Automorphisms of $X_0^*(p)$

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1. Introduction

Let $N$ be a positive integer, and let $X_0(N)$ be the modular curve corresponding to the congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod N \right\}.$$ 

It is known by [1] that the full normalizer $\mathfrak{N}(\Gamma_0(N))$ of $\Gamma_0(N)$ in $\text{GL}_2(\mathbb{Q}) = \{ \alpha \in \text{GL}_2(\mathbb{Q}) \mid \det \alpha > 0 \}$ contains the group $\Gamma_0^*(N)$ generated by all the matrices of the form

$$\begin{pmatrix} N'a & b \\ Nc & N'd \end{pmatrix}, \quad a, b, c, d \in \mathbb{Z}, \quad N'|N, \quad N'^2ad - Nbc = N',$$

and that the factor group $W(N) := \Gamma_0^*(N)/\Gamma_0(N)$ is an elementary 2-abelian group of order $2^{\omega(N)}$, where $\omega(N)$ is the number of distinct prime divisors of $N$. In case $N$ is square-free, it is also known [2] that $\Gamma_0^*(N)$ is maximal as a Fuchsian group (hence coincides with $\mathfrak{N}(\Gamma_0(N))$). Now put $B(N) = \mathfrak{N}(\Gamma_0(N))/\Gamma_0(N)$, which is naturally regarded as a subgroup of the automorphism group $\text{Aut}_X(N)$ of the curve $X_0(N)$. [3]

Recall that $\text{Aut}_X(N)$ has been completely determined [23][17][7]; it agrees with $B(N)$ whenever $X_0(N)$ has genus $g \geq 2$ and $N \neq 37, 63$. (For these two exceptional cases, the group $B(N)$ is of index 2 in $\text{Aut}_X(N)$.) In particular, if $N$ is square-free, then we have $\text{Aut}_X(N) = W(N)$ whenever $X_0(N)$ has genus $g \geq 2$ and $N \neq 37$.

Let $X_0^*(N)$ be the quotient curve of $X_0(N)$ by $W(N)$, and let $g^*(N)$ be the genus of $X_0^*(N)$. If $N$ is a prime power, then $X_0^*(N)$ coincides with $X_0^+(N)$, the quotient of $X_0(N)$ by the Atkin–Lehner involution. We are interested in determining the group $\text{Aut}_X^*(N)$ (for $g^*(N) \geq 2$). In the rest of this paper, we always assume $N$ to be square-free, unless otherwise specified. Then $\Gamma_0^*(N)$ is a maximal Fuchsian group, so every nontrivial automorphism of $X_0^*(N)$ is necessarily exceptional in the sense that it does not arise from a linear fractional transformation on the complex upper half plane. Thus, if we assume that $g^*(N) \geq 2$, we expect the group $\text{Aut}_X^*(N)$ to be very small.

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*Throughout this paper, if $X$ is a curve defined over a field $k$, we denote by $\text{Aut}_X$ the group $\text{Aut}_{k_s}(X \times_k k_s)$ of automorphisms of $X_{k_s}$ over a separable closure $k_s$ of $k$, where $X_{k_s} := X \times_k k_s$. By Lemma 2.2 below, this is the same as $\text{Aut}_K(X \times_k K)$ for any separably closed field extension $K$ of $k_s$. Similarly, if $A$ is an abelian variety over $k$, then $\text{End} A$ will be shorthand for $\text{End}_{k_s}(A \times_k k_s)$.
The purpose of this note is to prove that when $N$ is a prime number, this is indeed the case:

**Theorem 1.1.** Let $p$ be a prime such that $g^*(p) \geq 2$. Then

$$\text{Aut} \, X_0^*(p) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } g^*(p) = 2; \\ \{1\} & \text{if } g^*(p) > 2. \end{cases}$$

We remark that Ogg’s method in [23] for determining $\text{Aut} \, X_0(p)$ does not readily generalize to $X_0^*(p)$. Ogg’s proof makes essential use of the action of automorphisms on the cuspidal subgroup of $J_0(p)$. However, as $X_0^*(p)$ has only one cusp, the cuspidal group in $J_0^*(p)$ is trivial, so no information can be gained in this way. (Here $J_0(p)$ and $J_0^*(p)$ denote the Jacobians of $X_0(p)$ and $X_0^*(p)$, respectively.)

Our plan for proving Theorem 1.1 is as follows. In Section 2, we investigate the structure of $\text{Aut} \, X_0^*(N)$ for $N$ square-free. Using this, we show in Section 3 that $\text{Aut} \, X_0^*(p)$ is trivial for almost all primes $p$. Finally, by considering the reduction of $X_0^*(p)$ modulo various primes, we show in Sections 4 and 5 that $\text{Aut} \, X_0^*(p)$ is trivial for the remaining primes such that $g^*(p) > 2$.

To put Theorem 1.1 in a larger context, we note that little is known in general about the problem of determining the rational points on $X_0^*(p)$. (For a discussion of what is currently known about this problem, including the existence of certain “exceptional” rational points for $p = 191, 311$, see S. Galbraith’s paper [14].) On page 145 of his paper [11], Mazur writes (adapting his terminology to ours): “What further rational points [in addition to the cusp and the rational CM points] does the curve $X_0^*(p)$ possess? This diophantine question (when the genus $g^*(p) > 0$) is extremely interesting, since no known method appears to be applicable to it, for any value of $p$.” We hope that Theorem 1.1 will be useful in future investigations of this problem. In any case, if it had turned out that there were unexpected automorphisms of $X_0^*(p)$ for certain $p$, this might have “explained” the existence of certain exceptional rational points such as those found by Galbraith.

