Rastall’s gravity equations and Mach’s Principle

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Rastall\textsuperscript{1} generalized Einstein’s field equations relaxing the Einstein’s assumption that the covariant divergence of the energy-momentum tensor should vanish. His field equations contain a free parameter $\alpha$ and in an empty space, i.e. if $T_{\mu\nu} = 0$, they reduce to the Einstein’s equations of standard general relativity. We calculate the elements of the metric tensor given by Rastall’ equations for different $\alpha$ assuming that $T_{\mu\nu}$ to be that of a perfect fluid and analyse these model solutions from the point of view of Mach’s principle. Since the source terms in Rastall’s modified gravity equations include the common energy-momentum tensor $T_{\mu\nu}$ as well as the expressions of the form $(1-\alpha)g_{\mu\nu}T/2$, these source terms depend on metric determined by the mass and momentum distribution of the external space. Likewise, in the classical limit, the source term of the corresponding Poisson equation for $\alpha \neq 1$ depends on the gravitational potential in the sense of Mach’s principle.

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I. INTRODUCTION

General relativity assumes that particle rest masses are space-time constants independent of the space-time metric in which they occur. The experimental evidence relating to this assumption is fragmentary and far from establishing it on a firm basis. The space-time constancy of particle mass is a typical feature also of Newton’s mechanics. Mach in his critique of the Newtonian mechanics\textsuperscript{2} pointed out to the fact that mass and momentum distributions of the external space should be linked with inertia of a mass body. This approach to treat problems of mechanics is commonly called as Mach’s principle\textsuperscript{3}. Many versions of Mach’s principle can be found in the literature (see, e.g.\textsuperscript{3,5}) and unique, satisfactory “canonical” formulation of the principle does not seem to exist as yet. Nevertheless, there seem to be a fair consensus about the consequences which should follow from it. Mach’s principle requires that the presence and motions of large external masses in the universe must have some definite effect upon the mass objects occurring in it. As is well-known, Mach’s principle is not incorporated in the Einstein equations of classical general relativity. To be included, the standard general relativity equations have to be modified. In the literature, there are several attempts to modify the Einstein equations in such a way that they

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incorporate at least some requirements following from Mach’s principle (see, e.g. [10]).

One of the fundamental assumptions of the Einstein theory of gravitation is vanishing divergence of the energy-momentum tensor

\[ \nabla_\nu T^{\mu\nu} = 0. \]

(1)

However, as stressed by Rastall [1] and others (see e.g. [6]), the local conservation of the energy-momentum tensor expressed by this relation has not been specifically tested by observation. This is why Rastall [1] generalized the Einstein field equations by relaxing this assumption. Later, Al-Rawaf and Taha [7] found an alternative, equivalent form of Rastall’s equations introducing a free parameter \( \alpha \) and their modified equations used throughout this paper reduce to the Einstein ones when setting \( \alpha \) equal to 1. Rastall’s equations reduce to those of general relativity for empty space independently of the value of the parameter \( \alpha \). Thus, all the crucial tests of general relativity (the perihelion advance of planets in our solar system, the deflection of light, the gravitational red shift and the delay of radar echoes) remain valid also in Rastall’s theory.

Rastall’s modified equations becomes important mainly when treating general relativistic problems including sources. Since the source terms in Rastall’s modified gravity equations include the components of the common energy-momentum tensor \( T_{\mu\nu} \) as well as additional terms of the form \((1 - \alpha) g_{\mu\nu} T/2\), the source terms of Rastall’s equations depend on the metric determined by the mass and momentum distributions of the external space. This is consistent with an important requirement following from Mach’s principle [5].

In the Newtonian limit for \( \alpha \neq 1 \), Rastall’s equations with source terms lead to a modified Poisson equation whose source depends on the gravitational potential, which also puts a strong constraint on \( \alpha \). It should be approximately equal to 1 in order to provide the Poisson equation in common weak gravitational fields. However, in large mass concentration and in strong gravitational field, i.e. in the stellar dynamics, even this small deviation of \( \alpha \) from 1 might play a significant role.

