LOGARITHMIC KNOT INVARIANTS ARISING FROM RESTRICTED QUANTUM GROUPS

JUN MURAKAMI AND KIYOKAZU NAGATOMO

ABSTRACT. We construct knot invariants from the radical part of projective modules of the restricted quantum group \( \mathcal{U}_q(sl_2) \) at \( q = \exp(\pi \sqrt{-1}/p) \), and we also show a relation between these invariants and the colored Alexander invariants. These projective modules are related to logarithmic conformal field theories.

1. INTRODUCTION

Various knot invariants are constructed from the quantum \( R \)-matrix of the quantum group. However, most of them are constructed from semisimple algebras. Our concern in this note is constructing knot invariant arising from non-semisimple representations. We focus on the restricted quantum group \( \mathcal{U}_q(sl_2) \) and construct knot invariant which is understood as a derivative of the colored Alexander invariant \([1],[9]\).

Let \( Z \) and \( J \) be the center and the Jacobson radical of \( \mathcal{U}_q(sl_2) \) respectively. Then \( Z \) is a direct sum of \( Z(s) \) and \( Z(r) \), where \( Z(s) \) is the subalgebra of \( Z \) generated by the primitive idempotents and \( Z(r) = Z \cap J \). Let \( K \) be a knot in \( S^3 \). By using the idea of the universal invariant in \([10]\), we can associate an element \( z_K \in Z \) with \( K \). Then \( z_K \) is expressed as

\[
   z_K = z_K^{(s)} + z_K^{(r)} \quad (z_K^{(s)} \in Z^{(s)}, \ z_K^{(r)} \in Z^{(r)}).
\]

The first term \( z_K^{(s)} \) is a linear combination of the primitive idempotents and the coefficient of each idempotent corresponds to the colored Jones invariant. In this paper, we study about the knot invariants coming from \( z_K^{(r)} \). The space \( Z^{(r)} = Z \cap J \) has a natural basis corresponding to the indecomposable modules, and the coefficients of \( z_K^{(r)} \) with respect to this basis are also knot invariants.

In the construction of \( z_K \), we assume that \( K \) is a single component knot. For a multi-component link, the construction in this paper does not work well and we need another idea to extend \( z_K^{(r)} \) for link case.

A three-manifold invariant is constructed from the ‘integral’ of \( \mathcal{U}_q(sl_2) \) \([8],[4],[7],[11]\), which is defined for a finite dimensional Hopf algebra. Meanwhile, an action of \( SL(2,\mathbb{Z}) \) on the center \( Z \) of \( \mathcal{U}_q(sl_2) \) is given in \([6]\) and \([3]\). By using our invariants constructed here, these two theories can be combined as the usual topological quantum field theory, e.g. \([12],[2]\), related to \( \mathcal{U}_q(sl_2) \). The detail will be given elsewhere.

We review the definition of \( \mathcal{U}_q(sl_2) \) and its representations in Section 2. The construction of \( z_K^{(r)} \) is given in Section 3. In Sections 4 and 5, we show some property of the invariants coming from \( z_K \), especially the relation to the colored Alexander invariant in \([9]\).

2. RESTRICTED QUANTUM GROUPS

2.1. Definition. Let \( p \geq 2 \) be a positive integer and \( q = \exp(\pi \sqrt{-1}/p) \). The semistricted quantum group \( \mathcal{U}_q(sl_2) \) is the quotient of the usual quantum group \( \mathcal{U}_q(sl_2) \).
defined by the following generators and relations as an algebra.

\[ \hat{U}_q(sl_2) = \langle K, K^{-1}, E, F \mid K K^{-1} = K^{-1} K = 1, \]
\[ K E K^{-1} = q^2 E, \quad K F K^{-1} = q^{-2} F, \quad E F - F E = \frac{K - K^{-1}}{q - q^{-1}}, \]
\[ E^p = F^p = 0 \] .

