A NOTE ON PROJECTIVE DIMENSION OVER TWISTED COMMUTATIVE ALGEBRAS

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Abstract. Let $M$ be a finitely generated module over a free twisted commutative algebra $A$ that is finitely generated in degree one. We show that the projective dimension of $M(C^n)$ as an $A(C^n)$-module is eventually linear as a function of $n$. This confirms a conjecture of Le, Nagel, Nguyen, and Römer for a special class of modules.

1. Introduction

Fix a positive integer $d$ and let $A = C[x_{i,j} | 1 \leq i \leq d, 1 \leq j]$ be the infinite variable polynomial ring. One can picture the variables as the entries of a $d \times \infty$ matrix. The ring $A$ is obviously not noetherian, but it is known to be equivariantly noetherian with respect to the infinite symmetric group $\mathfrak{S}$ or the infinite general linear group $\text{GL}$; this means that the ascending chain condition holds for invariant ideals. The noetherian result for $\mathfrak{S}$ was proved by Cohen [Co]. The noetherian result for $\text{GL}$ follows from this, but also admits a direct (and easier) proof [SS2, §9.1.6].

Let $M$ be a module for $A$ that is equivariant with respect to $\mathfrak{S}$ or $\text{GL}$. We also assume that $M$ is finitely generated in the equivariant sense and “nice” as a representation. Taking invariants under an appropriate subgroup, one obtains a module $M_n$ over the finite variable polynomial ring $A_n = C[x_{i,j} | 1 \leq i \leq d, 1 \leq j \leq n]$. Given the above noetherian results, one might hope that this sequence of modules is well-behaved.

In the case of the symmetric group (and where $M$ is a homogeneous ideal of $A$), this has been investigated by Le, Nagel, Nguyen, and Römer. In [NR, Theorem 7.7], the authors show that the Hilbert series of $M_n$ behaves in a regular manner as $n$ varies: the generating function of this sequence of rational functions is itself a rational function in two variables. As a consequence, they show that the Krull dimension of $M_n$ is eventually linear [NR, Theorem 7.9]. In [LNNR1, Conjecture 1.1], the authors conjecture that the Castelnuovo–Mumford regularity of $M_n$ is eventually linear, and in [LNNR2, Conjecture 1.3] they conjecture the same for projective dimension.

In this paper, we consider the case of the general linear group. Since $\mathfrak{S}$ is a rather small subgroup of $\text{GL}$, it follows that $\text{GL}$-equivariant modules are much more constrained than $\mathfrak{S}$-equivariant modules. Unsurprisingly, many of the above results were previously known in the $\text{GL}$-case: for instance, very precise results are known on the Hilbert series, and it is known that regularity is eventually constant; see [NSS, SS1, SS3, SS4, SS5]. The main result of this paper (Theorem 4.1) shows that the projective dimension of $M$ is eventually linear. This confirms the conjecture of [LNNR2] in the $\text{GL}$ case. The key tools are the structure theory for modules developed in [SS3].

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2. Set-up

We work over the complex numbers. We assume general familiarity with Young diagrams, polynomial representations, polynomial functors, and Schur functors (denoted by $S_\lambda$ where $\lambda$ is an integer partition), and refer to [SS2] for the relevant background information and detailed references. Let $V = \bigcup_{n \geq 1} C^n$ and let $GL = \bigcup_{n \geq 1} GL_n$. Let $Rep^{pol}(GL)$ be the category of polynomial representations of $GL$. This is equivalent to the category of polynomial functors, and we freely pass between the two points of view. The simple objects of $Rep^{pol}(GL)$ are given by $S_\lambda(V)$ as $\lambda$ ranges over all partitions.

A twisted commutative algebra (tca) is a commutative algebra object in $Rep^{pol}(GL)$. Fix a $d$-dimensional vector space $E$, and put $A = Sym(V \otimes E)$. This is a tca. It is the same ring introduced in §1, but written in a coordinate-free manner.

By an $A$-module we always mean a module object for $A$ in $Rep^{pol}(GL)$. Explicitly, this is a module in the ordinary sense equipped with a compatible action of $GL$ under which it forms a polynomial representation. Suppose that $M$ is an $A$-module. Treating $M$ and $A$ as polynomial functions, $M(C^n)$ is an $A(C^n \otimes E)$-module; note that $A(C^n \otimes E) = Sym(C^n \otimes E)$ is a finite variable polynomial ring. These are the objects $M_n$ and $A_n$ from §1.

