Ranking Policy Gradient

Kaixiang Lin  
Department of Computer Science and Engineering  
Michigan State University  
East Lansing, MI 48824-4403, USA

Jiayu Zhou  
Department of Computer Science and Engineering  
Michigan State University  
East Lansing, MI 48824-4403, USA

Abstract

Sample inefficiency is a long-lasting problem in reinforcement learning (RL). The state-of-the-art uses value function to derive policy while it usually requires an extensive search over the state-action space, which is one reason for the inefficiency. Towards the sample-efficient RL, we propose ranking policy gradient (RPG), a policy gradient method that learns the optimal ranking of a set of discrete actions. To accelerate the learning of policy gradient methods, we describe a novel off-policy learning framework and establish the equivalence between maximizing the lower bound of return and imitating a near-optimal policy without accessing any oracles. These results lead to a general sample-efficient off-policy learning framework, which accelerates learning and reduces variance. Furthermore, the sample complexity of RPG does not depend on the dimension of state space, which enables RPG for large-scale problems. We conduct extensive experiments showing that when consolidating with the off-policy learning framework, RPG substantially reduces the sample complexity, comparing to the state-of-the-art.

1. Introduction

One of the major challenges in reinforcement learning (RL) is the high sample complexity (Kakade et al., 2003), which is the number of samples must be collected to conduct successful learning. There are different reasons leading to poor sample efficiency of RL (Yu, 2018). One often overlooked but decisive factor is the usage of value functions, which have been widely adopted in state-of-the-art (Van Hasselt et al., 2016; Hessel et al., 2017; Mnih et al., 2016; Schulman et al., 2017; Gruslys et al., 2018). Because algorithms directly optimizing return, such as REINFORCE Williams (1992), could suffer from high variance (Sutton and Barto, 2018), value function baselines were introduced by actor-critic methods to reduce the variance. However, since a value function is associated with a certain policy, the samples collected by former policies cannot be readily used without complicated manipulations (Degris et al., 2012) and extensive parameter tuning (Nachum et al., 2017). Such an on-policy requirement increases the difficulty of sample-efficient learning. On the other hand, off-policy methods, such as one-step $Q$-learning (Watkins and Dayan, 1992) and variants of deep $Q$ networks (DQN) (Hessel et al., 2017; Dabney et al., 2018; Van Hasselt et al., 2016), are currently among the most sample-efficient algorithms. These algorithms, however, often require extensive searching (Bertsekas and Tsitsiklis, 1996) over a usually large state-action
space to estimate the action value function. This requirement could limit the sample-efficiency of these algorithms.

To address the aforementioned challenge, we first revisit the decision process of RL. For a deterministic decision process, the action with the largest action value is chosen. For a stochastic decision process, the action values are normalized into a probability distribution, from which an action is sampled. In both cases, the crucial factor deciding which action to take is the relative relationship of actions rather than the absolute values. Therefore, instead of estimating the return of each action (i.e., the action-value function), we propose the ranking policy gradient (RPG), a policy gradient method that optimizes the ranking of actions with respect to the long-term reward by learning the pairwise relationship among actions.

Secondly, we propose an off-policy learning framework to improve the sample-efficiency of policy gradient methods, without relying on value function baselines. We establish the theoretical equivalence between RL optimizing the lower bound of the long-term reward and learning the near-optimal policy in a supervised manner. The central idea is that we separate policy learning into two stages: exploration and supervision. During the exploration stage, we use the near-optimal trajectories encountered to construct a training dataset that approximates the state-action pairs sampled from a near-optimal policy. During the supervision stage, we imitate the near-optimal policy with the training dataset. Such a separated design empowers the flexibility of off-policy learning so that we can smoothly incorporate various exploration strategies to improve sample-efficiency. Also, by using a supervised learning approach, the upper bound of gradient variance is largely reduced because of its independence of the horizon and reward scale, which does not hold for general policy gradient methods. This learning paradigm leads to a novel sample complexity analysis of large-scale MDP, in a non-tabular setting without the linear dependence on the state space. Besides, we empirically show that there is a trade-off between optimality and sample-efficiency. Last but not least, we demonstrate that the proposed approach, consolidating the RPG with off-policy learning, substantially outperforms the state-of-the-art (Hessel et al., 2017; Bellemare et al., 2017; Dabney et al., 2018; Mnih et al., 2015).

2. Related works

Sample Efficiency. The sample efficient reinforcement learning can be roughly divided into two categories. The first category includes variants of Q-learning (Mnih et al., 2015; Schaul et al., 2015; Van Hasselt et al., 2016; Hessel et al., 2017). The main advantage of Q-learning methods is the use of off-policy learning, which is essential towards sample efficiency. The representative DQN (Mnih et al., 2015) introduced deep neural network in Q-learning, which further inspired a track of successful DQN variants such as Double DQN (Van Hasselt et al., 2016), Dueling networks (Wang et al., 2015), prioritized experience replay (Schaul et al., 2015), and RAINBOW (Hessel et al., 2017). The second category is the actor-critic approaches. Most of recent works (Degris et al., 2012; Wang et al., 2016; Gruslys et al., 2018) in this category leveraged importance sampling by re-weighting the samples to correct the estimation bias and reduce variance. The main advantage is in the wall-clock times due to the distributed framework, firstly presented in (Mnih et al., 2016), instead of the sample-efficiency. As of the time of writing, the variants of DQN (Hessel et al., 2017; Dabney et al., 2018; Bellemare et al., 2017; Schaul et al., 2015; Van Hasselt et al., 2016)
are among the algorithms of most sample efficiency, which are adopted as our baselines for comparison.

**RL as Supervised Learning.** Many efforts have focused on developing the connections between RL and supervised learning, such as Expectation-Maximization algorithms (Dayan and Hinton, 1997; Peters and Schaal, 2007; Kober and Peters, 2009; Abdolmaleki et al., 2018), Entropy-Regularized RL (Oh et al., 2018; Haarnoja et al., 2018), and Interactive Imitation Learning (IIL) (Daumé et al., 2009; Syed and Schapire, 2010; Ross and Bagnell, 2010; Ross et al., 2011. Sun et al., 2017; Hester et al., 2018; Osa et al., 2018). EM-based approaches apply the probabilistic framework to formulate the RL problem maximizing a lower bound of the return as a re-weighted regression problem, while it requires on-policy estimation on the expectation step. Entropy-Regularized RL optimizing entropy augmented objectives can lead to off-policy learning without the usage of importance sampling while it converges to soft optimality (Haarnoja et al., 2018).

Of the three tracks in prior works, the IIL is most closely related to our work. The IIL works firstly pointed out the connection between imitation learning and reinforcement learning (Ross and Bagnell, 2010; Syed and Schapire, 2010; Ross et al., 2011) and explore the idea of facilitating reinforcement learning by imitating experts. However, most of imitation learning algorithms assume the access to the expert policy or demonstrations. The off-policy learning framework proposed in this paper can be interpreted as an online imitation learning approach that constructs expert demonstrations during the exploration without soliciting experts, and conducts supervised learning to maximize return at the same time.

In conclusion, our approach is different from prior arts in terms of at least one of the following aspects: objectives, oracle assumptions, the optimality of learned policy, and on-policy requirement. More concretely, the proposed method is able to learn optimal policy in terms of long-term reward, without access to the oracle (such as expert policy or expert demonstration) and it can be trained both empirically and theoretically in an off-policy fashion. A more detailed discussion of the related work is provided in Appendix A.

**PAC Analysis of RL.** Most existing studies on sample complexity analysis (Kakade et al., 2003; Strehl et al., 2006; Kearns et al., 2000; Strehl et al., 2009; Krishnamurthy et al., 2016; Jiang et al., 2017; Jiang and Agarwal, 2018; Zanette and Brunskill, 2019) are established on the value function estimation. The proposed approach leverages the probably approximately correct framework (Valiant, 1984) in a different way such that it does not rely on the value function. Such independence directly leads to a practically sample-efficient algorithm for large-scale MDP, as we demonstrated in the experiments.

### 3. Notations and Problem Setting

In this paper, we consider a finite horizon $T$, discrete time Markov Decision Process (MDP) with a finite discrete state space $\mathcal{S}$ and for each state $s \in \mathcal{S}$, the action space $\mathcal{A}_s$ is finite. The environment dynamics is denoted as $\mathcal{P} = \{p(s'|s,a), \forall s, s' \in \mathcal{S}, a \in \mathcal{A}_s\}$. We note that the dimension of action space can vary given different states. We use $m = \max_s |\mathcal{A}_s|$ to denote the maximal action dimension among all possible states. Our goal is to maximize the expected sum of positive rewards, or return $J(\theta) = \mathbb{E}_{\tau, \pi_{\theta}}[\sum_{t=1}^{T} r(s_t, a_t)]$, where $0 < r(s,a) < \infty, \forall s,a$. In this case, the optimal deterministic Markovian policy always exists (Puterman, 2014, Proposition 4.4.3). The upper bound of trajectory reward ($r(\tau)$) is denoted as
| Notations | Definition |
|-----------|------------|
| $Q_{ij}$  | The discrepancy of the value of action $i$ and action $j$. $Q_{ij} = Q_i - Q_j$, where $Q_i = Q(s, a_i)$. Notice that the value here is not the estimation of return, it represents which action will have relatively higher return if followed. |
| $p_{ij}$  | $p_{ij} = P(Q_i > Q_j)$ denotes the probability that $i$-th action is to be ranked higher than $j$-th action. Notice that $p_{ij}$ is controlled by $\theta$ through $Q_i, Q_j$. |
| $\tau$    | A trajectory $\tau = \{s(\tau, t), a(\tau, t)\}_{t=1}^{T}$ collected from the environment. It is worth noting that this trajectory is not associated with any policy. It only represents a series of state-action pairs. We also use the abbreviation $s_t = s(\tau, t), a_t = a(\tau, t)$. |
| $r(\tau)$ | The trajectory reward $r(\tau) = \sum_{t=1}^{T} r(s_t, a_t)$ is the sum of reward along one trajectory. |
| $\sum_{\tau}$ | The summation over all possible trajectories $\tau$. |
| $p(\tau)$ | The probability of a specific trajectory is collected from the environment given policy $\pi_\theta$. $p(\tau) = p(s_0)\Pi_{t=1}^{T} p(a_t|s_t)p(s_{t+1}|s_t, a_t)$ |
| $\mathcal{T}$ | The set of all possible near-optimal trajectories. $|\mathcal{T}|$ denotes the number of near-optimal trajectories in $\mathcal{T}$. |
| $n$ | The number of training samples or equivalently state action pairs sampled from uniformly (near)-optimal policy. |
| $m$ | The number of discrete actions. |

Table 1: Notations

$R_{\text{max}} = \max_{\tau} r(\tau)$. Table 1 provides a comprehensive reference describing the notations used throughout this work.

4. Ranking Policy Gradient

Value function estimation is widely used in advanced RL algorithms (Mnih et al., 2015, 2016; Schulman et al., 2017; Gruslys et al., 2018; Hessel et al., 2017; Dabney et al., 2018) to facilitate the learning process. In practice, the on-policy requirement of value function estimations in actor-critic methods has largely increased the difficulty of achieving sample-efficient learning (Degris et al., 2012; Gruslys et al., 2018). With the advantage of off-policy learning, the DQN (Mnih et al., 2015) variants are currently among the most sample-efficient algorithms (Hessel et al., 2017; Dabney et al., 2018; Bellemare et al., 2017). For complicated tasks, the value function can well align with the relative relationship of returns of actions, but the absolute values are hardly accurate (Mnih et al., 2015; Ilyas et al., 2018).

The above observations motivate us to look at the decision phase of RL from a different prospect: Given a state, the decision making is to perform a relative comparison over available actions and then choose the best action, which leads to a relatively higher return than others. Therefore, an alternative solution is to learn the ranking of the actions, instead of deriving policy from the action values. In this section, we show how to optimize the ranking of a
set of discrete actions to maximize the return, and thus avoid value function estimation. The discussion of the second question will be given in Section 5. In this section, the action value functions or $q$-values no longer represent the return, but are only used to illustrate the relative relationship of available actions.

In order to optimize the ranking, we formulate the action ranking problem as follows. Denote the action value as $Q(s, a_i) = Q_{\theta}(s, a_i)$, $\forall i = 1, \ldots, m$ given a state $s$, where the model parameter $\theta$ is omitted for concise presentation. Our goal is to optimize the action values such that the best action has the highest probability to be selected than others. Inspired by learning to rank (Burges et al., 2005), we consider the pairwise relationship among all actions, by modeling the probability (denoted as $p_{ij}$) of an action $a_i$ to be ranked higher than any action $a_j$ as follows:

$$p_{ij} = \frac{\exp(Q(s, a_i) - Q(s, a_j))}{1 + \exp(Q(s, a_i) - Q(s, a_j))}, \quad (1)$$

where $p_{ij} = 0.5$ means the action value of $a_i$ is same as that of the action $a_j$, $p_{ij} > 0.5$ indicates that the action $a_i$ is ranked higher than $a_j$. We would like to increase the probability of the optimal action $a_i$ such that it ranks higher than any other actions, which is the probability action $a_i$ is chosen. The pairwise ranking policy is defined in Eq (2), given the mild Assumption 1 is satisfied. Please refer to Appendix H for the discussions on Assumption 1.

