Purification of multipartite entanglement by local operations

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Abstract

Multipartite entanglement purification is revisited by using the Local operations and classical communications (LOCCs). We demonstrate our idea by considering the tripartite case, i.e. the purification of tripartite entanglement. We express the general tripartite entangled states in a special representation of total spin operators \( J_1^2 \) and \( J_2^2 \) of tripartite system with eigenvalues \( 15/4 \) and \( 2 \) respectively. This basis is a genuine basis because it consists of all the genuine entangled states of tripartite system. Our protocol is a recurrence one, and only two copies of the initial mixed tripartite entangled states are needed in each round. It is shown that if the initial fidelity is larger than a threshold \( 0.4 \), the purification process will succeed. The yield of the current protocol is higher than the previous multipartite entanglement purification protocols. As a by-product, we can get a bipartite pure Bell state when the purification protocol fails for W state. Our protocol also shows that there may be some special kind of tripartite entanglement which belongs to neither W-type entanglement nor GHZ-type entanglement.

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I. INTRODUCTION

Entanglement is a indispensable resource for many quantum information processing procedures, such as quantum teleportation\(^1\), quantum superdense coding\(^2\), quantum cryptography\(^3\) and quantum computation\(^4\) etc. Especially, the multipartite entanglement plays an important role in quantum computation\(^5\) and quantum communications\(^7, 8\). All of the above procedures can be perfectly implemented with maximally entangled bipartite or multipartite states only. But the inevitability of the coupling between the entangled systems to its environment will cause decoherence and disentanglement. Thus the entangled states available in real situation are all mixed states, which will decrease the efficiency and fidelity of quantum communication and quantum computation. So how to enhance the entanglement of the distributed mixed entangled states has been one of the highlighting research subjects in quantum information field since the initial entanglement purification protocol for bipartite mixed states\(^9\).

In the bipartite case, the basic purification idea is to extract smaller number of entangled pairs from large number of initial entangled pairs by Local operations and classical communications (LOCCs), and the extracted entangled pairs have more entanglement than the initial ones\(^9\). Purification of bipartite entanglement has been realized in linear optical system\(^10, 11, 12\) and ionic system\(^13, 14, 15\). Recently, the entanglement purification protocols for higher dimension mixed entangled states have also been proposed\(^16, 17, 18, 19\).

In the multipartite case, the situation is a little bit different. Multipartite system can be entangled in different ways, i.e. there exist many different kinds of genuine entanglements, and all states of each kind are considered to be equivalent under invertible local operations. In the three qubits case, there are two kinds of genuine tripartite entanglement, the Greenberger-Horne-Zeilinger (GHZ) entanglement and the W entanglement\(^20\). The local conversion between GHZ state and W state is only possible in an approximate way\(^21\). W. Dür et al generalized the bipartite entanglement purification to the multipartite case, such as purification protocols for generalized GHZ state, cluster state\(^6\) and various quantum error-correcting code, all of which belong to the class of graph states\(^22, 23, 24\). They also showed that the direct multipartite entanglement purification protocols are more efficient than the approaches based on bipartite entanglement purification. But these protocols does not apply to the mixed W state. The reasons are twofold. First, the entanglement of W state is different from the other types of entanglements\(^20\). Secondly, the expression of a general mixed W state is not unique due to the existence of different kinds of genuine multipartite entanglement. Akimasa Miyake and Hans J. Briegel constructed a W basis with all of its states being equivalent to each other under local unitary transformations, and the noisy W states resulting from typical decoherence were distilled by complementary stabilizer measurements rather than the standard CNOT operations and projection measurements used in the bipartite entanglement purification protocol\(^25\). In this paper, we will present an alternative construction of the expansion basis for a general mixed W state, which consists of the two kinds of genuine tripartite entangled states, GHZ states and W states. So we call this basis genuine basis. We generalize the standard bipartite entanglement purification protocol to the general mixed W state expressed in terms of the genuine basis. Our protocol is a recurrence one, and only two copies of the original mixed state are needed in each round. Every user will have two particles from the two different entangled triples undergo a CNOT operation. Then a projection measurement will be carried out on the target particle by each user. Through classical communication, all the users will compare their measurement results and decide whether keep the left particles or not. If the measurement results of the three users are same, the

