Dynamical symmetry algebra of the Calogero model

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Abstract

We study the dynamical symmetry algebra of the \( N \)-body Calogero model describing the structure of degenerate levels and demonstrate that the algebra is intrisically polynomial. We discuss some general properties of an algebra of \( S_N \)-symmetric operators acting on the \( S_N \)-symmetric subspace of the Fock space for any statistical parameter \( \nu \). In the bosonic case \( (\nu = 0) \) we find the algebra of generators for every \( N \). For \( \nu \neq 0 \), we explicitly reproduce the finite algebra for the 4-particle model, demonstrating some general features of our construction.

I. INTRODUCTION

The Calogero model \[1\] describes the system of \( N \) bosonic particles on a line interacting through the inverse square and harmonic potential. It is completely integrable, in both the classical and quantum case, the spectrum is known and the wave-functions are given implicitly. Since the model has connections with a host of physical problems, including recent proposal that the superconformal Calogero model provides a microscopic description of the extremal Reissner-Nordström black hole \[3\], there is considerable interest in finding the basis set of orthonormal eigenfunctions, and the structure of the dynamical algebra that characterizes the eigenstates of the system.

A lot of insight has been gained by investigating the algebraic properties of the Calogero model in terms of the \( S_N \)-extended Heisenberg algebra \[3\]. For the periodic version of the model \[4\], the orthonormal eigenfunctions in terms of Jack polynomials \[3\] have been constructed \[6\]. In the case on line, two- and three-particles systems have been considered \[7,8\]. The authors of Ref. \[7\] have shown that the dynamical symmetry algebra of the two-body model is a polynomial generalization of the \( SU(2) \) algebra. The same group of authors has also treated the three-body problem \[8\], obtaining the polynomial algebra and the action of its generators on the orthonormal basis. It has been shown that in the two-body case the

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polynomial $SU(2)$ algebra can be linearized, but an attempt to generalize this result to the $N$-body case has led to $(N - 1)$ linear $SU(2)$ subalgebras that operate only on subsets of the degenerate eigenspace [9].

The problem of constructing the algebra of symmetric one-particle operators for the Calogero model [10] resulted in a similar algebraic structure as that we discuss in Section 3. The algebra constructed in [10] is infinite, independent of particle number and the constant of interaction (statistical parameter), and is generally not known.

In this letter we consider the problem of dynamical algebra of the $N$-body Calogero model in a new way. We demonstrate that the algebra in question is intrinsically polynomial (except in the $N = 2$ case). We discuss some general properties of the algebra of operators acting on the $S_N$-symmetric subspace of the Fock space for any $\nu$. In the bosonic case ($\nu = 0$) we find the algebra of generators invariant under the $S_N$-permutation group. For $\nu \neq 0$, we discuss in detail the algebras for $N = 3, 4$, thus explaining some general features of our construction.

II. THE CALOGERO MODEL AND THE $S_N$-SYMMETRIC FOCK SPACE

The Calogero model is defined by the following Hamiltonian:

$$H = -\frac{1}{2} \sum_{i=1}^{N} \partial_i^2 + \frac{1}{2} \sum_{i=1}^{N} x_i^2 + \frac{\nu(\nu - 1)}{2} \sum_{i \neq j}^{N} \frac{1}{(x_i - x_j)^2},$$

(1)

For simplicity, we have set $\hbar$, the mass of particles and the frequency of harmonic oscillators equal to one. The dimensionless constant $\nu$ is the coupling constant ($\nu > -1/2$) and $N$ is the number of particles.

Let us introduce the following analogs of creation and annihilation operators [3]:

$$a_i^\dagger = \frac{1}{\sqrt{2}}(-D_i + x_i), \quad a_i = \frac{1}{\sqrt{2}}(D_i + x_i),$$

(2)

where

$$D_i = \partial_i + \nu \sum_{i \neq j}^{N} \frac{1}{x_i - x_j}(1 - K_{ij})$$

are Dunkl derivatives [11], and the operator $a_i$ annihilates the vacuum. The elementary generators $K_{ij}$ of the symmetry group $S_N$ exchange labels $i$ and $j$:

$$K_{ij}x_j = x_i K_{ij}, \quad K_{ij} = K_{ji}, \quad (K_{ij})^2 = 1,$$

$$K_{ij}K_{jl} = K_{jl}K_{il} = K_{il}K_{ij}, \quad \text{for} \ i \neq j, \ i \neq l, \ j \neq l,$$

