The optimal investment strategy of a DC pension plan under deposit loan spread and the O-U process

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Abstract

This paper is devoted to invest an optimal investment strategy for a defined-contribution (DC) pension plan under the Ornstein-Uhlenbeck (O-U) process and the loan. By considering risk-free asset, a risky asset driven by O-U process and a loan in the financial market, we firstly set up the dynamic equation and the asset market model which are instrumental in achieving the expected utility of ultimate wealth at retirement. Secondly, the corresponding Hamilton-Jacobi-Bellman (HJB) equation is derived by means of dynamic programming principle. The explicit expression for the optimal investment strategy is obtained by Legendre transform method. Finally, different parameters are selected to simulate the explicit solution and the financial interpretation of the optimal investment strategy is given.

Keywords: O-U process; loan; HJB equation; Legendre transform.
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1. Introduction

With the development of the global economy and society, pension is getting more important for the life of the elder. Besides, nowadays the aging of population is accelerating rapidly, and pension has become a focus. The enterprise annuity is divided into two basic modes: The defined benefit (DB) plan and the defined contribution (DC) plan. In the DC pension plan, it transfers the longevity and financial risks from the sponsor to the member and the DC pension plan is also playing an role in social security, which can

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not be ignored. Hence, asset allocation strategy is crucial to the distribution and deployment of DC pension funds.

In recent years, many scholars have focused on the optimal investment performance. Markowitz and Harry (1952) put forward the optimal portfolio problem for the first time and gave a theoretical proof. Interest rate was proposed by Duffie and Rui (1996) who described the general affine process. Boulier et al. (2001) studied the asset allocation problem of DC type enterprise fund where the interest rate obeys the framework of Vasicek, and obtained the analytic solution through the use of martingale method. The expected utility directly after the retirement pension was used in the paper of Blake et al. (2003). Through the expected utility maximization, Charupat and Milevsky (2002) found fixed and variable instantaneous annuities in the optimal combination of the different assumptions about mortality rates, and then made a comparison with the optimal situation of the cumulative period. Devolder et al. (2003) assumed that the price process of risky assets are networked with geometric Brownian motion (GBM), that is, the price fluctuation of risky assets is set as constant. Baev and Bondarev (2007) introduced O-U process instead of GBM. Moreover, inflation risk on the optimal DC pension was considered in Battocchio and Menoncin (2004). Gerrard et al. (2004) concentrated on income by using the stochastic optimal control technology. Afterwards, Jianwu et al. (2007) obtained an explicit solution by applying the CEV model. Gao (2008) examined the complete financial market with stochastic evolution of interest rate, and used Legendre transform to settle the optimal asset allocation strategy of DC type enterprise fund. Hsu et al. (2008) used the CEV model for asset pricing formulas. Gu et al. (2010) considered the optimal reinsurance and investment problem of Brownian motion with risk pricing process, and the assets are described by the constant elastic variance model. The CRRA utility maximization and mean-variance criteria were employed by Han and Hung (2012) to determine DC plan. Zhang and Rong (2013) further paid attention to the optimal allocation of DC pension with random wage under affine interest rate model. Guan and Liang (2016) studied the optimal allocation of DC pension under the framework of random interest rate and random fluctuation, in which the interest rate obeys an affine interest rate structure. Teng et al. (2016) came up with the interest rate which is subject to O-U process. Sun et al. (2017) proposed the expected investment goal based on deficit and surplus. Tang et al. (2018) made an on-the-spot investigation with two situations: random interest rate and annuity inflation. The opti-
mal allocation scheme with stochastic interest rate and stochastic volatility was characterised by Wang et al. (2018). Bian et al. (2018) paid more attention to a discrete-time model with mean-variance by a Markov chain. Guambe et al. (2019) further stated a investment problem, which consists of inflation and mortality risks. Optimal investment with transaction cost over an infinite horizon was developed by Blake and Sass (2002). Based on previous work, Mudzimbabwe (2019a) investigated a unsophisticated numerical solution method. Chen et al. (2019) construsted the investmal strategy for fund administers in a framework of Markov. A jump diffusion model was demonstrated by Mudzimbabwe (2019b). Dong and Zheng (2020) attempted to apply S-shaped utility. According to Zhang et al. (2020), mean-variance criterion and the Cox-Ingersoll-Ross (CIR) model were adopted. Most of the above literatures are involved with (CIR) model, Vasicek model, variance model and etc, however few of them apply O-U model. At the same time, they do not take loan into account in their financial market. We know that it is more accurate to adopt O-U process which reflects the fluctuation of asset price. What's more, with the upgrading and adjustment of China’s industry, capital driven by economic development will become the main thrust, that is to say, the era of capital economy has come when loan will be a normal state.