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entanglement of the left triple is enhanced i.e. the fidelity with respect to the standard W states can be amplified via LOCCs. If the measurement results are not same, the purification protocol for W state fails. But for some measurement results, two of the three source particles will be left in a pure bipartite Bell state. This will decrease the entanglement waste of the ordinary multipartite purification protocols. This paper is organized as follows: in section II, we will give the detailed genuine basis of a tripartite system and the general mixed W state expressed in terms of the genuine basis; The purification process for this general mixed W state will be discussed in section III. Some special cases about the mixed W state will be discussed in section IV; Section V is the conclusion section.

II. GENUINE BASIS

Three qubits can be entangled in two inequivalent ways, and GHZ states and W states are remarkable representatives of them. W state is more robust than GHZ state against the loss of one qubit. GHZ states and W states are the only two kinds of tripartite genuine entangled states. From these two genuine entangled states we construct the following genuine basis for tripartite systems:

\[ |GB^{1-3}\rangle = \left[ |001\rangle + \omega^n |010\rangle + \omega^{2n} |100\rangle \right] / \sqrt{3}, \quad (1a) \]

\[ |GB^{4-6}\rangle = \left[ |110\rangle + \omega^n |101\rangle + \omega^{2n} |011\rangle \right] / \sqrt{3}, \quad (1b) \]

\[ |GB^7\rangle = (|000\rangle + |111\rangle) / \sqrt{2}, \quad (1c) \]

\[ |GB^8\rangle = (|000\rangle - |111\rangle) / \sqrt{2}, \quad (1d) \]

where \( n = 0, 1, 2 \) denote the genuine basis 1, 2, 3 corresponding to \( n = 0, 1, 2 \), and the same convention applies to \( |GB^{4-6}\rangle \). \( \omega = \exp(-i2\pi/3) \). The \( |GB^7\rangle \) and \( |GB^8\rangle \) are two GHZ states. One can easily check that this set of states constructs a common orthogonal and complete set of basis for tripartite systems. The physical meaning of the above eight states can be understood in the following way. The first set of three states in Eq. (1a) are the standard W state with different phases, and they are orthogonal with each other; The second set of three states in Eq. (1b) are from the flipped W state with different phases, and they are orthogonal with each other; The last two states in Eqs. (1c) and (1d) are two standard GHZ states from two product states \(|000\rangle\) and \(|111\rangle\) with different phases. So the four basic states of the genuine basis are \( 1/\sqrt{3} (|001\rangle + |010\rangle + |100\rangle), 1/\sqrt{3} (|110\rangle + |101\rangle + |011\rangle), |000\rangle \) and \(|111\rangle\), which are the four simultaneous eigenvectors of total spin operators \( J^2_{123} \) and \( J^2_{12} \) of tripartite systems with eigenvalues \( 15/4 \) and \( 2 \) respectively.

Using the genuine basis we can write the general tripartite mixed entangled state in the following form:

\[ \rho = \sum_{i=1}^{8} C_i |GB^i\rangle \langle GB^i|, \quad (2) \]

which can describe the state of a distributed entangled quantum ensemble and \( C_i (i = 1, 2, \ldots, 8) \) are real numbers satisfying normalization condition. For the sake of simplicity, we suppose \( C_1 \) is the biggest coefficient. One can find that the first two states \( 1/\sqrt{3} ((|001\rangle + |010\rangle + |100\rangle) \) and \( 1/\sqrt{6} ((|110\rangle + |101\rangle + |011\rangle) \) of the four basic states of genuine basis are standard W states, so we call this general mixed state as mixed W state. Because W state has more robust entanglement than GHZ state, we are interested in extracting W state from this mixed state. This is another reason why we suppose \( C_1 \) is the biggest coefficient. For now, we use W fraction of the mixed state to indicate the entanglement of it, i.e. \( F_W = \max(W|\rho|W) \) with the maximum being taken over the first six state in the genuine basis. So the W fraction of the general mixed W state in Eq. (2) is \( F_W = C_1 \).

