Carleson Measure Spaces with Variable Exponents and Their Applications

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Abstract. In this paper, we introduce the Carleson measure spaces with variable exponents \( CMO^{p(\cdot)} \). By using discrete Littlewood–Paley–Stein analysis as well as Frazier and Jawerth’s \( \varphi \)-transform in the variable exponent settings, we show that the dual spaces of the variable Hardy spaces \( H^{p(\cdot)} \) are \( CMO^{p(\cdot)} \). As applications, we obtain that Carleson measure spaces with variable exponents \( CMO^{p(\cdot)} \), Campanato space with variable exponent \( L^{q,p(\cdot),d} \), and Hölder–Zygmund spaces with variable exponents \( \dot{H}^{p(\cdot)} \) coincide as sets and the corresponding norms are equivalent. Via using an argument of weak density property, we also prove that Calderón–Zygmund singular integral operators are bounded on \( CMO^{p(\cdot)} \).

Mathematics Subject Classification. Primary 42B25; Secondary 42B35, 46E30.

Keywords. Carleson measure spaces, Variable exponents, Dual spaces, Singular integrals.

1. Introduction

The Hardy and \( BMO \) spaces have been playing a crucial role in modern harmonic analysis since the early groundbreaking work in Hardy space theory came from Coifman, Fefferman, Stein and Weiss in [1,2,10]. In [10], Fefferman and Stein obtained that the space of functions of bounded mean oscillation, \( BMO \), is the dual space of the Hardy space \( H^1 \) and that the space \( BMO \) can be characterized by Carleson measure, which suggests that one could use the generalized Carleson measure to characterize the dual of the Hardy space. For this purpose, we introduce the Carleson measure spaces \( CMO^{p(\cdot)} \) that generalize \( CMO^p \) and \( BMO \). Note that Carleson measure spaces \( CMO^p \) have been studied in [15,18,19,21,22]. However, the theory of Carleson measure spaces is still unknown in the variable exponent settings.

The main goal of this paper is to develop a complete theory for the dual spaces of variable Hardy spaces \( H^{p(\cdot)} \). Before stating our main results,
we begin with the definitions of Lebesgue and Hardy spaces with variable exponents.

For any Lebesgue measurable function \( p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty] \) and for any measurable subset \( E \subset \mathbb{R}^n \), we denote \( p^{-}(E) = \inf_{x \in E} p(x) \) and \( p^{+}(E) = \sup_{x \in E} p(x) \). Especially, we denote \( p^{-} = p^{-}(\mathbb{R}^n) \) and \( p^{+} = p^{+}(\mathbb{R}^n) \). Let \( p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty) \) be a measurable function with \( 0 < p^{-} \leq p^{+} < \infty \) and \( \mathcal{P}^{0} \) be the set of all these \( p(\cdot) \).

**Definition 1.1.** [3, 8, 17] Let \( p(\cdot) : \mathbb{R}^n \rightarrow (0, \infty] \) be a Lebesgue measurable function. The variable Lebesgue space \( L^{p(\cdot)} \) consists of all Lebesgue measurable functions \( f \), for which the quantity \( \int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx \) is finite for some \( \varepsilon > 0 \) and 

\[
\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx \leq 1 \right\}.
\]

We also recall the following class of exponent functions, which can be found in [9]. Let \( \mathcal{B} \) be the set of \( p(\cdot) \in \mathcal{P} \) such that the Hardy–Littlewood maximal operator \( M \) is bounded on \( L^{p(\cdot)} \). An important subset of \( \mathcal{B} \) is the \( LH \) condition.

In the study of variable exponent function spaces it is common to assume that the exponent function \( p(\cdot) \) satisfies the \( LH \) condition. We say that \( p(\cdot) \in LH \), if \( p(\cdot) \) satisfies

\[ |p(x) - p(y)| \leq \frac{C}{-\log(|x-y|)}, \quad |x-y| \leq 1/2 \]

and

\[ |p(x) - p(y)| \leq \frac{C}{\log |x| + e}, \quad |y| \geq |x|. \]

It is well known that \( p(\cdot) \in \mathcal{B} \) if \( p(\cdot) \in \mathcal{P} \cap LH \). Denote by \( \mathcal{S} = \mathcal{S}(\mathbb{R}^n) \) the collection of rapidly decreasing \( C^\infty \) function on \( \mathbb{R}^n \). Also, denote by \( \mathcal{S}_\infty \) the functions \( f \in \mathcal{S} \) satisfying \( \int_{\mathbb{R}^n} f(x)x^\alpha \, dx = 0 \) for all multi-indices \( \alpha \in \mathbb{Z}_+^n := \{0,1,2,...\}^n \) and \( \mathcal{S}_\infty^\prime \) its topological dual space. For \( f \in \mathcal{S}_\infty^\prime \), we recall the definition of the Littlewood–Paley–Stein square function

\[
\mathcal{G}(f)(x) := \left( \sum_{j \in \mathbb{Z}} |\psi_j * f(x)|^2 \right)^{1/2},
\]

and the discrete Littlewood–Paley–Stein square function

\[
\mathcal{G}^d(f)(x) := \left( \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} |\psi_j * f(2^{-j}k)|^2 \chi_Q(x) \right)^{1/2},
\]

where \( Q \) denote dyadic cubes in \( \mathbb{R}^n \) with side-lengths \( 2^{-j} \) and the lower left-corners of \( Q \) are \( 2^{-j}k \). We also recall the definition of variable Hardy spaces \( H^{p(\cdot)} \) as follows.
Definition 1.2. ([4, 24]) Let $f \in \mathcal{S}'$, $\psi \in \mathcal{S}$, $p(\cdot) \in \mathcal{P}^0$ and $\psi_i(x) = t^{-n} \psi(t^{-1}x)$, $x \in \mathbb{R}^n$. Denote by $\mathcal{M}$ the grand maximal operator given by $\mathcal{M}f(x) = \sup \{ |\psi_t * f(x)| : t > 0, \psi \in \mathcal{F}_N \}$ for any fixed large integer $N$, where $\mathcal{F}_N = \{ \varphi \in \mathcal{S} : \int \varphi(x)dx = 1, \sum_{|\alpha| \leq N} \sup (1 + |x|^N)|\partial^\alpha \varphi(x)| \leq 1 \}$. The variable Hardy space $H^{p(\cdot)}$ is the set of all $f \in \mathcal{S}'$, for which the quantity
$$\|f\|_{H^{p(\cdot)}} = \|\mathcal{M}f\|_{L^{p(\cdot)}} < \infty.$$  

Throughout this paper, $C$ or $c$ denotes a positive constant that may vary at each occurrence but is independent to the main parameter, and $A \sim B$ means that there are constants $C_1 > 0$ and $C_2 > 0$ independent of the main parameter such that $C_1 B \leq A \leq C_2 B$. Given a measurable set $S \subset \mathbb{R}^n$, $|S|$ denotes the Lebesgue measure and $\chi_S$ means the characteristic function. We also use the notations $j \wedge j' = \min\{j, j'\}$ and $j \vee j' = \max\{j, j'\}$. Fix an integer $d \geq d_{p(\cdot)} \equiv \min\{d \in \mathbb{N} \cup \{0\} : p^{-}(n + d + 1) > n\}$. A function $a$ on $\mathbb{R}^n$ is called a $(p(\cdot), q)$-atom, if there exists a cube $Q$ such that supp $a \subset Q$;
$$\|a\|_{L^q} \leq \|\chi_Q\|_{L^{p(\cdot)}(\mathbb{R}^n)}; \int_{\mathbb{R}^n} a(x)x^\alpha dx = 0 \quad \text{for } |\alpha| \leq d.$$ We say that a cube $Q \subset \mathbb{R}^n$ is dyadic if $Q = Q_{jk} = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : 2^{-j}k_i \leq x_i < 2^{-j}(k_i + 1), i = 1, 2, \ldots, n\}$ for some $j \in \mathbb{Z}$ and $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$. Denote by $\ell(Q) = 2^{-j}$ the side length of $Q = Q_{jk}$. Denote by $z_Q = 2^{-j}k$ the left lower corner of $Q$ and by $x_Q$ is any point in $Q$ when $Q = Q_{jk}$. For any function $\psi$ defined on $\mathbb{R}^n$ and $Q = Q_{jk}$, set $\psi_j(x) = 2^{jn}\psi(2^jx)$, $\hat{\psi}(x) = \hat{\psi}(-x)$ $\psi_Q(x) = |Q|^{1/2}\psi_j(x - z_Q)$.

The remainder of this paper is organized as follows. In Sect. 2 we give the precise definition of the Carleson measure space $CMO^{p(\cdot)}$ and establish the Plancherel–Pòlya inequality for such space. In Sect. 3, we introduce sequence spaces with variable exponents $s^{p(\cdot)}$ and $c^{p(\cdot)}$ and obtain the duality of the variable Hardy space $H^{p(\cdot)}$ with $CMO^{p(\cdot)}$ by a constructive proof, which is the heart of the present paper. We show that Carleson measure spaces with variable exponents $CMO^{p(\cdot)}$, Campanato space with variable exponent $\mathcal{L}_{q,p(\cdot),d}$ and Hölder–Zygmund spaces with variable exponents $\mathcal{H}^{p(\cdot)}$ coincide as sets and the corresponding norms are equivalent in Sect. 4. In Sect. 5, we discuss the boundedness of Calderón–Zygmund singular integral operators on $CMO^{p(\cdot)}$ via using an argument of weak density property.

2. Carleson Measure Space with Variable Exponent

In this section, we introduce the Carleson measure space $CMO^{p(\cdot)}$. First we recall a wavelet Calderón reproducing formula developed by Deng and Han [7, Chapter 3.4]. The well-known discrete Calderón reproducing formula was first introduced by Frazier and Jawerth [11].

Lemma 2.1. Let $\psi \in \mathcal{S}$, and satisfy $\sum_j |\hat{\psi}(2^{-j}\xi)|^2 = 1$ for $\xi \neq 0$, and the moment conditions $\int_{\mathbb{R}^n} x^\alpha \psi(x)dx = 0$ for all multi-indices $\alpha$. Then we can choose a large $N$ depending on $\psi$ such that the following discrete Calderón reproducing identity

\[ \sum_{k=0}^{2N+1} \sum_{j=-N}^{N} \hat{\psi}(2^{-j}k) \phi_j(k) \phi_{2N+1-j}(x) = \psi(x) \]
\[ f(x) = \sum_j \sum_Q |Q| (\psi_j * f)(x_Q) \tilde{\psi}_j(x, x_Q), \]

where \( \tilde{\psi} \in \mathcal{S} \), \( Q \) are dyadic cubes with side-length \( \ell(Q) = 2^{-j-N} \) and \( x_Q \) is any fixed point in \( Q \), and where the series converges in \( \mathcal{S}_\infty \) for all \( f \in \mathcal{S}_\infty \). Furthermore, the convergence of the right-hand, as well as the equality, is in \( \mathcal{S}'_\infty \).

