Convexity of ratios of the modified Bessel functions of the first kind with applications

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Abstract
Let $I_\nu (x)$ be the modified Bessel function of the first kind of order $\nu$. Motivated by a conjecture on the convexity of the ratio $W_\nu (x) = x I_\nu (x) / I_{\nu+1} (x)$ for $\nu > -2$, using the monotonicity rules for a ratio of two power series and an elementary technique, we present fully the convexity of the functions $W_\nu (x)$, $W_\nu (x) - x^2 / (2\nu + 4)$ and $W_\nu (x^{1/\theta})$ for $\theta \geq 2$ on $(0, \infty)$ in different value ranges of $\nu$, which give an answer to the conjecture and extend known results. As consequences, some monotonicity results and new functional inequalities for $W_\nu (x)$ are established. As applications, an open problem and a conjectures are settled. Finally, a conjecture on the complete monotonicity of $W_\nu (x^{1/\theta})$ for $\theta \geq 2$ is proposed.

Keywords Modified Bessel function · Convexity · Monotonicity · Inequality

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Dedicated to people facing and battling COVID-19.

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1 Introduction

It is well-known that the second-order modified Bessel differential equation

\[ x^2 y''(x) + xy'(x) - \left( x^2 + u^2 \right) y(x) = 0, \quad (1.1) \]

has linearly independent solutions, that are, modified Bessel functions of the first and second, as usual, denoted by \( I_\nu \) and \( K_\nu \), respectively, see [1, p. 77]. The modified Bessel function of the first kind \( I_\nu (x) \) can be represented explicitly by the infinite series as

\[ I_\nu (x) = \sum_{n=0}^{\infty} \frac{(x/2)^{2n+\nu}}{n!\Gamma (\nu + n + 1)}, \quad x \in \mathbb{R}, \quad \nu \in \mathbb{R}\{−1, −2, \ldots\}, \quad (1.2) \]

while the modified Bessel function of the second kind \( K_\nu (x) \) is given by

\[ K_\nu (x) = \pi \frac{I_{-\nu} (x) - I_\nu (x)}{\sin (\nu \pi)}, \]

where the right-hand side of the formula above is replaced by its limiting value if \( \nu \) is an integer. It is known that \( I_\nu (x) \) satisfies the recurrence relations

\[ xI'_\nu (x) + \nu I_\nu (x) = xI_{\nu-1} (x), \quad (1.3) \]
\[ xI'_\nu (x) - \nu I_\nu (x) = xI_{\nu+1} (x), \quad (1.4) \]

which imply that

\[ I_{\nu-1} (x) - I_{\nu+1} (x) = \frac{2\nu}{x} I_\nu (x) \quad (1.5) \]

and

\[ y_\nu (x) := \frac{xI'_\nu (x)}{I_\nu (x)} = W_{\nu-1} (x) - \nu = \frac{x^2}{W_\nu (x)} + \nu, \quad (1.6) \]

where

\[ W_\nu (x) = \frac{xI_\nu (x)}{I_{\nu+1} (x)} \text{ for } \nu > -2 \text{ and } x > 0. \quad (1.7) \]

The ratio \( W_\nu (x) = xI_\nu (x) / I_{\nu+1} (x) \) plays an important role in finite elasticity [2, 3] and epidemiological models [4, 5], while another ratio \( R_\nu (x) = I_{\nu+1} (x) / I_\nu (x) \) appeared in probability and statistics [6–8], and was applied in chemical kinetics [9, 10], optics [11] and signal processing [12]. For this reason, the properties of the ratio \( W_\nu (x) \) have attracted many scholars’ interest.

In 1983 Watson [13] showed that \( W_\nu (x) \) is increasing on \((0, \infty)\) for all \( \nu \geq 0 \), which was used in the study of a Bayesian estimation problem by Robert [7]. This result was proven again in 1984 by Simpson and Spector [2] using another way. Baricz and Neuman [14, Theorem 2.2 (d)] in 2007 proved that \( W_\nu (x) \) is increasing on \((0, \infty)\) for all \( \nu > -2 \). Another proof of this monotonicity property for \( \nu > -1 \) can be seen
in [15, Theorem 2.2 (i)]. In this paper, Baricz [15, Theorem 2.2 (i)] proved that the function \( x \mapsto x^{\mu-v} I_v (x) / I_\mu (x) \) is strictly increasing on \((0, \infty)\) for all \( \mu > v > -1 \).

In [13] Watson also showed that the ratio \( R_v (x) = I_{v+1} (x) / I_v (x) \) is increasing and concave on \((0, \infty)\) for all \( v \geq -1/2 \). Using this result, Baricz [16, Eq. (2.12)] proved that the ratio \( y_v (x) = W_{v-1} (x) - v \) satisfies \( y_v'' (x) < 1 / (v+1) \) for \( v \geq -1/2 \), which implies that the function \( x \mapsto W_{v-1} (x) - x^2 / (2v + 2) \) is strictly concave on \((0, \infty)\) for \( v \geq -1/2 \). In 1984 Simpson and Spector [2] proved that the ratio \( W_v (x) \) is convex on \((0, \infty)\) for all \( v \geq 0 \). Based on numerical experiments, Baricz [15, p. 591] made the following conjecture.

**Conjecture 1** The function \( W_v (x) \) is strictly convex on \((0, \infty)\) for all \( v > -1 \).

**Remark 1** In a recent paper [17], Yang and Tian pointed out that Baricz’s conjecture may be modified as follows: \( W_v (x) \) is strictly convex on \((0, \infty)\) for all \( v \geq -1/2 \) and concave on \((0, \infty)\) if \(-3/2 \leq v < -1/2 \).

It is worth mentioning that, as early as in 1979, Ismail and Kelker [18] showed a more deep result which states that the function \( x \mapsto x^{(v-\mu)/2} I_\mu \left( \sqrt{x} \right) / I_v \left( \sqrt{x} \right) \) is strictly completely monotonic on \((0, \infty)\) for all \( \mu > v > -1 \). This, by choosing \( \mu = v+1 \), implies that the function \( x \mapsto 1 / W_v \left( \sqrt{x} \right) \) is strictly decreasing and convex on \((0, \infty)\) for \( v > -1 \).

Motivated by Baricz’s conjecture and the consequence of Ismail and Kelker’s result, the aim of this paper is to further investigate the convexity of the ratios \( W_v (x) \) and \( W_v \left( x^{1/\theta} \right) \) on \((0, \infty)\) for \( v > -2 \) instead of \( v > -1 \) with \( \theta \geq 2 \).

The rest of this paper is organized as follows. In the next section, we recall some tools which will be used in this paper and establish an asymptotic expansion of \( W_v (x) \). Our main results are presented in Sect. 3. In the first subsection, we present the monotonicity of the function \( x W_v' (x) - \theta W_v (x) \) for \( \theta = 1 \) and \( \theta \geq 2 \); in the second subsection, we find that the ratio \( W_v (x) \) is strictly convex on \((0, \infty)\) if and only if \( v \geq -1/2 \), which gives answer to Baricz’s conjecture; while the functions \( x \mapsto W_v \left( x^{1/\theta} \right) \) for \( \theta \geq 2 \) and \( x \mapsto W_v (x) - x^2 / (2v + 4) \) are both strictly concave on \((0, \infty)\) for \( v > -2 \), the former concavity improves a consequence of Ismail and Kelker; in the third subsection, we give necessary and sufficient conditions for which the functions \( x \mapsto [W_v (x) - p] / x \) where \( p = v+1/2 \) and \( 2 (v + 1) \) are respectively convex and concave on \((0, \infty)\). Some consequences of main results are given in Sect. 4, which include the monotonicity of three functions involving \( W_v (x) \) and several new functional inequalities for \( W_v (x) \). In Sect. 5, as applications, we give answers to Gaunt’s open problem and Baricz–Neuman’s conjecture. Finally, we make a conjecture that the function \( x \mapsto W_v \left( x^{1/\theta} \right) \) for \( \theta \geq 2 \) and \( v > -1 \) is a Bernstein function.

