The $A_5$ Hamiltonian

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We discuss in detail the example of the holomorphic $A_5$ Hamiltonian given in [1]. In particular, we use the finite group of discrete symmetries of the Hamiltonian to build a discrete subgroup of the 2-dimensional Euclidean group whose orbit space is affine model of an affine Riemann surface, which is a level set of the $A_5$ Hamiltonian. Our treatment differs from the one given in [2] because the map, which develops the affine Riemann surface, is not constructed from a Schwarz-Christoffel transformation. Also we do not describe the monodromy of the $A_5$ Hamiltonian, as it is fully discussed in [1].

1 The Hamiltonian system

Consider the holomorphic Hamiltonian system $(H, \mathbb{C}^2, \Omega = d\xi \wedge d\eta)$ with holomorphic Hamiltonian

$$H : \mathbb{C}^2 \to \mathbb{C} : (\xi, \eta) \mapsto \frac{1}{2}\eta^2 + \xi^6 - 1.$$  \hspace{1cm} (1)

The Hamiltonian vector field $X_H$ on $(\mathbb{C}^2, \Omega)$ corresponding to $H$ is $\eta \frac{\partial}{\partial \xi} - 6\xi^5 \frac{\partial}{\partial \eta}$.

A holomorphic integral curve $\gamma : \mathbb{C} \to \mathbb{C}^2 : \tau \mapsto (\xi(\tau), \eta(\tau))$ of $X_H$ satisfies the holomorphic form of Hamilton’s equations

$$\frac{d\xi}{d\tau} = \frac{\partial H}{\partial \eta} = \eta,$$

$$\frac{d\eta}{d\tau} = -\frac{\partial H}{\partial \xi} = -6\xi^5. \hspace{1cm} (2)$$

Because $H$ is constant on the holomorphic integral curves of $X_H$, $X_H$ is a holomorphic vector field on the affine Riemann surface

$$S = H^{-1}(0) = \{ (\xi, \eta) \in \mathbb{C}^2 | \frac{1}{2}\eta^2 = 1 - \xi^6 \}. \hspace{1cm} (3)$$

2 Topology of $S$

To begin the study of the geometry of the integral curves of $X_H$, we determine the topology of the affine Riemann surface $S \subseteq \mathbb{C}^2$. The projective Riemann surface $S \subseteq \mathbb{CP}^2$ corresponding to $S$ is defined by

$$G(\xi, \eta, \zeta) = \frac{1}{2}\eta^2 \zeta^4 + \xi^6 - \zeta^6 = 0.$$

The singular points of $S$ are determined by

$$\frac{\partial G}{\partial \xi} = 6\xi^5 = 0; \quad \frac{\partial G}{\partial \eta} = \zeta^4 \eta = 0; \quad \text{and} \quad \frac{\partial G}{\partial \zeta} = 2\zeta^3(\eta^2 - 3\zeta^2) = 0.$$
Thus $\xi = 0$. If $\zeta \neq 0$, then the second equation gives $\eta = 0$ and the third
$0 = \eta^2 - 3\zeta^2 = -3\zeta^2$, that is, $\zeta = 0$, which is a contradiction. So $\zeta = 0$
and $\eta$ is arbitrary. Hence $[0 : 1 : 0]$ is the only singular point of $S$. Thus
$S = S \cap \{\zeta = 1\} = S \setminus \{[0 : 1 : 0]\}$ is nonsingular.

**Lemma 2.1** The projection mapping

$$\pi : \mathbb{C}P^2 \to \mathbb{C}P : [\xi : \eta : \zeta] \to [\xi : \zeta],$$

when restricted to $S$ presents $S$ as a branched double covering of $\mathbb{C}P$ with branch
points at $\{[\xi : 0 : 1] \in \mathbb{C}P^2 | \xi^6 = 1\}$.

**Proof.** Observe that the affine map corresponding to $\pi$ is the projection map

$$\pi : \mathbb{C}P^2 \cap \{\zeta = 1\} = \mathbb{C}^2 \to \mathbb{C}P \cap \{\zeta = 1\} = C : (\xi, \eta) \mapsto \xi,$$

whose tangent at $(\xi, \eta)$ is $T_{(\xi, \eta)}\pi(X \frac{\partial}{\partial \xi} + Y \frac{\partial}{\partial \eta}) = X \frac{\partial}{\partial \xi}$. The complex tan-
gent space $T_{(\xi, \eta)}S$ to $S$ at $(\xi, \eta)$ is equal to $\ker dH(\xi, \eta)$, which is spanned by
$X_H(\xi, \eta) = \eta \frac{\partial}{\partial \xi} - 6\xi^5 \frac{\partial}{\partial \eta}$. Thus

$$(T_{(\xi, \eta)}\pi|_S)(X_H(\xi, \eta)) = \eta \frac{\partial}{\partial \xi}. \quad (5)$$

Hence $T_{(\xi, \eta)}\pi|_S$ is a complex linear isomorphism of $T_{(\xi, \eta)}S$ onto $T_{(\xi, \eta)}C$ if and
only if $(\xi, \eta) \in S^\dagger = S \setminus \{(\xi, 0) \in S | \xi^6 = 1\}$. So $\pi|_{S^\dagger}$ is a local
holomorphic diffeomorphism of $S^\dagger$ onto $C^\dagger = \mathbb{C} \setminus \{\xi^6 = 1\}$. It is a covering map
having two sheets, since $(\pi|_{S^\dagger})^{-1}(\xi) = \{(\xi, \pm \eta) \in S^\dagger\}$ for every $\xi \in C^\dagger$. The six
points $\xi_k = e^{2\pi ik/6}$ for $k = 0, \ldots, 5$ are branch values of the projection mapping
$\pi|_S$ because near $\xi_j$ we have

$$\eta = (1 - \xi^6)^{1/2} = \prod_{k=0}^5 (\xi - \xi_k)^{1/2}$$

$$= (\xi_j - \xi_k)^{1/2}(\xi - \xi_j)^{1/2} \prod_{k \neq j} (1 + (\xi_j - \xi_k)^{-1}(\xi - \xi_j))^{1/2}.$$

Each factor in the last product is a holomorphic function of $\xi - \xi_j$ when $|\xi - \xi_j|$ is
sufficiently small. \qed
The following argument shows that the genus of $S$ is 2. In figure 1 (right) the points at $\infty$ are identified and the edges from $\infty$ to a vertex of the regular hexagon are identified. With these identifications figure 1 (right) gives the triangulation of $S^2 = \mathbb{C}P^2$ in figure 1 (left) having $V = 8$ vertices; $E = 18$ edges; $F = 12$ faces. Thus the Euler characteristic $\chi$ of $S^2$ is $V - E + F = 8 - 18 + 12 = 2$, which is $2 - 2g$. Hence the genus $g$ of $S^2$ is 0, as expected. Taking two copies of figure 1 (right) with the same identifications as above and then identifying the darkened vertices of the regular hexagon gives a triangulation of $S$ having $V' = 10 = 2V - 6$ vertices; $E' = 36 = 2E$ edges; $F' = 24 = 2F$ faces. Thus the Euler characteristic $\chi'$ of $S$ is $\chi' = V' - E' + F' = -2$. Hence $-2 = 2 - 2g$, where $g$ is the genus of $S$. So $g = 2$.

