SARASON’S TOEPLITZ PRODUCT PROBLEM
FOR A CLASS OF FOCK SPACES

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ABSTRACT. Sarason’s Toeplitz product problem asks when the operator
\( T_u T_v \) is bounded on various Hilbert spaces of analytic functions, where
\( u \) and \( v \) are analytic. The problem is highly nontrivial for Toeplitz op-
erators on the Hardy space and the Bergman space (even in the case of
the unit disk). In this paper, we provide a complete solution to the prob-
lem for a class of Fock spaces on the complex plane. In particular, this
generalizes an earlier result of Cho, Park, and Zhu.

1. INTRODUCTION

Let \( \mathbb{D} \) be the open unit disk in the complex plane \( \mathbb{C} \) and let \( \mathbb{T} = \partial \mathbb{D} \) denote
the unit circle. The Hardy space \( H^2 \) consists of functions \( f \in L^2(\mathbb{T}) \) such
that its Fourier coefficients satisfy \( \hat{f}_n = 0 \) for all \( n < 0 \). Given a function
\( \varphi \in L^2(\mathbb{T}) \), the Toeplitz operator \( T_\varphi : H^2 \to H^2 \) is densely defined by
\( T_\varphi f = P(\varphi f) \), where \( P : L^2(\mathbb{T}) \to H^2 \) is the Riesz-Szegő projection.

The original problem that Sarason proposed in [14] was this: characterize
the pairs of outer functions \( u \) and \( v \) in \( H^2 \) such that the operator \( T_u T_v \) is
bounded on \( H^2 \). Inner factors can easily be disposed of, so it was only
necessary to consider outer functions in the Hardy space case. It was further
observed in [14] that a necessary condition for the boundedness of \( T_u T_v \) on
\( H^2 \) is that
\[
\sup_{w \in \mathbb{D}} P_w(|u|^2)P_w(|v|^2) < \infty,
\]
where \( P_w(f) \) means the Poisson transform of \( f \) at \( w \in \mathbb{D} \). In fact, the
arguments in [14] show that
\[
\sup_{w \in \mathbb{D}} P_w(|u|^2)P_w(|v|^2) \leq 4\|T_u T_v\|^2.
\]

Let \( A^2 \) denote the Bergman space consisting of analytic functions in
\( L^2(\mathbb{D}, dA) \), where \( dA \) is ordinary area measure on the unit disk. If \( P : \)
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$L^2(\mathbb{D}, dA) \to A^2$ is the Bergman projection, then Toeplitz operators $T_\varphi$ on $A^2$ are defined by $T_\varphi f = P(\varphi f)$. Sarason also posed a similar problem in [14] for the Bergman space: characterize functions $u$ and $v$ in $A^2$ such that the Toeplitz product $T_u T_v$ is bounded on $A^2$. It was shown in [17] that

$$\sup_{w \in \mathbb{D}} |\hat{u}|^2(w)|\hat{v}|^2(w) \leq 16\|T_u T_v\|^2$$

for all functions $u$ and $v$ in the Bergman space $A^2$, where $\hat{f}(w)$ is the so-called Berezin transform of $f$ at $w$. This provides a necessary condition for the boundedness of $T_u T_v$ on $A^2$ in terms of the Berezin transform.

The Berezin transform is well defined in many other different contexts. In particular, the classical Poisson transform is the Berezin transform in the context of the Hardy space $H^2$. So the estimates in (1) and (2) are in exactly the same spirit. Sarason stated in [14] that “it is tempting to conjecture that” $T_u T_v$ is bounded on $H^2$ or $A^2$ if and only if $|\hat{u}|^2(w)|\hat{v}|^2(w)$ is a bounded function on $\mathbb{D}$. It has by now become standard to call this “Sarason’s conjecture for Toeplitz products”.

It turns out that Sarason’s conjecture is false for both the Hardy space and the Bergman space of the unit disk, and the conjecture fails in a big way. See [1, 12] for counter-examples. In these cases, Sarason’s problem is naturally connected to certain two-weight norm inequalities in harmonic analysis, and counter-examples for Sarason’s conjecture were constructed by means of the dyadic model approach in harmonic analysis.

Another setting where Toeplitz operators have been widely studied is the Fock space. More specifically, we let $F^2$ be the space of all entire functions $f$ on $\mathbb{C}$ that are square-integrable with respect to the Gaussian measure

$$d\lambda(z) = \frac{1}{\pi} e^{-|z|^2} dA(z).$$

The function

$$K(z, w) = e^{z \overline{w}}, \quad z, w \in \mathbb{C},$$

is the reproducing kernel of $F^2$ and the orthogonal projection $P$ from $L^2(\mathbb{C}, d\lambda)$ onto $F^2$ is the integral operator defined by

$$Pf(z) = \int_{\mathbb{C}} K(z, w)f(w)d\lambda(w), \quad z \in \mathbb{C}.$$ 

If $\varphi$ is in $L^2(\mathbb{C}, d\lambda)$ such that the function $z \mapsto \varphi(z)K(z, w)$ belongs to $L^1(\mathbb{C}, d\lambda)$ for any $w \in \mathbb{C}$, we can define the Toeplitz operator $T_\varphi$ with symbol $\varphi$ by $T_\varphi f = P(\varphi f)$, or

$$T_\varphi f(z) = \int_{\mathbb{C}} K(z, w)\varphi(w)f(w) d\lambda(w), \quad z \in \mathbb{C},$$
when
\[
f(w) = \sum_{k=1}^{N} c_k K(w, c_k)
\]
is a finite linear combination of kernel functions. Since the set of all finite linear combinations of kernel functions is dense in \( F^2 \), the operator \( T_\varphi \) is densely defined and \( T_\varphi f \) is an entire function. See [19] for basic information about the Fock space and Toeplitz operators on it.

In a recent paper [8], Cho, Park and Zhu solved Sarason’s problem for the Fock space. More specifically, they obtained the following simple characterization for \( T_u T_v \) to be bounded on \( F^2 \): if \( u \) and \( v \) are functions in \( F^2 \), not identically zero, then \( T_u T_v \) is bounded on \( F^2 \) if and only if \( u = e^q \) and \( v = ce^{-q} \), where \( c \) is a nonzero constant and \( q \) is a complex linear polynomial. As a consequence of this, it can be shown that Sarason’s conjecture is actually true for Toeplitz products on \( F^2 \); see Section 5 below.

In this paper, we consider the weighted Fock space \( F_m^2 \), consisting of all entire functions in \( L^2(\mathbb{C}, d\lambda_m) \), where \( d\lambda_m \) are the generalized Gaussian measure defined by
\[
d\lambda_m(z) = e^{-|z|^{2m}} dA(z), \quad m \geq 1.
\]
Toeplitz operators on \( F_m^2 \) are defined exactly the same as the cases above, using the orthogonal projection \( P : L^2(\mathbb{C}, d\lambda_m) \to F_m^2 \).

We will solve Sarason’s problem and prove Sarason’s conjecture for the weighted Fock spaces \( F_m^2 \). Our main result can be stated as follows.

**Main Theorem.** Let \( u \) and \( v \) be in \( F_m^2 \), not identically zero. The following conditions are equivalent:

1. The product \( T = T_u T_v \) is bounded on \( F_m^2 \).
2. There exist a polynomial \( g \) of degree at most \( m \) and a nonzero complex constant \( c \) such that \( u(z) = e^{g(z)} \) and \( v(z) = ce^{-g(z)} \).
3. The product \( |u|^2(z) |v|^2(z) \) is a bounded function on \( \mathbb{C} \).

Furthermore, in the affirmative case, we have the following estimate of the norm:
\[
\|T\| \leq C_1 e^{C_2 \|g\|^2_{H^2}},
\]
where \( \|g\|_{H^2} \) is the norm in the Hardy space of the unit disc, and \( C_1 \) and \( C_2 \) are positive constants independent of \( g \).

Let us mention that [10] contains partial results related to Sarason’s conjecture on the Fock space. The arguments in [8] depend on the explicit form of the reproducing kernel and the Weyl operators induced by translations of the complex plane. Both of these are no longer available for the spaces \( F_m^2 \).
there is no simple formula for the reproducing kernel of \( F_m^2 \) and the translations on the complex plane do not induce nice operators on \( F_m^2 \). Therefore, we need to develop new techniques to tackle the problem.

2. Preliminary estimates

In this section we recall some properties of the Hilbert space \( F_m^2 \). It was shown in [6] that the reproducing kernel of \( F_m^2 \) is given by the formula

\[
K_m(z, w) = \frac{m}{\pi} \sum_{k=0}^{\infty} \frac{(z \bar{w})^k}{\Gamma \left( \frac{k+1}{m} \right)}.
\]  

(3)

In terms of the Mittag-Leffler function

\[
E_{\gamma, \beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\gamma k + \beta)}, \quad \gamma, \beta > 0,
\]

we can also write

\[
K_m(z, w) = \frac{m}{\pi} E_{1/m, 1/m}(z \bar{w}).
\]

(4)

Recall that the asymptotics of the Mittag-Leffler function \( E_{1/m, 1/m}(z) \) as \( |z| \to +\infty \) are given by

\[
E_{1/m, 1/m}(z) = \begin{cases} 
  mz^{m-1}e^{zm}(1 + o(1)), & |\arg z| \leq \frac{\pi}{2m}, \\
  O(\frac{1}{z}), & \frac{\pi}{2m} < |\arg z| \leq \pi
\end{cases}
\]

(5)

for \( m > \frac{1}{2} \), and by

\[
E_{1/m, 1/m}(z) = m \sum_{j=-N}^{N} z^{m-1}e^{\frac{2\pi ij}{m}(m-1)}e^{z^{m}e^{2\pi ijm}} + O(\frac{1}{z}), \quad -\pi < \arg z \leq \pi,
\]

for \( 0 < m \leq \frac{1}{2} \), where \( N \) is the integer satisfying \( N < \frac{1}{2m} \leq N + 1 \) and the powers \( z^{m-1} \) and \( z^m \) are the principal branches. See, for example, Bateman and Erdelyi [3], vol. III, 18.1, formulas (21)–(22).

The asymptotic estimates of the Mittag-Leffler function \( E_{1/m, 1/m}(z) \) provide the following estimates for the reproducing kernel \( K_m(z, w) \), which is a consequence of the results in [6] and Lemma 3.1 in [15].

