ON THE CHOW GROUPS OF HYPERSURFACES IN SYMPLECTIC 
GRASSMANNIANS

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ABSTRACT. Let $Y$ be a Plücker hypersurface in a symplectic Grassmannian $I_1 \text{Gr}(3, n)$ or a 
bisymplectic Grassmannian $I_2 \text{Gr}(3, n)$. We show that many Chow groups of $Y$ inject into coho-
mology.

1. INTRODUCTION

Given a smooth projective variety $Y$ over $\mathbb{C}$, let $A_i(Y) := CH_i(Y)_{\mathbb{Q}}$ denote the Chow groups of 
$Y$ (i.e. the groups of $i$-dimensional algebraic cycles on $Y$ with $\mathbb{Q}$-coefficients, modulo rational 
equivalence). Let $A_i^{\text{hom}}(Y) \subset A_i(Y)$ denote the subgroup of homologically trivial cycles.

The famous Bloch–Beilinson conjectures [8], [25] predict that the Hodge level of the coho-
mology of $Y$ should have an influence on the size of the Chow groups of $Y$. For surfaces, this is 
the notorious Bloch conjecture, which is still an open problem. For hypersurfaces in projective 
space, the precise prediction is as follows:

**Conjecture 1.1.** Let $Y \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d$. Then

$$A_i^{\text{hom}}(Y) = 0 \quad \forall \ i \leq \frac{n}{d} - 1.$$ 

Conjecture 1.1 is still open; partial results have been obtained in [16], [23], [19], [5], [6].

In [14], I considered a version of Conjecture 1.1 for Plücker hyperplane sections of Grass-
mannians. In this note, we look at the case of Plücker hyperplane sections of symplectic Grass-
mannians. Recall that inside the Grassmannian $\text{Gr}(3, n)$ (of 3-dimensional subspaces of an 
n-dimensional vector space), the symplectic Grassmannian $I_1 \text{Gr}(3, n) \subset \text{Gr}(3, n)$ parametrizes 
subspaces that are isotropic with respect to some fixed skew-symmetric 2-form. The precise 
prediction (cf. subsection 3.2 below) is as follows:

**Conjecture 1.2.** Let

$$Y := I_1 \text{Gr}(3, n) \cap H \subset \mathbb{P}(\mathbb{Z})^{-1}$$

be a smooth hyperplane section (with respect to the Plücker embedding). Then

$$A_i^{\text{hom}}(Y) = 0 \quad \forall \ i \leq n - 4.$$ 

The main result of this note is a partial verification of Conjecture 1.2.

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**Theorem (Theorem 3.4).** Let

\[ Y := I_1 \text{Gr}(3, n) \cap H \subset \mathbb{P}^{n-1} \]

be a smooth hyperplane section (with respect to the Plücker embedding). Then

\[ A_i^\text{hom}(Y) = 0 \quad \forall \ i \leq n - 5. \]

Moreover, in case \( n \leq 10 \) or \( n = 12 \) we have

\[ A_i^\text{hom}(Y) = 0 \quad \forall \ i \leq n - 4. \]

To prove Theorem 3.4, we rely on the recent notion of projections among (symplectic) Grassmannians [2]. Combined with the Chow-theoretic Cayley trick [9], this reduces Theorem 3.4 to understanding the Chow groups of a hyperplane section in an ordinary Grassmannian \( \text{Gr}(3, n + 1) \). This last problem was handled in [14].

As a consequence of Theorem 3.4, some instances of the generalized Hodge conjecture are verified:

**Corollary (Corollary 4.1).** Let \( Y \) be as in Theorem 3.4 and assume \( n \leq 10 \) or \( n = 12 \). Then \( H^{\dim Y}(Y, \mathbb{Q}) \) is supported on a subvariety of codimension \( n - 3 \).

Other consequences are as follows:

**Corollary (Corollary 4.2).** Let

\[ Y := I_1 \text{Gr}(3, n) \cap H \subset \mathbb{P}^{n-1} \]

be a smooth hyperplane section (with respect to the Plücker embedding).

