Quantization and simulation of Born-Infeld non-linear electrodynamics on a lattice

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Abstract

Born-Infeld non-linear electrodynamics arises naturally as a field theory description of the dynamics of strings and branes. Most analyses of this theory have been limited to studying it as a classical field theory. We quantize this theory on a Euclidean 4-dimensional space-time lattice and determine its properties using Monte-Carlo simulations. The electromagnetic field around a static point charge is measured using Lüscher-Weisz methods to overcome the sign problem associated with the introduction of this charge. The $D$ field appears identical to that of Maxwell QED. However, the $E$ field is enhanced by quantum fluctuations, while still showing the short distance screening observed in the classical theory. In addition, whereas for the classical theory, the screening increases without bound as the non-linearity increases, the quantum theory approaches a limiting conformal field theory.

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I. INTRODUCTION

Theories of strings and branes have shown much promise as theories unifying gravity with strong, weak and electromagnetic interactions. One limitation has been that most of the work on these theories has been performed on the classical theories. The most promising way to produce quantum theories of strings and branes has been to study field theories which describe their dynamics. The field theories which arise most naturally in this context are Born-Infeld non-linear electrodynamics and its extensions. Early papers which use Born-Infeld theories to describe the dynamics of strings and branes include. However, since much of the interesting physics arising from the non-linearities in Born-Infeld models is inaccessible to perturbation theory, most of the work on these theories has also been limited to the classical domain or to the small quantum fluctuations around these classical solutions. Beyond this, there have only been a few exploratory investigations of the general problem of quantizing this theory, for example, which examines quantum solutions representing propagation in a fixed direction.

Born-Infeld electrodynamics in \( n + 1 \) dimensions is described by the non-linear action

\[
S = b^2 \int d^{n+1}x \left[ 1 - \sqrt{-\det \left( g_{\mu\nu} + \frac{1}{b} F_{\mu\nu} \right)} \right].
\]  

(1)

To make the connection with strings and branes, one typically chooses \( n = 9 \) and dimensionally reduces the theory from \( 9 + 1 \) down to \( p + 1 \) dimensions. This describes a \( p \)-brane, and the extra \( 9 - p \) components of \( A_\mu \) are identified with the transverse degrees of freedom of the brane. The special case \( p = 1 \) describes a string. Here the non-linearity parameter \( b \) is related to the string tension through \( 1/b = 2\pi\alpha' \).

In this work we study the original Born-Infeld theory in \( 3 + 1 \) dimensions (\( n = 3 \)) with no dimensional reduction. In brane language we study a 3-brane with no transverse dimensions. This theory was originally proposed as a modification of QED in which the electric field of a static point particle was screened at short distances, rendering the self-energy of a point particle finite.

We note that, in Euclidean space, the action is positive so that quantization by the standard functional integral techniques is well defined. We then define this theory on a discrete space-time lattice, preserving gauge invariance. The positivity of the action allows the use of Metropolis Monte-Carlo simulations to extract the properties of the quantum
field theory. In particular, we measure the electromagnetic fields produced by a static point charge by including the effects of a Wilson Line (Polyakov Loop) in the action. This destroys the positivity, but we are able to use the methods of Lüscher and Weisz [12] (after Parisi, Petronzio and Rapuano [13]) to produce the exponential statistics required to overcome this lack of positivity.

We have performed simulations of the lattice Born-Infeld theory on $8^4$ and $12^4$ lattices. We present evidence that the $D$ field emanating from a point charge is identical with that of the Maxwell theory, for all values of $b$, up to lattice artifacts. The $E$ field shows short-distance screening as for the classical theory, but whereas classically the screening length increases without limit as $b \to 0$, the quantum field theory approaches a conformal field theory in this limit. In addition, the $E$ field is enhanced over its classical value by quantum fluctuations. Preliminary results for the smaller lattice were presented at Lattice 2005, Dublin [14].

In section 2 we summarise the salient features of the classical theory from the extensive literature on Born-Infeld electrodynamics. Section 3 discusses the lattice formulation in Euclidean space and simulation methods. We present the results of our simulations in section 4. These results are discussed and conclusions drawn in section 5.

