Collective Excitations in Thermal QED$_{3+1}$: Survival of the Fittest

I. V. Tyutin* and Vadim Zeitlin†

I. E. Tamm Theory Department, P. N. Lebedev Physical Institute Russia,
117924, Moscow, Leninsky prospect, 53

Abstract

The spectrum of collective fermionic excitations in a finite temperature QED$_{3+1}$ is studied in different regimes. It is shown that within the standard perturbation approach the one-loop dispersion equation, besides the ordinary one-particle excitation, has four new solutions. The additional excitations are gauge-dependent and two of them have nonphysical signs of residues in the propagator poles. The temperature evolution of the solutions is investigated and it is shown that the use of effective propagators leaves no more than one additional mode which becomes propagating at $T \gtrsim 10M$, when the gauge invariance is restored. The other three modes, including those with nonphysical residues in the propagator poles, are always strongly damped, thus the thermal effects do not produce pathologies in QED$_{3+1}$.

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*e-mail: tyutin@td.lpi.ac.ru

†e-mail: zeitlin@td.lpi.ac.ru
I. INTRODUCTION

A significant advancement in studying gauge theories at high temperatures has been achieved in recent years (see e.g. Ref. [1]). Hard thermal loops (HTL) resummation technique [2] has made possible to penetrate into the temperature region, where the effective coupling constant is large, \( gT/M \gg 1 \) (\( M \) is the vacuum fermion mass), and the ordinary perturbative series fail. The low-temperature limit, where the expansion in the vacuum coupling constant is valid, is also well-known (see, e.g. Ref. [3]). However, the intermediate range of temperatures comes as a surprise: the naive one-loop calculation in the framework of QED shows that at \( T \sim 4M \) new modes arise in the fermion spectrum.

At low temperature the modification of the fermion dispersion equation is obvious: thermal effects produce only a small shift of the vacuum mass shell of the one-particle excitation (OPE) and no new branches in the spectrum are formed. At high temperature the loop corrections may be of a tree-level order. For instance, when calculating the fermion self-energy every perturbation order brings the factor \( T^2 \) (which is typical for diagrams containing hard thermal loops, electron self-energy among them) and the effective coupling constant \( gT/M \) becomes (unlimitedly) large and one must sum out HTL. In practice, HTL summation implies substitution of the effective propagators, which are functions of the effective masses and dampings instead of the bare ones. In (extremely) high temperature limits, when the vacuum fermion mass is negligible, an additional hole-like state with a negative frequency arises in the electron branch, as well as a state with a positive frequency in the positron branch. Thus the number of excitations doubles as compared to the low-temperature case. The additional solutions (collective excitations, CE) were found in Refs. [4–6], where the one-loop self-energy of (massless) fermion was calculated (the spectrum structure at high temperatures has been studied also in Refs. [7,8]). Masses of the above excitations are of the order of \( gT \), and in the leading approximations the corresponding solutions to the dispersion equation are gauge-invariant [2,4].

Early in the 90th it was shown in Ref. [9] in the high-temperature limit and in Ref. [10]
at the "naive" one-loop level in QED with massive fermions that at $T > 4M$ in each branch a new low-frequency mode arises. Besides, it was demonstrated in Ref. [9] that the above excitation disappears after the damping of OPE is taken into account in the effective propagator.

Therefore, several questions arise: how high must the temperature be to let collective excitations come into existence, how do they exactly evolve with temperature, and are they physical excitations or not? In the present paper we shall study properties of the additional solutions to the dispersion equation, point out the physical ones, and estimate the temperature required to make them propagating. Within the naive perturbation theory the additional modes arise at rather low temperatures, $T \gtrsim 4M$. These modes are situated in the low frequency range ($\omega \sim M^3/T^2$ at zero momentum) and always arise in couples: two modes at positive and two at negative frequencies, both on electron and positron branches. However, the additional solutions are pathological: one excitation in each pair has the "wrong" sign of the residue. Besides, all the modes are (in leading order) gauge dependent. When the temperature is raised the additional positive-frequency modes (in the electron branch) and one of the negative-frequency modes (that with the wrong residue sign) remain in the low-frequency region, while the absolute value of frequency of the other modes rise and becomes $\sim gT$.

