DECAY RATE ESTIMATIONS FOR LINEAR QUADRATIC OPTIMAL REGULATORS

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Abstract. Let \( u(t) = -Fx(t) \) be the optimal control of the open-loop system \( x'(t) = Ax(t) + Bu(t) \) in a linear quadratic optimization problem. By using different complex variable arguments, we give several lower and upper estimates of the exponential decay rate of the closed-loop system \( x'(t) = (A - BF)x(t) \). Main attention is given to the case of a skew-Hermitian matrix \( A \).

Given an operator \( A \), for a class of cases, we find a matrix \( B \) that provides an almost optimal decay rate.

We show how our results can be applied to the problem of optimizing the decay rate for a large finite collection of control systems \( (A, B_j) \), \( j = 1, \ldots, N \), and illustrate this on an example of a concrete mechanical system. At the end of the article, we pose several questions concerning the decay rates in the context of linear quadratic optimization and in a more general context of the pole placement problem.

Highlights:

- We give several lower and upper estimates of the decay rate for the closed-loop system, arising from the linear quadratic optimal regulator problem for a system \( (A, B) \), where \( A \) is skew-Hermitian.
- For a class of cases, we find the control matrix \( B \) that provides an almost optimal decay rate.
- Numerical examples of tightness of our estimates are given.

1. Introduction

It is well-known that in many practical problems, an engineer has to optimize, in one or another sense, several performance parameters of a control system. The Linear Quadratic Optimal Regulator (LQR) problem searches a stabilizing feedback which optimizes some associated quadratic cost functional. Another important characteristic of stabilization is the exponential decay rate of the resulting closed-loop system. The main question we address in this article is to study in which situations the LQR provides good decay rates of the closed-loop system.

Recall that the standard Linear Quadratic Optimal Regulator problem concerns the dynamic system of the form

\[
  x'(t) = Ax(t) + Bu(t), \quad x(0) = x_0.
\]

The problem is to minimize the cost functional

\[
  J^u(x_0) = \int_0^\infty x(t)^*Qx(t) + u(t)^*Ru(t) \, dt.
\]

Here \( x(t) \in \mathbb{C}^n \) is the state of the system and \( u \in L^2_{\text{loc}}([0, +\infty), \mathbb{C}^m) \) is a control function. Matrices \( A, B, R, Q \) are complex and have suitable sizes. We assume that \( R \) and \( Q \) are positive definite. We are specially interested in the case when the dimension \( m \) of the control \( u(t) \) is less than \( n \), the dimension of the state \( x(t) \).

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As is well-known (see [23, 39]), the solution to the LQR problem is unique and the function $u(t)$, for which the minimum of the cost functional is attained is given by the feedback function

$$u(t) = -Fx(t) = -Fe^{(A-BF)t}x_0,$$

where $F = R^{-1}B^*X$ is the feedback matrix and $X$ is any nonnegative solution of the continuous Algebraic Riccati Equation

$$XBR^{-1}B^*X - XA - A^*X - Q = 0.$$

This solution $X$ is unique and positive definite, and the minimum cost functional is given by

$$J(x_0) = x_0^*XX_0.$$  

It is also notable that the feedback matrix $F$ does not depend on $x_0$. The closed-loop system is

$$x'(t) = A_{cl.loop}x(t),$$

where $A_{cl.loop} = A - BF$ is stable, that is, its spectrum $\sigma_{cl.loop}$ lies in the open left half-plane $\mathbb{C}_-$. We denote by $\| \cdot \|$ the euclidean norm of vectors in $\mathbb{C}^k$ and the induced norm of matrices.

The linear quadratic problem is one of the most widespread methods for stabilizing systems. In this work, we give various estimates of the quality of this stabilization in terms of the geometry of the spectrum of the open-loop system matrix $A$ and the characteristics of $B$. We remark that the pole placement problem is known to be very ill conditioned for control systems of large size and that the linear quadratic stabilization is one of the methods for overcoming this difficulty. We refer to [17, Section 4], [27], [6] and references therein for theoretical results and for a discussion of different aspects of the pole placement approach and its comparison with the linear quadratic approach to stabilization.

The exponential decay rate of the closed-loop system is given by

$$\gamma_{\text{decay}}(A,B) = \min \{ |\text{Re } \nu| : \nu \in \sigma_{cl.loop} \}.$$

It is well-known that

$$\gamma_{\text{decay}} = \sup \{ \varepsilon > 0 : \forall x_0 \exists K(\varepsilon, x_0) : \|x(t)\| \leq Ke^{-\varepsilon t}, \forall t \geq 0 \}.$$

Hence $\gamma_{\text{decay}}$ can be seen as a characteristic of the quality of the LQ control for large times $t$. The LQ regulator can be considered to be good in this sense if $\gamma_{\text{decay}}$ is big.

The main results of this article concern upper and lower estimates of $\gamma_{\text{decay}}$. This is done under the assumption that the matrix $A$ is skew-Hermitian: $A^* = -A$ (that is, $iA$ is Hermitian). This assumption just means that under the absence of control ($u(t) \equiv 0$), the energy $\|x(t)\|^2$ is conserved. Notice that if an open-loop linear system models a mechanical (or electrical) system where the energy is conserved, then we are in this situation.

We also will assume that

$$Q = I, \quad R = I.$$  

The assumption about $Q$ is rather natural in view of the above remark on the conservation of energy. The case of $Q = |p(A)|^2$, where $p$ is a polynomial, reduces easily to our setting. A general matrix weight $R > 0$ is converted to the the weight $R = I$ by making a linear substitution $\tilde{u}(t) = R^{1/2}u(t)$ in (11).

As we show, the upper and lower estimates of $\gamma_{\text{decay}}$ we give permit one to compare the performance of the LQ optimal regulators of control systems $(A, B_j)$, in which $A$ is fixed and there are several possibilities for the matrix $B$.

We are not aware of any previous work estimating $\gamma_{\text{decay}}$ for LQ optimal regulators. Other measures of the quality of control have been studied already. Among the most popular of them are the eigenvalues of $X$, $\|X\|$, trace $X$ and det $X$. Since $\tilde{J}(x_0) = x_0^*XX_0$, these measures are tightly related to the cost of the stabilized system. Indeed, $\|X\|$ has the sense of the worst case performance of the cost functional, for $x_0$ of fixed norm:

$$\|X\| = \max_{\|x_0\|=1} \tilde{J}(x_0).$$

Similarly, $n^{-1} \text{trace } X$ is the average value of $\tilde{J}(x_0)$ when $x_0$ ranges over the unit sphere. The larger is any of these measures of quality of the control, the worse is the LQ stabilization.
Estimates for all these measures are well known. See for instance the reviews by Mori and Derese [33], and Kwon, Moon and Ahn [22], the papers [21], [32], [40] and recent papers [8], [9], [24], [25], [26].

We observe the following easy relationship:

\[ \gamma_{\text{decay}} \geq \frac{1}{2\|X\|}. \]

This inequality is true because for any \( \nu \in \sigma_{\text{cl.loop}} \), if \((A - BF)x_0 = \nu x_0 \) and \( \|x_0\| = 1 \), then

\[ \|X\| \geq \langle Xx_0, x_0 \rangle \geq \int_0^\infty \|x(t)\|^2 dt = \frac{1}{2|\text{Re} \nu|}. \]

So any upper estimate of \( \|X\| \) implies a lower estimate of \( \gamma_{\text{decay}} \). Several works give upper bounds for \( \|X\| \), however, these bounds are given under assumptions that either \( A + A^* < 0 \) or that \( BB^* \) is invertible. All our results deal with the case when \( A + A^* = 0 \) and \( BB^* \) can be singular.

Notice that (5) shows that whenever the stabilization is bad in terms of the parameter \( \gamma_{\text{decay}} \), \( \|X\| \) also is large.

We put

\[ \sigma(A) = \{i\lambda_1, \ldots, i\lambda_n\} \]

(\( \lambda_j \in \mathbb{R} \)) and assume throughout the whole article that

\[ \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n. \]

Our estimates depend on the following numbers. The characteristic

\[ \delta(A) = \min_{j,k; j \neq k} |\lambda_j - \lambda_k| \]

gives the minimal separation of eigenvalues. We will write just \( \delta \) when the dependence on \( A \) is clear enough. For a fixed index \( k \), we put

\[ \delta_k = \min_{j; j \neq k} |\lambda_j - \lambda_k|, \]

which denotes the separation of the eigenvalue \( i\lambda_k \) of \( A \) from the rest. The number

\[ \Delta = \Delta(A) = \max_{j,k} |\lambda_j - \lambda_k| = \lambda_n - \lambda_1 \]

will also be used.

The skew-Hermitian matrix \( A \) can be diagonalized:

\[ Av_j = i\lambda_j v_j, \]

where \( \{v_j\} (1 \leq j \leq n) \) is an orthonormal basis of \( \mathbb{C}^n \). Put

\[ b_j = B^* v_j. \]

One of our main results can be stated as follows.

**Theorem 1.** Put

\[ \ell_{\text{est}} = \min_{1 \leq k \leq n} \frac{\|b_k\|}{\sqrt{2 \left(1 + 2 \frac{\|B\|}{\delta_k^2} \right)}}. \]

Then the following statements hold.

1. The eigenvalues \( \nu_j \) of the closed-loop system lie in the box \([-\|B\|, -\ell_{\text{est}}] \times [\lambda_1, \lambda_n]\).
2. If moreover, \( m \leq n \) and the smallest singular value \( \sigma_m \) of \( B \) satisfies \( \sigma_m > 2\sqrt{2}\Delta \), then exactly \( m \) eigenvalues lie in box \([-\|B\|, -\sqrt{3}\sigma_m] \times [\lambda_1, \lambda_n]\), and the other \( n - m \) eigenvalues lie in the box \((-\sqrt{3}\Delta, -\ell_{\text{est}}] \times [\lambda_1, \lambda_n]\).
3. In the case \( m = 1 \), the bound \( \ell_{\text{est}} \) in the above assertions can be improved by substituting it by a larger number

\[ \ell_{\text{est}}^1 = \min_{1 \leq k \leq n} \frac{\|b_k\|}{\sqrt{2 \left(1 + 2 \frac{\|B\|^2}{\delta_k^2} \right)}}. \]
In particular, it follows from this theorem that
\begin{equation}
\gamma_{\text{decay}} > \ell_{\text{est}} \quad (\gamma_{\text{decay}} > \ell_{\text{est}}^1 \text{ for } m = 1).
\end{equation}

If \( A \) has multiple eigenvalues, we put \( \ell_{\text{est}} = \ell_{\text{est}}^1 = 0 \). It follows from the proof of this theorem that all its statements remain true in this case.

It also follows from Theorem \( \text{I} \) that for \( m = 1 \), \( \gamma_{\text{decay}} \leq 2\sqrt{2} \Delta \), independently of the choice of the \( n \times 1 \) matrix \( B \). We will comment more on this phenomenon at the end of Section \( 2 \).

In Section \( 6.1 \), we give an example in low precision, which gives nice two-sided estimates of \( \gamma \) if \( \| \| \) is supposed to be small. (See Theorem \( 12 \) and Corollary \( 13 \).)

Notice that the appearance of the norms of vectors \( b_k = B^* v_k \) in this estimate is very natural. In fact, the quantity
\[ d_0(A, B) = \min_k \| b_k \| \]
can be taken for a kind of measure of controllability of the system \( \dot{x} = Ax + Bu \). In the case when all eigenvalues of \( A \) are distinct, the system is controllable if and only if \( \min_k \| b_k \| > 0 \). At the end of the Introduction, we will comment on the relation between \( d_0(A, B) \), the distance to uncontrollability \( d_{\text{uc}}(A, B) \), introduced by Eising, and \( \gamma_{\text{decay}}(A, B) \).

If \( A \) is not normal, then one should use eigenvectors of \( A^* \) instead of eigenvectors of \( A \) in the definition of the measure of controllability \( d_0(A, B) \).

We remark that if \( m = n \) and for some fixed \( \beta > 0 \) one can freely choose \( B \) with \( \| B \| = \beta \), then an optimal control with the best possible \( \gamma_{\text{decay}} \) can be given easily. If \( BB^* = \beta^2 I \) (for example take \( B = \beta I \)), then the solution to the associated continuous Algebraic Riccati Equation is \( X = \beta^{-1} I \). Hence, the closed-loop system matrix is \( A_{\text{cl.loop}} = A - \beta I \), and one can readily compute its eigenvalues. It follows that in this case, in the bound \( \gamma_{\text{decay}} \leq \| B \| \), which follows from Theorem \( \text{I} \) the equality is attained.

We also observe that the case \( m > n \) can be reduced to \( m \leq n \). In fact, the optimal feedback \( u(t) = -B^* X x(t) \) ranges over the space \( \text{Ran} B^* \). Therefore the linear quadratic problem for the pair \( (A, B) \) reduces to the same problem for the pair \( (A, B) \text{ Ran } B^* \); notice that \( \dim \text{Ran } B^* \leq n \). After this reduction, in place of \( B \), we get the operator \( B|\text{Ran } B^* \), which has trivial kernel.

