Geometrical approach to loop calculations and the $\varepsilon$-expansion of Feynman diagrams

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Abstract

Some problems related to the structure of higher terms of the $\varepsilon$-expansion of Feynman diagrams are discussed.

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1 Introduction

Present accuracy of experimental results \[1\] demands calculations of higher-order (two-, three- and, sometimes, four-loop) radiative corrections. Within dimensional regularization \[2\], with the space-time dimension \( n = 4 - 2\varepsilon \), one needs to evaluate higher terms of the \( \varepsilon \)-expansion of multiloop integrals. Dimensional regularization allows one to simultaneously regulate the ultraviolet (UV) and infrared (IR) singularities. The highest power of \( 1/\varepsilon \) contributing to the UV-singular part is connected with the number of loops \( L \). At the same time, IR-singularities or the reduction techniques \[3, 4\] may introduce additional powers of \( 1/\varepsilon \). For this reason, the higher-order terms of the \( \varepsilon \)-expansion of one- and two-loop diagrams become very important for multiloop calculations. In some cases, one can derive exact analytical results valid for an arbitrary space-time dimension \( n \), usually in terms of various hypergeometric functions \[5, 6\]. However, even in these cases it may be very difficult to extract from these functions the analytical form of the coefficients of the \( \varepsilon \)-expansion.

In this paper, we review some recent results devoted to construction of higher-order coefficients of the \( \varepsilon \)-expansion of (mainly) the single-scale diagrams, whose only mass scale, \( M \), can be easily factorized as \( (M^2)^{-L\varepsilon} \), \( L \) being the number of loops. After factorization, the coefficients in \( \varepsilon \) for this diagram are nothing but a combination of some (in general, irrational) numbers. In some cases, these numbers can be numerically calculated for each order of \( \varepsilon \), with a very high accuracy \[7\]. Nevertheless, only the exact analytical result would allow one to check important properties of quantum field theory, like gauge invariance of physical observables and the cancellation of the UV-poles \[8\]. Among the physically-important cases belonging to this class are renormalization group (RG) calculations within \(\overline{MS} \) scheme \[9\], on-shell calculations within QED and QCD \[10\] and some low-energy electroweak processes, where all external (internal) parameters are small with respect to internal (external) ones \[11, 12\].

2 Algebraic classification

Calculation of the RG functions, like \( \beta \)-function or anomalous dimension \( \gamma \), within \(\overline{MS} \) scheme can be reduced to the corresponding propagator-type massless diagrams. As it was shown in \[3\], all three-loop RG functions in arbitrary renormalizable models are expressible only in terms of \( \zeta \)-functions. The four-loop results for QCD and five-loop \( \beta \)-function for \( \phi^4 \)-model confirm this theorem \[3\]. At the higher orders (six and more loops) new, non-zeta terms appear \[13\]. They can be expressed in terms of the multiple Euler–Zagier sums \[14\],

\[
\zeta(s_1, \ldots, s_k; \sigma_1, \ldots, \sigma_k) = \sum_{n_1 > n_2 > \ldots > n_k > 0} \prod_{j=1}^{k} (\sigma_j)^{n_j} / n_j^{s_j},
\]

where \( \sigma_j = \pm 1 \) and \( s_j > 0 \). We will also use a short-hand notation

\[
\zeta_{s_1, \ldots, s_k} \equiv \zeta(s_1, \ldots, s_k; 1, \ldots, 1), \quad U_{a,b} \equiv \zeta(a, b; -1, -1).
\]

For sums of the type \( \zeta \), the weight \( j \) can be defined as \( \sum_{i=1}^{k} s_i \), whereas the value of \( k \) can be associated with the depth (see in \[13, 16\]). For lower cases, these sums correspond to the
ordinary $\zeta$-functions. The six-loop $\beta$-function for the scalar model contains $\zeta_{5,3}$ (this was recently confirmed in [16]). At the seven-loop order, a new transcendental number arises [13], $\zeta_{3,5,3}$. Note that $\zeta_{5,3}$ and another constant, $\zeta_{7,3}$, appear in the calculation of anomalous dimensions at $O(1/N^3)$ in the large-$N$ limit [18]. These results were confirmed and generalized in [19], where it was demonstrated that in odd dimension $3-2e$ the counterterms contain other constants $U_{a,b}$. A remarkable property of all these constants is their connection with knots [13, 18, 20]. All these results confirm miraculous “link” between QFT and the knot theory [20]: some Feynman diagrams can be connected with certain knots, so that the values (Euler–Zagier sums) of Feynman diagrams are also associated with knots. A detailed description of the alternating (non-alternating) Euler-Zagier sums is presented in Ref. [15].