The advantage of this formula is that it expresses \( f \) as a sum of molecular, or wavelet-like, functions \( \tilde{\psi}_j(x, x_Q) \) with coefficients \( |Q| (\psi_j * f)(x_Q) \). As a consequence, we can replace the coefficient \( |Q| (\psi_j * f)(x_Q) \) with either the supremum or infimum of such choices and retain appropriate estimates. This “wavelet” scheme is particularly useful in dealing with the Hardy spaces and Carleson measure spaces. We would like to point out that functions \( \psi \in \mathcal{S} \) used in the discrete Calderón identity do not have compact support. To prove the main result, we will also need the following new discrete Calderón-type identity, which, for the setting of spaces of homogeneous type, was first used in [7].

**Lemma 2.2.** [26] Suppose that \( p(\cdot) \in LH \). Let \( \phi \) be Schwartz functions with support on the unit ball satisfying the conditions: for all \( \xi \in \mathbb{R}^n \),

\[ \sum_{j \in \mathbb{Z}} |\hat{\phi}(2^{-j} \xi)|^2 = 1 \]

and \( \int_{\mathbb{R}^n} \phi(x) x^\alpha dx = 0 \) for all \( 0 \leq |\alpha| \leq M \). Then for all \( f \in H^{p(\cdot)} \cap L^q \), 1 < \( q < \infty \), there exists a function \( h \in H^{p(\cdot)} \cap L^q \) with

\[ \|f\|_{L^q} \sim \|h\|_{L^q} \quad \text{and} \quad \|f\|_{H^{p(\cdot)}} \sim \|h\|_{H^{p(\cdot)}} \]

such that for some large integer \( N \) depending on \( \phi \) and \( p(\cdot), q, M \),

\[ f(x) = \sum_j \sum_Q |Q| \phi_j * h(x_Q) \phi_j(x - x_Q), \]

where \( Q \) are dyadic cubes with side-length \( 2^{-j-N} \) and \( x_Q \) is the left lower point of \( Q \) and where the series converges in both norms of \( L^q \) and \( H^{p(\cdot)} \).

Hereafter, \( Q \) and \( Q' \) denote the dyadic cubes with side-length \( 2^{-j-N} \) and \( 2^{-j'-N} \) for some fixed positive large integer \( N \). We recall the key estimate for norms of characteristic functions on variable Lebesgue spaces.

**Lemma 2.3.** ([16]) Let \( p(\cdot) \in \mathcal{B} \), then there exist constant \( C > 0 \) such that for all balls \( B \) in \( \mathbb{R}^n \) and all measurable subsets \( S \subset B \),

\[ \frac{\|\chi_B\|_{L^{p(\cdot)}}}{\|\chi_S\|_{L^{p(\cdot)}}} \leq C \frac{|B|}{|S|}, \]

We also need the following generalized Hölder inequality on variable Lebesgue spaces.
Lemma 2.4. [3, 30] Given exponent function $p_i(\cdot) \in \mathcal{P}^0$, define $p(\cdot) \in \mathcal{P}^0$ by

$$\frac{1}{p(x)} = \sum_{i=1}^{m} \frac{1}{p_i(x)},$$

where $i = 1, 2, \ldots, m$. Then for all $f_i \in L^{p_i(\cdot)}$ and $f_i \in L^{p(\cdot)}$ and

$$\left\| \prod_{i=1}^{m} f_i \right\|_{p(\cdot)} \leq C \left\| \prod_{i=1}^{m} f_i \right\|_{p_i(\cdot)}.$$

Now we recall the following boundedness of the vector-valued maximal operator $M$.

Lemma 2.5. [5] Let $p(\cdot) \in LH \cap \mathcal{P}^0$. Then for any $q > 1$, $f = \{f_i\}_{i \in \mathbb{Z}}$, $f_i \in L_{loc}$, $i \in \mathbb{Z}$

$$\left\| \left\| M(f) \right\|_{L^q(\cdot)} \right\|_{L^{p(\cdot)}} \leq C \left\| f \right\|_{L^{p(\cdot)}},$$

where $M(f) = \{M(f_i)\}_{i \in \mathbb{Z}}$.

We now introduce a new space $CMO^{p(\cdot)}$ as follows.

Definition 2.6. Let $\psi$ satisfy the conditions in Lemma 2.1. Suppose that $0 < p^- \leq p^+ \leq 1$. The Carleson measure space $CMO^{p(\cdot)}$ is the collection of all $f \in S'_{\infty}$ fulfilling

$$\left\| f \right\|_{CMO^{p(\cdot)}} := \sup_{P} \left\{ \frac{|P|}{\| \chi_P \|_{p(\cdot)}} \left( \sum_{Q \subset P} |Q|^{-1} |\langle f, \psi_Q \rangle|^2 \chi_Q(x) dx \right)^{1/2} \right\} < \infty.$$  

The definition of $CMO^{p(\cdot)}$ is independent of the choice of $\{\psi_j\}_{j \in \mathbb{Z}}$ due to the following Plancherel–Pôlya inequality for $CMO^{p(\cdot)}$.

Theorem 2.7. Let $\{\varphi_Q\}_Q$ and $\{\varphi_Q\}_Q$ be any kernel functions defined above, and $p(\cdot) \in LH$, $0 < p^- \leq p^+ \leq 1$. Then for all $f \in S'_{\infty}$,

$$\sup_{P} \left\{ \frac{|P|}{\| \chi_P \|_{p(\cdot)}} \sum_{j} \sum_{Q \subset P} \left( \sup_{z \in Q} |\tilde{\phi}_j * f(z)| \right)^2 |Q| \right\}^{1/2} \sim \sup_{P} \left\{ \frac{|P|}{\| \chi_P \|_{p(\cdot)}} \sum_{j} \sum_{Q \subset P} \left( \inf_{z \in Q} |\tilde{\phi}_Q * f(z)| \right)^2 |Q| \right\}^{1/2}.$$

Proof. For any $f \in S'_{\infty}$, by Lemma 2.1 we have

$$f(x) = \sum_{j} \sum_{Q} |Q| (\varphi_j * f)(x_Q) \tilde{\varphi}_j(x, x_Q)$$

where the series converges in $L^2$, $S_{\infty}$ and $S'_{\infty}$. Then we rewrite $\tilde{\phi}_j * f(z)$ as $\tilde{\phi}_j * f(z)$

$$= \sum_{j' \in \mathbb{Z}} \sum_{Q'} |Q'|^{1/2} \left( f, |Q'|^{1/2} \varphi_{j'}(x_Q') \right) \int_{\mathbb{R}^n} \tilde{\phi}_j(z - x) \varphi_{j'}(x, x_Q') dx.$$
Here and below, we will apply the almost orthogonal estimate which can be found in many monographs. For example, please see [13] for more details.

To be more precise, for any given positive integers $L$ and $ψ, φ ∈ S$ satisfying cancellation conditions, then

$$|ψ_j * φ_{j'}(x)| \leq C \frac{2^{-|j-j'|L}2^{|j\wedge j'|}M}{(1 + 2^{|j\wedge j'|}|x|)(n+M)}.$$  \hspace{1cm} (2.1)

Using the inequality (2.1), for any given positive integers $L$, $M$, we obtain

$$\int_{\mathbb{R}^n} \tilde{φ}_j(z - x)\tilde{φ}_{j'}(x, x_{Q'})dx \leq C \frac{2^{-|j-j'|L}2^{|j\wedge j'|}M}{(2^{|j\wedge j'|} + |z - x_{Q'}|)(n+M)}.$$  \hspace{1cm} (2.2)

Therefore,

$$|\tilde{φ}_j * f(z)| \leq C \sum_{j' ∈ \mathbb{Z}} \sum_{Q'} |Q'| \langle f, φ_{j'}(x - x_{Q'})\rangle \frac{2^{-|j-j'|L}2^{|j\wedge j'|}M}{(2^{|j\wedge j'|} + |z - x_{Q'}|)(n+M)}.$$  

Hence, for $x ∈ Q$,

$$|\tilde{φ}_j * f(z)| \leq C \sum_{j' ∈ \mathbb{Z}} \sum_{Q'} |Q'| |\tilde{φ}_{j'} * f(x_{Q'})| \frac{2^{-|j-j'|L}2^{|j\wedge j'|}M}{(2^{|j\wedge j'|} + |x_{Q'} - x_{Q'}|)(n+M)}.$$  

Thus, applying Hölder’s inequality yields

$$(\sup_{z ∈ Q} |\tilde{φ}_j * f(z)|)^2 \leq C \left( \sum_{j' ∈ \mathbb{Z}} 2^{-|j-j'|L} \left\{ \sum_{Q'} |Q'| \left| \tilde{φ}_{j'} * f(x_{Q'}) \right|^2 \frac{2^{-|j\wedge j'|}}{(2^{|j\wedge j'|} + |x_{Q'} - x_{Q'}|)(n+1)} \right\}^{1/2} \right)^2$$  

Observe that

$$\sum_{Q'} |Q'| \left| \tilde{φ}_{j'} * f(x_{Q'}) \right|^2 \frac{2^{-|j\wedge j'|}}{(2^{|j\wedge j'|} + |x_{Q'} - x_{Q'}|)(n+1)} \leq C \sum_{Q'} \int_{\mathbb{R}^n} 2^{-|j\wedge j'|} \left( \frac{2^{-|j\wedge j'|}}{(2^{|j\wedge j'|} + |x_{Q'} - y|)(n+1)} \chi_{Q'}(y)dy \right) \leq C \int_{\mathbb{R}^n} 2^{-|j\wedge j'|} \left( \frac{2^{-|j\wedge j'|}}{(2^{|j\wedge j'|} + |x_{Q'} - y|)(n+1)} \right) dy \leq C.$$  \hspace{1cm} (2.3)

