Let $A$ and $B$ be two $N$ by $N$ deterministic Hermitian matrices and let $U$ be an $N$ by $N$ Haar distributed unitary matrix. It is well known that the spectral distribution of the sum $H = A + UBU^*$ converges weakly to the free additive convolution of the spectral distributions of $A$ and $B$, as $N$ tends to infinity. We establish the optimal convergence rate $1/N$ in the bulk of the spectrum.

1. Introduction

In the influential work [21], Voiculescu showed that two independent large Hermitian matrices are asymptotically free if one of them is conjugated by a Haar distributed unitary matrix. This observation identifies the law of the sum of two large Hermitian matrices in a randomly chosen relative basis. More specifically, if $A = A^{(N)}$ and $B = B^{(N)}$ are two sequences of deterministic $N$ by $N$ Hermitian matrices and $U$ is a Haar distributed unitary matrix, then the empirical eigenvalue distribution, $\mu_H$, of the random sum $H = A + UBU^*$ is asymptotically given by the free additive convolution, $\mu_A \boxplus \mu_B$, of the eigenvalue distributions of $A$ and $B$. A quantitative control of the closeness between $\mu_H$ and $\mu_A \boxplus \mu_B$, or the convergence rate of $\mu_H$, has been out of reach until very recently. The first convergence rate $(\log N)^{-1/2}$ was obtained by Kargin in [15] by using the Gromov-Milman concentration inequality for the Haar measure. Later, Kargin improved in [16] his result to $N^{-1/7}$ in the bulk of the spectrum by studying the Green function subordination property down to the scale $N^{-1/7}$. Recently, we used in [1] a bootstrap argument to successively localize the Gromov-Milman inequality from larger to smaller scales, whereby we improved the convergence rate to $N^{-2/3}$.

In the current paper, we establish the convergence rate $N^{-1+\gamma}$, for any given $\gamma > 0$, in the bulk regime. Since the typical eigenvalue spacing in the bulk of the spectrum is $N^{-1}$, our result is optimal, up to the $N^\gamma$ factor. In our recent work [2] on the local law of $H$ we showed that the Green function subordination property holds down to the optimal scale $N^{-1+\gamma}$; cf. Proposition 3.2 below. In particular, the fluctuations of the matrix elements of the Green function $G(z) = (H - z)^{-1}$ were shown to be of order $N^{-1/2+\gamma}$ for any fixed $z$ in the upper half plane, $\text{Im} \, z > 0$. To get the optimal convergence rate, we need to show that the fluctuations of the normalized trace of the Green function, $\frac{1}{N} \text{Tr} G$, are at most of order $N^{-1+\gamma}$. Thus the main task is to establish the fluctuation averaging of the diagonal entries of the Green function, i.e. that the fluctuations of the (weighted) average of the $G_{ii}$’s are typically as small as the square of the fluctuation of the $G_{ii}$’s; cf. (2.23).
Alongside with the convergence rate of $\mu_H$ to $\mu_A \boxtimes \mu_B$, the concentration rate of $\mu_H$ to its expectation $\mathbb{E}\mu_H$ is of interest. An order $N^{-1/2}$ estimate up to logarithmic corrections on the fluctuations of the distribution function was obtained by Chatterjee in [9] by studying mixing times of random walks on the unitary group. Using the Gromov-Milman concentration inequality, Kargin removed the logarithmic corrections [15]. More recently, a rate of order $N^{-2/3}$ in the $L^1$-Wasserstein distance was obtained by E. Meckes and M. Meckes in [18]. From our main result it follows that $\mu_H$, when restricted to the bulk, has concentration rate $N^{-1}$.

The fluctuation averaging mechanism is a key ingredient in proving the optimal convergence rate of local laws for random matrices. It was first introduced in [14] and substantially extended later in [12, 13] to generalized Wigner matrices. In all previous works, however, the proofs heavily relied on the independence (up to symmetry) of the matrix elements. Our matrix $H = A + UBU^*$ lacks this independence since the columns of a Haar unitary matrix are dependent. This fact was already a major obstacle in the proof of the optimal local law [2], where independence of columns was replaced with a specific partial randomness decomposition of the Haar unitaries; see Section 3.2. This decomposition, however, is not directly compatible with taking matrix elements of the Green function, thus the fluctuation averaging mechanism in the average $\frac{1}{N} \sum_{i=1}^N G_{ii}$ remains hidden. In fact, our proof does not attack this average directly, we first prove fluctuation averaging for an auxiliary quantity $Z_i$, a carefully chosen linear combination of $G_{ii}$ and $(UBU^*)_{ii}$; see (5.1). In the quantity $Z_i$ certain fluctuations of order $N^{-1/2}$ cancel for an algebraic reason. When passing from $Z_i$ to the original $G_{ii}$, we need to introduce an additional specially chosen quantity $\Upsilon$, see (4.1), that averages the effect of the fluctuations of order $N^{-1/2}$. Only a posteriori we show that $\Upsilon$ is in fact one order better than its naive size indicates. Identifying these somewhat counter-intuitive quantities for the fluctuation averaging is one of the main novelties of the current work.

Another key feature of the proof is that we do not directly compute high moments of the averages as it was customary in the previous proofs that led to involved expansions whose bookkeeping was quite tedious. Instead, we estimate the higher moments recursively, in terms of the lower moments, see Lemma 6.2, whose proof relies on integration by parts for Gaussian variables. This method to prove fluctuation averaging was recently introduced in [17] in the context of sparse Wigner matrices. In the current setup, circumventing the high moment calculation is a very important asset, due to the complexity of the partial randomness decomposition of the Haar measure and the numerous error terms involved in the necessary Gaussian approximation.

Notation: We use $C$ to denote strictly positive constants that do not depend on $N$. Their values may change from line to line. For $a, b \geq 0$, we write $a \lesssim b$, $a \gtrsim b$ if there is $C \geq 1$ such that $a \leq Cb$, $a \geq C^{-1}b$ respectively. We denote for $z \in \mathbb{C}^+$ the real part by $E = \Re z$ and the imaginary part by $\eta = \Im z$.

We use bold font for vectors in $\mathbb{C}^N$, denote their components by $v = (v_1, \ldots, v_N) \in \mathbb{C}^N$ and their Euclidean norm by $|v|_2$. The canonical basis of $\mathbb{C}^N$ is denoted by $(e_i)_{i=1}^N$. We denote by $M_N(\mathbb{C})$ the set of $N \times N$ matrices over $\mathbb{C}$. For $A \in M_N(\mathbb{C})$, we denote by $||A||$ its operator norm and by $||A||_2$ its Hilbert-Schmidt norm. The matrix entries of $A$ are denoted by $A_{ij} = e_i^* A e_j$. We use tr $A$ to denote the normalized trace of $A$, i.e. $\operatorname{tr} A = \frac{1}{N} \sum_{i=1}^N A_{ii}$.

Let $g = (g_1, \ldots, g_N)$ be a real or complex Gaussian vector. We write $g \sim \mathcal{N}_\sigma(0, \sigma^2 I_N)$ if $g_1, \ldots, g_N$ are i.i.d. $\mathcal{N}(0, \sigma^2)$ normal variables; and we write $g \sim \mathcal{N}_\sigma(0, \sigma^2 I_N)$ if $g_1, \ldots, g_N$ are i.i.d. $\mathcal{N}(0, \sigma^2)$ variables, where $g_i \sim \mathcal{N}_\sigma(0, \sigma^2)$ means that $\Re g_i$ and $\Im g_i$ are independent $\mathcal{N}(0, \frac{\sigma^2}{2})$ normal variables. Finally, we use double brackets to denote index sets, i.e. for $n_1, n_2 \in \mathbb{R}$, $[n_1, n_2] := [n_1, n_2] \cap \mathbb{Z}$.

2. Main results

2.1. Free additive convolution. For the reader’s convenience we recall from [1] some basic notions and results for the free additive convolution.

Given a probability measure $\mu$ on $\mathbb{R}$, its Stieltjes transform, $m_\mu$, on the complex upper half-plane $\mathbb{C}^+ := \{z \in \mathbb{C} : \Im z > 0\}$ is defined by

$$m_\mu(z) := \int_\mathbb{R} \frac{d\mu(x)}{x - z}, \quad z \in \mathbb{C}^+. $$

Note that $m_\mu : \mathbb{C}^+ \to \mathbb{C}^+$ is an analytic function such that

$$\lim_{\eta \to \infty} \im m_\mu(i\eta) = -1. $$

(2.1)
Conversely, if \( m : \mathbb{C}^+ \to \mathbb{C}^+ \) is an analytic function such that \( \lim_{\eta \to \infty} \text{Im}(m(i\eta)) = -1 \), then \( m \) is the Stieltjes transform of a probability measure \( \mu \). Let \( F_\mu \) be the negative reciprocal Stieltjes transform of \( \mu \),

\[
F_\mu(z) := -\frac{1}{m_\mu(z)}, \quad z \in \mathbb{C}^+.
\]

Observe that

\[
\lim_{\eta \to \infty} \frac{F_\mu(i\eta)}{i\eta} = 1,
\]

as follows from (2.1). Note, moreover, that \( F_\mu \) is analytic on \( \mathbb{C}^+ \) with nonnegative imaginary part.

The free additive convolution is the symmetric binary operation on probability measures on \( \mathbb{R} \) characterized by the following result.

**Proposition 2.1** (Theorem 4.1 in [3], Theorem 2.1 in [10]). Given two probability measures, \( \mu_1 \) and \( \mu_2 \), on \( \mathbb{R} \), there exist unique analytic functions, \( \omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+ \), such that,

(i) for all \( z \in \mathbb{C}^+ \), \( \text{Im}\omega_1(z)\), \( \text{Im}\omega_2(z) \geq \text{Im} z \), and

\[
\lim_{\eta \to \infty} \frac{\omega_1(i\eta)}{i\eta} = \lim_{\eta \to \infty} \frac{\omega_2(i\eta)}{i\eta} = 1;
\]

(ii) for all \( z \in \mathbb{C}^+ \),

\[
F_{\mu_1}(\omega_2(z)) = F_{\mu_2}(\omega_1(z)), \quad \omega_1(z) + \omega_2(z) - z = F_{\mu_1}(\omega_2(z)).
\]

It follows from (2.4) that the analytic function \( F : \mathbb{C}^+ \to \mathbb{C}^+ \) defined by

\[
F(z) := F_{\mu_1}(\omega_2(z)) = F_{\mu_2}(\omega_1(z)),
\]

satisfies the analogue of (2.3). Thus \( F \) is the negative reciprocal Stieltjes transform of a probability measure \( \mu \), called the free additive convolution of \( \mu_1 \) and \( \mu_2 \), usually denoted by \( \mu \equiv \mu_1 \boxplus \mu_2 \). The functions \( \omega_1 \) and \( \omega_2 \) of Proposition 2.1 are called subordination functions and \( F \) is said to be subordinated to \( F_{\mu_1} \), respectively to \( F_{\mu_2} \). To exclude trivial shifts of measures, we henceforth assume that both, \( \mu_1 \) and \( \mu_2 \), are supported at more than one point. Then the analytic functions \( F, \omega_1 \) and \( \omega_2 \) extend continuously to the real line [4, 5]. The subordination phenomenon was first observed by Voiculescu [22] in a generic situation and extended to full generality by Biane [8].

We next recall the notion of regular bulk of \( \mu_1 \boxplus \mu_2 \) introduced in [2]. Let

\[
U_{\mu_1 \boxplus \mu_2} := \text{int}\left\{\text{supp}(\mu_1 \boxplus \mu_2)^{ac} \setminus \{x \in \mathbb{R} : \lim_{\eta \to 0} F_{\mu_1 \boxplus \mu_2}(x + i\eta) = 0\}\right\},
\]

where \( \text{supp}(\mu_1 \boxplus \mu_2)^{ac} \) denotes the support of the absolutely continuous part of \( \mu_1 \boxplus \mu_2 \). We denote the density function of \( \mu_1 \boxplus \mu_2 \) by \( f_{\mu_1 \boxplus \mu_2} \). Then the regular bulk of \( \mu_1 \boxplus \mu_2 \) is defined as

\[
B_{\mu_1 \boxplus \mu_2} := U_{\mu_1 \boxplus \mu_2} \setminus \{x \in U_{\mu_1 \boxplus \mu_2} : f_{\mu_1 \boxplus \mu_2}(x) = 0\}.
\]

In short, regular bulk is the regime where the density is nonzero but finite. Finally, by general results of [6], the regular bulk contains at least one open interval; see Section 2.1 in [1] for detail.

**2.2. Random matrix model.** Let \( A \equiv A^{(N)} \) and \( B \equiv B^{(N)} \) be two sequences of deterministic real diagonal matrices in \( M_N(\mathbb{C}) \), whose empirical eigenvalue distributions are denoted by \( \mu_A \) and \( \mu_B \), respectively. More precisely,

\[
\mu_A := \frac{1}{N} \sum_{i=1}^{N} \delta_{a_i}, \quad \mu_B := \frac{1}{N} \sum_{i=1}^{N} \delta_{b_i},
\]

with \( A = \text{diag}(a_i), B = \text{diag}(b_i) \). The matrices \( A \) and \( B \) actually depend on \( N \), but we omit this from our notation. Throughout the paper, we assume that

\[
||A||, ||B|| \leq C,
\]

for some positive constant \( C \) uniform in \( N \). Proposition 2.1 asserts the existence of unique analytic functions \( \omega_A \) and \( \omega_B \) satisfying the analogue of (2.4) such that, for all \( z \in \mathbb{C}^+ \),

\[
F_{\mu_A}(\omega_B(z)) = F_{\mu_B}(\omega_A(z)), \quad \omega_A(z) + \omega_B(z) - z = F_{\mu_A}(\omega_B(z)).
\]
We will assume that there are deterministic probability measures $\mu_\alpha$ and $\mu_\beta$ on $\mathbb{R}$, neither of them being a single point mass, such that the empirical spectral distributions $\mu_A$ and $\mu_B$ converge weakly, as $N \to \infty$, to $\mu_\alpha$ and $\mu_\beta$, respectively. More precisely, we assume that
\[
d_\text{L}(\mu_A, \mu_\alpha) + d_\text{L}(\mu_B, \mu_\beta) \to 0,
\]
as $N \to \infty$, where $d_\text{L}$ denotes the Lévy distance. Proposition 2.1 asserts that there are unique analytic functions $\omega_\alpha$, $\omega_\beta$ satisfying the analogue of (2.4) such that, for all $z \in \mathbb{C}^+$,
\[
F_{\mu_\alpha}(\omega_\beta(z)) = F_{\mu_\beta}(\omega_\alpha(z)), \quad \omega_\alpha(z) + \omega_\beta(z) - z = F_{\mu_\alpha}(\omega_\beta(z)).
\]
Proposition 4.13 of [7] states that $d_\text{L}(\mu_A \boxplus \mu_B, \mu_\alpha \boxplus \mu_\beta) \leq d_\text{L}(\mu_A, \mu_\alpha) + d_\text{L}(\mu_B, \mu_\beta)$, i.e. the free additive convolution is continuous with respect to weak convergence of measures.

Denote by $U(N)$ the unitary group of degree $N$. Let $U \in U(N)$ be distributed according to Haar measure (in short a Haar unitary), and consider the random matrix
\[
H \equiv H^{(N)} := A + U B U^*.
\]
Our results also hold for the real setup when $U$ is Haar distributed on the orthogonal group, $O(N)$, of degree $N$. For definiteness, we work with the complex setup in this paper.

2.3. Statement of main results. To state our main results, we rely on the following definition for high-probability estimates, which was first used in [12].

