Suborbits of a point stabilizer in the orthogonal group on the last subconstituent of orthogonal dual polar graphs

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Abstract
As one of the serial papers on suborbits of point stabilizers in classical groups on the last subconstituent of dual polar graphs, the corresponding problem for orthogonal dual polar graphs over a finite field of odd characteristic is discussed in this paper. We determine all the suborbits of a point-stabilizer in the orthogonal group on the last subconstituent, and calculate the length of each suborbit. Moreover, we discuss the quasi-strongly regular graphs and the association schemes based on the last subconstituent, respectively.

Keywords: orthogonal group, suborbit, dual polar graph, subconstituent, quasi-strongly regular graph, association scheme

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1. Introduction

Let $F_q$ be a finite field with $q$ elements, where $q$ is an odd prime power. Let $F_q^n$ be the row vector space of dimension $n$ over $F_q$. The set of all $m \times n$ matrices over $F_q$ is denoted by $M_{mn}(F_q)$, and $M_{nn}(F_q)$ is denoted by $M_n(F_q)$ for simplicity. For any matrix $A = (a_{ij}) \in M_{mn}(F_q)$, we denote the transpose of $A$ by $A^t$.

Let $n = 2\nu + \delta$, where $\nu$ is a non-negative integer and $\delta = 0, 1$ or 2. Suppose

$$S_{2\nu+\delta,\Delta} = \begin{pmatrix} 0 & f(\nu) \\ f(\nu) & 0 \\ \Delta \end{pmatrix}, \quad \Delta = \begin{cases} \phi \text{ (disappear),} & \text{if } \delta = 0, \\ (1) \text{ or } (z), & \text{if } \delta = 1, \\ (1 - z) & \text{if } \delta = 2, \end{cases}$$

where $z$ is a fixed non-square element of $F_q$ such that $1 - z$ is a non-square element. When $\delta = 1$ or 2, $\Delta$ is definite in the sense that for any row vector $x \in F_q^n$, $x\Delta x^T = 0$ implies $x = 0$. Note that the set

$$\left\{ T \in GL_2(\mathbb{F}_q) \mid TS_{2\nu+\delta,\Delta}T^t = S_{2\nu+\delta,\Delta} \right\}$$

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forms a subgroup of $GL_{2r+\delta}(F_q)$, called the orthogonal group of degree $n = 2r + \delta$ with respect to $S_{2r+\delta,\Delta}$ over $F_q$, denoted by $O_{2r+\delta,\Delta}(F_q)$. The group $O_{2r+\delta,\Delta}(F_q)$ acts on $F_q^{2r+\delta}$ by the matrix multiplication. $F_q^{2r+\delta}$ together with this action is called the $(2r+\delta)$-dimensional orthogonal space over $F_q$ with respect to $S_{2r+\delta,\Delta}$. A matrix representation of a subspace $P$ is a matrix whose rows form a basis for $P$. When there is no danger of confusion, we use the same symbol to denote a subspace and its matrix representation. An $m$-dimensional subspace $P$ of $F_q^{2r+\delta}$ is called totally isotropic if $PS_{2r+\delta,\Delta}P = 0$. It is well-known that maximal totally isotropic subspaces of $F_q^{2r+\delta}$ are of dimension $\nu$.

Let $G$ be a group acting transitively on a finite set $X$. For a fixed element $a \in X$, the stabilizer $G_a$ is not transitive on $X$ in general. The orbits of $G_a$ on $X$ are said to be suborbits, and the number of such suborbits is the rank of this action. H. Wei and Y. Wang [15, 16, 17] studied the suborbits of the transitive set of all totally isotropic subspaces under finite classical groups. We discussed these problems in singular classical spaces in [5, 12].

Dual polar graphs are famous distance-regular graphs and have been well studied ([1, 2, 10]). The orthogonal dual polar graph $\Gamma$ (on the orthogonal space $F_q^{2r+\delta}$) has as vertices the maximal totally isotropic subspaces; two vertices $P$ and $Q$ are adjacent if and only if $\dim(P \cap Q) = \nu - 1$. It is well-known that $\Gamma$ is of diameter $\nu$. For any vertex $P$ of $\Gamma$, the $i$th subconstituent $\Gamma_i(P)$ with respect to $P$ is the induced graph on the set of vertices at distance $i$ from $P$ in $\Gamma$. A. Munemasa [9] initiated the study of the subconstituents of dual polar graphs in the orthogonal spaces, and characterized the first and last subconstituents. Subsequently, Y. Wang, F. Li and Y. Huo [6, 7, 13, 14] characterized all the subconstituents of dual polar graphs under finite classical groups, and proved that for any vertex $P$ of the dual polar graph $\Gamma$ in the $(2r+\delta)$-dimensional classical space (where $\delta = 0$, 1 or 2), the $m$th subconstituent $\Gamma_m(P)$ is isomorphic to $[\nu \cdot G^{(m, \delta)}]$, where $G^{(m, \delta)}$ is the graph with the vertex set consisting of the matrices $(X \in M_{m}(F_q), Z \in M_{\delta}(F_q))$; and two vertices $(X \in M_{m}(F_q), Z \in M_{\delta}(F_q))$ and $(X_1 \in M_{m}(F_q), Z_1 \in M_{\delta}(F_q))$ are adjacent if and only if $(X - X_1, Z - Z_1)$ is of rank 1. Note that the mapping

$$(X \in M_{m}(F_q), Z \in M_{\delta}(F_q)) \mapsto (X, I^{(m)} \in Z)$$

is an isomorphism from $G^{(m, \delta)}$ to the last subconstituent of the corresponding dual polar graph in the classical space $F_q^{2r+\delta}$. Therefore, the study of subconstituents of a dual polar graph may be reduced to that of the last subconstituent. In [3] we studied the suborbits of a point-stabilizer in the unitary group on the last subconstituent of Hermitian dual polar graphs. In this paper we discuss the corresponding problem for orthogonal dual polar graphs over a finite field of odd characteristic.

Let $\Gamma$ be the orthogonal dual polar graph. It is well-known that a point-stabilizer of $P$ of $\Gamma$ in $O_{2r+\delta,\Delta}(F_q)$ is transitive on the last subconstituent of $\Gamma$. In Section 2 we determine all the suborbits of this action, and calculate the rank and the lengths of these suborbits. As two applications of our results, in Sections 3 and 4, we discuss the quasi-strongly regular graphs and the association schemes based on the last subconstitute of $\Gamma$, respectively.
2. Suborbits

Let \( \Gamma \) be the dual polar graph in the orthogonal space \( \mathbb{F}^{2v+\delta}_p \). Note that the last subconstituent \( \Lambda \) is a coclique when \( \delta = 0 \) (see [7]), and \( \Lambda \) is studied in [7] when \( \delta = 1 \). So the case \( \delta = 2 \) is the main objective of this paper.

Denote by \([X_1, X_2, \ldots, X_r]\) the block diagonal matrix whose blocks along the main diagonal are matrices \( X_1, X_2, \ldots, X_r \) by \( \mathcal{A}_2 = [\mathcal{A}_1, \ldots, \mathcal{A}_2] \) the \( 2r \times 2r \) matrix of rank \( 2r \), in which \( \mathcal{A}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \), and by \((A \ B \ldots C)\) the matrix whose block entries are \( A, B, \ldots, C \). Suppose \( I^{(m)} \) denotes the identity matrix of order \( m \), and \( 0^{(p,q)} \) denotes the zero matrix of order \( p \times q \) when \( p = q \).