Since $x_{Q'}$ can be replaced by any point in $Q$ in the discrete Calderón identity, by Hölder’s inequality we get that

$$(\sup_{z ∈ Q} |\tilde{φ}_j * f(z)|)^2 \leq C \sum_{j' ∈ \mathbb{Z}} \sum_{Q'} 2^{-|j-j'|L} |Q'| \left( \frac{2^{-|j\wedge j'|}}{(2^{|j\wedge j'|} + |x_{Q'} - x_{Q'}|)(n+M)} \left( \inf_{z ∈ Q'} |\tilde{φ}_{j'} * f(z)| \right)^2 \right).$$
Given a dyadic cube $P$ with $\ell(P) = 2^{-k_0-N}$, we obtain

\[
\frac{|P|}{\|\chi_P\|_{p(.)}^2} \sum_{Q \subseteq P, \ell(Q) = 2^{-j-N}} \left( \sup_{z \in Q} |\tilde{\varphi}_j \ast f(z)| \right)^2 |Q| \\
\leq C \frac{|P|}{\|\chi_P\|_{p(.)}^2} \sum_{j=k_0}^\infty \sum_{Q \subseteq P, \ell(Q) = 2^{-j-N}} \sum_{j' = k_0}^\infty \sum_{Q' \subseteq P, \ell(Q') = 2^{-j'-N}} 2^{-|j-j'|L}|Q'| \\
\times \frac{2^{-(j \wedge j')}}{(2^{-(j \wedge j')} + |x_Q - x_Q'|)(n+M)} \left( \inf_{z \in Q'} |\tilde{\varphi}_{j'} \ast f(z)| \right)^2 |Q| \\
+ C \frac{|P|}{\|\chi_P\|_{p(.)}^2} \sum_{j=k_0}^\infty \sum_{Q \subseteq P, \ell(Q) = 2^{-j-N}} \sum_{j' = -\infty}^{k_0-1} \sum_{Q' \subseteq P, \ell(Q') = 2^{-j'-N}} 2^{-|j-j'|L}|Q'| \\
\times \frac{2^{-(j \wedge j')}}{(2^{-(j \wedge j')} + |x_Q - x_Q'|)(n+M)} \left( \inf_{z \in Q'} |\tilde{\varphi}_{j'} \ast f(z)| \right)^2 |Q| \\
=: I + II.
\]

To prove the desired result, it suffices to set $M = 1$ in the term $I$. Furthermore, $I$ can be decomposed as

\[
I = C \frac{|P|}{\|\chi_P\|_{p(.)}^2} \sum_{j=k_0}^\infty \sum_{Q \subseteq P, \ell(Q) = 2^{-j-N}} \sum_{j' = k_0}^\infty \sum_{Q' \subseteq P \cap \mathcal{Q} \neq \emptyset, \ell(Q') = 2^{-j'-N}} 2^{-|j-j'|L}|Q'| \\
\times \frac{2^{-(j \wedge j')}}{(2^{-(j \wedge j')} + |x_Q - x_Q'|)(n+1)} \left( \inf_{z \in Q'} |\tilde{\varphi}_{j'} \ast f(z)| \right)^2 |Q| \\
+ C \frac{|P|}{\|\chi_P\|_{p(.)}^2} \sum_{j=k_0}^\infty \sum_{Q \subseteq P, \ell(Q) = 2^{-j-N}} \sum_{j' = k_0}^\infty \sum_{Q' \cap \mathcal{Q} \neq \emptyset, \ell(Q') = 2^{-j'-N}} 2^{-|j-j'|L}|Q'| \\
\times \frac{2^{-(j \wedge j')}}{(2^{-(j \wedge j')} + |x_Q - x_Q'|)(n+1)} \left( \inf_{z \in Q'} |\tilde{\varphi}_{j'} \ast f(z)| \right)^2 |Q| \\
=: I_1 + I_2.
\]

Observe that

\[
\sum_{Q' \subseteq P \cap \mathcal{Q} \neq \emptyset, \ell(Q') = 2^{-j'-N}} \left( \inf_{z \in Q'} |\tilde{\varphi}_{j'} \ast f(z)| \right)^2 |Q'| = \sum_{Q' \subseteq P \cap \mathcal{Q} \neq \emptyset} \left( \inf_{z \in Q'} |\tilde{\varphi}_{j'} \ast f(z)| \right)^2 |Q'|
\]

\[
\leq C \sup_{P' \subseteq P \cap \mathcal{Q} \neq \emptyset} \sum_{Q' \subseteq P'} \left( \inf_{z \in Q'} |\tilde{\varphi}_{j'} \ast f(z)| \right)^2 |Q'|
\]
Thus,

\[
I_1 \leq C \frac{|P|}{\|X_P\|_{p(\cdot)}^2} \sum_{j=k_0}^{\infty} \sum_{Q \subset P} \sum_{j'=k_0}^{\infty} \sup_{\ell(Q) = 2^{-j-N}} \sum_{P' \subset 3P} \sum_{Q' \subset P'} 2^{-|j-j'|L/|Q'|} \\
\times \frac{2^{-(j \wedge j')}}{(2^{-(j \wedge j') + |x_Q - x_{Q'}|})} \left( \inf_{z \in Q'} |\tilde{\varphi}_{j'} * f(z)| \right)^2 |Q|
\]

\[
\leq C \frac{|P|}{\|X_P\|_{p(\cdot)}^2} \sum_{j=k_0}^{\infty} \sum_{j'=k_0}^{\infty} \sup_{\ell(P') = \ell(P)} \sum_{Q' \subset P'} 2^{-|j-j'|L/|Q'|} \left( \inf_{z \in Q'} |\tilde{\varphi}_{j'} * f(z)| \right)^2 |Q|
\]

\[
\leq C \sup_{P'} \frac{|P|}{\|X_P\|_{p(\cdot)}^2} \sum_{j=-\log_2 \ell(P')}^{\infty} \sum_{Q' \subset P'} |Q'| \left( \inf_{z \in Q'} |\tilde{\varphi}_{j'} * f(z)| \right)^2 |Q|
\]

where the first inequality follows from the estimate (2.3).

Next we decompose the set of dyadic cubes \( R : R \cap 3P = \emptyset, \ell(R) = \ell(P) \) into \( \{B_i\}_{i \in \mathbb{N}} \). Namely, for each \( i \in \mathbb{N} \),

\[
B_i := \{ P' : P' \cap 3P = \emptyset, \ell(P') = \ell(P), 2^{i-k_0-N} \leq |y_{P'} - y_P| \leq 2^{i-k_0-N+1} \},
\]

where \( y_{P'} \) and \( y_P \) denote the center of \( P' \) and \( P \), respectively. Then, we obtain

\[
I_2 = C \frac{|P|}{\|X_P\|_{p(\cdot)}^2} \sum_{j=k_0}^{\infty} \sum_{Q \subset P} \sum_{j'=k_0}^{\infty} \sum_{i=1}^{\infty} \sum_{P' \in B_i} \sum_{Q' \subset P'} 2^{-|j-j'|L/|Q'|} \\
\times \frac{2^{-(j \wedge j')}}{(2^{-(j \wedge j') + |x_Q - x_{Q'}|})} \left( \inf_{z \in Q'} |\tilde{\varphi}_{j'} * f(z)| \right)^2 |Q|
\]

\[
\leq C \sum_{i=1}^{\infty} \sum_{P' \in B_i} \frac{|P|}{\|X_P\|_{p(\cdot)}^2} \sum_{j=k_0}^{\infty} \sum_{Q \subset P} \sum_{j'=k_0}^{\infty} \sum_{Q' \subset P'} 2^{-|j-j'|L/|Q'|} \\
\times \frac{2^{-(j \wedge j')}}{(2^{-(j \wedge j') + |x_Q - x_{Q'}|})} \left( \inf_{z \in Q'} |\tilde{\varphi}_{j'} * f(z)| \right)^2 |Q|
\]

Observe that \( \sum_{Q \subset P} |Q| = |P| \) for each \( j \geq k_0 \) and there are at most \( 2^{i+1} \) cubes in \( B_i \). Hence,

\[
I_2 \leq C \sum_{i=1}^{\infty} \sum_{P' \in B_i} \frac{|P|}{2^{(i-k_0)(n+1)}} \sum_{j'=k_0}^{2^{k_0-(j \wedge j')-L|j-j'|}} \left( \sum_{j=k_0}^{\infty} 2^{k_0-(j \wedge j')-L|j-j'|} \right) \\
\times \left( \frac{|P'|}{\|X_{P'}\|_{p(\cdot)}^2} \sum_{Q' \subset P'} \left( \inf_{z \in Q'} |\tilde{\varphi}_{j'} * f(z)| \right)^2 |Q'| \right) \]
Let $P \in C_{\ell'}$, we denote $j' \equiv k_0 - m$, for $m \in \mathbb{N}$.

Let

$$E_m^0 = \{ Q' : \ell(Q') = 2^{m-k_0-N}; \ |y_P - y_{Q'}| \leq 2^{m-k_0-N-1} \}$$

and

$$E_m^i = \{ Q' : \ell(Q') = 2^{m-k_0-N}; 2^{i+m-k_0-N-1} < |y_P - y_{Q'}| \leq 2^{i+m-k_0-N} \}$$

for $i \in \mathbb{N}$. We rewrite

$$II \leq C \sup_{P''} \frac{|P'|}{\|x_P\|_{p(\cdot)}^2} \sum_{j' = k_0}^{\infty} \sum_{Q' \subset P'} \left( \inf_{z \in Q'} |\hat{f}_{j'} * f(z)| \right)^2 |Q'| \leq 2^{-|j-j'|L} |Q'| \times \frac{2^{-(j \wedge j')M}}{(2^{-(j \wedge j')} + |y_P - y_{Q'}|) (n+M)} \left( \inf_{z \in Q'} |\hat{f}_{j'} * f(z)| \right)^2$$

$$= C \frac{|P'|}{\|x_P\|_{p(\cdot)}^2} \sum_{j = k_0}^{\infty} 2^{-k_0 n} \sum_{m = 1}^{\infty} \sum_{i = 0}^{\infty} \sum_{E_m^i} 2^{-(j-k_0+m)L} |Q'| \times \frac{2^{(m-k_0)M}}{(2^{m-k_0} + |y_P - y_{Q'}|) (n+M)} \left( \inf_{z \in Q'} |\tilde{\varphi}_{k_0-m} * f(z)| \right)^2.$$

There are at most $2^{m} + 3^{n}$ dyadic cubes $Q' \in E_m^{i,j'}$, for $i = N \cup \{0\}$. We can choose $P' = 10002^{i+m} P$ such that $P' \supseteq Q'$ and $P' \supseteq P$. Therefore,

$$II \leq C \frac{|P'|}{\|x_P\|_{p(\cdot)}^2} \sum_{j = k_0}^{\infty} 2^{-k_0 n} \sum_{m = 1}^{\infty} \sum_{i = 0}^{\infty} \sum_{E_m^i} 2^{-(j-k_0+m)L} \times \frac{2^{(m-k_0)M}}{2^{(i+m-k_0)(n+M)}} \left( \inf_{z \in Q'} |\tilde{\varphi}_{k_0-m} * f(z)| \right)^2 |Q'|$$

$$\leq C \frac{|P'|}{\|x_P\|_{p(\cdot)}^2} \sum_{j = k_0}^{\infty} 2^{-k_0 n} \sum_{m = 1}^{\infty} \sum_{i = 0}^{\infty} \sum_{E_m^i} 2^{(i+m)(2^{i} + 3^{n})} 2^{-(j-k_0+m)L} \times \frac{2^{(m-k_0)M}}{2^{(i+m-k_0)(n+M)}} \left( \inf_{z \in Q'} |\tilde{\varphi}_{k_0-m} * f(z)| \right)^2 |Q'|$$
\[ \leq C \sum_{j=k_0}^{\infty} 2^{-kn} \sum_{m=1}^{\infty} \sum_{i=0}^{\infty} \left\| \chi_{P'} \right\|_{p(\cdot)}^2 \frac{|P|}{|P'|} \left(2^{in} + 3^n \right)2^{-(j-k_0+m)L} \frac{2^{(m-k_0)M}}{2^{(i+m-k_0)(n+M)}} \]