(i) If \( n \leq 8 \), then \( Y \) has finite-dimensional motive (in the sense of [11]).

(ii) If \( n \leq 9 \), then \( Y \) has trivial Griffiths groups (and so Voevodsky’s smash conjecture is true for \( Y \)).

(iii) If \( n \leq 10 \), the Hodge conjecture is true for \( Y \).

Applying the same method, we can also say something about hypersurfaces in bisymplectic Grassmannians. (Recall that the bisymplectic Grassmannian \( I_2 \text{Gr}(3, n) \subset \text{Gr}(3, n) \) is the locus of 3-spaces that are isotropic with respect to two fixed generic skew-forms.)

**Theorem (Theorem 3.5).** Assume \( n \) is even, and let

\[ Y := I_2 \text{Gr}(3, n) \cap H \subset \mathbb{P}^{n-1} \]

be a smooth hyperplane section (with respect to the Plücker embedding). Then

\[ A_i^\text{hom}(Y) = 0 \quad \forall \ i \leq n - 7. \]

Moreover, in case \( n \leq 10 \) we have

\[ A_i^\text{hom}(Y) = 0 \quad \forall \ i \leq n - 6. \]

The hyperplane sections \( I_1 \text{Gr}(3, 9) \cap H \) and \( I_2 \text{Gr}(3, 8) \cap H \) are of particular interest: they are Fano varieties of K3 type, and they are related to hyperplane sections \( \text{Gr}(3, 10) \cap H \) and hence to Debarre–Voisin hyperkähler fourfolds, cf. [2, Section 4].
Conventions. In this note, the word variety will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A subvariety is a (possibly reducible) reduced subscheme which is equidimensional.

All Chow groups will be with rational coefficients: we denote by $A_j(Y) := \text{CH}_j(Y)$ the Chow group of $j$-dimensional cycles on $Y$ with $\mathbb{Q}$-coefficients; for $Y$ smooth of dimension $n$ the notations $A_j(Y)$ and $A^{n-j}(Y)$ are used interchangeably. The notations $A^j_{\text{hom}}(Y)$ and $A^j_{\text{AJ}}(X)$ will be used to indicate the subgroup of homologically trivial (resp. Abel–Jacobi trivial) cycles.

The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [20], [17]) will be denoted $\mathcal{M}_{\text{rat}}$.

2. Preliminaries

2.1. Cayley’s trick and Chow groups.

Theorem 2.1 (Jiang [9]). Let $E \to U$ be a vector bundle of rank $r \geq 2$ over a projective variety $U$, and let $S := s^{-1}(0) \subseteq U$ be the zero locus of a regular section $s \in H^0(U, E)$ such that $S$ is smooth of dimension $\dim U - \text{rank } E$. Let $X := w^{-1}(0) \subseteq \mathbb{P}(E)$ be the zero locus of the regular section $w \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$ that corresponds to $s$ under the natural isomorphism $H^0(U, E) \cong H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(1))$. There is an isomorphism of integral Chow motives

$$h(X) \cong h(S)(1 - r) \bigoplus_{i=0}^{r-2} h(U)(-i) \quad \text{in } M^\mathbb{Z}_{\text{rat}}.$$

Proof. This is [9, Theorem 3.1]. Both the isomorphism and its inverse are explicitly described. □

Remark 2.2. In the set-up of Theorem 2.1, a cohomological relation between $X$, $S$ and $U$ was established in [12, Prop. 4.3] (cf. also [7, section 3.7], as well as [2, Proposition 46] for a generalization). A relation on the level of derived categories was established in [18, Theorem 2.10] (cf. also [10, Theorem 2.4] and [2, Proposition 47]).

