Spectral Flow, and the Spectrum of Multi-Center Solutions

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Abstract

We discuss “spectral flow” coordinate transformations that take asymptotically four-dimensional solutions into other asymptotically four-dimensional solutions. We find that spectral flow can relate smooth three-charge solutions with a multi-center Taub-NUT base to solutions where one or several Taub-NUT centers are replaced by two-charge supertubes, and vice versa. We further show that multi-parameter spectral flows can map such Taub-NUT centers to more singular centers that are either D2-D0 or pure D0-brane sources. Since supertubes can depend on arbitrary functions, we establish that the moduli space of smooth horizonless black hole microstate solutions is classically of infinite dimension. We also use the physics of supertubes to argue that some multi-center solutions that appear to be bound states from a four-dimensional perspective are in fact not bound states when considered from a five- or six-dimensional perspective.
1 Introduction

One of the most important problems in quantum gravity is understanding the origin of black hole entropy. In string theory one can match this entropy by studying brane configurations at small or vanishing string coupling \[1\], but in this limit the black hole simply does not exist. Until recently, very little was known about what gives rise to the black-hole entropy in the regime of parameters where there is, in fact, a classical black hole. However, the last few years has seen the development of a programme that addresses this issue and seeks to describe the black hole entropy in terms of huge number of “microstate geometries.” Such geometries are defined to be smooth, horizonless backgrounds that have the same charges and asymptotics as the original black hole. (See \[2,3\] for a review of this proposal.) Much progress has been made in constructing such microstate geometries for three-charge black holes in five dimensions, four-charge black holes in four dimensions and even for non-supersymmetric black holes. By now, it is clear that there exists a huge number of such horizonless geometries that have the same charges as BPS black holes and black rings with macroscopically large entropy, and that appear to be dual to CFT states belonging to the same sector as the CFT states that give the black hole entropy.

In the quest to understand black hole entropy in terms of microstate geometries, two problems appear to be most difficult to overcome. The first is to determine which of the microstate solutions are more “typical” than others. The second is to construct very large classes of microstate solutions whose counting can give the black hole entropy.

Spectral flow has proven to be a useful tool in addressing these kinds of questions. In the dual conformal field theory the spectral flow operation is initiated by redefining the \(R\)-charge current by mixing it with some other conserved \(U(1)\) current. This then requires a modification of the Hamiltonian in order to preserve the supersymmetry. In the bulk gravity theory, the \(U(1)\) \(R\)-current and the other conserved \(U(1)\) current are dual to isometries of the background and spectral flow can be achieved simply by a change of coordinates that mixes these two \(U(1)\) directions. One can then add an asymptotically flat region to this new geometry to obtain a geometry that has different charges from the original. This is an effective method of obtaining some five-dimensional three-charge and four-dimensional four-charge microstate geometries from two-charge geometries \[4,5,6,7\]. In addition, spectral flow can be used to determine exactly the CFT state dual to the black hole microstate one constructs, and hence is a useful tool in determining how typical a certain microstate geometry is.

Despite its usefulness, spectral flow appears to be a rather cumbersome operation on asymptotically flat five-dimensional geometries: One must first strip the geometry of its asymptotically-flat region, then perform the spectral flow, and then add back the asymptotically-flat geometry. The last step can be quite non-trivial, especially for geometries that do not have a large number of isometries (see, for example \[8\]).

In this paper we explore a simpler way to use spectral flow to generate asymptotically four-dimensional geometries starting from other asymptotically four-dimensional geometries, without stripping away the asymptotically flat region. This method has two immediate applications which we believe are quite useful in the programme of constructing microstates and finding their CFT dual. First, it allows us to use a known microstate solution to generate a huge number of other smooth microstate solutions. Secondly, it gives us new insights into which microstate geometries represent bound states in the CFT. Since a configuration that consists purely of
concentric (two-charge) supertubes is unbound, any spectral flow of this will give unbound states. In particular, we expect such solutions will not correspond to CFT states in the sector that is primarily responsible for the entropy. We will use this observation to examine the status of some of the microstate geometries that have been studied in the past.

The fact that one can relate bubbling solutions with a Gibbons-Hawking (GH), multi-centered Taub-NUT base to solutions with a supertube in a bubbling solution also indicates that in the vicinity of the black hole microstates with a GH base there exists a very large family of other, less symmetric microstate solutions with the same macroscopic charges. Indeed, we know from the Born-Infeld action that two-charge supertubes can have arbitrary shapes [9], and that these arbitrary shapes correspond (upon dualizing to the D1-D5-P duality frame) to smooth geometries [10, 11]. Hence, one can use spectral flow to transform a GH center into a supertube, wiggle the supertube, and undo the spectral flow, to obtain bubbling three-charge solutions that depend classically on several arbitrary continuous functions. Hence the dimension of the moduli space of smooth black hole microstate solutions is classically infinite. If, upon counting these solutions, one finds a black-hole-like entropy, this will be, in our opinion, compelling evidence that the microstates of black holes are given by horizonless configurations. In a forthcoming paper [12] we will indeed argue that for the deep, smooth microstate solutions of [13, 14] one can obtain an entropy with the correct charge dependence using the methods outlined here.

To clarify the relationship of the solutions discussed here with some earlier results, we note that it was shown in [15, 16, 17] that general BPS configurations with the same supersymmetries as a black hole or black ring require that the four-dimensional spatial base of the solution be hyper-Kähler. It should be remembered that in establishing this result it was assumed that the solution was independent of the “internal” directions of the compactification tori. The solutions that we discuss here, which come from the spectral flow of supertubes of arbitrary shape, necessarily depend upon one of these internal directions. Hence, they are more general than those considered in [17], corresponding to solutions of ungauged supergravity in six dimensions [18], and their base space is not hyper-Kähler but almost hyper-Kähler.

In section 2 we discuss general five-dimensional BPS solutions, their relation to solutions of six-dimensional supergravity, and the way in which the spectral flow transformation acts on a five-dimensional solution with a $U(1)$ isometry. In section 3 we specialize this to a translational $U(1)$ isometry where the solution has a multi-centered Taub-NUT (Gibbons-Hawking) base. We show that spectral flow acts by interchanging the harmonic functions underlying these solutions, while keeping the solutions smooth. The explicit transformation is given in equations (3.34) and (3.35). We also show that spectral flow is part of a larger $SL(2,\mathbb{Z})^3$ subgroup of the four-dimensional $E_{7(7)}$ U-duality group, and this particular subgroup of $E_{7(7)}$ is distinguished because, for generic parameters, it generates orbits of smooth solutions.

In section 4 we show that spectral flow can transform a configuration containing one or several supertubes in Taub-NUT into a multi-center bubbling solution; conversely, it can transform such a solution into a solution where at least one of the centers is replaced by a two-charge supertube. This demonstrates that the black hole microstates with a GH base constructed so far in the literature are part of an infinite-dimensional moduli space of smooth supersymmetric solutions. In section 5 we explore the action of generalized spectral flow on multi-center D6-D4-D2-D0 configurations and use the physics of supertubes to argue that some multi-center configurations that appear bound from a four-dimensional perspective are in fact not bound when seen as full
ten-dimensional solutions. Section 6 contains conclusions, and Appendix A contains a more
detailed discussion of the $SL(2, \mathbb{Z})^3$ group of generalized spectral flow transformations.

2 Three charge solutions

2.1 Five-dimensional BPS solutions

In the M-theory frame, the background that preserves the same supersymmetries as a BMPV
black hole or a supersymmetric black ring has a metric of the form \cite{16,17}:

$$ds^2_{11} = ds^2_5 + \left( Z_2 Z_3 Z_1^{-2} \right)^{1/3} (dx_5^2 + dx_6^2)$$
$$+ \left( Z_1 Z_3 Z_2^{-2} \right)^{1/3} (dx_7^2 + dx_8^2) + \left( Z_1 Z_2 Z_3^{-2} \right)^{1/3} (dx_9^2 + dx_{10}^2).$$  \hspace{1cm} (2.1)

The five dimensional metric is

$$ds^2_5 \equiv - (Z_1 Z_2 Z_3)^{-2/3} (dt + k)^2 + (Z_1 Z_2 Z_3)^{2/3} h_{\mu \nu} dx^\mu dx^\nu,$$  \hspace{1cm} (2.2)

and for simplicity we have assumed that the six-dimensional internal manifold is $T^6$. (In general,
this could be any compact Calabi-Yau three-fold.) Supersymmetry requires the metric, $h_{\mu \nu}$, to be
hyper-Kähler. The solutions (2.1) can be arranged to be either asymptotically five-dimensional
or four-dimensional. The three-form gauge field decomposes into three vector potentials, $A^{(I)}$,
for the Maxwell fields in the five-dimensional space-time

$$\mathcal{A} = A^{(1)} \wedge dx_1 \wedge dx_2 + A^{(2)} \wedge dx_3 \wedge dx_4 + A^{(3)} \wedge dx_5 \wedge dx_6.$$ \hspace{1cm} (2.3)

The complete solution is determined by a system of three “BPS equations”

$$\Theta^{(I)} = *_4 \Theta^{(I)},$$
$$\nabla^2 Z_I = \frac{1}{2} C_{IJK} *_4 (\Theta^{(J)} \wedge \Theta^{(K)}),$$
$$dk + *_4 dk = Z_I \Theta^{(I)},$$ \hspace{1cm} (2.4)

where we have used the “dipole field strengths” $\Theta^{(I)}$

$$\Theta^{(I)} = dA^{(I)} + d \left( \frac{dt + k}{Z_I} \right)$$ \hspace{1cm} (2.5)

and $*_4$ is the Hodge dual taken with respect to the four-dimensional hyper-Kähler metric $h_{\mu \nu}$.
The constants $C_{IJK}$ are the triple intersection numbers of the Calabi-Yau three-fold, and for $T^6$
we have simply $C_{IJK} = | \epsilon_{IJK} |$.

2.2 Six-dimensional BPS solutions and dimensional reduction

By dualizing one can recast the foregoing solution in the IIB frame in which the three fundamental
charges are those of the D1-D5-P system. In this form, the D5-brane wraps a four-torus, $T^4$,
while the D1-brane, the remaining spatial part of the D5-brane and the momentum follow a common $S^1$. The metric thus naturally decomposes into a six-dimensional part and the $T^4$-part:

$$ds_{10}^2 = ds_6^2 + Z_1^{1/2}Z_2^{-1/2}ds_{T^4},$$

(2.6)

with

$$ds_6^2 = -\frac{2}{H}(dv + \beta)\left(du + k + \frac{1}{2}F(dv + \beta)\right) + Hh_{\mu\nu}dx^\mu dx^\nu$$

(2.7)

and

$$H = \sqrt{Z_1Z_2}, \quad F = -Z_3, \quad d\beta = \Theta^{(3)}.$$  

(2.8)

In this formulation there is obviously no longer a symmetry between the three fundamental charges and, with the foregoing choices, $Z_1$ corresponds to the D1-charge, $Z_2$ to the D5-charge and $Z_3$ to the KK-momentum charge.

We have cast the six-dimensional metric in the form (2.7) because it affords the easiest comparison with previous work on the classification of all supersymmetric solutions of six-dimensional minimal supergravity, obtained in [18]. In the minimal theory, two of the $U(1)$ Maxwell fields, $\Theta^{(I)}$, are set equal and they appear in a three-form field strength:

$$G^{(3)} = d(H^{-1}(dv + \beta) \wedge (du + k)) + (dv + \beta) \wedge \mathcal{G}^+ + *_4dH$$

(2.9)

with

$$\mathcal{G}^+ = \Theta^{(1)} = \Theta^{(2)}, \quad d\beta = \Theta^{(3)}.$$ 

(2.10)

One also has $Z_1 = Z_2$. We will, however, not make this restriction here but this earlier work is of relevance here because it allowed more general backgrounds that could depend upon the extra background coordinate, $v$. The spectral flow operations that we wish to consider could generate such $v$-dependent solutions. See, for example, [8].