For this reason, we will assume throughout the paper that
\[ m \leq n \quad \text{and} \quad \ker B = 0. \]

Let us briefly overview the contents of the article by sections. Section \( 2 \) is devoted to the proof of Theorem \( \text{I} \) which implies Theorem \( \text{I} \) above.

In Section \( 3 \) we show that if the minimal separation \( \delta(A) \), defined in \( \text{I} \), is rather big in comparison with \( \| B \| \), then the closed-loop eigenvalues of the system can be located with good precision, which gives nice two-sided estimates of \( \gamma_{\text{decay}} \). In particular, Corollary \( 11 \) shows that if \( \| B \|/\delta(A) \) is rather small, then \( \gamma_{\text{decay}} \) is comparable with \( d_0(A, B) = \min \| b_k \| \). In many problems of the design of optimal controllers, the matrix \( B \) can be changed, up so some extent.

In this section, for a given \( A \), we find a “suboptimal” matrix \( B \) among all matrices with a fixed norm, which is supposed to be small. (See Theorem \( 12 \) and Corollary \( 13 \).)

For \( 1 \leq k \leq n - 1 \) we define
\begin{equation}
\Delta_k \overset{\text{def}}{=} \min_{1 \leq j \leq n-k} |\lambda_{j+k} - \lambda_j|.
\end{equation}

Observe that with this notation, \( \delta \) and \( \Delta \) defined in equations \( \text{I} \) and \( \text{II} \) are \( \delta = \Delta_1 \) and \( \Delta = \Delta_{n-1} \).

In Section \( 4 \), Theorem \( 14 \), we give an estimate of \( \gamma_{\text{decay}} \) in terms of \( \Delta_m \) (recall that \( m \) is the dimension of \( u(t) \)). For \( m > 1 \), this estimate may be much better than the estimate of Theorem \( \text{I} \) if some of eigenvalues of \( A \) are close to each other or coincide.

Section \( 5 \) contains a brief account of all our estimates of \( \gamma_{\text{decay}} \). In Section \( 6 \) some numerical examples that illustrate these estimates are given. In Subsection \( 6.1 \), we give an example in low dimension, which illustrates how our estimates compare in different cases. In Subsection \( 6.2 \) we discuss the problem of optimizing \( \gamma_{\text{decay}} \) among a finite family \( (A, B_j) \), with a fixed system matrix \( A \) and different possible choices for the control matrix \( B \). We give an algorithm which
uses our estimates to reduce the number of computations needed in the search. We illustrate this algorithm with a simple mechanical system.

In Section 7, we list some open questions, and in Section 8, we list the conclusions of this article.

In what follows, we use the notation $\|Y\|_F$ for the Frobenius norm of a matrix $Y$. It is given by

$$\|Y\|_F^2 = \text{trace}(Y^*Y);$$

this formula applies to rectangular matrices as well.

It is worth noticing that for general pairs of matrices $(A, B)$, Eising introduced in [12] the so-called “distance to uncontrollability”, given by

$$d_{uc}(A, B) = \inf \left\{ \left( \|\delta A\|_F^2 + \|\delta B\|_F^2 \right)^{1/2} : \delta A \in \mathbb{C}^{n \times n}, \delta B \in \mathbb{C}^{n \times m}, (A + \delta A, B + \delta B) \text{ uncontrollable} \right\}. \tag{16}$$

He proved that

$$d_{uc}(A, B) = \min_{\lambda \in \mathbb{C}} \sigma_{\min}([A - \lambda I, B]),$$

where $\sigma_{\min}$ stands for the minimal singular value. Estimates for the quantity $d_{uc}$ and methods for its computation have been studied further in numerous works, see [3], [11], [13], [16], [20], [37] and references therein. Related characteristics were studied in the works [38], [19] and others.

It is not difficult to show that for any normal matrix $A$ and for any $B$ such that $m < n$, one has an estimate

$$d_{uc}(A, B) \leq \min \left( d_0(A, B), r_{m+1}(\sigma(A)) \right), \tag{17}$$

where $r_k(\sigma(A))$ is the radius of the smallest disk containing at least $k$ points of $\sigma(A)$. (If $iA$ is Hermitian, then $r_{m+1}(\sigma(A)) = \frac{1}{2}\Delta_m$.)

One gets from it a certain relationship between $\gamma_{\text{decay}}$ and $d_{uc}$ for $m = 1$. Indeed, if $m = 1$, then by (14),

$$\gamma_{\text{decay}} > \frac{d_0(A, B)}{\sqrt{2} \sqrt{1 + 2 \|B\|_F^2 \delta(\sigma(A))}}. \tag{18}$$

By (17), $d_{uc} \leq \min \left( d_0(A, B), \delta(\sigma(A))/2 \right)$, and we get

$$\gamma_{\text{decay}} > \frac{d_{uc}^2}{\sqrt{\|B\|_F^2 + 2d_{uc}^2 \|B\|_F}} \geq \frac{d_{uc}^2}{\sqrt{3} \|B\|} \quad (m = 1). \tag{19}$$

We do not know whether an analogous estimate holds for $m > 1$. One can observe that the characteristics $d_0(A, B)$ and $\delta(\sigma)$ of the system $(A, B)$ are in some sense independent. Therefore the estimate (18), which uses both characteristics, gives in fact more information than (19).

Numerical methods for solving matrix algebraic Riccati equations, in fact, have been much investigated; see [5], [34] and books by Sima [36] and Datta [10]. We refer to [18], [35] and [4] for some other interesting aspects of the linear quadratic problem.

2. The main result on the location of closed-loop eigenvalues

The spectral theorem yields the decomposition

$$A = \sum_{j=1}^n i\lambda_j v_j v_j^*, \tag{20}$$

where the eigenvalues $i\lambda_j$ of $A$ are assumed to satisfy (6) and the eigenvectors $v_j$ form an orthonormal basis of $\mathbb{C}^n$ (see (10)). Moreover, $B$ and $B^*$ decompose as

$$B = \sum_{j=1}^n v_j b_j^* \quad B^* = \sum_{j=1}^n b_j v_j^*, \tag{21}$$

where the $b_j$’s have been defined in (11).
For any index $k$, $1 \leq k \leq n - 1$, we consider the closed right triangle $T_k$ in $\mathbb{C}$ with vertices at the points

$$i\lambda_k, \quad i\lambda_{k+1}, \quad -\frac{\lambda_{k+1} - \lambda_k}{2} + i\frac{\lambda_{k+1} + \lambda_k}{2}.$$ 

All these triangles lie in the half-plane $\text{Re} z < 0$ (see Figure 1). For any $k$, $1 \leq k \leq n$, we put

$$(22) \quad \rho_k = \frac{\|b_k\|}{1 + 2\|B\|^2 \delta_k}, \quad \rho_k^1 = \frac{\|b_k\|}{\sqrt{1 + 2\|B\|^2 \delta_k}}.$$ 

Our next goal is to prove the following result.

**Theorem 2.**

1. The eigenvalues $\nu_j$ of the closed-loop system lie in the box $[-\|B\|, 0) \times [\lambda_1, \lambda_n]$, outside the triangles $T_k$ and outside the closed disks centered in $i\lambda_k$ of radii $\rho_k$, given by (22).

2. If moreover, $m \leq n$ and the smallest singular value $\sigma_m$ of $B$ satisfies $\sigma_m > 2\sqrt{2}\Delta$, then exactly $m$ eigenvalues of the closed-loop system lie in box

$$[-\|B\|, -\frac{\sqrt{6}}{4} \sigma_m] \times [\lambda_1, \lambda_n],$$

and the other $n - m$ eigenvalues lie in the box

$$(-\sqrt{3}\Delta, 0) \times [\lambda_1, \lambda_n].$$

3. In the case $m = 1$, the assertion of (1) holds for disks with the same centra and larger radii $\rho_k^1$, instead of $\rho_k$.

Notice that $\ell_{est} = \min_k \frac{\rho_k}{\sqrt{2}}$, $\ell_{est}^1 = \min_k \frac{\rho_k^1}{\sqrt{2}}$ (see Figure 1). Therefore Theorem 1 is an immediate consequence of the above theorem.

**Remarks.**

1. Though we only deal with finite dimensional optimal control, we believe that the lower bounds for the decay rate $\gamma_{\text{decay}}$, given in Theorem 1, can be extended to well-posed systems with unbounded skew-symmetric operator $A$. Then, in order to get a nontrivial estimate, $B$ should be unbounded, but still can be finite dimensional. We refer to \[29\] and references therein for a discussion of exponential stabilization of the closed loop.
systems obtained by linear quadratic optimization. For infinite dimensional systems, the choice $Q = I$ (or, more generally, $Q = f(iA)$, where $f$ is a positive function on $\mathbb{R}$), is rather natural.

In [17], the same question was discussed for the collocated feedback $u = -B^*x$, which in many cases stabilizes the system. This choice of feedback is very common, for instance, in the control of flexible structures. In general, the decay rates of the corresponding closed loop systems are incomparable, and one can give examples when the collocated feedback yields much lower decay rate than the linear quadratic optimization.

(2) It should also be mentioned that (apart from the pole placement algorithms), there is a standard way to obtain a closed loop system with a prescribed decay rate. In application to our case, one has to fix some shift $\tau > 0$ and find a linear quadratic optimal feedback $F$ for the pair $(A + \tau I, B)$. Then the closed loop matrix $A - BF$ will have $\gamma_{\text{decay}} \geq \tau$.

See, for instance, the book by Zhou, Doyle and Glover [41, chapter 13.4] for a proof of this.

Before proving Theorem 2, we need some preliminaries and several lemmas.

2.1. The function $\Phi$ and its zeros. The rational matrix function defined as

\begin{equation}
\Phi(z) = I - B^*(zI - A)^{-2}B, \tag{23}
\end{equation}

is important in the control system theory. It is known that $\Phi$ factorizes as

$$
\Phi(z) = M(-\Xi)^*M(z),
$$

where

$$
M(z) = B^*X(zI - A)^{-1}B + I.
$$

The theory also shows that

$$
M(z)^{-1} = -B^*X(zI - (A - BB^*X))^{-1}B + I.
$$

See, for instance, the book by Zhou, Doyle and Glover [41, chapter 13.4] for a proof of this factorization.

Hence, the eigenvalues of $A - BB^*X$ are poles of $M(z)^{-1}$ in the sense that if $z_0$ is an eigenvalue of $A - BB^*X$ then $\det(M(z_0)^{-1}) = \infty$. It follows from the factorization of $\Phi(z)$ that the zeros of $\Phi(z)$ (in the sense that $\det(\Phi(z)) = 0$) are

$$
\{z \in \mathbb{C} : \det(\Phi(z)) = 0\} = \{z \in \mathbb{C} : z \in \sigma_{\text{cl.loop}} \text{ or } -\Xi \in \sigma_{\text{cl.loop}}\}.
$$

Definition. Let $\Phi(z)$ be as in (23) and $z_0 \in \mathbb{C}$ such that $\det(\Phi(z_0)) = 0$. If $\Re z_0 < 0$, then $z_0$ is called a stable zero of $\Phi(z)$. If $\Re z_0 > 0$, then $z_0$ is called an anti-stable zero of $\Phi(z)$.

So the stable zeros of $\Phi(z)$ are exactly the eigenvalues of $A_{\text{cl.loop}}$.

The function $\Phi$ will be very useful to make estimations of the cost characteristic $\gamma_{\text{decay}}$. The relation between $\gamma_{\text{decay}}$ and $\Phi(z)$ is

\begin{equation}
\gamma_{\text{decay}}(A, B) = \min \{|\Re z| : \det(\Phi(z)) = 0\}. \tag{24}
\end{equation}

If we define

\begin{equation}
f(\lambda; z) = \frac{1}{(z - \lambda)^2}, \tag{25}
\end{equation}

$$
\begin{align*}
g(\lambda; z) & = \frac{1}{(z - \lambda)^2} \\
& = \frac{1}{(z - \lambda)^2} \\
& = \frac{1}{(z - \lambda)^2}.
\end{align*}
$$
then, for fixed \( z \in \mathbb{C} \), \( f \) is holomorphic in \( \lambda \) on \( \mathbb{C} \setminus \{z\} \) and hence \( f(A; z) \) is well defined if \( z \notin \sigma(A) \). We can write
\[
\Phi(z) = I - B^* f(A; z) B.
\]

Using (20) and (21), we get
\[
(26) \quad \Phi(z) = I - \sum_{j=1}^{n} \frac{1}{(z-i\lambda_j)^2} b_j b_j^*.
\]

An important remark is that \( \Phi(z) \) is Hermitian and positive along the imaginary axis where it is defined. Indeed, we have \( A = iA_0 \), where \( A_0 = A_0^* \). Let \( t \in \mathbb{R} \), \( t \neq \lambda_j \), then
\[
\Phi(it) = I - B^* (itI - A)^{-2} B = I + B^*(tI - A_0)^{-2} B > I,
\]
because \( B^*(tI - A_0)^{-2} B \) is Hermitian and positive.

