The Whitney extension problem for Zygmund spaces and Lipschitz selections in hyperbolic jet-spaces

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Abstract
We study a variant of the Whitney extension problem [28, 29] for the space $C^k\Lambda^m_\omega(\mathbb{R}^n)$ of functions whose partial derivatives of order $k$ satisfy the generalized Zygmund condition. We identify $C^k\Lambda^m_\omega(\mathbb{R}^n)$ with a space of Lipschitz mappings from a metric space $(\mathbb{R}^{n+1}_+, \rho_\omega)$ supplied with a hyperbolic metric $\rho_\omega$ into a metric space $(\mathcal{P}_{k+m-1} \times \mathbb{R}^{n+1}_+, d_\omega)$ of polynomial fields on $\mathbb{R}^{n+1}_+$ equipped with a hyperbolic-type metric $d_\omega$. This identification allows us to reformulate the Whitney problem for $C^k\Lambda^m_\omega(\mathbb{R}^n)$ as a Lipschitz selection problem for set-valued mappings from $(\mathbb{R}^{n+1}_+, \rho_\omega)$ into a certain family of subsets of $\mathcal{P}_{k+m-1} \times \mathbb{R}^{n+1}_+$.

1. Introduction
Let $m$ be a non-negative integer. We let $\Omega_m$ denote the class of non-decreasing continuous functions $\omega: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\omega(0) = 0$ and the function $\omega(t)/t^m$ is non-increasing. Given non-negative integers $k$ and $m$ and $\omega \in \Omega_m$ we define the space $C^k\Lambda^m_\omega(\mathbb{R}^n)$ as follows: a function $f \in C^k(\mathbb{R}^n)$ belongs to $C^k\Lambda^m_\omega(\mathbb{R}^n)$ if there exists a constant $\lambda > 0$ such that for every multi-index $\alpha$, $|\alpha| = k$, and every $x, h \in \mathbb{R}^n$

$$|\Delta^m_h(D^\alpha f)(x)| \leq \lambda \omega(\|h\|).$$

Here as usual $\Delta^m_h f$ denotes the $m$-th difference of a function $f$ of step $h$, i.e., the quantity

$$\Delta^m_h f(x) := \sum_{i=0}^{m} (-1)^{m-i} \binom{m}{i} f(x + ih).$$

$C^k\Lambda^m_\omega(\mathbb{R}^n)$ is normed by

$$\|f\|_{C^k\Lambda^m_\omega(\mathbb{R}^n)} := \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |D^\alpha f(x)| + \sum_{|\alpha| = k} \sup_{x, h \in \mathbb{R}^n} \frac{|\Delta^m_h(D^\alpha f)(x)|}{\omega(\|h\|)}.$$
In particular, for \( m = 1 \) and \( \omega \in \Omega_1 \) the space \( C^k \Lambda^1_\omega(\mathbb{R}^n) \) coincides with the space \( C^{k,\omega}(\mathbb{R}^n) \) consisting of all functions \( f \in C^k(\mathbb{R}^n) \) whose partial derivatives of order \( k \) satisfy the Lipschitz condition (with respect to \( \omega \)):

\[
|D^n f(x) - D^n f(y)| \leq \omega(||x - y||), \quad x, y \in \mathbb{R}^n.
\]

In turn, the space \( \Lambda^m_\omega(\mathbb{R}^n) := C^0 \Lambda^m_\omega(\mathbb{R}^n) \), \( \omega \in \Omega_m \), coincides with the generalized Zygmund space of bounded functions \( f \) on \( \mathbb{R}^n \) whose modulus of smoothness of order \( m \), \( \omega_m(\cdot; f) \), satisfies the inequality

\[
\omega_m(t; f) \leq \lambda \omega(t), \quad t \geq 0.
\]

In particular, the space \( \Lambda^2_\omega(\mathbb{R}^n) \) with \( \omega(t) = t \) is the classical Zygmund space \( \mathcal{Z}(\mathbb{R}^n) \) of bounded functions satisfying the Zygmund condition: there is \( \lambda > 0 \) such that for all \( x, y \in \mathbb{R}^n \)

\[
|f(x) - 2f(t \frac{x + y}{2}) + f(y)| \leq \lambda ||x - y||.
\]

(See, e.g. Stein [27].)

Throughout the paper we let \( S \) denote an arbitrary closed subset of \( \mathbb{R}^n \).

In this paper we study the following extension problem.

**Problem 1.1** Given non-negative integers \( k \) and \( m \), a function \( \omega \in \Omega_m \), and an arbitrary function \( f : S \to \mathbb{R} \), what is a necessary and sufficient condition for \( f \) to be the restriction to \( S \) of a function \( F \in C^k \Lambda^m_\omega(\mathbb{R}^n) \)?

This is a variant of a classical problem which is known in the literature as the Whitney Extension Problem [28, 29]. It has attracted a lot of attention in recent years. We refer the reader to [4]-[7], [8]-[14], [1, 2] and [30, 31] and references therein for numerous results in this direction, and for a variety of techniques for obtaining them.

This note is devoted to the phenomenon of “finiteness” in the Whitney problem for the spaces \( C^k \Lambda^m_\omega(\mathbb{R}^n) \). It turns out that, in many cases, Whitney-type problems for different spaces of smooth functions can be reduced to the same kinds of problems, but for finite sets with prescribed numbers of points.

For the space \( \Lambda^2_\omega(\mathbb{R}^n) \), this phenomenon has been studied in the author's papers [21, 20, 22]. It was shown that a function \( f \) defined on \( S \) can be extended to a function \( F \in \Lambda^2_\omega(\mathbb{R}^n) \) with \( ||F||_{\Lambda^2_\omega(\mathbb{R}^n)} \leq \gamma = \gamma(n) \) provided its restriction \( f|_{S'} \) to every subset \( S' \subset S \) consisting of at most \( N(n) = 3 \cdot 2^{n-1} \) points can be extended to a function \( F_{S'} \in \Lambda^2_\omega(\mathbb{R}^n) \) with \( ||F_{S'}||_{\Lambda^2_\omega(\mathbb{R}^n)} \leq 1 \). (Moreover, the value \( 3 \cdot 2^{n-1} \) is sharp [22].)

This result is an example of “the finiteness property” of the space \( \Lambda^2_\omega(\mathbb{R}^n) \). We call the number \( N \) appearing in formulations of finiteness properties “the finiteness number”.

In his pioneering work [29], Whitney characterized the restriction of the space \( C^k(\mathbb{R}) \), \( k \geq 1 \), to an arbitrary subset \( S \subset \mathbb{R} \) in terms of divided differences of functions. An application of Whitney's method to the space \( C^{k,\omega}(\mathbb{R}) \) implies the finiteness property for this space with the finiteness number \( N = k + 2 \).

The restriction of the space \( C^k \Lambda^m_\omega(\mathbb{R}) \) to an arbitrary subset \( S \subset \mathbb{R} \) has been characterized by Jonsson [16] (\( m \) is arbitrary, \( k = 0 \), \( \omega(t) = t^{m-1} \)), Shevchuk [18, 19] (\( m, \omega \) are arbitrary, \( k = 0 \)), Galan [15] (the general case). These results imply the finiteness property for \( C^k \Lambda^m_\omega(\mathbb{R}) \) with the finiteness number \( N = m + k + 1 \).

For the space \( C^{1,\omega}(\mathbb{R}^n) \) the finiteness property (with the same finiteness number \( N(n) = 3 \cdot 2^{n-1} \)) has been proved in [7], see also [4].
An impressive breakthrough in the solution of the Whitney problem for $C^{k,\omega}$-spaces has recently been made by Fefferman [8]-[14]. In particular, one of his remarkable results states that the space $C^{k,\omega}(\mathbb{R}^n)$ possesses the finiteness property for all $k, n > 1$, see [8, 10]. (An upper bound for the finiteness number $N(k, n)$ is $N(k, n) \leq 2^{\dim \mathcal{P}_k}$, see Bierstone, Milman [3], and Shvartsman [26]. Here $\mathcal{P}_k$ stands for the space of polynomials of degree at most $k$ defined on $\mathbb{R}^n$. Recall that $\dim \mathcal{P}_k = \binom{n+k}{k}$.)

Thus for $m > 2$ or $m = 2$ and $k > 0$ the following problem is open.

**Problem 1.2** Whether the space $C^{k,\Lambda^m}(\mathbb{R}^n)$ possesses the finiteness property?

In this paper we develop an approach to Problem 1.1 which allows us to reformulate this problem as a purely geometric question about the existence of Lipschitz selections of set-valued mappings defined on metric spaces with certain hyperbolic structure.

### 2. Extensions of Zygmund functions and Lipschitz selections

We will demonstrate this approach for the case of the Zygmund space $Z_m(\mathbb{R}^n) := C^0\Lambda^m(\mathbb{R}^n)$ where $\omega(t) = t^{m-1}$. Thus $Z_m(\mathbb{R}^n)$ is defined by the finiteness of the norm

$$\|f\|_{Z_m(\mathbb{R}^n)} := \sup_{x \in \mathbb{R}^n} |f(x)| + \sup_{x, h \in \mathbb{R}^n} \left| \Delta^m_h f(x) \right| / \|h\|_{m-1}.$$  

The crucial ingredient of our approach is an isomorphism between the space $Z_m(\mathbb{R}^n)|_S$ and a space of Lipschitz mappings from the set $S \times \mathbb{R}_+^n$ into the product $\mathcal{P}_{m-1} \times \mathbb{R}_+^{n+1}$ equipped with certain metrics.

This isomorphism is motivated by a description of the restrictions of $C^{k,\Lambda^m}$-functions in terms of local approximations which we present in Section 3. Let us formulate this result for the space $Z_m(\mathbb{R}^n)$.

We will assume that all cubes in this paper are closed and have sides which are parallel to the coordinate axes. It will be convenient for us to measure distances in $\mathbb{R}^n$ in the uniform norm

$$\|x\| := \max\{|x_i| : i = 1, ..., n\}, \quad x = (x_1, ..., x_n) \in \mathbb{R}^n.$$  

Thus every cube

$$Q = Q(x,r) := \{y \in \mathbb{R}^n : \|y - x\| \leq r\}$$  

is a “ball” in the metric space $(\mathbb{R}^n, \|\cdot\|)$ of “radius” $r$ centered at $x$. We let $x_Q := x$ denote center of $Q$ and $r_Q := r$ its “radius”. Given a constant $\lambda > 0$, we let $\lambda Q$ denote the cube $Q(x, \lambda r)$.

We let

$$\mathcal{K}(S) := \{Q(x, r) : x \in S, \ r > 0\}$$

denote the family of all cubes centered in $S$. By $\mathcal{K}$ we denote the family of all cubes in $\mathbb{R}^n$; thus $\mathcal{K} = \mathcal{K}(\mathbb{R}^n)$.

**Theorem 2.1** (A) Let $f : S \to \mathbb{R}$. Suppose that there exists a function $F \in Z_m(\mathbb{R}^n)$ such that $F|_S = f$. Then there exists a constant $0 < \lambda \leq C\|f\|_{C^{k,\Lambda^m}(\mathbb{R}^n)}$ and a family of polynomials $\{P_Q \in \mathcal{P}_{m-1} : Q \in \mathcal{K}(S)\}$ such that:
(1) For every \( x \in S \) and every cube \( Q = Q(x, r), r > 0 \), we have \( P_Q(x) = f(x) \);

(2) For every \( Q \in \mathcal{K}(S) \) with \( r_Q \leq 1 \) and every \( \beta, ||\beta|| \leq m - 1 \),

\[
|D^\beta P_Q(x_Q)| \leq \lambda r_Q^{-|\beta|};
\]

(3) For every \( Q_1 = Q(x_1, r_1), Q_2 = Q(x_2, r_2) \in \mathcal{K}(S) \), and every \( \alpha, ||\alpha|| \leq m - 1 \), we have

\[
|D^\alpha P_{Q_1}(x_1) - D^\alpha P_{Q_2}(x_1)| \leq \lambda (\max\{r_1, r_2\} + \|x_1 - x_2\|)^{m-1-||\alpha||} \cdot \ln\left(1 + \frac{\max\{r_1, r_2\} + \|x_1 - x_2\|}{\min\{r_1, r_2\}}\right).
\]

(B) Conversely, suppose that there exists a constant \( \lambda > 0 \) and a family of polynomials \( \{P_Q \in \mathcal{P}_{m-1} : Q \in \mathcal{K}(S)\} \) such that conditions (2) and (3) are satisfied. Then for every \( x \in S \) there exists the limit

\[
f(x) = \lim_{x_Q = x, r_Q \to 0} P_Q(x).
\]

Moreover, there exists \( F \in \mathbb{Z}_m(\mathbb{R}^n) \) with \( \|F\|_{\mathbb{Z}_m(\mathbb{R}^n)} \leq C \lambda \) such that \( F|_S = f \).

Here \( C \) is a constant depending only on \( m \) and \( n \).

This result is a particular case of Theorems 3.2 and 3.7 proven in Section 3. It can be considered as a certain version of the classical Whitney extension theorem [28] for the Zygmund space \( \mathbb{Z}_m(\mathbb{R}^n) \) where the Taylor polynomials are replaced by corresponding approximation polynomials \( P_Q, Q \in \mathcal{K}(S) \).

Now our aim is to transform inequalities (2.3) into a certain Lipschitz condition for the mapping \( \mathcal{K}(S) \ni Q \to (P_Q, Q) \in \mathcal{P}_{m-1} \times \mathcal{K}(S) \). This is the crucial point of the approach.

We equip the family \( \mathcal{K} \) (of all cubes in \( \mathbb{R}^n \)) with the distance \( \rho : \mathcal{K} \times \mathcal{K} \to \mathbb{R}_+ \) defined by the following formula: if \( Q_1 = Q(x_1, r_1), Q_2 = Q(x_2, r_2) \in \mathcal{K}, \) and \( Q_1 \neq Q_2, \) then

\[
\rho(Q_1, Q_2) := \ln\left(1 + \frac{\max\{r_1, r_2\} + \|x_1 - x_2\|}{\min\{r_1, r_2\}}\right),
\]

and \( \rho(Q_1, Q_2) := 0 \) whenever \( Q_1 = Q_2 \). We prove that \( \rho \) is a metric on \( \mathcal{K} \); moreover, the metric space \( (\mathcal{K}, \rho) \) can be identified (up to a constant weight) with the classical Poincaré upper half-space model of the hyperbolic space \( H_{n+1} \), see Remark 2.2.

Now, for every \( \alpha, ||\alpha|| \leq m - 1 \), we will rewrite every inequality in (2.3) in such a way that its right-hand side will be precisely equal to \( \rho(Q_1, Q_2) \). By (2.3), we have

\[
\frac{1}{\lambda} \frac{|D^\alpha P_{Q_1}(x_1) - D^\alpha P_{Q_2}(x_1)|}{\min\{r_1, r_2\}^{m-1-||\alpha||}} \leq \left(\frac{\max\{r_1, r_2\} + \|x_1 - x_2\|}{\min\{r_1, r_2\}}\right)^{m-1-||\alpha||} \cdot \ln\left(1 + \frac{\max\{r_1, r_2\} + \|x_1 - x_2\|}{\min\{r_1, r_2\}}\right) = \left(e^{\rho(Q_1, Q_2)} - 1\right)^{m-1-||\alpha||} \rho(Q_1, Q_2).
\]

Put

\[
\psi_\alpha(t) := t(e^t - 1)^{m-1-||\alpha||},
\]
and by $\psi^{-1}_\alpha$ denote the inverse to $\psi_\alpha$. Then, by the latter inequality,
\[
\frac{1}{\lambda} \frac{|D^\alpha P_{Q_1}(x_1) - D^\alpha P_{Q_2}(x_1)|}{\min\{r_1, r_2\}^{m-1-|\alpha|}} \leq \psi_\alpha(\rho(Q_1, Q_2)),
\]
so that
\[
\psi^{-1}_\alpha \left( \frac{1}{\lambda} \frac{|D^\alpha P_{Q_2}(x_1) - D^\alpha P_{Q_2}(x_1)|}{\min\{r_1, r_2\}^{m-1-|\alpha|}} \right) \leq \rho(Q_1, Q_2).
\]

Of course, the same inequality holds for $x_2$ instead of $x_1$. Taking the maximum over $x_1, x_2$, and over all $\alpha$ with $|\alpha| \leq m - 1$, we obtain
\[
I \left( \frac{1}{\lambda} P_{Q_1}, \frac{1}{\lambda} P_{Q_2} \right) \leq \rho(Q_1, Q_2) \tag{2.6}
\]
where
\[
I(P_{Q_1}, P_{Q_2}) := \max_{i=1,2,|\alpha|\leq m-1} \psi^{-1}_\alpha \left( \frac{|D^\alpha P_{Q_1}(x_i) - D^\alpha P_{Q_2}(x_i)|}{\min\{r_1, r_2\}^{m-1-|\alpha|}} \right).
\tag{2.7}
\]

In general the quantity $I(\cdot, \cdot)$ does not satisfy the triangle inequality on the set
\[
\mathcal{P}_{m-1} \times \mathcal{K} := \{ T = (P, Q) : P \in \mathcal{P}_{m-1}, Q \in \mathcal{K} \}.
\]
However, after a simple, but important modification the function $I(\cdot, \cdot)$ transforms into a metric on $\mathcal{P}_{m-1} \times \mathcal{K}$.

Namely, let us add the quantity $\rho(Q_1, Q_2)$ to the maximum in the left-hand side of (2.7). The function obtained we denote by $\delta$. Thus for every $T_1 = (P_1, Q_1), T_2 = (P_2, Q_2) \in \mathcal{P}_{m-1} \times \mathcal{K}$ we put
\[
\delta(T_1, T_2) := \max\{\rho(Q_1, Q_2), I(P_1, P_2)\} \tag{2.8}
\]

Given $\gamma \in \mathbb{R}$ and $T = (P, Q) \in \mathcal{P}_{m-1} \times \mathcal{K}$ we put
\[
\gamma \circ T := (\gamma P, Q).
\]

Now inequality (2.6) is equivalent to the inequality
\[
\delta \left( \frac{1}{\lambda} \circ T_1, \frac{1}{\lambda} \circ T_2 \right) \leq \rho(Q_1, Q_2) \tag{2.9}
\]
(Actually, (2.9) is equality, but it will be more convenient for us to work with inequalities rather than equalities).

