Boundedness and Stability of Impulsively Perturbed Delay Differential Equations

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1 Introduction

It is characteristic for a linear ordinary differential equation that if any solution is bounded on the half-line for any bounded right-hand side then a solution of the corresponding homogeneous equation tends to zero exponentially [1]. The connection of boundedness with exponential behavior of solutions for impulsive differential equations is studied in [2,3] and many other papers. It turns out that for impulsive equations sometimes we can avoid checking the boundedness of solutions for any bounded right-hand side. In particular the following result is valid [4].

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Suppose the points of impulses $\tau_j$ are such that $0 < \rho \leq \tau_{j+1} - \tau_j \leq \sigma < \infty$ and a solution of the scalar impulsive equation

$$\dot{x}(t) + A(t)x(t) = 0,$$

(1)

$$x(\tau_j) = B_j x(\tau_j - 0) + \alpha_j, \ j = 1, 2, \ldots,$$

(2)

is bounded for each initial value $x(0)$ and each bounded sequence $\{\alpha_j\}$. (Here and in sequel we assume a solution is right continuous.)

Then the solution $X(t)$ of the homogeneous equation (1), $x(\tau_j) = B_j x(\tau_j - 0)$ satisfying $x(0) = 1$, has the exponential estimate

$$|X(t)| \leq Ne^{-\nu t},$$

(3)

with $N > 0$, $\nu > 0$, and any solution of the non-homogeneous equation

$$\dot{x}(t) + A(t)x(t) = f(t),$$

(4)

is bounded for $t \geq 0$, if $\sup_{t \geq 0} |f(t)| < \infty$, $\sup_j |\alpha_j| < \infty$.

The first result of this type for the equation

$$\dot{x}(t) + A(t)x(t) = \sum_{j=1}^{\infty} \alpha_j \delta(t - \tau_j),$$

(5)

where $\delta(t - \tau_j)$ are delta functions, was obtained in [6].

This paper is concerned with the problem whether this result is valid for impulsive delay differential equations. The point is that the proof of this fact for (1),(2) is essentially based on the representation [5]

$$x(t) = X(t)x(0) + \int_0^t G(t, s)f(s)ds + \sum_{0 < \tau_j \leq t} G(t, \tau_j)\alpha_j$$

(6)

for any solution of (5),(2), where

$$G(t, s) = X(t)X^{-1}(s).$$

(7)

For impulsive equations with delay we have a similar to (6) solution representation, but (7) generally speaking is not valid. Nevertheless as we demonstrate in this paper under certain conditions $|G(t, s)| \geq |X(t)X^{-1}(s)|$. Then
the above result is valid for equations with delay. It is to be emphasized that generally speaking the delay is unbounded.

Besides this we obtain an explicit relation (Theorem 5) connecting solutions of the impulsive equation with solutions of the non-impulsive delay differential equation. The corollary of Theorem 5 generalizes a result [7] on the existence of a non-oscillating solution of an impulsive delay differential equation.

2 Preliminaries

Let $l_{\infty}$ be a space of bounded sequences $\alpha = \{\alpha_j\}_{j=1}^{\infty}$, $\alpha_j \in \mathbb{R}$, $\|\alpha\|_{l_{\infty}} = \sup_j |\alpha_j|$.

We consider a scalar linear delay differential equation

$$\dot{x}(t) + \sum_{i=1}^{m} A_i(t)x[h_i(t)] = r(t), \quad t \geq 0, \quad x(\xi) = \varphi(\xi), \quad \xi < 0,$$

$$x(\tau_j) = B_j x(\tau_j - 0) + \alpha_j, \quad j = 1, 2, \ldots,$$

under the following assumptions

(a1) $0 = \tau_0 < \tau_1 < \tau_2 < \ldots$ are fixed points, $\lim_{j \to \infty} \tau_j = \infty$;

(a2) $A_i, r, i = 1, \ldots, m$ are Lebesgue measurable essentially bounded in any finite segment $[0, b]$ functions, $B_j \in \mathbb{R}, j = 1, \ldots$;

(a3) $h_i : [0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $h_i(t) \leq t$;

(a4) $\varphi : (-\infty, 0) \to \mathbb{R}$ is a Borel measurable bounded function.