We now study the suborbits of the stabilizer of each vertex \( P_0 \) in \( O_{2v+2,\Lambda}(\mathbb{F}_q) \) on \( \Lambda \). Since \( O_{2v+2,\Lambda}(\mathbb{F}_q) \) acts transitively on the subspaces of the same type, we may choose \( P_0 = (I^{(v)} \ 0^{(v,v)}) \). By (4), \( \Lambda \) consists of subspaces of form \((A \ I^{(v)} \ Z)\), where \( A \in M_r(\mathbb{F}_q) \) and \( Z \in M_{r^2}(\mathbb{F}_q) \) satisfy \( A + A^t + ZA^t = 0 \). Let \( G_0 \) be the stabilizer of \( P_0 \) in \( O_{2v+2,\Lambda}(\mathbb{F}_q) \). Then \( G_0 \) consists of matrices of the following form:

\[
\begin{pmatrix}
T_{11} & 0 & 0 \\
T_{21} & (T_{11}'^{-1}) & T_{23} \\
-S\Delta T_{23}' T_{11} & 0 & S
\end{pmatrix},
\]

where \( T_{11} \in GL_v(\mathbb{F}_q) \), \( T_{21} \in M_r(\mathbb{F}_q) \), \( T_{23} \in M_{r^2}(\mathbb{F}_q) \), \( S \in O_{2v+2,\Lambda}(\mathbb{F}_q) \) and

\[(T_{11}')^{-1} T_{21}' + T_{21} T_{11}^{-1} + T_{23} \Delta T_{23}' = 0.\]

It is well-known that \( G_0 \) acts transitively on \( \Lambda \). For any \( P_1 \in \Lambda \), the suborbits of \( G_0 \) are just the orbits of the point-stabilizer of \( P_1 \) in \( G_0 \) on \( \Lambda \). Let \( P_1 = (0^{(v)} \ 1^{(v)} \ 0^{(v,v)}) \) in \( \Lambda \) and \( G_0 \) be the stabilizer of \( P_0 \) and \( P_1 \) in \( O_{2v+2,\Lambda}(\mathbb{F}_q) \). Then \( G_{01} \) consists of matrices of the following form:

\[ [T, (T')^{-1}, S], \quad (1) \]

where \( T \in GL_v(\mathbb{F}_q) \) and \( S \in O_{2v+2,\Lambda}(\mathbb{F}_q) \). The action of \( O_{2v+2,\Lambda}(\mathbb{F}_q) \) on \( \mathbb{F}_q^{2v+2} \) induces an action \( G_{01} \) on \( \Lambda \):

\[ \Lambda \times G_{01} \rightarrow \Lambda \quad ((A \ I^{(v)} \ Z), [T, (T')^{-1}, S]) \mapsto (T'A^T I^{(v)} T'ZS). \]

Denote by \( K_v \) the set of all \( n \times n \) alternate matrices over \( \mathbb{F}_q \). In order to determine the orbits of \( G_{01} \) on \( \Lambda \), we need to introduce an action on \( K_v \). For \( i = 1, 2 \), let \( O_i \) denote the set of all matrices of the form

\[
\begin{pmatrix}
T_{11} & 0 \\
T_{21} & T_{22}
\end{pmatrix},
\]

where \( T_{11} \in GL_v(\mathbb{F}_q) \), \( T_{21} \in M_{r^2}(\mathbb{F}_q) \) and \( T_{22} \in GL_{r^2}(\mathbb{F}_q) \). Then \( O_i \) is a subgroup of \( GL_v(\mathbb{F}_q) \), and there is an action of \( O_i \) on \( K_v \):

\[ K_v \times O_i \rightarrow K_v \quad (X, T) \mapsto T'XT. \]

Note that \( (0^{(v)}) \) is the trivial orbit of \( O_i \) on \( K_v \) for \( i = 1, 2 \).
Theorem 2.1. (i) The nontrivial orbits of $O_1$ on $K_r$ have the following representatives:

$$[0, A_{2r}, 0^{(v-2r+1)}] \quad (1 \leq r \leq [(v-1)/2]), \quad [A_{2r}, 0^{(v-2r)}] \quad (1 \leq r \leq [v/2]).$$

(ii) The nontrivial orbits of $O_2$ on $K_r$ have the following representatives:

$$[0, A_{2r}, 0^{(v-2r+1)}] \quad (1 \leq r \leq [(v-1)/2]), \quad [0^{(2)}, A_{2r}, 0^{(v-2r-2)}] \quad (1 \leq r \leq [(v-2)/2]), \quad [A_{2r}, 0^{(v-2r)}] \quad (1 \leq r \leq [v/2]), \quad [K, A_{2r-4}, 0^{(v-2r)}] \quad (2 \leq r \leq [v/2]),$$

where $K = \begin{pmatrix} 0 & I^{(2)} \\ -I^{(2)} & 0 \end{pmatrix}$.

Proof. We only prove (ii), and (i) can be treated similarly. Let $X \in K_r$ with rank $2r > 0$. Write

$$X = \begin{pmatrix} xA_2 & X_{12} \\ -X_{12} & X_{22} \end{pmatrix},$$

where $X_{12} \in M_{2r-2}(F_q)$ and $X_{22} \in K_{v-2}$. Then rank $X_{22} = 2(r-i), i = 0, 1$ or 2. Hence there is a $T_{11} \in GL_{v-2}(F_q)$ such that $T_{11}'X_{22}T_{11} = [A_{2(r-i)}, 0^{(v-2r+2i-2)}]$. Let

$$X_{12}T_{11} = \begin{pmatrix} 2r-2i & v-2r+2i-2 \\ Y_{12} & Y_{13} \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} I^{(2)} \\ -A_{2(r-i)}Y_{12} \end{pmatrix} \begin{pmatrix} F^{(2r-2)} \\ T_{11} \end{pmatrix} \begin{pmatrix} F^{(2)} \\ T_{11}'Y_{12} \end{pmatrix} \begin{pmatrix} F^{(2r-2)} \\ T_{11}' \end{pmatrix}. $$

Then $T \in O_2$ and

$$T'XT = \begin{pmatrix} xA_2 & 0 & Y_{13} \\ 0 & A_{2(r-i)} & 0 \\ -Y_{13} & 0 & 0^{(v-2r+2i-2)} \end{pmatrix},$$

where $xA_2 = xA_2 - Y_{12}A_{2(r-i)}Y_{12}'$.

Case 1: rank $X_{22} = 2r$. Then $x = 0, Y_{13} = 0$ and $T'XT = [0^{(2)}, A_{2r}, 0^{(v-2r-2)}]$. 

