I. INTRODUCTION

Resonance is important in both classical and quantum mechanics. In classical mechanics, resonance is associated with an increase in amplitude when the frequency of an external force coincides with a natural frequency of the system. In quantum mechanics, resonances are often defined as peaks of the transmission coefficient. They are also defined as isolated poles of the S-matrix, or as rapid variations of the phase shift. In this article we explore these definitions of resonances using the analytic solutions of the one and three-dimensional potential well, and we show how this model can be used to illustrate some of the properties of atomic nuclei.

We begin by reviewing resonances in the familiar damped harmonic oscillator (Section II), and then consider the well-known bound and scattering solutions of the one-dimensional potential well in Sections III A and III B. The less known solution of the problem using the S-matrix formalism is developed in Section III C. In order to present resonances in the three-dimensional space we illustrate the phase shift in Section III D using the solution of the three-dimensional potential well.

In Section IV A we illustrate how the different kinds of poles of the S-matrix evolve from resonances to bound states for the one-dimensional potential well. In this section we also compare the poles of the S-matrix with the peaks of the transmission coefficient. In Section IV B we use the one-dimensional potential well as a toy model to show the characteristic behavior of the loosely bound state wave function of the deuteron. We also connect some of the poles of the previous section with the bound and anti-bound states of the deuteron. In the second part of Section IV B we illustrate the characteristic behavior of the resonant phase shift in the $^5$He nucleus using the three-dimensional potential well. Finally, in Sec. V we summarize our results.

II. DAMPED HARMONIC OSCILLATOR

We begin by reviewing the well-known solution to the classical damped harmonic oscillator so that in the following sections we can point out some commonalities with the quantum mechanical problem. The equation of motion for a particle of mass $m$ subject to a driving force $F(t)$ is

$$\frac{d^2 x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = \frac{F(t)}{m} = f(t),$$

where $\omega_0$ and $\gamma$ are the natural frequency and the damping coefficient of the system, respectively, and both have positive values.

With an oscillating driving force $f(t) = F_0 e^{-i\omega t}$, the solution is

$$x(t) = X_\omega e^{-i\omega t},$$

with

$$X_\omega = -\frac{F_0}{\omega^2 + 2i\gamma\omega - \omega_0^2} = F_0 G(\omega).$$

We are interested in the properties of the amplitude $G(\omega)$ of the response function because in this paper we will make connections between it and quantum scattering theory. First, we write $G(\omega)$ as follows

$$G(\omega) = -\frac{1}{(\omega - \omega_1)(\omega - \omega_2)},$$

with

$$\omega_{1,2} = \pm(\omega_0^2 - \gamma^2)^{1/2} - i\gamma.$$  

The complex poles of $G(\omega)$ imprint their signature in the shape of $|G(\omega)|$ and also in $\text{Arg}[G(\omega)]$. For a weak damping, i.e. $\gamma \ll \omega_0$, $|G(\omega)|^2$ has two resonant peaks with width $2\gamma$ centered at $\omega \approx \pm \omega_0$, while the function $\text{Arg}[G(\omega)]$ changes by $\pi$, centered at $\pi/2$ and $3\pi/2$, in a narrow range around each one of the resonant frequencies. These characteristics are shown in Fig. 1 for
\[ \gamma = 0.1\omega_0. \] Then we can say that a classical resonance is characterized by two isolated poles close to the real axis.

For \( \omega \approx \omega_0 \) we may write

\[
\left| \frac{G(\omega)}{G(\omega_0)} \right|^2 \approx \frac{\gamma^2}{(\omega - \omega_0)^2 + \gamma^2}, \quad (6)
\]

\[
\text{Arg}[G(\omega)] \approx \arctan \left( \frac{\gamma}{\omega - \omega_0} \right). \quad (7)
\]

The first expressions confirms that the width of the peak in \( |G(\omega)|^2 \) is \( 2\gamma \), while the second one shows the rapid variation of \( \text{Arg}[G(\omega)] \) in the vicinity of \( \omega \approx \omega_0 \).

