} \leq \begin{bmatrix}
P_1 \\
\vdots \\
P_N
\end{bmatrix}
\]
one can obtain that
\[
A^T Q A \leq \beta \sum_{i=1}^{N} \begin{bmatrix}
l_{i1} P_1^{-1} & \cdots & l_{iN} P_N^{-1}
\end{bmatrix} = \beta Q.
\]
Due to the positive definiteness of \( Q \), one can obtain that the matrix \( A \) is Schur stable. With [24, Th. 1], the estimation error covariance matrix will converge to the solution of the Lyapunov equation, i.e.,
\[
\lim_{k \to \infty} E \{ e_{k+1|k} e_{k+1|k}^T \} = \mathcal{P}
\]
where
\[
\mathcal{P} = \mathcal{P} A^T + \Gamma R T^T + 1_N T_N^T \otimes Q.
\]
the analysis in this subsection provides a new formulation for the steady-state performance of the estimation error covariance matrix of the CIDF algorithm. Specifically, we have derived the explicit form of a discrete-time Lyapunov equation (8), with the parameter matrices of the DLE obtained through solving the HCREs (4). This new result establishes a quantitative relationship between the filtering performance and the weighted parameter matrix $L$, providing essential performance metric for optimizing the parameters of CIDF algorithms.

**B. Unification of CI-Based Distributed Filtering With HCREs**

This subsection will utilize the HCRE framework to examine the structural similarities of several widely recognized CIDF algorithms proposed in [13], [14], and [15]. It will be demonstrated that the steady-state performances of all distributed algorithms based on CI information fusion technique can be unified as a solution to a discrete-time Lyapunov equation. By solving the corresponding HCREs (4), the parameter matrices of the DLE can be obtained.

Consider the following three matrix iterative laws proposed in [13], [14], and [15], respectively. The first one is the basic matrix iterative law of CIDF proposed in [14], i.e.,

$$P_{i,k+1} = AP_{i,k} A^T + Q,$$

and the estimation iterative law

$$\hat{x}_{i,k+1} = AP_{i,k} \left( \sum_{j=1}^{N} l_{ij} P_{j,k-1}^{-1} + l_{ij} C_j^T R_j^{-1} C_j \right)^{-1} \left[ \sum_{j=1}^{N} l_{ij} P_{j,k-1}^{-1} \hat{x}_{j,k-1} + l_{ij} C_j R_j^{-1} y_{j,k} \right].$$

The second one is the matrix iterative law of information weighted consensus filters (ICF) proposed in [13] ([13, Eqs. (33), (35), and (39)]), i.e.,

$$P_{i,k+1} = AP_{i,k} A^T + Q,$$

and the estimation iterative law

$$\hat{x}_{i,k+1} = AP_{i,k} \left( \sum_{j=1}^{N} l_{ij}^{(I)} P_{j,k-1}^{-1} + l_{ij}^{(I)} NC_j^T R_j^{-1} C_j \right)^{-1} \left[ \sum_{j=1}^{N} l_{ij}^{(I)} P_{j,k-1}^{-1} \hat{x}_{j,k-1} + l_{ij}^{(I)} NC_j R_j^{-1} y_{j,k} \right].$$

The third one is the matrix iterative law of the hybrid consensus algorithm on measurement and information (CMCI) algorithm proposed in [15], i.e.,

$$P_{i,k+1} = AP_{i,k} A^T + Q,$$

and the estimation iterative law

$$\hat{x}_{i,k+1} = AP_{i,k} \left( \sum_{j=1}^{N} l_{ij}^{(H)} P_{j,k-1}^{-1} + l_{ij}^{(H)} \omega_{j,k} C_j^T R_j^{-1} C_j \right)^{-1} \left[ \sum_{j=1}^{N} l_{ij}^{(H)} P_{j,k-1}^{-1} \hat{x}_{j,k-1} + l_{ij}^{(H)} \omega_{j,k} C_j R_j^{-1} y_{j,k} \right].$$

where $l_{ij}$ is also the $(i,j)$th element of $L^L$, and $\omega_{j,k}$ is the local weight parameter determined by each sensor node $j$. The estimation iterative law is proposed as

$$\hat{x}_{i,k+1} = AP_{i,k} \left( \sum_{j=1}^{N} l_{ij}^{(L)} P_{j,k-1}^{-1} \hat{x}_{j,k-1} + l_{ij}^{(L)} \omega_{j,k} C_j R_j^{-1} y_{j,k} \right).$$

