Bayesian inference on hierarchical nonlocal priors in generalized linear models

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Abstract

Variable selection methods with nonlocal priors have been widely studied in linear regression models, and their theoretical and empirical performances have been reported. However, the crucial model selection properties for hierarchical nonlocal priors in high-dimensional generalized linear regression have rarely been investigated. In this paper, we consider a hierarchical nonlocal prior for high-dimensional logistic regression models and investigate theoretical properties of the posterior distribution. Specifically, a product moment (pMOM) nonlocal prior is imposed over the regression coefficients with an Inverse-Gamma prior on the tuning parameter. Under standard regularity assumptions, we establish strong model selection consistency in a high-dimensional setting, where the number of covariates is allowed to increase at a sub-exponential rate with the sample size. We implement the Laplace approximation for computing the posterior probabilities, and a modified shotgun stochastic search procedure is suggested for efficiently exploring the model space. We demonstrate the validity of the proposed method through simulation studies and an RNA-sequencing dataset for stratifying disease risk.

Key words: High-dimensional, nonlocal prior, strong selection consistency

1 Introduction

The advance in modern technology has led to an increased ability to collect and store data on a large scale. This brings opportunities and, at the same time, tremendous challenges in analyzing

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data with a large number of covariates per observation, the so-called high-dimensional problem. In high-dimensional analysis, variable selection is one of the very important tasks and commonly used techniques, especially in radiological and genetic research, with the high-dimensional data naturally extracted from imaging scans and gene sequencing. There is an extensive frequentist literature on variable selection, especially ones that are based on regularization techniques enforcing sparsity through penalization functions that share the common property of shrinkage toward sparse models [Tibshirani, 1996, Fan and Li, 2001, Zhang, 2010]. On the other hand, Bayesian model selection expresses the sparsity through a sparse prior, such as the popular spike and slab prior [Ishwaran and Rao, 2005, George and McCulloch, 1993, Narisetty and He, 2014] and continuous shrinkage prior [Liang et al., 2008, Johnson and Rossell, 2012, Liang et al., 2013], i.e., a distribution that supports on the sparse model or model with sparse parameters, and inference is carried out through posterior inference.

In this paper, we are interested in nonlocal priors [Johnson and Rossell, 2010] that are identically zero whenever a model parameter is equal to its null value. Compared to local priors, nonlocal prior distributions have relatively appealing properties for Bayesian model selection. Specifically, nonlocal priors discard spurious covariates faster as the sample size grows, while preserving exponential learning rates to detect nontrivial coefficients [Johnson and Rossell, 2010]. Under the setup of linear regression with \( p \) predictors, Johnson and Rossell [2012] introduced the product moment (pMOM) nonlocal prior with density

\[
d_p(2\pi)^{-\frac{p}{2}}(\tau\sigma^2)^{-\frac{p}{2}}|U_p|^{\frac{1}{2}} \exp \left(-\frac{\beta^T U_p \beta_p}{2\tau\sigma^2}\right) \prod_{i=1}^{p} \beta_i^{2r}. \tag{1}
\]

Here \( U_p \) is a \( p \times p \) nonsingular matrix, \( r \) is a positive integer referred to as the order of the density and \( d_p \) is the normalizing constant independent of the scale parameter \( \tau \) and the variance \( \sigma^2 \). Variations of the density in (1), called the piMOM and peMOM density, have also been developed in Johnson and Rossell [2012] and Rossell et al. [2013]. Under regularity conditions, Johnson and Rossell [2012], Shin et al. [2018] and Cao and Lee [2020] demonstrated that the posterior distributions based on the pMOM and piMOM nonlocal prior densities can achieve strong model selection consistency in high-dimensional settings. It implies that the posterior probability assigned to the true model converges to 1 as the sample size grows. When the number of covariates is much smaller than the
sample size, Shi et al. [2019] established the posterior convergence rate of the probability regarding the Hellinger distance between the posterior model and the true model under pMOM priors in a logistic regression model.

In the pMOM prior (1), the hyperparameter $\tau$ controls the dispersion of the density around the origin, and thus implicitly determines the magnitude of the regression coefficients that will be shrunk to zero [Johnson and Rossell, 2012]. Wu et al. [2020] and Cao et al. [2020] extended the work in Johnson and Rossell [2012] and Shin et al. [2018] by proposing a fully Bayesian approach with the pMOM nonlocal prior and an appropriate Inverse-Gamma prior on the hyperparameter $\tau$ referred to as the hyper-pMOM prior. In particular, Wu et al. [2020] investigated model selection properties of hyper-pMOM priors in generalized linear models (GLMs) under a fixed dimension $p$, and Cao et al. [2020] established strong model selection consistency of hyper-pMOM priors in linear regression when $p$ is allowed to grow at a polynomial rate of $n$. For the hyper-piMOM priors composed of the mixture of piMOM and Inverse-Gamma densities, Bian and Wu [2017] established model selection consistency in generalized linear models under rather restrictive assumptions.

Despite recent developments in model selection using nonlocal priors, a rigorous Bayesian inference of hyper-pMOM priors in GLMs has not been undertaken to the best of our knowledge. Motivated by this gap, we establish model selection consistency of the hyper-pMOM prior on regression coefficients in a GLM, in particular, logistic regression when the number of covariates grows at a sub-exponential rate of the sample size (Theorems 3.1 to 3.3). Furthermore, it is known that the computation problem can arise for Bayesian approaches due to the non-conjugate nature of priors in GLMs. To address this issue, we obtain posterior probabilities via Laplace approximation and then implement a slightly modified shotgun stochastic search algorithm for exploring the sparsity pattern of the regression coefficients. We demonstrate that the proposed method can outperform existing state-of-the-art methods including both penalized likelihood and Bayesian approaches in various settings. Finally, the proposed method is applied to an RNA-sequencing dataset consisting of gene expression levels to identify differentially expressed genes for disease risk stratification.

The rest of paper is organized as follows. Section 2 provides background material regarding GLMs and revisits the hyper-pMOM distribution. We detail strong selection consistency results in Section 3, and proofs are provided in the supplement. The posterior computation algorithm is described in Section 4, and we show the performance of the proposed method and compare it with
other competitors through simulation studies in Section 5. In Section 6, we conduct a data analysis for predicting asthma and show that the hyper-pMOM prior yields better prediction performance compared with other contenders. We conclude with a discussion in Section 7.

2 Methodology

2.1 Variable Selection in Logistic Regression

We first describe the framework and introduce some notations for Bayesian variable selection in logistic regression. Let \( y \in \{0,1\}^n \) be the binary response vector and \( X \in \mathbb{R}^{n \times p} \) be the design matrix. Without loss of generality, we assume that the columns of \( X \) are standardized to have zero mean and unit variance. Let \( x_i \in \mathbb{R}^p \) denote the \( i \)th row vector of \( X \) that contains the covariates for the \( i \)th subject. Let \( \beta \) be the \( p \times 1 \) vector of regression coefficients. We first consider the standard logistic regression model:

\[
P(y_i = 1 \mid x_i, \beta) = \frac{\exp(x_i^\top \beta)}{1 + \exp(x_i^\top \beta)}, \quad \text{for } i = 1, 2, \ldots, n.
\] 

(2)

We present a scenario where the dimension of predictors, \( p \), grows with the sample size \( n \). Thus, the number of predictors is a function of \( n \), that is, \( p = p_n \), but we denote it as \( p \) for notational simplicity. The goal of this paper is variable selection, i.e., to correctly identify all the locations of nonzero regression coefficients.

We denote a model by \( k = \{k_1, k_2, \ldots, k_{|k|}\} \subseteq [p] =: \{1, 2, \ldots, p\} \) if and only if all the nonzero elements of \( \beta \) are \( \beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_{|k|}} \), where \( |k| \) is the cardinality of \( k \). For any \( \beta \in \mathbb{R}^p \) and \( k \subseteq [p] \), let \( \beta_k = (\beta_{k_1}, \beta_{k_2}, \ldots, \beta_{k_{|k|}})^\top \in \mathbb{R}^{|k|} \). Similarly, for any \( m \times p \) matrix \( A \) and \( k \subseteq [p] \), let \( A_k \in \mathbb{R}^{m \times |k|} \) denote the submatrix of \( A \) containing the columns of \( A \) indexed by model \( k \). In particular, for any \( 1 \leq i \leq n \) and \( k \subseteq [p] \), we denote \( x_{ik} \in \mathbb{R}^{|k|} \) as the subvector of \( x_i \in \mathbb{R}^p \) containing the entries of \( x_i \) corresponding to model \( k \).
2.2 Hierarchical Nonlocal Priors

The class of the following hierarchical nonlocal priors can be used for variable selection:

$$\pi(\beta_k | \tau, k) = d_k(2\pi)^{-|k|/2} \tau^{-|k|/2} |U_k|^{1/2} \exp \left( -\frac{\beta_k^\top U_k \beta_k}{2\tau}\right) \prod_{i=1}^{|k|} \beta_{k_i}^{2r}, \quad \beta_k \in \mathbb{R}^{|k|}, \quad (3)$$

$$\pi(\tau) = \frac{\lambda_1^{\lambda_1}}{\Gamma(\lambda_1)} \tau^{-\lambda_1-1} \exp \left( -\frac{\lambda_2}{\tau}\right), \quad \tau > 0, \quad (4)$$

where $U$ is a $p \times p$ nonsingular matrix, $r$ is a positive integer and $\lambda_1, \lambda_2$ are positive constants. We refer to the mixture of densities of pMOM and Inverse-Gamma in (3) and (4) as the hyper-pMOM prior [Wu et al., 2020, Cao et al., 2020]. It is easy to see that the marginal density of $\beta_k$, after integrating out $\tau$, has the following form:

$$\pi(\beta_k | k) = \int \pi(\beta_k | \tau, k)\pi(\tau)d\tau = \frac{\lambda_1^{\lambda_1}}{\Gamma(\lambda_1)} \left(\frac{\lambda_2}{\lambda_2 + \frac{1}{2} U_k \beta_k} \right)^{\frac{1}{2} |k| + \frac{1}{2}} \prod_{i=1}^{|k|} \beta_{k_i}^{2r},$$

Compared to the pMOM density in (3) with given $\tau$, $\pi(\beta_k | k)$ possesses thicker tails and could achieve better model selection performance especially for small samples. For more details, see Liang et al. [2008], for example, that investigates the finite sample performance of hyper-$g$ priors.

