In this study, we derive recursion formulas for the Kampé de Fériet hypergeometric matrix function. We also obtain some finite matrix and infinite matrix summation formulas for the Kampé de Fériet hypergeometric matrix function.

1. Introduction

The theory of special functions is closely related to the theory of Lie groups and Lie algebras, as well as certain topics in mathematical physics. Symbolic computation and engineering problems usually recognize the majority of special functions. Recently, there has been a surge in the study of recursion formulas for multivariable hypergeometric functions. Recursion formulas for the Appell function $F_2$ have been investigated by Opps et al. [1], followed by Wang [2], who presented the recursion relations for all Appell functions. Furthermore, recursion formulas for variant multivariable hypergeometric functions were presented in [3–6]. One can refer to various sources [7, 8] for the in-depth study of the hypergeometric functions for several variables.

The theory of generalized matrix special functions has witnessed a rather significant evolution during the last two decades. The reasons of interest have a manifold motivation. Restricting ourselves to the applicative field, we note that for some physical problems, the use of new classes of matrix special functions provided solutions hardly achievable with conventional analytical and numerical means. Special matrix functions appear in the literature related to statistics [9], Lie theory [10], and more recently in connection with the matrix version of Laguerre, Hermite, and Legendre differential equations and the corresponding polynomial families [11–13]. In [14], recursion formulas and matrix summation formulas for Srivastava’s triple hypergeometric matrix functions are obtained.

The study is organized in the following manner. In Section 2, we list basic definitions that are needed in the sequel. In Section 3, we obtain recursion formulas for the Kampé de Fériet hypergeometric matrix function (its abbreviation is K de FHMF). In Section 4, we present finite matrix summation formulas for the (K de FHMF) by applying a derivative operator. Finally, in Section 5, we establish infinite matrix summation formulas for the (K de FHMF).

2. Preliminaries

Let $\mathbb{C}^{n \times r}$ be the vector space of $r$ square matrices with complex entries. For any matrix $H \in \mathbb{C}^{n \times r}$, its spectrum $\sigma(H)$ is the set of eigenvalues of $H$. $H$ in $\mathbb{C}^{n \times r}$ is called a positive stable matrix if $\Re(\lambda) > 0$ for all $\lambda \in \sigma(H)$.

The reciprocal gamma function $\Gamma^{-1}(\theta) = 1/\Gamma(\theta)$ is an entire function of the complex variable $\theta$. The image of $\Gamma^{-1}(\theta)$ acting on $H$, denoted by $\Gamma^{-1}(H)$, is a well-defined matrix. If $H + \ell I$ is invertible for all integers $\ell \geq 0$, then the
The reciprocal gamma function \([\Gamma^{-1}(H)]\) is defined by
\[
\Gamma^{-1}(H) = (H)_{\ell} \Gamma^{-1}(H + \ell I),
\]
where \((H)_{\ell}\) is the shifted factorial matrix function for \(H \in \mathbb{C}^{r \times r}\) given as \([16]\)
\[
(H)_{\ell} = \begin{cases} 
1, & \ell = 0, \\
H(H + I), \ldots, (H + (\ell - 1)I), & \ell \geq 1.
\end{cases}
\]

In the sequel, consider \(I\) being the \(r\)-square identity matrix. If \(H \in \mathbb{C}^{r \times r}\) is a positive stable matrix and \(\ell \geq 1\), then by \([15]\), we have \(\Gamma(H) = \lim_{\ell \rightarrow -\infty} (\ell - 1)! (H)_{\ell}^{-1} e^{H}\).

The Gauss hypergeometric matrix function \([16]\) is defined by
\[
{_{2}F_{1}}(A, B; C; x) = \sum_{n=0}^{\infty} \frac{(A)_n (B)_n (C)_n^{-1}}{n!} x^n,
\]
for matrices \(A, B,\) and \(C\) in \(\mathbb{C}^{r \times r}\), such that \(C + kI\) is invertible for all \(k \geq 0\) and \(|x| \leq 1\).

