COMPLETELY REDUCIBLE LIE SUBALGEBRAS

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Abstract. Let $G$ be a connected and reductive group over the algebraically closed field $K$. J-P. Serre has introduced the notion of a $G$-completely reducible subgroup $H \subset G$. In this note, we give a notion of $G$-complete reducibility – $G$-cr for short – for Lie subalgebras of $\text{Lie}(G)$, and we show that if the closed subgroup $H \subset G$ is $G$-cr, then $\text{Lie}(H)$ is $G$-cr as well.

1. Introduction

Let $G$ be a connected and reductive group over the algebraically closed field $K$, and write $\mathfrak{g}$ for the Lie algebra of $G$. J-P. Serre has introduced the notion of a $G$-completely reducible subgroup; we state the definition here only for a closed subgroup $H \subset G$. We say $H$ is $G$-cr provided that whenever $H \subset P$ for a parabolic subgroup of $G$, there is a Levi factor $L \subset P$ such that $H \subset L$; cf. [Ser 05]. When $G = \text{GL}(V)$, the subgroup $H$ is $G$-cr if and only if $V$ is a semisimple $H$-module. Similarly, if the characteristic of $K$ is not 2 and $G$ is either the symplectic group $\text{Sp}(V)$ or the orthogonal group $\text{SO}(V)$, a subgroup $H$ of $G$ is $G$-cr if and only if $V$ is a semisimple $H$-module.

B. Martin [Ma 03] used some techniques from “geometric invariant theory” – due to G. Kempf and to G. Rousseau – to prove that if $H \subset G$ is $G$-cr, and if $N$ is a normal subgroup of $H$, then $N$ is $G$-cr as well; cf. [Ser 05 Théorème 3.6]. Martin’s result was obtained first for strongly reductive subgroups in the sense of Richardson; it follows from [BMR 05] that the strongly reductive subgroups of $G$ are precisely the $G$-cr subgroups. See also [Ser 05, §3.3] for an overview of these matters.

We are going to prove in this note a result related to that of Martin. If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra, say that $\mathfrak{h}$ is $G$-cr provided that whenever $\mathfrak{h} \subset \text{Lie}(P)$ for a parabolic subgroup $P$ of $G$, there is a Levi factor $L \subset P$ such that $\mathfrak{h} \subset \text{Lie}(L)$.

We will prove:

Theorem 1. Suppose that $G$ is a reductive group over the algebraically closed field $K$.

(1) Let $X_1, \ldots, X_d$ be a basis for the Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then $\mathfrak{h}$ is $G$-cr if and only if the $\text{Ad}(G)$-orbit of $(X_1, \ldots, X_d)$ is closed in $\bigoplus^d \mathfrak{g}$.

(2) If the closed subgroup $H \subset G$ is $G$-cr, then $\text{Lie}(H)$ is $G$-cr as well.

Our result – and our techniques – are related to those used by Richardson in [Ri 88], though he treats mainly the case of characteristic 0. See e.g. loc. cit. Theorem 3.6.

The converse to Theorem 1(2) is not true. Indeed, suppose the characteristic $p$ of $K$ is positive, and consider a finite subgroup $H \subset G$ whose order is a power of $p$. Then $\text{Lie}(H) = 0$ is clearly $G$-cr; however, if $G = \text{SL}(V)$ and if $H$ is non-trivial, then $V$ is not semisimple as an $H$-module, thus $H$ is not $G$-cr. The converse to Theorem 1(2) is even false for connected $H$: I thank Ben Martin for pointing out the following example. Take for $H$ any semisimple group

