BEHAVIOUR OF SINGULARITIES OF THE KERR–NEWMAN AND THE KERR–SEN SOLUTIONS BY ARBITRARY BOOST

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The behaviour of the singularities of the rotating black holes under an arbitrary boost is considered on the basis of a complex representation of the Kerr theorem. We give a simple algorithm allowing to get explicit expressions for the metric and the position of the singularities for arbitrary direction and magnitude of the boost, including the ultrarelativistic case. The non-smoothness of the ultrarelativistic limit is discussed. The Kerr-Sen BH-solution to low energy string theory is also analyzed.
1. INTRODUCTION

Recently, the problem of finding the ultrarelativistic limit of exact particle-like solutions of the Einstein field equations received considerable attention, especially in connection with some non-trivial gravitational effects which are expected to occur in the interparticle interactions at extreme energies due to the appearance of gravitational shock waves [1-8].

First results in this field were obtained by Aichelburg and Sexl [1], who considered the ultrarelativistic boost of the Schwarzschild solution to analyze the behaviour of the gravitational field of a massless point particle in the light-like limit.

A similar treatment for the Kerr geometry, which can be considered as a model of a spinning particle in general relativity, has to take into account the orientation of the angular momentum with respect to the boost [4-8].

There are three different physical situations connected with boosted BH-solutions. The first one is the original Aichelburg–Sexl problem of application of such solutions to describe the gravitational field of light–like particles with or without spin. The second application consists in modelling the gravitational field of elementary particles with finite rest mass under the boost, and it is connected with an analysis of possible effects generated in relativistic collisions. Finally, there are astrophysical applications, namely boosting black holes. In this case also the behaviour of the horizon and of the ergosphere under the boost are of interest.

The analysis of the boosted Kerr solution [4-8] exhibits some difficulties in the interpretation of the results in the limiting, ultrarelativistic case. In particular, there are technical difficulties due to the absence of smooth ultrarelativistic limits, as well as an ambiguity in performing the limits when more than one parameter is involved in the limiting procedure simultaneously (for example, the parameters $m$ and $a$ in the non–charged Kerr case).

In any of the above cited approaches, the boosted Kerr solution is given by approximate expressions so that one cannot obtain an invariant description of the behaviour of the singularity under the boost. We propose here a different method of description of the boosted Kerr solution based on the Debney, Kerr and Schild
formalism [9] (DKS) and on the Kerr theorem [9-13].

The advantage of this approach relies in the possibility of obtaining exact, explicit expressions for the metric and its singularities in the case of an arbitrary boost, namely a boost with an arbitrary orientation with respect to the angular momentum. In fact, being represented in the Kerr-Schild form, the boosted Kerr metric can be linked with an auxiliary Minkowski space having a “rigid” coordinate system. This allows us to represent shock waves and singularities in asymptotically flat Cartesian coordinates.

2. THE DKS-FORMALISM AND THE KERR THEOREM

In our notations we follow the work of Debney, Kerr and Schild [9]. All the BH-solutions in Einstein’s gravity can be described by the simple Kerr-Schild metric

\[ g_{\mu \nu} = \eta_{\mu \nu} + 2h e_\mu^3 e_\nu^3, \]  

(2.1)

where \( \eta_{\mu \nu} \) is the metric of an auxiliary Minkowski space \( M^4 \) with signature \((++++)\) and Cartesian coordinates \( t, x, y, z \). For a non-rotating BH the scalar function \( h \) has the form

\[ h = m/r - e^2/2r^2, \]

where \( m \) and \( e \) are the mass and the charge of the BH. The vector \( e^3\mu = (1, \vec{k}) \) is a field of principal null directions which is spherically symmetric (\( \vec{k} = (x, y, z)/r \) in the auxiliary Minkowski space with metric \( \eta_{\mu \nu} \)). In the case of rotating BH-solutions the metric is still of the Kerr-Schild form but its twisting structure is determined by a different null congruence \( e^3 \) and by a modification of the radial coordinate.

