Critical velocity of a mobile impurity in one-dimensional quantum liquids

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We study the notion of superfluid critical velocity in one spatial dimension. It is shown that for heavy impurities with mass \( M \) exceeding a critical mass \( M_c \), the dispersion develops periodic metastable branches resulting in dramatic changes of dynamics in the presence of an external driving force. In contrast to smooth Bloch Oscillations for \( M < M_c \), a heavy impurity climbs metastable branches until it reaches a branch termination point or undergoes a random tunneling event, both leading to an abrupt change in velocity and an energy loss. This is predicted to lead to a non-analytic dependence of the impurity drift velocity on small forces.

The study of the effect of impurities constitutes a major theme in condensed matter physics. In the solid state impurities are typically regarded as immobile, acting as static perturbations. Prime examples include the residual resistance of metals and the Kondo effect. In other (typically liquid) states mobile impurities play a vital role, with the influence of \(^3\)He impurities on the transport properties of superfluid \(^4\)He being perhaps the best studied example [1, 2].

The traditional goal has been to understand the effect of foreign particles or defects on an otherwise pure system. However, the converse problem – to determine the effect of the surrounding medium on the motion of impurities – is increasingly relevant to the study of ultracold atomic gases. There are several ways in which mobile impurities appear naturally in these systems: (i) a small fraction of atoms may be transferred to a different hyperfine state, creating a dilute gas of distinguishable impurities of the same mass [3–5]; (ii) mixtures of different atomic species (e.g., \(^{87}\)Rb and \(^{41}\)K in Ref. [6]) with one dilute component give a route to impurities of mass \( M \) different from the mass \( m \) of the host particles; (iii) ions of Yb\(^+\), Ba\(^+\) or Rb\(^+\) have been placed in a Bose-Einstein condensate of neutral \(^{87}\)Rb atoms [7, 8]. In each of these situations, impurity dynamics may be investigated with the help of external forces that act selectively on the impurity.

Owing to recent advances in cold atom trapping, systems in reduced dimensionalities are now well within experimental reach [5, 7, 8], bringing to light a wide range of rich and peculiar phenomena. In one spatial dimension (1D) Refs. [9, 10] established that even in the limit of strong interactions, there exists a timescale beyond which an impurity with quadratic dispersion, \( E(P) = P^2/2M^* \), may propagate in a quantum fluid. In this Letter we focus on the full dispersion \( E(P) \) of the resulting motion, which possesses several remarkable features not present in higher dimensions. The most prominent property is its periodicity, which is not evident from a quadratic, small \( P \) expansion, and may be explained as follows. The motion of \( N \) particles on a ring with circumference \( L \) results in the total momentum being quantized in units of \( 2\pi \hbar N/L = 2\pi \hbar n \), where \( n \) is the one-dimensional density of the fluid. On the the other hand, in the thermodynamic limit \( N, L \to \infty \), \( N/L = n \), this motion costs no kinetic energy due to infinite total mass \( Nm \). Thus, instead of paying macroscopic energy \( (2\pi \hbar n)^2/2M^* \) to
deliver this momentum to impurity motion, the system channels it to the superflow of the background liquid at no energy cost. As a result, the dispersion of the dressed impurity is a periodic function \( E(P) = E(P + 2\pi n\hbar) \). Owing to the quadratic dispersion near \( P = 0 \), impurity motion costs less energy for a given momentum than the softest excitations of the liquid, i.e., phonons with linear dispersion. This fact implies that impurity motion, while possessing a periodic dispersion, also defines the lower edge of the many-body continuum, above which (shaded grey, Fig. 1) there exist excitations comprised of the moving impurity and phonons [11–14]. Explicit examples demonstrating these features are furnished by models solvable via Bethe Ansatz [15–17]. A dramatic consequence of the dispersion periodicity is the possibility of observing Bloch oscillations of a driven impurity in 1D in the absence of a lattice [18, 19].

The same picture of momentum transfered to the fluid at no energy cost leads to the prediction of vanishing Landau critical velocity in 1D [20, 21]. It was shown that an infinitely massive object moving with any nonzero velocity with respect to the background nucleates phase slips corresponding to one quantum \( 2\pi n\hbar \) of total momentum transferred to the fluid. The finite rate of this process can then be related to the energy and momentum dissipation in the liquid, precluding superfluidity. On the other hand, a mobile impurity with finite mass \( M \) can propagate without dissipation at zero temperature with velocity given by slope of the dispersion \( V = \partial E/\partial P \).

