Mathematical description of recuperative heat transfer in a bioreactor during the filling of an apparatus

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Abstract. Despite the observed trend of a ubiquitous transition to continuously operating apparatuses, the specificity of the production technology of many microbiological products retains the advisability of using batch apparatuses. As an example, we mention the cooled bioreactors and yeast concentrate collectors used in the production of baker's yeast. The work considers the problem of the mathematical description of unsteady recuperative heat transfer in a batch capacitive apparatus (bioreactor) with a product distributor on a heat transfer surface during countercurrent movement of coolants with a filling level of the apparatus changing during the process. The problem is formulated by partial differential equations (in coordinate and time) with a moving boundary between sections with different heat transfer conditions. When formulating the problem, simplifying assumptions and boundary conditions were adopted. The mathematical problem posed is solved using the Laplace transforms. The mathematical description of recuperative heat transfer in bioreactors during the filling of the apparatus showed that with countercurrent movement of heat transfer liquids and coolants, the apparatus has an advantage in terms of the main indicator - heat transfer capacity and has a higher energy efficiency. The performed calculations make it possible to choose the apparatus design correctly and determine the rational thermal regime of its operation regarding related technological operations.

1. Introduction

The increasing demand for microbiological products is due to their exceptional value for humans and farm animals. The hardware design of production microbiological processes has its characteristics. In particular, despite the observed trend of a ubiquitous transition to continuously operating apparatuses, the specificity of the production technology of many microbiological products retains the advisability of using batch apparatuses[1, 2]. As an example, we point out the cooled bioreactors and yeast concentrate collectors used in the production of baker's yeast [3].

Typically, such devices are equipped with a mixer and a heat exchanger in the form of a jacket or a coil. Due to the large capacity of the apparatuses, the period of their filling with the initial product is a significant fraction of the total duration of the process. For intensifying heat transfer during the filling period of the apparatus, various devices are used for distributing the supplied product over the entire heat transfer surface, for example, made in the form of a fixed groove or disk mounted on the shaft of the mixer. The loaded product flows down the heat transfer surface with a thin film and is quickly
cooled by transferring heat through the wall to the coolant flow in the jacket. In the volume of the apparatus under the product level, the heat transfer conditions differ significantly from the heat transfer conditions in the flowing film. The mathematical description of the kinetics of heat transfer is complicated by the fact that during the filling of the apparatus, the level of the product is continually changing. This process leads to the need to formulate the problem of unsteady heat transfer by partial differential equations (in coordinate and time) with a moving boundary between sections with different heat transfer conditions. A mathematical description of such a process is an urgent task [4-8].

2. Statement of the problem

The sum represents the working volume $V_1$, occupied by the product in the filling apparatus $V_1 = V_0 + V_F$,

where $V_0$ is the part of $V_1$, located below the heat transfer surface (volume of the uncooled surface), $m^3$; $V_F = \pi D^2 H / 4$ – part of $V_1$ in contact with the heat transfer surface, $m^3$.

Accordingly, the period of filling the apparatus is presented as

$$\tau = \frac{V_1}{U} = \tau_0 + \tau_F,$$

where $U$ is the volumetric flow rate of the product, $m^3/s$; $\tau_0 = V_0 / U$ s the duration of filling the volume $V_0$, $c$; $\tau_F = \frac{V_F}{U}$ is the duration of filling the volume $V_F$, $s$.

The reference point of the current time is compatible with the moment of the beginning of supply to the apparatus, and the reference point of $x$ and $z$ coordinates is located at the upper level of the cooling jacket (Figure 1).

![Figure 1. Design scheme](image)
We make the following simplifying assumptions:

1) the thermogradient heat transfer in the film of the product and along the heat transfer liquids flow in the jacket is negligible compared to the heat transfer by the flow;

2) due to mixing, the product temperature in the filled part of the apparatus volume \( t_v \) (for \( x > z \)) depends only on time, but not on the coordinate, i.e. \( t_v = t_v(\tau) \neq t_v(x) \);

3) the heat transfer of the product and the coolant with the environment (including due to partial evaporation of the product) is negligible;

4) thermophysical parameters of heat carriers and heat transfer coefficients are constant within the considered heat exchange section of the cooling period;

5) the temperature of the product supplied to the apparatus and the temperature of the coolant supplied to the jacket, as well as their costs, are constant within the considered period.