**Lemma 3.** The zeros of \( \Phi \) lie in the box in the complex plane given by \( |\text{Re } z| \leq \|B\| \), \( \lambda_1 \leq \text{Im } z \leq \lambda_n \).

**Proof.** Recall that the real and imaginary parts of an operator \( T \) are defined by
\[
\text{Re } T = \frac{1}{2}(T + T^*) \quad \text{Im } T = \frac{1}{2i}(T - T^*).
\]
Put
\[
h(\lambda; z) = \frac{1}{2}(f(\lambda; z) + f(-\lambda; z)) \quad g(\lambda; z) = \frac{1}{2i}(f(\lambda; z) - f(-\lambda; z)).
\]
Then \( h \) and \( g \) are meromorphic in \( \lambda \) on the whole plane. It is easy to see that
\[
(27) \quad \text{Re } \Phi(z) = I - B^* h(A; z) B, \quad \text{Im } \Phi(z) = -B^* g(A; z) B.
\]

If \( z = x + iy \), \( \lambda \in \mathbb{R} \), a direct computation shows that
\[
(28) \quad h(i\lambda; z) = \frac{x^2 - (y - \lambda)^2}{(x^2 + (y - \lambda)^2)^2};
\]
\[
(29) \quad g(i\lambda; z) = \frac{-2x(y - \lambda)}{(x^2 + (y - \lambda)^2)^2}.
\]

First we show that if \( |\text{Re } z| > \|B\| \) then \( \text{Re } \Phi(z) > 0 \) so \( z \) is not a zero of \( \Phi(z) \). Let \( \xi \in \mathbb{C}^n \) with \( \|\xi\| = 1 \). Then
\[
(\text{Re } \Phi(z)\xi, \xi) = 1 - (h(A; z)B\xi, B\xi) \geq 1 - \|B\|^2 \max_{\lambda \in \sigma(A)} h(\lambda; z).
\]
Now, using (28), if \( |\text{Re } z| > \|B\| \), it follows
\[
\max_{\lambda \in \mathbb{R}} h(\lambda; z) < \frac{1}{\|B\|^2}
\]
and therefore \( (\text{Re } \Phi(z)\xi, \xi) > 0 \) for these \( z \).

Now observe that if either \( \text{Im } z < \lambda_1 \) or \( \text{Im } z > \lambda_n \), then (29) shows that \( g(\lambda; z) \) has constant sign for all \( \lambda \in \sigma(A) \) and therefore \( \text{Im } \Phi(z) \) is either positive or negative (since we may assume \( \ker B = 0 \)) so that \( z \) is not a zero of \( \Phi(z) \). \( \square \)

In Lemma 3 we have seen that the zeros of \( \Phi(z) \) cannot be too far from the imaginary axis. The next two lemmas imply that the zeros cannot be too close to the imaginary axis.

**Lemma 4.** Define the angles
\[
\mathcal{A}_k = \{ z \in \mathbb{C} : 3\pi/4 < \arg(z - i\lambda_k) < 5\pi/4 \}, \quad 1 \leq k \leq n.
\]
If \( z \) is in the left half-plane, but does not belong to the union of these angles, then \( \text{Re } \Phi(z) \geq I \).

**Proof.** Put \( z = x + iy \). It follows from the hypothesis on \( z \) that
\[
x^2 - (y - \lambda_j)^2 \leq 0, \quad j = 1, \ldots, n.
\]
Defining \( h(\lambda; z) \) as in the proof of Lemma 3 and using (28), we get that \( h(A; z) \leq 0 \), so that \( \text{Re } \Phi(z) \geq I \). \( \square \)
It follows from the above two lemmas that the stable zeros of $\Phi$ lie in the band $\lambda_1 \leq \text{Im } z \leq \lambda_n$ and outside the triangles $T_1, \ldots, T_{n-1}$.

**Lemma 5.** $\Phi$ has no zeros in the disks $\overline{D(i\lambda_k, \rho_k)}$, $1 \leq k \leq n$.

**Proof.** If $\rho_k = 0$ then the lemma is vacuously true for the corresponding $k$. Hence, assume $\rho_k > 0$. Suppose $\text{Re } z < 0$ and $|z - i\lambda_k| \leq \rho_k$ for some $k$. Fix this index $k$. Observe that $\frac{\delta_k}{\sqrt{2}}$ is the length of the legs of one of the triangles $T_\ell$, whose vertex is in $i\lambda_k$. Since

$$\rho_k^2 = \frac{||b_k||^2}{(1 + 2||B||^2)^2} < \frac{||b_k||^2\delta_k^2}{2||B||^2} \leq \frac{\delta_k^2}{2},$$

it follows that $z$ belongs to $D(i\lambda_k, \frac{\delta_k}{\sqrt{2}})$.

It can be shown geometrically that the intersection of $D(i\lambda_k, \frac{\delta_k}{\sqrt{2}})$ with $D(i\lambda_j, \frac{\delta_j}{\sqrt{2}})$ ($j \neq k$; notice that we take the same radii) and with the left half-plane is either empty or is contained in one of the triangles $T_\ell$. Therefore, if $z \in D(i\lambda_j, \frac{\delta_j}{\sqrt{2}})$ for some $j \neq k$, then $z$ is inside of one of the triangles, and it has already been shown that then $z$ will not be a zero of $\Phi$.

So let us assume that

$$|z - i\lambda_k| < \frac{\delta_k}{\sqrt{2}}, \quad \text{but } |z - i\lambda_j| > \frac{\delta_k}{\sqrt{2}} \quad \text{for } j \neq k. \quad (30)$$

Put

$$C(z) = \Phi(z) + \frac{1}{(z - i\lambda_k)^2}b_kb_k^* = I - B^*(\sum_{j \neq k} \frac{1}{(z - i\lambda_j)^2}v_jv_j^*)B$$

(see (20)). It follows from the first inequality in (30) that $z \notin \overline{D_j}$ for $j \neq k$. Hence Lemma 4 applied to the configuration of $n - 1$ points $\{i\lambda_1, i\lambda_2, \ldots, i\lambda_n\} \setminus \{i\lambda_k\}$ on the imaginary axis, gives that $\text{Re } C(z) > I$ (recall we have assumed $\ker B = 0$).

By (20), we also have

$$||C(z)|| \leq M \overset{\text{def}}{=} 1 + \frac{2||B||^2}{\delta_k^2}.$$ 

Next, let us show that the above properties $||C(z)|| \leq M$ and $\text{Re } C(z) > I$, imply that $\text{Re } C(z)^{-1} > \frac{1}{M^2}$. Indeed, take any $\xi \in \mathbb{C}^n$ with $||\xi|| = 1$ and set $\eta = C^{-1}(z)\xi$. Then $1 = ||\xi|| \leq ||C(z)||||\eta||$ so that $||\eta|| \geq \frac{1}{M}$. Hence,

$$\text{Re}(C(z)^{-1}\xi, \xi) = \text{Re}(C(z)\eta, \eta) > ||\eta||^2 \geq \frac{1}{M^2},$$

and the inequality $\text{Re } C(z)^{-1} > \frac{1}{M^2}$ follows.

Now suppose that $\Phi(z)\xi = 0$ for some fixed $\xi \in \mathbb{C}^m$, $\xi \neq 0$. Then,

$$(C(z) - \frac{1}{(z - i\lambda_k)^2}b_kb_k^*)\xi = 0.$$ 

Since $C(z)$ is invertible, we have $b_k^*\xi \neq 0$. Multiply the above equality by $(b_k^*\xi)^{-1}b_k^*C(z)^{-1}$ and regroup terms to yield

$$b_k^*C(z)^{-1}b_k = (z - i\lambda_k)^2.$$ 

Since $\text{Re } C(z)^{-1} > \frac{1}{M^2}$, we get

$$|z - i\lambda_k|^2 = |b_k^*C(z)^{-1}b_k| > \frac{||b_k||^2}{M^2} = \rho_k^2.$$ 

Therefore $|z - i\lambda_k| > \rho_k$, a contradiction. \hfill \Box

In the case $m = 1$, the above lemma can be strengthened.

**Lemma 6.** If $m = 1$, then $\Phi$ has no zeros in the disks $\overline{D(i\lambda_k, \rho_k)}$. 

Proof. Assume that \( m = 1, \Phi(z) = 0 \) and for some index \( k, z \in \mathcal{D}(i\lambda_k, \rho_k^1) \). Proceed as in the previous lemma to deduce that \( |z - i\lambda_j| > \frac{\delta_j}{\sqrt{2}} \) for \( j \neq k \). Now, since \( \Phi(z) = 0 \), we have

\[
\frac{|b_k|^2}{(z - i\lambda_k)^2} = 1 - \sum_{j \neq k} \frac{|b_j|^2}{(z - i\lambda_j)^2}
\]

(notice that now \( b_j \) are complex numbers). It follows that

\[
\frac{|b_k|^2}{|z - i\lambda_k|^2} \leq 1 + \sum_{j \neq k} \frac{|b_j|^2}{|z - i\lambda_j|^2} < 1 + \sum_{j=1}^{n} \frac{2|b_j|^2}{\delta_k^2} = 1 + 2 \frac{\|B\|^2}{\delta_k^2},
\]

so that \( |z - i\lambda_k| > \rho_k^1 \), a contradiction. \( \square \)

Lemma 7. Let \( \sigma_m \) be the minimum singular value of \( B \). If \( \sqrt{3}\Delta < \frac{\sqrt{6}}{4}\sigma_m \), then exactly \( m \) of the stable zeros of \( \Phi(z) \) lie in the box given by

\[-\|B\| \leq \text{Re} \; z \leq -\frac{\sqrt{6}}{4}\sigma_m, \quad \lambda_1 \leq \text{Im} \; z \leq \lambda_n,
\]

and the \( n - m \) remaining stable zeros lie all in the box given by

\[-\sqrt{3}\Delta < \text{Re} \; z < 0, \quad \lambda_1 \leq \text{Im} \; z \leq \lambda_n.
\]

In particular, no stable zero lies in the band \( \text{Re} \; z \in (-\frac{\sqrt{6}}{4}\sigma_m, -\sqrt{3}\Delta] \).

Proof. The restriction to \( \lambda_1 \leq \text{Im} \; z \leq \lambda_n \) comes from Lemma 3. To prove the statement about boxes, suppose that \( \sigma_m \) satisfies the hypothesis given.

Let \( \Gamma_c : [\alpha, \beta] \to \mathbb{C} \) be the closed positively oriented contour, traversing the boundary of the box \([-c, c] \times [-d, d] \). Since \( \Phi(\infty) = I \), \( d \) can be chosen large enough so that all the eigenvalues of \( \Phi(z) \) are arbitrarily close to 1 when \( z \) is on the horizontal segments of \( \Gamma_c \). We assume that \( \Gamma_c(\alpha) = \Gamma_c(\beta) = -c - i\alpha \).

Let \( \gamma_c \) be the right vertical segment of \( \Gamma_c \), going from \( -c - id \) to \( c + id \). We subdivide \( \gamma_c \) into three segments,

\[c - id, c + i\lambda_1, \quad [c + i\lambda_1, c + i\lambda_n], \quad [c + i\lambda_n, c + id].\]

We will use expressions (27) for the real and imaginary parts of \( \Phi(z) \). First observe that if \( z \in [c - id, c + i\lambda_1] \), then \( \text{Im} \; \Phi(z) < 0 \). Indeed, for these \( z \), \( \text{Re} \; z > 0 \) and \( \text{Im} \; z < \lambda_j \) for all \( j \). It follows that \( g(i\lambda_j; z) > 0 \) and therefore \( \text{Im} \; \Phi(z) < 0 \) (see (29)). Hence, all the eigenvalues of \( \Phi(z) \) lie in the open lower half-plane.

Similarly, if \( z \in [c + i\lambda_n, c + id] \), one has \( \text{Im} \; \Phi(z) > 0 \). Hence for these \( z \), all the eigenvalues of \( \Phi(z) \) lie in the upper half-plane.

Now we will show that if

\[
\sqrt{3}\Delta \leq c < \frac{\sqrt{6}}{4}\sigma_m
\]

then \( \text{Re} \; \Phi(z) < 0 \) for \( z \in [c + i\lambda_1, c + i\lambda_n] \). Write \( z = c + iy, \; y \in [\lambda_1, \lambda_n] \). Then, using (28) and (31), we get that for all \( j \),

\[
h(i\lambda_j; z) = \frac{c^2 - (y - \lambda_j)^2}{(c^2 + (y - \lambda_j)^2} \geq \frac{c^2 - \Delta^2}{(c^2 + \Delta^2)^2} \geq \frac{c^2 - c^2/3}{(c^2 + c^2/3)^2} = \frac{3}{8c^2} > \frac{1}{\sigma_m^2}.
\]

If \( \xi \in \mathbb{C}^m \) with \( \|\xi\| = 1 \), then

\[
\langle \text{Re} \; \Phi(z) \xi, \xi \rangle = 1 - \langle h(A; z) B\xi, B\xi \rangle \leq 1 - \left( \min_{\lambda \in \sigma(A)} h(\lambda; z) \right) \cdot \|B\xi\|^2 < 1 - \frac{1}{\sigma_m^2} \|B\xi\|^2 \leq 0,
\]

because \( \|B\xi\| \geq \sigma_m\|\xi\| \). Hence, \( \text{Re} \; \Phi(z) < 0 \).

