The associated map of the nonabelian Gauss–Manin connection

Ting Chen

1 Department of Mathematics, University of Pennsylvania, 209 South 33rd Street, Philadelphia, PA 19104, USA

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Abstract: The Gauss–Manin connection for nonabelian cohomology spaces is the isomonodromy flow. We write down explicitly the vector fields of the isomonodromy flow and calculate its induced vector fields on the associated graded space of the nonabelian Hodge filtration. The result turns out to be intimately related to the quadratic part of the Hitchin map.

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1. Introduction

The variation of Hodge structures for families of complex Kähler manifolds has been a much studied subject. Let \( \pi: X \to S \) be a proper holomorphic submersion of connected complex manifolds. Ehresmann’s lemma says that it is a locally trivial fiber bundle with respect to its underlying differentiable structure. In particular all the fibers of \( \pi \) are diffeomorphic. So if \( s \in S \) and \( X_s \) is the fiber of \( \pi \) over \( s \), \( X \to S \) can be viewed as a variation over \( S \) of complex structures on the underlying differentiable manifold of \( X_s \).

Let \( \pi^k: V \to S \) be the corresponding vector bundle of cohomologies whose fiber at \( s \in S \) is \( H^k(X_s, \mathbb{C}) \), \( k \in \mathbb{N} \). Since \( X \to S \) is locally trivial differentiably (and therefore topologically), there is an induced local identification of fibers of \( V \to S \). In other words, there is a flat connection on the vector bundle \( V \to S \). This connection is called the Gauss–Manin connection for the cohomologies of the family of complex Kähler manifolds \( X \to S \).

From Hodge theory we know there is a natural Hodge filtration on the vector bundle \( V \to S \): \( V = F^0 \supset F^1 \supset F^2 \supset \ldots \supset F^k \). Let \( \nabla \) be the Gauss–Manin connection, Griffiths transversality theorem says that \( \nabla F^p \subset F^{p-1} \otimes \Omega^1_S \).
So if \( \text{gr } V \) is the associated graded vector bundle of the filtered bundle \( V \), then the induced map \( \text{gr } \nabla \) of \( \nabla \) on \( \text{gr } V \) will be \( \mathcal{O}_X \)-linear. In fact, \( \text{gr } \nabla \) is equal to a certain Kodaira–Spencer map \([2]\). We call \( \text{gr } \nabla \) the \textit{associated map of the Gauss–Manin connection}.

The above has a nonabelian analogue. Let \( G \) be a complex algebraic group, \( X \) a smooth algebraic curve over \( \mathbb{C} \) of genus \( g \). Let \( \text{Conn}_X \) be the moduli space of principal \( G \)-bundles over \( X \) equipped with a flat connection. If we denote as \( H^1(X, G) \) the first řČech cohomology of \( X \) with coefficients the constant sheaf in \( G \), then \( \text{Conn}_X \) can be naturally identified with \( H^1(X, G) \), by considering the gluing data of flat \( G \)-bundles. Since the group \( G \) can be nonabelian, we call \( \text{Conn}_X \) the nonabelian cohomology space of \( X \).

Let \( M_g \) be the moduli space of genus \( g \) complex algebraic curves. The universal curve \( X \rightarrow M_g \) is (roughly) a variation of complex structures of the underlying real surface, and the universal moduli space of connections \( \text{Conn} \rightarrow M_g \) is the corresponding bundle of nonabelian cohomologies. For the same reason as before there is a Gauss–Manin connection on the bundle \( \text{Conn} \rightarrow M_g \). The local trivialization that defines it is often called the isomonodromy deformation, or the isomonodromy flow of \( \text{Conn} \) over \( M_g \).

There is also a nonabelian analogue of Hodge filtration which was determined by Carlos Simpson \([7]\), using a generalized definition of filtration of spaces. A vector space with filtration is equivalent, by the Rees construction, to a locally free sheaf over \( \mathbb{C} \) with a \( \mathbb{C}^\ast \)-action, and with the fiber over 1 being the vector space itself. To define the nonabelian Hodge “filtration” on the space \( \text{Conn} \) therefore, it would be reasonable to find a family of spaces over \( \mathbb{C} \) whose fiber over 1 is \( \text{Conn} \), together with a \( \mathbb{C}^\ast \)-action on the family. The way to do it in this case is to introduce the notion of \( \lambda \)-connections on a principal \( G \)-bundle on \( X \), for any \( \lambda \in \mathbb{C} \). It is a generalization of the notion of connections on a \( G \)-bundle. In particular a 1-connection is an ordinary connection, and a 0-connection is a so-called Higgs field, which is an object of much interest to people in complex geometry and high energy physics. The moduli space of principal \( G \)-bundles over \( X \) together with a Higgs field is called the Higgs moduli space over \( X \), and denoted as \( \text{Higgs}_X \). Simpson’s definition of nonabelian Hodge filtration immediately implies that the associated graded space of \( \text{Conn}_X \) is \( \text{Higgs}_X \). Then a question arises: what is the associated map of the nonabelian Gauss–Manin connection on the associated graded space? The answer is: it is a lifting\(^1\) of tangent vectors on the relative Higgs moduli space \( \text{Higgs} \rightarrow M_g \). On the other hand there is a well-known Hitchin map from \( \text{Higgs}_X \) to some vector spaces. The quadratic part of the Hitchin maps also induces a lifting of tangent vectors on \( \text{Higgs} \rightarrow M_g \). The fact that the two liftings agree is the content of our theorem.

**Theorem 1.1.**

The lifting of tangent vectors on \( \text{Higgs} \rightarrow M_g \) representing the associated map of the nonabelian Gauss–Manin connection is equal up to a constant multiple to the lifting of tangent vectors induced from the quadratic Hitchin map.

Closely related results have been obtained in \([1]\), where the authors apply localization for vertex algebras to the Segal–Sugawara construction of an internal action of the Virasoro algebra on affine Kac–Moody algebras to lift twisted differential operators from the moduli of curves to the moduli of curves with bundles. Their construction gives a uniform approach to several phenomena describing the geometry of the moduli spaces of bundles over varying curves, including a Hamiltonian description of the isomonodromy equations in terms of the quadratic part of Hitchin’s system. Our result and proof are much more elementary, avoiding the need for the vertex algebra machinery.

The organization of the paper is as follows. In Section 2 we give a detailed definition of all the objects concerned and a precise statement of the theorem. The rest of the sections are devoted to the proof. In Section 3 we recall the definition of Atiyah bundles and some of its properties that will be useful in the proof. In Section 4 we use deformation theory to write the tangent spaces to \( \text{Conn} \) as certain hypercohomology spaces. Section 5 gives an explicit description of the lifting of tangent vector on \( \text{Conn} \rightarrow M_g \) given by the isomonodromy flow. Section 6 extends the isomonodromy lifting to the moduli space of \( \lambda \)-connections for any \( \lambda \neq 0 \). Finally Section 7 takes the limit of the lifting at \( \lambda = 0 \), which is

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\(^1\) Here the word lifting has a slightly more general meaning: it means a map of tangent vectors in the opposite direction of the pushforward, without requiring its composition with pushforward being identity. In fact, the lifting here composed with pushforward is zero.
precisely the associated map of the nonabelian Gauss–Manin connection, and shows that it is equal up to a constant to
the quadratic Hitchin lifting of tangent vectors.