Since the difficulties described above jeopardize not an abstract model, but quantum electrodynamics, the theory describing the real world, one should believe the pathological modes (those which are gauge-variant and/or have indefinite metrics) should be exorcised by applying a correct calculation procedure.

We shall demonstrate that after inserting the damping of OPE into the bare propagators among positive frequency exitations only one mode survives at all temperatures. At $T \gtrsim 5M$ the real part of the inverse propagator has two zeroes at negative frequencies, but its imaginary part is larger than the frequency, thus the physical excitations are absent, and only at $T \gtrsim 10M$ one of the negative-frequency solutions overcomes the damping and becomes the propagating one. This is in a good agreement with the approach to the region $T \sim 10M$
from the higher temperature domain where the hard thermal loops results are valid.

The paper is organized as follows. In Sec. 2 the one–loop self–energy in an arbitrary relativistic gauge is calculated using the bare propagators. We show that the above naive calculation brings pathological modes into the model. In Sec. 3 we study how the damping may affect the collective excitations in QED plasma and show that the ”improved” naive approach and HTL scheme are in a good agreement with one another in the intermediate temperature range. A possible physical interpretation of the additional high-temperature excitation found in the fermion spectrum is given in Sec. 4.

II. ONE–LOOP ELECTRON SELF–ENERGY AT FINITE TEMPERATURE

In this section we shall study the spectrum of one–particle fermion excitations within the standard one-loop approximation, i.e. using the free electron and photon propagators. We shall use the thermal Green functions method \[3,11–13\] (of course, the real-time approach \[14,15,1\] produces the same results). The spectrum is defined as poles of the Fourier–transformed retarded Green function \(S_R(p_0, p) = S_R(p)\) or, equivalently, as zeroes of \(S_R^{-1}(p)\). The propagator \(S_R(p_0, p)\) is evaluated from the thermal Green function \(S(i\omega_n, p), \omega_n = (2n + 1)\pi\beta, \beta \equiv 1/T\) via analytical continuation \(i\omega_n \rightarrow \omega \equiv p_0\), holomorphic in the upper half-plane of \(\omega\) and having correct asymptotic behavior at \(\omega \rightarrow \infty\) (the fermion self-energy falls as \(1/\omega\) at high frequencies) \[3,13\].

The Lagrangian of QED\(_{3+1}\) in a relativistic gauge \(\alpha\) is

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_{\mu} A^{\mu})^2 + \bar{\psi}(i\hat{\partial} + g\hat{A} - M)\psi .
\]

(1)

The fermion self-energy \(\Sigma(p)\) is defined as \[3\]:

\[
S^{-1}(p) = S_0^{-1}(p) + \Sigma(p), \quad p_0 = i\omega_n ,
\]

(2)

\(^1\)We define the Green functions (propagators) in the temperature technique as average of the \(T\)–products in thermal time \(\tau\) of field operators.
\[ S_0(p) = -\frac{1}{\hat{p} - M} . \]

In the one–loop approximation it equal to

\[ \Sigma_1(p) = -ig^2T \sum_l \int \frac{d\mathbf{k}}{(2\pi)^3} \gamma_\mu S_0(p + k) \gamma_\nu D^{\mu\nu}(k) , \quad k_0 = \frac{2\pi l}{T} , \]

\[ D^{\mu\nu}(k) = \frac{1}{k^2} \left( g^{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) + \alpha \frac{k_\mu k_\nu}{k^4} . \]

The study of \( \Sigma_1(p) \) for arbitrary temperature, frequency and external 3–momentum \( \mathbf{p} \) is quite difficult and we consider the case of zero spatial momentum, \( \mathbf{p} = 0 \), only. The corresponding expression for \( \Sigma_1 \) may be written as (in Ref. [10] \( \Sigma_1(\omega) \) was calculated in the Feynman gauge):

\[ \Sigma_1(\omega, 0) \equiv \Sigma_1(\omega) = \gamma^0 \left( a(\omega) + (1 - \alpha)b(\omega) \right) + \frac{3 + \alpha}{4} c(\omega) , \]

