Quantum entanglement between two magnon modes via Kerr nonlinearity driven far from equilibrium

Zhang, Zhedong; Scully, Marlan O.; Agarwal, Girish S.

Published in: Physical Review Research

Published: 01/09/2019

Document Version: Final Published version, also known as Publisher's PDF, Publisher's Final version or Version of Record

License: CC BY

Published version (DOI): 10.1103/PhysRevResearch.1.023021

Publication details: Zhang, Z., Scully, M. O., & Agarwal, G. S. (2019). Quantum entanglement between two magnon modes via Kerr nonlinearity driven far from equilibrium. Physical Review Research, 1(2), [023021]. https://doi.org/10.1103/PhysRevResearch.1.023021

Citing this paper
Please note that where the full-text provided on CityU Scholars is the Post-print version (also known as Accepted Author Manuscript, Peer-reviewed or Author Final version), it may differ from the Final Published version. When citing, ensure that you check and use the publisher's definitive version for pagination and other details.

General rights
Copyright for the publications made accessible via the CityU Scholars portal is retained by the author(s) and/or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights. Users may not further distribute the material or use it for any profit-making activity or commercial gain.

Publisher permission
Permission for previously published items are in accordance with publisher's copyright policies sourced from the SHERPA RoMEO database. Links to full text versions (either Published or Post-print) are only available if corresponding publishers allow open access.

Take down policy
Contact lbscholars@cityu.edu.hk if you believe that this document breaches copyright and provide us with details. We will remove access to the work immediately and investigate your claim.
Quantum entanglement between two magnon modes via Kerr nonlinearity driven far from equilibrium

Zhedong Zhang,1,2 Marlan O. Scully,1,2,3 and Girish S. Agarwal1,4

1Texas A&M University, College Station, Texas 77843, USA
2Baylor University, Waco, Texas 76704, USA
3Princeton University, Princeton, New Jersey 08544, USA

(Received 20 May 2019; revised manuscript received 21 July 2019; published 19 September 2019)

We propose a scheme to entangle two magnon modes via the Kerr nonlinear effect when driving the systems far from equilibrium. We consider two macroscopic yttrium iron garnets interacting with a single-mode microcavity through the magnetic dipole coupling. The Kittel mode describing the collective excitations of a large number of spins is excited through the driving cavity with a strong microwave field. We demonstrate how Kerr nonlinearity creates the entangled quantum states between the two macroscopic ferromagnetic samples, when the microcavity is strongly driven by a blue-detuned microwave field. Such quantum entanglement survives at the steady state. Our work offers insights and guidance in designing experiments for observing entanglement in massive ferromagnetic materials. It can also find broad applications in macroscopic quantum effects and magnetic spintronics.

DOI: 10.1103/PhysRevResearch.1.023021

I. INTRODUCTION

Recent advances in ferromagnetic materials have drawn considerable attention in studies of quantum nature in magnetic systems, as the limitations of electrical circuitry are reached. Thanks to the low loss of the collective excitations of spins known as magnons in magnetic samples, the magnons offer a new paradigm for developing future generations of spintronic devices and quantum engineering [1–6]. Yttrium iron garnet (YIG) with a size of ∼100 μm as fabricated in recent experiments provides new insights for studying macroscopic quantum effects, such as entanglement and squeezing, that have raised widespread interest in different branches of physics during the last decade [7–12]. Quantum entanglement between massive mirror and optical cavity photons has been explored, in both theoretical and experimental aspects [13–18]. Several following ideas have suggested the extension of such an entangled quantum state to magnons in the microwave regime, due to their great potential for macroscopic spintronic devices. Many experimental efforts have been devoted to the quantum nature of magnon states, through hybridizing the spin waves with other degrees of freedom, e.g., superconducting qubits and phonon modes [19–22]. Compared to atoms and photons, magnonics holds the potential for implementing quantum states in more massive objects. This can be seen from the 320-μm-diameter YIG spheres implemented in recent experiments [23].

