On the disorder-driven quantum transition in three-dimensional relativistic metals

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The Weyl semimetals are topologically protected from a gap opening against weak disorder in three dimensions. However, a strong disorder drives this relativistic semimetal through a quantum transition towards a diffusive metallic phase characterized by a finite density of states at the band crossing. This transition is usually described by a perturbative renormalization group in a := 2 + ε of a \( U(N) \) Gross-Neveu model in the limit \( N \rightarrow 0 \). Unfortunately, this model is not multiplicatively renormalizable in \( 2 + \varepsilon \) dimensions: An infinite number of relevant operators are required to describe the critical behavior. Hence its use in a quantitative description of the transition beyond one-loop is at least questionable. We propose an alternative route, building on the correspondence between the Gross-Neveu and Gross-Neveu-Yukawa models developed in the context of high energy physics. It results in a model of Weyl fermions with a random non-Gaussian imaginary potential which allows one to study the critical properties of the transition within a \( a = 4 - \varepsilon \) expansion. We also discuss the characterization of the transition by the multifractal spectrum of wave functions.

Introduction. - After the discovery of graphene, materials with a relativistic-like spectrum of electronic excitations have become a popular subject which currently drives several hot topics in condensed matter physics. Examples include three dimensional materials such as Na\(_3\)Bi and Cd\(_3\)As\(_2\) which have been identified as Dirac semimetals \[1,2\]. The twofold band degeneracy of Dirac semimetals can be lifted by breaking time or inversion symmetry as it happens in Ta\(_3\)As and Nb\(_3\)As leading to the so called Weyl semimetal \[3,4\]. The latter is topologically protected from a gap opening against small perturbations. Indeed, real materials inevitably contain disorder of different kinds, which turn out to be irrelevant in the renormalization group (RG) sense. The weakly disordered materials remain in a semimetallic phase \[6–8\]. At the nodal point, the system is characterized by a density of states (DOS) vanishing quadratically with energy up to exponentially small corrections due to rare events \[9,10\]. It exhibits a vanishing zero-frequency optical conductivity \[11\] and a pseudoballistic transport \[12\]. However, as was pointed for the first time in Refs. \[13\], a strong enough disorder may drive the system into a diffusive phase with a finite DOS, optical conductivity and diffusive transport at zero energy. The semimetal to diffusive metal transition has been numerically studied for several models \[14–16\] including both the Dirac and Weyl semimetals. For the simplest scalar potential considered in this Rapid Communication all of them belong to the same universality class. However, the nature of the disordered phase and its protection against Anderson localization depends on the precise nature of the phase, e.g. Dirac versus Weyl semimetals \[17–21\]. It is now believed that the disorder-driven quantum transition from a single-cone Weyl semimetal to a diffusive phase is related to the chiral transition well studied in high energy physics and described by the 3D \( U(N) \) Gross-Neveu (GN) model, but in the unusual limit of a vanishing number of components \( N \rightarrow 0 \). This relation has been confirmed by direct calculations to two-loop order on the initial Weyl model using either super-symmetry \[22\] or replica methods \[23,24\]. The massless GN model possesses a chiral symmetry which is spontaneously broken for sufficiently strong interactions. For the disordered Weyl fermions this transition translates into the appearance of a finite DOS at the nodal point for disorder stronger than a critical value.

However, we recall here that the \( U(N) \) GN model is not multiplicatively renormalizable in dimension \( a = 2 + \varepsilon > 2 \). This manifests itself in the generation of an infinite number of relevant operators along the RG flow beyond two-loop order. Moreover, these relevant operators collapse into a few operators when extrapolating this technique to \( a = 3 \). This casts some doubts about the direct applicability of this approach to the Weyl fermion problem in \( a = 3 \). Taking into account the inherent difficulties of this \( a = 2 + \varepsilon \) expansion we propose a different approach based on a \( a = 4 - \varepsilon \) expansion to study the disorder-driven transition in the Weyl semimetals. In this approach we build on the known correspondence between the GN model and the \( U(N) \) Gross-Neveu-Yukawa (GNY) model for \( 2 \leq a \leq 4 \), which is similar to the relation of the \( O(N) \) non-linear \( \sigma \)-model with respect to the \( O(N) \) \( \phi^4 \) model \[25\]. Besides the fermionic field, the GNY model involves an additional scalar bosonic field. In the limit of \( N \rightarrow 0 \) it can be interpreted as a random non-Gaussian imaginary potential. The equivalence between this and the initial problem sheds light on the quantum transition and we discuss several of its possible consequences. For instance, it allows one to calculate the critical exponents in a systematic controllable way, since this model is renormalizable in \( 4 - \varepsilon \) dimensions.

Model. - The action of the \( d \)-dimensional relativistic fermions moving in the random disorder potential \( V(r) \) can be written as \[27\].

\[
S_{\text{Weyl}} = \int d^d r \int d \omega \bar{\psi} (r, -\omega) \left[ -i \partial^\mu + i \omega + V(r) \right] \psi (r, \omega),
\]

where \( \partial^\mu = \gamma^\mu \partial^\mu \) and \( \omega \) is a Matsubara frequency. The \( \gamma_\imath \) are elements of the Clifford algebra which satisfy the anticommutation relations: \( \gamma_\imath \gamma_j + \gamma_j \gamma_\imath = 2 \delta_{ij} \), and \( i, j =
where a summation over \( \alpha \) hence reads
\[
\int \prod_{\alpha=1}^{N} D\psi_\alpha e^{-S_{\text{ren}}^\alpha} = \int D\Psi V[\Psi] \prod_{\alpha=1}^{N} D\psi_\alpha e^{-S_{\text{ren}}^\alpha},
\]
where \( D\psi_\alpha = D\bar{\psi}_\alpha D\psi_\alpha \). We neglect the possible presence of long-range spatial correlations which can modify the critical properties \( \gamma_3 \) and take the distribution of disorder potential to be Gaussian, \( P[V] \sim e^{-\frac{1}{\lambda^2} \int d^dV(r)^2} \).

This yields
\[
S_{\text{ren}} = \int d^dV \int d\omega \left[ -\frac{\Delta_0}{2} \bar{\psi}_\alpha(r, -\omega) \bar{\psi}_\alpha(r, \omega) \right],
\]
where a summation over \( \alpha, \beta \) is implied and disorder generates an attractive interaction between different replicas.

It turns out that the Green’s functions computed for the action \( (2) \) at fixed energy \( \omega \) in the limit \( N \to 0 \) can be deduced from the \( d \)-dimensional \( U(N) \) GN model
\[
S_{\text{GN}} = -\int d^dV \left[ \bar{\chi} \cdot (\partial + \omega) \chi - \frac{\Delta_0}{2} \bar{\chi} \chi \right],
\]
which appears here with a negative (attractive) coupling constant in terms of new fields \( \bar{\chi} = i\psi(r, -\omega) \) and \( \chi = \psi(r, \omega) \).