The restricted quantum group \( \overline{U}_q(sl_2) \) is obtained from \( \hat{U}_q(sl_2) \) by inquiring one more relation \( K^{2p} = 1 \). The coproduct, counit and antipode of \( \hat{U}_q(sl_2) \) and \( \overline{U}_q(sl_2) \) are defined as follows.

\[ \Delta(K) = K \otimes K, \quad \Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \]
\[ \epsilon(E) = \epsilon(F) = 0, \quad \epsilon(K) = 1, \]
\[ S(E) = -E K^{-1}, \quad S(F) = -K F, \quad S(K) = K^{-1}. \]

2.2. \textbf{R-matrix.} By introducing a symbol \( k \) such that \( k^2 = K \), we can define an \( R \)-matrix \( R \) of \( \overline{U}_q(sl_2) \) satisfying

\[ R \Delta(X) R^{-1} = \overline{\Delta}(X), \]
where \( \overline{\Delta}(X) = \sum_i z_i \otimes y_i \) if \( \Delta(X) = \sum_i y_i \otimes z_i \). The explicit form of \( R \) is given as follows.

\[ R = q^{\frac{1}{2}H^\times H} \sum_{n=0}^{p-1} \left( \frac{q - q^{-1}}{n!} \right) q^{\frac{n(n-1)}{2}} (E^n \otimes F^n), \]

where \( [n] = \frac{q^n - q^{-n}}{q - q^{-1}} \) and \( [n]! = [n][n-1] \cdots [1] \), and \( H \) is an element such that \( q^H = K \).

2.3. \textbf{Irreducible modules.} The irreducible modules \( X^\alpha(s) \) of \( \overline{U}_q(sl_2) \) are labeled by \( \alpha = \pm 1 \) and \( 1 \leq s \leq p \), and is spanned by weight vectors \( |s, n\rangle^\pm, 0 \leq n \leq s - 1 \) with the action of \( \overline{U}_q(sl_2) \) given by

\[ K |s, n\rangle^\pm = \pm q^{s-1-2n} |s, n\rangle^\pm, \]
\[ E |s, n\rangle^\pm = \pm [n][s-n] |s, n-1\rangle^\pm, \]
\[ F |s, n\rangle^\pm = |s, n+1\rangle^\pm, \]
where \( |s, s\rangle^\pm = 0 \).

2.4. \textbf{Projective modules.} Projective modules \( \mathcal{P}^\pm(s) \) of \( \overline{U}_q(sl_2) \) which are fundamental to investigate the structure of the center of \( \mathcal{U}_q(sl_2) \) are labeled by \( 1 \leq s \leq p - 1 \). Note that a \( \overline{U}_q(sl_2) \) module is also naturally a \( \hat{U}_q(sl_2) \) module.

Let \( s \) be any integer \( 1 \leq s \leq p - 1 \). The projective module \( \mathcal{P}^+(s) \) has the basis

\[ \{ x_k^{(+,s)}, y_k^{(+,s)} \}_{0 \leq k \leq p-s-1} \cup \{ a_n^{(+,s)}, b_n^{(+,s)} \}_{0 \leq n \leq s-1}, \]
and the action of \( \overline{U}_q(sl_2) \) is given by

\[ K x_k^{(+,s)} = -q^{p-s-1-2k} x_k^{(+,s)}, \quad K y_k^{(+,s)} = -q^{p-s-1-2k} y_k^{(+,s)}, \quad 0 \leq k \leq p-s-1, \]
\[ K a_n^{(+,s)} = q^{s-1-2n} a_n^{(+,s)}, \quad K b_n^{(+,s)} = q^{s-1-2n} b_n^{(+,s)}, \quad 0 \leq n \leq s-1, \]
\[ E x_k^{(+,s)} = -[k][p-s-k] x_{k-1}^{(+,s)}, \quad 0 \leq k \leq p-s-1, \quad (\text{with } x_{-1}^{(+,s)} = 0), \]
and the action of the canonical central elements in $\mathfrak{z}$ as follows: Two special primitive idempotents

