NOTES ON THE CLUSTER MULTIPLICATION FORMULAS FOR 2-CALABI-YAU CATEGORIES

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ABSTRACT. Y. Palu has generalized the cluster multiplication formulas to 2-Calabi-Yau categories with cluster tilting objects ([Pa2]). The aim of this note is to construct a variant of Y. Palu’s formula and deduce a new version of the cluster multiplication formula ([XX]) for acyclic quivers in the context of cluster categories.

INTRODUCTION

Cluster algebras were introduced by S. Fomin and A. Zelevinsky ([FZ]) in order to develop a combinatorial approach to study problems of total positivity in algebraic groups and canonical bases in quantum groups. The link between acyclic cluster algebras and representation theory of quivers were first revealed in ([MRZ]). In ([BMRRT]), the authors introduced the cluster categories as the categorification of acyclic cluster algebras. In ([CC]), the authors introduced a certain structure of Hall algebra involving the cluster category by associating its objects to some variables given by an explicit map $X_?$ called the Caldero-Chapoton map. The images of the map are called generalized cluster variables. For simply laced Dynkin quivers, P. Caldero and B. Keller constructed a cluster multiplication formula (of finite type) between two generalized cluster variables ([CK]). The Caldero-Chapoton map and the Caldero-Keller cluster multiplication theorem open a way to construct cluster algebras from 2-Calabi-Yau categories. The cluster multiplication formula of finite type was generalized to affine type in ([Hu]) and any type in ([XX] and [Xu]). Y. Palu ([Pa2]) further extended the formula to 2-Calabi-Yau categories with cluster tilting objects.

The aim this note is twofold. One is to simplify the cluster multiplication formula in ([XX]) and ([Xu]). In practice, the formula is useful for constructing $\mathbb{Z}$-bases of cluster algebras of affine type ([DXX], [Du]). However, the formula is given in the context of module categories and a bit complicate. We construct a more ‘unified’ version in the context of cluster categories (Theorem 3.3). The other aim is to construct a variant of Y.Palu’s formula (Theorem 1.2). In particular, when applied to acyclic quivers, the variant is exactly the cluster multiplication formula in ([XX] and [Xu]). Since the variant implies Y.Palu’s formula, we can view it as a refinement of Y.Palu’s formula.

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1. The cluster multiplication formulas for 2-Calabi-Yau categories

We recall the notations used in [Pa1] [Pa2]. Let $k$ be the field of complex numbers and $\mathcal{C}$ be a Hom-finite, 2-Calabi-Yau and Krull-Schmidt $k$-linear triangulated category with a basic cluster tilting object $T$. We set $B = \text{End}_{\mathcal{C}}(T, T)$ and $F = \mathcal{C}(T, -)$. Thus following the hypotheses above, the functor $F : \mathcal{C} \to \text{mod } B$ induces an equivalence of categories:

$$\mathcal{C}/(T[1]) \cong \text{mod } B$$

where $(T[1])$ denotes the ideal of morphism of $\mathcal{C}$ which factor through a direct sum of copies of $T[1]$ and $[1]$ denote the shift functor. Let $T_1, \cdots, T_n$ be the pairwise non-isomorphic indecomposable direct summands of $T$ and $S_i$ be the simple tops of projective $B$-modules $P_i = FT_i$ for $i = 1, \cdots, n$. In [Pa1], the author generalized the Caldero-Chapoton map ([CC]) as follows. Define a map

$$X_t : \text{obj}(\mathcal{C}) \to \mathbb{Q}(x_1, \cdots, x_n)$$

by mapping any $M \in \text{obj}(\mathcal{C})$ to

$$X_M^t = \sum_{\varphi \in K_0(\text{mod } B)} \chi(\text{Gr}_B FM) \prod_{i=1}^n x_i^{\langle \varphi, e \rangle_i}$$

where $\varphi_i = \dim S_i$ for $i = 1, \cdots, n$ and we refer to [Pa1] for the definitions of the coindex $\text{coind } M$ of $M$ and the antisymmetric bilinear form $\langle - , - \rangle_a$ on $K_0(\text{mod } B)$.