Let $K$ be the closed stellated regular hexagon formed by repeatedly rotating the closed quadrilateral $Q' = OD'C'D'$ by $R$ through an angle $2\pi/6$, see figure 2. We now use the stellated regular hexagon $K$ to construct a Riemann surface of the same genus as $S^1$, see [3].

Let $R$ be a rotation of $\mathbb{C}$ given by multiplication by $e^{2\pi i/6}$ and let $U$ be the reflection given by complex conjugation. Let $G^\vee$ be the group generated by the reflections $S_k = R^k SR^{-k} = R^{2k+1}U$ for $k = 0, 1, \ldots, 5$. Here $S = RU = S_0$ is the reflection, which leaves the closed ray $\ell = \{te^{2\pi i/6} | t \in OD'\}$ fixed, and $S_k$ is the reflection, which leaves the ray $R^{2k}\ell$ fixed. Note that $G^\vee = \langle R, U \mid R^6 = e = U^2, RU = UR^{-1} \rangle$. Define an equivalence relation on $K$ by saying that two points $x$ and $y$ in $K$ are equivalent, $x \sim y$, if and only if 1) $x$ and $y$ lie on $\partial K$ with $x$ on the closed edge $E$ and $y$ on the closed edge of $K$, where $y = S_m(x) \in S_m(E)$ for some reflection $S_m \in G^\vee$ or 2) if $x$ and $y$ lie in the interior of $K$ and $x = y$. Two edges of $K$ are equivalent if they contain equivalent points. Geometrically two edges are equivalent if they extend to lines in $\mathbb{C}$ which are parallel. Let $K^\sim$ be the space of equivalence classes of points in $K$ and let

$$\rho : K \to K^\sim : p \mapsto [p]$$

be the identification map which sends a point $p \in K$ to its equivalence class $[p]$, which contains $p$. Give $K$ the topology induced from $\mathbb{C}$. Placing the quotient topology on $K^\sim$ turns it into a compact connected topological manifold without boundary, see the argument below. Let $K^*$ be $K$ less its vertices.

**Proposition 2.2** With $O$ the center of $K^*$ the identification space $(K^* \setminus O)^\sim = \rho(K^* \setminus O)$ is a connected 2-dimensional smooth manifold without boundary.

**Proof.** Let $E_+$ be an open edge of $K^*$. For $p_+ \in E_+$ let $D_{p_+}$ be a disk in $\mathbb{C}$ with center at $p_+$, which does not contain a vertex of $K$. Set $D^+_{p_+} = K^* \cap D_{p_+}$. Let $E_-$ be an open edge of $K^*$, which is equivalent to $E_+$ via the reflection $S_m$. Let $p_- = S_m(p_+)$ and set $D^-_{p_-} = S_m(D^+_{p_+})$. Then $V[p] = \rho(D^+_{p_+} \cup D^-_{p_-})$ is an open neighborhood of $[p] = [p_+] = [p_-]$ in $(K^* \setminus O)^\sim$, which is a smooth 2-disk, since the identification mapping $\rho$ is the identity on int $K^*$. It follows that $(K^* \setminus O)^\sim$ is a smooth 2-dimensional manifold without boundary. \(\square\)
Proposition 2.3 The identification space $K^\sim$ is a connected compact topological manifold with a conical singular point at each vertex of $K$ and at the center $O$ of $K$.

**Proof.** We now handle the vertices of $K$. Let $v_+$ be a vertex of $K^*$ and set $D_{v_+} = \overline{D} \cap K^*$, where $\overline{D}$ is a disk in $\mathbb{C}$ with center at the vertex $v_+ = r_0e^{i\pi\theta_0}$. The map

$$W_{v_+} : D_+ \subseteq \mathbb{C} \to D_{v_+} \subseteq \mathbb{C} : r e^{i\pi\theta} \mapsto |r - r_0|e^{i\pi s(\theta - \theta_0)}$$

with $r \geq 0$ and $0 \leq \theta \leq 1$ is a homeomorphism, which sends the wedge with angle $\pi$ to the wedge with angle $\pi s$. The latter wedge is formed by the closed edges $E_+^\prime$ and $E_-$ of $K^*$, which are adjacent at the vertex $v_+$ such that $e^{i\pi s}E_+^\prime = E_+$ with the edge $E_+^\prime$ being swept out through int $K^*$ during its rotation to the edge $E_+$. Because $K^*$ is a rational stellated regular hexagon, the value of $s$ is a rational number for each vertex of $K^*$. Let $E_- = S_m(E_+)$ be an edge of $K^*$, which is equivalent to $E_+$ and set $v_- = S_m(v_+)$. Then $v_-$ is a vertex of $K^*$, which is the center of the disk $D_{v_-} = S_m(D_{v_+})$. Set $D_+ = \overline{D}_+$. Then $D = D_+ \cup D_-$. The map $W : D \to \rho(D_{v_+} \cup D_{v_-})$, where $W|D_+ = \rho W_{v_+}$ and $W|D_- = \rho S_m \circ W_{v_+} \circ \rho^*$, is a homeomorphism of $D$ into a neighborhood $\rho(D_{v_+} \cup D_{v_-})$ of $[v] = [v_+] = [v_-]$ in $(K^*)^\sim$.

The center $O$ of $K$ is a conical singular point of $K^\sim$ because it is a fixed point of the linear action of $G^\gamma$ on $\mathbb{C}$.

Consequently, the identification space $(K^*)^\sim$ is a topological manifold with a conical singularity at a point corresponding to a vertex or the center $O$ of $K$.

Let $\tilde{G} = \langle R \mid R^6 = e \rangle$. Then $\tilde{G}$ is the abelian group $\mathbb{Z}$ mod 6. The usual $\tilde{G}$-action $\tilde{G} \times K \subseteq \tilde{G} \times \mathbb{C} \to K \subseteq \mathbb{C} : (g, z) \mapsto g(z)$ preserves equivalent edges of $K$ and is free on $K^* \setminus O$. Hence it induces a $\tilde{G}$ action on $(K^* \setminus O)^\sim$, which is free and proper with orbit map

$$\sigma : (K^* \setminus O)^\sim \to (K^* \setminus O)^\sim / \tilde{G} = \tilde{S}^\dagger : z \mapsto z\tilde{G}$$

Proposition 2.4 The orbit space $\tilde{S}^\dagger = \sigma((K^* \setminus O)^\sim)$ is a connected smooth complex manifold.