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in characteristic $p > 0$, let $\rho_i : H \to \text{SL}(V_i)$ be representations for $i = 1, 2$ with $\rho_1$ semisimple and $\rho_2$ not semisimple, and consider the representation $\rho : H \to G = \text{SL}(V_1 \oplus V_2)$ given by $h \mapsto \rho_1(h) \oplus \rho_2(F(h))$ where $F : H \to H$ is the Frobenius endomorphism. If $J$ denotes the image of $\rho$, then $J$ is not $G$-cr since $V_1 \oplus V_2$ is not semisimple as a $J$-module; however, $\text{Lie}(J) = \text{im} \, d\rho$ lies in the the Lie algebra of the subgroup $M = \text{SL}(V_1) \times \text{SL}(V_2)$; moreover, $M$ is a Levi factor of a parabolic subgroup of $G$, and $\text{Lie}(J) = \text{im} \, d\rho_1 \oplus 0 \subset \mathfrak{sl}(V_1) \oplus \mathfrak{sl}(V_2) = \text{Lie}(M)$. Since the image of $\rho_1 \times 1 : H \to M$ is $M$-cr (use [BMR 05, Lemma 2.12(i)]), the main result of this paper implies $\text{Lie}(J)$ to be $M$-cr, hence Lemma 4 below shows that $\text{Lie}(J)$ is $G$-cr as well.

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2. Preliminaries

We work throughout in the geometric setting; thus, $K$ is an algebraically closed field. A variety will mean a separated and reduced scheme of finite type over $K$. The group $G$ will be a connected and reductive algebraic group (over $K$). A closed subgroup $H \subseteq G$ is in particular a subvariety of $G$ and so $H$ is necessarily reduced — e.g. if $G$ acts on a variety $X$ and if $x \in X$, then $\text{Stab}_G(x)$ will mean the reduced subgroup determined by the “abstract group theoretic” stabilizer (even if the $G$-orbit of $x$ is not separable).

2.1. Closed orbits. Let $X$ be an affine $G$-variety, let $x \in X$ and choose a maximal torus $S \subseteq \text{Stab}_G(x)$ of the stabilizer in $G$ of $x$. Let $L = C_G(S)$; thus $L$ is a Levi factor of a parabolic subgroup of $G$.

**Proposition 2.** If the $G$-orbit $G \cdot x$ is closed in $X$, then the $L$-orbit $L \cdot x$ is closed in $X$.

**Proof.** The fixed point set $X^S$ is closed in $X$. Since by assumption $G \cdot x$ is closed in $X$, it follows that

$$(G \cdot x)^S = X^S \cap G \cdot x$$

is closed in $X$.

Let now $N = N_G(S)$ be the normalizer in $G$ of $S$. We claim that $(G \cdot x)^S = N \cdot x$. Indeed, let $g \in G$ and suppose $g \cdot x$ is fixed by $S$. The claim follows once we prove that $g \cdot x \in N \cdot x$. Well, for each $s \in S$ we have $sg \cdot x = g \cdot x$ so that $g^{-1}sg \in \text{Stab}_G(x)$. Thus $g^{-1}Sg$ is a maximal torus of $\text{Stab}_G(x)$. Since maximal tori are conjugate [Spr 95, Theorem 6.4.1], there is an element $h \in \text{Stab}_G(x)$ such that $g^{-1}Sg = hSh^{-1}$. But then $gh \in N$, and moreover, $g \cdot x = gh \cdot x$.

Note that $N$ contains $L$ as a normal subgroup. We now observe that the stabilizer in $L$ of a point $y$ of the orbit $N \cdot x$ is conjugate to $\text{Stab}_L(x)$ by an element of $N$. Indeed, choosing $h \in N$ such that $h \cdot x = y$, one knows that $h \cdot \text{Stab}_L(y) \cdot h^{-1} = \text{Stab}_L(x)$. It follows that all $L$-orbits in $N \cdot x$ have the same dimension.

Since the closure of any $L$-orbit must be the union of orbits of strictly smaller dimension, it follows that the $L$-orbits in $N \cdot x$ are closed.

Since $N \cdot x = X^S \cap G \cdot x$ is closed in $X$, it follows at once that $L \cdot x$ is closed in $X$, as required. \hfill $\Box$

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1. This can be seen more easily: it is straightforward to check that a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{sl}(V)$ is $\text{SL}(V)$-cr if and only if $V$ is a semisimple $\mathfrak{h}$-module.
Lemma 5.\[\text{Spr}\ 98,\ 13.4.2\] that $L$ has finite index in $N$. In particular, $(G \cdot x)^S$ is a finite union of $L$-orbits which are permuted transitively by $N$; moreover, these $L$-orbits are precisely the connected components of $(G \cdot x)^S$.

