Decentralized State Estimation In A Dimension-Reduced Linear Regression

Robin Forsling*, Fredrik Gustafsson†, Zoran Sjanic, and Gustaf Hendeby‡

* Student Member, IEEE, † Senior Member, IEEE, ‡ Fellow, IEEE

Dept. of Electrical Engineering, Linköping University, Linköping, Sweden

Abstract—Decentralized state estimation in a communication constrained sensor network is considered. To reduce the communication load only dimension-reduced estimates are exchanged between the networking agents. The considered dimension-reduction is restricted to be a linear mapping from a higher-dimensional space to a lower-dimensional space. The optimal, in the mean square error sense, linear mapping depends on the particular estimation method used. Several dimension-reducing algorithms are therefore proposed, where each algorithm corresponds to a commonly applied decentralized estimation method. All except one of the algorithms are shown to be optimal. For the remaining algorithm we provide a convergence analysis where it is theoretically shown that this algorithm converges to a stationary point and numerically shown that the convergence rate is fast. A message encoding solution is proposed that allows for efficient communication when using the proposed dimension-reduction techniques. Applicability of the different algorithms is illustrated by a numerical evaluation.

Index Terms—Decentralized estimation, communication constraints, dimension-reduction, cross-correlated estimates.

I. INTRODUCTION

DECENTRALIZED ESTIMATION has been studied extensively the last few decades. The primary goal is to improve estimates in a sensor network (SN) while maintaining the benefits of a decentralized design: robustness and modularity [1]. A disadvantage of a decentralized SN is that in general only suboptimal estimates [2] can be derived. This is in contrast to a centralized SN where it is possible to obtain mean square error (MSE) optimal estimates. However, centralized SNs has higher communication demands as all information extracted by the sensors is transmitted to a central agent for processing. Despite that the communication cost is reduced in decentralized SNs, due to local preprocessing of information, also decentralized SNs impose communication constraints [3]. Developing communication reducing techniques is therefore essential in general network-centric estimation.

A decentralized SN is illustrated in Fig. 1a. Multiple agents use local measurements to derive local estimates of a common target state $x_i$, i.e., $(y_i, R_i)$ where $y_i$ is the local estimate of the $i$th agent and $R_i$ is the covariance of $y_i$. Local estimates are exchanged between the agents for further improvement. To reduce the communication load an agent, e.g., agent 2, can instead exchange a linear combination of $y_2, \Psi y_2$ with covariance $\Psi R_2 \Psi^T$, where $\Psi$ is a wide matrix, see Fig. 1b. The problem is then about how to best choose $\Psi$.

In this paper a decentralized SN is considered where the task is to compute optimal estimates in a communication constrained SN. To cope with the communication constraints only dimension-reduced estimates are exchanged. This compression is accomplished by the matrix $\Psi$ and therefore the derivation of $\Psi$ becomes crucial for the overall estimation performance. The contributions are:

- We extend the previous work in [4] from row vector valued $\Psi$ to matrix valued $\Psi$.
- A convergence analysis is provided for the alternating minimization algorithm proposed for deriving $\Psi$ in the covariance intersection [5] case. Moreover, a numerical convergence rate analysis is given.
- The methodology used to select $\Psi$ is further extended to the largest ellipsoid method [6]. We formulate conditions for when the derived $\Psi$ is optimal.
- We propose a message encoding that encodes all data contained in $(\Psi y_2, \Psi R_2 \Psi^T, \Psi)$ for maximum communication efficiency.

This work has been supported by the Industry Excellence Center LINKSIC funded by The Swedish Governmental Agency for Innovation Systems (VINNOVA) and Saab AB, and by the project Scalable Kalman filters funded by the Swedish Research Council (VR). G. Hendeby has received funding from the Center for Industrial Information Technology at Linköping University (CENIIT) grant no. 17.12.
Fig. 2. Sensor networks. White and colored circles represent sensor nodes and fusion nodes, respectively. Solid and dotted arrows represent flow of measurements and preprocessed information, respectively.

II. BACKGROUND

In this section we briefly describe several important SN architectures. Then the related literature is reviewed.

A. Sensor Networks

Network-centric estimation is a well-studied subject [7]. By utilizing multiple sensors it is possible to extract additional information about a target. This information is then exchanged for further refinement. A centralized SN is illustrated in Fig. 2a. As independent information is transmitted without preprocessing from a sensor to a central fusion node it is possible to compute optimal estimates at the fusion node. Major drawbacks of the centralized SN include vulnerability to failure of critical nodes and high communication load as all measurements are transmitted to the central node.

To increase robustness a distributed SN design is often preferred. In a distributed SN the locally extracted information is preprocessed at an agent before it is distributed among the other agents. While enhancing robustness and scalability, the distributed SN has certain inherent assumptions about the network topology, flow of information and preprocessing schemes. For instance, the distributed Kalman filter algorithms of [8–10] assume that the implemented filters follow predefined schemes, and the diffusion-based schemes of [11, 12] are developed under certain assumptions about the network topology.

An even more robust and modular network design is the decentralized SN—illustrated in Fig. 2b. The decentralized SN is similar to a distributed SN w.r.t. preprocessing of information. Main differences lie in how ad hoc information is allowed to be exchanged and what restrictions there are on the network topology. The decentralized SN allows for heterogeneous agents, ad hoc networks and switching network topologies. This implies a high level of flexibility, scalability and robustness which are all desirable properties in a wide range of real-world applications [1].

B. Related Research

A main challenge in decentralized estimation is how to handle cross-correlated estimates since typically the full correlation structure is only partially available. If the cross-correlations are known it is possible to compute optimal estimates using Bar-Shalom- Campo (BSC, [13]) formulas. The generalized information matrix filter (GIMF, [14]) compensates for cross-correlations by explicit subtraction of previously used information, hence circumventing the problem with cross-correlations. If the cross-correlations are unknown a conservative estimation method is often required [2], where a conservative estimator does not underestimate the computed uncertainty. One of the most famous conservative methods is covariance intersection (CI, [5]). An alternative to CI is the largest ellipsoid (LE, [6]) method which is able to ensure conservative estimates under rather restrictive assumptions.

Decentralized estimation under communication constraints is studied using different approaches in [4, 15, 16]. This paper is mainly based on the work in [4], where estimates to be exchanged are reduced to scalars. In this paper we consider reducing the estimate dimensionality to an arbitrary size. The problem of deriving estimates from dimension-reduced data has however been studied earlier. In [17] it is shown that the problem of finding an optimal linear mapping that compresses measurements to be exchanged to a fusion node boils down to an eigenvalue problem. In [18] the problem is extended to the dynamic case. Estimation based on dimension-reduced data is handled in a distributed context in [19] and [20], where in the latter also a non-ideal communication channel is considered.

The authors of [21] address the problem of jointly assigning the dimensionality and deriving optimal data compression matrices. In [22] the authors propose communication efficient algorithms for distributed estimation based on transforming the data using linear mappings. Performance bounds of data compression techniques are analyzed in [23]. A framework for dimension-reduction including data denoising for distributed algorithms is proposed in [20].

Except for the previous work in [4], the dimension-reduction problem described above is so far untreated in the context of decentralized SNs. Hence, in this paper we derive a methodology for decentralized estimation in a dimension-reduced regression. A complicating factor in the decentralized case is the unknown cross-correlations. This problem will be addressed in the following sections by considering relevant methods of decentralized estimation.

Remark 1. An alternative strategy for reducing the communication load is to quantize the data to be exchanged, see, e.g., [24–26]. In this scope it is assumed that each data element is of high enough precision such that quantization effects are negligible. It should however be noted that the methods proposed in this work can be combined with quantization.

III. STATE ESTIMATION IN A DIMENSION-REDUCED LINEAR REGRESSION

We start by introducing necessary notation and mathematical preliminaries. Then the problem is stated. Finally we define the considered fusion methods.