2.2. Linear sections of $\text{Gr}(2, n)$.

Proposition 2.3. Let

$$Y := \text{Gr}(2, n) \cap H_1 \cap \cdots \cap H_s \subseteq \mathbb{P}^{n-1}$$

be a smooth dimensionally transverse intersection with $s$ hyperplanes (with respect to the Plücker embedding). Assume $s \leq 2$. Then

$$A^i_{\text{hom}}(Y) = 0 \quad \forall i.$$

Proof. This uses a geometric construction that can be found in [4]. Let $P \subseteq \mathbb{P}(V_n)$ be a fixed hyperplane, and consider (as in [4, Section 2.3]) the rational map

$$\text{Gr}(2, V_n) \dashrightarrow P$$
sending a line in \( \mathbb{P}(V_n) \) to its intersection with \( P \). This map is resolved by blowing up a subvariety \( \sigma_{11}(P) \cong \text{Gr}(2, n - 1) \), resulting in a morphism

\[
\Gamma : \widetilde{\text{Gr}} \to P
\]

(where \( \widetilde{\text{Gr}} \to \text{Gr}(2, V_n) \) denotes the blow-up with center \( \sigma_{11}(P) \)).

Let \( \tilde{Y} \to Y \) be the blow-up of \( Y \) with center \( \sigma_{11}(P) \cap Y \), and let us consider the morphism

\[
\Gamma_Y : \tilde{Y} \to P,
\]

obtained by restricting \( \Gamma \).

In case \( s = 1 \) and \( P \) is generic with respect to \( Y \), the morphism \( \Gamma_Y \) is a \( \mathbb{P}^{n-3} \)-fibration over \( P \). It follows that \( \tilde{Y} \), and hence \( Y \), has trivial Chow groups.

In case \( s = 2 \), and \( P \) chosen generically with respect to \( Y \), the morphism \( \Gamma_Y \) is generically a \( \mathbb{P}^{n-4} \)-fibration over \( P \), and there are finitely many points in \( P \) where the fiber is \( \mathbb{P}^{n-3} \). Applying Theorem 2.1, this implies that \( \tilde{Y} \), and hence also \( Y \), has trivial Chow groups. \( \square \)

### 2.3. Hyperplane sections of \( \text{Gr}(3, n) \).

**Theorem 2.4.** Let

\[
Y := \text{Gr}(3, n) \cap H \subset \mathbb{P}^{(n)}_{(5)} - 1
\]

be a smooth hyperplane section (with respect to the Plücker embedding). Then

\[
A^\text{hom}_i(Y) = 0 \quad \forall \ i \leq n - 3 .
\]

Moreover, in case \( n \leq 11 \) or \( n = 13 \) we have

\[
A^\text{hom}_i(Y) = 0 \quad \forall \ i \leq n - 2 .
\]

**Proof.** This is \([14\text{, Theorems 3.1 and 3.2}]\), which uses the notion of jumps between Grassmannians as developed in \([2]\). \( \square \)

### 3. Main results

#### 3.1. Projections.

As in \([2]\), let \( I_r \text{Gr}(k, n) \subset \text{Gr}(k, n) \) parametrize linear subspaces that are isotropic with respect to \( r \) fixed generic skew-forms. One has

\[
\dim I_r \text{Gr}(k, n) = k(n - k) - r \binom{k}{2} .
\]

For example, \( I_2 \text{Gr}(2, n) \) is just the intersection of \( \text{Gr}(2, n) \) with \( r \) Plücker hyperplanes. The case \( I_2 \text{Gr}(k, n) \) is studied in detail in \([1]\).

To relate hyperplane sections of different symplectic Grassmannians, Bernardara–Fatighenti–Manivel \([2]\) have developed a theory of projections. The starting point is a rational map

\[
\pi : \text{Gr}(k, n + 1) \longrightarrow \text{Gr}(k, n) ,
\]

determined by the choice of a line in the \( n + 1 \)-dimensional vector space. If \( Y' \) is a hyperplane section of \( I_r \text{Gr}(k, n + 1) \), one can restrict \( \pi \) to \( Y' \). A detailed analysis of the case \( k = 3 \) yields the following:
Theorem 3.1 ([2]). Assume $n$ is even and $r \leq 1$, or $n$ is odd and $r = 0$. Let

