Explicit Relation of Quantum Hall Effect and Calogero-Sutherland Model

Hiroo Azuma and Satoshi Iso

Department of Physics, Faculty of Science, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113, Japan

Abstract

Explicit relation between Laughlin state of the quantum Hall effect and one-dimensional (1D) model with long-ranged interaction \(1/r^2\) is discussed. By rewriting lowest Landau level wave functions in terms of 1D representation, Laughlin state can be written as a deformation of the ground state of Calogero-Sutherland model. Corresponding to Laughlin state on different geometries, different types of 1D \(1/r^2\) interaction models are derived.
1 Introduction

Recent studies of one-dimensional (1D) integrable models with $1/r^2$ interaction \cite{1} \cite{2} show that the model has a close similarity to the quantum Hall effect (QHE) \cite{3}. In both models, the ground state is given by the Jastrow form and the excited states are constructed by multiplying polynomials to the ground state. Besides wave functions, some properties common in both models have been discussed. It is pointed out that the excited states of Haldane-Shastry model (spin model with $1/r^2$ interaction) have fractional statistics as the quasi-particle excitation in QHE \cite{4}. Hierarchy extension of 1D model is also studied by Kawakami \cite{5}. He constructed a generalized 1D $1/r^2$ model with the same hierarchy as the QHE and showed that the matrix classifying the excitation is the same as the topological order matrix introduced in the QHE. And the same algebraic structure ($W_{1+\infty}$ algebra) has been studied in both models to characterize their universal structure \cite{6} \cite{7}. In spite of these similarities, it has not yet been clarified if there is any explicit relation between them.

In this paper, we show that there is indeed an explicit relationship between these two different models. If two-dimensional electrons in strong magnetic field are constrained in the lowest Landau level, two of four phase space degrees of freedom are frozen and effective degrees are reduced. Hence, wave functions in the lowest Landau level can be represented in 1D form. This makes it possible to relate 2D QHE with 1D $1/r^2$ model. In this 1D representation, Laughlin states of FQHE on different geometries are shown to correspond to different 1D models with $1/r^2$ interaction. Laughlin state on disk is shown to be rewritten as a one-parameter deformation of the ground state of Calogero model (1D integrable model with $1/r^2$ interaction in harmonic potential). The deformation parameter is the magnetic field $B$. The Laughlin wave function on cylinder is rewritten as a deformation of the ground state of Sutherland model (periodic 1D model with $1/\sin(r)^2$ interaction without harmonic potential).

By these correspondences, it is shown that both models have many common properties. Excitations in 1D $1/r^2$ model corresponding to quasi-holes (particles) in QHE must have fractional charge and statistics. Moreover, the same algebraic structure ($W_{1+\infty}$ algebra) will characterize their universal structure.
2 Kinematics

For a planar electron in a magnetic field normal to plane Hamiltonian is given by

$$H_0 = \sum_{i=1,2} \frac{(\Pi_i)^2}{2m}, \quad \Pi_i = p_i - A_i, \quad i = 1, 2. \quad (1)$$

Here we set $c = \hbar = e = 1$ and assume that the constant magnetic field is in the negative $z$ direction. By defining an annihilation operator

$$a = (\Pi_x - i\Pi_y)/\sqrt{2B}, \quad [a, a^\dagger] = 1, \quad (2)$$

$H_0$ is written as $H_0 = \omega_c(a^\dagger a + 1/2), \omega_c = B/m$. Heisenberg equation of motion

$$\pi_x = i[H_0, \pi_x] = -\omega_c\pi_y, \quad \pi_y = i[H_0, \pi_y] = \omega_c\pi_x \quad (3)$$

show that $(\pi_x, \pi_y)$ rotate with frequency $\omega_c$ and therefore represent cyclotron motion in a magnetic field. Spectrum of $H_0$ is quantized as Landau levels. States in the lowest Landau level (LLL) satisfy the LLL condition $a\phi_0 = 0$ and $a^\dagger$ creates states in higher Landau levels. The guiding center coordinates of the cyclotron motion are defined by

$$X = x - \frac{\Pi_y}{B}, \quad Y = y + \frac{\Pi_x}{B}, \quad [X, Y] = \frac{i}{B}. \quad (4)$$

They commute with $a$ and $a^\dagger$ and therefore with $H_0$. These coordinates describe degeneracy in each Landau level. These four variables $(\Pi_x, \Pi_y, X, Y)$ are more convenient phase space variables than $(p_x, p_y, x, y)$ in a constant magnetic field.

