Three-Weight Ternary Linear Codes from a Family of Monomials

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Abstract

Based on a generic construction, two classes of ternary three-weight linear codes are obtained from a family of power functions, including some APN power functions. The weight distributions of these linear codes are determined through studying the properties of some exponential sum related to the proposed power functions.

Index Terms linear code, weight distribution, exponential sum, quadratic form.

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1 Introduction

Throughout this paper, we assume that $p$ is an odd prime. For a positive integer $m$, let $\mathbb{F}_{p^m}$ denote the finite field with $p^m$ elements, $\mathbb{F}_{p^m}^* = \mathbb{F}_{p^m}\setminus\{0\}$ and $\alpha$ a primitive element of $\mathbb{F}_{p^m}$. Let $m$ and $k$ be two positive integers such that $\frac{m}{\gcd(k,m)} \geq 3$ is odd. Under these conditions, let $d$ be a positive integer satisfying

$$d(p^k + 1) \equiv 2 \pmod{p^m - 1}. \quad (1)$$

Let

$$D(a) = \{ x \in \mathbb{F}_{p^m}^* | \text{Tr}_{1}^{m}(x^d) = a \}, \quad a \in \mathbb{F}_p, \quad (2)$$

where $\text{Tr}_{1}^{m} (\cdot)$ is the trace function from $\mathbb{F}_{p^m}$ to $\mathbb{F}_p$ [16]. Assume $D(a)$ contains $l_a$ different elements $\beta_1, \beta_2, \cdots, \beta_{l_a}$. For each $a \in \mathbb{F}_p$, we define a linear code of length $l_a$ over $\mathbb{F}_p$ by

$$C_{D(a)} = \{ (\text{Tr}_{1}^{m}(\beta_1 x), \text{Tr}_{1}^{m}(\beta_2 x), \cdots, \text{Tr}_{1}^{m}(\beta_{l_a} x)) : x \in \mathbb{F}_{p^m} \} \quad (3)$$

and call $D(a)$ the defining set of this code. In this paper, we study the linear codes $C_{D(a)}$ defined by (1)-(3) and prove the following two theorems.

Theorem 1 For $p = 3$, $C_{D(0)}$ defined in (3) is a $[3^{m-1} - 1, m]$ linear code with weight distribution given in Table 1, where $e = \gcd(k,m)$.

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Table 1: Weight distribution of $C_{D(0)}$ in Theorem 1

| Weight | Frequency |
|--------|-----------|
| $0$    | $1$       |
| $2 \cdot (3^{m-2} - 3^{\frac{m+3}{2} - 2})$ | $3^{m-e} + 3^{\frac{m-e}{2}}$ |
| $2 \cdot (3^{m-2} + 3^{\frac{m+3}{2} - 2})$ | $3^{m-e} - 3^{\frac{m-e}{2}}$ |
| $2 \cdot 3^{m-2}$ | $3^{m-1} - 2 \cdot 3^{m-e}$ |

Table 2: Weight distribution of $C_{D(a)}$ in Theorem 2

| Weight | Frequency |
|--------|-----------|
| $0$    | $1$       |
| $2 \cdot 3^{m-2} - 3^{\frac{m+3}{2} - 2}$ | $3^{m-e} - 3^{\frac{m-e}{2}}$ |
| $2 \cdot 3^{m-2} + 3^{\frac{m+3}{2} - 2}$ | $3^{m-e} + 3^{\frac{m-e}{2}}$ |
| $2 \cdot 3^{m-2}$ | $3^{m-1} - 2 \cdot 3^{m-e}$ |

**Theorem 2** For $p = 3$ and $a \in \mathbb{F}_3^*$, (i) if $e = \gcd(k,m)$ is even, or $e = \gcd(k,m)$ is odd and $d \equiv 1 \pmod{3^e - 1}$, then $C_{D(a)}$ defined in (3) is a $[3^{m-1} - 1, m]$ linear code with weight distribution given in Table 2. (ii) if $e = \gcd(k,m)$ is odd and $d \equiv 1 + \frac{3^{e-1}}{2} \pmod{3^e - 1}$, then $C_{D(a)}$ is a $[3^{m-1} + (-1)^{\frac{m-1}{2}} 3^{\frac{m-1}{2}} (\frac{a}{3})], m]$ linear code and its possible nonzero weights are

\[
\begin{align*}
2 \cdot (3^{m-2} + (-1)^{\frac{m-1}{2}} 3^{\frac{m-3}{2}} (\frac{a}{3})) , \\
2 \cdot (3^{m-2} + (-1)^{\frac{m-1}{2}} 3^{\frac{m-3}{2}} (\frac{a}{3})) \pm 2 \cdot 3^{m-3} , \\
2 \cdot (3^{m-2} + (-1)^{\frac{m-1}{2}} 3^{\frac{m-3}{2}} (\frac{a}{3})) \pm (\frac{3^{m-3} - 3^{\frac{m+1-4}{2}}}{4}) , \\
2 \cdot (3^{m-2} + (-1)^{\frac{m-1}{2}} 3^{\frac{m-3}{2}} (\frac{a}{3})) \pm (\frac{3^{m-3} + 3^{\frac{m+1-4}{2}}}{4}) , \\
2 \cdot (3^{m-2} + (-1)^{\frac{m-1}{2}} 3^{\frac{m-3}{2}} (\frac{a}{3})) \pm (\frac{3^{m-3} - 3^{\frac{m+1-4}{2}}}{4}) , \\
2 \cdot (3^{m-2} + (-1)^{\frac{m-1}{2}} 3^{\frac{m-3}{2}} (\frac{a}{3})) \pm (\frac{3^{m-3} + 3^{\frac{m+1-4}{2}}}{4}) ,
\end{align*}
\]

where $(\frac{a}{3})$ denotes the quadratic character of $\mathbb{F}_3$.

The idea of constructing linear codes from a defining set $D$ was introduced in [1, 2]. Very recently several defining sets have been considered to generate linear codes with few weights [3, 4, 5, 6, 7]. The defining sets therein are constructed from Bent functions and quadratic functions. The defining set in this paper is constructed from the general power functions with $d$ defined in [11], which covers several APN power functions (see Corollary 2). The proposed linear codes also have applications in secret sharing [10, 11], authentication codes [9], association schemes [8], and strongly regular graphs [8].
The remainder of this paper is organized as follows. Section 2 gives some preliminaries and notation, including some useful lemmas. In Section 3 we calculate the weight distributions of two cyclic codes with three nonzero weights. Section 4 concludes the study.

2 Preliminaries

Let $e$ be a divisor of a positive integer $m$. The trace function from $\mathbb{F}_{p^m}$ to $\mathbb{F}_{p^e}$ is defined as

$$\text{Tr}_e^m(x) = \sum_{i=0}^{\frac{m}{e} - 1} x^{p^{ei}}.$$  

It is well known that $\text{Tr}_e^m(x) = \text{Tr}_1^e(\text{Tr}_e^m(x))$ for any $x \in \mathbb{F}_{p^m}$.