$Y' := I_r \text{Gr}(3, n+1) \cap H$

be a smooth hyperplane section. There exists a commutative diagram

$$
\begin{array}{ccc}
E & \hookrightarrow & \tilde{Y}' \\
\downarrow & & \downarrow \sigma \\
Z' & \hookrightarrow & Y'
\end{array}
\learrow
\begin{array}{ccc}
F & \twoheadrightarrow & \tilde{Y}' \\
\downarrow & & \downarrow \tau \\
Y & \twoheadrightarrow & Y'
\end{array}
$$

where $Y := I_{r+1} \text{Gr}(3, n) \cap H$ is a smooth hyperplane section. The morphism $\sigma$ is the blow-up with center $Z' \cong I_{r+1} \text{Gr}(2, n)$. The morphism $q$ is a $\mathbb{P}^3$-fibration, while $\tau$ is a $\mathbb{P}^2$-fibration over the complement of $Y$.

Proof. This is contained in [2, Section 3.2] (NB: note that our $n$ is $n-1$ in loc. cit.). As explained in loc. cit., the assumptions on $n$ and $r$ guarantee that the target $I_r \text{Gr}(3, n)$ and the hyperplane section $Y$ are generic and hence smooth. □

3.2. Motivating the conjecture. As a consequence of Theorem 3.1, one can compute the Hodge level of hyperplane sections $Y'$ of symplectic Grassmannians: surprisingly, it turns out that (at least for $n > 8$) $Y$ is of “Calabi–Yau type”:

Theorem 3.2 ([2]). Let

$Y := I_1 \text{Gr}(3, n) \cap H \subset \mathbb{P}^{(n^2)} - 1$

be a smooth hyperplane section (with respect to the Plücker embedding). Assume $n > 8$. Then $Y$ has Hodge coniveau $n - 3$. More precisely, the Hodge numbers verify

$$h^{p, \dim Y, -p}(Y) = \begin{cases} 
1 & \text{if } p = n - 3, \\
0 & \text{if } p < n - 3.
\end{cases}$$

Proof. This is implicit in [2], as we now explain. Let the set-up be as in Theorem 3.1. Then [2, Proposition 6] relates $Y$ and $Y'$ on the level of cohomology: one has an isomorphism of Hodge structures

$$H^{j-6}(Y, \mathbb{Q})(-3) \oplus \bigoplus_{i=0}^{2} H^{j-2i}(I_r \text{Gr}(3, n), \mathbb{Q})(-i) \xrightarrow{\cong} H^j(Y', \mathbb{Q}) \oplus \bigoplus_{i=1}^{c-1} H^{j-2i}(I_{r+1} \text{Gr}(2, n), \mathbb{Q})(-i)$$

(1)

(where $c$ denotes the codimension of $Z'$ in $Y'$). Setting $r = 0$ and combining with [2, Theorem 3] (which gives the Hodge numbers of $Y'$), plus the fact that $\text{Gr}(3, n)$ and $I_1 \text{Gr}(2, n)$ have algebraic cohomology, this gives the required Hodge numbers of $Y$. □
Theorem 3.2 motivates Conjecture 1.2. Indeed, the generalized Bloch conjecture [25, Conjecture 1.10] predicts that any variety $Y$ with Hodge coniveau $\geq c$ has

$$A_i^{hom}(Y) = 0 \quad \forall \ i < c .$$

Note that at least for $n > 8$, the bound of Conjecture 1.2 is optimal: assuming $A_i^{hom}(Y) = 0$ for $j \leq n - 3$ and applying the Bloch–Srinivas argument [3], one would get the vanishing $h^{n-3, \dim Y - n + 3}(Y) = 0$, contradicting Theorem 3.2.

3.3. A relation of motives. The cohomological relation (1) between $Y$ and $Y'$ also exists as (and actually is implied by) a relation on the level of the Grothendieck ring of varieties [2, Proposition 4], and on the level of derived categories [2, Proposition 5]. To complete the picture, we now lift the relation (1) to the level of Chow motives:

**Proposition 3.3.** Let notation and assumptions be as in Theorem 3.1. There is an isomorphism of integral Chow motives

$$h(Y)(-3) \oplus \bigoplus_{i=0}^{2} h(I_r \text{Gr}(3,n))(-i) \cong h(Y') \oplus \bigoplus_{i=1}^{c-1} h(I_{r+1} \text{Gr}(2,n))(-i)$$

in $\mathcal{M}_{rat}^Z$.