Classical and quantum resonances show some common characteristics; in particular, this kind of bump also appears in the experimentally-measured resonant scattering cross-section as a function of the energy in nuclear physics. In this context, a lorentzian parametrization like Eq. (6) is called Breit-Wigner distribution (see for example pp. 23-24 of Ref. 1). This characteristic behavior of \( \text{Arg}[G(\omega)] \) shown in Fig. 1 can also be reproduced in a simple quantum-mechanical model such as the one introduced in Sec. IIID.

For the limit \( \gamma \to \omega_0 \), we find that both poles converge at \( \omega_1 = \omega_2 = -i\omega_0 \). So the pole structure of \( G(\omega) \) changes from two isolated single poles to a second-order pole at \( \omega = -i\omega_0 \). Figure 2 shows this transition for three values of \( \gamma \). We can see the two peaks approach and merge into a single one. We also observe that the argument of \( G(\omega) \) changes from two jumps to a single smooth jump of \( 2\pi \).

Let us consider two particles of masses \( m_1 \) and \( m_2 \) limited to move in one dimension, subject to an attractive interaction \( V = -V_0 \) \((V_0 > 0)\) when their separation is less than \( R \). This problem can be mapped onto a one-dimensional Schrödinger equation for a single particle with reduced mass \( \mu = m_1 m_2/(m_1 + m_2) \):

\[
-\frac{\hbar^2}{2\mu} \frac{d^2u}{dx^2} + V(x)u(x) = \varepsilon u(x), \quad (8)
\]

with

\[
V(x) = \begin{cases} 
-V_0 & |x| \leq R \\
0 & |x| > R.
\end{cases}
\]

We look for the bound state solutions, i.e. \(-V_0 < \varepsilon < 0\), with the usual boundary conditions at \( x = \pm R \). With the requirement that \( u(x) \) is bounded, we find discrete energies, \( \varepsilon_n < 0 \), that are obtained from

\[
\rho_n = K_n \tan(K_n R), \quad (9)
\]

\[
-\rho_n = K_n \cot(K_n R). \quad (10)
\]

Equations (9) and (10) correspond to even and odd solutions, respectively. The parameters \( \rho_n \) and \( K_n \) are related to energy by

\[
\rho_n = \sqrt{-\frac{2\mu}{\hbar^2} \varepsilon_n}, \quad K_n = \sqrt{\frac{2\mu}{\hbar^2}(V_0 + \varepsilon_n)}. \quad (11)
\]

\section{III. QUANTUM MECHANICAL SCATTERING}

In this section we consider resonances in quantum mechanics. In Sections IIIA and IIIB we obtain the negative and positive-energy solutions of a one-dimensional Schrödinger equation for a square potential well. Then in Section IIIC we introduce the \( S \)-matrix for the one-dimensional system. Finally in Section IIID we consider resonances within the phase shift formalism for a 3-D model.
t = D'/A and reflection r = A'/A amplitudes:

\begin{align}
t(k) &= \frac{e^{-i2Rk}}{\cos(2RK) - i\frac{k^2 + 2\mu}{2\hbar^2}V_0 \sin(2RK)}, \\
r(k) &= \frac{i2\mu V_0}{2\hbar^2} \sin(2RK) e^{-i2Rk}/\cos(2RK) - i\frac{k^2 + 2\mu}{2\hbar^2} \sin(2RK).
\end{align}

In the rest of this paper, we will consider the \( k \) representation instead of the energy \( \varepsilon \) representation; i.e., we will consider \( k \) as the independent variable, so we rewrite

\[ K(k) = \sqrt{k^2 + 2\mu/\hbar^2 V_0}. \]

The transmission coefficient \( T = |t|^2 \) is

\[ T = \left[ 1 + \left( \frac{2\mu}{\hbar^2} V_0 \right)^2 \sin^2(2RK) \right]^{-1}, \]

where the maximums \( T = 1 \) occur for \( 2RK = \pi, 2\pi, \cdots \); that is, when there is an integer number of half-wavelengths of \( K \) inside the potential well. From a time-dependent picture a wave packet can be formed from plane waves and the maximums may be interpreted as the constructive interference inside the well.