Throughout this paper we assume that $q = p^e$ and $h = \frac{m}{e}$. Then $\mathbb{F}_{p^m} = \mathbb{F}_q^h$. By identifying the finite field $\mathbb{F}_q^h$ with the $h$-dimensional $\mathbb{F}_q$-vector space $\mathbb{F}_q^h$, a function from $\mathbb{F}_q^h$ to $\mathbb{F}_q$ can be regarded as an $h$-variable polynomial over $\mathbb{F}_q$. In this sense, a function $f(x)$ from $\mathbb{F}_q^h$ to $\mathbb{F}_q$ is called a quadratic form over $\mathbb{F}_q$ if it can be written as a homogeneous polynomial in $\mathbb{F}_q[x_1, x_2, \ldots, x_h]$ of degree 2 as

$$f(x_1, \ldots, x_h) = \sum_{1 \leq i \leq j \leq h} a_{ij} x_i x_j.$$  

The rank $r$ of the quadratic form $f(x)$ is defined as the codimension of the $\mathbb{F}_q$-vector space

$$W = \{z \in \mathbb{F}_q^h \mid f(x + z) = f(x) \text{ for all } x \in \mathbb{F}_q^h\},$$

namely, $|W| = q^{h-r}$.

For each nonzero quadratic form $f(x)$ from $\mathbb{F}_q^h$ to $\mathbb{F}_q$, there exists an $h \times h$ symmetric matrix $A$ such that $f(x) = X^T AX$, where $X$ is written as a column vector and its transpose is $X^T = (x_1, x_2, \ldots, x_h) \in \mathbb{F}_q^h$. By Theorem 6.21 of [10], there exists a nonsingular matrix $B$ of order $h$ such that $B^T AB$ is a diagonal matrix diag $(a_1, a_2, \ldots, a_r, 0, \ldots, 0)$, where $r$ is the rank of the quadratic form $f(x)$ and $a_1, a_2, \ldots, a_r \in \mathbb{F}_q^*$. By the nonsingular linear substitution $X = BY$ with $Y^T = (y_1, y_2, \ldots, y_h)$, the quadratic form $f(x)$ is transformed into a diagonal form as

$$f(x) = Y^T B^T ABY = \sum_{i=1}^{r} a_i y_i^2.$$  

(5)

Given a positive integer $t$, let $\eta^{(t)}(\cdot)$ and $\chi^{(t)}(\cdot)$ denote the quadratic character and the canonical
additive character of $\mathbb{F}_{p^t}$, respectively. Namely,

$$\eta^{(t)}(x) = \begin{cases} 
1, & \text{if } x \text{ is a square in } \mathbb{F}_{p^t}^*, \\
-1, & \text{if } x \text{ is a non-square in } \mathbb{F}_{p^t}^*, \\
0, & \text{if } x = 0,
\end{cases} \quad (6)$$

and

$$\chi^{(t)}(x) = \omega_p \text{Tr}_1^t(x), \quad x \in \mathbb{F}_{p^t}, \quad (7)$$

where $\omega_p = e^{\frac{2\pi \sqrt{-1}}{p}}$ is a primitive complex $p$-th root of unity. The following two results related to Gaussian sums are useful in the sequel.

**Lemma 1** ([16, Theorem 5.15 and Theorem 5.33]) Let $t$ be a positive integer and $\eta^{(t)}$ and $\chi^{(t)}$ defined in (6) and (7). For any element $a$ in $\mathbb{F}_{p^t}^*$,

$$\sum_{x \in \mathbb{F}_{p^t}} \omega_p \text{Tr}_1^t(ax^2) = \eta(a)G(\eta^{(t)}, \chi^{(t)}),$$

where $G(\eta^{(t)}, \chi^{(t)}) = \sum_{x \in \mathbb{F}_{p^t}^*} \eta^{(t)}(x)\chi^{(t)}(x)$ is the Gaussian sum given by

$$G(\eta^{(t)}, \chi^{(t)}) = \begin{cases} 
(-1)^{t-1}p^\frac{1}{2}, & \text{if } p \equiv 1 \pmod{4}, \\
(-1)^{t-1}(\sqrt{-1})^tp^\frac{1}{2}, & \text{if } p \equiv 3 \pmod{4},
\end{cases} \quad (8)$$

From Lemma 1 one can derive the following lemma immediately.

**Lemma 2** ([23, Lemma 1]) Let $m = eh$ and $f(x)$ be a quadratic form over $\mathbb{F}_{p^e}$ with rank $r$ and the diagonal form given in (5). Then,

$$\sum_{x \in \mathbb{F}_{p^e}} \omega_p \text{Tr}_1^r(f(x)) = \begin{cases} 
\eta^{(e)}(\Delta)(-1)^{(e^{-1})r}p^{m-\frac{r}{2}}, & \text{for } p \equiv 1 \pmod{4}, \\
\eta^{(e)}(\Delta)(-1)^{(e^{-1})r}(\sqrt{-1})^rp^{m-\frac{r}{2}}, & \text{for } p \equiv 3 \pmod{4},
\end{cases}$$

where $\Delta = a_1a_2\cdots a_r$ with $a_1, a_2, \ldots, a_r$ in (5).

From Lemma 2 it follows that for any quadratic form $f(x)$ over $\mathbb{F}_{q}$ with rank $r$,

$$\sum_{x \in \mathbb{F}_{p^e}} \omega_p \text{Tr}_1^r(f(x)) = \eta^{(e)}(\chi^r) \sum_{x \in \mathbb{F}_{p^e}} \omega_p \text{Tr}_1^r(f(x)), \quad \forall \lambda \in \mathbb{F}_{p^r}^e. \quad (9)$$

Furthermore, if $\lambda$ is a non-square in $\mathbb{F}_{p^r}$, one has

$$\sum_{x \in \mathbb{F}_{p^e}} \omega_p \text{Tr}_1^r(f(x)) + \sum_{x \in \mathbb{F}_{p^e}} \omega_p \text{Tr}_1^r(\lambda f(x)) = \begin{cases} 
\pm 2p^{m-\frac{r}{2}}, & r \text{ even}, \\
0, & r \text{ odd},
\end{cases}$$

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Table 3: Value distribution for $T(u,v)$

| Value               | Frequency (each)                                      |
|---------------------|------------------------------------------------------|
| $p^m$               | $1$                                                  |
| $\pm \epsilon p^m$ | $m$                                                   |
| $p^{m+1}$           | $(p^m-1)(p^{m-s}+p^{m-s+1})$                         |
| $-p^{m+s}$          | $(p^m-1)(p^{m-s}+p^{-s})$                            |
| $\pm \epsilon p^{m+s}$ | $(p^m-1)(p^{m-s+1})$                              |

