The linear unicyclic hypergraph with the second or third largest spectral radius

Chao Ding\textsuperscript{1,2}, Yi-Zheng Fan\textsuperscript{2}†, Jiang-Chao Wan\textsuperscript{2}

1. School of Mathematics and Computational Science, Anqing Normal University, Anqing 246133, P. R. China
2. School of Mathematical Sciences, Anhui University, Hefei 230601, P. R. China

Abstract: The spectral radius of a uniform hypergraph is defined to be that of the adjacency tensor of the hypergraph. It is known that the unique unicyclic hypergraph with the largest spectral radius is a nonlinear hypergraph, and the unique linear unicyclic hypergraph with the largest spectral radius is a power hypergraph. In this paper we determine the unique linear unicyclic hypergraph with the second or third largest spectral radius, where the former hypergraph is a power hypergraph and the latter hypergraph is a non-power hypergraph.

Keywords: Linear unicyclic hypergraph; adjacency tensor; spectral radius; weighted incidence matrix

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1 Introduction

A hypergraph \(G = (V, E)\) consists of a nonempty vertex set \(V = \{v_1, v_2, \cdots, v_n\}\) denoted by \(V(G)\) and an edge set \(E = \{e_1, e_2, \cdots, e_m\}\) denoted by \(E(G)\), where \(e_i \subseteq V\) for \(i \in [m] := \{1, 2, \cdots, m\}\). If \(|e_i| = k\) for each \(i \in [m]\) and \(k \geq 2\), then \(G\) is called a \(k\)-uniform hypergraph. In particular, the 2-uniform hypergraphs are exactly the classical simple graphs. The degree of a vertex is the number of edges containing the vertex. A vertex \(v\) of \(G\) is called a cored vertex if it has degree one. An edge \(e\) of \(G\) is called a pendant edge if it contains \(|e| - 1\) cored vertices. Sometimes a cored vertex in a pendant edge is also called a pendant vertex.

A walk \(W\) of length \(l\) in \(G\) is a sequence of alternate vertices and edges: \(v_0e_1v_1e_2 \cdots e_lv_l\), where \(\{v_i, v_{i+1}\} \subseteq e_i\) for \(i = 0, 1, \ldots, l - 1\). If \(v_0 = v_l\), then \(W\) is called a circuit. A walk of \(G\) is called a path if no vertices or edges are repeated. A circuit \(G\) is called a cycle if no vertices or edges are repeated except \(v_0 = v_l\). The hypergraph \(G\) is said to be connected if every two vertices are connected by a walk.

If \(G\) is connected and acyclic, then \(G\) is called a hypertree (also called supertree in \cite{16} and other literatures). It is known that a \(k\)-uniform hypertree on \(n\) vertices has \(\frac{n - 1}{k - 1}\) edges \cite[Proposition 4, p.392]{2}. If \(G\) is connected and contains exactly one cycle, then \(G\) is called a unicyclic hypergraph. A \(k\)-uniform unicyclic hypergraph on \(n\) vertices has \(\frac{n}{k + 1}\) edges \cite{7}.

Hu, Qi and Shao \cite{14} introduced a class of hypergraphs constructed from simple graphs. Let \(G = (V, E)\) be a simple graph. For any \(k \geq 3\), the \(k\)-th power of \(G\), denoted by \(G^k := (V^k, E^k)\), is defined as the \(k\)-uniform hypergraph with the set of vertices \(V^k := \{v_1, v_2, \cdots, v_n\}\) and the set of edges \(E^k\) consisting of all \(k\)-sets of \(V\) with the property that for any \(k\)-set \(S = \{v_1, v_2, \cdots, v_k\}\) of \(V\), if \(v_i, v_j \in V\) for \(i, j \in [k]\), then \(v_i = v_j\) if and only if \(v_i, v_j \in S\).

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†Corresponding author. E-mail address: fanyz@ahu.edu.cn(Y.-Z. Fan), dcmath@sina.cn (C. Ding), 1500256209@qq.com (J.-C. Wan).
V \cup \{i_e,1,\ldots,i_e,k-2|e \in E\} and the set of edges \(E^k := \{e \cup \{i_e,1,\ldots,i_e,k-2\}|e \in E\}\). If a hypergraph can be obtained from the power of a simple graph, then we will such hypergraph a \textit{power hypergraph}. If for any two distinct edges \(e_i, e_j\) of \(G\), \(|e_i \cap e_j| \leq 1\), then \(G\) is called a \textit{linear hypergraph}. It is known that all hypertrees and power hypergraphs are linear. For a unicyclic hypergraph \(G\), if \(G\) is linear, then the unique cycle of \(G\) is a power of a cycle (as a simple graph) of length at least 3; otherwise, \(G\) contains a pair of edges sharing exactly two vertices which yields the unique cycle of \(G\), and any other pair of edges shares at most one vertices.

