The centroid Banach-Mazur distance between the parallelogram and the triangle

Marek Lassak

Abstract. Let $C$ and $D$ be convex bodies in the Euclidean space $E^d$. We define the centroid Banach-Mazur distance $\delta_{\text{cen}}(C, D)$ similarly to the classic Banach-Mazur distance $\delta_{BM}(C, D)$, but with the extra requirement that the centroids of $C$ and an affine image of $D$ coincide. We prove that for the parallelogram $P$ and the triangle $T$ in $E^2$ we have $\delta_{BM}(P, T) = \frac{5}{2}$.

Keywords: Banach-Mazur distance, centroid Banach-Mazur distance, convex body, centroid, parallelogram, triangle

MSC: Primary: 52A21, Secondary 46B20, 52A10

1 Introduction

The classical definition of the Banach-Mazur distance of centrally symmetric convex bodies of the Euclidean $d$-space $E^d$ is given by Banach [2] in behalf of him and Mazur over nine decades ago. For over four decades this definition is considered also for a arbitrary convex bodies of $E^d$. Namely, for convex bodies $C, D$ of $E^n$ this extended Banach-Mazur distance sounds as follows

$$\delta_{BM}(C, D) = \inf_{a, h_\lambda} \{\lambda; \ a(D) \subset C \subset h_\lambda a(D)\},$$

where $a$ stands for an affine transformation and $h_\lambda$ denotes a homothety with a positive ratio $\lambda$. For the relationship of them see Claim of [3].

A survey on the Banach-Mazur distance is given in the book [11] by Tomczak-Jaegerman. Moreover, in Sections 3.2 and 3.3 of the book [12] by Toth, and in Section 4.1 of the book [1] by Aubrun and Szarek.

Here is the notion of the centroid Banach-Mazur distance of convex bodies $C, D$ of $E^d$:

$$\delta_{BM}^{\text{cen}}(C, D) = \inf_{a, h_\lambda} \{\lambda; \ a(D) \subset C \subset h_\lambda a(D) \text{ and } \text{cen}(a(D)) = \text{cen}(C)\},$$

where $a$ again stands for an affine transformation, but $h_\lambda$ means a homothety with the ratio $\lambda \geq 1$ whose center is at the centroids $\text{cen}(a(D)) = \text{cen}(C)$ of $a(D)$ and $C$. Observe that the centroids of $C, D$ in this definition take over the roles of the centers of the centrally-symmetric bodies in the original definition of Banach-Mazur distance. We easily show that $\delta_{BM}^{\text{cen}}(C, D) = \delta_{BM}^{\text{cen}}(D, C)$ for every $C$ and $D$. Recall that pioneer research on the centroid was provided by Neumann [10].
In Theorem we prove that \( \delta_{\text{cen}}^{BM}(P, T) = \frac{5}{2} \) for the parallelogram \( P \) and the triangle \( T \) in \( E^2 \). Our effort is put in order to show that \( \delta_{\text{cen}}^{BM}(P, T) \geq \frac{5}{2} \) since the opposite inequality immediately follows from easy examples.

At the end of the paper we present a few remarks. The first concerns the positions of our triangle with respect to the parallelogram for which the ratio \( \frac{5}{2} \) is realized. The second comments the dual version of Theorem. The third shows the following generalization of Theorem for a centrally-symmetric convex body \( M \) in place of \( P \): for every triangle \( T \) inscribed in \( M \) with the common centroid we have \( M \subset 3T \). We also propose a more general task to consider an arbitrary convex body instead of \( M \). Finally, we ask about a generalization of Theorem for \( E^d \).

As usual, by \( \text{int}(A) \) and \( \text{bd}(A) \) we denote the interior and boundary of a set \( A \).

## 2 The distance between the parallelogram and the triangle is \( \frac{5}{2} \)

**Theorem.** For the parallelogram \( P \) and the triangle \( T \) we have \( \delta_{\text{cen}}^{BM}(P, T) = \frac{5}{2} \).

**Proof.** In the proof, by \( \lambda C \) we mean the homothetic image of a set \( C \) with a positive ratio \( \lambda \) and the center at the origin \( o \) of \( E^2 \). By \( S \) denote the square with vertices \((1, \pm 1)\) and \((-1, \pm 1)\).

