Self-gravitating Brownian particles in two dimensions: 
the case of $N = 2$ particles

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Abstract. We study the motion of $N = 2$ overdamped Brownian particles in gravitational interaction in a space of dimension $d = 2$. This is equivalent to the simplified motion of two biological entities interacting via chemotaxis when time delay and degradation of the chemical are ignored. This problem also bears some similarities with the stochastic motion of two point vortices in viscous hydrodynamics [Agullo & Verga, Phys. Rev. E, 63, 056304 (2001)]. We analytically obtain the probability density of finding the particles at a distance $r$ from each other at time $t$. We also determine the probability that the particles have coalesced and formed a Dirac peak at time $t$ (i.e. the probability that the reduced particle has reached $r = 0$ at time $t$). Finally, we investigate the variance of the distribution ($r^2$) and discuss the proper form of the virial theorem for this system. The reduced particle has a normal diffusion behavior for small times with a gravity-modified diffusion coefficient ($r^2 = r_0^2 + (4k_B/\xi\mu)(T - T_c)t$, where $k_B T_c = GMm^2/2$ is a critical temperature, and an anomalous diffusion for large times ($r^2 \propto t^{1-T_c/T}$). As a by-product, our solution also describes the growth of the Dirac peak (condensate) that forms at large time in the post collapse regime of the Smoluchowski-Poisson system (or Keller-Segel model) for $T < T_c$. We find that the saturation of the mass of the condensate to the total mass is algebraic in an infinite domain and exponential in a bounded domain. Finally, we provide the general form of the virial theorem for Brownian particles with power law interactions.

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1 Introduction

Systems with long-range interactions have recently been the object of considerable interest \[\text{(1)}\]. One usually considers isolated systems in which the particles evolve according to deterministic Hamiltonian equations. These systems are described by the microcanonical ensemble. Examples of such systems include self-gravitating systems, two-dimensional point vortices, the Hamiltonian mean field (HMF) model etc. However, one may also consider dissipative systems in which the particles, in contact with a thermal bath, evolve according to stochastic Langevin equations. These systems are described by the canonical ensemble. The statistical mechanics of Hamiltonian and Brownian systems with long-range interactions is discussed in \[\text{(2)}\] at a general level. In this paper, we consider the case of Brownian particles in gravitational interaction. It is known that this system bears deep analogies with simple models of bacterial populations experiencing chemotaxis in biology \[\text{(3)}\] (see, e.g., Ref. \[\text{(3)}\] for a description of this analogy). In a proper thermodynamic limit $N \to +\infty$, the mean field approximation becomes exact and the dynamics of these systems is described by the Smoluchowski-Poisson system (gravity) \[\text{(4)}\] or by the Keller-Segel model (chemotaxis) \[\text{(5)}\]. These equations display rich phenomena such as collapse and evaporation. In particular, in $d = 2$ dimensions, the evolution leads to the formation of Dirac peaks if the temperature is below the critical value $k_B T_c = GMm^2/4$ \[\text{(5,9,10)}\]. These systems have been studied essentially in the mean field approximation, i.e. for a large number of particles. The case of a finite number of particles can be studied numerically by

\[\text{Footnote 1: These systems are isomorphic up to a change of notations. In this paper, we shall use the notations of astrophysics because they are closer to the notations that are familiar in physics and thermodynamics. However, our results can be transposed easily to the biological context. We refer to Perthame \[\text{(3)}\] for a complete bibliography of the chemotactic problem from the viewpoint of applied mathematics and to Chavanis & Sire for additional references in physics \[\text{(4)}\].}
solving the \( N \)-body stochastic equations. Numerical results will be presented in a companion paper \[11\]. In the present paper, we consider the extreme case of only \( N = 2 \) Brownian particles in gravitational interaction that can be solved analytically. We analytically obtain the probability density of finding the particles at a distance \( r \) from each other at time \( t \) and determine the probability that the particles have coalesced and formed a Dirac peak at time \( t \). We also investigate the variance of the distribution \( \langle r^2 \rangle \) and discuss the proper form of the virial theorem for this system. In particular, we show that the virial theorem obtained in \[12\] is only valid as long as the particles have not formed Dirac peaks.

The paper is organized as follows. In Sec. 2 we recall the \( N \)-body coupled stochastic equations describing the evolution of self-gravitating Brownian particles and specifically consider the case \( N = 2 \). We introduce the center of mass and the reduced particle. We show that the center of mass undergoes a pure Brownian motion and that the reduced particle undergoes a Brownian motion in a central potential \( U = Gm_1m_2/r \). We also recall the “naive” virial theorem obtained in \[12\] and discuss, with a new light, the distinction between the critical temperatures \( k_B T_c = Gm_1m_2/4 \) and \( k_B T_c = Gm_1m_2/2 \). In Sec. 3 we study the motion of a Brownian particle (reduced particle) in an attractive central potential \( U = Gm_1m_2 \ln r \) in \( d = 2 \). We show that the corresponding Fokker-Planck equation is equivalent to a Schrödinger equation (with imaginary time) with a potential \( V = -D(a/r)^2 \). This equation can be solved analytically in terms of Bessel functions. Then, we can obtain various analytical results such as the probability to find the reduced particle at position \( r \) at time \( t \), the probability that the particle reaches the origin for the first time between \( t \) and \( t + dt \), the probability that the particle has reached the origin at time \( t \) and the variance \( \langle r^2 \rangle \) of the distribution. We find that the reduced particle has a normal diffusion behavior for small times with a gravity-modified diffusion coefficient \( \langle r^2 \rangle = r_0^2 + (4k_B/\xi \mu)(T - T_c) \) and an anomalous diffusion for large times \( \langle r^2 \rangle \propto t^{1-T/T_c} \). In particular, the variance increases with time when \( T > T_c \) and tends to zero for \( t \to +\infty \) when \( T < T_c \). In Sec. 4 we consider the case of two self-gravitating Brownian particles in a bounded domain and discuss the differences with the case of an infinite domain. Finally, in Sec. 5 we show that our study also describes the large time asymptotics of the Smoluchowski-Poisson system (or Keller-Segel model) for \( T < T_c = GMm/(4k_B) \). Indeed, in the post-collapse regime, the system is made of a growing central Dirac peak (condensate) surrounded by a dilute halo whose dynamical evolution is eventually described by a Fokker-Planck equation similar to the one studied in the case of \( N = 2 \) particles. We find that the saturation of the mass of the condensate to the total mass is algebraic in an infinite domain and exponential in a bounded domain and we characterize it precisely. In Sec. 6, we briefly generalize our results to the logarithmic Fokker-Planck equation in \( d \) dimensions. The Appendices provide complements such as the deterministic limit \( T = 0 \) (Sec. C), the van Kampen classification (Sec. F), the correlation functions (Sec. G) and the general form of the virial theorem for Brownian particles with power law interaction (Sec. H).

We may note that our study bears some similarities with the stochastic motion (induced by viscosity) of two point vortices studied by Agullo & Verga \[13\]. However, there also exists crucial differences between the two problems since in our case the interaction is radial leading to the formation of Dirac peaks while in the case of point vortices the interactions is rotational leading to the formation of a spiral structure.

We may also note that the statistical mechanics of \( N = 2 \) particles in gravitational interaction has been considered by Padmanabhan \[14\] in \( d = 3 \) (and generalized by Chavanis \[12\] for the dimensions \( d = 1 \) and \( d = 2 \)) in the microcanonical and canonical ensembles. However, these authors consider the equilibrium statistical mechanics of \( N = 2 \) self-gravitating particles in a box, and with a small-scale cut-off, while we consider here the dynamical evolution of \( N = 2 \) self-gravitating Brownian particles in a finite or infinite domain without small-scale cut-off. Therefore, we address the time dependent problem and investigate the formation of Dirac peaks.

Finally, the particular character of the dimension \( d = 2 \) in gravity is well-known. We refer for example to \[15,16,17,18,19,20,21\] for more details and further references.

2 The position of the problem

2.1 The \( N \)-body problem

We consider a system of \( N \) overdamped Brownian particles with mass \( m_\alpha \) in gravitational interaction in a space of dimension \( d \). Their motion is described by the coupled stochastic equations \[12\]:

\[
\frac{d\mathbf{r}_\alpha}{dt} = -\frac{1}{\xi m_\alpha} \nabla_\alpha U(\mathbf{r}_1,...,\mathbf{r}_N) + \sqrt{2D_\alpha} \mathbf{B}_\alpha(t),
\]

with

\[
U(\mathbf{r}_1,...,\mathbf{r}_N) = -\frac{G}{d-2} \sum_{\alpha<\beta} \frac{m_\alpha m_\beta}{|\mathbf{r}_\alpha - \mathbf{r}_\beta|^{d-2}},
\]

for \( d \neq 2 \) and

\[
U(\mathbf{r}_1,...,\mathbf{r}_N) = \sum_{\alpha<\beta} m_\alpha m_\beta \ln |\mathbf{r}_\alpha - \mathbf{r}_\beta|,
\]

for \( d = 2 \). Here, \( \xi \) is the friction coefficient and \( \mathbf{B}_\alpha(t) \) is a white noise satisfying \( \langle \mathbf{B}_\alpha(t) \rangle = 0 \) and \( \langle B_{\alpha,i}(t)B_{\beta,j}(t') \rangle = \delta_{ij}\delta_{\alpha\beta}\delta(t - t') \) where \( \alpha = 1,...,N \) refers to the particles and \( i = 1,...,d \) to the coordinates of space. The diffusion coefficient is given by the Einstein relation

\[
D_\alpha = \frac{k_B T}{\xi m_\alpha},
\]

The deterministic limit \( T = 0 \) (Sec. C), the van Kampen
where $T$ is the temperature. We assume that the friction $\xi$ is the same for all the particles.

From these stochastic equations, it is possible to derive the Fokker-Planck equation for the $N$-body distribution $P_N(r_1, \ldots, r_N, t)$ and then write the BBGKY-hierarchy for the reduced distributions $P_N(1)$. Let us consider for brevity the single-species system. The proper thermodynamic limit corresponds to $N \to +\infty$ in such a way that the normalized temperature $\eta = \beta GM m / R^d$ is of order unity. In that limit, it can be shown that the mean field approximation becomes exact so that the $N$-body distribution factorizes in a product of $N$ one-body distributions: $P_N(r_1, \ldots, r_N, t) = \prod_P P_1(r\alpha, t)$ [2]. Furthermore, the one-body distribution, or equivalently the smooth density field $\rho(r, t) = N m P_1(r, t)$, is solution of the Smoluchowski-Poisson system [2]:

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left[ \frac{1}{\xi} \left( \frac{k_B T}{m} \nabla \rho + \rho \nabla \Phi \right) \right],$$

$$\Delta \Phi = S_d G \rho.$$  \hspace{1cm} (5)

The equations generalizing Eqs. (5)-(6) for the multispecies case are given in [12,22].

Up to a change of notations, these equations are isomorphic to a simplified version of the Keller-Segel model of chemotaxis that is valid in the limit of large diffusivity of the chemical and in the absence of degradation [5].

\subsection*{2.2 The case $N = 2$: the reduced particle}

From now on, we consider only $N = 2$ self-gravitating Brownian particles in $d = 2$. In that case, the stochastic equations [1-3] reduce to

$$\frac{d r_1}{d t} = -\frac{G m_2}{\xi} \frac{r_1 - r_2}{|r_1 - r_2|^2} + \sqrt{2 D_1} B_1(t),$$

$$\frac{d r_2}{d t} = \frac{G m_1}{\xi} \frac{r_1 - r_2}{|r_1 - r_2|^2} + \sqrt{2 D_2} B_2(t),$$

with $D_1 = k_B T / \xi m_1$ and $D_2 = k_B T / \xi m_2$. Like for the standard two-body problem, we introduce the center of mass

$$R = \frac{m_1 r_1 + m_2 r_2}{M}, \quad M = m_1 + m_2,$$  \hspace{1cm} (9)

and the reduced particle

$$r = r_2 - r_1, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}. \hspace{1cm} (10)$$

Concerning the motion of the center of mass, we have

$$\frac{d R}{d t} = \frac{1}{M} \left( m_1 \sqrt{2 D_1} B_1(t) + m_2 \sqrt{2 D_2} B_2(t) \right) \equiv Q(t),$$

where the noise satisfies

$$\langle Q(t) Q(t') \rangle = \frac{2 k_B T}{M \xi} \delta_{ij} \delta(t - t'). \hspace{1cm} (11)$$

Therefore, the center of mass undergoes a pure Brownian motion of the form

$$\frac{d R}{d t} = \sqrt{2 D_\mu} B(t),$$

with a diffusion coefficient

$$D_\mu = \frac{k_B T}{\xi M}. \hspace{1cm} (14)$$

Concerning the motion of the reduced particle, we have

$$\frac{d r}{d t} + \frac{G}{\xi} \frac{(m_1 + m_2) r}{r^2} = \sqrt{2 D_2} B_2(t) - \sqrt{2 D_1} B_1(t) \equiv S(t),$$

where the noise satisfies

$$\langle S(t) S(t') \rangle = \frac{2 k_B T}{\mu \xi} \delta_{ij} \delta(t - t'). \hspace{1cm} (16)$$

Therefore, the reduced particle undergoes a Brownian motion in a central potential of the form

$$\frac{d r}{d t} = -\frac{G m_1 m_2}{\xi \mu} \frac{r}{r^2} + \sqrt{2 D B(t)},$$

with a diffusion coefficient

$$D = \frac{k_B T}{\xi \mu}. \hspace{1cm} (18)$$

\subsection*{2.3 The naive virial theorem}

Let us introduce the total moment of inertia

$$I_{\text{tot}}(t) = m_1 (r_1^2) + m_2 (r_2^2).$$

In [12] (see also Appendix I), an exact closed expression of the virial theorem valid for an arbitrary number of self-gravitating Brownian particles in $d = 2$ has been obtained. For $N = 2$, it writes

$$\frac{1}{4} \xi \dot{I}_{\text{tot}} = 2 k_B (T - T_c),$$

with the critical temperature

$$k_B T_c = \frac{G m_1 m_2}{4}. \hspace{1cm} (21)$$

It is instructive to recover this result in a different manner. The positions of the particles 1 and 2 can be expressed in terms of $r$ (reduced particle) and $R$ (center of mass) as

$$r_1 = \frac{M R - m_2 r}{M}, \quad r_2 = \frac{M R + m_1 r}{M}.$$  \hspace{1cm} (22)

