Effective scalar potential in asymptotically safe quantum gravity

C. Wetterich

Institut für Theoretische Physik
Universität Heidelberg
Philosophenweg 16, D-69120 Heidelberg

We investigate the shape of the effective potential for scalar fields at and near the ultraviolet fixed point of asymptotically safe quantum gravity. We find scaling solutions with a completely flat potential and vanishing gauge and Yukawa couplings. Nonvanishing gauge couplings can induce spontaneous symmetry breaking due to a potential minimum at nonzero field values. In contrast, Yukawa couplings to fermions tend to stabilize a minimum at zero field values. A nonminimal coupling between the scalar field and gravity is associated to a field-dependent effective Planck mass. It can induce spontaneous symmetry breaking. In general, nonzero gauge, Yukawa or nonminimal couplings prevent the scaling potential to be completely flat. For the standard model coupled to asymptotically safe quantum gravity the non-minimal Higgs-couvature coupling is bound to be small, $\xi_\infty \lesssim 10^{-3}$, in contrast to large values $\xi_\infty > 1$ often assumed for Higgs inflation. We also discuss small modifications of the predicted value for the top quark mass due to nonzero gauge and Yukawa couplings.

1. Introduction

Within asymptotically safe quantum gravity [1, 2] the value of the Higgs boson mass has been predicted to be 126 GeV with a few GeV uncertainty [3]. This prediction relies on two assumptions. The first is a positive and substantial gravity-induced anomalous dimension $A$ that renders the quartic scalar coupling $\lambda_H$ an irrelevant parameter. Then $\lambda_H$ is predicted to have a very small value at and near the ultraviolet (UV) fixed point. The second assumes that once the metric fluctuations decouple at momenta sufficiently below the effective Planck mass the running of $\lambda_H$ is given by the standard model with at most small modifications. First indications for a positive $A$ have been seen in early investigations how matter couples to gravity in asymptotic freedom [4]. Physical gauge fixing, or a gauge invariant flow equation for a single metric [5], show a clear graviton domination of $A$ [6] and establish a positive $A$ [6–8], substantiating the prediction of the mass of the Higgs boson.

The prediction of the Higgs boson mass concerns the properties of the effective scalar potential at field values much smaller than the effective Planck mass $M$. In contrast, models of Higgs inflation [9, 10] explore instead properties of the potential at field values somewhat below $M$, or even exceeding $M$. Usually only particle fluctuations are included in the computation of the effective potential, while the contributions of metric fluctuations are neglected. It has been argued [11] that asymptotically safe quantum gravity may substantially influence the behavior of the Higgs potential at large fields. In this paper we aim for a more global view of the effective scalar potential, ranging from small field values to large ones exceeding $M$.

A global view on the effective potential for a scalar singlet field $\chi$ is also needed for models of cosmon inflation [12–14] and dynamical dark energy as quintessence [15]. It has been found [11] that this potential shows a rather rich structure, due to crossover between different fixed points. While “gravity scale symmetry” associated to the UV fixed point is responsible for the almost scale invariant primordial fluctuation spectrum, an infrared (IR) fixed point [6, 16] is reached for large values of $\chi$. The “cosmic scale symmetry” associated to the IR fixed point is spontaneously broken by any nonzero $\chi$. The associated pseudo-Goldstone boson (cosmon) has a very small mass for large $\chi$. It is responsible for dynamical dark energy [15].

The present paper addresses this issue as well. The parts concentrating on the fluctuations of a scalar singlet and the metric, with all other particles treated as massless (sects 3, 8) can be seen as a computation of the effective potential $U(\chi)$ for the scalar singlet $\chi$. Concerning the properties of the effective potential for other scalar fields as the Higgs doublet, we do not distinguish here between Quantum-Einstein gravity [2], where the Planck mass $M$ corresponds to a relevant parameter and constitutes an intrinsic mass scale breaking quantum scale symmetry explicitly, and dilaton quantum gravity [17, 18], where the effective Planck mass depends monotonically on $\chi$, such that for a suitable normalization of $\chi$ one has $M = \chi$ for large $\chi$. In the latter case quantum scale symmetry can be preserved, being only spontaneously broken by $\chi \neq 0$. Whenever we use $M^2$, the reader may substitute it by a function $F(\chi)$.

The renormalization flow describes the change of the effective scalar potential for increasing length scales, as more and more fluctuation effects are included. It is characterized by different regimes. The “quantum gravity regime” is associated to renormalization scales exceeding $M$. In this regime the fluctuations of the metric play an important role. The quantum gravity regime is associated to the UV fixed point defining quantum gravity as a nonperturbatively renormalizable quantum field theory (asymptotic safety). At the UV fixed point one has a scaling behavior

$$M^2(k) = 2w_* k^2,$$

with $w_*$ the fixed point value of the dimensionless coupling $w(k) = M^2(k)/(2k^2)$. (In case of additional scalar fields $\chi$ we may replace $w_*$ by a scaling function depending
also on $\chi^2/k^2$. In the quantum gravity regime the effective scalar potential takes a scaling form where the dimensionless potential $u = U/k^4$ only depends on dimensionless field ratios as $\tilde{\rho} = \rho/k^2$, with $\rho$ a typical quadratic invariant formed from scalar fields. (For the Higgs doublet one has $\rho = h^2$, with $h$ the renormalized scalar doublet, while for a scalar singlet $\chi$ we use $\rho = \chi^2/2$.) The main emphasis of the present paper is the computation of the “scaling potential” $u(\tilde{\rho})$. For this purpose we first treat $w_\ast$ as an unknown parameter. Our computation needs therefore to be supplemented by a computation of $w_\ast$. The latter depends on the precise particle content of the model. In sect. 8 we extend this to a fixed scaling function $w_\ast(\tilde{\rho}) = w_0 + \xi \tilde{\rho}/2$, with free parameters $w_0$ and $\xi$. Finally, in sects 9 and 10 we extend the truncation to simultaneous solutions of flow equations for both $u(\tilde{\rho})$ and $w(\tilde{\rho})$. This establishes a system of combined scaling functions $w_\ast(\tilde{\rho})$ and $w_\ast(\tilde{\rho})$.

A second “particle regime” concerns the flow for $k \ll M$. In this regime the metric fluctuations decouple effectively up to the flow of an overall constant in $U$, e.g. the cosmological constant. The flow equation for the field dependence of $U$ is governed by the effective particle theory for momenta below the Planck mass. This flow can be computed in perturbation theory. It obviously depends on the precise particle content of the the effective low energy theory. The flow in the particle regime may again be characterized by an approximate fixed point, and the associated “particle scale symmetry”. For the standard model as effective low energy theory this fixed point is associated to the (almost) second order character of the vacuum electroweak phase transition. A similar fixed point may exist for grand unified theories (GUT). The present paper will not deal with the flow in the particle regime which has to be added for $k \ll M$. The transition from the quantum gravity regime to the particle regime is modeled by a simple behavior for the $k$-dependent Planck mass,

$$M^2(k) = M^2 + 2w_\ast k^2, \quad (2)$$

where $M^2$ is associated to the observed Planck mass and could be given by a scalar field $M^2 = \chi^2$.

For extremely large field values $\rho/k^2$ one finally reaches the infrared regime. There graviton fluctuations may become again important due to a potential instability in the graviton propagator. A “graviton barrier” [6] prevents the potential to rise for large field values stronger than the field dependent squared Planck mass. We will not be concerned very much with the infrared regime in the present investigation.

The present paper concentrates on the quantum gravity regime. We are mainly interested in general characteristics of the scaling form of the effective potential, as the location of the minimum $\tilde{\rho}_0$ at $\tilde{\rho}_0 = 0$ or at $\tilde{\rho}_0 \neq 0$, and the general behavior as $\tilde{\rho}$ vanishes or increases beyond $\tilde{\rho}_0$. We put emphasis on the dependence on gauge couplings and Yukawa couplings that we treat here as constants. This covers two scenarios. Either the fixed point values of these couplings may be at nonzero values. In this case the gauge couplings and Yukawa couplings typically correspond to irrelevant parameters that can be predicted by quantum gravity [19]. Or the UV fixed point corresponds to zero values of these couplings, which are relevant parameters. The flow away from the fixed point is, however, very slow in the vicinity of the fixed point. For their observed small values the gauge and Yukawa couplings only increase rather slowly with decreasing $k$. To a good approximation they can be treated as constants in this regime. Our investigation of scaling solutions for constant gauge and Yukawa couplings describes then approximate scaling solutions in the vicinity of the UV fixed point.

For vanishing gauge and Yukawa couplings there exists a constant scaling solution” for which $u(\tilde{\rho})$ and $w(\tilde{\rho})$ are independent of $\tilde{\rho}$. This is the extended Reuter fixed point. We are interested in the possible existence of other fixed points, for which the scaling functions $u(\tilde{\rho})$ and $w(\tilde{\rho})$ are independent of $k$, but show a non-trivial dependence on $\tilde{\rho}$. This is typically induced by non-zero gauge and Yukawa couplings. It could also occur for vanishing gauge and Yukawa couplings. We consider first the regime where non-minimal couplings of the scalar field to gravity $\sim \xi \rho \tilde{R}$ can be neglected. (Here $\tilde{R}$ is the curvature scalar and $\xi$ the non-minimal coupling.) In this case our main findings for the global form of the scaling form for a possible non-constant dimensionless effective scalar potential $u(\tilde{\rho})$ are the following. For zero gauge and Yukawa couplings the potential interpolates between two constants,

$$u(\tilde{\rho} \to 0) = u_0, \quad u(\tilde{\rho} \to \infty) = u_\infty, \quad u_\infty > u_0. \quad (3)$$

The minimum is situated at the origin $\tilde{\rho} = 0$. This behavior occurs also for nonzero Yukawa couplings $\tilde{y}$ and zero gauge couplings. In contrast, nonzero gauge couplings $g$ can induce a potential minimum at $\tilde{\rho}_0 \neq 0$. The asymptotic behavior (3) remains valid. While for vanishing gauge and Yukawa couplings particular “constant scaling solutions” exist, with $\tilde{\rho}$-independent $u_\ast(\tilde{\rho}) = u_0$ or $u_\ast(\tilde{\rho}) = u_\infty$, this possibility is no longer given in our truncation for nonzero gauge or Yukawa couplings.

For $\xi \neq 0$ the asymptotic behavior of $u$ for $\tilde{\rho} \to \infty$ can change. For some solutions we still find a constant $u_\infty$. Alternatively, for asymptotically large $\tilde{\rho}$ the “IR-behavior” $u(\tilde{\rho} \to \infty) = \xi \tilde{\rho}/2$ is reached. The intermediate behavior can be rather complex. In particular, we find for $g = y = 0$ that the scaling potential can develop a minimum at $\tilde{\rho}_0 \neq 0$. For $\xi \neq 0$ no constant scaling solution exists.

For scaling solutions of the combined flow equations for $u(\tilde{\rho})$ and $w(\tilde{\rho})$ we focus on a family of candidate scaling solutions that depend on a continuous parameter $\xi_\infty$. For these solutions one has the asymptotic behavior

$$w(\tilde{\rho} \to \infty) = \frac{1}{2} \xi_\infty \tilde{\rho}, \quad u(\tilde{\rho} \to \infty) = u_\infty. \quad (4)$$

For the particle content of the standard model the minimum of the scaling potential occurs for $\tilde{\rho} = 0$. As $\xi_\infty \to 0$ the constant scaling solution is approached smoothly. For $\xi_\infty \gtrsim 10^{-3}$ the existence of the solution becomes questionable since the $\tilde{\rho}$-dependence of $u/w$ becomes strong, with
a rather irregular behavior of \( \partial u / \partial \rho \) and \( \partial v / \partial \rho \) in an intermediate region. For the solutions with \( \xi_\infty < 10^{-3} \) more elaborate numerical solutions should establish if these solutions exist for all \( \xi_\infty \) in this range or not.

As a general feature, the scaling solutions for the effective scalar potential are not approximated by polynomials. In this respect quantum gravity is rather different from perturbatively renormalizable quantum field theories as gauge theories or Yukawa-type theories. For example, a crossover between two constants as for eq. (3) can well happen with a positive mass term \( m_a^2 \) at the origin, but a negative quartic coupling \( \lambda_a \). The negative quartic coupling does not indicate any instability of the potential, but merely a decrease of \( m^2(\rho) \) as \( \rho \) increases. This is rather typical for a crossover between constants for \( \rho \to 0 \) and \( \rho \to \infty \). The perturbative experience that a negative quartic coupling \( \lambda \) indicates an instability or the presence of another potential minimum for larger field values is misleading in the context of quantum gravity.

The scaling potential \( u(\rho) \) is a function of the scale invariant variable \( \tilde{\rho} = \rho / k^2 \). In particular, a minimum at \( \tilde{\rho}_0 \) corresponds to a “sliding minimum” of the effective potential \( U = \tilde{\rho} k^4 \), at \( \tilde{\rho}_0 = \tilde{\rho}_0 k^2 \). The question arises which range of \( \tilde{\rho} \) is relevant for observations. For a rough estimate we make the simple ansatz that the scaling solution is valid for \( k > k_t \), with transition scale \( k_t \) determined by \( 2w(\tilde{\rho}) k_t^4 = M^2 + \xi \rho \) and \( M \) the observed Planck mass. We further assume that for \( k < k_t \) the metric fluctuations decouple and the effective low energy theory becomes valid. This approximation determines at \( k_t \) the field \( \rho = \tilde{\rho} k^2 \) as

\[
\frac{\rho}{M^2} = \frac{\tilde{\rho}}{2w_0},
\]

A minimum of the scaling potential at \( \tilde{\rho}_0 \) corresponds at \( k_t \) to \( \rho_0 = M^2 \tilde{\rho}_0 / (2w_0) \). Typically, \( \rho_0 \) continues to change in the low energy effective theory. Nevertheless, for \( 2w_0 \approx 0.1 \) a characteristic field \( \tilde{\rho} \) can be associated with field values \( \rho \approx 10^6 M^2 \). A typical GUT scale \( \rho \approx (10^{16} \text{ GeV})^2 \) corresponds to \( \tilde{\rho} \approx 10^{-5} \) or \( x = \ln(\tilde{\rho}) \approx -11.5 \). We often find the location of a minimum at \( x \) around \(-2\) which corresponds to \( \rho \) around \( M^2 \).

The present paper is organized such that the effects of different couplings are described separately. In sect. 2, we present the flow equation for the effective scalar potential, following closely [6, 7, 11]. The specific physical gauge fixing, equivalent in our truncation to the gauge invariant flow equation [5], makes the structure very apparent. The general features are similar to earlier investigations [4, 20–36]. In sect. 3 we concentrate on the scaling solution for “matter freedom”, which describes a situation where gauge and Yukawa couplings, as well as the nonminimal coupling \( \xi \), can be neglected. In this limit all particles are free except for their gravitational interactions. We find candidate scaling solutions characterized by a crossover from a fixed point with constant \( u = u_0 \) for \( \tilde{\rho} \to 0 \) to another one with constant \( u = u_\infty \) to \( \tilde{\rho} \to \infty \). Improvement of the numerical treatment would be needed in order to decide definitely if this truncation admits a scaling solution different from the constant scaling solutions. Sect. 4 addresses the flow in the vicinity of the scaling solution for matter freedom, and sect. 5 discusses the scalar mass term and quartic coupling.

In sect. 6 we take a first step beyond matter freedom by discussing nonvanishing gauge couplings, still keeping an approximation with constant \( w_\infty \). Typical scaling potentials show a minimum near \( \tilde{\rho} = 1 \). In sect. 7 we switch to non-vanishing Yukawa couplings for which the minimum of the scaling potential occurs for \( \tilde{\rho} = 0 \). For scalars with both gauge and Yukawa couplings the competition between the opposite tendencies for gauge and Yukawa couplings will be important. In sect. 8 we include a non-minimal coupling \( \xi \) of the scalar field to the curvature tensor. This changes the behavior for \( \tilde{\rho} \to \infty \).

In sects. 9 and 10 we extend the truncation by investigating solutions to the combined flow equations for \( u(\rho) \) and \( w(\rho) \). For the derivation of the flow equations we will follow ref. [8]. In sect. 9 we discuss general features and turn to the standard model coupled to quantum gravity in sect. 10. There we discuss in particular the issue of Higgs inflation and the prediction for the top quark mass. Sect. 11 contains our conclusions.

2. Flow equation for effective potential

The present work is based on the flow equation for the effective average action [37–41] for which the UV-cutoff in earlier formulation [42, 43] is replaced by an infrared cutoff and the focus is on the effective action. The simple one-loop form of the exact flow equation permits for successful non-perturbative approximations. Reviews on functional renormalization are refs. [44–51], and for its applications to quantum gravity see refs. [52–59].

Let us consider scalar fields \( \phi_a \), belonging to various representations of some symmetry group, and investigate the flow of the effective scalar potential \( U(\phi_a) \). Our truncation for the effective average action involves up to two derivatives

\[
\mathcal{L} = \sqrt{g} \left\{ -\frac{F(\phi_a)}{2} R + U(\phi_a) + \sum_a \frac{Z_a}{2} D^\mu \phi_a D_\mu \phi_a + \ldots \right\},
\]

where the dots denote parts involving gauge fields and fermions. The flow equation for \( U \) has contributions from fluctuations of various fields,

\[
\partial_t U = k \partial_k U = \tilde{\zeta} = \tilde{\pi} + \tilde{\pi}_s + \tilde{\pi}_\text{gauge} + \tilde{\pi}_f.
\]

For a physical gauge fixing or the gauge invariant flow equation the gravitational contribution takes a rather simple form

\[
\tilde{\pi}_\text{grav} = \frac{k^4}{24 \pi^2} \left( 1 - \eta \right) \frac{1}{8} \left( \frac{5}{1 - v} + \frac{1}{1 - v/4} \right) - \frac{k^4}{8 \pi^2}.
\]

The gravitational contribution depends on \( U \) and the coefficient \( F \) in front of the curvature scalar via the combination

\[
v = \frac{2U}{FK^2} = \frac{u}{w},
\]
with dimensionless functions \( u \) and \( w \) depending on the scalar fields \( \phi_a \),
\[
  u = \frac{U}{k^4}, \quad w = \frac{F}{2k^2}.
\]

We first approximate \( F \) by a field-independent running squared Planck mass. At the UV-fixed point it scales \( \sim k^2 \), with fixed dimensionless parameter \( w_0 \),
\[
  F = M_p^2(k) = 2w_0k^2.
\]
The overall scale is set by \( k^4 \), as appropriate for the dimension of \( U \), and \( 1/(32\pi^2) \) is a typical loop factor from the momentum integration. The first term in eq. (8) arises from the fluctuations of the graviton (five components of the traceless transversal tensor), the second from the physical scalar fluctuations. For a computation of the flow of \( U \) the effect of these fluctuations is evaluated in flat space. The minus sign in the denominator \((1-v)^{-1}\) reflects the negative mass-like term in the flat space graviton propagator for a positive \( U \), and similar for the physical scalar mode. The third “measure contribution” accounts for the gauge modes in the metric fluctuations and ghosts. It is independent of the scalar fields. For the specific form of the “threshold functions” appearing in eq. (8) we have employed a Litim cutoff function [60].

We also neglect the mixing between the physical scalar mode in the metric and the scalars \( \phi_a \), which only plays a very small role for our investigation. Finally, the gravitational anomalous dimension
\[
  \eta_g = -\partial_k \ln(w) \tag{12}
\]
reflects the choice of the IR-cutoff function proportional to \( F \). At the UV-fixed point \( \eta_g \) vanishes if \( w \) is field independent.

The contribution from scalar fluctuations \( \tilde{\pi}_s \) reads [61]
\[
  \tilde{\pi}_s = \frac{k^4}{32\pi^2} \sum_a \left(1 - \frac{\eta_A}{6}\right) (1 + \tilde{m}^2_A)^{-1}, \tag{13}
\]
with \( A \) labeling the eigenvalues \( M_A^2 \) of the (renormalized) scalar mass matrix
\[
  M^2_{ab} = (Z_a Z_b)^{-\frac{1}{2}} \frac{\partial^2 U}{\partial \phi_a \partial \phi_b}, \quad \tilde{m}^2_A = \frac{M_A^2}{k^2}. \tag{14}
\]

Here \( Z_a \) are the scalar wave functions, given by the coefficient of the kinetic term for \( \phi_a \). The factor \((1 + \tilde{m}^2_A)^{-1}\) is a threshold function that accounts for the suppression of contributions of particles with mass terms larger than \( k^2 \), ensuring decoupling automatically. The anomalous dimension \( \eta_A = -\partial_k \ln(Z_A) \) reflects the choice of an IR-cutoff function for the scalar proportional to \( Z_A \), with \( Z_A \) connected suitably to \( Z_a \). (In the case of scalars in a single representation one uses the same \( Z \) for the cutoff function and the definition of all renormalized fields.) Through the threshold function in the scalar contribution the flow equation for \( U \) involves field-derivatives of \( U \).

The contributions from gauge bosons \( \tilde{\pi}_\text{gauge} \) and the contributions from fermions \( \tilde{\pi}_f \) do not depend on the scalar fields in the limit of zero gauge couplings or Yukawa couplings, respectively. They will be specified later. The flow of mass terms and quartic couplings obtains by differentiating eq. (7) twice or four times with respect to \( \phi \).

The flow equation (7) holds for fixed values of \( \phi_a \). For the investigation of the scaling solution relevant for a fixed point one transforms this to a flow equation for \( u = U/k^4 \) at fixed dimensionless renormalized fields,
\[
  \tilde{\phi}_a = \frac{Z_a^2 \phi_a}{k}, \tag{15}
\]
where
\[
  \partial_t u = -4u + \sum_a \left(1 + \frac{\eta_a}{2}\right) \frac{\tilde{\phi}_a \partial \tilde{u}}{\partial \tilde{\phi}_a} + \frac{\zeta}{k^4}. \tag{16}
\]

Derivatives of \( u \) with respect to \( \tilde{\phi}_a \) define dimensionless renormalized couplings. For the scaling solution characterizing a fixed point the r.h.s. of eq. (16) has to vanish, resulting in a system of differential equations for \( u \).

3. Scaling solutions for matter freedom

We first discuss an approximation where \( w(\bar{\rho}) \) is taken as a constant \( w_0 \). All matter interactions are neglected. This approximation reveals some characteristic features of the effects of gravitational fluctuations on the scalar effective potential.

3.1. Flow equation for matter freedom

Let us first consider a situation where the values of gauge couplings, Yukawa couplings, dimensionless scalar mass terms, and quartic scalar couplings are sufficiently small such that the contribution of these fluctuations only matters for the flow of the field-independent part of \( u \). We call this approximation “matter freedom” since the interactions between matter components are neglected. Approximating \( \eta_g = 0, \eta_A = 0 \) one finds
\[
  \zeta = \frac{\zeta}{k^4} = \frac{1}{24\pi^2} \left( \frac{5}{1 - v} + \frac{1}{1 - v/4} \right) + 4b_v \tag{17}
\]
with constant
\[
  b_v = \frac{N - 4}{128\pi^2}. \tag{18}
\]
The effective number of degrees of freedom is given by
\[
  N = N_S + 2N_B - 2N_F, \tag{19}
\]
where \( N_S \) denotes the number of real scalars, \( N_B \) the number of gauge bosons \((N_B = 45 \text{ for } SO(10), N_B = 24 \text{ for } SU(5) \text{ and } N_B = 12 \text{ for the standard model} \) and \( N_F \) the number of Weyl fermions \((N_F = 48 \text{ for } SO(10), N_F = 45 \text{ for } SU(5) \text{ and the standard model} \). For the standard model one has \( N_S = 4 \), with much larger numbers of scalars for GUT models. For the standard model \( N = -62 \) is negative, while GUT models typically have positive \( N \). In the
counting only particles with masses much smaller than \( k \) are included and approximated by massless particles.