Let us rephrase our theorem as the conjunction of the two following sentences

(*) for every triangle \( \Delta \subset S \) with the centroid at the center \( o \) of \( S \) the interior of the triangle \( \frac{5}{2} \Delta \) does not contain \( S \),

(**) there exists a triangle \( \Delta_0 \subset S \) with the centroid in the center of \( S \) for which \( \frac{5}{2} \Delta_0 \) contains \( S \).

Taking as \( \Delta_0 \) the triangle with vertices \((1, \frac{1}{2}), (-1, \frac{1}{2})\) and \((0, -1)\) we see that (***) is true. Later, our aim is to show that (*) holds true.

The following obvious fact is applied soon two times in the proof.

(\( \triangleright \)) If (*) is true for a triangle centered at \( o \) and containing \( \Delta \), then (*) is true for \( \Delta \).

By (\( \triangleright \)) we may assume that \( \Delta \subset S \) has at least one vertex in the boundary of \( S \). Still if all the vertices of the triangle are in the interior of \( \text{int}(S) \), we can increase this triangle by a homothety with center \( o \) such that at least one of its vertices “arrives” to the boundary of \( S \).

Without losing the generality assume that such a vertex \( a \) is \((1, \alpha)\), where \( 0 \leq \alpha \leq 1 \).

Denote by \( b \) and \( c \) the endpoints of the opposite side of \( \Delta \) such that \( a, b, c \) are in the positive order. The midpoint of \( bc \) is \( d = (-\frac{1}{2}, -\frac{1}{2} \alpha) \). The reason is that the segment connecting \( d \) with \( a \) passes through \( o \) which is the centroid of \( \Delta \).

We may assume that \( b \) or \( c \) is in \( \text{bd}(S) \). Still if \( b \) and \( c \) are in \( \text{int}(S) \), we do not lose the generality by properly enlarging \( \Delta \). Namely, we increase \( bc \) by the homothety at its center \( d \) as long as an endpoint attains the boundary of \( S \), but clearly, the other must remain in \( S \). This follows by (\( \triangleright \)) and the observation that the center of the increased triangle is \( o \) again.
Case 1, when the vertex $b$ attains the boundary of $S$ not later than $c$.

Subcase 1.1, when $\alpha \in (0, 1]$. 

Since $\alpha > 0$, we have $-\frac{1}{2}\alpha < 0$. Thus $d$ is below the axis $y = 0$. Since $d$ is the midpoint of $bc$, this and the fact that $c$ is over or on the straight line $y = -1$ imply that $b$ must be below the straight line $y = 1$. Hence $b$ does not belong to the side $(1, 1)(-1, 1)$. Obviously, it also does not belong to the sides $(-1, -1)(1, -1)$ and $(1, -1)(1, 1)$. So $b$ belongs to the side $(-1, 1)(-1, -1)$ and has the form $b = (-1, \beta)$. See Figure 1. Clearly, $-1 \leq \beta \leq 1$.

Applying again the fact that $d = (-\frac{1}{2}, -\frac{1}{2}\alpha)$ is the middle of $bc$, we conclude that $c = (0, -\alpha - \beta)$. Since the second coordinate of $c$ is at least $-1$, we get $\alpha - \beta \geq -1$ which means that $1 - \alpha \geq \beta$. Since the order of vertices $a, b, c$ of $\Delta$ is positive, we get $\beta \geq -\alpha$ (for $\beta = -\alpha$ our triangle $\Delta$ degenerates to a segment). Resuming, $-\alpha \leq \beta \leq 1 - \alpha$ and thus $b \in gh$, where $g = (-1, 1 - \alpha)$ and $h = (-1, -\alpha)$.