Substituting these relations in Eq. (19), we obtain after straightforward algebra

$$I_{\text{tot}}(t) = M (R^2) + \mu (r^2),$$

$$k_B T_c = \frac{G m_1 m_2}{4}.$$
a relation which was of course expected. Now, the Fokker-
Planck equation associated with the stochastic motion
of the reduced particle is
\[ \frac{\partial P}{\partial t} = \nabla \cdot \left( k_B T \frac{\xi}{\mu} \nabla P + P \frac{Gm_1 m_2}{\mu} \frac{r}{r^2} \right). \]  

(24)

Taking the time derivative of
\[ \langle r^2 \rangle = \int P r^2 \, dr, \]
and using simple integrations by parts, we naively obtain
\[ \frac{1}{4} \frac{d\langle r^2 \rangle}{dt} = k_B T - \frac{Gm_1 m_2}{2}. \]

(26)

This relation exhibits a critical temperature
\[ k_B T_c = \frac{Gm_1 m_2}{2}. \]

(27)

Introducing the moment of inertia of the reduced particle
\( I(t) = \mu \langle r^2 \rangle \), we can rewrite Eq. (26) as
\[ \frac{1}{4} \frac{dI}{dt} = k_B (T - T_c). \]

(28)

The mean square displacement of the reduced particle satis-
\[ \langle r^2 \rangle = 4 k_B \mu (T - T_c) t + \langle r^2 \rangle_0. \]

(29)

This is a normal diffusion with a gravity modified diffusion
coefficient
\[ D(T) = \frac{k_B T}{\xi \mu} \left( 1 - \frac{T}{T_c} \right). \]

(30)

The variance increases for \( T > T_c \) and tends to zero in a
finite time for \( T < T_c \). On the other hand, the Fokker-
Planck equation associated to the stochastic motion of
the center of mass is simply
\[ \xi \frac{\partial P}{\partial t} = \frac{k_B T}{M} \Delta P, \]

(31)

and we classically obtain the relation
\[ \frac{1}{4} \frac{d \langle R^2 \rangle}{dt} = k_B T. \]

(32)

Finally, summing Eqs. (20) and (32) and using Eq. (29),
we recover Eq. (28). We now clearly see the origin of the
two temperatures \( T_c \) and \( T_\ast = 2 T_c \) that were reported in
[12]. In the case \( N = 2 \), the critical temperature \( T_c \) is asso-
ciated to the dynamics of the reduced particle while the
critical temperature \( T_\ast \) enters in the expression of virial
theorem for the total moment of inertia (reduced particle
and center of mass). This distinction is further discussed in
Appendix A in the general case of \( N \) particles.

2 We shall see later that this expression is in fact incorrect.

2.4 The problem

In fact, there is a flaw in the above derivation of the virial
theorem because we have naively assumed that the norm-
alization \( \int P(r, t) \, dr = 1 \) is conserved in time. However,
as we shall see, this is not correct. The normalization is
not conserved in time because the reduced particle can
reach the origin \( r = 0 \) and be “lost” by the system (if it
reaches the origin, it remains there for ever). This corre-
sponds to the coalescence of the two particles, resulting in
the formation of a Dirac peak, i.e. a new particle of mass
\( m_1 + m_2 \). As a result of these “trapping” events
\[ \int P(r, t) \, dr \neq 1, \]

(33)

and we must reconsider the problem in more detail.

3 Brownian particle in a Newtonian potential
in two dimensions

3.1 The Fokker-Planck equation

Let \( P(r, t) \) denote the probability density of finding the
reduced particle in \( r \) at time \( t \). The evolution of \( P(r, t) \) is
governed by the Fokker-Planck equation
\[ \xi \frac{\partial P}{\partial t} = \nabla \cdot \left( k_B T \frac{\xi}{\mu} \nabla P + P \frac{Gm_1 m_2}{\mu} \frac{r}{r^2} \right). \]

(34)

The initial distribution \( P_0(r) \) is normalized such that
\( \int P_0(r) \, dr = 1 \). Introducing
\[ D = \frac{k_B T}{\mu \xi}, \quad \beta = \frac{1}{k_B T}, \quad U = Gm_1 m_2 \ln r, \]

(35)

the Fokker-Planck equation can be rewritten
\[ \frac{\partial P}{\partial t} = \nabla \cdot [D (\nabla P + \beta P \nabla U)]. \]

(36)

In the absence of small and large scale cut-offs, this
equation has no steady state since the distribution \( P =
A/r^{3Gm_1 m_2} \) is not normalizable. We assume that the
initial distribution \( P_0(r) \) is radially symmetric, so that
\( P(r, t) \) is radially symmetric for all times. Therefore, we
can write the Fokker-Planck equation in the form
\[ \frac{\partial P}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( D r \frac{\partial P}{\partial r} + P \frac{\beta Gm_1 m_2}{r} \right). \]

(37)

As discussed previously, the probability is not conserved
because the reduced particle may reach the origin \( r = 0 \)
and form a Dirac peak (the two particles coalesce). The
probability that the particle has not reached \( r = 0 \) at time
\( t \) is
\[ \chi(t) = \int_0^t P(r, t) 2 \pi r \, dr. \]

(38)

3 Note that, for \( d = 1 \), the Fokker-Planck equation for the
reduced particle corresponds to a V-shaped potential \( U(x) =
Gm_1 m_2 |x| \) that relaxes to \( P_c(x) = \frac{1}{2} \beta Gm_1 m_2 e^{3Gm_1 m_2 |x|} \) [23].
Taking the time derivative of this quantity and using the Fokker-Planck equation \(37\) we obtain
\[ \chi(t) = -2\pi D\beta Gm_1m_2 P(0,t), \]  
which is non zero since \(P(0,t) \neq 0\). Therefore, the probability for the particle to form a Dirac peak between \(t\) and \(t + dt\) (i.e. to reach \(r = 0\) for the first time between \(t\) and \(t + dt\)) is
\[ \chi_D(t) = 2\pi D\beta Gm_1m_2 P(0,t), \]  
and the probability for the particle to have formed a Dirac peak at time \(t\) (i.e. to have reached \(r = 0\) at time \(t\)) is
\[ \chi_D(t) = 2\pi D\beta Gm_1m_2 \int_0^t P(0,\tau) d\tau. \]
We obviously have \(\chi_D(t) = 1 - \chi(t)\). We can now obtain the proper form of the virial theorem associated to the Fokker-Planck equation \(37\). Introducing the moment of inertia of the reduced particle
\[ I(t) = \int P \mu^2 \, dr, \]
we easily obtain the virial theorem
\[ \frac{1}{4} \frac{dI}{dt} = \chi(t) k_B (T - T_s), \]
instead of Eq. \(28\). It has to be noted that this relation is not closed since it depends on \(\chi(t)\) that must be obtained by solving the Fokker-Planck equation \(37\).

### 3.2 The associated Schrödinger equation

Let us consider a general Fokker-Planck equation of the form
\[ \frac{\partial P}{\partial t} = \frac{\partial}{\partial r} \left[ D(r) \frac{\partial P}{\partial r} + \beta P \frac{\partial U}{\partial r} \right]. \]
For a spherically symmetric distribution in \(d\) dimensions, it can be rewritten
\[ \frac{\partial P}{\partial t} = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left[ D(r) r^{d-1} \left( \frac{\partial P}{\partial r} + \beta P \frac{\partial U}{\partial r} \right) \right]. \]
As is well-known \(23\), we can transform this Fokker-Planck equation into a Schrödinger equation (with imaginary time) by setting
\[ P(r,t) = e^{-\frac{1}{2}\beta U(r)} \psi(r,t). \]
This yields
\[ \frac{\partial \psi}{\partial t} = \frac{\partial}{\partial r} \left( D(r) \frac{\partial \psi}{\partial r} \right) + V(r) \psi = (H + V) \psi, \]
with the potential
\[ V(r) = \frac{1}{2} \beta \frac{\partial}{\partial r} \left( D(r) \frac{\partial U}{\partial r} \right) - \frac{1}{4} D \beta^2 \left( \frac{\partial U}{\partial r} \right)^2. \]
For a spherically symmetric distribution, the Schrödinger equation \(47\) can be rewritten
\[ \frac{\partial \psi}{\partial t} = \frac{1}{i r^{d-1}} \frac{\partial}{\partial r} \left( r^{d-1} D(r) \frac{\partial \psi}{\partial r} \right) + V(r) \psi, \]
with
\[ V(r) = \frac{1}{2} \beta \frac{1}{r^{d-1}} \frac{d}{dr} \left( r^{d-1} D(r) \frac{dU}{dr} \right) - \frac{1}{4} \beta^2 D(r) \left( \frac{dU}{dr} \right)^2. \]
Making the separation of variables
\[ \psi = e^{-\frac{i}{2} \beta U} \phi(r), \]
we obtain the eigenvalue equation
\[ \frac{1}{r^{d-1}} \frac{d}{dr} \left( r^{d-1} D(r) \frac{d\phi}{dr} \right) + (V(r) + \lambda) \phi = 0. \]
Let us note \(\lambda_n\) the eigenvalues and \(\phi_n\) the corresponding eigenfunctions. The eigenfunctions are orthogonal with respect to the scalar product
\[ \langle f | g \rangle = \int f(r) g(r) S_d r^{d-1} \, dr. \]
We also normalize them so that \(\langle \phi_n | \phi_m \rangle = \delta_{nm}\). Then, any function can be expanded in the form
\[ h(r) = \sum_{n=1}^{+\infty} \langle h | \phi_n \rangle \phi_n(r). \]
If the spectrum is continuous, the sum over \(n\) must be replaced by an integral over \(\lambda \geq 0\).

### 3.3 The general solution

We consider the Green function \(P(r,t|r_0)\) which corresponds to the initial condition
\[ P(r,0|r_0) = \frac{\delta(r - r_0)}{S_d r_0^{d-1}}. \]
The solution on the Fokker-Planck equation \(45\) can be expanded on the eigenfunctions in the form
\[ P(r,t|r_0) = \sum_{n=1}^{+\infty} A_n e^{-\lambda_n t} e^{-\frac{1}{2} \beta U(r)} \phi_n(r). \]
Noting that
\[ \frac{\delta(r - r_0)}{S_d r_0^{d-1}} = \sum_{n=1}^{+\infty} \phi_n(r_0) \phi_n(r), \]
and using the initial condition \(45\), we finally obtain
\[ P(r,t|r_0) = e^{-\frac{1}{2} \beta (U(r) - U(r_0))} \sum_{n=1}^{+\infty} e^{-\lambda_n t} \phi_n(r_0) \phi_n(r). \]
3.4 The case of a logarithmic potential in $d = 2$

For the Fokker-Planck equation (67), we have $d = 2$, $D = k_B T / (\xi \mu)$ and $U = Gm_1 m_2 \ln r$. The potential $V(r)$ arising in the corresponding Schrödinger equation (49) is

$$ V(r) = -D \left( \frac{\beta G m_1 m_2}{2r} \right)^2. $$

(59)

Therefore, if we assume that initially

$$ P(r,0) = \frac{\delta(r-r_0)}{2\pi r_0}, $$

(60)

the solution of the Fokker-Planck equation (67) can be written

$$ P(r,t) = \left( \frac{r_0}{r} \right)^{\frac{d}{2} \frac{\beta G m_1 m_2}{2}} \int_0^r e^{-\lambda t} \phi_\lambda(r_0) \phi_\lambda(r) d\lambda, $$

(61)

where $\phi_\lambda(r)$ is solution of the differential equation

$$ r^2 \phi'' + r \phi' + \left( \frac{\lambda}{D} r^2 - a^2 \right) \phi = 0, $$

(62)

where

$$ a = \frac{\beta G m_1 m_2}{2} = \frac{T}{T}. $$

(63)

Equation (62) is a Bessel differential equation that can be solved analytically. The solutions that are finite at the origin are of the form

$$ \phi_\lambda(r) = J_\alpha(\sqrt{\lambda/D} r). $$

(64)

Substituting Eq. (64) in Eq. (61), the solution of the Fokker-Planck equation (67) can be written

$$ P(r,t) = 2D \left( \frac{r_0}{r} \right)^{\alpha} A^2 \int_0^r e^{-\lambda t} J_\alpha(\lambda r_0) J_\alpha(\lambda r) d\lambda, $$

(65)

where we have made the change of notation $\lambda \to D\lambda^2$ for convenience. The normalization constant $A$ is determined so as to recover the initial condition (60) as $t \to 0$. Taking $t = 0$ in Eq. (65), we get

$$ P(r,0) = 2D \left( \frac{r_0}{r} \right)^{\alpha} A^2 \int_0^r J_\alpha(\lambda r_0) J_\alpha(\lambda r) d\lambda. $$

(66)

Using the closure relation

$$ \int_0^{\infty} J_\alpha(u x) J_\alpha(v x) x dx = \frac{1}{u} \delta(u-v), $$

(67)

we obtain

$$ P(r,0) = \frac{2DA^2}{r_0} \delta(r-r_0). $$

(68)

Comparing with Eq. (60), we find that $A^2 = 1/(4\pi D)$. Therefore, the solution of the Fokker-Planck equation (67) with the initial condition (60) is

$$ P(r,t) = \frac{1}{2\pi} \left( \frac{r_0}{r} \right)^{\alpha} \int_0^{\infty} e^{-\lambda t} J_\alpha(\lambda r_0) J_\alpha(\lambda r) d\lambda. $$

(69)

Using the identity

$$ \int_0^{\infty} e^{-\gamma x^2} J_\alpha(\alpha x) J_\gamma(\beta x) x dx = \frac{1}{2\alpha^2} e^{-\gamma a^2} I_\alpha \left( \frac{\alpha \beta}{2} \right), $$

(70)

valid for $\gamma > -1$, we find that it can finally be written

$$ P(r,t) = \left( \frac{r_0}{r} \right)^{\alpha} \frac{1}{4\pi Dt} e^{-\frac{r_0^2}{4Dt}} I_\alpha \left( \frac{r_0^2}{2Dt} \right). $$

(71)

The distribution $P(r,t)$ is plotted in Fig. 1 at different times and for $T/T_s = 1/2$ (corresponding to $a = 2$).