Case 2: rank $X_{22} = 2(r-1)$. Then rank $Y_{13} = 0$ or 1. If rank $Y_{13} = 0$, then $x \neq 0, T_1 = [x^{-1}, F^{(v-1)}] \in O_2$ and $T_1T'XTT_1 = [A_{2r}, 0^{(v-2r)}]$. If rank $Y_{13} = 1$, then there exists a $T_{12} \in GL_{2}(F_q)$ and $T_{13} \in GL_{v-2}(F_q)$ such that

$$T_{12}Y_{13}T_{13} = \begin{pmatrix} 0 & 0^{(v-2r+1)} \\ 1 & 0^{(v-2r+1)} \end{pmatrix}. $$

Let $xA_2 = xT_{12}A_2T_{12}'$ and

$$T_2 = \begin{pmatrix} T_{12}' \\ F^{(2r-2)} \\ T_{13} \end{pmatrix} \begin{pmatrix} F^{(2)} \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0^{(v-3)} \\ 1 & 0 & 0^{(v-3)} \\ 1 & x_2 & 0^{(v-3)} \end{pmatrix}. $$

Then $TT_3 \in O_2$ and $(TT_3)^{T}X(TT_2) = [0, A_{2r}, 0^{(v-2r-1)}]$. 

Case 3: rank $X_{22} = 2(r-2)$. Then rank $Y_{13} = 2$; and so there exists a $T_{14} \in GL_{v-2r+2}(F_q)$ such that $Y_{13}T_{14} = (F^{(2)} 0^{(2r-2r)})$. Let

$$T_3 = \begin{pmatrix} F^{(2r-2)} \\ T_{14} \end{pmatrix} \begin{pmatrix} F^{(2)} \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0^{(v-3)} \\ 1 & 0 & 0^{(v-3)} \\ 1 & x_2 & 0^{(v-3)} \end{pmatrix}. $$
Then \(TT_3 \in O_2\) and \((TT_3)^t X(TT_3) = [K, \mathcal{A}_{2r-1}, 0^{(r-2)}]\).

Note that matrices of difference ranks cannot be in the same orbit. Now we show that any two distinct matrices in \(\mathcal{A}_{2r-1}, 0^{(r-2)}\) cannot fall into the same orbit of \(O_2\). Otherwise, there exists a \(T \in O_2\), which is of the form \(\mathcal{A}_{2r-1}, 0^{(r-2)}\), carrying \([0, \mathcal{A}_{2r-1}, 0^{(r-2)}] \to [0^{(r-2)}, \mathcal{A}_{2r-1}, 0^{(r-2)}]\), then \(T_{22}[0, \mathcal{A}_{2r-1}, 0^{(r-2)}]T_{22} = [\mathcal{A}_{2r-1}, 0^{(r-2)}]\), which is impossible since \(T_{22}\) is nonsingular. Similarly, the left cases may be handled.

By above discussion, the desired result follows. □

To determine the orbits of \(G_{01}\) on \(\Lambda\), we need the following two lemmas.

**Lemma 2.2.** Let \(a, b \in \mathbb{F}_q\) with \((a, b) \neq (0, 0)\). Then there exists a \(T \in O_{2 \times 0+2, \mathbb{F}_q}\) such that the subspace \((a, b)T\) has the matrix representation of the form \((1, 0)\) or \((1, 1)\) corresponding to \(a^2 - zb^2\) is a square element or not, respectively.

**Proof.** Note that \((a, b)\) is of type \((1, 1, 0, 1)\) or \((1, 1, 0, z)\) in \(\mathbb{F}_q^{2 \times 0+2}\) corresponding to \(a^2 - zb^2\) being a square element or not, respectively. The result follows from [11, Theorem 6.4]. □

**Lemma 2.3.** Any element of \(O_{2 \times 0+2, \mathbb{F}_q}\) has one of the following forms

\[
\begin{pmatrix}
  x & y \\
  yz & x
\end{pmatrix},
\begin{pmatrix}
  x & y \\
  -yz & -x
\end{pmatrix},
\]  

(5)

where \(x^2 - y^2z = 1\).

**Proof.** Let \(T \in O_{2 \times 0+2, \mathbb{F}_q}\) and write

\[T = \begin{pmatrix}
  x & y \\
  u & v
\end{pmatrix},\]

where \(x^2 - y^2z = 1\), \(ux - yvz = 0\) and \(u^2 - v^2z = -z\). If \(xyuv \neq 0\), then \(u = x^{-1}yvz\) and \(v^2 = x^2\), i.e., \(v = \pm x\) and \(u = \pm yz\). Then \(T\) has one of the form \(5\) with \(x^2 - y^2z = 1\). If \(xyuv = 0\), then \(T\) has one of the following forms

\[
\pm I^{(2)}\pm \begin{pmatrix}
  1 & 0 \\
  0 & -1
\end{pmatrix}, \begin{pmatrix}
  0 & y \\
  y^{-1} & 0
\end{pmatrix}, \begin{pmatrix}
  0 & y \\
  -y^{-1} & 0
\end{pmatrix}
\]

with \(-y^2z = 1\), which are of the form \(5\). □

Pick a fixed subset \(\Omega\) of \(\mathbb{F}_q^*\) such that \(\mathbb{F}_q^* = \Omega \cup -\Omega\), where \(-\Omega = \{-a | a \in \Omega\}\). Let \(E_i\) denote the \(v\)-dimensional column vector having 1 as its \(i\)-entry and other entries 0’s. Similar to [6, Theorem 4.1], any element of \(\Lambda\) is of the form \((X - 2^{-1}Z\Delta Z^T T^{(v)} Z)\), where \(X \in K_v\) and \(Z \in M_{2\times v}(\mathbb{F}_q)\). Note that \(\{\varphi_0\} = \{P_1\}\) is the trivial orbit of \(G_{01}\) on \(\Lambda\). We have
Theorem 2.4. The nontrivial orbits of \( G_{01} \) on \( \Lambda \) have the following representatives:

\[
\begin{align*}
\varphi_1(r) &= ([\mathcal{A}_2, 0^{(v-2)}] f^{(v)}(E_1 a E_1)) \\
\varphi_2(r, a) &= ([1 - 2^{-1}(1 - z a^2), \mathcal{A}_2, 0^{(v-2)}] f^{(v)}(E_1 a E_1)) \\
\varphi_3(r, a) &= ([\mathcal{A}_2, 0^{(v-2)}] + [1 - 2^{-1}(1 - z a^2), 0^{(v-1)}] f^{(v)}(E_1 a E_1)) \\
\varphi_4(r) &= ([0, \mathcal{A}_2, 0^{(v-2)}] + [-2^{-1} \Delta, 0^{(v)}] f^{(v)}(E_1 E_2)) \\
\varphi_5(r) &= ([\mathcal{A}_2, 0^{(v-2)}] + [-2^{-1} \Delta, 0^{(v-2)}] f^{(v)}(E_1 E_2)) \\
\varphi_6(r) &= ([2^{-1} \Delta, \mathcal{A}_2, 0^{(v-2)}] f^{(v)}(E_1 E_2)) \\
\varphi_7(r, b) &= ([b, \mathcal{A}_2, 2^{-1} \Delta, 0^{(v-2)}] f^{(v)}(E_1 E_2)) \\
\varphi_8(r) &= ([2^{-1} \Delta, \mathcal{A}_2, 0^{(v-2)}] + [-2^{-1} \Delta, 0^{(v-2)}] f^{(v)}(E_1 E_2))
\end{align*}
\]

where \( a \in [0, 1], b \in \Omega \).

\[
Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}
\]

and \( \mathcal{K} \) is given by Theorem 2.1. Moreover the rank of \( G_{01} \) on \( \Lambda \) is

\[ (q + 7)/2 \cdot [v/2] + 4 \cdot [(v - 1)/2] + [(v - 2)/2] + 3. \]

Proof. Suppose \( P \in \Lambda \setminus \{P_1\} \). Then \( P = (X - 2^{-1} \Delta A Z') f^{(v)}(Z) \), where \( \text{rank}(X - 2^{-1} \Delta A Z') > 0 \).