**Proof.** Since the action of $\tilde{G}$ on $(K^* \setminus O)^\sim$ is free and proper, its orbit space $\sigma((K^* \setminus O)^\sim)$ is a smooth 2-dimensional manifold that is the orientated. This orientation is induced from an orientation of $K^* \setminus O$, which comes from the orientation $\mathbb{C}$. So $\tilde{S}^\dagger$ has a complex structure, since each element of $\tilde{G}$ is a conformal mapping of $\mathbb{C}$ into itself. 

\[\square\]
Figure 2. The triangulation $\mathcal{T}_{cl(K^*)}$ of the regular stellated hexagon $cl(K^*)$.

Next we specify the topology of $\overline{S}^1$. The stellated regular hexagon $K^* \setminus O$ less the origin has a triangulation $T_{K^* \setminus O}$ made up of 12 open triangles $R^j(\triangle OCD')$ and $R^j(\triangle OCD')$ for $j = 0, 1, \ldots, 5$; 24 open edges $R^j(OC)$, $R^j(OCD')$, and $R^j(CD')$ for $j = 0, 1, \ldots, 5$; and 12 vertices $R^j(D')$ and $R^j(C)$ for $j = 0, 1, \ldots, 5$, see figure 2.

Consider the set $E$ of unordered pairs of equivalent closed edges of $K$, that is, $E$ is the set $[E, S_k(E)]$ for $k = 0, 1, \ldots, 5$, where $E$ is a closed edge of $K$ and $S_k \in G^\vee$. Table 1 lists the elements of $E$. $G^\vee$ acts on $E$, namely, $g \cdot [E, S_k(E)] = [g(E), gS_kg^{-1}(g(E))]$, for $g \in G^\vee$. This $G^\vee$ action is well defined, since $gS_kg^{-1}$ is a reflection in the line $g(R^k\ell)$. This argument shows that $G$ acts on $E$, since $G$ is a subgroup of $G^\vee$.

Table 1. Elements of the set $E$ of unordered pairs of equivalent closed edges of $K$. Here $D'_k = R^k(D')$ and $\overline{D}'_k = R^k(\overline{D}')$ for $k = 0, 2, 4$ and $C_k = R^k(C)$ for $k = \{0, 1, \ldots, 5\}$, see figure 2.

We now look at the $G^\vee$-orbits on $E$. We compute the $G^\vee$-orbit of $d \in E$ as follows.

$$(UR) \cdot d = [UR(\overline{D}'_2C_2), UR(S_2(\overline{D}'_2C_2))] = [UR(\overline{D}'_2C_2), UR(\overline{D}'_2C_3)]$$

$$(UR) \cdot d = [UR(\overline{D}'_2C_3), UR(D'_4C_4)] = [\overline{D}'_2C_3, D'_2C_2] = d,$n

$R^2 \cdot d = R^2 \cdot [\overline{D}'_2C_2, S_2(\overline{D}'_2C_2)] = [R^2(\overline{D}'_2C_2), R^2S_2R^{-2}(R^2(\overline{D}'_2C_2))]$$

$R^4 \cdot d = [R^4(\overline{D}'_2C_2), R^4S_2R^{-4}(R^4(\overline{D}'_2C_2)))]$
Thus the $G'$ orbit $G' \cdot d$ of $d \in \mathcal{E}$ is $H \cdot d = \{a, d, e\}$, which is a $\tilde{G}$ orbit, since $H = G'/\langle UR \rangle \langle UR \rangle^2 = e = (V = R^2 \mid V^3 = e)$ is a subgroup of $\tilde{G}$. Similarly, the $G'$-orbit $G' \cdot f$ of $f \in \mathcal{E}$ is $H \cdot f = \{b, c, f\}$. Since $G' \cdot d \cup G' \cdot f = \mathcal{E}$, we have found all $G'$-orbits and hence all the $\tilde{G}$-orbits on $\mathcal{E}$.

The end points of the elements of the orbit $G' \cdot d$ are $a = \{\mathcal{D}', C, \mathcal{D}'_2, C_1\}$; $d = \{\mathcal{D}_2', C_2, \mathcal{D}'_4, C_3\}$; and $e = \{\mathcal{D}_4', C_4; \mathcal{D}', C_5\}$, see figure 2. Thus $\{\mathcal{D}', \mathcal{D}'_2, \mathcal{D}'_4\}$, $\{C, C_2, C_4\}, \{\mathcal{D}_2', \mathcal{D}_4', \mathcal{D}'\}$ and $\{C_1, C_3, C_5\}$ are $G'$ orbits of vertices of $K$. They are also $G$ orbits. Similarly the end points of the orbit $G' \cdot f$ are $b = \{D', C_1, D'_3, C_2\}; c = \{D'_3, C_3, D'_4, C_4\}$ and $f = \{D'_4, C_5, D', C\}$. So $\{D', D'_2, D'_4\}, \{C_1, C_3, C_5\}, \{D'_3, D'_4, D'\}$ and $\{C_2, C_4, C\}$ are $G'$ orbits of vertices of $K$. They are also $G$ orbits.

To determine the topology of the $\tilde{G}$ orbit space $\tilde{S}^1$ we find a triangulation of its closure. Note that the triangulation $T_{K \setminus \tilde{O}}$ of $K^+ \setminus \tilde{O}$, illustrated in figure 2, is $\tilde{G}$-invariant. Its image under the identification map $\rho$ is a $\tilde{G}$-invariant triangulation $T_{(K^+ \setminus \tilde{O})^\circ}$ of $(K^+ \setminus \tilde{O})^\circ$ with vertices $\rho(v)$, where $v$ is a vertex of $K$; open edges $\rho(E)$ having $\rho(O)$ as an end point, where $E$ is an edge of $T_{K \setminus \tilde{O}}$ having $O$ as an end point; or $\rho([F, F'])$, where $[F, F']$ is an unordered pair of equivalent edges in $K$; open triangles $\rho(T)$, where $T$ is a triangle in $T_{K \setminus \tilde{O}}$. The triangulation $T_{(K^+ \setminus \tilde{O})^\circ}$ is invariant under the induced $\tilde{G}$ action on $(K^+ \setminus \tilde{O})^\circ$. It follows that $\sigma(\rho(v)), \sigma(\rho(E))$ or $\sigma(\rho([F, F']))$, and $\sigma(\rho(T))$ is a vertex, an open edge, and an open triangle, respectively, of a triangulation $T_{\tilde{S}}$ of $\tilde{S}$. The triangulation $T_{\tilde{S}}$ has 1) 4 vertices, corresponding to the $\sigma \circ \rho$ image of the $\tilde{G}$ orbits $\{D', D'_2, D'_4\}, \{C_1, C_3, C_5\}, \{\mathcal{D}_2, \mathcal{D}_4, \mathcal{D}'\}$ and $\{C_2, C_4, C\}$ of vertices of $K$. 2) 8 open edges corresponding to $\sigma \circ \rho$ image of the $\tilde{G}$ orbits of the two edge pairs $d$ and $f$ of $K$ and $\sigma \circ \rho$ image of the six $\tilde{G}$ orbits $\{R^j(OC)\}$ and $\{R^{2j}(OD')\}$ for $j = 0, 1, 2$ and $\{R^{j}(R(OC))\}$ and $\{R^{j}(R(OD'))\}$ for $j = 0, 1, 2$ and $\{R^{j}(CD')\}$ and $\{R^{j}(C_1D_2)\}$ for $j = 0, 1, 2$; 3) 2 open triangles corresponding to the $\sigma \circ \rho$ image of $\tilde{G}$ orbits of the triangles $\triangle OCD'$ and $\triangle OCD'$. Thus the Euler characteristic $\chi(\tilde{S}')$ of $\tilde{S}'$ is $4 - 2 + 2 - 2 = 2$. Since $\tilde{S}'$ is a 2-dimensional smooth real manifold, $\chi(\tilde{S}') = 2 - 2g$, where $g$ is the genus of $\tilde{S}'$. Hence $g = 2$. So $\tilde{S}'$ is a smooth 2-sphere with 2 handles, less a finite number of points, which lies in a compact topological space $K^\circ/G$, that is its closure.