(3)

and we choose $K_{ij}|0\rangle = |0\rangle$. One can easily check that the commutators of creation and annihilation operators (2) are

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \left(1 + \nu \sum_{k=1}^{N} K_{ik}\right) \delta_{ij} - \nu K_{ij}.$$

(4)

After performing a similarity transformation on the Hamiltonian (1), we obtain the reduced Hamiltonian
\[ H' = \left( \prod_{i<j}^{N} |x_i - x_j|^{-\nu} \right) H \left( \prod_{i<j}^{N} |x_i - x_j|^{\nu} \right) = \frac{1}{2} \sum_{i=1}^{N} \{a_i, a_i^\dagger\} = \sum_{i=1}^{N} a_i^\dagger a_i + E_0, \] (5)

acting on the space of symmetric functions. The constant \( E_0 \) is ground-state energy \( E_0 = N[1 + (N - 1)\nu]/2 \). We restrict the Fock space \( \{a_1^{n_1} \cdots a_N^{n_N} | 0 \} \) to the \( S_N \)-symmetric subspace \( F_{\text{symm}} \), where \( \mathcal{N} = \sum_{i=1}^{N} a_i^\dagger a_i \) acts as the total number operator. Next, we introduce the collective \( S_N \)-symmetric operators

\[ B_n = \sum_{i=1}^{N} a_i^n, \ n = 0, 1, \ldots, N, \] (6)

where \( B_0 \) is the constant \( N \) multiplied by the identity operator, and \( B_1 \) represents the center-of-mass operator (up to the constant \( N \)). The complete \( F_{\text{symm}} \) can be described as \( \{B_1^{n_1} B_2^{n_2} \cdots B_N^{n_N} | 0 \} \). Note that \( [B_1, B_k^\dagger] \neq 0 \), hence we wish to construct the operators \( X_k^\dagger \) such that \( B_1 \) commute with \( X_k^\dagger \) for every \( k \) greater than one. The general solution of this equation is described by any symmetric monomial polynomial \( m_N(a_1, \ldots, a_N) = \sum a_1^\lambda_1 a_2^\lambda_2 \cdots a_N^\lambda_N \), where \( a_i = a_i - B_i \), and the sum goes over all distinct permutations of \( \{\lambda_1, \lambda_2, \ldots, \lambda_N\} \). The set \( \{\lambda_1, \lambda_2, \ldots, \lambda_N\} \) denotes any partition of \( N \) such that \( \sum_{i=1}^{N} \lambda_i = N \) and \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0 \). The simplest choice of the \( (N - 1) \) operators commuting with \( B_1 \) is

\[ A_n \equiv B_n = \sum_{i=1}^{N} \left( a_i - \frac{B_1}{N} \right)^n, \ n = 2, \ldots, N. \] (7)

For later convenience, it is useful to define \( A_0 = N - 1 \) and \( A_1 = 0 \). The \( F_{\text{symm}} \) symmetric Fock space is now \( \{B_1^{n_1} A_2^{n_2} \cdots A_N^{n_N} | 0 \} \) and after removing \( B_1 \) it reduces to the Fock subspace \( \{A_2^{n_2} \cdots A_N^{n_N} | 0 \} \). We have reduced the problem to the \( (N - 1) \) Jacobi-type operators\[^1\]. However, two different states \( A_2^{n_2} \cdots A_N^{n_N} | 0 \) and \( A_2^{n_2'} \cdots A_N^{n_N} | 0 \) are generally not orthogonal. For example, \( \langle 0 | A_2^3 A_2^3 | 0 \rangle = \langle 0 | A_2^2 A_3^2 | 0 \rangle \neq 0 \). The total number operator on \( F_{\text{symm}} \) splits into

\[ \mathcal{N} = \mathcal{N}_1 + \mathcal{N}, \ \mathcal{N}_1^\dagger = \mathcal{N} = \sum_{i=1}^{N} a_i^\dagger a_i, \]

\[ \mathcal{N}_1 = \mathcal{N}_1 = \frac{1}{N} B_1^\dagger B_1, \]

\[ \mathcal{N} = \mathcal{N} = \mathcal{N} \equiv \sum_{k=2}^{N} k \mathcal{N}_k. \] (8)