The function $\delta$ generates the standard geodesic metric $d$ on $\mathcal{P}_{m-1} \times \mathcal{K}$ defined as follows: given $T, T' \in \mathcal{P}_{m-1} \times \mathcal{K}$ we put
\[
d(T, T') := \inf \sum_{i=0}^{M-1} \delta(T_i, T_{i+1}) \tag{2.10}
\]
where the infimum is taken over all finite families $\{T_0, T_1, ..., T_M\} \subset \mathcal{P}_{m-1} \times \mathcal{K}$ such that $T_0 = T$ and $T_M = T'$.

Our main result, Theorem 4.2, being applied to the case $k = 0$, $\omega(t) = t^{m-1}$, states that for every $T, T' \in \mathcal{P}_{m-1} \times \mathcal{K}$ the following inequality
\[
d(T, T') \leq \delta(T, T') \leq d(e^n \circ T, e^n \circ T')
\]
holds. (Of course, the first inequality is trivial and follows from definition (2.10).) This result allows us to reformulate Theorem 2.1 as follows: \( f \in Z_m(R^n) \) \( \Rightarrow \) there exists \( \lambda > 0 \) and a mapping \( T(Q) = (P_Q, Q) \) from \( (K(S), \rho) \) into \( (P_{m-1} \times K(S), d) \) such that

(i) for every \( Q \in K(S) \) with \( r_Q \leq 1 \) and every \( \beta, |\beta| \leq m - 1 \), we have

\[
|D^\beta P_Q(x_Q)| r_Q^{|\beta|} \leq \lambda;
\]

(ii) for every \( Q_1, Q_2 \in K(S) \)

\[
d\left( \frac{1}{\lambda} \circ T(Q_1), \frac{1}{\lambda} \circ T(Q_2) \right) \leq \rho(Q_1, Q_2);
\]

(iii) for every \( x \in S \) we have

\[
f(x) = \lim_{x_Q=x, r_Q \to 0} P_Q(x).
\]

Inequality (2.12) motivates us to introduce a Lipschitz-type space \( LO(K(S)) \) of mappings

\[
T : K(S) \to T := P_{m-1} \times K(S)
\]

defined by the finiteness of the following “seminorm”

\[
\|T\|_{LO(K(S))} := \inf\{ \lambda > 0 : \|\lambda^{-1} \circ T\|_{Lip(K(S), T)} \leq 1 \}.
\]

Also, inequality (2.11) motivates us to define a “norm”

\[
\|T\|^* := \sup\{ |D^\beta P_Q(x_Q)| r_Q^{|\beta|} : Q \in K(S), r_Q \leq 1, |\beta| \leq m - 1 \}.
\]

By \( LO(K(S)) \) we denote a subspace of \( LO(K(S)) \) defined by the finiteness of the “norm”

\[
\|T\|_{LO(K(S))}^* := \|T\|^* + \|T\|_{LO(K(S))}.
\]

See Section 5 for details.

Thus one can identify the space \( Z_m(R^n) \) with “limiting values” \( \lim_{x_Q=x, r_Q \to 0} P_Q(x) \) of mappings \( Q \in K(S) \to T(Q) = (P_Q, Q) \) from the space \( LO(K(S)) \).

**Remark 2.2** By the formula:

\[
K \ni Q = Q(x, r) \Leftrightarrow y = (x, r) \in R^{n+1}_+
\]

we identify the family \( K \) of all cubes in \( R^n \) with the upper half-space \( R^{n+1}_+ \)

\[
R^{n+1}_+ := R^n \times R_+ = \{ y = (y_1, \ldots, y_n, y_{n+1}) \in R^{n+1} : y_{n+1} > 0 \}.
\]

This identification and the metric (2.5) generate a metric \( \rho \) on \( R^{n+1}_+ \) defined by the following formula: for \( z_i = (x_i, r_i) \in R^{n+1}_+, i = 1, 2, \)

\[
\rho(z_1, z_2) := \ln \left( 1 + \frac{\max(r_1, r_2) + \|x_1 - x_2\|}{\min(r_1, r_2)} \right),
\]

if \( z_1 \neq z_2 \), and \( \rho(z_1, z_2) := 0 \) whenever \( z_1 = z_2 \).
Given \( z = (z_1, \ldots, z_{n+1}) \in \mathbb{R}_{+}^{n+1} \) we put \( \bar{z} := z = (z_1, \ldots, -z_{n+1}) \). Also, by \( \|z\|_2 \), we denote the Euclidean distance in \( \mathbb{R}_{+}^{n+1} \). We recall that the Poincaré metric on \( \mathbb{R}_{+}^{n+1} \) is defined by the formula

\[
\rho_H(z_1, z_2) := \ln \frac{\|z_1 - \bar{z}_2\|_2 + \|z_1 - z_2\|_2}{\|z_1 - \bar{z}_2\|_2 - \|z_1 - z_2\|_2}.
\]

This metric is the Riemannian metric for which the line element \( ds \) is given by

\[
ds := \sqrt{dx_1^2 + \ldots + dx_n^2 + dx_{n+1}^2}.
\]

It determines the classical Poincaré upper half-space model of the hyperbolic space \( H^{n+1} := (\mathbb{R}_{+}^{n+1}, \rho_H) \). It can be readily seen that

\[
\rho(z_1, z_2) \sim 1 + \rho_H(z_1, z_2), \quad z_1, z_2 \in \mathbb{R}_{+}^{n+1}.
\]

This equivalence, (2.15) and (2.5) show that the metric space \((\mathcal{K}, \rho)\) can be identified (up to a constant weight) with the hyperbolic space \( H^{n+1} \).

In view of this remark and identification (2.14), one can interpret the equality (2.13) as the restriction to \( \mathbb{R}^n \) of the mapping \( \mathbb{R}_{+}^{n+1} \ni z \rightarrow (P_z, z) \in \mathcal{P}_{m-1} \times \mathbb{R}_{+}^{n+1} \). This enables us to identify the Zygmund space \( \mathcal{Z}_m(\mathbb{R}^n) \) with the restriction to \( \mathbb{R}_{+}^{n+1} \) of all Lipschitz mappings of the form \( T(z) = (P_z, z), \quad z \in \mathbb{R}_{+}^{n+1} \), defined on the hyperbolic space \( H^{n+1} \) and taking their values in the metric space \( (\mathcal{P}_{m-1} \times \mathbb{R}_{+}^{n+1}, d) \).

In Section 3 and 4 we develop this approach for the general case of the space \( C_k \Lambda^m(\mathbb{R}^n) \) with \( k > 0 \). This enables us to reformulate the extension Problem 1.1 as a geometrical problem of the existence of Lipschitz selections of certain set-valued mappings from \( \mathbb{R}_{+}^{n+1} \) into a family of subsets in \( \mathcal{P}_{k+m-1} \times \mathbb{R}_{+}^{n+1} \). We will discuss a generalization of Problem 1.1 raised by C. Fefferman [11] (for the space \( C^k(\mathbb{R}^n) \)) and show how this problem can be reduced to the Lipschitz selection problem for certain jet-spaces generated by functions from \( C_k \Lambda^m(\mathbb{R}^n) \).

We observe that the Lipschitz selection method has been used for proving the finiteness property of the spaces \( \Lambda^2(\mathbb{R}^n) \) [22], \( C^1(\mathbb{R}^n) \) [7] and \( C^k \Lambda^2(\mathbb{R}^n) \) [6] (a jet-version). In [26] we used the same technique to prove a certain weak version of the finiteness property of the space \( C^k(\mathbb{R}^n) \).

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### 3. The space \( C_k \Lambda^m(\mathbb{R}^n) \) and local polynomial approximations

Our notation is fairly standard. Throughout the paper \( C, C_1, C_2, \ldots \) will be generic positive constants which depend only on \( k, m, n \). These constants can change even in a single string of estimates. The dependence of a constant on certain parameters is expressed, for example, by the notation \( C = C(k, m, n) \). We write \( A \approx B \) if there is a constant \( C \geq 1 \) such that \( A/C \leq B \leq CA \).
We let \( \mathcal{P}_\ell = \mathcal{P}_\ell(\mathbb{R}^n), \ell \geq 0 \), denote the space of all polynomials on \( \mathbb{R}^n \) of degree at most \( \ell \). Finally, given \( k \)-times differentiable function \( f \) and \( x \in \mathbb{R}^n \), we let \( T^k_x(f) \) denote the Taylor polynomial of \( f \) at \( x \) of degree at most \( k \):

\[
T^k_x(f)(y) := \sum_{|\alpha| \leq k} \frac{1}{\alpha!} (D^\alpha f)(x)(y-x)^\alpha, \quad y \in \mathbb{R}^n.
\]

Finally, we put \( L := k + m - 1 \).

**Theorem 3.1** Given a family of polynomials \( \{P_x \in \mathcal{P}_k : x \in S\} \) there is a function \( F \in C^k\Lambda_m^\omega(\mathbb{R}^n) \) such that \( T^k_x(F) = P_x \) for every \( x \in S \) if and only if there is a constant \( \lambda > 0 \) and a family of polynomials

\[
\{P_Q \in \mathcal{P}_L : Q \in \mathcal{K}(S)\}
\]

such that

1. For every cube \( Q \in \mathcal{K}(S) \) we have
   \[
   T^k_{x_Q}(P_Q) = P_{x_Q};
   \]
2. For every \( Q \in \mathcal{K}(S) \) with \( r_Q \leq 1 \) and every \( \alpha, |\alpha| \leq k \),
   \[
   \sup_Q |D^\alpha P_Q| \leq \lambda;
   \]
3. For every \( Q, Q' \in \mathcal{K}(S) \), such that \( Q' \subset Q \) we have
   \[
   \sup_{Q'} |P_{Q'} - P_Q| \leq \lambda (r_{Q'} + \|x_Q - x_{Q'}\|)k^\omega(r_Q).
   \]

Moreover,

\[
\inf \{\|F\|_{C^k\Lambda_m^\omega(\mathbb{R}^n)} : T^k_x(F) = P_x, x \in S\} \approx \inf \lambda
\]

with constants of equivalence depending only on \( k, m \) and \( n \).

For the homogeneous space \( C^k\Lambda_m^\omega(\mathbb{R}^n) \) (normed by the second item in (1.1)) a variant of this theorem has been proved in [6]. The present result can be obtained by a simple modification of the method of proof suggested in [6].

**Theorem 3.2** If a function \( F \in C^k\Lambda_m^\omega(\mathbb{R}^n) \), then there exists a family of polynomials

\[
\{P_Q \in \mathcal{P}_L : Q \in \mathcal{K}(S)\}
\]

such that:

1. For every cube \( Q \in \mathcal{K}(S) \) we have
   \[
   T^k_{x_Q}(P_Q) = T^k_{x_Q}(F);
   \]
2. For every \( Q \in \mathcal{K}(S) \) with \( r_Q \leq 1 \) and every \( \alpha, |\alpha| \leq k \), and \( \beta, |\beta| \leq L - |\alpha| \),
   \[
   |D^{\alpha+\beta} P_Q(x_Q)| \leq C\|F\|_{C^k\Lambda_m^\omega(\mathbb{R}^n)} r_Q^{-|\beta|};
   \]

   \[
   \]
Lemma 3.3  Let \( \omega \in \Omega_m \). Assume that a family of polynomials \( \{P_Q \in \mathcal{P}_L : Q \in \mathcal{K}(S)\} \) and a constant \( \lambda > 0 \) satisfy the inequality

\[
(3.1) \quad \sup_{Q'} |P_{Q'} - P_Q| \leq \lambda (r_{Q'} + \|x_Q - x_{Q'}\|)^k \omega(r_Q),
\]

for every \( Q = Q(x_Q, r_Q), Q' = Q(x_{Q'}, r_{Q'}) \in \mathcal{K}(S) \) such that \( Q' \subset Q \) and \( r_Q \leq 4r_{Q'} \).

Then for every \( Q, Q' \in \mathcal{K}(S), Q' \subset Q \), and every \( \alpha, |\alpha| \leq L, \) we have

\[
(3.2) \quad \sup_{Q'} |D^\alpha P_{Q'} - D^\alpha P_Q| \leq C \lambda \int_{r_{Q'}}^{2r_{Q'}} \frac{\omega(t)}{t^{|\alpha| - k}} \frac{dt}{t}
\]

where \( C = C(k, m, n) \).

Proof. Let \( x \in Q' \). We put

\[ Q_i := Q(x_{Q_i}, 2^i r_{Q_i}), \quad i = 0, ..., \ell, \]

where \( \ell := \left[ \ln \frac{r_Q}{r_{Q'}} \right] + 1 \). Then

\[ r_{Q_i} \leq 2^{\ell} r_{Q'} \leq 2r_Q < 2^{\ell+1} r_{Q'} = r_{Q_{i+1}}. \]

Since \( 2r_Q \leq r_{Q_i} := 2^i r_{Q'} \) and \( x_{Q_i} \in Q \cap Q_{i} \), we have

\[
(3.3) \quad Q \subset Q_{i}, \quad 2r_Q \leq r_{Q_i} \leq 4r_Q.
\]

By Markov’s inequality,

\[
\sup_{Q_i} |D^\alpha (P_{Q_i} - P_{Q_{i+1}})| \leq C \sup_{Q_i} r_{Q_i}^{2|\alpha|} \sup_{Q_i} |P_{Q_i} - P_{Q_{i+1}}|
\]

where \( C = C(k, m, n) \). Since \( r_{Q_{i+1}} = 2r_{Q_i} \) and \( x_{Q_{i+1}} = x_{Q_i} \), by (3.1) we obtain

\[
\sup_{Q_i} |D^\alpha (P_{Q_i} - P_{Q_{i+1}})| \leq C \lambda r_{Q_i}^{-|\alpha|} (r_{Q_i}^k \omega(r_{Q_{i+1}}) = C \lambda r_{Q_i}^{-|\alpha|} \omega(2r_{Q_i}).
\]

Since \( x \in Q' \cap Q_{i} \), we have

\[
|D^\alpha P_{Q_i}(x) - D^\alpha P_{Q_{i+1}}(x)| \leq C \lambda r_{Q_i}^{2(|\alpha| + 1)} \omega(2r_{Q_i}), \quad i = 0, ..., \ell - 1.
\]
Hence

\[ |D^\alpha P_{Q'}(x) - D^\alpha P_Q(x)| = |D^\alpha P_{Q_0}(x) - D^\alpha P_{Q_1}(x)| \leq \sum_{i=0}^{\ell-1} |D^\alpha P_{Q_i}(x) - D^\alpha P_{Q_{i+1}}(x)| \]

so that

\[ (3.4) \quad |D^\alpha P_{Q'}(x) - D^\alpha P_Q(x)| \leq C\lambda \sum_{i=0}^{\ell-1} r_{Q_i}^{k-|\alpha|} \omega(2r_{Q_i}). \]

Recall that \( Q \subset Q_\ell \) and \( r_{Q_\ell} \leq 4r_Q \), so that by (3.1) and Markov’s inequality

\[ |D^\alpha P_Q(x) - D^\alpha P_{Q_\ell}(x)| \leq \sup_Q |D^\alpha P_Q - D^\alpha P_{Q_\ell}| \]
\[ \leq C r_Q^{-|\alpha|} \sup_Q |P_Q - P_{Q_\ell}| \]
\[ \leq C \lambda (r_Q + \|x_Q - x_{Q'}\|)^k r_Q^{-|\alpha|} \omega(r_{Q_\ell}). \]

But \( x_{Q'} \in Q' \subset Q \) so that \( \|x_Q - x_{Q'}\| \leq r_Q \). Therefore by (3.3)

\[ |D^\alpha P_Q(x) - D^\alpha P_{Q_\ell}(x)| \leq 4^{|\alpha|} 2^k C \lambda r_Q^{-|\alpha|} \omega(r_{Q_\ell}) \]
\[ = 4^{|\alpha|} 2^k C \lambda (2r_{Q_{\ell-1}})^k r_{Q_{\ell-1}}^{-|\alpha|} \omega(2r_{Q_{\ell-1}}) \]
\[ = C_1(k,m) \lambda r_{Q_{\ell-1}}^{-|\alpha|} \omega(r_{Q_{\ell-1}}). \]

Since \( \omega \in \Omega_m \), we have \( \omega(2t) \leq 2^m \omega(t) \) so that by (3.3)

\[ |D^\alpha P_{Q'}(x) - D^\alpha P_Q(x)| \leq |D^\alpha P_{Q'}(x) - D^\alpha P_{Q_\ell}(x)| + |D^\alpha P_{Q_\ell}(x) - D^\alpha P_Q(x)| \]
\[ \leq C \lambda \left( \sum_{i=0}^{\ell-1} r_{Q_i}^{k-|\alpha|} \omega(2r_{Q_i}) \right) + C_1(k,m) \lambda r_{Q_{\ell-1}}^{-|\alpha|} \omega(r_{Q_{\ell-1}}) \]
\[ \leq C_2(k,m) \lambda \sum_{i=0}^{\ell-1} r_{Q_i}^{k-|\alpha|} \omega(r_{Q_i}). \]

Since \( \omega \) is non-decreasing, for every \( \alpha, |\alpha| \neq k \), and every \( a, b, 0 < a < b \), we have

\[ \int_a^b \frac{\omega(t)}{t^{|\alpha|-k}} \frac{dt}{t} \geq \omega(a) \int_a^b \frac{1}{t^{|\alpha|-k}} \frac{dt}{t} = (|\alpha| - k)^{-1} \omega(a)(a^{k-|\alpha|} - b^{k-|\alpha|}) \]

so that

\[ \frac{\omega(a)}{a^{|\alpha|-k}} \leq \frac{(|\alpha| - k)b^{|\alpha|-k}}{b^{|\alpha|-k} - a^{|\alpha|-k}} \int_a^b \frac{\omega(t)}{t^{|\alpha|-k}} \frac{dt}{t}. \]

