TRAVELING WAVES FOR A REACTION-DIFFUSION MODEL WITH A CYCLIC STRUCTURE

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Abstract. In this paper, a reaction-diffusion model with a cyclic structure is studied, which includes the SIS disease-transmission model and the nutrient-phytoplankton model. The minimal wave speed \( c^* \) of traveling wave solutions is given. The existence of traveling semi-fronts with \( c > c^* \) is proved by Schauder’s fixed-point theorem. The traveling semi-fronts are shown to be bounded by rescaling method and comparison principle. The existence of traveling semi-front with \( c = c^* \) is obtained by limit arguments. Finally, the traveling semi-fronts are shown to connect to the positive equilibrium by a Lyapunov function.

1. Introduction. In this paper, we will study the traveling waves of the following reaction-diffusion system:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_1 \Delta u + d(K - u) - f(u)v + \sigma v, \\
\frac{\partial v}{\partial t} &= d_2 \Delta v + \gamma f(u)v - \delta v,
\end{align*}
\]

where all coefficients \( d_1, d_2, d, K, \gamma, \delta, \sigma \) are positive constants, \( \Delta = \sum_{i=1}^{n_0} \frac{\partial^2}{\partial x_i^2} \), and \( f(u) \) satisfies

(A1): \( f(\cdot) \in C^2([0, +\infty)), f(0) = 0. \) \( f(u) \) is strictly increasing in \([0, +\infty)\).

If \( \gamma = 1, \delta > \sigma, f(u) = \beta u \), then system (1) denotes the famous SIS disease model with natural and disease-related death rates incorporated [16], where \( u \) denotes the susceptible and \( v \) is the infective. If \( d_1 = d_2, f(u) = bu/(a + u) \), system (1) becomes the diffusive nutrient-phytoplankton model in [5] without time delay, where \( u \) stands for nutrients and \( v \) is the phytoplankton. However, according to the arguments in [6], the nutrient uptake rate of phytoplankton \( f(u) \) can have a general form satisfying assumption (A1). Since the spreading or invasion speed is close linked with the minimal wave speed of traveling wave solutions [8, 12], it is necessary to study the existence of traveling wave solutions of system (1).

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There are few literatures on traveling wave solutions of system (1) [7, 20]. Ding et al. [7] studied a diffusive SIS disease-transmission model with standard disease incidence and Ma and Yuan [20] studied an SIRS disease-transmission model. Though methods in [7, 20] give important intuition into the studies of traveling wave solutions of reaction-diffusion systems with a cyclic structure, there are some restriction conditions on the models in [7, 20], i.e., natural and disease-related death rates are not considered in them and the diffusive coefficients are assumed to be equal in [20]. As what are pointed in [8, 18], it is necessary to incorporate death rates into the models and the diffusive coefficients may differ from each other.

Obviously, \( u \) has positive effects on \( v \), and \( v \) can also change into class \( u \) due to the term \( \sigma v \). Then we say that this structure of model (1) is cyclic if \( \sigma > 0 \). However, this cyclic structure vanishes when \( \sigma = 0 \). System (1) with \( \sigma = 0 \) is similar to diffusive predator-prey models or SI disease-transmission models, for which the traveling wave solutions have been deeply studied by lots of researchers (see [9, 2, 19, 10, 15, 23] and the references therein). In these literatures, Schauder’s fixed-point theorem method and geometric method are used for the existence of traveling waves. To see why it is difficult to study the traveling waves of system (1), denote

\[
F_1(u, v) := d(K - u) - f(u)v + \sigma v, \quad F_2(u, v) := \gamma f(u)v - \delta v.
\] (2)

If \( \sigma = 0 \), then \( \frac{\partial F_1}{\partial v} = -f(u) < 0 \) for \( u > 0 \). If \( \sigma > 0 \), \( \frac{\partial F_2}{\partial v} = \sigma - f(u) \) changes sign when \( u \) goes through \( f^{-1}(\sigma) \). This means that system (1) has weaker monotonicity than predator-prey models and SI disease-transmission models, which are non-cooperative reaction-diffusion systems. The weaker monotonicity can cause difficulties in constructing a pair of upper- and lower-solutions. Secondly, since, for fixed \( u > 0 \), \( f(u)v \) is unbounded with respect to \( v \in [0, +\infty) \), then the upper- and lower-solutions constructed for system (1) are also unbounded in \( \mathbb{R} \). This unboundedness can cause obstacles in limit arguments. Therefore, it seems that the methods in [9, 2, 19, 23] can not directly applied to system (1) since the counterparts of \( f(u)v \) in these literatures are bounded with respect to \( v \in [0, +\infty) \) and thus the upper- and lower-solutions in them are bounded. Fu and Tsai [10] constructed a pair of unbounded upper- and lower-solutions for a diffusive predator-prey model with unbounded interaction function and the traveling waves were shown to be bounded by constructing a Lyapunov function together with some analysis techniques. However, Fu and Tsai did not study the existence of traveling wave with minimal wave speed due to the lack of a uniform bound for a sequence of traveling waves. We will study the existence of traveling wave with minimal wave speed of (1). Consequently, the methods in [10] may need to be improved if one wants to use them to study system (1) with the cyclic structure. Zhao and Wang [24] and Zhao et al. [25] also studied the existence of traveling waves for two diffusive disease-transmission models with an unbounded interaction function, i.e. bilinear disease incidence, but there are some restrictions on the diffusive coefficients in their models. In this paper, we only suppose \( d_1 > 0, d_2 > 0 \). Therefore, it seems difficult to directly apply aforementioned methods to system (1) due to the cyclic structure and the unbounded interaction function.