Ferromagnetic materials provide a promising platform for studying the hybrid quantum systems in diversity, due to the rich interactions including magnetic dipole, magneto-optical, and magnetostrictive couplings that allow one to implement the interactions between drastically different physical systems [24]. Of particular interest are magnon polaritons, where strong and even ultrastrong light-matter couplings can be achieved, along with the fact of their high spin density and low dissipation rate [25–31]. This may serve as a potential candidate for implementing quantum information transducers and memories [31,32]. For the architecture of quantum magnonics, the macroscopic quantum effects are essentially worthy of being explored. The most recent work using driven-dissipation theory suggests magnon-photon-phonon entanglement and also the squeezing of magnon modes in which both the entanglement and squeezing are essentially transferred into the mechanical mode [33–35]. From a theoretical point of view, this macroscopic quantum nature of magnon modes stems from the nonlinearity that can be enhanced by driving the systems far from equilibrium. Two prominent schemes are responsible for introducing such nonlinearity: the magnetostrictive interaction and the Kerr effect, where the latter results from the magnetocrystalline anisotropy. Apart from the magnon-photon interaction, Kerr nonlinearity plays a significant role in magnon spintronics [5]. Recent experiments in YIG spheres demonstrated the multistability and photon-mediated control of spin current, due to the Kerr effect [36–38]. These studies are essentially semiclassical. In contrast, the current article is to show how Kerr nonlinearity can be used to produce remarkable quantum features of YIG bulks.

In this article, we propose a scheme of entangling magnon modes in two massive YIG spheres via Kerr nonlinearity.
The two magnon modes interact with a microcavity through the beam-splitter-like coupling, which cannot produce any entanglement. Nevertheless, activating Kerr nonlinearity via strong driving results in squeezing-like coupling which may let the magnon become entangled with cavity photons. This is reasonable as the magnetostatic field is along the $z$ axis. The static magnetic field for producing the Kittel mode is made of high-reflection material so that photons leak from inside the cavity are along the $z$ axis. The Holstein-Primakoff transform yields $S_{z}=S_{s}-m_{s}^{2}m_{s}$, $S_{+}=(2S_{s}-m_{s}^{2})^{1/2}m_{s}$, $S_{-}=m_{s}^{2}(2S_{s}-m_{s}^{2})^{1/2}$, where $S_{s}=\pm S_{s}$ and $m_{s}$ represents the bosonic annihilation operator [46]. For the yttrium iron garnets (YIGs) with diameter $d = 40 \mu m$, the density of the ferric ion $Fe^{3+}$ is $\rho = 4.22 \times 10^{27} m^{-3}$, which leads to the total spin $S = 7/2\rho V_{m} = 7.07 \times 10^{14}$. This is often much larger than the number of magnons, so that we can safely approximate $S_{+}\approx \sqrt{2S_{s}}m_{s}$, $S_{-}\approx \sqrt{2S_{s}}m_{s}$. In the presence of the external microwave driving field, the effective Hamiltonian of the hybrid magnon-cavity system is of the form [47]

$$H_{eff} = \hbar\omega_{c}a^{\dagger}a + \hbar\sum_{j=1}^{2}[\omega_{j}m_{j}^{2}m_{j} + g_{j}(m_{j}^{3}a^{\dagger} + m_{j}a)] + i\hbar\Omega(a^{\dagger}e^{-i\omega_{d}t} - ae^{i\omega_{d}t}),$$

(3)

where $\delta_{s} = \omega_{c} - \omega_{s}$, $\delta_{a} = \omega_{c} - \omega_{a}$, and the cavity frequency is denoted by $\omega_{c}$. The frequency of the Kittel mode is $\omega_{c} = \gamma B_{0} = 2\mu_{0}K_{m}^{(s)}\gamma S_{s}/2V_{m}$, with $\gamma/2\pi = 28$ GHz/T, $g_{s}$ gives the magnon-cavity coupling and $\Delta_{s} = \mu_{0}K_{m}^{(s)}\gamma S_{s}/2V_{m}$ gives the Kerr nonlinearity. The Rabi frequency $\Omega = \sqrt{2P_{2}/\hbar\omega_{c}}$ in the last term quantifies the strength of the field inside microcavity driven by the microwave magnetic field, where $P_{2}$ and $\omega_{c}$ represent the power and frequency of the microwave field, respectively. The quantum Langevin equations (QLEs) for the hybrid magnon-cavity system are given by

$$\dot{m}_{s} = -(i\delta_{s} + \gamma_{s})m_{s} - 2i\Delta_{s}m_{s}^{2}m_{s} - ig_{s}a + \sqrt{2\gamma_{s}}m_{s}^{\nu}(t),$$

$$\dot{a} = -(i\delta_{a} + \gamma_{a})a - i\sum_{j=1}^{2}g_{j}m_{j} + \Omega + \sqrt{2\gamma_{s}}a^{\nu}(t),$$