Note that \( \mathcal{N}_k \) are the number operators of \( A_k^\dagger \) but not of \( A_k \). Namely, \([\mathcal{N}_k, A_k^\dagger] = \delta_{kl} A_l^\dagger \), and \( \mathcal{N}_k(\cdots A_{k-1}^\dagger \cdots | 0 \rangle) = n_k(\cdots A_{k-1}^\dagger \cdots | 0 \rangle) \) for every \( k \) greater than one, but \( \mathcal{N}_k^\dagger \neq \mathcal{N}_k \). If \( \mathcal{N}_k \) were hermitian, then the eigenstates \( A_2^{n_2} \cdots A_N^{n_N} | 0 \) would be orthogonal, and vice versa.

\[^1\]Note that the algebraic sum of all coefficients of homogeneous monomials in any \( A_k \) is zero.
III. THE POLYNOMIAL ALGEBRA $\mathcal{B}_N(\nu)$

Here we construct the finite (for finite $N$) and closed algebra $\mathcal{B}_N(\nu)$, which appears naturally when we calculate the commutators between the operators $A_i$ and $A_i^\dagger$, defined in Eq. (7). Let us define $S_N$-symmetric operators $\mathcal{B}_{n,m}$:

$$\mathcal{B}_{n,m} = \sum_{i=1}^{N} \bar{a}_i^n a_i^m = \mathcal{B}_{m,n}, \; n, m \in \mathbb{N}_0. \tag{9}$$

There are $(N+4)(N-1)/2$ algebraically independent operators contained in the algebra $\mathcal{B}_N(\nu)$, namely $2(N-1)$ operators $\mathcal{B}_{n,0} = A_i^n$, for $n = 2, 3, \ldots, N$ and their hermitian conjugates, and $N(N-1)/2$ operators $\mathcal{B}_{n,m}$, for $n, m \geq 1, n + m \leq N$. $[N/2]$ of the latter are hermitian operators $\mathcal{B}_{n,n}$. Note that $\mathcal{B}_{0,0} = N \cdot \mathbb{I}$ and $\mathcal{B}_{1,0} = \mathcal{B}_{0,1} = 0$. One can express the operators $\mathcal{B}_{n,m}$ for $n + m > N$ in terms of algebraically independent operators $\mathcal{B}_{n,m}$ with $n + m \leq N$. The operators $\mathcal{B}_{n,m}$ can be represented in the symmetric Fock space:

$$\mathcal{B}_{n,m} A_2^{i_2} \cdots A_N^{i_N} |0\rangle \equiv \mathcal{B}_{n,m} (\prod A_i^{\dagger})^N |0\rangle = \sum (\prod A_i^{\dagger})^{N \pm n - m} |0\rangle.$$

The symbolical expression $(\prod \mathcal{O})^k$ denotes a product of operators $\mathcal{O}_i$ of the total order $k$ in $a_i (a_i^\dagger)$. Generally,

$$\begin{align*}
\mathcal{B}_{1,1} &= \mathcal{N}, \quad [\mathcal{B}_{1}, \mathcal{B}_{n,m}] = 0, \quad [\mathcal{B}_{1,1}, \mathcal{B}_{n,m}] = (n - m) \mathcal{B}_{n,m}, \\
[a_i, \mathcal{B}_{n,m}] &= n \left[ a_i^{(n-1)} a_i^m - \frac{1}{N} \mathcal{B}_{n-1,m} \right]. \tag{10}
\end{align*}$$

One can define operators $L_m = -B_{m+1,1}$ and $L_{-m} = -B_{1,m+1}$, for $m \geq 0$ satisfying the centerless Virasoro algebra [12]. It is easy to check that the ”bar” operators satisfy the following commutation relation, for any $\nu$:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m - n) \mathcal{L}_{m+n} + \frac{1}{N} [(n + 1) A_{m+1}^{\dagger} \mathcal{L}_{n-1} - (m + 1) A_{n+1}^{\dagger} \mathcal{L}_{m-1}].$$

In the limit of large $N$, it becomes the centerless Virasoro algebra.