Also, we remark that the curve $X_0^*(p^2)$ coincides with the curve which Mazur in [24] calls $X_{\text{split}}(p)$. The rational points on this family of curves are also not known in general (see [24] for some results in this direction). The problem of determining the rational points on $X_{\text{split}}(p)$ is related to a famous question formulated by Serre asking which elliptic curves $E/\mathbb{Q}$ have the image of their mod $p$ Galois representations contained in a proper subgroup of $\text{GL}_2(\mathbb{F}_p)$ (in this case the normalizer of a split Cartan subgroup). See [20] for further details. It would be especially interesting for this reason to prove a result similar to Theorem 1.1 in which the automorphism group of $X_0^*(p^2)$ is determined. Our methods shed only partial light on this problem.

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2. Structure of $\text{Aut} \, X_0^*(N)$

In this section, we investigate the structure of $\text{Aut} \, X_0^*(N)$ for $N$ square-free. It is known in this case (see [3]) that $J_0(N)$, and hence $J_0^*(N)$, is a semistable abelian variety over $\mathbb{Q}$. We begin by recalling the following fact:

**Lemma 2.1.** Let $X$ be a smooth, proper, and geometrically connected algebraic curve of genus $g \geq 2$ over a field $k$, and let $J = \text{Pic}^0_{X/k}$ be the Picard (Jacobian) variety of $X$ over $k$. Then $\text{Aut}_k(X)$ injects into $\text{End}_k(J)$. 

Proof. Suppose \( \phi \in \mathrm{Aut}_k(X) \) induces the identity map on \( J \). Then \( \phi \) acts trivially on the cotangent space at \( 0 \) in \( J \), which is canonically isomorphic to \( H^0(X, \Omega^1_X) \). It follows that \( \phi \) acts trivially on the image of \( X \) under the canonical map to projective space. Now if \( X \) is not hyperelliptic, then this canonical map is an embedding, which implies that \( \phi \) is identity on \( X \). If \( X \) is hyperelliptic, this argument shows that \( \phi \) must be either the identity or the hyperelliptic involution \( h \). Since \( h \) acts as \(-1\) on \( J \), it follows that the natural map from \( \mathrm{Aut}_k(X) \) to \( \mathrm{End}_k(J) \) is injective in the hyperelliptic case as well. \( \blacksquare \)

We also have the following general fact:

**Lemma 2.2.** Let \( X \) as above be a smooth, proper, geometrically connected curve of genus \( g \geq 2 \) over a field \( k \), and let \( K/k \) be an extension field. Let \( k_s \) denote the separable closure of \( k \) in \( K \). Then every automorphism of \( X_K \) is defined over \( k_s \).

Proof. Grothendieck’s theory of the Hilbert scheme (see [1]) implies that there is a scheme \( \mathrm{Aut}_{X/k} \), locally of finite type over \( k \), which represents the functor associating to each \( k \)-scheme \( S \) the set of \( S \)-isomorphisms from \( X \times_k S \) to itself. It is enough to prove that \( \mathrm{Aut}_{X/k} \) is etale over \( k \), since this implies that every \( K \)-point comes from a \( k_s \)-point. This is a special case of Lemma 1.11 of [3]. \( \blacksquare \)

**Corollary 2.3.** Let \( X, k, J \) be as in Lemma 2.1, and let \( K/k \) be an extension field. If \( \phi \) is a \( K \)-automorphism of \( X_K := X \times_k K \) such that the induced \( K \)-endomorphism \( \phi^* \) of \( J \times_k K = \mathrm{Pic}^0_{X/K} \) is defined over \( k \), then \( \phi \) is defined over \( k \).

Proof. We may clearly enlarge \( K \) so that it is assumed to be separably closed. If \( k_s \) denotes the separable closure of \( k \) in \( K \), then by Lemma 2.2, we may assume without loss of generality that \( K = k_s \). Descent theory now tells us that \( \phi \) is defined over \( k \) if and only if \( \phi^g = \phi \) for all \( g \in \mathrm{Gal}(k_s/k) \). Since the natural map from \( \mathrm{Aut}_{k_s}(X \times_k k_s) \) to \( \mathrm{End}_{k_s}(J \times_k k_s) \) is injective, it suffices to show that \( (\phi^g)^* = \phi^* \) on \( J \times_k k_s \) for all \( g \). But \((\phi^g)^* = (\phi^*)^g \) by the definition of Picard functoriality, so the hypothesis that \( \phi^* \) is defined over \( k \) gives us what we want. \( \blacksquare \)

Lemma 2.1 and Corollary 2.3 are particularly useful when the torsion part of \( \mathrm{End}(\mathrm{Pic}^0(X)) \) has a simple structure. For example, we obtain the following result:

**Proposition 2.4.** Let \( X \) be an algebraic curve over \( \mathbb{Q} \) of genus \( g \geq 1 \). Assume that its Jacobian \( J(X) \) is semistable over \( \mathbb{Q} \) and that \( \mathrm{End}(J(X)) \otimes \mathbb{Q} \) is a product of totally real fields. Then \( \mathrm{Aut}(X) \) is elementary 2-abelian: \( \mathrm{Aut}(X) \cong \mathbb{Z}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2\mathbb{Z} \). Moreover, every automorphism of \( X \) is defined over \( \mathbb{Q} \).

Proof. The first assertion follows from Lemma 2.1, since by assumption the torsion part of \( (\mathrm{End}(J(X)))^\times \) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/2\mathbb{Z} \). The latter part is a consequence of Corollary 2.3 and the following result of Ribet. \( \blacksquare \)

**Theorem 2.5** (Ribet 20). Let \( A \) be a semistable abelian variety over \( \mathbb{Q} \). Then every endomorphism of \( A \) is defined over \( \mathbb{Q} \).

