Topological full groups of line-like minimal group actions are amenable

Nóra Gabriella Szőke

Abstract
We consider a finitely generated group acting minimally on a compact space by homeomorphisms, and assume that the Schreier graph of at least one orbit is quasi-isometric to a line. We show that the topological full group of such an action is amenable.

1 Introduction
Consider a group $G$ and a compact Hausdorff topological space $\Sigma$. A group action $G \bowtie \Sigma$ by homeomorphisms is called minimal if $\Sigma$ has no proper $G$-invariant closed subset. The topological full group $[[G \bowtie \Sigma]]$ is the group of all homeomorphisms of $\Sigma$ that are piecewise given by the action of elements of $G$, where each piece is open in $\Sigma$.

The notion of topological full groups was first introduced for $\mathbb{Z}$-actions by Giordano, Putnam and Skau [2]. Among others, Matui and Nekrashevych investigated these groups ([11], [12], [13], [15]). Their results show that the derived subgroup of the topological full group is often simple, and in many cases it is also finitely generated.

In their groundbreaking paper [7], Juschenko and Monod developed a strategy for proving the amenability of topological full groups. They show that the topological full group of a minimal Cantor $\mathbb{Z}$-action is amenable. Combined with the results of Matui, their paper provides the first examples of finitely generated infinite simple amenable groups. A natural question arises: how far can we extend their technique? Several directions were investigated in [5], [8], [6], and by the author of the present paper in [16].

The goal of this paper is to further stretch the Juschenko-Monod result in a certain direction. Namely, we consider a minimal action of a finitely generated group such that there exists an orbit that is quasi-isometric to a line, and show that the topological full group of such an action is amenable. This is a generalization of Theorem A in [16], where the group was virtually cyclic.

Theorem 1.1. Let $G$ be a finitely generated group acting minimally on a compact Hausdorff topological space $\Sigma$ by homeomorphisms. Assume that there exists a $G$-orbit $X \subseteq \Sigma$, such that the Schreier graph of the action of $G$ on $X$ is quasi-isometric to $\mathbb{Z}$. Then the topological full group $[[G \bowtie \Sigma]]$ is amenable.
In order to illustrate the interest of our result, let us mention how to recover a result of Matte Bon about the Grigorchuk group. Let $G$ be the first Grigorchuk group (defined in [3]), which is usually defined as a transformation group of the binary rooted tree. Its action on the boundary of the tree - a Cantor set - is known to be minimal and its Schreier graphs are quasi-isometric to lines, as seen in [4].

Theorem 1.1 can be applied to deduce the following.

**Corollary 1.2** (Matte Bon, [10]). The topological full group of the Grigorchuk group acting on the boundary of the rooted binary tree is amenable.

This result was first proved by Matte Bon, who showed that the Grigorchuk group can be embedded in the topological full group of a minimal Cantor $\mathbb{Z}$-action ([10]).

There are more groups to which our Theorem 1.1 can be applied, for instance the groups defined by Nekrashevych in [14]. Let $a$ be an involution on a Cantor space $\Sigma$. We say that a finite group $A$ of homeomorphisms of $\Sigma$ is a **fragmentation** of $a$ if for all $h \in A$ and all $x \in \Sigma$, we have $h(x) = x$ or $h(x) = a(x)$ and for every $x \in \Sigma$ there exists $h \in A$ such that $h(x) = a(x)$. In [14] it is shown that for a fragmentation $A, B$ of a minimal action of the dihedral group $D_\infty = \langle a, b \rangle$, the action of the topological full group $G = \langle A, B \rangle$ is minimal. It is not difficult to see that the associated Schreier graphs are quasi-isometric to lines. Therefore, we get the following result of Nekrashevych as a corollary of Theorem 1.1.

**Corollary 1.3** (Nekrashevych, [14]). For any fragmentation $A, B$ of a minimal action of the dihedral group $D_\infty = \langle a, b \rangle$, the topological full group of $G = \langle A, B \rangle$ is amenable.

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## 2 Preliminaries

### 2.1 Quasi-isometry

If $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ is a connected graph, then we can think of $\mathcal{G}$ as a metric space. The distance $d : V(\mathcal{G}) \times V(\mathcal{G}) \to \mathbb{N}$ is defined to be the length of the shortest path between two vertices.

Let $\mathcal{G}_1, \mathcal{G}_2$ be two connected graphs with distance functions $d_1, d_2$ respectively. Recall that the map $f : V(\mathcal{G}_1) \to V(\mathcal{G}_2)$ is a **quasi-isometry** if there exist constants $\alpha \geq 1$, $\beta \geq 0$ and $\gamma \geq 0$ such that the following two properties hold.

1. For all $u, v \in V(\mathcal{G}_1)$ we have
   $$\alpha^{-1} \ d_1(u, v) - \beta \leq d_2(f(u), f(v)) \leq \alpha \ d_1(u, v) + \beta.$$

2. For every $w \in V(\mathcal{G}_2)$ there exists $u \in V(\mathcal{G}_1)$ such that $d_2(f(u), w) \leq \gamma$.

Two graphs are **quasi-isometric** if there exists a quasi-isometry between them.
2.2 Group actions and graphs

As a convention, throughout the paper we always consider groups acting from the left.

Let $G$ be a group acting on a set $X$. The piecewise group $\text{PW}(G \rtimes X)$ of the action is defined as follows. A bijection $\varphi : X \to X$ is a piecewise $G$ map, i.e., an element of the piecewise group, iff there exists a finite subset $S \subset G$ such that $\varphi(x) \in S \cdot x$ for every $x \in X$. In other words, we cut the space $X$ into finitely many pieces, and act on each of them with a group element. It is clear that the piecewise $G$ maps form a group.

If the group $G$ acts on a compact space by homeomorphisms, then the topological full group of this action is always a subgroup of its piecewise group. Indeed, by the compactness of the space a partition into open subsets is necessarily finite.

Let $G$ be a finitely generated group with a symmetric generating set $S$, and assume that $G$ acts on a set $X$. Recall that the Schreier graph of this action $\text{Sch}(G, X, S)$ is defined to be the graph with vertex set $X$ and edge set $\{(x, sx) : x \in X, s \in S\}$. Sometimes it is also called the graph of the action $G \ltimes X$.

Note that the Cayley graph of $G$ is the Schreier graph of its action on itself by (left) multiplication.

**Definition 2.1.** If $G$ is a finitely generated group with a fixed symmetric generating set $S$, then the length of a group element $g \in G$ is defined as

$$\text{len}(g) = \min\{n \in \mathbb{N} : g = s_1s_2\ldots s_n \text{ with } s_1, s_2, \ldots, s_n \in S\}.$$  

In other words, the length of an element is its distance from the identity element in the Cayley graph.