Based on the above settings, a risky asset is depicted by the O-U process in this paper. In the framework of a discrete-time, the business administrator is to make the expectation of the terminal wealth under the utility framework before the retirement. By adopting the theory of the stochastic control, the original nonlinear HJB equation is achieved which is hard to depict closed-form expressions. And then, we introduce the Legendre transform and separation of variables. In this case, nonlinear partial differential equations (NPDE) are transformed into linear partial differential equations. Finally, we derive the explicit expressions of the DC scheme. In summary, this article has two innovations: (i) we describe the optimal investment problem under the O-U process with CRRA utility function; (ii) Deposit and loan spreads are taken into account, and related financial explanation is presented.

The rest is laid out as bellow. Section 2 characterizes the assumptions of the model. Section 3 shows the definition of the value function and derives the corresponding HJB equations and by using principle of dynamic programming. Section 4 completes the closed-form solutions for the stochastic dynamic programming problem under Legendre transform and CRRA utility function. In section 5, we present the numerical simulation analysis. We
have made a summary in the final chapter.

2. The economy and model

We list the following assumptions for our model.

**Assumption 1.** Consider a financial market which ignores transaction fees. We use a finite-time horizon and continuous-time model. The uncertainty is represented by a complete probability space \((\Omega, F, P)\).

**Assumption 2.** Suppose that the financial market involves with three tradable assets: a bank account, a stock and a loan.

**Assumption 3.** We denote the price of the bank account at time \(t\) by \(B(t)\), such that

\[
  dB(t) = rB(t)dt, \quad B(0) = B_0, \quad r > 0,
\]

where \(r\) is a constant rate of interest.

**Assumption 4.** Let the price of the stock at time \(t\) be \(S(t)\), which is depicted by the stochastic differential equation (SDE).

By comparison with GBM, the O-U process is closer to the change of stock price. Here we use \(S(t)\) to express the price of risk assets at \(t\), which is described by O-U process

\[
  dS(t) = k(\theta - S(t))dt + \sigma dW(t), \quad S(0) = S_0,
\]

where \(\alpha > 0, \theta > 0\) and \(\sigma > 0\) represent the recovery rate, the response center and the volatility, respectively.

**Assumption 5.** Let \(R\) denote the lending rate, where \(0 < r < R < \mu\). \(V(t)\) is the pension wealth at time \(t\). \(V(t), B(t)\) and \(Y(t)\) are the total amount of money of the loan with interest, the risk-free asset and risky asset at time \(t\), respectively.

**Definition 1.** (Admissible strategy) If it meets the requirements as follow, investment loans are admissible.

1. \((L(t), B(t), Y(t))\) is \(\mathcal{F}_t\) measurable on a complete probability space;
2. \(\int_0^T L^2(t)dt < +\infty, \int_0^T B^2(t)dt < +\infty, \int_0^T Y^2(t)dt < +\infty\), a.s. \(T < \infty\);
3. For rational investors, with interest rates higher than the deposit rate, it’s impossible to choose between deposits and loans. That is, \(L(t)B(t) = 0\), with \(L(t) \geq 0, B(t) \geq 0\) and \(t \in [0, T]\). Assume that the set of all admissible investment and loan scheme \(((L(t), B(t), Y(t))\) are expressed by \(\pi = \{(L(t), B(t), Y(t)) : t \in [0, T]\}\).
Assumption 6. Define the retirement moment and the contribution rate of the enterprise annuity for \( T \) and \( c \), separately. Where \( T \) and \( c \) are the constants. Until retirement \( T \), \( cL(t) \) is supplied to the pension fund for each period. In order to simplify the model, the total salary is set as 1 dollar, and only one insured person is studied.

