Inf-convolution of $g_\Gamma$-solution and its applications

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Abstract

A risk-neutral method is always used to price and hedge claims in complete market, but another wildly used method in more general case is based on utility maximization or risk minimization. All kinds of risk measure have been used in literature. In this paper, We use a kind of risk measure induced by $g_\Gamma$-solution or the minimal solution of a Constrained Backward Stochastic Differential Equation (CBSDE) when constraints in investment comes to our consideration. We adopt the inf-convolution of convex risk measures to solve some optimization problem. A dynamic version risk measures defined through $g_\Gamma$-solution will be get. Just like the case without constraint, the inf-convolution of two minimal solutions of CBSDE with two different coefficients is equivalent to that of CBSDE with the inf-convolution of their coefficients. In this case, it is also possible to characterize the optimal risk transfer.

1 Introduction

The theory of Backward Stochastic Differential Equation (shortly BSDE) and risk measure are two wonderful tools to price and hedge claims in financial market. Useful reference about these can be found in Pardox and Peng [9] and Artzner et al. [2] and Delbaen [3]; Föllmer and Schied [5], [6], [7] and Frittelli and Rosazza [8], [9]. Unsurprisingly, one may wonder if there is some relationship between them, fortunately, Rossazza [4] has done this work, that is some kind of useful risk measure can be induced by g-expectation.

In a complete market, a kind of risk-neutral method is always used to price and hedge claims via equivalent martingale measure. However, when the market is incomplete or more generally when some constraints were put on wealth and portfolio process, one need to use super-hedging strategy to get upper price. In this paper, we define a risk measure via $g_\Gamma$-solution, which is a newly notation given by the author in Peng and Xu [14]. Interestingly, we prove such risk measure satisfies the important Fatou-property.

The risk measure induced by $g_\Gamma$-solution is different from the market modified risk measure used in Pauline Barrieu., Nicole El Karoui [11], [12]. In their paper, a market modified risk measure was defined as a inf-convolution of some risk measure and the risk measure generated by some convex set which usually can be viewed as some constraints in hedging problem. To make the risk measure generated by some set be well defined, one always ask the set to satisfy some additional conditions. A convenience to use the risk measure induced by $g_\Gamma$-solution is that we need not such conditions any more.

This paper is organized as follows: In section 2, we state the framework in Peng[13] and some propositions about $g_\Gamma$-solution. Under some mild assumptions, $g_\Gamma$-solution is well defined on $L^\infty(F)$, the space of (P)-essentially bounded variables on some probability space $(\Omega, \mathcal{F}, P)$. Some results about the risk measure induced by such solution and some applications of it are given in section 3.

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\section{BSDE and \(g_t\)-solution of CBSDE}

Given a probability space \((\Omega, \mathcal{F}, P)\) and \(\mathbb{R}^d\)-valued Brownian motion \(W(t)\), we consider a sequence \(\{(\mathcal{F}_t); t \in [0,T]\}\) of filtrations generated by Brownian motion \(W(t)\) and augmented by \(P\)-null sets. \(\mathcal{P}\) is the \(\sigma\)-field of predictable sets of \(\Omega \times [0,T]\). We use \(L^2_T(\mathbb{R}^d)\) to denote the space of all \(F_T\)-measurable random variables \(\xi: \Omega \rightarrow \mathbb{R}^d\) for which
\[
||\xi||^2 = E[|\xi|^2] < +\infty.
\]
and use \(H^2_T(\mathbb{R}^d)\) to denote the space of predictable process \(\varphi: \Omega \times [0,T] \rightarrow \mathbb{R}^d\) for which
\[
||\varphi||^2 = E[\int_0^T |\varphi|^2] < +\infty.
\]
The backward stochastic differential equation (shortly BSDE) driven by \(g(t,y,z)\) is given by
\[
-dy_t = g(t,y_t,z_t)dt - z_t^*dW(t)
\]
where \(y_t \in \mathbb{R}\) and \(W(t) \in \mathbb{R}^d\). Suppose that \(\xi \in L^2_T(\mathbb{R})\) and \(g\) satisfies
\[
|g(\omega, t, y_1, z_1) - g(\omega, t, y_2, z_2)| \leq M(|y_1 - y_2| + |z_1 - z_2|), \quad \forall (y_1, z_1, y_2, z_2)
\]
for some \(M > 0\) and
\[
g(\cdot, 0, 0) \in H^2_T(\mathbb{R})
\]
Pardoux and Peng \cite{PardouxPeng1992} proved the existence of adapted solution \((y(t), z(t))\) of such BSDE. We call \((g, \xi)\) standard parameters for the BSDE.

