On reduced density matrices for disjoint subsystems

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Abstract – We show that spin and fermion representations for solvable quantum chains lead in general to different reduced density matrices if the subsystem is not singly connected. We study the effect for two sites in XX and XY chains as well as for sublattices in XX and transverse Ising chains.

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Introduction. – The investigation of entanglement features in many-body quantum systems has been a topic of intense research in recent years [1]. In such studies, one divides the system into two parts in space and asks how these are coupled in the quantum state. This can be answered from the reduced density matrix (RDM) for one of the subsystems. For a number of solvable lattice models, these density matrices can be found exactly [2] and their spectra determine the entanglement entropy which is a simple measure of the coupling. In the ground state of typical systems with short-range interactions, it is connected with the interface between the two parts and proportional to its extent, although there are corrections to this “area law” for fermions [3,4].

Commonly, one divides the system into two parts which are singly connected, and most results were obtained for this geometry. However, there are a number of studies which treat multiply connected subsystems. These were investigated in one dimension for free fermions and bosons within a continuum approach [5,6] and for conformally invariant systems [4,7]. On a lattice, spin chains have been treated where the subsystem consisted of many single spins separated by a certain number of sites [8], of two blocks of spins, again separated [9,10], and of a whole sublattice [11]. The latter choice was motivated by the search for phase-transition indicators while for two blocks one can use the RDM to study the entanglement between them. In [8,9] the calculation was carried out by transforming the transverse Ising (TI) model into fermions and determining the density-matrix eigenvalues from the fermionic correlation functions [12–15]. The basic finding for isolated spins was that, due to the many “interfaces” with the surrounding, the entanglement entropy becomes extensive.

In this note, we want to point out that the reduced density matrices for the spin representation and the fermionic representation are, in general, not identical and that correspondingly also the entanglement entropies differ, if one deals with disjoint subsystems. This might be surprising at first, but is connected with the non-local structure of the Jordan-Wigner transformation. Thus, to obtain a transverse spin correlation function, one needs information about a whole string of sites in the fermionic picture [16]. This is not necessary if one asks only for fermionic correlations. Thus the two RDMs contain different information. In the following section, we demonstrate this for the simple and analytically solvable case of a subsystem of two sites in an XX chain. In the third section we indicate the generalization to the XY chain. The possibly more relevant case of sublattice RDMs is treated in the fourth section for XX and TI rings and the results are summarized in the last section.

Two sites in an XX chain. – In the following we consider the spin–one-half XX chain described by the Hamiltonian

\begin{equation}
H = -\frac{1}{2} \sum_{m} \left[ \sigma_{m}^{x} \sigma_{m+1}^{x} + \sigma_{m}^{y} \sigma_{m+1}^{y} \right].
\end{equation}

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Here the $\sigma^\alpha_m$ are Pauli matrices at site $m$. In terms of fermionic creation and annihilation operators $c^\dagger_m, c_m$, $H$ reads [16]

$$ H = -\sum_m \langle c^\dagger_m c_{m+1} + c^\dagger_{m+1} c_m \rangle $$

and corresponds to a simple hopping model. If the system forms a ring, one has to take care of a boundary term. In the following we always look at the ground state and choose as subsystem the sites $1$ and $n$.

**Spin RDM.** The RDM for two spins in XX, XY and TI chains has been discussed in many papers, see [17–24]. It is a $4 \times 4$ matrix and its elements can be expressed as expectation values of proper operators at the two sites, i.e. as two-point correlation functions. Since the XX ground state has fixed $S_z = 0$, all entries corresponding to a change of this value are zero. As a result, the RDM in the basis $|++\rangle, |--\rangle$, $|--\rangle$, $|+\rangle$ in the simple form

$$ \rho_S = \begin{pmatrix} a & b & c & d \\ b & c & d & a \\ c & d & a & b \\ d & a & b & c \end{pmatrix}, $$

where $2a + 2b = 1$ due to $\text{tr}(\rho_S) = 1$. The non-zero matrix elements are given by

$$ a = \rho_S(\cdot, \cdot) = \frac{1}{4} \langle 1 + \sigma_z^1 \sigma_z^n \rangle, $$

$$ b = \rho_S(\cdot, \cdot) = \frac{1}{4} \langle 1 + \sigma_x^1 \sigma_x^n \rangle, $$

$$ c = \rho_S(\cdot, \cdot) = \frac{1}{4} \langle \sigma_x^1 \sigma_x^n \rangle, $$