Hence, the maximum slope of the dispersion defines a non-zero critical velocity for light mobile impurities. The main result of this work is the prediction that the above regimes of heavy and light impurities are separated by a quantum phase transition taking place at a critical mass \( M_c \), given by Eq. (3). We show that above the critical mass the many-body ground state in a total momentum sector \( P \) has cusp at \( P = \pm \pi n, \pm 3\pi n, \ldots \), while the impurity dispersion develops a swallowtail structure with metastable branches shown in Fig. 1. In this case the true thermodynamic critical velocity \( V_c \), given by the maximal slope of the many-body ground state, should be distinguished from the dynamical critical velocity \( V_c \) obtained from the maximal slope of the metastable branch. It can be shown that for \( M \gg M_c \) the thermodynamic critical velocity is inversely proportional to \( M \) consistent with the infinite mass limit of Refs. [20, 21].

**Metastability and two definitions of critical velocity.** For a concrete example of how a metastable impurity dispersion can arise, let us take our fluid to be a weakly interacting Bose gas. At the mean field level, appropriate for weak interactions, the condensate wavefunction develops a phase drop \( \Phi \) across the impurity located at \( X \). In the limit of strong coupling between gas and impurity, the latter gives rise to the Josephson term

\[
H_J(\Phi) = -nV_c \cos \Phi
\]  

(we have set \( h = 1 \)) in the impurity energy. Here \( V_c \) is the dynamical critical velocity which depends on the details of the impurity. The phase drop inevitably creates a background supercurrent contributing a term \( n\Phi \) to the total momentum \( P \). The total energy of the impurity and the background can then be written as

\[
H(P, \Phi) = \frac{1}{2 M}(P - n\Phi)^2 + H_d(\Phi)
\]  

(2)

The first term is the kinetic energy of the impurity moving with velocity \( V = (P - n\Phi)/M \), where \( M \) is the total mass of the impurity and induced depletion cloud. As we shall see on we can assume \( M \simeq M \) for a sufficiently heavy impurity.

The phase drop \( \Phi \) represents a collective coordinate characterizing the state of the impurity equilibrated with the background [19]. Its value is determined from the requirement of the minimum of the total energy (2) leading to the matching \( P - n\Phi = MV_c \sin \Phi \) of the Josephson current \( nV_c \sin \Phi \) to the current \( nV \) experienced by the impurity in its rest frame. For sufficiently large mass \( M > M_c \) this equation has several solutions corresponding to different arrangements of the impurity and the background. The total energy \( H(P, \Phi) \) in Eq. (2) can therefore have minima for several values of the phase drop \( \Phi \) for a given value of the total momentum \( P \). Plotting the corresponding energies results in a typical swallowtail structure for the dispersion, see Fig. 1.

Following the global minimum \( E_-(P) \) of the dispersion one sees that it develops cusps at the crossing points \( P = \pm \pi n, \pm 3\pi n, \ldots \) corresponding to a first order transition as a function of the total momentum \( P \). In addition to the true ground state \( E_-(P) \) there is also a metastable branch \( E_+(P) \) corresponding to local minimum. The minima corresponding to \( E_-(P) \) and \( E_+(P) \) are separated by the maximum. At the termination points \( P_t \) the maximum merges with the local minimum at \( \Phi = \Phi_t \) and the metastable branch ceases to exist. The critical mass \( M_c \) for the swallowtail catastrophe can be estimated from the simple model Eq. (2) as

\[
M_c = \pi n/V_c = Km(c/V_c)
\]  

(3)

where \( K = \pi n/mc \) is the Luttinger parameter of the liquid depending on the speed of sound \( c \). For a strongly repulsive impurity in a weakly interacting background \( M_c \gg m \) since \( K \gg 1 \) and \( c \gg V_c \), justifying a posteriori the assumption \( M \approx M_c \).

For \( M \gg M_c \) the termination point corresponds to \( \sin \Phi_t = \pm 1 \) and the mass-independent velocity \( \partial E_+ / \partial P = \pm V_c \). In contrast, the thermodynamic critical velocity, defined with the help of the stable branch \( V_c = \partial E_- / \partial P \), behaves as \( 1/M \) resulting in zero critical velocity for infinitely heavy impurity [20, 21].