Using the identity

$$ I_\alpha(x) \sim \frac{e^x}{\sqrt{2\pi x}}, \quad (x \to +\infty) $$

(72)

we get for $t \to 0$:

$$ P(r,t) \sim \frac{1}{4\pi r_0 \sqrt{\pi Dt}} e^{-\frac{r_0^2}{4Dt}}, $$

(73)

which tends to Eq. (60) as expected. On the other hand, for $t \to +\infty$, the probability tends to zero meaning that the particle has been absorbed in $r = 0$ after a sufficiently long time so that it is ultimately lost by the system. For $r \to +\infty$, using the identity (72), we have

$$ P(r,t) \sim \left( \frac{r_0}{r} \right)^{\alpha} \frac{1}{4\pi \sqrt{\pi r_0 Dt}} e^{-\frac{r^2}{4Dt}}. $$

(74)

For $r = 0$, using the identity

$$ I_\alpha(x) \sim \frac{1}{\Gamma(a+1)} \left( \frac{x}{2} \right)^a, \quad (x \to 0) $$

(75)
we get
\[ P(0, t) = \left( \frac{T_0}{2} \right)^{2a} \frac{1}{4\pi^2} \frac{1}{(DT)^{1+a}} e^{-\frac{r^2}{4Dt}}. \]  

(76)

Finally, for \( \beta = 0 \), the probability density (71) becomes
\[ \chi(t) = 1 - \frac{\Gamma_a}{\Gamma(a)} \left( \frac{r^2}{4Dt} \right)^{-\frac{a}{2}} \frac{1}{\Gamma(a)(DT)^{1+a}} e^{-\frac{r^2}{4Dt}}. \]

(77)

which is the solution of the diffusion equation in \( d = 2 \). Indeed, in the limit of infinite temperature, the gravity is negligible with respect to diffusion.

### 3.5 The probability to form a Dirac peak

The probability that the particle has not reached \( r = 0 \) at time \( t \) is given by Eq. (85). For the distribution (71), the integral can be performed analytically and we obtain
\[ \chi(t) = 1 - \frac{\Gamma_a}{\Gamma(a)} \left( \frac{r^2}{4Dt} \right)^{-\frac{a}{2}} \frac{1}{\Gamma(a)(DT)^{1+a}} e^{-\frac{r^2}{4Dt}}. \]

(78)

where
\[ \Gamma_a(x) = \int_x^{+\infty} t^{a-1} e^{-t} dt, \]

(79)

is the incomplete Gamma function. The probability decays because, as time goes on, the particle has more and more chance to reach \( r = 0 \) and form a Dirac. The probability that the particle reaches \( r = 0 \) for the first time between \( t \) and \( t + dt \) is given by Eq. (86). Combining this relation with Eq. (77), we obtain
\[ \chi_D = D \left( \frac{T_0}{2} \right)^{2a} \frac{1}{4\pi^2} \frac{1}{\Gamma(a)(DT)^{1+a}} e^{-\frac{r^2}{4Dt}}. \]

(80)

Integrating Eq. (80), we obtain the probability that the particle has formed a Dirac peak at time \( t \):
\[ \chi_D(t) = \frac{\Gamma_a}{\Gamma(a)} \left( \frac{r^2}{4Dt} \right)^{-\frac{a}{2}} \frac{1}{\Gamma(a)(DT)^{1+a}} e^{-\frac{r^2}{4Dt}}. \]

(81)

and we check that \( \chi(t) = 1 - \chi_D(t) \) as expected. The evolution of \( \chi_D(t) \) for different values of the temperature is shown in Figs. 2 and 3.

For \( t \to 0 \), using the expansion
\[ \Gamma_a(x) \sim \Gamma(a) - \frac{x^a}{a}, \quad (x \to 0), \]

(82)

we obtain
\[ \chi(t) \sim \frac{1}{\Gamma(a)} \left( \frac{r^2}{4Dt} \right)^{-\frac{a}{2}} e^{-\frac{r^2}{4Dt}}, \quad (t \to 0). \]

(83)

We see that the probability \( \chi_D(t) \to 0 \) for \( t \to 0 \) due to the exponential factor. This tendency is reinforced by the algebraic factor for \( T > T_\star \) (\( a < 1 \)) while it is reduced for \( T < T_\star \) (\( a > 1 \)).

\textbf{Fig. 2.} Evolution of the function \( \chi_D(t) \), giving the probability that the particle has formed a Dirac peak at time \( t \), for different values of the temperature (we have taken \( T/T_\star = 1/2 \), \( T/T_\star = 1 \) and \( T/T_\star = 2 \)). The probability increases more rapidly at smaller temperatures.

\textbf{Fig. 3.} Same as Fig. 2 for larger times.

For \( t \to +\infty \), using the expansion
\[ \Gamma_a(x) \simeq \Gamma(a) - \frac{x^a}{a}, \quad (x \to 0), \]

(84)

we obtain
\[ \chi_D(t) = 1 - \frac{1}{\Gamma(a)} \left( \frac{r^2}{4Dt} \right)^a, \quad (t \to +\infty). \]

(85)

Therefore, the probability that the particle has not formed a Dirac at time \( t \) decreases \textit{algebraically} as \( t^{-a} \). Equation (85) can be written in the form
\[ \chi_D(t) = 1 - \left( \frac{t_*}{t} \right)^a, \]

(86)

where the time
\[ t_* = \frac{r^2}{4D[a\Gamma(a)]^{1/a}}, \]

(87)

gives an idea of the rapidity at which the Dirac forms as a function of the temperature \( a = T_\star/T \). The function
We readily check that

\[ \int \Psi \, d\mathbf{r} = \int_0^{+\infty} P(r,t) \mu r^2 2\pi r \, dr, \]

and the variance of the distribution (mean square displacement) is

\[ \langle r^2 \rangle = \frac{I(t)}{\mu}. \]

For the density distribution given by Eq. (71), the integral can be calculated explicitly yielding

\[ I(t) = \frac{4\mu}{T(a)} \left( \frac{r_0^2}{4Dt} \right)^a D t e^{-\frac{r_0^2}{4Dt}} + \mu (r_0^2 + 4D(1-a)t) \left[ 1 - \frac{\Gamma_a \left( \frac{r_0^2}{4\pi T} \right)}{\Gamma(a)} \right]. \]

In Appendix B, we check that this relation is consistent with the virial theorem (\ref{virial}). The evolution of the moment of inertia is represented in Fig. 5 for different values of the temperature. For \( T > T_* \), the moment of inertia increases, for \( T = T_* \) the moment of inertia is constant \( I(t) = \mu r_0^2 \) and for \( T < T_* \) the moment of inertia decreases. This will become clear from the asymptotic behaviors.

For \( t \to 0 \), we get

\[ I(t) \simeq \mu r_0^2 + 4D\mu(1-a)t, \]

which can be rewritten

\[ \langle r^2 \rangle \simeq r_0^2 + \frac{4kB}{\xi\mu}(T-T_*)t. \]

In that case, we have a normal diffusion with a gravity-modified diffusion coefficient

\[ D(T) = \frac{k_B}{\xi\mu}(T-T_*). \]

For \( T > T_* \), the variance increases with time while for \( T < T_* \), it decreases. This expression agrees with the naive virial theorem (\ref{virial}). Indeed, for small times, the probability for the particle to reach \( r = 0 \) is exponentially small so that the probability is conserved and the naive virial theorem holds since there is no Dirac peak.

For \( t \to +\infty \), using the expansion (\ref{asymptotic}), we get

\[ I(t) \sim \frac{4\mu}{aT(a)} \left( \frac{r_0}{2} \right)^a (D t)^{1-a}. \]

This corresponds to

\[ \langle r^2 \rangle \sim \frac{4}{aT(a)} \left( \frac{r_0}{2} \right)^a (D t)^{1-a}. \]

For \( T > T_* \), i.e. \( a < 1 \), the variance increases and goes to \( +\infty \) for large times. In that case, we have an anomalous diffusion \( \langle r^2 \rangle \sim t^\alpha \) with an exponent \( \alpha = 1-T_*/T \). The evolution is always sub-diffusive. The origin of the anomalous diffusion is related to the fact that the particle can be trapped at \( r = 0 \) (and form a Dirac peak). For \( T < T_* \), the variance decreases and goes to 0 for \( t \to +\infty \).
3.7 The most probable position

The most probable value \( r_P(t) \) of the distribution \( P(r,t) \) is obtained by maximizing \( P(r,t) \), or equivalently \( \ln P(r,t) \), with respect to \( r \). This gives

\[
\frac{2aDt}{r_P r_0} + \frac{r_P}{r_0} = \frac{I_a'(\frac{r_P r_0}{2Dt})}{I_a(\frac{r_P r_0}{2Dt})}.
\] (98)

Using the recurrence relation

\[
I_a'(x) = I_{a+1}(x) + \frac{a}{x} I_a(x),
\] (99)

we obtain

\[
\frac{r_P}{r_0} = \frac{I_{a+1}(\frac{r_P r_0}{2Dt})}{I_a(\frac{r_P r_0}{2Dt})}.
\] (100)

This equation can be rewritten in the parametric form

\[
x = \frac{r_P r_0}{2Dt} + \frac{2Dt}{r_0^2} = \frac{I_{a+1}(x)}{x I_a(x)},
\] (101)

or equivalently

\[
\frac{r_P}{r_0} = \frac{I_{a+1}(x)}{I_a(x)} + \frac{2Dt}{r_0^2} = \frac{I_{a+1}(x)}{x I_a(x)}.
\] (102)

which gives \( r_P(t) \). Using the asymptotic expansions of \( I_a(x) \), we find that

\[
\frac{r_P}{r_0} = 1 - \frac{a+1}{2} \frac{2Dt}{r_0}, \quad (t \to 0),
\] (103)

and

\[
\frac{r_P}{r_0} = \left( \frac{a+2}{a+1} \right)^{1/2} \sqrt{1-t/t_c}, \quad (t \to t_c),
\] (104)

where

\[
\frac{2Dt_c}{r_0^2} = \frac{1}{2(a+1)}.
\] (105)

Using Eq. (100), we get

\[
\frac{r_*}{r_0} = \frac{2(a-1)Dt}{r_* r_0} + \frac{r_*}{r_0} = \frac{I_a'(\frac{r_* r_0}{2Dt})}{I_a(\frac{r_* r_0}{2Dt})}.
\] (106)

This equation can be rewritten in the parametric form

\[
x = \frac{r_* r_0}{2Dt} = \frac{I_{a+1}(x)}{x I_a(x)} + \frac{1}{x^2},
\] (108)

or equivalently

\[
\frac{r_*}{r_0} = \frac{I_{a+1}(x)}{I_a(x)} + \frac{1}{x}, \quad \frac{2Dt}{r_0^2} = \frac{I_{a+1}(x)}{x I_a(x)} + \frac{1}{x^2},
\] (109)

Therefore, the radius \( r_P(t) \) is decreasing for any temperature and it goes to zero in a finite time \( t_c(a) \) depending on the temperature. Some curves are represented in Fig. 8.

The most probable value \( r_*(t) \) of the radial distribution \( P(r,t) \) is solution of

\[
\frac{2(a-1)Dt}{r_* r_0} + \frac{r_*}{r_0} = \frac{I_a'(\frac{r_* r_0}{2Dt})}{I_a(\frac{r_* r_0}{2Dt})}.
\] (106)
which gives \( r_*(t) \). Using the asymptotic expansions of \( I_\alpha(x) \), we find that
\[
\frac{r_*}{r_0} = 1 - \frac{2a - 1}{2} \frac{2Dt}{r_0^2}, \quad (t \to 0),
\]
and
\[
r_* \sim \sqrt{2Dt}, \quad (t \to +\infty).
\]
Therefore, the radius \( r_*(t) \) is always increasing for \( T > 2T_* \). For \( T < 2T_* \), it starts to decrease before finally increasing. For \( T = 2T_* \), we can use the identities
\[
I_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sinh(x),
\]
\[
I_{3/2}(x) = \frac{2}{\pi x} \left[ \cosh(x) - \frac{\sinh(x)}{x} \right],
\]
so that \( r_*/r_0 = 1/\tan(x) - 1/\sqrt{x} = L(x) \) (where \( L(x) \) is the Langevin function) and \( 2Dt/r_0^2 = 1/(x \tan(x)) - 1/x^2 \). Using the asymptotic expansion of \( \tan(x) \) for \( x \to 0 \) and \( x \to +\infty \), we find that
\[
\frac{r_*}{r_0} \simeq 1 + 2e^{-x^2}, \quad (t \to 0).
\]
Some curves are represented in Fig. 8.

### 3.8 Reflecting boundary conditions

In the previous sections, we have considered the case of absorbing boundary conditions at \( r = 0 \). They lead to the formation of a Dirac peak with an amplitude \( \chi_D(t) \) growing with time. We shall now consider the case of reflecting boundary conditions and determine their domain of existence. Reconsidering the calculations of Sec. 3.4 these boundary conditions correspond to solutions of the form
\[
\phi(x) = J_{-a}(\sqrt{\lambda/4D}r),
\]
that diverge at the origin. Substituting Eq. (116) in Eq. (61) and repeating the calculations of Sec. 3.4 we obtain
\[
P(r,t) = (\frac{\tau_0}{r})^a \frac{1}{4\pi Dt} e^{-\frac{r^2}{4\pi Dt}} I_{-a} \left( \frac{\tau r_0}{2Dt} \right).
\]
According to identity (210), this solution is valid only for \( a < 1 \) (i.e. \( T > T_* \)). This is confirmed by considering the equivalent of \( P(r,t) \) close to \( r = 0 \):
\[
P(r,t) \sim \frac{1}{r^{2\beta G_m + 2}} \frac{4^a}{4\pi Gm_2 ^2 (1-a)} \frac{1}{(4\pi Dt)^{1-a}} e^{-\frac{r^2}{4\pi Dt}}.
\]
This distribution diverges at the origin and it is normalized iff \( T > T_* \). Since the distribution \( P(r,t) \) diverges at the origin, we must replace Eq. (39) by
\[
\chi(t) = -2\pi D \lim_{r \to 0} r \left( \frac{\partial P}{\partial r} + P \frac{\beta G_m m_2}{r} \right).
\]

![Fig. 9. Probability density of finding the particles at a distance \( r \) from each other, at different times \( t \) and for a temperature \( T = 2T_* \) (\( a = 1/2 \)). From top to bottom: \( t = 0.01, 0.025, 0.05, 0.1, 0.2, 1 \).](image)

Using Eq. (116) and the expansion
\[
P_\alpha(x) = \frac{1}{\Gamma(a+1)} \left( \frac{x}{2} \right)^a + \frac{1}{\Gamma(a+2)} \left( \frac{x}{2} \right)^{a+2} + ..., \quad (119)
\]
valid for \( x \to 0 \), we find that \( \chi = 0 \). Therefore, the normalization is conserved in time and there is no Dirac peak formation.