\times \left( \sup_{P'} \frac{|P'|}{\|\chi_{P'}\|^2_{p(\cdot)}} \sum_{j'=-\log_{2} \ell(P')}^{\infty} \sum_{Q' \subset P'} \left( \inf_{z \in Q'} |\tilde{\varphi}_{j'} \ast f(z)| \right)^2 |Q'| \right). \]

By Lemma 2.3, we have

\[ \left\| \chi_{P'} \right\|_{p(\cdot)}^2 \frac{|P|}{|P'|} = \left\| \chi_{P'} \right\|_{p(\cdot)/p^-}^2 \frac{|P|}{|P'|} \leq \frac{|P'|^{2/p^- - 1}}{|P|} = C 2^{(2/p^- - 1)(i+m)n}. \]

Set \( L > \max\{1, n(2/p^- - 2)\} \) and \( M > n(2/p^- - 1) \). Observe that

\[ \sum_{i=0}^{\infty} 2^{in(2/p^- - 1)} \left(2^{in} + 3^n\right)2^{-i(n+M)} \leq C \]

and

\[ \sum_{j=k_0}^{\infty} 2^{-jL} \leq C 2^{-k_0 L} ; \quad \sum_{m=1}^{\infty} 2^{mn(2/p^- - 1)} 2^{-mL} 2^{-mn} \leq C. \]

Then we get

\[ \sum_{j=k_0}^{\infty} 2^{-kn} \sum_{m=1}^{\infty} \sum_{i=0}^{\infty} \left\| \chi_{P'} \right\|_{p(\cdot)}^2 \frac{|P|}{|P'|} \left(2^{in} + 3^n\right)2^{-(j-k_0+m)L} \frac{2^{(m-k_0)M}}{2^{(i+m-k_0)(n+M)}} \]

\[ \leq \sum_{j=k_0}^{\infty} 2^{-kn} \sum_{m=1}^{\infty} \sum_{i=0}^{\infty} 2^{(2/p^- - 1)(i+m)n} \left(2^{in} + 3^n\right)2^{-(j-k_0+m)L} \frac{2^{(m-k_0)M}}{2^{(i+m-k_0)(n+M)}} \]

\[ \leq 2^{-k_0(n+L)} \sum_{m=1}^{\infty} 2^{mn(2/p^- - 1)} 2^{-(k_0+m)L} \frac{2^{(m-k_0)M}}{2^{(m-k_0)(n+M)}} \leq C. \]

The proof of the Plancherel–Pólya inequality for \( CMO_{p(\cdot)} \) is complete. \( \Box \)

By Theorem 2.7, we immediately obtain the following discrete version of \( CMO^{p(\cdot)} \).

**Corollary 2.8.** Let \( \{\varphi_j\}_j \) be any kernel functions satisfying the same conditions in Lemma 2.1, and \( p(\cdot) \in LH, 0 < p^- \leq p^+ < \infty \). Then for all \( f \in CMO^{p(\cdot)} \),

\[ \|f\|_{CMO^{p(\cdot)}} \sim \sup_{P} \left\{ \frac{|P|}{\|\chi_{P}\|_{p(\cdot)}^2} \sum_{j} \sum_{Q \subset P} \varphi_j \ast f(x_Q)^2 |Q| \right\}^{1/2}, \]

where \( x_Q \) is any fixed point in \( Q \).
3. Duality of $H^{p(\cdot)}$ and $CMO^{p(\cdot)}$

Define a linear map $S_\varphi$ by

$$S_\varphi(f) = \{\langle f, \varphi_Q \rangle \}_{Q},$$

and another linear map $T_\psi$ by

$$T_\psi(s_Q) = \sum_Q s_Q \psi_Q.$$

For $g \in CMO^{p(\cdot)}$, define a linear functional $L_g$ by

$$L_g(f) = \langle S_\psi(g), S_\varphi(f) \rangle = \sum_Q \langle g, \psi_Q \rangle \langle f, \varphi_Q \rangle$$

for $f \in S_\infty$.

We now state the following main result in this section.

**Theorem 3.1.** Suppose that $p(\cdot) \in LH$, $0 < p^- \leq p^+ \leq 1$. The dual of $H^{p(\cdot)}$ is $CMO^{p(\cdot)}$ in the following sense.

- (1) For $g \in CMO^{p(\cdot)}$, the linear functional $L_g$, defined initially on $S_\infty$, extends to a continuous linear functional on $H^{p(\cdot)}$ with $\|L_g\| \leq C \|g\|_{CMO^{p(\cdot)}}$.
- (2) Conversely, every continuous linear functional $L$ on $H^{p(\cdot)}$ satisfies $L = L_g$ for some $g \in CMO^{p(\cdot)}$ with $\|g\|_{CMO^{p(\cdot)}} \leq C \|L\|$.

To prove this theorem, we first introduce sequence spaces with variable exponents. For $p(\cdot) \in LH$, $0 < p^- \leq p^+ \leq 1$, the sequence space $s^{p(\cdot)}$ consists all complex-value sequences

$$s^{p(\cdot)} = \left\{ s_Q : \|s_Q\|_{s^{p(\cdot)}} := \left\| \left\{ \sum_Q |s_Q|^2 |Q|^{-1} \chi_Q \right\}^{1/2} \right\|_{L^{p(\cdot)}} < \infty \right\};$$

the sequence space $c^{p(\cdot)}$ consists all complex-value sequences

$$c^{p(\cdot)} = \left\{ t_Q : \|t_Q\|_{c^{p(\cdot)}} := \sup_P \left\{ \left( \frac{|P|}{\|\chi_P\|_{L^{p(\cdot)}}^2} \sum_Q \|t_Q\|^2 \right)^{1/2} \right\} < \infty \right\}.$$

We mention that, the sequence spaces $s^p$ and $c^1$ were first introduced by Frazier and Jawerth ([12]), $c^p$ was introduced by Lee et al. ([19]), the sequence space $f_{p(\cdot), \varphi}^{s(\cdot)}$, corresponding to the space $L_{p(\cdot), \varphi}^{s(\cdot)}$ was introduced by Yang et al. ([31]). We also remark that Zhuo et al. ([32]) showed that $p(\cdot)$-Carleson measure characterizations are presented for the dual space of $H^{p(\cdot)}$. However, the main results and the methods are quite different. In order to prove our main result in this section, we need the following lemma.

**Lemma 3.2.** ([24]) Assume that $p^+ \leq 1$. For sequences of scalars $\{\lambda_j\}_{j=1}^{\infty}$ and sequences of $(p(\cdot), q)$-atoms $\{a_j\}$, we have

$$\sum_{j=1}^{\infty} |\lambda_j| \leq A(\{\lambda_j\}_{j=1}^{\infty}; \{Q_j\}_{j=1}^{\infty}),$$
where
\[ A(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) = \left\| \sum_j \left( \frac{|\lambda_j| \chi_{Q_j}}{\|\chi_{Q_j}\|_{L^p(\cdot)}} \right)^{1/p} \right\|_{L^p(\cdot)}. \]

To prove Theorem 3.1, we also need the following two propositions.

**Proposition 3.3.** Suppose that \( p(\cdot) \in LH, 0 < p^- \leq p^+ \leq 1 \) and \( \varphi, \psi \) satisfy the conditions in Lemma 2.1. The linear operator \( S_\varphi : H^{p(\cdot)} \mapsto s^{p(\cdot)} \) and \( T_\psi : s^{p(\cdot)} \mapsto H^{p(\cdot)}, \) respectively, are bounded. Furthermore, \( T_\psi \circ S_\varphi \) is the identity on \( H^{p(\cdot)}. \)

**Proposition 3.4.** Suppose that \( p(\cdot) \in LH, 0 < p^- \leq p^+ \leq 1 \) and \( \varphi, \psi \) satisfy the conditions in Lemma 2.1. The linear operator \( S_\varphi : CMO^{p(\cdot)} \mapsto c^{p(\cdot)} \) and \( T_\psi : c^{p(\cdot)} \mapsto CMO^{p(\cdot)}, \) respectively, are bounded. Furthermore, \( T_\psi \circ S_\varphi \) is the identity on \( CMO^{p(\cdot)}. \)

Assume the above two propositions first, then we return the proof of Theorem 3.1.

**Proof of Theorem 3.1.** By Lemma 2.2, for all \( f \in S_\infty \)
\[ f(x) = \sum_Q \langle h, \phi_Q \rangle \phi_Q(x), \]
where the series also converges in \( S_\infty \) and \( \phi \in S_\infty \) is defined in Lemma 2.2. Let \( g \in CMO^{p(\cdot)} \) and \( f \in H^{p(\cdot)}. \) Define a linear functional \( L_g \) on \( S_\infty \) by
\[ L_g(f) = \langle f, g \rangle = \sum_Q \langle h, \phi_Q \rangle \langle \phi_Q, g \rangle. \]

Now we need the maximal square function defined by
\[ G^d_\phi(h)(x) := \left( \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^n} \sup_{x_Q \in Q} |\phi_j * h(x_Q)|^2 \chi_{Q}(x) \right)^{1/2}. \]

Set
\[ \Omega_i = \{ x \in \mathbb{R}^n : G^d_\phi(h)(x) > 2^i \}. \]

and
\[ \tilde{\Omega}_i = \{ x \in \mathbb{R}^n : M(\chi_{\Omega_i})(x) > \frac{1}{10} \}, \]
where \( M \) is the Hardy–Littlewood maximal operator. Then \( \Omega_i \subset \tilde{\Omega}_i. \) By the \( L^2 \) boundedness of \( M, |\tilde{\Omega}_i| \leq C|\Omega_i|. \) Denote
\[ B_i = \{ Q : |Q \cap \Omega_i| > \frac{1}{2}|Q|, |Q \cap \Omega_{i+1}| \leq \frac{1}{2}|Q| \}. \]

