Attainability of a stationary Navier-Stokes flow around a rigid body rotating from rest

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Abstract. We consider the large time behavior of the three-dimensional Navier-Stokes flow around a rotating rigid body. Assume that the angular velocity of the body gradually increases until it reaches a small terminal one at a certain finite time and it is fixed afterwards. We then show that the fluid motion converges to a steady solution as time $t \to \infty$.

Key Words and Phrases. Navier-Stokes flow, rotating obstacle, attainability, starting problem, steady flow, evolution operator

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1 Introduction

We consider the large time behavior of a viscous incompressible flow around a rotating rigid body in $\mathbb{R}^3$. Assume that both a compact rigid body $\mathcal{O}$ and a viscous incompressible fluid which occupies the outside of $\mathcal{O}$ are initially at rest; then, the body starts to rotate with the angular velocity which gradually increases until it reaches a small terminal one at a certain finite time and it is fixed afterwards. We then show that the fluid motion converges to a steady solution obtained by Galdi [8] as time $t \to \infty$ (Theorem 2.1 in Subsection 2.2). This was conjectured by Hishida [13, Section 6], but it has remained open. Such a question is called the starting problem and it was originally raised by Finn [5], in which rotation was replaced by translation of the body. Finn’s starting problem was first studied by Heywood [11]; since his paper, a stationary solution is said to be attainable if the fluid motion converges to it as $t \to \infty$. Later on, by using Kato’s approach [17] (see also Fujita and Kato [6]) together with the $L^q$-$L^r$ estimates for the Oseen equation established by Kobayashi and Shibata [19], Finn’s starting problem was completely solved by Galdi, Heywood and Shibata [9].

Let us introduce the mathematical formulation. Let $\mathcal{O} \subset \mathbb{R}^3$ be a compact and connected set with non-empty interior. The motion of $\mathcal{O}$ mentioned above is described in terms of the angular velocity

$$\omega(t) = \psi(t)\omega_0, \quad \omega_0 = (0, 0, a)^\top$$ (1.1)
with a constant \( a \in \mathbb{R} \), where \( \psi \) is a function on \( \mathbb{R} \) satisfying the following conditions:

\[
\psi \in C^1(\mathbb{R}; \mathbb{R}), \quad |\psi(t)| \leq 1 \quad \text{for} \quad t \in \mathbb{R}, \quad \psi(t) = 0 \quad \text{for} \quad t \leq 0, \quad \psi(t) = 1 \quad \text{for} \quad t \geq 1. \quad (1.2)
\]

Here and hereafter, \((\cdot)\top\) denotes the transpose. Then the domain occupied by the fluid can be expressed as \( D(t) = \{ y = O(t)x; \; x \in D \} \), where \( D = \mathbb{R}^3 \setminus \mathcal{O} \) is assumed to be an exterior domain with smooth boundary \( \partial D \) and

\[
O(t) = \begin{pmatrix}
\cos \Psi(t) & -\sin \Psi(t) & 0 \\
\sin \Psi(t) & \cos \Psi(t) & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad \Psi(t) = \int_0^t \psi(s)a\,ds.
\]

We consider the initial boundary value problem for the Navier-Stokes equation

\[
\begin{aligned}
\partial_t w + w \cdot \nabla y w &= \Delta y w - \nabla y \pi, \quad y \in D(t), t > 0, \\
\nabla_y \cdot w &= 0, \quad y \in D(t), t \geq 0, \\
w|_{\partial D(t)} &= \psi(t)\omega_0 \times y, \quad t \geq 0, \\
w(y, t) &\to 0 \quad \text{as} |y| \to \infty, \\
w(y, 0) &= 0, \quad y \in D,
\end{aligned}
\]

(1.3)

where \( w = (w_1(y, t), w_2(y, t), w_3(y, t))\top \) and \( \pi = \pi(y, t) \) denote unknown velocity and pressure of the fluid, respectively. To reduce the problem to an equivalent one in the fixed domain \( D \), we take the frame \( x = O(t)\top y \) attached to the body and make the change of the unknown functions: \( u(x, t) = O(t)\top w(y, t), \; p(x, t) = \pi(y, t) \). Then the problem (1.3) is reduced to

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u &= \Delta u + (\psi(t)\omega_0 \times x) \cdot \nabla u - \psi(t)\omega_0 \times u - \nabla p, \quad x \in D, t > 0, \\
\nabla \cdot u &= 0, \quad x \in D, t \geq 0, \\
u|_{\partial D} &= \psi(t)\omega_0 \times x, \quad t \geq 0, \\
u &\to 0 \quad \text{as} \; |x| \to \infty, \\
u(x, 0) &= 0, \quad x \in D.
\end{aligned}
\]

(1.4)

The purpose of this paper is to show that (1.4) admits a global solution which tends to a solution \( u_s \) for the stationary problem

\[
\begin{aligned}
\nabla \cdot u_s &= 0, \quad x \in D, \\
u_s|_{\partial D} &= \omega_0 \times x, \\
u_s &\to 0 \quad \text{as} \; |x| \to \infty.
\end{aligned}
\]

(1.5)
The rate of convergence in $L^r$ with $r \in (3, \infty]$ is also studied. In [8], Galdi successfully proved that if $|\omega_0|$ is sufficiently small, problem (1.5) has a unique smooth solution $(u_s, p_s)$ with pointwise estimates
\[
|u_s(x)| \leq \frac{C|\omega_0|}{|x|}, \quad |\nabla u_s(x)| + |p_s(x)| \leq \frac{C|\omega_0|}{|x|^2}.
\] (1.6)

We note that the decay rate (1.6) is scale-critical, which is also captured in terms of the Lorentz space (weak-Lebesgue space) $L^{3,\infty}$. This was in fact done by Farwig and Hishida [3] even for the external force being in a Lorentz-Sobolev space of order $(-1)$.

Let us mention some difficulties of our problem and how to overcome them in this paper. In [9], the $L^q$-$L^r$ estimates for the Oseen semigroup play an important role. In the rotational case with constant angular velocity, Hishida and Shibata [10] also established the $L^q$-$L^r$ estimates of the semigroup generated by the Stokes operator with the additional term $(\omega_0 \times x) \cdot \nabla - \omega_0 \times$. If we use this semigroup as in [9], we have to treat the term $(\psi(t) - 1)(\omega_0 \times x) \cdot \nabla v$, which is however no longer perturbation from the semigroup on account of the unbounded coefficient $\omega_0 \times x$, where $v = u - \psi(t)u_s$. In this paper, we make use of the evolution operator $\{T(t, s)\}_{t \geq s > 0}$ on the solenoidal space $L^q_\sigma(D)$ ($1 < q < \infty$), which is a solution operator to the linear problem
\[
\begin{cases}
\partial_t u = \Delta u + (\psi(t)\omega_0 \times x) \cdot \nabla u - \psi(t)\omega_0 \times u - \nabla p, & x \in D, t > s, \\
\nabla \cdot u = 0, & x \in D, t \geq s, \\
u|_{\partial D} = 0, & t > s, \\
u \to 0 & \text{as } |x| \to \infty, \\
u(x, s) = f, & x \in D.
\end{cases}
\] (1.7)

Hansel and Rhandi [10] succeeded in the proof of generation of this evolution operator with the $L^q$-$L^r$ smoothing rate. They constructed the evolution operator in their own way since the corresponding semigroup is not analytic (Hishida [12], Farwig and Neustupa [4]). Recently, Hishida [14, 15] developed the $L^q$-$L^r$ decay estimates of the evolution operator, see Section 3. With those estimates, we solve the integral equation which perturbation from the stationary solution $u_s$ obeys. However, it is difficult to perform analysis with the standard Lebesgue space on account of the scale-critical pointwise estimates (1.6).

Thus, we first construct a solution for the weak formulation in the framework of Lorentz space by the strategy due to Yamazaki [23]. We next identify this solution with a local solution possessing better regularity in a neighborhood of each time. The later procedure is actually adopted by Kozono and Yamazaki [20]. Furthermore, we derive the $L^\infty$ decay which is not observed in [9]. When the stationary solution possesses the scale-critical rate $O(1/|x|)$, Koba [18] first derived the $L^\infty$ decay of perturbation with less sharp rate in the context of stability analysis, see also Remark 4.4. Although he used both the $L^1$-$L^r$ estimates of the Oseen semigroup $T(t)$ and the $L^q$-$L^r$ estimates (yielding the $L^q$-$L^\infty$ estimates) of the composite operator $T(t)P_{\text{div}}$, where $P$ denotes the Fujita-Kato projection (see Subsection 2.1), it turns out that either of them is enough to accomplish...
the proof. In this paper, we employ merely the $L^1$-$L^r$ estimates of the adjoint evolution operator $T(t,s)^*$ to simplify the argument.

The paper is organized as follows. In Section 2 we introduce the notation and give the main theorems. Section 3 is devoted to some preliminary results on the stationary problem and the evolution operator. In Section 4 we give the proof of the main theorems.

2 Main theorems

In this section, we first introduce some notation and after that, we give our main theorems.

2.1 Notation

We introduce some function spaces. Let $D \subset \mathbb{R}^3$ be an exterior domain with smooth boundary. By $C_0^\infty(D)$, we denote the set of all $C^\infty$ functions with compact support in $D$. For $1 \leq q \leq \infty$ and nonnegative integer $m$, $L^q(D)$ and $W^{m,q}(D)$ denote the standard Lebesgue and Sobolev spaces, respectively. We write the $L^q$ norm as $\| \cdot \|_q$. The completion of $C_0^\infty(D)$ in $W^{m,q}(D)$ is denoted by $W^{m,q}_0(D)$. Let $1 < q < \infty$ and $1 \leq r \leq \infty$. Then the Lorentz spaces $L^{q,r}(D)$ are defined by

$$L^{q,r}(D) = \{ f : \text{Lebesgue measurable function} \mid \| f \|^*_r < \infty \},$$

where

$$\| f \|^*_r = \begin{cases} \left( \int_0^\infty t^{1/r} \mu(\{ x \in D \mid | f(x) | > t \})^{1/q} \frac{dt}{t} \right)^{\frac{1}{r}} & 1 \leq r < \infty, \\ \sup_{t > 0} t^{1/r} \mu(\{ x \in D \mid | f(x) | > t \})^{1/q} & r = \infty \end{cases}$$

and $\mu(\cdot)$ denotes the Lebesgue measure on $\mathbb{R}^3$. The space $L^{q,r}(D)$ is a quasi-normed space and it is even a Banach space equipped with norm $\| \cdot \|_{q,r}$ equivalent to $\| \cdot \|^*_r$. The real interpolation functor is denoted by $(\cdot, \cdot)_{\theta,r}$, then we have

$$L^{q,r}(D) = (L^{q_0}(D), L^{q_1}(D))_{\theta,r},$$

where $1 \leq q_0 < q < q_1 \leq \infty$ and $0 < \theta < 1$ satisfy $1/q = (1 - \theta)/q_0 + \theta/q_1$, while $1 \leq r \leq \infty$, see Bergh-Löfström [11]. We note that if $1 \leq r < \infty$, the dual of the space $L^{q,r}(D)$ is $L^{q/(q-1),r/(r-1)}(D)$. It is well known that if $1 \leq r < \infty$, the space $C_0^\infty(D)$ is dense in $L^{q,r}(D)$, while the space $C_0^\infty(D)$ is not dense in $L^{q,\infty}(D)$.

