The Hausdorff dimension of the range of the Lévy multistable processes

R. Le Guével

Université de Rennes 2 - Haute Bretagne, Equipe de Statistique Irmar, UMR CNRS 6625
Place du Recteur Henri Le Moal, CS 24307, 35043 RENNES Cedex, France
ronan.leguevel@univ-rennes2.fr

Abstract

We compute the Hausdorff dimension of the image \(X(E)\) of a non random Borel set \(E \subset [0,1]\), where \(X\) is a Lévy multistable process in \(\mathbb{R}\). This extends the case where \(X\) is a classical stable Lévy process by letting the stability exponent \(\alpha\) be a smooth function, which leads to non-homogeneous processes because their increments are not stationary and not necessarily independent. Contrary to the situation where the stability parameter is a constant, the dimension depends on the version of the multistable Lévy motion when the process has an infinite first moment.

1 Introduction

For \((X_t)_t\) a stochastic process, we define the range of \(X\) on a non random Borel set \(E\) as the set \(X(E) = \{x : x = X_t \text{ for some } t \in E\}\). We already know that for a typical Lévy process \(X\), \(X(E)\) is a random fractal set. Many authors have been interested in producing the dimension properties of the sets \(X(E)\). The computation of \(\dim X(E)\) have been performed under various assumptions on \(X\) and \(E\), mainly if \(X\) is a stable process, a subordinator or a general Lévy process. For instance, see MacKean [19], Blumenthal and Getoor [4], Hawkes [10], Pruitt and Taylor [22], Hendricks [11] or Kahane [12] for stable processes, Millar [20], Pruitt [21] or Blumenthal and Getoor [5] for processes with stationary independent increments. More recently, some results on operator-stable sample paths or additive Lévy processes have been obtained for example in Becker-Kern, Meerschaert and Scheffler [3], Khoshnevisan, Xiao and Zhong [14] or Khoshnevisan and Xiao [13]. Our aim in this article is to present similar results with the assumption that \(X\) belongs to the class of multistable Lévy processes, a natural extension of the stable processes.

The multistable processes have been introduced by Falconer and Lévy-Véhel in 2009 [8]. Their distributions, their Hölderian regularity or their multifractal properties have been studied for instance in [1, 15, 10, 17, 9]. They provide useful models for all applications that deal with discontinuous processes where the intensity of jumps is non-constant. Most multistable processes are non-homogeneous in the sense that their increments are neither independent nor stationary. In this article, we consider only multistable Lévy motions which are the simplest examples of multistable processes.

The paper is organised as follows. Section 2 contains the notations. In Section 3 we present the main results on the computation of the Hausdorff dimension of the range. Section 4 is dedicated to statement of useful technical lemmas on multistable processes. All the proofs are gathered in Section 5.
2 Notations

We first summarise the basic notions of Hausdorff measures on the real line (see Falconer [7] for more details). For a subset $E$ of $[0,1]$, the diameter of $E$ is defined as $|E| = \sup\{|x-y| : x \in E, y \in E\}$. Let $\beta$ be a non-negative number. For any $\delta > 0$ we define

$$\mathcal{H}_\delta^\beta(E) = \inf \left\{ \sum_{i=1}^{+\infty} |U_i|^\beta : \{U_i\} \text{ is a } \delta - \text{cover of } E \right\}.$$ 

We call $\mathcal{H}^\beta(E) = \lim_{\delta \to 0} \mathcal{H}_\delta^\beta(E)$ the $\beta$-dimensional Hausdorff measure of $E$, and the Hausdorff dimension of $E$ is defined as

$$\dim(E) = \inf \left\{ \beta : \mathcal{H}^\beta(E) = 0 \right\} = \sup \left\{ \beta : \mathcal{H}^\beta(E) = \infty \right\}.$$ 

The convex hull of $E$ is denoted by $c(E)$, that is $c(E) = \{tx + (1-t)y : t \in [0,1], x \in E, y \in E\}$. $\hat{E}$ will be the interior of $E$, and $\mathcal{P}$ will represent the set of partitions of $[0,1]$. For $A \in \mathcal{P}$, we shall write $A = A^n$ if the number of intervals composing $A$ is $n$, and if $A^n = (A^n_i)_{i=1,\ldots,n}$ is such that $[0,1] = \bigcup_{i=1}^n A^n_i$ and $A^n_i \cap A^n_j = \emptyset$ for $i \neq j$, the mesh of $A^n$ is defined as $|A^n| = \max_{i=1}^n |A^n_i|$.

Without loss of generality, $A^n_1$ is assumed to be the first set, that is for all $n \geq 1, 0 \in A^n_1$.

We then introduce the multistable Lévy processes using their Ferguson-Klass-LePage representation. We need for that the following objects:

- $\alpha : [0,1] \to [\alpha_*, \alpha^*] \subset (0,2)$ a $C^1$ function,
- $(\Gamma_i)_{i \geq 1}$ a sequence of arrival times of a Poisson process with unit arrival time,
- $(V_i)_{i \geq 1}$ a sequence of i.i.d. random variables with uniform distribution on $[0,1]$,
- $(\gamma_i)_{i \geq 1}$ a sequence of i.i.d. random variables with distribution $P(\gamma_i = 1) = P(\gamma_i = -1) = 1/2$.