We concentrate on a particular \( N_s' \)-dimensional scalar representation and a potential \( u(\tilde{\rho}) \) depending only on the invariant

\[
\tilde{\rho} = \frac{1}{2} \sum_{a=1}^{N_s'} \tilde{\rho}_a^2.
\]

The other \( N_s - N_s' \) scalar fields may be set to zero. Alternatively, one may take fixed values for the dimensionless ratios as \( \chi^2/k^2 \). Our interest is the \( \tilde{\rho} \)-dependence of the potential. Neglecting the anomalous dimension \( \eta \) the flow equation for the potential reads

\[
\partial_t u = \beta_u = -4u + 2\tilde{\rho} \partial_{\tilde{\rho}}u + 4c_V ,
\]

with

\[
c_V = \frac{1}{96\pi^2} \left( \frac{5}{1 - v} + \frac{1}{1 - v/4} \right) + b_V.
\]

### 3.2. Constant scaling solution

We are interested in the scaling solution at the UV-fixed point for which \( \partial_t u \) vanishes. For any given \( w(\tilde{\rho}) \) this scaling solution for \( u(\tilde{\rho}) \) has to obey the nonlinear differential equation

\[
2\tilde{\rho} \partial_{\tilde{\rho}} u = 4u - \frac{1}{24\pi^2} \left( \frac{5}{1 - u/w} + \frac{1}{1 - u/4w} \right) - 4b_V.
\]

In general, \( w \) depends on \( \tilde{\rho} \). We first consider the case where the scaling form can be approximated by a constant \( w = w_0 \) and generalize this setting in sects. 8-10. For \( \tilde{\rho} \to 0 \) a simple scaling solution is a constant potential,

\[
u_s(\tilde{\rho} \to 0) = u_0.
\]

If \( \partial_u/\partial\tilde{\rho} \) remains finite for \( \tilde{\rho} \to 0 \) (or does not diverge too strongly), the constant \( u_0 \) obtains by setting the r.h.s. of eq. (23) to zero. Simplifying by approximating \((1 - v/4)^{-1}\) by \((1 - v)^{-1}\) yields a quadratic equation for \( v_0 = u_0/w_0 \), namely

\[
(v_0 + (N_0 - 4)z)(1 - v_0) = 8z, \quad z = \frac{1}{128\pi^2 w_0},
\]

with \( N_0 \) the effective particle number for \( \tilde{\rho} = 0 \).

Fixed point solutions for \( v_0 \) obey

\[
v_{\pm} = \frac{1}{2} \left( 1 + (N_0 - 4)z \pm \sqrt{(1 - (N_0 - 4)z)^2 - 32z} \right).
\]

They exist provided \( z \) is in a range where the argument of the square root is positive. For the special case \( N_0 = -4 \) the argument of the square root is positive for all \( z \). The two solutions are \( v_+ = 1 - 8z, v_- = 0 \). For \( N_0 < -4 \) the argument of the square root is again always positive and one finds \( v_+ > 1 - 8z, v_- < 0 \). Restrictions on \( z \) can arise for \( N_0 > -4 \). In this case \( z \) has to be outside the interval \([z_-, z_+]) \), given for \( N_0 \neq 4 \) by

\[
z_\pm = \frac{N_0 + 12 \pm 4\sqrt{2N_0 + 8}}{(N_0 - 4)^2}.
\]

For \( N_0 = 4 \) the condition reads \( z < 1/32 \). For \( N_0 \to 4 \) the lower boundary \( z_- \) approaches 1/32 while \( z_+ \) diverges.

The precise relation between \( v_0 \) and \( w_0 \) or \( z \) according to the solution of eq. (21) for \( \partial_t u = 0, \tilde{\rho} \partial_{\tilde{\rho}} u = 0 \) is algebraically less simple, but qualitatively and quantitatively similar [7]. We can infer it from Fig. 1 which plots the relation between \( v \) and \( w \) in the form of a function \( w(v) \).

The latter follows from eq. (21),

\[
w = \frac{c_V(v)}{v} = \frac{1}{96\pi^2v} \left( \frac{5}{1 - v} + \frac{1}{1 - v/4} \right) + \frac{b_V}{v}.
\]

Acceptable scaling solutions require \( v < 1, w > 0 \). For \( N > -4 \) the function \( w(v) \) is positive for the interval \( 0 < v < 1 \), diverging at both ends of the interval. This is the only allowed range. There is a minimum of \( w(v) \) at \( v_c \), with critical value \( w_c = w(v_c) \). For \( N > -4 \) scaling solutions exist only for \( w > w_c \). For \( N < -4 \) one finds positive \( w \) for negative \( v \), with \( w(v \to -\infty) \to 0 \). A second solution with positive \( w \) corresponds to a range of positive \( v \) sufficiently close, but still smaller than the pole at \( v = 1 \).

For an appropriate range of \( w \) one finds two solutions \( v_+ \) and \( v_- \). They correspond to the two solutions \( v_+ \) of the approximation (24), (25). For \( N > -4 \) this requires \( w > w_c \), corresponding to the restriction on \( z \) given by eq. (27). We conclude that acceptable scaling solutions exist in our truncation for matter freedom, except for very strong gravity \((w_0 < w_c) \) for \( N > -4 \).

So far we have computed \( u_0 \) and \( v_0 \) as the limiting behavior of the scaling solution for \( \tilde{\rho} \to 0 \). For any given allowed value of \( w_0 \) or \( z \) there are two possible values \( v_0 \) and therefore two possible solutions for \( u_0 \). For \( w(\tilde{\rho}) = u_0 \) independent of \( \rho \) eq. (23) actually admits a solution with constant \( u(\tilde{\rho}) = u_0 \) for all values of \( \tilde{\rho} \). For this simple solution the effective potential is completely flat

\[
U(\rho) = u_0 k^4.
\]

Solutions with \( \tilde{\rho} \)-independent \( u \) are called “constant scaling solutions”. We will see in sect. 9 that one of these constant scaling solutions corresponds to the extended Reuter fixed point.

### 3.3. Crossover scaling solutions

Since eq. (23) is a first order differential equation one may ask if there exist other scaling solutions with \( \partial_u/\partial\tilde{\rho} \neq 0 \). For these solutions the boundary conditions \( u(\tilde{\rho} \to 0) = u_0 \) should be obeyed. A numerical solution of eq. (23) indeed finds a family of scaling solutions that interpolate between the constant values \( v_{\pm} \), as shown in Figs 2, 3. For all values of \( w_0 \) compatible with the presence of two fixed points \( v_+ \) and \( v_- \), the generic scaling solution is a crossover from \( u(\tilde{\rho} \to 0) = v_- w \) to \( u(\tilde{\rho} \to \infty) = v_+ w \). We show the numerical solutions of
eq. (23) for different initial conditions (chosen arbitrarily at \( \hat{\rho} = 1 \)) in Fig. 2. The different curves can be obtained by a shift in \( x = \ln(\hat{\rho}) \). The crossover trajectory between the two fixed points is universal. The initial conditions only specify at which \( \hat{\rho} \) a given value of \( u \) on the crossover trajectory is reached. The possible shifts in \( x = \ln(\hat{\rho}) \) are arbitrary. Limiting cases are the constant scaling solutions \( u(\hat{\rho}) = v_- w \) or \( u(\hat{\rho}) = v_+ w \) for initial conditions with \( u(\hat{\rho}_i) \) outside the interval \([v_- w, v_+ w]\) no scaling solution exists. Local solutions of the differential equation do not reach finite values for \( \hat{\rho} \rightarrow 0 \) and \( \hat{\rho} \rightarrow \infty \). They typically diverge at some finite \( \hat{\rho} \).

As \( w \) is lowered, the interval \([v_-, v_+]\) shrinks. This is reflected by a shrinking of the distance between the boundary values \( u(\hat{\rho} \rightarrow 0) \) and \( u(\hat{\rho} \rightarrow \infty) \), as depicted in Fig. 3. This shrinking continues until one reaches \( w_c \) where \( v_-(w_c) = v_+(w_c) = v_c \). For \( w = w_c \) the unique scaling solution is a constant \( u(\hat{\rho}) = v_c w_c \). For \( w < w_c \) the scaling solution ceases to exist.

We also show in Fig. 4 the mass term \( \hat{m}^2 = \partial u / \partial \hat{\rho} \) for matter freedom. As the location of the crossover, which may be associated to the maximum of \( \hat{m}^2 \), moves to larger \( x \) the height of the maximum decreases. Matter domination could therefore provide for a rather accurate picture for the sub-family of scaling solutions where the crossover happens at large \( x = \ln(\hat{\rho}) \).

Let us next investigate small deviations \( \Delta u \) from the constant scaling solution. For the crossover solutions shown in Figs 2, 3 they describe the onset of the crossover region. A linear approximation in \( \Delta u \) will always become valid for \( \hat{\rho} \rightarrow 0 \) and \( \hat{\rho} \rightarrow \infty \). In the vicinity of a constant scaling solution we expand

\[
u(\hat{\rho}) = u_0 + \Delta u(\hat{\rho}), \tag{30}\]

where \( \Delta u(\hat{\rho}) \) obeys the linear differential equation

\[
2\hat{\rho} \frac{\partial \Delta u}{\partial \hat{\rho}} = (4 - A) \Delta u, \tag{31}\]

with

\[
A = 4 \frac{\partial c_v}{\partial u} = \frac{4}{w} \frac{\partial c_v}{\partial v}. \tag{32}\]

From eq. (22) one infers

\[
A = \frac{1}{96\pi^2 w} \left( \frac{20}{(1-v)^2} + \frac{1}{(1-v/4)^2} \right). \tag{33}\]

The quantity \( A \) will play an important role for the discussion of the gravitational effects on the scalar potential. It will act as a gravity-induced anomalous dimension for the flow of the scalar mass term and quartic coupling. For all \( v \) and positive \( w \) one finds \( A(v, w) \geq 0 \). For positive \( v \) the first term in eq. (33) dominates by more than a factor of 20, justifying the “graviton approximation” which keeps only the transversal traceless metric fluctuations [6].

We plot \( A \) as a function of \( v \) in Fig. 5. For this purpose we use \( w(v) \) according to the constant scaling solution (28), as shown in Fig. 1. Inversion leads to two values \( A_k \) for a given \( w_0 \), corresponding to the solutions \( v_k \). For \( N = -4 \) one finds values \( A < 4 \) for negative \( v \), not shown in Fig. 5. For \( v \rightarrow -\infty \) one reaches \( A = 0 \) (for \( N \leq -4 \)). Generically, \( A \) increases for decreasing \( N \) and fixed \( v \), and for increasing

![Fig. 1. Relation between \( v \) and \( w \) for three values \( N = 12 \) (upper curve), \( N = 4 \) (middle curve) and \( N = -4 \) (lower curve).](image1)

![Fig. 2. Scaling potential \( u \) as a function of \( x = \ln(\hat{\rho}) \). The three curves correspond to different initial conditions, which may be specified by \( u(x = 0) \). The parameters are \( N = 12 \) and \( w = 0.06 \).](image2)

![Fig. 3. Scaling potential \( u \) as a function of \( x = \ln(\hat{\rho}) \) for different values of \( w = 0.06, w = 0.05 \) and \( w = 0.045 \) (upper, middle and lower curve for large \( x \), respectively). We use \( N = 12 \).](image3)
This holds except for the special case \( A_0 = 2 \). For a diverging \( \lambda \) our approximation of neglecting the quartic scalar coupling and mass term, which leads to constant \( b_V \), no longer holds. The dimensionless scalar mass term

\[
\tilde{m}^2 = \frac{\partial u}{\partial \rho}
\]

behaves for \( \tilde{\rho} \to 0 \) as

\[
\tilde{m}^2 = c_0 \left( 2 - \frac{A_0}{2} \right) \tilde{\rho}^1 - \frac{A_0}{2}.
\]

For \( c_0 \neq 0 \) it diverges for \( A_0 > 2 \), and vanishes for \( A_0 < 2 \). In sect. 5.2 we include effects of nonzero \( \tilde{m}^2 \) and \( \lambda \) in the flow equations. This may cure the divergence of \( \tilde{\lambda} \) (and possibly \( \tilde{m}^2 \)) for \( \tilde{\rho} \to 0 \).

There is a particular situation for which the whole crossover can be described within the validity of the matter freedom approximation. In our truncation this occurs for the choice \( \tilde{w}_0 = \tilde{w}_0^{(2)} \) for which \( A(\tilde{w}_0^{(2)}) = 2 \). Then

\[
\Delta u = c_0 \tilde{\rho}
\]

leads for \( \tilde{\rho} \to 0 \) to \( \tilde{m}_0^2 = c_0 \) and \( \lambda_0 = 0 \). For small \( c_0 \) matter freedom remains valid for arbitrarily small \( \tilde{\rho} \).

There is a critical value \( w_c \) for which the two solutions \( v_+ \) and \( v_- \) merge. For \( N > -4 \) it corresponds to the minimum of the curves in Fig. 1. For the approximation (26) this happens when \( \sqrt{A} = 0 \) and \( \beta_u \) vanish simultaneously. At this merging point one has \( \partial c_V / \partial u = 4 \) and therefore

\[
A(w_c) = 4.
\]

For an appropriate range of \( w_0 \) the flow equation (21) has for \( \tilde{\rho} \)-independent \( u \) two fixed points where \( \beta_u = 0 \). They merge at the critical \( w_c \) for which \( \partial \beta_u / \partial u \) and \( \beta_u \) vanish simultaneously. At this merging point one has \( \partial c_V / \partial u = 4 \) and therefore

\[
A(w_c) = 4.
\]

This can be seen in Fig. 5. At \( v_c = v(w_c) \) one finds indeed \( A(v_c) = 4 \).

For \( w < w_c \) no fixed point remains. For \( w > w_c \) the solution \( v_+ \) has \( A > 4 \), while for the solution \( v_- \) one finds \( A < 4 \). This follows directly from the monotonic behavior of \( A(v) \). From eq. (34) we conclude that the fixed point \( v_- \) is attractive for \( \tilde{\rho} \to 0 \), while \( v_+ \) is repulsive.

Consider next the behavior of the scaling solution for \( \tilde{\rho} \to \infty \). For a valid scaling solution one needs \( v(\tilde{\rho}) < 1 \) for all \( \tilde{\rho} \), since the pole of \( \beta_u \) for \( v = 1 \) should not be crossed. Also negative \( v \) cannot diverge for \( \tilde{\rho} \to \infty \) — such a behavior would lead to an unbounded effective potential which is not compatible with a stable theory. For all scaling solutions one requires a finite constant \( v_\infty \)

\[
\lim_{\tilde{\rho} \to \infty} v(\tilde{\rho}) = v_\infty.
\]

In turn, this enforces

\[
\lim_{\tilde{\rho} \to \infty} \left( \tilde{\rho} \frac{\partial v}{\partial \tilde{\rho}} \right) = 0,
\]
since otherwise \( v(\rho) \) would diverge for \( \rho \to \infty \). With eq. (43) the computation of \( v_\infty \) is the same as for \( v_0 \), with possible solutions \( v_+ \) and \( v_- \) given by eq. (26). Also the discussion of solutions that approach \( u_\infty \) for \( \rho \to \infty \) remains unchanged, with \( A_0 \) and \( c_0 \) in eq. (34) replaced by \( A_\infty \) and \( c_\infty \). The constant \( u_\infty \) is approached by neighboring solutions for \( \rho \to \infty \) if \( A_\infty > 4 \). While \( v_- \) is attractive for \( \rho \to 0 \), \( v_+ \) is attractive for \( \rho \to \infty \). Taking things together, the general scaling solution for matter freedom is a crossover from the fixed point \( v_- \) for \( \rho \to 0 \) to the fixed point \( v_+ \) for \( \rho \to \infty \).

4. Flow in vicinity of scaling solution

We next turn to the flow with \( k \) at fixed \( \rho \). Defining \( \tilde{m}^2(\rho) \) and \( \lambda(\rho) \) by the first and second \( \rho \)-derivatives of \( u \) according to eqs. (37), (35), we first derive the flow equations in the limit of matter freedom where the contributions of vector bosons, fermions, and scalars can be approximated by a constant \( b \nu \). The flow of the mass term obeys

\[
\partial_t \tilde{m}^2(\rho) = (A(\rho) - 2)\tilde{m}^2(\rho) + 2\rho \lambda(\rho),
\]

with \( A(\rho) \) given by eq. (33) in terms of \( u_0 \) and \( v(\rho) = u(\rho)/u_0 \). This flow equation follows by taking a \( \rho \)-derivative of eq. (21). Taking a further \( \rho \)-derivative yields

\[
\partial_\lambda \lambda(\rho) = \frac{A(\rho) \lambda(\rho) + 2 \rho \partial_\rho \lambda(\rho) + \frac{1}{w} \partial_\nu \tilde{m}^4(\rho)}{\partial_\rho \nu}. \tag{45}
\]

Here \( \partial A/\partial \nu \) obtains from eq. (33), again evaluated for \( w = w_0 \) and \( v = v(\rho) = u(\rho)/u_0 \).

Consider first the limit where \( \rho \partial \lambda/\partial \rho \) can be neglected. In this case the fixed point of the flow occurs for

\[
\tilde{m}^2 = 0, \quad \lambda = 0. \tag{46}
\]

This is indeed realized for the scaling solution for \( \rho \to \infty \). Indeed, for an asymptotic behavior of the scaling solution for \( \rho \to \infty \),

\[
u_+(\rho) = u_\infty + c_\infty \rho \frac{A_\infty}{2} \tag{47}
\]

and \( A_\infty > 4 \) one finds that

\[
\tilde{m}^2_*(\rho) = c_\infty \left( 2 - \frac{A_\infty}{2} \right) \rho^{1 - \frac{A_\infty}{2}} \tag{48}
\]

and

\[
\lambda_*(\rho) = c_\infty \left( 2 - \frac{A_\infty}{2} \right) \left( 1 - \frac{A_\infty}{2} \right) \rho^{1 - \frac{A_\infty}{2}}. \tag{49}
\]

Both approach zero for \( \rho \to \infty \). (In this section, and more generally if needed, we denote by stars the scaling solutions or fixed points.)

Linearizing the flow in the vicinity of the fixed point (46) yields (in the approximation \( \partial \lambda/\partial \rho = 0 \))

\[
\partial_t \tilde{m}^2 = (A - 2)\tilde{m}^2 + 2\rho \lambda, \\
\partial_t \lambda = A \lambda. \tag{50}
\]

These flow equations hold strictly for \( \rho \to \infty \). Replacing \( \tilde{m}^2 \) by \( \Delta \tilde{m}^2 = \tilde{m}^2 - \tilde{m}^2_*(\rho) \), and \( \lambda \) by \( \Delta \lambda = \lambda - \lambda_*(\rho) \), they also hold for \( \Delta \tilde{m}^2 \) and \( \Delta \lambda \) at finite large \( \rho \) to a very good approximation. The solution for \( \Delta \lambda \)

\[
\Delta \lambda = \tilde{c}_\lambda k^A \tag{51}
\]

drives \( \Delta \lambda \) to its fixed point value \( \Delta \lambda_* = 0 \) as \( k \) is lowered. Thus \( \lambda \) is an irrelevant coupling at the quantum gravity fixed point. For a complete theory that can be continued to arbitrary large \( k \) according to the quantum gravity fixed point one predicts \( \lambda(\rho) \) to be given by the scaling solution \( \lambda_*(\rho) \). For \( \Delta \tilde{m}^2 \) one obtains the solution

\[
\Delta \tilde{m}^2 = \tilde{c}_m k^{A - 2} + \tilde{c}_\rho k^A. \tag{52}
\]

With \( A_\infty > 4 \) also \( \Delta \tilde{m}^2 \) is irrelevant and \( \tilde{m}^2_*(\rho) \) is predicted to be the scaling solution \( \tilde{m}^2_* \). These properties hold for the region of large \( \rho \) for which \( A(\rho) \) exceeds 4.

The situation is more complicated for \( \rho = 0 \). If \( \partial \lambda/\partial \rho \) remains finite or does not increase too rapidly for \( \rho \to 0 \), one finds again a fixed point \( \lambda_0 = 0, \tilde{m}^2_* = 0 \), and solutions in the vicinity of the fixed point

\[
\tilde{m}^2_0 = c_m k^{A - 2}, \quad \lambda_0 = c_\lambda k^A. \tag{53}
\]

Now \( A \) is given by \( A_0 \) and therefore smaller than four. Since \( A \) is positive, \( \lambda_0 \) is an irrelevant parameter and predicted to be at its fixed point value \( \lambda_* = 0 \). The mass term is irrelevant for \( A > 2 \), predicted to be \( \tilde{m}^2_* = 0 \) in this case. For \( A < 2 \) it is a relevant parameter. Its value cannot be predicted since it involves the free constant \( c_m \).

We will discuss in sect. 5.2 under which circumstances the scaling solution indeed leads to \( \tilde{m}^2_* = \lambda_* = 0 \), gauge and Yukawa couplings are neglected and \( w_* \) is a constant. Except for \( A_0 = 2 \) we have seen that the approximation of matter freedom leads to a divergence of \( \lambda(\rho) \) for \( \rho \to 0 \) such that the fixed point \( \tilde{m}^2_* = \lambda_* = 0 \) is not realized. The neglect of \( \rho \partial \lambda/\partial \rho \) is not justified in this case.

We observe a connection between the behavior of deviations from the scaling solution and the asymptotic behavior of the scaling solution itself for \( \rho \to 0 \) and \( \rho \to \infty \). Both are given by the same anomalous dimension \( A \). The root of this connection resides in the general form of the flow equation for \( u \) at fixed \( \rho \) that can be written in the form

\[
(\partial_t - 2\rho \partial_\rho) u = -4 (u - c_V) = \tilde{\beta}_u. \tag{54}
\]

A similar form holds for \( \rho \)-derivatives, as \( m^2(\rho) = \partial_\rho u(\rho) \),

\[
(\partial_t - 2\rho \partial_\rho) \tilde{m}^2 = -2\tilde{m}^2 + 4 \partial_\rho c_V = \tilde{\beta}_m. \tag{55}
\]

For the scaling solution one has

\[
2\rho \partial_\rho u = -\tilde{\beta}_u. \tag{56}
\]

On the other hand, if \( \rho \partial_\rho u \) can be neglected for \( \rho \to 0 \) or \( \rho \to \infty \), one finds for these limits

\[
\partial_t u = \tilde{\beta}_u. \tag{57}
\]

Both expressions (56) and (57) involve the same \( \beta \)-function \( \tilde{\beta}_u \), but with opposite sign. Thus \( k^2 \to 0 \) corresponds to increasing \( \rho \).
5. Scalar mass term and quartic coupling

For the general crossover solutions for matter freedom discussed previously one cannot neglect \( \dot{\rho} \partial \lambda / \partial \dot{\rho} \) for \( \dot{\rho} \to 0 \). According to eq. (36) both \( \dot{\rho} \partial \lambda / \partial \dot{\rho} \) and \( \lambda \) would diverge for \( \dot{\rho} \to 0 \). For large \( \lambda \) the approximation of the flow contribution from scalars by a constant is no longer valid. We can no longer neglect \( \tilde{m}_A^2 \) in eq. (13). This will lead to corrections for a small range of \( \dot{\rho} \) near zero. In this section we will keep the approximation of constant \( u(\dot{\rho}) = u_0 \) and vanishing gauge and Yukawa couplings. We include now on the r.l.s. of the flow equation the effects of \( \dot{\rho} \)-derivatives of \( u(\dot{\rho}) \).