Let $m, k$ be two positive integers such that $\frac{m}{e}$ is an odd integer larger than 1, where $e = \gcd(k, m)$. Define

$$Q_{u,v}(x) = \text{Tr}_e \left( u x^{p^k+1} + v x^2 \right), \quad u, v \in \mathbb{F}_{p^m},$$

and the associated exponential sum

$$T(u,v) = \sum_{x \in \mathbb{F}_{p^m}} \omega_p \text{Tr}^e(Q_{u,v}(x)) = \sum_{x \in \mathbb{F}_{p^m}} \omega_p \text{Tr}^e \left( u x^{p^k+1} + v x^2 \right).$$

Note that when $u, v$ are not simultaneously zero, $Q_{u,v}(x)$ is a nonzero quadratic form over $\mathbb{F}_{p^e}$. The properties of $Q_{u,v}(x)$ and the associated exponential sum $T(u,v)$ have been intensively studied in [15 21 23 17 18 14 22 24]. The following results are useful in the sequel.

**Lemma 3** Let $Q_{u,v}(x)$ be the quadratic form defined by (10), where $(u,v) \in \mathbb{F}_{p^m}^2 \setminus \{(0,0)\}$ and $h = \frac{m}{e}$.

(i) ([23, Lemma 2]) The rank of $Q_{u,v}(x)$ is $h$, $h-1$ or $h-2$. Especially, both $Q_{u,0}(x)$ with $u \in \mathbb{F}_{p^m}^*$ and $Q_{0,v}(x)$ with $v \in \mathbb{F}_{p^m}^*$ have rank $h$.

(ii) ([18, Lemma 6] and [24, Lemma 3.3]) For any given $(u,v) \in \mathbb{F}_{p^m}^2 \setminus \{(0,0)\}$, at least one of $Q_{u,v}(x)$ and $Q_{u,-v}(x)$ has rank $h$.

**Lemma 4** ([23, Theorem 1]) Let $T(u,v)$ be the exponential sum defined in (11) and $\epsilon = \sqrt{\eta^{(e)}(-1)}$. The value distribution of $T(u,v)$ as $(u,v)$ runs through $\mathbb{F}_{p^m}^2$ is given in Table 3. Moreover, $T(u,v) = p^m$ if and only if $(u,v) = (0,0)$.

In addition, when $p^e \equiv 3 \pmod{4}$, the value distribution of

$$\hat{T}(u,v) = (T(u,v), T(-u,v))$$

as $(u,v)$ runs through $\mathbb{F}_{p^m}^* \times \mathbb{F}_{p^m}^*$ can be settled in the following lemma.
Proof: When given in Table 4, where $c$ is a two distinct integers distribution of $\hat{p}$ and $d$.

Combining this equality and Lemma 1, the value distribution of $\hat{p}$ runs through $F_{p^m} \times F_{p^m}$ is given by

$$c_1 = \left\{ \begin{array}{ll} \sqrt[p^m]{\eta^{t^2}}(-1)p^{\frac{m+1}{2}}, & i = 0, 2, \\
p^{\frac{m-1}{2}}, & i = 1.
\end{array} \right. \quad (12)$$

**Lemma 5** For $p^e \equiv 3 \pmod{4}$, the value distribution of $\hat{T}(u, v)$ as $(u, v)$ runs through $F_{p^m}^* \times F_{p^m}^*$ is given in Table 4 where $c_i$, $i = 0, 1, 2$, are given by

$$c_1 = \left\{ \begin{array}{ll} \sqrt[p^m]{\eta^{t^2}}(-1)p^{\frac{m+1}{2}}, & i = 0, 2, \\
p^{\frac{m-1}{2}}, & i = 1.
\end{array} \right. \quad (12)$$

Proof: When $p^e \equiv 3 \pmod{4}$, $-1$ is a nonsquare in $F_{p^m}^*$. Then, by Theorem 1 of [14], the value distribution of $\hat{T}(u, v)$ as $(u, v)$ runs through $F_{p^m} \times F_{p^m}$ can be obtained. Note that $\hat{T}(0, 0) = (p^m, p^m)$.

Since $\frac{m}{\gcd(k, m)}$ is odd, $\gcd(p^k + 1, p^m - 1) = 2$. Then, for any $u \in F_{p^m}$, we have

$$\sum_{x \in F_{p^m}} \omega_p^m \left( u x^{p^k+1} \right) = \sum_{x \in F_{p^m}} \omega_p^m \left( u x^2 \right).$$

Combining this equality and Lemma [14] the value distribution of $\hat{T}(u, 0)$ as $u$ runs through $F_{p^m}^*$ can be determined. Similarly, the value distribution of $\hat{T}(0, v)$ as $v$ runs through $F_{p^m}^*$ can also be derived. Then, a straightforward calculation gives the desired result. \qed

**Lemma 6** ([14] Lemma 5) Given $m$ and $k$ satisfying the condition that $\frac{m}{\gcd(k, m)} \geq 3$ is odd, there are two distinct integers $d_1, d_2 \in \mathbb{Z}_{p^m - 1}$ satisfying [14], of which one satisfies $d \equiv 1 \pmod{p^e - 1}$, and the other satisfies $d \equiv 1 + \frac{p^e - 1}{2} \pmod{p^e - 1}$.

In the sequel, we always assume that $\theta$ is a fixed non-square in $F_{p^e}$. Then $\theta$ is also a non-square in $F_{p^m}$ since $\frac{m}{\gcd(m, k)}$ is odd.
Lemma 7 Denote by $S$ the set of all square elements in $\mathbb{F}^*_{p^m}$. When $x$ runs through $\mathbb{F}^*_{p^m}$ once, $x^{p^{k+1}}$ runs through $S$ twice. Moreover, $\mathbb{F}^*_{p^m} = S \cup \theta S$, where $\theta S = \{\theta x \mid x \in S\}$.

3 The weight distribution of $C_{D(a)}$

In this section, we will give the proofs of Theorems 1 and 2. Before we begin the proofs, we shall make some preparations.

Let $r_{u,v}$ denote the rank of the quadratic form $Q_{u,v}(x)$ defined in (10). It follows from (9) that for any $\lambda \in \mathbb{F}^*_{p^e}$,

$$T(\lambda u, \lambda v) = \eta(\lambda)T(u, v) = \begin{cases} -T(u, v), & \text{if } \lambda \text{ is a non-square, } r_{u,v} \text{ is odd,} \\ T(u, v), & \text{otherwise.} \end{cases}$$

(13)

This fact will be heavily used in the sequel.