Take into consideration the triangle $\frac{5}{2}\Delta$. Its corresponding vertices are $a' = (\frac{5}{2}, \frac{5}{2}\alpha)$, $b' = (-\frac{5}{2}, \frac{5}{2}\beta)$ and $c' = (0, -\frac{5}{2}\alpha - \frac{5}{2}\beta)$. Here are the equations of the straight lines $\ell_{a'c'}$ and $\ell_{b'c'}$ containing the sides $a'c'$ and $b'c'$, respectively:

$$\ell_{a'c'}: y - \frac{5}{2}\alpha = (2\alpha + \beta)(x - \frac{5}{2}),$$

$$\ell_{b'c'}: y + \frac{5}{2}\alpha + \frac{5}{2}\beta = (-\alpha - 2\beta)x.$$  

We intend to show that for every $\alpha \in (0, 1]$ and $\beta \in [-\alpha, 1 - \alpha]$ at least one of the straight lines $\ell_{a'c'}, \ell_{b'c'}$ intersects the square $S$. This would mean that the interior of $\frac{5}{2}\Delta$ does not contain the whole $S$. Consequently (*) would hold true.

Consider the parallelogram $V = \{(\alpha, \beta); 0 < \alpha \leq 1, \alpha \leq \beta \leq 1 - \alpha\}$ in the coordinate system $O\alpha\beta$, where $a = (0, 0)$, without the side $(0, 0)(0, 1)$. See Figure 2.

The below consideration shows that for every $(\alpha, \beta) \in R$ at least one of the following two sentences is true in the system $Oxy$:

- $\ell_{a'c'}$ intersects the side $(-1, -1)(1, -1)$.
- $\ell_{b'c'}$ intersects the side $(-1, -1)(-1, 1)$.

We intersect $\ell_{a'c'}$ with the line $y = -1$ containing the side $(-1, -1)(1, -1)$. The point of intersection is $e = \left(\frac{2 - 5\alpha + 5\beta}{4\alpha + 2\beta}, -1\right)$. If the line $\ell_{a'c'}$ intersects the side $(-1, -1)(1, -1)$, from $x \leq 1$ we obtain $\beta \leq \frac{2 - \alpha}{3}$. As a result we see the region $V(\ell_{a'c'})$ of points $(\alpha, \beta)$ in the coordinate system $O\alpha\beta$ for which the line $\ell_{a'c'}$ intersects the side $(-1, -1)(1, -1)$; it is the part of $R$ not above the straight line $\beta = \frac{2 - \alpha}{3}$.

The straight line $\ell_{b'c'}$ intersects the one $x = -1$ containing the side $(-1, 1)(-1, -1)$. The point $f$ of intersection is $(-1, -\frac{3}{2}\alpha - \frac{1}{2}\beta)$. If the line $\ell_{a'c'}$ intersects the side $(-1, 1)(1, -1)$, then from $y \geq -1$ we conclude that $\beta \leq 2 - 3\alpha$. The region $V(\ell_{b'c'})$ of points $(\alpha, \beta)$ in the system $O\alpha\beta$ for which the line $\ell_{b'c'}$ intersects the side $(-1, -1)(-1, 1)$ is the part of $R$ not above the straight line $\beta = 2 - 3\alpha$. 


These two straight lines $\beta = \frac{2-\alpha}{3}$ and $\beta = 2 - 3\alpha$ intersect at the point $(\frac{1}{2}, \frac{1}{2})$ of the system $\alpha\beta$ which is in the side $(1, 0)(0, 1)$ of $R$. From this and the two preceding paragraphs we see that $V \subset \ell_{a'b'} \cup V(\ell_{b'c'})$. Hence (*) holds true.

**Subcase 1.2**, when $\alpha = 0$.

We have $d = (\frac{-1}{2}, 0)$ which implies that $b \in (0, 1)(-1, 1)$ and $c \in (-1, -1)(0, -1)$ are symmetric with respect to $d$. We let the reader to check that the interior of the triangle $\frac{5}{2}\Delta$ does not contain $S$, so (*) holds true.

**Case 2**, when the vertex $c$ attains the boundary of $S$ not later than $b$.

**Subcase 2.1** when $\alpha \in (0, 1]$.

Observe that $c$ must be in the side $(-1, -1)(1, -1)$ (see Figure 3). So $c$ has the form $(\gamma, -1)$, where $-1 \leq \gamma \leq 1$. From $b \in S$ and the fact that $d$ has the first coordinate $\frac{-1}{2}$ we see that $-1 \leq \gamma \leq 0$. Since $d$ is the midpoint of $bc$, we have $b = (-1 - \gamma, 1 - \alpha)$.