(4)

where $\gamma_{s}$ and $\gamma_{a}$ represent the rates of cavity leakage and magnon dissipation, respectively. $m_{s}^{\nu}(t)$ and $a^{\nu}(t)$ are the input noise operators having zero mean and white noise: $\langle m_{s}^{\nu}(t)m_{s}^{\nu}(t')\rangle = \bar{n}_{s}\delta(t - t')$, $\langle m_{s}^{\nu}(t)m_{s}^{\nu}(t')\rangle = \langle n_{s} + 1\rangle\delta(t - t')$; $\langle a^{\nu}(t)a^{\nu}(t')\rangle = 0$, $\langle a^{\nu}(t)a^{\nu}(t')\rangle = \delta(t - t')$, where $\bar{n}_{s} = [\exp(\hbar\omega_{c}/kB T) - 1]^{-1}$ denotes the Planck factor of the $s$th magnon mode.

Since the microcavity is under strong driving by the microwave field, the beam-splitter-like coupling between

FIG. 1. Schematic of cavity magnons. Two YIG spheres are interacting with the basic mode of the microcavity in which the right mirror is made of high-reflection material so that photons leak from the left side. The static magnetic field for producing the Kittel mode is along the $z$ axis whereas the microwave driving and magnetic field inside the cavity are along the $x$ axis.
magnons and cavity leads to the large amplitudes of both magnon and cavity modes, namely, $| ⟨m_1⟩ |, | ⟨a⟩ | ≫ 1$. In this case, one can safely introduce the expansion $m_s = ⟨m_1⟩ + \delta m_s, a = ⟨a⟩ + \delta a$ in the vicinity of the steady state, by neglecting the higher-order fluctuations of the operators. We thereby obtain the linearized QLEs for the quadratures $δX_1, δY_1, δX, δY$ defined as $δX_1 = (δm_1 + δm_1^\dagger) / \sqrt{2}, δY_1 = (δm_1 - δm_1^\dagger) / \sqrt{2}, δX_2 = (δm_2 + δm_2^\dagger) / \sqrt{2}, δY_2 = (δm_2 - δm_2^\dagger) / \sqrt{2}$, $\delta X = (δa + δa^\dagger) / \sqrt{2}, \delta Y = (δa - δa^\dagger) / \sqrt{2}$.

$\sigma(t) = A \sigma(f(t)) + f(t)$.

$\sigma(t) = [δX_1(t), δY_1(t), δX_2(t), δY_2(t), δX(t), δY(t)]^T$

and $f(t) = [\sqrt{2}γ_2 X_1(t), \sqrt{2}γ_2 Y_1(t), \sqrt{2}γ_2 X_2(t), \sqrt{2}γ_2 Y_2(t), \sqrt{2}γ_2 X(t), \sqrt{2}γ_2 Y(t)]^T$ are the vectors for quantum fluctuations and noise, respectively. The drift matrix reads

$$A = \begin{pmatrix}
F_1 - γ_1 & δ_1 - G_1 & 0 & 0 & 0 & g_1 \\
δ_1 - G_1 & -F_1 - γ_1 & 0 & 0 & -g_1 & 0 \\
0 & 0 & F_2 - γ_2 & δ_2 - G_2 & 0 & g_2 \\
0 & 0 & δ_2 - G_2 & -F_2 - γ_2 & 0 & g_2 \\
-g_1 & 0 & g_2 & -g_2 & δ_1 - γ_1 & γ_c \\
g_1 & 0 & 0 & -δ_1 - γ_1 & γ_c & -g_2 \\
\end{pmatrix}$$

with the magnetocrystalline anisotropy quantified by $G_s = 2\Delta_s \text{Re}(m_s^2), F_s = 2\Delta_s \text{Im}(m_s^2)$, and the effective detuning of magnons $δ_1 = δ_s + 2\sqrt{G_s^2 + F_s^2} = δ_s + 4\Delta_s |⟨m_1⟩|^2$, which includes the frequency shift caused by Kerr nonlinearity. The means $(m_{1,2})$ are given by

$$⟨m_j⟩ = \frac{i g_j Ω}{(δ_j - iγ_1)(δ_{1,2} - iγ_2) - g_j^2 - \frac{G_s^2}{δ_{1,2} - iγ_2}},$$

and $(1 ↔ 2)$. Before studying entanglement, it is essential to elucidate the mechanism for optimizing the entanglement via Kerr nonlinearity. To this end, we proceed via the effective Hamiltonian for quantum fluctuations

$$H_{ct} = \hbar \sum_{j=1}^2 \left[ δ_2 δm_j δm_j^\dagger + \tilde{A}_j δm_j δm_j^\dagger + δ^*_j δm_j δm_j^\dagger \right] + h/δ_{2,1}^1 δa i δa,