Since \( \Phi(-z) = \Phi(z)^* \), \( \Phi(z) \) behaves similarly on the left vertical segment of \( \Gamma_c \).

Now choose \( c \) satisfying (31) and study the winding number of \( \det \Phi \circ \Gamma_c \) around 0. The \( m \) eigenvalues of \( \Phi(z) \), \( \varphi_1(z), \ldots, \varphi_m(z) \), can be numbered so that \( \varphi_j \circ \Gamma_c(t) \) are all continuous
functions of the parameter \( t, t \in [\alpha, \beta] \). Since \( \det \Phi \circ \Gamma_c = (\varphi_1 \circ \Gamma_c) \cdot (\varphi_2 \circ \Gamma_c) \cdot \ldots \cdot (\varphi_m \circ \Gamma_c) \), it follows that

\[
\text{index}(\det \Phi \circ \Gamma_c) = \sum_{j=1}^{m} \text{index}(\varphi_j \circ \Gamma_c).
\]

Let us calculate the winding number of the curves \( \varphi_j \circ \Gamma_c \).

When \( z = \Gamma_c(t) \) is in the lower horizontal segment, \( \varphi_j \) are all close to 1. Then, as \( z \) travels through \( \gamma_c \), \( \varphi_j \) first are all in the lower half-plane, then go to the left half-plane and then to the upper half-plane. When \( z \) is in the upper horizontal segment, all the numbers \( \varphi_j \) are again close to 1. It follows that by choosing \( d \) sufficiently large, we can make the winding number of each of the functions \( \varphi_j \) to be arbitrarily close to \(-1\) on each of two vertical segments of \( \Gamma_c \) and arbitrarily close to \( 0 \) on the two horizontal parts of \( \Gamma_c \). Since \( \det \Phi \circ \Gamma_c : [\alpha, \beta] \to \mathbb{C} \) is a closed curve, its winding number around 0 is an integer. By \((32)\), it is equal to \(-2m\). Using the argument principle and the fact that \( \det \Phi(z) \) has \( 2n \) poles counting multiplicities inside \( \Gamma_c \), one gets that \( \det \Phi(z) \) has \( n - m \) stable zeros inside \( \Gamma_c \). Hence, \( \Phi(z) \) has \( n - m \) stable zeros inside \( \Gamma_c \).

Setting \( c = \sqrt{3}\Delta \), we see that there are \( n - m \) stable zeros inside \((-\sqrt{3}\Delta, 0) \times [\lambda_1, \lambda_n] \). Letting \( c \to \frac{\sqrt{3}}{4}\sigma_m \), we obtain that again \( \Phi \) has \( n - m \) stable zeros inside the box \((-\frac{\sqrt{3}}{4}\sigma_m, 0) \times [\lambda_1, \lambda_n] \). The remaining \( m \) stable zeros must lie all outside this box, and by Lemma \( \Box \) they belong to the box to \([-\|B\|, -\frac{\sqrt{3}}{4}\sigma_m] \times [\lambda_1, \lambda_n] \). □

**Proof of Theorem 3** All the statements of this theorem follow from Lemmas 3, 7. □

**Proof of Theorem 7**. As we already pointed out just after the statement of Theorem 2, Theorem 1 is its direct consequence. □

Using Theorem 2, we can provide upper bounds for the value of \( \gamma_{\text{decay}} \).

**Corollary 8**. The following upper bound always holds for the value of \( \gamma_{\text{decay}} \):

\[
\gamma_{\text{decay}}(A, B) \leq \|B\|.
\]

If in addition, \( \sigma_m \), the smallest singular value of \( B \), satisfies \( \sigma_m > 2\sqrt{2}\Delta \) and \( m < n \), then

\[
\gamma_{\text{decay}}(A, B) < \sqrt{3}\Delta.
\]

If \( m = 1 \), then \( \gamma_{\text{decay}} \leq 2\sqrt{2}\Delta \) for any \( B \) such that the pair \( (A, B) \) is controllable.

**Proof.** The first bound comes from lemma 3. Under the conditions of the second bound, using lemma 7 it follows that \( \Phi(z) \) has at least one zero on \((-\sqrt{3}\Delta, 0) \times [\lambda_1, \lambda_n] \). If \( m = 1 \), then \( \sigma_m = \|B\| \), which gives the last statement. □

**Remark.** Upper and lower bounds for \( \gamma_{\text{decay}} \) given in Theorem 1 and in the above Corollary fail for a general (not skew-Hermitian) \( A \) with imaginary spectrum. Consider, for instance, matrices

\[
A_2 = i \begin{pmatrix} -1 & t \\ 0 & 1 \end{pmatrix}, \quad A_3 = i \begin{pmatrix} -1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

(so that \( \sigma(A_2) = \{-i, i\} \) and \( \sigma(A_3) = \{-i, 0, i\} \) for all \( t \in \mathbb{R} \)). Put \( B_2 = (1, 1)^T, B_3 = (1, 1, 1)^T \). Then numerical simulation shows that for large positive \( t \)'s, \( \gamma_{\text{decay}}(A_2, B_2) \) is very large (and does not satisfy \( \gamma_{\text{decay}} \leq \|B\| \)) and \( \gamma_{\text{decay}}(A_3, B_3) \) is very close to zero (and does not satisfy \( \gamma_{\text{decay}} \geq t_{\text{est}}^1 \)). In fact, the simulation suggests that \( \gamma_{\text{decay}}(A_2, B_2) \to +\infty \) and \( \gamma_{\text{decay}}(A_3, B_3) \to 0 \) as \( t \to +\infty \).

As we already mentioned before, Theorem 1 also implies lower estimates for \( \gamma_{\text{decay}} \), namely \( \gamma_{\text{decay}} > t_{\text{est}} \) and \( \gamma_{\text{decay}} > t_{\text{est}}^m \) for \( m = 1 \).

The following upper bound holds for \( t_{\text{est}} \):

\[
t_{\text{est}} = \min_{1 \leq k \leq n} \frac{\|b_k\|}{\sqrt{2} (1 + 2\|B\|^2/\delta_k)} \leq \frac{\sqrt{2}}{4} \sqrt{\frac{m}{n}} \delta^2 \|B\|.
\]
Indeed, the inequality
\[ \|b_k\| \leq \sqrt{\frac{m}{n}} \|B\|, \]
(see the proof of Theorem 12) implies that we get
\[ \ell_{est} \leq \frac{\sqrt{2}}{4} \min_k \frac{\|b_k\| \delta_k^2}{\|B\|^2} \leq \frac{\sqrt{2}}{4} \sqrt{\frac{m}{n}} \delta^2. \]
Similarly, \( \ell_{est}^1 \leq \delta/(2\sqrt{n}) \). One can guess that a matrix \( B \) in which all \( \|b_k\| \) are as big as possible can be used to ensure a nearly optimal stabilization of the system. A matrix with these characteristics will be given below in Theorem 12.

3. The estimate of \( \gamma_{\text{decay}} \) in the case of a sufficient separation of the open-loop spectrum

Here we will assume that the minimal separation \( \delta(A) \), defined in (7), is rather big in comparison with \( \|B\| \). We will use the following analogue of Rouché’s theorem for matrix-valued functions.

Lemma 9. Let \( F(z) \) and \( G(z) \) be meromorphic functions on some open subset \( D \subset \mathbb{C} \), whose values are \( m \times m \) complex matrices. Let \( \gamma \) be a closed curve in \( D \) such that \( \text{det} F(z) \) has no poles or zeros on \( \gamma \). If \( \|F^{-1}(z)G(z)\| < 1 \) for all \( z \in \gamma \), then the scalar functions \( \text{det} F(z) \) and \( \text{det}(F(z) + G(z)) \) have the same winding number along \( \gamma \).

This lemma is known. See, for instance, [15, Theorem 2.2] (for operator-valued functions) or [31].

The next theorem locates the points of the closed-loop spectrum inside disks of radii \( r_k \) such that \( r_k \to 0 \) as \( \frac{\|B\|^3}{\delta_k^2} \to 0 \). Recall that the zeros of \( \Phi \) which lie in the left half-plane coincide with the eigenvalues of the closed-loop system (see Subsection 2.1).

Theorem 10. Suppose \( k \) is an index such that \( \frac{\|B\|^2}{\delta_k^2} < \frac{(2-\sqrt{2})^2}{2} \). Then \( \Phi \) has exactly one zero in the open disk of centre \( z_k = -\|b_k\| + i\lambda_k \) and radius \( r_k = \frac{2}{(2-\sqrt{2})} \delta_k^2 \|b_k\| \).

Proof. Observe that \( r_k < \|b_k\| \) and consider the contour
\[ \gamma = \{z_k + r_k e^{i\theta} : 0 \leq \theta \leq 2\pi\} \]
and the functions
\[ F(z) = I - \frac{1}{(z - i\lambda_k)^2} b_k b_k^* \]
\[ G(z) = \Phi(z) - F(z). \]
These functions are holomorphic on \( \gamma \) and its interior. We will prove that \( \|G(z)\| < \|F(z)^{-1}\|^{-1} \) for \( z \in \gamma \), so that we can use the above version of Rouché’s theorem.

First observe that \( F(z) \) is normal, so that \( \|F(z)^{-1}\|^{-1} = \min_{\lambda \in \sigma(F(z))} |\lambda| \). The spectrum of \( F(z) \) can be computed easily:
\[ \sigma(F(z)) = \left\{ 1, 1 - \frac{\|b_k\|^2}{(z - i\lambda_k)^2} \right\}. \]

Take any \( z \) such that \( |z - z_k| = r_k \) and put \( z = z_k + r_k e^{i\theta} \). Notice that \( r_k < \|b_k\| \) implies \( r_k e^{i\theta} - 2\|b_k\| \geq \|b_k\| - r_k e^{i\theta} \). We get
\[ 1 - \frac{\|b_k\|^2}{(z - i\lambda_k)^2} = 1 - \frac{\|b_k\|^2}{(\|b_k\| - r_k e^{i\theta})^2} = \frac{r_k^2 e^{2i\theta} - 2r_k \|b_k\| e^{i\theta}}{(\|b_k\| - r_k e^{i\theta})^2} \]
\[ \geq \frac{r_k}{\|b_k\| - r_k e^{i\theta}} \geq \frac{r_k}{\|b_k\| + r_k} \geq \frac{r_k}{2 \|b_k\|.} \]
and so \(\|F(z)^{-1}\|^{-1} \geq \frac{r_k}{2\|b_k\|}\). Hence it will suffice to prove that \(\|G(z)\| < \frac{r_k}{2\|b_k\|}\). By 20, 

\[
G(z) = -B^* \left( \sum_{j \neq k} \frac{1}{(z - i\lambda_j)^2} v_j v_j^* \right) B.
\]

Then, observe that the condition \(\frac{\|B\|^2}{\delta_k^2} < \frac{(2 - \sqrt{2})^2}{2}\) implies \(r_k < (\sqrt{2} - 1) \delta_k\) since

\[
r_k < \|b_k\| \leq \|B\| < \frac{2 - \sqrt{2}}{\sqrt{2}} \delta_k = (\sqrt{2} - 1) \delta_k.
\]

Now we have for \(z \in \gamma\) and for all \(j \neq k\)

\[
|z - i\lambda_j| = |r_k e^{i\theta} - \|b_k\| + i\lambda_k - i\lambda_j| \geq |i\lambda_k - i\lambda_j - \|b_k\|| - r_k \\
\geq |\lambda_k - \lambda_j| - r_k > \delta_k - (\sqrt{2} - 1) \delta_k = (2 - \sqrt{2}) \delta_k.
\]

Hence it follows that

\[
\|G(z)\| < \frac{\|B\|^2}{(2 - \sqrt{2})^2 \delta_k^2} = \frac{r_k}{2\|b_k\|} \leq \|F(z)^{-1}\|^{-1}.
\]

So, by Rouché’s theorem, \(\det F(z)\) and \(\det(F(z) + G(z))\) have the same number of zeros inside \(\gamma\). The only zeros of \(\det F(z)\) are \(z_k\) and \(-\hat{z}_k\). Since \(\gamma\) lies completely in the left half-plane, \(\det F(z)\) has exactly one zero inside \(\gamma\). Therefore \(\det \Phi(z) = \det(F(z) + G(z))\) has exactly one zero inside \(\gamma\).

\[\square\]

**Corollary 11.** Set

\[
(35) \quad \varphi_k \overset{\text{def}}{=} \frac{2\|B\|^2}{(2 - \sqrt{2})^2 \delta_k^2}, \quad 1 \leq k \leq n.
\]

Suppose \(\varphi_k < 1\) for at least one index \(k\). Put

\[
\Gamma_+ = \min\{(1 + \varphi_k)\|b_k\| : \varphi_k < 1\}, \quad \Gamma_- = \min\{(1 - \varphi_k)\|b_k\|\}.
\]

Then \(\gamma_{\text{decay}} < \Gamma_+\). If moreover \(\varphi_k < 1\) for all \(k\), then

\[
\Gamma_- < \gamma_{\text{decay}} < \Gamma_+.
\]

**Proof.** If \(\varphi_k < 1\) for some \(k\), the preceding theorem shows that some eigenvalue \(\nu\) of the closed-loop system satisfies \(-\Re \nu < (1 + \varphi_k)\|b_k\|\), and the upper bound follows. If \(\varphi_k < 1\) for all \(k\) and \(\nu\) is any eigenvalue of the closed-loop system, then \(-\Re \nu > (1 - \varphi_k)\|b_k\|\) for some \(k\), so that the lower bound follows.