Let $L$ and $M$ be objects in $\mathcal{C}$. Given a subset $S \subseteq \text{obj}(\mathcal{C})$, we denote by $\mathcal{C}(L, M[1])_S$ the set of morphisms in $\mathcal{C}(L, M[1])$ with the middle terms belonging to $S$. For any object $Y$ in $\mathcal{C}$, we denote by $(Y)$ the set

$$\{ Y' \in \text{obj}(\mathcal{C}) | \text{coind } Y' = \text{coind } Y, \chi(\text{Gr}_B FY') = \chi(\text{Gr}_B FY) \text{ for any } \varphi \in K_0(\text{mod } B) \}$$

$Y'$ is the middle term of some morphism in $\mathcal{C}(L, M[1])$ or $\mathcal{C}(M, L[1])$.

If the cylinders over the morphisms $L \to M[1]$ and $M \to L[1]$ are constructible with respect to $T$ (see [Pa2] Section 1.3 for definition), then there exists a finite subset $\mathcal{Y}$ of the set of middle terms of morphisms in $\mathcal{C}(L, M[1])$ or $\mathcal{C}(M, L[1])$ such that the subsets $\mathcal{C}(L, M[1])_{(Y)}$ (resp. $\mathcal{C}(M, L[1])_{(Y)}$) are constructible in $\mathcal{C}(L, M[1])$ (resp. $\mathcal{C}(M, L[1])$) for $Y \in \mathcal{Y}$ and there are finite stratifications

$$\mathcal{C}(L, M[1]) = \bigcup_{Y \in \mathcal{Y}} \mathcal{C}(L, M[1])_{(Y)} \text{ and } \mathcal{C}(M, L[1]) = \bigcup_{Y \in \mathcal{Y}} \mathcal{C}(M, L[1])_{(Y)}$$

([Pa2] Proposition 9)).

**Theorem 1.1.** [Pa2] Theorem 1 With the above notation, assume that for any $L, M \in \text{obj}(\mathcal{C})$, the cylinders over the morphisms $L \to M[1]$ and $M \to L[1]$ are constructible with respect to $T$. Then we have

$$\chi(\mathcal{PC}(L, M[1])_{(Y)})X^T_MX^T_T = \sum_{Y \in \mathcal{Y}} (\chi(\mathcal{PC}(L, M[1])_{(Y)}) + \chi(\mathcal{PC}(M, L[1])_{(Y)}))X^T_Y.$$
and

\((T[1])(L, M[1])\) := \(C(L, M[1]) \cap (T[1])(L, M[1])\).

Then it is clear that \(C/(T[1])(L, M[1])\) (resp. \((T[1])(L, M[1])\)) are constructible subsets of \(C/(T[1])(L, M[1])\) (resp. \((T[1])(L, M[1])\)) and there are finite stratifications

\[ C/(T[1])(L, M[1]) = \bigcup_{Y \in \mathcal{Y}} C/(T[1])(L, M[1])_{(Y)} \]

and

\[ (T[1])(L, M[1]) = \bigcup_{Y \in \mathcal{Y}} (T[1])(L, M[1])_{(Y)}, (T[1])(M, L[1]) = \bigcup_{Y \in \mathcal{Y}} (T[1])(M, L[1])_{(Y)}. \]

We will prove its following variant which is a generalization of the cluster multiplication formula of any type in [XX] and [Xu] (see Theorem 3.2).

**Theorem 1.2.** With the notation and assumption of Theorem 1.1, we have

\[
\chi(\mathbb{P}(C/(T[1])(L, M[1])) \times \mathbb{P}(C/(T[1])(M, L[1]))) = \sum_{Y \in \mathcal{Y}} (\chi(\mathbb{P}(C/(T[1])(L, M[1])) - \chi(\mathbb{P}(C/(T[1])(M, L[1]))) \times \mathbb{P}(C/(T[1])(M, L[1])])
\]

where \((T[1])(M, L[1])\) denotes the subset of \(C(M, L[1])\) consisting of the morphisms in which factor through a direct sum of copies of \(T[1]\).