The \textit{adjacency tensor} of a \(k\)-uniform hypergraph \(G\) on \(n\) vertices is defined to be a \(k\)th order \(n\) dimensional tensor \(A(G) = (a_{i_1,i_2,\ldots,i_k})\), where

\[
a_{i_1,i_2,\ldots,i_k} = \begin{cases} \frac{1}{(k-1)!}, & \text{if } \{i_1, i_2, \cdots, i_k\} \in E(G), \\ 0, & \text{orthwise.} \end{cases}
\]

Qi \cite{20} introduces the eigenvalues of a supersymmetric tensor, from which one can get the definition of the eigenvalues of the adjacency tensor of a uniform hypergraph. The \textit{spectral radius} of a uniform hypergraph is the maximum modulus of the eigenvalues of its adjacency tensor; see more in Section 2.

The spectral hypergraph theory has emerged as a hot topic in algebraic graph theory \cite{1, 6, 8, 9, 17, 19, 25, 29, 31}. Among all uniform hypertrees with given number of vertices or edges, researchers worked on the ordering the hypertrees by their spectral radii. In 2015, Li, Shao and Qi \cite{16} determined the hypertrees with the largest and the second largest spectral radii respectively. In 2016, Yuan, Shao and Shan \cite{20} determined the first eight hypertrees with largest spectral radii, and in 2017 Yuan, Si and Zhang \cite{27} determined the ninth and tenth hypertrees with largest spectral radii. In 2016, Fan, Tan, Peng and Liu \cite{7} investigated the hypergraphs that attain largest spectral radii among all hypergraphs with given number of edges. They determined the unique unicyclic hypergraphs with the largest spectral radius, which is not a linear hypergraph; and they also determined the unique linear unicyclic hypergraph with the largest spectral radius, which is a power hypergraph. They proposed several candidates for the linear bicyclic hypergraph with the largest spectral radius. Later in 2018, Kang et al. \cite{15} confirmed a conjecture in \cite{7} which lead to the unique linear bicyclic hypergraph with the largest spectral radius. Recently, Ouyang, Qi and Yuan \cite{18} considered the nonlinear hypergraphs, and determined the first five unicyclic hypergraphs and first three bicyclic hypergraphs with largest spectral radii. Other works on the ordering of hypertrees or unicyclic hypergraphs can be referred to \cite{5, 11, 21, 22, 23, 24, 30}.

In this paper we continue the work on the ordering of linear unicyclic hypergraphs by their spectral radii, and determine the the unique linear unicyclic hypergraph with the second or third largest spectral radius, where the former hypergraph is a power hypergraph and the latter hypergraph is a non-power hypergraph.

## 2 Preliminaries

For integers \(k \geq 3\) and \(n \geq 2\), a real \textit{tensor} (also called \textit{hypermatrix}) \(T = (t_{i_1,\ldots,i_k})\) of order \(k\) and dimension \(n\) refers to a multidimensional array with entries \(t_{i_1,i_2,\ldots,i_k}\) such that \(t_{i_1,i_2,\ldots,i_k} \in \mathbb{R}\) for all \(i_j \in [n]\) and \(j \in [k]\). The tensor \(T\) is called \textit{symmetric} if its entries are invariant under any permutation of their indices. Given a vector \(x \in \mathbb{R}^n\), \(Tx^k\) is a real
number, and $T^{x^{k-1}}$ is an $n$-dimensional vector, which are defined as follows:

$$T^x = \sum_{i_1, i_2, \ldots, i_k \in [n]} t_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \cdots x_{i_k},$$

$$(T^{x^{k-1}})_i = \sum_{i_2, i_3, \ldots, i_k \in [n]} t_{i_2 i_3 \ldots i_k} x_{i_2} x_{i_3} \cdots x_{i_k}, \text{ for } i \in [n].$$

Let $I$ be the identity tensor of order $k$ and dimension $n$, that is, $II = 1$ if and only if $i = 1 = 2 = \cdots = k \in [n]$ and zero otherwise.

**Definition 2.1** \[20, 4\] Let $T$ be a $k$-th order $n$-dimensional real tensor. For some $\lambda \in \mathbb{C}$, if the polynomial system $(\lambda I - T)x^{k-1} = 0$, or equivalently $Tx^{k-1} = \lambda x^{[k-1]}$, has a solution $x \in \mathbb{C}^n \setminus \{0\}$, then $\lambda$ is called an eigenvalue of $T$ and $x$ is an eigenvector of $T$ associated with $\lambda$, where $x^{[k-1]} := (x_1^{k-1}, x_2^{k-1}, \ldots, x_n^{k-1}) \in \mathbb{C}^n$.

If $x$ is a real eigenvector of $T$, surely the corresponding eigenvalue $\lambda$ is real. In this case, $x$ is called an $H$-eigenvector and $\lambda$ is called an $H$-eigenvalue. The spectral radius of $T$ is defined as

$$\rho(T) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } T\}.$$  

By the Perron-Frobenius theorem for nonnegative tensors [3, 10, 28], the spectral radius of $A(G)$, also referred to the spectral radius of $G$, denoted by $\rho(G)$, is exactly the largest $H$-eigenvalue of $A(G)$. If $G$ is connected, there exists a unique positive eigenvector up to scales corresponding to $\rho(G)$, called the Perron vector of $G$.