Using Theorem 10 when \(\delta(A)\) is sufficiently large, we can give a matrix \(B\), in a sense close to optimal.

**Theorem 12.** Suppose \(m \leq n\). Let \(w\) be the primitive \(n\)-th root of 1 given by

\[
w = e^{-\frac{i\pi}{n}}.
\]

Let \(\beta > 0\).

Let the matrix \(\hat{B}\) be represented in the orthonormal basis given by \(\{v_j\}\), the eigenvectors of \(A\), as

\[
(36) \quad \hat{B} = \frac{\beta}{\sqrt{n}} \begin{bmatrix}
u^{0,0} & \nu^{0,1} & \cdots & \nu^{0,\langle m-1\rangle} \\
\nu^{1,0} & \nu^{1,1} & \cdots & \nu^{1,\langle m-1\rangle} \\
\vdots & \vdots & \ddots & \vdots \\
\nu^{\langle n-1\rangle,0} & \nu^{\langle n-1\rangle,1} & \cdots & \nu^{\langle n-1\rangle,\langle m-1\rangle}
\end{bmatrix}.
\]

Then, \(\|\hat{B}\| = \beta\) and for any \(\varepsilon > 0\), there exists \(K > 0\) such that if \(\delta(A) > K\), then

\[
\sup_{\|B\| = \beta} \gamma_{\text{decay}}(A, B) - \gamma_{\text{decay}}(A, \hat{B}) < \varepsilon.
\]
Proof. First observe that $\hat{B}$ is related to the unitary Discrete Fourier Transform. If $U \in \mathbb{C}^{n \times n}$ is the matrix of the unitary DFT, then

$$\hat{B} = \beta U \left[ \begin{array}{c} I \otimes \mathbf{v} \\ 0_{n-m \times m} \end{array} \right].$$

It follows that $\|\hat{B}\| = \beta$. Define $\hat{b}_j$ in the same way as in (11), that is, put $\hat{b}_j = \hat{B}^* v_j$. Then $\|\hat{b}_j\| = \beta \sqrt{\frac{m}{n}}$ for all $j = 1, \ldots, n$.

Let $\hat{B}$ be arbitrary with $\|\hat{B}\| = \beta$. Let $\varepsilon > 0$ be given. Define $K > 0$ from

$$K^2 = \max \left\{ \frac{2}{(2 - \sqrt{2})^2} \beta^2, \frac{4}{(2 - \sqrt{2})^2} \frac{\varepsilon^3}{\beta} \right\}.$$ 

Suppose $\delta^2 > K^2$. Then $\frac{\beta^2}{\delta^2} < \frac{(2 - \sqrt{2})^2}{2}$, and the hypothesis of Theorem 10 is satisfied for any index $k$. We obtain disks of radii $r_k$ such that the zeros of $\Phi$ lie inside this disks. Now,

$$r_k = \frac{2}{(2 - \sqrt{2})^2} \frac{\beta^2}{\delta^2} \|b_k\| \leq \frac{2}{(2 - \sqrt{2})^2} \frac{\beta^3}{\delta^2} < \frac{2}{(2 - \sqrt{2})^2} \frac{\beta^3}{\varepsilon} \leq \frac{\varepsilon}{2}.$$ 

Since there is a zero of $\Phi$ in each of these disks of centre $z_k = -\|b_k\| + i\lambda_k$, we have

$$\min \|b_k\| - \frac{\varepsilon}{2} < \gamma_{\text{decay}}(A, B) < \min \|b_k\| + \frac{\varepsilon}{2}.$$ 

Notice that

$$\sum_{j=1}^{n} \|b_j\|^2 = \|B\|^2 F \leq m \|B\|^2 = m \beta^2.$$ 

Hence, $\min \|b_j\|^2 \leq \beta^2 \frac{m}{n}$. Therefore

$$\sup_{\|B\| = \beta} \gamma_{\text{decay}}(A, B) \leq \beta \sqrt{\frac{m}{n} \frac{\varepsilon}{2}}.$$ 

Since $\hat{B}$ has $\|\hat{b}_j\| = \beta \sqrt{\frac{m}{n}}$ for all $j$,

$$\gamma_{\text{decay}}(A, \hat{B}) > \beta \sqrt{\frac{m}{n} - \frac{\varepsilon}{2}}$$

and the theorem follows. \hfill \square

Corollary 13. Assume that $m \leq n$. Let $\beta > 0$. If $A$ is such that

$$\delta^2(A) > \frac{6}{(2 - \sqrt{2})^2} \sqrt{\frac{n}{m} \beta^2},$$

then the matrix $\hat{B}$ given in (36) satisfies

$$\gamma_{\text{decay}}(A, \hat{B}) > \frac{1}{2} \sup_{\|B\| = \beta} \gamma_{\text{decay}}(A, B).$$

Proof. By applying Theorem 12 and its proof with $\varepsilon = \frac{3}{4} \beta \sqrt{\frac{m}{n}}$, one gets

$$K^2 = \frac{6}{(2 - \sqrt{2})^2} \sqrt{\frac{n}{m} \beta^2}.$$ 

Therefore, if $\delta^2(A) > K^2$, then

$$\gamma_{\text{decay}}(A, \hat{B}) > \beta \sqrt{\frac{m}{n} - \frac{\varepsilon}{2}} = \frac{2}{3} \beta \sqrt{\frac{m}{n}} = \frac{1}{2} \left( \beta \sqrt{\frac{m}{n} + \frac{\varepsilon}{2}} \right) \geq \frac{1}{2} \sup_{\|B\| = \beta} \gamma_{\text{decay}}(A, B).$$

\hfill \square
4. Estimates of decay rate in terms of $\Delta_m$ for $m > 1$

We begin with the following remark. Let $\theta > 0$. Consider the following function

$$f(\sigma) = \frac{\sigma - \theta}{\sigma + \sigma^2 + \cdots + \sigma^{m+1}},$$

which is positive on $(\theta, +\infty)$ and vanishes at $\theta$ and at $+\infty$. We denote by $\sigma_0(\theta)$ the point in $(\theta, +\infty)$ where $f$ takes its maximal value and by $\mu(\theta) = f(\sigma_0(\theta))$ this maximal value. If we put

$$P(\sigma) = \sigma + \sigma^2 + \cdots + \sigma^{m+1} - \left(\sum_{j=0}^{m} (j+1)\sigma^j\right)(\sigma - \theta)$$

then it is easy to see that $\sigma_0(\theta)$ is the unique root of $P$ in $(\theta, +\infty)$ (notice that $P'$ < 0 on $(\theta, +\infty)$).

In what follows, $P_{\text{Lin}\{w_1,\ldots, w_r\}}$ will stand for the orthogonal projection onto the linear span generated by vectors $w_1, \ldots, w_r$.

**Theorem 14.** Let $2 \leq m < n$. Let $v_j$ be the eigenvectors of $A$ (see (6) and (10)). Put

$$\Pi_k = P_{\text{Lin}\{v_k, \ldots, v_{k+m-1}\}} : \mathbb{C}^n \to \mathbb{C}^n, \quad k = 1, \ldots, n - m + 1.$$

Assume that positive constants $\gamma$, $K$ satisfy

(i) $\|B\| \leq K$;
(ii) $\|B^*\Pi_k d\| \geq \gamma \|\Pi_k d\|$ for all $d \in \mathbb{C}^n$ and for all $k$.

Define $\Delta_m$ from (16). Put $\sigma_0 = \sigma_0(K^2/\gamma^2)$,

$$\rho = \frac{1}{2} \min \left\{ \Delta_m \left(2 \frac{\sigma_0^{m+1} - 1}{\sigma_0 - 1} - 1 \right)^{-1/2}, \gamma \left[\mu\left(\frac{K^2}{\gamma^2}\right)\right]^{1/2} \right\}. \tag{39}$$

Then all eigenvalues of the closed-loop system lie in the half-plane $\text{Re} \, z \leq -\rho < 0$.

Notice that $\Delta_m$ can be positive even in the case when some of the eigenvalues of $A$ coincide. We do not exclude this case.

The rest of this Section is devoted to the proof of this theorem.

The plan of the proof is as follows. First we remark that it is easy to get from (i) and (ii) that $K/\gamma \geq 1$. Hence $\sigma_0 > 1$.

Fix some $z = x + iy \in \mathbb{C}$ such that $-\rho < x < 0$. We have to prove that $z \notin \sigma_{\text{cl.loop}}$. To do that, let us consider a reordering $\lambda_{r(1)}, \lambda_{r(2)}, \ldots, \lambda_{r(n)}$ of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\frac{1}{\gamma} A$ such that

$$|z - i\lambda_{r(1)}| \leq |z - i\lambda_{r(2)}| \leq \cdots \leq |z - i\lambda_{r(n)}|. \tag{40}$$

Let us assume that

$$|z - i\lambda_{r(1)}|^2 \leq 2x^2 \tag{41}$$

(if it is not true, then $z \notin \sigma_{\text{cl.loop}}$, due to Theorem [2]). We will divide the spectrum $\sigma(A)$ into two parts:

$$\sigma_{\text{close}}(A) = \{i\lambda_{r(1)}, \ldots, i\lambda_{r(s)}\}, \quad \sigma_{\text{far}}(A) = \{i\lambda_{r(s+1)}, \ldots, i\lambda_{r(n)}\},$$

where the index $s$ will be elected according to Lemma [13] below. (Notice that the reordering (40) and this partitioning of $\sigma(A)$ depend on the position of $z$.) Once this partition is chosen, we put

$$\eta_{\text{close}} = |z - i\lambda_{r(s)}|^2, \quad \eta_{\text{far}} = |z - i\lambda_{r(s+1)}|^2.$$ 

Introduce the notation

$$\omega = \gamma^2, \quad \kappa = K^2$$

(so that $\omega \leq \kappa$).

We will say that $\sigma_{\text{close}}(A)$ and $\sigma_{\text{far}}(A)$ are sufficiently separated (with respect to $z$) if

$$\eta_{\text{far}} > 2x^2 \quad \text{and} \quad \eta_{\text{close}} < \omega \frac{\eta_{\text{far}} - 2x^2}{\eta_{\text{far}} - 2x^2 + \kappa}. \tag{42}$$
Inequality \( \eta_{\text{far}} > 2x^2 \) implies that \( \eta_{\text{far}}(\omega - \kappa + 2x^2 - \eta_{\text{far}}) < 0 \). By using the second inequality in (12), one gets that the sufficient separation implies the strict inequality \( \eta_{\text{close}} < \eta_{\text{far}} \).

Before finishing the proof, we need three lemmas.

**Lemma 15.** For any \( z \) such that \(-\rho < \Re z < 0 \) and (11) holds there exists an index \( s, 1 \leq s \leq m \), such that the corresponding parts \( \sigma_{\text{far}}(A) \) and \( \sigma_{\text{close}}(A) \) of the spectrum of \( A \) are sufficiently separated.

**Proof.** Take some \( z \) that satisfies the hypotheses. Let \( \lambda_{\tau(1)}, \ldots, \lambda_{\tau(t)} \) be all point of the spectrum of \( A \) that satisfy
\[
\eta_j \overset{\text{def}}{=} |z - i\lambda_{\tau(j)}|^2 \leq 2x^2.
\]

Since
\[
|x| < \rho \leq \Delta m/2
\]
(see (39)), it follows that \( 1 \leq t \leq m \) and that \( \eta_{t+1} > 2x^2 \). Assume that the subdivision of the spectrum of \( A \) into two sufficiently separated parts is impossible. Then
\[
\eta_j \geq \omega - \frac{\eta_{j+1} - 2x^2}{\eta_{j+1} - 2x^2 + \kappa} \quad \text{for } j = t, \ldots, m.
\]

We will prove that this leads to a contradiction. Put \( \delta = \omega - \kappa/\sigma_0 \). Since \( \sigma_0 > K^2/\gamma^2 \), we have 
\( 0 < \delta < \omega \) and \( \sigma_0 = \frac{\omega}{\omega - \delta} \).