Take into account the triangle $\frac{5}{2}\Delta$. Its corresponding vertices are $a' = (\frac{5}{2}, \frac{5}{2}\alpha)$, $b' = (\frac{-\frac{5}{2} - \frac{5}{2}\gamma, \frac{5}{2} - \frac{5}{2}\alpha})$ and $c' = (\frac{\gamma}{2}, -\frac{\gamma}{2})$ (see Figure 3).

Here are the equations of the straight lines containing the sides of the triangle $a'b'c'$.

- $\ell_{a'b'} : y - \frac{5}{2}\alpha = \frac{-1+2\alpha}{2+\gamma}(x - \frac{5}{2})$,
- $\ell_{b'c'} : y + \frac{5}{2} = \frac{2\alpha-\gamma}{2-\gamma}(x - \frac{5}{2}\gamma)$,
- $\ell_{a'c'} : y - \frac{5}{2}\alpha = \frac{\alpha+1}{\gamma}(x - \frac{5}{2})$.

We intend to show that for every $\gamma \in [-1, 0]$ at least one of these lines $\ell_{a'b'}, \ell_{b'c'}$ and $\ell_{a'c'}$ intersects the square $S$. This would mean that (*) holds true.
Now let us deal with the set $W = \{(\alpha, \gamma) : 0 < \alpha \leq 1, -1 \leq \gamma \leq 0\}$ of points in the coordinate system $o\alpha\gamma$ (see Figure 4), where $o = (0,0)$.

The point $k$ of intersection of $\ell_{a'}$ with the straight line $x = 1$ containing the side \((1,1)(1, -1)\) has the second coordinate $y = -\frac{3}{2} - \frac{1 + 2\alpha}{2 + \gamma} + \frac{5}{2} \alpha$. If $\ell_{a'}$ intersects this side, then from $y \leq 1$ we get $\frac{3}{2} - \frac{1 + 2\alpha}{2 + \gamma} + \frac{5}{2} \alpha \leq 1$. Equivalently, $\gamma \geq \frac{1 - 4\alpha}{2 + 5\alpha}$. So the curve $\gamma = \frac{1 - 4\alpha}{2 + 5\alpha}$ in the coordinate system $o\alpha\gamma$ intersects the axis $o\alpha$ at $(\frac{1}{4}, 0)$ and the axis $o\gamma$ at $(0, -\frac{1}{2})$. This permits to see the subregion $W(\ell_{a'})$ of $W$ of points for which $\ell_{a'}$ intersects $S$. It is bounded by the piece of $\gamma = \frac{1 - 4\alpha}{2 + 5\alpha}$ for $0 \leq \alpha \leq \frac{1}{4}$ (marked by the dotted line in Figure 4) and the segments \((0,0)(0, \frac{1}{4})\) and \((0,0)(0, -\frac{1}{2})\) in the system $o\alpha\gamma$.

The intersection of $\ell_{b'}$ with the straight line $y = -1$ containing the side \((-1, -1)(1, -1)\) is at a point $l = (x, -1)$, where $x$ fulfills $\frac{3}{2} = \frac{2 - \alpha}{2 + \gamma}(x - \frac{5}{2} \gamma)$, this is $x = \frac{-3 - 6\alpha}{4 - 2\gamma} + \frac{5}{2} \gamma$. If $\ell_{b'}$ intersects the side \((1, -1)(-1, -1)\), then from $x \geq -1$ we obtain $-1 \leq \frac{-3 - 6\alpha}{4 - 2\gamma} + \frac{5}{2} \gamma$ which (for $\alpha \in [0, \frac{1}{2}]$) is equivalent to $\gamma \geq \frac{1 - 2\alpha}{4 + 5\alpha}$. We easily check that the curve $\gamma = \frac{1 - 2\alpha}{4 + 5\alpha}$ in the coordinate system $o\alpha\gamma$ intersects the axis $o\alpha$ at $(\frac{1}{4}, 0)$ and the axis $o\gamma$ at $(0, -\frac{1}{4})$. In Figure 4 we see the subregion $W(\ell_{b'})$. It is bounded by the piece of $\gamma = \frac{1 - 2\alpha}{4 + 5\alpha}$ for $0 \leq \alpha \leq \frac{1}{4}$ (marked by the dashed line in Figure 4) and the segments \((0,0)(0, \frac{1}{4})\) and \((0,0)(0, -\frac{1}{4})\) in the coordinate system $o\alpha\gamma$.

