SUPERPROCESSES OVER A STOCHASTIC FLOW WITH SPATIALLY DEPENDENT BRANCHING

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Abstract. This paper considers a generalized model of [G. Skoulakis and R. J. Adler, Ann. Appl. Probab., 11 (2001), pp.488-543]. We show the existence of superprocesses in a random medium (flow) with location dependent branching. Technically, we make use of a duality relation to establish the uniqueness of the martingale problem and to obtain the moment formulas which generalize those of [G. Skoulakis and R. J. Adler, Ann. Appl. Probab., 11 (2001), pp.488-543].

Key words. superprocess, stochastic flow, martingale problem, dual process, moments

1. Motivation and introduction. Superprocesses over a stochastic flow are introduced in [12], where the motions of all particles are subject to the common noise (flow) and the branching of particles is independent of their motions. The authors of [12] used branching particle systems approximation to prove the existence of superprocesses over a stochastic flow (flow superprocesses for short). Moreover, they made detailed use of the approximating processes to establish the moments of the flow superprocesses. It was mentioned in [12] that the moment formulas could also be obtained by a dual method, which “would undoubtedly be more elegant” but was not adopted since it “does not seem to be straightforward” ([12, p.497]). In this paper we will consider a generalized model of [12], in which the branching of particles is location dependent. A similar model on Polish space was studied in the first part of [10]. When considering superprocesses, a martingale problem is usually inevitable and the duality approach usually plays a key role in deriving the uniqueness for the martingale problem. To establish uniqueness, the approach used in [12] was to justify the duality conditions of [1] instead of constructing a dual process. We shall use the latter as in [2] to show uniqueness and derive moment formulas for the flow superprocess as well.

In the rest of this section, we give a concise description of our model; the reader is referred to [12] for a more specific one. The main results are given as well as proved in the next section. In the final section, a further extension is provided. Let \( N \) be a positive integer, which varies whenever necessary. Let \( E = \mathbb{R}^d \) with \( \Delta \) its infinity and write \( \overline{E} = E \cup \{ \Delta \} \), the one-point compactification of \( E \). \( C_b(E) \) is the space of bounded continuous real-valued functions. \( C_b(E) \) denotes the subspace of \( C_b(E) \) such that its members have limits at infinity. \( C_b^2(E) \) stands for the subspace of \( C_b(E) \) such that its members have two continuous derivatives which have limits at infinity. \( C_b^2(E) \) consists of the elements in \( C_b(E) \) possessing bounded first and second partial derivatives. \( M_b(E) \) is the space of finite Borel measures on \( E \) endowed with weak convergence topology. \( D_{M_b(E)}[0, \infty) \) is the well-known Skorokhod space and the meaning of \( C_{M_b(E)}[0, \infty) \) is obvious. Let \( \Rightarrow \) and \( = \) denote weak convergence and uniform convergence, respectively. The superscript + attached to a set will mean its non-negative subset. Write \( \mu(h) \equiv \langle h, \mu \rangle \) as the integral of \( h \) with respect to the measure \( \mu \). Throughout the paper, let \( \mathbf{P} \) always denote the probability measure for the probability space involved and \( \mathbb{E} \) the corresponding expectation.

Let \( I = \{ \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_k) : k \geq 0, \alpha_i \in \{1, 2, \ldots\}, 0 \leq i \leq k \} \) be the family of multi-indices, setting \( |\alpha| = \left| (\alpha_0, \alpha_1, \ldots, \alpha_k) \right| = k, \alpha - 1 = (\alpha_0, \ldots, \alpha_{|\alpha|-1}) \) and \( \alpha_i = (\alpha_0, \ldots, \alpha_i) \). Let \( n = 1, 2, \ldots \). Suppose at time zero that \( K_n \) (deterministic) particles, located separately at \( x^1_n, \ldots, x^K_n \in E \), are given. For \( t \geq 0 \), write \( \alpha \sim_n t \) if and only if \( |\alpha|/n \leq t < (1 + |\alpha|)/n \) and \( \alpha_0 \leq K_n \). Each particle in our model is labeled by a multi-index in \( I \). A particle with label \( \alpha \) is understood to be born at time \( |\alpha|/n \) and to die at \( (1 + |\alpha|)/n \) with \( N^{*\alpha} \) offspring reproduced. For \( \alpha \sim_n t \) (between branching), the motion \( Y^{\alpha,n}_t \) of
particle $\alpha$ is determined by

$$dY_{t}^{\alpha,n} = b(Y_{t}^{\alpha,n})dt + e(Y_{t}^{\alpha,n})dB_{t}^{\alpha,n} + c(Y_{t}^{\alpha,n})dW_{t}^{n}, \quad Y_{0}^{\alpha,n} = x_{\alpha_{0}},$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$, $c : \mathbb{R}^d \to \mathbb{R}^{d \times m}$, and $e : \mathbb{R}^d \to \mathbb{R}^{d \times d}$; $W^{n}$ is an $\mathbb{R}^{m}$-valued Brownian motion, random environment (flow), independent of which is the family $\{B^{\alpha,n} : \alpha_{0} \leq K_{n}\}$ of $E$-valued Brownian motions stopped at time $t = (|\alpha| + 1)/n$. For each $k$, members in $\{B^{\alpha,n} : \alpha_{0} \leq K_{n}, |\alpha| = k\}$ are conditionally independent given $\sigma\{B^{\alpha,n} : \alpha_{0} \leq K_{n}, |\alpha| < k\}$, and $B_{t}^{\alpha,n} = B_{t}^{\alpha-1,n}$ for $t \leq |\alpha|/n$. Let $k_{n} = k/n$ and $a_{n} = 1/n$. Define for $t \in [k_{n}, k_{n} + a_{n})$ and $k = 0, 1, \ldots$

$$\mathcal{F}_{t}^{n} = \sigma(B^{\alpha,n}, N^{\alpha,n} : |\alpha| < k) \bigvee_{r > t} \sigma(W_{s}^{n}, B_{s}^{\alpha,n} : s \leq r, |\alpha| = k)$$

and

$$\tilde{\mathcal{F}}_{k_{n}}^{n} = \mathcal{F}_{k_{n}}^{n} \bigvee_{s \leq k_{n} + a_{n}, |\alpha| = k} \sigma(W_{s}^{n}, B_{s}^{\alpha,n} : s \leq k_{n} + a_{n}, |\alpha| = k).$$

Assume that $\{N^{\alpha,n} : |\alpha| = k\}$ are conditionally independent given $\tilde{\mathcal{F}}_{k_{n}}^{n}$, and

$$\left\{ \begin{array}{l}
\mathbb{E}(N^{\alpha,n}|\tilde{\mathcal{F}}_{k_{n}}^{n}) = 1 + \gamma_{n}(Y_{k_{n}+a_{n}}^{\alpha,n})/n =: \beta_{n}(Y_{k_{n}+a_{n}}^{\alpha,n}) \\
\text{Var}(N^{\alpha,n}|\tilde{\mathcal{F}}_{k_{n}}^{n}) = \sigma_{n}(Y_{k_{n}+a_{n}}^{\alpha,n})^{2},
\end{array} \right.$$

where $\gamma_{n} \in C_{1}(E)$ and $\sigma_{n} \in C_{1}(E)^{+}$. Now define

$$X_{t}^{n}(B) = \frac{\text{number of particles in } B \text{ at time } t}{n},$$

where $B$ is a Borel subset of $E$. Intuitively, $X_{t}^{n}$ characterizes the mass distribution of the particle system at time $t$.