**Definition 2.2.** For any graph $G = (V, E)$ with distance function $d$ and a number $n \in \mathbb{N}$ we define the $n$-ball around a point $p \in V$ to be

$$B_n(p) = \{q \in V : d(p, q) \leq n\}.$$  

For a set $W \subseteq V$, the $n$-neighborhood of $W$ is

$$\Gamma_n(W) = \{q \in V : d(q, W) \leq n\}.$$  

If a group $G$ acts on the graph $G$, then for a set of elements $D \subseteq G$ the $D$-neighborhood of a point $p \in V$ is $D \cdot p = \{d \cdot p : d \in D\}$, and the $D$-neighborhood of a set $W \subseteq V$ is

$$D \cdot W = \bigcup_{q \in W} D \cdot q.$$  

Keep in mind that when $G$ is a Schreier graph of a $G$-action, and $G = \langle S \rangle$, then the $n$-ball around a point is exactly the $S^n$-neighborhood of that point and the $n$-neighborhood of a set is equal to its $S^n$-neighborhood.

2.3 Extensive amenability

A group action $G \ltimes X$ is amenable if there exists a $G$-invariant mean on $X$. In the proof of our result we will use a stronger property, the extensive amenability of an action.
Definition 2.3. For a set $X$, let us denote the set of all finite subsets of $X$ by $\mathcal{P}_f(X)$. Note that this set becomes an abelian group with the symmetric difference. If a group $G$ acts on $X$, it gives rise to a $G$-action on $\mathcal{P}_f(X)$.

We say that the action $G \curvearrowright X$ is extensively amenable if there exists a $G$-invariant mean on $\mathcal{P}_f(X)$ that gives full weight to the collection of sets containing any given finite subset of $X$.

Extensively amenable actions were first used (without a name) in [7]. The name was given in [6] and this concept turned out to be very useful for proving the amenability of topological full groups, see [7], [8], [9], or [16]. For a detailed introduction to extensive amenability, see Chapter 11 of [1]. The following two statements about extensive amenability will be among the core ingredients in our proof.

Proposition 2.4 (Proposition 3.6 in [16]). Let $G$ be a group acting on a set $X$. Assume that for all finitely generated subgroups $H \leq G$ and all $H$-orbits $Y \subseteq X$ the Schreier graph of the action $H \curvearrowright Y$ is recurrent. Then the action of the piecewise group $\text{PW}(G \curvearrowright X)$ on $X$ is extensively amenable.

Proposition 2.5 (Remark 1.5 in [6]). Let $G \curvearrowright X$ be an extensively amenable action. Assume that there exists an embedding $G \hookrightarrow \mathcal{P}_f(X), g \mapsto (c_g, g)$ such that the subgroup $\{g \in G : c_g = \emptyset\} \leq G$ is amenable. Then $G$ itself is also amenable.

In Proposition 2.5 such a map $c : G \to \mathcal{P}_f(X), g \mapsto c_g$ is called a cocycle with amenable kernel. Thus, we can rephrase the statement of the proposition as follows: If $G \curvearrowright X$ is an extensively amenable action, and there exists a cocycle on $G$ with amenable kernel, then $G$ is amenable. Let us denote the orbit of $p$ by $X$, i.e., $X = G \cdot p$.

3 The proof

In this section we consider a finitely generated group $G = \langle S \rangle$ acting minimally on a compact space $\Sigma$ satisfying the assumption in Theorem 1.1. Let $X$ be an orbit such that the Schreier graph $\text{Sch}(G, S, X)$ is quasi-isometric to $\mathbb{Z}$.

3.1 Action on the orbit $X$

The set $X$ is dense in $\Sigma$, since the action $G \curvearrowright \Sigma$ is minimal. Consider the restricted action of $G$ on $X$. We can define the embedding

$$\varepsilon_X : [[G \curvearrowright \Sigma]] \hookrightarrow \text{PW}(G \curvearrowright X); \quad \varphi \mapsto \varphi|_X.$$ Since $X$ is dense in $\Sigma$, the $\varphi$-action on $X$ determines the $\varphi$-action on $\Sigma$, so the map $\varepsilon_X$ is injective.

Definition 3.1. Let $d$ denote the distance function on the graph $X$. Let $f : X \to \mathbb{Z}$ be the quasi-isometry between $X$ and $\mathbb{Z}$. By definition, there exist constants $\alpha \geq 1$, $\beta \geq 0$ and $\gamma \geq 0$ such that
1. for all \( x, y \in X \) we have
\[
\alpha^{-1} \cdot d(x, y) - \beta \leq |f(x) - f(y)| \leq \alpha \cdot d(x, y) + \beta,
\]
2. for every \( n \in \mathbb{Z} \) there is \( x \in X \) such that \( |f(x) - n| \leq \gamma \).

**Lemma 3.2.** For every \( n \in \mathbb{Z} \), the set \( f^{-1}(n) \subseteq X \) is finite.

**Proof.** If \( f^{-1}(n) = \emptyset \), then it is finite. Now assume that it is non-empty. Let \( x \in f^{-1}(n) \), then by Definition 3.1, for any \( y \in f^{-1}(n) \) we have
\[
\alpha^{-1} d(x, y) - \beta \leq |f(x) - f(y)| = 0
\]
\[
d(x, y) \leq \alpha \beta.
\]
Hence, \( f^{-1}(n) \) is contained in the ball of radius \( \alpha \beta \) around \( x \). This ball is a finite set since the graph \( X \) is locally finite, so \( f^{-1}(n) \) is also finite. \( \square \)

The following two propositions are well-known for any graph that is quasi-isometric to \( \mathbb{Z} \), but we include their proofs for completeness.

**Proposition 3.3.** There exists a bi-infinite geodesic in \( X \).

**Lemma 3.4.** Let \( G \) be a locally finite graph, i.e., the degree of every vertex is finite. The following are equivalent for a vertex \( v \in G \).

1. For every \( n \in \mathbb{N} \), there exists a geodesic of length \( 2n \) with midpoint \( v \).
2. There exists a bi-infinite geodesic through the vertex \( v \).

**Proof.** The implication 2 \( \Rightarrow \) 1 is clear. For the other direction, consider a vertex \( v \) that satisfies the first statement.