If \( Z = 0 \), then \( \text{rank}(X) = 2r > 0 \), which implies that there exists a \( T \in GL_r(F_q) \) satisfying \( T^*X T = [\mathcal{A}_2, 0^{(v-2)}] \). Observe \( \{1, T^*T^{-1}, T f^{(v)}\} \in G_{01} \) carries \( P \) to \( \mathcal{A}_2 \).

If \( Z \neq 0 \), then \( \text{rank}(Z) = 1 \). Then there exists an \( S \in GL_r(F_q) \) such that \( S'Z = (x E_1 E_1) \), where \( (x, y) \neq (0, 0) \). By Theorem 2.1 there exists a \( T \in \mathcal{O}_1 \), which is of the form \( \mathcal{A}_2 \), such that \( T'(S'S)T' \) is \( 0^{(v)} \) or of form \( \mathcal{A}_2 \). By Lemma 2.1 there exists an \( S_{11} \in O_{2 \times 0 + 2 \Delta}(F_q) \) such that \( T'(S'S)_{11} = b(E_1 a E_1) \), where \( a \in [0, 1] \) and \( b \in F_q^* \). Observe

\[
(TS')^*(Z\Delta Z')ST = (T'S'Z_{11})S \Delta (S_{11}Z' S) = b^2(E_1 a E_1) \Delta (E_1 a E_1) = [b^2(1 - z a^2), 0^{(v-1)}].
\]

Let \( T_1 = [b^{-1}, f^{(v)}] \). Then \( [SSTT_1, ((SSTT_1)'^{-1})^{-1}, S_{11}] \in G_{01} \) carries \( P \) to

\[
((SSTT_1)'(X - 2^{-1} \Delta Z') SSTT_1) f^{(v)}(SSTT_1)'Z_{11}).
\]

Note that \( \text{rank}((SSTT_1)'(X - 2^{-1} \Delta Z') SSTT_1) f^{(v)}(SSTT_1)'Z_{11}) = \text{rank}(X - 2^{-1} \Delta A Z') \).

If \( (TS)'XST = 0^{(v)} \), then \( [SSTT_1, ((SSTT_1)'^{-1})^{-1}, S_{11}] \) carries \( P \) to \( \mathcal{A}_2 \) for \( r = 0 \).

If \( (TS)'XST = [0, \mathcal{A}_2, 0^{(v-2)}] \), then \( [SSTT_1, ((SSTT_1)'^{-1})^{-1}, S_{11}] \) carries \( P \) to \( \mathcal{A}_2 \) for \( r > 0 \).

If \( (TS)'XST = [\mathcal{A}_2, 0^{(v-2)}] \), then \( [SSTT_1, ((SSTT_1)'^{-1})^{-1}, S_{11}] \) carries \( P \) to

\[
(b^{-1} \mathcal{A}_2, \mathcal{A}_2, 2^{-1} \Delta, 0^{(v-2)}) f^{(v)}(E_1 a E_1). \]

Suppose \( T_2 = [1, b, f^{(v)}] \). Then \([SSTT_1T_2, ((SSTT_1T_2)'^{-1})^{-1}, S_{11}] \) carries \( P \) to \( \mathcal{A}_2 \).

Case 2: \( \text{rank}(Z) = 2 \). Then there exists an \( S \in GL_r(F_q) \) such that \( S'Z = (E_1 E_2) \). By Theorem 2.1 there exists a \( T \in \mathcal{O}_2 \), which is of the form \( \mathcal{A}_2 \) satisfying \( T'(S'S)T' \) is \( 0^{(v)} \) or of form \( \mathcal{A}_2 \). Let \( T_1 = [T_1^{-1}, f^{(v)}] \). Then \( [SSTT_1, ((SSTT_1)'^{-1})^{-1}, I_{22}] \in G_{01} \) carries \( P \) to

\[
((SSTT_1)'(X - 2^{-1} \Delta Z') SSTT_1) f^{(v)}(E_1 E_2)).
\]
Observe
\[(S T T t)'(Z Δ Z')S T T t = (T t' S') Z Δ (Z' S T T t) = [Δ, 0^{(r-2)}] \]
and
\[\text{rank}((S T T t)'(X - 2^{-1} Z Δ Z')S T T t) = \text{rank}(X - 2^{-1} Z Δ Z').\]

If \((S T)' X(S T) = 0^{(r)}\), then \([S T T t, ((S T T t)')^{-1}, 1^{(2)})\] carries \(P\) to \(\mathbb{F}_q\) for \(r = 0\).
If \((S T)' X(S T) = [0, A_2, 0^{(r-1-2r)}]\), then \([S T T t, ((S T T t)')^{-1}, 1^{(2)})\] carries \(P\) to
\[([Y_{u,v}, A_{2-2}, 0^{(r-1-2r)}] + [-2^{-1}Δ, 0^{(r-2)}] \ P^{(r)} (E_1 E_2)),\]
where
\[Y_{u,v} = \begin{pmatrix}
0 & 0 & u \\
0 & 0 & v \\
-u & -v & 0
\end{pmatrix}, \quad (T t')^{-1}{1, 0 \ 1}.\]

Take \(T_2 = [1^{(2)}, v^{(-1)}, 1^{(r-3)}]\) or \([1^{(2)}, u^{(-1)}, 1^{(r-3)}]\) according to \(u = 0\) or not, respectively. Then \([S T T t_2, ((S T T t_2)')^{-1}, 1^{(2)})\] carries \(P\) to \(\mathbb{F}_q\) for \(r > 0\), or
\[([Y_{1,c}, A_{2-2}, 0^{(r-1-2r)}] + [-2^{-1}Δ, 0^{(r-2)}] \ P^{(r)} (E_1 E_2)),\]
where \(c = u^{(-1)}v\). When \(c^2 - z\) is a square element, we may choose an \(s \in \mathbb{F}_q^\ast\) such that \(c^2 - z = s^2\).
Let \(A = [A_{11}, s^{-1}, 1^{(r-3)}],\) where
\[A_{11} = s^{-1}
\begin{pmatrix}
c & -z \\
-1 & c
\end{pmatrix}.
\]

By Lemma 2.3 \(A_1^{-1} \in O_{2 \times 0 + 2(\mathbb{F}_q)}\), and \([A, (A')^{-1}, (A_1')^{-1}] \in G_{01}\) carries
\[([Y_{1,c}, A_{2-2}, 0^{(r-1-1)}] + [-2^{-1}Δ, 0^{(r-2)}] \ P^{(r)} (E_1 E_2))\]
to \(\mathbb{F}_q\) for \(r > 0\). When \(c^2 - z\) is a non-square element, we may choose an \(s \in \mathbb{F}_q^\ast\) such that \(s^2(c^2 - z) = 1 - z\). Let \(B = [B_{11}, s, 1^{(r-3)}],\) where
\[B_{11} = \frac{1}{s(c^2 - z)}
\begin{pmatrix}
c - z & z(c - 1) \\
-1 & c - z
\end{pmatrix}.
\]

By Lemma 2.3 \(B_1^{-1} \in O_{2 \times 0 + 2(\mathbb{F}_q)}\), and \([B, (B')^{-1}, (B_1')^{-1}] \in G_{01}\) carries
\[([Y_{1,c}, A_{2-2}, 0^{(r-1-2r)}] + [-2^{-1}Δ, 0^{(r-2)}] \ P^{(r)} (E_1 E_2))\]
to \(\mathbb{F}_q\).