The three sequences $(\Gamma_i)_{i \geq 1}$, $(V_i)_{i \geq 1}$ and $(\gamma_i)_{i \geq 1}$ are assumed to be independent. The Lévy multistable motion defined in [15] is the process

$$X(t) = \sum_{i=1}^{\infty} \gamma_i \Gamma_i^{-1/\alpha(t)} 1_{[0,t]}(V_i)$$

and the Lévy multistable process resulting from the construction of multistable measures in Falconer-Liu [9] is

$$Z(t) = \sum_{i=1}^{\infty} \gamma_i C_i \Gamma_i^{-1/\alpha(V_i)} 1_{[0,t]}(V_i),$$

where $C_t = \left( \int_0^\infty x^{-\alpha(t)} \sin x \,dx \right)^{-1/(\alpha(t))}$. Both of them define the standard symmetric $\alpha$-stable Lévy motion when the function $\alpha$ is constant. We already know that the two processes are linked by the following formula (Falconer, Theorem 8):

$$X(t) = Y(t) + Z(t),$$

where $Y(t) = \int_0^t \sum_{i=1}^{+\infty} \gamma_i K_i(u) 1_{[0,u]}(V_i) \,du$ and $K_i(u) = \frac{d\left(C_i \Gamma_i^{-1/\alpha(s)}\right)}{ds}(u)$. Our results involve the following quantities: $\alpha_*(E) = \inf_{t \in E} (t, \alpha^*(E) = \sup_{t \in E} (t, \dim(E) = \max(1, \alpha_*(E)) \dim(E)$ and $d^*(E) = \max(1, \alpha^*(E)) \dim(E)$.  

2
Finally, in all the paper, for some parameter $\beta$, $K_{\beta}$ will mean a finite positive constant which depends only on $\beta$, and we will use the fact that there exists $K > 0$ such that for all $u \in U$ and all $i \geq 1$,
$$|K_i(u)| \leq K(1 + |\log \Gamma_i|)(\frac{1}{\Gamma_i^{1/\alpha^*(U)}} + \frac{1}{\Gamma_i^{1/\alpha^*(U)}}). \tag{2}$$

3 Main theorems

Theorem 1. Let $E$ be a subset of $[0, 1]$. Almost surely,
$$\dim Z(E) \geq \min(1, \alpha_* (c(E)) \dim(E)).$$
Suppose also that $\inf s > 0$, $\sup_{(s,t) \in E^2} \frac{|t-s|}{|\alpha(t) - \alpha(s)|} < +\infty$ and $\alpha^*(c(E)) - \alpha_*(c(E)) \leq \frac{\alpha^2}{2}$. Almost surely,
$$\dim X(E) \geq \min(1, d_*(E)).$$

Theorem 2. Let $E$ be a subset of $[0, 1]$. Almost surely,
$$\dim Z(E) \leq \alpha^*(E) \dim(E)$$
and
$$\dim X(E) \leq d^*(E).$$

Theorem 3. Let $(A^n)_{n \in \mathbb{N}}$ be a sequence of partitions of $\mathcal{P}$ such that $\lim_{n \to +\infty} |A^n| = 0$, $E$ a subset of $[0, 1]$. Almost surely,
$$\dim Z(E) = \min(1, \limsup_{n \to +\infty} \max_{i=1}^n \alpha^*(E \cap A^n_i) \dim(E \cap A^n_i)).$$

Theorem 4. Let $(A^n)_{n \in \mathbb{N}}$ be a sequence of partitions of $\mathcal{P}$ such that $\lim_{n \to +\infty} |A^n| = 0$, $E$ a subset of $[0, 1]$ such that $\inf s > 0$. Assume that $\exists n_0 \geq 1$ such that $\forall n \geq n_0$, $\forall i \in [1, n]$,
$$\sup_{(s,t) \in (E \cap A^n_i)^2} \frac{|t-s|}{|\alpha(t) - \alpha(s)|} < +\infty.$$ Almost surely,
$$\dim X(E) = \min(1, \limsup_{n \to +\infty} \max_{i=1}^n d^*(E \cap A^n_i)) = \min(1, \limsup_{n \to +\infty} \max_{i=1}^n d_*(E \cap A^n_i)).$$

4 Technical lemmas

Lemma 1. $\forall \beta \in (0, 1), \forall U \subset [0, 1], \exists K_{U,\beta} > 0$ such that $\forall (s,t) \in U^2$,
$$E[|X(t) - X(s)|^{-\beta}] \leq K_{U,\beta}|t - s|^{-\alpha_*(c(U)) - \beta}$$
and
$$E[|Z(t) - Z(s)|^{-\beta}] \leq K_{U,\beta}|t - s|^{-\alpha^*(c(U)) - \beta}.$$ If we assume also that $\inf s > 0$, $\sup_{(s,t) \in U^2} \frac{|t-s|}{|\alpha(t) - \alpha(s)|} < +\infty$ and $\alpha^*(c(U)) - \alpha_*(c(U)) \leq \frac{\alpha^2}{2}$, then $\exists K_{U,\beta} > 0$ such that $\forall (s,t) \in U^2$,
$$E[|X(t) - X(s)|^{-\beta}] \leq K_{U,\beta}|t - s|^{-\beta}.$$
Lemma 2. Let \((I_j)_j = ([a_j, b_j])_j\) be a collection of closed intervals of \([0, 1]\) and \(p \in (0, \inf \alpha_*(I_j))\). For all \(\varepsilon > 0\), \(\exists K_{p, \varepsilon} > 0\) such that \(\forall j\),

\[
E[\sup_{(s, t) \in I_j^2} |Z(t) - Z(s)|^p] \leq K_{p, \varepsilon} |I_j|^{\max\{1, \sup_j \alpha^*(I_j) + \varepsilon\}}
\]

and \(E[\sup_{(s, t) \in I_j^2} |X(t) - X(s)|^p] \leq K_{p, \varepsilon} |I_j|^{\max\{1, \sup_j \alpha^*(I_j) + \varepsilon\}}\).

Lemma 3. Let \(\{A_n\}_{n \in \mathbb{N}} \in \mathcal{P}^\mathbb{N}\) be a sequence of partitions of \([0, 1]\) such that \(\lim_{n \to +\infty} |A_n| = 0\). Then, for all subsets \(E\) of \([0, 1]\),

\[
\limsup_{n \to +\infty} \max_{i=1}^n d^*(E \cap A^n_i) = \limsup_{n \to +\infty} \max_{i=1}^n d_*(E \cap A^n_i)
\]

and

\[
\limsup_{n \to +\infty} \max_{i=1}^n \alpha^*(E \cap A^n_i) \dim(E \cap A^n_i) = \limsup_{n \to +\infty} \max_{i=1}^n \alpha_*(E \cap A^n_i) \dim(E \cap A^n_i)
\]

Furthermore, all these equalities also occur with \(\liminf_{n \to +\infty}\) instead of \(\limsup_{n \to +\infty}\).