$$E y_k^{(+,s)} = \begin{cases} -[k][p - s - k] y_{k-1}^{(+,s)}, & 1 \leq k \leq p - s - 1, \\ a_{s-1}^{(+,s)}, & k = 0, \end{cases}$$

$$E a_n^{(+,s)} = [n][s - n] a_{n-1}^{(+,s)}, \quad 0 \leq n \leq s - 1, \quad \text{(with } a_{-1}^{(+,s)} = 0),$$

$$E b_n^{(+,s)} = \begin{cases} [n][s - n] b_{n-1}^{(+,s)} + a_{n-1}^{(+,s)}, & 1 \leq n \leq s - 1, \\ x_{p-s-1}^{(+,s)}, & n = 0, \end{cases}$$

and

$$F x_k^{(+,s)} = \begin{cases} x_{k+1}^{(+,s)}, & 0 \leq k \leq p - s - 2, \\ a_0^{(+,s)}, & k = p - s - 1, \end{cases}$$

$$F y_k^{(+,s)} = y_{k+1}^{(+,s)}, \quad 0 \leq k \leq p - s - 1, \quad \text{(with } y_{p-s}^{(+,s)} = 0),$$

$$F a_n^{(+,s)} = a_{n+1}^{(+,s)}, \quad 0 \leq n \leq s - 1, \quad \text{(with } a_s^{(+,s)} = 0),$$

$$F b_n^{(+,s)} = \begin{cases} b_{n+1}^{(+,s)}, & 0 \leq n \leq s - 2, \\ y_0^{(+,s)}, & n = s - 1. \end{cases}$$

Let $s$ be an integer $1 \leq s \leq p - 1$. The projective module $P^{-}(p-s)$ has the basis

$$\{x_k^{(-,s)}, y_k^{(-,s)}\}_{0 \leq k \leq p-s-1} \cup \{a_n^{(-,s)}, b_n^{(-,s)}\}_{0 \leq n \leq s-1},$$

and the action of $\mathfrak{u}_q(sl_2)$ is given by

$$K x_k^{(-,s)} = -q^{p-s-1-2k} x_{k}^{(+,s)}, \quad K y_k^{(+,s)} = -q^{p-s-1-2k} y_k^{(-,s)}, \quad 0 \leq k \leq p - s - 1,$$

$$K a_n^{(-,s)} = q^{s-1-2n} a_{n}^{(-,s)}, \quad K b_n^{(-,s)} = q^{s-1-2n} b_{n}^{(-,s)}, \quad 0 \leq n \leq s - 1,$$

$$E x_k^{(-,s)} = -[k][p - s - k] x_{k-1}^{(-,s)}, \quad 0 \leq k \leq p - s - 1, \quad \text{(with } x_{-1}^{(-,s)} = 0)$$

$$E y_k^{(-,s)} = \begin{cases} -[k][p - s - k] y_{k-1}^{(-,s)} + x_{k-1}^{(-,s)}, & 1 \leq k \leq p - s - 1, \\ a_{s-1}^{(-,s)}, & k = 0, \end{cases}$$

$$E a_n^{(-,s)} = [n][s - n] a_{n-1}^{(-,s)}, \quad 0 \leq n \leq s - 1, \quad \text{(with } a_{-1}^{(-,s)} = 0),$$

$$E b_n^{(-,s)} = \begin{cases} [n][s - n] b_{n-1}^{(-,s)} + a_{n-1}^{(+,s)}, & 1 \leq n \leq s - 1, \\ x_{p-s-1}^{(-,s)}, & n = 0, \end{cases}$$

and

$$F x_k^{(-,s)} = x_{k+1}^{(-,s)}, \quad 0 \leq k \leq p - s - 1, \quad \text{(with } x_{p-s}^{(-,s)} = 0),$$

$$F y_k^{(-,s)} = \begin{cases} y_{k+1}^{(-,s)}, & 0 \leq k \leq p - s - 2, \\ b_0^{(-,s)}, & k = p - s - 1, \end{cases}$$

$$F a_n^{(-,s)} = \begin{cases} a_{n+1}^{(-,s)}, & 0 \leq n \leq s - 2, \\ x_0^{(-,s)}, & n = s - 1, \end{cases}$$

$$F b_n^{(-,s)} = b_{n+1}^{(-,s)}, \quad 0 \leq n \leq s - 1, \quad \text{(with } b_s^{(-,s)} = 0).$$

Note that the diagonal part is a direct sum of irreducible modules.