5.1. Flow equation

Let us consider a single real scalar field with discrete symmetry \( \phi \to -\phi \) and \( \rho = \phi^2/2 \). Neglecting the anomalous dimension, \( \eta_A = 0 \), it contributes to the flow of \( u \) by a term

\[
(\dot{\rho} u)_s = \frac{1}{32\pi^2} (1 + \tilde{m}^2 + 2\dot{\rho}\lambda)^{-1},
\]

where we employ for the squared scalar mass term

\[
\tilde{m}_A(\dot{\rho}) = \frac{\partial u}{\partial \dot{\rho}} + 2\dot{\rho} \frac{\partial^2 u}{\partial \dot{\rho}^2} = \tilde{m}^2(\rho) + 2\dot{\rho}\lambda(\dot{\rho}).
\]

The difference between the expression (58) and the value for \( \tilde{m}^2 = 0 \), \( \lambda = 0 \), which is incorporated in \( b_v \), yields to the flow equation for \( u \) an additional contribution

\[
\Delta_s \dot{\rho} u = \frac{\Delta \pi_s}{k^4} = 4c_{V,s} = \frac{1}{32\pi^2} \left[ \frac{1}{1 + \tilde{m}^2 + 2\dot{\rho}\lambda} - 1 \right] = -\frac{1}{32\pi^2} \tilde{m}^2 + 2\dot{\rho}\lambda.
\]

The particular fixed point solution \( u(\dot{\rho}) = u_0 = v \) corresponds to \( \tilde{m}^2 = 0 \), \( \lambda = 0 \). It is not changed by the additional term (60). While the particular constant scaling solutions remain valid in the presence of \( \Delta \pi_s \), this does not hold for the generic crossover scaling solutions for matter freedom. For the latter \( \Delta \pi_s \) will induce substantial corrections for \( \dot{\rho} \)-derivatives of \( u \) in the region of small \( \dot{\rho} \).

The correction to the flow of the mass term is given by the \( \dot{\rho} \)-derivative of eq. (60). Combining with eq. (44) one has

\[
\dot{\rho} \tilde{m}^2 = (A - 2)\tilde{m}^2 + 2\dot{\rho}\lambda - \frac{3\lambda + 2\dot{\rho}\lambda / \dot{\rho}}{32\pi^2 (1 + \tilde{m}^2 + 2\dot{\rho}\lambda)^2}.
\]

Possible scaling solutions for \( \tilde{m}^2 \) obtain if the r.h.s. of eq. (61) vanishes. With \( \lambda = \partial \tilde{m}^2 / \partial \dot{\rho} \) this condition amounts to a differential equation involving up to two \( \dot{\rho} \)-derivatives of \( \tilde{m}^2 \).

In the presence of \( \Delta \pi_s \) the structure of the differential equation for the scaling solutions for \( u \) gets more complicated since it involves up to two \( \dot{\rho} \)-derivatives of \( u \). Denoting by \( u' = \partial u / \partial \dot{\rho} \), \( u'' = \partial^2 u / \partial \dot{\rho}^2 \) the scaling solutions have to obey the differential equation

\[
2\ddot{\rho}u = 4u - \frac{1}{24\pi^2} \left( \frac{5}{1 - u/w} + \frac{1}{1 - u/4w} \right) - \frac{N - 5}{32\pi^2} \frac{1}{32\pi^2 (1 + u + 2\ddot{\rho}u \prime')}.
\]

For constant \( w = w_0 \) this is a nonlinear second order differential equation. The general solution involves two free integration constants, say \( u(\dot{\rho}_0) \) and \( u'(\dot{\rho}_0) \) at some suitably chosen \( \dot{\rho}_0 \). The question is which one of the local solutions remains valid for the whole range of \( \dot{\rho} \) without encountering a singularity. Finding the valid scaling solutions is not a simple task since the generic local solutions become singular at some \( \dot{\rho} \). The two constant scaling solutions are regular solutions of eq. (62) for the whole range of \( \dot{\rho} \). One would like to know if a one-parameter family of regular solutions can be found that replaces the one-parameter family of crossover solutions for the approximation of matter freedom.

A similar question arises for extensions to nonvanishing gauge couplings, Yukawa couplings or nonminimal scalar-gravity couplings that we will discuss in the following sections. Omitting \( \Delta \pi_s \) we will find one-parameter families of scaling solutions, similar to the crossover scaling solutions for matter freedom. Again the question will arise if a one-parameter family of regular solutions persists in the presence of \( \Delta \pi_s \). It is therefore worthwhile to discuss this question in some detail.

At this point we should emphasize that even for the existence of a one-parameter family of regular scaling solutions for \( u(\dot{\rho}) \) for a given \( w(\dot{\rho}) \), there is no guarantee that full quantum gravity admits a one-parameter family of scaling solutions. It is possible that the flow equations of \( w(\dot{\rho}) \) (and other invariants not considered here) are compatible only with a subclass of the regular scaling solutions for \( u(\dot{\rho}) \). It is well conceivable that the fixed point for quantum gravity only admits a single solution. The present investigation should therefore be seen as an exploration of possibilities rather than a definite determination of the scaling solution. On the other hand, any overall scaling solution of quantum gravity has to result in a regular scaling solution for \( u(\dot{\rho}) \) for all other couplings taking their values for the scaling solution. The general properties of \( u(\dot{\rho}) \) found in this paper therefore apply to any overall scaling solution for quantum gravity.

5.2. Scaling solution near the origin

If \( \dot{\rho} \partial \lambda / \partial \dot{\rho} \) vanishes for \( \dot{\rho} \to 0 \), e.g. for finite \( \lambda(\dot{\rho} = 0) \), the fixed point value of the mass term at \( \dot{\rho} = 0 \), \( \tilde{m}_0^2 = \tilde{m}^2(\dot{\rho} = 0) \), obeys for \( A_0 \neq 2 \)

\[
\tilde{m}_0^2 = \frac{3\lambda_0}{32\pi^2 (A_0 - 2) (1 + \tilde{m}_0^2)}.
\]

It vanishes only for \( \lambda_0 = \lambda(\dot{\rho} = 0) = 0 \). We may use \( \tilde{m}_0^2 \) as a parameter to characterize a possible family of scaling solutions. For a given \( \tilde{m}_0^2 \) the quartic coupling \( \lambda_0 \) is fixed and computable. This continues to higher order couplings.
The scaling solution for $\lambda_0$ fixes $\partial \lambda / \partial \rho (\hat{\rho} = 0)$ and so on. The particular solution $\tilde{m}_0^2 = 0$ is the flat scaling solution.

For a non-zero scalar mass term the approach of a general scaling solution to $u(\hat{\rho} = 0) = u_0$ gets an additional contribution, modifying eq. (31) to

$$2\hat{\rho} \frac{\partial \Delta u}{\partial \hat{\rho}} = (4 - A)\Delta u - 4c_{V,s}.$$  \hfill (64)

Let us consider the vicinity of the constant scaling solution. Employing

$$\tilde{m}^2 = \frac{\partial \Delta u}{\partial \hat{\rho}},$$  \hfill (65)

we may linearize in $\tilde{m}^2$ and $\lambda$

$$\left(2\hat{\rho} - \frac{1}{32\pi^2}\right) \frac{\partial \Delta u}{\partial \hat{\rho}} = (4 - A)\Delta u + \frac{\lambda \hat{\rho}}{16\pi^2}. \hfill (66)$$

We first consider the limit $\hat{\rho} \to 0$ where we neglect the last term $\sim \lambda \hat{\rho}$ in eq. (66). The solution,

$$\Delta u = \hat{c}_0 \left(2 - \frac{A_0}{2}\right) \left(\frac{1}{64\pi^2} - \hat{\rho}\right)^{2-\frac{A_0}{2}}, \hfill (67)$$

deviates substantially from eq. (34) in the range of small $\hat{\rho}$. Derivatives no longer diverge. For $\hat{\rho} \to 0$ one finds finite $\tilde{m}^2$,

$$\tilde{m}^2 = -\hat{c}_0 \left(2 - \frac{A_0}{2}\right) \left(\frac{1}{64\pi^2} - \hat{\rho}\right)^{1-\frac{A_0}{2}}. \hfill (68)$$

The constant $\hat{c}_0$ is related to $\tilde{m}_0^2 = \tilde{m}^2(\hat{\rho} = 0)$ in eq. (63) in the limit of small $\tilde{m}_0^2$.

For higher derivatives we have to take the term $\sim \lambda \hat{\rho}$ in eq. (66) into account. We make the ansatz

$$\frac{\lambda}{32\pi^2} = f\tilde{m}^2.$$  \hfill (69)

The resulting differential equation

$$\left(\hat{\rho} (1 - f) - \frac{1}{64\pi^2}\right) \frac{\partial \Delta u}{\partial \hat{\rho}} = \left(2 - \frac{A_0}{2}\right) \Delta u \hfill (70)$$

has the solution

$$\Delta u = c_0 \left(\frac{1}{64\pi^2} - (1 - f)\hat{\rho}\right)^{\frac{A_0}{2(1 - f)}}, \hfill (71)$$

Identification with eq. (67) for $\hat{\rho} = 0$ relates $c_0$ and $\hat{c}_0$

$$c_0 = \hat{c}_0 \left(\frac{1}{64\pi^2}\right)^{\frac{4 - A_0}{2(1 - f)}}. \hfill (72)$$

Taking a derivative of eq. (71),

$$\tilde{m}^2 = \frac{\partial \Delta u}{\partial \hat{\rho}} = -c_0 \left(2 - \frac{A_0}{2}\right) \left(\frac{1}{64\pi^2} - (1 - f)\hat{\rho}\right)^{\frac{4 - A_0}{2(1 - f)}} - 1, \hfill (73)$$

one observes that $\tilde{m}^2(\hat{\rho} = 0)$ indeed coincides with the value (68). Taking a further $\hat{\rho}$-derivative of eq. (73) and evaluating it at $\hat{\rho} = 0$, one finds for $\hat{\rho} = 0$ the relation

$$\frac{\lambda}{m^2} = 32\pi^2 (A_0 - 2 - 2f). \hfill (74)$$

Comparison with eq. (69) leads to a self-consistent determination of $f$,

$$f = \frac{A_0 - 2}{3}. \hfill (75)$$

This corresponds to eq. (63) for small $\tilde{m}^2$. We infer for the limiting behavior of $u$ for $\hat{\rho} \to 0$, $u_0 = v_\infty$,

$$u(\hat{\rho} \to 0) = u_0 + c_0 \left(\frac{1}{64\pi^2} - \frac{5 - A_0}{3}\hat{\rho}\right)^{\alpha} \hfill (76)$$

with

$$\alpha = 3 (4 - A_0) \frac{2}{(5 - A_0)}. \hfill (77)$$

We conclude that for $\hat{\rho} \to 0$ the inclusion of the correction term $\Delta_\pi$ cures the divergence of derivatives of the crossover scaling solution. We will next establish that the fixed point solution is now compatible with the flow of couplings at fixed $\hat{\rho}$.

### 5.3. Relevant and irrelevant couplings

For the flow of $\tilde{m}^2$ and $\lambda$ away from the fixed point we employ the $\hat{\rho}$-derivative of eq. (60)

$$\partial_t \tilde{m}^2 = (A - 2) \tilde{m}^2 + 2\hat{\rho}\lambda - \frac{3\lambda + 2\hat{\rho}u^{(3)}}{32\pi^2 (1 + \tilde{m}^2 + 2\hat{\rho}\lambda)}; \hfill (78)$$

with

$$u^{(3)}(\hat{\rho}) = \frac{\partial \lambda(\hat{\rho})}{\partial \hat{\rho}}. \hfill (79)$$

For $\hat{\rho} \to 0$ we take advantage of the finiteness of $\tilde{m}^2$, $\lambda$ and $u^{(3)}$ and evaluate the flow of $\tilde{m}_0^2 = \tilde{m}^2(\hat{\rho} = 0)$, $\lambda_0 = \lambda(\hat{\rho} = 0)$,

$$\partial_t \tilde{m}_0^2 = (A_0 - 2) \tilde{m}_0^2 - \frac{3\lambda_0}{32\pi^2 (1 + \tilde{m}_0^2)^2}. \hfill (80)$$

For the fixed point we find the relation

$$\lambda_{0,*} = \frac{32\pi^2 (A_0 - 2)}{3} \tilde{m}_{0,*}^2 \left(1 + \tilde{m}_{0,*}^2\right)^2. \hfill (81)$$

Linearizing for small $\lambda$ and $\tilde{m}^2$ this fixed point for the ratio $\lambda / \tilde{m}^2$ is indeed consistent with eqs. (69), (73). Taking a $\hat{\rho}$-derivative of eq. (78) and evaluating it at $\hat{\rho} = 0$ yields $u_{0,*}^{(3)}$ in terms of $\tilde{m}_{0,*}^2$ and $\lambda_{0,*}$.

Deviations from $\tilde{m}_{0,*}^2$ from the fixed point value $\tilde{m}_{0,*}^2$ are denoted by

$$\gamma = \tilde{m}_{0,*}^2 - \tilde{m}_{0,*}^2.$$  \hfill (82)
For small $\gamma$ the flow equations read,
\[
\partial_t \gamma = (A_0 - 2) \gamma + \frac{3\lambda_0, * \gamma}{32\pi^2 (1 + \tilde{m}_{0, *}^2)^2} - \frac{3 \delta \lambda}{32\pi^2 (1 + \tilde{m}_{0, *}^2)^2} + \tilde{m}_{0, *}^2 \frac{\partial A}{w_0 \partial\nu} \delta u , \tag{83}
\]
with
\[
\delta \lambda = \lambda_0 - \lambda_0, *, \quad \delta u = u_0 - u_0, * . \tag{84}
\]
Neglecting first $\delta \lambda$ and $\delta u$, the insertion of eq. (81) yields
\[
\partial_t \gamma = (A_0 - 2) \left( 1 + \frac{\tilde{m}_{0, *}^2}{(1 + \tilde{m}_{0, *}^2)^2} \right) \gamma = -\theta \gamma. \tag{85}
\]
This generalizes to more complex settings. The vacuum electroweak phase transition is almost of second order. For a second order phase transition the flow of $\gamma$ must vanish for $\gamma = 0$ [62–68]. This ensures naturalness of very small $\gamma$ due to the enhanced quantum scale symmetry for $\gamma = 0$.

For $A_0 < 2$ the deviation from the scaling solution $\gamma$ is a relevant parameter. The critical exponent $\theta$, is positive. For constant $\theta$, the flow of $\gamma$ obeys
\[
\gamma = \gamma_A \left( \frac{k}{\Lambda} \right)^{-\theta}, \tag{86}
\]

The initial value $\gamma_A = \gamma(k = \Lambda)$ specified at arbitrary chosen scale $\Lambda$ determines the particular flow trajectory. Restoring dimensions the renormalized scalar mass term at $\rho = 0$ behaves as
\[
m^2(k) = \tilde{m}_{0, *}^2 k^2 + \gamma_A \Lambda^{\theta / 2} k^{-\theta / 2}. \tag{87}
\]

For $\gamma_A > 0$ the mass term is positive for all $k$. The origin $\rho = 0$ is a local minimum of the effective potential. The model is typically in the “symmetric phase” with unbroken $Z_2$ symmetry $\phi \rightarrow -\phi$. In contrast, for $\gamma_A < 0$ the mass term turns negative for $k$ smaller than some critical $k_c$.

\[
k_c = \Lambda \left( -\frac{\gamma_A}{\tilde{m}_{0, *}^2} \right)^{\frac{1}{\theta}} . \tag{88}
\]

The origin at $\rho = 0$ becomes a local maximum for small enough $k$. This indicates a minimum of $U$ for $\rho = \rho_0 > 0$, and therefore a spontaneous breaking of the $Z_2$-symmetry. The phase transition at the boundary between the symmetric phase and the phase with spontaneous symmetry breaking is given by $\gamma_A = 0$. This is the scaling solution. The existence of a scaling solution is a general feature of a second order phase transition.

We observe that the critical trajectory $\gamma_A = 0$ is never crossed by any flow trajectory. This generalizes to the critical surface of a second order phase transition. It follows generally from the existence of a scaling solution and continuity. For the scaling solution, corresponding to the critical surface, the flow vanishes. For neighboring solutions the flow is very small by continuity, as exemplified by the flow of $\tilde{m}_{0, *}^2$ in the region where $\gamma$ remains small. Since the flow becomes arbitrarily slow if the scaling solution is approached arbitrarily close by, no flow trajectory can cross the critical surface.

For $\theta \gamma < 0$, typically requiring $A_0 > 2$, the deviation $\gamma$ from the critical surface becomes an irrelevant parameter. One observes self-tuned criticality towards the electroweak phase transition. This can explain the gauge hierarchy [69, 70].

These general statements apply to the flow according to eq. (83) for nonvanishing $\delta u$ and $\delta \lambda$ as well. The critical surface is now a two-dimensional hypersurface in the three-dimensional space of couplings $u_0, \tilde{m}_{0, *}, \lambda_0$. On the critical surface the couplings follow scaling solutions $u_{0, *}, \tilde{m}_{0, *}, \lambda_{0, *}$. Deviations from the scaling solution may be denoted by $\alpha_i = (\delta u, \gamma, \delta \lambda)$. For the scaling solution the flow of $\delta u, \gamma$ and $\delta \lambda$ vanishes. Critical exponents are the eigenvalues of the stability matrix $T$ which characterizes the linearized flow in the vicinity of the scaling solution
\[
\partial_t \alpha_i = -T_{ij} \alpha_j . \tag{89}
\]

For a computation of the stability matrix we need the flow of $\delta u$ and $\delta \lambda$. For $\delta u$ one finds from eqs. (21), (58)
\[
\partial_t \delta u = (A - 4) \delta u + 2 \rho \gamma - \frac{1}{32\pi^2} \left( 1 + \tilde{m}^2 + 2 \rho \lambda \right)^{-2} (\gamma + 2 \rho \delta \lambda) . \tag{90}
\]

For the flow of the quartic coupling we first take a $\rho$-derivative of eq. (78),
\[
\partial_t \lambda = A \lambda + 2 \tilde{m}^2 \frac{1}{w} \frac{\partial A}{\partial v} \tilde{m}^4 + \frac{1}{16\pi^2} \left( 3 \lambda + 2 \rho \tilde{m}^2 \right)^2 - \frac{1}{32\pi^2} \left( 5 \tilde{m}^4 + 2 \rho \tilde{m}^4 \right) , \tag{91}
\]

Neglecting $\tilde{m}^4$ and $\tilde{m}^4$ the linearized flow equation for $\delta \lambda$ becomes
\[
\partial_t \delta \lambda = A \delta \lambda + \frac{1}{w} \frac{\partial A}{\partial v} \left( \lambda_0 \delta u + 2 \tilde{m}^2 \gamma \right) + \frac{9 \lambda_0 \delta \lambda}{8\pi^2} \left( 1 + \tilde{m}^2 + 2 \rho \lambda \right)^3 - \frac{27 \lambda_0 (\gamma + 2 \rho \delta \lambda)}{16\pi^2 (1 + \tilde{m}^2 + 2 \rho \lambda)^4} . \tag{92}
\]

The stability matrix for $\rho = 0$ reads
\[
- T = \begin{pmatrix}
A - 4 & -d & 0 \\
B \tilde{m}^2 & A - 2 + 3d \lambda_0 & -3d \\
B \lambda_0 + C \tilde{m}^4 & 2B \tilde{m}^2 & \frac{54d \lambda_0}{(1 + \tilde{m}^2)^2} + A + \frac{36d \lambda_0}{(1 + \tilde{m}^2)^2}
\end{pmatrix} , \tag{93}
\]

with
\[
B = \frac{1}{w} \frac{\partial A}{\partial v} , \quad C = \frac{1}{w^2} \frac{\partial^2 A}{\partial v^2} , \quad d = \frac{1}{32\pi^2} \left( 1 + \tilde{m}^2 \right)^2 . \tag{94}
\]

For $A$ of the order one and $3d$ smaller than 0.01 the off-diagonal elements in the upper right corner are small.
For $d = 0$ the eigenvalues of $T$ are given by the diagonal elements. Also for small $\tilde{m}^2, \lambda$ the off-diagonal elements in the lower left corner are all small. Neglecting them, the eigenvalues are given by the diagonal elements even for $d \neq 0$. We conclude that to a very good approximation the critical exponents are given by the diagonal elements of $T$. Since $A_0 < 4$ there is always one relevant coupling, which is dominantly given by $\delta u$. A second coupling, dominantly $\gamma$, occurs for $A_0 < 2 - 3d \lambda$. This coupling becomes irrelevant for $A_0 > 2 - 3d \lambda$. The third coupling, dominantly $\delta \lambda$, is irrelevant.

If we keep also $\Delta u^{(3)}$ etc., the flow for a finite number of couplings will not be closed. In principle, the number of couplings is infinite, such that the stability matrix $T$ is infinite dimensional. The almost diagonal structure of $T$ continues if we extend it to higher order couplings. The critical exponent for $\delta u^{(3)}$ is approximately $A + 2$, while for $\Delta u^{(4)}$ it amounts to $A + 4$. Higher order couplings are all irrelevant parameters. In the absence of off-diagonal elements they would be predicted to take exactly the values for the scaling solution. In the presence of the off-diagonal elements the values of $\delta \lambda, \Delta u^{(3)}$ etc. remain predictable as a function of the relevant couplings that we may parameterize by $\delta u$ (and $\gamma$ if this is also relevant).

The flow away from the critical surface is determined by the relevant couplings. These are linear combinations of $\alpha_i$ that are eigenvectors to positive eigenvalues of $T$. The irrelevant couplings correspond to negative eigenvalues of $T$. They can be set to zero. In consequence, $\delta \lambda$ can always be expressed as a linear combination of $\gamma$ and $\delta u$. As an example, we take the flat scaling solution for which $\tilde{m}^2_{0,0} = 0$, and therefore also $\lambda_{0,0} = 0$, as well as $u_{0,0}^{(3)} = 0$ and similar for higher order couplings.

From

$$\partial_t \delta \lambda = A_0 \delta \lambda$$

we conclude that $\delta \lambda$ is an irrelevant coupling. This coincides with the findings of some of the early investigations [22, 27, 28, 32, 71, 72]. Setting $\delta \lambda = 0$ in the flow equation for $\gamma$ and $\delta u$ results in

$$\partial_t \gamma = (A_0 - 2) \gamma, \quad \partial_t \delta u_0 = (A_0 - 4) \delta u_0 - \frac{1}{32\pi^2} \gamma.$$  

The eigenvectors of $T$ are $\gamma$ and $\delta u_0 + \gamma/(64 \pi^2)$, with eigenvalues $2 - A_0$ and $4 - A_0$.

### 5.4. Asymptotic behavior for large scalar fields

For large values $\rho \to \infty$ the $\rho$-dependence of the scaling effective potential is given by eqs (47) – (49). With $A_\infty > 4$ both $\tilde{m}^2(\rho)$ and $\lambda(\rho)$ vanish for $\rho \to \infty$. We conclude that the contribution (60) becomes negligible for large $\rho$ for the scaling solution and the vicinity of it. The scalar field is free and massless in the range of very large $\rho$. For deviations from the scaling solution the stability matrix has vanishing off-diagonal elements in the lower left corner. The eigenvalues are the diagonal elements, with $A_{\infty} = 4, A_{\infty} - 2, A_{\infty} - 7, A_{\infty} - 13, \ldots$ all positive for $A_{\infty} > 4$. There are no relevant couplings in the scalar sector. As long as the dimensionless Planck mass $w$ can be approximated by a constant, the scalar potential for large $\rho$ is predicted to be given precisely by the scaling solution.

The family of possible scaling solutions can be parameterized by the coefficient $c_\infty$ in eqs. (47) – (49). It specifies at which $\rho$ the crossover from the vicinity of the constant fixed point potential for $\rho \to \infty$ to the fixed point potential for $\rho \to 0$ takes place, cf. Figs 2, 3. In the large $\rho$-region where $\tilde{m}^2$ and $\lambda$ can be neglected the differential equation (23) for the scaling solution for $u$ does not involve $x = \ln(\rho)$ explicitly

$$\partial u / \partial x = 2(u - c_V(u)).$$  

In this range the family of scaling solutions can be obtained by constant shifts in $x$ or multiplicative rescalings of $\rho$. The family of possible scaling solutions is characterized by a single parameter $c_\infty$. This extends to the full characterization of scaling solutions in the whole range of $\rho$, provided that for each $c_\infty$ the solution can be continued to $\rho \to 0$. If a certain range of $c_\infty$ cannot be continued, the allowed family of scaling solutions will be restricted to a range in $c_\infty$.