2 + \( \varepsilon_2 \) expansion. - We now show that a renormalization procedure based on the model \( (3) \) is inherently flawed beyond the two-loop order of previous studies \( \gamma_3 \). The problem is related to the extension of the Clifford algebra to arbitrary dimensions necessary within the renormalization scheme. Indeed, in \( 2 < d = 2 + \varepsilon_2 < 3 \), the product \( \gamma_1 \gamma_2 \) cannot be expressed as a linear combination of \( \gamma_1 \), so that the Clifford algebra becomes infinite-dimensional. It is then convenient to use antisymmetrized products such as \( \gamma^{(n)}_A = \Lambda_n[\gamma_{a_1} \ldots \gamma_{a_n}], \) where we have introduced the notation \( A = \{a_1, \ldots, a_n\} \), as a basis in this infinite-dimensional space so that one does not need any explicit representation of these objects to perform calculations. Thus, along the RG flow an infinite number of corresponding operators are generated, of the form \( V^{(n)} = \left( \bar{\chi}_\alpha \gamma_A^{(n)} \right) \chi_\alpha \), where a summation over \( \alpha, \beta \) and \( A \) is implied. The minimal multiplicatively renormalizable model replacing \( \text{(3)} \) hence reads
\[
S_{\text{GN}} = -\int d^dV \left[ \bar{\chi} \cdot (\partial + \omega) \chi - \frac{1}{2} \sum_{n=0}^{\infty} \Delta_n V^{(n)} \right].
\]

As an example, let us consider the three-loop order for which only the operators \( V^{(3)} \) and \( V^{(4)} \) are generated
\[
\text{(3)} \quad \text{GNY}\quad \text{model.}
\]

The corresponding RG flow equations are given in the limit \( N \to 0 \) by the \( \beta \)-functions,
\[
\frac{\partial \Delta_0}{\partial \ln L} = -\varepsilon_2 \Delta_0 + 4 \Delta_0^2 + 8 \Delta_0^3 + 28 \Delta_0^4, \quad (5a)
\]
\[
\frac{\partial \Delta_3}{\partial \ln L} = -\varepsilon_2 \Delta_3 + a \Delta_0^4 + 16 \Delta_0 \Delta_3 + 8 \Delta_0 \Delta_3, \quad (5b)
\]
\[
\frac{\partial \Delta_4}{\partial \ln L} = -\varepsilon_2 \Delta_4 - 4 \Delta_0 \Delta_3 - 12 \Delta_0 \Delta_4, \quad (5c)
\]
where \( a = -4 + \zeta(3) \) and \( \zeta(x) \) is the Riemann zeta function. To this order, the fixed point (FP) describing the transition reads \( \Delta^*_0 = \varepsilon_2/4 - \varepsilon_2^2/8 + \varepsilon_2^3/64 + O(\varepsilon_2^4), \Delta^*_3 = a \varepsilon_2^3/96 - 23a \varepsilon_2^3/1152 + O(\varepsilon_2^4), \Delta^*_4 = -a \varepsilon_2^3/384 + 49a \varepsilon_2^3/9216 + O(\varepsilon_2^4). \) Note the peculiarity of the limit \( N \to 0 \) where, while \( \Delta^*_0 \) is of order \( \varepsilon_2 \), the generated operators are of order \( \varepsilon_2^2 \) instead of \( \varepsilon_2^3 \) expected in the three-loop order. The critical exponent of the correlation length divergence at the transition as \( \xi \sim (\Delta - \Delta^*)^{-\nu} \) reads \( 1/\nu = 2 + 1/2 + 2/2 + O(\varepsilon_2^4). \) For a Weyl semimetal \( (\varepsilon_4 = 1) \) we find: \( \nu = 0.533 \) (direct substitution) \( \text{[31]} \).

Crucially, the validity of this renormalization picture directly in dimension \( d = 3 \) is questionable: The Clifford algebra is then of finite dimension. Hence all the operators \( V^{(n)} \) generated by the RG flow beyond three-loops either disappear (evanescent operators) or collapse on a few operators when extending \( d = 2 + \varepsilon_2 \to 3 \). Contrary to the two-dimensional case \( \text{[22]} \) no standard projecting procedure exists to reduce the \( \beta \)-functions for these evanescent operators to the \( \beta \)-function for the remaining operators in \( d = 3 \).

4 + \( \varepsilon_4 \) expansion. - Here we propose another way to describe the quantum transition alternative to the use of \( \text{(3)} \). This new approach is of interest beyond the quantitative calculations since it provides an example of a physical model possessing the same quantum critical properties as the disordered Weyl fermions. It is based on the well known correspondence between the critical properties of the \( U(N) \) GN and GNY models \( \text{[25]} \) which we transpose in the context of the disordered relativistic fermions associated with the \( N \to 0 \) limit. Contrary to the GN model the GNY model is renormalizable in dimension \( d = 4 - \varepsilon_4 \): critical properties of the transition can be obtained to any order without generating an infinite number of relevant operators. In the \( U(N) \) GNY model, an additional scalar field \( \phi \) is introduced, and the action reads
\[
S_{\text{GNY}} = \int d^dV \left[ -\bar{\chi}_\alpha (\partial + \sqrt{\nu} \phi) \chi_\alpha + \frac{1}{2} (\nabla \phi)^2 + \frac{\mu}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right].
\]

In terms of the initial fields \( \bar{\psi} = -i\bar{\chi}, \chi = \psi \), the GNY model \( \text{(6)} \) corresponds to the Weyl fermions at \( \omega = 0 \) coupled to an imaginary random potential \( \text{[23]} \)
\[
S^\alpha = \int d^dV \bar{\psi}_\alpha (-i\partial - i\sqrt{\nu} \phi) \psi_\alpha,
\]

as an example, let us consider the three-loop order for which only the operators \( V^{(3)} \) and \( V^{(4)} \) are generated.
with the random potential distribution given by

\[ P[\phi] \propto \exp \left( -\int d^d r \left[ \frac{1}{2} (\nabla \phi)^2 + \frac{\mu}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right] \right). \]  

Such a random imaginary potential is unusual: It can be interpreted as an effective inverse life-time (imaginary part of a self-energy), which appear to be randomly distributed. The transposition of the GN - GNY correspondence in the context of disordered Weyl fermions amounts to the equivalence between a random Gaussian scalar potential and a non-Gaussian imaginary field distributed according to Eq. (8). Studying the relevance of this correspondence beyond these simple distribution functions will be of great interest.

The transition within the GNY model can be understood at the mean-field level: (i) For \( \mu > 0 \), the typical (most probable) value of the scalar field \( \phi \) vanishes and we recover a theory of free fermions. This corresponds to a phase where the disorder potential \( \sqrt{2} \phi \) is Gaussian, distributed around \( \phi = 0 \); (ii) on the other hand, for \( \mu < 0 \), the scalar field acquires a finite typical value. This translates into a finite density of states of the Weyl fermions at zero energy, \( \rho(0) > 0 \). In this phase, the mean-field distribution of the disorder potential \( P[\phi] \) is peaked around opposite values (see Fig. 1) and the distribution is no longer Gaussian. In the context of the high energy physics the generation of a finite correlation length \( \xi \) defines the critical length exponent \( \nu \): \( |\delta \mu| \sim \xi^{-1/\nu} \). We find to two-loop order \( \xi = 1/\nu \):

\[ P[\phi] \propto \exp \left( -\int d^d r \left[ \frac{1}{2} (\nabla \phi)^2 + \frac{\mu}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right] \right). \]  

The numerical value of the exponent \( \nu \) to two-loop order is given by \( \nu = 0.65 \) (direct substitution \( \xi = 1 \)).

Discussion. - We studied the disorder driven transition of the 3D Weyl semimetals towards a diffusive metal. We showed that the description of this transition using the \( U(N) \) GN model in \( 2 + \epsilon_2 \) dimensions in the limit of \( N \to 0 \) encounters significant difficulties already beyond one-loop approximations. They are related to the multiplicative non-renormalizability of the model and generation of an infinite number of vertices whose three-loop corrections unexpectedly shift the fixed point to the order \( \epsilon_2 \). We have proposed an alternative approach based on the correspondence between the \( U(N) \) GNY and GN models. The previous numerical and analytical studies give values of the correlation length exponent \( \nu \) which lie in a broad range from 0.6 to 1.5, that can be related to the existence of a large number of relevant operators in the GN model. The GNY model has only one relevant operator and we find \( \nu \approx 0.65 - 0.67 \).