Proposition 1 Let $T(u, v)$ be the exponential sum defined in (11). Then, for each $\varepsilon \in \{1, -1\}$, the number of $u \in \mathbb{F}^*_{p^m}$ such that $T(u, 1) = \varepsilon p^m$ is equal to

$$\frac{p^m - \varepsilon p^{(m-\varepsilon)/2}}{2}.$$ 

Proof: By Lemmas 2 and 3 if $T(u, v) = \varepsilon p^{m+e}$, the rank of $\text{Tr}_e(u v^{p^{k+1}} + v x^2)$ is $\frac{m - e}{2}$ and $(u, v)$ belongs to $\mathbb{F}^*_{p^m} \times \mathbb{F}^*_{p^m}$. For convenience, with the notation introduced in Lemma 7, we define the following notation:

$$M_{1,\varepsilon} = \left\{(u, v) \in \mathbb{F}^*_{p^m} \times S : T(u, v) = \varepsilon p^{m+e}\right\}, \quad M_{\theta,\varepsilon} = \left\{(u, v) \in \mathbb{F}^*_{p^m} \times \theta S : T(u, v) = \varepsilon p^{m+e}\right\},$$

$$N_{1,\varepsilon} = \left\{u \in \mathbb{F}^*_{p^m} : T(u, 1) = \varepsilon p^{m+e}\right\}, \quad N_{\theta,\varepsilon} = \left\{u \in \mathbb{F}^*_{p^m} : T(u, \theta) = \varepsilon p^{m+e}\right\},$$

where $\varepsilon \in \{1, -1\}$. Recall that $\mathbb{F}^*_{p^m} = S \cup \theta S$. By Lemma 4 one has

$$|M_{1,\varepsilon}| + |M_{\theta,\varepsilon}| = (p^m - 1)\frac{p^m - \varepsilon p^{(m-\varepsilon)/2}}{2}, \quad \varepsilon \in \{1, -1\}.$$ (14)

In the following, we will prove two statements:

(i) $|N_{1,\varepsilon}| = |N_{\theta,\varepsilon}|$ for any $\varepsilon \in \{1, -1\}$;

(ii) $|M_{1,\varepsilon}| = \frac{p^{m-1}}{2}|N_{1,\varepsilon}|$, $|M_{\theta,\varepsilon}| = \frac{p^{m-1}}{2}|N_{\theta,\varepsilon}|$ for any $\varepsilon \in \{1, -1\}$.

For the first statement, let $u \in N_{1,\varepsilon}$, then we have $T(u, 1) = \varepsilon p^{m+e}$ and the rank $r_{u,1} = \frac{m - e}{\varepsilon}$ is even. It follows from (13) that

$$T(\theta u, \theta) = T(u, 1).$$
Thus, there is a one-to-one correspondence between these two sets and we have

\[ d \text{ when } \|

When

Proposition 2

\[ a \equiv \frac{m-1}{2} (\mod 4) \]

Therefore, \( u \in N_{1,\varepsilon} \) implies \( u \theta \in N_{0,\varepsilon} \). Similarly, we can prove the converse: if \( u \in N_{0,\varepsilon} \), then \( \frac{u}{\theta} \in N_{1,\varepsilon} \).

Thus, there is a one-to-one correspondence between these two sets and we have \( |N_{1,\varepsilon}| = |N_{0,\varepsilon}| \) for each \( \varepsilon \in \{1, -1\} \).

Now we prove the second statement. Let \((u, v) \in M_{1,\varepsilon}\), then \( v \) is a square element in \( \mathbb{F}_{p^m}^* \), and we have

\[ T(u, v) = T\left(\frac{u}{v^{(p^m+1)/2}}, 1\right) \quad (15) \]

From (15), for each fixed \( v \in S \), the number of \( u \in \mathbb{F}_{p^m}^* \) such that \( T(u, v) = \varepsilon p^{\frac{m-1}{2}} \) is equal to \( |N_{1,\varepsilon}| \).

Thus, we obtain \( |M_{1,\varepsilon}| = \frac{p^{m-1}}{2} |N_{1,\varepsilon}| \) for each \( \varepsilon \in \{1, -1\} \). Similarly, one has \( |M_{0,\varepsilon}| = \frac{p^{m-1}}{2} |N_{0,\varepsilon}| \) for each \( \varepsilon \in \{1, -1\} \).

The desired result follows from these two statements and (14). \( \square \)

**Proposition 2** Let \( d \) be the integer satisfying (11) and

\[ n_a = |\{x \in \mathbb{F}_{p^m} : \text{Tr}^m_1(x^d) = a\}|, \quad a \in \mathbb{F}_p. \quad (16) \]

When \( d \) satisfies \( d \equiv 1 (\mod p^e - 1) \),

\[ n_a = p^{m-1}; \]

When \( d \) satisfies \( d \equiv 1 + \frac{p^e-1}{2} (\mod p^e - 1) \),

\[
\begin{align*}
  n_a &= \begin{cases} 
    p^{m-1}, & \text{if } p^e \equiv 1 (\mod 4), \\
    p^{m-1}, & \text{if } a = 0 \text{ and } p^e \equiv 3 (\mod 4), \\
    p^{m-1} + (-1)^\frac{m-1}{2} \left(\frac{m}{2}\right) \eta(1)(a), & \text{if } a \neq 0 \text{ and } p^e \equiv 3 (\mod 4). 
  \end{cases}
\end{align*}
\]

**Proof.** Using the theory of exponential sums, one can express \( n_a \) as follows:

\[
\begin{align*}
  n_a &= \frac{1}{p} \sum_{x \in \mathbb{F}_{p^m}} \sum_{y \in \mathbb{F}_p} \omega_p^{\text{Tr}^m_1(x^d) - a} \\
  &= p^{m-1} + \frac{1}{2p} \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_{p^m}} \omega_p^{\text{Tr}^m_1(x^d) - a} \\
  &= p^{m-1} + \frac{1}{2p} \sum_{y \in \mathbb{F}_p} \omega^a \left[ \sum_{x \in \mathbb{F}_{p^m}} \omega_p^{\text{Tr}^m_1(x^2)} + \sum_{x \in \mathbb{F}_{p^m}} \omega_p^{\text{Tr}^m_1(\theta^d x^2)} \right] \\
  &= p^{m-1} + \frac{1}{2p} \sum_{y \in \mathbb{F}_p} \omega^a \left[ \eta(m)(y) + \eta(m)(\theta^d y) \right] G(\chi(m), \eta(m)) \\
  &= p^{m-1} + \frac{1}{2p} G(\chi(m), \eta(m)) \left[ 1 + \eta(m)(\theta^d) \right] \sum_{y \in \mathbb{F}_p} \omega^{-ay} \eta(m)(y) \\
  &= p^{m-1} + \frac{1}{2p} G(\chi(m), \eta(m)) \left[ 1 + \eta(m)(\theta^d) \right] \sum_{y \in \mathbb{F}_p} \omega^{-ay} \eta(e)(y),
\end{align*}
\]
where the third and the fifth equalities hold due to Lemmas 4 and 1 respectively, and the last equality holds since \( \eta^{(c)}(x) = \eta^{(c)}(x) \) for any \( x \in \mathbb{F}_p \). According to Lemma 3, we need to consider the following two cases.