**Definition 1** The pairwise ranking policy is defined as:

$$\pi(a = a_i|s) = \prod_{j=1, j\neq i}^m p_{ij},$$

where the $p_{ij}$ is defined in Eq (1), given current state $s$, or equivalently given the current action values $q = [Q_1, \ldots, Q_m]$.

**Assumption 1** Given a state $s$, the events that the action $a_i$ is ranked higher than the action $a_j$ are independent, for all $j \neq i$.

To increase the probability of selecting the best action, is equal to increase the joint probability that the best action is ranked higher than all other actions. With Assumption 1, we can decompose the joint probability as the multiplication of pairwise probability Eq (1). Our ultimate goal is to maximize the long-term reward through optimizing the pairwise relationship among the action pairs. To achieve this goal, we resort to the policy gradient method. Formally, we propose the ranking policy gradient method (RPG), as shown in Theorem 2.

**Theorem 2 (Ranking Policy Gradient Theorem)** For any MDP, the gradient of the expected long-term reward $J(\theta) = \sum_{\tau} p_{\theta}(\tau) r(\tau)$ w.r.t. the parameter $\theta$ of a pairwise ranking policy (Def 1) is given by:

$$\nabla_{\theta} J(\theta) = \mathbb{E}_{\tau \sim p_{\theta}} \left[ \sum_{t=1}^T \nabla_{\theta} \left( \sum_{j=1, j\neq i}^m (Q_i - Q_j)/2 \right) r(\tau) \right], \quad (3)$$

and the deterministic pairwise ranking policy $p_{\theta}$ is: $a = \arg \max_i Q_i$, $i = 1, \ldots, m$, where $Q_i$ denotes the action value of action $a_i$ ($Q_{\theta}(s_t, a_t)$, $a_i = a_t$), $s_t$ and $a_t$ denotes the $t$-th state-action pair in trajectory $\tau$, $Q_j, \forall j \neq i$ denote the action values of all other actions that were not taken given state $s_t$ in trajectory $\tau$, i.e., $Q_{\theta}(s_t, a_j)$, $\forall a_j \neq a_t$. 

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**Note:** The above text was generated as natural language and may need further refinement for clarity and coherence. The equations and notation are consistent with the provided text.
The proof of Theorem 2 is provided in Appendix B. Theorem 2 states that optimizing the discrepancy between the action values of the best action and all other actions, is optimizing the pairwise relationships that maximize the return. One limitation of RPG is that it is not convenient for the tasks where only optimal stochastic policies exist since the pairwise ranking policy takes extra efforts to construct a probability distribution [see Appendix B.1]. In order to learn the stochastic policy, we introduce Listwise Policy Gradient (LPG) that optimizes the probability of ranking a specific action on the top of a set of actions, with respect to the return. In the context of RL, this top one probability is the probability of action \( a_i \) to be chosen, which is equal to the sum of probability all possible permutations that map action \( a_i \) in the top. This probability is computationally prohibitive since we need to consider the probability of \( m! \) permutations. Inspired by listwise learning to rank approach (Cao et al., 2007), the top one probability can be modeled by the softmax function (see Theorem 3). Therefore, LPG is equivalent to the REINFORCE (Williams, 1992) algorithm with a softmax layer. LPG provides another interpretation of REINFORCE algorithm from the perspective of learning the optimal ranking and enables the learning of both deterministic policy and stochastic policy (see Theorem 4).

**Theorem 3** ((Cao et al., 2007), Theorem 6) Given the action values \( q = [Q_1, ..., Q_m] \), the probability of action \( i \) to be chosen (i.e. to be ranked on the top of the list) is:

\[
\pi(a_t = a_i | s_t) = \frac{\phi(Q_i)}{\sum_{j=1}^{m} \phi(Q_j)},
\]

where \( \phi(*) \) is any increasing, strictly positive function. A common choice of \( \phi \) is the exponential function.

**Theorem 4** (Listwise Policy Gradient Theorem) For any MDP, the gradient of the long-term reward \( J(\theta) = \sum_{\tau} p_{\theta}(\tau) r(\tau) \) w.r.t. the parameter \( \theta \) of listwise ranking policy takes the following form:

\[
\nabla_{\theta} J(\theta) = \mathbb{E}_{\tau \sim \pi_{\theta}} \left[ \sum_{t=1}^{T} \nabla_{\theta} \left( \log \frac{e^{Q_i}}{\sum_{j=1}^{m} e^{Q_j}} \right) r(\tau) \right],
\]

where the listwise ranking policy \( \pi_{\theta} \) parameterized by \( \theta \) is given by Eq (6) for tasks with deterministic optimal policies:

\[
a = \arg \max_{i} Q_i, \quad i = 1, \ldots, m
\]

or Eq (7) for stochastic optimal policies:

\[
a \sim \pi(*)|s), \quad i = 1, \ldots, m
\]

where the policy takes the form as in Eq (8)

\[
\pi(a = a_i|s_t) = \frac{e^{Q_i}}{\sum_{j=1}^{m} e^{Q_j}}
\]

is the probability that action \( i \) being ranked highest, given the current state and all the action values \( Q_1 \ldots Q_m \).
The proof of Theorem 4 exactly follows the direct policy differentiation (Peters and Schaal, 2008; Williams, 1992) by replacing the policy to the form of the Softmax function. The action probability $\pi(a_i|s), \forall i = 1, ..., m$ forms a probability distribution over the set of discrete actions Cao et al. (2007, Lemma 7). Theorem 4 states that the vanilla policy gradient (Williams, 1992) parameterized by Softmax layer is optimizing the probability of each action to be ranked highest, with respect to the long-term reward. Furthermore, it enables learning both of the deterministic policy and stochastic policy.

Comparing LPG with RPG, one advantage of LPG is that it is more convenient to model the stochastic policies since softmax directly constructs a valid probability distribution. However, forming a probability distribution also limits the expressive power of the model since the dimension of the action space is fixed. Since RPG is optimizing the pairwise relationship among actions, it is possible to use one model to solve the tasks with dynamic action spaces and multi-task RL, without the design of the action masking mechanism (Li et al., 2017) or adding each specific layer for each task (Rusu et al., 2015). Furthermore, RPG is more sample-efficient than LPG when we are learning a deterministic policy.

To this end, seeking sample-efficiency motivates us to learn the relative relationship (RPG in Theorem 2 and LPG in Theorem 4) of actions, instead of deriving policy based on value function estimations. However, both of RPG and LPG belong to policy gradient methods, which suffer from large variance and require on-policy learning (Sutton and Barto, 2018). Therefore, the intuitive implementations of RPG or LPG are still far from sample-efficient. In the next section, we will describe a general off-policy learning framework empowered by supervised learning, which provides an alternative way to accelerate learning other than using value function baselines.

5. Off-policy learning as supervised learning

In this section, we discuss the connections and discrepancies between RL and supervised learning, and our results lead to a sample-efficient off-policy learning paradigm for RL. The main result in this section is Theorem 10, which casts the problem of maximizing the lower bound of return into a supervised learning problem, given one relatively mild Assumption 2 and practical assumptions 1 3. It can be shown that these assumptions are valid in a range of common RL tasks, as discussed in Lemma 30 in Appendix G. The central idea is to collect only the near-optimal trajectories when the learning agent interacts with the environment, and imitate the near-optimal policy by maximizing the log likelihood of the state-action pairs from these near-optimal trajectories. With the road map in mind, we then begin to introduce our approach as follows.

In a discrete action MDP with finite states and horizon, given the near-optimal policy $\pi_\star$, the stationary state distribution is given by: $p_{\pi_\star}(s) = \sum_{\tau} p(s|\tau)p_{\pi_\star}(\tau)$, where $p(s|\tau)$ is the probability of a certain state given a specific trajectory $\tau$ and is not associated with any policies, and only $p_{\pi_\star}(\tau)$ is related to the policy parameters. The stationary distribution of state-action pairs is thus: $p_{\pi_\star}(s,a) = p_{\pi_\star}(s)\pi_\star(a|s)$. In this section, we consider the MDP that each initial state will lead to at least one (near)-optimal trajectory. For a more general case, please refer to the discussion in Appendix C. In order to connect supervised learning (i.e., imitating a near-optimal policy) with RL and enable sample-efficient off-policy learning, we first introduce the trajectory reward shaping (TRS), defined as follows:
Definition 5 (Trajectory Reward Shaping, TRS) Given a fixed trajectory $\tau$, its trajectory reward is shaped as follows:

$$w(\tau) = \begin{cases} 
1, & \text{if } r(\tau) \geq c \\
0, & \text{o.w.}
\end{cases}$$

where $c = R_{\text{max}} - \epsilon$ is a problem-dependent near-optimal trajectory reward threshold that indicates the least reward of near-optimal trajectory, $\epsilon \geq 0$ and $\epsilon \ll R_{\text{max}}$. We denote the set of all possible near-optimal trajectories as $\mathcal{T} = \{\tau | w(\tau) = 1\}$, i.e., $w(\tau) = 1, \forall \tau \in \mathcal{T}$.

Remark 6 The threshold $c$ indicates a trade-off between the sample-efficiency and the optimality. The higher the threshold, the less frequently it will hit the near-optimal trajectories during exploration, which means it has higher sample complexity, while the final performance is better (see Figure 3).

Remark 7 The trajectory reward can be reshaped to any positive functions that are not related to policy parameter $\theta$. For example, if we set $w(\tau) = r(\tau)$, the conclusions in this section still hold (see Eq (18) in Appendix D). For the sake of simplicity, we set $w(\tau) = 1$.

Different from the reward shaping work (Ng et al., 1999), where shaping happens at each step on $r(s_t, a_t)$, the proposed approach directly shapes the trajectory reward $r(\tau)$, which facilitates the smooth transform from RL to SL. After shaping the trajectory reward, we can transfer the goal of RL from maximizing the return to maximize the long-term performance (Def 8).

Definition 8 (Long-term Performance) The long-term performance is defined by the expected shaped trajectory reward:

$$\sum_{\tau} p_{\theta}(\tau) w(\tau).$$

According to Def 5, the expectation over all trajectories is the equal to that over the near-optimal trajectories in $\mathcal{T}$, i.e., $\sum_{\tau} p_{\theta}(\tau) w(\tau) = \sum_{\tau \in \mathcal{T}} p_{\theta}(\tau) w(\tau)$.

The optimality is preserved after trajectory reward shaping ($\epsilon = 0, c = R_{\text{max}}$) since the optimal policy $\pi^*_s$ maximizing long-term performance is also an optimal policy for the original MDP, i.e., $\sum_{\tau} p_{\pi^*_s}(\tau) r(\tau) = \sum_{\tau \in \mathcal{T}} p_{\pi^*_s}(\tau) r(\tau) = R_{\text{max}}$, where $\pi^*_s = \arg \max_{\pi_s} \sum_{\tau} p_{\pi_s}(\tau) w(\tau)$ and $p_{\pi_s}(\tau) = 0, \forall \tau \notin \mathcal{T}$ (see Lemma 26 in Appendix D). Similarly, when $\epsilon > 0$, the optimal policy after trajectory reward shaping is a near-optimal policy for original MDP. Note that most policy gradient methods use the softmax function, in which we have $\exists \tau \notin \mathcal{T}, p_{\pi^*_s}(\tau) > 0$ (see Lemma 27 in Appendix D). Therefore when softmax is used to model a policy, it will not converge to an exact optimal policy. On the other hand, ideally, the discrepancy of the performance between them can be arbitrarily small based on the universal approximation (Hornik et al. 1989) with general conditions on the activation function and Theorem 1 in Syed and Schapire (2010).

Essentially, we use TRS to filter out near-optimal trajectories and then we maximize the probabilities of near-optimal trajectories to maximize the long-term performance. This procedure can be approximated by maximizing the log-likelihood of near-optimal state-action pairs, which is a supervised learning problem. Before we state our main results, we first introduce the definition of uniformly near-optimal policy (Def 9) and a prerequisite (Asm. 2) specifying the applicability of the results.
**Definition 9 (Uniformly Near-Optimal Policy, UNOP)** The Uniformly Near-Optimal Policy $\pi_*$ is the policy whose probability distribution over near-optimal trajectories ($T$) is a uniform distribution, i.e.

$$p_{\pi_*}(\tau) = \frac{1}{|T|}, \forall \tau \in T,$$

where $|T|$ is the number of near-optimal trajectories. When we set $c = R_{\text{max}}$, it is an optimal policy in terms of both maximizing return and long-term performance. In the case of $c = R_{\text{max}}$, the corresponding uniform policy is an optimal policy, we denote this type of optimal policy as uniformly optimal policy (UOP).

**Assumption 2 (Existence of Uniformly Near-Optimal Policy)** We assume the existence of Uniformly Near-Optimal Policy (Def. 9).

