Quantization of the Reissner–Nordström Black Hole

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Abstract

The Reissner–Nordström family of solutions can be understood to arise from the spherically symmetric sector of a nonlinear SO(2,1)/SO(1,1) sigma model coupled to three dimensional Euclidean gravity. In this context a group theoretical quantization is performed. We identify the observables of the theory and calculate their spectra.
1 Introduction

The Einstein-Maxwell (EM) theory is the simplest member of a class of gravity theories involving Abelian vector fields, whose dimensional reduction to three dimensions can be dualized to a nonlinear sigma model. Depending on the signature of the Killing vector (KV) field generating the symmetry along the reduced dimension there results either a three dimensional Euclidean theory with a non-compact pseudo-Riemannian target space (timelike KV) or a Minkowskian one with a non-compact Riemannian target space (spacelike KV). In particular in the case of EM one obtains the coset spaces $SU(2,1)/S[U(1,1) \times U(1)]$ resp. $SU(2,1)/S[U(2) \times U(1)]$. The first case, which will be our starting point for the quantization of the RN family, is a 4-dimensional pseudo-Riemannian symmetric space of signature $(+ + - -)$. This space contains two important subspaces $SU(1,1)/U(1)$ corresponding to solutions of pure gravity without EM field and $SO(2,1)/SO(1,1)$ corresponding to static solutions.

In order to simplify matters even more we shall restrict ourselves to spherically symmetric solutions, an assumption justified by the spherical symmetry of the classical RN solutions as well as by a general theorem of Israel stating the spherical symmetry of any single static black hole of the EM theory. Under these assumptions the solutions depend only on one (arbitrary) radial variable $\rho$ and considered as a dynamical system the field theory is replaced by a mechanical system with a constraint. Taking this view-point the $\rho$–dependence of the solution is obtained evolving the mechanical system with respect to “time” $\rho$. The corresponding Hamiltonian involves the Casimir operator of $SO(2,1)$ expressed by the Noether currents of the sigma model. These currents are the “observables” of the model. The quotation marks are meant to indicate that the concept of observables is adapted to the dimensionally reduced theory. In order to arrive at the sigma model structure one has to use adapted coordinates and to restrict oneself to coordinate (gauge) transformations respecting this choice. Correspondingly the concept of gauge invariance is somewhat looser compared to the general covariance of the full theory. Requiring self-adjointness of the currents expressed as differential operators on the configuration space $SO(2,1)/SO(1,1)$ naturally leads to unitary representations of $SO(2,1)$ on a suitable $L^2$-space over the coset space. This provides a rather rigid group theoretical framework for the quantization of the model already employed for the vacuum theory in [3], [4].

The Wheeler–DeWitt equation of this “mini-superspace” model equation splits into a group theoretic and a 1–dimensional gravity part. Its solution requires the spectral decomposition of the Casimir on the $L^2$ over the coset space or its harmonic analysis to put it in mathematical terms. In general this is a complicated problem for non-compact groups, however, in the case of $SO(2,1)/SO(1,1)$ resp. $SU(2,1)/S[U(1,1)\times U(1)]$ the solution can be found in the mathematical literature [5], [6]. For reasons of simplicity we only treat the former case corresponding to static solutions.
Besides the Casimir we also analyze the spectra of the mass and the charge operator, the physical observables of the model. Since the corresponding currents do not commute, only one of them can be diagonalized simultaneously with the Casimir. Classically the Casimir equals the square of the “irreducible mass” $m^2 - q^2$. In contrast to the vacuum gravity case the Casimir has also a discrete spectrum for values with $m^2 - q^2 < 1/4$.