Below we will also need the following hypotheses

(a5) $M = \sup \|B_j\| < \infty$;

(a6) there exists $\rho > 0$ such that $\tau_{j+1} - \tau_j \geq \rho, \quad j = 1, 2, \ldots$;

(a7) there exists $\sigma > 0$ such that $\tau_{j+1} - \tau_j \leq \sigma, \quad j = 1, 2, \ldots$;

(a8) there exists $\delta > 0$ such that $t - \delta \leq h_i(t), \quad i = 1, \ldots, m$.

(a9) there exists $Q > 0$ such that

$$\int_{k}^{k+1} |A_i(t)| \ dt \leq Q, \quad k = 1, 2, \ldots, \quad i = 1, \ldots, m.$$

**Definition.** A function $x : [0, \infty) \to \mathbb{R}$ absolutely continuous in each $[\tau_j, \tau_{j+1})$ is a solution of the impulsive equation (8), (9), if for $t \neq \tau_j$ it satisfies (8) and for $t = \tau_j$ it satisfies (9).
A solution $X(t)$ of the homogeneous equation

$$\dot{x}(t) + \sum_{i=1}^{m} A_i(t)x[h_i(t)] = 0$$

(10)

for $t \geq 0$,

$$x(\xi) = 0$$

(11)

for $\xi < 0$, satisfying impulsive conditions

$$x(\tau_j) = B_j x(\tau_j - 0),$$

(12)

$j = 1, 2, \ldots$, and such that $x(0) = 1$ is said to be a fundamental solution.

A solution $G(t, s)$ of the homogeneous equation (10), $t \geq s$, (11), $\xi < s$, (12), $\tau_j > s$, satisfying $G(s, s) = 1$ is said to be a fundamental function (sometimes it is called a Green or an evolutionary function). We assume $G(t, s) = 0$, $0 \leq t < s$.

**Theorem 1** [8] Suppose (a1)-(a4) are satisfied. Then there exists one and only one solution of the problem (8), with the initial value

$$x(0) = \alpha_0$$

and impulsive conditions (9) and it can be presented as

$$x(t) = X(t)x(0) + \int_0^t G(t, s)r(s)ds -$$

$$- \sum_{i=1}^{m} \int_0^t G(t, s)A_i(s)\varphi(h_i(s))ds + \sum_{0<\tau_j \leq t} G(t, \tau_j)\alpha_j.$$  

(13)

Here $\varphi(\zeta) = 0$, if $\zeta \geq 0$, and $X(t) = G(t, 0)$.

The fundamental function $G(t, s)$ generally speaking does not satisfy (7). However in certain cases the inequality $|G(t, s)| \geq |X(t)X^{-1}(s)|$ holds as in the case of delay differential equations without impulses [9].

**Lemma 1** Suppose (a1)-(a4) hold. Let $G(t, s) \geq 0$, $X(t) > 0$, $A_i(t) \geq 0$ for any $t, s$, $0 \leq s \leq t < \infty$, $i = 1, \ldots, m$.

Then $G(t, s) \leq G(t, \zeta)G(\zeta, s)$ and $G(t, s) \geq X(t)X^{-1}(s)$, $0 \leq s \leq \zeta < t$. 