\text{where} \quad \tilde{A}_j = (G_j + i F_j) / 2. \quad \text{The quadratic terms} \quad δm_j δm_j^\dagger, \quad δm_j δm_j^\dagger \text{imply the effective magnon-magnon interaction induced by the magnetocrystalline anisotropy, which may be significantly enhanced by strong driving. This, in fact, is responsible for the entanglement. To make it elaborate, let us introduce the Bogoliubov transformation [48,49]} \quad δβ_s = u_s δm_s - v_s^* δm_s^\dagger, \quad δ^*_β_s = -v_s δm_s + u_s^* δm_s^\dagger, \quad \text{where} \quad u_s = \sqrt{1 + δ_{2,1}^2 (δ_{2,1}^2 - 1)}, \quad v_s e^{iω_s} = −\sqrt{1 + δ_{2,1}^2 (δ_{2,1}^2 - 1)}, \quad \alpha = \arctan(F_j / G_j), \quad \text{and} \quad ε_s = (δ_{2,1}^2 - 4 |A_j|^2)^{1/2}. \quad \text{Inserting these into Eq. (8) we find}$

$$H_{ct} = \hbar \sum_{j=1}^2 \left[ ε_s δ^*_β_s δ^*_β_s + g_s (v_s δβ_s + u_s δ^*β_s^\dagger) δa + (u_s^* δβ_s + v_s δ^*β_s^\dagger) δa^\dagger + hδ_s δa i δa, \quad \text{which shows that} \quad ε_s \simeq -δc \text{ is optimal for the entanglement, due to the magnon-photon squeezing term} \quad g_s (v_s δβ_s δa + v_s δ^*β_s^\dagger δa^\dagger). \text{This will be confirmed by the later numerical results when taking into account experimental parameters.}$

\section{III. \textbf{ENTANGLEMENT BETWEEN MAGNON MODES}}

Since we are using the linearized quantum Langevin equations, the Gaussian nature of the input states will be preserved during the time evolution of systems. The quantum fluctuations are thus the continuous three-mode Gaussian state, which is completely characterized by a $6 × 6$ covariance matrix (CM) defined as $C_i(t, t') = \frac{1}{2} ⟨σ_i(t)σ_i^T(t') + σ_i^T(t')σ_i(t))⟩$, $i, j = 1, 2, 3, 4, 5, 6$, where the average is taken over the system and bath degrees of freedoms. Suppose the drift matrix $A$ is negatively defined; the solution to Eq. (5) is $\sigma(t) = M(t)σ(0) + \int_0^t M(s)f(t-s)ds$, where $M(t) = \exp(At)$. This enables us to find the equation which the CM obeys,

$$C(t + τ, t) = AC(t + τ, t) + C(t + τ, t)A^T + e^{4tD},$$

for $τ > 0$. Thus the stationary CM can be straightforwardly obtained by letting $τ = 0$, $t → \infty$ in Eq. (10) that yields the Lyapunov equation

$$AC∞ + C∞A^T = -D,$$

where the diffusion matrix is $D = \text{diag}[γ_1(2δ_n + 1), γ_1(2δ_n + 1), γ_2(2δ_n + 1), γ_2(2δ_n + 1), γ_c(2δ_n + 1), γ_c(2δ_n + 1)]$ defined through $⟨f_i(t)f_j(t') + f_j(t')f_i(t)⟩ = 2D_{ij}δ(t - t')$.

\section{A. Magnon-magnon and magnon-photon entanglements}

To study the bipartite magnon-magnon and magnon-photon entanglements, we adopt the logarithmic negativity $E_N$ by computing the $4 × 4$ CM related to the two modes of interest. This can be achieved by defining $E_N = \text{max}[0, -\ln2v_−]$, where $v_− = \text{min}(\text{eig} \{p^1 \otimes \text{eig} \{p^2} \sum_{j=1}^2 (-σ_j)P_jC_jP_j^T\})$ and $σ_j$ is the Pauli matrix [50,51]. $C_j$ is the CM of two subsystems, obtained by removing in $C∞$ solved from Eq. (11) the rows and columns of the uninteresting modes. The matrix $P_j = σ_j ⊕ 1$ realizes the partial transposition at the level of the CM. In what follows, we will work in the monostable scheme of magnons. Furthermore, we will focus on the case of two identical magnons having $G_{1,2} = G, F_{1,2} = F, δ_{1,2} = δ, Λ_{1,2} = Λ, g_{1,2} = g$. 