### 5.5. Transition region

The approximations employed for the derivation of the scaling solution near $\rho = 0$ break down if $\rho$ comes close to the value $\rho_s = 1/(64 \pi^2)$. The region of $\rho$ around $\rho_s$ describes the transition from the UV-region near $\rho = 0$, where the details of the scalar fluctuations may matter, to the region of larger $\rho$ where $\tilde{m}^2$ and $\lambda$ can be neglected. For small $\Delta u$, $\tilde{m}^2 = \partial(\Delta u)/\partial \rho$ and $\lambda = \partial^2(\Delta u)/\partial \rho^2$, we can use eq. (66) as a second order differential equation

$$\left(\rho - \frac{1}{64\pi^2}\right) \frac{\partial \Delta u}{\partial \rho} = - \frac{A - 4}{2} \Delta u + \hat{\rho} \frac{\partial^2 \Delta u}{\partial \rho^2}.$$  

In terms of the variable

$$\hat{s} = 64 \pi^2 \rho$$

this reads

$$(s - 1) \frac{\partial \Delta u}{\partial s} + \frac{A - 4}{2} \Delta u - 2s \frac{\partial^2 \Delta u}{\partial s^2} = 0.$$  

Except for the constant scaling solution $\Delta u = 0$ the function $\Delta u$ grows outside the range of this linear approximation as $s$ increases. We also have investigated numerically the nonlinear second order differential equation

$$\hat{\rho} \frac{\partial u}{\partial \rho} = 2(u - c_V) ,$$

with $\tilde{m}^2 = \partial u / \partial \rho$, $\lambda = \partial^2 u / \partial \rho^2$ in eq. (60). The only scaling solutions extending for the whole range $0 \leq \rho < \infty$ that we have found so far are the constant scaling solutions.

The issues of finding scaling solutions for scalars coupled to gravity can be understood by neglecting the term $2 \lambda \rho$ in...
In this approximation the scaling solution has to obey the differential equation
\[
2\rho \frac{\partial u}{\partial \rho} = 4(u - c_V) + \frac{M}{32\pi^2} \left(1 - \frac{1}{1 + \frac{\rho}{\rho_0}}\right).
\] (102)

Here we consider \( M \) scalars with mass term \( m^2 = \partial u/\partial \rho \), while \( N - M \) accounts for additional massless particles as fermions, gauge bosons or further scalars. Eq. (102) can be written as a quadratic equation in \( \dot{\rho} \partial u/\partial \rho \) and therefore be transformed to
\[
\dot{\rho} \frac{\partial u}{\partial \rho} = \frac{1}{2} \left(2u - 2c_V + \frac{M}{64\pi^2} - \rho \pm \sqrt{W}\right),
\] (103)

with
\[
W = \left(2u - 2c_V + \frac{M}{64\pi^2} + \dot{\rho}\right)^2 - \frac{M\dot{\rho}}{16\pi^2}.
\] (104)

Let us first consider the limit \( \rho \to \infty \) where
\[
\sqrt{W} = \rho + 2u - 2c_V - \frac{M}{64\pi^2} + \frac{M(u - c_V)}{4\pi^2 \rho} + \ldots
\] (105)

Employing the solution with the plus sign in eq. (103) one finds
\[
\rho \frac{\partial u}{\partial \rho} = (2u - 2c_V) \left(1 + \frac{M}{64\pi^2 \rho}\right).
\] (106)

One sees again that the scaling solution approaches for \( \rho \to \infty \) the constant \( u = c_V \) discussed in sect. 3. For the opposite limit for \( \rho \to 0 \) one has
\[
\sqrt{W} = 2u - 2c_V + \frac{M}{64\pi^2} + \frac{u - c_V - \frac{M}{128\pi^2}}{u - c_V + \frac{M}{128\pi^2}}\rho.
\] (107)

If \( \partial u/\partial \rho \) remains finite for \( \rho \to 0 \) the relative minus sign in eq. (103) is appropriate, yielding
\[
\frac{\partial u}{\partial \rho} = \frac{u - c_V}{u - c_V + \frac{M}{128\pi^2}}.
\] (108)

The asymptotic behavior \( u = c_V \) for \( \rho = 0 \) implies a vanishing mass term \( \delta m^2 = \partial u/\partial \rho \to 0 \). With \( u_0 \) given by the solution \( u_0 = c_V(u_0) \) as in sect. 3, and \( u = u_0 + \Delta u \), the linear expansion of eq. (103) in \( \Delta u \) becomes
\[
\frac{\partial \Delta u}{\partial \rho} = -\frac{(4 - A)\Delta u}{(4 - A)\Delta u + \frac{M}{32\pi^2}} \approx -\frac{32\pi^2(4 - A)}{M} \Delta u,
\] (109)

in accordance with eq. (66) for \( \rho \to 0 \) for \( M = 1 \). The sign of the mass term at the origin is the opposite to the sign of \( \Delta u \).

For the differential equation (103) the relative plus sign for the square root applies for \( \rho \to \infty \), while for \( \rho \to 0 \) one needs the relative minus sign. There has to be a matching, which must occur for \( W = 0 \) for reasons of continuity. The value of \( \rho_t \) where \( W(\rho_t) = 0 \) obeys
\[
\rho_t = 2 \left(\sqrt{c_V - u \pm \sqrt{\frac{M}{128\pi^2}}}\right)^2.
\] (110)

A switch of sign of the \( \sqrt{W} \)-term is only possible in a region where \( u \leq c_V \). Continuity of the quartic coupling \( \lambda \) at \( \rho_t \) requires further
\[
\frac{\partial W}{\partial \rho}(\rho_t) = 0.
\] (111)

Combining eq. (111) with \( W(\rho_t) = 0 \) yields the condition
\[
\rho_t = \frac{M}{64\pi^2}.
\] (112)

Comparing further eqs (112) and (110) requires at \( \rho_t \)
\[
c_V - u = \left\{0, \frac{M}{32\pi^2}\right\},
\] (113)

corresponding to
\[
\delta m^2(\rho_t) = \{0, -2\}.
\] (114)

The second value violates the condition \( \delta m^2 > -1 \) needed for a stable scalar propagator, and we conclude that scaling solutions with a change of the sign of the term \( \pm \sqrt{W} \) in eq. (103) require
\[
\delta m^2(\rho) = \frac{M}{64\pi^2} = 0.
\] (115)

Generic local scaling solutions with arbitrary \( \delta m^2 \) do not obey eq. (115), as we have checked by a numerical solution of eq. (103).

The two flat scaling solutions discussed in sect. 3,
\[
u(\rho) = c_V(v_{\pm}),
\] (116)

with \( v_{\pm} \) given by eq. (26), obey eq. (115). For these particular solutions there is actually a change of sign in the term \( \pm \sqrt{W} \) in eq. (103) at \( \rho_t = M/(64\pi^2) \), since \( \sqrt{W} = \rho - \rho_t \) for \( \rho > \rho_t \) and \( \sqrt{W} = \rho_t - \rho \) for \( \rho < \rho_t \). It is not obvious if there exist other scaling solutions obeying the condition (115). If not, and if the inclusion of nonzero \( \lambda \) does not change in an important way the possible scaling solutions, the inclusion of the scalar mass term reduces the family of scaling solution for matter freedom discussed in sect. 3 to only two scaling solutions, both with a flat potential.

If we do not impose the condition (111) the matching of solutions with different signs \( \pm \sqrt{W} \) at \( \rho_t \) typically induces a discontinuity of \( \partial \delta m^2/\partial \rho \) at \( \rho_t \). Higher order couplings will then diverge and the approximation of neglecting them remains no longer valid. So far, it is not known if the inclusion of \( \lambda \) (and higher order couplings as \( u^{(3)} \)) can smoothen the discontinuity or not. At the present stage it is therefore not known if a continuous family of scaling solutions exist, or if this is reduced to a discrete subset.

As a general lesson, we conclude that the issue of the existence of a whole family of scaling solutions, or only a discrete subset, is typically decided in the transition region. Both limiting cases \( \rho \to 0 \) and \( \rho \to \infty \) admit families of scaling solutions characterized by a continuous parameter. The question is if the families of scaling solutions in the two
limiting cases can be matched to each other continuously in the transition region. For pure scalar models coupled to gravity with constant \( w \) it seems most likely that only a discrete subset of overall scaling solutions remains. We will see that non-vanishing gauge or Yukawa couplings, as well as non-trivial couplings to gravity encoded in the \( \rho \)-dependence of \( w(\rho) \), modify the properties of the transition region considerably.

6. Gauge couplings

In this section we investigate the impact of nonvanishing gauge couplings on the flow of the scalar effective potential. For nonzero values of scalar fields coupling to gauge bosons with a gauge coupling \( g \) the gauge bosons acquire a mass through the Higgs mechanism. This mass suppresses the contribution of gauge bosons to the flow of the scalar potential. As a result, for nonvanishing gauge couplings the flow generator \( \zeta \) in eq. (7) receives an additional contribution \( \Delta \tilde{\pi}_{\text{gauge}} \) given by

\[
\frac{\Delta \tilde{\pi}_{\text{gauge}}}{k^4} = \frac{3}{32\pi^2} \sum_{i=1}^{N_V} \left( \frac{1}{1 + w_i} - 1 \right). \tag{117}
\]

Here the sum is over all gauge bosons and \( w_i = m_i^2/k^2 \) are the dimensionless squared gauge boson masses for the corresponding values of the scalar field

\[
w_i = \frac{m_i^2}{k^2} = g^2 a_i(\phi_a)/k^2. \tag{118}
\]

Typically, \( a_i(\phi_a) \) is a quadratic form in the scalar fields \( \phi_a \). The factor \( (1 + w_i)^{-1} \) is a threshold function that suppresses the contribution from massive gauge bosons as compared to massless ones.

In order to keep the discussion of the structure of these modifications simple we consider a toy flow equation where the squared mass of \( N_V \) gauge bosons is \( c_g g^2 \rho \), while the other gauge bosons remain massless for the particular configuration of scalar fields that is used to define \( \rho \). From the difference of the fluctuation contribution from massless and massive fields one obtains the modification of \( \tilde{\pi}_{\text{gauge}} \),

\[
\frac{\Delta \tilde{\pi}_{\text{gauge}}}{k^4} = -\frac{3c_g \bar{N}_V g^4 \rho}{32\pi^2 (1 + c_g g^2 \rho)}. \tag{119}
\]

It vanishes for \( \rho = 0 \). The contribution to the flow of the scalar mass term at the origin \( \rho = 0 \) reads

\[
\partial_\rho \tilde{m}^2 = \frac{3c_g \bar{N}_V g^2}{32\pi^2} + \ldots, \tag{120}
\]

while the contribution to the quartic coupling at \( \rho = 0 \), \( \lambda = \partial^2 \hat{U}/\partial \rho^2 |_{\rho=0} \), becomes

\[
\partial_\rho \lambda = \frac{3c_g \bar{N}_V g^4}{16\pi^2} + \ldots \tag{121}
\]

Eq. (121) corresponds to the standard perturbative term \( \sim g^4 \) in the flow equation for quartic scalar couplings.

Due to the negative sign in eq. (119) a nonvanishing gauge coupling lowers the scaling solution for the effective potential for large \( \rho \) as compared to \( \rho = 0 \). Indeed, the differential equation (23) for the scaling solution receives an additional positive contribution

\[
2\rho \frac{\partial u}{\partial \rho} = 4(u - c_V) - \frac{\Delta \tilde{\pi}_{\text{gauge}}}{k^4}, \tag{122}
\]

enhancing the increase of \( u \) with \( \rho \). Initial conditions near \( \rho = 0 \) that would lead to a decrease of \( u \) with increasing \( \rho \) may be turned to an increase with \( \rho \) for larger \( \rho \). As a result, the effective potential can develop a minimum for \( \rho \neq 0 \). This is clearly seen for a numerical solution of eq. (122) in Fig. 6, shown in more detail in Fig. 7. We employ \( N = 10, N_B = 3, c_g = 1, \alpha = g^2/4\pi = 1/40 \) and set the initial condition by \( u(\ln(\rho) = 1) = 0.02 \). The three values of \( w_0 = 0.06, 0.05, 0.042 \) used in Figs 6, 7 correspond to \( A_0 = A(\rho = 0) = 0.68, 1.0, 1.64 \), and therefore all have \( A_0 < 2 \).

For a more detailed understanding we first consider the region of small \( \rho \). The flow equation for the mass term at the origin, \( \tilde{m}_0^2 = \tilde{m}^2(\rho = 0) \), reads, with \( \lambda_0 = \lambda(\rho = 0) \),

\[
\partial_\rho \tilde{m}_0^2 = (A_0 - 2) \tilde{m}_0^2 - \frac{3\lambda_0}{32\pi^2 (1 + \tilde{m}_0^2)} - \frac{3c_g \bar{N}_V g^2}{32\pi^2}. \tag{123}
\]

The fixed point occurs for \( (A_0 \neq 2) \)

\[
\tilde{m}^2_{0,*} = \frac{3}{32\pi^2 (A_0 - 2)} \left( c_g \bar{N}_V g^2 + \frac{\lambda_0}{(1 + \tilde{m}^2_{0,*})^2} \right). \tag{124}
\]

For small \( \lambda_{0,*} \) and \( A_0 < 2 \) the mass term is negative,

\[
\tilde{m}^2_{0,*} = -\frac{3c_g \bar{N}_V g^2}{32\pi^2 (2 - A_0)}, \tag{125}
\]

such that the origin at \( \tilde{\rho} = 0 \) is a local maximum of the effective potential. This is seen for the curves in Figs 6, 7 which indeed have all \( A_0 < 2 \). We show in Fig. 8 the mass term \( \tilde{m}^2(\rho) \) for the scaling solutions for the three sets of parameters used in Figs 6, 7. For \( \tilde{\rho} \to 0 \) the result for \( \tilde{m}^2(\rho \to 0) \) comes indeed very close to the value (125).

For \( A_0 > 2 \) a negative value of \( \tilde{m}^2_{0,*} \) remains possible if \( \lambda_{0,*} < 0 \). Indeed, the potential may have at the origin a local minimum or a maximum, depending on the relative size of the two terms on the r.h.s. of eq. (124). The fixed point value for \( \lambda_{0,*} \) is negative, given for a single scalar by

\[
\lambda_{0,*} = \frac{1}{A_0} \left( \frac{3c_g \bar{N}_V g^4}{16\pi^2} + \frac{9\lambda_{0,*}}{16\pi^2 (1 + \tilde{m}^2_{0,*})^3} \right)
- \frac{5\nu_{0,*}^{(3)}}{32\pi^2 (1 + \tilde{m}^2_{0,*})^2} + \frac{1}{w_0} \frac{\partial A}{\partial \nu} \tilde{m}^2_{0,*}. \tag{126}
\]

The size of a negative \( \lambda_{0,*} \), and its influence is increased as \( \tilde{m}^2_{0,*} \) comes close to \( -1 \). We show in Fig. 9 the numerical scaling solutions for the effective potential with parameters \( N = 20, N_V = 3, c_g = 1, w_0 = 0.05 \) and \( \alpha = 1/40 \). For these parameters one has \( A_0 = 2.91 \), and therefore \( A_0 > \)
FIG. 6. Effective potential $u(x)$ as function of $x = \ln(\tilde{\rho})$ for three values of $w_0$. The highest curve on the right (green) is for $w_0 = 0.06$, the middle curve (orange) for $w_0 = 0.05$ and the lowest curve (blue) for $w_0 = 0.042$. Initial values are set by $U(x = 1) = 0.02$. Parameters are $N = 10$, $N_V = 3$, $\alpha = g^2/4\pi = 1/40$.

FIG. 7. Minimum of effective potential $u(x)$. Parameters are the same as for Fig. 6. The lowest green curve is for $w_0 = 0.06$, the middle orange curve for $w_0 = 0.05$ and the highest blue curve for $w_0 = 0.042$.

2. The mass term is an irrelevant coupling in this case. Depending on the initial conditions we found solutions with a minimum at the origin or a minimum at $\tilde{\rho} \neq 0$.

Consider next the limit $\tilde{\rho} \to \infty$. In this limit the correction (119) approaches a constant

$$
\lim_{\tilde{\rho} \to \infty} \left( \frac{\Delta \tilde{\pi}_{\text{gauge}}}{k^4} \right) = -\frac{3N_V}{32\pi^2}.
$$

(127)

This simply reflects that in the range of large $\tilde{\rho}$ only a reduced number of gauge bosons contributes to the number of “active degrees of freedom” $N$ in eqs (18), (19). The gauge boson contribution to the running of $\tilde{m}^2(\tilde{\rho})$ and $\lambda(\tilde{\rho})$ is suppressed for large $\tilde{\rho}$ by threshold functions

$$
\frac{\partial_\tilde{\rho} \tilde{m}^2(\tilde{\rho})}{\partial_\tilde{\rho} \lambda(\tilde{\rho})} = -\frac{3c_gN_Vg^2}{32\pi^2(1 + c_gg^2\tilde{\rho})} + \ldots,
$$

(128)

that account for the decoupling of heavy degrees of freedom. As a consequence, the scaling solution for the effective potential reaches for $\tilde{\rho} \to \infty$ again a constant value, but with a different number of degrees of freedom $N_\infty$.

Denoting by $N_0$ the number of light degrees of freedom for $\tilde{\rho} \to 0$, and $N_\infty$ the corresponding one for $\tilde{\rho} \to \infty$, one has

$$
N_\infty = N_0 - 3N_V.
$$

(129)
In contrast, the constant value \( u_0 = u(\hat{\rho} \to 0) \) is only indirectly influenced by \( g^2 \neq 0 \) due to nonzero \( \tilde{m}_0 \).

We can formulate this issue more generally. Applying the defining differential equation (23) for the scaling form of the potential to a situation where \( u(\hat{\rho}) \) is approximated by a constant both for \( \hat{\rho} \to 0 \) and for \( \hat{\rho} \to \infty \) we obtain for \( u_0 \) and \( u_\infty = u(\hat{\rho} \to \infty) \)

\[
u_0 = \frac{1}{96\pi^2} C_0, \quad u_\infty = \frac{1}{96\pi^2} C_\infty, \quad (130)
\]

with

\[
C_0 = \frac{5}{1 - v_0} + \frac{1}{1 - v_0/4} + \frac{3 (N_0 - 4)}{4},
\]

\[
C_\infty = \frac{5}{1 - v_\infty} + \frac{1}{1 - v_\infty/4} + \frac{3 (N_\infty - 4)}{4}. \quad (131)
\]

Here \( v_0 = u_0/w_0 \), \( v_\infty = u_\infty/w_\infty \) with \( w_0 \) and \( w_\infty \) the dimensionless coefficients of the curvature scalar for \( \hat{\rho} \to 0 \) and \( \hat{\rho} \to \infty \), respectively. Similarly, \( N_0 \) and \( N_\infty \) denote the number of effective matter degrees of freedom in the two limits. The potential difference

\[
\Delta u_\infty = u_\infty - u_0 = \frac{1}{96\pi^2} (C_\infty - C_0) \quad (132)
\]

is positive if \( C_\infty \) is larger than \( C_0 \). If the scalar field represented by \( \hat{\rho} \) does not couple to fermions one has \( N_\infty < N_0 \) if some of the bosons decouple effectively for \( \hat{\rho} \to \infty \), as in the case of gauge bosons discussed above. If also \( v_\infty < v_0 \) one concludes \( u_\infty < u_0 \). In this case one typically encounters a minimum of the effective potential for \( \hat{\rho} \to \infty \). In contrast, for \( v_\infty > v_0 \) one may find \( u_0 < u_\infty \). This is the case for the examples shown in Figs 6 – 9. The minimum of \( u(\hat{\rho}) \) occurs now for \( \hat{\rho} = 0 \) or for finite \( \hat{\rho} \). For both cases the potential is flat for \( \hat{\rho} \to \infty \).

For a more quantitative understanding of the flow of the potential we may neglect the effect of the scalar mass term \( m^2 \) for a single gauge field, \( \bar{\rho} \).\( \rho \) obeys the differential equation

The scaling solution is defined by

\[
\lambda(\hat{\rho} \to \infty) = \frac{\partial \tilde{m}^2}{\partial \hat{\rho}} (\hat{\rho} \to \infty) = 0. \quad (137)
\]

For finite large \( \hat{\rho} \) we can approximate eq. (135) by

\[
\frac{\partial \tilde{m}^2}{\partial \hat{\rho}} = \frac{3g^2}{16 (6 - A_\infty) \pi^2 \hat{\rho} (1 + g^2 \hat{\rho})^2} \approx \frac{3}{16 (6 - A_\infty) \pi^2 g^2 \hat{\rho}^2}. \quad (138)
\]

Here, we employ \( \tilde{m}^2 \sim \hat{\rho}^{-2} \sim - (\hat{\rho} \partial_{\hat{\rho}} \tilde{m}^2)/2 \). The asymptotic scaling solution for \( \hat{\rho} \to \infty \) is

\[
\tilde{m}^2 = \frac{3}{32 (A_\infty - 6) \pi^2 g^2 \hat{\rho}^2}. \quad (139)
\]

One typically has \( A_\infty \geq 6 \), \( \tilde{m}^2 > 0 \), such that \( u_\infty \) is approached from below.

For the quartic coupling the flow equation reads

\[
\partial_\lambda \lambda = A \lambda + B \tilde{m}^4 + 2\hat{\rho} \frac{\partial \lambda}{\partial \hat{\rho}} + \frac{3g^4}{16\pi^2} (1 + g^2 \hat{\rho})^{-3}, \quad (140)
\]

with

\[
B = \frac{\partial A}{\partial u} = \frac{5}{12\pi^2 w^2} \left( \frac{1}{1 - v^3} + \frac{1}{8 (1 - v/4)^3} \right). \quad (141)
\]

The scaling solution for \( \lambda(\hat{\rho}) \) therefore obeys

\[
2\hat{\rho} \frac{\partial \lambda}{\partial \hat{\rho}} = -A \lambda - B \tilde{m}^4 + \frac{3g^4}{16\pi^2} (1 + g^2 \hat{\rho})^{-3}. \quad (142)
\]

We can interpret the running of the quartic coupling with \( \hat{\rho} \) as a type of gravitational Coleman-Weinberg mechanism. Starting from \( \hat{\rho} \to \infty \) with \( \lambda(\hat{\rho} \to \infty) = 0 \) and lowering \( \hat{\rho} \) the quartic coupling first gets negative. Indeed, with \( \lambda \sim \hat{\rho}^{-3} \), \( \tilde{m}^4 \sim \hat{\rho}^{-4} \) we can neglect the term \( B \tilde{m}^4 \) for large \( \hat{\rho} \) and employ \( \lambda = - (\hat{\rho} \partial_{\hat{\rho}} \lambda)/3 \), such that

\[
\lambda = - \frac{3}{16 (A_\infty - 6) \pi^2 g^2 \hat{\rho}^3}. \quad (143)
\]

This coincides with the \( \hat{\rho} \)-derivative of eq. (139), as it should be. As \( \hat{\rho} \) decreases, \( \lambda(\hat{\rho}) \) first gets more and more negative, such that \( \tilde{m}^2 \) increases to larger positive values. As \( A(\hat{\rho}) \) decreases for decreasing \( \hat{\rho} \), the influence of the first positive term \( -A \lambda \) in eq. (142) becomes less important and \( \lambda(\hat{\rho}) \) starts to increase due to the other negative terms. Once \( \lambda \) becomes positive, \( \tilde{m}^2(\hat{\rho}) \) starts to decrease.
until it reaches zero at some local minimum of the potential. For a rough qualitative estimate we replace \( A(\rho) \) in eqs (139), (143). The minimum occurs in a region where \( A(\rho_{\text{min}}) < 6 \), with positive \( \lambda(\rho_{\text{min}}) \). This qualitative behavior is well visible by taking the \( \rho \)-derivative of \( \tilde{m}^2(\rho) \) in Fig. 8 or the second \( \rho \)-derivative of \( u(\rho) \) in Figs 6, 7. The upper curve in Fig. 9 shows that the appearance of a minimum of \( u(\rho) \) is not the only possibility.