where $\sigma^z = \sigma^x \pm i \sigma^y$. In the ground state, $\langle \sigma_z^m \rangle = 0$ and the expressions become

$$ a = \frac{1}{4} + \frac{1}{4} \langle \sigma_z^1 \sigma_z^n \rangle, $$

$$ b = \frac{1}{4} - \frac{1}{4} \langle \sigma_z^1 \sigma_z^n \rangle, $$

$$ c = \frac{1}{2} \langle \sigma_x^1 \sigma_x^n \rangle. $$

In this way, the RDM is expressed completely in terms of standard spin correlation functions. This approach has also been used to determine RDMs in the XXZ model, see, e.g., [25] and references therein.

The correlation functions can be calculated in the fermionic representation. Then one has

$$ \frac{1}{2} \langle \sigma_x^1 \sigma_x^n \rangle = C_{1,1} C_{n,n} = - C_{1,n}^2, $$

where $C_{ij} = \langle c^\dagger_i c_j \rangle$ is the fermionic single-particle function. For the $xx$ correlations one finds the determinant [16]

$$ \frac{1}{2} \langle \sigma_x^1 \sigma_x^n \rangle = 2^{n-2} \begin{vmatrix} C_{1,2} & C_{1,3} & \cdots & C_{1,n} \\ C_{2,2} & C_{2,3} & \cdots & C_{2,n} \\ C_{3,2} & C_{3,3} & \cdots & C_{3,n} \\ \cdots & \cdots & \cdots & \cdots \\ C_{n-1,2} & C_{n-1,3} & \cdots & C_{n-1,n} \end{vmatrix}. $$

Here the quantities $C_{m,m}$ are given by $C_{m,m} = C_{m,m} - 1/2$ and vanish in the ground state, for which the fermionic system is half-filled. In two cases the expression reduces to a single term, namely if the two sites are nearest neighbours ($n = 2$), and if they are the ends of an open chain. Then the effect of the Jordan-Wigner strings disappears and

$$ \frac{1}{2} \langle \sigma_x^1 \sigma_x^n \rangle = C_{1,n}. $$

The “long-distance entanglement” in the latter case has been studied, e.g., in [22,26,27].

**Fermion RDM.** In the fermionic case, one can proceed in exactly the same way. The RDM is again a $4 \times 4$ matrix, this time in the basis specified by the occupation numbers $|11\rangle, |10\rangle, |01\rangle, |00\rangle$. Since the ground state has fixed total particle number $N$, all elements corresponding to a change of $N$ vanish. Therefore $\rho_F$ has again the form (3) and the elements are now given by

$$ a = \langle c^\dagger_i c_j c_j c_i \rangle = \frac{1}{4} - C_{1,n}^2, $$

$$ b = \langle c^\dagger_i c_j c_j c_i \rangle = \frac{1}{4} + C_{1,n}^2, $$

$$ c = \langle c^\dagger_i c_j \rangle = C_{1,n}. $$

One sees that $a$ and $b$ are the same as in the spin representation, but $c$ is in general different. Instead of the $xx$ spin correlation function, the fermionic one-particle correlation function $C_{1,n}$ appears. This is the basic difference between the two representations. Only for the two exceptional cases mentioned above, the expressions for $c$ coincide. Then the two spins are nearest neighbours, or can be considered as neighbours by bending the open chain to a ring, and correspondingly the subsystem is singly connected.

The appearance of $C_{1,n}$ is natural, since with $\rho_F$ one must in turn be able to calculate this correlator. That essentially only this quantity enters, could have been seen also from the general result for fermionic RDMs [2,12]

$$ \rho_F = \frac{1}{Z} \exp (-H) = \frac{1}{Z} \exp \left( - \sum_{i,j=1}^n H_{i,j} c^\dagger_i c_j \right), $$

where $Z$ ensures $\text{tr}(\rho_F) = 1$ and the matrix $H_{i,j}$ in the exponent follows from the correlation matrix involving the sites of the subsystem, in our case $C_{1,1} = C_{n,n} = 1/2$ and $C_{1,n}$. One can obtain the form (3) also from (16), but the route taken above is much simpler.