**Definition 2.2.** Let $X \equiv X^{(N)}$, $Y \equiv Y^{(N)}$ be $N$-dependent nonnegative random variables. We say that $Y$ stochastically dominates $X$ if, for all (small) $\epsilon > 0$ and (large) $D > 0$,
\[
P(X^{(N)} > N^\epsilon Y^{(N)}) \leq N^{-D},
\]
for sufficiently large $N \geq N_0(\epsilon, D)$, and we write $X \prec Y$. When $X^{(N)}$ and $Y^{(N)}$ depend on a parameter $v \in V$ (typically an index label or a spectral parameter), then $X(v) \prec Y(v)$, uniformly in $v \in V$, means that the threshold $N_0(\epsilon, D)$ can be chosen independently of $v$.

In Appendix A we collected some properties of the relation $\prec$.

Let $H$ be given in (2.14) and denote by $(\lambda_i)_{i=1}^N$ its eigenvalues. Let $\mu_H$ stand for the empirical eigenvalue distribution of $H$, i.e.
\[
\mu_H := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i}.
\]
Our result on the convergence rate of $\mu_H$ to $\mu_A \boxplus \mu_B$ in the bulk is as follows.

**Theorem 2.3** (Convergence rate). Let $\mu_\alpha$ and $\mu_\beta$ be two compactly supported probability measures on $\mathbb{R}$, and assume that neither is supported at a single point and that at least one of them is supported at more than two points. Assume that the sequence of matrices $A$ and $B$ in (2.14) satisfy (2.10). Fix any nonempty compact interval $I \subset B_{\mu_A, \mu_\beta}$. Then there is a (small) constant $b > 0$, depending only on the measures $\mu_\alpha$ and $\mu_\beta$, on the interval $I$ and on the constant $C$ in (2.10), such that whenever
\[
d_\text{L}(\mu_A, \mu_\alpha) + d_\text{L}(\mu_B, \mu_\beta) \leq b,
\]
then
\[
\sup_{I' \subset I} \left| \mu_H(I') - \mu_A \boxplus \mu_B(I') \right| < \frac{1}{N},
\]
where the supremum ranges over all subintervals of $I$.

The proof of Theorem 2.3 is based on an optimal local law for the Stieltjes transform of the empirical eigenvalue distribution of $H$ which is the main technical result of this paper. Denote the Green function (or the resolvent) of $H$ and its normalized trace by
\[
G(z) \equiv G_H(z) := \frac{1}{H - z}, \quad m(z) = m_H(z) := \text{tr} G(z) = \frac{1}{N} \sum_{i=1}^N G_{ii}(z), \quad z \in \mathbb{C}^+.
\]
where $G_{ij}(z)$ are the matrix entries of $G(z)$. Note that $m_H$ is the Stieltjes transform of $\mu_H$,

$$m_H(z) = \text{tr} G_H(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i - z} = \int_{\mathbb{R}} \frac{d\mu_H(x)}{x - z}, \quad z \in \mathbb{C}^+.$$  \hspace{1cm} (2.20)

To state our next result, we introduce the following domain of the spectral parameter $z$: For $b \geq a \geq 0$, and $\mathcal{I} \subset \mathbb{R}$, let

$$\mathcal{S}_\mathcal{I}(a, b) := \{ z = E + i\eta \in \mathbb{C}^+ : E \in \mathcal{I}, \ a < \eta \leq b \}.$$  \hspace{1cm} (2.21)

Throughout the paper, we use the control parameter

$$\Psi \equiv \Psi(z) := \frac{1}{\sqrt{N\eta}}, \quad z = E + i\eta \in \mathbb{C}^+.$$  

We next state the local law for the Stieltjes transform of $\mu_H$.

**Theorem 2.4** (Local law for the Stieltjes transform). Let $\mu_\alpha$, $\mu_\beta$, $A$ and $B$ satisfy the assumption of Theorem 2.3. Fix any nonempty compact interval $\mathcal{I} \subset \mathcal{B}_{\mu_\alpha \boxplus \mu_\beta}$. Let $d_1, \ldots, d_N \in \mathbb{C}$ be any deterministic complex numbers satisfying

$$\max_{i \in [1, N]} |d_i| \leq 1.$$  \hspace{1cm} (2.22)

Then there is a (small) constant $b > 0$, depending only on the measures $\mu_\alpha$ and $\mu_\beta$, on the interval $\mathcal{I}$ and on the constant $C$ in (2.10), such that whenever (2.17) holds, then

$$\left| \frac{1}{N} \sum_{i=1}^{N} d_i \left( G_{i,i}(z) - \frac{1}{a_i - \omega_B(z)} \right) \right| \prec \Psi^2,$$  \hspace{1cm} (2.23)

holds uniformly on $\mathcal{S}_\mathcal{I}(0,1)$. In particular, choosing $d_i = 1$ for all $i \in [1, N]$,

$$\left| m_H(z) - m_{\mu_\alpha \boxplus \mu_\beta}(z) \right| \prec \Psi^2,$$  \hspace{1cm} (2.24)

holds uniformly on $\mathcal{S}_\mathcal{I}(0,1)$.

**Remark 2.5.** The constant $b > 0$ in Theorem 2.3 and Theorem 2.4 is the same.

**Remark 2.6.** In (2.21) of [2] we obtained the bound $|m_H(z) - m_{\mu_\alpha \boxplus \mu_\beta}(z)| \prec \Psi$, uniformly on $\mathcal{S}_\mathcal{I}(0,1)$, under the same assumptions as above. The improvement to $\Psi^2$ in (2.24) is essentially due to the averaging of the fluctuation of $G_{i,i}$’s in the normalized trace of the Green function.

**Remark 2.7.** Note that the control parameter $\Psi(z)$ is small when the spectral parameter $z$ satisfies $\eta \gg N^{-1}$. Thus (2.23) and (2.24) are effective when $\eta$ is slightly above $N^{-1}$, while for even smaller $\eta$ the terms are simply estimated using monotonicity of the Green function. We further remark that the $N^\epsilon$ corrections in probability estimates $\prec$ can be improved to logarithmic corrections by pushing our estimates, yet we do not pursue this direction here.

**Remark 2.8.** In Theorem C.1 of Appendix C, we collect the counterparts of the results in Theorem 2.4 and Theorem 2.3 for the case that both $\mu_\alpha$ and $\mu_\beta$ are convex combinations of two points masses. In fact, the result is exactly the same as in the general case, unless $\mu_\alpha = \mu_\beta$ when a possible singularity at one particular energy $E$ needs to be incorporated in the estimates.

Theorem 2.3 follows from Theorem 2.4. It relies on the formula (2.20) and a standard application of the Helffer-Sjöstrand functional calculus. We omit the proof here and refer to, e.g., Section 7.1 of [13] for a very similar argument.

3. Preliminaries

In this section, we collect some basic tools and necessary results from [1] and [2].
3.1. Local stability of the system (2.11). We first consider (2.5) in a general setting: For generic probability measures \( \mu_1, \mu_2 \), let \( \Phi_{\mu_1, \mu_2} : (\mathbb{C}^+)^3 \to \mathbb{C}^2 \) be given by
\[
\Phi_{\mu_1, \mu_2}(\omega_1, \omega_2, z) := \begin{pmatrix}
F_{\mu_1}(\omega_2) - \omega_1 - \omega_2 + z \\
F_{\mu_2}(\omega_1) - \omega_1 - \omega_2 + z
\end{pmatrix},
\]
where \( F_{\mu_1}, F_{\mu_2} \) are the negative reciprocal Stieltjes transforms of \( \mu_1, \mu_2 \); see (2.2). Considering \( \mu_1, \mu_2 \) as fixed, the equation
\[
\Phi_{\mu_1, \mu_2}(\omega_1, \omega_2, z) = 0,
\]
is equivalent to (2.5) and, by Proposition 2.1, there are unique analytic functions \( \omega_1, \omega_2 : \mathbb{C}^+ \to \mathbb{C}^+, \ z \mapsto \omega_1(z), \omega_2(z) \) satisfying (2.4) that solve (3.2) in terms of \( z \). Choosing \( \mu_1 = \mu_2 = \mu_3 \) Equation (3.2) is equivalent to (2.13); choosing \( \mu_1 = \mu_A, \mu_2 = \mu_B \) it is equivalent to (2.11).

We call the system (3.2) linearly \( S \)-stable at \((\omega_1, \omega_2)\) if
\[
\left\| \begin{pmatrix}
-1 & F'_{\mu_2}(\omega_1) - 1 \\
-1 & -1
\end{pmatrix} \right\| \leq S,
\]
for some positive constant \( S \).

We recall a result from [1] showing that the system \( \Phi_{\mu_A, \mu_B}(\omega_A, \omega_B, z) = 0 \) is \( S \)-stable for all \( z \in \mathcal{S}_2(0,1) \). In Section 4 we will use Proposition 4.1 of [1], where we showed that \( S \)-stability implies linear stability of the system in the sense that if
\[
\Phi_{\mu_A, \mu_B}(\omega_1(z), \omega_2(z), z) = \tilde{r}(z)
\]
holds and \( \omega_1, \omega_2 \) are sufficiently close to \( \omega_A, \omega_B \) at some \( z_0 \in \mathcal{S}_2(0,1) \), then
\[
|\omega_1(z_0) - \omega_A(z_0)| \leq 2S\|\tilde{r}(z_0)\|_2, \quad |\omega_2(z_0) - \omega_B(z_0)| \leq 2S\|\tilde{r}(z_0)\|_2.
\]

**Lemma 3.1** (Lemma 5.1 and Corollary 5.2 of [1]). Let \( \mu_A, \mu_B \) be the probability measures from (2.9) satisfying the assumptions of Theorem 2.3. Let \( \omega_A, \omega_B \) denote the associated subfunctions of (2.11). Let \( \mathcal{I} \) be the interval in Theorem 2.3 and assume that (2.17) holds. Then for \( N \) sufficiently large, the system
\[
\Phi_{\mu_A, \mu_B}(\omega_A, \omega_B, z) = 0
\]
is \( S \)-stable with some positive constant \( S \), uniformly on \( \mathcal{S}_2(0,1) \). Further, we have
\[
\max_{z \in \mathcal{S}_2(0,1)} |\omega'_A(z)| \leq 2S, \quad \max_{z \in \mathcal{S}_2(0,1)} |\omega'_B(z)| \leq 2S,
\]
for \( N \) sufficiently large. Moreover, there exist two strictly positive constants \( k \) and \( K \) such that, for \( N \) sufficiently large,
\[
\max_{z \in \mathcal{S}_2(0,1)} |\omega_A(z)| \leq K, \quad \max_{z \in \mathcal{S}_2(0,1)} |\omega_B(z)| \leq K,
\]
\[
\min_{z \in \mathcal{S}_2(0,1)} \text{Im}\omega_A(z) \geq k, \quad \min_{z \in \mathcal{S}_2(0,1)} \text{Im}\omega_B(z) \geq k.
\]

3.2. Partial randomness decomposition. In the sequel, we recall some notations on the partial randomness decomposition and some related results from [2]. We use a decomposition of Haar measure on the unitary groups obtained in [11] (see also [19]): For any \( i \in \llbracket 1, N \rrbracket \), there exists an independent pair \((v_i, U^i)\), with \( v_i \in \mathcal{S}_N^{N-1} := \{ x \in \mathbb{C} : x^*x = 1 \} \) a uniformly distributed complex unit vector and with \( U^i \in \mathcal{U}(N-1) \) a Haar unitary matrix, such that
\[
U = e^{i\theta}R_{i}U^{(i)}, \quad r_i := \sqrt{2} \frac{e_i + e^{-i\theta}v_i}{\|e_i + e^{-i\theta}v_i\|_2}, \quad R_i := I - r_ir_i^*,
\]
where \( U^{(i)} \) is a unitary matrix with \( e_i \) as its \( i \)-th column and \( U^i \) as its \((i, i)\)-matrix minor, and where \( \theta_i \) is the argument of the \( i \)-th component of \( v_i \). Since \( U^{(i)}e_i = e_i \), one can easily check
\[
Ue_i = -e^{i\theta}R_i e_i = v_i
\]
using the definition of \( R_i \) in (3.7). Hence, \( v_i \) is actually the \( i \)-th column of \( U \), and \( R_i = R_i^* \) is the Householder reflection sending \( e_i \) to \( -e^{-i\theta}v_i \).

With the decomposition of \( U \) in (3.7), we can write
\[
H = A + \tilde{B} = A + R_i\tilde{B}^{(i)}R_i,
\]
for any $i \in [1,N]$, where we introduced the shorthand notations
\[ \tilde{B} := UBU^*, \quad \tilde{B}^{(i)} := U^{(i)}B(U^{(i)})^*. \] (3.9)

Clearly, we have $\tilde{B}^{(i)}e_i = b_ie_i$ and $e_i^*\tilde{B}^{(i)} = b_ie_i^*$. We further define
\[ H^{(i)} := A + \tilde{B}^{(i)}, \quad G^{(i)}(z) := (H^{(i)} - z)^{-1}, \quad z \in \mathbb{C}^+. \] (3.10)

Note that $B^{(i)}$, $H^{(i)}$ and $G^{(i)}$ are independent of $v_i$.

It is known that for the uniformly distributed complex unit vector $v_i \in S^{N-1}_\mathbb{C}$, there exists a Gaussian vector $g_i \sim \mathcal{N}_\mathbb{C}(0, N^{-1}I_N)$ such that
\[ v_i = \frac{g_i}{\|g_i\|_2}. \]

We further define
\[ g_i := e^{-i\theta_i} \tilde{g}_i, \quad h_i := \frac{g_i}{\|g_i\|_2} = e^{-i\theta_i} v_i, \quad \ell_i := \frac{\sqrt{2}}{\|e_i + h_i\|_2}. \] (3.11)

Note that the components of $g_i$ are independent. In addition, for $k \neq i$, $g_{ik}$ is a $\mathcal{N}_\mathbb{C}(0, \frac{1}{N})$ random variables while $g_{ii}$ is a $\chi$-distributed random variable with $\mathbb{E}[g_{ii}^2] = \frac{1}{N}$. With the above notations, we can write the vector $r_i$ defined in (3.7) as
\[ r_i = \ell_i(e_i + h_i). \] (3.12)

Two simple estimates are
\[ \left\|g_i\right\|_2 - 1 - \frac{1}{2}\left(\|g_i\|_2^2 - 1\right) \leq \frac{1}{N}, \quad \left|\ell_i^2 - (1 - g_{ii})\right| \sim \frac{1}{N}, \] (3.13)

where in the first estimate we used $\|g_i\|_2^2 = N^{-1/2}$ and in the second we used $\ell_i^2 = (1 + e_i^*h_i)^{-1}$; cf. (3.11). Moreover, according to (3.8), the fact $R_i^2 = I$, and the definition of $h_i$ in (3.11), we also have
\[ R_i e_i = -h_i, \quad R_i h_i = -e_i, \] (3.14)

which further imply the identities
\[ h_i^*\tilde{B}^{(i)} R_i = -e_i^*\tilde{B}, \quad e_i^*\tilde{B}^{(i)} R_i = -b_i h_i^* = -h_i^*\tilde{B}, \] (3.15)

where in the first step of the second equation above we used the fact $e_i^*\tilde{B}^{(i)} = b_i e_i^*$.