It is worth pointing out that compared to [12] p.493, the different parts in our model are on the one hand the equation (1), where $e$ is extended to be non-diagonal. On the other hand the significant difference lies in the branching mechanism, which is location dependent as indicated in (2).

Suppose that there exist $p > 2$ and $C > 0$ such that

$$\mathbb{E}[(N^{\alpha,n})^{p}] \leq C \text{ for all } \alpha \text{ and } n, \gamma_{n} \Rightarrow \gamma \in C_{1}(E) \text{ and } \sigma_{n} \Rightarrow \sigma \in C_{1}(E)^{+} \text{ as } n \to \infty.$$  \hspace{1cm} (3)

$\gamma$ is called the drift function and $\sigma^{2}$ the branching variance. Let $C_{\gamma}, C_{\sigma}$ be the constants such that $|\gamma_{n}|, |\beta_{n}| \leq C_{\gamma}$ and $\sigma_{n} \leq C_{\sigma}$ for all $n$.

**Remark 1.1.** For each $\gamma \in C_{1}(E)$ and $\sigma \in C_{1}(E)^{+}$, there exist random variables $\xi_{n}$ such that (2) and (3) hold (see [10] p.143), and $p$ can be very close to 2.

2. **Continuous spatially dependent branching.** Based on a dual method, we shall discuss in this section the existence and moment properties of a flow superprocess with aforementioned parameters $\gamma$ and $\sigma$.

**Hypotheses (LU)**

(L) $|b(x) - b(y)| + \|c(x) - c(y)\| + \|e(x) - e(y)\| \leq K|x - y|, \quad x, y \in E.$

(U) $b_{i}, c_{il}, e_{ik} \in C^{2}_{l}(E), \ i, k = 1, \ldots, d, l = 1, \ldots, m$, and for any $N \geq 1$ there exists $\lambda_{N} > 0$ such that

$$\sum_{p,q=1}^{N} \sum_{i,j=1}^{d} \xi_{p}^{l}dij(x_{p}, x_{q})\xi_{j}^{q} \geq \lambda_{N} \sum_{p=1}^{d} \sum_{i=1}^{\xi_{l}^{p}} (\xi_{l}^{p})^{2}$$

for $x_{1}, \ldots, x_{n} \in E$ and $\{\xi_{1}^{l}, \ldots, \xi_{m}^{l}; \xi_{1}^{d}, \ldots, \xi_{m}^{d}\} \in E^{N}$, where $dij(x, y) = \sum_{k=1}^{d} e_{ik}(x)e_{jk}(y) + c_{ij}^{(m)}(x, y)$ with $a_{ij}^{(m)}(x, y) = \sum_{l=1}^{m} c_{il}(x)c_{jl}(y)$.  \hspace{1cm} (2)
Let \( Y = (Y^1, \ldots, Y^N) \) be the solution to the stochastic differential equation:

\[
\begin{align*}
\begin{cases}
    dY_t^1 = b(Y_t^1)dt + c(Y_t^1)dB_t^1 + c(Y_t^1)dW_t \\
    \vspace{1em}
    \cdots \cdots \\
    dY_t^N = b(Y_t^N)dt + c(Y_t^N)dB_t^N + c(Y_t^N)dW_t,
\end{cases}
\]
\]

where \( W \) is an \( \mathbb{R}^m \)-valued Brownian motion, and \( B^1, \ldots, B^N \) are mutually independent \( E \)-valued Brownian motion, which are independent of \( W \). Let \((S^N_t)^t\) be the semigroup of the diffusion \( Y \) with generator \( G_N \). Then for \( f \in \mathcal{D}(G_N) \), domain of \( G_N \), it is easy to see that

\[
G_Nf(x_1, \ldots, x_N) = \sum_{p=1}^N \sum_{i=1}^d b_i(x_p) \frac{\partial f(x_1, \ldots, x_N)}{\partial x_{pi}} + \frac{1}{2} \sum_{p=1}^N \sum_{i,j=1}^d a_{ij}(x_p) \frac{\partial^2 f(x_1, \ldots, x_N)}{\partial x_{pi} \partial x_{pj}}.
\]

We stress that under hypotheses (LU), the transition semigroup \((S^N_t)^t\) of the \( Nd \)-dimensional diffusion \( Y \) has a transition density, say \((p^N(t,x,y))_{t>0}\), and the semigroup is both Feller and strong Feller; see, for instance, [11, p.164] and [3, p.227]. Moreover, if hypotheses (LU) hold, then one can modify the construction of \([12]\) to construct a dense subset \( D(E^N) \) of \( C_b(E^N) \), satisfying \( D(E^N)|_{E^N} \subset C_b^2(E^N) \), and extend \((S^N_t)^t\) to a strongly continuous contraction semigroup \((\tilde{S}^N_t)^t\) on \( C_b(E^N) \) such that \( D(E^N) \) is invariant under \((\tilde{S}^N_t)^t\); see [3]. Write \( D(E^N) = D(E^N)|_{E^N} \), class of functions restricted to \( E^N \). Note that functions in \( D(E^N) \) are subject to \( \lim_{x \to \infty} G_Nf(x) = 0 \). Then \( D(E) := D(E^N) \supset C_b^2(E) \) (space of functions together with their two continuous derivatives vanishing at infinity).

To analyze \( X^\alpha = \{X^\alpha_t : t \geq 0\} \), we associate to each \( \alpha \in I \) and \( n \) a stopping time

\[
\tau_{\alpha,n} = \begin{cases} 
0 \quad & \text{if } \alpha_0 > K_n \\
\min \left\{ \frac{1}{1+|\alpha|}, \frac{1}{n+1} \right\} \quad & \text{if this set is non-empty and } \alpha_0 \leq K_n \\
\alpha \quad & \text{otherwise}
\end{cases}
\]

and define

\[
X_t^{\alpha,n} = \begin{cases} 
Y_t^{\alpha,n} \quad & \text{if } t < \tau_{\alpha,n} \\
\Delta \quad & \text{if } t \geq \tau_{\alpha,n}.
\end{cases}
\]

As a result, we have \( X_t^\alpha(h) = \frac{1}{n} \sum_{\alpha_{\sim n}} \tilde{h}(X_t^{\alpha,n}) \) for measurable \( h \), where \( \tilde{h} := h \mid E \) and \( \tilde{h}(\Delta) := 0 \).

By Itô’s formula it is easily verified that for each \( \alpha \sim n \) \( k_n, t \in [k_n, k_n + a_n] \) and \( f \in D(E) \)

\[
M_t^{\alpha,n,k_n}(f) = 1_E(X_t^{\alpha,n}) \left[ f(Y_t^{\alpha,n}) - f(Y_{k_n}^{\alpha,n}) - \int_{k_n}^t Gf(Y_u^{\alpha,n})du \right]
\]

is an \( \mathcal{F}_t^n \)-martingale with \( G := G_1 \), moreover, from the construction of \( X^n \) we have

\[
X_t^n(f) = X_0^n(f) + M_t^{(n)}(f) + J_t^{(n)}(f) + N_t^{(n)}(f) + Z_t^{(n)}(f) + C_t^{(n)}(f) + H_t^{(n)}(f),
\]

where

\[
M_t^{(n)}(f) = n^{-1} \sum_{r < k_{n+1}} \sum_{\alpha_{\sim n} r_n} M_{r_n + a_n}(\alpha)(N_{r_n + a_n} - \beta_n(Y_{r_n + a_n}^{n})),
\]