In null coordinates

\[ 2^{1/2} \zeta = x + iy, \quad 2^{1/2} \bar{\zeta} = x - iy, \]
\[ 2^{1/2} u = z + t, \quad 2^{1/2} v = z - t, \]  

(2.2)

the null vector \( e^3 \) can be expressed via a scalar function \( Y(x) \) in the following way:

\[ e^3 = du + \bar{Y} d\zeta + Y d\bar{\zeta} - Y \bar{Y} dv. \]  

(2.3)
The determination of $e^3$ is possible since the principal null congruences of rotating BH solutions are geodesic and shear-free, and the Kerr Theorem [9-13] gives a rule to construct all such congruences: an arbitrary, geodesic shear-free null congruence in Minkowski space is defined by a function $Y(x)$ which is a solution of the equation

$$F = 0,$$  \hspace{1cm} (2.4)

where $F(\lambda_1, \lambda_2, Y)$ is an arbitrary analytic function of the projective twistor coordinates

$$\lambda_1 = \zeta - Yv, \quad \lambda_2 = u + Y\bar{\zeta}, \quad Y. \hspace{1cm} (2.5)$$

A consequence of the Kerr Theorem is also the expression for the complex radial coordinate

$$\tilde{r} \equiv PZ^{-1} = dF/dY,$$  \hspace{1cm} (2.6)

which characterizes “dilatation $+i$ twist” of the congruence. Correspondingly, the singular regions of the metrics are defined by the system of equations

$$F = 0, \quad dF/dY = 0.$$  \hspace{1cm} (2.7)

The BH-solutions belong to a class of metrics for which the singularities are contained in a bounded region of space. In this case the equation $F = 0$ can be solved in explicit form. Moreover, in this case there exists a complex representation of the function $F$ in which the congruence is defined by an effective ”source” moving in complex Minkowski space $CM^4$ along a complex world line. Such a complex representation was initially suggested by Lind and Newman [14,15] in the Newman-Penrose formalism. The field $e^3$ can be used as one of the vectors of null tetrad $e_1, e_2, e_3, e_4$ satisfying

$$g_{ab} = e^\mu_a e_{b\mu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = g^{\mu\nu}, \hspace{1cm} (2.8)$$

($e^3, e^4$ are real null vectors, $e^1, e^2$ are complex conjugates). The null tetrad $e^\mu_a$ can
be completed as follows:

\[ e^1 = d\zeta - Y dv; \]
\[ e^2 = d\bar{\zeta} - \bar{Y} dv; \]
\[ e^4 = dv - he^3. \] (2.9)

The inverse tetrad has the form

\[ \partial_1 = \partial_\zeta - \bar{Y} \partial_u; \]
\[ \partial_2 = \partial_{\bar{\zeta}} - Y \partial_u; \]
\[ \partial_3 = \partial_u - h \partial_4; \]
\[ \partial_4 = \partial_v + Y \partial_\zeta + \bar{Y} \partial_{\bar{\zeta}} - Y \bar{Y} \partial_u. \] (2.10)

The function \( h \) of the Kerr-Newman solution has the form

\[ h = m(Z + \bar{Z})/P^3 - e^2/(Z \bar{Z}), \] (2.11)

while the electromagnetic field can be obtained from the potential

\[ A = -e(Z + \bar{Z})e^3/(2P^2). \] (2.12)

### 3. WEAK STATIONARITY AND CONGRUENCES HAVING SINGULARITIES CONTAINED IN A BOUNDED REGION

The null congruence with tangent \( e^3 \) is stationary in \( M^4 \) if \( \partial_t e^3 = 0 \). However, in general one can also consider a “weak nonstationarity” corresponding to the fact that the stationarity can be restored by a Lorentz transformation. In this case there exists a real time-like vector field \( K \) such that

\[ KY = K\bar{Y} = 0, \] (3.1)

and consequently \( Ke^3 = 0 \). The congruences stationary in this weak sense and having singularities contained in a bounded region have been considered in [16-18]. In this case the function \( F \) must be at most quadratic in \( Y \),

\[ F \equiv a_0 + a_1 Y + a_2 Y^2 + (qY + c)\lambda_1 - (p\bar{Y} + \bar{q})\lambda_2, \] (3.2)
where the coefficients $c$ and $p$ are real constants and $a_0, a_1, a_2, q, \bar{q}$, are complex constants. The solutions of the equation $F = 0$ and the equations for the singularities can be found in this case in explicit form. The solution $Y(x)$ of the equation $F = 0$ satisfies the weak stationarity condition (3.1) if

$$K = c\partial_u + \bar{q}\partial_\zeta + q\partial_\bar{\zeta} - p\partial_v.$$  