We prove that
\[
\eta_j \leq 2x^2 \frac{\sigma_0^{j+1} - 1}{\sigma_0 - 1}
\]
for \( j = t, \ldots, m + 1 \) by induction in \( j \). The induction base, \( j = t \), follows from our assumptions. Assume that (44) holds for \( j = j_0, t \leq j_0 \leq m \). By using that \( |x| < \rho \) and (39), we get
\[
2x^2 < 2\rho^2 \leq \frac{\gamma^2}{2} \mu \left( \frac{\omega}{\omega - \sigma_0} \right) = \frac{\omega \sigma_0 - \kappa}{2(\sigma_0 + \sigma_0^2 + \cdots + \sigma_0^{m+1})}.
\]

Therefore
\[
\eta_{j_0} \leq 2x^2 \frac{\sigma_0^{m+1} - 1}{\sigma_0 - 1} \leq \frac{1}{2} (\omega - \kappa/\sigma_0) = \frac{\delta}{2} < \omega.
\]

We also have \( \eta_{j_0 + 1} > 2x^2 \). Hence (43) implies that
\[
\eta_{j_0 + 1} - 2x^2 \leq \kappa - \eta_{j_0} \frac{\kappa}{\omega - \eta_{j_0}} \leq \eta_{j_0} \frac{\kappa}{\omega - \delta} = \eta_{j_0} \sigma_0.
\]

It follows that
\[
\eta_{j_0 + 1} \leq \eta_{j_0} \sigma_0 + 2x^2 \leq 2x^2 \left( \frac{\sigma_0^{j_0 + t + 1} - 1}{\sigma_0 - 1} \right) = 2x^2 \frac{\sigma_0^{j_0 + t + 2} - 1}{\sigma_0 - 1}.
\]

This gives the induction step. Hence (44) holds for all \( j = t, \ldots, m + 1 \). In particular,
\[
\eta_{m+1} = |z - i\lambda_{\tau(m+1)}|^2 \leq 2x^2 \frac{\sigma_0^{m+1} - 1}{\sigma_0 - 1}.
\]

This gives a contradiction. Indeed, it follows that \( \lambda_{\tau(1)}, \ldots, \lambda_{\tau(m+1)} \) are contained in the interval \([y - \ell, y + \ell]\), where \( \ell = (\eta_{m+1} - x^2)^{1/2} \). Then
\[
\ell^2 \leq x^2 \left( 2 \frac{\sigma_0^{m+1} - 1}{\sigma_0 - 1} \right) < \rho^2 \left( 2 \frac{\sigma_0^{m+1} - 1}{\sigma_0 - 1} \right) \leq \frac{\Delta^2 m}{4}
\]
(the last inequality is due to (39)). Hence \( \ell^2 < \frac{\Delta^2 m}{4} \). We get a contradiction to the definition of \( \Delta m \). \( \square \)

Next, we take \( s = s(z) \) as in the above Lemma and put
\[
P_{\text{close}} = P_{\text{Lin}} \{ v_{\tau(1)}, \ldots, v_{\tau(s)} \}, \quad P_{\text{far}} = P_{\text{Lin}} \{ v_{\tau(s+1)}, \ldots, v_{\tau(n)} \}.
\]
(recall that \(v_j\) is an eigenvector of \(A\) corresponding to \(i\lambda_j\)). Then

\[
\Phi(z) = I - B^*(z - A)^{-2}B = \Psi + \Sigma,
\]

where

\[
\Psi = I - B^*P_{\text{far}}(z - A)^{-2}P_{\text{far}}B, \quad \Sigma = -B^*P_{\text{close}}(z - A)^{-2}P_{\text{close}}B.
\]

Put

\[
W = B^*P_{\text{far}},
\]

then

\[
\Psi = I - W(z - A)^{-2}W^* = I + \Psi_1,
\]

where

\[
\Psi_1 \overset{\text{def}}{=} -W(z - A)^{-2}W^*.
\]

Define \(\alpha > 0\) from the equation

\[
\eta_{\text{far}} = (2 + \alpha)x^2.
\]

Then

\[
(y - \lambda_k)^2 \geq (1 + \alpha)x^2 \quad \text{for all } i\lambda_k \in \sigma_{\text{far}}(A).
\]

Hence

\[
(yI + iA)^2 - x^2I \mid \text{Ran } P_{\text{far}} \geq \alpha x^2I \mid \text{Ran } P_{\text{far}}.
\]

**Lemma 16.** Suppose \(|B| \leq K\). Then

\[
\text{Re } \Psi^{-1} \geq \beta^{-1}I,
\]

where

\[
\beta = 1 + \frac{K^2}{\alpha x^2}.
\]

**Proof.** A calculation gives

\[
\text{Re } \Psi_1 = -\frac{1}{2}W[(z - A)^{-2} + (\bar{z} + A)^{-2}]W^* = WGW^*
\]

where

\[
G = \frac{(y + iA)^2 - x^2I}{(z - A)^2(\bar{z} + A)^2} \mid \text{Ran } P_{\text{far}}.
\]

We wish to prove that

\[
\Psi^{-1} + \Psi^{-1} \geq 2\beta^{-1}I.
\]

First let us check the inequality

\[
\Psi^*\Psi \leq \frac{\beta}{2}(\Psi + \Psi^*) = \beta \text{ Re } \Psi.
\]

Inequality (54) is obtained as follows:

\[
\Psi^*\Psi = (I + \Psi_1^*)(I + \Psi_1)
\]

\[
= I + 2\text{ Re } \Psi_1 + \Psi_1^*\Psi_1
\]

\[
= I + 2WGW^* + W(\bar{z} + A)^{-2}W^*W(z - A)^{-2}W^*
\]

\[
\leq I + 2WGW^* + K^2W(\bar{z} + A)^{-2}(z - A)^{-2}W^*
\]

\[
\leq I + (2 + \frac{K^2}{\alpha x^2})WGW^*.
\]

The last inequality is due to (47) and (52). By (51), this implies

\[
\Psi^*\Psi \leq I + (1 + \beta)\text{ Re } \Psi_1 = (1 + \beta)\text{ Re } \Psi - \beta I.
\]

Rewrite this inequality as

\[
(\Psi^* - \frac{\beta + 1}{2}I)(\Psi - \frac{\beta + 1}{2}I) \leq \left(\frac{\beta - 1}{2}\right)^2I.
\]
or, equivalently, \( \| \Psi - \frac{\beta+1}{2} I \| \leq \frac{\beta+1}{2} \). This gives the inequality \( \text{Re } \Psi \leq (\frac{\beta+1}{2} + \frac{\beta+1}{2}) I = \beta I \). Then by (55), \( \Psi^* \Psi \leq (\beta + 1) \text{Re } \Psi - \beta I \leq \beta \text{Re } \Psi \), and we get (54).

We also get that \( \sigma(\Psi) \subset \{ z : |z - \frac{\beta+1}{2}| \leq \frac{\beta+1}{2} \} \), which implies that \( \Psi \) is invertible. So (53) follows immediately from (54). \( \square \)

**Lemma 17.** Suppose \( 1 \leq s \leq m \), \( \Psi \) is an \( m \times m \) matrix satisfying \( \text{Re } \Psi^{-1} \geq \beta^{-1} I \), \( V \) is an \( m \times s \) matrix such that

\[
\| Vd \| \geq \gamma \| d \|, \quad d \in \mathbb{C}^s
\]

and \( \Lambda \) is an \( s \times s \) invertible matrix. If \( \beta, \gamma \) are positive and

\[
\beta^{-1} \gamma^2 \geq \| \Lambda^{-1} \|,
\]

then the matrix \( \Psi + V \Lambda V^* \) is invertible.

**Proof.** Suppose it is not,

\[
(\Psi + V \Lambda V^*) w = 0
\]

for some \( w \in \mathbb{C}^m \), \( w \neq 0 \). Put \( V^* w = c \in \mathbb{C}^s \). One gets

\[
w = -\Psi^{-1} V \Lambda c, \quad -V^* \Psi^{-1} V \Lambda c = c.
\]

Hence

\[
\langle c, \Lambda c \rangle = -(V^* \Psi^{-1} V \Lambda c, \Lambda c).
\]

Then, on one hand, one has

\[
|\langle c, \Lambda c \rangle| = |\langle \Lambda^{-1} \Lambda c, \Lambda c \rangle| \leq \| \Lambda^{-1} \| \cdot \| \Lambda c \|^2,
\]

and on the other,

\[
|\langle c, \Lambda c \rangle| = |\langle V^* \Psi^{-1} V \Lambda c, \Lambda c \rangle| \geq \text{Re}(V^* \Psi^{-1} V \Lambda c, \Lambda c) \geq \beta^{-1}\|V \Lambda c\|^2 \geq \beta^{-1} \gamma^2 \| \Lambda c \|^2.
\]

These inequalities and (57) imply that \( c = V^* w = 0 \). By (58), it follows that \( \Psi w = 0 \), so that \( w = 0 \), which contradicts the choice of \( w \). \( \square \)

**The end of the proof of Theorem 14.** As before, we assume that some \( z = x + iy \in \mathbb{C}_- \) with \( -\rho < x < 0 \) has been fixed. Lemma 15 gives us an index \( s \), \( s \leq m \), which defines a partition of \( \sigma(A) \) into two sufficiently separated parts, \( \sigma_{\text{close}}(A) \) and \( \sigma_{\text{far}}(A) \). Define \( \Psi \) and \( \Sigma \) from (46).

By Lemma 16, (50) holds. Put

\[
V = B^* \text{ Ran } P_{\text{close}}, \quad \Lambda = (z - A)^{-2} \text{ Ran } P_{\text{close}}
\]

and apply Lemma 17 to these two matrices and \( \Psi \). Since \( \eta_{\text{close}} < \eta_{\text{far}} \), it follows that there is some \( k, 1 \leq k \leq n - m - 1 \), such that all the indices \( \tau(1), \ldots, \tau(s) \) are contained in the set \( \{ k, k+1, \ldots, k+m-1 \} \) (this is true even if \( A \) has multiple eigenvalues). Therefore, by hypothesis (ii) of the Theorem, \( V \) satisfies (59). By (17) and (12), one has

\[
\beta^{-1} \gamma^2 = \gamma^2 \frac{\alpha x^2}{K^2} = \gamma^2 \frac{\eta_{\text{far}} - 2 x^2}{\eta_{\text{far}} - 2 x^2 + K^2} > \eta_{\text{close}} = \| \Lambda^{-1} \|.
\]

Hence (57) holds. So \( \Phi(z) = \Psi + V \Lambda V^* = \Psi + \Sigma \) is invertible, and therefore \( z \not\in \sigma_{\text{cl.loop}} \). This proves the Theorem. \( \square \)

5. A BRIEF ACCOUNT OF OUR ESTIMATES OF \( \gamma_{\text{decay}} \)

Here, for the reader’s convenience, we gather all the above estimates.

Theorem 1 for \( m \geq 1 \): \( \gamma_{\text{decay}} > \ell_{\text{est}} \); see (12) and (13):

Theorem 1 for \( m = 1 \): \( \gamma_{\text{decay}} > \ell_{\text{est}} \).

**Corollary 11** \( \gamma_{\text{decay}} \leq \Gamma_+ = \min \{ (1 + \varphi_k) \| b_k \| : \varphi_k < 1 \} \): \( \gamma_{\text{decay}} \geq \Gamma_- = \min_k (1 - \varphi_k) \| b_k \| \) if \( \varphi_k < 1 \) for all \( k \), where \( \varphi_k = \frac{1 + \beta^2}{2(1 - \sqrt{\beta^2})^2} \), \( b_k = B^* v_k \) and \( Av_k = i \lambda_k v_k, \| v_k \| = 1 \).

**Theorem 14** \( \gamma_{\text{decay}} \geq \rho \) for \( m \geq 2 \), where \( \rho \) is defined in (39).
We notice also that if these statements provide several lower or upper bounds for $\gamma_{\text{decay}}$, then, obviously, one can take the best one of these.

6. Numerical examples

6.1. An example with 4 states and 2 controls. Take $n = 4$, $m = 2$ and consider the matrices

\[
A = i \begin{bmatrix}
-b & 0 & 0 & 0 \\
0 & -a & 0 & 0 \\
0 & 0 & a & 0 \\
0 & 0 & 0 & b
\end{bmatrix}, \\
B = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1
\end{bmatrix},
\]

where $a, b$ are positive real numbers. Consider the LQR problem for $(A, B)$ with $Q = I$, $R = I$. Table 1 collects the values of $\|X\|$ and of $\gamma_{\text{decay}}$ for different values of $a, b$, obtained by numerical calculations. The last four columns of this Table show the values of $\Gamma_-, \Gamma_+, \ell_{\text{est}}$ and $\rho$, which are the lower and upper bounds for $\gamma_{\text{decay}}$ guaranteed by our theorems (see the previous section for a brief account).

Row 1 shows that the bounds $\Gamma_-, \Gamma_+$ for $\gamma_{\text{decay}}$ are very precise in the case of large separation of the spectrum of $A$. In rows 2 and 3, one can see that as the separation diminishes (and some $\varphi_k$’s approach to 1), the bounds $\Gamma_-, \Gamma_+$ become much more vague.