In what follows we try to show that solutions of Rastall’s equations for some model gravitational systems satisfy at least one requirement following from Mach’s principle, namely that the source terms in field equations depend, either on metric elements in case of general relativistic problems or on the gravitational potential in the classical Newtonian case. In this sense they seem to be more ‘Machian’ than their standard general relativistic counterparts.

The article is organized as follows. In Section II we briefly summarize Rastall’s theory rewriting his equations in the form in which the source terms are written as sums of ordinary energy-momentum tensor \( T_{\mu\nu} \) and an additional terms of the form \((1 - \alpha) g_{\mu\nu} T/2\), \( T = T^\mu_{\mu} \), \( \alpha \) being a free parameter. In Section III we solve Rastall’s equations for the isotropic perfect fluid calculating the components of metric tensor for arbitrary \( \alpha \). In Section IV we discuss the Machian property of Rastall’s equations pointing out to the fact that in the classical “Newtonian” limit the source term (i.e. mass density) necessarily depends on the gravitational potential provided \( \alpha \neq 1 \).

Throughout the text geometrized units in which \( c = 1, G = 1 \) are used. The metric signature \(- + + +\), notation and sign conventions follow [8, 9].
II. RASTALL’S GRAVITY EQUATIONS

The standard Einstein field equations read as (see, e.g. [9])

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi T_{\mu\nu}, \]  \hspace{1cm} (2)

where \( R = R^{\mu}_{\mu} \); an alternative form of Eq. (2) is

\[ R_{\mu\nu} = 8\pi \left( T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right). \]  \hspace{1cm} (3)

As it was demonstrated by Al-Rawaf and Taha [7], the Rastall’s modification of Einstein theory can be obtained by substituting a more general form of the Einstein tensor

\[ G_{\mu\nu} = \alpha R_{\mu\nu} + \beta g_{\mu\nu} R \]

for the left-hand side of the Eq. (2), where \( \alpha \) and \( \beta \) are constants. In standard general relativity the corresponding values \( \alpha = 1 \) and \( \beta = 1/2 \) are determined by the requirements that [10]

(i) Eq. (2) becomes the classical Poisson equation for gravitational potential in the Newtonian limit and that

(ii) energy-momentum is locally conserved, i.e. Eq. (1) holds.

However, if we relax the latter condition, then the constants \( \alpha \) and \( \beta \) are linked only through the relation [7]

\[ \beta = \frac{\alpha (\alpha - 2)}{2 (3 - 2\alpha)}, \hspace{0.5cm} \alpha \neq 0, \frac{3}{2}. \]  \hspace{1cm} (4)

Using Eq. (4) one obtains the generalized field equations first published by Rastall [1]

\[ R_{\mu\nu} - \frac{1}{2} \gamma g_{\mu\nu} R = k T_{\mu\nu}, \]  \hspace{1cm} (5)

where \( \gamma \) and \( k \) are the following functions of the parameter \( \alpha \)

\[ \gamma = \frac{2 - \alpha}{3 - 2\alpha}, \quad k = \frac{8\pi}{\alpha}. \]

Evidently, if \( \alpha = 1 \) (i.e. also \( \gamma = 1 \)), then Eq. (5) reduces to Eq. (2), so the Rastall’s theory includes standard general relativity as its special case.

