FUNCTION THEORY IN THE QUANTUM MATRIX BALL: AN INVARIANT INTEGRAL

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1. It is well known [4] that the unit ball $U$ in the space of rectangle complex matrices is a bounded symmetric domain. Our recent work [8] introduces q-analogues of all such domains. This work considers the q-analogue of the matrix ball and presents an explicit formula for a positive invariant integral.

A parameter $q$ involved into the formulations of the main results is assumed to be a number: $0 < q < 1$. However, in sections 2, 3 the ground field will be not $\mathbb{C}$ but $\mathbb{C}(q^{1/2})$, the field of rational functions of the parameter $q^{1/2}$ (cf. [5]).

2. The Hopf algebra $U_q sl_N$ is determined by its generators $\{E_i, F_i, K_i^{\pm1}\}_{i=1,...,N-1}$ and the well known Drinfeld-Jimbo relations (see [5]). The universal enveloping algebra $U sl_N$ could be derived from $U_q sl_N$ via the “change of variables” $q = e^{-h/2}$, $K_i^{\pm1} = e^{\pm h H_i/2}$, $j = 1, \ldots, N - 1$, and a formal passage to a limit as $h \to 0$ in the list of relations.

Everywhere in the sequel $m, n \in \mathbb{N}$, $N = m + n$. We use the standard notation $su_{nm}$ for the Lie algebra of the automorphism group of the unit matrix ball $U \subset Mat_{mn}$.

Equip the Hopf algebra $U_q sl_N$ with the involution, which is defined on the generators $K_j^{\pm1}$, $E_j$, $F_j$, $j = 1, \ldots, N - 1$ by

$$(K_j^{\pm1})^* = K_j^{\pm1}, \quad E_j^* = \begin{cases} K_j F_j, & j \neq n \\ -K_j F_j, & j = n \end{cases}, \quad F_j^* = \begin{cases} E_j K_j^{-1}, & j \neq n \\ -E_j K_j^{-1}, & j = n \end{cases}.$$ 

The Hopf $*$-algebra $U_q su_{nm} = (U_q sl_N, *)$ arising in this way is a q-analogue of the Hopf algebra $Usu_{nm}$.

Remind some well known definitions [2]. An algebra $F$ is said to be an $A$-module algebra if it is a module over a Hopf algebra $A$ and the multiplication $F \otimes F \to F$, $f_1 \otimes f_2 \mapsto f_1 \cdot f_2$, is a morphism of $A$-modules. In the case of a $*$-algebra $F$ and a Hopf-$*$-algebra $A$, there is an additional requirement that the involutions agree as follows:

$$(af)^* = (S(a))^* f^*, \quad a \in A, \ f \in F,$$

with $S : A \to A$ being the antipode of $A$.

3. In [8] a $U_q su_{nm}$-module algebra $Pol(Mat_{mn})_q$ and its $U_q sl_N$-module subalgebra $\mathbb{C}[Mat_{mn}]_q$ were introduced (the notation $\mathfrak{g}_{-1}$ was used in [8] instead of $Mat_{mn}$). These
algebras are q-analogues of polynomial algebras in the vector spaces Mat_{mn}. For that, the passage to dual coalgebras was implemented, which constitutes an approach of V. Drinfeld [9]. We present below a description of the algebra C[Mat_{mn}]_q in terms of generators and relations.

With the definitions of [9] as a background, one can prove the following two propositions.

**Proposition 1.** There exists a unique family \{z_a^\alpha\}_{a=1,...,n,\alpha=1,...,m} of elements of the \(U_q\mathfrak{sl}_N\)-module algebra \(C[Mat_{mn}]_q\) such that for all \(a = 1,\ldots, n; \alpha = 1,\ldots, m\)

\[
H_n z_a^\alpha = \begin{cases} 2z_a^\alpha & , \ a = n & \alpha = m \\ z_a^\alpha & , \ a = n & \alpha \neq m \quad \text{or} \quad a \neq n & \alpha = m \\ 0 & , \text{otherwise} \end{cases}
\]

(1)

\[
F_n z_a^\alpha = q^{1/2} \cdot \begin{cases} 1 & , \ a = n & \alpha = m \\ 0 & , \text{otherwise} \end{cases}
\]