2. Definitions and statement of the theorem

All objects and morphisms in this paper will be algebraic over \( \mathbb{C} \), unless otherwise mentioned.

2.1. Moduli space of connections and isomonodromy flow

Let \( g \) be a natural number greater or equal to 2, so that a generic curve of genus \( g \) has no automorphisms. The
moduli space of all genus \( g \) curves is a smooth Deligne–Mumford stack, but if we restrict to the curves that have no
automorphisms, the moduli space is actually a smooth scheme. Let \( M_g \) be this scheme. In this paper we will ignore all
the special loci of the moduli spaces (as explained below) and focus on local behaviors around generic points.

Let \( G \) be a semisimple Lie group, \( X \) a smooth curve of genus \( g \). Let \( \text{Bun}_X \) be the coarse moduli space of regular stable
\( G \)-bundles on \( X \). \( \text{Bun}_X \) is also a smooth scheme [6]. The total space of the cotangent bundle \( T^* \text{Bun}_X \) is an open
subscheme of the Higgs moduli space over \( X \) [3]. However, since we are only concerned with generic situations, we will
use Higgs to denote the open subscheme \( T^* \text{Bun}_X \).

Let \( \text{Conn}_X \) be the moduli space of pairs \((\mathcal{P}, \nabla)\), where \( \mathcal{P} \) is a stable \( G \)-bundle on \( X \), and \( \nabla \) is a connection on \( \mathcal{P} \). \( \nabla \) is
necessarily flat as the dimension of \( X \) is equal to 1. \( \text{Conn}_X \) is an affine bundle on \( \text{Bun}_X \) whose fiber over \( \mathcal{P} \in \text{Bun}_X \) is
a torsor for \( T^*_\mathcal{P} \text{Bun}_X \). So it is also a smooth scheme.

Let \( \text{Conn} \rightarrow M_g \) be the relative moduli space of pairs whose fiber at \( X \in M_g \) is \( \text{Conn}_X \). Let \( \text{Irrep}_X \) be the space
of all irreducible group homomorphisms \( \pi_1(X) \rightarrow G \), \( \text{Irrep}_X \) is a smooth scheme [4]. There is also the relative space
\( \text{Irrep} \rightarrow M_g \). The Riemann–Hilbert correspondence \( RH : \text{Conn}_S \rightarrow \text{Irrep}_X \) taking a flat connection to its monodromy
is an analytic (and therefore differentiable) inclusion. Let \( S \subseteq M_g \) be a small neighborhood of \( X \) in analytic topology.
By Ehresmann’s lemma the family of curves over \( S \) is a trivial family with respect to the differentiable structure. This
implies that the restriction of \( \text{Irrep} \) over \( S \) is a differentiable trivial family. The trivial sections or trivial flows induce a
flow on the restriction of \( \text{Conn} \) over \( S \), by the Riemann–Hilbert correspondence. This flow is called the isomonodromy
flow of \( \text{Conn} \) over \( M_g \).

2.2. \( \lambda \)-connections and nonabelian Hodge filtration

As explained in the last section, \( \text{Conn}_X \) is the nonabelian cohomology space of \( X \) with coefficient in \( G \), and the isomonodromy flow on \( \text{Conn} \rightarrow M_g \) is the nonabelian Gauss–Manin connection. To define Hodge filtration on \( \text{Conn}_X \) one need
to generalize the definition of a filtration. A filtration on a vector space \( V \) is equivalent, by the Rees construction [3], to
a locally free sheaf \( W \) on \( \mathbb{C} \) whose fiber at \( 1 \in \mathbb{C} \) is isomorphic to \( V \), together with a \( \mathbb{C}^* \)-action on \( W \) compatible with
the usual \( \mathbb{C}^* \)-action on \( C \). The fiber of \( W \) at \( 0 \in \mathbb{C} \) will be isomorphic to the associated graded vector space of \( V \).

This sheaf definition of filtrations can be generalized in an obvious way to define filtrations on a space that is not a
vector space. In our case the space is \( \text{Conn}_X \), and its Hodge filtration is constructed as follows. \( \text{Conn}_X \) parametrizes
pairs \((\mathcal{P}, \nabla)\). Let \( \mathcal{P} \) also denote the sheaf of sections of \( \mathcal{P} \) on \( X \), \( \text{ad} \mathcal{P} \) be the adjoint bundle of \( \mathcal{P} \) as well as the sheaf
of its sections, and \( g \) the Lie algebra of \( G \). A connection \( \nabla \) is a map of sheaves

\[
\nabla : \mathcal{P} \rightarrow \text{ad} \mathcal{P} \otimes \Omega^1_X
\]

that after choosing local coordinates for \( X \) and local trivialization for \( \mathcal{P} \) can be written as

\[
\left( \frac{\partial}{\partial x} + [A(x), \cdot] \right) \otimes dx,
\]
The quadratic Hitchin map is $Higgs$. 2.3. Quadratic Hitchin map and statement of the theorem

Let the moduli space of $\lambda$-connections be denoted as $\lambda \text{Conn}_X$. For $\lambda \neq 0$, $\nabla \leftrightarrow \lambda \cdot \nabla$ is a bijection between $\text{Conn}_X$ and $\lambda \text{Conn}_X$. For $\lambda = 0$, the definition of a 0-connection agrees with that of a Higgs field. So $0 \text{Conn}_X$ is just Higgs$_X$.

Let $T_X$ be the moduli space of all $\lambda$-connections for all $\lambda \in \mathbb{C}$. There is a natural map $T_X \to \mathbb{C}$ taking a $\lambda$-connection to $\lambda$, whose preimage at $1 \in \mathbb{C}$ is $\text{Conn}_X$. In fact, Simpson showed that the nonabelian Hodge filtration of $\text{Conn}_X$ is precisely the sheaf of sections of this map, with the $C^*$-action given by multiplication by $\lambda$ for $\lambda \in \mathbb{C}^*$ [3]. The $C^*$-action is algebraic and induces an isomorphism of $\text{Conn}_X$ and $\lambda \text{Conn}_X$.