Quantum fluctuations can smear the transition by producing a mechanism for the decay of the metastable branch [14]. The metastable branch \( E_+(P) \) is well defined as long as the decay rate \( \Gamma(P) \) to the lower branch satisfies \( \Gamma(P) \ll E_+(P) - E_-(P) \approx 2V_c|P - \pi n| \). We show below that \( \Gamma(P) \) vanishes as a power law near the crossing points, \( \Gamma(P) \propto |P - \pi n|^\alpha \) for \( P \rightarrow \pi n \) with the
exponent $\alpha$ given in Eq. (7). Self-consistency therefore requires $\alpha > 1$. Hence $\alpha = 1$ is the condition defining the transition between the two classes of dispersion curves, and corresponds to $M = M_c$ [14].

Before proceeding we note that the qualitative features discussed above in no way depend upon the specific model Eq. (1) and occur (for example) in the dispersion relation of a particle moving in a Bose gas described by the Gross–Pitaevskii equation, valid for arbitrary coupling between the gas and impurity [22].

Depleton model. We now introduce a framework to describe the periodic impurity dispersion, the metastable branch, and its decay in the general case. The moving depleton, i.e., impurity dressed by the liquid depletion, is characterized by a number of particles $N$ expelled from its core in addition to the phase drop $\Phi$. The existence of the two collective slow coordinates is associated with the presence of the two conservation laws: number of particles and momentum $\mathbf{p}$ [19]. The depleton’s dynamics is then specified by the following Hamiltonian

$$H(\Phi, N) = \frac{1}{2M}(P - n\Phi)^2 + \mu N + H_d(\Phi, N).$$  \hspace{1cm} (4)

Here $H_d$ is the $\Phi$-periodic energy function of the depleton cloud, $\mu$ is the chemical potential of the liquid in the absence of the impurity and $M = M - mN$. In the limit of strongly repulsive impurity the dynamics of $N$ is frozen and $H_d(\Phi, N)$ reduces to the simple Josephson form Eq. (1). If the mass $M$ is sufficiently large, the Hamiltonian (4) (being almost a periodic energy function of $\Phi$) possesses many metastable minima in addition to the absolute one.

We wish to determine the tunneling rate from the above mentioned metastable minima to the ground state branch at the same momentum $P$. Such tunneling is accompanied by a change in the phase drop $\Delta \Phi$ and number of depleted particles $\Delta N$, despite the momentum of both states being equal [23]. Therefore one must know the Lagrangian governing the dynamics of the collective variables $\Phi$ and $N$. As shown in Ref. [19], this requires the introduction of, and coupling to, the phonon subsystem. The latter may be described by small deviations of the density, $\rho(x, t)$, and velocity, $u(x, t)$, fields from their unperturbed values. It is convenient to introduce the superfluid phase $\varphi(x, t)$ and the displacement field $\vartheta(x, t)$ defined by $u = \partial_x \varphi / m$ and $\rho = \partial_x \vartheta / \pi$.

The effective action governing these harmonic degrees of freedom is that of the Luttinger liquid [25, 26]

$$S_{\text{LL}} = \frac{1}{\pi} \int \mathrm{d}t \mathrm{d}x \left[ -\partial_x \vartheta \partial_t \varphi - \frac{c}{2K} (\partial_x \vartheta)^2 - \frac{cK}{2} (\partial_x \varphi)^2 \right],$$

while their coupling to the depleton degrees of freedom takes a universal form [19, 24],

$$S_{\text{int}} = \int \mathrm{d}t \left[ -\Phi \vartheta(X, t) / \pi - \dot{N} \varphi(X, t) \right].$$

It is the phonon subsystem which must supply the macroscopic momentum $n \Delta \Phi$ and take away $\Delta N$ particles from the depleton. Near the point $P = \pi n$, the phonons have only a small amount of energy $\varepsilon$ with which to do this. This low-probability event, or ‘under-the-barrier’ evolution of the phononic system, can be described in imaginary time by performing the Wick rotation $t \rightarrow -i\tau$ in Eqs. (5), (6). We have delegated details of the tunneling calculation to a short section in the Supplemental Material [22]. We find for the tunneling rate

$$\Gamma(\varepsilon) \sim e^\alpha; \quad \alpha = 2K \left( \frac{(\Delta \Phi)^2}{2\pi} + \frac{(\Delta N)^2}{2K} \right) - 1.$$  \hspace{1cm} (7)

Writing $\varepsilon = 2\sqrt{c} |P - \pi n|$ gives the momentum dependent tunneling rate advertised earlier.