023021-3
illustrate the entanglement between the cavity and the second sphere (Fig. 2(a)). Figure 2(b) and (c) depict how our procedure is valid under the parameter regimes we have taken into account the experimentally feasible parameters [36] $\omega_{\delta_2}/2\pi = 10$ GHz, $\delta_2/2\pi = 1$ MHz, $\gamma_{1,2}/2\pi = 8.8$ MHz, and $\gamma_{\delta}/2\pi = 1.9$ MHz for the YIG bulk at low temperature $T = 10$ mK. First of all we observe from Figs. 2(a) and 2(b) that Kerr nonlinearity is responsible for creating the steady-state entanglement between two magnon modes, evident from the fact that the entanglement dies out when $G = F = 0$. This results from the dominated beam-splitter interaction between magnon mode and cavity photons, once $G = F = 0$. Thereby no magnon-cavity entanglement can be created, as seen in Fig. 2(b). We take the condition $\varepsilon_j \simeq -\delta_\epsilon$ for optimizing the magnon-photon entanglement, as illustrated in Fig. 2(d), where $\varepsilon_j \simeq \sqrt{3(G_j^2 + F_j^2)}$. The two-mode squeezing term $g_j(\nu_j \beta_j \delta_j + \nu_j^* \beta_j^* \delta_j^*)$ squeezes the joint state between one magnon mode and cavity photons, which results in the partial entanglement in between. Because the same type of interaction occurs when coupling the other magnon mode with the cavity, the two distanced magnon modes are expected to be entangled. This is confirmed in Fig. 2(e), manifesting the optimal magnon-magnon entanglement in the vicinity of $\varepsilon_j \simeq -\delta_\epsilon$. The elaborate transfer from magnon-photon entanglement to magnon-magnon entanglement is subsequently evident as the coupling of the cavity to another sphere is turned on. Since the biparticle entanglement originates from the Kerr nonlinearity quantified by $G_{1,2}$ and $F_{1,2}$, there must be an interplay between the couplings $G_j$, $F_j$, and $g_j$, which is depicted in Figs. 2(e) and 2(f). In Fig. 2(c) we take $\omega_{\delta_2}/2\pi = 41$ MHz and this implies $\delta_\epsilon/2\pi \simeq -0.03$ GHz for the optimal entanglement $E_{m_{m2}}$. We then adopt the magnitude of $\delta_\epsilon$ for plotting Figs. 2(a) and 2(b). Using $\sqrt{G^2 + F^2} = 2\Delta(|m_j|^2$ and Eq. (7) for the 40-$\mu$m-diameter YIG spheres, the optimal entanglement with $|G| = 0.038$ GHz, $|F| = 0.028$ GHz [see Fig. 2(a)] yields the Rabi frequency $\Omega = 1.06 \times 10^{15}$ Hz, corresponding to the drive power $P_d = 314$ mW. Indeed, the stronger nonlinearity will create more entanglement between the magnon modes. But we have to ensure the negatively defined matrix $A$ given in Eq. (6), and the condition $\langle m_j m_j \rangle \ll 2N_s = 5\rho V_{j m}$. It is easy to show how our procedure is valid under the parameter regimes we considered. Also, the experimental feasibility of an ultrastrong drive using microwave field needs consideration.