(3.3)

In the papers [16,17] another, equivalent form of $F$ was suggested. This form allows to represent the parameters of the function $F$ and the vector field $K$ as retarded-time fields starting from an “effective” complex world line $x^\mu_0(\tau)$ depending from a complex time parameter $\tau$. This form is the following

$$F \equiv (\lambda_1 - \lambda^0_1)K\lambda_2 - (\lambda_2 - \lambda^0_2)K\lambda_1.$$  

(3.4)

Here the twistor components with zero indices

$$\lambda^0_1(\tau) = \zeta_0(\tau) - Yv_0(\tau), \quad \lambda^0_2(\tau) = u_0(\tau) + Y\bar{\zeta}_0(\tau),$$

(3.6)

denote the values on the points of the complex world-line represented in null coordinates $\phi_0(\tau) = (\zeta_0, \bar{\zeta}_0, u_0, v_0)$ ($\bar{\zeta}_0$ and $\zeta_0$ are not necessarily complex conjugates). The vector $K$ can be expressed in the form

$$K(\tau) = \dot{x}^\mu_0(\tau)\partial_\mu,$$

(3.7)

where the dot denotes $\partial_\tau$.

The Kerr congruences with weak nonstationarity are determined by straight analytic world lines with constant 3-velocity $\bar{v}$:

$$x^\mu_0(\tau) = x^\mu_0(0) + \xi^\mu \tau; \quad \xi^\mu = (1, \bar{v}),$$

(3.8)

correspondingly, the vector $K = \xi^\mu \partial_\mu$ is a constant Killing vector of the solutions. The form (3.4) has the remarkable property that, in spite of an explicit dependence of the parameters of the function $F$ in (3.3) on $\tau$, this dependence is absent really, since in consequence of the relations

$$\lambda^0_1(x_0(\tau)) = \lambda^0_1(x_0(0)) + \tau K\lambda_1, \quad \lambda^0_2(x_0(\tau)) = \lambda^0_2(x_0(0)) + \tau K\lambda_2,$$

(3.9)
the terms proportional to \( \tau \) cancel. Therefore the expressions (3.1) and (3.4) are equivalent.

The relation (3.4) is very convenient in order to obtain explicit representation of the congruences of the boosted Kerr solution. By writing the function \( F \) in the form

\[
F = AY^2 + BY + C, \tag{3.10}
\]

where

\[
A = (\bar{\zeta} - \bar{\zeta}_0)v_0 - (v - v_0)\dot{\bar{\zeta}}_0;
\]

\[
B = (u - u_0)v_0 + (\zeta - \zeta_0)\dot{\bar{\zeta}}_0 - (\bar{\zeta} - \bar{\zeta}_0)\dot{\zeta}_0 - (v - v_0)\dot{u}_0;
\]

\[
C = (\zeta - \zeta_0)\dot{u}_0 - (u - u_0)\dot{\zeta}_0,
\]

one can find two explicit solutions for the function \( Y(x) \)

\[
Y_{1,2} = (-B \pm \Delta)/2A, \tag{3.12}
\]

where \( \Delta = (B^2 - 4AC)^{1/2} \).

On the other hand differentiating \( F = 0 \) and using (2.6) one finds

\[
Y = -(B + PZ^{-1})/2A, \tag{3.13}
\]

and consequently

\[
PZ^{-1} = \mp \Delta. \tag{3.14}
\]

One can find also

\[
P = \dot{x}^\mu_o(\tau)e^3_\mu. \tag{3.15}
\]

The field \( e^3 \) can be normalized by introducing \( l^\mu = e^{3\mu}/P \) so that \( \dot{x}^\mu_o l_\mu = 1 \), that yields the following form of the Kerr-Newman metric

\[
g_{\mu\nu} = \eta_{\mu\nu} + [m(\bar{r}^{-1} + \bar{\bar{r}}^{-1}) - e^2(\bar{r}\bar{\bar{r}})^{-1}]l_\mu l_\nu. \tag{3.16}
\]

where the complex radial coordinate \( \bar{r} \equiv PZ^{-1} \) is given by the expression (3.14) or can be represented in the form

\[
\bar{r} = -dF/dY = -B - 2AY. \tag{3.17}
\]
It is convenient to represent $\tilde{r}$ as a sum of the real radial distance $r$ and an angular coordinate $\tilde{r} = r + ia \cos \theta$. Then the equation (3.14) fixes the relation between the polar coordinates $r, \theta, \phi$ and the null Cartesian coordinates through the expressions (3.11) for the coefficients $A, B, C$.