### C. \( S \)-matrix

In the \( S \)-matrix approach the scattering process is divided into three stages. In the first one the particles move one towards the other, but they are so far apart that one can neglect their mutual interaction. In the second stage the particles are close enough to interact, while in the final stage they are far away and then they do not interact any more. The operator that transforms the initial stage into the final stage is called \( S \)-matrix or scattering matrix. In a time-dependent framework the initial and final stages correspond to the remote past and future, respectively. In our time-independent formulation, the \( S \)-matrix connects the wave functions of the two asymptotic regions: \( x < -R \) and \( x > R \). In this section we calculate the \( S \)-matrix for the one-dimensional potential well.

Let us start by considering the scattering states \( \varepsilon > 0 \) for \( |x| \geq R \) for particles incident from the left and right:

\[ u_L(x) = \begin{cases} e^{ikx} + re^{-ikx} & x \leq -R \\ te^{ikx} & x \geq R, \end{cases} \]

\[ u_R(x) = \begin{cases} t'e^{-ikx} & x \leq -R \\ e^{-ikx} + r'e^{ikx} & x \geq R. \end{cases} \]

The coefficients \( t \) and \( r \) correspond to the transmission and reflection amplitudes in the asymptotic regions \( |x| \geq R \) from Section III B, i.e., Eqs. (14) and (15). On the other hand \( t' \) and \( r' \) are the corresponding magnitudes for the reverse process, i.e. a particle coming from the right, with \( t' = t \) and \( r' = r \).

The general solution is a linear combination of \( u_L(x) \) and \( u_R(x) \):

\[ u(x) = Au_L(x) + A'u_R(x) \]

\[ = \begin{cases} Ae^{ikx} + \A'e^{-ikx} & x \leq -R \\ \A'e^{ikx} + A'e^{-ikx} & x \geq R, \end{cases} \]

where \( A \) and \( A' \) are the amplitudes of the wave function as it approaches to the interaction region, also called incoming amplitudes, while \( \A \) and \( \A' \) are the outgoing amplitudes, as shown in Fig. 3.

\[ \begin{align}
\lambda \\
\lambda'
\end{align} \]

Figure 3. Incoming amplitudes \( \lambda \) and \( \lambda' \), and outgoing amplitudes \( \A \) and \( \A' \) which define the \( S \)-matrix.

The relationship between the incoming and outgoing amplitudes defines the \( S \)-matrix:

\[ \begin{pmatrix} \A \\ \A' \end{pmatrix} = S \begin{pmatrix} A' \\ A \end{pmatrix}, \]

and we see from the earlier analysis:

\[ S = \begin{pmatrix} t & r' \\ r & t \end{pmatrix}. \]
Using Eqs. (14) and (15) for the transmission $t(k)$ and reflection $r(k)$ amplitudes, respectively, we can write the $S$-matrix as

$$S(k) = \frac{e^{-i2Rk}}{\cos(2RK) - i\frac{k^2 + K^2}{2kK} \sin(2RK)} \begin{pmatrix} 1 & \frac{2\mu}{\pi} \frac{V_0}{i\rhoK} \sin(2RK) \\ i\frac{2\mu}{\pi} \frac{V_0}{\rhoK} \sin(2RK) & 1 \end{pmatrix}. \quad (24)$$

Up to now, we have considered the momentum variable $k$ to be a real number, or, equivalently, we have considered real energies. However, it is very useful to consider the transmission and reflection amplitudes as functions of the complex wavelength or complex energy. Doing so, we get the analytic extensions of $t(k)$ and $r(k)$, and then, the analytic extension of the $S$-matrix. At the common complex zeros of the denominator of $t$ and $r$, each element of the matrix $S$ will have a pole; we refer to these common poles of the matrix elements as poles of the $S$-matrix.

The $S$-matrix in Eq. (22) connects the amplitudes of the wave function before and after the interaction, and thus it contains information about the system. In particular, its poles give the energies of the bound and unbound states of the system. For finite-range potentials (for further details the reader may consult Refs. [1] and [4]), the poles of the $S$-matrix can be classified into three categories as shown in Fig. 3: (i) bound states poles, which are sitting on the positive imaginary axis on the complex energy plane. Notice that while in section III A we found these relations for example chapter 2 of Ref. [5]. The bound states sitting on the positive imaginary $k$ axis are mapped to the negative real axis of the first sheet, called physical sheet (filled dots). The anti-bound states, found on the negative imaginary $k$ axis, are also mapped to the negatives real axis, but to the filled triangle on the second energy sheet.

