1. Introduction

We discuss a functional integral approach to construction of Lorentz-covariant quantum gauge theories on a noncommutative space-time. There have been quite a number of work in this direction, mostly using various Moyal-type star products to construct Lagrangians. One of the most influential works was that of Seiberg and Witten [8], where they, among many other things, noted that simple problems of evolution of a string in a background force field invariably leads to some kind of noncommutativity of space-time coordinates. The type of noncommutativity they were using led to violation of the Lorentz covariance. There was a lot of works discussing these violations and attempting to fix this problem. In our paper [3] we proposed a version of the Moyal-type approach based on a group earlier used by Doplicher-Fredenhagen-Roberts [4] for other reasons. We came to this group by contracting the group \( SO(4,1) \) used by Snyder [9] to treat noncommutative space-time.

Though our approach allowed to avoid Lorentz-covariance violations, there were other problems that were dealt with in quite artificial ways – the most important being related to treating gauge fields in the noncommutative setting.

Reflecting on this circle of ideas we were led to consideration of a functional integral methods, based on deformation of a commutative group to a family of non-commutative ones. This approach leads to a natural way of constructing functions of fields on a noncommutative space-time, and the formulas suggest that the space of probability measures on the classical space-time is a natural (though infinite-dimensional) noncommutative analogue of the classical space-time. The idea is to view measures as "fat points" which are indistinguishable from usual points of the space-time if the scale is not small enough.

We would like to note that this idea seems to be very close to the ideology of string theory. To some extent, it follows from our considerations that non-commutativity of space-time invariably leads to a version of string field theory.

This article is a preliminary exposition of our results, we plan to write a more comprehensive paper, where we shall explore the connections with other approaches.

I would like to thank Carl Carlson, Chris Carone, Josh Ehrlich and Gene Tracy for valuable and illuminating discussions of many questions arising in relation to the problems studied in this article.
2. DOPPLER-FREDENHAGEN-ROBERTS ALGEBRA

Consider the following 10-dimensional Lie algebra $\mathfrak{g}_\epsilon = \mathbb{R}^4 \oplus (\mathbb{R}^4 \wedge \mathbb{R}^4)$, $\epsilon > 0$, with the bracket

$$\forall x, y \in \mathbb{R}^4, \ a, b \in \mathbb{R}^4 \wedge \mathbb{R}^4 \quad [(x, a), (y, b)] = (0, \epsilon x \wedge y).$$

It is easy to check that this bracket indeed defines a Lie algebra structure on $\mathfrak{g}_\epsilon$, and that $\mathfrak{z}_{\mathfrak{g}_\epsilon} = \{(0, a) : a \in \mathbb{R}^4 \wedge \mathbb{R}^4\}$ is the center of this algebra. This Lie algebra is a two-step nilpotent Lie algebra.

The linear space $\mathbb{R}^4 \oplus (\mathbb{R}^4 \wedge \mathbb{R}^4)$ has another Lie algebra structure – that of a commutative Lie algebra, with all brackets equal to zero. We denote this commutative Lie algebra $\mathfrak{g}_0$.

Let $G_\epsilon, \epsilon > 0$ (resp., $G_0$) denote the connected simply connected Lie group having $\mathfrak{g}_\epsilon$ (resp., $\mathfrak{g}_0$) as its Lie algebra. Obviously, $G_\epsilon$ and $G_0$ coincide with $\mathbb{R}^{10}$ as sets, and the group operations are given by the following formulas:

in the group $G_\epsilon$

$$\forall X, Y \in \mathbb{R}^4, \ A, B \in \mathbb{R}^4 \wedge \mathbb{R}^4 \quad (X, A) \diamond (Y, B) = (X + Y, A + B + \frac{\epsilon}{2} X \wedge Y),$$

in the group $G_0$

$$\forall X, Y \in \mathbb{R}^4, \ A, B \in \mathbb{R}^4 \wedge \mathbb{R}^4 \quad (X, A) + (Y, B) = (X + Y, A + B).$$

It is easy to see that the following is true:

(i) both groups have the same neutral element $(0, 0)$,

(ii) both groups have the same group inversion operation $(X, A) \mapsto (-X, -A)$,

(iii) both groups have the same left- and right-invariant Haar measure – the Lebesgue measure $dH(X, A) = d^4X d^6A$.

The group $G_0$ is obviously commutative, while the group $G_\epsilon, \epsilon > 0$, is not, though $G_\epsilon$ is very close to a commutative group – it is a unipotent group. In particular, its finite dimensional irreducible representations have to be one-dimensional. The group $G_\epsilon$ has a huge center

$$Z_{G_\epsilon} = \{(0, A) : A \in \mathbb{R}^4 \wedge \mathbb{R}^4\}.$$ 

One can compute the sets of (equivalence classes) of unitary irreducible representations of both groups.

Obviously,

$$\widehat{G}_0 = \mathbb{R}^{*4} \oplus SS(\mathbb{R}^4),$$

where $SS(\mathbb{R}^4)$ denotes the set of skew symmetric bilinear forms on $\mathbb{R}^4$, the dual space to $\mathbb{R}^4 \wedge \mathbb{R}^4$. Since $G_0$ is commutative, all irreducible unitary representations are one-dimensional, and they are all given by the characters

$$\chi_{(\phi, \Phi)}(X, A) = \exp i(\phi(X) + \Phi(A)), \ (X, A) \in G_0, \ \phi \in \mathbb{R}^{*4}, \ \Phi \in SS(\mathbb{R}^4).$$

The Plancherel measure $dP_0(\phi, \Phi)$ on $\widehat{G}_0$ is an appropriately scaled Lebesgue measure on $\mathbb{R}^{*4} \oplus SS(\mathbb{R}^4)$, more precisely,

$$dP_0(\phi, \Phi) = (2\pi)^{-10}d^4\phi d^6\Phi.$$ 

A description of unitary irreducible representations of $G_\epsilon$ is less obvious, but it can be directly derived, e.g., using the Kirillov’s Orbit Method, see, e.g., [57, 11, 1]).
We actually do not need this description, therefore we only very briefly present it here.