Since $g_{ii}$ is $\chi$-distributed, rather than Gaussian as the $g_{ik}$’s, it is convenient to kick it out of many arguments in the sequel where Gaussian integration by parts is repeatedly used. To this end, we denote by $\tilde{g}_i$ the vector obtained from $g_i$ via replacing $g_{ii}$ by zero, i.e.
\[ \tilde{g}_i := g_i - g_{ii} e_i. \]

Correspondingly, we set
\[ \tilde{h}_i := \frac{\tilde{g}_i}{\|\tilde{g}_i\|_2}. \] (3.16)

Throughout the paper, without loss of generality, we assume that
\[ \text{tr } A = \text{tr } B = 0. \] (3.17)

3.3. Approximate subordination and weak local law. We next briefly discuss the approximate subordination property of the Green function. In addition to $H = A + UBU^*$, we also use
\[ \mathcal{H} := \mathcal{H}^{(N)} := U^*AU + B \]
and denote the Green function of $\mathcal{H}$ by
\[ G(z) \equiv G_{\mathcal{H}}(z) := (\mathcal{H} - z)^{-1}, \quad z \in \mathbb{C}^+. \] (3.18)

Note that the normalized traces of the Green functions $G$ and $\mathcal{G}$ are equal,
\[ m_H(z) := \text{tr } G(z) = \text{tr } \mathcal{G}(z), \] (3.19)

and agree with the Stieltjes transform of the empirical spectral measure $\mu_H$. Recall $\tilde{B}$ introduced in (3.9). For brevity, we set
\[ \tilde{A} := U^*AU. \] (3.20)
Following [2], we define the approximate subordination functions by
\[ \omega_A^\gamma(z) := z - \frac{\text{tr} \bar{A}G(z)}{m_H(z)}, \quad \omega_B^\gamma(z) := z - \frac{\text{tr} \bar{B}G(z)}{m_H(z)}, \quad z \in \mathbb{C}^+. \]  
(3.21)
These are slight modifications of the approximate subordination functions used by Pastur and Vasilchuck in [20] and by Kargin in [16]. By cyclicity of the trace, we also have
\[ \omega_A^\gamma(z) = z - \frac{\text{tr} \bar{A}G(z)}{m_H(z)}, \quad z \in \mathbb{C}^+. \]  
(3.22)
A simple observation from (3.21), (3.22) and the definition of the Green function is that
\[ -\frac{1}{m_H(z)} = z - \omega_A^\gamma(z) - \omega_B^\gamma(z). \]  
(3.23)
This suggests that \( \omega_A^\gamma \) and \( \omega_B^\gamma \) approximately solve (2.11). This is indeed the case as is confirmed by the next result obtained in [2]. We need some more notation. For any (small) \( \gamma > 0 \), set
\[ \eta_m \equiv \eta_m(\gamma) := N^{-1+\gamma}. \]  
(3.24)
**Proposition 3.2.** (Theorem 2.6 and (7.12) in [2]) Suppose that the assumptions in Theorem 2.3 and (2.17) hold. Fix any (small) \( \gamma > 0 \) and recall \( \eta_m \equiv \eta_m(\gamma) \) from (3.24). Then we have
\[ |\omega_A^\gamma(z) - \omega_A(z)| < \Psi, \quad |\omega_B^\gamma(z) - \omega_B(z)| < \Psi \]  
(3.25)
and
\[ \max_{i,j \in [1,N]} \left| G_{ij}(z) - \delta_{ij} \frac{1}{a_i - \omega_B(z)} \right| < \Psi, \]  
(3.26)
uniformly on \( S_T(\eta_m, 1) \).

From (3.26), we directly get the following non-optimal estimate by taking the normalized trace,
\[ |\text{tr} G(z) - m_{\mu_A \oplus \mu_B}(z)| < \Psi, \]  
(3.27)
uniformly on \( S_T(\eta_m, 1) \). While the estimate in (3.26) is essentially optimal, the estimate in (3.25) is improved by the fluctuation averaging as is asserted by the next result.

**Theorem 3.3.** Suppose that the assumptions in Theorem 2.3 and (2.17) hold. Fix (small) \( \gamma > 0 \). Then
\[ |\omega_A^\gamma(z) - \omega_A(z)| < \Psi^2, \quad |\omega_B^\gamma(z) - \omega_B(z)| < \Psi^2 \]  
(3.28)
hold uniformly on \( S_T(\eta_m, 1) \) with \( \eta_m \equiv \eta_m(\gamma) \); see (3.24).

Next, recalling the notations introduced in Section 3.2, we introduce the following key quantities
\[ S_i \equiv S_i(z) := h_i^*\tilde{B}^{(i)}G e_i, \quad T_i \equiv T_i(z) := h_i^*G e_i. \]  
(3.29)
Note that here \( S_i, T_i \) are slightly different from the counterparts in (5.1) of [2], where we used a Gaussian vector to approximate \( h_i \) and 1 to approximate \( \ell_i \). Such a modification of the definition does not alter the estimate on \( S_i \) and \( T_i \) obtained in [2]; see (3.31) below. More specifically, we have the following lemma.

**Lemma 3.4.** Suppose that the assumptions in Theorem 2.3 and (2.17) hold. Letting \( Q_i, Q'_i \) stand for the matrix 1 or \( \tilde{B}^{(i)} \), and letting \( \beta_i, \beta'_i \) stand for \( h_i \) or \( e_i \). Fix any (small) \( \gamma > 0 \) and recall \( \eta_m \equiv \eta_m(\gamma) \) from (3.24). Then, we have the bound
\[ \max_{i \in [1,N]} |\alpha_i^*Q_iG(z)Q'_i\beta_i| < 1 \]  
(3.30)
uniformly on \( S_T(\eta_m, 1) \). For \( S_i \) and \( T_i \), we have the more precise estimates
\[ \max_{i \in [1,N]} \left| S_i(z) + \frac{z - \omega_B(z)}{a_i - \omega_B(z)} \right| < \Psi, \quad \max_{i \in [1,N]} |T_i| < \Psi \]  
(3.31)
uniformly on \( S_T(\eta_m, 1) \).
Proof. Using the last inequality in (3.25) and the lower bound in (3.6), we see that (3.31) is equivalent to
\[ \max_{i \in [1, N]} \left| S_i(z) + \frac{z - \omega_B^i(z)}{a_i - \omega_B^i(z)} \right| < \Psi, \quad \max_{i \in [1, N]} | T_i | < \Psi. \]
(3.32)
The counterparts of (3.30) and (3.32) in [2], with \( h_i \) replaced by a Gaussian approximation and \( \ell_i \), replaced by 1 in the quantity \( \alpha_i^* Q(z) Q_i^* \beta_i \), are (5.43) and (6.3) of [2], respectively. Hence, it suffices to show that the replacement of \( h_i \) by its Gaussian approximation in [2] and \( \ell_i \) by 1 in the quantity \( \alpha_i^* Q(z) Q_i^* \beta_i \) only causes an error of order \( \Psi \). This estimate was obtained in Lemma 4.1 of [2] for the case \( \alpha = \beta = \epsilon \), and \( Q_i = Q = I \), i.e. \( \alpha_i^* Q(z) Q_i^* \beta_i = G_n \). For the other choices of \( \alpha_i, \beta_i, Q_i \), and \( Q' \), the proof is nearly the same. We leave the details to the reader. \( \square \)

4. Proof of Theorem 2.4

In this section, we prove Theorem 2.4 with the aid of the following Proposition 4.1, which will be proved in Section 5. We introduce the tracial quantity \( \Upsilon \) by setting
\[ \Upsilon \equiv \Upsilon(z) := \text{tr} (\tilde{B} G) - (\text{tr} (\tilde{B} G))^2 + \text{tr} G \text{ tr} (\tilde{B} \tilde{G} \tilde{B}). \]
(4.1)
Fix a (small) \( \gamma > 0 \). Using the identities
\[ \tilde{B} G = I - (A - z) G, \quad \tilde{B} \tilde{G} \tilde{B} = \tilde{B} - A + z + (A - z) G (A - z), \]
and the estimate in (3.26), it is straightforward to check the \textit{a priori} bound
\[ |\Upsilon| \prec \Psi, \]
(4.3)
uniformly on \( \mathcal{S}_T(\eta_m, 1) \), with \( \eta_m \) as in (3.24). Theorem 2.4 then follows from the following key estimate.

Proposition 4.1. Suppose that the assumptions in Theorem 2.3 and (2.17) hold. Fix any (small) \( \gamma > 0 \). Then,
\[ \left| \frac{1}{N} \sum_{i=1}^{N} d_i \left( G_{ii}(z) - \frac{1}{a_i - \omega_B^i(z)} \frac{1}{(a_i - \omega_B^i(z))} \right) \right| \prec \Psi^2, \]
(4.4)
uniformly on \( \mathcal{S}_T(\eta_m, 1) \) with \( \eta_m \equiv \eta_m(\gamma) \). By switching the roles of \( A \) and \( B \), a similar statement holds for \( G_{ii} \) defined in (3.18) if \( a_i \) and \( \omega_B^i \) are replaced with \( b_i \) and \( \omega_A^i \), respectively.

With Proposition 4.1, we prove Theorem 2.4 and Theorem 3.3 at once.

Proof of Theorem 2.4 and Theorem 3.3. Fix a (small) \( 0 < \gamma < 1/2 \). Recall the \textit{a priori} bound of \( \Upsilon \) in (4.3). First, with Proposition 4.1, we show that the improved bound
\[ |\Upsilon| \prec \Psi^2 \]
(4.5)
holds uniformly on \( \mathcal{S}_T(\eta_m, 1) \). Using the identities in (4.2), the convention (3.17), the \textit{a priori} bound (4.3), and the bound (4.4) with \( d_i = 1, a_i = z \) and \( (a_i - z)^2 \) in the estimate of \( \text{tr} G, \text{tr} (\tilde{B} G) \) and \( \text{tr} (\tilde{B} \tilde{G} \tilde{B}) \), respectively, we get
\[
\text{tr} G = \text{tr} \left( A - \omega_B^i \frac{\Upsilon}{\text{tr} G} \right)^{-1} + O_\prec(\Psi^2)
\]
\[= \frac{1}{N} \sum_{i=1}^{N} \frac{1}{a_i - \omega_B^i} + \frac{1}{N} \sum_{i=1}^{N} \frac{1}{(a_i - \omega_B^i)^2} \frac{\Upsilon}{\text{tr} G} + O_\prec(\Psi^2)
\]
\[= m_A(\omega_B^i) + m'_A(\omega_B^i) \frac{\Upsilon}{\text{tr} G} + O_\prec(\Psi^2), \]
(4.6)
\[
\text{tr} (\tilde{B} G) = 1 - \text{tr} \left( (A - z)(A - \omega_B^i - \frac{\Upsilon}{\text{tr} G}) \right)^{-1} + O_\prec(\Psi^2)
\]
\[= 1 - \frac{1}{N} \sum_{i=1}^{N} \frac{a_i - z}{a_i - \omega_B^i} - \frac{1}{N} \sum_{i=1}^{N} \frac{a_i - z}{(a_i - \omega_B^i)^2} \frac{\Upsilon}{\text{tr} G} + O_\prec(\Psi^2)
\]
\[= (z - \omega_B^i)m_A(\omega_B^i) - \left( m_A(\omega_B^i) + (\omega_B^i - z)m'_A(\omega_B^i) \right) \frac{\Upsilon}{\text{tr} G} + O_\prec(\Psi^2), \]
(4.7)
\[
\text{tr} (\tilde{B} \tilde{G} \tilde{B}) = z + \text{tr} \left( (A - z)^2 (A - \omega_B^i - \frac{\Upsilon}{\text{tr} G}) \right)^{-1} + O_\prec(\Psi^2)
\]
\[ z + \frac{1}{N} \sum_{i=1}^{N} \frac{(a_i - z)^2}{a_i - \omega^A_B(z)} + \frac{1}{N} \sum_{i=1}^{N} \frac{(a_i - z)^2}{a_i - \omega^B_B(z)} \operatorname{tr} G + O_{\omega}(\Psi^2) \]

\[ = \omega^A_B - z + (\omega^A_B - z)^2 m_A(\omega^A_B) + (1 + 2(\omega^A_B - z) m_A(\omega^A_B) + (\omega^A_B - z)^2 m'_A(\omega^A_B)) \frac{\Upsilon}{\operatorname{tr} G} + O_{\omega}(\Psi^2), \]

(4.8)

where we also used \( |a_i - \omega^A_B(z)|^{-1} \leq (\operatorname{Im} \omega^B_B(z))^{-1} \times 1 \) that follows from the facts \( |\omega^A_B(z) - \omega^B_B(z)| < \Psi \) and \( \operatorname{Im} \omega^B_B(z) \geq k \) uniformly on \( S_T(\eta_m, 1) \) from (3.25) and (3.6), respectively. Here, \( m'_A(z) \) denotes the derivative with respect to \( z \) of \( m_A(z) \).

Recall the definition of \( \Upsilon \) in (4.1). Using (4.6)–(4.8) and the \textit{a priori} bound \( |\Upsilon| < \Psi \) of (4.3), we write

\[ \Upsilon = \operatorname{tr} (\tilde{B}G) - (\operatorname{tr} (\tilde{B}G))^2 + \operatorname{tr} G \operatorname{tr} (BG\tilde{B}) =: C_1 + C_2 \frac{\Upsilon}{\operatorname{tr} G} + O_{\omega}(\Psi^2), \]

(4.9)

where \( C_1 \equiv C_1(z) \) and \( C_2 \equiv C_2(z) \) are coefficients collected from (4.6)–(4.8). It is easy to check that

\[ C_1(z) = (z - \omega^A_B(z)m_A(\omega^A_B) - (\omega^A_B - z)^2(m_A(\omega^A_B))^2 + m_A(\omega^A_B)(\omega^A_B - z + (\omega^A_B - z)^2 m_A(\omega^A_B)) = 0, \]

and

\[ C_2(z) = - \left( m_A(\omega^A_B) + (\omega^A_B - z)m'_A(\omega^A_B) \right) + 2(z - \omega^A_B)m_A(\omega^A_B) \left( m_A(\omega^A_B) + (\omega^A_B - z)m'_A(\omega^A_B) \right) \]

\[ + m_A(\omega^A_B) \left( 1 + 2(\omega^A_B - z)m_A(\omega^A_B) + (\omega^A_B - z)^2 m'_A(\omega^A_B) \right) \]

\[ + m'_A(\omega^A_B) (\omega^A_B - z + (\omega^A_B - z)^2 m_A(\omega^A_B)) = 0, \]

for all \( z \in \mathbb{C}^+ \), \textit{i.e.} \( C_1 \) and \( C_2 \) vanish identically. Hence, from (4.9) we verified (4.5).

Now, applying (4.5), the facts \( |\omega^A_B(z) - \omega^B_B(z)| < \Psi \), and \( \operatorname{Im} \omega^B_B(z) \geq k \) uniformly on \( S_T(\eta_m, 1) \) from (3.25) and (3.6), we see from (4.4) that

\[ \left| \frac{1}{N} \sum_{i=1}^{N} d_i \left( G_{ii}(z) - \frac{1}{a_i - \omega^A_B(z)} \right) \right| < \Psi^2. \]

(4.10)

Switching the rôles of \( A \) and \( B \), \( U \) and \( U^* \), we also have

\[ \left| \frac{1}{N} \sum_{i=1}^{N} d_i \left( G_{ii}(z) - \frac{1}{b_i - \omega^B_B(z)} \right) \right| < \Psi^2, \]

(4.11)

where \( \mathcal{G} \) is defined in (3.18).

Setting \( d_i \) to be 1 for all \( i \in [1, N] \) in (4.10) and (4.11), and using (3.19), we obtain

\[ m_H(z) - m_A(\omega^B_B(z)) = O_{\omega}(\Psi^2), \quad m_H(z) - m_B(\omega^A_B(z)) = O_{\omega}(\Psi^2). \]

(4.12)

Recalling (3.23) and applying the \textit{a priori} estimate on \( \omega^A_B(z) \) and \( \omega^B_B(z) \) in (3.25) and the lower bound for \( \operatorname{Im} \omega^A_B(z) \) and \( \operatorname{Im} \omega^B_B(z) \) in (3.6), we can rewrite (4.12) as

\[ \| \Phi_{\mu_A, \mu_B}(\omega^A_B(z), \omega^B_B(z)) \|_2 < \Psi^2, \]

where \( \Phi_{\mu_A, \mu_B} \) is defined in (3.1). Then, by Proposition 4.1 of [1], we have the improved bound

\[ |\omega^A_B(z) - \omega^B_B(z)| < \Psi^2, \quad |\omega^B_B(z) - \omega^A_B(z)| < \Psi^2. \]

(4.13)

This completes the proof of Theorem 3.3.