A. Preliminaries

Let $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ be the set of real numbers, the set of $n$-dimensional real-valued vectors and the set of $n \times m$ real-valued matrices, respectively. Let $\mathbb{S}_+^n$ and $\mathbb{S}_{++}^n$ be the set of

Optimality is here considered in the mean square error sense.
n \times n \text{ symmetric positive semidefinite (PSD) matrices and the set of } n \times n \text{ symmetric positive definite (PD) matrices, respectively. For } A, B \in \mathbb{S}_n^+ \text{ the inequalities } A \succeq B \text{ and } A \succ B \text{ are equivalent to } (A - B) \in \mathbb{S}^+_n \text{ and } (A - B) \in \mathbb{S}^+_{n+}, \text{ respectively.}

The expected value and the covariance of } a \text{ are given by } E \cdot a 
\text{ and } \text{cov}(a) = E(a - E(a))(a - E(a))^T, \text{ respectively. The cross-covariance of } a \text{ and } b \text{ is given by } \text{cov}(a, b) = \text{cov}(b, a)^T = E(a - E(a)(b - E(b))^T. The ellipsoid } \mathcal{E}(A) \text{ of } A \in \mathbb{S}^+_n \text{ is given by the set of points } \mathcal{E}(A) = \{x \in \mathbb{R}^n \mid z^T A^{-1} z \leq 1\}.

The state to be estimated is denoted by } x^0 \in \mathbb{R}^n. \text{ An estimate } \hat{x} \text{ of } x^0 \text{ is to be derived from } y_1 \in \mathbb{R}^n \text{ and } y_2 \in \mathbb{R}^p \text{ related to } x^0 \text{ according to a linear regression}
\begin{align}
y &= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} I \\ H \end{bmatrix} x^0 + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \\
R &= \begin{bmatrix} R_1 \\ R_{12} \\ R_2 \end{bmatrix},
\end{align}

where } v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}^T \text{ is noise and } H \in \mathbb{R}^{p \times n} \text{ where } \text{rank}(H) = \min(p, n). \text{ It is assumed } R \in \mathbb{S}^{n+p}_+ \text{ such that } R_1 \in \mathbb{S}^+_n \text{ and } R_2 \in \mathbb{S}^+_{n+}. \text{ The components } y_1 \text{ and } y_2 \text{ can be interpreted as both measurements and estimates. However, in this context } y_1 \text{ and } y_2 \text{ are assumed to be estimates of } x^0 \text{ and } H x^0, \text{ respectively.}

The covariance associated with } \hat{x} \text{ is denoted by } P. \text{ An estimate } \hat{x} \text{ is } \text{MSE optimal } \text{if it minimizes } \text{tr}(P), \text{ and is conservative if }
P \succeq E \hat{x} \hat{x}^T, \tag{2}

where } \hat{x} = \hat{x} - x^0. \text{ For brevity we sometimes denote a pair, e.g., } (\hat{x}, P) \text{ or } (y_1, R_k), \text{ an estimate. In such case it should be clear that the first entry of the pair is the state estimate itself and the second entry is the associated covariance.}

A generalized eigenvalue } \lambda(A, B) \text{ of matrices } A \text{ and } B \text{ are given by } \tag{27}
\lambda = Ax = \lambda Bx, \tag{3}

where } x \text{ is a generalized eigenvector associated with } \lambda. \text{ If } A, B \in \mathbb{S}^n_+ \text{, then the } \text{generalized eigenvalue problem (GEVP) in } \text{(3)} \text{ has } n \text{ solutions } \lambda, \text{ not necessarily unique. The } i \text{th generalized eigenvalue } \lambda_i \text{ and associated generalized eigenvector } x_i \text{ form the pair } (\lambda_i, x_i). \text{ If } A \in \mathbb{S}^n_+ \text{ it has an eigendecomposition}
A = U \Sigma U^T = \sum_{i=1}^n \lambda_i u_i^T u_i, \tag{4}

where } u_i^T u_j = \delta_{ij} \text{ and } \delta_{ij} \text{ is the } \text{Kronecker delta.} \text{ The matrix } \Sigma \in \mathbb{S}^n_+ \text{ is diagonal with the eigenvalues of } A \text{ on its diagonal and the orthogonal matrix } U \text{ contains the associated eigenvectors as columns.} \text{ (Generalized) eigenvalues are assumed to be given in ascending order, i.e.,}
\lambda_{\min} = \lambda_1 \leq \cdots \leq \lambda_n = \lambda_{\max}. \tag{5}

B. Problem Statement
Let } y_1 \text{ and } y_2 \text{ be given according to } \text{(1)}, \text{ with } R_1 = \text{cov}(v_1), \text{ and } R_2 = \text{cov}(v_2) \text{ and } R_{12} = \text{cov}(v_1, v_2). \text{ Let}
\begin{align}
y_{\Psi} &= \Psi y_2 = \Psi (H x^0 + v_2), \tag{6a} \\
R_\Psi &= \text{cov}(\Psi v_2) = \Psi R_2 \Psi^T, \tag{6b}
\end{align}

where } \Psi \in \mathbb{R}^{m \times p}, \text{ } m \leq p \text{ and } \text{rank}(\Psi) = m. \text{ The problem is to find } \Psi \text{ such that } \hat{x} \text{ derived from } y_1 \text{ and } y_\Psi \text{ is MSE optimal.} \text{ An optimal } \Psi \text{ is denoted by } \Psi^*.

In general it is possible to have either } p \geq n \text{ or } p \leq n. \text{ In the fusion setting we typically have } H = I \text{ and hence } p = n.

C. Fusion Methods
The fusion methods used are stated below. We consider fusion of } y_1 \text{ and } y_\Psi \text{ which are given in } \text{(1)} \text{ and } \text{(4)}, \text{ respectively.}

1) Kalman Fuser: \text{If } R_{12} = 0, \text{ then an MSE optimal estimate is given by}
\begin{align}
\hat{x} &= P \left( R_1^{-1} y_1 + H^T \Psi^T R_\Psi^{-1} y_\Psi \right), \tag{7a} \\
P &= \left( R_1^{-1} + H^T \Psi^T R_\Psi^{-1} \Psi H \right)^{-1}, \tag{7b}
\end{align}

which is equivalent to the measurement update equation of a Kalman filter \textbf{[28]}—hence the name Kalman fuser (KF).

2) Covariance Intersection: \text{An estimate is computed using CI according to}
\begin{align}
\hat{x} &= P \left( (\omega R_1^{-1} y_1 + (1-\omega)\Psi^T R_\Psi^{-1} y_\Psi \right), \tag{8a} \\
P &= \frac{R_1}{\omega} + \frac{R_1}{\omega} H^T \Psi^T \left( \Psi H R_1 H^T \Psi^T + \frac{\omega R_\Psi}{1-\omega} \right)^{-1} \Psi H R_1, \tag{8b}
\end{align}

where } \omega \in [0, 1] \text{ is found by minimizing } \text{tr}(P). \text{ CI produces conservative estimates even if } R_{12} \text{ is unknown. It is shown in } \text{[29]} \text{ that if } R_{12} \text{ is completely unknown, then CI is an MSE optimal conservative estimator.}

3) Largest Ellipsoid Method: \text{An estimate is computed using LE according to}
\begin{align}
\hat{x} &= P T^{-1} t, \tag{9a} \\
P &= \left( T^{-1} TT^{-1} \right)^{-1}, \tag{9b}
\end{align}

where } T = T_{21}, \text{ and}
\begin{align}
\epsilon_1 &= R_1^{-1} y_1, \quad T_1 = R_1^{-1}, \tag{10a} \\
\epsilon_\Psi &= H^T \Psi^T R_\Psi^{-1} y_\Psi, \quad T_\Psi = H^T \Psi^T R_\Psi^{-1} \Psi H, \tag{10b} \\
T_3 &= U_1 \Sigma_1 U_1^T, \quad T_1 = \Sigma_1^* U_1^T, \tag{10c} \\
T_3 &= \Psi \Sigma_1 U_\Psi, \quad T_2 = U_\Psi^T, \tag{10d} \\
\epsilon_3' &= T_l, \quad T_3 = T_l^T T_l = I, \tag{10e} \\
\epsilon_\Psi' &= T_l \Psi, \quad T_\Psi = T_l \Psi T_l^T, \tag{10f}
\end{align}

and
\begin{align}
([i]_i, [T]_i) &= \begin{cases} ([\epsilon_1]_i, [T_1]_i), & \text{if } 1 \geq [T_1]_i, \\
([\epsilon_\Psi]_i, [T_\Psi]_i), & \text{if } 1 \leq [T_\Psi]_i, \\
([\epsilon_3']_i, [T_3]_i), & \text{if } 1 \geq [T_3]_i, \\
([\epsilon_\Psi']_i, [T_\Psi']_i), & \text{if } 1 \leq [T_\Psi']_i,
\end{cases}
\end{align}

for each } i = 1, \ldots, n.