4. BEHAVIOUR OF SINGULARITIES OF THE KERR-NEWMAN SOLUTION BY THE BOOST

In the “gauge” $x^0_0 = \tau$ the complex world line (3.8) can be represented as $x^\mu_0(\tau) = \{\tau, \vec{x}_0(0) + \vec{v}\tau\}$. The complex initial displacement can be decomposed as $\vec{x}_0(0) = \vec{c} + i\vec{d}$, where $\vec{c}$ and $\vec{d}$ are real 3-vectors with respect to the space $O(3)$-rotation. The real part $\vec{c}$ defines the initial shift of the solution, and the imaginary part $\vec{d}$ defines the size and the position of the singular ring as well as the corresponding angular momentum. It can be easily shown that in the rest frame, when $\vec{V} = 0$, $\vec{d} = \vec{d}_0$, the singular ring lies in the plane orthogonal to $\vec{d}$ and has a radius $a = |\vec{d}_0|$. The corresponding angular momentum is $\vec{J} = m\vec{d}_0$.

In the case of a boost orthogonal to the direction of $\vec{d}$, this vector is not altered by Lorentz contraction ($\vec{d} = \vec{d}_0$, $|\vec{d}| = a$), while if $\vec{d}$ and $\vec{V}$ are collinear we have

$$\vec{d}_0 = \vec{d}/\sqrt{1 - |\vec{V}|^2}.$$ (4.1)

This shows that the parameter $a$ coincides with its rest value $a_0$ if $\vec{d}$ and $\vec{V}$ are orthogonal, while

$$a_0 = a/\sqrt{1 - |\vec{V}|^2},$$ (4.2)

if $\vec{V}$ and $\vec{d}$ are collinear.

In order to calculate the parameters $A, B, C$ it is convenient to express the complex world line in null coordinates

$$2^{1/4} \zeta_0 = x_0 + iy_0, \quad 2^{1/4} \bar{\zeta}_0 = x_0 - iy_0,$$

$$2^{1/4} u_0 = z_0 + t_0, \quad 2^{1/4} v_0 = z_0 - t_0.$$ (4.3)

The Killing vector of the solution will then be

$$\xi^\mu = 2^{-1/2}\{\dot{x}_0 - i\dot{y}_0, \dot{\zeta}_0 + \dot{\bar{\zeta}}_0, -i(\dot{\zeta}_0 + \dot{\bar{\zeta}}_0), \dot{u}_0 + \dot{v}_0\},$$

$$\xi^0 = 2^{-1/2}\{\dot{x}_0 + i\dot{y}_0, \dot{\bar{\zeta}}_0 - \dot{\zeta}_0, -i(\dot{\bar{\zeta}}_0 - \dot{\zeta}_0), \dot{u}_0 - \dot{v}_0\},$$

$$\xi^i = 2^{-1/2}\{\dot{u}_0, \dot{v}_0, \dot{\zeta}_0, \dot{\bar{\zeta}}_0\}.$$
while the functions $P$ takes the form

$$P = e^3 \dot{x}^\mu_0 = \dot{u}_0 + \ddot{Y} \dot{\zeta}_0 + Y \ddot{\zeta}_0 - Y \dot{Y} \dot{v}_0. \quad (4.4)$$

The complex radial coordinate $\tilde{r} \equiv P Z^{-1}$ is given by the expression (3.4). As for the unboosted Kerr solution, one can represent $\tilde{r}$ as a “sum” of the real radial distance $r$ and an angular coordinate. Then equation (3.4) can be used to fix the relation between the polar coordinates $r, \theta, \phi$ and the null Cartesian coordinates (4.3) through the expressions (3.3) for the coefficients $A, B, C$. Due to the formula (3.17), the singular regions are defined by the zeros of the function $\tilde{r}$. In what follows, we present some examples of boosted Kerr solutions and then discuss the general features exhibited by them.