III. PURIFICATION PROTOCOL FOR THE GENERAL MIXED W STATES

Because the local operation and classical communication can not creat remote entanglement, the idea of purifying this mixed state is to transfer entanglement from big ensembles to small ones with only LOCCs. Next, we will discuss the purification process in detail.

Firstly, three remote users Alice, Bob and Charlie possess a distributed entangled quantum ensemble whose state can be described by Eq. (2). To get a state with increased W fraction, the three users must operate on two triples of particles of this distributed ensemble:

\[ \rho^s_{ABC} = \sum_{i=1}^{8} C_i |GB^i\rangle^s \langle GB^i|, \quad (3a) \]

\[ \rho^t_{ABC} = \sum_{i=1}^{8} C_i |GB^i\rangle^t \langle GB^i|, \quad (3b) \]

where the subscripts \( A, B, C \) denote the three users Alice, Bob and Charlie, respectively, and the superscripts \( s, t \) denote the source and target triples, respectively. Each user will carry out a controlled not(CNOT) operation on the two particles from two mixed states, i.e. the trilateral XOR operation(TXOR) which is analogous to the bilateral XOR operation(BXOR) in the bipartite entanglement purification protocol[9]. After the TXOR operation the target particles will be measured, and the three users will compare their measurement results through classical communication and decide whether keep the source triple or not. To get the explicit expression of the state of the two triples after TXOR operation, we can treat the initial state of the two triples as the probabilistic mixture of 64 pure states: \( |GB^i\rangle^s \langle GB^i| \) with a probability \( C_i C_j \), \( i, j = 1, 2, \ldots, 8 \).

Straightforward calculation shows that if the measurement results of the three users are same, the left triple are still in a tripartite entangled mixed state, which can have a higher W
fraction provided some restrictions on the initial W fraction. That is to say the purification protocol can succeed with some probability. For measurement $|000\rangle^t_{ABC}$, we can get a new mixed W state with some probability and new W fraction:

$$P_{000} = \left[ 2 \left( C_1 + C_2 + C_3 \right)^2 + 2 \left( C_4 + C_5 + C_6 \right)^2 + 3 \left( C_7 + C_8 \right)^2 \right] / 6,$$ \hspace{1cm} (4a)

$$F_{W}^{000} = \frac{2C_1^2 + 4C_2C_3}{2 \left( C_1 + C_2 + C_3 \right)^2 + 2 \left( C_4 + C_5 + C_6 \right)^2 + 3 \left( C_7 + C_8 \right)^2}.$$ \hspace{1cm} (4b)

To simplify the calculation, without loss of generality, we can let $C_i = \frac{4}{i(i+1)}$, $i = 2, 3, \cdots, 8$ by a random trilateral rotation on the initial mixed state in Eq.(4), which is analogous to the random bilateral rotation of the bipartite entanglement purification protocol[7]. Under this situation, the initial mixed W state takes a concise form:

$$\rho_{cs} = C_1 |GB^1\rangle \langle GB^1| + \frac{1 - C_1}{t} \left( I - |GB^1\rangle \langle GB^1| \right).$$ \hspace{1cm} (5)

Now the W fraction in Eq.(4b) becomes: $F_{W}^{000} = \frac{51C_1^2 - 4C_1 + 2}{40C_1^2 - 10C_1 + 19}$. Obviously, if $C_1 > 2/5$, $F_{W}^{000} - F_W > 0$, which means an amplification of the entanglement of the initial mixed W state. The new W fraction $F_{W}^{000}$ and the yield($P_{000}/2$) varying with the initial W fraction $F_W$ are shown in Fig.(1a) and Fig.(1b) respectively. The Fig.(1a) shows that the fidelity of the purified mixed W state in Ref.[28] is higher than ours new W fraction, but the yield of the current protocol is larger than that of Ref.[28]. The high yield means a great reduction of the number of iterations before the W fraction approaching one.