This is an adapted version of an algorithm that was originally proposed in \textbf{[30]} \text{. In } \textbf{[2]} \text{ it is shown that the LE method is an MSE optimal conservative estimator under certain assumptions about the cross-correlations.}
IV. OPTIMAL DIMENSION-REDUCTION USING GENERALIZED EIGENVALUE OPTIMIZATION

In this section we propose a method for deriving \( \Psi^* \). We start by formulating the general loss function to be used, and then show how the problem boils down to solving a GEVP.

A. Loss Function

MSE optimal means that \( \text{tr}(P) \) is to be minimized. The MSE optimal fusion rule for merging of two estimates \( y_1 \) and \( y_2 \) according to (11) when \( R_{12} \) is known is given by the BSC formulas derived in (13). Since \( H \neq I \) and \( \Psi \neq I \) these formulas have to be modified. In Appendix A an optimal fusion rule is derived for merging of \( y_1 \) and \( y_2 \). The resulting MSE optimal estimate is computed as

\[
\hat{x} = K_1 y_1 + K_2 y_2, \tag{12a}
\]

\[
P = R_1 - K_2 \Psi S \Psi^T K_2^T, \tag{12b}
\]

where

\[
K_1 = I - K_2 \Psi H, \tag{13a}
\]

\[
K_2 = (R_1 H^T - R_{12}) \Psi^T (\Psi S \Psi^T)^{-1}, \tag{13b}
\]

\[
S = H R_1 H^T + R_2 - H R_{12} - R_{21} H^T. \tag{13c}
\]

If \( \Psi H = I \), then (12) reduces to the original BSC formulas (13).

The problem of deriving \( \Psi^* \) boils down to an MSE optimization problem

\[
\min_{\Psi} \text{tr}(P), \tag{14}
\]

where \( P \) is according to (12b). In Sec. [V] it is shown that the solution to (14) with \( P \) according to (12b) can be utilized to find the solution to (14) with \( P \) according to any of the methods of Sec. [III-C].

B. A Generalized Eigenvalue Problem

We now derive the solution to (14), which essentially is the same as in (14), except that now \( \Psi \) is a matrix instead of a row vector and the loss function is more general. Similar problems are solved in, e.g., [17, 19].

Let \( \Psi \in \mathbb{R}^{m \times p} \) where \( m \leq p \) and \( \text{rank}(\Psi) = m \). Let \( \Delta = R_1 H^T - R_{12} \) such that \( P \) in (12b) is given by

\[
P = R_1 - \Delta \Psi^T (\Psi S \Psi^T)^{-1} \Psi \Delta^T, \tag{15}
\]

where \( S \) is according to (13c). Since \( R_1 \) is constant, minimization of \( \text{tr}(P) \) is equivalent to

\[
\max_{\Psi} \text{tr} \left( \Delta \Psi^T (\Psi S \Psi^T)^{-1} \Psi \Delta^T \right). \tag{16}
\]

We now substitute \( \Psi \) by a temporary variable \( X^T \). Using the cyclic property of trace this problem can be recast as

\[
\max_{X} \text{tr} \left( (X^T S X)^{-1} X^T Q X \right), \tag{17}
\]

where \( Q = \Delta^T \Delta \). We first note that since \( S \in \mathbb{S}^{++}_m \) and \( \text{rank}(X) = m \) we have \( X^T S X \in \mathbb{S}^{++}_m \), which implies that there exists an invertible matrix \( T \) such that \( T X^T S X T^T = I \). Hence, without loss of generality (w.l.o.g.) we can assume

\[
X^T S X = I. \tag{18}
\]

We see that the solution to (18) is given by the eigenvectors corresponding to the largest eigenvalues of \( A \). Using the transformation \( Z = L^T X \) we arrive at the solution to the original problem in (17). The solution includes a GEVP and is summarized in the following theorem.

**Theorem 2.** Assume \( Q \in \mathbb{S}^p_+ \) and \( S \in \mathbb{S}^p_+ \). Let \( X \in \mathbb{R}^{p \times m} \) where \( m \leq p \) and \( \text{rank}(X) = m \). The solution to

\[
\max_{X} \text{tr} \left( (X^T S X)^{-1} X^T Q X \right), \tag{19}
\]

is given by \( X^* = [x_p \ldots x_{p-m+1}] \) where \( x_i \) is a generalized eigenvector associated with \( \lambda_i(Q,S) \) and \( \lambda_1 \leq \cdots \leq \lambda_p \).

Consider now an example with \( p = n = 2 \) and \( m = 1 \). Let \( R_1 \) and \( R_2 \) be defined according to their ellipses in Fig. [A]. Solving \( Q x = \lambda S X \), where \( Q = R_1^T \) and \( S = R_1 + R_2 \), yields two solutions \( \lambda_{\min} \) and \( \lambda_{\max} \) with associated generalized eigenvectors \( x_{\min} \) and \( x_{\max} \), respectively. The quantity of interest is \( x_{\max} \), or more specifically, the subspace \( V \) spanned by \( x_{\max} \).

C. Change of Basis

The linear span of the vectors \( a_1, \ldots, a_m \) is denoted by \( \text{span}(a_1, \ldots, a_m) \). The column span \( \text{colspan}(A) \) of a matrix \( A = [a_1 \ldots a_m] \) is defined as \( \text{colspan}(A) = \text{span}(a_1, \ldots, a_m) \).

In general the columns of \( X^* \in \mathbb{R}^{p \times m} \) span an \( m \)-dimensional subspace \( \mathcal{V} = \text{colspan}(X^*) \) of \( \mathbb{R}^p \). In this sense \( \Psi^* = (X^*)^T \) is interpreted as a transformation defined according to

\[
\Psi^*: \mathbb{R}^p \rightarrow \mathcal{V}. \tag{20}
\]

For any two solutions \( x_i \) and \( x_j \) to \( Q x = \lambda S X \) we have \( x_i^T S x_j = 0 \) while in general \( x_i^T x_j \neq 0 \) for \( i \neq j \) [27], see, e.g., Fig. [B]. Hence, \( X^* \) is not an orthogonal basis. Since we are not specifically interested in \( X^* \) but rather in \( \mathcal{V} \) it is tempting to transform \( X^* \). Lemma [3] states the intuitive result that if \( \text{colspan}(X) = \text{colspan}(V) = \mathcal{V} \), then \( (\Psi^*)^T R_{\Psi^*}^{-1} \Psi^* \) does not change if \( \Psi^* = X^T \) is replaced by \( \Psi^* = V^T \). This implies...

\[ \text{Fig. 3. Example of deriving } X^* \text{ by solving } Q x = \lambda S X. \text{ Since } m = 1 \text{ we have } X^* = x_{\max}. \text{ The vector } x_{\max} \text{ spans the one-dimensional subspace } \mathcal{V}. \text{ Principle axes of ellipses are illustrated by dashed lines.} \]
that $\hat{x}$ and $P$ computed by any of the methods in Sec. III-C are invariant to any change of basis of $\mathcal{V}$.

**Lemma 3.** Let $y_2 \in \mathbb{R}^p$ and $R_2 \in \mathbb{S}^{m}_{++}$. Let $X, V \in \mathbb{R}^{m \times p}$ be such that $\text{colspan}(X) = \text{colspan}(V)$. Then

$$X(X^TR_2X)^{-1}X^T = V(V^TR_2V)^{-1}V^T,$$

and trivially $X(X^TR_2X)^{-1}X^Ty_2 = V(V^TR_2V)^{-1}V^Ty_2$.

**Proof.** Since $\text{colspan}(X) = \text{colspan}(V)$ there exists an invertible matrix $T$ such that $V = XT$ [32]. Hence

$$V(V^TR_2V)^{-1}V^T = XT(T^TXT)^{-1}TT^TX = X(X^TR_2X)^{-1}X^T.$$

An orthonormal basis is a convenient representation of $\mathcal{V}$. Using a Gram-Schmidt algorithm [33] it is possible to derive an orthonormal basis $V \in \mathbb{R}^{p \times m}$ from $X^* \in \mathbb{R}^{p \times m}$ such that $\text{colspan}(V) = \text{colspan}(X^*) = \mathcal{V}$. A Gram-Schmidt procedure can be expressed as $V = X^*T$ [33] which, since $\Psi^* = VT = TX^*$, means that Lemma 3 applies.