Following the discrete Calderón reproducing formula and denoting \( \tilde{Q} \in B_i \) are maximal dyadic cubes in \( B_i, \) we rewrite
\[ f(x) = \sum_{i \in \mathbb{Z}} \sum_{Q \in B_i} \sum_{Q \subset Q} \phi_Q * f(x_Q) \phi_Q(x - x_Q). \]
From Corollary 2.8 and the Hölder inequality, it follows that

$$|L_g(f)| = \left| \sum_{Q} \langle h, \phi_Q \rangle \langle \phi_Q, g \rangle \right|$$

$$= \left| \sum_{j \in \mathbb{Z}} \sum_{Q} |Q| \phi_j * h(x_Q) \phi_j * g(x_Q) \right|$$

$$\leq \sum_{i \in \mathbb{Z}} \sum_{Q \in B_i} \sum_{Q \subset \tilde{Q}} |\phi_Q * h(x_Q)| |\phi_Q * g(x)|$$

$$\leq \sum_{i \in \mathbb{Z}} \sum_{Q \in B_i} \left\{ \sum_{Q \subset \tilde{Q}} |\phi_Q * h(x_Q)|^2 \right\} \left\{ \sum_{Q \subset \tilde{Q}} |\phi_Q * g(x)|^2 \right\}^{\frac{1}{2}}$$

$$\leq \sum_{i \in \mathbb{Z}} \sum_{Q \in B_i} \left\{ \sum_{Q \subset \tilde{Q}} |\phi_Q * f(x_Q)|^2 \right\} \left\{ \frac{|\tilde{Q}|}{\|\chi_{\tilde{Q}}\|_{L^p(\cdot)}} \sum_{Q \subset \tilde{Q}} |\phi_Q * g(x)|^2 \right\}^{\frac{1}{2}}$$

$$\leq C \sum_{i \in \mathbb{Z}} \sum_{Q \in B_i} \frac{\|\chi_{\tilde{Q}}\|_{L^p(\cdot)}}{|\tilde{Q}|^{1/2}} \left\{ \sum_{Q \subset \tilde{Q}} |\phi_Q * h(x_Q)|^2 \right\}^{\frac{1}{2}} \|g\|_{CMO^{p(\cdot)}}.$$ 

We denote that

$$\lambda_{\tilde{Q}} := \frac{\|\chi_{\tilde{Q}}\|_{L^p(\cdot)}}{|\tilde{Q}|^{1/2}} \left\{ \sum_{Q \subset \tilde{Q}} |\phi_Q * h(x_Q)|^2 \right\}^{\frac{1}{2}}.$$ 

Then

$$f(x) = \sum_{i \in \mathbb{Z}} \sum_{Q \in B_i} \sum_{Q \subset \tilde{Q}} \phi_Q * h(x_Q) \phi_Q(x - x_Q) =: \sum_{i} \sum_{Q \in B_i} \lambda_{\tilde{Q}} a_Q(x),$$

where

$$a_Q = \frac{1}{\lambda_{\tilde{Q}}} \sum_{Q \subset \tilde{Q}} \phi_Q * h(x_Q) \phi_Q(x - x_Q).$$

Here we have established the atomic decomposition for $H^{p(\cdot)}$. In fact, from the definition of $a_{\tilde{Q}}$ and the support of $\phi$, we get that $a_{\tilde{Q}}$ is supported in $5\tilde{Q}$.

We claim that

$$A(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) \leq C \|f\|_{H^{p(\cdot)}}.$$ 

To prove the claim, we first observe that when $1 < q < \infty$

$$A(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty})$$

$$= \left\| \left\{ \sum_{i} \sum_{Q \in B_i} \left( \frac{|\lambda_{\tilde{Q}}| \chi_{5\tilde{Q}}}{\|\chi_{5\tilde{Q}}\|_{L^p(\cdot)}} \right)^{-p} \right\} \right\|_{L^p(\cdot)}.$$
Applying Lemma 2.5, we have that

\[
\begin{align*}
\mathcal{A}(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) & \\
\leq C \left\| \sum_i \left( \left( \sum_{Q \subset \bar{Q}} |\phi_Q \ast h(x_Q)|^2 \right)^{\frac{1}{2}} |\bar{Q}|^{-\frac{1}{2}} \chi_{\bar{Q}} \right)^p \right\|_{L^{p(\cdot)}}^{\frac{1}{p^-}} \\
& \leq C \left\| \sum_i \left( \left( \sum_{Q \subset \bar{Q}} |\phi_Q \ast h(x_Q)|^2 \right)^{\frac{1}{2}} |\bar{Q}|^{-\frac{1}{2}} M^{\frac{2}{p^-}} \chi_{\bar{Q}} \right)^p \right\|_{L^{p(\cdot)}}^{\frac{1}{p^-}} \\
& \leq C \left\| \sum_i \left( \left( \sum_{Q \subset \bar{Q}} |\phi_Q \ast h(x_Q)|^2 \right)^{\frac{2}{p}} |\bar{Q}|^{-\frac{2}{p}} \chi_{\bar{Q}} \right)^{\frac{2}{p^-}} \right\|_{L^{2p(\cdot)}}^{\frac{1}{p^-}} \\
& \leq C \left\| \sum_i \left( \left( \sum_{Q \subset \bar{Q}} |\phi_Q \ast h(x_Q)|^2 \right)^{\frac{1}{p}} |\bar{Q}|^{-\frac{1}{p}} \chi_{\bar{Q}} \right)^p \right\|_{L^{p(\cdot)}}^{\frac{1}{p^-}}.
\end{align*}
\]

If \( x \in Q \in B_i \), then \( M \chi_{\bar{Q} \cap \Omega_i \setminus \Omega_{i+1}}(x) > \frac{1}{2} \). From this fact, we have

\[
\chi_{Q}(x) \leq 2M \chi_{\bar{Q} \cap \Omega_i \setminus \Omega_{i+1}}(x) \implies \chi_{Q}(x) \leq 4M^2(\chi_{\bar{Q} \cap \Omega_i \setminus \Omega_{i+1}})(x).
\]

Thus, by the Fefferman–Stein vector valued inequality,

\[
\begin{align*}
\left( \sum_{Q \subset \bar{Q}} |\phi_j \ast h(x_Q)|^2 |Q| \right)^{\frac{1}{2}} \\
\leq \left( \int_{\mathbb{R}^n} \sum_{Q \subset \bar{Q}} |\phi_Q \ast h(x_Q)|^2 \chi_{Q}(x) dx \right)^{\frac{1}{2}} \\
\leq C \left( \int_{\mathbb{R}^n} \sum_{Q \subset \bar{Q}} |\phi_Q \ast h(x_Q)M(\chi_{\bar{Q} \cap \Omega_i \setminus \Omega_{i+1}})(x)|^2 dx \right)^{\frac{1}{2}}.
\end{align*}
\]
\[ \leq C \left( \int_{\tilde{Q} \cap Q_i \setminus Q_{i+1}} \sum_{Q \subset \tilde{Q}} \sup_{x \in Q} |\phi_Q * h(x)\chi_Q(x)|^2 \, dx \right)^{1/2} \]

\[ \leq C2^i |\tilde{Q}|^{1/2}. \]

Observing that \( \Omega_{i+1} \subset \Omega_i \) and \( \bigcap_{i=1}^{\infty} \Omega_i = 0 \), then for a.e \( x \in \mathbb{R}^n \) we have

\[ \sum_{i=-\infty}^{\infty} 2^i \chi_{\Omega_i}(x) = \sum_{i=-\infty}^{\infty} 2^i \sum_{j=i}^{\infty} \chi_{\Omega_j \setminus \Omega_{j+1}}(x) = 2 \sum_{j=-\infty}^{\infty} 2^j \chi_{\Omega_j \setminus \Omega_{j+1}}(x). \]

Hence,

\[ A(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) \leq C \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \sum_{i} \frac{2^i \chi_{\Omega_i \setminus \Omega_{i+1}}}{\lambda} \right)^{p(x)} \, dx \leq 1 \right\} \]

\[ = C \inf \left\{ \lambda > 0 : \sum_{i} \int_{\Omega_i \setminus \Omega_{i+1}} \left( \frac{2^i}{\lambda} \right)^{p(x)} \, dx \leq 1 \right\} \]

\[ \leq C \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{G_{\phi} f(x)}{\lambda} \right)^{p(x)} \, dx \leq 1 \right\} \leq C\|f\|_{H^p(\cdot)}. \]

Therefore, we have proved the claim. Moreover, we can obtain that every \( a_Q \) is a \((p(\cdot), q)\)-atom.

Therefore,

\[ |L_g(f)| = \left| \sum_{Q} \langle h, \phi_Q \rangle \langle \phi_Q, g \rangle \right| \]

\[ \leq C \sum_{i \in \mathbb{Z}} \sum_{Q \in B_i} \|\chi_Q\|_{L^p(\cdot)} \left\{ \sum_{Q \subset \tilde{Q}} \|\phi_Q * h(x)\|_Q^{1/2} \right\}^{\frac{1}{2}} \|g\|_{CMO^p(\cdot)} \]

\[ \leq C \sum_{j=1}^{\infty} |\lambda_j| \|g\|_{CMO^p(\cdot)} \]

\[ \leq A(\{\lambda_j\}_{j=1}^{\infty}, \{Q_j\}_{j=1}^{\infty}) \|g\|_{CMO^p(\cdot)} \]

\[ \leq C\|f\|_{H^p(\cdot)} \|g\|_{CMO^p(\cdot)}. \]

This shows that \( g \in (H^p(\cdot))^* \) and

\[ \|L_g\| \leq C\|g\|_{CMO^p(\cdot)}. \]

Conversely, we first prove that every continuous linear functional \( \ell \) on \( s^{p(\cdot)} \) satisfies \( \ell = \ell_t \) for some \( t \in c^{p(\cdot)} \) with \( \|t\|_{c^{p(\cdot)}} \leq C\|\ell\| \). For \( s = \{s_Q\}_{Q} \in s^{p(\cdot)} \), let \( \ell(s) = \sum_{Q} s_Q t_Q \). Fix a dyadic cube \( P \) in \( \mathbb{R}^n \). Let \( X \) be the sequence space consisting of \( s = \{s_Q\}_{Q \subset P} \), and define a counting measure on dyadic cubes \( Q \subset P \) by \( d\sigma(Q) = \frac{|Q|}{|P|} \chi_{P} \|x_P\|_{L^p(\cdot)}^{2}. \)
Then

\[
\left( \frac{|P|}{\|\chi_P\|_{L^p(\cdot)}^2} \sum_{Q \subset P} |t_Q|^2 \right)^{1/2} = \left\| |t_Q||Q|^{-1/2} \right\|_{L^2(\mathcal{X},d\sigma)}
\]

\[
= \sup_{\|s\|_{L^2(\mathcal{X},d\sigma)} \leq 1} \left| \frac{|P|}{\|\chi_P\|_{L^p(\cdot)}^2} \sum_{Q \subset P} |Q||s_Q||Q|^{-1/2}|t_Q| \right|
\]

\[
= \sup_{\|s\|_{L^2(\mathcal{X},d\sigma)} \leq 1} \left| \ell \left( \frac{|P|}{\|\chi_P\|_{L^p(\cdot)}^2} |Q||s_Q||Q|^{-1/2} \right) \right|
\]