2.2. Complete reducibility. The interpretation of complete reducibility using the spherical building of $G$ permits one to prove the following:

Lemma 4. Let $G$ be reductive and let $M \subset G$ be a Levi factor of a parabolic subgroup of $G$. Suppose that $J \subset M$ is a subgroup, and that $\mathfrak{h} \subset \mathfrak{Lie}(M)$ is a Lie subalgebra. Then $J$ is G-cr if and only if $J$ is M-cr and $\mathfrak{h}$ is G-cr if and only if $\mathfrak{h}$ is M-cr.

Proof. The assertion for $J$ follows from \[\text{Ser}\ 05\ Proposition\ 3.2\]. The proof for $\mathfrak{h}$ is similar; let us give a sketch. Write $X$ for the building of $G$. The Lie subalgebra $\mathfrak{h}$ defines a subcomplex $Y$ of $X$: the simplices of $Y$ are those simplices in $X$ which correspond to parabolic subgroups $P$ with $\mathfrak{h} \subset \mathfrak{Lie}(P)$.

Recall \[\text{Bo}\ 91\ Corollary\ 14.13\] that the intersection $P \cap P'$ of two parabolic subgroups $P, P' \subset G$ contains a maximal torus of $G$. This implies that $\mathfrak{Lie}(P \cap P') = \mathfrak{Lie}(P) \cap \mathfrak{Lie}(P')$; see e.g. the argument in the first paragraph of \[\text{La}\ 04\ §10.3\].

If now $\mathfrak{h} \subset \mathfrak{Lie}(P) \cap \mathfrak{Lie}(P')$, it follows that $\mathfrak{h} \subset \mathfrak{Lie}(P \cap P')$. This shows that the subcomplex $Y'$ is convex; see \[\text{Ser}\ 05\ Prop. 3.1\]. Evidently $\mathfrak{h}$ is G-cr if and only if $Y'$ is X-cr in the sense of \[\text{Ser}\ 05\ §2.2\].

Choose a parabolic subgroup $Q$ for which $M$ is a Levi factor. Then we may identify the building of $M$ with the residual building of $X$ determined by the parabolic $Q$; cf. \[\text{Ser}\ 05\ 2.1.8 and 3.1.7\]. Now the claim follows from \[\text{Ser}\ 05\ Proposition\ 2.5\].

2.3. Cocharacters and parabolic subgroups. If $V$ is a variety and $f : G_m \to V$ is a morphism, we write $v = \lim_{t \to 0} f(t)$, and we say that the limit exists, if $f$ extends to a morphism $\tilde{f} : A^1 \to V$ with $\tilde{f}(0) = v$. If $G_m$ acts on $V$, a closed point $w \in V$ determines a morphism $f : G_m \to V$ via the rule $t \mapsto t \cdot w$; one writes $\lim_{t \to 0} t \cdot w$ as shorthand for $\lim_{t \to 0} f(t)$.

A cocharacter of an algebraic group $A$ is a $K$-homomorphism $\gamma : G_m \to A$. A linear $K$-representation $(\rho, V)$ of $A$ yields a linear $K$-representation $(\rho \circ \gamma, V)$ of $G_m$. Then $V$ is the direct sum of the weight spaces

$$V(\gamma; i) = \{ v \in V \mid (\rho \circ \gamma)(t)v = t^i v, \ \forall t \in G_m \}$$

for $i \in \mathbb{Z}$. We write $X_\ast(A)$ for the set of cocharacters of $A$.

Consider now the reductive group $G$. If $\gamma \in X_\ast(G)$, then

$$P_G(\gamma) = P(\gamma) = \{ x \in G \mid \lim_{t \to 0} \gamma(t)x\gamma(t^{-1}) \text{ exists} \}$$

is a parabolic subgroup of $G$ whose Lie algebra is $\mathfrak{p}(\gamma) = \sum_{i \geq 0} \mathfrak{g}(\gamma; i)$. Moreover, each parabolic subgroup of $G$ has the form $P(\gamma)$ for some cocharacter $\gamma$; for all this cf. \[\text{Spr}\ 98\ 3.2.15 and 8.4.5\].