The logarithmic behavior of the tunneling action [22] is valid only on the long time-scale $\tau \geq 1/c$, implying the validity of Eq. (7) is restricted to the regime of small $\varepsilon$ close to $P = \pi n$. In such a case the velocities of the upper and lower branches of the dispersion are essentially opposite and close to $\pm \sqrt{c}$, while the change in number of particles $\Delta N$, being an even function of velocity, is negligible. As a result one arrives at the exponent $\alpha = 2K (\Delta \Phi / 2\pi)^2 - 1$. This single collective coordinate limit, i.e., $\Delta N = 0$, agrees with the previous result of Ref. [28] (cf. Eq. (7.20) therein) based on a phenomenological model of tunneling between two minima in the presence of coupling to an environment. Their results also show that a finite temperature $T$ acts as a low energy cut-off, namely $\Gamma \sim T^\alpha$, $\varepsilon \ll T$ (see Eqs. (5.30), (7.18) of Ref. [28]).

Higher order processes involve tunneling back and forth between the two minima that become degenerate for $\varepsilon = 0$, leading to an effective spin-boson or anisotropic Kondo description [14]. The inclusion of such processes is only necessary if $\alpha < 1$, meaning that tunneling is a relevant perturbation. This leads to the rounding out of the cusp in the dispersion $E(\Phi)$ at $P = \pi n$, and the disappearance of the metastable branch. Thus $\alpha = 1$ is the exact condition determining the critical mass $M = M_c$.

In the infinite mass limit with fixed velocity $V = P / M$ Eq. (4) becomes essentially a washboard potential, Fig. 1c. $H \rightarrow nV \Phi + H_d(\Phi)$ (change $\Delta N$ between adjacent minima goes to zero in this limit). Thus the change $\Delta \Phi$ between the adjacent minima is exactly $2\pi$, while the energy difference is $\varepsilon = 2\pi n V$. As a result, one finds $\alpha = 2K - 1$ and the power law dependence of the tunneling rate $\Gamma \sim V^{2K-1}$ [29] on the velocity in full agreement with the result of Ref. [20, 21].

In the limit of large energy detuning, i.e., away from $P = \pi n$, the tunneling rate does not take the form (7). In this region the barrier is small and the impurity emerges from under it with only slightly less energy, despite the large detuning between the branches (see Fig. 1 insets). Therefore most of the descent from the upper branch to the absolute one.

The effective action governing these harmonic degrees of freedom is that of the Luttinger liquid [25, 26]
distribution is by applying an external force This is in stark contrast to the case of
result, the energy dissipated per cycle and thus drift and relaxes to the lower branch, emitting phonons. As
termination point if the force is sufficiently large. In such a case the ex-
ternal force drives the impurity all the way up to the
strong force the driven impurity overshoots the ground
state branch and follows the metastable branch until it
stronger momentum where metastable branches become important.

One way to explore the metastable branches of the dis-
persion is by applying an external force $F$ to the impu-
ity and studying the ensuing dynamics. For sufficiently
force the driven impurity overshoots the ground state branch and follows the metastable branch until it
either tunnels or reaches the termination point $P_t$ (see Fig. 1). The energy dissipated per cycle, $\varepsilon$, must be sup-
plied by external force: $FV_D = \varepsilon/T$ resulting in the drift velocity $V_D = \varepsilon/2\pi n$. Here $T = 2\pi n/F$ is the period of
the motion.

Tunneling is negligible when $\Gamma T \ll 1$, which occurs if the force is sufficiently large. In such a case the ex-
ternal force drives the impurity all the way up to the
termination point $P_t$. It then loses the energy minimum and relaxes to the lower branch, emitting phonons. As
a result, the energy dissipated per cycle and thus drift
velocity are essentially independent of the applied force.
This is in stark contrast to the case of $M < M_c$, where the phonon emission due to dipole radiation gives rise to
a linear in $F$ drift velocity and hence a finite impurity mobility $\sigma = V_D/F$ [19].