Let $V^TR_2V = U\Sigma U^T$ be an eigendecomposition where $\Sigma \in \mathbb{S}^{m}_{++}$. An even more convenient basis for $\mathcal{V}$ is given by a transformation such that $UU^TVU$ is diagonal and then use $\Psi^* = UT^TV$. If $\Psi^* = UT^TV$, then since $U^T$ is orthogonal and $V^T$ has orthonormal rows we have

$$\Psi^*(\Psi^*)^T = UT^TVU = U^TIU = I,$$

i.e., $\Psi^*$ has orthonormal rows. In Sec. VII we describe why this latter representation of $\mathcal{V}$ is useful for communication reduction.

Fig. 4 illustrates an example with a change of basis where $p = m = 3$ and $m = 2$, and

$$R_1 = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$  

After computing $X^*$ by solving $Qx = \lambda Sx$, $V$ is computed using the Gram-Schmidt procedure and $\Psi^* = UT^TV$ is computed according to above. Since $X^*$ does not have orthogonal columns $R_X = (X^*)^TR_XX^*$ differs in both shape and orientation from $R_V = V^TR_2V$ and $R_\Psi = \Psi^*R_2(\Psi^*)^T$. However, the information is invariant to any change of basis of $\mathcal{V}$, i.e.,

$$X^*R_X^{-1}(X^*)^T = V R_V^{-1}V^T = (\Psi^*)^TR_\Psi^{-1}\Psi^*.$$  

![Fig. 4. Change of basis example. The information $X^*R_X^{-1}(X^*)^T$ is invariant to change of basis. In the figure $X^*R_X^{-1}(X^*)^T$, $V R_V^{-1}V^T$ and $(\Psi^*)^TR_\Psi^{-1}\Psi^*$ are projected onto $\mathcal{V}$ by $\Psi^*$.](https://gitlab.com/robinforsling/dtt/)

**Algorithm 1 GEVO**

**Input:** $R_1 \in \mathbb{S}^{n}_{++}, R_2 \in \mathbb{S}^{m}_{++}, R_{12} \in \mathbb{R}^{n \times m}, H \in \mathbb{R}^{p \times n}$ and $m$

1. Let $Q = (R_1H^T - R_{12})(R_1H^T - R_{12})$ and $S = HR_1H^T + R_2 - HR_{12} - R_{12}H^T$.
2. Compute $\lambda_1, \ldots, \lambda_m$ and $x_1, \ldots, x_p$ by solving $Qx = \lambda Sx$.
3. Let $X^* = [x_p \ldots x_{p-m+1}]$, where $x_i$ is a generalized eigenvector associated with $\lambda_i$.
4. Compute $V = [v_1 \ldots v_m]$ such that $v_i^Tv_j = \delta_{ij}$ and $\text{colspan}(V) = \text{colspan}(X^*)$.
5. Compute $V^TR_2V = U\Sigma U^T$ and let $\Psi^* = UT^TV$.

**Output:** $\Psi^*$

**Algorithm 2 GEVO-KF**

**Input:** $R_1 \in \mathbb{S}^{n}_{++}, R_2 \in \mathbb{S}^{m}_{++}, H \in \mathbb{R}^{p \times n}$ and $m$

1. Let $Q = HR_1H^T$ and $S = HR_1H^T + R_2$.
2. Compute $\lambda_1, \ldots, \lambda_m$ and $x_1, \ldots, x_p$ by solving $Qx = \lambda Sx$.
3. Let $X^* = [x_p \ldots x_{p-m+1}]$, where $x_i$ is a generalized eigenvector associated with $\lambda_i$.
4. Compute $V = [v_1 \ldots v_m]$ such that $v_i^Tv_j = \delta_{ij}$ and $\text{colspan}(V) = \text{colspan}(X^*)$.
5. Compute $V^TR_2V = U\Sigma U^T$ and let $\Psi^* = UT^TV$.

**Output:** $\Psi^*$

### D. The Generalized Eigenvalue Optimization Method

The proposed method for deriving $\Psi^*$ is denoted the *generalized eigenvalue optimization* (GEVO) method and is a direct application of Theorem 2. The GEVO method is given in Algorithm 1. Step 5 corresponds to a change of basis such that $\Psi^*R_2(\Psi^*)^T$ is diagonal.

### V. GEVO FOR THE CONSIDERED FUSION METHODS

The GEVO method is defined in Algorithm 1. When using GEVO to compute $\Psi$ for fusion of $(y_1, R_1)$ and $(y_2, R_2)$ using a certain fusion method, it is necessary to use $P$ as computed by that particular method, otherwise optimality guarantees are lost. In this section we adapt the GEVO method for the considered fusion methods of Sec. III-C KF, CI and LE. Optimality of the computed $\Psi$ is guaranteed for KF and LE. A convergence analysis is provided for the CI case.

### A. Kalman Fuser

The GEVO method in Algorithm 1 applies directly for the KF case by setting $R_{12} = R_{21}^T = 0$. This results in

$$Q = HR_1H^T, \quad S = HR_1H^T + R_2. \quad (22)$$

For convenience and to be consistent with the other fusion methods, the KF case is provided in Algorithm 2.

2The GEVO method is available as **MATLAB** code at

[https://gitlab.com/robinforsling/dtt/](https://gitlab.com/robinforsling/dtt/)
**Algorithm 3 GEVO-CI**

**Input:** $\omega_0, R_1 \in S^n_{++}, R_2 \in S^p_{++}, H \in \mathbb{R}^{p \times n}$, $m$ and $k = 0$

1. Let $k \leftarrow k + 1$. Compute $X_1, \ldots, X_p$ and $x_1, \ldots, x_p$ by solving $Qx = \lambda Sx$,
   where $Q = HR_1^2H^T/\omega_{k-1}$ and $S = HR_1H^T/\omega_{k-1} + R_2/(1 - \omega_{k-1})$. Let $X_k = [x_p \ldots x_{p-m+1}]$, where $x_i$ is a generalized eigenvector associated with $\lambda_i$.

2. Let $R_k = X_k^T R_2 X_k$. Compute $\omega_k$ by solving
   $$\min_{\omega} \text{tr} \left( (\omega R_1^{-1} + (1 - \omega) H^T X_k R_2^{-1} X_k H)^{-1} \right).$$

3. Let $J_k$ be according to (24). If $(J_{k-1} - J_k)/J_k > \epsilon$, then go back to step 2. Otherwise continue to step 4.

4. Compute $V = [v_1 \ldots v_m]$ such that $v_i^T v_j = \delta_{ij}$ and $\text{colspan}(V) = \text{colspan}(X_k)$.

5. Compute $V^T R_2 V = U \Sigma U^T$ and let $\Psi = U^T V^T$.

**Output:** $\Psi$

**B. Covariance Intersection**

For fusion using CI the GEVO method does not apply directly. The reason is the dependency on $\omega$. In [23] an iterative algorithm based on alternating minimization (AM, [24]) is proposed. The basic idea is to alternate between keeping $w$ and $X$ fixed. A generalization for $m \geq 1$ of the algorithm in [4] is provided in Algorithm 3. The loss function value $J_k$ is defined in (24) below.

1) Algorithm Parameters: The parameter $\epsilon$ in Algorithm 3 is a design parameter chosen as a compromise between computational speed and exactness of the solution given by the final iterate. The initial value $\omega_0$ can be chosen using different strategies of which two are suggested below:

   (i) Let $\omega_0 = 1/2$.

   (ii) Let $\omega_0$ be given by the solution to
   $$\min_{\omega} \text{tr} \left( (\omega R_1^{-1} + (1 - \omega) H^T R_2^{-1} H)^{-1} \right).$$

2) Convergence Analysis: The AM method in Algorithm 3 alternates between solving two different kind of optimization problems. Each separate problem is well-posed where a global minimum is obtained. The solution to the problem in step 1 is given by the generalized eigenvalues associated with the largest generalized eigenvalues of a GEVP [21]. Step 2 involves solving a convex optimization problem [25] for which any local minimum is also a global minimum [26]. This does however not imply that the final iterate of Algorithm 3 is a global minimizer. Below it is shown that the iterations of Algorithm 3 converges to a stationary point.