3 The developing mapping

Consider the mapping

$$
F: \mathbb{C}^+ = \mathbb{C} \setminus \{\xi^6 = 1\} \to \mathbb{C}: \xi \mapsto z = \frac{1}{\sqrt{2}} \int_0^\xi \frac{dw}{\sqrt{1 - w^6}}.
$$

\footnote{Note that the end points of the edges of these six $\tilde{G}$ orbits are $\tilde{G}$ orbits of end points of equivalent edges or the center $O$ of $K$.}
$F$ is holomorphic except at the sixth roots of unity \( \{ \xi \in \mathbb{C} \mid \xi^6 = 1 \} \), since $dz = \frac{1}{\sqrt{1-\xi^6}} \, d\xi$. In addition, $F$ is a local holomorphic diffeomorphism on $\mathbb{C}^\dagger$, because $F' \neq 0$. Let $\omega = e^{2\pi i/6}$ and set $R : \mathbb{C} \to \mathbb{C} : \xi \mapsto \omega \xi$. Since

$$F(\omega \xi) = \int_0^{\omega \xi} \frac{dw}{\sqrt{1-w^6}} = \int_0^\xi \frac{dw'}{\sqrt{1-\omega^6 w'^6}}; \text{ where } w = \omega w',$$

the map $F$ intertwines the $\mathbb{Z}_6$ action on $\mathbb{C}^\dagger = \mathbb{C} \setminus \{ \xi^6 = 1 \}$ generated by $R$, that is, $F(R\xi) = R^{-1}(F(\xi))$ for every $\xi \in \mathbb{C}^\dagger$. So $F(\omega^k) = \omega^{-k} F(1)$ for $k = 0, 1, \ldots, 5$. Let $T$ be the closed triangle with edges $01$, $1\omega$, and $0\omega$, see figure 3 (left).

![Figure 3. The mapping $F$.](image)

The image of $T$ under the mapping $F$ is the triangle $T'$ with edges $F(01) = \overline{0C}$, $F(\omega 1) = C(-\omega)\overline{C}$, and $F(0\omega) = \omega^{-1} F(01) = 0(-\omega C)$, see figure 3 (right). Here $C = F(1) = \int_0^1 \frac{dw}{\sqrt{1-w^6}}$.

Let $H = \bigcup_{\nu=0}^5 R^\nu(T)$. Then $H$ is a closed regular hexagon whose vertices are at the sixth roots of unity. Since $H$ is simply connected, the complex square root is single valued there. Hence $D = (\pi_{|S^\dagger})^{-1}(H)$ is one sheet of the projection map $\pi_{|S^\dagger}$. Since $\pi_{|S^\dagger}$ is a twofold branched covering map, the other sheet is $D' = (\pi_{|S^\dagger})^{-1}(K \setminus H)$, where $K$ is the stellated regular hexagon formed by adding an equilateral triangle with an edge in common with the hexagon $H$ for each edge of $H$, see figure 1 (right). The image of $\mathbb{C}^\dagger$ under the mapping $F$ is the closed stellated regular hexagon $K$ in $\mathbb{C}$ with center at the origin and side length $C$.

Using (5) we get

$$(T \xi F \circ \pi_{|S^\dagger})(X_H(\xi, \eta)) = T \xi F (T(\xi,\eta) \pi_{|S^\dagger}(X_H(\xi, \eta))) = T \xi F(\eta \frac{\partial}{\partial \xi}) = \frac{\partial}{\partial z}, \quad (8)$$

since equation (7) gives $dz = dF = \frac{1}{\eta} \, d\xi$, where $\eta = \sqrt{2(1-\xi^6)}$. Thus the
mapping
\[ \delta = F \circ \pi_{S^1} : S^1 \subseteq \mathbb{C}^2 \rightarrow K \subseteq \mathbb{C}^\omega = \mathbb{C} \setminus \{ z^6 = 0 \} \subseteq \mathbb{C} \] (9)

straightens the holomorphic vector field \( X_H \) on \( S^1 \), that is, on \( D \) and \( D' \).
Moreover, \( \delta_{\mathcal{D}} \) and \( \delta_{\mathcal{D}'} \) are holomorphic diffeomorphisms of \( D \) and \( D' \) onto the hexagon \( H \) less its vertices and \( K \setminus H \), less its vertices, respectively.

Consider the hermitian metric \( \gamma = dz \odot d\bar{z} \) on \( \mathbb{C}^\omega \). Pulling \( \gamma\vert_{K} \) back by the map \( \delta \) gives the hermitian metric \( \Gamma = \frac{1}{\eta} d\xi \odot \frac{1}{\eta} d\bar{\xi} \) on \( S^1 \), since
\[
\Gamma(X_H(\xi, \eta), X_H(\xi, \eta)) = \frac{1}{\eta} \frac{d\xi}{d\xi} (\eta \frac{\partial}{\partial \xi} - 6\xi^5 \frac{\partial}{\partial \eta}) \odot \frac{1}{\eta} \frac{d\bar{\xi}}{d\xi} (\eta \frac{\partial}{\partial \xi} - 6\xi^5 \frac{\partial}{\partial \eta}) \\
= \frac{1}{\eta} \frac{d\xi}{d\xi} \odot \frac{1}{\eta} \frac{d\bar{\xi}}{d\xi} = \frac{d\xi}{d\xi} \odot \frac{d\bar{\xi}}{d\xi} = 1.
\]
The metric \( \Gamma \) on \( S^1 \) is flat, because the metric \( \gamma\vert_{K} \) on \( \mathbb{C}^\omega \) is flat. The map \( \delta \) is a developing map. Since the real integral curves of \( \alpha \frac{\partial}{\partial \xi} \), \( \alpha \in S^1 \), on \( \mathbb{C}^\omega, \gamma \) are geodesics, which are straight lines that make an angle \( \theta \), where \( \alpha = e^{i\theta} \), with the \( x \)-axis, the real integral curves of \( X_{\alpha H} \) on \( (S^1, \Gamma) \) are geodesics for the hermitian metric \( \Gamma \). Because the image under \( (T\theta)^{-1} \) of the vector field \( \alpha \frac{\partial}{\partial \xi} \) is the vector field \( X_{\alpha H} \) on \( S^1 \), the image of a real integral curve of \( \alpha \frac{\partial}{\partial \xi} \) is the image of a real integral curve of \( X_H \) and hence is a geodesic on \( (S^1, \Gamma) \).