In the ordinary Hodge theory, if one uses the sheaf definition of filtrations, then the associated map of the Gauss–Manin connection is obtained as follows. Start with the Gauss–Manin connection on $M$. Let the moduli space of $\lambda$-connections be denoted as $\lambda \text{Conn}_X$ and defines a lifting $L$, which takes $\lambda \in \mathbb{C}$ and $X \in \text{Conn}_X$ and defines a lifting $L\lambda$ of $\lambda \text{Conn}_X$. For a fixed vector $\vec{t} \in T_sS$, the images of $\vec{t}$ under all the $L_{\lambda}$, $\lambda \neq 0$, give a vector field on the total space of $\mathcal{W}$ away from $V_0$ which is the fiber over $0 \in \mathbb{C}$. The continuous limit of that vector field on $V_0$ exists, and therefore defines a lifting $L_0$: $T_sS \to T_0V_0$ on $V_0 \to S$. $L_0$ satisfies

$$\pi^k_0 \circ L_0 = 0;$$

i.e., the image of $\vec{t} \in T_sS$ under $L_0$ is a vector field on the fiber $V_0$, of $V_0$ over $s$. This vector field is in fact linear and defines a linear map on $V_0$. Also $V_0$ is identified with $\text{gr} \mathcal{V}$. From this we see $L_0$ really gives a vector bundle map $\text{gr} \mathcal{V} \to \text{gr} \mathcal{V} \otimes \Omega^1_s$, and this map is the associated map of Gauss–Manin that we started with.

So in nonabelian Hodge theory, in order to calculate the associated map of the nonabelian Gauss–Manin connection, we will start with the lifting $L$ induced from the isomonodromy flow on $\text{Conn} \to \mathcal{M}_g$ (by a slight abuse of notation we will use the same notations for the liftings, the meaning should be clear from the context), and try to find the associated limit lifting $L_0$. Specifically, let $\mathcal{T} \to \mathcal{M}_g$ be the relative moduli space whose fiber at $X \in \mathcal{M}_g$ is $T_X$. $\mathcal{T}$ maps to $\mathbb{C}$ and the fiber at $\lambda$ is the relative moduli space of $\lambda$-connections, which is denoted as $\lambda \text{Conn}$. There is clearly also a $C^*$-action on $\mathcal{T}$ compatible with the $C^*$-action on $\mathbb{C}$. Let $L_{\lambda}$ be analogously the lifting on $\lambda \text{Conn} \to \mathcal{M}_g$ induced by the lifting $L$ via the $C^*$-action and multiplied by $\lambda$. Then the limit lifting $L_0$ will be the associated map that we want to calculate. It will again be a vertical lifting, i.e. the images of $L_0$ will be vectors tangent to the fibers $\text{Higgs}_X$ of $\text{Higgs} \to \mathcal{M}_g$, $X \in \mathcal{M}_g$.

2.3. Quadratic Hitchin map and statement of the theorem

$\text{Higgs}_X$ has a symplectic structure as it is equal to $T^* \text{Bun}_X$. Let $\langle \cdot , \cdot \rangle$ be the Killing form on the Lie algebra $g$ of $G$, the quadratic Hitchin map is

$$q_{\text{h}}: \text{Higgs}_X \to \mathcal{H}(X, \Omega^{2d}), \quad (P, \theta) \mapsto (\theta, \theta),$$
where $\theta \in H^0\{X, \text{ad} P \otimes \Omega^1\}$ is a 0-connection or a Higgs field. One can define a lifting of tangent vectors associated to $\text{qh}$,

$$L_{\text{qh}}: T_X M \to T (P, \theta) \text{Higgs}_X, \quad f \mapsto H_{\text{qh}^*f}|_{P, \theta},$$

where $f \in T_X M$ is viewed as a linear function on $H^0\{X, \Omega^{\text{ad}}\}$ by Serre duality, and $H_{\text{qh}^*f}$ is the Hamiltonian vector field of $\text{qh}^*f$ on $\text{Higgs}_X$.

The theorem can now be stated more precisely as:

**Theorem 2.1 (precise version of Theorem 1.1).**

The limit lifting of tangent vectors $L_0$ associated to the isomonodromy lifting $L$ is equal to $L_{\text{qh}}/2$.

### 3. Atiyah bundles

Before starting to prove the theorem, we recall here some facts about Atiyah bundles which will be used later. As before let $X$ be a smooth curve of genus $g$, $G$ a semisimple Lie group, $p: P \to X$ a principal $G$-bundle over $X$.

#### 3.1. Atiyah bundle and its sections

Let $TP$ be the tangent bundle over $P$. $G$ acts on $P$ and has an induced action on $TP$. The action is free and compatible with the vector bundle structure of $TP \to P$, so the quotient will be a vector bundle $TP/G \to P/G = X$. This vector bundle over $X$ is called the Atiyah bundle associated to $P$, and denoted as $A_P$.

In fact, $TP$ is isomorphic to the fiber product of $P$ and $A_P$ over $X$. So any section $t$ of $A_P$ over $X$ has a unique lift $\tilde{t}$ that makes the diagram commute:

\[
\begin{array}{ccc}
TP & \xrightarrow{\iota_G} & A_P \\
\downarrow \iota & & \downarrow \iota \\
P & \xrightarrow{p} & X
\end{array}
\]

The lift $\tilde{t}$ can be viewed as a vector field on $P$ which is $G$-invariant. Conversely, any $G$-invariant vector field on $P$ defines a section $t$ in the quotient bundle. Therefore sections of $A_P$ over $X$ are the same as $G$-invariant vector fields on $P$.

#### 3.2. Atiyah sequence

The sequence of tangent bundles associated to $P \to X$ is

$$0 \to T_{P/X} \to TP \to p^*TX \to 0.$$

$G$ acts on the sequence, and the quotient is

$$0 \to \text{ad} P \to A_P \to TX \to 0.$$

This quotient sequence is called the Atiyah sequence of $A_P$. We will denote the map $A_P \to TX$ also as $p_*$. 


3.3. Relation to connections

If \( \nabla \) is a connection on \( P \), then \( \nabla \) must be flat since the dimension of \( X \) is 1. So over a small open subset \( U \subset X \), there is a natural trivialization of \( P \) associated to \( \nabla \),

\[
\tau: U \times F \to P|_U,
\]
given by the flat sections of \( \nabla \). Here \( F \) denotes a torsor for \( G \).

The local trivialization gives a local section \( \tilde{s}_U: p^*TU \to TP|_U \), which is the composition

\[
p^*TU \xrightarrow{\tau^{-1}_U} p^*_U TU \xrightarrow{(id,0)} p^*_U TU \oplus p^*_F TF \xrightarrow{\tau} TP|_U,
\]
where \( p^*_U \) and \( p^*_F \) are the projections of \( U \times F \) to \( U \) and \( F \).

Since \( \tilde{s}_U \) is canonically associated to \( \nabla \), for two such open subsets \( U \) and \( V \), \( \tilde{s}_U \) and \( \tilde{s}_V \) agree on their intersection. So there is a well-defined map \( \tilde{s}: p^*TU \to TP \). Since \( \tau \) is \( G \)-invariant and the map \( (id,0) \) is obviously \( G \)-invariant, \( \tilde{s}_U \) is \( G \)-invariant. So \( \tilde{s} \) is \( G \)-invariant and gives a map \( s: TX \to AP \). The map \( (id,0) \) in the definition of \( \tilde{s}_U \) implies that \( s \) is a splitting of \( p_\ast: AP \to TX \), i.e. \( p_\ast \circ s = id_{TX} \). We can also say that \( s \) is a splitting of the Atiyah sequence

\[
0 \to \text{ad} P \xrightarrow{\rho_\ast} AP \xrightarrow{s} TX \xrightarrow{0}.
\]

To summarize, for any connection \( \nabla \) on \( P \) there is a uniquely associated splitting \( s \) of the Atiyah sequence of \( P \). \( s \) is locally defined as the splitting \((id,0)\) with \( P \) (and therefore \( AP \)) locally trivialized by \( \nabla \).

