Hadronic light-by-light scattering contribution to $g_\mu - 2^*$

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We briefly review the current status of the hadronic light-by-light scattering correction to the muon $g_\mu - 2$. Then we present our semi-analytical evaluation of the pion-pole contribution, using a description of the pion-photon-photon transition form factor based on large-$N_C$ and short-distance properties of QCD. We derive a two-dimensional integral representation which allows to separate the generic features from the model dependence, in order to better control the latter. Finally, we sketch an effective field theory approach to hadronic light-by-light scattering which yields the leading logarithmic terms that are enhanced by a factor $N_C$. It also shows that the modeling of hadronic light-by-light scattering by a constituent quark loop is not consistent with QCD.

1. Introduction

The muon $g - 2$ is an important quantity that provides a stringent test of the Standard Model and which is potentially sensitive to new physics. However, for this purpose one first needs to well understand the hadronic contributions, i.e. vacuum polarization effects and light-by-light scattering. The present picture of hadronic light-by-light scattering is shown in Fig. 1 and the corresponding contributions to $a_\mu$ are listed in Table 1, taking into account the corrections made in the two full evaluations [1,2], after we had discovered the sign error in the pion-pole contribution [3,4].

![Diagram](image)

Figure 1. The hadronic light-by-light scattering contribution to the muon $g - 2$.

Table 1
Contributions to $a_\mu (\times 10^{10})$ according to Fig. 1. The last column gives the result when no form factors are used in the couplings to the photons.

| Type | Ref. [1] | Ref. [2] | Ref. [3] | Ref. [4] |
|------|----------|----------|----------|----------|
| (b)  | -0.5(0.8) | -1.9(1.3) | -4.5     |
| (c)  | 8.3(0.6)  | 8.5(1.3)  | 8.3(1.2) | +∞      |
| $f_0, a_1$ | 0.174$^a$ | -0.4(0.3) |          |
| (d)  | 1.0(1.1)  | 2.1(0.3)  | 6$^b$    |
| Total| 9.0(1.5)  | 8.3(3.2)  | 8(4)$^b$ |

$^a$ Only $a_1$ exchange.
$^b$ Our estimate, using Refs. [1–3].

There are three classes of contributions to the hadronic four-point function [Fig. 1(a)], which can be understood from an effective field theory (EFT) analysis of hadronic light-by-light scattering [5,4]: (1) a charged pion loop [Fig. 1(b)], where the coupling to photons is dressed by some form factor ($\rho$-meson exchange, e.g. via vector meson dominance (VMD)), (2) the pseudoscalar pole diagrams [Fig. 1(c)] together with the ex-
change of heavier resonances \((f_0, a_1, \ldots)\) and, finally, (3) the irreducible part of the four-point function which was modeled in Refs. \([1,2]\) by a constituent quark loop dressed again with VMD form factors [Fig. 1(d)]. The latter can be viewed as a local contribution \(\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}\) to \(a_\mu\). The two groups \([1,2]\) used similar, but not identical models which explains the slightly different results for the dressed charged pion and the dressed constituent quark loop, although their sum seems to cancel to a large extent and the final result is essentially given by the pseudoscalar exchange diagrams. We take the difference of the results as indication of the error due to the model dependence.

Our approach to this problem consists of making an ansatz for the relevant Green’s functions in the framework of large-\(N_C\) QCD. In this limit, an infinite set of narrow resonance states contributes in each channel. Then we perform a matching of the ansatz with chiral perturbation theory (ChPT) at low energies and with the operator product expansion (OPE) at high momenta in order to reduce the model dependence. In practice, it is sufficient to keep a few resonance states to reproduce the leading behavior in ChPT and the OPE. In this way we show in Section 2 that the pseudoscalar contribution now seems under control, due to our semi-analytical calculation \([3]\), using a pion-photon-photon form factor \(\mathcal{F}_{\pi^0\gamma\gamma\gamma}\) which fulfills the relevant QCD short-distance constraints \([6]\), in contrast to the form factors used in Refs. \([1,2]\). \(^2\) These findings are also corroborated by an EFT and large-\(N_C\) analysis \([4]\) which allows to calculate the leading and next-to-leading logarithms in \(a_\mu^{L=1,\text{had}}\) (Sec. 3).