In row 4, there is some $k$ with $\varphi_k < 1$, but we do not have $\varphi_k < 1$ for all $k$. Hence, only the upper bound $\Gamma_+$ from Corollary 11 holds, and $\Gamma_-$ is not defined.

In rows 5, 6 and 7, $\varphi_k \geq 1$ for all $k$. Hence Corollary 11 provides no bounds at all, and we do not show the values of $\Gamma_-, \Gamma_+$. In these rows one can see how the lower estimate $\rho$ for $\gamma_{\text{decay}}$ from Theorem 3 can give better results than $\ell_{\text{est}}$ from Theorem 1 especially if some eigenvalues of $A$ are close together in comparison with $\|B\|$.

Part (2) of Theorem 1 and Theorem 3 show that if the minimal singular value of $B$ is large in comparison with the diameter of the spectrum of $A$, then the closed-loop spectrum divides in two parts: $m$ eigenvalues are in the band $\text{Re} \ z \in [-\|B\|, -\sqrt{\frac{\sigma}{4}}\sigma_m]$ and the resting $n - m$ eigenvalues lie in the band $\text{Re} \ z \in (-\sqrt{\frac{3}{4}}\Delta, -\max(\ell_{\text{est}}, \rho))$. Within the values of $a, b$ in the table, this result only applies to rows 6 and 7. For instance, for row 7, Part (2) of Theorem 1 yields that two closed-loop eigenvalues lie in the band $\text{Re} \ z \in [-1.4142, -0.8660]$ and two others in the band $\text{Re} \ z \in (-0.3811, -0.0198)$. Numerical simulation shows that two eigenvalues of $A - BF$ satisfy $\text{Re} \ \nu_{1,2} \approx -1.4024$ and two others satisfy $\text{Re} \ \nu_{3,4} \approx -0.1062$.

Simulation also shows that in many cases, the relative error in the estimate $\gamma_{\text{decay}} > \ell^1_{\text{est}}$, which Theorem 1 gives for $m = 1$, is less than in the corresponding estimate for $m > 1$. (On the other hand, the quality of the control increases with the increase of $m$).

6.2. A control problem for a mechanical problem. In many practical problems there is a large choice of possible physical or geometric configurations of the controller, which might make it necessary to solve a large amount of LQR optimization problems, in order to find a good one in some alternative sense. We will be speaking about the search of an LQR optimal regulator, which is also good in the sense that it has the largest possible $\gamma_{\text{decay}}$.

| $a$ | $b$ | $\|X\|$ | $\gamma_{\text{decay}}$ | $\Gamma_-$ | $\Gamma_+$ | $\ell_{\text{est}}$ | $\rho$ |
|-----|-----|---------|-------------------------|-----------|-----------|----------------|------|
| 1   | 15  | 45      | 1.0171                 | 0.9999    | 0.9870    | 1.0130         | 0.7040 | 0.0806 |
| 2   | 5   | 15      | 1.0537                 | 0.9988    | 0.8834    | 1.1166         | 0.6799 | 0.0806 |
| 3   | 1.8 | 5.4     | 1.1667                 | 0.9910    | 0.1006    | 1.8994         | 0.5403 | 0.0806 |
| 4   | 1   | 10      | 1.1031                 | 0.9960    | –         | 1.1439         | 0.3536 | 0.0806 |
| 5   | 4   | 4.1     | 1.1456                 | 0.9928    | –         | –              | 0.0018 | 0.0806 |
| 6   | 0.2 | 0.22    | 5.4750                 | 0.2199    | –         | –              | 7.07 · 10^{-5} | 0.0396 |
| 7   | 0.1 | 0.11    | 10.2326                | 0.1062    | –         | –              | 1.77 · 10^{-5} | 0.0198 |

Table 1. Numerical results and bounds for the LQR problems, $m = 2$. 
In this subsection, we propose an algorithm which allows one to reduce drastically this search, by making use of our theoretical estimates. We will illustrate this algorithm on a simple mechanical system (a very similar example has been considered in [18] in the presence of damping). The same algorithm, in fact, can be applied to the following general class of problems: to optimize $\gamma_{\text{decay}}$ among a large finite family of LQR problems $(A, B_j)$, with $A$ skew-Hermitian. In other words, the system matrix $A$ is supposed to be fixed, but there are several possible choices for the control matrix $B$.

This is not the only application of our bounds. We believe that in many cases the control designer can apply our results to obtain some a priori information on the systems in study.

Consider a one-dimensional massless string. Attached to the string are $N$ equal point masses of mass $M$, that are placed along it at equal distances $h$. It is assumed that the unperturbed string occupies the interval $[0, (N + 1)h]$ of the $x$ axis in an $xy$ plane; the string is supposed to move only in this plane. The two endpoints of the string are fixed, and it has constant tension $\tau > 0$.

The problem is to stabilize the string using $m$ controls, where $1 \leq m \leq N$. Namely, we choose point masses with numbers $j_1, j_2, \ldots, j_m$, where $1 \leq j_1 < j_2 < \cdots < j_m \leq N$, and apply a force $u_k$ to the $j_k$ point mass in the direction $y$. Every configuration $(j_1, j_2, \ldots, j_m)$ of controls leads to its own linear quadratic control problem and to a corresponding stable closed-loop system, which is optimal in the linear quadratic sense. However, the exponential decay rates of these closed-loop systems will depend on the chosen configurations of the control. The problem we discuss here is to find the configuration $(j_1, j_2, \ldots, j_m)$ which leads to the best exponential decay rate.

In the experiment, we have chosen the parameters $\tau/h = 10$, $M = 50$ and $N = 30$. We tried the values $m = 1, 2, 3, 4, 5, 8$. One can observe that $\gamma_{\text{decay}}$ depends much on the choice of the configuration $J = (j_1, j_2, \ldots, j_m)$ (these are the numbers of the masses to which the control forces are applied). For example, if $m = 5$, then the best value of $\gamma_{\text{decay}}$ equals to $8.87 \cdot 10^{-4}$, which is attained, for instance, for $J = (2, 5, 11, 19, 27)$, while for $J = (1, 2, 3, 29, 30)$ one only gets $\gamma_{\text{decay}} = 2.80 \cdot 10^{-4}$, which is several times less.

There are $(\binom{N}{m})$ configurations, and theoretically, the problem can be solved by a “brute force” complete search among all of them. However, even for moderate values of $N$ and $m$, solving numerically $(\binom{N}{m})$ LQR problems will be very time-consuming.

If the position of the $j$-th point mass is $(jh, y_j)$, we obtain (in the linear approximation) the following system of ODEs:

\[
\begin{align*}
My''_j &= -\frac{\tau}{h} (2y_j - y_{j-1} - y_{j+1}), & j &= 1, \ldots, N, \ j \neq j_k, \\
My''_j &= -\frac{2\tau}{h} (y_{j-1} + y_{j+1} - 2y_j) + u_k, & j &= j_k, \ k &= 1, \ldots, M, \\
y_0 &= y_{N+1} = 0,
\end{align*}
\]

where $y_0, y_{N+1}$ have only been introduced for convenience in the notation.

Put

\[
A_0 = \begin{pmatrix}
2 & -1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 2 & -1 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & 2 & -1 \\
0 & 0 & 0 & 0 & \cdots & -1 & 2
\end{pmatrix}_{N \times N}, \quad A = \left( -\frac{\tau}{hM} A_0 I \right)_{N \times N}, \quad x = \begin{pmatrix}
y_1 \\
y_N \\
\vdots \\
y_N^T \end{pmatrix},
\]

\[
B = \frac{1}{M} B_0, \quad B_0 = \begin{pmatrix} 0_{N \times m} \\
e_j, \ldots, e_j \end{pmatrix}_{2N \times m},
\]

where $e_j \in \mathbb{C}^N$ is the $j$-th (column) vector of the canonical basis. Then we obtain the control system

\[x' = Ax + Bu.\]
The energy of the system can be defined in terms of the following inner product in $\mathbb{C}^{2N}$:

$$\langle f, g \rangle_E = \frac{1}{N+1} \left[ \tau \sum_{k=1}^{N-1} (f_{k+1} - f_k)(\overline{g}_{k+1} - \overline{g}_k) + f_1 \overline{g}_1 + f_N \overline{g}_N + \sum_{k=1}^{N} f_{N+k} \overline{g}_{N+k} \right].$$

The energy is $E(x) = \frac{1}{2} \|x\|_E^2$. It is easy to show that energy is conserved, so that $A$ is skew-Hermitian with respect to this inner product.

Now we apply the Linear Quadratic Regulator using the cost functional

$$J^\gamma(x_0) = \int_0^\infty \|x(t)\|_E^2 + \|u(t)\|^2 dt$$

in order to stabilize the system.

We can do a theoretical study of the system to obtain expressions to compute our estimates. Notice that our string is a very particular case of a nonhomogeneous string, whose spectral theory comes back to M.G. Krein, see [14, Section 8 of Chapter VI]. In our case, the eigenvalues of $A$ are

$$i\lambda_k = -2i \sqrt{\frac{\tau}{hM}} \sin \left( \frac{k\pi}{2(N+1)} \right), \quad -N \leq k \leq N, k \neq 0,$$

and the corresponding orthonormal eigenvectors are $v_k$, where

$$v_k = \left( \overline{w}_k, w_k \right), \quad w_k = \left( \sin \left( \frac{kl\pi}{N+1} \right) \right)_{1 \leq l \leq N}.$$ 

See the paper [28] by Micu, where the same matrix $A$ appeared in the context of a semidiscrete numerical scheme for 1D wave equation. We also refer to [2], [30] for a related inverse problem.

The operator $\sqrt{N+1} B_0$ maps the canonical basis of $\mathbb{C}^m$ onto an orthonormal system of $m$ vectors in $\mathbb{C}^{2N}$ (we use the inner product $\langle \cdot, \cdot \rangle_E$ in $\mathbb{C}^{2N}$ and the standard one in $\mathbb{C}^m$). Hence, $\sqrt{N+1} B_0$ is an isometry and it follows that

$$\|B\| = \sigma_m(B) = \frac{1}{M\sqrt{N+1}}.$$

Finally, the vectors $b_k = B^* v_k$ can be computed to obtain

$$\|b_k\|^2 = \frac{1}{M^2(N+1)^2} \sum_{l=1}^{m} \sin^2 \left( \frac{jlk\pi}{N+1} \right), \quad -N \leq k \leq N, k \neq 0.$$

Using Corollary [11], we can give an upper bound for $\gamma_{\text{decay}}$, assuming that some $\varphi_k < 1$. Theorem [1] and Corollary [11] (if it applies) can be used to obtain a lower bound for $\gamma_{\text{decay}}$. The following algorithm uses these bounds to reduce the number of LQR problems being computed. In the course of its execution, the upper and the lower theoretical bounds for all configurations are taken into account, but the LQ optimal regulator is actually computed for a fewer number of configurations.

The algorithm works as follows:

1. Calculate the eigenvalues $i\lambda_k$ and the corresponding eigenvectors $v_k$ of $A$.
2. For each control configuration $\mathcal{J} = (j_1, \ldots, j_m)$, compute the vectors $b_k$ and the quantities $U_\mathcal{J}$ and $L_\mathcal{J}$, which are the upper and the lower theoretical bounds for $\gamma_{\text{decay}}$. Set $U_\mathcal{J} = +\infty$ if an upper bound is not available.
3. Select the configuration $\mathcal{J}_0$ having the maximal $L_\mathcal{J}$. Solve the LQR problem numerically for this configuration and compute $\gamma_{\text{decay}}$.
4. Now we proceed to a search, defined recursively as follows. Let $\gamma$ be the best $\gamma_{\text{decay}}$ found so far. If for all configurations $\mathcal{J}$ whose corresponding $\gamma_{\text{decay}}$ has not been computed yet, $U_\mathcal{J} < \gamma$, the search stops, and this current value of $\gamma$ is taken for the optimal $\gamma_{\text{decay}}$. If there are configurations $\mathcal{J}$ whose $\gamma_{\text{decay}}$ has not been computed that have $U_\mathcal{J} \geq \gamma$, the algorithm selects the one having the greatest $U_\mathcal{J}$. For this configuration, it solves the LQR problem numerically, computes its $\gamma_{\text{decay}}$ and updates $\gamma$ according to the rule $\gamma := \max(\gamma, \gamma_{\text{decay}})$. This is the best $\gamma_{\text{decay}}$ found so far.
| $m$ | $\gamma_{\text{decay}}$ | Time (s) | LQRs computed | % computed |
|-----|----------------|-------------|----------------|------------|
| 1   | $6.53 \cdot 10^{-5}$ | 2           | 30             | 100        |
| 2   | $3.39 \cdot 10^{-4}$ | 4           | 76             | 17.5       |
| 3   | $5.82 \cdot 10^{-4}$ | 28          | 441            | 10.9       |
| 4   | $7.70 \cdot 10^{-4}$ | 123         | 481            | 1.76       |
| 5   | $8.87 \cdot 10^{-4}$ | 738         | 5505           | 3.86       |
| 8   | $1.2 \cdot 10^{-3}$ | 56675       | 198369         | 3.39       |

Table 2. Results for $N = 30$, $\tau/h = 10$, $M = 50$. 

