Comparison of three systems of soil hydrophysical functions on example of silt loam

R S Ginevsky¹, V A Lazarev¹, A G Topaj², A O Nikonorov¹*, I A Dunaieva³ and A V Terleeva⁴

¹Peter the Great St. Petersburg Polytechnic University, Polytechnicheskaya, 29, St. Petersburg, 195251, Russian Federation
²Agrophysical Research Institute, Grazhdanskii pr., 14, St. Petersburg, 195220, Russian Federation
³Research Institute of Agriculture of Crimea (Federal State Budget Scientific Institution), Kievskaya street, 150, Simferopol, 295543, Russian Federation
⁴St. Petersburg State University, Universitetskaya naberezhnaya 7/9, St. Petersburg, 199034, Russian Federation

Email: coolhabit@yandex.ru

Abstract. Three systems of functions are used to describe the water-retention capacity and the ratio of the hydraulic conductivity of the soil to the moisture filtration coefficient (relative hydraulic conductivity of the soil). Each of the three systems uses an appropriate set of parameters. These parameters are common to each pair of functions that make up the corresponding system. The parameters of the first system of functions are formal in nature and are used in the Mualem-Van Genuchten method. The parameters of the second and third systems have a physics-statistical interpretation. The main purpose of the considered systems is to predictively evaluate the values of the relative hydraulic conductivity function using parameters identified by the point approximation of data on the water-retention capacity of the soil. Using the soil example «3304 Touchet silt loam» from the Mualem catalog, three systems of functions are compared. In order to identify significant differences between the errors of the compared function systems with respect to the point approximation of data on water-retention capacity, as well as for predictive estimates of the values of the function of the relative hydraulic conductivity of the soil, the Williams-Kloot test is used. The influence of the additive parameter describing the capillary pressure of the «air inlet» on the errors of the second and third systems is estimated. The results of the study indicate the advantages of the second and third systems over the first system, the functions of which are used in the Mualem-Van Genuchten method. The third system is formulated in the form of fairly simple relationships using elementary mathematical functions and is recommended for modeling the hydrophysical properties of the soil and solving problems of ameliorative farming.

1. Introduction

A significant number of tasks of automation and the use of resource-saving technologies in hydraulic engineering construction and agriculture are closely related to the need of mathematical models usage [1-8]. The use of models describing the hydrophysical properties of the soil reduces the complexity of field surveys, allows to optimize the management of the water regime of agrocenoses [9-13]. In hydrophysics, there is a problem of describing the water-retention capacity (WRC) and the relative
hydraulic conductivity (RHC) of the soil using functions that share a common set of parameters. For the description of the water-retention capacity and the ratio of the hydraulic conductivity of the soil to the moisture filtration coefficient (relative hydraulic conductivity of the soil), the Mualem – Van Genuchten method is most widely used in world practice [14, 15]. The undoubted advantage of this method is that for the identification of the two functions general parameters, it is sufficient to use data on only one hydrophysical property of the soil. For example, to identify the parameters by point approximation, data on the hydrophysical property, which is described by the first function, are used, and then the identified parameters are used to estimate the values of the second function, which describes another hydrophysical property of the soil. However, this method has significant drawbacks. These include: the use of formal parameters, as well as the existence of computational constraints on the exponential parameter.

The purpose of this study is to describe the hydrophysical properties of the soil using functions that have common parameters, as well as verification of these functions and their comparison with the most famous world analogues that are used in the Mualem-Van Genuchten method. The specified goal is achieved by solving a number of tasks, such as:

- identification of parameters by point approximation of data on soil water-retention capacity;
- estimation of the values of the relative hydraulic conductivity of the soil function;
- comparison of errors of approximation results and estimates obtained using different systems of functions;
- selection of the best physically sound system of functions based on the identification of significant differences between the compared systems.