The range of minimum values \( \rho_{\text{min}} \) is restricted, as seen from the lowest curve in Fig. 9 which corresponds to a boundary curve for this type of scaling solutions. The question arises if \( \rho_{\text{min}} \) can be arbitrarily small. For \( \rho_{\text{min}} \to 0 \), one has \( \tilde{m}^2_{0,\ast} = 0 \). From eq. (124) we conclude that this requires negative \( \lambda_0 \),

\[
\lambda_0 = -c_y \tilde{N}_V y^2. \tag{144}
\]

At least for small \( g^2 \) and not too large \( \eta_{0,\ast}^{(3)} \) there seems to be a discrepancy with eq. (126). More generally, it is not clear if all solutions shown in Figs 6–9 can be extended to \( \rho \to 0 \) once the effects of nonzero \( \tilde{m}^2 \) and \( \lambda \) in eq. (60) are included.

7. Yukawa couplings

A fermion with a Yukawa coupling \( y \) to a scalar field \( \phi \) acquires a mass \( m = y \phi \). Similar to massive gauge bosons, the mass suppresses its contribution to the flow of \( u \). This induces a modification of the flow generator

\[
\frac{\Delta \tilde{p}_f}{k^4} = -\frac{1}{16\pi^2} \sum_{j=1}^{N_F} \left( \frac{1}{1 + w_j^2} - 1 \right). \tag{145}
\]

Here \( j \) labels the mass eigenstates with mass \( m_j \), and

\[
w_j = \frac{m_j^2}{k^2} = y^2 a_j^2(F) (\phi_0) / k^2 \tag{146}
\]

is the dimensionless squared mass of the fermion \( j \). The sum is over Majorana fermions, with Dirac fermions counting as two Majorana fermions with equal \( m_j^2 \). The minus sign reflects Fermi statistics and is the main difference as compared to the gauge boson contribution. In case of several independent Yukawa couplings the mass eigenvalues \( m_j \) depend on a linear combination of Yukawa couplings and fields.

We may again investigate a simple scenario where \( \tilde{N}_F \) fermions have an equal mass,

\[
m_j^2 = c_f y^2 \rho. \tag{147}
\]

Similar to eq. (119) this results in

\[
\frac{\Delta \tilde{p}_f}{k^4} = \frac{c_f \tilde{N}_F y^2 \rho}{16\pi^2 (1 + c_f y^2 \rho)}. \tag{148}
\]

We can therefore take over the computations for gauge couplings with the replacements

\[
c_y^2 g^2 \to c_f y^2, \quad \tilde{N}_V \to -\frac{2}{3} \tilde{N}_F. \tag{149}
\]

The essential new feature is the overall change of sign of the fermion contribution as compared to the gauge boson contribution. Combining with the gravitational contribution and contributions from other massless fields one obtains

\[
\partial_t u = -4u + 2\rho \frac{\partial u}{\partial \rho} + 4c_V + 4c_{V,f}, \tag{150}
\]

with

\[
c_{V,f} = \frac{\Delta \tilde{p}_f}{4k^4} = -\tilde{N}_F \frac{1}{64\pi^2} \left( \frac{1}{1 + y^2 \rho} - 1 \right) + \frac{\tilde{N}_F y^2 \rho}{64\pi^2 (1 + y^2 \rho)}. \tag{151}
\]

Here we take \( c_f = 1 \). We observe that \( c_{V,f} \) adds a positive contribution to \( c_V \).

The flow of the mass term \( m^2 = \partial u/\partial \rho \) is found by taking a \( \rho \)-derivative of eq. (150),

\[
\partial_t m^2 = -2m^2 + 2\rho \frac{\partial m^2}{\partial \rho} + \tilde{m}^2 + \frac{N_F y^2}{16\pi^2 (1 + y^2 \rho)} \tag{152}
\]

where we have omitted contributions from \( \Delta \tilde{p}_f \) and possible contributions from \( \Delta \tilde{p}_{gauge} \). If \( \lambda(\rho) = \partial \tilde{m}^2 / \partial \rho \) remains finite for \( \rho \to 0 \) the flow of the scalar mass term at the origin reads

\[
\partial_t \tilde{m}^2_0 = (A_0 - 2) \tilde{m}^2_0 + \frac{\tilde{N}_F y^2}{16\pi^2}. \tag{153}
\]

For \( A_0 < 2 \) the fixed point occurs for positive \( \tilde{m}^2_{0,\ast} \), such that the origin is a local minimum for the scaling solution

\[
\tilde{m}^2_{0,\ast} = \frac{\tilde{N}_F y^2}{16\pi^2 (2 - A_0)}. \tag{154}
\]

In contrast, for \( A_0 > 2 \) one has \( \tilde{m}^2_{0,\ast} < 0 \) and a local maximum at the origin for the scaling solution. (This holds provided the term \( \sim \lambda_0 \) from \( \Delta \tilde{p}_f \) is small.)

The flow of the quartic coupling \( \lambda(\rho) \) is obtained by taking a further \( \rho \)-derivative of eq. (152)

\[
\partial_t \lambda = A \lambda + 2\rho \frac{\partial \lambda}{\partial \rho} - \frac{\tilde{N}_F y^4}{8\pi^2 (1 + y^2 \rho)} + B \tilde{m}^4. \tag{155}
\]

For \( \rho \to 0 \) one recognizes the well known negative contribution from the Yukawa coupling \( \sim y^4 \). The fixed point for \( \lambda_0 = \lambda(\rho = 0) \) occurs for

\[
\lambda_{0,\ast} = \frac{\tilde{N}_F y^4}{8\pi^2 A_0} - \frac{B_0}{A_0} \tilde{m}^2_{0,\ast} = \frac{\tilde{N}_F y^4}{8\pi^2 A_0} \left( 1 - \frac{\tilde{N}_F B_0}{32\pi^2 (2 - A_0)^2} \right). \tag{156}
\]

The relative size of the second contribution \( \sim B_0 \) is typically small, such that \( \lambda_{0,\ast} > 0 \).

For the flow away from the fixed point we can keep \( \lambda \) close to the fixed point value

\[
\lambda_{0,\ast} \approx \frac{\tilde{N}_F y^4}{8\pi^2 A_0}, \tag{157}
\]
since it is an irrelevant parameter. For $A_0 < 2$ the mass term is relevant, however, and the deviation from the critical surface $\gamma = \tilde{m}_0^2 - \tilde{m}_0^2$ increases according to eq. (83). Indeed, the last term in eq. (153) shifts the value of $\tilde{m}_0^2$, but does not contribute to the flow of $\delta u$. The only difference to the discussion in sect. 5.3 are the different fixed point values for $\tilde{m}_0^2$ and $\lambda_{0,*}$. Omitting the irrelevant coupling $\delta \lambda$ matrix for $\alpha_i = (\delta u, \gamma)$ reads, cf. eq. (93),

$$- T = \left( \begin{array}{ccc} A - 4 & - d \\ B \tilde{m}_0^2 & A - 2 + 3d \lambda_{0,*} \end{array} \right).$$

Neglecting the small off-diagonal elements we can follow the discussion of eqs (85) – (87), resulting in

$$\tilde{m}^2(k) = \frac{\tilde{N}_F y^2 k^2}{16\pi^2 (2 - A_0)} + \gamma_\Lambda \Lambda^2 e^{k^2 - \theta_\gamma},$$

with

$$\theta_\gamma = 2 - A_0 - \frac{3\tilde{N}_F y^4}{(16\pi^2)^2 A_0 (1 + \tilde{m}_0^2)^2}.\quad (159)$$

The last term in eq. (160) is small and may be neglected, such that

$$m^2(k) = \frac{\tilde{N}_F y^2 k^2}{16\pi^2 (2 - A_0)} + \gamma_\Lambda \Lambda^2 - A_0 k A_0.$$  

(161)

The scaling solution with constant $A_0$ holds only as long as the gravitational fluctuations are effective. The running coupling $w_0(k)$ corresponds to a relevant parameter, with qualitative behavior

$$w_0(k) = w_{0,*} + \frac{M^2}{2k^2},$$

where $M$ is the observed reduced Planck mass. Once $k^2$ drops below $k^2_e = M^2/(2w_{0,*})$, the function $w_0(k)$ starts to increase rapidly. As a consequence, $A_0(k)$ decreases rapidly to zero for $k \ll k_e$. For $k^2 \ll k^2_e$ one has approximately

$$m_0^2(k) = m_0^2(k_e) - \frac{\tilde{N}_F y^2 (k^2 - k^2_e)}{32\pi^2}.$$\phantom{163}

(163)

For a suitable value of $\gamma_\Lambda$ one can obtain $m_0(k_e) = \tilde{N}_F y^2 k^2_e/(32\pi^2)$, such that $m_0^2(k = 0) = 0$. More generally, by suitable initial conditions for the relevant parameter $\gamma$ one can achieve arbitrary values of $m_0^2(k = 0)$. In turn, one can realize an arbitrary value of spontaneous symmetry breaking, e.g. an arbitrary location of the minimum of $U$ at $k = 0$,

$$\rho_0(k) = - \frac{m_0^2(k)}{\lambda_0(k)}.$$\phantom{164}

(164)

If $\rho_0(0) \neq 0$ is associated with a spontaneous breaking of a symmetry, the scale of this symmetry breaking is a free parameter. What is predicted, however, is the value of the quartic coupling, since it is associated to an irrelevant parameter. With “initial values”

$$\lambda_0(k_e) = \lambda_{0,*}, \quad y(k_e) = y,$$ \phantom{165}

one can follow for $k < k_e$ the “low energy flow” of $\lambda$ and $y$ to $k = 0$. The gravitational degrees of freedom do not contribute to the low energy flow.

So far we only have discussed the vicinity of $\tilde{\rho} = 0$. As long as the minimum $\rho_0(k)/k^2$ stays small, this is a reasonable local approximation. In order to be sure that no other minimum of $U$ occurs for large $\tilde{\rho}$ one needs the global solution for the whole range of $\tilde{\rho}$. We have numerically solved the differential equation for the scaling solution

$$2\tilde{\rho} \frac{\partial u}{\partial \tilde{\rho}} = 4(u - c_{V} - c_{V,f}).$$ \phantom{166}

(166)

The result is shown in Fig. 10 for different values of $w_0$. The only minimum occurs for $\tilde{\rho} = 0$. The mass term $\tilde{m}^2(\tilde{\rho})$ is positive and small for the whole range of $\tilde{\rho}$, as can be seen from Fig. 11. This justifies the omission of $c_{V,s}$ for the whole range of $\tilde{\rho}$, in contrast to the situation with vanishing Yukawa coupling. The qualitative situation does not depend sensitively on the initial conditions, as demonstrated in Fig. 12. It seems that the differential equation (166) admits a whole family of scaling solutions extended over the whole range of $\tilde{\rho}$. In view of the small value of $\tilde{m}^2$ we do not expect that this changes if $c_{V,s}$ is included.

8. Nonminimal gravitational coupling

The effective Planck mass may depend on the scalar field due to a nonminimal coupling $\xi$,

$$\mathcal{L} = - \frac{1}{2} \sqrt{g} \xi \rho R,$$ \phantom{187}

(167)
with $R$ the curvature scalar and $\rho$ a suitable scalar bilinear, as $\rho = h^2/h$ for the Higgs doublet. As a consequence, one has a field-dependent shift in the squared Planck mass $M^2 \to M^2 + \xi \rho$, or an additional field dependence in the dimensionless quantity $w(\tilde{\rho})$,

$$w(\tilde{\rho}) = w_0 + \frac{\xi}{2} \tilde{\rho}. \quad (168)$$

Here $w_0$ is the dimensionless squared Planck mass discussed in the previous section. We assume in this section that both $w_0$ and $\xi$ are given by $k$-independent fixed point values and take them as undetermined parameters. (These quantities may also depend on a further scalar singlet field $\chi$.) The nonminimal coupling $\xi$ does not affect the contribution from fermions and gauge bosons. Its main effect is a modification of the contribution from the metric fluctuations by replacing in eq. (17)

$$v = \frac{u(\tilde{\rho})}{w(\tilde{\rho})} = \frac{u(\tilde{\rho})}{w_0 + \xi \tilde{\rho}/2}. \quad (169)$$

The coupling $\xi$ further influences the mixing between the physical scalar fluctuations in the metric and other scalars. We neglect this mixing in the present paper such that $\xi$ does not change the flow contribution from scalar fields. Then the replacement (169) is the only modification for nonzero $\xi$. Similar to our treatment of $w$ before, we do not compute here the flow equation for $w(\tilde{\rho})$.

The $\tilde{\rho}$-dependence of $v(\tilde{\rho})$ obeys [7]

$$\frac{\partial v}{\partial \tilde{\rho}} = \frac{\tilde{m}^2}{w} - \frac{\xi v}{2w}. \quad (170)$$

As a result, the flow equation for $\tilde{m}^2 = \partial u/\partial \tilde{\rho}$ becomes

$$\partial_t \tilde{m}^2 = 2\tilde{\rho} \frac{\partial \tilde{m}^2}{\partial \tilde{\rho}} + (A - 2)\tilde{m}^2 - \frac{\xi A v^2}{2} + \ldots \quad (171)$$

where $A$ is given by eq. (33) with $v(\tilde{\rho})$ and $w(\tilde{\rho})$. The dots denote contributions from scalars, fermions and gauge bosons that are not modified for $\xi \neq 0$. In particular, one finds for $\tilde{m}^2_0 = \tilde{m}^2(\tilde{\rho} = 0)$

$$\partial_t \tilde{m}^2_0 = (A - 2) \tilde{m}^2_0 - \frac{\xi A_0 v_0^2}{2} - \frac{(N'_s + 2) \lambda_0}{32\mu^2 (1 + \tilde{m}^2_0)^2} - \frac{3N_F g^2}{32\pi^2} + \frac{\bar{N}_F y^2}{16\pi^2}. \quad (172)$$

where we have taken $N'_s$ scalars with $O(N'_s)$-symmetric potential and assumed that $\lambda_0$ and $\tilde{\rho} \partial \lambda/\partial \tilde{\rho}$ remain finite for $\tilde{\rho} \to 0$, as well as $c_q = 1$, $c_f = 1$. For $\xi = 0$ a flat potential ($\tilde{m}^2_0 = \lambda_0 = 0$) is no longer a scaling solution even for vanishing gauge and Yukawa couplings $g^2 = y^2 = 0$.

In Fig. 13 we show a numerical scaling solution of eq. (23), with $w(\tilde{\rho})$ given by eq. (168) and $g^2 = y^2 = 0$. We take $\xi = 0.1$ and plot four different values of $w_0 = 0.038, 0.042, 0.05, 0.06$. The corresponding values of $A_0$ are $A_0 = 2.74, 1.64, 1.0, 0.68$, such that the highest curve corresponds to $A_0 > 2$. For all curves $u(\tilde{\rho})$ reaches a constant for $\tilde{\rho} \to 0$ and increases as $w(\tilde{\rho}) \sim \xi \tilde{\rho}/2$ for $\tilde{\rho} \to \infty$.

A minimum at $\tilde{\rho}_0 \neq 0$ is clearly visible. The precise value of the minimum depends on the initial condition for the first order differential equation that we choose for Fig. 13 as $u(\tilde{\rho} = 1)$. We find indeed a whole family of scaling solutions that may be parameterized by $u(\tilde{\rho} = 1)$. As discussed in sect. 5 it is so far not known if all scaling solutions can consistently be continued to $\tilde{\rho} = 0$ once the contribution from scalar fluctuations is included.

We plot in Fig. 14 the value $v(\tilde{\rho}) = u(\tilde{\rho})/w(\tilde{\rho})$ for the same set of parameters. All curves approach for $\tilde{\rho} \to \infty$ the asymptotic behavior $v(\tilde{\rho} \to \infty) \to 1$. This is consistent with the graviton barrier discussed in ref. [6, 16].

Correspondingly, the increase of $u(\tilde{\rho})$ for $\tilde{\rho} \to \infty$ is bounded to be linear in $\tilde{\rho}$. This can be seen in Fig. 15 for $\tilde{m}^2(\tilde{\rho}) = \partial_x u(\tilde{\rho})$. For large $\tilde{\rho}$ one finds the asymptotic value $\tilde{m}^2(\tilde{\rho} \to \infty) = \tilde{\xi}/2$. In the next two sections we will
For a restricted range of \( \tilde{\rho} \), the discussion in sects. 3-7 can be relevant only for a restricted range of \( \tilde{\rho} \), i.e.

\[
\tilde{\rho} < \frac{2w_0}{\xi}.
\]  

For small enough \( \xi \) this range may be rather large. What we have called the asymptotic behavior in sects. 3-7 becomes in the presence of a small non-minimal coupling the range of large \( \tilde{\rho} \), that still obeys eq. (173).

In this section we extend the truncation to two functions \( u(\tilde{\rho}) \) and \( w(\tilde{\rho}) \). For vanishing gauge and Yukawa couplings we recover the constant scaling solution which corresponds to the extension of the Reuter fixed point [2, 73–75] of pure gravity to the presence of matter [4]. Our investigation is based on the gauge invariant flow equation [5] and the flow for \( u(\tilde{\rho}) \) and \( w(\tilde{\rho}) \) equations derived in ref. [8]. We discuss the vicinity of the constant scaling solutions as well as the behavior for large \( \tilde{\rho} \) and possible crossover scaling solutions. We include the case of nonzero gauge and Yukawa couplings.

9. Scaling solution with field-dependent Planck mass

In this section we extend the truncation to two functions \( u(\tilde{\rho}) \) and \( w(\tilde{\rho}) \). For vanishing gauge and Yukawa couplings we recover the constant scaling solution which corresponds to the extension of the Reuter fixed point [2, 73–75] of pure gravity to the presence of matter [4]. Our investigation is based on the gauge invariant flow equation [5] and the flow for \( u(\tilde{\rho}) \) and \( w(\tilde{\rho}) \) equations derived in ref. [8]. We discuss the vicinity of the constant scaling solutions as well as the behavior for large \( \tilde{\rho} \) and possible crossover scaling solutions. We include the case of nonzero gauge and Yukawa couplings.

9.1. Flowing Planck mass

So far we have made an ansatz for the function \( F(\tilde{\rho}) \), or the associated dimensionless quantity \( w(\tilde{\rho}) = F/(2k^2) \). In this section we investigate the combined system of flow equations for \( u(\tilde{\rho}) \) and \( w(\tilde{\rho}) \). For the truncation (6) with \( Z_A = 1 \) the flow equations have been computed from the gauge invariant flow equation [8],

\[
\partial_t u = 2\tilde{\rho} \partial_{\tilde{\rho}} u - 4u + \frac{5}{24\pi^2} \left( 1 - \frac{u}{w} \right)^{-1} + \frac{\tilde{N}_U}{32\pi^2},
\]

and

\[
\partial_t w = 2\tilde{\rho} \partial_{\tilde{\rho}} w - 2w + \frac{25}{64\pi^2} \left( 1 - \frac{u}{w} \right)^{-1} + \frac{\tilde{N}_M}{96\pi^2},
\]

with

\[
\tilde{N}_U = N_S + 2N_V - 2N_F - \frac{8}{3},
\]

\[
\tilde{N}_M = -N_S + 4N_V - N_F + \frac{43}{6} - \frac{3\xi}{2} N_\xi.
\]

The result employs the Litim cutoff and simplifies the subdominant sector of scalar metric fluctuations by neglecting the mixing with other scalar fields and omitting a
factor \((1 - u/(4w))^{-1}\). For vanishing gauge and Yukawa couplings and massless fields, one has constant \(N_s\), \(N_V\) and \(N_F\), which count the number of real scalars, gauge bosons and Weyl fermions, respectively. For \(g\) or \(y\) different from zero one has effective \(\tilde{\rho}\)-dependent particle numbers that obtain by multiplication with “threshold functions” \((1 + \tilde{m}_i^2)^{-1}\). The field-dependent mass terms \(\tilde{m}_i^2\) are of the type \(\tilde{m}_i^2 = g_i^2\tilde{\rho}\) for gauge bosons, \(\tilde{m}_i^2 = g_i^2\tilde{\rho}\) for fermions and \(\tilde{m}_i^2 = u' + 2\tilde{\rho}u''\) or \(\tilde{m}_i^2 = u'\) for scalars in the radial or Goldstone directions. Different species may have different effective couplings. The function \(\xi\) is defined by

\[
\xi = \frac{\partial^2 F}{\partial \phi^2}, \tag{177}
\]

and may again be different for different scalar fields. It generalizes the nonminimal gravitational coupling \(\xi\) of sect. 8, with \(N_\xi\) the number of states with this coupling. For a Taylor expansion in small \(\tilde{\rho} = \varphi^2/(2k^2)\) it reads

\[
\tilde{\xi} = \xi + 4w_2\tilde{\rho}, \quad w_2 = \frac{\partial^2 w}{\partial \tilde{\rho}^2}. \tag{178}
\]

In case of \(O(N)\)-symmetry, \(\tilde{\rho} = \varphi_u\varphi_a/(2k^2)\), there are in addition \(N-1\)-contributions from the Goldstone directions,

\[
\tilde{\xi} = N\xi + 4w_2\tilde{\rho}, \tag{179}
\]

such that for the Higgs-doublet with \(N=4\) one has

\[
N_\xi\tilde{\xi} = 4(\xi + \tilde{\rho}w_2). \tag{180}
\]

### 9.2. Scaling solutions

For a given model the system of equations (174), (175) is closed. We are interested in the scaling solution with \(\xi\). We conclude that out of the two constant scaling solutions for \(u\), only one is compatible with a simultaneous scaling solution

\[
u_0 = \frac{u(0)}{w(0)}. \tag{186}
\]

### 9.3. Constant scaling solution

For vanishing gauge and Yukawa couplings, \(g = y = 0\), the system of differential equations (182), (183) admits a constant scaling solution, \(u'(\tilde{\rho}) = 0\), \(w'(\tilde{\rho}) = 0\), which has been discussed extensively in ref. [8]. It is given by

\[
\begin{align*}
v_* &= 1 - \frac{1}{4N_M} \left(b + \sqrt{b^2 + 440N_M}\right), \tag{187}
\end{align*}
\]

\[
\begin{align*}
b &= 2N_U - 3N_V - 75, \\
N_{U,*} &= N_s + 2N_V - 2N_F - \frac{8}{3}, \\
N_{M,*} &= -N_s + 4N_V - N_F + \frac{43}{6}. \tag{188}
\end{align*}
\]

The corresponding dimensionless potential and squared Planck mass are independent of \(\tilde{\rho}\),

\[
\begin{align*}
u_* &= \frac{1}{128\pi^2} \left(\tilde{N}_{U,*} + \frac{20}{3(1 - \nu_*)}\right), \\
w_* &= \frac{1}{192\pi^2} \left(\tilde{N}_{M,*} + \frac{75}{2(1 - \nu_*)}\right). \tag{189}
\end{align*}
\]

A second constant scaling solution exists only in a small regime of \(\tilde{N}_U\) and \(\tilde{N}_M\) and will not be discussed here explicitly.

For the major part of the model space \((\tilde{N}_U, \tilde{N}_M)\) we conclude that out of the two constant scaling solutions for \(u\) for a fixed constant \(w\), that we have discussed in sect. 3.2, only one is compatible with a simultaneous scaling solution for \(w\). It corresponds to the extended Reuter fixed point [4]. As a consequence, the crossover scaling solution for \(u_*(\tilde{\rho})\) discussed in sect. 3.3 is not a valid scaling solution for the whole system. It could only be an approximation for a region of a scaling solution in which \(w_*(\tilde{\rho})\) does not
change much with \( \tilde{\rho} \). Generic crossover scaling solutions can still exist for ranges in the field content and parameters for which two different constant scaling solutions exist. We learn that even very rough features of scaling solutions as the number of possible solutions can depend on the truncation in an important way.