The following definitions is necessary to help us go on with our study.

\begin{definition}
A super-solution of a BSDE associated with the standard parameters \((g, \xi)\) is a vector process \((y_t, z_t, C_t)\) satisfying
\[
-dy_t = g(t,y_t,z_t)dt + dC_t - z_t^*dW(t), \quad y_T = \xi,
\]
or being equivalent to
\[
y_t = \xi + \int_t^T g(s,y_s,z_s)ds - \int_t^T z_s^*dW_s + \int_t^T dC_s,
\]
where \((C_t, t \in [0,T]\)) is an increasing, adapted, right-continuous process with \(C_0 = 0\) and \(z_t^*\) is the transpose of \(z_t\). When \(C_t = 0\), we call \((y_t, z_t)\) a \(g\)-solution.

Constraints like
\[
(y(t), z(t)) \in \Gamma
\]
where \(\Gamma = \{(y,z)|\phi(y,z) = 0\} \subset \mathbb{R} \times \mathbb{R}^d\) and \(\phi(y, z) : \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}^+\) is always considered in this paper. In such case, we give the following definition,

\begin{definition}
A \(g_t\)-solution or the minimal solution) A \(g\)-supersolution \((y_t, z_t, C_t)\) is said to be the \(g\)-minimal solution, given \(y_T = \xi\), subjected to the constraint \((C)\) if for any other \(g\)-supersolution \((y'_t, z'_t, C'_t)\) satisfying \((C)\) with \(y'_T = \xi\), we have \(y_t \leq y'_t\) a.e., a.s.. The minimal solution is denoted by \(E^g_0(\xi)\) and for convenience called as \(g\)-solution.

For any \(\xi \in L^2_T(\mathbb{R})\), we denote \(\mathcal{H}^\phi(\xi)\) as the set of \(g\)-supersolutions \((y_t, z_t, C_t)\) subjecting to \((C)\) with \(y_T = \xi\). When \(\mathcal{H}^\phi(\xi)\) is not empty, Peng \cite{Peng1995} proved that \(g\)-solution exists.

The convexity of \(E^g_0(\xi)\) can be easily deduced from the same proposition of solution of BSDE with convex generator function.
Proposition 2.1. Let \( \phi(t, y, z) \) be a function: \([0, T] \times R \times R^d \rightarrow R^+ \) and \( g(t, y, z) \) be a function: \([0, T] \times R \times R^d \rightarrow R \). Suppose \( \phi(t, y, z) \) and \( g(t, y, z) \) are both convex in \((y, z)\) and satisfy \((A1)\) and \((A2)\), then
\[
\mathcal{E}^{\phi \cdot g}_t(a\xi + (1-a)\eta) \leq a\mathcal{E}^{\phi \cdot g}_t(\xi) + (1-a)\mathcal{E}^{\phi \cdot g}_t(\eta) \quad \forall t \in [0, T]
\]
holds for any \( \xi, \eta \in L^2_\mathbb{F}(R) \) and \( a \in [0, 1] \).