\[
N_t^{(n)}(f) = n^{-1} \sum_{r < k_{n+1}} \sum_{\alpha_{\sim n} r_n} \int_{r_n + \tau_{\alpha,n}}^{r_n + a_n} G\tilde{f}(X_u^{n})du[N_{r_n + a_n} - \beta_n(Y_{r_n + a_n}^{n})],
\]
\[ J_t^{(n)}(f) = n^{-1} \sum_{\alpha \sim \sim k_n} M_t^{\alpha,k_n}(f) + n^{-1} \sum_{r < k \sim \sim r_n} \int_{r_n}^{r + a_n} \hat{g} f(X_{\alpha,n}^{\alpha,n})du[\beta_n(Y_{r_n + a_n}^{\alpha,n}) - 1] \]

\[ + n^{-1} \sum_{r < k \sim \sim r_n} \hat{f}(X_{\alpha,n}^{\alpha,n})[\beta_n(Y_{r_n + a_n}^{\alpha,n}) - \beta_n(Y_{r_n}^{\alpha,n})] \]

\[ + n^{-1} \sum_{r < k \sim \sim r_n} M_{r_n + a_n}^{r,n}(f)[\beta_n(Y_{r_n + a_n}^{\alpha,n}) - \beta_n(Y_{r_n}^{\alpha,n})], \]

\[ Z_t^{(n)}(f) = n^{-1} \sum_{r < k \sim \sim r_n} \hat{f}(X_{\alpha,n}^{\alpha,n})[N_{\alpha,n}^{\alpha,n} - \beta_n(Y_{r_n + a_n}^{\alpha,n})] + n^{-1} \sum_{r < k \sim \sim r_n} M_{r_n + a_n}^{r,n}(f)[\beta_n(Y_{r_n + a_n}^{\alpha,n}) - \beta_n(Y_{r_n}^{\alpha,n})], \]

\[ C_t^{(n)}(f) = n^{-1} \sum_{r < k \sim \sim r_n} \int_{r_n}^{r + a_n} \hat{g} f(X_{\alpha,n}^{\alpha,n})du + n^{-1} \sum_{\alpha \sim \sim k_n} \int_{k_n}^{t} \hat{g} f(X_{\alpha,n}^{\alpha,n})du = \int_{0}^{t} X_{\alpha,n}^{\alpha,n}(Gf)du, \]

\[ H_t^{(n)}(f) = n^{-1} \sum_{r < k \sim \sim r_n} \hat{f}(X_{\alpha,n}^{\alpha,n})[\beta_n(Y_{r_n + a_n}^{\alpha,n}) - 1] = \int_{0}^{k_n} X_{[n\sigma_n]}^{\alpha,n}(f)\gamma_n)ds, \]

where \( \hat{h} \) is defined as before. The major difference between (5) above and (A.3) of [12] is the term \( J^{(n)} \), which is here more general.

It is well-known that to show the weak convergence of \( \{X_n\} \) involves proving its tightness, deriving a martingale problem for its limits and showing the uniqueness of solutions to the martingale problem. Undoubtedly, the techniques of [12] in deriving both tightness and martingale characterizations are applicable here. For uniqueness, the dual conditions of [1] were used in [12], while for the purpose of constructing a flow superprocess with general branching variance and obtaining moment formulas as well, the method of constructing directly dual processes as in [2] is proved to be much more powerful.

The lemma below shall play a fundamental role in proving the tightness of \( \{X_n\} \). Since the corresponding proof was not given in [12], we provide one. Note that \( X_0^n = \frac{1}{n} \sum_{i=1}^{K_n} \delta_{x_i^n} \).

**Lemma 2.1.** Let \( p \) be as in (3) and \( T > 0 \). If \( X_0^n \Rightarrow \nu \in M_F(E) \), then

\[ C_T = \sup_{n \geq 1} \mathbb{E} \left( \sup_{0 \leq t \leq T} X_n^n(1)^2 \right) < \infty \quad \text{and} \quad C_T^p = \sup_{n \geq 1} \mathbb{E} \left( \sup_{0 \leq t \leq T} X_n^n(1)^p \right) < \infty. \]

**Proof.** Note that for non-negative Borel measurable function \( \phi \) on \( E \) and for \( t \in [k_n, k_n + a_n] \)

\[ (6) \quad \mathbb{E} X_n^n(\phi) \leq e^{[nT_n]}C_T \mathbb{E} [\phi(Y_t)], \]

where \( Y \) is as in (4) with \( N = 1 \), and \( \mathbb{E}_Y \) denotes the conditional expectation given \( Y_0 = y \) ([10] Lemma II.3.3(a))). It suffices to show that

\[ C_T^p = \sup_{n \geq 1} \mathbb{E} \left( \sup_{0 \leq t \leq T} X_n^n(1)^p \right) < \infty. \]

In the following, we let \( C(u_1, \ldots, u_k) \) denote a constant depending only on \( u_1, \ldots, u_k \). Clearly

\[ \sup_{0 \leq t \leq T} X_n^n(1)^p \leq C(T, \gamma, p) \left( X_0^n(1)^p + \sup_{0 \leq t \leq T} |Z_t^{(n)}(1)|^p + \int_{0}^{[nT_n]} X_{n\sigma_n}^{n}(1)^p ds \right). \]

From (2) and (6), it follows that \( \{(Z_{k_n}^{(n)}(1), \mathcal{F}_{k_n}^{n}) : k = 0, 1, \ldots\} \) is a martingale. Its predictable quadratic variation process is calculated to be

\[ \langle Z^{(n)}(1) \rangle_{k_n} := \sum_{i=1}^{k_n} \mathbb{E} \left( \left( Z_{t_i}^{(n)}(1) - Z_{t_{i-1}}^{(n)}(1) \right)^2 \right) ds = \int_{0}^{k_n} X_{n\sigma_n}^{n}(S_{\alpha_n}^{n}(\sigma_{n}))ds. \]
Immediately
\[
\langle Z^{(n)}(1) \rangle_{k_n}^{p/2} \leq C(\sigma, p) \left( 1 + \int_0^{k_n} X_{[n\alpha]_n}^n(1)^p ds \right).
\]

Then by Burkholder’s inequality ([11] p.152) and Hölder’s inequality, we get
\[
E \left( \sup_{0 \leq t \leq T} |Z_t^{(n)}(1)|^p \right) \\
\leq cE \left[ \left( \langle Z^{(n)}(1) \rangle_{[nT]_n} \right)^{p/2} \right] \\
+ cE \left[ \left( \max_{0 \leq k < [nT]} \left| n \sum_{\alpha=k_n}^{n-1} 1_E(X_{[k_n\alpha]_n}^n) \left[ N^{\alpha,n} - \beta_n(Y_{[k_n\alpha]_n}^{\alpha,n}) \right] \right|^p \right) \right] \\
\leq \frac{c}{2} C_p^p T^{p/(2q)} \left( 1 + \int_0^{[nT]_n} E \left[ X_{[n\alpha]_n}^n(1)^p \right] ds \right) \\
+ c \sum_{k=0}^{[nT]_n-1} E \left[ \left( \frac{1}{n} \sum_{\alpha=k_n}^{n-1} 1_E(X_{[k_n\alpha]_n}^n) \left[ N^{\alpha,n} - \beta_n(Y_{[k_n\alpha]_n}^{\alpha,n}) \right] \right|^p \right],
\]
where all the expectations above are allowed to be infinite, $c$ is some constant depending only on $p$ and $1/p + 1/q = 1$. We shall use a technique in [12] pp.531-532. Fix $k$ and $n$. Note that
\[
E \left[ \left( \sum_{\alpha=k_n}^{n-1} 1_E(X_{[k_n\alpha]_n}^n) \left[ N^{\alpha,n} - \beta_n(Y_{[k_n\alpha]_n}^{\alpha,n}) \right] \right|^p \right] = E \left[ \left( \sum_{i=1}^K \left[ N^{\alpha_i,n} - \beta_n(Y_{k_n+a_n}^{\alpha_i,n}) \right] \right|^p \right],
\]
where $K = nX_{k_n}^n(1)$, and $\alpha^1, \ldots, \alpha^K$ are the labels of $K$ particles alive at time $k_n$. For $i = 1, 2, \ldots, m$ and $\alpha_i \sim k_n$ such that $\alpha_i \neq \alpha_j$ if $i \neq j$ and define
\[
M_m(1) := n^{-1} \sum_{i=1}^m \left[ N^{\alpha_i,n} - \beta_n(Y_{k_n+a_n}^{\alpha_i,n}) \right] \text{ and } \mathcal{G}_m := \sigma(\{N^{\alpha_i,n} : i = 1, \ldots, m\}) \cup \mathcal{F}_{k_n}^n.
\]