**Entanglement entropies.** The structure of $\rho_S$ and $\rho_F$ one reads off the four eigenvalues

$$ w_1 = b + c, \quad w_2 = w_3 = a, \quad w_4 = b - c. $$

The entanglement entropy is then given by $S = - \sum w_k \ln w_k$. Since the correlations go to zero for large separations, all $w_k$ approach $1/4$ in this limit and $S$ goes to the value $S = 2 \ln 2$ in both representations.
This is the sum of two single-spin contributions and also the
maximum one can have.

The complete behaviour of $S$ is obtained by using the
correlation function of the infinite chain

$$C_{i,j} = \frac{\sin(\pi(i-j)/2)}{\pi(i-j)}$$

and calculating the eigenvalues $w_k$ numerically. The result
is shown in fig. 1. According to the previous remarks, the
two entropies coincide for $n=2$ and for $n \to \infty$. In
between, the spin entropy $S_S$ always lies below the
fermionic one, $S_F$, because the $xx$ correlations only decay
as $n^{-1/2}$. If the distance $n-1$ is even, $i.e.$ if the two
sites are on the same sublattice, the correlation $C_{1,n}$
vanishes and $S_F$ has the maximum value $2\ln 2$. Apart from
this feature, the asymptotic behaviour is determined by $c:

$$S \simeq 2 \ln 2 - 4c^2,$$

which gives a $1/n$ approach to the limit in the spin picture
and a $1/n^2$ variation in the fermion picture.

**Anisotropic chains.** – The previous considerations
can be generalized to the case of an XY or a TI chain.
Then the rotational symmetry is absent, but the ground
state is built from configurations with either an even or
an odd number of (+) spins. Then

$$\rho_S = \begin{pmatrix} a & b & c & d \\ b & c & a & d \\ c & a & d & b \\ d & b & c & a \end{pmatrix}$$

and the additional matrix element $d$ is given by

$$d = \rho_S(++,--) = \frac{1}{4} (\sigma^+_a \sigma^+_b).$$

In the fermionic picture, on the other hand

$$d = \rho_F(11,00) = \langle c^+_1 c^+_0 \rangle.$$

Again, both expressions are different, except for $n=2$ or
for end spins. The presence of $d$ splits the two degenerate
eigenvalues of $\rho$ and gives $w_{2,3} = a \pm d$.

An additional feature arises from the long-range order
which exists in the anisotropic XY model and in the
TI model with strong coupling. In the spin picture, the
coefficients $c$ and $d$ do not vanish for large separations
and therefore the entropy does not become the sum of
two single-site contributions. If $\gamma$ denotes the anisotropy
of the XY model with zero field, the asymptotic values are

$$c = d = \frac{1}{2} (\sigma^+_a \sigma^+_b) = \frac{1}{2} \sqrt{1 + \gamma}$$

and vary between $0$ for $\gamma = 0$ and $1/4$ for $\gamma = 1$. Correspondingly, $S_S$ varies between $2\ln 2$ and $\ln 2$. The fermionic
correlation functions, on the other hand, approach
zero for large separations, as in the XX model, and $S_F = 2\ln 2$ independent of $\gamma$. Thus the quantities $S_S$ and $S_F$
have different asymptotic values. Only for end spins in
open chains, the asymptotic values coincide and are in this
case related to the surface order. This happens also in
dimerized XX chains [22,27].

**Sublattice entanglement.** – We now turn to the case
where the subsystem consists of every second site in a
chain. For the infinite XX model, it then follows from (18)
that the fermionic correlations vanish for unequal sites
on the same sublattice. This holds also for finite rings,
as well as for non-homogeneous couplings. Therefore the
correlation matrix $C_{i,j}$ on one sublattice is diagonal,
$C_{i,j} = 1/2 \delta_{i,j}$. The RDM obtained via (16) is then also
diagonal, all eigenvalues $w_k$ are equal and $S_F = L \ln 2$ if
the system is a ring with $N = 2L$ sites. This is the extensivity
mentioned in the introduction, and the result has a simple
interpretation. The ground state $|\Psi\rangle$ is obtained by filling
the single-particle eigenstates of (2) for momenta $|q| < \pi/2$. But the corresponding operators can be decomposed as

$$c^+_q = \frac{1}{\sqrt{N}} \sum_{m=1}^L \exp(-iq2m) c^+_m,$$

$$+ \frac{1}{\sqrt{N}} \sum_{m=1}^L \exp(-iq(2m+1)) c^+_m,$$

$$= \frac{1}{\sqrt{2}} (a^+_q + b^+_q).$$

Therefore

$$|\Psi\rangle = \prod_{|q| < \pi/2} \frac{1}{\sqrt{2}} (a^+_q + b^+_q)|0\rangle$$

is a product of fermionic triplets formed from modes
with the same $q$ on different sublattices. Each of them
contributes $\ln 2$ to the entanglement entropy.