Let \( J_0^g(N) \) be the Jacobian variety of \( X_0^g(N) \). Since \( N \) is square-free, \( J_0^g(N) \) is semistable over \( \mathbb{Q} \) (see [3]). It then follows from Theorem 2.5 and the results of [27] that \( \mathrm{End}(J_0^g(N)) \otimes \mathbb{Q} \) is isomorphic to a product of totally real fields. Applying Proposition 2.4, we have

**Corollary 2.6.** The group \( \mathrm{Aut}(X_0^g(N)) \) is elementary 2-abelian. Moreover, every automorphism of \( X_0^g(N) \) is defined over \( \mathbb{Q} \).
As an application we can determine \( \text{Aut} X_0^*(N) \) when \( J_0^*(N) \) is simple. Recall that \( X_0^*(N) \) is hyperelliptic if and only if it is of genus 2 (cf. [13]). Therefore

**Corollary 2.7.** If \( J_0^*(N) \) is simple, then

\[
\text{Aut} X_0^*(N) = \begin{cases} 
\mathbb{Z}/2\mathbb{Z} & \text{if } g^*(N) = 2; \\
\{1\} & \text{if } g^*(N) > 2.
\end{cases}
\]

Now let \( p \) be a prime number. Then \( X_0^*(p) \) is hyperelliptic (i.e., \( g^*(p) = 2 \)) if and only if \( p = 67, 73, 103, 107, 167, 191 \).

Since \( J_0^*(p) \) is simple for these six primes (see [3, Table 5]), we conclude that Theorem 1.1 holds when \( g^*(p) = 2 \).

### 3. \text{Aut} X_0^*(p) is trivial for almost all \( p \)

In the previous section we saw that \( \text{Aut} X_0^*(N) \) is an elementary 2-abelian group. The purpose of this section is to show that \( \text{Aut} X_0^*(p) \) is trivial for almost all primes \( p \).

Throughout this section, let \( p \) be a prime number and let \( N \) be a square-free integer such that \( g^*(N) \) and \( g^*(p) \) are at least 2.

We need the following two lemmas.

**Lemma 3.1.** Let \( u \) be an automorphism of \( X_0^*(N) \). Then, as endomorphisms of \( J_0^*(N) \), we have

\[
u T_l = T_l u
\]

for each prime \( l \), where \( T_l \) is the \( l \)-th Hecke operator.

**Proof.** This follows from the fact that since \( N \) is assumed to be square-free, \( \text{End} J_0^*(N) \) is commutative (cf. [17, Lem. 2.6]). ■

Let \( \infty \) denote the cusp on \( X_0^*(N) \) which is the image of the cusp \( \infty \) on \( X_0(N) \) under the natural map.

**Lemma 3.2.** If \( u \) is a nontrivial automorphism of \( X_0^*(N) \), then \( u \infty \neq \infty \).

**Proof.** (See also [23].) The eigenvalues of \( u \) acting on the space of holomorphic differentials of \( X_0^*(N) \) are all \( \pm 1 \). If they are all \( +1 \), then an argument using the canonical map to projective space as in the proof of Lemma 2.1 shows that \( u = id \) on \( X_0^*(N) \). Similarly, if the eigenvalues are all \( -1 \), then \( X \) is hyperelliptic and \( u \) is the hyperelliptic involution. In this case, one can see explicitly using Weierstrass points that \( u \infty \neq \infty \). Recall that \( X_0^*(N) \) is hyperelliptic exactly when \( g^*(N) = 2 \). The Weierstrass points on a genus 2 curve \( X \) are the fixed points of the hyperelliptic involution, which is equivalent to saying that \( P \) is a Weierstrass point if and only if there is a nonzero holomorphic differential vanishing to order 2 at \( P \). In our situation, by computing the \( q \)-expansion at \( \infty \) of each Hecke eigendifferential in the cases where \( g^*(N) = 2 \), one finds that \( \infty \) is not a Weierstrass point on \( X_0^*(N) \), so it cannot be fixed by the hyperelliptic involution.

Suppose, then, that \( u \) has eigenvalues \( +1 \) and \( -1 \). Then one can choose two differentials \( \omega_1, \omega_2 \in H^0(X_0^*(N), \Omega^1) \) with \( u^* \omega_1 = \omega_1, u^* \omega_2 = -\omega_2 \) which are normalized eigendifferentials for the action of the Hecke algebra \( T \). The \( q \)-expansion at \( \infty \) of each \( \omega_i \) is of the form

\[
\omega_i = \left( \sum_{m=1}^{\infty} a_m q^m \right) \frac{dq}{q}
\]

with \( a_1 = 1 \).
We see that \( \omega = \omega_1 + \omega_2 \) does not vanish at \( \infty \), but its pullback \( u^* \omega = \omega_1 - \omega_2 \) does vanish at \( \infty \). Consequently, we must have \( u \infty \neq \infty \). ■

Note that since \( \infty \) is the only cusp on \( X_0^*(N) \) when \( N \) is square-free, the statement \( u \infty \neq \infty \) is equivalent to the statement that \( u \infty \) is not a cusp of \( X_0^*(N) \).

We now come to the following key lemma.

**Lemma 3.3.** Let \( p \) be a prime number such that \( g^*(p) \geq 2 \), and suppose there is a nontrivial automorphism \( u \) of \( X_0^*(p) \). Let \( \infty \) be the cusp of \( X_0^*(p) \), and \( P \) a noncuspidal geometric point of \( X_0^*(p) \). Then for each prime \( l \) such that \( l \neq p \), the divisor \( D_{l,P} := (uT_l - T_lu)(\infty - P) \) on \( X_0^*(p)_{\overline{Q}} \) is non-zero but linearly equivalent to zero.