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Proof. Let $0 < s < \zeta$. Suppose $y$ is a function satisfying (10), (12) for $t > \zeta$, $\tau_j > \zeta$. Besides this, let $y(\xi) = G(\xi, s)$, if $s < \xi < \zeta$, $y(\xi) = 0$, if $\xi < s$, and $y(\zeta) = G(\zeta, s)$. Then by Theorem 1

$$G(t, s) = y(t) = G(t, \zeta)G(\zeta, s) - \sum_{i=1}^{m} \int_{\zeta}^{t} G(t, \xi)A_i(\xi)G(h_i(\xi), s) d\xi,$$

where $G(h_k(\xi), s) = 0$, if $h_k(\xi) < s$ or $h_k(\xi) > \zeta$. As $G(t, s) \geq 0$, $A_i(t) \geq 0$, then the second term in the right-hand side is not positive. Therefore $G(t, s) \leq G(t, \zeta)G(\zeta, s), s \leq \zeta \leq t$,

$$X(t)X^{-1}(s) = G(t, 0)G^{-1}(s, 0) \leq G(t, s)G(s, 0)G^{-1}(s, 0) = G(t, s),$$

which completes the proof.

3 Main results

**Theorem 2** Suppose $A_i(t) \geq 0$, $G(t, s) \geq 0$, $X(t) > 0$, the hypotheses $(a1)$-$(a4),(a7)$ are satisfied, and the solution of the problem (10), $t \geq 0$, (11), $\xi < 0$, (9), $x(0) = 0$ is bounded for each $\alpha = \{\alpha_i\}_{i=1}^{\infty} \in l_\infty$.

Then there exist positive constants $N$ and $\nu$ such that (3) holds.

Proof. By Theorem 1 the solution of the problem (10), $t \geq 0$, (11), $\xi < 0$, (9), $x(0) = 0$ can be presented as

$$x(t) = \sum_{0 < \tau_i \leq t} G(t, \tau_i) \alpha_i.$$

The right hand side of this formula for any $t$ defines a bounded linear operator acting from $l_\infty$ to $R$ since

$$|x(t)| \leq \sum_{0 < \tau_i \leq t} |G(t, \tau_i)| \|\alpha\|_{l_\infty},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots) \in l_\infty$. By the hypothesis of the theorem for any sequence $\alpha \in l_\infty$ the solution $x(t)$ is bounded. Therefore the uniform boundedness principle implies that there exists $k > 0$ such that

$$\sum_{0 < \tau_i \leq t} G(t, \tau_i) \alpha_i \leq k \|\alpha\|_{l_\infty}.$$
Without loss of generality we can assume $k > 1$. Since $G(t, s) \geq 0$ then

$$\sum_{0 \leq \tau_i \leq t} G(t, \tau_i) \leq k.$$ 

By Lemma 1 $G(t, \tau_i) \geq X(t)X^{-1}(\tau_i)$. Thus for $t = \tau_2$

$$G(\tau_2, \tau_2) + G(\tau_2, \tau_1) = 1 + G(\tau_2, \tau_1) \leq k$$

implies

$$\frac{X(\tau_2)}{X(\tau_1)} \leq k - 1,$$

therefore $X(\tau_2) \leq (k - 1)X(\tau_1)$. By assuming

$$X(\tau_i) \leq X(\tau_1)(k - 1)^{i-1}/k^{i-2}, \quad (14)$$

$2 \leq i \leq j$, we obtain

$$k \geq \sum_{i=1}^{j+1} G(\tau_{j+1}, \tau_i) \geq 1 + \sum_{i=0}^{j-1} \frac{X(\tau_{j+1})}{X(\tau_{i+1})} \geq$$

$$\geq 1 + \frac{X(\tau_{j+1})}{X(\tau_1)} \left[ 1 + \sum_{i=0}^{j-2} \frac{k^i}{(k - 1)^{i+1}} \right] = 1 + \frac{X(\tau_{j+1})}{X(\tau_1)} \left[ 1 + \frac{k^{j+1} - 1}{(k - 1)^{j+1}} \right] =$$

$$= 1 + \frac{X(\tau_{j+1})}{X(\tau_1)} \left[ 1 + \frac{k^{j-1}}{(k - 1)^{j-1}} - 1 \right] = 1 + \frac{X(\tau_{j+1})}{X(\tau_1)} \frac{k^{j-1}}{(k - 1)^{j-1}}.$$