One can show that
\[ \hat{G}_\epsilon = \mathbb{R}^{*4} \cup SS(\mathbb{R}^4). \]
The \( \mathbb{R}^{*4} \) part describes one dimensional representations of \( G \), given by the formula
\[ \chi_{\phi}(X, A) = \exp i\phi(X), \quad \phi \in \mathbb{R}^{*4}, \]
while the \( SS(\mathbb{R}^4) \) part describes the infinite-dimensional representations. The Plancherel measure \( dP_\epsilon \) is supported on the \( SS(\mathbb{R}^4) \) part, and it can be shown that
\[ dP_\epsilon(\Phi) = \sqrt{\det \Phi} \ d^6\Phi. \]
This means that the Plancherel measure is supported by the nondegenerate skew symmetric bilinear forms, i.e., by symplectic forms on \( \mathbb{R}^4 \). Since we are going to consider the Fourier transform on \( G \), we are really interested only in the unitary irreducible representations associated with symplectic forms. To get these representations, consider a symplectic form \( \Phi \), then choose two complementary 2-dimensional \( \Phi \)-Lagrangian subspaces \( l, l' \) in \( \mathbb{R}^4 \), and consider the usual (infinite-dimensional) irreducible Heisenberg representation in \( L_2(l) \), see, e.g., [10].

3. \textbf{Fourier Transform and Quantization Mapping}

Consider a function \( f \in L_2(G_0, dH) = L_2(\mathbb{R}^{10}, d^{10}x) \). Let
\[ \bar{F}_0 : L_2(G_0, dH) \to L_2(\hat{G}_0, dP_0) \]
be the Fourier transform on the commutative group \( G_0 \), which is simply the usual Fourier transform on \( \mathbb{R}^{10} : \)
\[ (\bar{F}_0 f)(\phi, \Phi) = \int_{G_0} \chi_{(\phi, \Phi)}(X, A) f(X, A) dH(X, A) \]
\[ = \int_{\mathbb{R}^{10}} \exp(i(\phi(X) + \Phi(A))) f(X, A) d^4 X d^6 A. \]
This mapping is a unitary operator, and its inverse is well known.

We can also consider the mapping
\[ \bar{F} : L_2(G_\epsilon, dH) \to L_2(\hat{G}_\epsilon, \Lambda(G_\epsilon); dP_\epsilon) \]
– the Fourier transform on the noncommutative group \( G_\epsilon \), given by the formula
\[ (\bar{F} f)(\rho) = \int_{G_\epsilon} \rho((X, A)) f(X, A) dH(X, A) = \int_{\mathbb{R}^{10}} \rho((X, A)) f(X, A) d^4 X d^6 A. \]
This mapping sends a function \( f \) to a section \( F \) of the \textbf{dual bundle}
\[ \Lambda(G_\epsilon) = \bigcup_{\rho \in \hat{G}_\epsilon} HS(H_\rho), \]
where \( H_\rho \) is the Hilbert space of the unitary irreducible representation \( \rho \in \hat{G}_\epsilon \), and \( HS(H_\rho) \) is the space of Hilbert-Schmidt operators in this space. So, \( F = \bar{F} f \) is a function on \( \hat{G}_\epsilon \), whose value at \( \rho \in \hat{G}_\epsilon \) is a Hilbert-Schmidt operator on the Hilbert space \( H_\rho \). The space of such sections has a natural structure of the Hilbert space, with
\[ \langle F_1, F_2 \rangle = \int_{\hat{G}_\epsilon} \text{tr}(F_1(\rho) F_2^*(\rho)) dP_\epsilon(\rho), \]
where \(dP_\rho\) is the Plancherel measure on \(\hat{G}\). It is known that the mapping \(\mathfrak{F}_{G_\epsilon}\) is unitary and the inverse mapping is given by the formula

\[
(\mathfrak{F}_{G_\epsilon}^{-1}F)(X, A) = \int_{\hat{G}_\epsilon} \text{tr}(F(\rho)\rho(X, A)^*)dP_\rho(\rho).
\]

Now consider the quantization mapping

\[Q_\epsilon = \mathfrak{F}_{G_\epsilon}, \mathfrak{F}_{G_0}^{-1} : L_2(\hat{G}_0, dP_0) \to L_2(\hat{G}_\epsilon, \Lambda(G_\epsilon), dP_\epsilon).\]

This mapping is obviously unitary, it can be explicitly inverted. The inverse mapping is called the de-quantization or the symbol mapping. These mappings are closely related to Weyl symbols, etc., see, e.g., [5, 6, 2].

Let us note an important property of the quantization mapping: let \(1_0\) denote the function on \(\hat{G}_0\) whose value at each point of \(\hat{G}_0\) is 1, let \(1_\epsilon\) denote the section, whose value at each point \(\rho \in \hat{G}_\epsilon\) is the identity operator in \(H_\rho\). One can easily see that

\[Q_\epsilon(1_0) = 1_\epsilon.\]

This implies (via the unitarity of \(Q_\epsilon\)) that

\[
\forall f \in L_1(\hat{G}_0, dP_0) \int_{\hat{G}_0} f dP_0 = \langle f, 1_0 \rangle = \langle Q_\epsilon f, 1_\epsilon \rangle = \int_{\hat{G}_\epsilon} \text{tr}(Q_\epsilon f)dP_\epsilon.
\]

We treat functions \(f : \mathbb{R}^4 \to \mathbb{C}\) as classical scalar fields on \(\mathbb{R}^4\). They can also be viewed as functions on \(\hat{G}_0\), though not integrable with respect to \(dP_0\). Let us fix a weight function \(W : SS(\mathbb{R}^4) \to \mathbb{R}_+\), that is, a positive function, such that \(\int_{SS(\mathbb{R}^4)} W(\Phi)d^6\Phi = 1\).