For the prior over the model space, we suggest using the following uniform prior and restricting the analysis to models with a size of less than or equal to $m_n$:

$$\pi(k) \propto 1(|k| \leq m_n). \quad (5)$$

Similar structure has also been considered in Narisetty et al. [2019], Shin et al. [2018] and Cao et al. [2020]. As an alternative to the uniform prior (5), one may also consider the complexity prior [Castillo et al., 2015]. However, as noted in Shin et al. [2018], the penalty over large models can be derived directly from the nonlocal densities themselves without the extra penalization through the prior over the model space. In particular, Cao et al. [2020] conducted simulation studies to compare the model selection results under a uniform prior and a complexity prior, and they showed the superior performance of model selection under a uniform prior.
Note that in the hierarchical nonlocal prior (2) to (5), no specific conditions have yet been assigned to the hyperparameters. Some standard regularity assumptions on the hyperparameters will be provided in Section 3.

By the hierarchical model (2) to (5) and Bayes’ rule, the resulting posterior probability for model $k$ is denoted by

$$\pi(k \mid y) = \frac{\pi(k)}{m(y)} m_k(y),$$

where $m(y)$ is the marginal density of $y$, and $m_k(y)$ is the marginal density of $y$ under model $k$ given by

$$m_k(y) = \int \exp \left\{ L_n(\beta_k) \right\} \pi(\beta_k \mid \tau, k) \pi(\tau) \, d\tau d\beta_k = \int \exp \left\{ L_n(\beta_k) \right\} \pi(\beta_k \mid k) \, d\beta_k,$$

where

$$L_n(\beta_k) = \log \left( \prod_{i=1}^{n} \left\{ \frac{\exp(x_{ik}^\top \beta_k)}{1 + \exp(x_{ik}^\top \beta_k)} \right\}^{y_i} \left\{ \frac{1}{1 + \exp(x_{ik}^\top \beta_k)} \right\}^{1-y_i} \right)$$

is the log-likelihood function. The above marginal posterior probabilities for model $k$ can be used to find the posterior mode,

$$\hat{k} = \arg \max_k \pi(k \mid y).$$

The closed form of these posterior probabilities cannot be obtained due to the non-conjugate nature of nonlocal densities. Therefore, special efforts need to be devoted for both consistency results and computational strategy as we shall see in the following sections. In Section 4, we will adopt a (modified) stochastic search algorithm that utilizes posterior probabilities to target the mode in a more efficient way compared with Markov chain Monte Carlo (MCMC).

### 2.3 Extension to Generalized Linear Model

In this section, we extend our previous discussion on logistic regression to a GLM. Given predictors $x_i$ and an outcome $y_i$ for $1 \leq i \leq n$, a GLM has a probability density function or probability mass
function of the form
\[ p(y_i \mid \theta_i) = \exp \left\{ a(\theta_i)y_i + b(\theta_i) + c(y_i) \right\}, \]
in which \( a(\cdot) \) is a continuously differentiable function with respect to \( \theta \) with nonzero derivative, \( b(\cdot) \) is also a continuously differentiable function of \( \theta \), \( c(\cdot) \) is some constant function of \( y \), and \( \theta_i = \theta_i(\beta) = x_i^\top \beta \) is the natural parameter.

The class of hierarchical pMOM densities specified in (3) and (4) can still be used for model selection in the generalized setting by noting that the log-likelihood function in (6) and (7) now takes the general form of
\[ L_n(\beta_k) = \sum_{i=1}^{n} \left\{ a(\theta_i(\beta_k))y_i + b(\theta_i(\beta_k)) + c(y_i) \right\}. \]

Using similar techniques in Section 4, one can also develop efficient search algorithms based on different log-likelihood functions to navigate the posterior mode through the model space.

3 Main Results

In this section, we show that the hyper-pMOM prior enjoys desirable model selection properties in a GLM. Let \( t = \{t_1, t_2, \ldots, t_{|t|}\} \subseteq [p] \) be the true model, which means that the nonzero locations of the true coefficient vector are \( t = (j, j \in t) \). We consider \( |t| \) to be a fixed value. Let \( \beta_0 \in \mathbb{R}^p \) be the true coefficient vector and \( \beta_{0,t} \in \mathbb{R}^{|t|} \) be the vector of the true nonzero coefficients. For a given model \( k \subseteq [p] \), we denote \( L_n(\beta_k) \) and \( s_n(\beta_k) = \partial L_n(\beta_k)/\partial \beta_k \) as the log-likelihood and score function, respectively. In the following analysis, we will focus on logistic regression, but our argument can be extended to any other GLMs such as a probit regression model by imposing certain conditions on the design matrix to effectively bound the Hessian matrix. Let
\[ H_n(\beta_k) = -\frac{\partial^2 L_n(\beta_k)}{\partial \beta_k \partial \beta_k^\top} = \sum_{i=1}^{n} \sigma_i^2(\beta_k)x_{ik}x_{ik}^\top = X_k^\top \Sigma(\beta_k)X_k \]
be the negative Hessian of $L_n(\beta_k)$, where $\Sigma(\beta_k) \equiv \Sigma_k = \text{diag}(\sigma_1^2(\beta_k), \ldots, \sigma_n^2(\beta_k))$, $\sigma_i^2(\beta_k) = \mu_i(\beta_k)(1 - \mu_i(\beta_k))$ and

$$\mu_i(\beta_k) = \frac{\exp(x_{ik}^T \beta_k)}{1 + \exp(x_{ik}^T \beta_k)}.$$  

In the rest of the paper, we denote $\Sigma = \Sigma(\beta_{0,t})$ and $\sigma^2_i = \sigma^2_i(\beta_{0,t})$ for simplicity.

Before establishing our main results, we introduce the following notation. For any $a, b \in \mathbb{R}$, $a \lor b$ and $a \land b$ mean the maximum and minimum of $a$ and $b$, respectively. For any positive real sequences $a_n$ and $b_n$, we denote $a_n \lesssim b_n$, or equivalently $a_n = O(b_n)$, if there exists a constant $C > 0$ such that $|a_n| \leq C|b_n|$ for all large $n$. We denote $a_n \ll b_n$, or equivalently $a_n = o(b_n)$, if $a_n/b_n \to 0$ as $n \to \infty$. We denote $a_n \sim b_n$, if there exist constants $C_1 > C_2 > 0$ such that $C_2 < b_n/a_n \leq a_n/b_n < C_1$. The $\ell_2$-norm for a given vector $v = (v_1, v_2, \ldots, v_p)^T \in \mathbb{R}^p$ is defined as $\|v\|_2 = (\sum_{j=1}^p v_j^2)^{1/2}$. For any real symmetric matrix $A$, let $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ be maximum and minimum eigenvalue of $A$, respectively. We assume the following standard conditions for obtaining the asymptotic results:

**Condition (A1)** $\log n \lesssim \log p = o(n)$ and $m_n = O\left((n/\log p)^{-d'/2} \lor p\right)$, as $n \to \infty$.

**Condition (A2)** For some constant $C > 0$ and $0 \leq d < (1 + d)/2 \leq d' \leq 1$,

$$\max_{i,j} |x_{ij}| \leq C,$$

$$0 < \lambda \leq \min_{k:|k| \leq m_n} \lambda_{\min}\left(n^{-1}H_n(\beta_{0,k})\right) \leq \Lambda_{m_n} \leq C^2\left(\frac{n}{\log p} \lor \log p\right)^d,$$

and $\Lambda_\zeta = \max_{k:|k| \leq \zeta} \lambda_{\max}(n^{-1}X_k^T X_k)$ for any integer $\zeta > 0$. Furthermore, $\|\beta_{0,t}\|_2^2 = O((\log p)^d)$.

**Condition (A3)** For some constant $c_0 > 0$,

$$\min_{j \in t} \|\beta_{0,j}\|_2^2 \geq c_0|t|\Lambda_{|t|}\left(\frac{\log p}{n} \lor \frac{1}{\log p}\right).$$

**Condition (A4)** For some constants $\delta, a_1, a_2 > 0$, the hyperparameters satisfy

$$a_1 < \lambda_1 < a_2, \quad \lambda_{2}^{1/2} \sim n^{-1/2} p^{2+\delta} \quad \text{and} \quad a_1 < \lambda_{\min}(U) \leq \lambda_{\max}(U) < a_2.$$  

Condition (A1) ensures our proposed method can accommodate high dimensions where the number
of predictors grows at a sub-exponential rate of $n$. Condition (A1) also specifies the parameter $m_n$ in the uniform prior (5) that restricts our analysis on a set of *reasonably large* models. Similar assumptions restricting the model size have been commonly assumed in the sparse estimation literature \[\text{Liang et al., 2013, Narisetty et al., 2019, Shin et al., 2018, Lee et al., 2019}\].

Condition (A2) gives lower and upper bounds of $\lambda_{\min}(n^{-1}H_n(\beta_{0,k}))$ and $\lambda_{\max}(n^{-1}X_k^TX_k)$, respectively, where $k$ belongs to the set of reasonably large models. The lower bound condition can be seen as a restricted eigenvalue condition for $k$-sparse vectors and is satisfied with high probability for sub-Gaussian design matrices \[\text{Narisetty et al., 2019}\]. Similar conditions have been used in the linear regression literature \[\text{Ishwaran and Rao, 2005, Yang et al., 2016, Song and Liang, 2017}\].

Condition (A2) also allows the magnitude of true signals to increase to infinity but stay bounded above by $(\log p)^d$ up to some constant, while Condition (A3), the well-known beta-min condition, gives a lower bound for nonzero signals. In general, this type of condition is necessary to not neglect any small signals.

Condition (A4) suggests appropriate conditions for the hyperparameters in (3) and (4). Similar assumption has also been considered in \[\text{Shin et al. [2018], Johnson and Rossell [2012] and Cao et al. [2020]}\]. In particular, we extend the previous polynomial rate of the dimension in \[\text{Cao et al. [2020]}\] by considering a larger order of the hyperparameter $\lambda_2$.

### 3.1 Model Selection Consistency

**Theorem 3.1** (No super set). Under Conditions (A1), (A2) and (A4),

$$
\pi(k \supseteq t \mid y) \overset{P}{\to} 0, \quad \text{as } n \to \infty.
$$

Theorem 3.1 says that, asymptotically, our posterior does not overfit the model, i.e., it does not include unnecessarily many variables. Of course, the result does not guarantee that the posterior will concentrate on the true model. To capture every significant variable, we require the magnitudes of nonzero entries in $\beta_{0,t}$ not to be too small. Theorem 3.2 shows that with an appropriate lower bound specified in Condition (A3), the true model $t$ will be the mode of the posterior.