5 Proofs

Proof of theorem 1

Let \(\beta < \min\{1, \alpha_*(c(E)) \dim(E)\}\). Since \(\frac{\beta}{\alpha_*(c(E))} < \dim(E)\), \(\mathcal{H}_{\frac{\beta}{\alpha_*(c(E))}}^\alpha (E) = +\infty\). According to Davies theorem \([6]\), \(\exists F \subset E, F\) closed set such that \(\mathcal{H}_{\frac{\beta}{\alpha_*(c(E))}}^\alpha (F) > 0\). Then \(C_{\frac{\beta}{\alpha_*(c(E))}} (F) > 0\) by Frostman’s theorem. Let \(p_m\) a probability measure concentrated on \(F\) s.t.

\[
\int_F \int_F |x - y|^{-\frac{\beta}{\alpha_*(c(E))}} p_m(dx)p_m(dy) < +\infty.
\]

With Lemma 1

\[
E\left[\int_F \int_F |Z(x) - Z(y)|^{-\beta} p_m(dx)p_m(dy)\right] \leq K_{\beta,F} \int_F \int_F |x - y|^{-\frac{\beta}{\alpha_*(c(E))}} p_m(dx)p_m(dy)
\]

\[
\leq K_{\beta,F} \int_F \int_F |x - y|^{-\frac{\beta}{\alpha_*(c(E))}} p_m(dx)p_m(dy)
\]

\[
< +\infty.
\]

So \(\mathbb{P}\left(\mathcal{H}^\beta(Z(F)) > 0\right) = 1\), \(\mathbb{P}\left(\mathcal{H}^\beta(Z(E)) > 0\right) = 1\), and \(\dim(Z(E)) \geq \beta\).

Assume now that \(\inf s > 0\), \(\sup_{(s, t) \in E^2} \frac{|s - t|}{\alpha(t) - \alpha(s)} < +\infty\) and \(\alpha^*(c(E)) - \alpha_*(c(E)) \leq \frac{\alpha_*(c(E))}{2}\). The proof for the process \(X\) is similar to the previous one. Consider \(\beta < \min\{1, d_*(E)\}\) and \(\gamma_*(E) = \max\{1, \alpha_*(E)\}\). We obtain \(\dim(X(E)) \geq \beta\) replacing \(\alpha_*(c(E))\) by \(d_*(E)\) in the previous calculus and \(\alpha_*(c(E))\) by \(\gamma_*(E)\).

Proof of theorem 2
For a partition \((A_k)_{k=1}^{N}\), \(\dim X(E) = \max_{k=1}^{N} \dim X(E \cap A_k)\) therefore it is enough to show that for all \(k\),
\[
\dim X(E \cap A_k) \leq \max(1, \alpha^*(E \cap A_k)) \dim(E \cap A_k) \leq \max(1, \alpha^*(E)) \dim E
\]
and
\[
\dim Z(E \cap A_k) \leq \alpha^*(E \cap A_k) \dim(E \cap A_k) \leq \alpha^*(E) \dim E.
\]
Thus we may suppose that \(|\alpha^*(E) - \alpha_*(E)| \leq \varepsilon\) for \(\varepsilon > 0\) as small as we want.

Suppose first that \(\dim(E) < 1\).

Let \(\beta \in (\dim(E), 1)\) and \(n_0 \in \mathbb{N}\). For each \(n \geq n_0\), let \(\{I_{in}, i \geq 1\}\) be a cover of \(E\) by closed intervals such that \(\lim_{n \to +\infty} \sum_{i=1}^{+\infty} |I_{in}|^{\beta} = 0\). This can be done since \(H^\beta(E) = 0\). Suppose also that \(\varepsilon\) is small enough to have \(\beta < \frac{\inf_{i,n \geq n_0} \alpha_*(I_{in})}{\inf_{i,n \geq n_0} \alpha_*(I_{in}) + 2\varepsilon} < 1\), and that \(\sup \alpha^*(I_{in}) < \inf_{i,n \geq n_0} \alpha_*(I_{in}) + \varepsilon\).

We shall denote \(c = \inf_{i,n \geq n_0} \alpha_*(I_{in})\) and \(d = \sup \alpha^*(I_{in})\). Notice that for all \(i,n\), \(\beta(d + \varepsilon) < \beta(\varepsilon + 2\varepsilon) < c\).

Now for each \(n \geq n_0\), \(\{X(I_{in}), i \geq 1\}\) is a cover of \(X(E)\), and \(\{Z(I_{in}), i \geq 1\}\) a cover of \(Z(E)\). We consider two cases to finish the proof when \(\dim(E) < 1\).

(i): Case \(\alpha^*(E) \geq 1\).

We apply Lemma 2 to obtain
\[
E \left[ \sum_{i=1}^{+\infty} |X(I_{in})|^{\beta(d+\varepsilon)} \right] \leq K_{d,\beta,\varepsilon} \sum_{i=1}^{+\infty} |I_{in}|^{\beta}
\]
and
\[
E \left[ \sum_{i=1}^{+\infty} |Z(I_{in})|^{\beta(d+\varepsilon)} \right] \leq K_{d,\beta,\varepsilon} \sum_{i=1}^{+\infty} |I_{in}|^{\beta}.
\]

Then for a subsequence of \(n\)’s approaching \(\infty\), almost surely, \(\lim_{n \to +\infty} \sum_{i=1}^{+\infty} |X(I_{in})|^{\beta(d+\varepsilon)} = 0\), and \(\dim X(E) \leq \beta(\sup \alpha^*(I_{in}) + \varepsilon)\). Letting \(\varepsilon\) tend to 0, then letting \(n_0\) tend to infinity one finally obtains \(\dim X(E) \leq \beta \alpha^*(E)\). Since \(\beta\) was arbitrary, \(\dim X(E) \leq \alpha^*(E) \dim(E)\). Equation (4) leads also to \(\dim Z(E) \leq \alpha^*(E) \dim(E)\) for the same reasons.

