More on sg-compact spaces*

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Abstract

The aim of this paper is to continue the study of sg-compact spaces, a topological notion much stronger than hereditary compactness. We investigate the relations between sg-compact and $C_2$-spaces and the interrelations to hereditarily sg-closed sets.

1 Introduction

In 1995, sg-compact spaces were introduced independently by Caldas [2] and by Devi, Balachandran and Maki [4]. A topological space $(X, \tau)$ is called sg-compact [2] if every cover of $X$ by sg-open sets has a finite subcover. In [4], the term $SGO$-compact is used.

Recall that a subset $A$ of a topological space $(X, \tau)$ is called sg-open [2] if every semi-closed subset of $A$ is included in the semi-interior of $A$. A set $A$ is called semi-open if $A \subseteq \text{Int}A$ and semi-closed if $\text{Int}A \subseteq A$. The semi-interior of $A$, denoted by sInt($A$), is the union of all semi-open subsets of $A$ while the semi-closure of $A$, denoted by sCl($A$), is the intersection of all semi-closed supersets of $A$. It is well known that sInt($A$) = $A \cap \text{Int}A$ and sCl($A$) = $A \cup \text{Int}A$.

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Every topological space \((X, \tau)\) has a unique decomposition into two sets \(X_1\) and \(X_2\), where \(X_1 = \{x \in X: \{x\}\text{ is nowhere dense}\}\) and \(X_2 = \{x \in X: \{x\}\text{ is locally dense}\}\). This decomposition follows from a result of Janković and Reilly [13, Lemma 2]. Recall that a set \(A\) is said to be \(\text{locally dense}[3]\) (= \(\text{preopen}\)) if \(A \subseteq \text{Int}A\).

It is a fact that a subset \(A\) of \(X\) is \(\text{sg-closed}\) (= its complement is \(\text{sg-open}\)) if and only if \(X_1 \cap \text{sCl}(A) \subseteq A\), or equivalently if and only if \(X_1 \cap \text{Int}A \subseteq A\). By taking complements one easily observes that \(A\) is \(\text{sg-open}\) if and only if \(A \cap X_1 \subseteq \text{sInt}(A)\). Hence every subset of \(X_2\) is \(\text{sg-open}\).

2 \quad \text{Sg-compact spaces}

Let \(A\) be a \(\text{sg-closed}\) subset of a topological space \((X, \tau)\). If every subset of \(A\) is also \(\text{sg-closed}\) in \((X, \tau)\), then \(A\) will be called \(\text{hereditarily sg-closed}\) (= \(\text{hsg-closed}\)). Observe that every nowhere dense subset is \(\text{hsg-closed}\) but not vice versa.

**Proposition 2.1** For a subset \(A\) of a topological space \((X, \tau)\) the following conditions are equivalent:

1. \(A\) is \(\text{hsg-closed}\).
2. \(X_1 \cap \text{Int}\overline{A} = \emptyset\).

**Proof.** (1) \(\Rightarrow\) (2) Suppose that there exits \(x \in X_1 \cap \text{Int}\overline{A}\). Let \(V_x\) be an open set such that \(V_x \subseteq \overline{A}\) and let \(B = A \setminus \{x\}\). Since \(B\) is \(\text{sg-closed}\), i.e. \(X_1 \cap \text{sCl}(B) \subseteq B\), we have \(x \notin \text{sCl}(B)\), hence \(x \notin \text{Int}B\), and thus \(x \in X \setminus \overline{B}\). If \(H = V_x \cap (X \setminus \overline{B})\), then \(H\) is nonempty and open with \(H \subseteq \overline{A}\) and \(H \cap B = \emptyset\) and so \(H \cap A = \{x\}\). Hence \(\emptyset \neq H = H \cap \overline{A} \subseteq \text{Int}A \subseteq \{x\}\), i.e. \(\text{Int}\{x\} \neq \emptyset\). Thus \(x \in X_2\), a contradiction.

(2) \(\Rightarrow\) (1) Let \(B \subseteq A\). Then \(\text{Int}B \subseteq \text{Int}\overline{A}\) and \(X_1 \cap \text{Int}B = \emptyset\), i.e. \(B\) is \(\text{sg-closed}\). \(\Box\)

We will call a topological space \((X, \tau)\) a \(C_2\)-space [9] (resp. \(C_3\)-space) if every nowhere dense (resp. \(\text{hsg-closed}\)) set is finite. Clearly every \(C_3\)-space is a \(C_2\)-space. Also, a topological
space \((X, \tau)\) is indiscrete if and only if every subset of \(X\) is hsg-closed (since in that case \(X_1 = \emptyset\)).