Based on Lemma 30 in Appendix G, Assumption 2 is satisfied for certain MDPs that have deterministic dynamics. Other than Assumption 2, all other assumptions in this work (Assumptions 1-3) can almost always be satisfied in practice, based on empirical observations. With these relatively mild assumptions, we present the following long-term performance theorem, which shows the close connection between supervised learning and RL.

**Theorem 10 (Long-term Performance Theorem)** Maximizing the lower bound of expected long-term performance in Eq (9) is maximizing the log-likelihood of state-action pairs sampled from a uniformly (near)-optimal policy $\pi_*$, which is a supervised learning problem:

$$\arg \max_\theta \sum_{s \in S} \sum_{a \in A_s} p_{\pi_*}(s, a) \log \pi_{\theta}(a|s)$$

The optimal policy of maximizing the lower bound is also the optimal policy of maximizing the long-term performance and the return.

**Remark 11** It is worth noting that Theorem 10 does not require a uniformly near-optimal policy $\pi_*$ to be deterministic. The only requirement is the existence of a uniformly near-optimal policy.

**Remark 12** Maximizing the lower bound of long-term performance is maximizing the lower bound of long-term reward since we can set $w(\tau) = r(\tau)$ and $\sum_\tau p_\theta(\tau)r(\tau) \geq \sum_\tau p_\theta(\tau)w(\tau)$. An optimal policy that maximizes this lower bound is also an optimal policy maximizing the long-term performance when $c = R_{\text{max}}$, thus maximizing the return.

The proof of Theorem 10 can be found in Appendix D. Theorem 10 indicates that we break the dependency between current policy $\pi_\theta$ and the environment dynamics, which means off-policy learning is able to be conducted by the above supervised learning approach. Furthermore, we point out that there is a potential discrepancy between imitating UNOP by maximizing log likelihood (even when the optimal policy’s samples are given) and the reinforcement learning since we are maximizing a lower bound of expected long-term performance (or equivalently the return over the near-optimal trajectories only) instead of return over all trajectories. In practice, the state-action pairs from an optimal policy is hard to construct while the uniform characteristic of UNOP can alleviate this issue (see Sec 6). Towards sample-efficient RL, we apply Theorem 10 to RPG, which reduces the ranking policy gradient to a classification problem by Corollary 13.
Corollary 13 (Ranking performance policy gradient) Optimizing the lower bound of expected long-term performance (defined in Eq (9)) using pairwise ranking policy (Eq (2)) is equal to:

$$\min_\theta \sum_{s,a_i} p_{\pi_*}(s,a_i) \left( \sum_{j=1,j\neq i}^{m} \max(0, 1 + Q(s,a_j) - Q(s,a_i)) \right).$$  \hspace{1cm} (11)

The proof of Corollary 13 can be found in Appendix E. Similarly, we can reduce LPG to a classification problem (see Corollary 14). One advantage of casting RL to SL is variance reduction. With the proposed off-policy supervised learning, we can reduce the upper bound of the policy gradient variance, as shown in the Corollary 15.

Corollary 14 (Listwise performance policy gradient) Optimizing the lower bound of expected long-term performance by the listwise ranking policy (Eq (8)) is equivalent to:

$$\max_\theta \sum_s p_{\pi_*}(s) \sum_{i=1}^{m} \pi_*(a_i|s) \log \frac{e^{Q_i}}{\sum_{j=1}^{m} e^{Q_j}}.$$  \hspace{1cm} (12)

The proof of this Corollary is a direct application of theorem 10 by replacing policy with the softmax function.

Corollary 15 (Policy gradient variance reduction) Given a stationary policy, the upper bound of the variance of each dimension of policy gradient is $O(T^2C^2R_{max}^2)$. The upper bound of gradient variance of maximizing the lower bound of long-term performance Eq (10) is $O(C^2)$, where $C$ is the maximum norm of log gradient based on Assumption 3. The supervised learning has reduced the upper bound of gradient variance by an order of $O(T^2R_{max}^2)$ as compared to the regular policy gradient, considering $R_{max} \geq 1, T \geq 1$, which is a very common situation in practice.

The proof of Corollary 15 can be found in Appendix F. This corollary shows that the variance of regular policy gradient is upper-bounded by the square of time horizon and the maximum trajectory reward. It is aligned with our intuition and empirical observation: the longer the horizon the harder the learning. Also, the common reward shaping tricks such as truncating the reward to $[-1, 1]$ (Castro et al., 2018) can help the learning since it reduces variance by decreasing $R_{max}$. With supervised learning, we concentrate the difficulty of long-time horizon into the exploration phase, which is an inevitable issue for all RL algorithms, and we drop the dependence on $T$ and $R_{max}$ for policy variance. Thus, it is more stable and efficient to train the policy using supervised learning. One limitation of this method is that it requires specific domain knowledge. We need to explicitly define the trajectory reward threshold $c$ for different tasks, which is crucial to the final performance and sample-efficiency. In many applications such as dialogue systems (Li et al., 2017), recommender systems (Melville and Sindhwani, 2011), etc., the reward functions are crafted to guide the learning process, and in these scenarios the values of $c$ are naturally known. For the cases that we have no prior knowledge on the reward function of MDP, we treat $c$ as a tuning parameter to balance the optimality and efficiency. The major theoretical uncertainty on general tasks is the existence of a uniformly near-optimal policy, and however it is almost negligible to the empirical performance in practice, as demonstrated in our experiments. The rigorous theoretical analysis of this problem is beyond the scope of this work.
6. An algorithmic framework for off-policy learning

Based on the discussions in Section 5, we separate the training of a RL agent into two stages: exploration and supervision. The key idea is that although we have no access to the UNOP, we can approximate the state action pairs sampled from the environment follow UNOP by only collecting the near-optimal trajectories.

**Exploration Stage.** The goal of the exploration stage is to collect different near-optimal trajectories as frequently as possible. Under the off-policy framework, the exploration agent and the learning agent are separated, therefore, any existing RL algorithm can be used during the exploration. In fact, the principle of this framework is that we should use the most advanced RL agents in the exploration to collect as many near-optimal trajectories as possible and leave the policy learning to the supervision stage.

**Supervision Stage.** The goal of supervision is to learn near-optimal policy by maximizing the log-likelihood of the state-action pairs collected from the above exploration stage. When ranking policies are applied, this supervision stage is a classification problem as shown in Corollary 13 or Corollary 14.

The two-stage algorithmic framework can be directly incorporated in RPG and LPG to improve sample efficiency. The implementation of RPG is given in Algorithm 1, and LPG follows the same procedure except for the difference in the loss function. The main requirements of Alg. 1 are on exploration efficiency and the MDP structure. During the exploration stage, a sufficient amount of the different near-optimal trajectories need to be collected for constructing a representative supervised learning training dataset. Theoretically, this requirement always holds when a sufficient number of training episodes are conducted [see Appendix G, Lemma 31]. One practical concern of the proposed algorithm is that the number of episodes could be prohibitively large, which makes this algorithm sample-inefficient. However, according to our extensive empirical observations, we notice that a sufficient amount of near-optimal trajectories have almost always been collected, long before the value function based state-of-the-art converge to near-optimal performance.

Therefore, we point out that instead of deriving policies from value functions, it can be more sample-efficient to imitate UNOP by supervised learning and use value functions to facilitate the exploration. It is not necessary to explore the suboptimal state-action space or estimate value functions accurately while we already acquired enough samples to learn a policy that can perform (nearly) optimally. In some relatively simple tasks such as Pong, we can even imitate the uniformly optimal policy directly without relying on any advanced exploration algorithm or value function estimations.

7. Sample Complexity and Generalization Performance

In this section, we present a theoretical analysis on the sample complexity of RPG with off-policy learning framework in Section 6. The analysis leverages the results from the Probably Approximately Correct (PAC) framework, and provides an alternative approach to quantify sample complexity of RL from the perspective of the connection between RL and SL (see Theorem 10), which is significantly different from the existing approaches that use value function estimations (Kakade et al., 2003; Strehl et al., 2006; Kearns et al., 2000; Strehl et al., 2009; Krishnamurthy et al., 2016; Jiang et al., 2017; Jiang and Agarwal, 2018; Zanette
Algorithm 1 Off-Policy Learning for Ranking Policy Gradient (RPG)

Require: The near-optimal trajectory reward threshold \( c \), the number of maximal training episodes \( N_{\text{max}} \). Maximum number of time steps in each episode \( T \), and batch size \( b \).

1: while episode < \( N_{\text{max}} \) do
2: repeat
3: Retrieve state \( s_t \) and sample action \( a_t \) by the specified exploration agent (can be random, \( \epsilon \)-greedy, or any RL algorithms).
4: Collect the experience \( e_t = (s_t, a_t, r_t, s_{t+1}) \) and store to the replay buffer.
5: \( t = t + 1 \)
6: if \( t \% \text{update step} == 0 \) then
7: Sample a batch of experience \( \{e_j\}_{j=1}^b \) from the near-optimal replay buffer.
8: Update \( \pi_\theta \) based on the hinge loss Eq (11) for RPG.
9: Update the exploration agent using samples from the regular replay buffer (In simple MDPs such as PONG where near-optimal trajectories are encountered frequently, near-optimal replay buffer can be used to update the exploration agent).
10: end if
11: until terminal \( s_t \) or \( t - t_{\text{start}} >= T \)
12: if return \( \sum_{i=1}^{T} r_i \geq c \) then
13: Take the near-optimal trajectory \( e_t, t = 1, ..., T \) in the latest episode from the regular replay buffer, and insert the trajectory into the near-optimal replay buffer.
14: end if
15: if \( t \% \text{evaluation step} == 0 \) then
16: Evaluate the RPG agent by greedily choosing the action. If the best performance is reached, then stop training.
17: end if
18: end while

and Brunskill, 2019). We show that the sample complexity of RPG (Theorem 17) depends on the properties of MDP such as horizon, action space, dynamics, and the generalization performance of supervised learning. It is worth mentioning that the sample complexity of RPG has no linear dependence on the state-space, which makes it suitable for large-scale MDPs. Moreover, we also provide a formal quantitative definition (Def 19) on the exploration efficiency of RL.

Corresponding to the two-stage framework in Section 6, the sample complexity of RPG also splits into two problems:

- **Learning efficiency:** How many state-action pairs from the uniformly optimal policy do we need to collect, in order to achieve good generalization performance in RL?

- **Exploration efficiency:** For a certain type of MDPs, what is the probability of collecting \( n \) training samples (state-action pairs from the uniformly near-optimal policy) in the first \( k \) episodes in worst case? This question leads to a quantitative evaluation metric of different exploration methods.

The first stage is resolved by Theorem 17, which connects the lower bound of the generalization performance of RL to the supervised learning generalization performance. Then we discuss the exploration efficiency of the worst case performance for a binary tree MDP in Lemma 21. Jointly, we show how to link the two stages to give a general theorem that studies how many samples we need to collect in order to achieve certain performance in RL.
In this section, we restrict our discussion on the MDPs with a fixed action space and assume the existence of deterministic optimal policy. The policy \( \pi = \hat{\pi} = \arg \min_{h \in \mathcal{H}} \hat{\epsilon}(h) \) corresponds to the empirical risk minimizer (ERM) in the learning theory literature, which is the policy we obtained through learning on the training samples. \( \mathcal{H} \) denotes the hypothesis class from where we are selecting the policy. Given a hypothesis (policy) \( h \), the empirical risk is given by \( \hat{\epsilon}(h) = \sum_{i=1}^{n} \frac{1}{n} \mathbb{1}\{h(s_i) \neq a_i\} \). Without loss of generality, we can normalize the reward function to set the upper bound of trajectory reward equals to one (i.e., \( R_{\text{max}} = 1 \)), similar to the assumption in (Jiang and Agarwal, 2018). It is worth noting that the training samples are generated i.i.d. from an unknown distribution, which is perhaps the most important assumption in the statistical learning theory. i.i.d. is satisfied in this case since the state action pairs (training samples) are collected by filtering the samples during the learning stage, and we can manually manipulate the samples to follow the distribution of UOP (Def 9) by only storing the unique near-optimal trajectories.

### 7.1 Supervision stage: Learning efficiency

To simplify the presentation, we restrict our discussion on the finite hypothesis class (i.e. \( |\mathcal{H}| < \infty \)) since this dependence is not germane to our discussion. However, we note that the theoretical framework in this section is not limited to the finite hypothesis class. For example, we can simply use the VC dimension (Vapnik, 2006) or the Rademacher complexity (Bartlett and Mendelson, 2002) to generalize our discussion to the infinite hypothesis class, such as neural networks. For completeness, we first revisit the sample complexity result from the PAC learning in the context of RL.