For simplicity, let us first consider $N$ free harmonic oscillators with $\nu = 0$. We start from a commutator of bosonic operators, with the center-of-mass coordinate removed, i.e. $[\bar{b}_i, \bar{b}_j^\dagger] = \delta_{ij} - 1/N$. Using the following relations:

$$\begin{align*}
[\bar{b}_i^m, \bar{b}_j^{n\dagger}] &= \sum_{k=0}^{\min(n,m)} \beta_k(n, m) \bar{b}_j^{(n-k)} \bar{b}_i^{(m-k)} \left( \delta_{ij} - \frac{1}{N} \right)^k \\
&= \sum_{k=1}^{\min(n,m)} \beta_k(n, m) (-1)^{(k-1)} \bar{b}_i^{(n-k)} \bar{b}_j^{(m-k)} \left( \delta_{ij} - \frac{1}{N} \right)^k, \tag{11}
\end{align*}$$

where

$$\beta_k(n, m) = \frac{m! n!}{k!(m-k)!(n-k)!},$$

we find the general $\mathcal{B}_N(0)$-algebra relation
\[
[B_{m',m}, B_{n,n'}] = \sum_{k=1}^{\min(n,m)} \beta_k(m,n) \left( \frac{1}{N} \right)^k \left\{ [(N-1)^k+1] B_{m'+n-k,m+n'-k} \\
- \frac{1}{N} \sum_{s=1}^{\min(n-k,m-k)} \beta_s(m-k,n-k) B_{m'+n-k-s,m+n'-k-s} \left(1 - \frac{1}{N}\right)^s \right\} \\
- \{m' \leftrightarrow n, n' \leftrightarrow m\}. \tag{12}
\]

For finite \(N\), the r.h.s. of Eq. (12) can be written in terms of the operators \(B_{n,m}\) with \(n + m \leq N\). Specially, we find

\[
[A_m, A_n^\dagger] = \sum_{k=1}^{\min(n,m)} \beta_k(m,n) \left( \frac{1}{N} \right)^k \left\{ B_{n-k,m-k} \left(1 - \frac{1}{N}\right)^k \right\},
\]

\[
[B_{m,m}, B_{n,n}] = - \sum_{k=1}^{\min(n,m)} \beta_k(m,n) \left( \frac{1}{N} \right)^k \left\{ B_{m-k,n} B_{n-k,m} - B_{n-k,n} B_{m-k,m} \right\}. \tag{13}
\]

One can obtain another form of the first relation in (13) by simply putting \(m' = n' = 0\) in Eq (12). Note that

\[
[B_{m',m}, B_{n,n'}] = \sum_{k=1}^{\min(n,m)} \beta_k(m,n) B_{m'+n-k,m+n'-k} - \{m' \leftrightarrow n, n' \leftrightarrow m\},
\]

i.e., it is a linear \(W_{1+\infty}\) algebra \([13,14]\). In the limit \(N \rightarrow \infty\) the algebra \(B_N(0)\), Eq. (12) becomes also the \(W_{1+\infty}\) algebra.

For \(\nu \neq 0\), the structure of the algebra \(B_N(\nu)\) becomes more complicated. New polynomial terms of the form \((\prod \bar{B}_{n\alpha,m\alpha})\) with \(\sum \alpha n\alpha \leq n + m' - 1, \sum \alpha m\alpha \leq n' + m - 1\) appear on the r.h.s. of the commutation relation (12). The corresponding coefficients are polynomial in \(\nu\), vanishing when \(\nu\) goes to zero. The coefficients of the leading terms (\(k = 1\) in Eq. (12)) do not depend on \(\nu\), i.e., they are the same for any \(\nu\). For example, for arbitrary \(N\) and \(\nu\) we find:

\[
\begin{align*}
[A_2, A_n^\dagger] &= 2n \bar{B}_{n-1,1} + n \left( \frac{N-1}{N} \right) A_{n-2}^\dagger (n-1+\nu N) + n\nu \sum_{i=1}^{n-2} (A_{n-2-i}^\dagger A_i^\dagger - A_{n-2}^\dagger), \\
[A_3, A_n^\dagger] &= 3n \left( \bar{B}_{n-1,2} - \frac{1}{N} A_{n-1}^\dagger A_2 \right) + n \bar{B}_{n-2,1} \left[ 3(n-1) \left( \frac{N-2}{N} \right) + \nu (N+2) \right] + n(N-1)(N-2) A_{n-3}^\dagger \left( \frac{(n-1)(n-2)}{N^2} + \frac{2n-3}{N}\nu + \nu^2 \right) \\
&- n\nu \sum_{i=2}^{n-2} A_{n-3-i}^\dagger \left[ (n-1) \left( \frac{N-2}{N} \right) + (n-2) \left( \frac{N-1}{N} \right) - 1 \right] \\
&+ n\nu \sum_{i=2}^{n-3} \left[ (n-1) \left( \frac{N-2}{N} \right) + (n-2-i) \left( \frac{N-1}{N} \right) + A_{n-3-i}^\dagger \right] A_i^\dagger \\
&+ 3n\nu \sum_{i=2}^{n-2} \left( A_{i}^\dagger \bar{B}_{n-i-2,1} - \bar{B}_{n-2,1} \right) - n\nu^2 \sum_{i=0}^{n-3} \sum_{j=0}^{n-3} \left( A_{n-3-j}^\dagger A_j^\dagger - A_{n-3}^\dagger \right) \\
&+ n\nu^2 \sum_{i=0}^{n-3} \sum_{j=1}^{n-3-i} \left( A_{n-3-i-j}^\dagger A_j^\dagger - A_{n-3-i}^\dagger \right) - \sum_{i=0}^{n-3} \sum_{j=1}^{n-3} \left( A_{n-3-j}^\dagger A_j^\dagger - A_{n-3}^\dagger \right). \tag{14}
\end{align*}
\]
For $N \leq 5$, the above commutation relations have the same structure as for bosons with $
u = 0$, but with the coefficients depending polynomially on $
u$. For $N = 4$, we give the complete $\mathcal{B}_4(\nu)$ algebra. The minimal set of operators is \{ $A_2, A_3, A_4, \overline{B}_{11}, \overline{B}_{12}, \overline{B}_{13}, \overline{B}_{22}$ \} plus hermitian conjugates.