II. CLASSICAL BORN-INFELD ELECTRODYNAMICS

In this section we summarise those results from the literature on the classical Born-Infeld theory in $3+1$ dimensional Minkowski space which are relevant to our investigations. These results are condensed from the following references [15, 16, 17, 18]. Evaluating the determinant in equation 1 leads to the Lagrangian

$$L = b^2 \left[ 1 - \sqrt{1 + \frac{1}{2b^2} F_{\mu\nu} F^{\mu\nu} - \frac{1}{16b^4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2} \right]$$

$$= b^2 \left[ 1 - \sqrt{1 - b^{-2}(E^2 - B^2) - b^{-4}(E \cdot B)^2} \right].$$

Note that the requirement that the argument of the square root be positive, restricts the magnitude of $E$ which in turn gives rise to short-distance screening and leads to a finite self energy for a point charge. This was the original reason Born and Infeld proposed this non-linear extension of electrodynamics. When dimensionally reduced Born-Infeld theories
are used to describe strings or branes, this same mechanism restricts the energy density of the string/brane to be finite.

To proceed, it is customary to define $D$ and $H$ fields by

$$D = \frac{\partial L}{\partial E} = \frac{E + b^{-2}(E \cdot B)B}{\sqrt{1 - b^{-2}(E^2 - B^2) - b^{-4}(E \cdot B)^2}}$$

$$H = \frac{\partial L}{\partial B} = \frac{B - b^{-2}(E \cdot B)E}{\sqrt{1 - b^{-2}(E^2 - B^2) - b^{-4}(E \cdot B)^2}}. \tag{3}$$

In terms of $E$, $D$, $B$ and $H$, the field equations are the standard Maxwell equations

$$\nabla \cdot D = \rho$$

$$\nabla \cdot B = 0$$

$$\nabla \times E + \frac{\partial B}{\partial t} = 0$$

$$\nabla \times H - \frac{\partial D}{\partial t} = j, \tag{4}$$

where, as usual, the interaction with the electromagnetic current $j_{\mu}$ is introduced by adding a term $j_{\mu}A_{\mu}$ to the Lagrangian. All the non-linearity is hidden in the defining relations of equations 3.

For a static point charge $\rho = e\delta^3(r)$, $\nabla \cdot D = \rho$ has the spherically symmetric solution

$$D = \frac{e}{4\pi r^3}r \tag{5}$$

as for the Maxwell case. $B = H = 0$ and equations 3 then yield

$$E = \frac{e}{4\pi r} \frac{r}{\sqrt{r^4 + r_0^4}} \tag{6}$$

where

$$r_0 = \sqrt{\frac{|e|}{4\pi b}} \tag{7}$$

defines the screening length. Due to this short distance screening, the electric field at the origin has magnitude $b$ rather than being infinite, and the energy in the field of the point charge is finite.

III. EUCLIDEAN BORN-INFELD ELECTRODYNAMICS ON A LATTICE

In Euclidean space, the Born-Infeld action is

$$S = b^2 \int d^4x \left[ \sqrt{1 + \frac{1}{2b^2}F_{\mu\nu}F_{\mu\nu} + \frac{1}{16b^4}(F_{\mu\nu}F_{\mu\nu})^2} - 1 \right]$$
\[ b^2 \int d^4 x \left[ \sqrt{1 + b^{-2}(E^2 + B^2)} + b^{-4}(E \cdot B)^2 - 1 \right]. \]  

(8)

We note that this action is positive, so this theory can be simulated using importance sampling methods. The expressions for \( D \) and \( H \) are identical to those of equations 3 except that all signs are positive.

To enable simulations, we must first transcribe this field theory to a discrete space-time lattice. We choose a hypercubic lattice with lattice spacing \( a \). For convenience we define \( \beta = b^2 a^4 \). From now on we will work in lattice units where \( a \) is set to 1, except where we discuss the effects of varying \( a \). Because power counting suggests this theory is non-renormalizable, it is generally considered as an effective theory, requiring a cutoff. Hence we are not required to take the \( a \to 0 \) limit. We shall have more to say about this later.