Proof. According to Peng [13], the solutions \( y^m_t(\xi) \) of
\[
y^m_t(\xi) = \xi + \int_t^T g(y^m_s(\xi), z^m_s, s)ds + A^m_t - A^m_0 - \int_t^T z^m_s dW_s,
\]
is an increasing sequence and converges to \( \mathcal{E}^{\phi \cdot g}_t(\xi) \), where
\[
A^m_t := m \int_0^t \phi(y^m_s, z^m_s, s)ds.
\]
For any fixed \( m \), by the convexity of \( g \) and \( \phi \), \( y^m_t(\xi) \) is a convex in \( \xi \), that is
\[
y^m_t(a\xi + (1-a)\eta) \leq ay^m_t(\xi) + (1-a)y^m_t(\eta),
\]
taking limit as \( m \to \infty \), we get the required result. \( \square \)

Proposition 2.2. Under the same assumptions as above proposition, we have
\[
\mathcal{E}^{\phi \cdot g}_t(\eta) \geq \mathcal{E}^{\phi \cdot g}_t(\xi), \quad \forall t \in [0, T] \quad P - a.s.
\]
for any \( \xi, \eta \in L^2_\mathbb{F}(R) \) when \( P(\eta \geq \xi) = 1 \) and \( g_1 \geq g_2 \).

3 Risk measure via \( g_\mathbb{F}\)-solution and its applications

In this section, we study convex risk measure induced by \( g_\mathbb{F}\)-solution. First we give the concept of convex risk measure which can be got from many papers such as Föllmer and Schied [5].

Definition 3.1. Let \( L^\infty(\mathbb{P}) \) be the space of \((\mathbb{P})\)-essentially bounded functions on some probability space \((\Omega, \mathcal{F}, \mathbb{P})\). A functional \( \rho : L^\infty(\mathbb{P}) \rightarrow R \) is a monetary convex risk measure if, for any \( \xi \) and \( \eta \) in \( L^\infty(\mathbb{P}) \), it satisfies the following properties:

a) Convexity: \( \forall \lambda \in [0, 1] \quad \rho(\lambda \xi + (1-\lambda)\eta) \leq \lambda \rho(\xi) + (1-\lambda)\rho(\eta); \)

b) Monotonicity: \( \xi \leq \eta \quad a.s(\mathbb{P}) \Rightarrow \rho(\xi) \geq \rho(\eta); \)

c) Translation invariance: \( \forall m \in R \quad \rho(\xi + m) = \rho(\xi) - m. \)

A convex risk measure \( \rho \) is coherent if it satisfies also:

d) Homogeneity: \( \forall \lambda \in \mathbb{R}^+ \quad \rho(\lambda \xi) = \lambda \rho(\xi). \)

In order to generate a convex risk measure by \( g_\mathbb{F}\)-solution, we need some additional assumptions such as
\[
g \text{ is independent of } y \text{ and } g(\cdot, 0) = 0 \quad (A3)
\]

When \( g \) satisfying conditions \( A(i), i = 1, 2, 3 \), just as Rosazza [4] noted, some useful risk measure can be generated by \( g\)-expectation.

First, we prove a result that \( g_\mathbb{F}\)-solution can be well defined on the space \( L^\infty(\mathcal{F}_T) \) of \((\mathbb{P})\)-essentially bounded functions on some probability space \((\Omega, \mathcal{F}_T, \mathbb{P})\).
Proposition 3.1. Suppose that $g$ and $\phi$ satisfy assumptions $A(i), i = 1, 2, 3,$ then $\mathcal{E}_{l,t}^{g,\phi}($) is well defined on $L^\infty(F_T)$

Proof Since $g$ is independent of $y$ and $g(t,0) = 0$, $\phi(t,0) = 0,$ then for any fixed $C_0 > 0, \mu > 0,$ we have

$$g(t,y,0) \leq C_0 + \mu|y|, \quad (y,0) \in \Gamma_t, \quad \forall y \geq C_0.$$  

By Peng and Xu [14], the $g_l$-solution with terminal condition $y_T = \xi$ exists for any $\xi \in L^2_{+,\infty}(F_T),$ where

$$L^2_{+,\infty}(F_T) := \{\xi \in L^2(F_T), \xi^+ \in L^\infty(F_T)\}.$$

It is obvious $L^\infty(F_T) \subset L^2_{+,\infty}(F_T),$ thus $\mathcal{E}_{l,t}^{g,\phi}(\xi)$ exists for any $L^\infty(F_T)$.