Lemma 1. For arbitrary points \( x, r \in (0, +\infty) \) and \( \theta \in (-\pi, \pi) \) we have

\[
|K_m(x, re^{i\theta})| \lesssim \begin{cases} 
  (xr)^{m-1}e^{(xr)^{m} \cos(m\theta)} & |\theta| \leq \frac{\pi}{2m}, \\
  O \left( \frac{1}{xr} \right) & \frac{\pi}{2m} < |\theta| < \pi
\end{cases}
\]

as \( xr \to +\infty \). Moreover, there is a constant \( c > 0 \) such that for all \( |\theta| \leq c\theta_0(xr) \) we have

\[
|K_m(x, re^{i\theta})| \gtrsim (xr)^{m-1}e^{(xr)^m}
\]
as \( xr \to +\infty \), where \( \theta_0(r) = r^{-\frac{m}{2}}/m \).

On several occasions later on we will need to know the maximum order of a function in \( \mathcal{F}_m^2 \). For example, if we have a non-vanishing function \( f \) in \( \mathcal{F}_m^2 \) and if we know that the order of \( f \) is finite, then we can write \( f = e^q \) with \( q \) being a polynomial. The following estimate allows us to do this.

**Lemma 2.** If \( f \in \mathcal{F}_m^2 \), there is a constant \( C > 0 \) such that

\[
|f(z)| \leq C |z|^{m-1} e^{\frac{1}{2}|z|^{2m}}, \quad z \in \mathbb{C}.
\]

Consequently, the order of every function in \( \mathcal{F}_m^2 \) is at most \( 2m \).

**Proof.** By the reproducing property and Cauchy-Schwartz inequality, we have

\[
|f(z)| = \left| \int_{\mathbb{C}} f(w) K_m(z, w) d\lambda_m(w) \right| \leq \|f\| K_m(z, z)^{1/2}
\]

for all \( f \in \mathcal{F}_m^2 \) and all \( z \in \mathbb{C} \). The desired estimate then follows from Lemma 1. See [4] for more details.

Another consequence of the above lemma is that, for any function \( u \in \mathcal{F}_m^2 \), the Toeplitz operators \( T_u \) and \( T_\pi \) are both densely defined on \( \mathcal{F}_m^2 \).

### 3. Sarason’s Problem for \( \mathcal{F}_m^2 \)

In this section we prove the equivalence of conditions (1) and (2) in the main theorem stated in the introduction, which provides a simple and complete solution to Sarason’s problem for Toeplitz products on the Fock space \( \mathcal{F}_m^2 \). We break the proof into several lemmas.

**Lemma 3.** Suppose that \( u \) and \( v \) are functions in \( \mathcal{F}_m^2 \), each not identically zero, and that the operator \( T = T_uT_\pi \) is bounded on \( \mathcal{F}_m^2 \). Then there exists a polynomial \( g \) of degree at most \( m \) and a nonzero complex constant \( c \) such that \( u(z) = e^{g(z)} \) and \( v(z) = ce^{-g(z)} \).

**Proof.** If \( T = T_uT_\pi \) is bounded on \( \mathcal{F}_m^2 \), then the Berezin transform \( \tilde{T} \) is bounded, where

\[
\tilde{T}(z) = \langle T_uT_\pi k_z, k_z \rangle, \quad z \in \mathbb{C}.
\]

By the reproducing property of the kernel functions, it is easy to see that

\[
\tilde{T}(z) = u(z)v(z).
\]

Since each \( k_z \) is a unit vector, it follows from the Cauchy-Schwarz inequality that

\[
|u(z)v(z)| = |\tilde{T}(z)| \leq \|T\|
\]
for all \( z \in \mathbb{C} \). This together with Liouville’s theorem shows that there exist a constant \( c \) such that \( uv = c \). Since neither \( u \) nor \( v \) is identically zero, we have \( c \neq 0 \). Consequently, both \( u \) and \( v \) are non-vanishing.

Recall from Lemma 2 that the order of functions in \( \mathcal{F}_m^2 \) is at most \( 2^m \), so there is a polynomial of degree \( d \),\n
\[
g(z) = \sum_{k=0}^{d} a_k z^k, \quad d \leq 2^m,
\]

such that \( u = e^{g} \) and \( v = c e^{-g} \). It remains to show that \( d \leq m \).

Since \( T \) is bounded on \( \mathcal{F}_m^2 \), the function

\[
F(z, w) = \frac{\langle T(K_m(\cdot, w)), K_m(\cdot, z) \rangle}{\sqrt{K_m(z, z)} \sqrt{K_m(w, w)}}
\]

must be bounded on \( \mathbb{C}^2 \). On general reproducing Hilbert spaces, we always have

\[
\langle T_u T_v K_w, K_z \rangle = \langle T_v K_w, T_u K_z \rangle = \langle (\overline{v}(w) K_v, \overline{u}(z) K_u) \rangle = u(z) \overline{v}(w) K(z, w).
\]

It follows that

\[
F(z, w) = \frac{K_m(z, w)}{\sqrt{K_m(z, z)} \sqrt{K_m(w, w)}}.
\]

From Lemma 4 we deduce that

\[
|F(z, w)| \geq e^{\text{Re}(g(z) - g(w))} e^{-\frac{1}{2}(|z|^m - |w|^m)^2} \quad \text{(6)}
\]

for all \( |\arg(z \bar{w})| \leq c \theta_0(|zw|) \) as \( |zw| \) grows to infinity.

Choose \( x > 0 \) sufficiently large and set

\[
z(x) = xe^{i \frac{\pi}{2}} e^{-i \frac{\arg(u_d)}{d}}
\]

and

\[
w(x) = xe^{i \frac{\pi}{2}} e^{-i \frac{\arg(u_d) + \theta_0}{d} \frac{m^m}{m^m}}.
\]

Since

\[
\theta_0(|z(x)w(x)|) = \frac{1}{m^m},
\]

we can apply (6) to \( z(x) \) and \( w(x) \) to get

\[
e^{\text{Re}(g(z(x)) - g(w(x)))} \lesssim \sup_{(z, w) \in \mathbb{C}^2} |F(z, w)| < \infty \quad \text{(7)}
\]
as $x$ grows to infinity. On the other hand, a few computations show that

$$\text{Re} \left( g(z(x)) - g(w(x)) \right) = \sum_{j=0}^{d} x^j \text{Re} \left( a_j e^{i j \frac{\pi}{d} - i \frac{1}{d} \arg(a_d)} \left( 1 - e^{-i \frac{\pi}{2m x^m}} \right) \right)$$

$$= |a_d| x^d \sin \left( \frac{c}{2m x^m} \right) + g_{d-1}(x),$$

where

$$g_{d-1}(x) = \sum_{j=0}^{d-1} x^j \text{Re} \left( a_j e^{i j \frac{\pi}{d} - i \frac{1}{d} \arg(a_d)} \left( 1 - e^{-i \frac{\pi}{2m x^m}} \right) \right)$$

$$= - \sum_{j=0}^{d-1} |a_j| x^j \sin \left( \frac{j \pi}{2d} + \arg a_j - \frac{j}{d} \arg(a_d) \right) \sin \frac{c j}{2md x^m}$$

$$+ \sum_{j=0}^{d-1} |a_j| x^j \cos \left[ j \frac{\pi}{2d} + \arg a_j - \frac{j}{d} \arg(a_d) \right] \left[ 1 - \cos \frac{c j}{2md x^m} \right]$$

$$\lesssim x^{d-1-m}.$$

Therefore, there exist some $x_0 > 0$ and $\delta > 0$ such that

$$\text{Re} \left( g(z(x)) - g(w(x)) \right) \geq \delta |a_d| \frac{x^d}{x^m}$$

for all $x \geq x_0$. Since $a_d \neq 0$, it follows from (7) that $d \leq m$. □

On several occasions later on we will need to estimate the integral

$$I(a) = \int_{0}^{\infty} e^{-\frac{1}{2} r^{2m+ar^d} r^N} dr,$$

where $m > 0, 0 \leq d \leq m, N > -1$, and $a \geq 0$.

First, suppose $a > 1$. By various changes of variables, we have

$$I(a) = \int_{0}^{1} e^{-\frac{1}{2} r^{2m+ar^d} r^N} dr + \int_{1}^{\infty} e^{-\frac{1}{2} r^{2m+ar^d} r^N} dr$$

$$\leq e^a \int_{0}^{1} r^N dr + \int_{1}^{\infty} e^{-\frac{1}{2} r^{2m+ar^d} r^N} dr$$

$$= \frac{e^a}{N+1} + e^{\frac{a^2}{2}} \int_{1}^{\infty} e^{-\frac{1}{2} (r^{a-1})^2} r^N dr$$

$$= \frac{e^a}{N+1} + \frac{e^{\frac{a^2}{2}}}{m} \int_{t}^{\infty} e^{-\frac{1}{2} (t-a)^2} t^{N+1} dt.$$
If \( \frac{N+1}{m} - 1 \leq 0 \), then

\[
I(a) \leq \frac{e^a}{N + 1} + \frac{\sqrt{2\pi}}{m} e^\frac{a^2}{2} \leq \left( \frac{\sqrt{e}}{N + 1} + \frac{\sqrt{2\pi}}{m} \right) e^\frac{a^2}{2}.
\]

Otherwise, we have \( \frac{N+1}{m} - 1 > 0 \). Using the fact that \( u \mapsto u^{N+1/m-1} \) is increasing, we see that

\[
\int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-(t + a)\frac{N+1}{m}-1} dt \leq \left( \frac{3a}{2} \right)^{\frac{N+1}{m}-1} \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{\frac{t^2}{2}} dt \leq \sqrt{2\pi} \left( \frac{3a}{2} \right)^{\frac{N+1}{m}-1}.
\]

For the same reason we also have

\[
\int_{\frac{a}{2}}^{+\infty} e^{-(t + a)\frac{N+1}{m}-1} dt \leq \int_{\frac{a}{2}}^{+\infty} e^{\frac{t^2}{2}} (3t)^{\frac{N+1}{m}-1} dt \leq 3^{\frac{N+1}{m}-1} \int_{0}^{+\infty} t^{\frac{N+1}{m}-1} e^{-\frac{t^2}{2}} dt = \frac{\sqrt{2}}{2} \left( 3\sqrt{2} \right)^{\frac{N+1}{m}-1} \Gamma \left( \frac{N + 1}{2m} \right).
\]