At the three-loop level, however, new constants appear [12]. For their classification, Standard Model calculations produce diagrams with other mass distributions. Two-loop vacuum diagrams with equal masses [11, 21] yield the transcendental number $Cl_2 \left( \frac{\pi}{2} \right)$, where $Cl_j (\theta)$ is the Clausen function [22], $Cl_{2l} (\theta) = \text{Im} \, Li_{2l} (e^{i\theta})$ and $Cl_{2l+1} (\theta) = \text{Re} \, Li_{2l+1} (e^{i\theta})$. The same constant $Cl_2 \left( \frac{\pi}{2} \right)$ appears in the one-loop off-shell three-point diagram with massless internal lines, in the symmetric case when all external momenta squared are equal [20]. A generalization to the $L$-loop ladder case yields $Cl_{2L} \left( \frac{\pi}{2} \right)$ [24].

At the three-loop level, however, new constants appear [12]. For their classification, Broadhurst [16] has introduced the “sixth root of unity” basis connected with

$$\zeta \left( \begin{array}{c} s_1 \\ \lambda^{p_1} \\ \vdots \\ s_k \\ \lambda^{p_k} \end{array} \right) = \sum_{n_1, n_2, \ldots, n_k > 0} \frac{\prod_{j=1}^{k} \lambda^{p_j n_j}}{n_j^{\frac{1}{6}}} \quad (3)$$

where $\lambda = e^{i\pi/3}$ and $p_j \in \{0, 1, 2, 3, 4, 5\}$. For $p_j \in \{0, 3\}$ it coincides with the Euler–Zagier sums [11]. For the lowest weights $j$, the bases consist of the following elements:

| $j$ | Basis Elements |
|-----|----------------|
| 1   | $\pi$, $\ln 2$, $\ln 3$ |
| 2   | $\pi^2$, $\ln^2 2$, $\ln^2 3$, $Cl_2 \left( \frac{\pi}{2} \right)$, $Li_2 \left( \frac{1}{4} \right)$, $\ln 2 \ln 3$, $\pi \ln 2$, $\pi \ln 3$ |
| 3   | $\pi Cl_2 \left( \frac{\pi}{2} \right)$, $\pi^2 \ln 2$, $\pi^2 \ln 3$, $Cl_2 \left( \frac{\pi}{2} \right) \ln 2$, $Cl_2 \left( \frac{\pi}{2} \right) \ln 3$, $\pi^3$, $Li_2 \left( \frac{1}{4} \right) \ln 2$, $Li_2 \left( \frac{1}{4} \right) \ln 3$, $\ln^3 2$, $\ln^2 2 \ln 3$, $\ln 2 \ln^2 3$, $\ln^3 3$, $\pi Li_2 \left( \frac{1}{4} \right)$, $\pi \ln^2 2$, $\pi \ln^2 3$, $\pi \ln 2 \ln 3$, $\zeta_3$, $Li_3 \left( \frac{4}{3} \right)$, $Li_3 \left( \frac{1}{2} \right)$ |

Unfortunately, a large number of elements (more than 4000) makes it difficult to define the complete basis of “sixth root of unity” at the weight 4. So far, only the cases with the depth...
$k \leq 2$ have been examined\cite{16}. One of the remarkable results of Ref.\cite{16} is that all finite parts of three-loop vacuum integrals without subdivergences, with an arbitrary distribution of massive and massless lines, can be expressed in terms of four weight-4 constants: $\zeta_4$, $\left[\text{Cl}_2\left(\frac{\pi}{3}\right)\right]^2$, $U_{3,1}$, and $V_{3,1} = \sum_{p>k>0} \frac{(-1)^p}{p^k} \cos\left(\frac{\pi}{3} p k\right)$, which is an essentially new constant. Note that $\left[\text{Cl}_2\left(\frac{\pi}{3}\right)\right]^2$ also appears in the two-loop non-planar three-point diagram\cite{25}, when internal lines are massless, whereas all external momenta squared are off shell and equal.

