Global existence of small amplitude solution to nonlinear system of wave and Klein-Gordon equations in four space-time dimensions

Yue MA*

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Abstract

In this article one will develop a so-called hyperboloidal foliation method, which is an energy method based on a foliation of space-time into hyperboloidal hypersurfaces. This method permits to treat the wave equations and the Klein-Gordon equations in the same framework so that one can apply it to the coupled systems of wave and Klein-Gordon equations. As an application, one will establish the global-in-time existence of small amplitude solution to the coupled wave and Klein-Gordon equations with quadratic nonlinearity in four space-time dimensions under certain conditions. Compared with those introduced by S. Katayama, the conditions imposed in this article permit to include some important nonlinear terms. All of these suggests that this method may be a more natural way of regarding the wave operator.

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*Laboratoire Jacques-Louis Lions. Email: ma@ann.jussieu.fr
1 Introduction

One will consider Cauchy problems associated to a class of coupled nonlinear wave and Klein-Gordon equations. The following is a prototype:

\[
\begin{align*}
\Box u &= N(\partial u, \partial u) + Q_1(\partial u, \partial v) + Q_2(\partial v, \partial v), \\
\Box v + v &= Q_3(\partial u, \partial u) + Q_4(\partial u, \partial v) + Q_5(\partial v, \partial v), \\
\partial_t u(B + 1, x) &= \varepsilon u_0, \\
v(B + 1, x) &= \varepsilon v_0, \\
\partial_t v(B + 1, x) &= \varepsilon v_1.
\end{align*}
\]

(1.1)

Here \(u_i\) and \(v_i\) are regular functions supported on disc \(|x| \leq B\). \(N(\cdot, \cdot)\) is a standard null quadratic form and \(Q_i(\cdot, \cdot)\) are arbitrary quadratic forms.

The method introduced in this article is a type of commuting vector field approach. With this technique, Klainerman has firstly established the global-in-time existence of regular solution to nonlinear wave equations with null condition. (see [3] for details). But when one attempts to try this idea on coupled system of wave and Klein-Gordon equations, one will face some difficulties. The main difficulty is that, one of the conformal Killing vector field of the wave equation, the scaling vector field of Klein-Gordon equation, so that \(S\) can not be used any longer. One may call this “the difficulty of \(S\)”.

In [3], S. Katayama has established the global-in-time existence of regular solution to \((1.1)\). To overcome “the difficulty of \(S\)”, Katayama has used another version of Sobolev type estimate for replacing the classical Klainerman-Sobolev inequality, and an technical \(L^\infty - L^\infty\) estimates.

The main result of this article (see section 3) will improve part of the result of [3]. The method is not a generalization of the techniques introduced in [3] but a new type of energy method based on a foliation of space-time into hyperboloidal hypersurfaces. As far as the author is concerned, this foliation first appears in [2], where L. Hörmander has developed an “alternative energy method” for dealing the global existence of quasilinear Klein-Gordon equation. His observation is as follows. Consider the following Cauchy problem associated to the linear Klein-Gordon equation in \(\mathbb{R}^{n+1}\)

\[
\begin{align*}
\Box u + \alpha^2 u &= f, \\
u(B + 1, x) &= u_0, \\
u_t(B + 1, x) &= u_1,
\end{align*}
\]

(1.2)

where \(u_0, u_1\) are regular functions supported on \(\{(B + 1, x) : |x| \leq B\}\) and \(f\) is also a regular function supported on

\[\Lambda' := \{(t, x) : |x| \leq t - 1\},\]

with \(a, B > 0\) two fixed positive constants. By the Huygens’ principle, the regular solution of \((1.2)\) is supported in

\[\Lambda' \cap \{t \geq B + 1\}.
\]

One denotes by:

\[H_T := \{(t, x) : t^2 - x^2 = T^2, \ t > 0\}\]

and

\[G_{B+1} = \Lambda' \cap \{(t, x) : \sqrt{t^2 - x^2} \geq B + 1\},\]

one can develop a hyperboloidal foliation of \(G_{B+1}\), which is

\[G_{2B} = H_T \times [B + 1, \infty).\]

Then, taking \(\partial_t u\) as multiplier, the standard procedure of energy estimate leads one to the following energy inequality

\[E_m(T, u)^{1/2} \leq E_m(B + 1, u)^{1/2} + \int_{B+1}^T ds \left( \int_{H_T} f^2 \right)^{1/2},\]
where 
\[ E_m(T, u) := \int_{H_T} \sum_{i=1}^{3} \left( \left( \frac{x^i}{t} \partial_t u + \partial_i u \right)^2 + \left( \frac{T}{t} \partial_t u \right)^2 + \left( a/2 \right) u^2 \right) dx. \]

Then, Hörmander has developed a Sobolev type estimate, see the lemma 7.6.1 of [2]. Combined with the energy estimate, he has managed to establish the decay estimate:

\[ \sup_{H_T} t^{n/2} |u| \leq \sum_{|I| \leq m_0} E_m(H_T, Z^I u)^{1/2} \leq E_m(H_{B+1}, Z^I u)^{1/2} + \int_{B+1}^T ds \left( \int_{H_s} Z^I f^2 \right)^{1/2}, \]

where \( m_0 \) is the smallest integer bigger than \( n/2 \).

But in the proof of [2], it seems that the only used term of the energy \( E_m(H_T, u) \) is the last term \( u^2 \). The first two terms seem to be omitted, at least when doing decay estimates. The new observation in this article is that the first two terms of the energy can also be used for estimating some important derivatives of the solution. This leads one to the possibility of applying this method on the case where \( a = 0 \), which is the wave equation, so that the wave equations and the Klein-Gordon equations can be treated in the same framework. This is the key of dealing the coupled wave and Klein-Gordon equations, and one may call it hyperboloidal foliation method.

Here is the structure of this article. In section 2, one will introduce the basic theory of hyperboloidal foliation method including the energy estimates, the estimates on commutators and the decay estimates. For the convenience of proof, a new frame, the so-called “one frame” will be introduced to replace the classical “null frame”. The main result will be stated in section 3. In this article one will not cite any technical result. All tools used will be established in section 2.

2 Preliminaries

2.1 Notation

First, one makes the following important conventions of index. The Latin index \( a, b, c \) denote one of the positive integers 1, 2, 3. The Greek index \( \alpha, \beta, \gamma \) denote one of the integers 0, 1, 2, 3. The Einstein’s summation will be used. But to avoid possible confusion, the dummy index will be printed in red.

One denotes by \( H_T \) the hyperboloid \( \{(t, x) : t^2 - |x|^2 = T^2, t > 0\} \) with its hyperbolic radius equals to \( T \). \( \Lambda' \) denotes the cone \( \{(t, x) : |x| \leq t - 1\} \). One denotes by \( \mathcal{G}_{B+1} \) the region \( \{(t, x) : B + 1 \leq t^2 - x^2 \leq T^2\} \). Notice that in the region \( \Lambda' \cap \{|x| \leq t/2\}, t \leq 2 \sqrt{3T}. \)

One introduces the following vector fields:
\[
H_a := x^a \partial_t + t \partial_a, \\
\mathcal{G}_a := t^{-1} H_a = (x^a / t) \partial_t + \partial_a.
\]

One sets \( \mathcal{Z} \) a family of vector fields consists of \( Z_a \), where
\[
Z_a := \partial_a, \\
and \quad Z_{a+a} := H_a.
\]

One notices that for any \( Z, Z' \in \mathcal{Z} \),
\[
[Z, Z'] \in \mathcal{Z},
\]

which means \( \mathcal{Z} \) forms a Lie algebra. For a general multi-index \( I \), denote by \( Z^I \) a \( |I| - th \) order derivatives
\[
Z^I := Z_{I_1} \cdots Z_{I_{|I|}}.
\]
2.2 Energy estimates

One considers the following differential system:

\[
\begin{aligned}
\Box w_i + G_i^{\alpha \beta} \partial_{\alpha} w_j + D_i w_i &= F_i, \\
w_i|_{H_{B+1}} = w_{i0}, \quad \partial_t w_i|_{H_{B+1}} = w_{i1}.
\end{aligned}
\]

Here \(G_i^{\alpha \beta}\) and \(F_i\) are regular functions supported in \(\Lambda'\). \(w_{i0}, w_{i1}\) are regular functions supported on \(H_{B+1} \cap \Lambda'\). To guarantee the hyperbolicity, one supposes that

\[
G_i^{\alpha \beta} = G_j^{\alpha \beta}, \quad G_i^{\alpha \beta} = G_i^{\beta \alpha}.
\]

\(D_i\) are constants. \(D_i = 0\) with \(1 \leq i \leq j_0\) and \(D_i > \sigma > 0\) when \(j_0 + 1 \leq i \leq j_0 + k_0 =: n_0\).

One introduces the following “standard” energy on hyperboloid \(H_T\):

\[
E_m(T, w_i) := \int_{H_T} \left( |\partial_t w_i|^2 + \sum_a |\partial_a w_i|^2 + (2x^a / t) \partial_t w_i \partial_a w_i + 2(D_i w_i)^2 \right) dx,
\]

\[
= \int_{H_T} 2(D_i w_i)^2 + \sum_a |\partial_a w_i|^2 + ((T/t) \partial_t w_i)^2 dx,
\]

\[
= \int_{H_T} 2(D_i w_i)^2 + \sum_a ((T/t) \partial_a w_i)^2 + \sum_a ((r/t) \partial_a w_i + \omega^a \partial_t w_i)^2 dx.
\]

And the “curved” energy associated to the principal part of (2.2):

\[
E_G(s, w_i) := E_m(s, w_i) + 2 \int_{H_s} (\partial_t w_i \partial_{\alpha} w_j G_i^{\alpha \beta}) \cdot (1, -x^a / t) dx - \int_{H_s} (\partial_{\alpha} w_i \partial_{\beta} w_j G_i^{\alpha \beta}) dx,
\]

Here the second term on the right-hand-side is the Euclidian inner product of the vector \((\partial_t w_i \partial_{\beta} w_j G_i^{\alpha \beta})\) and the vector \((1, -x^a / t)\).

Remark 2.1. From the structure of the strand energy on hyperboloid, one sees that the \(L^2\) norm of \(\partial_t w\) and \((T/s) \partial_{\alpha} w\) are controlled directly. These are (relatively) good derivatives. In general they enjoy better decay than \(\partial_{\alpha} w\). Notice that

\[
\sum_a \omega^a ((r/t) \partial_a + \omega^a \partial_t) w = t^{-1} S_w.
\]

So the derivative \(t^{-1} S\) is also good. But in this article it will not be used.

Then in general the following result holds:

Lemma 2.2 (Energy estimates). Let \(\{w_i\}\) be regular solution of (2.2). If the following estimates hold:

\[
\sum_i E_m(s, w_i) \leq 3 \sum_i E_G(s, w_i),
\]

\[
\int_{H_s} \frac{s}{t} \left( \partial_{\alpha} G_i^{\alpha \beta} \partial_t w_i \partial_{\beta} w_j - \frac{1}{2} \partial_{\alpha} G_i^{\alpha \beta} \partial_{\beta} w_i \partial_{\alpha} w_j \right) dx \leq M(s) E_m(s, w_i)^{1/2},
\]

\[
\left( \int_{H_s} |F_i|^2 dx \right)^{1/2} ds \leq L_i(s).
\]

Then the following energy estimate hold:

\[
\left( \sum_i E_m(s, w_i) \right)^{1/2} \leq \left( \sum_i E_m(B + 1, w_i) \right)^{1/2} + \sqrt{3} \int_{B+1}^s \sum_i L_i(\tau) + \sqrt{m_0} M(\tau) d\tau.
\]
Remark 2.3. In general the initial data is imposed on a plan rather than on a hyperboloid. But in Appendix A one will see that $E_m(B + 1, w_i)$ is controlled by the $H^1(\mathbb{R}^3)$ norm of the initial data given on $\{B + 1 \times \mathbb{R}^3\}$.

Proof. Under the assumptions (2.3), taking $\partial_i w_j$ as multiplier, the standard energy estimate procedure gives

$$
\sum_i \left( \frac{1}{2} \partial_i \sum_{\alpha} (\partial_\alpha w_i)^2 + \sum_{\alpha} \partial_\alpha (\partial_\alpha w_i \partial_\alpha w_i) + \partial_\alpha (G_i^{\alpha \beta} \partial_\sigma w_i \partial_\sigma w_j) - \frac{1}{2} \partial_i (G_i^{\alpha \beta} \partial_\alpha w_i \partial_\beta w_j) \right)
= \sum_i \partial_i w_i F_i + \sum_i \left( \partial_i G_i^{\alpha \beta} \partial_\alpha w_i \partial_\beta w_j - \frac{1}{2} \partial_i G_i^{\alpha \beta} \partial_\alpha w_i \partial_\beta w_j \right).
$$

Then integrate in the region $B_{B+1}$ and use the Stokes formulae,

$$
\frac{1}{2} \sum_i \left( E_G(s, w_i) - E_G(B + 1, w_i) \right)
= \int_{B+1} \partial_i w_i F_i dx + \int_1^{B+1} \partial_\alpha G_i^{\alpha \beta} \partial_\alpha w_i \partial_\beta w_j - \frac{1}{2} \partial_i G_i^{\alpha \beta} \partial_\alpha w_i \partial_\beta w_j \ dx,
$$

which leads to

$$
\frac{d}{ds} \sum_i E_G(s, w_i) = 2 \sum_i \int_{H_s} (s/t) \partial_\alpha G_i^{\alpha \beta} \partial_\alpha w_i \partial_\beta w_j - (s/2t) \partial_i G_i^{\alpha \beta} \partial_\alpha w_i \partial_\beta w_j \ dx
+ 2 \int_{H_s} (s/t) \partial_i w_i F_i dx.
$$

So one gets

$$
\left( \sum_i E_G(s, w_i) \right)^{1/2} \frac{d}{ds} \left( \sum_i E_G(s, w_i) \right)^{1/2}
\leq \sum_i \left( \int_{H_s} |F_i|^2 dx \right)^{1/2} \sum_i E_m(s, w_i)^{1/2} + M(s) \sum_i E_m(s, w_i)^{1/2}
\leq \sqrt{3} \left( \sum_i \int_{H_s} |F_i|^2 dx \right)^{1/2} \left( \sum_i E_G(s, w_i) \right)^{1/2} + \sqrt{3} M(s) \sum_i E_G(s, w_i)^{1/2}
\leq \sqrt{3} \sum_i L_i(s) \left( \sum_i E_G(s, w_i) \right)^{1/2} + \sqrt{3} n_0 M(s) \left( \sum_i E_G(s, w_i) \right)^{1/2}
$$

which leads to

$$
\frac{d}{ds} \left( \sum_i E_G(s, w_i) \right)^{1/2} \leq \sqrt{3} \sum_i L_i(s) + \sqrt{3} n_0 M(s)
$$