The task is to determine the functions that describe the change in the product temperature in the apparatus volume \( T_v(\tau) \), the coolant temperature at the outlet of the jacket \( T_c(\tau) \) and the temperature in the coolant collector \( T_c0 \).

3. Problem-solving

We compose the heat balance equation for the elementary portion of the product film with a height \( \delta x \) (Figure 1). The enthalpy flow entering it is equal to \( U\rho i \), and the outgoing one is equal to the sum \( U\rho (i + \delta i) + k\pi D\delta x(t - t_c) \), where the addend represents the heat flow from the product to the coolant through the apparatus wall. Here: \( \rho \) is the density of the product, kg/m\(^3\); \( i \) is the specific enthalpy of the product, J/kg; \( k \) is the heat transfer coefficient from the crumbling wall to the coolant, W/(m\(^2\)°C); \( D \) is the diameter of the apparatus housing, m.

The difference between the incoming and outgoing flows is equal to the rate of change of the enthalpy of the product in the elementary section under consideration, i.e.

\[
-U\rho di - k\pi D\delta x(t - t_c) = \rho \delta \pi D\delta x i/\tau,
\]

whence it follows

\[
U\rho di/dx + \rho \delta \pi D\delta t/\tau = -k\pi D(t - t_c),
\]

where \( t, t_c \) are the temperatures of the product in the falling film and the coolant in the jacket, °C; \( \delta \) is the film thickness of the product, m.

Since \( i = i(t(x, \tau)) \), we find:

\[
di/dx = (di/dt)\delta t/\delta x = c\delta t/\delta x,
\]

\[
di/d\tau = (di/dt)\delta t/\delta \tau = c\delta t/\delta \tau,
\]

where \( c \) is the specific heat of the product, J/(kg·°C).

Given these relationships, we obtain:

\[
U\rho c\delta t/\delta x + \rho c\delta \pi D\delta t/\delta \tau = -k\pi D(t - t_c).
\] (1)

Considering the introductory sections of the coolant flow similarly and with different heat transfer conditions, we obtain:

\[
U_c\rho_c c_\delta \delta t_c/\delta y + \rho_c c_\delta \pi D\delta t_c/\delta \tau = -k_\delta \pi D(t_v - t_c),
\] (2)

\[
U_c\rho_c c_\delta \delta t_c/\delta y - \rho_c c_\delta \pi D\delta t_c/\delta \tau = -k_\delta \pi D(t_v - t_c),
\] (3)

where \( U_c, \rho_c, c_\delta, \delta_\delta \) are the volumetric flow rate, m\(^3\)/s; density, kg/m\(^3\); specific heat, J/(kg·°C) and layer thickness in the jacket of the coolant, m. \( k_\delta \) – heat transfer coefficient from the mass of the product under the filling level to the coolant, W/(m\(^2\)°C); \( t_v \) – is the current temperature of the product in the volume of the apparatus under the level of its filling, °C respectively.

Let us draw up the heat balance equation for the mass of the product \( M_i(\tau) \), which is under the level
of $H - z$. The enthalpy flow introduced by the falling film is equal to $Upi[t(z)]$, and the output heat flux due to heat transfer through the apparatus wall in the section $z \leq x \leq H$ is equal to $U_c \rho_c c_c [t_c(z) - t_{cH}]$, where $t(z), t_c(z)$ – values of $t$ and $t_c$ at the filling level, °C; $t_{cH}$ is the temperature of the coolant at the entrance to the jacket, °C.

The difference between these flows is the rate of change of the enthalpy of the product, i.e.

$$ Upi[t(z)] - U_c \rho_c c_c [t_c(z) - t_{cH}] = dI_v/d\tau, $$

where

$$ I_v = M_v (\tau) i[t_v(\tau)]. $$

Here $I_v$ is the total enthalpy of the product under the level of filling, J; $M_v$ is a product mass under the apparatus filling level (current), kg.

Since $M_v (\tau) = \tau Up$ and therefore

$$ dI_v/d\tau = Upi[t_v(\tau)] + \tau U_p c dt_v/d\tau, $$

the heat balance equation is reduced to the form:

$$ \frac{dI_v}{d\tau} + Upc[t_v - t(z)] = - U_c \rho_c c_c [t_c(z) - t_{cH}], $$

where

$$ z = H - (\tau - \tau_0)[4U/(\pi D^2)]. $$

Here $D$ is the diameter of the apparatus housing, m.