We choose the non-compact formulation of QED on the lattice, since this is closest to continuum QED (In the Maxwell case it defines a solvable, free field theory, as in the continuum). This is done by defining the lattice \( F_{\mu \nu} \) by

\[ F_{\mu \nu}(x + \frac{1}{2} \hat{\mu} + \frac{1}{2} \hat{\nu}) = A_\nu(x + \hat{\mu}) - A_\nu(x) - A_\mu(x + \hat{\nu}) + A_\mu(x), \]  

(9)

which is gauge invariant as is the continuum \( F_{\mu \nu} \). This indicates the first subtlety; \( F_{\mu \nu} \) is not associated with a single site \( x \) or link of the original lattice. Therefore to preserve the symmetries of the cubic lattice we average our action over the 16 choices of 6 plaquettes associated with a lattice site. These 16 choices are defined by choosing 1 unit vector in each of the 4 directions emanating from the site, in all possible ways. The set of plaquettes associated with a given choice of unit vectors is uniquely defined by requiring that 2 edges of each plaquette belong to the chosen set of unit vectors. Simulations are performed using Metropolis Monte-Carlo updates of the gauge fields \( (A_\mu) \) on each of the links of the lattice. We periodically perform subtractions of constant fields from each \( A_\mu \), since such constant fields have no physics, to prevent the lattice average of \( A_\mu \) becoming too large. Similarly, we periodically gauge-fix these fields to Landau gauge, to remove large gauge fluctuations.

To compare with the classical Born-Infeld results we need to measure the \( E \) and \( D \) fields for a point charge. Since we use periodic boundary conditions on the gauge fields, the total charge on the lattice must be zero. We choose the ‘Jellium’ solution from condensed matter physics to circumvent this difficulty. Here, if the magnitude of the point charge is \( e \), we distribute a charge \(-e\) uniformly over the lattice. Hence each site has charge \(-e/V\)
\( V = N_x N_y N_z \) is the spatial volume of the lattice, except for that site containing the point charge, whose charge is now reduced to \( e - e/V \). On the lattice this charge is introduced by including a Wilson Line (Polyakov loop) \( W(x) \) in the functional integral.

\[
W(x) = \exp \left\{ i e \sum_t \left[ A_4(x, t) - \frac{1}{V} \sum_y A_4(y, t) \right] \right\}.
\]

(10)

We calculate the \( \langle E \rangle \) and \( \langle D \rangle \) due to this charge as

\[
i \langle E \rangle_{\rho}(y - x) = \frac{\langle E(y, t)W(x) \rangle}{\langle W(x) \rangle},
\]

\[
i \langle D \rangle_{\rho}(y - x) = \frac{\langle D(y, t)W(x) \rangle}{\langle W(x) \rangle}.
\]

(11)

where the \( i \) converts our electric fields from Euclidean space to Minkowski space, and we use the subscript \( \rho \) to denote the expectation value in the presence of charge density \( \rho \).

The Wilson Line for fixed \( x \) on any given configuration has magnitude one, so

\[
\langle |W(x)| \rangle = 1,
\]

(12)

while it is expected that

\[
\langle W(x) \rangle = e^{-\text{const}\cdot N_t}.
\]

(13)

This indicates that the phase of \( W \) causes important cancellations between the Wilson lines for different configurations in our ensemble. Hence we have a sign problem. Averaging over the site \( x \) helps improve our statistics. Even if we measured the Wilson line averaged over all sites, every sweep, we will run into trouble if

\[
\langle W(x) \rangle \lesssim 1/\sqrt{\text{number of sweeps} \times V},
\]

(14)

in the most optimistic scenario where the phases of the Wilson lines on a given configuration are uncorrelated as are those on consecutive sweeps. For any of our \( 12^4 \) runs, the number of sweeps is 500,000 and \( V = 1728 \) and the right hand side of equation (14) is \( \approx 3.4 \times 10^{-5} \).

As we shall see later, we really need to do better.

Lüscher and Weisz [12] (following Parisi, Petronzio and Rapuano [13]) have pointed out that one can effectively enhance ones statistics by dividing the lattice into some number (call it \( n \)) of timeslices. Notice that, if one fixes the link fields on the boundaries of these timeslices, the Monte Carlo updates of those fields on the interiors of each slice are independent of the updates of the interiors of all other slices. Thus if one performs \( m \) updates of the interior
of each time-slice, it is equivalent to performing $m^n$ (restricted) updates of the lattice. In addition this process can be performed recursively over progressively coarser slicings of the lattice. In practice we have used thickness 1 and 2 time slices recursively.

