THE QUANTUM WAVE PACKET OF THE NON-LINEAR GROSS-PITAEVSKII EQUATION

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Abstract: In this paper we study the quantum wave packet of the non-linear Gross-Pitaevskii equation.

1. Introduction

In the present work we investigate the quantum wave packet of the one-dimensional non-linear Gross-Pitaevskii equation with the potential $V(x, t)$ given by:

$$V(x, t) = \frac{1}{2} m \omega^2(t) x^2 , \quad (1.1)$$

which is the time dependent harmonic oscillator potential.

2. Gross-Pitaevskii Equation

Em 1961[1,2], E. P. Gross and, independently, L. P. Pitaevskii proposed a non-linear Schrödinger equation to represent time dependent physical systems, given by:
where $\psi(x, t)$ is a wavefunction and $g$ is a constant.

Writing the wavefunction $\psi(x, t)$ in the polar form, defined by the Madelung-Bohm transformation[3,4], we get:

$$\psi(x, t) = \phi(x, t) e^{i S(x, t)}, \quad (2.2)$$

where $S(x, t)$ is the classical action and $\phi(x, t)$ will be defined in what follows.

Substituting Eq.(2.2) into Eq.(2.1) and taking the real and imaginary parts of the resulting equation, we get[5]:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_{qu})}{\partial x} = 0, \quad (2.3)$$

$$\hbar \frac{\partial S}{\partial t} + \frac{1}{2} m v_{qu}^2(t) + \frac{1}{2} m \omega^2(t) x^2 + V_{qu} + V_{GP} = 0, \quad (2.4)$$

$$\frac{\partial v_{qu}}{\partial t} + v_{qu} \frac{\partial v_{qu}}{\partial x} + \omega^2(t) x = - \frac{1}{m} \frac{\partial}{\partial x} (V_{qu} + V_{GP}), \quad (2.5)$$

where:

$$\rho(x, t) = \phi^2(x, t), \quad (2.6) \quad \text{(quantum mass density)}$$

$$v_{qu}(x, t) = \frac{\hbar}{m} \frac{\partial S(x, t)}{\partial x}, \quad (2.7) \quad \text{(quantum velocity)}$$

$$V_{qu}(x, t) = - \frac{\hbar^2}{2 m} \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial x^2} = - \frac{\hbar^2}{2 m} \phi \frac{\partial^2 \phi}{\partial x^2}, \quad (2.8a,b) \quad \text{(Bohm quantum potential)}$$

and

$$V_{GP} = g \rho. \quad (2.9) \quad \text{(Gross-Pitaevskii potential)}$$

### 3. Quantum Wave Packet

In 1909 [6], Einstein studied the black body radiation in thermodynamical equilibrium with matter. Starting from Planck’s equation, of 1900, of the radiation density and using the Fourier expansion technique to calculate its fluctuations, he showed that it exhibits, simultaneously, fluctuations which are characteristic of waves and particles. In 1916 [7], analyzing again the black body Planckian radiation, Einstein proposed that an electromagnetic radiation with wavelength $\lambda$ had a linear momentum $p$, given by the relation:

$$p = \frac{\hbar}{\lambda}, \quad (3.1)$$
where \( h \) is the Planck constant [8].

In works developed between 1923 and 1925 [9] de Broglie formulated his fundamental idea that the electron with mass \( m \), in its atomic orbital motion with velocity \( v \) and linear momentum \( p = m v \) is guided by a "matter wave" (pilot-wave) with wavelenght given by:

\[
\lambda = \frac{h}{p} . \tag{3.2}
\]

In 1926 [10], Schrödinger proposed that the "pilot-wave de Brogliean" ought to obey a differential equation, today know as the famous Schrödinger’s equation:

\[
i \hbar \frac{\partial}{\partial t} \Psi(\vec{r}, t) = \hat{H} \Psi(\vec{r}, t) , \tag{3.3a}
\]

where \( \hat{H} \) is Hamiltonian operator defined by:

\[
\hat{H} = \frac{\hat{p}^2}{2m} + V(\vec{r}, t) , (\hat{p} = -i \hbar \nabla) , \tag{3.3b,c}
\]

where \( V \) is the potential energy. In this same year of 1926 [11] Born interpreted the Schrödinger wave function \( \Psi \) as being an amplitude of probability.