Equations (1)-(5) in conjunction with the boundary conditions

$$ t(0, \tau) = t_i, t_c(0, \tau) = t_{ck}(\tau), t_c(H, \tau) = t_{cH} $$

(6), (7), (8)

and continuity conditions

$$ t_c(y, \tau)|_{y=0} = t_c(x, \tau)|_{x=x} = t_c[x(\tau)] $$

(9)

form a closed system solvable with respect to functions

$$ t(x, \tau), t_c(x, \tau), t_c(y, \tau), t_c[z(\tau)]u t_v(\tau). $$

Here $t_i$ is the initial temperature of the product, °C; $t_{cH}$ is the temperature of the coolant at the outlet of the jacket, °C; $t_c(x, \tau), t_c(y, \tau)$ is temperature of the coolant in the jacket, °C; $t(x, \tau)$ is the temperature of the product in the falling film, °C.

The functions $t_v(\tau)$ and $t_{ck}(y, \tau)$, are of practical interest, describing the kinetics of product cooling and the change in the temperature of the coolant at the outlet of the jacket, since the heat of the outgoing coolant is often used for technological purposes. For example, in yeast production for the preparation of culture media, you can use the water removed from the shirt of yeast-growing apparatus.

The current temperature in the coolant collector $t_{cv}(\tau)$ is found from the balanced equation

$$ U_c \rho_c c_c [t_{cv}(\tau) - t_{cH}] = Upc[t_i - t_v(\tau)], $$

whence it follows:

$$ t_{cv}(\tau) = t_{cH} + \frac{Upc}{U_c \rho_c c_c} [t_i - t_v(\tau)], $$

To solve this problem, we introduce dimensionless coordinates

$$ X = x/H, Y = y/H, Z = z/H, $$

(10)

dimensionless time
\[ \theta = \frac{\tau_{F}}{\tau}, \theta_{0} = \frac{\tau_{0}}{\tau_{F}}, \theta_{1} = \frac{\tau_{1}}{\tau_{F}} \]

and dimensionless temperatures

\[ T = \frac{t}{t_{i}}, T_{c} = \frac{t_{c}}{t_{i}}, T_{CH} = \frac{t_{CH}}{t_{i}}, \]

\[ T_{v} = \frac{t_{v}}{t_{i}}, T_{ck} = \frac{t_{ck}}{t_{i}}, T_{cv} = \frac{t_{cv}}{t_{i}}, \]

where \( \tau, \tau_{0}, \tau_{F}, \tau_{1} \) are the current time and duration of the periods of filling the volumes \( V_{0}, V_{F}, V_{i}, c. \)

In view of definitions (10)-(12) equations (1)-(9) are transformed to the form:

\[ \frac{N}{I} + \frac{ON}{L} = -PN - N \tag{13} \]

\[ \frac{N}{I} + \frac{ON}{L} = -P_{N} - N \tag{14} \]

\[ \frac{N}{J} + \frac{ON}{L} = -P_{N} - N \tag{15} \]

\[ \frac{L}{N} \frac{N}{I} - NK = -Q[N_{K} - N_{L}] \tag{16} \]

\[ Z = 1 + \theta_{0} - \theta \tag{17} \]

\[ T(0, \theta) = 1, T_{c}(0, \theta) = T_{ck}(\theta), T_{c}(1, \theta) = T_{CH}, \tag{18}, (19), (20) \]

\[ T_{c}(Y, \theta)|_{y=0} = T_{c}(X, \theta)|_{x=Z} = T_{c}[Z(\theta)], \tag{21} \]

where \( A, A_{c}, A_{v}, B, B_{c} \) and \( C \) are dimensionless constants:

\[ A = k\pi DH/(U\rho c); A_{c} = k\pi DH/(U_{c}\rho_{c} c_{c}); \tag{22}, (23) \]

\[ A_{v} = k_{v}\pi DH/(U_{c}\rho_{c} c_{c}); \tag{24} \]

\[ B = 4\delta/D; B_{c} = (4\delta/D)(U/U_{c}); \tag{25}, (26) \]

\[ C = A/A_{c} = U_{c}\rho_{c} c_{c}/(Upc). \tag{27} \]

In (24) \( k_{c} \) is the heat transfer coefficient from the mass of the product under the filling level to the coolant, \( W/(m^{2}\cdot K) \).