We note that $\gamma$ “exhibits” a Levi decomposition of $P = P(\gamma)$. Indeed, $P(\gamma)$ is the semi-direct product $Z(\gamma) \cdot U(\gamma)$, where $U(\gamma) = \{ x \in P \mid \lim_{t \to 0} \gamma(t)x\gamma(t^{-1}) = 1 \}$ is the unipotent radical of $P(\gamma)$, and the reductive subgroup $Z(\gamma) = C_G(\gamma(G_m))$ is a Levi factor in $P(\gamma)$; cf. \[\text{Spr}\ 98\ 13.4.2\].

Lemma 5. Let $P$ be a parabolic subgroup of $G$, let $L$ be a Levi factor of $P$, let $\gamma \in X_\ast(L)$ and assume that $P = P(\gamma)$. Then $L = Z(\gamma)$ and the image of $\gamma$ lies in the connected center of $L$.\[\square\]
Proof. Let $R$ be the radical of $P$. Then the Levi factors of $P$ are precisely the centralizers of the maximal tori of $R$; cf. [Bo 91] Cor. 14.19. Since the connected center of a Levi factor of $P$ evidently lies in $R$, we see that the connected center of each Levi factor is a maximal torus of $R$.

Now, the centralizer $L_1 = Z(\gamma)$ is a Levi factor of $P$, so that $\gamma$ is a cocharacter of the connected center of $L_1$; in particular, the image of $\gamma$ lies in $R$. Moreover, since $L_1 = Z(\gamma)$, the centralizer of the image of $\gamma$ in $R$ is a maximal torus $S$ of $R$. It follows that $S$ is the unique maximal torus of $R$ containing the image of $\gamma$.

Since the image of $\gamma$ lies in $L$ and in $R$, and since $L$ intersects $R$ in a maximal torus of $R$, it follows that $S = L \cap R$ so that $L = L_1$ as required.

2.4. Instability in invariant theory. Let $(\rho, V)$ be a linear representation [always assumed finite dimensional] of $G$, and fix a closed $G$-invariant subvariety $S \subset V$. We are going to describe a precise form – due to Kempf and Rousseau – of the Hilbert-Mumford criteria for the instability of a vector $v \in V$ under the action of $G$.

Let us first briefly describe our goal: given a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g} = Lie(G)$, fix a basis $\mathbf{X} = (X_1, \ldots, X_d)$ of $\mathfrak{h}$. If the $G$-orbit of $\mathbf{X}$ in $\bigoplus \mathfrak{g}^d$ is not closed – so that $\mathbf{X}$ is an unstable vector – the results of Kempf and Rousseau permit us to associate to $\mathfrak{h}$ a unique parabolic subgroup $P_{\mathfrak{h}}$; see Corollary 2 below. If $g \in G$ satisfies $Ad(g)\mathfrak{h} = \mathfrak{h}$, one of our main objectives is to show that $g \in P_{\mathfrak{h}}$. Using $g$, we get a new basis $Ad(g)\mathbf{X} = (Ad(g)X_1, \ldots, Ad(g)X_d)$ of $\mathfrak{h}$, and generalities show that $P_{Ad(g)} = gP_{\mathfrak{h}}g^{-1}$. So we want to prove the equality $P_{\mathfrak{h}} = P_{Ad(g)}$; it will then follow that $g \in P_{\mathfrak{h}}$, as desired.

Return now to our general setting: $V$ is any linear representation of $G$. For $v \in V$, put

$$|V, v| = \{ \lambda \in X_*(G) \mid \lim_{t \to 0} \rho(\lambda(t))v \text{ exists}\}. $$

Write $V = \bigoplus_{i \in \mathbb{Z}} V(\lambda; i)$ as in (2.4.1), and write $v = \sum_i v_i$ with $v_i \in V(\lambda; i)$. Then evidently

$$\lambda \in |V, v| \iff v_i = 0 \quad \forall i < 0; $$

if $\lambda \in |V, v|$ then of course $\lim_{t \to 0} \rho(\lambda(t))v = v_0$.