   Let $J(\omega, \Psi) = \text{tr}(P)$ with $P$ according to (8b), i.e.,
   $$J(\omega, \Psi) = \text{tr} \left( (\omega R_1^{-1} + (1 - \omega) H^T \Psi^T R_2^{-1} \Psi H)^{-1} \right).$$

   Define
   $$J_{k-\frac{1}{2}} = J(\omega_{k-1}, X_k^T), \quad J_k = J(\omega_k, X_k^T).$$

   Consider a sequence of $N$ iterations and hence $2N$ subiterations. Each iteration and subiteration have the same feasible set. Therefore, since in each subiteration a minimum is obtained it is concluded that
   $$J_{\frac{3}{2}} \geq J_1 \geq \cdots \geq J_{N-\frac{1}{2}} \geq J_N,$$
   which is a monotonically nondecreasing sequence $\{J_k\}$. Denote by $\{(\omega, X)^T\}$ the sequence of points generating $\{J_k\}$.

   The assumption $P \in S^n_{++}$ implies the existence of a lower bound $J_{low} > 0$ such that $J(\omega, \Psi) > J_{low}$ for all $(\omega, \Psi)$. Hence, the monotonic convergence theorem (27) is applicable stating that
   $$\lim_{k \to \infty} J_k = J(\bar{\omega}, \bar{X})^T,$$
   where $(\bar{\omega}, \bar{X})^T$ is the limit point. In the limit we have $J_{k-\frac{1}{2}} - J_{k+\frac{1}{2}} \to 0$ and $J_k - J_{k+1} \to 0$. This implies, since $J(\omega, \Psi)$ is differentiable w.r.t. $\omega$ and $\Psi$ on its domain, that
   $$\frac{\partial}{\partial \omega} J(\omega, \Psi)_{|\omega=\bar{\omega}} = 0,$$
   and hence $(\bar{\omega}, \bar{X})^T$ is a stationary point. The convergence results are summarized in Theorem 4. It should be emphasized that from Theorem 4 alone, it cannot be concluded if $(\bar{\omega}, \bar{X})^T$ is a local minimizer, global minimizer or a saddle point.

**Theorem 4 (Convergence of GEVO-CI).** Let $\{(\omega, X)^T\}$ be a sequence of points generated by Algorithm 3 and let $J(\omega, \Psi)$ be given by (25). Then $\{(\omega, X)^T\}$ converges to a stationary point $(\bar{\omega}, \bar{X})^T$ of $J(\omega, \Psi)$.

The convergence rate is evaluated numerically. In each simulation $R_1, R_2 \in \mathbb{S}^n_{++}$ are sampled matrices constructed as $R_\ell = L_\ell L_\ell^T$ for $\ell = 1, 2$, where each element of $L_\ell \in \mathbb{R}^{n \times n}$ is i.i.d. according to $[L_\ell]_{ij} \sim N(0, 1)$. Hence, $R_1, R_2 \sim W(I, n)$, where $W(I, n)$ is the Wishart distribution of $n$ degrees of freedoms [28]. If in a certain scenario $R_2 \geq R_1$, then $R_1$ is resampled until $R_2 \not\geq R_1$. This resampling is done because $R_2 \geq R_1$ would trivially yield $\omega = 1$ in Algorithm 3 for all feasible $\Psi$.

The evaluation is made for $n \in \{6, 9\}$ and $\epsilon \in \{0.1\%\, 0.01\%\}$. In each case and for each $m$ we run 100000 simulations. The GEVO-CI algorithm is initiated using strategy (i) of Sec. V.B1. The evaluation is made w.r.t. the number of iterations. The results are shown in Fig. 5 and the statistics are summarized in Table 1. The GEVO-CI algorithm converges very fast in general, e.g., for $n = 9$ and $\epsilon = 0.01\%$ with $m = 3$ the mean number of iterations is approximately 4.056 ± 0.634 and the typical value is 4. An interesting feature is that the convergence rate is improved by increasing $m$. The number of iterations increases as $n$ increases and as $\epsilon$ decreases.

**C. The Largest Ellipsoid Method**

To be able to use GEVO for the LE method case small adoptions are required. In [29] it is shown that the LE method makes an implicit assumption about $R_{12}$ if $H = I$. Based on this we now derive $R_{12}$ in the general case where $H$ is not necessarily identity. Let $\tilde{y}_i = y_i - E[y], \tilde{y}_i^T = y_i^T - E[y_i^T]$ and $\hat{\gamma} = \gamma - E[\gamma]$. Assume

- $y_1 = R_1\left((R_1^{-1})^{-1} y_1^T + \Gamma^+ \hat{\gamma}\right)$, $R_1^{-1} = (R_1^{-1})^{-1} + \Gamma^+$, $y_2 = R_2\left((R_2^{-1})^{-1} y_2^T + H^+ \hat{\gamma}\right)$, $R_2^{-1} = (R_2^{-1})^{-1} + H^+ H^T$. 


where $y_k^q$ and $R_k^q = \text{cov}(y_k^q)$ correspond to an estimate for which $E\hat{y}_1\hat{y}_1^T = 0$ and $E\hat{y}_2\hat{y}_2^T = 0$. The vector $\hat{\gamma}$ and the pseudoinverse $\Gamma^+$ denote a common estimate where $\Gamma^+$ is the common information and $(\Gamma^+) = \Gamma = \hat{\gamma}\hat{\gamma}^T$. After deriving $\Gamma^+$, $R_{12} = E\hat{y}_1\hat{y}_2^T$ is computed as

$$R_{12} = ER_1(R_1^{-1}\hat{y}_1\Gamma^+\hat{\gamma})((R_2)^{-1}\hat{y}_2\Gamma^+\hat{\gamma})^T R_2 = E R_1\Gamma^+\Gamma^+H^TH R_2 = R_1\Gamma^+H^TH R_2 = R_1\Gamma^+H^TH.$$ 

as a property of the pseudoinverse. The GEVO method for LE is summarized in Algorithm 4. If $\text{rank}(H) = n$ then both $I_1$ and $I_2$ in Algorithm 4 are full rank, and hence also $\Gamma^+$ is full rank, i.e., $\Gamma^+ = \Gamma^{-1}$. Optimality guarantees of the LE method are valid when the following conditions hold: (i) $H = I$, and (ii) $R_{12}$ follows a certain structure proposed in

**Algorithm 4 GEVO-LE**

**Input:** $R_1 \in \mathbb{S}^n_{++}, R_2 \in \mathbb{S}^m_{++}, H \in \mathbb{R}^{p \times n}$ and $m$

1: Transform into the information domain:

$$I_1 = R_1^{-1}, \quad I_2 = H^TH^{-1}.$$ 

2: Let $I_1 = U_1 \Sigma_1 U_1^T$ and define $T_1 = \Sigma_1 \gamma U_1^T$. Let $T_2I_2 I_2 = U_2 \Sigma_2 U_2^T$ and define $T = U_2^T T_1$. Transform according to:

$$I_1' = T I_1 T^T = I, \quad I_2' = T I_2 T^T.$$ 

3: Let $D$ be diagonal. For each $i = 1, \ldots, n$ compute:

$$[D]_ii = \min([I_1']_ii, [I_2']_ii).$$

4: Let $\Gamma^+ = T^{-1}DT^{-T}$ and $R_{12} = R_1 \Gamma^+ H^TH R_2$. Compute $\Psi$ using Algorithm 1 with inputs $R_1$, $R_2$, $R_{12}$, $H$ and $m$.

**Output:** $\Psi$

This means that $\Psi$ computed by Algorithm 4 is guaranteed to be optimal if (i) and (ii) hold simultaneously.

**VI. COMMUNICATION CONSIDERATIONS**

In this section we describe how a message containing $(y_\Psi, R_\Psi, \Psi)$ should be encoded for maximum communication-reduction with the GEVO method. This message encoding is inspired by previous work in [16]. A quantitative analysis of the communication-reduction is provided at the end.