The important point in going to the six-dimensional metric from the five-dimensional solution is that one of the $U(1)$ gauge fields has been converted to a six-dimensional Kaluza-Klein field, $\beta$. This then puts it on the same footing as a $U(1)$ isometry on the four-dimensional base. In particular, one can then mix these two directions with a coordinate transformation and, as we will see, this generates a spectral flow transformation. One should also note that one has the freedom to choose which of the three $U(1)$ Maxwell fields in five dimensions will become the six-dimensional Kaluza-Klein field and so there are three independent ways of generating the spectral flow. We now discuss this in detail.

### 2.3 Spectral Flow

From the six-dimensional perspective the operation of “spectral flow” is simply a coordinate change that mixes periodic coordinates on the base with the extra Kaluza-Klein coordinate, $v$ (see, for example [20]). When the base is asymptotic to $\mathbb{R}^4$, the size of the circles that are mixed with the Kaluza-Klein circle becomes infinite, and the spectral flow operation changes

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1This corresponds to solutions of six-dimensional, ungauged supergravity with one tensor multiplet, and was studied in [19].
the asymptotics of the solution. We will bypass this problem by focusing on solutions that are asymptotically $\mathbb{R}^3 \times S^1$.

If the base metric has an isometry then one can adapt the coordinate system to that isometry and take the metric to be invariant under translations of a coordinate, $\tau$. In particular, the base metric can be written in the form:

$$ds_4^2 = h_{\mu\nu} dx^\mu dx^\nu = V^{-1}(d\tau + A)^2 + V \gamma_{ij} dx^i dx^j,$$  \hspace{1cm} (2.11)

where $i, j = 1, 2, 3$ and every component of the metric is independent of $\tau$. The one-form, $A$, and the three-metric, $\gamma_{ij}$ are, \textit{a priori}, arbitrary.\footnote{However, the condition that the base metric be hyper-Kähler means that this metric can be completely determined by solving the $SU(\infty)$ Toda equation [21, 22, 23, 24]. This fact will not be needed here.}

We will also assume that the complete six-dimensional solution is invariant under $\tau$-translations and for simplicity we will also assume that the six-dimensional solution is independent of $v$ but neither of these assumptions is essential to the spectral flow transformations. It is convenient to decompose the one-forms, $k$ and $\beta$, according to:

$$k = \mu (d\tau + A) + \omega, \hspace{0.5cm} \beta = \nu (d\tau + A) + \sigma,$$  \hspace{1cm} (2.12)

where $\omega$ and $\sigma$ are one-forms in the three-dimensional space.

A spectral flow is then generated by the change of coordinate:

$$\tau \rightarrow \tau + \gamma v,$$  \hspace{1cm} (2.13)

for some parameter, $\gamma$. For this to be a well-defined coordinate transformation on the two circles, $\gamma$ must be properly quantized.\footnote{For a Gibbons-Hawking base $\tau$ has a period of $4\pi$ and $v$ has a period of $2\pi$ so $\gamma$ has to be an even integer.} More generally we could consider any global diffeomorphism in the $SL(2, \mathbb{Z})$ that acts on the two-torus defined by these $U(1)$'s. We will return to this in section 3. The important point is that because these mappings are diffeomorphisms, they map regular solutions without closed time-like curves (CTC’s) onto regular solutions without closed time-like curves.

Inserting (2.13) into (2.7), one can collect terms and restore the entire metric back to its canonical form, (2.7). One finds that this coordinate transformation is equivalent to:

$$ds_6^2 \rightarrow ds_6^2 \equiv -2\tilde{H}^{-1} (dv + \tilde{\beta}) (du + \tilde{k} + \frac{1}{2} \tilde{F} (dv + \tilde{\beta})) + \tilde{H} \tilde{d}s_4,$$  \hspace{1cm} (2.14)

where

$$\tilde{V} = (1 + \gamma \nu) V, \hspace{0.5cm} \tilde{A} = A - \gamma \sigma, \hspace{0.5cm} \tilde{H} = (1 + \gamma \nu)^{-1} H,$$

$$\tilde{\beta} = (1 + \gamma \nu)^{-1} \beta, \hspace{0.5cm} \tilde{F} = (1 + \gamma \nu) F + 2\gamma \mu + (1 + \gamma \nu)^{-1} V^{-1} \gamma^2 H^2,$$

$$\tilde{k} = k - \frac{\gamma \mu}{(1 + \gamma \nu)^2} \beta + \frac{\gamma^2 H^2}{V (1 + \gamma \nu)^2} \beta - \frac{\gamma H^2}{V (1 + \gamma \nu)} (d\tau + A).$$  \hspace{1cm} (2.15)
structures depend upon $v$. As a result, the metric, $d\tilde{s}_4^2$ is almost-hyper-Kähler [18] but not hyper-Kähler. On the other hand, if the $U(1)$ isometry is translational then the hyper-Kähler metric may be put into Gibbons-Hawking form [25] and all three complex structures are independent of $\tau$ and so $d\tilde{s}_4^2$ will also be hyper-Kähler with a translational $U(1)$ isometry and hence must have Gibbons-Hawking form [26]. We now investigate this in more detail.

3 Solutions with a Gibbons-Hawking base

3.1 Review of solutions with a Gibbons-Hawking bases

The Gibbons-Hawking metrics have the form

$$ h_{\mu\nu} = V^{-1}(d\tau + \tilde{A} \cdot d\tilde{y})^2 + V(dy_1^2 + dy_2^2 + dy_3^2) $$

(3.16)

where we write $\tilde{y} = (y_1, y_2, y_3)$ and where

$$ \nabla \times \tilde{A} = \nabla V. $$

(3.17)

This means that $V$ must be harmonic on the $\mathbb{R}^3$ spanned by $\tilde{y}$ and one should recall that to avoid orbifold singularities at singular points of $V$ the coordinate $\tau$ has to have the range $0 \leq \tau \leq 4\pi$. The solutions of equations (2.4) with a Gibbons-Hawking base were derived in [15, 27, 28, 29].

The metric (3.16) has a natural set of frames:

$$ \hat{e}^1 = V^{-\frac{1}{2}}(d\tau + A), \quad \hat{e}^a+1 = V^{\frac{1}{2}} dy^a, \quad a = 1, 2, 3, $$

(3.18)

where $A \equiv \tilde{A} \cdot d\tilde{y}$. There are also two natural sets of two-forms:

$$ \Omega_\pm^{(a)} \equiv \hat{e}^1 \wedge \hat{e}^a+1 \pm \frac{1}{2} \epsilon_{abc} \hat{e}^{b+1} \wedge \hat{e}^{c+1}, \quad a = 1, 2, 3. $$

(3.19)

The $\Omega^{(a)}_-$ are anti-self-dual and harmonic, defining the hyper-Kähler structure on the base. The forms, $\Omega^{(a)}_+$, are self-dual, and we can take the self-dual field strengths, $\Theta^{(I)}$, to be proportional to them:

$$ \Theta^{(I)} = -\sum_{a=1}^{3} (\partial_a (V^{-1}K^I)) \Omega_+^{(a)}. $$

(3.20)

For $\Theta^{(I)}$ to be closed, the functions $K^I$ have to be harmonic on $\mathbb{R}^3$. One can easily find potentials, $B^I$, with $\Theta^{(I)} = dB^I$:

$$ B^I = V^{-1}K^I (d\tau + A) + \tilde{\xi}^I \cdot d\tilde{y}, $$

(3.21)

where

$$ \nabla \times \tilde{\xi}^I = -\nabla K^I. $$

(3.22)

Hence, $\tilde{\xi}^I$ are vector potentials for magnetic monopoles located at the poles of $K^I$. The three self-dual Maxwell fields $\Theta^{(I)}$ are thus determined by the three harmonic functions $K^I$. Inserting this result in the right hand side of (2.4) we find:

$$ Z_I = \frac{1}{2} C_{IJK} V^{-1}K^J K^K + L_I $$

(3.23)
where \( L_I \) are three more independent harmonic functions.

We now write the one-form, \( k \), as:

\[
k = \mu (d\tau + A) + \omega
\]

and then the last equation in (2.4) implies:

\[
\begin{align*}
\mu &= \frac{1}{6} C_{IJ} V^{-2} K^I K^J K^K + \frac{1}{2} V^{-1} K^I L_I + M, \\
\bar{\nabla} \times \bar{\omega} &= V \bar{\nabla} M - M \bar{\nabla} V + \frac{1}{2} (K^I \bar{\nabla} L_I - L_I \bar{\nabla} K^I),
\end{align*}
\]

where \( M \) is another harmonic function.

The solution is therefore characterized by the eight \(^4\) harmonic functions \( V, K^I, L_I \) and \( M \). However the solution is invariant under the following shifts

\[
\begin{align*}
V &\rightarrow V, \quad K^I \rightarrow K^I + c^I V, \\
L_I &\rightarrow L_I - C_{IJ} c^J K^K - \frac{1}{2} C_{IK} c^I c^K V, \\
M &\rightarrow M - \frac{1}{2} c^I L_I + \frac{1}{12} C_{IJ} (c^I c^J c^K V + 3 c^I c^J K^K),
\end{align*}
\]

where \( c^I \) are three arbitrary constants. As can be seen from (3.20), these shifts do not modify the metric and field strengths and so should be viewed as a gauge transformation. One would therefore expect that any physical quantity will be invariant under this transformation. There is, however, a minor subtlety: when the topology of the base is \( \mathbb{R}^3 \times S^1 \), the Maxwell fields can have Wilson lines around the \( S^1 \) and then (3.27) can modify the Wilson lines in non-trivial ways.

The eight harmonic functions that give the solution may be identified with the eight fundamental basis elements of the fifty-six dimensional representation of \( E_7(7) \):

\[
\begin{align*}
x_{12} &= L_1, & x_{34} &= L_2, & x_{56} &= L_3, & x_{78} &= -V, \\
y_{12} &= K^1, & y_{34} &= K^2, & y_{56} &= K^3, & y_{78} &= 2M.
\end{align*}
\]

With these identifications, one can identify the right-hand side of (3.26) in terms of the symplectic invariant of the \( 56 \) of \( E_7(7) \):

\[
\bar{\nabla} \times \bar{\omega} = \frac{1}{4} \sum_{A,B} (y_{AB} \bar{\nabla} x_{AB} - x_{AB} \bar{\nabla} y_{AB}).
\]

The quartic invariant of the \( 56 \) of \( E_7(7) \) is determined by:

\[
J_4 = -\frac{1}{4} (x_{12} y_{12}^2 + x_{34} y_{34}^2 + x_{56} y_{56}^2 + x_{78} y_{78}^2) - (x_{12} x_{34} x_{56} x_{78} + y_{12} y_{34} y_{56} y_{78})^2
\]

\[
+ x_{12} x_{34} y_{12}^4 + x_{12} x_{56} y_{12}^4 + x_{12} y_{78} y_{12}^4 + x_{34} x_{56} y_{34}^4 + x_{34} y_{78} y_{34}^4
\]

\[
+ x_{34} x_{78} y_{34}^4 + x_{56} y_{78} y_{56}^4 + x_{78} x_{78} y_{78}^4 + x_{78} x_{78} y_{78}^4.
\]

\(^4\)For a general \( U(1)^N \) five-dimensional ungauged supergravity this number is \( 2N + 2 \).
which is also gauge invariant. This quantity plays a major role in determining the horizon area of four-dimensional black holes, and in formulating a necessary condition for the absence of closed time-like curves in a given solution \[29, 31\].