\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \]

\[ a = \frac{g^2 \omega}{\pi^2} \int_0^\infty \frac{dk}{4\omega^2 k^2 - (M^2 - \omega^2)^2} \left( \frac{k(M^2 - \omega^2 + 2k^2)}{e^{\beta k} - 1} + \frac{2k^2 \varepsilon}{e^{\beta \varepsilon} + 1} \right) , \]

\[ b = \frac{g^2 \omega(M^2 - \omega^2)}{2\pi^2} \int_0^\infty \frac{dk}{(4\omega^2 k^2 - (M^2 - \omega^2)^2)^2} \times \]

\[ \left( \frac{k(8\omega^2 k^2 - 4k^2 M^2 - (M^2 - \omega^2)^2)}{e^{\beta k} - 1} + \frac{k^2((M^2 + \omega^2)^2 - 4\varepsilon^2 M^2)}{e^{\beta \varepsilon} + 1} \right) , \]

\[ c = \frac{2g^2 M}{\pi^2} \int_0^\infty \frac{dk}{4\omega^2 k^2 - (M^2 - \omega^2)^2} \left( \frac{k(M^2 - \omega^2)}{e^{\beta k} - 1} + \frac{k^2(M^2 + \omega^2)}{\varepsilon (e^{\beta \varepsilon} + 1)} \right) , \]

( the vacuum contribution, which is insufficient at high temperatures, is omitted).

The spectrum is defined as nontrivial solutions to the equation

\[ (\gamma^0 \omega - M - \Sigma_1(\omega))\psi = 0 . \]
We suppose that an excitation belongs to the electron branch if $\psi$ may be written as

$$
\psi = \begin{pmatrix} u \\ 0 \end{pmatrix},
$$

and to the positron branch if $\psi$ is

$$
\psi = \begin{pmatrix} 0 \\ v \end{pmatrix}.
$$

We shall consider below the electron branch excitations only, since the positron branch solutions may be obtained by reversing the sign of $\omega$. The electron spectrum is determined as solutions to the following equation:

$$
\omega - M = a(\omega) + (1 - \alpha)b(\omega) + \frac{3 + \alpha}{4}c(\omega).
$$

In the vicinity of a pole the retarded propagator may be presented as

$$
S_R(\omega) \sim Z_E \frac{\psi_E \bar{\psi}_E}{\omega - E + i\delta},
$$

$\omega = E$ is a solution to the dispersion equation, $\psi_E$ is the solution to the Eq.(11) for $\omega = E$ normalized to unity, and the residue $Z_E$ is

$$
\frac{1}{Z_E} = \frac{\partial}{\partial \omega}(\omega - M - \Sigma_1(\omega)) \bigg|_{\omega=E}.
$$

At low temperature corrections to the free propagator are small and the dispersion equation has just one solution, $\omega \approx M$. When the temperature is raised, $\Sigma_1(\omega)$ increases as $T^2$, but for low frequencies $\Sigma_1(\omega)$ varies drastically due to the contribution to the integral of the momentum region corresponding to pole in the integrand (this happens since the poles in the photon and electron propagators in the expression for $\Sigma_1$, Eq.(4) are very close to one another).

In Fig. 1 the dependence of real part of the fermion self-energy on frequency for the electron branch in the Feynman gauge $\alpha = 1$ (i.e. the function $a(\omega) + c(\omega)$) is presented for $T = 3M$ and $5M$. The intersections of $\Re \Sigma_1$ with the straight line $\omega - M$ are solutions to the dispersion equation in the electron sector. One can see that starting with $T \gtrsim 4M$ there
are five solutions in each branch. It follows from the continuity of $\Sigma(\omega)$ as a function of $\omega$ that the additional solutions come by two (the plot of $\Re e \Sigma_1$ at $T = 5M$ in Fig.1 confirms this) and that the residues of two poles in each pair have opposite signs. Actually, the exact location and the residue of the poles depend on the gauge. This follows from the explicit form of the function $\Sigma_1(\omega)$ Eq.(4). Consider the pole closest to zero as an illustration. Its location and residue may be obtained by expanding $\Sigma_1(\omega)$ in series in $\omega$ (the coefficients are calculated in the leading order in $T/M$):

$$\omega = \frac{M + \frac{3+\alpha}{4}c}{1 + (2 - \alpha)q}, \quad Z = 1 + (2 - \alpha)q,$$