**Case 1**: \( d \equiv 1 \pmod{p^c - 1} \). Then, \( \theta^d = \theta \) and \( \eta^{(m)}(\theta^d) = \eta^{(m)}(\theta) = -1 \). Therefore, \( n_a = p^{m-1} \) for any \( a \in \mathbb{F}_p \).

**Case 2**: \( d \equiv 1 + \frac{p^c - 1}{2} \pmod{p^c - 1} \). Then, \( \theta^d = -\theta \) and one has

\[
\begin{align*}
    n_a &= p^{m-1} + \frac{1}{2p^c} G(\chi^{(m)}, \eta^{(m)}) \left[ 1 + \eta^{(m)}(-\theta) \right] \sum_{y \in \mathbb{F}_p^*} \omega^{-ay\eta^{(c)}}(y) \\
    &= p^{m-1} - \frac{1}{2p} G(\chi^{(m)}, \eta^{(m)}) \left[ 1 - \eta^{(c)}(-1) \right] \sum_{y \in \mathbb{F}_p^*} \omega^{-ay\eta^{(c)}}(y).
\end{align*}
\]

We consider the following two subcases.

**Subcase 2.1**: \( p^c \equiv 1 \pmod{4} \). Then \( \eta^{(c)}(-1) = 1 \) and \( n_a = p^{m-1} \) for any \( a \in \mathbb{F}_p \).

**Subcase 2.2**: \( p^c \equiv 3 \pmod{4} \). Then \( \eta^{(c)}(-1) = -1 \) and \( c \) must be odd. Thus, \( \eta^{(c)}(y) = \eta^{(1)}(y) \) for any \( y \in \mathbb{F}_p \) and \( y \not\in \mathbb{F}_p^* \). Therefore, \( n_a = p^{m-1} \) if \( a = 0 \) and otherwise,

\[
n_a = p^{m-1} + \frac{1}{p} \eta^{(1)}(a) G(\chi^{(m)}, \eta^{(m)}) G(\chi^{(1)}, \eta^{(1)}).
\]

From (8) in Lemma 1, the desired result follows. 

In the proof of Proposition 2 we actually have calculated the value distribution of the following exponential sums.

**Corollary 1** Let \( S(a) = \sum_{y \in \mathbb{F}_p} \sum_{x \in \mathbb{F}_p^m} \omega_p^{y \langle \text{Tr}_p^m(x^d) - a \rangle} \) and \( n_a \) be defined in (10). Then,

\[
S(a) = p m_a - p^m.
\]

**Proposition 3** Let \( T(u,v) \) be defined in (11) and \( n_a \) given in Proposition 2. Define

\[
N(a,b) = \left| \{ x \in \mathbb{F}_p : \text{Tr}_p^m(x^d) = a \text{ and } \text{Tr}_p^m(bx) = 0 \} \right|, \quad a \in \mathbb{F}_p, \ b \in \mathbb{F}_p^m.
\]

If \( d \equiv 1 + \frac{p^c - 1}{2} \pmod{p^c - 1}, \ p^c \equiv 3 \pmod{4} \) and \( a \in \mathbb{F}_p \), we have

\[
N(a,b) = \frac{1}{p} n_a + \frac{1}{p^c} \sum_{y \in \mathbb{F}_p^*} \omega_p^{-ya} \sum_{z \in \mathbb{F}_p} T(zyb, y).
\]

(17)

(18)
Otherwise,

\[ N(a, b) = p^{m-2} + \frac{1}{2p} \sum_{z \in \mathbb{F}_p^*} \left( T(zb, 1) + T(\theta zb, \theta) \right) \left( \sum_{y \in \mathbb{F}_p^*} \omega_p^{-ay} \right), \quad (19) \]

where \( \theta \) is a non-square in \( \mathbb{F}_{p^n}^* \).

**Proof:** Using the theory of exponential sums, \( N(a, b) \) can be expressed as

\[
N(a, b) = \frac{1}{p} \sum_{x \in \mathbb{F}_{p^m}} \sum_{y \in \mathbb{F}_p} \omega_p^{y(\text{Tr}_m(x^d) - a)} \sum_{z \in \mathbb{F}_p^*} \omega_p^{\text{Tr}_m(bz)}
\]

\[
= \frac{1}{p} \sum_{y \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_{p^m}} \sum_{z \in \mathbb{F}_p^*} \omega_p^{\text{Tr}_m(\theta x^d + zbx) - ya}
\]

\[
= p^{m-2} + \frac{1}{p} \sum_{y \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_{p^m}} \text{Tr}_m(\theta x^d) - ya + \frac{1}{p} \sum_{y \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_{p^m}} \omega_p^{\text{Tr}_m(\theta x^d + zbx) - ya}
\]

\[
= p^{m-2} + \frac{1}{p} R(a, b)
\]

where \( S(a) \) is given by Corollary 1 and

\[
R(a, b) = \sum_{y \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_{p^m}} \omega_p^{-ya} \omega_p^{\text{Tr}_m(\theta x^d + zbx)}.
\]

By Lemma 7, \( R(a, b) \) can be represented as

\[
R(a, b) = \frac{1}{2} \sum_{y \in \mathbb{F}_p^*} \sum_{x \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_{p^m}} \left[ \omega_p^{\text{Tr}_m(\theta x^d + zbx^{k+1})} + \omega_p^{\text{Tr}_m(\theta x^d + zbx^{k+1} + yz^2)} \right].
\]

According to Lemma 8, the exponential sum \( R(a, b) \) will be investigated in the following two cases.

**Case 1:** \( \theta^d = \theta \). In this case, by Proposition 2, one has \( n_a = p^{m-1} \). Moreover,

\[
R(a, b) = \frac{1}{2} \sum_{y \in \mathbb{F}_p^*} \omega_p^{-ya} \sum_{x \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_{p^m}} \left[ \omega_p^{\text{Tr}_m(\theta x^d + zbx^{k+1})} + \omega_p^{\text{Tr}_m(\theta x^d + zbx^{k+1} + yz^2)} \right]
\]

\[
= \frac{1}{2} \sum_{y \in \mathbb{F}_p^*} \omega_p^{-ya} \sum_{x \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_{p^m}} \left[ \omega_p^{\text{Tr}_m(\theta x^d + zbx^{k+1} + yz^2)} + \omega_p^{\text{Tr}_m(\theta x^d + zbx^{k+1} + yz^2)} \right]
\]

\[
= \frac{1}{2} \sum_{y \in \mathbb{F}_p^*} \omega_p^{-ya} \sum_{x \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_{p^m}} \left[ \omega_p^{\text{Tr}_m(\theta x^d + zbx^{k+1} + x^2)} + \omega_p^{\text{Tr}_m(\theta x^d + zbx^{k+1} + x^2)} \right]
\]

\[
= \frac{1}{2} \sum_{y \in \mathbb{F}_p^*} \omega_p^{-ya} \sum_{y \in \mathbb{F}_p^*} (T(\theta yz b, y) + T(\theta yz b, y)).
\]