Let us construct the rooted tree \( T \) as follows. The vertices of \( T \) are the finite geodesics in \( G \) of even length with midpoint \( v \). Two such geodesics are connected in \( T \) if their length difference is exactly 2 and the shorter one is a subset of the longer one. The root is the “geodesic” of length zero consisting only of the point \( v \), and the \( n \)-th level of the tree consists of the geodesics of length \( 2n \). By the local finiteness of \( G \), the rooted tree \( T \) is also locally finite.

By König’s lemma, there exists an infinite ray in \( T \) from the root, say \( \{v\} = \ell_0, \ell_1, \ell_2, \ell_3, \ldots \), where the length of \( \ell_i \) is \( 2i \) and \( \ell_i \subseteq \ell_{i+1} \) for every \( i \in \mathbb{N} \). Then the union \( \bigcup_{i \in \mathbb{N}} \ell_i = \ell \subseteq G \) is a bi-infinite geodesic in \( G \). This proves the implication 1 \( \Rightarrow \) 2. \( \square \)

**Proof of Proposition 3.3.** Let \( f : X \to \mathbb{Z} \) be the quasi-isometry with constants \( \alpha, \beta, \gamma \), and let us define \( B = f^{-1}([0, \alpha + \beta]) \). By Lemma 3.2 the set \( f^{-1}(z) \subseteq X \) is finite for every \( z \in \mathbb{Z} \). Consequently, \( B \) is finite.

Let
\[ B_i = \{ x \in B \mid x \text{ is the midpoint of a length } 2i \text{ geodesic in } X \} \]
for all \( i \in \mathbb{N} \). We have \( B_{i+1} \subseteq B_i \) for every \( i \).

We show that \( B_i \neq \emptyset \) for every \( i \in \mathbb{N} \). Consider a fixed \( i \in \mathbb{N} \) and take \( k \in \mathbb{N} \) such that \( k \geq \alpha \cdot i + \beta \). Note that this choice ensures that \( |f(x) - f(y)| > k \) implies \( d(x, y) > i \) for some \( x, y \in X \) by Definition 3.1. Take \( x_1, x_2 \in X \) such that
Proposition 3.5. There exists a constant \( m \in \mathbb{N} \) such that \( X \) is contained in the \( m \)-neighborhood of any bi-infinite geodesic in \( X \).

Proof. Let \( f : X \to \mathbb{Z} \) be the quasi-isometry with constants \( \alpha, \beta, \gamma \) as in Definition 3.1. Let \( m = \alpha^2 + 2\alpha \beta \). Let \( \ell \) be a bi-infinite geodesic in \( X \), we would like to show that \( X \) is contained in the \( m \)-neighborhood of \( \ell \).

First, note that if \( u, v \in X \) such that \( d(u, v) > m = \alpha^2 + 2\alpha \beta \), then we have

\[
\begin{align*}
\alpha^{-1}d(u, v) - \beta & \leq |f(u) - f(v)| \\
\alpha^{-1}m - \beta & < |f(u) - f(v)| \\
\alpha + \beta & < |f(u) - f(v)|
\end{align*}
\]

Take an arbitrary \( x \in X \), and suppose for contradiction that \( d(x, \ell) > m = \alpha^2 + 2\alpha \beta \). This implies that for every \( y \in \ell \), we have \( \alpha + \beta < |f(x) - f(y)| \) by (1).

Note that if \( u, v \in X \) are neighbors, then \( |f(u) - f(v)| \leq \alpha d(u, v) + \beta = \alpha + \beta \). Therefore, as we walk along the geodesic \( \ell \), the \( f \)-image cannot jump over the value \( f(x) \), since the distance of \( f(\ell) \) from \( f(x) \) is more than \( \alpha + \beta \). Hence, \( f(\ell) \) must be contained in a half-line, either \((\infty, f(x) - \alpha - \beta) \) or \((f(x) + \alpha + \beta, +\infty) \).

We will show that \( f(\ell) \) cannot be contained in a half-line. Without loss of generality, suppose that \( f(\ell) \subseteq (N, +\infty) \), such that \( N = \min f(\ell) \geq f(x) + \alpha + \beta \), and take \( x_0 \in \ell \) so that \( f(x_0) = N \). Let \( I = B_m(x_0) \cap \ell \) be a geodesic segment of length \( 2m \) on \( \ell \) around \( x_0 \).

Take \( y_1, y_2 \in \ell \setminus I \) be in different components of \( \ell \setminus I \) such that \( f(y_1) \leq f(y_2) \). Consider \([x_0, y_2]\), which denotes a shortest path between the two points in \( X \), in this case we may take the path that is contained in \( \ell \). We know that \( |f(u) - f(v)| \leq \alpha + \beta \) if \( u \) and \( v \) are neighbors, hence if we “walk” along the path \([x_0, y_2]\), the \( f \)-image changes by at most \( \alpha + \beta \) in each step. Since \( N = f(x_0) \leq f(y_1) \leq f(y_2) \), we can find \( y_3 \in [x_0, y_2] \), such that \( |f(y_3) - f(y_1)| \leq \alpha + \beta \). On the other hand, \( d(y_1, y_3) \geq d(y_1, x_0) > m = \alpha^2 + 2\alpha \beta \), which is a contradiction by (1).

Therefore, for every \( x \in X \), we have \( d(x, \ell) \leq m \), so \( X \) is contained in the \( m \)-neighborhood of \( \ell \).
3.2 Definition of the cocycle

**Definition 3.6.** Let \( f: X \to \mathbb{Z} \) be the quasi-isometry from Definition 3.1. Let us define
\[
Y = f^{-1}(\mathbb{N}) \subseteq X.
\]

For a subgraph \( H \) of \( X \) let us denote by \( \partial H \) the vertices on the boundary of \( H \), i.e., let
\[
\partial H = \{ x \in H : \text{there exists } y \in X \setminus H \text{ such that } (x, y) \in E(X) \} \subseteq H \subseteq X.
\]

**Lemma 3.7.** The set \( Y \) is infinite and \( \partial Y \) is finite.

**Proof.** By the second requirement in Definition 3.1 we have that \( f^{-1}(I) \neq \emptyset \) for every interval \( I \) of length at least \( 2\gamma \). Since \( \mathbb{N} \) contains infinitely many pairwise disjoint intervals of length \( 2\gamma \), the preimage \( f^{-1}(\mathbb{N}) = Y \) is infinite.