By integrating on the interval $[B + 1, s]$, the lemma is proved. \hfill \Box

2.3 Commutators

In this subsection one will establish the very important results of commutators. Firstly, because $Z_\alpha$ are Killing vector fields of $\Box$, the following commutative relations hold:

$$
[\partial_\alpha, \Box] = 0, \quad [H_\alpha, \Box] = 0.
$$
The commutative relations between $H_a$ and $\overline{\partial}_b$ are

\begin{equation}
H_a \overline{\partial}_b = \overline{\partial}_b H_a - \frac{x^b}{t} \overline{\partial}_a.
\end{equation}

The commutative relations between $\partial_\beta$ and $H_a$ are:

\begin{equation}
\begin{aligned}
H_a \partial_\beta &= \partial_\beta H_a - \delta^a_\beta \partial_t, \\
H_a \partial_t &= \partial_t H_a - \partial_a.
\end{aligned}
\end{equation}

The commutative relations between $H_j$ and $(T/t)\partial_\alpha$ are

\begin{equation}
\begin{aligned}
H_a \left( \frac{T}{t} \partial_\alpha u \right) &= -\frac{T}{t} \left( \partial_a u + \frac{x^a}{t} \partial_t u \right) + \frac{T}{t} \partial_t (H_a u), \\
H_a \left( \frac{T}{t} \partial_a u \right) &= -\frac{T}{t} \left( \partial^a_\alpha \partial_t u + \frac{x^a}{t} \partial_b u \right) + \frac{T}{t} \partial_t (H_a u).
\end{aligned}
\end{equation}

The commutative relations between $\partial_\alpha$ and $\overline{\partial}_a$ are:

\begin{equation}
\begin{aligned}
\partial_b \overline{\partial}_a &= \overline{\partial}_a \partial_b + \delta^a_\beta \partial_t, \\
\partial_t \overline{\partial}_a &= \overline{\partial}_a \partial_t - \partial_t \partial_a.
\end{aligned}
\end{equation}

The commutative relations between $\partial_\alpha$ and $(T/t)\overline{\partial}_a$ are:

\begin{equation}
\begin{aligned}
\partial_t ((T/t)\partial_\alpha) &= (T/t)\partial_\alpha \partial_t - t^{-1} (T/t) \partial_\alpha, \\
\partial_\alpha ((T/t)\partial_\alpha) &= (T/t)\partial_\alpha \partial_\alpha - \left( \frac{x^a}{T/t} \right) (T/t) \partial_\alpha.
\end{aligned}
\end{equation}

One also needs the following commutative relations between $H_a$ and $\partial_\alpha \partial_\beta$:

\begin{equation}
\begin{aligned}
H_a \partial_\beta \partial_\delta = \partial_\beta \partial_\delta H_a - \delta^a_\beta \partial_t \partial_\delta - \delta^a_\beta \partial_\delta, \\
H_a \partial_\delta \partial_\beta = \partial_\delta \partial_\beta H_a - \partial_t \partial_\delta - \delta^a_\delta \partial_\beta, \\
H_a \partial_\beta \partial_\gamma = \partial_\beta \partial_\gamma H_a - \partial_\gamma \partial_\beta - \delta^a_\gamma \partial_\beta.
\end{aligned}
\end{equation}

In general, one has the following estimates:

**Lemma 2.4.** For any regular function $u$ supported in $\Lambda'$, the following estimates hold

\begin{equation}
\begin{aligned}
|Z^I((T/t)\partial_\alpha u)| &\leq |(T/t)\partial_\alpha Z^I u| + C(n, |I|) \sum_{\beta, |J| < |I|} |(T/t)\partial_\beta Z^J u|, \\
|(T/t)Z^I \partial_\alpha u| &\leq |(T/t)\partial_\alpha Z^I u| + C(n, |I|) \sum_{\beta, |J| < |I|} |(T/t)\partial_\beta Z^J u|, \\
|Z^I \overline{\partial}_a u| &\leq |\overline{\partial}_a Z^I u| + C(n, |I|) \sum_{b, |J| < |I|} |\overline{\partial}_b Z^J u| + C(n, |I|) \sum_{\beta, |J| < |I|} |(T/t)\partial_\beta Z^J u|, \\
|Z^I \partial_\alpha \partial_\beta u| &\leq |\partial_\alpha \partial_\beta Z^I u| + C(n, |I|) \sum_{\gamma, |J| < |I|} |\partial_\gamma \partial_\alpha \partial_\beta Z^J u|.
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
|[Z^I, \partial_\alpha \partial_\beta] u| + |[Z^I, \partial_\alpha \partial_\beta] u| &\leq C(n, I) \sum_{|J| < |I|} |\partial_\alpha \partial_\beta Z^J u| + C(n, |I|) t^{-1} \sum_{|J| < |I|} |\partial_\alpha \partial_\beta Z^J u|.
\end{aligned}
\end{equation}
Proof. To prove (2.15), one needs the following identities:

\[
\begin{align*}
H_b(x^a/t) &= -(x^a/t)(x^b/t) + \delta_b^a, \\
\partial_t(x^a/t) &= -t^{-1}(x^a/t), \\
(2.19)
\end{align*}
\]

Notice that in the cone \( \Lambda \) \((2.20)\), here any \(|I| \leq N\). Its inverse is so that

\[
Z_1(T/t)\partial_a u = (T/t)\partial_a Z I u + A(I, a)^{\alpha}_a (T/t)\partial_{\alpha} Z^J u + B(I, a)^{\alpha}_a (x^a/t)\partial_{\alpha} Z^J u
\]

\[
(2.20)
\]

where \( A(I, a)^{\alpha}_a \), \( B(I, a)^{\alpha}_a \) and \( C(I, a)^{\alpha}_m \) are constants depending on \( I, a, \alpha, m \). When \(|I| = 1\), by \((2.11)\) and \((2.13)\),

\[
Z_i(T/t)\partial_a u = A(i, a)^{\alpha}_a (T/t)\partial_a u + B(i, a)^{\alpha}_a (x^i/t)\partial_a u + C(i, a)^{\alpha}_a t^{-1}(T/t)\partial_a u,
\]

where \( i = 0, 1, \cdots 7 \). Suppose that \((2.20)\) holds for all multi-index \(|I'| \leq N\), Then for any multi-index \(|I| = N + 1\),

\[
Z^I(T/t)\partial_a u = Z^I Z^I (T/t)\partial_a u
\]

\[
= Z^I \left( (T/t)\partial_a Z^I u + A(I, a)^{\alpha}_a (T/t)\partial_{\alpha} Z^J u + B(I, a)^{\alpha}_a (x^a/t)\partial_{\alpha} Z^J u \right)
\]

\[
+ Z^I \left( \sum_{m \leq |J|} C(I, a)^{\alpha}_m t^{-m}(T/t)\partial_{\alpha} Z^J u \right),
\]

here \( I_1 \) represents the first component of \( I \). Then by \((2.19)\), one concludes that \((2.20)\) holds for any \(|I| \leq N + 1\). Then by induction one concludes that the first estimate in \((2.15)\) holds.

The rest part of the lemma is proved in the same way. One omits the details. \( \square \)

2.4 Frames and Null conditions

In this part one will introduce a so-called “one frame”, denoted by \( \{ \partial_\alpha \} \). This one frame will take the place of the classical “null frame”. Define

\[
\partial_\alpha := \partial_\alpha, \quad \partial_a := \vec{\partial}_a.
\]

The transition matrix between one frame and the natural frame is

\[
\Phi := \begin{pmatrix}
1 & 0 & 0 & 0 \\
x^1/t & 1 & 0 & 0 \\
x^2/t & 0 & 1 & 0 \\
x^3/t & 0 & 0 & 1
\end{pmatrix}
\]

Its inverse is

\[
\Psi := \begin{pmatrix}
1 & 0 & 0 & 0 \\
-x^1/t & 1 & 0 & 0 \\
-x^2/t & 0 & 1 & 0 \\
-x^3/t & 0 & 0 & 1
\end{pmatrix},
\]

so that

\[
\partial_\alpha = \Psi^\beta_\alpha \partial_\beta.
\]
Remark 2.5. One notices that compared with the classical null frame, the norm of the only “bad”
direction $\partial_0$ is 1, while the only “bad” direction $\mathbf{L}$ in the null frame is a null vector. That is the
reason why one calls it “one-frame”. The advantage of this one frame, compared with the classical
null frame is that the components of the transition matrix are always regular in the cone $\Lambda'$.

One may write a two tensor $\mathcal{T}$ under one frame or under the natural frame:
$$\mathcal{T} = T^{\alpha\beta} \partial_\alpha \partial_\beta = T^{\alpha\beta} \partial_\alpha \partial_\beta,$$
where $T^{\alpha\beta}$ represent its components under one frame. In general one has the following estimates:

**Lemma 2.6.** For general two tensor $T$, in the region $\Lambda'$
$$|Z^T T^{\alpha\beta}| \leq \sum_{\alpha',\beta'} |Z^T T^{\alpha'\beta'}|$$

**Proof.** One notices that in $\Lambda'$, $|\partial_\alpha \Phi_\beta| \leq 1$. The proof is just a calculation. $\square$

Any two-order differential operator $T^{\alpha\beta} \partial_\alpha \partial_\beta$, can also be written under this one-frame.

(2.21) $T^{\alpha\beta} \partial_\alpha \partial_\beta u = T^{\alpha\beta} \partial_\alpha \partial_\beta u - T^{\alpha\beta} (\partial_\alpha \Phi_\beta') \partial_{\beta'} u.$

Especially for the wave operator, one has the following expression:
$$\Box u = m^{\alpha\beta} \partial_\alpha \partial_\beta u - m^{\alpha\beta} (\partial_\alpha \Phi_\beta') \partial_{\beta'} u,$$

Simple calculation gives $m^{00} = T^2/t^2$, so one gets the following important identity:

(2.22) $(T/t)^2 \partial_0 \partial_0 u = \Box u - m^{00} \partial_0 \partial_0 u - m^{0a} \partial_0 \partial_\alpha u - m^{ab} \partial_a \partial_b u + m^{\alpha\beta} (\partial_\alpha \Phi_\beta') \partial_{\beta'} u,$

here $m^{00} = x^a/t$, $m^{ab} = \delta_a^b$ and
$$|\partial_\alpha \Phi_\beta'| \leq C, \text{ in } \Lambda'.$$

So one concludes by the following lemma

**Lemma 2.7.** Let $u$ be a regular function supported in $\Lambda'$. Then
$$(T/t)^2 |\partial_0 \partial_0 u| \leq |\Box u| + 2 \sum_{\alpha,\beta} |\partial_\alpha \partial_\beta u| + Ct^{-1} \sum_{\beta} |\partial_\beta u|.$$

**Proof.** This is a direct result of (2.22) and (2.12). $\square$

Now a version of the classical null conditions will be introduced. A quadratic form $T^{\alpha\beta} \xi_\alpha \xi_\beta$ is
said to satisfy the null conditions if for any $\xi \in \mathbb{R}^4$ such that
$$\xi_0 \xi_0 - \sum_a \xi_a \xi_a = 0,$$
then
$$T^{\alpha\beta} \xi_\alpha \xi_\beta = 0.$$

Similarly, a cubic form $A^{\alpha\beta\gamma} \xi_\alpha \xi_\beta \xi_\gamma$ is said to verify the null conditions if
$$A^{\alpha\beta\gamma} \xi_\alpha \xi_\beta \xi_\gamma = 0.$$
Lemma 2.8. Suppose that $T^{\alpha\beta}_{\alpha\beta}$ is a quadratic form which satisfies the null conditions. If $|T^{\alpha\beta}| \leq K$, then for any multi-index, the following estimate holds in the cone $\Lambda'$:

$$|Z^I T^{00}| \leq CK (T/t)^2.$$ 

Similarly, if a cubic form $A^{\alpha\beta\gamma}_{\alpha\beta\gamma}$ who verifies the null conditions and $|A^{\alpha\beta\gamma}| \leq K$, then in the cone $\Lambda'$,

$$|Z^I A^{000}| \leq CK (T/t)^2.$$ 

Proof. One defines $\omega_\alpha = \omega^\alpha = x^\alpha/|x|$ and $\omega_0 = -\omega^0 = -1$. Then

$$\omega_\alpha \omega_\alpha - \sum_\alpha \omega_\alpha \omega_\alpha = 0.$$ 

Let $\chi(\cdot)$ be a $C^\infty$ function defined on $(0, +\infty)$, $\chi(x) = 0$ when $x \leq 1/3$ and $\chi(x) = 1$ when $x \geq 1/2$. Now consider the component $T^{00}$,

$$T^{00} = T^{\alpha\beta} \Psi^0_\alpha \Psi^0_\beta$$

$$= T^{\alpha\beta} \Psi^0_\alpha \Psi^0_\beta - \chi(r/t) T^{\alpha\beta} \omega_\alpha \omega_\beta$$

$$= T^{\alpha\beta} (\Psi^0_\alpha \Psi^0_\beta - \chi(r/t) \omega_\alpha \omega_\beta).$$

Taking into account the fact that when $r \geq t/3$, $Z^I \omega^\alpha \leq C$ and $H_a(r/t) = \omega^\alpha (T/t)^2$. One has, when $r \geq t/2$,

$$Z^I T^{00} = Z^I \left( - \sum_\alpha T^{ab}_\alpha \omega^a r - \frac{t}{t} \sum_\alpha \omega^a \omega^a \omega^b \frac{r^2 - t^2}{t^2} \right) \leq CK (T^2/t^2).$$

When $r \leq t/2$, by simple calculation, one has:

$$|Z^I T^{00}| \leq CK.$$ 

But because in $\Lambda' \cap \{r \leq t/2\}$, $t^2 \leq \frac{2\sqrt{3}}{3} T^2$, one concludes that

$$|Z^I T^{00}| \leq CK (T/t)^2.$$ 

The proof of the result on cubic forms is similar. One omits the details. 