**Proof of Theorem 1.1 by Theorem 1.2** Since \(C\) is 2-Calabi-Yau, we have

\[
C/(T[1])(M, L[1]) = D(T[1])(L, M[1])
\]

([Palu1, Lemma 10]). Then we obtain a decompositions of \(C(L, M[1])\):

\[
C(L, M[1]) \cong C/(T[1])(L, M[1]) \oplus C/(T[1])(M, L[1]).
\]

Hence, we have

\[
\chi(C(L, M[1])) = \chi(C/(T[1])(L, M[1])) + \chi(C/(T[1])(M, L[1])).
\]

We define an action of \(C^*\) on \(\mathbb{P}(C(L, M[1]))\) by

\[
t \cdot \mathbb{P}(\varepsilon_1, \varepsilon_2) = \mathbb{P}(t \varepsilon_1, t^2 \varepsilon_2)
\]

for \(t \in C^*\) and \(\mathbb{P}(\varepsilon_1, \varepsilon_2) \in \mathbb{P}(L, M[1])\). It is easily know that \(\mathbb{P}(\varepsilon_1, \varepsilon_2)\) is stable under this action if and only if either \(\varepsilon_1\) or \(\varepsilon_2\) vanishes. Hence, for \(Y \in \mathcal{Y}\), we have

\[
\chi(\mathbb{P}(C(L, M[1]))_{(Y)}) = \chi(\mathbb{P}(C/(T[1])(L, M[1]))_{(Y)}) + \chi(\mathbb{P}(C/(T[1])(M, L[1]))_{(Y)}).
\]

Dually, we also have the decomposition

\[
C(M, L[1]) \cong (T[1])(L, M[1]) \oplus (T[1])(M, L[1]).
\]

In the same way as deducing the equation (1.2), we obtain that

\[
\chi(\mathbb{P}(C(M, L[1]))_{(Y)}) = \chi(\mathbb{P}(T[1])(L, M[1])_{(Y)}) + \chi(\mathbb{P}(T[1])(M, L[1])_{(Y)})
\]

for \(Y \in \mathcal{Y}\).

By the equations (1.1), (1.2) and (1.3), it is easy to prove Theorem 1.1 by using Theorem 1.2.

Hence, the formula in Theorem 1.2 is a refinement of the formula in Theorem 1.1.
2. Proof of Theorem 1.2

Since the proof of Theorem 1.2 is similar to the proof of Theorem 1.1 in [Pa2], we just sketch the proof. Set
\[ \Sigma_1 = \sum_{Y \in \mathcal{Y}} \chi(P\mathcal{C}/(T[1])(L, M[1] \cap Y))X_Y. \]

By definition, it is equal to
\[ \sum_{Y \in \mathcal{Y}} \sum_{g \in \mathcal{G}} \chi(P\mathcal{C}/(T[1])(L, M[1] \cap Y)) \chi(\text{Gr}_g FY)\text{coind} Y \prod_{i=1}^n x_i^{(\kappa, g)}a. \]

Given \( \varepsilon \in C/(T[1])(L, M[1] \cap Y) \), then it induces a triangle \( M \xrightarrow{\varepsilon} Y' \xrightarrow{P} L \xrightarrow{\varepsilon} M[1] \). Set \( \Delta := \dim FL + \dim FM \). Denote by \( W_{LM}(g, f, g) \) the set
\[ \{ (P, E) \mid P \in P\mathcal{C}/(T[1])(L, M[1] \cap Y), E \in \text{Gr}_g(FY') \}, \]
\[ \dim F \times \dim(E) = \chi(\text{dim}(F \times I)^{-1}(E) = f \}. \]

It is a constructible subset of the set \( P\mathcal{C}/(T[1])(L, M[1] \cap Y) \times \prod_{i=1}^{n} \text{Gr}_g(k^{d_i}) \) where \( d = (d_i)_{i=1}^n \) and \( g = (g_i)_{i=1}^n \) (compared to [Pa2, Lemma 17]). We set
\[ W_{LM}^{Y}(g) = \bigcup_{\varepsilon \leq \dim FL, I \leq \dim FM} W_{LM}(g, f, g) \quad \text{and} \quad W_{LM}^{Y}(g, f) = \bigcup_{\varepsilon \leq \Delta} W_{LM}(g, f, g). \]