Li, Shao and Qi [16] introduce the operation of moving edges on hypergraphs. Let $r \geq 1$ and let $G$ be a hypergraph with $u \in V(G)$ and $e_1, \ldots, e_r \in E(G)$ such that $u \notin e_i$ for $i = 1, \ldots, r$. Suppose that $v_i \in e_i$ and write $e'_i = (e_i \setminus \{v_i\}) \cup \{u\}$ for $i = 1, \ldots, r$. Let $G'$ be the hypergraph with $V(G') = V(G)$ and $E(G') = (E \setminus \{e_1, \ldots, e_r\}) \cup \{e'_1, \ldots, e'_r\}$. We say that $G'$ is obtained from $G$ by moving edges $(e_1, \ldots, e_r)$ from $(v_1, \ldots, v_r)$ to $u$.

**Lemma 2.2** [16] Let $r \geq 1$ and let $G$ be a connected hypergraph. Let $G'$ be obtained from $G$ by moving edges $(e_1, \ldots, e_r)$ from $(v_1, \ldots, v_r)$ to $u$. Assume that $G'$ contains no multiple edges. If $x$ is a Perron vector of $G$ and $x_u \geq \max_{1 \leq i \leq r} x_{v_i}$, then $\rho(G') > \rho(G)$.

Fan et al. [7] introduced a special case of moving edges. Let $G_1, G_2$ be two vertex-disjoint hypergraphs, where $v_1, v_2$ are two distinct vertices of $G_1$ and $u$ is a vertex of $G_2$ (called the root of $G_2$). Let $G = G_1(v_2) \ast G_2(u)$ (respectively, $G' = G_1(v_1) \ast G_2(u)$) be the hypergraph obtained by identifying $v_2$ with $u$ (respectively, identifying $v_1$ with $u$); see the hypergraphs in Fig. 24. It is said that $G'$ is obtained from $G$ by relocating $G_2$ rooted at $u$ from $v_2$ to $v_1$.

**Lemma 2.3** [7] Let $G = G_1(v_2) \ast G_2(u)$ and $G' = G_1(v_1) \ast G_2(u)$ be two connected hypergraphs. If there exists a Perron vector $x$ of $G$ such that $x_{v_1} \geq x_{v_2}$, then $\rho(G') > \rho(G)$.

Yuan, Shao and Shan [20] defined an new type of edge-moving operation.

**Definition 2.4** [20] Let $e, f$ be two edges of a $k$-uniform connected hypergraph $G$ such that $e \cap f = V_i$, where $|V_i| = k - r$, $2 \leq r \leq k - 1$. Write $e \setminus V_i = \{u_1, \ldots, u_r\}$ and $f \setminus V_i = \{v_1, \ldots, v_r\}$, where $u_1$ and $v_1$ are non-pendent vertices, but $u_2, \ldots, u_r$ and $v_2, \ldots, v_r$ are all pendent vertices. Define $G_{e,f}$ be the hypergraph obtained from $G$ by moving all edges incident to $v_1$ except $f$ from $v_1$ to $u_2$. 
Lemma 2.5 \[26\] Let $G$ be a $k$-uniform connected hypergraph, and let $e, f$ be two edges of $G$ satisfying the condition in Definition 2.4. Then $\rho(G_{e,f}) > \rho(G)$.

Lu and Man [17] introduced a novel method for computing or comparing the spectral radii of hypergraphs.

Definition 2.6 [17] A weighted incidence matrix $B$ of a hypergraph $G = (V,E)$ is a $|V| \times |E|$ matrix such that for any vertex $v$ and any edge $e$, the entry $B(v,e) > 0$ if $v \in e$ and $B(v,e) = 0$ if $v \notin e$.

Definition 2.7 [17] Let $G$ be hypergraph with a weighted incidence matrix $B$.

(1) $G$ is called $\alpha$-normal if $B$ satisfies

(i) $\sum_{v \in e} B(v,e) = 1$, for any $v \in V(G)$,

(ii) $\prod_{v \in e} B(v,e) = \alpha$, for any $e \in E(G)$.

(2) $G$ is called $\alpha$-supernormal if $B$ satisfies

(i) $\sum_{v \in e} B(v,e) \geq 1$, for any $v \in V(G)$.

(ii) $\prod_{v \in e} B(v,e) \leq \alpha$, for any $e \in E(G)$.

Moreover $G$ is called strictly $\alpha$-supernormal if $G$ is $\alpha$-supernormal but not $\alpha$-normal.

(3) The incidence matrix $B$ is called consistent if for any cycle $v_0 e_1 v_1 e_2 \cdots e_{l-1} v_l$ ($v_l = v_0$)

$$\prod_{i=1}^{l} \frac{B(v_i,e_i)}{B(v_{i-1},e_i)} = 1.$$ 

In this case, we call $G$ consistently $\alpha$-normal (resp. consistently $\alpha$-supernormal) if $G$ is also $\alpha$-normal (resp. $\alpha$-supernormal).