\[
\begin{align*}
[A_3, A_3^\dagger] &= 9(\overline{B}_{2,2} - 1/4A_3^\dagger A_2 + (1 + 2\nu)\overline{B}_{1,1} + (1 + 2\nu)(1 + 4\nu)), \\
[A_4, A_4^\dagger] &= 16(\overline{B}_{3,3} - 1/4A_3^\dagger A_3) + \overline{B}_{2,2}(36 - 8\nu) + A_2^\dagger A_2(9/2 + 16\nu) + 8\nu \overline{B}_{1,1}^2, \\
&\quad + (56\nu^2 + 64\nu + 42)\overline{B}_{1,1} + 6(1 + 4\nu)(4\nu^2 + 13\nu + 21/4), \\
[A_3, A_0^\dagger] &= 12(\overline{B}_{3,2} - 1/4A_3^\dagger A_2) + (18 + 12\nu)\overline{B}_{2,1}, \\
[A_1, \overline{B}_{1,j}] &= i(A_{i+j-1} - 1/4A_j A_{i-1}), \ i = 2, 3, 4, \ j = 1, 2, 3, \\
[A_2, \overline{B}_{3,i}] &= 3\overline{B}_{1,1} + (3/2 + 6\nu)A_i, \ i = 0, 1, 2, \\
[A_2, \overline{B}_{3,3}] &= 6\overline{B}_{2,2} + (9/2 + 6\nu)\overline{B}_{1,1}, \\
[A_3, \overline{B}_{3,i}] &= 6(\overline{B}_{1,i+2} - 1/4\overline{B}_{1,i+2} A_2) + (3 + 6\nu)A_{i+1}, \ i = 0, 1, 2, \\
[A_3, \overline{B}_{3,3}] &= 9(\overline{B}_{2,3} - 1/4\overline{B}_{2,1} A_2) + (9 + 6\nu)\overline{B}_{1,2}, \\
[A_4, \overline{B}_{3,i}] &= 8(\overline{B}_{1,i+3} - 1/4\overline{B}_{1,i+3} A_3) + (6 + 4\nu)A_{i+2} + (3/4 + 4\nu)A_2 A_i, \ i = 0, 1, 2, \\
[A_4, \overline{B}_{3,3}] &= 12(\overline{B}_{2,4} - 1/4\overline{B}_{2,1} A_3) + (18 - 24\nu)\overline{B}_{1,3} + (9/4 + 12\nu)\overline{B}_{1,1} A_2 \\
&\quad + (21/2 + 26\nu + 8\nu^2)A_2, \\
\overline{B}_{1,2}, \overline{B}_{2,1} &= 3\overline{B}_{2,2} + 1/4A_3^\dagger A_2 + (2 - 6\nu)\overline{B}_{1,1} - \overline{B}_{1,1}^2, \\
\overline{B}_{1,2}, \overline{B}_{2,2} &= 2\overline{B}_{2,3} - \overline{B}_{1,1} \overline{B}_{1,2} + 1/2\overline{B}_{2,1} A_2 + (2 - 6\nu)\overline{B}_{1,2}, \\
\overline{B}_{1,2}, \overline{B}_{3,1} &= 5\overline{B}_{3,2} + 1/4A_3^\dagger A_2 - 3/2 \overline{B}_{1,1} \overline{B}_{1,2} + (6 - 14\nu)\overline{B}_{2,1}, \\
\overline{B}_{1,2}, \overline{B}_{3,3} &= -\overline{B}_{1,4} - 1/2\overline{B}_{1,1} A_3 + 3/4\overline{B}_{1,2} A_2, \\
\overline{B}_{1,3}, \overline{B}_{2,2} &= 4\overline{B}_{2,4} - 3/2 \overline{B}_{1,1} \overline{B}_{2,1} A_3 - (3/8 - 4\nu)\overline{B}_{1,1} A_2 + (6 - 14\nu)\overline{B}_{1,3}, \\
\overline{B}_{1,3}, \overline{B}_{3,1} &= 8\overline{B}_{3,3} + 1/4A_3^\dagger A_3 + (18 - 24\nu)\overline{B}_{2,2} + 8\nu A_3^\dagger A_2 \\
&\quad - 9/4 \overline{B}_{1,2} \overline{B}_{2,1} + (-9/8 + 4\nu)\overline{B}_{1,1}^2 + (6 - 21\nu + 14\nu^2)\overline{B}_{1,1}. \tag{15}
\end{align*}
\]