Finally, we note that there is a natural \(SL(2, \mathbb{Z})^4\) subgroup of the \(E_7(7)\) duality group in which each \(SL(2, \mathbb{Z})\) subgroup acts simultaneously on four pairs of the form \((x_{AB}, y_{CD})\). Details will be given in the Appendix, where we will show that three of these \(SL(2, \mathbb{Z})\)'s are generated by generalized spectral flow transformations and generalized electric-magnetic dualities.

3.2 Spectral flow in Gibbons-Hawking metrics

Elevating, or “oxidizing,” the five-dimensional solution to six dimensions puts one of the \(K^I\)'s on the same footing as the function \(V\). One should therefore expect that the gauge invariance (3.27) should be paralleled by similar shifts of \(V\) by \(K^I\) in the six-dimensional solution. This is precisely what spectral flow achieves: It is a completely trivial coordinate change in six dimensions but from the five-dimensional perspective it significantly modifies the underlying geometry and Maxwell fields.

To make this more explicit, it is useful to rewrite the six-dimensional supergravity solution with a GH base (2.7) as \[18\]:

\[
\begin{align*}
    ds_6^2 &= -\frac{F}{H} \left[ dv + \beta + \frac{1}{F} (du + k) \right]^2 + \frac{1}{HF} (du + k)^2 + H \left[ \frac{1}{V} (d\tau + A)^2 + V (dx^2 + dy^2 + dz^2) \right],
    
    \end{align*}
\]

where one should recall that \(H = \sqrt{Z_1 Z_2}\) and \(F = -Z_3\). As before we define:

\[
\begin{align*}
    &k = \mu (d\tau + A) + \omega, \quad \beta = \nu (d\tau + A) + \sigma. 
    
    \end{align*}
\]

Starting from M-theory on \(T^6\), one can choose to dualize to six dimensions so that any one of the \(K^I\) becomes the Kaluza-Klein potential, and if we take this to be \(K^3\) then one has:

\[
\begin{align*}
    &\nu = V^{-1} K^3, \quad \vec{\nabla} K^3 = -\vec{\nabla} \times \vec{\sigma}. 
    
    \end{align*}
\]

The spectral flow transformation (2.15) then corresponds to:

\[
\begin{align*}
    &\tilde{L}_3 = L_3 - 2\gamma M, \quad \tilde{L}_2 = L_2, \quad \tilde{L}_1 = L_1, 
    
    &\tilde{K}^1 = K^1 - \gamma L_2, \quad \tilde{K}^2 = K^2 - \gamma L_1, \quad \tilde{K}^3 = K^3, 
    
    &\tilde{V} = V + \gamma K^3, \quad \tilde{M} = M, \quad \tilde{\omega} = \tilde{\omega}. 
    
    \end{align*}
\]

We can also consider a more general process in which each of the \(K^I\)'s is successively chosen to be the special one, and a spectral flow, with parameter \(\gamma_I\), is made. The result is:

\[
\begin{align*}
    &\tilde{L}_I = L_I - 2\gamma_I M, \quad \tilde{M} = M, \quad \tilde{\omega} = \tilde{\omega}, 
    
    &\tilde{K}^I = K^I - C^{IJK} \gamma_J L_K + C^{IJK} \gamma_J \gamma_K M, 
    
    &\tilde{V} = V + \gamma_I K^I - \frac{1}{2} C^{IJK} \gamma_I \gamma_J L_K + \frac{1}{3} C^{IJK} \gamma_I \gamma_J \gamma_K M, 
    
    \end{align*}
\]

where \(C^{IJK} \equiv C_{IJK} \equiv |\epsilon_{IJK}|\). The fact that \(\vec{\omega}\) remains unchanged follows from the invariance of the source term in (3.29). By exchanging \(x_{AB} \leftrightarrow y_{CD}\) in (3.28), one can map this transformation

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onto the gauge transformations (3.27). Indeed, one such inversion can be achieved by \( \tau \leftrightarrow v \) and the gauge transformations (3.27) can be generated by coordinate changes of the form \( v \rightarrow v + c\tau \).

We will refer to the transformations (3.35) as “generalized spectral flow.” Unlike the transformation (3.34), which transforms smooth six-dimensional solutions into other smooth six-dimensional solutions, the generalized spectral flow may, in some instances, transform a smooth solution to a duality frame in which it is no longer smooth. We will discuss this further in section 5 while in the remainder of this section we examine the \( SL(2, \mathbb{Z}) \) actions in more detail and explicitly verify how \( SL(2, \mathbb{Z}) \) transformations preserve regularity.

### 3.3 \( SL(2, \mathbb{Z}) \) transformations of bubbling solutions

The spatial part of the metric (3.31) may be thought of as a \( T^2 \) fibration over \( \mathbb{R}^3 \), where \( \tau \) and \( v \) define the \( T^2 \) fiber. As we have seen, spectral flows are generated by the coordinate transformation (2.13). Similarly, it follows directly from (3.31) and (3.33) that the gauge transformations (3.27) with \( c^1 = c^2 = 0 \), \( c^3 = c \) can be obtained from the coordinate transformation:

\[
 v \rightarrow v + c \tau .
\]

(3.36)

More generally, one can make any \( SL(2, \mathbb{Z}) \) transformation in the global diffeomorphisms of the \( T^2 \) defined by \( (\tau, v) \):

\[
 \begin{pmatrix}
 \tilde{\tau} \\
 2\tilde{v}
 \end{pmatrix} = \mathcal{M} \begin{pmatrix}
 \tau \\
 2v
 \end{pmatrix} = \begin{pmatrix}
 m & n \\
 p & q
 \end{pmatrix} \begin{pmatrix}
 \tau \\
 2v
 \end{pmatrix} ,
\]

(3.37)

Here \( \mathcal{M} \in SL(2, \mathbb{Z}) \) and the factors of 2 insure the correct periodicities for the \( \tilde{\tau} \) and \( \tilde{v} \) coordinates. Since it is a diffeomorphism, any such transformation will take smooth (CTC-free) solutions to smooth (CTC-free) solutions.

If one uses this transformation in (3.31) one can easily recast the metric back into the same form:

\[
 ds_6^2 = -\frac{\tilde{F}}{\tilde{H}} \left[ d\tilde{v} + \tilde{\beta} + \frac{1}{\tilde{F}} (du + \tilde{k})^2 + \frac{1}{H F} (du + \tilde{k})^2 + \tilde{H} \left[ \frac{1}{V} (d\tilde{\tau} + \tilde{A})^2 + \tilde{V} (dx^2 + dy^2 + dz^2) \right] \right] ,
\]

where

\[
 \tilde{k} \equiv \tilde{\mu}(d\tilde{\tau} + \tilde{A}) + \tilde{\omega} , \\
 \tilde{\beta} \equiv \tilde{\nu}(d\tilde{\tau} + \tilde{A}) + \tilde{\sigma} ,
\]

and

\[
 \tilde{V} = (m - 2nv)V , \\
 \tilde{H} = \frac{H}{m - 2nv} , \\
 \tilde{F} = (m - 2nv)F - 4n\mu - 4n^2 \frac{H^2}{(m - 2nv)V} ,
\]

\[
 \tilde{\nu} = -\frac{\nu - qv}{m - 2nv} , \\
 \tilde{\mu} = \frac{1}{m - 2nv} \left( \mu + 2n \frac{H^2}{(m - 2nv)V} \right) ,
\]

\[
 \tilde{A} = mA + 2n\sigma , \\
 \tilde{\sigma} = q\sigma + \frac{\nu}{2} A , \\
 \tilde{\omega} = \omega .
\]

(3.40)
The effect of this \( SL(2,\mathbb{Z}) \) transformation on the functions determining the underlying five-dimensional solutions is:

\[
\tilde{V} = (m - 2n\nu)V, \quad \tilde{\mu} = \frac{V}{\tilde{V}} \left( \mu + 2n\frac{Z_1Z_2}{V} \right)
\]

\[
\tilde{Z}_1 = \frac{V}{\tilde{V}}Z_1, \quad \tilde{Z}_2 = \frac{V}{\tilde{V}}Z_2, \quad \tilde{Z}_3 = \frac{V}{\tilde{V}}Z_3 + 4n\mu + 4n^2\frac{Z_1Z_2}{V}.
\]

Note that because the functions \( Z_I \) are gauge invariant, their transformations only depend upon the spectral flow parameter, \( \gamma = -2n \).

Upon identifying the harmonic functions \( V, K^I, L^I \) and \( M \) that give the solution with the eight \( E_{7(7)} \) parameters \( x \) and \( y \) \((3.28)\), the \( SL(2,\mathbb{Z}) \) transformation becomes simply

\[
\begin{align*}
\begin{pmatrix}
\tilde{y}_{12} \\
2\tilde{x}_{34}
\end{pmatrix} &= \mathcal{M} \begin{pmatrix}
y_{12} \\
2x_{34}
\end{pmatrix}, \\
\begin{pmatrix}
\tilde{y}_{34} \\
2\tilde{x}_{12}
\end{pmatrix} &= \mathcal{M} \begin{pmatrix}
y_{34} \\
2x_{12}
\end{pmatrix}, \\
\begin{pmatrix}
\tilde{x}_{56} \\
2\tilde{y}_{78}
\end{pmatrix} &= \mathcal{M} \begin{pmatrix}
x_{56} \\
2y_{78}
\end{pmatrix}, \\
\begin{pmatrix}
\tilde{x}_{78} \\
2\tilde{y}_{56}
\end{pmatrix} &= \mathcal{M} \begin{pmatrix}
x_{78} \\
2y_{56}
\end{pmatrix}
\end{align*}
\]

\((3.42)\)

From the point of view of the five-dimensional solution, the transformation \((3.42)\) is simply a subgroup of the \( E_{7(7)}(\mathbb{Z}) \) duality group that takes solutions into solutions. Nevertheless, the important feature of this transformation is that it takes smooth solutions into smooth solutions. As we will discuss below, for generic parameters, \((3.42)\) transforms bubbling solutions into bubbling solutions, while for specific parameters it can transform them into bubbling solutions that contain one or several two-charge supertubes, with charges corresponding to \( x_{12} \) and \( x_{34} \). As we will see, these solutions are smooth in the six-dimensional duality frame \((2.7)\), but not in five-dimensions.

In order to arrive at the foregoing transformation we chose to dualize using the function \( K^3 \) to get the six-dimensional background. One can obviously use the other two functions, \( K^1 \) and \( K^2 \) and obtain two other \( SL(2,\mathbb{Z}) \) subgroups of \( E_{7(7)}(\mathbb{Z}) \). Indeed, these three \( SL(2,\mathbb{Z}) \)'s commute with one another and thus form an \( SL(2,\mathbb{Z})^3 \) subgroup of \( E_{7(7)}(\mathbb{Z}) \). As could be expected this general \( SL(2,\mathbb{Z})^3 \) transformation leaves the quartic invariant \( J_4 \) unchanged. We discuss this further in the Appendix, where we give the explicit forms of these transformations.

### 3.4 Regularity and the Bubble Equations

Suppose that the harmonic functions take their usual form for an ambi-polar Gibbons-Hawking base

\[
V = \varepsilon_0 + \sum_{j=1}^{N} \frac{q_j}{r_j}, \quad K^I = k^I_0 + \sum_{j=1}^{N} \frac{k^I_j}{r_j},
\]

\[(3.43)\]

\[
L^I = l^I_0 + \sum_{j=1}^{N} \frac{l^I_j}{r_j}, \quad M = m_0 + \sum_{j=1}^{N} \frac{m_j}{r_j},
\]

\[(3.44)\]
The constant terms, \( \varepsilon \), five-dimensional or six-dimensional metrics will decompactify. If one tunes the constants appropriately (e.g., introduce new CTC’s). One can see this explicitly from (3.41). Suppose that diffeomorphisms only involve the space-like sections of the metric and hence they should not befeomorphisms (3.37) precisely because they are diffeomorphisms of the torus. Moreover, these are preserved under the simple spectral flow (3.34) and, more generally, under the global diffeomorphisms (3.41).