Following Hodel \[14\], we say that a cellular family in a topological space \((X, \tau)\) is a collection of nonempty, pairwise disjoint open sets. The following result reveals an interesting property of \(C_2\)-spaces.

**Lemma 2.2** Let \((X, \tau)\) be a \(C_2\)-space. Then every infinite cellular family has an infinite subfamily whose union is contained in \(X_2\).

**Proof.** Let \(\{U_i; i \in \mathbb{N}\}\) be a cellular family. Suppose that for infinitely many \(i \in \mathbb{N}\) we have \(U_i \cap X_1 \neq \emptyset\). Without loss of generality we may assume that \(U_i \cap X_1 \neq \emptyset\) for each \(i \in \mathbb{N}\). Now pick \(x_i \in U_i \cap X_1\) for each \(i \in \mathbb{N}\) and partition \(\mathbb{N}\) into infinitely many disjoint infinite sets, \(\mathbb{N} = \bigcup_{k \in \mathbb{N}} N_k\). Let \(A_k = \{x_i; i \in N_k\}\). Since \(A_k \cap (\bigcup_{i \notin N_k} U_i) = \emptyset\) and \(A_k \subseteq \bigcup_{i \in N_k} U_i\) for each \(k\), it is easily checked that \(\{\text{Int}\overline{A_k}; k \in \mathbb{N}\}\) is a disjoint family of open sets. Since \(X\) is a \(C_2\)-space, \(A_k\) cannot be nowhere dense and so, for each \(k\), there exists \(p_k \in \text{Int}\overline{A_k}\) and the \(p_k\)'s are pairwise distinct. Also, since \(X\) is \(C_2\), \(\bigcup_{i \in \mathbb{N}} U_i = \bigcup_{i \in \mathbb{N}} (U_i) \cup F\), where \(F\) is finite. Since \(p_k \in \bigcup_{i \in \mathbb{N}} U_i\) for each \(k\), there exists \(k_0\) such that \(p_k \in \bigcup_{i \in \mathbb{N}} U_i\) for \(k \geq k_0\), and since \(\text{Int}\overline{A_k} \cap (\bigcup_{i \notin N_k} U_i) = \emptyset\), we have \(p_k \in \bigcup_{i \in N_k} U_i\) for \(k \geq k_0\). Now, for each \(k \geq k_0\) pick \(i_k \in N_k\) such that \(p_k \in U_{i_k}\), and so \(p_k \in W = U_{i_k} \cap \text{Int}\overline{A_k}\). Thus \(\emptyset \neq W \subseteq U_{i_k} \cap \overline{A_k} \subseteq \overline{U_{i_k} \cap A_k} = \{x_{i_k}\}\). Hence \(\{x_{i_k}\}\) is locally dense, a contradiction. This shows that only for finitely many \(i \in \mathbb{N}\) we have \(U_i \cap X_1 \neq \emptyset\). Thus the claim is proved. \(\square\)

The \(\alpha\)-topology \[16\] on a topological space \((X, \tau)\) is the collection of all sets of the form \(U \setminus N\), where \(U \in \tau\) and \(N\) is nowhere dense in \((X, \tau)\). Recall that topological spaces whose \(\alpha\)-topologies are hereditarily compact have been shown to be semi-compact \[11\]. The original definition of semi-compactness is in terms of semi-open sets and is due to Dorsett \[8\]. By definition a topological space \((X, \tau)\) is called semi-compact \[8\] if every cover of \(X\) by semi-open sets has a finite subcover.

**Remark 2.3** (i) The 1-point-compactification of an infinite discrete space is a \(C_2\)-space having an infinite cellular family.
(ii) A topological space \((X, \tau)\) is semi-compact if and only if \(X\) is a \(C_2\)-space and every cellular family is finite.

(iii) Every subspace of a semi-compact space is semi-compact (as a subspace).

**Lemma 2.4**

(i) Every \(C_3\)-space \((X, \tau)\) is semi-compact.

(ii) Every sg-compact space is semi-compact.

**Proof.** (i) All \(C_3\)-spaces are \(C_2\)-spaces. Thus in the notion of Remark 2.3 (ii) above we need to show that every cellular family in \(X\) is finite. Suppose that there exists an infinite cellular family \(\{U_i : i \in \mathbb{N}\}\). For each \(i \in \mathbb{N}\) pick \(x_i \in U_i\) and, as before, partition \(\mathbb{N} = \bigcup_k \mathbb{N}_k\) and set \(A_k = \{x_i : i \in \mathbb{N}_k\}\). Since \(X\) is a \(C_2\)-space, \(\{\text{Int} \overline{A_k} : k \in \mathbb{N}\}\) is a cellular family. By Lemma 2.2, there is a \(k \in \mathbb{N}\) such that \(\text{Int} \overline{A_k} \subseteq X_2\). Since \(A_k\) is not hsg-closed, we must have \(X_1 \cap \text{Int} \overline{A_k} \neq \emptyset\), a contradiction. So, every cellular family in \(X\) is finite and consequently \((X, \tau)\) is semi-compact.

