RECENT PROGRESS IN LATTICE QCD

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Abstract

I give a brief overview of the status of lattice QCD, concentrating on topics relevant to phenomenology. I discuss the calculation of the light quark spectrum, the lattice prediction of $\alpha_{\text{MS}}(M_Z)$, and the calculation of $f_B$.

Plenary talk given at the Meeting of the Division of Particles and Fields of the American Physical Society, Fermilab, November 10-14, 1992

PREPARED FOR THE U.S. DEPARTMENT OF ENERGY

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December 1992

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1 INTRODUCTION

Lattice gauge theory has been somewhat out of the mainstream of particle physics for the past decade. It seems to me, however, that the field is now coming of age. It has certainly grown rapidly: roughly 300 people attended the recent LATTICE ’92 conference, compared to about 130 at LATTICE ’86. That it has also matured is indicated by the breadth of the subjects being studied with lattice methods. These include the traditional—QCD spectrum and matrix elements; the more statistical mechanical—rigorous theorems on finite size scaling; the more abstract—random surfaces; and the exotic—finite temperature baryon number violation. More concrete evidence for maturation is that lattice results are becoming useful to the rest of the particle physics community. For example, in his summary of B-physics, David Cassel noted that the bounds on the elements of the CKM-matrix which follow from $B - \bar{B}$ mixing depend on the values of $f_B$ and $B_B$, and that lattice results are now used as one of the estimates of these numbers. But perhaps the most important piece of evidence is that lattice studies are beginning to produce results with no unknown systematic errors. The best example is the calculation of the full QCD running coupling constant $\alpha_{\text{MS}}$ by the Fermilab group [1]. As I discuss below, this number can be directly compared to experiment.

In summary, I would say that lattice studies are beginning to make themselves useful. The “beginning” in this claim is important—there is a very long way to go before we can calculate, say, the $K \to \pi\pi$ amplitudes from first principles. Thus this talk does not consist of results for a long list of matrix elements. Instead, I discuss a few topics in some detail. Because of lack of time, I concentrate entirely on results from lattice QCD (LQCD) which are relevant to phenomenology.

When thinking about the progress that has been made, it is useful to keep in mind the questions that one ultimately wishes to answer using LQCD:

1. Does QCD give rise to the observed spectrum of hadrons? Do these particles have the observed properties (charge radii, decay widths, etc)?

2. Does the same theory also describe the perturbative, high-energy jet physics?

3. What are the values of the hadronic matrix elements ($f_B, B_K$, etc.) which are needed to determine the poorly known elements of the CKM matrix?

4. What happens to QCD at finite temperature, and at finite density?

5. What are the properties of “exotic” states in the spectrum, e.g. glueballs?

6. What is the physics behind confinement and chiral symmetry breaking?

The last question is the hardest, and, while it is important to keep thinking about it, there have not been any breakthroughs. The other questions are being addressed using numerical simulations, complemented by a variety of analytic calculations. Most recent progress has concerned the first four questions, and I will discuss only the first three. For additional details see the reviews of Mackenzie (heavy quarks), Petersson (finite temperature), Sachrajda (weak matrix elements) and Ukawa (spectrum) at LATTICE ’92 [2].
To set the stage I begin with a brief summary of LQCD, eschewing as many details as possible. The action of QCD is the sum of a gauge term,

\[ S_g = \frac{6}{g^2} \int \frac{1}{12} \sum_{\mu\nu} \text{Tr}(G_{\mu\nu}G_{\mu\nu}), \quad (1) \]

with \( G_{\mu\nu} \) the gluon field strength, and a quark part

\[ S_q = \int \sum_q \bar{q}(\gamma_\mu D_\mu + m_q)q, \quad D_\mu = \partial_\mu + iA_\mu. \]

These expressions are valid in Euclidean space, where numerical lattice calculations are almost always done. QCD is put on a hypercubic lattice by placing the quarks on the sites, and gauge fields on the links which join these sites, in such a way that gauge invariance is maintained. The derivative in \( D_\mu \) can be discretized in many ways, leading to different types of lattice fermions. The most common choices are Wilson and staggered fermions. The coupling constant \( g \) has been absorbed in the gauge fields, and appears only as an overall factor in \( S_g \). It is conventional (see Eq. 1) to use the combination \( \beta = 6/g^2 \) to specify \( g \). In present simulations \( \beta \sim 6 \), so that \( g^2 \sim 1 \). The lattice spacing \( a \) is determined implicitly by the choice of \( g^2 \), as discussed below.

The prototypical quantity of interest is the two-point correlator, e.g.

\[ C(t) = \left\langle \sum_\vec{x} [\bar{b}\gamma_0\gamma_5d(t, \vec{x})][\bar{d}\gamma_0\gamma_5b(0)] \right\rangle = \text{sign}(t) \ f_B^2m_Be^{-m_B|t|} \left( 1 + O(e^{-(m_B'-m_B)|t|}) \right). \quad (2) \]

At large Euclidean time, \( t \), one picks out the lightest state (here the \( B \) meson). The contribution of excited states (beginning with the \( B' \)) is suppressed exponentially. Thus one can just read off the mass, \( m_B \), while the amplitude of the exponential gives the decay constant \( (f_B) \) up to kinematical factors.