4. The Quantum Wave Packet of the non-linear Gross-Pitaevskii equation

Initially, let us calculate the quantum trajectory \( (x_{qu}) \) of the physical system represented by the eq.(2.1). To do this, let us integrate the equation given by [12] [remember that \( q(t) = \langle x \rangle \)]:

\[
v_{qu}(x, t) = \frac{\dot{\sigma}(t)}{2 \sigma(t)} [x - q(t)] + \dot{q}(t) , \tag{4.1}
\]

where:

\[
\sigma^2(t) = \frac{\hbar^2}{m^2} k(t) + \frac{1}{\sigma(t)} \frac{2}{m \sqrt{\pi} \sigma(t)} , \tag{4.2}
\]

and:

\[
k(t) = \left[ \frac{\hbar^2}{m^2 \sigma^2(t)} - \frac{2g}{\sigma(t) m \sqrt{\pi} \sigma(t)} \right] . \tag{4.3}
\]

So, remembering that \( \int \frac{dz}{z} = \ln z, \ln \left( \frac{x}{y} \right) = \ln x - \ln y \), and \( \ln xy = \ln x + \ln y \), we have:

\[
v_{qu}(x, t) = \frac{dx_{qu}}{dt} = \frac{\dot{\sigma}(t)}{2 \sigma(t)} [x - q(t)] + \dot{q}(t) , \rightarrow
\]
\[
\frac{dx_{qu}}{dt} - \frac{dq}{dt} = \frac{\dot{\sigma}(t)}{2 \sigma(t)} [x - q(t)] \rightarrow \frac{dx_{qu}(t) - q(t)}{[x_{qu}(t) - q(t)]} = \frac{\dot{\sigma}(t) dt}{2 \sigma(t)} = \frac{d\sigma(t)}{2 \sigma(t)} \rightarrow
\]

\[
\int_0^t \frac{dx_{qu}(t') - q(t')}{[x_{qu}(t') - q(t')]} = \int_0^t \frac{d\sigma(t')}{2 \sigma(t')} \rightarrow
\]

\[
\ell n \left( \frac{x_{qu}(t) - q(t)}{x_{qu}(0) - q(0)} \right) = \frac{1}{2} \ell n \left[ \frac{\sigma(t)}{\sigma(0)} \right] = \ell n \left[ \frac{\sigma(t)}{\sigma(0)} \right]^{1/2} \rightarrow
\]

\[
x_{qu}(t) = q(t) + \left[ \frac{\sigma(t)}{\sigma(0)} \right]^{1/2} [x_{qu}(0) - q(0)] , \quad (4.4)
\]

that represent the looked for quantum trajectory.

To obtain the quantum wave packet of the non-linear Gross-Pitaevskii equation given by the eq.(2.2), let us expand the functions \(S(x, t), V(x, t)\) and \(V_{qu}(x, t)\) around of \(q(t)\) up to second Taylor order [5]. In this way we have:

\[
S(x, t) = S[q(t), t] + S'[q(t), t] [x - q(t)] + \frac{S''[q(t), t]}{2} [x - q(t)]^2 , \quad (4.5)
\]

\[
V(x, t) = V[q(t), t] + V'[q(t), t] [x - q(t)] + \frac{V''[q(t), t]}{2} [x - q(t)]^2 , \quad (4.6)
\]

\[
V_{qu}(x, t) = V_{qu}[q(t), t] + V'_{qu}[q(t), t] [x - q(t)] + \frac{V''_{qu}[q(t), t]}{2} [x - q(t)]^2 , \quad (4.7)
\]

\[
V_{GP}(x, t) = V_{GP}[q(t), t] + V'_{GP}[q(t), t] [x - q(t)] + \frac{V''_{GP}[q(t), t]}{2} [x - q(t)]^2 , \quad (4.8)
\]

where (') and ("') signifies, respectively, \(\frac{\partial}{\partial q}\) and \(\frac{\partial^2}{\partial q^2}\).