3. Model Formulation

3.1. Wealth process

Let \( V(t) = B(t) + Y(t) - L(t) + ct \) denote the pension wealth at time \( t \in [0, T] \). The dynamics of wealth has the following form:

\[
dV(t) = rB(t)dt + Y(t)\frac{dS(t)}{S(t)} - RL(t)dt + cdt. \tag{3}
\]

Based on (1) and (2), we rewrite (3) as

\[
dV(t) = \left\{ rX + \left[ k\left( \theta - \frac{s}{s} \right) - r \right]Y + (r - R)L - rct + c \right\}dt + \sigma Y dt. \tag{4}
\]

3.2. the HJB equation

Next, the goal is to maximize the expected discounted utility and ultimate wealth over a limited retirement period. That is to seek the optimum investment project \( Y(t) \).

Applying the stochastic control theory, we define the value function as

\[
H(t, s, v) = \max_Y E[U(v) \mid S(t) = s, V(t) = v], \quad 0 < t < T,
\]

where \( U(\cdot) \) is an increasing concave utility function and satisfies the conditions \( U'(+\infty) < 0 \) and \( U'(0) < +\infty \).

As described in Fleming and Soner (2006), by the aid of Itô’s formula, we have

\[
\sup_Y \{ H_t + [rx + \left( \frac{k(\theta - s)}{s} - r \right)Y + (r - R)L - rct + c]H_v + \frac{k(\theta - s)}{s}H_s \} \tag{5}
\]

\[+ \frac{1}{2} \frac{\sigma^2}{s^2} Y^2 H_{vv} + \frac{1}{2} \sigma^2 H_{ss} + \frac{\sigma^2}{s} Y H_{vs} \} = 0,
\]

and it’s accompanied by a boundary condition \( H(T, s, v) = U(v) \), where \( H_t, H_s, H_v, H_{vv}, H_{ss} \) and \( H_{sv} \) represent the different partial derivatives of \( H(T, s, v) \).
According to \( v = V(t) = B(t) + Y(t) - L(t) + ct \) and \( 0 < r < R < \mu \), if \( v > Y(t) + ct \), the investor will reject the loan; If \( v \leq Y(t) + ct \), the investor will choose to load, but the total amount will not exceed \( Y(t) + ct - v \), that is, \( L^*(t) = Y(t) + ct - v = \max\{0, Y(t) + ct - v\} \).

From the above setting, the two situations are discussed as follows:

(1) In the case of \( v \geq Y(t) + ct \), substituting \( L^*(t) = 0 \) back into (5), the HJB equation can be rewritten as

\[
\sup\{H_t + [rv + \left(\frac{k(\theta - s)}{s} - r\right)Y - rct + c]H_v + k(\sigma - s)H_s
\]
\[
+ \frac{1}{2}\sigma^2 \left(\frac{Y}{s} \right)^2 H_{vv} + \frac{1}{2}\sigma^2 H_{ss} + \frac{\sigma^2}{s} Y H_{vs}\} = 0
\]

\[H(T, s, v) = U(v).\]  

(2) In the case of \( v \geq Y(t) + ct \), putting \( L^*(t) = 0 \) in (5), the corresponding HJB equation can be rewritten as

\[
\sup\{H_t + [rv + \left(\frac{k(\theta - s)}{s} - r\right)Y - rct + c]H_v + k(\theta - s)H_s
\]
\[
+ \frac{1}{2}\sigma^2 \left(\frac{Y}{s} \right)^2 H_{vv} + \frac{1}{2}\sigma^2 H_{ss} + \frac{\sigma^2}{s} Y H_{vs}\} = 0
\]

\[H(T, s, v) = U(v).\]  

Take the derivative of (6) with respect to \( Y \) and we have

\[Y^*_1 = -\frac{k(\theta - s) - rs}{\sigma^2} \frac{H_v}{H_{vv}} - \frac{s}{s} \frac{H_{vs}}{H_{vv}}.\]  

Similarly, we can also get the efficient investment strategy of this problem (7)

\[Y^*_2 = -\frac{k(\theta - s) - Rs}{\sigma^2} \frac{H_v}{H_{vv}} - \frac{s}{s} \frac{H_{vs}}{H_{vv}}.\]  