**Proof.** The divisor \( D_{l,P} \) is linearly equivalent to zero by Lemma 3.2. Also, note that by Lemma 3.2, \( Q := u \infty \) is not a cusp. Now suppose that the divisor \( D_{l,P} \) is identically zero, i.e., that \( uT_l \infty + T_luP = T_lu \infty + uT_lP \). Since \( T_l \infty = (l + 1) \infty \), we see by applying \( u \) to both sides that there can be no cancellation between points in the support of \( uT_l \infty \) and \( uT_lP \), because all points in the support of \( T_lP \) are noncuspidal by the definition of the correspondence \( T_l \). Therefore we must have complete cancellation between \( uT_l \infty \) and \( T_lu \infty \). In other words, we must have \( (uT_l - T_lu)(\infty) = 0 \), i.e., \( (l + 1)Q = T_l(Q) \). It follows that \( Q \) is a rational point of \( X_0^*(p) \) represented by a CM elliptic curve. By [3, Thm. 3.2] (and the remarks following the proof of that theorem), we see that \( Q \) must be a Heegner point of class number one. (Note in particular that all noncuspidal rational points on \( X_0^*(p) \) coming from rational points on \( X_0(p) \) are Heegner points of class number one when \( g_0^*(p) \geq 2 \), and this happens only for \( p = 67 \) and \( p = 163 \)). Let \( (E,C) \) represent a geometric point of \( X_0^*(p) \) which projects to \( Q \), so that \( Q \) itself can be thought of as corresponding to the set \( \{P, w_P, P\} \) represented by the unordered pair \( \{(E,C), (E/C, E[p]/C)\} \). Since \( Q \) is a Heegner point of class number one, complex multiplication theory tells us that \( E/C \cong E \). If \( D \) is a cyclic subgroup of order \( l \) of \( E \), then \( E/D \) is isomorphic either to \( E \) or to \( E/C \), since \( (l + 1)Q = T_l(Q) \) as divisors on \( X_0^*(p) \). But \( E/C \) is isomorphic to \( E \), so we conclude that \( E/D \cong E \) for each of the \( l + 1 \) cyclic subgroups \( D \subset E \) of order \( l \). The resulting degree \( l \) maps from \( E \) to \( E/D \cong E \) give rise to \( l + 1 \) distinct elements of norm \( l \) in \( \text{End}(E)/\{\text{units}\} \). This is impossible, however. To see this, let \( \mathcal{O} = \text{End}(E) \), let \( K \) be the fraction field of \( \mathcal{O} \) (which is an imaginary quadratic field), and let \( R \) be the ring of integers (maximal order) of \( K \). If the units of \( \mathcal{O} \) coincide with those of \( R \) (which is automatic except when \( K = \mathbb{Q}(i) \) or \( \mathbb{Q}(\omega) \), where \( i \) and \( \omega \) are primitive fourth and third roots of unity, respectively), we would get \( l + 1 \) distinct ideals of norm \( l \) in \( R \), which is clearly impossible since \( l + 1 \geq 3 \). We are left with the possibility that \( R = \mathbb{Z}[i] \) or \( \mathbb{Z}[\omega] \) and \( \mathcal{O} \) is a nonmaximal order in \( K \). However, it is easy to see that these exceptional cases are also impossible using the fact that in \( \mathbb{Z}[i] \), 2 ramifies and 3 is inert, and in \( \mathbb{Z}[\omega] \), 3 ramifies and 2 and 5 are inert. ■

**Corollary 3.4.** Suppose there is a nontrivial automorphism \( u \) of \( X_0^*(p) \). Then there is a finite morphism \( f : X_0^*(p) \to \mathbb{P}^1 \) of degree at most 6, defined over \( \mathbb{Q} \). Moreover, every nontrivial automorphism of \( X_0^*(p) \) has at most 12 fixed points.

**Proof.** We let \( l = 2 \) in the above lemma. For the first assertion, take \( P = u \infty \). For the latter, take \( P \neq \infty \), \( u \infty \) and consider a morphism \( f : X_0^*(p) \to \mathbb{P}^1 \) such that \( (f) = D_{l,P} \). Then \( u^*f = f \), since a comparison of divisors shows that \( u^*f \) has a zero at \( \infty \) whereas \( f \) does not. Therefore the assertion follows from the following general fact. ■
Lemma 3.5. Let $X$ be a smooth projective curve over an algebraically closed field $k$, and let $u$ be a nontrivial automorphism of $X$ having $r$ fixed points. If there is a finite morphism $f : X \to \mathbb{P}^1$ of degree $d$ such that $u^*f \neq f$, then $r \leq 2d$.

Proof. Let us write $g = u^*f - f$. Assume first that $g$ is constant. Then all the fixed points of $u$ must be poles of $f$, so $2d \geq d \geq r$. Next assume that $g$ is non-constant. Then $2d - r' \geq \deg g$, where $r'$ is the number of fixed points of $u$ which are poles of $f$. Furthermore, since every fixed point of $u$ which is not a pole of $f$ is a zero of $g$, we must have $2d - r' \geq r - r'$.

Now let $l$ be a prime such that $l \neq p$. By abuse of notation, we let $X_0(p)$ denote the standard model of $X_0(p)\mathbb{Q}$ over $\mathbb{Z}[1/p]$, and we let $\tilde{X}_0(p)$ be the mod $l$ reduction of $X_0(p)$, which is again a smooth curve.

By Corollary 3.4 there must exist a nonconstant morphism $X_0(p) \to \mathbb{P}^1$ of degree at most 12 over $\mathbb{Q}$ whenever $\text{Aut} X_0^*(p)$ is nontrivial.

We claim that there necessarily exists a nonconstant morphism of degree at most 12 from $\tilde{X}_0(p)$ to $\mathbb{P}^1_{\mathbb{F}_l}$ defined over $\mathbb{F}_l$. This follows from the following general lemma (c.f. [19], Chapter III, Lemma 2.6):

Lemma 3.6. Let $R$ be a discrete valuation ring with field of fractions $K$, residue field $k$, and uniformizing parameter $\pi$. Let $X$ be a regular scheme which is proper and flat over $\text{Spec}(R)$ whose generic fiber $X_K$ is a smooth and geometrically connected curve over $K$. Let $X_k$ denote the (possibly singular) closed fiber of $X$. Then any line bundle $L_K$ of degree $d$ on $X_K$ extends to a line bundle $L$ on $X$, and the pullback $L_k$ to $X_k$ is a degree $d$ line bundle with $\dim_k H^0(X_K, L_K) \leq \dim_k H^0(X_k, L_k)$.