In this paper, we still adopt the Schauder’s fixed-point theorem idea for the existence of traveling waves. But the construction of upper- and lower-solutions has to be developed to adapt to the cyclic structure of system (1). The difficulty coming from the unbounded function \( f(u)v \) will be overcome by both rescaling method
and comparison principle. Rescaling method was used by Berestycki et al. [3] and Ducrot et al. [9] for the non-existence of traveling waves and by Lam et al. [17] and Girardin [11] for the existence of traveling waves. The methods in our paper are motivated by [17, 11]. But note that the boundedness of traveling waves is necessary when applying rescaling method and limit arguments, and this boundedness of traveling wave solutions in [11] can be guaranteed by density-dependent competitions of the model. At the same time, there does not exist a cyclic structure in the model of [17].

Actually, Under assumption \( \sigma < \delta/\gamma < f(K) \), there exists a simpler method than those used in this paper to study the existence of traveling waves for model (1). Under assumption \( \sigma < \delta/\gamma < f(K) \), the transformation

\[
\tilde{u} = u - f^{-1}(\sigma)
\]

converts system (1) into

\[
\frac{\partial \tilde{u}}{\partial t} = d_1 \Delta \tilde{u} + d(\tilde{K} - \tilde{u}) - \tilde{f}(\tilde{u})v,
\frac{\partial v}{\partial t} = d_2 \Delta v + \gamma \tilde{f}(\tilde{u})v - \tilde{\delta}v,
\]

where

\[
\tilde{f}(\tilde{u}) = f(u + f^{-1}(u)) - \sigma, \quad \tilde{K} = K - f(u), \quad \tilde{\delta} = \delta - \gamma \sigma,
\]

are positive for \( \tilde{u} > 0 \). This reduces the problem to finding traveling wave solutions for the classical diffusive predator-prey system. Hence the results in [17] and in Fu and Tsai [10] can be applied to obtain the results in Theorem 2.1(i), which is the main focus of this paper. However, the methods used in this paper still have its own values since it is possible for them to be improved for more general reaction-diffusion systems than model (1).

This paper is organized as follows. In Section 2, the main theorem is given and the minimal wave speed in one case is proved. Section 3 is devoted to the existence of minimal wave speed in another case. Specifically, in Section 3.1, the traveling semi-fronts with wave speed \( c > c^* \) are given by Schauder’s fixed-point theorem; in Section 3.2, the traveling semi-fronts are shown to be bounded by rescaling method and comparison principle; in Section 3.3, the existence of traveling semi-front with \( c = c^* \) is obtained by limit arguments; in Section 3.4, the traveling semi-fronts are shown to connect to the positive equilibrium by a Lyapunov function.

2. Main results. A solution \((u(x,t), v(x,t))\) of (1) is called a traveling wave solution if it has the form

\[
(u(x,t), v(x,t)) = (U(s), V(s)) =: (U, V)(s), \quad s = \nu \cdot x + ct,
\]

where \( c > 0 \) is the wave speed, and the unit vector \( \nu \in \mathbb{R}^n \) is the traveling direction. The positive solution (3) is called a traveling semi-front if it satisfies

\[
(U, V)(-\infty) = E_0(K, 0),
\]

where \( E_0 \) is the invasion-free equilibrium of (1).

Some other assumptions are needed.

\( \text{(A2): } \sigma < \delta/\gamma < f(K). \)
\( \text{(A3): } f(K) < \delta/\gamma < \sigma. \)

Noting that \( f(u) \) is strictly increasing in \([0, +\infty)\) by the assumption (A1), it is easy to get the following remark.
Remark 1. It follows from (A1) that the positive equilibrium $E^*(u^*, v^*)$ of system (1) has the form

$$u^* = f^{-1}\left(\frac{\delta}{\gamma}\right), \quad v^* = \frac{d(K - u^*)}{f(u^*) - \sigma}. \quad (5)$$

Then assumption (A2) is equivalent to $\sigma < f(u^*) < f(K)$ and assumption (A3) is equivalent to $f(K) < f(u^*) < \sigma$. By (A1) and (5), system (1) has a positive equilibrium if and only if (A2) or (A3) holds.

In order to study the invasion behaviours, we assume that the coexistence equilibrium $E^*$ exists, that is, (A2) or (A3) holds. Our main result is as follows.

**Theorem 2.1.** Denote $c^* = 2\sqrt{d_2(\gamma f(K) - \delta)}$ and suppose (A1) holds.

(i): If (A2) holds, then system (1) has a positive traveling wave solution $(U, V)$ $(x + ct)$ satisfying

$$(U, V)(-\infty) = E_0(K, 0), \quad (U, V)(+\infty) = E^* \quad (6)$$

if and only if $c \geq c^*$.

(ii): If (A3) holds, there exists a positive constant $\bar{c}^*$ such that system (1) has a positive traveling wave solution $(U, V)$ $(x + ct)$ satisfying (6) if and only if $c \geq \bar{c}^*$, where $U(s)$ and $V(s)$ are increasing on $\mathbb{R}$.

**Proof of Theorem 2.1(ii).** It follows from (A1), (A3) and Remark 1 that

$$\frac{\partial F_1}{\partial v} = \sigma - f(u) \geq 0, \quad \frac{\partial F_2}{\partial u} = \gamma f'(u)v \geq 0$$

for any $(u, v) \in [K, u^*] \times [0, v^*]$. Then Theorem 1.2 in [21, p. 158] completes the proof.

**Remark 2.** The constant $\bar{c}^*$ in Theorem 2.1(ii) can be determined by the constant $c^*$ in Lemma 2.3(ii) of Lam et al. [17] with replacing the matrix $P$ by $\frac{\partial (F_1(u,v),F_2(u,v))}{\partial (u,v)}|_{u=K,v=0}$. The proof of Theorem 2.1(ii) can also be completely similar to those in Hsu and Yang [13].