In the case when \( 1 - a < -\frac{a}{2} \) (or equivalently \( a > 2 \)),

\[
\int_{1-a}^{-\frac{a}{2}} e^{-(t + a)\frac{N+1}{m}-1} dt \leq \left( \frac{a}{2} \right)^{\frac{N+1}{m}-1} \int_{1-a}^{-\frac{a}{2}} e^{\frac{t^2}{2}} dt \leq \left( \frac{a}{2} \right)^{\frac{N+1}{m}-1} \int_{1-a}^{-\frac{a}{2}} e^{\frac{a^2}{8}} dt \leq 2 \left( \frac{a}{2} \right)^{\frac{N+1}{m}-1} \frac{e^{a^2}}{a^2}.
\]

It follows that there exists a constant \( C = C(m, N) > 0 \) such that

\[
\int_{1}^{\infty} e^{-\frac{1}{2}(t-a)^2} t^{\frac{N+1}{m}-1} dt = \int_{1-a}^{\infty} e^{-\frac{t^2}{2}} (t + a)^{\frac{N+1}{m}-1} dt \leq C (1 + a)^{\frac{N+1}{m}-1}
\]

for \( \frac{N+1}{m} - 1 > 0 \). It is then easy to find another positive constant \( C = C(m, N) \), independent of \( a \), such that

\[
I(a) \leq C (1 + a)^{\frac{N+1}{m}-1} e^{\frac{a^2}{2}}.
\]
for all \( a \geq 1 \) and \( \frac{N+1}{m} - 1 > 0 \). Therefore,
\[
\int_0^\infty e^{-\frac{1}{2}r^2m + ar^d} r^N \, dr \leq C (1 + a)^{\max(0, \frac{N+1}{m} - 1)} e^{\frac{a^2}{2}}
\]
(8)
for all \( a \geq 1 \). Since \( I(a) \) is increasing in \( a \), the estimate above holds for \( 0 \leq a \leq 1 \) as well.

**Lemma 4.** For any \( m > 0, \delta > 0, R \geq 1, N > -1, \) and \( p \geq 0, \) we can find a constant \( C > 0 \) (depending on \( R, \delta, p, N, m \) but not on \( a, d, x \)) such that
\[
x^{N+1-p} \int_\frac{R}{x}^\infty e^{-\frac{2m}{2} + a \delta r^d} r^N \, dr 
\leq C (1 + a)^{\max(0, \frac{N+1}{m} - 1)} e^{\frac{1}{2}a^2} a^2
\]
and
\[
x^{m} \int_\frac{R}{x}^\infty e^{-\frac{2m}{2} + a \delta r^d} r^m \, dr 
\leq C(1 + a)e^{\frac{a^2}{2}}
\]
for all \( x > 0, a > 0, \) and \( 0 \leq d \leq m. \)

**Proof.** Let \( I = I(m, N, p, R, x, a, d) \) denote the first integral that we are trying to estimate. If \( x \geq 1, \) we have
\[
I = x^{N+1-p} e^{-\frac{x^m}{2} + ax^d} \int_\frac{R}{x}^\infty e^{-\frac{(rx)^2}{2} + a\delta(rx)^d} r^N \, dr 
\leq x^{-p} e^{-\frac{x^m}{2} + ax^m} \int_\frac{R}{x}^\infty e^{-\frac{2m}{2} + a \delta r^d} r^N \, dr 
\leq e^{-\frac{1}{2}(x^m-a)^2 + \frac{a^2}{2}} \int_\frac{R}{x}^\infty \frac{r^p}{R^p} e^{-\frac{2m}{2} + a \delta r^d} r^N \, dr 
\leq \frac{e^{\frac{a^2}{2}}}{R^p} \int_\frac{R}{x}^\infty e^{-\frac{2m}{2} + a \delta r^d} r^{N+p} \, dr.
\]
The desired result then follows from (8).

If \( 0 < x < 1, \) we have
\[
I = x^{N+1-p} e^{-\frac{x^m}{2} + ax^d} \int_\frac{R}{x}^\infty e^{-\frac{(rx)^2}{2} + a\delta(rx)^d} r^N \, dr 
\leq e^a x^{-p} \int_\frac{R}{x}^\infty e^{-\frac{2m}{2} + a \delta r^d} r^N \, dr 
\leq \frac{e^{\frac{a^2}{2} + 1}}{R^p} \int_\frac{R}{x}^\infty e^{-\frac{2m}{2} + a \delta r^d} r^{N+p} \, dr.
\]
The desired estimate follows from (8) again.
To prove the second part of the lemma, denote by \( J = J(m, d, R, x, a) \) the second integral that we are trying to estimate. Then it is clear from a change of variables that for \( 0 < x < 1 \) we have

\[
J(m, d, R, x, a) = x^{m-1} \int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^m - r^m)^2 + a(x^d - r^d)} \frac{m}{r} \, dr
\]

\[
\leq \frac{e^a}{R} x^{m-1} \int_{\frac{R}{x}}^{+\infty} e^{-\frac{1}{2}(x^m - 2x^d + r^2m)} \frac{m}{r} \, dr
\]

\[
\leq \frac{e^a}{R} \int_{0}^{+\infty} e^{-\frac{r^2m}{2} + r^2m} \frac{m}{r} \, dr
\]

\[
= Ce^a \leq C'(1 + a)e^{\frac{a^2}{2}},
\]

where the constants \( C \) and \( C' \) only depend on \( R \) and \( m \).

Next assume that \( x \geq 1 \). In case \( R \leq x^2 \) we write \( J = J_1 + J_2 \), where

\[
J_1 = J_1(m, d, R, x, a) = x^{m} \int_{\frac{1}{x}}^{1} e^{-\frac{x^2m}{2}(1-r^m)^2 + ax^d(1-r^d)} \frac{m}{r} \, dr,
\]

and

\[
J_2 = J_2(m, d, R, x, a) = x^{m} \int_{1}^{+\infty} e^{-\frac{x^2m}{2}(1-r^m)^2 + ax^d(1-r^d)} \frac{m}{r} \, dr.
\]

Otherwise we just use \( J \leq J_2 \). So it suffices to estimate the two integrals above.

To handle \( J_1(m, d, R, x, a) \), we fix \( \varepsilon > 0 \) and consider two cases. In the case \( x^m \leq a(1 + \varepsilon) \), we have

\[
J_1(m, d, R, x, a) \leq x^{m} \int_{\frac{1}{x}}^{1} e^{-\frac{x^2m}{2}(1-r^m)^2 + ax^d(1-r^d)} \frac{m}{r} \, dr
\]

\[
\leq a(1 + \varepsilon)e^{\frac{a^2}{2}} \int_{\frac{1}{x}}^{1} e^{-\frac{1}{2}(x^m(1-r^m) - a)^2} \frac{m}{r} \, dr
\]

\[
\leq a(1 + \varepsilon)e^{\frac{a^2}{2}}.
\]

When \( x^m \geq a(1 + \varepsilon) \), we set \( y = x^m \) and \( \tau = (y - a)/2 \). Then we have

\[
\tau \geq \frac{\varepsilon}{2(1 + \varepsilon)} y \to +\infty
\]
as \( y \to +\infty \). By successive changes of variables we see that

\[
J_1(m, d, R, x, a) \leq x^m \int_{\frac{R}{x^2}}^1 e^{-x^2 (1-r^m)^2 + ax^m (1-r^m)} \frac{r}{x^2} \, dr
\]

\[
= \frac{y}{m} \int_0^{1-\frac{R^m}{y^2}} (1-r)^{\frac{1}{m} - \frac{1}{2}} e^{-\frac{r^2}{2} + ayr} \, dr
\]

\[
= \frac{1}{m} \int_0^{y-\frac{R^m}{y}} (1-r)^{\frac{1}{m} - \frac{1}{2}} e^{-\frac{r^2}{2} + ar} \, dr
\]

\[
= \frac{e^{\frac{a^2}{m}}}{m} \int_{-a}^{y-a-\frac{R^m}{y}} (1-a-y-r)^{\frac{1}{m} - \frac{1}{2}} e^{-\frac{r^2}{2}} \, dr.
\]

This shows that for \( 1 \leq m \leq 2 \) we have

\[
J_1 \leq \frac{e^{\frac{a^2}{m}}}{m} \int_{-a}^{y-a-\frac{R^m}{y}} e^{-\frac{r^2}{2}} \, dr \leq \frac{\sqrt{2\pi}}{m} e^{\frac{a^2}{m^2}}.
\]

Thus we suppose that \( m > 2 \). Then

\[
\int_{-\tau}^{\tau} \left( 1 - \frac{a}{y} - \frac{r}{y} \right)^{\frac{1}{m} - \frac{1}{2}} e^{-\frac{r^2}{2}} \, dr \leq \left( 1 - \frac{a}{y} - \frac{\tau}{y} \right)^{\frac{1}{m} - \frac{1}{2}} \int_{-\tau}^{\tau} e^{-\frac{r^2}{2}} \, dr
\]

\[
= \left( \frac{\tau}{2y} \right)^{\frac{1}{m} - \frac{1}{2}} \int_{-\tau}^{\tau} e^{-\frac{r^2}{2}} \, dr
\]

\[
\leq \sqrt{2\pi} \left( \frac{\varepsilon}{4(1+\varepsilon)} \right)^{\frac{1}{m} - \frac{1}{2}}.
\]

Moreover, in case \(-a < -\tau\), we have

\[
\int_{-\tau}^{-a} \left( 1 - \frac{a}{y} - \frac{r}{y} \right)^{\frac{1}{m} - \frac{1}{2}} e^{-\frac{r^2}{2}} \, dv \leq \left( 1 - \frac{a}{y} + \frac{\tau}{y} \right)^{\frac{1}{m} - \frac{1}{2}} \int_{-\tau}^{-a} e^{-\frac{r^2}{2}} \, dr
\]

\[
\leq 2 \left( \frac{3\varepsilon}{2(1+\varepsilon)} \right)^{\frac{1}{m} - \frac{1}{2}} e^{-\frac{\tau^2}{2}}
\]

\[
\leq 4 \left( \frac{3}{2} \right)^{\frac{1}{m} - \frac{1}{2}} \left( \frac{\varepsilon}{1+\varepsilon} \right)^{\frac{1}{m} - \frac{1}{2}} e^{-\frac{\varepsilon^2}{8(1+\varepsilon)^2}}.
\]
Similarly, in case \( y - a - \frac{R^m}{y} \geq \tau \), we have
\[
\int_{\tau}^{y-a-R^m/y} \left[ 1 - \frac{a}{y} - \frac{r}{y} \right]^{-\frac{1}{2}} e^{-\frac{r^2}{2}} dr \leq \left[ \frac{R^m}{y^2} \right]^{-\frac{1}{2}} \int_{\tau}^{y-a-R^m/y} e^{-\frac{r^2}{2}} dr
\]
\[
\leq 2R^{1-m} \left[ \frac{\varepsilon}{2(1+\varepsilon)} \right]^{\frac{2}{m}-1} \tau^{-\frac{2}{m}} e^{-\frac{\varepsilon^2}{2}}
\]
\[
\left( \text{since } \tau \geq \frac{\varepsilon}{2(1+\varepsilon)} \right) \leq 4R^{1-m} \frac{1+\varepsilon}{\varepsilon} e^{-\frac{\varepsilon^2}{8(1+\varepsilon)^2}}.
\]
The last three estimates yield
\[
J_1 \leq C(1+a)e^{\frac{a^2}{2}}
\]
for some \( C > 0 \) that is independent of \( x \) and \( a \).