Hence

$$\frac{X(\tau_{j+1})}{X(\tau_1)} \leq \frac{(k - 1)(k - 1)^{j-1}}{k^{j-1}} = \frac{(k - 1)^j}{k^{j-1}}.$$ 

Thus by induction we obtain (14) for any $i = 3, 4, \ldots$

Let $\tau_j < t \leq \tau_{j+1}$. As $X(t) > 0$, $A_i(t) \geq 0$, then in $[\tau_j, \tau_{j+1})$ $X(t)$ does not increase. Since by (a7) $\tau_j < (j + 1)\sigma$, i.e. $j \geq t/\sigma - 1$, then

$$\ln X(t) \leq \ln X(\tau_j) \leq \ln X(\tau_1) - (t/\sigma - 2) \ln[k/(k - 1)] + \ln k.$$ 

Therefore (3) holds, with

$$\nu = \ln[k/(k - 1)]/\sigma, \quad N = \max \left\{ X(\tau_1)k^3/(k - 1)^2, \sup_{0 \leq t \leq \tau_1} \exp(\nu t)X(t) \right\}.$$
Theorem 3. Suppose the hypotheses of Theorem 2 and (a5), (a9) hold. Then there exist positive constants $N$ and $\nu$ such that

$$| G(t, s) | \leq N \exp[-\nu(t - s)].$$

(15)

Proof. Similar to the proof of Theorem 2 we obtain

$$0 \leq G(t, s) \leq N_s \exp[-\nu(t - s)], \quad \nu = \ln[k/(k - 1)]/\sigma,$$

$N_s = \max\{G(\tau_p, s)k^3/(k - 1)^2, \quad \sup_{s \leq t \leq \tau_p} \exp(\nu(t - s))G(t, s)\}$,

where $\tau_p$ is the least of $\tau_j > s$. Hence $\tau_p - s \leq \sigma$ and

$$\sup_{s \leq t \leq \tau_p} \exp(\nu(t - s))G(t, s) \leq \exp(\nu\sigma) \sup_{s \leq t \leq \tau_p} G(t, s).$$

In Lemma 3.2 of the paper [8] the following estimate is established

$$| G(t, s) | \leq (1 + | B_p |) \exp \left\{ \int_s^t \sum_{i=1}^m | A_i(\zeta) | d\zeta \right\},$$

$\tau_{p-1} < s \leq t \leq \tau_p$. Therefore

$$G(t, s) \leq (1 + M) \exp[mQ(t - s)] \leq (1 + M) \exp(mQ\sigma)$$

for $\tau_{p-1} < s \leq t \leq \tau_p$. Hence by assuming

$$N = \max \left\{ \frac{(1 + M) \exp(mQ\sigma)k^3}{(k - 1)^2}, (1 + M) \exp[\sigma(\nu + mQ)] \right\},$$

we obtain the estimate (15) for $G(t, s)$.

Theorem 4. Suppose (a1)-(a9) are satisfied, $A_i(t) \geq 0$, $G(t, s) \geq 0$, $X(t) > 0$ and the solution of (10), $t \geq 0$, (11), $\xi > 0$, (9), $x(0) = 0$ is bounded on $[0, \infty)$ for any $\alpha = \{\alpha_i\}_{i=1}^\infty \in \mathbb{L}_\infty$.

(a) If $\sup | \alpha_i | < \infty$, $\sup_{t \geq 0} | r(t) | < \infty$, then any solution of (8), (9) is bounded on the half-line $[0, \infty)$.