3. **Proof of Theorem 2.1(i).** In this section we will prove Theorem 2.1(i). Therefore, we suppose (A1)-(A2) hold in the remainder of this paper. Set

$$H(\lambda) = d_2\lambda^2 - c\lambda + \gamma f(K) - \delta. \quad (7)$$

Let $\lambda_1, \lambda_2$ be the two zeros of $H(\lambda)$. Then $\lambda_1, \lambda_2 \in \mathbb{R}$ if and only if $c \geq c^*$. We assume without loss of generality that $\lambda_1 \leq \lambda_2$ if $c \geq c^*$.

It is easy to verify that (3) is a traveling wave solution of system (1) if and only if $(U, V)(s)$ satisfies

$$\begin{cases} cU' = d_1U'' + d(K - U) - f(U)V + \sigma V, \\ cV' = d_2V'' + \gamma f(U)V - \delta V, \end{cases} \quad (8)$$

where $'$ denotes the derivative with respect to $s$. To prove Theorem 2.1(i), we first consider the case $c < c^*$.

**Lemma 3.1.** For any $c \in (-\infty, c^*)$, system (1) has no traveling semi-fronts with wave speed $c$.

The proof of this lemma is completely similar to that of Ducrot et al. [9, Proposition 4.2], and it is omitted. Next we will study the existence of traveling wave solutions with wave speed $c \geq c^*$.
3.1. **Existence of traveling semi-fronts when** \( c > c^* \). In this subsection we assume \( c > c^* \), and thus \( 0 < \lambda_1 < \lambda_2 \). Define

\[
\begin{align*}
\bar{U}(s) &= K, \quad \bar{U}(s) = \max\{K - \alpha e^{\beta s}, f^{-1}(\sigma)\}, \\
\bar{V}(s) &= e^{\lambda_1 s}, \quad \bar{V}(s) = \max\{e^{\lambda_1 s}(1 - Me^{\varepsilon s}), 0\}, \\
\end{align*}
\tag{9}
\]

where positive constants \( \alpha, \beta, \varepsilon \) and \( M \) will be determined later.

**Lemma 3.2.** The function \( \bar{V}(s) \) satisfies inequality

\[
c\bar{V}' \geq d_2 \bar{V}'' + \bar{V}[\gamma f(\bar{U}) - \delta]
\]

for any \( s \in \mathbb{R} \).

**Proof.** It is easy to show that

\[
d_2 \bar{V}'' - c\bar{V}' + \bar{V}[\gamma f(\bar{U}) - \delta] = [d_2 \lambda_1^2 - c\lambda_1 + \gamma f(K) - \delta]\bar{V} = H(\lambda_1)\bar{V} = 0
\]

for all \( s \in \mathbb{R} \). The proof is completed. \( \square \)

**Lemma 3.3.** Assume

\[
\beta < \frac{1}{2} \min\left\{ \frac{c}{d_1}, \lambda_1 \right\}, \quad \alpha > \max\left\{ K - f^{-1}(\sigma), \frac{f(K)}{(c - d_1 \beta)\beta} \right\}. \tag{10}
\]

Then the function \( \bar{U}(s) \) satisfies

\[
c\bar{U}' \leq d_1 \bar{U}'' + d(K - \bar{U}) - f(\bar{U})\bar{V} + \sigma\bar{V}
\]

for any \( s \neq \tilde{s}_1 := \frac{1}{\beta} \ln \frac{K - f^{-1}(\sigma)}{\alpha} \).

**Proof.** By the choice of \( \alpha \), it follows that \( \tilde{s}_1 < 0 \). If \( s > \tilde{s}_1 \), then \( \bar{U}(s) = f^{-1}(\sigma) \). It follows from (A2) that the inequality in this lemma holds. Now assume \( s < \tilde{s}_1 < 0 \) and we thus have \( \bar{U}(s) = K - \alpha e^{\beta s}, \bar{V}(s) = e^{\lambda_1 s} \). Then

\[
d_1 \bar{U}'' - c\bar{U}' + d(K - \bar{U}) - f(\bar{U})\bar{V} + \sigma\bar{V}
\]

\[
= c\alpha e^{\beta s} - d_1 \alpha^2 e^{\beta s} + d\alpha e^{\beta s} - \bar{f}(\bar{U})e^{\lambda_1 s} + \sigma\bar{V}
\]

\[
= \left[ c\alpha - d_1 \alpha^2 - f(\bar{U})e^{(\lambda_1 - \beta)s} \right] e^{\beta s} + d\alpha e^{\beta s} + \sigma\bar{V}
\]

\[
\geq [(c - d_1 \alpha)\beta - f(K)] e^{\beta s} + d\alpha e^{\beta s} + \sigma\bar{V} \geq 0,
\]

where we have used the fact \( c > d_1 \beta \) and \( e^{(\lambda_1 - \beta)s} < 1 \) due to \( s < 0 \) and \( \beta < \lambda_1 \). \( \square \)

**Lemma 3.4.** Let \( \alpha, \beta \) satisfy (10) and assume \( \varepsilon < \beta < \min\{\lambda_1, \lambda_2 - \lambda_1\}/2 \). Then for \( M > 0 \) large enough, the function \( \bar{V}(s) \) satisfies

\[
c\bar{V}' \leq d_2 \bar{V}'' + \bar{V}[\gamma f(\bar{U}) - \delta]
\]

for any \( s \neq -\frac{\ln M}{\varepsilon} \).