Hence

\[ r_{Q_i}^{k-|\alpha|} \omega(r_{Q_i}) \leq \frac{(|\alpha| - k)(2r_{Q_i})^{|\alpha|-k}}{(2r_{Q_i})^{|\alpha|-k} - r_{Q_i}^{|\alpha|-k}} \int_{r_{Q_i}}^{2r_{Q_i}} \frac{\omega(t)}{t^{|\alpha|-k}} \frac{dt}{t} \]
\[ = \frac{(|\alpha| - k)2^{|\alpha|-k}}{2^{|\alpha|-k} - 1} \int_{r_{Q_i}}^{2r_{Q_i}} \frac{\omega(t)}{t^{|\alpha|-k}} \frac{dt}{t}. \]
In a similar way we show that
\[ \omega(r_{Q_i}) \leq \frac{1}{\ln 2} \int_{r_{Q_i}}^{2r_{Q_i}} \frac{\omega(t)}{t} \, dt. \]

Thus for every \( \alpha, |\alpha| \leq L \), we have
\[ r_{Q_i}^{k-|\alpha|} \omega(r_{Q_i}) \leq C_3(k, m) \int_{r_{Q_i}}^{2r_{Q_i}} \frac{\omega(t)}{t} \, dt. \]

Finally, we obtain
\[
|D^\alpha P_{Q'}(x) - D^\alpha P_Q(x)| \leq C_2 \lambda \sum_{i=0}^{\ell-1} r_{Q_i}^{k-|\alpha|} \omega(r_{Q_i}) \leq C_3 \lambda \sum_{i=0}^{\ell-1} \int_{r_{Q_i}}^{2r_{Q_i}} \frac{\omega(t)}{t} \, dt \leq C_3 \lambda \int_{r_{Q'}}^{4r_{Q}} \frac{\omega(t)}{t} \, dt.
\]

It can be easily seen that
\[
\int_{2r_{Q}}^{4r_{Q}} \frac{\omega(t)}{t} \, dt \leq C_4 \int_{r_{Q}}^{2r_{Q}} \frac{\omega(t)}{t} \, dt \leq C_4 \int_{r_{Q'}}^{2r_{Q}} \frac{\omega(t)}{t} \, dt.
\]

so that
\[
|D^\alpha P_{Q'}(x) - D^\alpha P_Q(x)| \leq C_3 \lambda \int_{r_{Q'}}^{4r_{Q}} \frac{\omega(t)}{t} \, dt \leq C_3(1 + C_4) \lambda \int_{r_{Q'}}^{2r_{Q}} \frac{\omega(t)}{t} \, dt
\]

proving the lemma. \( \square \)

**Lemma 3.4** Let \( \lambda > 0 \) and let \( \{ P_Q \in \mathcal{P}_L : Q \in \mathcal{K}(S) \} \) be a family of polynomials satisfying the conditions of Lemma 3.3. Then for every \( Q_1 = Q(x_1, r_1), Q_2 = Q(x_2, r_2) \in \mathcal{K}(S) \), and every \( \alpha, |\alpha| \leq L \), we have
\[
|D^\alpha P_{Q_1}(x_1) - D^\alpha P_{Q_2}(x_1)| \leq C \lambda (\max\{r_1, r_2\} + \|x_1 - x_2\|)^{L-|\alpha|} \int_{\min\{r_1, r_2\}}^{r_1+r_2+\|x_1-x_2\|} \frac{\omega(t)}{t^m} \, dt
\]

where \( C = C(k, m, n) \).

**Proof.** Put
\[
\tilde{r} := r_1 + r_2 + \|x_1 - x_2\|
\]

and
\[
\tilde{Q} := Q(x_2, \tilde{r}) = Q(x_2, r_1 + r_2 + \|x_1 - x_2\|)
\]

Then
\[
I := |D^\alpha P_{Q_1}(x_1) - D^\alpha P_{Q_2}(x_1)|
\]

\[= \sum_{|\beta| \leq p-\alpha} \frac{1}{|\beta|!} [D^\beta(D^\alpha(P_{Q_2} - P_Q))(x_2)](x_1 - x_2) \]

\[\leq \sum_{|\beta| \leq L-\alpha} \left| D^{\alpha+\beta}(P_{Q_2} - P_Q)(x_2) \right| \|x_1 - x_2\|^\beta. \]
By Lemma 3.3
\[ |D^{\alpha+\beta} P_Q(x_2) - D^{\alpha+\beta} P_Q(x_2)| \leq C \lambda \int_{r_2}^{2r} \frac{\omega(t)}{t^{\alpha+|\beta| - k}} dt \]
for every \( \alpha, \beta, |\alpha| + |\beta| \leq L \). Hence
\[
I \leq C \lambda \sum_{|\beta| \leq L - |\alpha|} \left( \int_{r_2}^{2r} \frac{\omega(t)}{t^{\alpha+|\beta| - k}} dt \right) \|x_1 - x_2\|^{|\beta|}
\]
\[
= C \lambda \int_{r_2}^{2r} \frac{\omega(t)}{t^m} \left( \sum_{|\beta| \leq L - |\alpha|} \|x_1 - x_2\|^{|\beta|} \right) dt.
\]
Since \( \|x_1 - x_2\| \leq \tilde{r} = r_1 + r_2 + \|x_1 - x_2\| \), we obtain
\[
I \leq C_1 \lambda \tilde{r}^{L - |\alpha|} \int_{r_2}^{2(r_1 + r_2 + \|x_1 - x_2\|)} \frac{\omega(t)}{t^m} dt.
\]
Since \( \omega(t)/t^m \) is non-increasing, for every \( 0 < a < b_0 \leq b_1 \) we have
\[
\frac{1}{b_1 - a} \int_a^{b_1} \frac{\omega(t)}{t^m} dt \leq \frac{1}{b_0 - a} \int_a^{b_0} \frac{\omega(t)}{t^m} dt
\]
so that
\[
\int_{\min\{r_1, r_2\}}^{2(r_1 + r_2 + \|x_1 - x_2\|)} \frac{\omega(t)}{t^m} dt \leq \frac{2(r_1 + r_2 + \|x_1 - x_2\|) - \min\{r_1, r_2\}}{r_1 + r_2 + \|x_1 - x_2\| - \min\{r_1, r_2\}} \omega(t) dt
\]
\[
= \frac{\min\{r_1, r_2\} + 2 \max\{r_1, r_2\} + 2 \|x_1 - x_2\|}{\max\{r_1, r_2\} + \|x_1 - x_2\|} \int_{\min\{r_1, r_2\}}^{r_1 + r_2 + \|x_1 - x_2\|} \frac{\omega(t)}{t^m} dt
\]
\[
\leq 3 \int_{\min\{r_1, r_2\}}^{r_1 + r_2 + \|x_1 - x_2\|} \frac{\omega(t)}{t^m} dt.
\]
Hence
\[
I \leq 3C_1 \lambda (r_1 + r_2 + \|x_1 - x_2\|)^{L - |\alpha|} \int_{\min\{r_1, r_2\}}^{r_1 + r_2 + \|x_1 - x_2\|} \frac{\omega(t)}{t^m} dt
\]
\[
\leq 3 \cdot 2^L C_1 \lambda (\max\{r_1, r_2\} + \|x_1 - x_2\|)^{L - |\alpha|} \int_{\min\{r_1, r_2\}}^{r_1 + r_2 + \|x_1 - x_2\|} \frac{\omega(t)}{t^m} dt.
\]
The lemma is proved. \( \square \)

Proof of Theorem 3.2. We put
\[
P_x := T_x^k(F), \quad x \in S.
\]
Then by Theorem 3.1 there is a family of polynomials
\[ \{P_Q \in \mathcal{P}_L : Q \in \mathcal{K}(S)\} \]
satisfying conditions (1)-(3) of this theorem with \( \lambda = C(k, m, n)\|F\|_{C^k\Lambda^m_\omega(\mathbb{R}^n)} \).

Then equality (1) of Theorem 3.2 immediately follows from that of Theorem 3.1. In turn, condition (2) of Theorem 3.1 and Markov’s inequality imply inequality (2) of Theorem 3.2. Finally, by Lemma 3.4, condition (3) of Theorem 3.1 implies inequality (3) of Theorem 3.2.

\[ \square \]

**Definition 3.5** We say that a continuous function \( \omega : \mathbb{R}_+ \to \mathbb{R}_+ \) is quasipower if there is a constant \( C_\omega > 0 \) such that
\[ \int_0^t \omega(s) \frac{ds}{s} \leq C_\omega \omega(t) \]
for all \( t > 0 \).

**Example 3.6** Every function \( \omega(t) = t\varphi(t) \) where \( \varphi \) is a non-decreasing function is quasipower (with \( C_\omega = 1 \)). In fact
\[ \int_0^t \omega(s) \frac{ds}{s} = \int_0^t \varphi(s) ds \leq t\varphi(t) = \omega(t). \]

**Theorem 3.7** Let \( \omega \in \Omega_m \) be a quasipower function. Suppose that a family of polynomials
\[ \{P_Q \in \mathcal{P}_L : Q \in \mathcal{K}(S)\} \]
and a constant \( \lambda > 0 \) satisfy the following conditions:

1. For every \( Q \in \mathcal{K}(S) \) with \( r_Q \leq 1 \) and every \( \alpha, |\alpha| \leq k, \) and \( \beta, |\beta| \leq L - |\alpha|, \) we have
\[ |D^{\alpha+\beta}P_Q(x_Q)| \leq \lambda r_Q^{-|\beta|}; \]

2. For every two cubes \( Q_1 = Q(x_1, r_1), Q_2 = Q(x_2, r_2) \in \mathcal{K}(S), \) and every \( \alpha, |\alpha| \leq L, \)
\[ |D^\alpha(P_{Q_1} - P_{Q_2})(x_1)| \leq \lambda \left( \max\{r_1, r_2\} + \|x_1 - x_2\| \right)^{L-|\alpha|} \int_{\min\{r_1, r_2\}}^{r_1+r_2+\|x_1-x_2\|} \frac{\omega(t)}{t^{m}} \, dt. \]

Then for every \( x \in S \) there exists the limit
\[ (3.6) \quad P_x = \lim_{x_Q = x, r_Q \to 0} T_x^k(P_Q). \]

Moreover, there exists a function \( F \in C^k\Lambda^m_\omega(\mathbb{R}^n) \) with \( \|F\|_{C^k\Lambda^m_\omega(\mathbb{R}^n)} \leq C\lambda \) such that
\[ P_x = T_x^k(F), \quad x \in S, \]
and for every \( Q = Q(x, r) \in \mathcal{K}(S) \) and \( |\alpha| \leq k \) we have
\[ |D^\alpha T_x^k(F)(x) - D^\alpha P_Q(x)| \leq C\lambda r^{k-|\alpha|}\omega(r). \]

Here \( C \) is a constant depending only on \( k, m, n \) and the constant \( C_\omega \) (see Definition 3.5).
Hence, by (3.8), for every $\alpha, |\alpha| \leq L$, and every two cubes $Q_1 := Q(x, r)$ and $Q_2 := Q(x, \tilde{r}) \in \mathcal{K}(S)$ with $r \leq \tilde{r} \leq 2r$, we have
\[
|D^\alpha P_{Q_1}(x) - D^\alpha P_{Q_2}(x)| \leq \lambda \tilde{r}^{L-|\alpha|} \int_{r}^{r+\tilde{r}} \frac{\omega(t)}{t^m} dt \leq 2^{L-|\alpha|} \lambda r^{L-|\alpha|} \int_{r}^{3r} \frac{\omega(t)}{t^m} dt.
\]
Since $\omega(t)/t^m$ is non-increasing and $L := k + m - 1$, we have
\[
|D^\alpha P_{Q_1}(x) - D^\alpha P_{Q_2}(x)| \leq 2^{L-|\alpha|} \lambda r^{L-|\alpha|} \omega(r) \int_{r}^{3r} \frac{1}{t^m} dt = 2^{k+m-|\alpha|} \lambda r^{k-|\alpha|} \omega(r).
\]
Consider now two cubes $Q' = Q(x, r')$ and $Q'' = Q(x, r''), r' < r''$. Put $\ell := \lfloor \ln(r''/r') \rfloor$ and $r_i := 2^i r'$, $Q_i := Q(x, r_i), \ i = 0, 1, ...$
Thus
\[
r_\ell := 2^\ell r' \leq r'' < 2^{\ell+1} r' := r_{\ell+1}.
\]
Hence, by (3.8),
\[
|D^\alpha P_{Q'}(x) - D^\alpha P_{Q''}(x)| \leq \sum_{i=0}^{\ell-1} |D^\alpha P_{Q_i}(x) - D^\alpha P_{Q_{i+1}}(x)| + |D^\alpha P_{Q_{\ell}}(x) - D^\alpha P_{Q''}(x)|
\leq 2^{k+m-|\alpha|} \lambda \left\{ \left( \sum_{i=0}^{\ell-1} r_i^{k-|\alpha|} \omega(r_i) \right) + r_\ell^{k-|\alpha|} \omega(r_\ell) \right\}
= 2^{k+m-|\alpha|} \lambda \sum_{i=0}^{\ell} r_i^{k-|\alpha|} \omega(r_i).
\]
But
\[
\int_{r_i}^{r_{i+1}} t^{k-|\alpha|} \omega(t) \frac{dt}{t} \geq \omega(r_i) \ln(r_{i+1}/r_i) = (\ln 2) \omega(r_i),
\]
whenever $|\alpha| = k$, and
\[
\int_{r_i}^{r_{i+1}} t^{k-|\alpha|} \omega(t) \frac{dt}{t} \geq \omega(r_i) \left( r_i^{k-|\alpha|} - r_{i+1}^{k-|\alpha|} \right) = \frac{2^{k-|\alpha|} - 1}{(k - |\alpha|)2^{k-|\alpha|}} r_i^{k-|\alpha|} \omega(r_i),
\]
if $|\alpha| \neq k$. Thus, for every $\alpha$ and every $i, 0 \leq i \leq \ell$, we have
\[
\omega(r_i) \leq C(k, m) \int_{r_i}^{r_{i+1}} t^{k-|\alpha|} \omega(t) \frac{dt}{t}.
\]
Hence,
\[
|D^\alpha P_{Q'}(x) - D^\alpha P_{Q''}(x)| \leq C\lambda \int_{r_0}^{r_\ell} t^{k-|\alpha|} \omega(t) \frac{dt}{t} \leq C\lambda \int_{r'}^{r''} t^{k-|\alpha|} \omega(t) \frac{dt}{t}.
\]
\[
(3.9) \quad |D^\alpha P_{Q'}(x) - D^\alpha P_{Q''}(x)| \leq C\lambda \int_{r_0}^{r_\ell} t^{k-|\alpha|} \omega(t) \frac{dt}{t} \leq C\lambda \int_{r'}^{r''} t^{k-|\alpha|} \omega(t) \frac{dt}{t}.
\]
Consider now the case $|\alpha| \leq k$. Recall that, by the assumption, $\omega$ is a quasipower function, so that
\[
\int_{r'}^{2r''} \omega(t) \frac{dt}{t} \leq C_\omega \omega(2r'') \leq 2^m C_\omega \omega(r'').
\]
Therefore, by (3.9), for every $\alpha$, $|\alpha| = k$, we have
\[
|D^\alpha P_{Q'}(x) - D^\alpha P_{Q''}(x)| \leq C \lambda \omega(r'')
\]
with $C = C(k, m, C_\omega)$.

If $|\alpha| < k$, we obtain
\[
|D^\alpha P_{Q'}(x) - D^\alpha P_{Q''}(x)| \leq C \lambda \int_{r'}^{2r''} t^{k-|\alpha|} \omega(t) \frac{dt}{t} \leq C \lambda \omega(2r'') \int_{r'}^{2r''} t^{k-|\alpha|} \frac{dt}{t}
\]
\[
= \frac{1}{k-|\alpha|} C \lambda \omega(2r'') \left( (2r'')^{k-|\alpha|} - (r')^{k-|\alpha|} \right)
\]
\[
\leq \frac{1}{k-|\alpha|} C \lambda (2r'')^{k-|\alpha|} \omega(2r'').
\]
Thus for every $\alpha$, $|\alpha| \leq k$, we have
\[
(3.10) \quad |D^\alpha P_{Q'}(x) - D^\alpha P_{Q''}(x)| \leq C \lambda (r'')^{k-|\alpha|} \omega(r''),
\]
where $C = C(k, m, C_\omega)$.

Since $\omega(r) \to 0$ as $r \to 0$, there exists the limit
\[
(3.11) \quad p_\alpha(x) := \lim_{x_Q = x, r_Q \to 0} D^\alpha P_Q(x).
\]

We put
\[
P_x(y) := \sum_{|\beta| \leq k} \frac{p_\beta(x)}{\beta!} (y - x)^\beta.
\]
Thus $P_x \in \mathcal{P}_k$ and
\[
(3.12) \quad D^\alpha P_x(x) = p_\alpha(x), \quad |\alpha| \leq k.
\]
Since $\mathcal{P}_k$ is finite dimensional, by (3.12)
\[
P_x = \lim_{x_Q = x, r_Q \to 0} T^k_x P_Q.
\]
proving (3.6).