We say that a function $f: N \to N$ is eventually linear (here $N$ denotes the set of non-negative integers) if there exists $a \in N$ and $b \in Z$ such that $f(n) = an + b$ for all $n \gg 0$; we then call $a$ the slope of $f$.

3. The key technical result

For a polynomial representation $M$ of $GL$, we let $\gamma_M(n)$ or $\gamma(M; n)$ be the maximum size of a partition with at most $n$ columns appearing in $M$. The following is the key technical result we need to prove our main theorem:

**Theorem 3.1.** If $M$ is a finitely generated $A$-module then $\gamma_M$ is eventually linear with slope at most $d$.

**Example 3.2.** Let $M = A/a_r$ be the coordinate ring of the rank $\leq r$ matrices in $E \otimes V$. Suppose that $\min(n, d) \geq r$. The Cauchy identity gives the decomposition

$$M(C^n) = \bigoplus_{\ell(\lambda) \leq r} S_\lambda(E) \otimes S_{\lambda}(C^n)$$

where the sum is over all partitions with at most $r$ many parts. Hence $\gamma_M(n) = rn$. □

It is possible to give an elementary proof of Theorem 3.1 (see Remark 3.5), but we will give a more conceptual proof based on the structure theory of $A$-modules from [SS3]. We define the formal character of a polynomial representation $M$ of $GL$, denoted $\Theta_M$, to be the formal series $\sum_\lambda m_\lambda s_\lambda$, where the sum is over partitions, $m_\lambda$ is the multiplicity of $S_\lambda(V)$ in $M$, and $s_\lambda$ is a formal symbol. Note that we can read off $\gamma_M$ from $\Theta_M$.

Let $a_r \subset A$ be the determinantal ideal, as in Example 3.2. Let $Mod_{A, \leq r}$ be the category of modules (set-theoretically) supported on $V(a_r)$, and let $Mod_{A, > r} = Mod_A / Mod_{A, \leq r}$ be the Serre quotient category. Let $T_{> r}: Mod_A \to Mod_{A, > r}$
be the quotient functor, let $S_{>r}$ be its right adjoint, and let $\Sigma_{>r} = S_{>r} \circ T_{>r}$ be the saturation functor. Also let

$$\Gamma_{\leq r}: \text{Mod}_A \to \text{Mod}_A,_{\leq r}$$

be the functor assigning to a module its maximal submodule supported on $V(\mathfrak{a}_r)$. By [SS3, Theorem 6.10], $\Sigma_{>r}$ and $\Gamma_{\leq r}$ preserve the finitely generated bounded derived categories.

Let $D(A)_{\leq r}$, resp. $D(A)_{>r}$, be the full subcategories of the derived category $D(A)$ spanned by modules $M$ with $\Sigma_{>r}(M) = 0$, resp. $\Gamma_{\leq r}(M) = 0$, and let

$$D(A)_r = D(A)_{\leq r} \cap D(A)_{>r}.$$ 

Then $D(A)$ admits a semi-orthogonal decomposition into the $D(A)_0, \ldots, D(A)_d$. This holds for the finitely generated bounded derived categories too [SS3, §4]. Letting $K(A)$ denote the Grothendieck group of the category of finitely generated $A$-modules, we have $K(A) = \bigoplus_{r=0}^d K(A)_r$, where $K(A)_r$ is the Grothendieck group of $D^b_r(A)_r$. By [SS3, Theorem 6.19], we have a natural isomorphism $K(A)_r = \Lambda \otimes K(\text{Gr}_r(E))$, where $\Lambda$ is the ring of symmetric functions and $\text{Gr}_r(E)$ is the Grassmannian of $r$-dimensional quotient spaces of $E$. We note that $\Theta$ defines an additive function on $K(A)$.

For a partition $\lambda$, we let $\lambda[n^r]$ be the partition $(n, \ldots, n, \lambda_1, \lambda_2, \ldots)$, where the first $r$ coordinates are $n$. This is a partition provided that $n \geq \lambda_1$.

**Lemma 3.3.** Let $c \in K(A)_r$ be the class $s_\lambda \otimes [\mathcal{F}]$, where $\mathcal{F}$ is a coherent sheaf on $\text{Gr}_r(E)$.