We next introduce some solenoidal function spaces. Let $C_0^{\infty,\sigma}(D)$ be the set of all $C_0^\infty$-vector fields $f$ which satisfy $\text{div} f = 0$ in $D$. For $1 < q < \infty$, $L^q_\sigma(D)$ denote the completion of $C_0^{\infty,\sigma}(D)$ in $L^q(D)$. For every $1 < q < \infty$, we have the following Helmholtz decomposition:

$$L^q(D) = L^q_\sigma(D) \oplus \{ \nabla p \in L^q(D) \mid p \in L^q_\text{loc}(\overline{D}) \},$$
see Fujiwara and Morimoto [7], Miyakawa [21], and Simader and Sohr [22]. Let $P_q$ denote the Fujita-Kato projection from $L^q(D)$ onto $L^q_s(D)$ associated with the decomposition. We remark that the adjoint operator of $P_q$ coincides with $P_{q/(q-1)}$. We simply write $P = P_q$.

By real interpolation, it is possible to extend $P$ to a bounded operator on $L^{q,r}(D)$. We then define the solenoidal Lorentz spaces $L^{q,r}_s(D)$ by

$$L^{q,r}_s(D) = PL^{q,r}(D) = (L^q_s(D), L^q_R(D))_{\theta,r},$$

where $1 < q_0 < q < q_1 < \infty$ and $0 < \theta < 1$ satisfy $1/q = (1 - \theta)/q_0 + \theta/q_1$, while $1 \leq r \leq \infty$, see Borchers and Miyakawa [2]. We then have the duality relation $L^{q,r}_s(D)^* = L^{q/(q-1),r/(r-1)}_s(D)$ for $1 < q < \infty$ and $1 \leq r < \infty$. We denote various constants by $C$ and they may change from line to line. The constant dependent on $A, B, \cdots$ is denoted by $C(A, B, \cdots)$. Finally, if there is no confusion, we use the same symbols for denoting spaces of scalar-valued functions and those of vector-valued ones.

### 2.2 Main theorems

It is reasonable to look for a solution to (1.4) of the form

$$u(x, t) = v(x, t) + \psi(t)u_s, \quad p(x, t) = \phi(x, t) + \psi(t)p_s.$$  

Then the perturbation $(v, \phi)$ satisfies the following initial boundary value problem

$$\begin{cases}
\partial_t v = \Delta v + (\psi(t)\omega_0 \times x) \cdot \nabla v - \psi(t)\omega_0 \times v \\
\nabla \cdot v = 0, \quad x \in D, t \geq 0, \\
v|_{\partial D} = 0, \quad t > 0, \\
v \to 0 \quad \text{as } |x| \to \infty, \\
v(x, 0) = 0, \quad x \in D,
\end{cases}$$  

(2.1)

where

$$
(Gv)(x, t) = -v \cdot \nabla v - \psi(t)v \cdot \nabla u_s - \psi(t)u_s \cdot \nabla v, \quad (2.2)
$$

$$
H(x, t) = \psi(t)(\psi(t) - 1)\{-u_s \cdot \nabla u_s - \omega_0 \times u_s + (\omega_0 \times x) \cdot \nabla u_s\} - \psi'(t)u_s. \quad (2.3)
$$

In what follows, we concentrate ourselves on the problem (2.1) instead of (1.4). In fact, if we obtain the solution $v$ of (2.1) which converges to 0 as $t \to \infty$, the solution $u$ of (1.4) converges to $u_s$ as $t \to \infty$. By using the evolution operator $\{T(t, s)\}_{t \geq s \geq 0}$ on $L^q_s(D)$ $(1 < q < \infty)$ associated with (1.7), problem (2.1) is converted into

$$v(t) = \int_0^t T(t, \tau)P[(Gv)(\tau) + H(\tau)]d\tau.$$  

(2.4)

We are now in a position to give our attainability theorem.
Theorem 2.1. Let $\psi$ be a function on $\mathbb{R}$ satisfying $\|\psi(t)\|_q$ and put $\alpha := \max_{t \in \mathbb{R}} |\psi'(t)|$. For $q \in (6, \infty)$, there exists a constant $\delta(q) > 0$ such that if $(\alpha + 1)|a| \leq \delta$, problem (2.3) admits a solution $v$ which possesses the following properties:

(i) $v \in BC_{w^*}((0, \infty); L^3_\sigma(\mathcal{D}))$, $\|v(t)\|_{3, \infty} \to 0$ as $t \to 0$, $\sup_{0 < t < \infty} \|v(t)\|_{3, \infty} \leq C(\alpha + 1)|a|$, where $BC_{w^*}(I; X)$ is the set of bounded and weak-$*$ continuous functions on the interval $I$ with values in $X$, the constant $C$ is independent of $a$ and $\psi$;

(ii) $v \in C((0, \infty); L^r_\sigma(\mathcal{D})) \cap C_{w^*}((0, \infty); L^\infty(\mathcal{D}))$, $\nabla v \in C((0, \infty); L^r(\mathcal{D}))$ for $r \in (3, \infty)$;

(iii) (Decay) $\|v(t)\|_r = O(t^{-\frac{1}{4} + \frac{2}{3}q})$ as $t \to \infty$ for all $r \in (3, q)$,

$$\|v(t)\|_{q, \infty} = O(t^{-\frac{1}{4} + \frac{2}{3}q}) \quad \text{as } t \to \infty,$$

$$\|v(t)\|_r = O(t^{-\frac{1}{4} + \frac{2}{3}q}) \quad \text{as } t \to \infty \quad \text{for all } r \in (q, \infty].$$

Remark 2.2. We can obtain the $L^q$ decay of $v(t)$ like $O(t^{-1/2 + 3/(2q)} \log t)$ as $t \to \infty$, but it is not clear whether $\|v(t)\|_q = O(t^{-1/2 + 3/(2q)})$ holds.

To prove Theorem 2.1, the key step is to construct a solution of the weak formulation

$$(v(t), \varphi) = \int_0^t \left( v(\tau) \otimes v(\tau) + \psi(\tau)\{v(\tau) \otimes u_s + u_s \otimes v(\tau)\}, \nabla T(t, \tau)^* \varphi \right) d\tau$$

$$+ \int_0^t (H(\tau), T(t, \tau)^* \varphi) d\tau, \quad \forall \varphi \in C^\infty_{0, \sigma}(\mathcal{D}) \quad (2.5)$$

as in Yamazaki [23], where $T(t, \tau)^*$ denotes the adjoint of $T(t, \tau)$ and, here and in what follows, $(\cdot, \cdot)^*$ stands for various duality pairings. In this paper, a function $v$ is called a solution of (2.5) if $v \in L^\infty_{loc}([0, \infty); L^3_\sigma(\mathcal{D}))$ satisfies (2.5) for a.e. $t$. By following Yamazaki’s approach, we can easily see that the solution obtained in Theorem 2.1 is unique in the small, see Proposition 1.2 In the following theorem, we give another result on the uniqueness without assuming the smallness of solutions.

Theorem 2.3. Let $q \in (3, \infty)$. Then there exists a constant $\tilde{\delta} > 0$ independent of $q$ and $\psi$ such that if $|a| \leq \tilde{\delta}$, problem (2.5) admits at most one solution within the class

$$\{ v \in L^\infty_{loc}([0, \infty); L^3_\sigma(\mathcal{D})) \cap L^\infty_{loc}(0, \infty; L_q^0(\mathcal{D})) \mid \lim_{t \to 0} \|v(t)\|_{3, \infty} = 0 \}.$$
**Proposition 3.1** ([8]). There exists a constant $\eta \in (0, 1]$ such that if $|\omega_0| = |a| \leq \eta$, the stationary problem [15] admits a unique solution $(u_s, p_s)$ with the estimate
\[
\sup_{x \in D} \{ (1 + |x|)|u_s(x)| \} + \sup_{x \in D} \{ (1 + |x|^2)(|\nabla u_s(x)| + |p_s(x)|) \} \leq C|a|,
\]
where the constant $C$ is independent of $a$.

From now on, we assume that the angular velocity $\omega_0 = (0, 0, a)^\top$ always satisfies $|\omega_0| = |a| \leq \eta$. Proposition 3.1 then yields
\[
u_s \in L^{3,\infty}(D) \cap L^\infty(D), \quad \nabla u_s \in L^{3,\infty}(D) \cap L^\infty(D), \quad |x|\nabla u_s \in L^{3,\infty}(D) \cap L^\infty(D)
\]
and
\[
H(t) \in L^{3,\infty}(D), \quad \|H(t)\|_{3,\infty} \leq C(a^2 + \alpha|a|) \tag{3.1}
\]
for all $t > 0$. Here, $H(t)$ is defined by (2.3) and $\alpha = \sup_{t \in \mathbb{R}} |\psi'(t)|$.

