On a Result of Atkin and Lehner

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1 Introduction

We wish to give a new proof of one of the main results of Atkin-Lehner [1]. That paper depends, among other things, on a slightly strengthened version of Theorem 1 below, which characterizes forms in $S_k(\Gamma_0(N))$ whose Fourier coefficients satisfy a certain vanishing condition. Our proof involves rephrasing this vanishing condition in terms of representation theory; this, together with an elementary linear algebra argument, allows us to rewrite our problem as a collection of local problems. Furthermore, the classical phrasing of Theorem 1 makes the resulting local problems trivial; this is in contrast to the method of Casselman [3], whose local problem relies upon knowledge of the structure of irreducible representations of $GL_2(\mathbb{Q}_p)$. Our proof is therefore much more accessible to mathematicians who aren’t specialists in the representation theory of $p$-adic groups; the method is also applicable to other Atkin-Lehner-style problems, such as the level structures that were considered in Carlton [2].

Our proof of Theorem 1 occupies Section 2. In Section 3, we explain the links between this Theorem and the rest of Atkin-Lehner theory; in particular, we show that Theorem 1, together with either the Global Result of Casselman [3] or Theorem 4 of Atkin-Lehner [1], can be used to derive all of the important results of Atkin-Lehner theory.

2 The Main Theorem

Recall that, if $N|M$ and $d|(M/N)$, there is a map $i_d: M_k(\Gamma_0(N)) \to M_k(\Gamma_0(M))$ defined by

$$c_m(i_d(f)) = \begin{cases} 0 & \text{if } d \not| m \\ c_{m/d}(f) & \text{if } d|m. \end{cases}$$

This map sends cusp forms to cusp forms and eigenforms to eigenforms (with the same eigenvalues); up to multiplication by a constant, it is given by $f \mapsto f|\left(\frac{d}{0 1}\right)$.

**Theorem 1.** Let $f \in M_k(\Gamma_0(N))$ be such that $c_m(f) = 0$ unless $(m, N) > 1$. Then $f = \sum_{p|N} i_p(f_p)$, where $p$ varies over the primes dividing $N$ and where
\( f_p \in M_k(\Gamma_0(N/p)) \). Furthermore, if \( f \) is a cusp form (resp. eigenform) then the \( f_p \)'s can be chosen to be cusp forms (resp. eigenforms with the same eigenvalues as \( f \)).

Our proof rests on two elementary linear algebra lemmas:

**Lemma 2.** Let \( V_1, \ldots, V_n \) be vector spaces and, for each \( i \), let \( f_i \) be an endomorphism of \( V_i \). Then
\[
\ker(f_1 \otimes \cdots \otimes f_n) = \sum_{i=1}^n V_1 \otimes \cdots \otimes (\ker f_i) \otimes \cdots \otimes V_n.
\]

**Proof.** We can easily reduce to the case \( n = 2 \). If we write \( V_i = (\ker f_i) \oplus V_i' \) then \( f_i|_{V_i'} \) is an isomorphism onto its image, and
\[
V_1 \otimes V_2 = ((\ker f_1) \otimes (\ker f_2)) \oplus ((\ker f_1) \otimes V_2') \oplus (V_1' \otimes (\ker f_2)) \oplus (V_1' \otimes V_2').
\]
We see that \( f_1 \otimes f_2 \) kills the first three factors, and is an isomorphism from the fourth factor onto its image; \( \ker(f_1 \otimes f_2) \) is therefore the sum of the first three factors, which is what we wanted to show. \( \square \)

**Lemma 3.** Let \( V_1, \ldots, V_n \) be vector spaces and, for each \( i \), let \( V_i' \) and \( V_i'' \) be subspaces of \( V_i \). Then
\[
\left( \sum_{i=1}^n V_1 \otimes \cdots \otimes V_i' \otimes \cdots \otimes V_n \right) \cap (V_1'' \otimes \cdots \otimes V_n')
= \sum_{i=1}^n V_1'' \otimes \cdots \otimes (V_i' \cap V_i'') \otimes \cdots \otimes V_n''.
\]