For the spin RDM, a calculation for $N=4$ already
shows that the $w_k$ are not all equal, but given by $w_1 =
1/2, w_2 = w_3 = 1/4, w_4 = 0$. The corresponding entropy is
Fig. 2: (Colour on-line) Sublattice entanglement entropy for rings of length $N = 2L$, calculated in the spin and in the fermion basis. Upper panel: XX model, lower panel: critical TI model. The straight lines represent the respective asymptotic behaviour, see the text.

$S_S = 3/2 \ln 2$ and thus smaller than $S_F = 2 \ln 2$. To investigate the size dependence, we have calculated $\rho_S$ numerically by finding the ground state of $H$ in the subspace $S_{\text{tot}} = 0$ for $N$ up to 16 sites. The resulting values are plotted in fig. 2 on the upper panel.

The data can be fitted for the larger $L$ by $S_S(L)/\ln 2 = s * L + s_0$ with $s = 0.6116(4)$ and $s_0 = 0.290(5)$. In addition, there are subleading corrections which are different for even and odd values of $L/2$, related to the slightly alternating behaviour of $\rho_S$. Thus the extensivity of $S$ holds also in the spin picture, but the value is reduced to about 60% of the fermionic one. For open chains, the values for $S_S$ lie about 0.5 higher than for the rings, but the slope $s$ is similar.

We have also considered the XX chain with enforced dimerization where the coupling between sites $m$ and $m+1$ is given by $J_m = 1 + \delta (-1)^m$. Then one has a finite correlation length if $\delta \neq 0$. However, the fermionic sublattice entanglement is not affected. The expectation values $\langle c_i^\dagger c_j \rangle$ still vanish on the same sublattice and one can interpret the result $S_F = L \ln 2$ as before. The spin entanglement, on the other hand, is an even function of $\delta$ and depends on the dimerization. In particular for $\delta = \pm 1$, where the system decomposes into coupled pairs of sites each of which contributes $\ln 2$, it becomes equal to $S_F$.

The detailed variation with $\delta$ for fixed $N$ is shown in fig. 3. Since $S/L$ is plotted, the large-$L$ result is basically the slope $s$ introduced above and seen to increase monotonously from 0.6116 to 1 as $\delta$ varies between 0 and 1. Near $\delta = 0$ one finds a special feature due to an eigenvalue $w_k = \delta^2$ of $\rho_S$ which leads to a non-analytic dependence $S_S(L, \delta) - S_S(L, 0) \sim \delta^2 \ln |\delta|$.

Finally, we have investigated the transverse Ising model on a ring with Hamiltonian

$$H = - \sum_{m=1}^{N} [\sigma_m^x \sigma_{m+1}^x + h \sigma_m^z]$$

and calculated the sublattice entanglement in both representations. The result for the critical case $h = 1$ is very similar to that for the XX model and shown in fig. 2 on the lower panel for up to $N = 14$ sites. The curves can again be fitted by a linear function and give $s = 0.258(2)$ and $s_0 = 0.700(1)$ in the spin picture while $S_F$ is exactly proportional to $L$ with a slope $s = 0.6001$. Thus both entropies are extensive but the fermionic value is more than twice the spin value.

More interesting is the non-critical case. Figure 4 shows on the upper panel the results for the spin representation. One sees that for the larger sizes, $S_S/L$ shows a maximum near the critical value $h = 1$. In the disordered region $h > 1$, the curves approach a limit and $S$ is therefore extensive, but in the ordered region $h < 1$, they depend on $L$ and $S$ is not extensive. Rather one has $S_S \to \ln 2$ for $h \to 0$. This is the same value as for a subsystem in the form of one block and has the same origin. The ground state in this limit is a superposition of the two states with all spins having $\sigma_x = +1$ and all having $\sigma_x = -1$,
respectively. This GHZ state leads to a RDM with two non-zero eigenvalues $w = 1/2$ and thus to $\ln 2$ for the entropy.