We first consider the case $t = 0,$ then $\rho(\xi) = \mathcal{E}_{0}^{g,\phi}(-\xi)$ generated a static convex risk measure when both $g$ and $\phi$ are convex functions satisfying assumptions $A(i), i = 1, 2, 3.$ Furthermore, we can prove $\rho$ satisfies the important Fatou property.

Theorem 3.1. When both $g$ and $\phi$ satisfy assumptions $A(1)$ and $A(2),$ then $\mathcal{E}_{0}^{g,\phi}(\xi)$ is continuous from below, etc., when $\{\xi_n \in L^\infty(F_T), n = 1, 2, \cdots\}$ is an increasing sequence comes from $L^\infty(F_T)$ and converges almost surely to $\xi \in L^\infty(F_T),$ then

$$\lim_{n \to \infty} \mathcal{E}_{0}^{g,\phi}(\xi_n) = \mathcal{E}_{0}^{g,\phi}(\xi).$$

Proof Taking $y^m_n(\xi)$ as in proposition 2.1. By proposition 2.2, $\{\mathcal{E}_{0}^{g,\phi}(\xi_n), n = 1, 2, \cdots\}$ is an increasing sequence. We denote its limit at $t = 0$ as $a,$ then $a \leq \mathcal{E}_{0}^{g,\phi}(\xi).$ Since $\xi_n$ converges almost surely to $\xi \in L^\infty(F_T),$ by dominated convergence theorem, it also converges strongly in $L^2(P),$ then by the continuous dependence property of $g_l$-supersolution, the limit of $\{y^m_n(\xi_n)\}^\infty_{n=1}$ is $y^m(\xi(\xi)$ for any fixed $m.$

We want to show that $a = \mathcal{E}_{0}^{g,\phi}(\xi).$ If on the contrary on has $a < \mathcal{E}_{0}^{g,\phi}(\xi),$ then there is some $\delta > 0$ such that $\mathcal{E}_{0}^{g,\phi}(\xi) - \mathcal{E}_{0}^{g,\phi}(\xi_n) > \delta$ for any $n.$ On the other hand, for any $\epsilon > 0,$ $0 \leq \mathcal{E}_{0}^{g,\phi}(\xi) - y^m_n(\xi_n) \leq \epsilon$ holds for some larger $m_0.$ Fixing $m_0,$ $\epsilon,$ there is some $n_0$ which depends on $m_0$ and $\epsilon$ such that $0 \leq y^m_n(\xi_n) - y^m_n(\xi_n) \leq \epsilon,$ so $\mathcal{E}_{0}^{g,\phi}(\xi) - y^m_n(\xi_n) \leq 2\epsilon,$ but we have $\mathcal{E}_{0}^{g,\phi}(\xi) - y^m_n(\xi_n) > \delta,$ this is impossible for $\epsilon < \frac{\delta}{2}.$

Thanks to this property and the work done by Föllmer, H., Schied [6], [7], the convex risk measure can be represented by a family of probabilities which are absolutely continuous with $P$.

We then go to some applications of $g_l$-solution. Here we use some notations in Pauline Barrieu, Nicole El Karoui [11]. Let $\eta \in L^\infty(P), \rho(\xi) = \mathcal{E}_{0}^{g,\phi}(-\xi)$ be a convex risk measure when both $g$ and $\phi$ are convex, our first problem is a minimizing problem by inf-convolution. More explicitly, suppose two agents who have convex risk measure generated by $\rho_i(\xi) = \mathcal{E}_{0}^{g_i,\phi_i}(-\xi), i = 1, 2$ respectively, we want to find an optimal value in $L^\infty(P)$ to attain

$$\inf_{\xi \in L^\infty(F_T)} \{\rho_1(\eta - \xi) + \rho_2(\xi)\}. \quad (3.1)$$

We first consider two simple cases.