9.4. Scaling solutions close to a constant scaling solution

We next address the question if the constant scaling solution is isolated or part of a continuous family of scaling solutions. For this purpose we discuss first possible scaling solutions in the vicinity of the constant scaling solution. In the vicinity of the constant scaling solution we may linearize the differential equations (181) for small

\[
\Delta u(\tilde{\rho}) = u(\tilde{\rho}) - u^* \quad \Delta w(\tilde{\rho}) = w(\tilde{\rho}) - w^*,
\]

with \( u^*, w^* \) the constant scaling solutions according to eq. (189). The linearized equations read

\[
2\tilde{\rho}\partial_{\tilde{\rho}} \Delta u = 4\Delta u - 4\Delta c_v \quad 2\tilde{\rho}\partial_{\tilde{\rho}} \Delta w = 2\Delta w - 2\Delta c_M,
\]

with

\[
\Delta c_v = c_v(u^* + \Delta u, w^* + \Delta w) - c_v(u^*, w^*),
\]

\[
\Delta c_M = c_M(u + \Delta u, w^* + \Delta w) - c_M(u^*, w^*).
\]

For \( g^2 = \tilde{g}^2 = 0 \) one finds

\[
\Delta c_v = \frac{5}{96\pi^2} \hat{\Delta} - \frac{N_S}{128\pi^2} \Delta u',
\]

\[
\Delta c_M = \frac{25}{128\pi^2} \hat{\Delta} + \frac{N_S}{192\pi^2} \Delta u' - \frac{N_S}{64\pi^2} \Delta w',
\]

where

\[
\hat{\Delta} = \frac{1}{1 - v} - \frac{1}{1 - v^*} = \frac{1}{(1 - v^*)^2} \left( \frac{w^*}{w^2} \right)^2 \Delta u - \Delta w.
\]

Eq. (193) is a coupled system of two linear differential equations,

\[
\left( \frac{2\tilde{\rho} - N_S}{32\pi^2} \right) \Delta u' = 4\Delta u - 5 \frac{2\pi^2}{24\pi^2} \hat{\Delta},
\]

\[
\left( \frac{2\tilde{\rho} - N_S}{32\pi^2} \right) \Delta w' = \frac{N_S}{96\pi^2} \Delta u' - 2\Delta w - 25 \frac{2\pi^2}{64\pi^2} \hat{\Delta}.
\]

We can write \( \hat{\Delta} \) in terms of the graviton-induced anomalous dimension \( A \) for which the second term in eq. (33) is neglected and \( v, w \) taken as \( v^*, w^* \),

\[
A = \frac{5}{24\pi^2 w^* (1 - v^* )^2},
\]

namely

\[
\frac{5}{24\pi^2} \hat{\Delta} = A (\Delta u - v^* \Delta w). \tag{197}
\]

This yields

\[
\left( \frac{2\tilde{\rho} - N_S}{32\pi^2} \right) \Delta u' = (4 - A) \Delta u + A v^* \Delta w,
\]

\[
\left( \frac{2\tilde{\rho} - N_S}{32\pi^2} \right) \Delta w' + \frac{N_S}{96\pi^2} \Delta u' = 2\Delta w + 15A \frac{2\pi^2}{8} (\Delta u - v^* \Delta w), \tag{198}
\]

to be compared with eq. (66) for \( \lambda = 0 \).

We may discuss separately three characteristic regions in \( \tilde{\rho} \). For \( \tilde{\rho} \ll N_S/(64\pi^2) \) one has the approximate equations

\[
\Delta y' = -M_0 \Delta y, \quad \Delta y = \left( \frac{\Delta u}{\Delta w} \right), \tag{199}
\]

with

\[
M_0 = \frac{32\pi^2}{N_S} \left( \frac{4 - A}{A v^*} \frac{4}{A v^*} \frac{A v^*}{4} \right).
\]

The two eigenvalues \( \lambda_\pm \) of \( M_0 \) are the solutions of the quadratic equation

\[
\lambda^2 - \lambda \left( 6 - A \left( 1 + \frac{37}{24} v^* \right) \right) + 8 - A \left( 2 + \frac{15}{2} v^* \right) = 0. \tag{201}
\]

For \( A = 0 \) one finds two possible eigenvalues \( \lambda_+ = 4, \lambda_- = 2 \). With respect to the flow with increasing \( \tilde{\rho} \) both \( \Delta u \) and \( \Delta w \) are relevant parameters. This extends to a range \( A > 0 \) with eigenvalues depending continuously on \( A \). The solution approaches for \( \tilde{\rho} \to 0 \) constant values, \( \Delta u(\tilde{\rho} = 0) = \Delta u_0, \Delta w(\tilde{\rho} = 0) = \Delta w_0 \). They are related to the derivatives by eqs (184), (185). This behavior is similar to the discussion in sect. 5.2.

A second region for \( \tilde{\rho} \gg N_S/(64\pi^2) \) obeys the approximate equations

\[
\tilde{\rho} \partial_{\tilde{\rho}} \Delta y = M_\infty \Delta y \tag{202}
\]

with

\[
M_\infty = \frac{1}{2} \left( 4 - A \frac{15A/8}{2 - \frac{15A}{8} v^*} \right). \tag{203}
\]

If the largest eigenvalue \( \lambda_+ \) of \( M_\infty \) is positive, the solution diverges for \( \tilde{\rho} \to \infty \) as

\[
\Delta y_\ast = c_+ \tilde{\rho}^{\lambda_+} + c_- \tilde{\rho}^{\lambda_-}. \tag{204}
\]

with \( c_\pm \) eigenvectors of \( \lambda_\pm \) specifying the particular solutions. We conclude that the constant scaling solution is “unstable” in the sense that generic local solutions do not approach the scaling solutions for \( \tilde{\rho} \to \infty \). If a whole family of scaling solutions exists, the constant scaling does not correspond to the generic solution of this family.

The third region is the “transition region” for \( \tilde{\rho} \approx N_S/(64\pi^2) \). One expects that the neglection of second derivatives \( u'' \) and \( w'' \) is no longer justified. The situation is similar to the discussion in sect. 5.5. If we continue to neglect \( u'' \) and \( w'' \) the scaling solution can cross the transition region only at the price of strong variations of \( u' \) and \( w' \). In the close vicinity of \( \tilde{\rho} = N_S/(64\pi^2) \) eq. (198) is approximated by

\[
\Delta w = -\frac{4 - A}{A v^*} \Delta u, \tag{205}
\]

and

\[
\Delta u' = \frac{48\pi^2}{N_S} \left( 15 - \frac{4(4 - A)}{A v^*} \right) \Delta u. \tag{206}
\]
We could specify the initial conditions for the solution of the linear differential equations by specifying \( \Delta u \) and \( \Delta w \) at \( \hat{\rho} = 1/(64\pi^2) \). The condition (17) implies that only a one-parameter family of scaling solutions in the close vicinity of the constant scaling solutions is possible.

For any smooth scaling solution we can write

\[
\Delta u = a \Delta w, \quad \Delta u' = a \Delta w' + a' \Delta w, \tag{207}
\]

with \( a(\rho) \) a function without very rapid variation with \( \hat{\rho} \).

The second equation (198) becomes

\[
\left[ 2\hat{\rho} - \frac{N_S}{32\pi^2} \left( 1 - \frac{a}{3} \right) \right] \Delta w' = B \Delta w, \tag{208}
\]

with

\[
B = 2 + \frac{15 A}{8} (a - v) - \frac{N_S}{96\pi^2} f a, \quad f = \frac{a'}{a}. \tag{209}
\]

The coefficient of \( \Delta w' \) vanishes for

\[
\hat{\rho}_w = \frac{N_S}{64\pi^2} \left( 1 - \frac{a}{3} \right), \tag{210}
\]

which is for \( a < 0 \) somewhat smaller than the location of the vanishing coefficient of \( \Delta u' \) in the first equation (198) at \( \hat{\rho}_u = N_S/(64\pi^2) \). This is compatible with a smooth behavior only if \( B(\hat{\rho}_w) = 0 \). For a smooth solution the term \( f \) is typically small as compared to other terms in \( B \). One infers different values of \( a \) at \( \hat{\rho}_w \) and \( \hat{\rho}_u \),

\[
a(\hat{\rho}_w) = v_\ast - \frac{16}{15 A}, \quad a(\hat{\rho}_u) = -\frac{4 A v_\ast}{4 - A}. \tag{211}
\]

In turn, the first equation (198) yields at \( \hat{\rho}_w \)

\[
\Delta u'(\hat{\rho}_w) = -\frac{96\pi^2}{N_S} \left( \frac{4 - A + \frac{4 A v_\ast}{a(\hat{\rho}_w)} \right) \Delta w \\
\approx -\frac{384\pi^2}{N_S} \left( 1 + \frac{4 A}{15 A v_\ast - 16} \right) \Delta w. \tag{212}
\]

The value \( \Delta u'/\Delta w \) becomes typically very large at \( \hat{\rho}_w \), contradicting the assumption of slow variation of the scaling solution.

Numerical solutions in the transition region are indeed characterized by strong variations of \( \Delta u' \) and \( \Delta w' \) in the transition region. Those solutions typically diverge outside the transition region before \( \hat{\rho} = 0 \) and \( \hat{\rho} \to \infty \) are reached, such that the linear approximation does not remain valid. Similarly to sect. 5.5 we remain with two possibilities. Either the inclusion of \( \Delta u'' \) and \( \Delta w'' \) changes the variations such that a family of slowly varying scaling solutions exists in the vicinity of the constant scaling solution. Or only a discrete number of scaling solutions remains close to the scaling solution over the whole range of \( \hat{\rho} \). It could turn out that the constant scaling solution is the only possible scaling solution for \( g^2 = y^2 = 0 \).

A possible scenario for a continuous family of solutions could be that all solutions that are close to the constant scaling solution for \( \hat{\rho} \to 0 \) always deviate substantially from the scaling solution for large enough \( \hat{\rho} \), such that the linear approximation no longer holds. This happens, in particular, if the asymptotic behavior of \( w(\hat{\rho} \to \infty) \) is not a constant, but rather involves the non-minimal coupling \( \xi \).

We will discuss solutions of this type below and in the next section.

### 9.5. Asymptotic scaling for large fields

An interesting class of scaling solutions for large \( \hat{\rho} \) takes the form

\[
u(\hat{\rho}) = u_\infty + \frac{u^{(1)}}{\hat{\rho}} + \frac{u^{(2)}}{\hat{\rho}^2} + \ldots
\]

\[
w(\hat{\rho}) = \frac{1}{2} \xi_\infty \hat{\rho} + w^{(0)} + \frac{w^{(1)}}{\hat{\rho}} + \frac{w^{(2)}}{\hat{\rho}^2} + \ldots, \tag{213}
\]

where dots denote higher-order terms in an expansion in \( \hat{\rho}^{-1} \). This is the type of scaling solution that has been investigated for dilaton quantum gravity [17, 18]. We expand

\[
c_V = c_{V,\infty} + \frac{c_V^{(1)}}{\hat{\rho}} + \frac{c_V^{(2)}}{\hat{\rho}^2} + \ldots
\]

\[
c_M = c_{M,\infty} + \frac{c_M^{(1)}}{\hat{\rho}} + \frac{c_M^{(2)}}{\hat{\rho}^2} + \ldots, \tag{214}
\]

such that the differential equation (181) for the scaling solution takes the form

\[
2(u_\infty - c_{V,\infty}) + \frac{3u^{(1)} - 2c_V^{(1)}}{\hat{\rho}} + \frac{4u^{(2)} - 2c_V^{(2)}}{\hat{\rho}^2} + \ldots = 0, \tag{215}
\]

and

\[
w^{(0)} - c_{M,\infty} + \frac{2w^{(1)} - c_M^{(1)}}{\hat{\rho}} + \frac{2w^{(2)} - c_M^{(2)}}{\hat{\rho}^2} + \ldots = 0. \tag{216}
\]

The solution expresses the coefficients \( u^{(i)}, w^{(i)} \) in terms of \( c_V^{(i)}, c_M^{(i)} \), with \( \xi_\infty \) a free integration constant. We therefore have a one-parameter family of asymptotic scaling solutions, parameterized by \( \xi_\infty \).

For \( g = y = 0 \) one has

\[
c_V = 5 \frac{1}{96\pi^2(1-v)} + \frac{1}{128\pi^2} \left[ \tilde{N}_U + N_S \left( \frac{1}{1 + u'} - 1 \right) \right],
\]

\[
c_M = \frac{25}{128\pi^2(1-v)}
\]

\[
+ \frac{1}{192\pi^2} \left[ \tilde{N}_{M,\ast} - N_S \left( \frac{1}{1 + u'} - 1 \right) - \frac{3N_S w}{(1 + u')^2} \right]. \tag{217}
\]

With

\[
(1 - v)^{-1} = 1 + \frac{2u_\infty}{\xi_\infty \hat{\rho}} + \frac{2}{\xi_\infty \hat{\rho}^2} \left( u^{(1)} + \frac{2u_\infty}{\xi_\infty} (u_\infty - w^{(0)}) \right) , \tag{218}
\]
one obtains
\[ u_\infty = c V_\infty = \frac{5}{96\pi^2} + \frac{\dot{N}_{U,\ast}}{128\pi^2}, \]
\[ w^{(0)} = c M_\infty = -\frac{25}{128\pi^2} + \frac{1}{192\pi^2} \left( N_{M,\ast} - \frac{3 N_S \xi_\infty}{2} \right). \]

In the next order one finds
\[ u^{(1)} = \frac{2}{3} c V^{(1)} = \frac{5 u_\infty}{12\pi^2} \xi_\infty, \]
\[ w^{(1)} = \frac{1}{3} c M^{(1)} = \frac{25 u_\infty}{128\pi^2} \xi_\infty. \]

Continuation to the terms \( \sim \tilde{\rho}^{-2} \) yields
\[ u^{(2)} = \frac{1}{2} c V^{(2)} = \frac{u^{(1)}}{768\pi^2} \left( 3 N_S + \frac{40}{\xi_\infty} \right) \]
\[ + \frac{5 u_\infty}{48\pi^2} \xi_\infty (u_\infty - w^{(0)}), \]
\[ w^{(2)} = \frac{1}{3} c M^{(2)} = \frac{1}{192\pi^2} \xi_\infty \left( \frac{25 N_S}{3} - N_S \xi_\infty \right) \]
\[ + \frac{25 u_\infty}{96\pi^2} \xi_\infty (u_\infty - w^{(0)}) + \frac{N_S w^{(1)}}{192\pi^2}. \]

The question arises which ones of these asymptotic solutions correspond to true scaling solutions for the whole range of \( \tilde{\rho} \). For a numerical investigation we employ initial conditions for large \( \tilde{\rho} \), \( \tilde{\rho} = \tilde{\rho}_{as} \), say \( \tilde{\rho}_{as} = 1000 \) or even larger. The initial conditions for \( u(\tilde{\rho}_{as}) \), \( w(\tilde{\rho}_{as}) \) are taken from the asymptotic solution (213), with \( \xi_\infty \) a free parameter. For these large values of \( \tilde{\rho}_{as} \) the asymptotic solution (213) obeys the full differential equation (181), (182) with an accuracy of \( 10^{-13} \) or better. We then solve the differential equation (181), (182) numerically towards smaller \( \tilde{\rho} \) and ask for which values of \( \xi_\infty \) the solution extends to \( \tilde{\rho} \rightarrow 0 \). For \( g'^2 = 2 = 0 \) and \( N_S = 1 \), \( N_V = 0 \), \( N_P = 0 \) we plot the solution for different values of \( \xi_\infty \) in Figs 16-18. This could be a typical setting for a scalar singlet associated to the inflaton of the cosm, the scalar field mediating dynamical dark energy as \( \chi \) moves to infinity. In Fig. 16 we show \( u(x) \) for \( \xi_\infty = 0.405, 0.7, 1.0 \) and 1.5. For smaller or larger \( \xi_\infty \) outside the range of the plotted values, the solutions typically diverge within the interval of \( x = \ln \tilde{\rho} \) shown. Only the solutions for \( \xi_\infty \) within the restricted interval are candidates for valid scaling solutions. In Fig. 17 we display \( u'(x) \) for the same values of \( \xi_\infty \). One observes a switch from positive \( w = \xi_\infty/2 \) for large \( \tilde{\rho} \) to negative \( u' \) for \( \tilde{\rho} \rightarrow 0 \). The mass term \( \tilde{m}^2 = u' \) is displayed in Fig. 18. While it remains small everywhere, including the transition region, it shows substantial variation in the transition region. It remains open if this variation is damped by including the neglected terms \( \sim u'', u''' \) in the flow equation, or not. In view of this open question it is not yet possible to decide if the asymptotic solutions can be extended to \( \tilde{\rho} = 0 \) or not.

### 9.6. Scaling solutions with gauge and Yukawa couplings

Nonvanishing gauge couplings change the character of the scaling solution. We discuss here the approximation that the gauge coupling \( g \) is independent of \( \tilde{\rho} \), and make the simplification that all \( N_V \) vector bosons have the same mass term \( g^2 \tilde{\rho} \). Constant scaling solutions no longer exist for the whole range of \( \tilde{\rho} \) if \( g'^2 > 0 \). This is connected to the simple property that the effective number of gauge bosons is not the same for \( \tilde{\rho} \rightarrow 0 \) and \( \tilde{\rho} \rightarrow \infty \). For small \( \tilde{\rho} \), \( g'^2 \tilde{\rho} \ll 1 \), the effective number of gauge bosons is \( N_{V,\infty} \). On the other hand, for large \( \tilde{\rho} \) the mass term suppresses effectively the number of gauge bosons. Only the gauge modes are massless, replacing effectively \( N_V \) by \( N_{V,\infty} = -N_V/2 \). As a consequence, the effective numbers \( \tilde{N}_L \) and
One may still have an almost flat potential for \( \tilde{\rho} \rightarrow 0 \) corresponds to a maximum of the potential at the origin. The small local minimum of \( u \) at \( x \approx -6.5 \) may be an artifact of the truncation.

\[ \bar{N}_M \] for \( \tilde{\rho} \rightarrow 0 \) are replaced for \( \tilde{\rho} \rightarrow \infty \) by

\[ \bar{N}_{U,\infty} = N_S - N_V - 2N_F - \frac{8}{3}, \]
\[ \bar{N}_{M,\infty} = -N_S - 2N_V - N_F + \frac{43}{6} - \frac{3\xi}{2}N_S. \] (224)

One may still have an almost flat potential for \( \tilde{\rho} \rightarrow \infty \) as well as for \( \tilde{\rho} \rightarrow 0 \). The values of the flat potentials in the two limits will be different, however. Scaling solutions have then to describe a crossover between the two flat solutions, similar to sect. 6, cf. Figs 6, 9.

The situation is similar for nonzero Yukawa couplings. For \( \tilde{\rho} \rightarrow \infty \) the effective number of fermions reduces to zero if we assume that all fermions have nonvanishing Yukawa couplings. We will consider a setting where all fermions have the same \( \tilde{\rho} \)-independent Yukawa coupling \( y \). Taking further \( N_\xi = N_S \), the effective particle numbers become for \( \tilde{\rho} \rightarrow \infty \)

\[ \bar{N}_{U,\infty} = N_S - N_V - \frac{8}{3}, \]
\[ \bar{N}_{M,\infty} = -N_S - 2N_V - N_F + \frac{43}{6} - \frac{3\xi}{2}N_S. \] (225)

We may again explore the asymptotic scaling solutions of the type (213). The coefficients \( u_{\infty}, w^{(0)} \) are now given by

\[ u_{\infty} = c_{V,\infty} = \frac{5}{96\pi^2} + \frac{\bar{N}_{U,\infty}}{128\pi^2}, \]
\[ w^{(0)} = c_{M,\infty} = \frac{25}{128\pi^2} + \frac{1}{192\pi^2}\bar{N}_{M,\infty}. \] (226)

For the coefficients of the terms \( \sim \tilde{\rho}^{-1} \) one obtains

\[ u^{(1)}(\tilde{\rho}) = \frac{2}{3}\tilde{\rho}^{(1)} = \frac{5}{72\pi^2} + \frac{1}{64\pi^2}\left( \frac{N_V}{g^2} - 2 \frac{N_F}{3y^2} \right), \]
\[ w^{(1)}(\tilde{\rho}) = \frac{1}{2}\tilde{\rho}^{(1)} = \frac{25}{128\pi^2} + \frac{1}{384\pi^2}\left( \frac{6N_V}{g^2} - \frac{N_F}{y^2} \right). \] (227)

Finally, the coefficients \( u^{(2)}, w^{(2)} \) acquire additional contributions as well,

\[ u^{(2)} = u_0^{(2)} + \Delta u^{(2)}, \quad w^{(2)} = w_0^{(2)} + \Delta w^{(2)}, \] (228)

where \( u^{(2)} \) and \( w_0^{(2)} \) are given by eqs (222) and (223), respectively, and

\[ \Delta u^{(2)} = \frac{1}{2} \Delta c_{V}^{(2)} = -\frac{1}{256\pi^2} \left( \frac{3N_V}{g^4} - 2 \frac{N_F}{g^2} \right), \]
\[ \Delta w^{(2)} = \frac{1}{2} \Delta c_{M}^{(2)} = -\frac{1}{192\pi^2} \left( 2 \frac{N_V}{g^4} - \frac{N_F}{y^2} \right). \] (229)

In Figs 19–21 we plot the scaling solutions that connect to the asymptotic scaling solutions for constant \( g^2/4\pi = y^2/4\pi = 1/40 \). We choose the particle content of the standard model, \( N_S = 4, N_V = 12, N_F = 45 \) and set the asymptotic initial conditions at \( \tilde{\rho}_0 = 5000 \). We choose two values \( \xi_\infty = 0.1 \) and 1.0 and compare the solutions with the corresponding solutions for \( g^2 = y^2 = 0 \). For \( \xi_\infty \gtrsim 1.5 \) \((g^2 > 0) \) or \( \xi_\infty \gtrsim 2 \((g^2 = 0) \) the coefficient \( w \) turns negative inside the interval of \( x \) shown. These solutions are not acceptable scaling solutions, such that the allowed range of scaling solutions does not admit large scalar-curvature couplings \( \xi_\infty \).

In Fig. 19 we plot the dimensionless effective potential \( u(x), x = \ln \tilde{\rho} \). The upper two curves for large \( x \) correspond to \( g^2/4\pi = 1/40 \), the two lower ones to \( g^2 = 0 \). We observe indeed a crossover between two regions of almost constant potential for \( \tilde{\rho} \rightarrow 0 \) and \( \tilde{\rho} \rightarrow \infty \). The potential is higher for \( \tilde{\rho} \rightarrow \infty \). For \( g^2 > 0 \) this corresponds to the effective particle numbers \( \bar{N}_{U,\infty} \) and \( \bar{N}_{M,\infty} \). The dependence on \( \xi_\infty \) is small except for the transition region. For the curves with \( g^2 \approx 0.3 \) the transition occurs for \( \tilde{\rho} \approx 3 \), according to \( g^2 \tilde{\rho} \approx 1 \). The behavior of the two curves with \( g^2 > 0 \) is clearly dominated by the effect of the gauge and Yukawa couplings that result in the variation of the effective degrees of freedom. For the two curves with \( g^2 = 0 \) the crossover occurs for smaller \( \tilde{\rho} \) as compared to the curves with \( g^2 > 0 \). For all curves the potential \( u(\tilde{\rho} \rightarrow 0) \) is close to the constant scaling solution \( u_s \).

In Fig. 20 we display the dimensionless mass term \( \tilde{m}^2(\tilde{\rho}) = u'(\tilde{\rho}) \). The parameters are the same as for Fig. 19. For all curves the mass term vanishes for \( \tilde{\rho} \rightarrow \infty \), as expected for the asymptotic scaling solution. For \( \tilde{\rho} \rightarrow 0 \) the mass term switches to positive values, indicating that for all curves the minimum of the effective potential is situated at \( \tilde{\rho} = 0 \). The two upper curves on the left hand side correspond to the value \( \xi_\infty = 1.0 \), the lower ones to \( \xi_\infty = 0.05 \). For \( \xi_\infty = 1 \) the mass term at \( \tilde{\rho} = 0 \) is substantial. It depends only mildly on the value of \( g^2 \). For small \( \xi_\infty = 0.05 \)
10. Scaling solutions for the standard model

The standard model of particle physics coupled to quantum gravity may be a consistent quantum field theory. This requires the presence of an UV fixed point, rendering the model asymptotically safe. The fixed point does not only concern a small number of couplings. It requires the existence of a scaling solution for functions as \( u(\tilde{\rho}) \) and \( w(\tilde{\rho}) \). Within our approximation such scaling solutions indeed exist. For the standard model coupled to gravity the gauge and Yukawa couplings can be asymptotically free. This is necessary for the non-abelian couplings, while for the abelian coupling \( g_1 \) another fixed point with \( g_1 \neq 0 \) may exist [76–81]. For the scaling solution we take here \( g^2 = y^2 = 0 \).