## 2 Preliminary results

To study the convexity of \( W_v (x) \) on \((0, \infty)\) for all \( v > -2 \), we introduce the function

\[
F_{\theta, v} (x) = x W_v' (x) - \theta W_v (x)
\]

(2.1)
on \((0, \infty)\) for \(\nu > -2\). An obvious fact is that

\[
F'_{\theta, \nu} (x) = x W''_\nu (x) - (\theta - 1) W'_{\nu} (x) = x^\theta \left[ x^{1-\theta} W'_{\nu} (x) \right]'.
\] (2.2)

In particular, \(F'_{1, \nu} (x) = x W''_\nu (x)\). Thus, to show the convexity of \(W_\nu (x)\), it suffices to investigate the monotonicity of the function \(F_{1, \nu} (x)\). It was shown in [2] that the ratio \(W_\nu (x)\) satisfies the Riccati equation

\[
x W'_{\nu} (x) = x^2 + 2 (\nu + 1) W_{\nu} (x) - W^2_{\nu} (x) .
\] (2.3)

Thus, \(F_{\theta, \nu} (x)\) can be expressed as

\[
F_{\theta, \nu} (x) = x^2 + 2 (\nu + 1 - \theta/2) W_{\nu} (x) - W^2_{\nu} (x) = -S_{\nu+1-\theta/2, \nu} (x) ,
\] (2.4)

where

\[
S_{p, \nu} (x) = W^2_{\nu} (x) - 2 p W_{\nu} (x) - x^2 .
\] (2.5)

The monotonicity of \(S_{p, \nu} (x)\) on \((0, \infty)\) for \(\nu \geq -3/2\) was shown in [19]. For the sake of convenience, we quote some relevant computed results here. Let

\[
f_1 (x) = x^2 I_{\nu} (x)^2 - 2 p x I_{\nu} (x) I_{\nu+1} (x) - x^2 I_{\nu+1} (x)^2 \quad \text{and} \quad f_2 (x) = I_{\nu+1} (x)^2 .
\] (2.6)

It was shown in [19] that

\[
S_{p, \nu} (x) = \frac{f_1 (x)}{f_2 (x)} = \frac{\sum_{n=0}^{\infty} a_n (x^2/4)^n}{\sum_{n=0}^{\infty} b_n (x^2/4)^n} ,
\] (2.7)

where

\[
a_n = 4 \frac{(2\nu - 2p + 1) n + (2\nu + 1) (\nu + 1 - p)}{(2n + 2\nu + 1) (n + \nu + 1)} \frac{\Gamma (\nu + 1)^2 \Gamma (2n + 2\nu + 2)}{n! \Gamma (n + \nu + 1)^2 \Gamma (n + 2\nu + 2)} ,
\] (2.8)

\[
b_n = \frac{2}{(n + \nu + 1) (n + 2\nu + 2)} \frac{\Gamma (\nu + 1)^2 \Gamma (2n + 2\nu + 2)}{n! \Gamma (n + \nu + 1)^2 \Gamma (n + 2\nu + 2)} ,
\] (2.9)

and

\[
a_n \quad b_n = 2 \frac{n + 2\nu + 2}{2n + 2\nu + 1} [(2\nu - 2p + 1) n + (2\nu + 1) (\nu + 1 - p)] .
\] (2.10)

It follows easily that

\[
d_n := \frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = -2 \left[ p - h_n (\nu) \right] ,
\] (2.11)
where
\[ h_n (v) = (2v + 1) \frac{2n^2 + 4 (v + 1) n + v (2v + 3)}{(2n + 2v + 1) (2n + 2v + 3)}. \]

Moreover, it has been shown in [19] that
\[ S_{p,ν} (0) = \lim_{x \to 0} \frac{f_1 (x)}{f_2 (x)} = \frac{a_0}{b_0} = 4 (ν + 1) (ν + 1 - p) , \tag{2.12} \]
\[ S_{ν+1/2,ν} (∞) = \lim_{x \to ∞} \frac{f_1 (x)}{f_2 (x)} = \lim_{n \to ∞} \frac{a_n}{b_n} = ν + \frac{1}{2} . \tag{2.13} \]

2.1 Three tools

To prove the monotonicity of \( F_{θ,ν} (x) \), we need three tools. The first tool is the so-called \( H \)-function \( H_{f, g} \) and two identities, which appeared in [20] and was called Yang’s \( H \)-function and Yang’s identities in [21] by Tian et. al. For \(-∞ < a < b ≤ ∞\), let \( f \) and \( g \) be differentiable on \((a, b)\) and \( g' \neq 0 \) on \((a, b)\). Then the function \( H_{f, g} \) is defined by
\[ H_{f, g} := \frac{f'}{g'} g - f . \tag{2.14} \]

Using Yang’s \( H \)-function, we have the following simple but useful identities:
\[ \left( \frac{f}{g} \right)' = \frac{g'}{g^2} \left( \frac{f'}{g'} g - f \right) = \frac{g'}{g^2} H_{f, g}, \tag{2.15} \]
\[ H'_{f, g} := \left( \frac{f'}{g'} \right)' g , \tag{2.16} \]

where \( f \) and \( g \) are twice differentiable on the interval \((a, b)\) with \( gg' \neq 0 \) on \((a, b)\).

The second tool is the monotonicity rules for the ratio of two power series proven first by Biernacki and Krzyz [22], which play an important role in dealing with the monotonicity of the ratio of power series. A special monotonicity rule of two power series can be seen [23].

**Lemma 1** ([22]) Let \( a_n \) and \( b_n \) \((n = 0, 1, 2, \ldots)\) be real numbers and let the power series \( A (t) = \sum_{n=1}^{∞} a_n t^n \) and \( B (t) = \sum_{n=1}^{∞} b_n t^n \) be convergent for \(|t| < r\). If \( b_n > 0 \) for \( n = 0, 1, 2, \ldots \), and \( a_n/b_n \) is strictly increasing (or decreasing) for \( n = 0, 1, 2, \ldots \), then the function \( t \mapsto A (t) / B (t) \) is strictly increasing (or decreasing) on \((0, r)\).

The third tool is the piecewise monotonicity rule presented by Yang, Chu and Wang in [24, Theorem 1], which is a nice complement to Biernacki and Krzyz’s monotonicity rule. This new rule has been applied in the study for special functions, see for example, means and trigonometric [25–27], Bessel functions [28, 29], hypergeometric functions [30–35]. As a corollary of [24, Theorem 1], the following piecewise monotonicity rule on \((0, ∞)\), Lemma 2, first appeared in [36, Lemma 6.4] without proof and was also strictly proven in [37]. The present version of Lemma 2 was from [38] which modified something that is not rigorous in the assumptions for the sequences \( \{a_k/b_k\}_{0≤k≤m} \).
and \( \{a_k/b_k\}_{k \geq m} \) in the original version [24, Corollary 2.6] and modified version [19, Lemma 2.3].

**Lemma 2** ([24, Corollary 2.6], [38]) Let \( A(t) = \sum_{k=0}^{\infty} a_k t^k \) and \( B(t) = \sum_{k=0}^{\infty} b_k t^k \) be two real power series converging on \( \mathbb{R} \) with \( b_k > 0 \) for all \( k \). If for certain \( m \in \mathbb{N} \), the non-constant sequences \( \{a_k/b_k\}_{0 \leq k \leq m} \) and \( \{a_k/b_k\}_{k \geq m} \) are respectively increasing (decreasing) and decreasing (increasing), then there is a unique \( t_0 \in (0, \infty) \) such that the function \( A/B \) is increasing (decreasing) on \( (0, t_0) \) and decreasing (increasing) on \( (t_0, \infty) \).

### 2.2 Asymptotic formulas involving \( W_\nu (x) \) and \( S_{p,\nu} (x) \)

To investigate the convexity of \( W_\nu (x) \) on \( (0, \infty) \) for all \( \nu > -2 \), we also need the asymptotic expansions of \( W_\nu (x) \) and \( S_{p,\nu} (x) \). Using the asymptotic expansion (see [39, Eq. (10.40.1)])

\[
I_\nu (x) \sim \frac{e^x}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} \frac{\lambda_n (\nu)}{x^n} \quad \text{as} \quad x \to \infty, \tag{2.17}
\]

where \( \lambda_0 (\nu) = 1 \) and for \( k \geq 1 \)

\[
\lambda_k (\nu) = \frac{(-1)^k}{k! 8^k} \prod_{j=1}^{k} \left( 4\nu^2 - (2j-1)^2 \right), \tag{2.18}
\]

we easily obtain

\[
\frac{I_\nu (x)}{I_{\nu+1} (x)} \sim \sum_{n=0}^{\infty} \frac{c_n}{x^n} \quad \text{as} \quad x \to \infty, \quad \text{with} \quad c_0 = 1 \quad \text{and for} \quad n \geq 1,
\]

\[
c_n = \lambda_n (\nu) - \sum_{k=1}^{n} c_{n-k} \lambda_k (\nu + 1). \tag{2.19}
\]

Then we obtain the following

**Lemma 3** The asymptotic formula

\[
W_\nu (x) = \frac{x I_\nu (x)}{I_{\nu+1} (x)} \sim \sum_{n=0}^{\infty} \frac{c_n}{x^{n-1}}, \quad \text{as} \quad x \to \infty, \tag{2.20}
\]
Convexity of ratios of the modified Bessel functions

holds. In particular, as $x \to \infty$,

$$W_\nu (x) \sim x + \nu + \frac{1}{2} + \frac{(2\nu + 1)(2\nu + 3)}{8x} + \frac{(2\nu + 1)(2\nu + 3)}{8x^2},$$  \hfill (2.21)

$$W_\nu' (x) \sim 1 - \frac{(2\nu + 1)(2\nu + 3)}{8x^2} - \frac{(2\nu + 1)(2\nu + 3)}{4x^3},$$  \hfill (2.22)

$$W_\nu'' (x) \sim \frac{(2\nu + 1)(2\nu + 3)}{4x^3} + \frac{3(2\nu + 1)(2\nu + 3)}{4x^4}.$$  \hfill (2.23)

**Remark 2** It follows that

$$\lim_{x \to \infty} (W_\nu (x) - x) = \nu + \frac{1}{2}, \quad \lim_{x \to \infty} W_\nu' (x) = 1, \quad \lim_{x \to \infty} W_\nu'' (x) = 0.$$  \hfill (2.24)

By (1.2) we see that

$$I_\nu (x) \sim \frac{(x/2)^\nu}{\Gamma(\nu + 1)} \left( 1 + \frac{x^2}{4\nu + 1} \right), \text{ as } x \to 0,$$

which leads to the following.