Then clearly $Y_{k_n}^{\alpha_i,n} \in \mathcal{G}_m$ for all $i \geq 1$ and $\{\{M_m(1), \mathcal{G}_m) : m = 1, 2, \ldots\}$ is a square integrable martingale by the fact that $N^{\alpha,m}$ is conditionally independent of $\{N^{\alpha_i,n} : i = 1, \ldots, m-1\}$ given $\mathcal{F}_{k_n}^n$. Similarly, we have
\[
E \left( |M_m(1)|^p \right) \leq cE \left( \left( \langle M(1) \rangle_{m} \right)^{p/2} \right) + cE \left( \max_{1 \leq i \leq m} |M_i(1) - M_{i-1}(1)|^p \right) \\
\leq cm^{-p} C_p^p m^{p/2} + c \sum_{i=1}^m E \left[ \left| N^{\alpha_i,n} - \beta_n(Y_{k_n+a_n}^{\alpha_i,n}) \right|^p \right] \leq C(\gamma, \sigma, p) \left( \frac{\sqrt{m}}{n} \right)^p,
\]
where $c$ is the same as the one in (5). Recall that $p$ can be very close to 2. Therefore by the Cauchy-Schwarz inequality and $C_T < \infty$ we obtain that
\[
\sum_{k=0}^{[nT]_n-1} E \left[ \left( \frac{1}{n} \sum_{\alpha=k_n}^{n-1} 1_E(X_{[k_n\alpha]_n}^n) \left[ N^{\alpha,n} - \beta_n(Y_{k_n+a_n}^{\alpha,n}) \right] \right|^p \right] \\
\leq cC(\gamma, \sigma, p, c) \frac{[nT]_n}{n^{p/2}} E \left[ \left( \sup_{0 \leq t \leq T} X_t^n(1) \right)^{p/2} \right] \leq cTC(\gamma, \sigma, p, c) n^{1-p/2} C_T^{p/4}.
\]

Now combine (5) and (6) to see that
\[
E \left( \sup_{0 \leq t \leq T} |Z_t^{(n)}(1)|^p \right) \leq C(T, \gamma, \sigma, p) + C(T, \gamma, \sigma, p) \int_0^{[nT]_n} E \left[ X_{[n\alpha]_n}^n(1)^p \right] ds,
\]

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then (7) follows by an analogous argument as in [10] Lemma II.4.6. The proof is complete.

The following theorem is obviously an analogue of a combination of Propositions A.3.10 and A.3.12 and Lemma A.3.13 of [12], the proof of which certainly applies here except that it suffices to prove (instead of the stronger square result)

$$\lim_{n \to \infty} \mathbb{E}\left( \sup_{0 \leq t \leq T} |J_t^{(n)}(f)| \right) = 0$$

to obtain that $J_t^{(n)}(f)$ converges weakly to the zero process in $D_R(0, \infty)$. Now we state the result, and the interested reader is referred to [4] for the detailed proof.

**Theorem 2.1.** Suppose that the hypothesis (L) holds. If $X^n_t \Rightarrow \nu \in M_F(E)$, then $\{X^n\}$ is tight in $D_{M_F(E)}(0, \infty)$, and each limit point $X \in C_{M_F(E)}(0, \infty)$ and is a solution to the following martingale problem: For any $f \in D(E)$,

$$Z_t(f) = X_t(f) - \nu(f) - \int_0^t X_s((G + \gamma) f) \, ds$$

is a continuous square integrable martingale with $Z_0(f) = 0$ and quadratic variation process

$$\langle Z(f) \rangle_t = \int_0^t X_s(\sigma^2 f^2) \, ds + \int_0^t (X_s \times X_s)(\Lambda f) \, ds,$$

where $\Lambda f(x, y) = \sum_{i,j=1}^d a_{ij}^{(m)}(x, y)f_i(x)f_j(y)$.

Then we shall prove the uniqueness of the martingale problem (10). Define for $F$ in some subset $\mathcal{D}(\mathcal{L})$ (to be specified) of the domain of an operator $\mathcal{L}$ as follows

$$\mathcal{L}F(\mu) \equiv (\mathcal{L}F)(\mu) := \int_E ((G + \gamma) \left( \frac{dF(\mu)}{d\mu(x)} \right) \mu(dx) + \frac{1}{2} \int_E \sigma(x)^2 \frac{d^2F(\mu)}{d\mu(x)^2} \mu(dx) + \frac{1}{2} \sum_{i,j=1}^d \int_E a_{ij}^{(m)}(x, y) \frac{\partial^2F(\mu)}{\partial x_i \partial y_j} \mu(dx) \mu(dy),$$

where $\frac{dF(\mu)}{d\mu(x)} := \lim_{r \to 0} \frac{1}{r} [F(\mu + r\delta_x) - F(\mu)]$, $x \in E$, and similarly $d^2F(\mu)/d\mu(x)d\mu(y)$ is defined with $F$ replaced by $dF(\mu)/d\mu(y)$. Let $X$ be a limit as in Theorem 2.1. We will show that $X$ satisfies the martingale problem for $\mathcal{L}$ and then construct the dual process of $X$ to prove uniqueness.

Let $\mathcal{D}(\mathcal{L}) = \mathcal{D}_1(\mathcal{L}) \cup \mathcal{D}_2(\mathcal{L})$, where $\mathcal{D}_1(\mathcal{L})$ consists of functions $F_f(\mu) = (f, \mu^N)$ with $f \in D(E^N)$, and $\mathcal{D}_2(\mathcal{L})$ denotes the class of functions $F_{f, \phi}(\mu) = f(\mu(\phi_1), \ldots, \mu(\phi_N))$ with $f \in C_b(\mathbb{R}^N)$ and $\phi = \{\phi_1, \ldots, \phi_N\} \subset D(E)$, and of functions $F_{f, \phi}(\mu) = f(\mu(\phi))$ with $f \in C_b^2([0, \infty])$ and $\phi \in D(E)^+$. Let $E_{\nu}$ denote the conditional expectation given $X_0 = \nu$.

**Lemma 2.2.** $E_{\nu}[X_t(1)^n]$ is locally bounded in $t$ for each $n \geq 1$. Furthermore, $X$ is also a solution to the martingale problem for $(\mathcal{L}, \mathcal{D}(\mathcal{L}), \nu)$. That is, for all $F \in \mathcal{D}(\mathcal{L})$

$$F(X_t) - F(\nu) - \int_0^t \mathcal{L}F(X_s) \, ds$$

is a continuous martingale with $X_0 = \nu$.