For \( x \in H \) and \( y \in X \setminus H \) we have \((x, y) \in E(X)\) if and only if \( d(x, y) = 1 \). By Definition 3.1, we have
\[
\alpha^{-1} - \beta = \alpha^{-1}d(x, y) - \beta \leq |f(x) - f(y)| \leq \alpha \cdot d(x, y) + \beta = \alpha + \beta.
\]
Therefore, since \( f(x) \in \mathbb{N} \) and \( f(y) \in \mathbb{Z} \setminus \mathbb{N} \), we must have \( 0 \leq f(x) \leq \alpha + \beta - 1 \), so \( \partial Y \subseteq f^{-1}([0, \alpha + \beta - 1]) \). The latter is a finite set by Lemma 3.2 and hence \( \partial Y \) is also finite. \( \square \)

**Lemma 3.8.** For every group element \( g \in G \), the set \( gY \setminus Y \) is finite.

**Proof.** Notice that for all \( x \in X \) we have \( d(x, gx) \leq \text{len}(g) \). Therefore, the set \( gY \setminus Y \) is contained in the \( \text{len}(g) \)-neighborhood of \( \partial Y \). Since \( \partial Y \) is finite by Lemma 3.7, the \( \text{len}(g) \)-neighborhood is also a finite set by the local finiteness of \( X \). Hence, \( gY \setminus Y \) is finite. \( \square \)

**Proposition 3.9.** For every piecewise map \( \varphi \in \text{PW}(G \curvearrowright X) \), the set \( Y \triangle \varphi(Y) \) is finite.

**Proof.** There exists a finite set \( T \subseteq G \) such that for every \( x \in X \) we have \( \varphi(x) \in T \cdot x \). Hence, we have the inclusion
\[
\varphi(Y) \setminus Y \subseteq \left( \bigcup_{t \in T} tY \right) \setminus Y = \bigcup_{t \in T} (tY \setminus Y).
\]
By Lemma 3.8 \( tY \setminus Y \) is finite for all \( t \in T \), so \( \varphi(Y) \setminus Y \) is also finite. The same argument works for \( \varphi^{-1}(Y) \setminus Y \), and hence \( \varphi(\varphi^{-1}(Y) \setminus Y) = Y \setminus \varphi(Y) \) is finite as well. This implies that the set
\[
Y \triangle \varphi(Y) = (Y \setminus \varphi(Y)) \cup (\varphi(Y) \setminus Y)
\]
is also finite, finishing the proof. \( \square \)

**Definition 3.10.** For \( \varphi \in \text{PW}(G \curvearrowright X) \) let us define
\[
c_{\varphi} = Y \triangle \varphi(Y) \in \mathcal{P}_f(X).
\]
Remark 3.11. We defined the map \( c : \text{PW}(G \curvearrowright X) \to \mathcal{P}_f(X) \). This gives rise to the cocycle \( c : \llbracket G \curvearrowright \Sigma \rrbracket \to \mathcal{P}_f(X) \). We would like to show that its kernel \( \{ \varphi \in \llbracket G \curvearrowright \Sigma \rrbracket : c_\varphi = \emptyset \} \) is amenable in order to use this cocycle in Proposition 2.5. Note that
\[
\ker c = \{ \varphi \in \llbracket G \curvearrowright \Sigma \rrbracket : c_\varphi = \emptyset \} = \{ \varphi \in \llbracket G \curvearrowright \Sigma \rrbracket : Y \vartriangle \varphi(Y) = \emptyset \} = \{ \varphi \in \llbracket G \curvearrowright \Sigma \rrbracket : \varphi(Y) = Y \} = \llbracket G \curvearrowright \Sigma \rrbracket_Y.
\]
Hence, the kernel of \( c \) is exactly the stabilizer of the set \( Y \) in the topological full group \( \llbracket G \curvearrowright \Sigma \rrbracket \). In the next sections we prove that this stabilizer is amenable.

3.3 Ubiquitous patterns in the action

Definition 3.12. Let \( G \) be a group acting on the space \( \Sigma \). Let \( D \subset G \) be a finite set containing the identity element. For an element \( \varphi \in \text{PW}(G \curvearrowright \Sigma) \) and for two points \( q_1, q_2 \in \Sigma \), we say that the \( \varphi \)-action is the same on the \( D \)-neighborhood of \( q_1 \) and \( q_2 \), if the \( D \)-neighborhoods of \( q_1 \) and \( q_2 \) are isomorphic, and for every \( d \in D \), \( \varphi \) acts by the same element of \( G \) on \( d \cdot q_1 \) and on \( d \cdot q_2 \), i.e., there exists \( g \in G \) such that \( \varphi(d \cdot q_1) = gd \cdot q_1 \) and \( \varphi(d \cdot q_2) = gd \cdot q_2 \).

Lemma 3.13. Let \( G = \langle S \rangle \) be a group acting minimally on the compact space \( \Sigma \) with a finite symmetric generating set \( S \), and take an arbitrary point \( q \in X \).

For every finite subset \( F \subset \llbracket G \curvearrowright \Sigma \rrbracket \) and every \( n \in \mathbb{N} \), there exists \( r = r(q, F, n) \in \mathbb{N} \) so that for every \( y \in X \) there exists \( z \in B_r(x) \) such that for all \( \varphi \in F \), the \( \varphi \)-action is the same on the \( S^n \)-neighborhood of \( q \) and \( z \).

Proof. Let us fix the elements \( \varphi_1, \ldots, \varphi_k \in \llbracket G \curvearrowright \sigma \rrbracket \) and a number \( n \in \mathbb{N} \).

Choose a finite partition \( \mathcal{P} \) of \( \Sigma \) such that every \( \varphi_i \) is acting with one element of \( G \) when restricted to any element of \( \mathcal{P} \). Then there exists an open neighborhood \( V \) of \( q \) such that the sets \( g \cdot V \) for \( g \in S^n \) are pairwise disjoint, and every \( g \cdot V \) is contained in some element of \( \mathcal{P} \). Since \( V \) is open and non-empty, the union
\[
\bigcup_{g \in G} g \cdot V = \bigcup_{j \geq 1} \bigcup_{g \in S^j} g \cdot V
\]
is non-empty, open and \( G \)-invariant, so by minimality we have
\[
\bigcup_{j \geq 1} \bigcup_{g \in S^j} g \cdot V = \Sigma.
\]
Due to the compactness of \( \Sigma \), already a finite union must cover it, so there exists \( j \in \mathbb{N} \) such that
\[
\bigcup_{g \in S^j} g \cdot V = \Sigma.
\]
Let \( r = j \). Now let \( y \in X = G \cdot q \) be an arbitrary point. Then \( y = h \cdot q \) for some \( h \in G \). We have
\[
\Sigma = h^{-1} \cdot \Sigma = \bigcup_{g \in S^r} h^{-1} g \cdot V.
\]
so there exists \( \hat{g} \in S^c \) such that \( q \in h^{-1} \hat{g} \cdot V \). This means that \( \hat{g}^{-1} h \cdot q \in V \). Let \( z = \hat{g}^{-1} h \cdot q \), and note that \( z = \hat{g}^{-1} h \cdot q \in B_r(h \cdot q) = B_r(y) \). On the other hand, \( q \) and \( z \) are both in \( V \), so for every \( g \in S^n \), the points \( g \cdot q \) and \( g \cdot z \) are in the same element of the partition \( \mathcal{P} \), so every \( \varphi_i \) acts with the same element of \( G \) on them. Therefore, for all \( i = 1, \ldots, k \), the \( \varphi_i \)-action is the same on the \( S^n \)-neighborhood of \( q \) and \( z \).