Plugging \( Y^*_1 \) and \( Y^*_2 \) into (6) and (7), we derive respectively

\[H_t + k(\theta - s)H_s + \frac{1}{2}\sigma^2 H_{ss} + (rx - rct + c)H_v - \left[\frac{k(\theta - s) - rs}{\sigma^2}\right] \frac{H_v^2}{H_{vv}} - \frac{1}{2}\sigma^2 \frac{H_{vs}^2}{H_{vv}} - \left[\frac{k(\theta - s) - rs}{\sigma^2}\right] \frac{H_v H_{vs}}{H_{vv}} = 0,\]
and

\[ H_t + k(\theta - s)H_s + \frac{1}{2}\sigma^2 H_{ss} + (Rx - Rct + c)H_v - \frac{[k(\theta - s) - Rs]^2}{\sigma^2} \frac{H^2_v}{H_{vv}} \]

- \frac{1}{2}\sigma^2 \frac{H^2_{vs}}{H_{vv}} - [k(\theta - s) - Rs] \frac{H_v H_{vs}}{H_{vv}} = 0.

(11)

Obviously, the stochastic control problem is transformed into a NPDE. Next, we alternate the NPDE into the linear PDE based on the dual transformation.

4. Model solution

4.1. The Legendre transform

**Definition 2.** Let \( f : R^n \rightarrow R \) be a convex function. Legendre transform can be defined as follows:

\[ L(z) = \sup_{x} \{ f(x) - zx \}, \quad 0 < t < T. \]  \hfill (12)

Then the function \( L(z) \) is called Legendre dual function of \( Legendre \).

With reference to Jose et al. (2006), a specific definition is proposed by

\[ \hat{H}(t, s, z) = \sup_{v > 0} \{ H(t, s, v) - zx \mid 0 < v < \infty \}, \quad 0 < t < T, \]

where \( z > 0 \) denotes the dual variable to \( v \).

The value of \( v \) where this optimum is denoted by \( g(t, s, z) \), so that,

\[ g(t, s, z) = \inf_{v > 0} \{ v \mid H(t, s, v) \geq zx + \hat{H}(t, s, v) \}, \quad 0 < t < T. \]

From the above equation, we can get

\[ \hat{H}(t, s, z) = H(t, s, g) - zg, \]  \hfill (13)

Where \( g(t, s, z) = v \) and \( H_v = z \).

The function \( \hat{H} \) is related to \( g \) by

\[ g = -\hat{H}_z. \]  \hfill (14)
By differentiating (14), we achieve
\[
H_t = \dot{H}_t, \quad H_s = \dot{H}_s, \quad H_{vv} = -\frac{1}{H_{zz}}, \quad H_{ss} = \dot{H}_{ss} - \frac{\dot{H}_{sz}^2}{H_{zz}}, \quad H_{sv} = -\frac{\dot{H}_{sz}}{H_{zz}}, \tag{15}
\]

At the terminal time \( T \), we define
\[
\hat{U}(z) = \sup_{v > 0} \{ U(z) - zv \mid 0 < v < \infty \}, \tag{16}
\]
\[
G(z) = \sup_{v > 0} \{ U(z) - zv \mid 0 < v < \infty \}.
\]

In addition, there exists \( g(T, s, z) = (U')^{-1} \), which is a boundary condition.

Plugging (15) into (10) and (11), we derive
\[
\dot{H}_t + k(\theta - s)\dot{H}_s + \frac{1}{2}\sigma^2 \dot{H}_{ss} + (rx - rct + c)z \tag{17}
\]
\[
+ \frac{[k(\theta - s) - rs]^2 z^2 \dot{H}_{zz}}{2\sigma^2} - [k(\theta - s) - rs]z\dot{H}_{sz} = 0.
\]

Differentiating both sides of (17) with respect to \( z \), we obtain
\[
\dot{H}_{tz} + [k(\theta - s) - rs]\dot{H}_{sz} + \frac{1}{2}\sigma^2 \dot{H}_{ssz} + (rx - rct + c) + rzg_z + \frac{[k(\theta - s) - rs]^2 z\dot{H}_{zz}}{\sigma^2} \tag{18}
\]
\[
+ \frac{[k(\theta - s) - rs]^2 z^2 \dot{H}_{zzz}}{2\sigma^2} - [k(\theta - s) - rs]z\dot{H}_{sz} - [k(\theta - s) - rs]z\dot{H}_{szz} = 0.
\]