In a strong magnetic field, all electrons are confined in LLL whose one-particle wave functions satisfy the LLL constraint $a\phi = 0$. If we constrain the Hilbert space onto the LLL, two of four phase space variables, $(\Pi_x, \Pi_y)$ are frozen and remaining degrees of freedom are the guiding center coordinates $(X, Y)$. Therefore, effective degrees of freedom in the LLL are reduced to the half of the total degrees of freedom and the two-dimensional $(X, Y)$-coordinate space can be seen as a phase space of 1D system [8]. That is, when $X$ is diagonalized $X|s\rangle = s|s\rangle$, Y is interpreted as its dual momentum $Y =
\( p_s/B \), eigenstate of \( X \), is uniquely determined with the LLL condition and forms a complete basis in the LLL;

\[
\int |s\rangle\langle s| ds = 1.
\] (5)

Since any LLL wave function can be written in terms of \(|s\rangle\), wave functions and Hamiltonian in LLL can be interpreted as those of 1D system whose coordinate is \( s \). In the following, we study \( s \)-representation (or 1D representation) of LLL wave functions.

3 1D representation of disk Laughlin state

First let’s consider 1D representation of one-particle LLL wave function in symmetric gauge; \( \mathbf{A} = (B y/2, -B x/2) \). In this gauge, the annihilation operator \( a \) is given by \( a = -i(\partial_z + \bar{z}/2) \) where \( z = \sqrt{B/2}(x + iy) \) and the LLL wave functions are written as \( \langle z\bar{z}|\Psi\rangle = \Psi(\bar{z})e^{-|z|^2/2} \). Normalized eigenfunction \(|s\rangle\) of the guiding center coordinate \( X = i\partial_y/B + x/2 \) in the LLL is given by

\[
\frac{1}{\sqrt{2\pi}}\langle z\bar{z}|s\rangle = \left(\frac{B}{\pi}\right)^{1/4} e^{-Bs^2/2} e^{\frac{\sqrt{2Bsz-\bar{z}^2}}{2}} e^{-|z|^2/2}.
\] (6)

From equation (6), 1D representation of LLL wave function \( \langle z\bar{z}|\Psi\rangle = \Psi(\bar{z})e^{-|z|^2/2} \) is given by

\[
\langle s|\Psi\rangle = \int \langle s|z\bar{z}\rangle \langle z\bar{z}|\Psi\rangle \frac{d^2z}{\pi}
\]

\[
= \frac{1}{\sqrt{2\pi}} \left(\frac{B}{\pi}\right)^{1/4} \int e^{-Bs^2/2} e^{\frac{\sqrt{2Bsz-\bar{z}^2}}{2}} e^{-|z|^2/2} \Psi(\bar{z}) \frac{d^2z}{\pi}
\]

\[
= \frac{1}{\sqrt{2\pi}} \left(\frac{B}{\pi}\right)^{1/4} e^{-Bs^2/2} e^{\frac{\sqrt{2Bsz-\bar{z}^2}}{2}} \Psi(\sqrt{2Bs}).
\] (7)

In the last equality, we used the coherent state identity

\[
\int e^{az-|z|^2} \Psi(z) \frac{d^2z}{\pi} = \Psi(a).
\] (8)
Using 1D representation of LLL wave functions, dynamics in LLL can be interpreted as dynamics of 1D system.

Now we generalize (9) to many-particles case. N-particles wave functions in the LLL are generally written as
\[
\langle z_1 \bar{z}_1 \cdots z_N \bar{z}_N | \Psi \rangle = \Psi(\bar{z}_1 \cdots \bar{z}_N) e^{-\sum_i |z_i|^2/2}.
\]
(9)

1D representation of the wave function (9) is
\[
\langle s_1 \cdots s_N | \Psi \rangle = \left(\frac{B}{4}\right)^{N/4} e^{-B \sum_i s_i^2/2} e^{-\frac{1}{4B} \sum_i \left(\frac{\partial}{\partial s_i}\right)^2} \Psi(\sqrt{2Bs_1}, \ldots, \sqrt{2Bs_N}).
\]
(10)