Original M-theory geometry is generically asymptotic to \( R^r \) where \( q \) determine the scales of the singular points, but the solution generated by the simple spectral flow will still be smooth in the skew-symmetry of \( \Pi^{ij} \) (3.38).

For \( i = 1, \ldots, N \), and where \( r_{ij} \equiv |\vec{y}^{(i)} - \vec{y}^{(j)}| \). Summing both sides of this equation and using the skew-symmetry of \( \Pi^{ij}_l \) leads to:

\[
m_0 = q_0^{-1} \left( \varepsilon_0 m_0 - \frac{1}{2} \sum_{j=1}^N \sum_{I} \left( \ell^I_0 k^I_j - k^I_0 \ell^I_j \right) \right),
\]

where \( q_0 \) is given by (3.45).

We expect that both the regularity of the six-dimensional solution and the bubble equations are preserved under the simple spectral flow (3.34) and, more generally, under the global diffeomorphisms (3.37) precisely because they are diffeomorphisms of the torus. Moreover, these diffeomorphisms only involve the space-like sections of the metric and hence they should not introduce new CTC’s. One can see this explicitly from (3.41). Suppose that \( n \) is generic so that \( \vec{V} \) and \( V \) have exactly the same singular points. Then \( V^\pm 1 \vec{V}^\pm 1 \) is regular and so if one starts with regular \( Z_I \) and \( \mu \) then one will end up with regular \( \vec{Z}_I \) and \( \vec{\mu} \). Moreover, if the bubble equations are satisfied then \( \mu \to 0 \) as \( r_j \to 0 \) and hence \( \vec{\mu} \to 0 \) as \( r_j \to 0 \). Thus the bubble equations are satisfied in the new solution.

This argument obviously generalizes to any combination of transformations in \( SL(2, \mathbb{Z})^3 \) that do not change the singular structure of \( V \). Therefore such transformations clearly map smooth bubbling solutions into smooth bubbling solutions and preserve the bubble equations.

If the spectral flow parameter, \( n \) is not generic, then \( V \) and \( \vec{V} \) can have different sets of singular points, but the solution generated by the simple spectral flow will still be smooth in
six dimensions, and its physics is the subject of the next section. It turns out that this feature does not generalize to non-generic many-parameter spectral flow transformations (3.35). These flows will take multi-center black hole solutions into other multi-center solutions, by preserving the bubble equations and not introducing closed timelike curves. However, they may transform microstate solutions that are smooth in supergravity into solutions that do not appear smooth in supergravity. This will be the subject of section 5.

4 Two-Charge Supertubes and Spectral Flow

Perhaps the most physically interesting spectral flow transformation occurs when $V$ and $\tilde{V}$ have different sets of singular points. Suppose that we start with a regular, bubbled solution and that we use the simple spectral flow (3.34) so that $\tilde{V}$ has (at least) one less singularity than $V$. It follows that $\tilde{Z}_1$, $\tilde{Z}_2$ and $\tilde{\mu}$ now develop singularities, but these singularities have a very special form. As we will show, these singularities correspond exactly to having a two-charge supertube at the location of the old pole (or poles) of $V$. Going in the opposite direction, one can start from a geometry containing one or several two-charge supertubes and obtain a bubbling solution by doing the inverse spectral flow.

It is well known that two-charge supertubes give smooth supergravity solutions when in the duality frame in which they have D1 and D5 charges and KKM dipole charge, both in flat space \cite{10,11} and in Taub-NUT \cite{6}. Since the standard regularity conditions only involve the local geometry around the supertube, one would expect two-charge supertubes to be regular in more generic three-charge backgrounds \cite{32}. Hence, the fact that the spectral flow transformation (3.35) takes smooth solutions into smooth solutions is not surprising; after all, from a six-dimensional perspective, the flow (3.35) is nothing but a coordinate transformation.

The effect of the spectral flow transformation may, at first, appear surprising from the geometric perspective of the four-dimensional base: GH-based solutions are bubbling geometries with fluxes threading topologically non-trivial cycles while supertubes are thought of as rotating supersymmetric ensembles of branes that do not involve topology. The spectral flow maps one picture into the other and, once again, from the six-dimensional perspective it is easy to see how this happens. Consider the (spatial) $U(1)$ fiber parametrized by $v$ in (2.7) over any disk that spans the closed loop of the supertube. At the supertube the function $H$ in (2.7) becomes singular and pinches-off the $U(1)$ fiber. The result is a topologically non-trivial 3-sphere and the three-form, (2.9), has a non-zero flux through this 3-cycle. In the metric with a GH base, this 3-cycle simply appears as a non-trivial $U(1)$ fibration (parametrized by $v$) over a non-trivial 2-cycle in the base. The spectral flow merely “undoes” the topology in the base at the cost of introducing an apparent singularity but both perspectives are equivalent, and describe the same, completely regular, six-dimensional solution.

\footnote{This is exactly the way in which the first three-charge microstates were obtained by Lunin, \cite{4} and independently by Giusto, Mathur, and Saxena \cite{5}.}
4.1 Bubbling Geometries from Supertubes - One Supertube in Taub-NUT

It is useful to begin by illustrating the spectral-flow procedure on the solution corresponding to one supertube in Taub-NUT [6]. The smooth six-dimensional solution describing two-charge supertubes can be written as a solution with a GH base using the following harmonic functions [31]:

\[ V = \epsilon_0 + \frac{1}{r}, \quad L_1 = 1 + \frac{Q_1}{4|\vec{r} - \vec{R}|}, \quad L_2 = 1 + \frac{Q_2}{4|\vec{r} - \vec{R}|}, \quad L_3 = 1, \quad (4.49) \]

\[ K_1 = 0, \quad K_2 = 0, \quad K_3 = -\frac{q_3}{2|\vec{r} - \vec{R}|}, \quad M = \frac{J_T}{16} \left( \frac{1}{R} - \frac{1}{|\vec{r} - \vec{R}|} \right). \quad (4.50) \]

where \( \vec{R} \) defines the position of a round supertube that is wrapping the fiber of the Taub-NUT metric. Not all the constant parts in the harmonic functions are independent. The absence of closed timelike curves requires that

\[ J_T \left( \epsilon_0 + \frac{1}{R} \right) = 4q_3 \quad (4.51) \]

Moreover, in six dimensions the metric constructed using (4.49) is smooth (up to harmless \( \mathbb{Z}_{q_3} \) orbifold singularities) if [6]:

\[ q_3 J_T = Q_1 Q_2. \quad (4.52) \]

This condition comes from the requirement that \( \omega \) in (3.26) has no Dirac-Misner strings.

Before performing the spectral flow, we should observe that the harmonic functions above can be shifted using a subset of the gauge transformation (3.27) that preserves \( K_1 = K_2 = 0 \) and that sets the sum of the coefficients of the poles in \( K_3 \) to be zero:

\[ V = \epsilon_0 + \frac{1}{r}, \quad L_1 = 1 + \frac{Q_1}{4|\vec{r} - \vec{R}|}, \quad L_2 = 1 + \frac{Q_2}{4|\vec{r} - \vec{R}|}, \quad L_3 = 1, \quad K_1 = 0, \quad (4.53) \]

\[ K_2 = 0, \quad K_3 = \frac{q_3 \epsilon_0}{2} + \frac{q_3}{2r} - \frac{q_3}{2|\vec{r} - \vec{R}|}, \quad M = \frac{J_T}{16} \left( \frac{1}{R} - \frac{1}{|\vec{r} - \vec{R}|} \right) - \frac{q_3}{4}. \quad (4.54) \]

Under a spectral flow with parameter \( \gamma_3 \) one obtains a new solution with the harmonic functions:

\[ V = \epsilon_0 \left( 1 + \frac{\gamma_3 q_3}{2} \right) + \frac{1}{r} \left( 1 + \frac{\gamma_3 q_3}{2} \right) - \frac{q_3 \gamma_3}{2|\vec{r} - \vec{R}|}, \quad K_1 = -\gamma_3 - \frac{\gamma_3 Q_2}{4|\vec{r} - \vec{R}|}, \quad (4.55) \]

\[ K_2 = -\gamma_3 - \frac{\gamma_3 Q_1}{4|\vec{r} - \vec{R}|}, \quad K_3 = \frac{q_3 \epsilon_0}{2} + \frac{q_3}{2r} - \frac{q_3}{2|\vec{r} - \vec{R}|}, \quad (4.56) \]

\[ L_1 = 1 + \frac{Q_1}{4|\vec{r} - \vec{R}|}, \quad L_2 = 1 + \frac{Q_2}{4|\vec{r} - \vec{R}|}, \quad (4.57) \]

\[ L_3 = 1 + \frac{\gamma_3 q_3}{2} - \frac{\gamma_3 J_T}{8} \left( \frac{1}{R} - \frac{1}{|\vec{r} - \vec{R}|} \right), \quad M = \frac{J_T}{16} \left( \frac{1}{R} - \frac{1}{|\vec{r} - \vec{R}|} \right) - \frac{q_3}{4}. \quad (4.58) \]
It is not hard to check that the harmonic functions above satisfy the condition \((3.46)\), and hence they give a smooth three-charge two-centered bubbling solution. Moreover, the equation that gives the radius of the supertube in Taub-NUT \((4.51)\) becomes exactly the “bubble equation” \((3.47)\) governing the two-center bubbling solution. Hence, a spectral flow transformation can be used to change a smooth two-charge supertube in six dimensions into a smooth three-charge bubbling solution. This solution has the same singular parts as the four-dimensional microstate solution obtained in [33], but has different constant parts in the harmonic functions.

Of course, to obtain asymptotically five-dimensional solutions from other asymptotically flat solutions using spectral flow is a little more complicated. These solutions must not have any constant term in the \(K_I\) [28, 29]. Nevertheless, the solution before the spectral flow necessarily has all the \(Z_I\) (and hence \(L_I\)) limiting to constant values. Hence, a spectral flow will necessarily introduce a constant term in at least one of the \(K_I\). The way this problem is usually remedied [4, 5, 6, 8] is to strip off the asymptotically-flat region of the solution to obtain an asymptotically \(AdS_3\) geometry, spectral flow this geometry, and then add back by hand the asymptotically-flat part of the solution.

On the other hand, by looking at the solutions that have four-dimensional asymptotics, there is no need to eliminate the constant terms in the \(K_I\) harmonic function. A spectral flow will simply match two solutions with different values of the moduli at infinity.

### 4.2 Bubbling Geometries from Supertubes - Many Supertubes in Taub-NUT

We can generalize the foregoing example by starting with a solution describing \(N\) two-charge supertubes in Taub-NUT. The solution is specified by eight harmonic functions which have the form

\[
V = \epsilon_0 + \frac{1}{r}, \quad K^1 = K^2 = 0, \quad K^3 = k_0^3 - \sum_{i=1}^{N} \frac{q_i^3}{2r_i},
\]

\[
L_1 = l_0^1 + \sum_{i=1}^{N} \frac{Q_i^1}{4r_i}, \quad L_2 = l_0^2 + \sum_{i=1}^{N} \frac{Q_i^2}{4r_i}, \quad L_3 = l_0^3 \quad (4.59)
\]

\[
M = m_0 - \sum_{i=1}^{N} \frac{J_i}{16r_i},
\]

where \(r_i = |\vec{r} - \vec{r}_i|\) and \(\vec{r}_i\) are the locations of the supertubes in the base space. We will also define \(R_i \equiv |\vec{r}_i|\).