(Here $c$ denotes the codimension of $Z'$ in $Y'$.)

**Proof.** The idea is to express the motive of $\tilde{Y}'$ in two different ways:

The blow-up formula expresses $h(\tilde{Y}')$ in terms of $h(Y')$; this gives the right-hand side of the relation.

Looking at [2, Section 3.2], one finds that $\tilde{Y}'$ is the total space of a projectivization $\mathbb{P}(E)$ where $E$ is the vector bundle $E := \mathcal{O} \oplus \mathcal{U}^*$ on $I_r \text{Gr}(3,n)$ (in the notation of loc. cit.), and $Y$ is given by a section of $E$. That is, we are in the setting of Cayley’s trick, and so Theorem 2.1 expresses $h(\tilde{Y}')$ in terms of $h(Y)$; this gives the left-hand side of the relation. \hfill \square

3.4. Hyperplane sections of $I_r \text{Gr}(3,n)$.

**Theorem 3.4.** Let

$$Y := I_1 \text{Gr}(3,n) \cap H \subset \mathbb{P}(3)^{-1}$$

be a smooth hyperplane section (with respect to the Plücker embedding). Then

$$A_i^{hom}(Y) = 0 \quad \forall \ i \leq n - 5 .$$

Moreover, in case $n \leq 10$ or $n = 12$ we have

$$A_i^{hom}(Y) = 0 \quad \forall \ i \leq n - 4 .$$

**Proof.** A generic hyperplane section $Y$ is attained by the construction of Theorem 3.1 with $r = 0$, i.e. there is a smooth hyperplane section

$$Y' := \text{Gr}(3, n + 1) \cap H ,$$

satisfying the conditions of Proposition 3.3. \hfill \square
related to $Y$ via the projection of Theorem 3.1. In this case, Proposition 3.3 implies that there is an injection of Chow groups

$$A^i_{\text{hom}}(Y) \hookrightarrow A^i_{\text{hom}}(Y') \oplus \bigoplus A^i_{\text{hom}}(I_1 \text{Gr}(2, n)) .$$

The symplectic Grassmannian $I_1 \text{Gr}(2, n)$ is nothing but a Plücker hyperplane section of $\text{Gr}(2, n)$, and so Proposition 2.3 gives the vanishing

$$A^*_{\text{hom}}(I_1 \text{Gr}(2, n)) = 0 .$$

The variety $Y'$ is a hyperplane section of $\text{Gr}(3, n + 1)$, and so Theorem 2.4 gives the vanishing

$$A^i_{\text{hom}}(Y') = 0 \ \forall \ i \leq n - 5 ,$$

with the additional vanishing for $i = n - 4$ for small $n$. This proves the theorem for generic sections $Y$.

A standard spread argument allows to extend to all smooth hyperplane sections: Let $Y \to B$ denote the universal family of all smooth hyperplane sections of $\text{Gr}(k, n)$, and let $B^o \subset B$ denote the Zariski open subset parametrizing smooth $Y$ verifying the set-up of Theorem 3.1. Doing the Bloch–Srinivas argument [3] (cf. also [13]), the above implies that for each $b \in B^o$ one has a decomposition of the diagonal

$$\Delta_{Y_b} = \gamma_b + \delta_b \ \text{in} \ A^{\dim Y_b}(Y_b \times Y_b)$$

where $\gamma_b$ is completely decomposed (i.e. $\gamma_b \in A^*(Y_b) \otimes A^*(Y_b)$) and $\delta_b$ is supported on $Y_b \times W_b$ with codim $W_b = n - 2$ (and codim $W_b = n - 1$ for small $n$). Using the Hilbert schemes argument of [2] Proposition 3.7] (cf. also [15] Proposition A.1) for the precise form used here), the $\gamma_b, \delta_b, W_b$ exist relatively, i.e. one can find a cycle $\gamma \in (p_1)^* A^*(Y) \cdot (p_2)^* A^*(Y)$, a subvariety $W \subset Y$ of codimension $n - 2$, and a cycle $\delta$ supported on $Y \times B^o W$ such that

$$\Delta_Y|_b = \gamma|_b + \delta|_b \ \text{in} \ A^{\dim Y_b}(Y_b \times Y_b) \ \forall \ b \in B^o .$$