Prove that the family of polynomials $\{P_x \in \mathcal{P}_k : x \in S\}$ satisfies the conditions of Theorem 3.1, i.e., there exists a family of polynomials
\[
\{\tilde{P}_Q \in \mathcal{P}_L : Q \in \mathcal{K}(S)\}
\]
such that:
(a). $T^k_{x_Q}(\tilde{P}_Q) = P_{x_Q}$ for every $Q \in \mathcal{K}(S)$;
(b). $\sup_Q |D^\alpha \tilde{P}_Q| \leq C \lambda$ for every cube $Q \in \mathcal{K}(S)$ with $r_Q \leq 1$ and every $\alpha, |\alpha| \leq k$;
(c). For every $Q, Q' \in \mathcal{K}(S)$, such that $Q' \subset Q$ we have

$$\sup_{Q'} |\widetilde{P}_{Q'} - \widetilde{P}_Q| \leq C \lambda (r_{Q'} + \|x_Q - x_{Q'}\|)^k \omega(r_Q).$$

Put

$$\widetilde{P}_Q := P_Q + P_x - T_x P_Q,$$

for every cube $Q = Q(x,r) \in \mathcal{K}(S)$. Then

$$D^\alpha \widetilde{P}_Q(x) = \begin{cases} D^\alpha P_x(x), & |\alpha| \leq k, \\ D^\alpha P_Q(x), & |\alpha| > k. \end{cases}$$

In particular, the condition (a) is satisfied.

Observe that tending $r'$ to 0 in (3.10) we obtain

$$|D^\alpha P_x(x) - D^\alpha P_Q(x)| \leq C \lambda r^{|\alpha|} \omega(r)$$

for every cube $Q = Q(x,r) \in \mathcal{K}(S)$ and every $\alpha, |\alpha| \leq k$. This proves (3.7).

Let us prove (b). Note that for every $\alpha, |\alpha| \leq k$, by property (1) of the theorem (with $\beta = 0$) we have

$$|D^\alpha P_Q(x_Q)| \leq \lambda$$

for every $Q \in \mathcal{K}(S)$ with $r_Q \leq 1$. Therefore by (3.11) and (3.12)

$$|D^\alpha P_x(x)| = |\lim_{r \to 0} D^\alpha P_Q(x_Q)| \leq \lambda, \quad x \in S.$$  

Hence, by (3.14), for every $\beta$ such that $|\alpha + \beta| \leq k$ we have

$$|D^{\alpha+\beta} \widetilde{P}_Q(x_Q)| = |D^{\alpha+\beta} P_{x_Q}(x_Q)| \leq \lambda.$$  

In turn, if $|\alpha + \beta| > k$, by (3.14) and condition (1) of the theorem

$$|D^{\alpha+\beta} \widetilde{P}_Q(x_Q)| = |D^{\alpha+\beta} P_Q(x_Q)| \leq \lambda r_Q^{-|\beta|}.$$  

Therefore for every $x \in Q$ we have

$$|D^\alpha \widetilde{P}_Q(x)| = \left| \sum_{|\beta| \leq k - |\alpha|} \frac{1}{\beta!} D^{\alpha+\beta} \widetilde{P}_Q(x_Q)(x - x_Q)^\beta \right|$$

$$\leq \sum_{|\beta| \leq k - |\alpha|} |D^{\alpha+\beta} \widetilde{P}_Q(x_Q)||x - x_Q|^{|\beta|}$$

$$= \sum_{|\beta| \leq k - |\alpha|} |D^{\alpha+\beta} \widetilde{P}_Q(x_Q)||x - x_Q|^{|\beta|}$$

$$+ \sum_{k - |\alpha| < |\beta| \leq L - |\alpha|} |D^{\alpha+\beta} \widetilde{P}_Q(x_Q)||x - x_Q|^{|\beta|}$$

$$\leq \sum_{|\beta| \leq k - |\alpha|} \lambda r_Q^{|\beta|} + \sum_{k - |\alpha| < |\beta| \leq L - |\alpha|} \lambda r_Q^{-|\beta|} r_Q^{|\beta|}.$$
Since \( r_Q \leq 1 \), we obtain

\[ |D^\alpha \tilde{P}_Q(x)| \leq C(k, m, n)\lambda, \quad x \in Q, \]

proving (b).

Let us prove (c). Put \( \bar{r} := r + r' + \|x' - x\| \) and

\[ K := Q(x', r + r' + \|x' - x\|) = Q(x', \bar{r}). \]

Then clearly, \( Q' \subset Q \subset K \), and also

\[ r \leq \bar{r} = r + r' + \|x' - x\| \leq r' + 2r \leq 3r. \]

By (3.15)

\[ |D^\alpha \tilde{P}_{Q'}(x') - D^\alpha \tilde{P}_K(x')| \leq C\lambda \bar{r}^{k-|\alpha|}\omega(\bar{r}) \leq C\lambda r^{k-|\alpha|}\omega(r), \quad |\alpha| \leq k. \]

On the other hand, by condition (2) of the theorem, for every \( \alpha, |\alpha| \leq L \), we have

\[ |D^\alpha P_K(x') - D^\alpha P_Q(x')| \leq \lambda(\max\{\bar{r}, r\} + \|x' - x\|)^{L-|\alpha|} \int_{\min\{\bar{r}, r\}}^{\bar{r} + r + \|x' - x\|} \frac{\omega(t)}{t^m} dt \]

\[ = \lambda(\bar{r} + \|x' - x\|)^{L-|\alpha|} \int_{r}^{\bar{r} + \|x' - x\|} \frac{\omega(t)}{t^m} dt. \]

Since \( r \leq \bar{r} \leq 3r, \quad \|x' - x\| \leq r \), we obtain

\[ |D^\alpha P_K(x') - D^\alpha P_Q(x')| \leq 4^{L-|\alpha|} \lambda r^{L-|\alpha|} \int_{r}^{5r} \frac{\omega(t)}{t^m} dt \]

\[ \leq 4^{L-|\alpha|} \lambda r^{L-|\alpha|} \int_{r}^{5r} \frac{\omega(r)}{r^m} dt \]

so that

\[ |D^\alpha P_K(x') - D^\alpha P_Q(x')| \leq 4^{k+m} \lambda r^{k-|\alpha|}\omega(r), \quad |\alpha| \leq L. \]

Observe also that by (3.15) for all \( \alpha \) with \( |\alpha| \leq k \) we have

\[ |D^\alpha P_Q(x') - D^\alpha \tilde{P}_Q(x')| \leq C\lambda r^{k-|\alpha|}\omega(r) = C\lambda r^{k-|\alpha|}\omega(r). \]

Hence,

\[ |D^\alpha \tilde{P}_Q(x') - D^\alpha \tilde{P}_Q(x')| \leq |D^\alpha \tilde{P}_{Q'}(x') - D^\alpha \tilde{P}_K(x')| + |D^\alpha P_K(x') - D^\alpha P_Q(x')| \]

\[ + |D^\alpha P_Q(x') - D^\alpha \tilde{P}_Q(x')| \]

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so that by (3.16), (3.17) and (3.18) we obtain

(3.19) \[ |D^\alpha \tilde{P}_{Q'}(x') - D^\alpha \tilde{P}_{Q}(x')| \leq C\lambda r^{k-|\alpha|}\omega(r) \quad \text{for all } \alpha, |\alpha| \leq k. \]

Now consider the case $k < |\alpha| \leq L$. By (3.14) and (3.9) we have

\[
|D^\alpha \tilde{P}_{Q'}(x') - D^\alpha \tilde{P}_{K}(x')| = |D^\alpha P_{Q'}(x') - D^\alpha P_{K}(x')| \\
\leq C\lambda \int_{r'}^{2\bar{r}} \frac{\omega(t)}{t^{|\alpha|-k}} \, dt \leq C\lambda \omega(2\bar{r}) \int_{r'}^{2\bar{r}} t^{k-|\alpha|} \, dt \\
\leq C\lambda (|\alpha| - k)^{-1}(r')^{k-|\alpha|}\omega(2\bar{r}).
\]

Since $\bar{r} \leq 3r$, we obtain

\[
|D^\alpha \tilde{P}_{Q'}(x') - D^\alpha \tilde{P}_{K}(x')| \leq C\lambda \frac{2^m}{|\alpha| - k}(r')^{k-|\alpha|}\omega(r) = C_1\lambda (r')^{k-|\alpha|}\omega(r).
\]

On the other hand, by (3.17) and (3.14)

\[
|D^\alpha \tilde{P}_{K}(x') - D^\alpha \tilde{P}_{Q}(x')| = |D^\alpha P_{K}(x') - D^\alpha P_{Q}(x')| \leq C\lambda r^{k-|\alpha|}\omega(r).
\]

Hence

\[
|D^\alpha \tilde{P}_{Q'}(x') - D^\alpha \tilde{P}_{Q}(x')| \leq |D^\alpha \tilde{P}_{Q'}(x') - D^\alpha \tilde{P}_{K}(x')| + |D^\alpha \tilde{P}_{K}(x') - D^\alpha \tilde{P}_{Q}(x')| \\
\leq C\lambda ((r')^{k-|\alpha|}\omega(r) + r^{k-|\alpha|}\omega(r)) \\
\leq C\lambda (r')^{k-|\alpha|}\omega(r), \quad k < |\alpha| \leq L.
\]

We have proved that for every two cubes $Q' = Q(x', r')$ and $Q = Q(x, r)$ such that $Q' \subset Q$, we have

(3.20) \[ |D^\alpha \tilde{P}_{Q'}(x') - D^\alpha \tilde{P}_{Q}(x')| \leq C\lambda \begin{cases} 
  r^{k-|\alpha|}\omega(r), & |\alpha| \leq k, \\
  (r')^{k-|\alpha|}\omega(r), & k < |\alpha| \leq L.
\end{cases} \]

To estimate $\sup_{Q'}|\tilde{P}_{Q'} - \tilde{P}_{Q}|$ we let $\tilde{Q}$ denote the cube $\tilde{Q} := Q(x, r' + \|x - x'\|)$. Since $Q' \subset Q$, we have $r' + \|x - x'\| \leq r$ so that $Q' \subset \tilde{Q} \subset Q$. Also, by (3.14),

\[
D^\alpha \tilde{P}_{\tilde{Q}}(x) = D^\alpha P_x(x) = D^\alpha \tilde{P}_{Q}(x)
\]

so that by (3.20)

\[
|D^\alpha \tilde{P}_{\tilde{Q}}(x') - D^\alpha \tilde{P}_{Q}(x)| \leq C\lambda \begin{cases} 
  0, & |\alpha| \leq k, \\
  (r' + \|x - x'\|)^{k-|\alpha|}\omega(r), & k < |\alpha| \leq L.
\end{cases}
\]
Therefore for every \( y \in \tilde{Q} \) we have

\[
|\tilde{P}_Q(y) - \tilde{P}_Q(y)| = \left| \sum_{|\alpha| \leq L} \frac{1}{\alpha!} \left( D^\alpha \tilde{P}_Q(x) - D^\alpha \tilde{P}_Q(x) \right) (y - x)^\alpha \right|
\]

\[
= \left| \sum_{k < |\alpha| \leq L} \frac{1}{\alpha!} \left( D^\alpha \tilde{P}_Q(x) - D^\alpha \tilde{P}_Q(x) \right) (y - x)^\alpha \right|
\]

\[
\leq \sum_{k < |\alpha| \leq L} \left| D^\alpha \tilde{P}_Q(x) - D^\alpha \tilde{P}_Q(x) \right| \|y - x\|^{|\alpha|}
\]

\[
\leq C \lambda \sum_{k < |\alpha| \leq L} (r' + \|x - x'\|)^{k-|\alpha|} \omega(r) r^{|\alpha|}_Q
\]

\[
\leq C_1 \lambda (r' + \|x - x'\|^k \omega(r).
\]

Thus

\[
\sup_{\tilde{Q}'} |\tilde{P}_Q - \tilde{P}_Q| \leq \sup_{\tilde{Q}} |\tilde{P}_Q - \tilde{P}_Q| \leq C \lambda (r' + \|x - x'\|^k \omega(r).
\]

Let us estimate \( \sup_{\tilde{Q}'} |\tilde{P}_Q' - \tilde{P}_Q| \). By (3.20) for every \( \alpha, |\alpha| \leq L \),

\[
|D^\alpha \tilde{P}_Q'(x') - D^\alpha \tilde{P}_Q(x)| \leq C \lambda \left\{ \begin{array}{l}
r^{|\alpha|}_Q \omega(r_Q), \quad |\alpha| \leq k, \\
(r')^{k-|\alpha|} \omega(r_Q), \quad k < |\alpha| \leq L.
\end{array} \right\
\]

Hence, for each \( y \in Q' \) we have

\[
|\tilde{P}_Q'(y) - \tilde{P}_Q(y)| = \left| \sum_{|\alpha| \leq L} \frac{1}{\alpha!} \left( D^\alpha \tilde{P}_Q'(x') - D^\alpha \tilde{P}_Q'(x') \right) (y - x')^\alpha \right|
\]

\[
\leq \sum_{|\alpha| \leq L} \left| D^\alpha \tilde{P}_Q'(x') - D^\alpha \tilde{P}_Q(x') \right| \|y - x'\|^{|\alpha|}
\]

\[
= \sum_{|\alpha| \leq k} \left| D^\alpha \tilde{P}_Q'(x') - D^\alpha \tilde{P}_Q(x') \right| \|y - x'\|^{|\alpha|}
\]

\[
+ \sum_{k < |\alpha| \leq L} \left| D^\alpha \tilde{P}_Q'(x') - D^\alpha \tilde{P}_Q(x') \right| \|y - x'\|^{|\alpha|}
\]

\[
\leq C \lambda \sum_{|\alpha| \leq k} (r_Q)^{k-|\alpha|} \omega(r_Q)(r')^{k-|\alpha|}
\]

\[
+ C \lambda \sum_{k < |\alpha| \leq L} (r')^{k-|\alpha|} \omega(r_Q)(r')^{|\alpha|}.
\]

Since \( r' \leq r_Q = r' + \|x - x'\| \), we obtain

\[
|\tilde{P}_Q'(y) - \tilde{P}_Q(y)| \leq C \lambda r_Q^{k} \omega(r_Q) = C \lambda (r' + \|x - x'\|^k \omega(r_Q)
\]

proving that

\[
\sup_{Q'} |\tilde{P}_Q' - \tilde{P}_Q| \leq C \lambda (r' + \|x - x'\|^k \omega(r_Q).
\]
Finally, since \( r_Q = r' + \|x - x'\| \leq r \), we have

\[
\sup_{Q'} |P_{Q'} - \tilde{P}_Q| \leq \sup_{Q'} |P_{Q'} - \tilde{P}_\tilde{Q}| + \sup_{Q'} |\tilde{P}_\tilde{Q} - \tilde{P}_Q|
\leq C \lambda (r' + \|x - x'\|)^k \omega(r) + C \lambda (r' + \|x - x'\|)^k \omega(r)
\leq 2C \lambda (r' + \|x - x'\|)^k \omega(r).
\]

Theorem 3.7 is completely proved.

\[\square\]

4. \( C^k \Lambda^m_\omega(\mathbb{R}^n) \) as a space of Lipschitz mappings

The point of departure for our approach is the inequality (3.5) of Theorem 3.7. This inequality motivates the definition of a certain metric on the set

\[ \mathcal{P}_L \times \mathcal{K} = \{ T = (P, Q) : P \in \mathcal{P}_L, Q \in \mathcal{K} \}. \]

This allows us to identify the restriction \( C^k \Lambda^m_\omega(\mathbb{R}^n)|_S \) with a space of Lipschitz mappings from \( \mathcal{K}(S) \) (equipped with a certain hyperbolic-type metric) into \( \mathcal{P}_L \times \mathcal{K} \).

Given \( v > 0 \) and a multiindex \( \alpha, |\alpha| \leq L \), we define a function \( \varphi_\alpha(\cdot; v) \) on \( \mathbb{R}_+ \) by letting

\[
(4.1) \quad \varphi_\alpha(t; v) := t^{L-|\alpha|} \int_v^{v+t} \frac{\omega(s)}{s^m} ds.
\]

(Recall that \( L := k + m - 1 \).) By \( \varphi_\alpha^{-1}(\cdot; v) \) we denote the inverse to the function \( \varphi_\alpha(\cdot; v) \) (i.e., the inverse to the function \( \varphi_\alpha \) with respect to the first argument). Since for every \( v > 0 \) the function \( \varphi_\alpha(\cdot; v) \) is strictly increasing, the function \( \varphi_\alpha^{-1}(\cdot; v) \) is well-defined.

Thus for every \( u \geq 0 \) we have

\[
(4.2) \quad \varphi_\alpha^{-1}(u; v)^{L-|\alpha|} \int_v^{u+\varphi_\alpha^{-1}(u; v)} \frac{\omega(s)}{s^m} ds = u.
\]

In particular,

\[
(4.3) \quad \int_v^{v+\varphi_\alpha^{-1}(u; v)} \frac{\omega(s)}{s^m} ds = u, \quad |\alpha| = L.
\]

Now fix two elements

\[ T_1 = (P_1, Q_1), \quad T_2 = (P_2, Q_2) \in \mathcal{P}_L \times \mathcal{K} \]

where \( Q_1 = Q(x_1, r_1), \quad Q_2 = Q(x_2, r_2) \in \mathcal{K} \) and \( P_i \in \mathcal{P}_L, i = 1, 2 \). Put

\[
\Delta(T_1, T_2) := \max \{ \max \{ r_1, r_2 \} + \|x_1 - x_2\|, \max_{|\alpha| \leq L, i = 1, 2} \varphi_\alpha^{-1}(\|D^\alpha(P_1 - P_2)(x_i)\|; \min \{ r_1, r_2 \}) \}
\]

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and

\[ \delta_\omega(T_1, T_2) := \int_{\min\{r_1, r_2\}}^{\min\{r_1, r_2\} + \Delta(T_1, T_2)} \frac{\omega(s)}{s^m} ds, \]  

if \( T_1 \neq T_2 \), and \( \delta_\omega(T_1, T_2) := 0 \), if \( T_1 = T_2 \).