4. Tangent spaces

Now we start to prove the theorem. In this section we will identify the tangent spaces of \( \mathcal{C}om\), and more generally \( \lambda \mathcal{C}om \), as some hypercohomology spaces, so that we may write down the isomonodromy lifting \( L \) and the extended liftings \( L_\lambda \) explicitly in the next two sections.

The tangent space to a moduli space at a regular point is identified with the infinitesimal deformations of the object corresponding to that point. So we are in fact looking at infinitesimal deformations of the objects parametrized by \( \mathcal{C}om \), which are triples \((X, P, \nabla)\). We start with deformations of pairs \((X, P)\).

4.1. Deformation of pairs

From deformation theory the following two propositions are well-known.

**Proposition 4.1.**
The tangent space to \( \mathcal{M}_g \) at a point \( X \) is naturally isomorphic to \( H^0(X, TX) \).

**Proposition 4.2.**
The tangent space to \( \text{Bun}_X \) at a point \( P \) is naturally isomorphic to \( H^1(X, \text{ad} P) \).

Let \( \text{Bun} \) be the moduli space of pairs \((X, P)\). We expect that generically the tangent space at a point \((X, P)\) would satisfy

\[
0 \to H^1(X, \text{ad} P) \to T_{(X, P)} \mathcal{B}un \to H^0(X, TX) \to 0.
\]
On the other hand, since the Atiyah sequence of $P$, $0 \to \text{ad } P \to A_P \to TX \to 0$, induces

$$0 \to H^1(X, \text{ad } P) \to H^1(X, A_P) \to H^1(X, TX) \to 0,$$

it is natural to guess that

**Proposition 4.3.**

$T_{X, P} \text{ Bun}$ is naturally isomorphic to $H^1(X, A_P)$.

**Proof.** The proof is a combination of standard proofs for Proposition 4.1 and Proposition 4.2. Let $\{U_i\}_{i \in I}$ be a Čech covering of $X$, $P_{\varepsilon} \to X_\varepsilon$ a family of principal $G$-bundles over $D_\varepsilon = \mathbb{C}[\varepsilon]/(\varepsilon^2)$, which restricts to $P \to X$ over the closed point. Over each $U_i$, let

$$\phi_i : P|_{U_i} \times D_\varepsilon \to P_{\varepsilon} \big|_{U_i} \left( \phi_i^\vee : O_{P|_{U_i}} \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \hookrightarrow O_{P_{\varepsilon}|_{U_i}} \right)$$

be an isomorphism of $G$-bundles. So it is compatible with the $G$-actions and descends to an isomorphism

$$\hat{\iota}_i : U_i \times D_\varepsilon \to X_{\varepsilon} \left( \iota_i^\vee : O_{U_i} \otimes \mathbb{C}[\varepsilon]/(\varepsilon^2) \hookrightarrow O_{X_{\varepsilon}|_{U_i}} \right).$$

Over $U_{ij} = U_i \cap U_j$, the transition functions are related as in the commutative diagram

$$\begin{array}{ccc}
P|_{U_{ij}} \times D_\varepsilon & \xrightarrow{\phi_i^\vee \circ \phi_j^\vee} & P|_{U_{ij}} \times D_\varepsilon \\
\rho & & \rho \\
U_{ij} \times D_\varepsilon & \xrightarrow{\hat{\iota}_i^\vee \circ \hat{\iota}_j^\vee} & U_{ij} \times D_\varepsilon.
\end{array}$$

Let $\xi_{ij} \in \Gamma(U_{ij}, TX)$ be the vector field on $U_{ij}$ such that $(\iota_i^{-1} \circ \iota_j)^\vee = \text{Id} + \varepsilon \xi_{ij}$, and $\eta_{ij} \in \Gamma(P|_{U_{ij}}, TP)$ be the vector field on $P|_{U_{ij}}$ such that $(\phi_j^{-1} \circ \phi_i)^\vee = \text{Id} + \varepsilon \eta_{ij}$. Because $\phi_i$ is $G$-invariant, $\eta_{ij}$ is $G$-invariant. So one can view it as $\eta_{ij} \in \Gamma(U_{ij}, A_P)$. $(\eta_{ij})_{i,j \in I}$ forms a Čech 1-cocycle on $X$ with coefficients in $A_P$.

$(\eta_{ij})_{i,j \in I}$ is closed because it comes from transition functions $\phi_j^{-1} \circ \phi_i$. Any closed cochain $(\eta_{ij})_{i,j \in I}$ comes from some family $D_\varepsilon$ of pairs. Also for a fixed family $D_\varepsilon$ of pairs, a different choice of $\phi_i$’s will result in a cocycle differing from $(\eta_{ij})_{i,j \in I}$ by an exact cocycle. And any exact cocycle is the result of different choices of $\phi_i$’s. Therefore the infinitesimal deformations of $(X, P)$ are in natural correspondence with $H^1(X, A_P)$, which proves the proposition.

### 4.2. Deformation of triples

Now we come to the infinitesimal deformations of a triple $(X, P, \nabla)$. First a notation related to the connection $\nabla$. As discussed in subsection 3.3, a connection $\nabla$ on $P$ is equivalent to a splitting of the Atiyah sequence

$$0 \longrightarrow \text{ad } P \longrightarrow A_P \overset{P_\varepsilon}{\longrightarrow} TX \longrightarrow 0.$$ 

Let $s \in H^0(X, A_P \otimes \Omega^1_X)$ denote the global section associated to the splitting map $s$. We see that $s \mapsto 1$ under the map $H^0(X, A_P \otimes \Omega^1_X) \to H^0(X, TX \otimes \Omega^1_X) \cong H^0(X, \mathcal{O}_X)$.

To find the deformation of the triple $(X, P, \nabla)$, let $(X_\varepsilon, P_{\varepsilon}, \nabla_\varepsilon)$ be a family of triples over $D_\varepsilon$ starting with it. Let $s_\varepsilon$ be the family of sections corresponding to $\nabla_\varepsilon$. As in the proof of Proposition 4.3, let $\{U_i\}_{i \in I}$ again be a Čech covering of
we get

\[ \text{The diagram is commutative, i.e. for every } \sigma \in \mathcal{C} \text{, and } \phi_i, \iota_i, i \in I, \text{ defined in the same way. Let } s_i : TU_i \to \mathcal{A}_P|_{U_i} \text{ and } \sigma_i : TU_i \to \text{ad } P|_{U_i} \text{ be sections such that the following diagram commutates:} \]

\[ \begin{array}{ccc}
A_P|_{U_i} \times D_\epsilon & \xrightarrow{d\phi_i} & A_P|_{U_i} \\
\downarrow s_i + c_\sigma & & \downarrow s_i |_{U_i} \\
TU_i \times D_\epsilon & \xrightarrow{d\eta} & TX_{\epsilon A_i}.
\end{array} \]

The target space of \( \sigma_i \) is \( \text{ad } P \) instead of \( \mathcal{A}_P \), because \( p_\ast \circ s = \text{id} \) for all \( s \), so \( \sigma_i \), being the derivative of \( s \) (locally on \( U_i \), under the trivialization of the family \( \phi_i \)), projects to \( 0 \) under \( p_\ast \).