Consider the projection
\[ \pi : W_{LM}^{Y}(g) \rightarrow P\mathcal{C}/(T[1])(L, M[1] \cap Y) \]
It is obvious that \( \pi^{-1}(\mathcal{P} \varepsilon) = \{ \mathcal{P} \varepsilon \times \text{Gr}_g FY' \) for any \( \mathcal{P} \varepsilon \in P\mathcal{C}/(T[1])(L, M[1] \cap Y). \)

Since \( \chi(\text{Gr}_g FY') = \chi(\text{Gr}_g FY) \), we have
\[ \chi(W_{LM}^{Y}(g)) = \chi(P\mathcal{C}/(T[1])(L, M[1] \cap Y)) \chi(\text{Gr}_g FY). \]

Hence, we obtain
\[ \Sigma_1 = \sum_{g, Y \in \mathcal{Y}} \chi(W_{LM}^{Y}(g))\text{coind} Y \prod_{i=1}^n x_i^{(\kappa, g)}a \]
\[ = \sum_{g, Y \in \mathcal{Y}} \chi(W_{LM}^{Y}(g, f, g))\text{coind} Y \prod_{i=1}^n x_i^{(\kappa, g)}a. \]

The following equality in \( K_0(\text{modB}) \) ([Pa1 Lemma 16])
\[ \sum_{i=1}^n (s_i + f_i)a[P_i] - \text{coind} (L \oplus M) = \sum_{i=1}^n (s_i, g)a[P_i] - \text{coind} Y \]
implies that
\[ \Sigma_1 = \sum_{g, Y \in \mathcal{Y}} \chi(W_{LM}^{Y}(g, f))\text{coind} (L \oplus M) \prod_{i=1}^n x_i^{(\kappa, g + f)}a. \]

In order to make the connection between \( \Sigma_1 \) and the left side of the equation in Theorem 1.2, it is natural to consider the following (constructible) map
\[ \psi_{L, f} : \bigcup_{Y \in \mathcal{Y}} W_{LM}^{Y}(g, f) \rightarrow P\mathcal{C}/(T[1])(L, M[1]) \times \text{Gr}_g FL \times \text{Gr}_f FM \]
defined by mapping \((\mathbb{P}\varepsilon, E)\) to \((\mathbb{P}\varepsilon, Fp(E), (F_i)^{-1}(E))\). Since the fibre of any point in \(\text{Im}\psi_{e,L}\) are affine spaces by \([\mathbb{C}, \mathbb{C}]\), \(\sum_{Y \in Y} \chi(W^Y_{LM}(e,f)) = \chi(\text{Im}\psi_{e,L})\). Then we have
\[
\Sigma_1 = \sum_{e \in L} \chi(\text{Im}\psi_{e,L})e^{-\text{coind}} (L \oplus M) \prod_{i=1}^{n} x_i^{(e_i, e_i + L^\vee)}.
\]

Set \(L(e,f) = \mathbb{P}C/(T[1])(L,M[1]) \times \text{Gr}_{e,F}L \times \text{Gr}_{f,F}M\). We denote by \(L_2(e,f)\) the complement of \(\text{Im}\psi_{e,L}\) in \(L(e,f)\).

Dually, we set
\[
\Sigma_2 = \sum_{Y \in Y} \chi(\mathbb{P}(T[1])(M,L[1]) \langle Y \rangle) X^T_Y.
\]

Given \(\varepsilon \in (T[1])(M,L[1]) \langle Y \rangle\), then it induces a triangle \(L \to Y \to M \to L[1]\).