The Appell matrix functions are defined by
\[
\begin{align*}
F_1(A, B, B'; C; x, y) &= \sum_{m,n=0}^{\infty} \frac{(A)_{m+n} (B)_{m} (B')_{n} (C)_{m+n}^{-1}}{m! n!} x^m y^n, \\
F_2(A, B, B'; C, C'; x, y) &= \sum_{m,n=0}^{\infty} \frac{(A)_{m+n} (B)_{m} (B')_{n} (C)_{m} (C')_{n}^{-1}}{m! n!} x^m y^n, \\
F_3(A, A', B, B'; C, x, y) &= \sum_{m,n=0}^{\infty} \frac{(A')_{n} (B)_{m} (B')_{n} (C)_{m+n}^{-1}}{m! n!} x^m y^n, \\
F_4(A, B; C, C'; x, y) &= \sum_{m,n=0}^{\infty} \frac{(A)_{m+n} (B)_{m} (C)_{m+n}^{-1} (C')_{n}^{-1}}{m! n!} x^m y^n,
\end{align*}
\]
where \(A, A', B, B', C\), and \(C'\) are the positive stable matrices in \(\mathbb{C}^{r \times r}\), so that \(C + kI\) and \(C' + kI\) are invertible for each integer \(k \geq 0\). For regions of convergence of equations (3)–(6), see \([17–19]\).

The Kampé de Fériet hypergeometric matrix function is given as \([18, 19]\)
\[
{_{p}F_{q}}(A; B_1, B_2, \ldots, B_p; C_1, C_2, \ldots, C_q; x, y) = \sum_{m,n=0}^{\infty} \frac{(A)_{m+n} (B_1)_{m} \ldots (B_p)_{m} (C_1)_{n} \ldots (C_q)_{n}^{-1}}{m! n!} x^m y^n,
\]
where \(A\) abbreviates the sequence of matrices \(A_1, \ldots, A_m\), and \(A_i, B_i, C_i, D_i, E_i\), and \(F_i\) are the positive stable matrices in \(\mathbb{C}^{r \times r}\), such that \(D_i + kI, E_i + kI\) and \(F_i + kI\) are invertible for all integers \(k \geq 0\).

Next, we recall the definition of the derivative operator
\[
D_y f(y) = \lim_{h \to 0} \frac{f(y + h) - f(y)}{h},
\]
provided \(f\) is differentiable at \(y\). Also, \(D_k^2 f(y) = D_y (D_k^1 f(y))\) for \(k = 0, 1, 2, \ldots\).

In the whole study, \(I\) is the identity matrix and \(s\) is a nonnegative integer. In the sequel, consider
\[
\begin{align*}
A + sI &= A_1 + sI, A_2 + sI, \ldots, A_m + sI, \\
A' &= A_1, A_2, \ldots, A_m, \\
A' + sI &= A_1 + sI, \ldots, A_m + sI, \\
A &= A_1, A_2, \ldots, A_m, \\
A + sI &= A_1 + sI, \ldots, A_m + sI.
\end{align*}
\]

Also, we denote
\[
\begin{align*}
[A + kI]_s &= \prod_{i=1}^{m} (A_i + kI)_s, \\
[A + kI]^{-1}_s &= \prod_{i=1}^{m} (A_i + kI)_s^{-1}, \\
[A' + kI]_s &= \prod_{i=1}^{m} (A_i + kI)_s, \\
[A' + kI]^{-1}_s &= \prod_{i=1}^{m} (A_i + kI)_s^{-1}.
\end{align*}
\]

3. Recursion Formulas for the Kampé de Fériet Hypergeometric Matrix Function (K de FHMFD)

In this section, we obtain the recursion formulas for the (K de FHMFD).
Theorem 1. Let $A_i+sI, i=1, \ldots, m_1$ be invertible for each integer $s \geq 0$. Then, the following recursion formula holds true for the $(K \text{ de FHMF})$:

\[
F^{m_1} = F^{m_1} + x[A]B \sum_{k=1}^{m_1} F^{m_1} + y[A]C \sum_{k=1}^{m_1} F^{m_1} - y[A]C \sum_{k=1}^{m_1} F^{m_1}.
\]