In Refs.\cite{26,27}, the central binomial sums were considered, $S(a) \equiv \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n!)^2} n^{-a}$. In particular, in\cite{27} one-fold integral representation was established. It was shown that these sums are connected with the multi-dimensional polylogarithm\cite{28} of “sixth root of unity”,

$$\text{Li}_{n_1,...,n_k}(\lambda) = \sum_{n_1>n_2>...>n_k>0} \frac{\lambda^{n_1}}{n_1^{n_1} \cdots n_k^{n_k}}, \tag{4}$$

and Euler-Zagier sums.

### 3 Geometrical approach

To predict types of functions (and the values of their arguments) which may appear in higher orders of the $\varepsilon$-expansion, a geometrical approach\cite{29} happens to be very useful. Using this approach, the results for all terms of the $\varepsilon$-expansion have been obtained for the one-loop two-point function with arbitrary masses\cite{30,31}. Moreover, all terms have been also obtained for the $\varepsilon$-expansion of one-loop three-point integrals with massless internal lines and arbitrary (off-shell) external momenta and two-loop vacuum diagrams with arbitrary masses\cite{31,32}, which are related to each other, due to the magic connection\cite{33}. All these results have been represented in terms of the log-sine integrals (see in\cite{22} and below), whose angular arguments have a rather transparent geometrical interpretation (angles of certain triangles). For instance, for the one-loop two-point function with masses $m_1$ and $m_2$ the relevant angles are (see in\cite{23,31}) $\tau_{01}' = \arccos\left(\frac{k^2+m_2^2-m_1^2}{2m_1 \sqrt{k^2}}\right)$ and $\tau_{02}' = \arccos\left(\frac{k^2-m_1^2+m_2^2}{2m_2 \sqrt{k^2}}\right)$, where $k$ is the external momentum. For single-scale diagram with $m_1 = m_2 \equiv m$ and $k^2 = m^2$, each of these angles is equal to $\frac{\pi}{3}$. In more complicated cases, like, e.g., the three-point function with general values of the momenta and masses, an arbitrary term of the $\varepsilon$-expansion can be represented in terms of one-fold angular integrals whose parameters can be related to the angles associated with a three-dimensional simplex\cite{29,30}.

In Ref.\cite{34}, the on-shell values of two-loop massive propagator-type integrals have been studied, and it was observed that the finite (as $\varepsilon \to 0$) parts of all such integrals without subdivergences can be expressed in terms of three weight-3 constants, two for the real part, $\zeta_3$ and $\pi \text{Cl}_2\left(\frac{\pi}{3}\right)$, and one for the imaginary part, $\pi \zeta_2$.

It is natural to assume that the elements occurring in the coefficients of the $\varepsilon$-expansion are connected with some properties of diagrams, like the structure of the cut(s). In this case, the numbers of elements for higher-order weights may be essentially reduced. Basing on this conjecture, in Ref.\cite{35} an ansatz was elaborated for constructing the “irrationalities” occurring in the $\varepsilon$-expansion of single-scale diagrams involving cut(s) with two massive particles.
This construction is closely related to the geometrically-inspired all-order $\varepsilon$-expansion of the one-loop propagator-type diagrams \[30, 31\], which was also used to fix the normalization factor $\frac{1}{\sqrt{3}}$. The procedure of constructing the ansatz is as follows: for each given weight $j$ the set $\{b_j\}$ of the basic transcendental numbers contains (i) all products of the lower-weight elements $\{b_{j-k}b_k\}$, $k = 1, 2, \ldots, j - 1$ and (ii) a set of new (non-factorizable) elements $\{\bar{b}_j\}$, which are associated with the quantities arising in the real and imaginary parts of the polylogarithms $L_j \left(e^{i\theta}\right)$ and $L_j \left(1 - e^{i\theta}\right)$, with $\theta = \frac{\pi}{4}$ or $\theta = \frac{2\pi}{4}$. They can be expressed in terms of the Clausen function $Cl_j (\theta)$, log-sine integrals $Ls_j (\theta)$ and generalized log-sine integrals $Ls_j^{(k)} (\theta)$, defined as (see in \[22\])

\[
Ls_j^{(k)} (\theta) = - \int_0^\theta d\phi \, e^{i\phi} \ln^{j-k-1} \left| 2 \sin \frac{\phi}{2} \right|, \quad Ls_j (\theta) = Ls_j^{(0)} (\theta).
\]  