**Theorem 3.2** (Posterior ratio consistency). Under Conditions (A1)-(A4) with $c_0 = \{(1-\epsilon_0)\lambda\}^{-1}[2(3+...}}
\( \delta + 5 \{(1 - \epsilon_0)\lambda\}^{-1} \) for some small constant \( \epsilon_0 > 0 \),

\[
\max_{k \neq t} \frac{\pi(k \mid y)}{\pi(t \mid y)} \xrightarrow{P} 0, \quad \text{as } n \to \infty.
\]

Posterior ratio consistency is a useful property especially when we are interested in the point estimation with the posterior mode, but does not provide how large is the probability that the posterior puts on the true model. In the following theorem, we state that our posterior achieves strong selection consistency. By strong selection consistency, we mean that the posterior probability assigned to the true model \( t \) converges to 1, which requires a slightly stronger condition on the lower bound for the magnitudes of nonzero entries in \( \beta_{0,t} \) compared to that in Theorem 3.2.

**Theorem 3.3** (Strong selection consistency). Under Conditions (A1)–(A4) with \( \epsilon_0 = \{(1 - \epsilon_0)\lambda\}^{-1}[2(9 + 2\delta) + 5\{(1 - \epsilon_0)\lambda\}^{-1}] \) for some small constant \( \epsilon_0 > 0 \), the following holds:

\[
\pi(t \mid y) \xrightarrow{P} 1, \quad \text{as } n \to \infty.
\]

### 3.2 Comparison with Existing Work

We compare our results and assumptions with those of existing methods using nonlocal priors in generalized linear regression. Shi et al. [2019] established the posterior convergence rate for nonlocal priors under the assumption of \( p \log(1/\epsilon_n^2) \ll n\epsilon_n^2 \) for some \( \epsilon_n \in (0, 1] \) satisfying \( n\epsilon_n^2 \gg 1 \), which indicates that \( p \) can increase with the sample size but slower than \( n \). Wu et al. [2020] investigated the model selection performance of hyper-nonlocal priors that combine the Fisher information matrix with the pMOM density and established asymptotic properties under a fixed dimension of predictors. Both works considered the setting of low to moderate dimensions, while we allow \( p \) to grow at a sub-exponential rate of \( n \), the so-called “ultra high-dimensional” setting [Shin et al., 2018].
Bian and Wu [2017] considered the following hyper-piMOM priors for regression coefficients in GLMs and established the high-dimensional model selection consistency:

\[
\beta_k | \tilde{\tau}_k \sim \prod_{i=1}^{|k|} \left( \frac{\tau_i \sigma^2}{\Gamma(\frac{r}{2})} \right)^{\frac{1}{2}} \beta_k \left| - \left( \frac{r+1}{2} \right)^{-1} \right| \exp \left( - \frac{\tau_i \sigma^2}{\beta_k^2} \right),
\]

\[
\tau_i \overset{i.i.d.}{\sim} \text{Inverse-Gamma} \left( \frac{r+1}{2}, \psi \right), \quad \text{for } i = 1, \ldots, |k|,
\]

where \( \tilde{\tau}_k = (\tau_1, \tau_2, \ldots, \tau_{|k|})^T \). In particular, the authors put an independent piMOM prior on each linear regression coefficient (conditional on the hyperparameter \( \tau_i \)) and an Inverse-Gamma prior on \( \tau_i \).

There are some fundamental differences between Bian and Wu [2017] and our work in terms of the models considered and corresponding analysis. Firstly, unlike the piMOM prior, the pMOM prior in our model does not in general correspond to assigning an independent prior to each entry of \( \beta_k \). In particular, pMOM distributions introduce correlations among the entries in \( \beta_k \) through \( U_k \) and create more theoretical challenges. Furthermore, the pMOM prior imposes exact sparsity in \( \beta_k \), which is not the case for the piMOM prior in Bian and Wu [2017], thus they are structurally different. Secondly, Bian and Wu [2017] assumed the eigenvalues of the Hessian matrix to be bounded below and above by some constants, while we allow the upper bound to grow with \( n \) (Condition (A2)). In addition, to prove the model selection consistency, Bian and Wu [2017] required the spectral norm of the difference between the Hessian matrices corresponding to any two models to be bounded above by a function of the \( \ell_2 \)-norm difference between the respective regression coefficients, and they assumed that the product of the response variables and the entries of design matrix are bounded by a constant, while these constraints are not imposed in our study. See assumptions B1, B2 and C1 in Bian and Wu [2017] for details. Thirdly, no simulation studies were conducted in Bian and Wu [2017], leaving the empirical validity of the proposed method in question, while we include the computational strategy in the following section and examine the practical utility of the hyper-pMOM prior in the context of gene expression analysis.
4 Posterior Computation

In this section, we describe how to approximate the marginal density of data and conduct the model selection procedure. The integral formulation in (6) cannot be calculated in a closed form. Hence, we use Laplace approximation to compute \( m_k(y) \) and \( \pi(k \mid y) \). Similar approaches to compute posterior probabilities have been used in Johnson and Rossell [2012], Shi et al. [2019] and Shin et al. [2018].

For any model \( k \), when \( U_k = I_k \), the normalization constant \( d_k \) in (3) is given by

\[
\frac{(2r-1)!!}{|k|}. \]  

Let

\[
f(\beta_k) = \log \left( \exp \left\{ L_n(\beta_k) \right\} \pi(\beta_k \mid k) \right) = \sum_{i=1}^{n} \left\{ y_i x_{i,k}^\top \beta_k - \log (1 + \exp(x_{i,k}^\top \beta_k)) \right\} - |k| \log ((2r-1)!!) - \frac{|k|}{2} \log(2\pi) + \log \left( \frac{\lambda_1}{\Gamma(\lambda_1)} \right) - \left( r|k| + \frac{|k|}{2} + \frac{1}{2} \right) \log \left( \lambda_2 + \frac{1}{2} \|\beta_k\|_2^2 \right) + 2r \sum_{i=1}^{|k|} \log (|\beta_{k,i}|) + \log \Gamma \left( r|k| + \frac{|k|}{2} + \lambda_1 \right). \]

For any model \( k \), the Laplace approximation of \( m_k(y) \) is given by

\[
(2\pi)^{|k|/2} \exp \left\{ f(\hat{\beta}_k) \right\} |V(\hat{\beta}_k)|^{-1/2}, \tag{9} \]

where \( \hat{\beta}_k = \arg \max_{\beta_k} f(\beta_k) \) is obtained via the optimization function \texttt{optim} in R using a quasi-Newton method, and \( V(\beta_k) \) is a \(|k| \times |k|\) symmetric matrix defined as

\[
V(\beta_k) = -\sum_{i=1}^{n} \frac{x_{i,k} x_{i,k}^\top \exp(x_{i,k}^\top \beta_k)}{\left\{ 1 + \exp(x_{i,k}^\top \beta_k) \right\}^2} - \text{diag} \left( \frac{2r}{\beta_{k,1}^2}, \ldots, \frac{2r}{\beta_{k,|k|}^2} \right) - \left( r|k| + \frac{|k|}{2} + \frac{1}{2} \right) \left\{ \frac{1}{\lambda_2 + \|\beta_k\|_2^2 / 2} I_{|k|} - \frac{1}{(\lambda_2 + \|\beta_k\|_2^2 / 2)^2} \beta_k \beta_k^\top \right\}. \]

The above Laplace approximation can be used to compute the posterior probability ratio between two models.

The shotgun stochastic search (SSS) algorithm [Hans et al., 2007, Shin et al., 2018] is inspired by MCMC but enables much more efficient identification of probable models by swiftly moving around in the model space as the dimension escalates. The SSS algorithm explores high-dimensional model
Algorithm 1 Shotgun Stochastic Search (SSS)

Set an initial model $k^{(1)}$

for $i = 1$ to $i = N - 1$ do

(a) Compute $\pi(k | y)$ using (9) for all $k \in \text{nbd}(k^{(i)})$

(b) Sample $k^+, k^-$ and $k^0$ from $\Gamma^+_k$, $\Gamma^-_k$ and $\Gamma^0_k$ with probabilities proportional to $\pi(k | y)$

(c) Sample the next model $k^{(i+1)}$ from $\{k^+, k^-, k^0\}$ with probability proportional to $\{\pi(k^+ | y), \pi(k^- | y), \pi(k^0 | y)\}$

end for

spaces and quickly identifies “interesting” regions of high posterior probability over models. The SSS evaluates numerous models guided by the unnormalized posterior probabilities that can be approximated using the Laplace approximations of the marginal probabilities in (9). Let $\text{nbd}(k) = \{\Gamma^+_k, \Gamma^-_k, \Gamma^0_k\}$ containing all the neighbors of model $k$, in which $\Gamma^+_k = \{k \cup \{j\} : j \notin k\}$, $\Gamma^-_k = \{k \setminus \{j\} : j \in k\}$ and $\Gamma^0_k = \{k \setminus \{j\} \cup \{l\} : j \in k, l \notin k\}$. Algorithm 1 describes the SSS procedure.

However, as pointed out by Shin et al. [2018], the SSS algorithm can be computationally expensive in high-dimensional settings. The computational bottleneck is calculating the Laplace approximations of the marginal probabilities for the models in $\Gamma^+_k$ and $\Gamma^0_k$, whose cardinalities are $p - |k|$ and $(p - |k|)|k|$, respectively. To alleviate computational burden, we slightly modify the SSS algorithm by reducing the number of entries in $\Gamma^+_k$ and $\Gamma^0_k$. Specifically, we reduce the number of models in $\Gamma^+_k$ by selecting only (1) the top $K_1$ variables having large absolute sample correlation with $y$ and (2) $K_2$ randomly selected variables, and we define the resulting set as $\Gamma^+_{R,k}$. Similarly, we define a reduced set $\Gamma^0_{R,k} = \{k \setminus \{j\} \cup \{l\} : j \in k, l \in \Gamma^+_{R,k}\}$ and replace $\Gamma^0_k$ with $\Gamma^0_{R,k}$ in the algorithm. By doing so, we can efficiently reduce the computational complexity of the algorithm. Note that the cardinalities of $\Gamma^+_{R,k}$ and $\Gamma^0_{R,k}$ are $K_1 + K_2$ and $(K_1 + K_2)|k|$, respectively. We call this modified algorithm the reduced SSS (RSSS) algorithm and describe it in Algorithm 2. Note that the RSSS algorithm is different from the simplified shotgun stochastic search with screening (S5) algorithm [Shin et al., 2018]. The two main differences are that the RSSS algorithm does not completely ignore the set $\Gamma^0_k$ and does not introduce temperature parameters. In the subsequent simulation study and real data analysis, the RSSS algorithm with $K_1 = K_2 = 10$ is adopted for the posterior inference of the hyper-pMOM prior.