### 2. Pion-pole contribution

The contribution from the neutral pion intermediate state is given by the following two-loop integral (see Ref. \([3]\) for all the details)

\[
a_\mu^{L=1,\text{had}} = -e^6 \int \frac{d^4q_1}{(2\pi)^4} \int \frac{d^4q_2}{(2\pi)^4} \frac{1}{q_1^2 q_2^2 (q_1 + q_2)^2} \times \frac{1}{(p + q_1)^2 - m^2} \frac{1}{(p - q_2)^2 - m^2}.
\]

\(^2\)Furthermore, the calculations in Refs. \([1,2]\) were based purely on numerical approaches.

\[
\begin{align*}
&\times \left[ \frac{\mathcal{F}_{\pi^0\gamma\gamma\gamma}(q_1^2, (q_1 + q_2)^2) \mathcal{F}_{\pi^0\gamma\gamma\gamma}(q_2^2, 0)}{q_2^2 - M_{\pi^0}^2} T_1 \\
&+ \frac{\mathcal{F}_{\pi^0\gamma\gamma\gamma}(q_1^2, q_2^2) \mathcal{F}_{\pi^0\gamma\gamma\gamma}((q_1 + q_2)^2, 0)}{(q_1 + q_2)^2 - M_{\pi^0}^2} T_2 \right],
\end{align*}
\]

that involves the convolution of two pion-photon-photon transition form factors, see Fig. 1(c). The \(T_i\) are polynomials of up to sixth order in the momenta \(p, q_1\), and \(q_2\), with \(p^2 = m^2\).

Since no data on the doubly off-shell form factor \(\mathcal{F}_{\pi^0\gamma\gamma\gamma}(q_1^2, q_2^2)\) is available, one has to resort to models. In order to proceed with the analytical evaluation of the two-loop integrals, we considered a certain class of form factors which includes the ones based on large-\(N_C\) QCD that we studied in Ref. \([6]\). For comparison, we have also used a vector meson dominance (VMD) and a constant form factor, derived from the Wess-Zumino-Witten (WZW) term.

In large-\(N_C\) QCD, the pion-photon-photon form factor is described by a sum over an infinite set of narrow vector resonances, involving arbitrary couplings, although there are constraints at long and short distances. The normalization is given by the WZW term, \(\mathcal{F}_{\pi^0\gamma\gamma\gamma}(0, 0) = -N_C/(12\pi^2 F_\pi)\), whereas the OPE tells us that

\[
\lim \lambda \rightarrow \infty \mathcal{F}_{\pi^0\gamma\gamma\gamma}(\lambda q^2, (p - \lambda q)^2) = \frac{2 F_\pi}{3 q^2} \left\{ \frac{1}{\lambda^2} + \frac{1}{\lambda^4} \frac{q \cdot p}{q^2} + \mathcal{O}\left(\frac{1}{\lambda^6}\right) \right\}.
\]

In the following, we consider the form factors that are obtained by truncation of the infinite sum in large-\(N_C\) QCD to one (lowest meson dominance, LMD), and two (LMD+V), vector resonances per channel, respectively:

\[
\mathcal{F}^{\text{LMD}}_{\pi^0\gamma\gamma\gamma}(q_1^2, q_2^2) = \frac{F_\pi}{3} \left( \frac{q_1^2 + q_2^2 - c_V}{(q_1^2 - M_{V_1}^2)(q_2^2 - M_{V_2}^2)} \right),
\]

\[
\mathcal{F}^{\text{LMD+V}}_{\pi^0\gamma\gamma\gamma}(q_1^2, q_2^2) = \frac{F_\pi}{3} \left\{ \left( \frac{q_1^2 q_2^2 (q_1^2 + q_2^2) + h_1 (q_1^2 + q_2^2)}{(q_1^2 - M_{V_1}^2)(q_2^2 - M_{V_2}^2)} \right)
\]

\[
+ h_2 q_1^2 q_2^2 + h_3 (q_1^2 + q_2^2) + h_7 \right\} / \left((q_1^2 - M_{V_1}^2)(q_2^2 - M_{V_2}^2) \right). \]
with the constants \( c_V = N_C M_V^4 / (4 \pi^2 F_\pi^2) \) and \( h_\gamma = -N_C M_V^4 M_\gamma^2 / (4 \pi^2 F_\pi^2) \). The parameters \( h_1, h_2, \) and \( h_5 \) in the LMD+V form factor are not fixed by the normalization and the leading term in the OPE. We have determined these coefficients phenomenologically \([6,3]\). In particular, \( F_{\pi^0, \gamma^0} \) with one photon on-shell behaves like \( 1/Q^2 \) for large spacelike momenta, \( Q^2 = -q^2 \). Whereas the LMD form factor does not have such a behavior, it can be reproduced with the LMD+V ansatz, provided that \( h_1 = 0 \). A fit to the data yields moreover \( h_2 = 6.93 \pm 0.26 \) GeV\(^4\). Analyzing the experimental data for the decay \( \pi^0 \to e^+e^- \) leads to the loose bound \(|h_2| \lesssim 20 \) GeV\(^2\).

