Connectivity and Convexity Properties of the Momentum Map for Group Actions on Hilbert Manifolds

by

Kathleen Smith

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Graduate Department of Mathematics
University of Toronto

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Abstract

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Kathleen Smith
Doctor of Philosophy
Graduate Department of Mathematics
University of Toronto
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In the early 1980s a landmark result was obtained by Atiyah and independently Guillemin and Sternberg: the image of the momentum map for a torus action on a compact symplectic manifold is a convex polyhedron. Atiyah’s proof makes use of the fact that level sets of the momentum map are connected. These proofs work in the setting of finite-dimensional compact symplectic manifolds. One can ask how these results generalize. A well-known example of an infinite-dimensional symplectic manifold with a finite-dimensional torus action is the based loop group. Atiyah and Pressley proved convexity for this example, but not connectedness of level sets. A proof of connectedness of level sets for the based loop group was provided by Harada, Holm, Jeffrey and Mare in 2006.

In this thesis we study Hilbert manifolds equipped with a strong symplectic structure and a finite-dimensional group action preserving the strong symplectic structure. We prove connectedness of regular generic level sets of the momentum map. We use this to prove convexity of the image of the momentum map.
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Chapter 1

Introduction
Chapter 1. Introduction

In the early 1980s Atiyah [6] and independently and simultaneously Guillemin and Sternberg [17] arrived at a now famous finite-dimensional abelian convexity result. Their result is:

**Theorem 1.0.1 (Atiyah-Guillemin-Sternberg).** Let \((M, \omega)\) be a compact connected symplectic manifold. Let \(T\) be an \(n\)-torus and let \(\lambda: T \times M \to M\) be a Hamiltonian action of \(T\) on \(M\) with momentum mapping \(\mu: M \to t^*\). Let \(M_T\) denote the fixed point set of \(\lambda\). Then

1. The image \(\mu(M_T)\) is a finite subset of \(t^*\);

2. \(\mu(M)\) is the convex hull of \(\mu(M_T)\).

In particular the image \(\mu(M)\) is a convex polyhedron.

Atiyah’s proof of Theorem 1.0.1 makes use of the following connectivity result: Under the same hypotheses as Theorem 1.0.1,

**Theorem 1.0.2.** For every \(c \in t^*\), the level \(\mu^{-1}(c)\) is connected (or empty).

He deduces Theorem 1.0.1 from Theorem 1.0.2.

Over the last 30 years there has been considerable interest in various infinite-dimensional Hamiltonian systems, namely, infinite-dimensional symplectic manifolds equipped with actions of finite-dimensional tori. For example, Atiyah in [6] asked whether Theorem 1.0.1 could be extended in any interesting way to infinite-dimensions. Atiyah and Pressley [7] answered this question in the affirmative. They proved an extension of Theorem 1.0.1 for the based loop group, an infinite-dimensional symplectic manifold, with a finite-dimensional torus action. Before we state this result more precisely we need the following definitions.

Let \(G\) be a compact, connected and simply connected Lie group. Fix a \(G\)-invariant inner product on the Lie algebra \(\mathfrak{g}\). The **loop group** is defined as the set of maps from
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\[ S^1 \rightarrow G \] that are of Sobolev class \( H^1 \). We will denote the loop group by \( M_1 \). So

\[ M_1 = H^1(S^1, G). \]

The subset \( \Omega G \) of \( M_1 \) consisting of those loops \( f: S^1 \rightarrow G \) for which \( f(1) \) is the identity element in \( G \) is called the based loop group. We refer the reader to Chapter 6 for more details regarding the loop group and the based loop group.

Atiyah and Pressley in [7] prove:

**Theorem 1.0.3.** Let \( G \) be a compact, connected and simply connected Lie group with maximal torus \( T \). Let \( \Omega G \) be the based loop group. Let \( R := T \times S^1 \) act on \( \Omega G \) where

(i) the rotation group \( S^1 \) acts on \( \Omega G \) by “rotating the loop”:

if \( \gamma \in \Omega G \) and \( e^{i\theta} \in S^1 \), \( \theta \in [0, 2\pi] \), then \( (e^{i\theta} \gamma)(s) := \gamma(s + \theta) \gamma(\theta)^{-1} \), and;

(ii) the maximal torus acts on \( \Omega G \) by conjugation:

if \( \gamma \in \Omega G \) and \( t \in T \), then \( (t\gamma)(s) := t\gamma(s)t^{-1} \).

Note that these actions commute. Then the image of the momentum map is convex and it is the convex hull of the images of the fixed points.

**Remark 1.0.4.** Atiyah points out that the requirement that \( G \) be simply connected may be weakened to semi-simple. Notice that \( \Omega G \) then has several connected components. In this case the image of each component of \( \Omega G \) is a convex polyhedron; it is the convex hull of the images corresponding to the fixed points in that particular component.

We will not go into the very detailed proof of Theorem 1.0.3 which is specific to this example of the based loop group. We do nevertheless note that Atiyah and Pressley in [7] remark that their Theorem 1.0.3 could be proved by extending the method of proof of Theorem 1.0.1 so as to cover their infinite-dimensional situation. They do not carry out this argument nor do they provide any hints on what might be required to do so.

In 2006 in [30], Harada, Holm, Jeffrey, and Mare proved infinite-dimensional analogues (with respect to the based loop group \( \Omega G \) example of Atiyah [7]) of the well-known
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Theorem 1.0.2 result in finite-dimensional symplectic geometry. Before we can recall these specific results we need another definition.

The set $\Omega_{alg}$, the algebraic based loop group, is the subset of the based loop group $\Omega G$ consisting of loops which have a finite Fourier series (when $G$ is identified with a group of matrices).

The main results of [30] that we are concerned with are:

**Theorem 1.0.5.** Any level set of the momentum map $\mu$ of the $T \times S^1$ action restricted to $\Omega_{alg}$ is connected (for regular or singular values of the momentum map).

**Theorem 1.0.6.** Let $\mu$ be the momentum map for the $T \times S^1$ action on $\Omega G$. The level set $\mu^{-1}(c)$ of the momentum map is connected, provided that $c$ is a regular value.

**Remark 1.0.7.** The space $\Omega G$, being a Hilbert manifold, in particular has a topology. Theorem 1.0.6 refers to the topology of $\Omega G$ as a Hilbert manifold. The subset $\Omega_{alg}$ of $\Omega G$ can also be equipped with a topology. Theorem 1.0.5 refers to the direct limit topology on $\Omega_{alg}$. We direct the reader to Chapter 6 for further details.

**Remark 1.0.8.** The extra hypothesis that $c$ be a regular value of the momentum map in Theorem 1.0.6 is needed so that Morse-theoretic arguments in infinite-dimensions can be used in the proof. In later years, Mare in [28] was able to eliminate the regular value hypothesis for the momentum map $\mu$. Mare proved that the singular level sets of $\mu$ for the $T \times S^1$ action on $\Omega G$ are connected. His argument works for the space of $C^\infty$ loops and also for the space of loops of Sobolev class $H^s$ for any $s \geq 1$.

1.0.1 Thesis Outline

The main results of this thesis are infinite-dimensional analogues of well-known connectedness and convexity results in finite-dimensional symplectic geometry. Namely, we establish an analogue of Theorem 1.0.1 and Theorem 1.0.2. We prove:
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**Theorem 5.4.4.** (Connectivity Theorem). Let $M$ be a connected strongly symplectic Hilbert manifold. Suppose that we have an almost periodic $\mathbb{R}^n$ action on $M$ with momentum map $\mu : M \to \mathbb{R}^n$. Suppose that the $\mathbb{R}^n$ action has isolated fixed points. Suppose that there exists a complete invariant Riemannian metric on $M$ such that there exists a hyperplane $H$ of $\mathbb{R}^n$ such that for all $\xi \in \mathbb{R}^n \setminus H$ the map $\mu^\xi : M \to \mathbb{R}$ is bounded from one side and satisfies Condition (C) (See section 4.2). Then the momentum mapping $\mu$ satisfies

(A) The set $\{ c \in \mathbb{R}^n \mid c$ is a regular value of $\mu$ and $\mu^{-1}(c)$ is connected $\} \subseteq \mathbb{R}^n$ is residual.

**Theorem 5.4.5.** (Convexity Theorem). Let $M$ be a connected strongly symplectic Hilbert manifold. Suppose that we have an almost periodic $\mathbb{R}^n$ action on $M$ with momentum map $\mu : M \to \mathbb{R}^n$. Suppose that the $\mathbb{R}^n$ action has isolated fixed points and suppose that $\mu(M)$ is closed. Suppose that there exists a complete invariant Riemannian metric on $M$ such that there exists a hyperplane $H$ of $\mathbb{R}^n$ such that for all $\xi \in \mathbb{R}^n \setminus H$ the map $\mu^\xi : M \to \mathbb{R}$ is bounded from one side and satisfies Condition (C). Then the momentum mapping $\mu$ satisfies

(B) the image $\mu(M)$ is convex.

Note that the Palais-Smale compactness condition, namely Condition (C) (see section 4.2), is an important hypothesis for our connectedness and convexity theorems, Theorems 5.4.4, 5.4.5. Condition (C) is a “compactness condition” on real-valued functions of class $C^1$ defined on a Riemannian manifold modelled upon a Hilbert space. It is needed in order to extend Morse theory to our infinite-dimensional setting.

Let us now highlight the contents of each chapter in this thesis and, where appropriate, briefly explain how the respective material contributes to the main thesis results, the Convexity Theorem 5.4.5.
Chapter 2, Background and Preliminaries, provides a basic review of relevant known facts and definitions from the theory of differential topology. Throughout this thesis our manifold $M$ will always be a Hausdorff, paracompact Hilbert manifold modelled on a real separable Hilbert space. That is, $M$ is equipped with an equivalence class of smooth (meaning $C^\infty$) atlases such that all charts take values in an infinite-dimensional separable real Hilbert space.

The purpose of Chapter 3, Normal Forms, is to extend the existing theory on local normal forms for Hamiltonian group actions to infinite-dimensional Banach manifolds. More specifically, we formalize the local linearization theorem for compact group actions on Banach manifolds (Theorem 3.1.1) originally noted by Weinstein (without proof) in [50]. We also establish a symplectic version of this local linearization theorem (Theorem 3.1.2). In so doing, we provide a $G$-equivariant version of Moser’s argument (Lemma 3.2.3) suitable for our goal. It is the symplectic version of the local linearization theorem that is needed later in the thesis to help prove Theorem 5.1.7 which is an infinite-dimensional analogue of a lemma of Atiyah [6, Lemma 2.2] and Guillemin and Sternberg [17, Theorem 5.3].

Chapter 4, Connectedness - The Base Case, introduces the notion of what it means for a Riemannian metric on a Hilbert manifold $M$ to be standard near each critical point of a smooth real-valued function on $M$. Suppose that we are given a complete Riemannian metric $g$ on a Hilbert manifold $M$ and let $f: M \to \mathbb{R}$ be a smooth function. For $g$ to be standard (near each critical point $p$ of $f$) means that $g$ coincides with some Riemannian metric on $M$ whose gradient vector field is standard near each $p$. For a complete and precise definition see Definition 4.1.6 and the subsequent Remark 4.1.7. With this “standard” hypothesis on the Riemannian metric we are able to provide an alternate proof of the known Global (Un) Stable Manifold Theorem, Theorem 4.2.3, which tells us that the stable and unstable sets of $p$ are in fact manifolds. However, the main feature of Chapter 4 is the Connected Levels Theorem, Theorem 4.3.5.
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Theorem 4.3.5. (Connected Levels). Let $M$ be a connected Hilbert manifold and let $f : M \to \mathbb{R}$ be a Morse function that is bounded from below and none of whose critical points have index or coindex equal to 1. Suppose that there exists a complete Riemannian metric on $M$ such that $f$ satisfies Condition (C). Then the level set $f^{-1}(c) \subset M$ is connected for every $c$ in $\mathbb{R}$.

This result is interesting in its own right. Its proof relies on Morse theoretic arguments that follow from the fact that there exists a complete Riemannian metric on $M$ for which $f$ satisfies the Palais Smale Condition (C) and such that the negative gradient field of $f$ is standard near each critical point of $f$. Notice that Theorem 4.3.5 establishes the connectivity of all level sets of $f$. The $n = 1$ case of the Connectivity Theorem, Theorem 5.4.5, will follow from Theorem 4.3.5; details of this $n = 1$ claim are provided in the next chapter within the proof of Theorem 5.4.5.

Chapter 5, Convexity and Connectedness, defines one of the main ingredients in the Connectivity and Convexity Theorems. Specifically, the chapter begins by defining what is meant by an almost periodic $\mathbb{R}^n$ action on a Hilbert manifold $M$. See Definitions 5.1.1 and 5.1.2. The reader may think of an almost periodic $\mathbb{R}^n$ action as a generalization of a torus action. We prove that in the presence of an almost periodic $\mathbb{R}^n$ action on $M$, the set of singular values of the resulting momentum map is contained in a countable union of hyperplanes (Theorem 5.4.1). (In particular, the set of regular values of the momentum map is residual in $\mathbb{R}^n$.) Then, the chapter ends with the statement and proof of the thesis main results, the Connectivity Theorem 5.4.4 and Convexity Theorem 5.4.5. Following the method of Atiyah [6], the Connectivity Theorem is established by induction on the dimension of the almost periodic $\mathbb{R}^n$ action on $M$. Note that in the finite-dimensional convexity result, Theorem 1.0.1 Guillemin and Sternberg prove convexity but not through connectedness (see [17]). They do not provide any results for connectedness. Atiyah proves convexity using connectedness (see [6]) but there is a gap in his argument for connectedness. This occurs in his induction step where he claims that
the connectedness of the regular level sets of the momentum map implies that all level sets of the momentum map are connected by continuity. A nice example to illustrate the problem is provided below.

**Example 1.0.9.** Let $h: S^2 \to S^1$ be the map that sends $(x_1, x_2, x_3) \mapsto e^{i\pi x_3}$. This map has exactly one singular value (at $x_3 = -1$). All the regular level sets are connected; they are circles. But the singular level set above $-1$, namely $\{(0, 0, 1), (0, 0, -1)\}$, is disconnected. See Figure 1.1.

![Figure 1.1: Singular level set of $h$ above $-1$ for example 1.0.9.](image)

This by continuity matter was resolved by Lerman and Tolman in [25, sections §4 and §5].

Lastly, Chapter 6 illustrates that the Convexity Theorem reproduces known infinite-dimensional convexity results for a significant example (see [30], [7]). Namely, it reproduces the connectivity and convexity results with regards to the based loop group.
Chapter 2

Background and Preliminaries
This chapter consists of two parts. We review a selection of well known results and some standard definitions from the theory of differentiable manifolds, differential topology and point set topology. As well, we declare some notational conventions.

The material of these sections borrows from many sources. We use Lang [24], Palais [34] and Royden [40] for basic foundational results.

2.1 Function-Analytic Preliminaries

Let $M$ be a Hausdorff, paracompact Hilbert manifold modelled on a real separable Hilbert space $(\mathbb{H}, \langle \cdot, \cdot \rangle)$. That is, $M$ is equipped with an equivalence class of $C^\infty$ atlases such that all charts take values in a separable real Hilbert space $\mathbb{H}$.

Recall that a smooth vector field, say $X$, on $M$ is a smooth cross-section of the tangent bundle $TM$, i.e., a smooth map $X: M \to TM$ such that $\pi \circ X = id$.

**Definition 2.1.1.** Let $M$ be a Hilbert manifold.

For each $x \in M$ a strongly nondegenerate inner product $g_x$ on $T_xM$ is a positive-definite, symmetric, bilinear form $g_x(\cdot, \cdot): T_xM \times T_xM \to \mathbb{R}$ such that the norm $\| \cdot \|_x = g_x(\cdot, \cdot)^{\frac{1}{2}}$ defines the topology of $T_xM$. Moreover, we require that $g_x$ determine a bounded, invertible operator $T_xM \to (T_xM)^*$ with bounded inverse.

For each point in $M$ there exists a neighbourhood $D \subseteq M$ and a chart with target a Hilbert space. Let $\phi$ be a chart in $M$ having as target a Hilbert space $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ such that the following holds: for each $x \in D$ we define the operator $G(x): \mathbb{H} \to \mathbb{H}$ as follows:

Identify $T_xM$ with $\mathbb{H}$ by the Hilbert space isomorphism

$$(d\phi|_x)^{-1}: \mathbb{H} \to T_xM.$$  

Then

$$\langle G(x)u, v \rangle = g \left( (d\phi|_x)^{-1}(u), (d\phi|_x)^{-1}(v) \right) \text{ for all } u, v \in \mathbb{H}.$$
Thus \( x \mapsto G(x) \) is a map from \( D \) to the space of positive definite symmetric bounded operators on \( H \) with the operator norm. If we require the map \( x \mapsto G(x) \) to be smooth with respect to the operator topology (it follows that \( x \mapsto G^{-1}(x) \) is also smooth) then we call \( x \mapsto g_x(\cdot,\cdot) \) a (smooth) Riemannian metric (or (smooth) Riemannian structure) on \( M \).

A (strong) Riemannian manifold \((M,g)\) is a manifold \( M \) equipped with a smooth Riemannian metric \( g \).

Note that we require a strong Riemannian metric on \( M \). Fix one such metric on \( M \). For each \( x \in M \), we will denote by \( \langle \cdot,\cdot \rangle_x \) the inner product in the tangent space \( T_xM \).

Remark 2.1.2. Note that the topology given by the smooth Riemannian metric is the given topology of \( M \) (see [34, pg. 311]).

Let \( f: M \to \mathbb{R} \) be a smooth function on \( M \). Then \( df: TM \to \mathbb{R} \), the differential of \( f \), is a cross-section of the cotangent bundle, \( T^*M \), of \( M \). Hence, there is a uniquely determined vector field \( \nabla f: M \to TM \), the gradient of \( f \), such that \( df_x(v) = \langle v, \nabla f(x) \rangle_x \) for all \( x \in M, v \in T_xM \).

The reader should note that \( \nabla f \) will play a central role throughout this thesis.

Recall that a critical point of \( f \) is a point \( x \in M \) such that \( df_x: T_xM \to \mathbb{R} \) satisfies \( df_x = 0 \), equivalently where \( \nabla f_x \) vanishes. Throughout this thesis let us denote the set of critical points of \( f \) by \( \text{Crit}(f) \), i.e.,

\[
\text{Crit}(f) := \{ x \in M \mid df_x = 0 \}.
\]

If \( df_x \neq 0 \) then the point \( x \in M \) is called a regular point of \( f \). Let \( c \in \mathbb{R} \). If the level set \( f^{-1}(c) \) consists only of regular points of \( f \) then \( c \) is a regular value of \( f \). If the level set \( f^{-1}(c) \) contains at least one critical point of \( f \) then we say that \( c \) is a critical value of \( f \).

Definition 2.1.3. At a critical point \( p \) of \( f \) there is a uniquely determined continuous bilinear form \( H_p(f): T_pM \times T_pM \to \mathbb{R} \), the Hessian of \( f \) at \( p \), such that if \( \phi \) is any
chart around \( p \)

\[
H_p(f)(u,v) = d^2(f \circ \phi^{-1})(d\phi|_p(u), d\phi|_p(v)) ,
\]

where \( d^2 \) is defined below.

**Remark 2.1.4.**

1. Suppose that \( h \) is a continuously differentiable mapping of an open set \( W \) of a Hilbert space \( E \) into \( \mathbb{R} \). Then \( dh \) is a continuous mapping of \( W \) into the Hilbert space \( \mathcal{L}(E; \mathbb{R}) \). If that mapping is differentiable at a point \( x \in W \), recall that \( h \) is **twice differentiable** at \( x \), and the derivative of \( dh \) at \( x \) is called the **second derivative** of \( h \) at \( x \), and written \( d^2 h |_x \). This is an element of \( \mathcal{L}(E; \mathcal{L}(E; \mathbb{R})) \). We make the canonical identification of \( \mathcal{L}(E; \mathcal{L}(E; \mathbb{R})) \) with the space \( \mathcal{L}(E \times E; \mathbb{R}) \) of continuous bilinear mappings of \( E \times E \) into \( \mathbb{R} \): we recall that this is done by identifying \( u \in \mathcal{L}(E; \mathcal{L}(E; \mathbb{R})) \) with the bilinear mapping \((s,t) \to (u \cdot s) \cdot t\).

2. Note that the Hessian quadratic form in Definition 2.1.3 is independent of the choice of chart \( \phi \). Moreover, \( H_p(f) \) determines a **bounded** operator \( A: T_p M \to T_p M \) by

\[
H_p(f)(u,v) = \langle Au, v \rangle_p
\]

Because \( H_p(f) \) is symmetric, the operator \( A \) is self-adjoint.

In what follows, we choose a smooth Riemannian metric and then identify \( H_p(f) \) with the operator \( A \). The interpretation will be clear from the context.

The critical point \( p \) is called (strongly) **nondegenerate** if \( A \) is invertible with bounded inverse. Henceforth, we assume that \( f \) has only nondegenerate critical points.

**Definition 2.1.5.** Let \( p \in \text{Crit}(f) \). The **index** (coindex) of \( p \) is the index (coindex) of the Hessian \( H_p(f) \), i.e., the supremum of the dimensions of all linear spaces where \( H_p(f) \) is negative (positive) definite. We shall denote the index of \( p \) by \( \text{index}_p(f) \) and the coindex by \( \text{coindex}_p(f) \).

**Example 2.1.6.** Let \( \mathbb{H} \) be a Hilbert space and let \( \mathbb{H}_+ \subset \mathbb{H} \) be closed subspaces such that \( \mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_- \). Let \( x := (x_+, x_-) \in \mathbb{H} \) and let \( f: \mathbb{H} \to \mathbb{R} \) be a smooth function defined
by $f^H(x) = ||x_+||^2 - ||x_-||^2$. For $p = 0 \in \text{Crit}(f^H)$ we see that $\text{index}_p(f^H) = \dim(\mathcal{H}_-)$ and $\text{coindex}_p(f^H) = \dim(\mathcal{H}_+)$.

### 2.2 Two Important Theorems

#### 2.2.1 Baire Category Theorem

**Definition 2.2.1.** Let $M$ be a topological space. A set $E \subset M$ is said to be **nowhere dense** if $(E)^c = \emptyset$, i.e., $E$ has empty interior.

Notice that $E$ is nowhere dense is equivalent to

$$M = \left( (E)^c \right)^c = ((E^c)^c).$$

That is to say that $E$ is nowhere dense if and only if $E^c$ has dense interior.

**Theorem 2.2.2** (Baire Category Theorem). Let $M$ be a complete metric space.

(i) If $\{V_n\}_{n=1}^\infty$ is a sequence of dense open sets, then $\bigcap_{n=1}^\infty V_n$ is dense in $M$.

(ii) If $\{E_n\}_{n=1}^\infty$ is a sequence of nowhere dense sets, then $M \neq \bigcup_{n=1}^\infty E_n$.