Proof. With our hypotheses, it follows from [12, II, 6.11] that there is a natural isomorphism between the divisor class group of $X$ and the group $\text{Pic}(X)$. The surjectivity of the natural map $\text{Pic}(X) \to \text{Pic}(X_K)$ then follows from [12, II, 6.5(a)]. (If $X/R$ is smooth then this map is also injective.) Let $\mathcal{L}$ be a line bundle on $X$ extending the degree $d$ line bundle $L_K$ on $X_K$. If we define the degree of a line bundle $\mathcal{L}$ to be $\chi(\mathcal{L}) - \chi(\mathcal{O}_X)$, then this degree is constant on fibers because $X/R$ is flat; by Riemann–Roch this degree must be $d$.

We have $\dim_K H^0(X_K, L_K) = \dim_K H^0(X, \mathcal{L}) \otimes_R K$ because cohomology commutes with the flat base extension $R \to K$ by [12, III, 9.3].

The pullback (restriction) $L_k$ of $\mathcal{L}$ is a line bundle of degree $d$ on $X_k$, and we have $\dim_K H^0(X, \mathcal{L}) \otimes_R K \leq \dim_k H^0(X_k, L_k)$ by the semicontinuity theorem (c.f. [12, III, 12.8]).

Corollary 3.7. Let $X$ be as in the above lemma, and assume furthermore that the closed fiber $X_k$ is smooth. Suppose $X_K$ admits a morphism of degree at most $d$ to $\mathbb{P}^1$ defined over $K$. Then $X_k$ admits a morphism of degree at most $d$ to $\mathbb{P}^1$ defined over $k$.

Proof. This follows directly from the lemma, since if $Y$ is any smooth, proper, geometrically connected curve over a field $k$, then there exists a morphism of degree at most $d$ from $Y$ to $\mathbb{P}^1$ defined over $k$ iff there exists a line bundle $\mathcal{L}$ of degree at most $d$ on $Y$ such that $\dim_k H^0(Y, \mathcal{L}) \geq 2$.

In our situation, since $\sharp \mathbb{P}^1(\mathbb{F}_l) = l^2 + 1$, we see that the number $\sharp \tilde{X}_0(p)(\mathbb{F}_l)$ cannot exceed $12(l^2 + 1)$. On the other hand, Ogg [22] gives the following estimate:

$$\sharp \tilde{X}_0(p)(\mathbb{F}_l) \geq \frac{l - 1}{12}(p + 1) + 2.$$
Therefore, if $\text{Aut} \, X_0^*(p)$ is nontrivial, we must have the inequality
\[12(l^2 + 1) \geq \frac{l - 1}{12} (p + 1) + 2.\]
Setting $l = 2$, we obtain the following:

**Corollary 3.8.** If $p > 695$, then $\text{Aut} \, X_0^*(p)$ is trivial.

The next result follows from Corollary 3.4.

**Corollary 3.9.** If $J_0^*(p)$ has a $\mathbb{Q}$-simple factor of dimension larger than $(g^*(p) + 5)/2$, then $\text{Aut} \, X_0^*(p)$ is trivial.

Proof. Suppose $u$ is a nontrivial automorphism of $X_0^*(p)$, which we know must have order 2. By Corollary 3.4, the number $r$ of fixed points of $u$ is at most 12. Let $X'$ be the quotient of $X_0^*(p)$ by $u$, let $g'$ be its genus, and let $J'$ be the Jacobian of $X'$. Applying the Riemann–Hurwitz formula to the degree 2 map $X_0^*(p) \to X'$, we find that $2g' - 1 \leq g^*(p) \leq 2g' + 5$, so that
\[(1) \quad (g^*(p) - 5)/2 \leq g' \leq (g^*(p) + 1)/2.\]
In particular, if $m > (g^*(p) + 1)/2$ and if $J_0^*(p)$ has a $\mathbb{Q}$-simple factor $A$ of dimension equal to $m$, then $g^*(p) = \dim J_0^*(p) \geq m + \dim J' \geq m + (g^*(p) - 5)/2$, which implies that $m \leq (g^*(p) + 5)/2$ as desired. \[\blacksquare\]

**Remark 3.10.** The $\mathbb{Q}$-simple factors of $J_0^*(N)$ are the same as the $\overline{\mathbb{Q}}$-simple factors of $J_0^*(N)$ whenever $N$ is square-free, since in this case $J_0^*(N)$ is semistable over $\mathbb{Q}$ (c.f. [26]; see also Theorem 2.3).

**Remark 3.11.** Let $N$ be a positive integer. We may compute the $\mathbb{Q}$-simple splitting of $J_0^*(N)$ by factoring characteristic polynomials of Hecke operators acting on the space $S_2^*(N)$ of cuspforms of weight 2 on $\Gamma_0^*(N)$. For instance, suppose we are given an element $T$ of the Hecke ring $\mathbb{Z}[\{T_l\}_{l|N}]$ whose action on $S_2^*(N)$ has square-free characteristic polynomial $\Psi_T(x)$. Let $\Psi_T = \Psi_1 \cdot \Psi_2 \cdots \Psi_s$ be the factorization of $\Psi_T$ over $\mathbb{Q}$.

\[\deg \Psi_1 + \deg \Psi_2 + \cdots + \deg \Psi_s\]
gives the $\mathbb{Q}$-simple splitting of $J_0^*(N)$.

