Gravitational Instantons from Minimal Surfaces

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Physical properties of gravitational instantons which are derivable from minimal surfaces in 3-dimensional Euclidean space are examined using the Newman-Penrose formalism for Euclidean signature. The gravitational instanton that corresponds to the helicoid minimal surface is investigated in detail. This is a metric of Bianchi Type $VII_0$, or $E(2)$ which admits a hidden symmetry due to the existence of a quadratic Killing tensor. It leads to a complete separation of variables in the Hamilton-Jacobi equation for geodesics, as well as in Laplace's equation for a massless scalar field. The scalar Green function can be obtained in closed form which enables us to calculate the vacuum fluctuations of a massless scalar field in the background of this instanton.

\section{Introduction}

The interest in instantons originates from the discovery of finite-action solutions of the classical Yang-Mills equations, instantons \cite{1}-\cite{3}, which are localized in imaginary time. They provide the dominant contribution to the path-integral in the quantization of the Yang-Mills fields. The expectation that their gravitational counterparts should play a similar role in the path-integral approach to quantum gravity has been a constant stimulus for research on gravitational instantons. Recently gravitational instantons which are described by hyper-Kähler metrics were studied extensively in the framework of supergravity and $M$-theory as well as Seiberg-Witten theory \cite{4}-\cite{7}.

Gravitational instantons are given by regular complete metrics with Euclidean signature and self-dual curvature which implies that they satisfy the
vacuum Einstein field equations [8]-[12]. The simplest examples of gravitational instantons are obtained from the Schwarzschild-Kerr and Taub-NUT solutions by analytically continuing them to the Euclidean sector [8],[9]. Through this construction we obtain nonsingular positive-definite metrics which are asymptotically flat at spatial infinity and periodic in imaginary time. These properties play a central role in the Euclidean path-integral approach to the quantum mechanics of black holes [13]. The principal class of physically interesting gravitational instantons consist of Gibbons-Hawking multi Taub-NUT metrics [9] which are asymptotically locally Euclidean metrics with self-dual curvature. The Eguchi-Hanson instanton [14],[15] is a distinguished member of this class.

In the Euclidean path-integral approach to quantum gravity one is interested in the evaluation of the functional integral over all stationary phase metrics with appropriate boundary conditions. Gravitational instantons should provide the dominant contribution to the path integral and mediate the quantum tunnelling between two homotopically distinct vacua. Therefore the search for new instanton solutions of the Einstein field equations is of great interest physically, as well as mathematically. Recently it was observed [16] that minimal surfaces in Euclidean space can be used in the construction of instanton solutions, even as in the case of Yang-Mills instantons [17]. For every minimal surface in 3-dimensional Euclidean space there exists a gravitational instanton which is an exact solution of the Einstein field equations with Euclidean signature and self-dual curvature. If the surface is defined by the Monge ansatz \( \phi = \phi(x, t) \), then the metric establishing this correspondence is given by

\[
\begin{align*}
\text{ds}^2 &= \frac{1}{\sqrt{1 + \phi^2_t + \phi^2_x}} \left[ (1 + \phi^2_t)(dt^2 + dy^2) \\
&\quad + (1 + \phi^2_x)(dx^2 + dz^2) + 2\phi_t \phi_x (dt \, dx + dy \, dz) \right]
\end{align*}
\]

whereby the Einstein field equations reduce to the classical equation

\[
(1 + \phi^2_x) \phi_{tt} - 2\phi_t \phi_x \phi_{tx} + (1 + \phi^2_t) \phi_{xx} = 0
\]

governing minimal surfaces in \( \mathbb{R}^3 \). These are Kähler metrics obtained from Jörgens’ correspondence [18] between the equation for minimal surfaces and the real elliptic Monge-Ampère equation in 2-dimensions. Gravitational instantons that follow from this construction will admit at least two commuting
Killing vectors $\partial_y, \partial_z$ which implies that they may be complete non-compact Ricci-flat Kähler metrics which have been considered in the context of stringy cosmic strings [19], [20]. The general gravitational instanton metric that results from Weierstrass’ general local solution [21] for minimal surfaces has been constructed in [22] using the correspondence (1) between minimal surfaces and gravitational instantons.