Due to (14), we get
\[
v = g = -\dot{H}_z, \quad \dot{H}_{tz} = -g_t, \quad \dot{H}_{sz} = -g_s, \quad \dot{H}_{zz} = -g_z, \quad \dot{H}_{ssz} = -g_{ss}, \quad \dot{H}_{szz} = -g_{sz}. \tag{19}
\]

We recover (18) by using (19), and then obtain the following partial differential equation
\[
g_t + [k(\theta - s) - rs]g_s + \frac{1}{2}\sigma^2 g_{ss} - (rg - rct + c) - rzg_z + \frac{[k(\theta - s) - rs]^2 zg_z}{\sigma^2} \tag{20}
\]
\[
+ \frac{[k(\theta - s) - rs]^2 z^2 g_{zz}}{2\sigma^2} - [k(\theta - s) - rs]g_s - [k(\theta - s) - rs]zg_{sz} = 0.
\]
Through the dual transformation, (10) has been transformed into a linear PDE. Moreover, we obtain the optimal portfolio selection $Y_1^*$

$$Y_1^* = -\frac{k(\theta - s) - rs}{\sigma^2}g_z + sg_s. \quad (21)$$

4.2. The solution under the logarithmic utility function

**Theorem 4.2.1.** If the price of the risk-free asset, the price of the risk asset and the wealth process follow (1)-(3) respectively, the optimal portfolio of the enterprise annuity is specified by according to (22)-(30)

$$Y^*(t) = \begin{cases}
\frac{k(\theta - s) - Rs}{\sigma^2} + ct, & v \leq ct + \\
\frac{k(\theta - s) - Rs}{\sigma^2} + ct + cT e^{r(T-t)}s, & ct < v < \frac{k(\theta - s) - Rs}{\sigma^2} + ct + cT e^{r(T-t)}s \quad (22)
\end{cases}$$

**Proof.** In the light of the logarithmic utility function, a definition is also provided

$$U(x) = \ln x, \quad x > 0,$$

Depending on the form of logarithmic utility function, we have

$$g(T, s, z) = \frac{1}{z}.$$

In response to (20), we construct its corresponding solution

$$g(t, s, z) = \frac{1}{z}f(s_t) + \varphi(t). \quad (23)$$

In the meantime, we quote the boundary conditions by $f(s_T) = 1$ and $\varphi(T) = 0$.

Suppose it is a convex function and we can attain

$$\frac{df(x)}{f(x)} - z = 0.$$

Assume $x_0$ is the optimum point, and there is $L(z) = f(x_0) - zx_0$. If $f(x) = \ln x$, we have $x_0 = \frac{1}{z}$.

As a result,

$$L(z) = f\left(\frac{1}{z}\right) - 1 = \ln \frac{1}{z} - 1 = -\ln z - 1$$
Taking the partial derivative of (22), we get

\[
\begin{align*}
g_t &= \varphi_t, \quad g_s = \frac{1}{z} f_s + s, \quad g_z = -\frac{1}{z^2} f, \\
g_{ss} &= \frac{1}{z} f_{ss}, \quad g_{sz} = -\frac{1}{z^2} f_s, \quad g_{zz} = \frac{2}{z^3} f. 
\end{align*}
\] (24)

Substituting (23) back into (20), we obtain

\[
\varphi_t + k(\theta - s)\frac{1}{z} f_s + \frac{\sigma^2}{2} \frac{1}{z} f_{ss} + rct - c - r\varphi = 0. 
\] (25)

By observation, (24) can be decomposed into two equations, which is supplied to eliminate the dependence on \( s \). Furthermore, since the boundary conditions are \( f(s_T) = 1 \) and \( \varphi(T) = 0 \), we have

\[
\begin{align*}
\left\{ \begin{array}{l}
k(\theta - s)\frac{1}{z} f_s + \frac{\sigma^2}{2} \frac{1}{z} f_{ss} = 0 \\
f(s_T) = 1,
\end{array} \right.
\] (26)

and

\[
\begin{align*}
\left\{ \begin{array}{l}
\varphi_t - r\varphi + rct - c = 0 \\
\varphi(T) = 0.
\end{array} \right.
\] (27)

By integrating the two equations, we derive the solution to (25)

\[
f(s_t) = 1.
\]