Denote by \(W^Y_{ML}(f,e,g)\) the set
\[
\{(\mathbb{P}\varepsilon, E) | \mathbb{P}\varepsilon \in \mathbb{P}(T[1])(M,L[1]) \langle Y \rangle, E \in \text{Gr}_e(FY)\},
\]
\[
\dim Fp(E) = e, \dim (F_i)^{-1}(E) = f.
\]

Similarly, we set
\[
W^Y_{ML}(g) = \bigsqcup_{e \leq \dim ML, f \leq \dim FM} W^Y_{ML}(f,e,g) \quad \text{and} \quad W^Y_{ML}(f,e) = \bigsqcup_{g \leq \Delta} W^Y_{ML}(f,e,g).
\]

Then we have
\[
\Sigma_2 = \sum_{e \in L, f \in Y \in Y} \chi(W^Y_{ML}(f,e,g))e^{-\text{coind} Y} \prod_{i=1}^{n} x_i^{(e_i, f_i, g_i)}.
\]

Since \(\mathcal{C}\) is a 2-Calabi-Yau category and by [Palu1, Lemma 10], there is a non degenerate bifunctorial pairing
\[
\phi : \mathcal{C}/(T)(L[-1], M) \times (T[1])(M, L[1]) \to k.
\]

Let \(V \overset{i_V}{\to} L\) and \(U \overset{i_U}{\to} M\) be two morphisms such that \(F(i_V)\) and \(F(i_U)\) are monomorphisms. Similar to [Pa1], we define the map
\[
\alpha : \mathcal{C}(L[-1], U) \oplus \mathcal{C}/(T)(L[-1], M) \\
\mathcal{C}(T)(V[-1], U) \oplus \mathcal{C}(V[-1], M) \oplus \mathcal{C}/(T[1])(L[-1], M)
\]
by mapping \((a,b)\) to \(( ai_V[-1], i_U ai_V[-1] - bi_V[-1], i_U a - b)\). Its dual map is
\[
\alpha' : (T[1])(u,V[1]) \oplus (M, V[1]) \oplus \mathcal{C}/(T[2])(M, L[1]) \\
\mathcal{C}(u, L[1]) \oplus (T[1])(M, L[1])
\]
\[
(a, b, c) \to (i_V[1]a + ci_U[1] + iv[1]b, -c - iV[1]b)
\]
where \(V \in \text{Gr}_{e,F}L, U \in \text{Gr}_f,F\). Denote by \(C_{e,L}(Y,g)\) the set
\[
\{(\mathbb{P}\varepsilon, V, U), (\mathbb{P}\eta, E)\} \in L_2(e,L) \times W^Y_{ML}(g) | \phi(e[-1], \eta) \neq 0,
\]
\[
(F_i)^{-1} E = V, (Fp)(E) = U, \text{and} i, p \text{ are given by} \eta
\]
and
\[
C_{e,L} = \bigsqcup_{Y \in Y, g \in K_0(\text{mod} B)} C_{e,L}(Y,g).
\]

The following propositions also hold for \(\alpha\) and \(\alpha'\) defined here. We refer to [Pa2] for the proofs.
Proposition 2.1. [CK Proposition 3], [Pa2 Proposition 19] With the above notations, the following assertions are equivalent:
(i) \((P, V, U) \in L_2(e, f)\).
(ii) \(\varepsilon[-1]\) is not orthogonal to \((T[1])(M, L[1]) \cap \text{Im}\alpha'\).
(iii) There is an \(\eta \in (T[1])(M, L[1])\) such that \(\phi(\varepsilon[-1], \eta) \neq 0\) and such that if \(L \xrightarrow{\varepsilon} N \xrightarrow{\eta} M \xrightarrow{\delta} L[1]\)
is a triangle in \(\mathcal{C}\), then there exists the submodule \(E\) of \(FN\) such that \((F, \varepsilon) = V\), \((F, \delta) = U\).

Proposition 2.2. [CK Proposition 4], [Pa2 Proposition 20] (a) The projection \(C_{e, f} \xrightarrow{\pi_1} L_2(e, f)\) is surjective and the Euler characteristic of any fibre of \(\pi_1\) is 1.
(b) The projection \(C_{e, f}(Y, g) \xrightarrow{\pi_2} W_{M, L}(f, e, g)\) is surjective and its fibres are affine spaces.
(c) If \(C_{e, f}(Y, g)\) is not empty, then we have
\[
\sum_{i=1}^{n} (\varepsilon, e + f)_{a}[P_i] - \text{coind} \ (L \oplus M) = \sum_{i=1}^{n} (\varepsilon, g)_{a}[P_i] - \text{coind} \ Y.
\]

As a consequence, we have the following corollary.