As we have already mentioned, all operators $\overline{B}_{i,j}$ with $i + j > 4$ are algebraically dependent and can be expressed in terms of the minimal set of operators. For example, $A_5 = 5/6A_2 A_3$, $\overline{B}_{4,1} = 1/3A_4^\dagger A_{1,1} + 1/2A_2^\dagger A_{3,1}$, for $N \leq 4$.

For $N = 3$, the $\mathcal{B}_3(\nu)$ algebra is in full agreement with Ref. [2]. The exact correspondence between $Y_s$ and $J$ defined in Ref. [2] and our operators $\overline{B}_{i,j}$ is

\[
Y_1 = -2A_3^\dagger, \ Y_{3/2} = \frac{12}{\sqrt{6}}A_3^\dagger, \ Y_{1/2} = -2\sqrt{6} \overline{B}_{2,1}, \ J_3 = N_1, \ J = \frac{1}{2}\sqrt{N} + \frac{3}{2}\nu. \tag{16}
\]

The commutation relations which define the $\mathcal{B}_3(\nu)$ algebra are new and shed more light on the structure of the $\mathcal{B}_N(\nu)$ algebra and its representations.

Alternatively, the $\mathcal{B}_N(\nu)$ algebra can be constructed by grouping the generators into $sl(2)$-spin multiplets. Note that $J_+ = 1/2A_2^\dagger$, $J_- = 1/2A_2$, $J_0 = 1/8[A_2, A_2^\dagger]$ generate the $sl(2)$-algebra. The complete set of generators spanning the $\mathcal{B}_N(\nu)$ algebra is given by $(N - 1)$ nondegenerate spin multiplets with $s = 1, 3/2, 2, \ldots, N/2$. The unique generator with spin $s$ and projection $s_z$ is defined as $J_{s,s_z} = \sqrt{(s + s_z)!/[8^s(s - s_z)!(s + s_z - s)!]}[J_+, \ldots, [J_-, A_2^\dagger, \ldots, A_2]_{s_z}]$, for all statistical parameters $\nu$. Detail of this construction will be presented elsewhere [14].
IV. DYNAMICAL SYMMETRY OF THE CALOGERO MODEL

The dynamical symmetry $C_N(\nu)$ of the Calogero model is defined as maximal algebra commuting with the Hamiltonian (3). The generators of the algebra $C_N(\nu)$ act among the degenerate states with fixed energy $E = \mathcal{N} + E_0$, $\mathcal{N}$ a non-negative integer. Starting from any of degenerate states with energy $E$, all other states can be reached by applying generators $X_{i,j}$ of the algebra. Degeneracy appears for $\mathcal{N} \geq 2$. The vacuum $|0\rangle$ and the first excited state $B_1|0\rangle$ are nondegenerate. For $\mathcal{N} = 2$ the degenerate states are $B_1^2|0\rangle$ and $A_2^1|0\rangle$; for $\mathcal{N} = 3$ the degenerate states are $B_1^2|0\rangle$, $B_1^1A_2^1|0\rangle$ and $A_3^1|0\rangle$, etc. The number of degenerate states $\mathcal{N}$ is given by partitions $\mathcal{N}_1, \ldots, \mathcal{N}_k$ of $\mathcal{N}$ such that $\mathcal{N} = \sum_k k\mathcal{N}_k$.