Observe that definition (4.4) and equality (4.3) imply the following explicit formula for \( \delta_\omega(T_1, T_2) \), \( T_1 \neq T_2 \):

\[ \delta_\omega(T_1, T_2) := \max \left\{ \int_{\min\{r_1, r_2\}}^{\min\{r_1, r_2\} + \Delta(T_1, T_2)} \frac{\omega(s)}{s^m} ds, \max_{|\alpha| = L} |D^\alpha P_1 - D^\alpha P_2|, \right. \]
\[ \left. \max_{|\alpha| < L, i = 1, 2} \int_{\min\{r_1, r_2\}}^{\min\{r_1, r_2\} + \varphi^{-1}_\omega(|(D^\alpha(P_1 - P_2)(x_i)); \min\{r_1, r_2\})} \frac{\omega(s)}{s^m} ds \right\}. \]

Let us introduce a function \( \rho_\omega : \mathcal{K} \times \mathcal{K} \rightarrow \mathbb{R}_+ \) by letting

\[ \rho_\omega(Q_1, Q_2) := \begin{cases} \int_{\min{r_1, r_2}}^{r_1 + r_2 + \|x_1 - x_2\|} \frac{\omega(s)}{s^m} ds, & Q_1 \neq Q_2, \\ 0, & Q_1 = Q_2. \end{cases} \]

Here \( Q_i = Q(x_i, r_i), \ i = 1, 2 \). As we shall see below, \( \rho_\omega \) is a metric on \( \mathcal{K} \), see Remark 4.4.

Observe that for every \( P \in \mathcal{P}_L \) and \( Q_i \in \mathcal{K}, i = 1, 2 \), we have

\[ \delta_\omega((P, Q_1), (P, Q_2)) = \rho_\omega(Q_1, Q_2). \]

In these settings the inequality (3.5) of Theorem 3.7 can be reformulated in the following way.

**Claim 4.1** Given a family of polynomials

\[ \{P_Q \in \mathcal{P}_L : Q \in \mathcal{K}(S)\} \]

and a constant \( \lambda > 0 \) the following two statements are equivalent:

(i). For every two cubes \( Q_1 = Q(x_1, r_1), Q_2 = Q(x_2, r_2) \in \mathcal{K}(S) \), and every \( \alpha, |\alpha| \leq L, \)

\[ |(D^\alpha P_{Q_1} - D^\alpha P_{Q_2})(x_1)| \leq \lambda (\max\{r_1, r_2\} + \|x_1 - x_2\|)^L \int_{\min\{r_1, r_2\}}^{r_1 + r_2 + \|x_1 - x_2\|} \frac{\omega(t)}{t^m} dt. \]

(Recall that this is inequality (3.5) of Theorem 3.7).

(ii). Let \( T : \mathcal{K}(S) \rightarrow \mathcal{P}_L \times \mathcal{K} \) be a mapping defined by the formula \( T(Q) := (P_Q, Q) \), \( Q \in \mathcal{K}(S) \). Then

\[ \delta_\omega(\lambda^{-1} \circ T(Q_1), \lambda^{-1} \circ T(Q_2)) \leq \rho_\omega(Q_1, Q_2), \ Q_1, Q_2 \in \mathcal{K}(S). \]
(Recall that $\lambda \circ T := (\lambda P, Q)$ provided $T = (P, Q) \in \mathcal{P}_L \times \mathcal{K}$ and $\lambda \in \mathbb{R}$.)

**Proof.** Put

$$A_i := |(D^n(\lambda^{-1}P_{Q_i}) - D^n(\lambda^{-1}P_{Q_2}))(x_i)|, \quad i = 1, 2.$$  

By definition (4.2) inequality (4.8) can be reformulated as follows:

$$A_1 \leq \varphi_\alpha(\max\{r_1, r_2\} + \|x_1 - x_2\|; \min\{r_1, r_2\}).$$

Hence

$$\varphi^{-1}_\alpha(A_1; \min\{r_1, r_2\}) \leq \max\{r_1, r_2\} + \|x_1 - x_2\|.$$  

Changing the order of cubes in this inequality and taking the maximum over all $\alpha, |\alpha| \leq L$, we conclude that (4.8) is equivalent to the following inequality:

$$\max_{|\alpha| \leq L, i = 1, 2} \varphi^{-1}_\alpha(A_i; \min\{r_1, r_2\}) \leq \max\{r_1, r_2\} + \|x_1 - x_2\|.$$  

In turn, by definition (4.4), this inequality is equivalent to the next one:

$$\Delta\left(\frac{1}{\lambda} \circ T(Q_1), \frac{1}{\lambda} \circ T(Q_2)\right) = \max_{|\alpha| \leq L, i = 1, 2} \{\max\{r_1, r_2\} + \|x_1 - x_2\|, \varphi^{-1}_\alpha(A_i; \min\{r_1, r_2\})\}$$

$$\leq \max\{r_1, r_2\} + \|x_1 - x_2\|.$$  

Since the function $t \rightarrow \int_v^{v+t} \frac{\omega(s)}{s^m} ds$ is strictly increasing, the inequality

$$\Delta(\lambda^{-1} \circ T(Q_1), \lambda^{-1} \circ T(Q_2)) \leq \max\{r_1, r_2\} + \|x_1 - x_2\|$$

is equivalent to

$$\delta_\omega(\lambda^{-1} \circ T(Q_1), \lambda^{-1} \circ T(Q_2)) := \int_{\min\{r_1, r_2\}}^{\min\{r_1, r_2\} + \Delta(\lambda^{-1} \circ T(Q_1), \lambda^{-1} \circ T(Q_2))} \frac{\omega(s)}{s^m} ds$$

$$\leq \int_{\min\{r_1, r_2\}}^{\min\{r_1, r_2\} + \max\{r_1, r_2\} + \|x_1 - x_2\|} \frac{\omega(s)}{s^m} ds$$

$$= \int_{\min\{r_1, r_2\}}^{\min\{r_1 + r_2 + \|x_1 - x_2\|}} \frac{\omega(s)}{s^m} ds = \rho_\omega(Q_1, Q_2).$$

The claim is proved. \(\Box\)

Let us define a metric on $\mathcal{P}_L \times \mathcal{K}$ as a geodesic metric generated by the function $\delta_\omega$. Given $T, T' \in \mathcal{P}_L \times \mathcal{K}$ we put

$$d_\omega(T, T') := \inf\sum_{i=0}^{M-1} \delta_\omega(T_i, T_{i+1})$$

where the infimum is taken over all finite families $\{T_0, T_1, ..., T_M\} \subset \mathcal{P}_L \times \mathcal{K}$ such that $T_0 = T$ and $T_M = T'$.

Observe several elementary properties of $d_\omega$. In particular, as we have noted above, the function $\rho_\omega : \mathcal{K} \times \mathcal{K} \to \mathbb{R}_+$ is a metric on $\mathcal{K}$, see Remark 4.4. This property of $\rho_\omega$ and (4.5)
and (4.4) immediately imply the following inequality: for every $P_i \in \mathcal{P}_L, Q_i \in \mathcal{K}, i = 1, 2$, we have
\[ d_\omega((P_1, Q_1), (P_2, Q_2)) \geq \rho_\omega(Q_1, Q_2). \]
In turn, this inequality and (4.7) imply the following:
\[ d_\omega((P, Q_1), (P, Q_2)) = \rho_\omega(Q_1, Q_2), \quad P \in \mathcal{P}_L, Q_i \in \mathcal{K}, \ i = 1, 2. \]

The main result of the section is the following

**Theorem 4.2** For every $T, T' \in \mathcal{P}_L \times \mathcal{K}$ we have
\[ d_\omega(T, T') \leq \delta_\omega(T, T') \leq d_\omega(e^n \circ T, e^n \circ T'). \]

The proof of this theorem relies on a series of auxiliary lemmas.

**Lemma 4.3** For every $b_0, b_1, \ldots, b_{\ell} > 0$, $a_0, a_1, \ldots, a_{\ell-1} \geq 0$ and $c_0, c_1, \ldots, c_{\ell-1} \geq 0$ we have
\[
\begin{align*}
\max \left\{ \begin{array}{c}
\frac{b_0+b_\ell+\sum_{i=0}^{\ell-1} a_i}{\min\{b_0,b_\ell\}} \int_{\min\{b_0,b_\ell\}}^{\min\{b_0,b_\ell\}+\sum_{i=0}^{\ell-1} c_i} \frac{\omega(t)}{t^m} dt, \\
\frac{\min\{b_0,b_\ell\}+\sum_{i=0}^{\ell-1} c_i}{\min\{b_0,b_\ell\}} \int_{\min\{b_0,b_\ell\}}^{\min\{b_0,b_\ell\}+\sum_{i=0}^{\ell-1} c_i} \frac{\omega(t)}{t^m} dt
\end{array} \right\} \\
\leq \sum_{i=0}^{\ell-1} \max \left\{ \begin{array}{c}
\frac{b_i+b_{i+1}+a_i}{\min\{b_i,b_{i+1}\}} \int_{\min\{b_i,b_{i+1}\}}^{\min\{b_i,b_{i+1}\}+c_i} \frac{\omega(t)}{t^m} dt, \\
\frac{\min\{b_i,b_{i+1}\}+c_i}{\min\{b_i,b_{i+1}\}} \int_{\min\{b_i,b_{i+1}\}}^{\min\{b_i,b_{i+1}\}+c_i} \frac{\omega(t)}{t^m} dt
\end{array} \right\}.
\end{align*}
\]

**Proof.** We put $s_{-1} = a_{-1} = c_{-1} := 0,$
\[ s_i := \max\{ \max\{b_i, b_{i+1}\} + a_i, c_i \}, \quad i = 0, \ldots, \ell - 1, \]
and
\[ I := [\min\{b_0, b_\ell\}, \min\{b_0, b_\ell\} + \max\{\max\{b_0, b_\ell\} + \sum_{i=0}^{\ell-1} a_i, \sum_{i=0}^{\ell-1} c_i\}] \]

Then the inequality of the lemma is equivalent to the following one:
\[
\int_{I} \frac{\omega(t)}{t^m} dt \leq \sum_{i=0}^{\ell-1} \int_{\min\{b_i,b_{i+1}\}}^{\min\{b_i,b_{i+1}\}+s_i} \frac{\omega(t)}{t^m} dt.
\]

To prove this inequality we put
\[ I_i := [\min\{b_0, b_\ell\} + \sum_{j=-1}^{i-1} s_j, \min\{b_0, b_\ell\} + \sum_{j=-1}^{i} s_j], \quad i = 0, \ldots, \ell - 1. \]
Then $|I_i| = s_i$ and
\[
\sum_{j=1}^{l-1} s_j = \sum_{j=1}^{l-1} \max\{\max\{b_j, b_{j+1}\} + a_j, c_j\}
\geq \max\{\sum_{j=1}^{l-1} \max\{b_j, b_{j+1}\} + \sum_{j=1}^{l-1} a_j, \sum_{j=1}^{l-1} c_j\}
\geq \max\{\max\{b_0, b_\ell\} + \sum_{j=1}^{l-1} a_j, \sum_{j=1}^{l-1} c_j\}.
\]

Thus
\[
\bigcup_{i=0}^{\ell-1} I_i \supset I
\]
so that
\[
(4.12) \quad \int_{I} \frac{\omega(t)}{t^m} dt \leq \sum_{i=0}^{\ell-1} \int_{I_i} \frac{\omega(t)}{t^m} dt.
\]

Put
\[
A_i := [\min\{b_i, b_{i+1}\}, \min\{b_i, b_{i+1}\} + s_i], \quad i = 0, \ldots, \ell - 1.
\]

Then $|A_i| = |I_i| = s_i$. But the left end of the segment $I_i$ is bigger than the left end of the segment $A_i$. In fact,
\[
\min\{b_0, b_\ell\} + \sum_{j=1}^{i-1} s_j \geq s_{i-1} = \max\{\max\{b_{i-1}, b_i\} + a_{i-1}, c_{i-1}\} \geq b_i \geq \min\{b_i, b_{i+1}\}.
\]

Thus the segment $A_i$ is a shift of $I_i$ to the left. Since $\omega(t)/t^m$ is non-increasing, this implies
\[
\int_{I_i} \frac{\omega(t)}{t^m} dt \leq \int_{A_i} \frac{\omega(t)}{t^m} dt = \int_{\min\{b_i, b_{i+1}\}}^{\min\{b_i, b_{i+1}\} + s_i} \frac{\omega(t)}{t^m} dt.
\]

This inequality and inequality (4.12) imply (4.11). The lemma is proved. \(\square\)

**Remark 4.4** We put in Lemma 4.3 $\ell = 2$, $c_0 = c_1 = c_2 := 0$ and get
\[
(4.13) \quad \int_{\min\{b_0, b_2\}}^{b_0 + b_2 + a_0 + a_1} \frac{\omega(t)}{t^m} dt \leq \int_{\min\{b_0, b_1\}}^{b_0 + b_1 + a_0} \frac{\omega(t)}{t^m} dt + \int_{\min\{b_1, b_2\}}^{b_1 + b_2 + a_1} \frac{\omega(t)}{t^m} dt.
\]

This inequality easily implies the triangle inequality for the function $\rho_\omega : K \times K \to R_+$ defined by (4.6). In fact, for every cubes $Q_i = Q(x_i, r_i) \in K, i = 0, 1, 2$, we have
\[
\rho_\omega(Q_0, Q_2) \leq \int_{\min\{r_0, r_1\}}^{r_0 + r_2 + \|x_0 - x_2\|} \frac{\omega(s)}{s^m} ds \leq \int_{\min\{r_0, r_2\}}^{r_0 + r_2 + \|x_0 - x_1\| + \|x_1 - x_2\|} \frac{\omega(s)}{s^m} ds
\]
\[
\int_{\min\{r_0, r_2\}}^{r_0 + r_2 + \|x_0 - x_2\|} \frac{\omega(s)}{s^m} ds
\]
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so that by (4.13)
\[ \rho_\omega(Q_0, Q_2) \leq \int_{\min\{r_0, r_1\}}^{r_0 + r_1 + \|x_0 - x_1\|} \frac{\omega(s)}{s^m} \, ds + \int_{\min\{r_1, r_2\}}^{r_1 + r_2 + \|x_1 - x_2\|} \frac{\omega(s)}{s^m} \, ds = \rho_\omega(Q_0, Q_1) + \rho_\omega(Q_1, Q_2). \]

**Lemma 4.5** Let \( \{P_0, P_1, ..., P_\ell\} \) be a finite subfamily of \( \mathcal{P}_L \) and let \( \{x_0, x_1, ..., x_\ell\} \) be a subset of \( \mathbb{R}^n \).

Then for every \( \alpha, |\alpha| \leq L \), we have
\[
|D^\alpha(P_0 - P_\ell)(x_0)| \leq \epsilon^n \max_{|\beta| \leq L - |\alpha|} \left( \sum_{i=0}^{\ell-1} |D^{\alpha + \beta}(P_i - P_{i+1})(x_i)| \right) \cdot \left( \sum_{i=0}^{\ell-1} \|x_i - x_{i+1}\| \right)^{|\beta|}.
\]

**Proof.** We have
\[
|D^\alpha(P_i - P_{i+1})(x')| = \left| \sum_{|\beta| \leq L - |\alpha|} \frac{1}{\beta!} D^{\alpha + \beta}(P_i - P_{i+1})(x_i) \cdot (x' - x_i)^\beta \right|
\leq \sum_{|\beta| \leq L - |\alpha|} \frac{1}{\beta!} |D^{\alpha + \beta}(P_i - P_{i+1})(x_i)| \cdot \|x' - x_i\|^{|\beta|}
\leq \sum_{|\beta| \leq L - |\alpha|} \frac{1}{\beta!} \left( \sum_{i=0}^{\ell-1} |D^{\alpha + \beta}(P_i - P_{i+1})(x_i)| \cdot \|x' - x_i\|^{|\beta|} \right).
\]

so that
\[
|D^\alpha(P_0 - P_\ell)(x')| \leq \sum_{i=0}^{\ell-1} |D^\alpha(P_i - P_{i+1})(x')|
\leq \sum_{i=0}^{\ell-1} \sum_{|\beta| \leq L - |\alpha|} \frac{1}{\beta!} |D^{\alpha + \beta}(P_i - P_{i+1})(x_i)| \cdot \|x' - x_i\|^{|\beta|}
= \sum_{|\beta| \leq L - |\alpha|} \frac{1}{\beta!} \left( \sum_{i=0}^{\ell-1} |D^{\alpha + \beta}(P_i - P_{i+1})(x_i)| \cdot \|x' - x_i\|^{|\beta|} \right).
\]

Hence,
\[
|D^\alpha(P_0 - P_\ell)(x')| \leq \left( \sum_{|\beta| \leq L} \frac{1}{\beta!} \right) \max_{|\beta| \leq L - |\alpha|} \sum_{i=0}^{\ell-1} |D^{\alpha + \beta}(P_i - P_{i+1})(x_i)| \cdot \|x' - x_i\|^{|\beta|}
\leq \epsilon^n \max_{|\beta| \leq L - |\alpha|} \sum_{i=0}^{\ell-1} |D^{\alpha + \beta}(P_i - P_{i+1})(x_i)| \cdot \|x' - x_i\|^{|\beta|}.
\]

It remains to note that
\[
\|x' - x_i\| = \|x_0 - x_i\| \leq \sum_{j=0}^{\ell-1} \|x_j - x_{j+1}\|
\]
and the lemma follows. \(\square\)
Lemma 4.6 For every $v, R, t > 0$ and every $\alpha, \beta, |\alpha + \beta| \leq L$, we have
\[
\int_v^{v+\varphi_{-\alpha}^{-1}(R^{[\beta]}, t; v)} \frac{\omega(s)}{s^m} ds \leq \max \left\{ \int_v^{v+R} \frac{\omega(s)}{s^m} ds, \int_v^{v+\varphi_{-\alpha}^{-1}(R^{[\beta]}, t; v)} \frac{\omega(s)}{s^m} ds \right\}.
\]