Theorem 3.2. If both $g$ and $\phi$ satisfy assumptions $A(i), i = 1, 2, 3$ and

$$h(z_1 + z_2) \leq h(z_1) + h(z_2), \forall z_1, z_2$$

holds for $h = g, \phi,$ then $\xi = 0$ is a optimal value for problem (3.1) when $g_i = g, \phi_i = \phi, i = 1, 2.$
Proof Suppose that \( y(t) = \mathcal{E}_t^{g, \phi}(\xi - \eta), z(t), C(t) \) and \( \tilde{y}(t) = \mathcal{E}_t^{g, \phi}(-\xi), \tilde{z}(t), \tilde{C}(t) \) are \( \Gamma \)-solutions with terminal value \( \xi - \eta \) and \( -\xi \) respectively, that is
\[
y_t = \xi - \eta + \int_t^T g(s, z_s)ds - \int_t^T z_s^* dW_s + \int_t^T dC_s, \tag{3.2}
\]
\[
\tilde{y}_t = -\xi + \int_t^T g(s, \tilde{z}_s)ds - \int_t^T \tilde{z}_s^* dW_s + \int_t^T d\tilde{C}_s. \tag{3.3}
\]
Add (3.2) and (3.3) together, we have
\[
\mathcal{E}_t^{g, \phi}(\xi - \eta) + \mathcal{E}_t^{g, \phi}(-\xi) \geq -\eta + \int_t^T (g(s, z_s) + g(s, \tilde{z}_s))ds - \int_0^T (z_s^* + \tilde{z}_s^*)dW_s + \int_t^T d(C_s + \tilde{C}_s). \tag{3.4}
\]
By the assumption, we have furthermore that
\[
\gamma(t) + \tilde{y}(t) \geq \tilde{y}(t) := -\eta + \int_t^T g(s, z_s + \tilde{z}_s)ds - \int_0^T (z_s^* + \tilde{z}_s^*)dW_s + \int_t^T d(C_s + \tilde{C}_s). \tag{3.5}
\]
and \( 0 \leq \phi(z_s + \tilde{z}_s) \leq \phi(z_s) + \phi(\tilde{z}_s) = 0 \).
This means that \( (\gamma(t), z(t) + \tilde{z}(t), C(t) + \tilde{C}(t)) \) is a super-solution with terminal value \( -\eta \).
By (3.5) and the definition of \( \Gamma \)-solution, we have
\[
\mathcal{E}_t^{g, \phi}(\xi - \eta) + \mathcal{E}_t^{g, \phi}(-\xi) \geq \mathcal{E}_t^{g, \phi}(-\eta).
\]
Take \( t = 0 \), we have
\[
\rho(\eta - \xi) + \rho(\xi) \geq \rho(\eta), \quad \forall \xi \in L_\infty^\infty(P).
\]
This means \( \xi = 0 \) is an optimal value for problem (3.1). \( \square \)

The result of above tells us that if two agents having risk measure induced by same coefficients, then one rational way of them to transfer risk is doing nothing.

We then go to consider another interesting case concerning a useful operator of risk measure.

For any \( \lambda > 0 \), which always be considered as the risk tolerance coefficient, we can define the dilatation of convex risk measure \( \rho(\xi) \) as \( \rho_\lambda = \lambda \rho(\xi/\lambda) \). Our first result is that under some mild assumptions, the dilatation risk measure of \( \Gamma \)-solution coincides with the minimal solution of the dilatation of coefficients.