If \((S T)' X(S T) = [0^{(2)}, A_2, 0^{(r-2r)}]\), then \([S T T t_1, ((S T T t_1)')^{-1}, 1^{(2)})\] carries \(P\) to \(\mathbb{F}_q\).
If \((S T)' X(S T) = [A_2, 0^{(r-2r)}]\), then \([S T T t_1, ((S T T t_1)')^{-1}, 1^{(2)})\] carries \(P\) to \(φ(r, b),\) where \((T t_1')^{-1} A_2 T t_1^{-1} = hA_2\) for some \(h \in \mathbb{F}_q\). Note that \([-1, 1^{(r-1)}, -1, 1^{(r-1)}, -1, 1] \in G_{01}\) carries \(φ(r, b)\) to \(φ(-r, -b).\) So we may choose \(b \in Ω\).
If \((S T)' X(S T) = [K, A_{2-4}, 0^{(r-2r)}]\), then \([S T T t_1, ((S T T t_1)')^{-1}, 1^{(2)})\] carries \(P\) to
\[([U, A_{2-4}, 0^{(r-2r)}] + [-2^{-1}Δ, 0^{(r-2)}] \ P^{(r)} (E_1 E_2)),\]
where
\[U = \begin{pmatrix}
0 & (T t_1')^{-1} \\
-T t_1^{-1} & 0
\end{pmatrix}.\]
Pick $T_3 = [I^2, T_{11}', I^{v-4}]$. Then $[S TT T_3, (S TT T_3)^{-1}, f^{(2)}]$ carries $P$ to $(2)$.

What is left to show that no two subspaces in (6) - (13) can fall into the same orbit. As an example, we show that any two distinct $\varphi_2(r, a)$ and $\varphi_2(r, a')$ can’t fall into the same orbit, and the rest cases may be handled in a similar way. If there exists an element of $G_{01}$ of form (1) carrying $\varphi_2(r, a)$ to $\varphi_2(r, a')$, then $T$ is of the form

$$T = \begin{pmatrix} t & 0 \\ T_{21} & T_{22} \end{pmatrix},$$

where $t(1, a)S = (1, a')$. By [11, Theorem 6.4], the subspaces $(1, a)$ and $(1, a')$ is of the same type. Since $a, a' \in [0, 1]$, we have $a = a'$, a contradiction.

Therefore, the desired result follows.

For each vertex $Q$ of $\Lambda$, the symbol $\overline{Q}$ denotes the suborbit containing $Q$. By [11, Theorem 1.6, Theorem 3.16, Theorem 6.21], we have

$$|GL_{v}(F_{q})| = q^{v(v-1)/2} \prod_{i=1}^{v} (q^{i} - 1),$$

$$|S p_{2v}(F_{q})| = q^{v^2} \prod_{i=1}^{v} (q^{2i} - 1),$$

$$|O_{2v+2, \Delta}(F_{q})| = q^{v(v+1)} \prod_{i=1}^{v} (q^{i} - 1) \prod_{i=0}^{v+1} (q^{i} + 1).$$

**Theorem 2.5.** The nontrivial orbits of $G_{01}$ on $\Lambda$ have lengths as following:

$$|\varphi_1(r)| = \frac{|GL_{v}(F_{q})|}{|S p_{2v}(F_{q})| \cdot |GL_{v-2v}(F_{q})| \cdot q^{2(v-2v)}},$$

$$|\varphi_2(r, a)| = \frac{|S p_{2v}(F_{q})| \cdot |GL_{v-2v}(F_{q})| \cdot 2q^{2(v+1)v-2v-1}}{(q + 1)|GL_{v}(F_{q})|},$$

$$|\varphi_3(r, a)| = \frac{|S p_{2v-2v}(F_{q})| \cdot |GL_{v-2v}(F_{q})| \cdot 2q^{2v^2 + 2v - 2}}{(q + 1)|GL_{v}(F_{q})|},$$

$$|\varphi_4(r)| = \frac{|S p_{2v-2v}(F_{q})| \cdot |GL_{v-2v}(F_{q})| \cdot 2q^{2v^2 + 2v - 2}}{(q + 1)|GL_{v}(F_{q})|},$$

$$|\varphi_5(r)| = \frac{|S p_{2v-2v}(F_{q})| \cdot |GL_{v-2v}(F_{q})| \cdot 2q^{2v^2 + 2v - 2}}{|GL_{v}(F_{q})|},$$

$$|\varphi_6(r)| = \frac{|S p_{2v-2v}(F_{q})| \cdot |GL_{v-2v}(F_{q})| \cdot q^{2v^2 + 2v - 2}}{2|GL_{v}(F_{q})|},$$

$$|\varphi_7(r)| = \frac{|S p_{2v-2v}(F_{q})| \cdot |GL_{v-2v}(F_{q})| \cdot q^{2v^2 + 2v - 2}}{|GL_{v}(F_{q})|}.$$

**Proof.** We only calculate $|\varphi_3(r, a)|$ and $|\varphi_7(r, b)|$. The length of other suborbits may be computed in a similar way.

Let $G_{0}(r, a)$ be the stabilizer of $\varphi_3(r, a)$ in $G_{01}$, and let $[T, (T')^{-1}, S]$ be any element of $G_{0}(r, a)$. Then

$$T'([A_{2v}, 0^{v-2v}]) = [2^{-1}(1 - za^2), 0^{v-1})] \cdot T = [A_{2v}, 0^{v-2v}]) - [2^{-1}(1 - za^2), 0^{v-1})]$$
and $T(E_1 aE_1)S = (E_1 aE_1)$, which imply that $\mu(1\ a)S = (1\ a)$ and
\[
T = \begin{pmatrix}
1 & 1 & 2r-2 & r-\nu-2r \\
\mu & 0 & 0 & 0 \\
t & \mu & -\mu T_{31} T_{32} T_{33} & 0 \\
T_{31} & T_{32} & T_{33} & 0 \\
T_{41} & T_{42} & T_{43} & T_{44}
\end{pmatrix}
\]
where $\mu^2 = 1$ and $T'_{33} T_{32} T_{33} = A_{2r-2}$. By Lemma \[2.3\] $\mu(1\ a)S = (1\ a)$ implies that $S$ is one of the following forms

\[
\begin{cases}
\mu I(2), \mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } a = 0, \\
\mu I(2), \mu \begin{pmatrix} 1 + z & 2 \\ -2z & -(1 + z) \end{pmatrix} & \text{if } a = 1.
\end{cases}
\]

Hence $|G_3(r, a)| = |S p_{2r-2}(\mathbb{F}_q)| \cdot |GL_{2r-2}(\mathbb{F}_q)| \cdot 4q^{2(r-2r)}$ and

\[
|\varphi_3(r, a)| = |G_0 \setminus G_3(r, a)| = \frac{(q + 1)|GL_r(\mathbb{F}_q)|}{|S p_{2r-2}(\mathbb{F}_q)| \cdot |GL_{2r-2}(\mathbb{F}_q)| \cdot 2q^{2(r-2r)}}.
\]