For the Laughlin state \(\Psi_m(\bar{z}_1 \cdots \bar{z}_N) = \prod_{i<j}(\bar{z}_i - \bar{z}_j)^m\), 1D representation becomes
\[
\langle s_1 \cdots s_N | \Psi \rangle = e^{-B \sum_i s_i^2/2} e^{-\frac{1}{4B} \sum_i \left(\frac{\partial}{\partial s_i}\right)^2} \prod_{i<j}(s_i - s_j)^m.
\]
(11)

Here we neglected a constant normalization factor for simplicity. For \(m = 1\), as is expected, this is the Slater determinant of the lowest N eigenstates of a harmonic oscillator.

The 1D representation of Laughlin state (11) has an interesting property. Momentum representation of (11) is given by the same form
\[
\langle t_1 \cdots t_N | \Psi \rangle = e^{-B \sum_i t_i^2/2} e^{-\frac{1}{4B} \sum_i \left(\frac{\partial}{\partial t_i}\right)^2} \prod_{i<j}(t_i - t_j)^m
\]
(12)

where \(\langle s|t\rangle = e^{i\hbar s_1/\sqrt{2\pi}} (\text{momentum } p \text{ is set by } p_i = Bt_i)\). This duality is due to rotational invariance of the Laughlin state on disk (circular droplet). In momentum representation, guiding center coordinate \(Y\) is diagonalized by \(Y|t\rangle = t|t\rangle\) and therefore 1D representation (11) (\(X\) is taken as a 1D coordinate) and (12) (\(Y\) is taken as a 1D coordinate) must have the same form. Here we comment on rotational invariance. Rotation generator for guiding center coordinates is given by
\[
R = \sum_i \left\{ \frac{B}{2}(X_i^2 + Y_i^2) - \frac{1}{2} \right\}, \quad [R, X_i] = -iY_i, \quad [R, Y_i] = iX_i.
\]
(13)
It is easy to prove that
\[ \text{Re}^{-B \sum_i s_i^2/2} \prod_i \left( \frac{\partial}{\partial s_i} \right)^2 f(s_1, \ldots, s_N) \]
\[ = e^{-B \sum_i s_i^2/2} e^{-\frac{1}{4B} \sum_i \left( \frac{\partial}{\partial s_i} \right)^2 \left( \sum_i s_i \left( \frac{\partial}{\partial s_i} \right) \right)} f(s_1, \ldots, s_N). \] (14)

Therefore LLL state \( e^{-B \sum_i s_i^2/2} e^{-\frac{1}{4B} \sum_i \left( \frac{\partial}{\partial s_i} \right)^2 \left( \sum_i s_i \left( \frac{\partial}{\partial s_i} \right) \right)} f(s_1, \ldots, s_N) \) is rotational invariant if \( f(s_1, \ldots, s_N) \) is a homogeneous function of \( s_i \).

Rewriting (11) by dimensionless parameter \( \tilde{s} = \sqrt{B} s \), no dimensionful parameter as magnetic field \( B \) exists. Long distance behaviour (\( \sqrt{B} |s_i - s_j| = |\tilde{s}_i - \tilde{s}_j| \gg 1 \)) of the wave function eq.(11) is described by the wave function \( (B \to \infty \text{ limit of (11)}) \)
\[ \langle s_1 \cdots s_N | \Psi \rangle = e^{-B \sum_i s_i^2/2} \prod_{i<j} (s_i - s_j)^m. \] (15)

It is an exact form for \( m = 1 \) state (11). This is the well-known groundstate wave function of 1D integrable model with \( 1/r^2 \) interaction in harmonic potential (Calogero model);
\[ H = \sum_i \frac{p_i^2}{2} + \sum_{i<j} \frac{m^2 - m}{(s_i - s_j)^2} + \sum_i \frac{B^2 s_i^2}{2}. \] (16)

Short distance behaviour (\( |\tilde{s}_i - \tilde{s}_j| \ll 1 \)), on the other hand, is described by the ground state of Calogero model in \( t \)-space [9].