The expectation value in Eq. 2 indicates a functional integral over quarks, antiquarks and gluons weighted by \( \exp(-S_g - S_q) \). Doing the quark and antiquark integrals, one obtains

\[ C(t) = -\int [dA] e^{-S_g} \left[ \Pi_q \det(\hat{\mathcal{D}} + m_q) \right] \sum_\vec{x} \text{Tr}[\gamma_0\gamma_5G_d(t, \vec{x}; 0)\gamma_0\gamma_5G_b(0; t, \vec{x})]. \quad (3) \]

Here \([dA]\) is shorthand for the functional integral over lattice gauge fields, and the \( G \) are quark propagators \( G_q = (\hat{\mathcal{D}} + m_q)^{-1} \). The integral is normalized to give unity in the absence of the trace term. I have shown Eq. 3 diagrammatically in Fig. 1, where the lines are quark propagators, and the primed expectation value includes the determinant.

To make the calculations numerically tractable, they must be done in a finite volume of \( N_s^3 \times N_t \) points. The functional integral is then of large but finite dimension, and can be done by Monte-Carlo methods. Similarly, the propagators are obtained by inverting a finite matrix. Available computer speed and memory limits the number of sites in this four dimensional world. In fact, it is speed that limits present simulations: the bottleneck is the inclusion of \( \det(\hat{\mathcal{D}} + m_q) \) in the measure. To make progress, many calculations use the
“quenched” approximation in which the determinant in Eq. 3 is set equal to unity. This is equivalent to dropping all internal quark loops, so that there are only valence quarks. I discuss the consequences of this drastic approximation below.

It is important to realize that numerical LQCD calculations which do not use the quenched approximation become exact when the lattice spacing vanishes, the volume goes to infinity, and the statistical errors vanish. They are thus non-perturbative calculations from first principles, which have errors that can be systematically reduced. They are not “model” calculations. On the other hand, calculations in the quenched approximation are more similar to those in a model. Certain physics is being left out, and one does not know a priori how large an error is being made. As I will show, present calculations suggest that the error is small, at least for a range of quantities.

In the following three sections I discuss the spectrum, the calculation of $\alpha_{\text{MS}}$, and that of $f_B$. I will focus mainly on quenched results since these are more extensive. I will only make some brief comments on results for “full QCD”, i.e. QCD including the determinant in the measure.

2 SPECTRUM

I begin by discussing the status of calculations of the spectrum of light hadrons. I concentrate on $m_\pi$, $m_\rho$ and $m_N$ ($N$ refers to the nucleon), since these have the smallest errors. To give an idea of the rate of progress, I will compare with results from March 1990 [3]. The two and a half year interval since then is long enough to allow substantial changes. For example, computer power has increased as roughly $CPU \propto e^{\text{year}}$, so that, for a four dimensional theory such as QCD, the linear dimensions could have increased by a factor of $\sim e^{2.5/4} = 1.8$ in this period. In fact, the extra time has only partly been used in this way.

To display the data I follow the “APE” group and plot $m_N/m_\rho$ versus $(m_\pi/m_\rho)^2$ (see Figs. 3). To understand the significance of these plots, recall the following. In a lattice calculation, we can dial the values of the quark masses. Ignoring for the moment the strange quark, and assuming degenerate up and down quarks, we then have a single light quark mass, $m_l$, at our disposal. Each value of $m_l$ corresponds to a possible theory, each with different values for dimensionless mass ratios such as $m_\pi^2/m_\rho^2$, $m_\rho/m_N$, $f_\pi/m_\rho$, $m_\Delta/m_N$, etc. We would like to fix $m_l$ using one of these ratios, and then predict the others. In practice, it is technically very difficult to do a simulation with small enough $m_l$, and so we must extrapolate. The APE plot is one way of displaying how well this extrapolation agrees with

\[
C(t) = \sum_{x} \langle \Psi(t,\vec{x}) | \Phi(0) \rangle
\]

Figure 1: Diagrammatic representation of Eq. 3
the experimental masses. As \( m_t \) varies from 0 → ∞, \( m_π^2/m_ρ^2 \) varies monotonically from 0 → 1. Thus the theory maps out a curve in this plot, which we know must pass through the infinite mass point (\( m_π = m_ρ = 2m_N/3 \); shown by a square in the plots). The issue is whether this curve passes through the experimental point, indicated by a “?” on the plots.

The solid lines if Figs. 2 are the predictions of two phenomenological models. The curve for light quark masses uses chiral perturbation theory, with certain assumptions, and is constrained to go through the physical point. The curve at heavier masses is based on a potential model. For more discussion see Ref. [3].

What do we expect for the lattice results? There will be corrections to physical quantities which vanish in the continuum limit as powers of \( a \) (up to logarithms, which I will ignore), e.g.

\[
(m_N/m_ρ)_{\text{latt}} = (m_N/m_ρ)_{\text{cont}}[1 + aΛ + O(a^2)].
\]

Thus, for finite \( a \), the lattice curve should not pass through the experimental point. Similarly, if the physical size of the lattice (\( L = N_s a \)) becomes too small the masses will be shifted from their infinite volume values. Thus, in addition to extrapolating in the quark mass, one must attempt to extrapolate both \( a \to 0 \) and \( L \to ∞ \).