All of the abovementioned matrix iterative laws of $P_{i,k+1}$ can be simplified as the following HCREs matrix iterative law with different parameter matrices:

$$P_{i,k+1} = A \left( \sum_{j=1}^{N} l_{ij} P_{j,k-1}^{-1} + \nu_{ij} C_j^T R_j^{-1} C_j \right)^{-1} A^T + Q \tag{9}$$

where $L \equiv \{l_{ij}\}$, $\nu \equiv \{\nu_{ij}\}$ are parameter matrices corresponding with the communication graph $G$. In addition, the estimation iterative law can also be unified as

$$\hat{x}_{i,k+1} = AP_{i,k} \left( \sum_{j=1}^{N} l_{ij} P_{j,k-1}^{-1} \hat{x}_{j,k-1} + \nu_{ij} C_j R_j^{-1} y_{j,k} \right).$$

Both of $L$ and $\nu$ are irreducible and $L$ is row-stochastic. Based on the above analysis, the property of such an iterative law is mainly discussed in this subsection.

Let $L^{(m)} \equiv \{l_{ij}^{(m)}\}$, $\nu^{(m)} \equiv \{\nu_{ij}^{(m)}\}$, where $L^{(0)} = I$ and $\nu^{(1)} = \nu$. With [14, Lemma 1], one can obtain that

$$P_{i,k+1} \geq \beta A^{T-m} \left[ \sum_{j=1}^{N} l_{ij} P_{j,k-1}^{1-m} + \nu_{ij} C_j^T R_j^{-1} C_j \right] A^{-m}$$

$$\geq \beta^m \sum_{j=1}^{N} l_{ij} \left( A^{-1} \right)^m P_{j,k}^{1-m} \left( A^{-m} \right)^m$$

$$+ \sum_{k=1}^{m} \beta^k \sum_{j=1}^{N} \nu_{ij} \left( A^{-1} \right)^k C_j^T R_j^{-1} C_j \left( A^{-m} \right)^k$$

where $\beta < 1$. Note that both of $L$ and $\nu$ are irreducible, one has that for any $k \geq N$, there is $\omega_{i,j}^{(k)} > 0, i,j \in \mathcal{V}$, and $N$ is the number of the nodes. Together with the invertibility of matrix $A$ and the observability of $(A, C)$, one can obtain that the term $P_{i,k}$ of matrix iterative law (9) is uniformly bounded, i.e., there exists a matrix $P$ and a sufficient large number $\bar{k}$, such that $P_{i,k} \leq P \quad \forall i \in \mathcal{V}, k \geq \bar{k}$.

With the uniform boundedness of the term $P_{i,k+1}$ in (9), similar to the derivation procedure of Section III-A and III-B, one has the following lemma.

**Lemma 5:** For any given matrices $A, C, Q, R, L$ that satisfy Assumptions 1 and 2, if $\nu$ is primitive, the modified HCREs

$$P_{i} = A \left( \sum_{j=1}^{N} l_{ij} P_{j}^{-1} + \nu_{ij} C_j^T R_j^{-1} C_j \right)^{-1} \left( A^T + Q \right) \tag{10}$$

has a unique group of solution $\{P_{i}\}$. Moreover, the term $P_{i,k}$ of iterative law (9) converges to the unique solution $P_{i}$ with $k \rightarrow \infty$ for any $i \in \mathcal{V}$. 