Beyond offering a well defined framework for an unambiguous description of the critical properties it relates the quantum transition of disordered Weyl fermions with chemical potential fluctuations to that of a model with spatially correlated and non Gaussian imaginary disorder. We are confident that this novel correspondence between two models of identical fermions with distinct disorder potentials opens interesting perspectives for further investigations such as functional renormalization group studies of this transition.

Let us discuss these results in view of recent work on the relevance of rare disorder realizations around the transition. This is an important issue, since the semimetal phase can be destabilized not only by finite doping but also by the zero energy states emergent from these rare disorder configurations. The stability of a disordered fixed point with respect to fluctuations effects is
related to the extended Harris criterion $\nu_{FS} > 2/d$ [38] for the correlation length critical exponent. The values of $\nu$ for both the GN and GNY models violate this criterion at the order considered. However, this inequality has to be satisfied by the finite size correlation exponent, while there is no restriction on the intrinsic exponent usually probed by the RG methods. In principle, it can be different from the first one [39].

On the other hand, the relevance of rare fluctuations around the transition can manifest itself in the RG context by the development of a strong deviation from the Gaussian distribution of disorder. The corresponding cumulants are related to the composite operators $O_q = (\psi_\alpha \psi_\alpha)^q$. We find to order one-loop in the GN model the scaling dimension of these operators $[O_q] = (d-1)q - 2q^2 \Delta_0^* + O(\Delta_0^{*2})$. Thus, these operators with $q \geq 4$ become naively relevant at the FP of $\xi_2 > 2/5$. For this observation suggests that strong deviations from the Gaussian distribution of disorder develop in $d = 3$ ($\xi_2 = 1$), which could explain the importance of rare disorder realizations. Indeed, in Refs. [9, 11] it was shown that the average DOS at zero energy can be finite in the semi-metallic phase due to contributions from rare events that lead to an avoided quantum transition. For the GNY model we also find instanton-like solutions similar to those observed in the GN model [9, 10] and which are responsible for the contribution of the rare events to the DOS. Whether such instanton solutions can be accounted for by a more refined renormalization of the distribution of disorder beyond the GNY model remains a question of interest.

Let us note, however, that an alternative characterization of the transition exists, less sensitive to the rare effects, through the scaling properties of the critical wavefunction in a similar fashion to the Anderson transition [40]. The disorder averaged inverse participation ratios (IPR’s), $P_q = \int d^d r |\psi|^q$ are expected to scale with the size of the system $L$ as $P_q \sim L^{-\tau_q}$, where the $\tau_q$ describe the multifractal spectrum of the wavefunctions.

In the semi-metallic phase the only possible states at the nodal point are the algebraically-decaying instanton-like solutions predicted in Ref. [9] and observed numerically in Ref. [10]. Since these states, if present with finite density, are localized, we still obtain $\tau_q = 0$ in the semi-metallic phase, at least for small $q$. In the diffusive metal phase the system has a finite density of extended states at zero energy that results in $\tau_q = d(q-1)$. Exactly at the transition the exponent modifies to $\tau_q = d(q-1) + \Delta_q$, where $\Delta_q$ also governs the scaling of the moments of the local DOS (LDOS), $\tilde{\rho}^q \sim L^{-\Delta_q}$. It is related by $\Delta_q = x_q^* - q x_1^*$ to the scaling dimension $x_q^*$ of the local composite operator representing the $q$th moment of the LDOS. Fortunately, the scaling dimension of this operator has been calculated within the GN model to two-loop order in Ref. [11] and reads $x_q^* = (d-1)q - 2q \Delta_0^* - 2\Delta_0^{*2}[3q(q-1)+q]$ with $\Delta_0^*$ obtained from [39]. Note that $\Delta_q = \frac{q}{2} \epsilon q(1-q) \xi_2^2$ satisfies the convexity inequality $\partial^2 x_q^*/\partial q^2 \leq 0$ [12] and the identity $\Delta_q = \Delta_{1-q}$, i.e. $\rho^q \sim \rho^{1-q}$ with $\rho = \rho/\tilde{\rho}$. The latter holds for the multifractal exponents in the different Wigner-Dyson classes [39] and follows from a very general symmetry of the LDOS distribution $P_q(\rho) = \rho^{-3}P_q(\tilde{\rho}^{-1})$ [11]. Then the two-point correlation function is expected to scale as $\rho^{1/2}(r)\tilde{\rho}^1(\tilde{r}) \sim (r/a)^{-\Delta_p - \Delta_{1-q}(r/L)\Delta_{p+q}}$, where $a$ is the microscopic cutoff. Crucially, this description of the multifractal spectrum of the critical wavefunctions, at least for small $q$ is weakly sensitive to the presence of rare events and indeed characterizes the underlying avoided critical point.

Note added. During the final completion of this paper, we became aware of the recent preprint [45] where the authors independently came to the same conclusions about the multifractality at the transition.

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Supplemental Material
On the disorder-driven quantum transition in three-dimensional relativistic metals
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I. GENERALIZED GROSS-NEVEU MODEL: \(2 + \varepsilon\) EXPANSION

The minimal action of the Weyl fermions in \(d\) dimensions can be rewritten in Fourier space as

\[
S = \sum_{\alpha=1}^{N} \int k \tilde{\psi}_\alpha(-k)(\gamma k - i\omega)\psi_\alpha(k)
- \sum_{\alpha,\beta=1}^{N} \sum_{A, B=1}^{\Delta_n} \frac{\Delta_n}{2} \int k\left[\tilde{\psi}_\alpha(k_1)\gamma_A^{(n)}\psi_\alpha(k_2)\right] \\
\times \left[\tilde{\psi}_\beta(k_3)\gamma_A^{(n)}\psi_\beta(-k_1 - k_2 - k_3)\right].
\]

One can build up a perturbation theory in small disorder calculating all correlation and vertex functions perturbatively in \(\Delta_n\). Each term can be represented as a Feynman diagram. In these diagrams the solid lines stand for the bare propagator

\[
(\tilde{\psi}_\alpha(k,\omega)\psi_\beta(-k, -\omega))_0 = \delta_{\alpha\beta}\frac{\gamma k + i\omega}{k^2 + \omega^2},
\]

and the dashed line corresponds to one of the vertex \(\frac{1}{2}\Delta_n\). Note that the dashed line transmit only momenta but not frequency. These terms turn out to be diverging in \(d = 2\) which is the lower critical dimension of the transition. Simple scaling analysis shows that weak disorder is irrelevant for \(d > 2\). To make the theory finite we use the dimensional regularization and compute all integrals in \(d = 2 + \varepsilon_2\). At the end we put \(\varepsilon_2 = 1\). To render the divergences we employ the minimal subtraction scheme and collect all poles in \(\varepsilon_2\) in the \(Z\)-factors: \(Z_\omega, Z_\psi\) and \(Z_n\) so that the correlation function calculated with the renormalized action

\[
S_R = \sum_{\alpha=1}^{N} \int k \tilde{\psi}_\alpha(-k)(Z_\psi\gamma k - Z_\omega i\omega)\psi_\alpha(k)
- \sum_{\alpha,\beta=1}^{N} \sum_{A, B=1}^{\Delta_n} \frac{\mu^{-\varepsilon} \Delta_n}{K_d} \int k\left[\tilde{\psi}_\alpha^{(n)}(\gamma A)\psi_\alpha^{(n)}\right] \left[\tilde{\psi}_\beta^{(n)}(\gamma A)\psi_\beta^{(n)}\right]
\]