(ii): Case \(\alpha^*(E) < 1\).

Suppose that \(\forall i, \forall n \geq n_0, \alpha^*(I_{in}) + \varepsilon < 1\). With equations (1) and (2),
\[
|X(I_{in})| = \sup_{(s,t)\in I_{in}^2} |X(t) - X(s)| \leq \sup_{(s,t)\in I_{in}^2} \int_s^t \sum_{j=1}^{+\infty} K(1 + |\log \Gamma_j|)(\frac{1}{\Gamma_j^{1/c}} + \frac{1}{\Gamma_j^{1/d}})ds + |Z(I_{in})|,
\]
so
\[
\sum_{i=1}^{+\infty} |X(I_{in})|^\beta \leq \left( \sum_{j=1}^{+\infty} K(1 + |\log \Gamma_j|)(\frac{1}{\Gamma_j^{1/c}} + \frac{1}{\Gamma_j^{1/d}}) \right)^\beta \sum_{i=1}^{+\infty} |I_{in}|^\beta + \sum_{i=1}^{+\infty} |Z(I_{in})|^\beta.
\]

Since \(\sum_{j=1}^{+\infty} K(1 + |\log \Gamma_j|)(\frac{1}{\Gamma_j^{1/c}} + \frac{1}{\Gamma_j^{1/d}}) < +\infty\) and \(\lim_{n \to +\infty} \sum_{i=1}^{+\infty} |I_{in}|^\beta = 0\), almost surely,
\[
\lim_{n \to +\infty} \left( \sum_{j=1}^{+\infty} K(1 + |\log \Gamma_j|)(\frac{1}{\Gamma_j^{1/c}} + \frac{1}{\Gamma_j^{1/d}}) \right)^\beta \sum_{i=1}^{+\infty} |I_{in}|^\beta = 0.
\]
Let us show that \( \lim_{n \to +\infty} \sum_{i=1}^{+\infty} |Z(I_{in})|^\beta = 0 \) where the convergence is in probability.

\[
|Z(t) - Z(s)| \leq K \sum_{j=1}^{+\infty} \left( \frac{1}{\Gamma(1/c)} + \frac{1}{\Gamma(1/d)} \right) 1_{[s,t]}(V_j).
\]

Let \( D_\alpha(t) = \sum_{j=1}^{+\infty} \frac{1}{\Gamma_j} 1_{[0,t]}(V_j) \) so that

\[
|Z(I_{in})|^\beta \leq K_\beta |D_c(I_{in})|^\beta + K_\beta |D_d(I_{in})|^\beta.
\]

(6) 

\( D_d \) is a stable-subordinator so \( |D_d(I_{in})|^\beta \) is distributed as \( |I_{in}|^{\beta/d} |D_d(1)|^\beta \). Since \( \beta/d > \beta \), \( \sum_{i=1}^{+\infty} |D_d(I_{in})|^\beta \) tends to 0 in probability. For the same reasons, \( \lim_{n \to +\infty} \sum_{i=1}^{+\infty} |D_c(I_{in})|^\beta = 0 \) in probability, which entails with (5) that \( \lim_{n \to +\infty} \sum_{i=1}^{+\infty} |X(I_{in})|^\beta \) is a stable-subordinator so \( |X(I_{in})|^\beta = 0 \). Then for a subsequence of \( n \)'s approaching \( \infty \), almost surely, \( \lim_{n \to +\infty} \sum_{i=1}^{+\infty} |X(I_{in})|^\beta = 0 \), and \( \dim X(E) \leq \beta \). Since \( \beta \) was arbitrary, \( \dim X(E) \leq \dim(E) \).

Replacing \( \beta \) by \( \beta d \) in the equation (6), we obtain \( \lim_{n \to +\infty} \sum_{i=1}^{+\infty} |Z(I_{in})|^{\beta d} = 0 \), and \( \dim Z(E) \leq \beta d \). Letting \( n_0 \) tend to infinity one finally obtains \( \dim Z(E) \leq \beta \alpha^*(E) \).

**Suppose now that \( \dim(E) = 1 \).**

The result is obvious for the process \( X \) and for the process \( Z \) if \( \alpha^*(E) \geq 1 \) so we consider only the case \( \alpha^*(E) < 1 \). As previously, the result is a consequence of the equation (6). Let \( \beta > 1 \), \( n_0 \in \mathbb{N} \), \( n \geq n_0 \) and \( \{I_{in}, i \geq 1\} \) be a cover of \( E \) by closed intervals such that \( \lim_{n \to +\infty} \sum_{i=1}^{+\infty} |I_{in}|^\beta = 0 \).

Suppose also that \( d = \sup_{i,n \geq n_0} \alpha^*(I_{in}) < 1 \). Equation (6) and its consequences are still available:

\[
|Z(I_{in})|^{\beta d} \leq K_{\beta,d} |D_c(I_{in})|^{\beta d} + K_{\beta,d} |D_d(I_{in})|^{\beta d}
\]

leads to \( \lim_{n \to +\infty} \sum_{i=1}^{+\infty} |Z(I_{in})|^{\beta d} = 0 \), \( \dim Z(E) \leq \beta d \), and \( \dim Z(E) \leq \alpha^*(E) \).

**Proof of Theorem 3 and Theorem 4**

Let us prove Theorem 4 first. Suppose that \( 0 \in A^0_1 \) for all \( n \geq 1 \). Since \( \inf s > 0 \), for \( n \) large enough, \( E \cap A^0_1 = \emptyset \). We use Theorem 2 to obtain

\[
\dim X(E) = \max_{i=1}^{n} \dim X(E \cap A^0_i) \leq \max_{i=1}^{n} d^*(E \cap A^0_i)
\]

and

\[
\dim X(E) \leq \liminf_{n \to +\infty} \max_{i=1}^{n} d^*(E \cap A^0_i) \leq \limsup_{n \to +\infty} \max_{i=1}^{n} d^*(E \cap A^0_i).
\]

(7) 

Let us show that \( \dim X(E) \geq \min(1, \limsup_{n \to +\infty} \max_{i=1}^{n} d^*(E \cap A^0_i)) \). Theorem 1 gives

\[
\dim X(E) = \max_{i=2}^{n} \dim X(E \cap A^0_i) \geq \max_{i=2}^{n} \min(1, d_*(E \cap A^0_i)).
\]

(8) 

Then we consider three cases.