To demonstrate the computational efficiency of the RSSS algorithm and compare it with the SSS algorithm, we conduct a simulation study. We generate the data from the model (2) with


**Algorithm 2 Reduced Shotgun Stochastic Search (RSSS)**

Set an initial model $k^{(1)}$

\[ \text{for } i = 1 \text{ to } i = N - 1 \text{ do} \]

(a) Compute $\pi(k | y)$ using (9) for all $k \in \text{nbd}_R(k^{(i)}) = \{ \Gamma^+_{R,k}, \Gamma^-_{R,k}, \Gamma^0_{R,k} \}$

(b) Sample $k^+, k^-$ and $k^0$ from $\Gamma^+_{R,k}, \Gamma^-_{R,k}$ and $\Gamma^0_{R,k}$ with probabilities proportional to $\pi(k | y)$

(c) Sample the next model $k^{(i+1)}$ from $\{ k^+, k^-, k^0 \}$ with probability proportional to $\{ \pi(k^+ | y), \pi(k^- | y), \pi(k^0 | y) \}$

\[ \text{end for} \]

---

Figure 1: The average number of models searched before visiting the posterior mode.

the true coefficient $\beta_0 = (1, 1, 1, 0, \ldots, 0)^\top \in \mathbb{R}^p$ and design matrix $X = (x_1, \ldots, x_n)^\top \in \mathbb{R}^{n \times p}$, where $x_i \overset{i.i.d.}{\sim} \mathcal{N}_{p}(0, I_p)$ for $i = 1, \ldots, n$. The number of samples is fixed at $n = 100$, while the number of variables varies over $p \in \{100, \ldots, 500\}$. Figure 1 shows the average number of models searched before visiting the posterior mode for each $p$, where the averages are calculated based on 10 repetitions. When compared with the SSS algorithm, the RSSS algorithm investigates a far less number of models before hitting the posterior mode, while both algorithms found the same posterior mode for all data sets in our simulations. Therefore, the RSSS algorithm can achieve nearly identical performance to the SSS algorithm while boosting computing efficiency.

---

5 Simulation Studies

In this section, we investigate the performance of the hyper-pMOM prior for logistic regression models. For given $n = 100$ and $p \in \{100, 300\}$, simulated data sets are generated from (2) with
the true coefficient vector $\beta_0$ and design matrix $X = (x_1, \ldots, x_n)^\top \in \mathbb{R}^{n \times p}$. We set the index set for nonzero values in $\beta_0$ at $t = \{1, 2, 3\}$, where nonzero coefficients $\beta_{0,t}$ are generated under the following two different settings:

- Setting 1 (Weak signals): All the entries of $\beta_{0,t}$ are set to 1.
- Setting 2 (Moderate signals): All the entries of $\beta_{0,t}$ are set to 2.

We generate covariate vectors as $x_i \sim_{i.i.d.} N_p(0, \Sigma)$ for $i = 1, \ldots, n$, under the following cases of $\Sigma$:

- Case 1 (Isotropic design): $\Sigma = I_p$
- Case 2 (Correlated design): $\Sigma = (\Sigma_{ij})$, where $\Sigma_{ij} = 0.3^{|i-j|}$ for any $1 \leq i \leq j \leq p$.

We also generate test samples \{(y_{test,1}, x_{test,1}), \ldots, (y_{test,n_{test}}, x_{test,n_{test}})\} with $n_{test} = 50$ to evaluate the prediction performance.

In the various scenarios mentioned above, we compare the performance of our method with existing variable selection methods. As Bayesian contenders, we consider the nonlocal pMOM prior [Cao and Lee, 2020], spike and slab prior [Tüchler, 2008] and empirical Bayesian Lasso (EBLasso) [Cai et al., 2011], while we consider Lasso [Friedman et al., 2010] and and SCAD [Breheny and Huang, 2011] as frequentist competitors.

The R codes for implementing the hyper-pMOM prior are publicly available at https://github.com/leekjstat/Hierarchical-nonlocal. The hyperparameters in (3) and (4) are set at $U = I_p$, $r = 1$, $\lambda_1 = 1$ and $\lambda_2 = 10^2 n^{-1/3} p (2 + 0.001)^2 / 3$, which satisfy Condition (A4). For the implementation of the pMOM prior, the R codes available at https://github.com/xuan-cao/Nonlocal-Logistic-Selection are used, where the hyperparameters are set at $U = I_p$, $r = 1$ and $\tau = n^{-1/2} p^{2+0.05}$. For both RSSS and SSS procedures, initial models are set by randomly taking three nonzero entries. For the regularization approaches, the tuning parameters are chosen by 5-fold cross-validation.

To examine the performance of each method, the values of the precision, sensitivity, specificity, Matthews correlation coefficient (MCC) [Matthews, 1975] and mean squared prediction error (MSPE) are used. These criteria are defined as

$$\text{Precision} = \frac{TP}{TP + FP}, \quad \text{Sensitivity} = \frac{TP}{TP + FN}, \quad \text{Specificity} = \frac{TN}{TN + FP},$$

$$\text{MCC} = \frac{TP \times TN - FP \times FN}{\sqrt{(TP + FP)(TP + FN)(TN + FP)(TN + FN)}}, \quad \text{MSPE} = \frac{1}{n_{test}} \sum_{i=1}^{n_{test}} (\hat{y}_i - y_{test,i})^2,$$
Table 1: The summary statistics for Case 1 (isotropic design) when $p = 100$.

| Setting 1         | Hyper-pMOM | pMOM | Spike and slab | EBLasso | Lasso | SCAD |
|-------------------|------------|------|----------------|---------|-------|------|
| Precision         | 1.000      | 1.000| 0.917          | 0.500   | 0.364 | 0.389|
| Sensitivity       | 0.667      | 0.667| 0.700          | 0.733   | 0.900 | 0.915|
| Specificity       | 1.000      | 1.000| 0.997          | 0.979   | 0.889 | 0.915|
| MCC               | 0.812      | 0.812| 0.744          | 0.618   | 0.438 | 0.470|
| MSPE              | 0.210      | 0.206| 0.227          | 0.256   | 0.210 | 0.208|

| Setting 2         | Hyper-pMOM | pMOM | Spike and slab | EBLasso | Lasso | SCAD |
|-------------------|------------|------|----------------|---------|-------|------|
| Precision         | 1.000      | 0.967| 0.933          | 0.832   | 0.241 | 0.337|
| Sensitivity       | 0.967      | 1.000| 0.994          | 0.990   | 0.877 | 0.929|
| Specificity       | 1.000      | 0.994| 0.996          | 0.953   | 0.877 | 0.929|
| MCC               | 0.981      | 0.956| 0.934          | 0.953   | 0.543 | 0.533|
| MSPE              | 0.121      | 0.132| 0.124          | 0.227   | 0.147 | 0.125|

Table 2: The summary statistics for Case 2 (correlated design) when $p = 100$.

| Setting 1         | Hyper-pMOM | pMOM | Spike and slab | EBLasso | Lasso | SCAD |
|-------------------|------------|------|----------------|---------|-------|------|
| Precision         | 1.000      | 0.667| 0.927          | 0.753   | 0.392 | 0.452|
| Sensitivity       | 0.667      | 1.000| 0.867          | 0.800   | 0.920 | 0.938|
| Specificity       | 1.000      | 0.997| 0.997          | 0.986   | 0.920 | 0.938|
| MCC               | 0.812      | 0.956| 0.997          | 0.749   | 0.566 | 0.619|
| MSPE              | 0.177      | 0.132| 0.182          | 0.235   | 0.172 | 0.167|

| Setting 2         | Hyper-pMOM | pMOM | Spike and slab | EBLasso | Lasso | SCAD |
|-------------------|------------|------|----------------|---------|-------|------|
| Precision         | 0.975      | 0.800| 0.833          | 0.753   | 0.452 | 0.307|
| Sensitivity       | 0.800      | 0.999| 0.998          | 0.992   | 0.938 | 0.921|
| Specificity       | 0.999      | 0.999| 0.998          | 0.992   | 0.921 | 0.921|
| MCC               | 0.874      | 0.878| 0.886          | 0.886   | 0.433 | 0.528|
| MSPE              | 0.134      | 0.135| 0.116          | 0.208   | 0.125 | 0.113|

where $TP$, $TN$, $FP$ and $FN$ are true positive, true negative, false positive and false negative, respectively. Here, $\hat{y}_i = \exp(x_{test,i}^\top \hat{\beta})/\{1 + \exp(x_{test,i}^\top \hat{\beta})\}$, where $\hat{\beta}$ is the estimated coefficient vector.

For the hyper-pMOM and pMOM priors, the nonzero part in $\hat{\beta}$ is chosen as the posterior mode with the estimated model $\hat{k}$, i.e., $\hat{\beta}_k = \arg \max_{\hat{k}} f(\beta_\hat{k})$. For the spike and slab prior, the posterior mean based on 2,000 posterior samples is used as $\hat{\beta}$. The averages of each criterion based on 10 repetitions are summarized in Tables 1–4.

