Discrete Pluriharmonic Functions as Solutions of Linear Pluri-Lagrangian Systems

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Received: 12 March 2014 / Accepted: 14 July 2014
Published online: 27 November 2014 – © Springer-Verlag Berlin Heidelberg 2014

Abstract: Pluri-Lagrangian systems are variational systems with the multi-dimensional consistency property. This notion has its roots in the theory of pluriharmonic functions, in the Z-invariant models of statistical mechanics, in the theory of variational symmetries going back to Noether and in the theory of discrete integrable systems. A d-dimensional pluri-Lagrangian problem can be described as follows: given a d-form $L$ on an m-dimensional space, $m > d$, whose coefficients depend on a function $u$ of $m$ independent variables (called field), find those fields $u$ which deliver critical points to the action functionals $S_\Sigma = \int_\Sigma L$ for any d-dimensional manifold $\Sigma$ in the m-dimensional space. We investigate discrete 2-dimensional linear pluri-Lagrangian systems, i.e., those with quadratic Lagrangians $L$. The action is a discrete analogue of the Dirichlet energy, and solutions are called discrete pluriharmonic functions. We classify linear pluri-Lagrangian systems with Lagrangians depending on diagonals. They are described by generalizations of the star-triangle map. Examples of more general quadratic Lagrangians are also considered.

1. Introduction

In the last decade, a new understanding of integrability of discrete systems as their multi-dimensional consistency has been a major breakthrough [8,22]. This led to classification of discrete 2-dimensional integrable systems (ABS list) [1], which turned out to be rather influential. According to the concept of multi-dimensional consistency, integrable two-dimensional systems can be imposed in a consistent way on all two-dimensional sublattices of a lattice $\mathbb{Z}^m$ of arbitrary dimension. This means that the resulting multi-dimensional system possesses solutions whose restrictions to any two-dimensional sublattice are generic solutions of the corresponding two-dimensional system. To put this idea differently, one can impose the two-dimensional equations on any quad-surface in $\mathbb{Z}^m$ (i.e., a surface composed of elementary squares), and transfer solutions from one
such surface to another one, if they are related by a sequence of local moves, each one involving one three-dimensional cube, like the moves shown of Fig. 1.

A further fundamental conceptual development was initiated by Lobb and Nijhoff [17] and further generalized in various directions in [10,12,18,19,31]. This development deals with variational (Lagrangian) formulation of discrete multi-dimensionally consistent systems. Its main idea can be summarized as follows: solutions of integrable systems deliver critical points simultaneously for actions along all possible manifolds of the corresponding dimension in multi-time; the Lagrangian form is closed on solutions. This idea is, doubtless, rather inventive (not to say exotic) in the framework of the classical calculus of variations. However, it has significant precursors. These are:

- Theory of pluriharmonic functions and, more generally, of pluriharmonic maps [13,24,25]. By definition, a pluriharmonic function of several complex variables \(f : \mathbb{C}^m \to \mathbb{R}\) minimizes the Dirichlet functional\(E_\Gamma = \int_\Gamma |(f \circ \Gamma)_z|^2 dz \wedge d\bar{z}\) along any holomorphic curve in its domain \(\Gamma : \mathbb{C} \to \mathbb{C}^m\). Differential equations governing pluriharmonic functions (and maps) are heavily overdetermined. Therefore it is not surprising that they belong to the theory of integrable systems.

- Baxter’s Z-invariance of solvable models of statistical mechanics [3,4]. This concept is based on invariance of the partition function of solvable models under elementary local transformations of the underlying planar graph. It is well known (see, e.g., [7]) that one can associate the planar graphs underlying these models with quad-surfaces in \(\mathbb{Z}^m\). On the other hand, the classical mechanical analogue of the partition function is the action functional. This makes the relation of Z-invariance to the concept of closedness of the Lagrangian 2-form rather natural, at least at the heuristic level. Moreover, this relation has been made mathematically precise for a number of models, through the quasiclassical limit, in the work of Bazhanov et al. [5,6].

- The classical notion of variational symmetries, going back to the seminal work of Noether [23], turns out to be directly related to the idea of the closedness of the Lagrangian form in the multi-time. This was further elucidated in [30].

Especially the relation with the pluriharmonic functions motivates a novel term we have introduced to describe the situation we are interested in, namely: given a \(d\)-form \(\mathcal{L}\) in the \(m\)-dimensional space \((d < m)\), depending on a function \(u\) of \(m\) variables, one looks for functions \(u\) which deliver critical points to actions \(S_\Sigma = \int_\Sigma \mathcal{L}\) corresponding to any \(d\)-dimensional manifold \(\Sigma\). We call this a pluri-Lagrangian problem and claim that integrability of variational systems should be understood as the existence of the pluri-Lagrangian structure. We envisage that this notion will play a very important role in the future development of the theory of integrable systems.

A general theory of one-dimensional pluri-Lagrangian systems has been developed in [29]. It was demonstrated that for \(d = 1\) the pluri-Lagrangian property is characteristic for commutativity of Hamiltonian flows in the continuous time case and of symplectic maps in the discrete time case. This property yields that the exterior derivative of the