C*-ALGEBRAS OF GROUPOID MORPHISMS

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Abstract. In this paper we suggest a definition for a C*-algebra attached to an injective morphism of some Étale groupoid. We take into account all the peculiarities of such objects and present some interesting relations with already well-known C*-algebras from the literature.

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1. INTRODUCTION

The study of C*-algebras representing distinct mathematical objects is a huge field of research, since all types of the latters can be, somehow, represented as operators over some suitable operator space ([9], [10], [11], [2]).

In [6], Hirshberg constructed a C*-algebra associated with endomorphisms of groups with finite cokernel. His inspiration was to see the group elements as unitary operators on a convenient Hilbert space and the endomorphism as one isometry acting on the same object and, among other results, he proved that it can be seen as a crossed product too. In a subsequent paper, Vieira constructed the same C*-algebra for endomorphisms with infinite cokernel (see [14]).

Particularly, the use of groupoids and semigroupoids in these processes became convenient, since almost every structure can be viewed as one of them ([7], [8]). In this paper, we propose a definition replacing groups by groupoids. The notion of an abstract groupoid was introduced by Brandt in 1927 ([1]) and since then it has been vastly...
studied. In particular, we are interested in a special kind of groupoid called Étale groupoid, because of its interesting and convenient properties (see [13] for more details).

With this picture in mind, we define a $C^*$-algebra associated with an injective morphism $\varphi : G \to G$, of an Étale groupoid $G$. This case differs to that one of groups because, in a groupoid, we can not multiply every pair of elements. Therefore, the groupoid should be represented in some $C^*$-algebra respecting this condition.

Analyzing a concrete representation of the groupoid and the morphism inside $\mathcal{B}(l^2(G))$, it is an easy task to suggest the definition of a universal $C^*$-algebra $\mathbb{U}[\varphi]$, via generators and relations. After, we present and prove some relations between this object and other already known ones.

2. Groupoids and semigroupoids

In order to maximize our intuition, we review the concepts of a groupoid and a semigroupoid as presented in [12] and [3], beginning with an informal explanation of what a groupoid is, so that the reader has a natural idea of the concept. Then, we will follow to a formal definition.

In general, a groupoid $G$ is very similar to a group just as a semigroupoid $S$ is very similar to a semigroup, except that it is not possible to multiply every two elements of $G$ (or $S$). To see this in a more natural way, it is usual to consider them over some base space $M$: we use two projections $\sigma$ and $\tau$ from $G$ (or $S$) to $M$, called source and target projections. With them, it is easier to see elements $g$ of $G$ (or $S$) over the base space $M$ as arrows

$$
\tau(g) \quad \sigma(g)
$$

starting at the source $\sigma(g)$ and ending at the target $\tau(g)$, in $M$. Intuitively, the multiplication is just the composition of two arrows and it is only allowed when the target of the first coincides with the source of the second one:

$$
\tau(g) \overset{g}{\frown} \sigma(g) = \tau(h) \overset{h}{\frown} \sigma(h)
$$
Similarly to groups (or semigroups), it is required that the multiplication of three elements, when well defined, is associative:

\[
\begin{align*}
\tau(f) \cdot \sigma(g) &= \tau(g) \\
\tau(g) \cdot \sigma(h) &= \tau(h)
\end{align*}
\]

Since some elements are not multipliable in groupoids, its unit is, in fact, a family of elements parameterized by the elements of the base space \(M\). In other words, it can be seen as a section \(1 : M \rightarrow G\), associating each \(x \in M\) with an arrow starting and finishing at \(x\) such that the multiplication of this arrow with any other arrow (that starts or finishes at \(x\)) gives the second arrow as result:

\[
\begin{align*}
\tau(g) \cdot 1_{\tau(g)} &= \sigma(g) \\
\tau(g) \cdot 1_{\sigma(g)} &= \sigma(g)
\end{align*}
\]

A semigroupoid, like a semigroup, does not need to have a unit. When the semigroupoid has it, we will make it explicit.

Finally, in the groupoid, each arrow has an inverse arrow:

\[
\tau(g) \cdot g^{-1} = \sigma(g)
\]

Formally:

**Definition 2.1.** A groupoid over a set \(M\) is a non-empty set \(G\) equipped with a source projection \(\sigma : G \rightarrow M\), a target projection \(\tau : G \rightarrow M\) and a multiplication

\[
\cdot : G^{(2)} \rightarrow G \\
(g,h) \mapsto gh
\]

defined on the subset

\[
G^{(2)} := \{(g,h) \in G \times G \mid \sigma(g) = \tau(h)\}
\]

of \(G \times G\), such that

\[
\sigma(gh) = \sigma(h) , \quad \tau(gh) = \tau(g)
\]
and satisfying the associativity
\[ f(gh) = (fg)h \]
when \( \sigma(f) = \tau(g) \) and \( \sigma(g) = \tau(h) \). Also it has a unit
\[
1 : M \rightarrow G
\]
\[
x \mapsto 1_x
\]
such that
\[
\sigma(1_x) = x = \tau(1_x)
\]
and
\[
g1_{\sigma(g)} = g = 1_{\tau(g)} g
\]
and, finally, an inversion
\[
\iota : G \rightarrow G
\]
\[
g \mapsto g^{-1}
\]
such that
\[
\sigma(g^{-1}) = \tau(g) , \tau(g^{-1}) = \sigma(g)
\]
and
\[
g^{-1} g = 1_{\sigma(g)} , g g^{-1} = 1_{\tau(g)}.
\]

In the next two definitions we present another key concepts that we need.

