\[ PT\ symmetry\ and\ supersymmetry\]

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Abstract. Pseudo-Hermitian (so called PT symmetric) Hamiltonians are featured within a re-formulated Witten’s supersymmetric quantum mechanics. An unusual form of the supersymmetric partnership between the spiked harmonic oscillators is described.

1. Supersymmetry for pedestrians, or bosons and fermions in Witten’s picture

In the review [1] of the Witten’s (or so called supersymmetric) quantum mechanics one finds the specific Fock vacuum

\[ \exp(\frac{q^2}{2}) = \frac{\mathcal{P}}{\mathcal{Q}} - \frac{\mathcal{P}}{\mathcal{Q}} \]

(1)

In the upper line we recognize the ground state of the one-dimensional harmonic oscillator \( H(\text{HO}) = p^2 + q^2 \) and see that “bosons” may be created/annihilated by the action of the respective operators \( \partial_x + q \) and \( \partial_x + q \) (to be denoted as \( B(1) \) and \( A(1) \) here). The solvable harmonic-oscillator character of this elementary model enables us to define the three two-by-two matrices

\[ F_Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad N_F = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

(2)

which create and annihilate a “fermion” and determine the fermionic number, respectively. Formally, this enables us to introduce the two partner Hamiltonians \( H(\text{L}) = H(\text{HO}) - 1 \), \( H(\text{R}) = H(\text{HO}) + 1 \) and identify the underlying symmetry with the superalgebra \( \text{sl}(1=1) \) generated by the following three operator matrices

\[ H = \begin{pmatrix} 0 & 1 \\ H(\text{L}) & 0 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad Q' = \begin{pmatrix} 0 & B \\ A & 0 \end{pmatrix} \]

(3)

One easily verifies that \( fQ;Q'g = H \) while \( fQ;Qg = fQ';Q'g = 0 \) and \( [H;Q] = [H;Q'] = 0 \).

Our forthcoming considerations may by summarized as a generalization of the above harmonic-oscillator-based model to \( D \) dimensions. Beyond its obvious phenomenological and methodical appeal, the mathematical motivation for such a construction stems from the well known requirements of absence of the centrifugal-type singularities in the general Witten’s formalism [1]. Indeed, within the standard, Hermitian quantum mechanics, all ways of suppression of this difficulty seem to remain unclear up to these days [2]. In contrast, the weakening of the Hermiticity (to the so called \( PT\ ) symmetry – see below) appears to be amazingly efficient in this context [3, 4].
2. \( PT \) symmetry for pedestrians, interpreted as a regularization of a spike in the force

Our key idea dates back to Buslaev and Grecchi [5] who proposed a specific regularization of the non-vanishing centrifugal term in \( D \neq 1 \) dimensions (in their case, for some specific anharmonic oscillator examples) via a constant complex shift of the coordinate \( r = x + i \theta \) better understood as a transition to the (increasingly popular [6]) non-Hermitian formalism of the so called \( PT \) symmetric quantum mechanics [7]. In its present implementation, this merely means an extension of the real-line symmetry \( P \) to the complex plane of \( r \in \mathbb{C} \). Thus, we require the invariance of our Hamiltonian with respect to the parity \( P \) multiplied by the time reversal mimicked by the complex conjugation, \( T i = i \).

2.1. Illustration: \( PT \) symmetric version of the \( D \) dimensional harmonic oscillator

For the sake of brevity we shall only assign here our supersymmetry generators to the generalized class of the spiked harmonic oscillator Hamiltonians

\[
H^{(i)} = \frac{\partial^2}{\partial r^2} + \frac{2}{r^2} + r^2; \quad > 0;
\]

Their Buslaev’s and Grecchi’s regularization will use \( r = x + i \theta \) with real \( x \) and has thoroughly been studied in ref. [8]. Its normalizable wave functions

\[
\langle r \rangle L_n^{(\theta)} = \hbar r N; \quad \beta = \frac{N!}{N + \beta + 1} \exp(\! x^2 \! - \! 2) \quad L_n^{(\beta)}(r^2)
\]

and energies

\[
E = E N^{(\theta)} = 4N + 2\theta + 2; \quad \beta = \theta
\]

are both labeled by an additional quantum number \( \theta = 1 \) of the so called quasi-parity. In the context of fields, this concept proves closely related to the well known charge-conjugation symmetry \( C \) [3].

3. \( PT \) symmetric supersymmetry and the pedestrian’s spiked harmonic oscillator

We noticed in ref. [4] that the complexification of \( r \) regularizes the Witten’s spiked harmonic oscillator (SHO) superpotential

\[
W^{(i)}(r) = \frac{\partial}{\partial r} + \frac{1}{r}; \quad i = r; \quad + 1=2
\]

as well as all the related operator matrix elements in the \( sl(1|1) \) generators (3).

\[
A^{(i)} = \Theta + W^{(i)}; \quad B^{(i)} = \Theta + W^{(i)}; \quad \Theta; \quad 0; \quad 1; \ldots;
\]

This returns us to the \( D = 1 \) oscillator of section 1 at the “exceptional” value of \( \beta = 1=2 \) while at all the complex we have the generalized SUSY partners,

\[
H^{(i)} = H^{(0)} 2 2; \quad H^{(i)} = H^{(0)} 2 j j = j + 1 j;
\]

Whenever \( 2 ( 1 ; 1 ) \) is real (while \( j j \) and \( j + 1 j \) are defined as positive), we have to distinguish between the following three different SUSY regimes characterized by the unbroken \( PT \) symmetry.