Usually \( U \) and \( U_{c} \) are values of the same order, and \( \delta, \delta_{c} \ll D. \) Therefore, as can be seen from formulas (25) and (26), the coefficients \( B, B_{c} \ll 1. \) The magnitude of the changes in \( X, Y \) and \( \theta \) during the filling period of the apparatus coincide and are equal to 1. It follows that in equations (13)-(15) the terms containing the partial derivatives of the temperatures to time \( \theta \), are small in comparison with the derivatives of the temperatures with respect to the coordinates \( X \) and \( Y \). Also, it must be taken into account that the coefficients \( k_{a} \) and \( k_{v} \), which are part of formulas (22)-(24) for calculating the parameters \( A, A_{c}, \) and \( A_{v}, \) are usually determined very roughly. The above justifies the use of (13)…(15) in the quasistationary approximation:

\[ \frac{dT}{dX} = -A(T - T_{c}), \tag{28} \]

\[ \frac{dT_{c}}{dX} = -A_{c}(T - T_{c}), \tag{29} \]

\[ \frac{dT_{c}}{dY} = -A_{v}(T_{v} - T_{c}). \tag{30} \]

Applying the Laplace transforms [9-12] to the coordinate \( X \) to (28) and (29), taking into account conditions (18) and (19), we obtain:

\[ L(s) - 1 = -A[L(s) - L_{c}(s)], \]

\[ L_{c}(s) - T_{ck} = -A_{c}[L(s) - L_{c}(s)], \]

where we find Laplace transformation:

\[ L(s) = \frac{s - A_{c} + AT_{ck}}{s(s + A - A_{c})}. \]
\[
L_x(s) = \frac{T_{ck}}{s - A_c} - \frac{A_c(s - A_c + A T_{ck})}{s(s - A_c)(s + A - A_c)}.
\]

and then the originals of the desired functions on the section \(0 \leq X \leq Z:\)

\[
T(X) = \frac{1}{1 - c} \left[ 1 - C \exp((A_c - A)X) - C T_{ck} \left[ 1 - \exp((A_c - A)X) \right] \right]
\]

(31)

\[
T_c(X) = \frac{1}{1 - c} \left[ 1 - \exp((A_c - A)X) - T_{ck} \left[ C - \exp((A_c - A)X) \right] \right]
\]

(32)

Applying the Laplace transforms to the \(Y\), coordinate to (30), taking into account conditions (21), we obtain:

\[
L_c(s) - T_c(Z) = -A_v T_v s^{-1} + A_v L_c(s),
\]

where we find the transformation:

\[
L_c(s) = \frac{s T_c(Z) - A_v T_v}{s(s - A_v)}
\]

and corresponding original

\[
T_c(Y) = T_v + [T_c(Z) - T_v] \exp(A_v Y).
\]

Excluding from consideration the \(Y\) coordinate using the equality \(Y = X - Z\), for the section \(Z \leq X \leq 1\) we get:

\[
T_c(X) = T_v + [T_c(Z) - T_v] \exp(A_v (X - Z)).
\]

(33)

From (33), taking into account condition (20), we find:

\[
T_c(Z) = T_v - [T_v - T_{ck}] \exp(A_v (1 - Z)).
\]

(34)

From formula (32) for \(X = Z\) we find:

\[
T_{ck} = \frac{1 - \exp((A_c - A)Z)(1 - c) T_c(Z)}{C - \exp((A_c - A)Z)}
\]

(35)

From the formulas (31) and (32) it follows:

\[
T(Z) = 1 - C [T_{ck} - T_c(Z)].
\]

(36)

Substituting \(T_{ck}\) from (35) into (36) and then substituting \(T(Z)\) from (36) and \(T_c(Z)\) from (34) into equation (16) taking into account (17), we obtain in the interval \(1 \leq \theta_0 \leq \theta \leq \theta_1:\

\[
\theta dT_v / d\theta + P(\theta) T_v = Q(\theta),
\]