Choose that $0 < r(x) < \infty$ such that $\frac{1}{p(x)} = 1 + \frac{1}{r(x)}$. By Lemmas 2.4 and 2.3 and the Hölder inequality, we have

\[
\left\| \left\{ \left( \frac{|P||s_Q||Q|^{1/2}}{\|\chi_P\|_{L^p(\cdot)}^2} \right)_{Q \subset P} \right\}_{s_P(\cdot)} \right\|_{s_P(\cdot)}
\]

\[
= \left\| \left\{ \sum_{Q \subset P} \frac{|P|^2|s_Q|^2}{\|\chi_P\|_{L^p(\cdot)}^4} \chi_Q \right\}_{Q \subset P} \right\|_{L^p(\cdot)}^{1/2}
\]

\[
\leq C \left\| \left\{ \sum_{Q \subset P} \frac{|P|^2|s_Q|^2}{\|\chi_P\|_{L^p(\cdot)}^4} \chi_Q \right\}_{Q \subset P} \right\|_{L^1}^{1/2}
\]

\[
= C|P| \left\{ \frac{1}{|P|} \int_P \left( \sum_{Q \subset P} \frac{|s_Q|^2}{\|\chi_P\|_{L^p(\cdot)}^2 |P|^{-1/2}} \chi_Q(x) \right)^{1/2} \right\} \left\| \chi_Q \right\|_{L^p(\cdot)} |Q|^{-1}
\]

\[
\leq C|P| \left\{ \int_P \left( \sum_{Q \subset P} \frac{|s_Q|^2}{\|\chi_P\|_{L^p(\cdot)}^2 |P|^{-1/2}} \chi_Q(x) \right) \right\}^{1/2} \left\| \chi_Q \right\|_{L^p(\cdot)} |Q|^{-1}
\]

Then,

\[
\left\| \left\{ \left( \frac{|P||s_Q||Q|^{1/2}}{\|\chi_P\|_{L^p(\cdot)}^2} \right)_{Q \subset P} \right\}_{s_P(\cdot)} \right\|_{s_P(\cdot)}
\]

\[
\leq C \left\{ \int_P \left( \sum_{Q \subset P} \frac{|s_Q|^2}{\|\chi_P\|_{L^p(\cdot)}^2 |P|^{-1/2}} \chi_Q(x) \right) \right\}^{1/2} \left\| \chi_Q \right\|_{L^p(\cdot)} |P| \left\| \chi_P \right\|_{L^p(\cdot)} |Q|
\]

\[
\leq C \left\{ \int_P \left( \sum_{Q \subset P} \frac{|s_Q|^2}{\|\chi_P\|_{L^p(\cdot)}^2 |P|^{-1/2}} \chi_Q(x) \right) \right\}^{1/2}
\]
Lemma 3.5. \[ \|s\|_{l^2(X,d\sigma)} \leq C \left( \sum_{Q \subset P} \frac{|s_Q|^2|Q|}{\|x_P\|_{L^p(\cdot)}^2|P|^{-1}} \right)^{1/2} \]

Thus,

\[ \left( \frac{|P|}{\|x_P\|_{L^p(\cdot)}^2} \sum_{Q \subset P} |t_Q|^2 \right)^{1/2} \leq \|\ell\|. \]

Then let \( L \in (H^{p(\cdot)})' \) and define \( \ell = L \circ T_\psi \). By proposition 3.3, \( \ell \in (s^{p(\cdot)})' \). Thus, there exists \( t = \{t_Q\}_Q \in c^p(\cdot) \) such that

\[ \ell(\{s_Q\}_Q) = \sum_Q s_Q t_Q \quad \text{for} \quad f \in s^{p(\cdot)} \]

and \( \|t\|_{c^p(\cdot)} \sim \|\ell\| \leq C\|L\| \). For \( f \in c^p(\cdot) \), we have

\[ \ell \circ S_\varphi(f) = L \circ T_\psi \circ S_\varphi(f) = L(f). \]

Thus, for \( f \in S_0 \) and letting \( g = T_\psi(t) = \sum_Q t_Q \psi_Q \),

\[ L(f) = \ell \circ S_\varphi(f) = \sum_Q \langle f, \varphi_Q \rangle t_Q = \langle t, S_\varphi(f) \rangle. \]

Observe that \( \langle g, f \rangle = \langle S_\psi(g), S_\varphi(f) \rangle \) and \( \langle t, S_\varphi(f) \rangle = \langle T_\psi(t), f \rangle \). Then we have,

\[ L(f) = \langle T_\psi(t), f \rangle = L_g(f) \quad \text{for} \quad f \in S_0. \]

Therefore, by Proposition 3.4

\[ \|g\|_{CMO^{p(\cdot)}} \leq C\|t\|_{c^p(\cdot)} \leq C\|L\| \]

and the proof is complete. \( \square \)

Before we give the proofs to the above two propositions, we need the following equivalent characterizations of \( H^{p(\cdot)} \), which, for the case of inhomogeneous variable Hardy spaces, was studied in [27].

Lemma 3.5. [26] Let \( p(\cdot) \in LH \). Then for all \( f \in S_\infty' \),

\[ \|f\|_{H^{p(\cdot)}} \sim \|G(f)\|_{L^p(\cdot)} \sim \|G^d(f)\|_{L^p(\cdot)}. \]

We now are ready to prove Propositions 3.3 and 3.4.

**Proof of Proposition 3.3.** By Lemma 3.5, for \( f \in H^{p(\cdot)} \),

\[ \|S_\varphi(f)\|_{s^{p(\cdot)}} = \left\{ \left( \sum_Q \langle f, \varphi_Q \rangle^2 |Q|^{-1} \chi_Q \right) \right\}_{L^p(\cdot)}^{1/2} \leq \|f\|_{H^{p(\cdot)}}. \]

For \( \{s_Q\} \in s^{p(\cdot)} \),

\[ \|T_\psi(\{s_Q\})\|_{H^{p(\cdot)}} = \left\{ \left( \sum_{j' \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}^n} |\psi_{j'} \ast (T_\psi(\{s_Q\}))(2^{-j}k')|^2 \chi_{Q'} \right) \right\}_{L^p(\cdot)}^{1/2} \]
\[
\left\| \left( \sum_{j' \in \mathbb{Z}} \sum_{k' \in \mathbb{Z}^n} \psi_{j'} \ast \left( \sum_{Q} s_Q \psi_Q \right) (2^{-j} k')^2 \chi_{Q'} \right) \right\|_{L^p(\cdot)}^{1/2}.
\]

The rest of the proof is closely related to [26, Proposition 2.3], that is, it follows the similar routine as the proof of [26, Proposition 2.3]. Namely, by the almost-orthogonality estimates, the estimate in [12, pp. 147, 148], H"older’s inequality and the Fefferman-Stein vector-valued maximal function inequality in Proposition 2.5, we get

\[
\| T_\psi(\{s_Q\}) \|_{H^p(\cdot)} \leq C \| \{s_Q\} \|_{c(p)}.
\]

Finally, it is easy to check that from the discrete Calderón identity introduced by Frazier and Jawerth in Lemma 2.1, \( T_\psi \circ S_\varphi \) is the identity on \( H^p(\cdot) \).

\[\square\]

**Proof of Proposition 3.4.** For any \( g \in \text{CMO}^p(\cdot) \), applying Corollary 2.8 yields

\[
\| \{S_\varphi(g)\} \|_{c(p)} = \sup_P \left\{ \frac{|P|}{\|\chi_P\|_{L^p(\cdot)}^2} \sum_{Q \subset P} |\langle g, \varphi_Q \rangle|^2 \right\}^{1/2} = \sup_P \left\{ \frac{|P|}{\|\chi_P\|_{L^p(\cdot)}^2} \sum_{j = -\log_2 \ell(P)}^{\infty} \sum_{\ell(Q) = 2^{-j-1}} \sum_{Q \subset P} \frac{\|\varphi \ast f(x_Q)|^2 |Q|}{\ell(P)} \right\}^{1/2} \leq C \| f \|_{\text{CMO}^p(\cdot)}.
\]

For \( \{s_Q\} \in c(p) \),

\[
\| T_\psi(\{s_Q\}) \|_{\text{CMO}^p(\cdot)} = \sup_P \left\{ \frac{|P|}{\|\chi_P\|_{L^p(\cdot)}^2} \int_{\mathbb{R}^n} \sum_{Q \subset P} \chi_Q^{-1} \left| \langle T_\psi(\{s_Q\}), \psi_Q \rangle \right|^2 \chi_Q(x)dx \right\}^{1/2} = \sup_P \left\{ \frac{|P|}{\|\chi_P\|_{L^p(\cdot)}^2} \int_{\mathbb{R}^n} \sum_{Q \subset P} \chi_Q^{-1} \left| \sum_{Q'} s_Q \psi_Q' \right|^2 \chi_Q(x)dx \right\}^{1/2}.
\]

The rest of this proof is similar to that of Theorem 2.7. A same argument as the proof of Theorem 2.7, we have

\[
\| T_\psi(\{s_Q\}) \|_{\text{CMO}^p(\cdot)} \leq C \| \{s_Q\} \|_{c(p)}.
\]

Finally, by Lemma 2.1 we can easily get that from the Calderón reproducing formula \( T_\psi \circ S_\varphi \) is the identity operator on \( \text{CMO}^p(\cdot) \).

\[\square\]

**4. The Equivalence of \( \text{CMO}^p(\cdot) \)**

In this section, we will see that Carleson measure spaces with variable exponents \( \text{CMO}^p(\cdot) \), Campanato space with variable exponent \( \mathfrak{L}_{q,p(\cdot),d} \) and
Hölder–Zygmund spaces with variable exponents $H^{p(·)}_d$ coincide as sets and the corresponding norms are equivalent.

We first recall some definitions and lemmas below in [24]. Recall that the definition of atomic Hardy space with variable exponent $H^{p(·),q}_{atom}$. Let $1 < q \leq \infty$ and $p(·) \in \mathcal{P}^0 \cap LH$. The function space $H^{p(·),q}_{atom}$ is defined to be the set of all distributions $f \in S'$ which can be written as $f = \sum_j \lambda_j a_j$ in $S'$, where $\{a_j,Q_j\} \subset \mathcal{A}(p(·),q)$ with the quantities

$$\mathcal{A}(\{\lambda_j\}_{j=1}^\infty,\{Q_j\}_{j=1}^\infty) < \infty.$$  

One define

$$\|f\|_{H^{p(·),q}_{atom}} = \inf \mathcal{A}(\{\lambda_j\}_{j=1}^\infty,\{Q_j\}_{j=1}^\infty).$$

Let $q \gg 1$ and $p(·) \in \mathcal{P}^0 \cap LH$. It is well known that

$$H^{p(·)} = H^{p(·),q}_{atom}.$$

We also recall the notion of the Campanato space with variable exponent $\mathcal{L}_{q,p(·),d}$. Write that $\mathcal{P}^s$ is the set of all polynomials having degree at most $d$.