2.5 Decay estimates

To turn the $L^2$ type energy estimates into the $L^\infty$ type estimates, one needs the following Sobolev inequality, which is introduced as lemma 7.6.1 of [2].

Lemma 2.9 (Sobolev-type estimate on hyperboloid). Let $p(n)$ be the smallest integer $> n/2$. Any $C^\infty$ function defined on $\mathbb{R}^{1+n}$ satisfies

\[ \sup_{H_T} t^n |u(t,x)|^2 \leq C(n) \sum_{t \leq |p(n)|} \|Z^I u\|^2_{L^2(H_T)} \]

where $C(n) > 0$ is a constant depending only on dimension $n$.

Combine this Lemma with the lemma of commutators [2,4] one gets the following results

\[^{1}\text{Arising the index by Minkowski metric.}\]
Lemma 2.10. Let $u$ be a regular function supported in $\mathcal{N}$. Then the following estimates hold:

\[
\sup_{H_T} |t^{(n-2)/2} T^{-1} \partial_{\alpha} u|^2 \leq \sum_{|I| \leq p(n)} E_m(T, Z^I u),
\]
\[
\sup_{H_T} |t^{n/2} \overline{\partial}_{\alpha} u|^2 \leq \sum_{|I| \leq p(n)} E_m(T, Z^I u),
\]
\[
\sup_{H_T} |D_t u|^2 \leq \sum_{|I| \leq p(n)} E_m(T, Z^I u).
\]

Proof. One recalls the equation (2.14). Then it is a trivial result by lemma 2.4 $\square$

Now let us consider the homogeneous linear wave equation

\[
\Box w = 0, \quad w|_{H^{n+1}} = w_0, \quad \partial_t w|_{H^{n+1}} = w_1.
\]

where $w_i$ are regular functions supported on $H^{n+1} \cap \mathcal{N}$. By energy estimate lemma 2.2, the associated energy $E_m(s, Z^I w)$ is conserved. Then by estimates of commutators 2.4 and the sobolev lemma 2.3 one gets

\[
|\overline{\partial}_\alpha w| \leq C(n)t^{-n/2}, \quad |\partial_{\alpha} w| \leq C(n)t^{-n/2+1}T^{-1}.
\]

This is exactly the classical result. But one notices that neither the explicit expression of the solution nor the scaling vector field $S = r\partial_r + t\partial_t$ is used.

Now one will turn to the energy and decay estimates of the some “good” second-order derivatives, which are the derivatives such as $\overline{\partial}_\alpha \partial_{\alpha} u$. As we will see, these derivatives have better decay than that of $\partial_{\alpha} u$ or even $\overline{\partial}_\alpha u$. In general one has

Lemma 2.11. Let $u$ be a regular function supported in the region $\mathcal{N}$. The following estimates hold:

\[
\sup_{H_T} |t^{n/2} \overline{\partial}_\alpha \overline{\partial}_{\alpha} u|^2 + \sup_{H_T} |t^{n/2} \overline{\partial}_\alpha \overline{\partial}_{\alpha} u|^2 \leq C(n) \sum_{|I| \leq p(n)+1} E_m(T, Z^I u),
\]
\[
\int_{H_T} |\overline{\partial}_\alpha \overline{\partial}_{\alpha} u|^2 dx + \int_{H_T} |\overline{\partial}_\alpha \overline{\partial}_{\alpha} u|^2 dx \leq C \sum_{|I| \leq 1} E_m(T, Z^I u).
\]

Proof. Notice that

\[
\overline{\partial}_\alpha = \overline{\partial}_{\alpha} = t^{-1}H_{\alpha},
\]

one gets

\[
|\overline{\partial}_\alpha \partial_{\alpha} u| \leq t^{-1}|H_{\alpha} \partial u|.
\]

Then by lemma 2.10 one gets the first estimate. The second is a trivial result of the expression (2.24). $\square$

Remark 2.12. The energy estimates and decay estimates of $\overline{\partial}_\alpha \partial_{\alpha} u$ will concern the wave equation it-self. Roughly saying, by lemma 2.4. From here one can see that for wave equation, all the second-order derivatives do enjoy better decay compared with the gradient of the solution.

At the end of this section, one gives the decay and energy estimates of the solution it-self.

Lemma 2.13. Let $u$ be a regular function supported in the cone $\mathcal{N}$. Then for any multi-index $|I| \geq 1$,

\[
(2.24) \quad \int_{H_s} |t^{-1} Z^I u|^2 dx \leq C \sum_{|I| \leq p(n)-1} E_m(s, Z^I u).
\]

For any multi-index $J$, if $\sum_{|I| \leq |\alpha|+p(n)} E_m(s, Z^I u)^{1/2} \leq C'T^\varepsilon$ for an $\varepsilon \geq 0$, then

\[
(2.25) \quad |Z^J u| \leq CC't^{-n/2}T^{1+\varepsilon}.
\]
When all of the factor of $Z^l$ such as $\hat{i}, \hat{j}, \hat{k}, \hat{l}$ denote one of the integer $1, \hat{i}, \hat{j}, \hat{k}, \hat{l}$, notice that $t^{-1}H_a = \overline{\partial}_a$, by (2.4)

\[
\int_{H_{\hat{a}}} |(s/t)Z^l u|^2 dx \leq C \sum_{|\alpha\beta| \leq |\gamma|} E_m(s, Z^l u).
\]

When all of the factor of $Z^l$ are $H_a$, impose the following null conditions $\sum_{|\alpha\beta\gamma\delta| \leq |\gamma|} E_m(s, Z^l u)^{1/2}$ is bounded by $C(\varepsilon T^*\varepsilon)$, by lemma 2.10 in the cone $\Lambda'$, for $|\partial_t u| \leq C(n) t^{-(n-1-\varepsilon)/2}(t-r)^{-(1-\varepsilon)/2}$.

then the proof of (2.24) is a integration along the radical direction. □

### 3 Main result

One will consider the Cauchy problem associated to the following coupled wave and Klein-Gordon equations with quadratic nonlinearity:

\[
\begin{cases}
\Box u_i + G_i^{\alpha\beta}(w, \partial w)\partial_{\alpha\beta}w_j + D_i^2w_i = F_i(w, \partial w), \\
w_i(B + 1, x) = \varepsilon w_0, \quad w_i(B + 1, x) = \varepsilon w_1.
\end{cases}
\]

(3.1)

Here $1 \leq i \leq n_0$. $D_i$ are constants. $D_i = 0$ for $1 \leq i \leq j_0$ and $D_i > 0$ for $j_0 + 1 \leq i \leq n_0$.

For the convenience of proof, one makes the following conventions of index. The Latin index $i, j, k, l$ denote one of the integer $1, 2, 3, \cdots, n_0$. The Latin index with a circumflex accent on it such as $\hat{i}, \hat{j}, \hat{k}, \hat{l}$ denote one of the integer $1, 2, 3, \cdots, j_0$. The Latin index with a hacek on it such as $\hat{i}, \hat{j}, \hat{k}, \hat{l}$ denote one of the integer $j_0 + 1, j_0 + 2, \cdots, n_0$.

$G_i^{\alpha\beta}(\cdot, \cdot)$ and $F_i(\cdot, \cdot)$ are regular functions such that:

\[
G_i^{\alpha\beta}(w, \partial w) = A_i^{\alpha\beta\gamma\delta} w_j, \quad F_i(w, \partial w) = P_i^{\alpha\beta\gamma\delta} w_j.
\]

Here $A_i^{\alpha\beta\gamma\delta}, B_i^{\alpha\beta\gamma\delta}, P_i^{\alpha\beta\gamma\delta}, Q_i^{\alpha\beta\gamma\delta}, R_i^{\alpha\beta\gamma\delta}$ are constants with absolute value bounded by $K$. One impose the following null conditions

\[
(3.2) \quad A_i^{\alpha\beta\gamma\delta} \xi_\alpha \xi_\beta \xi_\gamma \xi_\delta = B_i^{\alpha\beta\gamma\delta} \xi_\alpha \xi_\beta = P_i^{\alpha\beta\gamma\delta} \xi_\alpha \xi_\beta = 0, \quad \text{for any } \xi_0 \xi_0 - \sum a \xi_a a = 0.
\]

One also supposes that

\[
(3.3) \quad B_i^{\alpha\beta\gamma\delta} = Q_i^{\alpha\beta\gamma\delta} = R_i^{\alpha\beta\gamma\delta} = 0.
\]

The initial data $w_i(0)$ and $w_i(1)$ are supposed to be $(C^\infty)$ regular functions supported on the disc $\{|x| \leq B + 1\}$.

Now one is ready to state the main theorem.

**Theorem 3.1.** Suppose (3.2) and (3.3) hold. Then there exists an $\varepsilon_0 > 0$ such that for any $0 \leq \varepsilon \leq \varepsilon_0$, (3.1) has a unique global-in-time regular solution. In this case, the hyperbolic energy associated to the wave components is conserved.

\[
\sum_{i, |\gamma| \leq 3} E_m(s, Z^l u_i)^{1/2} \leq C(\varepsilon_0).
\]
Remark 3.2. One improvement is that in [3], the system is not allowed to contain the term \( \partial u_{2i} \partial_0 u_j \) and \( \partial_1^2 w_{2i} \). In this article this restriction is relaxed. One only demand classical null conditions on these terms.

The most important improvement is that in the proof one will used nothing technical but only the tools one has prepared in section 2.

Remark 3.3. This theorem also holds in the case where \( w_{0i} \) and \( w_{1j} \) are not \( C^\infty \) (but still compactly supported). This is by a standard procedure of approximation. In general one only demand that \( w_{0i}, w_{1j} \in H^3 \). That is because following proof consults only the derivatives of solution of order \( \leq 4 \). This is also an improvement compared with [3], where the proof consults at least the 19th-order derivatives.

Remark 3.4. When \( n_0 = j_0 \), the theorem reduced to the classical result of global existence of regular solution to quasilinear wave equation with null conditions, see [2] for example. One can check that with out the Klein-Gordon components \( v_i \), the following proof becomes very short and trivial which is simpler than the classical one. Furthermore, the energy \( E_m(s, Z^I w_i) \) is conserved. This means the global solution is not only a “small amplitude solution” but also “small energy solution”.

4 Proof of main result

4.1 Structure of the proof

The proof is a standard boot-strap argument deviled into five parts. In the first part one supposes that in an interval \([0, T] \) the energy \( E_m(s, Z^I w_i) \) is bounded for \( 0 \leq s \leq T \). By lemma 2.10, lemma 2.11 and lemma 2.13 one gets the decay estimates of \( w_j, \partial w_j \) and \( \partial^2 w_j \). In part two with theorem 4.1, the following proof becomes very short and trivial which is simpler than the classical one. Furthermore, the energy \( E_m(s, Z^I w_i) \) is conserved. Then with all of these preparation, one will finally establish the main result in the last part.

4.2 Part one – Energy assumption

Suppose that on a interval \([B + 1, T^*] \), the following energy assumptions hold with \( 0 < \delta < 1/6 \):

\[
\begin{align*}
E_m(s, Z^I v_j)^{1/2} &\leq C_1 \varepsilon s^\delta, \quad \text{for} \quad j_0 + 1 \leq j \leq n_0, \quad 0 \leq |I^*| \leq 4, \\
E_m(s, Z^I w_i)^{1/2} &\leq C_1 \varepsilon s^\delta, \quad \text{for} \quad 1 \leq i \leq j_0, \quad |I^*| = 4, \\
E_m(s, Z^I w_i)^{1/2} &\leq C_1 \varepsilon, \quad \text{for} \quad 1 \leq i \leq j_0, \quad |I| \leq 3.
\end{align*}
\]

By theorem 4.1, for any \( C_1 \varepsilon \), one can choose \( \varepsilon' \) small enough such that

\[
\sum_{|I^*| \leq 4} E_m(B + 1, Z^I w_i)^{1/2} < C_1 \varepsilon,
\]

so that by continuity, \( T^* > B + 1 \).

From (4.1) and lemma 2.4, the following \( L^2 \) estimates hold on \([B + 1, T^*] \):

\[
\begin{align*}
\sum_{I^*, \alpha \leq s/t} \left( \int_{H_s} \left| \frac{Z^I}{Z^0} \partial_0 u_i \right|^2 dx \right)^{1/2} + \sum_{I^*, \alpha} \left( \int_{H_s} \left| Z^I \partial_0 u_i \right|^2 dx \right)^{1/2} &\leq CC_1 \varepsilon, \\
\sum_{|I^*| \leq 4} \left( \int_{H_s} \left| (s/t) Z^I \partial_0 u_i \right|^2 dx \right)^{1/2} + \sum_{|I^*| \leq 4} \left( \int_{H_s} \left| Z^I \partial_0 u_i \right|^2 dx \right)^{1/2} &\leq CC_1 \varepsilon s^\delta, \\
\sum_{|I^*| \leq 4} \left( \int_{H_s} \left| Z^I v_j \right|^2 dx \right)^{1/2} &\leq CC_1 \varepsilon s^\delta.
\end{align*}
\]
By lemma \ref{lem:2.11} one also has the following $L^2$ estimates:

\begin{align}
\sum_{a,\partial \beta} \left( \int_{H_s} |sZ^I \partial_{\partial \beta} u_i|^2 \, dx \right)^{1/2} + \sum_{a,\partial \beta} \left( \int_{H_s} |sZ^I \partial_{\partial \beta} u_i|^2 \, dx \right)^{1/2} & \leq CC_1 \epsilon, \\
\sum_{a,\partial \beta} \left( \int_{H_s} |sZ^I \partial_{\partial \beta} u_i|^2 \, dx \right)^{1/2} + \sum_{a,\partial \beta} \left( \int_{H_s} |sZ^I \partial_{\partial \beta} u_i|^2 \, dx \right)^{1/2} & \leq CC_1 \epsilon^s. \tag{4.3}
\end{align}

The following decay estimates come from lemma \ref{lem:2.10} For $|J^*| \leq 2$ and $|J| \leq 1$,

\begin{align}
\sup_{H_s} \left( |st^{1/2} \partial_{\partial} Z^J w_j| \right) + \sup_{H_s} \left( |t^{3/2} | \partial_{\partial} Z^J w_j| + t^{3/2} |Z^J v_k| \right) & \leq CC_1 \epsilon^s, \\
\sup_{H_s} \left( |st^{1/2} \partial_{\partial} Z^J u_k| \right) + \sup_{H_s} \left( |t^{3/2} | \partial_{\partial} Z^J u_k| \right) & \leq CC_1 \epsilon. \tag{4.4}
\end{align}

The following decay estimates will be more often used in the proof. The first inequality is due to \ref{4.4} and lemma \ref{lem:2.10}. The second is due to \ref{4.4} and \ref{2.15}. The last one is due to \ref{2.15}.

\begin{align}
\sup_{H_s} |st^{1/2} Z^J \partial_{\partial} u_j| + \sup_{H_s} |t^{3/2} Z^J \partial_{\partial} u_j| & \leq CC_1 \epsilon, \\
\sup_{H_s} \left( |st^{1/2} Z^J \partial_{\partial} v_k| \right) + \sup_{H_s} \left( |t^{3/2} | Z^J \partial_{\partial} v_k| + t^{3/2} |Z^J v_k| \right) & \leq CC_1 \epsilon^s, \\
\sup_{H_s} |st^{3/2} \partial_{\partial} Z^J u_j| + \sup_{H_s} |st^{3/2} \partial_{\partial} Z^J u_j| & \leq CC_1 \epsilon^s, \\
\sup_{H_s} |st^{3/2} \partial_{\partial} Z^J u_j| + \sup_{H_s} |st^{3/2} \partial_{\partial} Z^J u_j| & \leq CC_1 \epsilon^s. \tag{4.5}
\end{align}

### 4.3 Part two – $L^2$ estimates

In this step one will give $L^2$ type estimates of some quadratic terms which are components of the source terms. In general one has the following $L^2$ estimates.