What has happened in the last two years is that these extrapolations have become more reliable. I illustrate this with results using Wilson fermions in the quenched approximation (details of the data set are given in Table 1). Figure 2a shows the state-of-the-art in March 1990. The upper two sets of points are from a large lattice spacing (\( a ≈ 1/6 \text{ fm} \)) with two different lattice sizes (\( L ≈ 2 \) and 4 fm). The results agree within errors, and I concluded that we knew the infinite volume curve for \( a = 1/6 \text{ fm} \) with a precision of about 5%. This curve appeared not to pass through the physical point. The lower two sets of points are from a smaller lattice spacing (\( a ≈ 1/10 \text{ fm} \)). They suggested a downward shift in the data with decreasing lattice spacing, but it was difficult to draw definite conclusions given the large errors.

The present results are shown in Fig. 2b. To avoid clutter, I show only data with
Table 1: Parameters of lattices used to produce data shown in Fig. 2

| Year | Ref. | $\beta$ | $a$(fm) | $\pi/a$(GeV) | $N_s$ | $L$(fm) | Lattices |
|------|------|---------|--------|-------------|-------|--------|----------|
| 1990 | [4]  | 5.7     | 1/6    | 3.8         | 12,24 | 2.4    | 294,50   |
|      | [4]  | 6       | 1/10   | 6.3         | 18,24 | 1.8,2.4| 104,33   |
| 1992 | [5]  | 5.93    | 1/9    | 5.7         | 24    | 2.7    | 217      |
|      | [6]  | 6       | 1/10   | 6.3         | 24    | 2.4    | 78       |
|      | [7]  | 6.17    | 1/12   | 7.5         | 32    | 2.7    | 219      |
|      | [6]  | 6.3     | 1/16   | 10          | 24    | 1.5    | 128      |

$a \leq 1/9$fm\[6\] These are all consistent with a single curve lying below that at $a = 1/6$fm and close to the phenomenological predictions. The improvements in the results have come from using more lattices to approximate the functional integral, which reduces the statistical errors (see Table [4]). Furthermore, the decrease in $a$ has been compensated by an increase in $N_s$, so that (with the exception of the APE results at $\beta = 6.3$) the physical extent of the lattices exceeds $L = 2$fm. The results at $\beta = 5.7$ imply that this is large enough to get within a few percent of the infinite volume results.

The GF11 group have used their results at $a \approx 1/6, 1/9, 1/12$fm to do an extrapolation both to physical quark masses and to $a = 0$ [5]. They find that, for a range of quantities, the results are consistent with their experimental values within $\sim 5\%$ errors. For example, $m_N/m_\rho = 1.284^{+0.071}_{-0.065}$ (cf 1.222 expt.) and $m_\Delta/m_\rho = 1.627^{+0.051}_{-0.092}$ (cf 1.604 expt.). It should be recalled that a few years ago the quenched results for $m_N/m_\rho$ were thought to be larger than the experimental value, while $m_\Delta - m_N$ was smaller. What we have learned is that both the errors introduced by working in finite volume and at finite lattice spacing shift the curve in the APE plot upwards, so that one can easily be misled by results on small lattices.

We seem, then, to know the spectrum of light hadrons in the quenched approximation fairly well, and it looks a lot like the observed spectrum. There is not, however, complete agreement on the numerical results. The QCDPAX collaboration, working at lattice spacings comparable to those in Fig. [2]b, finds significantly larger values for $m_N/m_\rho$ at smaller $m_\pi^2/m_\rho^2$ [8]. They suggest that the disagreement may be due to a contamination from excited states in the correlators (see Eq. [2]). This disagreement will get cleared up in the next year or so. What is needed is operators which couple more strongly to the ground state, and less to excited states. There has been considerable improvement in such operators, for systems involving a heavy quark [9], but less so for light quark hadrons.

Let me assume that the results of Fig. [2]b are correct, so that the quenched spectrum does agree with the experiment to within $\sim 5\%$. Does this imply that we can trust quenched calculations of other quantities to this accuracy? I do not think so. A priori, I would not have expected the quenched approximation to work so well, because it leaves out so much important physics. For example, the quenched rho cannot decay into two pions, unlike the

\[\text{The new results at } a = 1/6 \text{fm } [6] \text{ confirm, with reduced errors, those in Fig. 2a.}\]
physical rho, which might lead to a 10% underestimate of its mass \[10\]. Also, the pion cloud around the light hadrons is much different in the quenched approximation than in full QCD, which should shift \(m_ρ\) and \(m_N\) in different ways. While it is possible that these effects largely cancel for the spectrum, I see no reason for them to do so in other quantities. This argument can be made more concrete for the charge radii \[11\]. Futhermore, as I mention below, there are reasons to think that the approach to the chiral limit in the quenched approximation is singular.