Note that the usual VMD form factor \( F_{\pi^0, \gamma^0} \sim 1/[(q_1^2 - M_1^2)(q_2^2 - M_2^2)] \) does not correctly reproduce the OPE in Eq. (2).

For the form factors discussed above one can perform all angular integrations in the two-loop integrals analytically \([3]\) using the method of Gegenbauer polynomials \([7]\). The key observation is that all form factors can be written as follows

\[
F_{\pi^0, \gamma^0}(q_1^2, q_2^2) = \hat{f}(q_1^2) - \sum_{M_\chi} \hat{g}_{M_\chi}(q_1^2) / (q_1^2 - M_\chi^2). \tag{5}
\]

This allows to cancel all dependences on \( q_1, q_2 \) in the numerators in \( F_{\pi^0, \gamma^0}(q_1^2, q_2^2) \) in the loop integrals in Eq. (1). Then one writes the propagators as (for Euclidean momenta \( K, L \))

\[
\frac{1}{(K-L)^2 + M^2} = \frac{Z_{K,L}^M}{|K||L|} \sum_{n=0}^\infty (Z_{K,L}^M)^n C_n(K \cdot L), \tag{6}
\]

with \( Z_{K,L}^M = (K^2 + L^2 + M^2 - [(K^2 + L^2 + M^2)^2 - 4K^2 L^2]^{1/2}) / (2|K||L|) \), and uses the orthogonality properties of the Gegenbauer polynomials

\[
\int d\Omega(K) C_n(\hat{Q}_1 \cdot \hat{K}) C_m(\hat{K} \cdot \hat{Q}_2) = 2\pi^2 \frac{\delta_{nm}}{n+1} C_n(\hat{Q}_1 \cdot \hat{Q}_2), \tag{7}
\]

where for instance \( \hat{Q}_1 \cdot \hat{K} \) denotes the cosine of the angle between the four-dimensional vectors \( Q_1 \) and \( K \). After performing the angular integrations in this way, the pion-exchange contribution to \( a_\mu \) can be written as a two-dimensional integral representation, where the integration runs over the moduli of the Euclidean momenta

\[
a_{\mu}^{\text{WZW}, e, 0} = \int_0^\infty dQ_1 \int_0^\infty dQ_2 \times \sum_i w_i(Q_1, Q_2) f_i(Q_1, Q_2), \tag{8}
\]

with universal [for the above class of form factors] weight functions \( w_i \) (rational functions, square roots and logarithms) \([3]\). The dependence on the form factors resides in \( f_i \). In this way we could separate the generic features of the pion-pole contribution from the model dependence and thus better control the latter. This is not possible anymore in the final analytical result (as a series expansion) for \( a_{\mu}^{\text{WZW}, e, 0} \) derived in Ref. \([8]\). Note that the analytical result has not the same status here as for instance in QED. One has to keep in mind that there is an intrinsic uncertainty in the form factor of \( 10 \leq 30 \% \), furthermore the VMD form factor used in that reference has the wrong high-energy behavior.

The weight functions \( w_i \) in the main contribution are positive and peaked around momenta of the order of 0.5 GeV. There is, however, a tail in one of these functions, which produces the constant WZW form factor a divergence of the form \( \ln^2 \Lambda \) for some UV-cutoff \( \Lambda \). Other weight functions have positive and negative contributions in the low-energy region, which lead to a strong cancellation in the corresponding integrals.

In Table 2 we present the numerical results for the different form factors. All form factors lead to very similar results (apart from WZW). Judging from the shape of the weight functions described

| Form factor          | \( a_{\mu}^{\text{WZW}, e, 0} \times 10^{10} \) |
|----------------------|---------------------------------------------|
| WZW                  | 12.2                                       |
| VMD                  | 5.6                                        |
| LMD                  | 7.3                                        |
| LMD+V (\( h_2 = 0 \) GeV\(^2\)) | 5.8                                        |
above, it seems more important to correctly reproduce the slope of the form factor at the origin and the available data at intermediate energies. On the other hand, the asymptotic behavior at large $Q$, seems not very relevant. The results for the LMD+V form factor are rather stable under the variation of the parameters, except for $h_2$. If all other parameters are kept fixed, our result changes in the range $|h_2| < 20$ GeV$^2$ by $\pm 0.9 \times 10^{-10}$ from the value for $h_2 = 0$.