For a given \( y \in \mathbb{F}_{p^n}^* \), one of \( y \) and \( y\theta \) is a square in \( \mathbb{F}_{p^n}^* \) and the other is a non-square since \( \eta^{(c)}(y)\eta^{(c)}(y\theta) = -1 \). Thus, it follows from (13) that

\[
T(\theta yz b, y) + T(\theta yz b, y\theta) = T(zb, 1) + T(\theta zb, \theta)
\]

(24)
for given $b \in \mathbb{F}_p^{*m}$, $y \in \mathbb{F}_p^{*}$ and $z \in \mathbb{F}_p^{*}$. From \textbf{20}, \textbf{23} and \textbf{24}, the desired result \textbf{19} in this case follows.

\textbf{Case 2:} $\theta^4 = -\theta$. In this case,

\begin{align*}
R(a,b) &= \frac{1}{2} \sum_{y \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_p^{*m}} \left[ \omega_p \left( \text{Tr}_1 \left( y^{x^2 + zbx^h + 1} \right) \right) + \omega_p \left( -y^{x^2 + zbdx^h + 1} \right) \right], \\
&= \frac{1}{2} \sum_{y \in \mathbb{F}_p^*} \sum_{u \in \mathbb{F}_p^{*m}} \sum_{x \in \mathbb{F}_p^{*m}} \left[ \omega_p \left( \text{Tr}_1 \left( y^{x^2 + ybx^h + 1} \right) \right) + \omega_p \left( -y^{x^2 - ybdx^h + 1} \right) \right] \\
&= \frac{1}{2} \sum_{y \in \mathbb{F}_p^*} \sum_{y \in \mathbb{F}_p^{*m}} \sum_{y \in \mathbb{F}_p^{*m}} \left[ \omega_p \left( y \text{Tr}_1 \left( zbx^h + 1 + x^2 \right) \right) + \omega_p \left( -y \text{Tr}_1 \left( zbx^h + 1 + x^2 \right) \right) \right] \\
&= \frac{1}{2} \sum_{y \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_p^{*m}} \left( T(yzb, y) + T(-y\theta zb, -y\theta) \right).
\end{align*}

\textbf{Subcase 2.1:} $p^e \equiv 1 \mod 4$. Then, $-1$ is a square in $\mathbb{F}_p^{*e}$. It follows from Proposition \textbf{2} that $n_a = p^{m-1}$. Moreover, since $-\theta$ is a non-square in $\mathbb{F}_p^{*}$, we also have a result similar to \textbf{24} as follows

$$T(yzb, y) + T(-y\theta zb, -y\theta) = T(zb, 1) + T(\theta zb, \theta)$$

for given $b \in \mathbb{F}_p^{*m}$, $y \in \mathbb{F}_p^{*}$ and $z \in \mathbb{F}_p^{*}$. This equality together with \textbf{23} and \textbf{20} leads to the desired result \textbf{19}.

\textbf{Subcase 2.2:} $p^e \equiv 3 \mod 4$ and $a = 0$. Then, $n_a = p^{m-1}$. By \textbf{25}, we have

\begin{align*}
R(a,b) &= \frac{1}{2} \sum_{y \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_p^{*m}} \left( T(yzb, y) + T(-y\theta zb, -y\theta) \right) \\
&= \frac{1}{2} \sum_{y \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_p^{*m}} T(yzb, y) + \frac{1}{2} \sum_{u \in \mathbb{F}_p^{*m}} T(\theta zb, u\theta) \\
&= \frac{1}{2} \sum_{y \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_p^{*m}} \left( T(yzb, y) + T(\theta zb, y\theta) \right) \\
&= \frac{1}{2} \left( \sum_{z \in \mathbb{F}_p^{*m}} \left( T(\theta zb, 1) + T(\theta zb, \theta) \right) \right) \left( \sum_{y \in \mathbb{F}_p^*} 1 \right),
\end{align*}

where the last equality also follows from \textbf{24}. By \textbf{20}, the desired result \textbf{19} then follows.

\textbf{Subcase 2.3:} $p^e \equiv 3 \mod 4$ and $a \neq 0$. Then, $-1$ is a non-square in $\mathbb{F}_p^e$ and then $-\theta$ is a square element in $\mathbb{F}_p^e$. By \textbf{13}, we have $T(-y\theta zb, -y\theta) = T(yzb, y)$. Thus,

\begin{align*}
R(a,b) &= \frac{1}{2} \sum_{y \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_p^{*m}} \left( T(yzb, y) + T(yzb, y) \right) = \sum_{y \in \mathbb{F}_p^*} \sum_{z \in \mathbb{F}_p^{*m}} T(yzb, y),
\end{align*}

This together with \textbf{20} implies the desired result \textbf{18}.

With the above preparations, we can give the proofs of Theorems \textbf{1} and \textbf{2}.\hfill\square
Proof of Theorem I. Let $C_{D(0)}$ be the linear code defined in (3) and
\[
\mathbf{c}^0_b = (\text{Tr}_1^m(\beta_1 b), \text{Tr}_1^m(\beta_2 b), \ldots, \text{Tr}_1^m(\beta_n b)) \in C_{D(0)}.
\]
Note the length $l_0$ of this code is equal to $n_0 - 1$, where $n_0$ is given in Proposition II. Denote the weight of $\mathbf{c}^0_b$ by $\text{wt}(\mathbf{c}^0_b)$. It is obvious that $\text{wt}(\mathbf{c}^0_b) = 0$. In the following, we assume that $b \neq 0$. Then, using the notation in Propositions II and III we have
\[
\begin{align*}
\text{wt}(\mathbf{c}^0_b) &= |\{ x \in \mathbb{F}_{3^m} : \text{Tr}_1^m(x^d) = 0 \}| - |\{ x \in \mathbb{F}_{3^m} : \text{Tr}_1^m(x^d) = 0 \text{ and } \text{Tr}_1^m(bx) = 0 \}| \\
&= |\{ x \in \mathbb{F}_{3^m} : \text{Tr}_1^m(x^d) = 0 \}| - |\{ x \in \mathbb{F}_{3^m} : \text{Tr}_1^m(x^d) = 0 \text{ and } \text{Tr}_1^m(bx) = 0 \}| \\
&= n_0 - N(0, b).
\end{align*}
\]
When $p = 3$, by Proposition III
\[
N(0, b) = 3^{m-2} + \frac{1}{3^m} \left( \sum_{z \in \mathbb{F}_3} (T(zb, 1) + T(\theta b, \theta)) \right) \left( \sum_{y \in \mathbb{F}_3^*} 1 \right) = 3^{m-2} + \frac{1}{3^m} \left[ T(b, 1) + T(\theta b, \theta) + T(-b, 1) + T(-\theta b, \theta) \right].
\]
By (13), $T(b, 1) + T(\theta b, \theta) \neq 0$ only if the rank of the quadratic form $Q_{b,1}(x)$ is even, i.e., $r_{b,1} = \frac{m+1}{2} - 1$. In this case,
\[
T(b, 1) + T(\theta b, \theta) \in \left\{ -2 \cdot 3^{\frac{m+1}{2}}, 2 \cdot 3^{\frac{m+1}{2}} \right\}.
\]
In a similar way, we have
\[
T(-b, 1) + T(-\theta b, \theta) \in \left\{ -2 \cdot 3^{\frac{m+1}{2}}, 2 \cdot 3^{\frac{m+1}{2}} \right\}
\]
only if the rank of $Q_{-b,1}(x)$ equals $\frac{m}{2} - 1$.