A deformation of the triple should contain the information about the deformation of the pair \((X, P)\) as well as the deformation of \( \nabla \). So the data associated to the infinitesimal family \((X_\epsilon, P_\epsilon, \nabla_\epsilon)\) should be the pair

\[ (n_{ij})_{i,j \in I}, \quad (\sigma_i)_{i \in I}, \]

where \((n_{ij})_{i,j \in I}\) is defined in the proof of Proposition 4.3 and shown to characterize the deformation of the pair \((X, P)\), and \((\sigma_i)_{i \in I}\) describes the deformation of \( \nabla \).

The data \((n_{ij})_{i,j \in I}, (\sigma_i)_{i \in I}\) looks like a 1-cocycle in defining the hypercohomology of some complex of sheaves. Recall that the tangent space to \( \text{Higgs}_X \) at a point \((P, \theta)\) is

\[ \mathbb{H}^1 \left( X, \text{ad } P \xrightarrow{\theta} \text{ad } P \otimes \Omega^1_X \right). \]

We will prove an analogous result about the tangent spaces to \( \mathcal{C} \text{Conn} \).

On \( U_{ij} \), the transition relations are expressed in the following diagram:

\[ \begin{array}{ccc}
A_P|_{U_i} \times D_\epsilon & \xrightarrow{d(\phi_i^{-1} \phi_j)} & A_P|_{U_i} \times D_\epsilon \\
\downarrow s_i + c_\sigma & & \downarrow s_i |_{U_i} \\
TU_{ij} \times D_\epsilon & \xrightarrow{d(\eta_i^{-1} \eta_j)} & TU_{ij} \times D_\epsilon.
\end{array} \]

Since \((\iota_i^{-1} \circ \iota_i)^\vee = \text{id} + \epsilon \xi_i\), we can write down the two horizontal maps more explicitly. For every \( Y + \epsilon Y_1 \in TU_i \times D_\epsilon \), its image \( Y' + \epsilon Y'_1 \) under \( d(\iota_i^{-1} \circ \iota_i) \) is determined by

\[ (Y' + \epsilon Y'_1)(f) = (l + \epsilon \xi_i)(Y + \epsilon Y_1)(f - \epsilon \xi_i)(f) \quad \text{for any function } f \text{ on } U_{ij}. \]

After simplification we get \( Y' = Y, Y'_1 = Y_1 + [\xi_i, Y] \) where the bracket is the Lie bracket of vector fields on \( U_{ij} \).

Similarly, for every \( Z + \epsilon Z_1 \in \mathcal{A}_P|_{U_{ij}} \times D_\epsilon \), we get \( d(\phi_i^{-1} \phi_j)(Z + \epsilon Z_1) = Z + \epsilon \{Z_1 + [\eta_i, Z]\} \), where the bracket is the Lie bracket of \((G\text{-invariant})\) vector fields on \( P|_{U_{ij}} \).

The diagram is commutative, i.e. for every \( Y + \epsilon Y_1 \in TU_i \times D_\epsilon \),

\[ d(\phi_i^{-1} \phi_j) \circ (s_i + \epsilon \sigma_i)(Y + \epsilon Y_1) = (s_j + \epsilon \sigma_j) \circ d(\iota_i^{-1} \circ \iota_i)(Y + \epsilon Y_1). \]
After simplification we get

\[ s_i(Y) = s_i(Y), \]
\[ (\sigma_j - \sigma_i)(Y) = [\eta_{ij}, s_i(Y)] - s_j([\xi_{ij}, Y]). \quad (1) \]

So if we use \( \hat{\partial}_j \in H^0(X, ad P \otimes \Omega^1_X) \) to denote the global section associated to \( \sigma_i \), the pair

\[ \left\{ (\eta_{ij}), (\hat{\partial}_i), \hat{\partial}_j \right\} \]

is a hyper Čech 1-cochain on \( X \) with coefficients in

\[ Ap \xrightarrow{[\hat{\partial}_j]} ad P \otimes \Omega^1_X. \]

The map \([\cdot, \hat{s}]\) is defined as: if \( \hat{s} = s' \otimes \omega \), where \( s' \in H^0(X, Ap) \), \( \omega \in H^0(X, \Omega^1_X) \), then \([\cdot, \hat{s}] = [\cdot, s'] \otimes \omega - s' \otimes [p_\omega(), \omega] \).

**Proposition 4.4.**

\( T_{(X, P, \nabla)}^{\lambda} Conn \) is naturally isomorphic to

\[ H^1 \left( X, Ap \xrightarrow{[\hat{\partial}_j]} ad P \otimes \Omega^1_X \right). \quad (2) \]

**Proof.** To any family of triples \( (X_i, P_i, \nabla_i) \) over \( D_\lambda \), a hyper 1-cochain \( \left\{ (\eta_{ij}), (\hat{\partial}_i), \hat{\partial}_j \right\} \) is associated by the above discussion. It is closed because of three facts: first, \( (\eta_{ij}) \) is a closed Čech 1-cochain with coefficients in \( Ap \); it is closed again because it comes from the transition function \( \phi_i^{-1} \circ \phi_j \); second, because of (1); third, the complex \( Ap \xrightarrow{[\hat{\partial}_j]} ad P \otimes \Omega^1_X \) has only two nonzero terms. These three facts imply that \( \left\{ (\eta_{ij}), (\hat{\partial}_i), \hat{\partial}_j \right\} \) is closed. Any closed hyper 1-cochain comes from some family of triples \( D_\lambda \). Also for a fixed family of triples \( D_\lambda \), a different choice of the \( \phi_i \)'s will result in a hyper cocycle differing from \( \left\{ (\eta_{ij}), (\hat{\partial}_i), \hat{\partial}_j \right\} \) by an exact hyper cocycle. And any exact hyper cocycle is the result of different choices of the \( \phi_i \)'s. Therefore the infinitesimal deformations of \( (X, P, \nabla) \) are in natural correspondence with (2), which is what we needed to prove.

\[ \square \]

### 4.3. Tangent spaces to \( \lambda Conn \)

Let \( \lambda \in \mathbb{C} \) be a fixed complex number. For the moduli space \( \lambda Conn \) of triples \( (X, P, \nabla) \) where \( \nabla \) is a \( \lambda \)-connection, the statement about its tangent spaces is completely analogous to that when \( \lambda = 1 \).

