A Coordinate System for Graphs

A New and Efficient Framework for the Graph Isomorphism Problem

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Abstract
In this paper, a function on any pair of graphs is defined whose properties are similar to the properties of dot product in vector space. This function enables us to define graph orthogonality and, also, a new metric on isomorphism classes of \( n \)-vertex graphs. Using dot product of graphs, a coordinate system for graphs is provided which benefits us in graph isomorphism and related problems.

Keywords: graph isomorphism problem; coordinate system of graphs; graph metric; orthogonal graphs; graph dot product.

1 Introduction
Graph isomorphism problem and its derivatives, such as graph matching [3] (in pattern recognition) and graph similarity problem have numerous applications in many areas such as biology, chemistry [11], pattern recognition[3] and web structure mining [2, 4]. Graph matching has been the topic of many studies in computer science over the last decades. In graph matching problem, the goal is to find maximum corresponding regions in the given graphs. The graph isomorphism is in fact an exact graph matching. In the graph similarity problem, the main interest is to assign an overall similarity score to indicate the level of similarity between two graphs [15]. Maximum common subgraph [1], edit distance methods [8] and measuring distance based
on some operations [7] are some approaches to compare the similarity of two graphs. Another approach is using graph kernels [9, 13].
Solving the graph isomorphism problem and related problems is entirely based on our perception to graph structure. Therefore, the main step towards an effective solution for these problems is finding a good framework to represent graphs.

This paper provides a structural representation for graphs. The set of graphs is equipped by a dot product which provides us with a graph coordinate representation. Furthermore, the defined dot product donates a new metric to graph space and, also, introduces graph orthogonality. These facilities provides us with a better intuition about graph structure which benefits us in graph isomorphism problem or graph matching in pattern recognition.

Representing a graph by its adjacency matrix, completely, depends on the vertex ordering. Hence, a graph finds different presentations, due to different reordering of its vertices. This fact stimulates us to ask whether it is possible to have a conceptual description for graphs which is unique for all isomorphic graphs. Such representation for graphs should be based on the graph structure instead of defining the edges states which, extremely, depends on vertices ordering. A complete set of graph invariants is a conceptual description for graphs. Also, graph spectrum is, roughly, what we want, but it does not specify each graph isomorphism class, uniquely, and the relation of structure of a graph and its spectrum is not very clear.

A real smooth function $F : (a, b) \to R$ can be represented by defining the value of $F$ on any point of the domain. Another possible representation for $F$ is defining it in terms of the basis functions, such as $\sin(.)$ and $\cos(.)$. However, it is not always easy to compute the Fourier series presentation, but it provides facilities which makes some complicated problems trivial. Fourier series representation offers a conceptual and structural view to functions which is the base of some technologies such as optics, telecommunications and mechanics engineering (vibration).

The representation of a function in terms of the basis functions enables us to be more strong in dealing with functions, either in theory or applications. Can we develop a similar tool for graphs? The first questions which, naturally, arise are: How can we define a basis for graphs? How can we measure how much a graph is close to a basis element? In this paper, we try to find an answer for these questions and to define a coordinate representation for graphs. In the second section, a function is defined on any pair of $n$-vertex graph. The properties of this function on graphs resemble to properties of the
dot product in vector space. Using this function, the orthogonality is defined for graphs which reveals the different structure of two graphs. In the third section, a new metric is defined on isomorphism classes of $n$-vertex graphs. In Section 4, a coordinates system is defined for graphs which benefit us in the graph isomorphism and related problems.

## 2 Dot product

The set of graphs is not a Hilbert space to have a dot product. But, we need something similar to dot product which enables us to define a basis for graphs and, also, to measure how much a graph is close to a basis element. The idea of kernel function embeds the set of graphs in a larger Hilbert space which is equipped with a dot product [14, 9, 13]. Here, we want to define something similar to a dot product on any pairs of $n$-vertex graphs, directly.

We wish this dot product measures the structural resemblance of any two graphs.

**Remark 1.** In this paper, the matrix representation of a graph $G$ is an $n \times n$ matrix $A_G$ in which there exists $+1$ in $(i, j)$ entry when $v_i$ and $v_j$ are adjacent and $-1$ otherwise $(i \neq j)$. Diagonal elements are zero.

Please note that according to this matrix representation for graphs, we have $A_G^T = -A_G$. The trace and transpose of a matrix $A$ is denoted by $tr(A)$ and $A^T$, respectively.