**Definition 2.2.3.** A subset $E \subset M$ is of **first Baire category** (or is **meager**) if

$$E = \bigcup_{n=1}^\infty E_n$$

where each $E_n$ is nowhere dense. A set $F$ is called **residual** if $F^c$ is of first Baire category.

**Remark 2.2.4.** The reader should think of first Baire category as being the topological analogue of sets of measure zero (so “small”), and residual as being the topological analogue of sets of full measure (so “big”).

Let us collect some facts about residual sets and meager sets. Let $M$ be a complete metric space.
1. A set \( F \subset M \) is residual if and only if \( F \) contains a countable intersection of open dense sets.

Indeed, if \( F \) is residual then there exist nowhere dense sets \( \{ E_n \} \) such that

\[
F^c = \bigcup_{n=1}^{\infty} E_n \subset \bigcup_{n=1}^{\infty} \overline{E_n}.
\]

Taking complements of this equation yields

\[
\bigcap_{n=1}^{\infty} (\overline{E_n})^c \subset F,
\]

i.e., \( F \) contains a set of the form \( \cap_{n=1}^{\infty} V_n \) where each \( V_n := (\overline{E_n})^c \) is an open dense subset of \( M \).

2. A countable union of sets of first Baire category is of first Baire category.

3. If a set is of first Baire category then any subset of this set also is of first Baire category.

4. A countable intersection of residual sets is residual.

**Remark 2.2.5.** The Baire Category Theorem [2.2.2] may now be re-stated as follows. If \( M \) is a complete metric space, then

(i) all residual sets are dense in \( M \), and

(ii) \( M \) is not of first Baire category.

### 2.2.2 Existence and Uniqueness Theorem for ODEs

Let \( M \) be an infinite-dimensional Hilbert manifold modelled on a real separable Hilbert space \((\mathbb{H}, \langle \cdot, \cdot \rangle)\). Recall that given a smooth (meaning \( C^\infty \)) map \( F: M \to \mathbb{R}^n \), a point \( x \in M \) is called a **regular point of** \( F \) if the linear map \( dF_x: T_x M \to T_{F(x)} \mathbb{R}^n \) is surjective.

A point \( x \in M \) is called a **singular point of** \( F \) if it is not regular. A point \( y \in \mathbb{R}^n \) is called a **singular value** of \( F \) if at least one point \( x \in F^{-1}(y) \) is a singular point of \( F \) and
is called a **regular value** of $F$ if every $x \in F^{-1}(y)$ is a regular point of $F$, i.e., $y \in \mathbb{R}^n$ is called a regular value of $F$ if it is not a singular value for $F$. Note that if $F^{-1}(y) = \emptyset$, then $y$ is considered to be a regular value of $F$ because the definition of regular value is vacuously true. By the Implicit Function Theorem (see [24] Chapter 1, §5 page 19), if $x$ is a regular point of $F$ and $y = F(x)$, then there is a neighbourhood $U_x \subset M$ of $x$ such that $U_x \cap F^{-1}(y)$ is a smooth submanifold of $M$. Thus, if $y$ is a regular value of $F$ then $F^{-1}(y)$ is a smooth submanifold of $M$.

Recall that if $X$ is a smooth vector field on $M$ then a **solution curve** for $X$ is a smooth map $\sigma$ of an open interval $(a, b) \subseteq \mathbb{R}$ into $X$ such that $\sigma'(t) = (X \circ \sigma)(t)$ for all $t \in (a, b)$. If $0 \in (a, b)$ and $x := \sigma(0)$ then we call $x$ the **initial condition** of the solution $\sigma$.

The next theorem is commonly called the local existence and uniqueness theorem for ordinary differential equations (or vector fields). A detailed exposition of this fundamental theorem is presented in Chapter IV of [24] or Palais [34] §2.

**Theorem 2.2.6** (Local Existence and Uniqueness for Ordinary Differential Equations). Let $X$ be a smooth vector field on an open set $\mathcal{O}$ in a Hilbert space $\mathbb{H}$. Given $x \in \mathcal{O}$ there is a neighbourhood $U$ of $x$ included in $\mathcal{O}$, an $\epsilon > 0$, and a smooth map $\phi: U \times (-\epsilon, \epsilon) \to \mathbb{H}$ such that:

1. If $x' \in U$ then the map $\sigma_{x'}: (-\epsilon, \epsilon) \to \mathbb{H}$ defined by $\sigma_{x'}(t) = \phi(x', t)$ is a solution of $X$ with initial condition $x'$;

2. If $\sigma: (a, b) \to \mathbb{H}$ is a solution curve of $X$ with initial condition $x' \in U$ then $\sigma(t) = \sigma_{x'}(t)$ for all $t \in (a, b) \cap (-\epsilon, \epsilon)$.

**Proof.** See Palais [34] §2 or Lang [24] Chapter IV. 

The next result is a consequence of Theorem 2.2.6 for vector fields.
Lemma 2.2.7. Let $M$ be a Hilbert manifold and let $X$ be a smooth vector field on $M$. For each $x \in M$ there exists a unique solution curve $\sigma_x$ of $X$ with initial condition $x$ such that every solution curve of $X$ with initial condition $x$ is a restriction of $\sigma_x$.

Proof. See Palais [34] §6. \qed

The solution curve $\sigma_x$ above in Lemma 2.2.7 is called the maximum solution curve of $X$ with initial condition $x$. Define $\alpha: M \to (0, \infty]$ and $\beta: M \to [-\infty, 0)$ by the requirement that the domain of $\sigma_x$ is $(\alpha(x), \beta(x))$. The function $\alpha$ and $\beta$ are called respectively the positive and negative escape time functions for $X$.

Definition 2.2.8. Let $M$ be a Hilbert manifold. Let

$$D := D(X) = \{(x, t) \in M \times \mathbb{R} \mid \alpha(x) < t < \beta(x)\}$$

and for each $t \in \mathbb{R}$ let $D_t := D_t(X) = \{x \in M \mid (x, t) \in D\}$. Define $\phi: D \to M$ by $\phi(x, t) = \sigma_x(t)$ and $\phi_t: D_t \to M$ by $\phi_t(x) = \sigma_x(t)$. The set $\{\phi_t\}$ is called the maximum local one parameter group generated by $X$ or the flow generated by $X$.

Theorem 2.2.9. In the set up of Definition 2.2.8 $D$ is open in $M \times \mathbb{R}$ and $\phi: D \to M$ is smooth. For each $t \in \mathbb{R}$ the set $D_t$ is open in $M$ and $\phi_t$ is a smooth diffeomorphism of $D_t$ onto $D_{-t}$ having $\phi_{-t}$ as its inverse. If $x \in D_t$ and $\phi_t(x) \in D_s$ then $x \in D_{t+s}$ and $\phi_{t+s}(x) = \phi_s(\phi_t(x))$. 

Chapter 3

Normal Forms
The purpose of this chapter is to extend the existing theory on local normal forms for Hamiltonian group actions to infinite-dimensional Banach manifolds. More specifically, we formalize the local linearization theorem for compact group actions on Banach manifolds and establish a symplectic version of this local linearization theorem. In so doing, we provide a $G$-equivariant version of Moser's argument suitable for our goal.

### 3.1 Statements

Our initial result is similar to the finite-dimensional local linearization theorem for compact group actions, found in [22]. In fact, in [50] Weinstein notes without proof that the local linearization theorem holds for smooth actions of compact groups on Banach manifolds. Following this lead (and for the sake of completeness here), we state and prove the following version of the local linearization theorem.

**Theorem 3.1.1** (The Local Linearization Theorem). Let a compact Lie group $G$ act on a real Banach manifold $M$ and let $m$ be a fixed point. Then there exists a $G$-equivariant diffeomorphism $f$ from an invariant neighbourhood of the origin in $T_mM$ onto an invariant neighbourhood of $m$ in $M$.

We shall now review some relevant definitions and notions to be used in a symplectic version of the local linearization theorem, Theorem 3.1.2. In the process we will point out differences from the finite-dimensional case when necessary.

To begin, we wish to call attention to the fact that there exist various definitions of differential forms and other related such concepts. For example, see [23, Chapter VIII: Infinite Dimensional Differential Geometry]. For our purposes, it is enough to use the definitions found in [24, p.61 and p.124]. That is, if $E$ is a real Banach space and $U$ an open chart of $E$, then a **differential form** of degree $r$ (or simply an $r$-**form**) on $U$ is an $r$-multilinear and alternating (in the last $r$ variables) smooth map $U \times E \times \cdots \times E \to E$. Let $L^\otimes_r(TU)$ denote the bundle of $r$-multilinear continuous alternating forms on $U$. Then
$L^r_a(TU)$ is equal to $U \times L^r_a(E)$. Thus, a differential form of degree $r$ on $U$ is a section of $L^r_a(TU)$ and is entirely determined by the projection on the second factor $L^r_a(E)$. The usual definition of the exterior derivative, and the proof of the Poincaré lemma, apply without modification [24].

Next, recall that on a vector space $E$, a bilinear form $\omega : E \times E \to \mathbb{R}$ is said to be weakly nondegenerate if for every $v \in E$,

$$\omega(v, w) = 0 \quad \forall w \in E \Rightarrow v = 0. \quad (3.1)$$

Now assume $E$ is a Banach space. Its dual, $E^*$, is the space of bounded linear functionals on $E$. Recall also that $\omega$ defines a linear map $\omega^\#: E \to E^*: u \mapsto \omega(u, \cdot)$. So weak nondegeneracy means $\ker(E \to E^*) = 0$, this is, $E \to E^*$ is injective. If this map is also surjective, then $\omega$ is said to be strongly nondegenerate.

In what follows we require our symplectic form to be nondegenerate in the strong sense. Let $M$ be a Banach manifold endowed with a closed differential 2-form $\omega$, which at each $m$ in $M$ is strongly nondegenerate as a bilinear form on $T_mM$. Said in other words, $T_mM \to T^*_mM$ is a linear homeomorphism. Notice that continuity of the inverse of this map is equivalent to the openness of $T_mM \to T^*_mM$, which immediately follows from the Open Mapping theorem as $T_mM \to T^*_mM$ is surjective here.

**Theorem 3.1.2** (The Local Linearization Theorem - symplectic version). Let a compact Lie group $G$ act on a strongly symplectic Banach manifold $(M, \omega)$. Let $m$ be a fixed point. Then there exists a $G$-equivariant symplectomorphism $f$ from an invariant neighbourhood of the origin in $T_mM$ onto an invariant neighbourhood of $m$ in $M$.

### 3.2 Proofs

The proof of Theorem 3.1.1 is obtained by analogy with the finite-dimensional argument. We begin with a simple lemma. Suppose $G$ is a compact Lie group, and that $G$ acts on a Banach manifold $M$. 
Lemma 3.2.1. Let $F$ be a diffeomorphism from an invariant neighbourhood of $m$ in $M$ onto a neighbourhood of the origin in a vector space $V$ such that $F(m) = 0$. Suppose that $G$ acts on $V$. Define the $G$-average of $F$ as $	ilde{F}(u) := \int_{g \in G} (gF(g^{-1}u))dg$, where $dg$ is the normalized Haar measure on $G$. Then the average $	ilde{F}$ is $G$-equivariant.

Proof. Let $U \subset M$ be an invariant neighbourhood of $m$ and let $F : U \to V$ be any diffeomorphism onto a neighbourhood of the origin in $V$. To ensure the existence of such a diffeomorphism we use that there exists a chart near $m$ and that every open neighbourhood of $m$ contains an invariant open neighbourhood of $m$. Let $	ilde{F} : U \to V$ given by $\tilde{F}(u) = \int_{g \in G} (gF(g^{-1}u))dg$ be its average. We want to show $\tilde{F}(h \cdot u) = h \cdot \tilde{F}(u)$ $\forall h \in G, u \in U$.

Consider

$$\tilde{F}(h \cdot u) = \int_{g \in G} (gF(g^{-1}h \cdot u)) dg \ ,$$

by definition of $\tilde{F}$

$$= h \left( \int_{g \in G} (h^{-1}gF(g^{-1}u)) dg \right)$$

$$= h \left( \int_{g \in G} h^{-1}gF((h^{-1}g)^{-1}u) dg \right)$$

$$= h \left( \int_{g \in G} jF(j^{-1}u) dj \right) \ ,$$

where $j = h^{-1}g$. Note $dg$ is invariant under $g \mapsto j$

$$= h \cdot \tilde{F}(u) \ ,$$

as wanted.

\[ \square \]

Lemma 3.2.2. Let $F$ be a diffeomorphism from an invariant neighbourhood of $m$ in $M$ to a neighbourhood of the origin in $V = T_mM$ with the isotropy action. Let $\bar{F}$ be its average. Suppose the derivative of $F$ at $m$ is the identity mapping on $T_mM$. Then $d\bar{F}|_m : T_mM \to T_mM$ is the identity.

Proof. Let $U \subset M$ be an invariant neighbourhood of $m$ and let $F : U \to T_mM$ be any diffeomorphism onto a neighbourhood of the origin in $T_mM$. 

We have for all $g \in G$, $g : U \to U$ and $g_* : T_m M \to T_m M$. By definition $dg|_m = g_*$ and $dg_*|_0 = g_*$ because $g_*$ is a linear map and $dg_*$ is also linear.

So the average of $F$ is $\tilde{f} := \int_{g \in G} (g_* F (g^{-1} \cdot u)) \, dg$. Therefore,

$$
d\tilde{f}|_m(\cdot) = \int_{g \in G} (g_* F g^{-1})|_m(\cdot) \, dg \\
= \int_{g \in G} (dg_*|_0 \circ dF|_m \circ dg^{-1}|_m)(\cdot) \, dg, \text{ by the chain rule} \\
= \int_{g \in G} (g_* \circ dF|_m \circ g^{-1})(\cdot) \, dg, \text{ by the above choice of notation and since } dg_*|_0 = g_* \\
= \int_{g \in G} (g_* \circ g^{-1})(\cdot) \, dg, \text{ because } dF|_m = \text{identity by assumption} \\
= \int_{g \in G} (\cdot) \, dg = \text{identity} \\
$$

Proof of Theorem 3.1.1. Let $U \subset M$ be an invariant neighbourhood of $m$. Let $F : U \to T_m M$ be any smooth map such that $dF|_m : T_m M \to T_m M$ is the identity mapping.

Take any $g \in G$. Note $g$ acts on both $U$ and $T_m M$; $g : U \to U$ and $g_* : T_m M \to T_m M$. Let $dg|_m = g_*$ and $g_*|_0 = g_*$. 

Consider $g_* \circ F \circ g^{-1} : U \to T_m M$. By construction, this map is also a diffeomorphism such that its derivative at $m$ is the identity mapping on $T_m M$. The average $\tilde{f} : U \to T_m M$, which is defined by $\tilde{f}(u) := \int_{g \in G} (g_* F (g^{-1} \cdot u)) \, dg$ where $dg$ is the invariant Haar measure on $G$, is a $G$-equivariant diffeomorphism such that $d\tilde{f}|_m = \text{identity}_{T_m M}$ by lemma 3.2.1 with $V = T_m M$ and lemma 3.2.2.

By the inverse function theorem for Banach manifolds (see [24]) we can invert $\tilde{f}$ on a neighbourhood of $m$ to obtain the desired diffeomorphism $f$, as required.
remark as to how to establish an equivariant version of the Darboux-Weinstein theorem is made. To help in the analysis in the proof of Theorem 3.1.2, we will need an equivariant local version of Moser’s theorem. Toward this end, and using similar techniques found in [50] and [51], we will employ the next lemma.

**Lemma 3.2.3** (Moser’s Theorem). Let $M$ be a Banach manifold with strongly symplectic forms $\omega_0$ and $\omega_1$. Let $m$ be in $M$. Assume $\omega_0$ and $\omega_1$ coincide on $T_mM$. Then there exists a neighbourhood $U$ of $m$ and there exists a diffeomorphism $\psi$ from $U$ to an open subset of $M$ such that $\psi^*\omega_1 = \omega_0|_U$.

**Proof.** Denote $\omega_t := (1-t)\omega_0 + t\omega_1$, where $\omega_0 := \psi^*\omega|_m$ and $\omega_1 := \omega$. By the Poincaré Lemma [24], there exists a 1-form $\sigma$ on $U$ such that $\omega_1 - \omega_0 = d\sigma$. Observe that we can arrange for $\sigma|_{T_mM} = 0$. We now look for a smooth, time dependent, vector field $X_t : M \to M$ on a neighbourhood of $m$ with $X_t|m = 0$ and $\iota(X_t)\omega_t = -\sigma$.

The main idea is to determine a family of diffeomorphisms $\psi_t \in \text{Maps}((U \to M)$ with $\psi_t^*\omega_t = \omega_0|_U$ by representing them as the flow of a family of time-dependent vector fields $X_t$ on a neighbourhood of $m$. Thus we suppose that

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \quad \psi_0 = \text{id}. \quad (3.2)$$

So we know

$$\psi_t^*\omega_t = \omega \iff \frac{d}{dt}(\psi_t^*\omega_t) = 0, \text{ for all } t$$

$$\iff \psi_t^* \left( \frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t \right) = 0, \text{ where } \mathcal{L}_{X_t} \text{ is the Lie derivative of } \omega_t \text{ along } X_t$$

$$\iff \psi_t^* (d\sigma + \iota(X_t)d\omega_t + d(\iota(X_t)\omega_t)) = 0, \text{ by using Cartan’s formula and the choice of } \sigma$$

$$\iff \psi_t^* (d\sigma + d(\iota(X_t)\omega_t)) = 0, \text{ since } \omega_t \text{ is closed by assumption}$$

$$\iff X_t \text{ satisfies the linear (over } \mathbb{R}) \text{ equation } d\sigma + d(\iota(X_t)\omega_t) = 0$$

$$\iff d(\sigma + \iota(X_t)\omega_t) = 0.$$.  


This last identity will hold if

\[ \sigma + \iota(X_t)\omega_t = 0. \quad (3.3) \]

Observe that for all \( t \), \( \omega_t \) is strongly nondegenerate at \( m \). Thus, there exists a neighbourhood \( U \) of \( m \) such that for all \( t \) \( \omega_t \) is strongly nondegenerate on \( U \). Let \( \omega_t(X_t, \cdot) = -\sigma \) where \( \omega_t : T_m M \to T^*_m M, (\sigma)_m \in T^*_m M \). Recall that if \( s \mapsto A_s \) is a smooth family of invertible operators then the family \( A_s^{-1} \) of inverses is smooth. So \( X_t = -(\omega_t)^{-1}\sigma \) is a smooth (and also smooth in \( t \)), time-dependent vector field taking values in \( M \). So, for any choice of 1-form \( \sigma \) equation (3.3) can always be solved for \( X_t \). Therefore, (reading this argument backwards) we see that we can always find an \( X_t \) that satisfies \( \frac{d}{dt}\omega_t + L_{X_t}\omega_t = 0 \).

Hence, by integrating \( X_t \) (and shrinking \( U \) again if necessary), there exists a family \( \psi_t \) of diffeomorphisms such that (3.2) holds. From this we easily deduce \( \psi_t^*\omega_t = \omega_0|_U \) and accordingly the required conditions are satisfied. Let \( \psi = \psi_1 \). That is to say, there exists an isotopy \( \psi : U \times [0, 1] \to M : (q, t) \mapsto \psi_t(q) \), \( \psi_t \in \text{Maps}(U \to M) \), and \( \psi_0 = \text{id} \) with \( \psi^*\omega_t = \omega_0 \) for all \( t \in [0, 1] \).

**Proof of Theorem 3.1.2.** Let \( U \subset M \) be an invariant neighbourhood of \( m \). Proceeding in the same manner as the proof of Theorem 3.1.1 let \( F : U \to T_m M \) be any smooth map such that \( dF|_m = \text{identity}_{T_m M} \). The average \( \psi : U \to T_m M \), given by

\[ \psi(u) := \int_{g \in G} (g_*F(g^{-1} \cdot u)) \, dg \]

where \( dg \) is the Haar measure on \( G \), is smooth, \( G \)-equivariant (c.f. Lemma 3.2.1), and satisfies \( d\psi|_m = \text{identity}_{T_m M} \) (c.f. Lemma 3.2.2).

Given a symplectic form \( \omega \) on \( M \), let \( \omega_0 := \psi^*(\omega|_m) \) and \( \omega_1 := \omega \). These are \( G \)-invariant symplectic forms on \( U \subset M \). Notice that \( \omega_0 \) and \( \omega_1 \) coincide on \( T_m M \) becuase

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1 See [24] chapters IV and V for explicit conditions that guarantee integrability of a vector field on a Banach manifold
$d\psi|_m = id_{T_xM}$. Consider now the family $\omega_t := (1-t)\omega_0 + t\omega_1$ of closed 2-forms on $U$. We can assume that $\omega_t$ is a symplectic form for all $t \in [0, 1]$ by shrinking $U$ if necessary. We want a $G$-equivariant map $\psi_t : U \to T_mM$ such that $\frac{d}{dt}\psi_t^*\omega_t = 0$. That is, we need a local equivariant Moser’s theorem. This map is obtained by Lemma 3.2.3 (applied to a neighbourhood of $m$) with an additional restriction. The $G$-equivariance of the $\psi_t$ provided in 3.2.3 can be achieved by restricting the choice of $\sigma$ to $G$-invariant $\sigma$; all of the constructions can then be made ‘equivariantly’ with respect to $G$.

Therefore, by the inverse function theorem [24] we invert $\psi$ on a neighbourhood of $m$ to get the desired symplectomorphism $f$.\thmbox
Chapter 4

Connectedness - The Base Case
We begin this chapter by collecting some facts on Morse Theory and gradient flows which are relevant and needed to prove the main results of this chapter, the Connected Levels Theorem (Theorem 4.3.5).