**Proof.** Let $T_k = \inf\{t \geq 0 : X_t(1) \geq k\}$. Then $\{T_k\}$ is a non-decreasing sequence of stopping times. It is easily seen from Lemma 2.1 that $T_k \to \infty$ as $k \to \infty$. Fix $n \geq 1$. For each $k$, by Itô's formula we have

$$X_{t \wedge T_k}(1)^n = X_0(1)^n + n \int_0^{t \wedge T_k} X_s(1)^{n-1} dX_s(1) + \frac{n(n-1)}{2} \int_0^{t \wedge T_k} X_s(1)^{n-2} d\langle Z(1) \rangle_s$$
of [9] and Theorem 7.13 of [8]. Roughly speaking, the martingale problems (10) and (12) are equivalent.

This completes the proof.

It follows that

$$\mathbb{E}[X_{t \wedge T_k}(1)^n] \leq X_0(1)^n + \frac{n(n - 1)t}{2} C_\sigma^2 + \left(nC_\gamma + \frac{n(n - 1)}{2} C_\sigma^2\right) \int_0^t \mathbb{E}[X_{s \wedge T_k}(1)^n]ds.$$  

An application of Gronwall’s inequality and Fatou’s lemma implies the local boundedness of $\mathbb{E}[X_t(1)^n]$ in $t$. The martingale property for $(\mathcal{L}', \mathcal{F}_1(\mathcal{L}), \nu)$ is actually implied in [12] pp.537-539. It is sufficient to consider $F_{f,\phi} = f(\mu(\phi_1), \ldots, \mu(\phi_N))$. Note that

$$\mathcal{L} F_{f,\phi}(\mu) = \sum_{p=1}^N f_p'(\mu(\phi_1), \ldots, \mu(\phi_N))\mu((G + \gamma)\phi_p) + \frac{1}{2} \sum_{p,q=1}^N f_{pq}''(\mu(\phi_1), \ldots, \mu(\phi_N))\mu(\sigma^2 \phi_p \phi_q) + \frac{1}{2} \sum_{p,q=1}^N f_{pq}''(\mu(\phi_1), \ldots, \mu(\phi_N)) \int_E \int_E \sum_{i,j=1}^d a_{ij}^{(m)}(x, y) \frac{\partial \phi_p(x)}{\partial x_i} \frac{\partial \phi_q(y)}{\partial y_j} \mu(dx) \mu(dy).$$

By the martingale property for $(\mathcal{L}', \mathcal{F}_2(\mathcal{L}), \nu)$ and Itô’s formula we have

$$f(X_t(\phi_1), \ldots, X_t(\phi_N)) = f(X_0(\phi_1), \ldots, X_0(\phi_N)) + \text{mart.}$$

$$+ \int_0^t \int_E \int_E \sum_{i,j=1}^d a_{ij}^{(m)}(x, y) \frac{\partial \phi_p(x)}{\partial x_i} \frac{\partial \phi_q(y)}{\partial y_j} X_s(dx) X_s(dy)ds.$$  

Then the martingale property for $(\mathcal{L}, \mathcal{F}_2(\mathcal{L}), \nu)$ follows once we use polarization to see that

$$(Z(\phi_p), Z(\phi_q))_t = \int_0^t X_s(\sigma^2 \phi_p \phi_q)ds + \int_0^t \int_E \int_E \sum_{i,j=1}^d a_{ij}^{(m)}(x, y) \frac{\partial \phi_p(x)}{\partial x_i} \frac{\partial \phi_q(y)}{\partial y_j} X_s(dx) X_s(dy)ds.$$  

This completes the proof.

It is worthwhile to notice that every solution to the martingale problem (12) is also such that (10) is a continuous local martingale with quadratic variation process given by (11); see, for instance, Theorem 4.8 of [13] and Theorem 7.13 of [5]. Roughly speaking, the martingale problems (10) and (12) are equivalent.

Before turning to our construction, observe that for $f \in D(E^N)$

$$(13) \quad \mathcal{L} F_f(\mu) = F_{G_N f}(\mu) + 1/2 \sum_{p,q=1}^N [F_{\Phi_{p,q} f}(\mu) - F_f(\mu)] + 1/2 \sum_{p \neq q}^N [F_{\Phi_{p,q} f}(\mu) - F_f(\mu)] + 1/2 N^2 F_f(\mu)$$

$$= F_\mu(G_N f, N) + 1/2 \sum_{p,q=1}^N [F_\mu(\Phi_{p,q} f, N - 1) - F_\mu(f, N)] + 1/2 N^2 F_\mu(f, N).$$
where for \( h \in B(E^n) \) and \( x = (x_1, \ldots, x_n) \in E^n \), \( F_\mu(h, n) := F_h(\mu) \), \( \Phi_{p,q} : B(E^n) \to B(E^{n-1}) \) is given by

\[
(14) \quad \Phi_{p,q} h(x_1, \ldots, x_{n-1}) := \sigma(x_{n-1})^2 h(x_1, \ldots, x_{n-1}, \ldots, x_{n-1}, \ldots, x_{n-1})
\]

with \( x_{n-1} \) in the positions of the \( p \)th and the \( q \)th variables of \( h \), and \( \Phi_p : B(E^n) \to B(E^n) \) by

\[
(15) \quad \Phi_p h(x) := 2\gamma(x_p)h(x).
\]

Based on \[13\], we now construct a function-valued process of \( X \). Let \( \mathbb{N} := \{1, 2, \ldots\} \). Let \( \mathcal{B} := \cup_{n=0}^\infty B(E^n) \) be endowed with bounded pointwise convergence on each \( B(E^n) \), where \( B(E^0) := \mathbb{R} \) and the union is required to be disjoint union and so we do not view \( B(E^k) \) as a subset of \( B(E^l) \) if \( k < l \). Assume \( \{e_1, e_2, \ldots\} \) is a sequence of mutually independent unit exponential random variables with \( e_0 := 0 \). Define a sequence \( \Gamma = \{\Gamma_k : k = 1, 2, \ldots\} \) of random operators on \( \mathcal{B} \) and a \( \mathcal{B} \)-valued càdlàg process \( L = \{L_t : t \geq 0\} \) as follows: Given a \( \mathcal{B} \)-valued random variable \( L_0 \), independent of \( \{e_1, e_2, \ldots\} \), define recursively

\[
\left\{ \begin{array}{l}
L_t = S_{t-\tau_{k-1}}^{N(L_{\tau_{k-1}})} \Gamma_k S_{\eta_{k+1}}^{N(L_{\eta_{k+1}})} \cdots \Gamma_2 S_{\eta_2}^{N(L_{\eta_2})} \Gamma_1 S_{\eta_1}^{N(L_{\eta_1})} L_{\tau_{k+1}}, \quad \text{if } \tau_k \leq t < \tau_{k+1} \\
\mathbb{P}\{\tau_{k+1} = \Phi_{p,q} N(L_{\tau_k}) = n_{k+1}\} = \mathbb{P}\{\tau_{k+1} = \Phi_{p,q} N(L_{\eta_k}) = n_{k+1}\} = \frac{1}{2} \quad \text{for } 1 \leq p \neq q \leq n_{k+1} \\
L_{\tau_{k+1}} = \Gamma_{k+1} S_{\eta_{k+1}}^{N(L_{\eta_{k+1}})} \Gamma_k S_{\eta_k}^{N(L_{\eta_k})} \cdots \Gamma_2 S_{\eta_2}^{N(L_{\eta_2})} \Gamma_1 S_{\eta_1}^{N(L_{\eta_1})} L_{\tau_{k+1}}, \quad k = 0, 1, 2, \ldots,
\end{array} \right.
\]