**Proof.** It is clear that \( \bar{U}(s) = f^{-1}(\sigma) \) if and only if \( s \geq \tilde{s}_1, \bar{V}(s) = 0 \) if and only if \( s \geq -\ln M/\varepsilon \), and \( -\ln M/\varepsilon < \tilde{s}_1 \) if and only if \( M > e^{-\tilde{s}_1} \). Assume \( M > e^{-\tilde{s}_1} \) in this proof. If \( s > -\ln M/\varepsilon \), then \( e^{\lambda_1 s}(1 - Me^{\varepsilon s}) < 0, \bar{V}(s) = 0 \) and the inequality in this lemma holds.

We now suppose \( s < -\ln M/\varepsilon \) (\( < \tilde{s}_1 < 0 \)), then

\[
\bar{U}(s) = K - \alpha e^{\beta s} > f^{-1}(\sigma), \quad \bar{V}(s) = e^{\lambda_1 s}(1 - Me^{\varepsilon s}) > 0.
\]

To prove this lemma, we only need to show

\[
e^{-\lambda_1 s} [d_2 \bar{V}'' - c\bar{V}' + \bar{V}[\gamma f(\bar{U}) - \delta]] \geq 0.
\]
Set
\[ M_0 := \max_{u \in [0, K]} f'(u), \quad M > \frac{\gamma M_0 \alpha}{-H(\lambda_1 + \varepsilon)}. \]

Then by the Mean Value Theorem, we have
\[
e^{-\lambda_1 \varepsilon} \left[ d_2 V'' - cV' + \gamma f(U)V - \delta V \right] = d_2 \lambda_1^2 - d_2 M(\lambda_1 + \varepsilon)^2 e^{\varepsilon \sigma} - c \lambda_1 + c M(\lambda_1 + \varepsilon) e^{\varepsilon \sigma} - \delta + \delta M e^{\varepsilon \sigma} + \gamma f'(K) - f'(\hat{u}) \alpha e^{\beta \varepsilon} (1 - M e^{\varepsilon \sigma})
\]
\[
\geq -M H(\lambda_1 + \varepsilon) e^{\varepsilon \beta \varepsilon} (1 - M e^{\varepsilon \sigma})
\]
\[
\geq -M H(\lambda_1 + \varepsilon) - \gamma M_0 \alpha e^{(\beta - \varepsilon)s} e^{\varepsilon \sigma} \geq 0,
\]
where \( f^{-1}(\sigma) < \hat{u} < K \), and the following facts are used:
\[
H(\lambda_1) = 0, \quad H(\lambda_1 + \varepsilon) < 0, \quad 0 < 1 - M e^{\varepsilon \sigma} \leq 1, \quad e^{(\beta - \varepsilon)s} \leq 1.
\]
The proof is completed. \( \square \)

Remark 3. It is obvious that positive constants \( \alpha, \beta, \varepsilon \) and \( M \) can be chosen such that Lemmas 3.2, 3.3 and 3.4 hold.

For \( a > 0 \), consider the following boundary-value problem:
\[
\begin{aligned}
d_1 U'' - cU' + d(K-U) - f(U)V + \sigma V &= 0, \quad s \in (-a, a), \\
d_2 V'' - cV' + \gamma f(U)V - \delta V &= 0, \quad s \in (-a, a), \\
(U,V)(-a) &= (U,V)(a), \quad (U,V)(s) = (\hat{U},\hat{V})(s), \\
\end{aligned}
\tag{11}
\]

Define
\[
\Gamma_a = \{(U,V)(\cdot) \in C([-a,a], \mathbb{R}^2) : U(s) \leq U(s) \leq \hat{U}(s), \hat{V}(s) \leq V(s) \leq \hat{V}(s), \quad s \in [-a,a] \}.
\]

Lemma 3.5. For \( a > 0 \) sufficiently large, boundary-value problem (11) has a solution \((U,V)(\cdot) \in C^2([-a,a], \mathbb{R}^2) \cap \Gamma_a \).

Proof. Set
\[
\rho = \max_{(u,v) \in \Gamma} \left( \left| \frac{\partial F_1}{\partial u}(u,v) \right| + \left| \frac{\partial F_2}{\partial v}(u,v) \right| \right),
\]
where
\[
\Gamma^* = \{(u,v) \in \mathbb{R}^2 : f^{-1}(\sigma) \leq u \leq K, 0 \leq v \leq \hat{V}(a) \}.
\]

Define the operator \( T : \Gamma_a \to C([-a,a], \mathbb{R}^2) \) by \( T(U^0, V^0) = (U,V) \) where \((U,V)(s)\) is the unique solution to
\[
\begin{cases}
- d_1 U'' + cU' + \rho U = \rho U^0 + F_1(U^0,V^0) = G_1(U^0,V^0), \quad s \in (-a,a), \\
- d_2 V'' + cV' + \rho V = \rho V^0 + F_2(U^0,V^0) = G_2(U^0,V^0), \quad s \in (-a,a), \\
(U,V)(-a) = (\hat{U},\hat{V})(-a), \quad (U,V)(a) = 0.
\end{cases}
\tag{12}
\]

Here \( F_i(u,v), i = 1, 2 \) are defined by (2). Regularity estimate for elliptic equation shows that \((U,V) \in C^2([-a,a], \mathbb{R}^2)\). From the choice of \( \rho \), we have, for all \((u,v) \in \Gamma^*\), that \( G_1(u,v) \) is increasing in \( u \) and decreasing in \( v \), and \( G_2(u,v) \) is increasing in both \( u \) and \( v \).