In this paper we shall use the Newman-Penrose formalism for Euclidean signature developed in [23], which will henceforth be referred to as I, to investigate the physical properties of gravitational instantons which are derivable from minimal surfaces. Among all such solutions, the gravitational instanton that corresponds to the helicoid minimal surface is of particular interest, however, it is incomplete and has a curvature singularity. The physical interpretation of this metric as an instanton must therefore await an analysis of its global structure. In this connection it is worth noting that singular hyper-Kähler instanton metrics sometimes admit an $M$-theory resolution [4] and are therefore of interest in supergravity. It is possible that a supergravity extension of the metric corresponding to helicoid may also be of interest and therefore we shall investigate its properties in some detail. We shall show that the self-dual metric derived from helicoid has a number of remarkable properties. It admits a 3-parameter group of motions $E(2)$, namely rotation and boosts on the Euclidean plane, which is also known as $G_3$, and the metric is of Bianchi Type $VII_0$. Furthermore, it admits a hidden symmetry which stems from the existence of a quadratic Killing tensor. Next we shall show that the Hamilton-Jacobi equation for geodesics, as well as Laplace’s equation for a massless scalar field are separable in the background of the metric generated by the helicoid. We construct the scalar Green function in closed form and calculate the vacuum fluctuations of a massless scalar field in the background of this instanton. Using the point-splitting procedure we obtain the renormalized expression for vacuum expectation value of the stress-energy tensor.
2 General minimal surface solution

The instanton metric obtained from Weierstrass’ general local solution for minimal surfaces is given by [22]

\[ ds^2 = \left(1 - |g|^4 \right) |f|^{2} d\zeta d\bar{\zeta} + \frac{1}{1 - |g|^4} |d\sigma - \bar{g}^2 d\bar{\sigma}|^2 \]  

(3)

where \( \sigma = y + iz \) is a complex coordinate and \( f, g \) are arbitrary holomorphic functions of \( \zeta \) which replaces \( t, x \) as a new complex coordinate through the Weierstrass formulae [21], [22]. The metric (3) contains two arbitrary holomorphic functions but only one of them, namely \( g \), is geometrically significant. Nevertheless, for some purposes it may be useful to keep \( f \).

In order to investigate the geometrical properties of the metric (3) we shall use the Newman-Penrose formalism for Euclidean signature and refer to the results of I. For this purpose we first need to specify a tetrad which will be given as an isotropic complex dyad defined by the vectors \( l, m \) together with their complex conjugates subject to the normalization conditions

\[ l_\mu \bar{l}^\mu = 1 \quad m_\mu \bar{m}^\mu = 1 \]  

(4)

with all others vanishing. The power of the Newman-Penrose formalism becomes evident if the legs of this complex dyad are chosen along an isotropic geodesic congruence. Using eqs.(I.19) and (I.20) we obtain

\[ l_{\mu;\nu}^\nu = (\epsilon - \bar{\gamma}) l_\mu + \kappa \bar{m}_\mu - \bar{\nu} m_\mu \]  

(5)

\[ m_{\mu;\nu}^\nu = (\bar{\alpha} - \beta) m_\mu + \mu l_\mu - \sigma \bar{l}_\mu \]  

(6)

where semicolon denotes covariant differentiation. From these equations it follows that if the spin coefficients \( \kappa = \nu = 0 \), or \( \mu = \sigma = 0 \), then the vector fields with components \( l^\mu \), or \( m^\mu \) determine the corresponding isotropic geodesic congruence up to affine reparametrization.

For the instanton metric (3) the obvious co-frame is given

\[ l = \frac{1}{\sqrt{2}} (1 - |g|^4)^{1/2} |f| d\zeta, \]  

\[ m = \frac{1}{\sqrt{2}} (1 - |g|^4)^{-1/2} \left( d\sigma - \bar{g}^2 d\bar{\sigma} \right) \]  

(7)
and its inverse will provide a convenient choice of the complex dyad. In this case there exists an isotropic geodesic congruence formed by only one basis vector, namely \( l^\mu \). To make this explicit we take the exterior derivative of the basis 1-forms (8) and compare the result with eqs.\((I.21)\) to obtain the spin coefficients for the metric \((3)\). Thus we have

\[
\kappa = \nu = \alpha = \beta = \pi = \tau = \mu = \lambda = \rho = 0,
\]

\[
\epsilon = \frac{1}{2\sqrt{2}|f|^3}(1 - |g|^4)^{-3/2} \left[ (1 - |g|^4) f \bar{f}' - 4|f|^2 |g|^2 g \bar{g}' \right],
\]

\[
\gamma = -\frac{1}{2\sqrt{2}|f|^3}(1 - |g|^4)^{-1/2} f \bar{f}',
\]

\[
\sigma = -2\sqrt{2}|f|^{-1}(1 - |g|^4)^{-3/2} \bar{g} \bar{g}',
\]

where prime denotes derivative with respect to the argument. We see that the spin coefficients \( \kappa \) and \( \nu \) vanish so that the vector field \( l \) forms an isotropic geodesic congruence. We note that for the choice \( f = 1 \) the spin coefficient \( \gamma \) also vanishes resulting in an anti-self-dual gauge, \( \text{cf} \) eqs.\((I.45)\). Next, using the spin coefficients (8) in the Ricci identities \((I.93,95)\) we find that the set of Weyl scalars labelled with tilde \( \tilde{\Psi}_i = 0, \ i = 0, 1, .., 4 \) reflecting the anti-self duality of the curvature tensor. For the remaining set of Weyl scalars we find

\[
\Psi_1 = -\tilde{\Psi}_3 = 0,
\]