If we choose all \(\vec{r}_i\) to lie on the negative \(z\) axis (in GH coordinates) this will correspond to a configuration of \(N\) concentric supertubes of “radius” \(R_i\). It is clear that the straightforward generalization of the analysis in [6] will imply that, in the duality frame where the two charges of the supertubes are D1 and D5 charges, the type IIB supergravity solution will be smooth if

\[\text{6This is the distance from the Taub-NUT center to the supertube as measured in the three-dimensional base, and not the physical radius of the supertube.}\]
is satisfied for each center:
\[ Q_1^i Q_2^i = q_3^i j_i. \] (4.60)

These \( N \) conditions guarantee that the full metric is completely regular (again up to \( \mathbb{Z}_q^3 \) orbifold singularities). The solution should be free of CTC’s and imposing this condition at the locations of the supertubes and at the origin of the four-dimensional base yields \( N + 1 \) equations: \( N \) expressions that give the radius of each supertube and generalize (4.51), as well as a relation that fixes the parameter \( m_0 \):
\[
\left( \epsilon_0 + \frac{1}{R_i} \right) j_i = 4 l_0^3 q_3^i , \quad m_0 = \frac{1}{16} \sum_{i=1}^{N} \frac{J_i}{R_i}.
\] (4.61)

We can use the gauge freedom (3.27) to fix a gauge in which \( \sum_{i=1}^{N+1} q_3^i = 0 \):
\[
V \rightarrow V , \quad K^1 \rightarrow K^1 , \quad K^2 \rightarrow K^2 , \quad K^3 \rightarrow K^3 + c V ,
\]
\[
L_1 \rightarrow L_1 - c K^2 = L_1 , \quad L_2 \rightarrow L_2 - c K^1 = L_2 , \quad L_3 \rightarrow L_3 ,
\]
\[
M \rightarrow M - \frac{c}{2} L_3 ,
\] (4.62)

where
\[
c = \sum_{i=1}^{N} \frac{q_3^i}{2}.
\] (4.63)

This will ensure that the sum of the GH charges of the solution will remain the same after the spectral flow. After the gauge transformation, the harmonic functions take the following form:
\[
V = \epsilon_0 + \frac{1}{r} , \quad K^1 = K^2 = 0 , \quad K^3 = l_0^3 + \frac{c}{r} - \sum_{i=1}^{N} \frac{q_3^i}{2 r_i} ,
\] (4.64)
\[
L_1 = l_0^1 + \sum_{i=1}^{N} \frac{Q_1^i}{4 r_i} , \quad L_2 = l_0^2 + \sum_{i=1}^{N} \frac{Q_2^i}{4 r_i} , \quad L_3 = l_0^3
\]
\[
M = m_0 - \frac{c l_0^3}{2} + \sum_{i=1}^{N} \frac{J_i}{16 r_i}.
\]

To transform the solution corresponding to many supertubes to a bubbling solution with an ambipolar Gibbons-Hawking base, we perform a spectral flow transformation (3.34) with parameter \( \gamma \) to obtain.
\[
\tilde{V} = V + \gamma K_4 , \quad \tilde{K}^1 = K^1 - \gamma L_2 , \quad \tilde{K}^2 = K^2 - \gamma L_1 , \quad \tilde{K}^3 = K^3 ,
\]
\[
\tilde{L}_1 = L_1 , \quad \tilde{L}_2 = L_2 , \quad \tilde{L}_3 = L_3 - 2 \gamma M , \quad \tilde{M} = M.
\] (4.65)
The GH base space of the transformed solution has $N+1$ centers. The new harmonic functions:

\[
\tilde{V} = \tilde{\epsilon}_0 + \sum_{j=1}^{N+1} \tilde{q}_j \tilde{r}_j, \quad \tilde{K}^I = \tilde{k}_0^I + \sum_{j=1}^{N+1} \tilde{k}_j^I \tilde{r}_j, \quad \tilde{L}^I = \tilde{l}_0^I + \sum_{j=1}^{N+1} \tilde{l}_j^I \tilde{r}_j, \quad \tilde{M} = \tilde{m}_0 + \sum_{j=1}^{N+1} \tilde{m}_j \tilde{r}_j,
\]

(4.66)

can be found straightforwardly from (4.64) and (4.65). It is also straightforward to check that (4.60), which insures the regularity of the supertubes, implies that the constants in these harmonic functions satisfy (3.46) for any value of $\gamma$. Moreover, the bubble equations (3.47) are equivalent to the $N+1$ equations (4.61) that give the radii of the $N$ supertubes and the value of the $m_0$ parameter. This establishes explicitly that for any even integer $\gamma$ the spectral flow transformation (4.65) maps smooth solutions containing supertubes to smooth multi-center GH bubbling solutions.

4.3 Supertubes from Bubbling Geometries

Having shown that a solution corresponding to many concentric supertubes can be transformed into a GH bubbling solutions, it is interesting to investigate the opposite transformation - that of a bubbling solution into a solution containing supertubes.

It is not hard to see that given a generic smooth bubbling solution, whose parameters respect (3.46) and (3.47), one can perform a spectral flow (3.34) with parameter $\gamma = -\frac{q_i}{k^3_i}$ to obtain a solution in which there is no GH charge at the $i^{th}$ point. Equations (3.36) then insure that the functions $K^1, K^2$ and $L_3$ will also not have a pole at the position of the $i^{th}$ point. The poles of the other harmonic functions are

\[
K_3 \sim \frac{k^3_i}{r_i}, \quad L_1 \sim -\frac{k^2_i k^3_i}{r_i}, \quad L_2 \sim -\frac{k^2_i k^3_i}{r_i}, \quad M \sim \frac{k^1_i k^2_i k^3_i}{2r_i}.
\]

(4.67)

This solution corresponds to an object with two charges, one dipole charge, and angular momentum, and it is simply a circular\footnote{The circle is along the $U(1)$ fiber of the ambi-polar Taub-NUT base.} two-charge supertube at position $\tilde{r}_i$.

It is clear that this solution will be smooth from a six-dimensional perspective, simply because spectral flow takes smooth solutions into smooth solutions. Moreover, the coefficients of the singular parts of $L_1, L_2, K_3$ and $M$ satisfy the same relation, (4.52), as do the coefficients in the smooth two-charge supertube solutions in $\mathbb{R}^4$ or Taub-NUT. In upcoming work\footnote{The circle is along the $U(1)$ fiber of the ambi-polar Taub-NUT base.} we will show that the smoothness conditions coming from the supergravity analysis coincide with the equations of motion for a two-charge supertube in a GH background that one obtains using the Born-Infeld action of the supertube. Hence, a spectral flow transformation with a well-chosen parameter can transform any multi-center Gibbons-Hawking solution to a solution where one (or several) of the centers has been replaced by a two-charge supertube.
5 Generalized Spectral Flow

It is also interesting to consider generalized spectral flow transformations that can take a GH center into an even simpler configuration. We begin by exploring the orbit of generalized spectral flow. We then use the physics of supertubes to argue that many multi-center configurations that appear to be bound states from a four-dimensional perspective do so only because of the limited supergravity Ansatz used to study their stability. When one explores them using a more complete supergravity Ansatz, based on the underlying holographic dual, they are in fact unbound.

5.1 Multi-center D6-D4-D2-D0 Configurations

In order to describe generalized spectral flow on multi-center solutions it is convenient to work in the five-dimensional duality frame in which the electric charges of the solution are those of three sets of M2 branes \( \mathbf{2.1} \). When the base space of these solutions is ambi-polar, multi-center Taub-NUT, they can be reduced to four-dimensional multi-center solutions. The M2 charges correspond to D2 charges, the M5 dipole charges correspond to D4 charges, the Kaluza-Klein momentum along the Taub-NUT fiber becomes the D0 charge and the geometric GH charges correspond to D6 branes. The sources that appear in the eight harmonic functions that determine the solutions thus correspond exactly to the four-dimensional D6, D4, D2 and D0 charges.

A smooth multi-center, five-dimensional solution corresponds, in four dimensions, to a multi-center solution where each center is a “primitive” D6 brane, that is, a D6 brane that has non-trivial world-volume flux and locally preserves sixteen supercharges. From the perspective of the D-brane world-volume, primitivity places non-trivial constraints upon the fluxes. In the supergravity background these constraints amount to imposing smoothness, which fixes the flux parameters as in \( \mathbf{3.46} \). In the same manner, a two-charge supertube, which is also smooth in the D1-D5-P duality frame, has D4, D2 and D0 charges that satisfy \( \mathbf{4.52} \). Thus it corresponds to a “primitive” D4 brane - a D4 brane with non-trivial world-volume flux that locally preserves sixteen supercharges.

In section 3 we have established that spectral flow generically takes multi-center, primitive D6 configurations into other such configurations. Moreover, in section 4 we have seen that for some specially-chosen parameters it can transform a primitive D6 center into a primitive D4 center. One can take this further, and consider two- and three-parameter generalized spectral flow. A two-parameter spectral flow, with

\[
\gamma_1 = -\frac{q}{k^1}, \quad \gamma_2 = -\frac{q}{k^2} \quad (5.68)
\]

can take a GH center with GH charge \( q \) into a center with only two singular harmonic functions, \( K_3 \) and \( M \). This corresponds to a set of primitive D2 branes that have a non-trivial D0 brane charge. Furthermore, one can perform another spectral flow to take this center into a center that only has a non-zero \( M \), and hence corresponds to a collection of D0 branes. The parameters of this flow are:

\[
\gamma_1 = -\frac{q}{k^1}, \quad \gamma_2 = -\frac{q}{k^2}, \quad \gamma_3 = -\frac{q}{k^3} \quad (5.69)
\]

\(^8\)Only four of those supercharges are common to all the D6 branes, and thus common to the complete multi-center solution.
Since this last configuration consists of only a single species of D-brane, primitivity (the local preservation of sixteen supercharges) is now manifest. One should note that each successive spectral flow decreases the number of types of D-brane charge possessed by the brane and that this reduction critically depends upon the selection of parameters, (3.46), that made the original fluxed D6-brane smooth. By reversing these multiple spectral flows one thus obtains another way to understand the primitivity of the original D6-brane configuration.

Unlike the primitive D6 branes, which give smooth five-dimensional solutions in all duality frames, or the primitive D4 branes, which are smooth in the D1-D5-P frame, the primitive D2 and D0 branes are not smooth in supergravity in any duality frame. This is not unexpected, because the U-duality group in four dimensions can take smooth solutions into singular ones, and generalized spectral flow is nothing but a three-parameter family of this group. This is also not in conflict with the fact that each spectral flow can be realized by a six-dimensional coordinate change and therefore will preserve the regularity of six-dimensional solutions. The point is that one cannot realize all three independent spectral flows as coordinate changes of a single regular metric and so concatenating spectral flows can generate singular solutions.

One can also extend spectral flow to $U(1)^N$ five-dimensional ungauged supergravities compactified on a GH space (or, after dimensional reduction, $N = 2$ supergravities in four dimensions), that come from M-theory compactified on a CY manifold. The equation that gives generalized spectral flow (3.35) is written in a way that trivially expands to such supergravities.

For such solutions a six-dimensional lift of the solution, and the smooth supertube interpretation of the primitive D4 centers are not straightforward (unless the CY is $K^3 \times T^2$). Nevertheless, for generic parameters the generalized spectral flow still takes smooth solutions into smooth solutions, while for special choices of parameters it can interpolate between solutions with primitive D6, D4, D2 and D0 centers.