**Proof.** Again, we can assume that \( n = 2 \). Write \( V_i = V_{i1} \oplus V_{i2} \oplus V_{i3} \oplus V_{i4} \) where \( V_{i1} = V_i' \cap V_i'' \), \( V_i' = V_{i1} \oplus V_{i2} \), and \( V_i'' = V_{i1} \oplus V_{i3} \). Then \( V_1' \otimes V_2 + V_1 \otimes V_2' \) is the direct sum of those \( V_1' \otimes V_2 \)'s where at least one of \( j \) or \( k \) is in the set \( \{1, 2\} \). Also, \( V_1'' \otimes V_2'' \) is the direct sum of the \( V_1' \otimes V_2 \)'s where \( j \) and \( k \) are both in the set \( \{1, 3\} \). Thus, their intersection is \( (V_{11} \otimes V_{21}) \oplus (V_{11} \otimes V_{23}) \oplus (V_{13} \otimes V_{21}) \), as claimed. \( \square \)

**Proof of Theorem 1.** If \( f \in M_k(\Gamma_0(N)) \) then \( f|_{\left( \begin{smallmatrix} N^{-1} & 0 \\ 0 & 1 \end{smallmatrix} \right)} \in M_k(\Gamma^0(N)) \), where we define the group \( \Gamma^0(N) \) by
\[
\Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \ \middle| \ b \equiv 0 \pmod{N} \right\}.
\]
Furthermore, up to multiplication by a constant, \( f|_{\left( \begin{smallmatrix} N^{-1} & 0 \\ 0 & 1 \end{smallmatrix} \right)} \) has the same Fourier coefficients as \( f \), except that we have to take the \( q \)-expansion with respect
to $e^{2\pi \sqrt{-1}/N}$ instead of $e^{2\pi \sqrt{-1}}$. Our Theorem, then, is equivalent to the statement that, if $f \in M_k(\Gamma^0(N))$ satisfies the condition

$$c_m(f) = 0 \text{ unless } (m, N) > 1$$

then $f = \sum_{p\mid N} f_p$ where $f_p \in M_k(\Gamma^0(N/p))$.

Let $M = M_k(\Gamma(N))$; it comes with an action of $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$. If $f \in M$ and $d\mid N$, define $\pi_d(f)$ to be $\sum_{d\mid m} c_m(f)q^m$. Then $\pi_d(f) \in M$: in fact,

$$\pi_d(f) = \frac{1}{d} \sum_{b=0}^{d-1} f|\begin{pmatrix} 1 & bN/d \\ 0 & 1 \end{pmatrix}.$$

The principle of inclusion and exclusion implies that $f$ satisfies (1) if and only if

$$f = \sum_{p\mid N} \pi_p(f) - \sum_{p_1, p_2\mid N \text{ and } p_1 < p_2} \pi_{p_1p_2}(f) + \cdots.$$

Thus, if $V$ is an irreducible $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$-representation contained in $M$, it suffices to prove our Theorem for a form in $V$, since the conditions of our Theorem can be expressed in terms of the action of $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$.

Let $N = \prod_{i=1}^n p_i^{n_i}$ be the prime factorization of $N$. Then $\text{SL}_2(\mathbb{Z}/N\mathbb{Z}) = \prod_i \text{SL}_2(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$, so $V = \bigotimes_i V_i$ where $V_i$ is a representation of $\text{SL}_2(\mathbb{Z}/p_i^{n_i}\mathbb{Z})$. Also, $\pi_{p_i}$ acts as the identity on the $V_j$ for $j \neq i$. So if we define

$$\pi(f) = f - \sum_{p\mid N} \pi_p(f) + \sum_{p_1, p_2\mid N \text{ and } p_1 < p_2} \pi_{p_1p_2}(f) - \cdots$$

then $\pi = (1 - \pi_{p_1}) \otimes \cdots \otimes (1 - \pi_{p_n})$ and $\ker(\pi)$ is the space of forms satisfying (1). Thus, Lemma 3 implies that

$$\ker(\pi) = \sum_{i=1}^n V_i \otimes \cdots \otimes (\ker(1 - \pi_{p_i})) \otimes \cdots \otimes V_n.$$

Turning now to the question of a form’s being in $M_k(\Gamma^0(N))$, that is the case if and only if the form is both in $M_k(\Gamma(N))$ and is invariant under the image $B(N)$ of $\Gamma^0(N)$ in $\text{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Also, $B(N) = \prod_i B(p_i)$. Thus, setting $V_i'$ to be the space of $B(p_i)$-invariant elements of $V_i$, Lemma 3 implies that an element of $V$ is both in $\ker \pi$ and invariant under $B(N)$ if and only if it is in

$$\sum_{i=1}^n V_i'' \otimes \cdots \otimes (V_i' \cap V_i'') \otimes \cdots \otimes V_n''.$$