The corresponding solution of (26) is given by

\[
\varphi(t) = ct - cTe^{r(t-T)},
\]

consequently,

\[
g = \frac{1}{z} + ct - cTe^{r(t-T)}. 
\] (28)

Due to \( g(t, s, v) = v \), we derive

\[
\frac{1}{z} = v - ct + cTe^{r(t-T)}. 
\] (29)

Finally, the optimal strategy \( Y_1^* \) can be rewritten as
From the equivalence of \( r \) and \( R \), we can get another optimal investment strategy \( Y_2^* \):

\[
Y_2^* = \frac{[k(\theta - s) - Rs][v - ct + cTe^{r(t-T)}]s}{\sigma^2}.
\]

The above results are discussed as follows.

(1) If \( v \leq \frac{[k(\theta - s) - rs][v - ct + cTe^{r(t-T)}]s}{\sigma^2} + ct \), then

\[
Y_1^* = \frac{[k(\theta - s) - rs][v - ct + cTe^{r(t-T)}]s}{\sigma^2}.
\]

(2) If \( v \leq \frac{[k(\theta - s) - Rs][v - ct + cTe^{r(t-T)}]s}{\sigma^2} + ct \), then

\[
Y_2^* = \frac{[k(\theta - s) - Rs][v - ct + cTe^{r(t-T)}]s}{\sigma^2}.
\]

(3) If \( \frac{[k(\theta - s) - Rs][v - ct + cTe^{r(t-T)}]s}{\sigma^2} + ct < v < \frac{[k(\theta - s) - rs][v - ct + cTe^{r(t-T)}]s}{\sigma^2} + ct \), we will proceed with two cases.

(i) With \( Y(t) + ct \in \left[ \frac{[k(\theta - s) - Rs][v - ct + cTe^{r(t-T)}]s}{\sigma^2} + ct, v \right] \), since \( L^*(t) = Y(t) + ct - v = \max\{0, Y(t) + ct - v\} \), then \( L^*(t) = 0 \). It means that the investment refuses to lend in this case. Let the left bracket of (6) be \( \phi_1(Y) \). Because \( \phi_1(Y) \) is increasing with respect to \( Y \leq \frac{[k(\theta - s) - Rs][v - ct + cTe^{r(t-T)}]s}{\sigma^2} + ct \), \( \phi_1(Y) \) attains its maximum at \( Y^*(t) = v - ct \). (ii) With \( Y(t) + ct \in \left[ v, \frac{[k(\theta - s) - rs][v - ct + cTe^{r(t-T)}]s}{\sigma^2} + ct \right] \), since \( L^*(t) = Y(t) + ct - v = \max\{0, Y(t) + ct - v\} \), then \( L^*(t) = v - ct \). Denote (7) the left bracket by \( \phi_2(Y) \). Considering \( \phi_2(Y) \) decreases of \( Y \) in the interval \( \left[ v - ct, \frac{[k(\theta - s) - rs][v - ct + cTe^{r(t-T)}]s}{\sigma^2} \right] \), we have that \( \phi_2(Y) \) reaches the maximum at \( Y^*(t) = v - ct \). Hence, in the interval \( \left[ \frac{[k(\theta - s) - Rs][v - ct + cTe^{r(t-T)}]s}{\sigma^2}, \frac{[k(\theta - s) - rs][v - ct + cTe^{r(t-T)}]s}{\sigma^2} \right] \), \( Y^*(t) = v - ct \).
In overall, the optimal investment strategy \( Y^*(T) \) can be expressed as

\[
Y^*(t) = \begin{cases} 
  \left[ k(\theta - s) - Rs \right] s \sigma^2, & v \leq c + ct \left[ k(\theta - s) - Rs \right] s \\
  v - ct, & c + ct < v < \left[ k(\theta - s) - Rs \right] s \sigma^2 \\
  \left[ k(\theta - s) - Rs \right] s \sigma^2, & v \geq c + ct \end{cases}
\]

Theorem 4.2.2. If the price of the risk-free asset, the price of the risk asset and the wealth process follow (1)-(3) respectively, the expected maximum utility of the enterprise annuity for problem (10) and (11) is

(1) In the case of \( v \geq Y(t) + ct \),

\[
H_1 = \ln(v - ct + cTe^{\alpha(t-T)}).
\]

(2) In the case of \( v < Y(t) + ct \),

\[
H_2 = \ln(v - ct + cTe^{\alpha(t-T)}).
\]

Proof. We first prove the first case. Combining (27) with \( -\hat{H}_z(t, s, v) = g \), we have

\[
\hat{H}_z = -\frac{1}{z} - ct + cTe^{\alpha(t-T)}.
\]

From (34), integrating yields

\[
\hat{H} = -lnz + -ctz + czTe^{\alpha(t-T)} + m,
\]

Where \( m \) is a constant.