**Definition 2.2.** Let \( G \) be a groupoid over \( M \) and \( H \) a groupoid over \( N \) with source projections \( \sigma_G \) and \( \sigma_H \) and target projections \( \tau_G \) and \( \tau_H \), respectively. A **groupoid morphism**, from \( G \) to \( H \), is a pair of maps \( \varphi : G \rightarrow H \) and \( \phi : M \rightarrow N \) satisfying
\[
\sigma_H \circ \varphi = \phi \circ \sigma_G \quad \text{and} \quad \tau_H \circ \varphi = \phi \circ \tau_G,
\]
that is, the diagram
\[
\begin{array}{ccc}
G & \xrightarrow{\varphi} & H \\
\sigma_G \downarrow & & \downarrow \tau_H \\
M & \xrightarrow{\phi} & N
\end{array}
\]
commutes and \( \varphi \) preserves product: for all \((g, h) \in G^{(2)}\),
\[
\varphi(gh) = \varphi(g) \varphi(h).
\]
Observe that \((\varphi(g), \varphi(h)) \in H^{(2)}\), because
\[
\sigma_H(\varphi(g)) = \phi(\sigma_G(g)) = \phi(\tau_G(h)) = \tau_H(\varphi(h)).
\]
We say that \( \varphi \) is a morphism over \( \phi \). If \( M = N \) and \( \phi \) is the identity, we say \( \varphi \) is a strict morphism.
**Definition 2.3.** A semigroupoid is a triple \((\Lambda, \Lambda^{(2)}, \cdot)\) such that \(\Lambda\) is a set, \(\Lambda^{(2)}\) is a subset of \(\Lambda \times \Lambda\) and \(\cdot : \Lambda^{(2)} \to \Lambda\) is an operation \((r, s) \mapsto rs\) such that if any of the following three items holds

(i) \((r, s), (s, t) \in \Lambda^{(2)}\);
(ii) \((r, s) \in \Lambda^{(2)}\) and \((rs, t) \in \Lambda^{(2)}\);
(iii) \((s, t) \in \Lambda^{(2)}\) and \((r, st) \in \Lambda^{(2)}\);

then \((r, s), (s, t), (rs, t)\) and \((r, st)\) lie in \(\Lambda^{(2)}\) and \((rs)t = r(st)\).

To finish this section, we present three definitions with some contents that will be used in the next sections.

**Definition 2.4.** Let \(r, t \in \Lambda\) where \(\Lambda\) is a semigroupoid. We shall say that \(r\) divides \(t\), or that \(t\) is a multiple of \(r\), in symbols \(r \mid t\), if either

(i) \(r = t\) or
(ii) there exists \(s \in \Lambda\) such that \((r, s) \in \Lambda^{(2)}\) and \(rs = t\).

The item (i) above is necessary since \(\Lambda\) may not have a unit.

**Definition 2.5.** We say that \(r\) and \(s\) in \(\Lambda\) intersect if they admit a common multiple, that is, there exists \(t \in \Lambda\) such that \(r \mid t\) and \(s \mid t\). In this case, we write \(r \triangleright s\) and, otherwise, we say that \(r\) and \(s\) are disjoint and write \(r \perp s\).

For our last definition, consider \(\Lambda^r := \{s \in \Lambda : (r, s) \in \Lambda^{(2)}\}\).

**Definition 2.6.** An element \(r \in \Lambda\) is monic if for every \(s, t \in \Lambda^r\) we have that \(rs = rt\) implies \(s = t\) and we say that \(r \in \Lambda\) is a spring when \(\Lambda^r = \emptyset\).

3. **Semigroupoid semidirect product and its \(C^\ast\)-algebra**

Starting from an action of the semigroup \(N\) on a groupoid \(G\), we show in this section how to construct the semidirect product \(G \rtimes_{\varphi} N\), where \(\varphi : G \to G\) is an injective groupoid morphism over \(\phi : M \to M\). We then prove that it is in fact a semigroupoid and exhibit its \(C^\ast\)-algebra as defined in [3].