Claim 1. \( T(\Gamma_a) \subset \Gamma_a \) for \( a > 0 \) sufficiently large.
Assume \((U^0, V^0) \in \Gamma_c\) and \((U, V) = \mathcal{T}(U^0, V^0)\). Since \(a > 0\) is sufficiently large, we can assume that \(-a < s_1 < a\), where \(s_1\) is defined in Lemma 3.3. Define \(\phi(s) = U(s) - \overline{U}(s)\). Then Lemma 3.3 and the first equality of (12) give
\[-d_1\phi'' + c\phi' + \rho\phi \geq G_1(U^0, V^0) - G_1(U, V) \geq 0, \quad s \in (-a, s_1) \cup (s_1, a),\]
where the monotonicity property of \(G_1(u, v)\) is used for the last inequality. We next show \(\phi(s) \geq 0\) in \(s \in [-a, a]\). Assume to the contrary that there exists \(s^* \in [-a, a]\) such that \(\phi(s^*) < 0\). The boundary condition in (12) shows \(s^* \in (-a, a)\). Strong maximum principle implies \(\min_{[-a,a]} \phi(s) = \phi(s_1) < 0\), and Hopf’s Lemma yields that
\[U'(s_1) - U'(s_1) = \phi'(s_1) < 0, \quad U'(s_1) - U'(s_1) = \phi'(s_1) > 0,\]
implying
\[0 = U'(s_1) < U'(s_1) < U'(s_1) < 0,\]
a contradiction. This proves that \(\phi(s) \geq 0\) in \(s \in [-a, a]\), and therefore \(U(s) \geq \overline{U}(s)\) in \(s \in [-a, a]\). It can be similarly shown that \(U(s) \leq \overline{U}(s), V(s) \leq \overline{V}(s)\) for \(s \in [-a, a]\) and thus \((U, V) \in \Gamma_c\). The proof of this claim is completed.

Elliptic estimates imply that \(\mathcal{T} : \Gamma_c \to \Gamma_a\) is continuous and compact with respect to the norm of \(C([-a, a], \mathbb{R}^2)\). Obviously, \(\Gamma_a\) is closed and convex. Then Schauder’s fixed-point theorem shows that \(\mathcal{T}\) has a fixed point in \(\Gamma_a\), which is a non-negative solution of (11).

**Lemma 3.6.** System (8) with \(c > c^*\) has a positive solution \((U, V)(s), s \in \mathbb{R}\) satisfying boundary condition (4).

**Proof.** Lemma 3.5 shows that (8) has a solution \((U_k, V_k)(\cdot) \in C^2([-k, k], \mathbb{R}^2) \cap \Gamma_k\) for any positive integer \(k\) sufficiently large. Elliptic estimates show that, by passing to a (diagonal) subsequence,
\[(U_k, V_k)(s) \to (U_\infty, V_\infty)(s)\text{ in } C^2_{\text{loc}}(\mathbb{R}, \mathbb{R}^2),\]
where \((U_\infty, V_\infty)(\cdot) \in \Gamma_\infty\) is a non-negative solution of (8). Since \(\overline{U} \leq U_\infty \leq U, \overline{V} \leq V_\infty \leq \overline{V}\) in \(\mathbb{R}\), it follows from \((U, V)(\cdot) = E_0 = (\overline{U}, \overline{V})(\cdot)\) that \((U_\infty, V_\infty)(s)\) satisfies (4). Since \(U_\infty(s) \geq \overline{U}(s) > 0, V_\infty(s) \geq \overline{V}(s) > 0\) for \(-s\) sufficiently large, the strong maximum principle shows that \((U_\infty, V_\infty)(s)\) is positive on \(\mathbb{R}\). \(\square\)

### 3.2. Boundedness of traveling semi-fronts when \(c > c^*\).

**Lemma 3.7.** For any bounded sequence \(\{c_k\} \subset (c^*, +\infty)\), \(\|(U_k, V_k)(\cdot)\|_{C(\mathbb{R})}\) is bounded with respect to \(k\), where \((U_k, V_k)(s)\) is the positive solution of (8) with \(c = c_k\) obtained in Lemma 3.6.

**Proof.** We suppose to the contrary that this lemma is false. Then, by passing to a subsequence, we have \(\|(U_k, V_k)(\cdot)\|_{C(\mathbb{R})} \to +\infty\) and \(c_k \to c_\infty\) for some constant \(c_\infty \in [c^*, +\infty)\). Since \(U_k(s) \leq K\) for all \(k\) and \(s \in \mathbb{R}\) by the construction of (9), thus \(\mathcal{M}_k := \|V_k(\cdot)\|_{C(\mathbb{R})} \to +\infty\).

**Step 1.** If \(V_k(s_k) \to +\infty\) for some sequence \(\{s_k\}\), then
\[\hat{U}_k(s) := U_k(s + s_k) \to f^{-1}(\sigma)\text{ in } C_{\text{loc}}(\mathbb{R})\text{ as } k \to \infty.\]

Applying the Harnack’s inequality [1, Theorem 2.1] to the second equation of (8) yields that
\[\min_{s \in [-L,L]} V_k(s_k + s) \to +\infty\] (13)
for any $L > 0$. Then, for any $\epsilon > 0$, let $\overline{u}(s) \in C^2([-L, L])$ satisfy

$$\overline{u}(\pm L) = K, \quad \overline{u}(s) \geq \epsilon + f^{-1}(\sigma) \text{ in } [-L, L], \text{ and } \overline{u}(s) = \epsilon + f^{-1}(\sigma) \text{ in } [-L/2, L/2].$$