(i): Case \( \limsup_{n \to +\infty} \max_{i=1}^{n} d^*(E \cap A^0_i) < 1 \).
With the two inequalities (7) and (8), 
\[ \max_{i=1}^{n} \min(1, d_*(E \cap A^n_i)) < 1 \]
so for all \( n \geq 1 \) and all \( i = 1, \ldots, n \), \( d_*(E \cap A^n_i) < 1 \) and
\[ \max_{i=1}^{n} \min(1, d_*(E \cap A^n_i)) = \max_{i=1}^{n} d_*(E \cap A^n_i), \]
i.e.
\[ \dim X(E) \geq \max_{i=1}^{n} d_*(E \cap A^n_i). \]

Finally, \( \dim X(E) \geq \limsup_{n \to +\infty} \max_{i=1}^{n} d_*(E \cap A^n_i) \) and the result comes from Lemma 3

(ii): Case \( \limsup_{n \to +\infty} \max_{i=1}^{n} d_*(E \cap A^n_i) = 1. \)

If for all \( n \geq 1 \) and all \( i = 1, \ldots, n \), \( d_*(E \cap A^n_i) < 1 \), we obtain as previously
\[ \dim X(E) \geq \limsup_{n \to +\infty} \max_{i=1}^{n} d_*(E \cap A^n_i) = 1. \]

Otherwise, there exists \( n_0 \in \mathbb{N} \) and \( i_0 \in [1, n_0] \) such that \( d_*(E \cap A^{n_0}_{i_0}) \geq 1 \). Then
\[ \dim X(E) \geq \dim X(E \cap A^{n_0}_{i_0}) \]
\[ \geq \min(1, d_*(E \cap A^{n_0}_{i_0})) \]
\[ = 1. \]

(iii): Case \( \limsup_{n \to +\infty} \max_{i=1}^{n} d_*(E \cap A^n_i) > 1. \)

With Lemma 3 \( \limsup_{n \to +\infty} \max_{i=1}^{n} d_*(E \cap A^n_i) > 1 \) so there exists \( n_0 \in \mathbb{N} \) and \( i_0 \in [1, n_0] \) such that
\( d_*(E \cap A^{n_0}_{i_0}) \geq 1 \). As previously stated, \( \dim X(E) \geq 1. \)

In order to get Theorem 3 replace \( X \) by \( Z \) and \( d(E \cap A^n_i) \) by \( \alpha(E \cap A^n_i) \dim(E \cap A^n_i) \) in the proof of Theorem 4.

Remarks:

- Notice that if \( \dim X(E) = \limsup_{n \to +\infty} \max_{i=1}^{n} d_*(E \cap A^n_i) \) \(< 1 \), then \( \limsup_{n \to +\infty} \max_{i=1}^{n} d_*(E \cap A^n_i) \) exists and is equal to \( \dim X(E) \): indeed the inequality (7) becomes
\[ \dim X(E) \leq \liminf_{n \to +\infty} \max_{i=1}^{n} d_*(E \cap A^n_i) \leq \limsup_{n \to +\infty} \max_{i=1}^{n} d_*(E \cap A^n_i) = \dim X(E). \]

Lemma 3 gives also in that case \( \dim X(E) = \lim_{n \to +\infty} \max_{i=1}^{n} d_*(E \cap A^n_i). \)

- If \( \alpha^*(E) \leq 1 \), almost surely, \( \dim(X(E)) = \dim(E) \), which is not true if \( \alpha \) is constant. Else, if \( \alpha^*(E) > 1 \), almost surely, \( \dim(X(E)) = \dim(Z(E)) \).

Proof of Lemma 1

By Proposition 6.1 of [15], the logarithm of the characteristic function of \( X(t) - X(s) \) satisfies for \( s \leq t \):
\[ \log \phi_{X(t) - X(s)}(\theta) = -2s \int_{0}^{\infty} \sin^2 \left( \frac{\theta}{2} \left[ \frac{C_t}{y^{1/\alpha(t)}} - \frac{C_s}{y^{1/\alpha(s)}} \right] \right) dy -(t-s)|\theta|^\alpha(t), \]
and by Proposition 2 of [18],
\[ \log \phi_{Z(t) - Z(s)}(\theta) = -\int_{s}^{t} |\theta|^\alpha(u) du. \]
Accordingly for $|\theta| \geq 1$ and $(s,t) \in U^2$,

$$
\phi_{X(t)-X(s)}(\theta) \leq e^{-|t-s|/\alpha^{(\nu)}(U)},
$$

(9)

and for all $\theta$,

$$
\phi_{X(t)-X(s)}(\theta) \leq e^{-|t-s|/\alpha^{(\nu)}(U)} \leq e^{-|t-s|/\alpha^{(c(U))}}
$$

(10)

We obtain then for $(s,t) \in U^2$, using the Parseval’s formula:

$$
|t-s|^{\beta/\alpha^{(c(U))}} E[|X(t) - X(s)|^{-\beta}] = \int_0^\infty P(|X(t) - X(s)| \leq |t-s|/x^{1/\beta}) dx
$$

$$
\leq 1 + \frac{1}{\pi} \int_1^\infty \frac{dx}{x^{1/\beta}} \phi_{X(t)-X(s)}(\xi) d\xi
$$

$$
= 1 + \frac{1}{\pi} \int_1^\infty \sin\left(\frac{\xi}{\sqrt{\pi}}\right) \phi_{X(t)-X(s)}\left(\frac{\theta}{|t-s|/\alpha^{(U)}}\right) d\theta dx
$$

$$
\leq 1 + \frac{1}{\pi} \int_1^\infty \frac{dx}{x^{1/\beta}} \int_R \phi_{X(t)-X(s)}\left(\frac{\theta}{|t-s|/\alpha^{(U)}}\right) d\theta
$$

$$
\leq 1 + \frac{1}{\pi} \left(2 + 2 \int_0^\infty e^{-|t-s|/\alpha^{(c(U))}} d\theta\right).
$$