**Definition 3.1.** Consider \(G\) a groupoid and \(\varphi : G \to G\) an injective groupoid morphism over \(\phi : M \to M\). We define the semidirect product \(G \rtimes_{\varphi} N\) to be the set \(\Lambda = G \times N\) together with

\[\Lambda^{(2)} := \{((g, m), (h, n)) \in \Lambda \times \Lambda : (g, \varphi^m(h)) \in G^{(2)}\}\]

The product \(\cdot : \Lambda^{(2)} \to \Lambda\) is defined as follows: given \(((g, m), (h, n)) \in \Lambda^{(2)}\),

\[(g, m)(h, n) := (g\varphi^m(h), m + n).\]
Consider the source projection \( \sigma \) and target projection \( \tau \) in \( G \) and define the functions \( \overline{\sigma} : D_{\overline{\sigma}} \rightarrow M \) and \( \overline{\tau} : D_{\overline{\tau}} \rightarrow M \) as

\[
\begin{align*}
\overline{\sigma}(g, m) & := \phi^{-m}(\sigma(g)) \\
\overline{\tau}(g, m) & := \tau(g).
\end{align*}
\]

(1)

**Proposition 3.1.** The functions \( \overline{\sigma} \) and \( \overline{\tau} \) have domains

\[
D_{\overline{\sigma}} = \{ (g, m) \in \Lambda : \sigma(g) \in \phi^m(M) \}
\]

and \( D_{\overline{\tau}} = \Lambda \). Also, it holds that

\[
((g, m), (h, n)) \in \Lambda^{(2)} \iff \sigma(g) \in \phi^m(M) \text{ and } \overline{\sigma}(g, m) = \overline{\tau}(h, n).
\]

**Proof.** We observe that \( (g, m) \in D_{\overline{\sigma}} \) if, and only if, there exists \( x \in M \) such that \( x = \phi^{-m}(\sigma(g)) \), that is, \( \sigma(g) = \phi^m(x) \in \phi^m(M) \). It is straightforward that \( D_{\overline{\tau}} = \Lambda \).

Besides, note that \( ((g, m), (h, n)) \in \Lambda^{(2)} \) if, and only if, \( (g, \varphi^m(h)) \in G^{(2)} \), that is equivalent to \( \sigma(g) = \tau(\varphi^m(h)) = \phi^m(\tau(h)) \) and so \( \sigma(g) \in \phi^m(M) \). Now, since \( \varphi \) is injective – and then \( \phi \) is injective – we obtain \( \phi^{-m}(\sigma(g)) = \tau(h) \), that is, \( \overline{\sigma}(g, m) = \overline{\tau}(h, n) \).

Now, let’s take a look at the set \( \Lambda \) and, for this, fix an element \( g \in G \) (an arrow that connects \( \sigma(g) \) and \( \tau(g) \), which are points in the base space). Note first that the arrow \( (g, 0) \) in the semigroupoid \( \Lambda \) can be identified with \( g \) since they have the same source and target in \( M \). The arrow \( (g, 1) \) is obtained from \( (g, 0) \) by changing the source \( \sigma(g) \) of \( (g, 0) \) to \( \phi^{-1}(\sigma(g)) \), if it is possible, and fixing the target. In general, the arrow \( (g, m + n) \) is obtained from \( (g, m) \) by applying \( \phi^{-n} \) on the source of \( (g, m) \).

On the other hand, the element \( (\varphi^n(g), 0) \) is obtained from \( (g, 0) \) by applying \( \phi^n \) on both source and target of \( (g, 0) \), since \( \varphi \) is a morphism. Finally, the element \( (\varphi^n(g), m) \) for \( m \leq n \) is obtained from \( (g, 0) \) by applying \( \phi^n \) on the target and \( \phi^{n-m} \) on the source of \( (g, 0) \).

**Proposition 3.2.** If \( ((g, m), (h, n)) \in \Lambda^{(2)} \) and \( (h, n) \in D_{\overline{\tau}} \) then

\[
(g, m)(h, n) \in D_{\overline{\sigma}},
\]

\[
\overline{\sigma}((g, m)(h, n)) = \overline{\sigma}(h, n) \text{ and } \overline{\tau}((g, m)(h, n)) = \overline{\tau}(g, m).
\]

**Proof.** Note that if \( (h, n) \in D_{\overline{\sigma}} \) then \( \sigma(h) \in \phi^n(M) \). This implies that \( \phi^m(\sigma(h)) \in \phi^{m+n}(M) \).