This proves the statement of the lemma for \( r \).

\[ \square \]

**Lemma 3.14.** For every piecewise map \( \varphi \in \text{PW}(G \curvearrowright X) \), there exists a number \( d_\varphi \in \mathbb{N} \), such that for every \( x \in X \), \( d(x, \varphi(x)) \leq d_\varphi \).

**Proof.** There exists a finite set \( T \subseteq G \) such that for every \( x \in X \), we have \( \varphi(x) \in T \cdot x \). The statement of the lemma holds for \( d_\varphi = \max \{ \text{len}(t) : t \in T \} \).

\[ \square \]

**Definition 3.15.** Let \( m \in \mathbb{N} \) be the constant from Proposition 3.5. Let us fix a bi-infinite geodesic \( \ell \) in \( X \) (it exists by Lemma 3.4), and a point \( p \in \ell \).

Let us denote the two ends of \( \ell \) by \( +\infty \) and \( -\infty \). For a set \( A \subseteq X \) we will say that \( +\infty \in A \), if there exists a point \( x \in \ell \) such that \( [x, +\infty] \subseteq A \), where \( [x, +\infty] \subseteq \ell \) denotes the half-line from \( x \) towards \(+\infty\). Similarly, \( -\infty \in A \) if there exists \( x \in \ell \) such that \( [x, -\infty] \subseteq A \).

Let \( R \in \mathbb{N} \) be such that the \( R \)-ball around the point \( p \) contains both \( \partial Y \) and \( \partial Y^c \) (such a radius exists since \( \partial Y \) is finite by Lemma 3.7, and hence \( \partial Y^c \) is also finite).

For a piecewise map \( \varphi \in \text{PW}(G \curvearrowright X) \) let us define the number

\[ N_\varphi = 6m + R + 2d_\varphi. \]

**Lemma 3.16.** If a set \( A \subseteq X \) and its complement \( A^c \) are both infinite, but its boundary \( \partial A \) is finite, then it contains exactly one end of \( \ell \).

**Proof.** It is enough to prove that if \( A \) is infinite and \( \partial A \) is finite, then it contains at least one end of \( \ell \). Indeed, we can apply this statement to both \( A \) and \( A^c \) – since \( \partial A^c \) is also finite – to prove the statement of the lemma.

Suppose for contradiction that \( A \subseteq X \) is an infinite set with finite boundary such that \( +\infty, -\infty \notin A \). Since its boundary is finite, there exists a ball \( B_r(x) \) with finite radius such that \( \partial A \subseteq B_r(x) \). By the definition of the boundary, a connected component of the set \( \ell \setminus B_r(x) \) must entirely belong either to \( A \) or to \( A^c \). Since \( B_r(x) \) is finite, there exists a connected component of \( \ell \setminus B_r(x) \) containing \( +\infty \), and there is one (possibly the same) containing \( -\infty \). Since we assumed that \( +\infty, -\infty \notin A \), we have \( +\infty, -\infty \notin A^c \).

Now consider an arbitrary point \( y \in X \setminus B_{r+m}(x) \), where \( m \) is the constant from Proposition 3.5. By Proposition 3.3, there exists a point \( \hat{y} \in \ell \) such that \( d(y, \hat{y}) \leq m \). Since \( \hat{y} \) is connected to either \( +\infty \) or \( -\infty \) outside of \( B_r(x) \) (and \( \partial A \subseteq B_r(x) \)), we must have \( \hat{y} \in A^c \). We have \( y \in X \setminus B_r(x) \), but we can say even more: there is a path of length at most \( m \) connecting \( y \) and \( \hat{y} \) that lies outside of the ball \( B_r(x) \).

Since \( y \) is connected to \( \hat{y} \) outside of \( B_r(x) \), it must also belong to \( A^c \). Therefore, \( X \setminus B_{r+m}(x) \subseteq A^c \), and hence \( A \subseteq B_{r+m}(x) \). This contradicts the assumption that \( A \) is infinite, so we must have \( +\infty \in A \) or \( -\infty \in A \).

\[ \square \]

**Corollary 3.17.** The set \( Y \) contains exactly one end of \( \ell \), we can assume that \( +\infty \in Y \), but \( -\infty \notin Y \).
Lemma 3.18. Let us fix a finite subset $F \subseteq \left[\left[ G \cap \Sigma \right]\right]_Y$ of the stabilizer of $Y$ and a number $n > \max\{N_\varphi : \varphi \in F\}$. Assume that there is a point $z \in X$ such that the $\varphi$-action is the same on the $S^n$-neighborhood of $p$ and of $z$ for every $\varphi \in F$. Then there exists a set $Y_z \subseteq X$, such that $\partial Y_z \subseteq B_R(z)$, $+\infty \in Y_z$, $-\infty \in Y_z^c$ and $F \subseteq \left[\left[ G \cap \Sigma \right]\right]_{Y_z}$.

**Proof.** Since the $\varphi$-action is the same on the $S^n$-neighborhood of $p$ and $z$ for every $\varphi \in F$, there exists a bijection

$h : S^n p \rightarrow S^n z,$

such that for every $x \in S^n p = B_n(p)$, $\varphi$ acts by the same group element on the points $x$ and $h(x)$. Let us define $B^+ = h(S^n p \cap Y)$ and $B^- = h(S^n p \cap Y^c)$, and let

$A^+ = \{x \in X : \text{there exists a path from } x \text{ to } B^+ \text{ that does not intersect } B^-\},$

$A^- = \{x \in X : \text{there exists a path from } x \text{ to } B^- \text{ that does not intersect } B^+\}.$

We will show that setting $Y_z = A^+$ or $Y_z = A^-$ satisfies the statement of the lemma.