For a \( \lambda \)-connection \( \nabla_\lambda \) on \( P \), \( \lambda \neq 0 \), \( \nabla_\lambda/\lambda \) is an ordinary connection, therefore corresponds to a splitting \( s_{\nabla, \lambda} \) of the Atiyah sequence of \( P \). Let \( s_\lambda = \lambda \cdot s_{\nabla, \lambda} \), so \( s_\lambda \) is a "\( \lambda \)-splitting" of the Atiyah sequence of \( P \), i.e. \( p_\lambda \circ s_\lambda = \lambda \cdot id_{TX} \).

Therefore to any \( \lambda \)-connection \( \nabla_\lambda \), \( \lambda \neq 0 \), a \( \lambda \)-splitting of the Atiyah bundle is associated. Notice that this is true for \( \lambda = 0 \) as well, as a 0-splitting of the Atiyah bundle of \( P \) is exactly a Higgs field on \( P \).

Let \( \tilde{s}_j \in H^0(X, Ap \otimes \Omega^1_X) \) be the global section associated to \( s_j \), we see \( \tilde{s}_j \mapsto \lambda \) under the map \( H^0 \left( X, Ap \otimes \Omega^1_X \right) \rightarrow H^0 \left( X, TX \otimes \Omega^1_X \right) \cong H^0 \left( X, \mathcal{O}_X \right) \). The arguments in the last subsection can be repeated with slight changes (replace 1 by \( \lambda \) at appropriate places) to give the following statement.

**Proposition 4.5.**

\( T_{(X, P, \nabla)}^{\lambda} Conn \) is naturally isomorphic to

\[ H^1 \left( X, Ap \xrightarrow{[\tilde{s}_j]} ad P \otimes \Omega^1_X \right), \quad \lambda \in \mathbb{C}. \]

**Remark.**

When \( \lambda = 0 \), the result agrees with the previous results about tangent spaces to the Higgs moduli space.
5. **Isomonodromy vector field**

The nonabelian Gauss–Manin connection on \( \text{Conn} \to \mathcal{M}_g \) is the isomonodromy flow. The local trivialization of \( \text{Conn} \to \mathcal{M}_g \) given by the flow induces a lifting of tangent vectors \( L : T_X \mathcal{M}_g \to T_{(X,P,\nabla)} \text{Conn} \). We have identified these tangent spaces as (hyper)cohomology spaces in the last section, now we will write down the map \( L \) as a map of cohomology spaces. We start with a useful fact about an isomonodromy family of connections.

5.1. **Universal connection of an isomonodromy family**

In [4] Inaba et al. constructed the moduli space of triples \( (X,P,\nabla) \) and a universal \( G \)-bundle on the universal curve with a universal connection. Though they did it for a special case (rank 2 parabolic vector bundle on \( \mathbb{P}^1 \) with 4 points), the more general case can be done similarly. The universal connection, when restricted to an isomonodromy family of triples, has the following important property.

**Proposition.**

If \( (X, P, \nabla) \) is an isomonodromy family of triples over a complex line \( D = \text{Spec} \mathbb{C}[t] \), then the restriction of the universal connection on \( P_t \) (viewed as a \( G \)-bundle over the total space of \( X_t \)) is flat.

**Proof.** If we look only at the underlying differentiable structure, the isomonodromy family over \( D = \mathbb{C}[t] \) is a trivial family of triples. The trivial family structure gives a flat connection on \( P_t \), which must be equal to the restriction of the universal connection on \( P_t \) since they are equal on each fiber of the family.

5.2. **Isomonodromy lifting of tangent vectors**

For \( \lambda \in \mathbb{C} \), let \( \pi_\lambda \) be the projection

\[
\pi_\lambda : \lambda \text{Conn} \to \mathcal{M}_g, \quad (X,P,\nabla) \mapsto X.
\]

From the proof of Proposition 4.3 and the discussions preceding Proposition 4.4 it is not hard to see that the differential of \( \pi_\lambda \),

\[
\begin{array}{ccc}
T_{(X,P,\nabla)} \lambda \text{Conn} & \cong & H^1 \left( X, A_P \xrightarrow{[\partial \lambda]} \text{ad } P \otimes \Omega^1_X \right) \\
\downarrow \pi_\lambda & & \\
T_X \mathcal{M}_g & \cong & H^0(X, TX) \cong H^0(X, TX \to 0),
\end{array}
\]

is induced from the map \( (\rho_*, 0) \) of complexes of sheaves

\[
\begin{array}{ccc}
A_P & \xrightarrow{[\partial \lambda]} & \text{ad } P \otimes \Omega^1_X \\
\rho_* \downarrow & & 0 \\
TX & \xrightarrow{0} & 0.
\end{array}
\]

The lifting of tangent vectors induced from the isomonodromy flow is a splitting of the map \( \pi_\lambda \).
Notice that the splitting map \( s: TX \to \mathcal{A}_P \) associated to \( \nabla \) gives a map of the complexes\

\[
\begin{array}{c}
\begin{array}{ccc}
\mathcal{A}_P & \xrightarrow{[\cdot, \cdot]} & \text{ad} P \otimes \Omega^1_X \\
\downarrow s & & \downarrow \text{id} \\
TX & \xrightarrow{\text{id}} & 0
\end{array}
\end{array}
\]

The diagram is commutative because \([\cdot, \cdot] \circ s\) is basically bracketing \( s\) with itself and therefore is equal to 0. The map of complexes \((s, 0)\) is obviously a splitting of the map \((\rho_*, 0)\).

The map \((s, 0)\) of the complexes of sheaves induces a map on the first hypercohomology, which we denote as \( H^1(s)\).

**Proposition 5.1.**
The isomonodromy lifting \( L \) is equal to\

\[
H^1(s): H^1(X, TX) \longrightarrow \mathbb{H}^1\left( X, \mathcal{A}_P \xrightarrow{[\cdot, \cdot]} \text{ad} P \otimes \Omega^1_X \right).
\]

**Proof.** At a point \((X, P, \nabla)\) of \( \text{Conn} \), let \((X_\varepsilon, P_\varepsilon, \nabla_\varepsilon)\) be an isomonodromy family of triples over \( D_\varepsilon \) starting with it. Again let \( \{U_i\}_{i \in I}\) be a Čech covering of \( X\).