(2)

\[
E_n z_a^\alpha = -q^{1/2} \cdot \begin{cases} q^{-1}z_a^\alpha & , \ a \neq n & \alpha \neq m \\ (z_n^m)^{\alpha} & , \ a = n & \alpha = m \\ z_n^m & , \text{otherwise} \end{cases}
\]

(3)

and with \(k \neq n\)

\[
H_k z_a^\alpha = \begin{cases} z_a^\alpha & , \ k < n & a = k \quad \text{or} \quad k > n & \alpha = N - k \\ -z_a^\alpha & , \ k < n & a = k + 1 \quad \text{or} \quad k > n & \alpha = N - k + 1 \\ 0 & , \text{otherwise} \end{cases}
\]

\[
F_k z_a^\alpha = q^{1/2} \cdot \begin{cases} z_{a+1}^\alpha & , \ k < n & a = k \\ z_a^{\alpha+1} & , \ k > n & \alpha = N - k \\ 0 & , \text{otherwise} \end{cases}
\]

\[
E_k z_a^\alpha = q^{-1/2} \cdot \begin{cases} z_{a-1}^\alpha & , \ k < n & a = k + 1 \\ z_a^{\alpha-1} & , \ k > n & \alpha = N - k + 1 \\ 0 & , \text{otherwise} \end{cases}
\]

**Proposition 2.** \{z_a^\alpha\}_{a=1,...,n,\alpha=1,...,m} generate \(C[Mat_{mn}]_q\) as an algebra and \(Pol(Mat_{mn})_q\) as a *-algebra. The complete list of relations is following:

\[
z_a^\alpha z_b^\beta = \begin{cases} qz_b^\beta z_a^\alpha & , \ a = b & \alpha < \beta \quad \text{or} \quad a < b & \alpha = \beta \\ z_b^\beta z_a^\alpha & , \ a < b & \alpha > \beta \\ z_b^\beta z_a^\alpha + (q - q^{-1})z_a^\beta z_b^\alpha & , \ a < b & \alpha < \beta \end{cases}
\]

\[
(z_b^\beta)^* z_a^\alpha = q^2 \cdot \sum_{a',b'=1}^{n} \sum_{\alpha',\beta'=1}^{m} R_{ba}^{a'b'} R_{\beta'\alpha'}^{\beta\alpha} z_{a'}^{\alpha'} (z_{b'}^\beta)^* + (1 - q^2)\delta_{ab}\delta^{\alpha\beta},
\]

with \(\delta_{ab}, \delta^{\alpha\beta}\) being the Kronecker symbols and

\[
R_{ba}^{a'b'} = \begin{cases} q^{-1} & , \ a \neq b & b = b' & a = a' \\ 1 & , a = b = a' = b' \\ -(q^{-2} - 1) & , a = b & a' = b' & a > a \\ 0 & , \text{otherwise} \end{cases}
\]
\( F^{\beta \alpha}_{\beta' \alpha'} = \begin{cases} 
q^{-1}, & \alpha \neq \beta \land \beta = \beta' \land \alpha = \alpha' \\
1, & \alpha = \beta = \beta' = \alpha' \\
-(q^{-2} - 1), & \alpha = \beta \land \alpha' = \beta' \land \alpha > \alpha \\
0, & \text{otherwise} \end{cases} \)

**Corollary 3.**

\((z^m_n)^* z_n^m = q^2 z_n^m (z_n^m)^* + 1 - q^2. \) \( \quad (4) \)

Note that \( z_n^m \) generates a \( U_q \mathfrak{su}_{11} \)-module algebra \( \text{Pol}(\mathbb{C})_q \) determined by relations \( (1) - (4) \) (see [4]). Commutation relations similar to those given in proposition 2 appear in a different context in [6].

**4.** Everywhere in the sequel we assume \( q \in (0,1) \), and \( \mathbb{C} \) is considered as a ground field. We keep the notation \( U_q \mathfrak{su}_{nm} \), \( \text{Pol}(\text{Mat}_{nm})_q \) for the Hopf \( * \)-algebra and the covariant \( * \)-algebra determined by the generators \{\( E_j, F_j, K_j^{1+} \}_{j=1,\ldots,N-1} \) and \{\( \alpha_a^\nu \}_{\alpha=1,\ldots,n; a=1,\ldots,m} \) respectively and the relations as above.

