On the Metric Dimension of Imprimitive Distance-Regular Graphs

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Abstract. A resolving set for a graph $\Gamma$ is a collection of vertices $S$, chosen so that for each vertex $v$, the list of distances from $v$ to the members of $S$ uniquely specifies $v$. The metric dimension of $\Gamma$ is the smallest size of a resolving set for $\Gamma$. Much attention has been paid to the metric dimension of distance-regular graphs. Work of Babai from the early 1980s yields general bounds on the metric dimension of primitive distance-regular graphs in terms of their parameters. We show how the metric dimension of an imprimitive distance-regular graph can be related to that of its halved and folded graphs. We also consider infinite families (including Taylor graphs and the incidence graphs of certain symmetric designs) where more precise results are possible.

Keywords: metric dimension, resolving set, distance-regular graph, imprimitive, halved graph, folded graph, bipartite double, Taylor graph, incidence graph

1. Introduction

Let $\Gamma = (V, E)$ be a finite, undirected graph without loops or multiple edges. For $u, v \in V$, the distance from $u$ to $v$ is the least number of edges in a path from $u$ to $v$, and is denoted $d_{\Gamma}(u, v)$ (or simply $d(u, v)$ if $\Gamma$ is clear from the context).

A resolving set for a graph $\Gamma = (V, E)$ is a set of vertices $R = \{v_1, \ldots, v_k\}$ such that for each vertex $w \in V$, the list of distances $(d(w, v_1), \ldots, d(w, v_k))$ uniquely determines $w$. Equivalently, $R$ is a resolving set for $\Gamma$ if, for any pair of vertices $u, w \in V$, there exists $v_i \in R$ such that $d(u, v_i) \neq d(w, v_i)$; we say that $v_i$ resolves $u$ and $w$. The metric dimension of $\Gamma$ is the smallest size of a resolving set for $\Gamma$. This concept was introduced to the graph theory literature in the 1970s by Harary and Melter [28] and, independently, Slater [36]; however, in the context of arbitrary metric spaces, the concept dates back at least as far as the 1950s (see Blumenthal [11], for instance). For further details, the reader is referred to the survey [7].
When studying metric dimension, distance-regular graphs are a natural class of graphs to consider. A graph $\Gamma$ with diameter $d$ is distance-regular if, for all $i$ with $0 \leq i \leq d$ and any vertices $u, v$ with $d(u, v) = i$, the number of neighbours of $v$ at distances $i - 1$, $i$, and $i + 1$ from $u$ depend only on the distance $i$, and not on the choices of $u$ and $v$. These numbers are denoted by $c_i, a_i$, and $b_i$ respectively, and are known as the parameters of $\Gamma$. It is easy to see that $c_0$, $b_d$ are undefined, $a_0 = 0$, $c_1 = 1$ and $c_1 + a_1 + b_1 = k$ (where $k$ is the valency of $\Gamma$). We put the parameters into an array, called the intersection array of $\Gamma$,

$$
\begin{pmatrix}
* & 1 & c_2 & \cdots & c_{d-1} & c_d \\
0 & a_1 & a_2 & \cdots & a_{d-1} & a_d \\
k & b_1 & b_2 & \cdots & b_{d-1} & *
\end{pmatrix}.
$$

In the case where $\Gamma$ has diameter 2, we have a strongly regular graph, and the intersection array may be determined from the number of vertices $n$, valency $k$, and the parameters $a = a_1$ and $c = c_2$; in this case, we say $(n, k, a, c)$ are the parameters of the strongly regular graph. Another important special case of distance-regular graphs are the distance-transitive graphs, i.e., those graphs $\Gamma$ with the property that for any vertices $u, v, u', v'$ such that $d(u, v) = d(u', v')$, there exists an automorphism $g$ such that $u^g = u'$ and $v^g = v'$. For more information about distance-regular graphs, see the book of Brouwer, Cohen, and Neumaier [12] and the recent survey by van Dam, Koolen, and Tanaka [16]. In recent years, a number of papers have been written on the subject of the metric dimension of distance-regular graphs (and on the related problem of class dimension of association schemes), by the present author and others: see [5–8, 10, 14, 17, 22–26, 29, 34], for instance. In this paper, we shall focus on various classes of imprimitive distance-regular graphs, which are explained below.

1.1. Primitive and Imprimitive Graphs

A distance regular graph $\Gamma$ with diameter $d$ is primitive if and only if each of its distance-$i$ graphs (for $0 < i \leq d$) is connected, and is imprimitive otherwise. For $d = 2$, i.e., strongly regular graphs, the only imprimitive examples are the complete multipartite graphs $sK_t$ (with $s > 1$ parts of size $t$). For valency $k \geq 3$, a result known as Smith’s Theorem (after Smith, who proved it for the distance-transitive case [37]) states that there are two ways for a distance-regular graph to be imprimitive: either the graph is bipartite, or is antipodal. The latter case arises when the distance-$d$ graph consists of a disjoint union of cliques, so that the relation of being at distance 0 or $d$ in $\Gamma$ is an equivalence relation on the vertex set. The vertices of these cliques are referred to as antipodal classes; if the antipodal classes have size $t$, then we say that $\Gamma$ is $t$-antipodal. It is possible for a graph to be both bipartite and antipodal, with the hypercubes providing straightforward examples.

If $\Gamma$ is a bipartite distance-regular graph, the distance-2 graph has two connected components; these components are called the halved graphs of $\Gamma$. If $\Gamma$ is $t$-antipodal, the folded graph, denoted $\bar{\Gamma}$, of $\Gamma$ is defined as having the antipodal classes of $\Gamma$ as vertices, with two classes being adjacent in $\bar{\Gamma}$ if and only if they contain adjacent vertices in $\Gamma$. The folded graph $\bar{\Gamma}$ is also known as an antipodal quotient of $\Gamma$; conversely, $\Gamma$ is an antipodal $t$-cover of $\bar{\Gamma}$. We note that $\Gamma$ and $\bar{\Gamma}$ have equal valency; a