The intersection of $\ell_{c'}$ with the straight line $y = -1$ is at a point $m = (x, -1)$, where $x$ fulfills $-1 - \frac{5}{2} \alpha = \frac{\alpha + 1}{1 + \gamma}(x - \frac{5}{2} \gamma)$, this is $x = \frac{5}{2} + \frac{2 - 5\alpha}{2 + 2\alpha}(1 - \gamma)$. If $\ell_{c'}$ intersects the side \((-1, -1)(1, -1)\), then from $x \leq 1$ we obtain $\frac{3}{2} \leq \frac{2 + 5\alpha}{2 + \alpha}(1 - \gamma)$ which, for our positive $\alpha$ is...
equivalent to $\gamma \leq \frac{-1+2\alpha}{2+5\alpha}$. We easily check that the curve $\gamma = \frac{-1+2\alpha}{2+5\alpha}$ in the coordinate system $o\alpha\gamma$ intersects the axis $o\alpha$ at $(\frac{1}{2}, 0)$ and the axis $o\gamma$ at $(0, -\frac{1}{2})$. In Figure 4 we see this subregion $W(\ell_{a'c'})$ of $W$ of points for which $\ell_{a'c'}$ intersects $S$. This subregion is bounded by the piece of the curve $\gamma = \frac{-1+2\alpha}{2+5\alpha}$ for $0 < \alpha \leq \frac{1}{2}$ (marked by the solid line in Figure 4) and by the segments connecting the succeeding pairs of points $(0, -\frac{1}{2}), (0, -1), (1, -1), (1, 0)$ and $(\frac{1}{2}, 0)$ in the coordinate system $o\alpha\gamma$. Clearly, here the first of these segments is not in $W(\ell_{a'c'})$.

Observe that the point $(\frac{1}{3}, -\frac{1}{3})$ belongs to the three pieces of curves bounding our three considered subregions. Moreover, $-\frac{1+2\alpha}{2+5\alpha} \geq \frac{-4\alpha}{2+5\alpha}$ for $0 < \alpha \leq \frac{1}{3}$ and $-\frac{1+\alpha}{2+5\alpha} \geq \frac{-2\alpha}{2+5\alpha}$ for $\frac{1}{3} \leq \alpha \leq \frac{1}{2}$ (see Figure 4). These facts and the three preceding paragraphs imply that $W \subset W(\ell_{a'c'}) \cup W(\ell_{b'c'}) \cup W(\ell_{a'c'})$.

**Subcase 2.2** for $\alpha = 0$.

Now we have $d = (-\frac{1}{2}, 0)$. This implies that $c \in (-1, -1)(0, -1)$ and $b \in (0, 1)(-1, 1)$ are symmetric with respect to $d$. It is easy to check that the interior of the triangle $\frac{5}{2}\triangle$ does not contain $S$. So (*) holds true.

From Cases 1 and 2 we see that always $\text{int}(\frac{5}{2}\triangle)$ does not contain $S$, i.e., (*) is fulfilled. $\square$

The fact proved in Theorem is claimed without a proof at the bottom of p. 259 of [4].

### 3 Final remarks

Let us comment the positions of the triangle $\Delta_0$ in $S$ (drawn by a thick line in Figure 5) from the proof of Theorem, for which (**) holds true. As it follows from our considerations, the only two such triangles $\Delta_0 \subset S$ (up to symmetric positions) are the following. The first has vertices $a_1 = (1, \frac{1}{2}), b_1 = (-1, \frac{1}{2})$ and $c_1 = (0, -1)$ (by the way, two symmetric positions are also seen for $\alpha = 0$ in Subcases 1.2 and 2.2 of the proof of Theorem). The second has vertices $a_2 = (1, \frac{1}{5}), b_2 = (-\frac{1}{5}, \frac{1}{5})$ and $c_2 = (-\frac{1}{5}, -1)$. We see both and also corresponding $\frac{5}{2}\triangle_0$ in Figure 5.