![Figure 4](image)

Figure 4. Mapping between the $k$ plane and the two energy sheets based on the dispersion relation $\varepsilon = \hbar^2 k^2/2\mu$, Eq. (11).

To illustrate property (i) for the potential well let us search for the zeros of the common denominator of the elements of the $S$-matrix of Eq. (24) in the complex $k$-plane

$$\cos(2RK) - i\frac{k^2 + K^2}{2kK} \sin(2RK) = 0. \quad (25)$$

For the special case of bound states it is convenient to replace $k \to i\rho$, with $\rho \in \mathbb{R}^+$, then

$$\cos(2RK) - \frac{K^2 - \rho^2}{2\rho K} \sin(2RK) = 0 \quad (26)$$

with $K = \sqrt{\frac{2\mu}{\hbar^2}V_0 - \rho^2}$. Notice that, if we use the dispersion relation $\varepsilon = \hbar^2 k^2$, with $k = i\rho$, the magnitude $K$ is the same as the one in Eq. (13) for bound states. By using the identity $2\cot(2u) = \cot(u) - \tan(u)$ we obtain,

$$\cot(RK) - \tan(RK) = \frac{K}{\rho} - \frac{\rho}{K}. \quad (27)$$

The above equation has the structure $s - s^{-1} = f - f^{-1}$, with $s = \cot(RK)$ and $f = K/\rho$, which can be cast into the form $(s - f)(1 + sf) = 0$, with the solutions $s - f = 0$ and $1 + sf = 0$. Using these solutions for Eq. (27) we get

$$\rho = K \tan(KR),$$
$$\rho = -K \cot(KR),$$

which are the equations we found in section III A for the bound solutions of the potential well. As a curiosity, notice that while in section III A we found these relations by searching for even and odd solutions independently, here we get both of them from a single condition, Eq. (26).

In Section IV A we will illustrate numerically properties (ii) and (iii) by searching the zeros of Eq. (26) for anti-bound states and resonances. Now let us come back to the interpretation of the poles of the $S$-matrix in the complex energy plane. Using the dispersion relation we can map the upper and lower halves of the wave number $k$-plane onto two complex energy planes, see Fig. 4. These two energy planes may be combined to form a single surface with two sheets called Riemann surface (see, for example chapter 2 of Ref. [3]). The bound states sitting on the positive imaginary $k$ axis are mapped to the negative real axis of the first sheet, called physical sheet (filled dots). The anti-bound states, found on the negative imaginary $k$ axis, are also mapped to the negatives real axis, but to the filled triangle on the second energy plane

$$\cos(2RK) - \frac{k^2 + K^2}{2kK} \sin(2RK) = 0. \quad (25)$$
As usual, we write the three-dimensional Laplacian of the spherical potential well ( \( r \) with \( L \) the sum of two terms, where \( L \) of the spherical harmonics show up as sharp maxima in the cross section as a function of the energy or as a rapid variation of the phase shift (we will see an example in the second part of Sec. [IVB]). Anti-bound states and resonances represent unbound systems that maximum will be sharp if the phase shift rapidly becomes a bound state.\(^7,8\) Resonances show up as sharp maxima in the cross section (we will see an example in the first part of Sec. [IVB]). The physical interpretation of the anti-bound states is more elusive. We have mentioned that they have negative energies, like other bound states, but they are not normalizable. We found that they lie in the second Riemann sheet, which makes clear why this sheet is called nonphysical. An interpretation of an anti-bound or virtual state is that if the interaction were a bit stronger, the anti-bound state would become a bound state.\(^9\) In Sec. [IV A] we will illustrate this transition for the potential well.