2. Materials and methods

In world hydropysics, the model of soil water-retention capacity proposed by Van Genuchten [15] is currently the most widely used:

$$S_e = \left(1 + \frac{-\alpha \psi}{n}\right)^{(1-1/n)} \psi < 0;$$

where $S_e = (\theta_s - \theta_r)/\theta_r$ – effective moisture saturation of soil; $\theta_r$ [cm$^3$·cm$^{-3}$] – volumetric water content at full water saturation of the soil; $\theta_s$ [cm$^3$·cm$^{-3}$] – volumetric water content corresponding to the minimum specific volume of moisture as a fluid in the soil; $n > 1$ and $\alpha$ [cm of water column]$^{-1}$ – empirical parameters.

Model (1) is described by a curve that has a sigmoidal shape, which is characteristic for most soils. For large values of the exponential parameter $n$ (in the passage to the limit), relation (1) reduces to the function proposed by Haverkamp et al [16].

An article by Van Genuchten [15] using the Mualem formula [14] and model (1) describes the function of the relative hydraulic conductivity of the soil:

$$k = \frac{\sqrt{S_e}}{k_1}, \text{ at } \theta_0 = \theta_r;$$

$$1. \text{ at } \theta_0 = \theta_r.$$
where \( \text{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \) – additional error function; \( S_e = (\theta - \theta_s)/(\theta_e - \theta_s) \) – effective moisture saturation of soil; \( \theta_s \) [cm\(^3\)-cm\(^{-3}\)] – volumetric water content at full water saturation of the soil; \( \theta_e \) [cm\(^3\)-cm\(^{-3}\)] – volumetric water content corresponding to the minimum specific volume of moisture as a fluid in the soil; \( n = 4/(\sigma \sqrt{2\pi}) \) – exponential parameter; \( \alpha = -1/(\psi_0 - \psi_e) \) [cm H\(_2\)O \(^{-1}\)] – multiplicative parameter; \( \psi_e \) [cm H\(_2\)O] – additive parameter, taking into account the capillary pressure of the «air inlet»; \( \psi_0 \) [cm H\(_2\)O] – capillary pressure at which the probability distribution density over the values of the applied random variable reaches a maximum (\( \psi_0 < \psi_e \)).

Model (4) is also described by a curve that has a sigmoidal shape, which is characteristic for most soils. This particular case model (when \( \psi_e = 0 \)) generates the function of water-retention capacity of the soil proposed by Kosugi [18].

In article [17], using the model (3) and the Mualem formula, a functional description of the relative hydraulic conductivity of the soil is proposed:

\[
\frac{k}{k_s} = \left[ \sqrt{\frac{\psi_e}{4 \pi \alpha \text{erfc}(2\psi_e - \frac{\psi_0}{\sqrt{\alpha \psi_e}}))} \right]^{\theta - \theta_0};
\]

(4)

where \( \text{erfc}(\text{erfc}(x)) = x \).

Model (4) as a special case (when \( \psi_e = 0 \)) generates the function of relative hydraulic conductivity of the soil proposed by Kosugi [19].

In article [17], relation (3) is approximated using the simplified Vinitsky formula [20] and elementary mathematical functions:

\[
S_e = \left[ 1 + \left( \frac{\alpha (\psi - \psi_e)}{\psi_0} \right)^{\psi - \psi_e} \right]^{-1}, \quad \psi < \psi_e;
\]

(5)

Model (4) is also described by a curve that has a sigmoidal shape, which is characteristic for most soils. This particular case model (when \( \psi_e = 0 \)) generates the function of water-retention capacity of the soil, proposed by Haverkamp et al. [16].

In [17], relations (4) are approximated using the simplified Vinitsky formula [20] and elementary mathematical functions:

\[
\frac{k}{k_s} = \left[ \sqrt{\frac{\psi_e}{4 \pi \alpha \text{erfc}(\psi_e - \frac{\psi_0}{\sqrt{\alpha \psi_e}}))} \right]^2, \quad 0 < \theta_0;
\]

(6)

Formulas (3) - (6) describe the hydrophysical functions of the soil with interpreted parameters. These ratios represent an alternative to the Mualem-Van Genuchten method.