Note that $Ls_2 (\theta) = Cl_2 (\theta)$. Therefore, in our case the non-factorizable part of the basis can be expressed in terms of the generalized log-sine integrals of two possible angles, $\theta = \frac{\pi}{4}$ and $\theta = \frac{2\pi}{4}$. To establish the basis of the weight $4$ within such an ansatz, one needs to analyze no more than 100 elements. It should be noted that this basis is not uniquely defined, since there are several relations between polylogarithmic functions $Cl_j (\theta)$, $Ls_j (\theta)$ and $Ls_j^{(k)} (\theta)$ of these arguments. After excluding all linearly-dependent terms (see Appendix A of \[36\]), the basis contains the following non-factorizable constants: $Ls_j \left(\frac{2\pi}{3}\right)\big|_{j=3,4,5}$, $Ls_j \left(\frac{\pi}{3}\right)\big|_{j=2,4,5}$, $Ls_j^{(1)} \left(\frac{2\pi}{3}\right)\big|_{j=4,5}$ and $Ls_j^{(2)} \left(\frac{2\pi}{5}\right)$. This set of elements is called the odd basis \[35\]. The numerical values of these constants are given in Appendix A of \[35\]. In particular, it was found that the constant $V_{3,1}$ can be expressed in terms of the weight-$4$ elements of the odd basis, $V_{3,1} = \frac{1}{3} \left[Cl_2 \left(\frac{\pi}{3}\right)\right]^2 - \frac{1}{4} \pi Ls_3 \left(\frac{2\pi}{3}\right) + \frac{13}{24} \zeta_3 \ln 3 - \frac{259}{108} \zeta_4 + \frac{3}{8} Ls_4^{(1)} \left(\frac{2\pi}{3}\right).

By analogy with the odd basis introduced in \[35\], it is possible to consider the even basis connected with the angles $\frac{\pi}{2}$ and $\pi$. Apart from the well-known elements $\pi$, $\ln 2$, $\zeta_j$ and the Catalan’s constant $G = Cl_2 \left(\frac{\pi}{2}\right) = Ls_2 \left(\frac{\pi}{2}\right)$, this basis also contains (up to the weight $5$) $Li_j \left(\frac{\pi}{2}\right)\big|_{j=4,5}$ (see also in \[15\]), $Ls_j \left(\frac{\pi}{2}\right)\big|_{j=3,4,5}$, $Cl_4 \left(\frac{\pi}{2}\right)$ and $Ls_5^{(2)} \left(\frac{\pi}{2}\right)$. An example of a physical calculation where the constant $Ls_3 \left(\frac{\pi}{2}\right)$ arises is given in \[37\]. Instead of $Li_4 \left(\frac{1}{2}\right)$ and $Li_5 \left(\frac{1}{2}\right)$, one could take, e.g., $Ls_4^{(1)} \left(\frac{\pi}{2}\right)$ and $Ls_5^{(1)} \left(\frac{\pi}{2}\right)$ (or $Ls_4^{(1)} (\pi)$ and $Ls_5^{(1)} (\pi)$) (see Appendix A of \[36\]). The constructed odd and even bases have an interesting property: the number $N_j$ of the basic irrational constants of a weight $j$ satisfies a simple “empirical” relation $N_j = 2^j$, which has been checked up to weight $4$. The situation with weight-$5$ bases is discussed below.

### 4 Searching for new elements

A natural generalization of the ansatz proposed in \[35\] could be the inclusion of the generalized (Nielsen) polylogarithms \[38\], $S_{a,b}(z)$, where again we consider the cases $z = e^{i\theta}$ and
\[ z = 1 - e^{i\theta}, \text{ with } \theta = \frac{\pi}{3} \text{ or } \frac{2\pi}{3}. \] It is easy to show that
\[ S_{a,b} \left( e^{i\theta} \right) = \frac{i^a \left( -1 \right)^b}{(a-1)!b!} \int_0^\theta d\phi \left( \theta - \phi \right)^{a-1} \left[ \ln \left| 2 \sin \frac{\phi}{2} \right| - \frac{1}{2}i(\pi - \phi) \right]^b + S_{a,b} \left( 1 \right), \]
 reduces to a combination of the generalized log-sine integrals \( S \). The function \( S_{a,b} \left( 1 - e^{i\theta} \right) \) can be reduced to \( S_{a,b} \left( e^{i\theta} \right) \) plus products of lower-order \( S_{a,b} \left( e^{i\theta} \right) \) and powers of \( \ln \left( 2 \sin \frac{\theta}{2} \right) \). Therefore, such Nielsen polylogarithms do not generate new elements. Note that \( S_{a,b} \left( 1 \right) \) is a combination of \( \zeta \)-functions \( \zeta \).