Now let $S \subset V$ be a $G$-invariant closed subvariety and suppose that $v \not\in S$. Given $\lambda \in |V, v|$, write $v_0 = \lim_{t \to 0} \rho(\lambda(t))v$. If $v_0 \in S$, write $\alpha_{S,v}(\lambda)$ for the order of vanishing of the regular function $(t \mapsto \rho(\lambda(t))v - v_0) : \mathbb{A}^1 \to V$, otherwise write $\alpha_{S,v}(\lambda) = 0$; see [Ke 78] §3 for more details. Then $\alpha_{S,v}(\lambda)$ is a non-negative integer, and $\alpha_{S,v}(\lambda) > 0$ if and only if $v_0 \in S$. Moreover, if $v = \sum_{i \in \mathbb{Z}} v_i$ with $v_i \in V(\lambda; i)$ as before, then

$$v_0 \in S \implies \alpha_{S,v}(\lambda) = \alpha_{\{v_0\}, v}(\lambda) = \min\{j > 0 \mid v_j \not= 0\}. $$

Suppose that $W \subset V$ is a subspace of dimension $d = \dim W$. Let $w_1, \ldots, w_d$ be a basis of $W$, and consider the point $x = (w_1, \ldots, w_d)$ of the linear space $\mathfrak{X} = \bigoplus V$; abusing notation somewhat, we write also $\rho$ for the diagonal action $\bigoplus^d \rho$ of $G$ on $X$. We observe for $\lambda \in X_*(G)$ that we have

(2.4.3) $$\lambda \in |X, x| \iff W \subset \sum_{j \geq 0} V(\lambda; j). $$

Fix $S \subset X = \bigoplus V$ a closed and $\rho(G)$-invariant subvariety, and assume that $x = (w_1, \ldots, w_d) \not\in S$. In this setting one may compute the function $\alpha_{S,x}$ for the diagonal action on $X$ using functions $\alpha_{\{w_0\}, v}$ for the $G$-representation $V$. More precisely, we have:

Lemma 6. Let $\lambda \in |X, x|$ and suppose $\alpha_{S,x}(\lambda) > 0$. For $w \in W$, write $w_0 = \lim_{t \to 0} \rho(\lambda(t))w$. Then

$$\alpha_{S,\lambda}(\lambda) = \min_{w \in W} \alpha_{\{w_0\}, w}(\lambda). $$
Proof. For $1 \leq i \leq d$ write $x = \sum_{j} x^{j}$ with $x^{j} \in X(\lambda; j)$.

By assumption, $\lambda \in |X, x|$; by (2.4.1) we see that $x^{j} = 0$ if $j < 0$. Moreover, using (2.4.2) we see that

$$\alpha_{S,x}(\lambda) = \alpha_{(v^{0}),x}(\lambda) = \min(j > 0 \mid x^{j} \neq 0).$$

If we now write $R = \min_{v \in W} \alpha_{(v^{0}),v}(\lambda)$ for the right hand side of (*), then upon considering the components in $V$ of the vectors $x^{j} \in X = \bigoplus_{j} V$, one uses (2.4.3) to see that $\alpha_{S,x}(\lambda) \geq R$.

On the other hand, we may choose $v \in W$ such that $R = \alpha_{(v^{0}),v}(\lambda)$. Writing $v = \sum_{j \geq 0} v^{j}$ with $v^{j} \in V(\lambda; j)$, we see that

$$R = \alpha_{(v^{0}),v}(\lambda) = \min(j > 0 \mid v^{j} \neq 0)$$

by (2.4.1). Now write

$$v = \sum_{i} \beta_{i} w_{i}$$

for scalars $\beta_{i} \in K$.

Now, $v^{R} \neq 0$ implies that $x^{R} \neq 0$; it follows from (2.4.1) that $R \geq \alpha_{S,x}(\lambda)$, and the Lemma is proved. \hfill \Box

Fix a basis $\{w_{i}\}$ for $W$ and let $x = (w_{1}, \ldots, w_{d}) \in X$. Write

$$S = \overline{\rho(G)x} = \rho(G)x$$

then $S$ is closed in $X$. Notice that $S$ is a closed subset, since $\rho(G)x$ is open in $\overline{\rho(G)x}$, and $S$ is $G$-invariant. We suppose that $\rho(G)x$ is not closed, or equivalently that $S$ is non-empty.