To establish the estimate for \( J_2 \), we perform a change of variables to obtain
\[
J_2 \leq x^m \int_{1}^{+\infty} e^{-\frac{2m}{m-1} \left( 1-r^m \right) r} dr = \frac{1}{m} \int_{0}^{+\infty} e^{-\frac{2}{m} \left( \frac{r}{x^m} + 1 \right)} \frac{1}{m-1} \frac{1}{2} dr.
\]
If \( m \geq 2 \), we have
\[
J_2 \leq \frac{1}{m} \int_{0}^{+\infty} e^{-\frac{2}{m} \left( r + 1 \right)} \frac{1}{m-1} \frac{1}{2} dr,
\]
and if \( 1 \leq m < 2 \), we have
\[
J_2 \leq \frac{1}{m} \int_{0}^{+\infty} e^{-\frac{2}{m} \left( r + 1 \right)} \frac{1}{m-1} \frac{1}{2} dr.
\]
Therefore, \( J_2 \leq C \) for some \( C > 0 \) that is independent of \( x \) and \( a \). This completes the proof of the lemma.

In the proof of the main theorem, we will have to estimate the following two integrals:
\[
I(x, r) = \int_{|\theta| \leq \frac{\pi}{2m}} e^{-(xr)^m + 2ar^d \sin^2 \left( \frac{d\theta}{2m} \right)} |K_m(x, re^{i\theta})| d\theta,
\]
and
\[
J(x, r) = \int_{|\theta| \geq \frac{\pi}{2m}} e^{-(xr)^m + a(x^d + r^d)} |K_m(x, re^{i\theta})| d\theta,
\]
where \( x, r, a \in (0, +\infty) \) and \( 0 \leq d \leq m \).

**Lemma 5.** For any \( m > 0 \) there exist positive constants \( C = C(m) \) and \( R = R(m) \) such that
\[
I(x, r) \leq C(xr)^{m-1} \int_{0}^{1} e^{-((xr)^m - ar^d)t^2} dt
\]
and
\[ J(x, r) \leq \frac{Ce^{-(mx^d+ar^d)}}{xr} \]
for all \( a > 0, \) \( 0 \leq d \leq m, \) and \( x > 0 \) with \( xr > R. \)

**Proof.** It follows from Lemma 1 that there exist positive constants \( C = C(m) \) and \( R = R(m) \) such that for all \( a > 0 \) and \( xr > R \) we have
\[
I(x, r) \leq C(xr)^{m-1} \int_{|\theta| \leq \frac{\pi}{2m}} e^{-(mx^d+ar^d)d\theta} \sin^2\left(\frac{\theta d}{2}\right) d\theta
\]
\[
= 2C(xr)^{m-1} \int_{0}^{\frac{\pi}{2m}} e^{-2(mx^d+ar^d)\sin^2\left(\frac{\theta d}{2}\right)} \cos^2\left(\frac{\theta d}{2}\right) d\theta
\]
\[
\leq 2C(xr)^{m-1} \int_{0}^{\frac{\pi}{2m}} e^{-2(mx^d+ar^d)\sin^2\left(\frac{\theta d}{2}\right)} d\theta
\]
\[
= \frac{4C}{m} (xr)^{m-1} \int_{0}^{\frac{\pi}{2m}} e^{-2((mx^d+ar^d)t^2)\theta} \sin\left(\frac{\theta d}{2}\right) d\theta
\]
\[
\leq \frac{4\sqrt{2}C}{m} (xr)^{m-1} \int_{0}^{\frac{\pi}{2m}} e^{-2((mx^d+ar^d)t^2)\theta} \sin\left(\frac{\theta d}{2}\right) d\theta
\]
\[
\leq \frac{4\sqrt{2}C}{m} (xr)^{m-1} \int_{0}^{\frac{\pi}{2m}} e^{-2((mx^d+ar^d)t^2)\theta} d\theta
\]
The estimate
\[ J(x, r) \leq \frac{Ce^{-(mx^d+ar^d)}}{xr}, \quad xr > R, \]
also follows from Lemma 1. \( \square \)

**Lemma 6.** For any \( m \geq 1 \) there exist constants \( R = R(m) > 1 \) and \( C = C(m) > 0 \) such that
\[
\int_{\frac{\pi}{2}}^{\pi} e^{-\frac{1}{2}(mx^d-\theta^2) + a(x^d+y^d)} I(x, r) \theta d\theta \leq C (1 + a)^{\frac{m-1}{2}} e^{a^2}
\]
and
\[
\int_{\frac{\pi}{2}}^{\pi} e^{-\frac{1}{2}(mx^d-\theta^2)} J(x, r) \theta d\theta \leq C (1 + a)^{\max(0, \frac{m-1}{2})} e^{a^2}
\]
for all \( x > 0, a > 0, \) and \( 0 \leq d \leq m. \)

**Proof.** For convenience we write
\[ A_f(x, r) = e^{-\frac{1}{2}(mx^d-\theta^2) + a(x^d+y^d)} I(x, r), \]
and

\[ A_J(x, r) = e^{-\frac{1}{2}(x^m - r^m)^2} J(x, r). \]

Let \( R \) and \( C \) be the constants from Lemma 5. In the integrands we have \( r > R/x \), or \( xr > R \), so according to Lemma 5,

\[ I(x, r) \leq C(xr)^{m-1} \int_0^1 e^{-(xr)^m t^2 + ar^2 t^2} \, dt. \]

If, in addition, \( x \leq 1 \), then

\[ I(x, r) \leq Cr^{m-1}e^{ar}, \]

and

\[ A_I(x, r) = e^{-\frac{1}{2}(x^m - r^m)^2} e^{ax - ar} I(x, r) r \leq Cr^{m} e^{-\frac{1}{2}(x^m - r^m)^2} e^{ax - ar} I(x, r) r. \]

It follows that

\[
\int_R^\infty A_I(x, r) \, dr \leq C e^a \int_R^\infty r^m e^{-\frac{1}{2}(x^m - r^m)^2} \, dr \\
\leq C e^a \int_0^\infty r^m e^{-\frac{1}{2}x^{2m} + x^{m}r^{m} - \frac{1}{2}r^{2m}} \, dr \\
\leq C e^a \int_0^\infty r^m e^{-\frac{1}{2}r^{2m}} \, dr \\
\leq C' (1 + a) \frac{1}{m-1} e^a^2.
\]

for all \( a > 0 \) and \( 0 < x \leq 1 \).

Similarly, if \( x \leq 1 \) (and \( xr > R \)), we deduce from Lemma 5 and (8) that

\[
\int_R^\infty A_J(x, r) \, dr \leq \frac{C}{R} \int_R^\infty e^{-\frac{1}{2}(x^m - r^m)^2} e^{-(xr)^m + ax - ar} \, dr \\
\leq \frac{Ce^a}{R} \int_R^\infty e^{-\frac{1}{2}r^{2m} + ar^2} \, dr \\
\leq C'(1 + a)^{\max(0, \frac{1}{m}-1)} e^a^2.
\]