Also, if $A_i-kI$ is invertible for integers $k \leq s$, then

\[
F^{m_1} = F^{m_1} + x[A]B \sum_{k=1}^{m_1} F^{m_1} + y[A]C \sum_{k=1}^{m_1} F^{m_1} - y[A]C \sum_{k=1}^{m_1} F^{m_1}.
\]

where $A_i, B_i, C_i, D_i, E_i$, and $F_i$ are the positive stable matrices in $C^{nxn}$, such that $A_iA_i = A_iA_i$, $A_iB_i = B_iA_i$, $A_iC_i = C_iA_i$, $B_iC_i = C_iB_i$, $F_iE_i = E_iF_i$, $F_iD_i = D_iF_i$; and $D_iE_i = E_iD_i$, and $D_i + kI$, $E_i + kI$, and $F_i + kI$ are invertible for each integer $k \geq 0$.

Proof. In view of equation (7) and the fact that

\[
(A_i + I)_{m_1} = A_i^{-1} (A_i)_{m_1} (A_i + mI + nI),
\]

we get the following contiguous matrix relation:

\[
F^{m_1} = F^{m_1} + x[A]B \sum_{k=1}^{m_1} F^{m_1} + y[A]C \sum_{k=1}^{m_1} F^{m_1} - y[A]C \sum_{k=1}^{m_1} F^{m_1}.
\]

Replacing $A_i$ with $A_i + I$ in equation (14), we have the following contiguous matrix relation:

\[
F^{m_1} = F^{m_1} + x[A]B \sum_{k=1}^{m_1} F^{m_1} + y[A]C \sum_{k=1}^{m_1} F^{m_1} - y[A]C \sum_{k=1}^{m_1} F^{m_1}.
\]
Iterating this process \( s \)-times, we get equation (11). For the proof of equation (12), replace the matrix \( A_i \) with \( A_i - I \) in equation (14). As \( A_i - I \) is invertible, we have

\[
F_{m_1,m_2,n_1,n_2}^n(A_i; x, y) = F_{m_1,m_2,n_1,n_2}^n(A_i, B_i; x, y) - x[A_i^t][B_i]F_{m_1,m_2,n_1,n_2}^n(A_i+1, B_i+1; x, y)[D_i^{-1}][E_i^{-1}]
\]

Iteratively, we get equation (12).

Using contiguous matrix relations equations (14) and (16), we get the following forms of the recursion formulas for the (K de FHMF).

**Theorem 2.** Let \( A_i + sI, i = 1, \ldots, n_i \) be invertible for each integer \( s \geq 0 \). Then, the following recursion formula holds true for the (K de FHMF):

\[
F_{m_1,m_2,n_1,n_2}^n(A_i, B_i, C_i; x, y) = \sum_{k_1 + k_2 \leq s} \left( \begin{array}{c} s \\ k_1, k_2 \end{array} \right) [A_i]_{k_1+k_2} [B_i]_{k_1} [C_i]_{k_2} 
\]

\[
\times (-x)^{k_1} (-y)^{k_2} F_{m_1,m_2,n_1,n_2}^{n_i}(A_i; x, y) [D_i]_{k_1+k_2} [E_i]_{k_1} [F_i]_{k_2}^{-1}
\]

where \( A_i, B_i, C_i, D_i, E_i, F_i \), and \( F_i \) are the positive stable matrices in \( \mathbb{C}^{r \times r} \), such that \( A_iA_j = A_jA_i, A_iB_j = B_jA_i, A_iC_j = C_jA_i, B_iC_j = C_jB_i, F_iE_i = E_iF_i, F_iD_i = D_iF_i, \) and \( D_iE_i = E_iD_i \), and \( D_i + kI, E_i + kI, \) and \( F_i + kI \) are invertible for each integer \( k \geq 0 \).