The introduction of the weight function allows one to integrate classical scalar fields with respect to the weighted measure, and so that

\[
\int_{\hat{G}_0} f(\phi)W(\Phi)dP_0(\phi, \Phi) = \int_{\mathbb{R}^4} f(\phi)d^4\phi.
\]

Note that since \(W\) does not depend upon \(\phi \in \mathbb{R}^4\), then \(\mathfrak{F}_{G_0}^{-1}W\) is supported by the subspace \(\mathbb{R}^4 \land \mathbb{R}^4\) in \(G_0 = G_\epsilon\), and therefore

\[
(Q_\epsilon W)(\rho) = \int_{\hat{G}_\epsilon} \rho(X, A)(\mathfrak{F}_{G_0}^{-1}W)(X, A)dH(X, A)
\]

\[
= \int_{\mathbb{R}^4 \land \mathbb{R}^4} \rho(0, A)(\mathfrak{F}_{G_0}^{-1}W)(0, A)d^6A,
\]

so since the elements \((0, A)\) belong to the center of the group \(G_\epsilon\) then the operators \(\rho(0, A)\) are scalar operators, \(\rho(0, A) = \lambda(A)\text{id}\), so for any \(\rho \in \hat{G}_\epsilon\) the operator \((Q_\epsilon W)(\rho)\) is scalar. Because \(W\) is real valued we also have

\[
\forall \rho \in \hat{G}_\epsilon \quad (Q_\epsilon W)(\rho) = (Q_\epsilon \overline{W})(\rho) = (Q_\epsilon W)(\rho)^*.
\]

So, this is a real valued scalar operator. To sum up, \(Q_\epsilon W\) can be treated as a real valued function on \(\hat{G}_\epsilon\). So \((Q_\epsilon W)(\rho) = (\text{function of } \rho)\text{id}_\rho\). Slightly abusing notation, we identify this function with \(Q_\epsilon W\).

Obviously,

\[
\int_{\mathbb{R}^4} f(\phi)d^4\phi = \int_{\hat{G}_0} f(\phi)W(\Phi)dP_0(\phi, \Phi) = \int_{\hat{G}_\epsilon} \text{tr}((Q_\epsilon f)(\rho)(Q_\epsilon W)(\rho))dP_\epsilon(\rho).
\]
\[\int_{\hat{G}_0} \text{tr}((Q_\epsilon f)(\rho))(Q_\epsilon W)(\rho) dP_\epsilon(\rho).\]

4. Symbol of a function of a section

The quantization mapping establishes a one-to-one linear correspondence between functions on \(\hat{G}_0\) and sections of the dual bundle \(\Lambda(G)\). But taking functions of functions and functions of sections is quite different.

Let \(f\) be a real-valued function on \(\hat{G}_0\). Choose \(k : \mathbb{R} \to \mathbb{C}\) be a function of one variable. Then one can consider the function \(k(f)\) as usual – \(k(f)(\phi, \Phi) = k(f(\phi, \Phi))\), and get a new function on the same set. We treat this obvious way of computing a function of a function as the "commutative way".

Let \(F = Q_\epsilon f\) be a section with, say, self-adjoint values. Choose any reasonable function \(k : \mathbb{R} \to \mathbb{C}\), and then one can define a new section \(k(F)\) as follows:

\[(k(F))(\rho) = k(F(\rho)),\]

assuming that the function \(k\) of the operator \(F(\rho)\) (acting on the Hilbert space \(H_\rho\)) makes sense for every \(\rho \in \hat{G}_\epsilon\).

One can easily see that \(Q_\epsilon^{-1}k(\epsilon f) \neq k(Q_\epsilon f)\).

So \(Q_\epsilon^{-1}k(\epsilon f)\) can be viewed as a different way of computing a function of a function – a "noncommutative way".

As a matter of fact what we really need to compute is an action functional which in this simplest case is defined to be

\[S_W(F) = \int_{\hat{G}_\epsilon} \text{tr}(k(F)(\rho))(Q_\epsilon W)(\rho) dP_\epsilon(\rho).\]

So we try first to compute the symbol \(Q_\epsilon^{-1}k(F)\) and then use the above mentioned fact that

\[\int_{\hat{G}_\epsilon} \text{tr}(k(F)(\rho))(Q_\epsilon W)(\rho) dP_\epsilon(\rho) = \int_{\hat{G}_0} (Q_\epsilon^{-1}k(F))(\phi, \Phi)W(\Phi)dP_0(\phi, \Phi).\]

Since the main goal of our computations is to get a hint for our definitions to be presented below, we are rather formal in our computations – we freely interchange limits and integrals, the order of integration, do not pay much attention to the questions of convergence, etc. We feel that all computations below can be made precise under some additional rather mild assumptions.

Let us first note that it is enough to compute \(Q^{-1}(\exp(itF))\) for every \(t \in \mathbb{R}\) since then one can calculate

\[Q^{-1}(k(F)) = \int_{\mathbb{R}} Q^{-1}(\exp(itF))\hat{k}(t)dt,\]

where \(\hat{k}\) is the usual inverse Fourier transform of the function \(k : \mathbb{R} \to \mathbb{C}\):

\[k(s) = \int_{\mathbb{R}} e^{its}\hat{k}(t)dt.\]

Of course, we assume that all integrals make sense (this is actually a part of the definition of a reasonable function \(k\)).

Let us recall several simple facts related to general Fourier transforms:

(i) the inverse Fourier transforms of the functions \(1_\epsilon, \epsilon \geq 0\), equals \(\delta_0(X,Y)\) (the delta function supported at the common neutral element of all groups in question),
(ii) the $G_\varepsilon$-convolution of functions $h_1, h_2, \ldots, h_N$ on $G_\varepsilon$ is given by the formula

$$(h_1 *_\varepsilon \cdots *_\varepsilon h_N)(X, A) = \int_{G_\varepsilon^N} \prod_{j=1}^N h_j(X_j, A_j) \delta(X_j, A_j) \delta_j(X_j, A_j) \prod_{j=1}^N dH(X_j, A_j),$$

(iii) the Fourier transform of a $G_\varepsilon$-convolution of several functions on a group $G_\varepsilon$ equals the product of the Fourier transforms of these functions.