where \( \eta_0 = 0, \eta_n = \frac{2e_n}{N(L_{\tau_{n-1}})}, \tau_k = \sum_{i=0}^k \eta_i \) and \( N(h) := l \) if \( h \in B(E^l) \). Note that given \( L_0 \in \mathcal{B} \), \( \tau_k \to \infty \) almost surely as \( k \to \infty \) and thus \( L_t \) is defined for all \( t > 0 \). Set \( M_t = N(L_t) \). Then \( (L, M) \) is a \( \mathcal{B} \times \mathbb{N} \)-valued strong Markov process and shall serve as the dual process of \( X \). Let \( \mathbb{E}_{h,n} \) denote the conditional expectation given \( (L_0, M_0) = (h, n) \in \mathcal{B} \times \mathbb{N} \) with \( N(h) = n \) and let \( \mathcal{L}^* \) be the generator of \( (L, M) \). Then from the previous construction, one can verify, with elementary arguments, that

\[
E_{h,n} \left[ \langle L_t, \mu^M \rangle \exp \left\{ \frac{1}{2} \int_0^t M_s^2 ds \right\} \right]
= \langle S_t^h, \mu^n \rangle + \frac{1}{2} \sum_{p,q=1}^n \int_0^t E_{\Phi_p S_p h, n-1} \left[ \langle L_{t-s}, \mu^{M_{t-s}} \rangle \exp \left\{ \frac{1}{2} \int_0^{t-s} M_u^2 du \right\} \right] ds
- \frac{1}{2} \sum_{p=1}^n \int_0^t E_{\Phi_p S_p h, n} \left[ \langle L_{t-s}, \mu^{M_{t-s}} \rangle \exp \left\{ \frac{1}{2} \int_0^{t-s} M_u^2 du \right\} \right] ds
\]

and that \( \mathcal{L}^* F_\mu(f, N) = \mathcal{L} F_\mu(f) - 1/2N^2 F_\mu(f, N) \) for \( f \in D(E^N) \).

**Theorem 2.2.** Suppose that hypotheses (LU) hold. Then for all \( n \geq 1, t \geq 0 \) and \( h \in B(E^n) \) we have

\[
E \left[ \| h, X_t^n \| \right] = E_{h,n} \left[ \langle L_t, \mu^M \rangle \exp \left\{ \frac{1}{2} \int_0^t M_s^2 ds \right\} \right],
\]

where \( X_t^n = X_t \times \cdots \times X_t \in M_F(E^n) \). Moreover, uniqueness holds for the martingale problem (12) and hence for the martingale problem (11).

**Proof.** In terms of Theorem 2.1, Lemma 2.2 and the relation (11), the assertions follow in much the same way as the proofs of Theorems 2.1 and 2.2 of [2] p.7. It was pointed out in Remark 2.2 of He [7] that there is a gap in the proof of Theorem 2.1 of [2] if \( \sigma \) is just bounded and measurable. This is because the function-valued process \( L \) is not always valued in the domain of \( G_N \). However, if \( \sigma, \gamma, L_0 \in C_0(E) \), then since the transition semigroup of the underlying motion has regular transition density, hence \( L_t (t \notin \{\tau_k\}) \) does belong to the domain of \( G_N \) and so we can still use the associated martingale relation between \( \mathcal{L} \) and \( \mathcal{L}^* \). Consequently, the discussions in proving Theorem 2.1 of [2] are applicable here since \( \xi, \delta \in C_l(E) \).
Note that one may use Lemma 2.2 to verify that if $\gamma$ and $\sigma$ are constants, then the total mass process $X(1)$ is a diffusion process with generator $\gamma x \frac{d}{dx} + \frac{\sigma^2 x}{2} \frac{d^2}{dx^2}$ and $E e^{-\rho X_t(1)} = e^{-x \Psi(t)}$, where $x = X_0(1)$ and $\Psi(t) = \frac{\rho e^{\gamma t}}{1 + \frac{\rho \gamma t}{2}}$ (see [12, p.540]).

It is natural to call an adapted càdlàg process in $M_P(E)$ which satisfies the martingale problem a superprocess over a stochastic flow, or simply flow superprocess $(G, \gamma, \sigma)$.

In the remainder of this section, we shall derive the moment formulas for any bounded linear functionals $(G, \gamma, \sigma)$ given by Theorems 2.1 and 2.2. Let $Y = (Y^1, \ldots, Y^N)$ be the $Nd$ dimensional diffusion process given by (1), its semigroup $S^N$. For $h \in B(E^N)$, define an operator $U(h)$ by

$$U(h) = \frac{1}{2} \sum_{p \neq q \in \left\{1, \ldots, N\right\}} \Phi_{p,q} h,$$

and a semigroup $T^N$ as follows

$$T^N_t h(y) = E_y \left[ \exp \left( \int_0^t \sum_{p=1}^N \gamma(Y^p(s)) ds \right) h(Y(t)) \right].$$

**Theorem 2.3.** For $h \in B(E^n)$ and each $n \geq 1$

$$E([h, X^n_t]) = (T^n_t h, \nu^n) + \sum_{i=1}^{n-1} \left( \int_0^t dt_1 \int_0^{t_1} \cdots \int_0^{t_{i-1}} T^{n-i} \Pi^{n-i}(i-1)t; hdt_i, \nu^{n-i} \right),$$

where $\Pi^{n}(i-1); t = (U^{n-i}; t)_{t_i=0}^{t; \Pi^{n-i}}$. Similarly $\Pi^{n}(i; t)$ is defined for $0 \leq t_{i+1} \leq t_i \leq \cdots \leq t_1 \leq t$ the operators $V^{n}(i; t)$ respectively by

$$V^{n}(i; t) = \frac{1}{2} \sum_{p=1}^N \Phi_{p,q} h, \quad \text{and } \pi^{n}(i; t) = \left( V^{n}(S^n_{t_{i-1}}) \cdots V^{n}(S^n_{t_i}) \right) V^{n}(S^n_{t_{i-1}}) h,$$

with $\pi^{n}(0; t) := V^{n}(S^n_{t_{i-1}})$. Similarly $\pi^{n}(k; s)$ is defined for $0 \leq s_{k+1} \leq s_k \leq \cdots \leq s_1 \leq s$. To simplify notation, write $V^{n}(x) = \frac{1}{2} \sum_{p=1}^N 2\gamma(x_p)$. Then $V^{n}(h)(x) = V^{n}(x)(h(x))$. We first show that for any bounded linear functionals $(V_t)_{t \geq 0}$ on $B(E^N)$

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{i-1}} V_{t+1} \left( S^n_{t_{i-1}} \pi^{n}(i-1; t) h \right) dt_{i+1} = \int_0^t V_{t+1} \left( E_{\frac{1}{2}} \left( \int_{t_{i+1}}^t V^{n}(Y(s-t_{i+1})) ds \right)^i h(Y(t-t_{i+1})) \right) dt_{i+1},$$

and

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{i-1}} S^n_{t_{i-1}} \pi^{n}(i-1; t) hdt_i = E_{\frac{1}{2}} \left( \int_0^t V^{n}(Y(u)) du \right)^i h(Y(t)) \right].$$

We only consider (19). Since $S^n_{t_{i-1}} \pi^{n}(i-1; t) h(x) = E_x \left[ V^{n}(Y(t_{i-2}))(h(Y(t-t_{i-2})) \right]$ by the Markov property of $Y$ as in (3), so by Fubini’s theorem (19) is clearly true for $i = 1$. Suppose that it holds for
some $i \geq 1$. Then

\[
\int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{i+1}} V_{i+1} \left( S_{t_{i+1}-t_i+2}^N (i; t) h \right) dt_{i+2} \\
= \int_0^t dt_1 \int_0^{t_1} V_{i+2} \left( E \left[ \frac{1}{i!} \left( \int_0^{t_i} V^N (Y(s-t_i+2)) ds \right)^i \left( V^N (S_{t_i}^N) h \right) \right] \right) dt_{i+2} \\
= \int_0^t dt_{i+2} V_{i+2} \left( E \left[ \frac{1}{i!} \left( \int_0^{t_i} V^N (Y(s-t_i+2)) ds \right)^i V^N (Y(t-1)) dt_1 h(Y(t-t_i+2)) \right] \right) dt_{i+2} \\
= \int_0^t V_{i+2} \left( E \left[ \frac{1}{(i+1)!} \left( \int_0^t V^N (Y(s-t_i+2)) ds \right)^{i+1} h(Y(t-t_i+2)) \right] \right) dt_{i+2}.
\]

Thus (19) is true for $i+1$ and hence for all $i \geq 1$.