Using the properties of \( \sigma \) and the fact that \( \varphi \) is a groupoid morphism, we obtain

\[
\phi^{m+n}(M) \ni \phi^m(\sigma(h)) = \sigma(\varphi^m(h)) = \sigma(g \varphi^m(h))
\]
since \((g, \varphi^m(h)) \in G^{(2)}\). This implies that \((g, m)(h, n) = (g\varphi^m(h), m + n) \in D_{\sigma}\). Now,

\[
\mathcal{P}((g, m)(h, n)) = \phi^{-(m+n)}(\sigma(g\varphi^m(h)))
\]
\[
= \phi^{-(m+n)}(\sigma(\varphi^m(h)))
\]
\[
= \phi^{n}(\sigma(h))
\]
\[
= \mathcal{P}(h, n).
\]

For \(\mathcal{P}\) we have

\[
\mathcal{P}((g, m)(h, n)) = \mathcal{P}(g\varphi^m(h), m + n) = \tau(g\varphi^m(h)) = \tau(g) = \mathcal{P}(g, m).
\]

The next lemma will be important to show that \(\Lambda\), as constructed here, is a semigroupoid.

**Lemma 3.3.** If \(((g, m), (h, n)) \in \Lambda^{(2)}\) and \((g, m)(h, n) \in D_{\sigma}\), then \((h, n) \in D_{\sigma}\).

**Proof.** We have, using the injectivity of \(\phi\), that

\[(g, m)(h, n) \in D_{\sigma} \Rightarrow (g\varphi^m(h), m + n) \in D_{\sigma}
\]
\[
\Rightarrow \sigma(g\varphi^m(h)) \in \varphi^{m+n}(M)
\]
\[
\Rightarrow \sigma(\varphi^m(h)) \in \varphi^{m+n}(M)
\]
\[
\Rightarrow \phi^m(\sigma(h)) \in \varphi^{m+n}(M)
\]
\[
\Rightarrow \sigma(h) \in \varphi^n(M)
\]
\[
\Rightarrow (h, n) \in D_{\sigma}.
\]

\(\square\)

**Theorem 3.4.** The triple \((\Lambda, \Lambda^{(2)}, \cdot)\) is a semigroupoid.

**Proof.** By Proposition 3.2 and Lemma 3.3, given \(((f, l), (g, m)),
((g, m), (h, n)), \((f, l)(g, m), (h, n))\) and \(((f, l), (g, m)(h, n))\) in \(\Lambda \times \Lambda\), if any of the items of the Definition 2.3 hold, then \(((f, l), (g, m))\),
\(((g, m), (h, n)), \((f, l)(g, m), (h, n))\) and \(((f, l), (g, m)(h, n))\) lie in \(\Lambda^{(2)}\).
Moreover, since \(G\) is a groupoid,

\[
[(f, l)(g, m)](h, n) = (f\varphi^l(g), l + m)(h, n)
\]
\[
= (f\varphi^l(g)\varphi^{l+m}(h), l + m + n)
\]
\[
= (f\varphi^l(g\varphi^m(h)), l + m + n)
\]
\[
= (f, l)(g\varphi^m(h), m + n)
\]
\[
= (f, l)[(g, m)(h, n)].
\]

\(\square\)
In the following, we present some interesting properties of $\Lambda$, since non monic elements and springs can not be properly represented in a C*-algebra.

**Proposition 3.5.** Every element of $\Lambda$ is monic.

*Proof.* Suppose that $(g, m), (h, n) \in \Lambda^{(f, l)}$ with $(f, l)(g, m) = (f, l)(h, n)$. Then $(f\varphi^l(g), l + m) = (f\varphi^l(h), l + n)$, that is, $m = n$ and $f\varphi^l(g) = f\varphi^l(h)$. Using $f^{-1}$ and the injectivity of $\varphi$, we conclude that $g = h$ and $(f, l)$ is monic. □

**Proposition 3.6.** An element $(g, m) \in \Lambda$ is a spring if, and only if, $(g, m) \notin D_{\sigma}$.

*Proof.* Observe that $(g, m)$ is a spring if, and only if, $\sigma(g) \neq \phi^m(\tau(h))$ for all $h \in G$, which means that $(g, m) \notin D_{\sigma}$. □

Therefore we will consider $D_{\sigma} = \Lambda$ because, in this setup, the semigroupoid $\Lambda$ will have no springs. This is equivalent to say that the function $\phi$ in the Definition 2.2 is bijective.

**Example 3.7.** Given a set $M$, consider the cartesian product $M \times M$ of two copies of $M$ and define the source projection $\sigma(y, x) := x$ and target projection $\tau(y, x) := y$, multiplication $(z, y)(y, x) := (z, x)$, the diagonal as unit, $1_x := (x, x)$, and switch as inversion, $(y, x)^{-1} := (x, y)$. Then $M \times M$ is a groupoid over $M$, called the pair groupoid of $M$. Given a bijective map $\phi : M \rightarrow M$ we can construct a groupoid morphism $\varphi : M \times M \rightarrow M \times M$ given by:

$$\varphi(y, x) := (\phi(y), \phi(x))$$

where the following diagram commutes

$$\begin{array}{ccc}
M \times M & \xrightarrow{\varphi} & M \times M \\
\sigma \downarrow & & \sigma \downarrow \\
M & \xrightarrow{\phi} & M \\
\tau \downarrow & & \tau \downarrow \\
M & \xrightarrow{\phi} & M
\end{array}$$

Observe that, since $\phi$ is injective, $\varphi$ is injective as well.