Using these facts plus the fact that all groups $G_\varepsilon$, $\varepsilon \geq 0$, coincide as sets, and have the same invariant measures $dH(X, A)$, we compute

$$Q_{\varepsilon}^{-1}(\exp(itF))(\phi, \Phi) = (\hat{\delta}_{G_\varepsilon} \hat{\delta}_{G_\varepsilon}^{-1}(\lim_{N \to \infty} (1_\varepsilon + \frac{itF}{N})))(\phi, \Phi)$$

$$= \lim_{N \to \infty} (\hat{\delta}_{G_\varepsilon}((\delta_0(X, A) + \frac{it(\hat{\delta}_{G_\varepsilon}^{-1}F)(X, A)}{N})))(\phi, \Phi)$$

$$= \lim_{N \to \infty} \int_{G_\varepsilon} \exp(i(\phi(X) + \Phi(A)))dH(X, A) \times$$

$$\times \left( \int_{G_\varepsilon^N} \prod_{j=1}^N \left( \delta_0(X_j, A_j) + \frac{it(\hat{\delta}_{G_\varepsilon}^{-1}F)(X_j, A_j)}{N} \right) \delta(X_j, A_j) \delta_j(X_j, A_j) \prod_{j=1}^N dH(X_j, A_j) \right)$$

$$= \lim_{N \to \infty} \int_{G_\varepsilon^N} \prod_{j=1}^N \left( \delta_0(X_j, A_j) + \frac{it(\hat{\delta}_{G_\varepsilon}^{-1}F)(X_j, A_j)}{N} \right) \delta(X_j, A_j) \delta_j(X_j, A_j) \prod_{j=1}^N dH(X_j, A_j) \times$$

$$\times \exp(i((\phi, \Phi)(\delta_j^{N-1}(X_j, A_j))) \prod_{j=1}^N dH(X_j, A_j)$$

$$= \lim_{N \to \infty} \int_{G_\varepsilon^N} \prod_{j=1}^N \left( \delta_0(X_j, A_j) + \frac{it(\hat{\delta}_{G_\varepsilon}^{-1}F)(X_j, A_j)}{N} \right) \delta(X_j, A_j) \delta_j(X_j, A_j) \prod_{j=1}^N dH(X_j, A_j) \times$$

$$\times \exp(-i((\phi, \Phi)(\delta_j^{N-1}(X_j, A_j))) \prod_{j=1}^N dH(X_j, A_j)$$

Applying the Plancherel identity ( = unitarity of the Fourier transform) for the group $G_\varepsilon^N$, we see that

$$Q_{\varepsilon}^{-1}(\exp(itF))(\phi, \Phi) = \lim_{N \to \infty} \int_{G_\varepsilon^N} \prod_{j=1}^N \left( 1 + \frac{it(Q_{\varepsilon}^{-1}F)(\phi_j, A_j)}{N} \right) \prod_{j=1}^N dP_0(\phi_j, A_j) \times$$

$$\times \int_{G_\varepsilon^N} \exp \left( i \sum_{j=1}^N \phi_j(X_j) + \Phi_j(A_j) \right) \exp \left( -i(\phi, \Phi)(\delta_j^{N-1}(X_j, A_j)) \prod_{j=1}^N dH(X_j, A_j) \right)$$

$$= \lim_{N \to \infty} \int_{G_\varepsilon^N} \prod_{j=1}^N \left( 1 + \frac{it(Q_{\varepsilon}^{-1}F)(\phi_j, A_j)}{N} \right) \prod_{j=1}^N dP_0(\phi_j, A_j) \times$$
\[
\times \int_{G^N_0} \exp \left( -i \sum_{j=1}^{N} \phi_j(X_j) + \Phi_j(A_j) \right) \exp \left( i(\phi, \Phi)(\hat{\diamond}_{j=1}^{N}(X_j, A_j)) \right) \prod_{j=1}^{N} dH(X_j, A_j).
\]

4.1. **Simplifications.** We need to simplify the term in the last line:

\[
\tilde{\Psi}_N(\phi; \Phi; \phi_1, \ldots, \phi_N; \Phi_1, \ldots, \Phi_N)
\]

\[
= \int_{G^N_0} \exp \left( -i \sum_{j=1}^{N} \phi_j(X_j) + \Phi_j(A_j) \right) \exp \left( i(\phi, \Phi)(\hat{\diamond}_{j=1}^{N}(X_j, A_j)) \right) \prod_{j=1}^{N} dH(X_j, A_j).
\]

Let us perform the following substitution in this integral:

\[
\hat{\diamond}_{j=1}^{k}(X_j, A_j) = (Y_k, B_k), \quad k = 1, 2, \ldots, N,
\]

let \((Y_0, B_0) = (0, 0)\)

Then

\[
(Y_{k-1}, B_{k-1})\hat{\diamond}(X_k, A_k) = (Y_k, B_k),
\]

so

\[
(X_k, A_k) = (Y_k, B_k)\hat{\diamond}(-Y_{k-1}, -B_{k-1}) = (Y_k - Y_{k-1}, A_k - A_{k-1} - \frac{\epsilon}{2} Y_j \wedge Y_{j-1}),
\]

\[
k = 1, 2, \ldots, N.
\]

So the substitution is one-to-one, its Jacobian is apparently 1, so we get

\[
\tilde{\Psi}_N(\phi; \Phi; \phi_1, \ldots, \phi_N; \Phi_1, \ldots, \Phi_N)
\]

\[
= \int_{G^N_0} \exp (-i) \left( \sum_{j=1}^{N} \phi_j(Y_j - Y_{j-1}) + \Phi_j(B_j - B_{j-1} - \frac{\epsilon}{2} Y_j \wedge Y_{j-1}) \right) \times
\]

\[
\times \exp i(\phi(Y_N) + \Phi(B_N)) \prod_{j=1}^{N} dH(Y_j, B_j)
\]

\[
= \int_{\mathbb{R}^{6N}} \exp i \left( (\Phi - \Phi_N)(B_N) + \sum_{j=1}^{N-1} (\Phi_{j+1} - \Phi_j)(B_j) \right) \prod_{j=1}^{N-1} d^6 B_j \left(2\pi\right)^{6N} \times
\]

\[
\times \int_{\mathbb{R}^{4N}} \exp i \left( (\phi - \phi_N)(Y_N) + \sum_{j=1}^{N-1} (\phi_{j+1} - \phi_j)(Y_j) + \frac{\epsilon}{2} \sum_{j=1}^{N} \Phi_j(Y_j \wedge Y_{j-1}) \right) \prod_{j=1}^{N} d^4 Y_j \left(2\pi\right)^{4N} \times
\]

\[
\delta_0(\Phi - \Phi_N) \prod_{j=1}^{N-1} \delta_0(\Phi_{j+1} - \Phi_j) \times
\]

\[
\times \int_{\mathbb{R}^{4N}} \exp i \left( (\phi - \phi_N)(Y_N) + \sum_{j=1}^{N-1} (\phi_{j+1} - \phi_j)(Y_j) + \frac{\epsilon}{2} \Phi \left( \sum_{j=1}^{N} Y_j \wedge Y_{j-1} \right) \right) \prod_{j=1}^{N} d^4 Y_j \left(2\pi\right)^{4N}.
\]