10.1. Constant scaling solution

The constant scaling solution with \( \tilde{\rho} \)-independent \( u \) and \( w \) is a viable scaling solution. This scaling solution predicts a vanishing quartic scalar coupling at the fixed point. Close to the fixed point the quartic scalar coupling is an irrelevant parameter with critical exponent given by \(-A\). In a complete theory it can therefore be predicted to take its fixed point value. The gauge and Yukawa couplings are relevant parameters at the fixed point. They will increase as the flow moves away from the fixed point towards the infrared. The flowing gauge and Yukawa couplings generate, in turn, a nonzero value for the quartic scalar coupling.
As long as the graviton fluctuations remain important this value remains very small. A more substantial increase happens only for scales below the Planck mass. This simple picture has successfully predicted [3] the mass of the Higgs boson in the range that has later been observed [82–84].

For the constant scaling solution the effective potential is flat and corresponds to a cosmological constant
\[ U(\rho) = U_0 = u_\ast k^4. \]

The cosmological constant is negative, \( u_\ast < 0 \), and vanishes for \( k \to 0 \). The cosmological constant is a relevant parameter. Its flow away from the fixed point leads to a value of \( U(\rho = 0) \) different from the one for the scaling solution (230). It is a free parameter and can be chosen arbitrarily, for example to coincide with the present observed dark energy density. Also the mass term for the Higgs potentially is a relevant parameter at the fixed point. Its value at \( k = 0 \) can be chosen such that the expectation value of the Higgs scalar coincides with the observed Fermi scale.

The constant scaling solution cannot account for Higgs inflation, however. The relevant coupling corresponding to the cosmological constant has to be chosen such that for \( k \to 0 \) the cosmological constant is very small. This implies that for \( k \) larger than the Fermi scale, \( U(k) \) is given by the scaling solution. Even if we could somehow identify \( U(k) \) with the cosmological constant for a scale \( k \) corresponding to the Hubble parameter \( H \), (such an identification is far from obvious,) the value of \( U(k) \) is negative and cannot describe cosmology close to de Sitter space. We extend the discussion of this issue to other candidate scaling solutions in sect. 10.3, with a similar negative outcome. A possible alternative for inflation for the pure standard model coupled to gravity could be a large coefficient of the term \( \sim R^2 \) in the effective action, leading to Starobinski inflation [85].

10.2. Crossover potential

We may explore the possible existence of other scaling solutions beyond the constant scaling solution. For large \( \tilde{\rho} \), scaling solutions different from the constant scaling solution can take the form of the asymptotic scaling solution (213). A numerical investigation shows that such scaling solutions indeed seem to exist. As before, we fix initial conditions at some large \( \tilde{\rho}_{\text{in}} \) as a function of the free parameter \( \xi_\infty \). We find two ranges of solutions, one for small \( \xi_\infty \) in the range \( 0 \leq \xi_\infty \lesssim 1.5 \), the other for large \( \xi_\infty \gtrsim 1000 \). In the range \( 1.5 \lesssim \xi_\infty \lesssim 1000 \) the coupling \( w \) turns negative, not consistent with stable gravity.

For this type of solution the potential is a crossover potential, as shown in Fig. 22 for low values of \( \xi_\infty = 0.1, 1, 10^3, 10^4 \). All curves show a crossover from larger values of \( u \) for \( \tilde{\rho} \to \infty (x \to \infty) \) than for \( \tilde{\rho} = 0 (x \to -\infty) \). The minimum of the effective potential is at the origin, \( \tilde{\rho} = 0 \). For \( \tilde{\rho} \to \infty \), all crossover scaling solutions approach a common constant, given by \( u_\ast = 0 \) according to eq. (219), with \( \tilde{N}_{U,\ast} = N_S + 2N_V - 2N_F - 8/3 \). This crossover behavior continues for smaller values \( \xi_\infty < 0.1 \), with a location of the crossover shifted further to the right, according to the relevant parameter being given by \( \tilde{\rho}/\xi_\infty \). We show \( u(x) \) in Fig. 23 in a smaller range around \( x = \ln(1/(16\pi^2)) \) for

\[ \xi_\infty = 2 \cdot 10^{-5}, 10^{-4}, 10^{-3}, 0.03. \] As \( \xi_\infty \) approaches zero, the curves approach the constant scaling solution which is also shown as the horizontal straight line. Simultaneously, the location of the crossover to larger values moves to \( \tilde{\rho}_{\text{cross}} \to \infty \) as \( \xi_\infty \to 0 \), realizing a smooth limit. We show the corresponding scaling solutions \( w(x) \) for the same small values of \( \xi_\infty \) in Fig. 24, with constant scaling solution \( w_\ast \) given by the horizontal line. We observe that the scaling solutions for small \( \xi_\infty \) all meet in a common point \( \tilde{\rho} \approx -5.05 \), both for \( u \) and for \( w \). Around this point the linearized differential equation (198) is valid.

For all solutions the mass term \( u'(\tilde{\rho}) \) increases as \( \tilde{\rho} \) decreases, as shown in Fig. 25 for \( \xi_\infty = 0.01, 1.0, 10^3, 10^4 \) from bottom to top. We display \( m_0^2 = u'(\tilde{\rho} = 0) \), corresponding to the asymptotic limit \( x \to -\infty \), in table I. (For very small \( \tilde{\rho} \) our numerical solution starts to be unstable, and we take in practice \( u'(\tilde{\rho} = 10^{-11}) \).) Also \( \xi(\tilde{\rho}) = 2 u'(\tilde{\rho}) \)
increases as $\dot{\rho}$ decreases. The values $\xi_0$ for $\dot{\rho} \to 0$ are shown as well in table I. The numerical solutions for $\xi_\infty$ in the range between $10^{-3}$ and 1 show a very narrow spike for a value of $x$ smaller than the point where all curves for $w$ and $m^2$ meet. So far we have not attempted for a better resolution of the spike. It is doubtful that the solutions in this range are acceptable scaling solutions.

10.3. Higgs inflation

Higgs inflation [9, 10] has been proposed as a possibility to accomodate the inflationary universe within the standard model. The original proposal has employed rather large values of the nonminimal scalar-gravity coupling $\xi$. Smaller values seem also possible. In the presence of quantum gravity effects, even small $\xi \ll 1$ could be compatible with realistic inflation [11].

Discussing these proposals in the light of the scaling solutions for quantum gravity, one encounters a major problem: The scaling potential remains negative for the whole range of $\dot{\rho}$, while a positive potential would be required for inflation. The relevant quantity for inflation is actually the potential in the Einstein frame (with $\hat{M}$ the observed fixed Planck mass)

$$V_E = \frac{\hat{M}^4 U}{F^2} = \frac{\hat{M}^4 u}{4 w^2}. \quad (231)$$

We display $V_E$ in Fig. 26 for $\xi_\infty = 2 \cdot 10^{-5}, 10^{-4}, 10^{-3}$ and 0.003 from right to left. It has a flat tail for $\rho \to \infty$, as suitable for inflation,

$$V_E(\rho \to \infty) = \frac{\hat{M}^4 u_\infty}{\xi_\infty^2 \rho^2} = \frac{u_\infty \hat{M}^4 k^4}{\xi_\infty^2 \rho^2}. \quad (232)$$

Successful inflation would need, however, a shift to positive values.

For a connection to observable quantities it is useful to transform all quantities to the Einstein frame with a constant Planck mass $\hat{M}$. The ratio

$$\dot{V} = \frac{U}{F^2} = \frac{V_E}{\hat{M}^4} = \frac{u}{4 w^2} \quad (233)$$


FIG. 24. Dimensionless squared Planck mass $w$ as function of $x = \ln \dot{\rho}$ for $\xi_\infty = 2 \cdot 10^{-5}$ (blue), $10^{-4}$ (orange), $10^{-3}$ (green), 0.003 (red), from top to bottom on the left. The horizontal line denotes the scaling solution which is approached for $\xi_\infty \to 0$. All curves meet in a common point at $x \approx -5.05$.

FIG. 25. Dimensionless mass term $\tilde{m}^2 = u'$ as function of $x = \ln \dot{\rho}$, for $\xi_\infty = 0.1$ (blue), 1 (orange), $10^3$ (green) and $10^4$ (red), from bottom to top.

FIG. 26. Potential in the Einstein frame $V_E$ as function of $x = \ln \dot{\rho}$ for $\xi_\infty = 2 \cdot 10^{-5}$ (blue), $10^{-4}$ (orange), $10^{-3}$ (green), 0.003 (red), from right to left.

| $\xi_\infty$ | $\tilde{m}_0^2$ | $\xi_0$ | $\lambda_{H0}$ |
|-------------|----------------|--------|----------------|
| $2 \cdot 10^{-5}$ | $8.6 \cdot 10^{-4}$ | $1.6 \cdot 10^{-3}$ | $1.3 \cdot 10^{-3}$ |
| $10^{-4}$ | $1.6 \cdot 10^{-3}$ | $2.5 \cdot 10^{-3}$ | $3.2 \cdot 10^{-3}$ |
| $10^{-3}$ | $0.019$ | $0.035$ | $0.63$ |
| 0.003 | $0.034$ | $0.065$ | $2.28$ |
| 1 | $0.073$ | $0.156$ |
| 10 | $0.43$ | $2.4$ |
| $10^3$ | $3.29$ | $28.8$ |
| $10^4$ | $4.4$ | $40.5$ |
is a frame-invariant quantity. It does not change under a Weyl transformation of the metric, i.e.

\[(g_E)_{\mu\nu} = \frac{F}{M^2} g_{\mu\nu} \,. \tag{234}\]

With \( K \) the prefactor of the kinetic term, \((K = 1 \text{ in our truncation})\) the frame-invariant expression for the kinetic term is \([86], [18]\)

\[
\hat{K} = \frac{K k^2}{F^2} + \frac{6\rho k^2}{F^2} \left( \frac{\partial E}{\partial \rho} \right)^2 = \frac{1}{2w} + \frac{6}{\rho} \left( \frac{\partial \ln w}{\partial \ln \rho} \right)^2 \,. \tag{235}\]

In the Einstein frame the kinetic term for the Higgs doublet \( h \) reads \((\rho = h^1 h)\)

\[
\mathcal{L}_{\text{kin}} = \frac{M^2}{k^2} \hat{K} \partial^\mu h^\dagger \partial^\mu h = \partial^\mu \bar{h}_E^\dagger \partial^\mu \bar{h}_E \, , \tag{236}\]

with \( \bar{h}_E \) the canonically normalized field in the Einstein frame, related to \( h \) by

\[
\frac{\partial h_E}{\partial h} = \frac{M}{k} \sqrt{\hat{K}} \,. \tag{237}\]

For the Einstein frame we define the dimensionless invariant

\[
\hat{\rho} = \frac{\rho_E}{M^2} = \frac{\bar{h}_E^\dagger \bar{h}_E}{M^2} \,. \tag{238}\]

For a formulation with canonical kinetic terms we need to express \( V_E/M^4 \) in terms of \( \hat{\rho} \).

For the relation between \( \hat{\rho} \) and \( \hat{\rho}_E \) we integrate eq. (237),

\[
\int_0^{\hat{\rho}} d\hat{\rho}' \sqrt{\hat{K} / \hat{\rho}'} \,. \tag{239}\]

The dimensionless invariant \( \hat{\rho}_E \) only depends on the dimensionless variable \( \hat{\rho} \), without any explicit dependence on \( k \). For the scaling solution, \( \hat{V} \) and \( \hat{K} \) are functions of \( \hat{\rho} \) without explicit dependence on \( k \). Thus \( k \) completely disappears in the Einstein frame. If a model is defined precisely on the fixed point exact quantum scale symmetry is realized \([11]\). In this case the relevant cosmological field equations in the Einstein frame can be directly extracted from the field equation derived from the action

\[
S = \int_x \sqrt{g_E} \left\{ -\frac{M^2}{2} R_E + \frac{1}{2} \partial^\mu \bar{h}_E^\dagger \partial^\mu \bar{h}_E + M^4 \hat{V}(h_E) \right\} \,. \tag{240}\]

They do not involve the scale \( k \), which therefore does not need to be specified.

For our scaling solutions eq. (239) can be solved easily for limiting cases. For \( \hat{\rho} \to 0 \) one has \( w = w_0 + \xi_0 \hat{\rho}/2 \) and therefore

\[
\hat{K} = \frac{1}{2w_0} + (6\xi_0 - 1) \frac{\xi_0}{4w_0^2} \hat{\rho} \,. \tag{241}\]

In leading order this yields

\[
\hat{\rho} = 2w_0 \hat{\rho}_E \,. \tag{242}\]

In the limit of large \( \hat{\rho} \) we use \( w = \xi_\infty \hat{\rho}/2 \) and

\[
\hat{K} = \left( 6 + \frac{1}{\xi_\infty} \right) \hat{\rho}^{-1} \,. \tag{243}\]

This implies for \( \hat{\rho} \to \infty \)

\[
\hat{\rho}_E = \frac{1}{4} \left( 6 + \frac{1}{\xi_\infty} \right) \ln^2 \frac{\hat{\rho}}{\rho_0} = \frac{1}{4} \left( 6 + \frac{1}{\xi_\infty} \right) (x - x_0)^2 \,. \tag{244}\]

Up to a constant factor the variable \( x \) used in our figures can be associated directly with \( |h_E| \) in the region of large \( x \).

In the large-field region \( \rho_\infty \gg \bar{M}^2 \) the potential in the Einstein frame approaches exponentially zero,

\[
V_E = \frac{u_\infty \bar{M}^4}{\xi_\infty^2} \exp(-2x_0) \exp \left\{ -4 \frac{\xi_\infty}{6\xi_\infty + 1} \sqrt{\bar{h}_E^\dagger \bar{h}_E} / \bar{M} \right\} \,. \tag{245}\]

If one could shift \( V_E \) by a positive constant this would be a flat region suitable for inflation. No such shift is possible, however, since \( \hat{V} \) has to go to zero for \( \hat{\rho} \to \infty \). Since \( u_\infty \ll 0 \) the potential in the Einstein frame approaches zero from below. The scaling potential for the standard model is not compatible with Higgs inflation.

The quantum field theory for the standard model and gravity is defined, however, only by an asymptotic approach to the fixed point in the ultraviolet. The values of relevant parameters play a role. The Planck mass and the cosmological constant are relevant parameters. They deviate from the scaling solution for small \( k \). The leading relevant parameter is the Planck mass, typically in the form

\[
F = M^2 + 2 w_\ast \left( \frac{\rho}{k^2} \right)^2 \,. \tag{246}\]

with \( w_\ast(\hat{\rho}) \) given by the scaling solution. For \( k \to 0 \) it assumes the form

\[
F = M^2 + \xi_\infty \rho \,, \tag{247}\]

and we associate the integration constant \( \bar{M} \) with the observed Planck mass. The transition scale \( k_t \) for the crossover from the scaling solution to the solution (247) for \( k \to 0 \) depends on \( \rho \) according to

\[
\frac{\bar{M}^2}{k_t^2} = 2w^{(0)} + \xi_\infty \frac{\rho}{k_t^2} \,. \tag{248}\]

which yields

\[
k_t^2(\rho) = \frac{\bar{M} - \xi_\infty \rho}{2w_0} \theta(\bar{M} - \xi_\infty \rho) \,. \tag{249}\]

For large \( \rho \gg \bar{M}^2/\xi_\infty \) the integration constant \( \bar{M}^2 \) plays no role and we can employ the scaling solution, e.g. \( k_t(\rho) = 0 \). In particular, there cannot be any constant shift in the behavior of \( u(\hat{\rho} \to \infty) \).

For \( w \) the flow away from the scaling solution implies a strong increase of \( w \) for \( k \to 0 \)

\[
w = w_\ast(\hat{\rho}) + \frac{\bar{M}^2}{2k^2} \approx w_\ast + \frac{\xi_\infty \rho + \bar{M}^2}{2k^2} \,. \tag{250}\]
where we have parameterized an approximate form of $w_*(\tilde{\rho}) \approx w_{0*} + \xi_{\infty}\tilde{\rho}/2$. This increase is responsible for the decoupling of gravity for low $k$.

For the scalar potential in the Einstein frame $V_E$ away from the scaling solution we may use the ansatz

$$U = u_*(\tilde{\rho})k^4 + V(\rho),$$

with $V = 0$ for the scaling solution. This results in

$$V_E = \frac{V + u_*(\tilde{\rho})k^4}{(1 + \xi_{\infty}\rho/M^2 + 2w_0k^2/M^2)^2}$$

If we assume that for the relevant epochs in cosmology we can take $k \to 0$ we remain with a potential that vanishes for $\rho \to \infty$, provided $V(\rho)$ does not increase too rapidly with $\rho$.

$$V_E = \frac{V(\rho)}{(1 + \xi_{\infty}\rho/M^2)^2}$$

What is needed is an understanding of $V(\rho)$. This function corresponds to a solution of the flow equation for $U$ as $k \to 0$. At fixed $\rho$ one has

$$\partial_k U = 4C_V(\tilde{\rho})k^4.$$ 

With the ansatz (251) one finds

$$\partial_k U = 4u_\ast k^4 - 2\tilde{\rho}\partial_{\tilde{\rho}}u_\ast k^4 + \partial_k V = 4C_{V*}(\tilde{\rho})k^4 + \partial_k V,$$

where we have inserted the equation for the scaling solution $u_*(\tilde{\rho})$. Comparison with eq. (254) yields

$$\partial_k V = 4(c_{V*}(\tilde{\rho}) - c_{V*(\tilde{\rho}))}k^4.$$ 

We recover the scaling solution for $V = 0$ and $c_{V*}(\tilde{\rho}) = c_{V*(\tilde{\rho})}$. For small deviations from the scaling solution one can linearize the flow equation. In this regime $\partial_k V$ is characterized by a critical exponent. Typically $V$ corresponds to a relevant coupling which implies at least one free parameter for the general solution for $V$.

For $k^2 \ll M^2$ the deviations from the scaling solution are not small. Understanding the $\rho$-dependencies of $V$ will require a numerical solution of the flow equation, which also takes into account the flow of other couplings, as well as Yukawa couplings, away from their fixed points. This is outside the scope of this paper. For $\rho \ll M^2$ one expects that $V$ approaches the perturbative, almost quartic potential of the standard model. It is, however, the behavior of the potential at much higher $\rho$, typically in the order $k^2$, that is relevant for Higgs inflation. For this region not much can be said at the present level of our investigation. The only direct consequence of the scaling solution remains the value of $\xi_{\infty}$. Given the restrictions on the asymptotic behavior for $\tilde{\rho} \to \infty$, however, compatibility with Higgs inflation seems unlikely.

### 10.4. Non-minimal Higgs-curvature coupling and prediction for the mass of the Higgs boson or top quark

The quartic coupling $\lambda$ of the Higgs self-interaction corresponds to an irrelevant parameter at the fixed point. It can therefore be predicted to take at short distances its fixed point value. The corresponding prediction for the mass of the Higgs boson to be 126 GeV with a few GeV uncertainty [3] agrees well with the experimental value of 125 GeV found later. The central value of the prediction depends on the pole mass of the top quark $m_t$. For $m_t = 171$ GeV the prediction for the central value is lowered to 125 GeV.

The fixed point value of $\lambda$ is influenced by the non-minimal Higgs-curvature coupling $\xi_0 = \xi(\tilde{\rho} \to 0)$ [11]. This was assumed to be negligible for the prediction in ref. [3]. Since the scaling solutions restrict the possible values of $\xi_0$, we investigate here the influence of $\xi_0$ on the prediction of the mass of the Higgs boson. More generally, we investigate the influence of metric fluctuations on the position of the fixed point value $\lambda_c$.

The couplings $\lambda$ and $\xi_0$ are defined by

$$\lambda = \frac{\partial^2 u}{\partial \rho^2} |_{\tilde{\rho} = 0}, \quad \xi_0 = \frac{2\partial w}{\partial \rho} |_{\tilde{\rho} = 0}.$$ 

The flow equations for $\lambda$ and $\xi_0$ can be obtained by taking suitable $\tilde{\rho}$-derivatives of eqs. (174), (175), evaluated at $\tilde{\rho} = 0$. For the flow of the quartic coupling one finds [7, 11]

$$\partial_t \lambda = A\lambda + \beta_{\lambda}(\rho) - C_g,$$

with $\beta_{\lambda}(\rho)$ the part induced by fluctuations of gauge bosons, fermions and scalars, and $C_g$ a gravitational contribution.

For $\beta_{\lambda}(\rho)$ we may employ here the approximate one-loop expression

$$\beta_{\lambda}(\rho) = \frac{3y_t^2}{4\pi^2} + \frac{171\alpha^2}{50},$$

that is the same as in standard perturbation theory. Here $y_t$ is the Yukawa coupling of the top quark, and $\alpha = g_2^2/(4\pi)$, with $g_2$ the SU(2)-gauge coupling of the standard model. (We have taken for the hypercharge coupling $g_1$ the approximation $g_1 = g_2$, neglected all Yukawa couplings to fermions except for the top quark, as well as small contributions $\sim \lambda\alpha, \lambda y_t^2, \lambda^2$.)

For $C_g$ one finds [7, 11]

$$C_g = \frac{A\xi_0}{w} \left( \frac{m^2 - \xi_0 v}{2} \right) - \frac{2A}{1-v} \left( \frac{m^2 - \xi_0 v}{2} \right)^2 + Aw_2.$$ 

Here all quantities have to be evaluated at $\tilde{\rho} = 0$ and

$$w_2 = \frac{\partial^2 w}{\partial \rho^2} |_{\tilde{\rho} = 0}.$$ 

With

$$V_1 = \frac{\partial v}{\partial \rho} |_{\tilde{\rho} = 0} = \left( \frac{m^2 - \xi_0 v}{2} \right) / w$$

the restrictions on the asymptotic behavior for $\tilde{\rho} \to \infty$, however, compatibility with Higgs inflation seems unlikely.
we observe that $C_g$ vanishes if $v$ is independent of $\tilde{\rho}$ and $w_2 = 0$,

$$C_g = A \left( \xi_0 - \frac{2w}{1 - v} \right) v_1 + Awv_2. \quad (263)$$

We may neglect $w_2$ and investigate the flow equations for $\tilde{m}^2$, $\xi_0$ and $v_1$. For the crossover scaling solutions one observes a very rapid increase of $v_1$ from $v_1(\xi_{\infty} = 10^{-3}) \approx 3$ to $v_1(\xi_{\infty} = 10^{-5}) \approx 40$. We doubt that a strong increase of $\partial v/\partial \rho \gg 10$ is compatible with a consistent scaling solution.

In the gravity-dominated regime for $k^2 \gg M^2$ and for small $\beta_\lambda^{(p)}$ the flow of $\lambda(k)$ is characterized by an approximate partial fixed point

$$\lambda_* = \frac{3y^2}{4\pi^2 A} - \frac{171\alpha^2}{50A} + \frac{C_g}{A} \quad (264)$$

This partial fixed point is valid for $k > k_i$ and constitutes the "initial value" for the flow in the low-energy regime $k < k_i$, for which gravitational effects vanish rapidly due to decreasing $A$, and only $\beta_\lambda^{(p)}$ survives effectively. For $v$ and $w$ we can take the values for the scaling solution at the UV-fixed point, while $y$ and $\alpha$ can be found from extrapolating the observed low energy couplings to $k_i$ by use of the perturbative renormalization group.

It is our aim to investigate the effect of the non-vanishing Higgs-curvature coupling $\xi_0$, and more generally $\Delta g$, on the prediction of the mass of the Higgs boson. Since the prediction of $\lambda(k_i)$ actually results in a prediction of the ratio of the mass of the Higgs boson compared to the mass of the top quark [11], and the mass of the Higgs boson is accurately measured, we may turn this to an investigation of the effect of

$$\Delta \lambda = \frac{C_g}{A} \quad (265)$$

on the prediction for the mass of the top quark $m_t$. For a quantitative analysis we employ the estimate [11] that a value $\Delta \lambda = 0.014$ decreases $m_t$ by 1 GeV,

$$m_t - m_{t,0} \approx -71.4 \Delta \lambda, \quad (266)$$

with $m_{t,0}$ the prediction for $C_g = 0$.