Based on the simulation results, the proposed hyper-pMOM prior tends to achieve high precision and specificity in most settings, which means that the hyper-pMOM prior produces low false
Table 3: The summary statistics for Case 1 (isotropic design) when $p = 300$.

| Setting | Method     | Precision | Sensitivity | Specificity | MCC     | MSPE    |
|---------|------------|-----------|-------------|-------------|---------|---------|
| 1       | Hyper-pMOM | 0.900     | 0.600       | 0.999       | 0.733   | 0.227   |
|         | pMOM       | 0.850     | 0.567       | 0.999       | 0.692   | 0.232   |
|         | Spike and slab | 0.925 | 0.567 | 0.999 | 0.661 | 0.238 |
|         | EBLasso    | 0.660     | 0.533       | 0.996       | 0.563   | 0.252   |
|         | Lasso      | 0.191     | 0.967       | 0.949       | 0.410   | 0.215   |
|         | SCAD       | 0.199     | 0.967       | 0.953       | 0.420   | 0.219   |

Table 4: The summary statistics for Case 2 (correlated design) when $p = 300$.

| Setting | Method     | Precision | Sensitivity | Specificity | MCC     | MSPE    |
|---------|------------|-----------|-------------|-------------|---------|---------|
| 1       | Hyper-pMOM | 1.000     | 0.800       | 1.000       | 0.889   | 0.164   |
|         | pMOM       | 0.943     | 0.800       | 0.999       | 0.854   | 0.166   |
|         | Spike and slab | 0.875 | 0.967 | 0.998 | 0.917 | 0.150 |
|         | EBLasso    | 0.925     | 0.900       | 0.999       | 0.901   | 0.224   |
|         | Lasso      | 0.184     | 1.000       | 0.941       | 0.406   | 0.153   |
|         | SCAD       | 0.198     | 1.000       | 0.956       | 0.433   | 0.139   |

| Setting | Method     | Precision | Sensitivity | Specificity | MCC     | MSPE    |
|---------|------------|-----------|-------------|-------------|---------|---------|
| 2       | Hyper-pMOM | 1.000     | 0.800       | 1.000       | 0.812   | 0.177   |
|         | pMOM       | 1.000     | 0.667       | 1.000       | 0.812   | 0.177   |
|         | Spike and slab | 0.927 | 0.867 | 0.997 | 0.881 | 0.182 |
|         | EBLasso    | 0.753     | 0.800       | 0.986       | 0.749   | 0.235   |
|         | Lasso      | 0.392     | 0.967       | 0.920       | 0.566   | 0.172   |
|         | SCAD       | 0.452     | 0.967       | 0.938       | 0.619   | 0.167   |

positives. The hyper-pMOM prior also tends to have high MCC especially under the isotropic design (Case 1), and even under the correlated design (Case 2), overall higher MCC than the EBLasso and frequentist methods. Furthermore, the hyper-pMOM prior outperforms the pMOM prior in the majority of situations, demonstrating its relative superiority to the pMOM prior. Overall, the Bayesian methods achieve high precision, specificity and MCC, while the frequentist methods have high sensitivity and MSPE. Similar observations have also been discussed in the literature [Meinshausen and Bühlmann, 2006, Cao and Lee, 2020].
6 Application to the Analysis of Differentially Expressed Genes

Asthma has been recognized as a systemic disease consisting of networks of genes showing inflammatory changes involving a broad spectrum of adaptive and innate immune systems. Utilizing measurable characteristics of asthmatic patients, including biologic gene expression markers, can help to identify phenotypic categories in asthma. Identification of these phenotypes may help develop strategies for preventing progression of disease severity [Carr and Bleecker, 2016]. We aim to apply the proposed variable selection method to develop an RNA-seq-based risk score for asthma stratification.

To construct the risk score, gene expression analysis is performed using an asthma RNA-seq dataset GSE146046 in the Gene Expression Omnibus (GEO) database [Seumois et al., 2020]. There are 95 individuals in the GSE146046 dataset including 51 asthmatic subjects and 44 non-asthmatic subjects. The gene expression levels of all the 95 individuals are first randomly split into 2/3 as training and 1/3 as test data while maintaining the same ratio between asthma and control groups. Next we conduct the analysis of differentially expressed genes (DEG) based on the training set and construct data tables containing raw count values for approximately 20,000 unique genes, with genes in rows and sample GEO accession numbers in columns. DESeq2 R package is used to store the read counts and the intermediate estimated quantities during statistical analysis [Love et al., 2014]. We extract summary statistics including \( p \)-values for all genes and retain a total of 180 DEGs with \( p \)-values less than \( 10^{-4} \) visualized in a Manhattan plot (Figure 2). The proposed method and other contenders are applied to the resulting dataset with \( p = 180 \). The hyperparameters for all the methods are set as in the simulation studies.

In Figure 3, we draw the receiver operating characteristic (ROC) curves for all the methods. The results are further summarized in Table 5 where a common cutoff value 0.5 is adopted for thresholding prediction. From Table 5 and Figure 3, we can tell that the hyper-pMOM prior has overall better prediction performance compared with other methods. Of the 180 genes, four genes, namely, TRIM26, CAPSL, FOXA3 and PYY, are selected by the proposed method. These identified genes seem plausible and have been established in the asthma GWAS catalog [Schoettler et al., 2019], which may help better understand the omics architecture that drives complex diseases.
Figure 2: Manhattan plot for association between RNA-seq and asthma. The points beyond the blue line represents the 180 selected genes with $p < 10^{-4}$ based on the DEG analysis.

| Method         | Precision | Sensitivity | Specificity | MCC  | MSPE |
|----------------|-----------|-------------|-------------|------|------|
| Hyper-pMOM     | 0.750     | 0.643       | 0.824       | 0.477| 0.225|
| pMOM           | 0.565     | 0.929       | 0.412       | 0.387| 0.355|
| Spike and slab | 0.571     | 0.857       | 0.471       | 0.349| 0.233|
| EBLasso        | 0.583     | 0.500       | 0.706       | 0.210| 0.238|
| Lasso          | 0.545     | 0.857       | 0.412       | 0.295| 0.233|
| SCAD           | 0.565     | 0.929       | 0.412       | 0.387| 0.224|

7 Discussion

In this paper, we consider the hyper-pMOM prior and investigate asymptotic properties of the resulting posterior distribution. Although the hyper-pMOM prior possesses thicker tails than the pMOM prior by adopting the hyperprior (4) rather than a fixed $\tau$, it still has the hyperparameters, $\lambda_1$ and $\lambda_2$. Because the choice of hyperparameters could affect variable selection performance, a cross-validation-based selection approach for $\lambda_1$ and $\lambda_2$ will be worth exploring. In this case, studying the theoretical properties of the posterior based on the hyperparameters chosen by the cross-validation procedure will be a challenging but important task.

Furthermore, as mentioned in Section 3, deriving strong model selection consistency in a broader class of GLMs is an interesting future research direction. Note that, in this work, we focus on
logistic regression models when proving strong model selection consistency of the posterior. An extension to general GLMs might require more conditions on the design matrix based on the current techniques used in the proof, due to more complicated structure of the Hessian matrix for other GLMs compared with that for the logistic regression model.
Supplementary Material

Throughout the Supplementary Material, we assume that for any

\[ u \in \{ u \in \mathbb{R}^n : u \text{ is in the space spanned by the columns of } \Sigma^{1/2}X_k \} \]

and any model \( k \in \{ k \subseteq [r] : |k| \leq m_n + |t| \} \), there exists \( \delta^* > 0 \) such that

\[
\mathbb{E} \left[ \exp \left\{ u^\top \Sigma^{-1/2}(y - \mu) \right\} \right] \leq \exp \left\{ \frac{(1 + \delta^*)u^\top u}{2} \right\}, \tag{10}
\]

for any \( n \geq N(\delta^*) \). However, as stated in Narisetty et al. [2019], there always exists \( \delta^* > 0 \) satisfying inequality (10), so it is not really a restriction. Since we will focus on sufficiently large \( n \), \( \delta^* \) can be considered an arbitrarily small constant, so we can always assume that \( \delta > \delta^* \).

Proof of Theorem 3.1. Let \( M_1 = \{ k : k \supseteq t, |k| \leq m_n \} \) and

\[
PR(k, t) = \frac{\pi(k | y)}{\pi(t | y)},
\]

where \( t \subseteq [r] \) is the true model. We will show that

\[
\sum_{k : k \in M_1} PR(k, t) \xrightarrow{P} 0 \quad \text{as } n \to \infty. \tag{11}
\]

By Taylor’s expansion of \( L_n(\beta_k) \) around \( \hat{\beta}_k \), which is the MLE of \( \beta_k \) under the model \( k \), we have

\[
L_n(\beta_k) = L_n(\hat{\beta}_k) - \frac{1}{2}(\beta_k - \hat{\beta}_k)^\top H_n(\hat{\beta}_k)(\beta_k - \hat{\beta}_k)
\]

for some \( \hat{\beta}_k \) such that \( \| \beta_k - \hat{\beta}_k \|_2 \leq \| \beta_k - \hat{\beta}_k \|_2 \). Furthermore, by Lemmas A.1 and A.3 in Narisetty et al. [2019] and Condition (A2), with probability tending to 1,

\[
-\frac{1 + \epsilon}{2} (\beta_k - \hat{\beta}_k)^\top H_n(\beta_{0,k})(\beta_k - \hat{\beta}_k) \leq L_n(\beta_k) - L_n(\hat{\beta}_k) \leq -\frac{1 + \epsilon}{2} (\beta_k - \hat{\beta}_k)^\top H_n(\beta_{0,k})(\beta_k - \hat{\beta}_k)
\]

for any \( k \in M_1 \) and \( \beta_k \) such that \( \| \beta_k - \beta_{0,k} \|_2 < c\sqrt{|k|L_k} \log p/n =: cw_n \), where \( \epsilon = \epsilon_n := \)
\(c' \sqrt{m^2 \Lambda_m \log p/n} = o(1)\), for some constants \(c, c' > 0\). Note that for \(\beta_k\) such that \(\|\beta_k - \hat{\beta}_k\|_2 = cw_n/2\),

\[
L_n(\beta_k) - L_n(\hat{\beta}_k) \leq -\frac{1}{2} \frac{c}{\epsilon \beta^2 w_n^2} n \lambda = -\frac{1}{8} c^2 \lambda \|k| \Lambda|k| \log p \to -\infty \quad \text{as } n \to \infty,
\]

where the second inequality holds due to Condition (A2). It also holds for any \(\beta_k\) such that \(\|\beta_k - \hat{\beta}_k\|_2 > cw_n/2\) by concavity of \(L_n(\cdot)\) and the fact that \(\hat{\beta}_k\) maximizes \(L_n(\beta_k)\).