So, letting \(\phi_{N+1} = \phi, \Phi_{N+1} = \Phi\), if necessary, we see that

\[
\tilde{\Psi}_N(\phi; \Phi; \phi_1, \ldots, \phi_N; \Phi_1, \ldots, \Phi_N)
\]

\[
= \prod_{j=1}^{N} \delta_0(\Phi - \Phi_j) \int_{\mathbb{R}^{4N}} \exp i \left( \sum_{j=1}^{N} (\phi_{j+1} - \phi_j)(Y_j) \right) \exp i\Phi \left( \frac{\epsilon}{2} \sum_{j=1}^{N} Y_j \wedge Y_{j-1} \right) \prod_{j=1}^{N} d^4 Y_j \left(2\pi\right)^{4N}.
\]
5. Computation of the simplest action functional

Now let us assume that the function $f$ does not depend upon $\Phi$, i.e., $f$ is a classical scalar field. Let us now compute the action functional

$$S_W(Q^{-1}_e(\exp itQ, f)) = \int_{G_0} Q^{-1}_e(\exp itQ, f)(\phi, \Phi) W(\Phi) dP_0(\phi, \Phi)$$

$$= \lim_{N \to \infty} \int_{G_0} W(\Phi) dP_0(\phi, \Phi) \int_{R^{4N}} \prod_{j=1}^{N} \left( 1 + \frac{itf(\phi_j)}{N} \right) \prod_{j=1}^{N} \frac{d^4\phi_j}{(2\pi)^4N} \times$$

$$\times \int_{R^{4N}} \exp i \left( \sum_{j=1}^{N} (\phi_{j+1} - \phi_j)(Y_j) \right) \exp i\Phi \left( \sum_{j=1}^{N} \frac{\epsilon}{2} Y_j \wedge Y_{j-1} \right) \prod_{j=1}^{N} \frac{d^4Y_j}{(2\pi)^4N}$$

$$= \lim_{N \to \infty} \int_{R^{4N}} \prod_{j=1}^{N} \left( 1 + \frac{itf(\phi_j)}{N} \right) \prod_{j=1}^{N} \frac{d^4\phi_j}{(2\pi)^4N} \times$$

$$\times \int_{R^{4N}} \frac{d^4\phi_{N+1}}{(2\pi)^4} \int_{R^{4N}} \exp i \left( \sum_{j=1}^{N} (\phi_{j+1} - \phi_j)(Y_j) \right) \prod_{j=1}^{N} \frac{d^4Y_j}{(2\pi)^4N}$$

For $A \in R^4 \wedge R^4$ denote

$$w(A) = \int_{SS(R^4)} W(\Phi) \exp i\Phi(A) \frac{d^6\Phi}{(2\pi)^6}.$$

Obviously,

$$w(0) = \int_{SS(R^4)} W(\Phi) \frac{d^6\Phi}{(2\pi)^6} = 1.$$

Then we get

$$S_W(Q^{-1}_e(\exp itQ, f)) = \lim_{N \to \infty} \int_{R^{4N}} \prod_{j=1}^{N} \left( 1 + \frac{itf(\phi_j)}{N} \right) \prod_{j=1}^{N} \frac{d^4\phi_j}{(2\pi)^4N} \times$$

$$\times \int_{R^{4N}} \frac{d^4\phi_{N+1}}{(2\pi)^4} \int_{R^{4N}} \exp i \left( \sum_{j=1}^{N} (\phi_{j+1} - \phi_j)(Y_j) \right) \prod_{j=1}^{N} \frac{d^4Y_j}{(2\pi)^4N}.$$

Integrating with respect to $\phi_{N+1}$ we get a factor of $\delta_0(Y_N)$ and therefore we finally obtain

$$S_W(Q^{-1}_e(\exp itQ, f)) = \lim_{N \to \infty} \int_{R^{4N}} \prod_{j=1}^{N} \left( 1 + \frac{itf(\phi_j)}{N} \right) \prod_{j=1}^{N} \frac{d^4\phi_j}{(2\pi)^4N} \times$$

$$\times \int_{R^{4(N-1)}} \exp i \left( \sum_{j=1}^{N-1} (\phi_{j+1} - \phi_j)(Y_j) \right) \prod_{j=1}^{N-1} \frac{d^4Y_j}{(2\pi)^4(N-1)}.$$
If \( \epsilon \to 0 \), then the function \( w \left( \frac{\epsilon}{2} \sum_{j=1}^{N-1} Y_j \land Y_{j-1} \right) \) goes to the constant function equal to \( w(0) = 1 \), the Fourier transform of this function goes to \( \delta_0 \), so
\[
\int_{\mathbb{R}^{(N-1)}} \exp \left( i \left( \sum_{j=1}^{N-1} \phi_{j+1} - \phi_j \right)(Y_j) \right) w \left( \frac{\epsilon}{2} \sum_{j=1}^{N-1} Y_j \land Y_{j-1} \right) \prod_{j=1}^{N-1} d^4 Y_j \to \prod_{j=1}^{N} \delta(\phi_{j+1} - \phi_j),
\]
and so
\[
\lim_{\epsilon \to 0} S_W(Q_\epsilon^{-1}(\exp itQ_\epsilon f)) = \lim_{N \to \infty} \int_{\mathbb{R}^{N+1}} \prod_{j=1}^{N} \left( 1 + \frac{if(\phi_j)}{N} \right) \prod_{j=1}^{N} \delta(\phi_{j+1} - \phi_j) \prod_{j=1}^{N} d^4 \phi_j \left( \frac{2\pi}{4} \right)^4,
\]
which is what one should have expected.

5.1. Formal passage to the limit as \( N \to \infty \). Now let us extract some hints from this formula. Since these are going to be only hints (or, better to say, motivations for our subsequent definitions) we proceed very formally.