To recapitulate, from a five-dimensional perspective a spectral flow with one parameter can take a smooth GH solution into a smooth solution that contains a two-charge supertube in a GH background. Furthermore, two-parameter and three-parameter generalized spectral flows can transform a GH center into a singular configuration, that from a four-dimensional perspective has D2-D0 and pure D0 charges respectively.

### 5.2 When is a multi-center solution a bound state?

In studying the microstates of a black-hole one obviously wants to ensure that one is studying a single black hole and not merely an ensemble, or gas, of unbound BPS black holes and black rings. That is, one should only consider a system as being a single black hole if there are no “separation moduli” that can be used to physically deform the system into widely separated components without changing the energy or other asymptotic charges.

Establishing whether a solution is bound or unbound can be rather subtle. Consider for example an asymptotically-flat five-dimensional solution containing two two-charge supertubes. These do not interact with each other, and one can move them arbitrary far apart at no energy cost (without affecting the asymptotic charges and angular momenta). Hence, this configuration has flat directions, and is unbound. However, when considered as a four-dimensional multi-center

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9For a general CY compactification $C_{IJK}$ are the triple intersection numbers of the CY three-fold.
solution, this solution has three centers that have a nontrivial four-dimensional $\vec{E} \times \vec{B}$ interaction, and appears bound. Of course, the answer to this puzzle is that the separation moduli of the five-dimensional solution break its tri-holomorphic $U(1)$ isometry, and hence are not visible in four dimensions.

If one’s purpose is to describe microstates of five-dimensional three-charge black holes or black rings, the ultimate arbiter of whether a multi-center solution is bound is to dualize it to the D1-D5-P duality frame, and take the limit in which it becomes asymptotic to $AdS_3 \times S^3$ [32, 34]. If the six-dimensional solution has separation moduli, the solution is not bound. As we will see below, these separation modes are often not visible if one constructs and analyzes the solution using a more limited four- or five-dimensional Ansatz [10].

It is also possible that a certain multi-center solution, which is unbound when embedded in an asymptotically $AdS_3 \times S^3$ spacetime, can become bound when embedded in an asymptotically $\mathbb{R}^{3,1} \times S^1 \times S^1$ spacetime. The simplest example is again that of two concentric two-charge supertubes in Taub-NUT. These supertubes have no zero-modes [36] because of the constraining nature of the Taub-NUT geometry. However, when taking the limit in which their solution becomes asymptotically $AdS_3 \times S^3$, the base of the solution becomes $\mathbb{R}^4$, and the two supertubes become indistinguishable from two supertubes in an asymptotically-flat five-dimensional space, which are unbound. The same analysis extends trivially to more concentric supertubes in Taub-NUT.

Hence, two or more supertubes in Taub-NUT do not form a true bound state. Rather they are \textit{geometrically bound}: their lack of separation moduli is a result of the compactification geometry rather than of binding interactions. Intuitively, one should think of such geometrically-bound configurations as being the analogue of an ideal gas in a box: there is no binding energy between the atoms, but the system cannot be deformed into widely separated components because of the walls of the box.

As we have seen in section 4, using spectral flow one can transform a solution that contains concentric two-charge supertubes to a bubbling multi-center solution. The analysis above implies that such a bubbling multi-center solution is not a bound state. Indeed, upon spectral flow, a solution where the supertubes are not concentric anymore becomes a $v$-dependent six-dimensional solution (2.7) of the type described in section 2.2 [18, 19]. Hence, as explained above, multi-center bubbling solutions that can be obtained by spectral flow from a concentric-supertube configuration only appear bound from a four- and five-dimensional perspective because of the limited supergravity Ansatz used to describe them. Upon embedding it in the correct holographic background this configuration has at least one zero-mode and this involves making the six-dimensional lift of the solution (2.7) $v$-dependent.

It is also important to realize that starting from concentric supertubes, a spectral flow transformation only generates a very specific type of bubbling geometries. Indeed, spectral flow leaves the bubble equations [11] invariant, and hence the bubble equations governing the unbound bubbling solutions do not contain any terms that depend on the distance between any two of the GH centers that come from the supertubes. Hence, from a four-dimensional perspective these

\[ \text{footnote 10: It would be interesting to explore if the unbound nature of certain multi-center solutions is also visible when they are embedded in asymptotically $AdS_3 \times S^2$ solutions [35].} \]

\[ \text{footnote 11: Or the “integrability conditions” in the case of more general centers [37].} \]
GH points are free to move on two-spheres of radius, \( R_i \) (given by (4.61)) around a central GH point. In the quiver language, used to describe multi-center four-dimensional solutions [37], these bubbling solutions can be depicted as “hedgehog” quivers, with all the arrows originating from one of the nodes, and joining it to all the other nodes.

It is also possible to argue that, at least for a large enough number of centers, spectral flow can be used to generate all the hedgehog multi-center GH solutions in Figure (1) from two-charge round supertubes in flat space. Indeed, by simple parameter counting, a solution with \( N+1 \) GH centers has \( 4N+1 \) parameters (three \( k_i^I \)'s and one GH charge \( q_i \) for each point minus three gauge transformations). Requiring the vanishing of \( \Pi^{(I)}_{ij} \Pi^{(2)}_{ij} \Pi^{(3)}_{ij} \) between any two of \( N \) centers naively imposes \( N(N-1) \) constraints, which, in general, cannot be satisfied. Nevertheless, since \( \Pi^{(I)}_{ij} \) are given by (3.45), it is not hard to see that one can also have all of them zero if for one of the \( I \), the value of \( k_i^I/q_i \) is the same for all the \( N \) points. Choosing, for example, \( I = 2 \), this implies

\[
k_i = (k_i^1, q_i \kappa_2, k_i^3),
\]

and imposes \( N-1 \) conditions, leaving \( 3N+2 \) independent parameters. This is exactly the number of parameters that describe all possible spectral flows of \( N \) round supertubes of arbitrary charge in a Taub-NUT space: three independent parameters \( (Q_1, Q_2, d_3) \) for each supertube, one for the Taub-NUT center and one spectral flow parameter, \( \gamma \). It is not hard to see that a spectral flow with parameter \( \gamma_2 = -1/\kappa^2 \) transforms the foregoing set of \( N \) GH centers into concentric two-charge supertubes.

There exists another way to make all the fluxes between the \( N \) GH points vanish: one can divide them in three sets, \( A, B, C \), that have fluxes

\[
\begin{align*}
k_i &= (k_i^1, q_i \kappa_2, k_i^3), \quad \text{for } i \in A \\
k_i &= (q_i \kappa_1, k_i^2, q_i \kappa_3), \quad \text{for } i \in B \\
k_i &= (q_i \kappa_1, q_i \kappa_2, k_i^3), \quad \text{for } i \in C
\end{align*}
\]

where \( \kappa_1, \kappa_2, \kappa_3 \) are constants, and \( k_i^1, k_i^2 \) and \( k_i^3 \) can be arbitrary for the GH centers in the...
A, B and C set respectively. A two-parameter spectral flow with parameters \( \gamma_1 = -1/\kappa^1 \) and \( \gamma_2 = -1/\kappa^2 \) transforms the GH centers in the A and B sets into two-charge supertubes of different type, and transforms the GH centers in the C set into singular D2-D0 centers. Normally, different kinds of two-charge supertubes have an \( \vec{E} \times \vec{B} \)-type electric-magnetic interaction, and cannot go arbitrarily far away from each other without changing the asymptotic charges of the solution. Therefore such a solution is generically a bound state. However, for the particular supertubes that are created from the spectral flow of (5.71) the \( \vec{E} \times \vec{B} \)-type interaction vanishes, and hence they can move freely away from each other. Hence this type of configuration also corresponds to an unbound state. It is quite clear that for \( N \) sufficiently large all hedgehog quivers can be only of the type (5.70) or (5.71), and hence they are all unbound from the point of view of six-dimensional supergravity.

Another intuitive way to think of our formulation of bound state classification is as follows. Bound states generically emerge through \( \vec{E} \times \vec{B} \) interactions. In four dimensions there are four independent \( U(1) \) Maxwell fields and thus many ways in which to generate the interaction. In five dimensions there are only three \( U(1) \) Maxwell fields and thus some of the four-dimensional \( \vec{E} \times \vec{B} \) interactions become trivial upon oxidation to five dimensions. Indeed, this is precisely what happens with the hedgehog quiver: One can map the sources of the four-dimensional \( \vec{E} \times \vec{B} \) interaction to a single D6 at the center of the quiver and D0 charges on the nodes. Upon lifting to five dimensions, the D6 brane disappears (becoming the “center of space”) and all the nodes become free.

One should note that our analysis here indicates that the hedgehog quiver describes unbound states only when the center of the quiver is primitive. Indeed, one can consider quivers in which the exterior nodes have charges corresponding to black rings, and the center is a primitive (fluxed) D6 brane\(^{12}\). This solution can be lifted to one or many concentric black rings on an \( \mathbb{R}^4 \) base. As with supertubes, the absence of arrows between the exterior nodes of the quiver is equivalent to the absence of \( \vec{E} \times \vec{B} \)-like interactions between the black rings. Hence, in the asymptotically five-dimensional solution, these rings can slide away from each other, and the configuration has zero modes and is unbound. However, if the center node is not a fluxed D6 brane, but a BMPV black hole, the sliding away of the rings becomes impossible. Indeed, as shown in [38], one cannot take a BMPV black hole away from the center of a black ring without modifying the asymptotic angular momenta of the solution. In that case, the black ring and the black hole interact via \( \vec{E} \times \vec{B} \)-type interactions, that render the sliding-away mode massive.

To summarize, our arguments indicate that all asymptotically five-dimensional solutions given by hedgehog quivers are unbound when their centers are primitive branes. Although a more detailed analysis is needed, this also seems to be true for hedgehog quivers whose central node is primitive and the outside nodes are not. However, the system may well be a bound state when the central node is not primitive.

A few examples of quivers describing unbound states include, for example the “Hall halo” configurations with a primitive center discussed in [37], the three-center configuration discussed in Section 6 of [29], and possibly also the “foaming quiver" (with charges equal to those of a maximally-spinning BMPV black hole) considered in [39]. As we stressed earlier, the unbound

\(^{12}\)One could also consider spectrally-flowed version of this configuration, in which the D6 brane is transformed into another primitive brane.
status of such geometrically bound systems can only be seen when considering asymptotically-flat five-dimensional solutions, or asymptotically $AdS_3 \times S^3$ solutions in six dimensions.

This analysis of bound and unbound systems is also in agreement with the recent findings that only quivers with closed loops can give solutions that have the charges of black holes and black rings with classically large horizon radius \cite{13, 14}, and that at weak coupling only these quivers give a macroscopic (black-hole-like) entropy \cite{40}. Based on this, one expects that the closed, deep or scaling quivers describe bound states, which they indeed do \cite{13}. Intuitively, one can think about the microstates that come from hedgehog quivers (and are necessarily “shallow”), and possibly about other “shallow” microstates as unbound or very weakly bound; conversely, the deep $AdS$ throat, which is the hallmark of the scaling solution, is a direct manifestation of the binding of the geometric components of the microstate geometry.

6 Conclusions

We have investigated spectral flow - a transformation that takes multi-center solutions into other multi-center solutions, by shifting the underlying harmonic functions. For generic parameters, this transformation takes bubbling solutions that have multiple GH centers into other bubbling solutions with GH centers. However, for specially-chosen parameters, a spectral flow transformation can take a bubbling solution into a smooth solution that contains one or several supertubes in a GH multi-center background.

The fact that spectral flow can be used to interchange solutions containing two-charge supertubes and bubbling solutions is a very powerful fact, which we have exploited throughout this paper, and will continue to use in upcoming work.