Applying (4.13) to (4.10), we further get (2.23) on \( S_T(\eta_m, 1) \). To extend the conclusion to all of \( S_T(0, 1) \), we use the monotonicity of the Green function. Since \( G_{ii}(z) = \sum_{k=1}^{N} G_{ik}(z) G_{ki}(z) \), we have

\[ \left| G_{ii}(z) \right| \leq \sum_{k=1}^{N} |G_{ik}(z)|^2 = \frac{\operatorname{Im} G_{ii}(z)}{\eta}, \]

as follows from the spectral decomposition of \( H \). Note next that the function \( s \to \operatorname{sIm} G_{ii}(E + is) \) is monotone increasing. Thus for any \( \eta \in (0, \eta_m) \), we have

\[ \left| d_i G_{ii}(E + i\eta) - d_i G_{ii}(E + i\eta_m) \right| \leq |d_i| \int_{\eta}^{\eta_m} \frac{s \operatorname{Im} G_{ii}(E + is)}{s^2} \, ds \]
with high probability, for any $E \in \mathcal{I}$, where we used Proposition 3.2 to bound $\text{Im} G_{ii}(z) < 1$, $z \in \mathcal{S}_\mathcal{I}(\eta_m, 1)$. On the other hand, by Lemma 3.1, $\omega_A(z)$ is uniformly bounded from above on $\mathcal{S}_\mathcal{I}(0, 1)$ and $|a_i - \omega_B(z)|$ is uniformly bounded from below on $\mathcal{S}_\mathcal{I}(0, 1)$. Thus

$$
\frac{1}{N} \sum_{i=1}^{N} \frac{d_i}{a_i - \omega_A(E + i\eta)} - \frac{1}{N} \sum_{i=1}^{N} \frac{d_i}{a_i - \omega_A(E + i\eta_m)} \leq C(\eta_m - \eta) \leq \Psi^2, \quad \eta \in (0, \eta_m], \quad E \in \mathcal{I},
$$

(4.15)

since $\gamma < 1/2$. Hence, from (4.15) and (4.14), we conclude by triangle inequality that (2.23) holds uniformly on $\mathcal{S}_\mathcal{I}(0, 1)$ since it holds on $\mathcal{S}_\mathcal{I}(\eta_m, 1)$. This proves (2.23) and concludes the proof of Theorem 2.4. □

5. PROOF OF PROPOSITION 4.1

In this section, we prove Proposition 4.1, assuming the validity of Lemma 5.1 below, whose proof is postponed to Section 6. Let us introduce the notation

$$
Z_i := (\tilde{B}G)_{ii} \text{tr} G - G_{ii}(\text{tr} \tilde{B}G - \Upsilon).
$$

(5.1)

We have the following lemma.

**Lemma 5.1.** Suppose that the assumptions in Theorem 2.3 and (2.17) hold. Then, for any fixed integer $p \geq 2$, we have

$$
\mathbb{E} \left[ \left| \frac{1}{N} \sum_{i=1}^{N} d_i Z_i \right|^2 \right] \leq \Psi^p,
$$

(5.2)

uniformly on $\mathcal{S}_\mathcal{I}(\eta_m, 1)$.

Next, we prove Proposition 4.1, with the aid of Lemma 5.1.

**Proof of Proposition 4.1.** Recall the definition of $\omega_A^c(z)$ in (3.21). Using the identity

$$
(a_i - z)G_{ii} = - (\tilde{B}G)_{ii} + 1,
$$

(5.3)

we can write

$$
\frac{1}{N} \sum_{i=1}^{N} d_i \left( G_{ii}(z) - \frac{1}{a_i - \omega_A^c(z)} - \frac{1}{\text{tr} G} \right) = \frac{1}{N} \sum_{i=1}^{N} d_i \left( G_{ii}(\text{tr} \tilde{B}G - \Upsilon) - (\tilde{B}G)_{ii} \text{tr} G \right).
$$

(5.4)

Recall the definition of $\Upsilon$ in (4.1) and the a priori bound (4.3). Using (5.3), (4.2) and (3.26), it is straightforward to check that

$$
|\text{tr} \tilde{B}G - (z - \omega_B)m_A(\omega_B)| \lesssim \Psi, \quad |Z_i| = \left| G_{ii}(\text{tr} \tilde{B}G - \Upsilon) - (\tilde{B}G)_{ii} \text{tr} G \right| \lesssim \Psi.
$$

(5.5)

Hence, using (3.25), (4.3) and (5.5), we obtain from (5.4) that

$$
\frac{1}{N} \sum_{i=1}^{N} d_i \left( G_{ii}(z) - \frac{1}{a_i - \omega_A^c(z)} - \frac{1}{\text{tr} G} \right)
= \frac{1}{N} \sum_{i=1}^{N} \frac{d_i}{(a_i - \omega_B)m_A(\omega_B)} \left( G_{ii}(\text{tr} \tilde{B}G - \Upsilon) - (\tilde{B}G)_{ii} \text{tr} G \right) + O_{\Psi} (\Psi^2)
= \frac{1}{N} \sum_{i=1}^{N} \frac{d_i}{(\omega_B - a_i)m_A(\omega_B)} Z_i + O_{\Psi} (\Psi^2).
$$

From Lemma 3.1 we have $\text{Im} \omega_A(z) \geq k$ and $m_A(\omega_B) \gtrsim 1$ uniformly on $\mathcal{S}_\mathcal{I}(0, 1)$, which imply

$$
\frac{d_i}{(\omega_B - a_i)m_A(\omega_B)} \lesssim 1,
$$

uniformly on $\mathcal{S}_\mathcal{I}(0, 1)$. Thus to prove (2.23), we need to show that, for any deterministic numbers $\tilde{d}_1, \ldots, \tilde{d}_N \in \mathbb{C}$ satisfying $\max_i |\tilde{d}_i| \leq 1$,

$$
\frac{1}{N} \sum_{i=1}^{N} \tilde{d}_i Z_i \lesssim \Psi^2
$$

(5.6)
holds uniformly on $\mathcal{S}_T(\eta_m, 1)$.

For fixed $z \in \mathcal{S}_T(\eta_m, 1)$, the estimate (5.6) follows from Lemma 5.1 and Markov’s inequality. To get a uniform bound on $\mathcal{S}_T(\eta_m, 1)$, we choose $|z|N^8$ lattice points $z_1, z_2, \ldots, z_{|z|N^8}$ in $\mathcal{S}_T(\eta_m, 1)$ such that for any $z \in \mathcal{S}_T(\eta_m, 1)$ there exists $z_n$ satisfying $|z - z_n| \leq N^{-4}$. Then using the Lipschitz continuity of $Z_i(z)$ in $z$ with Lipschitz constant bounded by $C\eta^{-3}$, for $C$ sufficiently large, and using (5.6) for all lattice points we get (5.6) uniformly on $\mathcal{S}_T(\eta_m, 1)$ from a union bound. This completes the proof of Proposition 4.1. \hfill \square

6. Proof of Lemma 5.1

In this section, we prove Lemma 5.1. Let $Z_i$ and $d_i$ be as in Lemma 5.1. For $k, l \in \mathbb{N}$, set

$$q(k, l) := \left(\frac{1}{N} \sum_{i=1}^{N} d_i Z_i\right)^k \left(\frac{1}{N} \sum_{i=1}^{N} d_i Z_i\right)^l. \quad (6.1)$$

To prove Lemma 5.1 we then need to show that $\mathbb{E}[q(p, p)] \prec \Psi^{4p}$, uniformly on $\mathcal{S}_T(\eta_m, 1)$. This is accomplished by using a recursive estimate for $\mathbb{E}[q(p, p)]$, see Proposition 6.1 below. The use of recursive moment estimates for the fluctuation averaging mechanism was introduced in [17].

In the rest of the paper, we use the following convention: the notation $O_\prec(\Psi^k)$, for any given positive integer $k$, stands for a generic (possibly) $\varepsilon$-dependent random variable $X \equiv X(z)$ that satisfies

$$X \prec \Psi^k \quad \text{and} \quad \mathbb{E}[|X|^q] \prec \Psi^{qk}, \quad (6.2)$$

for any given positive integer $q$. In the earlier works, the notation $O_\prec(\Psi^k)$ referred only to the first bound, $X \prec \Psi^k$, but in this paper it is convenient to require the second one as well. Nevertheless, in the sequel, we usually only check the first bound in (6.2) for various $X$’s. It will be clear that the second bound in (6.2) follows from the first one in all our applications. The reason is that the random variables $X$ to be estimated below are either bounded by $O(\eta^{-k_1}) = O(N^{k_1})$ for some nonnegative constant $k_1$ deterministically, or finite products of quadratic forms of the form

$$f(z)\alpha^\ast Q(z)/\beta.$$ 

Here $f(z) : \mathbb{C}^+ \to \mathbb{C}$ is a generic function satisfying $|f(z)| \leq C\eta^{-k_2}$ and $Q(z) : \mathbb{C}^+ \to M_N(\mathbb{C})$ satisfying $\|Q\| \leq C\eta^{-k_3}$ for some finite positive constants $C, k_2$ and $k_3$, and where $\alpha$ and $\beta$ are either Gaussian or deterministically bounded in the $\|\cdot\|_2$-norm. Then it is elementary to get the second bound in (6.2) from the first one by using the definition of $\prec$ in (2.2) together with the above deterministic bounds or the Gaussian tail of $\alpha$ or $\beta$.

Our main aim in this section is to show the following proposition.

Proposition 6.1. (Recursive moment estimate) Suppose that the assumptions in Theorem 2.3 and (2.17) hold. For any fixed integer $p \geq 2$, we have

$$\mathbb{E}[q(p, p)] = \mathbb{E}[O_\prec(\Psi^2)q(p - 1, p)] + \mathbb{E}[O_\prec(\Psi^4)q(p - 2, p)] + \mathbb{E}[O\prec(\Psi^4)q(p - 1, p - 1)]. \quad (6.3)$$

Proof of Proposition 6.1. According to (3.14), we see that $e_i^\ast R_i = -h_i^\ast$. Hence, using the decomposition (3.7) with (3.12) and recalling the notations defined in (3.9) and (3.29), we have

$$(\tilde{B}G)_{ii} = e_i^\ast R_i \tilde{B}^{(i)} R_i Ge_i = -h_i^\ast \tilde{B}^{(i)} R_i Ge_i$$

$$= -h_i^\ast \tilde{B}^{(i)} (I - \ell_i^2 e_i e_i^\ast - \ell_i^2 h_i e_i^\ast - \ell_i^2 e_i^\ast h_i - \ell_i^2 h_i^\ast) Ge_i$$

$$= -S_i + \ell_i^2 (h_i^\ast \tilde{B}^{(i)} e_i + h_i^\ast \tilde{B}^{(i)} h_i) (G_{ii} + T_i). \quad (6.4)$$

Now, recalling the definition of $h_i$ in (3.11) and using the large deviation inequalities in (A.1), we get

$$|h_i^\ast \tilde{B}^{(i)} e_i| \prec \frac{1}{\sqrt{N}}, \quad |h_i^\ast \tilde{B}^{(i)} h_i| \prec \frac{1}{\sqrt{N}}, \quad (6.5)$$

where we also used the convention $\text{tr} \tilde{B}^{(i)} = \text{tr} B = 0$ from (3.17). According to Lemma 3.4, (3.26) and Lemma 3.1, we also have

$$|T_i| \prec \Psi, \quad |G_{ii}| \prec 1. \quad (6.6)$$

In addition, by (3.13), we have the elementary estimate
\[ \ell_i = 1 + O_\prec\left(\frac{1}{\sqrt{N}}\right). \]  
(6.7)

Now, using (6.5)-(6.7) to bound several small terms in (6.4), we obtain
\[ (\widetilde{B}G)_{ii} = -S_i + h_i^* \widetilde{B}^{(i)} e_i G_{ii} + h_i^* \widetilde{B}^{(i)} h_i G_{ii} + O_\prec(\Psi^2). \]  
(6.8)

Moreover, using the fact \( \tilde{B}^{(i)} e_i = b_i e_i \), we can write
\[ (\tilde{B}G)_{ii} = - \sum_{h \neq i} h_k e_k \tilde{B}^{(i)} G_{ii} + h_i^* \tilde{B}^{(i)} h_i G_{ii} + O_\prec(\Psi^2) \]
\[ = h_i^* \tilde{B}^{(i)} G_{ii} + h_i^* \tilde{B}^{(i)} h_i G_{ii} + O_\prec(\Psi^2), \]  
(6.9)

where in the last step we used the notation introduced in (3.16).

Recalling the definition of \( Z_i \) in (5.1), with (6.9), we can write
\[
\mathbb{E}[q(p,p)] = \mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} d_i Z_i \right) q(p-1,p) \right]
\]
\[
= \mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} d_i (\tilde{B}G)_{ii} \text{tr} G \right) q(p-1,p) \right] - \mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} d_i G_{ii} (\text{tr} \tilde{B}G - \Upsilon) \right) q(p-1,p) \right]
\]
\[
= - \mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} d_i h_i^* \tilde{B}^{(i)} G_{ii} \text{tr} G \right) q(p-1,p) \right] - \mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} d_i G_{ii} (\text{tr} \tilde{B}G - \Upsilon) \right) q(p-1,p) \right]
\]
\[
+ \mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} d_i h_i^* \tilde{B}^{(i)} h_i G_{ii} \text{tr} G \right) q(p-1,p) \right] + \mathbb{E}\left[ O_\prec(\Psi^2) q(p-1,p) \right]. \]  
(6.10)

Next, we claim that the following lemma holds.

**Lemma 6.2.** Suppose that the assumptions in Theorem 2.3 and (2.17) hold. Then, for any fixed integer \( p \geq 2 \), we have
\[
\mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} d_i h_i^* \tilde{B}^{(i)} G_{ii} \text{tr} G \right) q(p-1,p) \right] + \mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} d_i G_{ii} (\text{tr} \tilde{B}G - \Upsilon) \right) q(p-1,p) \right]
\]
\[
= \mathbb{E}\left[ O_\prec(\Psi^2) q(p-1,p) \right] + \mathbb{E}\left[ O_\prec(\Psi^4) q(p-2,p) \right] + \mathbb{E}\left[ O_\prec(\Psi^4) q(p-1,p-1) \right]. \]  
(6.11)

Similarly, we have
\[
\mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} d_i h_i^* \tilde{B}^{(i)} h_i G_{ii} \text{tr} G \right) q(p-1,p) \right]
\]
\[
= \mathbb{E}\left[ O_\prec(\Psi^2) q(p-1,p) \right] + \mathbb{E}\left[ O_\prec(\Psi^4) q(p-2,p) \right] + \mathbb{E}\left[ O_\prec(\Psi^4) q(p-1,p-1) \right]. \]  
(6.12)

The proof of Lemma 6.2 will be postponed. Combining (6.10), (6.11) and (6.12), we can conclude the proof of Proposition 6.1. \( \square \)

With Proposition 6.1, we can prove Lemma 5.1.

**Proof of Lemma 5.1.** Fix any (small) \( \epsilon > 0 \). Then applying Young’s inequality to (6.3) we get
\[
\mathbb{E}[q(p,p)] \leq 3 \frac{1}{2p} \mathbb{E}\left[ O_\prec(N^{2\epsilon} p^{-\frac{4p}{2p}}) \right] + 3 \frac{2p - 1}{2p} N^{-\frac{2p}{2p}} \mathbb{E}[q(p,p)]. \]  
(6.13)

Hence absorbing the second term on the right side into the left side and recalling (6.2) we get
\[
\mathbb{E}[q(p,p)] \prec \Psi^4, \]  
(6.14)
uniformly on \( S_\epsilon(\eta_0,1) \), since \( \epsilon > 0 \) was arbitrary. \( \square \)

In the rest of this section, we prove Lemma 6.2.
Proof of Lemma 6.2. We use integration by parts for the Gaussian variables: regarding \( g \) and \( \tilde{g} \) as independent variables for computing \( \partial_g f(g, \tilde{g}) \), we have
\[
\int_C \tilde{g} f(g, \tilde{g}) e^{\frac{-|\tilde{g}|^2}{2}} dg \wedge d\tilde{g} = \sigma^2 \int_C \partial_g f(g, \tilde{g}) e^{\frac{-|\tilde{g}|^2}{2}} dg \wedge d\tilde{g}, \tag{6.15}
\]
for differentiable functions \( f : \mathbb{C}^2 \to \mathbb{C} \).