Before defining dot product of graphs, the scaler product for graphs is defined.

**Definition 1.** Let $G$ be a simple graph and $r \in R$. $rG$ is a weighted graph, where the weight $e_{u,v}$ is $+r$ for adjacent $\{u, v\}$ in $G$ and is $-r$, otherwise. The matrix representation of $rG$ is $rA_G$ where $A_G$ is the matrix representation of the graph $G$.

**Definition 2.** Let $A_G$, $A_H$ be the matrix representations of two $n$-vertex graphs $G$ and $H$, respectively. We define $G.H := \max_P(\text{tr}(A_G P A_H P^T))$

Where $P$ is a permutation matrix. Let $\text{Phase}(G, H)$ be the number of permutation matrices $P$ such that $G.H = \text{tr}(A_G P A_H P^T)$. 

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Clearly, we have $1 \leq \text{Phase}(G, H) \leq n!$.

The above dot product is defined in a natural way. A graph is permuted to
other graph as long as the best placement, maximum number of edge on edge
and not edge on not edge assignment, is found. Two graphs match exactly,
if they are isomorphic. We emphasize that this dot product is not exactly
a real dot product. But, it is a function on graphs similar to a dot product
with desired properties. For instance, it provides a metric on graphs. The
following properties are resulted directly from the definition.

**Lemma 1.** Let $G$ and $H$ be two arbitrary graphs on $n$ vertices and $\|G\|^2 = 
G.G$. We have,

a) $G.H = H.G$.

b) $rG.H = G.rH = r(G.H)$ for any $r \in R$.

c) $G.H = \overline{G.H}$.

d) $G.\overline{H} = \overline{G.H}$.

e) $\frac{G}{\|G\|} : \frac{H}{\|H\|} = 1$ if and only if $G \cong H$ .

*Proof.* Let $A_G$ and $A_H$ be the representation matrices of graphs $G$ and $H$,
respectively.

a) Since $\text{tr}(A) = \text{tr}(A^T)$ and $\text{tr}(AB) = \text{tr}(BA)$, we have
$\text{tr}(A_GP_AH^TP^T) = \text{tr}((A_GP_AH^TP^T)^T) = \text{tr}(PA_HP^T A_G) = \text{tr}(A_HP^T A_G P)$

b) $\text{tr}((rA_G)PA_H^TP^T) = \text{tr}(A_GP(rA_H)^P_P^T) = r.\text{tr}(A_GPA_H^TP^T)$.

c) $\text{tr}(A_GPA_H^TP^T) = \text{tr}((-A_G)P(-A_H)^P^T)$

d) $\text{tr}(A_GP(-A_H^P^T)) = \text{tr}((-A_G)PA_H^TP^T)$

e) If $G \cong H$, there exists a permutation matrix $P$ such that $A_G = PA_H^TP^T$.

Thus, $\frac{G}{\|G\|} : \frac{H}{\|H\|} = \frac{1}{\|G\|\|H\|}\text{max}_P\text{tr}(A_G^TPA_H^TP^T) = \frac{\|G\|\|H\|}{\|G\|\|H\|}\text{max}_P\text{tr}(A_GA_G) = 1$.

If $\frac{G}{\|G\|} : \frac{H}{\|H\|} = 1$, then $\text{max}_P(\text{tr}(A_GPA_H^TP^T)) = \|G\|.\|H\| = \text{tr}(A_G^2)$. Thus,
$2\text{max}_P\text{tr}(A_G^2PA_H^TP^T) = \text{tr}(A_G^2) + \text{tr}(A_H^2)$. Consequently,

$\text{max}_P(\text{tr}(A_G^2 + A_H^2 - 2PA_H^TP^T)) = \text{max}_P(\text{tr}(A_G - PA_H^TP^T)^2) = 0$.

Since $A_G - PA_H^TP^T = (A_G - PA_H^TP^T)^T$, we have

$\text{max}_P(\text{tr}(A_G - PA_H^TP^T)(A_G - PA_H^TP^T)^T) = 0$

We know that $\text{tr}(AA^T) = 0$ if and only if $A$ is a zero matrix. Thus,
$A_G - PA_H^TP^T = 0$. It means that graph $G$ is isomorphic to graph $H$.