By (16), it is simple to see that (18) holds for $t = 0$. Suppose that it is valid for some $n \geq 1$. For $t \geq 0$, write $H_t = \langle L_t, \nu^{h,N} \rangle \exp \left( \sum_{j=1}^n M_j^2 ds \right)$ and $W^N_t(h) = E_{h,N}[H_t]$ with $h \in B(E^N)$. Then $W^N_t$ is a bounded linear functional on $B(E^N)$. In the remainder of this proof, take $N = n + 1$ and it is enough to consider $h \in B(E^{n+1})$. By (16), we have

\[
E_{h,n+1}[H_s] = (S^{n+1}_n h) + \int_0^s E_{V^{(n+1)} S^{n+1}_{s-t-1} h,n+1} [H_s] ds_1 + \int_0^s E_{V^{(n+1)} S^{n+1}_{s-t+1} h,n+1} [H_s] ds_1 \\
=: W_s(h) + \int_0^s E_{V^{(n+1)} S^{n+1}_{s-t+1} h,n+1} [H_s] ds_1, \quad s \geq 0.
\]

Make repeated use of the above relation to conclude that

\[(21) \quad E_{h,n+1}[H_s] = W_s(h) + \sum_{i=1}^k \int_0^s ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{i-1}} W_{s_i} \left( \pi^{(n+1)}(i-1; s) h \right) ds_i \\
+ \int_0^s ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_k} E_{\pi^{(n+1)}(k; s) h,n+1} [H_{s+k}] ds_{k+1}, \quad k \geq 1.
\]

Since by (17) as well as the induction assumption

\[
W_{s_i} \left( \pi^{(n+1)}(i-1; s) h \right) = (S^{n+1}_{s_i} \left( \pi^{(n+1)}(i-1; s) h \right), \nu^{n+1}) \\
+ \int_0^{s_i} W_u \left( U^{(n+1)} S^{(n+1)}_{s_i-u} (\pi^{(n+1)}(i-1; s) h) \right) du,
\]

hence in terms of (19) and (20), we see that

\[
\sum_{i=1}^k \int_0^s ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{i-1}} W_{s_i} \left( \pi^{(n+1)}(i-1; s) h \right) ds_i \\
= \sum_{i=1}^k \int_0^s ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{i-1}} S^{n+1}_{s_i} \left( \pi^{(n+1)}(i-1; s) h \right) ds_i, \quad \nu^{n+1}) \\
+ \sum_{i=1}^k \int_0^s ds_1 \int_0^{s_1} ds_2 \cdots \int_0^{s_{i-1}} W_{s_i} \left( U^{(n+1)} S^{(n+1)}_{s_i-s_i+1} (\pi^{(n+1)}(i-1; s) h) \right) ds_{i+1} \\
= \sum_{i=1}^k E \left[ \frac{1}{i!} \left( \int_0^s V^{(n+1)}(Y(u)) du \right)^i h(Y(s)) \right] \nu^{n+1}) \\
+ \sum_{i=1}^k \int_0^s W_{s_i} \left( U^{(n+1)} \left( E \left[ \frac{1}{i!} \left( \int_0^{s_i} V^{(n+1)}(Y_{s_i-s_i+1}) ds_i \right)^i h(Y(s-s_i+1)) \right] \right) ds_{i+1}.
\]
for \( \lambda = \lim_{t \to \infty} \langle X, r \rangle \), replaced by \( L \).

Lemma 3.1.

The first moment of \( X \) is given by

\[
E_h, n \left[ (S^{n+1}_h) \right] = E_h, n \left[ \sum_{i=1}^{\infty} \frac{1}{i!} \left( \int_0^s V^{(n+1)}(Y(u))du \right)^i h(Y(s)) \right], \nu^{n+1}.
\]

Now a simple variable change together with (17) implies that (18) holds for \( n + 1 \). This completes the proof.

The formulas below were established in [12] under the condition of binary branching, using branching particle systems approximation. They are immediate from Theorem 2.3 and the Markov property of \( X \).

Corollary 2.1. If \( \gamma \) and \( \sigma \) are constants, then for \( h, h_1, h_2 \in B(E) \) and \( 0 \leq s \leq t \)

\[
E_\nu \left[ X_t(h) \right] = e^{\gamma t} \nu(S^1_t h),
\]

and

\[
E_\nu \left[ X_s(h_1)X_t(h_2) \right] = e^{\gamma(s+t)} \langle S^2_s(h_1S^1_{t-s}h_2), \nu \rangle + \frac{\sigma^2}{2} e^{\gamma(s+t)} \int_0^s e^{-\gamma u} \langle S^1_u[S^2_{u-s}(h_1S^1_{t-s}h_2)], \nu \rangle du,
\]

where \( S^1_u[S^2_{u-s}(h_1S^1_{t-s}h_2)](x) := \int_E \int_E \int_E \int_E h_1(w)S^1_{t-s}h_2(z)S^2_{u-s}(y, y; dw; dz)S^1_u(x, dy) \).

We note that by constructing a stochastic integral similarly as in [2], it is quite simple to obtain the first moment of \( X \) while it does not seem obvious to derive higher moments.

3. Measurable spatially dependent branching. In this part we shall construct, via approximation, flow superprocesses with drift function \( \gamma \in B(E) \) and branching variance \( \sigma^2 \in B(E)^+ \). To this aim, choose functions \( \{\gamma^{(i)}\} \subset C_l(E) \) and \( \{\sigma^{(i)}\} \subset C_l(E)^+ \) such that \( \gamma^{(i)}(x) \to \gamma(x), \sigma^{(i)}(x) \to \sigma(x) \) as \( i \to \infty \) for \( \lambda \)-a.e. \( x \in E \). There \( \lambda \) is Lebesgue measure on \( E \). Let \{\( X^{(i)} \)\} be the flow superprocesses \( (G, \gamma^{(i)}, \sigma^{(i)}) \) given by Theorems 2.1 and 2.2. Proceed as in the previous section to construct a function-valued process \( \{L_t : t \geq 0\} \) based on \( (\gamma, \sigma) \). Define naturally mappings \( \Phi_{p,q}^{(i)} \) and \( \Phi_{p}^{(i)} \) as in [13] and [15] with \( \gamma \) and \( \sigma \) replaced by \( \gamma^{(i)} \) and \( \sigma^{(i)} \), respectively. In an obvious way we can construct operators \( \Gamma^{(i)} = \{\Gamma_t^{(i)}\} \) and function-valued processes \( L^{(i)} = \{L_t^{(i)} : t \geq 0\} \). Define \( M_t^{(i)} = N(L_t^{(i)}) \), which is clearly independent of \( i \) and hence write \( M_t^{(i)} =: M_t \).