Lemma 2.8 [17] Let $G$ be a connected $k$-uniform hypergraph. Then the following results hold.

(1) $G$ is consistently $\alpha$-normal if and only if $\rho(G) = \alpha^{-1/k}$.

(2) If $G$ is strictly and consistently $\alpha$-supernormal, then $\rho(G) > \alpha^{-1/k}$.

Lemma 2.9 [31] Let $G^k$ be the $k$-th power of a simple graph $G$. Then $\rho(G^k) = \rho(G)^{\frac{k}{2}}$.

3 Main results

We first introduce some special graphs and hypergraphs. Let $K_{1,s}$ be a star on $1+s$ vertices, and let $C_n$ be cycle of length $n$, both as simple graphs. Let $S_{m,g}$ be a unicyclic graph obtained from a cycle $C_g$ by attaching a star $K_{1,m-g}$ at some vertex. Let $T_{m,1}$ (respectively, $T_{m,2}$) be
obtained from $S_{m-1,3}$ (respectively, $S_{m-2,3}$) by attaching one pendent edge (respectively, two pendent edges) at some vertex of degree 2. Let $U_{m,1}$ be obtained from $S_{m-1,3}$ by attaching one pendent edge at some pendent vertex.

The $k$-th power $K_{1,s}^k$ of $K_{1,s}$, is called a hyperstar with $s$ edges, where the vertex of maximum degree is called the center of the hyperstar. Let $O_m$ be the $k$-uniform hypergraph obtained from the power $C_{3}^k$ by attaching a hyperstar $K_{1,m-3}^k$ with its center at some cored vertex. Let $Q_m$ (respectively, $P_m$) be the $k$-uniform hypergraph obtained from the power $S_{m-1,3}^k$ by attaching a pendent edge to a cored vertex on the cycle adjacent to (respectively, not adjacent to) the vertex with maximum degree of $S_{m-1,3}^k$.

**Lemma 3.1** For $m \geq 4$, $\rho(Q_m) < \rho(T_{m,1}^k)$.

**Proof.** Label the partial vertices and edges of $Q_m$ as in Fig. 3.1 where $v_1e_1v_2e_2v_3e_3v_1$ is the 3-cycle, and $w$ is the vertex of the edge $e_2$ to which a pendent edge is attached. Let $x$ be a Perron vector of $Q_m$. If $x_v \geq x_w$, moving the pendent edge attached at $w$ from $w$ to $v_3$, we arrive at the hypergraph $T_{m,1}^k$. By Lemma 2.2, $\rho(Q_m) < \rho(T_{m,1}^k)$. Otherwise, $x_w > x_v$, moving the edge $e_3$ from $v_3$ to $w$, we also arrive at the hypergraph $T_{m,1}^k$. So, By Lemma 2.2, $\rho(Q_m) < \rho(T_{m,1}^k)$. The result follows. \hfill $\square$

**Lemma 3.2** For $m \geq 5$, $\rho(P_m) < \rho(Q_m)$.

**Proof.** Label the partial vertices and edges of $P_m$ as in Fig. 3.1 where $v_1e_1v_2e_2v_3e_3v_1$ is the 3-cycle, and $w$ is the vertex of the edge $e_3$ to which a pendent edge is attached. We first construct a consistently $\alpha$-normal weighted incidence matrix $B$ of $P_m$. Let $r := m - 4 \geq 1$, the number of pendent edges attached at $v_2$. For each cored vertex $v$ incident to the unique edge $e$, $B(v,e) = 1$. For each pendent edge $e$ attached at $v \in \{v_2, w\}$, define $B(v,e) = \alpha$.

Define

\[
B(w,e_3) = 1 - \alpha, B(v_1,e_3) = B(v_3,e_3) = \sqrt{\frac{\alpha}{1-\alpha}} =: \beta, B(v_1,e_1) = B(v_3,e_2) = 1 - \beta, B(v_2,e_1) = B(v_2,e_2) = \frac{\alpha}{1-\beta}.
\]
Then $B$ is consistent. Let
\[ f_P(\alpha) := \frac{2\alpha}{1 - \sqrt{1 - \alpha}} + r\alpha. \]  
(3.1)

Then $P_m$ is consistently $\alpha_0$-normal if $f_P(\alpha) = 1$ has a solution $\alpha_0 \in (0, \frac{1}{2})$. Observe that $f_P(\alpha) \to 0^+$ if $\alpha \to 0^+$, $f_P(\alpha) \to +\infty$ if $\alpha \to \frac{1}{2}^-$, and $f_P(\alpha)$ is strictly increasing in $(0, \frac{1}{2})$.