Corollary 2.3. \(\chi(C_{e, f}) = \chi(L_2(e, f))\) and \(\chi(C_{e, f}(Y, g)) = \chi(W_{M, L}(f, e, g))\).

Then by Proposition 2.2 and Corollary 2.3 we obtain
\[
\Sigma_2 = \sum_{\varepsilon, f} \chi(L_2(e, f))e^{-\text{coind} \ (L \oplus M)} \prod_{i=1}^{n} x_i^{\varepsilon, e + f}.
\]
Then, we have
\[
\Sigma_1 + \Sigma_2 = \sum_{\varepsilon, f} \chi(L(e, f))e^{-\text{coind} \ (L \oplus M)} \prod_{i=1}^{n} x_i^{\varepsilon, e + f} = \chi(\mathbb{P}C/(T[1])(L, M[1]))X_{L}^{T}X_{M}^{T}.
\]

This completes the proof of Theorem 1.2.

3. Application to acyclic quivers

In this section, we assume \(\mathcal{C}\) is the cluster category of an acyclic quiver \(Q\) and \(T = kQ\). Thus the cluster character defined in [Pa1] is exactly the Caldero-Chapoton map. So we simply write \(X_{T}\) instead of \(X_{T}^{T}\).

Proposition 3.1. Assume \(L, M\) are \(kQ\)-modules and \(M\) has no projective modules as its direct summand, then we have the following isomorphisms between vector spaces
\[
\text{Hom}_{kQ}(L, \tau M) \cong \text{Ext}_{kQ}^{1}(M, L) \cong (T[1])(M, L[1]) \cong C/(T[1])(L, M[1]).
\]

Proof. The first isomorphism is an application of the Auslander-Reiten formula. The third isomorphism follows from [Pa1 Lemma 10]. It is enough to prove \(\text{Ext}_{kQ}^{1}(M, L) \cong (T[1])(M, L[1])\). Consider the projective resolution of \(M\) short exact sequence:
\[
0 \rightarrow P_0 \rightarrow P_1 \rightarrow M \rightarrow 0
\]
such that $kQ \subseteq P_0$. Applying the functor $\text{Hom}_{kQ}(-, L)$ and $C(-, L)$ on it, we obtain long exact sequences

$$0 \rightarrow \text{Hom}_{kQ}(M, L) \rightarrow \text{Hom}_{kQ}(P_1, L) \rightarrow \text{Hom}_{kQ}(P_0, L) \overset{f}{\rightarrow} \text{Ext}^1_{kQ}(M, L) \rightarrow 0$$

and

$$\cdots \rightarrow C(P_1, L) \rightarrow C(P_0, L) \overset{g}{\rightarrow} C(M[-1], L) \rightarrow \cdots .$$

Note that $\text{Hom}_{kQ}(P_1, L) \cong C(P_1, L)$ and $\text{Hom}_{kQ}(P_0, L) \cong C(P_0, L)$ by \cite{BMRRT}, thus there is a natural mono map $i : \text{Ext}^1_{kQ}(M, L) \rightarrow C(M[-1], L)$ induces if $\cong g$. Since $f$ is surjective, we have

$$\text{Ext}^1_{kQ}(M, L) \cong \text{Im} \ g \cong (P_0)(M[-1], L) \cong (P_0[1])(M, L[1]) = (T[1])(M, L[1]).$$

Using Theorem 1.2 and Proposition 3.1 we can obtain the following cluster multiplication formulas for acyclic quivers of any type in the context of module categories.