My final remark on the quenched spectrum concerns the possibility of improving the approach to the continuum limit \[12\]. The gauge action is accurate to \(O(a^2)\), but the Wilson fermion action has \(O(a)\) corrections, as in Eq. \[4\]. It is possible to systematically “improve” the fermion action so that the corrections are successively reduced by powers of \(g^2/4\pi\). The idea is that by using a more complicated action one can work at a larger lattice spacing. The first results of this program are encouraging. Ref. \[13\] finds that an improved action shifts the results at \(a = 1/6\ \text{fm}\) downwards so as to agree with those at \(a \approx 1/10\ \text{fm}\), and that both agree with the “unimproved” results at \(a \leq 1/9\ \text{fm}\) shown in Fig. 2b. Ref. \[14\] finds that the improved action makes no difference at a smaller lattice spacing, \(a \approx 1/14\ \text{fm}\). All this suggests that it may be possible use lattice spacings as large as 1/6 fm with an improved Wilson fermion action. Further evidence for this comes from lattice studies of charmonium \[15, 16\]. With staggered fermions, on the other hand, the prospects are less rosy. Although the corrections are of \(O(a^2)\), and thus parametrically smaller than with Wilson fermions, they are in fact large at \(a = 1/6\ \text{fm}\ \[17\].

Ultimately, we must repeat the calculation using full QCD. Much of the increase in computer time in the last few years has gone into simulations which include quark loops. Nevertheless, it is too early to discuss physical predictions, since it is not yet possible to reliably extrapolate \(m_q \to 0\) or \(a \to 0\). The limit \(L \to \infty\) is, however, well understood and some interesting results have been obtained. The Kyoto-Tsukuba group finds good fits to the form \(m = m_\infty(1 + c/L^3)\), and argue that this can be understood as due to the “squeezing” of the hadrons in the finite box \[18\]. The MILC collaboration finds that at \(L = 2.7\ \text{fm}\) the finite volume effects are smaller than 2\% \[20\]. Both these results are consistent with the finite volume effects observed in the quenched approximation \[3\].

### 3  \(\alpha_{\overline{\text{MS}}}\) FROM THE LATTICE

It is straightforward, in principle, to extract \(\alpha_{\overline{\text{MS}}}\) from lattice calculations:

1. Pick a value of \(\beta = 6/g^2\).
2. Calculate a physically measurable quantity, e.g. \(m_ρ\) or \(f_π\).
3. Compare lattice and physical values and extract \(a\) using, e.g.

\[
(m_ρ)_{\text{latt}} = (m_ρ)_{\text{phys}} \times a \ (1 + O(a))
\]

The \(O(a)\) terms are not included when extracting \(a\).

\[\dagger\] This is not the asymptotic form: for large enough \(L\) the power law becomes an exponential \[19\].

\[\S\] The values for \(a\) quoted above were obtained in this way, using a variety of physical quantities.
4. Convert from the lattice bare coupling constant \( g_a^2 \) to \( \alpha_{\overline{\text{MS}}}(q = \pi/a) \) using perturbation theory. The result can then be evolved to other scales, e.g. \( M_Z \), using the renormalization group equation.

5. Repeat for a variety of values of \( \beta \), and extrapolate to the continuum limit \( a = 0 \). In this way the \( O(a) \) corrections in Eq. 5 are removed.

If this program were to be carried through, then the lattice result for \( \alpha_{\overline{\text{MS}}} \) would allow absolute predictions of jet cross sections, \( R(e^+e^-) \), etc. (modulo the effects of hadronization). If these predictions were successful, it would demonstrate that QCD could simultaneously explain widely disparate phenomena occurring over a large range of mass scales. While such success has not yet been achieved, there has been considerable progress in the last year or so. I will explain the two major problems, and the extent to which they have been resolved. I then discuss the present results.

3.1 Reliability of Perturbation Theory

The fourth step in the program requires that perturbation theory be valid at the scale of the lattice cut-off, which is roughly \( \pi/a \) in momentum space. On present lattices this ranges from \( 5 - 12 \text{ GeV} \) (the values are given in Table I). It turns out that these values are not large enough for perturbation theory in the bare coupling constant to be accurate, because there are large higher order corrections. These are exemplified by the relation needed in step 4 above \[ 21 \]

\[
\frac{1}{\alpha_{\overline{\text{MS}}} (\pi/a)} = \frac{1}{\alpha_{\text{latt}} (a)} - 3.880 + O(\alpha) ; \quad (\alpha = g^2/4\pi) .
\] (6)

Since \( \alpha_{\text{latt}} \approx 1/13 \), the first order correction is large, and higher order terms are likely to be important.

This problem has been understood by Lepage and Mackenzie \[ 22 \]. The large corrections arise from fluctuations of the lattice gauge fields, and in particular from “tadpole” diagrams which are present on the lattice but absent in the continuum. The solution is to express perturbative results in terms of a “continuum-like” coupling constant, e.g. \( \alpha_{\text{MOM}} \) or \( \alpha_{\overline{\text{MS}}} \), with the scale \( q \approx \pi/a \). When this is done the higher order coefficients are considerably reduced. This is similar to what happens in the continuum when one shifts from the MS to the \( \overline{\text{MS}} \) scheme.