The fermionic case is different and can be treated analytically, although the ground state is more complicated than in the XX model. Following [2,12], one has to find the eigenvalues $(2\zeta - 1)^2$ of the matrix

$$ M = (2C - 1 - 2F)(2C - 1 + 2F), $$

where $F_{q,i} = \langle c_i^c c_i^\dagger \rangle$ and all sites are on one sublattice. However, due to the translational invariance of the subsystems, one can work in momentum space. Then the matrix $M$ decomposes into $2 \times 2$ blocks involving the correlation functions $C_q = \langle a_i^\dagger a_q \rangle$, $C_{q,-q} = \langle a_i^\dagger a_q a_i a_{-q} \rangle$ and $F_q = \langle a_i^\dagger a_{-q} \rangle$. As a result, one finds

$$ (2\zeta - 1)^2 = \frac{1}{2} \left[ 1 + \frac{h^2 - 1}{(1 + h^2)^2 - 4h^2 \cos^2 q} \right], $$

from which the entanglement entropy is obtained as

$$ S_F = - \sum_{0 < q < \pi/2} \left[ \zeta_q \ln \zeta_q + (1 - \zeta_q) \ln(1 - \zeta_q) \right]. $$

This is always extensive and shown in fig. 4 on the lower panel. For $h \gtrsim 2.5$ the curve practically coincides with the one in the spin representation. This can be attributed to the short correlation length whereby only neighbouring sites see each other. The asymptotic form is $S/L \sim \ln h/h^2$ since $\zeta_q = 1 - O(1/h^2)$ for large $h$. As $h$ is reduced, however, the curve rises continuously to one, corresponding to the value $S_F = L \ln 2$, with no sign of the long-range order. For three values of $h$, the $\zeta_q$ are independent of $q$, namely for $h = 0$ ($\zeta_q = 1/2$, $S_F = L \ln 2$), for $h = \infty$ ($\zeta_q = 1$, $S_F = 0$) and for $h = 1$ where $\zeta_q = (1 + 1/\sqrt{2})/2$ and

$$ S_F = L \left[ \frac{3}{2} \ln 2 - \frac{1}{\sqrt{2}} \ln(1 + \sqrt{2}) \right]. $$

This gives the value $s = 0.6001$ cited above. Near $h = 1$, the entropy per site varies asymptotically as $(h - 1) \ln(h - 1)$ and thus has infinite slope. This is the signature of the phase transition in $S_F$. The interpretation of the extensivity is similar as in the XX model, but in the ground state

$$ |\Psi\rangle = \prod_{0 < q < \pi} \left[ u_q + v_q c_i^c c_q^\dagger c_{-q} \right] |0\rangle $$

one has now a coupling of the two single-particle states $(q,-q)$ in the two sublattices if one inserts (24). The corresponding operators appear twice in the product, since $c_i^c c_q^\dagger c_{-q} = (a_i a_{-q})/\sqrt{2}$.

Since TI and XX chains are related by a dual transformation, one could expect a relation between the corresponding entanglements. If the subsystem is a block in a ring, such a connection indeed exists [29]. In the sublattice case, however, we have not found a similar result.

**Conclusion.** — We have studied the entanglement in spin chains for the case that the subsystem is not singly connected. We demonstrated that working in the spin and the fermion representation leads in general to different RDMs and to different entanglement entropies. We did this by looking at two extreme cases, namely a subsystem of only two sites and one in the form of a whole sublattice. The first one displays the effect particularly clearly, while the second one provides an example, where it is particularly large.

At the level of the wave function, there is no difference between the two representations. One can rewrite the spin expression directly into the occupation-number form. However, the operators sample different information. In the fermion picture, the spin function $\langle \sigma_i^x \sigma_j^x \rangle$ needs all sites between $i$ and $j$, whereas in the spin picture the same is true for the fermion function $\langle c_i^c c_j \rangle$. If some of

Fig. 4: (Colour on-line) Sublattice entanglement entropy per site in TI rings as a function of the transverse field. Upper panel: calculated in the spin basis for rings of different size $N$. Lower panel: calculated in the fermion basis in the large-$N$ limit. Note the different vertical scales.

On reduced density matrices
these sites do not belong to the chosen subsystem, the reduced density matrices giving these expectation values will usually not coincide. If one determines them by integrating out degrees of freedom, this difference arises, because one needs Grassmann variables in the fermionic case [15,30]. This can lead to sign changes in the terms contributing to a particular matrix element, as compared to the spin calculation, and thus to a different final result. One can see this explicitly by considering an XX chain with four sites. A reordering of the sites would avoid this, but also spoil the simplicity of the fermionic Hamiltonian. Concluding, one can say that while the eigenvalues of the spin RDM are directly related to the coefficients in the Schmidt decomposition of the state, this does not necessarily hold for those of the fermion RDM. They and the resulting entropy measure the entanglement in a somewhat different way.