It is well known that in the classical case \( q = 1 \) the positive \( SU_{nm} \)-invariant measure on the matrix ball is infinite. Thus the positive invariant integral could not be defined on the polynomial algebra. In the quantum case this obstacle is still in effect, therefore we need to extend the \( U_q \mathfrak{su}_{nm} \)-module algebra \( \text{Pol}(\text{Mat}_{mn})_q \) up to the \( U_q \mathfrak{su}_{nm} \)-module algebra \( \text{Fun}(U)_q \).

The construction we produce below can seem non-substantiated. We refer the reader to the work [3], where it is described in more details in an important special case of the quantum disc (i.e. \( m=n=1 \)).

Consider the \( * \)-algebra \( \text{Fun}(U)_q \supset \text{Pol}(\text{Mat}_{mn})_q \) derived from \( \text{Pol}(\text{Mat}_{mn})_q \) by adjunction an additional generator \( f_0 \) such that \( f_0 = f_0^2 = f_0^* \) and \( \left( \alpha_a^\nu \right)^* f_0 = f_0 \alpha_a^\nu = 0 \), \( \alpha = 1,\ldots,n; \alpha = 1,\ldots,m \).

((5) allows one to treat \( f_0 \) as a \( q \)-analogue of the delta-function at zero.)

Extend the structure of a \( U_q \mathfrak{su}_{nm} \)-module algebra from \( \text{Pol}(\text{Mat}_{mn})_q \) onto \( \text{Fun}(U)_q \) via the relations (6) (which were obtained in our earlier work [3] in the special case \( m = n = 1 \)) and (7). The following proposition is a consequence of the definition of the \( * \)-algebra \( \text{Fun}(U)_q \), the relation (4), and the quasicommutativity of \( z_n^m \) with \( \left( \alpha_a^\nu \right)^* \) for \( (n,m) \neq (a,\alpha) \).

**Proposition 4.** There exists a unique extension of the structure of a \( U_q \mathfrak{su}_{nm} \)-module algebra from \( \text{Pol}(\text{Mat}_{mn})_q \) onto \( \text{Fun}(U)_q \) such that

\[ H_n f_0 = 0, \quad F_n f_0 = -\frac{q^{1/2}}{q^{-2} - 1} f_0 \cdot (z_n^m)^*, \quad E_n f_0 = -\frac{q^{1/2}}{1 - q^2} z_n^m \cdot f_0 \]

\( \text{and with } k \neq n \)

\[ H_k f_0 = F_k f_0 = E_k f_0 = 0. \]

**5.** The two-sided ideal \( D(U)_q \) defined \( \text{Fun}(U)_q f_0 \text{Fun}(U)_q \) is a \( U_q \mathfrak{su}_{nm} \)-module algebra. Its elements will be called the finite functions in the quantum matrix ball.

Our purpose is to produce a positive invariant integral on the algebra of finite functions

\[ D(U)_q \to \mathbb{C}, \quad f \mapsto \int_{U_q} f \, d\nu, \]
i.e. such linear functional that \( \int_{U_q} f^* f d\nu > 0 \) for all \( f \neq 0 \), and \( \int_{U_q} (\xi f) d\nu = \varepsilon(\xi) \int_{U_q} f d\nu \) for all \( \xi \in U_q \mathfrak{sl}_N \), \( f \in D(U)_q \). Here \( \varepsilon \) is the counit of the Hopf algebra \( U_q \mathfrak{sl}_N \).

Consider the vector space \( \mathcal{H} = \mathbb{C}[\text{Mat}_{mn}]/f_0 \subset D(U)_q \). Equip it with the grading (see [8]):

\[
\mathcal{H} = \bigoplus_{j=0}^\infty \mathcal{H}_j, \quad \mathcal{H}_j \overset{\text{def}}{=} \{ v \in \mathcal{H} \mid H_0 v = 2jv \}
\]

with

\[
H_0 = \frac{2}{m+n} \left( m \sum_{j=1}^{n-1} jH_j + n \sum_{j=1}^{m-1} jH_N-j + mnH_n \right).
\]

Evidently, \( \dim \mathcal{H}_j < \infty \) for all \( j \in \mathbb{Z}_+ \).