The conservation condition (1) is not generally satisfied in Rastall’s theory and according to Eq. (3) it is replaced by [7]

\[ \nabla_\nu T^\nu_\mu = -\frac{1 - \alpha}{2k (3 - 2\alpha)} \partial_\mu R = -\frac{1 - \alpha}{2} \partial_\mu T. \]

To make use of already known solutions of general relativity it is convenient to rewrite Eq. (5) into the conventional form employing the standard Einstein tensor

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = k T_{\mu\nu} = \frac{8\pi}{\alpha} T_{\mu\nu}, \]  \hspace{1cm} (6)
TABLE I: The functions $R$ and $P$ for various values of the parameter $\alpha$; let us remind that the values $\alpha = 0, 3/2$ are excluded by Eq. (4).

\begin{center}
\begin{tabular}{|c|c|c|}
\hline
$\alpha$ & $P(r, \alpha)$ & $R(r, \alpha)$ \\
\hline
$-1$ & $4p - \varrho$ & $-3p + 2\varrho$ \\
$1/2$ & $7p/4 - \varrho/4$ & $-3p/4 + 5\varrho/4$ \\
$1$ & $p$ & $\varrho$ \\
$2$ & $-p/2 + \varrho$ & $3p/2 + \varrho/2$ \\
$3$ & $-2p + \varrho$ & $3p$ \\
\hline
\end{tabular}
\end{center}

The linear dependence of the functions $R$ and $P$ on $\varrho$, $p$ for some values of $\alpha$ is given in Table I; it will be demonstrated below that the value $\alpha = 3$ is not acceptable for our observed physical universe. Moreover, the Bianchi identities and Eq. (4) ensure the vanishing divergence of introduced tensor $T_{\mu\nu}$, i.e.

$$\nabla_\nu T^{\mu\nu} = 0.$$ (12)

Therefore perfect fluid solutions of Rastall’s equations (5) or (6) are formally equivalent to the solutions of Einstein equations (2) for the fluid with “density” $R$ and “pressure” $P$ given by (10) and (11) respectively. We exploit this fact when studying particular cases.

In the following sections we assume that $\alpha$ is a universal parameter with the same value for all physical systems. It is always important to check that obtained results reduce to known solutions of standard general relativity for $\alpha = 1$.

\section*{III. Static Spherical Symmetric Case}

First, we find the static isotropic metric fulfilling the Rastall’s equations (5) assuming that the energy-momentum tensor $T_{\mu\nu}$ corresponds to the perfect fluid described by Eq. (8). Formally, we will follow calculations analogical to those presented e.g. in [9, 11].
The metric of a static, spherically symmetric spacetime can be written in the form

\[ ds^2 = - \exp \left[ 2B(r, \alpha) \right] dt^2 + \exp \left[ 2A(r, \alpha) \right] dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \] (13)

where we introduce two functions \( A(r, \alpha) \) a \( B(r, \alpha) \) and employ standard spherical coordinates of a distant observer (we implicitly suppose the asymptotic flatness of the spacetime). Then, the non-null components of the standard Einstein tensor read as

\[
G_{tt} = \frac{\exp(2B)}{r^2} \left\{ r \left[ 1 - \exp(-2A) \right] \right\}, \quad (14)
\]

\[
G_{rr} = - \frac{\exp(2A)}{r^2} \left[ 1 - \exp(-2A) \right] + \frac{2}{r} \frac{dB}{dr}, \quad (15)
\]

\[
G_{\theta\theta} = r^2 \exp(-2A) \left[ \frac{d^2B}{dr^2} + \left( \frac{dB}{dr} \right)^2 + \frac{1}{r} \frac{dB}{dr} - \frac{1}{r} \frac{dA}{dr} - \frac{dA}{dr} \frac{dB}{dr} \right], \quad (16)
\]

\[
G_{\varphi\varphi} = \sin^2 \theta \frac{dA}{dr}. \quad (17)
\]

For a static fluid it also holds \( u^t = \exp(-B), \ u_t = \exp(B) \) and \( u^\theta = u^\varphi = 0 \), therefore according to Eq. (8) we obtain non-vanishing components of energy-momentum tensor

\[
T_{tt} = \rho \exp(2B), \quad T_{rr} = p \exp(2A), \quad T_{\theta\theta} = p r^2, \quad T_{\varphi\varphi} = \sin^2 \vartheta T_{\theta\theta} \quad (18)
\]

which can be substituted into Rastall’s equations. Is is convenient to solve the \((t, t)\) component of these equations through introducing a different unknown function \( M(r) \) defined as

\[
M(r) = \frac{r}{2} \left[ 1 - \exp(-2A) \right] \quad (19)
\]

or compared with the metric (13)

\[
g_{rr} = \exp(2A) = \left( 1 - \frac{2M}{r} \right)^{-1}. \quad (20)
\]