Let us start with (6.11). First, we can get rid of the \( g \)-dependence of the factor \( \text{tr} G \), namely,
\[
\mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^N \partial h^* \tilde{B}^{(i)} Ge_i \text{tr} G \right) q(p - 1, p) \right] = \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^N \partial h^* \tilde{B}^{(i)} Ge_i \text{tr} G^{(i)} \right) q(p - 1, p) \right]
+ \mathbb{E} \left[ O_{\prec} (\Psi^2) q(p - 1, p) \right], \tag{6.16}
\]
where we used the finite rank perturbation estimate in (A.3) and
\[
|\partial h^* \tilde{B}^{(i)} Ge_i| < 1, \quad \forall i \in [1, N], \tag{6.17}
\]
which follows from \( \partial h^* \tilde{B}^{(i)} Ge_i = S_i - h_i h_i G_{ii} \) and the bounds in Lemma 3.4. Further, for brevity, we let
\[
d_{i, 1} \equiv d_{i, 1}(z) := d_i \text{tr} G^{(i)}. \tag{6.18}
\]
Recalling the definition in (3.16) and using the integration by parts formula (6.15) for the Gaussian variables \( \tilde{g}_{ik}, i \neq k \), we get
\[
\mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^N \partial h^* \tilde{B}^{(i)} Ge_i \right) q(p - 1, p) \right]
= \mathbb{E} \left[ \left( \frac{1}{N^2} \sum_{i=1}^N d_{i, 1} \frac{1}{|g_i|^2} \sum_{k, k \neq i} \partial (\epsilon^* \tilde{B}^{(i)} Ge_i) \right) q(p - 1, p) \right]
+ \mathbb{E} \left[ \left( \frac{1}{N^2} \sum_{i=1}^N d_{i, 1} \sum_{k, k \neq i} \partial \frac{|g_i|^2}{|\partial g_{ik}|} (\epsilon^* \tilde{B}^{(i)} Ge_i) \right) q(p - 1, p) \right]
+ \mathbb{E} \left[ \left( \frac{p - 1}{N^3} \sum_{i=1}^N d_{i, 1} \frac{1}{|g_i|^2} \sum_{k, k \neq i} \epsilon^* \tilde{B}^{(i)} Ge_i \sum_{j=1}^N d_j \frac{\partial Z_j}{\partial g_{ik}} \right) q(p - 2, p) \right]
+ \mathbb{E} \left[ \left( \frac{p}{N^3} \sum_{i=1}^N d_{i, 1} \frac{1}{|g_i|^2} \sum_{k, k \neq i} \epsilon^* \tilde{B}^{(i)} Ge_i \sum_{j=1}^N \frac{\partial Z_j}{\partial g_{ik}} \right) q(p - 1, p - 1) \right]. \tag{6.19}
\]

Using (3.7) and (3.12), it is elementary to compute
\[
\frac{\partial R_i}{\partial g_{ik}} = -\frac{\ell^2}{2|g_i|^2} e_k (e_i + h_i)^* + \frac{\ell^2}{2|g_i|^2} \tilde{g}_{ik} (e_i h_i^* + h_i e_i^* + 2 h_i h_i^*) = -\frac{\ell^4}{2|g_i|^2} \tilde{g}_{ik} e_k (e_i + h_i)^* + \Delta_R(i, k), \tag{6.20}
\]
where we introduced
\[
\Delta_R(i, k) := \frac{\ell^2}{2|g_i|^2} \tilde{g}_{ik} (e_i h_i^* + h_i e_i^* + 2 h_i h_i^*) - \frac{\ell^4}{2|g_i|^2} \tilde{g}_{ik} e_k (e_i + h_i)^*. \tag{6.21}
\]

The \( \Delta_R(i, k) \)'s are irrelevant error terms. Their estimates will be easy and kept separate in Appendix B. We focus on the other terms in the sequel. For convenience, we introduce
\[
c_i := \frac{\ell^2}{|g_i|^2} = \frac{1}{|g_i|^2} - g_{ii} + O_{\prec} (\frac{1}{N}) = |g_i|^2 - g_{ii} - (|g_i|^2 - 1) + O_{\prec} (\frac{1}{N}), \tag{6.22}
\]
where the last step follows from (3.13). Using (6.20), we have
\[
\frac{\partial G}{\partial g_{ik}} = -G \frac{\partial \tilde{B}}{\partial g_{ik}} G = -G \frac{\partial R_i}{\partial g_{ik}} \tilde{B}^{(i)} R_i G - GR_i \tilde{B}^{(i)} \frac{\partial R_i}{\partial g_{ik}} G
=: c_i \left[ G e_k (e_i + h_i)^* \tilde{B}^{(i)} R_i G + GR_i \tilde{B}^{(i)} e_k (e_i + h_i)^* G \right] + \Delta_G(i, k), \tag{6.23}
\]
where we set
\[ \Delta_G(i, k) := -G\Delta_R(i, k)\tilde{B}^{(i)} R_i G - GR_i \tilde{B}^{(i)} \Delta_R(i, k) G. \] (6.24)

Hence, applying (6.23), we obtain, for any \( i \in [1, N] \),
\[ \frac{1}{N} \sum_{k : k \neq i} e_i k \frac{\partial (e_i k \tilde{B}^{(i)} G e_i)}{\partial g_{ii}} = \frac{1}{N} \sum_{k : k \neq i} e_i k \frac{\partial G}{\partial g_{ik}} e_i \]
\[ = c_i \left[ \text{tr} (\tilde{B}^{(i)} G (e_i + h_i)^* \tilde{B}^{(i)} R_i G e_i + \text{tr} (\tilde{B}^{(i)} GR_i \tilde{B}^{(i)}) (e_i + h_i)^* G e_i) \right] \]
\[ - \frac{1}{N} \sum_{k : k \neq i} e_i k \frac{\partial (e_i k \tilde{B}^{(i)} G e_i)}{\partial g_{ik}} e_i \]
\[ = c_i \left[ \text{tr} (\tilde{B}^{(i)} G (b_i T_i + (\tilde{B}G)_{ii}) + \text{tr} (\tilde{B}^{(i)} GR_i \tilde{B}^{(i)}) (G_{ii} + T_i) \right] \]
\[ + \frac{1}{N} \sum_{k : k \neq i} e_i k \frac{\partial \Delta_G(i, k) e_i}{\partial g_{ik}} + O_\omega (\Psi^2), \] (6.25)

where in the last step we used (3.15) and thus
\[ e_i k \tilde{B}^{(i)} R_i G e_i = -b_i T_i, \quad h_i k \tilde{B}^{(i)} R_i G e_i = - (\tilde{B}G)_{ii}, \quad e_i k \tilde{B}^{(i)} GR_i \tilde{B}^{(i)} e_i = -2 \omega e_i k G h_i, \] (6.26)
whose bounds can be obtained from Lemma 3.4 and the identity \((\tilde{B}G)_{ii} = 1 - (a_i - z)G_{ii}\).

For the second term of the right side of (6.25), we use the next lemma, which is proved in Appendix B.

**Lemma 6.3.** Suppose that the assumptions in Theorem 2.3 and (2.17) hold, we have
\[ \frac{1}{N} \sum_{k : k \neq i} e_i k \tilde{B}^{(i)} \Delta_G(i, k) e_i = O_\omega (\Psi^2). \] (6.27)

With the aid of Lemma 6.3, we get from (6.25) that
\[ \frac{1}{N} \sum_{k : k \neq i} e_i k \frac{\partial (e_i k \tilde{B}^{(i)} G e_i)}{\partial g_{ik}} = c_i \left[ \text{tr} (\tilde{B}^{(i)} GR_i \tilde{B}^{(i)}) (G_{ii} + T_i) - \text{tr} (\tilde{B}^{(i)}) (b_i T_i + (\tilde{B}G)_{ii}) \right] + O_\omega (\Psi^2) \]
\[ = c_i \left[ \text{tr} (\tilde{B} G \tilde{B}) (G_{ii} + T_i) - \text{tr} (\tilde{B} G) (b_i T_i + (\tilde{B}G)_{ii}) \right] + O_\omega (\Psi^2), \] (6.28)

where in the last step we used the estimates in (6.6) and the facts that the differences \(\text{tr} (\tilde{B} G) - \text{tr} (\tilde{B}^{(i)} G)\) and \(\text{tr} (\tilde{B} \tilde{G}) - \text{tr} \tilde{B}^{(i)} G \tilde{B}^{(i)}\) can be written as the linear combination of the terms of the form \(\frac{1}{N} r_i Q_i G Q_i r_i\) for \(Q_i, Q_i' = I\) or \(\tilde{B}^{(i)}\), which implies according to (3.30) that
\[ \text{tr} (\tilde{B} G) - \text{tr} (\tilde{B}^{(i)} G) = O_\omega (\frac{1}{N}), \quad \text{tr} (\tilde{B} \tilde{G}) - \text{tr} \tilde{B}^{(i)} G \tilde{B}^{(i)} = O_\omega (\frac{1}{N}). \]

Analogously to (6.28), we can get
\[ \frac{1}{N} \sum_{k : k \neq i} e_i k \frac{\partial (e_i k G e_i)}{\partial g_{ik}} = c_i \left[ \text{tr} (\tilde{B} G) (G_{ii} + T_i) - (\text{tr} G) (b_i T_i + (\tilde{B}G)_{ii}) \right] + O_\omega (\Psi^2). \] (6.29)

Combining (6.28) and (6.29), and recalling the definition of \(\Upsilon\) in (4.1), we obtain
\[ \frac{1}{N} \sum_{k : k \neq i} e_i k \frac{\partial (e_i k \tilde{B}^{(i)} G e_i)}{\partial g_{ik}} \text{tr} G - \frac{1}{N} \sum_{k : k \neq i} e_i k \frac{\partial (e_i k G e_i)}{\partial g_{ik}} \text{tr} (\tilde{B} G) \]
\[ = c_i (G_{ii} + T_i) \left( \text{tr} G \text{tr} (\tilde{B} G) - (\text{tr} \tilde{B} G)^2 \right) + O_\omega (\Psi^2) \]
\[ = -c_i (G_{ii} + T_i) (\text{tr} \tilde{B} G - \Upsilon) + O_\omega (\Psi^2). \] (6.30)

Now, we set
\[ T_i := \tilde{g}_i k G e_i = \sum_{k : k \neq i} \tilde{g}_i k e_i k G e_i = T_i - g_{ii} G_{ii} + O_\omega (\Psi^2), \] (6.31)
where in the last step we used the definition of $T_i$ in (3.29), the bound $|T_i| \prec \Psi$ from (3.31), and the estimate $\|g_i^\alpha\|^{-1} = 1 + O_\prec(N^{-1/2})$. Using (6.22) and (6.31), we rewrite (6.30) as

$$\frac{1}{N} \sum_{k,k \neq i} \frac{\partial (e^*_{ik}B^{(i)}Ge_i)}{\partial g_{ik}} \text{tr} G$$

$$= - c_i (G_{ii} + T_i) (\text{tr} \tilde{B}G - \Psi) + \tilde{T}_i \text{tr} (\tilde{B}G) + \left( \frac{1}{N} \sum_{k,k \neq i} \frac{\partial (e^*_{ik}Ge_i)}{\partial g_{ik}} - \tilde{T}_i \right) \text{tr} (\tilde{B}G) + O_\prec(\Psi^2)$$

$$= - \left( \|g_i\|^2 - g_{ii} - (\|g_i\|^2 - 1) \right) (G_{ii} + T_i) (\text{tr} (\tilde{B}G - \Psi) + \left( T_i - g_{ii} G_{ii} \right) \text{tr} (\tilde{B}G)$$

$$+ \left( \frac{1}{N} \sum_{k,k \neq i} \frac{\partial (e^*_{ik}Ge_i)}{\partial g_{ik}} - \tilde{T}_i \right) \text{tr} (\tilde{B}G) + O_\prec(\Psi^2)$$

$$= - \|g_i\|^2 G_{ii} (\text{tr} \tilde{B}G - \Psi) + \left( \frac{1}{N} \sum_{k,k \neq i} \frac{\partial (e^*_{ik}Ge_i)}{\partial g_{ik}} - \tilde{T}_i \right) \text{tr} (\tilde{B}G)$$

$$+ \left( \|g_i\|^2 - 1 \right) G_{ii} \text{tr} (\tilde{B}G) + O_\prec(\Psi^2).$$

(6.32)

where in the last step we used the bound $|T_i| \prec \Psi, |\tilde{T}_i| \prec \Psi$ from (3.31) and (4.3), and $|g_{ii}| \prec N^{-1/2}$ and $\|g_i\|^2 = 1 + O_\prec(N^{-1/2})$. Notice that the two potentially dangerous terms $g_{ii} G_{ii} (\text{tr} (\tilde{B}G))$ of order $N^{-1/2}$ cancel exactly. Recalling from (6.18) that $d_{i,1} = d_i \text{tr} G^{(i)} = d_i \text{tr} G + O(\Psi^2)$, and using (6.32), we have

$$\frac{1}{N^2} \sum_{i=1}^N d_{i,1} \frac{1}{\|g_i\|^2} \sum_{k,k \neq i} \frac{\partial (e^*_{ik}B^{(i)}Ge_i)}{\partial g_{ik}} = - \frac{1}{N} \sum_{i=1}^N d_i G_{ii} (\text{tr} \tilde{B}G - \Psi)$$

$$+ \frac{1}{N} \sum_{i=1}^N d_i (\|g_i\|^2 - 1) \text{tr} \tilde{B}G$$

$$+ \frac{1}{N} \sum_{i=1}^N d_i G_{ii} \text{tr} \tilde{B}G + O_\prec(\Psi^2).$$

(6.33)

Substituting (6.33) into (6.19) and recalling (6.16), we obtain

$$\mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^N d_i h^*_{ik} \tilde{B}^{(i)} Ge_i \text{tr} G \right) q(p-1,p) \right] + \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^N d_i G_{ii} (\text{tr} \tilde{B}G - \Psi) \right) q(p-1,p) \right]$$

$$= \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^N d_i \left( \frac{1}{\|g_i\|^2} \sum_{k,k \neq i} \frac{\partial (e^*_{ik}Ge_i)}{\partial g_{ik}} - \tilde{T}_i \right) \text{tr} \tilde{B}G \right) q(p-1,p) \right]$$

$$+ \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^N d_i \frac{(\|g_i\|^2 - 1)}{\|g_i\|^2} \text{tr} \tilde{B}G \right) q(p-1,p) \right]$$

$$+ \mathbb{E} \left[ \left( \frac{1}{N^2} \sum_{i=1}^N d_{i,1} \sum_{k,k \neq i} \frac{\partial (\|g_i\|^2)}{\partial g_{ik}} (e^*_{ik}B^{(i)}Ge_i) \right) q(p-1,p) \right]$$

$$+ \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^N d_{i,1} \frac{1}{\|g_i\|^2} \sum_{k,k \neq i} e^*_{ik} \tilde{B}^{(i)} Ge_i \sum_{j=1}^N d_j \frac{\partial Z_j}{\partial g_{ik}} \right) q(p-2,p) \right]$$

$$+ \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^N d_{i,1} \frac{1}{\|g_i\|^2} \sum_{k,k \neq i} e^*_{ik} \tilde{B}^{(i)} Ge_i \sum_{j=1}^N d_j \frac{\partial Z_j}{\partial g_{ik}} \right) q(p-1,p-1) \right]$$

$$+ \mathbb{E} \left[ O_\prec(\Psi^2) q(p-1,p) \right].$$

(6.34)