D. Phase shift

The last picture we will use to illustrate resonances is the phase shift in a three-dimensional spherical potential well. As for the one-dimensional model, we write the Schrödinger equation in terms of the reduced mass and, for simplicity, we neglect the spin-orbit interaction:

\[
\left[ -\frac{\hbar^2}{2\mu} \nabla^2 + V(r) \right] \Psi(r) = \varepsilon \Psi(r). \tag{28}
\]

As usual, we write the three-dimensional Laplacian \( \nabla^2 \) as the sum of two terms, where \( L \) is the angular momentum operator:

\[
\nabla^2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) - \frac{L^2 / \hbar^2}{r^2}.
\]

The eigenfunction \( \Psi(r) \) can be written in terms of the spherical harmonics \( Y_{lm} \) which are eigenfunctions of \( \nabla^2 \), with eigenvalues \( l(l+1)\hbar^2 \). By writing \( \Psi(r) = r^{-1} u_l(r) Y_{lm}(\theta, \phi) \) in Eq. (28) we obtain the following radial Schrödinger equation for \( u_l(r) \):

\[
\left[ -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + V(r) \right] u_l(r) = \varepsilon u_l(r), \tag{29}
\]

where

\[
V(r) = \begin{cases} -V_0 & r \leq R \\ 0 & r > R \end{cases}
\]

with \( V_0 > 0 \) and \( R \) the strength and width, respectively, of the spherical potential well (\( r \geq 0 \)).

The above equation looks very much like the one-dimensional Schrödinger equation with an extra term called centrifugal interaction. This term depends on the orbital angular momentum \( l \), with \( l = 0, 1, 2, \ldots \). In order to introduce the phase shift we search for the scattering solutions \( u_l(r) \) for \( \varepsilon > 0 \), that satisfy \( u_l(r = 0) = 0 \) (this condition makes \( \Psi(r) \) be finite at zero), and \( u_l \) and \( u'_l \) continuous at \( r = R \). Since the interaction does not couple solutions for different orbital angular momentum, we can write for each \( l \)

\[
u_l(r) = \begin{cases} A_l j_l(Kr) & 0 \leq r \leq R \\ h_l^-(kr) - S_l h_l^+(kr) & r > R \end{cases}, \tag{30}
\]

with \( k \) and \( K \) as defined in Eq. (13) for the one-dimensional potential well. The scattering wave function is written in terms of \( j_l \) and \( h_l^\pm \), i.e., the spherical Bessel and Hankel functions, respectively.\(^9\) The coefficients \( S_l \) for the partial waves \( l \) correspond to the diagonal elements of the \( S \)-matrix. From Eq. (30) we have \( |S_l|^2 = 1 \), then we can write

\[
S_l = e^{2i\delta_l}, \tag{31}
\]

with \( \delta_l \) called the partial wave phase shift, for a reason that will be shown shortly. Requiring continuity of the wave function and its derivative at \( r = R \) allows us to find the phase shift.

\[
\delta_l = \arctan \left[ \frac{k j_l'(kR)j_l(KR) - K j_l(kR)j_l'(KR)}{k j_l(kR)j_l'(KR) - K j_l'(kR)j_l(KR)} \right], \tag{32}
\]

with \( \eta \) the Neumann functions. We will illustrate the behavior of a phase shift as a function of the wave number \( k \) in section [IVB].

In order to visualize that \( \delta_l \) is associated to a shift in phase, let us replace the asymptotic expressions of the Hankel functions

\[
h_l^\pm(kr \to \infty) = e^{\pm i(kr - \frac{\pi}{2} l \mp \delta_l)}.
\]

in Eq. (30). Then, together with Eq. (31) we get

\[
u_l(r \to \infty) = -2ie^{i\delta_l} \sin(kr - \frac{\pi}{2} l \pm \delta_l), \tag{33}
\]

which justifies the name of phase shift for \( \delta_l \) (for more details about the physical interpretation of the phase shift, see for example, Sec. VIII.C in Ref. [11]).