It follows from the limit (13) that if $k$ is large enough we have

$$d_1 \overline{u}'' - \sigma \overline{u} + d(K - \overline{u}) - [f(\overline{u}) - \sigma]V_k(s + s_k) \leq 0 \quad \text{for } s \in (-L, L),$$

where we used the fact $f(\overline{u}) - \sigma > \rho > 0$ for some constant $\rho$ (by the monotonicity property of $f(\cdot)$ in the assumption (A1)). Denote $\hat{w}(s) := \hat{u}(s) - U_k(s + s_k)$, so that $\hat{w}(\pm L) \geq 0$. Then by the first equality of (8) we get for all $s \in (-L, L)$ and large $k$ that

$$d_1 \hat{w}'' - \sigma \hat{w} - d \hat{w} - [f(\hat{u}) - f(U_k(s + s_k))]V_k(s + s_k) = d_1 \hat{w}'' - \sigma \hat{w} - [d + f'(\hat{u})d_k(s + s_k)]w \leq 0,$$

where $\hat{u}$ is between $\overline{u}$ and $U_k(s + s_k)$. Since $f'(-\hat{u}) \geq 0$, the comparison principle shows that, for sufficiently large $k$, $\hat{U}_k(s) \leq \overline{u}(s)$ in $s \in [-L, L]$ and thus $\hat{U}_k(s) \leq \epsilon + f^{-1}(\sigma)$ in $s \in [-L/2, L/2]$. Since $\epsilon$ and $L$ are arbitrary and $\hat{U}_k(s) \geq f^{-1}(\sigma)$, we have proved that $\hat{U}_k(s) \to f^{-1}(\sigma)$ as $k \to \infty$ in $C_{\text{loc}}(\mathbb{R})$. 

**Step 2.** It is impossible that there exists a sequence $s_k \in \mathbb{R}$ such that

$$V_k(s_k) \to +\infty, \quad V_k''(s_k) \leq 0. \tag{14}$$

Suppose to the contrary that (14) holds for some sequence $s_k \in \mathbb{R}$. Define $\hat{V}_k(s) := V_k(s_k + s)/V_k(s_k)$, implying $\hat{V}_k(0) = 1$. The Harnack’s inequality [1, Theorem 2.1] applied to the second equation of (8) shows that $\hat{V}_k(s)$ is bounded in $s \in [-L, L]$ uniformly with respect to $k$ for fixed $L > 0$. Then it follows from [22, Lemma 3.7], Step 1 and the second equation of (8) that, by passing to a subsequence, $\hat{V}_k(s) \to \hat{V}_\infty(s)$ in $C_{\text{loc}}^2(\mathbb{R})$, where $\hat{V}_\infty(s)$ is a non-negative solution of

$$d_2 \hat{V}'' - c_{\infty} \hat{V}' + (\gamma \sigma - \delta)\hat{V} = 0, \quad s \in \mathbb{R}.$$

By the non-negativity we have $\hat{V}_\infty(s) = a_1 e^{\lambda_1 s} + a_2 e^{\lambda_2 s}$, where $a_1 \geq 0, a_2 \geq 0, a_1 + a_2 = 1$, and $\lambda_1 < \lambda_2$ are the two roots of

$$d_2 \lambda^2 - c_{\infty} \lambda + \gamma \sigma - \delta = 0. \tag{15}$$

It follows from (A2) that $\gamma \sigma - \delta < 0$, implying that $\lambda_1 < 0 < \lambda_2$. Obviously, $\hat{V}_\infty''(0) = a_1 \lambda_1^2 + a_2 \lambda_2^2 > 0$, contradicting (14).

**Step 3.** $V_k(s)$ is strictly increasing for large $s$ and large $k$, and $\mathcal{M}_k = V_k(+\infty) \to +\infty$ as $k \to \infty$.

It follows from Step 2 and $\mathcal{M}_k = \|V_k(\cdot)\|_{C(\mathbb{R})} \to +\infty$ that $\mathcal{M}_k \neq \max_{s \in \mathbb{R}} V_k(s)$ for all large $k$ (by passing to a subsequence). Since $V_k(-\infty) = 0$ (see (4)), we have $\mathcal{M}_k = \limsup_{s \to +\infty} V_k(s) \to +\infty$. Again it follows from Step 2 that $V_k(s)$ is strictly increasing for large $s$ and large $k$. This proves Step 3.

**Step 4.** $\mathcal{M}_k = +\infty$ for all large $k$.

Suppose that this is false. Then there exists a subsequence $\{k_j\}$ such that $\mathcal{M}_{k_j} < +\infty$ for all $j$. Step 3 implies that there exists a sequence $s_{k_j} \to +\infty$ such that $V_{k_j}(s_{k_j}) \to +\infty$, $V_{k_j}''(s_{k_j}) \leq 0$, contradicting Step 2. This proves Step 4.

Now fix $k_0$ to be a large positive integer. Steps 3 and 4 yield that $V_{k_0}(s)$ is strictly increasing for large $s$ and $V_{k_0}(+\infty) = +\infty$. For any sequence $s_j \to +\infty$
Lemma 3.8. Existence of traveling semi-front when $c > c^\star$.

We will complete this proof by some steps.

**Step 1.** Let $(U, V)(s)$ be the positive solution of (8) with $c = c^\star$ obtained in Lemma 3.6. Then $\liminf_{s \to +\infty} U(s) \leq u^\star$ ($< K$), where $u^\star$ is defined by (5).