The impurity velocity takes the form of a saw-tooth
trajectory, experiencing a slow build-up and a sudden
drop when the velocity reaches $V_c$, see Fig. 2. As a
function of the mass, $V_D$ saturates to the critical ve-
locity in the limit $M \to \infty$. This is most easily seen by noting that in such a limit the energy drop occurs
between two adjacent and essentially parallel parabolas:
$\varepsilon = 2\pi n V_D = P^2/2M - (P - 2\pi n)^2/2M \to 2\pi n V_c$. On
the same grounds, the amplitude of the saw-tooth oscil-
lations is a decreasing function of $M$ going to zero in the
infinite mass limit.

Quantum effects become dominant only if the time
spent on a metastable branch is comparable with the
inverse tunneling rate: $\Gamma T \geq 1$. Thus, the effect of quan-
tum tunneling plays a role only for very small forces. In
this limit, the impurity is likely to tunnel at a moment-
immediately above $P = \pi n$, right when it enters the metastable branch. Using the rate Eq. (7) to calculate
the tunneling-averaged energy drop $\langle \varepsilon \rangle$, we find for the drift velocity [22]

$$ V_D = \frac{c}{K} \left( \frac{F}{F_c} \right)^{\frac{1}{1+\alpha}}. \quad (9) $$

The non-analytic dependence of the drift on small forces
is the result of the cusp of the ground state energy at
$P = \pi n$ and implies the divergence of the mobility $\sigma = V_D/F$ as $F \to 0$ across the $M = M_c$ transition (see inset
of Fig. 2). At finite temperature $T$ the mobility is not
given by $\sigma \sim T^{-\alpha}$ as might be expected due to the finite
tunneling rate $\Gamma \sim T^n$ near the cusp. Strictly speaking, it is given by $\sigma \sim T^{-4}$ due to two-phonon thermal Raman
scattering [18, 19, 31] which results in velocity saturation
below $V_c$ as $F \to 0$. In practice, therefore, it is necessary to
work at sufficiently low temperatures and finite drives
to overcome such saturation and allow the impurity to
explore the full range of momentum where metastable
branches become important.

In conclusion, we have shown that the dispersion re-
lation of a mobile impurity undergoes a sudden change
when the mass of the impurity exceeds a certain critical
value. The latter depends on the interaction parameter
within the host liquid and thus may be varied experiment-
ally. The predicted transition can be probed by applying
an external force and monitoring either impurity velocity
or excitations of the host liquid. The change of disper-
sion also affects the oscillation frequency and dissipation
of the impurity in the trapping potential. With investiga-
tions of impurity motion in 1D atomic gases now un-
derway [5, 7, 8], we hope that these predictions will guide
future experiments.

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FIG. 2. Impurity velocity (solid) in units of the dynamic critical velocity $V_c$ as a function of time in units of the period $T = 2\pi n/F$. The drift velocity is denoted by $V_D$ (dashed) and is displayed in the inset as a function of the applied force. Due
to quantum tunneling near $P = \pi n$, the drift velocity goes to
zero as a power law with exponent $1/(\alpha + 1)$ (solid, inset)
given by Eq. (9) for $F < F_c$ and approaches the semi-classical
result for $F > F_c$ (dashed, inset) [22].
I. SUPPLEMENTAL MATERIAL

Impurity coupled to a weakly interacting liquid - To model the heavy impurity in the presence of a weakly interacting superfluid at \( T = 0 \), we employ the Gross-Pitaevskii equation (GPE) using a point description of the impurity wavefunction with coordinate \( X \),

\[ i\hbar \frac{\partial \Psi}{\partial t} = \left( -\frac{\partial^2}{2m} + g|\Psi|^2 - \mu + G\delta(x - X) \right) \Psi. \tag{10} \]