(ii) is obvious since every semi-open set is sg-open. \(\Box\)

**Remark 2.5**

(i) It is known that sg-open sets are \(\beta\)-open, i.e. they are dense in some regular closed subspace \([5]\). Note that \(\beta\)-compact spaces, i.e. the spaces in which every cover by \(\beta\)-open sets has a finite subcover are finite \([10]\). However, one can easily find an example of an infinite sg-compact space – the real line with the cofinite topology is such a space.

(ii) In semi-\(T_D\)-spaces the concepts of sg-compactness and semi-compactness coincide. Recall that a topological space \((X, \tau)\) is called a *semi-\(T_D\)-space* \([13]\) if each singleton is either open or nowhere dense, i.e. if every sg-closed set is semi-closed.

**Theorem 2.6** For a topological space \((X, \tau)\) the following conditions are equivalent:

1. \(X\) is sg-compact.
2. \(X\) is a \(C_3\)-space.
Proof. (1) $\Rightarrow$ (2) Suppose that there exists an infinite hsg-closed set $A$ and set $B = X \setminus A$. Observe that for each $x \in A$, the set $B \cup \{x\}$ is sg-open in $X$. Thus $\{B \cup \{x\}; x \in A\}$ is a sg-open cover of $X$ with no finite subcover. Thus $(X, \tau)$ is $C_3$.

(2) $\Rightarrow$ (1) Let $X = \bigcup_{i \in I} A_i$, where each $A_i$ is sg-open. Let $S_i = \text{slint}(A_i)$ for each $i \in I$ and let $S = \bigcup_{i \in I} S_i$. Then $S$ is a semi-open subset of $X$ and each $S_i$ is a semi-open subset of $(S, \tau|S)$. Since $X$ is a $C_3$-space, $(X, \tau)$ is semi-compact and hence $(S, \tau|S)$ is a semi-compact subspace of $X$ (by Remark 2.3 (iii)). So we may say that $S = S_{i_1} \cup \ldots \cup S_{i_k}$. Since $A_i$ is sg-open, we have $X_1 \cap A_i \subseteq S_i$ for each index $i$ and so $X_1 = X_1 \cap (\bigcup A_i) \subseteq X_1 \cap S \subseteq S_{i_1} \cup \ldots \cup S_{i_k} = S$. Hence $X \setminus S$ is semi-closed and $X \setminus S \subseteq X_2$. Since $\text{Int}(X \setminus S) \subseteq X \setminus S \subseteq X_2$, we conclude that $X \setminus S$ is hsg-closed and thus finite. This shows that $X = S_{i_1} \cup \ldots \cup S_{i_k} \cup (X \setminus S) = A_{i_1} \cup \ldots \cup A_{i_k} \cup F$, where $F$ is finite, i.e. $(X, \tau)$ is sg-compact. ✷

Remark 2.7 (i) If $X_1 = X$, then $(X, \tau)$ is sg-compact if and only if $(X, \tau)$ is semi-compact. Observe that in this case sg-closedness and semi-closedness coincide.

(ii) Every infinite set endowed with the cofinite topology is (hereditarily) sg-compact.

It is known that an arbitrary intersection of sg-closed sets is also an sg-closed set [6]. The following result provides an answer to the question about the additivity of sg-closed sets.

Proposition 2.8 (i) If $A$ is sg-closed and $B$ is closed, then $A \cup B$ is also sg-closed.

(ii) The intersection of a sg-open and an open set is always sg-open.

(iii) The union of a sg-closed and a semi-closed set need not be sg-closed, in particular, even finite union of sg-closed sets need not be sg-closed.

Proof. (i) Let $A \cup B \subseteq U$, where $U$ is semi-open. Since $A$ is sg-closed, we have $\text{sCl}(A \cup B) = (A \cup B) \cup \text{Int}(\overline{A \cup B}) \subseteq U \cup \text{Int}(\overline{A \cup B}) \subseteq U \cup (\text{Int}\overline{A \cup B}) \subseteq U \cup (U \cup B) = U$.

(ii) follows from (i).