Look at the dual situation. Putting $C = P$ and $D = T$ in $\delta_{BM}^{\text{cen}}(C, D) = \delta_{BM}^{\text{cen}}(D, C)$, by Theorem we get $\delta_{BM}^{\text{cen}}(T, P) = \frac{5}{2}$. In particular, for a given equilateral triangle $T$ (marked by a thick line in Figure 6) there are only two extreme positions (up to symmetric ones) of a parallelogram $P_0$ for which $\frac{5}{2}P_0$ contains $T$.

In connection to our Theorem recall Theorem from [7] shows that every centrally symmetric convex body $M \subset E^2$ permits to inscribe a triangle $\Delta$ whose centroid is at the center of symmetry of $M$ such that $M \subset \frac{5}{2}\Delta$ (our present Theorem explains that this ratio cannot be lessened when $M$ is a parallelogram.) Here is a claim which considers any inscribed triangle.

**Claim.** Let $M \subset E^2$ be a centrally-symmetric convex body. For every inscribed triangle $\Delta$ in $M$ with the common centroid we have $M \subset 3\Delta$.

**Proof.** Let $\Delta = abc$. Take the symmetric triangle $\Delta_s = a_s b_s c_s$ with respect to $o$. The convex
hull $H$ of $\Delta \cup \Delta_s$ is an affine-regular hexagon inscribed in $M$. Prolonging the sides $ab_s, bc_s, ca_s$ we get three points of intersection. Denote by $S(H)$ the star being union of the triangle with these three vertices and the symmetric triangle with respect to $o$. By the convexity of $M$ we conclude that $M \subset S(H)$. Moreover, observe that $S(H) \subset 3\Delta$. Consequently, $M \subset 3\Delta$. 

By the way, Figure 3 of [5] shows an analogous situation of extreme homothetic parallelograms $S \subset \Delta$ and $(\sqrt{2} + 1)S \supset \Delta$ without the requirement that centroids of $S$ and $\Delta$ coincide. The ratio 3 in Claim cannot be lessened as it shows the example of the square with vertices $(-1, \pm 1), (1, \pm 1)$ and the triangle with vertices $(1, 1), (-1, 0), (0, -1)$.

Theorem of [7] says the following. Let $M \subset E^2$ be a centrally symmetric convex body. In $M$ it can be inscribed a triangle $\Delta$ whose centroid is the center of symmetry of $M$ such that $M \subset \frac{5}{2}\Delta$. From our Theorem it follows that this ratio $\frac{5}{2}$ cannot be lessened.

What about considering an arbitrary convex body in place of $M$ in Claim? The author conjectures that for every planar convex body $C$ and an arbitrary inscribed triangle $\Delta$ with the common centroid we have $C \subset 4\Delta$. Possibly the result of Neumann [10] would be a good tool. The the ratio 4 cannot be lessened for $C$ being a triangle with the centroid at $o$ and $\Delta = -\frac{1}{4}C$.

The above remarks can be seen in the wider context of approximation by triangles in [3].

Finally, let us ask about a higher dimensional generalization of Theorem. In $E^3$ the parallelepiped $Q$ permits to inscribe (thus also put inside) a simplex $S$ whose centroid is in the center of $Q$ and $Q \subset 3S$. Just put the vertices of $S$ at some four non-neighboring vertices of $Q$. Thus $\delta_{cen}^{BM}(Q, P) \leq 3$. The author believes that the equality holds true here, but possible
evaluation seems to be complicated. For higher dimensions the task of finding or at least estimating $\delta_{\text{cen}}^{BM}(Q, P)$ remains open. From [8] we only conclude that $\delta_{\text{cen}}^{BM}(Q, P) \leq 2n - 1$. Since $\delta_{\text{cen}}^{BM}(P, Q) = \delta_{\text{cen}}^{BM}(Q, P)$, the same also follows from the last paragraph of [6].

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