remain finite in the limit \(\varepsilon_2 \to 0\). Here we have introduced the renormalized fermionic fields \(\tilde{\psi}, \tilde{\psi}\) and the renormalized dimensionless coupling constants \(\Delta_n\) on the mass scale \(\mu\), which are related to the parameters by

\[
\tilde{\psi} = Z_\psi^{1/2} \psi, \quad \tilde{\psi} = Z_\psi^{1/2} \psi,
\]

\[
\tilde{\omega} = Z_\omega Z_\psi^{-1} \omega, \quad \tilde{\Delta}_n = \frac{2\mu^{-\varepsilon} Z_n}{K_d Z_\psi} \Delta_n,
\]

where \(K_d = 2\pi^{d/2}/((2\pi)^d \Gamma(d/2))\) is the area of the \(d\)-dimensional unite sphere divided by \((2\pi)^d\). The renormalized and the bare vertex and Green functions are related by

\[
\tilde{\Gamma}^{(N)}(p_i, \omega, \tilde{\Delta}) = Z_\psi^{N/2} \Gamma^{(n)}(p_i, \omega, \Delta, \mu),
\]

\[
\tilde{G}^{(N)}(p_i, \omega, \tilde{\Delta}) = Z_\psi^{N/2} G^{(n)}(p_i, \omega, \Delta, \mu),
\]

where \(\tilde{\Delta}\) stands for all \(\Delta_n\). Using that the bare functions \(\Gamma^{(N)}\) and \(G^{(n)}\) do not depend on the renormalization scale \(\mu\) we take the derivative of Eqs. (7) and (8) with respect to \(\mu\) and obtain the RG flow equations for the renormalized Green and vertex functions:

\[
\left[\frac{\partial}{\partial \Delta} - \sum_n \beta_n(\Delta) \frac{\partial}{\partial \Delta_n} - N \frac{\partial}{2 \eta_\psi(\Delta)}
\frac{-\gamma(\Delta)\omega}{\partial \mu} \Gamma^{(n)}(p_i, \omega, \Delta) = 0,
\]

\[
\left[\frac{\partial}{\partial \mu} - \sum_n \beta_n(\Delta) \frac{\partial}{\partial \Delta_n} + N \frac{\partial}{2 \eta_\psi(\Delta)}
\frac{-\gamma(\Delta)\omega}{\partial \omega} \Gamma^{(n)}(p_i, \omega, \Delta) = 0.
\]

Here we have defined the scaling functions

\[
\beta_n(\Delta) = - \frac{\partial \Delta_n}{\partial \mu} \bigg|_\Delta,
\]

\[
\eta_\psi(\Delta) = - \sum_n \beta_n(\Delta) \frac{\partial \ln Z_\psi}{\partial \Delta_n},
\]

\[
\eta_\omega(\Delta) = - \sum_n \beta_n(\Delta) \frac{\partial \ln Z_\omega}{\partial \Delta_n},
\]

\[
\gamma(\Delta) = \eta_\omega(\Delta) - \eta_\psi(\Delta).
\]

Dimensional analysis gives the following rescaling formulas

\[
\Gamma^{(N)}(p_i, \omega, \Delta, \mu) = \lambda^{-d+N(d-1)/2}
\]

\[
\times \Gamma^{(N)}(\lambda p_i, \lambda \omega, \Delta, \lambda \mu),
\]

\[
G^{(N)}(p_i, \omega, \Delta, \mu) = \lambda^{d(N-1)-N(d-1)/2}
\]

\[
\times G^{(N)}(\lambda p_i, \lambda \omega, \Delta, \lambda \mu),
\]
which can be rewritten in an infinitesimal form as

\[
\begin{align*}
\sum p_i \frac{\partial}{\partial p_i} + \phi \frac{\partial}{\partial \phi} &= -d + \frac{N(d-1)}{2} \Gamma^{(N)}(p_i, \omega, \Delta) = 0, \\
\sum p_i \frac{\partial}{\partial p_i} + \omega \frac{\partial}{\partial \omega} &= +d(N-1) - \frac{N(d-1)}{2} \Gamma^{(N)}(p_i, \omega, \Delta) = 0.
\end{align*}
\]  

(17)

(18)

Subtracting Eqs. (9) and (10) from Eqs. (17) and (18) we obtain

\[
\begin{align*}
\sum p_i \frac{\partial}{\partial p_i} + \omega \frac{\partial}{\partial \omega} &= -(d-1 + \eta_\psi(\Delta)) \Gamma^{(N)}(p_i, \omega, \Delta) = 0, \\
\sum p_i \frac{\partial}{\partial p_i} + (1 + \gamma(\Delta)) \omega \frac{\partial}{\partial \omega} &= +d(N-1) - \frac{N(d-1)}{2} \Gamma^{(N)}(p_i, \omega, \Delta) = 0.
\end{align*}
\]  

(19)

(20)

The solutions of Eqs. (19) and (20) can be found by using the method of characteristics. The characteristics, i.e., lines in the space of \( p_i, \omega, \) and \( \Delta_n \), parameterized by auxiliary parameter \( \xi \) which below will be identified with the correlation length, can be found from the equations

\[
\begin{align*}
\frac{dp_i(\xi)}{d\ln \xi} &= p_i(\xi), \\
\frac{d\Delta_n(\xi)}{d\ln \xi} &= \beta_n(\Delta(\xi)), \\
\frac{d\omega(\xi)}{d\ln \xi} &= (1 + \gamma(\Delta(\xi)))\omega(\xi),
\end{align*}
\]  

(21)

(22)

(23)

with initial conditions \( \Delta_n(1) = \Delta_n, \) \( p_i(1) = p_i, \) and \( \omega(1) = \omega. \) The solution of Eqs. (19) and (20) then propagate along the characteristics according to the equations

\[
\begin{align*}
\frac{dM_n(\xi)}{d\ln \xi} &= [-d + \frac{N}{2}(d-1 + \eta_\psi(\Delta(\xi)))] M_n(\xi), \\
\frac{dH_n(\xi)}{d\ln \xi} &= \left[ d(N-1) - \frac{N}{2}(d-1 + \eta_\psi(\Delta(\xi))) \right] H_n(\xi),
\end{align*}
\]  

(24)

with the initial conditions \( M_n(1) = H_n(1) = 1. \) Thus the solutions of Eqs. (19) and (20) satisfy

\[
\begin{align*}
\Gamma^{(N)}(p_i, \omega, \Delta) &= M_n(\xi) \Gamma^{(N)}(p_i(\xi), \omega(\xi), \Delta(\xi)), \\
G^{(N)}(p_i, \omega, \Delta) &= H_n(\xi) G^{(N)}(p_i(\xi), \omega(\xi), \Delta(\xi)).
\end{align*}
\]  

(25)

(26)

We assume that the \( \beta \)-function have a fixed point (FP)

\[
\beta(\Delta^*) = 0,
\]  

(27)

with a single unstable direction \( \delta = \Delta - \Delta^*, \) i.e. the stability matrix

\[
\mathcal{M}_{nm} = \left. \frac{\partial \beta_n(\Delta)}{\partial \Delta_m} \right|_{\Delta^*},
\]  

(28)

has only one positive eigenvalue \( \lambda^{(+)} \) associated with the direction \( \delta. \) Then the solutions (25) and (20) in the vicinity of the FP (27) can be rewritten as

\[
\begin{align*}
\Gamma^{(N)}(p_i, \omega, \delta) &= \xi^{N\nu} \omega dN \Gamma^{(N)}(p_i, \omega, \xi^z, \delta^{1/\nu}), \\
G^{(N)}(p_i, \omega, \delta) &= \xi^{d(N-1) - N\nu} g_N(p_i, \omega, \xi^z, \delta^{1/\nu}),
\end{align*}
\]  

(29)

(30)

where we defined the critical exponents \( \nu, z, d_\psi. \) The parameter \( \xi \) can be identified with the correlation length that gives the critical exponent for the correlation length

\[
\xi \sim \delta^{-\nu}, \quad \frac{1}{\nu} = \lambda^{(+)}_1,
\]  

(31)

and the dynamic critical exponent

\[
\omega \sim k^z, \quad z = 1 + \gamma(\Delta^*).
\]  

(32)

The anomalous dimension of the fields \( \psi \) and \( \tilde{\psi} \) reads

\[
d_\psi = \frac{1}{2}[d-1 + \eta_\psi(\Delta^*)].
\]  

(33)

Note, that the exponent \( \eta_\psi \) characterizes the scaling behavior of the two-point function

\[
G^{(2)}(p) = \langle \psi(p) \psi(-p) \rangle \sim p^{-1 + \eta_\psi(\Delta^*)},
\]  

(34)

which can be viewed as the momentum distribution of fermions at the transition.