**Lemma 4.1.** Let \{w_i\} be regular solution of \eqref{3.1}. Suppose that \eqref{1.5}, \eqref{1.2} and \eqref{1.2} hold. Let \(A_3\) be any of the following terms:

\[ v_k v_j, \quad v_k \partial_{\partial} w_j, \quad \partial_{\partial} v_j \partial_{\partial} v_k, \quad \partial_{\partial} u_j \partial_{\partial} u_k. \]

Then for any $|I| \leq 3$,

\[ \left( \int_{H_s} |Z^I A_3|^2 \, dx \right)^{1/2} \leq C(C_1 \epsilon)^2 s^{-3/2+2\delta}. \]

Furthermore, if \(\Gamma(t,x)\) is a regular function such that for any multi-index $J$, the following estimate holds in $\mathcal{N}$:

\[ |Z^J \Gamma| \leq C(J)(s/t), \]

then for any $|I| \leq 4$,

\[ \left( \int_{H_s} |Z^I (\partial_{\partial} u_j \partial_{\partial} u_j)|^2 \, dx \right)^2 \leq C(C_1 \epsilon)^2 s^{-3/2+2\delta}. \]

Especially, for any $|I| \leq 3$,

\[ \left( \int_{H_s} |Z^I F_1|^2 \, dx \right)^{1/2} \leq C(C_1 \epsilon)^2 s^{-3/2+2\delta}, \]

\[ \left( \int_{H_s} |Z^I F_1|^2 \, dx \right)^{1/2} \leq C(C_1 \epsilon)^2 s^{-3/2+2\delta}, \]
Proof. One begins with the estimates on $A_3$. The proof is mainly a substitution of (4.2), (4.3) and (4.5) into the corresponding expressions. Notice that when a product of derivatives $Z^I$ acts on a product of two factors, there is always one factor derived less than $|I|/2$ times which may be controlled by (4.5). Then the $L^2$ norm of the hole product can be controlled by (4.2) or (4.3) when $|I|/2 \geq p(3)$. One writes the proof on $\partial_\alpha v_j \partial_\beta w_k$ and $\partial_\alpha u_j \partial_\beta u_j$ in detail and omits the others. Suppose that $|I| \leq 3$.

\[ \left( \int_{H_+} |Z^I(\partial_\alpha v_j \partial_\beta w_k)|^2 \, dx \right)^{1/2} \]

\[ \leq \sum_{|I_1| \leq 3} \left( \int_{H_+} |Z^{I_1} \partial_\alpha v_j Z^{I_2} \partial_\beta w_k|^2 \, dx \right)^{1/2} + \left( \int_{H_+} |Z^I \partial_\alpha v_j \partial_\beta w_k|^2 \, dx \right)^{1/2} \]

\[ \leq \sum_{|I_1| \leq 3} C(C_1 \varepsilon s^{-3/2+\delta} \left( \int_{H_+} |s/t Z^{I_2} \partial_\beta w_k|^2 \, dx \right)^{1/2} + C(C_1 \varepsilon) s^{-3/2+\delta} \left( \int_{H_+} |Z^I \partial_\gamma v_j|^2 \, dx \right)^{1/2} \]

\[ \leq C(C_1 \varepsilon)^2 s^{-3/2+2\delta}. \]

\[ \left( \int_{H_+} |Z^I(\partial_\alpha u_j \partial_\beta u_j)|^2 \, dx \right)^{1/2} \]

\[ \leq \sum_{|I_1| \leq 3} \left( \int_{H_+} |Z^{I_1} \partial_\alpha u_j Z^{I_2} \partial_\beta u_j|^2 \, dx \right)^{1/2} + \left( \int_{H_+} |Z^I \partial_\alpha u_j \partial_\beta u_j|^2 \, dx \right)^{1/2} \]

\[ \leq \sum_{|I_1| \leq 3} C(C_1 \varepsilon t^{-3/2}s^{1/2}|(t/s)|Z^{I_2} \partial_\beta u_j|^2 \, dx \right)^{1/2} \]

\[ + C(C_1 \varepsilon) s^{-3/2} \left( \int_{H_+} |Z^I \partial_\gamma u_j|^2 \, dx \right)^{1/2} \]

\[ \leq \sum_{|I_1| \leq 3} C(C_1 \varepsilon t^{-1/2}s^{-1+\delta}|(t/s)|Z^{I_2} \partial_\beta u_j|^2 \, dx \right)^{1/2} + C(C_1 \varepsilon) s^{-3/2} \]

\[ \leq C(C_1 \varepsilon)^2 s^{-3/2+2\delta}. \]

Now one turns to the estimates of $\Gamma \partial_\alpha u_j \partial_\beta u_j$. This quadratic forms is composed purely by the “bad” derivative $\partial_0$. But with the additional decay provided by $\Gamma$, the $L^2$ estimates are still trivial:
\[
\left( \int_{H_s} |Z^I (\Gamma \partial_0 u_i \partial_0 u_j)|^2 \, dx \right)^{1/2}
\leq \sum_{i_1 + i_2 + i_3 = i} \left( \int_{H_s} |Z^I \Gamma Z^I \partial_0 u_i Z^I \partial_0 u_j|^2 \, dx \right)^{1/2}
+ \left( \int_{H_s} |\Gamma Z^I \partial_0 u_i \partial_0 u_j|^2 \, dx \right)^{1/2}
\leq C \sum_{i_1 + i_2 + i_3 = i} \left( \int_{H_s} |CC_1 \xi t^{-1/2} s^{-1/2} \| (s/t) Z^I \partial_0 u_i \partial_0 u_j \|^2 \, dx \right)^{1/2}
+ C \left( \int_{H_s} |CC_1 \xi t^{-1/2} s^{-1/2} \| (s/t) Z^I \partial_0 u_i |^2 \, dx \right)^{1/2}
\leq C (C_1 \xi)^2 s^{-3/2 + \delta}.
\]

The estimate on \( Z^I F_i \) will consult the \( L^2 \) estimates proved. By definition,
\[
F_i = P^{i_0 j k} \partial_0 u_i \partial_3 u_k + P^{i_0 j k} \partial_0 u_i \partial_1 u_k + P^{i_0 j k} \partial_0 u_i \partial_2 u_k + P^{i_0 j k} \partial_0 u_i \partial_2 u_k + Q^{i_0 j k} \partial_0 u_i \partial_2 u_k + R^{i_0 j k} \partial_0 u_i \partial_2 u_k.
\]
The first term can be written under one-frame:
\[
P^{i_0 j k} \partial_0 u_i \partial_3 u_k = P^{i_0 j k} \partial_0 u_i \partial_3 u_k = P^{i_0 j k} \partial_0 u_i \partial_3 u_k + P^{i_0 j k} \partial_0 u_i \partial_3 u_k + P^{i_0 j k} \partial_0 u_i \partial_3 u_k + P^{i_0 j k} \partial_0 u_i \partial_3 u_k.
\]
By the null conditions (3.2) and lemma 2.8,
\[
|Z^I (P^{i_0 j k})| \leq C(I)(s/t)^2 \leq C(I)(s/t).
\]
So
\[
\left( \int_{H_s} |Z^I (P^{i_0 j k})| | Z^I \partial_0 u_i \partial_0 u_k |^2 \, dx \right)^{1/2} \leq C (C_1 \xi)^2 s^{-3/2 + 2 \delta}.
\]
The rest terms of \( F_i \) have been already estimated by the estimates on \( A_3 \) terms. This completes the proof.

Define
\[
\tilde{G}_i^{\alpha \beta} \partial_0 u_j := G_i^{\alpha \beta} \partial_0 u_j - B_i^{\alpha \beta} u_k \partial_0 \partial_0 u_j
\]
and
\[
\tilde{G}_i^{\alpha \beta} \partial_0 u_j := G_i^{\alpha \beta} \partial_0 u_j + G_i^{\alpha \beta} \partial_0 u_j.
\]
This is the “good” part of \( G \)

**Lemma 4.2.** Let \( \{ w_i \} \) be regular solution of (3.1). Suppose that (1.5), (1.2) and (1.3) hold. Then for any \( |I| \leq 3 \),
\[
\left( \int_{H_s} |[\tilde{G}_i^{\alpha \beta} \partial_0 u_j, Z^I] w_j |^2 \, dx \right)^{1/2} \leq C (C_1 \xi)^2 s^{-3/2 + 2 \delta}.
\]

**Proof.** Notice the following decomposition:
\[
(4.6) \quad [\tilde{G}_i^{\alpha \beta} \partial_0 u_j, Z^I] w_j = [\tilde{G}_i^{\alpha \beta} \partial_0 u_j, Z^I] w_j + [G_i^{\alpha \beta} \partial_0 u_j, Z^I] v_j.
\]
The second term is decomposed as following:

\[ (4.7) \quad [v_i^{\alpha \beta} \partial_\alpha \partial_\beta, Z^I]v_j = \sum_{I_i + I_2 = I} Z^{I_i} G_i^{\alpha \beta} Z^{I_2} \partial_\alpha \partial_\beta v_j + G_i^{\alpha \beta}[\partial_\alpha \partial_\beta, Z^I]v_j \]

Recall that

\[ |Z^I G_i^{\alpha \beta}| \leq C(J)K \sum_{k, \gamma} |Z^I \partial_\gamma w_k|. \]

The \( L^2 \) norm of the first term in right-hand-side of (4.7) can be estimated as follows:

\[
\sum_{|I_1| \geq 1 \atop I_1 + I_2 = I} \left( \int_{H_s} \left| Z^{I_1} G_i^{\alpha \beta} Z^{I_2} \partial_\alpha \partial_\beta v_j \right|^2 dx \right)^{1/2}
\]

\[
\leq \sum_{|I_1| = 1 \atop I_1 + I_2 = I} \left( \int_{H_s} \left| Z^{I_1} G_i^{\alpha \beta} Z^{I_2} \partial_\alpha \partial_\beta v_j \right|^2 dx \right)^{1/2}
\]

\[
\leq \sum_{\alpha, \beta, |I| = 1 \atop I_1 + I_2 = I} \left( \int_{H_s} \left| CC_1 \hat{t}^{-1/2} s^{-1} \cdot Z^{I_1} \partial_\alpha \partial_\beta v_j \right|^2 dx \right)^{1/2}
\]

\[
+ \sum_{\alpha, \beta, |I| \geq 2 \atop I_1 + I_2 = I} \left( \int_{H_s} \left| K(s/t) Z^{I_1} \partial_\alpha \partial_\beta v_j \right|^2 dx \right)^{1/2}
\]

\[
\leq C(C_1 \varepsilon)^2 s^{-3/2 + 2 \delta}. \]

The second term in right-hand-side of (4.7) is estimated as follows. One notices that in the cone \( \Lambda' \)

\[
\left( \int_{H_s} \left| G_i^{\alpha \beta} [\partial_\alpha \partial_\beta, Z^I]v_j \right|^2 dx \right)^{1/2} \leq C \sum_{|I| \leq 1} |\partial_\alpha \partial_\beta, Z^I| v_j |\]

So

\[
\left( \int_{H_s} |G_i^{\alpha \beta} [\partial_\alpha \partial_\beta, Z^I]v_j|^2 dx \right)^{1/2} \leq C \sum_{|I| \leq 4} \left( \int_{H_s} \left| K \partial_\gamma v_j \partial_\alpha \partial_\beta , Z^I \right| v_j |^2 dx \right)^{1/2}
\]

\[
\leq KC(C_1 \varepsilon)^2 s^{-3/2 + 2 \delta}. \]

The first term in right hand side of (4.9) is decomposed as follows:

\[
[\hat{G}_i^{\alpha \beta} \partial_\alpha \partial_\beta, Z^I]u_j = [A_i^{\alpha \gamma} \partial_\gamma u_k \partial_\alpha \partial_\beta, Z^I]u_j + [B_i^{\alpha \beta} v_k \partial_\alpha \partial_\beta, Z^I]u_j + [C_i^{\alpha \beta} \partial_\alpha \partial_\beta u_k \partial_\beta, Z^I]u_j
\]

The last two terms are finite linear combinations of \( Z^{I_1} v_j Z^{I_2} \partial_\alpha \partial_\beta u_k \) and \( Z^{I_1} \partial_\gamma v_j Z^{I_2} \partial_\alpha \partial_\beta u_k \) with \( I_1 + I_2 = I \) and \( |I_1| \geq 1 \). As in (4.8), their \( L^2 \) norms on \( H_s \) can be estimated by \( C(C_1 \varepsilon)^2 s^{-3/2 + 2 \delta} \).