Corollary 7. Let $h \in G$ satisfy $\rho(h)W = W$. If $x' = \rho(h)x$, then we have $|X, x| = |X, x'|$. Moreover,

$$\alpha_{S,x}(\lambda) = \alpha_{S,x'}(\lambda)$$

for each $\lambda \in |X, x|$. \hfill \Box

Proof. Since by (2.4.1) the sets $|X, x|$ and $|X, x'|$ both consist of all cocharacters $\lambda$ for which $W \subset \sum_{j \geq 0} V(\lambda; j)$, we have that $|X, x| = |X, x'|$.

Now write $x_{0} = \lim_{t \to 0} \rho(\lambda(t))x$ and $x_{0}' = \lim_{t \to 0} \rho(\lambda(t))x'$. We first claim that $x_{0} \in S$ if and only if $x_{0}' \in S$.

Well, assume that $x_{0} \not\in S$. Since $x_{0}$ lies in the closure of $\rho(G)x$ but not in $S$, it actually lies in $\rho(G)x$; thus (i) $x_{0} = \rho(g)x$ for some $g \in G$.

Since the components in $V$ of the vector $x \in X = \bigoplus_{j} V$ form a basis of $W$, one concludes from (i) that

$$\lim_{t \to 0} \rho(\lambda(t))y = \rho(g)y$$

for each $y \in \bigoplus_{j} W \subset X$. This shows in particular that $x_{0}' = \rho(g)x' = \rho(gh)x$, so that $x_{0}' \not\in S$. Since the argument just given is symmetric in $x$ and $x'$, it follows that $x_{0} \in S$ if and only if $x_{0}' \in S$.

Recall that $\alpha_{S,x}(\lambda) > 0$ if and only if $x_{0} \in S$ and that $\alpha_{S,x'}(\lambda) > 0$ if and only if $x_{0}' \in S$. Thus to prove the final equality asserted by the corollary, we may suppose that $x_{0}, x_{0}' \in S$. Now, according to (*) of Lemma 3 we have

$$\alpha_{S,x}(\lambda) = \min_{w \in W} \alpha_{(v^{0}),w}(\lambda) = \alpha_{S,x'}(\lambda)$$

as required. \hfill \Box

Fix a real-valued $G$-invariant length function $\lambda \mapsto \|\lambda\|$ on the set $X_{*}(G)$ of cocharacters of $G$. 
Then the non-trivial elements of \( \rho(G) \cap S \) is non-empty. Then the function \( \alpha_{S,z}(\lambda)/\|\lambda\| \) assumes a maximal value \( B > 0 \) on the non-trivial elements of \( |X, z| \). Let

\[
\Delta_{S,z} = \{ \lambda \in |X, z| \mid \alpha_{S,z}(\lambda) = B \cdot \|\lambda\| \text{ and } \lambda \text{ is indivisible} \}.
\]

Then

1. \( \Delta_{S,z} \) is non-empty,
2. there is a parabolic subgroup \( P_{S,z} \) of \( G \) such that \( P_{S,z} = P(\lambda) \) for each \( \lambda \in \Delta_{S,z} \),
3. \( \Delta_{S,z} \) is a principal homogeneous space under \( R_u P_{S,z} \), and
4. any maximal torus of \( P_{S,z} \) contains a unique cocharacter which lies in \( \Delta_{S,z} \).

Let \( H \subset G \) be a subgroup and suppose that \( W \) is \( \rho(H) \) invariant. Let \( x = (w_1, \ldots, w_d) \in X \) for a basis \( \{w_i\} \) of \( W \).

**Corollary 9.** Assume that \( \rho(G)x \) is not closed in \( X \), and let

\[
S = \rho(G)x - \rho(G)x.
\]

Then

1. \( P_{S,x} \) is a proper parabolic subgroup of \( G \),
2. \( H \subset P_{S,x} \), and
3. if \( L \subset P_{S,x} \) is a Levi factor, there is a cocharacter \( \lambda \) of the connected center \( Z \) of \( L \) which lies in \( \Delta(S, x) \).