(b) if $\lim_{n \to \infty} | \alpha_n | = 0$, $\lim_{t \to \infty} | r(t) | = 0$, then for any solution $x$ of (8), (9) $\lim_{t \to \infty} | x(t) | = 0$;

(c) if there exist positive constants $P$ and $\lambda$ such that $| \alpha_n | \leq Pe^{-\lambda n}$ and $| r(t) | \leq Pe^{-\lambda t}$, then for any solution $x$ of (8), (9) there exist $P_0 > 0$, $\lambda_0 > 0$ such that $| x(t) | \leq P_0 e^{-\lambda_0 t}$.
Proof. The similar result for impulsive equations without delay is obtained in [4]. Comparing solution representations (6) and (13) we obtain that only the terms

$$\int_0^t G(t, s)A_i(s)\varphi(h_i(s))ds, \quad \varphi(\xi) = 0, \xi \geq 0,$$

have to be estimated. Since by (a8) $\varphi(h_i(s)) = 0$, $s \geq \delta$, then by Theorem 3

$$|\int_0^t G(t, s)A_i(s)\varphi(h_i(s))ds| \leq \int_0^\delta |G(t, s)| |A_i(s)| |\varphi(h_i(s))| \, ds \leq$$

$$\leq \sup_{s \in [0, \delta]} |A_i(s)| \sup_{s < 0} |\varphi(s)| \int_0^\delta Ne^{-\nu(t-s)} \, ds \leq \sup_{s \in [0, \delta]} |A_i(s)| \sup_{s < 0} |\varphi(s)| \frac{N}{\nu} e^{\nu \delta} e^{-\nu t},$$

which is obviously bounded, tends to zero as $t$ tends to infinity and has an exponential estimate.

Remark. (c) yields that under the hypotheses of Theorem 4 the equation (10), (12) is exponentially stable, i.e. for its solution $x$ the following inequality holds

$$|x(t)| \leq Ne^{-\nu t} \left(|x(0)| + \sup_{s < 0} |\varphi(s)| \right).$$

All the hypotheses of Theorems 2-4 are easily verified except the conditions $G(t, s) \geq 0$, $X(t) > 0$. For $G(t, s)$ the representation (7) is not valid. However sufficient conditions for $C(t, s)$ being positive are known [10,11], where $C(t, s)$ is a fundamental function of the equation (8) without impulses. For instance, $C(t, s)$ is positive if

$$\sum_{i=1}^m \int_{h^+_i(t)}^t A^+_i(s)ds \leq 1/e,$$

(16)

where $a^+ = \max\{a, 0\}$.

We study the following problem: is $G(t, s)$ positive for $C(t, s)$ being positive? Generally speaking it is not true. For an ordinary differential equation this assertion is valid if and only if $B_j > 0$, $j = 1, 2, \ldots$. For an impulsive delay differential equation the positiveness of $C(t, s)$, $B_j$, $j = 1, 2, \ldots$, does not imply the positiveness of $G(t, s)$.
Example. By (16) the equation

\[ \dot{x}(t) + x(t - \frac{1}{3}) = f \]

has \( C(t, s) > 0 \). Consider this equation with impulsive conditions

\[ x\left(\frac{j}{3}\right) = \frac{1}{6} x\left(\frac{j}{3} - 0\right), \quad j = 1, 2, \ldots. \]

Then \( G(t, 0) = 1, \quad t \in [0, \frac{1}{3}) \), \( G(t, 0) = \frac{1}{6} - (t - \frac{1}{3}), \quad t \in [\frac{1}{3}, \frac{2}{3}) \). Thus \( G(t, 0) < 0 \) for \( \frac{1}{2} < t < \frac{2}{3} \).