The first term can be written under one-frame:

\[
(4.9) \quad [A_i^{\alpha \gamma} \partial_\gamma u_k \partial_\alpha \partial_\beta, Z^I]u_j = [A_i^{\alpha \gamma \beta} \partial_\gamma u_k \partial_\alpha \partial_\beta, Z^I]u_j + [A_i^{\alpha \gamma} \partial_\alpha \partial_\beta u_k \partial_\beta, Z^I]u_j + [A_i^{\alpha \gamma} \partial_\alpha \partial_\beta u_k \partial_\gamma, Z^I]u_j
\]

Recall that by null conditions (3.2) and lemma 2.8 one has

\[
|Z^I \hat{\Delta}^{000 \gamma \beta}| \leq C(I)(s/t)^2.
\]

The first term can be controlled as follows:

\[
|\hat{A}^{\alpha \gamma \beta} \partial_\gamma u_k \partial_\alpha \partial_\beta, Z^I| u_j
\]

\[
\leq |\hat{A}^{000 \gamma \beta} \partial_\gamma u_k \partial_\alpha \partial_\beta, Z^I| u_j + [\hat{A}^{\alpha \gamma \beta} \partial_\gamma u_k \partial_\alpha \partial_\beta, Z^I]u_j + [\hat{A}^{\alpha \gamma \beta} \partial_\gamma u_k \partial_\alpha \partial_\beta, Z^I]u_j.
\]
Then as in (4.8), its $L^2$ norm on $H_s$ is controlled by $C(C_1\varepsilon)^2s^{-3/2+2\delta}$.

To estimate the $L^2$ norm on $H_s$ of the last term in right-hand-side of (4.9), one recalls that from definition, $|Z^{l_i} \partial_\alpha \Phi^\beta_{\alpha s}| \leq Ct^{-1}$. Taking this into consideration and run the same method of (4.8) one sees that its $L^2$ norm on $H_s$ is also controlled by $C(C_1\varepsilon)^2s^{-5/2+\delta}$. Then finally the lemma is proved when taking the assumption $\delta < 1/6$ into account.

Lemma 4.3. Let $\{w_1\}$ be regular solution of (3.1). Suppose that (4.5), (4.2) and (4.2) hold. Let $A_4$ be any of the following terms:

$$v_k v_j, \quad v_k \partial_\alpha w_j, \quad \partial_\alpha w_j \partial_\beta w_k.$$  

Then for any $|I^*| \leq 4$

$$\left( \int_{H_s} |Z^{l^*}_{I^*} A_4|^2 dx \right)^{1/2} \leq C(C_1\varepsilon)^2s^{-1+\delta}.$$  

Especially, for any $|I^*| \leq 4$

$$\left( \int_{H_s} |Z^{l^*}_{I^*} F_1|^2 dx \right)^{1/2} \leq C(C_1\varepsilon)^2s^{-1+\delta}.$$  

Proof. The $L^2$ estimates of the terms $A_4$ are nearly the same of those in the proof of lemma 4.1.

One will only prove the case where $A_4 = \partial_\alpha w_j \partial_\beta w_k$. To estimate the term consulting $\partial_\alpha w_j \partial_\beta u_k$.

\[
\left( \int_{H_s} |Z^{l^*_{I^*}} (\partial_\alpha v_j \partial_\beta u_k)|^2 dx \right)^{1/2} \leq \sum_{|l|_1 \leq 2 \atop t_1^i + t_2^i = 1^*} \left( \int_{H_s} |Z^{l^*_{I^*}} \partial_\alpha v_j Z^{l_1^*} \partial_\beta u_k|^2 dx \right)^{1/2} + \sum_{|l|_1 \leq 1 \atop t_1^i + t_2^i = 1^*} \left( \int_{H_s} |Z^{l^*_{I^*}} \partial_\alpha v_j Z^{l_1^*} \partial_\beta u_k|^2 dx \right)^{1/2} \leq C C_1 \varepsilon \sum_{|l|_1 \leq 2 \atop t_1^i + t_2^i = 1^*} \left( \int_{H_s} |t^{-3/2}s^{\delta}(t/s)|^2 \cdot |(s/t)Z^{l_1^*} \partial_\beta u_k|^2 dx \right)^{1/2} + C C_1 \varepsilon \sum_{|l|_1 \leq 1 \atop t_1^i + t_2^i = 1^*} \left( \int_{H_s} |(s/t)Z^{l_1^*} \partial_\alpha v_j|^2 (t/s)^2 \cdot |t^{-1/2}s^{-1}|^2 dx \right)^{1/2} \leq C(C_1\varepsilon)^2(s^{-3/2+2\delta} + s^{-1+\delta}).
\]

The terms $\partial_\alpha u_j \partial_\beta u_k$ are estimated as follows:
\[
\left( \int_{H_s} |Z^{I^*} (\partial_\alpha w_j \partial_\beta v_k)|^2 \, dx \right)^{1/2} \\
\leq \sum_{1 \leq |I^*| \leq 1} \left( \int_{H_s} |Z^{I^*_1} \partial_\alpha w_j Z^{I^*_2} \partial_\beta w_k|^2 \, dx \right)^{1/2} + \sum_{1 \leq |I^*| \leq 1} \left( \int_{H_s} |Z^{I^*_1} \partial_\alpha v_j Z^{I^*_2} \partial_\beta v_k|^2 \, dx \right)^{1/2} \\
+ \sum_{1 \leq |I^*| \leq 2} \left( \int_{H_s} |Z^{I^*_1} \partial_\alpha w_j Z^{I^*_2} \partial_\beta v_k|^2 \, dx \right)^{1/2}
\]

\[
\leq \sum_{1 \leq |I^*| \leq 1} \left( \int_{H_s} |CC \varepsilon t^{-1/2} s^{-1} |^2 \cdot (t/s)^2 |(s/t) Z^{I^*_2} \partial_\beta v_k|^2 \, dx \right)^{1/2} \\
+ \sum_{1 \leq |I^*| \leq 1} \left( \int_{H_s} |(s/t) Z^{I^*_1} \partial_\alpha w_j|^2 (t/s)^2 \cdot |CC \varepsilon t^{-1/2} s^{-1}|^2 \, dx \right)^{1/2} \\
+ \sum_{1 \leq |I^*| \leq 2} \left( \int_{H_s} |(s/t) Z^{I^*_1} \partial_\alpha w_j|^2 (t/s)^2 \cdot |CC \varepsilon t^{-1/2} s^{-1}|^2 \, dx \right)^{1/2}
\]

\[
\leq C(C_1 \varepsilon)^2 s^{-1/2} \leq C(C_1 \varepsilon)^2 s^{-1/2}.
\]

The terms \(\partial_\alpha v_j \partial_\beta v_k\) are estimated as follows:

\[
\left( \int_{H_s} |Z^{I^*} (\partial_\alpha v_j \partial_\beta v_k)|^2 \, dx \right)^{1/2} \\
\leq \sum_{1 \leq |I^*| \leq 2} \left( \int_{H_s} |Z^{I^*_1} \partial_\alpha v_j Z^{I^*_2} \partial_\beta v_k|^2 \, dx \right)^{1/2} + \sum_{1 \leq |I^*| \leq 1} \left( \int_{H_s} |Z^{I^*_1} \partial_\alpha v_j Z^{I^*_2} \partial_\beta v_k|^2 \, dx \right)^{1/2} \\
\leq \sum_{1 \leq |I^*| \leq 2} \left( \int_{H_s} |CC \varepsilon t^{-3/2} s^{-3/2} |^2 \cdot (t/s)^2 |(s/t) Z^{I^*_2} \partial_\beta v_k|^2 \, dx \right)^{1/2} \\
+ \sum_{1 \leq |I^*| \leq 2} \left( \int_{H_s} |(s/t) Z^{I^*_1} \partial_\alpha v_j|^2 (t/s)^2 \cdot |CC \varepsilon t^{-3/2} s^{-3/2}|^2 \, dx \right)^{1/2}
\]

\[
\leq C(C_1 \varepsilon)^2 s^{-3/2} \leq C(C_1 \varepsilon)^2 s^{-1/2}.
\]

One observes that \(F_i\) is a finite linear combination of \(\partial_\alpha w_j \partial_\beta v_k, v_j \partial_\alpha w_k\) and \(v_j v_k\). By the estimates just proved, the last estimate on \(Z^I F_i\) is trivial.

**Lemma 4.4.** Let \(\{w_j\}\) be regular solution of (3.1). Suppose that (4.2), (4.3) and (4.4) hold. Then for any \(|I^*| \leq 4\),

\[
\left( \int_{H_s} |\tilde{G}_{i}^{j} \partial_\alpha \partial_\beta, Z^{I^*} w_j|^2 \, dx \right)^{1/2} \leq C(C_1 \varepsilon)^2 s^{-1/2}.
\]

**Proof.** The proof is exactly the same to that of lemma 4.2. The only thing one should pay attention to is that when \(|I^*| = 3\) the decay estimates and \(L^2\) estimates on \(Z^{I^*} \partial_\alpha \partial_\beta v_j\) provided by (4.5) and (4.6) is not as good as in the case where \(|I^*| \leq 2\) which is the case in the proof of lemma 4.2. So here one has only a decay rate as \(s^{-1/2}\).
4.4 Part three – Energy and decay estimates of “bad” derivatives

In this part one will give the energy and decay estimates of “bad” second-order derivatives, which are the terms $\partial_i \partial_\nu Z^I u_j$. The following result is an expression of $\partial_i \partial_\nu Z^I u_j$ given by other “good” terms, which is an algebraic transform of (3.1).

Lemma 4.5. Let $\{w_i\}$ be solution of (3.1), then for any multi-index $I$ the following identity holds

\[
(s/t)^2 \partial I \partial_0 Z^I u_i + (t/s)^2 u_k \partial (t/s)^2 \partial_0 Z^I u_j \\
= Z^I F_i - Z^I \left( \tilde{G}^{\alpha\beta} \partial_\nu w_j \right) + \left[ B_i^{00k} u_0 \partial_0 Z^I u_j \right] - Z^I \left( B_i^{0\alpha\beta} u_0 \partial_\nu u_j \right) + \left[ B_i^{\alpha\beta} u_0 w_j \right] - \left( m^{\alpha\beta} \partial_\nu Z^I u_j \right) + m^{\alpha\beta} \partial_\nu \partial_\beta Z^I u_j \\
= \mathcal{R}_I.
\]

(4.10)

Furthermore, there exists a universal constant $C^*$ such that when $|u_i| \leq K^{-1} C^* \leq 1$, the following estimate holds

\[
|\partial I \partial_0 Z^I u_i| \leq C \max_i |\mathcal{R}_i|,
\]

where $C$ is a universal constant

Proof. One can write (3.1) under the following form:

\[
\Box u_i + B_i^{\alpha\beta} \partial_\alpha \partial_\beta u_j = F_i - \tilde{G}^{\alpha\beta} \partial_\nu w_j.
\]

Then write the term $B_i^{\alpha\beta} \partial_\alpha \partial_\beta u_j$ under one frame, by (2.21),

\[
\Box u_i + B_i^{\alpha\beta} \partial_\alpha \partial_\beta u_j = F_i - \tilde{G}^{\alpha\beta} \partial_\nu w_j + B_i^{\alpha\beta} u_k \partial_\alpha \Phi^\beta_\nu \partial_\nu w_j.
\]

Then derive it with respect to an arbitrary product $Z^I$:

\[
\Box Z^I u_i + B_i^{\alpha\beta} u_k \partial_\alpha \partial_\beta Z^I u_j = - Z^I \left( B_i^{\alpha\beta} \partial_\alpha \partial_\beta Z^I u_j \right) + Z^I F_i - Z^I \left( \tilde{G}^{\alpha\beta} \partial_\nu w_j \right) + Z^I \left( B_i^{\alpha\beta} u_k \partial_\alpha \Phi^\beta_\nu \partial_\nu w_j \right) + \left[ B_i^{00k} u_0 \partial_0 Z^I u_j \right].
\]

By (2.22), one gets (4.10).

Consider the linear algebraic equations given by (4.10),

\[
(s/t)^2 \partial I \partial_0 Z^I u_i + (t/s)^2 \partial I \partial_0 Z^I u_j = \mathcal{R}_i.
\]

By lemma 2.3 and 4.5,

\[
|\partial I \partial_0 Z^I u_i| \leq C K \max_k |u_k|,
\]

where $C$ is a universal constant. When $CK \max_k |u_k| \leq 1/2$, that is max_k |u_k| \leq (2KC)^{-1}, by basic linear algebra, one has the following estimates:

\[
|\partial I \partial_0 Z^I u_i| \leq C \max_i |\mathcal{R}_i|,
\]

where $C^*$ is also a universal constant. 

\[
\square
\]
Remark 4.6. In the expression of $R$, the first term is a linear term while the rest are quadratic terms. The linear part are composed by the “good” second-order derivatives so that one can deduce from here a better decay of $(s/t)^2\partial_u\partial_v Z'u$.

Now one turns to the decay estimate of $\partial_u\partial_v Z'u$.

Lemma 4.7. Let $\{w_i\}$ a regular solution of (3.1). Suppose that (4.5) and (4.2) hold with $C_1\varepsilon \leq 1$. Then for any $|J| \leq 1$, 

$$|(s/t)^2\partial_u\partial_v Z^j u_j| \leq C(K+1)C_1\varepsilon t^{-3/2}s^{-1+2^6},$$

and

$$|(s/t)^2Z^j_0 \partial_u u_j| \leq C(K+1)C_1\varepsilon t^{-3/2}s^{-1+2^6}.$$ 

Proof. Take the notation of in lemma 4.7 The proof is mainly a $L^\infty$ estimate of $R_i$.