2.5. **Center.** The dimension of the center $\mathcal{Z}$ of $\mathfrak{u}_q(sl_2)$ is $3p - 1$. The basis of $\mathcal{Z}$ is given by the canonical central elements in $\mathfrak{z}$ as follows: Two special primitive idempotents $e_0$ and $e_p$, other primitive idempotents $e_s$, ($1 \leq s \leq p - 1$), and $2(p - 1)$ elements $w_1^{\pm}$.
(1 \leq s \leq p - 1) corresponding to the radical part. These basis satisfy the following relations.

\begin{align}
    e_s e_{s'} &= \delta_{s,s'} e_s, \\
    e_s w^\pm_s &= \delta_{s,s'} w^\pm_{s'}, \\
    w^\pm_s w^\mp_{s'} &= 0,
\end{align}

3. Logarithmic invariants of knots

3.1. Knots and (1,1)-tangles. In this paper, knots and tangles are oriented and framed. For a connected (1,1)-tangle \( T \), let \( K_T \) be the knot obtained by joining the two open ends as in Figure 1. For two tangles \( T \) and \( T' \), it is known that \( K_T \) and \( K_{T'} \) are isotopic as framed knots if and only if \( T \) and \( T' \) are isotopic as framed tangles. So, in the following, we sometimes mix up invariants of connected (1,1)-tangles and invariants of knots.

![Figure 1. Cosure of a framed tangle.](image)

3.2. Framed braid. Framed braid group on \( n \) strings \( FB_n \) is defined by the following generators and relations.

\[ FB_n = \langle \sigma_1, \sigma_2, \ldots, \sigma_{n-1}, \tau_1, \tau_2, \ldots, \tau_n \mid \]
\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad (i = 1, 2, \ldots, n - 2), \]
\[ \sigma_i \sigma_j = \sigma_j \sigma_i, \quad (|i - j| > 1), \]
\[ \tau_i^{\pm 1} \sigma_i = \sigma_i \tau_{i+1}^{\pm 1}, \quad \tau_{i+1} \sigma_i = \sigma_i \tau_i^{\pm 1}, \quad (i = 1, 2, \ldots, n - 1), \]
\[ \sigma_i \tau_j = \tau_j \sigma_i, \quad \tau_i \tau_j = \tau_j \tau_i \quad (|i - j| > 1) \rangle. \]

The generators \( \sigma_i^{\pm 1} \) correspond to the positive and negative crossings, and \( \tau_i^{\pm 1} \) represent the blackboard framing corresponding to the twist as in Figure 2. Let \( S_n \) be the symmetric group of \( n \) letters \( \{1, 2, \ldots, n\} \), and \( \pi \) be the group homomorphism from \( FB_n \) to \( S_n \) sending \( \sigma_i \) to the transposition \( (i, i+1) \) and \( \tau_i \) to the identity for \( i = 1, 2, \ldots, n - 1 \).

3.3. Alexander’s and Markov’s theorems. Then we have the framed versions of Alexander’s and Markov’s theorem as follows.

**Theorem 3.3.1.** Any framed link is isotopic to the closure \( \hat{b} \) of some framed braid \( b \in FB_n \).

**Theorem 3.3.2.** Two framed braids \( b_1, b_2 \) have isotopic closures if and only if \( b_1 \) can be transformed to \( b_2 \) by a finite sequence of moves of the following two types.