Figures 3(a) and 3(b) illustrate the entanglement between two magnon modes versus some controllable parameters by considering the 40-$\mu$m-diameter YIG sphere experiment, where $\omega_{\delta_2}/2\pi = 10$ GHz, $\delta_2/2\pi = -1$ MHz, $\gamma_{\delta}/2\pi = 1$ MHz, $\Delta_{1,2}/2\pi = 1$ GHz, $\gamma_{1,2}/2\pi = 41$ MHz, $\gamma_{\delta}/2\pi = 8.8$ MHz, and $\gamma_{\delta}/2\pi = 1.9$ MHz have been taken according to Ref. [36]. We observe in Fig. 3(a) that for fixed driving power, the magnon-magnon entanglement is quite sensitive to cavity detuning $\delta_\epsilon \equiv \omega_\epsilon - \omega_\delta$, reaching its maximum at $\delta_\epsilon/2\pi \simeq -0.03$ GHz. This is consistent with the condition $\varepsilon_j \simeq -\delta_\epsilon$, as clarified for optimizing the entanglement. Figure 3(b) shows considerable entanglement when the system is driven far from equilibrium. This is reasonable because the strong external driving significantly enhances the Kerr nonlinearity that is responsible for both magnon-cavity squeezing and...
and 3(b) show that the weaker magnon-magnon entanglement defined as the squared logarithmic negativity involves one or two modes. This is a proper entanglement for Gaussian states, quantified by the logarithmic negativity for bipartite entanglement.

\[ E_{ij|k} = \max[0, -\ln 2 v_{ij|k}] \]  

along the line with the logarithmic negativity for bipartite entanglement. \( v_{ij|k} = \min \{ \text{eig}(\Omega_2 C) \} \) is the minimum symplectic eigenvalue of the 6 × 6 CM \( C = P_{ij|k} C'_{ij|k} \) with the symplectic matrix \( \Omega_2 = \oplus_{j=1}^3 i\sigma^z \), \( C \) is the 6 × 6 CM of the full system. \( P_{ij|k} = \sigma^x \oplus 1 \oplus 1, \) \( P_{2|3} = 1 \oplus \sigma^x \oplus 1, \) and \( P_{3|2} = 1 \oplus 1 \oplus \sigma^x \) are the matrices for partial transposition at the level of 6 × 6 CM. The residue contangle satisfies the monogamy of quantum entanglement, that is, \( R_{ijk} \geq 0 \) and \( C_{ij|k} \geq C_{ij} + C_{ijk} \), which is reminiscent of the Coffman-Kundu-Wootters monogamy inequality that holds for a system consisting of three qubits. A *bona fide* quantification of tripartite entanglement for Gaussian states is given by the minimum residual contangle

\[ R_{\text{min}} = \min \left[ R_{m_{ij|s}}, R_{m_{j|s}}, R_{m_{i|ms}} \right] \]  

which guarantees the invariance of tripartite entanglement under all permutations of the modes.

In Figs. 3(c) and 3(d), one can observe the important role that Kerr nonlinearity plays in tripartite entanglement, besides the bipartite entanglement discussed above. Figure 3(c) shows that the tripartite entanglement is quite sensitive to cavity detuning \( \delta_c \), reaching its maximum in the vicinity of \( \delta_c \approx -3.03 \text{ GHz} \). This is consistent with the condition \( \epsilon_j \approx \delta_c \), as clarified for optimizing the entanglement in Eq. (9). Figure 3(d) shows the considerable tripartite entanglement when the system is driven far from equilibrium. This is attributed to the significant enhancement of Kerr nonlinearity by strong driving that is responsible for the magnon-cavity squeezing and entanglement. Furthermore, under the same parameter regime as in magnon-magnon entanglement, the magnon-magnon-photon entanglement is also quite robust against the cavity leakage.

The time-resolved detection of the photons emitting off the cavity axis may offer an alternative scheme for entanglement measurement. This leakage to the side is denoted by the blue wavy lines in Fig. 1. These photons arise from the decay of YIG’s excitations quantified by \( \gamma_c \). \( s = 1, 2 \), in Eq. (4). The quadrature information of magnon modes can be transferred to the time-gated emitted photons, which can be homodyne detected by interfering with an extra microwave field. This quantum-light-probe scheme may take advantage of being a noninvasive detection for entanglement measurement.

IV. CONCLUSION AND REMARKS

In conclusion, we have proposed a protocol for entangling the magnon modes in two massive YIG spheres, through the Kerr nonlinearity that originates from the magnetocrystalline anisotropy. We show that such nonlinearity has to be essentially included to produce the entanglement. Our work demonstrates the stationary entanglement between two macroscopic YIG spheres driven far from equilibrium, within the experimentally feasible parameter regime. The amount of entanglement is quantified by the logarithmic negativity and is surprisingly robust against the cavity leakage: the entangled quantum state may persist with a low-quality cavity.
giving weak magnon-cavity coupling. This may be helpful to experimental design for entanglement measurement.