**Example I.**

Spinning particle moves with speed of the light in the positive direction of the $z$-axis, 3-vector $\vec{d} = (0, 0, a)$ is also directed along the $z$-axis. We have the following coordinates of complex world line

$$x_0^0(\tau) \equiv \tau, \quad z_0(\tau) = ia + \tau, \quad x_0(\tau) = y_0(\tau) = 0.$$

In the null coordinates it gives

$$\sqrt{2} u_0 = z_0 + \tau = ia + 2\tau; \quad \sqrt{2} v_0 = z_0 - \tau = ia, \quad \zeta_0 = \bar{\zeta}_0 = 0, \quad (I.1)$$

that yields

$$\dot{u}_0 = \sqrt{2}, \quad \dot{v}_0 = 0, \quad \dot{\zeta}_0 = \ddot{\zeta}_0 = 0, \quad \text{and}$$

$$u - u_0 = (z - ia + t - 2\tau)/\sqrt{2},$$

$$v - v_0 = (z - ia - t)/\sqrt{2}, \quad (I.2)$$

$$\zeta - \zeta_0 = \zeta, \quad \bar{\zeta} - \bar{\zeta}_0 = \bar{\zeta}$$

Coefficients $A, B, C$ calculated from (3.6) will be

$$A = 0; \quad B = t - z + ia; \quad C = x + iy. \quad (I.3)$$
As a result the function $F$ acquires the form $F = x + iy - Y(z - ia - t)$, and solution of the equation $F = 0$ is

$$Y = (x + iy)/(z - ia - t). \quad (I.4)$$

The function

$$\tilde{r} = -dF/dY = z - ia - t. \quad (I.5)$$

One can see that there is no singularity if $a \neq 0$ since there is no real solutions to the system of equations $F = F_Y = 0$.

On the other hand, setting $a = 0$ we have got the case of spinless particle, and a moving singular plane which is placed at $z = t$. Therefore there is no smooth limit by $a \to 0$.

**Example II.**

The motion with speed of the light in the positive direction of the $x$-axis, orthogonal to the 3-vector $\vec{d}$ which defines the direction and the value of the angular momentum $\vec{J} = m(0,0,a)$, $a = |\vec{d}|$. We have the complex world line $x_0(\tau) = \tau$, $y_0(\tau) = 0$, $z_0(\tau) = ia$, $t_0 = \tau$. Correspondingly, the world line in the null coordinates is

$$\sqrt{2}u_0 = ia + \tau, \quad \sqrt{2}v_0 = ia - \tau, \quad \sqrt{2}\zeta_0 = \tau, \quad \sqrt{2}\bar{\zeta}_0 = \tau; \quad (II.1)$$

and the velocities are

$$\sqrt{2}\dot{u}_0 = 1, \quad \sqrt{2}\dot{v}_0 = -1, \quad \sqrt{2}\dot{\zeta}_0 = 1, \quad \sqrt{2}\dot{\bar{\zeta}}_0 = 1.$$

We have also

$$\sqrt{2}(u - u_0) = z + t - ia - \tau, \quad \sqrt{2}(v - v_0) = z - t - ia + \tau,$$

$$\sqrt{2}(\zeta - \zeta_0) = x + iy - \tau, \quad \sqrt{2}(\bar{\zeta} - \bar{\zeta}_0) = x - iy - \tau. \quad (II.2)$$

Coefficients $A, B, C$ take the form

$$A = (-x + iy - z + t + ia)/2; \quad B = ia + iy - z; \quad C = (x + iy - z - t + ia)/2. \quad (II.3)$$
The function $\tilde{r} \equiv PZ^{-1}$ takes the form
\[ PZ^{-1} = -dF/dY = x - t. \] (II.4)

There is therefore a moving singular plane placed at $x = t$. The function $Y$ will be
\[ Y = (dF/dY - B)/2A = (x + iy - z - t + ia)/(x - iy + z - t - ia). \] (II.5)

**Example III.**

To understand better the absence of smooth limit in the example I we consider here an intermediate case with a boost with a speed $v = \alpha c$, $\alpha \leq 1$ in the positive direction of the $z$-axis, and then we will consider the limit $\alpha \to 1$. As in the example I, the 3-vector $\vec{d} = (0, 0, a)$ is directed along the $z$-axis. We have the complex world line
\[ x_0(\tau) = y_0(\tau) = 0, \quad z_0(\tau) = ia + \alpha \tau, \quad t_0 = \tau. \] (III.1)

In null coordinates we have $\sqrt{2}\dot{u}_0 = 1 + \alpha$, $\sqrt{2}\dot{v}_0 = -1 + \alpha$, $\sqrt{2}\dot{\xi}_0 = 0$, $\sqrt{2}\dot{\bar{\xi}}_0 = 0$.