Suppose now that \( x \geq 1 \) and \( rx > R \). By Lemma 5 again,

\[ A_I(x, r) \leq Cr(rx)^{m-1} e^{-\frac{1}{2}(x^m - r^m)^2 + a(x^d - r^d)} \int_0^1 e^{-t^2((rx)^m - ar^d)} \, dt. \]
Fix a sufficiently small $\varepsilon \in (0, 1)$. If $(xr)^m \geq ar^d(1 + \varepsilon)$, then
\[
\int_0^1 e^{-t^2(xr)^m - ar^d} \, dt = \frac{1}{\sqrt{(xr)^m - ar^d}} \int_0^{\sqrt{(xr)^m - ar^d}} e^{-s^2} \, ds \\
\leq \frac{1}{\sqrt{(xr)^m - ar^d}} \int_0^\infty e^{-s^2} \, ds \\
= \frac{\sqrt{\pi}}{2} \frac{(x)^{-\frac{m}{2}}}{\sqrt{1 - (ar^d/(xr)^m)}} \\
\leq \frac{\sqrt{\pi(1 + \varepsilon)}}{4\varepsilon} (x)^{-\frac{m}{2}},
\]
so there exists a constant $C = C(m)$ such that
\[
A_f(x, r) \leq C r^{\frac{m}{2} - 1} e^{-\frac{1}{2}((x^m - r^m) + a(x^d - r^d)}.
\]
If $(xr)^m \leq ar^d(1 + \varepsilon)$, we have
\[
A_f(x, r) \leq a^{m-1} \int_0^1 e^{-\frac{1}{2}((x^m - r^m) + a(x^d - r^d)} \, dt \\
\leq a^{m-1} \int_0^1 e^{-\frac{1}{2}((x^m - r^m) + a(x^d - r^d)} \, dt.
\]
It follows that
\[
\int_{\mathbb{R}}^+ A_f(x, r) \, dr \leq a^{m-1} \int_{\mathbb{R}}^+ e^{-\frac{1}{2}((x^m - r^m) + a(x^d - r^d)} \, dr \\
+ a^{m-1} \int_{\mathbb{R}}^+ r^{\frac{d(m-1)+m}{m}} e^{-\frac{1}{2}((x^m + r^2) + a(x^d + \varepsilon r^d)} \, dr.
\]
The change of variables $r \mapsto xr$ along with the second part of Lemma 4 shows that
\[
x^{\frac{m}{2} - 1} \int_{\mathbb{R}}^+ e^{-\frac{1}{2}((x^m - r^m) + a(x^d - r^d)} \, dr \leq C(1 + a)e^{\frac{a^2}{2}}.
\]
Similarly, the change of variables $r \mapsto xr$ together with the first part Lemma 4 shows that
\[
\int_{\mathbb{R}}^+ r^{\frac{d(m-1)+m}{m}} e^{-\frac{1}{2}((x^m + r^2) + a(x^d + \varepsilon r^d)} \, dr \leq C(1 + a)^{\frac{d(m-1)+1}{m}} e^{\frac{1+a^2}{2} a^2}.
\]
We may assume that $\varepsilon < 1$. Then we can find a positive constant $C$ such that
\[
a^{m-1} \int_{\mathbb{R}}^+ r^{\frac{d(m-1)+m}{m}} e^{-\frac{1}{2}((x^m + r^2) + a(x^d + \varepsilon r^d)} \, dr \leq C(1 + a)^{\frac{1}{m}-1} e^{a^2}.
\]
It follows that
\[
\int_{\mathbb{R}_+} A_{f}(x, r) \, dr \leq C \left(1 + a \right)^{\frac{1}{m} - 1} e^{a^2}
\]
for some other positive constant \(C\) that is independent of \(a\) and \(x\). This proves the first estimate of the lemma.

To establish the second estimate of the lemma, we use Lemma 5 to get
\[
x A_{f}(x, xr) = x^2 e^{-\frac{2m}{2}(1-r^m)} J(x, xr) \leq C e^{-\frac{2m}{2}(1+r^m) + a x d(1+r^d)}.
\]
It follows from this and Lemma 4 that
\[
\int_{\mathbb{R}_+} A_{f}(x, r) \, dr = x \int_{\mathbb{R}_+} A_{f}(x, xr) \, dr \leq C(1 + a)^{\max(0, \frac{1}{m} - 1)} e^{a^2}.
\]
This completes the proof of the lemma.

**Lemma 7.** If \(u(z) = e^{g(z)}\) and \(v(z) = e^{-g(z)}\), where \(g\) is a polynomial of degree at most \(m\), then the operator \(T = T_u T_v\) is bounded on \(F^2_m\).

**Proof.** To prove the boundedness of \(T = T_u T_v\), we shall use a standard technique known as Schur’s test [18, p.42]. Since
\[
T f(z) = \int_{\mathbb{C}} K_m(z, w) e^{g(z) - g(w)} f(w) e^{-\frac{1}{2} |w|^2 m} \, dA(w),
\]
we have
\[
|T f(z)| e^{-\frac{1}{2} |z|^2 m} \leq \int_{\mathbb{C}} H_g(z, w) |f(w)| e^{-\frac{1}{2} |w|^2 m} \, dA(w),
\]
where
\[
H_g(z, w) := |K_m(z, w)| e^{-\frac{1}{2} (|z|^2 m + |w|^2 m) + \operatorname{Re}(g(z) - g(w))}.
\]
Thus \(T\) will be bounded on \(F^2_m\) if the integral operator \(S_g\) defined by
\[
S_g f(z) = \int_{\mathbb{C}} (H_g(z, w) + H_g(w, z)) f(w) \, dA(w)
\]
is bounded on \(L^2(\mathbb{C}, dA)\). Let
\[
H_g(z) = \int_{\mathbb{C}} H_g(z, w) \, dA(w), \quad z \in \mathbb{C}.
\]
Since
\[
H_{-g}(z) = \int_{\mathbb{C}} H_g(w, z) \, dA(w),
\]
for all \(z \in \mathbb{C}\), by Schur’s test, the operator \(S_g\) is bounded on \(L^2(\mathbb{C}, dA)\) if we can find a positive constant \(C\) such that
\[
H_g(z) + H_{-g}(z) \leq C, \quad z \in \mathbb{C}.
\]
By the Cauchy-Schwarz inequality, we have

\[ H_{g_1+g_2}(z) \leq \sqrt{H_{2g_1}(z)H_{2g_2}(z)} \]

for all \( z \in \mathbb{C} \) and holomorphic polynomials \( g_1 \) and \( g_2 \). Moreover, if

\[ U_\theta(z) = e^{i\theta}z, \quad z \in \mathbb{C}, \theta \in [-\pi, \pi], \]

then

\[ H_{g \circ U_\theta} = H_g \circ U_\theta \]

for all \( z \in \mathbb{C}, \theta \in [-\pi, \pi] \), and holomorphic polynomials \( g \). Therefore, we only need prove the theorem for \( g(z) = az^d \) with some \( a > 0 \) and \( d \leq m \) and establish that

\[ \sup_{x \geq 0} H_g(x) \leq C_1 e^{C_2 a^2}, \quad (9) \]

where \( C_k \) are positive constants independent of \( a \) and \( d \) (but dependent on \( m \)). We will see that \( C_2 \) can be chosen as any constant greater than 1.

It is also easy to see that we only need to prove (9) for \( x \geq 1 \). This will allow us to use the inequality \( x^d \leq x^m \) for the rest of this proof.

For \( R > 0 \) sufficiently large (we will specify the requirement on \( R \) later) we write

\[ H_g(x) = \int_{|xw| \leq R} H_g(x, w) \, dA(w) + \int_{|xw| \geq R} H_g(x, w) \, dA(w). \]

We will show that both integrals are, up to a multiplicative constant, bounded above by \( e^{(1+\varepsilon)a^2} \).

By properties of the Mittag-Leffler function, we have

\[ |K_m(x, w)| \leq \frac{m}{\pi} E_{\frac{1}{m} + \frac{1}{m}}(R) := C_R, \quad |xw| \leq R. \]

It follows that the integral

\[ I_1 = \int_{|xw| \leq R} H_g(x, w) \, dA(w) \]
satisfies
\[ I_1 = \int_{|w| \leq R} |K_m(z, w)| e^{-\frac{1}{2}(|z|^{2m} + |w|^{2m}) + a \text{Re}(x^d - w^d)} \, dA(w) \]
\[ \leq C_R \int_{|w| \leq R} e^{-\frac{1}{2}(|z|^{2m} + |w|^{2m}) + a \text{Re}(x^d - w^d)} \, dA(w) \]
\[ \leq C_R e^{-\frac{1}{2}x^{2m} + ax^d} \int_{|w| \leq R} e^{-\frac{1}{2}w^{2m} + a|w|^d} \, dA(w) \]
\[ \leq 2\pi C_R e^{-\frac{1}{2}x^{2m} + ax^d} \int_0^{+\infty} e^{-\frac{1}{2}r^{2m} + ar^d} r \, dr \]
\[ \leq 2\pi C_R e^{-\frac{1}{2}x^{2m} + ax^d} \int_0^{+\infty} e^{-\frac{1}{2}r^{2m} + ar^d} r \, dr \]
\[ \leq C(1 + a)^{\max(0, \frac{1}{2} - 1)} e^{e^2}, \]

where the last inequality follows from (8).

We now focus on the integral
\[ I_2 = \int_{|w| \geq R} H_g(x, w) \, dA(w). \]

Observe that for all \( x, r, \) and \( \theta \) we have
\[ \text{Re}\left(x^d - r^d e^{i\theta}\right) = x^d - r^d \cos(d\theta) \]
\[ = x^d - r^d + r^d (1 - \cos(d\theta)) \]
\[ = x^d - r^d + 2r^d \sin^2\left(\frac{d\theta}{2}\right). \]

It follows from polar coordinates that
\[ I_2 = \int_{R}^{+\infty} \int_{-\pi}^{\pi} H_g(x, re^{i\theta}) r \, d\theta \, dr \]
\[ = \int_{R}^{+\infty} \int_{-\pi}^{\pi} e^{-\frac{1}{2}(x^{2m} + r^{2m}) + a(x^d - r^d \cos(d\theta))} |K_m(x, re^{i\theta})| r \, d\theta \, dr \]
\[ = \int_{R}^{+\infty} e^{-\frac{1}{2}(x^{2m} - r^{2m}) + a(x^d - r^d - (xr)^m)} r \, dr \int_{-\pi}^{\pi} e^{2ar^d \sin^2\left(\frac{d\theta}{2}\right)} |K_m(x, re^{i\theta})| \, d\theta \]
\[ \leq \int_{R}^{+\infty} e^{-\frac{1}{2}(x^{2m} - r^{2m})} r \left(e^{a(x^d - r^d)} I(x, r) + J(x, r)\right) \, dr, \]

where
\[ I(x, r) = \int_{|\theta| \leq \frac{\pi}{2m}} e^{-(xr)^m + 2ar^d \sin^2\left(\frac{d\theta}{2}\right)} |K_m(x, re^{i\theta})| \, d\theta, \]
and

\[ J(x, r) = \int_{|\theta| \geq \frac{\pi}{2m}} e^{-(x^2 + r^2)|K_m(x, re^{i\theta})|} d\theta. \]

By Lemma 6, there exists another constant \( C > 0 \) such that

\[ I_2 \leq C(1 + a)^{\text{max}(0, \frac{2}{m} - 1)}e^{a^2}. \]

Therefore,

\[ \sup_{z \in \mathbb{C}} \int_{\mathbb{C}} H_g(z, w) dA(w) \leq C(1 + a)^{\text{max}(0, \frac{2}{m} - 1)}e^{a^2} \]

for yet another constant \( C \) that is independent of \( a \) and \( d \). Similarly, we also have

\[ \sup_{z \in \mathbb{C}} \int_{\mathbb{C}} H_{-g}(z, w) dA(w) \leq C(1 + a)^{\text{max}(0, \frac{2}{m} - 1)}e^{a^2} \]

This yields (9) and proves the lemma. \( \square \)

4. Sarason’s Conjecture for \( F^2_m \)

In this section we show that Sarason’s conjecture is true for Toeplitz products on the Fock type space \( F^2_m \). More specifically, we will prove that condition (3) in the main theorem stated in the introduction is equivalent to conditions (1) and (2). Again we will break the proof down into several lemmas.