**A. Message Coding**

Assume $H \in \mathbb{R}^{p \times n}$ and $y_\Psi \in \mathbb{R}^m$, and that $R_\Psi \in \mathbb{S}^m_{++}$ is diagonal as a result of the change of basis procedure described in Sec. IV-C. Suppose now that agent 2 is transmitting the dimension-reduced estimate $(y_\Psi, R_\Psi)$ to agent 1 who is to fuse $(y_\Psi, R_\Psi)$ with its local estimate. To be able to use $(y_\Psi, R_\Psi)$ agent 1 also requires $\Psi$, but to simply transmit $(y_\Psi, R_\Psi, \Psi)$ is costly. A more efficient way is therefore to transmit $(y_\Psi, \Phi)$ where

$$\Phi = \begin{bmatrix} \phi_1 & \psi_1 \\ \vdots & \vdots \\ \phi_m & \psi_m \end{bmatrix} = \Psi R_\Psi = \begin{bmatrix} r_1 \psi_1 \\ \vdots \\ r_m \psi_m \end{bmatrix},$$

where $r_i = [R_\Psi]_{ii}$, and $\phi_i$ and $\psi_i$ represent the $i$th row of $\Phi$ and $\Psi$, respectively. When agent 1 receives $(y_\Psi, \Phi)$, $R_\Psi$ and $\Phi$ are computed as

$$R_\Psi = \text{diag} ([\phi_1, \ldots, \phi_m]), \quad \Psi = \Phi R_\Psi^{-1}.$$ 

The number of exchanged parameters can be further reduced by exploiting the fact that $\psi_i \psi_i^T = \delta_{ij}$, and hence $\phi_i \phi_i^T = 0$ if $i \neq j$. For $\Phi \in \mathbb{R}^{m \times p}$ we need to transmit $m$ components of $\phi_i$, but only $m-i+1$ components of $\phi_i$. The components not transmitted are given by utilizing $\phi_i \phi_j^T = 0$ for $i \neq j$ sequentially. For example, assume $m = 3$ and let $\phi_3 = [\phi_3]$. Assume that agent 1 does not have access to $\phi_2, \phi_3, \text{ and } \phi_3$. The missing components are given by first solving for $\phi_{2,1}$ in $\phi_1 \phi_2^T = 0$ which is equivalent to solving $A_2 \phi_{2,1} = b_2$ where

$$A_2 = \phi_{1,1}, \quad b_2 = -\sum_{j=2}^p \phi_{1,j} \phi_{2,j}.$$
We then compute φ_{3,1} and φ_{3,2} by solving $A_3 \begin{bmatrix} φ_{3,1} \\ φ_{3,2} \end{bmatrix} = b_3$ where

$$A_3 = \begin{bmatrix} φ_{1,1} & φ_{1,2} \\ φ_{2,1} & φ_{2,2} \end{bmatrix}, \quad b_3 = -\left[ \sum_{j=3}^{p} φ_{1,j} φ_{2,j} \right].$$

This sequential approach generalizes to arbitrary $i$. Let superscripts $M_i$ and $K_i$, e.g., $a^M_i$ and $a^K_i$, denote a vector that contains the components of a vector, e.g., $a$, corresponding to the missing and available components of $φ_i$, respectively. In particular, the union and intersection of the components of $φ_{i1}^K_i$ and $φ_{i2}^M_i$ correspond to $φ_i$ and an empty set, respectively. For arbitrary $i \in \{2, \ldots, p\}$ we compute the missing components $φ_{i1}^M_i$ by solving

$$A_i φ_{i1}^M_i = b_i, \quad A_i = \begin{bmatrix} φ_{1}^M_i \\ \vdots \\ φ_{i-1}^M_i \\ φ_{i+1}^{K_i} \\ \vdots \\ φ_{p}^{K_i} \end{bmatrix}, \quad b_i = -\left[ \sum_{j=3}^{p} φ_{1,j} φ_{j+1,j} \right]. \quad (29)$$

We denote $Φ^K$ the parts of $Φ$ that are actually transmitted using the proposed message encoding, i.e.,

$$φ^K = [φ_1 φ_2^K \ldots φ_p^K]. \quad (30)$$

Care must be taken to ensure $\det(A_i) \neq 0$, otherwise the system of equations in (29) is not solvable. The condition $\det(A_i) \neq 0$ have to be checked at the transmitting agent. This means that it is not possible by default to always skip transmitting the first components of $φ_i$ since this could potentially imply that $\det(A_i) = 0$. Hence, alongside transmission of the information contained in $φ_i$, i.e., $φ^K_i$, we must also transmit information about which components that are not transmitted, i.e., which elements of $φ_i$ that are included in $φ_i^M_i$. This requires transmission of a few extra parameters. For example, in case of $m = 3$ we must transmit $1 + 2 = 3$ extra parameters that indicate the components of $φ_1$ and $φ_2$ that are not transmitted. For general $m$, a number of $N_{ex} = \sum_{i=1}^{m} (i-1) = \frac{m(m-1)}{2}$ extra parameters must be included in the message. However, as described below, to enumerate the excluded components of the $φ_i$, where $i = 1, \ldots, m$, only requires a few extra bits which is negligible compared to the remaining parameters to be exchanged.

In summary, the transmitted message is given by $(γ_{ψ}, Φ^K, J)$, where $J$ is an $N_{ex}$-dimensional vector containing the indices of the missing components of $φ_i$, where $i = 1, \ldots, m$. For clarity, MATLAB® code for the message coding is available at [https://gitlab.com/robinforsling/dtt](https://gitlab.com/robinforsling/dtt).

**B. Communication Reduction**

The total number of components required to be transmitted when exchanging $(γ_{ψ}, Φ^K)$ is given by

$$N_{red} = m + \frac{p(p+1)}{2} - \frac{(p-m)(p-m+1)}{2} \quad (32)$$

$$= \frac{2mp - m^2 + 3m}{2},$$

where $m$ is due to $γ_{ψ}$ and the last two terms of the r.h.s. of the first line is due to $Φ$. The ratio between $N_{red}$ and the total number of components $N_{full} = \frac{p(p+3)}{2}$ of a full estimate $(γ_{ψ}, R_2)$ is

$$ζ(m, p) = \frac{N_{red}}{N_{full}} = \frac{2mp - m^2 + 3m}{p(p+3)}. \quad (33)$$

The quantity $ζ$ of (33) as a function of $p$ is illustrated in Fig. 6 for different values of $m$. For example, if $p = 9$ then the communication savings are approximately 81% for $m = 1$ and 50% for $m = 3$. If $p = 15$ then the communication savings are approximately 88% for $m = 1$ and 67% for $m = 3$.

To continue the discussion about the $N_{ex}$ extra parameters to be exchanged, assume that the size of each parameter in $(γ_{ψ}, Φ^K)$ is 32 bits. Moreover, assume that size of each parameter in $J$ is 4 bits, which would be valid for $p \leq 16$. Then $(γ_{ψ}, Φ^K)$ and $J$ consist of $32N_{red}$ bits and $4N_{ex}$ bits, respectively. In Table II the ratio

$$\frac{4N_{ex}}{32N_{red}} = \frac{m-1}{8(2p-m+3)}, \quad (34)$$

is illustrated for a number of values of $m$ and $p$. The amount of extra bits required to transmit $J$ is marginal in this configuration. We hence have that the bit size of $(γ_{ψ}, Φ^K)$ and the bit size of $(γ_{ψ}, Φ^K, J)$ are approximately equal.

**Table II**

| $m$ | $p$ | $\frac{N_{ex}}{N_{red}}$ | $m$ | $p$ | $\frac{N_{ex}}{N_{red}}$ | $m$ | $p$ | $\frac{N_{ex}}{N_{red}}$ |
|-----|-----|--------------------------|-----|-----|--------------------------|-----|-----|--------------------------|
| 2   | 4   | 1.39%                    | 3   | 9   | 1.39%                    | 3   | 12  | 1.04%                    |
| 2   | 6   | 0.96%                    | 5   | 9   | 3.12%                    | 6   | 12  | 2.98%                    |
| 3   | 6   | 2.08%                    | 7   | 9   | 5.36%                    | 9   | 15  | 4.17%                    |

**VII. Method Evaluation**

In Sec. VI the GEVO method for computing an optimal $Ψ$ is derived. Based on GEVO, the three submethods, GEVO-KF, GEVO-CI and GEVO-LE are proposed in Sec. VI. In this section GEVO-KF, GEVO-CI and GEVO-LE are evaluated using a numerical example.

**A. Comparison Scope**

Each of GEVO-KF, GEVO-CI and GEVO-LE correspond to a particular fusion method assumed when fusing $(γ_1, R_1)$
and \((y_\Phi, R_\Phi)\). Each of the fusion methods is important in decentralized estimation:

- KF assumes zero cross-correlations and is therefore applicable when some kind of decorrelation procedure is utilized. One example of this is the GIMF [14].
- CI and LE are able to work even if the cross-correlations are nonzero and unknown. CI provides conservative results for any degree of cross-correlations while LE is conservative only under rather restrictive assumptions about the cross-correlations [2].