4 Metric geometry of \( S^1 \)

In this section we discuss some properties of geodesics on the smooth affine Riemann surface \( S^1 \).

Lemma 4.1 The mappings \( \mathcal{R} : S^1 \subseteq \mathbb{C}^2 \rightarrow S^1 \subseteq \mathbb{C}^2 : (\xi, \eta) \mapsto (\omega \xi, \eta) \), where \( \omega = e^{2\pi i/6} \) and \( \mathcal{U} : S^1 \subseteq \mathbb{C}^2 \rightarrow S^1 \subseteq \mathbb{C}^2 : (\xi, \eta) \mapsto (\bar{\xi}, \bar{\eta}) \) are isometries of \( (S^1, \Gamma) \).

Proof. We compute.
\[
\mathcal{R}^* \Gamma(\xi, \eta) = \mathcal{R}^* \left( \frac{1}{\eta} d\xi \odot \frac{1}{\eta} d\bar{\xi} \right) = \frac{1}{\eta} d(\omega \xi) \odot \frac{1}{\eta} d(\bar{\omega \xi}) = \frac{1}{\eta} d\xi \odot \frac{1}{\eta} d\bar{\xi} = \Gamma(\xi, \eta).
\]
\( \mathcal{U} \) maps \( H^{-1}(0) \) into itself, for if \( (\xi, \eta) \in H^{-1}(0) \), then \( \frac{1}{2} \xi^2 + \xi^6 - 1 = \frac{1}{2} \eta^2 + \xi^6 - 1 \)
\( = 0 \). So \( (\bar{\xi}, \bar{\eta}) \in H^{-1}(0) \). The set \( \{ \xi^6 = 1 \} \) is mapped onto itself by \( \mathcal{U} \). Hence \( \mathcal{U} \) preserves \( S^1 \). Now
\[
\mathcal{U}^* \Gamma(\xi, \eta) = \mathcal{U}^* \left( \frac{1}{\eta} d\xi \odot \frac{1}{\eta} d\bar{\xi} \right) = \frac{1}{\eta} d\xi \odot \frac{1}{\eta} d\bar{\xi} = \frac{1}{\eta} d\xi \odot \frac{1}{\eta} d\bar{\xi} = \Gamma(\xi, \eta).
\]
Thus \( \mathcal{U} \) is an isometry of \((S^1, \Gamma)\). \( \square \)
So the group \( \hat{G} = \langle R, U \mid R^6 = U^2 = e, \& RU = UR^{-1} \rangle \) is a group of isometries of \((S^1, \Gamma)\).

**Lemma 4.2** The image of a geodesic on \((S^1, \Gamma)\) under the action of the group of isometries \( \hat{G} \) is a geodesic on \( S^1 \).

**Proof.** This follows because a geodesic is locally length minimizing, which is a property preserved by an isometry. \( \square \)

Let \( \hat{G} \) be the group of invertible linear maps of \( \mathbb{C} \) into itself generated by

\[
R : \mathbb{C} \to \mathbb{C} : z \mapsto \omega z \quad \text{and} \quad U : \mathbb{C} \to \mathbb{C} : z \mapsto \overline{z},
\]

which satisfy the relations \( R^6 = U^2 = e \) and \( RU = UR^{-1} \). The elements of \( \hat{G} \) preserve \( \mathbb{C}' = \mathbb{C} \setminus \{z^0 = C^0\} \).

**Claim 4.3** The developing map \( \delta : S^1 \subseteq \mathbb{C}' \to \mathbb{C}' \) intertwines the \( \hat{G} \) action \( \Phi \) on \( S^1 \) with the \( \hat{G} \) action \( \phi \) on \( \mathbb{C}' \). Specifically, for every \( g \in \hat{G} \) and every \( (\xi, \eta) \in S^1 \) we have

\[
\delta(\Phi_g(\xi, \eta)) = \phi_{\psi(g)}(\delta(\xi, \eta)),
\]

where \( \psi : \hat{G} \to \hat{G} \) is the isomorphism defined by \( \psi(R) = R \) and \( \psi(U) = U \).

**Proof.** The following computation shows that equation \((10)\) holds when \( g \) is \( R \) and \( U \), respectively.

1. \( g = R \).

\[
\delta(\Phi_R(\xi, \eta)) = F(\omega \xi) = \omega F(\xi) = R(\delta(\xi, \eta)) = \phi_{\psi(R)}(\delta(\xi, \eta)).
\]

2. \( g = U \).

\[
\delta(\Phi_U(\xi, \eta)) = \delta(\overline{\xi}, \overline{\eta}) = F(\pi(\overline{\xi}, \overline{\eta})) = F(\overline{\xi}) = U(\delta(\xi, \eta)) = \phi_{\psi(U)}(\delta(\xi, \eta)).
\]

The fourth equality above follows by changing the variable \( w \) to \( \overline{w} \) in the integral

\[
\frac{1}{\sqrt{2}} \int_0^\xi \frac{dw}{\sqrt{1 - w^2}}
\]

\( \square \).

**Corollary 4.3A** The developing map \( \delta \) intertwines the action of \( \hat{G} \) on \( S^1 \) with the action \( \phi \) of \( \hat{G} \) restricted to the stellated hexagon \( K \).

**Corollary 4.3B** \((S^1, \Gamma)\) is geodesically incomplete.

**Proof.** Since \((S^1, \Gamma)\) is isometric to \((\mathbb{C}', dz \circ d\overline{z})\), which is \(((\mathbb{R}^2)', \varepsilon = dx \circ dx + dy \circ dy)\), where

\[
(\mathbb{R}^2)' = \mathbb{R}^2 \setminus \{(C \cos \frac{2\pi k}{6}, C \sin \frac{2\pi k}{6}) \in \mathbb{R}^2 \mid k = 0, 1, \ldots, 5\},
\]

it suffices to show that there are integral curves of the geodesic vector field \( \frac{\partial}{\partial x} \) on \(((\mathbb{R}^2)', \varepsilon)\) which run off \((\mathbb{R}^2)'\) in finite time. Consider the horizontal line segment

\[
\gamma : [0, 1] \to \mathbb{R}^2 : t \mapsto (1 - t)(C \cos \frac{4\pi}{6}, C \sin \frac{4\pi}{6}) + t(C \cos \frac{5\pi}{6}, C \sin \frac{5\pi}{6})
\]

\[
= C(-\frac{1}{2} + t, \frac{\sqrt{3}}{2}),
\]
which is a horizontal side of a regular hexagon centered at \((0,0)\) with side length \(C\). Since \(\gamma\) is an integral curve of \(\frac{\partial}{\partial x}\) and hence is a geodesic on \(((\mathbb{R}^2)^\gamma, \varepsilon)\), it takes time \(C = \int_0^1 |\frac{d\gamma}{dt}| \, dt\) to go from \((C \cos \frac{4\pi}{6}, C \sin \frac{4\pi}{6})\) to \((C \cos \frac{5\pi}{6}, C \sin \frac{5\pi}{6})\). Thus \(\gamma\) runs off \(((\mathbb{R}^2)^\gamma, \varepsilon)\) in finite time. □