These results are consistent with the van Kampen classification of singularities (see Appendix E). For \( T < T_* \) (i.e. \( a > 1 \) or \( \beta G_m m_2 > 2 \), the singularity at \( r = 0 \) behaves as an adhesive boundary. In that case, the solution is unique and no boundary condition has to be fixed by hand. It is given by Eq. (27) leading to a Dirac peak (\( \chi \neq 0 \)). On the other hand, for \( T > T_* \) (i.e. \( a < 1 \) or \( \beta G_m m_2 < 2 \), the singularity at \( r = 0 \) behaves as a regular boundary. In that case, the boundary condition can be absorbing, reflecting or mixed. It has to be fixed by hand. In the previous sections, we considered a purely absorbing boundary condition and, in the present section, we considered a purely reflecting boundary condition. In fact, for \( a < 1 \) (i.e. \( T > T_* \)), the general solution can be written as a “mixture” of the two previous solutions
\[
P(r,t) = \nu P_+(r,t) + (1-\nu)P_-(r,t),
\]
where \( P_+(r,t) \) is the distribution (21), \( P_-(r,t) \) is the distribution (116), and \( \nu \) is a parameter taking values between \( \nu = 1 \) (purely absorbing) and \( \nu = 0 \) (purely reflecting).

Let us specifically consider the distribution (116). It is plotted in Fig. 9 at different times for \( T/T_* = 2 \) (corresponding to \( a = 1/2 \)). Substituting Eq. (116) in the expression (20) defining the moment of inertia of the reduced particle, and carrying out the integrations, we find that
\[
I(t) = \mu r^2 - 4 D \mu (1-a)t.
\]
This expression agrees with the naive virial theorem (28). Indeed, in the present case \( \chi(t) = 1 \) since there is no Dirac peak formation, and the exact expression (23) of the virial theorem reduces to Eq. (28).
On the other hand, the most probable value $r_P(t)$ of the distribution $P(r, t)$ is obtained by maximizing $P(r, t)$, or equivalently $\ln P(r, t)$ with respect to $r$. In fact, since this distribution diverges at $r = 0$, the global maximum (infinite) is $r_P(t) = 0$. However, for sufficiently short times, the distribution $P(r, t)$ also admits a local maximum $r_P^{(1)}(t)$ and a local minimum $r_P^{(2)}(t)$. Proceeding as in Sec. 3.7, they are the solutions of the equation

$$\frac{4aDt}{r_P r_0} + \frac{r_P}{r_0} = \frac{I_{1-a}(\frac{r_P r_0}{2aDt})}{I_{-a}(\frac{r_P r_0}{2aDt})}$$  \hspace{1cm} (122)$$

Setting $x = \frac{r_P r_0}{2aDt}$, this equation can be rewritten in the parametric form

$$\frac{r_P}{r_0} = \frac{I_{1-a}(x)}{I_{-a}(x)} - \frac{2a}{x}, \hspace{1cm} \frac{2Dt}{r_0^2} = \frac{I_{1-a}(x)}{x I_{-a}(x)} - \frac{2a}{x^2},$$  \hspace{1cm} (123)

which gives $r_P(t)$. Using the asymptotic expansion of $I_a(x)$ for $x \to +\infty$, we find that

$$\frac{r_P^{(1)}}{r_0} = 1 - \frac{2a + 1}{2} \frac{2Dt}{r_0^2}, \hspace{1cm} (t \to 0).$$  \hspace{1cm} (124)

The radius $r_P^{(1)}(t)$ of the local maximum is decreasing for any temperature and it disappears at a time $t_{end}(a)$ which depends on the temperature. The curve corresponding to $a = 1/2$ is represented in Fig. 11.

Let us now consider the most probable value $r_*(t)$ of the radial distribution $P(r, t)$. This distribution diverges at the origin $r = 0$ for $1/2 < a < 1$, vanishes at the origin for $0 \leq a < 1/2$ and is finite at the origin for $a = 1/2$. In the first case, the global maximum (infinite) is at $r_P = 0$. However, for sufficiently short times, the distribution $P(r, t)$ also admits a local maximum $r_P^{(1)}$ and a local minimum $r_P^{(2)}$. Proceeding as in Sec. 3.7, they are the solutions of the equation

$$\frac{2(2a - 1)Dt}{r_* r_0} + \frac{r_*}{r_0} = \frac{I_{1-a}(\frac{r_* r_0}{2aDt})}{I_{-a}(\frac{r_* r_0}{2aDt})}$$  \hspace{1cm} (125)$$

Setting $x = \frac{r_* r_0}{2aDt}$, this equation can be rewritten in the parametric form

$$\frac{r_*}{r_0} = \frac{I_{1-a}(x)}{I_{-a}(x)} - \frac{2a - 1}{x}, \hspace{1cm} \frac{2Dt}{r_0^2} = \frac{I_{1-a}(x)}{x I_{-a}(x)} - \frac{2a - 1}{x^2},$$  \hspace{1cm} (126)

which gives $r_*(t)$. Using the asymptotic expansion of $I_a(x)$ for $x \to +\infty$, we find that

$$\frac{r_*^{(1)}}{r_0} = 1 + \frac{1 - 2a}{2} \frac{2Dt}{r_0^2}, \hspace{1cm} (t \to 0).$$  \hspace{1cm} (127)

For $1/2 < a < 1$, the radius $r_*^{(1)}(t)$ of the local maximum is decreasing and it disappears at a time $t_{end}(a)$ which depends on the temperature. For $0 \leq a < 1/2$, the radius $r_*^{(1)}(t)$ increases initially and behaves for large times as

$$r_* \sim \sqrt{2(1 - 2a)Dt}, \hspace{1cm} (t \to +\infty).$$  \hspace{1cm} (128)

For $a = 1/2$, we can use the identities (112) and

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cosh(x),$$  \hspace{1cm} (129)$$

so that $r_*/r_0 = \tanh(x)$ and $2Dt/r_0^2 = \tanh(x)/x$. Using the asymptotic expansions of $\tanh(x)$ for $x \to 0$ and $x \to +\infty$, we find that

$$\frac{r_*}{r_0} \simeq 1 - 2e^{-r_0^2/4Dt}, \hspace{1cm} (t \to 0),$$  \hspace{1cm} (130)$$

$$\frac{r_*}{r_0} \simeq \sqrt{3 \left(1 - \frac{2Dt}{r_0^2}\right)^{1/2}}, \hspace{1cm} (t \to \frac{r_0^2}{2Dt}),$$  \hspace{1cm} (131)$$

establishing $t_{end} = t_c = \frac{r_0^2}{2Dt}$ for $a = 1/2$. Some curves are represented in Fig. 11.
4 The case of a bounded domain

In the previous sections, we considered the case where the particles are free to move in an infinite domain. We shall now consider the case where the particles are confined in a bounded domain. As an idealization, we shall consider that the reduced particle evolves in a spherical box of radius $R$. We first look for the existence of an equilibrium state. The steady solution of the Fokker-Planck equation \[ \nabla P_{\text{eq}} + \beta Gm_1 m_2 P_{\text{eq}} \frac{p}{r^2} = 0, \] (132) which is integrated into \[ P_{\text{eq}} = \frac{A}{r^{3\beta Gm_1 m_2}}, \] (133) It is clear that this distribution is normalizable iff $\beta Gm_1 m_2 < 2$ so that an equilibrium state exists iff $T > T_\ast$. For $T < T_\ast$, the singularity at $r = 0$ is adhesive and a Dirac peak grows, ultimately absorbing the particle with probability one. For $T > T_\ast$, the singularity at $r = 0$ is regular and its nature (absorbing, reflecting or mixed) has to be fixed by hand. In the case of an absorbing boundary, we ultimately obtain a Dirac peak of amplitude $\chi_D(+\infty) = 1$, so that there is no equilibrium state outside the Dirac. In the case of a reflecting boundary, there is no Dirac peak and the system reaches an equilibrium state, given by Eq. (133), with normalization $\int P_{\text{eq}} \, dr = 1$. In the mixed case, we ultimately obtain a Dirac peak with amplitude $\chi_D(+\infty) = \nu$ and an equilibrium state outside the Dirac, given by Eq. (133), with normalization $\int P_{\text{eq}} \, dr = 1 - \nu$.

Let us now solve the Fokker-Planck equation \[ \nabla P_{\text{eq}} + \beta Gm_1 m_2 P_{\text{eq}} \frac{p}{r^2} = 0, \] (134) in $r = R$ meaning that there is no flux of probability at the boundary. In terms of the eigenfunction $\phi$ defined in Eqs. (10) and (51), this can be rewritten
\[ \phi'(R) \big/ \phi(R) = -\frac{1}{2} \beta U'(R). \] (135) For the potential $U = Gm_1 m_2 \ln r$, we obtain
\[ \frac{\phi'(R)}{\phi(R)} = -\frac{\beta Gm_1 m_2}{2R}. \] (136)

This boundary condition implies that the eigenvalues are quantized. Let us first consider purely absorbing boundary conditions at $r = 0$. Making the change of notations $\lambda \to D\lambda^2$, the eigenfunctions that are solution of Eq. (132) and that are finite at the origin are given by
\[ \phi_n(r) = A_n J_\lambda(\lambda_n r). \] (137)
Substituting this solution in Eq. (132), we find that the eigenvalues are determined by
\[ \frac{\lambda_n R J_\lambda(\lambda_n R)}{J_\lambda'(\lambda_n R)} = -a. \] (138)
Using the recurrence relation
\[ J_n'(x) = J_{n-1}(x) - \frac{a}{x} J_n(x), \] (139)
the foregoing equation can be rewritten
\[ J_n(\lambda_n R) = 0, \] (140)
with $\lambda_n \neq 0$. Therefore, the eigenvalues $R\lambda_n(a)$ are the zeros of $J_{a-1}(x)$. Using the general identity
\[ \int_0^R J_n^2(\lambda r) r \, dr = \frac{R^2}{2} \left[ J_n^2(\lambda R)^2 + \left( 1 - \frac{a^2}{\lambda^2 R^2} \right) J_n^2(\lambda R) \right], \] (141) together with the relation (138), we obtain
\[ \int_0^R J_n^2(\lambda_n r) r \, dr = \frac{R^2}{2} J_n^2(\lambda_n R). \] (142)
Therefore, the normalized eigenfunctions are
\[ \phi_n(r) = \frac{1}{\sqrt{\pi R} J_n(\lambda_n R)} J_n(\lambda_n r). \] (143)
Finally, using Eq. (55), the solution of the Fokker-Planck equation (51) in a bounded domain can be written
\[ P(r, t) = \left( \frac{r_0}{r} \right)^a \sum_n e^{-D\lambda_n^2 t} \phi_n(r_0) \phi_n(r). \] (144)
We can now redo the preceding analysis except that the results will be less explicit since they will be expressed in the form of series.

The probability $\chi_D(t)$ for the particle to have formed a Dirac peak at time $t$ is given by Eq. (11). Using Eq. (144) and the equivalent
\[ J_n(x) \sim \frac{1}{\Gamma(n + 1)} \left( \frac{x}{2} \right)^n, \quad (x \to 0), \] (145)
we obtain
\[ P(0, t) = \frac{1}{\pi R^2} \left( \frac{r_0}{2} \right)^a \frac{1}{\Gamma(a + 1)} \sum_n \lambda_n^a J_n(\lambda_n r_0) J_n(\lambda_n R) e^{-D\lambda_n^2 t}. \] (146)
Therefore, the probability for the particle to have formed a Dirac peak at time $t$ can be written
\[ \chi_D(t) = 1 - \sum_n B_n e^{-D\lambda_n^2 t}, \] (147)
with
\[ B_n = \frac{2\beta Gm_1 m_2}{R^2} \left( \frac{r_0}{2} \right)^a \frac{1}{\Gamma(a + 1)} \lambda_n^{a-2} J_n(\lambda_n r_0) J_n^2(\lambda_n R). \] (148)
For $t \to +\infty$, we obtain
\[ \chi_D(t) \simeq 1 - B_1 e^{-D\lambda_n^2 t}, \]  
so that the probability that the particle has not formed a Dirac at time $t$ decreases exponentially as $e^{-D\lambda_n^2 t}$ instead of algebraically in an unbounded domain (see Sec. 5.5). The exponential decay is controlled by the first eigenvalue $\lambda_1(T)$ (fundamental) of the Schrödinger equation.