Thus, with the LMD+V form factor, we get

$$a_{\mu}^{\text{LbyL, }\pi^0} = +5.8 \ (1.0) \times 10^{-10},$$

where the error includes the variation of the parameters and the intrinsic model dependence. A similar short-distance analysis in the framework of large-$N_C$ QCD and including quark mass corrections for the form factors for the $\eta$ and $\eta'$ was beyond the scope of Ref. [3]. We therefore used VMD form factors fitted to the available data for $F_{\pi^0, \gamma^*\gamma^*}(-Q^2,0)$ to obtain our final estimate

$$a_{\mu}^{\text{LbyL,PS}} = a_{\mu}^{\text{LbyL, }\pi^0} + a_{\mu}^{\text{LbyL, }\eta}_{\text{VMD}} + a_{\mu}^{\text{LbyL, }\eta'}_{\text{VMD}}$$

$$= +8.3 \ (1.2) \times 10^{-10},$$

An error of 15% for the pseudoscalar pole contribution seems reasonable, since we impose many theoretical constraints from long and short distances on the form factors. Furthermore, we use experimental information whenever available.

3. EFT approach to $a_{\mu}^{\text{LbyL, had}}$

In Ref. [4] we discussed an EFT approach to hadronic light-by-light scattering based on an effective Lagrangian that describes the physics of the Standard Model well below 1 GeV. It includes photons, light leptons, and the pseudoscalar mesons and obeys chiral symmetry and $U(1)$ gauge invariance.

The leading contribution to $a_{\mu}^{\text{LbyL, had}}$, of order $p^6$, is given by a finite loop of charged pions with point-like electromagnetic vertices, see Fig. 1(b). Since this contribution involves a loop of hadrons, it is subleading in the large-$N_C$ expansion.

At order $p^8$ and at leading order in $N_C$, we encounter the divergent pion-pole contribution, diagrams (a) and (b) of Fig. 2, involving two WZW vertices. The diagram (c) is actually finite. The divergences of the triangular subgraphs in the diagrams (a) and (b) are removed by inserting the counterterm $\chi$ from the Lagrangian $\mathcal{L}^{(6)} = (\alpha^2/4\pi^2 F_0) \chi \bar{\psi} \gamma_\mu \gamma_5 \psi \partial^\mu \pi^0 + \cdots$, see the one-loop diagrams (d) and (e). Finally, there is an overall divergence of the two-loop diagrams (a) and (b) that is removed by a local counterterm, diagram (f). Since the EFT involves such a local contribution, we will not be able to give a precise numerical prediction for $a_{\mu}^{\text{LbyL, had}}$.

![Figure 2. The graphs contributing to $a_{\mu}^{\text{LbyL, }\pi^0}$ at lowest order in the effective field theory.](image)

Nevertheless, it is interesting to consider the leading and next-to-leading logarithms that are in addition enhanced by a factor $N_C$ and which can be calculated using the renormalization group [4]. The EFT and large-$N_C$ analysis tells us that

$$a_{\mu}^{\text{LbyL, had}} = \left(\frac{\alpha}{\pi}\right)^3 \left\{ f \left(\frac{M_{\pi^\pm}}{m_\mu}, \frac{M_{K^\pm}}{m_\mu}\right) + N_C \left\{ \frac{m_\mu^2}{16\pi^2 F_\pi^2} \cdot \frac{N_C}{3} \left[ \ln^2 \frac{\mu_0}{m_\mu} + c_1 \ln \frac{\mu_0}{m_\mu} + c_0 \right] + O \left(\frac{m_\mu^2}{\mu_0^3} \times \ln s + \frac{m_\mu^4}{\mu_0^5} N_C \times \log s\right) \right\},$$

where $f(M_{\pi^\pm}/m_\mu, M_{K^\pm}/m_\mu) = -0.038$ represents the charged pion and kaon-loop that is formally of order one in the chiral and $N_C$ counting and $\mu_0$ denotes some hadronic scale, e.g. $M_\rho$. The coefficient $C$ of the log-square term in the second line is universal and of order $N_C$, since $F_\pi = O(\sqrt{N_C})$. 