Introduce a mapping \( \phi : [0, 1] \to \mathbb{R}^4 \), and let \( \phi_j = \phi(j/N), \ j = 1, 2, \cdots, N \). The set of such mappings is denoted \( T(\mathbb{R}^4) \). Also, introduce a mapping \( \zeta : [0, 1] \to \mathbb{R}^4 \), and let \( \zeta_j = \zeta(j/N), \ j = 1, 2, \cdots, N - 1 \). The set of such mappings is denoted \( T(\mathbb{R}) \).

Then
\[
S_W(Q_\epsilon^{-1}(\exp itQ_\epsilon f)) = \lim_{N \to \infty} \int_{\mathbb{R}^{N+1}} \exp it \left( \sum_{j=1}^{N} \frac{1}{N} f(\phi(j/N)) \right) \prod_{j=1}^{N} d^4 \phi(j/N) \left( \frac{2\pi}{4} \right)^4 \times
\]
\[
\times \int_{\mathbb{R}^{(N-1)}} \exp \left( i \left( \sum_{j=1}^{N-1} \frac{\phi((j+1)/N) - \phi(j/N)}{1/N} \right) Z(j/N) \right) \prod_{j=1}^{N-1} \prod_{j=1}^{N} d^4 Z(j/N) \left( \frac{2\pi}{4} \right)^4 (N-1) \times
\]
\[
\times w \left( \frac{\epsilon}{2} \sum_{j=1}^{N-1} \frac{Z(j/N) - Z((j-1)/N)}{1/N} \right) \prod_{j=1}^{N-1} d^4 Z(j/N) \left( \frac{2\pi}{4} \right)^4 (N-1) \times
\]
\[
\times \int_{T(\mathbb{R}^4)} \exp \left( it \int_0^1 f(\phi(\sigma)) d\sigma \right) D\phi(\cdot) \times
\]
\[
\times \int_{T(\mathbb{R}^4)} \exp \left( i \int_0^1 \phi'(\sigma)(Z(\sigma)) d\sigma \right) w \left( \int_0^1 \frac{\epsilon}{2} Z'(\sigma) \land Z(\sigma) d\sigma \right) DZ(\cdot).
\]

5.2. From parametrized paths to measures. A continuous parametrized path \( \phi : [0, 1] \to \mathbb{R}^4 \) can have an almost arbitrary form (in particular, it can fill a whole cube in \( \mathbb{R}^4 \), due to the famous Peano example), we prefer to treat a parametrized curve \( \phi \) as a probability measure on \( \mathbb{R}^4 \), supported on (the closure of) the range of \( \phi \), with the measure of a piece of the curve equal to the one dimensional measure of its pre-image. Then \( \int_0^1 f(\phi(\sigma)) d\sigma \) is simply the integral of \( f \) against this measure, we denote it \( \langle f, \phi \rangle \).

Accordingly, the space \( T(\mathbb{R}^4) \) can be treated as the space of probability measures on \( \mathbb{R}^4 \). Let us drop the condition that \( \langle 1, \phi \rangle = 1 \), instead we require that the measure \( \phi \in T(\mathbb{R}^4) \) is nonnegative and \( \langle 1, \phi \rangle \leq 1 \).
We would like to treat expression
\[ D\phi(\cdot) \int_{T(R^4)} \exp \left( i \int_0^1 \phi'(\sigma)(Z(\sigma))d\sigma \right) w \left( \int_0^1 \frac{1}{2} Z'(\sigma) \wedge Z(\sigma)d\sigma \right) DZ(\cdot) \]
as a measure on the (weakly compact) set \( T(R^4) \) of measures on \( R^4 \). However, this is not likely, e.g., this expression is explicitly complex-valued, which causes a lot of difficulties. However, the infinite-dimensional integral is over the space of parametrized curves in \( R^4 \), but many of these curves lead to the same measures on \( T(R^4) \). On the other hand the expression \( k(\langle f, \phi \rangle) \) depends upon a measure. We presume (though have not yet proven this) that a "symmetrized" version of the expression \( D\phi(\cdot) \int_{T(R^4)} \exp \left( i \int_0^1 \phi'(\sigma)(Z(\sigma))d\sigma \right) w \left( \int_0^1 \frac{1}{2} Z'(\sigma) \wedge Z(\sigma)d\sigma \right) DZ(\cdot) \) obtained by summation over all parametrized curves leading to the same measure, is likely to define a positive measure on the space \( T(R^4) \). We have some promising developments in the finite-dimensional case associated with finite analogues of the groups in question.

Let \( dM_{W,\epsilon}(\phi) \) denote the hypothetical positive measure on the space \( T(R^4) \) such that
\[ S_W(Q^{-1}_\epsilon(\exp i\epsilon Q_\epsilon f)) = \int_{T(R^4)} \exp(i\epsilon \langle f, \phi \rangle) dM_{W,\epsilon}(\phi). \]

Keeping in mind that any function \( k : R \to C \) is an integral over exponentials, we see that for any function \( k \) we have
\[ S_W(Q^{-1}_\epsilon(k(Q_\epsilon f))) = \int_{T(R^4)} k(\langle f, \phi \rangle) dM_{W,\epsilon}(\phi). \]

One can easily show that a natural extension of this formula holds for vector valued fields \( \vec{f} : R^4 \to R^m \), and for functions \( k : R^k \to C \):
\[ S_W(Q^{-1}_\epsilon(k(Q_\epsilon \vec{f}))) = \int_{\hat{G}_\epsilon} \text{tr}(k(Q_\epsilon \vec{f})) Q_\epsilon W dP_\epsilon = \int_{T(R^4)} k(\langle \vec{f}, \phi \rangle) dM_{W,\epsilon}(\phi), \]
where \( Q_\epsilon \vec{f} = (Q_\epsilon f_i)_{i=1}^m \), the function \( k(F_1, \cdots, F_m) \) of several operators is defined in the symmetric (Weyl) way (one first defines the exponential functions \( k = \exp \sum_{i=1}^m \epsilon_i x_i \) of the operators – the problem of ordering does not exist for these functions, then any function is represented as an integral over exponentials), \( \langle \vec{f}, \phi \rangle \in R^4 \).