$\square$
Definition 3. We define the normalized dot product of two graphs $G$ and $H$ as $G \cdot H = \frac{G \cdot H}{\|G\| \cdot \|H\|}$.

Clearly, we have $-1 \leq G \cdot H \leq 1$. We saw that $G \cdot H = 1$ if and only if $G$ and $H$ are isomorphic. The normalized dot product of graphs on 4 vertices are shown in Table 1.

|     | 1   | 2/3 | 1/3 | 0   | 0   | -1/3 | -1/3 | -2/3 | -1 |
|-----|-----|-----|-----|-----|-----|------|------|------|----|
| 1/3 | 2/3 | 1   | 2/3 | 1/3 | 0   | 2/3  | 1/3  | 0    | -1/3|
| 0   | 1/3 | 2/3 | 1   | 0   | 1/3 | 1/3  | 2/3  | 0    | 1/3 |
| 1/3 | 0   | 1/3 | 0   | 1   | 0   | 2/3  | 1/3  | 1/3  | 0   |
| 0   | 1/3 | 0   | 1/3 | 0   | 1   | 1/3  | 2/3  | 0    | 1/3 |
| 0   | 1/3 | 2/3 | 1/3 | 2/3 | 1/3 | 1    | 2/3  | 1/3  | 0   |
| -1/3| 0   | 1/3 | 2/3 | 1/3 | 2/3 | 1    | 1/3  | 2/3  | 1/3 |
| -1/3| 0   | 1/3 | 0   | 1/3 | 0   | 2/3  | 1/3  | 1    | 2/3 |
| -2/3| -1/3| 0   | 1/3 | 0   | 1/3 | 1/3  | 2/3  | 2/3  | 1   |
| -1  | -2/3| -1/3| 0   | -1/3| 0   | 1/3  | 1/3  | 2/3  | 1   |

Table 1: The normalized dot product of 4-vertex graphs

Now, we define graph orthogonality. This new concept offers a new perception about graph structure. The study of orthogonal graphs seems to be essential to make our understanding complete about graph structure.

Definition 4. Two graphs $G$ and $H$ are orthogonal, if $G \cdot H = G \cdot \overline{H} = 0$.

For $n \neq 4k, 4k + 1$, it is not possible to have $G \cdot H = 0$ and the minimum possible value for $|G \cdot H|$ is 1. We call two matrices are quasi-orthogonal, if for two graphs $G$ and $H$ with $n \neq 4k, 4k + 1$, we have $G \cdot H = G \cdot \overline{H} = \pm 1$. If $n = 4k, 4k + 1$, the value of zero for dot product of two graphs is possible. Trying to put two orthogonal graphs on another, at most half of the edge to edge assignments are successful (edge on edge and not edge on not edge).
Two orthogonal graphs are shown in Fig. 1. According to the following lemma, two graphs are orthogonal if and only if $tr(A_GPA_HP^T)$ is invariant to the selection of matrix $P$ and it, constantly, equals to zero. Orthogonality of two graphs reveals perfect different structure of them.

![Figure 1: A pair of orthogonal graphs](image)

**Lemma 2.** If $G$ and $H$ are orthogonal, then $\text{Phase}(G, H) = n!$.

**Proof:** According to definition, if $G.H = G.H = 0$, then $\max_P(A_GPA_HP^T) = \max_P(-A_GPA_HP^T) = 0$. Thus, for any permutation matrix $P$, we have

$$A_GPA_HP^T \leq 0 \quad \text{and} \quad -A_GPA_HP^T \leq 0$$

It follows that $A_GPA_HP^T = 0$ for any permutation matrix $P$. □

### 3 A metric space for graphs

Some different distances are defined on the set of isomorphism classes of graphs which only some of them are metrics. For instances, the distances defined in [15, 1] are metrics which are based on maximum common subgraph. In [12], a metric is defined for the cut of graphs. Here, a new metric on the set of $n$-vertex graphs is introduced.

**Definition 5.** We define for any two $n$-vertex graphs $G$ and $H$.

$$d(G, H) := \|G\|^2 + \|H\|^2 - 2G.H$$

where $\|G\| = \sqrt{G.G}$.