Differentiating the expression (4.5) in the variable \(x\), multiplying the result by \(\frac{\hbar}{m}\), using the relations (2.6) and (4.1), and taking into account the polynomial identity property, we obtain:

\[
\frac{\hbar}{m} \frac{\partial S(x, t)}{\partial x} = \frac{\hbar}{m} \left( S'[q(t), t] + S''[q(t), t] [x - q(t)] \right) =
\]

\[
= v_{qu}(x, t) = \left[ \frac{\dot{\sigma}(t)}{2 \sigma(t)} \right] [x - q(t)] + \dot{q}(t) = \rightarrow
\]

\[
S'[q(t), t] = \frac{m}{\hbar} \frac{\dot{\sigma}(t)}{\sigma(t)} , \quad S''[q(t), t] = \frac{m}{\hbar} \left[ \frac{\dot{\sigma}(t)}{2 \sigma(t)} \right] . \quad (4.9a,b)
\]

Substituting the expressions (4.9a,b) in the equation (4.5), results:

\[
S(x, t) = S_o(t) + \frac{m}{\hbar} \frac{\dot{\sigma}(t)}{\sigma(t)} [x - q(t)] + \frac{m}{4 \hbar} \left[ \frac{\dot{\sigma}(t)}{2 \sigma(t)} \right] [x - q(t)]^2 , \quad (4.10)
\]

where:
\[ S_0(t) \equiv S[q(t), t], \quad (4.11) \]

is the quantum action.

Differentiating the eq.(4.10) in relation to the time \( t \), we obtain (remembering that \( \frac{\partial x}{\partial t} = 0 \)):

\[
\frac{\partial S}{\partial t} = \dot{S}_0(t) + \frac{\partial}{\partial t} \left( \frac{m \dot{q}(t)}{\hbar} [x - q(t)] \right) + \frac{\partial}{\partial t} \left( \frac{m}{4\hbar} \left[ \frac{\dot{\sigma}(t)}{\sigma(t)} \right] [x - q(t)]^2 \right) \to \\
\frac{\partial S}{\partial t} = \dot{S}_0(t) + \frac{m \dot{q}(t)}{\hbar} [x - q(t)] - \frac{m \dot{q}(t)^2}{\hbar} + \\
+ \frac{m}{4\hbar} \left[ \frac{\ddot{\sigma}(t)}{\sigma(t)} - \frac{\dot{\sigma}(t)}{\sigma^2(t)} \right] [x - q(t)]^2 - \frac{m \dot{q}(t)}{2 \hbar} \left( \frac{\dot{\sigma}(t)}{\sigma(t)} \right) [x - q(t)]. \quad (4.12)
\]

Considering that [12]:

\[
\phi(x, t) = \sqrt{\rho(x, t)} = \left[ \pi \sigma(t) \right]^{-1/4} e^{-\frac{|x - q(t)|^2}{2\sigma(t)}} \to \\
\rho(x, t) = \left[ \pi \sigma(t) \right]^{-1/2} e^{-\frac{|x - q(t)|^2}{\sigma(t)}}, \quad (4.13)
\]

let us write \( V_{qu} \) in terms of \([x - q(t)]\). Initially, we calculate the following differentiations:

\[
\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x} \left( \left[ \pi \sigma(t) \right]^{-1/4} e^{-\frac{|x - q(t)|^2}{2\sigma(t)}} \right) = \left[ \pi \sigma(t) \right]^{-1/4} e^{-\frac{|x - q(t)|^2}{2\sigma(t)}} \frac{\partial}{\partial x} \left( - \frac{|x - q(t)|^2}{2\sigma(t)} \right) \to \\
\frac{\partial \phi}{\partial x} = - \left[ \pi \sigma(t) \right]^{-1/4} e^{-\frac{|x - q(t)|^2}{2\sigma(t)}} \frac{|x - q(t)|}{\sigma(t)},
\]