Another possibility is the multi-dimensional polylogarithm \( \Omega \). Performing PSLQ analysis \( \Omega \) with an accuracy of 100 decimals we established that, up to weight 5, all constants \( \Omega \) are expressible in terms of the odd basis.

Let us consider the following hypergeometric function:
\[ _{p+1}F_p \left( \frac{3}{2} + b\varepsilon, 2 + c_1\varepsilon, \ldots, 2 + c_R\varepsilon, 1 + d_1\varepsilon, \ldots, 1 + d_{P-R+1}\varepsilon \right| \frac{z}{4} \right), \] (6)

where \( 0 \leq z \leq 4 \). The one-, two-, and three-loop Feynman diagrams which are expressible via hypergeometric function of this type are given in \( [5, 36] \). The \( \varepsilon \)-expansion of the function \( \Omega \) can be written as \( \Omega \) (for details, see Appendix B of Ref. \( [30] \))

\[ \frac{2(1+2b\varepsilon)}{z} \left[ \frac{R}{\prod_{i=1}^{R} (1 + c_i\varepsilon)} \right] \sum_{j=1}^{\infty} \frac{(j!)^2}{(2j)!} \frac{z^j}{j^{R+1}} \left\{ 1 - \varepsilon \left[ S_1 T_1 + \frac{D_1}{j} + 2b S_1 \right] + \varepsilon^2 \left[ \frac{1}{2j^2} \left( D_2 + D_1 \right) \right] \right. \]
\[ \left. + 2b S_1 S_1 T_1 + \frac{D_1}{j} \left( S_1 T_1 + 2b S_1 \right) + 2b^2 \left( S_1 + S_1 \right) + \frac{1}{2} S_1 T_1 + \frac{1}{2} S_1 T_1 \right\} + O(\varepsilon^3) \right), \] (7)

where \( A_k \equiv \sum_{i=1}^{P+1} a_i^k \), \( C_k \equiv \sum_{i=1}^{R} c_i^k \), \( D_k \equiv \sum_{i=1}^{P-R+1} d_i^k \) and \( T_k \equiv C_k + D_k - A_k - b^k \). It is implied that \( C_k = 0 \) and \( D_k = 0 \) when \( P-R-1 = 0 \). Here and below, we also use the short-hand notations \( S_a \equiv S_a \left( n - 1 \right) \) and \( \bar{S}_a \equiv S_a \left( 2n - 1 \right), \) where \( S_a \left( n \right) = \sum_{j=1}^{n} j^{-a} \) is the harmonic sum. In this way, the \( \varepsilon \)-expansion of the hypergeometric function \( \Omega \) is reduced to series of the following type:
\[ \Sigma \left( \begin{array}{cc} i_1, & i_2, \ldots, i_q \\ a_1, & a_2, \ldots, a_q \\ b_1, & b_2, \ldots, b_q \end{array} \right) (z) \equiv \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{z^n}{n^k} \left( S_{a_1} \right)^{i_1} \ldots \left( S_{a_q} \right)^{i_q} \left( \bar{S}_{b_1} \right)^{j_1} \ldots \left( \bar{S}_{b_q} \right)^{j_q}. \] (8)

For \( p = q = 0 \), the analytical result is available \( [40] \)
\[ \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{z^n}{n^k} = - \sum_{j=0}^{k-2} \frac{(-2)^j}{(k-2-j)!} \left( \ln z \right)^{k-2-j} \left( \frac{1}{2} S_{j+2} \left( \theta_z \right) \right), \quad \theta_z \equiv 2 \arcsin \frac{\sqrt{z}}{2}. \] (9)