\section{A. Critical exponents to three-loop order}

To renormalize the theory we use the minimal subtraction scheme

\[
Z_\psi \Gamma^{(2)}(p, \omega) = Z_\omega Z_\psi^{-1} \mu, \Delta(\Delta) = \text{finite},
\]  

(35)

\[
Z_\psi^2 \Gamma^{(4)}(p_i = 0, \omega) = Z_\omega Z_\psi^{-1} \mu, \Delta(\Delta) = \text{finite},
\]  

(36)

where \( \Delta(\Delta) \) is given by Eq. (3) and \( \Gamma^{(4)} \) is the renormalized vertex \( V_n. \) The three-loop corrections to the vertex \( V_0 \) have been many times discussed in the literature in the context of the GN model [1]. The corresponding \( \beta_0 \)-function defined in Eq. (11) reads

\[
\beta_0 = -\epsilon_2 \Delta_0 - 2 \Delta_0^2 (N-2) - 4 \Delta_0^3 (N-2) + 2 \Delta_0^4 (N-2) (N-7),
\]  

(37)

where we kept the dependence on \( N. \) The 24 diagrams derived from the diagram (b) shown in Fig. 1 by permutation of the dashed line ends which were neglected in Ref. [2] generate the vertex \( V_3 [3]. \) Other diagrams which
one has to take into account in calculation to order of $\varepsilon_2^3$ are the diagrams (c)-(e) shown in Fig. 1. These diagrams with lines corresponding to $V_0$ and $V_3$ contribute to $V_4$ and with lines corresponding to $V_0$ and $V_3$ contribute to $V_4$. Since the contributions of the diagrams (b) are of order $\Delta_0^4$ one may naively conclude that while $\Delta_0$ is of order $\varepsilon_2$, the two over vertices $\Delta_3$ and $\Delta_4$ are of order $\varepsilon_2^3$. Indeed, the corresponding $\beta$-functions
\[
\beta_3 = -\varepsilon_2 \Delta_3 + a \Delta_0^4 + 16 \Delta_0 \Delta_4 + 8 \Delta_0 \Delta_3,
\]
\[
\beta_4 = -\varepsilon_2 \Delta_4 - 4 \Delta_0 \Delta_3 - 12 \Delta_0 \Delta_4,
\]
have the fixed point
\[
\Delta_0^* = \frac{\varepsilon_2}{4 - N} - \frac{\varepsilon_2^2}{2(2 - N)^2} + \frac{(1 + N)\varepsilon_2^3}{8(2 - N)^3} + O(\varepsilon_2^4),
\]
\[
\Delta_3^* = \frac{\alpha\varepsilon_2^3}{16N(N - 6)(N - 2)^3} + O(\varepsilon_2^4),
\]
\[
\Delta_4^* = \frac{\alpha\varepsilon_2^3}{8N(N - 6)(N - 2)^3} + O(\varepsilon_2^4),
\]
which has non analytic behavior in the limit $N \to 0$. Taking first the limit $N \to 0$ in the $\beta$-functions one finds the fixed point
\[
\Delta_0^* = \frac{\varepsilon_2}{4} - \frac{\varepsilon_2^2}{8} \varepsilon_2 + \frac{\varepsilon_2^4}{64} + O(\varepsilon_2^4),
\]
\[
\Delta_3^* = \frac{\alpha\varepsilon_2^3}{96} - \frac{23\alpha\varepsilon_2^3}{1152} + O(\varepsilon_2^4),
\]
\[
\Delta_4^* = -\frac{\alpha\varepsilon_2^3}{384} + \frac{49\alpha\varepsilon_2^3}{9216} + O(\varepsilon_2^4),
\]
similar to $\sqrt{\varepsilon}$ expansion for the diluted Ising model [4].

The stability of the FP can be described by the eigenvalues of the stability matrix $\Delta^*_i$, $i, j \in 0, 3, 4$. Since one expects that the transition is controlled by an unstable IR FP, the stability matrix is expected to have only one positive eigenvalue which is related to the critical exponent $1/\nu = \lambda_1^{(+)}$. The stability eigenvalues read:
\[
\lambda_1^{(+)} = \varepsilon_4 + \frac{\varepsilon_2^2}{2} + \frac{3\varepsilon_2^3}{8} + O(\varepsilon_2^4),
\]
\[
\lambda_2^{(-)} = -3\varepsilon_4 + \frac{\varepsilon_2^2}{2} + \frac{3\varepsilon_2^3}{8} + O(\varepsilon_2^4),
\]
\[
\lambda_3^{(-)} = -\frac{\varepsilon_2^2}{2} + \frac{\varepsilon_2^3}{16} + O(\varepsilon_2^4).
\]

Only the first eigenvalue $\lambda_1^{(+)}$ associated with a single instability direction is positive.

The generation of vertices $\Delta_3$ and $\Delta_4$ at three-loop order might renormalize $\omega$ and thus give a correction to the other critical exponents via diagrams of the type of diagram (a) of Fig. 1. The combinatorial factor associated to this diagram is 2. The contribution will take the form $(n = 3, 4)$:

\[
I_n = \sum_{A = \{i_1, ..., i_n\}} \gamma_{A}^{n} \int_{k} \frac{\gamma_{k} + i\omega}{k^2 + \omega^2} \gamma_{A}^{(n)}
\]
\[
= \sum_{A = \{i_1, ..., i_n\}} \gamma_{A}^{n} \gamma_{A}^{(n)} \int_{k} \frac{i\omega}{k^2 + \omega^2},
\]
besides
\[
\gamma_A^{(n)} = A i_{\gamma_1, ..., \gamma_n},
\]
where the set of indices is set and $\epsilon^{i_{1}..i_{n}}$ is the corresponding element of the n-th Levi Civita tensor. Therefore (no contraction on A is implied here):
\[
(\gamma_{A}^{(n)} )^2 = (\epsilon^{i_{1}..i_{n}})^2 \gamma_{i_1, ..., \gamma_n} \gamma_{i_1, ..., \gamma_n}
\]
\[
= \gamma_{i_1, ..., \gamma_n} \gamma_{i_1, ..., \gamma_n}
\]
\[
= (-1)^{n-1} \gamma_{i_2, ..., \gamma_n} (\gamma_{i_1}^2) \gamma_{i_2, ..., \gamma_n}
\]
\[
= (-1)^{(n-1)!}
\]
using the anticommutation relation $\{\gamma_{\mu}, \gamma_{\nu}\} = 2\delta_{\mu\nu}$ and assuming all indices $i_1, ..., i_n$ are distinct (otherwise, $\gamma_{A}^{(n)}$ vanishes trivially.) Note that since $n = 3$ or 4, $(-1)^{(n-1)!} = 1$. Performing the sum in (49) thus yields:
\[
I_n = \left(\frac{d}{n}\right) \int_{k} \frac{i\omega}{k^2 + \omega^2} \propto (d - 2) \int {d^d k} \frac{i\omega}{(2\pi)^d k^2 + \omega^2}
\]
For $n = 3, 4$, $(d - 2) \propto O(\varepsilon)$, and thus the binomial coefficient cancels the pole in the integral, making the contribution $I_n = O(1)$ finite. At the end of the day we find that this diagram will give no contribution to the frequency renormalisation and a fortiori to the $z$ exponent. Thus the critical exponents to three-loop order are given by