**Lemma 4** We have

$$W_\nu (x) = \frac{x I_\nu (x)}{I_{\nu+1} (x)} \sim 2(\nu + 2) \frac{x^2 + 4\nu + 4}{x^2 + 4\nu + 8}, \text{ as } x \to 0,$$  \hfill (2.25)

$$W_\nu' (x) \sim 16(\nu + 2) \frac{x}{(x^2 + 4\nu + 8)^2} \to 0 \text{ as } x \to 0,$$  \hfill (2.26)

$$W_\nu'' (x) \sim 16(\nu + 2) \frac{4\nu + 8 - 3x^2}{(4\nu + 8 + x^2)^3} \to \frac{1}{\nu + 2} \text{ as } x \to 0.$$  \hfill (2.27)

**Remark 3** Using L’Hospital rule we have

$$\lim_{x \to 0} \frac{W_\nu (x) - 2(\nu + 1)}{x^2/2} = \lim_{x \to 0} \frac{W_\nu' (x)}{x} = \lim_{x \to 0} W_\nu'' (x) = \frac{1}{\nu + 2}.$$  \hfill (2.28)

**Lemma 5** Let $p = \nu + 1/2$. As $x \to \infty$, we have

$$\left[ \begin{array}{c} f_1 (x) \\ f_2 (x) \end{array} \right] \sim -\frac{(2\nu + 3)(2\nu + 1)}{4x^2}.$$  \hfill (2.29)

And therefore, if $-2 < \nu < -3/2$ then

$$H_{f_1, f_2} (x) = \frac{f_1' (x)}{f_2' (x)} f_2 (x) - f_1 (x) \to -\infty \text{ as } x \to \infty.$$  \hfill (2.30)
**Proof** We give first the asymptotic formula of the function

\[ S_{p,v} (x) = W_v^2 (x) - 2p W_v (x) - x^2 = \frac{f_1 (x)}{f_2 (x)}. \]

Using the asymptotic formula (2.21) yields, as \( x \to \infty \),

\[
\frac{f_1 (x)}{f_2 (x)} \sim \left( x + v + \frac{1}{2} + \frac{(2v + 1)(2v + 3) + (2v + 1)(2v + 3)}{8x} \right)^2 - 2p \left( x + v + \frac{1}{2} + \frac{(2v + 1)(2v + 3) + (2v + 1)(2v + 3)}{8x} \right) - x^2
\]

\[
\sim (2v + 1 - 2p)x + (2v + 1)(v + 1 - p) - \frac{(2v + 3)(2v + 1)(2p - 2v - 3)}{8x}.
\]

Particularly, when \( p = v + 1/2 \),

\[
\frac{f_1 (x)}{f_2 (x)} \sim v + \frac{1}{2} + \frac{(2v + 3)(2v + 1)}{4x} \text{ as } x \to \infty,
\]

and it is deduced that

\[
\left[ \frac{f_1 (x)}{f_2 (x)} \right]' \sim -\frac{(2v + 3)(2v + 1)}{4x^2} \text{ as } x \to \infty.
\]

Since as \( x \to \infty \),

\[
f_2 (x) = I_{v+1} (x)^2 \sim \frac{e^{2x}}{2\pi x} \text{ and } f_2' (x) \sim \left[ \frac{e^{2x}}{2\pi x} \right]' = \frac{(2x - 1)e^{2x}}{2\pi x^2},
\]

\[
\frac{f_2 (x)^2}{f_2' (x)} \sim \left( \frac{e^{2x}}{2\pi x} \right)^2 \left( \frac{2x - 1)e^{2x}}{2\pi x^2} \right) \sim \frac{e^{2x}}{4\pi x}.
\]

From the formula (2.15) we arrive at

\[
H_{f_1, f_2} (x) = \frac{f_2 (x)^2}{f_2' (x)} \left[ \frac{f_1 (x)}{f_2 (x)} \right]' \sim -\frac{e^{2x}}{4\pi x} \frac{(2v + 3)(2v + 1)}{4x^2} \to -\infty \text{ as } x \to \infty,
\]

if \(-2 < v < -3/2\), which proves (2.30) and the proof is done. \( \Box \)

### 3 Main results

We state and prove our main results in this section beginning with giving the monotonicity properties of the function \( x \mapsto F_{\theta,v} (x) \) on \((0, \infty)\) for \( \theta = 1 \) and \( \theta \geq 2 \).
3.1 The monotonicity of $F_{\theta,v}(x) = xW'_v(x) - \theta W_v(x)$

We are now in a position to prove the monotonicity of $F_{\theta,v}(x) = xW'_v(x) - \theta W_v(x)$.

**Theorem 1** Let $\nu > -2$. The following statements are valid.

(i) If $\nu \geq -1/2$, then the function $F_{1,v}(x) = xW'_v(x) - W_v(x)$ is increasing from $(0, \infty)$ onto $(-2v - 2, -\nu - 1/2)$, and therefore, the double inequality

$$-2(\nu + 1) < xW'_v(x) - W_v(x) < -(\nu + 1)$$

holds for $x > 0$.

(ii) If $-3/2 \leq \nu < -1/2$, then there exists an $x_0 > 0$ such that $F_{1,v}(x)$ is increasing on $(0, x_0)$ and decreasing on $(x_0, \infty)$. Moreover, the inequalities

$$-2(\nu + 1) < xW'_v(x) - W_v(x) \leq F_{1,v}(x_0) < -(2v + 1)$$

hold for $x > 0$.

(iii) If $-2 < \nu < -3/2$, then there exist $x_2 > x_1 > 0$ such that $F_{1,v}(x)$ is first increasing on $(0, x_1)$ then decreasing on $(x_1, x_2)$ and finally increasing on $(x_2, \infty)$. Consequently, the inequalities

$$-\frac{2(2\nu + 1)(\nu + 2)}{2v + 5} < \min \{ -2(\nu + 1), F_{1,v}(x_2) \} \leq xW'_v(x) - W_v(x) \leq \max \{ -(\nu + 1/2), F_{1,v}(x_1) \} < -(2\nu + 1)$$

hold for $x > 0$.

**Proof** Using the relation (2.4) we have $F_{1,v}(x) = -S_{v+1/2,v}(x)$.

(i) When $\nu \geq -1/2$, by [19, Theorem 3.1 (ii)], we see that the function $S_{v+1/2,v}(x)$ is decreasing from $(0, \infty)$ onto $(\nu + 1/2, 2(\nu + 1))$, which implies the first assertion.

(ii) When $-3/2 \leq \nu < -1/2$, it was shown in [19, Theorem 3.4 (iii)] that, for $-3/2 < \nu < -1/2$, there exists an $x_0 > 0$ such that $S_{v+1/2,v}(x)$ is decreasing on $(0, x_0)$ and increasing on $(x_0, \infty)$ with the estimates

$$S_{v+1/2,v}(x_0) \leq S_{v+1/2,v}(x) < 2\nu + 2$$

for all $x > 0$, where $x_0$ is a unique solution of the equation $S_{v+1/2,v}'(x) = 0$ on $(0, \infty)$, and then the monotonicity of the function $F_{1,v}(x) = -S_{v+1/2,v}(x)$ follows. It is easy to check that this conclusion is also true for $\nu = -3/2$ (the same below). To prove the third inequality of (3.2), it suffices to prove $S_{v+1/2,v}(x_0) > 2\nu + 1$. Substituting $p = \nu + 1/2$ into the formula (2.10) yields

$$\frac{a_n}{b_n} = (2\nu + 1) \frac{n + 2\nu + 2}{2n + 2\nu + 1},$$

(3.4)
and then
\[
\frac{a_n}{b_n} - (2v + 1) = (2v + 1) \frac{n + 2v + 2}{2n + 2v + 1} = -\frac{(2v + 1)(n - 1)}{2n + 2v + 1} \geq 0 \quad (3.5)
\]

for all \(n \geq 0\) and \(-2 < \nu < -1/2\). Since \(b_n > 0\) for all \(n \geq 0\), it follows from the formula (2.7) that

\[
S_{\nu+1/2, \nu} (x) - (2v + 1) = \sum_{n=0}^{\infty} a_n \left( \frac{x^2}{4} \right)^n - (2v + 1)
\]

\[
= \sum_{n=0}^{\infty} \left[ \frac{a_n}{b_n} - (2v + 1) \right] b_n \left( \frac{x^2}{4} \right)^n > 0
\]

(3.6)

for all \(x > 0\) and \(-2 < \nu < -1/2\), which completes the proof of the second assertion of this theorem.