**Claim 3.19.** We have $A^- = (A^+)^c$.

**Proof.** Since $X$ is connected, every $x \in X$ is connected to some point of $B^+ \cup B^- = S^n z = B_n(z)$, and hence $A^+ \cup A^- = X$. Therefore, we have to prove $A^+ \cap A^- = \emptyset$.

Suppose for contradiction that there exists a point that can be connected to both $B^+$ and $B^-$ without intersecting the other. This means that we can find points $z_+ \in B^+$ and $z_- \in B^-$ that are connected by a path outside of $B_n(z)$, i.e., there exists a path $z_+ = x_0, x_1, x_2, \ldots, x_{k-1}, x_k = z_-$, such that $x_i \in X \setminus B_n(z)$ for $i = 1, \ldots, k - 1$.

For a point $x \in X$, we will denote its ‘projection’ to $\ell$ by $\hat{x}$, i.e., the closest point to $x$ on $\ell$. If there are several such points, let us choose the closest one to the end $-\infty$. By Proposition 3.5 for every $x \in X$, we have $d(x, \ell) = d(x, \hat{x}) \leq m$.

We know that $d(z, x_i) \geq n > N_\varphi$ for $i = 0, 1, \ldots, k$. By the triangle inequality, we have

$$4m + R + 2d_\varphi = N_\varphi - 2m < n - 2m \leq d(\hat{z}, \hat{x}_i) \leq n + 2m. \quad (3)$$

The two projections $\hat{z}_+$ and $\hat{z}_-$ are either separated by $\hat{z}$ on $\ell$ or they are on the same side of it. In both cases, we get a contradiction:

1. Suppose that $\hat{z}_+$ and $\hat{z}_-$ are separated by $\hat{z}$ on $\ell$. Consider the points $x_i$ of the path connecting $z_+$ and $z_-$, and their projections $\hat{x}_i$. Since $\hat{z}_+ = \hat{x}_0$ and $\hat{z}_- = \hat{x}_k$ are separated by $\hat{z}$, there exists $i$ such that $\hat{x}_i$ and $\hat{x}_{i+1}$ are also separated by $\hat{z}$ on $\ell$. For this $i$, we must have

$$2(4m + R + 2d_\varphi) \leq d(\hat{x}_i, \hat{x}_{i+1})$$

by (3). On the other hand, $d(\hat{x}_i, x_i) \leq m$, and $d(x_{i+1}, \hat{x}_{i+1}) \leq m$, and hence

$$d(\hat{x}_i, \hat{x}_{i+1}) \leq 2m + 1,$$

this gives a contradiction.
2. Suppose that $\hat{z}_+$ and $\hat{z}_-$ are on the same side of $\hat{z}$ on $\ell$. Our goal is to find a path of length at most $12m$ connecting $z_+$ and $z_-$ that lies in the ball $B_n(z)$.

Since $d(z_+, \hat{z}) = n$, we can choose $y_+$ such that $d(z_+, y_+) = 3m$ and $d(y_+, \hat{z}) = n - 3m$. Let $[z_+, y_+]$ denote a shortest path between these two points in the graph $X$. Clearly this path lies in $B_n(z)$. We define $y_-$ similarly for $z_-$. Consider the projection $\hat{y}_+$. First, note that $d(y_+, \hat{y}_+) \leq m$ and $d(y_+, \hat{z}) = n - 3m$, and hence $[y_+, \hat{y}_+] \subseteq B_n(z)$. By the triangle inequality, we have $d(\hat{z}_+, \hat{y}_+) \leq 5m$, so $\hat{y}_+$ cannot be separated from $\hat{z}_+$ by $\hat{z}$ on $\ell$. Similarly, the point $\hat{y}_-$ is also on the same side of $\hat{z}$ and $[y_-, \hat{y}_-] \subseteq B_n(z)$.

Again by the triangle inequality, we have that
\[
n - 5m \leq d(\hat{y}_+, \hat{z}) \leq n - m, \\
- 5m \leq d(\hat{y}_-, \hat{z}) \leq n - m.
\]

Since $\hat{y}_+$ and $\hat{y}_-$ are not separated by $\hat{z}$, we must have $d(\hat{y}_+, \hat{y}_-) \leq 4m$, and a shortest path connecting them lies on the geodesic $\ell$, so it is contained in $B_n(z)$.

Now look at the path
\[P = [z_+, y_+] \cup [y_+, \hat{y}_+] \cup [\hat{y}_+, y_-] \cup [y_-, z_-].\]

We have seen that all sections of this path are contained in $B_n(z)$. Its length is at most $3m + m + 4m + m + 3m = 12m$.

Therefore, there exists a path $P$ of length at most $12m$ connecting $z_+$ with $z_-$ that lies in $B_n(z)$. Since $d(z_+, \hat{z}) = n > N_\varepsilon = 6m + R + 2d_\varepsilon$, and also $d(z_-, \hat{z}) > 6m + R + 2d_\varepsilon$, we have that $d(P, \hat{z}) > R + 2d_\varepsilon$. Consequently, taking the $h$-preimage of the path $P$, we have that $d(h^{-1}(P), p) > R + 2d_\varepsilon$. Since $\partial Y \subseteq B_R(p)$ (by Definition 3.15), the path $h^{-1}(P)$ cannot intersect the boundary of $Y$, so it must lie entirely in $Y$ or in $Y^c$. This contradicts the assumption that $z_+ \in B^+ = h(Y \cap B_n(p))$ and $z_- \in B^- = h(Y^c \cap B_n(p))$.

This concludes the proof of the fact that $A^+ \cap A^- = \emptyset$. \hfill \Box

**Claim 3.20.** We have $\partial A^+ = h(\partial Y)$ (and similarly $\partial A^- = h(\partial Y^c)$).

**Proof.** Since $\partial Y \subseteq B_R(p)$ and $R < n$, we have $h(\partial Y) \subseteq \partial A^+$. For the other direction, consider a point $x \in \partial D^+$. Then $x$ has a neighbor $y \in A^-$. There are four possibilities.

1. If $x, y \notin B_n(z)$, then there is a path from $x$ (going through $y$) to $B^-$ without touching $B^+$, so $x \in A^-$, this contradicts the fact that $A^+ \cap A^- = \emptyset$.

2. If $x \notin B_n(z), y \in B_n(z)$, then $x, y$ is a path from $x$ to $B^-$ that does not intersect $B^+$, and hence $x \in A^-$. This is again a contradiction.

3. If $x \in B_n(z), y \notin B_n(z)$, then $y, x$ is a path from $y$ to $B^+$ without going through $B^-$, so $y \in A^+$, which is also a contradiction.