**Lemma 16 (Supervised Learning Sample Complexity (Mohri et al., 2018))** Let \( |\mathcal{H}| < \infty \), and let \( \delta, \gamma \) be fixed, the inequality \( \epsilon(\hat{h}) \leq (\min_{h \in \mathcal{H}} \epsilon(h)) + 2\gamma = \eta \) holds with probability at least \( 1 - \delta \), when the training set size \( n \) satisfies:

\[
n \geq \frac{1}{2\gamma^2} \log \frac{2|\mathcal{H}|}{\delta}, \tag{13}
\]

where the generalization error (expected risk) of a hypothesis \( \hat{h} \) is defined as:

\[
\epsilon(\hat{h}) = \sum_{s,a} p_{\pi_*}(s,a) \mathbb{1}\{\hat{h}(s) \neq a\}.
\]

**Condition 1 (Action values)** We restrict the action values of RPG in certain range, i.e., \( Q_i \in [0, c_q] \), where \( c_q \) is a positive constant.

This condition can be easily satisfied, for example, we can use a sigmoid to cast the action values into \([0, 1]\). We can impose this constraint since in RPG we only focus on the relative relationship of action values. Given the mild condition and established on the prior work in statistical learning theory, we introduce the following results that connect the supervised learning and reinforcement learning.

**Theorem 17 (Generalization Performance)** Given a MDP where the UOP (Def 9) is deterministic, let \( |\mathcal{H}| \) denote the size of hypothesis space, and \( \delta, n \) be fixed, the following inequality holds with probability at least \( 1 - \delta \):

\[
\sum_{\tau} p_{\theta}(\tau) r(\tau) \geq D(1 + e)^{\eta(1-m)T},
\]
where $D = |T| (\Pi_{\tau \in T} p_d(\tau))^{|\Pi|}$, $p_d(\tau) = p(s_1) \Pi_{t=1}^{T-1} p(s_{t+1} | s_t, a_t)$ denotes the environment dynamics. $\eta$ is the upper bound of supervised learning generalization performance, defined as $\eta = (\min_{h \in \mathcal{H}} \epsilon(h)) + 2 \sqrt{\frac{1}{2n} \log \frac{2|\mathcal{H}|}{\delta}} = 2 \sqrt{\frac{1}{2n} \log \frac{2|\mathcal{H}|}{\delta}}$.

**Corollary 18 (Sample Complexity)** Given a MDP where the UOP (Def 9) is deterministic, let $|\mathcal{H}|$ denotes the size of hypothesis space, and let $\delta$ be fixed. Then for the following inequality to hold with probability at least $1 - \delta$:

$$\sum_{\tau} p_{\theta}(\tau)r(\tau) \geq 1 - \epsilon,$$

it suffices that the number of state action pairs (training sample size $n$) from the uniformly optimal policy satisfies:

$$n \geq \frac{2(m - 1)^2 T^2}{(\log_{1+\epsilon} \frac{D}{1 - \epsilon})^2} \log \frac{2|\mathcal{H}|}{\delta} = O \left( \frac{m^2 T^2}{(\log \frac{D}{1 - \epsilon})^2} \log \frac{|\mathcal{H}|}{\delta} \right).$$

The proofs of Theorem 17 and Corollary 18 are provided in Appendix I. Theorem 17 establishes the connection between the generalization performance of RL and the sample complexity of supervised learning. The lower bound of generalization performance decreases exponentially with respect to the horizon $T$ and action space dimension $m$. This is aligned with our empirical observation that it is more difficult to learn the MDPs with a longer horizon and/or a larger action space. Furthermore, the generalization performance has a linear dependence on $D$, the transition probability of optimal trajectories. Therefore, $T$, $m$, and $D$ jointly determines the difficulty of learning of the given MDP. As pointed out by Corollary 18, the smaller the $D$ is, the higher the sample complexity. Note that $T$, $m$, and $D$ all characterize intrinsic properties of MDPs, which cannot be improved by our learning algorithms. One advantage of RPG is that its sample complexity has no dependence on the state space, which enables the RPG to resolve large-scale complicated MDPs, as demonstrated in our experiments. In the supervision stage, our goal is the same as in the traditional supervised learning: to achieve better generalization performance $\eta$.

### 7.2 Exploration stage: Exploration efficiency

The exploration efficiency is highly related to the MDP properties and the exploration strategy. To provide interpretation on how the MDP properties (state space dimension, action space dimension, horizon) affect the sample complexity through exploration efficiency, we characterize a simplified MDP as in (Sun et al., 2017), in which we explicitly compute the exploration efficiency of a stationary policy (random exploration), as shown in Figure 1.

**Definition 19 (Exploration Efficiency)** We define the exploration efficiency of a certain exploration algorithm ($A$) within a MDP ($\mathcal{M}$) as the probability of sampling and collecting $i$ distinct optimal trajectories in the first $k$ episodes. We denote the exploration efficiency as $p_{A,\mathcal{M}}(n_{\text{traj}} \geq i | k)$. When $\mathcal{M}$, $k$, $i$ and optimality threshold $c$ are fixed, the higher the $p_{A,\mathcal{M}}(n_{\text{traj}} \geq i | k)$, the better the exploration efficiency. We use $n_{\text{traj}}$ to denote the number...
Figure 1: The binary tree structure MDP ($\mathcal{M}_1$) with one initial state, similar as discussed in (Sun et al., 2017). In this work, we assume there is no duplicated states in the tree, the uniform initial state distribution if the MDP has multiple initial states and the deterministic environment dynamics. For $\mathcal{M}_1$ the worst case exploration is random exploration and each trajectory will be visited at same probability under random exploration. Note that in this type of MDP, the Assumption 2 is satisfied.

of near-optimal trajectories in this subsection. If the exploration algorithm derives a series of learning policies, then we have $p_{A,\mathcal{M}}(n_{\text{traj}} \geq i|k) = p_{\{\pi_i\}_{i=0}^{t}}(n_{\text{traj}} \geq i|k)$, where $t$ is the number of steps the algorithm $A$ updated the policy. If we would like to study the exploration efficiency of a stationary policy, then we have $p_{A,\mathcal{M}}(n_{\text{traj}} \geq i|k) = p_{\pi,\mathcal{M}}(n \geq i|k)$.

**Definition 20 (Expected Exploration Efficiency)** The expected exploration efficiency of a certain exploration algorithm ($A$) within a MDP ($\mathcal{M}$) is defined as:

$$E_{A,k,\mathcal{M}} = \sum_{i=0}^{k} p_{A,\mathcal{M}}(n_{\text{traj}} = i|k)i.$$  

The definitions provide a quantitative metric to evaluate the quality of exploration. Intuitively, the quality of exploration should be determined by how frequently it will hit different good trajectories. We use Def 19 for theoretical analysis and Def 20 for practical evaluation.

**Lemma 21 (The Worst Case Exploration Efficiency)** The Exploration Efficiency of random exploration policy in a binary tree MDP ($\mathcal{M}_1$) is given as:

$$p_{\pi_r,\mathcal{M}}(n_{\text{traj}} \geq i|k) = 1 - \sum_{i'=0}^{i-1} C_{i'}^{i} \frac{\sum_{j=0}^{i'} (-1)^j C_j^i (N - |\mathcal{T}| + i' - j)^k}{N^k},$$

where $N$ denotes the total number of different trajectories in the MDP. In binary tree MDP $\mathcal{M}_1$, $N = |S_0||A|^{|\mathcal{T}|}$, where $|S_0|$ denotes the number of distinct initial states. $|\mathcal{T}|$ denotes the number of optimal trajectories. $\pi_r$ denotes the random exploration policy, which means the probability of hitting each trajectory in $\mathcal{M}_1$ is equal.

The proof of Lemma 21 is available in Appendix J.
7.3 Joint Analysis Combining Exploration and Supervision

In this section, we jointly consider the learning efficiency and exploration efficiency to study the generalization performance. Concretely, we would like to study if we interact with the environment a certain number of episodes, what is the worst generalization performance we can expect with certain probability, if RPG is applied.

**Corollary 22 (RL Generalization Performance)** Given a MDP where the UOP (Def 9) is deterministic, let $|H|$ be the size of the hypothesis space, and let $\delta, n, k$ be fixed, the following inequality holds with probability at least $1 - \delta'$:

$$\sum_{\tau} p_{\theta}(\tau)r(\tau) \geq D(1 + e)\gamma(1-m)T,$$

where $k$ is the number of episodes we have explored in the MDP, $n$ is the number of distinct optimal state-action pairs we needed from the UOP (i.e., size of training data). $n'$ denotes the number of distinct optimal state-action pairs collected by the random exploration. $\eta = 2\sqrt{\frac{1}{2n}(\log \frac{2|H|p_{\pi_\theta},M(n' \geq n|k)}{1-\delta})}.$

The proof of Corollary 22 is provided in Appendix K. Corollary 22 states that the probability of sampling optimal trajectories is the main bottleneck of exploration and generalization, instead of state space dimension. In general, the optimal exploration strategy depends on the properties of MDPs. In this work, we focus on improving learning efficiency, i.e., learning optimal ranking instead of estimating value functions. The discussion of optimal exploration is beyond the scope of this work.

8. Applications and Empirical Results

In this section, we present our empirical results of sample complexity on comparing RPG (combined with different exploration strategies) and LPG agents with the state-of-the-art baselines. The set of baselines we evaluated include DQN (Mnih et al., 2015), C51 (Bellemare et al., 2017), Implicit Quantile (Dabney et al., 2018), and Rainbow (Hessel et al., 2017). Implementations of all methods are provided in the Dopamine framework (Castro et al., 2018). The proposed methods, including EPG, LPG, and RPG, are also implemented based on Dopamine with the following adaptations:

**EPG:** EPG denotes the stochastic listwise policy gradient with off-policy supervised learning, which is the vanilla policy gradient trained with off-policy supervised learning. The exploration and supervision agent is parameterized by the same neural network. The supervision agent minimizes the cross-entropy loss (see Eq (12)) over the near-optimal trajectories collected in an online fashion.

**LPG:** LPG denotes the deterministic listwise policy gradient with off-policy supervised learning. We choose an action greedily based on the value of logits during the evaluation, and it stochastically explores the environment as does by EPG.

**RPG:** RPG explores the environment using a separate EPG agent in Pong and Implicit Quantile in other games. Then RPG conducts supervised learning by minimizing the hinge loss Eq (11). It is worth mentioning that the exploration agent (EPG or Implicit Quantile) can be replaced by any other existing exploration method.

1. Code is available at https://github.com/illidanlab/rpg.
Figure 2: The training curves of the proposed RPG, LPG and state-of-the-art on eight Atari games. All results are averaged over random seeds from 1 to 5. The x-axis represents the number of training iterations (each iteration consists of 250,000 interactions.) and the y-axis represents the averaged training episodic return. The last figure plots the expected exploration efficiency of state-of-the-art, the results are averaged over random seeds from 1 to 10.

The testbeds are eight Atari 2600 games (Pong, Breakout, Bowling, BankHeist, DoubleDunk, Pitfall, Boxing, Robotank) from the Arcade Learning Environment (Bellemare et al., 2013) without randomly repeating the previous action. In our experiments, we simply collect all trajectories with the trajectory reward no less than the threshold $c$ without eliminating the duplicated trajectories, which we found it is an empirically reasonable simplification. As the results shown in Figure 2 Pong, the proposed RPG and LPG achieve higher sample efficiency than competing baselines, and converge to the optimal deterministic policy. We see that RPG converges faster than LPG, which may be due to that hinge loss is more sample-efficient than cross-entropy in terms of imitating a deterministic policy.

Comparing EPG with LPG, we notice that since Pong exists a deterministic optimal policy, EPG that uses a stochastic policy to fit an optimal policy is not as effective as LPG, let along RPG. Except for Pong, we use implicit quantile as our exploration agent (denoted as implicit_quantilerpg) and found it is the most efficient exploration strategy for RPG.
Figure 3: The trade-off between sample efficiency and optimality on DOUBLE-DUNK, BREAKOUT, BANKHEIST. The three curves refer to returns of algorithms that use different near-optimal trajectory reward thresholds $c$. The thresholds are denoted by the numbers at the end of legends.

the results shown in Figure 2, RPG configurations with different exploration strategies are more sample-efficient than the state-of-the-art.

**Ablation Study: On the Trade-off between Sample-Efficiency and Optimality.** Results in Figure 3 show that there is a trade-off between sample efficiency and optimality, which is controlled by the trajectory reward threshold $c$. Recall that $c$ determines how close is the learned UNOP to optimal policies. A higher value of $c$ leads to a less frequency of near-optimal trajectories being collected and thus a lower sample efficiency, and however the algorithm is expected to converge to a strategy of better performance. We note that $c$ is the only parameter we tuned across all experiments.

**Ablation Study: Exploration Efficiency.** We empirically evaluate the Expected Exploration Efficiency (Def 19) of the state-of-the-art on PONG. It is worth noting that the RL generalization performance is determined by both of learning efficiency and exploration efficiency. Therefore, higher exploration efficiency does not necessarily lead to more sample efficient algorithm due to the learning inefficiency, as demonstrated by RAINBOW and DQN (see the last subfigure in Figure 2). Also, the Implicit Quantile achieves the best performance among baselines, since its exploration efficiency largely surpasses other baselines.