As a final observation before moving to the applications, let us connect the properties of the amplitude \( G(\omega) \) of Section [III] with the cross section for a quantum resonance. The partial wave cross section \( \sigma_l \) can be expressed in terms of the phase shift\(^1\)

\[
\sigma_l = \frac{4\pi}{k^2} (2l + 1) \sin^2(\delta_l) \cdot \tag{34}
\]

For \( \delta_l = \pi/2 \), the cross section will be a maximum, and that maximum will be sharp if the phase shift rapidly
changes around this value. These are the properties shown in Fig. 1 for a resonance in the classical damped harmonic oscillator. Then, it is possible to write the phase shift as the right side of Eq. 7 with \( \gamma \rightarrow \Gamma / 2 \), and \( \omega_0 \rightarrow \varepsilon_r \), where \( \varepsilon_r \) and \( \Gamma \) are the energy and width, respectively of the resonance,

\[
\delta_l = \arctan \left( \frac{\Gamma / 2}{\varepsilon - \varepsilon_r} \right) .
\]

By replacing this expression of the phase shift in Eq. 34, we get

\[
\sigma_l = \frac{4\pi}{k^2} (2l + 1) \frac{\Gamma^2}{4} \frac{4}{(\varepsilon - \varepsilon_r)^2 - \Gamma^2/4}.
\]

This is the so called Breit-Wigner formula. Notice that this parametrization resembles that of the right side of Eq. 6. The condition \( \Gamma / \varepsilon_r \ll 1 \) is equivalent to a sharp structure of the cross section. This ratio is like the one between the damping coefficient \( \gamma \) and the resonant frequency \( \omega_0 \) we considered in Sec. 11, where we found that the sharp structure occurs when \( \gamma / \omega_0 \ll 1 \).

In summary, at the resonant energy, the phase shift takes the value \( \pi / 2 \) and it rapidly changes in a small interval of \( k \) or energy, while the cross section shows a sharp maximum around the resonant energy as a function of the energy. In Sec. 4 we will see some examples of phase shift in the atomic nucleus.

### IV. APPLICATIONS

In this section we are going to illustrate the concepts developed in the previous one using some realistic parametrization for the potentials well. In Sec. 4A we study the movement of the poles of the S-matrix and the behavior of the transmission coefficient for the one-dimensional potential well. While in Sec. IV B we use the three-dimensional potential well to illustrate the characteristic of the loosely bound state of the deuteron and the resonant phase shift of the nucleus \(^5\)He.

#### A. Moving poles in the one-dimensional potential well

The goal of this section is to illustrate the three categories of poles of the S-matrix presented in Section III C. We also show the transitions between bound, anti-bound and resonant states. We will calculate the poles of the S-matrix Eq. (24) as a function of the potential strength \( V_0 \) with \( R = 2.1 \) fm and \( 2\mu / h^2 = 0.024 \text{ MeV}^{-1} \text{ fm}^{-2} \).

We start by calculating the first four bound states using Eqs. 39 and 10, which are shown in Fig. 5 as a function of \( V_0 \). We see that for any value of the strength there is always a bound state with even parity; as the strength increases this state becomes more bound. A new bound state, with odd parity, appears for \( V_0 \approx 23.3 \) MeV. As the strength increases, more states appear with interleaved parity.

Next, we study the locations of the first three states of Fig. 5 in the complex \( k \)-plane by finding the zeros of Eq. 25 for different values of \( V_0 \). The left panel of Fig. 6 shows the movement of the first even state (x’s) and first odd state (circles), while the right panel shows the evolution of the second even state (triangles). The first even bound state starts with \( k = 0 \) and moves along the positive imaginary axis as \( V_0 \) increases, implying a state that becomes increasingly more bound as the well gets deeper. The first odd state, however, starts at a large negative value along the imaginary axis. It moves upwards along the negative imaginary axis as \( V_0 \) increases, reaching the origin and becoming a bound state at \( V_0 = 23.29 \) MeV, consistent with Fig. 5.

The second even state is missing as a bound state in Fig. 5 for small values of the strength. In order to find its evolution we search for complex zeros of Eq. 25 with non zero real part for \( V_0 \) small, i.e., we search for resonances. The right panel of Fig. 6 shows the evolution of a pair of symmetric poles (\( n = 3 \) of Ref. 12) which moves upwards in the third and fourth quadrant of the \( k \)-plane. This pair of poles merges on the negative imaginary axis at \( k \approx -0.5 \) fm \(^{-1} \) (\( V_0 \approx 83 \) MeV) into a second-order pole. Afterward, one of the poles moves up and the other moves down, both along the imaginary axis. Eventually, as the strength \( V_0 \) keeps growing, one pole moves to positive imaginary values of \( k \) and becomes the second even bound state (right triangle) that appears in Fig. 5 for \( V_0 \approx 83 \) MeV.