Hence, for (6.11), it suffices to estimate the right side of (6.34). We start with the first term on the right side of (6.34). First, by (6.29), one can easily check that $\frac{1}{N} \sum_{k,k \neq i} \frac{\partial (e^*_{ik}Ge_i)}{\partial g_{ik}} = O_\prec(\Psi)$ for any $i$ from the estimate of $G_{ii}$'s and $T_i$'s (cf. (3.26) and (3.31)), and the first identity in (4.2) that expresses $(\tilde{B}G)_{ii}$ in terms of $G_{ii}$. In addition, we also have $\tilde{T}_i = O_\prec(\Psi)$ from (6.31) and the estimate of $G_{ii}$ and $T_i$ (cf. (3.26).
and (3.31)). These facts, together with \( \|g_i\|_2 = 1 + O_{\prec}(\frac{1}{N}) \) and the finite rank perturbation bound for the tracial quantities of Green function in Corollary A.3, we have

\[
\mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} d_i \|g_i\|_2 \left( \frac{1}{N} \sum_{k \neq i} \frac{\partial (e_i^* Ge_i)}{\partial g_{ik}} - \tilde{T}_i \right) \text{tr}(\tilde{B}G) \right) q(p-1, p) \right]
\]

\[
= \mathbb{E}\left[ \left( \frac{1}{N^2} \sum_{i=1}^{N} \sum_{k \neq i} \left( d_i \text{tr}(\tilde{B}^{(i)}G^{(i)}) \right) \frac{\partial (e_i^* Ge_i)}{\partial g_{ik}} \right) q(p-1, p) \right]
\]

\[
- \mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} \left( d_i \text{tr}(\tilde{B}^{(i)}G^{(i)}) \right) \tilde{T}_i \right) q(p-1, p) \right] + \mathbb{E}\left[ O_{\prec}(\Psi^2) q(p-1, p) \right], \tag{6.35}
\]

For brevity, for each \( i \in [1, N] \), we set

\[
d_{i,2} \equiv d_{i,2}(z) := d_i \text{tr}(\tilde{B}^{(i)}G^{(i)}),
\]

which is independent of \( g_i \). Recall the definition of \( \tilde{T}_i \) in (6.31). Using the integration by parts formula (6.15) for the second term on the right side of (6.35), we have

\[
\mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} d_{i,2} \tilde{T}_i \right) q(p-1, p) \right] = \mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} d_{i,2} \tilde{g}_{ik} e_i^* Ge_i \right) q(p-1, p) \right]
\]

\[
= \mathbb{E}\left[ \left( \frac{1}{N^2} \sum_{i=1}^{N} \sum_{k \neq i} d_{i,2} \frac{\partial (e_i^* Ge_i)}{\partial g_{ik}} \right) q(p-1, p) \right]
\]

\[
+ \mathbb{E}\left[ \left( \frac{p-1}{N^3} \sum_{i=1}^{N} \sum_{k \neq i} d_{i,2} e_i^* G e_i \sum_{j=1}^{N} d_j \frac{\partial Z_j}{\partial g_{ik}} \right) q(p-2, p) \right]
\]

\[
+ \mathbb{E}\left[ \left( \frac{p}{N^2} \sum_{i=1}^{N} \sum_{k \neq i} d_{i,2} e_i^* G e_i \sum_{j=1}^{N} d_j \frac{\partial Z_j}{\partial g_{ik}} \right) q(p-1, p-1) \right], \tag{6.36}
\]

where the first term on the right side cancels the first term on the right side of (6.35). Hence, for (6.11), it suffices to estimate the second term to the fifth term of the right side of (6.34) and the last two terms of (6.36). Note that the fourth and fifth terms of the right side of (6.34) have a very similar form as the last two terms in (6.36), respectively. In addition, for the second term on the right side of (6.34) we use

\[
\frac{\|g_i\|_2^2 - 1}{\|g_i\|_2} = g_i^* g_i - 1 + O_{\prec}(\frac{1}{N}).
\]

Moreover, we can replace \( \text{tr} \tilde{B}G \) by \( \text{tr} \tilde{B}^{(i)}G^{(i)} \) in the second term on the right side of (6.34), up to an error \( O_{\prec}(\Psi^2) \), according to Corollary A.3. Let \( Q_i = I \) or \( \tilde{B}^{(i)} \). In addition, we use the notation \( \tilde{Q}_i \) to denote the matrix obtained from \( Q_i \) via replacing its \((i, i)\)-th entry by zero. Choosing \( Q_i = I \), we see that for the second term on the right side of (6.34) is of the form

\[
\mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} \tilde{d}_i (\tilde{g}_i^* Q_i g_i - \tilde{Q}_i G_{ii}) \right) q(p-1, p) \right] + \mathbb{E}\left[ O_{\prec}(\Psi^3) q(p-1, p) \right], \tag{6.37}
\]

for some \( g_i \)-independent quantities \( \tilde{d}_i \equiv \tilde{d}_i(z) \) satisfying \( |\tilde{d}_i(z)| \ll 1 \) uniformly on \( S_F(\eta_m, 1) \) and in \( i \in [1, N] \). Now, using Lemma 6.4 below to estimate third term to the fifth term of the right side of (6.34) and the last two terms of (6.36), and using Lemma 6.5 below to estimate the second term on the right side of (6.34), we can conclude the proof of (6.11).

To prove (6.12), we use the approximation

\[
\tilde{h}_i B^{(i)} h_i = \tilde{g}_i^* \tilde{B}^{(i)} g_i + O_{\prec}(\frac{1}{N}).
\]

Moreover, we can replace \( \text{tr} G \) by \( \text{tr} G^{(i)} \) in the left side (6.12), up to any error \( O_{\prec}(\Psi^2) \), according to Corollary A.3. Similarly, choosing \( Q_i = \tilde{B}^{(i)} \), we see that the left side of (6.12) is of the form (6.37), in
light of the fact \( \text{tr} \tilde{B}^{(i)} = \text{tr} B = 0 \). Hence, (6.12) follows from Lemma 6.5 below directly. This completes the proof of Lemma 6.2. \( \square \)

It remains is to prove the following two lemmas.

**Lemma 6.4.** Suppose that the assumptions in Theorem 2.3 and (2.17) hold. Letting \( \tilde{d}_1, \cdots, \tilde{d}_N \in \mathbb{C} \) be any possibly \( z \)-dependent random variables satisfying \( \max_{i \in [1, N]} |\tilde{d}_i| < 1 \) uniformly on \( S_z(\eta_m, 1) \), and letting \( Q_i = I \) or \( \tilde{B}^{(i)} \), we have the estimates

\[
\frac{1}{N^2} \sum_{i=1}^{N} \sum_{k, k \neq i} \tilde{d}_i \frac{\partial}{\partial g_{ik}} e_k^* \tilde{B}^{(i)} Ge_i = O_\prec(\Psi^2),
\]

(6.38)

\[
\frac{1}{N^3} \sum_{i=1}^{N} \sum_{k, k \neq i} \tilde{d}_i e_k^* Q_i Ge_i \sum_{j=1}^{N} d_j \frac{\partial Z_j}{\partial g_{jk}} = O_\prec(\Psi^4),
\]

(6.39)

and the same estimates hold if we replace \( d_i \) and \( Z_j \) by their complex conjugates in (6.39).

**Lemma 6.5.** Suppose that the assumptions in Theorem 2.3 and (2.17) hold. Let \( \tilde{d}_1, \cdots, \tilde{d}_N \in \mathbb{C} \) be any possibly \( z \)-dependent random variables satisfying \( \max_{i \in [1, N]} |\tilde{d}_i| < 1 \) uniformly on \( S_z(\eta_m, 1) \). Assume that \( \tilde{d}_i \) is independent of \( g_i \) for each \( i \in [1, N] \). Let \( Q_i = I \) or \( \tilde{B}^{(i)} \). We have the estimate

\[
\mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^{N} \tilde{d}_i (g_i^* Q_i g_i - \text{tr} Q_i) G_{ii} \right) q(p-1, p) \right]
= \mathbb{E} \left[ O_\prec(\Psi^2) q(p-1, p) \right] + \mathbb{E} \left[ O_\prec(\Psi^4) q(p-2, p) \right] + \mathbb{E} \left[ O_\prec(\Psi^4) q(p-1, p-1) \right].
\]

**Proof of Lemma 6.4.** For (6.38), we have

\[
\frac{1}{N^2} \sum_{i=1}^{N} \sum_{k, k \neq i} \tilde{d}_i \frac{\partial}{\partial g_{ik}} e_k^* \tilde{B}^{(i)} Ge_i = -\frac{1}{2N^2} \sum_{i=1}^{N} \sum_{k, k \neq i} \tilde{d}_i \frac{\partial}{\partial g_{ik}} (g_i^2) \tilde{e}^* \tilde{B}^{(i)} Ge_i
= -\frac{1}{2N^2} \sum_{i=1}^{N} \tilde{d}_i g_i^* \tilde{B}^{(i)} Ge_i = O_\prec \left( \frac{1}{N} \right) = O_\prec(\Psi^2),
\]

where in the third step we used (6.17).

In the sequel, we prove (6.39). By the definition of \( Z_j \) in (5.1) and \( Y \) in (4.1), and the identities in (4.2), we can write

\[
Z_j = (\tilde{B}G)_{jj} \text{tr} G - G_{jj} (\text{tr} (\tilde{B}G) - Y) = (\tilde{B}G)_{jj} \text{tr} G - G_{jj} \left( (\text{tr} (\tilde{B}G))^2 - \text{tr} G \text{tr} (\tilde{B}G) \right)
= \text{tr} G - G_{jj} \left( 1 - \text{tr} ((A-z)G) \right)^2 - \text{tr} G \left( (A-z)^2G - a_j + 2z \right)
\]

Hence, we have

\[
\frac{\partial Z_j}{\partial g_{ik}} = \text{tr} \left( \frac{\partial G}{\partial g_{ik}} \right) - e_j^* \frac{\partial G}{\partial g_{ik}} e_j (A_1 + a_j G) + A_2 G_{jj} \text{tr} \left( (A-z) \frac{\partial G}{\partial g_{ik}} \right)
+ G_{jj} \text{tr} \left( \frac{\partial G}{\partial g_{ik}} \right) (A_3 - a_j) + G_{jj} \text{tr} G \text{tr} \left( (A-z)^2 \frac{\partial G}{\partial g_{ik}} \right),
\]

(6.40)

where we introduced the shorthand notations

\[
A_1 \equiv A_1(z) := \left( 1 - \text{tr} ((A-z)G) \right)^2 - \text{tr} G (\text{tr} ((A-z)^2G) + 2z),
\]

\[
A_2 \equiv A_2(z) := 2 \left( 1 - \text{tr} ((A-z)G) \right),
\]

\[
A_3 \equiv A_3(z) := \text{tr} ((A-z)^2G) + 2z
\]

to denote some \( O_\prec(1) \) tracial quantities whose explicit formulas are irrelevant for our analysis below. In addition, recalling the notation \( \Delta G(i, k) \) from (6.24), we denote

\[
\Delta Z_j(i, k) := \text{tr} \left( \Delta G(i, k) \right) - e_j^* \Delta G(i, k) e_j (A_1 + a_j G) + A_2 G_{jj} \text{tr} \left( (A-z) \Delta G(i, k) \right)
+ G_{jj} \text{tr} \left( \Delta G(i, k) \right) (A_3 - a_j) + G_{jj} \text{tr} G \text{tr} ((A-z)^2 \Delta G(i, k)),
\]

(6.41)
For convenience, we introduce the matrix
\[ D := \text{diag}(d_i), \]
and the shorthand notation
\[ w_i := c_i(e_i + h_i), \]
where \( c_i \) is defined in (6.22).

Substituting (6.23) into (6.40) and using the notations defined in (6.41)-(6.43), we obtain
\[
\sum_{j=1}^{N} d_j \frac{\partial Z_j}{\partial g_{jk}} = \left( w_i^* \tilde{B}^{(i)} R_i G^2 e_k + w_i^* G R_i \tilde{B}^{(i)} e_k \right) \left( \text{tr} D + \text{tr} (DG) A_3 - \text{tr} (ADG) \right) \\
- \left( w_i^* \tilde{B}^{(i)} R_i G D G e_k + w_i^* G D G R_i \tilde{B}^{(i)} e_k \right) A_1 \\
- \left( w_i^* \tilde{B}^{(i)} R_i G A D G e_k + w_i^* G A D G R_i \tilde{B}^{(i)} e_k \right) \text{tr} G \\
+ \left( w_i^* \tilde{B}^{(i)} R_i G (A-z)G e_k + w_i^* G (A-z)G R_i \tilde{B}^{(i)} e_k \right) \text{tr} (DG) A_2 \\
+ \left( w_i^* \tilde{B}^{(i)} R_i G (A-z)^2G e_k + w_i^* G (A-z)^2G R_i \tilde{B}^{(i)} e_k \right) \text{tr} G \text{tr} (DG) \\
+ \sum_{j=1}^{N} d_j \Delta Z_j(i,k). \tag{6.44}
\]

Since \( |G_{ii}| \ll 1 \) for all \( i \in [1, N] \) (cf. (3.30)), we have \( |\text{tr} (QG)| \ll 1 \) for all diagonal matrix satisfying \( \|Q\| \ll 1 \). Therefore, except for the last term, all the other terms in (6.44) are of the form
\[ \tilde{d} \left( w_i^* \tilde{B}^{(i)} R_i G Q G e_k + w_i^* G Q G R_i \tilde{B}^{(i)} e_k \right) \]
for some \( z \)-dependent quantity \( \tilde{d} \equiv \tilde{d}(z) \) satisfying \( |\tilde{d}| \ll 1 \) uniformly on \( S_{\Gamma}(\eta_m, 1) \), and some diagonal matrix \( Q \) with \( \|Q\| \ll 1 \), which can be \( I, D, AD, A-z \) or \( (A-z)^2 \). Hence, to establish (6.39), it suffices to estimate
\[
\frac{1}{N^3} \sum_{i=1}^{N} \sum_{k, k \neq i} \tilde{d}_i e_i^* Q_i G e_i \left( w_i^* \tilde{B}^{(i)} R_i G Q G e_k + w_i^* G Q G R_i \tilde{B}^{(i)} e_k \right) \tag{6.45}
\]
and
\[
\frac{1}{N^3} \sum_{i=1}^{N} \sum_{k, k \neq i} \tilde{d}_i e_i^* Q_i G e_i \sum_{j=1}^{N} d_j \Delta Z_j(i,k) \tag{6.46}
\]
for any possibly \( z \)-dependent random variables \( \tilde{d}_1, \ldots, \tilde{d}_N \in \mathbb{C} \) which satisfy \( \max_{i \in [1, N]} |\tilde{d}_i| \ll 1 \) uniformly on \( S_{\Gamma}(\eta_m, 1) \).

The following lemma provides the bound on the quantity in (6.46).

**Lemma 6.6.** Suppose that the assumptions in Theorem 2.3 and (2.17) hold. Letting \( \tilde{d}_1, \ldots, \tilde{d}_N \in \mathbb{C} \) be any possibly \( z \)-dependent random variables satisfying \( \max_{i \in [1, N]} |\tilde{d}_i| \ll 1 \) uniformly on \( S_{\Gamma}(\eta_m, 1) \), and letting \( Q_i = I \) or \( \tilde{B}^{(i)} \), we have
\[
\frac{1}{N^3} \sum_{i=1}^{N} \sum_{k, k \neq i} \tilde{d}_i e_i^* Q_i G e_i \sum_{j=1}^{N} d_j \Delta Z_j(i,k) = O_{\omega}(\Psi^3) \tag{6.47}
\]
uniformly on \( S_{\Gamma}(\eta_m, 1) \).