The following number of notation should be introduced. Function (1) is denoted as WRC-VG. Function (3) is denoted as WRC-KT in case of \( \psi_e \neq 0 \) and WRC-KT\(_0\) in case of \( \psi_e = 0 \). Function (5) is denoted WRC-HT in case of \( \psi_e \neq 0 \) and WRC-HT\(_0\) in case of \( \psi_e = 0 \). Function (2) is denoted as RHC-MVG. Function (4) is denoted as RHC-MKT in case of \( \psi_e \neq 0 \) and RHC-MKT\(_0\) in case of \( \psi_e = 0 \). Function (6) is denoted as RHC-MT in case of \( \psi_e \neq 0 \) and RHC-MT\(_0\) in case of \( \psi_e = 0 \).

The listed functions are usually used in pairs (\( S_e(\psi) \)) and \( k(\theta)/k_s \), so they were grouped into the next systems:

- system #1 (WRC-VG and RHC-MVG);
- system #2 (WRC-KT and RHC-MKT or WRC-KT₀ and RHC-MKT₀);
- system #3 (WRC-HT and RHC-MT or WRC-HT₀ and RHC-MT₀).

For verification and comparative analysis of three systems of functions that describe the water-retention capacity and relative hydraulic conductivity of the soil, with common parameters, a scenario of a computational experiment using soil data «3304 Touchet silt loam» from an authoritative literary source was developed [21]. In the framework of this experiment, the parameters of the compared systems are identified by the point approximation of experimental data on the water-retention capacity of the soil (32 points using the «SoilHydrophysics-v.1.0» computer program developed by the authors [22]. The obtained parameters are used for predictive estimation of the relative hydraulic conductivity function of the studied soil. The following are the results of this experiment.

3. Results and discussions
The parameter values identified by the point approximation of the data on the water-retention capacity of the studied soil for the three systems are given in Table 1.

| System # | Model parameters | α   | n   |
|----------|------------------|-----|-----|
| 1        | 0.480 0.187      | 0.0101 | 6.266 |
| 2        | 0.480 0.142 70.41 | 0.0272 | 1.403 |
| 3        | 0.480 0.191      | 0.0098 | 5.791 |

The root mean square errors (RMSE) of the point approximation of water-retention data and estimates of the relative hydraulic conductivity of the studied soil are given in Table 2, and the results of comparing these errors by the Williams-Kloot test [23, 24] with confidence probabilities of 0.95 and 0.975 are given in Table 3 and 4.

| System #1 | System #2 | System #3 |
|-----------|-----------|-----------|
| WRC-VG RHC-MVG | WRC-KT RHC-MKT | WRC-HT RHC-MT |
| 0.0096 0.0980 | 0.0029 0.0715 | 0.0029 0.0680 |

The data from the Table. 3 indicate that almost all the differences in the errors of the point approximation of the water-retention capacity of the soil are significant. Errors of systems #2 and #3 in case of \( \psi_e \neq 0 \) approximately three times lower compared to the error of system #1, as well as compared to the error of systems #2 and #3 in case of \( \psi_e = 0 \). The usage of the additive parameter \( \psi_e \) significantly reduces errors. The data from the Table. 4 indicate that systems #2 and #3 in case of \( \psi_e \neq 0 \) have significantly lower errors in predictive estimates of the relative hydraulic conductivity of the soil than system #1, and the use of the additive parameter \( \psi_e \) also reliably reduces errors. Moreover, the functions of system #3, being approximations of the functions of system #2, do not have significant differences between the errors in case of \( \psi_e \neq 0 \).

The results of a point approximation of experimental data on water-retention capacity (WRC) performed to identify the parameters of hydrophysical functions, as well as the results of a predictive estimation of the values of the relative hydraulic conductivity (RHC) of the soil using the identified parameters of the three compared function systems for the case of \( \psi_e = 0 \) and for the case of \( \psi_e \neq 0 \) are
presented in the Figure 1. The solid curves depict the results of a computational experiment (the colors used are: red for system #1; blue for system #2; green for system #3); round and square dots depict experimental data on the water-retention capacity and relative hydraulic conductivity of the studied soil, respectively.