We next collect some results on the evolution operator associated with (1.7). We define the linear operator by
\[
D_0(L(t)) = \{ u \in L^q_0(D) \cap W^{1,q}_0(D) \cap W^{2,q}_0(D) \mid (\omega_0 \times x) \cdot \nabla u \in L^q(D) \}, \quad L(t)u = -P[\nabla u + (\psi(t)\omega_0 \times x) \cdot \nabla u - \psi(t)\omega_0 \times u].
\]
Then the problem (1.7) is formulated as
\[
\partial_t u + L(t)u = 0, \quad t \in (s, \infty); \quad u(s) = f \tag{3.2}
\]
in $L^q_0(D)$. We can see that (1.2) implies
\[
\psi(t)\omega_0 \in C^\theta([0, \infty); \mathbb{R}^3) \cap L^\infty(0, \infty; \mathbb{R}^3) \tag{3.3}
\]
for all $\theta \in (0, 1)$. In fact, we have
\[
\sup_{0 \leq t < s} |\psi(t)\omega_0| = |a|, \quad \sup_{0 \leq s < t < \infty} \frac{|\psi(t)\omega_0 - \psi(s)\omega_0|}{(t-s)^\theta} \leq |a| \max_{t \in \mathbb{R}} |\psi'(t)| \tag{3.4}
\]
for all $\theta \in (0, 1)$. We fix, for instance, $\theta = 1/2$. Under merely the local Hölder continuity of the angular velocity, Hansel and Rhandi [10] proved the following proposition (see also Hishida [15] concerning the assertion 1). Indeed they did not derive the assertion 4, but it directly follows from the real interpolation. For completeness, we give its proof.

**Proposition 3.2** ([10]). Let $1 < q < \infty$. Suppose (1.2). The operator family $\{L(t)\}_{t \geq 0}$ generates a strongly continuous evolution operator $\{T(t,s)\}_{t \geq s \geq 0}$ on $L^q_0(D)$ with the following properties:
1. Let \( q \in (3/2, \infty) \) and \( s \geq 0 \). For every \( f \in Z_q(D) \) and \( t \in (s, \infty) \), we have \( T(t, s)f \in Y_q(D) \) and \( T(\cdot, s)f \in C^1((s, \infty); L^q_s(D)) \) with

\[
\partial_t T(t, s)f + L(t)T(t, s)f = 0, \quad t \in (s, \infty)
\]

in \( L^q_s(D) \), where

\[
Y_q(D) = \{ u \in L^q_s(D) \cap W_0^{1,q}(D) \cap W^{2,q}(D) \mid \| x \| \nabla u \in L^q(D) \},
\]

\[
Z_q(D) = \{ u \in L^q_s(D) \cap W^{1,q}(D) \mid \| x \| \nabla u \in L^q(D) \}.
\]

2. For every \( f \in Y_q(D) \) and \( t > 0 \), we have \( T(t, \cdot)f \in C^1([0, t]; L^q_s(D)) \) with

\[
\partial_s T(t, s)f = T(t, s)L(s)f \quad s \in [0, t]
\]

in \( L^q_s(D) \).

3. Let \( 1 < q \leq r < \infty \), and \( m, \mathcal{T} \in (0, \infty) \). There is a constant \( C = C(q, r, m, \mathcal{T}, D) \) such that

\[
\| \nabla T(t, s)f \|_r \leq C(t - s)^{-\frac{1}{2} \left( \frac{1}{q} - \frac{1}{r} \right) - \frac{1}{2}} \| f \|_q
\]

holds for all \( 0 \leq s < t \leq \mathcal{T} \) and \( f \in L^q_s(D) \) whenever

\[
(1 + \max_{t \in \mathbb{R}} \| \psi(t) \|) \| a \| \leq m
\]

is satisfied.

4. Let \( 1 < q < r < \infty \), \( 1 \leq \rho_1, \rho_2 \leq \infty \) and \( m, \mathcal{T} \in (0, \infty) \). There is a constant \( C = C(q, r, \rho_1, \rho_2, m, \mathcal{T}, D) \) such that

\[
\| \nabla T(t, s)f \|_{r, \rho_2} \leq C(t - s)^{-\frac{1}{2} \left( \frac{1}{q} - \frac{1}{r} \right) - \frac{1}{2}} \| f \|_{q, \rho_1}
\]

holds for all \( 0 \leq s < t \leq \mathcal{T} \) and \( f \in L^{q, \rho_1}(D) \) whenever (3.6) is satisfied.

**Proof of the assertion 4.** We choose \( r_0, r_1 \) such that \( 1 < q < r_0 < r < r_1 < \infty \). From the assertion 3 and the real interpolation, we have

\[
\| \nabla T(t, s)f \|_{r_0, \rho_1} \leq C(t - s)^{-\frac{1}{2} \left( \frac{1}{q} - \frac{1}{r_0} \right) - \frac{1}{2}} \| f \|_{q, \rho_1},
\]

\[
\| \nabla T(t, s)f \|_{r_1, \rho_1} \leq C(t - s)^{-\frac{1}{2} \left( \frac{1}{q} - \frac{1}{r_1} \right) - \frac{1}{2}} \| f \|_{q, \rho_1}.
\]

By the reiteration theorem for real interpolation (see for instance [1, Theorem 3.5.3]), we obtain

\[
L^{r, \rho_2}_\sigma(D) = (L^{r_0, \rho_1}_\sigma(D), L^{r_1, \rho_1}_\sigma(D))_{\beta/\rho_2},\quad \| u \|_{r, \rho_2} \leq C \| u \|_{r_0, \rho_1}^{1-\beta} \| u \|_{r_1, \rho_1}^\beta,\quad \frac{1}{r} = \frac{1}{r_0} + \frac{\beta}{r_1}
\]

which combined with (3.8) and (3.9) concludes (3.7).
We know that the adjoint operator $T(t, s)^*$ is also a strongly continuous evolution operator and satisfies the backward semigroup property

$$T(\tau, s)T(t, \tau)^* = T(t, s)^* \quad (t \geq \tau \geq s \geq 0), \quad T(t, t)^* = I,$$

see Hishida [14] Subsection 2.3. Under the assumption (3.3) with some $\theta \in (0, 1)$, Hishida [14,15] established the following $L^q$-$L^r$ decay estimates. The assertion 3 is not found there but can be proved in the same way as above. We note that the idea of deduction of (3.21) below is due to Yamazaki [23] once we have the assertion 5.

**Proposition 3.3** ([14,15]). Let $m \in (0, \infty)$ and suppose (1.2).

1. Let $1 < q \leq r \leq \infty$ ($q \neq \infty$). Then there exists a constant $C = C(m, q, r, D)$ such that

$$\|T(t, s)f\|_r \leq C(t - s)^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{2})}\|f\|_q \quad (3.11)$$

$$\|T(t, s)^* g\|_r \leq C(t - s)^{-\frac{1}{2}(\frac{1}{r} - \frac{1}{2})}\|g\|_q \quad (3.12)$$

hold for all $t > s \geq 0$ and $f, g \in L_q^r(D)$ whenever (3.6) is satisfied.

2. Let $1 < q \leq r < \infty, 1 \leq \rho \leq \infty$. Then there exists a constant $C = C(m, q, r, \rho, D)$ such that

$$\|T(t, s)f\|_{r, \rho} \leq C(t - s)^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{2})}\|f\|_{q, \rho} \quad (3.13)$$

$$\|T(t, s)^* g\|_{r, \rho} \leq C(t - s)^{-\frac{1}{2}(\frac{1}{r} - \frac{1}{2})}\|g\|_{q, \rho} \quad (3.14)$$

hold for all $t > s \geq 0$ and $f, g \in L_q^r(D)$ whenever (3.6) is satisfied.

3. Let $1 < q < r < \infty, 1 \leq \rho_1, \rho_2 \leq \infty$. Then there exists a constant $C = C(m, q, r, \rho_1, \rho_2, D)$ such that

$$\|T(t, s)f\|_{r, \rho_2} \leq C(t - s)^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{2})}\|f\|_{q, \rho_1} \quad (3.15)$$

$$\|T(t, s)^* g\|_{r, \rho_2} \leq C(t - s)^{-\frac{1}{2}(\frac{1}{r} - \frac{1}{2})}\|g\|_{q, \rho_1} \quad (3.16)$$

hold for all $t > s \geq 0$ and $f, g \in L_q^r(D)$ whenever (3.6) is satisfied.

4. Let $1 < q \leq r \leq 3$. Then there exists a constant $C = C(m, q, r, D)$ such that

$$\|\nabla T(t, s)f||_r \leq C(t - s)^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{2}) - \frac{1}{2}}\|f\|_q \quad (3.17)$$

$$\|\nabla T(t, s)^* g||_r \leq C(t - s)^{-\frac{1}{2}(\frac{1}{r} - \frac{1}{2}) - \frac{1}{2}}\|g\|_q \quad (3.18)$$

hold for all $t > s \geq 0$ and $f, g \in L_q^r(D)$ whenever (3.6) is satisfied.

5. Let $1 < q \leq r \leq 3, 1 \leq \rho < \infty$. Then there exists a constant $C = C(m, q, r, \rho, D)$ such that

$$\|\nabla T(t, s)f||_{r, \rho} \leq C(t - s)^{-\frac{1}{2}(\frac{1}{q} - \frac{1}{2}) - \frac{1}{2}}\|f\|_{q, \rho} \quad (3.19)$$

$$\|\nabla T(t, s)^* g||_{r, \rho} \leq C(t - s)^{-\frac{1}{2}(\frac{1}{r} - \frac{1}{2}) - \frac{1}{2}}\|g\|_{q, \rho} \quad (3.20)$$

hold for all $t > s \geq 0$ and $f, g \in L_q^r(D)$ whenever (3.6) is satisfied.
6. Let $1 < q \leq r \leq 3$ with $1/q - 1/r = 1/3$. Then there exists a constant $C = C(m, q, D)$ such that
\[
\int_0^t \|\nabla T(t, s)^*g\|_{r, 1} \, ds \leq C\|g\|_{q, 1} \tag{3.21}
\]
holds for all $t > 0$ and $g \in L^{q, 1}_r(D)$ whenever (3.6) is satisfied.

To prove the $L^\infty$ decay estimate in Theorem 2.1, we also prepare the following $L^1$-$L^r$ estimates. The following estimates for data being in $C_0^\infty(D)^3$ are enough for later use, but it is clear that, for instance, the composite operator $T(t, s)P$ extends to a bounded operator from $L^1(D)$ to $L^{r}_r(D)$ with the same estimate.