We look for a solution of the traveling wave form \( \Psi(x,t) = \Psi(x-Vt) \) and \( X(t) = Vt \). The usual way of dealing with a delta function perturbation is to attempt a solution constructed from two homogeneous (i.e., \( G = 0 \)) solutions that must satisfy the proper boundary conditions at \( x = X \). This strategy is facilitated by the fact that the bare GPE \( (G = 0) \) admits a one-parameter family of solutions

\[ \Psi_s(x-Vt) = \sqrt{a} \left( \frac{V}{c} - i\sqrt{1 - \frac{V^2}{c^2}} \tanh \left( \frac{x-Vt}{l} \right) \right), \tag{11} \]

known as grey solitons [32, 33]. They can be visualized as a density dip moving with velocity \( V \), having a core size \( l = \xi(1-V^2/c^2)^{-1/2} \), where \( c = \sqrt{ng/m} \) is the speed of sound, \( n = \mu/g \) is the 1D density and \( \xi = 1/mc \) is the healing length (the condition of weak inter-particle coupling implies \( n\xi \gg 1 \)). Thus, by appropriately matching two solitonic solutions at \( x = Vt \), one formally solves Eq. (10) with \( \Psi(x-Vt) \) given by

\[ \Psi(y) = \begin{cases} 
\Psi_s(y + x_0)e^{i\phi_0/2}, & y > 0 \\
\Psi_s(y - x_0)e^{-i\phi_0/2}, & y < 0
\end{cases} \tag{12} \]

where \( y = x-Vt \). Here we introduced two "matching parameters" \( x_0 \) and \( \phi_0 \) which must be chosen to satisfy the two boundary conditions: \( \Psi(0^+) = \Psi(0^-), \) \( \Psi(x=0^+) = 2mG\Psi(0) \). This results in the following two equations for \( \phi_0 \) and \( z = \tanh(x_0/l) \)

\[ \tan \frac{\phi_0}{2} = z \tan \frac{\phi_s}{2}, \tag{13} \]

\[ \sin^3 \frac{\Phi_s}{2} (1 - z^2) z = G \frac{c}{\sqrt{2m}} \left[ \cos^2 \frac{\phi_s}{2} + z^2 \sin^2 \frac{\phi_s}{2} \right], \tag{14} \]

where we have parameterized the velocity \( V \) by the solitonic phase \( \phi_s \) via \( V/c = \cos \frac{\phi_s}{2} \).

Equations (13), (14) permit a solution only for \( V < V_c \) where \( V_c = V_c(G) < c \) is the dynamic critical velocity that depends only upon the parameter \( G \) [34, 35]. This may be seen by plotting the right and left hand sides of Eq. (14). While the left hand side is bounded by a maximum, the right hand side grows quadratically with \( z \) and therefore the solution exists only for a limited range of \( \Phi_s \). For this reason we choose to parameterize the solution, Eq. (12), by the total phase drop across the impurity, \( \Phi = \Phi_s - \phi_0 \), which permits a solution for any \( \Phi \).

Upon solving Eqs. (13), (14) one thus finds the periodic relations \( z = z(\Phi, G/c) \) and \( \Phi_s = \Phi_s(\Phi, G/c) \) (i.e., they are invariant with respect to \( \Phi \rightarrow \Phi + 2\pi j \) for integer \( j \)).

We may use these relations to compute the dressed impurity energy \( E = \frac{1}{2} MV^2 + E_d \) and momentum \( P = MV + P_d \), where \( E_d \) and \( P_d \) are the corresponding depletion cloud contributions given by

\[ E_d = \int dx \left[ \frac{\partial_x |\Psi|^2}{2m} + \frac{g}{2}(n - |\Psi|^2) \right] + G|\Psi(X)|^2 \]

\[ = \frac{4}{3} n c \sin^3 \frac{\phi_s}{2} \left[ 1 - \frac{3}{4} z - \frac{1}{4} z^3 \right], \tag{15} \]

\[ P_d = \text{Im} \int dx \left[ \Psi^* \partial_x \Psi \right] - n \arg \Psi(x = \pm \infty) \]

\[ = n\Phi - mnV. \tag{16} \]