One notices that $Z^j F_i$ is a finite linear combination of $\partial w_i \partial w_j$, $v_k \partial w_j$ and $v_j v_j$. By (4.5), one easily gets

$$|Z^j F_i| \leq C(C_1\varepsilon)^2(t^{-1}s^{-2+2^6} + t^{-2}s^{-1+2^6} + t^{-3}s^{2^6}) \leq C(C_1\varepsilon)^2t^{-1}s^{-2+2^6}.$$ 

Similarly, $\partial^i\partial^jw_j$ is a finite linear combination of $v_k\partial_{\alpha\beta}w_j$ and $\partial_j w_j \partial_{ij}w_k$. By (4.5)

$$|Z^j (\partial^i\partial^j w_j)| \leq C(K(C_1\varepsilon)^2(t^{-2}s^{-2+2^6} + t^{-1}s^{-2+2^6}) \leq C(K(C_1\varepsilon)^2t^{-1}s^{-2+2^6}.$$ 

By lemma 2.8 $|Z^j \partial^i \partial^j \partial_{\alpha\beta}| \leq C$. Then by the last inequality of (4.5), one has,

$$|Z^j (\partial^i \partial^j \partial_{\alpha\beta})u_j| + |Z^j (\partial^i \partial^j \partial_{\alpha\beta})u_j| \leq C(K(C_1\varepsilon)^2t^{-3}s^{2^5}.$$ 

Similarly,

$$|Z^j \partial_{\alpha\beta} Z^i u_j| + |Z^j \partial_{\alpha\beta} Z^i u_j| \leq C(K(C_1\varepsilon)^2t^{-3}s^{2^5}.$$ 

One also notices that $|Z^j (\partial_{\alpha\beta} (\Phi^\gamma))| \leq Ct^{-1}$. Then

$$|Z^j (\partial_{\alpha\beta} (\Phi^\gamma) \partial_{\gamma\delta} Z^j u_j)| \leq C(K(C_1\varepsilon)^2t^{-3}s^{2^5}.$$ 

Similarly,

$$|Z^j (\partial_{\alpha\beta} (\Phi^\gamma) \partial_{\gamma\delta} Z^j u_j)| \leq C(K(C_1\varepsilon)^2t^{-3}s^{2^5}.$$ 

Now one will consider the term $|Z^j_{\alpha\beta} u_k, Z^j|u_j$. In general one has the following decomposition,

$$|Z^j_{\alpha\beta} u_k, Z^j|u_j = \sum_{\substack{J_1 = J_2 = J \lor J_2 \subseteq J \lor J_2 \velopment J_1 \subseteq J}} |Z^j_{\alpha\beta} u_k, Z^j|u_j.$$ 

By (4.5) and lemma 2.8

$$\sum_{\substack{J_1 + J_2 = J \lor J_2 \subseteq J \lor J_2 \?option{= J} \subseteq J \subseteq J}} |Z^j_{\alpha\beta} u_k, Z^j|u_j| \leq (Ks^2t^{-2}) \cdot (CC_1\varepsilon t^{-3/2}s) \cdot \sum_{\substack{J_1 \subseteq J \subseteq J \subseteq J \subseteq J}} |\partial_{\alpha\beta} Z^j u_k|.$$ 

Similarly by (2.A), the second term can be bounded as follows

$$|Z^j_{\alpha\beta} u_k, Z^j|u_j| \leq (Ks^2t^{-2}) \cdot (CC_1\varepsilon t^{-3/2}s) \cdot \sum_{\substack{J_1 \subseteq J \subseteq J \subseteq J}} |\partial_{\alpha\beta} Z^j u_k|.$$ 

Notice that in the cone $\Lambda'$

$$|\partial_{\alpha\beta} u_1| \leq \sum_{\alpha, \beta} |\partial_{\alpha\beta} u_1|.$$ 

(4.12)
by (4.5), one has
\[
|L^{106}_{t_k} u_k, Z^I u_j| \leq CK(C_1 \varepsilon)^2 t^{-3/2} s^{-1+2\delta} + CKC_1 \varepsilon t^{-1/2} \sum_{|J| \leq |I|-1} |\partial_{u_k} Z^I u_k|.
\]

One concludes by
\[
|(s/t)^2 \partial_{u_k} Z^I u_i| \leq C(K + 1)C_1 \varepsilon t^{-3/2} s^{-1+2\delta} + CKC_1 \varepsilon t^{-1/2} \sum_{|J| \leq |I|-1} |\partial_{u_k} Z^I u_k|,
\]
when \(|J| = 0\), the last term in right-hand-side disappears. Then by induction, the first inequality is proved.

For the second inequality, one observes that it is a trivial result of the first inequality, (4.12) and (2.17).

At the end of this section, one will give the \(L^2\) estimates of the “bad” derivatives.

**Lemma 4.8.** Let \(\{w_i\}\) be solution of (3.1) and suppose that (3.2), (3.5) hold. Then for any \(|I| \leq 2\) the following estimates hold with \(C_1 \varepsilon \leq 1\).
\[
\left( \int_{H_s} |s^{3}t^{-2} \partial_{u_k} Z^I u_i|^{2} dx \right)^{1/2} \leq C(K + 1)C_1 \varepsilon,
\]
for any \(|I^*| \leq 3\)
\[
\left( \int_{H_s} |s^{3}t^{-2} \partial_{u_k} Z^{I^*} u_i|^{2} dx \right)^{1/2} \leq C(K + 1)C_1 \varepsilon s^{3}\delta.
\]

**Proof.** Similar to that of lemma 4.7, the proof is mainly a \(L^2\) estimate of \(\mathcal{R}_1\). Lemma 4.1 (lemma 4.3) gives the \(L^2\) estimate of \(Z^{I} F_i\) (respectively \(Z^{I^*} F_i\)).

\(\tilde{G}^{\alpha \beta} \partial_{\alpha \beta} w_j\) can be decomposed as follows:
\[
\tilde{G}^{\alpha \beta} \partial_{\alpha \beta} w_j = A_i^{\alpha \beta} \partial_{\alpha \beta} w_j + A_i^{\alpha \beta} v_k \partial_{\alpha \beta} w_j + B_i^{\alpha \beta} v_k \partial_{\alpha \beta} w_j + C_i^{\alpha \beta} \partial_{\alpha \beta} v_j.
\]
The last three terms are finite linear combinations of \(v_j \partial_{\alpha \beta} w_j, \partial_{\alpha \omega} \partial_{\beta \gamma} v_j\) and \(\partial_{\alpha \omega} \partial_{\beta \gamma} v_j\).

When \(|I^*| \leq 3\),
\[
\left( \int_{H_s} |Z^{I^*} (v_j \partial_{\alpha \beta} w_j)|^{2} dx \right)^{1/2} \leq C \int_{H_s} |Z^{I^*} \partial_{\alpha \beta} w_j|^{2} dx^{1/2} + C \int_{H_s} |Z^{I^*} v_j \partial_{\alpha \beta} w_j|^{2} dx^{1/2} \leq C \int_{H_s} |Z^{I^*} v_j|^{2} dx^{1/2} \leq C \int_{H_s} |Z^{I^*} v_j|^{2} dx^{1/2} \leq C \int_{H_s} |Z^{I^*} v_j|^{2} dx^{1/2} \leq C(K_1 C_1 \varepsilon)^2 s^{-3/2]}
\]

Similarly,
\[
\left( \int_{H_s} |Z^I (\partial_{\alpha \omega} \partial_{\beta \gamma} v_j)|^{2} dx \right)^{1/2} \leq C(K_1 C_1 \varepsilon)^2 s^{-3/2].
\]

Of course, when \(|I| \leq 2\),
\[
\left( \int_{H_s} |Z^I (v_j \partial_{\alpha \beta} w_j)|^{2} dx \right)^{1/2} \leq C(K_1 C_1 \varepsilon)^2 s^{-3/2+2\delta}.
\]
The term $A^i_\alpha \partial_w u_j$ can be written under one-frame:

$$A^i_\alpha \partial_w u_j = A^i_\alpha \partial_w u_j - A^i_\alpha \partial_w u_j.$$ 

Recall that for any multi-index $J$, $|Z^J \partial_w \Phi_{\beta'}| \leq C(J)t^{-1}$ and $|Z^J \partial_w \Phi_{\beta'}| \leq C(J)K$. One has for $|I| \leq 3$,

$$\left( \int_{H_s} |Z^I (A^i_\alpha \partial_w u_j | \partial_w \Phi_{\beta'})^2 dx \right)^{1/2} \leq CK(C_1\varepsilon)^2 s^{-3/2}.$$

Take the null conditions (3.2) into consideration, for any multi-index $I$,

$$|Z^I (A^i_\alpha \partial_w u_j | \partial_w \Phi_{\beta'}) | \leq C(I)(s/t)^2.$$

Then,

$$|Z^I (A^i_\alpha \partial_w u_j | \partial_w \Phi_{\beta'}) | \leq \sum_{l_1, l_2, l_3} |Z^{I_1} A^i_\alpha \partial_w u_j | |Z^{I_2} \partial_w u_j | |Z^{I_3} \partial_w u_j | + \sum_{l_1, l_2, l_3} |Z^{I_1} A^i_\alpha \partial_w u_j | |Z^{I_2} \partial_w u_j | |Z^{I_3} \partial_w u_j | \leq CK \sum_{l_1, l_2, l_3} (s/t)^2 |Z^{I_1} \partial_w u_j | |Z^{I_2} \partial_w u_j | + CK \sum_{l_1, l_2, l_3} |Z^{I_1} \partial_w u_j | |Z^{I_2} \partial_w u_j | \leq : M_0 + M_1 + M_2 + M_3.$$

By the same argument of (4.14), one has

$$\sum_{k=0}^3 \left( \int_{H_s} |M_k|^2 dx \right)^{1/2} \leq C(C_1\varepsilon)^2 s^{-3/2}, \quad \text{when } |I| \leq 2,$$

and

$$\sum_{k=0}^3 \left( \int_{H_s} |M_k|^2 dx \right)^{1/2} \leq C(C_1\varepsilon)^2 s^{-1+\delta}, \quad \text{when } |I| \leq 3.$$

So one concludes by

$$\left( \int_{H_s} |Z^I (\tilde{G}_{i_1}^{i_\alpha \beta} \partial_w u_j | \partial_w \Phi_{\beta'})^2 dx \right)^{1/2} \leq KC(C_1\varepsilon)^2 s^{-1+\delta}, \quad \text{for } |I| \leq 3$$

and

$$\left( \int_{H_s} |Z^I (\tilde{G}_{i_1}^{i_\alpha \beta} \partial_w u_j | \partial_w \Phi_{\beta'})^2 dx \right)^{1/2} \leq KC(C_1\varepsilon)^2 s^{-3/2+2\delta}, \quad \text{for } |I| \leq 2.$$

Recall that $|Z^J \partial_w \Phi_{\beta'}| \leq Ct^{-1}$ and $|Z^J \partial_w \Phi_{\beta'}| \leq K$. One has for $|I^*| \leq 3$,

$$\left( \int_{H_s} |Z^I (\tilde{G}_{i_1}^{i_\alpha \beta} \partial_w u_j | \partial_w \Phi_{\beta'})^2 dx \right)^{1/2} \leq C(C_1\varepsilon)^2 s^{-3/2}.$$
Similarly, for $|I^*| \leq 3$,

$$
\left( \int_{H_*} \left| m^{\alpha \beta} (\partial_{\alpha} \Phi^\nu) \partial_{\beta} Z^{I^*} u_i \right|^2 \, dx \right) \leq CC_1 \delta s^{-1+\delta}
$$

and for $|I| \leq 2$

$$
\left( \int_{H_*} \left| m^{\alpha \beta} (\partial_{\alpha} \Phi^\nu) \partial_{\beta} Z^I u_i \right|^2 \, dx \right) \leq CC_1 \delta s^{-1}.
$$

Now one turns to the term $[B_{100}^{100} u_k \partial_{\alpha} Z^I] u_j$. Recall the following decomposition

$$
(4.14) \quad [B_{100}^{100} u_k \partial_{\alpha} Z^I] u_j = \sum_{t_1+t_2=t \atop |t_2| \leq |I|-1} Z^{t_1} (B_{100}^{100} u_k) Z^{t_2} \partial_{\alpha} Z^I u_j + B_{100}^{100} u_k [\partial_{\alpha} Z^I] u_j
$$

Notice that

$$
|[\partial_{\alpha} Z^I] u_j| \leq C \sum_{\alpha \beta} |\partial_{\alpha \beta} Z^I u_j| \leq C \sum_{|\alpha| \leq |I|-1} |\partial_{\alpha} Z^I u_j| + C \sum_{|\alpha| \leq |I|-1} |\partial_{\alpha} Z^I u_j|.
$$

The first term in right-hand-side of (4.14) is estimated as follows: when $|I| \leq 3$,

$$
\sum_{t_1+t_2=t \atop |t_2| \leq |I|-1} \left( \int_{H_*} |Z^{t_1} (B_{100}^{100} u_k) Z^{t_2} \partial_{\alpha} Z^I u_j|^2 \, dx \right)^{1/2}
\leq \sum_{t_1+t_2=t \atop |t_2| \leq |I|-1} \left( \int_{H_*} |Z^{t_1} (B_{100}^{100} u_k) \partial_{\alpha} Z^{t_2} u_j|^2 \, dx \right)^{1/2}
\quad + \sum_{t_1+t_2=t \atop |t_2| \leq |I|-1 \atop \alpha \beta} \left( \int_{H_*} |Z^{t_1} (B_{100}^{100} u_k) \partial_{\alpha \beta} Z^{t_2} u_j|^2 \, dx \right)^{1/2}
$$

For the first term:

$$
\sum_{t_1+t_2=t \atop |t_2| \leq |I|-1} \left( \int_{H_*} |Z^{t_1} (B_{100}^{100} u_k) \partial_{\alpha} Z^{t_2} u_j|^2 \, dx \right)^{1/2}
\leq \sum_{t_1+t_2=t \atop |t_2| \leq |I|-1} + \sum_{t_1+t_2=t \atop |t_2| \leq |I|-1} \left( \int_{H_*} |Z^{t_1} (B_{100}^{100} u_k) \partial_{\alpha} Z^{t_2} u_j|^2 \, dx \right)^{1/2}
\leq \sum_{t_1+t_2=t \atop |t_2| \leq |I|-1} \left( \int_{H_*} C K C_1 t^{-3/2} \cdot s(s/t)^2 \partial_{\alpha} Z^{t_2} u_j|^2 \, dx \right)^{1/2}
\quad + \sum_{t_1+t_2=t \atop |t_2| \leq |I|-1} \left( \int_{H_*} C K C_1 t^{-3/2} \cdot s^{-1+2\delta} Z^{t_1} u_k|^2 \, dx \right)^{1/2}
\leq C K (C_1 \delta)^2 s^{-3/2+2\delta} + C K C_1 \delta s^{-3/2} \sum_{|\alpha| \leq |I|-1} \left( \int_{H_*} s^3 t^{-2} \partial_{\alpha} Z^I u_j \right)^{1/2}.
$$
The second term in right-hand-side of (4.14) is estimated as follows: when $|I| \leq 3$, 
\[
\left( \int_{H_s} \left| \frac{\partial}{\partial t} w_{i}^{000} Z^I u_j \right|^2 dx \right)^{1/2} 
\leq CK C \varepsilon \sum_{|J| \leq |I| - 1} \left( \int_{H_s} t^{-3} |s \partial_0 Z^I u_j|^2 dx \right)^{1/2} + CK C \varepsilon s^{-3/2} \sum_{|J| \leq |I| - 1} \left( \int_{H_s} |s^3 t^{-2} \partial_0 Z^I u_k|^2 dx \right)^{1/2}
\]
\[
\leq CK C \varepsilon s^{-3/2} \sum_{|J| \leq |I| - 1} E(s, Z^I u_j)^{1/2} + CK C \varepsilon s^{-3/2} \sum_{|J| \leq |I| - 1} \left( \int_{H_s} |s^3 t^{-2} \partial_0 Z^I u_k|^2 dx \right)^{1/2}
\]
\[
\leq CK C \varepsilon^2 s^{-3/2 + \delta} + CK C \varepsilon s^{-3/2} \sum_{|J| \leq |I| - 1} \left( \int_{H_s} |s^3 t^{-2} \partial_0 Z^I u_k|^2 dx \right)^{1/2}
\]