**Lemma 3.4.** Let $m \in (0, \infty)$ and suppose (1.2).

1. Let $1 < r < \infty$. Then there is a constant $C = C(m, r, D) > 0$ such that
\[
\|T(t, s)Pf\|_{r, \rho} \leq C(t - s)^{-\frac{2}{3}(1-\frac{1}{r})}\|f\|_1, \tag{3.22}
\]
\[
\|T(t, s)^*Pg\|_{r, \rho} \leq C(t - s)^{-\frac{2}{3}(1-\frac{1}{r})}\|g\|_1 \tag{3.23}
\]
for all $t > s \geq 0$ and $f, g \in C_0^\infty(D)^3$ whenever (3.6) is satisfied.

2. Let $1 < r < \infty$ and $1 \leq \rho \leq \infty$. Then there is a constant $C = C(m, r, \rho, D) > 0$ such that
\[
\|T(t, s)Pf\|_{r, \rho} \leq C(t - s)^{-\frac{2}{3}(1-\frac{1}{r})}\|f\|_1, \tag{3.24}
\]
\[
\|T(t, s)^*Pg\|_{r, \rho} \leq C(t - s)^{-\frac{2}{3}(1-\frac{1}{r})}\|g\|_1 \tag{3.25}
\]
for all $t > s \geq 0$ and $f, g \in C_0^\infty(D)^3$ whenever (3.6) is satisfied.

3. Let $1 < r \leq 3$, $1 \leq \rho < \infty$. Then there is a constant $C = C(m, r, \rho, D) > 0$ such that
\[
\|\nabla T(t, s)Pf\|_{r, \rho} \leq C(t - s)^{-\frac{2}{3}(1-\frac{1}{r})-\frac{1}{2}}\|f\|_1, \tag{3.26}
\]
\[
\|\nabla T(t, s)^*Pg\|_{r, \rho} \leq C(t - s)^{-\frac{2}{3}(1-\frac{1}{r})-\frac{1}{2}}\|g\|_1 \tag{3.27}
\]
for all $t > s \geq 0$ and $f, g \in C_0^\infty(D)^3$ whenever (3.6) is satisfied.

**Proof.** The proof is simply based on duality argument (see Koba [13] Lemma 2.15), however, we give it for completeness. Let $1 < q \leq r < \infty$ and $1/r + 1/r' = 1$. By using (3.12), we see that
\[
|(T(t, s)Pf, \varphi)| = |(f, T(t, s)^*\varphi)| \leq \|f\|_1\|T(t, s)^*\varphi\|_\infty \leq C(t - s)^{-\frac{2}{3}(1-\frac{1}{r})}\|f\|_1\|\varphi\|_{r'}, \tag{3.28}
\]
for all $\varphi \in L^{r'}_q(D)$, which implies (3.22). We next show (3.24). We fix $q$ such that $1 < q < r$. Combining the estimate (3.15) with (3.22), we have
\[
\|T(t, s)Pf\|_{r, \rho} \leq C(t - s)^{-\frac{2}{3}(1-\frac{1}{r})}\left\|T\left(\frac{t+s}{2}, s\right)Pf\right\|_q \leq C(t - s)^{-\frac{2}{3}(1-\frac{1}{r})}\|f\|_1.
\]
Finally, in view of (3.19) and (3.21), we have
\[ \| \nabla T(t,s)Pf \|_{r,\rho} \leq C(t-s)^{-\frac{1}{2}} \left\| T \left( \frac{t+s}{2},s \right) Pf \right\|_{r,\rho} \leq C(t-s)^{-\frac{1}{2}(1-\frac{1}{q})-\frac{1}{2}} \| f \|_1 \]
which implies (3.26). The proof for the adjoint \( T(t,s)^* \) is accomplished in the same way. \( \square \)

4 Proof of the main theorems

In this section we prove the main theorems (Theorem 2.1 and Theorem 2.3). We first give some key estimates and then show Theorem 2.3. After that, following Yamazaki [23], we construct a solution with some decay properties for (2.5) and then derive the \( L^\infty \) decay of the solution. We finally identify the solution above with a local solution possessing better regularity for the integral equation (2.4) in a neighborhood of each time \( t > 0 \).

Let us define the function spaces
\[ X = \{ v \in BC_w^q((0, \infty); L^\infty_\sigma(D)) \mid \lim_{t \to 0} \| v(t) \|_{3,\infty} = 0 \}, \]
\[ X_q = \{ v \in X \mid t^{\frac{1}{2}-\frac{3}{2q}} v \in BC_w^q((0, \infty); L^\infty_\sigma(D)) \}, \quad 3 < q < \infty. \]
Both are Banach spaces endowed with norms \( \| \cdot \|_X = \| \cdot \|_{X,\infty} \) and \( \| \cdot \|_{X_q} = \| \cdot \|_{X_q,\infty} \), respectively, where
\[ \| v \|_{X,t} = \sup_{0<\tau<t} \| v(\tau) \|_{3,\infty}, \quad \| v \|_{X_q,t} = \| v \|_{X,t} + \| v \|_{q,t}, \quad \| v \|_{q,t} = \sup_{0<\tau<t} \left( \frac{1}{\tau^{\frac{1}{2}-\frac{3}{2q}} \| v(\tau) \|_{q,\infty}} \right) \]
for \( t \in (0, \infty] \).

Lemma 4.1. 1. Let \( v, w \in X \) and set
\[ \langle I(v,w)(t), \varphi \rangle := \int_0^t (v(\tau) \otimes w(\tau), \nabla T(t,\tau)^* \varphi) d\tau, \]
\[ \langle J(v)(t), \varphi \rangle := \int_0^t (\psi(\tau)\{v(\tau) \otimes u_s + u_s \otimes v(\tau)\}, \nabla T(t,\tau)^* \varphi) d\tau \]
for all \( \varphi \in C^\infty_0(D) \). Then \( I(v,w), J(v) \in X \) and there exists a positive constant \( C \) such that
\[ \| I(v,w) \|_{X,t} \leq C \| v \|_{X,t} \| w \|_{X,t}, \quad \| J(v) \|_{X,t} \leq C \| u_s \|_{3,\infty} \| v \|_{X,t} \]
hold for any \( v, w \in X \) and \( t \in (0, \infty] \).

2. Let \( q \in (3, \infty) \). If \( v \in X_q, w \in X \), then \( I(v,w), J(v) \in X_q \) and there exists a positive constant \( C = C(q) \) such that
\[ \| I(v,w) \|_{X_q,t} \leq C \| v \|_{X_q,t} \| w \|_{X,t}, \quad \| J(v) \|_{X_q,t} \leq C \| u_s \|_{3,\infty} \| v \|_{X_q,t} \]
hold for every \( v \in X_q, w \in X \) and \( t \in (0, \infty] \).
3. We set

$$\langle K(t), \varphi \rangle := \int_0^t (H(\tau), T(t, \tau)^* \varphi) \, d\tau$$

for $\varphi \in C_{0,\sigma}^\infty(D)$. Let $q \in (3, \infty)$. Then $K \in X_q$ and there exist positive constants $C$ independent of $q$ and $C' = C'(q)$ such that

$$\|K\|_{X, t} \leq C(a^2 + \alpha |a|), \quad \|K\|_{X_q, t} \leq C'(a^2 + \alpha |a|) \quad (4.3)$$

hold for every $t \in (0, \infty)$.

Proof. Estimates (4.1) and (4.2) can be proved in the same way as done by Yamazaki [23, Lemma 6.1.], see also Hishida and Shibata [16, Section 8], however, we briefly give the proof of (4.1) and (4.2). By (3.21), we have

$$| \langle I(v, w)(t), \varphi \rangle | \leq \|v\|_{X,t} \|w\|_{X,t} \int_0^t \|\nabla T(t, \tau)^* \varphi\|_{3,1} \leq C \|v\|_{X,t} \|w\|_{X,t} \|\varphi\|_{\frac{3}{2},1},$$

which yields (4.1). We choose $r$ such that $1/3 + 1/q + 1/r = 1$ to find

$$| \langle I(v, w)(t), \varphi \rangle | \leq [v]_{X,q} \|w\|_{X,t} \int_0^t \tau^{-\frac{1}{2} + \frac{3q}{2q}} \|\nabla T(t, \tau)^* \varphi\|_{r,1} \, d\tau = [v]_{q} \|w\|_{X,t} \left( \int_0^t + \int_t^t \right);$$

In view of (3.20), we have

$$\int_0^\frac{t}{2} \leq C \int_0^\frac{t}{2} \tau^{-\frac{1}{2} + \frac{3q}{2q}} (t - \tau)^{-1} \, d\tau \|\varphi\|_{q',1} \leq Ct^{-\frac{1}{2} + \frac{3q}{2q}} \|\varphi\|_{q',1},$$

where $1/q + 1/q' = 1$, whereas, (3.21) implies

$$\int_\frac{t}{2}^t \leq \left( \frac{t}{2} \right)^{-\frac{1}{2} + \frac{3q}{2q}} \int_0^t \|\nabla T(t, \tau)^* \varphi\|_{r,1} \, d\tau \leq Ct^{-\frac{1}{2} + \frac{3q}{2q}} \|\varphi\|_{q',1}$$

from which together with (4.1), we obtain (4.2). The estimate (4.1) leads us to

$$\lim_{t \to 0} \|I(v, w)(t)\|_{3,\infty} = 0, \quad \lim_{t \to 0} \|J(v)(t)\|_{3,\infty} = 0.$$
for all $0 < t < \infty$ and $\varphi \in C_{0,\sigma}^{\infty}(D)$. Let $0 < \sigma < t$. By using the backward semigroup property, we have

$$\left| \langle I(v, w)(t) - I(v, w)(\sigma), \varphi \rangle \right| \leq \int_0^\sigma \left| (v(\tau) \otimes w(\tau), \nabla T(\sigma, \tau)^* (T(t, \tau)^* \varphi - \varphi) \right| d\tau$$

$$+ \int_\sigma^t \left| (v(\tau) \otimes w(\tau), \nabla T(t, \tau)^* \varphi) \right| d\tau =: I_1 + I_2.$$  

The estimate (3.21) yields

$$I_1 \leq \|v\|_{X,t} \|w\|_{X,t} \int_0^\sigma \|\nabla T(\sigma, \tau)^* (T(t, \tau)^* \varphi - \varphi)\|_{3,1} d\tau$$

$$\leq C \|v\|_{X,t} \|w\|_{X,t} \|T(t, \sigma)^* \varphi - \varphi\|_{\frac{3}{2},1} \to 0 \text{ as } \sigma \to t.$$  

Furthermore, (3.20) yields

$$I_2 \leq \|v\|_{X,t} \|w\|_{X,t} \int_\sigma^t \|\nabla T(t, \tau)^* \varphi\|_{3,1} d\tau \leq C \|v\|_{X,t} \|w\|_{X,t} (t - \sigma)^{\frac{1}{2}} \|\varphi\|_{3,1} \to 0 \text{ as } \sigma \to t.$$  

We can discuss the other case $0 < t < \sigma$ similarly and thus we obtain (4.4). By the same manner, we can obtain the desired weak-\(*\) continuity of $J$. We thus conclude the assertion 1 and 2.