Excited states also correspond between QHE and Calogero model in 1D. Quasi-holes in QHE are constructed by multiplying \( \prod_i (\bar{z}_i - \bar{z}_0) \) on the Laughlin state. Then, in 1D representation, this excited state is constructed by multiplying \( \prod_i (\sqrt{2}B\tilde{s}_i - \bar{z}_0) \) on the Calogero ground state. Since the quasi-hole has \( 1/m \) fractional statistics and fractional charge independent of the magnetic field or the shape of the droplet, corresponding excited states in 1D \( 1/r^2 \) model also have the same fractional statistics and fractional charge.

4 1D representation of cylinder Laughlin state

Next let’s consider 1D representation of Laughlin state on cylinder. Here we use Landau gauge for convenience \( A = (By, 0) \). In this gauge, LLL
wave functions are written as \( \Psi(\bar{z}) e^{-By^2/2} \). We impose a periodic boundary condition for \( x \) with period \( L_x \). Then an anti-holomorphic part of a LLL wave function can be written as a linear combination of \( \exp[2\pi i n(x - iy)/L_x] = \omega^n \) where \( \omega \equiv \exp[2\pi i (x - iy)/L_x] \). Filling factor \( \nu = 1 \) state is given by

\[
\prod_{i<j} (\omega_i - \omega_j) e^{-B \sum_s \bar{y}_i^2/2}.
\]  

(17)

Since \( \prod_{i<j} (\omega_i - \omega_j) \) is a Slater determinant of \( (1, \omega, ..., \omega^{N-1}) \), there are two boundaries at \( y = 0 \) and \( y = 2\pi (N - 1)/BL_x \). Laughlin state with filling factor \( \nu = 1/m \) can be constructed as

\[
\prod_{i<j} (\omega_i - \omega_j)^m e^{-B \sum_s \bar{y}_i^2/2}.
\]  

(18)

Note that its short distance behaviour is the same as the disk Laughlin state. Filled region is expanded by \( m \)-times and boundaries are located at \( y = 0 \) and \( y = 2\pi m (N - 1)/BL_x \).

Now let’s consider 1D representation of eq.(18). In Landau gauge, eigenstate of \( X = x + i\partial_y/B \) in LLL is given by

\[
\langle \bar{z} \bar{z} | s \rangle = \frac{1}{\sqrt{2\pi}} \left( \frac{B}{\pi} \right)^{1/4} e^{-Bs^2/2} e^{\sqrt{2Bs} \bar{z} - \bar{z}^2} e^{-By^2/2}.
\]  

(19)

Since we identify \( s \) and \( s + L_x \). we must sum all \( s \) mod \( L_x \);

\[
|s\rangle \rightarrow |s\rangle_{\text{per.}} \equiv \sum_n |s + nL_x\rangle.
\]  

(20)

Then \( \langle \bar{z} \bar{z} | s \rangle_{\text{per.}} \) is shown to be under a shift \( x \rightarrow x + L_x \) and can be written as a linear combination of \( \omega^n \). 1D representation of LLL wave function \( \langle \bar{z} \bar{z} | \Psi \rangle = \Psi(\bar{z}) e^{-By^2/2} \) is

\[
\langle s | \Psi \rangle = \int \langle s | \bar{z} \bar{z} \rangle \langle \bar{z} \bar{z} | \Psi \rangle \frac{d^2 z}{\pi}
\]

\[
= \frac{1}{\sqrt{2\pi}} \left( \frac{B}{\pi} \right)^{1/4} \int e^{-Bs^2/2} e^{\sqrt{2Bs} \bar{z} - \bar{z}^2} e^{-|z|^2} e^{s^2/2} \Psi(\bar{z}) \frac{d^2 z}{\pi}
\]

\[
= \frac{1}{\sqrt{2\pi}} \left( \frac{B}{\pi} \right)^{1/4} e^{-Bs^2/2} e^{- \frac{1}{4\pi} \left( \frac{d}{dz} \right)^2} e^{Bs^2} \Psi(\sqrt{2Bs})
\]
\[
\langle s_1 \cdots s_N | \Psi \rangle = e^{-\frac{1}{2} \sum_i (\frac{\theta_i}{\pi})^2} \prod_{i<j} (e^{i 2 \pi s_i / L_x} - e^{i 2 \pi s_j / L_x})^m. \tag{22}
\]