We remove the incompleteness of the geodesic vector field on \((K, \gamma|_{K})\) by imposing the following condition: when a geodesic starting at a point in \(\text{int} \, K\) meets \(\partial K\) at a point on an open edge, it undergoes a reflection in that edge; otherwise it meets \(K\) at a vertex, where it reverses its motion. Such motions in the stellated regular hexagon \(K\), called billiard motions, are defined for all time.

5 An affine model of \(S^\dagger\)

In this section we construct an affine model of the affine Riemann surface \(S^\dagger\). In other words, we find a discrete subgroup \(\mathcal{G}\) of the 2-dimensional Euclidean group \(E(2)\), which acts freely and properly on \(\mathbb{C} \setminus \mathbb{V}^+\) and has the stellated regular hexagon \(K\) less its vertices and center, as its fundamental domain. Here \(\mathbb{V}^+\) is a discrete subset of \(\mathbb{C}\) formed by translating the vertices of \(K\) and its center. After constructing the identification space \((\mathbb{C} \setminus \mathbb{V}^+)\sim\), we show that its \(\mathcal{G}\) orbit space \((\mathbb{C} \setminus \mathbb{V}^+)\sim/\mathcal{G}\) is holomorphically diffeomorphic to \(S^\dagger\).

First we specify the set \(\mathbb{V}^+\). We label the edges of the regular hexagon \(H\) of the stellated regular hexagon \(K\) in order by \(\{0, 1, \ldots, 5\}\). Reflecting the regular hexagon \(H\) in its edge \(k_0\) gives the regular hexagon \(H_{k_0}\), which uniquely determines the reflected stellated regular hexagon \(K_{k_0}\), see figure 4.

![Figure 4. The reflection of the regular stellated hexagon K in the edge e of the regular hexagon H gives the regular stellated hexagon K_e.](image)

Then we reflect in the edge \(k_1\) of \(K_{k_0}^*\) giving the stellated hexagon \(K_{k_0k_1}^*\). After \(n + 1\) repetitions we get \(K_{k_0k_1 \ldots k_n}^*\). Repeat this indefinitely and obtain the triangle tiling in figure 5.
Consider the translations
\[ \tau_k : C \rightarrow C : z \mapsto z + u_k, \]  
where \( u_k = \sqrt{3}CR^h \) for \( k = 0, 1, \ldots, 5 \).

The set of centers of the iteratively reflected stellated regular hexagons is
\[ \{(\tau_0^{\ell_0} \circ \cdots \circ \tau_5^{\ell_5})(0) \in C \mid (\ell_0, \ell_1, \ldots, \ell_5) \in \mathbb{Z}^5\}. \]

The set of vertices of \( K \) is
\[ V = \{V_2 = CR^{2\ell}, V_{2\ell-1} = \frac{\sqrt{3}}{2}CR^{2\ell-1} \text{, for } \ell = 0, 1, \ldots, 5\}. \]

The set of centers and vertices the hexagonal tiling of \( C \) is
\[ V^+ = \{\tau_0^{\ell_0} \circ \cdots \circ \tau_5^{\ell_5}(V^+) \in C \mid V^+ = V \cup \{0\} \& (\ell_0, \ldots, \ell_5) \in \mathbb{Z}^5\}, \]
see figure 5.

Let \( T \) be the abelian subgroup of \( E(2) \) generated by the translations \( \tau_k \), \( k = 0, \ldots, 5 \), see (11). \( T \) is isomorphic to the additive group \( \mathbb{Z}^5 \). By definition \( K^* \setminus O \) is a fundamental domain for the action of \( T \) on \( C \setminus V^+ \). Let \( \mathfrak{G} = G \ltimes T \subseteq G \times T \). An element of \( \mathfrak{G} \) is the affine mapping
\[ (R^j, u_k) : C \rightarrow C : z \mapsto R^jz + u_k, \text{ for } j, k \in \{0, \ldots, 5\}. \]

Multiplication \( \cdot \) in \( \mathfrak{G} \) is given by
\[ (R^h, u_\ell) \cdot (R^j, u_k) = (R^{(h+j) \mod 6}, R^hu_k + u_\ell) = (R^{(h+j) \mod 6}, u_{(k+\ell) \mod 6} + u_k). \]

The group \( \mathfrak{G} \) acts on \( C \) just as \( E(2) \) does, namely, by affine orthogonal mappings. Denote this action by
\[ \psi : \mathfrak{G} \times C \rightarrow C : ((g, \tau), z) \mapsto \tau(g(z)). \]
Lemma 5.1 The set $V^+$ is invariant under the $\mathfrak{G}$ action $\psi$.

**Proof.** Let $v \in V^+$. For some $(k_0, \ldots, k_n) \in \{0, \ldots, 5\}^{n+1}$ and some $w \in V^+$

$$v = \tau_{k_n} \circ \cdots \circ \tau_{k_0}(w) = \psi(e, u)(w),$$

where $u = \sum_{j=0}^{5} \ell_j u_j$ with $(\ell_0, \ldots, \ell_5) \in \mathbb{Z}^6$. Each $\ell_j$ is uniquely determined by $(k_0, \ldots, k_n) \in \{0, \ldots, 5\}^{n+1}$. For $(R^j, u') \in \mathfrak{G}$ with $j = 0, \ldots, 5$ and $u' \in \mathcal{T}$

$$\psi_{(R^j, u')}(v) = \psi_{(R^j, u')}(\psi(e, u)(w)) = \psi_{(R^j, u')}(w) = \psi_{(e, R^j u + u')}(w) = \psi_{(e, R^j u + u')}(w'),$$

where $w' = \psi_{(R^j, 0)}(w) \in V^+$. Since $(e, u') \in \mathcal{T}$ and

$$R^j u = \sum_{k=0}^{5} \ell_k R^j u_k = \sum_{k=0}^{5} \ell_k u_{(k+j) \mod 6},$$

it follows that $(e, R^j u + u') \in \mathcal{T}$. Hence $\psi_{(R^j, u)}(v) \in V^+$, as desired. \hfill $\square$

Lemma 5.2 The action of $\mathfrak{G}$ on $\mathbb{C} \setminus V^+$ is free.