**Proof.** We prove equation (17) by applying a mathematical induction on \( s \). For \( s = 1 \), the result equation (17) is true due to equation (14). Assume equation (17) is true for \( s = t \), that is,

\[
F_{m_1,m_2,n_1,n_2}^n(A_i, B_i, C_i; x, y) = \sum_{k_1 + k_2 \leq s} \left( \begin{array}{c} t \\ k_1, k_2 \end{array} \right) [A_i]_{k_1+k_2} [B_i]_{k_1} [C_i]_{k_2} 
\]

\[
\times (-x)^{k_1} (-y)^{k_2} F_{m_1,m_2,n_1,n_2}^{n_i}(A_i, B_i, C_i; x, y) [D_i]_{k_1+k_2} [E_i]_{k_1} [F_i]_{k_2}^{-1}
\]

Replacing \( A_i \) with \( A_i + I \) in equation (19) and using the contiguous matrix relation equation (14), we get

\[
F_{m_1,m_2,n_1,n_2}^n(A_i+1, B_i+1, C_i+1; x, y) = \sum_{k_1 + k_2 \leq s} \left( \begin{array}{c} t + 1 \\ k_1, k_2 \end{array} \right) [A_i]_{k_1+k_2} [B_i]_{k_1} [C_i]_{k_2} 
\]

\[
\times (-x)^{k_1} (-y)^{k_2} F_{m_1,m_2,n_1,n_2}^{n_i}(A_i; x, y) [D_i]_{k_1+k_2} [E_i]_{k_1} [F_i]_{k_2}^{-1}
\]

Applying the known relation \( \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \) and \( \binom{n}{k} = 0 \) (for \( k > n \) or \( k < 0 \), the above identity can be reduced to the following result:
Theorem 3. Let \( B_i = sI, i = 1, \ldots, n_1 \) be invertible for each integer \( s \geq 0 \). Then, the following recursion formula is satisfied for the (K de FHMF):

\[
F_{m_1, n_1}^{(i)}(A; B_i + sI, C; D, E; F; x, y) = F_{m_1, n_1}^{(i)}(A; B_i, C; D, E; F; x, y) + x[A][B] \sum_{k=1}^{s} F_{m_1, n_1}^{(i)}(A + B_k; C; D, E; F; x, y) [D]_{k}^{-1} [E]_{k}^{-1}.
\]

(22)

In addition, if \( B_i - kI \) is invertible for integers \( k \leq s \), one gets

\[
F_{m_1, n_1}^{(i)}(A; B_i - kI, C; D, E; F; x, y) = F_{m_1, n_1}^{(i)}(A; B_i, C; D, E; F; x, y) - x[A][B] \sum_{k=1}^{s} F_{m_1, n_1}^{(i)}(A + B_k; C; D, E; F; x, y) [D]_{k}^{-1} [E]_{k}^{-1}.
\]

(23)

where \( A_i, B_i, C_i, D_i, E_i \), and \( F_i \) are the positive stable matrices in \( C^{\times r} \), such that \( A_i B_i = B_i A_i; B_i B_j = B_j B_i; F_i D_j = D_j F_i; D_i E_j = E_j D_i; F_i E_j = E_j F_i \), and \( D_i + kI, E_i + kI, F_i + kI \) are invertible for each integer \( k \geq 0 \).

The recursion formulas for \( F_{m_1, n_1}^{(i)}(A; B_i C_i + sI; D, E; F; x, y) \) are obtained by replacing \( B \rightarrow C, E \rightarrow F, \) and \( x \rightarrow y \) in Theorems 3 and 4.

Next, we state the recursion formulas for the matrix \( D_i \) of the (K de FHMF).

Theorem 5. Let \( D_i = sI, i = 1, \ldots, m_2 \) be invertible for each integer \( s \geq 0 \). Then, the following recursion formula holds true for the (K de FHMF):

\[
F_{m_1, n_1}^{(i)}(A; B_i C_i + sI; D, E; F; x, y) = F_{m_1, n_1}^{(i)}(A; B_i, C_i; D, E; F; x, y) + x[A][B] \sum_{k=1}^{s} F_{m_1, n_1}^{(i)}(A + B_k; C_i; D, E; F; x, y) [D_i]_{k}^{-1} [E_i]_{k}^{-1}.
\]

(26)
where $A_i, B_i, C_i, D_i, E_i,$ and $F_i$ are the positive stable matrices in $\mathbb{C}^{\infty \times \infty}$, such that $A_i B_i = B_i A_i$; $A_i C_i = C_i A_i$; $B_i C_i = C_i B_i$; $F_i E_j = E_j F_i$; $D_i D_j = D_j D_i$; $D_i F_j = F_j D_i$; and $D_i E_j = E_j D_i$, and $D_i + k I, E_i + k I,$ and $F_i + k I$ are invertible for each integer $k \geq 0$.