**9. Conclusion and Discussions**

In this work, we introduced ranking policy gradient methods that, for the first time, approach the RL problem from a ranking perspective. Furthermore, towards the sample-efficient RL, we propose an off-policy learning framework, which trains RL agents in a supervised learning manner and thus largely facilitates the learning efficiency. The off-policy learning framework separates RL into exploration and supervision stages, and enables the flexibility to integrate a variety of advanced exploration algorithms. Besides, we provide an alternative approach to analyze the sample complexity of RL, and show that the sample complexity of RPG has no dependency on the state space dimension. Last but not least, the RPG with the off-policy learning framework achieves surprising empirical results as compared to the state-of-the-art.
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Appendix A. Discussion of Existing Efforts on Connecting Reinforcement Learning to Supervised Learning.

There are two main distinctions between supervised learning and reinforcement learning. In supervised learning, the data distribution $D$ is static and training samples are assumed to be sampled i.i.d. from $D$. On the contrary, the data distribution is dynamic in reinforcement learning and the sampling procedure is not independent. First, since the data distribution in RL is determined by both environment dynamics and the learning policy, and the policy keeps being updated during the learning process. This updated policy results in dynamic data distribution in reinforcement learning. Second, policy learning depends on previously collected samples, which in turn determines the sampling probability of incoming data. Therefore, the training samples we collected are not independently distributed. These intrinsic difficulties of reinforcement learning directly cause the sample-inefficient and unstable performance of current algorithms.

On the other hand, most state-of-the-art reinforcement learning algorithms can be shown to have a supervised learning equivalent. To see this, recall that most reinforcement learning algorithms eventually acquire the policy either explicitly or implicitly, which is a mapping from a state to an action or a probability distribution over the action space. The use of such a mapping implies that ultimately there exists a supervised learning equivalent to the original reinforcement learning problem, if optimal policies exist. The paradox is that it is almost impossible to construct this supervised learning equivalent on the fly, without knowing any optimal policy.

Although the question of how to construct and apply proper supervision is still an open problem in the community, there are many existing efforts providing insightful approaches to reduce reinforcement learning into its supervised learning counterpart over the past several decades. Roughly, we can classify the existing efforts into the following categories:

- **Expectation-Maximization (EM):** Dayan and Hinton (1997); Peters and Schaal (2007); Kober and Peters (2009); Abdolmaleki et al. (2018), etc.

- **Entropy-Regularized RL (ERL):** Oh et al. (2018); Haarnoja et al. (2018), etc.

- **Interactive Imitation Learning (IIL):** Daumé et al. (2009); Syed and Schapire (2010); Ross and Bagnell (2010); Ross et al. (2011); Sun et al. (2017), etc.

The early approaches in the EM track applied Jensen’s inequality and approximation techniques to transform the reinforcement learning objective. Algorithms are then derived from the transformed objective, which resemble the Expectation-Maximization procedure and provide policy improvement guarantee (Dayan and Hinton, 1997). These approaches typically focus on a simplified RL setting, such as assuming that the reward function is not associated with the state (Dayan and Hinton, 1997), approximating the goal to maximize the expected immediate reward and the state distribution is assumed to be fixed (Peters and Schaal, 2008). Later on in Kober and Peters (2009), the authors extended the EM framework from targeting immediate reward into episodic return. Recently, Abdolmaleki et al. (2018) used the EM-framework on a relative entropy objective, which adds a parameter prior as regularization. It has been found that the estimation step using Retrace (Munos et al., 2016) can be unstable even with a linear function approximation (Touati et al., 2017). In general, the estimation step in EM-based algorithms involves on-policy evaluation, which is one challenge shared among policy gradient methods. On the other hand, off-policy learning usually leads to a much better sample efficiency, and is one main motivation that we want to reformulate RL into a supervised learning task.

On the track of entropy regularization, Soft Actor-Critic (Haarnoja et al., 2018) used the framework of entropy-regularized RL to improve sample-efficiency and achieve faster convergence. It was shown to converge to the optimal policy that optimizes the composite objective including the long-term reward and policy entropy. This approach provides a rather efficient way to model suboptimal behavior, and lead to empirically sound policies. However, the entropy term in the objective leads to
Table 2: A comparison of studies reducing RL to SL. The Objective column denotes whether the goal is to maximize long-term reward. The Cont. Action column denotes whether the method is applicable to both continuous and discrete action spaces. The Optimality denotes whether the algorithms can model the optimal policy. ✓† denotes the optimality achieved by ERL is w.r.t. the entropy regularize objective instead of the original objective on return. The Off-Policy column denotes if the algorithms enable off-policy learning. The No Oracle column denotes if the algorithms need to access to a certain type of oracle (expert policy or expert demonstrations).

On the other hand, Oh et al. (2018) shared similarity to our work in terms of the method we collecting the samples, but radically different from the proposed approach in terms of theoretical formation. The approach collects good samples based on the past experience and then conduct the imitation learning w.r.t. those good samples. However, this self-imitation learning procedure was eventually connected to lower-bound-soft-Q-learning, which belongs to entropy-regularized reinforcement learning. Indeed, there is a trade-off between sample-efficiency and modeling suboptimal behavior: a more strict requirement on the samples being collected will lead to less chance to hit satisfactory samples while the resulting policy is more close to imitate the optimal behavior.

From the track of interactive imitation learning, early efforts such as (Ross and Bagnell, 2010; Ross et al., 2011) pointed out that the main discrepancy between imitation learning and reinforcement learning is the violation of i.i.d. assumption. SMILE (Ross and Bagnell, 2010) and DAgGER (Ross et al., 2011) are proposed to overcome the distribution mismatch. Theorem 2.1 in Ross and Bagnell (2010) quantified the performance degradation from the expert considering that the learned policy fails to imitate the expert with a certain probability. The theorem seems to resemble the long-term performance theorem (Thm. 10) in this paper. However, it studied the scenario that the learning policy is trained through a state distribution induced by the expert, instead of state-action distribution as considered in Theorem 10. As such, Theorem 2.1 in Ross and Bagnell (2010) may be more applicable to the situation where an interactive procedure is needed, such as querying the expert during the training process. On the contrary, the proposed work focuses on directly applying supervised learning without having access to the expert to label the data. The optimal state-action pairs are collected during exploration and conducting supervised learning on the replay buffer will provide a performance guarantee in terms of long-term expected reward. Concurrently, a resemble of Theorem 2.1 in (Ross and Bagnell, 2010) is Theorem 1 in (Syed and Schapire, 2010), where the authors reduced the apprenticeship learning to classification, under the assumption that the apprentice policy is deterministic and the misclassification rate is bounded at all time steps. In this work, we show that it is possible to circumvent such a strong assumption and reduce RL to its SL. Furthermore, our theoretical framework also leads to an alternative analysis of sample-complexity. Later on AggreVAte (Ross and Bagnell, 2014) was proposed to incorporate the information of action costs to facilitate imitation learning, and its differentiable version AggreVAteD (Sun et al., 2017) was developed in succession and achieved impressive empirical results. Recently, hinge loss
was introduced to regular Q-learning as a pre-training step for learning from demonstration (Hester et al. 2018), or as a surrogate loss for imitating optimal trajectories (Osa et al., 2018). In this work, we show that hinge loss constructs a new type of policy gradient method and can be used to learn optimal policy directly.

In conclusion, our method approaches the problem of reducing RL to SL from a unique perspective that is different from all prior work. With our reformulation from RL to SL, the samples collected in the replay buffer satisfy the i.i.d. assumption, since the state-action pairs are now sampled from the data distribution of UNOP. A multi-aspect comparison between the proposed method and relevant prior studies is summarized in Table 2.

Appendix B. Ranking Policy Gradient Theorem

The Ranking Policy Gradient Theorem (Theorem 2) formulates the optimization of long-term reward using a ranking objective. The proof below illustrates the formulation process.

Proof The following proof is based on direct policy differentiation (Peters and Schaal, 2008; Williams, 1992). For a concise presentation, the subscript $t$ for action value $Q_i, Q_j$, and $p_{ij}$ is omitted.

$$
\nabla_\theta J(\theta) = \nabla_\theta \sum_\tau p_\theta(\tau) r(\tau)
$$

$$
= \sum_\tau p_\theta(\tau) \nabla_\theta \log p_\theta(\tau) r(\tau)
$$

$$
= \sum_\tau p_\theta(\tau) \nabla_\theta \log \left( \prod_{i=1}^T \pi_\theta(a_t | s_t) p(s_{t+1} | s_t, a_t) \right) r(\tau)
$$

$$
= \sum_\tau p_\theta(\tau) \sum_{t=1}^T \nabla_\theta \log \pi_\theta(a_t | s_t) r(\tau)
$$

$$
= \mathbb{E}_{\tau \sim \pi_\theta} \left[ \sum_{t=1}^T \nabla_\theta \log \pi_\theta(a_t | s_t) r(\tau) \right]
$$

$$
= \mathbb{E}_{\tau \sim \pi_\theta} \left[ \sum_{t=1}^T \nabla_\theta \log \left( \prod_{j=1, j \neq i}^m p_{ij} \right) r(\tau) \right]
$$

$$
= \mathbb{E}_{\tau \sim \pi_\theta} \left[ \sum_{t=1}^T \nabla_\theta \log \left( \prod_{j=1, j \neq i}^m e^{Q_{ij}} \right) r(\tau) \right]
$$

$$
= \mathbb{E}_{\tau \sim \pi_\theta} \left[ \sum_{t=1}^T \nabla_\theta \log \left( \prod_{j=1, j \neq i}^m \left( \frac{1}{1 + e^{Q_{ij}}} \right) \right) r(\tau) \right]
$$

$$
\approx \mathbb{E}_{\tau \sim \pi_\theta} \left[ \sum_{t=1}^T \nabla_\theta \left( \sum_{j=1, j \neq i}^m (Q_i - Q_j)/2 \right) r(\tau) \right],
$$

where the trajectory is a series of state-action pairs from $t = 1, ..., T$, i.e. $\tau = s_1, a_1, s_2, a_2, ..., s_T$. From Eq (14) to Eq (15), we use the first-order Taylor expansion of $\log(1 + e^x)|_{x=0} = \log 2 + \frac{1}{2} x + O(x^2)$ to further simplify the ranking policy gradient.

B.1 Probability Distribution in Ranking Policy Gradient

In this section, we discuss the output property of the pairwise ranking policy. We show in Corollary 23 that the pairwise ranking policy gives a valid probability distribution when the dimension of the action space $m = 2$. For cases when $m > 2$ and the range of $q$-value satisfies Condition 2, we show in Corollary 24 how to construct a valid probability distribution.

Corollary 23 The pairwise ranking policy as shown in Eq (2) constructs a probability distribution over the set of actions when the action space $m$ is equal to 2, given any action values $Q_i, i = 1, 2$. For the cases with $m > 2$, this conclusion does not hold in general.
It is easy to verify that $\pi(a_i|s) > 0$, $\sum_{i=1}^2 \pi(a_i|s) = 1$ holds and the same conclusion cannot be applied to $m > 2$ by constructing counterexamples. However, we can introduce a dummy action $a'$ to form a probability distribution for RPG. During policy learning, the algorithm increases the probability of best actions and the probability of dummy action decreases. Ideally, if RPG converges to an optimal deterministic policy, the probability of taking best action is equal to 1 and $\pi(a'|s) = 0$. Similarly, we can introduce a dummy trajectory $\tau'$ with the trajectory reward $r(\tau') = 0$ and $p_\theta(\tau') = 1 - \sum_\tau p_\theta(\tau)$. The trajectory probability forms a probability distribution since $\sum_\tau p_\theta(\tau) + p_\theta(\tau') = 1$ and $p_\theta(\tau) \geq 0 \forall \tau$ and $p_\theta(\tau') \geq 0$. The proof of a valid trajectory probability is similar to the following proof on $\pi(a|s)$ to be a valid probability distribution with a dummy action. Its practical influence is negligible since our goal is to increase the probability of (near)-optimal trajectories. To present in a clear way, we avoid mentioning dummy trajectory $\tau'$ in Proof B while it can be seamlessly included.

**Condition 2 (The range of q-value)** We restrict the range of $q$-values in RPG so that it satisfies $Q_m \geq \ln(m^{\frac{1}{m-1}} - 1)$, where $Q_m = \min_{i,j} Q_{ji}$ and $m$ is the dimension of the action space.

This condition can be easily satisfied since in RPG we only focus on the relative relationship of $Q$ and we can constrain the range of $q$-values so that $Q_m$ satisfies the condition 2. Furthermore, since we can see that $m^{\frac{1}{m-1}} > 1$ is decreasing w.r.t to action dimension $m$. The larger the action dimension, the less constraint we have on the action values.

**Corollary 24** Given Condition 2, we introduce a dummy action $a'$ and set $\pi(a = a'|s) = 1 - \sum_i \pi(a = a_i|s)$, which constructs a valid probability distribution $(\pi(a|s))$ over the action space $A \cup a'$.