Let us now examine the limiting cases, first where $b \to \infty$, and second where $b \to 0$. As $b \to \infty$, the Euclidean Lagrangian approaches its Maxwell (free field) form

$$\mathcal{L}_E = \frac{1}{2}(E^2 + B^2).$$

(15)

In the limit $b \to 0$ the Euclidean Lagrangian approaches the limiting form

$$\mathcal{L}_E = |E \cdot B|,$$

(16)

which describes a 4-dimensional conformal field theory. [Note that the Minkowski Lagrangian (equation 11) does not exist in this limit. However, the Hamiltonian formulation is valid in this limit, and leads to the Hamiltonian

$$\mathcal{H} = |D \times B|,$$

(17)

which also describes a conformal field theory [15, 19].]

IV. SIMULATION DETAILS AND RESULTS

We have simulated lattice Born-Infeld QED on $8^4$ and $12^4$ lattices. On both $8^4$ and $12^4$ lattices we performed runs of 500,000 10-hit Metropolis sweeps of the lattice at each of $\beta = 0.0001, 0.01, 1, 100$. On the $8^4$ lattice we ran an additional 100,000 sweeps at each of $\beta = 0.1, 0.2, 0.5, 2, 5$. Measurements were made every 100 sweeps. On the $8^4$ lattices the Wilson lines and the on-axis fields they produced were measured in the Lüscher-Weisz scheme where we used 10 10-hit Metropolis updates of the interiors of each thickness-1 timeslice for each of 10 10-hit Metropolis updates of the interiors of each thickness-2 timeslice. For the $12^4$ lattice we increased the number of updates at each level to 20. This means that we effectively used $10^{12}$ configurations every 100 sweeps for our $8^4$ lattice simulations and $2.62144 \times 10^{23}$ configurations every 100 sweeps for our $12^4$ runs, for our measurements of the Wilson lines and electric fields. Admittedly these configurations generated in the Lüscher-Weisz scheme are not all independent.

In figure 1 we present the average action density $\langle S \rangle / V$, where $V$ is the space-time volume, as a function of $\beta = b^2 a^4$ for our runs on both $8^4$ and $12^4$ lattices. We notice that there is
good agreement between the 2 lattice sizes, indicating that that the finite volume effects are small. As $\beta \to \infty$, where the theory becomes the linear Maxwell theory, the equipartition theorem predicts that the action density is $\frac{3}{2}$. We note that the action density is within a percent of this value by $\beta = 100$, indicating that for $\beta \gtrsim 100$ the theory is close to the Maxwell theory. In the $\beta \to 0$ limit the scaling properties of the action again lead to an action density of $\frac{3}{2}$. We see that our $\beta = 0.0001$ action density is very close to this value, while the $\beta = 0.01$ value is only a little over 2% higher. This suggests that lattice Born-Infeld QED is close to the limit described by equation 16 for $\beta \lesssim 0.01$. [Note that, because the zero modes do not contribute to the action, the limiting values $\frac{3}{2}$ have corrections $O(1/V)$]. In
between the 2 limits the action rises to a maximum of magnitude $< 2$ for $\beta \sim 1$.

Figure 2 shows the Wilson Lines as functions of electric charge $e$ for 16 values of this charge ranging from 0.0078125 to 4.0 and for the 4 beta values $\beta = 0.0001, 0.01, 1.0, 100.0$. At each $\beta$, $\langle W \rangle$ starts from 1 at $e = 0$, and falls faster than exponentially with charge $e$ as $e$ is increased. The falloff becomes faster as $\beta$ (or $b$) is decreased, and the field theory becomes more non-linear. We note, however, that the values of the Wilson lines for $\beta = 0.0001$ and $\beta = 0.01$ are almost identical. This is further evidence that the theory is close to the $\beta = 0$ theory for $\beta \lesssim 0.01$. This also gives us the first evidence for differences between the electrostatics of the quantum and classical theories. Classical electrostatics has $B$ and $H$ identically zero, the contribution of equation 16 vanishes and the argument the square root of equation 1 reduces to $1 - b^{-2}E^2$ for all $b$. In the quantum theory it is not even consistent to set $B = 0$, and the result is that the action still reduces to the $b(\beta)$ independent form of equation 16 as $b \to 0$. We note that the Lüscher-Weisz prescription yields an excellent signal for the Wilson line with small error bars over almost 11 orders of magnitude.