We suppose to the contrary that $\liminf_{s \to +\infty} U(s) > u^\star$, from which it follows that $\gamma f(U(s)) - \delta > 0$ for large $s$. This implies that $V(s)$ is strictly monotonic for large $s$. Otherwise, there exists a sequence $s_k \to +\infty$ such that $V'(s_k) = 0, V''(s_k) \geq 0$, contradicting the second equation of (8). We further claim that $V(s)$ is strictly increasing for large $s$. Otherwise, there exists $s^\star > 1$ such that $V'(s) \leq 0$ for all $s > s^\star$, which, together with the second equality of (8), implies $V''(s_k) < 0$ for all $s > s^\star$ and thus $V(+\infty) = -\infty$. This contradicts the positivity of $V(s)$ in $\mathbb{R}$. Therefore, $V(s)$ is strictly increasing for large $s$ and $0 < V(+\infty) := \lim_{s \to +\infty} V(s) < +\infty$. It follows from the boundedness of $V'(s)$ and $V''(s)$ on $\mathbb{R}$ that $V'(+\infty) = 0 = V''(+\infty)$, which, together with the second equality of (8), gives that $U(+\infty) = u^\star$. However, this contradicts the assumption $\liminf_{s \to +\infty} U(s) > u^\star$. Step 1 is proved.

Let $c_k = c^\star + 1/k$, and let $(U_k, V_k)(s)$ be the positive solution of (8) with $c = c_k$ obtained in Lemma 3.6. By Step 1, boundary condition (4) and translations, we can assume that

$$U_k(0) = \hat{u} := \frac{K + u^\star}{2}, \ U_k(s) \geq \hat{u} \ \forall s < 0.$$  

From Lemma 3.7, [22, Lemma 3.7] and by passing to a subsequence, $(U_k, V_k)(s) \to (U_\infty, V_\infty)(s)$ in $C^2_{\text{loc}}(\mathbb{R})$, where $(U_\infty, V_\infty)(s)$ is a non-negative solution of (8) with $c = c^\star$ such that

$$U_\infty(0) = \hat{u}, \ U_\infty(s) \geq \hat{u} \ \forall s < 0.$$  

**Step 2.** $U_\infty(s) > 0, V_\infty(s) > 0$ for all $s \in \mathbb{R}$.  

Define $\tilde{V}(s) := V_k(s + s_j)/V_k(s_j)$. Then, by passing to a subsequence and similar to the proof of Step 2, we have

$$\tilde{V}(s) \to \tilde{V}_\infty(s) = a_1 e^{\lambda_1 s} + a_2 e^{\lambda_2 s} \ \text{in} \ C^2_{\text{loc}}(\mathbb{R}),$$

where $a_1 \geq 0$, $a_2 \geq 0$, $a_1 + a_2 = 1$, and $\lambda_1^2 < 0 < \lambda_2^2$ are given by (15). Since $k_0$ is large enough, and $\gamma f(K) - \delta > 0, \gamma\sigma - \delta < 0$ (by assumption (A2)), it follows that $\lambda_1^2 < 0 < \lambda_2^2 < \lambda_2^2$, where $\lambda_1$ and $\lambda_2$ are the two roots of $d_2 \lambda^2 - c_k \lambda + \gamma f(K) - \delta$. Then Step 3 shows that $\tilde{V}_\infty(s)$ is non-decreasing in $\mathbb{R}$, implying $a_1 = 0$ and $\tilde{V}_\infty(s) = e^{\lambda_2 s}$. Since $\lambda_2^2 > \lambda_1$, we have

$$\lim_{s \to -\infty} \tilde{V}'(s) = \lim_{s \to -\infty} \tilde{V}'(0) = \tilde{V}'(0) = \lambda_2 > \frac{\lambda_2^2 + \lambda_1}{2} > \lambda_1.$$ 

The arbitrariness of $\{s_j\}$ gives $\liminf_{j \to \infty} \tilde{V}(s_j) > \tilde{V}_\infty(s_j) > \tilde{V}_\infty(s) > \tilde{V}_\infty(s) > \tilde{V}_\infty(s) > \tilde{V}_\infty(s)$. Therefore, there exists $s^\star > 0$ such that

$$V_k(s) > V_k(s^\star) e^{\tilde{V}(s-s^\star)} = e^{\lambda_1 s} V_k(s^\star) e^{(\lambda_1-\lambda_2) s - \tilde{V}_\infty(s)} \ \text{for} \ s \geq s^\star. \ (16)$$

However, the construction in (9) gives $V_k(s) \leq e^{\lambda_1 s}$ for all $s \in \mathbb{R}$, contradicting (16).
It follows from the construction of (9) that \( U_\infty(s) \geq f^{-1}(\sigma) > 0 \) for all \( s \in \mathbb{R} \).
Now assume there exists \( s_0 \) such that \( V_\infty(s_0) = 0 \). The Harnack’s inequality gives that \( V_\infty(s) \equiv 0 \) on \( \mathbb{R} \). This means that \( U_\infty(s) \) satisfies \( c'U''_\infty = d_1U''_\infty + d(K - U_\infty) \), implying that
\[
U_\infty(s) = a_1e^{\lambda_1s} + a_2e^{\lambda_2s} + K,
\]
where \( \lambda_1 < 0 < \lambda_2 \) are the roots of \( d_1\lambda^2 - c'\lambda - d = 0 \). Then the boundedness of \( U_\infty(s) \) over \( \mathbb{R} \) gives that \( U_\infty(s) \equiv K \), contradicting \( U_\infty(0) = \hat{u} < K \). We thus have that \( V_\infty(s) > 0 \) for all \( s \in \mathbb{R} \). This proves Step 2.

To complete the proof of this lemma, we only need to show \( (U_\infty, V_\infty)(-\infty) = E_0 \). However, noting that \( U_\infty(s) \geq \hat{u} > u^* \) for all \( s \leq 0 \), we can, similar to the proof of Step 1, prove \( (U_\infty, V_\infty)(-\infty) = E_0 \).

### 3.4. Connecting to \( E^* \)
With Lemmas 3.6 and 3.8, we only need to show \( (U, V)(+\infty) = E^* \) to complete the proof of Theorem 2.1(i).