The proof of Lemma 6.6 will also be postponed to Appendix B. With Lemma 6.6, it suffices to estimate (6.45) below. Note that
\[
\frac{1}{N^3} \sum_{i=1}^{N} \sum_{k, k \neq i} \tilde{d}_i e_i^* Q_i G e_i \left( w_i^* \tilde{B}^{(i)} R_i G Q G e_k + w_i^* G Q G R_i \tilde{B}^{(i)} e_k \right)
\]
\[
\frac{1}{N^3} \sum_{i=1}^{N} \tilde{d}_i \left( w_i^* \tilde{B}^{(i)} R_i GGQG_i Ge_i + w_i^* GGQR_i \tilde{B}^{(i)} Q_i Ge_i \right) \\
- \frac{1}{N^3} \sum_{i=1}^{N} \tilde{d}_i e_i^* Q_i Ge_i \left( w_i^* \tilde{B}^{(i)} R_i GGGe_i + w_i^* GGQR_i \tilde{B}^{(i)} e_i \right).
\]
(6.48)

Recall the fact that \(Q_i = I\) or \(\tilde{B}^{(i)}\). Now, using the facts \(\tilde{B}^{(i)} = R_i \tilde{B} R_i\) and \(R_i^2 = I\), we have the following relations
\[
\tilde{B}^{(i)} = B - r_i r_i^* B - B r_i r_i^* + r_i r_i^* B r_i r_i^* , \\
R_i \tilde{B}^{(i)} = B - B r_i r_i^* , \\
R_i (\tilde{B}^{(i)})^2 = B - B^2 r_i r_i^* ,
\]
(6.49)
i.e. the \(i\)-dependence of these quantities is shifted to \(r_i\). Recalling the notations \(w_i = c_i (e_i + h_i)\) and \(r_i = \ell_i (e_i + h_i)\), and using (3.15) and (6.49) to (6.48) for either \(Q_i = I\) or \(\tilde{B}^{(i)}\), it is not difficult to check that the right side of (6.48) is the sum of terms in the form
\[
\frac{1}{N^3} \sum_{i=1}^{N} y_i e_i^* (\ast G G \ast G) e_i ,
\]
(6.50)
\[
\frac{1}{N^3} \sum_{i=1}^{N} y_i h_i^* (\ast G G \ast G) e_i ,
\]
(6.51)
where \(y_i, \ldots, y_N \in \mathbb{C}\) are some random variables (which can be different from line to line) satisfying \(\max_{i \in [1,N]} |y_i| \ll 1\), and where each \(\ast\) either stands for one of the matrices \(I, A, A - z, \tilde{B}, D\) or the product of some of them (which can be different from one to another), but are all \(i\)-independent and their operator norms are \(O_\prec(1)\). In addition, \(\alpha, \beta, e_i, h_i, Q_i = I, \tilde{B}^{(i)}\) or \(B\) in (6.51).

Now, recall the fact that \(v_i\) is the \(i\)-th column of \(U\), i.e. \(U e_i = v_i\), and \(h_i = e^{-i\theta_i} v_i\) from (3.11). Therefore we have the following identities: for any diagonal matrix \(Y := \text{diag}(y_i)\),
\[
\sum_{i=1}^{N} y_i e_i e_i^* = Y , \quad \sum_{i=1}^{N} y_i h_i h_i^* = U Y U^* , \quad \sum_{i=1}^{N} y_i h_i e_i^* = U Y \Theta^* , \quad \sum_{i=1}^{N} y_i e_i h_i^* = Y \Theta U^* ,
\]
(6.52)
where \(\Theta := \text{diag}(e^{i\theta_i})\). Applying (6.52) to the quantities in (6.50), and using \(\|G(z)\| \leq \eta^{-1}\), we get
\[
\frac{1}{N^3} \sum_{i=1}^{N} y_i e_i^* (\ast G G \ast G) e_i = \frac{1}{N^2} \text{tr} (\ast G G \ast G Y) = O_\prec \left( \frac{\text{tr} |G|^2}{N^2 \eta^2} \right) = \frac{\text{Im} \text{tr} G}{N^2 \eta^2} = O(\Psi^4),
\]
\[
\frac{1}{N^3} \sum_{i=1}^{N} y_i h_i^* (\ast G G \ast G) e_i = \frac{1}{N^2} \text{tr} (\ast G G \ast G Y \Theta U^*) = O_\prec \left( \frac{\text{tr} |G|^2}{N^2 \eta^2} \right) = O(\Psi^4).
\]

For the terms in (6.51), we set \(\tilde{y}_i = y_i \beta_i^* Q_i Ge_i\) and \(\tilde{Y} := \text{diag}(\tilde{y}_i)\). First, we claim \(|\tilde{y}_i| \ll 1\). Such a bound follows from (3.30) in case \(Q_i = I\) or \(\tilde{B}^{(i)}\). In case of \(\beta_i = e_i\) and \(Q_i = B\), we have \(\tilde{y}_i = y_i (\tilde{B} G)_{ii}\), which is \(O_\prec(1)\), according to \((\tilde{B} G)_{ii} = 1 - (a_i - z) G_{ii}\) and \(|G_{ii}| \ll 1\); in case of \(\beta_i = h_i\) and \(Q_i = B\), we can use (3.15) to get \(\tilde{y}_i = y_i h_i^* B G e_i = -y_i h_i^* R_i G e_i = y_i h_i G_{ii}\) and thus \(|\tilde{y}_i| \ll 1\). Consequently, we have \(\|\tilde{Y}\| \ll 1\). Applying this fact together with (6.52) to the quantities in (6.51), we obtain
\[
\frac{1}{N^3} \sum_{i=1}^{N} y_i e_i^* (\ast G G \ast G) \alpha_i \beta_i^* Q_i Ge_i = \frac{1}{N^2} \sum_{i=1}^{N} \tilde{y}_i e_i^* (\ast G G \ast G) \alpha_i = \frac{1}{N^2} \text{tr} \left( \ast G G \ast G \left( \sum_{i=1}^{N} \tilde{y}_i \alpha_i e_i^* \right) \right) = O_\prec \left( \frac{\text{Im} \text{tr} G}{N^2 \eta^2} \right) = O(\Psi^4),
\]
and
\[
\frac{1}{N^3} \sum_{i=1}^{N} y_i h_i^* (\ast G G \ast G) \alpha_i \beta_i^* Q_i Ge_i = \frac{1}{N^2} \sum_{i=1}^{N} \tilde{y}_i h_i^* (\ast G G \ast G) \alpha_i
\]
where we used the fact $\alpha_i = e_i$ or $h_i$ and the identities in (6.52) to show $\| \sum_{i=1}^{N} \tilde{y}_i \alpha_i \beta_i^* \| < 1$ for $\alpha_i, \beta_i = e_i$ or $h_i$. Hence, we conclude the proof of Lemma 6.4.

Proof of Lemma 6.5. By assumption, both $\tilde{d}_i$ and $Q_i$ are independent of $g_i$ for each $i \in [1, N]$. Hence, using integration by parts formula (6.15), we obtain

$$
\mathbb{E}\left[ \frac{1}{N} \sum_{i=1}^{N} \tilde{d}_i (\tilde{g}_i \cdot Q_i - \text{tr} \tilde{Q}_i) G_{ii} \right] q(p-1, p)
$$

$$= \mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} \tilde{d}_i \sum_{k, k \neq i} \tilde{g}_{ik} e_i^* Q_i g_i G_{ii} \right) q(p-1, p) \right] - \mathbb{E}\left[ \left( \frac{1}{N} \sum_{i=1}^{N} \tilde{d}_i \text{tr} \tilde{Q}_i G_{ii} \right) q(p-1, p) \right]
$$

$$= \mathbb{E}\left[ \left( \frac{1}{N^2} \sum_{i=1}^{N} \tilde{d}_i \sum_{k, k \neq i} e_i^* Q_i g_i G_{ii} \frac{\partial G}{\partial g_{ik}} e_i \right) q(p-1, p) \right]
$$

$$+ \mathbb{E}\left[ \left( \frac{p-1}{N^2} \sum_{i=1}^{N} \tilde{d}_i \sum_{k, k \neq i} e_i^* Q_i g_i G_{ii} \frac{1}{N} \sum_{j=1}^{N} d_j \frac{\partial Z_j}{\partial g_{ik}} \right) q(p-2, p) \right]
$$

$$+ \mathbb{E}\left[ \left( \frac{p}{N^2} \sum_{i=1}^{N} \tilde{d}_i \sum_{k, k \neq i} e_i^* Q_i g_i G_{ii} \frac{1}{N} \sum_{j=1}^{N} d_j \frac{\partial Z_j}{\partial g_{ik}} \right) q(p-1, p-1) \right].
$$

(6.53)

We start with the first term of the right side of (6.53). Recalling (6.22) and (6.23), and using the shorthand notation $\tilde{d}_i := \tilde{d}_i c_i$, we have

$$
\frac{1}{N^2} \sum_{i=1}^{N} \tilde{d}_i \sum_{k, k \neq i} e_i^* Q_i g_i \frac{\partial G}{\partial g_{ik}} e_i
$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} \tilde{d}_i \sum_{k, k \neq i} e_i^* Q_i g_i e_i \Delta_G(i, k) e_i
$$

$$+ \frac{1}{N^2} \sum_{i=1}^{N} \tilde{d}_i \sum_{k, k \neq i} e_i^* Q_i g_i e_i \Delta_G(i, k) e_i
$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} \tilde{d}_i \left[ - e_i^* GQ_i (b_i T_i + (\tilde{B} G)_{ii}) + e_i^* G R_i \tilde{B}^{(i)} Q_i g_i (G_{ii} + T_i) \right]
$$

$$- \frac{1}{N^2} \sum_{i=1}^{N} \tilde{d}_i e_i^* Q_i g_i \left[ - G_{ii} (b_i T_i + (\tilde{B} G)_{ii}) - b_i e_i^* G h_i (G_{ii} + T_i) \right]
$$

$$+ \frac{1}{N^2} \sum_{i=1}^{N} \tilde{d}_i \sum_{k, k \neq i} e_i^* Q_i g_i e_i \Delta_G(i, k) e_i
$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} \tilde{d}_i \sum_{k, k \neq i} e_i^* Q_i g_i e_i \Delta_G(i, k) e_i + O_{\infty}(\Psi^2),
$$

where in the second step we separated the sum $\sum_{i} \sum_{k, k \neq i} = \sum_{k, i} - \sum_{k = i}$ and used (6.26), and in the last step we used the bound (3.30) again. Then the estimate

$$
\frac{1}{N^2} \sum_{i=1}^{N} \tilde{d}_i \sum_{k, k \neq i} e_i^* Q_i g_i e_i \frac{\partial G}{\partial g_{ik}} e_i = O_{\infty}(\Psi^2)
$$

(6.54)

is implied by the following lemma, whose proof will be postponed to Appendix B.
Lemma 6.7. Suppose that the assumptions of Theorem 2.3 and (2.17) hold. Let \( \hat{d}_1, \ldots, \hat{d}_N \) be any possibly \( z \)-dependent complex random variables satisfying \( \max_{i \in [1, N]} |\hat{d}_i| < 1 \) uniformly on \( S_T(\eta_0, 1) \), and let \( Q_i = I \) or \( \tilde{B}^{(i)} \). Then,

\[
\frac{1}{N^2} \sum_{i=1}^{N} \hat{d}_i \sum_{k, k \neq i} e_i^* Q_i g_i e_i^* \Delta_G(i, k) e_i = O_\ast(\Psi^2). \tag{6.55}
\]

Now we investigate the last two terms of (6.53). Let \( \hat{d}_1, \ldots, \hat{d}_N \) be any possibly \( z \)-dependent complex random variables satisfying \( |\hat{d}_i| < 1 \) uniformly on \( S_T(\eta_0, 1) \). Let \( Q_i = I \) or \( \tilde{B}^{(i)} \). We claim that

\[
\frac{1}{N^4} \sum_{i=1}^{N} \sum_{k, k \neq i} \hat{d}_i e_i^* Q_i g_i \sum_{j=1}^{N} \hat{d}_j \frac{\partial g_j}{\partial g_{jk}} = O_\ast(\Psi^4) \tag{6.56}
\]

holds uniformly on \( S_T(\eta_0, 1) \), and the same estimate holds if we replace \( d_j \) and \( Z_j \) by their complex conjugates. The proof of (6.56) is nearly the same as (6.39). The only difference is a missing \( G \) in the factor \( e_i^* Q_i g_i \), which played no essential rôle in the proof of (6.39). We omit the details of the proof of (6.56).

Using (6.54) and (6.56) to (6.53), we can conclude the proof of Lemma 6.5. \( \square \)

APPENDIX A.

In this appendix, we collect some basic tools from random matrix theory.

A.1. Stochastic domination and large deviation properties. Recall the stochastic domination in Definition 2.2. The relation \( \prec \) is a partial ordering: it is transitive and it satisfies the arithmetic rule \( s \prec t \) implies \( s + t \prec s \), \( s \ast t \prec s \), \( s - t \prec s \), \( s - t \prec s \), \( s \ast t \prec s \). Further assume that \( \Phi(v) \geq N^{-C} \) is deterministic and that \( Y(v) \) is a nonnegative random variable satisfying \( \mathbb{E}[Y(v)]^2 \leq N^C \) for all \( v \). Then \( Y(v) \prec \Phi(v) \), uniformly in \( v \), implies \( \mathbb{E}[Y(v)] \prec \Phi(v) \), uniformly in \( v \).

Gaussian vectors have well-known large deviation properties. We will use them in the following form whose proof is standard.

**Lemma A.1.** Let \( X = (x_{ij}) \in M_N(\mathbb{C}) \) be a deterministic matrix and let \( y = (y_i) \in \mathbb{C}^N \) be a deterministic complex vector. For a Gaussian real or complex random vector \( g = (g_1, \ldots, g_N) \in \mathscr{N}(0, \sigma^2 I_N) \) or \( \mathscr{N}(0, \sigma^2 I_N) \), we have

\[
|y^* g| < \sigma \|y\|_2, \quad |y^* X g - \sigma^2 N \text{tr} X| < \sigma^2 \|X\|_2. \tag{A.1}
\]

A.2. Rank-one perturbation formula. At various places, we use the following fundamental perturbation formula: for \( \alpha, \beta \in \mathbb{C}^N \) and an invertible \( D \in M_N(\mathbb{C}) \), we have

\[
(D + \alpha \beta^*)^{-1} = D^{-1} - \frac{D^{-1} \alpha \beta^* D^{-1}}{1 + \beta^* D^{-1} \alpha}, \tag{A.2}
\]

as can be checked readily. A standard application of (A.2) is recorded in the following lemma.

**Lemma A.2.** Let \( D \in M_N(\mathbb{C}) \) be Hermitian and let \( Q \in M_N(\mathbb{C}) \) be arbitrary. Then, for any finite-rank Hermitian matrix \( R \in M_N(\mathbb{C}) \), we have

\[
\left| \text{tr} \left( Q(D + R - z)^{-1} \right) - \text{tr} \left( Q(D - z)^{-1} \right) \right| \leq \frac{\text{rank}(R) \|Q\|}{N \eta}, \quad z = E + i\eta \in \mathbb{C}^+. \tag{A.3}
\]

Using Lemma A.2, we also have the following corollary.