6. Noncommutative space-time and fields on it

A vector field \( \vec{f} \) on \( R^4 \) gives rise to a linear vector valued function \( \phi \to \langle \vec{f}, \phi \rangle \) on \( T(R^4) \). Then we take a scalar function of this linear vector valued function and integrate it over the space-time to obtain the quantities needed to form the action functional.

So if we take the classical vector fields on \( R^4 \), transplant them to \( \hat{G}_\epsilon \), so that when computing a function \( k \) of the field we are able to take the noncommutativity of the space-time into account, and then form the noncommutative version of the action functional of such classical fields, we arrive at an infinite-dimensional integral of a scalar function of a linear vector valued function on \( T(R^4) \). However, there is a serious problem arising here: let the function \( k \) be invariant under action of a subgroup \( \Theta \subset GL(m) \):
\[ k(x) = k(gx), \forall g \in \Theta, x \in R^m. \]
Then the function \( k(\tilde{f}) \) is invariant under the gauge group of continuous mappings from \( \mathbb{R}^{4*} \) to \( \mathcal{G} \), i.e., \( k(f(\psi)) = k(g(\psi)) \), \( \forall \psi \in \mathbb{R}^{4*} \), \( g(\psi) \in \mathcal{G} \). Apparently the function \( k((\tilde{f}, \phi)) \) on \( T(\mathbb{R}^{4*}) \) is not invariant under this gauge group. However, this function \( k((\tilde{f}, \phi)) \) is obviously invariant under the action of the **extended gauge group** consisting of continuous mappings from \( T(\mathbb{R}^{4*}) \) to \( \mathcal{G} \) (the set \( T(\mathbb{R}^{4*}) \) is endowed with the weak topology):

\[
  k((\tilde{f}, \phi)) = k(g(\phi)(\tilde{f}, \phi)), \quad \forall \phi \in T(\mathbb{R}^{4*}) \quad g(\phi) \in \mathcal{G}.
\]

But in this case the vector valued function \( g(\mu)(\tilde{f}, \phi) \) is **not linear** in \( \phi \).

Restricting ourselves only to linear vector valued functions \( \phi \mapsto (\tilde{f}, \phi) \) was caused by the fact that we were considering only usual vector fields \( \tilde{f} \) on the space-time \( \mathbb{R}^{4*} \).

The extended gauge transformations produce nonlinear vector valued functions on \( T(\mathbb{R}^{4*}) \), which obviously do not come from the classical vector fields on \( \mathbb{R}^{4*} \).

In quantum field theory the quantities of interest arise when one integrates the exponentials of the action functional (as a function of a field) over all fields. So a field is a “silent” variable, being integrated over, and therefore it seems to be not a big deal if we broaden the set of fields.

Let us very significantly broaden the notion of a vector field in this (noncommutative) context, allowing any (not necessarily linear) vector valued continuous function on the weakly compact set of measures \( T(\mathbb{R}^{4*}) \). The extended gauge group naturally acts on such vector fields. The action functional associated with such field is defined as

\[
  S_W(k(\tilde{f})) = \int_{T(\mathbb{R}^{4*})} k((\tilde{f}, \phi)) dM_{W, \epsilon}(\phi).
\]

So we define

**the noncommutative space-time** = the set \( T(\mathbb{R}^{4*}) \) of measures on the commutative space-time \( \mathbb{R}^{4*} \).

the noncommutative space-time \( T(\mathbb{R}^{4*}) \) is a weakly compact subset in an infinite-dimensional dual Banach space. The usual space-time \( \mathbb{R}^{4*} \) is imbedded into the noncommutative space-time by the mapping \( \mathbb{R}^{4*} \ni \psi \mapsto \delta_\psi = \phi \in T(\mathbb{R}^{4*}) \). Since the delta measures form kind of a basis in the linear space of measures on \( \mathbb{R}^{4*} \), then a classical scalar field is an arbitrary function on this basis, and it naturally extends to the whole linear space of measures as a linear function. However, there exist many extensions of the same function on the basis to nonlinear functions on the whole space of measures. These nonlinear functions are not distinguishable from linear functions if we observe only their values on the basis. Since for a very small \( \epsilon \) our space-time is almost commutative, which means that the support of the measure \( dM_{W, \epsilon}(\phi) \) is very close to the set of delta measures, then all nonlinear extensions are indistinguishable from linear extensions, and we in fact are able to observe only functions on the commutative space-time. In other words, the classical fields on the commutative space-time \( \mathbb{R}^{4*} \) are only shadows of fields on the noncommutative space-time \( T(\mathbb{R}^{4*}) \), and for sufficiently small \( \epsilon \) we observe only these shadows.

7. Geometry on the noncommutative space-time

7.1. **Tangent and cotangent bundles over the noncommutative space-time.** By our definition, the noncommutative space-time \( T(\mathbb{R}^{4*}) \) is a simplex in the infinite dimensional dual Banach space \( \mathfrak{M}_4 \) of measures (not necessarily positive,
and of any total variation) on $\mathbb{R}^4$. This space $\mathcal{M}_4$ is dual to the space $C_0(\mathbb{R}^4)$ of continuous compactly supported functions on $\mathbb{R}^4$.

As usual, the tangent space to a linear space at any point coincides with this linear space. So, the tangent space $\mathfrak{T}_\phi T(\mathbb{R}^4)$ to $T(\mathbb{R}^4)$ at each point $\phi \in T(\mathbb{R}^4)$ is simply $\mathcal{M}_4$. So the tangent bundle $\mathfrak{T}T(\mathbb{R}^4)$ is simply $T(\mathbb{R}^4) \times \mathcal{M}_4$.

Therefore a tangent vector field $\Delta$ on $T(\mathbb{R}^4)$ – a section of the tangent bundle – is a rule assigning a measure $\nu \in \mathcal{M}_4$ to each $\phi \in T(\mathbb{R}^4)$: $\Delta(\phi) = \nu$.

Let $A_0$ denote the algebra of complex-valued functions on $T(\mathbb{R}^4)$.