(iii) When \(-2 < \nu < -3/2\), since \(F_{1, \nu} (x) = -S_{\nu+1/2, \nu} (x)\), it suffices to prove that there exist \(x_2 > x_1 > 0\) such that \(S_{p, \nu} (x) = f_1 (x) / f_2 (x)\) for \(p = \nu + 1/2\) is first decreasing on \((0, x_1)\) then increasing on \((x_1, x_2)\) and finally decreasing on \((x_2, \infty)\). Substituting \(p = \nu + 1/2\) into (2.11) gives

\[
d_n = \frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} = -\left[ v + \frac{1}{2} - h_n (v) \right] = -\frac{(2v + 1)(2v + 3)}{(2n + 2v + 1)(2n + 2v + 3)}.
\]

A simple computation yields \(d_0 = -1 < 0\), \(d_1 = -(2v + 1) / (2v + 5) > 0\) and \(d_n < 0\) for \(n \geq 2\), which implies that the sequence \(\{a_n / b_n\}\) is decreasing for \(n = 0, 1\), then increasing for \(n = 1, 2\), and decreasing for \(n \geq 2\). Clearly, Lemmas 1 and 2 do not directly apply in such case. For this, we let

\[
f_i (x) = g_i \left( \frac{x^2}{4} \right), \; i = 1, 2.
\]

By (2.7), we have

\[
g_1 (t) = \sum_{n=0}^{\infty} a_n t^n
\]

\[
g_2 (t) = \sum_{n=0}^{\infty} b_n t^n.
\]

Differentiation yields

\[
g_1' (t) = \sum_{n=1}^{\infty} n a_n t^{n-1}
\]

\[
g_2' (t) = \sum_{n=1}^{\infty} n b_n t^{n-1} = \sum_{n=0}^{\infty} (n + 1) a_{n+1} t^n
\]

\[
= \sum_{n=0}^{\infty} (n + 1) b_{n+1} t^n.
\]

Obviously, the sequence \(\{(n + 1) a_{n+1} / (n + 1) b_{n+1}\} = \{a_{n+1} / b_{n+1}\}\) is increasing for \(n = 0, 1\) and decreasing for \(n \geq 1\). By Lemma 2, there is a unique \(t_0 \in (0, \infty)\) such that the function \(g_1 (t) / g_2 (t)\) is increasing on \((0, t_0)\) and decreasing on \((t_0, \infty)\).

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Using the formula (2.16), we see that $H'_{g_1, g_2} = (g'_1 / g'_2) g_2 > 0$ for $t \in (0, t_0)$ and $H'_{g_1, g_2} < 0$ for $t \in (t_0, \infty)$ due to $g_2 > 0$. Since

$$H_{g_1, g_2}(0) = \frac{g'_1(0)}{g'_2(0)} g_2(0) - g_1(0) = \frac{a_1}{b_1} b_0 - a_0 = -\frac{1}{(v + 1)^2} < 0,$$

and by Lemma 5,

$$\lim_{t \to \infty} H_{g_1, g_2}(t) = \lim_{t \to \infty} \left[ \frac{g'_1(t)}{g'_2(t)} g_2(t) - g_1(t) \right] = \lim_{x \to \infty} \left[ \frac{f'_1(x)}{f'_2(x)} f_2(x) - f_1(x) \right] = -\infty,$$

where $x = 2\sqrt{t}$, we claim that $H_{g_1, g_2}(t_0) > 0$. If not, that is, $H_{g_1, g_2}(t_0) \leq 0$, then $H_{g_1, g_2}(t) \leq 0$ for all $t > 0$. This, in view of $g'_2 > 0$, in combination with the relation $(g'_1 / g'_2) = (g'_2 / g'_2) H_{g_1, g_2}$ yields $(g'_1 / g'_2) \leq 0$, that is, $g'_1 / g'_2$ is decreasing on $(0, \infty)$. It thus follows that

$$2(v + 1) = \frac{a_0}{b_0} = \lim_{t \to 0} \frac{g_1(t)}{g_2(t)} \geq \lim_{t \to \infty} \frac{g_1(t)}{g_2(t)} = \lim_{n \to \infty} \frac{a_n}{b_n} = v + \frac{1}{2},$$

which implies that $v \geq -3/2$. This is in contradiction with the assumption that $-2 < v < -3/2$. Consequently, there are two positive numbers $t_1$ and $t_2$ with $t_1 < t_2$ for which $H_{g_1, g_2}(t) < 0$ for $t \in (0, t_1) \cup (t_2, \infty)$ and $H_{g_1, g_2}(t) > 0$ for $t \in (t_1, t_2)$. Then

$$(g'_1(t) / g'_2(t)) < 0 \quad \text{for} \quad t \in (0, t_1) \cup (t_2, \infty) \quad \text{and} \quad (g'_1(t) / g'_2(t)) > 0 \quad \text{for} \quad t \in (t_1, t_2),$$

and therefore, $(f'_1(x) / f'_2(x)) < 0$ for $x \in (0, x_1) \cup (x_2, \infty)$ and $(f'_1(x) / f'_2(x)) > 0$ for $x \in (x_1, x_2)$, where $x_i = 2\sqrt{i}$, $i = 1, 2$, which proves the required piecewise monotonicity. From the monotonicity of the function $S_{p,v}(x) = f'_1(x) / f'_2(x)$ for $p = v + 1/2$ on $(0, \infty)$ we have

$$\min \left\{ S_{v+1/2,v}(x_1), S_{v+1/2,v}(\infty) \right\} \leq S_{v+1/2,v}(x) \leq \max \left\{ S_{v+1/2,v}(0), S_{v+1/2,v}(x_2) \right\}$$

for all $x > 0$. Since the inequality (3.5) holds for all $n \geq 0$ and $v < -1/2$, it thus holds that

$$\min \left\{ S_{v+1/2,v}(x_1), S_{v+1/2,v}(\infty) \right\} > \min \{ 2v + 1, v + 1/2 \} = 2v + 1.$$

Finally, using the formula (3.4) leads to

$$\frac{a_n}{b_n} - 2(v + 1)(v + 2) = -\frac{(2v + 1)(2v + 3)(n - 2)}{(2v + 5)(2n + 2v + 1)} \leq 0.$$
for all \( n \geq 0 \) and \(-2 < \nu < -3/2\). It then follows that

\[
S_{\nu+1/2,\nu}(x) - \frac{2(2\nu + 1)(\nu + 2)}{2\nu + 5} = \frac{\sum_{n=0}^{\infty} a_n (x^2/4)^n}{\sum_{n=0}^{\infty} b_n (x^2/4)^n} - (2\nu + 1)\]

\[
= \frac{1}{\sum_{n=0}^{\infty} b_n (x^2/4)^n} \sum_{n=0}^{\infty} \left[ a_n - \frac{2(2\nu + 1)(\nu + 2)}{2\nu + 5} b_n (x^2/4)^n \right] < 0
\]

for all \( x > 0 \) and \(-2 < \nu < -3/2\), which indicates that

\[
\max \left\{ S_{\nu+1/2,\nu}(0), S_{\nu+1/2,\nu}(x_2) \right\} \leq \max \left\{ 2\nu + 2, \frac{2(2\nu + 1)(\nu + 2)}{2\nu + 5} \right\}
\]

\[
= \frac{2(2\nu + 1)(\nu + 2)}{2\nu + 5}.
\]

Due to \( F_{1,\nu}(x) = x W'_\nu(x) - W_\nu(x) = -S_{\nu+1/2,\nu}(x) \), the inequalities (3.3) follow, and the proof is completed. \( \square \)

**Theorem 2** Let \( \theta \geq 2 \) and \( \nu > -2 \). The function \( F_{\theta,\nu}(x) = x W'_\nu(x) - \theta W_\nu(x) \) is decreasing from \((0, \infty)\) onto \((-\infty, -2\theta(\nu + 1))\), and hence, the inequality

\[
x W'_\nu(x) - \theta W_\nu(x) < -2\theta(\nu + 1)
\]

holds for \( x > 0 \).

**Proof** From [19, Theorem 3.1 (iv) and Theorem 3.4 (v)] we see that if \( p = \nu + 1 - \theta/2 \leq \nu \), that is, \( \theta \geq 2 \), then \( F_{\theta,\nu}(x) = -S_{\nu+1-\theta/2,\nu}(x) \) is decreasing on \((0, \infty)\) for \( \nu \geq -3/2 \). We prove that this conclusion is also valid for all \( \nu > -2 \). In fact, since \( b_0 = 1/(\nu + 1)^2 > 0 \) and \( b_n > 0 \) for all \( n \geq 1 \) and \( \nu > -2 \), if \( p \leq \nu \) then

\[
\frac{a_{n+1}}{b_{n+1}} - \frac{a_n}{b_n} \geq -2[v - h_n(\nu)] = \frac{4n(2n + 2\nu + 2)}{(2n + 2\nu + 1)(2n + 2\nu + 3)} \geq 0
\]

for all \( n \geq 0 \) and \( \nu > -2 \), that is, the sequence \( \{a_n/b_n\}_{n \geq 0} \) is increasing. By Lemma 1 the function \( S_{p,\nu}(x) \) for \( p \leq \nu \) is increasing on \((0, \infty)\), which implies that \( F_{\theta,\nu}(x) = -S_{\nu+1-\theta/2,\nu}(x) \) is decreasing on \((0, \infty)\). The desired inequality follows from \( F'_{\theta,\nu}(x) < 0 \) for \( x > 0 \). This completes the proof. \( \square \)

### 3.2 The convexity of \( W_\nu(x) \)

We are now in a position to state and prove the convexity of the ratio \( W_\nu \). First, we prove the following theorem, which describes fully the convex pattern of the ratio \( W_\nu \) and gives an answer to Baricz’s conjecture stated in Introduction.