So $f_P(\alpha) = 1$ has a unique solution $\alpha_0 \in (0, \frac{1}{2})$, and $\rho(P_m) = \alpha_0^{-\frac{1}{2}}$ by Lemma 2.8. As $f_P(\frac{1}{2}) \geq 1$ and $f_P(\frac{1}{2^+}) > 1$,
\[ \alpha_0 \leq \frac{1}{5}, \alpha_0 < \frac{1}{r + 2}. \]  
(3.2)

We next define a weighted incident matrix $\bar{B}$ of $Q_m$. For each cored vertex $v$ incident to the unique edge $e$, $\bar{B}(v, e) = 1$. For each pendent edge $e$ attached at $v \in \{v_2, w\}$, define $\bar{B}(v, e) = \alpha$. Define
\[ \bar{B}(v_3, e_2) = x, \bar{B}(v_2, e_2) = \frac{\alpha}{(1 - \alpha)x}, \bar{B}(v_3, e_3) = 1 - x, \]
\[ \bar{B}(v_1, e_3) = \frac{\alpha}{1 - x} =: \beta, \bar{B}(v_1, e_1) = 1 - \beta, \bar{B}(v_2, e_1) = \frac{\alpha}{1 - \beta}. \]

To make $\bar{B}$ be strictly consistently $\alpha$-supernormal, we need
\[ (1 - \beta) \cdot \frac{\alpha}{(1 - \alpha)x} \cdot (1 - x) = \beta \cdot \frac{\alpha}{1 - \beta} \cdot x. \]  
(3.3)

\[ h(x) := \frac{\alpha}{1 - \beta} + \frac{\alpha}{(1 - \alpha)x} + r\alpha > 1, \]  
(3.4)

By Eq. (3.3), we have
\[ x = \frac{1 - \alpha}{1 + \sqrt{\alpha(1 - \alpha)}} =: \gamma. \]  
(3.5)

Now substituting (3.5) to $h(x)$ by taking $\alpha = \alpha_0$, and combining Eq. (3.1) and the fact $f_P(\alpha_0) = 1$, we have
\[ h(\gamma) = \frac{2\alpha_0}{1 - \alpha_0} \sqrt{\frac{\alpha_0}{1 - \alpha_0}} + \frac{\alpha_0(2 - \alpha_0)}{(1 - \alpha_0)^2} + r\alpha_0 \]
\[ = 1 + \frac{2\alpha_0}{1 - \alpha_0} \sqrt{\frac{\alpha_0}{1 - \alpha_0}} + \frac{\alpha_0(2 - \alpha_0)}{(1 - \alpha_0)^2} - \frac{2\alpha_0}{1 - \sqrt{1 - \alpha_0}} \]
\[ = 1 + \frac{\alpha_0^2}{(1 - \alpha_0)^2(1 - 2\alpha_0)} \left(1 - 4\alpha_0 + 2\alpha_0^2 - 2\alpha_0\sqrt{\alpha_0(1 - \alpha_0)}\right) \]
\[ = 1 + \frac{\alpha_0^2}{(1 - \alpha_0)^2(1 - 2\alpha_0)} \left(-1 + 2\sqrt{1 - \alpha_0 \left((1 - \alpha_0)^{\frac{1}{2}} - \alpha_0^{\frac{1}{2}}\right)}\right) . \]

Let $\phi(\alpha) := -1 + 2\sqrt{1 - \alpha \left((1 - \alpha)^{\frac{1}{2}} - \alpha^{\frac{1}{2}}\right)}$. As $\alpha_0 \leq \frac{1}{5}$ by (3.2), $\phi(\alpha_0) \geq \phi\left(\frac{1}{5}\right) = \frac{2}{25} > 0$, and hence $h(\gamma) > 1$. So, $Q_m$ is strictly consistently $\alpha_0$-supernormal, and by Lemma 2.8
\[ \rho(Q_m) > \alpha_0^{-\frac{1}{2}} = \rho(P_m) . \]

The result follows. \(\square\)

**Lemma 3.3** For $m \geq 5$, $\rho(O_m) < \rho(P_m)$.

**Proof.** Label the partial vertices and edges of $O_m$ as in Fig. 3.1 where $v_1e_1v_2e_2v_3e_3v_1$ is the 3-cycle, and $w$ is the vertex of the edge $e_3$ to which a hyperstar is attached. We define a weighted incidence matrix $B$ of $O_m$ as follows. For each pendent vertex $v$ incident to the
unique edge \( e \), \( B(v,e) = 1 \). For each pendent edge \( e \) attached at \( w \), define \( B(w,e) = \alpha \). Let \( r := m - 4 \geq 1 \). Define
\[
\begin{align*}
B(w,e_3) &= 1 - (r + 1)\alpha, B(v_1,e_3) = B(v_2,e_3) = \beta, \\
B(v_1,e_1) &= B(v_3,e_2) = 1 - \beta, B(v_2,e_1) = B(v_2,e_2) = \frac{\alpha}{1-\beta}.
\end{align*}
\]
It is easily seen \( B \) is consistent. To make \( O_m \) be \( \alpha \)-normal, we require
\[
\beta = \sqrt{\frac{\alpha}{1 - (r + 1)\alpha}}, \quad \frac{2\alpha}{1 - \beta} = 1.
\]
Let
\[
f_{O}(\alpha) := \frac{2\alpha}{1 - \sqrt{\frac{\alpha}{1 - (r + 1)\alpha}}}, \quad \tag{3.6}
\]
Observe that \( f_{O}(\alpha) \to 0 + \) if \( \alpha \to 0 + \), \( f_{O}(\alpha) \to +\infty \) if \( \alpha \to \frac{1}{r+2} - \), and \( f_{O}(\alpha) \) is strictly increasing in \( \alpha \in (0, \frac{1}{r+2}) \). So, there exists a unique \( \alpha_1 \in (0, \frac{1}{r+2}) \) such that \( f_{O}(\alpha_1) = 1 \). Hence \( O_m \) is consistently \( \alpha_1 \)-normal, and \( \rho(O_m) = \alpha_1^{-\frac{1}{2}} \) by Lemma 2.8.