This leads us to the construction of a semigroupoid $(M \times M) \rtimes_{\varphi} \mathbb{N}$, the semidirect product of $M \times M$ with $\mathbb{N}$. Namely, using the maps defined in (1) and multiplication from Definition 3.1 we obtain, for $(y, x, n) \in (M \times M) \rtimes_{\varphi} \mathbb{N}$, a source projection

$$\bar{\sigma}(y, x, n) = \phi^{-n}(\sigma(y, x)) = \phi^{-n}(x),$$

da target projection

$$\bar{\tau}(y, x, n) = \tau(y, x) = y.$$
and multiplication
\[(y_2, x_2, n_2)(y_1, x_1, n_1) = ((y_2, x_2)\phi^{n_2}(y_1, x_1), n_1 + n_2)\]
\[= ((y_2, x_2)(\phi^{n_2}(y_1), \phi^{n_2}(x_1)), n_1 + n_2)\]
\[= (y_2, \phi^{n_2}(x_1), n_1 + n_2),\]
where \(y_1 = \phi^{-n_2}(x_2)\). The set \((M \times M) \rtimes \varphi \mathbb{N}\) endowed with this structure is a semigroupoid.

Since we want to present an isomorphism involving the C*-algebra of a semigroupoid, we repeat its construction (from [3]) in the following two definitions.

**Definition 3.2.** Let \(\Lambda\) be a semigroupoid and \(B\) a unital C*-algebra. A mapping \(S : \Lambda \to B\) given by \(r \mapsto S_r\) will be called a representation of \(\Lambda\) in \(B\) if, for every \(r, t \in \Lambda\), denoting by \(P_r = S_rS_r^*\) and \(Q_r = S_r^*S_r\) the initial and final projections, respectively, we have:

1. \(S_r\) is a partial isometry;
2. \(S_rS_t = \begin{cases} S_{rt} & \text{if } (r, t) \in \Lambda^{(2)} \\ 0 & \text{otherwise} \end{cases}\)
3. \(P_rP_t = 0\) if \(r \perp t\);
4. \(Q_rP_t = P_t\) if \((r, t) \in \Lambda^{(2)}\).

In order to define the C*-algebra of a semigroupoid, normally one considers tight representations. However, we have observed that, in [3], the main reason to choose tight representations (and not merely a representation) is to ensure an isomorphism between the C*-algebra of the Markov semigroupoid and the unital Cuntz-Krieger algebra introduced in [4]. Therefore, we will not use tight representations to define the C*-algebra of a semigroupoid, since it is not needed to derive our results.

**Definition 3.3.** Given a semigroupoid \(\Lambda\), its C*-algebra \(C^*(\Lambda)\) is the universal unital C*-algebra generated by a family of partial isometries \(\{S_r\}_{r \in \Lambda}\) such that \(S\) is a representation of \(\Lambda\).

### 4. The C*-Algebra of a Groupoid Morphism

Based on the similarities of groupoids and groups, we adapt the definition of the C*-algebra of a group endomorphism to our setup. Let \(G\) be an Étale groupoid and consider the Hilbert space \(l^2(G)\) with canonical orthonormal basis \(\{\xi_h : h \in G\}\). For each \(g \in G\), define
$U_g \in B(l^2(G))$ as

$$U_g(\xi_h) := \begin{cases} 
\xi_{gh}, & \text{if } (g, h) \in G^{(2)} \\
0, & \text{otherwise}.
\end{cases}$$

It is easy to prove that they are partial isometries. Besides, the endomorphism induces an isometry $S$ acting on $l^2(G)$ by

$$S(\xi_h) := \xi_{\varphi(h)}.$$

The $C^*$-algebra generated by them (inside $B(l^2(G))$) is called the reduced $C^*$-algebra of $\varphi$ and its elements satisfy interesting properties which are the base of our following definition.

**Definition 4.1.** Consider an injective endomorphism $\varphi$ of an Étale groupoid $G$. We define $\mathbb{U}[\varphi]$ as the universal $C^*$-algebra generated by partial isometries $\{u_g : g \in G\}$ and an isometry $s$

satisfying

(i) $u_gu_h = \begin{cases} 
u_{gh}, & \text{if } (g, h) \in G^{(2)} \\
0, & \text{otherwise};\end{cases}$

(ii) $u_g^* = u_{g^{-1}}$;

(iii) $su_g = u_{\varphi(g)}s$;

(iv) $\sum_{x \in M} u_{1_x}s = s$.

Furthermore, the elements $u_g s^m s^m u_{g^{-1}}$ and $u_h s^n s^n u_{h^{-1}}$ are projections that commute and if $m \leq n$ then

(v) $u_g s^m s^m u_{g^{-1}} u_h s^n s^n u_{h^{-1}} = 0$, if $g^{-1}h \notin \varphi^m(G)$.