**Definition 4.1.** Let $p(·) \in \mathcal{P}$, $d$ be a nonnegative integer and $1 \leq q < \infty$. Then the Campanato space with variable exponent $\mathcal{L}_{q,p(·),d}$ is defined to be the set of all $f \in L^q_{loc}$ such that

$$\|f\|_{\mathcal{L}_{q,p(·),d}} = \sup_{Q \subset \mathbb{R}^n} \frac{|Q|}{\|Q\|_{L^{p(·)}}} \left[ \frac{1}{|Q|} \int_Q |f(x) - P^d_Q f(x)|^q dx \right]^{\frac{1}{q}} < \infty,$$

where $P^d_Q f$ denotes the unique polynomial $P \in \mathcal{P}^d$ such that, for all $h \in \mathcal{P}^d$, $\int_Q [f(x) - P(x)] h(x) dx = 0$.

Let $L^q_{comp} d$ be all the set of all $L^q$–functions with compact support. For a nonnegative integer $d$, let

$$L^q_{comp} d = \left\{ f \in L^q_{comp} : \int_{\mathbb{R}^n} f(x) x^\alpha dx = 0, |\alpha| \leq d \right\}.$$

**Lemma 4.2.** Suppose that $p(·) \in LH$, $0 < p^- \leq p^+ \leq 1$, $q > p^+$ and $d$ is a nonnegative integer such that $d \in (n/p^+ - n - 1, \infty)$. The dual of $H^{p(·),q}_{atom}$, denoted by $(H^{p(·),q}_{atom})'$ is $\mathcal{L}_{q',p(·),d}$ in the following sense.

1. For any $b \in \mathcal{L}_{q',p(·),d}$, the linear functional $L_b := \int_{\mathbb{R}^n} b(x) dx$, defined initially on $L^q_{comp} d$, has a bounded extension on $H^{p(·),q}_{atom}$ with $\|L_b\| \leq C\|b\|_{\mathcal{L}_{q',p(·),d}}$.

2. Conversely, every continuous linear functional $L$ on $H^{p(·),q}_{atom}$ satisfies $L = L_b$ for some $b \in \mathcal{L}_{q',p(·),d}$ with $\|b\|_{\mathcal{L}_{q',p(·),d}} \leq C\|L\|$.

Define $\Delta^1_h$ to be a difference operator, which is defined inductively by

$$\Delta^1_h f = \Delta_h f \equiv f(· + h) - f, \quad \Delta^k h \equiv \Delta^1_h \circ \Delta^{k-1}_h, \quad k \geq 2.$$
Definition 4.3. Let \( p(\cdot) \in \mathcal{P} \), and \( d \in \mathbb{N} \cup \{0\} \). Then the Hölder–Zygmund spaces with variable exponents, \( \dot{H}^p_d(\cdot) \), is defined to be the set of all continuous functions \( f \) such that
\[
\|f\|_{\dot{H}^p_d(\cdot)} = \sup_{x \in \mathbb{R}^n, h \neq 0} \frac{|Q|}{\|\chi_Q\|_{L^p(\cdot)}} |\Delta_h^{d+1} f(x)| < \infty,
\]
where \( Q = Q(x,|h|) \).

Note that we still use \( \dot{H}^p_d(\cdot) \) to denote the above function space modulo the polynomials of degree \( d \).

Lemma 4.4. Suppose that \( p(\cdot) \in LH \), \( 0 < p^- \leq p^+ \leq 1 \). Then the function spaces \( \dot{H}^p_d(\cdot) \) and \( \mathcal{L}_{q,p(\cdot),d} \) are isomorphic in the following sense.

1. For any \( f \in \dot{H}^p_d(\cdot) \) we have \( \|f\|_{\mathcal{L}_{q,p(\cdot),d}} \leq C \|f\|_{\dot{H}^p_d(\cdot)} \).
2. Any element in \( \mathcal{L}_{q,p(\cdot),d} \) has a continuous representative. Moreover, whenever continuous functions \( f \in \mathcal{L}_{q,p(\cdot),d} \) then \( f \in \dot{H}^p_d(\cdot) \) and we have \( \|f\|_{\dot{H}^p_d(\cdot)} \leq C \|f\|_{\mathcal{L}_{q,p(\cdot),d}} \).

Now we state the main result in this section.

Theorem 4.5. Suppose that \( p(\cdot) \in LH \), \( 0 < p^- \leq p^+ \leq 1 \), \( 1 < q < \infty \) and \( d \) is a nonnegative integer such that \( d \in (n/p^--n-1,\infty) \). Then Carleson measure spaces with variable exponents \( \mathcal{C}MO^p(\cdot) \), Campanato space with variable exponent \( \mathcal{L}_{q,p(\cdot),d} \) and Hölder–Zygmund spaces with variable exponents \( \dot{H}^p_d(\cdot) \) coincide as sets and
\[
\|f\|_{\mathcal{C}MO^p(\cdot)} \sim \|f\|_{\mathcal{L}_{q,p(\cdot),d}} \sim \|f\|_{\dot{H}^p_d(\cdot)}.
\]

Proof. Applying Theorem 3.1 and Lemma 4.2 yields
\[
(H^p_{\text{atom}})^{q'} = \mathcal{L}_{q',p(\cdot),d}, \quad (H^p(\cdot))^q = \mathcal{C}MO^p(\cdot).
\]

Assume that \( q \geq 1 \) is sufficiently large and \( p(\cdot) \in \mathcal{P}^0 \cap LH \). Then by [24, Theorem 4.6] we have
\[
H^p(\cdot) = H^p_{\text{atom}},
\]
with \( \|f\|_{H^p(\cdot)} \sim \|f\|_{H^p_{\text{atom}}} \). For any given \( f \in \mathcal{L}_{q',p(\cdot),d} \), we see that \( f \) is a linear and continuous on \( H^p_{\text{atom}} \). That is, \( \|x_n - x\|_{H^p_{\text{atom}}} \to 0 \) implies \( |f(x_n) - f(x)| \to 0 \). On the other hand, for any \( \|x_n - x\|_{H^p(\cdot)} \to 0 \), we have \( \|x_n - x\|_{H^p(\cdot)} \to 0 \). Then we have \( |f(x_n) - f(x)| \to 0 \). So \( f \) is also a linear and continuous on \( H^p(\cdot) \) and \( f \in \mathcal{C}MO^p(\cdot) \). Therefore,
\[
\mathcal{L}_{q',p(\cdot),d} \subset \mathcal{C}MO^p(\cdot).
\]

Moreover,
\[
\|f\|_{\mathcal{C}MO^p(\cdot)} \sim \|L_f\|_{(H^p(\cdot))^{q'}} = \sup_{\|f\|_{H^p(\cdot)} \leq 1} |L_f(h)| \leq \sup_{\|Cf\|_{H^p_{\text{atom}}} \leq 1} |L_f(h)| = C\|L_f\|_{(H^p_{\text{atom}})^q} \sim \|f\|_{\mathcal{L}_{q',p(\cdot),d'}}.
\]
Similarly, we also have
\[\text{CMO}^p(\cdot) \subset \mathcal{L}_{q', p(\cdot), d}\]
and
\[\|f\|_{\mathcal{L}_{q', p(\cdot), d}} \leq C\|f\|_{\text{CMO}^p(\cdot)}.\]
Thus, \(\text{CMO}^p(\cdot)\) and \(\mathcal{L}_{q', p(\cdot), d}\) coincide as sets and
\[\|f\|_{\text{CMO}^p(\cdot)} \sim \|f\|_{\mathcal{L}_{q', p(\cdot), d}}.\] (4.1)

According to [32, Corollary 2.22], for any \(1 < q < \infty\), \(f \in \mathcal{L}_{q, p(\cdot), d}\) if and only if \(f \in \mathcal{L}_{1, p(\cdot), d}\) and
\[\|f\|_{\mathcal{L}_{q, p(\cdot), d}} \sim \|f\|_{\mathcal{L}_{1, p(\cdot), d}}.\] (4.2)

Furthermore, by Lemma 4.4, \(\dot{\mathcal{H}}_{d}^{p(\cdot)}\) and \(\mathcal{L}_{q, p(\cdot), d}\) are isomorphic and
\[\|f\|_{\dot{\mathcal{H}}_{d}^{p(\cdot)}} \sim \|f\|_{\mathcal{L}_{q, p(\cdot), d}}.\] (4.3)

Combining both (4.1), (4.2) and (4.3), we immediately obtain that
\[\|f\|_{\text{CMO}^p(\cdot)} \sim \|f\|_{\mathcal{L}_{q, p(\cdot), d}} \sim \|f\|_{\dot{\mathcal{H}}_{d}^{p(\cdot)}}.\]
This proves Theorem 4.5.

□

5. Applications

In this section we show that Calderón–Zygmund singular integral operators are bounded on \(\text{CMO}^p(\cdot)\) via using an argument of weak density property. Note that the weak density property is very useful when we deal with the boundedness of operators on Carleson measure type spaces or Lipschitz type spaces (see [14,20–22,28,29]). First we recall some necessary notations and definitions.

We say \(K(x, y) \in C^\infty_c\) is a standard kernel, if it is defined for \(x \neq y\), and satisfies the following weaker version of the differential inequalities:
\[|K(x, y)| \leq C|x - y|^{-n};\]
if \(|x - y| \geq 2|y - y'||,
\[|K(x, y) - k(x, y')| \leq C|y - y'|^\gamma|x - y|^{-n-\gamma}\]
and if \(|x - y| \geq 2|x - x'||,
\[|K(x, y) - k(x', y)| \leq C|x - x'|^\gamma|x - y|^{-n-\gamma};\]
where \(0 < \gamma \leq 1\). We denote \(K \in SK(\gamma)\).

When \(K \in SK(\gamma)\), if \(\varphi, \psi \in C^\infty_c\), supp \(\varphi \cap\) supp \(\psi = \emptyset\), then
\[\langle T\varphi, \psi \rangle = \int_{\mathbb{R}^n \times \mathbb{R}^n} K(x, y)\varphi(y)\psi(y)dydx.\]
That is, for any \(x \notin\) supp \(\varphi\)
\[T(\varphi)(x) = \int_{\mathbb{R}^n} K(x, y)\varphi(y)dy.\]
Suppose that $T$ is bounded on $L^2$. The relation between $K$ and $T$ is that $f \in L^2$ has compact support, then, outside the support of $f$, the distribution $Tf$ agrees with the function

$$T(f)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy, \quad x \notin \text{supp}(f).$$

Then $T$ is Calderón–Zygmund operator. We denote $T \in CZO(\gamma)$.