As \( f_{O}(\alpha_1) = 1 \), by Eq. (3.6), we have
\[
\frac{r\alpha_1}{1 - \alpha_1} = 1 - \frac{\alpha_1}{(1 - 2\alpha_1)^2}. \tag{3.7}
\]
Substituting (3.7) into Eq. (3.1), we have
\[
f_{P}(\alpha_1) = \frac{2\alpha_1}{1 - \sqrt{\frac{\alpha_1}{1 - (1 - 2\alpha_1)^2}}} + 1 = 1 - \frac{\alpha_1}{(1 - 2\alpha_1)^2}
\]
\[
= 1 + \frac{2\alpha_1(1 - 2\alpha_1) \sqrt{\alpha_1 (1 - \alpha_1)} - \alpha_1}{(1 - 2\alpha_1)^2}
\]
Let \( \psi(\alpha) := (1 - 2\alpha) \sqrt{\alpha_1 (1 - \alpha)} - \alpha \). When \( \alpha \in (0, \frac{1}{2}) \), \( \psi(\alpha) > 0 \) if and only if
\[
1 - \alpha - \frac{\alpha}{(1 - 2\alpha)^2} > 0.
\]
By Eq. (3.7), surely \( \psi(\alpha_1) > 0 \). So, \( f_{P}(\alpha_1) > 1 \), and hence \( \alpha_0 < \alpha_1 \) as \( f_{O}(\alpha) \) and \( f_{P}(\alpha) \) are strictly increasing in \( (0, \frac{1}{r+2}) \). By Lemma 2.8
\[
\rho(O_m) = \alpha_1^{-\frac{1}{2}} < \alpha_0^{-\frac{1}{2}} = \rho(P_m).
\]
The result follows. \( \Box \).

**Lemma 3.4** For \( m \geq 4 \), \( \rho(S_{m,4}^k) < \rho(O_m) \).

**Proof.** Label the partial vertices and edges of \( S_{m,4}^k \) as in Fig. 3.1. Let \( e_1, e_2 \) be two non-pendent edges of \( S_{m,4}^k \) incident to the vertex of maximum degree, and let \( \{u_1, u_2\} \subseteq e_1 \) and \( \{v_1, v_2\} \subseteq e_2 \), where \( u_2, v_2 \) are pendent (cored) and \( u_1, v_1 \) are non-pendent. Let \( e_3 \) be the edge incident to \( v_1 \) except \( e_2 \). Then \( e_1, e_2 \) satisfy the condition in Definition 2.4 and by moving \( e_3 \) from \( v_1 \) to \( u_2 \), we get a hypergraph \( S_{m,4, e_1, e_2}^k \) which is isomorphic to \( O_m \). By Lemma 2.6, we have
\[
\rho(S_{m,4}^k) < \rho(S_{m,4, e_1, e_2}^k) = \rho(O_m).
\]
The result follows. \( \Box \).

**Lemma 3.5** For \( m \geq 8 \), \( \rho(T_{m,3}^k) < \rho(U_{m,1}^k) \).

\[
\]
Lemma 3.6 For \( m \geq 5 \), \( \rho(U_{m,1}^k) < \rho(Q_m) \).

Proof. Label the partial vertices and edges of \( U_{m,1}^k \) as in Fig. 3.1. Let \( e_4 \) be the pendent edge incident to \( w \), and let \( e_5 \) be the non-pendent edge incident to \( w \). Now \( e_2, e_5 \) satisfy the condition in Definition 2.4. Let \( u \) be a pendent (cored) vertex of \( e_2 \). Moving \( e_4 \) from \( w \) to \( u \), we get a hypergraph \( U_{m,1}^k e_2, e_5 \) isomorphic to \( Q_m \). By Lemma 2.5, we have

\[
\rho(U_{m,1}^k) < \rho(U_{m,1}^k e_2, e_5) = \rho(Q_m).
\]

The result follows.