4. The only remaining possibility is $x, y \in B_n(z)$. In this case we have $x \in B^+$, $y \in B^-$, so $h^{-1}(x) \in Y$, $h^{-1}(y) \in Y^c$, and hence $x \in h(\partial Y)$. 

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We have proved the equality $\partial A^+ = h(\partial Y)$. Similarly, one can prove that $\partial A^- = h(\partial Y^-)$. \hfill \square

**Claim 3.21.** The sets $A^+$ and $A^-$ are both invariant under the action of $F$.

**Proof.** It is enough to prove the $F$-invariance of $A^+$, the other statement follows from this.

Take a point $x \in A^+$ and let us fix $\varphi \in F$. We distinguish three cases.

1. If $x \notin B_n(z)$, then $d(x, A^-) > d_\varphi$, since $\partial A^+ \subseteq B_R(z)$ and $n > R + d_\varphi$.

   We know that the distance of $x$ and $\varphi(x)$ is at most $d_\varphi$, so we must have $\varphi(x) \in A^+$. Similarly, we have $\varphi^{-1}(x) \in A^+$.

2. If $x \in B_n(z)$, but $\varphi(x) \notin B_n(z)$, then we must have $d(x, X \setminus B_n(z)) \leq d_\varphi$.

   Hence, $d(x, B_R(z)) > d_\varphi$ (since $n > R + 2d_\varphi$), so we get $d(x, A^-) > d_\varphi$ again. Therefore, we have $\varphi(x) \in A^+$ and $\varphi^{-1}(x) \in A^+$.

3. If $x, \varphi(x) \in B_n(z)$, then we have $\varphi(h^{-1}(x)) = h^{-1}(\varphi(x))$ since the $\varphi$-action is the same on the $S^n$-neighborhood of $p$ and of $z$. We know that $h^{-1}(x) \in Y$, so $\varphi(h^{-1}(x)) \in Y$ by the $\varphi$-invariance of $Y$. Hence, we have $\varphi(x) \in h(Y \cap B_n(p)) \subseteq A^+$. Similarly, $\varphi^{-1}(x) \in A^+$.

Therefore, we have $\varphi \in [[G \acts \Sigma]]_{A+}$ and hence also $\varphi \in [[G \acts \Sigma]]_{A-}$. \hfill \square

**Claim 3.22.** $A^+$ and $A^-$ are both infinite.

**Proof.** Suppose for contradiction that $A^+$ is finite. This implies that all infinite components of $\ell \setminus B_R(z)$ belong to $A^-$ (the number of such components is one or two). Hence, we have that $\ell \cap (B_{n+m}(z) \setminus B_{n-2m}((z)) \subseteq A^-$. Now consider any point $x \in B_n(z) \setminus B_{n-m}(z)$. There exists a projection $\hat{x}$, such that $d(x, \hat{x}) \leq m$. Therefore, by the triangle inequality, we have

$$\hat{x} \in \ell \cap (B_{n+m}(z) \setminus B_{n-2m}((z)) \subseteq A^-.$$}

Furthermore, the shortest path connecting $x$ to $\hat{x}$ lies outside of $B_R(z)$, so it cannot intersect $\partial A^+$, and hence $x \in A^-$. We proved that $B_n(z) \setminus B_{n-m}(z) \subseteq A^-$. Therefore, $B_n(z) \setminus B_{n-m}(z) \subseteq B^-$, so $B_n(p) \setminus B_{n-m}(p) \subseteq Y^-$. Since $\partial Y^- \subseteq B_n(p)$, this implies that $X \setminus B_n(p) \subseteq Y^c$, so $Y$ is finite. This is a contradiction, hence $A^+$ is infinite. We can prove the same way that $A^-$ is also infinite. \hfill \square

We showed that $A^- = (A^+)^c$, and that $A^+$ and $A^-$ are both infinite. Therefore, they both contain exactly one end of the geodesic $\ell$ by Lemma 8.10. Let us define

$$Y_z = \begin{cases} A^+ & \text{if } + \infty \in A^+, \\ A^- & \text{if } + \infty \in A^- \end{cases}$$

In both cases, we have $+ \infty \in Y_z$, $- \infty \in Y_z^c$, $\partial Y_z \subseteq B_R(z)$ (by Claim 3.21) and $F \subseteq [[G \acts \Sigma]]_{Y_z}$ (by Claim 3.21). This concludes the proof of the lemma. \hfill \square
3.4 Amenable kernel

**Proposition 3.23.** The stabilizer \([G \curvearrowright \Sigma]_Y\) is locally finite.

**Proof.** Consider a finite set \(F \subseteq [G \curvearrowright \Sigma]_Y\), our goal is to prove that the subgroup \(\langle F \rangle\) is also finite. Define \(N_F = \max \{N_\varphi : \varphi \in F\}\).

Let \(n > N_F\). Let \(r = r(p, F, n)\) from Lemma 3.18 for the point \(p\), the finite set \(F\) and the number \(n\). Let \(y_0 = z_0 = p\), and pick \(y_i \in \ell\) for all \(i \in \mathbb{Z} \setminus \{0\}\) such that \(d(y_i, y_{i+1}) = 2r + 2n + 2m + 2\) and \(y_i\) is closer to \(-\infty\) than \(y_{i+1}\) for every \(i \in \mathbb{Z}\). Now for every \(i \in \mathbb{Z} \setminus \{0\}\) let us use Lemma 3.18 for the point \(y_i\). Thus, we get the points \(z_i \in B_i(y_i)\) (for all \(i \in \mathbb{Z}\)) such that for every \(\varphi \in F\), the \(\varphi\)-action is the same on the \(S^a\)-neighborhood of \(p\) and \(z_i\). Note that due to the choice of the \(y_i\)'s, the \(n\)-balls around the points \(z_i\) are pairwise disjoint.

Let \(Y_0 = Y\) and for every \(i \in \mathbb{Z} \setminus \{0\}\) let us use Lemma 3.18 for \(F\), the point \(z_i\) and the number \(n\). For every \(i\), there exists an infinite set \(Y_i = Y_{z_i} \subseteq X\), such that we have \(+\infty \in Y_i\), \(-\infty \in Y_i\), furthermore \(\partial Y_i \subseteq B_R(z_i)\) and \(F \subseteq [G \curvearrowright \Sigma]_Y\).

**Claim 3.24.** For every \(i \in \mathbb{Z}\), the set \(Y_i\) contains \(Y_{i+1}\).

**Proof.** Let us denote by \(v\) the midpoint between \(y_i\) and \(y_{i+1}\) on \(\ell\). Note that we have \(d(v, B_n(z_i)) \geq m + 1\) and \(d(v, B_n(z_{i+1})) \geq m + 1\) by the choice of the distance between \(y_i\) and \(y_{i+1}\).