In (iii), observe that $u_{g^{-1}}s^* = s^* u_{\varphi(g^{-1})}$.

Our goal now is to show that the $C^*$-algebra $\mathbb{U}[\varphi]$ constructed in this section is isomorphic to the $C^*$-algebra of the semigroupoid $\Lambda$ presented in Section 3. For this, we need an easy lemma.

**Lemma 4.1.** It holds that $g^{-1}h \in \varphi^m(G)$ if, and only if, $(g, m) \bowtie (h, n)$.

**Proof.** If $g^{-1}h \in \varphi^m(G)$ then there exists $f \in G$ such that $g^{-1}h = \varphi^m(f)$, i.e., $h = g\varphi^m(f)$. Also, note that there exist $l, k \in \mathbb{N}$ such that
\[ m + l = n + k. \] So,
\[
(g, m)(f, l) = (g \varphi^m(f), m + l)
= (h, m + l)
= (h1_{\sigma(h)}, m + l)
= (h1_{\phi^n(x)}, m + l)
= (h\varphi^n(1_x), m + l)
= (h\varphi^n(1_x), n + k)
= (h, n)(1_x, k).
\]

It means that there exist \((f, l), (1_x, k) \in \Lambda\) such that \((g, m)(f, l) = (h, n)(1_x, k)\), i.e., \((g, m) \cap (h, n)\). Now, if \((g, m) \cap (h, n)\) it is obvious that \(g^{-1}h \in \varphi^m(G)\).

**Theorem 4.2.** \(\mathbb{U}[\varphi]\) is isomorphic to \(C^*(\Lambda)\).

**Proof.** To start, let us use the notation of Definition 3.2 to define
\[
\Phi : \mathbb{U}[\varphi] \longrightarrow C^*(\Lambda),
\]
given by \(\Phi(u_g) := S_{(g,0)}\) and \(\Phi(s) := \sum_{x \in M} S_{(1_x,1)}\). We need to show that these representatives satisfy the relations generating \(\mathbb{U}[\varphi]\).

Using the condition (ii) of Definition 3.2 relation (i) generating \(\mathbb{U}[\varphi]\) follows by easy calculations.

Condition (ii) follows from condition (i) of Definition 3.2 on one hand, since \(S_{(g,0)}\) is a partial isometry, \(S_{(g,0)}S_{(g,0)}^*S_{(g,0)} = S_{(g,0)}\). On the other hand,
\[
S_{(g,0)}S_{(g^{-1},0)}S_{(g,0)} = S_{(g,0)(g^{-1},0)}S_{(g,0)}
= S_{(1_{g^{-1}},0)}S_{(g,0)}
= S_{(1_{(g^{-1},0)},0)}S_{(g,0)}
= S_{(g,0)}.
\]

Therefore, \(S_{(g,0)}^* = S_{(g^{-1},0)}\).

Finally, conditions (iii) and (iv) are straightforward and condition (v) follows from Lemma 4.1 and condition (iii) of the Definition 3.2.

For the inverse map, consider
\[
\Psi : C^*(\Lambda) \longrightarrow \mathbb{U}[\varphi],
\]
given by \(\Psi(S_{(g,m)}) := u_g s^m\). Let us show that the elements \(u_g s^m\) satisfy the four relations generating \(C^*(\Lambda)\).
To prove condition (i) note that
\[
(u_g s^m)(u_g s^m)^* (u_g s^m) = u_g s^m s^* m u_g^{-1} u_g s^m \\
= u_g s^m s^* m u_{1_x(s)} s^m \\
= u_g s^m \varphi^{-m(1_{\sigma(g)})} s^* m s^m \\
= u_g s^m \varphi^{-m(1_{\sigma(g)})} \\
= u_g u_{1_x(s)} s^m \\
= u_g s^m.
\]
Likewise, we can prove that
\[
(u_g s^m)^* (u_g s^m)(u_g s^m)^* = (u_g s^m)^*.
\]
The condition (ii) follows from (i) of Definition 4.1.
For (iii), if \(\tau(g) \neq \tau(h)\) then
\[
(u_g s^m s^* m u_g^{-1})(u_h s^n s^* n u_h^{-1}) = 0,
\]
as we want to proof. Now, if \(\tau(g) = \tau(h)\) and \((g, m) \perp (h, n)\), Lemma 4.1 implies that \(g^{-1} h \notin \varphi^m(G)\). In this case,
\[
u_g s^m s^* m u_g^{-1} u_h s^n s^* n u_h^{-1} = 0.
\]
To show (iv) consider \(((g, m), (h, n)) \in \Lambda^{(2)}\). Then,
\[
(s^* m u_g^{-1} u_g s^m)(u_h s^n s^* n u_h^{-1}) = s^* m u_g^{-1} u_g \varphi^m(h)s^m s^* n u_h^{-1} \\
= s^* m \varphi^m(h)s^m s^* n u_h^{-1} \\
= s^* m s^* u_h s^n s^* n u_h^{-1} \\
= u_h s^n s^* n u_h^{-1}.
\]
Finally, a simple calculation shows that \(\Psi\) is the inverse map of \(\Phi\):
\[
\Psi(\Phi(u_g)) = \Psi(S_{(g,0)}) = u_g,
\]
\[
\Psi(\Phi(s)) = \Psi \left( \sum_{x \in M} S_{(1_x,1)} \right) = \sum_{x \in M} u_{1_x} s = s.
\]
Besides, an induction argument implies that
\[
\Phi(\Psi(S_{(g,m)})) = S_{(g,m)}.
\]
5. Crossed product description