Unfortunately, although the logarithm is sizeable, in \( a_{\mu,LbyL;PS}^{\text{VMD}} \) there occurs a cancellation between the log-square and the log-term. If we fit our result for the VMD form factor for large \( M_\rho \) to an expression as given in Eq. (11), we obtain

\[
a_{\mu,LbyL;VMD}^{\text{LbyL}} \equiv \left( \frac{\alpha}{\pi} \right)^3 C \left[ \ln\frac{M_\rho}{m_\mu} + c_1 \ln \frac{M_\rho}{m_\mu} + c_0 \right]
\]

\[
\equiv \left( \frac{\alpha}{\pi} \right)^3 C \left[ 3.94 - 3.30 + 1.08 \right]
\]

\[
= [12.3 - 10.3 + 3.4] \times 10^{-10}
\]

\[
= 5.4 \times 10^{-10},
\]

which is confirmed by the analytical result of Ref. [8] (setting for simplicity \( M_\rho = m_\mu \)):

\[
a_{\mu,LbyL;VMD}^{\text{LbyL}} = [12 - 8.0 + 1.7] \times 10^{-10} = 5.7 \times 10^{-10}.
\]

This cancellation is now also visible in the revised version of Ref. [9]. In that paper the remaining parts of \( c_1 \) have been calculated: \( c_1 = -2\chi(\mu_0)/3 + 0.237 = -0.93^{+0.67}_{-0.83} \), with our conventions for \( \chi \) and \( \chi(M_\rho)_{\text{exp}} = 1.75_{-1.00} \).

Finally, the EFT analysis shows that the modeling of hadronic light-by-light scattering by a constituent quark loop is not consistent with QCD. The latter has a priori nothing to do with the full quark loop in QCD which is dual to the corresponding contribution in terms of hadronic degrees of freedom. Equation (11) tells us that at leading order in \( N_C \) any model of QCD has to show the behavior \( a_{\mu,LbyL;had}^{\text{LbyL}} \sim (\alpha/\pi)^3 N_C [N_C m_\mu^2 / (48\pi^2 F_\pi^2)] \ln^2 \Lambda \), with a universal coefficient, if one sends the cutoff \( \Lambda \) to infinity. From the analytical result given in Ref. [10] for the quark loop, one obtains the behavior \( a_{\mu,LbyL;CQM}^{\text{LbyL}} \sim (\alpha/\pi)^3 N_C (m_\mu^2 / M_Q^2) + \ldots \), for \( M_Q \gg m_\mu \), if we interpret the constituent quark mass \( M_Q \) as a hadronic cutoff. Even though one may argue that \( N_C / 48\pi^2 F_\pi^2 \) can be replaced by \( 1/M_Q^2 \), the log-square term is not correctly reproduced with this model. Therefore, the constituent quark model (CQM) cannot serve as a reliable description for the dominant contribution to \( a_{\mu,LbyL;had}^{\text{LbyL}} \), in particular, its sign. Moreover, we note that the pion-pole contribution is infrared finite in the chiral limit\(^3\), whereas the quark loop shows an infrared divergence \( \ln(M_Q/m_\mu) \) for \( M_Q \to 0 \) [10].

4. Conclusions

The pseudoscalar pole contribution \( a_{\mu,LbyL;PS}^{\text{LbyL}} \) seems to be under control at the 15\% level. Moreover, the EFT and large-\( N_C \) analysis provides a systematic approach to \( a_{\mu,LbyL,\text{had}}^{\text{LbyL}} \) and yields the leading and next-to-leading logarithmic terms, enhanced by a factor \( N_C \). It also shows that the modeling of hadronic light-by-light scattering by a constituent quark loop is not consistent with QCD. Since model calculations for the dressed charged pion and the dressed constituent quark loop yield slightly different results, our present estimate reads (by adding the errors linearly)

\[
a_{\mu,LbyL;\text{had}}^{\text{LbyL}} = +8 (4) \times 10^{-10}.
\]

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\(^3\)This can be shown by studying the low momentum behavior of the weight functions \( w_i \) corresponding to the two-loop diagrams 2(a)–(c) and the one-loop diagrams 2(d)+(e) (given in Ref. [4]) for \( M_{\rho,0} \to 0 \), see also Ref. [9].