\[
\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x} \left( - \left[ \pi \sigma(t) \right]^{-1/4} e^{-\frac{|x - q(t)|^2}{2\sigma(t)}} \frac{|x - q(t)|}{\sigma(t)} \right) = \\
= - \left[ \pi \sigma(t) \right]^{-1/4} e^{-\frac{|x - q(t)|^2}{2\sigma(t)}} \frac{\partial}{\partial x} \left( \frac{|x - q(t)|}{\sigma(t)} \right) - \\
- \left[ \pi \sigma(t) \right]^{-1/4} e^{-\frac{|x - q(t)|^2}{2\sigma(t)}} \frac{\partial}{\partial x} \left( \frac{|x - q(t)|^2}{2\sigma(t)} \right) \left( \frac{|x - q(t)|}{\sigma(t)} \right) \to \\
\frac{\partial^2 \phi}{\partial x^2} = - \left[ \pi \sigma(t) \right]^{-1/4} e^{-\frac{|x - q(t)|^2}{2\sigma(t)}} \frac{1}{\sigma(t)} + \left[ \pi \sigma(t) \right]^{-1/4} e^{-\frac{|x - q(t)|^2}{2\sigma^2(t)}} \frac{|x - q(t)|^2}{2\sigma^2(t)} = \\
= - \phi \frac{1}{\sigma(t)} + \phi \frac{|x - q(t)|^2}{\sigma^2(t)} \to \frac{1}{\phi} \frac{\partial^2 \phi}{\partial x^2} = - \frac{2m}{\hbar^2} V_{qu}(x, t) = \frac{|x - q(t)|^2}{\sigma^2(t)} - \frac{1}{\sigma(t)} \quad (4.14)
\]
Substituting the relation (4.14) in the equation (2.7b), taking into account the expression (4.7), results:

\[ V_{qu}(x, t) = V_{qu}[q(t), t] + V_{qu}'[q(t), t] [x - q(t)] + \frac{V_{qu}''[q(t), t]}{2} [x - q(t)]^2 \rightarrow \]

\[ V_{qu}(x, t) = \frac{h^2}{2 m \sigma(t)} [x - q(t)]^2 - \frac{h^2}{2 m \sigma^2(t)} [x - q(t)]^2 \quad (4.15) \]

Comparing the eqs. (4.7) and (4.15), we have:

\[ V_{qu}[q(t), t] = \frac{h^2}{2 m \sigma(t)}; \quad V_{qu}'[q(t), t] = 0; \quad V_{qu}''[q(t), t] = - \frac{h^2}{m \sigma^2(t)}. \quad (4.16a,b,c) \]

Besides this the eq.(4.6) will be written, using the eq.(1.1) in the form:

\[ V(x, t) = V[q(t), t] + V'[q(t), t] [x - q(t)] + \frac{V''[q(t), t]}{2} [x - q(t)]^2 \rightarrow \]

\[ V(x, t) = \frac{1}{2} m \omega^2(t) q^2(t) + \]

\[ + \left( m \omega^2(t) q(t) \right) [x - q(t)] + \frac{m}{2} \omega^2(t) [x - q(t)]^2. \quad (4.17) \]

Comparing the eqs. (4.6) and (4.17), results:

\[ V[q(t), t] = \frac{1}{2} m \omega^2(t) q^2(t); \quad V'[q(t), t] = m \omega^2(t) q(t); \quad (4.18a,b) \]

\[ V''[q(t), t] = m \omega^2(t). \quad (4.18c) \]

Now, let us expand the eq. (4.13) around of q(t) to second Taylor order:

\[ \rho(x, t) = \rho[q(t), t] + \rho'[q(t), t] [x - q(t)] + \frac{\rho''[q(t), t]}{2} [x - q(t)]^2 \rightarrow \]

\[ \rho(x, t) = [\pi \sigma(t)]^{-1/2} e^{-\frac{|x - q(t)|^2}{\sigma(t)}} \quad [\pi \sigma(t)]^{-1/2} e^{-\frac{|x - q(t)|^2}{\sigma(t)}} \frac{1}{\sigma(t)} \]

\[ + \left[ [\pi \sigma(t)]^{-1/2} e^{-\frac{|x - q(t)|^2}{\sigma(t)}} \left( - \frac{1}{\sigma(t)} + \frac{2 |x - q(t)|^2}{\sigma^2(t)} \right) \right] [x - q(t)]^2. \]