By Lemma III (ii), at least one of the quadratic forms $Q_{b,1}(x)$ and $Q_{-b,1}(x)$ has rank $\frac{m}{2}$. When one of $Q_{b,1}(x)$ and $Q_{-b,1}(x)$ has rank $\frac{m}{2} - 1$, the other one must has rank $\frac{m}{2}$. Consequently, the sums $T(b, 1) + T(\theta b, \theta)$ and $T(-b, 1) + T(-\theta b, \theta)$ cannot be nonzero simultaneously. Thus, by (27), we have
\[
N(0, b) = \left\{ 3^{m-2} - 2 \cdot 3^{\frac{m+1}{2}-2}, 3^{m-2} + 2 \cdot 3^{\frac{m+1}{2}-2} \right\}.
\]
When $b$ runs through $\mathbb{F}_{3^m}$, for each $\varepsilon \in \{1, -1\}$, the number of $b$ such that $N(0, b) = 3^{m-2} + 2\varepsilon 3^{\frac{m+1}{2}-2}$ is equal to the number of $b$ such that $T(b, 1) = \varepsilon 3^{\frac{m+1}{2}}$ or $T(-b, 1) = \varepsilon 3^{\frac{m+1}{2}}$. By Proposition I we can conclude the number of such $b$ is equal to $3^{m-\varepsilon} + \varepsilon 3^{\frac{m+1}{2}-\varepsilon}$, $\varepsilon \in \{1, -1\}$. This result together with (20) and Proposition II gives the desired result. \qed
Proof of Theorem 2. For each $a \in \mathbb{F}_3^*$, let $C_{D(a)}$ be the linear code defined in (3) and $c^a_b$ a codeword of $C_{D(a)}$ given by

$$(\text{Tr}_1^m(\beta_1 a), \text{Tr}_1^m(\beta_2 a), \ldots, \text{Tr}_1^m(\beta_n a)).$$

Denote the weight of $c^a_b$ by $\text{wt}(c^a_b)$. It is easily seen that $\text{wt}(c^a_0) = 0$. In the sequel, we compute $\text{wt}(c^a_b)$ for $b \neq 0$. By Proposition 2, the length of this code is $n$ and

$$\text{wt}(c^a_b) = |\{ x \in \mathbb{F}_3^m : \text{Tr}_1^m(x^d) = a \} | - |\{ x \in \mathbb{F}_3^m : \text{Tr}_1^m(x^d) = a \text{ and } \text{Tr}_1^m(bx) = 0 \} | = n_a - N(a, b).$$

The following two cases are considered.

Case 1: $e$ is even, or $e$ is odd and $d \equiv 1 \pmod{3^e - 1}$. Then, by Propositions 2 and 3 we have $n_a = 3^{m-1}$ for each $a \in \mathbb{F}_3^*$ and

$$N(a, b) = 3^{m-2} - \frac{1}{2^{3^e}} \sum_{z \in \mathbb{F}_3^*} (T(zb, 1) + T(\theta zb, \theta)) = 3^{m-2} - \frac{1}{2^{3^e}} \left( (T(b, 1) + T(\theta b, \theta)) + (T(-b, 1) + T(-\theta b, \theta)) \right).$$

A similar analysis as for (27) in the proof of Theorem 1 yields the desired result.

Case 2: $e$ is odd and $d \equiv 1 + \frac{3^e - 1}{2} \pmod{3^e - 1}$. Note that $a \in \mathbb{F}_3^*$. Then, by Proposition 2 we have $n_a = 3^{m-1} + (-1)^{\frac{m-1}{2}} \cdot 3^{\frac{m-1}{2}} \cdot \eta(1)(a)$. By Proposition 3

$$N(a, b) = \frac{1}{2} n_a + \frac{1}{2} \sum_{y \in \mathbb{F}_3^*} \sum_{z \in \mathbb{F}_3^*} T(yzb, y) = \frac{1}{2} n_a + \frac{1}{2} \sum_{y \in \mathbb{F}_3^*} \omega_3^{-ya} (T(b, 1) + T(-b, 1)) + \frac{1}{2} \omega_3^a (T(b, -1) + T(-b, -1)).$$

Note that $-1$ is a non-square in $\mathbb{F}_3^*$ in this case. For each $b \in \mathbb{F}_3^m$, if $(T(b, 1), T(-b, 1))$ is given, then the ranks of $Q_{b,1}(x)$ and $Q_{-b,1}(x)$ are determined. Consequently, by (13), the value of $(T(-b, -1), T(b, -1))$ is uniquely determined. Then, by Lemma 5 and (29), we can calculate the possible values of $N(a, b)$ which are given in Table 5 where $c_i$, $i = 0, 1, 2$, are defined by (12).

Take $\omega_3 = \frac{-1 + \sqrt{-3}}{2}$. Then, by Table 5 and (28), the possible weights of $C_{D(1)}$ in (3) is obtained. Similarly, one can obtain the possible weights of $C_{D(-1)}$ when $b \neq 0$. \(\square\)