We should note that our idea for entangling magnon modes may be potentially extended to other complex systems, such as molecular aggregates and clusters, along with the fact of similar forms of nonlinear couplings $b^\dagger b^\dagger + \Delta b^\dagger b^\dagger b$. With the scaled-up parameters, the long-range entanglement in molecular aggregates would be anticipated, in that the exciton-exciton interaction is several orders of magnitude higher than the Kerr nonlinearity resulting from the magnetocrystalline anisotropy. For instance, the two-exiton coupling in J-aggregates and light-harvesting antennas takes the value of $\sim 50$ cm$^{-1}$, which is $\sim 0.3\%$ of the exciton frequency. This is a much stronger nonlinearity than that in YIGs with Kerr coefficient $K \sim 0.1$ nHz, that is, $\sim 10^{-11}$ of its Kittel frequency. Recent developments in both ultrafast spectroscopy and synthesis have revealed the important role of quantum coherence which may significantly modify the functions of complex molecules and may help the design of polaritonic molecular devices as well as polariton chemistry. Hence entangling the molecular aggregates may help studies of quantum phenomena in complex molecules.

**ACKNOWLEDGMENTS**

We gratefully acknowledge the support of AFOSR Award No. FA-9550-18-1-0141, ONR Award No. N00014-16-1-3054, and the Robert A. Welch Foundation (Awards No. A-1261 and No. A-1943-20180324). We also thank Jie Li and Tao Peng for useful discussions.