4.1 Morse Functions and Their Gradient Flows

Lemma 4.1.1. Let $\mathbb{H}$ be a Hilbert space and let $\mathbb{H}_+ \subset \mathbb{H}$ be closed subspaces such that $\mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_-$. Let $f^\mathbb{H} : \mathbb{H} \to \mathbb{R}$ be defined by

$$f(x_+, x_-) = \|x_+\|^2 - \|x_-\|^2.$$ 

Then the trajectory of $-\nabla f^\mathbb{H}$ starting at $x = (x_+, x_-) \in \mathbb{H}$ is given by

$$t \mapsto (e^{-2t}x_+, e^{2t}x_-).$$

Proof. Note that $(e^{-2t}x_+, e^{2t}x_-)|_{t=0} = x$. It is enough to show that

$$\frac{d}{dt} \bigg|_{t=0} (e^{-2t}x_+, e^{2t}x_-) = \langle \nabla f^\mathbb{H} |_x, v \rangle.$$ 

Recall that the gradient vector field $\nabla f^\mathbb{H}$ on $\mathbb{H}$ is defined by the property that for all $x, v \in \mathbb{H}$, $df |_x (v) = \langle \nabla f^\mathbb{H} |_x, v \rangle$. So it is enough to show that for all $x, v \in \mathbb{H}$,

$$df |_x (v) = -\langle (2x_+, 2x_-), v \rangle.$$ 

Let $x = (x_+, x_-) \in \mathbb{H}$ and $v = (v_+, v_-) \in \mathbb{H}$. Then

$$df |_x (v) = df |_{(x_+, x_-)} (v_+, v_-)$$

$$= D_{v_+} (\|x_+\|^2) - D_{v_-} (\|x_-\|^2) \text{ because } f^\mathbb{H}(x) = \|x_+\|^2 - \|x_-\|^2$$

$$= \frac{d}{dt} \bigg|_{t=0} \|x_+ + tv_+\|^2 - \frac{d}{dt} \bigg|_{t=0} \|x_- + tv_-\|^2$$

$$= \frac{d}{dt} \bigg|_{t=0} (\|x_+\|^2 + 2t\langle x_+, v_+ \rangle + t^2\|v_+\|^2) - \frac{d}{dt} \bigg|_{t=0} (\|x_-\|^2 + 2t\langle x_-, v_- \rangle + t^2\|v_-\|^2)$$

$$= 2\langle x_+, v_+ \rangle - 2\langle x_-, v_- \rangle$$
\[ -\langle (-2x_+, 2x_-), v \rangle \]

Therefore, \((e^{-2t}x_+, e^{2t}x_-)\) gives the desired flow. \(\square\)

**Definition 4.1.2.** A smooth function \(f: M \to \mathbb{R}\) on a Hilbert manifold \(M\) is called a **Morse function** if all of its critical points are strongly nondegenerate. That is, for every \(x \in \text{Crit}(f)\), the operator \(\nabla^2 f|_x: T_x M \to T_x M\) obtained from the Hessian via the Riemannian metric is a linear isomorphism.

**Remark 4.1.3.**

1. Note that whether or not a function is Morse is independent of a choice of Riemannian metric.

2. Some references in the literature have weak nondegeneracy, that is the Hessian \(H_p(f)\) induces only an injective map \(\nabla^2 f(x): T_x M \to T_x M\), i.e. \(\ker(\nabla^2 f|_x) = 0\), in their definition of a Morse function.

In Morse theory, the Morse lemma introduces special coordinates around a critical point. We recall this fundamental lemma now for Hilbert manifolds.

**Lemma 4.1.4** (The Morse Lemma). Let \(f: M \to \mathbb{R}\) be a smooth function and let \(p \in \text{Crit}(f)\). Suppose that \(p\) is strongly nondegenerate. Then there exists an open neighbourhood \(B \subset M\) of \(p\) and a chart \(\phi: B \to \mathbb{H}\) around \(p\) with target a Hilbert space \(\mathbb{H}\) such that \(\phi(p) = 0\) and \((f \circ \phi^{-1})(v) = \|Pv\|^2 - \|(I - P)v\|^2\) on \(\phi(B)\), where \(P\) is an orthogonal projection in \(\mathbb{H}\) to a closed subspace (i.e., \(Pv \in \mathbb{H}_+\) and \((I - P)v \in \mathbb{H}_-\) where \(\mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_-\)).

**Proof.** See Palais [34] page 307. \(\square\)

**Remark 4.1.5.**

1. It is an immediate consequence of the Morse Lemma that a nondegenerate critical point of a smooth function, say \(f\), on a Hilbert manifold is isolated in \(\text{Crit}(f)\). In particular, if \(f\) is a Morse function then the set \(\text{Crit}(f)\) is discrete.
2. Note that weak nondegeneracy does not work in this setting; in fact weakly non-degenerate critical points need not be isolated in \( \text{Crit}(f) \). For example let \( M = \ell_2 = \left\{ \{ x_k \} \subseteq \mathbb{R} \mid \sum_{k=1}^{\infty} |x_k|^2 < \infty \right\} \). Define \( f: \mathbb{H} \rightarrow \mathbb{R} \) by \( f(x) = -\sum_{k=1}^{\infty} \frac{\cos(kx_k)}{k^4} \) (\( f \) is smooth). Then \( 0 \in \text{Crit}(f) \). Moreover \( 0 \) is weakly nondegenerate. But any neighbourhood of \( 0 \) has infinitely many critical points. See [48], pg. 51 for details.

In the Morse Lemma [4.1.4] the coordinate chart \( \phi \) is called a \textbf{Morse chart} for the function \( f \). Note that the index at \( p \) equals the dimension of the range of \( I - P \) and the coindex of \( p \) equals the dimension of the range of \( P \), where \( P \) is the projection from Lemma [4.1.4] ([34] pg. 303).

**Definition 4.1.6.** Let \( X \) be a vector field on a manifold \( M \). The vector field \( X \) is said to be \textbf{standard near a point} \( p \) in \( M \) if there exists a chart \( \phi: U_p \rightarrow B_0 \subseteq \mathbb{H} \), where \( B_0 \) is a neighbourhood of \( 0 \) in \( \mathbb{H} \), such that \( p \mapsto 0 \) and there exists a decomposition \( \mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_- \) such that \( \phi \) intertwines the vector field near \( p \) with the vector field on \( \mathbb{H} \) whose value at the point \( (x_+, x_-) \) is equal to \( (-2x_+, 2x_-) \).

**Remark 4.1.7.**
1. Let \( (M, g) \) be a Riemannian manifold and let \( f \) be a smooth function on \( M \). If there exists a neighbourhood of a point \( p \in M \) and a Morse chart near \( p \) which is also an isometry (with respect to the metric on \( \mathbb{H} \)), then the gradient vector field \( \nabla_g f \) of \( f \) is standard near \( p \).

We will say that the Riemannian metric is standard (near each critical point \( p \) of \( f \)) if the gradient vector field with respect to this metric is standard near each \( p \).

2. Note that the flow generated by a smooth vector field which is standard near a point \( p \) is locally conjugate to the flow generated by its linearization.

Suppose that we are given a complete Riemannian metric \( g \) on a Hilbert manifold \( M \) and let \( f: M \rightarrow \mathbb{R} \) be a smooth real-valued function on \( M \). Let us collect together some basic properties of \( -\nabla_g f \), the negative gradient of \( f \) with respect to \( g \):
1. $-\nabla_g f$ has the property that $((\nabla_g f)f)(p) = 0$ if and only if $p \in \text{Crit}(f) \subset M$.

Therefore $\text{Crit}(f)$ is the set of zeros of the real-valued function $||\nabla_g f||$;

2. The flow of the vector field $-\nabla_g f$ is a one-parameter group of diffeomorphisms $\rho^M_t : D_t \to M$ for $t \in \mathbb{R}$. We require that $\rho^M_0 = \text{id}$ and $\frac{d\rho^M_t}{dt}|_m = -\nabla_g f|_{\rho^M_t(m)}$.

3. The value of $f$ decreases along any non-constant flow line, $t \mapsto \rho^M_t$, of $-\nabla_g f$. We can easily see this, by Rolle’s theorem, from the following calculation:

$$\frac{d}{dt} f(\rho^M_t(\cdot)) = df(\dot{\rho}^M_t(\cdot)) \text{ by def of } df$$

$$= \langle \nabla_g f(\cdot), \dot{\rho}^M_t(\cdot) \rangle \text{ by def of } \nabla_g f$$

$$= \langle \nabla_g f(\cdot), -\nabla_g f(\cdot) \rangle \text{ by def of } \rho^M_t$$

$$= -||\nabla_g f(\cdot)||^2 \leq 0$$

with equality only if $p \in \text{Crit}(f)$. That is, by Rolle’s theorem, $(-\nabla_g f)(f)$ is negative off the critical set of $f$.

Next we establish that a Morse chart that is also an isometry intertwines the negative gradient flow on the neighbourhood with the negative gradient flow on the vector space.

**Lemma 4.1.8.** Let $M$ be a Hilbert manifold and let $f : M \to \mathbb{R}$ be a Morse function. Let $p \in \text{Crit}(f)$ and let $U_p \subset M$ be a neighbourhood of $p$. Let $\mathbb{H}$ be a Hilbert space and let $\mathbb{H}_\pm \subset \mathbb{H}$ be closed subspaces such that $\mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_-$. Let $\phi : U_p \to \mathbb{H}$ be an isometry such that $\phi(U_p) = B_+ \times B_-$ where $B_\pm \subset \mathbb{H}_\pm$ are unit balls in $\mathbb{H}_\pm$ respectively. Assume that $\phi$ is a Morse chart. Let $\rho^M_t$ be the gradient flow of $-f$ on $M$. By Lemma 4.1.1 the negative gradient flow of $f^\mathbb{H}(x) = ||x_+||^2 - ||x_-||^2$ on $\mathbb{H}$ is

$$\rho^\mathbb{H}_t(x_+, x_-) = (e^{-2t}x_+, e^{2t}x_-).$$

Then for all $t \in \mathbb{R}$ and for any $m \in U_p \cap (\rho^M_t)^{-1}(U_p)$,

$$\phi(\rho^M_t(m)) = \rho^\mathbb{H}_t(\phi(m)).$$
Proof. Let $t \in \mathbb{R}$. Let $m \in U_p \cap (\rho_t^M)^{-1}(U_p)$.

We first show that $\phi$ intertwines the vector field $-\nabla_g f$ on $M$ with the vector field $(x \mapsto (-2x_+, 2x_-))$ on $\mathbb{H}$. That is, we need to show that

$$d\phi_m (-\nabla_g f|m) = (x \mapsto (-2x_+, 2x_-))_{\phi(m)}.$$

Consider $d\phi_m: T_m U_p \to T_{\phi(m)} \mathbb{H}$. Note that $T_{\phi(m)} \mathbb{H} = \mathbb{H}$ and that $T_m U_p = T_m M$ because $U_p$ is open. So $d\phi_m$ is a bijective linear map between $T_m M$ and $\mathbb{H}$. It follows that $d\phi_m(-\nabla_g f|m) \in \mathbb{H}$. But recall $\phi$ is a Morse chart and that $f^\mathbb{H}(x_+, x_-) = ||x_+||^2 - ||x_-||^2$ by hypothesis. Hence, $d\phi_m(-\nabla_g f|m)$ decomposes into a positive and negative part. Namely, $d\phi_m(-\nabla_g f|m) = -(2x_+, -2x_-)$. Since $\phi$ is an isometry it follows that

$$d\phi_m (-\nabla_g f|m) = (x \mapsto (-2x_+, 2x_-))_{\phi(m)}$$

as wanted.

Next we show that $\phi$ intertwines the flow $\rho_t^M$ on $U_p \subset M$ with the flow $\rho_t^\mathbb{H}$ on $B_+ \times B_- \subset \mathbb{H}$. Assume that $t > 0$. The case $t < 0$ is similar. Let $\gamma: [0, t] \to M$ be a maximal trajectory for $-\nabla_g f$ such that $\gamma(0), \gamma(t) \in U_p$. Note that $\gamma^{-1}(U_p)$ is an interval.
The diffeomorphism $\phi$ takes $\gamma$ to a maximal trajectory, say $\gamma^* := \gamma \circ \phi$, in $B_+ \times B_-$ for $(x \mapsto (-2x_+^2, 2x_-))|_{B_+ \times B_-}$. Since $\phi$ is a diffeomorphism between $U_p$ and $B_+ \times B_-$ that is also an isometry, we have that

$$\rho^M_t(m) \in U_p \text{ if and only if } \rho^H_t(\phi(m)) \in B_+ \times B_- \text{ for all } t' \in [0, t].$$

That is, the “entry” and “exit” values of $f$ (with respect to the flow $\rho^M|_{U_p}$) and $f^H$ (with respect to the flow $\rho^H|_{B_+ \times B_-}$) are the same.

Figure 4.1: Intertwining Gradient Flows

On $M$: Consider $\rho^M_t$, an arbitrary flow line of $-\nabla_g f$ on $M$. We know by definition
that \( \frac{d}{dt} \rho_t^M(m) = -\nabla_g f|_{\rho_t^M(m)} \). So we have that

\[
\frac{d}{dt} f (\rho_t^M(m)) = ((-\nabla_g f) f)(m) = -||\nabla_g f(m)||^2.
\]

This implies that \( f (\rho_t^M(m)) \) is (monotonically) decreasing in \( t \), i.e., \( f \) is decreasing along non-constant flow lines of \(-\nabla_g f\). We define the entry time of \( \rho_t^M \) on \( U_p \) as the point

\[
t_\rho := \inf \{ \tau \mid [0, \tau] \subseteq \gamma^{-1}(U_p) \}.
\]

Then the entry point of \( \rho_t^M \) on \( U_p \) is \( x_\rho := \gamma(t_\rho) \in \overline{U_p} \subseteq M \). Similarly, we define the exit time of \( \rho_t^M \) on \( U_p \) as the point \( \bar{t}_\rho := \sup \{ \tau \mid [\tau, t] \subseteq \gamma^{-1}(U_p) \} \). Then the exit point of \( \rho_t^M \) on \( U_p \) is \( y_\rho := \gamma(\bar{t}_\rho) \in \overline{U_p} \subseteq M \).

From these entry/exit point definitions we see that \( f(x_\rho) > f(y_\rho) \) since \( f \) is decreasing along \( \rho_t^M \).

On \( \mathbb{H} \): Recall again that by assumption, \( \phi \) is a diffeomorphism between \( U_p \) and its image \( \phi(U_p) = B_+ \times B_- \). So \( d\phi \) is a bijective linear map between the sets \{ vector fields on \( M \) \} and \{ vector fields on \( \mathbb{H} \) \}. Consequently, all that remains is to consider \( \rho_t^\mathbb{H} \), the corresponding flow of \(-\nabla_g f\) on \( \mathbb{H} \). Recall that \( \rho_t^\mathbb{H}(x_+, x_-) = (e^{-2t}x_+, e^{2t}x_-) \) by Lemma 4.1.1. Suppose that \( ||B_+|| = ||B_-|| = 1 \). Observe that \( \rho_t^\mathbb{H} \) meets \( B_+ \times \partial B_- \) at the point \( (||x_-||x_+, \frac{1}{||x_-||}x_-) \). Also observe that \( \rho_t^\mathbb{H} \) meets \( \partial B_+ \times B_- \) at the point \( (\frac{1}{||x_+||}x_+, ||x_+||x_-) \).

We define the entry point, respectively exit point, of \( \rho_t^\mathbb{H} \) with \( B_+ \times B_- \) as follows:

Case 1: If both \( x_+ \) and \( x_- \) are nonzero, then the entry point is \( (\frac{1}{||x_+||}x_+, ||x_+||x_-) \) and the exit point is \( (||x_-||x_+, \frac{1}{||x_-||}x_-) \).

Case 2: If \( x_+ = 0 \) but \( x_- \neq 0 \), then \( \rho_t^\mathbb{H}(x_+, x_-) = (0, e^{2t}x_-) \). Therefore, for large \( t \) \( \rho_t^\mathbb{H} \) never meets \( \partial B_+ \times B_- \). That is, \( \rho_t^\mathbb{H} \) never meets \( B_+ \times B_- \). For \( t \ll 0 \), the entry point is \( (0, \frac{1}{||x_-||}x_-) \) and there is no exit point. That is, \( \rho_t^\mathbb{H} \) enters \( B_+ \times B_- \) and converges to \( \phi(p) = (0, 0) \in B_+ \times B_- \).

Case 3: If \( x_- = 0 \) but \( x_+ \neq 0 \), then \( \rho_t^\mathbb{H}(x_+, x_-) = (e^{-2t}x_+, 0) \). So for large \( t \), the entry point of \( \rho_t^\mathbb{H} \) is \( (\frac{1}{||x_+||}x_+, 0) \) and there is no exit point because \( \rho_t^\mathbb{H} \) never meets \( B_+ \times \partial B_- \). That is, \( \rho_t^\mathbb{H} \) enters \( B_+ \times B_- \) and converges to \( \phi(p) = (0, 0) \), i.e., \( \rho_t^\mathbb{H} \) never exits. For \( t \ll 0 \), \( \rho_t^\mathbb{H} \) never meets \( \partial B_+ \times B_- \). That is, \( \rho_t^\mathbb{H} \) never meets \( B_+ \times B_- \).
Case 4. If \((x_+, x_-) = (0, 0)\) then \(\rho_{t}^{H}(x_+, x_-)\) is constant. For all \(t \in \mathbb{R}\), \(\rho_{t}^{H}\) will either never meet \(B_+ \times B_-\) or it will enter at the point \(\left(\frac{1}{\|x_+\|}x_+, \|x_+\|x_-\right)\) and exit at the point \(\left(\|x_-\|x_+, \frac{1}{\|x_-\|}x_\right)\).

Thus, \(f^{H} > 0\) at each entry point and \(f^{H} < 0\) at each exit point for the flow on \(B_+ \times B_-\). Consequently, \(f > 0\) at each entry point and \(f < 0\) at each exit point for the flow on \(U_p\). Hence, if any trajectory on \(M\) exits \(U_p\) it does not return.

It now follows from the local existence and uniqueness results for ODEs (see Lang \[24\] Chapter IV), that our result \(\phi_\left(\rho_{t}^{M}(m)\right) = \rho_{t}^{H}(\phi(m))\) holds. \(\square\)

The last lemma shows us that near each critical point of \(f\) we can always modify a Riemannian metric on \(M\) so that the negative gradient vector field of \(f\) is standard near each critical point of \(f\). Stated more precisely,

**Lemma 4.1.9.** Let \(M\) be a Hilbert manifold. Let \(f: M \to \mathbb{R}\) be a Morse function. Let \(g\) be a Riemannian metric on \(M\). For each \(p \in \text{Crit}(f)\), let \(U_p\) be a neighbourhood of \(p\). Then there exists a Riemannian metric \(\tilde{g}\) on \(M\) such that:

(i) for all \(p \in \text{Crit}(f)\) there is a neighbourhood \(V_p\) of \(p\) in \(U_p\) such that \(-\nabla_{\tilde{g}} f\) is standard on \(V_p\);

(ii) \(\tilde{g}\) coincides with \(g\) outside of \(\bigcup_{p \in \text{Crit}(f)} U_p\)

**Remark 4.1.10.** This lemma serves as motivation for Lemma \[4.3.3\] in Section §4.3 (Connected Levels) which gives a direct proof of a stronger result.

**Proof.** We can shrink \(U_p\) such that the \(\overline{U_p}\) are disjoint. Let \(p \in \text{Crit}(f)\). By the Morse Lemma \[4.1.4\], there exists a neighbourhood \(B_p \subseteq U_p\) of \(p\) and a Morse chart \(\phi_p: B_p \to \mathbb{H}\) such that \(\phi_p(p) = 0\) and \((f \circ \phi_p^{-1})(v) = \|Pv\|^2 - \|(I - P)v\|^2\) on \(\phi_p(B_p)\).

Let \(\lambda_p: \mathbb{H} \to \mathbb{R}\) be a bump function. That is, let \(\lambda_p\) be a smooth function satisfying:

- \(0 \leq \lambda_p(x) \leq 1\), and
- \(\lambda_p(x) = 1\) near 0, and
• supp(\(\lambda_p(x)\)) \(\subseteq \phi_p(B_p)\).

Let \(m \in B_p\) and \(X, Y \in T_m B_p\). Then define the new metric
\[
\tilde{g}|_m(X, Y) = \begin{cases} 
(1 - \lambda_p(\phi(m)) \langle X, Y \rangle \phi(m) + \lambda_p(\phi(m)) \langle X_m, Y_m \rangle \phi(m) & \text{if } m \in U_p \\
g|m & \text{if } m \not\in \cup_{p \in \text{Crit}(p)} U_p
\end{cases}
\]

where \(\langle \cdot, \cdot \rangle_{\phi(m)}\) denotes the inner product coming from \(\mathbb{H}\).

By construction this new metric \(\tilde{g}\) satisfies properties (i) and (ii), as wanted. \(\square\)

### 4.2 Stable and Unstable Manifolds

Let us start this section by reviewing some known definitions and giving some important assumptions. We will then state and prove the Global (Un)Stable Manifold Theorem \[4.2.3\]. Lastly, we finish this section by examining a couple of additional results pertaining to the stable manifold.

**Definition 4.2.1.** Let \(M\) be a Hilbert manifold. Let \(f: M \to \mathbb{R}\) and let \(p \in \text{Crit}(f)\). Fix a metric \(g\) on \(M\). The **stable set** \(W^s(p)\) of \(p\) is defined to be the set of all points \(x \in M\) such that the \(- (\nabla g f)\)-trajectory \(\rho^M_t(x)\) starting at \(x\) is defined for all \(t \in \mathbb{R}^+\) and \(\lim_{t \to \infty} \rho^M_t(x) = p\). That is,
\[
W^s(p) = \{ x \in M \mid \rho^M_t(x) \text{ is defined for all } t \in \mathbb{R}^+ \text{ and } \lim_{t \to \infty} \rho^M_t(x) = p \}.
\]

The **unstable set** \(W^u(p)\) of \(p\) is defined to be the set of all points \(x \in M\) such that the \(- (\nabla g f)\)-trajectory \(\rho^M_t(x)\) starting at \(x\) is defined for all \(t \in \mathbb{R}^-\) and \(\lim_{t \to -\infty} \rho^M_t(x) = p\). That is,
\[
W^u(p) = \{ x \in M \mid \rho^M_t(x) \text{ is defined for all } t \in \mathbb{R}^- \text{ and } \lim_{t \to -\infty} \rho^M_t(x) = p \}.
\]

In the rest of this section we assume that \(M\) is a complete Riemannian Hilbert manifold (see below) and \(f: M \to \mathbb{R}\) is a Morse function that is bounded from below and
Chapter 4. Connectedness - The Base Case

satisfies Condition (C). By complete we mean that \( M \) is a complete metric space in the
metric induced from the Riemannian metric.

For the reader’s convenience we recall how this metric on \( M \) is defined. Given \( x \) and
\( y \) in \( M \) we define

\[
\rho(x, y) = \inf \int_0^1 \|\sigma'(t)\|dt
\]

where the infimum is over all \( C^1 \) paths \( \sigma: [0, 1] \rightarrow M \) such that \( \sigma(0) = x \) and \( \sigma(1) = y \).

Just as in the finite dimensional case one shows that \( \rho \) is a metric on \( M \) which is consistent
with the manifold topology (see Palais [34], §9 pg. 311).

We recall Condition (C) of Palais and Smale for \( f \):

Condition (C) (Palais-Smale condition):

If \( \{x_n\} \subset M \) is any sequence in \( M \) for which \( |f(x_n)| \) is bounded and for which
\( ||df|_{x_n}|| \rightarrow 0 \), then \( \{x_n\} \) has a convergent subsequence \( \{x_{n_k}\} \rightarrow p \)

Remark 4.2.2. 1. If \( M \) is finite dimensional and compact then for any choice of Riemannian metric for \( M \) the completeness, the boundedness below and the Condition (C) assumptions are automatically satisfied. Note also that if \( M \) is finite dimensional but not necessarily compact then Condition (C) for a smooth real-valued function is satisfied automatically for proper maps.

