On transcendental numbers: new results and a little history

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Abstract: Attempting to create a general framework for studying new results on transcendental numbers, this paper begins with a survey on transcendental numbers and transcendence, it then presents several properties of the transcendental numbers $e$ and $\pi$, and then it gives the proofs of new inequalities and identities for transcendental numbers. Also, in relationship with these topics, we study some implications for the theory of the Yang-Baxter equations, and we propose some open problems.

Keywords: Euler’s relation, transcendental numbers; transcendental operations / functions; transcendence; Yang-Baxter equation

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1. Introduction

One of the most famous formulas in mathematics, the Euler’s relation:

$$e^{\pi i} + 1 = 0,$$  \hspace{1cm} (1)

contains the transcendental numbers $e$ and $\pi$, the imaginary number $i$, the constants 0 and 1, and (transcendental) operations. Beautiful, powerful and surprising, it has changed the mathematics forever.

We will unveil profound aspects related to it, and we will propose a counterpart:

$$|e^i - \pi| < e,$$  \hspace{1cm} (2)
In some sense, the formulas (1) and (2) can be unified, and we will explain this in Section 3.

The next section of the current paper is a survey on transcendental numbers, transcendental functions, transcendental operations and transcendence (following the ideas from [1,2]). Section 3 deals with identities and inequalities of transcendental numbers. In Section 4 we start with some comments, we propose some open problems, and then we prove our new results. Section 5 contains results related to the Yang-Baxter equation. In the last section we draw the final conclusions.

2. Transcendence and transcendental numbers in mathematics

The term “transcendence” has been introduced to express issues related to the Divinity. Implying initially the aforementioned attributes, it has produces an inflation of uses nowadays: movies, novels, jazz albums, corporations, game brands, poems, internet websites, internet blogs, names of all kinds of doctrines, philosophies, etc. The “trans” prefix comes from Latin, implying passing through a certain area, and it is all around: translation, transport, transmission, transformation, transplant, transparent, transgress, transdisciplinarity etc.

One talked for the first time about transcendence in mathematics in the 18th century, and it was Leonhard Euler who initiated this discussion (see [3]). Leibniz was the first referring to transcendence in mathematics (see [4]). Euler does not refer directly to transcendental numbers (as it happened, subsequently, in the 19th century), but he refers to transcendental operations. The non-transcendental operations are: the addition, the subtraction, the multiplication, the division, the exponentiation and the rooting (applied to integers and to those numbers that are obtained from them by such operations).

Why all these operations are considered to be non-transcendent? They are considered to be algebraic operations, because all numbers generated in this way are roots of some algebraic equations; we refer to those equations expressed by polynomial structures, where all mentioned operations are applied for finite number of times (focus on finite) and, as soon as this is not the case, we enter in the field of transcendence.

Therefore, we are suddenly suggested that, in mathematics, the idea of transcendence is essentially related to the idea of infinity. But not to any kind of infinity. For Euler, the entry in mathematical transcendence is in the differential and integral calculus. Why is Euler considering the exponentiation a to x, the logarithm, the trigonometric sinus-cosine functions as being all transcendental functions and operations? Because they are immediately associated, in a way or another, to the mathematical idea of integral, they are immediately associated to expressions including an integral (for instance, the logarithm
is associated to the integral of \( \frac{dx}{x} \)). For Euler, the function \( \log(x) + 7 \) is transcendent, but \( \log(7) + x \) is algebraic.

The passing from non-transcendent to transcendent is gradual. Such an assertion deviates us from mathematical rigor, but it consolidates our understanding. We have in mathematics the situation of irrational numbers, where the word irrational would suggest transcendent, but it is immediately completed by the word algebraic the negation of transcendent, such as the square root of 2. We may consider the algebraic irrationals as a passing bridge, therefore intermediary when passing from non-transcendent to transcendent. It is not rigorous, but it is suggestive. We must add another thing: we talk about rational numbers (such as \( \frac{2}{3} \) or \(-\frac{3}{7}\)) and irrational numbers, and we are tempted to associate such irrational numbers as square root of 2 with something beyond reason. But the Latin etymology clarifies the fact that rational is sending here to ratio as fraction; so, in modern understanding, as a number of the form \( \frac{a}{b} \), where \( a \) and \( b \) are integers, with \( b \) different from zero.