(5) The algorithm stops after having exhausted all possible configurations. It returns the last value of $\gamma$, which is equal to the maximum of the values of $\gamma_{\text{decay}}$ over all possible configurations.

Observe that this algorithm also allows one to compute all the configurations having the optimal $\gamma_{\text{decay}}$.

The results of the execution of the algorithm are shown on Table 2. The computations were done on a modern desktop computer. Recall that we have chosen the total number of masses $N = 30$. The table shows that the decay rate $\gamma_{\text{decay}}$ improves when $m$ increases. The fourth column collects the number of LQRs the algorithm had to solve, and the fifth column shows the ratio between the total of $\binom{N}{n}$ possible configurations and the number of configurations that were actually processed. One can see that in many cases, our algorithm reduces drastically the amount of computations.

The values $\tau/h = 10$ and $M = 50$ have been chosen for these computations because they provide a moderate separation of the spectrum of $A$ with respect to $\|B\|$. If we fix $M = 50$ and increase $\tau/h$ (say $\tau/h = 1000$), then the number of computations is further reduced, since the separation of the spectrum of $A$ increases and we obtain tighter theoretical bounds. On the other hand, if one sets $\tau/h$ to a small enough value while maintaining $M$ fixed, our algorithm will not provide much save in the computations.

7. Some open questions

**Question 1.** Assume that $m < n$, $R = I$, $Q = I$ and that a skew-Hermitian matrix $A$ is fixed. Does it follow that there is a constant $C = C(A)$ such that $\gamma_{\text{decay}} \leq C$, independently of $B$? As we already mentioned in Corollary 8, it is true if $m = 1$, with $C(A) = 2\sqrt{2}\Delta$. More generally, part (2) of Theorem 1 shows that it is also true if, for instance, $\sigma_m(B) \geq \frac{1}{2}\|B\|$, or even if we assume that $\sigma_m(B) \geq f(\|B\|)$, where $f$ is any function on $[0, +\infty)$ such that $\lim_{x \to \infty} f(x) = +\infty$. We conjecture that it is true in general.

**Question 2.** We can pose a somewhat related question concerning the general pole placement problem for a general complex matrix $A$. Suppose that $m < n$, and let $\gamma_{\text{decay}}$ denote the decay rate of the matrix of a stable closed loop system $A_{cl\text{-}loop} = A - BF$, which is obtained by (an arbitrary) state space control $u(t) = -Fx(t)$. Can one assert that the cost matrix $X_0 = f_0^\infty \exp(A_{cl\text{-}loop}^* \exp(A_{cl\text{-}loop}) \exp(A_{cl\text{-}loop}) \exp(A_{cl\text{-}loop}) dt$ is large every time when $\gamma_{\text{decay}}$ is large? We conjecture that it is so. Then, it would be interesting to find an explicit function $G(x)$ (which may depend only on $n, m, A$), that goes to infinity as $x \to +\infty$ and satisfies $\|X_0\| \geq G(\gamma_{\text{decay}})$ for all $B, F$ such that $A_{cl\text{-}loop}$ is stable. A weaker version of this question is whether there is such function $G$ that may depend on both $A$ and $B$.

**Question 3.** Corollary 8 can be used to obtain an upper bound for $\gamma_{\text{decay}}$. However, if $\varphi_k < 1$ for some $k$, then Corollary 11 gives a much tighter bound. Can one give a tighter upper bound even when $\varphi_k \geq 1$ for every $k$?
8. Conclusions

- The bounds $\ell_{\text{est}}, \ell^1_{\text{est}}$ given in Theorem 1 can be applied only if all the eigenvalues of $A$ are different.
- The lower bound $\rho$ given in Theorem 14 is the one which can be used in a more general setting (namely $\Delta_m > 0$, which allows some eigenvalues of $A$ to coincide). There are cases when it is the best bound available. It happens, in particular, if some eigenvalues of $A$ are close together (compared with $\|B\|$).
- The two-sided bound given in Corollary 11 holds only when $\varphi_k < 1$ for all $k$, i.e., when the spectrum of $A$ is separated enough.
- If all $\varphi_k$ are small, this two-sided bound is very tight and one can take $d_0 = \min \|b_k\|$ as a good approximation for $\gamma_{\text{decay}}$.
- When all $\varphi_k$ are small, one can also use Theorem 10 to locate with precision all the eigenvalues of the closed-loop system.
- Corollary 8 shows that if $m < n$ and the diameter $\Delta$ of the spectrum of $A$ is much smaller than all singular values of $B$, then $\gamma_{\text{decay}}$ is less than $\sqrt{3} \Delta$.
- One can observe that, as a rule, if the separation of the eigenvalues of $A$ increases or the number of controls $m$ increases, then $\gamma_{\text{decay}}$ grows.
- If one has to find an optimal $\gamma_{\text{decay}}$ among a large finite family of LQR control problems, our estimates permit one to design an algorithm to reduce the search (in some situations, drastically; see Subsection 6.2).
- By now, we only have estimates of $\gamma_{\text{decay}}$ for the case of a skew-Hermitian matrix $A$. It would be very desirable to give good estimates of $\gamma_{\text{decay}}$ and $\|X\|$ for non-skew Hermitian matrices, or at least for the case of matrices $A$ such that $\text{Re} A \geq 0$. Another interesting subclass are normal matrices $A$, for which some modifications of our methods could apply. This can also be interesting for the stabilization method we mentioned in Remark 2 after Theorem 2.

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References

[1] B.D.O. Anderson, J.B. Moore, Optimal Control: Linear Quadratic Methods, Prentice-Hall, 1989.
[2] M.I. Belishev, M.V. Putov, A finite-dimensional inverse spectral problem for a pencil of Hermitian quadratic forms. (Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 186 (1990), Mat. Vopr. Teor. Rasprostr. Voln. 20, 32–36, 180–181; translation in J. Math. Sci. 73 (1995), no. 3, 317–319
[3] D. Boley and W-S. Lu, Measuring how far a controllable system is from an uncontrollable one, IEEE Transactions on Automatic Control, 31:3 (1986), 249–251.
[4] F. Borrelli, T. Keviczky, Distributed LQR Design for Identical Dynamically Decoupled Systems, IEEE Transactions on Automatic Control, 53:8 (2008), 1901–1912.
[5] J.A. Burns, E.W. Sachs, L. Zietsman, Mesh independence of Kleinman-Newton iterations for Riccati equations in Hilbert space. SIAM J. Control Optim. 47:5 (2008), 2663–2692.
[6] M.T. Chu, Inverse eigenvalue problems, SIAM Rev. 40 (1998), no. 1, 1–39.
[7] R.F. Curtain, G. Weiss, Exponential stabilization of well-posed systems by colocated feedback, SIAM J. Control Optim. 45 (2006), no. 1, 273–297.
[8] R. Davies, P. Shi, R. Wiltshire, New lower solution bounds of the continuous algebraic Riccati matrix equation, Linear Algebra and its Applications, 427 (2007), 242–255.
[9] R. Davies, P. Shi, R. Wiltshire, New upper matrix bounds for the solution of the continuous algebraic Riccati matrix equation, International Journal of Control, Automation, and Systems, 6:5 (2008), 776–784.
[10] B.N. Datta, Numerical methods for linear control systems. Design and analysis, Elsevier Academic Press, 2004. 695 pp.
[11] J. Demmel, A lower bound on the distance to the nearest uncontrollable system, Technical Report No. 272, Courant Inst., New York Univ., New York, 1987.
[12] R. Eising, Between controllable and uncontrollable, Systems Control Lett. 4:5 (1984), 263–264.
[13] P. Gahinet, A. J. Laub, Algebraic Riccati equations and the distance to the nearest uncontrollable pair, SIAM J. Control Optim. 30:4 (1992), 765–786.
[14] I. Gohberg, M. Krein, Theory and Applications of Volterra Operators in Hilbert Space, Amer. Math. Soc., Providence, R.I., 1970.
[15] I. C. Gohberg, E. L. Sigal, An operator generalization of the logarithmic residue theorem and the theorem of Rouché, Mat. Sbornik 84:126 (1971), 607–629 (Russian), Engl. Transl., Math USSR Sbornik 13 (1971), 603–625.
[16] C. He, On the distance to uncontrollability and the distance to instability and their relation to some condition numbers in control, Numer. Math. 76 (1997), no. 4, 463-477.
[17] C. He, A. J. Laub, V. Mehrmann, Placing plenty of poles is pretty preposterous, Preprint SPC 95-17, Forschungsgruppe “Scientific Parallel Computing”, Fakultät für Mathematik, TU Chemnitz-Zwickau, 30 pp.
[18] J. J. Hench, C. He, V. Kučera, V. Mehrmann, Dampening controllers via a Riccati equation approach, IEEE Trans. Automat. Control 43 (1998), no. 9, 1280–1284.
[19] D. Hinrichsen, A. J. Pritchard, Stability radius for structured perturbations and the algebraic Riccati equation, Systems Control Lett. 8:2 (1986), 105–113.
[20] M. Karow and D. Kressner, On the structured distance to uncontrollability, Systems Control Lett. 58:2 (2009), 128–132.
[21] N. Komaroff, Simultaneous eigenvalue lower bounds for the Riccati matrix equation, IEEE Transactions on Automatic Control, 34 (1989), 175–177.
[22] W. H. Kwon, Y. S. Moon and S. C. Ahn, Bounds in algebraic Riccati equations: a survey on some new results, International Journal of Control, 64:3 (1996), 377–389.
[23] P. Lancaster, L. Rodman, Algebraic Riccati equations, Oxford University Press, 1995.
[24] C-H. Lee, Solution bounds of the continuous Riccati matrix equation, IEEE Transactions on Automatic Control, 48:8 (2003), 1409–1413.
[25] C-H. Lee, New upper solution bounds of the continuous algebraic Riccati matrix equation, IEEE Transactions on Automatic Control, 51:2 (2006), 330–334.
[26] J. Liu, J. Zhang and Y. Liu, A new upper bound for the eigenvalues of the continuous algebraic Riccati equation, Electronic Journal of Linear Algebra, 20 (2010), 314–321.
[27] V. Mehrmann, H. Xu, Numerical methods in control, J. Comput. Appl. Math. 123 (2000), no. 1-2, 371–394.
[28] S. Micu, Uniform boundary controllability of a semi-discrete 1-D wave equation. Numer. Math. 91 (2002), no. 4, 723–768.
[29] K. M. Mikkola, State-feedback stabilization of well-posed linear systems. Integral Equations Operator Theory 55 (2006), no. 2, 249–271.
[30] M. Yu. Mitrofanov, O. A. Tarakanov, A discrete string that models a musical scale. (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 270 (2000), Issled. po Linein. Oper. i Teor. Funkts. 28, 242–252.
[31] Y. Monden, S. Arimoto, Generalized Rouché’s theorem and its application to multivariate autoregressions. IEEE Trans. Acoust. Speech Signal Process. 28:6 (1980), 733–738.
[32] T. Mori, On some bounds in the algebraic Riccati and Lyapunov equations, IEEE Transactions on Automatic Control, 30 (1985), 162–164.
[33] T. Mori and I. A. Derese, A brief summary of the bounds on the solution of the algebraic matrix equations in control theory, International Journal of Control, 39:2 (1984), 247–256.
[34] P. Pandey, C. Kenney, A. J. Laub, Solving the algebraic Riccati equation on supercomputers. Recent advances in mathematical theory of systems, control, networks and signal processing, II (Kobe, 1991), 3–8, Mita, Tokyo, 1992.
[35] M. N. Safonov, M. Athans, Gain and phase margin for multiloop LQG regulators, IEEE Transactions on Automatic Control, 22:2 (1977), 173–179.
[36] V. Simon, Algorithms for linear-quadratic optimization, Chapman and Hall/CRC, March 1996.
[37] N. K. Son, D. D. Thuan, The structured distance to uncontrollability under multi-perturbations, Systems Control Lett. 59:10 (2010), 676–483.
[38] C. F. van Loan, How near is a stable matrix to an unstable matrix? in: Linear algebra and its role in systems theory (Brunswick, Maine, 1984), Contemp. Math., vol. 47, Amer. Math. Soc., Providence, RI, 1985, pp. 465-478.
[39] J. Willems, Least squares stationary optimal control and the algebraic Riccati equation, IEEE Transactions on Automatic Control, 16:6 (1971), 621–624.
[40] K. Yasuda, K. Hirai, Upper and lower bounds on the solution of the algebraic Riccati equation, IEEE Transactions on Automatic Control, 24 (1979), 483–487.
[41] K. Zhou, J. C. Doyle, K. Glover, Robust and optimal control, Prentice Hall, 1996.
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