**Proof** Since we have $\pi(a = a_i|s) > 0 \forall i = 1, \ldots, m$ and $\sum_i \pi(a = a_i|s) + \pi(a = a'|s) = 1$. To prove that this is a valid probability distribution, we only need to show that $\pi(a = a'|s) \geq 0$, $\forall m \geq 2$, i.e. $\sum_i \pi(a = a_i|s) \leq 1$, $\forall m \geq 2$. Let $Q_m = \min_{i,j} Q_{ji},$

\[
\sum_i \pi(a = a_i|s) = \sum_i \prod_{j=1, j\neq i}^m p_{ij} = \sum_i \prod_{j=1, j\neq i}^m \frac{1}{1 + e^{Q_{ji}}} \leq \sum_i \prod_{j=1, j\neq i}^m \frac{1}{1 + e^{Q_m}} = m \left( \frac{1}{1 + e^{Q_m}} \right)^{m-1} \leq 1 \quad \text{(Condition 2)}.
\]

This thus concludes the proof.

### Appendix C. Condition of Preserving Optimality

The following condition describes what types of MDPs are directly applicable to the trajectory reward shaping (TRS, Def 5):

**Condition 3 (Initial States)** The (near)-optimal trajectories will cover all initial states of MDP. i.e. $\{s(\tau, 1) \mid \forall \tau \in T\} = \{s(\tau, 1) \mid \forall \tau\}$, where $T = \{\tau|w(\tau) = 1\} = \{\tau|r(\tau) \geq c\}$. 

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Figure 4: The binary tree structure MDP with two initial states \( S_1 = \{ s_1, s'_1 \} \), similar as discussed in (Sun et al., 2017). Each path from a root to a leaf node denotes one possible trajectory in the MDP.

The MDPs satisfying this condition cover a wide range of tasks such as Dialogue System (Li et al., 2017), Go (Silver et al., 2017), video games (Bellemare et al., 2013) and all MDPs with only one initial state. If we want to preserve the optimality by TRS, the optimal trajectories of a MDP need to cover all initial states or equivalently, all initial states must lead to at least one optimal trajectory. Similarly, the near-optimality is preserved for all MDPs that its near-optimal trajectories cover all initial states.

Theoretically, it is possible to transfer more general MDPs to satisfy Condition 3 and preserve the optimality with potential-based reward shaping (Ng et al., 1999). More concretely, consider the deterministic binary tree MDP \( \mathcal{M}_1 \) with the set of initial states \( S_1 = \{ s_1, s'_1 \} \) as defined in Figure 4. There are eight possible trajectories in \( \mathcal{M}_1 \). Let \( r(\tau_1) = 10 = R_{\text{max}}, r(\tau_8) = 3, r(\tau_i) = 2, \forall i = 2, \ldots , 7 \). Therefore, this MDP does not satisfy Condition 3. We can compensate the trajectory reward of the best trajectory starting from \( s'_1 \) to the \( R_{\text{max}} \) by shaping the reward with the potential-based function \( \phi(s_7') = 7 \) and \( \phi(s) = 0, \forall s \neq s_7' \). This reward shaping requires more prior knowledge, which may not be feasible in practice. A more realistic method is to design a dynamic trajectory reward shaping approach. In the beginning, we set \( c(s) = \min_{s \in S_1} r(\tau|s(\tau, 1) = s), \forall s \in S_1 \). Take \( \mathcal{M}_1 \) as an example, \( c(s) = 3, \forall s \in S_1 \). During the exploration stage, we track the current best trajectory of each initial state and update \( c(s) \) with its trajectory reward.

Nevertheless, if the Condition 3 is not satisfied, we need more sophisticated prior knowledge other than a predefined trajectory reward threshold \( c \) to construct the replay buffer (training dataset of UNOP). The practical implementation of trajectory reward shaping and rigorously theoretical study for general MDPs are beyond the scope of this work.

Appendix D. Proof of Long-term Performance Theorem 10

**Lemma 25** Given a specific trajectory \( \tau \), the log-likelihood of state-action pairs over horizon \( T \) is equal to the weighted sum over the entire state-action space, i.e.:

\[
\frac{1}{T} \sum_{t=1}^{T} \log \pi_\theta(a_t|s_t) = \sum_{s,a} p(s,a|\tau) \log \pi_\theta(a|s),
\]

where the sum in the right hand side is the summation over all possible state-action pairs. It is worth noting that \( p(s,a|\tau) \) is not related to any policy parameters. It is the probability of a specific state-action pair \( (s,a) \) in a specific trajectory \( \tau \).
Proof Given a trajectory $\tau = \{(s(\tau, 1), a(\tau, 1)), \ldots, (s(\tau, T), a(\tau, T))\}$, denote the unique state-action pairs in this trajectory as $U(\tau) = \{(s_i, a_i)\}_{i=1}^n$, where $n$ is the number of unique state-action pairs in $\tau$ and $n \leq T$. The number of occurrences of a state-action pair $(s_i, a_i)$ in the trajectory $\tau$ is denoted as $|\{(s_i, a_i)\}|$. Then we have the following:

$$\frac{1}{T} \sum_{t=1}^T \log \pi_\theta(a_i|s_t) = \sum_{i=1}^n \frac{|\{(s_i, a_i)\}|}{T} \log \pi_\theta(a_i|s_i) = \sum_{(s, a) \in U(\tau)} p(s, a|\tau) \log \pi_\theta(a|s) = \sum_{(s, a) \in U(\tau)} p(s, a|\tau) \log \pi_\theta(a|s) \tag{16}$$

$$\sum_{(s, a) \in U(\tau)} p(s, a|\tau) = \sum_{i=1}^n p(s_i, a_i|\tau) = \sum_{i=1}^n \frac{|\{(s_i, a_i)\}|}{T} = 1, \tag{17}$$

From Eq (16) to Eq (17) we used the fact:

$$\sum_{(s, a) \in U(\tau)} p(s, a|\tau) = \sum_{i=1}^n p(s_i, a_i|\tau) = \sum_{i=1}^n \frac{|\{(s_i, a_i)\}|}{T} = 1,$$

and therefore we have $p(s, a|\tau) = 0, \forall (s, a) \notin U(\tau)$. This thus completes the proof. ■

Now we are ready to prove the Theorem 10:

Proof The following proof holds for an arbitrary subset of trajectories $T$ determined by the threshold $c$ in Def 9. The $\pi_*$ is associated with $c$ and this subset of trajectories. We present the following lower bound of the expected long-term performance:

$$\arg \max _\theta \sum_{\tau \in T} p_\theta(\tau)w(\tau)$$

$$\therefore w(\tau) = 0, \text{if } \tau \notin T$$

$$= \arg \max _\theta \frac{1}{|T|} \sum_{\tau \in T} p_\theta(\tau)w(\tau)$$

use Lemma 27 $\therefore p_\theta(\tau) > 0$ and $w(\tau) > 0$, $\therefore \sum_{\tau \in T} p_\theta(\tau)w(\tau) > 0$

$$= \arg \max _\theta \log \left( \frac{1}{|T|} \sum_{\tau \in T} p_\theta(\tau)w(\tau) \right)$$

$$\therefore \log \left( \sum_{i=1}^n x_i/n \right) \geq \sum_{i=1}^n \log(x_i)/n, \forall i, x_i > 0, \text{we have:}$$

$$\log \left( \frac{1}{|T|} \sum_{\tau \in T} p_\theta(\tau)w(\tau) \right) \geq \sum_{\tau \in T} \frac{1}{|T|} \log p_\theta(\tau)w(\tau),$$

where the lower bound holds when $p_\theta(\tau)w(\tau) = \frac{1}{|T|}, \forall \tau \in T$. To this end, we maximize the lower bound of the expected long-term performance:

$$\arg \max _\theta \sum_{\tau \in T} \frac{1}{|T|} \log p_\theta(\tau)w(\tau)$$

$$= \arg \max _\theta \sum_{\tau \in T} \log(p(s_1) \prod_{t=1}^T (\pi_\theta(a_t|s_t)p(s_{t+1}|s_t, a_t))w(\tau))$$

$$= \arg \max _\theta \sum_{\tau \in T} \log \left( p(s_1) \prod_{t=1}^T \pi_\theta(a_t|s_t) \prod_{t=1}^T p(s_{t+1}|s_t, a_t)w(\tau) \right)$$

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\[
\begin{align*}
&= \arg \max_{\theta} \sum_{\tau \in T} \left( \log p(s_1) + \sum_{t=1}^{T} \log p(s_{t+1}|s_t, a_t) + \sum_{t=1}^{T} \log \pi_\theta(a_t|s_t) + \log w(\tau) \right) \tag{18}
\end{align*}
\]

The above shows that \(w(\tau)\) can be set as an arbitrary positive constant

\[
\begin{align*}
&= \arg \max_{\theta} \frac{1}{|T|} \sum_{\tau \in T} \sum_{t=1}^{T} \log \prod_\theta (a_t|s_t) \\
&= \arg \max_{\theta} \frac{1}{|T|} \sum_{\tau \in T} \sum_{t=1}^{T} \log \prod_\theta (a_t|s_t) \\
&= \arg \max_{\theta} \frac{1}{|T|} \sum_{\tau \in T} \frac{1}{T} \sum_{t=1}^{T} \log \pi_\theta(a_t|s_t) \quad \text{(the existence of UNOP in Assumption 2)} \\
&= \arg \max_{\theta} \sum_{\tau \in T} \mathbb{E}_{p_\pi}(\tau) \frac{1}{T} \left( \sum_{t=1}^{T} \log \pi_\theta(a_t|s_t) \right)
\end{align*}
\]

where \(\pi_*\) is a UNOP (Def 9) \(\Rightarrow p_{\pi_*}(\tau) = 0 \quad \forall \tau \notin T\)

Eq (20) can be established based on \(\sum_{\tau \in T} p_{\pi_*}(\tau) = \sum_{\tau \in T} 1/|T| = 1\)

\[
\begin{align*}
&= \arg \max_{\theta} \sum_{\tau} \mathbb{E}_{p_\pi}(\tau) \frac{1}{T} \left( \sum_{t=1}^{T} \log \pi_\theta(a_t|s_t) \right) \quad \text{(Lemma 25)} \\
&= \arg \max_{\theta} \sum_{\tau} \mathbb{E}_{p_\pi}(\tau) \sum_{s,a} p(s,a|\tau) \log \pi_\theta(a|s)
\end{align*}
\]

The 2nd sum is over all possible state-action pairs. \((s,a)\) represents a specific state-action pair.

\[
\begin{align*}
&= \arg \max_{\theta} \sum_{\tau} \sum_{s,a} \mathbb{E}_{p_\pi}(\tau) p(s,a|\tau) \log \pi_\theta(a|s) \\
&= \arg \max_{\theta} \sum_{s,a} \sum_{\tau} \mathbb{E}_{p_\pi}(\tau) p(s,a|\tau) \log \pi_\theta(a|s) \\
&= \arg \max_{\theta} \sum_{s,a} \mathbb{E}_{p_\pi}(s,a) \log \pi_\theta(a|s) \quad \text{(21)}
\end{align*}
\]

In this proof we use \(s_t = s(\tau,t)\) and \(a_t = a(\tau,t)\) as abbreviations, which denote the \(t\)-th state and action in the trajectory \(\tau\), respectively. \(|T|\) denotes the number of trajectories in \(T\). We also use the definition of \(w(\tau)\) to only focus on near-optimal trajectories. We set \(w(\tau) = 1\) for simplicity but it will not affect the conclusion if set to other constants.

**Optimality:** Furthermore, the optimal solution for the objective function Eq (21) is a uniformly (near)-optimal policy \(\pi_*\).

\[
\begin{align*}
&= \arg \max_{\theta} \sum_{s,a} \mathbb{E}_{p_\pi}(s,a) \log \pi_\theta(a|s) \\
&= \arg \max_{\theta} \sum_{s} \mathbb{E}_{p_\pi}(s) \sum_{a} \pi_*(a|s) \log \pi_\theta(a|s) \\
&= \arg \max_{\theta} \sum_{s} \mathbb{E}_{p_\pi}(s) \sum_{a} \pi_*(a|s) \log \pi_\theta(a|s) - \sum_{s} \mathbb{E}_{p_\pi}(s) \sum_{a} \log \pi_*(a|s) \\
&= \arg \max_{\theta} \sum_{s} \mathbb{E}_{p_\pi}(s) \sum_{a} \pi_*(a|s) \log \frac{\pi_\theta(a|s)}{\pi_*(a|s)} \\
&= \arg \max_{\theta} \sum_{s} \mathbb{E}_{p_\pi}(s) \sum_{a} -KL(\pi_*(a|s)||\pi_\theta(a|s)) = \pi_*
\end{align*}
\]

Therefore, the optimal solution of Eq (21) is also the (near)-optimal solution for the original RL problem since \(\sum_{\tau} \mathbb{E}_{p_\pi}(\tau)r(\tau) = \sum_{\tau \in T} \frac{1}{|T|} r(\tau) \geq c = R_{\text{max}} - \epsilon\). The optimal solution is obtained when we set \(c = R_{\text{max}}\). 

\[
\]
**Lemma 26** Given any optimal policy $\pi$ of MDP satisfying Condition 3, $\forall \tau \notin T$, we have $p_\pi(\tau) = 0$, where $T$ denotes the set of all possible optimal trajectories in this lemma. If $\exists \tau \notin T$, such that $p_\pi(\tau) > 0$, then $\pi$ is not an optimal policy.