Finally, we mention that a global rotation in spin space can change the eigenvalues of $p_F$ and thus $S_F$ in the studied geometry, since it is a non-local transformation in the fermions. One can see that explicitly by studying an XX chain of four sites. However, such a rotation will destroy the free-particle nature of the fermionic formulation which is the very reason for working with it.

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Additional remark: The difference between the two representations discussed here has also been noted in a recent preprint by Alba et al. [31].

REFERENCES

[1] Amico L., Fazio R., Osterloh A. and Vedral V., Rev. Mod. Phys., 80 (2008) 517.
[2] Peschel I. and Eisler V., J. Phys. A, 42 (2009) 504003.
[3] Eisert J., Cramer M. and Plenio M. B., Rev. Mod. Phys., 82 (2010) 277.
[4] Calabrese P. and Cardy J., J. Phys. A, 42 (2009) 504005.
[5] Casini H. and Huerta M., Class. Quantum Grav., 26 (2009) 185005, JHEP, 0903 (2009) 048.
[6] Markovitch S., Retzker A., Plenio M. B. and Reznik B., Phys. Rev. A, 80 (2009) 012325.
[7] Calabrese P., Cardy J. and Tonni E., J. Stat. Mech.: Theory Exp. (2009) P11001.
[8] Keating J. P., Mezzadri F. and Novais M., Phys. Rev. A, 74 (2006) 012311.
[9] Fachini P., Florio G., Invernizzi C. and Pascacio S., Phys. Rev. A, 78 (2008) 052302.
[10] Wichterich H., Molina-Vilaplana J. and Bose S., Phys. Rev. A, 80 (2009) 010304(R).
[11] Chen Y., Zanardi P., Wang Z. D. and Zhang F. C., New J. Phys., 8 (2006) 97.
[12] Peschel I., J. Phys. A: Math. Gen., 36 (2003) L205.
[13] Vidal G., Latorre J. I., Rico E. and Kitaev A., Phys. Rev. Lett., 90 (2003) 227902.
[14] Latorre J. I. E., Rico E. and Vidal G., Quantum Inf. Comput., 4 (2004) 48, preprint quant-ph/0304098.
[15] Cheong S. A. and Henley C. L., Phys. Rev. B, 69 (2004) 075111, preprint cond-mat/0206196.
[16] Lieb E. H., Schultz T. D. and Mattis D. C., Ann. Phys. (N.Y.), 16 (1961) 407.
[17] Osterloh A., Amico L., Falci G. and Fazio R., Nature, 416 (2002) 608.
[18] Osborne T. J. and Nielsen M. A., Phys. Rev. A, 66 (2002) 032110.
[19] Jin B.-Q. and Korepin V. E., Phys. Rev. A, 69 (2004) 062314.
[20] de Oliveira T. R., Rigolin G. and de Oliveira M. C., Phys. Rev. A, 73 (2006) 010305(R).
[21] de Oliveira T. R., Rigolin G., de Oliveira M. C. and Miranda E., Phys. Rev. Lett., 97 (2006) 170401.
[22] Campos Venuti L., Giampaolo S. M., Illuminati F. and Zanardi P., Phys. Rev. A, 76 (2007) 052328.
[23] Chen H. D., J. Phys. A: Math. Theor., 40 (2007) 10215.
[24] Stauber T. and Guinea F., Ann. Phys. (Berlin), 18 (2009) 561.
[25] Sato J. and Shiroishi M., J. Phys. A: Math. Theor., 40 (2007) 8739.
[26] Wichterich H. and Bose S., Phys. Rev. A, 79 (2009) 060302(R).
[27] Giampaolo S. M. and Illuminati F., preprint arXiv:0910.0016.
[28] McCoy B. M., Phys. Rev., 173 (1968) 531.
[29] Iglói F. and Juhász R., EPL, 81 (2008) 57003.
[30] Chung M. C. and Peschel I., Phys. Rev. B, 64 (2001) 064412.
[31] Alba V., Tagliacozzo L. and Calabrese P., preprint arXiv:0910.0706, to be published in Phys. Rev. B.