Using the same inequalities and (10) instead of (9), we obtain also

$$
|t-s|^{\beta/\alpha^{(c(U))}} E[|Z(t) - Z(s)|^{-\beta}] \leq 1 + \frac{1}{\pi} \frac{\beta}{1-\beta} \left(2 + 2 \int_0^\infty e^{-|t-s|/\alpha^{(c(U))}} d\theta\right).
$$

Assume now that $\inf_{s \in U} s > 0$, $\sup_{(s,t) \in U^2} |t-s|/\alpha(t) - \alpha(s)| < +\infty$ and $\alpha^+(c(U)) - \alpha^+(c(U)) \leq \frac{\alpha^2}{2}$. Notice that $C_t = h \circ \alpha(t)$ where $h(v) = \left(\int_0^\infty x^{-v} \sin x \, dx\right)^{-1/v}$ is a continuously differentiable function on $[\alpha^*, \alpha^*]$. Property 1.2.15 of [23] gives an explicit formula of $h$. Then there exists $\omega_y \in [\alpha(t), \alpha(s)]$ (or $[\alpha(s), \alpha(t)]$) such that

$$
\frac{C_t}{y^{1/\alpha(t)}} - \frac{C_s}{y^{1/\alpha(s)}} = (\alpha(t) - \alpha(s)) \left[ h'(\omega_y) + h(\omega_y) \frac{\log(y)}{\omega_y^2} \right].
$$

Now the previous calculus gives

$$
|t-s|^\beta E[|X(t) - X(s)|^{-\beta}] \leq 1 + \frac{1}{\pi} \left(2 + 2 \int_0^\infty e^{-|t-s|/\alpha^{(c(U))}} d\theta\right)
$$

$$
\leq 1 + K_\beta \int R e^{-2\nu \int_0^\infty \sin^2\left(\frac{\theta}{2|x-t|}\right) \left[\omega_y^2 h'(\omega_y) + h(\omega_y) : \frac{\log(y)}{\omega_y^2 y^{1/\alpha}}\right] d\theta}
$$

8
where \( \nu = \inf_{s \in U} s > 0 \). Changing the variable \(|\theta|\) according to the formula \( \xi = \theta^{2(t) - \alpha(s)} \) leads to

\[
|t - s|^\beta E[|X(t) - X(s)|^{-\beta}] \leq 1 + K_\beta \sup_{(s,t) \in I^2} \frac{|t - s|}{|\alpha(t) - \alpha(s)|} \int e^{-2\nu \int_0^\infty \sin^2\left(\frac{e^{\alpha(t) + \alpha(s) \log(y)}}{\omega^\beta y^{1/\omega}}\right) dy} d\xi.
\]

Let \( \varepsilon \in (0, \frac{\alpha_2}{2}) \). Using the fact that for \(|x|\) small enough, \( \sin^2(x) \geq \frac{1}{2}x^2 \), and the inequality \( \inf_{x \in [\alpha, \alpha^*]} |h(x)| > 0 \), we may choose \( K_1 > 1 \) and \( K_2 > 1 \) such that for all \(|\xi| \geq 1\),

\[
y \geq K_1|\xi|^{\frac{\alpha^*(c(U))}{1 - \varepsilon}} \Rightarrow \sin^2\left(\frac{\omega^2 \xi (\alpha^*)(c(U))}{\omega^2 y^{1/\omega}}\right) \geq K_2|\xi|^2 y^{-\frac{2}{\alpha^*(c(U))}}.
\]

Now

\[
\int_0^\infty \sin^2\left(\frac{\omega^2 h'(\omega)y + h(\omega_y) \log(y)}{\omega^2 y^{1/\omega}}\right) dy \geq K_2|\xi|^2 \int_{y \geq K_1|\xi|^{1-\frac{\alpha^*(c(U))}{1 - \varepsilon}}} y^{-\frac{2}{\alpha^*(c(U))}} dy
\]

\[
\geq K|\xi|^{2 + \frac{2}{\alpha^*(c(U))} - \frac{2}{\alpha^*(c(U))}} \geq |\xi|^{-\frac{2}{\alpha^*(c(U))}} \text{ and } \int e^{-2\nu \int_0^\infty \sin^2\left(\frac{e^{\alpha(t) + \alpha(s) \log(y)}}{\omega^\beta y^{1/\omega}}\right) dy} d\xi \leq K_{\varepsilon,U} < +\infty.
\]

**Proof of Lemma 2**

Let \( p \in (0, \inf_{j} \alpha_i(I_j)) \), \( \varepsilon > 0 \) and \( n_0 \in \mathbb{N} \) large enough to have \( n_0 \alpha_* > 2 \). Let \( p' \in (\max_{j} \sup_{j} \alpha_i(I_j), 2) \), \( c = \inf_{j} \alpha_i(I_j) \) and \( d = \sup_{j} \alpha_i(I_j) \). Equation (11) can be written

\[
X(t) = \int_0^t W_1(u) du + \int_0^t W_2(u) du + Z(t)
\]

with \( W_1(u) = \sum_{i=1}^{n_0} \gamma_i K_i(u) 1_{[0,u]}(V_i) \). Then there exists a constant \( K > 0 \) such that

\[
\sup_{(s,t) \in I^2_j} |X(t) - X(s)|^p \leq K \left( \int_{a_j}^{b_j} |W_1(u)| du \right)^p + K \left( \int_{a_j}^{b_j} |W_2(u)| du \right)^p + K \sup_{(s,t) \in I^2_j} |Z(t) - Z(s)|^p.
\]