In this section we want to show that $\mathbb{U}[\varphi]$ can be viewed as a quotient of a crossed product. Due to the particularities of our object, the crossed product that we will construct is a semigroup one, in which the initial $C^*$-algebra is a groupoid crossed product.

This last construction proceeds as follows: given a $C^*$-algebra $A$ and a groupoid $G$ over a locally compact Hausdorff space $M$, we can construct a groupoid crossed product $A \rtimes_\alpha G$ by following the steps on [5]. Basically, because of the fibered nature of a groupoid, $A$ must have a “fibration” so that we can define an action $\alpha$ of $G$ on $A$.

If $A$ is a $C_0(M)$-space, we can construct the associated upper-semicontinuous $C^*$-bundle $\mathcal{D}[\varphi]$ over $M$ such that $\mathcal{D}[\varphi]$ is isomorphic to the space of sections $\Gamma_0(\mathcal{D}[\varphi])$ that vanishes at infinity. Then, the groupoid dynamical system $(A, G, \alpha)$ is defined by choosing an action $\alpha$ of $G$ on $A$ and the groupoid crossed product $A \rtimes_\alpha G$ is defined as the completion of $\Gamma_0(G, \tau^*A)$ with respect to an specific norm.

In our present case, we consider directly the fibration

$$D_x[\varphi] := C^*\{u_h s^n s^n u_h^{-1} : h \in G, n \in \mathbb{N} \text{ and } \tau(h) = x\}$$

which gives us the $C^*$-bundle

$$\mathcal{D}[\varphi] := \bigcup_{x \in M} D_x[\varphi]$$

whose projection $\pi : \mathcal{D}[\varphi] \to M$ is defined by $\pi(u_h s^n s^n u_h^{-1}) := \tau(h)$.

Following [5], the notation is $\mathcal{D}[\varphi] = \mathfrak{A}$ and, by Theorem C.26 of [15] the $C^*$-algebra $A$ is $\Gamma_0(M, \mathcal{D}[\varphi])$, which we will denote by $D[\varphi]$ to simplify. Here, the pull-back of $\mathcal{D}[\varphi]$ by $\tau$ is given by

$$\tau^*\mathcal{D}[\varphi] := \{(P, g) \in \mathcal{D}[\varphi] \times G; \tau(g) = \pi(P)\}$$

represented by the commutative diagram

$$\begin{array}{ccc}
\tau^*\mathcal{D}[\varphi] & \longrightarrow & \mathcal{D}[\varphi] \\
\downarrow & & \downarrow \pi \\
G & \longrightarrow & M \\
\tau & & \\
\end{array}$$

Furthermore, we define the action $\alpha$ from $G$ on the fibred $C^*$-algebra $\mathcal{D}[\varphi]$ as the family of functions $\{\alpha_g\}_{g \in G}$ such that

$$\alpha_g(u_h s^n s^n u_h^{-1}) := u_g u_h s^n s^n u_h^{-1} u_g^{-1} = u_{gh} s^n s^n u_{(gh)^{-1}}.$$  

**Proposition 5.1.** Let $\alpha$ be defined as previously. Then:

(i) For all $(f, g) \in G^{(2)}$, we have $\alpha_{fg} = \alpha_f \circ \alpha_g$;

(ii) Given $f \in G$, $\alpha_f : \mathcal{D}_{\sigma(f)}[\varphi] \to \mathcal{D}_{\tau(f)}[\varphi]$ is an isomorphism.
That is, $\alpha$ is an action from $G$ on the $C^*$-algebra $D[\varphi]$.