Proof. Applying the definition of the (K de FHMF) and the fact that

$$ (D_i - I)^{-1} = (D_i)^{-1} + n(D_i - I)^{-1}, $$

(27)

the following contiguous matrix relation is obtained:

$$ F^{m_i, n_i}_{m_j, n_j}(A;B,C;D,E,F;x,y) = F^{m_i, n_i}_{m_j, n_j}(A;B,C;D,E,F;x,y) + x[A]B \left[ F^{m_i, n_i}_{m_j, n_j}(A;B,C;D,E,F;x,y) D_i - I \right]^{-1} [D_i - I]^{-1} E_i^{-1} + y[A]C \left[ F^{m_i, n_i}_{m_j, n_j}(A;B,C;D,E,F;x,y) D_i - I \right]^{-1} [D_i - I]^{-1} F_i^{-1}.$$

(28)

$$ F^{m_i, n_i}_{m_j, n_j}(A;B,C;D,E,F;x,y) = F^{m_i, n_i}_{m_j, n_j}(A;B,C;D+E,F;x,y) + \sum_{k=1}^{s} F^{m_i, n_i}_{m_j, n_j}(A;B,C;D+E,F;x,y) [D_i - I]^{-1} [D_i - (k - 1)I]^{-1} [E_i - kI]^{-1} (E_i - (k - 1)I) [D_i - I]^{-1} E_i^{-1} F_i^{-1},$$

(29)

and $D_i F_j = F_j D_i$, and $D_i + k I, E_i + k I,$ and $F_i + k I$ are invertible for each integer $k \geq 0$.

By using the generalized Leibnitz formula,

$$ D_y^k (f(y)g(y)) = \sum_{k=0}^{p} \binom{p}{k} D_y^k f(y) D_y^k g(y),$$

(31)

and equation (30), we derive the following finite matrix summation formulas of the (K de FHMF).

Theorem 6. Let $E_i - sI, i = 1, \ldots, n_3$ be invertible for each integer $s \geq 0$. Then, the following recursion formula holds true for the (K de FHMF):

$$ E_i - sI = [E_i - sI]^{-1} [E_i - (s-1)I]^{-1} [E_i - (s-2)I]^{-1} \cdots [E_i - I]^{-1} [E_i - 0I]^{-1},$$

and $D_i F_j = F_j D_i$, and $D_i + k I, E_i + k I,$ and $F_i + k I$ are invertible for each integer $k \geq 0$.

Theorem 7. Let $A_i, B_i, C_i, D_i, E_i,$ and $F_i$ be the positive stable matrices in $\mathbb{C}^{\infty \times \infty}$, such that $A_i B_i = B_i A_i$; $A_i C_i = C_i A_i$; $B_i C_i = C_i B_i$; $F_i E_j = E_j F_i$; $D_i D_j = D_j D_i$; $D_i F_j = F_j D_i$; and $D_i E_j = E_j D_i$, and $D_i + k I, E_i + k I,$ and $F_i + k I$ are invertible for all integers $k \geq 0$. Then, the following finite matrix summation formulas hold for the (K de FHMF):

$$ \sum_{k=0}^{p} \binom{p}{k} [A_i] k [C_i] x F^{m_i, n_i}_{m_j, n_j}(A;B,C;D+E,F;x,y) [D_i - kI]^{-1} [F_i]^{-1},$$

(32)

Proof. From definition of the (K de FHMF) and the generalized Leibnitz formula for differentiation of a product of two functions, we have

4. Finite Matrix Summation Formulas for the Kampé de Fériet Hypergeometric Function by a Derivative Operator

In this section, we obtain the finite matrix summation formulas for the (K de FHMF) by a derivative operator. These formulas are analogues for some summation formulas of double hypergeometric functions [8]. The $p^{th}$ derivative on $y$ of the (K de FHMF) is obtained as follows:

$$ D_y^p F^{m_i, n_i}_{m_j, n_j}(A;B,C;D+E,F;x,y) = [A_i] [C_i] x F^{m_i, n_i}_{m_j, n_j}(A;B,C;D+E,F;x,y) [D_i - kI]^{-1} [F_i]^{-1},$$