**Proof.** Suppose that for $v \in \mathbb{C} \setminus V^+$ there is $(R^j, u) \in \mathfrak{G}$ such that $v = \psi_{(R^j, u)}(v)$. Then $v$ lies in the stellated hexagon $K^*_{k_0 \cdots k_n}$ given by reflecting $K^*$ in the successive edges $k_0, k_1, \ldots, k_n$. So for some $v' \in K^*$

$$v = (\tau_{k_n} \circ \cdots \circ \tau_{k_0})(v') = \psi(e, u')(v'),$$

where $u' = \sum_{j=0}^{5} \ell_j u_j$ for some $(\ell_0, \ldots, \ell_5) \in \mathbb{Z}^6$, which is uniquely determined by $(k_0, \ldots, k_n) \in \{0, \ldots, 5\}^{n+1}$. Thus

$$\psi_{(e, u')}(v') = \psi_{(R^j, u)}(\psi_{(e, u')}(v')) = \psi_{(R^j u, R^j u + u)}(v'),$$

which implies $R^j = e$, that is, $j = 0 \mod 6$ and $u' = R^j u' + u = u' + u$, that is, $u = 0$. Hence $(R^j, u) = (e, 0)$, which is the identity element of $\mathfrak{G}$. \hfill $\square$

Lemma 5.3 The action of $\mathcal{T}$ (and hence of $\mathfrak{G}$) on

$$\mathbb{C} \setminus V^+ = K^* \cup \{ \cup_{n \geq 0} \cup_{0 \leq j \leq n} \cup_{0 \leq k_j \leq 5} K^*_{k_0 \cdots k_n} \}$$

is transitive.

**Proof.** Let $K^*_{k_0 \cdots k_n}$ and $K^*_{k'_0 \cdots k'_n}$ lie in $\mathbb{C} \setminus V^+$. Since $K^*_{k_0 \cdots k_n} = (\tau_{k_n} \circ \cdots \circ \tau_{k_0})(K^*)$ and $K^*_{k'_0 \cdots k'_n} = (\tau_{k_n} \circ \cdots \circ \tau_{k_0})(K^*)$, it follows that

$$K^*_{k_0 \cdots k_n} = (\tau_{k_n} \circ \cdots \circ \tau_{k_0})(K^*_{k_n \cdots k'_n} \circ \tau_{k_n} \circ \cdots \circ \tau_{k_0})^{-1}(K^*),$$

Thus the action of $\mathcal{T}$ on $\mathbb{C} \setminus V^+$ is transitive. \hfill $\square$
The action of \( \mathcal{G} \) on \( \mathbb{C} \setminus \mathbb{V}^+ \) is proper because \( \mathcal{G} \) is a discrete subgroup of \( E(2) \), having no accumulation points.

Let \( E_{k_0 \ldots k_n} = (\tau_{k_n} \circ \ldots \circ \tau_{k_0})(E) \in K_{k_0 \ldots k_n}^* \). Then \( E_{k_0 \ldots k_n} \) is an open edge in \( \mathbb{C} \setminus \mathbb{V}^+ \) of the stellated hexagon \( K_{k_0 \ldots k_n}^* \). So \( \mathcal{E} = \{ E_{k_0 \ldots k_n} \mid n \geq 0, \ 0 \leq j \leq n \ & 0 \leq k_j \leq 5 \} \) is the set of open edges of \( \mathbb{C} \setminus \mathbb{V}^+ \). By definition \( \mathcal{E} \) is invariant under translations in \( T \).

**Lemma 5.4** The \( \mathcal{G} \) action \( \psi [12] \) leaves the subset \( \mathcal{E} \) of \( \mathbb{C} \setminus \mathbb{V}^+ \) invariant.

**Proof.** Let \( F \in \mathcal{E} \). For some \((k_0, \ldots, k_n) \in \{0, \ldots, 5\}^{n+1} \) and some open edge \( E \) of \( K^* \)

\[
F = (\tau_{k_n} \circ \ldots \circ \tau_{k_0})(E) = \psi(e,u')(E),
\]

where \( u' = \sum_{j=0}^5 \ell_j u_j \) with \((\ell_0, \ldots, \ell_5) \in \mathbb{Z}^6 \). Each \( \ell_j \) is uniquely determined by \((k_0, \ldots, k_n) \in \{0, \ldots, 5\}^{n+1} \). For \((R^j, u) \in \mathcal{G} \) with \( j = 0, 1, \ldots, 5 \) and \( u \in T \)

\[
\begin{align*}
\psi(R^j, u)(F) &= \psi(R^j, u) \circ \psi(e,u')(E) = \psi(R^j, u)(\psi(e,u')(E)) \\
&= \psi(R^j, R^j u' + u)(E) = \psi(e, R^j u' + u)(R^j, 0)(E) \\
&= \psi(e, R^j u' + u)(\psi(R^j, 0)(E)) = \psi(e, R^j u' + u)(E'),
\end{align*}
\]

where \( E' = \psi(R^j, 0)(E) = R^j E \) is an open edge of \( K^* \). Since \((e, u) \in \mathcal{G} \) and

\[
R^j u' = \sum_{i=0}^5 \ell_i R^j u_i = \sum_{i=0}^5 \ell_i u_{(j+i) \bmod 6},
\]

it follows that \((e, R^j u' + u) \in T \). Hence \( \psi(e, R^j u' + u)(E') \in \mathcal{E} \). So \( \psi(R, u)(F) \in \mathcal{E} \), as desired. \( \square \)

We say that two open edges in \( \mathcal{E} \) are equivalent if they are parallel in \( \mathcal{C} \). The \( \mathcal{G} \) action \( \psi [12] \) on \( \mathcal{E} \) preserves the relation of equivalence of edges.

**Proposition 5.5** The identification space \((\mathcal{C} \times \mathbb{V}^+)\sim \) formed by identifying equivalent open edges in \( \mathcal{E} \) is equal to the identification space \((K^* \setminus O)\sim \) formed by identifying equivalent open edges of the stellated regular hexagon \( K^* \) less its vertices.

**Proof.** This follows from the observation that every equivalence class of open edges in \( \mathcal{E} \) contains a unique equivalence class of open edges of \( K^* \).

**Corollary 5.5A** The \( \mathcal{G} \) orbit space \((\mathcal{C} \times \mathbb{V}^+)\sim / \mathcal{G} \) is equal to the \( \tilde{G} \) orbit space \((K^* \setminus O)\sim / \tilde{G} \), which is a 1-dimensional complex manifold.

**Proof.** Since the action of \( \mathcal{G} \) on \( \mathbb{C} \times \mathbb{V}^+ \) is proper and free, it induces a free and proper action on the identification space \((\mathcal{C} \times \mathbb{V}^+)\sim \). Hence the orbit space \((\mathcal{C} \times \mathbb{V}^+)\sim / \mathcal{G} \) is a complex manifold, which is equal to the orbit space \((K^* \setminus O)\sim / \tilde{G} \). \( \square \)

The orbit space \((\mathcal{C} \times \mathbb{V}^+)\sim / \mathcal{G} = \tilde{S}^1 \) is an affine model of the affine Riemann surface \( S^1 \), since \((K^* \setminus O)\sim / \tilde{G} \) is holomorphically diffeomorphic to \( S^1 \).
Theorem 5.6 The image of a $\mathcal{G}$ invariant geodesic on $(S^r, \Gamma)$ under the developing map $\delta$ is a $G$ invariant billiard motion in $K^*$. 