Let $\bar{\gamma}, \bar{\delta} \in A^{\dim Y_b}(Y \times B Y)$ be cycles that restrict to $\gamma$ resp. $\delta$. The spread lemma [25, Lemma 3.2] implies that

$$\Delta_Y|_b = \bar{\gamma}|_b + \bar{\delta}|_b \ \text{in} \ A^{\dim Y_b}(Y_b \times Y_b) \ \forall \ b \in B .$$

Given any $b_1 \in B \setminus B^o$, using the moving lemma, one can find representatives for $\bar{\gamma}$ and $\bar{\delta}$ in general position with respect to the fiber $Y_{b_1} \times Y_{b_1}$. Restricting to the fiber, this implies that the diagonal of $Y_{b_1}$ has a decomposition as in (2). Letting the decomposition (2) act on Chow groups, this shows that

$$A^i_{\text{hom}}(Y_b) = 0 \ \forall \ i \leq n - 5 , \ \forall \ b \in B$$

(with the additional vanishing for $i = n - 4$ for small $n$).

3.5. Hyperplane sections of $I_2 \text{Gr}(3, n)$.

**Theorem 3.5.** Assume $n$ is even, and let

$$Y := I_2 \text{Gr}(3, n) \cap H \subset \mathbb{P}^{(5)}-1$$

be a smooth hyperplane section (with respect to the Plücker embedding). Then

$$A^i_{\text{hom}}(Y) = 0 \ \forall \ i \leq n - 7 .$$
Moreover, in case \( n \leq 10 \) we have
\[
A_i^{\text{hom}}(Y) = 0 \quad \forall \ i \leq n - 6 .
\]

**Proof.** For \( n \) even, a generic hyperplane section \( Y \) is attained by the construction of Theorem 3.1 with \( r = 1 \), i.e. there is a smooth hyperplane section
\[
Y' := I_1 \text{Gr}(3, n + 1) \cap H ,
\]
related to \( Y \) via the projection of Theorem 3.1. In this case, Proposition 3.3 implies that there is an injection of Chow groups
\[
A_i^{\text{hom}}(Y) \hookrightarrow A_{i+3}^{\text{hom}}(Y') \oplus \bigoplus A_i^{\text{hom}}(I_2 \text{Gr}(2, n)) .
\]
The bisymplectic Grassmannian \( I_2 \text{Gr}(2, n) \) is nothing but an intersection \( \text{Gr}(2, n) \cap H_1 \cap H_2 \) (where the \( H_j \) are Plücker hyperplanes), and so Proposition 2.3 gives the vanishing
\[
A_i^{\text{hom}}(I_2 \text{Gr}(2, n)) = 0 .
\]
The variety \( Y' \) is a hyperplane section of \( I_1 \text{Gr}(3, n + 1) \), and so Theorem 3.4 gives the vanishing
\[
A_i^{\text{hom}}(Y') = 0 \quad \forall \ i \leq n - 7 ,
\]
with the additional vanishing for \( i = n - 6 \) for small \( n \). This proves the theorem for generic sections \( Y \).

The extension to all smooth hyperplane sections \( Y \) is done just as in the proof of Theorem 3.4. \( \square \)

**4. Some Consequences**

**Corollary 4.1.** (i) Let \( Y \) be as in Theorem 3.4 and \( n \leq 10 \) or \( n = 12 \), or as in Theorem 3.5 and \( n \leq 10 \). Then \( H^{\dim Y}(Y, \mathbb{Q}) \) is supported on a subvariety of codimension \( n - 3 \).