(i) \( \beta_1 \beta_2 \leftrightarrow \beta_2 \beta_1 \) for \( \beta_1, \beta_2 \in FB_n \).
$\sigma_i \quad \sigma_i^{-1} \quad \tau_i \quad \tau_i^{-1}$

\[ \begin{array}{cccc}
\cdots & \times & \cdots & \times \\
 i & i+1 & i & i+1 \\
\cdots & \times & \cdots & \times \\
 i & i & i & i \\
\end{array} \]

Figure 2. Generators of the framed braid group $FB_n$.

(ii) $\beta \tau_n^{\pm 1} \leftrightarrow i(\beta) \sigma_n^{\pm 1}$ for $\beta \in FB_n \xrightarrow{i} FB_{n+1}$.

3.4. **Representation of $FB_n$ on** $\otimes^n U_q(sl_2)$. The universal $R$-matrix satisfies the Yang-Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} : \otimes^3 U_q(sl_2) \rightarrow \otimes^3 U_q(sl_2),$$

where $\otimes^3 U_q(sl_2) = U_q(sl_2) \otimes U_q(sl_2) \otimes U_q(sl_2)$ and $R_{ij}$ acts on the $i$-th and $j$-th components of the tensor product. Let

$$R = \sum_i r_i' \otimes r_i'' : U_q(sl_2) \otimes U_q(sl_2) \rightarrow U_q(sl_2) \otimes U_q(sl_2),$$

and

$$v = \sum_i r_i'' K^{N-1} r_i' : U_q(sl_2) \rightarrow U_q(sl_2).$$

Then $(U_q(sl_2), R, v)$ is a ribbon Hopf algebra with the ribbon element $v$. Therefore, we can define a homomorphism $\rho$ from $FB_n$ to $\text{End}(\otimes^n U_q(sl_2))$ by

$$\rho(\sigma_i)(x_1 \otimes \ldots \otimes x_i \otimes x_{i+1} \otimes \ldots \otimes x_n) = \sum_i x_1 \otimes \ldots \otimes r_i' x_{i+1} \otimes r_i'' x_i \otimes \ldots \otimes x_n,$$

$$\rho(\tau_i)(x_1 \otimes \ldots \otimes x_i \otimes \ldots \otimes x_n) = x_1 \otimes \ldots \otimes v x_i \otimes \ldots \otimes x_n.$$

3.5. **Universal invariant.** Let $K$ be a knot and let $b_K$ be a framed braid whose closure is equivalent to $K$. Then $\rho(b_K) \in \text{End}(\otimes^n U_q(sl_2))$ is expressed as follows.

$$\rho(b_K) = \sum_i a_{1,i} \otimes a_{2,i} \otimes \ldots \otimes a_{n,i}.$$

Let $T$ be a $(1,1)$-tangle corresponding to $K$. The element $z_T \in U_q(sl_2)$ corresponding to $T$ is defined by

$$z_T = a_{\pi(b_K)^{n-1}(1),i} K^{N-1} a_{\pi(b_K)^{n-2}(1),i} \ldots a_{\pi(b_K)^2(1),i} K^{N-1} a_{\pi(b_K)(1),i} K^{N-1} a_{1,i}.$$

The element $z_T \in U_q(sl_2)$ commutes with any elements of $U_q(sl_2)$ and is in the center $Z$. Therefore, we have

$$z_T = \sum_{s=0}^p (a_s(T) e_s + b^s_+(T) w^+_s + b^s_-(T) w^-_s),$$

where $a_s(T), b^s_+(T)$ are scalars and are invariants of the closure $K$ of $T$. Hence we can also denote them by $a_s(K), b^s_+(K)$. 


We show some property of $a_s(K)$ and $b_s^\pm(K)$.