2. Condition (C) is a condition on \( f \) that for many purposes can replace the compactness of the manifold. As a rule in extending finite dimensional results in differential topology to infinite dimensions, we transfer the compactness condition from the space \( M \) itself to the function on \( M \).

The Global (Un)Stable Manifold Theorem, Theorem 4.2.3 is an important result that
tells us that the sets \( W^s(p) \) and \( W^u(p) \) are (immersed) submanifolds of \( M \) that have the
same codimension as the stable and unstable subspaces, respectively, of the linearization
of \( f \) at \( p \). The proof of Theorem 4.2.3 is an adaptation of the proof presented in [32]
Chapter 1, §1.7].
Lemma 4.2.3 (The Global (Un)Stable Manifold Theorem). Let $M$ be a Hilbert manifold. Let $f: M \to \mathbb{R}$ be a Morse function and let $p \in \text{Crit}(f)$. Fix a Riemannian metric on $M$ such that the negative gradient vector field of $f$ is standard near $p$. Then $W^s(p)$ is a connected submanifold of $M$ of codimension equal to $\text{index}_p(f)$ and $W^u(p)$ is a connected submanifold of $M$ of codimension equal to $\text{coindex}_p(f)$.

Proof. Let $p \in \text{Crit}(f)$ and let $U \subset M$ be a neighbourhood of $p$. Let $\rho^M_t$ be the negative gradient flow of $f$ on $M$.

The local stable set of $p$ (relative to $U$) is defined as the set

$$W^s_{\text{loc}}(p) := \{ x \in U \mid \rho^M_t(x) \text{ is defined for all } t \geq 0, \rho^M_t(x) \in U \text{ for all } t \geq 0 \text{ and } \lim_{t \to \infty} \rho^M_t(x) = p \}$$

where $\rho_U^t$ is the negative gradient flow of $f$ on $U$.

Let $D^U_t$ be the domain of definition of $\rho^U_t$. Then $W^s_{\text{loc}}(p)$ may be equivalently expressed as the set

$$\{ x \in U \mid x \in D^U_t \text{ for all } t \geq 0 \text{ and } \lim_{t \to \infty} \rho^U_t(x) = p \}$$

Similarly, the local unstable set of $p$ is defined as the set

$$W^u_{\text{loc}}(p) := \{ x \in U \mid \rho^U_t(x) \text{ is defined for all } t \leq 0, \rho^U_t(x) \in U \text{ for all } t \leq 0 \text{ and } \lim_{t \to -\infty} \rho^U_t(x) = p \}.$$

where $\rho^U_t$ is the negative gradient flow of $f$ on $U$.

Let $H$ be a Hilbert space and let $H^\pm \subset H$ be closed subspaces such that $H = H^+ \oplus H^-$. We shall identify a neighbourhood of $p$ with a neighbourhood of 0 in $H$. Let $\phi: U \to H$ be a Morse chart with properties:

- $\phi$ is an isometry, and
- $\phi(U) = B_+ \times B_-$ where $B_\pm \subset H_\pm$ are unit balls in $H_\pm$ respectively.
Note that we have

\[ H_+ := \{ x \in H \mid \lim_{t \to \infty} \rho_t^H(x) = 0 \} \]

\[ H_- := \{ x \in H \mid \lim_{t \to -\infty} \rho_t^H(x) = 0 \} \]

where \( \rho_t^H(x) = ||x_+||^2 - ||x_-||^2. \)

\[ M \supset U \xymatrix{ & B_+ \times B_- \subset H_+ \oplus H_- \\ & \mathbb{R} \ar@{|->}[ur]^{||x_+||^2 - ||x_-||^2} } \]

It follows that \( W^{H,s}_{H} (\phi(p)) = W^{H,s}_{H}(0) = H_+ \) and \( W^{H,u}_{H} (\phi(p)) = W^{H,u}_{H}(0) = H_. \)

The proof of this Lemma requires that:

**Step 1:** We must show that \( W^{s}_{\text{loc}}(p) \) (respectively, \( W^{u}_{\text{loc}}(p) \)) is a manifold.

**Step 2:** We must extend the local results of Step 1 to \( W^{s}(p) \) (respectively \( W^{u}(p) \)).

**Step 1:** We wish to show that the set \( W^{s}_{\text{loc}}(p) \) is a submanifold of \( U \).

By Lemma 4.1.8 recall that \( \phi \) intertwines the flow on \( U \subset M \) with the flow on \( B_+ \times B_- \subset H \). More precisely, \( \phi: U \to B_+ \times B_- \) is a diffeomorphism such that for all \( t \in \mathbb{R} \) and for any \( m \in U_p \cap (\rho_t^M)^{-1}(U_p) \) we have that

\[ \phi (\rho_t^M(m)) = \rho_t^H(\phi(m)) \]

\[ = \rho_t^H(x_+, x_-) \text{ because } \phi(m) = (x_+, x_-) \in B_+ \times B_- \]

\[ = (e^{-2t}x_+, e^{2t}x_-), \text{ by Lemma 4.1.1} \]

Thus, it is sufficient to show that \( W^{H,s}_{\text{loc}}(\phi(p)) = B_+ \times \{0\}. \)
Note that $W_{loc}^{\mathbb{H},s}(\phi(p)) \subseteq W_{\mathbb{H},s}^{\mathbb{H},s}(\phi(p))$. Moreover, recall that $W_{loc}^{\mathbb{H},s}(\phi(p)) = \mathbb{H}_+$ and that $W_{loc}^{\mathbb{H},s}(\phi(p)) = W_{\mathbb{H},s}^{\mathbb{H},s}(\phi(p)) \cap \phi(U)$. Therefore,

$$W_{loc}^{\mathbb{H},s}(\phi(p)) = \mathbb{H}_+ \cap (B_+ \times B_-)$$

$$= B_+ \times \{0\}$$

as wanted. By the properties of $\phi$, observe that $W_{loc}^{\mathbb{s}}(\phi(p))$ is connected.

Therefore $W_{loc}^{s}(p)$ is a connected submanifold of $M$ which contains $p$ with codimension $index_p(f)$.

Step 2: By using $\rho_t^M$, the negative gradient flow of $f$ on $M$, we wish to extend the local results of Step 1 to the global stable manifolds $W^s(p)$ and $W^u(p)$.

Fix an $x \in M$. Fix a time $T \in \mathbb{R}$. Suppose that $\rho_t^M: (\rho_T^M)^{-1}(U) \to U \cap D_M^{-T}$ is a diffeomorphism.
Note that the set \((\rho_T^M)^{-1}(U)\) is open because \(\rho_T^M\) is continuous. To prove Step 2 it is enough to show that \(W^s(p) \cap (\rho_T^M)^{-1}(U)\) is a submanifold of \(M\).

We claim that:

\[ q \in W^s(p) \cap (\rho_T^M)^{-1}(U) \text{ if and only if } \rho_T^M(q) \in W^s_{loc}(p). \]

It will follow from the claim that the image under \(\rho^M\) of \(W^s(p) \cap (\rho_T^M)^{-1}(U)\) is equal to the submanifold \(W^s_{loc}(p) \cap (U \cap D_{-T}^M)\). In other words, the set \(W^s(p)\) inherits the structure of a manifold from that of \(W^s_{loc}(p)\) by the set of maps \(\{\rho^M(t, \cdot)\}\). Therefore \(W^s(p)\) is a connected submanifold of \(M\) which contains \(p\) with codimension \(\text{index}_p(f)\).

Proof of claim: \((\Rightarrow)\) Let \(q \in W^s(p) \cap (\rho_T^M)^{-1}(U)\). Then \(q \in W^s(p)\) and \(q \in (\rho_T^M)^{-1}(U)\). This implies, respectively, that \(\rho_T^M(q) \in W^s(p)\) and \(\rho_T^M(q) \in U\). So \(\rho_T^M(q) \in W^s(p) \cap U\). But \(W^s(p) \cap U = W^s_{loc}(p)\) (this follows from the fact that all entry values of \(f\) (with respect to \(\rho^M\)) are bigger than all exit values. This fact appeared in the proof of Lemma [4.1.8].
\( \Rightarrow \) Let \( \rho^M_T(q) \in W^s_{loc}(p) \). That is, \( q \in \rho^M_{-T}(W^s_{loc}(p)) \). However
\[
\rho^M_{-T}(W^s_{loc}(p)) = (\rho^M_T)^{-1}(W^s_{loc}(p)) \text{, by Theorem 2.2.9 (} \rho^M_T)^{-1} = \rho^M_{-T} \]
\[
= (\rho^M_T)^{-1}(W^s(p) \cap U) \\
= W^s(p) \cap (\rho^M_T)^{-1}(U)
\]
Thus \( q \in W^s(p) \cap (\rho^M_T)^{-1}(U) \), and completing the proof of the claim.

It follows that \( W^s(p) \) is a submanifold of \( M \).

The analogous results for \( W^u(p) \) follows by giving all of the same arguments as above but by considering the vector field \( \nabla_g f \) (instead of \( -\nabla_g f \)).

Remark 4.2.4. Both a Local (Un) Stable Manifold theorem and a Global (Un)Stable Manifold theorem for Banach manifolds exist in the literature ([32, Chapter 1], [43, Chapters 5 and 6]). These references do not assume that the vector field is standard a point in the manifold. Let us briefly review what is known:

1. Known proofs of the Local (Un)Stable Manifold theorem are based on methods such as the “graph transform method” or the “orbit space method”. A brief description of these methods is provided below.

   • For detailed information on the so called “graph transform method” see [43]; 1987, Chapter 5. The Hadamard approach, this so called “graph transform method”, to proving the Local (Un)Stable Manifold theorem uses what is known as a graph transform. This method constructs the stable and unstable manifolds as graphs over the linearized stable and unstable spaces, respectively. This method is more geometrical in nature than the next Liapunov-Perron orbit space method.

   • For detailed information on the so called “orbit space method” see [32]: Chapter 1. The Liapunov-Perron orbit space method is another approach used to prove the Local (Un)Stable Manifold theorem. This method (in the
context of ordinary differential equations) deals with the integral equation formulation of the ordinary differential equations and constructs the invariant manifolds as a fixed point of an operator that is derived from the integral equation of a function whose elements have the appropriate interpretations as stable and unstable manifolds.

2. A complete proof for the **Global (Un)Stable Manifold Theorem** is also given in [32]: Chapter 1, Section § 1.7. This proof identifies $W^s(p)$ and $W^u(p)$ as particular images of injective immersions of manifolds. Note, again, that all of the aforementioned results are established for Banach manifolds. In particular they are true for Hilbert manifolds. Their proofs become simpler in the Hilbert manifold setting. For example, if $M$ is a Hilbert manifold in the Global (Un)Stable Manifold Theorem [32], then the regularity of the norm implies that $W^s(p)$ and $W^u(p)$ are actually images of the tangent space to $W^s(p)$, say $E^s_p$, and the tangent space to $W^u(p)$, say $E^u_p$, (respectively) where $T_p M = E^s_p \oplus E^u_p$.

**Lemma 4.2.5.** Let $M$ be a Riemannian Hilbert manifold and $f: M \to \mathbb{R}$ a Morse function. Let $x$ be a regular point for $f$. Fix a Riemannian metric on $M$ such that for every critical point $p$ of $f$ the negative gradient vector field of $f$ with respect to that Riemannian metric is standard near $p$. Then there exists a neighbourhood $U_x$ of $x$ in $M$ such that $U_x \cap f^{-1}(f(x))$ is a manifold. Moreover, let $p \in \text{Crit}(f)$. Then, after possibly shrinking $U_x$, the set $(U_x \cap f^{-1}(f(x))) \cap W^s(p)$ is a submanifold of $U_x \cap f^{-1}(f(x))$ with codimension equal to $\text{index}_p(f)$. This submanifold either passes through $x$ or is empty.

**Proof.** By the Implicit Function theorem we know that there exists a neighbourhood $U_x$ of $x$ in $M$ such that $U_x \cap f^{-1}(f(x))$ is a smooth manifold and that

$$T_x f^{-1}(f(x)) = \ker(df|_x : T_x M \to \mathbb{R}).$$

Let $p \in \text{Crit}(f)$. If $W^s(p) \cap \{x\} \neq \emptyset$ then we claim that $U_x \cap f^{-1}(f(x))$ is transverse
to $W^s(p)$ at $x$ (hence, near $x$). By the definition of transversality, it suffices to find a $v \in T_x W^s$ such that $d_x f(v) \neq 0$. Take $v = -\nabla_g f_x$, the negative $g$-gradient of $f$ at $x$.

From transversality, it follows that after possibly shrinking the neighbourhood $U_x$, the set $(U_x \cap f^{-1}(f(x))) \cap W^s(p)$ is a smooth submanifold of $U_x \cap f^{-1}(f(x))$ and that the codimension of $(U_x \cap f^{-1}(f(x))) \cap W^s(p)$ in $U_x \cap f^{-1}(f(x))$ is equal to the codimension of $W^s(p)$ in $M$. This codimension is equal to $\text{index}_p(f)$.

Recall that $M$ is a connected Riemannian Hilbert manifold and $f: M \to \mathbb{R}$ a Morse function. Fix a Riemannian metric on $M$ such that $f$ is bounded from below and satisfies Condition (C). Let $\{p_i\}, i \in I$ be the set of critical points of index equal to 0. Define

$$M_0 := \bigsqcup_{i \in I} W^s(p_i).$$

Thus, $M_0$ is the disjoint union of the (open) stable manifolds with index zero.

**Lemma 4.2.6.** Let $M$ be a complete connected Riemannian manifold and $f: M \to \mathbb{R}$ a Morse function that is bounded from below. Fix a Riemannian metric on $M$ such that $f$ satisfies Condition (C) and that for every critical point $p$ of $f$ the negative gradient vector field of $f$ with respect to the Riemannian is standard near $p$. Suppose that none of the critical points of $f$ have index equal to 1. Then the complement of $M_0$ is a locally finite union of submanifolds of codimension at least two.

**Remark 4.2.7.** Recall that a collection of subsets of a topological space is said to be **locally finite**, if each point in the space has a neighbourhood that intersects only finitely many of the sets in the collection.

**Proof.** From Palais [34] we know that:

(i) (Prop. 1 pg.314) if $a, b \in \mathbb{R}$ then there is at most a finite number of critical points $p$ of $f$ that satisfy $a < f(p) < b$. 
(ii) (Prop. 3 pg.321) if $\sigma_t(x)$ is any maximal solution curve of $-\nabla g f$ starting at the point $x$, then $\sigma_t(x)$ is defined for all $t > 0$, and $\lim_{t \to \infty} \sigma_t(x)$ exists and is a critical point of $f$.

Note that for each $c \in \mathbb{R}$, the set $\{ x \in M \mid f(x) < c \}$ is open in $M$ because $f$ is continuous. Moreover, each point $x \in M$ is contained in at least one of these sets. Thus for all $c \in \mathbb{R}$,

$$\{ x \in M \mid f(x) < c \} \cap (M \setminus M_0) \supseteq \{ x \in M \mid f(x) < c \} \cap \bigcup_{p \in a} W^s(p).$$

where $a = \text{Crit}(f)$ such that $\text{index}_p(f) \geq 2$ and $f(p) < c$ by (i) the union is finite.

But recall, by Lemma 4.2.3 we know that $\text{codim}(W^s(p)) = \text{index}_p(f)$, which is greater than or equal to two. Therefore, $M \setminus M_0$ is a locally finite union of submanifolds with codimension at least two. \hfill \square

**Lemma 4.2.8.** Let $M$ be a complete connected Riemannian manifold and $f: M \to \mathbb{R}$ a Morse function that is bounded from below. Fix a Riemannian metric on $M$ such that $f$ satisfies Condition (C) and that for every critical point $p$ of $f$ the negative gradient vector field of $f$ with respect to the Riemannian is standard near $p$. Suppose that none of the critical points of $f$ have index equal to 1. Then $M_0$ is connected.

**Proof.** Let $M_0$ be as in Lemma 4.2.6. Recall that $M_0 = \sqcup_{i \in I} W^s(p_i)$ where $\{p_i\} (i \in I)$ is the set of critical points of index equal to zero. Lemma 4.2.6 ensures that $I \neq \emptyset$. By hypothesis, no critical points of $f$ have index equal to 1, so $M_0^c$ is a locally finite union of submanifolds of codimension at least 2 by Lemma 4.2.6. This implies that $M_0$ is connected. We give more details:

For each $x \in M$, there exists a neighbourhood $U_x$ of $x$ such that $U_x \cap M_0$ is path connected and dense in $U_x$. This can be established by using Lemma 4.2.6 and the definition of a submanifold.
Let \( p, q \in M_0, p \neq q \). Let \( \gamma : [0,1] \to M \) be such that \( \gamma(0) = p \) and \( \gamma(1) = q \). Choose \( U_{x_i} \) as above, \( i = 0, \ldots, N - 1 \), such that

- the collection of \( U_{x_i} \) cover the path \( \gamma \), and
- \( U_{x_i} \cap U_{x_{i+1}} \neq \emptyset \) for all \( i \), and
- \( p \in U_{x_0}, q \in U_{x_N} \).

![Figure 4.2: Construction of a path \( \tilde{\gamma} \) which avoids \( M_0^c \)](image)

It follows that for all \( x \), \( U_{x_i} \cap U_{x_{i+1}} \cap M_0 \) is nonempty. Let \( q_0 = p, q_N = q \). For each \( i = 0, \ldots, N - 2 \) choose a point \( q_{i+1} \in U_{x_i} \cap U_{x_{i+1}} \cap M_0 \). For each \( i = 0, \ldots, N - 1 \), we may construct a path \( \gamma_{i+1} \) connecting \( q_i \) to \( q_{i+1} \) in \( U_{x_i} \cap M_0 \). We can do so because \( U_{x_i} \cap M_0 \) is path connected. Now so as to finish concatenate the \( \gamma_{i+1} \) to construct a path \( \tilde{\gamma} := \gamma_1 \gamma_2 \cdots \gamma_N \). Notice that \( \tilde{\gamma} \) is a path between \( p \) and \( q \) which does not intersect \( M_0^c \), by construction. That is, \( \tilde{\gamma} \) is a path in \( M_0 \). Hence, the open set \( M_0 \subset M \) is path connected and so also connected.

\[ \square \]

**Remark 4.2.9.** 1. In the set-up of Lemma 4.2.8, \( f \) attains its global minimum since the critical point set of \( f \) is discrete (see [34] Section \( \S 15 \), Theorem 4, Corollary 2).
The following notation will from here on will be used throughout this thesis: from remark 4.2.9 let \( p_0 \in \text{Crit}(f) \) denote the unique critical point of \( f \) with index zero and let \( f(p_0) := c_0 \) denote the global minimum value of \( f \) on \( M \).

### 4.3 Connected Levels

Let us start with a couple of technical lemmas. The first lemma provides a list of properties satisfied by a metric \( g \) whose gradient vector field, \( \nabla_g f \), is standard near a critical point \( p \) of \( f \).

**Lemma 4.3.1.** Let \( M \) be a complete connected Riemannian Hilbert manifold and \( f: M \to \mathbb{R} \) a Morse function that is bounded from below and satisfies Condition (C). Let \( M_0 \) be the open stable manifold with index zero. Suppose that, for every critical point \( p \) of \( f \) not in \( M_0 \), the Riemannian metric on \( M \) is standard near \( p \). Suppose that none of the critical points of \( f \) have index equal to 1. Then for each \( x \in M \) there exists a connected neighbourhood \( U_x \) of \( x \) such that

\[(i) \quad U_x \cap M_0 \text{ is open, connected, and dense in } U_x, \text{ and}

\[(ii) \quad \text{if } x \text{ is a regular point of } f \text{ then for all } c \in \mathbb{R}, \ (U_x \cap M_0) \cap f^{-1}(c) \text{ is open, connected, and dense in } U_x \cap f^{-1}(c).\]

**Proof.** Recall that \( M_0 = W_s(p_0) \) where \( p_0 \in \text{Crit}(f) \) is the unique critical point of index zero. Then \( M_0 \) is open and connected by Lemma 4.2.3. Let \( x \in M \). Choose a connected neighbourhood \( U_x \) of \( x \).

For property (i): Let \( E = M_0^c \). Observe that \( U_x \cap M_0 = U_x \setminus E \). The \( U_x \cap M_0 \) is open because \( M_0 \) is open because \( M_0 \) is. It follows that \( U_x \cap M_0 \) is open in \( U_x \). Also note that \( E \) is a locally finite union of submanifolds of \( M \) of codimension 2 or more, by Lemma 4.2.6. Hence, \( U_x \cap M_0 \subset U_x \) is connected and dense in \( U_x \).

For property (ii): Let \( c \in \mathbb{R} \).
If \( x \) is a regular point of \( f \) then \( U_x \cap M_0 \cap f^{-1}(f(x)) \) and \( f^{-1}(f(x)) \cap U_x \) are path connected by the implicit function theorem. It follows that

\[
(U_x \cap M_0) \cap f^{-1}(f(x)) \subset U_x \cap f^{-1}(f(x))
\]

is open (in the relative topology). By Lemma 4.2.5 we know that the set \( (U_x \cap M_0) \cap f^{-1}(f(x)) \) is a smooth submanifold of \( U_x \cap f^{-1}(f(x)) \) with codimension equal to \( \text{index}_{p_0}(f) \geq 2 \). Then it follows that

\[
(U_x \cap M_0) \cap f^{-1}(f(x)) \subset U_x \cap f^{-1}(f(x))
\]

is connected and dense because its complement has codimension at least 2.

\[\square\]

Let \( f \) be a Morse function on a connected Riemannian manifold \( M \). Let \( d \) be the distance function coming from the Riemannian metric \( g \) on \( M \). Note that the set \( \text{Crit}(f) \) has no accumulation points. This follows by the Morse Lemma \[4.1.4\] applied to \( f \).

For each point in \( M \) there exists a neighbourhood \( D \subset M \) and a chart with target a Hilbert space. Let \( \phi \) be a chart in \( M \) having as target a Hilbert space \( (\mathbb{H}, \langle \cdot, \cdot \rangle) \). For each \( x \in D \) let \( G(x): \mathbb{H} \to \mathbb{H} \) be an operator defined as in Definition \[2.1.1\]. Then each \( G(x) \) is an invertible linear operator that is bounded with bounded inverse. Recall that by Lemma \[4.1.9\] for each critical point \( p \in \text{Crit}(f) \), there exists a neighbourhood \( U_p \subset M \) of \( p \) on which there is a standard metric (cf. remark \[4.1.7\]) \( g_p \). For each \( x \in U_p(:= D) \) we define the operator \( G_p(x): \mathbb{H} \to \mathbb{H} \) as above. Using ingredients similar to Palais \[34, \text{Lemma 2 pg 311}\], we can shrink \( U_p \) such that there exist constants \( a_p := \|G_p\|, b_p := \|G_p^{-1}\| > 0 \) such that throughout the neighbourhood

\[
\frac{1}{b_p} \|v\|_{g_p} \leq \|v\|_g \leq a_p \|v\|_{g_p}
\]

for all \( x \in U_p \), for all \( v \in T_x M \).