Having re-expressed all perturbative expressions in terms of, say, \( \alpha_{\overline{\text{MS}}} \), there remains the problem of finding the value of this coupling in terms of \( \alpha_{\text{latt}} \), since Eq. 3 is not reliable. One has to use a non-perturbative definition of coupling constant which automatically sums up the large contributions, and which is related to \( \alpha_{\overline{\text{MS}}} \) in a reliable way. One choice is \[ 22 \]

\[
\alpha_P = -\frac{3 \ln \langle \text{Tr} U_P \rangle}{4\pi} ; \quad \frac{1}{\alpha_{\overline{\text{MS}}} (\pi/a)} = \frac{1}{\alpha_P} - 0.5 + O(\alpha) ,
\] (7)

where \( U_P \) is the product of gauge links around an elementary square. One first determines \( \alpha_P \) from the numerical value of \( \text{Tr} U_P \), and then converts this to \( \alpha_{\overline{\text{MS}}} \), which is then used in perturbative expressions. Lepage and Mackenzie find that the resulting numerical predictions
of lattice perturbation theory work well for quantities (such as small Wilson loops) which are
dominated by short-distance perturbative contributions. This is true for lattice spacings as
large as 1/9 fm, and perhaps 1/6 fm. Thus the determination of $\alpha_{\overline{\text{MS}}}$ using Eq. 7 is reliable,
with errors probably no larger than a few percent.

3.2 Errors Introduced by the Quenched Approximation

The dominant source of uncertainty in present calculations of $\alpha_{\overline{\text{MS}}}$ is the use of the quenched
approximation. The problem is the lack of a “physical” quenched theory to use in the
comparison of step 3 above. This shows up in two ways. First, the value of $\alpha$ depends on
the quantity chosen in the comparison: using $m_{\rho}$ gives one value, $f_{\pi}$ another. Second, the
coupling constant that one obtains is for a theory with zero flavors, $\alpha^{(0)}_{\overline{\text{MS}}}$, and must somehow
be related to the physical coupling constant $\alpha_{\overline{\text{MS}}}$. However, such a relationship involves
non-perturbative physics, and so can only be determined by a calculation using full QCD!
The best that we can hope for at present is a good estimate of the relationship between the
couplings.

Recently, the FNAL group have made such an estimate for the coupling determined
using the $1P - 1S$ splitting in charmonium [1]. The crucial simplifying feature is that
charmonium is described reasonably well by a potential model, so one need only estimate
the effects of quark loops on the potential itself. In outline, this is done as follows. Matching
the lattice and continuum $1P - 1S$ splittings makes the quenched and full QCD potentials
similar at separations $R \sim 0.5$ fm. The potentials will, however, differ at smaller separations.
We understand this difference at small enough $R$, where the Coulomb term dominates, i.e.
$V \propto -\alpha_{\overline{\text{MS}}} (1/R)/R$. In the quenched approximation $\alpha$ varies more rapidly with $R$ because of
the absence of fermion loops, which means that the quenched potential is steeper. Assuming
that this is true all the way out to 0.5 fm, where the potentials match, implies that the
quenched potential must lie below that for full QCD at short distances. It follows that
$\alpha_{\overline{\text{MS}}} > \alpha^{(0)}_{\overline{\text{MS}}}$ at short distances. A quantitative estimate is given in Ref. [1].

Unfortunately, there is no such simple way of estimating the effects of the quenched
approximation on the values of $\alpha^{(0)}_{\overline{\text{MS}}}$ extracted from the properties of light quark hadrons.

3.3 Results

Various physical quantities have been used to calculate $\alpha^{(0)}_{\overline{\text{MS}}}$, and I collect the most accurate
results in Table 2. I also give the “experimental” number obtained from an average of various
perturbative QCD fits to data [23]. I quote the coupling at the scale $M_Z$, to allow comparison
with Fig. 1 of the 1992 Review of Particle Properties (RPP) [23]. The second row gives the
results of the FNAL group, including the correction for quenching. The remaining results
use the “string tension”, $\sigma$. This is the coefficient of the linear term in the heavy quark-
antiquark potential: $V(R) \to \sigma R$ for $R \to \infty$. All groups use the “improved” perturbation
theory explained above. I have not shown results from light hadron masses as they are less
accurate.

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6 Actually, since light hadron mass ratios are well reproduced by the quenched approximation, the variation
of $\alpha$ is small for such quantities. This need not be true in general.
Table 2: Results for $\alpha_{\text{MS}}$ in quenched and full QCD. Errors are statistical.

| Quantity             | Ref. | $\alpha_{\text{MS}}^{(0)}(M_Z)$ | $\alpha_{\text{MS}}(M_Z)$ |
|----------------------|------|----------------------------------|---------------------------|
| “Experiment”         | 23   | 0.1134(35)                       |                           |
| $M_{1P} - M_{1S}$    | 1    | 0.0790(9)                        | 0.105(4)                  |
| String tension       | 24   | 0.0796(3)                        |                           |
| String tension       | 25   | 0.0801(9)                        |                           |

The results for $\alpha_{\text{MS}}^{(0)}$ obtained using $\sigma$ have the smallest statistical errors. Indeed, it is a triumph of LQCD that the linear term in the quenched potential is very well established, for it is this term that causes confinement. The difficulty is that $\sigma$ is not a physical quantity. The “physical” value, $\sqrt{\sigma} = 0.44$ GeV, is extracted from the potential needed to fit the $\bar{c}c$ and $\bar{b}b$ spectra. These systems do not, however, probe the region of the potential where the linear term dominates. Furthermore, at large $R$ the full QCD potential flattens out due to quark-pair creation, while the quenched potential continues its linear rise. Thus the extracted value of $\sigma$ is somewhat uncertain, and it is difficult to relate $\alpha_{\text{MS}}^{(0)}$ obtained using $\sigma$ to the full QCD coupling. Nevertheless, it seems to me possible to make a rough estimate, and it would certainly be interesting to try.

