Tropical Flag Varieties

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Example. $\text{Gr}(2,4) \& U_{2,4}$

The Grassmannian $\text{Gr}(2,4) \hookrightarrow \mathbb{P}^{15}_2$ via the Plücker embedding:

$$A = \begin{bmatrix}
    a_{11} & a_{12} & a_{13} & a_{14} \\
    a_{21} & a_{22} & a_{23} & a_{24}
\end{bmatrix} \mapsto \left( p_{ij} = \det \begin{bmatrix}
    a_{1i} & a_{4j} \\
    a_{2i} & a_{2j}
\end{bmatrix} : i,j \in [4] \right)$$

and is cut out by the equation $p_{14}p_{23} - p_{13}p_{24} + p_{12}p_{34}$.

Points in $\text{Gr}(2,4)^0$ are realizations of the uniform matroid.
\[ \text{trop} (\text{Gr}(2,4)^{{}\circ}) \text{ is a \ linear \ space.} \]

- \text{trop} (\text{Gr}(2,4)^{{}\circ}) \text{ is weight vectors inducing a matroidal subdivision of } B(M) \]

\[ B(M) = \]

- \text{trop}(\text{Gr}(2,4)^{{}\circ}) \text{ is tropical lines in } \mathbb{TP}^3 : \]
Dressians

M rank \( r \) matroid on \([n]\)

\( \text{Gr}(r,n) \hookrightarrow \mathbb{P}(r^n) \) cut out by Grassman-Plücker relations. Setting variables indexing non-bases of \( M \) to 0 gives equations cutting out the points of \( \text{Gr}(r,n) \) realizing \( M \).

Theorem [Dress-Wentzel, Speyer, Hermann-Jensen-Joswig-Sturmfels]

The Dressian of \( M \) is:

- tropical prevariety
- matroidal subdivisions

- valuated matroids
- tropical linear spaces
Example. \((U_{1,4}, U_{2,4}) \hookrightarrow \text{flag matroid} \)

(A) \(\text{Fl}(1,2;4) \hookrightarrow \mathbb{P}^3 \times \mathbb{P}^5 \) cut out by:

\[
\begin{align*}
(P_{14}P_{23} - P_{13}P_{24} + P_{12}P_{34}, & \quad P_4P_{23} - P_3P_{24} + P_2P_{34}, \quad P_4P_{13} - P_3P_{14} + P_1P_{34}, \quad P_4P_{12} - P_2P_{14} + P_1P_{24}, \quad P_3P_{12} - P_2P_{13} + P_1P_{23})
\end{align*}
\]

GP on \(U_{2,4}\)

\[
\text{tropical prevariety:}
\]

\[
\text{IP putting point \& line} \quad \hookrightarrow \text{flag Dressian}
\]

(C) Base Polytope: Subdivisions into flag matroid polytopes:
(D) tropical flags of a point in a line in $\mathbb{TP}^3$:
Flag Matroids

Definition. A flag matroid is a sequence of matroids 

\[(M_1, \ldots, M_k)\]

of ranks \((r_1, \ldots, r_k)\) on \([n]\) such that every circuit of \(M_j\) is a union of circuits of \(M_i\) for \(i < j\).

Example. \((U_{1,4}, U_{2,4})\) is a flag matroid of rank \((1,2)\) on \([4]\).
Valuated Flag Matroids

**Definition.** A **valuated flag matroid** is a flag matroid 
$(M_1, \ldots, M_k)$ together with functions $(v_1, \ldots, v_k)$, 

$$v_i : \mathcal{B}(M_i) \to \mathbb{R}$$

such that for $j \leq i$, $J \subseteq \mathcal{B}(M_j)$, $I \subseteq \mathcal{B}(M_i)$, and $e \in J \setminus I$, there exists $f \in I \setminus J$ such that

$$v(J) + v(I) \geq v(J \setminus \{e\} \cup \{f\}) + v(I \setminus \{f\} \cup \{e\}).$$
Let \((V,M)\) be a valued matroid. (Valuated flag matroid length 1)

This data gives a tropical linear space

\[
\operatorname{trop}(V) = \bigcap \left\{ u \in \mathbb{R}^n \mid \text{the min in } \{ u_j + V(T_j) \} \text{ is attained twice} \right\}
\]

\[
\operatorname{trop}(V_1) \ldots \operatorname{trop}(V_k) \quad |T| = r + 1
\]
Base Polytopes of Flag Matroids \((C)\) \((M_1, \ldots, M_k)\)

\[
\sum_{1 \leq i \leq k} \text{Conv}(e_B \mid B \in B(M_i)) \subset \mathbb{R}^n
\]

Weights: \((v_1, \ldots, v_k) \in \mathbb{R}^{\left|B(M_1)\right|} \times \cdots \times \mathbb{R}^{\left|B(M_k)\right|}

\[
(v_1, \ldots, v_k) \mapsto \min_{\text{vertex}} \{v_1(B_1) + \cdots + v_k(B_k) \mid P = e_{B_1} + \cdots + e_{B_k}\}
\]
Main Theorem. The following are equivalent:

(A) $(v_1, \ldots, v_k)$ is a point on a tropical prevariety called the Flag Dressian.

(B) $(v_1, \ldots, v_k)$ is a valuated flag matroid on the underlying flag matroid $(M_1, \ldots, M_k)$.

(C) $(v_1, \ldots, v_k)$ induces a subdivision of the base polytope of $(M_1, \ldots, M_k)$ into base polytopes of flag matroids.

(D) The tropical linear spaces form a flag: $\text{trop}(v_1) < \cdots < \text{trop}(v_k)$.
Theorem. Every valuated flag matroid on $n \leq 5$ is realizable.

The tropicalization of $Fl(r_1, \ldots, r_k; n)$ equals the flag Dressian.

Example. $Fl(1,2,3;4)$. The tropicalization of $Fl(1,2,3;4)$ is the flag Dressian which parameterizes a point in a line in a plane in $\mathbb{R}^3$. Thinking of the plane dually, this is 2 points in a line.

$f$-vector of $\text{trop}(Fl(1,2,3;4)) = (1, 20, 79, 78)$
Happy Friday & Thanks to the organizers!!
Example: [failure of realizability for n=6].

Consider $(U_{2,6}, M_4)$: It's Dressian:

All points in $Dr(U_{2,6})$ and $Dr(M_4)$ are realizable. So, points in the triangle correspond to two realizable valuated matroids that fail to form a realizable quotient (as pictured).
Points in the triangle have the tree:

So, over the residue field, we need
\[ 1 = c, \quad 3 = 4, \quad 2 = 5 \]

dually, this means finding a line intersecting at
\[ 6\eta_1, \quad 5\eta_2, \quad 4\eta_3. \]
(can only do in \( \text{char} = 2 \))
Theorem A:

proof

\[ (A) \leftrightarrow (B) \]

\[ (C) \leftrightarrow (D) \]

Not too bad

Prevairity

defitional

valuations

medium

hardest

trop linear flags