For the \( p + q = 1 \) terms, the following one-fold integral representation can be constructed:
\[ \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{z^n}{n^k} \left[ \alpha S_1 \left( n - 1 \right) + S_1 \left( 2n - 1 \right) \right] = \frac{2}{(k-2)!} \int_0^{\theta_z} d\theta \left[ \ln z - 2 \ln \left( 2 \sin \frac{\theta}{2} \right) \right]^{k-2} \]
\[ \times \left\{ \frac{1}{2} \left( S_2 \left( \theta \right) + \frac{1}{2} \theta \ln \left( 2 \sin \frac{\theta}{2} \right) - (1 + \alpha) \left[ L_{s_2} \left( \pi + \theta \right) + \theta \ln \left( 2 \cos \frac{\theta}{2} \right) \right] \right) \right\}, \] (10)
where $k \geq 2$, $0 < z < 4$ and $\alpha$ is an arbitrary constant. For all other types of the sums (8), the results are available only for particular values $z = 1, 2, 3$ [36, 38, 40]. It should be noted that not all sums (8) can be expressed separately in terms of the even and odd bases, and vice versa, not all basis elements are expressible in terms of the binomial sums.

Testing the $\varepsilon$-expansion of a one-loop three-point diagram contributing to the $H \to \gamma\gamma$ decay (see details in [36]) we revealed that new elements $\chi_5$ and $\tilde{\chi}_5$ appear at the weight-5 level in the odd and even bases, respectively. They can be presented as special types of the inverse binomial sum (8),

$$
\chi_5 = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{1}{n^2} [S_1(n-1)]^3, \quad \tilde{\chi}_5 = \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \frac{2^n}{n^2} [S_1(n-1)]^3.
$$

The r.h.s. of Eq. (10) gives us an idea about a possible integral representation for new functions arising in higher orders of the $\varepsilon$-expansion. Apart from $L_{S_j}^{(i)}(\theta)$, we meet

$$
L_{\text{sc}_{i,j}}(\theta) = -\int_0^\theta d\phi \ln^{i-1} \left| 2 \sin \frac{\phi}{2} \right| \ln^{j-1} \left| 2 \cos \frac{\phi}{2} \right|.
$$

Some properties of these functions were studied in Appendix A of [36]. In particular, a careful PSLQ analysis showed that a new element for the even basis should be added, $L_{\text{sc}_{2,4}}(\pi/2)$. In this way we restore, accidentally, the “empirical” relation $N_j = 2^j$ for the $j = 5$ level of the odd and even bases. However, the class of possible new functions is not restricted only by $L_{\text{sc}_{i,j}}(\theta)$ [12]. We have not tested the harmonic polylogarithms [11] and their multiple-value generalization [12], which may be related to the $\varepsilon$-expansion of certain Appell functions (see, e.g., in [13]). Another possibility is to test all types of the integrals occurring in (10):

$$
\int_0^\theta d\phi \phi^k \ln^i \left| 2 \sin \frac{\phi}{2} \right| \ln^j \left| 2 \cos \frac{\phi}{2} \right|, \quad \int_0^\theta d\phi L_{S_2}(\phi) \ln^i \left| 2 \sin \frac{\phi}{2} \right| \ln^j \left| 2 \cos \frac{\phi}{2} \right|.
$$

All these issues require a more careful analysis. We present here one-fold integrals related to our elements $\chi_5$ and $\tilde{\chi}_5$ obtained by using a PSLQ analysis with an accuracy of 200 decimals:

$$
\int_0^{2\pi/3} d\phi L_{S_2}(\phi) \ln^2 \left( 2 \sin \frac{\phi}{2} \right) = -\frac{55}{324} \pi \zeta_2 L_{S_2} \left( \frac{\pi}{3} \right) + \frac{1225}{1296} \zeta_2 \zeta_3 + \frac{4621}{1296} \zeta_5 + \frac{23}{972} \pi L_{S_4} \left( \frac{\pi}{3} \right)
- \frac{5}{18} \pi L_{S_4} \left( \frac{2\pi}{3} \right) - \frac{1}{3} L_{S_2} \left( \frac{\pi}{3} \right) L_{S_3} \left( \frac{2\pi}{3} \right) + \frac{1}{3} L_{S_5}^{(1)} \left( \frac{2\pi}{3} \right) - \frac{1}{48} \chi_5
= 0.744148409838194515377332007 \ldots,
$$