Considering only quadratics terms, results:

\[ \rho(x, t) = [\pi \sigma(t)]^{-1/2} e^{-\frac{|x - q(t)|^2}{\sigma(t)}} \left( 1 + \frac{|x - q(t)|^2}{\sigma(t)} \right). \quad (4.19) \]

By using the eqs. (2.9), (4.8) and (4.19), we have:
\[ V_{GP}[q(t), t] = g [\pi \sigma(t)]^{-1/2} e^{-\frac{|x - q(t)|^2}{\sigma(t)}} \left( 1 + \frac{|x - q(t)|^2}{\sigma(t)} \right) = \]

\[ = V_{GP}[q(t), t] + V'_{GP}[q(t), t] [x - q(t)] + \frac{V''_{GP}[q(t), t]}{2} [x - q(t)]^2 \quad \rightarrow \]

\[ V_{GP}[q(t), t] = g [\pi \sigma(t)]^{-1/2} e^{-\frac{|x - q(t)|^2}{\sigma(t)}}, \quad V'_{GP}[q(t), t] = 0; \quad (4.20a,b) \]

\[ V''_{GP}[q(t), t] = 2 g [\pi \sigma(t)]^{-1/2} e^{-\frac{|x - q(t)|^2}{\sigma(t)}}. \quad (4.20c) \]

Inserting the eqs. (2.7), (4.1; 4.5-8; 4.12; 4.14; 4.16a-c; 4.18a-c; 4.20a-c) into the eq.(2.4), we obtain, remembering that \( S_o(t), \sigma(t) \) and \( q(t) \):

\[ \hbar \frac{\partial \sigma}{\partial t} + \frac{1}{2} m v_{qu}^2 + V + V_{qu} + V_{GP} = \]

\[ = \hbar \left[ \dot{S}_o + \frac{m \dot{q}}{\hbar} (x - q) - \frac{m \dot{q}^2}{\hbar} + \frac{m}{4 \hbar} \left( \frac{\dot{\sigma}}{\sigma} - \frac{\dot{\sigma}^2}{\sigma^2} \right) \right] (x - q)^2 - \]

\[ - \frac{m \dot{q}}{2 \hbar} \left( \frac{\dot{\sigma}}{\sigma} \right) (x - q) + \frac{1}{2} m \left[ \left( \frac{\dot{\sigma}}{2 \sigma} \right) (x - q) + \dot{q} \right]^2 + \]

\[ + \frac{1}{2} m \omega^2 q^2 + m \omega^2 q^2 (x - q) + \frac{m}{2} \omega^2 (x - q)^2 + \frac{\hbar}{2 m \sigma} - \]

\[ - \frac{\hbar^2}{2 m \sigma^2} (x - q)^2 + g [\pi \sigma(t)]^{-1/2} e^{-\frac{|x - \sigma|^2}{\sigma(t)}} \left( 1 + \frac{2 |x - q|^2}{\sigma(t)} \right) = 0. \quad (4.21) \]

Since \((x - q)^o = 1\), we can gather together the above expression in potencies of \((x - q)\), obtaining:

\[ \left[ \hbar \dot{S}_o - m \dot{q}^2 + \frac{1}{2} m \dot{q}^2 + \frac{1}{2} m \omega^2 q^2 + \frac{\hbar^2}{2 m \sigma^2} + g [\pi \sigma(t)]^{-1/2} e^{-\frac{|x - \sigma|^2}{\sigma(t)}} \right] (x - q)^o + \]

\[ + \left[ m \ddot{q} - m \dot{q} \frac{\dot{\sigma}}{2 \sigma} + m \dot{q} \frac{\dot{\sigma}}{2 \sigma} + m \omega^2 q \right] (x - q) + \left[ \frac{m \dot{\sigma}}{2 \sigma} - \frac{m \dot{\sigma}^2}{2 \sigma^2} + \right. \]

\[ + \frac{m \ddot{\sigma}^2}{8 \sigma^2} + \frac{m \omega^2}{8 \sigma^2} - \frac{\hbar^2}{2 m \sigma^2} + 2 g [\pi \sigma(t)]^{-1/2} e^{-\frac{|x - \sigma|^2}{\sigma(t)}} \left( x - q \right)^2 = 0. \quad (4.22) \]