(a) Every partition appearing in $\Theta_c$ is contained in $\lambda[n^r]$ for some $n$.

(b) For $n \geq \lambda_1$, the coefficient of $\lambda[n^r]$ in $\Theta_c$ is $h_{\mathcal{F}}(n)$, where $h_{\mathcal{F}}$ is the Hilbert polynomial of $\mathcal{F}$ with respect to the Plücker embedding.

**Proof.** Let $Q$ be the rank $r$ tautological quotient bundle on $X = \text{Gr}_r(E)$ and let $B = \text{Sym}(V \otimes Q)$, which can be thought of as a tca on $X$. If $M$ is a $B$-module then $\Gamma(X, M)$ is naturally an $A$-module [SS3, §6.2]. Under the description of $K(A)$ given above, $c$ is the class of the complex $\Gamma(X, M)$ where $M = S_\lambda(V) \otimes \mathcal{F} \otimes B$ (see [SS3, §6.6]). Using the Cauchy decomposition for $B$, we have

$$H^i(X, M) = S_\lambda(V) \otimes \bigoplus_{\ell(\mu) \leq r} (S_\mu(V) \otimes H^i(X, \mathcal{F} \otimes S_\mu(Q))).$$

Note that the cohomology group above is just a vector space; the $\text{GL}$ action comes from the first two Schur functors. Since $\mu$ has at most $r$ rows, the Littlewood–Richardson rule shows that all partitions appearing in $S_\lambda \otimes S_\mu$ are contained in $\lambda[n^r]$ for some $n$. This proves (a). The Littlewood–Richardson rule also shows that $\lambda[n^r]$ appears with multiplicity one in $S_\lambda \otimes S_{(n^r)}$ for $n \geq \lambda_1$, and does not appear in any other $S_\lambda \otimes S_\mu$ with $\ell(\mu) \leq r$. Note that $S_{(n^r)}(Q) = \det(Q)^{\otimes n}$ and $\det(Q)$ is the Plücker bundle. We thus see that the coefficient of $\lambda[n^r]$ in $\Theta_c$ is

$$\sum_{i \geq 0} (-1)^i \dim H^i(X, \mathcal{F}(n)) = h_{\mathcal{F}}(n),$$

which proves (b).

**Proof of Theorem 3.1.** Let $M$ be a finitely generated $A$-module, and suppose that $M$ is supported on $V(\mathfrak{a}_r)$ with $r$ minimal. By [SS3, Theorem 6.19], we then have the following:

- In $K(A)$, we have $[M] = c_0 + \cdots + c_r$ with $c_i \in K(A)_i$. Write $c_i = \sum_{\lambda} c_{i,\lambda}$ where $c_{i,\lambda} = s_\lambda \otimes [\mathcal{F}_{i,\lambda}]$ and $\mathcal{F}_{i,\lambda}$ is a coherent complex on $\text{Gr}_i(E)$.
- The class $[\mathcal{F}_{r,\lambda}]$ is effective, i.e., we can assume $\mathcal{F}_{r,\lambda}$ is a coherent sheaf.
There is a partition $\lambda$ such that $[F_{r,\lambda}] \neq 0$.

By Lemma 3.3(a) a partition with $\leq n$ columns appearing with non-zero coefficient in $\Theta_{c,\mu}$ has size $\leq in + |\mu|$. We thus see that $\gamma_M(n) \leq rn + b$ where $b$ is the maximal size of a partition $\lambda$ with $F_{r,\lambda} \neq 0$, at least for $n \gg 0$.

Now, let $\lambda$ be a partition of size $b$ with $F_{r,\lambda}$ non-zero. By Lemma 3.3(b), $\lambda[n^r]$ appears with positive coefficient in $\Theta_{c,\lambda}$ for $n \gg 0$. Furthermore, the lemma shows that $\lambda[n^r]$ does not appear in $\Theta_{c,\mu}$ for any $(i, \mu) \neq (r, \lambda)$ and for $n \gg 0$. We thus see that $\lambda[n^r]$ has positive coefficient in $\Theta_M$, and so $\gamma_M(n) \geq rn + b$. This completes the proof. \hfill $\Box$

Remark 3.4. The proof shows that the slope of $\gamma_M$ is the minimal $r$ such that $M$ is supported on $V(a_r)$. \hfill $\Box$