Let $G_1(r, b)$ be the stabilizer of $\varphi_1(r, b)$ in $G_0$. Then $G_1(r, b)$ consists of matrices $[T, (T')^{-1}, (T_1')^{-1}]$, where
\[
T = \begin{pmatrix}
0 & 2r-2 & r-\nu-2r \\
T_{11} & 0 & 0 \\
0 & T_{22} & 0 \\
T_{31} & T_{32} & T_{33}
\end{pmatrix}
\]

$T_{11} \in O_{2r+0+2,2}(\mathbb{F}_q)$, $T_{11} A_2 T_{11} = A_2$ and $T_{11} T_{22} A_{2r-2} T_{2} = A_{2r-2}$. By Lemma \[2.3\] the matrix $T_{11}'$ satisfying $T_{11}' \in O_{2r+0+2,2}(\mathbb{F}_q)$ and $T_{11}' A_2 T_{11}' = A_2$ is of the form
\[
T_{11}' = \begin{pmatrix} x & y & \nu \\ y & x & \nu \\
\nu & \nu & x \end{pmatrix},
\]

where $x^2 - y^2 = 1$. By \[11\] Lemma 1.28], the number of solutions of the equation $x^2 - y^2 = 1$ is $q + 1$. Hence $|G_1(r, b)| = |S p_{2r-2}(\mathbb{F}_q)| \cdot |GL_{2r-2}(\mathbb{F}_q)| \cdot (q + 1)q^{2(r-2r)}$ and

\[
|\varphi_1(r, b)| = |G_0 \setminus G_3(r, b)| = \frac{2|GL_r(\mathbb{F}_q)|}{|S p_{2r-2}(\mathbb{F}_q)| \cdot |GL_{2r-2}(\mathbb{F}_q)| \cdot q^{2(r-2r)}}.
\]

3. Quasi-strongly regular graphs

As a generalization of strongly regular graphs, quasi-strongly regular graphs were discussed by W. Golightly, W. Hayworth and D.G. Sarvate \[4\] and F. Goldberg \[5\]. Let $c_1, c_2, \ldots, c_p$ be distinct non-negative integers. A connected graph of degree $k$ on $n$ vertices is quasi-strongly regular with parameters $(n, k, \lambda, c_1, c_2, \ldots, c_p)$ if any two adjacent vertices have $\lambda$ common neighbors, and any two non-adjacent vertices have $c_i$ common neighbors for some $i (1 \leq i \leq p)$. 


Since $\Gamma$ is a regular near polygon, the induced subgraph on $\Lambda$ is edge regular, denoted by the same symbol $\Lambda$. Therefore, $\Lambda$ is quasi-strongly regular. In this section we compute all the parameters of $\Lambda$.

Let $\Lambda(P)$ be the set of neighbors of $P$ in $\Lambda$. Clearly, $|\Lambda(P) \cup \Lambda(Q)| = 0$ whenever $\dim(P + Q) > v + 2$. Note that for the vertex $P_1$ in $\Lambda$ as in Section 2, the subspace $Q \in \Lambda$ satisfying $\dim(Q + P_1) = v + 2$ lies in the set

$$\varphi(1) \cup \varphi(0) \cup \varphi(1, a) \cup \varphi(1, b).$$

To study $|\Lambda(P) \cap \Lambda(Q)|$ for any two vertices $P$ and $Q$ with $\dim(P + Q) = v + 2$, by Theorem 2.3 it suffices to consider $|\Lambda(P_1) \cap \Lambda(Q)|$, where $Q \in \{\varphi(1), \varphi(1, a), \varphi(0), \varphi(1, b)\}, a \in \{0, 1\}$ and $b \in \Omega$.

**Lemma 3.1.** For any vertex $R = (A - 2^{-1}C\Delta C, \Gamma)$ of $\Lambda$, the neighborhood of $R$ is

$$\Lambda(R) = \{(A - 2^{-1}(C\Delta C + D\Delta F + 2D\Delta C)) \Gamma + D) \in M_{2}(\mathbb{F}_q), \text{ rank } D = 1\}.$$  

**Proof.** Note that $\Lambda(R)$ consists of matrices with the form $(A - 2^{-1}(C\Delta C - X \Gamma) + D)$, where $X \in M_{2}(\mathbb{F}_q)$, $D \in M_{2}(\mathbb{F}_q)$, rank $(X, D) = 1$ and $X + Y + C\Delta F + D\Delta C + D\Delta F = 0$. It follows that rank $D = 1$. So we may write $D = D_0(x, y)$ and $X = D_0W$, where $0 \neq D_0 \in M_{2}(\mathbb{F}_q), (x, y) \neq (0, 0)$ and $W \in M_{2}(\mathbb{F}_q)$. Let $C = (C_1, C_2)$ and $T(D_0) = E_1$ for some $T \in GL_2(\mathbb{F}_q)$. Then

$$E_1(T(W + xC_1 - yC_2) + T(W + xC_1 - yC_2)E_1^\dagger + (x^2 - y^2z^2)E_1E_1^\dagger = 0.$$

It follows that $T(W + xC_1 - yC_2) = -2^{-1}(x^2 - y^2z^2)E_1$. So $W = -2^{-1}(x^2 - y^2z^2)D_0 - xC_1 + yC_2$ and $X = -2^{-1}(D\Delta F + 2D\Delta C)$. The desired result follows. \hfill $\square$

Note that when $v = 1$, any element of $\Lambda$ is of the form $(-2^{-1}(a^2 - zb^2) + 1, (a, b))$, where $a, b \in \mathbb{F}_q$. Then $\Lambda$ is a clique with $q^2$ vertices.

**Lemma 3.2.** Let $P$ and $Q$ be any two vertices of $\Lambda$ with $\dim(P + Q) = v + 2$. If $v \geq 2$, then $|\Lambda(P) \cap \Lambda(Q)|$ is equal to $0, q^2, q^2 - 1$ or $q^2 + q$.

**Proof.** For any $Q \in \{\varphi(1), \varphi(1, a), \varphi(0), \varphi(1, b)\}$, it suffices to show that $|\Lambda(P_1) \cap \Lambda(Q)| = 0, q^2, q^2 - 1$ or $q^2 + q$. We only compute $|\Lambda(P_1) \cap \Lambda(\varphi(1, a))$, and the others are treated similarly.