Joseph Liouville is the first who managed to encounter an example of transcendental numbers. This important discovery happened in the year 1844. Liouville introduced a class of real numbers wearing subsequently his name. A real number \( x \) is a Liouville number if there exist an integer \( b \) higher or equal to 2 and an infinite sequence of integers \( (a_1, a_2, \ldots, a_n, \ldots) \) so that \( x \) is the sum of the series having as general term the ratio between \( a_k \) and the power of exponent \( k! \) of \( b \). (For \( b = 10 \) and \( a_k = 1 \) for any \( k \), \( x \) becomes Liouville’s constant.) Concerning his numbers, Liouville proves that they are not algebraic; they are transcendental. It was the first example of non-algebraic real numbers. Another presentation of Liouville’s numbers stands on the manner of approximation by rational numbers: \( x \) is a Liouville number if for any natural number \( n \) there are integers \( p \) and \( q \) with \( q \) higher or equal to 2, such that the absolute value of the difference between \( x \) and \( p/q \) is strictly between zero and the fraction having 1 at the nominator and the value of \( q \) power \( n \), at the denominator. It is thus clear that Liouville’s numbers have the privilege of a tight approximation by rational numbers. This fact is against our intuitive expectations, because it shows that in some respects transcendental numbers are nearer to rational numbers than algebraic irrationals.

The manner how the rational numbers are diligently running to get close to the transcendent numbers urges us to see the transcendence process in exactly the infinite range of approximations of transcendental numbers. This situation suggests us to generally regard transcendence as result of an asymptotic process when all stages are in the terrestrial universe, but the infinity of the number of stages makes impossible to be crossed in real time. Such a view increases the rightfulness of the hypothesis of absence of a sharp border between transcendent and terrestrial. The relation between Liouville numbers and other important classes of real numbers has been recently studied by [5]. It is noticed that the Liouville numbers are encountered in remarkable classes of real numbers, but as rare exceptional phenomenon.

There are other philosophical enigmas related to mathematical transcendence. One would expect that transcendental numbers, such as \( \pi \) or \( e \), to be somehow farther from rational numbers, than irrational algebraic numbers. But in terms of approximation by rational numbers, it seems that this does not happen. It must be noticed, as well, that transcendental numbers are more than algebraic numbers, but more means here a metaphoric extension of such qualification, from finite to infinite. The mathematicians
are expressing this by the following statement: “The algebraic numbers form a countable set, while the transcendental numbers form an uncountable set; it is a set of the power of the continuum”.

3. Transcendental numbers: identities and inequalities

The following identities which contain the transcendental numbers \( e \) and \( \pi \) are well-known:

\[
\int_{-\infty}^{+\infty} e^{-x^2} \, dx = \sqrt{\pi},
\]

\[
\int_{-\infty}^{+\infty} e^{-ix^2} \, dx = \frac{\pi}{\sqrt{2}} (1 - i).
\]

These formulas, unified in the paper \([6]\), can be proved by contour integration.