Define the set \(B := \{\beta_k : \|\beta_k - \hat{\beta}_k\|_2 \leq cw_n/2\}\), then we have \(B \subset \{\beta_k : \|\beta_k - \beta_{0,k}\|_2 \leq cw_n\}\) for some large \(c > 0\) and any \(k \in M_1\), with probability tending to 1.

\[
m_k(y) = \int \int \exp \{L_n(\beta_k)\} \pi(\beta_k \mid k) \, d\beta_k d\tau \leq \int \int \exp \{L_n(\beta_k)\} d_k(2\pi)^{-|k|/2} \tau^{-r\|k|-|k|/2} |U_k|^{1/2} \exp \left( -\frac{\beta_k^\top U_k \beta_k}{2\tau} \right) \prod_{i=1}^{|k|} \beta_{k_i}^{2r} \\
\times \frac{\lambda^2}{\Gamma(\lambda_1)} \tau^{-\lambda_1 - 1} \exp \left( -\frac{\lambda_2}{\tau} \right) \, d\beta_k d\tau \leq \frac{\lambda^2}{\Gamma(\lambda_1)} d_k(2\pi)^{-|k|/2} |U_k|^{1/2} \exp \{L_n(\hat{\beta}_k)\} \times \left\{ \int \int B \, d\beta_k d\tau + \exp \left( -\frac{1}{8} c^2 \lambda \|k| \Lambda|k| \log p \right) \right\} \int \int M_2 \, d\beta_k d\tau \right\},
\]

where

\[
M_1 = \exp \left( -\frac{\lambda_2}{\tau} \right) \exp \left\{ -\frac{1}{2} (\beta_k - \hat{\beta}_k)^\top H_n(\beta_{0,k})(\beta_k - \hat{\beta}_k) - \frac{\beta_k^\top U_k \beta_k}{2\tau} \right\} \prod_{i=1}^{|k|} \beta_{k_i}^{2r}
\]

and

\[
M_2 = \tau^{-r\|k|-|k|/2} \lambda_1^{-1} \exp \left( -\frac{\lambda_2}{\tau} \right) \exp \left( -\frac{\beta_k^\top U_k \beta_k}{2\tau} \right) \prod_{i=1}^{|k|} \beta_{k_i}^{2r}.
\]
Note that for $A_k = (1 - \epsilon)H_n(\beta_{0,k})$ and $\beta_k^* = (A_k + U_k/\tau)^{-1}A_k\hat{\beta}_k$, we have

$$\int \int_B M_1 \, d\beta_k \, d\tau \leq \int \int \tau^{-r|k|-|k|/2-\lambda_1-1} \exp \left( -\frac{\lambda_2}{\tau} \right) \exp \left\{ -\frac{1}{2} (\beta_k - \beta_k^*)^\top (A_k + U_k/\tau)(\beta_k - \beta_k^*) \right\} \prod_{i=1}^{\frac{|k|}{2}} \beta_i^{2r} \times \exp \left\{ -\frac{1}{2} \beta_k^\top (A_k - A_k(A_k + U_k/\tau)^{-1}A_k) \beta_k \right\} \, d\beta_k \, d\tau$$

$$= \int (2\pi)^{|k|/2} \det (A_k + U_k/\tau)^{-1/2} \tau^{-r|k|-|k|/2-\lambda_1-1} \exp \left( -\frac{\lambda_2}{\tau} \right) E_k \left( \prod_{i=1}^{\frac{|k|}{2}} \beta_i^{2r} \right) \times \exp \left\{ -\frac{1}{2} \beta_k^\top (A_k - A_k(A_k + U_k/\tau)^{-1}A_k) \beta_k \right\} \, d\tau,$$

where $E_k(.)$ denotes the expectation with respect to a multivariate normal distribution with mean $\beta_k^*$ and covariance matrix $A_k + U_k/\tau$. It follows from Lemma 6 in the supplementary material for Johnson and Rossell [2012] that

$$E_k \left( \prod_{i=1}^{\frac{|k|}{2}} \beta_i^{2r} \right) \leq \left( \frac{nA_l|k| + \tau^{-1}a_2}{n\lambda + \tau^{-1}a_1} \right)^{|k|/2} \left\{ \frac{4V}{|k|} + \frac{4[(2r-1)!!]^2}{n(\lambda + \tau^{-2})} \right\}^{r|k|} \leq \left( \frac{nA_l|k|}{n\lambda + \tau^{-1}a_1} \right)^{|k|/2} 2^{r|k|-1} \left\{ \left( \frac{4V}{|k|} \right)^{r|k|} + \left( \frac{4[(2r-1)!!]^2}{n(\lambda + \tau^{-1}a_1)} \right)^{r|k|} \right\},$$

where $V = \|\beta_k^*\|_2^2$ and the last inequality follows from $1 + x \leq \exp(x)$. Next, note that it follows from Lemma A.3 in the supplemental material for Narisetty et al., [2019] that

$$V = \|\beta_k^*\|_2^2 \leq \|\hat{\beta}_k\|_2^2 \leq \left( \|\hat{\beta}_k - \beta_{0,k}\|_2 + \|\beta_{0,k}\|_2 \right)^2 \leq \left( \frac{\sqrt{|k|A_l|k| \log p}}{n} + \sqrt{\log p} \right)^2 \leq 2 \left( \frac{\sqrt{|k|A_l|k| \log p}}{n} + \log p \right).$$

Therefore,

$$\left( \frac{V}{|k|} \right)^{r|k|} \leq \left( 2 \frac{\log p}{|k|} + o(1) \right)^{r|k|}.$$
Therefore, by noting that \( \exp \left\{ -1/2 \beta_k^T (A_k - A_k (A_k + U_k/\tau)^{-1} A_k) \beta_k \right\} \geq 1 \), it follows from (12) that, for some constant \( C_1 > 0 \),

\[
\int \int_B M_1 \, d\beta_k \, d\tau \leq (C_1 \Lambda_k)^{|k|/2} \left( \frac{\log p}{|k|} \right)^{r|k|} \times \int \det (A_k + U_k/\tau)^{-1/2} \tau^{-r|k|/2 - \lambda_1 - 1} \exp \left\{ - \frac{\lambda_2 - a_2 |k|/(2n\Lambda_k)}{\tau} \right\} d\tau \\
\leq \left( \frac{C_1 \Lambda_k}{n} \right)^{|k|/2} \left( \frac{\log p}{|k|} \right)^{r|k|} \det (A_k)^{-1/2} \times \int \tau^{-r|k|/2 - \lambda_1 - 1} \exp \left\{ - \frac{\lambda_2 - a_2 |k|/(2n\Lambda_k)}{\tau} \right\} d\tau \\
\leq \left( \frac{C_1 \Lambda_k}{n} \right)^{|k|/2} \left( \frac{\log p}{|k|} \right)^{r|k|} \det (n^{-1} A_k)^{-1/2} \times \{ \lambda_2 - a_2 |k|/(2n\Lambda_k) \}^{-(r|k|+|k|/2+\lambda_1)} \Gamma(r|k| + |k|/2 + \lambda_1). \quad (13)
\]

Next, note that

\[
\int \int_{B^c} M_2 \, d\beta_k \, d\tau \leq \int \int \tau^{-r|k|/2 - \lambda_1 - 1} \exp \left( - \frac{\lambda_2}{\tau} \right) \exp \left( - \frac{\beta_k^T U_k \beta_k}{2\tau} \right) \prod_{i=1}^{|k|} \beta_i^{2r_i} d\beta_k \, d\tau \\
\leq \int \tau^{-r|k|/2 - \lambda_1 - 1} \exp \left( - \frac{\lambda_2}{\tau} \right) \left\{ 2\pi (2r - 1)^2 / a_1 \right\}^{|k|/2} \left( \tau/a_1 \right)^{r|k|} d\tau \\
\leq (C_2)^{|k|} \lambda_2^{-\lambda_1} \Gamma(\lambda_1).
\]

Combining with (12) and using the Stirling approximation for the gamma function, we obtain the following upper bound for \( m_k(y_n) \),

\[
m_k(y) \leq C_3^{[k]} \exp \left\{ L_n(\beta_k) \right\} n^{-|k|/2} \det (n^{-1} A_k)^{-1/2} \lambda_2^{(r|k|+|k|/2)} \Lambda_k^{1/2} \left( [k]^{1/r} \log p \right)^{r|k|} \\
+ C_3^{[k]} \exp \left\{ L_n(\beta_k) \right\} \exp \left( - \frac{1 - \epsilon}{8} c^2 \lambda |k| \log p \right) \\
\leq C_3^{[k]} \exp \left\{ L_n(\beta_k) \right\} n^{-|k|/2} \det (n^{-1} A_k)^{-1/2} \lambda_2^{(r|k|+|k|/2)} \Lambda_k^{1/2} \left( [k]^{1/r} \log p \right)^{r|k|}, \quad (14)
\]

for any \( k \in M_1 \) and some constant \( C_3 > 0 \), by noting that

\[
\det (A_k)^{1/2} \lambda_2^{r|k|+|k|/2} \Lambda_k^{-|k|/2} \left( [k]^{1/r} \log p \right)^{-r|k|} \ll \exp \left\{ \frac{1 - \epsilon}{8} c^2 \lambda |k| \log p \right\}
\]

by Conditions (A1), (A2) and (A4).
Similarly, by Lemma 4 in the supplemental material for Johnson and Rossell [2012] and the similar arguments leading up to (13), with probability tending to 1, we have, for some constant $C_4 > 0$, the marginal conditional density $m_t(y)$ will be bounded below by,

$$m_t(y) = \int \int \exp \{L_n(\beta_t)\} d\tau|U_t|^\frac{1}{2} \exp \left(-\frac{\beta_t^T U_t \beta_t}{2}\right) \prod_{i=1}^{\|t\|^2} \beta^2_{i, t}$$

$$\times \frac{\lambda_2^\lambda_1}{\Gamma(\lambda_1)} \tau^{-\lambda_1-1} \exp \left(-\frac{\lambda_2}{\tau}\right) d\beta_i d\tau$$

$$\geq \frac{\lambda_2^\lambda_1}{\Gamma(\lambda_1)} d\tau|U_t|^\frac{1}{2} \exp \{L_n(\beta_t)\}$$

$$\times \int \int \tau^{-r|t|/2-\lambda_1-1} \exp \left(-\frac{\lambda_2}{\tau}\right) \exp \left\{-\frac{1}{2} (\beta_t - \beta^*_{t,i})^T (A'_{i} + U_t/\tau) (\beta_t - \beta^*_{t,i})\right\} \prod_{i=1}^{\|t\|^2} \beta^2_{i, t}$$