FIG. 1. Diagrams entering the renormalization of the generalized GN action. Solid lines stands for fermionic propagators and dashed lines for disorder vertices.
\[ z = 1 + \frac{\varepsilon_2}{2} - \frac{\varepsilon_2^2}{8} + \frac{3\varepsilon_2^3}{32} + O(\varepsilon_2^4), \]
\[ \eta = -\frac{\varepsilon_2^3}{8} + \frac{3\varepsilon_2^3}{16} - \frac{25\varepsilon_2^4}{128} + O(\varepsilon_2^5). \]
\[ d_\psi = \frac{1}{2} [d - 1 + \eta_\psi]. \]
\[ = \frac{1}{2} + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{16} + \frac{3\varepsilon^3}{32} \frac{25\varepsilon^4}{256} + O(\varepsilon^5). \]

To estimate numerical values of the exponents in \( d = 3 \) we use direct evaluation at \( \varepsilon_2 = 1 \) (D), Padé approximant \( \text{P}[M/L] \) and Padé-Borel resummation \( \text{PB}[M/L] \).

We find \( z = 1.409 \) (D), \( z = 1.429 \) (P[2/1]) and \( z = 1.425 \) (PB[2/1]): \( \eta = 0.0625 \) (D) to three loop and \( \eta = -0.133 \) (D) to four loop.

**B. Renormalization of composite operators**

We now discuss the renormalization of the composite operators

\[ \mathcal{O}_\psi(r) := (\bar{\psi}_\alpha(r) \psi_\alpha(r))^q. \]

which are related to the deviation of the disorder distribution from the Gaussian distribution. The bare scaling dimension of operators \( \mathcal{O}_\psi \) is \( \sum \mathcal{O}_\psi \) and Padé-Borel resummation \( \text{PB}[M/L] \).

We use direct evaluation at \( \mathcal{O}_\psi = 1 \) (D), Padé approximant \( \text{P}[M/L] \) and Padé-Borel resummation \( \text{PB}[M/L] \).

We find \( \eta \) to one loop order

\[ \mathcal{O}_\psi = \mathcal{O}_\psi \mathcal{Z}_\psi^{-\eta} \mathcal{O}_\psi. \]

which has to render the divergence of the correlation functions involving operators \( \mathcal{O}_\psi \). To one loop order the diagrams contributing to the \( \mathcal{Z}_\psi \) factor are shown in Fig. 2. We find to one-loop order

\[ \mathcal{Z}_\psi = 1 + 2[q + q(q - 1)] \frac{\Delta \varphi}{\varepsilon_2}. \]

The corresponding scaling function

\[ \eta_\psi(\Delta) = -\sum_n \beta_n(\Delta) \frac{\partial \ln \mathcal{Z}_\psi}{\partial \Delta}. \]

gives the scaling dimension of the composite operators \( \mathcal{O}_\psi \)

\[ \mathcal{O}_\psi = (d - 1 + \eta_\psi) q - \eta_\psi(\Delta^*). \]

To one loop order this yields

\[ \mathcal{O}_\psi = (1 + \varepsilon_2) q - \frac{1}{2} q^2 \varepsilon_2 + O(\varepsilon_2^3). \]

that is consistent with the conformal theory results of [3].

In order to calculate the scaling behavior of the local DOS \( \rho(\omega, \delta) \) it is enough to consider renormalization of the composite operator \( \mathcal{O}_1 \). The corresponding \( \mathcal{Z} \) factor is not independent and is related to \( \mathcal{Z}_\omega \) by

\[ \mathcal{O}_1 = \mathcal{Z}_\omega \mathcal{Z}_\psi^{-1} \mathcal{O}_1. \]

We can write the flow equation for the local DOS as

\[ \sum_n \beta_n(\Delta) \frac{\partial \mathcal{Z}_\psi}{\partial \Delta} + (1 + \gamma(\Delta)) \omega \frac{\partial \mathcal{Z}_\psi}{\partial \omega} \]

\[ - (d - 1) + \eta_\psi(\Delta) - \eta_\psi(\Delta) \rho(\omega, \Delta) = 0. \]

The solution of Eq. (63) in the vicinity of the FP (27) has the form

\[ \rho(\omega) = \xi^{-d} \rho_0(\omega \xi^d, \delta \xi^{1/2}). \]

with \( z = \mathcal{O}_1 \) given to three-loop order by [53].

**II. GROSS-NEVEU-YUKAWA MODEL: 4 - \epsilon EXPANSION**

**A. Model**

The action for the \( U(N) \) GNY model is given by

\[ S_{\text{GNY}} = \int d^4 x [\bar{\chi}_\alpha(\gamma \cdot \nabla) + \sqrt{g} \phi] \chi_\alpha + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} \mu \phi^2 + \frac{\lambda}{4!} \phi^4. \]

We are interested in the \( N \to 0 \) limit. In Fourier space \((-\gamma \cdot \nabla \to \gamma \cdot k)\), the bare fermionic and bosonic propagators read

\[ \langle \chi_\alpha(k) \chi_\alpha(-k') \rangle = \frac{i \gamma \cdot k}{k^2}. \]

\[ \langle \phi(q) \phi(-q) \rangle = \frac{1}{q^2 + \mu}. \]

**B. Renormalization**

We perform a perturbative expansion of correlation and vertex functions in the disorder parameters \( g \) and
from the \( \phi^4 \) theory not shown.

Fig. 3 shows the diverging diagrams in \( d = 4 \) that involve fermionic-bosonic vertices. Other diverging diagrams come from the expansion in \( \lambda \) and are known from the \( \phi^4 \) theory. We use dimensional regularization in \( d = 4 - \varepsilon \) and put \( \varepsilon = 1 \) at the end of the day.

Following the minimal subtraction scheme, we introduce the renormalization constants \( Z_\chi, Z_\phi, Z_\mu, Z_g \) and \( Z_\lambda \). Calling \( \Lambda \) the renormalization scale, the renormalized action reads:

\[
S = \int d^d r [\bar{\chi} \gamma \cdot \nabla \chi + \Lambda^{\varepsilon/2} \sqrt{g} \phi \chi] + \frac{1}{2} Z_\phi (\nabla \phi)^2 + \frac{1}{2} [\mu, Z_\phi + \Lambda^2 Z_\mu] \frac{\delta \phi^2}{\delta \phi^2} + \Lambda \chi Z_\chi \frac{\phi^4}{4!}.
\]

The renormalized fields are related to the bare ones through \( \tilde{\chi} = Z_\chi^{1/2} \chi \), and \( \phi = Z_\phi^{1/2} \phi \). Similarly, we define the renormalized bosonic mass \( \tilde{\mu} = \mu_c + \Lambda^2 Z_\mu Z_\phi^{-1} \delta \mu \). The relations between bare and renormalized couplings read \( \tilde{g} = \Lambda^{\varepsilon/2} Z_g Z_\phi^{-2} Z_\chi^{-1} g \) and \( \tilde{\lambda} = \Lambda^{\varepsilon/2} Z_\lambda Z_\phi^{-2} \lambda \), where we have introduced the renormalization scale \( \Lambda \) to render the renormalized couplings dimensionless.