Since \( \text{Crit}(f) \) has no accumulation points, for each \( p \in \text{Crit}(f) \) there exists \( R_p > 0 \) which is less than half the distance (in the distance function \( d \)) from \( p \) to any other point.
in $\text{Crit}(f)$. Thus the balls of radius $R_p$ (in the metric space $(M, d)$) about $p$ do not intersect.

Since $\text{Crit}(f)$ is countable, write $\text{Crit}(f) = \{p_1, p_2, p_3, \ldots, p_j, \ldots\}$. For each $j = 1, \ldots \infty$, let $U_{p_j}$ be the open ball of radius $r_j$ about $p_j$ (in the distance $d$) where $r_j$ is chosen to be sufficiently small so that

$$r_j < \min \{R_{p_j}, \frac{1}{2j}\}$$

and $U_{p_j}$ is contained in the domain of $g_{p_j}$.

Set $U := \bigcup_{j=1}^{\infty} U_{p_j}$, $\hat{U} := \bigcup_{j=1}^{\infty} \overline{U_{p_j}}$, and $V := M \setminus \hat{U}$.

**Lemma 4.3.2.** $V$ is open.

**Proof.** Suppose not. Then there exists a convergent sequence $(x_m) \to x$ such that $x \in V$ and $x_m \in \hat{U}$ for all $m$.

Each set $\overline{U_{p_j}}$ can contain only finitely many points from the sequence $(x_m)$ since otherwise the limit $x$ would lie in $\overline{U_{p_j}}$.

For each $m$, find $j_m$ such that $x_m \in \overline{U_{p_{j_m}}}$.

Given $n$, since $(x_m) \to x$ there exist infinitely many $m$ such that $d(x, x_m) < \frac{1}{2m}$. In particular, since only finitely many $x_m$ lie in any $\overline{U_{p_j}}$, there exists $m$ such that $d(x, x_m) < \frac{1}{2m}$ and $j_m > n$.

Since $x_m \in \overline{U_{p_{j_m}}}$, we have

$$d(x_m, p_{j_m}) < r_{j_m} < \frac{1}{2j_m} < \frac{1}{2n}.$$  

Thus

$$d(x, p_{j_m}) \leq d(x, x_m) + d(x_m, p_{j_m}) < \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}.$$  

However the existence for each $n$ of an element of $\text{Crit}(f)$ whose distance to $x$ is less than $1/n$ shows that $x$ is an accumulation point of $\text{Crit}(f)$, contrary to the fact that $\text{Crit}(f)$ has no accumulation points. Therefore there is no such sequence $(x_m) \to x$ and so $V$ is open. □
Given a connected Riemannian Hilbert manifold \((M, g)\) and a Morse function \(f\) on \(M\) that satisfies Condition (C) with respect to \(g\), the next technical lemma shows us that for each critical point \(p\) of \(f\), we can modify the metric \(g\) in a neighbourhood of \(p\) so that Condition (C) continues to hold for \(f\) with respect to this new metric on this neighbourhood.

**Lemma 4.3.3.** Let \(M\) be a connected Hilbert manifold. Let \(f: M \to \mathbb{R}\) be a Morse function and let \(g\) be a complete Riemannian metric on \(M\) such that \(f\) satisfies condition (C). Then there exist neighbourhoods \(U_p\) of \(p\) for each \(p \in \text{Crit}(f)\) such that the \(U_p\) are disjoint and there exists a Riemannian metric \(g_{\text{new}}\) on \(M\) such that:

(i) \(g_{\text{new}}\) is standard.

(ii) \(g_{\text{new}}\) coincides with \(g\) outside of \(U_p\).

(iii) \(g_{\text{new}}\) is complete, and \(f\) satisfies condition (C) with respect to \(g_{\text{new}}\).

**Proof.** Choose neighbourhoods \(U_p\) so that Lemma 4.3.2 applies. By Lemma 4.1.9 the existence of standard metrics \((4.1.7)\) \(g_p\) on neighbourhoods of \(p\) is guaranteed. As in Palais [34, Lemma 2 pg 311], we use similar ingredients to show that, for each \(p \in \text{Crit}(f)\) we can shrink \(U_p\) such that there exist constants \(a_p, b_p > 0\) such that

\[
\frac{1}{b_p} ||v||_{g_p} \leq ||v||_{g} \leq a_p ||v||_{g_p}
\]

for all \(x \in U_p\), for all \(v \in T_xM\).

For each \(p\), choose a bump function \(\kappa_p: M \to \mathbb{R}\) for the neighbourhood \(U_p\). That is, let \(\kappa_p\) be a smooth function with:

- \(0 \leq \kappa_p(x) \leq 1\), and
- \(\kappa_p(x) = 1\) near \(p\), and
- \(\text{supp}(\kappa_p(x)) \subseteq U_p\).

For \(x \in M\), define a new metric by
Then $g_{new}$ satisfies (i)–(iii) by construction. (Note that $g_{new}$ is a Riemannian metric because $V$ as defined in Lemma 4.3.2 is open).

Claim 4.3.4. $\| \cdot \|_{g_{new}} \geq \| \cdot \|_{g}$.

Proof. If $x \in U_p$ then

$$\| \cdot \|_{g_{new}}^2 |_x = (1 - \kappa_p(x)) \| \cdot \|_{g}^2 |_x + a_p^2 \kappa_p(x) \| \cdot \|_{g_p}^2 |_x \geq \kappa_p(x) \| \cdot \|_{g}^2 |_x$$

$$\geq (1 - \kappa_p(x)) \| \cdot \|_{g}^2 |_x + \kappa_p(x) \| \cdot \|_{g}^2 |_x$$

$$= (1 - \kappa_p(x) + \kappa_p(x)) \| \cdot \|_{g}^2 |_x$$

$$= \| \cdot \|_{g}^2 |_x.$$

and if $x$ lies outside $U_p$ for every $p$ then $g_{new}|_x = g_x$. So $\| \cdot \|_{g_{new}}|_x \geq \| \cdot \|_{g}|_x$. 

To show $g_{new}$ is complete

Let $(x_m)$ be a Cauchy sequence in the distance function coming from $g_{new}$. By Claim 4.3.4 the sequence $(x_m)$ is also a Cauchy sequence in the distance function coming from $g$. Since $(M,d)$ is a complete metric space, there exists $y \in M$ such that $(x_m) \to y$ in the distance function $d$. Recall that convergence with respect to one of these metrics implies convergence with respect to the other because the topology induced by these two metrics is the same (see 2.1.2).

To show $f$ satisfies condition $(C')$ with respect to $g_{new}$

Let $\{x_n\} \subset M$ be a sequence for which $|f(x_n)|$ is bounded. Suppose that

$$\| df|_{x_n} \|_{g_{new}}^2 := \langle df|_{x_n}, df|_{x_n} \rangle_{g_{new}} \to 0.$$
We wish to show that \((x_n)\) has a subsequence which converges to a critical point.

By Lemma 4.3.3,

\[ \|df|_{x_n}\|_{g_{new}}^2 \geq \|df|_{x_n}\|_g^2 \]

and so

\[ \|df|_{x_n}\|_g^2 \to 0. \]

The fact that \(f\) satisfies condition \((C)\) with respect to \(g\) gives a subsequence \((x_{n_k})\) of \((x_n)\) which converges to a critical point \(y\). Say \((x_{n_k}) \to y\). Again recall the fact that convergence with respect to one of these metrics implies convergence with respect to the other because the topology induced by these two metrics is the same (see 2.1.2). So \((x_n)\) has a convergent subsequence, as desired.

**End of Proof of Lemma 4.3.3**

We are now prepared to prove the connectivity for each level set of \(f\).

**Theorem 4.3.5** (Connected Levels). *Let \(M\) be a connected Hilbert manifold and let \(f: M \to \mathbb{R}\) be a Morse function that is bounded from below and none of whose critical points have index or coindex equal to 1. Suppose that there exists a complete Riemannian metric on \(M\) such that \(f\) satisfies condition \((C)\). Then the level set \(f^{-1}(c) \subset M\) is connected for every \(c\) in \(\mathbb{R}\).*

**Proof.** By the definition of a Morse function, each of the critical points of \(f\) is (strongly) nondegenerate. By the Morse Lemma for Hilbert manifolds [34], each critical point of \(f\) on \(M\) is isolated.

By Lemma 4.3.3 there exists a complete Riemannian metric, call it \(g\), on \(M\) for which \(f\) satisfies Condition \((C)\) and such that \(-\nabla_g f\) is standard near each critical point. Consider the vector field \(-\nabla_g f\).

Recall that \(f\) has only one critical point of index zero, say \(p_0\). Moreover, \(f\) attains its global minimum value \(c_0 := f(p_0)\) on \(M\) (see remark 4.2.9). Also recall that Palais (see
Proposition 1 pg 314) proves that if \(a, b \in \mathbb{R}\) then there is at most a finite number of critical points \(p\) of \(f\) satisfying \(a < f(p) < b\). Hence, the critical values of \(f\) are isolated and there are at most a finite number of critical points of \(f\) below any critical level since \(f\) is bounded from below by assumption. Let \(c_0 < c_1 < c_2 < \cdots\) be the critical values of \(f\).

Let \(c \in \text{Im}(f)\) such that \(c > c_0\).

**Case I:** for any regular point of \(f\) in \(f^{-1}(c)\), \(f^{-1}(c)\) is connected

Let \(E = M_0\). Note that by Lemma 4.3.1(ii), any regular point in \(f^{-1}(c)\) can be connected by a continuous path in \(f^{-1}(c)\) to a point that belongs to \(M_0\) (a ‘totally descending point’) of \(f^{-1}(c)\). Thus, following the method of Bryant [10], to prove the connectedness of \(f^{-1}(c)\) it suffices to show that any two totally descending points of \(f^{-1}(c)\) can be joined by a continuous path in \(f^{-1}(c)\). Let us give more details.

Suppose that \(x, y \in f^{-1}(c)\) are regular points of \(f\) such that \(x \neq y\). Then there exist neighbourhoods \(U_x\) and \(U_y\) of \(x\) and \(y\), respectively, that satisfy the properties of Lemma 4.3.1(ii). So we can choose totally descending points, say \(x'\) and \(y'\), in \(f^{-1}(c)\), which connect to \(x\) and \(y\) in \(f^{-1}(c)\). Moreover, by Lemma 4.3.1(i) and the Morse Lemma 4.1.4, there exists a (‘controlled’) neighbourhood \(U_0\) of \(p_0 \in M\) such that for all \(c\), the set \(U_0 \cap f^{-1}(c)\) is connected and such that \(U_0 \subset M_0\).

We pass to the normalized gradient flow. Note that by Palais [34] there exists a time \(t \in \mathbb{R}\) such that the normalized forward (downward) flow lines of \(x'\) and \(y'\) belong to \(U_0\).

The fact that the gradient flow lines are normalized means that their speed of descent is one, and therefore level sets map to level sets. We make explicit use of this fact throughout this proof.

More precisely, let \(\psi_t\) denote the downward normalized flow. There are points \(x'' := \psi_t(x')\) and \(y'' := \psi_t(y')\) (i.e., \(x''\) lies on the forward normalized flow line of \(-\nabla_g f\) through
there are a finite number of critical points between \( c \) and \( x \).

Chapter 4. Connectedness - The Base Case

We may choose a path \( \gamma \) such that \( \text{index}_x \gamma = \text{codimension} \) equal to \( 1 \) because the coindex of \( f \) cannot equal one for any critical point. Consequently, we may choose a path \( \gamma^*: [0, 1] \to U_0 \cap f^{-1}(c'') \) with \( \gamma^*(0) = x'' \) and \( \gamma^*(1) = y'' \) such that it is transverse to each of the unstable manifolds \( W^u(p_i), 1 \leq i \leq k \).

We can now use the gradient flow to move this path \( \gamma^* \) back up to the level of \( f^{-1}(c) \); Recall that \( \psi_t \) denotes the downward normalized flow. Let \( (t, m) \mapsto \psi_t(m) \) be defined on an open subset \( U \subseteq \mathbb{R} \times M \). Define an open subset \( U_t := \{ m \in M \mid (t, m) \in U \} \subseteq M \). So by Lang (Chapter IV Theorem 2.9), for all \( t \) the map \( \psi_t: U_t \to U_{-t} \) is a diffeomorphism with inverse \( \psi_{-t} \). Note that \( \psi_t \) restricts to a diffeomorphism \( U_t \cap f^{-1}(c) \to U_{-t} \cap f^{-1}(c'') \) with inverse the restriction of \( \psi_{-t} \) when \( t = c - c'' \). It then follows that for \( t = c - c'' \), the set \( \psi_{-t}(\gamma^*(\cdot)) \) is an open and dense subset of \( f^{-1}(c) \); this can be arranged because from Palais it follows that

\[
 f^{-1}(c'') \setminus (U_{-t} \cap f^{-1}(c'')) = f^{-1}(c'') \cap \bigcup_{p \in \text{Crit}(f) \text{ such that } c' \leq f(p) \leq c} W^u(p).
\]

where \( c'' \) is a regular value of \( f \). Now \( x' \) and \( y' \) can be joined by a path in \( f^{-1}(c) \), as desired.

This proves that \( x' \) and \( y' \) can be joined by a path in \( f^{-1}(c) \). Therefore, we may conclude that \( f^{-1}(c) \) is connected for every \( c \in \mathbb{R} \) in this case.

**Case II:** for any critical point of \( f \) in \( f^{-1}(c) \), \( f^{-1}(c) \) is connected

We first prove that \( f^{-1}(c_0) \) is connected. Recall that \( c_0 \) is the global minimum value
Figure 4.3: $f^{-1}(c)$ connected for any $c \in \mathbb{R}$
of $f$ (See remark 4.0.35, $f(p_0) = c_0$). We know that $\text{index}_{p_0}(f) = 0$. By Lemma 4.2.3, the stable manifold of $p_0$, $W^s(p_0) \subset M$, is connected. Hence, $f^{-1}(c_0) \subseteq W^s(p_0)$ must also be connected. To see this suppose that $f^{-1}(c_0)$ is not connected, i.e. $f^{-1}(c_0) = U \sqcup V$ such that $U, V \neq \emptyset$ and $U \neq V$. Then $W^s(p_0) = W^s(U) \sqcup W^s(V)$ with both $W^s(U), W^s(V)$ nonempty and not equal to each other. But this means that $W^s(p_0)$ is not connected, a contradiction. Therefore, $f^{-1}(c_0)$ is connected.

Next, note that for every singular point in $M$ there exists a regular point of $f$ in the same level set such that we can connect them through a path that lies entirely within the level set. Then we may connect any two regular points in this level of $f$ as in Case I, so as to obtain that $f^{-1}(c)$ is connected. Thus it is sufficient to show that a critical point can be connected to a regular point within the level. Let us provide more details.

Suppose that $x \in f^{-1}(c)$ is a singular point of $f$. By the Morse Lemma 4.1.3, there exists a neighbourhood $U_x$ of $x$ and a chart $\phi$ such that $\phi(x) = 0$, $f^{\mathbb{H}}(x_+, x_-) = ||x_+||^2 - ||x_-||^2$ on $\phi(U_x)$ and that $\mathbb{H} = \mathbb{H}_+ \oplus \mathbb{H}_-$. Fix such a Morse chart $\phi: U_x \to B_0 \subset \mathbb{H}$ (where $B_0$ is a neighbourhood of 0) of $x$ with the properties:

- $\phi$ is an isometry,
- $\phi(U_x) = B_+ \times B_-$, where $B_+ \subset \mathbb{H}_+$ and $B_- \subset \mathbb{H}_-$ are unit balls in $\mathbb{H}_+$ respectively.

So $(B_+ \times B_-) \cap (f^{\mathbb{H}})^{-1}(0) = \{(x_+, x_-) \in \mathbb{H} \mid ||x_+||^2 = ||x_-||^2\}$. Observe that this is homeomorphic to a cone on $S_+(1) \times S_-(1)$, where $S_\pm(1)$ are unit spheres in $\mathbb{H}_\pm$ respectively. Note that the set $(B_+ \times B_-) \cap (f^{\mathbb{H}})^{-1}(0)$ collapses at the origin to give a cone over $S_+(1) \times S_-(1))$.

Recall that the critical points of $f$ are isolated. If we start at the origin $(0, 0)$ then $\{(tx_+, tx_-) \mid 0 \leq t \leq 1\}$ is the path connecting $(0, 0)$ to a regular point, say $\phi(x')$, in $(B_+ \times B_-) \cap (f^{\mathbb{H}})^{-1}(0)$. This implies that we can connect $x$ to a regular point of $f$, say $x'$, in $f^{-1}(f(x))$ in $M_0 \cap f^{-1}(f(x))$, as desired.

This proves that $f^{-1}(c)$ is connected in this case.
Figure 4.4: Cone over $S_+(1) \times S_-(1)$.

Taken as a whole, we see that the proof is complete.
Chapter 5

Convexity and Connectedness
In this chapter we will state and prove the main results of this thesis.

5.1 Almost Periodic $\mathbb{R}^n$ Actions and Complex Structures

**Definition 5.1.1.** An $\mathbb{R}$-action on a manifold $M$ is said to be **almost periodic** if there exists a torus action $(S^1)^N \circ M$ and a one-parameter subgroup $\mathbb{R} \to (S^1)^N$ such that the $\mathbb{R}$-action is the composition $(\mathbb{R}, +) \to (S^1)^N \circ M$.

**Definition 5.1.2.** An $\mathbb{R}^n$-action on a manifold $M$ is said to be **almost periodic** if there exists a torus action $(S^1)^N \circ M$ and a homomorphism $(\mathbb{R}^n, +) \to (S^1)^N$ such that the $\mathbb{R}^n$-action is the composition $\mathbb{R}^n \to (S^1)^N \circ M$.

**Remark 5.1.3.** Let $T$ be the closure of the image of the homomorphism $(\mathbb{R}^n, +) \to (S^1)^N$.

**Definition 5.1.4.** In the notation of Remark 5.1.3, we define the **generated torus action** on $M$ to be $T$ with its action on $M$.

From now let $M$ be a connected strongly symplectic Hilbert manifold. Suppose that we have an almost periodic $\mathbb{R}^n$ action on $M$ with momentum map $\mu: M \to \mathbb{R}^n$. Suppose that the $\mathbb{R}^n$ action has isolated fixed points. Fix a $\xi \in \mathbb{R}^n$ such that the momentum map component $\mu^\xi := \langle \mu(\cdot), \xi \rangle: M \to \mathbb{R}$ has only nondegenerate critical points (i.e., $\mu^\xi$ is a Morse function). Recall that

(i) the critical point set of every Morse component is fixed by $T$, the generated torus action on $M$; and conversely

(ii) if the set of critical points of a component of $\mu$ is fixed by $T$ then that component is Morse.

Let $x \in M^T$, the fixed point set of the generated torus action. By continuity, $M^T = M^{\mathbb{R}^n}$, the fixed point set of the almost periodic $\mathbb{R}^n$ action on $M$. Note that $M^{\mathbb{R}^n}$ only
depends on \( \mu \). In what follows, we will show that there exists a \( T \)-invariant compatible complex structure on the symplectic vector space \((T_x M, \omega)\). We will also establish that no critical points of \( \mu^\xi: M \to \mathbb{R} \) have index or coindex equal to one.

**Lemma 5.1.5.** Let \((M, \omega)\) be a connected strongly symplectic Hilbert manifold. Suppose that we have an almost periodic \( \mathbb{R}^n \) action on \( M \) with momentum map \( \mu : M \to \mathbb{R}^n \). Suppose that this \( \mathbb{R}^n \) action has isolated fixed points. Let \( T \) be the torus generated by the almost periodic \( \mathbb{R}^n \)-action and let \( p \in M^T \). There exists an \( \omega \)-compatible and \( T \)-invariant complex structure \( J \) on \( T_p M \).

We establish Lemma 5.1.5 in a manner similar to Weinstein [52, Lecture 2, pg 8].

**Proof.** By averaging over the torus \( T \), we may choose a positive \( T \)-invariant inner product \( \langle \cdot, \cdot \rangle \) on \( T_p M \). Observe that \( T_p M \) is a strongly symplectic (real) vector space since it is equipped with a strongly symplectic nondegenerate 2-form \( \omega \). Since \( \omega \) and \( \langle \cdot, \cdot \rangle \) are nondegenerate,

\[
\begin{align*}
  u \in T_p M & \mapsto \omega(u, \cdot) \in T^*_p M \\
  v \in T_p M & \mapsto \langle v, \cdot \rangle \in T^*_p M
\end{align*}
\]

are isomorphisms between \( T_p M \) and \( T^*_p M \). Hence, \( \omega \) can be represented by some linear (skew-adjoint) operator \( A: T_p M \to T_p M \), i.e., \( \omega(u, v) = \langle Au, v \rangle \) for \( u, v \in T_p M \). Note that \( A \) is skew-adjoint (with respect to \( \langle \cdot, \cdot \rangle \) ) because

\[
\langle A^T u, v \rangle = \langle u, Av \rangle , \text{by definition of } A^T
\]

\[
= \langle Av, u \rangle , \text{since } \langle \cdot, \cdot \rangle \text{ is symmetric}
\]

\[
= \omega(v, u) , \text{by definition of } A
\]

\[
= -\omega(u, v) , \text{since } \omega \text{ is skew-symmetric}
\]

\[
= -\langle Au, v \rangle , \text{by definition of } A.
\]

We wish to find a \( T \)-invariant and \( \omega \)-compatible complex structure \( J \) on \( T_p M \). We claim that: \( J = \sqrt{(AA^T)^{-1}} A \) has these properties.
Note that \((AA^T)^{-1}\) is an operator on \(T_pM\) that is positive definite and symmetric with respect to \(\langle \cdot, \cdot \rangle\). By the Spectral Theorem we can obtain an operator \(\sqrt{(AA^T)^{-1}}\) such that \((\sqrt{(AA^T)^{-1}})^2 = (AA^T)^{-1}\). Moreover, \(\sqrt{(AA^T)^{-1}}\) commutes with every operator that commutes with \((AA^T)^{-1}\); See [13 Chap. 4, Prop. 4.33 page 86]. In particular, since \(A\) commutes with \((AA^T)^{-1} = -(A^2)^{-1}\), \(\sqrt{AA^T}^{-1}\) comutes with \(A\). Moreover \(\sqrt{(AA^T)^{-1}}\) is symmetric and positive definite. Let

\[
J := (AA^T)^{-\frac{1}{2}} A.
\]

\(J\) is orthogonal (with respect to \(\langle \cdot, \cdot \rangle\) because

\[
\langle Ju, Jv \rangle = \langle (AA^T)^{-\frac{1}{2}} Au, (AA^T)^{-\frac{1}{2}} Av \rangle, \text{ by definition of } J
\]

\[
= \langle Au, (AA^T)^{-1} Av \rangle, \text{ since } (AA^T)^{-\frac{1}{2}} \text{ is symmetric}
\]

\[
= \langle Au, (A^T)^{-1} A^{-1} Av \rangle
\]

\[
= \langle Au, (A^T)^{-1} v \rangle
\]

\[
= \langle u, A^T (A^T)^{-1} v \rangle
\]

\[
= \langle u, v \rangle
\]

From \(A\) skew-adjoint \((A^T = -A)\), we can deduce that \(J^T = -J\):

\[
A^T = -A \Rightarrow (AA^T)^{-\frac{1}{2}} A^T = -(AA^T)^{-\frac{1}{2}} A = -J
\]

\[
\Leftrightarrow \left( A(AA^T)^{-\frac{1}{2}} \right)^T = -J, \text{ since } (AA^T)^T = AA^T
\]

\[
\Leftrightarrow \left( (AA^T)^{-\frac{1}{2}} A \right)^T = -J, \text{ as } A \text{ and } (AA^T)^{-\frac{1}{2}} \text{ commute}
\]

\[
\Leftrightarrow J^T = -J
\]

Hence,

\[
J^2 = J(-J^T), \text{ because } J^T = -J
\]

\[
= -(AA^T)^{-\frac{1}{2}} A \left( (AA^T)^{-\frac{1}{2}} A \right)^T
\]

\[
= -(AA^T)^{-\frac{1}{2}} AA^T (AA^T)^{-\frac{1}{2}}, \text{ since } (AA^T)^T = AA^T
\]
= -AA^T (AA^T)^{-\frac{1}{2}} (AA^T)^{-\frac{1}{2}} , as AA^T and (AA^T)^{-\frac{1}{2}} commute
= -AA^T (AA^T)^{-1}
= -\text{Id}

That is, \( J \) is a complex structure on \( T_pM \). Moreover, \( J \) is \( T \)-invariant (because \( \langle \cdot, \cdot \rangle \) is and \( \omega \) is) and \( \omega \)-compatible because

\[
\omega(Ju,Jv) = \langle AJu,Jv \rangle , \text{ by definition of } A \\
= \langle JAu,Jv \rangle , \text{ since } J \text{ and } A \text{ commute} \\
= \langle Au,v \rangle , \text{ since } J \text{ is orthogonal} \\
= \omega(u,v) , \text{ by definition of } A
\]

\[
\omega(u,Ju) = \langle Au,Ju \rangle , \text{ by definition of } A \\
= \langle JAu,J^2u \rangle , \text{ since } J \text{ is orthogonal} \\
= \langle JAu,-u \rangle , \text{ since } J^2 = -\text{Id} \\
= -\langle -\sqrt{AA^T}u,u \rangle , \text{ using definition of } J \text{ in terms of } A \\
= \langle \sqrt{AA^T}u,u \rangle \\
> 0 , \text{ for } u \neq 0
\]

Therefore, \( J \) is a \( T \)-invariant and \( \omega \)-compatible complex structure on \( T_pM \) as wanted.

\[\square\]

**Remark 5.1.6.** 1. The factorization \( \sqrt{(AA^T)}J = A \) (equivalently, \( J = (AA^T)^{-\frac{1}{2}}A \) as written in the proof) is known as the *polar decomposition* of \( A \).