The adjoint operator $T^*$ is defined by

$$\langle T^*f, g \rangle = \langle f, Tg \rangle, \quad f, g \in S.$$  

It is associated with the standard kernel $\tilde{K}(x,y) = K(y,x)$.

Note that $T \in CZO(\gamma)$ can be extended to a bounded linear operator on $H^q$ for all $\frac{n}{n+\gamma} < q < \infty$, when $T^*(1) = 0$, and that its adjoint $T^*$ also can be extended to a bounded linear operator on $H^q$ for all $\frac{n}{n+\gamma} < q < \infty$, when $T(1) = 0$, for a proof, see [6, Section 10, Theorem 1.1]. In [25, Theorem 1.2], we proved the following result:

**Proposition 5.1.** Suppose that $T \in CZO(\gamma)$, $p(\cdot) \in LH$ and $\frac{n}{n+\gamma} < p^- \leq p^+ < \infty$. If $T^*(1) = 0$, then $T$ is a bounded linear operator on $H^{p(\cdot)}$. Similarly, if $T(1) = 0$, then $T^*$ is a bounded linear operator on $H^{p(\cdot)}$.

Our main result in this section is the following theorem.

**Theorem 5.2.** Suppose that $T \in CZO(\gamma)$, $p(\cdot) \in LH$ and $\frac{n}{n+\gamma} < p^- \leq p^+ \leq 1$. If $T(1) = 0$, then $T$ admits a bounded extension from $CMO^{p(\cdot)}$ to itself.

The following proposition on the weak density property for $CMO^{p(\cdot)}$ plays a key role in the proof of Theorem 5.2.

**Proposition 5.3.** Let $p(\cdot) \in LH$ and $0 < p^- \leq p^+ \leq 1$. If $f \in CMO^{p(\cdot)}$, then there exist a sequence $\{f_m\} \in CMO^{p(\cdot)} \cap L^2$ such that $f_m$ converges to $f$ in the distribution sense. Furthermore,

$$\|f_m\|_{CMO^{p(\cdot)}} \leq C\|f\|_{CMO^{p(\cdot)}}, \quad \text{for} \quad f \in CMO^{p(\cdot)}.$$  

**Proof.** By Lemma 2.1, we have the following Calderón identity

$$f(x) = \sum_{j \in \mathbb{Z}} \psi_j \ast \psi_j \ast f(x) \quad \text{in} \quad S'_{\infty}.$$  

The partial sum of the above series will be denoted by $f_m$ and is given by

$$f_m(x) = \sum_{|j| \leq m} \psi_j \ast \psi_j \ast f(x).$$

Then we claim that

$$\|f_m\|_{L^2} < \infty.$$  

In fact, applying the fact that $|\psi_j \ast f(x)| \leq C$ proved in Theorem 2.7 yields that

$$\|\psi_j \ast \psi_j \ast f\|_{L^2} \leq C.$$
For any \( g \in \mathcal{S}_\infty \), we obtain that
\[
|\langle f - f_m, g \rangle| = \lim_{n \to \infty} |\langle f_n - f_m, g \rangle| \\
\leq \liminf_{n \to \infty} \left| \sum_{m < |j| \leq n} \psi_j * \psi_j * f, g \right| \to 0, \quad \text{as } m \to 0.
\]
Thus, \( f_m \in L^2 \) and converges to \( f \) in the distribution sense.

To conclude the proof, note that
\[
\psi_j * f_m(x) = \sum_{|j'| \leq m} \psi_j * \psi_j * \psi_j * f(x)
\]
and by Corollary 2.8, it follows that
\[
\|f_m\|_{CMO^{p(\cdot)}(\cdot)} \sim \sup_P \left\{ \frac{|P|}{\|\chi_P\|_{p(\cdot)}^2} \sum_{Q \subset P} \int_{\ell(Q) = 2^{-j} - N} \left| \langle T_f, \psi_Q \rangle \right|^2 \chi_Q(x) dx \right\}^{1/2}.
\]
Again applying the almost orthogonal estimate and Corollary 2.8, for any \( m \geq 0 \) we have \( \|f_m\|_{CMO^{p(\cdot)}(\cdot)} \leq C\|f\|_{CMO^{p(\cdot)}(\cdot)} \).

Therefore, the proof of Proposition 5.3 is completed. \( \square \)

Now we prove of Theorem 5.2.

**Proof of Theorem 5.2.** First we show that the Calderón–Zygmund operator \( T \) is a bounded operator on \( CMO^{p(\cdot)}(\cdot) \cap L^2 \). Applying Theorem 3.1 and Proposition 5.1 yield that
\[
|\langle Tf, g \rangle| = |\langle f, T^* g \rangle| \leq \|f\|_{CMO^{p(\cdot)}(\cdot)} \|T^* g\|_{H^{p(\cdot)}} \leq C\|f\|_{CMO^{p(\cdot)}(\cdot)} \|g\|_{H^{p(\cdot)}}.
\]
That is, for each \( f \in CMO^{p(\cdot)}(\cdot) \cap L^2 \), \( L_f(g) = \langle Tf, g \rangle \) is a continuous linear functional on \( H^{p(\cdot)} \cap L^2 \). Since \( H^{p(\cdot)} \cap L^2 \) is dense in \( H^{p(\cdot)} \), \( L_f \) can be extended to a continuous linear functional on \( H^{p(\cdot)} \) with
\[
\|L_f\| \leq C\|f\|_{CMO^{p(\cdot)}(\cdot)}.
\]
Conversely, by Theorem 3.1 again, there exists \( h \in CMO^{p(\cdot)}(\cdot) \) such that \( \langle Tf, g \rangle = \langle h, g \rangle \) for \( g \in H^{p(\cdot)} \cap L^2 \) with
\[
\|h\|_{CMO^{p(\cdot)}(\cdot)} \leq C\|L_f\|.
\]

Thus,
\[
\|Tf\|_{CMO^{p(\cdot)}(\cdot)} = \sup_P \left\{ \frac{|P|}{\|\chi_P\|_{p(\cdot)}^2} \int_{\mathbb{R}^n} \sum_{Q \subset P} |Q|^{-1} |\langle T_f, \psi_Q \rangle|^2 \chi_Q(x) dx \right\}^{1/2} \\\n= \sup_P \left\{ \frac{|P|}{\|\chi_P\|_{p(\cdot)}^2} \int_{\mathbb{R}^n} \sum_{Q \subset P} |Q|^{-1} |\langle h, \psi_Q \rangle|^2 \chi_Q(x) dx \right\}^{1/2} \\\n= \|h\|_{CMO^{p(\cdot)}(\cdot)} \leq C\|L_f\| \leq C\|f\|_{CMO^{p(\cdot)}(\cdot)}.
\]
Next we extend this result to $CMO^p(\cdot)$ via an argument of weak density property. Suppose that $f \in CMO^p(\cdot)$. By Proposition 5.3, we can choose a sequence $\{f_m\} \subset CMO^p(\cdot) \cap L^2$ with
$$
\|f_m\|_{CMO^p(\cdot)} \leq C\|f\|_{CMO^p(\cdot)}
$$
such that $f_m$ converges to $f$ in the distribution sense. Therefore, for $f \in CMO^p(\cdot)$, define
$$
\langle Tf, g \rangle = \lim_{m \to \infty} \langle Tf_m, g \rangle, \quad \text{for } g \in H^p(\cdot) \cap L^2.
$$
In fact, we have $\langle Tf, g \rangle = \langle f_i - f_j, T^*(g) \rangle$, where $f_i - f_j$ and $g$ belong to $L^2$. By Proposition 5.1, we have $T^* \in H^{p(\cdot)} \cap L^2$. Applying Proposition 5.3 again, we get that
$$
\langle T(f_i - f_j), g \rangle = \langle f_i - f_j, T^*(g) \rangle \to 0
$$
as $j, k \to \infty$. Therefore, $Tf$ is well defined and
$$
\langle Tf, g \rangle = \lim_{m \to \infty} \langle Tf_m, g \rangle
$$
for any $g \in H^p(\cdot) \cap L^2$ and $f_m \in CMO^p(\cdot) \cap L^2$. Then by Fatou’s Lemma, for each dyadic cube $P$ in $\mathbb{R}^n$,
$$
\left\{ \frac{|P|}{\|\chi_P\|_{L^p(\cdot)}} \int_{\mathbb{R}^n} \sum_{Q \subset P} |Q|^{-1} |\langle Tf, \psi_Q \rangle|^2 \chi_Q(x) dx \right\}^{1/2}
\leq \liminf_{m \to \infty} \left\{ \frac{|P|}{\|\chi_P\|_{L^p(\cdot)}} \int_{\mathbb{R}^n} \sum_{Q \subset P} |Q|^{-1} |\langle f_m, \psi_Q \rangle|^2 \chi_Q(x) dx \right\}^{1/2}.
$$

Hence,
$$
\|Tf\|_{CMO^p(\cdot)} \leq \liminf_{m \to \infty} \|Tf_m\|_{CMO^p(\cdot)} \leq C\|f_m\|_{CMO^p(\cdot)} \leq C\|f\|_{CMO^p(\cdot)}.
$$
This completes this proof. \qed

As a corollary, we obtain that the convolution type Calderón-Zygmund singular integrals are bounded on $CMO^p(\cdot)$.

**Corollary 5.4.** Assume that $p(\cdot) \in LH$ and $0 < p^- \leq p^+ \leq 1$. Let $k \in S$ and
$$
\sup_{x \in \mathbb{R}^n} |x|^{n+m} |\nabla^m k(x)| < \infty \quad (m \in \mathbb{N} \cup \{0\}).
$$

Define a convolution operator $T$ by
$$
Tf(x) = k \ast f(x) \quad (f \in L^2).
$$

Then $T$ admits a bounded extension from $CMO^p(\cdot)$ to itself.

Combining both Theorems 4.5 and 5.2, we also have the following corollary.

**Corollary 5.5.** Suppose that $T \in CZO(\gamma)$. Let $p(\cdot) \in LH$, $\frac{n}{n+\gamma} < p^- \leq p^+ \leq 1$, $1 < q < \infty$ and $d$ is a nonnegative integer such that $d \in (n/p^- - n - 1, \infty)$. If $T(1) = 0$, then $T$ can be extended to a bounded operator on $\mathcal{L}_{q,p(\cdot),d}$ or $\mathcal{H}^p_d$. 

Acknowledgements
The Project is sponsored by Natural Science Foundation of Jiangsu Province of China (Grant No. BK20180734), Natural Science Research of Jiangsu Higher Education Institutions of China (Grant No. 18KJB110022) and Nanjing University of Posts and Telecommunications Science Foundation (Grant No. NY219114). The author wishes to express his heartfelt thanks to the referee for careful reading.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

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Received: December 2, 2018.
Revised: July 26, 2019.