Remark 1 When $p = 3$ and $e = \gcd(k, m)$ is odd, the weight distribution of $C_{D(a)}$ with $a \in \mathbb{F}_p^*$ is dependent on the value distribution of $\hat{T}(b, 1) = (T(-b, -1), T(b, -1))$ as $b$ runs through $\mathbb{F}_3^m$. If we can determine the value distribution of $\hat{T}(b, 1)$, then the weight distribution of $C_{D(a)}$ will be determined. Lemma 5 is necessary for determining the value distribution of $\hat{T}(b, 1)$. However, in order to find the
Table 5: Possible values of $N(a, b)$

| $(T(b, 1), T(-b, 1))$ | $(T(-b, -1), T(b, -1))$ | $N(a, b)$ |
|------------------------|--------------------------|-----------|
| $(c_0, c_0)$           | $(-c_0, -c_0)$           | $\frac{1}{2}n_a + \frac{2}{3}c_0(\omega_3^{-a} - \omega_3^{b})$ |
| $(-c_0, -c_0)$         | $(c_0, c_0)$             | $\frac{1}{2}n_a + \frac{2}{3}c_0(\omega_3^{a} - \omega_3^{-b})$ |
| $(-c_0, c_0)$          | $(c_0, -c_0)$            | $\frac{1}{2}n_a$ |
| $(c_0, c_1)$           | $(-c_0, c_1)$            | $\frac{1}{2}n_a + \frac{2}{3}c_0(\omega_3^{a} - \omega_3^{-b}) + \frac{2}{3}(\omega_3^{-a} + \omega_3^{b})$ |
| $(c_1, c_0)$           | $(c_0, -c_0)$            | $\frac{1}{2}n_a + \frac{2}{3}c_0(\omega_3^{a} - \omega_3^{-b}) + \frac{2}{3}(\omega_3^{-a} + \omega_3^{b})$ |
| $(-c_0, -c_1)$         | $(c_0, c_1)$             | $\frac{1}{2}n_a + \frac{2}{3}c_0(\omega_3^{-a} - \omega_3^{b}) - \frac{2}{3}(\omega_3^{-a} + \omega_3^{b})$ |
| $(-c_1, c_0)$          | $(c_0, -c_1)$            | $\frac{1}{2}n_a + \frac{2}{3}c_0(\omega_3^{-a} - \omega_3^{b}) - \frac{2}{3}(\omega_3^{-a} + \omega_3^{b})$ |
| $(c_0, c_2)$           | $(-c_0, c_2)$            | $\frac{1}{2}n_a + \frac{2}{3}c_0(\omega_3^{-a} - \omega_3^{b}) + \frac{2}{3}(\omega_3^{-a} + \omega_3^{b})$ |
| $(c_2, c_0)$           | $(-c_0, -c_0)$           | $\frac{1}{2}n_a + \frac{2}{3}c_0(\omega_3^{-a} - \omega_3^{b}) + \frac{2}{3}(\omega_3^{-a} + \omega_3^{b})$ |
| $(-c_0, -c_2)$         | $(c_0, c_2)$             | $\frac{1}{2}n_a + \frac{2}{3}c_0(\omega_3^{-a} - \omega_3^{b}) + \frac{2}{3}(\omega_3^{-a} + \omega_3^{b})$ |
| $(-c_2, c_0)$          | $(c_0, -c_2)$            | $\frac{1}{2}n_a + \frac{2}{3}c_0(\omega_3^{-a} - \omega_3^{b}) + \frac{2}{3}(\omega_3^{-a} + \omega_3^{b})$ |

value distribution of $\hat{T}(b, 1)$, we need to explore the relation between $\hat{T}(b, 1)$ and $\hat{T}(u, v)$. It seems to be a difficult problem.

As documented in [22], there are a number of integers satisfying the congruence (1), where three classes of APN exponents are also covered. From Theorem 1, we have the following corollary.

**Corollary 2** Let $m \geq 3$ be odd and $x^d$ be an APN function over $\mathbb{F}_{3^m}$ with

(i) $d = \frac{3^{m+1} + 1}{2}$; or

(ii) $d = \frac{3^m - 1}{2}$; or

(iii) $d = \left(\frac{3^{m+1} + 1}{2}\right) \left(\frac{3^{m+1}}{2} + 1\right)$ for $m \equiv 3 \mod 4$.

Then, these APN functions satisfy the congruence (1) and can be used for constructing $[3^{m-1} - 1, m]$ linear codes $C_{D(0)}$ with weight distribution given in Table 4.

The following examples are provided for verifying the main results in Theorems 1 and 2 and they are confirmed by Magma.

**Example 1** Let $p = 3$, $m = 5$ and $k = 2$. Then, $e = \gcd(k, m) = 1$, $p^e \equiv 3 \mod 4$ and the congruence $d(p^k + 1) \equiv 2 \mod p^m - 1$ has two solutions in $\mathbb{Z}_{p^m - 1}$: $d_1 = 97$, $d_2 = 218$. Using $d_1$ and $d_2$ in the construction given by (2) and (5), the obtained linear code $C_{D(0)}$ has length 80 and the weight enumerator is $1 + 90x^{48} + 80x^{54} + 72x^{60}$.
Example 2 Let \( p = 3, m = 6 \) and \( k = 2 \). Then, \( e = \gcd(k, m) = 2, p^e \equiv 1 \pmod{4} \) and the congruence \( d(p^k + 1) \equiv 2 \pmod{p^m - 1} \) has two solutions in \( \mathbb{Z}_{p^m-1} \): \( d_1 = 73, d_2 = 437 \). Using these two integers in the construction for \( a = 1 \) or \( a = 2 \), the obtained linear codes have length 243 and they share the same weight enumerator \( 1 + 72x^{153} + 566x^{162} + 90x^{171} \).

Example 3 Let \( p = 3, m = 9 \) and \( k = 3 \). Then, \( e = \gcd(k, m) = 3, p^e \equiv 3 \pmod{4} \) and the congruence \( d(p^k + 1) \equiv 2 \pmod{p^m - 1} \) has two solutions in \( \mathbb{Z}_{p^m-1} \): \( d_1 = 703, d_2 = 10544 \). Using \( d_1 \) in the construction for \( a = 1 \) or \( a = 2 \), the obtained linear codes have length 6561 and they share the same weight enumerator \( 1 + 702x^{4293} + 18224x^{4374} + 756x^{4455} \).

Using \( d_2 \) in the construction for \( a = 1 \) (resp. \( a = 2 \)), denote the obtained linear code by \( C_1 \) (resp. \( C_2 \)). Then, \( C_1 \) has length 6642 and its weight enumerator is

\[
1 + 2x^{5184} + 414x^{4536} + 4848x^{4482} + 9138x^{4428} + 4938x^{4374} + 342x^{4320},
\]

and \( C_2 \) has length 6480 and the weight enumerator is

\[
1 + 2x^{3564} + 342x^{4428} + 4992x^{4374} + 9138x^{4320} + 4848x^{4266} + 360x^{4212}.
\]

Note that not all the possible weights of \( C_1 \) (resp. \( C_2 \)) disappear.

4 Conclusion

In this paper, two classes of three-weight ternary linear codes are obtained. Compared with the work in [3, 4, 5, 6, 7], we utilized a family of power functions \( x^d \), including three classes of APN power functions, to construct the defining sets. We proceed our study mostly for general odd primes \( p \), and obtained three-weight linear codes in the ternary case. The techniques and results in Propositions 1-3 would be useful for studying the weight distributions of other \( p \)-ary codes.

Acknowledgment

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