In many cases we can construct the fundamental function \( C(t, s) \) of the non-impulsive equation (8). Now we will obtain a relation connecting the fundamental function of the impulsive equation \( G(t, s) \), with \( C(t, s) \). The corresponding relation for impulsive equations without delay is [5]

\[
G(t, s) = \begin{cases} 
C(t, s), & \tau_i \leq t, s < \tau_{i+1}, \\
C(t, \tau_i) \left[ \prod_{j=k+1}^{\infty} B_j C(\tau_j, \tau_{j-1}) \right] B_k C(\tau_k, s), & \tau_{k-1} \leq s < \tau_k \leq \tau_i \leq t < \tau_{i+1}.
\end{cases}
\] (17)

Let \( k \) and \( j \) be positive integers, \( k \leq j \). Denote by \( \Omega_{k,j} \) the set of all non-empty ordered subsets \( e = \{n_1, \ldots, n_i\} \) of the set \( \{k, k+1, \ldots, j-1, j\} \), with natural order. Denote by \( \max(e) = n_i \) and \( \min(e) = n_1 \) a minimal and a maximal elements of \( e \) correspondingly. Let for any \( e \in \Omega_{k,j} \) a set \( \Lambda_e \) be the corresponding set of pairs \( \{(n_i, n_i-1), \ldots, (n_2, n_1)\} \). \( \Lambda_e = \emptyset \) if \( e \) is a one-point set.

Theorem 5 Suppose (a1)-(a4) hold and \( C(t, s) \) is a fundamental function of the non-impulsive delay differential equation (8).

Then \( G(t, s) = C(t, s), \quad \tau_i \leq t, s < \tau_{i+1}, \)

\[
G(t, s) = C(t, s) + \sum_{e \in \Omega_{k,i}} C(\tau_{\max(e)}(t)) \left[ B_{n_p} - 1 \right] C(\tau_{n_p}, \tau_{n_p-1}) \prod_{(n_p, n_{p-1}) \in \Lambda_e} \left( B_{n_{p-1}} \right) C(\tau_{\min(e)}, s), \quad \tau_{k-1} \leq s < \tau_k \leq \tau_i \leq t < \tau_{i+1}.
\] (18)

Here \( \prod_{(n_p, n_{p-1}) \in \Lambda_e} = 1 \) if \( e \) is a one-point set.
Proof. For $s < t < \tau_1$ solution $G(t, s)$ of the impulsive equation and $C(t, s)$ of the equation without impulses obviously coincide. If $t = \tau_1$, then

$$G(\tau_1, s) = B_1 C(\tau_1, s).$$

Let $0 \leq s < \tau_1 < t < \tau_2$. Then the solution $G(t, s)$ of the impulsive equation (10), $t \geq s$, (12) can be treated as the solution of the equation (10) on the segment $[\tau_1, \tau_2]$ with the initial value $G(\tau_1, s) = B_1 C(\tau_1, s)$ and the initial function $C(t, s)$. By Theorem 1

$$G(t, s) = C(t, \tau_1) B_1 C(\tau_1, s) - \sum_{i=1}^{m} \int_{\tau_1}^{t} C(t, \xi) A_i(\xi) C(h_i(\xi), s) d\xi, \quad (19)$$

where $C(h_i(\xi), s) = 0$, if $h_i(\xi) > \tau_1$.

On the other hand, $C(t, s)$ for $0 \leq s < \tau_1 < t < \tau_2$ can also be treated as a solution of (10) on the segment $[\tau_1, \tau_2]$ with the same initial function $C(\zeta, s), \zeta \in [s, \tau_1)$ and the initial value $C(\zeta, s)$. Therefore

$$C(t, s) = C(t, \tau_1) C(\tau_1, s) - \sum_{i=1}^{m} \int_{\tau_1}^{t} C(t, \xi) A_i(\xi) C(h_i(\xi), s) d\xi. \quad (20)$$

By subtracting (19) from (18) one obtains

$$G(t, s) - C(t, s) = C(t, \tau_1)(B_1 - 1) C(\tau_1, s),$$

hence for $0 \leq s < \tau_1 \leq t < \tau_2$

$$G(t, s) = C(t, s) + C(t, \tau_1)(B_1 - 1) C(\tau_1, s).$$

In fact the latter formula defines how the fundamental function changes when it overcomes the point $\tau_1$. Let $G_i(t, s)$ be a function satisfying the homogeneous equation (10), $t \geq s$, and the impulsive conditions

$$G_i(\tau_i, s) = B_i G_i(\tau_i - 0, s), \quad i = 1, \ldots, l - 1, \quad \tau_i > s.$$ 