We next consider $K(t)$. We use (3.14) as well as (3.1) to obtain

$$\left| \langle K(t), \varphi \rangle \right| \leq C(a^2 + \alpha|a|) \int_0^{\min(1,t)} \|T(t, \tau)^* \varphi\|_{\frac{3}{2},1} d\tau \leq C(a^2 + \alpha|a|) \min\{1, t\} \|\varphi\|_{\frac{3}{2},1}$$

for $\varphi \in C_{0,\sigma}^{\infty}(D)$ and $t > 0$ which yields $K(t) \in L^3_{\sigma,\infty}(D)$ with

$$\|K\|_{X,t} \leq C(a^2 + \alpha|a|) \text{ for } t \in (0, \infty], \lim_{t \to 0} \|K(t)\|_{3,\infty} = 0.$$  

To derive the estimate $[K]_{q,t} \leq C(a^2 + \alpha|a|)$, we consider two cases: $0 < t \leq 2$ and $t \geq 2$. For $0 < t \leq 2$, (3.14) yields

$$\left| \langle K(t), \varphi \rangle \right| \leq C(a^2 + \alpha|a|) \int_0^t \|T(t, \tau)^* \varphi\|_{\frac{3}{2},1} d\tau \leq C(a^2 + \alpha|a|) \|\varphi\|_{q',1}.$$  

For $t \geq 2$, we have

$$\left| \langle K(t), \varphi \rangle \right| \leq C(a^2 + \alpha|a|) \int_0^1 \|T(t, \tau)^* \varphi\|_{\frac{3}{2},1} d\tau \leq C(a^2 + \alpha|a|) t^{-\frac{1}{2} + \frac{3}{2q}} \|\varphi\|_{q',1}.$$  

We thus obtain (4.3). It remains to show the weak-\(*\) continuity. To this end, it is sufficient to show that

$$\left| \langle K(t) - K(\sigma), \varphi \rangle \right| \to 0 \text{ as } \sigma \to t \quad (4.5)$$
for all \( t \in (0, \infty) \) due to (1.3). To prove (1.5), we suppose \( 0 < \sigma < t \). We use the backward semigroup property to observe

\[
\langle K(t) - K(\sigma), \varphi \rangle = \int_0^\sigma (H(\tau), T(\sigma, \tau)^* (T(t, \sigma)^* \varphi - \varphi)) \, d\tau + \int_\sigma^t (H(\tau), T(t, \tau)^* \varphi) \, d\tau.
\]

By applying (3.14), we find that

\[
\| \varphi \|_{\sigma} \leq C(a^2 + |\alpha| |a|) \| T(t, \sigma)^* \varphi - \varphi \|_{\frac{1}{2},1} \rightarrow 0 \quad \text{as } \sigma \rightarrow t,
\]

\[
\| \varphi \|_{t} \leq C(a^2 + |\alpha| t(\sigma - \tau)) \| \varphi \|_{\frac{1}{2},1} \rightarrow 0 \quad \text{as } \sigma \rightarrow t.
\]

The other case \( t < \sigma \) is discussed similarly. Hence we have (4.5). The proof is complete. \( \square \)

**Proof of Theorem 2.3.** The idea of the proof is traced back to Fujita and Kato [6, Theorem 3.1.]. Let \( v_1 \) and \( v_2 \) be the solutions of (2.5). Then we have

\[
(v_1(t) - v_2(t), \varphi) = \int_0^t \left( (v_1(\tau) \otimes (v_1(\tau) - v_2(\tau)) + (v_1(\tau) - v_2(\tau)) \otimes v_2(\tau) + \psi(\tau)(v_1(\tau) - v_2(\tau)) \otimes u_s + \psi(\tau)u_s \otimes (v_1(\tau) - v_2(\tau)) \right), \nabla T(t, \tau)^* \varphi \, d\tau
\]

for \( \varphi \in C^{\infty}_{0,\sigma}(D) \). By applying (4.11) to (4.6) and by Proposition 3.1, we have

\[
\| v_1 - v_2 \|_{X_t} \leq C(\| v_1 \|_{X_{t_0}} + \| v_2 \|_{X_{t_0}} + \| u_s \|_{3,\infty}) \| v_1 - v_2 \|_{X_t}.
\]

Suppose

\[
|a| < \frac{1}{2C} =: \tilde{\delta}.
\]

Since \( \| v_j(t) \|_{3,\infty} \rightarrow 0 \) as \( t \rightarrow 0 \) \((j = 1, 2)\), one can choose \( t_0 > 0 \) such that \( C(\| v_1 \|_{X_{t_0}} + \| v_2 \|_{X_{t_0}}) < 1/2 \), which implies \( v_1 = v_2 \) on \((0, t_0]\). Hence, (4.6) is written as

\[
(v_1(t) - v_2(t), \varphi) = \int_{t_0}^t \left( (v_1(\tau) \otimes (v_1(\tau) - v_2(\tau)) + (v_1(\tau) - v_2(\tau)) \otimes v_2(\tau) + \psi(\tau)(v_1(\tau) - v_2(\tau)) \otimes u_s + \psi(\tau)u_s \otimes (v_1(\tau) - v_2(\tau)) \right), \nabla T(t, \tau)^* \varphi \, d\tau.
\]

We fix \( T \in (t_0, \infty) \) and set \([v]_{T_0,T} = \sup_{t_0 \leq \tau \leq T} \| v(\tau) \|_q \) for \( t \in (t_0, T) \). It follows from (4.7) that

\[
[v_1 - v_2]_{T_0,T} \leq C_\ast (t - t_0)^{\frac{1}{2} - \frac{1}{q}} [v_1 - v_2]_{T_0,T}, \quad t \in (t_0, T),
\]

(4.8)
Lemma 4.1 implies that \( \Phi \) yields

\[
\int_{t_0}^{t} \left| (v_1(\tau) \otimes (v_1(\tau) - v_2(\tau)), \nabla T(t, \tau)^* \varphi) \right| d\tau \\
\leq C[v_1(\tau)_{q,t_0,\tau}] [v_1 - v_2]_{q,t_0,t} \int_{t_0}^{t} \| \nabla T(t, \tau)^* \varphi \| (1 - \frac{\tau}{q})^{-1} d\tau \\
\leq C[v_1(\tau)_{q,t_0,\tau}] [v_1 - v_2]_{q,t_0,t}(t - t_0)^{\frac{1}{2} - \frac{q}{4}} \| \varphi \| (1 - \frac{\tau}{q})^{-1}
\]

for all \( \varphi \in C_{0,\sigma}^\infty (D) \) and \( t \in (t_0, T) \). Since the other terms in (4.7) are treated similarly, we obtain (4.8). We take

\[
\xi = \min \left\{ \left( \frac{1}{2C_*} \right)^{(\frac{1}{2} - \frac{q}{4})^{-1}}, T - t_0 \right\}
\]

which leads us to \( v_1 = v_2 \) on \( (0, t_0 + \xi) \). Even though we replace \( t_0 \) by \( t_0 + \xi, t_0 + 2\xi, \ldots \), we can discuss similarly. Hence, \( v_1 = v_2 \) on \( (0, T) \). Since \( T \) is arbitrary, we conclude \( v_1 = v_2 \).

To prove Theorem 2.1 we begin to construct a solution of (2.5) by applying Lemma 4.1.

**Proposition 4.2.** Let \( \psi \) be a function on \( \mathbb{R} \) satisfying (1.2). We put \( \alpha = \max_{t \in \mathbb{R}} |\psi'(t)| \).

1. There exists \( \delta_1 > 0 \) such that if \((\alpha + 1)|a| \leq \delta_1\), problem (2.5) admits a unique solution within the class

\[
\left\{ v \in BC_{w^*}((0, \infty); L_{3,\sigma}^{3,\infty}(D)) \mid \lim_{t \to 0} \| v(t) \|_{3,\infty} = 0, \sup_{0 < \tau < \infty} \| v(\tau) \|_{3,\infty} \leq C(\alpha + 1)|a| \right\},
\]

where \( C > 0 \) is independent of \( a \) and \( \psi \).

2. Let \( 3 < q < \infty \). Then there exists \( \delta_2(q) \in (0, \delta_1] \) such that if \((\alpha + 1)|a| \leq \delta_2\),

\[
t^{\frac{3}{2} - \frac{q}{4}} v \in BC_{w^*}((0, \infty); L_{q,\sigma}^{q,\infty}(D)),
\]

where \( v(t) \) is the solution obtained above.

**Proof.** We first show the assertion 1 by the contraction mapping principle. Given \( v \in X \), we define

\[
\langle (\Phi v)(t), \varphi \rangle = \text{the RHS of (2.5)}, \quad \varphi \in C_{0,\sigma}^\infty (D).
\]

Lemma 4.1 implies that \( \Phi v \in X \) with

\[
\| \Phi v \|_X \leq C_1 \| v \|^2_X + C_2 |a| \| v \|_X + C_3 (a^2 + \alpha |a|), \quad (4.9)
\]

\[
\| \Phi v - \Phi w \|_X \leq (C_1 \| v \|_X + C_1 |w| \| w \|_X + C_2 |a|) \| v - w \|_X \quad (4.10)
\]
for every \( v, w \in X \). Here, \( C_1, C_2, C_3, C_4 \) are constants independent of \( v, w, a \) and \( \psi \). Hence, if we take \( a \) satisfying

\[
\frac{1}{2} (\alpha + 1)|a| < \min \left\{ \frac{1}{2C_2}, \frac{1}{16C_1C_3}, \eta \right\} =: \delta_1,
\]

where \( \eta \in (0, 1] \) is a constant given in Proposition 3.1, then we obtain a unique solution \( v \) within the class

\[
\{ v \in X \mid \| v \| \leq 4C_3(\alpha + 1)|a| \}
\]

which completes the proof of the assertion 1.