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In light of the analysis presented above, it can be concluded that all of the matrix iterative laws proposed in [13], [14], and [15] can be unified into the same iterative law (9) with different parameter matrices $\nu$. This implies that the differences between the three information fusion mechanisms are ultimately reflected in the values of the parameter matrix $\nu$. Furthermore, the steady-state performances $P_i$ of all three iterative laws can be simplified as the solution to the modified HCReEs (10).

As for the real estimation error covariance matrix, with the unified iterative law of estimation, the steady-state performance of all kinds of CDF algorithms can also be simplified as the solution $\mathcal{P}$ to the modification of (8) with the term $\Gamma$ replaced by $\bar{\Gamma}$, where

$$
\bar{\Gamma} \triangleq \begin{bmatrix}
\nu_{11} A \bar{P}_1 C_1 R_1^{-1} & \cdots & \nu_{1N} A \bar{P}_1 C_N R_N^{-1} \\
\vdots & \ddots & \vdots \\
\nu_{N1} A \bar{P}_N C_1 R_1^{-1} & \cdots & \nu_{NN} A \bar{P}_N C_N R_N^{-1}
\end{bmatrix}.
$$

C. Asymptotic Analysis of Filtering Performance With Parameters $L$ and $\mathcal{C}$

This subsection will delve into the asymptotic properties of HCReEs. Specifically, the matrix performances of various CDF algorithms as $L \to \infty$. It is assumed that the parameter matrices $\mathcal{C}$ and $\nu$ are row stochastic to obtain the necessary asymptotic results for sufficiently large fusion steps $L$.

For primitive and row stochastic matrices $\mathcal{C}$ and $\nu$, there exist vectors $\mu_1, \mu_2, \mu_3,$ and $\mu_4$, such that

$$
\mathcal{C} \mu_1 = \mu_1, \quad \mu_2^T \mathcal{C} = \mu_2^T, \quad \nu \mu_3 = \mu_3, \quad \mu_3^T \nu = \mu_4^T
$$

where $\mu_2^T \mu_1 = \mu_4^T \mu_3 = 1$. From [34, Th. 8.5.1], one has

$$
\lim_{k \to \infty} \mathcal{C}^k = \mu_1 \mu_2^T, \quad \lim_{k \to \infty} \nu^k = \mu_3 \mu_4^T.
$$

Meanwhile, since $\mathcal{C}$ and $\nu$ are row stochastic, one has $\nu_1 = \nu_3 = 1_N$, and

$$
\lim_{k \to \infty} \nu_1^{(k)} = 1_N, \quad \lim_{k \to \infty} \nu_3^{(k)} = 1_N
$$

where $\nu_{1,j}$, $\nu_{3,j}$ is the $j$th element of vectors $\nu_1$ and $\nu_3$, respectively.

Through performing the information fusion step for $L$ times, one can rewrite the modified HCReEs (9) as

$$
P_{i,k+1|i} = A \left( \sum_{j=1}^{N} \nu_{ij}^{(L)} P_{j,k|k-1}^{-1} + \nu_{ij}^{(L)} C_j^T R_j^{-1} C_j \right)^{-1} A^T + Q
$$

and the asymptotic form of the iteration law with $L \to \infty$ as

$$
P_{i,k+1|i} = A \left( \sum_{j=1}^{N} \mu_{2,j} P_{j,k|k-1}^{-1} + \mu_{4,j} C_j^T R_j^{-1} C_j \right)^{-1} A^T + Q.
$$

Note that with $L \to \infty$, the iteration law of each sensor $i$ will tend to be the same with each other due to the fact that $\mathcal{C}$ and $\nu$ are row stochastic. Therefore, even if the initial value $P_{i,0}$ is different with respect to $i$, there is $P_{i,k+1|i} = P_{j,k+1|i}$ $\forall i, j \in \mathcal{I}, k \geq 1$. With the property that $\mu_2^T \mu_1 = \mu_4^T 1_N = 1$, one can further rewrite the above iteration law as

$$
P_{i,k+1|i} = A \left( P_{i,k|k-1}^{-1} + \sum_{j=1}^{N} \mu_{4,j} C_j^T R_j^{-1} C_j \right)^{-1} A^T + Q.
$$

(11)

With the obtained asymptotic iterative law (11), one can further compare the asymptotic performance of the consensus-based distributed filtering algorithm proposed in [13], [14], and [15], respectively.