The values of $v$ and $w$ at the fixed point are given for the standard model as

$$u_* = -0.0507 \approx -0.0508, \quad v_* = -10.05 \approx -10.27, \quad w_* = 0.00505 (0.00495). \quad (267)$$

Here the first value corresponds to the constant scaling solution, as computed in ref. [8], while the value in brackets corresponds to a typical crossover scaling solution as described in this section. Since the two values are rather similar only a modest uncertainty is related to the difference of these values. The gravity induced anomalous dimension reads

$$A = \frac{5}{24\pi^2 w(1 - v)^2} + \frac{1}{96\pi^2 w(1 + v/4)^2} \approx 0.051, \quad (268)$$

where the second term arises from the physical scalar fluctuation in the metric. Due to the large negative value of $v$ for the standard model, one finds that $A$ is substantially smaller than one, and the second term contributes of similar size as the first term, in contrast to $v > 0$.

Let us first discuss the value of $\Delta \lambda$ for the scaling solution with vanishing gauge and Yukawa couplings. For the constant scaling solution one has $\Delta \lambda = 0$. This solution has a vanishing Higgs-curvature coupling, $\xi_0 = \xi_{\infty} = 0$. For the possible crossover scaling solutions $\Delta \lambda$ corresponds to $\lambda_{H0}$ in table I. There is a one-parameter family of crossover solutions parameterized by $\xi_{\infty}$. Only a range of very small $\xi_{\infty}$ of the order of a few times $10^{-3}$ or less is consistent with the observed mass ratio between Higgs boson and top quark mass, even if we admit an uncertainty in the present experimental determination of the pole mass for the top quark of one or two GeV and theoretical uncertainties of a similar order. In particular, large values of $\xi_{\infty} \gg 1$, as often used for Higgs inflation, are not compatible with asymptotic safety for quantum gravity coupled to the standard model. This points towards the constant scaling solution, which is the only one which is firmly established within our truncation.

For the constant scaling solution (or crossover scaling solutions with small $\xi_{\infty}$) the flow away from the fixed point could induce a more sizable $\Delta \lambda$ due to the effects of gauge and Yukawa couplings. We therefore include next the effect of gauge and Yukawa couplings to the flow of $\xi_0$. They may lead to a value of $\xi_0(k_i)$ that differs from the fixed point value given in table I. The $\tilde{\rho}$- derivative of eq. (175) at $\tilde{\rho} = 0$, yields

$$\partial_i \xi = \frac{25}{128\pi^2 w(1 - v)^2} \left( \tilde{m}^2 - \frac{\xi v}{2} \right) - \frac{\tilde{m}^2 - \frac{\xi v}{2}}{1152\pi^2 w \partial v} \left( \frac{8 + \frac{9}{16}v^2}{(1 - \frac{v}{4})^2} - \frac{3w_2}{32\pi^2} + \frac{1}{192\pi^2} \left\{ 4\partial_i N_V - \partial_i N_S - \partial_i N_F \right\} \right) \quad (269)$$

Here the first term arises from graviton (transverse traceless tensor) fluctuations, the second term from the physical scalar in the metric in the approximation of neglected mixing with other scalars, and the third accounts for the non-minimal coupling to gravity for the scalar fluctuations. The remaining parts reflect the contribution from gauge couplings, Yukawa couplings and scalar mass terms and self interactions.

The particle contributions result from the reduction of effective particle numbers due to mass terms. For gauge bosons with squared masses $m_i^2 = g_i^2 \rho$ one has

$$\partial_i N_V = \frac{3}{2} \sum_i g_i^2 \frac{1}{1 + g_i^2 \rho} \left( 1 + g_i^2 \rho \right)^2 \quad (270)$$

For the standard model this results in a contribution

$$\partial_i \xi^{(g)} = -\frac{3}{32\pi^2} (2g_w^2 + g_z^2) = -\frac{3}{64\pi^2} \left( g_w^2 + \frac{g_z^2}{5} \right), \quad (271)$$

where $g_w$ and $g_z$ are the weak and electromagnetic couplings.
where we use \( g_{w}^2 = g_{1}^2/2 = 2\pi\alpha \) and \( g_{v}^2 = g_{2}^2/2 + 3g_{1}^2/10 \).

For the top quark with \( m_{t}^2 = h_{t}^2/\rho \) one has
\[
\partial_{\rho}N_{F} = 6\partial_{\rho}\frac{1}{1 + y_{t}^2/\bar{\rho}} = -\frac{6y_{t}^2}{(1 + y_{t}^2/\bar{\rho})^2},
\]
resulting in
\[
\partial_{\xi}(\xi(t)) = \frac{y_{t}^2}{32\pi^2}.
\]

Finally, for the scalar fluctuations one has
\[
\partial_{\rho}N_{S} = \partial_{\rho}\left(\frac{3}{1 + \bar{m}^2} + \frac{1}{1 + \bar{m}^2 + 2\rho\lambda}\right) = -\frac{3\lambda}{(1 + \bar{m}^2)} - \frac{3\lambda}{(1 + \bar{m}^2 + 2\rho\lambda)},
\]
and therefore
\[
\partial_{\xi}(\xi(s)) = \frac{\lambda}{32\pi^2(1 + m^2)}.
\]

The results (271), (273) and (275) for \( \bar{m}^2 = 0 \) and \( \bar{\rho} = 0 \) can also be obtained from one-loop perturbation theory.

We will neglect the scalar contribution (275) as compared to the much larger top-quark contribution (273). Furthermore, the physical scalar metric fluctuations contribute for large negative \( v \) similar to the graviton fluctuations with 25/128 replaced by \(-7/144\). Neglecting \( w_{2} \) we obtain the approximate flow equation for \( \xi_{0} \).
\[
\partial_{t}\xi_{0} = \frac{19v_{1}}{128\pi^2(1 - v)} + \frac{y_{t}^2}{32\pi^2} - \frac{9\alpha}{40\pi}.
\]

With \( y_{t}(10^{18}\text{GeV}) \approx 0.38 \), \( y_{t}^2/(32\pi^2) \approx 4.6 \cdot 10^{-4} \), \( 9\alpha/(40\pi) \approx 1.8 \cdot 10^{-3} \), the gauge boson fluctuations tend to induce for \( v_{1} = 0 \) a small positive \( \xi_{0} \) as \( \xi \) flows towards the IR.

We further need the flow equation for \( v_{1} \)
\[
\partial_{t}v_{1} = \frac{1}{w}\left\{ \partial_{t}\bar{m}^2 - \frac{\xi_{0}}{2w}\partial_{t}u - \frac{v}{2}\partial_{t}\xi_{0} + \left(\frac{\xi_{0}v}{2w} - v_{1}\right)\partial_{t}w\right\}.
\]

The flow of \( u \) and \( w \) at \( \bar{\rho} = 0 \) does not depend directly on the gauge and Yukawa couplings. It vanishes for the fixed point until the transition region near \( k_{t} \) is reached. We may neglect \( \partial_{t}u \) and \( \partial_{t}w \) in eq. (277), such that the influence of the gauge and Yukawa couplings arises from the flow of \( \bar{m}^2 \) and \( \xi_{0} \). We can employ
\[
\partial_{t}\bar{m}^2 = \left(\frac{A}{2} - 2\right)\bar{m}^2 - \frac{1}{2}A\xi_{0}\nu
\]
\[
+ \frac{1}{32\pi^2}(\partial_{\rho}N_{S} + 2\partial_{\rho}N_{V} - 2\partial_{\rho}N_{F})
\]
\[
\approx -2\bar{m}^2 + Awv_{1} + \frac{3y_{t}^2}{8\pi^2} - \frac{9}{64\pi^2}\left(\frac{g_{1}^2 + g_{2}^2}{5}\right).
\]

The constant scaling solution has \( \bar{m}^2 = \xi_{0} = v_{1} = 0 \). If we assume a bound for the effective contribution of gauge and Yukawa couplings as
\[
|v_{1}| < \frac{3c_{1}y_{t}^2}{8\pi^2w}, \quad |\xi_{0}| < \frac{y_{t}^2c_{2}}{32\pi^2},
\]
we conclude that the contribution of flowing gauge and Yukawa couplings to \( \Delta \lambda \) is bounded by
\[
|\Delta \lambda|_{g,h} < \frac{3c_{1}c_{2}y_{t}^4}{256\pi^3w},
\]
which is of the order of a few times \( c_{1}c_{2}10^{-6} \). Given that \( c_{1} \) and \( c_{2} \) are typically smaller than one due to cancellations between Yukawa and gauge couplings, and the flow of \( g_{2} \) and \( y_{t} \) deviating substantially from zero only in vicinity of \( k_{t} \), we conclude that the effect of the flowing gauge and Yukawa couplings is too small for influencing the prediction of the top quark mass.

For the flow away from the constant scaling solution of the standard model coupled to gravity the dominant contribution to \( \Delta \lambda \) seems to arise from the particle fluctuations,
\[
\Delta \lambda = C_{\nu} = \frac{1}{A} \left(\frac{3y_{t}^4}{4\pi^2} - \frac{171\alpha^2}{50}\right) = -\beta_{\lambda}/A.
\]

Typically, \( \beta_{\lambda}(k_{t}) \) is slightly positive, with details depending on \( m_{t} \) [87, 88]. With the unusually small value of \( A \) for the standard model, \( 1/A \approx 20 \), a value \( \beta_{\lambda} \approx 10^{-3} \) enhances the central value of the prediction for the top quark mass by around 1.5 GeV. If the theoretical uncertainties can be reduced below this level, a precision measurement of the pole mass for the top quark could distinguish between the asymptotically safe standard model coupled to gravity with its small value of \( A \), and other models as grand unification which typically have \( A \) of the order one or even larger. A dedicated solution of the combined set of flow equations in the threshold region around \( k_{t} \), together with a matching to three-loop running for \( k \ll k_{t} \), should improve our rough estimates in this case.

### 11. Conclusions

An ultraviolet fixed point defines a consistent quantum field theory for gravity coupled to particle physics. In this asymptotic safety scenario not only a few couplings as the dimensionless Planck mass or cosmological constant take fixed values. Whole functions, as the dimensionless effective potential \( u = U/k^4 \), take a fixed form as functions of dimensionless invariants formed from scalar fields. Typical invariants are \( \bar{\rho} = \chi^2/(2k^2) \) for a singlet field \( \chi \) or \( \bar{\rho} = h^4/h^2 \) for the Higgs doublet \( h \). This fixed form is the scaling potential \( u_{s}(\bar{\rho}) \) which does not depend on \( k \). Another important scaling function is \( w_{s}(\bar{\rho}) = F/(2k^2) \), with \( F \) the field-dependent coefficient of the curvature scalar in the effective action, corresponding to a field- and scale-dependent squared Planck mass. A fixed point is characterized by infinitely many such scaling functions that constitute the scaling functional, corresponding to a \( k \)-independent effective average action expressed in terms of dimensionless variables.

In this paper we have investigated the form of the scaling potential \( u_{s}(\bar{\rho}) \), together with \( w_{s}(\bar{\rho}) \). We also discuss the flow of \( u(\bar{\rho}) \) and \( w(\bar{\rho}) \) away from the ultraviolet
fixed point. In the vicinity of the fixed point one can linearize the flow of small deviations of couplings from their fixed point values, encoded in \( \delta u(\tilde{\rho}) = u(\tilde{\rho}) - u_*(\tilde{\rho}) \) and \( \delta w(\tilde{\rho}) = w(\tilde{\rho}) - w_*(\tilde{\rho}) \). A few relevant parameters, that describe deviations that increase for the flow away from the fixed point towards the infrared, determine all observable quantities of a given model. If there are less relevant parameters as compared to the number of renormalizable couplings in the standard model, relations between the standard model couplings become predictable. Due to the presence of relevant parameters, the observable effective potential \( U(\rho) = k^2(u(\rho)/k^2) \) for \( k \to 0 \), evaluated in units of the Planck mass, \( U/F^2 = u/(4w^2) \), differs from the scaling potential \( u_*/(4w^2) \). Nevertheless the scaling form determines many properties of the observable effective potential in the Einstein frame \( V_E = M^2 U/F^2 \) for \( k \to 0 \). The scaling form is the boundary value or “initial value” of the flow for \( k \to \infty \). For example, it determines the UV-value of all quartic scalar couplings in GUT-models which therefore become predictable for a given model \([89]\).

A particular scaling solution is the constant scaling solution for which \( u_*(\tilde{\rho}) \) and \( w_*(\tilde{\rho}) \) are independent of \( \tilde{\rho} \), while gauge and Yukawa couplings vanish. This constant scaling solution is the simplest extension of the Gaussian fixed point in particle physics, for which all particles are massless free particles, to quantum gravity where the gravitational couplings do not vanish. If there is a unique constant scaling solution it corresponds to the extended Reuter fixed point \([4]\). For a truncation with two scale dependent functions \( u(\tilde{\rho}; k), w(\tilde{\rho}; k) \) this fixed point has been studied in detail for many particle physics models in ref. \([8]\).

In the present paper we ask if there could be other fixed points distinct from the extended Reuter fixed point with constant scaling solutions. Non-trivial scaling functions \( u_*(\tilde{\rho}) \) and \( w_*(\tilde{\rho}) \) may be induced by non-vanishing gauge or Yukawa couplings. In our truncation with two functions \( u(\tilde{\rho}) \) and \( w(\tilde{\rho}) \), plus constant gauge couplings \( g \) and Yukawa couplings \( y \), any nonzero \( g \) or \( y \) necessarily induces a non-trivial \( \tilde{\rho} \)-dependence of \( u_*(\tilde{\rho}) \) and \( w_*(\tilde{\rho}) \). Thus any fixed point that is not asymptotically free in the particle sector \( g_* = y_* = 0 \) leads to nonzero \( \partial_\tilde{\rho} u_*(\tilde{\rho}) \) and \( \partial_\tilde{\rho} w_*(\tilde{\rho}) \) in this truncation. Even if the fixed point is the extended Reuter fixed point, the flow away from the fixed point will involve nonzero \( g \) and \( y \). If the flow of \( g \) and \( y \) is slow, the flow functions \( u(\tilde{\rho}) \) and \( w(\tilde{\rho}) \) will still be characterized by partial fixed points that close to a relative with scaling functions for nonzero \( g \) and \( y \). Understanding the \( \tilde{\rho} \)-dependence of possible scaling functions or approximate scaling functions is crucial for a connection to the observable quantities for \( k \to 0 \).

A complete answer to the question of the possible shapes of \( u(\tilde{\rho}) \) and \( w(\tilde{\rho}) \) in the presence of gauge and Yukawa couplings is rather complex. At the present stage of the investigation we find several important general features of candidate scaling functions \( u_*(\tilde{\rho}), w_*(\tilde{\rho}) \) which have a non-trivial \( \tilde{\rho} \)-dependence:

1. The scaling functions \( u_*(\tilde{\rho}) \) and \( w_*(\tilde{\rho}) \) do not have a polynomial form.

2. The scaling potential \( u_*(\tilde{\rho}) \) is often characterized by a crossover between a constant \( u_0 \) for \( \tilde{\rho} \to 0 \) to a different constant \( u_\infty \), for \( \tilde{\rho} \to \infty \).

3. A negative quartic coupling \( \lambda = \partial^2 u/\partial \tilde{\rho}^2 \) at \( \tilde{\rho} = 0 \) is no sign of instability. It only characterizes the Taylor expansion around \( \tilde{\rho} = 0 \), which does not describe the overall potential if the scaling potential does not have a polynomial form. An example is the crossover of a potential with a minimum at \( \tilde{\rho} = 0 \) and \( \tilde{m}^2 = \partial u/\partial \tilde{\rho} \) at \( \tilde{\rho} = 0 \) being positive, to a constant value \( u_\infty > u_0 \) for \( \tilde{\rho} \to \infty \). A negative value of \( \lambda \) only implies that \( \tilde{m}^2(\tilde{\rho}) \) decreases for increasing \( \tilde{\rho} \), which is typical for a crossover.

4. It is possible that families of scaling solutions exist, characterized by one or even several continuous parameters. This would be in contrast to a simple UV-fixed point or a discrete set of such fixed points. We have found candidates for one-parameter families of crossover scaling solutions within our truncation. The present numerical approximation to the flow equations for \( u \) and \( w \) is, however, not sufficient for the clarification if all candidates are really overall solutions of the corresponding system of differential equations for the scaling functions. Furthermore, it is not guaranteed that all members of such families “survive” extended truncations. This is demonstrated by a family of crossover solutions for \( u_*(\tilde{\rho}) \) obtained at constant \( w \) (truncation 1), for which only particular members solve the combined system of differential equations for \( u_*(\tilde{\rho}) \) and \( w_*(\tilde{\rho}) \) (truncation 2).

5. Non-vanishing gauge couplings \( g^2 > 0 \) have the tendency to create a minimum of \( u_*(\tilde{\rho}) \) at \( \tilde{\rho}_0 \neq 0 \). This implies spontaneous symmetry breaking for the scaling solution. This tendency dominates if the non-minimal scalar-curvature coupling \( \xi \) is small enough. This observation is relevant for GUT-models where certain scalar fields couple to the gauge bosons while no Yukawa couplings to the fermions are allowed. In particular, for a fixed point with \( g^2 > 0 \) \([19]\) this could account for partial symmetry breaking of the GUT-symmetry close to the Planck mass.

6. Non-zero Yukawa couplings \( y^2 > 0 \) have the opposite tendency of generating a minimum of \( u_*(\tilde{\rho}) \) at \( \tilde{\rho} = 0 \). The competition between gauge and Yukawa couplings is relevant both for the standard model and GUT-models.

7. A non-minimal scalar-curvature coupling \( \xi > 0 \) favors a minimum of \( u_*(\tilde{\rho}) \) at \( \tilde{\rho} = 0 \) or at \( \tilde{\rho} \to \infty \), depending on the particle content, cf. figs. 16, 19, 22. We have found candidate scaling solutions with constant asymptotic behavior, \( u_*(\tilde{\rho} \to \infty) = u_\infty \). Together with \( w_*(\tilde{\rho}) = \xi_\infty \tilde{\rho}^2/2 \) this implies for the potential in the Einstein frame

\[
\frac{V_E(\tilde{\rho} \to \infty)}{M^4} = \frac{u_\infty}{\xi_\infty \tilde{\rho}^2} = \frac{u_\infty k^4}{\xi_\infty \tilde{\rho}^2}. 
\]

The flow away from the fixed point may stop the flow of \( V_E \) at some characteristic \( \tilde{k} \). (In this case \( M^2 \) does not denote the integration constant in the flow of \( \omega \), but rather is introduced only as a unit in the Weyl scaling.) The decrease of
$V_E$ to zero for $\rho \to \infty$ can solve the cosmological constant problem dynamically by a "runaway cosmology" leading to dynamical dark energy or quintessence [15]. We have not investigated here another possible asymptotic behavior allowed by the graviton barrier [7], namely $u(\tilde{\rho} \to \infty) \sim \tilde{\rho}$, $v(\tilde{\rho} \to \infty) = u(\tilde{\rho} \to \infty)/w(\tilde{\rho} \to \infty) \leq 1$. We refer to ref. [6, 16] for a detailed discussion.

Already at the present stage of the investigation it is apparent that the understanding of the scaling form of the effective potential has important consequences for particle physics and cosmology. The effective potential is the central ingredient for the phenomenon of spontaneous symmetry breaking as well as for inflation and dynamical dark energy. We highlight here three important consequences for the standard model coupled to quantum gravity:

1. For a minimal model of the standard model coupled to quantum gravity the non-minimal Higgs-curvature coupling has to be small, $\xi < 10^{-3}$. Higgs inflation with a large coupling $\xi > 1$, as usually assumed, is not possible. The scaling potential $u_*(\tilde{\rho})$ and associated Einstein frame potential $V_E(h^h)$ does not allow for Higgs inflation. A positive $V_E$ for large $h^h$ could only be generated by the flow away from the scaling solution. This may not seem likely, but a detailed study of the flow away from the fixed point is necessary in order to settle this question. The minimal model may still permit Starobinski inflation [85] if the coefficient of the squared curvature scalar $R^2$ in the effective action is large enough.

2. The ratio between the Higgs scalar mass $m_H$ and the top quark mass $m_t$ can be predicted [3]. For the extended Reuter fixed point with constant scaling solution the flow away from the fixed point affects the prediction of $m_H/m_t$ only very mildly. The non-minimal Higgs-curvature coupling $\xi$ generated by the flow due to non-zero gauge and Yukawa couplings is too small to affect the predicted value. Possible crossover scaling solutions, if established, could lead to somewhat larger $\xi$. In the parameter region where such crossover solutions are reasonable candidates for scaling solutions the small value of $\xi$ seems to have only a small effect on the ratio $m_H/m_t$. With future possible precision measurements of the pole mass for the top quark it will become important to settle even the size of small effects for the predicted value $m_H/m_t$.

3. It is an interesting question if a constant scaling solution is possible also for non-zero fixed point values of gauge and Yukawa couplings, $g_1^2 > 0$, $g_2^2 > 0$. This concerns GUT-models [19] as well as the standard model if the hypercharge coupling $g_1$ takes a value $g_1 \neq 0$ [76–81]. If such a fixed point exists, the contributions of particle fluctuations to the scaling form of $\partial_\rho u_*(\tilde{\rho})$ and $\partial_\rho w_*(\tilde{\rho})$ have to vanish. For the Higgs potential this implies that $\beta_\lambda^{(p)}$ in eqs. (258), (259) has to vanish, since a constant scaling solution has $\lambda_* = 0$, $C_g = 0$. As a direct consequence, the Yukawa coupling of the top quark and therefore $m_t$ can be predicted as a function of the gauge coupling. In this case not only the ratio $m_b/m_t$, but $m_t$ and $m_H$ separately are predicted! The extrapolation of the running Yukawa and gauge couplings to the vicinity of the Planck mass yields indeed a very small value of $\beta_\lambda$. For $m_t$ near 171 GeV both $\lambda$ and $\beta_\lambda$ vanish in this region. Keeping in mind small corrections from the flow away from the fixed point the prediction agrees with the observation. For this type of prediction it is actually sufficient that the scaling solution is constant in the region near $\tilde{\rho} = 0$.

This scenario is not compatible with a minimal standard model since the weak and strong gauge couplings have to flow away from their vanishing fixed point values substantially before $k_1$ is reached. It could be realized in GUT-models however, where all gauge couplings of the standard model take a common fixed point value $\alpha \approx 1/40$. A realization of this scenario needs a truncation beyond the present one. Within our truncation one has for a constant scaling solution
\[
\frac{\partial u_*}{\partial \rho} = 0 \Leftrightarrow \partial_\rho N_V = \partial_\rho N_F
\] (283)
and
\[
\frac{\partial w_*}{\partial \rho} = 0 \Leftrightarrow 4 \partial_\rho N_V = \partial_\rho N_F.
\] (284)
This implies that $\partial_\rho N_V$ and $\partial_\rho N_F$ both have to vanish, and therefore zero gauge and Yukawa couplings. In an extended truncation other couplings may contribute to $\partial_\rho w_*$ and a constant scaling solution could become possible with $\partial_\rho N_V \neq 0$, $\partial_\rho N_F \neq 0$.

The present investigation of ultraviolet fixed points with non-constant scaling functions and/or non-zero gauge and Yukawa couplings leaves many questions open. Already now it shows that simple extrapolations of perturbative features, as approximately polynomial potentials, to the quantum gravity regime are not correct. It will be necessary to gain understanding and intuition for central quantities as the effective potential in the quantum gravity regime. Features that seem "unnatural" from a perturbative point of view, as the gauge hierarchy or the tiny value of dark energy, may find explanations in the genuinely non-perturbative setting of quantum gravity.
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