Then the \((t, t)\) Rastall equation implies

\[
\frac{dM}{dr} = \frac{k}{2} r^2 \mathcal{R} = \frac{4\pi}{\alpha} r^2 \mathcal{R}, \quad (21)
\]

substituting for \( \mathcal{R} \) and integrating with the condition \( M(0) = 0 \), one gets

\[
M(R) = \int_0^R \frac{4\pi}{\alpha} r^2 \mathcal{R} dr = \int_0^R \frac{4\pi}{\alpha} r^2 \left[ \frac{1}{2} (3 - \alpha) \varrho - \frac{3}{2} (1 - \alpha) p \right] dr. \quad (22)
\]

Evidently, setting \( \alpha = 1 \) we come to the familiar result of general relativity

\[
M(R) = m(R) = \int_0^R 4\pi \varrho r^2 dr,
\]
where \( m(R) \) is often called as mass function and outside the fluid, i.e. for a distant observer, it determines the mass of the studied spherical symmetric body with a surface at \( r = R \).

Using this result we can transform the \((r, r)\) component of Eq. (6) into the form

\[
\frac{dB}{dr} = \frac{M + 4\pi P r^3/\alpha}{r (r - 2M)}.
\] (23)

As we can see, the two remaining Rastall’s equations for \((\vartheta, \vartheta)\) and \((\varphi, \varphi)\) components are completely equivalent. Instead of solving them explicitly, commonly another equation is employed instead; substituting from Eq. (18) into formal “conservation law” (12) one gets

\[
(R + P) \frac{dB}{dr} = -\frac{dP}{dr}, \text{ i.e. } (\vartheta + p) \frac{dB}{dr} = -\frac{1}{2} (1 - \alpha) \vartheta + \frac{1}{2} (5 - 3\alpha) p.
\] (24)

Again, setting \( \alpha = 1 \) one arrives at the general relativistic counterpart of the above equation

\[
(\vartheta + p) \frac{dB}{dr} = -\frac{dp}{dr}.
\]

Moreover, combining Eq. (23) and (24) we obtain the Rastall’s counterpart of the Oppenheimer-Volkov equation (see e.g. [11, 12])

\[
\frac{dP}{dr} = -(R + P) \frac{M + 4\pi P r^3/\alpha}{r (r - 2M)}.
\] (25)

Here, \( p \) is always understood to be related to \( \vartheta \) by the equation of state, which also determines the relation between \( P \) and \( R \). In full analogy to general relativity equations (21), (23) and (25) may be collectively referred to as the equations of structure for spherical stars in Rastall’s generalization of the Einstein gravitational theory.

Let us look for the counterpart of the famous Schwarzschild solution. Let \( p, \vartheta \neq 0 \) only for \( r < R \), where \( R \) corresponds to a surface of a spherical star. Then Eq. (22) gives

\[
\mathcal{M}(R) = \frac{4\pi}{\alpha} \int_0^R R r^2 dr = \int_0^R 4\pi \varrho r^2 dr + \frac{3}{2} \frac{1 - \alpha}{\alpha} \int_0^R 4\pi (\vartheta - p) r^2 dr,
\]

where the first part corresponds to the standard Schwarzschild mass \( M_{\text{schw}} \) and the second represents “Rastall’s” correction. For a distant observer at \( r > R \) the value \( \mathcal{M} = \mathcal{M}(R) \) is constant and the metric coefficients read as

\[
g_{rr} = \exp [2A(r, \alpha)] = \frac{1}{1 - 2\mathcal{M}/r}, \quad g_{tt} = -\exp [2B(r, \alpha)] = -\left( 1 - \frac{2\mathcal{M}}{r} \right),
\]

where

\[
\mathcal{M} = M_{\text{schw}} + \frac{3}{2} \frac{1 - \alpha}{\alpha} \int_0^R 4\pi (\vartheta - p) r^2 dr.
\]