Over \( U_i\), let

\[
\tau_{i, \varepsilon}: X_{\varepsilon}|_{U_i} \times F \to P_\varepsilon|_{U_i}
\]

be the trivialization of \( P_\varepsilon|_{U_i}\) over \( X_{\varepsilon}|_{U_i}\) determined by the flat universal connection (see subsection 5.1) on \( P_\varepsilon|_{U_i}\), and \( \tau_i\) be its restriction at \( \varepsilon = 0\). Let

\[
t_i: U_i \times D_\varepsilon \to X_{\varepsilon}|_{U_i}
\]

be an isomorphism and define

\[
\phi_i: P|_{U_i} \times D_\varepsilon \to P_\varepsilon|_{U_i}
\]

as the composition

\[
P|_{U_i} \times D_\varepsilon \xrightarrow{\phi^{-1}_i \circ \text{id}_{D_\varepsilon}} U_i \times D_\varepsilon \times F \xrightarrow{\text{id}_{U_i} \circ \tau_i} X_{\varepsilon}|_{U_i} \times F \xrightarrow{\tau_{i, \varepsilon}} P_\varepsilon|_{U_i}.
\]

Let \( \xi_{ij}, \eta_{ij}, s_e, s_i, \) and \( \alpha_i \) be all defined as before in the proof of Proposition 4.3 and in subsection 4.2. Note that since the local trivializations of the \( G\)-bundles are canonically given by the flat universal connection, \( \tau_{i, \varepsilon}\) and \( \tau_{j, \varepsilon}\) agree on \( U_{ij}\), i.e. on \( U_{ij}\),

\[
\tau_{i, \varepsilon} = \tau_{j, \varepsilon}, \quad \tau_i = \tau_j.
\]

Therefore over \( U_{ij}\), the transition map \( \phi^{-1}_i \circ \phi_i\) fits in the diagram

\[
\begin{array}{c}
\begin{array}{ccc}
U_{ij} \times F \times D_\varepsilon & \xrightarrow{\left(\phi^{-1}_i \circ \phi_i\right)} & U_{ij} \times F \times D_\varepsilon \\
\downarrow \left(\tau_{i, \varepsilon} \circ \text{id}_{D_\varepsilon}\right) & \uparrow \left(\tau_{j, \varepsilon} \circ \text{id}_{D_\varepsilon}\right) \\
P|_{U_{ij}} \times D_\varepsilon & \xrightarrow{\phi^{-1}_i \circ \phi_i} & P|_{U_{ij}} \times D_\varepsilon
\end{array}
\end{array}
\]

In other words, with the local trivializations \( \left(\tau_{i, \varepsilon} \circ \text{id}_{D_\varepsilon}\right) \) and \( \left(\tau_{j, \varepsilon} \circ \text{id}_{D_\varepsilon}\right)\), the transition map \( \phi^{-1}_i \circ \phi_i\) corresponds to \( \left(\tau^{-1}_i \circ \text{id}_{F}\right)\). Let \( \left(\phi^{-1}_i \circ \phi_i\right)'\) and \( \eta_{ij}'\) be \( \phi^{-1}_i \circ \phi_i\) and \( \eta_{ij}\) under the local trivializations, then

\[
\left(\phi^{-1}_i \circ \phi_i\right)' = \left(\tau^{-1}_i \circ \text{id}_{F}\right).
\]
and therefore
\[ \text{Id} + \varepsilon \eta' = (\text{Id} + \varepsilon \xi_j, \text{Id}_i). \]
Comparing the coefficients of \( \varepsilon \) we get
\[ \eta'_j = (\xi_j, 0). \]
According to the last paragraph in subsection 3.3, we see this means precisely that \( a_{ij} = s(\xi_j) \).

With \( \phi: P_{\xi_j} \times D_\varepsilon \rightarrow P_{\xi_j} \) defined as above, \( s_{\varepsilon|U_i} : TX_{\varepsilon|U_i} \rightarrow A_{\varepsilon|U_i} \) corresponds to the section \( s_i : D_\varepsilon \rightarrow A_\varepsilon \times D_\varepsilon \) constant along \( D_\varepsilon \), i.e. \( \sigma_i = 0 \). Therefore \( \tilde{a}_i = 0 \), and the pair
\[ (a_i)_{i,j \in I}, \quad (\tilde{a}_i)_{i \in I} \]
is exactly the hyper 1-cocycle which is the image of \((\xi_j, 0)\) under the map \( H^1(s) \), which finishes the proof. \( \square \)

6. Extended isomonodromy lifting

The associated lifting \( L_\lambda \) is obtained by extending the isomonodromy lifting \( L \) to \( \lambda \text{Conn} \rightarrow \mathcal{M}_g \) by the \( \mathbb{C}^* \)-action, and multiplying by \( \lambda \). For a fixed \( \lambda, \lambda \neq 0 \), the \( \mathbb{C}^* \)-action gives an isomorphism
\[ \text{Conn} \leftrightarrow \lambda \text{Conn}, \quad \nabla \leftrightarrow \lambda \cdot \nabla. \]
The induced lifting on \( \lambda \text{Conn} \rightarrow \mathcal{M}_g \) by \( L \) via the isomorphism, called the extended isomonodromy lifting, can be written very similarly as \( L \). In the same way that the splitting map \( s \) associated to a connection \( \nabla \) induces a map \( H^1(s) \) of hypercohomologies, the \( \lambda \)-splitting map \( s_\lambda \) associated to a \( \lambda \)-connection \( \nabla_\lambda \) induces a map of the corresponding hypercohomology spaces, which will be denoted as \( H^1(s_\lambda) \).

**Proposition 6.1.**
The extended isomonodromy lifting of tangent vector on \( \lambda \text{Conn} \rightarrow \mathcal{M}_g \) is given by:
\[ \frac{1}{\lambda} H^1(s_\lambda) : H^1(X, TX) \longrightarrow \mathbb{H}^1(X, A_\lambda \xrightarrow{[\lambda]} \text{ad } P \otimes \Omega^1_\lambda). \]

**Proof.** Since the map of moduli spaces is \( \nabla \mapsto \lambda \cdot \nabla \) (or \( s \mapsto \lambda s, \hat{s} \mapsto \lambda \hat{s} \)), the induced map on the tangent spaces \( T_{[X,P,\nabla]} \text{Conn} \rightarrow T_{[X,P,\lambda \nabla]} \lambda \text{Conn} \) is
\[ \mathbb{H}^1(X, A_\lambda \xrightarrow{[\lambda]} \text{ad } P \otimes \Omega^1_\lambda) \xrightarrow{H^1([\lambda])} \mathbb{H}^1(X, A_\lambda \xrightarrow{[\lambda]} \text{ad } P \otimes \Omega^1_\lambda), \]
where \( ([\lambda], \lambda) \) is the map of complexes of sheaves
\[
\begin{align*}
\begin{pmatrix}
A_P & \xrightarrow{[\lambda]} & \text{ad } P \otimes \Omega^1_\lambda \\
\text{id} & & \\
\end{pmatrix}
\end{align*}
\]
and \( H^1([\lambda]) \) is the induced map on hypercohomology.
So to get the corresponding lifting on \( \mathcal{O}_{\text{Conn}} \), i.e. to make the following diagram commutate, the vertical map on the right must be \( H^1(\lambda \mathcal{L})/\lambda \).

\[
\begin{array}{ccc}
H^1(X, A_p \xrightarrow{[s]} \text{ad } P \otimes \Omega^1_X) & \xrightarrow{H^1(\mathcal{L})} & H^1(X, A_p \xrightarrow{[\lambda s]} \text{ad } P \otimes \Omega^1_X) \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \\
H^1(X, TX) & \xrightarrow{id} & H^1(X, TX).
\end{array}
\]

Since \( L_\lambda \) is the extended isomonodromy lifting multiplied by \( \lambda \), \( L_\lambda = H^1(\lambda \mathcal{L}) \). \( L_\lambda \) is a \( \lambda \)-lifting of tangent vectors.