**Theorem 3.2.** (\cite{XX}, \cite{XX} Theorem 4.1) Let $kQ$ be an acyclic quiver. Then

1. For any $kQ$–modules $L, M, N$ and $M$ has no projective modules as its direct summands, we have

$$\dim_k \text{Ext}^1_{kQ}(M, L) \cdot X_L X_M = \sum_{Y \in \mathcal{R}(\mathfrak{e})} (\chi(\text{PExt}_{kQ}^1(M, L)(Y)) + \chi(\text{PHom}_{kQ}(L, \tau M)(Y)))X_Y$$

where $\mathfrak{e} = \dim M + \dim N$.

2. For any $kQ$–module $M, N$, and projective module $P$, we have

$$\dim_k \text{Hom}_{kQ}(P, M) \cdot X_M X_P[1] = \sum_{Y \in \mathcal{S}} (\chi(\text{PHom}_{kQ}(M, I)(Y)) + \chi(\text{PHom}_{kQ}(P, M)(Y)))X_Y$$

where $I = \text{DHom}_{kQ}(P, kQ)$.

Here, $\mathcal{R}(\mathfrak{e})$ and $\mathcal{S}$ are some finite sets defined as the finite set $\mathcal{Y}$ in Theorem 1.1 we refer to \cite{XX} for details.

**Proof.** (1) By Proposition 3.1 the first formula in Theorem 1.2 and the fact that $\chi(\text{PExt}_{kQ}^1(M, L)) = \dim_k \text{Ext}^1_{kQ}(M, L)$.

(2) We know $I = P[2] \in \text{obj}(\mathcal{C})$ and then

$$C/(T[1])(M, P[2]) \cong C/(T[1])(M, I) \cong \text{Hom}_{kQ}(M, I)(Y).$$

We also have

$$\text{D}C/(T[1])(M, P[2]) \cong (T[1])(P[1], M[1]) \cong (T)(P, M) \cong \text{Hom}_{kQ}(P, M).$$

Therefore the proof follows from Theorem 1.2 and the fact that $\chi(\text{PHom}_{kQ}(P, M)) = \dim_k \text{Hom}_{kQ}(P, M)$.

In the same way as in the proof of Theorem 1.1 by Theorem 1.2 we obtain a ‘unified’ version of the formulas in Theorem 3.2 in the context of cluster categories. It is clear that Theorem 2.1 is a generalization of Theorem 3.3.

**Theorem 3.3.** Let $kQ$ be an acyclic quiver and $\mathcal{C}$ be the cluster category of $kQ$. For any $M, N \in \text{obj}(\mathcal{C})$, if $\text{Ext}^1_{kQ}(M, N) := C(M, N[1]) \neq 0$, we have

$$\chi(\text{PExt}_{kQ}^1(M, N)) \cdot X_M X_N = \sum_{Y \in \mathcal{Y}} (\chi(\text{PExt}_{kQ}^1(M, N)(Y)) + \chi(\text{PExt}_{kQ}^1(M, N)(Y)))X_Y.$$
If either Ext$_{Q}^1(M, L)$ or Ext$_{K}^1(L, M)$ vanishes, then Theorem 3.2 coincides with Theorem 3.3 (see [Hu]). In general, it is possible that both of two extension spaces are not zero. For example, consider the affine quiver $Q = \tilde{D}_4$ with a fixed orientation. Let $C$ be the cluster category of type $\tilde{D}_4$. The Auslander-Reiten quiver of $\tilde{D}_4$ contains three nonhomogeneous tubes, denoted by $T_0, T_1, T_\infty$. Assume that $E_1, E_2$ are two regular simple modules in $T_0$. Then we have
\[ \dim_k \text{Ext}^1_{Q}(E_1, E_2) = \dim_k \text{Ext}^1_{Q}(E_2, E_1) = 1, \quad \dim_k \text{Ext}^1_{C}(E_1, E_2) = 2. \]

By Theorem 3.2 we obtain
\[ X_{E_1} X_{E_2} = X_{E_1[2]} + 1. \]
where $E_1[2]$ denotes the extension of $E_1$ by $E_2$. By Theorem 3.3 we obtain
\[ 2X_{E_1} X_{E_2} = 2X_{E_1[2]} + 2. \]

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