Which method to be used depends on the situation and it is not possible to say something general about the performance in arbitrary configurations. Hence, in the following comparison the relevance of each method is illustrated while we put no value into which method to use in general. The latter decision is up to the user.

B. Simulation Specifications

The problem is to fuse \((y_1, R_1)\) and \((y_\Phi, R_\Phi)\), where \(y_\Phi = \Psi R_1\) and \(R_\Phi = \Psi R_2\Psi^T\). The mapping \(\Psi\) is computed using GEVO-KF, GEVO-CI and GEVO-LE, and after that \((y_1, R_1)\) and \((y_\Phi, R_\Phi)\) are fused using KF, CI and LE, respectively. It is assumed \(H = I\) such that \(p = n\) where \(n = 6\). In this comparison we are only interested in the covariances, e.g., \(R_1\) and \(P\). Hence, the state estimates, e.g., \(y_1\) and \(\hat{x}\), are never explicitly computed below.

The scenario is parametrized in \(\rho \in [0, 1]\) according to

\[
R_1^{-1} = (1 - \rho)A^{-1} + \rho \Gamma^{-1}, \quad R_2^{-1} = (1 - \rho)B^{-1} + \rho \Gamma^{-1},
\]

where

\[
A^{-1} = \text{diag}(64, 32, 16, 8, 4, 2), \quad B^{-1} = \text{diag}(5, 8, 13, 21, 34, 55),
\]

and

\[
\Gamma^{-1} = \begin{bmatrix}
16 & 4 & 4 & 0 & -2 & 0 \\
4 & 20 & 8 & -8 & 4 & -4 \\
4 & 30 & 0 & -4 & -4 & 0 \\
0 & -8 & 0 & 50 & 0 & 0 \\
-2 & -4 & 4 & 0 & 10 & 0 \\
0 & -4 & -4 & 0 & 0 & 20
\end{bmatrix}.
\]

We interpret \(\rho \Gamma^{-1}\) as common information, cf. Sec. [V.C] and have \(R_{12} = \rho R_1 \Gamma^{-1} R_2\). The parameter \(\rho\) is varied in the interval \([0, 1]\) and \((\hat{x}, P)\) is computed for each \(\rho\) and for each of the considered methods. As \(\rho\) increases \(y_1\) and \(y_2\) becomes more correlated to eventually become fully correlated at \(\rho = 1\).

As pointed out earlier, applying KF in a decentralized SN requires some sort of decorrelation mechanism as otherwise the independency assumed in KF is in general violated. In the comparison KF will be used based on two different assumptions: (i) \(y_2\) is decorrelated by removal of common information such that \(R_2^{-1} = (1 - \rho)B^{-1}\). This case is denoted decorrelated KF (dKF). (ii) \(y_1\) and \(y_2\) are uncorrelated. This case is denoted naïveKF (nKF). Table [III] provides a summary of how different quantities are computed in the simulations.

### Table III

**Computation of Simulated Quantities**

| Method | \(R_{12}^{-1}\) | \(R_{12}\) | \(\Psi\) | \(P\) |
|--------|----------------|------------|--------|------|
| dKF    | \((1 - \rho)B^{-1}\) | 0          | Alg. [2] | 70   |
| CI     | \((1 - \rho)B^{-1} + \rho \Gamma^{-1}\) | \(\rho R_1 \Gamma^{-1} R_2\) | Alg. [3] | 85   |
| LE     | \((1 - \rho)B^{-1} + \rho \Gamma^{-1}\) | \(\rho R_1 \Gamma^{-1} R_2\) | Alg. [4] | 90   |
| nKF    | \((1 - \rho)B^{-1} + \rho \Gamma^{-1}\) | \(\rho R_1 \Gamma^{-1} R_2\) | Alg. [2] | 75   |

C. Performance and Conservativeness Measures

Performance is evaluated w.r.t. \(\text{tr}(P)\). Since \(R_{12}\) is known in the simulations it is possible to compute \(\Psi^*\) using Algorithm [I] and an optimal \(P^{\text{BSC}}\) according to [12b]. This baseline is used for normalizing \(\text{tr}(P)\) such that the performance measure considered here is given by

\[
J = \frac{\text{tr}(P)}{\text{tr}(P^{\text{BSC}})}.
\]

For an optimal estimator \(J = 1\). Note, if \(P\) is computed with \(m = m\)' then \(P^{\text{BSC}}\) is computed with \(m = m\)'.

A conservative estimator satisfies \(P \preceq \rho \text{true}\), where \(\rho \text{true} = E \hat{x}\hat{x}^T\). Conservativeness is commonly evaluated using average normalized estimation error squared (ANEES, [40]). In the current scenario we have access to \(\rho \text{true}\) and hence ANEES can be computed as

\[
\varepsilon = \frac{1}{n} \text{tr}(P^{-1} \rho \text{true}).
\]

For a conservative estimator \(\varepsilon \leq 1\) but the opposite does not hold in general since it is possible to simultaneously have \(P \not\preceq \rho \text{true}\) and \(\varepsilon \leq 1\). Below we propose an additional measure for conservativeness.

Let \(P = LL^T\). Since \(P \preceq \rho \text{true}\) implies that \(I \succeq L^{-1} \rho \text{true} L^{-T}\), for a conservative estimator the largest eigenvalue of \(L^{-1} \rho \text{true} L^{-T}\) is at maximum 1. Hence, the measure

\[
\eta = \lambda_{\text{max}}(L^{-1} \rho \text{true} L^{-T}),
\]

will also be used to evaluate conservativeness. For a conservative estimator \(\eta \leq 1\). If \(\eta > 1\) then this estimator is not conservative in the strict sense of \(P \preceq \rho \text{true}\).

D. Results

The results are visualized in Fig. [I]. Different methods are distinguished by different colors. Solid, dashed and dotted lines refer to \(m = 1, m = 2\) and \(m = 3\), respectively. Since \(n = p = 6\) the communication savings for \(m = 1, m = 2\) and \(m = 3\) are approximately 74%, 52% and 33%, respectively.

KF: dKF provides optimal results for all values of \(\rho\) since the decorrelation procedure perfectly decorrelates the estimates. dKF is also conservative w.r.t. both \(\varepsilon\) and \(\eta\). On the other hand nKF provides \(J \leq 1\) but at the cost of not being conservative for \(\rho > 0\). nKF hence utilizes more information than is actually available in the problem. This effect becomes more prominent for large \(m\).

As seen in Fig. [I] dKF is able to achieve \(J\) slightly below 1. This is possible because dKF, using decorrelation, exploits more structure in the problem compared to what is possible using the linear MSE optimal estimator in [12].
If decorrelation is possible before fusion, then GEVO-KF is the most efficient criteria are considered. The comparison also provides values marginally larger than 1. GEVO methods is relevant if both performance and conservativeness criteria are considered. The comparison also provides values marginally larger than 1. LE: The LE method provides relatively good performance for all $\rho$. LE is conservative w.r.t. $\varepsilon$ but not w.r.t. $\eta$ for which it provides values marginally larger than 1.

E. Discussion

The numerical comparison shows that each of the proposed GEVO methods is relevant if both performance and conservativeness criteria are considered. The comparison also provides an indication under which conditions each method is favorable. If decorrelation is possible before fusion, then GEVO-KF performs well. If it is not possible to decorrelate the estimates, then naïve usage of GEVO-KF will typically result in a non-conservative estimate. If the cross-correlations are unknown but relatively small or some level of non-conservativeness is allowed, then GEVO-LE is a relevant option. If the cross-correlations are unknown and possible strong, or if conservativeness is not negotiable, then GEVO-CI provides the most robust solution.