(30)

where $A_i, B_i, C_i, D_i, E_i,$ and $F_i$ are the positive stable matrices in $\mathbb{C}^{\infty \times \infty}$, such that $A_i C_j = C_j A_i$; $B_i C_j = C_j B_i$; $F_i E_j = E_j F_i$; and $D_i D_j = D_j D_i$; $D_i F_j = F_j D_i$; and $D_i E_j = E_j D_i$, and $D_i + k I, E_i + k I,$ and $F_i + k I$ are invertible for each integer $k \geq 0$.
We used equation (30) and some simplification in the second equality. Next, we combine $y^{C_\nu + (p-1)I}$ with the variable $y$ in the (K de FHMF) and apply the derivative operator $p$-times on $y$ to get the following result:

\[
D_y^p \left\{ y^{C_\nu + (p-1)I} f_{m_1,n_1}^{n_2} (A,B,C,D,E,F; x, y) \right\} = \sum_{k=0}^{p} \binom{p}{k} D_y^{p-k} \left\{ y^{C_\nu + (p-k-1)I} f_{m_1,n_1}^{n_2} (A,B,C,D,E,F; x, y) \right\} = (C_\nu) p y^{C_\nu - I} f_{m_1,n_1}^{n_2} (A,B,C,D,E,F; x, y) \left[ D_0^{-1} \right]_k \]

Equating the above two relations leads to equation (32).

**Theorem 8.** Let $A_i, B_i, C_i, D_i, E_i,$ and $F_i$ be the positive stable matrices in $C^{n \times r},$ such that $A_i C_i = C_i A_i;$ $B_i C_i = C_i B_i;$ $D_i E_i = E_i D_i;$ and $F_i = F_i F_i$ for all integers $k \geq 0.$ Then, the following finite matrix summation formulas of the (K de FHMF) hold true:

\[
\sum_{k=0}^{p} \binom{p}{k} [A_k]_k y^{k} f_{m_1,n_1}^{n_2} (A+kI,B,C+kl,D+kI,E+kI,F+kI; x, y) (F_i - pI)_k^{-1} \left[ D_0^{-1} \right]_k \]

where $F_i + (k - p)I$ is an invertible matrix for $0 \leq k \leq p$ and $i = 1, \ldots, n_2.$

**Proof.** Applying the derivative operator and some transformations, we can get the finite matrix summation formulas of the (K de FHMF) as follows.

**Theorem 9.** Let $A_i, B_i, C_i, D_i, E_i,$ and $F_i$ be the positive stable matrices in $C^{n \times r},$ such that $A_i C_i = C_i A_i;$ $B_i C_i = C_i B_i;$ $D_i E_i = E_i D_i;$ and $F_i = F_i F_i$ for all integers $k \geq 0.$ Then, the following finite matrix summation formulas of the (K de FHMF) hold true:

\[
\sum_{k=0}^{p} \binom{p}{k} [A_k]_k y^{k} f_{m_1,n_1}^{n_2} (A+kI,B,C+kl,D+kI,E+kI,F+kI; x, y) (F_i - pI)_k^{-1} \left[ D_0^{-1} \right]_k \]

Applying a derivative operator and some transformations, we can get the finite matrix summation formulas of the (K de FHMF) as follows.
\(D_iE_j = E_iD_j;\) and \(D_iF_j = F_iD_j;\) and \(D_i + kI,\ E_i + kI,\) and \(F_i + kI\) are invertible for each integer \(k \geq 0.\) Then, the following finite matrix summation formulas of the (K de FHMF) hold true:

\[
\sum_{k=0}^{r} \binom{r}{k} (-1)^k F_{m_1,m_2,n_1}^{m_1,m_2,n_1}(A;C_1,B_1,C;D,E,F_r;\{x,y\})(I - F_i)_k ((2 - r)I - F_i)_k^{-1}
\]