2 Quantization of the Reissner–Nordstrøm Solutions

As is well known the stationary solutions of the EM theory can be derived from the dimensionally reduced 3-dimensional action describing a nonlinear sigma model with target space $SU(2,1)/SU(1,1) \times U(1)$ coupled minimally to 3-dimensional Euclidean gravity \cite{7,8,1}

$$L = \sqrt{h} \left( \frac{1}{2} (3) R - \frac{1}{2\Delta^2} \left[ (\partial_m \Delta)^2 + \omega_m^2 \right] + \frac{1}{\Delta} \left[ (\partial_m \varphi)^2 + (\partial_m \xi)^2 \right] \right)$$

$$= \left( \frac{1}{2} (3) R - \frac{h^{mn}}{8} \mathrm{Tr} \left( \chi^{-1} \partial_m \chi \chi^{-1} \partial_n \chi \right) \right),$$

(1)

with $\omega_m = \partial_m \omega + 2 \xi \partial_m \varphi - 2 \varphi \partial_m \xi$. $\Delta$ and $\omega$ are the gravitational potential $g_{tt}$ (in adapted coordinates) and the “twist” potential, resp. $\xi$ is the magnetic and $\varphi$ is the electric potential. $(3) R$ denotes the 3-dimensional Euclidean scalar curvature and $\sqrt{h}$ is the square root of the 3-dimensional metric. The sigma model matrix $\chi$ will be specified for the restriction to the Reissner–Nordstrøm family of solutions below. In addition to the stationarity of the solution we require its spherical symmetry. Then the 3-dimensional space splits into a (warped) product of 2-spheres and the positive real line. The metric can be transformed into the standard form

$$ds^2 = h_{mn} dx^m dx^n = N^2(\rho) \, d\rho^2 + r^2(\rho) \, d\Omega^2,$$

where $\rho$ is an arbitrary radial coordinate and $d\Omega^2$ the invariant metric of the 2-sphere. Then the Lagrangian \cite{11} simplifies to

$$L = N \left[ \frac{r^2}{N^2} + 1 - \frac{r^2}{8N^2} \mathrm{Tr} \left( \chi^{-1} \chi' \chi^{-1} \chi' \right) \right].$$

(2)

All the fields only depend on the remaining spacelike coordinate $\rho$, where the prime denotes the derivative with respect to $\rho$.

By reduction of the four dimensional sigma model target space to a two dimensional subspace the Einstein-Maxwell theory can be simplified \cite{11} to a space of constant curvature. In this context two of them are especially interesting: setting $\varphi$ and $\xi$ equal to zero reduces the theory to the stationary spherically symmetric pure gravitational case, i.e. a $SU(1,1)/U(1)$ nonlinear sigma model and restricting to $\omega = \xi = 0$ leads to the static truncation of the $SU(2,1)/SU(1,1) \times U(1)$.
U(1)] nonlinear coset sigma model to SO(2,1)/SO(1,1), the Reissner–Nordström family. [3], [4] already dealt with the quantization of the former case. Here we discuss the quantization of the Reissner–Nordström family.

The matrix $\chi$ simplifies to

$$\chi = \begin{pmatrix} \Delta - 2 \varphi^2 + \frac{\varphi^4}{\Delta} & i\sqrt{2} (\varphi - \frac{\varphi^3}{\Delta}) & -i \frac{\varphi^2}{\Delta} \\ i\sqrt{2} (\varphi - \frac{\varphi^3}{\Delta}) & 1 - 2 \varphi^2 & -\sqrt{2} \frac{\varphi}{\Delta} \\ i \frac{\varphi^2}{\Delta} & \sqrt{2} \frac{\varphi}{\Delta} & \frac{1}{\Delta} \end{pmatrix}.$$ 

The Lagrangian $\mathcal{L}$ is invariant under SO(2,1) transformations acting on $\chi$. The matrix of Noether currents has the structure

$$J = \frac{f^2}{2N^2} \chi^{-1} \chi' = \begin{pmatrix} J_S & iJ_H & 0 \\ iJ_G & 0 & -J_H \\ 0 & J_G & -J_S \end{pmatrix}.$$ (3)

The currents are the essential dynamical objects of the model obeying the (Euclidean) field equations. It is not difficult to confirm that the currents indeed form an so(2,1) algebra.

The meaning of the various elements in the Lie algebra is found in [5]: $J_G$ corresponds to a gauge transformations, which does not change the classical solution. $J_H$ denotes a Harrison transformation, which leads to a change of $\Delta$ into $\varphi$ and $J_S$ is a scale transformation.