Naturally, putting \( \alpha = 1 \) we get the standard Schwarzschild solution. Considering only dust \((p = 0)\) solution, which is theoretically interesting, but physically highly non-realistic case, one come to the result

\[
\mathcal{M} = M_{\text{schw}} + \frac{3}{2} \frac{1 - \alpha}{\alpha} \int_0^R 4\pi \varrho r^2 dr = \frac{3 - \alpha}{2\alpha} M_{\text{schw}}.
\]
This puts a strong upper bound for the value of $\alpha$ – evidently it must hold that $\alpha < 3$ to ensure $M \geq 0$.

The inconvenience of the value $\alpha = 3$ can be illustrated by the following example. Let us calculate the metric elements $A(r, \alpha = 3)$ and $B(r, \alpha = 3)$ for an ideal perfect fluid with zero pressure $p = 0$ (i.e. for a dust) as a source. In this case Rastall’s equations turn out to be

$$G_{\mu\nu} = \frac{8\pi}{3} \left( T_{\mu\nu} - g_{\mu\nu} T \right)$$

with $T = 3p - \varrho = \varrho$ and functions $R, P$ listed in the last row of Table I particularly $R = 3p = 0$ and $P = 2p + \varrho = \varrho$. Solving Eq. (21) we come to the condition $dM/dr = 0$, which is identically valid for all $r$, thus taking into account asymptotic flatness we get $M = 0$, $A(r, \alpha = 3) = \ln(1)/2 = 0$, $g_{rr} = \exp(2A) = 1$, which means the space-like part of the metric is flat.

On the other hand, Eq. (23) leads to

$$\frac{dB}{dr} = \frac{4\pi}{3} \varrho r.$$ 

Setting the conditions of asymptotic flatness $B(0) = B(\infty) = 0$ again, one obtains

$$\int_0^\infty \frac{dB}{dr'} dr' = B(\infty) - B(0) = 0 = \int_0^r \frac{4\pi}{3} \varrho r' dr' + \int_r^\infty \frac{4\pi}{3} \varrho r' dr' = \int_0^r \frac{4\pi}{3} \varrho r' dr' - B(r)$$

and

$$B(r) = \int_0^r \frac{4\pi}{3} \varrho r' dr', \quad g_{tt} = -\exp(2B) = -\exp \left( \int_0^r \frac{4\pi}{3} \varrho r' dr' \right).$$

Thus the fluid affects only the $(t, t)$-part of the metric, while the space remains Euclidean, which is physically acceptable only as an approximation of weak gravitational fields.

### IV. MACHIAN PROPERTY OF RASTALL’S FIELD EQUATIONS

An important consequence of Mach’s principle is that inertial properties of a mass body are determined by the distribution of mass-energy in the neighbouring space. Here, the problem arises how to adequately characterize the physical state of this space. It seems that the simplest way to do it is either to take the gravitational potential in classical gravity, or the metric in general relativity. That is, to suppose that the source of gravitation, either mass-density or the elements of energy-momentum tensor, depend either on the gravitational potential or on the corresponding metric, respectively.

Let us recall Rastall’s equations in the form

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi}{\alpha} \left( T_{\mu\nu} + \frac{1 - \alpha}{2} g_{\mu\nu} T \right) = \frac{8\pi}{\alpha} \left[ T_{\mu\nu} + a(\alpha) g_{\mu\nu} T \right],$$

where $a(\alpha) = (1 - \alpha)/2$. We see that the source contains elements of metric tensor. This is in accord with the conclusion of Mach’s principle which says that the inert mass of a mass body depends on the magnitude and mutual distance of the neighbouring mass fields.
distribution. Thus, from the point of view of Mach’s principle, Rastall’s field equations seem to be more “Machian” than those of the Einstein general relativity.