(17)

with

$$q = \frac{2g^2}{\pi^2} \frac{T^4}{M^4} \int_0^\infty k^3 dk \left( \frac{1}{e^k - 1} + \frac{1}{e^k + 1} \right) \approx -a'(0) \approx -b'(0),$$

(18)

$$c = \frac{2g^2}{\pi^2} \frac{T^2}{M} \int_0^\infty k dk \left( \frac{1}{e^k - 1} + \frac{1}{e^k + 1} \right) \approx c(0).$$

(19)

i.e. parameters of this excitation do depend on $\alpha$.

Thus we see that the additional excitations arising in the perturbation theory are pathological: their characteristics are gauge–variant and some residues are negative, which implies indefinite metrics. One should believe that when a correct calculational procedure is used the excitations with unacceptable properties would vanish from the physical spectrum.

When the temperature increases excitations behave in a differentl way. Energies of the middle three remain small. This may be easily understood: the above excitations arise due to large oscillations of $\Sigma_1$ in the small frequency region. As we have discussed above, those oscillations are due to large contribution of the momenta corresponding to the pole of the integrand in $\Sigma_1$. It is well-known that at high temperatures the effective mass of OPE becomes much greater than the vacuum one. If one neglects the vacuum mass while calculating $\Sigma_1$, the dispersion equation has two solutions for the electron branch (and two for the positron one), then three middle excitations vanish, and the energies of the remaining modes grow when the temperature is raised and the corresponding poles prove to be gauge-invariant in the leading order in temperature \[2,4\].
Therefore, the following picture may be expected: within a correct calculational procedure the pole in the integrand of $\Sigma$ would be somehow regularized, and the regularization would affect the low frequency region only. At low and medium energies the dispersion equation would have only one solution for the electron and only one for the positron branch. When the temperature grows an additional solution will appear, the same as in the high temperature limit.

### III. ACCOUNT TAKEN OF DAMPING

When calculating the Feynman diagram in quantum field theory or in the real-time approach to the quantum statistics a correct definition (regularization) of propagator poles is necessary. This is achieved by adding infinitesimal imaginary term $i\delta$ to $p_0$ or $p^2$ depending on the propagator type, that makes the propagator free of singularities when $\delta \neq 0$. Thus a natural method of regularization is the use of propagators with nonzero imaginary part (this approach to regularization of the integrand in $\Sigma$ was proposed in [9]). The above propagator arises in the following way. The exact expression for the fermion self-energy is

$$\Sigma(p) = g^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu S(p - k)\Gamma^\nu(p - k, p)D_{\mu\nu}(k),$$

(20)

where $S$, $D_{\mu\nu}$, $\Gamma^\nu$ are the exact fermion and photon propagators and the vertex (within the temperature technique the 4–dimensional integral $\int d^4k$ must be replaced by $-2\pi T \sum_n \int d^3k$). At high temperature, $M \ll |\omega| \ll T$, where HTL approach is valid, the free vertex $\gamma^\nu$ is used as $\Gamma^\nu$, and propagators are free effective propagators, which are functions of effective masses and fermion damping (see below). At low temperature the self-energy must be calculated using the standard perturbation expansion. However, we shall replace the free fermion propagator by free damped fermion propagator, i.e. that containing OPE damping $\gamma$. For the retarded propagator it is equivalent to the substitution $p_0 \rightarrow p_0 + i\gamma$, thus in the temperature technique we have the standard equation with $\omega_n$ replaced by $\omega_n + \gamma$. For $\gamma$ at low temperature we shall take the high temperature expression [9].
\[ \gamma = -\frac{g^2 T}{12} \ln g. \quad (21) \]

As a result, for \( \Sigma(\omega) \) at low temperatures one obtains equations (6) – (9) again, where the substitution \( \omega \rightarrow \omega + i\gamma \) is made (we have neglected the imaginary term \( i\gamma \) in the thermal distribution functions, since the principal contribution to the integral comes from the momenta \( k \sim T \), and \( \gamma \ll k \)).