**Theorem 4.0.1.**

1. For the connected sum of two knots $K_1$ and $K_2$,
   
   $$a_s(K_1 \# K_2) = a_s(K_1) a_s(K_2), \quad b_s^+(K_1 \# K_2) = a_s(K_1) b_s^+(K_2) + b_s^+(K_1) a_s(K_2).$$

2. For a knot $K$, the invariant $a_s(K)$ ($1 \leq s \leq p$) is equal to the colored Jones invariant $J_s(K)$ corresponding to the $s$-dimensional irreducible module $\mathcal{X}^+(s)$ normalized as
   
   $$J_s(\text{unknot}) = 1.$$

**Proof.** For two tangles $T_1$, $T_2$, let $T_1 \cdot T_2$ be the tangle obtained by joining $T_2$ below $T_1$. Then, for two knots $K_1$ and $K_2$, $T_{K_1} \cdot T_{K_2}$ is a tangle representing the connected sum $K_1 \# K_2$. Therefore, by using (2.2) and (3.1), we have

   $$z_{K_1 \# K_2} = z_{K_1} z_{K_2} = \sum_{s=0}^{p} (a_s(K_1) a_s(K_2)) e_s + (b_s^+(K_1) a_s(K_2) + a_s(K_1) b_s^+(K_2)) w_s + (b_s^-(K_1) a_s(K_2) + a_s(K_1) b_s^-(K_2)) w_s$$

   and we obtain (1).

   The center $e_s$ acts on $\mathcal{X}(s)$ as identity, and the other basis $e_t$ ($t \neq s$), and $w_t^+$ acts on $\mathcal{X}(s)$ as zero. Hence $a_s(K)$ corresponds to the scalar representing the action of $z_K$ on $\mathcal{X}(s)$, which is equal to the colored Jones invariant and we get (2). \qed

5. **Relation to the colored Alexander invariant**

5.1. **Relation.** Let $K$ be a framed knot and $T_K$ be the corresponding framed tangle. Let $O_s^p(T_K)$ be the scalar multiple of the colored Alexander invariant defined in [9]. Then we have the following.

**Theorem 5.1.1.** The invariants $a_s(K)$, $b_s^+(K)$, $b_s^-(K)$ are given by the colored Alexander invariants as follows.

   $$a_s(K) = O_s^{p-1}(T_K), \quad 0 \leq s \leq p,$$

   $$b_s^+(T_K) = -\frac{p \sin^2 \frac{\pi}{p}}{\sin \frac{2\pi}{p}} \left( \frac{d O_s^p(T_K)}{d\lambda} \bigg|_{\lambda = 2p-s-1} - \frac{d O_s^p(T_K)}{d\lambda} \bigg|_{\lambda = s-1} \right), \quad 1 \leq s \leq p-1,$$

   $$b_s^-(K) = \frac{p \sin^2 \frac{\pi}{p}}{\sin \frac{2\pi}{p}} \left( \frac{d O_s^p(T_K)}{d\lambda} \bigg|_{\lambda = s-1} - \frac{d O_s^p(T_K)}{d\lambda} \bigg|_{\lambda = -s-1} \right), \quad 1 \leq s \leq p-1.$$

Before proving the above, we introduce some representations of $\hat{U}_q(sl_2)$.

5.2. **Non-integral representations.** We introduce highest weight representations of $\hat{U}_q(sl_2)$ for non-integral weights and obtain the projective modules $P^\pm$ as a specialization of certain non-irreducible module.

First, we define the irreducible module for non-integer number $\lambda$ as follows. Let $\mathcal{X}(\lambda)$ be the $\hat{U}_q(sl_2)$ module spanned by weight vectors $v_n^\lambda$, $0 \leq n \leq p-1$. The action of $\hat{U}_q(sl_2)$ to $\mathcal{X}(\lambda)$ is given by

   $$K v_n^\lambda = q^{\lambda-1-2n} v_n^\lambda, \quad E v_n^\lambda = [n][\lambda - n] v_{n-1}^\lambda, \quad F v_n^\lambda = v_{n+1}^\lambda,$$

   where $v_p^\lambda = 0$. 