The lattice prediction for $\alpha_{\text{MS}}$ is slightly less than $2\sigma$ below the RPP average. In his summary talk, Keith Ellis quoted an updated average, $\alpha_{\text{MS}}(M_Z) = 0.119(4)$, which is almost $3\sigma$ higher than the lattice result. I think that this near agreement is a success of LQCD, and that it is too early to make anything out of the small discrepancy. For one thing, the dominant error in the lattice result is the uncertainty in the conversion from $\alpha_{\text{MS}}^{(0)}$ to $\alpha_{\text{MS}}$, and this might be underestimated. It is important to realize, however, that this uncertainty will be gradually reduced as simulations of full QCD improve.

I end this section with a general comment on the calculations. For the method to work, both non-perturbative and perturbative physics must be included in a single lattice simulation. At long distances the quarks are confined in hadrons, while at the scale of the lattice spacing their interactions must be described by perturbation theory. On a finite lattice, these two requirements pull in opposite directions. For example, if perturbation theory requires $a < 1/9$ fm, while finite volume effects require $L > 2.7$ fm (as may be true for light hadrons), the lattice must have $N_s \geq 30$, and so present lattices are barely large enough. This is another reason for using charmonium to determine $\alpha_{\text{MS}}$. The $\bar{c}c$ states are smaller than light hadrons, and it turns out that $N_s = 24$ is large enough to reduce the finite volume errors below 1% [1].

Nevertheless, it would be nice to extend the range of scales so as to provide a detailed test of the dependence on $a$. To accomplish this with present resources requires the use of multiple lattices having a range of sizes. Lüscher et al. have proposed such a program, and done the calculation for SU(2) pure gauge theory [26].
4 The B-meson decay constant $f_B$

One of the most important numbers that LQCD can provide phenomenologists is $f_B$. Analyses of the constraints due to $\bar{K} - K$ and $\bar{B} - B$ mixing find (for $m_t \geq 140$ GeV) two types of solutions for CP-violation in the CKM-matrix (see, e.g. Ref. [27]). These are distinguished by whether $f_B$ is “small” (100 − 150 MeV) or “large” (160 − 340 MeV). In the former case, CP-violation in $B \to K_S J/\psi$ is small; in the latter it is much larger. We would like the lattice to resolve this ambiguity.

The calculation of $f_B$ is also interesting as a testing ground for the “heavy quark effective theory” (HQET) [28]. The B-meson consists of heavy $b$ quark, and a light quark ($u$ or $d$). Imagine that we were free to vary the heavy quark mass, $m_Q$. The mass of the pseudoscalar meson (which I call $m_P$ to distinguish it from the physical $B$ meson mass, $m_B$), and its decay constant $f_P$, would depend upon $m_Q$. As $m_Q \to \infty$ one can show that [29]

$$\phi_P = f_P \sqrt{m_P} \left[ \frac{\alpha(m_P)}{\alpha(m_B)} \right]^{2/\beta_0} = \phi_\infty \left( 1 + \frac{A}{m_P} + \frac{B}{m_P^2} + \ldots \right).$$

(8)

Here $\beta_0$ is the first term in the QCD $\beta$-function, and $A$ and $B$ are constants except for a weak logarithmic dependence on $m_P$. The issue for HQET is the size of the $1/m_P$ corrections at the $B$ and $D$ masses, for these might be indicative of the size of the corrections in other applications of HQET, e.g. $B \to D$ transitions.

It is important to realize that, while the $m_Q \to \infty$ limit simplifies the kinematics of the heavy quark, it does not simplify the dynamics of the light quark. In particular, the quenched approximation is no better in the heavy quark limit, nor is the dynamics more perturbative. Thus one needs a LQCD calculation here just as much as for light quark hadrons.

What we would like to do is map out the curve of $\phi_P$ versus $1/m_P$, and read off the values of $f_B$ and $f_D$. Examples of present results are shown in Figs. 3, and in Table 3 I give the present numerical values for $f_B$ and $f_D$. I also quote $f_B(\text{static}) = \phi_\infty / \sqrt{m_B}$, which is the value of $f_B$ ignoring the $1/m_P$ corrections in Eq. 8. Unfortunately, the situation is less straightforward than the figures imply, and I will spend the remainder of this section explaining and evaluating these results.

There are three major causes of uncertainty in $f_B$. The first concerns the overall scale. To extract a result in physical units we need to know the lattice spacing. As discussed above, the value we obtain depends on the physical quantity that we use, particularly at finite lattice spacing. $f_B$ is more sensitive to the uncertainty in $a$ than are light hadron masses, since one is calculating $a^{3/2} \phi_P$ rather than $a m$. To illustrate this sensitivity, I include in Table 3 results from Refs. [4, 30] using two different determination of $a$. For the other results, the uncertainty is about 15%. Ultimately, to remove this uncertainty one must repeat the calculation with full QCD.