We now determine the linear unicyclic hypergraph with the second or third largest spectral radius among all linear unicyclic hypergraph with \( m \) edges. We need the following result.

Lemma 3.7 \cite{7} (1) Among all unicyclic linear \( k \)-uniform hypergraphs with \( m \geq 4 \) edges and girth \( g \), the power hypergraph \( S_{m,g}^k \) is the unique maximizing hypergraph.

(2) For \( g \geq 4 \), \( \rho(S_{m,g}^k) < \rho(S_{m,g-1}^k) \).

(3) Among all unicyclic linear \( k \)-uniform hypergraphs with \( m \geq 4 \) edges, \( S_{m,3}^k \) is the unique maximizing hypergraph.

Theorem 3.8 Among all linear unicyclic \( k \)-uniform hypergraphs with \( m \geq 5 \) edges, \( T_{m,1}^k \) is the unique hypergraph with the second largest spectral radius.

Proof. Let \( G \) be a hypergraph with the second largest spectral radius among all linear unicyclic \( k \)-uniform hypergraphs with \( m > 3 \) edges. Surely \( G \neq S_{m,3}^k \) by Lemma 3.7. In the following we call a hypergraph proper if it is not equal to \( S_{m,3}^k \). Suppose \( G \) has girth \( g \). We assert that \( g = 3 \); otherwise, by Lemma 3.7 and Lemma 3.4

\[
\rho(G) \leq \rho(S_{m,g}^k) \leq \rho(S_{m,3}^k) < \rho(O_m).
\]

(3.8)

So \( G \) is obtained from a cycle \( C \) of length 3 by attached some hypertrees at its vertices.

Let \( x \) be a Perron vector of \( G \). The result will follows by the following cases.

Case 1. Exactly one hypertree is attached to some vertex of \( C \). Let \( T_u \) be such hypertree attached at \( u \) of \( C \). Write \( G = C(u) * T_u(u) \).

Case 1.1. \( u \) is a cored vertex of \( C \). Then \( u \) is the unique vertex of \( T_u \) such that \( x_u = \max\{x_v : v \in V(T_u)\} \). Otherwise, let \( v \in V(T_u) \setminus \{u\} \) such that \( x_v \geq x_u \). Relocating \( C \) from \( u \) to \( v \), we will get a proper hypergraph \( G' \) which holds \( \rho(G') > \rho(G) \) by Lemma 2.3 a contradiction.

We assert \( T_u \) is a hyperstar with center \( u \). Otherwise there exists a pendent edge \( e \) of \( T_u \) incident to a non-cored vertex \( w \neq u \). Relocating the edge \( e \) from \( w \) to \( u \), we also get a proper hypergraph but with a larger spectral radius, a contradiction. So \( G = O_m \) in this case. However, by Lemma 3.3 \( \rho(O_m) < \rho(P_m) \). So this case cannot happen.

Case 1.2. \( u \) is a vertex of \( C \) of degree two. Then \( u \) is the unique vertex of \( T_u \) such that \( x_u = \max\{x_v : v \in V(T_u)\} \). Otherwise, let \( v \in V(T_u) \setminus \{u\} \) such that \( x_v \geq x_u \), and let \( e \) be an edge of \( C \) incident to \( u \). Moving \( e \) from \( u \) to \( v \), we also get a proper hypergraph \( G' \) with girth at least 4 and a larger spectral radius, a contradiction. We assert \( G = U_{m,1}^k \).

Otherwise, as \( G \notin \{S_{m,3}^k, U_{m,1}^k\} \), there exists a pendent edge \( e \) of \( T_u \) incident to a non-cored
vertex \( w \neq u \). Relocating the edge \( e \) from \( w \) to \( u \), we also get a proper hypergraph but with a larger spectral radius, a contradiction. However, by Lemma 3.6 \( \rho(U_{m,1}^k) < \rho(Q_m) \). So this case cannot happen.

**Case 2.** At least two hypertrees are attached at different vertices of \( C \). Suppose that there are \( s \) vertices, say \( v_1, \ldots, v_s \) of \( C \), are attached \( s \) hypertrees, where \( s \geq 3 \). Without loss of generality, assume that \( x_{v_1} = \max \{ x_{v_i} : i = 1, \ldots, s \} \). Relocating the hypertree attached at \( v_s \) from \( v_s \) to \( v_1 \), we will get a proper hypergraph with larger spectral radius, a contradiction. So, there are exactly two hypertrees, say \( T_u \) and \( T_v \), attached at \( u \) and \( v \) of \( C \) respectively. By a similar discussion as in Case 1.1, \( T_u \) and \( T_v \) are hyperstars with center \( u \) and \( v \) respectively.

**Case 2.1.** Both \( u \) and \( v \) are cored vertices of \( C \). Without loss of generality, \( x_u \geq x_v \). Relocating \( T_v \) from \( v \) to \( u \), we will get the hypergraph \( O_m \) with larger spectral radius, \( \rho(O_m) < \rho(Q_m) \), implying this case also cannot happen.