It yields
\[ \sqrt{2}(u - u_0) = z + t - ia - (\alpha + 1)\tau, \quad \sqrt{2}(v - v_0) = z - t - ia + (1 - \alpha)\tau, \]
\[ \sqrt{2}(\xi - \xi_0) = x + iy, \quad \sqrt{2}(\bar{\xi} - \bar{\xi}_0) = x - iy. \] (III.2)

Coefficients $A, B, C$ will be
\[ A = -(x - iy)(1 - \alpha)/2; \quad B = ia - z + \alpha t; \quad C = (x + iy)(1 + \alpha)/2. \] (III.3)

The expression for complex radial distance (3.9) yields
\[ \tilde{r}^2 = B^2 - 4AC = (z - \alpha t)^2 + (1 - \alpha^2)(x^2 + y^2) - a^2 - 2ia(z - \alpha t). \] (III.4)

Like to the standard Kerr solution one can represent the complex radial coordinate $\tilde{r}$ as a sum of the real radial distance $r$ and an angular coordinate $\tilde{r} = r + ia \cos \theta$. 
Then, selecting the real and imagine parts of the expression (III.4) one obtains the following relations between the polar coordinates $r, \theta, \phi$ and Cartesian coordinates of the auxiliary Minkowski space

\[ x + iy = (r + ia)e^{i\phi} \sin \theta / \sqrt{1 - \alpha^2}, \]
\[ z - \alpha t = r \cos \theta. \]  

(III.4)

For the case $\alpha = 0$ this coincides with the coordinate relations of the standard Kerr solution. Setting $r = \cos \theta = 0$ we obtain the equation of singular ring

\[ x^2 + y^2 = a^2/(1 - \alpha^2), \quad z - \alpha t = 0. \]  

(III.5)

It may be seen that size of the ring grows by the increasing of $\alpha$, and in the limiting case $\alpha = 1$ the singularity is placed on infinity.

The cause of this effect is the above mentioned relation (4.2) $a_0 = a/\sqrt{1 - \alpha^2}$. The increasing of singular ring by $\alpha = v/c \to 1$ is a seeming effect connected with using the parameter $a$ instead of its rest value $a_0$. Being to expressed via the rest value singular region takes the form of moving ring of constant radius $a_0$, however, if we consider a light-like particle its rest mass is infinitely small and singularity is to be placed on infinity.

**Example IV.**

Intermediate case clarifying the limiting result of example II. The boost with a speed $v = \alpha c$, $\alpha \leq 1$ in the direction $x$, orthogonal to direction of angular momentum $\vec{d} = (0,0,a)$.

We have the complex world line

\[ x_0(\tau) = \alpha \tau, \quad y_0(\tau) = 0, \quad z_0(\tau) = ia, \quad t_0 = \tau; \]  

(IV.1)

In the null coordinates it yields

\[ \sqrt{2}u_0 = 1, \quad \sqrt{2}v_0 = -1, \quad \sqrt{2}\xi_0 = \alpha, \quad \sqrt{2}\bar{\xi}_0 = \alpha. \]

\[ \sqrt{2}(u - u_0) = z + t - ia - \tau, \quad \sqrt{2}(v - v_0) = z - t - ia + \tau, \]
\[ \sqrt{2}(\xi - \xi_0) = x + iy - \alpha \tau, \quad \sqrt{2}(\bar{\xi} - \bar{\xi}_0) = x - iy - \alpha \tau. \]  

(IV.2)
Coefficients $A, B, C$ will be the following

$$A = -(x-iy)/2 - (z-t-ia)\alpha/2; \quad B = ia - z + iy\alpha; \quad C = (x+iy)/2 - \alpha(z+t-ia)/2.$$  \hspace{1cm} (IV.3)