**Theorem 3** Let \( \nu > -2 \). (i) The ratio \( W_\nu(x) \) is strictly convex on \((0, \infty)\) if and only if \( \nu \geq -1/2 \); (ii) if \( -3/2 \leq \nu < -1/2 \), then \( W_\nu(x) \) is convex then concave on \((0, \infty)\); (iii) while \(-2 < \nu < -3/2\), \( W_\nu(x) \) is first convex then concave and finally convex on \((0, \infty)\).

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Both of the functions

\begin{proof}
(i) Using the asymptotic relation (2.23), the necessary condition for which \( W_v(x) \) is strictly convex on \((0, \infty)\) is that: \( v \geq -1/2 \). Since \( F_{1,v}(x) = xW'_v(x) - W_v(x) \) is strictly increasing on \((0, \infty)\) by Theorem 1 (i), we immediately get that \( F^{'}_{1,v}(x) = xW''_v(x) > 0 \) for \( x > 0 \), which implies that the ratio \( W_v(x) \) is strictly convex on \((0, \infty)\), and the sufficiency follows.

(ii) If \(-3/2 \leq v < -1/2\), then by Theorem 1 (ii), there exists an \( x_0 > 0 \) such that \( F^{'}_{1,v}(x) = xW''_v(x) > 0 \) for \( x \in (0, x_0) \) and \( F^{''}_{1,v}(x) = xW''_v(x) < 0 \) for \( x \in (x_0, \infty) \), which implies the second required claim.

(iii) If \(-2 < v < -3/2\), then using Theorem 1 (iii) we can prove the required assertion, and the proof is done. \( \square \)

Due to the relation (2.2), that is, \( F^{'}_{\theta,v}(x) = x^\theta [x^{1-\theta}W'_v(x)]' \), we can obtain an interesting concavity of the ratio \( W_v(x^{1/\theta}) \) for \( \theta \geq 2 \) and the function \( x \mapsto W_v(x) - x^2/(2v + 4) \) on \((0, \infty)\) for \( v > -2 \).

**Theorem 4** Let \( \theta \geq 2 \) and \( v \geq -2 \). Then the following statements are valid.

(i) Then function \( x \mapsto x^{1-\theta}W'_v(x) \) is strictly decreasing on \((0, \infty)\). Taking \( \theta = 2 \), the double inequality

\[
\max \{ W''_v(x), 0 \} < \frac{W'_v(x)}{x} < \frac{1}{\nu + 2},
\]

holds for \( x > 0 \).

(ii) Both of the functions

\[
x \mapsto W_v(x^{1/\theta}) \quad \text{and} \quad x \mapsto W_v(x) - \frac{x^2}{2(\nu + 2)}
\]

are strictly concave on \((0, \infty)\).

\begin{proof}
(i) The decreasing property of the function \( x \mapsto x^{1-\theta}W'_v(x) \) on \((0, \infty)\) follows from Theorem 2 and the relation (2.2). This property for \( \theta = 2 \) implies that

\[
0 = \lim_{x \to \infty} \frac{W'_v(x)}{x} < \lim_{x \to 0} \frac{W'_v(x)}{x} = \frac{1}{\nu + 2}
\]

for \( x > 0 \); while \( F^{''}_{2,v}(x) = xW''_v(x) - W_v(x) < 0 \) for \( x > 0 \) implies that \( W''_v(x) < W'_v(x)/x \) for \( x > 0 \). The double inequality (3.8) thus follows.

(ii) Differentiation leads to

\[
\frac{d}{dx} W_v(x^{1/\theta}) = \frac{1}{\theta} x^{1/\theta-1} W'_v(x^{1/\theta}) = \frac{1}{\theta} t^{1-\theta} W'_v(t), \quad \text{where} \quad t = x^{1/\theta},
\]

\[
\frac{d^2}{dx^2} W_v(x^{1/\theta}) = \frac{t^{1-\theta}}{\theta^2} \left[ t^{1-\theta} W'_v(t) \right] < 0
\]

\( \square \) Springer
for $x > 0$. Using the inequalities (3.8) yields

$$
\left[ W_\nu (x) - \frac{x^2}{2(v+2)} \right]'' = W''_\nu (x) - \frac{1}{v+2} < 0
$$

for $x > 0$, thereby completing the proof. \( \square \)

**Remark 4** From Theorem 4 (ii) we immediately deduce that the functions

$$
x \mapsto \frac{1}{W_\nu \left( x^{1/\theta} \right)} \quad \text{and} \quad x \mapsto \ln W_\nu \left( x^{1/\theta} \right)
$$

are both strictly convex on $(0, \infty)$ for $\theta \geq 2$ and $\nu > -1$. In particular, the convexity of the ratio $1/W_\nu \left( \sqrt{x} \right)$ is a direct consequence of Ismail and Kelker's deep result in [18].

**Remark 5** It was shown in Baricz [16, Eq. (2.12)] that the inequalities $y''_\nu (x) < 1/(\nu+1)$ and $y'_\nu (x) < x/(\nu+1)$ holds for $x > 0$ and $\nu \geq -1/2$. Due to the relation 1.6, that is, $y'_\nu (x) = W_{\nu-1} (x) - \nu$, Theorem 4 (ii) implies that the two inequalities hold for $x > 0$ and $\nu > -1$, which extends the value range of $\nu$ from $\nu \geq -1/2$ to $\nu > -1$.

### 3.3 The convexity of $\xi_{p,\nu} (x) = \left[ W_\nu (x) - p \right]/x$

**Theorem 5** Let $p \in \mathbb{R}$, $\nu > -2$ and let the function $\xi_{p,\nu}$ by defined on $(0, \infty)$ by

$$
\xi_{p,\nu} (x) = \frac{W_\nu (x) - p}{x}. \quad (3.9)
$$

The function $\xi_{\nu+1/2,\nu} (x)$ is convex on $(0, \infty)$ if and only if $\nu \geq -1/2$, while $\xi_{2\nu+2,\nu} (x)$ is concave on $(0, \infty)$ if and only if $\nu \geq -3/2$.

**Proof** Differentiation yields

$$
x^2 \xi'_{p,\nu} (x) = xW'_\nu (x) - W_\nu (x) + p, \quad (3.10)
$$

$$
x^3 \xi''_{p,\nu} (x) = x^2W''_\nu (x) - 2 \left[ xW'_\nu (x) - W_\nu (x) + p \right]. \quad (3.11)
$$

Using the asymptotic formulas (2.21)–(2.23) we have

$$
x^3 \xi''_{p,\nu} (x) \sim 2\nu + 1 - 2p + \frac{3(2\nu+1)(2\nu+3)}{4x}, \text{ as } x \to \infty,
$$

which implies that

$$
\lim_{x \to \infty} \left[ x^4 \xi'_{\nu+1/2,\nu} (x) \right] = 3 \left( \nu + \frac{1}{2} \right) \left( \nu + \frac{3}{2} \right),
$$

$$
\lim_{x \to \infty} \left[ x^3 \xi''_{2\nu+2,\nu} (x) \right] = -(2\nu + 3).
$$
(i) If \( \xi_{\nu+1/2,v} (x) \) is convex on \((0, \infty)\), then \( \lim_{x \to \infty} \left[ x^4 \xi''_{\nu+1/2,v} (x) \right] \geq 0 \), which implies that \( \nu \geq -1/2 \). Conversely, if \( \nu \geq -1/2 \), then by the right hand side inequality of (3.1),
\[
x W'_v (x) - W_v (x) + \nu + \frac{1}{2} < 0
\]
for \( x > 0 \), which, in combination with \( W''_v (x) > 0 \) for \( x > 0 \) by Theorem 3, yields
\[
x^3 \xi''_{\nu+1/2,v} (x) = x^2 W''_v (x) - 2 \left[ x W'_v (x) - W_v (x) + \nu + \frac{1}{2} \right] > 0
\]
for \( x > 0 \), and therefore, \( \xi''_{\nu+1/2,v} (x) > 0 \) for \( x > 0 \).

(ii) If \( \xi_{2\nu+2,v} (x) \) is concave on \((0, \infty)\), then \( \lim_{x \to \infty} \left[ x^4 \xi''_{\nu+1/2,v} (x) \right] \leq 0 \), which implies that \( \nu \geq -3/2 \). To prove the sufficiency, we use the relation (1.6), which implies
\[
W_{v-1} (x) = \frac{x^2}{W_v (x)} + 2v,
\]
to write \( \xi_{2\nu+2,v} (x) \) as
\[
\xi_{2\nu+2,v} (x) = \frac{W_v (x) - 2 (\nu + 1)}{x} = \frac{x}{W_{v+1} (x)}.
\]
(3.12)

Differentiation again yields
\[
\xi'_{2\nu+2,v} (x) = \frac{W_{v+1} (x) - x W'_{v+1} (x)}{W_{v+1} (x)^2},
\]
\[
\xi''_{2\nu+2,v} (x) = \frac{-x W_{v+1} (x) W''_{v+1} (x) + 2 \left[ x W'_{v+1} (x) - W_{v+1} (x) \right] W'_{v+1} (x)}{W_{v+1} (x)^3}.
\]

Since \( W_{v+1}^{(i)} (x) > 0 \) \((i = 0, 1, 2)\) for \( x > 0 \) and \( \nu + 1 \geq -1/2 \), and \( x W'_{v+1} (x) - W_{v+1} (x) < - (\nu + 3/2) \leq 0 \) for \( x > 0 \) and \( \nu + 1 \geq -1/2 \) due to the right hand side inequality of (3.1), we immediately get \( \xi''_{2\nu+2,v} (x) < 0 \) for \( x > 0 \) and \( \nu \geq -3/2 \). This completes the proof. \[\square\]

**Remark 6** The relation (3.12) implies that \( \xi_{2\nu+2,v} (x) = I_{\nu+2} (x) / I_{\nu+1} (x) = R_{\nu+1} (x) \). The second assertion of Theorem 5 means that the ratio \( R_v (x) \) is concave on \((0, \infty)\) if and only if \( \nu \geq -1/2 \). This result was due to Watson [13].