In summary, we have shown in Fig. 6 how the different kinds of poles of the S-matrix evolve as a function of the strength. Let us remark on a detail which will be relevant for the next section: for a one-dimensional potential well there always exists at least one bound state for any value of \( V_0 \); this is not the case for a three-dimensional well.

Let us now study the behavior of the transmission coefficient \( T \), Eq. (17), as a function of the real wave number \( k \), for two values of the strength, \( V_0 = 33.84 \) and 21.76 MeV (these figures corresponds to the bound and
anti-bound strengths for the deuteron of Sec. IV B. The left and right panels of Fig. 7 show $T$ for $V_0 = 33.84$ and 21.76 MeV, respectively, with the scale on the right. Both curves show the characteristic behavior of $T$, with some resonant peaks for which $T = 1$; these peaks widen as $k$ increases. Notice that the right figure has a very narrow peak close to $k = 0$.

In this section we will connect the bound and anti-bound states found in Fig. 6 with the corresponding states of the deuteron. We also will show the behavior of the phase shift for the nucleus $^5\text{He}$.

First, let us consider the one-dimensional potential well as an approximation for the interaction in the deuteron, the nucleus of the heavy hydrogen atom, $^2\text{H}$. It can be solved by reducing the two-body proton-neutron system to a one-body problem with reduced mass $\mu$, with $2\mu/\hbar^2 = 0.024 \text{ MeV}^{-1}\text{ fm}^{-2}$. The strength $V_0$ measures the interaction between the proton and the neutron. For the parameter $R$ of the well we take the experimental radius of the deuteron $R = 2.1\text{ fm}$ (p. 40 in Ref. 15). Notice that these are the parameters used in the previous section to study the positions of the poles of the $S$-matrix. The experimental binding energy of the deuteron is $\varepsilon = -2.225 \text{ MeV}$.

In our one-dimensional potential well the first bound state is an even state then, in order to describe the deuteron we have to consider the second bound state of the one-dimensional potential well, i.e. the first odd state. Taken for the strength the value $V_0 = 33.84 \text{ MeV}$ we reproduce the ground state energy $-2.225 \text{ MeV}$ of the deuteron. It appear as a bound state pole in the left panel of Fig. 7 at the value $k = +i 0.231 \text{ fm}^{-1}$. The deuteron has an unbound state at $\varepsilon = -0.066 \text{ MeV}$ which is found, in our one-dimensional potential well for $V_0 = 21.76 \text{ MeV}$. This state appears as an anti-bound pole on the right panel in Fig. 7 at the value $k = -i 0.040 \text{ fm}^{-1}$. Different values of $V_0$ were needed to describe the bound and anti-bound states of the deuteron because of our oversimplified model. A more realistic description of the deuteron would require a more complex interaction.

The deuteron is an example of the so called loosely bound nuclei. The characteristic of these nuclei is that the wave function extends well beyond the radius $R$. In order to have an idea about this property, we artificially increases the strength to $V_0 = 46 \text{ MeV}$; the energy of this
As a final example of real nucleus, let us consider the \(^5\)He isotope. Experimentally, it is found that the separation energy of this nucleus is negative\(^{16}\), meaning that it is unbound. From the nuclear shell model the \(^5\)He nucleus is modeled as a two-body system, a \(^4\)He core plus a neutron with intrinsic spin \(s = 1/2\). The spin \(s\) couples to the orbital angular momentum \(l\) to give the total angular momentum \(j = l + s\). In this picture the valence neutron may be in the \(p_{3/2}\) or \(p_{1/2}\) (with \(p\) indicating the orbital angular momentum \(l = 1\) and \(j = 3/2\) or \(j = 1/2\)). The reduced mass of the \(^4\)He plus neutron system yields \(2\mu/h^2 = 0.0835\) MeV\(^{-1}\)fm\(^{-2}\), while the value for the parameter \(R\) was set equal to 2.15 fm.