$$
\int_0^{\pi/2} d\phi L_{S_2}(\phi) \ln^2 \left( 2 \sin \frac{\phi}{2} \right) = -\frac{1}{48} \ln^6 2 + \frac{5}{48} \zeta_2 \ln^3 2 - \frac{35}{128} \zeta_3 \ln^2 2 - \frac{5}{8} \ln 2 - \frac{5}{8} \ln 5 \left( \frac{1}{2} \right)
+ \frac{475}{1024} \zeta_2 \zeta_3 + \frac{3379}{2048} \zeta_5 - \frac{17}{64} \pi \zeta_2 L_{S_2} \left( \frac{\pi}{2} \right) - \frac{1}{16} \pi L_{S_4} \left( \frac{\pi}{2} \right) + \frac{3}{16} \pi C_4 \left( \frac{\pi}{2} \right) - \frac{1}{2} L_{S_2} \left( \frac{\pi}{2} \right) L_{S_3} \left( \frac{\pi}{2} \right) - \frac{1}{48} \tilde{\chi}_5
= 0.6565469031185897863954386 \ldots.
$$

5 Concluding remarks

Let us return to the idea about the connection between the structures occurring in the $\varepsilon$-expansion and the structure of the cut(s). Careful analysis of existing results [14] for
the single-mass-scale diagrams allows us to formulate the following conjecture [40]: the \( \varepsilon \)-expansion of diagrams with cuts involving 0, 1 or 3 massive lines (or their combinations) are expressible in terms of the even basis, whereas the diagrams with a cut involving 2 massive lines (or its mixing with a 0 or 1 massive cut) produce elements of the odd basis. For all other cases the problem is open. Recent results [45] confirm this conjecture. We would expect that the four-loop numerical results in QED [46] can be represented in terms of the even basis.

Another interesting example is the one-loop triangle diagram \( J_3 \) with three massive lines with \( m_i = m \), when all scales (masses and momenta) are equal to each other. The geometrical approach [29, 30] yields the following representation:

\[
J_3(1, 1, 1; m)|_{\vec{p}^2=m^2} = -\frac{i\pi^{2-\varepsilon}}{m^{2+2\varepsilon}} \Gamma(1+\varepsilon)2^{1-\varepsilon}3^{1/2+\varepsilon} \sum_{j=0}^{\infty} \frac{(-\varepsilon)^j}{(j+1)!} \int_0^{\pi/3} d\phi \ln^{j+1} \left( 1 + \frac{1}{8\cos^2\phi} \right) \tag{16}
\]

where (see also in Refs. [47, 48, 49])

\[
\int_0^{\pi/3} d\phi \ln \left( 1 + \frac{1}{8\cos^2\phi} \right) = \frac{1}{6} \left[ \pi \ln 2 - \text{Cl}_2 \left( \frac{\pi}{3} \right) \right] , \tag{17}
\]

\[
\int_0^{\pi/3} d\phi \ln^2 \left( 1 + \frac{1}{8\cos^2\phi} \right) = \frac{1}{3} \text{Cl}_2 \left( \frac{\pi}{3} \right) \ln 2 + \frac{1}{3} \pi \ln^2 2 + \frac{1}{3} \pi \text{Li}_2 \left( \frac{1}{4} \right) - \frac{1}{2} \text{Li}_3 \left( \frac{2\pi}{3} \right) + 2\text{Im} \left[ \text{Li}_3 \left( \frac{1}{2} e^{i\pi/3} \right) \right] - \frac{31}{12} \pi \zeta_2 . \tag{18}
\]

One can see that the \( \varepsilon \)-term is expressed in terms of “sixth root of unity” basis [16]. However, it is not sufficient to have only the elements from the odd/even bases. To explain this situation, we should recall that this vertex diagram also possesses an anomalous threshold. If we put all \( \vec{p}^2 = p^2 \) then this threshold would correspond to \( p^2/m^2 = 3 \). In other words, this triangle diagram has a mixture of an anomalous cut corresponding to \( p^2/m^2 = 3 \) and two-particle cuts (with respect to each leg) at \( p^2/m^2 = 4 \). A similar situation is also observed for the case when \( p^2/m^2 = 4 \) and \( p^2/m^2 = 9 \) cuts are present [49].

Finally, let us summarize the situation with the even and odd bases. Although our construction is incomplete, it has allowed us to obtain new results for several one-, two- and three-loop master integrals implemented in different packages [50]. In this way, we found several interesting relations between generalized log-sine integrals and multiple zeta functions. Moreover, a new relation between \( 3F_2 \) and \( 2F_1 \) hypergeometric functions of argument \( \frac{1}{4} \) was established [31].

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