2. In general (as indicated in the proof), the positive inner product defined by \( \omega(u,Jv) = \langle \sqrt{AA^T}u,v \rangle \) is different from \( \langle u,v \rangle \).

3. This construction of \( J \) is canonical after an initial choice of Riemannian metric \( M \).
We are now ready to examine a Morse component of the momentum map \( \mu: M \to \mathbb{R}^n \). The next lemma shows us that no critical points of this component \( \mu^\xi \) have index or coindex equal to one.

**Theorem 5.1.7.** Let \( M \) be a strongly symplectic Hilbert manifold. Suppose that we have an almost periodic \( \mathbb{R}^n \) action on \( M \) with momentum map \( \mu: M \to \mathbb{R}^n \). Fix a \( \xi \in \mathbb{R}^n \) such that \( \mu^\xi := \langle \mu(\cdot), \xi \rangle \) is a Morse function. Then none of the critical points of \( \mu^\xi \) have index or coindex equal to 1.

**Remark 5.1.8.** This Lemma is the infinite-dimensional analogue of a lemma in Atiyah, [6, Lemma (2.2)] and Guillemin-Sternberg [17, Theorem 5.3].

**Proof of Lemma 5.1.7.** Let \( T \) be the torus generated by the almost periodic \( \mathbb{R}^n \)-action on \( M \). The critical points of \( \mu^\xi \) are the fixed points of \( T \), i.e., \( \text{Crit}(\mu^\xi) = M^T \). Let \( p \in M^T \) and let \( \mathbb{H} \) be a strongly symplectic (real) Hilbert space on which \( M \) is modelled. By an appropriate choice of charts we may identify \( T_p M \) with \( \mathbb{H} \). Note that different charts induce on \( T_p M \) different inner products (but with the same topology).

We will show that we may choose symplectic coordinates which linearize the action. In such coordinates, \( \mu^\xi \) looks like a quadratic. Note in particular that the eigenspaces of the Hessian of \( \mu^\xi \) at \( p \) are even-dimensional. The details are as follows:

**Step 1:** existence of a \( T \)-invariant metric on \( T_p M \)

Fix some Riemannian metric on \( M \). Choose a \( T \)-invariant inner product, say \( \langle \cdot, \cdot \rangle \), on \( T_p M \). Observe that \( T_p M \) is a strongly symplectic (real) vector space since it is equipped with a strongly nondegenerate 2-form \( \omega \). Then \( \omega \) can be identified with some skew-adjoint operator \( A: T_p M \to T_p M \) such that \( \omega(u, v) = \langle Au, v \rangle \).

**Step 2:** obtain a \( T \)-invariant, \( \omega \)-compatible complex structure on \( T_p M \)

By Theorem 5.1.5, there exists a \( T \)-invariant and \( \omega \)-compatible complex structure \( J \) on \( T_p M \). Namely, \( J = (AA^T)^{-1} A \).
Step 3: obtain a $J$-invariant orthogonal decomposition of $T_pM$

Given a complex structure $J$, $T_pM$ becomes a complex vector space where the Hermitian inner product is $T$-invariant. We may now decompose $T_pM$ into irreducible complex representations according to the weights associated with the linear isotropy representation of $T$ on $T_pM$.

We obtain a $J$-invariant orthogonal decomposition

$$T_pM = \left( \bigoplus_{\alpha \in \mathbb{Z}^*_+} V_\alpha \right) \bigoplus \left( \bigoplus_{\alpha \in \mathbb{Z}^*_+} V_\alpha \right)$$

where each $V_\alpha$, for $\alpha > 0$, corresponds to a non-trivial character of $T$ while the vector space $V_0 = T_pM^T$ and is fixed by $T$. Note that the summands in the above decomposition of $T_pM$ are orthogonal with respect to $\omega$ as well as with respect to the inner product.

Step 4: $\mu^\xi$ is a quadratic

For each $z \in V_\alpha$ we claim that $\mu^\xi(z) = -\frac{1}{2}||z||^2\alpha$ (meaning that the Hessian $H(z, z) = \langle z, z \rangle$ by compatibility). To see this note that the $S^1$ action on $V_\alpha$ is generated by

$$\frac{d}{dt} \bigg|_{t=0} (e^{it} \cdot z) = ie^{it}z|_{t=0} = iz = Jz.$$ 

Let $X$ be the vector field (associated to the linearized flow) on $T_pM$ that satisfies $X|_z = Jz$. Now, by the Local Linearization Theorem 3.1.2 there is a $G$-equivariant symplectomorphism (say $\phi$) from an invariant neighbourhood of the origin in $T_pM$ onto an invariant neighbourhood of $p \in M$.

It follows that

$$d\mu^\xi|_z(v) = -\langle z, v \rangle$$
\[= -\omega(v, Jz) = \omega(Jz, v) = \omega_p|_z(X, v).\]

In other words, the momentum map is given by \(\langle \alpha, \xi \rangle\)

\[\mu^\xi(z) = \sum_{\alpha \in \mathbb{t}} \frac{-1}{2} \|z\|^2 \alpha\]

in a coordinate system on \(T_pM\). Hence, all of the eigenspaces of the Hessian of \(\mu^\xi\) at any \(p \in \text{Crit}(\mu^\xi)\) are even-dimensional. This proves that the critical points of \(\mu^\xi\) have even index and coindex. In particular, \(\text{index}_p(\mu^\xi)\) and \(\text{coindex}_p(\mu^\xi)\) are not equal to one, as wanted.

\[\square\]

**Corollary 5.1.9.** Let \(M\) be a connected strongly symplectic Hilbert manifold. Suppose that we have an almost periodic \(\mathbb{R}\)-action on \(M\) with momentum map \(\mu: M \to \mathbb{R}\). Suppose that the \(\mathbb{R}\) action has isolated fixed points. Suppose that there exists a complete invariant Riemannian metric on \(M\) such that either the map \(\mu\) or \(-\mu: M \to \mathbb{R}\) is bounded from below and satisfies Condition (C). Then for every \(c \in \mathbb{R}\), the level set \(\mu^{-1}(c)\) is connected (or empty).

**Remark 5.1.10.** Note that Corollary 5.1.9 is a stronger version of the main Convexity Theorem, Theorem 5.4.5, where \(n = 1\) and \(H = \{0\}\).

**Proof of Lemma 5.1.7.** We have an almost periodic \(\mathbb{R}\)-action and thus \(t = \mathbb{R}\) and \(t^* = \mathbb{R}\). Hence, the momentum mapping \(\mu: M \to \mathbb{R}\) is a smooth \(\mathbb{R}\)-valued function. Without loss of generality, suppose that \(\mu\) is bounded from below (otherwise apply the below argument to \(-\mu\)). Since the critical points of \(\mu\) are nondegenerate (by assumption) note that \(\mu\) is a Morse function. By Theorem 5.1.7, none of the critical points of \(\mu\) have index or coindex equal to 1. Therefore, by Theorem 4.3.5 the level set \(\mu^{-1}(c)\) is connected for every \(c \in \mathbb{R}\).
5.2 Rational Independence and Consequences

Definition 5.2.1. A collection of real numbers $\theta_1, \ldots, \theta_n$ is said to be **rationally independent** over $\mathbb{Q}$ if the only $n$-tuple of integers $s_1, \ldots, s_n$ such that $s_1\theta_1 + \cdots + s_n\theta_n = 0$ is the trivial solution in which every $s_i = 0$.

Example 5.2.2. $3, \sqrt{8}, 1 + \sqrt{2}$ are rationally independent.

$rationaly independent$

Example 5.2.2. $3, \sqrt{8}, 1 + \sqrt{2}$ \hspace{1cm} {rationaly dependent}

Definition 5.2.3. Let $T$ be an $N$-dimensional torus. Choose a splitting of $T$, then $t = \mathbb{R}^N$ and $\ker(\exp) = \mathbb{Z}^N$. Let $\theta \in \mathbb{R}^N$. We say that $\theta := (\theta_1, \ldots, \theta_N)$ has **rationally independent components** if the numbers $\theta_1, \ldots, \theta_N$ are rationally independent over $\mathbb{Q}$.

Remark 5.2.4. Definition 5.2.3 is independent of the choice of splitting. Observe that if definition 5.2.3 is satisfied with respect to one splitting of $T$ then it is satisfied with respect to every splitting of $T$ since they differ by a linear invertible map over $\mathbb{Q}$.

Definition 5.2.5. Suppose that we have an almost periodic $\mathbb{R}^n$ action on $M$. Let $T$ be the $N$-dimensional generated torus action on $M$ (where $n \leq N$). We say that $\theta \in \mathbb{R}^n$ has **rationaly independent components** with respect to the almost periodic $\mathbb{R}^n$ action if the image of $\theta$ in $t \cong \mathbb{R}^N$ has rationally independent components.

\[ \begin{array}{c}
\mathbb{R}^n \xrightarrow{\text{linear map}} \mathbb{R}^N \\
\downarrow \exp \quad \downarrow \exp \\
T = \mathbb{R}^N/\mathbb{Z}^N
\end{array} \]

The following Lemma 5.2.6 shows us that if the components of $\theta \in \mathbb{R}^n$ are rationally independent then the $\theta$ component of $\mu$, $\mu^\theta$, satisfies the two equivalent conditions (i)
and (ii) in section §5.1, i.e., that $\mu^\theta$ is Morse and its critical point set is fixed by $T$. This result will play an important role in establishing our convexity result, Theorem 5.4.5, for a generic set of regular values of the momentum map (Cf. Lemma 5.3.1). Moreover, this lemma will illustrate another consequence of the complex structure from the prior section, §5.1, when we prove that the critical point set of these components of the momentum map are themselves symplectic submanifolds of $M$. In our case these are just points.

**Lemma 5.2.6.** Let $M$ be a connected strongly symplectic Hilbert manifold. Suppose that we have an almost periodic $\mathbb{R}^n$ action on $M$ with momentum map $\mu : M \to \mathbb{R}^n$. Let $T$ be the torus generated by the almost periodic $\mathbb{R}^n$ action. For every $\theta \in \mathbb{R}^n$, let $\mu^\theta : M \to \mathbb{R}$ where $\mu^\theta(\cdot) := \langle \mu(\cdot), \theta \rangle$ be the corresponding component of the momentum map. If $\theta$ has rationally independent components, then the critical set of $\mu^\theta$ is equal to the fixed point set $M^T$, and $\text{Crit}(\mu^\theta)$ is a symplectic submanifold of $M$.

**Remark 5.2.7.** This Lemma 5.2.6 is the almost periodic $\mathbb{R}^n$ action analogue of the well known torus action result [29] pg 186: Let $(\mathcal{M}, \omega)$ be a compact connected symplectic manifold and $\mathbb{T}^n$ be a torus action on $\mathcal{M}$ with momentum map $\mu : \mathcal{M} \to \mathbb{R}^n$. Then for every $\theta \in \mathbb{R}^n$ with rationally independent components, the critical set of the function $H_\theta := \langle \mu, \theta \rangle : \mathcal{M} \to \mathbb{R}$ is fixed under the $\mathbb{T}^n$ action. Moreover, the critical set of $H_\theta$ is a symplectic submanifold of $\mathcal{M}$.

**Proof of Lemma 5.2.6.** Let $X, Y \in \mathfrak{t} = \mathbb{R}^N$. Note that

$$
\mu^{kX}(\cdot) = \langle \mu(\cdot), kX \rangle \\
= k \langle \mu(\cdot), X \rangle \\
= k \mu^X(\cdot)
$$

for all $k \in \mathbb{Z}$, so $\text{Crit}(\mu^{kX}) = \text{Crit}(\mu^X)$.

Let $\theta \in \mathbb{R}^n$ such that $\theta$ has rationally independent components. Recall that if $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ has rationally independent components then we can choose a lattice $\Lambda \subset \mathbb{R}^n$. 


of $t$ such that the closure of the one parameter subgroup $\{\exp(s\text{Im}(\theta)) \mid s \in \mathbb{R}\}$ is $T \cong U(1)^N$. Said another way, the set of vectors $\{s\text{Im}(\theta) + k \mid \text{for all } s \in \mathbb{R} \text{ and } k \in \mathbb{Z}^N\}$ form a dense set in $\mathbb{R}^N$. Then, since $\{s\theta + k \mid s \in \mathbb{R}, \ k \in \mathbb{Z}^N\} = \mathbb{R}^N$, we may conclude that

$$\text{Crit}(\mu^\theta) = \bigcap_{t \in T} \text{Crit}(\mu^t).$$

But for $\mathbb{R}$-valued momentum maps a critical point of the momentum map is the same as a fixed point of the action. Therefore,

$$\text{Crit}(\mu^\theta) = \bigcap_{t \in T} \text{Crit}(\mu^t)$$

$$= \bigcap_{t \in T} \text{Fix}(\mu^t), \text{ where Fix}(\mu^t) \text{ are the fixed points}$$

$$= M^T, \text{ where } M^T \text{ denotes the } T\text{-fixed points in } M$$

as desired.

We can use this to prove that $\text{Crit}(\mu^\theta)$ is a symplectic manifold: Since $M^T$ is a discrete set it is a symplectic submanifold. It then follows that $\text{Crit}(\mu^\theta) = M^T$ is a symplectic submanifold of $M$. □

5.3 Good Projections

In this section we use the notation $\text{Fix}(\ast)$ to denote the fixed point set of the $\mathbb{R}^n$ action whose momentum map is the function $\ast$.

Lemma 5.3.1. Let $M$ be a connected strongly symplectic Hilbert manifold. Suppose that we have an almost periodic $\mathbb{R}^{n+1}$ action on $M$ with momentum map $\mu : M \to \mathbb{R}^{n+1}$. Suppose that the $\mathbb{R}^{n+1}$ action has isolated fixed points. Suppose that there exists a complete invariant Riemannian metric on $M$ such that there exists a hyperplane $H$ of $\mathbb{R}^{n+1}$ such that for all $\xi \in \mathbb{R}^{n+1} \setminus H$ the component $\mu^\xi := \langle \mu, \xi \rangle : M \to \mathbb{R}$ is bounded from one side and satisfies Condition (C). Then there exists a projection $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ satisfying
(i) the $\mathbb{R}^n$ action generated by $\mu' := \pi \circ \mu$ is almost periodic and has isolated fixed points; and

(ii) there exists a hyperplane $H' \subset \mathbb{R}^n$ such that for all $\xi' \in \mathbb{R}^n \setminus H'$ the component $(\mu')^{\xi'}: M \to \mathbb{R}$ is bounded from one side and satisfies condition (C).

Proof. We first prove Lemma 5.3.1 in the special case of a torus action on $M$, that is, in the case when we have a periodic $\mathbb{R}^n$ action on $M$. This is in preparation to set up for the almost periodic case.

For property (i): Let $A_{RI} \subset \mathbb{R}^{n+1}$ be the set of elements whose members are rationally independent in $\mathbb{R}^{n+1}$ and denote its complement by $A_{RD} \subset \mathbb{R}^{n+1}$, i.e.

$$A_{RD} = \mathbb{R}^{n+1} \setminus A_{RI}$$

$$= \{ v \in \mathbb{R}^{n+1} \mid \exists s_1, \ldots, s_{n+1} \in \mathbb{Q}, \text{not all zero, such that } \sum s_i v_i = 0 \}$$

$$= \{ v \in \mathbb{R}^{n+1} \mid \exists w \in \mathbb{Q}^{n+1} \setminus \{0\} \text{ such that } \langle v, w \rangle = 0 \}.$$

• Proposition 1: Let $\pi: \mathbb{R}^{n+1} \to \mathbb{R}^n$. Write $\ker(\pi) = \langle p \rangle$. Suppose that there exists $\theta \in A_{RI}$ such that $\theta \perp p$ (i.e. $\langle \theta, p \rangle = 0$). Then $Fix(\mu') = M^T$.

Proof: We know that $M^T \subseteq Fix(\mu')$. So we need to show that the opposite containment holds, i.e., show $Fix(\mu') \subseteq M^T$. Choose $\theta$ as in the hypothesis. Let $x \in Fix(\mu')$. Then $0 = d\mu'_x = \pi \circ d\mu_x$. So $\text{Im}(d\mu_x) \subseteq \ker(\pi)$. But $\theta \perp \text{Im}(d\mu_x)$ by hypothesis. That is, $\langle d\mu_x(\cdot), \theta \rangle = 0$, i.e., $d\mu^\theta_x = 0$. Hence $x \in Fix(\mu^\theta) = \text{Crit}(\mu^\theta)$.

Then we have that $\text{Crit}(\mu^\theta) = Fix(\mu^\theta) \subseteq M^T$. However, the inclusion is an equality because $\text{Crit}(\mu^\theta) = M^T$ since $\theta$ has rationally independent components by Lemma 5.2.6. Thus $Fix(\mu') = M^T$ ending the proof of Proposition 1. ■

Let

$$S = \{ p \in \mathbb{R}^{n+1} \mid p^\perp \subseteq A_{RD} \}$$

$$= \{ p \in \mathbb{R}^{n+1} \mid \forall a \in p^\perp, \exists q \in \mathbb{Q}^{n+1} \setminus \{0\} \text{ with } \langle q, a \rangle = 0 \}.$$
\[ p \in \mathbb{R}^{n+1} \mid p^\perp \subseteq \bigcup_{q \in Q^{n+1} \setminus \{0\}} q^\perp \] \subseteq \mathbb{R}^{n+1}.

By Proposition 1, in order to show that there is a projection \( \pi \) such that the periodic \( \mathbb{R}^n \) action generated by \( \mu' \) has isolated fixed points, it suffices to show that the set \( S \) has measure zero. The complement of \( S \) is the union of kernels \( \langle p \rangle \) of desired projections. So if \( S \) has measure zero then its complement must be nonempty.

\[ \text{Proposition 2}: \text{The set } S \text{ has measure zero in } \mathbb{R}^{n+1}. \]

\textit{Proof:} To prove this we will require a preliminary result. Let \( H \) and \( \{H_i\}_{i=1}^\infty \) be hyperplanes in \( \mathbb{R}^{n+1} \). Suppose that \( H \subseteq \bigcup_{i=1}^\infty H_i \). Then there exists an \( i \in \mathbb{N} \) such that \( H \subset H_i \). To prove this, first note that

\[ H = \bigcup_{i=1}^\infty (H \cap H_i). \]

Suppose for contradiction that for all \( i \) we have that \( H \cap H_i \subsetneq H \). If \( H \cap H_i \neq H \) then \( H \cap H_i \) has measure zero in \( H \). So if there is no \( H_i \) with \( H \cap H_i = H \), then \( H \) is a countable union of sets of measure zero in \( H \), which means that \( H \) itself has measure zero in \( H \). This is a contradiction. Thus \( H \subset H_i \) for some \( i \) and this ends the proof of the preliminary result.

It follows that

\[ S = \{p \mid p^\perp \subseteq \bigcup_{q \in Q^{n+1} \setminus \{0\}} q^\perp \} \]

\[ = \bigcup_{q \in Q^{n+1} \setminus \{0\}} \{p \mid p^\perp \subseteq q^\perp \}, \text{ by the above claim} \]

\[ = \bigcup_{q \in Q^{n+1} \setminus \{0\}} \{p \mid p^\perp = q^\perp \}, \text{ since } p^\perp \text{ cannot be a proper subset of } q^\perp \]

This is a countable union of lines. This completes the proof of Proposition 2. \( \blacksquare \)
We now generalize the preceding arguments to establish the almost periodic $\mathbb{R}^{n+1}$ action on $M$ case.

Let $i: \mathbb{R}^{n+1} \to \mathbb{R}^N$ be a linear map such that the composition

$$\mathbb{R}^{n+1} \to \mathbb{R}^N \to \mathbb{R}^N/\mathbb{Z}^N := T$$

has dense image in $T$.

Let $\tilde{A}_{RI} \subseteq \mathbb{R}^{n+1}$ be the set of rationally independent elements in $\mathbb{R}^{n+1}$. That is,

$$\tilde{A}_{RI} = \{ \theta \in \mathbb{R}^{n+1} | \ (i \circ \pi)(\theta) \in \mathbb{R}^N \text{ has rationally independent components} \}$$

Let $\tilde{A}_{RD} \subseteq \mathbb{R}^{n+1}$ denote its complement.