Let $R \in \Lambda(P_1) \cap \Lambda(\varphi(1, a))$. From $R \in \Lambda(P_1)$ and Lemma 3.1 we know that $R$ is of the form $R = (-2^{-1}D\Delta F + \Gamma) D_1)$, where $D \in M_{2}(\mathbb{F}_q)$ and rank $D = 1$. Again from $R \in \Lambda(\varphi(1, a))$ and Lemma 3.1 we know that $R = ([A_2, 0^{(q - 2)}] - [2^{-1}(1 - za^2), 0^{(q - 1)}] - 2^{-1}(D_1\Delta D_1^\dagger + 2D_1\Delta(E_1 aE_1) + D_1)$, where $D_1 \in M_{2}(\mathbb{F}_q)$ and rank $D_1 = 1$. Therefore, $D = (E_1 aE_1) + D_1$ and

$$-2^{-1}D\Delta F = [A_2, 0^{(q - 2)}] - [2^{-1}(1 - za^2), 0^{(q - 1)}] - 2^{-1}(D_1\Delta D_1^\dagger + 2D_1\Delta(E_1 aE_1) + D_1).$$

It follows that $-2^{-1}(E_1 aE_1)\Delta D_1^\dagger = [A_2, 0^{(q - 2)}] - 2^{-1}D_1\Delta(E_1 aE_1)$; and so $D$ is of the form

$$D = \left( \begin{array}{cccc} c + 1 & zd_2 - 2 & 0 & \cdots & 0 \\ a + d_1 & d_2 & 0 & \cdots & 0 \end{array} \right).$$

where $cd_2 - d_1(zd_2 - 2) = 0$. Observe the number of solutions $(c, d_1, d_2)$ satisfying $cd_2 - d_1(zd_2 - 2) = 0$ is $q + (q - 1)q = q^2$. Hence $|\Lambda(P_1) \cap \Lambda(\varphi(1, a))| = q^2$. \hfill $\square$
Theorem 3.3. Let \( \nu \geq 2 \). Then \( \Lambda \) is a quasi-strongly regular graph with parameters
\[
(q^{(\nu+3)/2}, (q^\nu - 1)(q + 1), q^\nu + q^2 - q - 1; 0, q^2, q^2 - 1, q^2 + q).
\]

Proof. Since \( \Lambda \) consists of the vertices as the form \((X - 2^{-1}Z \Delta Z^t \; t^v \; Z)\), where \( X \) is a \( \nu \times \nu \) alternate matrix, and \( Z \in M_{\nu, 2}(\mathbb{F}_q) \), we have \( n = q^{(\nu+3)/2} \). By Theorem 2.5,
\[
k = |\varphi_2(0, 0)| + |\varphi_2(0, 1)| = 2|\varphi_2(0, 0)| = (q^\nu - 1)(q + 1).
\]

Note that \( \varphi_2(0, a) \in \Lambda(P_1) \). In order to compute the parameter \( \lambda \), it suffices to compute the size of the common neighbors of \( P_1 \) and \( \varphi_2(0, a) \). Let \( R \in \Lambda(P_1) \cap \Lambda(\varphi_2(0, a)) \). From \( R \in \Lambda(P_1) \) and Lemma 3.1 we know that \( R \) is of the form \( R = (-2D\Delta D^t \; t^v \; D) \), where \( D \in M_{\nu, 2}(\mathbb{F}_q) \) and \( \text{rank } D = 1 \). Similar to the proof of Case 2 in Lemma 3.2, \( D \) is of rank 1 with the form
\[
D = \begin{pmatrix}
c + 1 & azd_2 & \cdots & azd_r \\
d_1 + 1 & d_2 & \cdots & d_r
\end{pmatrix}.
\]

Observe the number of matrices \( D \) satisfying \((d_2, \ldots, d_r) = (0, \ldots, 0)\) and \((d_2, \ldots, d_r) \neq (0, \ldots, 0)\) are \( q^2 - 1 \) and \((q^\nu - 1)q\), respectively. So \( \lambda = |\Lambda(P_1) \cap \Lambda(\varphi_2(0, a))| = q^\nu + q^2 - q - 1 \). The rest parameters of \( \Lambda \) are listed in Lemma 3.2. \( \square \)

4. Association schemes

In this section we discuss the association scheme based on \( \Lambda \) when \( \nu = 2 \).

A \( d \)-class association scheme \( \mathcal{X} \) is a pair \((X, [R_i]_{i \in I})\), where \( X \) is a finite set, and each \( R_i \) is a nonempty subset of \( X \times X \) satisfying the following axioms:

(i) \( R_0 = \{(x, x) \mid x \in X\} \);
(ii) \( X \times X = R_0 \cup R_1 \cup \cdots \cup R_d \); \( R_i \cap R_j = \emptyset \) \( (i \neq j) \);
(iii) \( R_i = R_{i'} \) for some \( i' \in \{0, 1, \ldots, d\} \), where \( R_{i'} = \{(y, x) \mid (x, y) \in R_i\} \);
(iv) for all \( i, j, k \in \{0, 1, \ldots, d\} \), there exists an integer \( p^k_{ij} = |\{z \in X \mid (x, z) \in R_k, (z, y) \in R_j\}| \)

for every \((x, y) \in R_i\).

The integers \( p^k_{ij} \) are called the intersection numbers of \( \mathcal{X} \), and \( k_i (= p^i_0) \) is called the valency of \( R_i \). Furthermore, \( \mathcal{X} \) is called symmetric if \( i' = i \) for all \( i \). As for more information concerning association schemes, the readers may consult [1][2].

Let \( G \) be a transitive permutation group on a finite set \( X \), and \( R_0, R_1, \ldots, R_d \) be the orbits of the induced action of \( G \) on \( X \times X \). It is well known that \((X, [R_i]_{i \in I})\) is an association scheme ([1], §2.2]).

Note that the action of \( G_0 \) on \( \Lambda \times \Lambda \) determines an association scheme. We shall discuss the association scheme in the case \( \nu = 2 \).

In the rest we always assume that \( \nu = 2 \). By Theorem 2.3 the orbits of \( G_0 \) on \( \Lambda \) have the following representatives:

\[
\varphi_0, \varphi_1(1), \varphi_2(0, a), \varphi_3(1, a), \varphi_4(0), \varphi_7(1, b),
\]

where \( a \in \{0, 1\}, b \in \Omega \). For the action of \( G_0 \) on \( \Lambda \times \Lambda \), let \( R_0, R_1, R_2, R_3, R_4, R_5 \) denote the orbits containing \((\varphi_0, \varphi_0), (\varphi_0, \varphi_1(1)), (\varphi_0, \varphi_2(0, a)), (\varphi_0, \varphi_3(1, a)), (\varphi_0, \varphi_4(0), (\varphi_0, \varphi_7(1, b)), \)

respectively. Then \( R_0, R_1, R_2, R_3, R_4, R_5 \) are all the orbits of the action of \( G_0 \) on \( \Lambda \times \Lambda \).
Lemma 4.3. Let $c \in \Omega$. Then the number of $(T, S)$ satisfying $T' \in S \{F_q \mid |O_{2\times 0+2\Delta}(F_q)| = 2q(q - 1)(q + 1)^2 \}
$ and $T'[c, 1]S = [c, 1]$ is $q + 1$.

Proof. Since $T'[c, 1]S = [c, 1]$, by Lemma 2.3, $T = \begin{pmatrix} \mu & c^{-2}s^2 \\ s & \mu \end{pmatrix}$, $S = \begin{pmatrix} \mu & -c^{-1}s \\ -c^{-1}s & \mu \end{pmatrix}$, where $\mu, s, c \in F_q$ and $\mu^2 - c^{-2}s^2 = 1$. By [11, Lemma 1.28], the number of $(\mu, c)$ satisfying $\mu^2 - c^{-2}s^2 = 1$ is $g + 1$, as desired. \qed

Lemma 4.4. Let $c \in \Omega$. Then the number of $(T, S)$ satisfying $T' \in S \{F_q \mid |O_{2\times 0+2\Delta}(F_q)| = 2q(q - 1)(q + 1)^2 \}
$ and $T'[c, 1]S = [c, 1]$ is $q + 1$.