To estimate the only linear terms, 
\[
\frac{\partial}{\partial t} w_{i}^{000} Z^I u + \frac{\partial}{\partial t} w_{i}^{000} Z^I u + \frac{\partial}{\partial t} w_{i}^{000} Z^I u_i
\]
one notice that they are “good derivatives”. By using directly the last inequality of (4.5), one has when $|I| \leq 2$, 
\[
\left( \int_{H_s} \left| \frac{\partial}{\partial t} w_{i}^{000} Z^I u_j \right|^2 dx \right)^{1/2} \leq CC \varepsilon s^{-1}.
\]

When $|I^*| \leq 3$, 
\[
\left( \int_{H_s} \left| \frac{\partial}{\partial t} w_{i}^{000} Z^I u_j \right|^2 dx \right)^{1/2} \leq CC \varepsilon s^{-1 + \delta}.
\]

So finally one gets, when $|I| \leq 2$ 
\[
\left( \int_{H_s} \left| s^3 t^{-2} \partial_0 Z^I u_k \right|^2 dx \right)^{1/2}
\]
\[
\leq \max_J ||w_j||_{L^2(H_s)}
\]
\[
\leq CK C \varepsilon^2 s^{-3/2 + 2\delta} + CC \varepsilon s^{-1} + CK C \varepsilon s^{-3/2} \sum_{|J| \leq |I| - 1} \left( \int_{H_s} |s^3 t^{-2} \partial_0 u_k|^2 dx \right)^{1/2}.
\]

By induction, one gets the desired result. The case where $|I^*| \leq 3$ can be proved similarly, one omits the details. \hfill \Box

4.5 Part four – Estimates of other source terms

Lemma 4.9. Let $\{w_i\}$ be regular solution of (5.1). Suppose that (4.2) and (4.3), then for any $|I^*| \leq 4$, the following estimate holds:
\[
\left( \int_{H_s} \left| \frac{\partial}{\partial t} w_{i}^{000} \frac{\partial}{\partial x} w_{i}^{000} Z^I u_j \right|^2 dx \right)^{1/2} \leq C(C \varepsilon^2) K s^{-1 + \delta}.
\]

For any $|I| \leq 3$ the following estimate holds:
\[
\left( \int_{H_s} \left| \frac{\partial}{\partial t} w_{i}^{000} \frac{\partial}{\partial x} w_{i}^{000} Z^I u_j \right|^2 dx \right)^{1/2} \leq C(C \varepsilon^2) K s^{-3/2 + 2\delta}.
\]
Proof. By definition,
\[ [\gamma_i^{\alpha\beta}\partial_\alpha\partial_\beta, Z^I] w_j = [\hat{G}_i^{\alpha\beta}\partial_\alpha\partial_\beta, Z^I] w_j + [B_i^{\alpha\beta}\partial_\alpha\partial_\beta, Z^I] w_j \]
The first component is controlled by lemma 4.4. The second term is estimated similarly. First one rewrite it under one-frame.

(4.15) \[ [B_i^{\alpha\beta}\xi_k \partial_\alpha\partial_\beta, Z^I] w_j = [B_i^{\alpha\beta}\xi_k \partial_\alpha\partial_\beta, Z^I] w_j + [B_i^{\alpha\beta}\xi_k \partial_\alpha, \Phi^\beta, Z^I] w_j \]

One recalls that
\[ \left| Z^I \partial_\alpha \Phi^\beta \right| \leq C t^{-1}. \]

With this additional decay, as in the proof of lemma 4.2, the \( L^2 \) norm of the second term in right-hand-side of (4.15) on \( H_s \) is bounded by \( C(C_1 \varepsilon)^{2} s^{-3/2+\delta} \).

One notices that \( B_i^{\alpha\beta}\xi_k \xi_\beta \) is a null form so following the lemma 4.8 one has
\[ \left| Z^I B_i^{\alpha\beta}\xi_k \right| \leq C(I)(s/t)^2, \]

Then exactly as in the proof of lemma 4.2 and take the estimates given by lemma 4.4 and lemma 4.8 into consideration, the \( L^2 \) norm of the first term in right-hand-side of (4.15) on \( H_s \) is bounded by \( C(C_1 \varepsilon)^{2} s^{-3/2+\delta} \).

The proof of the second estimate is the same. One omits the details. \( \square \)

**Lemma 4.10.** Suppose (4.5) and (4.2) hold, then for any \( |I'| \leq 4 \) the following estimates is true:

(4.16) \[ \left| \int_{H_s} \left( (s/t)(\partial_\alpha C_i^{\alpha\beta}) \partial_\beta Z^I w_i \partial_\gamma Z^I w_j - \frac{1}{2}(\partial_\alpha C_i^{\alpha\beta}) \partial_\beta Z^I w_i \partial_\gamma Z^I w_j \right) dx \right| \]
\[ \leq C C_1 \varepsilon s^{-1+\delta} E_m(s, Z^I w_i)^{1/2}. \]

Proof. The proof is mainly a substitution of (4.5) and (4.2). One writes the estimate of
\[ \left| \int_{H_s} (s/t)(\partial_\alpha C_i^{\alpha\beta}) \partial_\beta Z^I w_i \partial_\gamma Z^I w_j dx \right| \]
in detail and omits the rest part. First one notices that
\[ \left| \partial_\alpha C_i^{\alpha\beta} \right| \leq C \sum_{j} \left| \partial_\alpha w_j \right| + C \sum_{j, \beta} \left| \partial_\alpha \partial_\beta w_j \right| \leq C(C_1 \varepsilon)^{2} s^{-1/2} s^{-1}. \]
Substitute this into the expression,
\[ \left| \int_{H_s} (s/t)(\partial_\alpha C_i^{\alpha\beta}) \partial_\beta Z^I w_i \partial_\gamma Z^I w_j dx \right| \]
\[ \leq \int_{H_s} (|s/t)| \partial_\alpha C_i^{\alpha\beta} |(s/t)| \partial_\beta Z^I w_i |(s/t)| \partial_\gamma Z^I w_j dx \]
\[ \leq \sum_{j, \beta} \int_{H_s} C(C_1 \varepsilon)^{t^{1/2} s^{-2}} |(s/t)| \partial_\alpha Z^I w_i |(s/t)| \partial_\beta Z^I w_j dx \]
\[ \leq C(C_1 \varepsilon)^{2} s^{-1} \sum_{j, \beta} \left( |(s/t)| \partial_\gamma Z^I w_i |(s/t)| \partial_\beta Z^I w_j \right) \]
\[ \leq C(C_1 \varepsilon)^{2} s^{-1+\delta} \left( \int_{H_s} |(s/t)| \partial_\gamma Z^I w_i |^2 dx \right)^{1/2} \left( \int_{H_s} |(s/t)| \partial_\beta Z^I w_j |^2 dx \right)^{1/2} \]
\[ \leq C(C_1 \varepsilon)^{2} s^{-1+\delta} E_m(s, Z^I w_i)^{1/2}. \]
\( \square \)
Lemma 4.11. Suppose (4.15) and (4.2) hold, then for any $|I| \leq 3$ the following estimates is true:

\[
\left| \int_{H_s} \left( (\partial_{\alpha} G_i^{\alpha \beta}) \partial_{\beta} Z^{l} u_i \partial_{\beta} Z^{l} w_j - \frac{1}{2} (\partial_{\alpha} G_i^{\alpha \gamma}) \partial_{\gamma} Z^{l} u_i \partial_{\gamma} Z^{l} w_j \right) dx \right| \\
\leq C C_{\varepsilon} s^{-3/2+2\varepsilon} E_m(s, Z^{l} u_i)^{1/2}.
\]

**Proof.** Again one will only write the estimate on $(\partial_{\alpha} G_i^{\alpha \beta}) \partial_{\beta} Z^{l} u_i \partial_{\beta} Z^{l} w_j$ in detail. By definition,

\[
(\partial_{\alpha} G_i^{\alpha \beta}) \partial_{\beta} Z^{l} u_i \partial_{\beta} Z^{l} w_j = (\partial_{\alpha} G_i^{\alpha \beta}) \partial_{\beta} Z^{l} u_i \partial_{\gamma} Z^{l} v_j + (\partial_{\alpha} G_i^{\alpha \beta}) \partial_{\beta} Z^{l} u_i \partial_{\gamma} Z^{l} u_j.
\]

The second term in right-hand-side is decomposed again as follows:

\[
\int_{H_s} (s/t) (\partial_{\alpha} G_i^{\alpha \beta}) \partial_{\beta} Z^{l} u_i \partial_{\beta} Z^{l} w_j dx
\]

\[
\leq \int_{H_s} C K C_{\varepsilon} t^{-1/2} s^{-1} (s/t) \partial_{\beta} Z^{l} u_i \partial_{\beta} Z^{l} w_j dx
\]

\[
\leq C K C_{\varepsilon} s^{-3/2} \sum_{\beta, j} \left( \int_{H_s} |(s/t) \partial_{\beta} Z^{l} u_i|^2 dx \right)^{1/2} \cdot \left( \int_{H_s} |\partial_{\beta} Z^{l} w_j|^2 dx \right)^{1/2}
\]

\[
\leq C K (s/t)^{2} s^{-3/2+2\varepsilon} \left( \int_{H_s} |(s/t) \partial_{\beta} Z^{l} u_i|^2 dx \right)^{1/2}
\]

\[
\leq C K (s/t)^{2} s^{-3/2+2\varepsilon} E_m(s, Z^{l} u_i)^{1/2}.
\]

To estimate the last term of (4.19), one needs the null condition (4.2). First, one writes this term under one-frame:

\[
\partial_{\alpha} (A_i^{\alpha \beta} \partial_{\beta} Z^{l} u_i + B_i^{\alpha \beta} u_k) \partial_{\beta} Z^{l} u_i \partial_{\beta} Z^{l} w_j
\]

\[
= \partial_{\alpha} A_i^{\alpha \beta} \partial_{\beta} Z^{l} u_i + B_i^{\alpha \beta} \partial_{\beta} Z^{l} u_i \partial_{\beta} Z^{l} w_j
\]

\[
+ \partial_{\alpha} (\Phi^{\alpha \beta}_\alpha \Phi^{\beta \gamma}_\beta) (A_i^{\alpha \beta} \partial_{\beta} Z^{l} u_i + B_i^{\alpha \beta} u_k) \partial_{\beta} Z^{l} u_i \partial_{\gamma} Z^{l} w_j.
\]

Taking into account the fact that $|\partial_{\gamma} (\Phi^{\alpha \beta}_\alpha \Phi^{\beta \gamma}_\beta)| \leq C t^{-1}$ as in the proof of lemma 4.2 one can easily prove that

\[
\left| \int_{H_s} (s/t) (\partial_{\alpha} (\Phi^{\alpha \beta}_\alpha \Phi^{\beta \gamma}_\beta) (A_i^{\alpha \beta} \partial_{\beta} Z^{l} u_i + B_i^{\alpha \beta} u_k) \partial_{\beta} Z^{l} u_i \partial_{\gamma} Z^{l} w_j dx \right|
\]

\[
\leq C (s/t)^{2} s^{-3/2} E_m(s, Z^{l} u_i)^{1/2}.
\]
Now the most difficult term, the first term in right-hand-side of (4.20) will be considered.

\[
\partial_t \left( A^{\alpha \beta \gamma \delta}_{\alpha \beta \gamma \delta} \partial_t u_k + B^{\alpha \beta \delta}_{\alpha \beta \delta} u_k \right) \cdot \partial_t Z^I u_i
\]

\[
= \partial_t \left( A^{\alpha \beta \gamma \delta}_{\alpha \beta \gamma \delta} \partial_t u_k \partial_t Z^I u_j + B^{\alpha \beta \delta}_{\alpha \beta \delta} u_k \partial_t Z^I u_j \right) \cdot \partial_t Z^I u_i
\]

\[
+ \partial_t \left( A^{\alpha \beta \gamma \delta}_{\alpha \beta \gamma \delta} \partial_t u_k \partial_t Z^I u_j + B^{\alpha \beta \delta}_{\alpha \beta \delta} u_k \partial_t Z^I u_j \right) \cdot \partial_t Z^I u_i
\]

\[
+ \partial_t \left( A^{\alpha \beta \gamma \delta}_{\alpha \beta \gamma \delta} \partial_t u_k \partial_t Z^I u_j + B^{\alpha \beta \delta}_{\alpha \beta \delta} u_k \partial_t Z^I u_j \right) \cdot \partial_t Z^I u_i
\]

\[
:= \partial_t \left( A^{\alpha \beta \gamma \delta}_{\alpha \beta \gamma \delta} \partial_t u_k \partial_t Z^I u_j + B^{\alpha \beta \delta}_{\alpha \beta \delta} u_k \partial_t Z^I u_j \right) \cdot \partial_t Z^I u_i + \mathcal{N} \cdot \partial_t Z^I u_i.
\]

Notice that \( \mathcal{N} \) is a linear combination of
\[
\Gamma_\alpha \partial_\beta u_j \partial_\gamma Z^I u_k \partial_\delta u_i, \quad \Gamma_\alpha \partial_\beta u_j \partial_\gamma Z^I u_k \partial_\delta u_i, \quad \Gamma_\alpha \partial_\beta u_j \partial_\gamma Z^I u_k \partial_\delta u_i, \quad \Gamma_\alpha \partial_\beta u_j \partial_\gamma Z^I u_k \partial_\delta u_i
\]

with \( \Gamma \) a function bounded by \( CK \). By (4.5) and (4.3), one sees easily that
\[
\left| \int_{H_T} (s/t) \mathcal{N} \cdot \partial_t Z^I u_i \, dx \right| \leq C (C_1 \epsilon)^2 s^{-3/2 + 2\delta} E_m(s, Z^I u_i)^{1/2}.
\]

Taking into account the null condition (3.2), one has by lemma 2.8
\[
|A^{\alpha \beta \gamma \delta}_{\alpha \beta \gamma \delta}| + |B^{\alpha \beta \delta}_{\alpha \beta \delta}| \leq C(s/t)^2.
\]

Then by lemma 4.7 and lemma 4.8 one can show that
\[
\left| \int_{H_T} (s/t) \partial_t \left( A^{\alpha \beta \gamma \delta}_{\alpha \beta \gamma \delta} \partial_t u_k \partial_t Z^I u_j + B^{\alpha \beta \delta}_{\alpha \beta \delta} u_k \partial_t Z^I u_j \right) \cdot \partial_t Z^I u_i \right| \leq C (C_1 \epsilon)^2 s^{-3/2 + 2\delta} E_m(s, Z^I u_i)^{1/2}.
\]

So finally the desired result is proved.