As we have seen, $\lambda_1(a)R$ is the first zero of $J_{a-1}(x)$. This is valid for any temperature. It is instructive, however, to determine the asymptotic behavior of $\lambda_1$ for $a \to 0$ (i.e. $T \to +\infty$). In that limit, $\lambda_1 \to 0$. Substituting the expansion
\[ J_n(x) = \left( \frac{x}{2} \right)^n \left[ \frac{1}{\Gamma(n+1)} - \frac{1}{\Gamma(n+2)} \left( \frac{x}{2} \right)^2 + \ldots \right], \]
valid for $x \to 0$ in Eq. (140), we obtain
\[ (\lambda_1 R)^2 \sim 4a \sim \frac{2G m_1 m_2}{k_B T}. \]  
On the other hand, in this limit, $B_1 \to 1$. Therefore, Eq. (149) becomes
\[ \chi_D(t) \simeq 1 - e^{-\frac{2G m_1 m_2}{k_B T^2} t}. \]  

Using Eq. (58), it can be rewritten
\[ \chi_D(t) \simeq 1 - e^{-\frac{2G m_1 m_2}{k_B T^2} \cdot \frac{t}{\lambda_1}}. \]  
For $T = T_* \ (i.e. \ a = 1)$, we find that $\lambda_1 R = j_{01}$ where $j_{01} \simeq 2.40482\ldots$ is the first zero of $J_0(x)$. Finally, for $T \to 0$, we find that
\[ \lambda_1 R \simeq 2 \sim \frac{G m_1 m_2}{2k_B T}. \]

Let us now consider purely reflecting boundary conditions at $r = 0$. Making the change of notations $\lambda \to D\lambda^2$, the eigenfunctions that are solution of Eq. (42) and that diverge at the origin are given by
\[ \phi_n(r) = A_n J_{-a}(\lambda_n r). \]
Substituting this solution in Eq. (136), we find that the eigenvalues are determined by
\[ \frac{\lambda_n R J_{-a}(\lambda_n R)}{J_{-a}(\lambda_n R)} = -a. \]
Using the recurrence relation
\[ J'_a(x) = -J_{a+1}(x) + \frac{a}{x} J_a(x), \]
the foregoing equation can be rewritten
\[ J_{-a}(\lambda_n R) = 0, \]  
with $\lambda_n \neq 0$. Therefore, the eigenvalues $R\lambda_n(a)$ are the zeros of $J_{-a}(x)$. Using the general identity [141] together with the relation [150], we find that the normalized eigenfunctions are
\[ \phi_n(r) = \frac{1}{\sqrt{\pi} R J_{-a}(\lambda_n R)} J_{-a}(\lambda_n r). \]  
This expression is valid for $\lambda_n \neq 0$. We must also add the eigenmode corresponding to $\lambda_0 = 0$ whose normalized expression is
\[ \phi_0(r) = \sqrt{\frac{1 - a}{\pi R} \left( \frac{R}{r} \right)^a}. \]  
Finally, using Eq. (58), the solution of the Fokker-Planck equation [40] in a bounded domain can be written
\[ P(r, t) = P_{eq}(r) + \left( \frac{r_0}{r} \right)^a \sum_n e^{-D\lambda_n^2 t} \phi_n(r_0) \phi_n(r), \]  
where
\[ P_{eq}(r) = \frac{1 - a}{\pi R^2 \Gamma(1 - a)} \frac{2r_0^a}{r} \sum_n \lambda_n^2 \frac{J_{-a}(\lambda_n r_0)}{J_{-a}(\lambda_n R)} e^{-D\lambda_n^2 t}, \]  
This expression is valid for $\lambda_n \neq 0$. We must also add the eigenmode corresponding to $\lambda_0 = 0$ whose normalized expression is
\[ \phi_0(r) = \sqrt{\frac{1 - a}{\pi R} \left( \frac{R}{r} \right)^a}. \]  
Finally, using Eq. (58), the solution of the Fokker-Planck equation [40] in a bounded domain can be written
\[ P(r, t) = P_{eq}(r) + \left( \frac{r_0}{r} \right)^a \sum_n e^{-D\lambda_n^2 t} \phi_n(r_0) \phi_n(r), \]  
where
\[ P_{eq}(r) = \frac{1 - a}{\pi R^2 \Gamma(1 - a)} \frac{2r_0^a}{r} \sum_n \lambda_n^2 \frac{J_{-a}(\lambda_n r_0)}{J_{-a}(\lambda_n R)} e^{-D\lambda_n^2 t}, \]  
so that the distribution converges exponentially rapidly towards the equilibrium state as $e^{-D\lambda_n^2 t}$. The exponential relaxation time is controlled by the first eigenvalue $\lambda_1(a)$ (fundamental) of the Schrödinger equation. As we have seen, $R\lambda_1(a)$ is the first zero of $J_{-a}(x)$. This is valid for any temperature $T > T_*$. For $T \to +\infty$ (i.e. $a \to 0$), we find that $\lambda_1 R \to j_{11}$ where $j_{11} \simeq 3.83171\ldots$ is the first zero of $J_1(x)$. For $T = T_* (i.e. \ a = 1)$, we find that $\lambda_1 R = j_{01}$ where $j_{01} \simeq 2.40482\ldots$ is the first zero of $J_0(x)$.

Remark: We could also consider the case of $N = 2$ self-gravitating Brownian particles with a short-range regularization. This could be due to a softened potential $U = Gm_1 m_2 ln(\sqrt{x^2 + x^2})$, to a hard core $a$ or to an exclusion principle such as the Pauli exclusion principle for fermions in quantum mechanics. In that case, there is no Dirac peak formation. There exists an equilibrium state for any temperature in a box and for $T < T_*$ in an infinite domain. For low temperatures, the two particles are at a typical distance $a$ from each other so that the equilibrium state is controlled by the small-scale cut-off. For high temperatures, the particles are at a typical distance $R$ (the box radius) from each other in a bounded domain or tend to “evaporate” to infinity in an unbounded domain.
5 By-product: post-collapse of the Smoluchowski-Poisson system

There is an interesting by-product of the previous study. Indeed, the preceding analysis can be used to obtain new results concerning the post-collapse dynamics of the Smoluchowski-Poisson system (or Keller-Segel model). In $d = 2$ dimensions and for $T < T_c = GMm/(4k_B)$, it is known that the Smoluchowski-Poisson system forms a Dirac peak of mass $M_0 = (T/T_c)M$ in a finite time $t_{\text{coll}}$ \[4\]. For $t > t_{\text{coll}}$ the Dirac continues to grow by accretion of the surrounding matter.\[4\] For $T = T_c$ or for $t \to +\infty$ and any $T < T_c$, the Dirac peak has accreted most of the mass. As a result, the system is formed by a Dirac peak of mass $M_0(t) \simeq M$ surrounded by a dilute halo containing the remaining mass $\epsilon = M - M_0(t)$. Now, it is possible to neglect the self-gravity of the halo and consider that the particles of the halo are only subject to the gravity of the central peak of mass $\sim M$. Therefore, the evolution of the halo density is governed by a Fokker-Planck equation of the form

\[ \xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \frac{k_B T}{m} \nabla \rho + \rho \frac{GM \rho}{r^2} \right). \]  

This equation has been studied in \[24\] in a box. However, it was not realized that the corresponding eigenvalue equation could be solved analytically in $d = 2$. Indeed, Eq. (\[165\]) is equivalent to Eq. (\[54\]) up to a change of notation and we can therefore apply the results of the previous sections with now

\[ a = \frac{\beta GM m}{2} = 2T_c T. \]  

In an infinite domain, using Eq. (\[85\]), we find that the mass of the Dirac peak saturates to $M$ algebraically rapidly as

\[ 1 - \frac{M_0(t)}{M} \sim t^{-a}. \]  

In a bounded domain, using Eq. (\[149\]), we find that the mass of the Dirac peak saturates to $M$ exponentially rapidly as

\[ 1 - \frac{M_0(t)}{M} \sim e^{-DA_\lambda_1(a)t}, \]  

where $R\lambda_1(a)$ is the first zero of $J_{\alpha - 1}(x)$. This result was previously found in \[24\] but the exponential rate (eigenvalue) was not obtained explicitly (except in the asymptotic limit $T \to 0$). Note finally that for $T = 0$, $M_0(t)$ saturates to $M$ in a finite time

\[ t_{\text{end}} = \frac{R^2}{2GM}. \]  

6 The logarithmic Fokker-Planck equation in $d$ dimensions

In this section, we briefly generalize the previous results to the logarithmic Fokker-Planck equation in $d$ dimensions

\[ \xi \frac{\partial P}{\partial t} = \nabla \cdot \left( \frac{k_B T}{\mu} \nabla P + \frac{P \cdot (GM_{m2} r)}{\mu r^2} \right). \]  

The previous results are recovered for $d = 2$. Taking the time derivative of the moment of inertia of the reduced particle \[12\], and using Eq. (\[170\]), we obtain after an integration by parts

\[ \xi \frac{dI}{dt} = -2 \int r \cdot \left( k_B T \nabla P + PGM_{m2} \frac{r}{r^2} \right) \, dr. \]  

It can be shown, using the following expressions for $P(r, t)$, that the boundary terms at $r = 0$ and $r = +\infty$ vanish. Integrating again by parts and introducing the notation $\chi(t) = \int P(r, t) \, dr$ taking into account the possibility that the normalization of $P(r, t)$ is not conserved (due to the formation of a Dirac peak at $r = 0$), we obtain

\[ \xi \frac{dI}{dt} = 2k_B T d \chi(t) - 2GM_{m2} \chi(t). \]  

Finally, this can be rewritten

\[ \frac{1}{2d} \xi \frac{dI}{dt} = \chi(t) k_B T - T_c, \]  

where we have introduced the critical temperature

\[ k_B T_c = \frac{GM_{m2}}{d}. \]  

We emphasize that this relation is valid whether $P(r, t)$ is spherically symmetric or not.

We now consider a spherically symmetric evolution. Using the notations of Eq. (\[92\]), we can write the logarithmic Fokker-Planck equation (\[170\]) in the form

\[ \frac{\partial P}{\partial t} = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left( D_r r^{d-1} \left( \frac{\partial P}{\partial r} + P \frac{\beta GM_{m2}}{r} \right) \right). \]  

The time derivative of the normalization is

\[ \chi(t) = -SD \lim_{r \to 0} r^{d-1} \left( \frac{\partial P}{\partial r} + P \frac{\beta GM_{m2}}{r} \right). \]  

The Fokker-Planck equation (\[175\]) can be transformed into a Schrödinger equation (\[49\]) with a potential

\[ V(r) = -\frac{D}{r^2} \left( a^2 + \frac{(d-2)^2}{4} \right), \]  

where $D$ is a parameter.
where we have defined
\[ a = \frac{\beta G m_1 m_2 - (d - 2)}{2}. \] (178)

When \( d > 2 \), we see that \( a = 0 \) at the new critical temperature
\[ k_B T^*_c = \frac{G m_1 m_2}{d - 2}. \] (179)

In the following discussion, it is implicit that \( T^*_c = +\infty \) if \( d \leq 2 \). The eigenvalue equation (180) takes the form
\[ r^2 \phi'' + (d - 1) r \phi' + \left[ \frac{\lambda}{D} r^2 - a^2 + \frac{(d - 2)^2}{4} \right] \phi = 0, \] (180)
and it can be solved analytically in terms of Bessel functions.

Let us first consider the solution
\[ \phi_+(r) = r^{\frac{d - 2}{2}} J_{|a|} (\sqrt{\lambda} D t r). \] (181)

Repeating the calculations of Sec. 3.4, we find that the solution of the logarithmic Fokker-Planck equation (175) is
\[ P_+(r, t) = \frac{1}{r^{d - 2}} \left( \frac{r_0}{r} \right)^a \frac{1}{2 S_d D t} e^{-\frac{r^2}{4 S_d D t}} I_{|a|} \left( \frac{r r_0}{2 D t} \right). \] (182)

We need to distinguish two cases. For \( a > 0 \) (i.e. \( \beta G m_1 m_2 > d - 2 \) or \( T > T^*_c \)), the behavior of the distribution close to \( r = 0 \) is
\[ P_+(r, t) \sim \frac{1}{r^{d - 2}} \left( \frac{r_0}{2} \right)^{2a} \frac{1}{2 S_d D (a + 1)} \frac{1}{(D t)^a + 1} e^{-\frac{r^2}{4 S_d D t}}. \] (183)

Substituting this equivalent in Eq. (176), we find that
\[ \chi_+ = -D \left( \frac{r_0}{2} \right)^{2a} \frac{1}{2 S_d D (a + 1)} \frac{1}{(D t)^a + 1} e^{-\frac{r^2}{4 S_d D t}}. \] (184)

In that case, we have an absorbing boundary condition at \( r = 0 \) and the growth of a Dirac peak. For \( a < 0 \) (i.e. \( \beta G m_1 m_2 < d - 2 \) or \( T > T^*_c \)), the behavior of the distribution close to \( r = 0 \) is
\[ P_+(r, t) \sim \frac{1}{r^{d - 2} \beta G m_1 m_2} \frac{1}{4 |a|} \frac{1}{2 S_d D (1 + |a|)} \frac{1}{(D t)^{|a|} + 1} e^{-\frac{r^2}{4 S_d D t}}. \] (185)

Substituting this equivalent in Eq. (176), we find that \( \chi_+ = 0 \). In that case, the normalization condition is conserved and there is no Dirac peak.

Let us now consider the solution
\[ \phi_-(r) = r^{\frac{d - 2}{2}} J_{-|a|} (\sqrt{\lambda} D t r). \] (186)

Repeating the calculations of Sec. 3.4, we find that the solution of the logarithmic Fokker-Planck equation (180) with initial condition (181) is
\[ P_-(r, t) = \frac{1}{r^{d - 2}} \left( \frac{r_0}{r} \right)^a \frac{1}{2 S_d D t} e^{-\frac{r^2}{4 S_d D t}} I_{-|a|} \left( \frac{r r_0}{2 D t} \right). \] (187)

provided that \(|a| < 1\), according to identity (184). We need to distinguish two cases. For \( 0 < a < 1 \) (i.e. \( d - 2 < \beta G m_1 m_2 \) or \( T_c < T < T^*_c \)), the behavior of the distribution close to \( r = 0 \) is
\[ P_-(r, t) \sim \frac{1}{r \beta G m_1 m_2} \frac{4^a}{2 S_d D t (1 - a)} \frac{1}{(D t)^{a - 1}} e^{-\frac{r^2}{4 S_d D t}}. \] (188)

Substituting this equivalent in Eq. (176), we find that \( \chi_- = 0 \). In that case, the normalization condition is conserved and there is no Dirac peak. For \(-1 < a < 0 \) (i.e. \( d - 4 < \beta G m_1 m_2 \) or \( T_c < T < T^*_c \) with \( k_B T^*_c = G m_1 m_2 /(d - 4) \)), the behavior of the distribution close to \( r = 0 \) is
\[ P_-(r, t) \sim \frac{1}{r^{d - 2}} \left( \frac{2}{r_0} \right)^{2|a|} \frac{1}{2 S_d D (1 - |a|)} \frac{1}{(D t)^{|a|} + 1} e^{-\frac{r^2}{4 S_d D t}}. \] (189)

Substituting this equivalent in Eq. (176), we find that \( \chi_- > 0 \) which is not physically possible (the normalization of \( P(r, t) \) can decrease if the particle is absorbed at \( r = 0 \), but it cannot spontaneously increase). Therefore, this solution must be rejected.