Here we introduced the number of expelled particles \( N = \frac{2n}{mc} \sin \frac{\phi_s}{2} (1 - z) \). Written in this way, the equilibrium expressions \( E(\Phi + 2\pi j) = E(\Phi) \) and \( P(\Phi + 2\pi j) = 2\pi nj + P(\Phi) \) are functions of the total phase drop \( \Phi \) across the impurity, whose functional forms depend only on the
\[ M = M_c - 12 \]

\[ M = M_c \]

\[ M = 2M_c \]

\[ 0 \leq \Phi \leq 2\pi \]

\[ P(\Phi) \]

\[ \Phi = \Phi(P) \]

\[ E(\Phi(P)) \]

\[ E(P) = E(P + 2\pi n). \]

\[ \frac{\partial P}{\partial \Phi} \bigg|_{\Phi=\pi} = 0 \quad \Rightarrow \quad M_c = \frac{2n_c}{c} \left( 1 + \frac{G}{c} \right) \approx \frac{mK_c}{\pi V_c}. \]

\[ G/c = 0.05 \]

\[ G/c = 0.2 \]

\[ G/c = 0.5 \]

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\[ G/c = 0.5 \]

\[ G/c = 0.2 \]

\[ G/c = 0.05 \]
the main text.

Weakly coupled, heavy impurity – The case of weak impurity coupling does not lend itself to convenient analytical analysis. For given $G/c < 1$, we thus solve the boundary Eqs. (13), (14) numerically to produce the dispersion law $E(P)$, see Fig. 5. For comparison to the strong coupling case, we also plot $P(\Phi)$ for various couplings $G/c < 1$. In the limit $G \to 0$, the impurity is dressed by an essentially unperturbed gray soliton (i.e., the two solitons of Fig. 3 are separated by a very small distance). This approximation breaks down near $\Phi = 2\pi nj$ for integer $j$ since we must have $P = 2\pi nj$. Consequently, there is a sharp deviation from the soliton configuration (dashed line in Fig. 5) near these points, making the functional form of $P(\Phi)$ difficult to characterize analytically. As a result of the numerical analysis, we see that even in the regime of weak coupling the appearance of multiple momenta extrema give rise to the observed swallowtail catastrophe of the energy dispersion and is thus not restricted to the Josephson limit.

Quantum tunneling and non-analytic drift – We wish to calculate the tunneling-averaged energy drop $\langle \varepsilon \rangle$ as a function of the (small) force $F = P/t$ using the momentum dependent tunneling rate $\Gamma(P)$. To this end we establish the validity of the result (7) in the main text.

The tunneling rate $\Gamma(\tau)$ is given by the exponentiated action $S(\tau)$ evaluated on the trajectory where the system tunnels for an imaginary time $\tau$ from one minimum to the other. For $\alpha > 1$ where tunneling is an irrelevant perturbation, we may safely neglect higher order processes involving tunneling back and forth between the minima and focus on the single tunneling event. Starting from Eqs. (5), (6) of the main text (after the Wick rotation $t \to -i\tau$) one integrates the imaginary time Luttinger liquid action with linear coupling to arrive at the tunneling action

$$S(\tau) = -\frac{1}{2} \int dx dx' dr \tau \tilde{J}_{x,r}(\tau) \tilde{D}(x, -x', \tau - \tau') \tilde{J}_{x,r}(\tau),$$

(19)

where $\tilde{J}_{x,r}(\tau) = \delta(x - X(\tau))(\Phi, \tilde{N}) = \delta(x - X(\tau))(\delta(\tau - \delta(\tau - \tau'))))|\Delta \Phi, \Delta N|$ describes the tunneling trajectory between minima having configurations which differ in phase by $\Delta \Phi$ and number of expelled particles $\Delta N$ and $\tilde{D}(x, \tau)$ is the matrix imaginary time phononic propagator. The latter is determined by inverting the matrix operator of the quadratic Luttinger liquid action, Eq. (5) of the main text, in the form $S_{LL} = -\frac{1}{2} \int d\tau \chi^{\dagger} \tilde{D}^{-1} \chi$, where $\chi^{\dagger} = (\theta/\pi, \varphi)$.

Substituting the above tunneling trajectories into Eq. (19) with the imaginary time propagator

$$\tilde{D}(x, \tau) = \frac{1}{4\pi} \left[ \frac{k}{\pi} \ln \left( \frac{\pi^2 \tau^2 + x^2}{\pi^2 \tau^2 + \pi^2} \right) - \ln \left( \frac{\pi^2 \tau^2 + x^2}{\pi^2 \tau^2 + \pi^2} \right) \right]$$

(20)

we find $S(\tau) = (1 + \alpha)\ln(\varepsilon/\ell)$ where $\ell \gtrsim (mc^{-1})$ is a short distance cut-off beyond which the point-particle description of the depletional breaks down and $\alpha = 2K \left( \left( \frac{2\pi}{2\pi} \right)^2 + \left( \frac{2\pi}{2\pi} \right)^2 - 1 \right)$. The X dependence of the action is neglected because it enters as $X(\tau) - X(\tau') \approx V(\tau - \tau') \ll c(\tau - \tau')$.