[1] Y. Kajiwara et al., *Nature (London)* **464**, 262 (2010).
[2] L. J. Cornelissen, J. Liu, R. A. Duine, J. B. Youssef, and B. J. van Wees, *Nat. Phys.* **11**, 1022 (2015).
[3] N. Zhu et al., *Appl. Phys. Lett.* **109**, 082402 (2016).
[4] T. An et al., *Nat. Mater.* **12**, 549 (2013).
[5] A. V. Chumak, V. I. Vasyuchka, A. A. Serga, and B. Hillebrands, *Nat. Phys.* **11**, 453 (2015).
[6] A. V. Chumak, A. A. Serga, and B. Hillebrands, *Nat. Commun.* **5**, 4700 (2014).
[7] M. Collet et al., *Nat. Commun.* **7**, 10377 (2016).
[8] M. Ho, E. Oudot, J.-D. Bancel, and N. Sangouard, *Phys. Rev. Lett.* **121**, 023602 (2018).
[9] R. Riedinger et al., *Nature (London)* **556**, 473 (2018).
[10] H. Y. Yuan and M.-H. Yung, *Phys. Rev. B* **97**, 060405(R) (2018).
[11] T. Morimae, A. Sugita, and A. Shimizu, *Phys. Rev. A* **71**, 032317 (2005).
[12] C. F. Ockeloen-Korppi et al., *Nature (London)* **556**, 478 (2018).
[13] S. Gröblacher, K. Hammerer, M. Vanner, and M. Aspelmeyer, *Nature (London)* **460**, 724 (2009).
[14] M. Aspelmeyer, T. J. Kippenberg, and F. Marquardt, *Rev. Mod. Phys.* **86**, 1391 (2014).
[15] C. Genes, D. Vitali, P. Tombesi, S. Gigan, and M. Aspelmeyer, *Phys. Rev. A* **77**, 033804 (2008).
[16] E. Verhagen, S. Deleglise, S. Weis, A. Schliesser, and T. J. Kippenberg, *Nature (London)* **482**, 63 (2012).
[17] T. A. Palomaki, J. D. Teufel, R. W. Simmonds, and K. W. Lehnert, *Science* **342**, 710 (2013).
[18] D. Vitali, S. Gigan, A. Ferreira, H.R. Bohm, P. Tombesi, A. Guerreiro, V. Vedral, A. Zeilinger, and M. Aspelmeyer, *Phys. Rev. Lett.* **98**, 030405 (2007).
[19] B. Julsgaard, A. Kozhelkin, and E. S. Polzik, *Nature (London)* **413**, 400 (2001).
[20] A. J. Berkley et al., *Science* **300**, 1548 (2003).
[21] D. Lachance-Quirion et al., *Sci. Adv.* **3**, e1603150 (2017).
[22] X. Zhang, C.-L. Zou, L. Jiang, and H. X. Tang, *Sci. Adv.* **2**, e1501286 (2016).
[23] D. Zhang et al., *npj Quantum Inf.* **1**, 15001 (2015).
[24] D. Lachance-Quirion, Y. Tabuchi, A. Gloppe, K. Usami, and Y. Nakamura, *Appl. Phys. Express* **12**, 070101 (2019).
[25] Y. Tabuchi, S. Ishino, T. Ishikawa, R. Yamazaki, K. Usami, and Y. Nakamura, *Phys. Rev. Lett.* **113**, 083603 (2014).
[26] Ó. O. Soykal and M. E. Flatte, *Phys. Rev. Lett.* **104**, 077202 (2010).
[27] X. Zhang, C.-L. Zou, L. Jiang, and H. X. Tang, *Phys. Rev. Lett.* **113**, 156401 (2014).
[28] J. Bourhill, N. Kostylev, M. Goryachev, D. L. Creedon, and M. E. Tobar, *Phys. Rev. B* **93**, 144420 (2016).
[29] Y. Tabuchi et al., *Science* **349**, 405 (2015).
[30] M. Hadar, Y. Yang, B. M. Yao, C. H. Yu, J. W. Rao, Y. S. Gui, R. L. Stamps, and C. M. Hu, *Phys. Rev. Lett.* **121**, 137203 (2018).
[31] B. Yao et al., *Nat. Commun.* **8**, 1437 (2017).
[32] X. Zhang et al., *Nat. Commun.* **6**, 8914 (2015).
[33] J. Li, S.-Y. Zhu, and G. S. Agarwal, *Phys. Rev. Lett.* **121**, 203601 (2018).
[34] J. Li, S.-Y. Zhu, and G. S. Agarwal, *Phys. Rev. A* **99**, 021801(R) (2019).
[35] J. Li and S.-Y. Zhu, *New J. Phys.* **21**, 085001 (2019).
[36] Y. P. Wang, G. Q. Zhang, D. Zhang, T. F. Li, C. M. Hu, and J. Q. You, *Phys. Rev. Lett.* **120**, 057202 (2018).
[37] L. Bai et al., *Phys. Rev. Lett.* **118**, 217201 (2017).
[38] P. Hyde, B. M. Yao, Y. S. Gui, G.-Q. Zhang, J. Q. You, and C.-M. Hu, *Phys. Rev. B* **98**, 174423 (2018).
[39] M. Sarovar, A. Ishizaki, G. R. Fleming, and K. Birgitta Whaley, *Nat. Phys.* **6**, 462 (2010).
[40] Z. D. Zhang and J. Wang, *Sci. Rep.* **6**, 37629 (2016).
[41] D. M. Coles et al., *Nat. Mater.* **13**, 712 (2014).
[42] Z. D. Zhang, P. Saurabh, K. E. Dorfman, A. Debnath, and S. Mukamel, *J. Chem. Phys.* **148**, 074302 (2018).
[43] M. Kowalewski, K. Bennett, and S. Munakel, *J. Phys. Chem. Lett.* **7**, 2050 (2016).
[44] C. Kittel, *Phys. Rev.* **73**, 155 (1948).
[45] There is also an exception as in Ref. [23], where the cavity design can achieve the simultaneous excitation of the Kittel and magnetostatic modes. Nevertheless, it is worth noting that the magnetostatic modes may undergo an incoherent intraband relaxation process which results in a dramatically larger dephasing than the Kittel mode. The MS can be thereby ignored when working within the timescale of $\sim 1\mu s$.
[46] O. Madelung and T. C. Taylor, *Introduction to Solid-State Theory* (Springer-Verlag, Berlin, 1978).
[47] The magnetic dipole-dipole interaction between two YIG bulks is neglected, because such an interaction obeys the
scaling of $(2\pi r/\lambda)^{-3}$ with respect to distance. In recent experiments, the wavelength is about $30$ mm and hence we can safely drop the dipolar coupling between YIGs when $r \gtrsim 1$ mm.

[48] E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics: Theory of the Condensate State*, Part 2, revised ed. (Butterworth-Heinemann, Oxford, 1980).

[49] A. L. Fetter and J. D. Walecka, *Quantum Theory of Many-Particle Systems* (Dover Publications, Mineola, 2003).

[50] G. Vidal and R. F. Werner, *Phys. Rev. A* 65, 032314 (2002).

[51] R. Simon, *Phys. Rev. Lett.* 84, 2726 (2000).

[52] G. Adesso and F. Illuminati, *New J. Phys.* 8, 15 (2006).

[53] G. Adesso and F. Illuminati, *J. Phys. A: Math. Theor.* 40, 7821 (2007).