These results are consistent with the van Kampen classification of singularities (see Appendix E). For \( T < T_c \) (i.e. \( a > 1 \) or \( \beta G m_1 m_2 > d \)), the singularity at \( r = 0 \) behaves as an adhesive boundary. In that case, the solution is unique and no boundary condition has to be fixed by hand. It is given by Eq. (182) leading to a Dirac peak (\( \chi_+ \neq 0 \)). For \( T_c < T < T^*_c \) (i.e. \( 0 < a < 1 \) or \( d - 2 < \beta G m_1 m_2 \)), the singularity at \( r = 0 \) behaves as a regular boundary. In that case, the boundary condition can be absorbing, reflecting or mixed. It has to be fixed by hand. The general solution can be written as a “mixture” (120) of the two solutions (182) and (185). The solution (182) leads to a Dirac peak (\( \chi_+ \neq 0 \)) contrary to the solution (185) for which the normalization is conserved (\( \chi_- = 0 \)). For \( T > T^*_c \) (i.e. \( a < 0 \) or \( \beta G m_1 m_2 < d - 2 \)), the singularity at \( r = 0 \) behaves as a natural repulsive boundary. In that case, the solution is unique and no boundary condition has to be fixed by hand. It is given by Eq. (185) which does not lead to a Dirac peak (\( \chi_+ = 0 \)).

7 Conclusion

In this paper, we have analytically studied the evolution of \( N = 2 \) Brownian particles in gravitational interaction in a space of \( d = 2 \) dimensions. Up to a change of notations, this is equivalent to the simplified motion of two biological entities interacting via chemotaxis (in which case the dimension \( d = 2 \) is physically relevant). Of course, the consideration of only \( N = 2 \) particles is an extreme limit but the problem is already involved and shows that the dynamics is complex since the particles can coalesce to form Dirac peaks. The same phenomenon (collapse and Dirac peaks) occurs for a larger number of particles and has been investigated analytically in the mean field limit \( N \to +\infty \) (9,10). It shares some analogies with the Bose-Einstein condensation (25). The case of a finite number of
particles will be investigated numerically in a forthcoming paper \cite{11}.

Finally, we would like to point out some analogies with the transport of passive particles by a stochastic turbulent flow characterized by scale invariant structure functions \cite{20,21,22,23} or, more generally, with correlated Brownian motions with scale invariant correlations \cite{29}. In particular, implosive collapse of trajectories has been found by Gawędzki & Vergassola \cite{20} for strongly compressible flows. These authors determined the statistics of inter-trajectory distances and observed a lack of normalization when the diffusivity tends to zero. Like in our problem, a defect of probability concentrates for strongly compressible flows. These authors determined the statistics of inter-trajectory distances and observed a lack of normalization when the diffusivity tends to zero. In particular, implosive collapse of trajectories has been found by Gawędzki & Vergassola \cite{20}.

Indeed, if all the particles collapse in a single point, this point will be the center of mass of the reduced particle. The center of mass has a pure Brownian motion

\[
\frac{d\mathbf{R}}{dt} = \sqrt{2D_\alpha} \mathbf{B}(t),
\]

with a diffusion coefficient

\[
D_\alpha = \frac{k_B T}{\xi M}.
\]

It satisfies a relation of the form

\[
\frac{1}{4} \xi M \frac{d(R^2)}{dt} = k_B T - P_{eff} V,
\]

where \(P_{eff}\) is an effective pressure on the boundary of the box due to the center of mass. Therefore, the virial theorem expressed in terms of \(I\) is

\[
\frac{1}{4} \frac{dI}{dt} = N k_B (T - T_\alpha) - k_B T - \Delta PV,
\]

where \(\Delta P = P - P_{eff}\). Since the motion of the center of mass is completely decoupled, it is as if we had only

\[
\frac{dI}{dt} = N k_B (T - T_\alpha) - k_B T - \Delta PV.
\]

According to the discussion of Sec. \cite{24} we now know that this relation ceases to be exact when the particles form Dirac peaks. However, we shall not address this problem here.

Acknowledgment: We thank the referee for mentioning the connection of our study with the transport of a passive particle by a stochastic turbulent flow (Refs. \cite{20,21,22,23}) and indicating to us the van Kampen classification. This led to a more detailed analysis of our model.

A Moment of inertia and critical temperatures

It is shown in \cite{12} (see also Appendix \cite{B}) that, in \(d = 2\), the total moment of inertia of self-gravitating Brownian particles

\[
I_{tot} = \sum_\alpha \langle m_\alpha r_\alpha^2 \rangle,
\]

satisfies the virial theorem\footnote{According to the discussion of Sec. \cite{24} we now know that this relation ceases to be exact when the particles form Dirac peaks. However, we shall not address this problem here.}
For a single Brownian particle (the Dirac) at temperature $T_c$, we have

$$T_c = \frac{N}{N-1} k_B T_c.$$  

(202)

For equal mass particles, we have

$$k_B T_c = N \frac{G m^2}{4}.$$  

(203)

Furthermore, if we consider an infinite domain and measure the displacement of the particles relative to the center of mass, using Eq. (201), we find that the mean square displacement of the reduced particle, we have

$$\langle \Delta r^2 \rangle = 2 k_B T_c.$$  

(204)

Integrating this relation, we get

$$I(t) = 4 D(1-a) \mu \int_0^t \chi(\tau) d\tau + I(0),$$  

(205)

with $I(0) = M r_0^2$ in our case. Using Eq. (7.3), we have

$$\int_0^t \chi(\tau) d\tau = t - \int_0^t \frac{\Gamma_a}{\Gamma(a)} \frac{r_0^2}{4D\tau} d\tau.$$  

(206)

Setting $x = r_0^2/(4Dt)$, this can be rewritten

$$\int_0^t \chi(\tau) d\tau = t - \frac{1}{\Gamma(a)} \frac{r_0^2}{4D} \int_{a^{-1}}^{a} \Gamma_a(x) dx.$$  

(207)

Let us consider the integral

$$K(s) = \int_s^{+\infty} \frac{\Gamma_a(x)}{x^2} dx,$$  

(208)

where we recall that

$$\Gamma_a(x) = \int_x^{+\infty} t^{a-1} e^{-t} dt.$$  

(209)

Integrating by parts, we get

$$K(s) = \frac{\Gamma_a(s)}{s} - \int_s^{+\infty} x^{a-2} e^{-x} dx.$$  

(210)

Integrating by parts again, we obtain

$$K(s) = \left( \frac{1}{s} + \frac{1}{1-a} \right) \Gamma_a(s) - \frac{s^{a-1} e^{-s}}{1-a}.$$  

(211)

We now have

$$I(t) = \mu r_0^2 + 4D(1-a) \mu \left[ t - \frac{1}{\Gamma(a)} \frac{r_0^2}{4D} \int_{a^{-1}}^{a} \Gamma_a(x) dx \right].$$  

(212)

Substituting Eq. (214) in (215), we recover Eq. (216).

### C The case $T \neq 0$

In this Appendix, we check that the relation (2.22) is consistent with the virial theorem (10). The virial theorem (10) can be rewritten

$$I = 4 D \mu \chi(t)(1-a).$$  

(213)

This study can be easily generalized in $d$ dimensions.
Therefore, the particle reaches the origin at a time
\[ t(a) = \frac{a^2}{2k}. \]  (219)

Equivalently, the particle that reaches the origin at time \( t \) was located initially at
\[ a(t) = \sqrt{2kt}. \]  (220)

Let \( P_0(a) \) be the initial probability density to find the particle in \( a \). The conservation of the probability density imposes
\[ P(r, t)2\pi rdr = P_0(a)2\pi da. \]  (221)

Now, according to the equation of motion (213), we have \( rdr = ada \) so that \( P(r, t) = P_0(a) \). Therefore, the probability density to find the particle in \( r \) at time \( t \) is
\[ P(r, t) = P_0 \left( \sqrt{r^2 + 2kt} \right). \]  (222)

We can check by direct substitution that this is indeed the solution of the Fokker-Planck equation (214) at \( T = 0 \):
\[ \frac{\partial P}{\partial t} = k \frac{\partial P}{\partial r} \tau. \]  (223)

The probability that the particle has not reached \( r = 0 \) at time \( t \) is
\[ \chi(t) = \int_0^{+\infty} P(r, t)2\pi rdr. \]  (224)

Using the distribution (222) and performing the change of variables (218), we get
\[ \chi(t) = \int_0^{+\infty} P_0(a)2\pi a da = 1 - \int_0^{\sqrt{2kt}} P_0(a)2\pi a da. \]  (225)

Therefore, the probability that the particle has formed a Dirac peak at time \( t \) is
\[ \chi_D(t) = \int_0^{\sqrt{2kt}} P_0(a)2\pi a da. \]  (226)

This corresponds to probability to find initially the particle in the disk of radius \( \sqrt{2kt} \). For consistency, let us derive this result in a different manner. Starting from the relation
\[ \dot{\chi}_D = 2\pi kP(0, t), \]  (227)
and using Eq. (222), we get
\[ \dot{\chi}_D = 2\pi kP_0 \left( \sqrt{2kt} \right). \]  (228)

Integrating this relation, we find that
\[ \chi_D(t) = 2\pi \int_0^t P_0 \left( \sqrt{2k\tau} \right) d\tau = \int_0^{\sqrt{2kt}} P_0(a)2\pi a da. \]  (229)

This returns Eq. (226) as it should. Finally, the moment of inertia is
\[ I(t) = \int_0^{+\infty} P(r, t)\mu r^2 2\pi r dr. \]  (230)

Substituting the probability density (222) in Eq. (230) and performing the change of variables (218), we obtain
\[ I(t) = \int_0^{+\infty} P_0(a)\mu(a^2 - 2kt)2\pi a da. \]  (231)

This can be written equivalently
\[ I(t) = \int_0^{+\infty} P_0(a)\mu^2 2\pi a da - 2k\mu\chi(t)t. \]  (232)

In Appendix D, we check that this relation is consistent with the virial theorem (43).

Let us specifically apply these results to an initial distribution of the form (60). Since the motion is deterministic, the particle initially located at \( r_0 \) will be located at \( r(t) = \sqrt{r_0^2 - 2kt} \) at time \( t \). It will reach the origin in a finite time \( t_c = r_0^2/(2k) \) (at that time, the two original particles stick together and remain tightly bound). Therefore, the probability that the particle has formed a Dirac at time \( t \) is a Heaviside function: \( \chi_D = 0 \) if \( t < t_c \) and \( \chi_D = 1 \) if \( t > t_c \). The moment of inertia is \( I(t) = \mu r(t)^2 \) if \( t \leq t_c \) and \( I(0) = 0 \) if \( t \geq t_c \).

## D Check of consistency for \( T = 0 \)

For \( T = 0 \), the virial theorem (43) becomes
\[ \dot{I} = -2k\mu\chi(t). \]  (233)

Integrating this relation, we get
\[ I(t) = I(0) - 2k\mu\int_0^t \chi(t) d\tau. \]  (234)

Integrating by parts, we have
\[ \int_0^t \chi(t) d\tau = t\chi(t) - \int_0^t \dot{\chi}(t)\tau d\tau. \]  (235)

According to Eqs. (30) and (222), we have
\[ \dot{\chi}(t) = -2\pi kP_0(\sqrt{2kt}). \]  (236)

Therefore
\[ I(t) = I(0) - 2k\mu\chi(t)t - 4\pi k^2 \mu \int_0^t P_0(\sqrt{2kt})\tau d\tau. \]  (237)

Setting \( a = \sqrt{2kt} \), we obtain
\[ I(t) = I(0) - 2k\mu\chi(t)t - 2\pi \mu \int_0^{\sqrt{2kt}} P_0(a)a^3 da. \]  (238)

Since
\[ I(0) = 2\pi \mu \int_0^{+\infty} P_0(a)a^3 da, \]  (239)
we finally recover Eq. (226).
E Asymptotic behaviors

In this Appendix, we determine the asymptotic behaviors of the function

$$f(x) = \frac{1}{x \Gamma(x)^{1/x}}, \quad (240)$$

appearing in Eq. (247). We first determine the behavior of the Gamma function for small $x$. Expanding the identity

$$\Gamma(1 - x) \Gamma(x) = \frac{\pi}{\sin(\pi x)}, \quad (241)$$

for $x \to 0$, and using

$$\Gamma'(1) = −\gamma \simeq −0.57721..., \quad (242)$$

where $\gamma$ is the Euler constant, we obtain

$$x \Gamma'(x) \sim 1 - \gamma x, \quad (x \to 0). \quad (243)$$

From the previous result, we deduce that

$$\ln \left\{ \frac{1}{x \Gamma(x)^{1/x}} \right\} \sim \frac{1}{x} \ln(1 - \gamma x) \to -\gamma. \quad (244)$$

Therefore, for $x \to 0$, we find that

$$\frac{1}{x \Gamma(x)^{1/x}} \to e^\gamma \simeq 1.78107... \quad (245)$$

On the other hand, using the equivalent

$$\Gamma(x) \sim \sqrt{2\pi e^{-x}} x^{x - \frac{1}{2}}, \quad (x \to +\infty), \quad (246)$$

we find, for $x \to +\infty$, that

$$\frac{1}{x \Gamma(x)^{1/x}} \sim \frac{e}{x}. \quad (247)$$

Considering now the function defined by Eq. (258) and using the equivalent

$$\Gamma(x) \sim \frac{1}{x}, \quad (x \to 0), \quad (248)$$

we find, for $a \to 0$, that

$$\chi_D(1) \sim a \Gamma_0(1) \sim 0.219384a. \quad (249)$$

To determine its asymptotic behavior for $a \to +\infty$, we first note that $\Gamma_n(x) = \Gamma(a) - \gamma(a, x)$ where

$$\gamma(a, x) = \int_0^x e^{-t} a^{t-1} dt. \quad (250)$$

For $a \to +\infty$, we have the asymptotic expansion

$$\gamma(a, x) \sim \sum_{n=0}^{+\infty} \frac{(-1)^n x^a + n}{(a + n)n!}. \quad (251)$$

This implies $\gamma(a, 1) \sim 1/(ae)$. On the other hand,

$$\Gamma(a) \sim \sqrt{2\pi e^{-a}} a^{a - \frac{1}{2}}. \quad (252)$$

Combining these results and recalling that $\chi_D(1) = 1 - \gamma(a, 1)/\Gamma(a)$, we obtain

$$1 - \chi_D(1) \sim \frac{1}{\sqrt{2\pi} a^{a+1/2}}, \quad (253)$$

for $a \to +\infty$.