The bare and renormalized correlation and vertex functions are related as follows:

\[
\Gamma^{(n,l)}(p_i, q_j, \mu - \mu_c, \tilde{g}, \tilde{\lambda}) = Z_\chi^{-n/2} Z_\phi^{-l/2} \Gamma^{(n,l)}(p_i, q_j, \delta \mu, g, \lambda, \Lambda),
\]

\[
\tilde{\Gamma}^{(n,l)}(p_i, q_j, \mu - \mu_c, \tilde{g}, \tilde{\lambda}) = Z_\chi^{-n/2} Z_\phi^{-l/2} \tilde{\Gamma}^{(n,l)}(p_i, q_j, \delta \mu, g, \lambda, \Lambda).
\]

From Eq. (68a) we derive the RG flow equation for the vertex functions:

\[
\left[ \Lambda \frac{\partial}{\partial \Lambda} - \beta_g \frac{\partial}{\partial g} - \beta_\lambda \frac{\partial}{\partial \lambda} - \frac{n}{2} \eta_\chi - \frac{l}{2} \eta_\phi - \gamma_\mu \frac{\partial}{\partial \delta \mu} \right] \Gamma^{(n,l)}(p_i, q_j, \delta \mu, g, \lambda, \Lambda) = 0,
\]

with the scaling functions:

\[
\beta_g(g, \lambda) = - \Lambda \frac{\partial g}{\partial \Lambda},
\]

\[
\beta_\lambda(g, \lambda) = - \Lambda \frac{\partial \lambda}{\partial \Lambda},
\]

\[
\eta_\chi(g, \lambda) = - \sum_{u=\lambda,g} \beta_u \frac{1}{\partial u} \ln Z_\chi,
\]

\[
\eta_\phi(g, \lambda) = - \sum_{u=\lambda,g} \beta_u \frac{1}{\partial u} \ln Z_\phi,
\]

\[
\eta_\mu(g, \lambda) = - \sum_{u=\lambda,g} \beta_u \frac{1}{\partial u} \ln Z_\mu,
\]

\[
\gamma_\mu(g, \lambda) = 2 + \eta_\mu - \eta_\phi.
\]

Besides, dimensional analysis gives

\[
\Gamma^{(n,l)}(p_i, q_j, \delta \mu, g, \lambda, \Lambda) = X^{-d+n(d-1)/2+l(1-d-2)/2} \times \Gamma^{(n,l)}(X p_i, X q_j, \delta \mu, g, \lambda, \chi X \Lambda),
\]

\[
G^{(n,l)}(p_i, q_j, \delta \mu, g, \lambda, \Lambda) = X^{n-1-d-2(d-1)/2} \times G^{(n,l)}(X p_i, X q_j, \delta \mu, g, \lambda, \chi X \Lambda).
\]

We rewrite the relation (71a) in an infinitesimal form as

\[
\left[ \Lambda \frac{\partial}{\partial \Lambda} + p_i \frac{\partial}{\partial p_i} + q_j \frac{\partial}{\partial q_j} - d + \frac{n(d-1)}{2} \right] \Gamma^{(n,l)}(p_i, q_j, \delta \mu, g, \lambda, \Lambda) + \frac{l(d-2)}{2} \Gamma^{(n,l)}(\chi X p_i, \chi X q_j, \delta \mu, g, \lambda, \chi X \Lambda) = 0.
\]

Subtracting (71a) from (72) to get rid of the derivative with respect to \( \Lambda \) we obtain

\[
\left[ \beta_g \frac{\partial}{\partial g} + \beta_\lambda \frac{\partial}{\partial \lambda} + p_i \frac{\partial}{\partial p_i} + q_j \frac{\partial}{\partial q_j} + \gamma_\mu \frac{\partial}{\partial \delta \mu} - d + \frac{n}{2} (\eta_\chi + d-1) + \frac{l}{2} (\eta_\phi + d-2) \right] \Gamma^{(n,l)}(p_i, q_j, \delta \mu, g, \lambda, \Lambda) = 0.
\]

The solutions to Eq. (73) can be found using the method of characteristics. These solutions propagate along specific lines in the space of \( p_i, q_j, \delta \mu, g \) and \( \lambda \) called the characteristics. The characteristics are parametrized by an auxiliary parameter \( L \), which can be identified with a length scale; they are determined by the following set of RG flow equations:

\[
\frac{dp_i}{d \ln L} = p_i(L),
\]

\[
\frac{dq_i}{d \ln L} = q_i(L),
\]

\[
\frac{d\delta \mu}{d \ln L} = \gamma_\mu \delta \mu(L),
\]

\[
\frac{dg}{d \ln L} = \beta_g(g(L)),
\]

\[
\frac{d\lambda}{d \ln L} = \beta_\lambda(\lambda(L)),
\]

\[
\frac{d\ln L}{d \ln L} = \beta_\lambda(\lambda(L)).
\]
with initial conditions \( p_i(1) = p_i, q_i(1) = q_i, \delta \mu(1) = \delta \mu, g(1) = g, \lambda(1) = \lambda \). Thus the solutions of (73) satisfy

\[
\Gamma^{(n,i)}(p_i, q_i, \delta \mu, g, \lambda) = \mathcal{M}(L) \Gamma^{(n,i)}(p_i(L), q_i(L), \delta \mu(L), g(L), \lambda(L)).
\]

with

\[
\frac{d \ln \mathcal{M}_{n,i}(L)}{d \ln L} = \frac{n}{2}(\eta + d - 1) + \frac{l}{2}(\eta_0 + d - 2) - d.
\]

In the vicinity of the critical point the RG flow parameter \( L \) can be identified with the correlation length \( \xi \) in (74), allowing one to calculate the critical exponents from the RG equations.

### C. Critical exponents

Calculation of the one-loop diagrams shown in Fig. 3 in the limit \( N \to 0 \) gives [1]:

\[
\Gamma^{(2,0)} = \langle \chi(k) \chi(-k) \rangle^{-1} = i \gamma p Z_k + i \gamma p \frac{g K_d}{2 \varepsilon_4}.
\]

\[
\Gamma^{(2,1)} = \langle \chi(k_1) \chi(k_2) \phi(p) \rangle_{1PI} = \sqrt{\frac{g Z_k}{g}} - g^{\beta/2} \frac{K_d}{\varepsilon_4}.
\]

\[
\Gamma^{(0,2)} = \langle \phi(k) \phi(-k) \rangle^{-1} = Z_\phi k^2 + Z_\mu \delta \mu \frac{K_d}{\varepsilon_4},
\]

\[
\Gamma^{(0,4)} = \langle \phi(k_1) \phi(k_2) \phi(k_3) \phi(k_4) \rangle_{1PI} = Z_\lambda \lambda - \frac{3}{2} \lambda^2 \frac{K_d}{\varepsilon_4}.
\]

To make these functions finite, we define the renormalization constants as follows:

\[
Z_k = 1 - \frac{1}{2} g \frac{K_d}{\varepsilon_4},
\]

\[
Z_\phi = 1,
\]

\[
Z_\mu = 1 + \frac{\lambda K_d}{2 \varepsilon_4},
\]

\[
Z_g = 1 + \frac{2 g K_d}{\varepsilon_4},
\]

\[
Z_\lambda = 1 + \frac{3 \lambda^2 K_d}{2 \varepsilon_4}.
\]

It is convenient to include \( K_d/2 \) in the redefinition of \( g \) and \( \lambda \). The \( \beta \)-functions read

\[
\beta_g(g, \lambda) = \varepsilon_4 g - 6 g^2,
\]

\[
\beta_\lambda(g, \lambda) = \varepsilon_4 \lambda - 3 \lambda^2,
\]