Other inequalities for \( e \) and \( \pi \) (from \([7]\)) are quite new; for example, we list just two of them:

\[
|e^{1-z} + e^z| > \pi \quad \forall z \in \mathbb{C},
\]

\[
\int_a^b e^{-x^2} \, dx < \frac{e^z}{\pi} \left( \frac{1}{e^{x_a}} - \frac{1}{e^{x_b}} \right).
\]

Still other relations will be proved in this paper: formula (2) and

\[
|\pi^i - i^\pi| = 2 \sin\left(\frac{\pi^2}{4} - \ln \sqrt{\pi}\right).
\]

Let us consider the two variable complex function \( f : \mathbb{C} \times \mathbb{C} \to \mathbb{R}, \, f(z, w) = |e^z + e^w| \). The formulas (1), (2) and (5) can be unified using the function \( f(z, w) \):

\[
f(\pi i, 0) = 0, \quad f(1 - z, \bar{z}) > \pi \quad \forall z \in \mathbb{C}, \quad f(i, \pi i + \ln \pi) < e.
\]

4. Further comments and proofs for our new results

There exist real solutions for the equations

\[
x^2 - \pi x + (1 + \frac{1}{r})^r = 0, \quad r \in \mathbb{Q}^*,
\]

for \( r \) sufficiently small, but there are no real solutions for the “limit” equation

\[
x^2 - \pi x + e = 0,
\]

because \( \Delta = \pi^2 - 4e < 0 \). The question if \( \Delta = \pi^2 - 4e \) is a transcendental number is an open problem!

Resembling the problem of squaring the circle, the geometrical interpretation of the formula \( \pi^2 < 4e \) could be stated as: “The length of the circle with diameter \( \pi \) is almost equal (and less) to the perimeter of a square with edges of length \( e \)” (see the Figure 2). In this case, the area of the above circle is greater than the area of the above square, because \( \pi^2 > 4e^2 \).
**Figure 2.** A circle with diameter $\pi$ and a square with edges of length $e$ have almost the same perimeter.

OPEN PROBLEMS (related to $\pi$). For an arbitrary closed curve, we consider the smallest diameter ($d$) and the maximum diameter ($D$). (These can be found by considering the center of mass of a body which corresponds to the domain inside the given curve.)

(i) If $L$ is the length of the given curve and the domain inside the given curve is a convex set, then we conjecture that:

$$\frac{L}{D} \leq \pi \leq \frac{L}{d} .$$

(ii) Moreover, the first inequality becomes equality if and only if the second inequality becomes equality if and only if the given curve is a circle.

(iii) If the area of the domain inside the given curve is $A$, then $d \cdot D > A$.

(iv) The equation $x^2 - \frac{L}{d} x + A = 0$ is not completely solved; for example, if the given curve is an ellipse, solving this equation is an unsolved problem.

Solving the equation $x^2 + \sum_{k=0}^{\infty} 10^{-k!} x + \sum_{k=1}^{\infty} 10^{-k!} = 0$, in which two coefficients are Liouville numbers, by using the quadratic formula is hard; however, one can use the formula $Ax^2 + (A+C)x + C = (Ax + C)(x + 1)$ in order to solve this equation. The solution $x = -1$ could be observed directly.

**Proof for the identity (6):**

$$|\pi^i - i^\pi| = |e^{i \ln \pi} - e^{\pi \ln i}| = |e^{i \ln \pi} - e^{i \pi \frac{\pi}{2}}| = \sqrt{2 - 2(\cos(\ln \pi) \cos(\frac{\pi^2}{2}) + \sin(\ln \pi) \sin(\frac{\pi^2}{2}))} = \sqrt{2 - 2 \cos(\ln \pi - \frac{\pi^2}{2})} = \sqrt{4 \sin^2(\frac{\ln \pi - \frac{\pi^2}{2}}{2})} = 2 \sin(\frac{\pi^2}{4} - \ln \sqrt{\pi}) \approx 0.95.$$  

In connection with the formula (6), one could consider the equation

$$x^i = i^x \quad x \in \mathbb{R}^*_+, $$

which is equivalent to

$$e^{\frac{\pi}{2}} = x^{\frac{1}{\pi}} \quad x \in \mathbb{R}^*_+, $$

and it has no real solution, because $\frac{\pi}{2} > \frac{1}{e}$.