(ii) Let \( Y \) be as in Theorem 3.5 and \( n \leq 10 \). Then \( H^{\dim Y}(Y, \mathbb{Q}) \) is supported on a subvariety of codimension \( n - 5 \).

**Proof.** This follows in standard fashion from the Bloch–Srinivas argument [3]. Let us treat (i) (the argument for (ii) is the same). The vanishing
\[
A_i^{\text{hom}}(Y) = 0 \quad \forall \ i \leq n - 4
\]
(Theorem 3.4) is equivalent to the decomposition
\[
\Delta_Y = \gamma + \delta \quad \text{in} \quad A^{\dim Y}(Y \times Y) ,
\]
where \( \gamma \) is a completely decomposed cycle (i.e. \( \gamma \in A^*(Y) \otimes A^*(Y) \)), and \( \delta \) has support on \( Y \times W \) with \( W \subset Y \) of codimension \( n - 3 \) (to see this equivalence, one can look for instance at [12, Theorem 1.7]). Let \( H_{tr}^{\dim Y}(Y, \mathbb{Q}) \) denote the transcendental cohomology (i.e. the complement of the algebraic part under the cup product pairing). The cycle \( \gamma \) does not act on \( H_{tr}^{\dim Y}(Y, \mathbb{Q}) \). The action of \( \delta \) on \( H_{tr}^{\dim Y}(Y, \mathbb{Q}) \) factors over \( W \), and so
\[
H_{tr}^{\dim Y}(Y, \mathbb{Q}) \subset H_{W}^{\dim Y}(Y, \mathbb{Q}) .
\]
Since the algebraic part of \( H^{\dim Y}(Y, \mathbb{Q}) \) is (by definition) supported in codimension \( \dim Y/2 \), this settles the corollary. \( \square \)
Corollary 4.2. Let

\[ Y := I_1 \text{Gr}(3,n) \cap H \subset \mathbb{P}^{(n)-1} \]

be a smooth hyperplane section (with respect to the Plücker embedding).

(i) If \( n \leq 8 \), then \( Y \) has finite-dimensional motive (in the sense of \([11]\)).

(ii) If \( n \leq 9 \), then \( Y \) has trivial Griffiths groups (and so Voevodsky’s smash conjecture \([22]\) is true for \( Y \), i.e. numerical equivalence and smash-equivalence coincide on \( Y \)).

(iii) If \( n \leq 10 \), the Hodge conjecture is true for \( Y \).

Proof. This is similar to the argument of Corollary 4.1

(i) The vanishing

\[ A_i^{\text{hom}}(Y) = 0 \quad \forall \ i \leq n - 4 \]

(Theorem 3.4) is equivalent to the decomposition of the diagonal

\[ \Delta_Y = \gamma + \delta \quad \text{in} \quad A^{\dim Y}(Y \times Y), \]

where \( \gamma \) is a completely decomposed cycle, and \( \delta \) has support on \( Y \times W \) with \( W \subset Y \) of codimension \( n - 3 \) (cf. \([3]\) or \([13]\)). The dimension of \( Y \) is \( 3n - 13 \), and so (looking at the action of the diagonal) one finds that

\[ A^*_Y(Y) = 0 \]

as long as \( n \leq 8 \). This implies Kimura finite-dimensionality of \( Y \) \([21, \text{Theorem 4}]\).

(ii) The vanishing of Theorem 3.4 implies that

\[ \text{Niveau}(A_*(Y)) \leq 2 \]

(in the sense of \([13]\)), i.e. the motive of \( Y \) factors over a surface. Since surfaces have trivial Griffiths groups, the conclusion follows.

(iii) The vanishing of Theorem 3.4 implies that

\[ \text{Niveau}(A_*(Y)) \leq 3 \]

(in the sense of \([13]\)), i.e. the motive of \( Y \) factors over a threefold. Since threefolds verify the Hodge conjecture, the conclusion follows. \( \Box \)

We leave it to the zealous reader to formulate and prove a version of Corollary 4.2 for bisymplectic Grassmannians.

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