The FP solution is given by

\[
g_* = \frac{\varepsilon_4}{6}, \lambda_* = \frac{\varepsilon_4}{3}.
\]

The total RG flow in the three parameter space is shown in Fig. 3.
and therefore we get \( z = \varepsilon_4/2 + \beta/\nu \). Moreover, the exponent \( \beta \) is related to the exponent \( \nu \) through the scaling relation
\[
\nu d = 2\beta + (2 - \eta_0)\nu.
\]

From (78e) we know \( Z_\phi = 1 \), which gives \( \eta_0 = O(\varepsilon_4^2) \). Hence we get
\[
\beta = \nu \frac{2 - \varepsilon_4}{2},
\]
and finally we find for the critical dynamic exponent \( z \) to one loop order:
\[
z = \frac{\varepsilon_4}{2} + \frac{2 - \varepsilon_4}{2} = 1 + O(\varepsilon_4^2).
\]

The two-loop order contribution can be calculated using the two loop expression of \( 2 - \eta_0 \) \[3\]:
\[
2 - \eta_0 = 2 - \frac{\varepsilon_4^2}{54} + O(\varepsilon_4^4),
\]
which gives
\[
z = \frac{\varepsilon_4}{2} + \frac{\beta}{\nu} = \frac{\varepsilon_4}{2} + \frac{1}{2}(d - 2 + \eta_0)
\]
\[
= 1 + \frac{\varepsilon_4^2}{108} + O(\varepsilon_4^4).
\]

D. Instanton solutions

We now show the existence of localized instanton solutions to the GN action in the limit \( N \to 0 \) that can give a non-zero contribution to the zero-energy DOS in the semimetallic phase, similar to that found for the GN model in Ref. \[4\]. Following \[8\] we start by rewriting the average DOS at the Dirac point directly in \( d = 3 \) in the form:
\[
\langle \rho(E = 0) \rangle_N = \frac{1}{L^3} \int D[V, \chi, \Psi, \Upsilon] \exp[-S],
\]
where \( \Psi(x) \) is a Lagrange multiplier field selecting solutions to the Dirac equation and \( \Upsilon \) is a Lagrange multiplier enforcing normalization of \( \Psi(x) \) and the action is given by
\[
S = \int d^3x \left[ (\nabla \phi)^2 + \mu \phi(x)^2 + \frac{\lambda}{3!} \phi(x)^4 \right]
- \int d^3x \Psi(x)(\sigma \cdot \nabla + \sqrt{g} \phi(x))\chi(x)
+ \Upsilon \left[ \int d^3x \chi^\dagger(x)\chi(x) - 1 \right],
\]
where \( \sigma = \sigma_x, \sigma_y, \sigma_z \) are the Pauli matrices. We now look for a saddle-point solution to the classical equations of motion. To obtain the latter we vary the action \[92\] with respect to \( \phi, \chi, \chi^\dagger, \Psi^\dagger, \text{and} \ \Upsilon \).

![Image of instanton wavefunction components](image-url)

FIG. 5. The instanton wavefunction components \( (f_1: \text{blue, continuous}, \ f_2: \text{red, dashed}) \) \[98\] and scalar field \( \phi \) (black, dot-dashed) solution of \[100\]-\[102\] computed numerically using the expansion \[103\]-\[105\] for \( A = 1, \mu = -0.1, \lambda = 0.001, \ g = 0.001, \text{and} \ \Psi_0 = 1 \).

This yields
\[
-\nabla^2 \phi(x) + \mu \phi(x) + \frac{\lambda}{3!} \phi(x)^3 = \sqrt{g} \Psi^\dagger(x)\chi(x),
\]
\[
\Psi^\dagger(x)[\sigma \cdot \nabla + \sqrt{g} \phi(x)] = 0,
\]
\[
[\sigma \cdot \nabla + \sqrt{g} \phi(x)]\chi(x) = 0,
\]
\[
\int d^3x \chi^\dagger(x)\chi(x) = 1,
\]
\[
\Upsilon \chi(x) = 0.
\]
From Eq. \[97\] it follows that \( \Upsilon = 0 \) and thus we can take \( \Psi^\dagger(x) = \Psi_0\chi^\dagger(x) \) where \( \Psi_0 \) is a real number. Since the disorder distribution is isotropic it is naturally to assume that the DOS is dominated by a spherically symmetric saddle-point solution. This drastically simplifies the solution of the classical equations of motion \[93\]-\[97\] since they can be reduced to the problem of a Dirac particle in a self-consistent central potential. The solution to this problem can be factorized in the radial and angular parts according to
\[
\chi = f_1(r)\varphi^- - f_2(r)\varphi^+, \quad \varphi^+ \text{ and } \varphi^- \text{ are two-component spinors with total angular momentum } j, \text{angular momentum along } z \text{ and orbital angular momentum } l = j \mp 1/2. \text{ We have:}
\]
\[
\sigma \cdot \nabla f_1(r)\varphi^\pm = \sigma \cdot \hat{r} \left( \partial_\tau + \frac{\sigma \cdot L}{r} \right) f_1(r)\varphi^\pm
\]
\[
= \left( \partial_\tau + \frac{1}{r} - \frac{\kappa}{r} \right) f_1(r) \sigma \cdot \hat{r} \varphi^\pm
\]
\[
= \left( \partial_\tau + \frac{1}{r} - \frac{\kappa}{r} \right) f_1(r)\varphi^\mp, \quad \text{ (99)}
\]
with $\kappa = \pm (j + 1/2) = \pm 1$ for the lowest level $j = 1/2$. Thus we get the following system:

$$\partial_r f_2(r) = f_1(r) \sqrt{g} \phi(r),$$

(100)

$$\left(\partial_r + \frac{2}{r}\right) f_1(r) = f_2(r) \sqrt{g} \phi(r),$$

(101)

$$-(\partial_r^2 + \frac{2}{r} \partial_r - \mu) \phi + \frac{\lambda}{3!} \phi^3 = \sqrt{g} \Psi_0 \left[|f_1|^2 + |f_2|^2\right].$$

(102)

The large $r$ expansion of the Eqs. (100)-(102) gives the following asymptotic behavior

$$f_1(r) = \frac{A}{r^2} + \frac{A^5 g}{30 \mu^2} \frac{1}{r^8} + O(\frac{A^9 g^4}{\mu^4 r^{12}}),$$

(103)

$$f_2(r) = -\frac{A^3 g}{5 \mu} \frac{1}{r^5} - \frac{A^7 g^3}{550 \mu^3} \frac{1}{r^{11}} + O(\frac{A^{11} g^5}{\mu^5 r^{17}}),$$

(104)

$$\phi(r) = \frac{A^2}{\mu} \frac{1}{r^4} + O(\frac{A^6 g^{5/2}}{\mu^3 r^{10}}),$$

(105)

which depends on a single free parameter $A$. A typical solution to Eqs. (100)-(102) obtained numerically using the asymptotic behavior (103)-(105) is shown in Fig. 5. The wave function and disorder distribution both exhibit a singular behavior at $r = 0$ and thus require a regularization \[\text{[7]}\]. Moreover, to obtain the full instanton contribution to the DOS (using either GN or GNY models) one has to expand around the instanton solution and calculate the corresponding Gaussian integral which gives a prefactor to the exponential behavior. It is known that in the case of quadratic dispersion this prefactor can be expressed in the form of a ratio of two functional determinants which diverges in $d > 1$. Thus, in this case the instanton solution requires renormalization \[\text{[9]}\]. The regularization and renormalization of the instanton solution in the case of a Dirac particles in disordered potential is an interesting open question which is left for the future.

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