Now consider the case $g^*(p) > 2$. (For a discussion of how to compute $g^*(p)$, see [23, p. 20]). Using the above observation, we find that there are 39 primes in the range $p < 695$ for which $J_0^*(p)$ is not simple (see Table 3). Applying Corollary 3.4 to these cases, we see that $\text{Aut} \, X_0^*(p)$ is trivial except possibly for

\[p = 163, 193, 197, 211, 223, 227, 229, 269, 331, 347, 359, 383, 389, 431, 461, 563, 571, 607.\]

4. Reduction of $X_0^*(p)$ modulo $l$

In this section we consider the reduction of $X_0^*(p)$ and its reduction modulo $l$, where $l$ is a prime such that $l \neq p$. Let $S_2^*(p)$ be the space of cuspforms of weight 2 on $\Gamma_0^*(p)$. For each Hecke-stable subspace $S$ of $S_2^*(p)$, define
\[\nu_{S,v} = 1 + l^n - \text{tr} \, T_{v^n} | S + \begin{cases} l \cdot \text{tr} \, T_{v^n} | S & \text{if } n \geq 2; \\ 0 & \text{otherwise}. \end{cases}\]
Formulas for computing the traces of Hecke operators are given in [15, 30].
Table 1. Simple splitting of $J_0^*(p)$

| $p$  | Splitting | $p$  | Splitting | $p$  | Splitting | $p$  | Splitting |
|------|-----------|------|-----------|------|-----------|------|-----------|
| 163  | 1 + 5     | 347  | 1 + 2 + 7 | 461  | 2 + 3 + 7 | 587  | 5 + 13    |
| 193  | 2 + 5     | 359  | 1 + 1 + 4 | 467  | 1 + 12    | 593  | 1 + 18    |
| 197  | 1 + 5     | 373  | 1 + 12    | 491  | 2 + 10    | 599  | 2 + 11    |
| 211  | 3 + 3     | 383  | 2 + 6     | 499  | 2 + 16    | 607  | 5 + 7 + 7 |
| 223  | 2 + 4     | 389  | 2 + 3 + 6 | 503  | 1 + 10    | 613  | 5 + 18    |
| 227  | 2 + 3     | 397  | 2 + 13    | 523  | 2 + 15    | 647  | 2 + 14    |
| 229  | 1 + 6     | 431  | 1 + 3 + 4 | 547  | 2 + 18    | 653  | 7 + 17    |
| 269  | 1 + 5     | 439  | 2 + 9     | 557  | 1 + 18    | 659  | 1 + 16    |
| 277  | 1 + 9     | 443  | 1 + 1 + 12| 563  | 3 + 3 + 9 | 677  | 1 + 2 + 18|
| 331  | 1 + 3 + 7 | 457  | 2 + 15    | 571  | 3 + 6 + 10|      |           |

We note that since the $\mathbb{Q}$-simple factors of $J_0(p)$ (and hence of $J_0^*(p)$) are Hecke-stable and appear with multiplicity one (see [13, Chapter II, Section 10]), it follows that if $Y$ is any curve of positive genus covered by $X_0^*(p)$, then the image of the Jacobian of $Y$ in $J_0^*(p)$ via Picard functoriality is Hecke-stable. This implies that the subspace $S$ of $S_2^*(p)$ (which is canonically identified with the cotangent space of $J_0^*(p)$ at the origin) corresponding to the cotangent space of $J$ at the origin is Hecke-stable. We say colloquially in this situation that the Jacobian of $Y$ “comes from” $S$.

Before giving the next proposition, we recall the following fact, which is proved, for example, in [13]. Suppose $R$ is a discrete valuation ring with fraction field $K$, and that $f : X \to Y$ is a finite morphism of projective, smooth, geometrically connected curves over $K$. Assume furthermore that $X(K)$ is nonempty. Then if $X$ has good reduction, $Y$ does as well. (We say a curve $X$ as above has good reduction if there exists a scheme of finite type which is proper and smooth over $R$ and whose generic fiber is $X$.)

**Proposition 4.1.** Suppose there is a nontrivial automorphism $u$ of $X_0^*(p)$. (By the remarks at the end of section 2, we may assume that $g^*(p) > 2$, so that $X_0^*(p)$ is not hyperelliptic.) Let $Y = X_0^*(p)/\langle u \rangle$ be the quotient of $X_0^*(p)$ by the action of $u$, which is a curve of genus $g \geq 1$, and let $\bar{Y}$ be the reduction of $Y$ modulo $l$. If the Jacobian $J$ of $Y$ comes from $S$, we must have $\sharp\bar{Y}(\mathbb{F}_l^n) = \nu_{S,l^n}$.

Proof. We can write $\sharp\bar{Y}(\mathbb{F}_l^n) = 1 + l^n - \sum_{i=1}^g (\alpha_i^n + \overline{\alpha}_i^n)$, where the $(\alpha_i, \overline{\alpha}_i)$ are conjugate pairs of eigenvalues of Frobenius acting on the $l'$-adic Tate module of $J$ for any prime $l' \neq l$. On the other hand, since $u$ is defined over $\mathbb{Q}$, the Jacobian of $Y$ may be considered, up to $\mathbb{Q}$-isogeny, as an abelian subvariety over $\mathbb{Q}$ of $J_0^*(p)$. Thus from the Eichler–Shimura congruence relation we conclude that the set $\{\alpha_i + \overline{\alpha}_i\}_{i=1}^g$ gives the eigenvalues of $T_l|S$. Using the standard recurrence relations for the Hecke eigenvalues, we obtain the desired equality.