F The van Kampen classification

In this Appendix, we apply to our system the boundary classification introduced by van Kampen [28]. For a summary, we refer to Gabrielli and Cecconi [29]. In order to avoid repetitions, we shall use their notations and we refer the reader to their paper for more details.

For the sake of generality, we consider the logarithmic Fokker-Planck equation in $d$ dimensions

$$\frac{\partial P}{\partial t} = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} \left\{ Dr^{d-1} \left( \frac{\partial P}{\partial r} + P \beta Gm_{1/m_2} \right) \right\}, \quad (254)$$

To apply the van Kampen classification, we need first to transform the Fokker-Planck equation (254) into a one dimensional Fokker-Planck equation. To that purpose, we set $r = \sqrt{D} \chi$ and $f(x, t) = \sqrt{D} P(r, t)|_{S\chi}$. This transforms Eq. (254) into

$$\frac{\partial f}{\partial t} = - \frac{d}{dx} \left( \frac{d - 1}{x} f + \beta Gm_{1/m_2} f \right), \quad (255)$$

with the normalization condition $\int_0^{+\infty} f(x, t) \, dx = 1$. This is a one dimensional Fokker-Planck equation of the form

$$\frac{\partial f}{\partial t} - \frac{1}{2} \frac{\partial^2}{\partial x^2} (D(x)f) - \frac{\partial}{\partial x} (K(x)f), \quad (256)$$

with $D(x) = 2$ and $K(x) = (d - 1 - \beta Gm_{1/m_2})/x$. Van Kampen’s classification for a singularity $x = 0$ is based on the analysis of the behavior for $\epsilon \to 0$ of the integrals

$$L_1 = \int_0^{x_0} dx \, e^{\phi(x)}, \quad (257)$$

$$L_2 = \int_\epsilon^{x_0} dx \, e^{\phi(x)} \int_0^x dx' \frac{e^{-\phi(x')}}{D(x')}, \quad (258)$$

$$L_3 = \int_\epsilon^{x_0} dx \, e^{-\phi(x)} D(x), \quad (259)$$

where

$$\phi(x) = -2 \int_0^x dx' \frac{K(x')}{D(x')}. \quad (260)$$

and with $x_0 > 0$. For the logarithmic Fokker-Planck equation (255), we have $D(x) = 2$ and

$$\phi(x) = -(d - 1 - \beta Gm_{1/m_2}) \ln \left( \frac{x}{x_0} \right). \quad (261)$$

Considering the limit $\epsilon \to 0$, it is easy to see that: (i) $L_1 < +\infty$ iff $\beta Gm_{1/m_2} > d - 2$, (ii) $L_2 < +\infty$ iff $\beta Gm_{1/m_2} > d - 2$, (iii) $L_3 < +\infty$ iff $\beta Gm_{1/m_2} < d$. Therefore, according to van Kampen’s classification, we need to consider three cases (it is useful to introduce the critical temperatures $T_s = Gm_{1/m_2}/d$ and $T_s' = Gm_{1/m_2}/(d - 2)$):
\( \text{(i) if } \beta G_m m_2 < d - 2, \text{ i.e. } T > T^*_s, \text{ the singularity } x = 0 \text{ behaves as a natural repulsive boundary } (L_1 \to +\infty). \text{ The particle run away from the singularity never touching it. The solution is unique.} \)

\( \text{(ii) if } d - 2 < \beta G_m m_2 < d, \text{ i.e. } T_s < T < T^*_s, \text{ the singularity } x = 0 \text{ behaves as a regular boundary } (L_1, L_2, L_3 < +\infty). \text{ In that case, an absorbing or reflecting boundary condition has to be fixed by hand to determine the solution of the equation.} \)

\( \text{(iii) if } \beta G_m m_2 > d, \text{ i.e. } T < T_s, \text{ the singularity } x = 0 \text{ behaves as an attractive adhesive boundary } (L_1, L_2 < +\infty \text{ and } L_3 \to +\infty). \text{ In that case, } f(x, t) \text{ develops a Dirac peak at } x = 0 \text{ with a time increasing coefficient. The solution is unique.} \)

For \( d \leq 2 \), we only have a transition at temperature \( T_s = G m_2 / d \). For \( d > 2 \), we have two transitions at temperatures \( T_s = G m_2 / d \) and \( T^*_s = G m_2 / (d - 2) \).

\section*{G Temporal correlation functions and front structure of the logarithmic Fokker-Planck equation}

We consider the logarithmic Fokker-Planck equation \((254)\) in a space of dimension \( d \). We assume that the domain is unbounded. In order to have an equilibrium state \( P_\star(x) \), the potential must be regularized at short distances. Therefore, we assume that the potential has a logarithmic behavior for sufficiently large \( r \) and that it tends to a finite constant for \( r \to 0 \). In that case, \( f(x, t) \) develops a Dirac peak at \( x = 0 \) with a time increasing coefficient. The solution is unique.

The total density profile of the self-gravitating Brownian gas can be written

\[ \rho(r, t) = M_D(t) \delta(r) + \rho(r, t), \quad (265) \]

where the first term takes into account the possible formation of a Dirac peak at \( r = 0 \) and the second term is the (regular) density profile excluding the Dirac. The total mass is \( M = M_D(t) + M(t) \) where \( M_D(t) \) is the mass contained in the Dirac peak and \( M(t) = \int \rho(r, t) \, d\mathbf{r} \) is the mass outside the Dirac. The Smoluchowski-Poisson system accounting for the presence of a Dirac peak can be written

\[ \xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \frac{k_B T}{m} \nabla \rho + \frac{G M_D(t)}{r^d} \mathbf{r} + \rho \nabla \Phi \right), \quad (266) \]

\[ \Delta \Phi = S_0 G \rho. \quad (267) \]

Using \( M_D = -M \) and integrating Eq. \((266)\) on the infinite space, we find that the mass accumulated in the Dirac peak by unit of time is

\[ \frac{dM_D}{dt} = \frac{S_0 G}{\xi} M_D(t) \rho(0, t). \quad (268) \]

Equations \((266) - (268)\) form a closed system describing the evolution of the system in the pre and post collapse regimes. These equations have been studied in \([24]\).

We now specialize on the 2D Smoluchowski-Poisson system. Taking the time derivative of the moment of inertia

\[ I = \int \rho \, d\mathbf{r}, \quad (269) \]

and using Eq. \((266)\), we obtain after integrations by parts

\[ \xi \frac{dI}{dt} = \frac{4k_B T}{m} M(t) - 2G M(t) M_D(t) + 2W_{ii}, \quad (270) \]

where \( W_{ii} = -\int \rho \mathbf{r} \cdot \nabla \Phi \, d\mathbf{r} \) is the virial of the gravitational force. In \( d = 2 \) dimensions, it is equal to \( W_{ii} \)
\(-GM(t)^2/2\) (see [6]) and Appendix [1]. Therefore, we obtain the virial theorem
\[
\frac{1}{4} \frac{dI}{dt} = M(t) \left( \frac{k_BT}{m} - \frac{GM_D(t)}{2} - \frac{GM(t)}{4} \right),
\]
that is valid in all the regimes of the dynamics. For \(T > T_c\) or in the pre-collapse regime \(t < t_{coll}\) for \(T < T_c\), there is no Dirac peak at \(r = 0\). In that case, \(M_D(t) = 0\), \(M(t) = M\), and the virial theorem [271] reduces to
\[
\frac{1}{4} \frac{dI}{dt} = Nk_B(T - T_c),
\]
where \(k_BT_c = GMm/4\). This returns the result of [6]. Let us now consider the post collapse regime \(t > t_{coll}\) for \(T < T_c\). When \(t \to +\infty\), almost all the mass is in the Dirac so that \(M_D(t) \simeq M\) and \(M(t) = M - M_D(t) \simeq 0\). In that case, the Smoluchowski equation [265] can be approximated by
\[
\xi \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \frac{k_BT}{m} \nabla \rho + \frac{GM}{r^2} \rho \right),
\]
and Eq. [268] becomes
\[
\frac{dM_D}{dt} = -S_dG \xi \rho(0, t).
\]
This is equivalent to Eqs. [33] and [40] studied in this paper, with a simple change of notations discussed in Sec. [4]. In that case, the virial theorem [271] becomes
\[
\frac{1}{4} \frac{dI}{dt} = N(t)k_B(T - T_c),
\]
where \(k_BT_c = GMm/2\) (i.e. \(T_c = 2T_c\)). This is equivalent to the virial theorem [43].

I Virial theorem for power-law interactions

In this Appendix, we provide the general form of the virial theorem for Brownian particles with power law interactions in \(d\) dimensions. We only give the final expressions, and refer to [6] for more details on their derivation. As explained in Sec. [2.3] the following expressions are valid as long as there are no Dirac peaks.

Let us consider \(N\) Brownian particles with individual mass \(m_\alpha\) in a space of dimension \(d\). We assume that the particles are subject to an external harmonic potential \(V(r) = \frac{1}{2}m_\alpha^2r^2\) and that they interact through an algebraic potential \(u(\xi) = \frac{1}{\xi^{d+\gamma}}\) if \(\gamma \neq 2 - d\) and through a logarithmic potential \(u(\xi) = G\ln \xi\) if \(\gamma = 2 - d\), both corresponding to a force \(-u'(\xi) = -G/\xi^{d+\gamma-1}\). The gravitational potential is recovered for \(\gamma = 0\). The case studied in the present paper is very particular because it corresponds to a logarithmic (\(\gamma = 2 - d\)) and a Newtonian (\(\gamma = 0\)) interaction. The stochastic equations of motion of the particles are
\[
\ddot{x}_{i\alpha} = \sum_{\beta \neq \alpha} \frac{Gm_\beta(x_{i\beta} - x_{i\alpha})}{|r_{\beta} - r_{\alpha}|^{d+\gamma}} - \omega_0^2 x_{i\alpha} - \xi \dot{x}_{i\alpha} + \sqrt{2D_xB_{i\alpha}}(t),
\]
where \(B_{i\alpha}(t)\) is a white noise. Here, the Greek letters refer to the particles and the Latin letters to the coordinates of space. The diffusion coefficient is given by the Einstein formula \(D_x = \frac{Gk_BT}{m_\alpha}\). The moment of inertia tensor is defined by
\[
I_{ij} = \sum_\alpha m_\alpha x_{i\alpha} x_{j\alpha}.
\]
We introduce the kinetic energy tensor
\[
K_{ij} = \frac{1}{2} \sum_\alpha m_\alpha x_{i\alpha} x_{j\alpha},
\]
and the potential energy tensor
\[
W_{ij} = G \sum_{\alpha \neq \beta} m_\alpha m_\beta \frac{(x_{i\alpha} - x_{i\beta})(x_{j\beta} - x_{j\alpha})}{|r_{\beta} - r_{\alpha}|^{d+\gamma}},
\]
where the second equality results from simple algebraic manipulations obtained by interchanging the dummy variables \(\alpha\) and \(\beta\) and summing the resulting expressions. The tensor virial theorem associated with the stochastic equations [276] is
\[
\frac{1}{2} \dot{I}_{ij} + \frac{1}{2} \xi \dot{I}_{ij} + \omega_0^2 I_{ij} = 2K_{ij} + W_{ij} - \frac{1}{2} \int (P_{ik}x_j + P_{jk}x_i) dS_k,
\]
where the last term takes into account pressure forces at the boundary of the system. For Brownian particles, it is implicitly assumed that the quantities appearing in Eq. [280] are averaged over the noise and over statistical realizations, while for Hamiltonian systems (\(\xi = D_\alpha = 0\), Eq. [280]) is exact without averages. The scalar virial theorem is obtained by contracting the indices leading to
\[
\frac{1}{2} \dot{I} + \frac{1}{2} \xi \dot{I} + \omega_0^2 I = 2K + W_{ii} - \int P_{ik}x_j dS_k,
\]
where
\[
I = \sum_\alpha m_\alpha x_{i\alpha}^2, \quad K = \frac{1}{2} \sum_\alpha m_\alpha v_{i\alpha}^2,
\]
are the moment of inertia and the kinetic energy. On the other hand, \(W_{ii}\) is the virial which takes the form
\[
W_{ii} = -\frac{1}{2} G \sum_{\alpha \neq \beta} \frac{m_\alpha m_\beta}{|r_{\beta} - r_{\alpha}|^{d+\gamma-2}}.
\]
For \(\gamma \neq 2 - d\), we find that
\[
W_{ii} = (d + \gamma - 2)W,
\]
where \(W\) is the potential energy
\[
W = -\frac{G}{2(d + \gamma - 2)} \sum_{\alpha \neq \beta} \frac{m_\alpha m_\beta}{|r_{\beta} - r_{\alpha}|^{d+\gamma-2}}.
\]
In that case, the scalar virial theorem reads
\[
\frac{1}{2} \ddot{\xi} + \frac{1}{2} \dot{\xi}^2 + \omega_0^2 I = 2K + (d + \gamma - 2)W - \oint P_{ik} \dot{x}_i dS_k.
\] (286)

For Hamiltonian systems ($D = \xi = 0$), the scalar virial theorem in an unbounded domain ($P = 0$) reduces to:
\[
\frac{1}{2} \ddot{\xi} + \omega_0^2 I = 2K + (d + \gamma - 2)W.
\] (287)

Since the total energy $E = K + W + \frac{1}{2} \omega_0^2 I$ is conserved, we obtain
\[
\frac{1}{2} \ddot{\xi} + 2\omega_0^2 I = 2E + (d + \gamma - 4)W.
\] (288)

For the index $\gamma = 4 - d$, we get
\[
\ddot{\xi} + 4\omega_0^2 I = 4E.
\] (289)

When $\omega_0^2 = 0$, we obtain $\ddot{\xi} = 4E$ which yields after integration $I = 2Et^2 + C_1 t + C_2$. For $E > 0$, $\ddot{\xi} \rightarrow +\infty$ indicating that the system evaporates. For $E < 0$, $\ddot{\xi}$ goes to zero in a finite time, indicating that the system forms a Dirac peak in a finite time. When $\omega_0^2 > 0$, the moment of inertia $I$ oscillates with pulsation $2\omega_0 \omega$.