**Lemma 3.9.** The traveling semi-fronts obtained in Lemmas 3.6 and 3.8 satisfy \( (U, V)(+\infty) = E^* \).

**Proof.** We first claim that
\[
\liminf_{s \to +\infty} U(s) > f^{-1}(\sigma), \quad \frac{\|V'(\cdot)\|}{\|V(\cdot)\|_{C(\mathbb{R})}} < +\infty. \tag{17}
\]
It follows from [22, Lemma 3.7] and \( \|(U(\cdot), V(\cdot))\|_{C(\mathbb{R})} < +\infty \) that \( \|(U(\cdot), V(\cdot))\|_{C^2(\mathbb{R})} < +\infty \). By the construction of (9), we have \( U(s) \geq f^{-1}(\sigma) \) for all \( s \in \mathbb{R} \). We to the contrary assume that \( \liminf_{s \to +\infty} U(s) = f^{-1}(\sigma) \). If \( \liminf_{s \to +\infty} U(s) < \limsup_{s \to +\infty} U(s) \), there exists a sequence \( s_k \to +\infty \) such that
\[
U(s_k) \to f^{-1}(\sigma), \quad U'(s_k) = 0, \quad U''(s_k) > 0.
\]
By passing to a subsequence, it can be supposed that \( U''(s_k) \to U'' \geq 0 \). Then it follows from the first equality of (8) that \( 0 = d_1U''_\infty + d(K - f^{-1}(\sigma)) \), implying \( f^{-1}(\sigma) > K \). However, this contradicts assumption (A2). Thus we have
\[
\liminf_{s \to +\infty} U(s) = \limsup_{s \to +\infty} U(s) =: U(+\infty) = f^{-1}(\sigma).
\]
\( \|(U(\cdot), V(\cdot))\|_{C^2(\mathbb{R})} \) further gives \( U'(+\infty) = U''(+\infty) = 0 \). Then the first equality of (8) yields \( 0 = d(K - f^{-1}(\sigma)) \), contradicting assumption (A2). We thus have proved \( \liminf_{s \to +\infty} U(s) > f^{-1}(\sigma) \). The second inequality of (17) can be deduced from [22, Lemma 3.7]. Thus (17) was proved.

System (8) can be written as
\[
\begin{aligned}
U' &= W, \\
d_1W' &= cW - F_1(U, V), \\
V' &= Z, \\
d_2Z' &= cZ - F_2(U, V),
\end{aligned}
\tag{18}
\]
where \( F_1(U, V), F_2(U, V) \) are defined by (2). Similar to the expression (3.3) of Huang [14], define
\[
L_1 = c \int_{u^*}^{U} \frac{f(\xi) - f(u^*)}{f(\xi) - \sigma} d\xi - d_1Wf(U) - f(u^*), \quad L_2 = c \int_{V}^{V} \frac{\xi - v^*}{\xi} d\xi - d_2ZV - v^*.
\]
Since $U'(s), V'(s)$ are bounded on $\mathbb{R}$, (17) implies that $L_1$ is bounded in $s \in [s^*, +\infty)$ for some $s^* \gg 1$ and $L_2$ is bounded below in $s \in [s^*, +\infty)$. Calculating the derivative of $L_1$ along the traveling wave solution $(U(s), V(s))$ gives
\[
\frac{dL_1}{ds} = (cU' - d_1W')\frac{f(U) - f(u^*)}{f(U) - \sigma} - d_1Wf'(U)\frac{f(u^*) - \sigma}{[f(U) - \sigma]^2}.
\]
Noting that $d(K - u^*) - f(u^*)v^* + \sigma v^* = 0$, we deduce that
\[
F_1(U, V) = d(u^* - U) + (f(u^*) - f(U))v^* + (f(U) - \sigma)(v^* - V).
\]
Then we have
\[
\frac{dL_1}{ds} = -\frac{d[f(U) - f(u^*)][U - u^*]}{f(U) - \sigma} - \frac{v^*[f(U) - f(u^*)]^2}{f(U) - \sigma} + \frac{[f(U) - f(u^*)](v^* - V) - d_1W^2f'(U)(f(u^*) - \sigma)}{[f(U) - \sigma]^2}.
\]
Similarly, along the traveling wave solution $(U(s), V(s))$, we have
\[
\frac{dL_2}{ds} = \gamma(V - v^*)[f(U) - f(u^*)] - \frac{d_2v^*Z^2}{V^2}.
\]
Define $L = L_1 + L_2/\gamma$. Then, along the traveling wave solution $(U(s), V(s))$, we have
\[
\frac{dL}{ds} = -\frac{d[f(U) - f(u^*)][U - u^*]}{f(U) - \sigma} - \frac{v^*[f(U) - f(u^*)]^2}{f(U) - \sigma} - \frac{d_1W^2f'(U)(f(u^*) - \sigma)}{[f(U) - \sigma]^2} - \frac{d_2v^*Z^2}{\gamma V^2}.
\]
Since $f(u^*) > \sigma$ by Remark 1, $f'(U) \geq 0$ by (A1), and $f(U) - \sigma > 0$ for all large $s$ by (17), the monotonicity of $f(U)$ in (A1) gives that $\frac{dL}{ds} \leq 0$ and the maximal invariant set of $\{(U, W, V, Z) : L' = 0\}$ consists of only the equilibrium $(u^*, 0, v^*, 0)$. Then the LaSalle’s invariance principle completes the proof.

Proof of Theorem 2.1(i). Combination of Lemmas 3.1, 3.6, 3.8 and 3.9 completes the proof of Theorem 2.1(i).

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