**Theorem 2.** $d$ is a metric on the set of isomorphism classes of $n$-vertex graphs.
Proof. We should check the following properties hold true for any graphs $G, G_1, G_2$ and $H$

1. $d(G, H) \geq 0$

2. $d(G, H) = 0$ if and only if $G \cong H$

3. $d(G, G_2) \leq d(G, H) + d(G_2, H)$

Please note that

\[
d(G, H) = \|G\|^2 + \|H\|^2 - 2G.H = \|G\|^2 + \|H\|^2 - 2\max_P(\tr(AGPA_HP^T))
\]

\[
= \min_P\{\tr(A_G^2 + A_H^2 - 2AGPA_HP^T)\} = \min_P\{\tr(A_G^2 + A_H^2 - AGPA_HP^T - PA_HP^T A_G)\}
\]

Thus,

\[
d(G, H) = \min_P\{\tr(A_G - PA_HP^T)^2\}
\]

(1) We know that $\tr(AA^T) \geq 0$ for any real matrix $A$. Since $d(G, H) = \min_P\{\tr((A_G - PA_HP^T)(A_G - PA_HP^T)^T)\}$, we have $d(G, H) \geq 0$.

(2) We know that $\tr(AA^T) = 0$, if and only if $A$ is a zero matrix. Thus, $d(G, H) = 0$, if and only if $A_G - PA_HP^T$ is a zero matrix. It means that there exists a permutation matrix $P$ such that $A_G = PA_HP^T$, i.e. $G$ is isomorphic to $H$.

(3) Let $P$ and $Q$ be the permutation matrices such that $H.G_1 = tr(A_HPA_GP^T)$ and $H.G_2 = tr(A_HQA_GQ^T)$. The possible values for the entries of matrix $C = A_H - PA_GP^T$ are $2, -2$ and $0$. We have $c_{ij} = 0$, if $(A_H)_{ij} = (PA_GP^T)_{ij}$. If $(A_H)_{ij} \neq (PA_GP^T)_{ij}$, then $c_{ij} = -2$ or $2$. Thus,

\[
\frac{1}{4} \sum_{ij} c_{ij}^2 = \frac{1}{4} \tr(CC^T) = \frac{1}{4} \tr((A_H - PA_GP^T)^2)
\]

indicates the number of non-zeros entries of $C = A_H - PA_GP^T$, i.e. the number of entries which $A_H$ is different from $PA_GP^T$. Also, $\frac{1}{4} \tr((A_H - QA_GQ^T)^2)$ indicates the number of entries which $A_H$ is different from $QA_GQ^T$.

Clearly, if $(A_H)_{ij} = (PA_GP^T)_{ij}$ and $(A_H)_{ij} = (QA_GQ^T)_{ij}$, then $(PA_GP^T)_{ij} = (QA_GQ^T)_{ij}$. In opposite, we have $(PA_GP^T)_{ij} \neq (QA_GQ^T)_{ij}$, if $(PA_GP^T)_{ij} \neq (A_H)_{ij}$ or $(QA_GQ^T)_{ij} \neq (A_H)_{ij}$. Thus, the number of entries of $PA_GP^T$
which are different from $QA_{G_2}Q^T$ is at most $\frac{1}{4}tr((A_H - PA_G)^2) + \frac{1}{4}tr((A_H - QA_{G_2}Q^T)^2)$. In other words,

$$\frac{1}{4}tr((PA_G P^T - QA_{G_2}Q^T)^2) < \frac{1}{4}tr((A_H - PA_G)^2) + \frac{1}{4}tr((A_H - QA_{G_2}Q^T)^2)$$

$$tr(PA_G P^T - QA_{G_2}Q^T)^2 \leq tr(A_H - PA_G)^2 + tr(A_H - QA_{G_2}Q^T)^2$$

By manipulating,

$$tr((A_G - P^TQA_{G_2}Q^TP)^2) \leq tr((A_H - PA_G)^2) + tr((A_H - QA_{G_2}Q^T)^2)$$

Clearly, $P^TQ$ is a permutation matrix. Substituting $P^TQ$ by $S$, we have

$$tr((A_G - SQA_{G_2}S^T)^2) \leq tr((A_H - PA_G)^2) + tr((A_H - QA_{G_2}Q^T)^2)$$

According to definition, we have $d(G_1, G_2) = min_P(tr(A_G - PA_G)^2)$. Thus, we have $d(G_1, G_2) \leq tr((A_G - SQA_{G_2}S^T)^2)$ for any permutation matrix $S$. Therefore,

$$d(G_1, G_2) \leq tr((A_G - SQA_{G_2}S^T)^2) \leq d(G_1, H) + d(G_2, H)$$

The study of topology that metric $d$ induces on the set of graphs is suggested for the future work.