**Theorem 3.3.** Suppose \( g \) and \( \phi \) satisfy the assumptions \( A(i), i = 1, 2, 3, \phi(\lambda z) = \lambda \phi(z) \) holds for any \( 0 < \lambda \). Let \( \rho(\xi) = \mathcal{E}_0^{g, \phi}(\xi) \), \( g_\lambda(z) = \lambda g(z/\lambda) \), then we have
\[
\lambda \rho(\xi/\lambda) = \mathcal{E}_0^{g_\lambda, \phi}(\xi)
\]
Proof Suppose that \( (y(t), z(t), C(t)) \) is the \( \Gamma \)-solution with terminal value \( \xi/\lambda \),
\[
\mathcal{E}_t^{g_\lambda, \phi}(\xi/\lambda) = y_t = \xi/\lambda + \int_t^T g(s, z_s)ds - \int_t^T z_s^* dW_s + \int_t^T dC_s, \tag{3.6}
\]
then
\[
\lambda \mathcal{E}_t^{g, \phi}(\xi/\lambda) = \lambda y_t = \xi + \int_t^T \lambda g(s, z_s)ds - \int_t^T \lambda z_s^* dW_s + \int_t^T d\lambda C_s. \tag{3.7}
\]
At the same time we suppose that \( (\tilde{y}(t), \tilde{z}(t), \tilde{C}(t)) \) is the minimal solution with coefficient \( g_\lambda = \lambda g(z/\lambda) \) and terminal value \( \xi \) satisfying constraint (C),
\[
\mathcal{E}_t^{g_\lambda, \phi}(\xi) = \tilde{y}_t = \xi + \int_t^T g_\lambda(s, \tilde{z}_s)ds - \int_t^T \tilde{z}_s^* dW_s + \int_t^T d\tilde{C}_s. \tag{3.8}
\]
By (3.7), we can see that \((\lambda y_t, \lambda z_t, \lambda C_t)\) is a \(g\)-supersolution with terminal value \(\xi\) satisfying constraint (C), thus we have

\[
\lambda \mathcal{E}_t^{g,\phi}(\xi) \geq \mathcal{E}_t^{g,\phi}(\xi) \quad a.e \quad a.s. \tag{3.9}
\]

Similarly, by (3.8), \((\tilde{y}(t)/\lambda, \tilde{z}(t)/\lambda, \tilde{C}(t)/\lambda)\) is a \(g\)-supersolution with terminal value \(\xi\) satisfying constraint (C), thus we have

\[
\mathcal{E}_t^{g,\phi}(\xi) \leq \mathcal{E}_t^{g,\phi}(\xi)/\lambda \quad a.e \quad a.s. \tag{3.10}
\]

Put (3.9) and (3.10) together, we get

\[
\lambda \mathcal{E}_t^{g,\phi}(\xi) = \mathcal{E}_t^{g,\phi}(\xi) \quad a.e \quad a.s.
\]

Specially

\[
\lambda \rho(\xi/\lambda) = \mathcal{E}_0^{g,\phi}(\xi)
\]

holds.

Thanks to this result and the wonderful result in Pauline Barrieu, Nicole El Karoui [11], we have the following result.

**Theorem 3.4.** Suppose \(g\) and \(\phi\) satisfy the assumptions \(A(i), i = 1, 2, 3\), two agents have risk measure with different risk tolerance coefficient \(g_\lambda\) and \(g_\gamma\) respectively, then one optimal value of problem (3.1) is

\[
\xi = \frac{\gamma}{\gamma + \lambda}\eta.
\]

When one consider the optimal problem (3.1) with general coefficients \(g_i, i = 1, 2\), we need more concepts.

**Definition 3.2.** Let \(X\) be a Banach space, \(X^*\) is its dual space and \(\varphi : X \to R\) is a convex functional. For any \(\xi \in X\), define

\[
\partial \varphi(\xi) = \{f \in X^*, f(\eta) \leq \varphi(\xi + \eta) - \varphi(\xi), \forall \eta \in X\}
\]

as the subdifferential of \(\varphi\) at \(\xi\), every member of \(\partial \varphi(\xi)\) is called a subgradient of subdifferential of \(\varphi\) at \(\xi\).

The following result is basic in convex analysis.

**Proposition 3.2.** Suppose \(\varphi\) is a continuous convex functional on \(X\), then for any \(\xi \in X\), \(\partial \varphi(\xi)\) is not empty.