Let us choose $(N + 4)(N - 1)/2$ algebraically independent generators $X_{i,j}$, $(i + j \leq N)$ in the following way:

$$X_{i,j} = \overline{B}_{i,j} \left( \frac{B_1}{\sqrt{N}} \right)^{(i-j)}, \quad X_{j,i} = X_{i,j}^\dagger, \quad X_{i,i} = \overline{B}_{j,i} \left( \frac{B_1^\dagger}{\sqrt{N}} \right)^{(i-j)}, \quad i \geq j. \quad (17)$$

For example,

$$X_{i,0} = A_i^1 \left( \frac{B_1}{\sqrt{N}} \right)^i, \quad X_{0,i} = A_i \left( \frac{B_1^\dagger}{\sqrt{N}} \right)^i,$$

$$[X_{i,0}, X_{j,0}] = [X_{0,i}, X_{0,j}] = 0, \quad X_{i,i}^\dagger = X_{i,i} = \overline{B}_{i,i}. \quad (18)$$

The generators $X_{i,j}$ are hermitian but they do not commute, since $\overline{B}_{i,j}$’s do not commute (see Eq. (13)). On the other hand, the number operators $\mathcal{N}_k$ (8) commute but are not hermitian since the states $A_2^1 \cdots A_N^{\nu N} |0\rangle$ are not mutually orthogonal. Generally,

$$[\mathcal{N}_1, X_{i,j}] = -(i-j)X_{i,j}, \quad [\mathcal{N}, X_{i,j}] = (i-j)X_{i,j},$$

$$[H, X_{i,j}] = [\mathcal{N}, X_{i,j}] = 0, \quad \text{for all } i,j. \quad (19)$$

The general structure of the commutation relations for $i \geq j$, $k \geq l$ is

$$[X_{i,j}, X_{k,l}] = \left[ \overline{B}_{i,j}, \overline{B}_{k,l} \right] \left( \frac{B_1}{\sqrt{N}} \right)^{(i-j)+(k-l)} = \sum \left[ \prod_\alpha X_{n_\alpha, m_\alpha} \right], \quad (20)$$

and for $i > j$, $k < l$

$$[X_{i,j}, X_{k,l}] = \sum \left[ \prod_\alpha X_{n_\alpha, m_\alpha} g_{n_\alpha, m_\alpha} (\mathcal{N}_1) \right] + X_{i,j}X_{k,l}f_{ijkl}(\mathcal{N}_1), \quad (21)$$

with the restriction $0 \leq \sum m_\alpha \leq j + l - 1$, $0 \leq \sum n_\alpha \leq i + k - 1$, and similarly for hermitian conjugate relations. The functions $f$ and $g$ are generally rational functions of $\mathcal{N}_1$, with the finite action on all states. One can show that for $i \geq j$,

$$\left( \frac{B_1}{\sqrt{N}} \right)^i \left( \frac{B_1^\dagger}{\sqrt{N}} \right)^j = \left( \frac{B_1}{\sqrt{N}} \right)^{(i-j)} (\mathcal{N}_1 + 1) \cdots (\mathcal{N}_1 + j)$$

$$= (\mathcal{N}_1 + 1 + i - j) \cdots (\mathcal{N}_1 + i) \left( \frac{B_1}{\sqrt{N}} \right)^{(i-j)},$$

which is the basis of the dynamical symmetry of the Calogero model.
and
\[
\left( \frac{B_i^1}{\sqrt{N}} \right)^j \left( \frac{B_i^1}{\sqrt{N}} \right)^i = \mathcal{N}_1(\mathcal{N}_1 - 1) \cdots (\mathcal{N}_1 - j + 1) \left( \frac{B_i^1}{\sqrt{N}} \right)^{(i-j)} = \left( \frac{B_i^1}{\sqrt{N}} \right)^{(i-j)} (\mathcal{N}_1 - i + j) \cdots (\mathcal{N}_1 - i + 1),
\]
and similarly for \( i < j \). Now it is easy to see that
\[
\left[ \left( \frac{B_i^1}{\sqrt{N}} \right)^i, \left( \frac{B_i^1}{\sqrt{N}} \right)^j \right] = \sum_{k=1}^{\min(i,j)} \beta_{k(i,j)} \left( \frac{B_i^1}{\sqrt{N}} \right)^{(j-k)} \left( \frac{B_i^1}{\sqrt{N}} \right)^{(i-k)}.
\]
(22)