Lemma 8. Suppose \( u \) and \( v \) are functions in \( F^2_m \), not identically zero, such that the operator \( T_uT_v = T_uT_v^*T_vT_u \) is bounded on \( F^2_m \). Then the function \( |u|^2(z)|v|^2(z) \) is bounded on the complex plane.

Proof. Since \( T_uT_v \) is bounded on \( F^2_m \), the operator \( (T_uT_v)^*T_uT_v \) and the products \( (T_uT_v)^*T_uT_v \) are also bounded on \( F^2_m \). Consequently, their Berezin transforms are all bounded functions on \( \mathbb{C} \).

For any \( z \in \mathbb{C} \) we let \( k_z \) denote the normalized reproducing kernel of \( F^2_m \) at \( z \). Then

\[ \langle (T_uT_v)^*T_uT_vk_z, k_z \rangle = \langle T_uT_vk_z, T_uT_vk_z \rangle \]

\[ = \langle uv(z)k_z, uv(z)k_z \rangle \]

\[ = |u(z)|^2 |v(z)|^2 \]

is bounded on \( \mathbb{C} \). Similarly \( |u(z)|^2 |v|^2(z) \) is bounded on \( \mathbb{C} \). By the proof of Lemma 8, the product \( uv \) is a non-zero complex constant, say, \( u(z)v(z) = C \). It follows that the function

\[ |v|^2(z)|u|^2(z) = |u(z)|^2 |v|^2(z) |v(z)|^2 \frac{1}{|C|^2} \]
is bounded as well.

To complete the proof of Sarason’s conjecture, we will need to find a lower bound for the function

\[ B(z) = |v|^2(z) |u(z)|^2, \]

where \( u = e^g, v = e^{-g} \), and \( g \) is a polynomial of degree \( d \). We write

\[ g(z) = a_d z^d + g_{d-1}(z), \]

where

\[ a_d = a e^{i\alpha_d}, \quad a > 0, \]

and

\[ g_{d-1}(z) = \sum_{l=0}^{d-1} a_l z^l. \]

In the remainder of this section we will have to handle several integrals of the form

\[ I(x) = \int_J S_x(r) e^{-g_x(r)} dr, \]

where \( S_x \) and \( g_x \) are \( C^2 \)-functions on the interval \( J \), and the real number \( x \) tends to \( +\infty \). We will make use of the following variant of the Laplace method (see [15]).

**Lemma 9.** Suppose that

(a) \( g_x \) attains its minimum at a point \( r_x \), which tends to \( +\infty \) as \( x \) tends to \( +\infty \), with \( c_x = g_x''(r_x) > 0 \);

(b) there exists \( \tau_x \) such that for \( |r - r_x| < \tau_x \), \( g_x''(r) = c_x(1 + o(1)) \) as \( x \) tends to \( +\infty \);

(c) for \( |r - r_x| < \tau_x \), \( S_x(r) \sim S_x(r_x) \);

(d) we have

\[ \int_J S_x(r) e^{-g_x(r)} dr = (1 + o(1)) \int_{|r-r_x|<\tau_x} S_x(r) e^{-g_x(r)} dr. \]

Then we have the following estimate

\[ I(x) = \left( \sqrt{2\pi} + o(1) \right) [c_x]^{-1/2} S_x(r_x) e^{-g_x(r_x)}, \quad x \to +\infty. \] (10)

The computations in [15] ensure that, under the assumptions on \( g_x \) and \( S_x \), we have

\[ \int_{|r-r_x|>\tau_x} S_x(r) e^{-g_x(r)} dr \lesssim (c_x \tau_x)^{-1} \int_{|t|>\tau_x} e^{-\frac{1}{2} c_x t^2} dt. \] (11)

In particular, if one of the two conditions \( c_x \tau_x^2 \to +\infty \) and \( c_x \tau_x \to +\infty \) is satisfied, then hypothesis (d) in Lemma 9 holds.
The study of $\mathcal{B}(z)$ will require some additional technical lemmas.

**Lemma 10.** For $z = xe^{i\phi}$, with $x > 0$ and $e^{i(\alpha_d + d\phi)} = 1$, we have

$$\mathcal{B}(z) \geq \int_0^{+\infty} (r x)^{-\frac{d}{2}} r^{2m-1} e^{-h_x(r)} \, dr$$

as $x \to +\infty$, where

$$h_x(r) = (r^m - x^m)^2 - 2a (x^d - r^d) + C (r^{d-1} + x^{d-1} + 1),$$

for some positive constant $C$.

**Proof.** It is easy to see that

$$\mathcal{B}(z) = \int_{\mathbb{C}} |K_m(w, z)|^2 e^{2\text{Re}(g(z)-g(w))} |K_m(z, z)|^{-1} e^{-|w|^{2m}} \, dA(w),$$

which, in terms of polar coordinates, can be rewritten as

$$\int_0^{+\infty} \int_{-\pi}^{\pi} |K_m(r e^{i\theta}, z)|^2 e^{2\text{Re}(g(z)-g(r e^{i\theta}))} |K_m(x, x)|^{-1} e^{-r^{2m}} r \, dr \, d\theta.$$

By Lemma [10], $\mathcal{B}(z)$ is greater than or equal to

$$\int_0^{+\infty} \int_{|\theta - \phi| \leq \epsilon \theta_0(r_x)} |K_m(r e^{i\theta}, z)|^2 e^{2\text{Re}(g(z)-g(r e^{i\theta}))} |K_m(x, x)|^{-1} e^{-r^{2m}} r \, dr \, d\theta.$$

This together with Lemma [11] shows that

$$\mathcal{B}(z) \geq \int_0^{+\infty} r^{2(m-1)} e^{-(r^m - x^m)^2} I(r, z) r \, dr,$$

where

$$I(r, z) = \int_{|\theta - \phi| \leq \epsilon \theta_0(r_x)} e^{2\text{Re}(g(z)-g(r e^{i\theta}))} \, d\theta.$$

Note that

$$I(r, z) = \int_{|\theta - \phi| \leq \epsilon \theta_0(r_x)} e^{2\text{Re}(a x^{\alpha_d} (x^d - r^d e^{i\theta})) + 2\text{Re}(g_{d-1}(z) - g_{d-1}(r e^{i\theta}))} \, d\theta$$

$$= \int_{|\theta - \phi| \leq \epsilon \theta_0(r_x)} e^{2\text{Re}(a (x^d - r^d e^{i\theta})) + 2\text{Re}(g_{d-1}(z) - g_{d-1}(r e^{i\theta}))} \, d\theta.$$

The condition on $\phi$ yields

$$I(r, z) = \int_{|\theta| \leq \epsilon \theta_0(r_x)} e^{2\text{Re}(a (x^d - r^d e^{i\theta})) + 2\text{Re}(g_{d-1}(z) - g_{d-1}(r e^{i\theta + \phi}))} \, d\theta.$$

Since

$$g_{d-1}(z) - g_{d-1}(r e^{i(\theta + \phi)}) = \sum_{l=0}^{d-1} a_l \left( x^l e^{il\phi} - r^l e^{il(\theta + \phi)} \right),$$
we have
\[ \text{Re} \left[ g_{d-1}(z) - g_{d-1}(re^{i(\theta + \phi)}) \right] \geq -C \left( r^{d-1} + x^{d-1} + 1 \right) \]
for some constant \( C \). It follows that
\[ I(r, z) \geq e^{-C(r^{d-1} + x^{d-1} + 1)} \int_{|\theta| \leq c\theta_0(rx)} e^{2a\text{Re}[x^d re^{i\theta}]} d\theta. \]

For the integral we have
\[ J(r, z) := \int_{|\theta| \leq c\theta_0(rx)} e^{2a\text{Re}[x^d re^{i\theta}]} d\theta \]
\[ = \int_{|\theta| \leq c\theta_0(rx)} e^{2a(x^d - r^d \cos(\theta))} d\theta \]
\[ = \int_{|\theta| \leq c\theta_0(rx)} e^{2a(x^d - r^d + (-\cos(\theta) + 1)r^d)} d\theta \]
\[ \geq e^{2a(x^d - r^d)} \int_{|\theta| \leq c\theta_0(rx)} e^{4|a_d|\sin(\frac{\theta}{2})^2} r^d d\theta \]
\[ \geq e^{2a(x^d - r^d)} \int_{|\theta| \leq c\theta_0(rx)} d\theta \]
\[ \geq e^{2a(x^d - r^d)} (rx)^{-\frac{d}{2}}, \]
which completes the proof of the lemma.