In Fig. 2 plots of imaginary and real parts of the electron branch of the self-energy calculated in the Feynman gauge at \( T = 5M, 8M \) and \( 10M \) are presented. One can notice that there are no additional solutions at positive frequency. At negative frequencies the real part of the inverse propagator has two zeroes, which would imply existence of two additional solutions. However, the right (closest to \( \omega = 0 \)) solution to the equation \( \Re(\omega - M - \Sigma) = 0 \) is always within the range of large (negative) imaginary part of \( \Sigma(\omega) \), \( \Re E \ll \Im mE \), and thus the physical excitations are absent. The left zero at \( T = 5M \) and \( T = 8M \) (Fig. 2 a, b) is also strongly damped. However, when the temperature is increased the corresponding point moves leftward and at \( T \gtrsim 10M \) escapes the high damping domain (Fig. 2 c). Therefore, the temperature for which an additional (propagating) mode appears is approximately \( 10M \).

At \( T \gg |\omega| \gg M \) the fermion self-energy \( \Sigma \) should be calculated using the effective electron propagator \( S_{\text{eff}} \)

\[
S_{\text{eff}}(p) = -\frac{\gamma^\mu p_\mu}{p_0^2 - p_i^2 - M_{\text{eff}}^2}, \quad p_0 = i\frac{(2n + 1)\pi}{T} + \gamma, \quad M_{\text{eff}}^2 = \frac{g^2 T^2}{4} \quad (22)
\]

and the photon propagator \( D_{\mu\nu}^{\text{eff}}(k) \)

\[
D_{\mu\nu}^{\text{eff}}(k) = \frac{g_{\mu\nu}}{k^2 - \mu_{\text{eff}}^2}, \quad k_0 = i\frac{2n\pi}{T}, \quad \mu_{\text{eff}}^2 = \frac{g^2 T^2}{12}. \quad (23)
\]

In the above temperature range the self-energy in the leading order in \( T \) is

\[ \Sigma(\omega) = \frac{M_{\text{eff}}^2}{2\omega} \quad (24) \]

---

\(^2\) In Ref. [4] the photon propagator was taken in the \( A_0 = 0 \) gauge. We are using the Feynman gauge, since the results are leading order gauge-invariant.
and the dispersion equation looks like

\[ \omega = \frac{M_{\text{eff}}^2}{2\omega} \]  \hspace{1cm} (25)

This provides the following value of OPE mass \( \omega_+ \) (energy at zero momentum)

\[ \omega_+ = \frac{1}{\sqrt{2}} M_{\text{eff}} = \frac{gT}{2\sqrt{2}} \]  \hspace{1cm} (26)

and CE mass \( \omega_- \)

\[ \omega_- = -\frac{1}{\sqrt{2}} M_{\text{eff}} = -\frac{gT}{2\sqrt{2}} \]  \hspace{1cm} (27)

To evaluate the localization of the additional low–frequency excitation localization one may use Eq. (17) with the substitution \( M \to M_{\text{eff}} \sim gT \), which gives

\[ \frac{|\omega|}{M_{\text{eff}}} \lesssim g^2 \quad , \quad |\omega| \lesssim g^3 T \]  \hspace{1cm} (28)

The plot of \( \Im m \Sigma(\omega) \) presented in Fig. 3 demonstrates that the imaginary part of \( \Sigma \) is large in this frequency domain (the fact that insertion of OPE damping into the effective propagator makes \( \Im m \Sigma(0) \sim T \) was mentioned in Ref. [9]).