We next show the assertion 2. By applying Lemma 4.1 we see that \( \Phi v \in X_q \) together with (4.9)–(4.10) in which \( X \) norm was replaced by \( X_q \) norm and the constants \( C_i \) \((i = 1, 2, 3)\) are also replaced by some others \( \tilde{C}_i(q)(\geq C_i) \). If we assume

\[
(\alpha + 1)|a| < \min \left\{ \frac{1}{2\tilde{C}_2}, \frac{1}{16\tilde{C}_1\tilde{C}_3}, \eta \right\} =: \delta_2 \leq \delta_1,
\]

we can obtain a unique solution \( \hat{v} \) within the class

\[
\{ v \in X_q \mid \| v \|_q \leq 4\tilde{C}_3(\alpha + 1)|a| \}.
\]

Under the condition (4.11), let \( v \) be the solution obtained in the assertion 1. Then we have (4.7) in which \( v_1, v_2 \) are replaced by \( v \) and \( \hat{v} \). By applying (4.11), we see that

\[
\| v - \hat{v} \|_X \leq \{ C_1(\| v \|_X + \| \hat{v} \|_X) + C_2|a| \}\| v - \hat{v} \|_X \leq \{ 8\tilde{C}_1\tilde{C}_3(1 + \alpha)|a| + \tilde{C}_2|a| \}\| v - \hat{v} \|_X.
\]

Furthermore, the condition (4.11) yields

\[
8\tilde{C}_1\tilde{C}_3(1 + \alpha)|a| + \tilde{C}_2|a| < 1
\]

which leads us to \( v = \hat{v} \). The proof is complete.

We note that Proposition 4.2 implies

\[
t^{\frac{1}{2} - \frac{3}{4q}} v \in BC_w((0, \infty); L^r(D))
\]

for all \( r \in (3, q) \) by the interpolation inequality

\[
\| f \|_r \leq C\| f \|_{3, \infty}^{1 - \beta} \| f \|_{q, \infty}^\beta, \quad \frac{1}{r} = \frac{1 - \beta}{3} + \frac{\beta}{q},
\]

see (3.10).

Let \( q \in (6, \infty) \), then the solution obtained in Proposition 4.2 also fulfills the following decay properties.
Proposition 4.3. Let \( \psi \) be a function on \( \mathbb{R} \) satisfying (1.2) and we put \( \alpha := \max_{t \in \mathbb{R}} |\psi'(t)| \).
Suppose that \( 6 < q < \infty \). Then, under the same condition as in the latter part of Proposition 1.2, the solution \( v \) obtained in Proposition 4.2 satisfies \( v(t) \in L^r(D) \) \( (t > 0) \) with

\[
\|v(t)\|_r = O(t^{-\frac{1}{2} + \frac{\alpha}{q}}) \quad \text{as } t \to \infty
\]

for \( r \in (q, \infty) \).

Proof. We first show (4.13) with \( r = \infty \), that is, \( v(t) \in L^\infty(D) \) for \( t > 0 \) with

\[
\|v(t)\|_\infty = O(t^{-\frac{1}{2} + \frac{\alpha}{q}}) \quad \text{as } t \to \infty.
\]

We note by continuity that \( C_{0,\alpha}^\infty(D) \) can be replaced by \( PC_{0,\alpha}^\infty(D) \) as the class of test functions in (2.5). Hence, it follows that

\[
\sup_{\varphi \in C_{0,\alpha}^\infty(D), \|\varphi\|_1 \leq 1} |(v(t), \varphi)| \leq N_1 + N_2 + N_3 + N_4,
\]

where

\[
N_1 = \sup_{\varphi \in C_{0,\alpha}^\infty(D), \|\varphi\|_1 \leq 1} \int_0^t \|v(\tau) \otimes v(\tau), \nabla T(t, \tau)^* P\varphi\| \, d\tau,
\]

\[
N_2 = \sup_{\varphi \in C_{0,\alpha}^\infty(D), \|\varphi\|_1 \leq 1} \int_0^t \|v(\tau) \otimes u_s, \nabla T(t, \tau)^* P\varphi\| \, d\tau,
\]

\[
N_3 = \sup_{\varphi \in C_{0,\alpha}^\infty(D), \|\varphi\|_1 \leq 1} \int_0^t \|u_s \otimes v(\tau), \nabla T(t, \tau)^* P\varphi\| \, d\tau,
\]

\[
N_4 = \sup_{\varphi \in C_{0,\alpha}^\infty(D), \|\varphi\|_1 \leq 1} \int_0^t |(H(\tau), T(t, \tau)^* P\varphi)| \, d\tau.
\]

We begin by considering \( N_1 \). In view of (3.27), we have

\[
\int_0^t \|v(\tau) \otimes v(\tau), \nabla T(t, \tau)^* P\varphi\| \, d\tau \leq C[v]_{q,\infty}^2 \int_0^t \tau^{-1 + \frac{3}{q}} \|\nabla T(t, \tau)^* P\varphi\|_1 \, d\tau
\]

\[
\leq C[v]_{q,\infty}^2 \int_0^t \tau^{-1 + \frac{3}{q}} (t - \tau)^{-\frac{3}{4} - \frac{1}{2}} \, d\tau \leq C[v]_{q,\infty}^2 t^{-\frac{3}{2}} \|\varphi\|_1
\]

for all \( \varphi \in C_{0,\alpha}^\infty(D) \) and \( t > 0 \). Here, the integrability is ensured because of \( q \in (6, \infty) \). Hence we obtain

\[
N_1 \leq C t^{-\frac{3}{2}} \quad \text{for } t > 0.
\]

We next consider \( N_2 \). By applying (3.27), it follows that

\[
\int_0^t \|v(\tau) \otimes u_s, \nabla T(t, \tau)^* P\varphi\| \, d\tau \leq [v]_{q,\infty} \|u_s\|_{q,\infty} \int_0^t \tau^{-\frac{1}{2} + \frac{3}{q}} \|\nabla T(t, \tau)^* P\varphi\|_1 \, d\tau
\]

\[
\leq C[v]_{q,\infty} \|u_s\|_{q,\infty} t^{-\frac{3}{2}} \|\varphi\|_1
\]

\[17\]
for $t > 0$. We thus have

$$N_2 \leq Ct^{-\frac{3}{2q}} \quad \text{for } t > 0.$$  \hfill (4.17)

We next intend to derive the rate of decay $N_2$ as fast as possible. To this end, we split the integral into

$$\int_0^t |(v(\tau) \otimes u_s, \nabla T(t, \tau)^* P \varphi)| \, d\tau = \int_0^\frac{t}{2} + \int_{\frac{t}{2}}^{t-1} + \int_{t-1}^t$$

for $t > 2$. We apply (3.27) again to find

$$\int_0^\frac{t}{2} \leq \|u_s\|_{3, \infty} \|v\|_X \int_0^{\frac{t}{2}} \|\nabla T(t, \tau)^* P \varphi\|_{3, 1} \, d\tau \leq Ct^{-\frac{3}{2}} \|\varphi\|_1,$$

$$\int_{\frac{t}{2}}^{t-1} \leq \|u_s\|_{3, \infty} |v|_{q, \infty} \int_{\frac{t}{2}}^{t-1} \tau^{-\frac{3}{2} + \frac{3}{2q}} \|\nabla T(t, \tau)^* P \varphi\|_{(1-\frac{1}{q})^{-1}, 1} \, d\tau \leq Ct^{-\frac{1}{2} + \frac{3}{2q}} \|\varphi\|_1$$

and

$$\int_{t-1}^t \leq \|u_s\|_{q, \infty} |v|_{q, \infty} \int_{t-1}^t \tau^{-\frac{3}{2} + \frac{3}{2q}} \|\nabla T(t, \tau)^* P \varphi\|_{(1-\frac{1}{q})^{-1}, 1} \, d\tau \leq Ct^{-\frac{1}{2} + \frac{3}{2q}} \|\varphi\|_1$$

for all $\varphi \in C_0^\infty(D)$ and $t > 2$. Summing up the estimates above, we are led to

$$N_2 \leq Ct^{-\frac{1}{2} + \frac{3}{2q}} \quad \text{for } t > 2.$$  \hfill (4.19)

Similarly, we have

$$N_3 \leq Ct^{-\frac{1}{2} + \frac{3}{2q}} \quad \text{for } t > 0,$$  \hfill (4.20)

$$N_3 \leq Ct^{-\frac{1}{2} + \frac{3}{2q}} \quad \text{for } t > 2.$$  \hfill (4.21)

It is easily seen from (1.2), (3.1) and (3.25) that

$$\int_0^t |(H(\tau), T(t, \tau)^* P \varphi)| \, d\tau \leq C(a^2 + \alpha |a|) \int_0^{\min\{1, t\}} \|T(t, \tau)^* P \varphi\|_{\frac{3}{2}, 1} \, d\tau \leq C(a^2 + \alpha |a|) \int_0^{\min\{1, t\}} (t - \tau)^{-\frac{1}{2}} \, d\tau \|\varphi\|_1$$

for all $\varphi \in C_0^\infty(D)$ and $t > 0$, which yields

$$N_4 \leq Ct^{\frac{3}{2}} \quad \text{for } t > 0,$$  \hfill (4.22)

$$N_4 \leq Ct^{-\frac{1}{2}} \quad \text{for } t > 2.$$  \hfill (4.23)

Combining (4.15)–(4.23) implies $v(t) \in L^\infty(D) \ (t > 0)$ and (4.14). In view of the interpolation relation

$$(L^{q, \infty}(D), L^\infty(D))_{1-\frac{2}{r}, r} = L^r(D), \quad \|f\|_r \leq C \|f\|_{q, \infty} \|f\|_{1-\frac{2}{r}}, \quad q < r < \infty,$$

we obtain (4.13) for $r \in (q, \infty]$ as well. This completes the proof. \hfill \square
Remark 4.4. When the stationary solution possesses the scale-critical rate $O(1/|x|)$, the $L^\infty$ decay of perturbation with less sharp rate $O(t^{-\frac{4}{3}+\varepsilon})$ was derived first by Koba [18] in the context of stability analysis, where $\varepsilon > 0$ is arbitrary. If we have a look only at the $L^\infty$ decay rate, our rate is comparable with his result since $q \in (6, \infty)$ is arbitrary. However, we are not able to prove Proposition 4.3 by his method. This is because he doesn’t split the integrals in $N_2$ and $N_3$, so that the rate of $L^\infty$ decay is slower than the one of $L^{q,\infty}$ decay. From this point of view, Proposition 4.3 is regarded as a slight improvement of his result.