\[
= (-1)^r [A]_r [C]_r [D]_r^{(r + 1)} [F]_r^{-1},
\]

\[
\sum_{k=0}^{r} \binom{r}{k} (-1)^k F_{m_1,m_2,n_1}^{m_1,m_2,n_1}(A;C_1,B_1,C;D,E,F_r;\{x,y\})(F_i + (r - 1)I)_k (F_i + rI)_k^{-1}
\]

where \((2 + k - r)I - F_i, F_i - kI,\) and \(F_i + (k - 1)I\) is an invertible matrix for \(0 \leq k \leq r\) in (36); \(F_i + rI\) is an invertible matrix in equation (37), \(i = 1, \ldots, n_2^3.

\[
\sum_{k=0}^{r} \binom{r}{k} (-1)^k F_{m_1,m_2,n_1}^{m_1,m_2,n_1}(A;C_1,B_1,C;D,E,F_r;\{x,y\})(F_i - I)_r (I - F_i)_k (2I - F_i - rI)_k^{-1} y^{r - k}
\]

Now, using the derivative operator on the (K de FHMF) for \(r\)-times directly and equating with the above equality gives equation (36) after some simplifications. Next, applying the operator \(D_y^r\) on

\[
F_{m_1,m_2,n_1}^{m_1,m_2,n_1}(A;C_1,B_1,C;D,E,F_r;\{x,y\}) y^{rF_r - r} x y^{rF_r - r},
\]

and proceeding as in the proof of equation (36) gives the result equation (37).

\[
D_y^r \left\{ F_{m_1,m_2,n_1}^{m_1,m_2,n_1}(A;C_1,B_1,C;D,E,F_r;\{x,y\}) y^{rF_r - r} x y^{rF_r - r} \right\}
\]

where \(i = 1, \ldots, m_1;\)

\[
\sum_{k=0}^{\infty} \binom{m}{k} [A]_k [B]_k [C]_k [D,E,F]_k \left\{ (A_i + kI;B_i,C;D,E,F) x y \right\}
\]

\[
= \frac{1}{1-t} [A]_m x y,\]

and

\[
\sum_{k=0}^{\infty} \binom{m}{k} [B]_k [C]_k [D,E,F]_k \left\{ (A_i + kI;B_i,C;D,E,F) x y \right\}
\]

\[
= \frac{1}{1-t} [B]_m x y,\]

where \(i = 1, \ldots, n_1;\)

**5. Infinite Summation Formulas for the Kampé de Fériet Hypergeometric Matrix Function**

In this section, we will establish the infinite summation formulas of the (K de FHMF).

**Theorem 10.** Let \(A_0, B_0, C_0, D_0, E_0,\) and \(F_0\) be the positive stable matrices in \(C^{n \times n},\) such that \(D_i + kI, E_i + kI,\) and \(F_i + kI\) are invertible for each integer \(k \geq 0.\) Then, the following infinite summation formulas of the (K de FHMF) hold true:

\[
\sum_{k=0}^{\infty} \binom{m}{k} [A]_k [B]_k [C]_k [D,E,F]_k \left\{ (A_i + kI;B_i,C;D,E,F) x y \right\}
\]

where \(i = 1, \ldots, n_1;\)

**Proof:** We shall prove equation (40). We apply the definition of the (K de FHMF) and transformation,

\[
\left( A_i \right)_{k} \left( A_i + kI \right)_{m \times n} = \left( A_i \right)_{m \times n} \left( A_i + (m + n)I \right)_{k},
\]

to get that the left side of equation (40) is written as
Let \( A_i \) be the positive stable matrices in \( \mathbb{C}^{m \times r} \), such that \( A_i B_j = B_j A_i ; D_i E_j = E_j D_i \) and \( F_i F_j = F_j F_i \), and all \( A_i \), \( E_i \), \( kI \), and \( F_i + kI \) are invertible for each integer \( k \geq 0 \). Then, the infinite summation formulas of the (K de FHMF) hold true:

\[
\sum_{m,n=0}^{\infty} F_0^{A_i} \mid x, y \rangle \langle x, y \mid t \rangle = (1 - t)^{-A_i}, \tag{44}
\]

and after simplifications, the right side of equation (40) is obtained. This ends the proof of equation (40). The identity equation (41) is proved in a similar manner. \( \square \)