### 3.1 Dot product of graphs with different order

In Section 2, the dot product of graphs with the same order was defined. Here, the graph dot product is extended for graphs with different order. In [10, 5], the number of subgraph of a graph is merely, counted as subgraph algebra. Here, there is a more general approach.

**Definition 6.** Let $A_G, A_H$ be, respectively, the matrix representations of two graphs $G$ and $H$ with respectively $n$ and $k$ vertices such that $n > k$. First, we add extra zero rows and columns to matrix $A_H$ to have two matrices with the same size, then we define

$$G.H := \max_P(tr(A_G P A_H P^T))$$

where $P$ is a permutation matrix.

Assuming that $f$ is a mapping from $V(H)$ to $V(G)$ and $H_f$ is the induced subgraph on $f(V(H))$, $Phase(G, H)$ denotes the number of mapping $f : V(H) \rightarrow V(G)$ such that $G.H_f = G.H$. 

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Lemma 3. Let $G$ and $H$ be arbitrary graphs.

a) $G.H = H.H$ if and only if $H$ is a subgraph of $G$.

b) If $H$ is a subgraph of $G$, then $\text{Phase}(G,H)/\text{Aut}(H)$ is the number of occurrences of the subgraph $H$ in $G$.

Proof.

a) It can be easily checked that $\max_P(tr(A_G P A_H P^T)) = \max_{H_f}(H.H_f)$ where $H_f$ is a $k$-vertex subgraph of graph $G$. Thus, the maximum value of $G.H$ occurs if and only if there exists a subgraph $H_f$ in $G$ isomorphic to $H$. In opposite, if there is a subgraph $H_f$ isomorphic to $H$ in $G$, then $G.H = H.H_f = H.H$.

b) If $H$ is a subgraph of graph $G$, we have $\text{Phase}(G,H) = |\{H_f|H_f \text{ is a } k\text{-vertex subgraph of } G \text{ isomorphic to } H\}|.|\text{Aut}(H)|$

\qed

4 Coordinate Representation

We saw how graph dot product reveals significant information about the structure of a pair of graphs, such as being isomorphic, close to isomorphic or completely different structure by orthogonality. Now, we want to identify a graph according to its dot product by a a set of graphs.

Definition 7. Let $S = (H_1, ..., H_t)$ be an ordered set of graphs. The coordinates of a graph $G$ with respect to the set $S$ is $(G.H_1, \cdots, G.H_t)$.

Please note that for any graph $H_i$, the $\text{Phase}(H_i, G)$ is also computed.

Definition 8. Let $\Gamma$ be a set of graphs. A set $S$ of graphs is a basis for $\Gamma$, if any graph in $\Gamma$ has a unique coordinates with respect to $S$.

A set of graphs is a basis for itself. A basis for 6-vertex graphs is shown in Fig. 2. The coordinate representation of graphs is a useful tool to deal with graphs. Two isomorphic graphs share the same coordinates. Thus, to test the isomorphism of two graphs, it is sufficient to check their coordinates. Also, the coordinates of graphs indicate how similar two graphs are.
In the conventional approach for checking the isomorphism of two graphs, one tries to find a one to one correspondence between the vertices of two graphs. Against, in a coordinate system, it is sufficient to compute the coordinates and compare them.

Although, computing the dot product of two graphs, in general, is as hard as graph isomorphism problem, but, the basis is a fixed set of graphs. The fixedness of basis is the outstanding advantage of this new approach. The fixedness of basis elements makes it possible to have some pre-computations, if it is needed, or have a dedicated physical infrastructure which computes the dot product of any graphs with a fixed basis element. The implementation of this infrastructure is given in Fig. 3.

More importantly, the basis elements can be chosen cleverly to decrease the computational complexity. For example, there are families of graphs that the computation of their dot product to any arbitrary graph can be done in polynomial time, such as bounded order graphs, star graphs $S_k$, $K_r \cup K_{n-r}$ or $K_r \cup \overline{K}_{n-r}$ ($r$ is a fixed integer).

To study the computational complexity of the graph isomorphism problem, we should find a suitable basis for graphs. For instance, if we can find a basis for $n$-vertex graphs whose cardinality is a polynomial in terms of $n$ and the dot product of any graph with basis elements can be done in polynomial time in terms of $n$, in fact, we have found a polynomial time algorithm for the graph isomorphism problem.