From the equation (4.4) we obtain

$$\tilde{r}^2 = B^2 - 4AC = (x - \alpha t)^2 + (1 - \alpha^2)[y^2 + (z - ia)^2].$$  \hspace{1cm} (IV.4)

Representing $\tilde{r} = r + ia\sqrt{1 - \alpha^2}\cos\theta$ and selecting the real and imaginary parts of (IV.4) one obtains the following coordinate relations which generalize corresponding relations of the stationary Kerr solution

$$(x - \alpha t)/\sqrt{1 - \alpha^2} + iy = (r/\sqrt{1 - \alpha^2} + ia)e^{i\phi}\sin\theta,$$

$$z = r\cos\theta/\sqrt{1 - \alpha^2}.\hspace{1cm} (IV.5)$$

Singular region $r = \cos\theta = 0$ will be

$$z = 0; \quad (x - \alpha t)^2 + (1 - \alpha^2)y^2 = a^2(1 - \alpha^2).\hspace{1cm} (IV.6)$$

This is a moving ring placing in the $z = 0$ plane. It is oblate in $x$ direction with the Lorentz factor $\sqrt{1 - \alpha^2}$.

Therefore, in the limit $\alpha = 1$ singular region will be moving segment of the line $z = 0, x = t; -a \leq y \leq a$, which is parallel to the $y$-axis. One can mention that this limit is also non-smooth.

So, setting $\alpha = 1$ in the equations (IV.4),(IV.5) we come to equation $x = t$, coordinate $y$ is not restricted, however, the coordinate $z$ is indefinite $\sim 0/0$. It will be equal to zero if we set first $r = \cos\theta = 0$ corresponding the singular region and then take the limit $\alpha \to 1$, however it is equal to infinity if the limit $\alpha = 1$ is taken first.

**Example V.**

We finally consider the general case in which the value of the velocity is arbitrary as well as its direction with respect to the angular momentum. Without lost of
generality, we can consider the boost performed with a parameter $\alpha$ in the $z$-direction ($\alpha = v_z/c$), and a parameter $\beta$ in the $x$-direction ($\beta = v_x/c$), while the angular momentum is defined by $\vec{d} = (0, 0, a)$. Denoting $w^2 = \alpha^2 + \beta^2$ the following general formula for the coordinate relations can be easily obtained:

\[
(x - \beta t)\sqrt{1 - \alpha^2} + iy\sqrt{1 - w^2} = (r + ia\sqrt{1 - \beta^2})e^{i\phi}\sin \theta, \quad (V.1)
\]

\[
z - \alpha t = -r \cos \theta / \sqrt{1 - \beta^2}.
\]

The singular region $r = 0, \cos \theta = 0$ is placed on the plane $z = \alpha t$ and is described by

\[
(1 - \alpha^2)(x - \beta t)^2 + (1 - w^2)y^2 = a^2(1 - \beta^2). \quad (V.2)
\]

The singularity is a moving ring distorted in the $x$ direction by a factor $\sqrt{(1 - \beta^2)/(1 - \alpha^2)}$ and in the $y$ direction by a factor $\sqrt{(1 - \beta^2)/(1 - w^2)}$. The ultrarelativistic limit corresponds to $w = 1$ and the singular region is a couple of straight lines parallel to the $y$ axis.

Therefore, we can conclude that the non-smoothness and the non-commutativeness of the limiting procedure is a general feature of the boosted Kerr solutions.

The main consequences of the considered examples are the non-smoothness and the non-commutativeness of the limiting procedure as well as an unexpected behaviour of the Kerr singular ring which is connected with the definition of the parameters of the solution after the boost and shows that such parameters must be “renormalized” by the boost.

It is interesting also to observe from the coordinate relations that the coordinate $r$ is ‘scaled’ by the boost with respect to the asymptotically flat Cartesian coordinates $x, y, z$; as a consequence the region of small values of $r$ (and big values of $h$) is stretched to big values of the $x, y, z$-coordinates, this is the origin of the shock waves in the ultrarelativistic limit since the region of big $h$ can be very far from the “centre” of the solution.

In astrophysical applications, the behaviour of the horizon and of the ergosphere after the boost also has a physical interest. It can be easily shown that in the above suggested coordinates $r$ and $\theta$ the horizon as well as the ergosphere are simply
given by the known formulae for the Kerr case where the mass parameter \( m \) must be scaled by the Lorentz factor.