4.6 Last Part – Bootstrap argument

First one verifies (2.6) by the following lemma:

**Lemma 4.12.** Suppose (4.5) holds with \( KC_1 \epsilon \) small enough. Then following estimate holds

\[
\sum_i E_g(s, Z^I u_i) \leq 3 \sum_i E_m(s, Z^I u_i).
\]

**Proof.** One notice that

\[
\sum_{i,j,\alpha,\beta} |C^{\alpha \beta}_{\alpha \beta}| \leq CK \sum_i (|\partial w_i| + |w_i|).
\]

Then by simple calculation
\[
\sum_i |E_G(s, w_i) - E_m(s, w_i)| = 2 \int_{H_s} \left( \partial_t w_i \partial_3 w_j G_{i}^{j,0,3} \right) \cdot (1 - x^3/t) dx - \int_{H_s} \left( \partial_t w_i \partial_3 w_j G_{i}^{j,0,3} \right) dx \\
\leq 2 \int_{H_s} \left( \sum_{i,j,\alpha,\beta} |G_{i}^{j,0,3}| \right) \cdot \left( \sum_{\alpha,k} |\partial_\alpha w_k|^2 \right) dx \\
\leq 2CK \int_{H_s} \sum_i (|\partial w_i| + |w_i|) \cdot \left( \sum_{\alpha,k} |\partial_\alpha w_k|^2 \right) dx \\
\leq 2CKC_1 \varepsilon \int_{H_s} \left( t^{-3/2}s^3 + t^{-1/2}s^{-1} + t^{-3/2} \right) (s/t)^2 \cdot \left( \sum_{\alpha,k} |(s/t)\partial_\alpha w_k|^2 \right) dx \\
= 2CKC_1 \varepsilon \int_{H_s} \left( t^{1/2}s^{-2+2\delta} + t^{3/2}s^{-3} + t^{1/2}s^{-1} \right) \cdot \left( \sum_{\alpha,k} |(s/t)\partial_\alpha w_k|^2 \right) dx \\
\leq CKC_1 \varepsilon \sum_i E_m(s, w_i).
\]

Here one takes \( CKC_1 \varepsilon \leq 2/3 \) with \( C \) a universal constant, then the lemma is proved. \( \square \)

With all those preparations above, one is now ready to prove the main theorem.

**Proof of theorem 4.1.** Suppose that \( \{w_i\} \) is the unique regular local-in-time solution of (3.1). Let \( C_1, \varepsilon \) be positive constants. Suppose that \( [B + 1, T^*] \) is the largest interval containing \( B + 1 \) on which (4.1) holds for any \( B + 1 \leq s \leq T^* \). As one discussed in section 4.1, there always exists an \( \varepsilon' \) such that, when \( E^*_\varepsilon(B + 1, Z^{1*} w_j)^{1/2} \leq \varepsilon', \) \( E^*_\varepsilon(B + 1, Z^{1*} w_j)^{1/2} \leq (1/4)C_1 \varepsilon \). So by continuity \( T^* > 0 \) when \( \varepsilon' \) sufficiently small.

Also, if \( T^* > +\infty \), then when \( s = T^* \), at least one of the three inequalities of (4.1) will be replaced by an equality. That is at least on of the following equations holds:

\[
E_m(T^*, Z^{1*} w_j)^{1/2} = C_1 \varepsilon s^\delta, \quad \text{for } j_0 + 1 \leq j \leq n_0, \quad 0 \leq |I^*| \leq 4, \\
E_m(T^*, Z^{1*} w_i)^{1/2} = C_1 \varepsilon s^\delta, \quad \text{for } 1 \leq i \leq j_0, \quad |I| = 4, \\
E_m(T^*, Z^{1*} w_i)^{1/2} = C_1 \varepsilon , \quad \text{for } 1 \leq i \leq j_0, \quad |I| \leq 3.
\]

One derives the equation (3.1) with respect to \( Z^{1*} \) with \( |I^*| \leq 4 \):

\[
\box{Z^{1*} w_i} + G_{i}^{j,0,3} \partial_\alpha \partial_\beta Z^{1*} w_j + D^2 w_i = [G_{i}^{j,0,3} \partial_\alpha \partial_\beta, Z^{1*}] w_j + Z^{1*} F_i.
\]

By energy lemma 2.2 and lemma 4.12 using the notation of lemma 2.2 when \( |I^*| \leq 4 \),

\[
\left( \sum_i E_m(s, Z^{1*} w_i) \right)^{1/2} \leq \left( \sum_i E_m(B + 1, Z^{1*} w_i) \right)^{1/2} + \int_{B+1} \sqrt{3} \sum_i L_i(\tau) + \sqrt{3} M(\tau) d\tau.
\]

By lemma 4.3 and 4.4

\[
\sum_i L_i(s) + M(s) \leq C(C_1 \varepsilon)^2 s^{-1+\delta}
\]

with \( C \) a universal constant. Then

\[
\left( \sum_i E_m(s, Z^{1*} w_i) \right)^{1/2} \leq C_m(C_1 \varepsilon)^2 s^{-1+\delta} + (1/4)C_1 \varepsilon.
\]

When \( \varepsilon \leq \frac{\delta}{4C_1 C_m} \),

\[
\left( \sum_i E_m(s, Z^{1*} w_i) \right)^{1/2} \leq (1/2)C_1 \varepsilon s^\delta,
\]

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which leads to

$$E_m(s, Z^I w_i) \leq (1/2)C_1 \varepsilon s^\delta.$$  

Similarly, one derives 3.1 with respect to \(Z^I\) with \(|I| \leq 3:\)

$$\Box Z^I u_i + G^{\alpha\beta}_i \partial_{\alpha\beta} w_j = [G^{\alpha\beta}_i \partial_{\alpha\beta} Z^I] w_j + Z^I F_i.$$  

Also by lemma 2.2

$$\left( \sum_i E_m(s, Z^I w_i) \right)^{1/2} \leq \left( \sum_i E_m(B + 1, Z^I w_i) \right)^{1/2} + \int_{B+1}^s \sqrt{3} \sum_i L_i(\tau) + \sqrt{3n_0} M(\tau) d\tau.$$  

By lemma 4.1, lemma 4.9 and lemma 4.11

$$\sum_i L_i(s) + M(s) \leq C(C_1 \varepsilon)^2 s^{-3/2 + \delta}.$$  

When \(\varepsilon \leq \frac{1 - 4s}{4C_1 C_{n_0}},\)

$$E_m(s, Z^I w_i)^{1/2} \leq (1/2)C_1 \varepsilon.$$  

When taking \(\varepsilon \leq \min \left\{ \frac{4}{4C_1 C_{n_0}}, \frac{1 - 4s}{4C_1 C_{n_0}} \right\} \), non of the equality of (4.21) holds. This contradiction leads to the desired result. \(\Box\)

### A Local existence for small initial data

One will establish the following local-in-time existence result for small initial data. The interest is to control the energy on hyperboloid \(H_{B+1}\) by the energy on plan \(\{t = 0\}\). Consider the Cauchy problem on \(\mathbb{R}^{n+1}:\)

\[(A.1)\]

\[
\begin{align*}
g^{\alpha\beta}(w, \partial w) \partial_{\alpha\beta} w_i + D^2_i w_i &= F_i(w, \partial w), \\
w_i(B + 1, x) &= \varepsilon w_i, \\
\partial_t w_i(B + 1, x) &= \varepsilon' w_i.
\end{align*}
\]

Here

$$g_i(w, \partial w) = m^{\alpha\beta} + A^{\alpha\beta\gamma}_i \partial_{\alpha\beta} y_j + B^{\alpha\beta}_i y_j,$$

$$F_i(w, \partial w) = P^{\gamma\beta\gamma}_i \partial_{\alpha\beta} y_j + Q^{\gamma\beta}_i \partial_{\alpha\beta} y_j + R^{\gamma\beta}_i y_j.$$  

These \(A^{\alpha\beta\gamma}_i, B^{\alpha\beta}_i, P^{\gamma\beta\gamma}_i, Q^{\gamma\beta}_i, R^{\gamma\beta}_i\) are constants. \((w_{i0}, w_{i1}) \in H^{n+1} \times H^s\) functions and supported on the disc \(|x| \leq B\). In general the following local-in-time existence holds

**Theorem A.1.** For any integer \(s \geq 2p(n) - 1\), there exists a time interval \([0, T(\varepsilon')]\) on which the cauchy problem (A.1) has an unique solution in sense of distribution \(w_i(t, x)\). Further more

$$w_i(t, x) \in C([0, T(\varepsilon')], H^{n+1}) \cap C^1([0, T(\varepsilon')], H^s),$$

and when \(\varepsilon'\) sufficiently small,

$$T(\varepsilon') \geq C(A\varepsilon')^{-1/2}$$

where \(A\) is a constant depending only on \(w_{i0}\) and \(w_{i1}\). Let \(E_g(T, w_i)\) be the hyperbolic energy defined in the section 2.2. For any \(\varepsilon, C_1 > 0\), there exists an \(\varepsilon'\) such that

$$\sum_1 E_g(B + 1, w_i) \leq C_1 \varepsilon.$$

---

2 Also, taking \(C_1 \varepsilon < 1\) small enough such that lemma 4.3 and lemma 4.12 hold.
Proof. The proof is just a classical iteration procedure. One will not give the details but the key steps. One defines the standard energy associated to a curved metric $g$

$$E_g^*(s, w_i) := \int_{\mathbb{R}^n} (g^{00}(\partial_i u)^2 - g^{ij}\partial_i \partial_j u) dx.$$ 

One takes the following iteration procedure:

(A.2)

$$\begin{cases}
g_i^{\alpha\beta}(w^k, \partial w^k)\partial_{\alpha\beta} w_i^{k+1} = F(w^k, \partial w^k), \\
w_i^{k+1}(0, x) = \varepsilon' w_{i0}, \quad \partial_t w_i^{k+1}(0, x) = \varepsilon' w_{i1},
\end{cases}$$

and take $w_i^0$ as the solution of the following linear Cauchy problem:

$$\begin{cases}
\Box w_i = 0, \\
w_i(0, x) = \varepsilon' w_{i0}, \quad \partial_t w_i(0, x) = \varepsilon' w_{i1}.
\end{cases}$$

Suppose that for any $|I| \leq 2p(n) - 1$,

(A.3)

$$\varepsilon'A \geq e \cdot E_g^*(B+1, \partial^I w_i^k)^{1/2},$$

$$\varepsilon'A \geq E_g^*(t, \partial^I w_i^k)^{1/2}.$$ 

Taking the size of the support of the solution $w_i^k(t, \cdot)$ into consideration, by Sobolev’s inequality, for any $|J| \leq p(n) - 1$,

(A.4)

$$|\partial^J w_i^k|(t, x) \leq C(t + B + 1)\varepsilon'A.$$

Now one wants to get the energy estimate on $\partial^I w_i^{k+1}$. By the same method used in [6], one gets

$$E_g^*(t, \partial^I w_i^{k+1})^{1/2} \leq E_g^*(t, \partial^I w_i^k) \exp \left( CA\varepsilon' \int_{B+1}^t (\tau + B + 1) d\tau \right)$$

$$\leq e^{-1}\varepsilon'A \exp \left( CA\varepsilon' \int_{B+1}^t (\tau + B + 1) d\tau \right)$$

When

$$\sqrt{CA\varepsilon} \leq (B + 1)^{-1}$$

and

$$t \leq \frac{1}{3}(CA\varepsilon')^{-1/2},$$

one gets that

$$E_g^*(t, \partial^I w_i^{k+1})^{1/2} \leq \varepsilon'A.$$ 

Then by an standard method presented in the proof of theorem ... of [6],

$$\lim_{k \to \infty} w_i^k = w_i$$

is the unique solution of (A.1), and $w_i \in C([0, T(\varepsilon')], H^{s+1}) \cap C^1[0, T(\varepsilon'), H^s)$. Here one can take

$$T(\varepsilon') = C(A\varepsilon)^{-1/2}$$

To estimate $E_g(B + 1, Z^I w_i)$, one takes $\partial_t w_i$ as the multiplier and by the standard procedure of energy estimate,

$$E_g(B + 1, Z^I w_i) - E_g^*(B + 1, Z^I w_i) = \int_{V(B)} (Z^I F_i(w, \partial w)\partial_t w_i - [Z^I, g^{\alpha\beta}\partial_{\alpha\beta}] w_i \cdot \partial_t w_i) dx$$

$$+ \int_{V(B)} (\partial_\alpha g^{\alpha\beta} \partial_\beta w_i \partial_t w_i - \frac{1}{2} \partial_h g^{\alpha\beta} \partial_\alpha w_i \partial_\beta w_i) dx,$$
where $V(B) := \{(t, x) : t \geq B + 1, t^2 - |x|^2 \leq B + 1\} \cap \Lambda'$. When $\epsilon \leq (B + 1)^{-2}$, thanks to (A.4) and (A.3), the right hand side can be controlled by $C\epsilon'$. Then one gets

$$E_g(B + 1, Z^Iw_i) \leq C\epsilon'.$$

\[\square\]

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