In the Newtonian gravity limit the metric in the Cartesian coordinates can be put down form 

$$ds^2 = -(1 + 2\phi) \, dt^2 + (1 - 2\phi) \left( dx^2 + dy^2 + dz^2 \right),$$

where $\phi$ is the classical gravitational potential (typically, $\phi = -M/r$ outside a spherical star of the mass $M$). Taking into account the approximation of the energy momentum tensor

$$T_{tt} = \rho, \quad T = -\rho$$

we derive from Eq. (6)

$$G_{tt} \approx 2\Delta \phi = \frac{8\pi}{\alpha} \left( T_{tt} + \frac{1 - \alpha}{2} g_{tt} T \right) = \frac{8\pi}{\alpha} \rho \left[ 1 + \frac{1 - \alpha}{2} (1 + 2\phi) \right].$$

Apparently, setting $\alpha = 1$ we get the standard Poisson equation $\Delta \phi = 4\pi \rho$, otherwise

$$\Delta \phi(\vec{r}) = 4\pi \rho(\vec{r}) \left[ A + B\phi(\vec{r}) \right], \quad (27)$$

where

$$A = 1 + \frac{3(1 - \alpha)}{2\alpha} \quad \text{and} \quad B = \frac{1 - \alpha}{\alpha} \phi(\vec{r}).$$

Eq. (27) shows that in the classical limit the source term on the right-hand side of Rastall’s equations depends on the potential of gravitational field in a specific way, which puts a strong constraint on $\alpha$. It should be approximately equal to 1 in order to obtain the familiar Poisson equation for weak gravitational fields. However, for large mass concentrations and strong gravitational fields Eq. (27) might play a significant role in astrophysics and cosmology [13], even for small deviation of $\alpha$ from 1. The fact that the source term of Eq. (27) depends on the potential $\phi(\vec{r})$ might be understood as an effect consistent with Mach’s principle in non-relativity physics because the mass density $\rho$ depends on the all neighbouring mass-energy distribution through the scalar potential $\phi(\vec{r})$. Moreover, Eq. (27) as well as Eq. (6) shows that for $\alpha \neq 1$ the source terms depend, contrary to general relativity, on spacetime coordinates, i.e. in Rastall’s theory the concept of variable mass appears.

The concept of variable rest mass in the theory of gravity is not new. It appears in Dicke’s reformulation of Brans-Dicke theory in which the metric obeys Einstein’s equations, but in which rest masses vary in particular way [14] or in Bekenstein’s theory of the universal scalar field [15] to mention only two examples. The introduction of variable rest masses in the theory of gravitation is part of the general program for the possible incorporating some features of Mach’s principle into general relativity which may be reached mainly by the following assumptions: (i) that particles interact with some external fields as they move on trajectories [15] or by Mannheim’s [16] assumption of an external scalar field, which gives a position dependent coupling to motion; (ii) that particles move on standard geodesics but the Christoffel symbols are modified because the equation of motion for the metric is different from the standard one. This can be reached either through a fundamental replacement of the Einstein curvature tensor $G_{\mu\nu}$ by another equally covariant second-rank tensor, or through introducing a more complicated energy-momentum tensor $T_{\mu\nu}$ than it is conventionally considered, keeping the common Einstein tensor at the same time.
Concluding remark

Rastall modified Einstein’ equations by replacing the second term in the Einstein tensor. The advantage of this substitution is that Rastall’s equations reduce to Einstein’s one for empty space-time. Differences between Einstein’s and Rastall’s equations arise when one investigates gravitating systems with non-zero sources. We have showed that considerable differences between both theories exists also regarding the incorporation of Mach’s principle in the theory of gravitation. Hence, Rastall’s equations represent a set of gravity equations which includes also the Einstein ones. In this sense, they can be considered as a generalization of general relativity.

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