**Corollary A.3.** With the notations in (3.9) and (3.10), we have

\[
|\text{tr} G - \text{tr} G^{(i)}| \leq C \Psi^2, \quad |\text{tr} \tilde{B}^{(i)} G^{(i)} - \tilde{B} G| \leq C \Psi^2, \quad |\text{tr} \tilde{B}^{(i)} G^{(i)} \tilde{B}^{(i)} - \tilde{B} G \tilde{B}| \leq C \Psi^2. \tag{A.4}
\]

**Proof.** Recalling the Hermitian matrix \( H^{(i)} \) defined in (3.10), we see that \( H \) is a finite rank perturbation of \( H^{(i)} \) and the perturbation \( H - H^{(i)} \) is obviously Hermitian. Using (A.3) with \( Q = I, D = H^{(i)} \) and \( R = H - H^{(i)} = B - \tilde{B}^{(i)} \), it is straightforward to get the first bound in (A.4). For the second bound, at first, we see that

\[
\text{tr} (\tilde{B}^{(i)} G) - \text{tr} (\tilde{B} G) = \text{tr} (\tilde{B}^{(i)} G) - \text{tr} (R_i \tilde{B}^{(i)} R_i G)
\]
\[ = \frac{1}{N} r_i^* \tilde{B}^{(i)} G r_i + \frac{1}{N} r_i^* G \tilde{B}^{(i)} r_i - \frac{1}{N} r_i^* \tilde{B}^{(i)} r_i G r_i = O_\prec (\frac{1}{N}), \tag{A.5} \]

where in the last step we used the fact \( r_i = \ell_i (e_i + h_i) \), the estimates in (6.5), and the bound in (3.30).

Then applying (A.3) with \( Q = \tilde{B}^{(i)} \), \( D = H^{(i)} \) and \( R = H - H^{(i)} = B - \tilde{B}^{(i)} \), we obtain
\[
\| \text{tr} (\tilde{B}^{(i)} G^{(i)}) - \text{tr} (\tilde{B}^{(i)} G) \| \leq C \Psi^2. \tag{A.6} \]

Combining (A.5) and (A.6) yields the second estimate in (A.4). The third one in (A.4) can be verified similarly. We omit the details. So we complete the proof of Corollary A.3.

\[ \square \]

**Appendix B.**

In this appendix, we estimate the terms with \( \Delta_R (i, k) \)'s involved. More specifically, we will prove Lemmas 6.3, 6.6 and 6.7.

According to (6.21), we see that \( \Delta_R (i, k) \) is the sum of terms of the form
\[
\hat{a}_i \hat{g}_{ik} \alpha_i \beta_i^*,
\]
for some \( \hat{a}_i \in \mathbb{C} \) satisfying \( |\hat{a}_i| \ll 1 \) uniformly on \( \mathcal{S}_r (\eta_m, 1) \), and \( \alpha_i, \beta_i = e_i \) or \( h_i \). Hereafter \( \hat{a}_i \) can change from line to line, up to the bound \( |\hat{a}_i| \ll 1 \) uniformly on \( \mathcal{S}_r (\eta_m, 1) \). Then, by (6.24), we see that \( \Delta_G (i, k) \) is a sum of the terms of the form
\[
\hat{a}_i \hat{g}_{ik} G \alpha_i \beta_i^* \tilde{B}^{(i)} R_i G, \quad \hat{a}_i \hat{g}_{ik} G R_i \tilde{B}^{(i)} \alpha_i \beta_i^* G. \tag{B.1} \]

Recalling the definition of \( \Delta_{Z_j} (i, k) \) in (6.41) and the matrix \( D \) in (6.42), we see that
\[
\sum_{j=1}^N d_j \Delta_{Z_j} (i, k) = N \text{tr} \Delta_G (i, k) \text{tr} D - N A_1 \text{tr} (\Delta_G (i, k) D)
\]
\[
- N \text{tr} (\Delta_G (i, k) \text{tr} D + N A_2 \text{tr} (D G) (\text{tr} ((A - z) \Delta_G (i, k)))
\]
\[
+ N \text{tr} \Delta_G (i, k) \left( A_3 \text{tr} (D G) - \text{tr} (G A D) \right)
\]
\[
+ N \text{tr} (G \text{tr} (D G)) \text{tr} ((A - z)^2 \Delta_G (i, k)). \tag{B.2} \]

Then, according to (B.1) and (B.2), we see that \( \sum_j d_j \Delta_{Z_j} (i, k) \) is the sum of the terms of the form
\[
N \hat{a}_i \hat{g}_{ik} \text{tr} (Q G \alpha_i \beta_i^* \tilde{B}^{(i)} R_i G) = \hat{a}_i \hat{g}_{ik} \beta_i^* \tilde{B}^{(i)} R_i G Q G \alpha_i,
\]
\[
N \hat{a}_i \hat{g}_{ik} \text{tr} (Q G R_i \tilde{B}^{(i)} \alpha_i \beta_i^* G) = \hat{a}_i \hat{g}_{ik} \alpha_i \tilde{B}^{(i)} G Q G R_i \beta_i^* G. \tag{B.3} \]

for some random variables \( \hat{a}_i \), with \( |\hat{a}_i| \ll 1 \), for all \( i \in [1, N] \), and some i-independent diagonal matrix \( Q \) with \( ||Q|| \ll 1 \), which can be \( A, D, A - z, (A - z)^2 \) or the product of some of them.

With the above facts, we can prove Lemmas 6.3, 6.6 and 6.7 in the sequel.

**Proof of Lemma 6.3.** Using the fact that \( \Delta_G (i, k) \) is a sum of the terms of the form in (B.1), we see that the left side of (6.27) is the sum of the terms of the form
\[
\frac{1}{N} \sum_{k, k \neq i} \hat{g}_{ik} e_k^* \tilde{B}^{(i)} G \alpha_i \beta_i^* \tilde{B}^{(i)} R_i G e_i = \frac{1}{N} \hat{g}_{ik} e_k^* \tilde{B}^{(i)} G \alpha_i \beta_i^* \tilde{B}^{(i)} R_i G e_i = O_\prec (\frac{1}{N}),
\]
\[
\frac{1}{N} \sum_{k, k \neq i} \hat{g}_{ik} e_k^* \tilde{B}^{(i)} G R_i \alpha_i \beta_i^* G e_i = \frac{1}{N} \hat{g}_{ik} e_k^* \tilde{B}^{(i)} G R_i \alpha_i \beta_i^* G e_i = O_\prec (\frac{1}{N}),
\]
where we used \( \hat{g}_{ik} = g_{ik} - g_{ii} e_i \), the identities in (3.15) and the bound in (3.30). Hence, we conclude the proof of Lemma 6.3.

**Proof of Lemma 6.6.** Using that \( \sum_j d_j \Delta_{Z_j} (i, k) \) is a sum of such terms as in (B.3) and \( \sum_{k, k \neq i} \hat{g}_{ik} e_k^* = \hat{g}_{ii}^* \), we see that the left side of (6.47) is the sum of the terms of the form
\[
\frac{1}{N} \sum_{i=1}^N \hat{a}_i \hat{g}_{ii}^* Q_i G e_i \beta_i^* \tilde{B}^{(i)} R_i G Q G G e_i = O_\prec (\Psi^4),
\]
\[
\frac{1}{N^s} \sum_{i=1}^{N} \tilde{a}_i G_i Q_i G e_i \beta_\ast_i G Q G R_i \tilde{B}^{(i)} \alpha_i = O_{\prec}(\Psi^4),
\]
where we used \( \tilde{g}_i = g_i - g_{ii} e_i \), (3.15) and (3.30). This completes the proof of Lemma 6.6. \( \square \)

**Proof of Lemma 6.7.** Using that \( \Delta Q(i, k) \) is the sum of such terms as in (B.1) and \( \sum_{k \neq i} \tilde{g}_{ik} e_k^* = \tilde{g}_i \), we see that the left side of (6.55) is the sum of the terms of the form

\[
\frac{1}{N^s} \sum_{i=1}^{N} \tilde{a}_i G_i Q_i g_i e_i^* G \alpha_i \beta_\ast_i \tilde{B}^{(i)} R_i G e_i = O(\frac{1}{N}),
\]

\[
\frac{1}{N^s} \sum_{i=1}^{N} \tilde{a}_i G_i Q_i g_i e_i^* G R_i \tilde{B}^{(i)} \alpha_i \beta_\ast_i G e_i = O(\frac{1}{N}),
\]

where we used the fact \( \tilde{g}_i = g_i - g_{ii} e_i \), (3.15) and (3.30). Hence, we conclude the proof of Lemma 6.7. \( \square \)

**Appendix C.**

In this appendix, we discuss the case when both \( \mu_\alpha \) and \( \mu_\beta \) (cf. (2.12)) are convex combinations of two point masses. Without loss of generality (up to shifting and scaling), we may assume that \( \mu_\alpha \) and \( \mu_\beta \) have the form

\[
\mu_\alpha = \xi \delta_1 + (1 - \xi) \delta_0, \quad \mu_\beta = \zeta \delta_\theta + (1 - \zeta) \delta_0,
\]

with real parameters \( \xi, \zeta \) and \( \theta \) satisfying

\[
\theta \neq 0, \quad \xi, \zeta \in \left( \frac{1}{2}, 1 \right], \quad \xi \leq \zeta, \quad (\theta, \xi, \zeta) \neq \left( -1, \frac{1}{2}, \frac{1}{2} \right).
\]

Recall the domains \( S_\ast(a, b) \) in (2.21). For given (small) \( \varsigma, \gamma > 0 \), we set

\[
S_\ast(a, b) := \{ z \in S_\ast(a, b) : \varsigma |z - 1| \geq \max \left\{ \sqrt{d_L(\mu_A, \mu_\alpha)}, \sqrt{d_L(\mu_B, \mu_\beta)} \right\} \}
\]

\[
\tilde{S}_\ast(a, b) := S_\ast(a, b) \cap \left\{ z \in \mathbb{C} : |z - 1| \geq \frac{N^\gamma}{(N \eta)^{\frac{\varsigma}{2}}} \right\}.
\]

The following theorem presents the local law under the setting (C.1).

**Theorem C.1** (Local law in the two point masses case). Let \( \mu_\alpha, \mu_\beta \) be as in (C.1), with fixed \( \xi, \zeta \) and \( \theta \). Assume that the sequence of matrices \( A \) and \( B \) satisfy (2.10). Fix any compact nonempty interval \( \mathcal{I} \subset B_{\mu_\alpha, \mu_\beta} \). Then there is a constant \( b > 0 \) such that if

\[
d_L(\mu_A, \mu_\alpha) + d_L(\mu_B, \mu_\beta) \leq b,
\]

holds, then the following statements hold:

(i) If \( \mu_\alpha \neq \mu_\beta \), then

\[
\left| \frac{1}{N} \sum_{i=1}^{N} d_i (G_i(z) - \frac{1}{a_i - \omega_B(z)}) \right| \prec \Psi^2
\]

holds uniformly for all \( z \in S_\ast(0, 1) \). Consequently,

\[
\sup_{\mathcal{I} \subseteq S_\ast} \left| \mu_H(\mathcal{I}') - \mu_A \boxplus \mu_B(\mathcal{I}') \right| \prec \frac{1}{N},
\]

where the supremum is over all subintervals of \( \mathcal{I} \).

(ii) If \( \mu_\alpha = \mu_\beta \), then, for sufficiently small \( \varsigma > 0 \),

\[
\left| \frac{1}{N} \sum_{i=1}^{N} d_i (G_i(z) - \frac{1}{a_i - \omega_B(z)}) \right| \prec \frac{\Psi^2}{|z - 1|^2}
\]

holds uniformly for all \( z \in \tilde{S}_\ast(0, 1) \). Thus, for any nonempty compact interval \( \tilde{\mathcal{I}} \subseteq \mathcal{I} \setminus \{1\} \),

\[
\sup_{\mathcal{I} \subseteq \tilde{\mathcal{I}}} \left| \mu_H(\mathcal{I}') - \mu_A \boxplus \mu_B(\mathcal{I}') \right| \prec \frac{1}{N},
\]

where the supremum is over all subintervals of \( \tilde{\mathcal{I}} \).
Notice that the result deviates from the general case from only if $\mu_\alpha = \mu_\beta$ due to an instability at $z = 1$ in the free convolution $\mu_\alpha \boxplus \mu_\beta$.

Remark C.2. For $\mu_\alpha, \mu_\beta$ given in (C.1), the regular bulk $\mathcal{B}_{\mu_\alpha \boxplus \mu_\beta}$ can be written down explicitly, in terms of $\xi, \zeta$ and $\theta$, see (B.2) and (B.3) in [2] for more detail.

Proof. For (C.5) and (C.7), analogously to (2.23), one needs to exploit the fluctuation average of the $G_{ii}$’s, namely, that the fluctuation of the (weighted) average of $G_{ii}$’s is typically as small as the square of the fluctuation of $G_{ii}$’s. Note that the estimate of the individual $G_{ii}$’s of the two point masses case has been obtained in Proposition B.1 of [2]. Since the proofs of (C.5) and (C.7) are nearly the same as (2.23), given Proposition B.1 of [2], we omit the details. Then the convergence rates (C.6) and (C.8) follow from (C.5) and (C.7), respectively, via a routine application of the Helffer-Sjöstrand functional calculus; see e.g. Section 7.1 of [13]. This completes the proof. □

References

[1] Bao, Z. G., Erdős, L., Schnelli, K.: Local stability of the free additive convolution, J. Funct. Anal. 271(3), 672-719 (2016).
[2] Bao, Z. G., Erdős, L., Schnelli, K.: Local law of addition of Random Matrices on optimal scale, arXiv:1509.07080 (2015).
[3] Belinschi, S., Bercovici, H.: A new approach to subordination results in free probability, J. Anal. Math. 101(1), 357-365 (2007).
[4] Belinschi, S.: A note on regularity for free convolutions, Ann. Inst. Henri Poincaré Probab. Stat. 42(5), 635-648 (2006).
[5] Belinschi, S.: The Lebesgue decomposition of the free additive convolution of two probability distributions, Probab. Theory Related Fields 142(1-2), 125-150 (2008).
[6] Belinschi, S.: $L^\infty$-boundedness of density for free additive convolutions, Rev. Roumaine Math. Pures Appl. 59(2), 173-184 (2014).
[7] Bercovici, H., Voiculescu, D.: Free convolution of measures with unbounded support, Indiana Univ. Math. J. 42, 733-773 (1993).
[8] Biane, P.: Processes with free increments, Math. Z. 227(1), 143-174 (1998).
[9] Chatterjee, S.: Concentration of Haar measures, with an application to random matrices, J. Funct. Anal. 245(2), 379-389 (2007).
[10] Chistyakov, G. P., Götze, F.: The arithmetic of distributions in free probability theory, Cent. Euro. J. Math. 9, 997-1050 (2011).
[11] Diaconis, P., Shahshahani, M.: The subgroup algorithm for generating uniform random variables, Probab. Engrg. Inform. Sci. 1(01), 15-32 (1987).
[12] Erdős, L., Knowles, A., Yau, H.-T.: Averaging fluctuations in resolvents of random band matrices, Ann. Henri Poincaré 14, 1837-1926 (2013).
[13] Erdős, L., Knowles, A., Yau, H.-T., Yin, J.: The local semicircle law for a general class of random matrices, Electron. J. Probab., 18(59), 1-58, (2013).
[14] Erdős, L., Yau, H.-T., Yin, J.: Universality for generalized Wigner matrices with Bernoulli distribution, J. Comb. 2(1), 15-85 (2011).
[15] Kargin, V.: A concentration inequality and a local law for the sum of two random matrices, Prob. Theory Related Fields 154, 677-702 (2012).
[16] Kargin, V.: Subordination for the sum of two random matrices, Ann. Probab. 43(4), 2119-2150 (2015).
[17] Lee, J. O., Schnelli, K.: Local law and Tracy-Widom limit for sparse random matrices, arXiv:1605.08767 (2016).
[18] Meckes, E. S., Meckes, M. W.: Concentration and convergence rates for spectral measures of random matrices, Probab. Theory Related Fields 156(1-2), 145164 (2013).
[19] Mezzadri F.: How to generate random matrices from the classical compact groups, Notices Amer. Math. Soc. 54(5), 592-604 (2007).
[20] Pastur, L., Vasilchuk. V.: On the law of addition of random matrices, Comm. Math. Phys. 214(2), 249-286 (2000).
[21] Voiculescu, D.: Limit laws for random matrices and free products, Invent. Math. 104(1), 201-220 (1991).
[22] Voiculescu, D.: The analogues of entropy and of Fisher’s information measure in free probability theory I, Comm. Math. Phys. 156(1), 71-92 (1993).