Since we are not considering the spin-orbit interaction, the states \(p_{3/2}\) and \(p_{1/2}\) are degenerate in a single \(p\) state. The strength of the potential well \(V_0 = 53.18\) MeV was chosen to have \(\delta_p = \pi/2\) at the experimental resonant energy \(\varepsilon_{3/2^-} = 0.735\) MeV. Using Eq. \((42)\) we calculate the phase shift \(\delta_p\) shown by a solid line in Fig. \(9\). The profile of the calculated phase shift \(\delta_p\) is reminiscent of the function \(Arg(G)\) of the damped harmonic oscillator of Fig. \(1\). From the experimentally-measured cross-section as a function of the energy one may extract the phase shift for the different quantum numbers. In Fig. \(7\) we show the experimental phase shift for \(p_{3/2}\) and \(p_{1/2}\) (data from Ref. \(17\)). We observe that the calculated phase shift \(\delta_p\) in our simplified model follows closer the shape of the experimental phase shift \(\delta_{p_{3/2}}\), i.e. the resonant behaviour of the \(^5\)He nucleus. Better agreement with the experimental data would require the inclusion of the spin-orbit interaction.

The condition \(\Gamma/\varepsilon_r < 1\) is usually used to characterize an observable resonance\(^{17}\), with \(\varepsilon_r\) the energy of the resonance and \(\Gamma\) its width. From Fig. \(9\) we can calculate the width\(^{18}\)

\[
\Gamma = 2\left(\frac{d\delta}{d\varepsilon}\right)^{-1}_{\varepsilon_r} = 0.373\text{ MeV}.
\]

Then, by taking the ratio between the width \(\Gamma\) and the experimental resonant energy \(\varepsilon_r = 0.735\) MeV\(^{16}\) for the \(^5\)He, we get \(\Gamma/\varepsilon_r = 0.51\) MeV. Since this ratio is not as small as for the harmonic oscillator, the jump in Fig. \(9\) is somewhat less pronounced than in Fig. \(1\).

V. SUMMARY

We started by reviewing the behavior of the amplitude and phase of the classical damped harmonic oscillator near resonance. Then we applied the usual matching procedure to find the bound and scattering solutions of the one-dimensional well potential. After that we defined the \(S\)-matrix for the one dimensional potential well and introduced the its analytic extension to the complex wave number plane. We classified the poles of the scattering matrix and made the connection between its poles in the complex wave number and energy planes. Finally we introduced the phase shift in the three-dimensional spherical potential well.

In the applications we showed the movement of the first few poles of scattering matrix for the one-dimensional
potential well in the complex wave number plane as a function of the strength of the potential. We showed how the first state is always present for any value of the strength and how the second state becomes a bound state starting as an anti-bound state. The movement of these two states in the energy plane is as follows: the first state moves always on the physical (first) Riemann sheet along the negative real axis from zero towards negative infinity as the strength increases, while the second state also moves along the negative real axis but it starts on the nonphysical (second) Riemann sheet and then continues along the negative real axis on the physical Riemann sheet. The third bound state has an anti-bound partner; both these states started as resonances from small values of the strength and they merged on the negative imaginary axis, as it was found for the classical harmonic oscillator. Also, for the one-dimensional potential well, we illustrated the relation between the isolated poles of the scattering matrix and the maximums of the transmission coefficient. We found that the agreement departed as the imaginary part of the poles widens. We showed that the anti-bound state showed up as a narrow peak in the transmission coefficient very close to the origin.

By using the one-dimensional potential well we compared the loosely bound state wave function of the deuteron with a more tightly bound state and showed how the former extends much more in the space than the latter. As an example of an experimentally observable resonance in nuclear physics, we used the three-dimensional potential well as an approximate model to illustrated the unbound $^3\text{He}$ nucleus. We found the usual rapid variation of the phase shift that characterizes a resonance.

The use of the potential well in one and three dimensions allowed us to build toy models of real nuclei which were analytically solved. The concepts and equations presented in this article will allow advanced students and instructors to understand resonant properties in real nuclei by themselves, varying the parameters that define the Schrödinger equation.

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