The proof for the inequality (2) follows in a similar fashion, using the Taylor series for $\cos x$ in order to approximate $\cos 1$, and three digit approximations for $e$ and $\pi$. 
5. Transcendental numbers in mathematical physics

In our special issues on Hopf algebras, quantum groups and Yang-Baxter equations, several papers [8–17], as well the feature paper [18], covered many topics related to the Yang-Baxter equation, ranging from mathematical physics to Hopf algebras, and from Azumaya Monads to quantum computing.

The terminology of this section is compatible with the above cited papers. Let \( V \) be a complex vector space, and let \( I_j : V \otimes j \rightarrow V \otimes j \) \( \forall j \in \{1, 2\} \) be identity maps. We consider \( J : V \otimes 2 \rightarrow V \otimes 2 \) a linear map which satisfies \( J \circ J = -I_2 \) and \( J^{12} \circ J^{23} = J^{23} \circ J^{12} \), where \( J^{12} = J \otimes I_1 \) and \( J^{23} = I_1 \otimes J \).

Then, \( R(x) = \cos x I_2 + \sin x J \) satisfies the colored Yang-Baxter equation:

\[
R^{12}(x) \circ R^{23}(x + y) \circ R^{12}(y) = R^{23}(y) \circ R^{12}(x + y) \circ R^{23}(x).
\]

The proof of (7) could be done by writing \( R(x) = e^{x J} \), and checking that (7) reduces to

\[
x J^{12} + (x + y) J^{23} + y J^{12} = y J^{23} + (x + y) J^{12} + x J^{23}.
\]

Such an operator \( J \) could have, in dimension two, the following matrix form (for \( \alpha \in \mathbb{R} \)):

\[
\begin{pmatrix}
0 & 0 & 0 & \frac{1}{\alpha} i \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
\alpha i & 0 & 0 & 0
\end{pmatrix}
\]

Based on results from the previous section, a counterpart for the formula

\[
e^{\pi J} + I_4 = 0_4 \quad J, I_4, 0_4 \in \mathcal{M}_4(\mathbb{C})
\]

could be the following inequality:

\[
X^2 + eI_2 > \pi X \quad \forall X \in \mathcal{M}_2(\mathbb{R}^+_*) , \quad trace(X) > \pi.
\]

Replacing the above condition \( J^{12} \circ J^{23} = J^{23} \circ J^{12} \) with \( J^{12} \circ J^{23} = -J^{23} \circ J^{12} \), the authors of [19] obtained interesting results (a new realization of doubling degeneracy based on emergent Majorana operator, new solutions for the Yang-Baxter equation, etc). For example, in dimension two, the matrix form of this new operator \( J \) could be:

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}
\]

With this case we enter into the world of the quaternions and Clifford algebras.
6. Conclusions

The author of [20] considers two types of scientists: birds (they resemble scientists with a broad vision, who try to unify theories, who obtain results of interest for a large readership) and frogs (which are less influential). Solomon Marcus used the terms of Francis Bacon (Novum Organum), bees versus ants, while describing mathematicians who are involved in many different areas of research versus the mathematicians who work on a restricted domain.

Because there is a huge number of new disciplines, it is important to have a transdisciplinary understanding of the world: a transdisciplinary approach (see [21–23]) attempts to discover what is between disciplines, across different disciplines, and beyond all disciplines. Transcendence is a concept which plays an important role in theology, in science and in art; it can be considered beyond all disciplines. The Yang-Baxter equation appears across different disciplines. Mathematical Physics is at the border of two disciplines.

Our paper is written in transdisciplinary fashion, and it has some kind of editorial flavor. We used results and concepts from algebra, mathematical analysis, mathematical physics, geometry, history of mathematics, numerical analysis, epistemology, philosophy etc. Attempting to continue the approach of our recent papers and talks on the transcendental numbers (see [1], [2], [7]), we brought together our investigations and we presented new results.

It is our pleasure to conclude with some historical facts relevant for our journal: Euler was born in Basel (on 15 April 1707), and he received his Master of Philosophy from University of Basel.

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