In \( B \to \infty \) limit, this wave function reduces to the ground state of Sutherland model (periodic 1D model with \( 1 / \sin^2(\pi (s_i - s_j) / L_x) \) interaction) \[10\]:

\[
\prod_{i<j} (e^{i 2 \pi s_i / L_x} - e^{i 2 \pi s_j / L_x})^m. \tag{23}
\]

For \( m = 1 \), \( \sum_i (\partial_i)^2 \) becomes a constant and this is exact. In strong magnetic field limit, the width of cylinder Laughlin state \( \delta y = 2 \pi (N - 1) m / BL_x \) becomes infinitesimal. In disk case, Laughlin state is reduced to 1D system in harmonic potential. But Laughlin state on cylinder is reduced to 1D system without external potential. This is due to the difference in shape of the droplet on two-dimensional phase space.

Now let’s study the momentum representation, or \( Y \)-diagonalized representation of (21). Set \( |t\rangle \) by \( \langle s | t \rangle = e^{i B Ts} / \sqrt{2 \pi} \) as before. Then Fourier-transformation of (21) gives

\[
\langle t | \Psi \rangle = \frac{1}{\sqrt{2 \pi B}} \left( \frac{B}{\pi} \right)^{1/4} e^{-B t^2 / 2} e^{-\frac{1}{4 \pi} (\frac{\theta}{\pi})^2} e^{-B t^2} \Psi(-i \sqrt{2 B} t). \tag{24}
\]

To derive it, we used the second form of eq. (21). Rewriting (24), it becomes

\[
\langle t | \Psi \rangle = \frac{1}{\sqrt{2 \pi B}} \left( \frac{B}{\pi} \right)^{1/4} g_0 e^{-\frac{1}{2 \pi} (\frac{\theta}{\pi})^2} e^{-B t^2 / 2} \Psi(-i \sqrt{2 B} t). \tag{25}
\]
By the replacement $\bar{z} \rightarrow -i\sqrt{B/2t}$, $\omega = \exp[2\pi i/L_x(x-iy)]$ becomes $\exp(2\pi t/L_x)$.

Therefore, Laughlin state on cylinder (18) has the following $t$-representation;

$$\langle t_1 \cdots t_N | \Psi \rangle = e^{-\frac{1}{2}B \sum_i \left(\frac{\partial}{\partial t_i}\right)^2} e^{-B \sum_i t_i^2/2} \prod_{i<j} (e^{2\pi t_i/L_x} - e^{2\pi t_j/L_x})^m$$

$$\propto e^{-\frac{1}{2B} \sum_i \left(\frac{\partial}{\partial t_i}\right)^2} e^{-B \sum_i (t_i - t_j)^2/2} \prod_{i<j} (\sinh \pi (t_i - t_j)/L_x)^m,$$

(26)

where $t_0 \equiv m(N-1)\pi/BL_x$. In large $B$ limit, this becomes

$$e^{-B \sum_i (t_i - t_0)^2/2} \prod_{i<j} (\sinh \pi (t_i - t_j)/L_x)^m.$$  

(27)

Note that $t_0$ is the center of the two boundaries (at $Y = 0$ and $Y = 2\pi m(N-1)/BL_x$) of the cylinder Laughlin droplet.

5 Conclusion

In this letter, we presented an explicit relation between Laughlin state and one-dimensional integrable model with $1/r^2$ interaction. In one-dimensional representation of lowest Landau level wave functions, Laughlin state can be written as a one-parameter deformation of the ground state of 1D model $1/r^2$ model. Different types of 1D models are derived ($1/r^2$, $1/\sin^2 r$ and $1/\sinh^2 r$) corresponding to Laughlin state on different geometries. The deformation parameter is magnetic field $B$.

Finally we list some topics which we will discuss in separate papers [11].

(1) Laughlin state on torus and its 1D model
(2) $W_{1+\infty}$ algebra in QHE and 1D $1/r^2$ model
(3) extension to hierarchy and $SU(N)$ generalization of 1D model
(4) Tomonaga-Luttinger liquid behaviour of Laughlin state and relation with edge state
(5) $X$-$Y$ duality in two-dimensions and duality of long-distance and short distance physics in one-dimension.

We gratefully acknowledge helpful discussion with B. Sakita on QHE and with K. Hikami on Calogero-Sutherland model. S. Iso is supported by Grant-
in Aid for Scientific Research from the Ministry of Education, Science and Culture in Japan.
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