Table 3. The reliability of differences between the errors of point approximation of data on the soil water-retention capacity according to the Williams-Kloot test.

| y: (y_1+y_2)/2 = λ(y_1-y_2), where y - experimental data |
|------------------|------------------|------------------|------------------|
| y_1: WRC-VG, y_2: WRC-KT | y_1: WRC-VG, y_2: WRC-HT | y_1: WRC-KT, y_2: WRC-HT |
| λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ | λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ | λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ |
| -0.453     | 0.221 0.266   | -0.490     | 0.235 0.282   | 0.512     | 1.533 1.845   |
| y_1: WRC-VG, y_2: WRC-KT₀ | y_1: WRC-VG, y_2: WRC-HT₀ | y_1: WRC-KT, y_2: WRC-HT₀ |
| λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ | λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ | λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ |
| 1.555     | 2.302 2.770   | 7.263     | 3.545 4.266   | 1.580     | 2.372 2.854   |
| y_1 and y_2 are equal | y_1: WRC-KT, y_2: WRC-HT₀ | y_1: WRC-KT, y_2: WRC-HT₀ |
| λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ | λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ | λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ |
| 0.471     | 0.124 0.149   | 0.488     | 0.129 0.155   | 0.454     | 0.117 0.141   |

Table 4. The reliability of differences in errors of estimating the soil relative hydraulic conductivity according to the Williams-Kloot test.

| y: (y_1+y_2)/2 = λ(y_1-y_2), where y - experimental data |
|------------------|------------------|------------------|------------------|
| y_1: RHC-MVG, y_2: RHC-MKT | y_1: RHC-MVG, y_2: RHC-MT | y_1: RHC-MKT, y_2: RHC-MT |
| λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ | λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ | λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ |
| 0.168     | 0.074 0.089   | 0.185     | 0.087 0.105   | 0.031     | 1.198 1.442   |
| y_1: RHC-MVG, y_2: RHC-MKT₀ | y_1: RHC-MVG, y_2: RHC-MT₀ | y_1: RHC-MKT, y_2: RHC-MT₀ |
| λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ | λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ | λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ |
| 21.757    | 3.018 3.631   | 8.562     | 0.675 0.812   | 13.558    | 1.465 1.763   |
| y_1 more accurate than y_2 | y_1: RHC-MKT, y_2: RHC-MKT₀ | y_1: RHC-MKT, y_2: RHC-MT₀ |
| λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ | λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ | λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ |
| 0.178     | 0.074 0.089   | 0.215     | 0.085 0.102   | 0.195     | 0.072 0.086   |
| y_1 more accurate than y_2 | y_1: RHC-MKT, y_2: RHC-MKT₀ | y_1: RHC-MKT, y_2: RHC-MT₀ |
| λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ | λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ | λ          | λₐ₀.₉₅ λₐ₀.₉₇₅ |
| 0.471     | 0.124 0.149   | 0.488     | 0.129 0.155   | 0.454     | 0.117 0.141   |
4. Conclusions

Three systems are described, each of which is formed by a pair of functions describing the water-retention capacity and relative hydraulic conductivity of the soil with common parameters. The presence of common parameters reduces the complexity of parametric identification of soil-hydrophysical functions. Experimental data on the water-retention capacity of loamy soil «3304 Touchet silt loam» [21] were used to identify the parameters of three function systems using the «SoilHydrophysics-v.1.0» computer program [22]. The following conclusions follow from the results:

1. systems of functions #2 and #3 have significantly lower errors in the case of point approximation of experimental data on water-retention capacity, and in the predictive estimation of the values of the relative hydraulic conductivity function of the studied soil;
2. the application of the additive parameter reduces the errors of the point approximation of experimental data on the water-retention capacity of the soil, as well as the errors in the estimates of the values of the function of the relative hydraulic conductivity of the soil;
3. the program «SoilHydrophysics-v.1.0» allows to automate the calculation of hydrophysical parameters of the soil and the identification of the parameters of the functions that describe these properties;
4. since the errors of systems #2 and #3 do not reliably differ, but system #3 is formulated in the form of fairly simple relations using elementary mathematical functions, the advantage of system #3 with respect to modeling the hydrophysical properties of the soil seems obvious.

The results of the study are of great importance in studying the hydrogeological conditions of the territory, conducting engineering surveys in the design, construction, operation and reconstruction of irrigation and drainage systems, as well as in modeling the dynamics of soil moisture to predict the productivity of agrocenoses.

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