The energy-dependent tunneling rate is found with the help of the Fourier transformation, $\Gamma(\varepsilon) \sim \int d\varepsilon \exp(-it\varepsilon)$, where the analytical continuation to real energy $\varepsilon \to \varepsilon$ is accompanied by the Wick rotation $\tau \to it$. Substituting $S(\tau)$ given above and using the identity $\int d\varepsilon \exp(-i\varepsilon) \theta(1+\alpha) = \frac{\pi^{1/2} \Gamma(1+\alpha)}{\Gamma(1+\alpha)}$, we arrive at Eq. (7) of the main text.

To compute $\langle \varepsilon \rangle$ one must know the probability to tunnel per unit time: $\dot{P}(t) = \Gamma(t) \exp \left[ -\int_0^t d\tau' \Gamma(t') \right]$, which follows from the assumption that the process is Markovian. By parameterizing the average over time by an average over momenta one finds

$$\langle \varepsilon \rangle = \int_{n\pi}^{P_s} dP \varepsilon(P) \frac{\Gamma(P)}{F} \exp \left[ -\int_{n\pi}^{P_s} dP \frac{\Gamma(P')}{F} \right]$$

(21)

$$= \int_0^{F_F/F_s} d\varepsilon \varepsilon(P(x)) e^{-\frac{x}{\tau}}.$$  

(22)

Here we introduced the dimensionless variable $x(P) = \int_{n\pi}^{P_s} dP \Gamma(P')/F$ along with $F_s = \int_{n\pi}^{P_s} dP \Gamma(P')$. Due to the exponential decay of the kernel in Eq. (21), the integral is cut at $x_c = \min\{1, F_s/F \}$. As a result, one obtains the approximation $\langle \varepsilon \rangle \sim x_c \varepsilon(P_c)$, where $P_c$ satisfies $x_c = \int_{n\pi}^{P_c} dP \Gamma(P)/F$. For $F \ll F_s$, we have $x_c = 1$ and hence $P_c$ may be solved by writing $P_c = n\pi + i\delta P$ for small, yet undetermined, $\delta P$. Expanding $\Gamma$ near $\eta$ as $\Gamma = \Gamma(\eta) + (P - n\pi) \varepsilon_c / mc^2$, we find $\delta P = (mc^2 \varepsilon_c / (F/F_c))^{1/m}$. Here we used $F_c = \frac{\Gamma_m}{\varepsilon_c n^{2m} / \varepsilon_c n^{2m} / 2K}$ for large $M$. Thus,

$$\langle \varepsilon \rangle = mc^2 \left( \frac{F}{F_c} \right)^\frac{1}{m} \Rightarrow V_D = \frac{c}{K} \left( \frac{F}{F_c} \right)^\frac{1}{m},$$

(23)

establishing Eq. (9) of the main text.

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[22] See Supplemental Material at [URL] for a detailed analysis within the Gross-Pitaevskii framework and a brief derivation of Eqs. (7), (9).

[23] While the impurity gains the supercurrent momentum $n\Delta\Phi$, it loses a compensating amount of core momentum associated with the change in velocity.

[24] A derivation of Eq. (6) can also be obtained from the effective field theory model of Refs. [11–13]. It consists of replacing $d^\dagger d \rightarrow \delta(x - X)$ with $X$ the impurity coordinate, while also performing a canonical transformation to phononic fields in the presence of the equilibrated dressed impurity, namely $\rho = \tilde{\rho} - N\delta(x - X); u = \tilde{u} - \frac{\Delta}{m}\delta(x - X)$. This transformation generates Eq. (6) with the understanding that $\Phi, N$ are related by a linear combination to the chiral phase shifts $\delta_\pm, \beta_\pm$ defined in Refs. [12, 13], Ref. [11], respectively.

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