The second problem concerns the isolation of the lightest state. In principle, calculating $\phi_P$ is straightforward. One simply studies the long time behavior of the two point function of the local axial current, Eq. 2, and reads off $f_P \sqrt{m_P}$. In practice, to obtain a signal it is necessary to use extended operators, which couple more strongly to the lightest state than the local current. This is particularly important at large $m_Q$. With a less than optimal operator there are likely to be systematic errors introduced by contamination from excited
states. Indeed, there are disagreements between results using different operators. This is illustrated in Table 3 by the variation in $f_B$(static). It seems to me, however, that the operators of Ref. [31] are close to optimal, and that the discrepancies will go away as other groups optimize their operators.

The third and most difficult problem concerns putting very heavy quarks on the lattice. As the quark’s mass increases, the ratio of its Compton wavelength to the lattice spacing, $1/(m_Q a)$, decreases. For $m_Q a > 1$, its propagation through the lattice will be severely affected by lattice artifacts. There are, however, no such difficulties for an infinitely massive quark, for such a quark remains at rest both in the continuum and on the lattice [29]. Thus it seems that we are forced to interpolate between $m_Q \sim 1/a$ and $m_Q = \infty$.

To illustrate the problems that this introduces, consider the situation about two years ago. The smallest lattice spacing was $a = 1/10$ fm, so that $m_c a \approx 0.75$ and $m_b \approx 2.5$. Typical results are those represented by squares in Fig. 3a. The quark mass has been restricted to

Table 3: Results for decays constants in the quenched approximation. The normalization is such that $f_\pi = 132$ MeV. Only statistical errors are shown; systematic errors are discussed in the text.

| Ref. | $\beta$   | Scale from | $f_B$(static)(MeV) | $f_B$(MeV) | $f_D$(MeV) |
|------|-----------|------------|-------------------|------------|------------|
| [32] | 6.0-6.4   | $f_\pi, m_\rho$ | 310(25)          | 205(40)    | 210(15)    |
| [34] | 6.2       | $f_\pi$    | 183(10)          | 198(5)     |            |
| [33] | 6.3       | $f_\pi$    | 230(15)          | 187(10)    | 208(9)     |
| [4]  | 5.9       | $M(1P) - M(1S)$ | 319(11)       |            |            |
|      |           | $\sigma$   | 265(10)          |            |            |
| [31] | 5.74-6.26 | $\sigma$   | 230(22)          |            |            |
|      |           | $m_\rho$   | 256(78)          |            |            |
satisfy \( m_Q a \leq 0.7 \) to avoid large artifacts (an arbitrary but reasonable choice for an upper bound), and thus lie to the right of the \( D \) meson line. Clearly it is difficult to convincingly interpolate using Eq. \( 8 \): the variation in \( \phi_P \) is so large that one cannot truncate the \( 1/m_P \) expansion. Nevertheless, if one assumes that the curvature is not too large, the fact that we know \( \phi_\infty \) does allow a rough estimate of \( f_B \).

There has been considerable progress in the last two years. The lattice spacing has been reduced to \( 1/17 \) fm, which allows one to use heavier quarks while keeping \( m_Q a < 1 \). This is illustrated by the remaining points in Fig. 3\( a \), all of which have \( m_Q a \leq 0.7 \). These points can now be fit in a more reliable way to the asymptotic form of Eq. \( 8 \). The curve shows such a fit, the results from which are given in Table 3. Other groups, however, find results in apparent disagreement. For example, the “uncorrected” points in Fig. 3\( b \) should agree with those in 3\( a \), but instead are lower. This disagreement is, I suspect, mainly due to inadequate isolation of the lightest \( B \) meson by one or both groups.

Another development has been the use an improved fermion action \cite{34}. Since this has smaller \( O(a) \) corrections, one should be able to work at larger values of \( m_Q a \). I give the results in Table 3.

Despite these improvements, it would be much better if we could work at any value of \( m_Q a \), and simply map out the entire curve. There is a dispute about whether this is possible, and I will attempt to give a summary of the arguments.

I begin by noting that the errors in \( \phi_P \) do not keep growing as \( m_Q a \) increases. This is because, if \( m_Q \gg \Lambda_{\text{QCD}} \), the quark is non-relativistic. Its dynamics can then be expanded in powers of \( 1/m_Q \) \cite{35}

\[
\mathcal{L} = \overline{\psi} \left[ iD_0 + \frac{D^2}{2m_1} + c(g^2)\frac{\sigma_i B_i}{2m_2} + O(1/m_Q^2) \right] \psi , \tag{9}
\]

where \( \psi \) is a two component field, and \( c \) is a perturbative coefficient. In the continuum, \( m_1 = m_2 = m_Q \). The discrete rotational symmetries are sufficient to ensure that a heavy lattice quark will be described by the same Lagrangian, except that neither \( m_1 \) nor \( m_2 \) are equal to the bare lattice quark mass \( m_Q \). Nevertheless, although the lattice heavy quarks have the wrong dynamics, this should only introduce errors of \( O(\Lambda_{\text{QCD}}/m_Q) \). Thus, if \( a \) is small enough that \( m_Q a < 1 \) when the quark becomes non-relativistic, the errors will be small for all \( m_Q a \).