Lemma 11. Assume \( d = 2m \). For \( z = xe^{i\phi} \), where \( x > 0 \) and \( e^{i(\alpha_d + d\phi)} = 1 \), we have
\[ B(z) \geq e^{(1+o(1))\frac{2a}{1+2a}x^{2m}}, \quad x \to +\infty. \]

Proof. For \( x \) large enough, the function \( h_x \) defined in (12) is convex on some interval \([M_x, +\infty)\) and attains its minimum at some point \( r_x \). In order to bound \( B(z) \) from below, we shall use the modified Laplace method from Lemma[9] Since
\[ h_x'(r) = 2mr^{m-1} (r^m - x^m) + 2adr^{d-1} + C(d-1)r^{d-2}, \]
we have
\[ h_x'(r) = 2m(1 + 2a)r^{2m-1} - 2mx^m r^{m-1} + C(d-1)r^{d-2}, \]
and
\[ h_x''(r) = 2m(2m-1)(1+2a)r^{2m-2} - 2m(m-1)x^m r^{m-2} + C(d-1)(d-2)r^{d-3}. \]
Writing $h'_x(r_x) = 0$ and letting $x$ tend to $+\infty$, we obtain

$$m(1 + 2a)(r_x)^{2m-1} \sim mx^m r_x^{m-1},$$
or

$$r_x \sim (1 + 2a)^{-\frac{1}{m}} x.$$ (14)

Thus there exists $\rho_x$, which tends to 0 as $x$ tends to $+\infty$, such that

$$r_x = (1 + 2a)^{-\frac{1}{m}} x(1 + \rho_x).$$ (15)

When $x$ tends to $+\infty$, we have

$$h_x(r_x) \sim (r_x^m - x^m)^2 + 2a (r_x^{2m} - x^{2m})$$
$$\sim (r_x^m - x^m) [(r_x^m - x^m) + 2a (r_x^m + x^m)]$$
$$\sim x^{2m} [(1 + 2a)^{-1}(1 + \rho_x)^m - 1]$$
$$\sim -x^{2m} \frac{2a}{(1 + 2a)}.$$ or

$$-h_x(r_x) \sim x^{2m} \frac{2a}{(1 + 2a)}.$$ (16)

In order to estimate $c_x := h''_x(r_x)$, we compute that

$$h''_x(r_x) \sim 2m^2(1 + 2a)^{-1} + \frac{2}{m} x^{2m-2}.$$ Thus we get

$$c_x \approx x^{2m-2}.$$ (17)

For $r$ in a neighborhood of $r_x$, we set $r = (1 + \sigma_x) r_x$, where $\sigma_x = \sigma_x(r) \to 0$ as $x \to +\infty$; a little computation shows that

$$h''_x(r_x) \sim h''_x(r_x)$$
as $x \to +\infty$. Taking $\tau_x = r_x^{1/2}$ and $|r - r_x| < \tau_x$, we have $h''_x(r) = (1 + o(1)) c_x$, so

$$h_x(r) - h_x(r_x) = \frac{1}{2} c_x (r - r_x)^2 (1 + o(1)).$$

Thus

$$\int_{|r - r_x| < \tau_x} e^{-\frac{1}{2} c_x (r - r_x)^2 (1 + o(1))} dr = \int_{|t| < \tau_x} e^{-\frac{1}{2} c_x t^2 (1 + o(1))} dt$$
$$\sim \frac{1}{\sqrt{c_x}} \int_{|y| < \tau_x \sqrt{c_x}} e^{-\frac{1}{2} y^2} dy$$
$$\approx \frac{1}{\sqrt{c_x}},$$
because \( c_x \tau_x^2 \approx r_x^{2m-1} \) tends to \(+\infty\) as \( x \) tends to \(+\infty\). Finally, the estimates
\[
\mathcal{B}(z) \gtrsim \int_{|r-r_x|<\tau_x} (rx)^{-\frac{m}{2}} r^{2m-1} e^{-h_x(r)} dr
\]
\[
= \int_{|r-r_x|<\tau_x} (rx)^{-\frac{m}{2}} r^{2m-1} e^{-h_x(r_x)} e^{-\frac{1}{2} c_x (r-r_x) (1+o(1))} dr
\]
\[
\approx e^{-h_x(r_x)} r_x^{\frac{3m-1}{2}} x^{-\frac{m}{2}} \int_{|r-r_x|<\tau_x} e^{-\frac{1}{2} c_x (r-r_x) (1+o(1))} dr
\]
\[
\approx e^{-h_x(r_x)} r_x^{\frac{3m-1}{2}} x^{-\frac{m}{2}} \frac{1}{\sqrt{c_x}}
\]
along with (14), (16), and (17) give the lemma. \( \square \)

**Lemma 12.** Assume \( d < 2m \). For \( z = xe^{i\phi} \), with \( x > 0 \) and \( e^{i(\alpha_x + d\phi)} = 1 \), we have
\[
\mathcal{B}(z) \gtrsim e^{(1+o(1)) \frac{x^{2d-2m} - C x^{d-1-m}}{m}} \quad x \to +\infty
\]
for some positive constant \( C \).

**Proof.** Let \( \tau_x = o(x) \) be a positive real number that will be specified later. As in the proof of Lemma 10 we have
\[
\mathcal{B}(z) \gtrsim \int_0^{+\infty} r^{2(m-1)} e^{-(r^m - x^m)^2} I(r, z) r dr
\]
\[
\gtrsim \int_{|r-x|\leq \tau_x} r^{2(m-1)} e^{-(r^m - x^m)^2} I(r, z) r dr,
\]
where
\[
I(r, z) = \int_{|\theta-\phi|\leq \epsilon \theta_0(r_x)} e^{2\Re(g(z)-g(re^{i\theta}))} d\theta.
\]
There exists \( c' > 0 \) such that for \( |r-x| \leq \tau_x \) we have
\[
I(r, z) \geq \int_{|\theta-\phi|\leq c' \theta_0(x^2)} e^{2\Re(g(z)-g(re^{i\theta}))} d\theta
\]
\[
= \int_{|\theta|\leq c' \theta_0(x^2)} e^{2a \Re(x^d - r^d e^{i\theta}) + 2\Re(g_{d-1}(z) - g_{d-1}(re^{i\theta}))} d\theta
\]
\[
= \int_{|\theta|\leq c' \theta_0(x^2)} e^{2a \Re(x^d - r^d e^{i\theta}) - 2 \sum_{l=0}^{d-1} |a_l| |x^l - r^l e^{i\theta}|} d\theta.
\]
Now for $|r - x| \leq \tau_x$, we write $r = (1 + \sigma)x$, where $\sigma$ tends to 0 as $x \to +\infty$. Thus for $0 \leq l \leq d - 1$ and $|\theta| \leq c'\theta_0(x^2)$, we obtain

$$|x^l - r^l e^{i\theta}|^2 = x^{2l} \left[ 1 - 2(1 + \sigma)^l \cos(\theta) + (1 + \sigma)^{2l} \right]$$

$$= x^{2l} \left[ 1 - 2 \left( 1 + l\sigma + O(\sigma^2) \right) \cos(\theta) + 1 + 2l\sigma + O(\sigma^2) \right]$$

$$\lesssim x^{2l} \left[ \sin^2 \left( \frac{l\theta}{2} \right) + \sigma^2 \right]$$

$$\lesssim x^{2l} \left[ \theta^2 + \sigma^2 \right].$$

Next choosing $|\sigma| \leq x^{-m}$, we get

$$|x^l - r^l e^{i\theta}| \lesssim x^{2l} x^{-2m} \lesssim x^{2(d-1) - 2m}$$

or

$$|x^l - r^l e^{i\theta}| \lesssim x^{d-1-m}.$$
where \( \rho_x \) tends to 0 as \( x \to +\infty \). Using the fact that \( h'_x(r_x) = 0 \), we have
\[
2m x^{2m-1}(1 + \rho_x)^{m-1} [(1 + \rho_x)^m - 1] \sim -2ad x^{d-1}(1 + \rho_x)^{d-1},
\]
and
\[
2m x^{2m-1} \rho_x \sim -2ad x^{d-1}.
\]
Therefore,
\[
\rho_x \sim \frac{ad}{m^2} x^{d-2m}.
\] (19)

Since
\[
h''_x(r) = 2m(2m-1)r^{2m-2} - 2m(m-1)x^m r^{m-2}
\]
and \( d < 2m \), we get
\[
h''_x(r_x) \sim 2m x^{2m-2} \left[ (2m-1)(1 + \rho_x)^{2m-2} - (m-1)(1 + \rho_x)^{m-2} \right]
\]
\[
\sim 2m^2 x^{2m-2}.
\]

Also,
\[
h_x(r_x) \sim x^{2m} [(1 + \rho_x)^m - 1]^2 + 2ax^d [(1 + \rho_x)^d - 1] + C(x^{d-1} + r_x^{d-1} + 1)
\]
\[
\sim m^2 \rho_x^2 x^{2m} + 2ax^d \rho_x
\]

It follows that
\[
c_x \sim 2m^2 x^{2m-2},
\] (20)

and
\[
-h_x(r_x) \sim \frac{a^2 d^2}{m^2} x^{2d-2m}.
\] (21)

Reasoning as in the proof of Lemma 11 we arrive at
\[
\mathcal{B}(z) \gtrsim x^{-m} \frac{1}{\sqrt{c_x}}
\]

The desired estimate then follows from (21), and (20).

**Lemma 13.** Suppose \( u \) and \( v \) are functions in \( \mathcal{F}_m^2 \), not identically zero, such that \( |u|^2(z)|v|^2(z) \) is bounded on the complex plane. Then there exists a nonzero constant \( C \) and a polynomial \( g \) of degree at most \( m \) such that \( u(z) = e^{g(z)} \) and \( v(z) = C e^{-g(z)} \).

**Proof.** It is easy to check that for \( u \in \mathcal{F}_m^2 \) we have
\[
u(z) = \int_{\mathbb{C}} u(x)|k_x(x)|^2 d\lambda_m(x) = \tilde{u}(z).
\]

Also, it follows from the Cauchy-Schwarz inequality that \( |u(z)|^2 \) \( |v|^2(z) \) is bounded on \( \mathbb{C} \), then \( \mathcal{B}(z) \) and \( |u(z)v(z)|^2 \) are also
bounded. Consequently, $uv$ is a constant, and as in section 3, there is a non-zero constant $C$ and a polynomial $g$ such that $u = e^g$ and $v = Ce^{-g}$. The condition $u \in \mathcal{F}_m^2$ implies that the degree $d$ of $g$ is at most $2m$; see Lemma 2.

Without loss of generality we shall consider the case where $u(z) = e^{g(z)}$ and $v(z) = e^{-g(z)}$. We will show that that the boundedness of $B(z)$ implies $d \leq m$. If $2m$ is an integer, Lemma 11 shows that we must have $d < 2m$. Thus, in any case ($2m$ being an integer or not), a necessary condition is $d < 2m$. The desired result now follows from Lemma 12.

5. Further Remarks

In this final section we specialize to the case $m = 1$ and make several additional remarks. For convenience we will alter notation somewhat here. Thus for any $\alpha > 0$ we let $F^2_\alpha$ denote the Fock space of entire functions $f$ on the complex plane $\mathbb{C}$ such that

$$\int_{\mathbb{C}} |f(z)|^2 \, d\lambda_\alpha(z) < \infty,$$

where

$$d\lambda_\alpha(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2} \, dA(z).$$

Toeplitz operators on $F^2_\alpha$ are defined exactly the same as before using the orthogonal projection $P_\alpha : L^2(\mathbb{C}, d\lambda_\alpha) \to F^2_\alpha$.