For the index $\gamma = 4 - d$, corresponding to a logarithmic potential in $d$ dimensions, we have the simple exact result
\[
W_{ii} = -\frac{1}{2} G \sum_{\alpha \neq \beta} m_\alpha m_\beta.
\] (290)

It is interesting to note that this expression only depends on the mass of the particles and not on their position. For equal mass particles,
\[
W_{ii} = -\frac{1}{2} G N (N - 1) m^2.
\] (291)

Since
\[
\sum_{\alpha \neq \beta} m_\alpha m_\beta = M^2 - \sum_\alpha m_\alpha^2,
\] (292)

we see that the first term is of order $N^2 \overline{m}^2$ and the second of order $N \overline{m}^2$ (where $\overline{m}$ is a typical mass). Therefore, in the mean-field limit $N \rightarrow +\infty$, we obtain
\[
W_{ii}^{m,f} = -\frac{G M^2}{2},
\] (293)

whatever the number of species in the system.

At equilibrium, the scalar virial theorem (284) reduces to
\[
2K + W_{ii} - \omega_0^2 I = \oint P_{ik} \dot{x}_i dS_k.
\] (294)

For Hamiltonian systems, this relation is valid for a steady state after time averages, or averages over statistical realizations, have been made. If the system is at statistical equilibrium, then $K = \frac{4}{3} Nk_B T$ and $P_{ij} = \rho \delta_{ij}$ with $p = \sum \rho_i k_B T / m_i$, where $\rho_i$ refers to the density of the different species. Introducing the notation $P = \frac{1}{2} \rho \rho \cdot \delta_{ij} dS$, we get
\[
dk_B T + W_{ii} - \omega_0^2 I = dPV.
\] (295)

For an ideal gas without interaction ($W_{ii} = 0$), we recover the perfect gas law $PV + \omega_0^2 I / d = Nk_B T$ in the presence of a harmonic potential (when $\omega_0^2 > 0$, we get in an unbounded domain $I = dNk_B T / \omega_0^2$, and when $\omega_0 = 0$, we get $PV = Nk_B T$). Alternatively, for a gas with logarithmic interactions ($\gamma = 2 - d$), using Eq. (290), we obtain the exact equation of state
\[
PV + \frac{\omega_0^2 I}{d} = Nk_B (T - T_c),
\] with the exact critical temperature
\[
k_B T_c = \frac{G \sum_{\alpha \neq \beta} m_\alpha m_\beta}{2dN}.
\] (297)

For equal mass particles, we get
\[
k_B T_c = (N - 1) \frac{Gm^2}{2d}.
\] (298)

In the mean-field limit
\[
k_B T_c^{m,f} = \frac{G M^2}{2dN}
\] (299)

If $\omega_0 = 0$, the equation of state (296) reduces to
\[
PV = Nk_B (T - T_c).
\] (300)

When $\omega_0^2 \geq 0$, according to Eq. (296), an equilibrium state can possibly exist only for $T \geq T_c$ (since $P \geq 0$ and $I \geq 0$). For $T = T_c$, we have $P = 0$ if $\omega_0^2 = 0$, $P = I = 0$ if $\omega_0^2 > 0$ and $I = dPV / \Omega_0^2$ if $\omega_0^2 = -\Omega_0^2 < 0$. In an unbounded domain ($P = 0$), we get
\[
\frac{\omega_0^2 I}{d} = Nk_B (T - T_c).
\] (301)

If $\omega_0 = 0$, an equilibrium state can possibly exist only for $T = T_c$. If $\omega_0^2 = -\Omega_0^2 < 0$, an equilibrium state can possibly exist only for $T < T_c$. For $T = T_c$ and $\omega_0 \neq 0$, we must have $I = 0$.

We now consider the strong friction limit $\xi \rightarrow +\infty$ where inertial effects are negligible. In that limit, the velocities thermalize on a timescale of order $1/\xi$. In that case, $K_{ij} = \frac{1}{2} Nk_B T \delta_{ij}$ and $P_{ij} = \rho \delta_{ij}$ with $p = \frac{1}{2} Nk_B T \delta_{ij}$ and $P_{ij} = \rho \delta_{ij}$.
\[ \sum_{a} \rho_a k_B T / m_a \text{ even if the system has not yet reached a state of mechanical equilibrium} \text{ [6]. From Eq. (287), we obtain the overdamped virial theorem for a self-gravitating Brownian gas} \]

\[ \frac{1}{2} \xi I_{ij} + \omega_0^2 I_{ij} = N k_B T \delta_{ij} + W_{ij} - \frac{1}{2} \iint p(x, dS_j + x_j dS_i). \tag{302} \]

We can obtain this result in a different manner. In the strong friction limit \( \xi \to +\infty \), the inertial term in Eq. (276) can be neglected so that the stochastic equations of motion reduce to

\[ \dot{x}_i^\alpha = \mu_\alpha m_\alpha \sum_{\beta \neq \alpha} G_{\beta \alpha}(x^\beta - x^\alpha) + \omega_\alpha^2 \dot{x}_i^0 + \sqrt{2D_\alpha B_\alpha^0}(t), \tag{303} \]

where \( D_\alpha = k_B T \mu_\alpha \) is the diffusion coefficient is physical space and \( \mu_\alpha = 1 / (\xi m_\alpha) \) the mobility. The overdamped virial theorem [202] can be directly obtained from these stochastic equations [6]. The scalar virial theorem reads

\[ \frac{1}{2} \xi I + \omega_0^2 I = dNk_BT + W_{ii} - dPV. \tag{304} \]

For \( \gamma = 2 - d \), using Eq. (290), we obtain

\[ \frac{1}{2} \xi I + \omega_0^2 I = dNk_B(T - T_c) - dPV, \tag{305} \]

with the exact critical temperature [207]. In an infinite domain \((P = 0)\), this relation reduces to

\[ \frac{1}{2} \xi I + \omega_0^2 I = dNk_B(T - T_c). \tag{306} \]

This is a closed equation that can be solved analytically. For \( \omega_0 \neq 0 \), the solution is

\[ I(t) = \left[ I(0) - \frac{dNk_B}{\omega_0^2}(T - T_c) \right] e^{-\frac{2\omega_0^2}{\xi} t} + \frac{dNk_B}{\omega_0^2}(T - T_c). \tag{307} \]

Let us introduce the new critical temperature

\[ k_B T_\omega = k_BT_c + \frac{2}{\omega_0^2} \frac{I(0)}{dN}, \tag{308} \]

depending on the initial value of moment of inertia (note that \( T_\omega > T_c \)). This is the value at which the term in bracket in Eq. (307) vanishes. If \( T > T_c \), the system tends to an equilibrium state corresponding to \( I_{eq} = \frac{dNk_B}{\omega_0^2}(T - T_c) \), see Eq. (301). More precisely, for \( T > T_\omega \), the moment of inertia increases, for \( T_c < T < T_\omega \) it decreases and for \( T = T_\omega \) it remains constant. If \( T = T_c \), we find that \( I(t) \to 0 \) for \( t \to +\infty \) implying a collapse in infinite time. If \( T < T_c \), the moment of inertia vanishes at a time

\[ t_{end} = \frac{\xi I(0)}{2dNk_B(T_c - T)}. \tag{309} \]

implying the finite time collapse of the system. If \( \omega_0^2 = -\Omega^2 < 0 \) (repulsive harmonic potential or rotation), the picture is different. For \( T > T_\omega \) (note that now \( T_\omega < T_c \)), the system evaporates. For \( T < T_\omega \), which is possible iff \( I(0) < dNk_B T_c / (-\omega_0^2) \), the moment of inertia vanishes at a time \( t_{end} \), given by Eq. (301), implying finite time collapse. Finally, for \( T = T_\omega \), the moment of inertia is conserved. Some representative curves of these different evolutions are given in Figs. [12] [13] and [14].

We now give the proper form of virial theorem corresponding to the generalized Smoluchowski equation [6]:

\[ \frac{\partial p}{\partial t} = \nabla \cdot \left[ \frac{\xi}{\rho} \left( \nabla p + \rho \nabla \Phi + \rho \omega_0^2 \gamma \right) \right], \tag{312} \]

where the pressure \( p = p(r,t) \) is given by an arbitrary barotropic equation of state \( p = p(\rho) \). We assume that
the particles are subject to an external harmonic potential $V(\mathbf{r}) = \frac{1}{2} \omega_0^2 \mathbf{r}^2$ and that they interact through an algebraic potential

$$\Phi(\mathbf{r}, t) = -\frac{G}{d + \gamma - 2} \int \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|^{d+\gamma-2}} d\mathbf{r}',$$

(313)

if $\gamma \neq 2 - d$, or a logarithmic potential

$$\Phi(\mathbf{r}, t) = G \int \rho(\mathbf{r}', t) \ln |\mathbf{r} - \mathbf{r}'| d\mathbf{r'},$$

(314)

if $\gamma = 2 - d$. The mean force of interaction acting on a particle in $\mathbf{r}$ is

$$\mathbf{F} = -\nabla \Phi = -G \int \frac{\rho(\mathbf{r}', t) \mathbf{r}' - \mathbf{r}}{|\mathbf{r} - \mathbf{r}'|^{d+\gamma}} d\mathbf{r}'. $$

(315)

For simplicity, we assume that the particles have the same mass $m$. The Lyapunov functional associated with the Smoluchowski equation is the free energy

$$F = \int \rho \int_0^\rho \frac{p(\rho')}{\rho^2} d\rho' d\mathbf{r} + \frac{1}{2} \int \rho \Phi d\mathbf{r} + \int \rho V d\mathbf{r},$$

(316)

and it satisfies an $H$-theorem, i.e. $\dot{F} \leq 0$. The Smoluchowski equation with an isothermal equation of state $p(\mathbf{r}, t) = \rho(\mathbf{r}, t) k_B T/m$ is the mean field Fokker-Planck equation associated with the overdamped stochastic process. The potential energy tensor is defined by

$$W_{ij} = -\int \rho x_i \frac{\partial \Phi}{\partial x_j} d\mathbf{r},$$

(317)

while the virial is

$$W_{ii} = -\int \rho \mathbf{r} \cdot \nabla \Phi d\mathbf{r}.$$  

(318)

Substituting Eq. (315) in Eq. (317) and using the usual symmetrization procedure, it can be rewritten

$$W_{ij} = -\frac{1}{2} G \int \rho(\mathbf{r}) \rho(\mathbf{r}') \frac{(x_i - x_i')(x_j - x_j')}{|\mathbf{r} - \mathbf{r}'|^{d+\gamma}} d\mathbf{r} d\mathbf{r}'.$$  

(319)

Contracting the indices, we get

$$W_{ii} = -\frac{1}{2} G \int \frac{\rho(\mathbf{r}) \rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^{d+\gamma}} d\mathbf{r} d\mathbf{r}'.$$  

(320)

For $\gamma \neq 2 - d$, we obtain

$$W_{ii} = (d + \gamma - 2) W,$$

(321)

where $W = \frac{1}{4} \int \rho \Phi d\mathbf{r}$ is the mean field potential energy. For $\gamma = 2 - d$, we obtain

$$W_{ii} = -\frac{G \gamma}{2}.$$  

(322)

Introducing the moment of inertia tensor

$$I_{ij} = \int \rho x_i x_j d\mathbf{r},$$

(323)

we find that the tensor virial theorem associated with the generalized Smoluchowski equation is given by

$$\frac{1}{2} \dot{I}_{ij} + \omega_0^2 I_{ij} = \delta_{ij} \int p d\mathbf{r} + W_{ij} - \frac{1}{2} \int p(x_i dS_j + x_j dS_i).$$

(324)

The scalar virial theorem, obtained by contracting the indices, takes the form

$$\frac{1}{2} \dot{I} + \omega_0^2 I = \int p d\mathbf{r} + W_i - dP V,$$

(325)

where

$$I = \int \rho \mathbf{r}^2 d\mathbf{r},$$

(326)

is the moment of inertia. The equilibrium scalar virial theorem is

$$\omega_0^2 I = \int p d\mathbf{r} + W_i - dP V.$$  

(327)

For an isothermal equation of state $p = \rho k_B T/m$, we recover Eq. (304) where $W_i$ is now given by Eq. (322). For a logarithmic potential ($\gamma = 2 - d$), we recover Eq. (305) where $T_c$ is given by

$$k_B T_c = \frac{G \gamma n m^2}{2d}.$$  

(328)

At equilibrium, we recover Eq. (293). Finally, we give the proper form of virial theorem for the damped barotropic Euler equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

(329)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{\rho} \nabla p - \nabla \Phi - \xi \mathbf{u} - \omega_0^2 \mathbf{r},$$

(330)

under the same conditions as before. The tensor virial theorem is given by

$$\frac{1}{2} \dot{I}_{ij} + \frac{1}{2} \dot{I}_{ij} + \omega_0^2 I_{ij} = \int \rho u_i u_j d\mathbf{r} + \delta_{ij} \int p d\mathbf{r}$$

$$+ W_{ij} - \frac{1}{2} \int p(x_i dS_j + x_j dS_i),$$

(331)
and the scalar virial theorem by
\[ \frac{1}{2} \ddot{I} + \frac{1}{2} \xi \dot{I} + \omega_0^2 I = \int \rho u^2 \, dr + d \int p \, dr + W_{ii} - dPV. \]
(332)

At equilibrium, we obtain Eq. (327). The virial theorem for the barotropic Euler equations is recovered by taking \( \xi = 0 \) [40,41]. The Lyapunov functional associated with the damped Euler equations (329)-(330) is the free energy
\[ F = \int \rho \int_0^\rho \frac{p(r')}{\rho^2} \, dr' + \frac{1}{2} \int \rho \dot{\phi} \, dr + \int \rho V \, dr + \int \rho \frac{u^2}{2} \, dr, \]
(333)
and it satisfies an \( H \)-theorem, i.e. \( \dot{F} \leq 0 \) if \( \xi \neq 0 \). For the Euler equations (\( \xi = 0 \)), the energy functional (333) is conserved \( \dot{F} = 0 \).

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