We measure the $E$ field for on axis separations (for definiteness consider this to be in the $z$ direction) from the electric charge. Here we need only measure $E_z$. For this we need to measure plaquettes in the $(z, t)$ plane containing the Wilson Line. We call the $z$ separation of the centre of such a plaquette from the Wilson line $Z$. Clearly $Z$ takes the values $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ..., N_z - \frac{1}{2}$. It is also clear that

\[
\langle E_z(Z) \rangle = -\langle E_z(N_z - Z) \rangle \\
\langle D_z(Z) \rangle = -\langle D_z(N_z - Z) \rangle.
\] (18)

The lattice $D_z$ field is obtained by calculating $\frac{\partial S}{\partial E_z}$, where $E_z$ is a $(z, t)$ plaquette and $S$ is the lattice action. The $\langle E_z \rangle_\rho$ and $\langle D_z \rangle_\rho$ are obtained using equation 14. Again, the Lüscher-Weisz method yields a good signal. In what follows, we will drop the subscripts $\rho$ and $z$ for convenience.

We will start by considering $D$. Since the equation of motion $\nabla \cdot D = \rho$ holds for all $\beta$, this means that, to the extent that we have rotational invariance, $\langle D \rangle$ should be independent of $\beta$, and equal to the classical value. Of course, since we are on the lattice, rotational invariance is at best approximate, and we expect departures from this expectation.

Figure 3 shows the ratios of the $D$ fields produced by a point charge $e$ on both $8^4$ and $12^4$ lattices for $\beta = 0.0001$, where we expect the action to closely approximate the small $\beta$
FIG. 2: Wilson Lines as functions of charge $e$, for a range of $\beta = a^4 b^2$ values: a) $8^4$ lattice; b) $12^4$ lattice.
FIG. 3: Ratios of $D$ to their Maxwell (free field) values for $\beta = 0.0001$ as functions of electric charge $e$, for accessible values of the on-axis separation $Z$: a) for an $8^4$ lattice, b) for a $12^4$ lattice.
limit of equation 16 to their free field values. For the minimum separation of source and field \((Z = \frac{1}{2})\) \(D\) is consistent with its free-field (Maxwell) value, evaluated on the same-size lattice. This is because, at this separation, the discrete Gauss’ equation plus cubic symmetry is sufficient to uniquely determine \(D\); in fact \(D = \frac{1}{6}(1 - 1/V)e\). At separation \(Z = \frac{3}{2}\), \(D\) has fallen to 86–87% of its free-field value. Gauss’ law and cubic symmetry are no longer sufficient to determine the \(D\) field uniquely, and we do not have the simple relation \(D = E\) and hence \(\nabla \times D + \partial B/\partial t = 0\) of the Maxwell theory to help us. It is thus reassuring that \(D\) is as close to its free-field value as it is. As the separation increases, the ratio of \(D\) to its free-field value appears to be approaching 1. This is an indication that rotational symmetry is restored at length scales much larger than the lattice spacing. The fact that the \(D\) for this highly non-linear theory appears identical to that of the Maxwell theory, apart from lattice artifacts, is non-trivial. For the Maxwell theory, \(D = E\), while in the \(\beta \to 0\) limit \(D = \varepsilon(E \cdot B)B\), where \(\varepsilon\) is the familiar sign function.