**Case 2.2.** One of \( u, v \) is a cored vertex and the other is non-cored vertex of \( C \). Without loss of generality, \( u \) is cored and \( v \) is non-cored. First assume that \( u, v \) are adjacent. If \( x_u \geq x_v \), relocating \( T_v \) from \( v \) to \( u \), we will arrive the hypergraph \( O_m \), a contradiction. Otherwise, \( x_u < x_v \), moving all pendent edges except one arbitrarily specified edge incident with \( u \) from \( u \) to \( v \) if there exists more than one pendent edges incident to \( u \), we will get the hypergraph \( Q_m \) with larger spectral radius. So \( G = Q_m \) in this case. However, by Lemma 3.1 \( \rho(Q_m) < \rho(T_{m,1}^k) \), implying this case also cannot happen.

Secondly assume that \( u, v \) are not adjacent. Similarly, if \( x_u \geq x_v \), relocating \( T_v \) from \( v \) to \( u \), we will arrive the hypergraph \( O_m \), a contradiction. If \( x_u < x_v \), moving all pendent edges except one arbitrarily specified edge incident with \( u \) from \( u \) to \( v \) if there exists more than one pendent edges incident to \( u \), we will get the hypergraph \( P_m \) with larger spectral radius. So \( G = P_m \) in this case. However, by Lemma 3.2 \( \rho(P_m) < \rho(Q_m) \), implying this case also cannot happen.

**Case 2.3.** Both \( u, v \) are non-cored vertices of \( C \). Without loss of generality, assume that \( x_u \geq x_v \). Moving all pendent edges incident with \( v \) except one arbitrarily specified edge from \( v \) to \( u \) (if there exists more than one pendent edges incident to \( v \), we will get the hypergraph \( T_{m,1}^k \) with larger spectral radius. So \( G = T_{m,1}^k \) in this case.

**Theorem 3.9** Among all linear unicyclic \( k \)-uniform hypergraphs with \( m \geq 8 \) edges, \( Q_m \) is the unique hypergraph with the third largest spectral radius.

**Proof.** Let \( G \) be a hypergraph with the third largest spectral radius among all linear unicyclic \( k \)-uniform hypergraphs with \( m \geq 8 \) edges. Surely \( G \notin \{ S_{m,3}^k, T_{m,1}^k \} \) by Lemma 3.7 and Lemma 3.8. In the following we call a hypergraph proper if it is not equal to \( S_{m,3}^k \) or \( T_{m,1}^k \). By (3.8), we know \( G \) has girth 3, and \( G \) is obtained from a cycle \( C \) of length 3 by attached some hypertrees at its vertices. Let \( x \) be a Perron vector of \( G \). We now follow the routine of the proof of Theorem 3.8.

**Case 1.** Exactly one hypertree is attached to some vertex, say \( u \), of \( C \). We assert \( u \) is a non-cored vertex, and hence \( G = U_{m,1}^k \) by Case 1.2 in the proof of Theorem 3.8. However, by Lemma 3.6 \( \rho(U_{m,1}^k) < \rho(Q_m) \), a contradiction. Otherwise, if \( u \) is a cored vertex, then from Case 1.1 of Theorem 3.8 \( G = O_m \) and \( \rho(O_m) < \rho(P_m) \) by Lemma 3.3 a contradiction.

**Case 2.** At least two hypertrees are attached at different vertices of \( C \). By a similar discussion as in Case 2 of Theorem 3.8 \( G \) is obtained from \( C \) by attaching two hyperstars \( T_u, T_v \) at \( u, v \) of \( C \), where \( u, v \) are the centers of \( T_u, T_v \) respectively.
Case 2.1. If $u,v$ are both cored vertices of $C$, then $\rho(G) < \rho(O_m)$, a contradiction.

Case 2.2. Suppose that $u$ is cored and $v$ is non-cored. If $u,v$ are adjacent, then $\rho(G) < \rho(O_m)$ or $G = Q_m$ depending on whether $x_u \geq x_v$ or not. So $G = Q_m$ in this case. If $u,v$ are not adjacent, then $\rho(G) < \rho(O_m)$ or $G = P_m$ depending on whether $x_u \geq x_v$ or not. However, $\rho(P_m) < \rho(Q_m)$ by Lemma 3.2 a contradiction.

Case 2.3. Finally suppose that $u,v$ are non-cored vertices of $C$. As $G \notin \{S_{m,3}^{k}, T_{m,1}^{k}\}$, both $T_u, T_v$ have at least 2 pendent edges. Without loss of generality, assume that $x_u \geq x_v$. Moving all pendent edges incident with $v$ except two arbitrarily specified edge from $v$ to $u$ if there exist more than two pendent edges incident to $v$, we will get the hypergraph $T_{m,2}^{k}$ with a larger spectral radius. So $G = T_{m,2}^{k}$. So $\rho(T_{m,2}^{k}) < \rho(U_{m,1}^{k})$ by Lemma 3.5 a contradiction.

The result now follows by the above discussion. \[\square\]

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