Taking into account \( \hat{H} = H - zg \) and the terminal condition \( m = -1 \), we obtain

\[
H_1 = \ln(v - ct + cTe^{\alpha(t-T)}).
\]

By the same token, we derive

\[
H_2 = \ln(v - ct + cTe^{\alpha(t-T)}).
\]

5. Numerical analysis

Based on these simulation results, we provide some economic explanations and discuss the behavioral features related to loss aversion, and contribution rate. We take the initial time \( t = 5 \), and the investor will retire at \( T = 20 \).
In the financial market, other parameters are \( r = 0.03, R = 0.06, \sigma_1 = 0.005 \) and \( c = 0.2 \).

In Fig. 1 and Fig. 2, the volatility \( \sigma \) on the optimal investment strategy \( Y^*(t) \) is taken into account. Assume that the wealth value is 500, 1200 at time \( t \), respectively, and the volatility \( \sigma \) varies at \([1, 2.4]\). If \( v = 500 \), then \( v \leq ct + \frac{[k(\theta - s) - Rs][v - ct + cTe^R(t-T)]s}{\sigma^2} \). If \( \sigma \) varies in the range \([1, 2.4]\) and \( v = 1200 \), then \( v \geq ct + \frac{[k(\theta - s) - rs][v - ct + cTe^r(t-T)]s}{\sigma^2} \).

![Fig. 1. Effect of parameter \( \sigma \) on \( Y^* \) when \( v = 500 \)](image)

![Fig. 2. Effect of parameter \( \sigma \) on \( Y^* \) when \( v = 1200 \)](image)

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Fig. 2 presents the impacts of $\sigma$ on $Y^*$ invested in the risky asset. We can see, from Fig. 1, $Y^*$ reduces while $\sigma$ grows. In an economic sense, the more drastic the stock price varies, the more uncertain the market is. Investors are afraid to take risks and will engage in conservative risk-averse behaviors, that is, they will increase their investment in risk-free assets.

![Graph](image1)

**Fig. 3. Effect of parameter $v$ on $Y^*$ when $R > r$**

Fig. 3 shows $v$ on $Y^*$ invested in the risky asset. We adopt the instantaneous volatility $\sigma = 0.2$. From Fig. 3, we may find that $Y^*$ increases with the initial wealth $v$. This can be explained by the fact that employees become richer, they become more capable of taking risks. Therefore, the pension manager tend to spend money on risky assets to get more return.

![Graph](image2)

**Fig. 4. Effects of parameters $\sigma$ and $\mu$ on $Y^*$**
Fig. 4 displays $\sigma$ and $\mu$ on the robust optimal investment strategy. As shown in Fig. 4, $Y^*$ decreases with regard to $\sigma$. A higher $\sigma$ leads to a larger expected drop in volatility and an increased probability of a large adverse movement in the risky assets price. In addition, under the elasticity coefficient $\sigma$ is fixed, when $\mu$ is raising, $Y^*$ also rises. This is because that an increase in the expected instantaneous rate makes the member improve his ability to resist risk, and hence she invests more in the stock.

Fig. 5 displays that as the lending rate $R$ increases, the proportion of wealth invested in the stock becomes larger. With the increase of $R$, there is more risk in the market and it takes a lot of time to invest. As a result, the manager will reduce the amount of money on risk assets which is also in line with the economic market.

Fig. 5. Effect of parameter $R$ on $Y^*$

6. Conclusion

Optimal portfolio has always been the core of financial market research. We do the research about problem with the the O-U process under $CRRA$ utility. With the help of dynamic programming principle and dual transform method, the closed-form of optimal asset allocation strategy is obtained. Finally, MATLAB software is used for programming. More importantly, we do an analysis of the volatility of the stocks, the initial wealth, the elasticity coefficient and the lending rate on the investment behavior.

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