Next, we define a non-irreducible module which is isomorphic to direct sum of two non-integral highest modules. Let \( t \) be an integer with \( 1 \leq s \leq p \) and \( \hat{\mathcal{Y}}(\lambda, s) \) be the \( \hat{\mathcal{U}}_q(sl_2) \) module which is spanned by weight vectors \( c_j^{(\lambda,s)} \) and \( d_j^{(\lambda,s)} \) for \( 0 \leq j \leq p-1 \). The action of \( \hat{\mathcal{U}}_q(sl_2) \) is given by

\[
K c_n^{(\lambda,s)} = q^{\lambda-1-2n} c_n^{(\lambda,s)}, \quad K d_n^{(\lambda,s)} = q^{\lambda-1-2s-2n} d_n^{(\lambda,s)}, \quad 0 \leq n \leq p-1,
\]

\[
E c_n^{(\lambda,s)} = \begin{cases} 0, & n = 0, \\ [n][\lambda-n] c_{n-1}^{(\lambda,s)}, & 1 \leq n \leq p-1, \end{cases}
\]

\[
E d_n^{(\lambda,s)} = \begin{cases} [n][\lambda-2s-n] d_{n-1}^{(\lambda,s)} + c_{n-s-1}^{(\lambda,s)}, & 1 \leq n \leq p-s, \\ [n][\lambda-2s-n] d_{n-1}^{(\lambda,s)}, & p-s+1 \leq n \leq p-1, \end{cases}
\]

\[
F c_n^{(\lambda,s)} = \begin{cases} c_{n+1}^{(\lambda,s)}, & 0 \leq n \leq p-2, \\ 0, & n = p-1, \end{cases}
\]

\[
F d_n^{(\lambda,s)} = \begin{cases} d_{n+1}^{(\lambda,s)}, & 0 \leq n \leq p-2, \\ 0, & n = p-1. \end{cases}
\]

### 5.3. Colored Alexander invariant

Let \( K \) be a framed knot, \( T_K \) be the corresponding framed tangle, and \( \hat{z}_K \) be the corresponding central element in the semi-restricted quantum group \( \hat{\mathcal{U}}_q(sl_2) \) which is defined as \( z_K \) by using the universal R-matrix of \( \hat{\mathcal{U}}_q(sl_2) \) given by (2.1). Let \( Z_K^{(\lambda,s)} \) be the representation matrix of \( \hat{z}_K \) on \( \mathcal{Y}(\lambda, s) \) with respect to the above basis \( \{c_n^{(\lambda,s)}, d_n^{(\lambda,s)}; 0 \leq n \leq p-1\} \). Then the diagonal element corresponding to \( c_n^{(\lambda,s)} \) and \( d_n^{(\lambda,s)} \) \( (0 \leq n \leq p-1) \) are equal to \( O_{\lambda-1}(T_K) \) and \( O_{\lambda-1-2s}(T_K) \) respectively, where \( O_{\lambda}^{p}(T_K) \) is given in [9] as the scalar corresponding to the tangle \( T_K \). Note that \( O_{\lambda}^{p}(T_K) \) is a scalar multiple of the colored alexander invariant \( \Phi_K^{p}(\lambda) \) and \( O_{\lambda}^{p}(T_K) \) itself is also an invariant of \( K \) if \( K \) is a single component knot.

### 5.4. Proof of Theorem 4

The matrix \( Z_K^{(\lambda,s)} \) has off-diagonal elements at \( (c_{n+s}, d_n) \) components for \( 0 \leq n \leq p-s \). Let \( x \) be the \((c_s, d_0)\) component of \( Z_K^{(\lambda,s)} \). Then

\[
(5.1) \quad Z_{K}^{(\lambda,s)} c_s^{(\lambda,s)} = O_{\lambda-1}^{p}(T_K) c_s^{(\lambda,s)}, \quad Z_{K}^{(\lambda,s)} d_0^{(\lambda,s)} = O_{\lambda-1-2s}^{p}(T_K) d_0^{(\lambda,s)} + x c_s^{(\lambda,s)}.
\]

Let

\[
h = c_s^{(\lambda,s)} - [s][\lambda-s] d_0^{(\lambda,s)}.
\]