For the iterative law proposed in [14], for the special case that the matrix $\nu$ is doubly stochastic, $\mu_4 = 1_N$. Hence, with the fusion step $L$ tending to infinity, the asymptotic form of the iterative law can be rewritten as

$$
P_{i,k+1|i} = A \left( P_{i,k|k-1}^{-1} + \sum_{j=1}^{N} C_j^T R_j^{-1} C_j \right)^{-1} A^T + Q.
$$

For the iterative law proposed in [13], with the parameter $N$ to compensate for the underconfidence of the information matrix, one can finally rewrite the asymptotic iterative law as

$$
P_{i,k+1|i} = A \left( P_{i,k|k-1}^{-1} + \sum_{j=1}^{N} \omega_{ij,k} C_j^T R_j^{-1} C_j \right)^{-1} A^T + Q.
$$

Hence, with sufficiently large fusion step $L$, the performance of the IKF proposed in [13] converges to the centralized optimal performance.

For the iterative law proposed in [15], for the special case that the matrix $\nu$ is doubly stochastic, one can finally rewrite the asymptotic iterative law as

$$
P_{i,k+1|i} = A \left( P_{i,k|k-1}^{-1} + \sum_{j=1}^{N} \omega_{ij,k} C_j^T R_j^{-1} C_j \right)^{-1} A^T + Q.
$$

where the parameter $\omega_{ij,k}$ can be tuned by each sensor node.

Based on the analysis presented above, it is evident that the performance gap between the CDF algorithm proposed in [14] and the centralized optimal case increases significantly with large sensor numbers $N$ and fusion steps $L$. This phenomenon has also been discussed in [14], where it was emphasized that the information fusion operation in the CDF requires a cautious strategy to ensure robustness against data incest (i.e., the repeated usage of the same observation). The compensation strategy proposed in [13] can ensure asymptotic optimality of filtering performance, but for finite fusion steps $L$, the algorithm may suffer from inconsistency induced by overconfidence in the observation information, where $N \nu_{ij}^{(L)}$ is much larger than 1. Therefore, future research should focus on developing effective parameter tuning techniques to maintain a proper balance between the information fusion weights and estimation performance.
V. SIMULATION

This section includes two numerical experiments aimed at validating the theory proposed in this article. The first experiment verifies the HCREs theory presented in Section III, which contains the uniqueness of the solution to HCREs and the convergence of the iterative law of CIDF. The second experiment validates the theory proposed in Section IV, including the closed-form of the steady-state performance of the error covariance matrix of the CIDF algorithm and performance comparison between different CIDF algorithms.

In the first numerical experiment, in order to fully validate the theory of HCREs, the matrices $A, C, Q, R$ are all randomly generated, with the expression as

$$
A = \begin{bmatrix}
0.3836 & 0.2558 & 0.2525 & 0.1766 & 0.4524 & 0.3534 \\
0.1978 & 0.2351 & 0.4546 & 0.5642 & 0.1739 & 0.4899 \\
0.3322 & 0.4508 & 0.4779 & 0.4064 & 0.5716 & 0.4073 \\
0.5927 & 0.4560 & 0.5109 & 0.6161 & 0.2135 & 0.1504 \\
0.6139 & 0.4898 & 0.3574 & 0.3858 & 0.6741 & 0.6985 \\
0.5316 & 0.0479 & 0.0019 & 0.5526 & 0.0543 & 0.4081
\end{bmatrix}
$$

The maximum eigenvalue of $A$ is 2.31, which indicates that the matrix $A$ is not Schur stable.