### 4 Consequences

In this section, we give some direct consequences of the convexity of the ratios \( W_v (x) \) and \( W_v (x^{1/\theta}) \) for \( \theta \geq 2 \), including some monotonicity and new functional inequalities for \( W_v (x) \).
4.1 Three monotonicity results involving $W_\nu(x)$

First, we consider the difference $W_\nu(x) - x$, which appeared in the study of generalized Marcum Q-function, see [40, Lemma 1]. The monotonicity of $x \mapsto y_\nu(x) - x$ on $(0, \infty)$ for $\nu \geq 1/2$ was proven in [41], which, since $y_\nu(x) = W_{\nu-1}(x) - \nu$, implies that the difference $W_\nu(x) - x$ has the same monotonicity as the function $x \mapsto y_{\nu+1}(x) - x$ for $\nu \geq -1/2$. In the following corollary, we present a new proof of this monotonicity for $\nu \geq -1/2$ and discuss the piecewise monotonicity for $-2 < \nu < -1/2$ using the convexity of the ratio $W_\nu(x)$ on $(0, \infty)$ for $\nu > -2$.

**Corollary 1** Let the function $\delta_\nu$, defined on $(0, \infty)$ for $\nu > -2$ by $\delta_\nu(x) = W_\nu(x) - x$. The following statements are valid:

(i) If $\nu \geq -1/2$, then the function $\delta_\nu(x)$ is decreasing from $(0, \infty)$ onto $(\nu + 1/2, 2\nu + 2)$. Consequently, the double inequality

$$x + \nu + \frac{1}{2} < W_\nu(x) < x + 2(\nu + 1) \quad (4.1)$$

holds for $x > 0$.

(ii) If $-3/2 \leq \nu < -1/2$, then the function $\delta_\nu(x)$ is decreasing then increasing on $(0, \infty)$, and therefore, the right hand side inequality of (4.1) holds for $x > 0$.

(iii) If $-2 < \nu < -3/2$, then the function $\delta_\nu(x)$ is first decreasing then increasing and finally decreasing on $(0, \infty)$.

**Proof** Using the relations (2.25) and (2.21) gives

$$\delta_\nu(0) = 2(\nu + 1) \quad \text{and} \quad \delta_\nu(\infty) = \lim_{x \to \infty} \delta_\nu(x) = \nu + 1/2. \quad (4.2)$$

(i) In the case of $\nu \geq -1/2$, due to $W_\nu''(x) > 0$ for $x > 0$ by Theorem 3 (i), we derive that

$$0 = W_\nu'(0) < W_\nu'(x) < \lim_{x \to \infty} W_\nu'(x) = 1, \quad (4.3)$$

where $W_\nu'(0) = 0$ and $\lim_{x \to \infty} W_\nu'(x) = 1$ are as in (2.26) and (2.24). This implies that $\delta_\nu'(x) = W_\nu'(x) - 1 < 0$ for $x > 0$, that is, the function $\delta_\nu(x) = W_\nu(x) - x$ is strictly decreasing on $(0, \infty)$. By virtue of the decreasing property of $\delta_\nu(x)$ on $(0, \infty)$, the double inequality (4.1) follows.

(ii) In the case of $-3/2 \leq \nu < -1/2$, by Theorem 3 (i) there is an $x_2 > 0$ such that $\delta_\nu''(x) = W_\nu''(x) > 0$ for $x \in (0, x_2)$ and $\delta_\nu''(x) = W_\nu''(x) < 0$ for $x \in (x_2, \infty)$, that is, $\delta_\nu'(x)$ is increasing on $(0, x_2)$ and decreasing on $(x_2, \infty)$. This in conjunction with facts that

$$\delta_\nu'(x) = W_\nu'(0) - 1 = -1 < 0 \quad \text{and} \quad \lim_{x \to \infty} \delta_\nu'(x) = \lim_{x \to \infty} W_\nu'(x) - 1 = 0$$
leads to that there is an \( x_1 \in (x_2, \infty) \) for which \( \delta'_v(x) < 0 \) for \( x \in (0, x_1) \) and \( \delta''_v(x) > 0 \) for \( x \in (x_1, \infty) \). We thus obtain

\[
\delta_v(x) < \max \{ \delta_v(0), \delta_v(\infty) \} = \max \left\{ 2(v + 1), v + \frac{1}{2} \right\} = 2(v + 1),
\]

which indicates that the right hand side inequality of (4.1) holds for \( x > 0 \).

(iii) In the case of \(-2 < v < -3/2\), it is seen from Theorem 3 (ii) that there are \( x_4 > x_3 > 0 \) such that \( \delta''_v(x) = W''_v(x) > 0 \) for \( x \in (0, x_3) \cup (x_4, \infty) \) and \( \delta''_v(x) = W''_v(x) < 0 \) for \( x \in (x_3, x_4) \). Since \( \delta'_v(0) = -1 < 0 \) and \( \delta'_v(\infty) = 0 \), we have \( \delta'_v(x_4) < \delta'_v(\infty) = 0 \). Moreover, we have \( \delta'_v(x_3) > 0 \). If not, that is, \( \delta'_v(x_3) \leq 0 \), then \( \delta'_v(x) \leq 0 \) for all \( x > 0 \), which means that \( \delta_v(x) \) is decreasing on \((0, \infty)\), and therefore,

\[
2(v + 1) = \delta_v(0) > \delta_v(\infty) = v + \frac{1}{2},
\]

namely, \( v > -3/2 \). This is clearly in contradiction with the assumption that \(-2 < v < -3/2\). Thus, there are two \( x_1 \in (0, x_3) \) and \( x_2 \in (x_3, x_4) \) such that \( \delta'_v(x) < 0 \) for \( x \in (0, x_1) \cup (x_2, \infty) \) and \( \delta'_v(x) > 0 \) for \( x \in (x_1, x_2) \).

We thus complete the proof. \( \square \)

**Remark 7** Due to the relation (1.6), the first inequality of (4.1) is equivalent to

\[
x - \frac{1}{2} < \frac{xI_v'(x)}{I_v(x)} = y_v(x)
\]

for \( x > 0 \) and \( v \geq 1/2 \). This was proven in [41] by Gronwall. The second inequality of (4.1) was shown in [42] by Nåsell.

**Remark 8** Using the monotonicity of the function \( x \mapsto W_v(x) - x \), the proof of Baricz’s Lemma 1 in [40] can be greatly simplified. Moreover, a version of the modified Bessel function of the second of Corollary 1 was shown in [43, Proposition 2] by Yang and Tian.

Second, using Theorems 5 and 1 we have the following monotonicity results of the function \( \xi_{p,v}(x) \) defined by (3.9) on \((0, \infty)\) for \( p = v + 1/2, 2(v + 1) \).

**Corollary 2** Let \( v > -2 \). The following statements are valid:

(i) The function \( \xi_{v+1/2,v}(x) \) is strictly decreasing on \((0, \infty)\) if and only if \( v \geq -1/2 \); when \(-3/2 < v < -1/2\), it is decreasing then increasing on \((0, \infty)\); while \(-2 < v < -3/2\), it is increasing then decreasing on \((0, \infty)\); it is strictly increasing on \((0, \infty)\) if and only if \( v = -3/2 \).

(ii) The function \( \xi_{2v+2,v}(x) \) is strictly increasing if and only if \( v \geq -3/2 \), while \(-2 < v < -3/2\), it is increasing then decreasing on \((0, \infty)\).

Using the concavity of the function \( x \mapsto W_v(x) - x^2/(2v + 4) \) on \((0, \infty)\) for \( v > -2 \) and the inequality (3.7) for \( \theta = 2 \), we have
Corollary 3 Let $\nu > -2$. The functions $x \mapsto W_\nu(x) - x^2/(2\nu + 4)$ and $x \mapsto [W_\nu(x) - 2(\nu + 1)]/x^2$ are both strictly decreasing from $(0, \infty)$ onto $(-\infty, 0)$. Consequently, the double inequality

$$2(\nu + 1) < W_\nu(x) < 2(\nu + 1) + \frac{x^2}{2(\nu + 2)} \quad (4.4)$$

holds for $x > 0$.