**Proof** In the product space \(X \times R\), let \(D = \{(\xi, t) | \varphi(\xi) \leq t\}\) be the upper semi-graph of \(\varphi\). For any fixed point \(\xi_0 \in X\), since \(\varphi(\cdot)\) is continuous at \(\xi_0\), \((\xi_0, \varphi(\xi_0) + 1)\) is an interior point of \(D\). Note that

\[
\{(\xi_0, \varphi(\xi_0))\} \cap \bar{D} = \emptyset,
\]

then by separating theorem of convex sets in Banach space, there is some no zero point \((g, a) \in X^* \times R\) such that

\[
g(\xi_0) + a\varphi(\xi_0) \leq g(x) + at \quad \forall (x, t) \in D.
\]

It is not hard to check that \(a > 0\), then if we take \(f = -g/a\), then \(f \in \partial \varphi(\xi)\). \(\square\)

The next result gives us a sufficient condition for a convex functional to be continuous, for its proof, we refer to Aubin[1].
Proposition 3.3. Let $X$ be a Banach space, $\varphi : X \to \mathbb{R}$ be a convex functional. If $\varphi$ is lower semi-continuous on $X$, then it is continuous on $X$.

A useful result has been obtained in our previous paper.

Theorem 3.5. Suppose $g$ and $\phi$ satisfy the assumptions $A(i), i = 1, 2, 3$, then $E_0^{g, \phi}(\xi)$ is lower semi-continuous on $L^\infty(F_T)$.

Proof. See Wu [15] for reference. □

We then have a general result when two agents have risk measure generated by general coefficients $g_i, \phi_i, i = 1, 2$.

Theorem 3.6. Suppose $g_i, \phi_i, i = 1, 2$ are convex functions satisfying the assumptions $A(i), i = 1, 2, 3$ and there is some $a, b \in \mathbb{R}$ such that $g_i(t, z) \geq az + b, i = 1, 2$. If there is some $\xi \in L^\infty(F_T)$ and some finite additive measure $Q \in \partial \rho(\eta) \cap \partial \rho_1(\eta - \xi) \cap \partial \rho_2(\xi)$, then $\xi$ is optimal for problem (3.1), where

$$
\rho_i(\cdot) = E_0^{g_i, \phi_i}(\cdot), i = 1, 2; \quad \rho(\cdot) = \inf_{\xi \in L^\infty(F_T)} \{\rho_1(\cdot - \xi) + \rho_2(\xi)\}.
$$

Proof. By the assumption that $g_i(z) \geq az + b, i = 1, 2$, we have that the inf-convolution $\rho$ is well defined on $L^\infty(F_T)$. The rest of the proof can be found in Pauline Barrieu, Nicole El Karoui [12].

At last, we state a dynamic version of inf-convolution of $g_1$-solution.

Theorem 3.7. Suppose $g_i, i = 1, 2, \phi$ are convex functions satisfying the assumptions $A(i), i = 1, 2, 3$, $\phi(t, z_1, z_2) \leq \phi(t, z_1) + \phi(t, z_2), \forall z_1, z_2$ and there is some $a, b \in \mathbb{R}$ such that $g_i(t, z) \geq az + b, i = 1, 2$. The inf-convolution of $g_1$ and $g_2$ is given by

$$
g_3(t, z) = g_1 \Box g_2(t, z) = \inf_y \{g_1(t, z - y) + g_2(t, y)\}.
$$

Let $(E_t^{g_3, \phi}(\eta), \hat{z}_3(t), \hat{C}_3(t))$ be the $g_1$-solution with terminal value $\xi \in L^\infty(F_T)$ satisfying constraint (C) and $\hat{z}$ be a measurable process such that $\hat{z} = \arg\min_y \{g_1(t, \hat{z}_3(t) - y) + g_2(t, y)\}$ dt $\times$ d$P$ a.s., then the following results hold:

1. For any $t \in [0, T]$ and any $\xi \in L^\infty(F_T)$,

$$
E_t^{g_3, \phi}(\eta) \leq E_t^{g_1, \phi}(\eta - \xi) + E_t^{g_2, \phi}(\xi).
$$