**Proof** We prove this by contradiction. We assume $\pi$ is an optimal policy. If $\exists \tau' \notin T$, such that 1) $p_\pi(\tau') \neq 0$, or equivalently: $p_\pi(\tau') > 0$ since $p_\pi(\tau') \in [1, 0]$. and 2) $\tau' \notin T$. We can find a better policy $\pi'$ by satisfying the following three conditions:

$$p_\pi(\tau') = 0$$
$$p_\pi(\tau_1) = p_\pi(\tau_1) + p_\pi(\tau'), \tau_1 \in T \text{ and}$$
$$p_\pi(\tau) = p_\pi(\tau), \forall \tau \notin \{\tau', \tau_1\}$$

Since $p_\pi(\tau) \geq 0, \forall \tau$ and $\sum_{\tau} p_\pi(\tau) = 1$, therefore $p_\pi, \pi'$ constructs a valid probability distribution. Then the expected long-term performance of $\pi'$ is greater than that of $\pi$:

$$\sum_{\tau} p_\pi(\tau)w(\tau) - \sum_{\tau} p_\pi(\tau)w(\tau)$$
$$= \sum_{\tau \notin \{\tau', \tau_1\}} p_\pi(\tau)w(\tau) + p_\pi(\tau_1)w(\tau_1) + p_\pi(\tau')w(\tau')$$
$$- \left( \sum_{\tau \notin \{\tau', \tau_1\}} p_\pi(\tau)w(\tau) + p_\pi(\tau_1)w(\tau_1) + p_\pi(\tau')w(\tau') \right)$$
$$= p_\pi(\tau_1)w(\tau_1) + p_\pi(\tau')w(\tau') - (p_\pi(\tau_1)w(\tau_1) + p_\pi(\tau')w(\tau'))$$
$$\therefore \tau' \notin T, \therefore w(\tau') = 0 \text{ and } \tau_1 \in T, \therefore w(\tau) = 1$$
$$= p_\pi(\tau_1) - p_\pi(\tau)$$
$$= p_\pi(\tau_1) + p_\pi(\tau') - p_\pi(\tau_1) = p_\pi(\tau') > 0.$$

Essentially, we can find a policy $\pi'$ that has higher probability on the optimal trajectory $\tau_1$ and zero probability on $\tau'$. This indicates that it is a better policy than $\pi$. Therefore, $\pi$ is not an optimal policy and it contradicts our assumption, which proves that such $\tau'$ does not exist. Therefore, $\forall \tau \notin T$, we have $p_\pi(\tau) = 0$. ■

**Lemma 27 (Policy Performance)** If the policy takes the form as in Eq (4) or Eq (2), then we have $\forall \tau$, $p_\theta(\tau) > 0$. This means for all possible trajectories allowed by the environment, the policy takes the form of either ranking policy or softmax will generate this trajectory with probability $p_\theta(\tau) > 0$. Note that because of this property, $\pi_\theta$ is not an optimal policy according to Lemma 26, though it can be arbitrarily close to an optimal policy.

**Proof**

The trajectory probability is defined as: $p(\tau) = p(s_1)\prod_{t=1}^{T}(\pi_\theta(a_t|s_t)p(s_{t+1}|s_t, a_t))$

Then we have:

The policy takes the form as in Eq (4) or Eq (2) $\Rightarrow \pi_\theta(a_t|s_t) > 0$.

$p(s_1) > 0, p(s_{t+1}|s_t, a_t) > 0 \Rightarrow p_\theta(\tau) = 0$.

$p(s_{t+1}|s_t, a_t) = 0$ or $p(s_1) = 0 \Rightarrow p_\theta(\tau) = 0$, which means $\tau$ is not a possible trajectory.

In summary, for all possible trajectories, $p_\theta(\tau) > 0$.

This thus completes the proof. ■
Appendix E. Proof of Corollary 13

Corollary 28 (Ranking performance policy gradient) Optimizing the objective lower bound of expected long-term performance by ranking policy is equivalent to the following:

$$\min_{\theta} \sum_{s,a} p_{\pi_{\ast}}(s,a_i)L(s_i,a_i)$$

where the pair-wise loss $L(s,a_i)$ is defined as:

$$L(s,a_i) = \sum_{|A|}^{j=1,j\neq i} \max(0, 1 + Q(s,a_j) - Q(s,a_i))$$

**Proof** In RPG, the policy $\pi_{\theta}(a|s)$ is defined as in Eq (2). We then replace the action probability distribution in Eq (10) with the RPG policy.

\[\pi(a = a_i|s) = \Pi_{j=1,j\neq i}^m p_{ij}\]

Because RPG is fitting a deterministic optimal policy, we denote the optimal action given state $s$ as $a_i$, then we have

\[\max_{\theta} \sum_{s,a} p_{\pi_{\ast}}(s,a_i) \log \pi(a_i|s) = \max_{\theta} \sum_{s,a} p_{\pi_{\ast}}(s,a_i) \log(\Pi_{j=1,j\neq i}^m p_{ij})\]

\[= \max_{\theta} \sum_{s,a} p_{\pi_{\ast}}(s,a_i) \log(\frac{1}{1 + e^{Q_j - Q_i}})\]  

(First order Taylor expansion.)

\[\approx \min_{\theta} \sum_{s,a} p_{\pi_{\ast}}(s,a_i) \sum_{j\neq i,j=1}^m Q_j = \min_{\theta} \sum_{s,a} p_{\pi_{\ast}}(s,a_i) \sum_{j\neq i,j=1}^m (Q_j - Q_i)\]

Since in RPG, we are fitting a deterministic policy, as long as we have $Q_j < Q_i$, when the $i$-th action is the optimal action, we obtained our optimal policy. Therefore, we can use the hinge loss to replace the above objective function.

\[\Rightarrow \min_{\theta} \sum_{s,a} p_{\pi_{\ast}}(s,a_i)L(s_i,a_i) \quad (Eq \ (22))\]

where the pair-wise loss $L(s,a_i)$ is defined as:

$$L(s,a_i) = \sum_{|A|}^{j=1,j\neq i} \max(0, 1 + Q(s,a_j) - Q(s,a_i))$$

Appendix F. Policy gradient variance reduction

Before we prove the corollary, we first introduce two mild assumptions as follows:

**Assumption 3** we assume the existence of maximum norm of log gradient over all possible state-action pairs, i.e.

$$C = \max_{s,a} ||\nabla_{\theta} \log \pi_{\theta}(a|s)||_\infty$$
Assumption 4 We assume the trajectory return is bounded \( R_{\text{max}} = \max_\tau r(\tau) \).

Corollary 29 (Variance reduction) Given a stationary policy, the upper bound of the variance of each dimension of policy gradient is \( O(T^2C^2R_{\text{max}}^2) \). The upper bound of gradient variance of maximizing the lower bound of long-term performance Eq (10) is \( O(C^2) \), where \( C \) is the maximum norm of log gradient based on Assumption 3. The supervised learning has reduced the upper bound of gradient variance by an order of \( O(T^2R_{\text{max}}^2) \) as compared to the regular policy gradient, considering \( R_{\text{max}} \geq 1, T \geq 1 \), which is a very common situation in practice.

Proof The regular policy gradient of policy \( \pi_\theta \) is given as (Williams, 1992):

\[
\sum_\tau p_\theta(\tau) \left[ \sum_{t=1}^T \nabla_\theta \log(\pi_\theta(a(\tau,t)|s(\tau,t)))r(\tau) \right]
\]

The regular policy gradient variance of the \( i \)-th dimension is denoted as follows:

\[
\text{Var} \left( \sum_{t=1}^T \nabla_\theta \log(\pi_\theta(a(\tau,t)|s(\tau,t)))_i r(\tau) \right)
\]

We denote \( x_i(\tau) = \sum_{t=1}^T \nabla_\theta \log(\pi_\theta(a(\tau,t)|s(\tau,t)))_i r(\tau) \) for convenience. Therefore, \( x_i \) is a random variable. Then apply \( \text{var}(x) = \mathbb{E}_{p_\theta(\tau)}[x^2] - \mathbb{E}_{p_\theta(\tau)}[x]^2 \), we have:

\[
\text{Var} \left( \sum_{t=1}^T \nabla_\theta \log(\pi_\theta(a(\tau,t)|s(\tau,t)))_i r(\tau) \right)
= \text{Var} (x_i(\tau)) \\
= \sum_\tau p_\theta(\tau) x_i(\tau)^2 - \left[ \sum_\tau p_\theta(\tau) x_i(\tau) \right]^2 \\
\leq \sum_\tau p_\theta(\tau) x_i(\tau)^2 \\
= \sum_\tau p_\theta(\tau) \left[ \sum_{t=1}^T \nabla_\theta \log(\pi_\theta(a(\tau,t)|s(\tau,t)))_i r(\tau) \right]^2 \quad \text{(Assumption 4)} \\
\leq \sum_\tau p_\theta(\tau) \left[ \sum_{t=1}^T \nabla_\theta \log(\pi_\theta(a(\tau,t)|s(\tau,t)))_i \right]^2 R_{\text{max}}^2 \\
= R_{\text{max}}^2 \sum_\tau p_\theta(\tau) \left[ \sum_{t=1}^T \sum_{k=1}^T \nabla_\theta \log(\pi_\theta(a(\tau,t)|s(\tau,t)))_i \nabla_\theta \log(\pi_\theta(a(\tau,k)|s(\tau,k))) \right] \quad \text{(Assumption 3)} \\
\leq R_{\text{max}}^2 \sum_\tau p_\theta(\tau) \left[ \sum_{t=1}^T \sum_{k=1}^T C^2 \right] \\
= R_{\text{max}}^2 \sum_\tau p_\theta(\tau) T^2 C^2 \\
= T^2 C^2 R_{\text{max}}^2
\]

The policy gradient of long-term performance (Def 8)

\[
\sum_{s,a} p_{\pi_*}(s,a) \nabla_\theta \log \pi_\theta(a|s)
\]

The policy gradient variance of the \( i \)-th dimension is denoted as

\[
\text{var}(\nabla_\theta \log \pi_\theta(a|s)_i)
\]

Then the upper bound is given by

\[
\text{var}(\nabla_\theta \log \pi_\theta(a|s)_i)
\]
\[
\begin{align*}
&= \sum_{s,a} p_{\pi_*}(s,a) [\nabla_{\theta} \log \pi_\theta(a|s)]^2 - [\sum_{s,a} p_{\pi_*}(s,a) \nabla_{\theta} \log \pi_\theta(a|s)]^2 \\
&\leq \sum_{s,a} p_{\pi_*}(s,a) [\nabla_{\theta} \log \pi_\theta(a|s)]^2 \\
&\leq \sum_{s,a} p_{\pi_*}(s,a) C^2 \\
&= C^2
\end{align*}
\]

This thus completes the proof.

\section{Appendix G. Discussions of Assumption 2}

\textbf{Lemma 30}  For MDPs defined in Section 3 satisfying the following conditions:

- \textit{Condition 3.}
- \textit{|s_1| = |T|, where T denotes the set of optimal trajectories in this lemma.}
- \textit{Deterministic transitions, i.e., p(s'|s,a) \in \{0,1\}.}
- \textit{Uniform initial state distribution, i.e., p(s_1) = \frac{1}{|T|}.}

Then we have: \( \exists \pi_*, \text{ where } s.t. \ p_{\pi_*}(\tau) = \frac{1}{|T|}, \forall \tau \in T \). It means that a deterministic uniformly optimal policy always exists for this MDP.

\textbf{Proof}  We can prove this by construction. The following analysis applies for any \( \tau \in T \).

\begin{align*}
p_{\pi_*}(\tau) &= \frac{1}{|T|} \\
&\iff \log p_{\pi_*}(\tau) = - \log |T| \\
&\iff \log p(s_1) + \sum_{t=1}^T \log p(s_{t+1}|s_t,a_t) + \sum_{t=1}^T \log \pi_*(a_t|s_t) = - \log |T| \\
&\iff \sum_{t=1}^T \log \pi_*(a_t|s_t) = - \log p(s_1) - \sum_{t=1}^T \log p(s_{t+1}|s_t,a_t) - \log |T| \\
&\text{where we use } a_t, s_t \text{ as abbreviations of } a(\tau,t), s(\tau,t). \\
&\text{We denote } D(\tau) = - \log p(s_1) - \sum_{t=1}^T \log p(s_{t+1}|s_t,a_t) > 0 \\
&\iff \sum_{t=1}^T \log \pi_*(a_t|s_t) = D(\tau) - \log |T| \\
&\therefore \text{we can obtain a uniformly optimal policy by solving the nonlinear programming:} \\
&\sum_{t=1}^T \log \pi_*(a(\tau,t)|s(\tau,t)) = D(\tau) - \log |T| \ \forall \tau \in T \quad (23) \\
&\log \pi_*(a(\tau,t)|s(\tau,t)) = 0, \forall \tau \in T, t = 1, ..., T \quad (24) \\
&\sum_{i=1}^m \pi_*(a_i|s(\tau,t)) = 1, \forall \tau \in T, t = 1, ..., T \quad (25)
\end{align*}

Use the condition \( p(s_1) = \frac{1}{|T|} \), then we have:

\begin{align*}
\therefore \sum_{t=1}^T \log \pi_*(a(\tau,t)|s(\tau,t)) &= \sum_{t=1}^T \log 1 = 0 \text{ (LHS of Eq (23))} \\
\therefore - \log p(s_1) - \sum_{t=1}^T \log p(s_{t+1}|s_t,a_t) - \log |T| &= \log |T| - 0 - \log |T| = 0 \text{ (RHS of Eq (23))} \\
\therefore D(\tau) - \log |T| &= \sum_{t=1}^T \log \pi_*(a(\tau,t)|s(\tau,t)), \forall \tau \in T.
\end{align*}
Figure 5: The directed graph that describes the conditional independence of pairwise relationship of actions, where $Q_1$ denotes the return of taking action $a_1$ at state $s$, following policy $\pi$ in $\mathcal{M}$, i.e., $Q_\mathcal{M}^\pi(s, a_1)$. $I_{1,2}$ is a random variable that denotes the pairwise relationship of $Q_1$ and $Q_2$, i.e., $I_{1,2} = 1$, i.i.f. $Q_1 \geq Q_2$, o.w. $I_{1,2} = 0$.