Figure 4 shows the \(e\) dependence of \(D\), for \(Z < N_z/2\), for \(\beta = 1.0\). This \(\beta\) is close to that which maximizes \(\langle S \rangle/V\). Thus, for this value of \(\beta\), Born-Infeld QED should be far from regular (Maxwell) QED, and also far from the non-linear limit of equation 16. Again we see that at minimum separation, \(D\) is at its free field value. The ratio of \(D\) to its free-field value falls to a minimum for \(1.5 Z \lesssim 2.5\), after which it begins to increase again. For the \(8^4\) lattice, the finite lattice extent tends to suppress this increase while the larger \(12^4\) lattice shows a clear increase. Hence we see evidence that rotational symmetry is being restored at large distances, just as for small \(\beta\). Again we believe that the difference between the \(\beta = 1.0\) \(D\) fields and their free-field counterparts is a lattice artifact.

In figure 5 we graph the ratios of \(D\) to its free field value for \(\beta = 100.0\), where we expect Born-Infeld QED to closely approximate standard (Maxwell) QED. We note again that at minimum separation \(D\) is consistent with its free field value. At larger separations there are small but significant departures from the Maxwell theory. However, since these departures are always less than (and often considerably less than) one percent, this is further evidence that \(\beta = 100.0\) Born-Infeld QED is close to standard QED. Because these departures comparable in size with our error-bars, it is difficult to make any more quantitative observations, although it does appear that the \(8^4\) and \(12^4\) values are consistent for the ranges of \(Z\) which they share.

We now turn our attention to the \(E\) field. Here, unlike the case of the \(D\) field, Maxwell’s
FIG. 4: Ratios of $D$ to their Maxwell (free field) values for $\beta = 1.0$ as functions of electric charge $e$, for accessible values of the on-axis separation $Z$: a) for an $8^4$ lattice, b) for a $12^4$ lattice.
FIG. 5: Ratios of $D$ to their Maxwell (free field) values for $\beta = 100.0$ as functions of electric charge $e$, for accessible values of the on-axis separation $Z$: a) for an $8^4$ lattice, b) for a $12^4$ lattice.
equations are of little help, because the relations of equations 3 are highly non-linear. What we are interested in knowing is how similar these $E$ fields are to their classical counterparts and, in particular, whether they show the short-distance screening of classical Born-Infeld electrodynamics. We shall use our experience with the $D$ fields as a guide to how much the results that we observe are affected by lattice artifacts.

First we look at the case where $\beta = 0.0001$ which we expect to be close to the non-linear limit of equation 16. Figure 6 shows the ratios of $E$ to its free-field value for this $\beta$ as functions of $e$ for all accessible values of $Z$. The similar graph (not shown) for $\beta = 0.01$ is almost identical, which justifies our claim that we are close to the non-linear limit. We first note that the $8^4$ and $12^4$ graphs are very similar for those $Z$ values common to each, indicating that we should be able to extract information which is not dominated by finite volume effects from lattices of this size. Secondly we notice that for all accessible $Z$s and over most of the range of charges considered ($0 < e \leq 3$), the electric fields are larger than their free field counterparts by a factor of 2–3. This contrasts with the classical case where $E \leq D$. This enhancement of $E$ is due to quantum fluctuations which are clearly quite large. At $Z = \frac{1}{2}$, it is clear that $E$ is screened, and that this screening increases with increasing $e$. The $Z = \frac{3}{2}$ field also shows such screening, but the falloff with increasing $e$ is much less pronounced than at $Z = \frac{1}{2}$, while there is no clear signal for screening for larger $Z$. This behaviour is qualitatively similar to what is observed classically. However, classically its screening is determined by the ratio $e/b$ (or $r_0$), whereas in the quantum theory, for small enough $b$ (and hence $\beta$) the screening becomes $b$ independent. Since the limit $\beta \to 0$ can be achieved by taking $a \to 0$ at fixed $b$, rather than $b \to 0$, the non-linear limit of equation 16 could be expected to describe the continuum limit. The question arises as to whether this limiting field theory is trivial or not. If it is trivial, the $E$ field of a point charge (a propagator of this field theory) would be proportional to the free field propagator. Of course, this would only be true of this propagator at finite distances in physical units, which as $a \to 0$ means large lattice distances. Thus the fact that this ratio on the $12^4$ lattice varies by only of order $5\%$ for $Z > 0.5$, at least for small charges, could be an indication that this limit is trivial.