**Proposition 3:** Let $\pi: \mathbb{R}^{n+1} \to \mathbb{R}^n$. Write $\ker(\pi) = \langle p \rangle$. Suppose that there exists $\theta \in \tilde{A}_{RI}$ such that $\theta \perp p$ (i.e. $\langle \theta, p \rangle = 0$). Then $\text{Fix}(\mu') = M^T$.

**Proof:** We know that $M^T \subseteq \text{Fix}(\mu')$. So we need to show that the opposite containment holds. Choose $\theta$ as in the hypothesis. Let $x \in \text{Fix}(\mu')$. Then $0 = d\mu'_x = \pi \circ d\mu_x$. So $\text{Im}(d\mu_x) \subseteq \ker(\pi)$. But $\theta \perp \text{Im}(d\mu_x)$ by hypothesis. That is, $\langle d\mu_x(\cdot), \theta \rangle = 0$, i.e., $d\mu^\theta_x = 0$. Hence $x \in \text{Fix}(\mu^\theta) = \text{Crit}(\mu^\theta)$ since $\mu^\theta$ is a real-valued function. But $\text{Crit}(\mu^\theta) = M^T$ by Lemma 5.2.6 because $\theta$ is rationally independent. Thus, $\text{Fix}(\mu^\theta) = \text{Crit}(\mu^\theta) \subseteq M^T$ ending the proof of Proposition 3. ■

By Proposition 3, in order to show that there is a projection $\pi$ such that the almost periodic $\mathbb{R}^n$ action generated by $\mu'$ has isolated fixed points, it suffices to show that the set $\tilde{A}_{RD}$ has measure zero in $\mathbb{R}^{n+1}$. This is sufficient because if $\tilde{A}_{RD}$ has measure zero in $\mathbb{R}^{n+1}$ then its complement $\tilde{A}_{RI}$ must be nonempty.
• **Proposition 4:** The complement of the set

\[ \tilde{A}_{RI} = \{ \theta \in \mathbb{R}^{n+1} \mid (i \circ \pi)(\theta) \in \mathbb{R}^N \text{ has rationally independent components} \} \]

has measure zero in \( \mathbb{R}^{n+1} \).

**Proof:** Let \( \theta \in \mathbb{R}^{n+1} \). Denote its image \((i \circ \pi)(\theta)\) by \((i \circ \pi)(\theta) := \tilde{\theta} = (\tilde{\theta}_1, \ldots, \tilde{\theta}_N)\).

Note that

\[ \tilde{A}_{RD} = \mathbb{R}^{n+1} \setminus \tilde{A}_{RI} = \{ \theta \in \mathbb{R}^{n+1} \mid \exists c \in \mathbb{Z}^N \setminus \{0\} \text{ such that } \langle c, \tilde{\theta} \rangle := \Sigma_{j=1}^N c_j \tilde{\theta}_j = 0 \} \]

is a countable union of hyperplanes in \( \mathbb{R}^{n+1} \). Hence the complement of \( \tilde{A}_{RI} \) has measure zero in \( \mathbb{R}^{n+1} \). ■

To summarize what we have done, Proposition 3 shows us that to establish \((i)\) it is sufficient to show that the set \( \tilde{A}_{RD} \subset \mathbb{R}^{n+1} \) has measure zero in \( \mathbb{R}^{n+1} \). Then by Proposition 4 we know that \( \tilde{A}_{RD} \) has measure zero. This completes the proof of \((i)\).

For \((ii)\): Let \( \pi: \mathbb{R}^{n+1} \to \mathbb{R}^n \) be any projection such that \( \pi^*(\mathbb{R}^n) \neq H \) (in \( \mathbb{R}^{n+1} \)), where \( \pi^* := i: \mathbb{R}^n \to \mathbb{R}^{n+1} \). Choose hyperplane \( H' = (\pi^*)^{-1} H \) (the pre-image of \( H \)) = \( \{ \xi' \in \mathbb{R}^n \mid \pi^*(\xi') \in H \} \subset \mathbb{R}^n \). Observe that \( H' \) has dimension \( n - 1 \). Let \( \xi' \in (H')^c \). Let \( \xi = \pi^* \xi' \). Then the component

\[ \mu^\xi = \langle \mu, \xi \rangle = \langle \mu, \pi^* \xi' \rangle = \langle \pi \mu, \xi' \rangle = \langle \mu', \xi' \rangle = (\mu')^{\xi'} \]

We claim that \( \xi = \pi^* \xi' \in (H)^c \).
This is clear from the above diagram together with the definition of $H'$.

Thus by hypothesis $\mu^\xi$ is bounded from below and satisfies condition (C). But we saw that $\mu^\xi = (\mu')^\xi'$. Therefore (ii) holds as wanted.

**End of Proof of Lemma 5.3.1.**

### 5.4 The Connectivity and Convexity Theorems

The next Theorem, Theorem 5.4.1, may be of independent interest. We prove that in the presence of an almost periodic $\mathbb{R}^n$ action on $M$, the set of singular values of the resulting momentum map is contained in a countable union of hyperplanes. In particular, the set of regular values of the momentum map is residual in $\mathbb{R}^n$. It is tempting to use the Sard-Smale Theorem \[44\], an infinite-dimensional versions of Sard’s Theorem, but we cannot in the setting of this thesis. The Sard-Smale Theorem requires that the map be Fredholm.

**Theorem 5.4.1.** Let $M$ be a connected strongly symplectic Hilbert manifold. Suppose that we have an almost periodic $\mathbb{R}^n$ action on $M$ with momentum map $\mu: M \to \mathbb{R}^n$. Suppose that the $\mathbb{R}^n$ action has isolated fixed points. Then the set of singular values of $\mu$ is contained in a countable union of hyperplanes. In particular, the set of regular values of $\mu$ is residual in $\mathbb{R}^n$.

**Remark 5.4.2.** We will only use that the regular values of the momentum map are residual in $\mathbb{R}^n$ for the purpose of this thesis.

**Proof.** Let $T$ be the $N$-dimensional generated torus action on $M$ and let $H \subset T$ be a connected subgroup with $\text{dim}(H) > 0$. Note that $H$ must be a torus. Let $x \in M$. 

Note that the critical points of \( \mu \) are exactly those points whose stabilizer has positive dimension, and a connected component of the set of points with a fixed stabilizer of positive dimension gets mapped into a proper affine subspace of \( t^* \cong \mathbb{R}^N \). Because \( M \) is second countable, it is sufficient to show that each point in \( M \) has a neighbourhood in which at most countably many stabilizers occur. Recall that

- a linear representation of a compact abelian Lie group decomposes into a direct sum (in the Hilbert space sense) of subspaces, on each of which the group acts through a homomorphism to \( S^1 \); and

- a strictly decreasing sequence of subgroups of a compact abelian group must be finite.

First, the fixed point set of \( H \), denoted \( M^H \), coincides with that of the closure of \( H \) (by continuity), so we can assume that \( H \) is closed. Consider a connected component \( N \) of \( M^H \), and \( x \in N \). By the Local Linearization Theorem 3.1.1, \( M^H \) is a locally finite disjoint union of closed connected submanifolds. It follows that

\[
\text{Crit}(\mu) = \bigcup_{\text{subtori } H \subseteq T} M^H
\]
is a countable union.

Let \( j: \mathbb{R}^n \rightarrow T \). Observe that \( Stab_{\mathbb{R}^n}(x) = j^{-1}(H) \) where \( H = Stab_T(x) \). Then by definition of the momentum map

\[
\text{CritValues}(\mu) = \bigcup_{\text{subtori } H \subseteq T \text{ such that } j^{-1}(H) \subseteq \mathbb{R}^n \text{ and dim } (j^{-1}(H)) > 0} \bigcup_{\text{components } N \text{ of } M^H} \mu(N) .
\]

Note that each \( \mu(N) \) is contained in an affine subspace of \( \mathbb{R}^n \) of positive codimension. It follows that the complement of the set \( \text{CritValues}(\mu) \) is a countable intersection of residual sets, and hence residual. That is, the regular values of \( \mu \) are residual. \( \blacksquare \)
We require one last ingredient for the proof of the Convexity Theorem, Theorem 5.4.5. Namely, we require a lemma which makes explicit the relationship between statements $(A_n)$ and $(B_n)$ below. We now state and prove this result.

**Lemma 5.4.3.** For every $n \in \mathbb{N}$, consider the following two statements

$(A_n)$ Let $M$ be a connected strongly symplectic Hilbert manifold. Suppose that we have an almost periodic $\mathbb{R}^n$ action on $M$ with momentum map $\mu : M \to \mathbb{R}^n$. Suppose that the $\mathbb{R}^n$ action has isolated fixed points. Suppose that there exists a complete invariant Riemannian metric on $M$ such that there exists a hyperplane $H$ of $\mathbb{R}^n$ such that for all $\xi \in \mathbb{R}^n \setminus H$ the map $\mu^\xi : M \to \mathbb{R}$ is bounded from one side and satisfies Condition (C). Then the set

$$\{ c \in \mathbb{R}^n \mid c \text{ is a regular value of } \mu \text{ and } \mu^{-1}(c) \text{ is connected} \} \subseteq \mathbb{R}^n$$

is residual;

$(B_n)$ Let $M$ be a connected strongly symplectic Hilbert manifold. Suppose that we have an almost periodic $\mathbb{R}^n$ action on $M$ with momentum map $\mu : M \to \mathbb{R}^n$. Suppose that the $\mathbb{R}^n$ action has isolated fixed points and suppose that $\mu(M)$ is closed. Suppose that there exists a complete invariant Riemannian metric on $M$ such that there exists a hyperplane $H$ of $\mathbb{R}^n$ such that for all $\xi \in \mathbb{R}^n \setminus H$ the map $\mu^\xi : M \to \mathbb{R}$ is bounded from one side and satisfies Condition (C). Then the image $\mu(M) \subset \mathbb{R}^n$ is convex.

Suppose that $(A_n)$ is true for all $n$. Then $(B_n)$ is true for all $n$.

**Proof.** Note that $(B_1)$ trivially holds; For an almost periodic $\mathbb{R}$ action the momentum mapping $\mu : M \to \mathbb{R}$ is continuous. Since $M$ is connected, it follows that $\mu(M) \subset \mathbb{R}$ is connected; $\mu(M)$ is an interval. But connectedness is convexity in $\mathbb{R}$. Therefore, $(B_1)$ is true.
We want to show that \((B_{n+1})\) is true, i.e., we want to show that given any two distinct points in \(\mu(M) \subset \mathbb{R}^{n+1}\) then the line segment joining them is also in \(\mu(M)\). This proof follows the method of McDuff and Salamon [29].

**Case 1:** The “regular value” case

Choose an injective matrix \(A \in \mathbb{R}^{(n+1) \times n}\) such that (good projection) \(\pi := A^T; \mathbb{R}^{n+1} \to \mathbb{R}^n\) satisfies conditions (i) and (ii) of Lemma 5.3.1 and such that \(c' \in \mathbb{R}^n\) is a regular value of the restricted momentum map and is in the (residual) set of values for which the restricted momentum map is connected. Consider the restricted almost periodic \(\mathbb{R}^n\) action on \(M\). This action is Hamiltonian with momentum map \(\mu_A := A^T \circ \mu; M \to \mathbb{R}^n\).

\[
\begin{array}{ccc}
M & \xrightarrow{\mu} & \mathbb{R}^{n+1} \\
\mu_A \downarrow & & \downarrow A^T \\
\mathbb{R}^n & & 
\end{array}
\]

Choose \(x'_0 \in M\) such that it is in the \(c'\) level set of \(\mu_A\). Notice that \(x \in \mu_A^{-1}(c') \Leftrightarrow A^T \mu_A(x) = c' = A^T \mu_A(x'_0)\). Therefore the set \(\mu_A^{-1}(c')\) can be written in the form

\[
\mu_A^{-1}(c') = \{x \in M \mid \mu(x) - \mu(x'_0) \in \ker(A^T)\}.
\]

By assumption, \(\mu_A^{-1}(c')\) is connected, in fact path connected.

Let \(x'_1 \in \mu_A^{-1}(c')\) be another point in the same level set. If \(\mu(x'_1) - \mu(x'_0) \in \ker(A^T)\) then every convex combination of \(\mu(x'_0)\) and \(\mu(x'_1)\) is in \(\mu(M)\). We provide the details:

Let \(\gamma: [0,1] \to \mu_A^{-1}(c')\) with \(\gamma(0) = x'_0\), \(\gamma(1) = x'_1\) be the path connecting \(x'_0\) and \(x'_1\). Observe that \(\dim(\ker(A^T)) = 1\) because \(A\) is injective by hypothesis. This implies that \(A^T\) is surjective. Then \(\mu(\gamma(t)) - \mu(x'_0) \in \ker(A^T)\) for each \(t \in [0,1]\). Hence, every convex combination of \(\mu(x'_0)\) and \(\mu(x'_1)\) must lie in \(\mu(M)\), thus completing the proof of Case 1.

**Case 2:** The “general” case

Let \(x_0, x_1\) be distinct arbitrary points in \(M\).

We claim that \(x_0\) and \(x_1\) can be approximated arbitrarily closely by points \(x'_0, x'_1\) with the property that \(\mu(x'_1) - \mu(x'_0) \in \ker(A^T)\) for some injective matrix \(A \in \mathbb{R}^{(n+1) \times n}\) such
that $\pi := A^T$ satisfies conditions (i) and (ii) of Lemma 5.3.1. With a further perturbation we may assume that $A^T \mu(x_0')$ is a regular value of $\mu_A$ and is in the (residual) set of values for which the level set of $\mu_A$ is connected (by applying hypothesis $(A_n)$ to $\mu_A$). To see this, first recall that the set of regular values of $\mu$ is residual in $\mathbb{R}^{n+1}$ by Theorem 5.4.1. But a residual set in a complete metric space (such as $\mathbb{R}^{n+1}$) is dense in $\mathbb{R}^{n+1}$. It follows that the set of regular values of $\mu$ is dense in $\mu(M)$. By a similar argument applied to $\mu_A$ it can be established that the set of regular values of $\mu_A$ is dense in $\mu_A(M)$; Note that our assumptions imply that this restricted almost periodic $\mathbb{R}^n$ action on $M$ with momentum map $\mu_A$ satisfies conditions (i) and (ii) of Lemma 5.3.1 (in particular, $\mu_A$ has isolated fixed points). Moreover, note that the intersection of the image of $\mu_A$ with the residual set described in $(A_n)$ is dense in the momentum image.

Now, by Case 1, every convex combination of $\mu(x_0')$ and $\mu(x_1')$ lies in $\mu(M)$. Then our convexity result follows; since the image of $\mu$ is closed, by taking limits as $x_0' \to x_0$ and $x_1' \to x_1$ we obtain that $(1 - t)\mu(x_0) + t\mu(x_1) \in \mu(M)$ for all $0 \leq t \leq 1$.

Taken as a whole, the statement $(B_n)$ holds. \hfill \Box

Our main result is the following.
**Theorem 5.4.4** (Connectivity Theorem). Let $M$ be a connected strongly symplectic Hilbert manifold. Suppose that we have an almost periodic $\mathbb{R}^n$ action on $M$ with momentum map $\mu : M \to \mathbb{R}^n$. Suppose that the $\mathbb{R}^n$ action has isolated fixed points. Suppose that there exists a complete invariant Riemannian metric on $M$ such that there exists a hyperplane $H$ of $\mathbb{R}^n$ such that for all $\xi \in \mathbb{R}^n \setminus H$ the map $\mu^\xi : M \to \mathbb{R}$ is bounded from one side and satisfies Condition (C). Then the momentum mapping $\mu$ satisfies

\[(A)\] The set $\{ c \in \mathbb{R}^n \mid c$ is a regular value of $\mu$ and $\mu^{-1}(c)$ is connected $\} \subseteq \mathbb{R}^n$ is residual.

**Theorem 5.4.5** (Convexity Theorem). Let $M$ be a connected strongly symplectic Hilbert manifold. Suppose that we have an almost periodic $\mathbb{R}^n$ action on $M$ with momentum map $\mu : M \to \mathbb{R}^n$. Suppose that the $\mathbb{R}^n$ action has isolated fixed points and suppose that $\mu(M)$ is closed. Suppose that there exists a complete invariant Riemannian metric on $M$ such that there exists a hyperplane $H$ of $\mathbb{R}^n$ such that for all $\xi \in \mathbb{R}^n \setminus H$ the map $\mu^\xi : M \to \mathbb{R}$ is bounded from one side and satisfies Condition (C). Then the momentum mapping $\mu$ satisfies

\[(B)\] the image $\mu(M)$ is convex.

**Remark 5.4.6.** The Convexity Theorem, Theorem [5.4.5] applies to finite-dimensional connected symplectic manifolds but eliminates the compactness assumption in the Atiyah-Guillemin-Sternberg Convexity Theorem [1.0.1].

We are ready to prove the main result of this thesis, the Connectivity Theorem [5.4.4].

**Proof of Theorem [5.4.4].** Consider the statement

\[(A_n)\]: Let $M$ be a connected strongly symplectic Hilbert manifold. Suppose that we have an almost periodic $\mathbb{R}^n$ action on $M$ with momentum map $\mu : M \to \mathbb{R}^n$. Suppose that the $\mathbb{R}^n$ action has isolated fixed points. Suppose that there exists a
complete invariant Riemannian metric on $M$ such that there exists a hyperplane $H$ of $\mathbb{R}^n$ such that for all $\xi \in \mathbb{R}^n \setminus H$ the map $\mu^\xi : M \to \mathbb{R}$ is bounded from one side and satisfies Condition (C). Then the set

$$\{ c \in \mathbb{R}^n \mid c \text{ is a regular value of } \mu \text{ and } \mu^{-1}(c) \text{ is connected} \} \subseteq \mathbb{R}^n$$

is residual.

Notice that $(A_n)$ applies to all $M$ and every $\mu$ on $M$. By Lemma 5.4.3 it is sufficient to prove statement $(A_n)$ holds for all $n \in \mathbb{N}$. We proceed by induction on $n$.

Base Case: In the case $n = 1$, we have an almost periodic $\mathbb{R}$-action and thus $t = \mathbb{R}$ and $t^* = \mathbb{R}$, hence the momentum mapping $\mu : M \to \mathbb{R}$ is a smooth $\mathbb{R}$-valued function. By Corollary 5.1.9, $\mu^{-1}(c)$ is connected for every $c \in \mathbb{R}$, i.e., the set

$$\{ c \in \mathbb{R} \mid \mu^{-1}(c) \text{ is connected} \} = \mathbb{R}.$$

Then the base case $(A_1)$ holds because the set $\{ c \in \mathbb{R} \mid c \text{ is a regular value of } \mu \text{ and } \mu^{-1}(c) \text{ is connected} \}$ is residual.

Induction Step: Let $k \in \mathbb{N}$ be arbitrary. Assume that $(A_k)$ is true for all possible almost periodic $\mathbb{R}^k$ actions on $M$ and let $\mu_1, \mu_2, \ldots, \mu_{k+1}$ be the components of a momentum mapping $\mu : M \to \mathbb{R}^{k+1}$ satisfying the hypothesis conditions of Theorem 5.4.5. We want to show that $(A_{k+1})$ is true. We have two cases to consider:

1. $\mu$ is reducible; and
2. $\mu$ is irreducible.

We say that $\mu$ is said to be irreducible if the 1-forms $d\mu_1, d\mu_2, \ldots, d\mu_{n+1}$ are linearly independent, i.e.,

$$\alpha_1 d\mu_1(m)(v) + \cdots + \alpha_{n+1} d\mu_{n+1}(m)(v) = 0$$

at all points $m \in M$ and all vectors $v \in T_m M$ if and only if $\alpha_1 = \cdots = \alpha_{n+1} = 0$. We say that $\mu$ is reducible otherwise.

If $\mu$ is reducible, then we are finished; in this case there exists an $i \in \mathbb{N}$, $1 \leq i \leq k+1$, such that $d\mu_i$ is a linear combination of the other 1-forms. So we can drop $d\mu_i$ and apply
our inductive hypothesis. Thus, by the induction hypothesis the set of $c \in \mathbb{R}^{k+1}$ such that $c$ is a regular value of $\mu$ and $\mu^{-1}(c)$ is connected, is residual in $\mathbb{R}^{k+1}$.

Let us assume that $\mu$ is irreducible. By Lemma 5.3.1, there exists a projection $\pi := A^T : \mathbb{R}^{k+1} \to \mathbb{R}^k$ such that the restricted momentum map $\mu' := \pi \circ \mu$ satisfies all of the properties (i) and (ii) in Lemma 5.3.1. Fix such a projection $\pi$. Let

$$G_{\mu'} := \{ c' \in \mathbb{R}^k \mid c' \text{ is a regular value of } \mu' \text{ and } (\mu')^{-1}(c') \text{ is connected} \} \subseteq \mathbb{R}^k.$$

Notice that $\mu'$ is the momentum map of a restricted almost periodic $\mathbb{R}^k$ action on $M$. Note that there exists a basis of $\mathbb{R}^{k+1}$ so that $\pi$ drops the last coordinate. Without loss of generality we may assume this is the standard basis.

Let $c = (c_1, \ldots, c_{k+1}) \in \mathbb{R}^{k+1}$. Consider $N := \mu_1^{-1}(c_1) \cap \cdots \cap \mu_k^{-1}(c_k)$. Suppose that $c'$ is a regular value of $\mu'$. It follows that:

- the subset $N \subset M$ is a submanifold (of codimension $k$) in $M$ by the Implicit Function Theorem, and

- the 1-forms $(d\mu_i)(x)$, $1 \leq i \leq k$, are linearly independent for all $x \in N$.

Moreover, suppose that $\pi(c) \in G_{\mu'}$. Then $N$ is connected by the definition of $G_{\mu'}$.

Next, let us consider the restricted function $\mu_{k+1}|_N : N \to \mathbb{R}$.

**Proposition:** The function $\mu_{k+1}|_N$ is a Morse function none of whose critical points have index or coindex equal to one in $N$.

**Step 1:** We define a function $\phi : M \to \mathbb{R}$ and show that it has nondegenerate critical points of even index and coindex in $M$. 
Note that given some $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ and $\mu'(x) = (\mu_i(x), \ldots, \mu_k(x)) \in \mathbb{R}^k$ then $\langle \mu'(x), \lambda \rangle = \sum_{i=1}^{k} \lambda_i \mu_i(x)$. Recall that a point $x \in N$ is a critical point of $\mu_{k+1}|_N$ if and only if there exist some constant $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ such that

$$d\mu_{k+1}(x)(v) + \sum_{i=1}^{k} \lambda_i d\mu_i(x)(v) = 0$$

for all $v \in T_xM$. Therefore, $x$ is a critical point on $M$ for the function $\phi := \langle \mu, \lambda \rangle: M \to \mathbb{R}$ where $\lambda = (\lambda_1, \ldots, \lambda_k, 1) \in \mathbb{R}^{k+1}$. That is,

$$\phi = \mu_{k+1} + \sum_{i=1}^{k} \lambda_i \mu_i.$$

Notice that $\phi$ is a Morse function because it has nondegenerate critical points (since $\mu_{k+1}$ has nondegenerate critical points and $\mu_{k+1}$ and $\phi$ differ only by the constant $\sum_{i=1}^{k} \lambda_i \mu_i$). Thus, by Lemma 5.1.7 we know that no critical points of $\phi$ have index (coindex) equal to one in $M$.