Using the identity,

\[
\sum_{m,n=0}^{\infty} \prod_{i=1}^{n_1} (A_i)_{m \times r} \prod_{i=1}^{n_2} (B_i)_{m \times r} \prod_{i=1}^{n_3} (C_i)_{n \times m} \prod_{i=1}^{n_4} (D_i)_{n \times r} \prod_{i=1}^{n_5} (E_i)_{n \times r} \prod_{i=1}^{n_6} (F_i)_{n \times r} \frac{x^m y^n}{m! n!} \tag{43}
\]

Proof: From the definition of the (K de FHMF) and the transformation \((A)_k ((A + kI)_m = (A)_{k+m}, the right side of equation (45) is expressed as

\[
\sum_{k,m,n=0}^{\infty} \prod_{i=1}^{n_1} (A_i)_{m \times r} \prod_{i=1}^{n_2} (B_i)_{m \times r} \prod_{i=1}^{n_3} (C_i)_{n \times m} \prod_{i=1}^{n_4} (D_i)_{n \times r} \prod_{i=1}^{n_5} (E_i)_{n \times r} \prod_{i=1}^{n_6} (F_i)_{n \times r} \frac{x^m y^n}{m! n!} k! \tag{46}
\]

Replace \( m + k \) by \( l \) in the above result. After some simplifications, we have

\[
\sum_{l=0}^{\infty} \prod_{i=1}^{n_1} (A_i)_{l+m \times r} \prod_{i=1}^{n_2} (B_i)_{l+m \times r} \prod_{i=1}^{n_3} (C_i)_{l+n \times m} \prod_{i=1}^{n_4} (D_i)_{l+n \times r} \prod_{i=1}^{n_5} (E_i)_{l+n \times r} \prod_{i=1}^{n_6} (F_i)_{l+n \times r} \frac{x^m y^n}{m! n!} k! \tag{47}
\]

Using the relation

\[
\sum_{k=0}^{l} \binom{l}{k} x^k y^{l-k} = (x + y)^l, \tag{48}
\]

in the inner summation, finishes the proof of equation (45). \( \square \)

Theorem 12. Let \( A_i \), \( B_j \), \( C_i \), \( D_i \), \( E_i \), and \( F_i \) be the positive stable matrices in \( \mathbb{C}^{m \times r} \), such that \( D_i + kI \), \( E_i + kI \), and \( F_i + kI \) are invertible for each integer \( k \geq 0 \). Then, the infinite summation formulas of the (K de FHMF) hold true:

\[
\sum_{k=0}^{\infty} \binom{n_1}{k} (-1)^k F_0^{A_i B_j C_i D_i E_i F_j} \frac{(1 + t)}{x, y} \tag{49}
\]

where \( A_i B_j = B_j A_i, i = 1, \ldots, n_1 \).
Proof: From the definition of the (K de FHMF), the left side of equation (49) can be expressed as

\[
\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(B_i)_k(-kl)^m}{m!n!} \prod_{i=1}^{m} (A_i)_{m+n} \prod_{j=1}^{m} (B_i)_m \prod_{i=1}^{n} (C_i)_n \prod_{i=1}^{m} (D_i)_{m+n} \prod_{i=1}^{n} (E_i)_m \prod_{i=1}^{n} (F_i)^{-1}_n (1 + t)^x y^n (-t)^k. (50)
\]

Taking \( k = m + l \), changing the summation order and simplifying, we get

\[
\sum_{m=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-m)^l}{l!} \prod_{i=1}^{m} (A_i)_{m+n} \prod_{j=1}^{m} (B_i)_m \prod_{i=1}^{n} (C_i)_n \prod_{i=1}^{m} (D_i)_{m+n} \prod_{i=1}^{n} (E_i)_m \prod_{i=1}^{n} (F_i)^{-1}_n (1 + t)^x y^n (1 + t)^m. (51)
\]

Authors’ Contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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