Figure 7 shows $E/E_{\text{free}}$ as functions of $e$ for all accessible separations $Z$, for $\beta = 1.0$. Again, we note the similarity between the results from the $8^4$ and $12^4$ lattices up until $Z$ approaches $N_z/2$ where the effect of the finite volume becomes apparent, which gives us confidence that the finite volume effects are under control. Despite the fact that we are now
FIG. 6: Ratios of $E$ to their Maxwell (free field) values for $\beta = 0.0001$ as functions of electric charge $e$, for accessible values of the on-axis separation $Z$: a) for an $8^4$ lattice, b) for a $12^4$ lattice.
FIG. 7: Ratios of $E$ to their Maxwell (free field) values for $\beta = 1.0$ as functions of electric charge $e$, for accessible values of the on-axis separation $Z$: a) for an $8^4$ lattice, b) for a $12^4$ lattice.
far from both the small and large $\beta$ limits, these graphs are qualitatively similar to those
for $\beta = 0.0001$, except for the scale on the vertical axis. The enhancement of this electric
field over its Maxwell value due to quantum fluctuations is smaller than that for $\beta = 0.0001$.
The screening which we observe at $Z = \frac{1}{2}$ (and $Z = \frac{3}{2}$) is also less by roughly a factor of 3
over the same range of charges $e$.

In the last of this series of figures showing the $e$ and $Z$ dependence of the ratio of $E$
to its free field value, figure 8, we show the results for $\beta = 100$ where we expect Born-Infeld
QED to be close to its Maxwell (standard QED) limit. What is clear from these graphs is
that, since the $E$ never differs from its free field value by more than 2% over the range of
$e$ considered, Born-Infeld QED is close to its free-field limit for $\beta = 100$. We do, however,
observes a small increase in $E$ over its free-field value due to quantum fluctuations, and see
clear evidence for weak screening for $Z = \frac{1}{2}$. This is good evidence that expanding the
square root in the action in powers of $1/\beta$, and using a perturbative analysis, would be valid
in this domain.

In an effort to get a more quantitative understanding of the screening of the $E$ field that
we have observed, we look again at the screening in classical Born-Infeld electrodynamics.
Here we can expose the screening length ($r_0$) by looking at

$$\left\{ \frac{[E(r)/e]_{e=0}}{[E(r)/e]} \right\}^2 = 1 + \left( \frac{r}{r_0} \right)^4 = 1 + \frac{1}{16\pi^2 b^2 r^4} e^2. \quad (19)$$

So, for the classical case, plotting the left hand side of this equation for fixed $r$ against $e^2$
(or $1/r_0^4$) would give a straight line. Figure 9 shows this quantity from our simulations at
$\beta = 0.0001$. While straight line fits to the $Z = \frac{1}{2}$ and $Z = \frac{3}{2}$ are impossible, the straight lines
we have drawn to guide the eye indicate that the $e^2$ dependence is almost linear. Perhaps
the observed departures are due to lattice artifacts. What this behaviour does suggest is
that the screening increases without bound as $e^2$ is increased, as is true classically.

Finally, for completeness, we show the Wilson Line correlations for each of the 16 charges
for $\beta = 100$ and $\beta = 0.0001$ on the $12^4$ lattice in figure 10. These correlation functions
show the interactions between charge $+e$ and charge $-e$ separated by distance $Z$. What
these indicate is that the effect of screening on such interactions is to reduce the correlation
functions as $\beta$ is decreased (which as observed above, increases screening). One expects that
the Wilson Line correlation function will behave as

$$\langle W(x)W^\dagger(x + Z) \rangle = C(Z) \exp[-V(Z)T]. \quad (20)$$
FIG. 8: Ratios of $E$ to their Maxwell (free field) values for $\beta = 100.0$ as functions of electric charge $e$, for accessible values of the on-axis separation $Z$: a) for an $8^4$ lattice, b) for a $12^4$ lattice.
FIG. 9: $e^2$ dependence of the inverse electric field: a) on an $8^4$ lattice b) on a $12^4$ lattice. Note that the straight lines drawn to aid the eye on (a) and (b) are identical.
Although the electric field shows screening, $E$ for $\beta = 0.0001$ exceeds that for the free field and hence that for $\beta = 100.0$ for the same charge, over the range of charges considered. Hence, the potential energy $V(Z)$ at $\beta = 0.0001$ exceeds that for $\beta = 100.0$. It is for this reason that this correlation function falls more rapidly for the smaller $\beta$.