For measurement result $|111\rangle^t_{ABC}$, although we can get a tripartite entangled mixed state with new W fraction $F_{W}^{111}$, the new W fraction $F_{W}^{111}$ never exceeds the initial W fraction $F_W$ because of the assumption that $C_1$ is the biggest coefficient. So this result does not contribute to the purification of general mixed W states.

For other six measurement results, the state of the left triple only has bipartite entanglement, which indicates the fail of the purification for mixed tripartite W states.

IV. SOME DISCUSSIONS ON THE RESULTS OF THE PURIFICATION PROTOCOL

Although the tripartite purification will fail for the above mentioned six unwanted measurement results, the left triple of particles will still have bipartite entanglement in these six cases. In general, this bipartite entanglement is a mixed entanglement. From the detailed results of the purification protocol in section III, we can see that if the initial state satisfies some conditions, this left bipartite entanglement can be pure Bell entanglement without measurement on the third one. For example, if the initial state is a mixture of two states $|GB^1\rangle$, $|GB^4\rangle$ (or $|GB^2\rangle$, $|GB^5\rangle$ or $|GB^3\rangle$, $|GB^6\rangle$), only $C_1$, $C_4$ (or $C_2$, $C_5$ or $C_3$, $C_6$) are not zero, and all the other coefficients are vanishing. Then, for the measurement results $|100\rangle^t_{ABC}$, $|010\rangle^t_{ABC}$, $|001\rangle^t_{ABC}$, the two particles of the left triple can be in a pure Bell state with the third particle being factorized out naturally. Another example is a very interesting one where we suppose $C_1 = C_4 = 1/2$ (or $C_2 = C_5$, or $C_3 = C_6$) and the other coefficients are vanishing. In these cases, the initial states are all in a total mixture of two orthogonal basis states from the first six states of the genuine basis (For example, $\rho_{mix} = \frac{1}{2} \left( |GB^1\rangle \langle GB^1| + |GB^4\rangle \langle GB^4| \right)$). Intuitively, there should be no entanglement in these total mixtures. Counterintuitively, if we apply the above purification protocol on these two mixtures, we will get pure bipartite Bell state for the measurement results on the targets $|110\rangle^t_{ABC}$, $|011\rangle^t_{ABC}$, $|101\rangle^t_{ABC}$. This point can be understood from analyzing the entanglement property of the total mixture $\rho_{mix}$. Although, up to now, we do not have a very rigorous definition of the entanglement measure of multipartite system, we can detect whether there is some kind of entanglement or not by using entanglement witness[26, 27]. This is done by constructing a witness operator that can detect whether the state has the desired genuine entanglement $|\psi\rangle$, and this witness operator takes the form $W_\psi = \alpha I - |\psi\rangle \langle \psi|$, with
\( \alpha = \max_{\phi \in B} |\langle \phi | \psi \rangle|^2 \). Here \( B \) denotes the set of bispaliable states. If the expectation value of the witness operator is negative in some state, we can say this state has multipartite entanglement of the kind of \( |\psi\rangle \), otherwise there is no genuine entanglement in this state. For example, ones used to use the witness operator \( W_G = \frac{1}{2} \mathbb{I} - \rho_{\text{GHZ}} \langle \text{GHZ} \rangle \) for tripartite W state, and \( W_{\text{GHZ}} = \frac{1}{2} \mathbb{I} - \rho_{\text{GHZ}} \langle \text{GHZ} \rangle \) for tripartite GHZ state \([26, 27]\). If we apply these two tripartite entanglement witness operators on the above mentioned total mixture states \( \rho_{\text{mix}} \), we will find that these total mixture \( \rho_{\text{mix}} \) do not carry any genuine tripartite entanglement. But, if we trace out one of the three particles, bipartite entanglement is there between the two left particles. So, it is possible to extract a pure Bell state from the above mentioned two total mixture states \( \rho_{\text{mix}} \).