Let us check whether the results for CE energy at \( T = 10M \) where the effective HTL scheme (a descent from high temperatures to \( T = 10M \)) is consistent with that obtained within the “improved” naive approach (an ascent from low temperatures to \( T = 10M \)). In Fig. 4 the plot of \( \Re e \Sigma(\omega)/M_{\text{eff}} \) (the quantity independent of temperature) calculated within the HTL method, and the straight line \( (\omega - M)/M_{\text{eff}} \) for \( T = 10 \) are presented. (One cannot yet neglect the vacuum mass in the dispersion equation at this temperature since \( M_{\text{eff}} \) and \( M \) are of same order.) It follows from Fig. 2c and Fig. 4 that the values of CE (and OPE) energies are perfectly matched.

Let us emphasize that although the exact propagator has two poles one should take into account only one (OPE pole) when calculating \( \Sigma \), since the residue in CE pole at \( k \sim T \) is exponentially small.
IV. PHYSICAL INTERPRETATION

A satisfactory mechanism giving rise to an additional collective excitation is not revealed yet. In Ref. [5] the appearance of CE was supposed to be a result of interaction between the electron and a sea of electron-positron pairs. We shall show below that in the model describing ”nonrelativistic” electron and photon, when antiparticles are absent, a CE similar to that in quantum electrodynamics still arises.

Consider the Lagrangian

\[ L = \psi^\dagger (i\partial_t - \varepsilon (i\vec{\partial})) \psi + \frac{1}{2} (\partial_t \varphi \partial_t \varphi - \varphi \kappa^2 (i\vec{\partial}) \varphi) + \psi^\dagger \psi f (i\vec{\partial}) \varphi \quad . \]  

(29)

The self-energy \( \Sigma(p) \) of the field \( \psi \) is defined as

\[ S^{-1}(p) = S_0^{-1}(p) + \Sigma(p) \quad , \]  

(30)

where \( S \) and \( S_0 \) are the exact and the free propagators of the field \( \psi \). At the one–loop level the self–energy is

\[ \Sigma_1(p) = \int \frac{d^3 k}{(2\pi)^3} \frac{f^2(k)}{\kappa^2(k) - (p_0 - \varepsilon(p - k))^2} \left[ \varepsilon(p - k) - p_0 \frac{1}{\kappa(k)} - \frac{1}{e^{\beta \kappa(k)} - 1} + \frac{1}{e^{\beta \varepsilon(p-k)} + 1} \right] \]  

(31)

(the vacuum contribution is omitted).

First, suppose that

\[ f(k) \equiv g, \quad \varepsilon(k) = \sqrt{k^2 + M^2} \quad , \quad \kappa(k) = \sqrt{k^2 + \mu^2} \quad . \]  

(32)

In this case Eq.(29) describes a model with relativistic kinematics like electrodynamics, but no antifermions. Let the external momentum \( p \) be zero and consider the frequency range \( M, \mu \ll |\omega| \ll T, \omega \equiv p_0 \). Taking into account the fact that the principal contribution comes from the integration over \( k \sim T \) one obtains in the leading order in \( T \):

\[ \Sigma_1(\omega) = \frac{1}{\omega} \frac{g^2 T^2}{4 \pi^2} \int_0^\infty x dx \left[ \frac{1}{e^x - 1} + \frac{1}{e^x + 1} \right] = \frac{g^2 T^2}{16 \omega} \quad . \]  

(33)

The corresponding dispersion equation may be written as
\[
\omega = \frac{\omega_0^2}{\omega}, \quad \omega_0^2 = \frac{g^2 T^2}{16},
\] (34)

which coincides with the similar equation in QED, Eq.(25).