The end of the proof consists of showing the inequality for these three terms. For the first term, \( \int_{a_j}^{b_j} |W_1(u)| du \leq (b_j - a_j) \sum_{i=1}^{n_0} \sup_{u \in I_j} |K_i(u)| \) so inequality (2) gives:

\[
\left( \int_{a_j}^{b_j} |W_1(u)| du \right)^p \leq K_{n_0} |b_j - a_j|^p \sum_{i=1}^{n_0} \sup_{u \in I_j} |K_i(u)|^p
\]

\[
\leq K_{p} |b_j - a_j|^p \sum_{i=1}^{n_0} (1 + |\log \Gamma_i|)^p \left( \frac{1}{\Gamma_i^{1/c}} + \frac{1}{\Gamma_i^{1/d}} \right)^p,
\]

9
hence \( E \left( \int_{a_j}^{b_j} |W_2(u)|^p du \right)^p \leq K_{n_0,p} |I_j|^p \). For the second term, we obtain by Hölder and Jensen inequalities

\[
E \left( \int_{a_j}^{b_j} |W_2(u)|^p du \right)^p \leq E \left( \int_{a_j}^{b_j} |W_2(u)|^{p'} du' \right)^{\frac{p}{p'}} \leq |b_j - a_j|^p \left( \sup_{u \in I_j} E[|W_2(u)|^{p'}] \right)^{\frac{p}{p'}}.
\]

Since \( K_i(u) 1_{[0,u]}(V_i) \) is independent of \( \gamma_i \), we obtain with Theorem 2 of [2] that for all \( u \in I_j \),

\[
E[|W_2(u)|^{p'}] \leq \sum_{i > n_0} E[|K_i(u)|^{p'}].
\]

Then inequality (2) leads to \( \sup_{u \in I_j} \left( \sup_{u \in I_j} E[|W_2(u)|^{p'}] \right)^{\frac{p}{p'}} < +\infty \).

Let us consider the process \( Z \). Proposition 5 of [13] yields that \( Z \) is a semi-martingale and can be decomposed into \( A + M \) where \( M \) is a martingale and

\[
M(t) = \sum_{i=1, \Gamma_i \geq 1}^{\infty} \gamma_i C_{V_i} \Gamma_i^{-1/\alpha(V_i)} 1_{[0,t]}(V_i).
\]

Let \( N = \text{Card}\{i \geq 1 | \Gamma_i < 1\} \) and \( K_i = C_{V_i} \Gamma_i^{-1/\alpha(V_i)} \). We will use the following inequality: if \( V_i \in I_j, \ K_i \leq K \left( \frac{1}{\Gamma_i} + \frac{1}{\Gamma_i^{\alpha}} \right) \) for some constant \( K \) and the fact that \( N \) is distributed as a Poisson random variable with unit mean. For all \((s,t) \in [a_j, b_j]^2\),

\[
|A(t) - A(s)|^p = \sum_{n=0}^{+\infty} \left( \sum_{i=1}^{n} \gamma_i K_i 1_{[s,t]}(V_i) \right)^p 1_{N=n} \leq K \sum_{n=1}^{+\infty} n \left( \frac{1}{\Gamma_1} + \frac{1}{\Gamma_1^{\alpha}} \right)^p \left( \sum_{i=1}^{n} 1_{[a_j,b_j]}(V_i) \right) 1_{N=n}.
\]

Using the fact that \( V_i \) is independent of \( \Gamma_1 \) and \( N \),

\[
E \left[ \sup_{(s,t) \in I_j^2} |A(t) - A(s)|^p \right] \leq K(b_j - a_j) \sum_{n=1}^{+\infty} n^2 E \left[ \left( \frac{1}{\Gamma_1} + \frac{1}{\Gamma_1^{\alpha}} \right)^p 1_{N=n} \right].
\]

Since \( \sum_{n=1}^{+\infty} n^2 E \left[ \left( \frac{1}{\Gamma_1} + \frac{1}{\Gamma_1^{\alpha}} \right)^p 1_{N=n} \right] < +\infty \),

\[
E \left[ \sup_{(s,t) \in I_j^2} |A(t) - A(s)|^p \right] \leq K(b_j - a_j).
\]

The last step of the proof is to show the inequality for the martingale \( M \). Let \( p' = \max(1, d) + \varepsilon \). We apply the Hölder inequality to get

\[
E \left[ \sup_{(s,t) \in I_j^2} |M(t) - M(s)|^{p'} \right] \leq E \left[ \sup_{(s,t) \in I_j^2} |M(t) - M(s)|^p \right]^{\frac{p'}{p}}.
\]

By the Doob’s martingale inequality, there exists \( K_{p'} > 0 \) such that

\[
E \left[ \sup_{(s,t) \in I_j^2} |M(t) - M(s)|^{p'} \right] \leq K_{p'} \sup_{(s,t) \in I_j^2} E[|M(t) - M(s)|^p].
\]
Now for every \((s, t) \in I_j^2\), Theorem 2 of [2] leads again to
\[
E[|M(t) - M(s)|^{p^*}] \leq \sum_{i \geq 1} E[|K_i|^{p^*} \mathbf{1}_{[s, t]}(V_i)] \\
\leq K_{p'} |b_j - a_j| \sum_{i=1}^{+\infty} E\left[\left(\frac{1}{\Gamma_i} + \frac{1}{\Gamma_i^{1/d}}\right)^{p^*} \mathbf{1}_{\Gamma_i \geq 1}\right].
\]
Since \(\sum_{i=1}^{+\infty} E\left[\left(\frac{1}{\Gamma_i} + \frac{1}{\Gamma_i^{1/d}}\right)^{p^*} \mathbf{1}_{\Gamma_i \geq 1}\right] \leq +\infty\), \((\sup_{(s, t) \in I_j^2} E[|M(t) - M(s)|^{p^*}])^{\frac{p}{p'}} \leq K_{p, c} |b_j - a_j|^{\frac{p}{p'}}\) which is the result of the Lemma.