Proof. Put $s := \varphi_{\alpha + \beta}^{-1}(t; v)$ and $q := \max\{R, s\}$. Since $R \leq q$, we obtain
\[
R^{[\beta]} q^{L-|\alpha + \beta|} \leq q^{L-|\alpha|},
\]
so that
\[
R^{[\beta]} q^{L-|\alpha + \beta|} \int_v^{v+q} \frac{\omega(s)}{s^m} ds \leq q^{L-|\alpha|} \int_v^{v+q} \frac{\omega(s)}{s^m} ds.
\]
By definition, see (4.1),
\[
\varphi_{\alpha}(u; v) := u^{L-|\alpha|} \int_v^{v+u} \frac{\omega(s)}{s^m} ds, \quad u > 0,
\]
so that the latter inequality can be written in the following form:
\[
R^{[\beta]} \varphi_{\alpha + \beta}(q; v) \leq \varphi_{\alpha}(q; v).
\]
Since $s \leq q$ and $\varphi_{\alpha + \beta}$ is increasing, this inequality implies the following
\[
R^{[\beta]} \varphi_{\alpha + \beta}(s; v) \leq \varphi_{\alpha}(q; v).
\]
But $\varphi_{\alpha + \beta}(s; v) = t$ so that $R^{[\beta]} t \leq \varphi_{\alpha}(q; v)$. Since $\varphi_{\alpha}^{-1}$ is increasing, we have
\[
\varphi_{\alpha}^{-1}(R^{[\beta]} t; v) \leq q = \max\{R, \varphi_{\alpha + \beta}^{-1}(t; v)\}
\]
proving the lemma. □

Lemma 4.7 For every $b_0, b_1, ..., b_\ell > 0$, $u_0, u_1, ..., u_{\ell-1} \geq 0$ and every $\alpha, |\alpha| \leq L$, we have
\[
\sum_{i=0}^{\ell-1} \max \left\{ \int_{\min\{b_i, b_{i+1}\}}^{b_i + b_{i+1} + 1} \frac{\omega(t)}{t^m} dt, \int_{\min\{b_i, b_{i+1}\}}^{\min\{b_{i+1}, b_{i+1} + 1\} + \varphi_{\alpha}^{-1}(u_i; \min\{b_i, b_{i+1}\})} \frac{\omega(t)}{t^m} dt \right\}
\]

Proof. Put
\[
A := \varphi_{\alpha}^{-1} \left( \sum_{i=0}^{\ell-1} u_i; \min\{b_0, b_\ell\} \right)
\]
and
\[
c_i := \varphi_{\alpha}^{-1}(u_i; \min\{b_i, b_{i+1}\}), \quad i = 0, ..., \ell - 1.
\]
Assume that
\[ A \leq B := \sum_{i=0}^{\ell-1} c_i = \sum_{i=0}^{\ell-1} \varphi^{-1}_\alpha(u_i; \min\{b_i, b_i+1\}). \]

Hence
\[
I := \int_{\min\{b_0, b_\ell\}}^{\min\{b_0, b_\ell\}+A} \frac{\omega(t)}{t^m} \, dt = \int_{\min\{b_0, b_\ell\}}^{\min\{b_0, b_\ell\}+A} \frac{\varphi^{-1}(u_i; \min\{b_0, b_\ell\}) + A}{\min\{b_0, b_\ell\}} \, dt = \sum_{i=0}^{\ell-1} u_i
\]
so that by Lemma 4.3 (with \(a_i = 0, i = 0, \ldots, \ell - 1\)) we obtain
\[
I \leq \sum_{i=0}^{\ell-1} \max\left\{ \int_{\min\{b_i, b_i+1\}}^{\min\{b_i, b_i+1\}+c_i} \frac{\omega(t)}{t^m} \, dt, \int_{\min\{b_i, b_i+1\}}^{\min\{b_i, b_i+1\}+c_i} \frac{\varphi^{-1}(u_i; \min\{b_0, b_\ell\}) + A}{\min\{b_0, b_\ell\}} \, dt \right\}.
\]
This proves the lemma under the assumption \(A \leq B\).

Suppose that \(A > B\). By identity (4.2)
\[
\left(\varphi^{-1}_\alpha\left(\sum_{i=0}^{\ell-1} u_i; \min\{b_0, b_\ell\}\right)\right)^{L-|\alpha|} \int_{\min\{b_0, b_\ell\}}^{\min\{b_0, b_\ell\}+A} \frac{\varphi^{-1}(u_i; \min\{b_0, b_\ell\})}{\min\{b_0, b_\ell\}} \, dt = \sum_{i=0}^{\ell-1} u_i
\]
so that
\[
A^{L-|\alpha|} \int_{\min\{b_0, b_\ell\}}^{\min\{b_0, b_\ell\}+A} \frac{\omega(t)}{t^m} \, dt = \sum_{i=0}^{\ell-1} u_i,
\]
Hence
\[
I = \int_{\min\{b_0, b_\ell\}}^{\min\{b_0, b_\ell\}+A} \frac{\omega(t)}{t^m} \, dt = A^{L-|\alpha|} \int_{\min\{b_0, b_\ell\}}^{\min\{b_0, b_\ell\}+A} \frac{\omega(t)}{t^m} \, dt = A^{L-|\alpha|} \sum_{i=0}^{\ell-1} u_i.
\]
Since \(A > B\) and \(|\alpha| \leq L\), we obtain
\[
(4.14)\quad I = A^{L-|\alpha|} \sum_{i=0}^{\ell-1} u_i \leq B^{L-|\alpha|} \sum_{i=0}^{\ell-1} u_i.
\]
Again, by identity (4.2) for every \(i = 0, \ldots, \ell - 1\), we have
\[
u_i = (\varphi^{-1}_\alpha(u_i; \min\{b_i, b_i+1\}))^{L-|\alpha|} \int_{\min\{b_i, b_i+1\}}^{\min\{b_i, b_i+1\}+A} \frac{\omega(t)}{t^m} \, dt
\]
so that by (4.14)

\[ I \leq \sum_{i=0}^{\ell-1} \left( \frac{\varphi^{-1}_\alpha(u_i; \min\{b_i, b_{i+1}\})}{B} \right)^{L-|\alpha|} \frac{\omega(t)}{t^m} \ dt. \]

But

\[ \varphi^{-1}_\alpha(u_i; \min\{b_i, b_{i+1}\}) \leq B := \sum_{j=0}^{\ell-1} \varphi^{-1}_\alpha(u_j; \min\{b_j, b_{j+1}\}) \]

for every \( i = 0, \ldots, \ell - 1 \), so that

\[ I \leq \sum_{i=0}^{\ell-1} \min\{b_i, b_{i+1}\} \varphi^{-1}_\alpha(u_i; \min\{b_i, b_{i+1}\}) \frac{\omega(t)}{t^m} \ dt. \]

The lemma is proved. \( \square \)

**Proof of Theorem 3.1.** The inequality \( d_\omega(T, T') \leq \delta_\omega(T, T') \) trivially follows from definition (4.10) of the metric \( d_\omega \).

In turn, the inequality \( \delta_\omega(T, T') \leq d_\omega(e^n \circ T, e^n \circ T') \) is equivalent to the following statement: Let

\[ \{T_i = (P_i, Q_i) \in P_L \times K : i = 0, \ldots, \ell\} \]

where \( Q_i = Q(x_i, r_i) \), be a subfamily of \( P_L \times K \) such that \( T_0 = T, T_\ell = T' \). Put

\[ (4.15) \quad I := \int_{\min\{r_0, r_\ell\}}^{\min\{r_0, r_\ell\} + \Delta(T_0, T_\ell)} \frac{\omega(t)}{t^m} \ dt, \]

and

\[ A := \sum_{i=0}^{\ell-1} \int_{\min\{r_i, r_{i+1}\}}^{\min\{r_i, r_{i+1}\} + \Delta(e^n \circ T_i, e^n \circ T_{i+1})} \frac{\omega(t)}{t^m} \ dt. \]

Then

\[ (4.16) \quad I \leq A. \]

For the sake of brevity we put

\[ v := \min\{r_0, r_\ell\}. \]

Recall that

\[ \Delta(T_0, T_\ell) := \max_{|\alpha| \leq L} \left\{ \max\{r_0, r_\ell\} + \|x_0 - x_\ell\|, \varphi^{-1}_\alpha(|D_\alpha(P_0 - P_\ell)(x_0)|; v), \varphi^{-1}_\alpha(|D_\alpha(P_0 - P_\ell)(x_\ell)|; v) \right\}. \]

Given multiindex \( \alpha, |\alpha| \leq L \), we put

\[ I_\alpha := \int_{v}^{v+\varphi^{-1}_\alpha(|D_\alpha(P_0 - P_\ell)(x_0)|; v)} \frac{\omega(t)}{t^m} \ dt, \]

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and
\[ J_\alpha := \int \frac{\omega(t)}{t^m} dt. \]

Then by (4.15)
\[ I := \int \frac{\omega(t)}{t^m} dt = \max_{|\alpha| \leq L} \left\{ \int \frac{\omega(t)}{t^m} dt, I_\alpha, J_\alpha \right\}. \]

Prove that
\[ \int \frac{\omega(t)}{t^m} dt \leq A. \]

In fact,
\[ \int \frac{\omega(t)}{t^m} dt = \min_{\omega, r_x} \left\{ \int \frac{\omega(t)}{t^m} dt \right\}. \]

so that applying Lemma 4.3 (with \( c_i = 0, i = 0, \ldots, \ell - 1 \)) we obtain
\[ \int \frac{\omega(t)}{t^m} dt \leq \sum_{i=0}^{\ell-1} \min_{r_i, r_{i+1}} \int \frac{\omega(t)}{t^m} dt. \]

By definition (4.4) for every \( i = 0, \ldots, \ell - 1 \), we have
\[ \max\{r_i, r_{i+1}\} + \|x_i - x_{i+1}\| \leq \Delta(e^h \circ T_i, e^h \circ T_{i+1}) \]
so that
\[ r_i + r_{i+1} + \|x_i - x_{i+1}\| = \min\{r_i, r_{i+1}\} + \max\{r_i, r_{i+1}\} + \|x_i - x_{i+1}\| \leq \min\{r_i, r_{i+1}\} + \Delta(e^h \circ T_i, e^h \circ T_{i+1}). \]

Hence
\[ \int \frac{\omega(t)}{t^m} dt \leq \int \frac{\omega(t)}{t^m} dt \]

which implies the following inequality
\[ \int \frac{\omega(t)}{t^m} dt \leq \sum_{i=0}^{\ell-1} \min_{r_i, r_{i+1}} \int \frac{\omega(t)}{t^m} dt = A. \]
Thus
\[ v^+\|x_0-x_l\| \leq \int_v^{v+\sum_{i=0}^{\ell-1}\|x_i-x_{i+1}\|} \frac{\omega(t)}{t^m} dt \leq A. \]
proving (4.18).

Now prove that
\[ I_\alpha \leq A, \quad |\alpha| \leq L. \tag{4.21} \]

To this end given multiindex \( \gamma \) and \( i = 0, ..., \ell - 1 \), we put
\[ U_{\gamma,i} := |D^\gamma(e^n P_i - e^n P_{i+1})(x_i)| \tag{4.22} \]
and
\[ U_\gamma := \sum_{i=0}^{\ell-1} U_{\gamma,i}. \]

We also set
\[ R := \sum_{i=0}^{\ell-1} \|x_i - x_{i+1}\|. \]

Then by Lemma 4.5
\[
\left| D^\alpha(P_0 - P_\ell)(x_0) \right| \leq \max_{|\beta| \leq L - |\alpha|} \left( \sum_{i=0}^{\ell-1} |D^{\alpha+\beta}(e^n P_i - e^n P_{i+1})(x_i)| \right) \cdot \left( \sum_{i=0}^{\ell-1} \|x_i - x_{i+1}\| \right) \tag{4.23}
\]
Hence
\[
I_\alpha \leq \int_v^{v+\varphi^{-1}_\alpha(\max_{|\beta| \leq L - |\alpha|} R^{\beta}U_{\alpha+\beta})} \frac{\omega(t)}{t^m} dt = \max_{|\beta| \leq L - |\alpha|} \int_v^{v+\varphi^{-1}_\alpha(R^{\beta}U_{\alpha+\beta})} \frac{\omega(t)}{t^m} dt
\]
so that by Lemma 4.6
\[
I_\alpha \leq \max_{|\beta| \leq L - |\alpha|} \left\{ \int_v^{v+R} \frac{\omega(t)}{t^m} dt, \int_v^{v+\varphi^{-1}_\alpha(U_{\alpha+\beta})} \frac{\omega(t)}{t^m} dt \right\}. \tag{4.23}
\]

By (4.20)
\[
\int_v^{v+R} \frac{\omega(t)}{t^m} dt = \int_v^{v+\sum_{i=0}^{\ell-1}\|x_i-x_{i+1}\|} \frac{\omega(t)}{t^m} dt \leq A. \tag{4.24}
\]
Prove that
\[ v + \varphi_{\alpha + \beta}^{-1}(U_{\alpha + \beta}; v) \]
\[ \int_v \frac{\omega(t)}{t^m} dt \leq A. \]  
(4.25)

For the sake of brevity we put
\[ v_i := \min\{r_i, r_{i+1}\}, \quad i = 0, \ldots, \ell - 1. \]

Then by Lemma 4.7
\[ v + \varphi_{\alpha + \beta}^{-1}(U_{\alpha + \beta}; v) \]
\[ \int_v \frac{\omega(t)}{t^m} dt = \min\{r_0, r_\ell\} + \varphi_{\alpha + \beta}^{-1}(\sum_{i=0}^{\ell-1} U_{\alpha + \beta,i}; \min\{r_0, r_\ell\}) \]
\[ \int_v \frac{\omega(t)}{t^m} dt \leq \sum_{i=0}^{\ell-1} \max \left\{ \int_{v_i} \frac{\omega(t)}{t^m} dt, \int_{v_i} \frac{\omega(t)}{t^m} dt \right\}. \]

By (4.19)
\[ (4.26) \int_{v_i} \frac{\omega(t)}{t^m} dt = \min\{r_i, r_{i+1}\} \int_{v_i} \frac{\omega(t)}{t^m} dt \leq \min\{r_i, r_{i+1}\} \int_{v_i} \frac{\omega(t)}{t^m} dt. \]

In turn, by (4.22) and definition (4.4)
\[ \varphi_{\alpha + \beta}^{-1}(U_{\alpha + \beta}; v_i) = \varphi_{\alpha + \beta}^{-1}(|D^n(e^n P_i - e^n P_{i+1})(x_i)|; \min\{r_i, r_{i+1}\}) \leq \Delta(e^n \circ T_i, e^n \circ T_{i+1})\]
so that
\[ \int_{v_i} \frac{\omega(t)}{t^m} dt \leq \min\{r_i, r_{i+1}\} \int_{v_i} \frac{\omega(t)}{t^m} dt. \]

Hence
\[ (4.25) \int_v \frac{\omega(t)}{t^m} dt \leq \sum_{i=0}^{\ell-1} \min\{r_i, r_{i+1}\} \int_{v_i} \frac{\omega(t)}{t^m} dt = A, \]
proving (4.25).

Now inequality (4.21) follows from (4.23), (4.24) and (4.25). In the same way we prove that
\[ J_\alpha \leq A, \quad |\alpha| \leq L. \]
Finally, this inequality, (4.25), (4.18) and (4.17) imply the required inequality (4.16).

Theorem 3.1 is proved. \[ \square \]

We present several results related to calculation of the function \( \delta_\omega \), see (4.4) and (4.5), and the metric \( d_\omega \), see (4.10).
Let \( T_1 = (P_1, Q_1), T_2 = (P_2, Q_2) \in \mathcal{P}_L \times \mathcal{K} \), where \( Q_1 = Q(x_1, r_1), Q_2 = Q(x_2, r_2) \in \mathcal{K} \) and \( P_i \in \mathcal{P}_L, i = 1, 2 \). Fix a point \( y \in \mathbb{R}^n \) and put

\[
\Delta(T_1, T_2; y) := \max \{ \max \{ r_1, r_2 \} + \| x_1 - x_2 \|, \max \varphi^{-1}(\{ |D^\alpha(P_1 - P_2)(y)|; \min \{ r_1, r_2 \} \}) \}.
\]

Thus, we define \( \Delta(T_1, T_2; y) \) by replacing in (4.4) the points \( x_i, i = 1, 2 \), by \( y \).