If $f \in A_0$ then we define its derivative at a point $\phi \in T(\mathbb{R}^4)$ with respect to a tangent vector $\nu \in \mathfrak{T}_\phi T(\mathbb{R}^4)$ in the usual way:

$$(\partial_\nu f)(\phi) = \lim_{t \to 0} \frac{1}{t}(f(\phi + t\nu) - f(\phi)).$$

We say that $f$ is weakly Frechet differentiable at $\phi$ if there exists a function $df_\phi \in C_0(\mathbb{R}^4)$ such that for any $\nu \in \mathfrak{T}_\phi T(\mathbb{R}^4)$ we have

$$(\partial_\nu f)(\phi) = (df_\phi, \nu).$$

Therefore we define $\mathfrak{T}_\phi^* T(\mathbb{R}^4)$ – the cotangent space to $T(\mathbb{R}^4)$ at $\phi \in T(\mathbb{R}^4)$ – as $C_0(\mathbb{R}^4)$ (it is a pre-dual to the tangent space, rather than the dual).

A 1-form (or a cotangent vector field) on $T(\mathbb{R}^4)$ is a section of the cotangent bundle, i.e., a rule assigning a function $h_\phi \in C_0(\mathbb{R}^4)$ to each $\phi \in T(\mathbb{R}^4)$. Pairing a cotangent vector field $h$ and a tangent vector field $\nu$ yields a function on $T(\mathbb{R}^4)$:

$$\langle h, \nu \rangle(\phi) = \langle h_\phi, \nu_\phi \rangle.$$

For any tangent vector field $\Delta$ on $T(\mathbb{R}^4)$ we define an operator on $A_0$, which we continue to denote $\Delta$ despite an obvious abuse of notation:

$$(\Delta f)(\phi) = (\partial_{\Delta(\phi)} f)(\phi).$$

For a weakly Frechet differentiable function $f$ we have

$$(\Delta f)(\phi) = (df_\phi, \Delta(\phi)).$$

This operator is obviously linear, it also satisfies the Leibniz rule, so it is a local differentiation of the algebra $A_0$ (locality means that the support of $\Delta(f)$ is contained in the support of $f$). Therefore the space of tangent vector fields is also denoted $Diff A_0$.

### 7.2. Gauge fields on the noncommutative space-time

There is a natural basis in the linear space $\mathcal{M}_4$ – it consists of all $\delta$ measures, $\delta_x$, $x \in \mathbb{R}^4$. For each $x \in \mathbb{R}^4$ we let $\partial_x$ denote differentiation with respect to $\delta_x$, it can be viewed as a (translation invariant) tangent vector field on $T(\mathbb{R}^4)$. One can show that

$$\forall x, y \in \mathbb{R}^4, x \neq y, \ [\partial_x, \partial_y] = 0.$$ 

Let $E = T(\mathbb{R}^4) \times C^m$ be the trivial vector bundle over the noncommutative space-time. Let

$$\Gamma(E) = \{ \bar{f} = (f_i)_{i=1}^m : \forall i, 1 \leq i \leq m, f_i : T(\mathbb{R}^4) \to C \}$$

denote the space of its sections – mappings from $T(\mathbb{R}^4)$ to $C^m$. Then a tangent vector field $\Delta$ still defines an operator on $\Gamma(E)$:

$$\Delta \bar{f} = (\Delta f_i)_{i=1}^m.$$
In particular, standard tangent vector fields $\partial_x$ define operators on $\Gamma(E)$. For each $x \in \mathbb{R}^4$ and for each $\phi \in T(\mathbb{R}^4)$ choose a linear operator $A_x(\phi) : \mathbb{C}^m \to \mathbb{C}^m$. Consider the operators

$$\nabla_x = \partial_x + A_x(\phi)$$

on $\Gamma(E)$:

$$(\nabla_x \tilde{f})(\phi) = (\partial_x \tilde{f})(\phi) + A_x(\phi) \tilde{f}(\phi).$$

They are called the covariant differentiations, or connections, or gauge fields.

Easy to see that

$$[\nabla_x, \nabla_y] = \partial_x A_y - \partial_y A_x + [A_x, A_y],$$

i.e., $([\nabla_x, \nabla_y] \tilde{f})(\phi)$ is multiplication by an operator in $\mathbb{C}^m$. This operator is denoted $F_{xy}(\phi)$ and is called the curvature of the connection $\nabla$.

Let $\mathfrak{G}$ be a Lie group and let $\rho$ be its irreducible representation on $\mathbb{C}^m$. Let $\gamma$ be its Lie algebra. We may consider the connections such that $\forall x \in \mathbb{R}^4, \forall \phi \in T(\mathbb{R}^4) A_x(\phi) \in \rho'(\gamma)$. Such covariant differentiations help ensure that the related expressions are gauge invariant, with the gauge group $\mathfrak{G}$. This means that the expressions are invariant under the transformations $\tilde{f}(\phi) \mapsto g(\phi) \tilde{f}(\phi)$ for an arbitrary function $\phi \mapsto g(\phi) \in \mathfrak{G}$.

We are mostly interested in the following types of expressions:

$$K_1(\nabla; \phi) = \int_{\mathbb{R}^4} \langle F_{xy}(\phi), F_{zw}(\phi) \rangle_{\gamma} B(x, z) B(y, w) d^4 x d^4 y d^4 z d^4 w,$$

where $\langle \cdot, \cdot \rangle_{\gamma}$ is the Killing form - a $\mathfrak{G}$-invariant bilinear form on $\gamma$, $B(\cdot, \cdot)$ is a Lorentz invariant function;

$$K_2(\nabla, \tilde{f}; \phi) = \int_{\mathbb{R}^4} \langle (\nabla_x \tilde{f})(\phi), (\nabla_y \tilde{f})(\phi) \rangle_m B(x, y) d^4 x d^4 y,$$

where $\langle \cdot, \cdot \rangle_m$ is a $\mathfrak{G}$-invariant bilinear form on $\mathbb{C}^m$.

Now we can construct the main object of the quantum gauge field theory on the noncommutative space time:

$$W[\vec{J}] = \int Df D\nabla \times \times \exp i \int_{T(\mathbb{R}^4)} \left( K_1(\nabla; \phi) + K_2(\nabla, \tilde{f}; \phi) + V(\tilde{f}(\phi)) - \langle \tilde{J}(\phi), \tilde{f}(\phi) \rangle \right) dM_{\mathbb{G},\epsilon}(\phi).$$

We believe that it is possible to develop Feynman rules for computation of this integral, since they are quite algebraic and do not depend too much on the finite dimensionality of the classical. We hope to deal with this problem in future publications.

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