There are three kinds of the observation matrices as follows:

$$
C^{(1)} = \begin{bmatrix} 0.3711, 0.4438, 0.2733, 0.3920, 0.3768, 0.1424 \end{bmatrix}
$$

$$
C^{(2)} = \begin{bmatrix} 0.7154, 0.3439, 0.4017, 0.9339, 0.1471, 0.2543 \end{bmatrix}
$$

$$
C^{(3)} = \begin{bmatrix} 0, 0, 0, 0, 0, 0 \end{bmatrix}
$$

The value of $L$ is obtained with a randomly generated communication network. The whole network corresponding to the weighting matrix $L$ in HCREs (4) consists of 50 nodes, including 3 nodes of kind $C^{(1)}$, 3 nodes of kind $C^{(2)}$, and 44 nodes of kind $C^{(3)}$. The locations of the nodes are randomly set in a 500 × 500 region and each node is with a communication radius of 110. Hence, both of the communication topology of the network and the structure of weighting matrix $L$ are randomly generated in the numerical experiments, as presented in Fig. 1.

The covariance matrix $Q$ and $R_i$ takes the form as

$$
Q = \begin{bmatrix}
1.79 & -0.69 & 0.48 & -0.39 & -0.26 & -0.25 \\
-0.69 & 1.45 & -0.07 & 0.01 & 0.56 & 0.05 \\
0.48 & -0.07 & 2.12 & -0.11 & -0.61 & -0.61 \\
-0.39 & 0.01 & -0.11 & 1.88 & 0.49 & 0.46 \\
-0.26 & 0.56 & -0.61 & 0.49 & 2.37 & 0.20 \\
-0.25 & 0.05 & -0.61 & 0.46 & 0.20 & 1.24
\end{bmatrix}
$$

$$
R_i = 0.3818 \quad \forall i \in \mathcal{V}.
$$

The explicit value of $L$ is determined with $l_{ij} = \frac{a_{ij}}{d_{ii}}$, where $a_{ij} = 1$ if $(i, j) \in \mathcal{E}$, and $d_{ii} = \sum_{j=1}^{N} a_{ij}$. Hence, the matrix $L$ obtained in this way is row-stochastic and primitive. Through performing the iterative law (5), one can obtain the solution to the HCREs (4).

In Fig. 2, the iterative values of six $P_{i,k}$ are presented, where the notation $P_i$ here denotes the iteration of the trace of matrix $P_{i,k}$. It is shown that each $P_{i,k}$ converges to the steady-state form, i.e., the solution to HCREs (4), through the iterative law (5). In Fig. 3, three of the iterative values $P_{i,k}$ with three different initial values $P_{i,0}$ are presented. One can find that the fixed-point $P_i$ has no relation with the initial value, which verifies the uniqueness of the solution to HCREs.

In addition, a target tracking numerical experiment is also provided to verify the theory proposed in Section IV related to CIDF. The target is assumed to move on a plane and the state of the target takes the form as $[d_1, v_1, d_2, v_2]^T$, where $d_1, d_2$ denotes the location of the target system and $v_1, v_2$ denotes the velocity of the target system. The state transition matrix has the expression

$$
A_k = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} a_k & 0_{2 \times 2} \\ 0_{2 \times 2} & a_k \end{pmatrix}
$$

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The noise covariance matrix $Q$ takes the form of

$$G = \begin{pmatrix} T^3 & T^2 \\ T^2 & T \end{pmatrix}, \quad Q = \begin{pmatrix} G & 0.5G \\ 0.5G & G \end{pmatrix}$$

where the sample interval is set to be $T = 1$. The observation models of the three kinds of sensors are modified as follows:

$$C^{(1)} = [1, 0, 0, 0]$$
$$C^{(2)} = [0, 0, 1, 0]$$
$$C^{(3)} = [0, 0, 0, 0].$$

The structure of the network remains to be the same as that of the first numerical experiment, including 3 sensors of kind $C^{(1)}$, 3 sensors of kind $C^{(2)}$, 44 sensors of kind $C^{(3)}$, and $R_i = 1 \quad \forall i \in V$. The expression of three kinds of observation matrices indicates that each kind of sensor is only able to measure some partial states of the system but none of the pair $(A, C_i)$ is fully observable. Thus, the numerical experiments can fully illustrate the effectiveness of the CIDF algorithm. The $C^{(3)}$ also implies the existence of naive nodes [18], which is more realistic.