Lemma 3.1. Suppose that \( \mu_i \Rightarrow \mu \) in \( M_F(E) \). Then for \( t \geq 0 \) and \( h \in B \) with \( N(h) = n \)

\[
\lim_{i \to \infty} E_{h,n} \left[ (L_t^{(i)}, \mu_i^{M_t}) \exp \left\{ \frac{1}{2} \int_0^t M_s^2 ds \right\} \right] = E_{h,n} \left[ (L_t, \mu^{M_t}) \exp \left\{ \frac{1}{2} \int_0^t M_s^2 ds \right\} \right].
\]
Proof. Let \( L_{(h,n)} = \{ L_{(h,n)}(t) : t \geq 0 \} \) denote the process \( L \) with \( L_0 = h \) and \( N(h) = n \). Let \( L^{(i)}_{(h,n)} = \{ L^{(i)}_{(h,n)}(t) : t \geq 0 \} \) stand for the process \( L^{(i)} \) with initial value \( L^{(i)}_0 = h \) and \( N(h) = n \). By (16), we obtain that

\[
\mathbb{E}_{k,n} \left[ \left(L^{(i)}_{\tau_t^k}, \mu^{M_i}_t\right) \exp \left\{ \frac{1}{2} \int_0^t M_s^2 ds \right\} \right]
\]

\[
= \mathbb{E} \left[ (L^{(i)}_{(h,n)}(t), \mu^{M_i}_t) \exp \left\{ \frac{1}{2} \int_0^t M_s^2 ds \right\} \right]
\]

\[
= \left(S^n_{\Phi^{(2,n)}}(h), \mu^n_{\Phi^{(2,n)}}\right) + \mathbb{E} \left[ \int_0^t \left(L^{(i)}_{\Phi^{(2,n)}(h,n-1)}(s), \mu^{M_i}_s\right) \exp \left\{ \frac{1}{2} \int_0^s M_r^2 du \right\} ds \right]
\]

\[
+ \mathbb{E} \left[ \int_0^t \left(L^{(i)}_{\Phi^{(1,n)}(h,n)}(s), \mu^{M_i}_s\right) \exp \left\{ \frac{1}{2} \int_0^s M_r^2 du \right\} ds \right],
\]

where \( \Phi^{(2,n)} = \frac{1}{2} \sum_{p \neq q \in \{1, \ldots, n\}} \Phi^{(i)}_{p,q} S^n_{\Phi^{(i)}_{p,q}} \) and \( \Phi^{(1,n)} = \frac{1}{2} \sum_{p=1}^n \Phi^{(i)}_{p,s} S^n_{\Phi^{(i)}_{p,s}} \). For notational simplicity, write \( n_k = N(L^{(i)}_{\tau_k^1}) = N(L^{(i)}_{\tau_k^1}) \). Since \( \{N^N(y)\} \geq 0 \) has transition density \( (p^N(x,y))_{t>0} \) and since \( \gamma^{(i)}(x) \to \gamma(x) \) and \( \sigma^{(i)}(x) \to \sigma(x) \) a.e. \( x \), we can use dominated convergence and induction to see that for \( \tau_k < t < \tau_{k+1} \) with \( k = 0, 1, \ldots \)

\[
(S^n_{\Phi^{(2,n)}} S^n_{\Gamma^{(i)}_{\tau_k^1, \tau_k^1}} \cdots S^n_{\Gamma^{(i)}_{\tau_k^1, \tau_k}} L^{(i)}_{\tau_k^1, \tau_k^1}) (z) \longrightarrow (S^n_{\Phi^{(1,n)}} S^n_{\Gamma^{(i)}_{\tau_k^1, \tau_k}} L^{(i)}_{\tau_k^1, \tau_k}) (z)
\]

for all \( z \in E^{n_k} \), and

\[
(\Gamma^{(i)}_{k+1} S^n_{\Gamma^{(i)}_{\tau_k^1, \tau_k}} \cdots \Gamma^{(i)}_{1} S^n_{\Gamma^{(i)}_{\tau_k^1, \tau_k}} L^{(i)}_{\tau_k^1, \tau_k}) (z) \longrightarrow (\Gamma^{(i)}_{k+1} S^n_{\Gamma^{(i)}_{\tau_k^1, \tau_k}} \cdots \Gamma^{(i)}_{1} S^n_{\Gamma^{(i)}_{\tau_k^1, \tau_k}} L^{(i)}_{\tau_k^1, \tau_k}) (z)
\]

for \( \lambda_{k+1} \)-a.e. \( z \in E^{n_{k+1}} \), where \( \lambda_{k+1} \) denotes Lebesgue measure on \( E^{n_{k+1}} \). Now fix an arbitrary \( k \). Then, for \( \mu \in C_b(E^{n_k} \times E^{n_k}) \) with compact support, by Fubini’s theorem and dominated convergence again we have

\[
\int_{E^{n_k}} \int_{E^{n_k}} g(x, y) \left( \Gamma^{(i)}_{k} S^n_{\Gamma^{(i)}_{\tau_k^1, \tau_k}} \cdots \Gamma^{(i)}_{1} S^n_{\Gamma^{(i)}_{\tau_k^1, \tau_k}} L^{(i)}_{\tau_k^1, \tau_k} \right) (y)p^n_k(x, y)dy\mu^n_k(dx)
\]

\[
= \int_{E^{n_k}} \int_{E^{n_k}} g(x, y)p^n_k(x, y)\mu^n_k(dx) \left( \Gamma^{(i)}_{k} S^n_{\Gamma^{(i)}_{\tau_k^1, \tau_k}} \cdots \Gamma^{(i)}_{1} S^n_{\Gamma^{(i)}_{\tau_k^1, \tau_k}} L^{(i)}_{\tau_k^1, \tau_k} \right) (y)dy
\]

\[
- \int_{E^{n_k}} \int_{E^{n_k}} g(x, y)p^n_k(x, y)\mu^n_k(dx) \left( \Gamma_{k} S^n_{\Gamma^{(i)}_{\tau_k^1, \tau_k}} \cdots \Gamma_{1} S^n_{\Gamma^{(i)}_{\tau_k^1, \tau_k}} L^{(i)}_{\tau_k^1, \tau_k} \right) (y)dy.
\]

By [3, Proposition 4.4, p.112], for each \( k \) the measures \( \{(\Gamma^{(i)}_{k} S^n_{\Gamma^{(i)}_{\tau_k^1, \tau_k}} \cdots \Gamma^{(i)}_{1} S^n_{\Gamma^{(i)}_{\tau_k^1, \tau_k}})(y)p^n_k(x, y)\mu^n_k(dx)\} \) on \( E^{n_k} \times E^{n_k} \) are weakly convergent. It is then evident that the required result follows from (16) and (22) and hence the proof is finished.

The following theorem asserts the existence of flow superprocesses with bounded branching variance and drift function.

**Theorem 3.1.** \( X^{(i)} \Rightarrow X \) in \( C_{M_{\text{loc}}[0, \infty)} \) and \( X \) satisfies the martingale problem (12).

**Proof.** The proof can be proceeded as the one of Theorem 5.2 of [2] and hence is omitted.

**Remark 3.1.** It is clear that the moments of flow superprocess \( (G, \gamma, \sigma) \), \( X \), are still given by Theorem 2.3. Moreover, both its mean measure \( m_1(A) : = EX_t(A) \) and covariance measure \( m_2(A \times B) : = E(X_t(A)X_t(B)) \) are absolutely continuous with respect to Lebesgue measure since \( \gamma \) is bounded and the semigroups \( S^1 \) and \( S^2 \) have densities.

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