Imposing proper boundary conditions at infinity on the sigma model fields there exists an asymptotic multipole expansion of $\chi$.

$$\chi \sim \sum_{n=0}^{\infty} \rho^{-n} \chi_n(\theta, \phi).$$

A suitable choice of coordinates leads to the asymptotic behaviour of the fields

$$\Delta = 1 - \frac{2m}{\rho} + O(\frac{1}{\rho^2}) \quad \varphi = \frac{q}{\rho} + O(\frac{1}{\rho^2}),$$

which is used to calculate the Noether charges of the theory:

$$Q = \chi_0^{-1} \chi_1 = \begin{pmatrix} -2m & i\sqrt{2}q & 0 \\ i\sqrt{2}q & 0 & -\sqrt{2}q \\ 0 & \sqrt{2}q & 2m \end{pmatrix}.$$ 

4 the scalar product is taken with respect to the matrix

$$\eta = \begin{pmatrix} 0 & 0 & i \\ 0 & 1 & 0 \\ -i & 0 & 0 \end{pmatrix}.$$
The mass $m$ corresponds to $J_S$ and the electric charge is related with a linear combination of a Harrison transformation $J_H$ and a gauge transformation $J_G$.

We now switch to a modified Hamiltonian formalism. The slicing is performed according to $\rho$, the spacelike coordinate [9], [10], [11]. The application of the usual quantization algorithm – associate multiplication operators with the fields and differentiation operators with the momenta – requires some care. We aim at identifying a complete set of observables. There is a one-to-one correspondence between the observables of the theory and the initial data $m$ and $q$. (The equations of motion define a geodesic motion on the coset space. The tangent vector at $\rho = \infty$ is determined by the mass and the electric charge). Therefore we expect the currents to be the observables of the quantized theory. As a manifold the coset space SO(2,1)/SO(1,1) is the De–Sitter space, which can be illustrated by a hyperboloid. Our coordinates, which are given by the dimensional reduction cover only half of the hyperboloid as illustrated in [12] (p. 125). Therefore globally $\Delta$ and $\varphi$ are not a good choice of coordinates. We need coordinates which cover the whole coset space. As a part of the quantization procedure we therefore postulate a change of coordinates and in particular we consider the coordinates $x, y, z$ defined by $x^2 + y^2 - z^2 = 1$ or coordinates $t$ and $\chi$ with $x = \text{ch} t \sin \chi, y = \text{ch} t \cos \chi$ and $z = \text{sh} t$. Let us first deal with the $t, \chi$–coordinate system. The application of the usual quantization procedure supplemented by the change of coordinates yields the Hamiltonian operator

$$\hat{H} = \frac{1}{4} \partial_t^2 + 1 - \frac{1}{8f^2} \hat{\text{Tr}} J^2,$$

which defines the Wheeler–DeWitt equation $\hat{H}\psi = 0$. The current and the Casimir operators are calculated to be

$$\hat{J}_S = i(- \cos \chi \partial_t + \sin \chi \text{th} t \partial_\chi)$$
$$\hat{J}_H = -\frac{i}{\sqrt{2}} [\sin \chi \partial_t + (\cos \chi \text{th} t + 1) \partial_\chi]$$
$$\hat{J}_G = \frac{i}{\sqrt{2}} [\sin \chi \partial_t + (\cos \chi \text{th} t - 1) \partial_\chi]$$
$$\hat{\text{Tr}} J^2 = -\partial_t^2 - \text{th} t \partial_t + \frac{1}{\text{ch}^2 t} \partial_\chi^2.$$

$\hat{H}$ commutes with the currents and $\hat{\text{Tr}} J^2$.

$$[\hat{H}, \hat{J}_S] = 0, \quad [\hat{H}, \hat{J}_G] = 0, \quad [\hat{H}, \hat{J}_H] = 0, \quad [\hat{H}, \hat{\text{Tr}} J^2] = 0. \quad (4)$$