Suppose $u$ and $v$ are functions in $F^2_\alpha$, not identically zero. It was proved in [8] that $T_u T_v$ is bounded on the Fock space $F^2_\alpha$ if and only if there is a point $a \in \mathbb{C}$ such that

$$u(z) = be^{\alpha_1 z}, \quad v(z) = ce^{-\alpha_2 z}, \quad (22)$$

where $b$ and $c$ are nonzero constants. This certainly solves Sarason’s problem for Toeplitz products on the space $F^2_\alpha$. But the paper [8] somehow did not address Sarason’s conjecture, which now of course follows from our main result.

We want to make two points here. First, the proof of Sarason’s conjecture for $F^2_\alpha$ is relatively simple after Sarason’s problem is solved. Second, Sarason’s conjecture holds for the Fock space $F^2_\alpha$ for completely different reasons than was originally thought, namely, the motivation for Sarason’s conjecture provided in [14] for the cases of Hardy and Bergman spaces is no longer valid for the Fock space. It is therefore somewhat amusing that Sarason’s conjecture turns out to be true for the Fock space but fails for the Hardy and Bergman spaces.
Suppose \( u \) and \( v \) are given by (22). We have
\[
|f_{k_z}|^2 = \int_C |f(w)e^{\alpha w \bar{z} - (\alpha/2)|z|^2}|^2 \, d\lambda_\alpha(w)
\]
\[
= |b|^2 e^{-\alpha|z|^2} \int_C |e^{\alpha w(\bar{a} + \bar{z})}|^2 \, d\lambda_\alpha(w)
\]
\[
= |b|^2 e^{-\alpha|z|^2 + |\alpha a + \bar{z}|^2}
\]
\[
= |b|^2 e^{\alpha(\alpha |a|^2 + \bar{\pi}z + \alpha \bar{\tau})}.
\]
Similarly,
\[
|v|^2 = |c|^2 e^{\alpha(\alpha |a|^2 - \pi z - \alpha \tau)}.
\]
It follows that
\[
|u|^2(z) \cdot |v|^2(z) = |bc|^2 e^{2\alpha|a|^2}
\]
is a constant and hence a bounded function on \( \mathbb{C} \).

On the other hand, it follows from Hölder’s inequality that we always have
\[
|u(z)|^2 \leq |u|^2(z), \quad u \in F^2_\alpha, z \in \mathbb{C}.
\]
Therefore, if \( |u|^2|v|^2 \) is a bounded function on \( \mathbb{C} \), then there exists a positive constant \( M \) such that
\[
|u(z)v(z)|^2 \leq |u|^2(z)|v|^2(z) \leq M
\]
for all \( z \in \mathbb{C} \). Thus, as a bounded entire function, \( uv \) must be constant, say \( u(z)v(z) = C \) for all \( z \in \mathbb{C} \). Since \( u \) and \( v \) are not identically zero, we must have \( C \neq 0 \). Since functions in \( F^2_\alpha \) must have order less than or equal to 2, we can write \( u(z) = e^{p(z)} \), where
\[
p(z) = az^2 + bz + c
\]
is a polynomial of degree less than or equal to 2. But \( u(z)v(z) \) is constant, so \( v(z) = e^{q(z)} \), where
\[
q(z) = -az^2 - bz + d
\]
is another polynomial of degree less than or equal to 2.

We will show that \( a = 0 \). To do this, we will estimate the Berezin transform \( |u|^2 \) when \( u \) is a quadratic exponential function as given above. More specifically, for \( C_1 = |c|^2 \), we have
\[
|u|^2(z) = C_1 \left| \int_C e^{a(z+w)^2 + b(z+w)} \right|^2 d\lambda_\alpha(w)
\]
\[
= C_1 \left| e^{az^2 + bz} \int_C e^{aw^2 + (b+2az)w} \right|^2 d\lambda_\alpha(w).
\]
Write $b + 2az = \alpha \zeta$. Then it follows from the inequality $|F|^2 \geq |\tilde{F}|^2$ for $F \in \mathbb{F}^2_{\alpha}$ again that
\[
|u|^2(z) = C \left| e^{\alpha z^2 + bz} \right|^2 e^{a|z|^2} \int_{\mathbb{C}} |e^{aw^2} k_{\alpha}(w)|^2 d\lambda_{\alpha}(w)
\geq C \left| e^{\alpha z^2 + bz} \right|^2 e^{a|z|^2} e^{a|z|^2}.\]
If we do the same estimate for the function $v$, the result is
\[
|\tilde{u}|^2(z) \geq C_2 \left| e^{-a z^2 - bz} \right|^2 e^{a|z|^2} e^{-a|z|^2},
\]
where $\zeta$ is the same as before and $C_2 = |e^d|^2$. It follows that
\[
|\tilde{u}|^2(z) |\tilde{v}|^2(z) \geq C_1 C_2 e^{2a|\zeta|^2} = C_1 C_2 e^{2b + 2az|z|^2/\alpha}.
\]
This shows that $|\tilde{u}|^2 |\tilde{v}|^2$ is unbounded unless $a = 0$.

Therefore, the boundedness of $|u|^2 |v|^2$ implies that
\[
u(z) = e^{bz + c}, \quad v(z) = e^{-bz + d}.
\]
By [8], the product $T_u T_{\tau}$ is bounded on $\mathbb{F}^2_{\alpha}$. In fact, $T_u T_{\tau}$ is a constant times a unitary operator.

Combining the arguments above and the main result of [8] we have actually proved that the following conditions are equivalent for $u$ and $v$ in $\mathbb{F}^2_{\alpha}$:

(a) $T_u T_{\tau}$ is bounded on $\mathbb{F}^2_{\alpha}$.
(b) $T_u T_{\tau}$ is a constant multiple of a unitary operator.
(c) $|u|^2 |v|^2$ is bounded on $\mathbb{C}$.
(d) $|u|^2 |v|^2$ is constant on $\mathbb{C}$.

Recall that in the case of Hardy and Bergman spaces, there is actually an absolute constant $C$ (4 for the Hardy space and 16 for the Bergman space) such that
\[
|\tilde{u}|^2(z) |\tilde{v}|^2(z) \leq C \|T_u T_{\tau}\|^2
\]
for all $u$, $v$, and $z$. We now show that such an estimate is not possible for the Fock space. To see this, consider the functions
\[
u(z) = e^{\alpha z}, \quad v(z) = e^{-\alpha z}.
\]
By calculations done in [8], we have
\[
T_u T_{\tau} = e^{a|\alpha|^2/2} W_{\alpha},
\]
where $W_{\alpha}$ is the Weyl unitary operator defined by $W_{\alpha} f(z) = f(z - \alpha) k_{\alpha}(z)$.

On the other hand, by calculations done earlier, we have
\[
|\tilde{u}|^2(z) |\tilde{v}|^2(z) = e^{2a|\alpha|^2}.
\]
It is then clear that there is NO constant $C$ such that
\[ e^{2\alpha|a|^2} \leq C e^{\alpha|a|^2/2} \]
for all $a \in \mathbb{C}$. Therefore, there is NO constant $C$ such that
\[ \sup_{z \in \mathbb{C}} |\overline{u}(z)||\overline{v}(z)| \leq C \|T_u T_v\|^2 \]
for all $u$ and $v$. In other words, the easy direction for Sarason’s conjecture in the cases of Hardy and Bergman spaces becomes difficult for Fock spaces.

REFERENCES

[1] A. Aleman, S. Pott, and C. Reguera, Sarason’s conjecture on the Bergman space, preprint, 2013.
[2] H. Bateman and A. Erdélyi, Higher transcendental functions, Vol. I, McGraw-Hill, New York-Toronto-London, 1953.
[3] H. Bateman and A. Erdélyi, Higher transcendental functions, Vol. III, McGraw-Hill, New York-Toronto-London, 1955.
[4] H. Bommier-Hato, Lipschitz estimates for the Berezin transform, Journ. Funct. Space. Appl. 8, no 2 (2010), 103–128.
[5] H. Bommier-Hato, M. Engliš, and E.-H. Youssfi, Dixmier trace and the Fock space, Bull. Sci. Math. 138 (2014), 199–224.
[6] H. Bommier-Hato and E.H. Youssfi: Hankel operators on weighted Fock spaces, Integ. Eqs. Oper. Theory 59 (2007), 1–17.
[7] H. Bommier-Hato and E. H. Youssfi, Hankel operators and the Stieltjes moment problem, J. Funct. Anal. 258 (2010), no 3, 978–998.
[8] H.R. Cho, J.D. Park, and K. Zhu, Products of Toeplitz operators on the Fock space, Proc. Amer. Math. Soc. 142 (2014), 2483–2489.
[9] F. Holland and R. Rochberg, Bergman kernel asymptotics for generalized Fock spaces, J. Analyse Math. 83 (2001), 207–242.
[10] J. Isralowitz, J. of Oper. Theory, 71, Issue 2, (2014), 381–410.
[11] W. Magnus, F. Oberhettinger, and R. P. Soni, Formulas and Theorems for the Special Functions of Mathematical Physics, Springer Verlag, 1966.
[12] F. Nazarov, A counterexample to Sarason’s conjecture, preprint, 1997.
[13] W. Rudin, Function Theory in the Unit Ball, Springer, New York, 1980.
[14] D. Sarason, Products of Toeplitz operators, pages 318-319 in Linear and Complex Analysis Problem Book 3. Part I (V.P. Khavin and N.K. Nikolskii, editors), Lecture Notes in Mathematics 1573, Springer, Berlin, 1994.
[15] K. Seip and E. H. Youssfi, Hankel operators on Fock spaces and related Bergman kernel estimates, J. Geom. Anal. 23 (2013), 170–201.
[16] T. Stieltjes, Recherches sur les fractions continues, Annales de la Faculté des Sciences de Toulouse. 8 (1894), 1-122, 9 (1895), 5–47.
[17] K. Stroethoff and D. Zheng, Products of Hankel and Toeplitz operators on the Bergman space, J. Funct. Anal. 169 (1999), 289–313.
[18] K. Zhu, Operator Theory on Function spaces, Second Edition, Math. Surveys and Monographs 138, American Mathematical Society, Providence, Rhode Island, 2007.
[19] K. Zhu, Analysis on Fock Spaces, Springer-Verlag, New York, 2012.
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