Obviously $G_i(t, s) = G(t, s)$, $t < \tau_i$. Then for $0 \leq s < \tau_1 \leq t < \tau_{l+1}$ we similarly obtain

$$G(t, s) = G_i(t, s) + G_i(t, \tau_1)(B_1 - 1) G_i(\tau_1, s). \quad (21)$$

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We prove the equality (18) by induction in \( l \). For \( l = 1 \) this equality was obtained above.

Suppose that for \( 0 \leq \tau_{k-1} \leq s < \tau_k \leq \tau_{l-1} \leq t < \tau_l \) this equality is valid. Then \( G_l(t, s) \) for \( t \in [\tau_l, \tau_{l+1}) \) can be presented as

\[
G_l(t, s) = \sum_{e \in \Omega_{k,l-1}} C(t, \tau_{\max(e)}) \prod_{(n, n_{p-1}) \in \Lambda_e} (B_{n_p} - 1)C(\tau_{n_p}, \tau_{n_p-1}) (B_{\min(e)} - 1)C(\tau_{\min(e)}, s).
\]

Let \( 0 \leq \tau_{k-1} \leq s < \tau_k \leq \tau_l \leq t < \tau_{l+1} \).

Obviously \( \Omega_{k,l} = \Omega_{k,l-1} \cup \{l\} \cup \{e, l\} \mid e \in \Omega_{k,l-1} \) and \( G_l(t, \tau_l) = C(t, \tau_l) \).

Therefore by substituting this in (21) we obtain

\[
G(t, s) = G_l(t, s) + G_l(t, \tau_l)(B_{l-1} - 1)G_l(\tau_l, s) = C(t, s) +
\]

\[
+ \sum_{e \in \Omega_{k,l-1}} C(t, \tau_{\max(e)}) \prod_{(n, n_{p-1}) \in \Lambda_e} (B_{n_p} - 1)C(\tau_{n_p}, \tau_{n_p-1}) (B_{\min(e)} - 1)C(\tau_{\min(e)}, s) +
\]

\[
+C(t, \tau_l)(B_{l-1} - 1)C(\tau_l, s) + \sum_{e \in \Omega_{k,l-1}} C(t, \tau_l)(B_{l-1} - 1)C(\tau_l, \tau_{\max(e)}) \times
\]

\[
\prod_{(n, n_{p-1}) \in \Lambda_e} (B_{n_p} - 1)C(\tau_{n_p}, \tau_{n_p-1}) (B_{\min(e)} - 1)C(\tau_{\min(e)}, s) = C(t, s) +
\]

\[
+ \sum_{e \in \Omega_{k,l}} C(t, \tau_{\max(e)}) \prod_{(n, n_{p-1}) \in \Lambda_e} (B_{n_p} - 1)C(\tau_{n_p}, \tau_{n_p-1}) (B_{\min(e)} - 1)C(\tau_{\min(e)}, s).
\]

The proof of the theorem is complete.

**Corollary 1.** Suppose the fundamental function of the non-impulsive equation \( C(t, s) > 0, \ B_i \geq 1, \ i = 1, 2, \ldots \) Then the fundamental function of the impulsive equation \( G(t, s) > 0 \).

**Corollary 2.** Suppose (16) holds, \( B_j \geq 1 \). Then the fundamental solution is non-oscillating.

**Remarks.**

1. By the straightforward verification one can prove that (18) implies (17) in the case of an impulsive differential equation without delay.

2. In the vector case Theorem 5 is valid if in (18) we change unit by the unit matrix.

3. Corollary 2 generalizes the result obtained in [7] on the existence of a non-oscillating solution of an impulsive equation with constant delay and constant coefficients.
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