Therefore, \(v \in Y_i\) since \(+\infty \in Y_i\) and the half line \([v, +\infty]\) does not intersect \(\partial Y_i \subseteq B_n(z_i)\). Similarly, we have \(v \in Y_i\) since \(-\infty \in Y_i\) and \([v, -\infty]\) does not intersect \(\partial Y_{i+1} \subseteq B_n(z_i)\).

Suppose for contradiction that there exists a point \(x \in Y_{i+1} \setminus Y_i\). Let \(\hat{x}\) be the closest point to \(x\) on \(\ell\), and let \([x, \hat{x}]\) denote a shortest path between them. We have \(d(\hat{x}, x) \leq m\) by Proposition 3.39. There are two possibilities.

1. If \(\hat{x} \in [v, +\infty]\), then the path \([x, \hat{x}] \cup [\hat{x}, v]\) does not intersect \(B_n(z_i)\) since \(d(\hat{x}, B_n(z_i)) \geq m + 1\). However, \(x \notin Y_i\) and \(v \in Y_i\), so any path between the two must intersect \(\partial Y_i \subseteq B_n(z_i)\). Hence, we get a contradiction.

2. If \(\hat{x} \in [-\infty, v]\), then we can use a similar argument: We have \(x \in Y_{i+1}\) but \(v \notin Y_{i+1}\), so any path between them must intersect \(\partial Y_{i+1} \subseteq B_n(z_{i+1})\). However, the path \([x, \hat{x}] \cup [\hat{x}, v]\) does not intersect \(B_n(z_{i+1})\), leading to a contradiction.

We get a contradiction in both cases, so such a point \(x\) cannot exist. This proves that \(Y_{i+1} \subseteq Y_i\).

**Claim 3.25.** For every \(i \in \mathbb{Z}\), the set \(Y_i \setminus Y_{i+1}\) is a finite set. Moreover, there is a uniform bound on the cardinality of the sets \(Y_i \setminus Y_{i+1}\).

**Proof.** Consider the \(m\)-neighborhood of the segment \([y_{i-1}, y_{i+2}] \subseteq \ell\), denoted by \(B_m([y_{i-1}, y_{i+2}])\). First of all, the size of these sets has a uniform bound, since the graph \(X\) is regular, and hence the size of the \(m\)-neighborhood of a set of \(3(2r + 2n + 2m + 2) + 1\) points is uniformly bounded.

We show that \(Y_i \setminus Y_{i+1} \subseteq B_m([y_{i-1}, y_{i+2}])\). Take any point \(x \in Y_i \setminus Y_{i+1}\), let \(\hat{x}\) denote its projection to \(\ell\). Suppose for contradiction that \(\hat{x} \in [y_{i+2}, +\infty]\), then \([\hat{x}, +\infty]\) does not intersect \(\partial Y_{i+1}\) since \(\partial Y_{i+1} \subseteq B_n(z_{i+1}) \subseteq B_{n+r}(y_{i+1})\). Hence, we have \(\hat{x} \in Y_{i+1}\), and also \(x \in Y_{i+1}\), since \([x, \hat{x}]\) cannot intersect \(\partial Y_{i+1}\) either. This contradicts the face that \(x \in Y_i \setminus Y_{i+1}\). Therefore, we must have \(\hat{x} \in [-\infty, y_{i+2}]\).
Similarly, one can prove that if \( \hat{x} \in (-\infty, y_{i-1}] \), then \( x \notin Y_i \), leading to a contradiction again. This proves that \( \hat{x} \in [y_{i-1}, y_{i+2}] \). Recall that \( d(x, \hat{x}) \leq m \), and hence \( x \in B_m([y_{i-1}, y_{i+2}]) \).

We proved that \( Y_i \setminus Y_{i+1} \subseteq B_m([y_{i-1}, y_{i+2}]) \), this shows that the set \( Y_i \setminus Y_{i+1} \) is finite for every \( i \in \mathbb{Z} \), and that there is a uniform bound on their cardinalities.

Notice that due to the \( F \)-invariance of every \( Y_i \), the sets \( Y_i \setminus Y_{i+1} \) are also \( F \)-invariant for all \( i \in \mathbb{Z} \). Their union is the whole graph \( X \), therefore, we can embed \( \langle F \rangle \) into the direct product of the finite symmetric groups on the sets \( Y_i \setminus Y_{i+1} \). By Claim 3.25, the size of these finite symmetric groups is uniformly bounded, and hence their direct product is locally finite.

Since \( \langle F \rangle \) can be embedded into a locally finite group, and is finitely generated, it must be finite. This concludes the proof of the proposition.

Finally, we are ready to prove our main theorem.

**Proof of Theorem 1.** Consider a minimal action \( G \curvearrowright \Sigma \) of the finitely generated group \( G \), such that there exists an orbit \( X \) which is quasi-isometric to \( \mathbb{Z} \).

First, we show that the action \( \langle [G \curvearrowright \Sigma] \rangle \curvearrowright X \) is extensively amenable. The Schreier graph of the action \( G \curvearrowright X \) is quasi-isometric to \( \mathbb{Z} \), and hence it is recurrent. As a corollary of Rayleigh’s monotonicity principle, all connected subgraphs of a recurrent graph are also recurrent (for a proof see [9], Chapter 2). Thus, by Proposition 2.5, the action \( PW(G \curvearrowright X) \curvearrowright X \) is extensively amenable. It follows easily from the definition of extensive amenability that the action of any subgroup of \( PW(G \curvearrowright X) \) on \( X \) is also extensively amenable. Therefore, \( \langle [G \curvearrowright \Sigma] \rangle \curvearrowright X \) is extensively amenable, as desired.

Next, we apply Proposition 2.5 for \( \langle [G \curvearrowright \Sigma] \rangle \) and \( X \), with the cocycle \( c \) defined in Definition 3.10. By (2) in Remark 3.11, we have \( \ker c = \langle [G \curvearrowright \Sigma] \rangle_Y \). By Proposition 3.23, this stabilizer \( \langle [G \curvearrowright \Sigma] \rangle_Y \) is locally finite, and hence it is amenable. Therefore, the conditions of Proposition 2.5 are satisfied: the action \( \langle [G \curvearrowright \Sigma] \rangle \curvearrowright X \) is extensively amenable, and the kernel of the cocycle \( c \) is amenable. This proves that the topological full group \( \langle [G \curvearrowright \Sigma] \rangle \) is also amenable.

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