4.2 Several new inequalities for $W_\nu(x)$

There are various existing inequalities for the ratio $W_\nu(x)$ for $x > 0$ and different range of values of $\nu$, which mainly include Amos type inequality that $W_\nu(x) < (>) p + \sqrt{x^2 + q^2}$ (see [44, 45]), Simpson-Spector type inequality that $W_\nu(x) < (>) S_{p,\nu}(x)$ (see [2, 19, 46]), Yang-Zheng type inequality that $W_\nu(x) < (>) p + r\sqrt{x^2 + q^2}$ ($r > 0$) (see [28]). More inequalities for the ratio $W_\nu(x)$ can be found in [15, 16, 29, 42, 47–53].

Using our main results, we can deduce some known inequalities for the ratio $W_\nu(x)$. For example, using the second inequality of (3.8) and Riccati Eq. (2.3) gives

$$W_\nu'(x) = \frac{x^2 + 2(\nu + 1)W_\nu(x) - W_\nu^2(x)}{x^2} < \frac{1}{\nu + 2},$$

which implies

$$\nu + 1 + \sqrt{\frac{\nu + 1}{\nu + 2}x^2 + (\nu + 1)^2} < W_\nu(x)$$

for $x > 0$ and $\nu > -1$. This inequality was first proven by Segura [49, Eq. (61)] (see also [28, Eq. (4.16)]) . Since

$$\frac{W_{\nu-1}'(x)}{x} = \frac{I_\nu(x)^2 - I_{\nu+1}(x)I_{\nu-1}(x)}{I_\nu(x)^2},$$

the double inequality (3.8) implies the Turán type inequality

$$0 < I_\nu(x)^2 - I_{\nu+1}(x)I_{\nu-1}(x) < \frac{1}{\nu + 1}I_\nu(x)^2$$

for $x > 0$ and $\nu > -1$, which was first proven by Thiruvenkatachar and Nanjundiah [54]. While the inequality (4.3), or equivalently, $0 < W_{\nu-1}(x) < 1$ for $x > 0$ and $\nu \geq 1/2$, implies another Turán type inequality

$$0 < I_\nu(x)^2 - I_{\nu+1}(x)I_{\nu-1}(x) < \frac{1}{x}I_\nu(x)^2$$

for $x > 0$ and $\nu \geq 1/2$, which was due to Baricz [16, Eq. (2.5)].
In what follows, we list several new inequalities. The following corollary gives the lower and upper bounds for the function $W_{\nu}(x) - x W'_{\nu}(x)$ in the case of $\nu > -2$, which are combined by the inequalities (3.1), (3.2) and (3.3).

**Corollary 4** Let $\nu > -2$. The double inequality

$$
\min \left\{ \frac{\nu+1}{2}, 2\nu + 1 \right\} < W_{\nu}(x) - x W'_{\nu}(x) < \max \left\{ 2(\nu + 1), \frac{2(2\nu + 1)(\nu + 2)}{2\nu + 5} \right\}
$$

(4.5)

holds for all $x > 0$.

Whence, we can deduce the following

**Corollary 5** Let $\nu > -2$. The double inequality

$$
x + \min \left\{ \frac{\nu+1}{2}, 2\nu + 1 \right\} < W_{\nu}(x) < x + \max \left\{ 2(\nu + 1), \frac{2(2\nu + 1)(\nu + 2)}{2\nu + 5} \right\}
$$

(4.6)

holds for all $x > 0$.

**Proof** It follows from Corollary 1 that the first inequality of (4.6) for $\nu \geq -1/2$ and the second one of (4.6) for $\nu \geq -3/2$ hold for $x > 0$.

We next prove the inequality

$$
x + 2\nu + 1 < W_{\nu}(x)
$$

(4.7)

for $x > 0$ in the case of $-2 < \nu < -1/2$. In this case, the left hand side inequality of (4.5) becomes

$$
2\nu + 1 < W_{\nu}(x) - x W'_{\nu}(x)
$$

for $x > 0$, which is equivalent to

$$
\left[ -\frac{W_{\nu}(x)}{x} \right]' = -\frac{x W'_{\nu}(x) - W_{\nu}(x)}{x^2} > \frac{2\nu + 1}{x^2} = \left( -\frac{2\nu + 1}{x} \right)',
$$

which in turn implies that the function $x \mapsto [W_{\nu}(x) - (2\nu + 1)]/x$ is strictly decreasing on $(0, \infty)$. It thus follows that

$$
\frac{W_{\nu}(x) - (2\nu + 1)}{x} > \lim_{x \to \infty} \frac{W_{\nu}(x) - (2\nu + 1)}{x} = 1
$$

for $x > 0$, and the inequality (4.7) follows.

Finally, we prove the inequality

$$
W_{\nu}(x) < x + \frac{2(2\nu + 1)(\nu + 2)}{2\nu + 5}
$$

(4.8)
for $x > 0$ in the case of $-2 < \nu < -3/2$. In this case the right hand side inequality of (4.5) changes to

$$W_\nu(x) - x W'_\nu(x) < \frac{2(2\nu + 1)(\nu + 2)}{2\nu + 5}$$

for $x > 0$. Similarly, it implies that the function

$$x \mapsto \frac{1}{x} \left[ W_\nu(x) - \frac{2(2\nu + 1)(\nu + 2)}{2\nu + 5} \right]$$

is strictly increasing on $(0, \infty)$, which leads to (4.8), and the proof is completed. □

**Remark 9** In 1974 Näsell [55] proved a nice inequality

$$\left(1 + \frac{\nu}{x}\right) I_{\nu+1}(x) < I_\nu(x), \text{ or equivalently, } W_\nu(x) > x + \nu$$

for all $\nu > -1$ and $x > 0$. Now using the first inequality of (4.6) we have

$$W_\nu(x) > x + \nu + \min \left\{ \frac{1}{2}, \nu + 1 \right\} > x + \nu$$

for $x > 0$ if $\nu > -1$. This shows that our inequality is a refinement of Näsell’s.

Taking into account the inequalities (4.4), (3.1) and (4.3) we obtain the following chain of inequalities.

**Corollary 6** Let $\nu \geq -1/2$ and $x > 0$. The inequalities

$$0 < \frac{W_\nu(x) - 2(\nu + 1)}{x} < W'_\nu(x) < 1 < \frac{W_\nu(x) - (\nu + 1/2)}{x}$$

(4.9)

hold. The first and second inequalities of (4.9) also hold for $-3/2 \leq \nu < -1/2$.

**Corollary 7** The inequality

$$W_\nu(x) > \nu + 1 + \sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\nu + \frac{1}{2}\right)\left(\nu + \frac{3}{2}\right)}$$

(4.10)

holds for $x > 0$ and $\nu \geq -1/2$.

**Proof** Taking into account the inequality $W'_\nu(x) < 1$ and Riccati Eq. (2.3) gives

$$x W'_\nu(x) = x^2 + 2(\nu + 1)W_\nu(x) - W^2_\nu(x) < x$$

for $x > 0$ and $\nu \geq -1/2$. This implies the required inequality. □
Remark 10 Simpson and Spector [2, Theorem 2] proved that the inequality

\[ W_\nu(x) > \nu + \frac{1}{2} + \sqrt{x^2 + \left(\nu + \frac{1}{2}\right)^2 + \frac{3}{2}} \]  

holds for \( x > 0 \) and \( \nu \geq 0 \). Hornik and Grün [45] showed that this inequality is sharp and extended the range \( \nu \geq 0 \) to \( \nu \geq -\frac{1}{2} \). It is easy to check that the inequality (4.10) is stronger than (4.11). In fact, let \( L \) and \( l \) denote the lower bounds given in (4.10) and (4.11), respectively. Then

\[
\left( L - \nu - \frac{1}{2} \right)^2 - \left( l - \nu - \frac{1}{2} \right)^2 = \sqrt{x - \nu + \frac{1}{2}} \left( \nu + \frac{3}{2} \right) - \left( x - \frac{1}{2} \right) \geq 0
\]

for \( x > 0 \) and \( \nu \geq -\frac{1}{2} \).

Applying Jensen’s inequality to the convex and concave functions given in Theorems 3 and 4, the following consequences are immediate.

Corollary 8 Let \( \theta \geq 2 \) and \( \nu > -2 \). The inequalities

\[
0 < W_\nu(x) + W_\nu(y) - W_\nu\left(\frac{x + y}{2}\right) < \frac{1}{8} (x - y)^2,
\]

\[
W_\nu\left(\frac{x + y}{2}\right) < \frac{W_\nu(x) + W_\nu(y)}{2} < W_\nu\left(\frac{x^\theta + y^\theta}{2}^{1/\theta}\right)
\]

hold for \( x, y > 0 \) with \( x \neq y \) and \( \nu \geq -\frac{1}{2} \), where the right hand side inequalities hold for \( \nu > -2 \).

5 Applications

5.1 An answer to Gaunt’s open problem

The variance-gamma (VG) distribution introduced into the financial literature in [56], also known as the generalized Laplace or Bessel function distribution, is a continuous statistical distribution defined and supported on the set of real numbers. For variance-gamma approximation via Stein’s method [57] (see also [58]), it is crucial to obtain the uniform bounds for four expressions of the type listed in [59, Eq. (1.1), (1.2)]. Gaunt has obtained uniform bounds for them, where the bound for the third follows from [59, Eq. (2.30)], which in turn depends on the upper bound for the integral

\[
\int_0^x e^{\beta t^\nu} I_\nu(t) \, dt \text{ for } \nu > -\frac{1}{2}, \beta \in (-1, 0).
\]
For this reason, he proved Theorem 2.7 in [59] by using the inequality

\[ I_{v+1}(x) < (1 - a_v) I_v(x) + a_v I_{v+2}(x) \quad (5.1) \]

for \( x > 0 \), where \( a_v \) is the largest number in the interval \([0, 1]\). Gaunt claimed that “the inequality has not been studied in the literature” and left for readers as an open problem.