Proof. Because $R^j$ and $R^j$ are isometries of $(S^r, \Gamma)$ and $(K^*, \gamma_{|K^*})$, respectively, it follows that the surjective map $\delta : (S^r, \Gamma) \rightarrow (K^*, \gamma_{|K^*})$ is an isometry, because $\delta$ pulls back the metric $\gamma_{|K^*}$ to the metric $\Gamma$. Hence $\delta$ is a local developing map. Using the local inverse of $\delta$, it follows that a billiard motion in $\text{int}(K^* \setminus O)$ is mapped onto a geodesic in $(S^r, \Gamma)$, which is possibly broken at the points $(\xi_i, \eta_i) = \delta^{-1}(p_i)$. Here $p_i \in \partial K^*$ are the points where the billiard motion undergoes a reflection or a reversal. But the geodesic on $S^r$ is smooth at $(\xi_i, \eta_i)$ since the geodesic vector field is holomorphic on $S^r$. Thus the image of a geodesic under the developing map $\delta$ is a billiard motion. If the geodesic is $\mathcal{G}$ invariant, then the image billiard motion is $\mathcal{G}$ invariant, because the developing map intertwines the $\mathcal{G}$ action on $S^r$ with the $G$ action on $K^*$. \hfill $\square$

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure6.png}
\caption{Reflect the stellated regular hexagon $K^*$ with center $O$ in the line $\ell$, which is an extension of its edge $e$. This gives another stellated hexagon $K^e$ with center $O_e$. The reflection of the straight line segment geodesic $s$ in the edge $E$ of $K^*$ gives a straight line segment geodesic $s_e$, which extends $s$ so that $s \cup s_e$ is a straight line segment.}
\end{figure}

Theorem 5.7 Under the restriction of the mapping
\begin{equation}
\nu = \sigma \rho : \mathbb{C} \setminus V^+ \rightarrow (\mathbb{C} \setminus V^+)\sim /\mathfrak{G} = \tilde{S}^r \tag{13}
\end{equation}
to $K^* \setminus O$ the image of a $\mathcal{G}$ invariant billiard motion $\lambda_z$ is a smooth geodesic $\tilde{\lambda}_{\nu(z)}$ on $(\tilde{S}^r, \tilde{\gamma})$. Here $\tilde{\gamma}$ is the metric on $\tilde{S}^r$ whose pull back under the mapping $\nu$ is the metric $\gamma_{|K^*}$. 

Proof. Since the Riemannian metric $\gamma$ on $\mathbb{C}$ is invariant under the group of Euclidean motions, the Riemannian metric $\gamma_{|K^* \setminus O}$ on $K^* \setminus O$ is $G$-invariant. Hence $\gamma_{|K^* \setminus O}$ is invariant under the reflection $S_m$ for $m \in \{0, 1, \ldots, 5\}$. So $\gamma_{|K^* \setminus O}$ piece together to give a Riemannian metric $\gamma^\sim$ on the identification space $(K^* \setminus O)^\sim$. In other words, the pull back of $\gamma^\sim$ under the map $\rho_{|K^* \setminus O} : K^* \setminus O \rightarrow (K^* \setminus O)^\sim$, which identifies equivalent edges of $K^*$, is the metric $\gamma_{|K^* \setminus O}$. Since $\rho_{|K^* \setminus O}$ intertwines the $\mathcal{G}$-action on $K^* \setminus O$ with the $G$-action on $K^* \setminus O$. 

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(\(K^* \setminus O\))^\sim, the metric \(\gamma^\sim\) is \(\tilde{\Gamma}\)-invariant. It is flat because the metric \(\gamma\) is flat.

Since the billiard motion \(\lambda_z\) is a \(\tilde{\Gamma}\)-invariant broken geodesic on \((K^* \setminus O, \gamma_{K^* \setminus O})\), it gives rise to a continuous broken geodesic \(\lambda^\sim_{\rho(z)}\) on \(((K^* \setminus O)^\sim, \gamma^\sim)\), which is \(\tilde{\Gamma}\)-invariant. Thus \(\tilde{\lambda}_{\nu(z)} = \nu(\lambda_z)\) is a piecewise smooth geodesic on the smooth \(\tilde{\Gamma}\)-orbit space \(((K^* \setminus O)^\sim/\tilde{\Gamma} = \tilde{S}^\dagger, \tilde{\gamma})\). We need only show that \(\tilde{\lambda}_{\nu(z)}\) is smooth.

To see this we argue as follows. Let \(s \subseteq K^*\) be a closed segment of a billiard motion \(\gamma_z\), that does not meet a vertex of \(K\). Then we may assume that \(s\) is a horizontal straight line motion in \(K^*\), perpendicular to the direction \(u_k\), which is first met by \(s\) and let \(P_k\) be the meeting point. Let \(S_k\) be the reflection in \(E_k\). The continuation of the motion \(s\) at \(P_k\) is the horizontal line \(RS_k(s)\) in \(K^*\). Recall that \(K^*\) is the translation of \(K^*\) by \(\tau_k\). Using a suitable sequence of reflections in the edges of a suitable \(K^*\), each followed by a rotation \(R\) and then a translation in \(T\) corresponding to their origins, we extend \(s\) to a smooth straight line \(\lambda\) in \(\mathbb{C} \setminus \mathbb{V}^+\), see figure 6.

The line \(\lambda\) is a geodesic in \((\mathbb{C} \setminus \mathbb{V}^+, \gamma|_{\mathbb{C} \setminus \mathbb{V}^+})\), which in \(K^*\) has image \(\tilde{\lambda}_{\nu(z)}\) under the \(\mathfrak{G}\)-orbit map \(\nu\) that is a smooth geodesic on \((\tilde{S}^\dagger, \tilde{\gamma})\). The geodesic \(\nu(\lambda)\) starts at \(\nu(z)\). Thus the smooth geodesic \(\nu(\lambda)\) and the geodesic \(\tilde{\lambda}_{\nu(z)}\) are equal.

In other words, \(\tilde{\lambda}_{\nu(z)}\) is a smooth geodesic.

The affine orbit space \(\tilde{S}^\dagger = (\mathbb{C} \setminus \mathbb{V}^+)\sim/\mathfrak{G}\) with flat Riemannian metric \(\tilde{\gamma}\) is the affine analogue of the Poincaré model of the Riemann surface \(S^\dagger\) as an orbit space of a discrete subgroup of \(\text{PGl}(2, \mathbb{C})\) acting on the unit disk in \(\mathbb{C}\) with the Poincaré metric, see [4].

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