We next show that the solution $v$ obtained in Proposition 4.2 actually satisfies the integral equation (2.4) by identifying $v$ with a local solution $\tilde{v}$ of (2.4) in a neighborhood of each time $t > 0$. To this end, we need the following lemma on the uniqueness. The proof is similar to the argument in the second half (after (4.7)) of the proof of Theorem 2.3 and thus we may omit it.

Lemma 4.5. Let $3 < r < \infty$, $0 \leq t_0 < t_1 < \infty$ and $v_0 \in L^r_{\sigma}(D)$. Then the solution to the problem

$$(v(t), \varphi) = (v_0, T(t, t_0)^* \varphi) + \int_{t_0}^{t} \left( v(\tau) \otimes v(\tau) + \psi(\tau) \{ v(\tau) \otimes u_s + u_s \otimes v(\tau) \}, \nabla T(t, \tau)^* \varphi \right) d\tau$$

$$+ \int_{t_0}^{t} (H(\tau), T(t, \tau)^* \varphi) d\tau, \quad \forall \varphi \in C^\infty_0(D)$$

(4.24)

on $(t_0, t_1)$ admits at most one solution within the class $L^\infty(t_0, t_1; L^r_{\sigma}(D))$. Here, $H$ is given by (2.3).

Given $v_0 \in L^r_{\sigma}(D)$ with $r \in (3, \infty)$, let us construct a local solution of the integral equation

$$v(t) = T(t, t_0)v_0 + \int_{t_0}^{t} T(t, \tau)P[(Gv)(\tau) + H(\tau)] d\tau,$$

(4.25)

where $G$ and $H$ are defined by (2.2) and (2.3), respectively. For $0 \leq t_0 < t_1 < \infty$ and $r \in (3, \infty)$, we define the function space

$$Y_r(t_0, t_1) = \{ v \in C([t_0, t_1]; L^r_{\sigma}(D)) \mid (\cdot - t_0)\frac{1}{2} \nabla v(\cdot) \in BC_w((t_0, t_1]; L^r(D)) \}$$

(4.26)

which is a Banach space equipped with norm

$$\|v\|_{Y_r(t_0, t_1)} = \sup_{0 \leq t_0 < \tau \leq t_1} \|v(\tau)\|_r + \sup_{t_0 \leq \tau \leq t_1} (\tau - t_0)^{\frac{1}{2}} \|\nabla v(\tau)\|_r$$

(4.27)

and set

$$U_1(v, w)(t) = \int_{t_0}^{t} T(t, \tau)P[v(\tau) \cdot \nabla w(\tau)] d\tau, \quad U_2(v)(t) = \int_{t_0}^{t} T(t, \tau)P[\psi(\tau)v(\tau) \cdot \nabla u_s] d\tau,$$

$$U_3(v)(t) = \int_{t_0}^{t} T(t, \tau)P[\psi(\tau)u_s \cdot \nabla v(\tau)] d\tau, \quad U_4(t) = \int_{t_0}^{t} T(t, \tau)PH(\tau) d\tau.$$

(4.28)
Lemma 4.6. Let $3 < r < \infty$ and $0 \leq t_0 < t_1 \leq t_0 + 1$. Suppose that $v, w \in Y_r(t_0, t_1)$. Then $U_1(v, w), U_2(v), U_3(v), U_4 \in Y_r(t_0, t_1)$. Furthermore, there exists a constant $C = C(r, t_0)$ such that

$$
\|U_1(v, w)\|_{Y_r(t_0, t)} \leq C(t - t_0)^{\frac{1}{2} - \frac{3}{2r}} \|v\|_{Y_r(t_0, t)} \|w\|_{Y_r(t_0, t)},
$$

(4.29)

$$
\|U_2(v)\|_{Y_r(t_0, t)} \leq C(t - t_0)^{1 - \frac{3}{r}} \|\nabla u_s\|_r \|v\|_{Y_r(t_0, t)},
$$

(4.30)

$$
\|U_3(v)\|_{Y_r(t_0, t)} \leq C(t - t_0)^{\frac{1}{2} - \frac{1}{r}} \|u_s\|_r \|v\|_{Y_r(t_0, t)},
$$

(4.31)

$$
\|U_4\|_{Y_r(t_0, t)} \leq C(t - t_0)^{\frac{1}{2} + \frac{3}{2r}} (a^2 + a|a|)
$$

(4.32)

for all $t \in (t_0, t_1]$.

Proof. In view of (3.11), we have

$$
\|U_1(t)\|_r \leq C \int_{t_0}^t (t - \tau)^{-\frac{3}{r}} \|v\|_r \|w\|_r \, d\tau \leq C(t - t_0)^{-\frac{3}{r} + \frac{1}{2}} \|v\|_{Y_r(t_0, t)} \|w\|_{Y_r(t_0, t)}.
$$

(4.33)

Furthermore, (3.5) with $T = t_0 + 1$ yields

$$
\|\nabla U_1(t)\|_r \leq C(t - t_0)^{-\frac{3}{r} + \frac{1}{2}} \|v\|_{Y_r(t_0, t)} \|w\|_{Y_r(t_0, t)}.
$$

(4.34)

By (4.33) and (4.34), we obtain (4.29). Similarly, we can show (4.30)–(4.32). We note that the estimate (4.32) follows from (3.7) with $T = t_0 + 1$ together with (3.1).

We next show the continuity of $U_1$ with respect to $t$. Let $t_2 \in [t_0, t_1]$. If $t_2 < t$, we have

$$
U_1(t) - U_1(t_2) = \int_{t_0}^{t_2} (T(t, t_2) - 1) T(t_2, \tau) P[v(\tau) \cdot \nabla w(\tau)] \, d\tau + \int_{t_0}^{t_2} T(t, \tau) P[v(\tau) \cdot \nabla w(\tau)] \, d\tau
$$

$$
=: U_{11}(t) + U_{12}(t).
$$

Lebesgue’s convergence theorem yields that $\|U_{11}(t)\|_r \to 0$ as $t \to t_2$, while

$$
\|U_{12}(t)\|_r \leq C(t - t_2)^{\frac{1}{2} - \frac{3}{2r}} \|v\|_{Y_r(t_0, t_1)} \|w\|_{Y_r(t_0, t_1)} \to 0 \quad \text{as} \quad t \to t_2.
$$

To discuss the case $t < t_2$, we need the following device. Let $(t_0 + t_2)/2 \leq \hat{t} < t_2$, where $\hat{t}$ will be determined later, then

$$
U_1(t) - U_1(t_2) = \left( \int_{t_0}^{\hat{t}} + \int_{\hat{t}}^{t_2} \right) T(t, \tau) P[v(\tau) \cdot \nabla w(\tau)] \, d\tau
$$

$$
- \left( \int_{t_0}^{\hat{t}} + \int_{\hat{t}}^{t_2} \right) T(t_2, \tau) P[v(\tau) \cdot \nabla w(\tau)] \, d\tau.
$$

We observe that

$$
\int_{\hat{t}}^{t} \|T(t, \tau) P[v(\tau) \cdot \nabla w(\tau)]\|_r \, d\tau + \int_{\hat{t}}^{t_2} \|T(t_2, \tau) P[v(\tau) \cdot \nabla w(\tau)]\|_r \, d\tau
$$

$$
\leq C \|v\|_{Y_r(t_0, t_1)} \|w\|_{Y_r(t_0, t_1)} \left( \int_{\hat{t}}^{t} (t - \tau)^{-\frac{3}{r}} (\tau - t_0)^{-\frac{1}{2}} \, d\tau + \int_{\hat{t}}^{t_2} (t_2 - \tau)^{-\frac{3}{r}} (\tau - t_0)^{-\frac{1}{2}} \, d\tau \right)
$$

$$
\leq \frac{2C}{1 - \frac{3}{2r}} \|v\|_{Y_r(t_0, t_1)} \|w\|_{Y_r(t_0, t_1)} \left( \frac{t_0 + t_2}{2} - t_0 \right)^{\frac{1}{2}} (t_2 - \hat{t})^{1 - \frac{3}{2r}}.
$$

20
Proof. We put 
\[ \frac{2C}{1 - \frac{1}{2r} \|v\|_{Y_{r}(t_0,t_1)}} \|w\|_{Y_{r}(t_0,t_1)} \left( \frac{t_0 + t_2}{2} - t_0 \right)^{-\frac{1}{2}} (t_2 - \tilde{t})^{1 - \frac{3}{2r}} < \varepsilon \]
which yields
\[ \|U_1(t) - U_1(t_2)\|_r \leq \int_{t_0}^{\tilde{t}} \|(T(t, \tau) - T(t_2, \tau)) P[v(\tau) \cdot \nabla w(\tau)]\|_r d\tau + \varepsilon \quad \text{for } \tilde{t} < t < t_2 \]
and therefore,
\[ \limsup_{t \to t_2} \|U_1(t) - U_1(t_2)\|_r \leq \limsup_{t \to t_2} \int_{t_0}^{\tilde{t}} \|(T(t, \tau) - T(t_2, \tau)) P[v(\tau) \cdot \nabla w(\tau)]\|_r d\tau + \varepsilon. \quad (4.35) \]
Since \( \|(T(t, \tau) - T(t_2, \tau)) P[v(\tau) \cdot \nabla w(\tau)]\|_r = \|(T(t, \tilde{t}) - T(t_2, \tilde{t})) T(\tilde{t}, \tau) P[v(\tau) \cdot \nabla w(\tau)]\|_r \)
tends to 0 as \( t \to t_2 \) for \( t_0 < \tau < \tilde{t} \), it follows from Lebesgue’s convergence theorem that the integral term in (4.35) tends to 0 as \( t \to t_2 \). Since \( \varepsilon > 0 \) is arbitrary, we have
\[ U_1 \in C([t_0, t_1]; L^r_\sigma(D)). \quad (4.36) \]
Furthermore, we find \( \nabla U_1 \in C_w((t_0, t_1]; L^r(D)) \) on account of (4.34) and (4.36) together with the relation
\[ (\nabla U_1(t) - \nabla U_1(t_2), \varphi) = -(U_1(t) - U_1(t_2), \nabla \cdot \varphi) \]
for all \( t_2 \in (t_0, t_1] \) and \( \varphi \in C_0^{\infty}(D)^{3 \times 3} \). Since \( U_2, U_3 \) and \( U_4 \) are discussed similarly, the proof is complete. \( \square \)

The following proposition provides a local solution of (4.25).