V. DISCUSSION AND CONCLUSIONS

Euclidean Born-Infeld electrodynamics has been quantized on a discrete space-time lattice. The Euclidean action is positive, allowing us to simulate this theory by importance sampling techniques. We have simulated the gauge-invariant non-compact lattice implementation on $8^4$ and $12^4$ lattices using standard Metropolis Monte-Carlo methods for several values of the non-linearity parameter $b$ [or the dimensionless $\beta = b^2a^4$ ($a$ is the lattice spacing)]. We measure the electric fields produced by a static point charge $e$, as functions of that charge. On the lattice this charge is introduced by the inclusion of a Wilson line (Polyakov loop). The phase of this Wilson line introduces a sign problem. However, this sign problem is overcome using the method of Lüscher and Weisz, which uses partial factorization of the functional integral to produce the exponential statistics required to circumvent such sign problems. With this, we were able to obtain excellent signals for the Wilson line, $E$ and $D$ fields, over an appreciable range of $e$ values.

We performed simulations with $\beta$ values ranging from those large enough to approximate the Maxwell (free-field) theory to the those where Born-Infeld QED is well approximated by a $\beta(b)$-independent conformal field theory with Euclidean Lagrangian $\mathcal{L}_E = |E \cdot B|$ or (in Minkowski space) by the Hamiltonian $\mathcal{H} = |D \times B|$. For all $\beta$ values, the electric displacement $D$ appears to differ from its value in the Maxwell theory only by lattice artifacts. The electric field $E$ shows considerable enhancement over its Maxwell value by as much as a factor of $\approx 3$ due to quantum fluctuations. Away from the Maxwell limit, $E$ exhibits short-distance screening with a screening length whose 4th power is approximately linear in $e^2$, as in the classical theory. Also as in the classical theory, screening is enhanced as $\beta(b)$ is increased. However, unlike in the classical case where the screening at fixed $e$ increases without limit as $b$ is increased, in the quantum theory this screening approaches a limit described by the conformal field theory mentioned at the beginning of this paragraph. Once this limit is achieved, the only way to increase the screening length is to increase $|e|$. 

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FIG. 10: Wilson line correlation functions for charges $e$ ranging from $0.0078125$ (top curve) to $4$ (bottom curve) a) for $\beta = 100.0$ and b) for $\beta = 0.0001$. 

$12^4$ Lattice -- $\beta=100.0$

$12^4$ Lattice -- $\beta=0.0001$

FIG. 10: Wilson line correlation functions for charges $e$ ranging from $0.0078125$ (top curve) to $4$ (bottom curve) a) for $\beta = 100.0$ and b) for $\beta = 0.0001$. 
Since our chief reason for studying Born-Infeld QED is because of its connection to strings and branes, we only need to consider it as an effective field theory with a momentum cutoff, or in our language, a finite lattice spacing. However, it is interesting to consider whether one can remove the momentum cutoff (take $a \to 0$) in such a way as to define a renormalizable field theory. If we fix $b$ and take $a$ to zero, we are taking the $\beta \to 0$ limit, where the field theory is described by the Euclidean Lagrangian $L_E = |E \cdot B|$. Hence this theory describes the infinite-momentum-cutoff limit of Born-Infeld QED. What we do not know, is whether this quantum field theory is non-trivial. Our simulations hint that the propagators of this theory might show free-field scaling. However, the evidence is far from convincing, especially if we consider the possibility that deviations from free-field scaling could well be logarithmic.

It would be useful to perform simulations of Born-Infeld QED on larger lattices, possibly restricting these simulations to those using the action based on the Lagrangian $L_E = |E \cdot B|$. Of more interest would be simulations using those actions obtained from higher dimensional Born-Infeld theories by dimensional reduction, which describe strings or branes with transverse degrees of freedom. In these actions, the square root contains an additional term $(\partial_\mu X_a)^2$, where the scalar fields $X_a$ are the $n - p$ additional components of the $A$ field corresponding to the reduced dimensions and are interpreted as the transverse displacements of the $p$-brane. We plan such simulations in the near future.

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