2. If $\phi(t, \hat{z}(t)) = 0, \phi(t, \hat{z}_3(t) - \hat{z}(t)) = 0$ and

$$
\xi^* := \int_0^T g_2(s, \hat{z}_s)ds - \int_0^T \hat{z}_s dW_s \in L^\infty(F_T),
$$

then $\xi^*$ is an optimal value for problem (3.1), further, we have

$$
E_t^{g_3, \phi}(\eta) = E_t^{g_1, \phi} \Box E_t^{g_2, \phi}(\eta), \quad \forall t \in [0, T].
$$

Proof. (1) By the same argument of proposition 3.1, $E_t^{g_3, \phi}(\eta)$ exists for any $\eta \in L^\infty(F_T)$.

Suppose that $(g_i(t), z_i(t), C_i(t)), i = 1, 2$ is the minimal solution with terminal value $\eta - \xi$ and $\xi$ for CBSDE with coefficients $g_i$ satisfying constraint (C), that is

$$
E_t^{g_i, \phi}(\eta - \xi) = y_1(t) = \eta - \xi + \int_t^T g(s, z_1(s))ds - \int_t^T z_1^*(s)dW_s + \int_t^T dC_1(s).
$$

(3.11)
\[ E_t^{g_1, \phi}(\xi) = y_2(t) = \xi + \int_t^T g(s, z_2(s))ds - \int_t^T z_2^*(s)dW_s + \int_t^T dC_2(s). \quad (3.12) \]

Put (3.11) and (3.12) together, by the comparison property of proposition 2.2, we have

\[ E_t^{g_1, \phi}(\eta - \xi) + E_t^{g_2, \phi}(\xi) \geq y_3(t) = \eta + \int_t^T g_3(s, z_3(s))ds - \int_t^T z_3^*(s)dW_s + \int_t^T dC_3(s). \]

where \( z_3(t) = z_1(t) + z_2(t), C_3(t) = C_1(t) + C_2(t). \)

But \((y_3(t), z_3(t), C_3(t))\) is a \( g_3 \)-supersolution satisfying constraint (C), we have

\[ E_t^{g_3, \phi}(\eta) \leq E_t^{g_1, \phi}(\eta - \xi) + E_t^{g_2, \phi}(\xi). \quad (3.13) \]

(2)

Since

\[ E_t^{g_1, \phi}(\eta) = \eta + \int_t^T g_1(s, \hat{\xi}_3(s))ds - \int_t^T \hat{\xi}_3^*(s)dW_s + \int_t^T dC_3(s). \quad (3.13) \]

But \( g_3(t, \hat{\xi}_3(t)) = g_1(t, \hat{\xi}_3(t) - \hat{\xi}(t)) + g_2(\hat{\xi}(t)) \). Let

\[ \hat{y}(t) = -\int_0^t g_2(s, \hat{\xi}(s))ds + \int_0^t \hat{\xi}^*(s)dW_s, \]

that is

\[ \hat{y}(t) = \xi^* + \int_t^T g_2(s, \hat{\xi}(s))ds - \int_t^T \hat{\xi}^*(s)dW_s \]

and it is obvious that \( E_t^{g_2, \phi}(\xi^*) = \hat{y}(t). \) By (3.14), \((E_t^{g_1, \phi}(\eta) - E_t^{g_2, \phi}(\xi^*), \hat{\xi}_3(t) - \hat{\xi}(t), \hat{C}_3(t))\) is a \( g_1 \)-supersolution with terminal value \( \eta - \xi^* \) satisfying constraint (C), so

\[ E_t^{g_3, \phi}(\eta) - E_t^{g_2, \phi}(\xi^*) \geq E_t^{g_1, \phi}(\eta - \xi^*). \quad (3.15) \]

By (3.13) and (3.15), we get

\[ E_t^{g_3, \phi}(\eta) = E_t^{g_1, \phi}(\eta - \xi^*) + E_t^{g_2, \phi}(\xi^*) = E_t^{g_1, \phi} & E_t^{g_2, \phi}(\eta). \]

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