Strictly speaking, to bring the lattice Lagrangian into the form of Eq. 8 one must perform a wavefunction renormalization on the heavy field, which changes the normalization of the \( \bar{b}u \) axial current. This has been calculated in perturbation theory keeping only the “tadpole” diagrams which are thought to be dominant \cite{36}. The result is illustrated in Fig. 3\( b \): upon renormalization the “uncorrected” points are shifted upwards into those labeled “corrected”. With this modification, the curve for finite \( m_Q a \) is guaranteed to pass through the \( \phi_\infty \). Without it, the curve will bend over and eventually pass through the origin. In fact, the uncorrected points in Fig. 3\( b \) do appear to be flattening out at the smallest values of \( 1/m_P \), although those of Fig. 3\( a \) do not.

A second correction has also been applied in Fig. 3\( b \). It is possible to partly account for the difference between \( m_1 \) and \( m_Q \) by an appropriate shift to smaller \( 1/m_Q \). The size of this shift has been calculated keeping “tadpole” diagrams \cite{37}. This correction ensures that the
data approach $\phi_\infty$ with a slope which is linear in $1/m_P$. This slope will not, however, be correct since the $1/m^2$ term in Eq. 1 has the wrong normalization. Nevertheless, in contrast to the “uncorrected” points, the shifted points fit well to the form of Eq. 3. The fit is shown by the curve, and the resulting values for decay constants are given in Table 3.

The controversial issue is how to further modify the calculation so as to obtain the correct $1/m_P$ terms. Kronfeld, Lepage and Mackenzie have suggested a program for removing the error [36]. The idea is to add new terms to the lattice action, and to choose the parameters so that one obtains the correct Lagrangian at $O(1/m_Q)$. This is essentially a complicated way of putting non-relativistic QCD on the lattice [35]. The difficult question in both cases is how to fix the parameters.

Kronfeld et al propose that this can be done using perturbation theory. Maiani et al. argue, however, that non-perturbative contributions may be important [37]. Normally, such contributions are suppressed by powers of $a$, but here they are enhanced by factors of $1/a$ due to linear divergences. (A clear explanation of this effect is given in Ref. [38].) The result is that, with parameters fixed using perturbation theory, there will be errors in $\phi_P$ which are of $O(A_{QCD}/m_Q)$, i.e. of the same size as the terms one is trying to calculate. The point of disagreement is whether these non-perturbative terms are large or small. If they are large, the only solution would be to fix parameters using non-perturbative normalization conditions. In general this would reduce predictive power.

Clearly more work is needed to resolve this dispute. One way to do this is to push the tests of perturbation theory [22] to the level at which non-perturbative terms show up.

It is fortunate, however, that this uncertainty has only a small effect on $f_B$ and $f_D$. This is shown by the good agreement in Table 3 despite the different methods being used. The results favor the “large” solution for $f_B$. The only way this could change is if the systematic error due to the quenched approximation turns out to be large.

We can also estimate the size of the $O(1/M)$ corrections to the heavy quark limit. Taking the data of Ref. [33] as an example, they are $\sim 15\%$ for $f_B$, and $\sim 45\%$ for $f_D$. These numbers increase further if one uses the larger values of $\phi_\infty$ found by Ref. [9].

5 OUTLOOK

There are many interesting developments that I have not have had time or space to cover. One which impinges on much of the work discussed above concerns the accuracy of the quenched approximation [39]. In this approximation there are nine pseudo-Goldstone bosons, rather than the eight of QCD, the extra one being the $\eta'$. This means that quenched particles have an $\eta'$ cloud, not present in full QCD. It turns out that this gives rise to singularities in the chiral limit, much as pion loops give singularities in the charge radii of pions and nucleons [11]. For example, the quark condensate diverges. The implications of these divergences are not yet clear. The most optimistic view is that the effects of $\eta'$ loops are small as long as we work above a certain quark mass. This is supported by the absence of numerical evidence to date for the divergences.

Unfortunately, the quenched approximation is likely to be with us for a number of years. At present, the fastest computers simulating LQCD are running at $1 - 10$ GFlops. Even a
TeraFlop machine, such as that proposed by the TeraFlop collaboration [40], will focus on quenched lattices. These will have $N_s \approx 100$, and should give definitive quenched results for a reasonable number of interesting quantities. The major problem with simulations of full QCD is that the CPU time scales as $m_{\pi}^{-10.5}$ with present algorithms. Clearly, it is crucial that effort go into improving these algorithms.

ACKNOWLEDGEMENTS

I thank Don Weingarten Jim Labrenz and Akira Ukawa for providing me with data, and Estia Eichten, Aida El-Khadra, Rajan Gupta, Brian Hill, Andreas Kronfeld, Jim Labrenz and Paul Mackenzie for discussions. This work was supported by the DOE under contract DE-FG09-91ER40614, and by an Alfred P. Sloan Fellowship.

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