Proof. Since $T'[c, 1]S = [c, 1]$, by Lemma 2.3, $T = \begin{pmatrix} \mu & c^{-2}s^2 \\ s & \mu \end{pmatrix}$, $S = \begin{pmatrix} \mu & -c^{-1}s \\ -c^{-1}s & \mu \end{pmatrix}$, where $\mu, s, c \in F_q$ and $\mu^2 - c^{-2}s^2 = 1$. By [11, Lemma 1.28], the number of $(\mu, c)$ satisfying $\mu^2 - c^{-2}s^2 = 1$ is $g + 1$, as desired. \qed

Lemma 4.5. The representatives $\phi_{1x}, \phi_{2x, a}, \phi_{3x, c}$ listed in Lemma 4.2 satisfy

\[\begin{align*}
(\varphi_0, \varphi_0) &\in R_0, \\
(\varphi_0, \varphi_1) &\in R_1, \\
(\varphi_0, \varphi_2) &\in R_2, \\
(\varphi_0, \varphi_3) &\in R_3, \\
(\varphi_0, \varphi_4) &\in R_4, \\
(\varphi_0, \varphi_5) &\in R_5_{s^{-1}}, \\
(\varphi_0, \varphi_6) &\in R_5_{s^{-1}}, \\
(\varphi_0, \varphi_7) &\in R_5_{a^{-1}}, \\
(\varphi_0, \varphi_8) &\in R_5_{a^{-1}},
\end{align*}\n
where $d \in F_q \backslash \{0, 1\}$, $\varepsilon, \varepsilon_1, \varepsilon_2 \in \{1, -1\}$, $\varepsilon_1 \varepsilon^{-1}, \varepsilon_1 \varepsilon^{-1}d, \varepsilon_2 \varepsilon^{-1}(1 - d) \in \Omega$.

Proof. We only show $\phi_{30, c} \in R_4$ and $(\varphi_{30, c}, \varphi_1) \in R_{5_{s^{-1}}}$. The left cases may be treated similarly, and will be omitted. Note that $[c^{-1}, 1, c, 1, F^2] \in G_0$ carries $\varphi_0$ and $\varphi_{30, c}$ to $\varphi_0$ and $\varphi_{30, c}$, respectively, so $(\varphi_0, \varphi_{30, c}) \in R_4$. Let $\varepsilon = 1$ or $-1$ according to $c^{-1} \in \Omega$ or $-c^{-1} \in \Omega$, respectively. Then

\[\begin{pmatrix}
-c^{-1} & 0 \\
0 & -\varepsilon \\
\frac{c}{2} & 0 & -c & 0 & -c \\
0 & -\frac{c}{2} & 0 & -\varepsilon & 0 & -\varepsilon \\
-1 & 0 & 1 & 0 \\
0 & \varepsilon & 0 & \varepsilon 
\end{pmatrix} \in G_0\n\]
carries $\varphi_{30, c}$ and $\varphi_1$ to $\varphi_0$ and $\varphi_7(1, \varepsilon c^{-1})$, respectively, which implies $(\varphi_{30, c}, \varphi_1) \in R_{5_{s^{-1}}}$. \qed
Theorem 4.4. The configuration $\mathcal{X} = (\Lambda, \{R_0, R_1, R_2, R_3, R_4, R_5\}_{a \in \{0,1,0\}^2})$ is a symmetric association scheme with class $(q + 11)/2$, whose non-zero intersection numbers $p_{ij}^1$ are given by

$$
p_{01}^1 = p_{10}^1 = 1, \ \ p_{11}^1 = q - 2, \ \ p_{21,3}^1 = p_{3,2}^1 = (q - 1)(q + 1)^2/2, \ \ p_{1,3}^1 = (q - 2)(q - 1)(q + 1)^1/2, \ \ p_{4,5}^1 = p_{5,4}^1 = p_{2,5}^1 = 2q(q^2 - 1),
$$

where $d \in \mathbb{F}_q \setminus \{0, 1\}$, $\varepsilon \in \{1, -1\}$ and $sbd^{-1}(1 - d) \in \Omega$.

Proof. By Theorem 2.4, $\mathcal{X}$ forms an association scheme of class $(q + 11)/2$.

Now we prove $\mathcal{X}$ is symmetric. Since

$$(1 \ 0 \\
0 \ -1 \\
0 \ 1 \ 1 \ 0 \\
1 \ 0 \ 0 \ -1) \in G_0^{(2)}$$

interchanges $\varphi_1(1)$ and $\varphi_0$, $R_1 = R_1$. The left cases can be treated similarly, and will be omitted.

In order to compute non-zero intersection numbers $p_{ij}^1$ of $\mathcal{X}$, we need consider the cases listed in Lemma 4.3. Here we only calculate $p_{2,1,3}^1$ and $p_{4,5}^1$ by the way of examples.

Let $G_{\varphi_0u}$ be the stabilizer of $\varphi_{20u}$ in $G_{\varphi_1}$, by Lemma 4.3, $p_{2,1,3}^1 = [G_{\varphi_1} : G_{\varphi_0u}]$, the index of $G_{\varphi_0u}$ in $G_{\varphi_1}$. Note that $G_{\varphi_0u}$ consists of matrices $[T, (T')^{-1}, S]$, where

$$T' \in S_p(F_q), \ \ T'[1, 0]T = [1, 0], \ S \in O_{2 \times 0 + 2A}(F_q) \ \ and \ \ T'(E_1 \ aE_1)S = (E_1 \ aE_1).$$

It follows that

$$T = \begin{pmatrix} \mu & 0 \\ t & \mu \end{pmatrix}, \ S = \mu T^{(2)} \ \ or \ \ S = \frac{\mu}{1 - za^2} \begin{pmatrix} 1 + za^2 & 2a \\ -2za & -1 + za^2 \end{pmatrix},$$

where $\mu^2 = 1$. Therefore, $|G_{\varphi_0u}| = 4q$ and

$$p_{2,1,3}^1 = |G_{\varphi_0u}| = |G_{\varphi_1} : G_{\varphi_0u}] = (q - 1)(q + 1)^2/2.$$

Let $G_{\varphi_0u}$ be the stabilizer of $\varphi_{30u}$ in $G_{\varphi_1}$, where $c \in \Omega$. Then $G_{\varphi_0u}$ consists of matrices $[T, (T')^{-1}, S]$, where

$$T' \in S_p(F_q), \ \ T'[c^2, -z]T = [c^2, -z], \ S \in O_{2 \times 0 + 2A}(F_q) \ \ and \ \ T'[c, 1]S = [c, 1].$$

Note that $T'[c, 1]S = [c, 1]$ implies $T'[c^2, -z]T = [c^2, -z]$. By Lemma 4.3, $|G_{\varphi_0u}| = q + 1$; and by Lemma 4.3,

$$p_{4,5}^1 = [G_{\varphi_1} : G_{\varphi_0u}] = |G_{\varphi_1}/G_{\varphi_0u}| = 2q(q^2 - 1),$$

which is independent of choices of $c \in \Omega$. Hence $p_{4,5}^1 = 2q(q^2 - 1)$.

Remarks. All the valencies of $\mathcal{X}$ are given by Theorem 2.5. By a similar method in this section, all the intersection numbers $p_{ij}^1$ of $\mathcal{X}$ can be calculated.
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