**Proof of Lemma 3**

Notice that
\[
|d_*(E \cap A^n_i^\alpha) - d^*(E \cap A^n_i^\alpha)| \leq |(\alpha_*(E \cap A^n_i^\alpha) - \alpha^*(E \cap A^n_i^\alpha)| \dim(E \cap A^n_i^\alpha).
\]
\(\alpha\) is a \(C^1\) function so there exists \(K > 0\) such that the following inequalities hold:
\[
|d_*(E \cap A^n_i^\alpha) - d^*(E \cap A^n_i^\alpha)| \leq K |A^n|, \quad (11)
|\alpha_*(E \cap A^n_i^\alpha) - \alpha^*(E \cap A^n_i^\alpha)| \dim(E \cap A^n_i^\alpha) \leq K |A^n|, \quad (12)
|\alpha_*(c(E \cap A^n_i^\alpha)) - \alpha^*(c(E \cap A^n_i^\alpha))| \dim(E \cap A^n_i^\alpha) \leq K |A^n|, \quad (13)
|\alpha^*(E \cap A^n_i) - \alpha^*(c(E \cap A^n_i))| \dim(E \cap A^n_i) \leq K |A^n|, \quad (14)
\]

Then, in order to prove equality (3), we use the inequality (11) to obtain \(d^*(E \cap A^n_i^\alpha) \leq K |A^n| + d_*(E \cap A^n_i^\alpha).\) This implies that
\[
\lim_{n \to +\infty} \max_{i=1}^{n} d^*(E \cap A^n_i^\alpha) \leq \lim_{n \to +\infty} \max_{i=1}^{n} d_*(E \cap A^n_i^\alpha)
\]
and
\[
\lim_{n \to +\infty} \min_{i=1}^{n} d^*(E \cap A^n_i^\alpha) \leq \lim_{n \to +\infty} \min_{i=1}^{n} d_*(E \cap A^n_i^\alpha).
\]
Equality (3) comes from the fact that \(d_* \leq d^*\). To obtain the second result of Lemma 3, we may replace \(d\) by \(\alpha\) using (12), (13) and (14) instead of (11) ●

**References**

[1] Ayache, A. (2013) Sharp estimates on the tail behavior of a multistable distribution, *Statistics and Probability Letters*, 83, (3), p. 680–688.

[2] Von Bahr, B. AND Essen, C.G. (1965) Inequalities for the rth Absolute Moment of a Sum of Random Variables, 1 <= r <= 2. *The Annals of Mathematical Statistics* 36, (1), 299–303.

[3] Becker-Kern, P., Meerschaert, M. M. and Scheffler, H.-P. (2003) Hausdorff dimension of operator-stable sample paths. *Monatsh. Math.* 14, 91-101.

[4] Blumenthal, R. M. AND Getoor, R. K. (1960) A dimension theorem for sample functions of stable processes. *Illinois J. Math.* 4 370-375.

[5] Blumenthal, R. M. AND Getoor, R. K. (1961) Sample functions of stochastic processes with stationary independent increments. *J. Math. Mech.*, 10 493-516.
[6] Davies, R.O. (1952) Subsets of finite measure in analytic sets, *Indag. Math.*, vol.14 (1952), pp. 488-489.

[7] Falconer, K. (1990) Fractal Geometry: Mathematical Foundations and Applications. *John Wiley*, New York.

[8] Falconer, K. and Lévy Véhel, J. (2009) Multifractional, multistable, and other processes with prescribed local form. *J. Theoret. Probab.*, 22 p. 375-401.

[9] Falconer, K. J. and Liu, L. (2012) Multistable Processes and Localisability. *Stochastic Models*, 28 (2012): 503-526.

[10] Hawkes, J. (1971) On the Hausdorff dimension of the intersection of the range of a stable process with a Borel set. *Z. Wahrsch. Verw. Gebiete*, 19, 90-102.

[11] Hendricks, W. J. (1972) Hausdorff dimension theorem in a processes with stable components – An interesting counterexample. *Ann. Math. Stat.*, 43, 690-694.

[12] Kahane, J.-P. (1985) Ensembles aléatoires et dimensions. *Recent Progress in Fourier Analysis*, El Escorial 1983, 65-121. North-Holland, Amsterdam.

[13] Khoshnevisan, D. and Xiao, Y. (2005) Lévy processes: capacity and Hausdorff dimension. *Ann. Probab.*, 33, 841-878, doi : 10.1214/009117904000001026.

[14] Khoshnevisan, D., Xiao, Y. and Zhong, Y. (2003) Measuring the range of an additive Lévy process. *Ann. Probab.*, 31, 1097-1141.

[15] Le Guével, R. and Lévy Véhel, J. (2012) A Ferguson - Klass - LePage series representation of multistable multifractional processes and related processes. *Bernoulli*, 18 (4) (2012): 1099-1127.

[16] Le Guével, R. and Lévy Véhel J. (2013) Incremental moments and Hölder exponents of multifractional multistable processes. *ESAIM PS*. DOI: [http://dx.doi.org/10.1051/ps/2011151](http://dx.doi.org/10.1051/ps/2011151)

[17] Le Guével, R. and Lévy Véhel J. (2014) Hausdorff, Large Deviation and Legendre Multifractal Spectra of Lévy Multistable Processes. *Submitted*. Arxiv: [http://arxiv.org/abs/1412.0599](http://arxiv.org/abs/1412.0599)

[18] Le Guével, R., Lévy-Véhel, J. and Lining, L. (2012) On two multistable extensions of stable Lévy motion and their semimartingale representation. *J. Theoret. Probab.*, doi: 10.1007/s10959-013-0528-6.

[19] McKean, H. P. Jr. (1955) Sample functions of Stable Processes. *Annals of Mathematics*, Second Series, Vol.61, No. 3 (May,1955), pp. 564-579.

[20] Millar, P. W. (1971) Path behavior of processes with stationary independent increments. *Z. Wahrsch. Verw. Gebiete*, 17, pp. 53-73.

[21] Pruitt, W. E. (1969) The Hausdorff dimension of the range of a process with stationary independent increments. *J. Math. Mech.*, 19, pp. 371-378.

[22] Pruitt, W. E. and Taylor, S. J. (1969) Sample path properties of processes with stable components. *Z. Wahrsch. Verw. Gebiete*, 12, 267-289.

[23] Samorodnitsky, G. and Taqqu, M.S. (1994) *Stable Non-Gaussian Random Processes*, Chapman and Hall.