We also put

\[
\delta_\omega(T_1, T_2; y) := \int_{\min \{ r_1, r_2 \}}^{\min \{ r_1, r_2 \} + \Delta(T_1, T_2; y)} \frac{\omega(s)}{s^m} ds,
\]

for \( T_1 \neq T_2 \), and \( \delta_\omega(T_1, T_2; y) := 0 \), whenever \( T_1 = T_2 \). Clearly,

\[
\Delta(T_1, T_2) = \max \{ \Delta(T_1, T_2; x_1), \Delta(T_1, T_2; x_2) \},
\]

and

\[
\delta_\omega(T_1, T_2) = \max \{ \delta_\omega(T_1, T_2; x_1), \delta_\omega(T_1, T_2; x_2) \}.
\]

**Proposition 4.8** For every \( y, z \in \mathbb{R}^n \) and every \( T_i = (P_i, Q_i) \in \mathcal{P}_L \times \mathcal{K} \), where \( Q_i = Q(x_i, r_i) \in \mathcal{K}, i = 1, 2 \), we have

\[
\delta_\omega(T_1, T_2; z) \leq \delta_\omega(\gamma \circ T_1, \gamma \circ T_2; y),
\]

where

\[
\gamma = \max \left\{ 1, \frac{e^n \| y - z \|^L}{(\max \{ r_1, r_2 \} + \| x_1 - x_2 \|)^L} \right\}.
\]

**Proof.** Fix a multiindex \( \alpha, |\alpha| \leq L \), and put

\[
\widetilde{P}_0 := P_1, \quad \widetilde{P}_1 := P_1, \quad \widetilde{P}_2 := P_2, \quad \text{and} \quad \widetilde{x}_0 := z, \quad \widetilde{x}_1 := y, \quad \widetilde{x}_2 := y.
\]

Let us apply Lemma 4.5 to polynomials \( \{ \widetilde{P}_0, \widetilde{P}_1, \widetilde{P}_2 \} \) and points \( \{ \widetilde{x}_0, \widetilde{x}_1, \widetilde{x}_2 \} \). We have

\[
|D^\alpha(P_1 - P_2)(z)| = |D^\alpha(\widetilde{P}_0 - \widetilde{P}_2)(\widetilde{x}_0)|
\]

\[
\leq e^n \max_{|\beta| \leq L - |\alpha|} \left( \sum_{i=0}^{1} |D^{\alpha + \beta}(\widetilde{P}_i - \widetilde{P}_{i+1})(\widetilde{x}_i)| \right) \cdot \left( \sum_{i=0}^{1} \| \widetilde{x}_i - \widetilde{x}_{i+1} \| \right)^{|\beta|}
\]

\[
= e^n \max_{|\beta| \leq L - |\alpha|} |D^{\alpha + \beta}(P_1 - P_2)(y)| \cdot \| z - y \|^{|\beta|}.
\]

Put

\[
U_{\alpha + \beta} := |D^{\alpha + \beta}(\gamma P_1 - \gamma P_2)(y)|.
\]

Then

\[
|D^\alpha(P_1 - P_2)(z)| \leq \max_{|\beta| \leq L - |\alpha|} |D^{\alpha + \beta}(P_1 - P_2)(y)|
\]

\[
\cdot e^n \left( \frac{\| z - y \|}{(\max \{ r_1, r_2 \} + \| x_1 - x_2 \|)^L} \right)^{|\beta|} \left( \max \{ r_1, r_2 \} + \| x_1 - x_2 \| \right)^{|\beta|}
\]

\[
\leq \max_{|\beta| \leq L - |\alpha|} |D^{\alpha + \beta}(\gamma P_1 - \gamma P_2)(y)| \cdot (\max \{ r_1, r_2 \} + \| x_1 - x_2 \|)^{|\beta|}
\]

\[
= \max_{|\beta| \leq L - |\alpha|} (\max \{ r_1, r_2 \} + \| x_1 - x_2 \|)^{|\beta|} U_{\alpha + \beta}.
\]

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We also put
\[ v := \min\{r_1, r_2\}, \quad A := \max\{r_1, r_2\}. \]

Then
\[ I_\alpha := \int_v^{v+\varphi^{-1}_\alpha(|D^\alpha(P_1-P_2)(z)|;v)} \frac{\omega(s)}{s^m} \, ds \leq \max_{|\beta| \leq L-|\alpha|} \int_v^{v+\varphi^{-1}_\alpha((A+\|x_1-x_2\|)|\beta|U_{\alpha+\beta};v)} \frac{\omega(s)}{s^m} \, ds, \]

so that by Lemma 4.6
\[ I_\alpha \leq \max_{|\beta| \leq L-|\alpha|} \left\{ \int_v^{v+A+\|x_1-x_2\|} \frac{\omega(s)}{s^m} \, ds, \int_v^{v+\varphi^{-1}_\alpha(U_{\alpha+\beta};v)} \frac{\omega(s)}{s^m} \, ds \right\}. \]

But by (4.27)
\[ A + \|x_1 - x_2\| = \max\{r_1, r_2\} + \|x_1 - x_2\| \leq \Delta(\gamma \circ T_1, \gamma \circ T_2; y), \]

and
\[ \varphi^{-1}_{\alpha+\beta}(U_{\alpha+\beta}; v) = \varphi^{-1}_{\alpha+\beta}(|D^{\alpha+\beta}(\gamma P_1 - \gamma P_2)(y)|; \min\{r_1, r_2\}) \leq \Delta(\gamma \circ T_1, \gamma \circ T_2; y). \]

Hence
\[ I_\alpha \leq \int_v^{v+\Delta(\gamma \circ T_1, \gamma \circ T_2; y)} \frac{\omega(s)}{s^m} \, ds = \delta_\omega(\gamma \circ T_1, \gamma \circ T_2; y), \]

proving the proposition. \(\square\)

Proposition 4.8 and Theorem 4.2 imply the following

**Corollary 4.9** Let \( \theta \geq 1 \). For every \( T_i = (P_i, Q_i) \in P_L \times K \), where \( Q_i = Q(x_i, r_i) \in K \), \( i = 1, 2 \), and every point \( y \in \mathbb{R}^n \) such that
\[ \|y - x_1\| \leq \theta (r_1 + r_2 + \|x_1 - x_2\|) \]

we have
\[ \delta_\omega(\gamma^{-1} \circ T_1, \gamma^{-1} \circ T_2; y) \leq \delta_\omega(T_1, T_2) \leq \delta_\omega(\gamma \circ T_1, \gamma \circ T_2; y), \]

and
\[ \delta_\omega(\gamma^{-1} \circ T_1, \gamma^{-1} \circ T_2; y) \leq d_\omega(T_1, T_2) \leq \delta_\omega(\gamma \circ T_1, \gamma \circ T_2; y). \]

Here \( \gamma = \gamma(n, \theta) \) is a constant depending only on \( n \) and \( \theta \).

For instance, by this corollary,
\[ (4.29) \quad \delta_\omega(\gamma^{-1} \circ T_1, \gamma^{-1} \circ T_2; x_i) \leq d_\omega(T_1, T_2) \leq \delta_\omega(\gamma \circ T_1, \gamma \circ T_2; x_i), \quad i = 1, 2, \]

or
\[ \delta_\omega\left(\gamma^{-1} \circ T_1, \gamma^{-1} \circ T_2; \frac{x_1 + x_2}{2}\right) \leq d_\omega(T_1, T_2) \leq \delta_\omega\left(\gamma \circ T_1, \gamma \circ T_2; \frac{x_1 + x_2}{2}\right), \]

with \( \gamma = \gamma(n) \) depending only on \( n \).
Let us present one more formula for calculation of $\delta_\omega$. Given $v > 0$ and multiindex $\alpha, |\alpha| \leq L$, we put
\[
h(t; v) := \int_{v}^{v+t} \frac{\omega(s)}{s^m} ds, \quad t > 0,
\]
and
\[
\psi_\alpha(u; v) := \left( t \left[ h^{-1}(t; v) \right]^{L-|\alpha|} \right)^{-1}(u), \quad u > 0.
\]
Thus
\[
t \left[ h^{-1}(t; v) \right]^{L-|\alpha|} = \psi_\alpha^{-1}(t; v), \quad t > 0.
\]
(4.30)

**Proposition 4.10** For every $T_i = (P_i, Q_i) \in P_L \times K$, where $Q_i = Q(x_i, r_i) \in K$, $i = 1, 2$, we have
\[
\delta_\omega(T_1, T_2; x_1) = \max_{|\alpha| \leq L} \left\{ \frac{r_1 + r_2 + \|x_1 - x_2\|}{\min\{r_1, r_2\}} \left( \int_{v}^{v+\varphi^{-1}_\alpha(U; v)} \frac{\omega(s)}{s^m} ds, \psi_\alpha(|D^\alpha(P_1 - P_2)(x_1)|; \min\{r_1, r_2\}) \right) \right\}.
\]

*Proof.* We put $v := \min\{r_1, r_2\}$,
\[
U_\alpha := |D^\alpha(P_1 - P_2)(x_1)|,
\]
and
\[
I_\alpha := \int_{v}^{v+\varphi^{-1}_\alpha(U; v)} \frac{\omega(s)}{s^m} ds.
\]
Recall that
\[
\delta_\omega(T_1, T_2; x_1) = \max_{|\alpha| \leq L} \left\{ \frac{r_1 + r_2 + \|x_1 - x_2\|}{\min\{r_1, r_2\}} \left( \int_{v}^{v+\varphi^{-1}_\alpha(U; v)} \frac{\omega(s)}{s^m} ds, I_\alpha \right) \right\}.
\]
Prove that
\[
(4.31) \quad I_\alpha = \psi_\alpha(U_\alpha; v).
\]
Recall that by (4.2)
\[
(4.32) \quad \varphi^{-1}_\alpha(u; v)^{L-|\alpha|} \int_{v}^{v+\varphi^{-1}_\alpha(U; v)} \frac{\omega(s)}{s^m} ds = u
\]
so that
\[
I_\alpha = \frac{\varphi^{-1}_\alpha(U_\alpha; v)^{L-|\alpha|}}{\varphi^{-1}_\alpha(U_\alpha; v)^{L-|\alpha|}} \int_{v}^{v+\varphi^{-1}_\alpha(U; v)} \frac{\omega(s)}{s^m} ds = \frac{U_\alpha}{\varphi^{-1}_\alpha(U_\alpha; v)^{L-|\alpha|}}.
\]
Hence
\[
\varphi^{-1}_\alpha(U_\alpha; v) = \left( \frac{U_\alpha}{I_\alpha} \right)^{\frac{1}{L-|\alpha|}}
\]
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so that by (4.32)

\[
\left(\frac{U_\alpha}{T_\alpha}\right)^{\frac{1}{L-\lvert\alpha\rvert}} L^{-\lvert\alpha\rvert} v^{+\left(\frac{1}{L-\lvert\alpha\rvert}\right)^{\frac{1}{L-\lvert\alpha\rvert}}} \int_v^{v+\left(\frac{1}{L-\lvert\alpha\rvert}\right)^{\frac{1}{L-\lvert\alpha\rvert}}} \frac{\omega(s)}{s^m} ds = U_\alpha.
\]

We obtain

\[
\frac{U_\alpha}{T_\alpha} \int_v^{v+\left(\frac{1}{L-\lvert\alpha\rvert}\right)^{\frac{1}{L-\lvert\alpha\rvert}}} \frac{\omega(s)}{s^m} ds = U_\alpha
\]

which implies

\[
I_\alpha = \int_v^{v+\left(\frac{1}{L-\lvert\alpha\rvert}\right)^{\frac{1}{L-\lvert\alpha\rvert}}} \frac{\omega(s)}{s^m} ds = h((U_\alpha/I_\alpha)^{\frac{1}{L-\lvert\alpha\rvert}}; v).
\]

Hence \(h^{-1}(I_\alpha; v) = (U_\alpha/I_\alpha)^{\frac{1}{L-\lvert\alpha\rvert}}\), so that \(I_\alpha [h^{-1}(I_\alpha; v)]^{L-\lvert\alpha\rvert} = U_\alpha\). By (4.30) this implies \(\psi^{-1}_\alpha(I_\alpha; v) = U_\alpha\) proving (4.31) and the proposition. \(\square\)

In particular, by this proposition and (4.29), for every \(T_i = (P_i, Q_i) \in \mathcal{P}_L \times \mathcal{K}\), where \(Q_i = Q(x_i, r_i) \in \mathcal{K}, \ i = 1, 2\), we have

\[
d_\omega \left(\frac{1}{\gamma} \circ T_1, \frac{1}{\gamma} \circ T_2\right) \leq \max_{\lvert\alpha\rvert \leq L} \left\{ \int_{\min\{r_1, r_2\}}^{r_1 + r_2 + \|x_1 - x_2\|} \frac{\omega(s)}{s^m} ds, \psi_\alpha(|D^\alpha(P_1 - P_2)(x_1)|; \min\{r_1, r_2\}) \right\}
\]

\[
\leq d_\omega(\gamma \circ T_1, \gamma \circ T_2)
\]

where \(\gamma = \gamma(n)\) depends only on \(n\).

Let us present two examples.

**Example 4.11** Consider the case of the Zygmund space \(Z_m(\mathbb{R}^n) := \Lambda_m(\mathbb{R}^n)\) with \(\omega(t) = t^{m-1}\), see (2.1). In this case

\[
h(t; v) := \int_v^{v+t} \frac{1}{s} ds = \ln \left(1 + \frac{t}{v}\right), \ t > 0,
\]

so that \(h^{-1}(s; v) = v(e^s - 1)\). Hence

\[
\psi_\alpha(u; v) := [sv^{-1-\lvert\alpha\rvert}(e^s - 1)^{m-1-\lvert\alpha\rvert}]^{-1}(u) = [s(e^s - 1)^{m-1-\lvert\alpha\rvert}]^{-1} \left(\frac{u}{v^{m-1-\lvert\alpha\rvert}}\right).
\]

Thus in this case for every \(T_i = (P_i, Q_i) \in \mathcal{P}_L \times \mathcal{K}\), where \(Q_i = Q(x_i, r_i) \in \mathcal{K}, \ i = 1, 2\), we have

\[
\delta_\omega(T_1, T_2) = \max_{\lvert\alpha\rvert \leq m-1, i=1,2} \left\{ \ln \left(1 + \frac{\max\{r_1, r_2\} + \|x_1 - x_2\|}{\min\{r_1, r_2\}}\right), \left[\frac{s(e^s - 1)^{m-1-\lvert\alpha\rvert}}{\min\{r_1, r_2\}^{m-1-\lvert\alpha\rvert}}\right]\right\}.
\]
so that we again obtain the formula (2.8) for $\delta_\omega$.

**Example 4.12** Consider the space $C^k_\omega(R^n) = C^k\Lambda^1_\omega(R^n)$ with $\omega(t) = t$. This is the Sobolev space $W^{k+1}_\infty(R^n)$ of bounded functions $f \in C^k(R^n)$ whose partial derivatives of order $k$ satisfy the Lipschitz condition:

$$|D^\alpha f(x) - D^\alpha f(y)| \leq \lambda \|x - y\|, \quad x, y \in R^n, \quad |\alpha| = k.$$  

Since $m = 1$ and $L = k$, we have

$$h(t; v) := \int_0^t 1 ds = t, \quad t > 0,$$

and

$$\psi_\alpha(u; v) := [s s^{k-|\alpha|}]^{-1}(u) = \frac{s^{1-|\alpha|}}{u^{1-|\alpha|}}.$$  

Thus, in this case for every $T_i = (P_i, Q_i) \in P_k \times K$, where $Q_i = Q(x_i, r_i) \in K$, $i = 1, 2$, we have

$$\delta_\omega(T_1, T_2) = \max_{|\alpha| \leq m-1, i = 1, 2} \left\{ \max\{r_1, r_2\} + \|x_1 - x_2\|, |D^\alpha(P_1 - P_2)(x_i)|^{\frac{1}{m-|\alpha|}} \right\}.$$  

A function $\delta_\omega$ of such a kind and the metric $d_\omega$ generated by $\delta_\omega$ have been studied in [26].

Now by Claim 4.1 and Theorem 4.2, the results of Theorem 3.2 and Theorem 3.7 can be formulated in the following form.

**Theorem 4.13** Let $F \in C^k\Lambda^m_\omega(R^n)$ and let $\lambda := \|F\|_{C^k\Lambda^m_\omega(R^n)}$. There exists a family of polynomials $\{P_Q \in P_L : Q \in K(S)\}$ such that:

1. $T^k_{x_Q}(P_Q) = T^k_{x_Q}(F)$ for every cube $Q \in K(S)$;
2. For every $Q \in K(S)$ with $r_Q \leq 1$ and every $\alpha, |\alpha| \leq k$, and $\beta, |\beta| \leq L$,

$$|D^{\alpha + \beta}P_Q(x_Q)| \leq C \lambda r_Q^{-|\beta|};$$

3. The mapping $T(Q) := (P_Q, Q)$, $Q \in K(S)$, satisfies the Lipschitz condition

$$d_\omega((C\lambda)^{-1} \circ T(Q_1), (C\lambda)^{-1} \circ T(Q_2)) \leq \rho_\omega(Q_1, Q_2), \quad Q_1, Q_2 \in K(S).$$  

Here $C$ is a constant depending only on $k, m$ and $n$.

**Theorem 4.14** Let $\omega \in \Omega_m$ be a quasipower function. Assume that a mapping

$$T(Q) = (P_Q, Q), \quad Q \in K(S),$$

from $K(S)$ into $P_L \times K$ and a constant $\lambda > 0$ satisfy the following conditions:

1. For every $Q \in K(S)$ with $r_Q \leq 1$ and every $\alpha, |\alpha| \leq k$, and $\beta, |\beta| \leq L - |\alpha|$, we have

$$|D^{\alpha + \beta}P_Q(x_Q)| \leq \lambda r_Q^{-|\beta|};$$

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Moreover, for every $Q \in \mathcal{Q}$ the metric space $\mathcal{U}$ and taking their values in the metric space $\mathcal{U}$.

We introduce a Lipschitz-type space

\begin{equation}
\| T \|_{Lip(\mathcal{M}, \mathcal{T}_\omega)} := \sup_{x,y \in \mathcal{M}, x \neq y} \frac{d_\omega(T(x), T(y))}{\rho(x,y)}.
\end{equation}

We introduce a Lipschitz-type space $LO(\mathcal{M}, \mathcal{T}_\omega)$ of mappings $T : \mathcal{M} \to \mathcal{P}_L \times \mathcal{K}$ defined by the finiteness of the following seminorm:

\begin{equation}
\| T \|_{LO(\mathcal{M}, \mathcal{T}_\omega)} := \inf \{ \lambda > 0 : \| \lambda^{-1} \circ T \|_{Lip(\mathcal{M}, \mathcal{T}_\omega)} \leq 1 \}.
\end{equation}

Thus $T \in LO(\mathcal{M}, \mathcal{T}_\omega)$ whenever there exists a constant $\lambda > 0$ such that the mapping $\lambda^{-1} \circ T$ belongs to the unit ball of the Lipschitz space $Lip(\mathcal{M}, \mathcal{T}_\omega)$. In other words, the quantity $\| \cdot \|_{LO(\mathcal{M}, \mathcal{T}_\omega)}$ is the standard Luxemburg norm with respect to the unit ball of $Lip(\mathcal{M}, \mathcal{T}_\omega)$ (and the “multiplication” operation $\circ$) and $LO(\mathcal{M}, \mathcal{T}_\omega)$ is the corresponding Orlicz space determined by this norm, see, e.g. [17].