$$\geq \frac{\lambda_2^\lambda_1}{\Gamma(\lambda_1)} d\tau|U_t|^\frac{1}{2} \exp \{L_n(\beta_t)\} \exp \left(-\frac{1}{2} \beta^*_{t,i} (A'_{i} - A'_{i} + U_t/\tau)^{-1} A'_{i} \beta^*_{t,i}\right)$$

$$\times \frac{\lambda_2^\lambda_1}{\Gamma(\lambda_1)} \tau^{-\lambda_1-1} \exp \left(-\frac{\lambda_2}{\tau}\right) \prod_{i=1}^{\|t\|^2} \beta^2_{i, t}$$

$$\geq \frac{\lambda_2^\lambda_1}{\Gamma(\lambda_1)} d\tau|U_t|^\frac{1}{2} \exp \{L_n(\beta_t)\} (\log p)^{-2|t|/2} \det(A'_{i})^{-1/2}$$

$$\times \int \tau^{-r|t|/2-\lambda_1-1} \exp \left(-\frac{\lambda_2}{\tau}\right) \exp \left\{-\frac{|t|}{2(1+\epsilon)\lambda}\right\} d\tau$$

$$\geq C_4 \exp \{L_n(\beta_t)\} n^{-|t|/2} \det(n^{-1} A'_{i})^{-1/2} \lambda_2^{-(r|t|+|t|/2-1)} (\log p)^{-2d-2r|t|},$$

by Lemma 7.1, where $A'_{i} = (1 + \epsilon) H_n(\beta_{0,i})$. Therefore, with probability tending to 1, combining with (14), for some constant $C^* > 0$,

$$\frac{m_k(y)}{m_t(y)} \leq \left\{ C^* n^{1/2} \right\}^{-r+1/2} \lambda^{r+1/2} \exp \{L_n(\beta_k) - L_n(\beta_t)\} \frac{\det(n^{-1} A'_{i})^{-1/2}}{\det(n^{-1} A_k)^{1/2}}$$

$$\times \lambda_k^{\frac{|k|/2}{2d} (\log p)^r |k| (\log p)^{2d+2r|t|} \lambda_2^{-1}}$$

$$\leq (C^*)^{-(2+\delta/2)(|k|-|t|)} p^{(1+\delta^*)(1+2w)(|k|-|t|)} ,$$

(15)

for any $k \in M_1$, where the second inequality holds by Lemma 7.2 in [Lee and Cao, 2020] and by noting that it follows from Lemma 7.3 in [Lee and Cao, 2020],

$$L_n(\beta_k) - L_n(\beta_t) \leq b_n(|k| - |t|)$$

(16)

for any $k \in M_1$ with probability tending to 1, where $b_n = (1 + \delta^*)(1 + 2w) \log p$ such that $1 + \delta/2 > (1 + \delta^*)(1 + 2w)$.
Hence, with probability tending to 1, it follows from (15) and (16) that

$$\sum_{k \in M_1} PR(k, t) = \sum_{k \ni t} \frac{\pi(k)m_k(y)}{\pi(t)m_t(y)} \leq \sum_{k \ni t} \frac{m_k(y)}{m_t(y)} \leq \sum_{|k|-|t|=1} \left( \frac{p - |t|}{|k| - |t|} \right) (C^s p)^{-(1+\delta')(|k|-|t|)}, \quad (17)$$

for some constant $\delta' > 0$. Using $(\frac{p}{|k|}) \leq p^{|k|}$ and (17), we get

$$\sum_{k \in M_1} PR(k, t) = o(1).$$

Thus, we have proved the desired result (11). \qed

Proof of Theorem 3.2. Let $M_2 = \{ k : k \ni t, |k| \leq m_n \}$. For any $k \in M_2$, let $k^* = k \cup t$, so that $k^* \in M_1$. Let $\beta_{k^*}$ be the $|k^*|$-dimensional vector including $\beta_k$ for $k$ and zeros for $t \setminus k$. Then by Taylor’s expansion and Lemmas A.1 and A.3 in Narisetty et al. [2019], with probability tending to 1,

$$L_n(\beta_{k^*}) = L_n(\tilde{\beta}_{k^*}) - \frac{1}{2} (\beta_{k^*} - \tilde{\beta}_{k^*})^T H_n(\tilde{\beta}_{k^*})(\beta_{k^*} - \tilde{\beta}_{k^*})$$

$$\leq L_n(\tilde{\beta}_{k^*}) - \frac{1 - \epsilon}{2} (\beta_{k^*} - \tilde{\beta}_{k^*})^T H_n(\beta_{0,k^*})(\beta_{k^*} - \tilde{\beta}_{k^*})$$

$$\leq L_n(\tilde{\beta}_{k^*}) - \frac{n(1 - \epsilon)\lambda}{2} \| \beta_{k^*} - \tilde{\beta}_{k^*} \|^2$$

for any $\beta_{k^*}$ such that $\| \beta_{k^*} - \beta_{0,k^*} \|_2 \leq c \sqrt{|k^*|\lambda |k^*| \log p/n} = cw_n$ for some large constant $c > 0$. Let $B_k = n(1 - \epsilon)\lambda I_k$ and $\beta_k^* = (B_k + U_k/\tau)^{-1} B_k \tilde{\beta}_k$. Define the set $B_* := \{ \beta_k : \| \beta_{k^*} - \tilde{\beta}_{k^*} \|_2 \leq cw_n/2 \}$, for some large constant $c > 0$, then by similar arguments used for super sets, with probability tending to 1,

$$\pi(k \mid y) = \int \int \exp \{ L_n(\beta_{k^*}) \} d_k(2\pi)^{-|k|/2 - r}|k| - |k|/2 |U_k|^{1/2} \exp \left( -\frac{\beta_k^T U_k \beta_k}{2\tau} \right) \prod_{i=1}^{|k|} \beta_{k_i}^{2r}$$

$$\leq \frac{\lambda_2^{\lambda_1}}{\Gamma(\lambda_1)} d_k(2\pi)^{-|k|/2} |U_k|^{1/2} \exp \left\{ L_n(\tilde{\beta}_{k^*}) \right\}$$

$$\times \left\{ \int \int_{B_*} N_1 d\beta_k d\tau + \exp \left( -\frac{1 - \epsilon}{8} c^2 \lambda |k^*| \Lambda |k^*| \log p \right) \int \int_{B_*} N_2 d\beta_k d\tau \right\},$$

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where

\[ N_1 = \exp \left( -\frac{\lambda_2}{\tau} \right) \exp \left\{ -\frac{n(1-\epsilon)\lambda}{2} \| \beta_k^* - \hat{\beta}_k^* \|^2_2 - \frac{\beta_k^\top U_k \beta_k}{2\tau} \right\} \prod_{i=1}^{\vert k \vert} \beta_{k_i}^{2r} \]

and

\[ M_2 = \tau^{-r\vert k \vert - |k|/2 - \lambda_1^{-1}} \exp \left( -\frac{\lambda_2}{\tau} \right) \exp \left\{ -\frac{\beta_k^\top U_k \beta_k}{2\tau} \right\} \prod_{i=1}^{\vert k \vert} \beta_{k_i}^{2r}. \]

Note that

\[
\int \exp \left\{ -\frac{n(1-\epsilon)\lambda}{2} \| \beta_k^* - \hat{\beta}_k^* \|^2_2 \right\} \exp \left( -\frac{\beta_k^\top U_k \beta_k}{2\tau} \right) \prod_{i=1}^{\vert k \vert} \beta_{k_i}^{2r} \, d\beta_k \\
= \int \exp \left\{ -\frac{n(1-\epsilon)\lambda}{2} \| \beta_k - \hat{\beta}_k \|^2_2 - \frac{1}{2\tau} \beta_k^\top U_k \beta_k \right\} \prod_{i=1}^{\vert k \vert} \beta_{k_i}^{2r} \, d\beta_k \exp \left\{ -\frac{n(1-\epsilon)\lambda}{2} \| \hat{\beta}_{v \setminus k} \|^2_2 \right\} \\
= (2\pi)^{\vert k \vert} \det(B_k + U_k/\tau)^{-1/2} \exp \left\{ -\frac{1}{2} \hat{\beta}_k^\top (B_k - B_k(B_k + U_k/\tau)^{-1}B_k) \hat{\beta}_k \right\} E_k \left( \prod_{i=1}^{\vert k \vert} \beta_{k_i}^{2r} \right) \\
\times \exp \left\{ -\frac{n(1-\epsilon)\lambda}{2} \| \hat{\beta}_{v \setminus k} \|^2_2 \right\},
\]

where \( E_k(\cdot) \) denotes the expectation with respect to a multivariate normal distribution with mean \( \beta_k^* \) and covariance matrix \( B_k + U_k/\tau \). It follows from Lemma 6 in the supplementary material for Johnson and Rossell [2012] that

\[
E_k \left( \prod_{i=1}^{\vert k \vert} \beta_{k_i}^{2r} \right) \leq \left( \frac{n\lambda + \tau^{-1}a_2}{n\lambda + \tau^{-1}a_1} \right)^{\vert k \vert/2} \left\{ 4V^{|k|} \left( \begin{array}{c} 4 \left[ (2r-1)!! \right]^{1/2} \\ n(\lambda + \tau^{-1}a_1) \end{array} \right)^{r|k|} \right\} \\
\leq \exp \left( \frac{a_2|k|}{2n\lambda\tau} \right) 2^{r|k|-1} \left\{ \left( \frac{4V}{|k|} \right)^{r|k|} + \left( \frac{4 \left[ (2r-1)!! \right]^{1/2}}{n\lambda} \right)^{r|k|} \right\},
\]

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where $V = \|\beta_k^*\|^2$. Therefore, for some constant $T_1 > 0$, we have

$$
\int \int_{B_\tau} N_k \, d\beta_k \, d\tau \leq \left( T_1 \frac{\log p}{|k|} \right)^{|r|/|k|} \exp \left\{ \frac{n(1 - \epsilon)\lambda}{2} \|\tilde{\beta}_{i\setminus k}\|^2 \right\} \times \int \det \left( B_k + U_k / \tau \right)^{-1/2} \exp \left\{ - \frac{\lambda_2 - a_2 |k|/(2n\lambda)}{\tau} \right\} d\tau
$$

$$
\leq T_1^{1/2} \int \det \left( n^{-1}B_k \right)^{-1/2} \exp \left\{ - \frac{n(1 - \epsilon)\lambda}{2} \|\tilde{\beta}_{i\setminus k}\|^2 \right\} \times \left\{ \lambda_2 - a_2 |k|/(2n\lambda) \right\}^{-|r|/|k|} \Gamma \left( r + |k|/2 + \lambda_1 \right).
$$