**Step 2:** Show that the restricted function $\phi|_N$ is a Morse function

Let $C := \text{Crit}(\phi) \subset M$ be the critical point set of $\phi$. Let $x \in N$. We wish to demonstrate that the manifold $C$ intersects $N$ transversally at $x$ (i.e. $T_xM = T_xC + T_xN$). This means that the 1-forms $d\mu_i(x): T_xM \to \mathbb{R}$, $1 \leq i \leq k$, remain linearly independent when restricted to the subspace $T_xC$ (because this would show that the dual vector space to $T_xN + T_xC$ has the same codimension as $T_xM$ since the $d\mu_i(x)$, $1 \leq i \leq k$, vanish on $T_xN$). Thus, it is sufficient to prove that $d\mu_1(x), \ldots, d\mu_k(x)$ remain linearly independent on $T_xC$.

To begin with observe that

- the vector fields $X_i := X_{\mu_i}$ (given by $d\mu_i = \iota_{X_i}$) for $i = 1, \ldots, k$ must all lie tangent to $C$;

We have that

$$0 = d\mu_i(X_\phi)$$
Thus, $\phi$ is constant on the level curves of $\mu_i$. But then the Hamiltonian flow of $\mu_i$ must preserve $C$. Therefore the (Hamiltonian) vector fields $X_i$ are tangent to $C$.

Thus $X_i(x) \in T_xC$ for $i = 1, \ldots, k$.

- $T_xC$ is a symplectic vector space;

$C$ is a symplectic submanifold of $M$ by Lemma 5.2.6 because $C$ is a fixed point set of a torus action. Therefore $T_xC$ is a symplectic vector space.

This means that $\omega_x$ is nondegenerate on $T_xC$. So for all $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ with not all $\lambda_i$ zero, there exists a nonzero vector $v \in T_xC$ such that

$$0 \neq \omega_x \left( \sum_{i=1}^{k} \lambda_i X_i(x), v \right)$$

$$= \sum_{i=1}^{k} \lambda_i \iota_{X_i(x)} \omega_x(v)$$

$$= \sum_{i=1}^{k} \lambda_i d\mu_i(x)(v).$$

Hence $d\mu_i(x)$, for $i = 1, \ldots, k$, are linearly independent on $T_xC$. Therefore $C$ is transverse to $N$.

Now the fact that $T_xM = T_xN + T_xC$ implies that $(T_xC)^\perp \subseteq T_xN$. From this notice that $H_x(\phi)$, the Hessian of $\phi$ at $x$, is nondegenerate on $T_xN \cap (T_xC)^\perp$ because $T_xM \cap (T_xC)^\perp = T_xN \cap (T_xC)^\perp$ and so

$$T_xN = T_xN \cap T_xC + T_xN \cap (T_xC)^\perp.$$
In particular, this means that the restricted function $\phi|_N: N \to \mathbb{R}$ is a Morse function with critical point set $C \cap N$.

**Step 3**: Show that the function $\mu_{k+1}|_N$ has no critical points of index or coindex equal to one in $N$

Observe that by Lemma 5.1.7, the function $\phi|_N$ has critical points of even index and coindex since $\phi|_N$ has nondegenerate critical points (by Step 2). It then follows that $\mu_{k+1}|_N$ has nondegenerate critical points with even index and coindex because $\mu_{k+1}|_N$ only differs from $\phi$ by a constant, namely the constant $\sum_{i=1}^{k} \lambda_ic_i$, by definition of $\phi$. This completes the proof of the proposition.

By the proposition and by Theorem 4.3.5, the level set of $\mu_{k+1}|_N$ is connected for every $c_{k+1} \in \mathbb{R}$, i.e., $(\mu_{k+1}|_N)^{-1}(c_{k+1}) \subseteq N$ is connected for all $c_{k+1} \in \mathbb{R}$. Hence

$$\mu^{-1}(c) = N \cap \mu_{k+1}^{-1}(c_{k+1})$$

is connected for all $c \in \pi^{-1}(c)$. So the level set $\mu^{-1}(c)$ is connected for all $c \in \pi^{-1}(G_{\mu'})$. But by the induction hypothesis the set $G_{\mu'}$ is residual in $\mathbb{R}^k$. This implies that the set

$$\pi^{-1}(G_{\mu'}) \subseteq \mathbb{R}^{k+1}$$

is residual in $\mathbb{R}^{k+1}$ because $\pi^{-1}(G_{\mu'})$ is homeomorphic to $G_{\mu'} \times \mathbb{R}$.

Let $G_{\mu} := \{ c \in \mathbb{R}^{k+1} \mid c$ is a regular value of $\mu$ and $\mu^{-1}(c)$ is connected $\} \subseteq \mathbb{R}^{k+1}$.

By the definition of $G_{\mu'}$, the result just proven, and the definition of $G_{\mu}$, the set

$$\pi^{-1}(G_{\mu'}) \bigcap \{ \text{regular values of } \mu \} \subseteq G_{\mu}.$$ 

It follows that $G_{\mu}$ is residual in $\mathbb{R}^{k+1}$.

\[\square\]

**Proof of Theorem 5.4.5** This proof follows the method of Atiyah [6] where $n = \dim(\mathbb{R}^n)$. Consider the statements $(A_n)$ and $(B_n)$ of Lemma 5.4.3.
Then the statement “image of $\mu$ is convex” holds if and only if $(B_n)$ holds for all $n$.

Note that $(A_n)$ holds for all $n$ by the Connectedness Theorem, Theorem 5.4.4. It follows that $(B_n)$ holds for all $n$ by Lemma 5.4.3. Hence, the image $\mu(M)$ is convex.

Remark 5.4.7. The results of the Connectivity Theorem 5.4.4 and the Convexity Theorem 5.4.5 also apply to finite-dimensions where the manifold is not required to be compact or where the map is not required to be proper.

Remark 5.4.8. We wonder whether the assumptions of our Connectedness Theorem, Theorem 5.4.4 imply that the image of the momentum map is closed. We do not know counterexamples. Moreover, from Palais we know that for real-valued functions many consequences that follow from the image being closed are true.

Remark 5.4.9. In light of the Connectivity Theorem 5.4.4 and the Convexity Theorem 5.4.5 directions for future research could include:

- establishing connectivity of the level set $\mu^{-1}(c)$ for all regular values $c$ of the momentum map $\mu$;
- establishing connectivity of the level set $\mu^{-1}(c)$ for all critical values $c$ of the momentum map $\mu$;
- generalizing the Connectivity and Convexity Theorems so as to apply to Morse-Bott functions;
- developing an infinite-dimensional non-abelian convexity result.
Chapter 6

Example - The Based Loop Group
The purpose of this chapter is to provide examples of Theorem 5.4.5, the convexity main theorem.

6.1 Example: The Based Loop Group

The Loop Group

Let $G$ be a compact, connected and simply connected Lie group. Fix a $G$-invariant inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$. The loop group, which we denote by $M_1$, is defined as the set of maps $S^1 \to G$ that are Sobolev class $H^1$. Recall that a map $f: S^1 \to G$ is said to be Sobolev class $H^1$ if $f$ is absolutely continuous and $f^{-1} f' \in L^2(S^1, \mathfrak{g})$.

The space $M_1 = H^1(S^1, G)$ is an infinite-dimensional Hilbert manifold (cf. [34, section §13] and [37, Section §3]). It carries a left invariant Riemannian metric, called the $H^1$ metric. The $H^1$ metric is uniquely determined by its restriction to the Lie algebra of $M_1$ which is $H^1(S^1, \mathfrak{g})$ (the tangent space at the constant loop $e$). That is, if we fix an $Ad(G)$-invariant metric, $(\cdot, \cdot)$, on $\mathfrak{g}$ then the $H^1$ metric is determined by

$$\langle \gamma, \eta \rangle_e = \frac{1}{2\pi} \int_0^{2\pi} \langle \gamma(\theta), \eta(\theta) \rangle d\theta + \frac{1}{2\pi} \int_0^{2\pi} \langle \gamma'(\theta), \eta'(\theta) \rangle d\theta,$$

for $\gamma, \eta \in \text{Lie}(M_1) = H^1(S^1, \mathfrak{g})$.

The Based Loop Group

The subset $\Omega G$ of $M_1$ consisting of those loops $f:S^1 \to G$ for which $f(1) (= e)$ is the identity element in $G$ is called the based loop group. Notice that $\Omega G$ is a closed submanifold of $M_1$ whose Lie algebra consists of those maps $\tilde{f}: S^1 \to \mathfrak{g}$ such that $\tilde{f}(1) = 0$, i.e., $T_f \Omega G \cong H^1(S^1, \mathfrak{g})/\mathfrak{g}$. Moreover, the $H^1$ metric defined on $M_1$ induces a complete metric on $\Omega G$ (which we will denote by $\langle \cdot, \cdot \rangle$). See Palais [34, Section §13 Theorem 6]. So $\Omega G$ is a connected Riemannian Hilbert manifold.
It can be seen (see [7, Atiyah-Pressley, Section §2]) that the formula
\[
\omega(\gamma, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \gamma'(\theta), \eta(\theta) \rangle \, d\theta
\]
where \( \gamma, \eta \in H^1(S^1, g) \), defines a skew-symmetric bilinear form on \( H^1(S^1, g) \). Moreover, \( \omega \) is strongly nondegenerate. Extending \( \omega \) by left translations gives a left invariant closed 2-form \( \omega \) on \( \Omega G \) (cf. [37], [7, Section §4]). Thus, \( (\Omega G, \omega) \) is strongly symplectic.

**Group Actions on \( \Omega G \)**

The rotation group \( S^1 \) acts on \( \Omega G \) by “rotating the loop”:

if \( \gamma \in \Omega G \) and \( e^{i\theta} \in S^1, \theta \in [0, 2\pi] \), then \( (e^{i\theta} \gamma)(s) := \gamma(s + \theta)\gamma(\theta)^{-1} \).

Let \( T \) be the maximal torus of \( G \). Then \( T \) acts on \( \Omega G \) by conjugation:

if \( \gamma \in \Omega G \) and \( t \in T \), then \( (t\gamma)(s) := t\gamma(s)t^{-1} \).

Note that these actions commute and they are Hamiltonian [37].

**Remark 6.1.1.** The action \( T \times S^1 \circlearrowright \Omega G \) is a special case of an almost periodic \( \mathbb{R}^n \) action on \( \Omega G \).

The resulting \( T \times S^1 \) momentum map \( \mu: \Omega G \to \text{Lie}(T \times S^1) \cong \mathfrak{t}^* \oplus \mathbb{R}^* \cong \mathfrak{t} \oplus \mathbb{R} \) is given by \( \mu = p \oplus E \) with

\[
E(f) := \frac{1}{4\pi} \int_0^{2\pi} \|f(\theta)^{-1}f'(\theta)\|^2 \, d\theta \quad \text{Energy Functional}
\]

\[
p(f) := \text{pr}_t \left( \frac{1}{2\pi} \int_0^{2\pi} f(\theta)^{-1}f'(\theta) \, d\theta \right) \quad \text{Momentum Functional}
\]

where \( \text{pr}_t: g \to \mathfrak{t} \) is the projection onto the Lie algebra of \( T \).

**Morse Theory for the Components of \( \mu \)**

In this subsection we discuss the fact that a certain set of components of the momentum map \( \mu: \Omega G \to \mathfrak{t} \oplus \mathbb{R} \) satisfy Condition (C) with respect to the \( H^1 \) metric.

Note that the image of the momentum map \( \mu = p \oplus E \) lies in \( \mathfrak{t} \oplus \mathbb{R} \) which we can identify with its dual and with \( \mathbb{R}^{N-1} \oplus \mathbb{R} \cong \mathbb{R}^N \). Choose a hyperplane \( H \subset \mathbb{R}^N \) such
that $H = \{ x \in \mathbb{R}^N \mid x = (0, x_2, \ldots, x_N) \}$. Then observe that for each $\xi \in \mathbb{R}^N \setminus H$ the $\mu^\xi$ component of the momentum map may be written as

$$
\mu^\xi(f) = x_1 E(f) + \sum_{i=2}^{N} x_i p_i(f),
$$

where $x_1 \neq 0$ and $f \in \Omega G$. The fact that for each $\xi \in \mathbb{R}^N \setminus H$, $\mu^\xi$ is bounded from one side and satisfies Condition (C) follows from [30, Proposition 2.9] whose proof relies on results of [46].

**Connectedness of Level Sets**

Let us briefly review what is known about the connectivity with regards to the based loop group.

Recall that in [30] Harada, Holm, Jeffrey, and Mare proved that any level set of the momentum map $\mu$ of the $T \times S^1$ action restricted to $\Omega_{\text{alg}}$ is connected (for regular or singular values of the momentum map) [1.0.5]. Note that the subset $\Omega_{\text{alg}}$ of $\Omega G$ could be equipped with the *subspace topology* induced from the inclusion $\Omega_{\text{alg}} \hookrightarrow \Omega G$. However, $\Omega_{\text{alg}}$ can also be equipped with a *direct limit topology* induced by the Grassmannian model (see [30], Section §2) for the algebraic loop group. It turns out that the direct limit topology on $\Omega_{\text{alg}}$ is the appropriate topology for Theorem [1.0.5]. Harada, Holm, Jeffrey, and Mare also proved in [30] that any level set of the momentum map $\mu$ for the $T \times S^1$ action on $\Omega G$ is connected provided that $c$ is a regular value of $\mu$ (with respect to the $H^1$ metric) [1.0.6]. In [28] Mare proved that the level set of $\mu^{-1}(c)$ of the momentum map for the $T \times S^1$ action on $\Omega G$ is connected for singular values of $\mu$. His argument works for the space of $C^\infty$ loops and also for the space of loops of Sobolev class $H^s$ for any $s \geq 1$.

In terms of the results for this thesis, the Connectivity Theorem [5.4.4] establishes that in the presence of an almost periodic $\mathbb{R}^n$ action on $\Omega G$ (with momentum map $\mu$), the set 

$$
\{ c \in \mathbb{R}^n \mid c \text{ is a regular value of } \mu \text{ and } \mu^{-1}(c) \text{ is connected} \} \subseteq \mathbb{R}^n
$$

is residual.
Convexity

Let $R := T \times S^1$ act on $\Omega G$ as described above in the subsection “Group Actions on $\Omega G$”. Atiyah and Pressley [7] showed in Theorem 1.0.3 that the image of the momentum map $\mu = p \oplus E$ is convex. So the Convexity Theorem 5.4.5 reproduces this known convexity result when $M = \Omega G$. 
Bibliography

[1] A. Abbondandolo and P. Majer. Morse Homology on Hilbert Spaces. *Communications on Pure and Applied Mathematics*, 54:689–760, 2001.

[2] A. Abbondandolo and P. Majer. Lectures on the Morse complex for infinite dimensional manifolds. [http://www.dm.unipi.it/abbondandolo/preprints/montreal.pdf](http://www.dm.unipi.it/abbondandolo/preprints/montreal.pdf), 2004.

[3] A. Abbondandolo and P. Majer. A Morse complex for infinite dimensional manifolds - part I. *Advances in Mathematics*, 197:321–410, 2005.

[4] R. Abraham and J. Robbin. *Transversal Mappings and Flows*. W. A. Benjamin, 1967.

[5] J. Frank Adams. *Lectures on Lie Groups*. Mathematical Lecture Note Series. W. A. Benjamin, Inc., 1969.

[6] M. F. Atiyah. Convexity and Commuting Hamiltonians. *Bulletin of the London Mathematical Society*, 14:1–15, 1982.

[7] M. F. Atiyah and A.N. Pressley. Convexity and loop groups. *Progress in Mathematics*, 36:33–64, 1983.

[8] Dario Bambusi. On Darboux Theorem for Weak Symplectic Manifolds. *Proceedings of the American Mathematical Society*, 127:3383–3391, 1999.
[9] A. Banyaga and D. Hurtubise. *Lectures on Morse Homology*. Kluwer Academic Publishers, 2004.

[10] R. Bryant. COURSE Topics in Differential Geometry (Symplectic Geometry). http://www.math.duke.edu/ bryant/268/, 2003.

[11] Daniel Bump. *Lie Group*, volume 225 of *Graduate Texts in Mathematics*. Springer, 2004.

[12] Ana Cannas da Silva. *Lectures in Symplectic Geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag Berlin Heidelberg, 2001.

[13] Ronald G. Douglas. *Banach Algebra Techniques in Operator Theory*, volume 179 of *Graduate Texts in Mathematics*. Springer, 1998.

[14] E. Meinrenken E. Lerman, S. Tolman and C. Woodward. Non-abelian Convexity by Symplectic Cuts. *Topology*, 37:245–259, 1998.

[15] V. Guillemin and A. Pollack. *Differential Topology*. Prentice-Hall, 1974.

[16] V. Guillemin and R. Sjamaar. *Convexity Properties of Hamiltonian Group Actions*, volume 26 of *CRM Proceedings and Lecture Notes*. American Mathematical Society, 2005.

[17] V. Guillemin and S. Sternberg. Convexity Properties of the Moment Mapping. *Inventiones Mathematicae*, 67:491–513, 1982.

[18] V. Guillemin and S. Sternberg. *Symplectic Techniques in Physics*. Cambridge Univ. Press, 1984.

[19] M. Willem J. Mawhin. Origin and evolution of the Palias-Smale condition in critical theory. *Journal of Fixed Point Theory and Applications*, 7:265–290, 2010.
[20] L. Jeffrey. *Connectedness of Level Sets of the Moment Map for Torus Actions on the Based Loop Group*, volume 50 of *CRM Proceedings and Lecture Notes*, pages 181–184. American Mathematical Society, 2010.

[21] Y. Karshon and Christina Bjoerndahl. Revisiting Tietze-Nakajima - local and global convexity for maps. *Canadian Journal of Mathematics*, 2008.

[22] J.-L. Koszul. Sur certains groupes de transformations de Lie. *Géométrie différentielle*, pages 137–141, 1953.

[23] A. Kriegl and P. Michor. *The Convenient Setting of Global Analysis*, volume 53 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1997.

[24] Serge Lang. *Fundamentals of Differential Geometry*. Graduate Texts in Mathematics. Springer, second edition, 2001.

[25] E. Lerman and S. Tolman. Hamiltonian Torus Actions on Symplectic Orbifolds and Toric Varieties. *Transactions of the American Mathematical Society*, 349:4201–4230, 1997.

[26] W. Lui. Convexity of the moment polytopes of algebraic varieties. *Proceedings of the American Mathematical Society*, 131:2921–2932, 2003.

[27] P. Dazord M. Condevaux and P. Molino. Geometrie du moment. *Sem. Sud-Rhodanien*, 1988.

[28] A-L. Mare. Connectedness of Levels for Moment Maps on Various Classes of Loop Groups. *Osaka J. Math.*, 47:609–626, 2010.

[29] D. McDuff and D. Salamon. *Introduction to Symplectic Topology*. Oxford Mathematical Monographs. Oxford University Press, second edition, 1998.

[30] L.C. Jeffrey M.Harada, T.S.Holm and A.-L. Mare. Connectivity Properties of Moment Maps and Based Loop Groups. *Geometry and Topology*, 10:1607–1664, 2006.
[31] J. Milnor. *Morse theory*. Annals of Mathematics Studies. Princeton University Press, 1969.

[32] O. Cornea P. Biran and F. Lalonde. *Morse Theoretic Methods in Nonlinear Analysis and in Symplectic Topology*, volume 216 of *Series II: Mathematics, Physics and Chemistry*. Springer, 2006.

[33] J-P. Ortega P. Birtea and T. S. Ratiu. Openness and convexity for moment maps. *Trans. Amer. Math. Soc.*, 361(2):603–630, 2009.

[34] R. S. Palais. Morse theory on Hilbert manifolds. *Topology*, 2:299–340, 1963.

[35] R. S. Palais and S. Smale. A generalized Morse theory. *Bulletin of the American Mathematical Society*, 70(1):165–172, 1964.

[36] R. S. Palais and C. Terng. *Critical point theory and submanifold geometry*. Lecture Notes in Mathematics. Springer-Verlag, 1988.

[37] A. Pressley and G. Segal. *Loop Groups*. Oxford Mathematical Monographs. Oxford University Press, 2003.

[38] J. E. Marsden R. Abraham and T. Ratiu. *Manifolds, Tensor Analysis, and Applications*. Addison-Wesley Publishing Company, first edition, 1983.

[39] E. H Rothe. *Introduction to Various Aspects of Degree Theory in Banach Spaces*, volume 23 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1986.

[40] H.L. Royden. *Real Analysis*. Prentice Hall, 1988.

[41] I. Mezic S. Wiggins, G. Haller. *Normally Hyperbolic Invariant Manifolds in Dynamical Systems*, volume 105 of *Applied Mathematical Sciences*. Springer-Verlag, 1994.
[42] Hermann Schichl. *On the existence of slice theorems for moduli spaces on fiber bundles*. PhD thesis, University of Vienna, 1996.

[43] Michael Shub. *Global Stability of Dynamical Systems*. Springer-Verlag, 1987.

[44] S. Smale. Infinite Dimensional Version of Sard’s Theorem. *American Journal of Mathematics*, 87(4):861–866, 1965.

[45] Kathleen Smith. Symplectic Geometry and Topology. Master’s thesis, University of Toronto, 2005.

[46] Chuu-Lian Terng. Proper Fredholm submanifolds of Hilbert space. *Journal of Differential Geometry*, 29:1297–1432, 1989.

[47] Chuu-Lian Terng. Convexity theorem for infinite dimensional isoparametric submanifolds. *Inventiones mathematicae*, 112:9–22, 1993.

[48] A. J. Tromba. A General Approach to Morse Theory. *Journal of Differential Geometry*, 12:47–85, 1977.

[49] V. Guillemin V.L. Ginzburg and Y. Karshon. *Moment maps, Cobordisms, and Hamiltonian Group Actions*, volume 98 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1997.

[50] Alan Weinstein. Symplectic structures on Banach manifolds. *Bulletin of the American Mathematical Society*, 75:1040–1041, 1969.

[51] Alan Weinstein. Symplectic manifolds and their Lagrangian submanifolds. *Advances in Mathematics*, 6:329–346, 1971.

[52] Alan Weinstein. *Lectures on Symplectic manifolds*. Number 29 in Regional Conference Series in Mathematics. Bulletin of the American Mathematical Society, 1979.