The Hamiltonian generates the gauge transformations such that (4) defines the current operators and the Casimir operator to be gauge invariant quantities, i.e. they are observables. The Casimir commutes with the currents. Therefore the Casimir operator and any linear combination of the currents can be diagonalized simultaneously. The currents constitute an so(2,1) algebra

$$[\hat{J}_S, \hat{J}_H] = \hat{J}_H, \quad [\hat{J}_S, \hat{J}_G] = -\hat{J}_G, \quad [\hat{J}_H, \hat{J}_G] = -\hat{J}_S.$$
Hence it is not possible to “measure” the mass and the charge of the black hole simultaneously. The current and the Casimir operators are per construction self-adjoint operators with respect to the SO(2,1) invariant measure. The scalar product is given by

\[(\psi, \psi) = \int_0^\pi d\chi \int_{-\infty}^{\infty} dt \ ch t |\psi(t, \chi)|^2.\]

The spectrum of the Casimir operator can be read off in the paper by Rossmann [5]. It consists of a discrete part, namely the eigenvalues \(\lambda = \frac{1}{4} - \nu^2, \nu \in \frac{1}{2} \mathbb{N}_0\) (here we assume that the wave function is single valued on the hyperboloid) and a continuous part with eigenvalues \(\lambda \in (\frac{1}{4}, \infty)\). However, it is instructive to look at the Casimir more closely. We notice that \(\hat{J}^H + \hat{J}^G\) is proportional to the differential operator \(\partial_\chi\). With the ansatz \(\psi(t, \chi) = e^{iq\chi} \Phi(t)\) and the transformation \(\Phi(t) \rightarrow \frac{1}{\sqrt{cht}} \phi(t)\) the eigenvalue equation

\[(-\partial^2_t + th t \partial_t + \frac{1}{cht^2} \partial^2_\chi) \psi(t, \chi) = \lambda \psi(t, \chi)\]

becomes a standard Schrödinger operator

\[-\partial^2_t \phi + V(t) \phi = (\lambda - \frac{1}{4}) \phi, \quad \text{with} \quad V(t) = \frac{1}{cht^2} - \frac{q^2}{cht}\]

For \(|q| < 1/2\) the potential is repulsive. For \(|q| > 1/2\) one finds a potential valley and expects to get a discrete spectrum as well. The discrete spectrum can be generated algebraically. For this the coordinates \(x, y, z\) are the best choice. Let us define \(\hat{J}_x, \hat{J}_y, \hat{J}_z, \hat{J}_+\) and \(\hat{J}_-\) by

\[
\begin{align*}
\hat{J}_z &= i (x \partial_y - y \partial_x), \\
\hat{J}_+ &= \hat{J}_x + i \hat{J}_y, \\
\hat{J}_- &= \hat{J}_x - i \hat{J}_y,
\end{align*}
\]

with \(\hat{J}_x = i(y \partial_z + z \partial_y)\) and \(\hat{J}_y = i(z \partial_x + x \partial_z)\). \(\hat{J}_x, \hat{J}_y\) and \(\hat{J}_z\) are the current operators adopted to the structure of the G/H coset space. \(\hat{J}_x\) parametrizes the non-compact part of the group and is proportional to \(\hat{J}_S\), \(\hat{J}_z\) belongs to the maximal compact subgroup and is the electric charge operator. \(\hat{J}_y\) corresponds to \(H\). In terms of the “physical” operators we find

\[
\begin{align*}
\hat{J}_z &= \frac{1}{\sqrt{2}} \left( \hat{J}_G + \hat{J}_H \right), \\
\hat{J}_+ &= -\hat{J}_S + \frac{i}{\sqrt{2}} \left( \hat{J}_G - \hat{J}_H \right), \\
\hat{J}_- &= -\hat{J}_S - \frac{i}{\sqrt{2}} \left( \hat{J}_G - \hat{J}_H \right),
\end{align*}
\]

which constitute the algebra

\[[\hat{J}_z, \hat{J}_+] = \hat{J}_+, \quad [\hat{J}_z, \hat{J}_-] = -\hat{J}_-, \quad [\hat{J}_+, \hat{J}_-] = -2\hat{J}_z.\]
The Casimir operator is