**Proposition 4.7.** Let \( 3 < r < \infty, \ t_0 \geq 0 \) and \( v_0 \in L^r_\sigma(D) \). There exists \( t_1 \in (t_0, t_0 + 1] \) such that (4.25) admits a unique solution \( v \in Y_{r}(t_0, t_1) \). Moreover, the length of the existence interval can be estimated from below by
\[ t_1 - t_0 \geq \zeta(\|v_0\|_r), \]
where \( \zeta(\cdot) : [0, \infty) \to (0, 1) \) is a non-increasing function defined by (1.40) below.

**Proof.** We put
\[ (\Psi v)(t) = \text{the RHS of (4.25)}. \]
By applying Lemma 4.6 we have
\[ \|\Psi v\|_{Y_{r}(t_0, t)} \leq (C_1\|v\|_{Y_{r}(t_0, t)}^2 + C_2\|v\|_{Y_{r}(t_0, t)} + C_3)(t - t_0)^{\frac{1}{2} - \frac{3}{2r}} + C_4\|v_0\|_r, \]
\[ \|\Psi v - \Psi w\|_{Y_{r}(t_0, t)} \leq (C_1(\|v\|_{Y_{r}(t_0, t)} + \|w\|_{Y_{r}(t_0, t)}) + C_2)(t - t_0)^{\frac{1}{2} - \frac{3}{2r}} \|v - w\|_{Y_{r}(t_0, t)} \]
for all \( t \in (t_0, t_0+1) \) and \( v, w \in Y(t_0, t) \). We note that the constants \( C_i \) may be dependent on \( \|u_s\|_r, \|\nabla u_s\|_r, \alpha \) and \( a \). We choose \( t_1 \in (t_0, t_0 + 1] \) such that

\[
C_2(t_1 - t_0)^{1-\frac{3}{r}} < \frac{1}{2},
\]

\[
8C_1(t_1 - t_0)^{1-\frac{3}{r}} \{ C_3(t_1 - t_0)^{1-\frac{3}{r}} + C_4\|v_0\|_r \} < \frac{1}{2}
\]

which imply

\[
\lambda := \left\{ 1 - C_2(t_1 - t_0)^{1-\frac{3}{r}} \right\}^2 - 4C_1(t_1 - t_0)^{1-\frac{3}{r}} \{ C_3(t_1 - t_0)^{1-\frac{3}{r}} + C_4\|v_0\|_r \} > 0.
\]

We set

\[
\Lambda := \frac{1 - C_2(t_1 - t_0)^{1-\frac{3}{r}} - \sqrt{\lambda}}{2C_1(t_1 - t_0)^{1-\frac{3}{r}}} < 4(C_3 + C_4\|v_0\|_r),
\]

\[
Y_{r, \Lambda}(t_0, t_1) := \{ v \in Y_{r}(t_0, t_1) | \|v\|_{Y_{r}(t_0, t_1)} \leq \Lambda \}.
\]

Then we find that the map \( \Psi : Y_{r, \Lambda}(t_0, t_1) \to Y_{r, \Lambda}(t_0, t_1) \) is well-defined and also contractive. Hence we obtain a local solution. Indeed, the conditions (4.37) and (4.38) are accomplished by

\[
t_1 - t_0 < \min \left\{ 1, \left( \frac{1}{2C_2} \right)^{\frac{1}{1-\frac{3}{r}}} \right\}, \left( \frac{1}{16C_1(C_3 + C_4\|v_0\|_r)} \right)^{\frac{1}{1-\frac{3}{r}}} \right\}.
\]

Thus, it is possible to take \( t_1 \) such that

\[
t_1 - t_0 \geq \frac{1}{2} \min \left\{ 1, \left( \frac{1}{2C_2} \right)^{\frac{1}{1-\frac{3}{r}}} \right\}, \left( \frac{1}{16C_1(C_3 + C_4\|v_0\|_r)} \right)^{\frac{1}{1-\frac{3}{r}}} \right\} =: \zeta(\|v_0\|_r).
\]

The proof is complete. \( \square \)

**Lemma 4.8.** Let \( 3 < r < \infty, t_0 \geq 0 \) and \( v_0 \in L^r_\sigma(D) \). The local solution \( v \) obtained in Proposition 4.7 also possesses the following properties:

\[
v \in C((t_0, t_1]; L^r_\sigma(D)) \cap C_w((t_0, t_1]; L^\infty(D))
\]

for every \( \kappa \in (r, \infty) \) and

\[
\nabla v \in C_w((t_0, t_1]; L^\gamma(D))
\]

for every \( \gamma \in (r, \infty) \) satisfying

\[
\frac{2}{r} - \frac{1}{\gamma} < \frac{1}{3}.
\]
Proof. By using (3.11) and (3.15) and the semigroup property, we find $v(t) \in L^\infty(D)$ with

$$
\|v(t)\|_\infty \leq C(t - t_0)^{-\frac{\gamma}{2}} \left\{ \|v_0\|_r + \|v\|_{Y_r((t_0, t_1]}^2 + \|v\|_{Y_r((t_0, t_1]} (\|u_s\|_r + \|\nabla u_s\|_r) + (\alpha^2 + \alpha|a|) \right\}
$$

(4.44)

for all $t \in (t_0, t_1]$. Moreover, for each $t_2 \in (t_0, t_1]$, we know from $v \in C([t_0, t_1]; L^r_s(D))$ that

$$
(v(t), \varphi) - (v(t_2), \varphi) \to 0 \quad \text{as} \quad t \to t_2
$$

(4.45)

for all $\varphi \in C^\infty_0(D)$, which combined with (4.44) yields $v \in C_w((t_0, t_1]; L^\infty(D))$. Since

$$
\|v(t) - v(t_2)\|_\kappa \leq \left\{ \|v(t) - v(t_2)\|^{\frac{\gamma}{2}} \right\} \|v(t) - v(t_2)\|^{\frac{1}{2}}
$$

for $\kappa \in (r, \infty)$ and $t_2 \in (t_0, t_1]$, it follows from (4.44) that

$$
v \in C((t_0, t_1]; L^\infty_s(D)) \quad \text{for} \quad \kappa \in (r, \infty).
$$

(4.46)

The estimates (3.34) and (3.7) with $T = t_0 + 1$ imply that if we assume (4.43), we have $\nabla v(t) \in L^\gamma(D)$ with

$$
\|\nabla v(t)\|_\gamma \leq C(t_1 - t_0)^{-\frac{\gamma}{2}} \frac{\gamma}{\kappa - \frac{\gamma}{2}} \left\{ \|v_0\|_r + \|v\|_{Y_r((t_0, t_1]}^2 + \|v\|_{Y_r((t_0, t_1]} (\|u_s\|_r + \|\nabla u_s\|_r) + (\alpha^2 + \alpha|a|) \right\}
$$

(4.47)

for all $t \in (t_0, t_1]$. Here, we note that (4.43) is needed for estimates of $\nabla U_1$ and $\nabla U_3$ given in (4.25). On account of (4.46), (4.47) and

$$
(\nabla v(t) - \nabla v(t_2), \varphi) = -(v(t) - v(t_2), \nabla \cdot \varphi)
$$

for all $t_2 \in (t_0, t_1]$ and $\varphi \in C^\infty_0(D)^{3 \times 3}$, we find the weak continuity of $\nabla v$ with values in $L^\gamma(D)$. The proof is complete.

We close the paper with completion of the proof of Theorem 2.1.

Proof of Theorem 2.1 It remains to show that the solution $v$ obtained in Proposition 4.2 also satisfies (2.1) with

$$
v \in C((0, \infty); L^\infty_s(D)) \cap C_w((0, \infty); L^\infty(D)), \quad \nabla v \in C_w((0, \infty); L^\infty(D))
$$

(4.48)

for all $3 < \kappa < \infty$. Let $t_\ast \in (0, \infty)$. By applying Proposition 4.7 and Lemma 4.8 with $r = 6$, we can see that for each $t_0 \in [t_\ast/2, t_\ast)$, there exists $\tilde{v} \in Y_6((t_0, t_1]$) which satisfies (4.25) and therefore, (4.24) with $v_0 = v(t_0)$ such that

$$
\tilde{v} \in C((t_0, t_1]; L^6_s(D)) \cap C_w((t_0, t_1]; L^\infty(D)), \quad \nabla \tilde{v} \in C_w((t_0, t_1]; L^\infty(D))
$$

for all $\kappa \in [6, \infty)$. Moreover, the length of the existence interval can be estimated by

$$
t_1 - t_0 \geq \zeta(\|v(t_0)\|_6) \geq \zeta \left( C_5 \left( \frac{t_\ast}{2} \right)^{-\frac{1}{4}} \right) =: \varepsilon,
$$

23
where $\zeta(\cdot)$ is the non-increasing function in Proposition 4.7 because of
\[
\|v(t)\|_6 \leq C_5 \left( \frac{t_*}{2} \right)^{-\frac{\kappa}{4}}
\]
for all $t \geq t_*/2$, see (4.12). We note that the solution $v$ obtained in Proposition 4.2 also satisfies (4.24) with $v_0 = v(t_0)$ since $C_0^\infty(\sigma) (D)$ can be replaced by $L_6^{6/5}(D)$ as the class of test functions in (2.5). Let us take $t_0 := \max\{t_*/2, t_* - \varepsilon/2\}$ so that $t_* \in (t_0, t_1)$, in which $v = \tilde{v}$ on account of Lemma 4.5. Since $t_*$ is arbitrary, we conclude (4.48) for $\kappa \in [6, \infty)$. It is also proved by applying Proposition 4.7 with $r \in (3, 6)$ that the solution belongs to the class (4.48) for $\kappa \in (3, 6)$ as well. The proof is complete.

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