The first problem we have addressed using this tool is to understand which of the three-charge bubbling solutions constructed in the literature are bound states, and which are not. We have found that the solutions that correspond to quivers without loops or bifurcations can be transformed into solutions that in the five-dimensional lift can be taken apart. Therefore they should not correspond to bound states in the dual CFT.

The second use of spectral flow has been to prove the existence of three-charge smooth BPS solutions that depend on arbitrary functions in the vicinity of any multi-center GH solution. Indeed, any GH center can be related to a round supertube via spectral flow. Furthermore, supertubes can have arbitrary shape while still remaining regular and supersymmetric. Hence the inverse spectral flow of a wiggly supertube gives a new smooth black-hole microstate solution, that does not have a GH base, and that can depend on arbitrary functions.

Even without knowing the explicit form of these BPS solutions, it is still possible to investigate their physics (at least in the vicinity of GH solutions), analyze their moduli space, and count their entropy by using supertube counting techniques \cite{11, 12, 42}. The fact that GH solutions can be deformed to BPS solutions that depend on arbitrary functions establishes the existence of families of black hole microstates that depend on an infinite number of continuous parameters. In upcoming work \cite{12} we will explore these microstates, and argue that they can have a macroscopically large (black-hole-like) entropy.

We have also explored a larger class of spectral flow transformations (called generalized spectral flow) that for generic parameters transform multi-center bubbling solutions into other multi-
center bubbling solutions, but for special parameters can transform one or several of the centers of a bubbling solution into a two-charge supertube, D2-D0 or D0 center.

By taking the limit of parameters in which the D2-D0, or the D0 branes do not back-react on the geometry, one can study them using their (non-abelian) Born-Infeld action, and count their entropy. In fact, such a counting has been performed in several circumstances. For example, in [44, 45] it was found that the entropy coming from D0 branes in a D6-D0 background (which lifts to a multi-center GH solution in five dimensions) is of the same order as the would-be black hole entropy. One could then use generalized spectral flow to transform the D0-D6-D6 system considered there into a multi-center D6-D6 configuration, which (unlike the system with D0 branes) is well-described by supergravity in the regime of parameters where the classical black hole exists. It would be very interesting to follow the spectral flow, and find the description of the D0 configurations that give the black-hole-like entropy [44, 45] in the multi-center D6 frame, and to find whether these configurations are still smooth in supergravity.

Moreover, in [40] the entropy of a three-center D6-D6-D6 configuration was computed at intermediate coupling (using the quiver quantum mechanics describing these branes) and found to give a macroscopic, black-hole-like answer when the branes form a scaling solution. It would be interesting to spectrally flow one of the D6 branes into a two-charge supertube, and to see if the modes that give the macroscopic entropy of the D6-D6-D6 are the same as the fluctuation modes of the supertube. This would also help find a realization of smooth BPS black hole microstates in the regime of parameters where the black hole exists.

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Appendix A. $SL(2, \mathbb{Z})^3$ transformations

If one applies three $SL(2, \mathbb{Z})$ transformations (3.42) on the three pairs formed by $x_{78}$ and $y_{12}$, $y_{34}$ and $y_{56}$ respectively (and permutations thereof), one obtains

\[
\tilde{X} = (B_3^{-1}A_3B_3B_2^{-1}A_2B_2B_1^{-1}A_1B_1) \cdot X
\]

(A.1)

where

\[
\tilde{X} \equiv \{\tilde{x}_{12}, \tilde{x}_{34}, \tilde{x}_{56}, \tilde{x}_{78}, \tilde{y}_{12}, \tilde{y}_{34}, \tilde{y}_{56}, \tilde{y}_{78}\}, \quad X \equiv \{x_{12}, x_{34}, x_{56}, x_{78}, y_{12}, y_{34}, y_{56}, y_{78}\}
\]

(A.2)

and

\[
B_1 = \text{diag}(1, 2, 2, 1, 2, 1, 1, 2) \quad B_2 = \text{diag}(2, 1, 2, 1, 2, 1, 2, 1) \quad B_3 = \text{diag}(2, 2, 1, 1, 1, 1, 2, 2).
\]

(A.3)
The diagonal matrices \( B_I \) fix the correct periodicity for the \( \tau \) and \( v \) coordinates, namely \( 0 \leq \tau < 4\pi \) and \( 0 \leq v < 2\pi \). The matrices \( A_I \) are three commuting, rank 8 matrices composed of the matrix elements of the three \( SL(2,\mathbb{Z}) \) matrices \( \mathcal{M}_I \). They have the explicit form

\[
A_1 = \begin{pmatrix}
  m_1 & 0 & 0 & 0 & 0 & 0 & 0 & n_1 \\
  0 & q_1 & 0 & 0 & 0 & 0 & 0 & p_1 \\
  0 & 0 & q_1 & 0 & 0 & 0 & 0 & p_1 \\
  0 & 0 & 0 & m_1 & n_1 & 0 & 0 & 0 \\
  0 & 0 & 0 & p_1 & q_1 & 0 & 0 & 0 \\
  0 & 0 & n_1 & 0 & 0 & m_1 & 0 & 0 \\
  0 & n_1 & 0 & 0 & 0 & 0 & m_1 & 0 \\
  p_1 & 0 & 0 & 0 & 0 & 0 & 0 & q_1
\end{pmatrix}
\]

(A.4)

\[
A_2 = \begin{pmatrix}
  q_2 & 0 & 0 & 0 & 0 & 0 & 0 & p_2 \\
  0 & m_2 & 0 & 0 & 0 & 0 & 0 & n_2 \\
  0 & 0 & q_2 & 0 & 0 & 0 & 0 & p_2 \\
  0 & 0 & 0 & m_2 & 0 & n_2 & 0 & 0 \\
  0 & 0 & n_2 & 0 & m_2 & 0 & 0 & 0 \\
  0 & 0 & 0 & p_2 & 0 & q_2 & 0 & 0 \\
  n_2 & 0 & 0 & 0 & 0 & 0 & m_2 & 0 \\
  0 & p_2 & 0 & 0 & 0 & 0 & 0 & q_2
\end{pmatrix}
\]

(A.5)

\[
A_3 = \begin{pmatrix}
  q_3 & 0 & 0 & 0 & 0 & 0 & 0 & p_3 \\
  0 & q_3 & 0 & 0 & 0 & 0 & 0 & p_3 \\
  0 & 0 & m_3 & 0 & 0 & 0 & 0 & n_3 \\
  0 & 0 & 0 & m_3 & 0 & 0 & n_3 & 0 \\
  0 & n_3 & 0 & 0 & m_3 & 0 & 0 & 0 \\
  n_3 & 0 & 0 & 0 & 0 & m_3 & 0 & 0 \\
  0 & 0 & p_3 & 0 & 0 & 0 & q_3 & 0 \\
  0 & 0 & p_3 & 0 & 0 & 0 & 0 & q_3
\end{pmatrix}
\]

(A.6)

Note that

\[
\text{Det} A_I = (m_I q_I - n_I p_I)^4 = 1, \quad \text{and} \quad \text{Tr} A_I = 4(m_I + q_I) = 4\text{Tr} \mathcal{M}_I
\]

(A.7)

and also

\[
\text{Tr}(A_1 A_2 A_3) = (m_1 + q_1)(m_2 + q_2)(m_3 + q_3) = \frac{1}{64}(\text{Tr} A_1)(\text{Tr} A_2)(\text{Tr} A_3).
\]

(A.8)

Thus, in general, the new solution obtained after spectral flow is determined by the eight harmonic functions \( \{ \tilde{V}, \tilde{K}_I, \tilde{L}_I, \tilde{M} \} \) each of which is a linear combination of the eight harmonic functions that determine the original solution \( \{ V, K_I, L_I, M \} \).

For non-zero \( m_I \) and \( q_I \) one can use the gauge transformations (3.27), which leave the physical solution invariant, to set \( p_1 = p_2 = p_3 = 0 \). Then the most general transformation on the eight
harmonic functions is:

\[ \tilde{L}_I = \frac{1}{2} \delta_{I,J} C^{JKL} m_J q_K q_L L_J + 2 \delta_{I,J} C^{JKL} n_J q_K q_L M, \quad \tilde{M} = \frac{1}{6} C^{JKL} q_J q_K q_L M, \quad \tilde{\omega} = \tilde{\omega} \]

\[ \tilde{K}^I = \frac{1}{2} \delta^I_J C^{JKL} q_J m_K m_L K^J + 2 \delta^I_J C^{JKL} q_J m_K n_L L_K + 4 \delta^I_J C^{JKL} q_J n_K n_L M \]

\[ \tilde{\nu} = \frac{1}{6} C^{JKL} m_J m_K m_L v - C^{JKL} n_J m_K m_L K^J - 2 C^{JKL} n_J n_K n_L L^J - \frac{8}{3} C^{JKL} n_J n_K n_L M \] (A.9)

For \( m_I = q_I = 1 \) one obtains the generalized spectral flow transformation (3.35) (using \( \gamma_I = -2 n_I \)).

For non-trivial transformations with \( m_I = 0 \) or \( q_I = 0 \), one has \( n_I = -p_I = \pm 1 \), and one cannot use the gauge transformation (3.27) to set \( p_1 = p_2 = p_3 = 0 \). Hence the \( SL(2, \mathbb{Z})^3 \) transformation does not reduce to a generalized spectral flow. For example, when both \( m_I = 0 \) and \( q_I = 0 \), the new harmonic functions are:

\[ \tilde{\nu} = -\frac{8}{3} C^{IJK} n_I n_J n_K M = \pm 16 M, \quad \tilde{M} = -\frac{1}{12} C^{IJK} \frac{p_J p_K}{8} V = \pm \frac{1}{16} V, \]

\[ \tilde{L}_I = -\frac{1}{2} \delta_{I,J} C^{JKL} n_J p_K p_L K^J = \pm \frac{1}{2} K^I, \quad \tilde{K}^I = -\frac{1}{2} \delta_{I,J} C^{JKL} 2 p_J m_K n_L L_J = \pm 2 L_I, \] (A.10)

which is an interchange of the \( x \) and \( y \) parameters of \( E_7(7) \), and in four dimensions corresponds to an electric-magnetic duality.

It is also interesting to note that although the transformations (A.9) are obtained for \( U(1)^3 \) ungauged supergravity in five dimensions, they are also valid for a more general \( U(1)^N \) supergravity obtained from M-theory compactified on a Calabi-Yau 3-fold.