It follows from this proposition that $S$ cannot correspond to the Jacobian of $Y$ once we know $\nu_{l^n} := \sharp X_0^*(p)(\mathbb{F}_{l^n}) > 2 \cdot \nu_{S,l^n}$ for some prime power $l^n$. Checking this inequality for all $S \neq \{1\}$ such that $(g^*(p) - 5)/2 \leq g_S \leq (g^*(p) + 1)/2$ (see (8) above), where $g_S$ is the dimension of $S$, we see that $\text{Aut } X_0^*(p)$ is trivial for $p = 163, 193, 197, 277, 331, 347, 359, 383, 461, 563, 607$ (Table 3).
The quotient of $\mathcal{X}_0(p)$ by $w_p$ is an arithmetic surface $\mathcal{X}^*_0(p)$ whose generic fiber is the smooth curve $\mathcal{X}^*_0(p)$, and whose closed fiber $\mathcal{X}^*_0(p)$ is a copy of the $j$-line that intersects itself transversely at the $g^* = g^*(p)$ geometric points corresponding to conjugate pairs of supersingular elliptic curves defined over $\mathbf{F}_{p^2}$ but not $\mathbf{F}_p$. The fact that the closed fiber of $\mathcal{X}^*_0(p) = \mathcal{X}_0(p)/w_p$ is the quotient of $\mathcal{X}_0(p)$ by the action of $w_p$ follows from the discussion in [16, Appendix A7], especially Prop. A7.1.3.

One can see that the fixed points of $\tilde{w}_p$ on $\mathcal{X}_0(p)$ get mapped to smooth points of $\mathcal{X}^*_0(p)$ by using the fact that the components $Y_1$ and $Y_2$ of $\mathcal{X}^*_0(p)$ intersect transversely, plus the following local calculation: if $k$ is any field, the subring of $k[[x,y]]/(xy)$ consisting of all elements left invariant by the automorphism which interchanges $x$ and $y$ is isomorphic to $k[[t]]$, a power series ring in one variable.

We assume for the rest of this section that $g^*(p) > 0$.

Note that if $p \not\equiv 1 \pmod{12}$, then $\mathcal{X}_0(p)$ is not regular: the supersingular points corresponding to elliptic curves with extra automorphisms, i.e., to elliptic curves with $j$-invariant 0 or 1728, are not regular. However, the quotient of $\mathcal{X}_0(p)$ by $w_p$ behaves better in this respect:

\[\text{Specifically, we are using the fact that (a) the order 2 of the group generated by } w_p \text{ is invertible in } \mathbf{F}_p \text{ since } p \not\equiv 2, \text{ and (b) the order of all stabilizer groups of all geometric points of } \mathcal{X}_0(p) \text{ are invertible in } \mathbf{F}_p \text{ since } p \not\equiv 2,3. \text{ The reason we need (b) is that the Deligne–Rapoport model is only a coarse moduli scheme, but one can show under the hypothesis in (b) that formation of the coarse moduli scheme associated to an algebraic stack commutes with the base change in question.}\]
Lemma 5.1. The arithmetic surface $X_0^*(p)$ is the minimal proper regular model over $\mathbb{Z}_{(p)}$ for $X_0^*(p)$.

Proof. We first claim that $X_0^*(p)$ is indeed regular. Note that $X_0^*(p)$ is normal and its closed fiber is generically smooth. Since the generic fiber $X_0^*(p)$ is smooth, the only points which are potentially not regular are the finitely many singular points on the closed fiber. Let $s \in \tilde{X}_0^*(p)(\mathbb{F}_{p^2})$ be such a point, corresponding to a supersingular $j$-invariant $j(s)$ defined over $\mathbb{F}_{p^2}$ but not $\mathbb{F}_p$. In particular, $j(s)$ is not 0 or 1728, and therefore $\pi^{-1}(s)$ consists of two distinct regular points $s_1, s_2$ of $X_0^*(p)$, where $\pi$ is the canonical map from $X_0^*(p)$ to $X_0^*(p)$. Since $s_1$ and $s_2$ are not fixed points of $\omega_p$, it follows from [16, Thm. A7.1.1] that $\pi$ is étale over $s$. Therefore $s$, being the image of a regular point by an étale map, is regular.

To see that $X_0^*(p)$ is minimal, it suffices by Castelnuovo’s criterion ([4, Thm. 3.1]) to note that $X_0^*(p)$ has no exceptional divisors in the sense of [3, Def. 1.3]. This follows immediately from the fact that the special fiber is irreducible, and therefore by [4, Cor. 4.1] has self-intersection zero.

Let $u$ be an automorphism of $X_0^*(p)$. By the defining property of the fact that $X_0^*(p)$ is a minimal proper regular model (see [4, Def. 1.1]), $u$ extends to an automorphism of the scheme $X_0^*(p)$. It therefore induces an automorphism $\tilde{u}$ of the special fiber $\tilde{X}_0^*(p)$ and an automorphism $\alpha$ of the normalization $\mathbb{P}_{\tilde{X}_0^*(p)}^1$ which permutes the set of $2g^*$ supersingular geometric points defined over $\tilde{F}_{p^2}$ but not $\tilde{F}_p$.

We claim that if $u$ is not the identity map on the generic fiber $X_0^*(p)$, then $\tilde{u}$ is not the identity on the special fiber.

This follows from Lemma 2.1 and the fact that $J_0^*(p)$ has semistable reduction at $p$, together with the following two basic results:

Theorem 5.2 (Raynaud). Let $R$ be a discrete valuation ring with fraction field $K$. Let $X/R$ be a proper, flat, and regular curve whose generic fiber is smooth and geometrically irreducible and whose closed fiber is geometrically reduced. Let $X$ be the generic fiber of $X$, and let $J$ be the Néron model of the Jacobian of $X$. Then the connected component $J_0$ of the identity in $J$ coincides naturally with the relative Picard scheme $\text{Pic}^0_{X/R}$.

In particular, every automorphism $\psi$ of $X$ induces an endomorphism of $J_0$, and if $\psi$ is the identity map on the special fiber of $X$ then it induces the identity map on the connected component of the closed fiber of $J$.

Proof. This is a special case of [3, Theorem 4, Section 9.5].

Lemma 5.3. If $G$ is a semiabelian group scheme over a discrete valuation ring $R$, then an endomorphism of $G$ inducing the identity map on the special fiber is necessarily the identity.

Proof. We may assume without loss of generality that $R$ is complete, and we denote by $m$ its maximal ideal.