Remark 3.5. Here is how one can prove Theorem 3.1 without using the theory of [SS3]. For a polynomial representation $M$, let $M[n]$ be the sum of the $\lambda$-isotypic pieces of $M$ over those $\lambda$ of size at least $n$ and with at most $n$ columns, and let $M = \bigoplus_{n \geq 0} M[n]$. Suppose $M$ is a finitely generated $A$-module. One then shows that $M'$ is a finitely generated $A^\dagger$-module, and from this deduces the structure of the bi-variate Hilbert series of $M'$ (note that $M'$ is bi-graded since each $M[n]$ is graded). One can deduce the theorem from this, as the Hilbert series determine $\gamma_M$. \hfill $\Box$

4. Depth and Projective Dimension

Let $M$ be an $A$-module. We write $\operatorname{depth}_{A}(n)$ or $\operatorname{depth}(M; n)$ for the depth of $M(C^n)$ as an $A(C^n)$-module, and $\operatorname{pdim}_{M}(n)$ or $\operatorname{pdim}(M; n)$ for the projective dimension of $M(C^n)$ as an $A(C^n)$-module. Our main result is the following theorem:

Theorem 4.1. If $M$ is a finitely generated $A$-module then $\operatorname{pdim}_{M}$ and $\operatorname{depth}_{M}$ are eventually linear with slope at most $d$.

Example 4.2. Let $M = A/a_r$ be the coordinate ring of the rank $\leq r$ matrices, as in Example 3.2. Suppose that $\min(n, d) \geq r$. Then $M(C^n)$ has codimension $(d-r)(n-r)$ and is Cohen–Macaulay, so its projective dimension is $\operatorname{pdim}_{M}(n) = (d-r)n - (d-r)r$ and by the Auslander–Buchsbaum formula, its depth is $\operatorname{depth}_{M}(n) = rn + r(d-r)$. \hfill $\Box$

We now prove Theorem 4.1. The Auslander–Buchsbaum formula states that

$$\operatorname{depth}_{M}(n) + \operatorname{pdim}_{M}(n) = dn,$$

which allows us to deduce the result for depth from that for pdim.

Using [SS3, Theorem 7.7], there are finitely generated $A$-modules $F_k(M)$ that can be extracted from the linear strands of the minimal free resolution of $M$; its graded components are given by

$$F_k(M)_{p+k} = \operatorname{Tor}_{p}^{A}(M, C)_{p+k},$$

where $\vee$ is the duality on polynomial functors which fixes simple objects (see [SS2, (6.1.6)]), and $\dag$ is the equivalence on polynomial functors which interchanges the usual symmetric structure with the graded symmetric structure, and in particular has the effect $S^\dag_k = S_k$ (see [SS2, (6.1.5)]). There are only finitely many values of $k$ for which $F_k(M)$ is non-zero.

The theorem is now a consequence of Theorem 3.1 and the following lemma:

Lemma 4.3. Let $M$ be a finitely generated $A$-module. Then

$$\operatorname{pdim}_{M}(n) = \max_{k}(\gamma(F_k(M); n) - k).$$
Proof. Fix $n$, and let $N$ be the maximum appearing on the right side of the above equation. For this proof, write $T_i(M)$ for $\text{Tor}^A_i(M, C)$. By definition, we have

$$T_p(M) = \bigoplus_k F_k(M)_{p+k}.$$  

We thus see that $T_q(M)(C^n) \neq 0$ for some $q \geq p$ if and only if there exists some $k$ such that $F_k(M)$ has a partition of size at least $p + k$ with at most $n$ columns, that is, $\gamma(F_k(M); n) \geq p + k$. Therefore, the maximum $p$ for which $T_p(M)(C^n) \neq 0$ is $p = N$, and the result follows since $\text{pdim}_A(n)$ is the maximum $p$ for which

$$T_p(M)(C^n) = \text{Tor}^A_p(C^n)(M(C^n), C)$$

is non-zero.

5. Krull dimension

Let $B$ be a quotient tca of $A$. Define $\delta_B(n)$ to be the Krull dimension of the ring $B(C^n)$. Since the defining ideal for $B$ is stable under the infinite symmetric group $\mathfrak{S}$, it follows from [NR, Theorem 7.9] that $\delta_B$ is eventually linear. We now give an easy proof of a more precise result by leveraging the theory from [SS3].