For general \( N \), we present several typical commutators which demonstrate the general structure given by Eqs. (20-21):
\[
\begin{align*}
[X_{2,0}, X_{0,2}] &= -2 [2X_{1,1} + (N - 1)(1 + N\nu)] \mathcal{N}_1 + 1) \mathcal{N}_1 + 2) + 2X_{2,0}X_{0,2} \frac{2\mathcal{N}_1 + 1}{(\mathcal{N}_1 + 1)(\mathcal{N}_1 + 2)}, \\
[X_{2,0}, X_{0,3}] &= -6X_{1,2}(\mathcal{N}_1 + 2)(\mathcal{N}_1 + 3) + 6X_{2,0}X_{0,3} \frac{\mathcal{N}_1 + 1}{\mathcal{N}_1(\mathcal{N}_1 - 1)}, \\
[X_{2,0}, X_{2,1}] &= -2X_{3,0}, \\
[X_{2,0}, X_{1,2}] &= -4X_{2,1}(\mathcal{N}_1 + 1) + 2X_{2,0}X_{1,2} \frac{1}{\mathcal{N}_1 + 1}, \\
[X_{4,0}, X_{3,1}] &= -4(X_{6,0} - \frac{1}{N}X_{3,0}^2), \\
[X_{4,0}, X_{0,2}] &= -8X_{3,1} - 4 \left[ 3 \left( \frac{N - 1}{N} \right) + \nu(2N - 3) \right] X_{2,0}(\mathcal{N}_1 + 1)(\mathcal{N}_1 + 2) + 3X_{4,0}X_{0,2} \frac{2\mathcal{N}_1 + 1}{(\mathcal{N}_1 + 1)(\mathcal{N}_1 + 2)}.
\end{align*}
\]
(23)

The \( C_N(\nu) \) algebra is intrinsically polynomial. For \( N = 2 \), the \( C_2(\nu) \)-Calogero algebra is the \( SU(2) \)-polynomial (cubic) algebra [7], i.e., \([X_{2,0}, X_{0,2}] = P_3(\mathcal{N}_1, \mathcal{N}) \). In this case, the algebra \( C_2(\nu) \) can be linearized to the ordinary \( SU(2) \) algebra owing to the fact that there are two independent, uncoupled oscillators \( B_1 \) and \( A_2 \), which can be mapped to two ordinary Bose oscillators [13]. The \( SU(2) \) generators are
\[
J_+ = \frac{1}{4\sqrt{(\mathcal{N}_1 - 1)(\mathcal{N} + 1 + 2\nu)}} B_{12}^2 A_2,
\]
\[
(J_+)^\dagger = J_- = A_2^\dagger B_1^2 \frac{1}{4\sqrt{(\mathcal{N}_1 - 1)(\mathcal{N} + 1 + 2\nu)}},
\]
\[
J_0 = \frac{1}{16} \left( \frac{1}{(\mathcal{N}_1 - 1)} B_{12}^2 B_1^2 - \frac{4}{(\mathcal{N} + 1 + 2\nu)} A_2 A_2 \right) = \frac{1}{4} \left( \mathcal{N}_1 - \mathcal{N} \right),
\]
(24)
satisfying \([J_+, J_-] = 2J_0, [J_0, J_\pm] = \pm J_\pm \). The generators \( J_+ \) and \( J_- \) are hermitian conjugates to each other and in this respect differ from the construction done in Ref. [8].
$N = 3$, the $\mathcal{C}_3(\nu)$ algebra in Eqs. (20, 21) is the same as in Ref. [8]. One can easily find the exact correspondence using Eq. (16). For general $N$, our construction can be viewed as a generalization of the polynomial algebras for $N = 2$ [7] and $N = 3$ [8, 17], using the algebra $\mathcal{B}_N(\nu)$.

Finally, it would be interesting to construct some different sets of generators of dynamical algebra, and to discuss the relation between bosonic realization of the $SU(N)$ algebra and the $\mathcal{C}_N(\nu)$ algebra, generalizing Eq. (24) for any $N$. We plan to answer some of these open questions in the forthcoming publication [15].

Acknowledgment

We would like to thank M. Mileković and D. Svrtan for useful discussions. this work was supported by the Ministry of Science and Technology of the Republic of Croatia under contract No. 00980103.
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