- I: \( \text{large negative} = < 1; \text{dominant} = + 1 \)
- II: \( \text{small negative} = > 1; \text{both small} + 1 = 1 \)
- III: \( \text{positive} = > 0; \text{dominant} = + 1 \)
The energies (arranged in the descending order) form a (once degenerate) **completely real** quadruplet at each $N$.

| SUSY partner energies | I:         | II:         | III:        |
|------------------------|------------|-------------|-------------|
| $E_{(L)}^{(\cdot)}$    | $4N + 4$   | $4N + 4$    | $4N + 4$    |
| $E_{(L)}^{(\cdot)}$    | $4N + 4$   | $4N + 4$    | $4N$        |
| $E_{(L)}^{(\cdot)}$    | $4N + 4$   | $4N + 4$    | $4N$        |
| $E_{(L)}^{(\cdot)}$    | $4N$       | $4N$        | $4N$        |

It is amusing to notice that up to the regular case (with $\lambda = 1=2$) there always exist two alternative $= \, \text{to a given } < 0$. Thus, each also has the two different partners such that $1 = j \, \text{and } 1j < 2 = \, \text{and } + 1$. Finally, in the domain II, all our SUSY construction remains perfectly valid even in the Hermitian limit $0$ [10].

4. **Non-standard P T symmetric supersymmetries**

4.1. **Working at a fixed parameter**

The respective annihilation and creation of ref. [4] was mediated by the second-order differential operators

$$
A^{(\cdot)}(L) = A^{(\cdot)}(R) = A^{(\cdot)}(L) = A^{(\cdot)}(R),
$$

$$
B^{(\cdot)}(L) = B^{(\cdot)}(R) = B^{(\cdot)}(L) = B^{(\cdot)}(R),
$$

with the “norm” $c_5(N; \lambda) = 4(N + 1)(N + \lambda + 1)$ and property

$$
A^{(\cdot)}(L)\left(\begin{array}{c}
L_{N+1}^{(\cdot)}
\end{array}\right) = c_2(N; \lambda)\left(\begin{array}{c}
L_{N}^{(\cdot)}
\end{array}\right), \quad B^{(\cdot)}(L)\left(\begin{array}{c}
L_{N+1}^{(\cdot)}
\end{array}\right) = c_2(N; \lambda)\left(\begin{array}{c}
L_{N}^{(\cdot)}
\end{array}\right):
$$

Hamiltonian $H^{(\cdot)} = [A^{(\cdot)}B^{(\cdot)}B^{(\cdot)}A^{(\cdot)}] = 8$ satisfies commutation relations

$$
A^{(\cdot)}H^{(\cdot)} = H^{(\cdot)}A^{(\cdot)}; \quad B^{(\cdot)}H^{(\cdot)} = H^{(\cdot)}B^{(\cdot)}; \quad B^{(\cdot)}H^{(\cdot)} = 4B^{(\cdot)}(L); \quad H^{(\cdot)}B^{(\cdot)} = B^{(\cdot)}H^{(\cdot)} = 4B^{(\cdot)}(L)
$$

of the Lie algebra $\mathfrak{sl}(2;\mathbb{R})$ with the normalized generators $A^{(\cdot)} = \frac{P}{32}, B^{(\cdot)} = \frac{P}{32}$ and $H^{(\cdot)} = 4$. As a consequence, the new, P T SUSY results from eq. (3), with $A^{(\cdot)}B^{(\cdot)}B^{(\cdot)}A^{(\cdot)} = H^{(\cdot)} = 2^2 = 4^2$, respectively. The SHO eigenvectors themselves may then be obtained as solutions of the differential equations of the fourth order (cf. ref. [11]) which, in our case, read

$$
G^{(\cdot)}_{(L)} N^{(\cdot)} = N^{(\cdot)} E^{(\cdot)}; \quad G^{(\cdot)}_{(R)} N^{(\cdot)} = N^{(\cdot)} E^{(\cdot)};
$$

where $N^{(\cdot)} = 16N (N + \lambda)$. This is our present main result.
4.2. SUSY constructions at the complex

Marginally, let us note that even the complex choice of \( \mathcal{P} \mathcal{T} \) symmetry itself is broken may lead to the partially real SUSY spectrum of energies. In order to show that, one has to derive a few identities for the Laguerre polynomials in \([4]\) showing that the operators \([5]\) change merely the subscripts or superscripts \([4]\). In the regime with the spontaneously broken \( \mathcal{P} \mathcal{T} \) symmetry we may distinguish between the two options,

\[
\begin{align*}
& > 0 \text{ in } = i; \quad = i; \quad = 1 + \\
& > 0 \text{ in } = i; \quad = i; \quad = 1
\end{align*}
\]

and get the partially real energy multiplets

\[
\begin{align*}
8 & \quad < E^{(-)}_{(L)} \quad = 4n; \quad E^{(-)}_{(R)} = 4n + 4 \\
& \quad > E^{(+)}_{(L)} \quad = 4n + 4; \quad E^{(+)}_{(R)} = 4n + 4
\end{align*}
\]

Similarly, at \( = N + iq \) with \( q = 1 \) and \( > 0, \) i.e., with no \( \mathcal{P} \mathcal{T} \) symmetry at all, we get

\[
\begin{align*}
& \quad \text{E}_{(L)} = 4n + 4; \quad \text{E}_{(R)} = 4n + 4
\end{align*}
\]

for the indices \( 1 = N + 1 + i, \quad = N + i \) and \( 2 = N + 1 + i, \) still giving the partially real energy spectra.

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