Another application of the defined graph coordinate system is graph clustering, i.e. classifying a set of graphs according to their class of isomorphism. Assume we have a set of $m$ graphs with $n$ vertices. We want to classify them into isomorphism classes. We need to compare any pair of graphs. Hence, the algorithm of checking isomorphism of two graphs should be called $O(m^2)$ times. In opposite, using the graphs coordinates, it is sufficient to compute the graph coordinates and compare them. Thus, clustering of graphs which needs $O(m^2)$ comparison of graphs reduces to a sorting problem with $O(\log(m))$ time complexity.
The graph coordinate representation is useful not only in graph isomorphism problem, but also in graph matching and graph similarity problems. The closer coordinates, the more similar structure. Therefore, the coordinate system of graphs, also, benefits us in classifying and clustering graphs in inexact cases.

4.1 A basis for almost all $n$-vertex graphs

In the probability space of graphs on $n$ labeled vertices in which the edges are chosen independently, with probability $p = 1/2$, we say that almost every graph $G$ has a property $Q$ if the probability that $G$ has $Q$ tends to 1 as $n \to \infty$.

It has been shown that almost every $n$-vertex graph is, uniquely, determined by the number of occurrence of its $3 \log_2 n$-vertex subgraphs [6]. According to Lemma 3, $G.H$ indicates whether $H$ is a subgraph of $G$. Additionally, if $H$ is a subgraph of $G$, the number of copies of subgraph $H$ occurred in $G$ can be obtained from $Phase(G, H)$. Therefore, the set of graphs with $3 \log_2 n$ vertices is a basis for almost all $n$-vertex graphs.

References

[1] Horst Bunke and Kim Shearer. A graph distance metric based on the maximal common subgraph. Pattern recognition letters, 19(3):255–259, 1998.

[2] Soumen Chakrabarti. Mining the Web: Discovering knowledge from hypertext data. Morgan Kaufmann, 2003.
[3] Donatello Conte, Pasquale Foggia, Carlo Sansone, and Mario Vento. Thirty years of graph matching in pattern recognition. *International journal of pattern recognition and artificial intelligence, 18*(03):265–298, 2004.

[4] Matthias Dehmer, Frank Emmert-Streib, and Olaf Wolkenhauer. Perspectives of graph mining techniques. *Rostocker Informatik Berichte, 30*(2):47–57, 2006.

[5] Paul Erdős, László Lovász, and Joel Spencer. Strong independence of graphcopy functions. *Graph Theory and Related Topics*, pages 165–172, 1979.

[6] Ameneh Farhadian. Almost every n-vertex graph is determined by its \(3\log(n)\)-vertex subgraphs. *arXiv preprint arXiv:1805.05387*, 2018.

[7] Ralph J Faudree, Richard H Schelp, L Lesniak, András Gyárfás, and Jenő Lehel. On the rotation distance of graphs. *Discrete Mathematics, 126*(1):121–135, 1994.

[8] Xinbo Gao, Bing Xiao, Dacheng Tao, and Xuelong Li. A survey of graph edit distance. *Pattern Analysis and applications, 13*(1):113–129, 2010.

[9] Thomas Gärtner, Peter Flach, and Stefan Wrobel. On graph kernels: Hardness results and efficient alternatives. In *Learning Theory and Kernel Machines*, pages 129–143. Springer, 2003.

[10] WL Kocay. Some new methods in reconstruction theory. In *Combinatorial Mathematics IX*, pages 89–114. Springer, 1982.

[11] Si Quang Le, Tu Bao Ho, and TT Hang Phan. A novel graph-based similarity measure for 2d chemical structures. *GENOME INFORMATICS SERIES, 15*(2):82, 2004.

[12] László Lovász. *Large networks and graph limits*, volume 60. American Mathematical Soc., 2012.

[13] Michel Neuhaus and Horst Bunke. *Bridging the gap between graph edit distance and kernel machines*. World Scientific Publishing Co., Inc., 2007.
[14] Kaspar Riesen and Horst Bunke. *Graph classification and clustering based on vector space embedding*. World Scientific Publishing Co., Inc., 2010.

[15] Bohdan Zelinka. On a certain distance between isomorphism classes of graphs. *Časopis pro pěstování matematiky*, 100(4):371–373, 1975.