For negative frequencies the self-energy is a negative quantity (at \( M > \mu \)), in particular,

\[
\Sigma_1(0) = \frac{g^2}{2\pi^2(\mu^2 - M^2)} \int_0^\infty k^2 dk \left[ \frac{\varepsilon(k)}{\kappa(k)} \frac{1}{e^{\beta \kappa(k)} - 1} + \frac{1}{e^{\beta \varepsilon(k)} + 1} \right] \approx \frac{7g^2 T^3 \zeta(3)}{4\pi^2(\mu^2 - M^2)}
\] (35)

(the approximate equality is valid for \( T \gg M, \mu \)). Since \( \Sigma_1(\omega) \) is positive at high positive frequencies (see Eq.(33)) and \(|\Sigma_1(0)|\) grows and becomes greater than \( M \) when the temperature increases, the situation is analogous to QED: at low temperature \( \Sigma_1(\omega) \) is small and only one solution (corresponding to OPE) exists. When the temperature is growing two additional solutions appear, one of them having negative residue in the propagator pole. When the temperature raised further on, one of the additional solutions is shifting to large negative frequencies (and it would leave the damping domain if we take the OPE damping into account when calculating \( \Sigma \)). The frequency of the second solution remains small and has negative residue (the imaginary part of \( \Sigma \) would be large there if we took the damping into account).

Second, consider the electron–phonon interaction model

\[
f^2(k) = g^2 \kappa^2(k) \theta(k_D - k), \quad g^2 = \frac{2\pi^2}{k_D M}, \quad \varepsilon(k) = \frac{k^2}{2M} - \mu, \quad \kappa(k) = ck
\] (36)

where \( M \) is the electron mass, \( c \) is the speed of sound, \( k_D \sim p_F, \mu \sim p_F^2/2M, p_F \) is Fermi momentum. Let the external momentum be \( p = p_F \) and let us examine the frequency range \(|\omega| \gg p_F^2/M\). In this case in the leading order in \( T \) one has

\[
\Sigma_1(\omega) = \frac{p_F^2 T}{M} \frac{1}{\omega}
\]
(37)

and the corresponding dispersion equation is

\[
\omega = \frac{\omega_0^2}{\omega}, \quad \omega_0^2 = \frac{p_F^2}{M} T
\] . (38)

Therefore, starting with \( T \gg p_F^2/M \) the exact propagator of the field \( \psi \) has poles in symmetric points:
\[ \omega = \pm \omega_0 = \pm \sqrt{\frac{p_F^2}{M}} T \quad . \]  

(39)

A typical Fermi energy is few eV, thus a pole in the symmetric point \( \omega = -\omega_0 \) would appear at \( T \gtrsim 10^4 - 10^5 K \). At these temperatures the model considered is apparently senseless, moreover even ”solids” do not exist. But we come to an important conclusion: just in any model, not only relativistic ones, at sufficiently high temperatures a pole in a symmetric point \( \omega = -\omega_0 \) arises. That symmetric pole is a result of the growth of \( \Sigma \) and looks like an increase of the effective coupling constant. The latter may be explained as follows: when the temperature is raised the characteristic momentum grows, \( k \sim T \), and the number of particles participating in the interaction (phase volume) increases drastically.

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FIG. 1. Real part of one–loop fermion self-energy at $T = 3M$ and $T = 5M$ and $\omega - M$ (straight line). Frequency and self-energy are normalized by fermion vacuum mass.
FIG. 2. Imaginary and real parts of \( \Sigma \) at \( T = 5M, 8M \) and \( 10M \) (frequency and self-energy are normalized by fermion vacuum mass).
FIG. 3. $\Im m \Sigma$ as function of frequency ($\omega$ and $\Sigma$ are normalized by $M_{\text{eff}} = gT/2$). Within low frequency range $|\omega|/M_{\text{eff}} < g^2 \sim 0.01$, $\Im m \Sigma / M_{\text{eff}}$ is large, $|\Im m \Sigma|/M_{\text{eff}} \sim 5$.
FIG. 4. $\Re \Sigma$ as function of frequency ($\omega$ and $\Sigma$ are normalized by $M_{eff}$). Left solution $|\omega_-|/M_{eff} \approx 0.43$ or $|\omega_-| = 0.64M$. The latter should be compared with corresponding solution on Fig. 2c $|\omega_-| \approx 0.66M$. 