Using the Monte Carlo method, the filtering process is run 100 steps for each simulation and 1000 times in total. In this experiment, the performance of the CIDF algorithm is evaluated by the mean square error (mse), i.e.,

$$\text{MSE}_{i,k} = \frac{1}{1000} \sum_{l=1}^{1000} \left\| \hat{x}_{i,k}^{(l)} - x_k^{(l)} \right\|_2$$

and the mean of the mse of all sensors at time step $k$ takes the expression as

$$\text{MSE}_k = \frac{1}{N} \sum_{i=1}^{N} \text{MSE}_{i,k}$$

where $\hat{x}_{i,k}^{(l)}$ and $x_k^{(l)}$ denote the estimated state and real state at time step $k$ in the $l$th simulation, respectively. With the result obtained in Section IV-A, in each simulation, the estimation error covariance matrix of the CIDF algorithm converges to the steady-state performance, which can be simplified as the solution $P$ to DLE (8). Hence, one can compare the trace of $\frac{1}{N}P$ and $\text{MSE}_k$ to verify the theory proposed in Section IV-A.

In Fig. 4, the comparison between the real states of the target and estimations from different sensors is proposed, where $x^{(3)}$ denotes the iteration of the third element of the target state, i.e., $d_2$, and $\hat{x}_i^{(3)}$ denotes the estimation of the state from sensor $i$. From Fig. 4, one can find that the estimation of each sensor can keep with the target state and then verify the effectiveness of the CIDF algorithm.

In Fig. 5, the performance comparison between the classical CIDF algorithm proposed in [14] and the ICF algorithm proposed in [13] are mainly compared with numerical experiment,
where MSE (simulation) and IMSE (simulation) denote the average mse of the CIDF and ICF computed through (13). MSE (theory) and IMSE (theory) are obtained from the trace of solutions $P$ to the corresponding DLEs (8). From Fig. 5, one can find that the mse will converge to the steady-state performance with $k \to \infty$, which verifies the correctness of the theory proposed in Section IV-A. Moreover, as shown in Fig. 5, the performance of the ICF is better than CIDF when the observable nodes are sparsely located in the sensor network, which indicates that one can optimize the performance of CIDF algorithm through locally tuning the parameter $\omega_{i,k}$. Note that the closed-form of the performance of the CIDF algorithm is formulated as the solution to DLE (8), future work may contain a further investigation of the connection between the gain $\omega_{i,k}$ and the property of the solution $P$ to DLE (8) to obtain efficient parameter optimization techniques.

VI. CONCLUSION

In this article, we have investigated the properties of the solution to newly formulated HCRESs. We have shown that the uniqueness of the solution to HCRESs can be guaranteed with the collective observability and primitiveness of the weighting matrix $\mathcal{L}$. In addition, we have demonstrated that the matrix iterative law proposed in the CIDF algorithm converges to the solution to HCRESs as the iteration step tends to infinity. Leveraging the newly discovered properties, we have simplified the closed form of the steady-state estimation error covariance matrix of the CIDF algorithms as the solution to a DLE, the parameters of which are determined by solving the corresponding HCRESs. Moreover, we have shown that the performance analysis of some well-known CIDF algorithms can also be unified under the framework of HCRESs.

Future research will focus on investigating the relationship between the weighting parameters and the solution to HCRESs and developing effective parameter tuning techniques to control the performance of the CIDF algorithm.

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