VIII. CONCLUSIONS

In this paper we have considered the problem of decentralized estimation under communication constraints. By a linear mapping, dimension-reduced estimates are obtained which allows for less data to be transmitted. This dimension-reduction methodology is named generalized eigenvalue optimization (GEVO). The GEVO method was based on a generalization of the Bar-Shalom-Campo (BSC) formulas. The main focus has been on three variants of GEVO corresponding to the following fusion methods: the Kalman filter (KF), covariance intersection (CI) and the largest ellipsoid (LE) method. We have shown how the optimal linear mapping is computed for BSC, KF and LE. For CI an alternating minimization algorithm has been proposed which was shown to converge to stationary point. To allow for efficient communication a message coding solution was provided. Numerical examples illustrate the usability of the proposed dimension-reducing techniques. Moreover, it shows that the different proposed GEVO methods are preferable in different problems, and none of the proposed GEVO methods is generally superior the other ones.

In this paper it has been assumed that $R_1$ is available for deriving $\Psi$. In real-world problems this is in general an unrealistic assumption, particularly in arbitrary decentralized sensor networks. For future research it is therefore important to investigate how $R_1$ can be replaced without a significant performance degradation.

APPENDIX A

DERIVATION OF AN MSE OPTIMAL FUSION METHOD FOR CORRELATED ESTIMATES

The underlying model is given in (1). Let $y_\psi = \Psi v_2$ and $R_\psi = \Psi R_1 \Psi^T$, where $\Psi \in \mathbb{R}^{m \times p}$, $m \leq p$ and $\text{rank}(\Psi) = m$, such that $R_{1\psi} = \text{cov}(v_1, \Psi v_2) = R_{12} \Psi^T$. Let $\Psi_H = \Psi H$. The goal is to compute an estimate $\hat{x} = \tilde{K} [y_1^T \ y_\psi^T]^T$, where $\tilde{K} = \Psi^T \tilde{H}$ that minimizes trace of

$$ P = \tilde{K} \begin{bmatrix} R_1 & R_{1\psi} \\ R_{\psi_1} & R_\psi \end{bmatrix} \tilde{K}^T. $$

By defining $\tilde{K} = [K_1 \ K_2]$ we have $\tilde{K} \tilde{H} = I \implies K_1 = I - K_2 \Psi_H$. It is assumed $\tilde{R} \succ 0$ such that $S = HR_1 H^T + R_2 - HR_{12} - R_{21} H^T \succ 0$ and $\Psi S \Psi^T \succ 0$.

Let $K_2 = K$, then $\tilde{K} = [I - K \Psi_H \ K]$ and

$$ P = R_1 - R_1 \Psi_H K^T - K \Psi_H R_1 + K \Psi_H R_1 \Psi_H^T K + K R_{1\psi} - K R_{1\psi} \Psi_H^T K^T + R_{1\psi} K^T - K \Psi H R_{1\psi} K^T + K R_{1\psi} K^T - R_\psi K^T $$

$$ = R_1 - K (\Psi_H R_1 - R_{1\psi}) - (R_{1\psi}^T - R_\psi) R_{1\psi} K^T + (\Psi_H R_1 \Psi_H^T + R_\psi - \Psi_H R_{1\psi}) K^T - R_{1\psi} \Psi_H^T K^T $$

$$ = R_1 - K A - A^T K^T + KBK^T, $$

where

$$ A = \Psi_H R_1 - R_{1\psi} = \Psi (HR_1 - R_{21}), $$

$$ B = \Psi_H R_1 \Psi_H^T + R_\psi - \Psi_H R_{1\psi} - R_{1\psi} \Psi_H^T $$

$$ = \Psi (HR_1 H^T + R_2 - HR_{12} - R_{21} H^T) \Psi^T = \Psi S \Psi^T. $$

[Fig. 7. Results of the method comparison. Different methods are distinguished by different colors. Solid, dashed and dotted lines refer to $m = 1$, $m = 2$ and $m = 3$, respectively. The lines for dKF lie approximately constantly around 1.]
By completing the square we have
\[ P = R_1 - A^T B^{-1} A + (K - A^T B^{-1}) B (K^T - B^{-1} A), \]
which is minimized w.r.t. \( K \) when \( K = A^T B^{-1} \). An MSE optimal estimate is given by
\[ \hat{x} = K_1 y_1 + K_2 y_2, \]
\[ K_1 = I - K_2 \Psi H, \quad K_2 = (R_H^T - R_{12}) (\Psi^T \Psi)^{-1}, \]
where \( S = H R_1 H^T + R_2 - H R_{12} - R_2 H^T \).

REFERENCES

[1] J. K. Uhlmann, “Covariance consistency methods for fault-tolerant distributed data fusion,” *Information Fusion*, vol. 4, no. 3, pp. 201–215, 2003.

[2] R. Forsling, A. Hansson, F. Gustafsson, Z. Sjanic, J. Löfberg, and G. Hendeby, “Conservative linear unbiased estimation under partially known covariances,” *IEEE Transactions on Signal Processing*, vol. 70, pp. 3123–3135, Jun. 2022.

[3] N. Kimura and S. Latifi, “A survey on data compression in wireless sensor networks,” in *Proceedings of the IEEE International Conference on Information Technology: Coding and Computing*, vol. 2, Las Vegas, NV, USA, April 2005, pp. 8–13.

[4] R. Forsling, Z. Sjanic, F. Gustafsson, and G. Hendeby, “Optimal linear fusion of dimension-reduced estimates using eigenvalue optimization,” in *Proceedings of the 25th IEEE International Conference on Information Fusion*, Linköping, Sweden, Jul. 2022.

[5] S. J. Julier and J. K. Uhlmann, “A non-divergent estimation algorithm in the presence of unknown correlations,” in *Proceedings of the 1997 American Control Conference*, Albuquerque, NM, USA, Jun. 1997, pp. 2369–2373.

[6] A. R. Benaskeur, “Consistent fusion of correlated data sources,” in *Proceedings of the 28th Annual Conference of the IEEE Industrial Electronics Society*, Seville, Spain, Nov. 2002, pp. 2652–2656.

[7] M. E. Liggins and K.-C. Chang, “Distributed fusion architectures, algorithms, and performance within a network-centric architecture,” in *Handbook of Multisensor Data Fusion: Theory and Practice*, M. Liggins, D. Hall, and J. Llinas, Eds. Boca Raton, FL, USA: CRC Press, 2009, ch. 17.

[8] U. A. Khan and J. M. F. Moura, “Distributing the Kalman filter for large-scale systems,” *IEEE Transactions on Signal Processing*, vol. 56, no. 10, pp. 4919–4935, Oct. 2008.

[9] F. Govaers and W. Koch, “Distributed Kalman filter fusion at arbitrary instants of time,” in *Proceedings of the 13th IEEE International Conference on Information Fusion*, Edinburgh, Scotland, Jul. 2010, pp. 1–8.

[10] M. Reinhardt, B. Noack, and U. D. Hanebeck, “The hypothesizing distributed Kalman filter,” in *Proceedings of the IEEE International Conference on Multisensor Fusion and Integration*, Hamburg, Germany, Sep. 2012.

[11] C. G. Lopes and A. H. Sayed, “Diffusion least-mean squares over networks,” in *Proceedings of the 56th IEEE International Conference on Decision and Control*, Dec. 2017, pp. 6676–6681.

[12] F. S. Cattivelli and A. H. Sayed, “Diffusion LMS strategies for distributed estimation,” *IEEE Transactions on Signal Processing*, vol. 58, no. 3, pp. 1035–1048, Mar. 2010.

[13] Y. Bar-Shalom and L. Campo, “The effect of the common process noise on the two-sensor fused-track covariance,” *IEEE Transactions on Aerospace and Electronic Systems*, vol. 22, no. 6, pp. 803–805, Nov. 1986.

[14] X. Tian and Y. Bar-Shalom, “On algorithms for asynchronous track-to-track fusion,” in *Proceedings of the 13th IEEE International Conference on Information Fusion*, Edinburgh, Scotland, Jul. 2010.

[15] R. Forsling, Z. Sjanic, F. Gustafsson, and G. Hendeby, “Consistent distributed track fusion under communication constraints,” in *Proceedings of the 22nd IEEE International Conference on Information Fusion*, Ottawa, Canada, Jul. 2019.

[16] , “Communication efficient decentralized track fusion using selective information extraction,” in *Proceedings of the 23rd IEEE International Conference on Information Fusion*, Virtual Conference, Jul. 2020.

[17] R. Forsling, X. R. Liu, and J. Llinas, “Optimal linear estimation fusion - part VI: Sensor data compression,” in *Proceedings of the 6th IEEE International Conference on Information Fusion*, vol. 1, Cairns, Queensland, Australia, Jul. 2003, pp. 221–228.