**Proof.** Since the image of any \( \lambda \in |X, x| \) with \( \alpha_{S,x}(\lambda) > 0 \) is not central in \( G \), (1) is immediate.

Since the parabolic subgroup \( P = P_{S,x} \) is self-normalizing, (2) will follow if we show that \( hP h^{-1} = P \) for each \( h \in H(k) \). But \( h P_{S,x} h^{-1} = P_{S,\rho(h)x} \); see e.g. [Ke 78, Cor. 3.5]. Since \( \rho(h)W = W \), Corollary 7 shows that \( |X, x| = |X, \rho(h)x| \) and that \( \alpha_{S,x}(\lambda) = \alpha_{S,\rho(h)x}(\lambda) \) for all \( \lambda \in |X, x| = |X, \rho(h)x| \); thus \( \Delta_{S,x} = \Delta_{S,\rho(h)x} \) so that \( P_{S,x} = P_{S,\rho(h)x} \) by Theorem 8 Thus indeed \( H \subset P_{S,x} \).

Finally, for (3) let \( S \) be a maximal torus of \( L \) and hence of \( P_{S,x} \). By (3) of Theorem 8 \( S \) has a cocharacter \( \lambda \) which lies in \( \Delta_{S,x} \). Since \( P_{S,x} = P(\lambda) \), it follows from Lemma 5 that the image of \( \lambda \) lies in the connected center of \( L \), as required. \( \square \)

Finally, we record:

**Lemma 10.** Assume that \( \rho(G)x \) is closed in \( X \) and that \( \lambda \in |X, x| \). Then the subset

\[
\lim_{t \to 0} \rho(\lambda(t))W = \left\{ \lim_{t \to 0} \rho(\lambda(t))w \mid w \in W \right\}
\]

satisfies

\[
\lim_{t \to 0} \rho(\lambda(t))W = \rho(g)W
\]

for some \( g \in G \).

**Proof.** Since \( \lambda \in |X, x| \), the limit \( x_\lambda = \lim_{t \to 0} \rho(\lambda(t))x \) exists. Since the orbit \( \rho(G)x \) is closed, we have \( \rho(g)x = x_\lambda \) for some \( g \in G \). Since \( w_1, \ldots, w_d \) is a basis of \( W \), it follows that \( \text{Ad}(g)w = \lim_{t \to 0} \rho(\lambda(t))w \) for each \( w \in W \), whence the Lemma. \( \square \)
3. Proof of the main theorem

Recall that $G$ is a reductive group with Lie algebra $\mathfrak{g}$, and that $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra. Fix a basis $X_1, \ldots, X_d \in \mathfrak{h}$, and let $X = (X_1, \ldots, X_d) \in \bigoplus^d \mathfrak{g} = Y$. We write $(\text{Ad}, Y)$ for the representation $(\bigoplus^d \text{Ad}, \bigoplus^d \mathfrak{g})$ of $G$.

Proof of part (1) of Theorem 1. Recall that we must show: the Lie algebra $\mathfrak{h}$ is $G$-cr if and only if the $G$-orbit of $X$ is closed in $Y = \bigoplus^d \mathfrak{g}$.

We first suppose that $\text{Ad}(G)X$ is closed, and we show that $\mathfrak{h}$ is $G$-cr. Let $S$ be a maximal torus of the centralizer $C_G(\mathfrak{h})$. Then $\mathfrak{h} \subset \text{Lie}(L)$ where $L = C_G(S)$; moreover, $L$ is a Levi factor of a parabolic subgroup of $G$. It follows from Lemma 9 that $\mathfrak{h}$ is $G$-cr if and only if $\mathfrak{h}$ is $L$-cr.

Moreover, it follows from Proposition 2 that $\text{Ad}(L)X$ is closed in $Y$. Thus we may replace $G$ by $L$ and so suppose that any torus in $G$ which centralizes $\mathfrak{h}$ is central in $G$. [Equivalently: $\mathfrak{h}$ is not contained in the Lie algebra of any Levi factor of a proper parabolic subgroup of $G$.]