Also the deterministic optimal policy satisfies the conditions in Eq (24 25). Therefore, the deterministic optimal policy is a uniformly optimal policy. This lemma describes one type of MDP in which UOP exists. From the above reasoning, we can see that as long as the system of non-linear equations Eq (23 24 25) has a solution, the uniformly (near)-optimal policy exists. ■

Lemma 31 (Hit optimal trajectory) The probability that a specific optimal trajectory was not encountered given an arbitrary softmax policy $\pi_\theta$ is exponentially decreasing with respect to the number of training episodes. No matter a MDP has deterministic or probabilistic dynamics.

Proof Given a specific optimal trajectory $\tau = \{s(\tau, t), a(\tau, t)\}_{t=1}^T$, and an arbitrary stationary policy $\pi_\theta$, the probability that has never encountered at the $n$-th episode is $[1 - p_\theta(\tau)]^n = \xi^n$, based on lemma 27 we have $p_\theta(\tau) > 0$, therefore we have $\xi \in [0, 1)$.

Appendix H. Discussions of Assumption 1

Intuitively, given a state and a stationary policy $\pi$, the relative relationships among actions can be independent, considering a fixed MDP $\mathcal{M}$. The relative relationship among actions is the relative relationship of actions’ return. Starting from the same state, following a stationary policy, the actions’ return is determined by MDP properties such as environment dynamics, reward function, etc.

More concretely, we consider a MDP with three actions $(a_1, a_2, a_3)$ for each state. The action value $Q_\mathcal{M}^\pi$ satisfies the Bellman equation in Eq (26). Notice that in this subsection, we use $Q_\mathcal{M}^\pi$ to denote the action value that estimates the absolute value of return in $\mathcal{M}$.

$$Q_\mathcal{M}^\pi(s, a_i) = r(s, a_i) + \max_a E_{s' \sim P(s|s, a)} Q_\mathcal{M}^\pi(s', a), \forall i = 1, 2, 3.$$ (26)

As we can see from Eq (26), $Q_\mathcal{M}^\pi(s, a_i), i = 1, 2, 3$ is only related to $s, \pi,$ and environment dynamics $P$. It means if $\pi, \mathcal{M}$ and $s$ are given, the action values of three actions are determined. Therefore, we can use a directed graph (Bishop, 2006) to model the relationship of action values, as shown in Figure 5 (a). Similarly, if we only consider the ranking of actions, this ranking is consistent with the relationship of actions’ return, which is also determined by $s, \pi,$ and $P$. Therefore, the pairwise
Appendix I. The proof of Theorem 17

Proof  The proof mainly establishes on the proof for long term performance Theorem 10 and connects the generalization bound in PAC framework to the lower bound of return.

\[
\log(\frac{1}{|T|} \sum_{\tau \in T} p_\theta(\tau) w(\tau)) \geq \frac{1}{|T|} \sum_{\tau \in T} \log p_\theta(\tau) w(\tau)
\]

\[
\Leftrightarrow \sum_{\tau \in T} p_\theta(\tau) w(\tau) \geq |T| \exp(\frac{1}{|T|} \sum_{\tau \in T} \log p_\theta(\tau) w(\tau))
\]

denote \( F = \sum_\tau p_\theta(\tau) w(\tau) = \sum_{\tau \in T} p_\theta(\tau) w(\tau) \)

\[
\Leftrightarrow F \geq |T| \exp(\frac{1}{|T|} \sum_{\tau \in T} \log p_\theta(\tau) w(\tau))
\]

\[
= |T| \exp \left( \frac{1}{|T|} \sum_{\tau \in T} \left( \log p(s_1) + \sum_{t=1}^T \log p(s_{t+1}|s_t, a_t) + \sum_{t=1}^T \log \pi_\theta(a_t|s_t) + \log w(\tau) \right) \right)
\]

\[
= |T| \exp \left( \frac{1}{|T|} \sum_{\tau \in T} \left( \log p(s_1) + \sum_{t=1}^T \log p(s_{t+1}|s_t, a_t) + \sum_{t=1}^T \log \pi_\theta(a_t|s_t) \right) \right)
\]

Denote the dynamics of a trajectory as \( p_d(\tau) = p(s_1)\prod_{t=1}^T p(s_{t+1}|s_t, a_t) \)

Notice that \( p_d(\tau) \) is environment dynamics, which is fixed given a specific MDP.

\[
\Leftrightarrow F \geq |T| \exp \left( \frac{1}{|T|} \sum_{\tau \in T} \log p_d(\tau) \right) \exp \left( \frac{1}{|T|} \sum_{\tau \in T} \left( \sum_{t=1}^T \log \pi_\theta(a_t|s_t) \right) \right)
\]

\[
= |T| \left( \prod_{\tau \in T} p_d(\tau) \right) \frac{1}{|T|} \exp \left( \frac{1}{|T|} \sum_{\tau \in T} \left( \sum_{t=1}^T \log \pi_\theta(a_t|s_t) \right) |T| \right)
\]

Use the same reasoning from Eq (19) to Eq (21).

\[
= |T| \left( \prod_{\tau \in T} p_d(\tau) \right) \frac{1}{|T|} \exp \left( T \sum_{s,a} p_{\pi_\theta}(s,a) \log \pi_\theta(a|s) \right)
\]

Apply Corollary 13 and denote \( L = \sum_{s,a} p_{\pi_\theta}(s,a) \log \pi_\theta(a|s) \).

\( L \) is the only term that is related to the policy parameter \( \theta \)

\[
L = \sum_{s,a \in U_w} p_{\pi_\theta}(s,a) \log \pi_\theta(a|s) + \sum_{s,a \notin U_w} p_{\pi_\theta}(s,a) \log \pi_\theta(a|s)
\]

\[
\therefore \text{with current policy classifier } \theta, \forall s,a \notin U_w, \pi_\theta(a|s) = 1
\]

\[
= \sum_{s,a \in U_w} p_{\pi_\theta}(s,a) \log \pi_\theta(a|s)
\]

If we use RPG as our policy parameterization, then with Eq (2)

\[
= \sum_{s,a \in U_w} p_{\pi_\theta}(s,a) \log (\Pi_{j \neq i}^m \theta_{ij})
\]
= \sum_{s,a_i \in U_w} p_{\pi_\ast}(s,a_i) \sum_{j \neq i,j = 1}^m \log \frac{1}{1+e^{Q_{ji}}}

By Condition 1 which can be easily satisfied in practice. Then we have: 

\[ Q_{ij} < 2c_q \leq 1 \]

Given \( h = \pi_\ast \), misclassified state action pairs set \( U_w = \{s,a|h(s) \neq a,(s,a) \sim p_\ast(s,a)\} \)

Apply Lemma 16, the misclassified rate is at most \( \eta \).

\[ \geq \sum_{s,a_i \in U_w} p_{\pi_\ast}(s,a_i)(m-1) \log(1+e) \]

\[ = \eta(m-1) \log(1+e) \]

\[ = \eta \]

\[ \geq D(1+e) \eta(1-m)T \]

From generalization performance to sample complexity:

\[ n \geq \frac{1}{2\gamma^2} \log \frac{2|H|}{\delta} \geq \frac{2(m-1)^2T^2}{(\log_2 \frac{D}{1-\epsilon})^2} \log \frac{2|H|}{\delta} \]

Bridge the long-term reward and long-term performance:

\[ \sum_{\tau} p_\theta(\tau)r(\tau) \text{ In Section 7, } r(\tau) \in (0,1], \forall \tau. \]

\[ \geq \sum_{\tau} p_\theta(\tau)w(\tau) \text{ Since we focus on UOP Def 9, } c = 1 \text{ in TSR Def 5} \]

\[ = \sum_{\tau \in T} p_\theta(\tau)w(\tau) \geq 1 - \epsilon \]

This thus concludes the proof.

**Assumption 5 (Realizable)** We assume there exists a hypothesis \( h_\ast \in H \) that obtains zero expected risk, i.e. \( \exists h_\ast \in H \Rightarrow \sum_{s,a} p_{\pi_\ast}(s,a) I\{h_\ast(s) \neq a\} = 0 \).

The Assumption 5 is not necessary for the proof of Theorem 17. For the proof of Corollary 18, we introduce this assumption to achieve more concise conclusion. In finite MDP, the realizable
Appendix J. The proof of Lemma 21

**Proof** Let \( e_{=i} \) denotes the event \( n = i|k \), i.e. the number of different optimal trajectories in first \( k \) episodes is equal to \( i \). Similarly, \( e_{\geq i} \) denotes the event \( n \geq i|k \). Since the events \( e_{=i} \) and \( e_{=j} \) are mutually exclusive when \( i \neq j \), therefore, \( p(e_{\geq i}) = p(e_{=i}, e_{=i+1},..., e_{=|T|}) = \sum_{j=i}^{|T|} p(e_{=j}) \). Further more, we know that \( \sum_{i=0}^{|T|} p(e_{=i}) = 1 \) since \( \{e_{=i}, i = 0,...,|T|\} \) constructs an universal set. For example, \( p(e_{\geq 1}) = p_{\pi,\mathcal{M}}(n \geq 1|k) = 1 - p_{\pi,\mathcal{M}}(n = 0|k) = 1 - \left( \frac{N-|T|}{N} \right)^k \).

\[
p_{\pi,\mathcal{M}}(n \geq i|k) = 1 - \sum_{i'=0}^{i-1} p_{\pi,\mathcal{M}}(n = i'|k) = 1 - \sum_{i'=0}^{i-1} C_{|T|}^{i'} \sum_{j=0}^{i'-1} (-1)^j C_j^i (N - |T| + i' - j)^k N^k
\]

In Eq (27), we use the inclusion-exclusion principle (Kahn et al., 1996) to have the following equality.

\[
p_{\pi,\mathcal{M}}(n = i'|k) = C_{|T|}^{i'} p(e_{\tau_1,\tau_2,...,\tau_{i'}}) = C_{|T|}^{i'} \sum_{j=0}^{i'-1} (-1)^j C_j^i (N - |T| + i' - j)^k N^k
\]

\( e_{\tau_1,\tau_2,...,\tau_{i'}} \) denotes the event: in first \( k \) episodes, a certain set of \( i' \) optimal trajectories \( \tau_1, \tau_2, ..., \tau_{i'}, i' \leq |T| \) is sampled.

Appendix K. The proof of Corollary 22

**Proof** The Corollary 22 is a direct application of Lemma 21 and Theorem 17. First, we reformat Theorem 17 as follows:

\[
p(A|B) \geq 1 - \delta
\]

where event \( A \) denotes \( \sum_{\tau} p_\theta(\tau) r(\tau) \geq D(1+\epsilon)^{p(1-n)m}|T| \), event \( B \) denotes the number of state-action pairs \( n' \) from UOP 9 satisfying \( n' \geq n \), given fixed \( \delta \). With Lemma 21, we have \( p(B) \geq p_{\pi,\mathcal{M}}(n' \geq n|k) \). Then, \( P(A) = P(A|B)P(B) \geq (1 - \delta)p_{\pi,\mathcal{M}}(n' \geq n|k) \).

Set \( (1 - \delta)p_{\pi,\mathcal{M}}(n' \geq n|k) = 1 - \delta' \) we have \( P(A) \geq 1 - \delta' \)

\[
\delta = \frac{1 - \delta'}{p_{\pi,\mathcal{M}}(n' \geq n|k)}
\]

\[
\eta = 2\sqrt{\frac{1}{2n} \log \frac{2|\mathcal{H}|}{\delta}}
\]

\[
= 2\sqrt{\frac{1}{2n} \log \frac{2|\mathcal{H}|p_{\pi,\mathcal{M}}(n' \geq n|k)}{1 - \delta'}}
\]