But if $v_i \in V_i$ is in $V_i' \cap V_i''$ then it is invariant both under $B(p_i)$ and under projection to the subspace of invariants under the cyclic subgroup generated by $(\begin{smallmatrix} 1 & p_i^{n_i-1} \\ 0 & 1 \end{smallmatrix})$; this last condition is equivalent to its being invariant under
Thus, our vector $v_i$ is invariant under

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}/p_i^n\mathbb{Z}) \mid b \equiv 0 \pmod{p_i^{n-1}} \right\},$$

and $V_i' \otimes \cdots \otimes (V_i' \cap V_i'') \otimes \cdots \otimes V_n''$ is the set of invariants under

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}/N\mathbb{Z}) \mid b \equiv 0 \pmod{N/p_i} \right\},$$

i.e. the elements of $V \cap M_k(\Gamma_0(N/p_i))$, completing our proof.

The cusp form case is similar, replacing $M$ by the space of cusp forms. The eigenform case then follows from the facts that the Hecke operators are simultaneously diagonalizable and that their action is preserved by the operators $i_p$. □

3 Newforms, Oldforms, and All That

In this Section, we explain the relation between Theorem 1 and the rest of Atkin-Lehner theory. We shall see that the whole theory follows from Theorem 1 together with facts about $L$-series associated to modular forms, as expressed by Theorem 4 of Atkin-Lehner [1] or the Global Result of Casselman [3]. We claim no originality in the methods used in this Section.

Define $K_0(N)$ to be the subspace of $f \in S_k(\Gamma_0(N))$ such that $c_m(f) = 0$ unless $(m, N) > 1$: thus, $K_0(N)$ is the subspace characterized in Theorem 1. Define $S_k(\Gamma_0(N))$ to be $S_k(\Gamma_0(N))/K_0(N)$; for $f \in S_k(\Gamma_0(N))$, $c_m(f)$ is well-defined exactly when $(m, N) = 1$. Also, let $T^N$ be the free polynomial algebra over $\mathbb{C}$ generated by commuting operators $T_m$ for $(m, N) = 1$. Then $T^N$ acts on $S_k(\Gamma_0(N))$ (where $T_m$ acts as the $m$'th Hecke operator), and its action is diagonalizable; it is easy to see that its action descends to $\overline{S_k(\Gamma_0(N))}$. (For example, $T_m$ commutes with the action of the operators $\pi_d$ defined in the proof of Theorem 4.)

**Proposition 4.** The $T^N$-eigenspaces in $\overline{S_k(\Gamma_0(N))}$ are one-dimensional; furthermore, an eigenform $f \in \overline{S_k(\Gamma_0(N))}$ is zero if and only if $c_1(f) = 0$.

**Proof.** If $f \in \overline{S_k(\Gamma_0(N))}$ is an eigenform for $T_m$ with eigenvalue $\lambda_m(f)$ then $c_m(f) = \lambda_m(f)c_1(f)$. Thus, if $f$ is a $T^N$-eigenform then it is determined by its eigenvalues and by $c_1(f)$. □

This Proposition, together with Theorem 1, sometimes allows one to reduce questions about the spaces $S_k(\Gamma_0(N))$ to spaces whose eigenspaces are one-dimensional.

**Proposition 5.** If $f$ and $g$ are eigenforms in $\overline{S_k(\Gamma_0(N))}$ such that, for some $D$, they have the same eigenvalues $\lambda_m$ for all $m$ with $(m, ND) = 1$, then they have the same eigenvalues for all $m$ with $(m, N) = 1$. 

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Proof. This is part of the Global Result of Casselman \[3\], or of Theorem 4 of Atkin-Lehner \[1\].

We should also point out that our Theorem 1 isn’t quite the same as Theorem 1 of Atkin-Lehner \[1\]. Their Theorem 1 assumes that \( c_m(f) = 0 \) unless \((m, ND) = 1\), and thus breaks down into two parts: showing that you can assume that \( D = 1 \), and our Theorem 1. It is easy to show that the first part is equivalent to Proposition 3, at least in the eigenform case; the cusp form case takes a bit more work.

We now present what is traditionally thought of as the core of Atkin-Lehner theory.