**Open Problem 1** ([59, Open Problem 2.11]) Let \( v > -1/2 \). Establish lower and upper bounds for \( a_v \) that improve on the trivial estimate \( 0 \leq a_v \leq 1 \), where \( a_v \) satisfies the inequality (5.1).

We now give an answer to the above open problem using Corollary 1.

**Proposition 1** Let \( v \geq -1/2 \) and \( 0 < a_v < 1 \). The inequality (5.1) holds for \( x > 0 \) if and only if \( 0 < a_v \leq (v + 1/2) / (2v + 2) \).

**Proof** The inequality (5.1) is equivalent to

\[ a_v < \frac{I_v(x) - I_{v+1}(x)}{I_v(x) - I_{v+2}(x)} := q_v(x) \]

for \( x > 0 \). Using the recurrence formula (1.5) gives

\[ q_v(x) = \frac{I_v(x) - I_{v+1}(x)}{2(v + 1)x^{-1}I_{v+1}(x)} = \frac{W_v(x) - x}{2(v + 1)}. \]

The necessary condition follows from

\[ 0 < a_v \leq \lim_{x \to \infty} q_v(x) = \frac{v + 1/2}{2(v + 1)}. \]

By Corollary 1 (i) we deduce that the function \( q_v(x) \) is strictly decreasing on \((0, \infty)\) with

\[ q_v(0) = 1 \text{ and } q_v(\infty) = \lim_{x \to \infty} q_v(x) = \frac{v + 1/2}{2(v + 1)}. \]

If \( 0 < a_v \leq (v + 1/2) / (2v + 2) \), then by Corollary 1 (i), the function \( q_v(x) \) is strictly decreasing on \((0, \infty)\), and therefore,

\[ a_v \leq \frac{v + 1/2}{2(v + 1)} = \lim_{x \to \infty} q_v(x) < q_v(x) \]

for \( x > 0 \), thereby completing the proof. \( \square \)

**Remark 11** Taking \( a_v = (v + 1/2) / (2v + 2) \), Gaunt’s Theorem 2.7 in [59] can be improved; correspondingly, Theorem 3.2 (iv) [59] can also be slightly improved.
**Remark 12** Gaunt [59, Open Problem 2.12] also proposed an analogous problem for the modified Bessel function $K_v(x)$: Let $v > -1/2$. Establish lower and upper bounds for $b_v$ that improve on the trivial estimate $0 \leq b_v \leq 1$ such that, for all $x > 0$, the inequality

$$K_{v+1}(x) < b_v K_v(x) + (1 - b_v) K_{v+2}(x)$$

holds. Using Proposition 2 in [43] we easily get the best constant $b_v = (2v + 3) / (4v + 4)$.

### 5.2 An answer to Baricz and Neuman’s conjecture

Nàsell [42] proved that the function $x \mapsto x^{-v} e^{-x} I_v (x)$ is completely monotonic on $(0, \infty)$ for $v \geq -1/2$. This implies that the function $x \mapsto I_v (x) = 2^v \Gamma (v + 1) x^{-v} I_v (x)$ is log-convex on $(0, \infty)$ for $v \geq -1/2$, since a completely monotonic function is log-convex. Other proofs of this result can be found in [14, 46, 60]. In 2007, Baricz and Neuman [14] made the following conjecture.

**Conjecture 2** The function $x \mapsto I_v (x) = 2^v \Gamma (v + 1) x^{-v} I_v (x)$ is strictly log-convex on $\mathbb{R}$ for all $v > -1$ while the function $x \mapsto I_v (x)$ is strictly log-concave on $(0, \infty)$ for $v > 0$.

We now give an answer to this conjecture.

**Proposition 2** Let $v > -1$. The following statements are true.

(i) The function $x \mapsto I_v (x) = 2^v \Gamma (v + 1) x^{-v} I_v (x)$ is log-convex on $(0, \infty)$ if and only if $v \geq -1/2$, and it is log-convex then log-concave on $(0, \infty)$ if $v \in (-1, -1/2)$.

(ii) When $v > 0$, the function $x \mapsto I_v (x)$ is log-concave then log-convex on $(0, \infty)$.

**Proof** Let

$$h_{p,v} (x) = x W'_v (x) - W_v (x) + p.$$

Differentiation yields

$$x^2 [\ln I_v (x)]'' = x W'_{v-1} (x) - W_{v-1} (x) + 2v = h_{2v,v-1} (x),$$

$$x^2 [\ln I_v (x)]'' = x W'_{v-1} (x) - W_{v-1} (x) + v = h_{v,v-1} (x).$$

It is easy to check that

$$h_{2v,v-1} (0) = 0 \text{ and } h_{2v,v-1} (\infty) = \lim_{x \to \infty} h_{2v,v-1} (x) = v + \frac{1}{2};$$

$$h_{v,v-1} (0) = -v \text{ and } h_{v,v-1} (\infty) = \lim_{x \to \infty} h_{v,v-1} (x) = \frac{1}{2}.$$

(i) Clearly, the necessary condition for $[\ln I_v (x)]'' > 0$ is: $v \geq -1/2$. Conversely, if $v \geq -1/2$, i.e. $v - 1 \geq -3/2$, then by the left hand side inequalities of (3.1) and (3.2) $h_{2v,v-1} (x) \geq -2v + 2v = 0$ for $x > 0$, which proves the sufficiency.
When \( \nu \in (-1, -1/2) \), by Theorem 1 (ii) it is seen that the function \( h_{2\nu, \nu-1}(x) \) is first increasing then decreasing and finally increasing on \((0, \infty)\). This together with \( h_{2\nu, \nu-1}(0) = 0 \) and \( h_{2\nu, \nu-1}(\infty) = \nu + 1/2 < 0 \) indicates that the sign of \( h_{2\nu, \nu-1}(x) \) changes from positivity to negativity, and hence, the function \( x \mapsto \ln I_\nu(x) \) is convex then concave on \((0, \infty)\).

(ii) Similarly, we can prove the function \( x \mapsto \ln I_\nu(x) \) is convex then concave on \((0, \infty)\). \(\square\)

**Remark 13** Using the same method as in the above proof, we can show the necessary and sufficient conditions for which the function \( \phi_{p, \nu}(x) = x^p I_\nu(x) \) is log-convex or log-concave on \((0, \infty)\) as follows.

(i) If \( \nu \geq -1/2 \), then the function \( \phi_{p, \nu}(x) \) is log-convex on \((0, \infty)\) if and only if \( p \leq -\nu \).

(ii) If \( \nu \geq 1/2 \), then the function \( \phi_{p, \nu}(x) \) is log-concave on \((0, \infty)\) if and only if \( p \geq 1/2 \).

(iii) If \( \nu \geq -1/2 \) and \( -\nu < p < 1/2 \), then the function \( \phi_{p, \nu}(x) \) is log-concave then log-convex on \((0, \infty)\).

### 6 Conclusions

In this paper, using the monotonicity rules for a ratio of power series and an elementary auxiliary function we proved several convexity results:

- For \( \nu > -2 \), the functions \( x \mapsto W_\nu(x) \) and \( x \mapsto [W_\nu(x) - (\nu + 1/2)]/x \) are both strictly convex on \((0, \infty)\) if and only if \( \nu \geq -1/2 \).
- For \( \theta \geq 2 \) and \( \nu > -2 \), the functions \( x \mapsto W_\nu(x^{1/\theta}) \) and \( x \mapsto W_\nu(x) - x^2/(2\nu+4) \) are both strictly concave on \((0, \infty)\), while \( x \mapsto [W_\nu(x) - 2(\nu+1)]/x \) is strictly concave on \((0, \infty)\) if and only if \( \nu \geq -3/2 \).

As direct consequences of the above main results, we describe the monotonic pattern of the difference \( W_\nu(x) - x \) on \((0, \infty)\) for \( \nu \) in the intervals \([-1/2, \infty), [-3/2, -1/2)\) and \((-2, -3/2)\), and establish several new inequalities for the ratio \( W_\nu(x) \). Finally, as applications, we give answers to an open problem and a conjecture.

Finally, let us recall that an infinitely differentiable function \( f : I \to [0, \infty) \) is called a Bernstein function on an interval \( I \) if \( f' \) is completely monotonic on \( I \) (see [61, Definition 3.1]). Now, inspired the facts that \( W_\nu(x) \), \( W'_\nu(x) > 0 \) and \( [W_\nu(x^{1/\theta})]' \) \( < 0 \) for \( x > 0 \) and \( \nu > -1 \) with \( \theta \geq 2 \), we propose a conjecture as follows.

**Conjecture 3** Let \( \tau \in (0, 1/2] \) and \( \nu > -1 \). The function \( x \mapsto W_\nu(x^\tau) \) is a Bernstein function on \((0, \infty)\).

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