One may consider also other \( SL(2, \mathbb{Z}) \) transformations on the harmonic functions \( x_{a,a+1}, y_{a,a+1} \). However these will not preserve the regularity of the initial bubbling solution. Consider for example the transformation

\[ \begin{pmatrix} \tilde{x}_{12} \\ 2 \tilde{y}_{12} \end{pmatrix} = M_4 \begin{pmatrix} x_{12} \\ 2 y_{12} \end{pmatrix}, \quad \begin{pmatrix} \tilde{x}_{34} \\ 2 \tilde{y}_{34} \end{pmatrix} = M_4 \begin{pmatrix} x_{34} \\ 2 y_{34} \end{pmatrix}, \]

\[ \begin{pmatrix} \tilde{x}_{56} \\ 2 \tilde{y}_{56} \end{pmatrix} = M_4 \begin{pmatrix} x_{56} \\ 2 y_{56} \end{pmatrix}, \quad \begin{pmatrix} \tilde{x}_{78} \\ 2 \tilde{y}_{78} \end{pmatrix} = M_4 \begin{pmatrix} x_{78} \\ 2 y_{78} \end{pmatrix}, \] (A.11)

which could be expressed as

\[ \tilde{X} = (B_4^{-1} A_4 B_4) \cdot X, \] (A.12)

where

\[ B_4 = \text{diag}(1, 1, 1, 1, 2, 2, 2, 2) \] (A.13)
and

\[ A_4 = \begin{pmatrix}
  m_4 & 0 & 0 & n_4 & 0 & 0 & 0 \\
  0 & m_4 & 0 & 0 & n_4 & 0 & 0 \\
  0 & 0 & m_4 & 0 & 0 & n_4 & 0 \\
  0 & 0 & 0 & m_4 & 0 & 0 & n_4 \\
  p_4 & 0 & 0 & 0 & q_4 & 0 & 0 \\
  0 & p_4 & 0 & 0 & 0 & q_4 & 0 \\
  0 & 0 & p_4 & 0 & 0 & 0 & q_4 \\
  0 & 0 & 0 & p_4 & 0 & 0 & 0 & q_4
\end{pmatrix} . \]  

(A.14)

As before we have the relations

\[ \text{Det}A_4 = (m_4 q_4 - n_4 p_4)^4 = 1, \quad \text{and} \quad \text{Tr}A_4 = 4(m_4 + q_4) = 4\text{Tr}\mathcal{M}_4 . \]  

(A.15)

Note that the matrix \( A_4 \) does not commute with the other three matrices \( A_I \). More importantly, the new solution specified by the eight harmonic functions \( \tilde{x}_{a,a+1}, \tilde{y}_{a,a+1} \) is not regular because it fails to satisfy the regularity conditions:

\[ \tilde{l}_i = -\frac{1}{2}C_{iJK}\tilde{k}_J^i\tilde{k}_K^i \quad \text{and} \quad \tilde{m}_i = \frac{1}{2} \frac{\tilde{k}_1^i\tilde{k}_2^i\tilde{k}_3^i}{(\tilde{q}_i)^2} . \]  

(A.16)

One can, of course, apply other \( SL(2,\mathbb{Z}) \) transformations that are subgroups of the \( E_{7(7)} \) U-duality group of the supergravity theory. However such transformations will generically convert a regular bubbling solution to an singular one. It is interesting to observe that the \( SL(2,\mathbb{Z}) \) transformation which produced a singular solution (\( \mathcal{M}_4 \) above) leads to an \( 8 \times 8 \) matrix that does not commute with the three matrices \( A_I \) that correspond to the \( SL(2,\mathbb{Z})^3 \) that generically takes smooth solutions into smooth solutions. One could speculate that there might exist a relation between the fact that the matrices \( A_1, A_2 \) and \( A_3 \) commute and the fact that they generate the largest subgroup of \( E_{7(7)} \) that preserves smoothness, but we will leave the investigation of this to future work.

References

[1] A. Strominger and C. Vafa, “Microscopic Origin of the Bekenstein-Hawking Entropy,” Phys. Lett. B 379, 99 (1996) [arXiv:hep-th/9601029].

[2] S. D. Mathur, “The fuzzball proposal for black holes: An elementary review,” Fortsch. Phys. 53, 793 (2005) [arXiv:hep-th/0502050].

[3] I. Bena and N. P. Warner, “Black holes, black rings and their microstates,” [arXiv:hep-th/0701216].

[4] O. Lunin, “Adding momentum to D1-D5 system,” JHEP 0404, 054 (2004) [arXiv:hep-th/0404006].
[5] S. Giusto, S. D. Mathur and A. Saxena, “Dual geometries for a set of 3-charge microstates,” Nucl. Phys. B 701, 357 (2004) [arXiv:hep-th/0405017].
S. Giusto, S. D. Mathur and A. Saxena, “3-charge geometries and their CFT duals,” Nucl. Phys. B 710, 425 (2005) [arXiv:hep-th/0406103].
S. Giusto and S. D. Mathur, “Geometry of D1-D5-P bound states,” Nucl. Phys. B 729, 203 (2005) [arXiv:hep-th/0409067].

[6] I. Bena and P. Kraus, “Microstates of the D1-D5-KK system,” Phys. Rev. D 72, 025007 (2005) [arXiv:hep-th/0503053].

[7] V. Jejjala, O. Madden, S. F. Ross and G. Titchener, “Non-supersymmetric smooth geometries and D1-D5-P bound states,” Phys. Rev. D 71, 124030 (2005) [arXiv:hep-th/0504181].

[8] J. Ford, S. Giusto and A. Saxena, “A class of BPS time-dependent 3-charge microstates from spectral flow,” Nucl. Phys. B 790, 258 (2008) [arXiv:hep-th/0612227].

[9] D. Mateos and P. K. Townsend, “Supertubes,” Phys. Rev. Lett. 87, 011602 (2001) [arXiv:hep-th/0103030].

[10] O. Lunin and S. D. Mathur, “AdS/CFT duality and the black hole information paradox,” Nucl. Phys. B 623, 342 (2002) [arXiv:hep-th/0109154].

[11] O. Lunin, J. M. Maldacena and L. Maoz, “Gravity solutions for the D1-D5 system with angular momentum,” [arXiv:hep-th/0212210].

[12] I. Bena, N. Bobev, C. Ruef and N. P. Warner, “Entropy Enhancement and Black Hole Microstates”, to appear.

[13] I. Bena, C. W. Wang and N. P. Warner, “Mergers and typical black hole microstates,” JHEP 0611, 042 (2006) [arXiv:hep-th/0608217].

[14] I. Bena, C. W. Wang and N. P. Warner, “Plumbing the Abyss: Black Ring Microstates,” [arXiv:0706.3786 [hep-th]].

[15] J. P. Gauntlett, J. B. Gutowski, C. M. Hull, S. Pakis and H. S. Reall, “All supersymmetric solutions of minimal supergravity in five dimensions,” Class. Quant. Grav. 20, 4587 (2003) [arXiv:hep-th/0209114].

[16] J. B. Gutowski and H. S. Reall, “General supersymmetric $AdS_5$ black holes,” JHEP 0404, 048 (2004) [arXiv:hep-th/0401129].

[17] I. Bena and N. P. Warner, “One ring to rule them all ... and in the darkness bind them?,” Adv. Theor. Math. Phys. 9, 667 (2005) [arXiv:hep-th/0408106].

[18] J. B. Gutowski, D. Martelli and H. S. Reall, “All supersymmetric solutions of minimal supergravity in six dimensions,” Class. Quant. Grav. 20, 5049 (2003) [arXiv:hep-th/0306235].
[19] M. Cariglia and O. A. P. Mac Conamhna, “The general form of supersymmetric solutions of \( N = (1,0) \) \( U(1) \) and \( SU(2) \) gauged supergravities in six dimensions,” Class. Quant. Grav. 21, 3171 (2004) [arXiv:hep-th/0402055].

[20] V. Balasubramanian, J. de Boer, E. Keski-Vakkuri and S. F. Ross, “Supersymmetric conical defects: Towards a string theoretic description of black hole formation,” Phys. Rev. D 64, 064011 (2001) [arXiv:hep-th/0011217].

J. M. Maldacena and L. Maoz, “De-singularization by rotation,” JHEP 0212, 055 (2002) [arXiv:hep-th/0201205].

[21] C. P. Boyer and J. D. . Finley, “Killing Vectors In Selfdual, Euclidean Einstein Spaces,” J. Math. Phys. 23, 1126 (1982).

[22] A. Das and J. Gegenberg, “Stationary Riemannian space-times with self-dual curvature,” Gen. Rel. Grav. 16, (1984) 817.

[23] I. Bakas and K. Sfetsos, “Toda fields of \( \text{SO}(3) \) hyper-Kähler metrics and free field realizations,” Int. J. Mod. Phys. A 12, 2585 (1997) [arXiv:hep-th/9604003].

[24] I. Bena, N. Bobev and N. P. Warner, “Bubbles on Manifolds with a \( U(1) \) Isometry,” JHEP 0708, 004 (2007) [arXiv:0705.3641 [hep-th]].

[25] G. W. Gibbons and S. W. Hawking, “Gravitational Multi - Instantons,” Phys. Lett. B 78, 430 (1978).

[26] G.W. Gibbons and P.J. Ruback, “The Hidden Symmetries of Multi-Center Metrics,” Commun. Math. Phys. 115, 267 (1988).

[27] J. P. Gauntlett and J. B. Gutowski, “General concentric black rings,” Phys. Rev. D 71, 045002 (2005) [arXiv:hep-th/0408122].

[28] I. Bena and N. P. Warner, “Bubbling supertubes and foaming black holes,” Phys. Rev. D 74, 066001 (2006) [arXiv:hep-th/0505166].

[29] P. Berglund, E. G. Gimon and T. S. Levi, “Supergravity microstates for BPS black holes and black rings,” JHEP 0606, 007 (2006) [arXiv:hep-th/0505167].

[30] E. G. Gimon, T. S. Levi and S. F. Ross, “Geometry of non-supersymmetric three-charge bound states,” JHEP 0708, 055 (2007) [arXiv:0705.1238 [hep-th]].

[31] I. Bena, P. Kraus and N. P. Warner, “Black rings in Taub-NUT,” Phys. Rev. D 72, 084019 (2005) [arXiv:hep-th/0504142].

[32] I. Bena, N. Bobev, C. Ruef and N. P. Warner, in preparation

[33] A. Saxena, G. Potvin, S. Giusto and A. W. Peet, “Smooth geometries with four charges in four dimensions,” JHEP 0604, 010 (2006) [arXiv:hep-th/0509214].
1. I. Bena and P. Kraus, “Microscopic description of black rings in AdS/CFT,” JHEP 0412, 070 (2004) [arXiv:hep-th/0408186].

2. J. de Boer, F. Denef, S. El-Showk, I. Messamah and D. V. d. Bleeken, “Black hole bound states in $AdS_3 \times S^2$,” [arXiv:0802.2257 [hep-th]].

3. Y. K. Srivastava, “Bound states of KK monopole and momentum,” [arXiv:hep-th/0611124].

4. F. Denef, “Supergravity flows and D-brane stability,” JHEP 0008, 050 (2000) [arXiv:hep-th/0005049].

5. B. Bates and F. Denef, “Exact solutions for supersymmetric stationary black hole composites,” [arXiv:hep-th/0304094].

6. I. Bena, C. W. Wang and N. P. Warner, “Sliding rings and spinning holes,” JHEP 0605, 075 (2006) [arXiv:hep-th/0512157].

7. I. Bena, C. W. Wang and N. P. Warner, “The foaming three-charge black hole,” Phys. Rev. D 75, 124026 (2007) [arXiv:hep-th/0604110].

8. F. Denef and G. W. Moore, “Split states, entropy enigmas, holes and halos,” [arXiv:hep-th/0702146].

9. B. C. Palmer and D. Marolf, “Counting supertubes,” JHEP 0406, 028 (2004) [arXiv:hep-th/0403025].

10. D. Bak, Y. Hyakutake, S. Kim and N. Ohta, “A geometric look on the microstates of supertubes,” Nucl. Phys. B 712, 115 (2005) [arXiv:hep-th/0407253].

11. L. Grant, L. Maoz, J. Marsano, K. Papadodimas and V. S. Rychkov, “Minisuperspace quantization of ’bubbling AdS’ and free fermion droplets,” JHEP 0508, 025 (2005) [arXiv:hep-th/0505079]. V. S. Rychkov, “D1-D5 black hole microstate counting from supergravity,” JHEP 0601, 063 (2006) [arXiv:hep-th/0512053].

12. F. Denef, D. Gaiotto, A. Strominger, D. Van den Bleeken and X. Yin, “Black hole deconstruction,” [arXiv:hep-th/0703252].

13. E. G. Gimon and T. S. Levi, “Black Ring Deconstruction,” [arXiv:0706.3394 [hep-th]].