Let $u \in \text{End} G$ be such that $u_s$ (the induced map on the closed fiber $G_s$ of $G$) is the identity map. We want to show that $u = 1$ on all of $G$. It is more convenient to work with the endomorphism $v := u - 1$ which satisfies $v_s = 0$. We want to show that $v = 0$.

Fix an auxiliary prime $l$ not equal to the residue characteristic of $R$, and let $H_n = G[l^n]$ for each integer $n \geq 1$. Since $G$ is semiabelian, it follows from [3, Lemma 2, Section 7.3] that $H_n$ is a quasi-finite, flat, étale, and separated group scheme over $R$ for all $n$. Since $R$ is complete, and therefore henselian, we may consider, for each $n$, the
finite part \((H_n)^f\) of \(H_n\), which is a finite étale open and closed subscheme of \(H_n\) having the same closed fiber as \(H_n\) (see [8, Section 7.3, p. 179]). As \(v_s = 0\) on the closed fiber of \((H_n)^f\), and \((H_n)^f/R\) is finite étale, it follows that \(v = 0\) on \((H_n)^f\).

In particular, \(v = 0\) on all infinitesimal closed fibers of \((H_n)^f\) (i.e., on \((H_n)^f \times_R R/m^k\) for all integers \(k \geq 1\)). We claim that \(v = 0\) on all infinitesimal closed fibers of \(G\). To see this, note that the infinitesimal closed fibers of \((H_n)^f\) and of \(H_n\) are the same by the construction of \((H_n)^f\). Also, since \(G_s\) is semiabelian, the collection of subschemes \((-\{H_n\}_s\)} is topologically dense in \(G_s\). It follows by [10, Proposition 11.10.1, Theorem 11.10.9] that the collection of subschemes induced by \(-\{H_n\}\) is schematically dense in each infinitesimal closed fiber of \(G\). This proves our claim that \(v = 0\) on each infinitesimal closed fiber of \(G/R\).

It now follows that \(v\) induces the zero map on the formal completion of \(G\) along the identity section. Therefore \(v = 0\) on the generic fiber of \(G\), and since \(G\) is flat over \(R\), it follows that \(v = 0\) on \(G\) as desired.

Denote by \(Y\) the normalization of \(\widetilde{X}_0^s(p)\), which as we have seen is isomorphic to \(\mathbb{P}_F^1\), and let \(\{e_i, e'_i\}\) be the conjugate pairs of \(j\)-values of supersingular elliptic curves which are identified in the map \(Y \to \widetilde{X}_0^s(p)\) \((i = 1, \ldots, g^*)\).

We conclude from our above discussion that if there is a nontrivial element \(u \in \text{Aut}X_0^s(p)\), then there is an element \(\alpha \in \text{Aut}(\mathbb{P}_F^1) = \text{PGL}(2, \mathbb{F}_p)\) of order 2 which permutes the set \(\{e_i, e'_i\}\) of \(2g^*\) supersingular \(j\)-invariants which are not \(\mathbb{F}_p\)-rational. Now if \(g \geq 3\) is an integer, then there is no nontrivial automorphism of \(\mathbb{P}^1\) permuting a general set of \(2g\) points. So if \(g^* \geq 3\), it is unlikely that an automorphism \(\alpha\) as above exists; in any case, for each specific value of \(p\) one can compute whether or not such an \(\alpha\) exists, since the set \(\{e_i, e'_i\}\) can be computed explicitly (see Table 3).

We therefore check by an explicit computation that there is no such element \(\alpha \in \text{PGL}(2, \mathbb{F}_p)\) for the remaining seven cases \(p = 211, 223, 227, 229, 389, 431, 571\). It follows that \(\text{Aut}X_0^s(p) = \{1\}\) for these values of \(p\). This concludes our proof of Theorem 1.1.

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Table 3. Quadratic supersingular $j$-invariants for the remaining cases

| $p$ | Quadratic supersingular $j$-invariants |
|-----|--------------------------------------|
| 211 | $(j^2+162j+146)(j^2+186j+97)(j^2+56j+23)$  
|     | $\times (j^2+152j+88)(j^2+184j+86)(j^2+121j+206)$ |
| 223 | $(j^2+87j+193)(j^2+198j+7)(j^2+197j+1)$  
|     | $\times (j^2+174j+49)(j^2+137j+95)(j^2+12j+54)$ |
| 227 | $(j^2+168j+208)(j^2+102j+201)(j^2+81j+63)(j^2+73j+163)(j^2+223j+186)$ |
| 229 | $(j^2+55j+175)(j^2+14j+84)(j^2+162j+16)(j^2+32j+86)$  
|     | $\times (j^2+51j+122)(j^2+20j+2)(j^2+63j+216)$ |
| 389 | $(j^2+6j+326)(j^2+233j+57)(j^2+43j+344)(j^2+93j+99)(j^2+210j+377)(j^2+240)$  
|     | $\times (j^2+91j+374)(j^2+27j+173)(j^2+293j+337)(j^2+165j+59)(j^2+84j+36)$ |
| 431 | $(j^2+195j+282)(j^2+104j+148)(j^2+301j+11)(j^2+232j+279)$  
|     | $\times (j^2+149j+95)(j^2+51j+138)(j^2+10j+134)(j^2+254j+361)$ |
| 571 | $(j^2+513j+217)(j^2+34j+160)(j^2+319j+194)(j^2+38j+51)(j^2+284j+540)$  
|     | $\times (j^2+12j+355)(j^2+472j+23)(j^2+502j+522)(j^2+322j+545)(j^2+25j+23)$  
|     | $\times (j^2+307j+466)(j^2+455j+306)(j^2+326j+7)(j^2+154j+51)(j^2+30j+38)$  
|     | $\times (j^2+222j+354)(j^2+116j+446)(j^2+162j+68)(j^2+212j+138)$ |

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