$$\hat{\text{Tr}} J^2 = -\hat{J}_z^2 + \frac{1}{2} \left( \hat{J}_+ \hat{J}_- + \hat{J}_- \hat{J}_+ \right)$$

$$= -\hat{J}_z^2 + \hat{J}_z \hat{J}_- = -\hat{J}_z - \hat{J}_z \hat{J}_+.$$ 

As $\hat{J}_\pm = \hat{J}_\mp$ we have

$$< \hat{J}_\pm \psi | \hat{J}_\pm \psi > = < \psi | \hat{J}_\mp \hat{J}_\pm \psi > \geq 0$$

and consequently

$$\lambda + q^2 + q \geq 0 \quad \text{and} \quad \lambda + q^2 - q \geq 0.$$ 

These inequalities lead to $\lambda \leq \frac{1}{4}$ and to $q \geq s$ or $q \leq -s$, where $\lambda$ is defined to be equal to $s(1-s)$. $\hat{J}_\pm \psi$ is an eigenfunction of $\hat{J}_z$ but with $q$-value increased by unity. In analogy to the algebra of the angular momentum operators in quantum mechanics we find for every eigenvalue $\lambda$ of the Casimir operator a sequence of eigenvalues of the current operators $\hat{J}_z$. The $q$-values are not confined in an interval but range from a finite value $q = s$ or $q = -s$ in integral steps to $\infty$ or $-\infty$ resp. The wave functions can be constructed explicitly. Let us label the wavefunction by $s, q$. With $\hat{J}_- \psi_{ss} = 0$ we find $\psi_{ss}(t, \chi) = C e^{-is\chi} \sin^s t$. A wave function with given value of $s$ and $q > s$ can be constructed by successive application of the raising operator $\hat{J}_+$. Application of the raising operator $\hat{J}_-$ on the wave functions $\psi_{s(1-s)}$ gives zero. Operating with $\hat{J}_-$ on the solution of this differential equation one derives the wave functions with given values of $s$ and $q < -s$.

Classically the Reissner–Nordström family of solutions can be parametrized by the value of the Casimir operator $\hat{\text{Tr}} J^2$, which is evaluated on the the solutions equal to $m^2 - q^2$. Due to the indefiniteness of the metric of the coset space $m^2 - q^2$ can have either sign. $m^2 - q^2 > 0$ corresponds to the Reissner–Nordström solutions, $m^2 = q^2$ gives the extrem Reissner–Nordström solutions and $m^2 - q^2 < 0$ label the so called over extreme solutions. They are obtained by analytic continuation from the solutions with $m^2 - q^2 < 0$ and are naked singularities. In the quantized version model the Reissner–Nordström solutions belong to the continuous and the over extreme solutions belong to the discrete spectrum of the Casimir operator. The extremal Reissner–Nordström solution is characterized by $\lambda = \frac{1}{4}$. In this case the potential of the standard Schrödinger problem vanishes identically.

To summarize: dimensional reduction of the Einstein–Maxwell theory with respect to a timelike KV yields a 4-dimensional Pseudo–Riemannian symmetric space. The subspace of static solutions is Pseudo–Riemannian as well. In contrast to the pure gravity coset space, which is Riemannian, the spectrum of the Casimir operator consists of a continuous and a discrete part. The usual quantization procedure has to be supplemented by a change of coordinates. As the mass operator refers to a non–compact direction in the Lie algebra the spectrum is continuous. The charge operator belongs to a compact direction.
and has a discrete spectrum imposing the wave function to be single valued on the hyperboloid. But as the group SO(2,1) is not simply connected we do not know up to now of any method to decide whether the spectrum has integral or fractional values. Allowing for the limiting case of the infinite covering group the spectrum would even be continuous. We should like to mention that the papers by Rossmann and Schlichtkrull [5], [6] provide the mathematical tools to quantize the complete stationary sector of the Einstein–Maxwell theory and to deal with higher dimensional hyperbolic spaces.

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