Using similar arguments leading up to (14), since the lower bound for $\pi(t \mid y)$ can be derived as before, we obtain the following upper bound for the posterior ratio, for some constants $T_2 > 0$ and $c > 0$,

$$
PR(k, t) \lesssim \left( T_2 n^{1/(2r+1)} \lambda_2 \right)^{-|r+1/2|(|k| - |t|)} \det \left\{ (1 + \epsilon)n^{-1}H_n(\beta_{0, t}) \right\}^{1/2} \frac{\det \left\{ (1 - \epsilon)\lambda M_k \right\}^{1/2}}{\det \left\{ (1 - \epsilon)\lambda I_k \right\}^{1/2}} \times \exp \left\{ L_n(\tilde{\beta}_k) - L_n(\tilde{\beta}_t) \right\} \exp \left\{ - \frac{n(1 - \epsilon)\lambda}{2} \|\tilde{\beta}_{i\setminus k}\|^2 \right\} + \left( T_2 n^{1/(2r+1)} \lambda_2 \right)^{-|r+1/2|(|k| - |t|)} \det \left\{ (1 + \epsilon)n^{-1}H_n(\beta_{0, t}) + (n\tau)^{-1}U_t \right\}^{1/2} \times \exp \left\{ L_n(\tilde{\beta}_k) - L_n(\tilde{\beta}_t) \right\} \exp \left\{ - cC|k^*|/\Lambda_{|k^*|} \log p \right\}
$$

for any $k \in M_2$ with probability tending to 1.

We first focus on (19). Note that

$$
\frac{\det \left\{ (1 + \epsilon)n^{-1}H_n(\beta_{0, t}) \right\}^{1/2}}{\det \left\{ (1 - \epsilon)\lambda M_k \right\}^{1/2}} \leq \left\{ (1 + \epsilon)\Lambda_{|t|} \right\}^{1/2} \left\{ (1 - \epsilon)\lambda \right\}^{1/2} \leq \exp \left\{ T_3 |t| \log \Lambda_{|t|} \right\} \left\{ \frac{1}{(1 - \epsilon)\lambda} \right\}^{(|k| - |t|)/2}
$$

for some constant $T_3 > 0$. Furthermore, by the same arguments used in (16), we have

$$
L_n(\tilde{\beta}_k) - L_n(\tilde{\beta}_t) \lesssim C_* (|k^*| - |t|) \log p
$$

$$
= C_* |t \setminus k| \log p + C_* (|k| - |t|) \log p
$$

for some constant $C_* > 0$ and for any $k \in M_2$ with probability tending to 1. Here we choose
$C_* = (1 + \delta^*)(1 + 2w)$ if $|k| > |t|$ or $C_* = 3 + \delta$ if $|k| \leq |t|$ so that

$$(T_2 p)^{-\frac{|k| - |t|}{2k}} \left( \frac{1}{(1 - \epsilon)^\lambda} \right)^{rac{|k| - |t|}{2k}} p^{C_* (|k| - |t|)}$$

\[ \lesssim p^{(C_* - 2\epsilon)(|k| - |t|)} = o(1), \]

where the inequality holds by Condition (A4). To be more specific, we divide $M_2$ into two disjoint sets $M_2' = \{ k : k \in M_2, |t| < |k| \leq m_n \}$ and $M_2^* = \{ k : k \in M_2, |k| \leq |t| \}$, and will show that $\sum_{k \in M_2'} PR(k, t) + \sum_{k \in M_2^*} PR(k, t) \to 0$ as $n \to \infty$ with probability tending to 1. Thus, we can choose different $C_*$ for $M_2'$ and $M_2^*$ as long as $C_* \geq (1 + \delta^*)(1 + 2w)$. On the other hand, with probability tending to 1, by Condition (A3),

\[
\exp \left\{ - \frac{n(1 - \epsilon)\lambda}{2} \| \hat{\beta}_{t,k} \|^2 \right\} \leq \exp \left\{ - \frac{n(1 - \epsilon)\lambda}{2} \left\{ \| \hat{\beta}_{0,t,k} \|^2 - \| \hat{\beta}_{t,k} - \beta_{0,t,k} \|^2 \right\} \right\} \\
\leq \exp \left\{ - \frac{n(1 - \epsilon)\lambda}{2} \left\{ |t \setminus k|^2 \min_{j \in t} \| \beta_{0,j} \|^2 - c' w_n^2 \right\} \right\} \\
\leq \exp \left\{ - \frac{(1 - \epsilon)\lambda}{2} (c_0 |t \setminus k|^2 |t| - c' |t \setminus k|) \Lambda_{|t|} \log p \right\} \\
\leq \exp \left\{ - \frac{(1 - \epsilon)\lambda}{2} (c_0 - c') |t \setminus k|^2 |t| \Lambda_{|t|} \log p \right\}
\]

for any $k \in M_2$ and some large constants $c_0 > c' > 0$, where $w_n^2 = |t \setminus k| \Lambda_{|t|} \log p / n$. Here, $c' = 5\lambda^{-2}(1 - \epsilon)^{-2}$ by the proof of Lemma A.3 in Narisetty et al. [2019].

Hence, (19) for any $k \in M_2$ is bounded above by

\[
(T_2 p)^{-\frac{|k| - |t|}{2k}} \exp \left\{ |t| \Lambda_{|t|} + C_* |t \setminus k| \log p - \frac{(1 - \epsilon)\lambda}{2} (c_0 - c') |t \setminus k|^2 |t| \Lambda_{|t|} \log p \right\} \\
\leq \exp \left\{ - \left( \frac{(1 - \epsilon)\lambda}{2} (c_0 - c') - C_* - o(1) \right) |t \setminus k|^2 |t| \Lambda_{|t|} \log p \right\} \\
\leq \exp \left\{ - \left( \frac{(1 - \epsilon)\lambda}{2} (c_0 - c') - C_* - o(1) \right) |t \setminus k|^2 |t| \Lambda_{|t|} \log p \right\} \\
\leq \exp \left\{ - \left( \frac{(1 - \epsilon)\lambda}{2} (c_0 - c') - C_* - o(1) \right) |t| \Lambda_{|t|} \log p \right\}
\]

with probability tending to 1, where the last term is of order $o(1)$ because we assume $c_0 = \frac{1}{(1 - \epsilon)\lambda} \{ 2(3 + \delta) + \frac{5}{(1 - \epsilon)\lambda} \} > \frac{2}{(1 - \epsilon)\lambda} (C_* + o(1)) + c'$.

It is easy to see that the maximum (20) over $k \in M_2$ is also of order $o(1)$ with probability tending to 1 by the similar arguments. Since we have (11) in the proof of Theorem 3.1, it completes the
Proof of Theorem 3.3. Let \( M_2 = \{ k : k \not\in t, |k| \leq m_n \} \). Since we have Theorem 3.1, it suffices to show that

\[
\sum_{k:k \in M_2} PR(k, t) \xrightarrow{P} 0 \quad \text{as } n \to \infty.
\] (21)

By the proof of Theorem 3.2, the summation of (19) over \( k \in M_2 \) is bounded above by

\[
\sum_{k \in M_2} (T_2 \lambda_2)^{-r(|k| - |t|)} \exp \left\{ - \frac{(1 - \epsilon) \lambda}{2} (c_0 - c') - C_\ast - o(1) \right\} |t \setminus k|^2 |t| |\Lambda_{|t|} \log p \}
\]

\[
\leq \sum_{|k|=0}^{p} \sum_{v=0}^{(|t| - 1) \wedge |k|} (|t|)^{p - |t|} |k|^{-v} \exp \left\{ - \frac{(1 - \epsilon) \lambda}{2} (c_0 - c') - C_\ast - o(1) \right\} (|t| - v)^2 |t| |\Lambda_{|t|} \log p \}
\]

with probability tending to 1, where \( C_\ast \leq 3 + \delta \) is defined in the proof of Theorem 3.2. Note that the last term is of order \( o(1) \) because we assume \( c_0 = \frac{1}{(1 - \epsilon_0) \lambda} \left\{ 2(9 + 2\delta) + \frac{5}{(1 - \epsilon_0) \lambda} \right\} > \frac{2}{(1 - \epsilon) \lambda} (C_\ast + 6 + o(1)) + c' \). It is easy to see that the summation of (20) over \( k \in M_2 \) is also of order \( o(1) \) with probability tending to 1 by the similar arguments. \( \square \)

**Lemma 7.1.** Under Condition (A2), we have

\[
\exp \left\{ \frac{1}{2} \tilde{\beta}_t^T (A_t' - A_t'(A_t' + \tau^{-1}U_t)^{-1}A_t') \tilde{\beta}_t \right\} \lesssim \tau^{-1}(\log p)^{2d}
\]

with probability tending to 1.

**Proof.** Note that, by Condition (A2),

\[
(A_t' + \tau^{-1}U_t)^{-1} \geq (A_t' + a_2 (n\tau \lambda)^{-1} A_t')^{-1} = \frac{n\tau \lambda}{n\tau \lambda + a_2 A_t^{-1}}
\]
which implies that
\[
\frac{1}{2} \hat{\beta}_t^\top (A'_t - A'_t(A'_t + \tau^{-1}U_t)^{-1}A'_t) \hat{\beta}_t \leq \frac{a_2}{2(n\tau \lambda + a_2)} \hat{\beta}_t^\top A'_t \hat{\beta}_t.
\]

Thus, we complete the proof if we show that
\[
\frac{1}{n\tau} \hat{\beta}_t^\top H_n(\beta_{0,t}) \hat{\beta}_t \lesssim \tau^{-1}(\log p)^2d
\]
for any \( t \in M_1 \) with probability tending to 1. By Lemma A.3 in Narisetty et al. [2019] and Condition (A2),
\[
\frac{1}{n\tau} \hat{\beta}_t^\top H_n(\beta_{0,t}) \hat{\beta}_t \leq \frac{1}{\tau} \lambda_{\max}\{n^{-1}H_n(\beta_{0,t})\}\|\hat{\beta}_t\|_2^2 \\
\leq \frac{1}{\tau}(\log p)^d(\|\beta_{0,t}\|_2^2 + o(1)) = O(\tau^{-1}(\log p)^2d)
\]
with probability tending to 1.

\[ \square \]

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