5. BOOST OF THE KERR-SEN SOLUTION

Recently, rotating BH-solutions received attention also in string theory. The Kerr-Sen BH-solution is a generalization of the Kerr solution to low energy string theory [19] (or to axion-dilaton gravity). We would like to show that the above formalism is also applicable to the Kerr-Sen solution. The metric of the Kerr-Sen BH may be written in the form [20]

\[
ds^2_{\text{dil}} = 2e^{-2(\Phi - \Phi_0)}\tilde{e}_1\tilde{e}_2^2 + 2\tilde{e}_3\tilde{e}_4,
\]

where

\[
\tilde{e}_1 = (PZ)^{-1}dY, \quad \tilde{e}_2 = (P\bar{Z})^{-1}d\bar{Y},
\]

\[
\tilde{e}_3 = P^{-1}e^3,
\]

\[
\tilde{e}_4 = dr + iaP^{-2}(Yd\bar{Y} - \bar{Y}dY) + (H_{\text{dil}} - 1/2)e^3,
\]

and

\[
H_{\text{dil}} = Mr/\Sigma_{\text{dil}}; \quad \Sigma_{\text{dil}} = e^{-2(\Phi - \Phi_0)}(Z\bar{Z})^{-1};
\]

\[
e^{-2(\Phi - \Phi_0)} = 1 + (Q^2/2M)(Z + \bar{Z}); \quad Z^{-1} \equiv \tilde{r}.
\]

The field of principal null directions is \( \tilde{e}^3 \). Following eq.(6.1) of [9] this tetrad is related to the Kerr-Schild tetrad (2.3),(2.9) as follows

\[
\tilde{e}_1 = e_1 - P^{-1}P\bar{e}^3, \quad \tilde{e}_2 = e_2 - P^{-1}P\bar{e}^3,
\]

\[
\tilde{e}_3 = P^{-1}e^3,
\]

\[
\tilde{e}_4 = P\bar{e}_4^{\text{dil}} + P\bar{e}^1 + P\bar{e}^2 - P\bar{e}\bar{P}P^{-1}e^3.
\]

Therefore the Kerr-Sen metric (5.1) may be reexpressed in the form containing the Kerr-Schild tetrad \( e^a \), the dilaton factor \( e^{-2(\Phi - \Phi_0)} \), and a “deformed” function

\[
H_{\text{dil}} = he^{2(\Phi - \Phi_0)}
\]
instead of the function $h$.

It was shown in [20] that the Kerr-Sen metric is of type I contrary to the Kerr solution which is type D. However one of the principal null directions $e^3$ of the Kerr and the Kerr-Newman solutions survives in the Kerr-Sen solution and retains the property of being geodesic and shear free. It means that the Kerr theorem is applicable to this solution, since it has just the same principal null congruence and positions of caustics. Therefore the above analysis can be extended to the Kerr-Sen solution.

6. CONCLUSIONS

We discussed here a method allowing to describe in explicit form the metric and the behaviour of the singular region of the Kerr solution under arbitrary boost and with arbitrary orientations of angular momentum. In particular, we have shown that the Kerr theorem automatically allows to obtain an asymptotically flat coordinate system and the equations describing the singularities in these coordinates. These results throw some light on the somewhat mysterious “standard” procedure commonly used to obtain shock waves metrics. In fact this procedure gives only approximate expressions before taking the ultrarelativistic limits [5-8]. Of course, the tail of the shock wave (logarithmic term) cannot be obtained using the present method, because first of all we work always with vacuum, singular solutions of the Einstein field equations. To obtain the profile of the wave located on the delta–like singularity of the metric, one must first re–interpret the metric itself as being created by a singular distribution of matter on an extended manifold.

Our results are not very encouraging as far as the physical content of all such ultrarelativistic solutions is concerned. In fact, we obtained a quite general picture of non-smoothness and non-commutativeness of the limits $a \to 0$, $v \to 1$ and $r \to 0$. The absence of a smooth limit explains the well known fact that the limiting solution belongs to a completely different class with respect to the starting one: it is type $N$ and not type $D$ as the original Kerr solution, it is not asymptotically flat and has another group of symmetry.
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