Using the witness operator we can detect the entanglement of the initial general mixed W state in Eq. (5). Direct calculations show that \( \text{Tr} (W_W \rho_{\text{cs}}) < 0 \) is found when \( C_1 > \frac{2}{\sqrt{3}} \). That is to say, only when the \( W \) fraction is over \( \frac{13}{26} \) of the state in Eq. (5) possesses W-type entanglement, which coincides with the result of Ref. [24]. But the threshold of our purification protocol is \( \frac{1}{2} \), which is smaller than \( \frac{2}{\sqrt{3}} \). The fact is that, by using our purification protocol, one always can get a mixed W state with \( W \) fraction over \( \frac{13}{26} \), which means that we distill W-type entanglement from non-W-type entanglement. In this sense, we say that the initial state with \( W \) fraction being in the range \( \frac{1}{2} < F_W < \frac{13}{26} \) has some special kind of tripartite entanglement which belongs to neither W-type entanglement nor GHZ-type entanglement (\( \text{Tr} (W_{\text{GHZ}} \rho_{\text{cs}}) > 0 \) for \( \frac{13}{26} < C_1 < 1 \)). At the same time, our threshold is also different from that (\( \frac{1}{4} \)) of Ref. [25], and this point may caused by the reason that we used a type of initial entangled mixed W state which is different from that of Ref. [25]. This may be another evidence of the existence of the above mentioned special tripartite entanglement.

Generalization of the current tripartite purification protocol to the multipartite case is not straightforward, because it is not easy to find the genuine basis for the multipartite case. Anyway, we found the genuine basis for the four-particle system:

\[
\begin{align*}
|GB_1^1\rangle &= \frac{1}{2} \left( |0001\rangle + |0010\rangle + |0100\rangle + |1000\rangle \right), \quad (6a) \\
|GB_1^2\rangle &= \frac{1}{2} \left( |0001\rangle + |0010\rangle - |0100\rangle - |1000\rangle \right), \quad (6b) \\
|GB_1^3\rangle &= \frac{1}{2} \left( |0001\rangle - |0010\rangle + |0100\rangle - |1000\rangle \right), \quad (6c) \\
|GB_1^4\rangle &= \frac{1}{2} \left( |1110\rangle + |1101\rangle + |1011\rangle + |0111\rangle \right), \quad (6d) \\
|GB_1^5\rangle &= \frac{1}{2} \left( |1110\rangle + |1101\rangle - |1011\rangle - |0111\rangle \right), \quad (6e) \\
|GB_1^6\rangle &= \frac{1}{2} \left( |1110\rangle - |1101\rangle + |1011\rangle - |0111\rangle \right), \quad (6f) \\
|GB_1^7\rangle &= \frac{1}{2} \left( |1110\rangle - |1101\rangle - |1011\rangle + |0111\rangle \right), \quad (6g) \\
|GB_1^{15}\rangle &= \left( |0000\rangle + |1111\rangle \right)/\sqrt{2}, \quad (6i) \\
|GB_1^{16}\rangle &= \left( |0000\rangle - |1111\rangle \right)/\sqrt{2}, \quad (6j)
\end{align*}
\]

where \( \omega = \exp \left( \frac{i \pi}{6} \right) \). One can easily design the purification protocol for the four-partite mixed entangled states along the lines we suggested here.

V. CONCLUSIONS

We construct a complete basis from all the two kinds tripartite genuine entangled states, and we call it genuine basis. The general mixed W state of tripartite system is expressed in terms of this genuine basis. The \( W \) fraction can be enhanced by LOCCs, i.e. it can be purified. Multipartite entanglement witness shows that there exists a special kind of tripartite entanglement which belongs to neither W-type entanglement nor GHZ-type entanglement, but we can extract W-type entanglement from this kind of entangled states. The yield of the protocol is higher than the previous multiparticle entanglement purification protocol for mixed W states. As a by-product, we can get bipartite pure Bell states when the purification protocols fails for tripartite entangled states, depending on the initial mixed states.
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