We first recall some relevant information from [SS3, §3]. Let $C$ be any tca. An ideal $I$ of $C$ is prime if, given any other ideals $J, J'$ of $C$, we have that $JJ' \subseteq I$ if and only if $J \subseteq I$ or $J' \subseteq I$. (Note that, by definition, all ideals are GL-stable.) The spectrum $\text{Spec}(C)$ is defined to be the set of prime ideals of $C$, and is equipped with the Zariski topology (defined in the same way as for ordinary rings).

Next, let $\text{Gr}_r(E)$ denote the underlying topological space of the Grassmannian (thought of as a scheme) parametrizing rank $r$ quotients of $E$. The total Grassmannian of $E$, denoted $\text{Gr}(E)$, is $\bigsqcup_{r=0}^d \text{Gr}_r(E)$ as a set. We topologize $\text{Gr}(E)$ by defining a subset $Z \subset \text{Gr}(E)$ to be closed if and only if

- $Z \cap \text{Gr}_r(E)$ is closed for all $r$, and
- $Z$ is closed under taking quotients: if $E \to U$ is in $Z$, then so is $E \to U'$ for any quotient space $U'$ of $U$.

By [SS3, Theorem 3.3], we have a homeomorphism $\text{Spec}(A) \cong \text{Gr}(E)$, and hence $\text{Spec}(B)$ can be identified with a closed subset of $\text{Gr}(E)$. If $Z \subset \text{Gr}_r(E)$ is a Zariski closed irreducible subset, then its closure in $\text{Gr}(E)$ is irreducible, and every irreducible closed subset of $\text{Gr}(E)$ is of this form [SS3, Proposition 3.2]. Hence we can label irreducible closed subsets of $\text{Gr}(E)$ by pairs $(r, Z)$ where $Z \subset \text{Gr}_r(E)$ is a Zariski closed irreducible subset.

We then have the following result:

Theorem 5.1. Let $B$ be as above, and recall that $d = \text{dim}(E)$.

(a) There exist integers $0 \leq a \leq d$ and $0 \leq b \leq (d - a)a$ such that $\delta_B(n) = an + b$ for all $n \gg 0$.

Now assume that $\text{Spec}(B)$ is irreducible.

(b) If $\text{Spec}(B)$ corresponds to the pair $(r, Z)$, then $a = r$ and $b = \text{dim} Z$.

(c) If $b = 0$ then $\text{Spec}(B) = V(I)$ where $I$ is generated by linear forms.

(d) If $b = (d - a)a$ then $\text{Spec}(B)$ is the determinantal variety of rank $\leq a$ maps.
Proof. By noetherianity of $A$, Spec$(B)$ has finitely many irreducible components, so it suffices to prove (a) when Spec$(B)$ is irreducible. We will assume that from the beginning. Suppose Spec$(B)$ corresponds to $(r, Z)$. Let $Y_n \subset$ Spec$(A(C^n))$ be the space of maps of rank exactly $r$. Then the natural map $\pi_n : Y_n \to Gr_r(E)$ is a fibration of relative dimension $rn$. Furthermore, Spec$(B(C^n))$ is the inverse image of $Z$ under $\pi_n$ (see [SS3, Lemma 3.7]). This proves (a) and (b). If $b = 0$ then $Z$ is a point, while if $b = (d - a)a$ then $Z$ is all of $Gr_r(E)$; (c) and (d) follow. □

6. Local cohomology

Let $M$ be a finitely generated $A$-module. We have seen that depth$_M$ is eventually linear (Theorem 4.1). This tells us that the local cohomology groups $H^i_{m(C^n)}(M(C^n))$ have some uniformity as $n$ varies, where $m$ is the maximal ideal of $A$. In [SS3, §6], we defined local cohomology groups $H^i_{a_r}(M)$ for $M$ at each determinantal ideal $a_r$, and proved a finiteness result about them. A natural problem is to try and relate these two local cohomology groups. This is addressed by the following theorem:

Theorem 6.1. Let $M$ be a bounded $A$-module. Then there is a canonical isomorphism of $A(C^n)$-modules $H^i_{a_r}(M)(C^n) \cong H^i_{a_r}(C^n)(M(C^n))$ for $n \geq \max(\ell(M), i + d)$.

The proof of this result is somewhat lengthy, so we have not included it.

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