To show that $\mathfrak{h}$ is $G$-cr we will show that $\mathfrak{h}$ is not contained in $\text{Lie}(P)$ for any proper parabolic subgroup $P$ of $G$.

Suppose that $\mathfrak{h} \subset \text{Lie}(P)$ for a parabolic subgroup $P \subset G$; we will show that $P = G$. Write $P = P(\phi)$ for some cocharacter $\phi$ of $G$, and write $L = L(\phi)$ for the centralizer in $G$ of the image of $\phi$; then $L$ is a Levi factor of $P$.

Since the $G$-orbit of $X$ is closed, Lemma 10 shows that

$$\lim_{t \to 0} \text{Ad}(\phi(t))h = \text{Ad}(g)\mathfrak{h}$$

for some $g \in G$. Since $\lim_{t \to 0} \text{Ad}(\phi(t))H \in \text{Lie}(L)$ for each $H \in \mathfrak{h}$, we conclude that $\mathfrak{h} \subset \text{Ad}(g^{-1})\text{Lie}(L)$. But then the image of the cocharacter $\text{Int}(g^{-1}) \circ \phi$ is a torus centralizing $\mathfrak{h}$; hence the image of $\phi$ is central in $G$ so that $P = G$. This proves that $\mathfrak{h}$ is indeed $G$-cr.

To complete the proof of (i), it remains to show: if the orbit $\text{Ad}(G)X$ is not closed, then $\mathfrak{h}$ is not $G$-cr. As in Corollary 9 let $S = \overline{\text{Ad}(G)X} = \text{Ad}(G)X$; our assumption means that $S$ is non-empty so that $\alpha_{\mathfrak{s},X}(\lambda) > 0$ for each $\lambda \in \Delta_{S,X}$. Moreover, $P = P_{S,X}$ is a proper parabolic subgroup of $G$.

We have $\mathfrak{h} \subset \text{Lie}(P)$ by Proposition 2. To complete the proof, we suppose $\mathfrak{h}$ is $G$-cr, and find a contradiction.

Since $\mathfrak{h}$ is $G$-cr, there is a Levi factor $L$ of $P$ with $\mathfrak{h} \subset \text{Lie}(L)$. By Corollary 9 there is a cocharacter $\lambda$ of the connected center of $L$ which lies in $\Delta_{S,X}$. Since $\mathfrak{h} \subset \text{Lie}(L)$, we have $\mathfrak{h} \subset \mathfrak{g}(\lambda; 0)$; thus $X \in X(\lambda; 0)$. But then $\alpha_{\mathfrak{s},X}(\lambda) = 0$, which is impossible since $\lambda \in \Delta_{S,X}$. □

Proof of part (2) of Theorem 1. Recall that if $H \subset G$ is a subgroup which is $G$-cr, we must prove that $\mathfrak{h} = \text{Lie}(H)$ is $G$-cr.

Let $S \subset C_G(H)$ be a maximal torus. Then $H \subset L = C_G(S)$ and $\mathfrak{h} \subset \text{Lie}(L)$. Applying Lemma 9 it is enough to show that $\mathfrak{h}$ is $L$-cr; thus we replace $G$ by $L$ and so suppose that $H$ is not contained in a Levi factor of any proper parabolic subgroup of $G$. Since $H$ is $G$-cr, we conclude that $H$ is contained in no proper parabolic subgroup of $G$.

To show that $\mathfrak{h}$ is $G$-cr, we use part (1) of Theorem 1 it is enough to show that $\text{Ad}(G)X$ is closed in $Y$. In fact, we are going to suppose that $\text{Ad}(G)X$ is not closed and obtain a contradiction. Let $S = \overline{\text{Ad}(G)X} = \text{Ad}(G)X$ and let $P = P_{S,X}$. Since $S$ is assumed non-empty, Corollary 9 shows that $P$ is a proper parabolic subgroup. Moreover, since $\text{Ad}(H)$ leaves $\mathfrak{h}$ invariant, that same corollary shows that $H \subset P$. This contradiction completes the proof. □
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