We call the space $LO(\mathcal{M}, \mathcal{T}_\omega)$ the Lipschitz-Orlicz space. We use it to define a second “norm”: given a mapping $T(z) = (P_z, Q_z)$, $z \in \mathcal{M}$, we put

\[ \| T \|^* := \sup \{ |D^{\alpha + \beta} P_z(x_{Q_z})| r_{Q_z}^{|\beta|} : z \in \mathcal{M}, r_{Q_z} \leq 1, \; |\alpha| \leq k, \; |\beta| \leq L - |\alpha| \}, \]

and

\begin{equation}
\| T \|_{LO(\mathcal{M}, \mathcal{T}_\omega)} := \| T \|^* + \| T \|_{LO(\mathcal{M}, \mathcal{T}_\omega)}.
\end{equation}

We let $LO(\mathcal{M}, \mathcal{T}_\omega)$ denote the subspace of $LO(\mathcal{M}, \mathcal{T}_\omega)$ of “bounded” Lipschitz mappings defined by the finiteness of the “norm” (4.36).

Given closed subset $S \subset \mathbb{R}^n$ we consider the family of cubes $\mathcal{K}(S)$ (i.e., the cubes centered at $S$) as a metric space equipped with the metric $\rho_\omega$, i.e., as a subspace of the metric space $\mathcal{K}_\omega = (\mathcal{K}, \rho_\omega)$. Now Theorem 4.13 and Theorem 4.14 imply the following
Theorem 4.15  (a). For every function $F \in C^k \Lambda^m_\omega(\mathbb{R}^n)$ there exists a mapping

$$T(Q) = (P_Q, Q), \quad Q \in \mathcal{K}(S),$$

from $\text{LO}(\mathcal{K}(S), T_\omega)$ with $\|T\|_{\text{LO}(\mathcal{K}(S), T_\omega)} \leq C\|F\|_{C^k \Lambda^m_\omega(\mathbb{R}^n)}$ such that $T^k_x(P_Q) = T^k_x(F)$ for every cube $Q = Q(x, r) \in \mathcal{K}(S)$.

(b). Conversely, let $\omega \in \Omega_m$ be a quasipower function. Assume that a mapping $T(Q) = (P_Q, Q), \quad Q \in \mathcal{K}(S)$, belongs to $\text{LO}(\mathcal{K}(S), T_\omega)$. Then there exists a function $F \in C^k \Lambda^m_\omega(\mathbb{R}^n)$ with $\|F\|_{C^k \Lambda^m_\omega(\mathbb{R}^n)} \leq C\|T\|_{\text{LO}(\mathcal{K}(S), T_\omega)}$ such that for all $x \in S$

$$(4.37) \quad T^k_x(F) = \lim_{x_Q = x, r_Q \to 0} T^k_x(P_Q).$$

Moreover, for every $Q = Q(x, r) \in \mathcal{K}(S)$ and $\alpha$, $|\alpha| \leq k$,

$$(4.38) \quad |D^\alpha T^k_x(F)(x) - D^\alpha P_Q(x)| \leq C\|T\|_{\text{LO}(\mathcal{K}(S), T_\omega)} r^{|\alpha|} \omega(r).$$

Here $C$ is a constant depending only on $k, m, n$ and the constant $C_\omega$.

5. Lipschitz selections of polynomial-set valued mappings

The ideas and results presented in Section 4 show that even though Whitney’s problem for $C^k \Lambda^m_\omega(\mathbb{R}^n)$ deals with restrictions of $k$-times differentiable functions, it is also a problem about Lipschitz mappings defined on subsets of $\mathcal{K}$ and taking values in a very non-linear metric space $T_\omega = (P_L \times \mathcal{K}, d_\omega)$. More specifically, the Whitney problem can be reformulated as a problem about Lipschitz selections of set-valued mappings from $\mathcal{K}(S)$ into $2^{T_\omega}$.

We recall some relevant definitions: Let $X = (\mathcal{M}, \rho)$ and $Y = (T, d)$ be metric spaces and let $G : \mathcal{M} \to 2^T$ be a set-valued mapping, i.e., a mapping which assigns a subset $G(x) \subset T$ to each $x \in \mathcal{M}$. A function $g : \mathcal{M} \to T$ is said to be a selection of $G$ if $g(x) \in G(x)$ for all $x \in \mathcal{M}$. If a selection $g$ is an element of Lip$(X, Y)$ then it is said to be a Lipschitz selection of the mapping $G$. (For various results and techniques related to the problem of the existence of Lipschitz selections in the case where $Y = (T, d)$ is a Banach space, we refer the reader to [23, 24, 25] and references therein.)

In [11] C. Fefferman considered the following version of the Whitney problem: Let $\{G(x) : x \in S\}$ be a family of convex centrally-symmetric subsets of $\mathcal{P}_k$.

How can we decide whether there exist $F \in C^k(\mathbb{R}^n)$ and a constant $A > 0$ such that $T^k_x(F) \in A \oplus G(x)$ for all $x \in S$? Here $A \oplus G(x)$ denotes the dilation of $G(x)$ with respect to its center by a factor of $A$.

Let $P_x \in \mathcal{P}_k$ be the center of the set $G(x)$. This means that $G(x)$ can be represented in the form $G(x) = P_x + \sigma(x)$ where $\sigma(x) \subset \mathcal{P}_k$ is a convex family of polynomials which is centrally symmetric with respect to 0. It is shown in [11] that, under certain conditions on the sets $\sigma(x)$ (the so-called condition of Whitney’s $\omega$-convexity), the finiteness property holds. The approach described in Section 4, see Example 4.12, and certain ideas related to Lipschitz selections in Banach spaces [24], allows us to give an upper bound for a finiteness number in Fefferman’s theorem [11]:

$$N(k, n) = 2^{\min\{\ell + 1, \dim \mathcal{P}_k\}},$$

where $\ell = \max_{x \in S} \dim \sigma(x)$, see [26].

This improvement of the finiteness number follows from Fefferman’s result [11] and the following
Theorem 5.1 ([26]) Let $G$ be a mapping defined on a finite set $S \subset \mathbb{R}^n$ which assigns a convex set of polynomials $G(x) \subset \mathcal{P}_k$ of dimension at most $\ell$ to every point $x$ of $S$. Suppose that, for every subset $S'$ of $S$ consisting of at most $2^{\min\{\ell+1, \dim \mathcal{P}_k\}}$ points, there exists a function $F_{S'} \in \mathcal{C}^{k,\omega}(\mathbb{R}^n)$ such that $\|F_{S'}\|_{\mathcal{C}^{k,\omega}(\mathbb{R}^n)} \leq 1$ and $T^k_x(F_{S'}) \in G(x)$ for all $x \in S'$. Then there is a function $F \in \mathcal{C}^{k,\omega}(\mathbb{R}^n)$, satisfying $\|F\|_{\mathcal{C}^{k,\omega}(\mathbb{R}^n)} \leq \gamma$ and

$$T^k_x(F) \in G(x) \quad \text{for all} \quad x \in S.$$

Here $\gamma$ depends only on $k$, $n$ and $\text{card } S$.

We use the rather informal and imprecise terminology “$\mathcal{C}^{k,\omega}(\mathbb{R}^n)$ has the weak finiteness property” to express the kind of result where $\gamma$ depends on the number of points of $S$. The weak finiteness property also provides an upper bound for the finiteness constant whenever the strong finiteness property holds. For instance, Fefferman’s theorems in [11] reduce the problem to a set of cardinality at most $N(k,n)$ while the weak finiteness property decreases this number to $2^{\min\{\ell+1, \dim \mathcal{P}_k\}}$.

In [26], Theorem 1.10, we show that, in turn, the “weak finiteness” theorem, is equivalent to a certain Helly-type criterion for the existence of a certain Lipschitz selection of the set-valued mapping $G(x) = (G(x), x), x \in S$. An analog of this result for set-valued mappings from $\mathcal{K}_\omega(S) := (\mathcal{K}(S), \rho_\omega)$ into $2^{\mathcal{P}_L \times \mathcal{K}}$ where $m = 1$ and $\omega(t) = t$, is presented in Theorem 5.4 below.

Let us see how these ideas and results can be generalized for the space $\mathcal{C}^{k,\omega}(\mathbb{R}^n)$ with $m > 1$, and what kind of difficulties appear in this way. We will consider the following general version of the problem raised in [11].

**Problem 5.2** Let $\{G(x) : x \in S\}$ be a family of convex closed subsets of $\mathcal{P}_k$. How can we decide whether there exist a function $F \in \mathcal{C}^{k,\omega}(\mathbb{R}^n)$ such that

$$T^k_x(F) \in G(x) \quad \text{for all} \quad x \in S?$$

In particular, if $G(x) = \{P \in \mathcal{P}_k : P(x) = f(x)\}, x \in S$, where $f$ is a function defined on $S$, this problem is equivalent to the Whitney Problem 1.1 for $\mathcal{C}^{k,\omega}(\mathbb{R}^n)$.

First, let us show that Problem 5.2 is equivalent to a Lipschitz selection problem for set-valued mappings from $\mathcal{K}(S)_\omega := (\mathcal{K}(S), \rho_\omega)$ into a certain family of subsets of $\mathcal{T}_\omega := (\mathcal{P}_L \times \mathcal{K}, d_\omega)$. To this end, given a cube $Q = Q(x, r) \in \mathcal{K}(S)$ and $\lambda > 0$ we let $H_\lambda(Q)$ denote the set of all polynomials $P \in \mathcal{P}_L$ satisfying the following condition:

*There exists $\tilde{P} \in G(x)$ such that for every $\alpha$, $|\alpha| \leq k$, \begin{equation} |D^\alpha \tilde{P}(x) - D^\alpha P(x)| \leq \lambda r^{k - |\alpha|} \omega(r). \end{equation} \begin{equation} H_0(Q) := \{P \in \mathcal{P}_L : T^k_x(P) \in G(x)\}, \quad Q = Q(x, r) \in \mathcal{K}(S). \end{equation} Clearly, $H_\lambda(Q)$ is a convex closed subset of $\mathcal{P}_L$.*

By $\mathcal{H}_\lambda$ we denote the set-valued mapping from $\mathcal{K}(S)$ into $2^{\mathcal{P}_L \times \mathcal{K}}$ defined by the following formula:

$$\mathcal{H}_\lambda(Q) := (H_\lambda(Q), Q), \quad Q \in \mathcal{K}(S).$$
Theorem 5.1. Suppose that $F \in C^k \Lambda_m^w(\mathbb{R}^n)$ and $T^k_x(F) \in G(x)$ for every $x \in S$. Then the set-valued mapping $H_0$ has a selection $T \in LO(K(S), T_\omega)$ with $\|T\|_{LO(K(S), T_\omega)} \leq C \|F\|_{C^k \Lambda^w_m(\mathbb{R}^n)}$.

(b) Conversely, let $\omega \in \Omega_m$ be a quasipower function. Suppose that there exists $\lambda > 0$ such that $H_\lambda$ has a selection $T \in LO(K(S), T_\omega)$ with $\|T\|_{LO(K(S), T_\omega)} \leq \lambda$. Then there exists $F \in C^k \Lambda^m_w(\mathbb{R}^n)$ with $\|F\|_{C^k \Lambda^w_m(\mathbb{R}^n)} \leq C\lambda$ such that $T^k_x(F) \in G(x)$ for every $x \in S$.

Here $C$ is a constant depending only on $k, m, n$ and the constant $C_\omega$.

Proof. (a) By part (a) of Theorem 4.15 there exists a mapping $T : K(S) \rightarrow P_L \times K$ of the form $T(Q) = (P_Q, Q)$, $Q \in K(S)$, satisfying the following conditions: $T \in LO(K(S), T_\omega)$, $\|T\|_{LO(K(S), T_\omega)} \leq C \|F\|_{C^k \Lambda^w_m(\mathbb{R}^n)}$, and $T^k_x(P_Q) = T^k_x(F)$ for every cube $Q(x, r) \in K(S)$.

But $T^k_x(F) \in G(x)$ so that $T^k_x(P_Q) \in G(x)$ as well, proving that $P_Q \in H_0$, see (5.2). Thus, the mapping $T$ is a selection of the set-valued mapping $H_0$, and part (a) of the theorem is proved.

(b) Since $T$ is a selection of $H_\lambda$, it can be written in the form

$$T(Q) = (P_Q, Q) \in K(S), \ P_Q \in P_L.$$ 

By part Theorem 4.15, part (b), there exists $F \in C^k \Lambda^m_w(\mathbb{R}^n)$ with $\|F\|_{C^k \Lambda^w_m(\mathbb{R}^n)} \leq C\lambda$ such that (4.37) and (4.38) are satisfied.

Prove that $T^k_x(F) \in G(x)$, $x \in S$. Since $T$ is a selection of $H_\lambda$, for each cube $Q = Q(x, r) \in K(S)$ we have $P_Q \in H_\lambda(Q)$, so that, by (5.1), there exists $\tilde{P} \in G(x)$ such that

$$|D^\alpha \tilde{P}(x) - D^\alpha P_Q(x)| \leq \lambda r^{k-|\alpha|} \omega(r), \ |\alpha| \leq k.$$ 

We have

$$|D^\alpha T^k_x(F)(x) - D^\alpha \tilde{P}(x)| \leq |D^\alpha T^k_x(F)(x) - D^\alpha P_Q(x)| + |D^\alpha P_Q(x) - D^\alpha \tilde{P}(x)|$$

$$\leq |D^\alpha T^k_x(F)(x) - D^\alpha P_Q(x)| + \lambda r^{k-|\alpha|} \omega(r), \ |\alpha| \leq k.$$ 

Since $\|T\|_{LO(K(S), T_\omega)} \leq \lambda$, by (4.38),

$$|D^\alpha T^k_x(F)(x) - D^\alpha \tilde{P}(x)| \leq C\lambda r^{k-|\alpha|} \omega(r) \quad \text{for all} \quad |\alpha| \leq k.$$ 

Clearly, given $x \in S$ the quantity $\|P\| := \max\{|D^\alpha P(x) : |\alpha| \leq k\}$ presents an equivalent norm on the finite dimensional space $P_k$. By inequality (5.3), the distance from $T^k_x(F)$ to $G(x)$ in this norm tends to 0 as $r \rightarrow 0$. Since $G(x)$ is closed, $T^k_x(F) \in G(x)$ proving the theorem.

As we have noted above, a geometrical background of the weak finiteness property, Theorem 5.1, is a certain Helly-type criterion for the existence of a Lipschitz selection. Let us formulate a version of this result for set-valued mappings defined on finite families of cubes in $\mathbb{R}^n$.

Theorem 5.4. Let $m = 1$ and let $\omega \in \Omega_1$ be a quasipower function. Let $K \subset K$ be a finite set of cubes in $\mathbb{R}^n$ and let $H(Q) = (H(Q), Q), Q \in K$, be a set-valued mapping such that for each $Q \in K$ the set $H(Q) \subset P_k$ is a convex set of polynomials of dimension at most $\ell$. Suppose that there exists a constant $A > 0$ such that, for every subset $K' \subset K$ consisting of at most $2^{\min\ell+1, \dim P_k}$ elements, the restriction $H|_{K'}$ has a selection $h_{K'} \in LO(K', T_\omega)$ with $\|h_{K'}\|_{LO(K', T_\omega)} \leq A$.

Then $H$, considered as a map on all of $K$, has a Lipschitz selection $h \in LO(K, T_\omega)$ with $\|h\|_{LO(K, T_\omega)} \leq \gamma A$. Here the constant $\gamma$ depends only on $k, n$ and card $K$.
We recall that for $m = 1$ we have $L := k + m - 1 = k$ so that $T_\omega := (P_L \times K, d_\omega) = (P_k, d_\omega)$. The proof of this result follows precisely the same scheme as in [26].

It would be very useful to have such a criterion for arbitrary $m > 1$ which, in view of Theorem 5.3, would immediately lead to the weak finiteness property for the space $C^k\Lambda^m_\omega(\mathbb{R}^n)$. However, the straightforward application of the method of proof given in [26] to this case meets certain difficulties. In particular, one of the crucial ingredients of the proof in [26] is “consistency” of the metrics $\rho_1(x, y) := \|x - y\|$ and $\rho_2(x, y) := \omega(\|x - y\|)$ in the following sense: for every $x_1, x_2, x_3, x_4 \in \mathbb{R}^n$ the inequality $\rho_1(x_1, x_2) \leq \rho_1(x_3, x_4)$ imply the inequality $\rho_2(x_1, x_2) \leq \rho_2(x_3, x_4)$ (and vice versa).

For instance, a corresponding analog of this property for the space $Z^m(\mathbb{R}^n)$, see (2.1), is “consistency” of the distance

$$\rho_1(Q_1, Q_2) := \max\{r_1, r_2\} + \|x_1 - x_2\|, \quad Q_i = Q(x_i, r_i), i = 1, 2,$$

defined on the family $\mathcal{K}$ of all cubes in $\mathbb{R}^n$, and the metric $\rho$ defined by (2.5). However, in general, such a “consistency” does not hold. For example, consider the family $\{Q_i = Q(0, r_i), i = 1, 2, \ldots\}$ of cubes in $\mathbb{R}^n$ with $r_i = 2^{-i^2}$. Clearly, $\rho_1(Q_i, Q_{i+1}) = 2^{-i^2} \rightarrow 0$ while $\rho_2(Q_i, Q_{i+1}) = \ln(1 + r_i/r_{i+1}) \rightarrow +\infty$ as $i \rightarrow \infty$.

This simple example shows that the set $S = \{0\} \cup \{x_i, i = 1, 2, \ldots\}$ where $x_i$ are points in $\mathbb{R}^n$ with $\|x_i\| = 2^{-i^2}$, could play a role of a test-set in proving the weak finiteness property for the space $Z_m(\mathbb{R}^n)$.

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