Let \( T \) be the representation of \( D(U)_q \) in \( \mathcal{H} \) given by \( T_f \psi = f \cdot \psi \), \( f \in D(U)_q \), \( \psi \in \mathcal{H} \subset D(U)_q \). The following statements are direct consequences of the definitions.

**Lemma 5.** For any element \( f \in D(U)_q \) there exists such positive integer \( M(f) \) that \( T_f \mathcal{H}_j = 0 \) for all \( j \geq M(f) \).

**Corollary 6.** All the operators \( T_f \), \( f \in D(U)_q \), are finite dimensional.

It is easy to prove the existence and uniqueness of such scalar product in \( \mathcal{H} \) that \( (f_0, f_0) = 1 \) and \( (T_f \psi_1, \psi_2) = (\psi_1, T_f^* \psi_2) \) for all \( f \in D(U)_q \), \( \psi_1, \psi_2 \in \mathcal{H} \).

The representation \( T \) is faithful \( (T(f) \neq 0 \text{ for } f \neq 0) \), and one also has \( (\psi, \psi) > 0 \) for all \( \psi \in \mathcal{H} \), \( \psi \neq 0 \).

6. Let \( U_q \mathfrak{p}_+ \subset U_q \mathfrak{sl}_N \) be the Hopf subalgebra generated by all the generators \( E_j, F_j, K_j^{\pm 1}, j = 1, \ldots, N-1 \) except \( F_n \).

(6), (7) imply that the subspace \( \mathcal{H} \) is a submodule of the \( U_q \mathfrak{p}_+ \)-module \( D(U)_q \). Let \( \Gamma \) stand for the representation of \( U_q \mathfrak{p}_+ \) in \( \mathcal{H} \). Let also \( \bar{\rho} = \frac{1}{2} \sum_{j=1}^{N-1} j(N-j)H_j \). Then \( e^{\bar{\rho}} = \prod_{j=1}^{N-1} K_j^{-j(N-j)} \), and the operator \( \Gamma(e^{\bar{\rho}}) \) in \( \mathcal{H} \) is well defined and takes each \( \mathcal{H}_j \), \( j \in \mathbb{Z}_+ \) into itself. Hence by lemma 5 the trace \( \text{Tr}(T(f)\Gamma(e^{\bar{\rho}})) \) is well defined for all \( f \in D(U)_q \). An application of theorem 7 yields

**Theorem 8.** The linear functional

\[
\int_{U_q} f d\nu \overset{\text{def}}{=} \text{Tr}(T(f)\Gamma(e^{\bar{\rho}})) \quad \text{(8)}
\]

on the algebra \( D(U)_q \) of finite functions in the quantum matrix ball is positive and invariant.

7. Consider a class of those \( U_q \mathfrak{su}_{nm} \)-module algebras for which an analogue of theorem 8 is valid due to obvious reasons. Let \( \mathcal{H} \) be a unitarizable Harish-Chandra module with the lowest weight over the Hopf algebra \( U_q \mathfrak{su}_{nm} \), and let \( \Gamma \) be the associated representation of \( U_q \mathfrak{su}_{nm} \) in \( \mathcal{H} \). The canonical embedding \( \mathcal{H} \otimes \mathcal{H}^* \hookrightarrow \text{End}_{\mathbb{C}}(\mathcal{H}) \) allows one to equip \( F = \mathcal{H} \otimes \mathcal{H}^* \) with a structure of a \( U_q \mathfrak{su}_{nm} \)-module algebra. Let \( T \) be the associated representation of \( F \) in \( \mathcal{H} \). It is well known and can be easily proved that the linear functional \( \text{Tr}_q T(f) \overset{\text{def}}{=} \text{Tr}(T(f)\Gamma(e^{\bar{\rho}})) \) on \( F \) is a positive invariant integral. It was shown in [8] for the case of quantum disc that
the $U_q\mathfrak{su}_{nm}$-module algebra $D(U)_q$ is a "limit point of the set of" $U_q\mathfrak{su}_{nm}$-module algebras of the class described above. This substantiates invariance and positivity of the integral (8).

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