**Proof.** (i): Let $u_h s^n s^n u_{h^{-1}} \in D[\varphi]$. Then,

$$
\alpha_f(u_h s^n s^n u_{h^{-1}}) = u_f(u_g(u_h s^n s^n u_{h^{-1}})u_{g^{-1}})u_f^{-1} = \alpha_f \circ \alpha_g(u_h s^n s^n u_{h^{-1}}).
$$

(ii) Item (i) already ensures that $\alpha_f$ is bijective with inverse $\alpha_f^{-1}$. To prove that it is a morphism, consider $u_g s^m s^m u_{g^{-1}}$ and $u_h s^n s^n u_{h^{-1}}$ in $D[\sigma(f)][\varphi]$. Then,

$$
\alpha_f((u_g s^m s^m u_{g^{-1}})(u_h s^n s^n u_{h^{-1}})) = u_f(u_g s^m s^m u_{g^{-1}}u_h s^n s^n u_{h^{-1}})u_f^{-1} = u_f u_g s^m s^m u_{g^{-1}}u_f^{-1}u_f u_h s^n s^n u_{h^{-1}}u_f^{-1} = (u_f u_g s^m s^m u_{g^{-1}}u_f^{-1})(u_f u_h s^n s^n u_{h^{-1}}u_f^{-1}) = \alpha_f(u_g s^m s^m u_{g^{-1}})\alpha_f(u_h s^n s^n u_{h^{-1}}).
$$

Also

$$
\alpha_f((u_g s^m s^m u_{g^{-1}})^*) = \alpha_f(u_g s^m s^m u_{g^{-1}}) = u_f(u_g s^m s^m u_{g^{-1}})u_f^{-1} = (u_f(u_g s^m s^m u_{g^{-1}}u_f^{-1}))^* = (\alpha_f(u_g s^m s^m u_{g^{-1}}))^*.
$$

With this, we are able to present the groupoid dynamical system $(D[\varphi], G, \alpha)$ and, finally, to construct the groupoid crossed product $D[\varphi] \rtimes_\alpha G$ as the universal enveloping algebra of $\Gamma_c(G, \tau^*D[\varphi])$.

The next step is to construct the crossed product of $D[\varphi] \rtimes_\alpha G$ with $\mathbb{N}$. Given $n \in \mathbb{N}$, consider $\beta_n : D[\varphi] \rtimes_\alpha G \rightarrow D[\varphi] \rtimes_\alpha G$ defined by

$$
\beta_n(u_g s^m s^m u_{g^{-1}}, h) := (s^n u_g s^m s^m u_{g^{-1}}s^n, \varphi^n(h)).
$$

Then, we construct the crossed product $(D[\varphi] \rtimes_\alpha G) \rtimes_\beta \mathbb{N}$, using the definitions of $[9]$.

**Theorem 5.2.** The $C^*$-algebra $\mathbb{U}[\varphi]$ is a quotient of $(D[\varphi] \rtimes_\alpha G) \rtimes_\beta \mathbb{N}$.

**Proof.** The definition of semigroup crossed products ensures the existence of a unital $*$-homomorphism

$$
i_D : D[\varphi] \rtimes_\alpha G \rightarrow (D[\varphi] \rtimes_\alpha G) \rtimes_\beta \mathbb{N}
$$

and a homomorphism of semigroups

$$
i_N : \mathbb{N} \rightarrow Isom((D[\varphi] \rtimes_\alpha G) \rtimes_\beta \mathbb{N})$$
satisfying
\[ i_D(\beta_n(x)) = i_N(n)i_D(x)(i_N(n))^*. \]
But, if we consider the maps
\[ \varpi : D[\varphi] \rtimes_\alpha G \to U[\varphi] \]
\[ (u_g s^m s^m u_{g^{-1}}, h) \mapsto (u_g s^m s^m u_{g^{-1}}) u_h \]
and
\[ \rho : \mathbb{N} \to Isom(U[\varphi]), \]
\[ n \mapsto s^n \]
we observe that
\[ \rho(n)\varpi(u_g s^m s^m u_{g^{-1}}, h)(\rho(n))^* = s^n((u_g s^m s^m u_{g^{-1}}) u_h)(s^n)^* \]
\[ = s^n u_g s^m s^m u_{g^{-1}} s^n u_{\varphi^n(h)} \]
\[ = \varpi(s^n u_g s^m s^m u_{g^{-1}} s^n, \varphi^n(h)) \]
\[ = \varpi(\beta_n(u_g s^m s^m u_{g^{-1}}, h)), \]
that is, \((U[\varphi], \varpi, \rho)\) is a covariant representation of \(((D[\varphi] \rtimes_\alpha G), \mathbb{N}, \beta)\).
Therefore, there exists a \(*\)-homomorphism
\[ \delta : (D[\varphi] \rtimes_\alpha G) \rtimes_\beta \mathbb{N} \to U[\varphi] \]
such that \(\delta \circ i_D = \varpi\) and \(\delta \circ i_N = \rho. \)

It is natural to ask if \(\delta\) above is injective, which would imply that those two objects are isomorphic. We tried to find elements inside \(U[\varphi]\) which behave just like the images of \(i_D\) and \(i_N\), but it seems to be impossible since we would need that \(su_{1_{r(h)}}s^* = u_{1_{r(\varphi(h))}}\).
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