Topology and Confinement in SU($N$) Gauge Theories

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The large $N$ limit of SU($N$) gauge theories in 3+1 dimensions is investigated on the lattice by extrapolating results obtained for $2 \leq N \leq 5$. A numerical determination of the masses of the lowest-lying glueball states and of the topological susceptibility in the limit $N \to \infty$ is provided. Ratios of the tensions of stable $k$-strings over the tension of the fundamental string are investigated in various regimes and the results are compared with expectations based on several scenarios – in particular MQCD and Casimir scaling. While not conclusive at zero temperature in D=3+1, in the other cases investigated our data seem to favour the latter.

1. Introduction

A lattice investigation of the large $N$ limit of SU($N$) gauge theories \cite{1} is interesting in many respects: it can improve our understanding of SU(3) gauge theory as a base for full QCD and allows a direct comparison with predictions derived from theories beyond the Standard Model, often rigorous only when $N \to \infty$.

Existing lattice results suggest that SU(2) and SU(3) may be close to the large $N$ limit. For this reason, we have investigated SU($N$) gauge theories in 3+1 dimensions for $N = 2, 3, 4, 5$ \cite{2}. If as $N \to \infty$ a smooth limit exists, corrections to this limit can be expressed as a power series in $1/N^2$.

Of course, a priori it is unknown whether observables at small $N$ can be continued analytically to $N = \infty$ and how many terms of the series are needed to relate quantities at finite $N$ to their $N = \infty$ values. To answer those questions, we have studied the dependence on $N$ of the fundamental string (Sect. 2), of the mass of the $0^{++}$ and $2^{++}$ glueballs (Sect. 3) and of the topological susceptibility (Sect. 4) \cite{2}.

As soon as $N = 4$ other strings appear in the spectrum that are stable \cite{3}. These strings are denoted as $k$-strings, $k = 1, ..., N/2$ being an integer identifying the class of the representations of the gauge group under which sources connected by a $k$-string transform. The values of the ratios of the string tensions $\sigma_k/\sigma_f$ (the denominator being the tension of the fundamental string) give insights on the mechanism of colour confinement. A numerical investigation of those ratios \cite{4,5} is reported in Sect. 5.

2. The string tension

In our calculations we have used the Wilson action $S = \beta \sum_{\mu > \nu, i} \left( 1 - \frac{1}{2N} \text{Tr} (U_{\mu\nu}(i) + U_{\mu\nu}^\dagger(i)) \right)$, with $\beta = 2N/g^2$, $g$ being the coupling constant.

The string tension has been extracted by looking at the decay of correlation functions of blocked Polyakov loops. To the lowest order in $1/L$, the string tension $\sigma$ is related to the smallest mass governing this decay by the formula

$$m = \sigma L - c_s \frac{\pi(D - 2)}{6L},$$

where $L$ is the length of the string and $D$ is the number of dimensions. If the infrared behaviour of the string can be described by an effective string theory, the coefficient $c_s$ is universal \cite{6} and depends only on the number of bosonic and fermionic modes propagating along the string. A reliable extraction of the string tension must keep into account the term proportional to $1/L$. For this reason, we have determined $c_s$ for the gauge group SU(2) in $D = 2 + 1$ and $D = 3 + 1$ \cite{5}. Our fits give $c_s = 1.066 \pm 0.036$ and $c_s = 0.94 \pm 0.04$ respectively. Those values are compatible at the $2\sigma$ level with an effective string theory of the bosonic type and are incompatible with other simple string models. In the following we will...
assume $c_s = 1$, i.e. that the long range fluctuations of the string are described by an effective string theory and that the string is bosonic.

The string tension is used to set the scale of physical quantities at different $N$. Equivalent quantities are identified by equivalent couplings. In the original 't Hooft’s idea, equivalent couplings in the large $N$ limit correspond to fixed $\lambda = g^2 N$. We have verified that the string tensions for different $N$ lie on an universal curve as a function of $\lambda I = g^2 I N$, $g^2 I$ being the tadpole improved coupling [2].

A fit of the form $\hat{m}(N) = \hat{M} + a/N^2$ works for all these states and all the way down to SU(2). Hence for these quantities a smooth $N \to \infty$ limit exists and the first expected correction explains the finite $N$ results all the way down to SU(2).

The $N = \infty$ values extracted for $\hat{M} = M/\sqrt{\sigma}$ are $\hat{M} = 3.380(69)$ for the $0^{++}$, $\hat{M} = 4.94(13)$ for the $2^{++}$ and $\hat{M} = 6.74(16)$ for the $0^{++*}$.

### 4. Topological susceptibility

Large $N$ arguments were the framework used to derive the Witten-Veneziano formula [7], which relates the topological susceptibility $\chi_t$ to the mass of the $\eta$. This formula does work even when the large $N$ limit of the susceptibility is replaced by its SU(3) value. This fact possibly implies that SU(3) is close to SU($\infty$). In Fig. 2 we report results for $\chi_t^{1/4}/\sqrt{\sigma}$, which has discretisation artifacts proportional to $a^2 \sigma$. Our fit gives $\chi_t^{1/4}/\sqrt{\sigma} = 0.3739(59)+0.439/N^2$ for $2 \leq N \leq 5$.

As $N$ increases and for large values of the lattice size large correlations appear in the time se-

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1In this work, our error bars refer to a confidence level of 68%, while in Ref. [2] a confidence level of 75% was taken.
We have studied the ratio $\sigma_k/\sigma_f$ in both $D=3+1$ and $D=2+1$ [4,5]. Our results for its zero temperature value at various $N$ and $k$ are reported in Table 1. Those values should be compared to predictions based on various scenarios for the confining vacuum. Our data immediately exclude unbound strings ($\sigma_k \propto k \sigma_f$); also the bag model [8] ($\sigma_k \propto \sqrt{k(N-K)(N+1)/2N}$) is excluded. Our results in $D=3+1$ are in agreement at the $2\sigma$ level with both the Casimir scaling [9] ($\sigma_k/\sigma_f = (k(N-k))/(N-1)$) and the MQCD [3] ($\sigma_k/\sigma_f = \sin(k\pi/N)/\sin(\pi/N)$) predictions. Other calculations for $D=3+1$ SU(6) find a better agreement with MQCD [10]. Our results in $D=2+1$ seem to favour Casimir scaling.

At high temperature the spatial $k$-string tensions are found to be closer to Casimir scaling than to MQCD (the evidence being more striking in $D=2+1$). Our $D=3+1$ high $T$ results for the spatial $k$-string tensions agree with a recent model calculation [11]. Other evidences for Casimir scaling in $D=2+1$ are discussed in [8].

6. Conclusions

We have reported results from the first modern lattice calculation aimed at investigating the large $N$ limit of SU($N$) gauge theories in $D=3+1$. We have shown that this limit seems to exist. Moreover, the values of observables like the masses of the lowest-lying glueballs and the topological susceptibility at finite $N$ are related to their $N=\infty$ counterpart by a simple $1/N^2$ correction.

We have also investigated the ratios $\sigma_k/\sigma_f$ for stable $k$-strings. While those ratios seems to fulfill Casimir scaling in $D=2+1$ in various regimes and possibly in $D=3+1$ for the high $T$ spatial string tensions, our data can not resolve between that prediction and the one based on MQCD at zero temperature in $D=3+1$.

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