(iii) Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$. Note that the two sets $A = \{a\}$ and $B = \{b\}$ are semi-closed but their union $\{a, b\}$ is not sg-closed. ✷
Theorem 3 from [1] states that if \( B \subseteq A \subseteq (X, \tau) \) and \( A \) is open and sg-closed, then \( B \) is sg-closed in the subspace \( A \) if and only if \( B \) is sg-closed in \( X \). Since a subset is regular open if and only if it is \( \alpha \)-open and sg-closed [7], by using Proposition 2.8, we obtain the following result:

**Proposition 2.9** Let \( R \) be a regular open subset of a topological space \((X, \tau)\). If \( A \subseteq R \) and \( A \) is sg-open in \((R, \tau|_R)\), then \( A \) is sg-open in \( X \). \( \Box \)

**Proof.** Since \( B = R \setminus A \) is sg-closed in \((R, \tau|_R)\), \( B \) is sg-closed in \( X \) by [1, Theorem 3]. Thus \( X \setminus B \) is sg-open in \( X \) and by Proposition 2.8 (ii), \( R \cap (X \setminus B) = A \) is sg-open in \( X \). \( \Box \)

Recall that a subset \( A \) of a topological space \((X, \tau)\) is called \( \delta \)-open [8] if \( A \) is a union of regular open sets. The collection of all \( \delta \)-open subsets of a topological space \((X, \tau)\) forms the so called semi-regularization topology.

**Corollary 2.10** If \( A \subseteq B \subseteq (X, \tau) \) such that \( B \) is \( \delta \)-open in \( X \) and \( A \) is sg-open in \( B \), then \( A \) is sg-open in \( X \).

**Proof.** Let \( B = \bigcup_{i \in I} B_i \), where each \( B_i \) is regular open in \((X, \tau)\). Clearly, each \( B_i \) is regular open also in \((B, \tau|B)\). By Proposition 2.8 (ii), \( A \cap B_i \) is sg-open in \((B, \tau|B)\) for each \( i \in I \). In the notion of Proposition 2.9, \( B \setminus (A \cap B_i) \) is sg-closed in \((X, \tau)\) for each \( i \in I \). Hence \( X \setminus (B \setminus (A \cap B_i)) = (A \cap B_i) \cup (X \setminus B) \) is sg-open in \((X, \tau)\). Again by Proposition 2.8 (ii), \( B \cap ((A \cap B_i) \cup (X \setminus B)) = A \cap B_i \) is sg-open in \((X, \tau)\). Since any union of sg-open sets is always sg-open, we have \( A = \bigcup_{i \in I} (A \cap B_i) \) is sg-open in \((X, \tau)\). \( \Box \)

**Proposition 2.11** Every \( \delta \)-open subset of a sg-compact space \((X, \tau)\) is sg-compact, in particular, sg-compactness is hereditary with respect to regular open sets.

**Proof.** Let \( A \subseteq X \) be \( \delta \)-open. If \( \{U_i : i \in I\} \) is a sg-open cover of \((S, \tau|S)\), then by Corollary 2.10, each \( U_i \) is sg-open in \( X \). Then, \( \{U_i : i \in I\} \) along with \( X \setminus A \) forms a sg-open cover of \( X \). Since \( X \) is sg-compact, there exists a finite \( F \subseteq I \) such that \( \{U_i : i \in F\} \) covers \( A \). \( \Box \)
Example 2.12 Let $A$ be an infinite set with $p \notin A$. Let $X = A \cup \{p\}$ and $\tau = \{\emptyset, A, X\}$.

(i) Clearly, $X_1 = \{p\}$, $X_2 = A$ and for each infinite $B \subseteq X$, we have $\overline{B} = X$. Hence $X_1 \cap \text{Int} B \neq \emptyset$, so $B$ is not hsg-closed. Thus $(X, \tau)$ is a $C_3$-space, so sg-compact. But the open subspace $A$ is an infinite indiscrete space which is not sg-compact. This shows that (1) hereditary sg-compactness is a strictly stronger concept than sg-compactness and (2) in Proposition 2.11, ‘$\delta$-open’ cannot be replaced with ‘open’.

(ii) Observe that $X \times X$ contains an infinite nowhere dense subset, namely $X \times X \setminus A \times A$. This shows that even the finite product of two sg-compact spaces need not be sg-compact, not even a $C_2$-space.

(iii) [15] If the nonempty product of two spaces is sg-compact $T_{gs}$-space (see [13]), then each factor space is sg-compact.

Recall that a function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called pre-sg-continuous [17] if $f^{-1}(F)$ is sg-closed in $X$ for every semi-closed subset $F \subseteq Y$.

Proposition 2.13 (i) The property ‘sg-compact’ is topological.

(ii) Pre-sg-continuous images of sg-compact spaces are semi-compact. □

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