As the above relation is an identically null polynomial, the coefficients of the potencies must be all equal to zero, that is:

\[ \dot{S}_o(t) = \frac{1}{\hbar} \left[ \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega^2 q^2 - \frac{\hbar^2}{2 m \sigma^2} - g [\pi \sigma(t)]^{-1/2} e^{-\frac{|x - \sigma|^2}{\sigma(t)}} \right] \rightarrow \]
\[
\dot{S}_o(t) = \frac{1}{\hbar} \left[ \frac{1}{2} m \dot{q}^2 - \frac{1}{2} m \omega_q^2 q^2 - \frac{\hbar^2}{2 m \sigma^2} - g \rho \right], \quad (4.23)
\]

\[
\ddot{q} + \omega_q^2(t) q = 0, \quad (4.24)
\]

\[
\frac{\ddot{\sigma}}{2 \sigma} - \frac{\dot{\sigma}^2}{4 \sigma^2} + \omega^2 + \frac{2 g \rho}{m \sigma} = \frac{\hbar^2}{2 m \sigma^2}. \quad (4.25)
\]

Assuming that the following initial conditions are obeyed:

\[
q(0) = x_o, \quad \dot{q}(0) = v_o, \quad \sigma(0) = a_o, \quad \dot{\sigma}(0) = b_o, \quad (4.26a-d)
\]

and that [see eq.(2.7)]:

\[
S_o(0) = \frac{m v_o x_o}{\hbar}, \quad (4.27)
\]

the integration of the expression (4.23) will be given by:

\[
S_o(t) = \frac{1}{\hbar} \int_t^0 dt' \left[ \frac{1}{2} m \dot{q}^2(t') - \frac{1}{2} m \omega_q^2(t') q^2(t') - \frac{\hbar^2}{2 m \sigma(t')} - g \rho(x, t') \right] + \frac{m v_o x_o}{\hbar}. \quad (4.28)
\]

Taking into account the expressions (4.9a,b) and (4.28) in the equation (4.10) results:

\[
S(x, t) = \frac{1}{\hbar} \int_t^0 dt' \left[ \frac{1}{2} m \dot{q}^2(t') - \frac{1}{2} m \omega_q^2(t') q^2(t') - \frac{\hbar^2}{2 m \sigma(t')} - g \rho(x, t') \right] +
\]

\[
+ \frac{m v_o x_o}{\hbar} + \frac{m \dot{\sigma}(t)}{\hbar} [x - q(t)] + \frac{m \dot{\sigma}}{4 \hbar} \sigma [x - q(t)]^2. \quad (4.29)
\]

This result obtained above permit us, finally, to obtain the wave packet for of the non-linear Gross-Pitaevskii equation. Indeed, considering the equations (2.2), (.6), (4.13) and (4.29), we get:

\[
\Psi(x, t) = [\pi \sigma(t)]^{-1/4} \exp \left[ \left( i \frac{m}{\hbar} \frac{\dot{\sigma}(t)}{\sigma(t)} - \frac{1}{2} \frac{\dot{\sigma}}{\sigma(t)} \right) [x - q(t)]^2 \right] \times
\]

\[
\times \exp \left[ i \frac{m}{\hbar} \dot{q}(t) [x - q(t)] + i \frac{m v_o x_o}{\hbar} \right] \times
\]

\[
\times \exp \left[ i \frac{\hbar}{\hbar} \int_t^0 dt' \left[ \frac{1}{2} m \dot{q}^2(t') - \frac{1}{2} m \omega_q^2(t') q^2(t') - \frac{\hbar^2}{2 m \sigma(t')} - g \rho(x, t') \right] \right]. \quad (4.30)
\]

The eq. (4.30) we show that when \( q = 0 \), then we obtains the eq. (3.3.2.25) of the Reference [5], if \( \sigma(t) = 2 a^2(t), q(t) = X(t) \) and \( \frac{1}{2} m \omega_q^2(t) q^2(t) = V[X(t)] \).
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