**Theorem 6.** If \( \{\lambda_m\} \) is a set of eigenvalues (for all \( m \) relatively prime to a finite set of primes) that occurs in some space \( S_k(\Gamma_0(N)) \) then there is a unique minimal such \( N \) (with respect to division) for which those eigenvalues occur, and the corresponding eigenspace is one-dimensional. If \( f \) is a basis element for that eigenspace and if \( M \) is a multiple of \( N \) then the corresponding eigenspace in \( S_k(\Gamma_0(M)) \) has a basis given by the forms \( id(f) \) where \( d \) varies over the (positive) divisors of \( M/N \).

Proof. For any positive integer \( M \), write \( V_0(M) \) for the set of eigenforms in \( S_k(\Gamma_0(M)) \) with eigenvalues \( \{\lambda_m\} \). By Proposition 3, we don’t have to worry exactly about which primes are avoided in our set of eigenvalues, so this notation makes sense. Furthermore, let \( N \) be a minimal level such that \( V_0(N) \) is nonzero. By Proposition 3, the image of \( V_0(N) \) in \( \overline{S}_k(\Gamma_0(N)) \) is one-dimensional. Theorem 3 shows that any element of the kernel of the map from \( V_0(N) \) to \( \overline{S}_k(\Gamma_0(N)) \) is of the form \( \sum_{p|N} i_p(f_p) \), where \( f_p \in V_0(N/p) \). But the minimality of \( N \) shows that there aren’t any such forms; the kernel is therefore zero, so \( V_0(N) \) is one-dimensional.

To see that \( N \) is unique, let \( S_k \) be the space of adelic cusp forms of weight \( k \) but of arbitrary level structure; it comes with an action of \( \text{GL}_2(\mathbf{A}_\infty) \), and elements of \( S_k(\Gamma_0(M)) \) correspond to elements of \( S_k \) invariant under the action of a certain subgroup \( U_0(M) = \prod_p U_0(p^{m_p}) \), where \( p \) varies over the set of all primes and \( p^{m_p} \) is the highest power of \( p \) that divides \( M \). Casselman’s Global Result says that the set \( V \) of forms in \( S_k \) with eigenvalues \( \{\lambda_m\} \) gives an irreducible representation of \( \text{GL}_2(\mathbf{A}_\infty) \); thus, it can be written as a restricted tensor product \( V = \bigotimes_p V^{U_0(p^{m_p})} \), and

\[
V_0(M) = \bigotimes_p V^{U_0(p^{m_p})}.
\]

Since \( U_0(p^{m_p}) \) contains \( U_0(p^{m+1}) \), for each \( p \) it is the case that, if for some power \( m_p \), \( V^{U_0(p^{m_p})} \) is nonzero, then there is a minimal such power. Thus, taking \( N \) to be the product of those minimal powers of \( p \), we see that, if for some \( M \), \( V_0(M) \) is nonzero, then it is nonzero for a unique minimal \( M \), namely our \( N \). (Alternatively, the uniqueness of the minimal level is part of Theorem 4 of Atkin-Lehner \[1\].)
Finally, to see that the eigenspace grows as indicated, let $f$ be a nonzero element of $V_0(N)$ for $N$ minimal. By Proposition 4, we can assume that $c_1(f) = 1$, since our argument above showed that the image of $f$ in $\mathcal{S}_k(\Gamma_0(N))$ is nonzero. Fix some multiple $M$ of $N$, and assume that we have shown that, for all proper divisors $M'$ of $M$ with $N \mid M'$,

$$V_0(M') = \bigoplus_{d \mid (M'/N)} i_d(f) \cdot \mathbb{C}. \tag{2}$$

We then want to show that the same statement holds with $M$ in place of $M'$. Thus, let $g$ be an element of $V_0(M)$. By Proposition 3, the image of $g - c_1(g)i_1(f)$ in $\mathcal{S}_k(\Gamma_0(M))$ is zero, so by Theorem 1

$$g = c_1(g)i_1(f) + \sum_{p \mid M} i_p(g_p)$$

for some forms $g_p \in V_0(M/p)$. Also, $g_p = 0$ unless $p \mid (M/N)$, since otherwise $N$ wouldn’t divide $M/p$, contradicting the unique minimality of $N$. But then (2) implies that each $g_p$, and hence $g$, can be written as a linear combination of the forms $i_d(f)$ for $d \mid (M/N)$; it is easy to see that such an expression for $g$ is unique.

References

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