### 7. Limit lifting at \( \lambda = 0 \)

The continuous limit of \( L_\lambda \) at \( \lambda = 0 \) is a 0-lifting \( L_0 : T_X \mathcal{M}_g \rightarrow T_{X,P,s_0} \mathcal{Higgs} \). Since \( L_\lambda = H^1(\lambda \mathcal{L}) \), by continuity \( L_0 \) is equal to

\[
H^1(s_0) : H^1(X, TX) \rightarrow H^1(X, A_p \xrightarrow{[s]} \text{ad } P \otimes \Omega^1_X),
\]

where \( s_0 \) is the 0-splitting of the Atiyah bundle of \( P \) associated to the 0-connection (or Higgs field) \( \nabla_0 \) on \( P \). Because \( \pi_s \circ H^1(s_0) = 0 \), in fact \( H^1(s_0) \) can be written as

\[
H^1(s_0) : H^1(X, TX) \rightarrow H^1(X, \text{ad } P \xrightarrow{[s]} \text{ad } P \otimes \Omega^1_X).
\]

The images of a vector in \( T \in T_X \mathcal{M}_g \) under \( H^1(s_0) \) form a vector field on the fiber \( \mathcal{Higgs}_X \) of \( \pi_0 \).

Recall that the quadratic Hitchin map on \( \mathcal{Higgs}_X \) is

\[
\text{qh} : \mathcal{Higgs}_X \rightarrow H^1(X, \Omega^{\otimes 2}), \quad (P, s_0) \mapsto (\hat{s}_0, \tilde{s}_0),
\]

and its associated lifting of tangent vectors is

\[
L_{\text{qh}} : H^1(X, TX) \rightarrow H^1(X, \text{ad } P \xrightarrow{[\tilde{s}_0]} \text{ad } P \otimes \Omega^1_X), \quad \tilde{l} \mapsto H^1_{\text{qh}}(\tilde{l}|(P, s_0)).
\]

The main theorem (Theorem 2.1) is that \( H^1(s_0) \) is equal to \( L_{\text{qh}}/2 \). To prove it we need two lemmas. For the first lemma, let \( ((\eta_i)_{i \in I}, (\tilde{\eta}_i)_{i \in I}) \) be a representative of an arbitrary element

\[
\nu \in H^1(X, \text{ad } P \xrightarrow{[\tilde{s}_0]} \text{ad } P \otimes \Omega^1_X). \tag{3}
\]

Because on \( U_{ij} \), \( \langle \tilde{s}_0, \tilde{\eta}_j - \tilde{\eta}_i \rangle = \langle \tilde{s}_0, [\eta_j, \tilde{s}_0] \rangle = -\langle [\tilde{s}_0, \tilde{s}_0], \eta_j \rangle = 0 \), we have

\[
\langle \tilde{s}_0, \tilde{\eta}_i \rangle = \langle \tilde{s}_0, \tilde{\eta}_i \rangle.
\]

Let \( (\tilde{s}_0, \tilde{\eta}) \in H^1(X, \Omega^{\otimes 2}) \) denote the resulting global quadratic differential form.
Lemma 7.1.

Using the above notations, the differential of the map $q_h$ is equal to

$$q_h : \mathbb{H}^1 \left( X, \text{ad } P \xrightarrow{[\cdot, \cdot]} \text{ad } P \otimes \Omega_1^X \right) \to H^0 \left( X, \Omega^2 \right), \quad \nu \mapsto 2 \langle \delta_0, \delta \rangle.$$

Proof. Let $\{ U_i \}_{i \in I}$ be the Čech covering of the curve $X$, $(P_x, s_x)$ the family of Higgs bundles over $D_x$ that correspond to $\nu$, i.e. for some $\phi_i: P_{U_i} \times D_x \to P_{U_i}$, some $s_i: T U_i \to \text{ad } P_{U_i}$ and the given $\sigma_i: T U_i \to \text{ad } P_{U_i}$, the diagram

$$
\begin{array}{ccc}
\text{ad } P_{U_i} \times D_x & \xrightarrow{\text{ad } \phi_i} & \text{ad } P_{U_i} \\
\downarrow s_i + \epsilon \sigma_i & & \downarrow s_i \downarrow \sigma_i \\
T U_i \times D_x & \xrightarrow{\text{id}} & T U_i \times D_x
\end{array}
$$

is commutative. Because $q_h: (P_x, s_x) \mapsto \langle \delta_x, \delta_x \rangle$, and that over $U_i$, $\langle \delta_x, \delta_x \rangle = \langle \delta_x + \epsilon \phi_i, \delta_x + \epsilon \phi_i \rangle = \langle \delta_x, \delta_x \rangle + 2 \langle \delta_x, \delta \rangle \epsilon$, we conclude $q_h: (P_x, s_x) \mapsto \langle \delta_0, \delta_0 \rangle + 2 \langle \delta_0, \delta \rangle \epsilon$. Taking the coefficient of $\epsilon$, we see that $q_h$ maps $\nu$ to $2 \langle \delta_0, \delta \rangle$. \qed

For the second lemma, let $\omega_H$ be the symplectic 2-form on $Higgs_X, \{(\eta_{ij})_{i,j \in I}, (\delta_{ij})_{i,j \in I}\}$ and $\{(\eta_{ij}')_{i,j \in I}, (\delta_{ij}')_{i,j \in I}\}$ representatives of two vectors

$$\nu, \nu' \in \mathbb{H}^1 \left( X, \text{ad } P \xrightarrow{[\cdot, \cdot]} \text{ad } P \otimes \Omega_1^X \right).$$

Lemma 7.2 ([5, Proposition 7.12]).

Let $f: H^1(X, \Omega_1^X) \to \mathbb{C}$ be the canonical map, then

$$\omega_H(\nu, \nu') = \int \left( \eta_{ij} \cup \delta_{ij}' + \eta_{ij}' \cup \delta_{ij} \right)$$

where $\cup$ means the cup product $\cup$ of Čech cochains composed with the Killing form $\langle \cdot, \cdot \rangle$.

Theorem.

$H^1(s_0)$ is equal to $L_{q_h}/2$.

Proof. For $f \in H^1(X, TX)$, we want to show that $L_{q_h}(f) = 2H^1(s_0)(f)$. Let $\{(\eta_{ij}')_{i,j \in I}, (\delta_{ij}')_{i,j \in I}\}$ be a representative of an element (3). Using Lemma 7.1,

$$\omega_H(L_{q_h}(f), \nu) = d(q_h^* f)(\nu) = df(\nu) = df(2 \langle \delta_0, \sigma \rangle) = f(2 \langle \delta_0, \sigma \rangle).$$

Using Lemma 7.2,

$$\omega_H \left( H^1(s_0)(f), \nu \right) = \omega_H \left( \langle s_0(f), 0 \rangle, (\eta_{ij}, \sigma) \right) = f(\delta_0, \sigma).$$

So $L_{q_h}(f) = 2H^1(s_0)(f)$ for all $f \in H^1(X, TX)$. Therefore $H^1(s_0) = L_{q_h}/2$. \qed

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