Then \( E h = 0 \) and so \( h \) is a highest weight vector of weight \( \lambda - 1 - 2s \). Therefore, on the one hand,

\[
Z_{K}^{(\lambda,s)} h = O_{\lambda-1-2s}^{p}(T_K) h = O_{\lambda-1-2s}^{p}(T_K) c_s^{(\lambda,s)} - O_{\lambda-1-2s}^{p}(T_K) [s][\lambda-s] d_0^{(\lambda,s)}.
\]

On the other hand, from (5.1),

\[
Z_{K}^{(\lambda,s)} h = O_{\lambda-1}^{p}(T_K) c_0^{(\lambda,s)} - [s][\lambda-s] x c_0^{(\lambda,s)} - O_{\lambda-1-2s}^{p}(T_K) [s][\lambda-s] d_0^{(\lambda,s)}.
\]

Thus we have

\[
(5.2) \quad O_{\lambda-2s-1}^{p}(T_K) = O_{\lambda-1}^{p}(T_K) - [s][\lambda-s] x,
\]
and then

\[ x = \frac{O^p_{\lambda-1}(T) - O^p_{\lambda-1-2s}(T)}{[s][\lambda - s]} . \]

Hence, we get

\[ \lim_{\lambda \to s + mp} x = (-1)^m \frac{p \sin \frac{\pi}{p}}{\pi |s|} \left( \left| \frac{dO^p_{\lambda-1}(T)}{d\lambda} \right|_{\lambda=2p-1} - \left| \frac{dO^p_{\lambda-1-2s}(T)}{d\lambda} \right|_{\lambda=2mp} \right), \quad m \in \mathbb{Z}. \]

The projective module \( P^+(s) \) is identical to \( \mathcal{Y}(2p - s, p - s) \) by the correspondence of the basis \( x^{(s)}(s_n) \mapsto c^{(2p-s,p-s)}_n, x^{(-s)}(n+s) \mapsto c^{(p-s,p-s)}_{n+p}, b^{(+,s)}_n \mapsto d^{(2p-s,p-s)}_n, y^{(+,s)}_n \mapsto d^{(2p-s,p-s)}_{n+s} \).

Therefore, by substituting \( 2p - s \) to \( \lambda, p - s \) to \( s \), and \( 1 \) to \( m \) for (5.3), we have

\[ a_s(T) = O^p_{2p-s-1}(T), \]

and

\[ b^+_s(T) = -\frac{p \sin \frac{2\pi}{p}}{\pi |s|} \left( \left| \frac{dO^p_{\lambda}(T)}{d\lambda} \right|_{\lambda=2p-1} - \left| \frac{dO^p_{\lambda}(T)}{d\lambda} \right|_{\lambda=s-1} \right), \quad 1 \leq s \leq p - 1. \]

Similarly, the projective module \( P^-(p - s) \) is identical to \( \mathcal{Y}(s, s) \) by the correspondence of the basis \( d^{(s,s)}(n) \mapsto c^{(s,s)}_{n}, y^{(-s)}(n+s) \mapsto d^{(s,s)}_n, b^{(-s)}_n \mapsto d^{(s,s)}_{n+p} \). Hence we have

\[ b^-_{p-s}(T) = \frac{p \sin \frac{2\pi}{p}}{\pi |s|} \left( \left| \frac{dO^p_{\lambda}(T)}{d\lambda} \right|_{\lambda=1} - \left| \frac{dO^p_{\lambda}(T)}{d\lambda} \right|_{\lambda=-s-1} \right), \quad 1 \leq s \leq p - 1. \]

The formula (5.2) implies that

\[ O^{mp+s-1}_{\lambda}(T) = O^{mp-s-1}_{\lambda}(T). \]

By putting \( m = 1 \) and \( s = p - s \), we have

\[ a_s(T) = O^{2p-s-1}_{\lambda}(T) = O^{s-1}_{\lambda}(T). \]

\[ \square \]

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Department of Mathematics, Faculty of Science and Engineering, Waseda University, 3-4-1 Ohkubo, Shinjyuku-ku, Tokyo 169-8555, JAPAN., E-mail: murakami@waseda.jp

Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, JAPAN.