(37)

where

\[
P(\theta) = 1 - C (1 - C) \left[ 1 \exp(A_v (\theta_0 - \theta)) / C - \exp((A_c - A)(\theta_1 - \theta)) \right]
\]

\[
Q(\theta) = C \left[ 1 + (1 - C) \exp(A_v (\theta_0 - \theta)) / C - \exp((A_c - A)(\theta_1 - \theta)) \right] T_{ck} - (1 - c) \exp((A_c - A)(\theta_1 - \theta)) / C - \exp((A_c - A)(\theta_1 - \theta))
\]

Equation (37) has a solution expressed in quadratures

\[
T_v(\theta) = \exp \left[ - \int_{\theta_0}^{\theta} \theta^{-1} P(\theta) d\theta \right] \left[ T_v(\theta_0) + \int_{\theta_0}^{\theta} \theta^{-1} Q(\theta) \exp \left[ \int_{\theta_0}^{\theta} \theta^{-1} P(\theta) d\theta \right] d\theta \right],
\]

(38)

where \(T_v(\theta_0)\) is found using formulas (35) and (36), taking \(Z = 1\) in them:

\[
T_v(\theta_0) = 1 - \frac{C (1 - T_{ck}) [1 - \exp((A_c - A))] / C - \exp((A_c - A))}.\]

(39)
In the case \( A = A_c \) or \( C = 1 \) (which is almost unlikely, but theoretically possible), an uncertainty of type 0/0 appears in formula (39). For this case, using the rule of disclosing uncertainties (according to Lopital), we obtain:

\[
T_p(\theta_0) = 1 - \frac{C(1 - T_{ch})}{C - \exp[A_c(1 - C)]} = 1 - \frac{A_c(1 - T_{ch})}{A_c + 1}.
\]

After the function \( T_p(\theta) \) is calculated using formulas (38) and (39), the function \( T_{ck}(\theta) \) is calculated by the formula:

\[
T_{ck}(\theta) = \{1 - \exp[(A_c - A)(\theta_1 - \theta)] - (1 - C)T_p(\theta)[1 - \exp(A_c(\theta_1 - \theta))]/[C - \exp[(A_c - A)(\theta_1 - \theta)]\}.
\]

In the case \( C = 1 \), instead of (40) we have:

\[
T_{ck}(\theta) = A_c(\theta_1 - \theta) + T_{ch}\exp[A_v(\theta_0 - \theta)] + T_p(\theta)[1 - \exp(A_v(\theta_0 - \theta))]
= A_c(\theta_1 - \theta) + T_p(\theta) + [T_{ch} - T_p(\theta)]\exp[A_v(\theta_0 - \theta)].
\]

Formula (40) follows from (35), (34), (17). The formula calculates the function \( T_{ck}(\theta) \):

The formula calculates the function:

\[
T_{ck}(\theta) = T_{ch} + C^{-1}[1 - T_p(\theta)].
\]

following from (39).

Taking in (40) \( \theta = \theta_0 \), we find:

\[
T_{ck}(\theta_0) = \frac{1 - \exp([A_c - A])}{C - \exp(A_c - A)}.
\]

For \( C = 1 \) instead of (42) we have:

\[
T_{ck}(\theta_0) = \frac{T_{ch} + A_c}{A_c + 1}.
\]

From the previous, it is clear that at the stage of filling the volume \( V_0 \), i.e. at \( 0 \leq \theta \leq \theta_0 \), temperatures \( T_p, T_{ck}, T_{cv} \) keep constant values:

\[
T_p = T_p(\theta_0), T_{ck} = T_{cv} = T_{ck}(\theta_0),
\]

defined by formulas (39), (41), and (42).

Thus, we determined the functions that describe the change in the product temperature in the volume of the apparatus \( T_v(\tau) \), the temperature of the coolant at the outlet of the jacket \( T_{ck}(\tau) \) and the temperature in the coolant collector \( T_{cv} \).

The results obtained above are valid not only for the product cooling process but also in the case of its heating.

4. Conclusion
The performed mathematical description of recuperative heat transfer in batch bioreactors with a product distributor on a heat transfer surface during the filling of the apparatus showed that with countercurrent movement of heat transfer liquids and coolants, the apparatus has an advantage in terms of the main indicator – heat transfer capacity and has a higher exergy efficiency. The performed calculations make it possible to choose the apparatus design correctly and determine the rational thermal regime of its operation regarding related technological operations [13].
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