\[
\Psi_2 = \frac{8}{|f|^2 (1 - |g|^4)^2} |g \bar{g}'|^2
\]

\[
\Psi_0 = \tilde{\Psi}_4 = -\frac{4}{|f|^2 (1 - |g|^4)^2} \left[ \bar{g} \bar{g}'' - \frac{f f'}{|f|^2} \bar{g} \bar{g}' + \frac{1 + 5|g|^4}{1 - |g|^4} (\bar{g}')^2 \right]
\]

which shows that, \( \text{cf} \) eqs.\((I.121)\), the instanton metric \((3)\) obtained from Weierstrass’ formulae for minimal surfaces is of Petrov Type \( I \). Finally, we note that

\[
|g|^4 = 1
\]

defines the locus of curvature singularities.
Among the various particular realizations of the general metric (3), the instanton metric that corresponds to the helicoid minimal surface appears to be the one with greatest interest. The graph of the helicoid is given by

$$\phi = a \tan^{-1} \left( \frac{x}{t} \right)$$

(11)

and by introducing new coordinates

$$x = r \cos \theta, \quad t = r \sin \theta,$$

(12)

the corresponding instanton metric becomes [16]

$$ds^2 = \frac{1}{\sqrt{1 + a^2 r^2}} \left[ dr^2 + (r^2 + a^2) d\theta^2 + \left( 1 + \frac{a^2}{r^2} \sin^2 \theta \right) dy^2 
- \frac{a^2}{r^2} \sin 2\theta \, dy \, dz + \left( 1 + \frac{a^2}{r^2} \cos^2 \theta \right) dz^2 \right]$$

(13)

where the coordinates $y$ and $z$ along the Killing directions will be taken to be periodic, coordinates on a 2-torus, as in the discussion of stringy cosmic strings [14]. There are two asymptotic regions $r \to \pm \infty$ which must be identified. The singularity in the metric (13) at $r = 0$ is a curvature singularity because we shall show that curvature scalars are singular there. In this paper we shall not discuss the global properties of the metric (13) but instead point out some of its remarkable properties such as a hidden symmetry in addition to Bianchi $VII_0$ symmetry. This is described by a Killing tensor and leads to a complete separation of variables in the Hamilton-Jacobi equation for geodesics, as well as in Laplace’s equation for a massless scalar field.

The coordinates $r, \theta$ are inherited from the standard description of the helicoid and we can immediately recognize the first fundamental form of the helicoid $I = dr^2 + (r^2 + a^2) d\theta^2$ in the metric (13). The helicoid is a ruled surface. There are further helicoids here, namely the constant $r = r_0, z = z_0$ sections of the metric (13)

$$dl^2 = (r_0^2 + a^2) \, d\theta^2 + \left( 1 + \frac{a^2}{r_0^2} \sin^2 \theta \right) dy^2$$

(14)
are helicoids in the torus $S^2 \times S^1$ as shown in Fig. 1 and similarly, the 2-sections defined by constant $r, y$ are also helicoid surfaces in another $S^2 \times S^1$. These helicoids are ruled by a geodesic of $S^2$.

An alternative coordinate system for the metric (13) obtained by letting $r = a \sinh \chi$ results in the following form of the metric

$$ds^2 = \frac{a^2}{2} \sinh 2\chi (d\chi^2 + d\theta^2) + \frac{2}{\sinh 2\chi} \left[ (\sinh^2 \chi + \sin^2 \theta) dy^2 
- \sin 2\theta dydz + (\sinh^2 \chi + \cos^2 \theta) dz^2 \right]$$

which can be obtained directly from the general minimal surface solution (3) through the choice of holomorphic functions

$$f = a e^\zeta, \quad g = e^{-\zeta}$$

where $\zeta = \chi + i\theta$.

The metric (16) is of Bianchi Type $VII_0$. It is a homogeneous anisotropic manifold that admits a 3-parameter group of motions which is the same as $E(2)$, the group of motions on the Euclidean plane. The metric (16) can therefore be written in the compact form

$$ds^2 = \frac{a^2}{2} \sinh 2\chi \left[ d\chi^2 + (\sigma^3)^2 \right] + \tanh \chi (\sigma^1)^2 + \coth \chi (\sigma^2)^2$$

using the left-invariant 1-forms

$$\sigma^1 = \cos \theta dy + \sin \theta dz,$$
$$\sigma^2 = -\sin \theta dy + \cos \theta dz,$$
$$\sigma^3 = d\theta,$$

of Bianchi Type $VII_0$ [24]. They satisfy the Maurer-Cartan equations of structure

$$d\sigma^i = \frac{1}{2} c^{ijk}_{\sigma^j \sigma^k}$$

where

$$c_{1}^{1} = -c_{2}^{1} = c_{1}^{2} = -c_{3}^{2} = 1$$
are the only non-vanishing structure constants. A different representation of the metric (18) is obtained if in place of the left-invariant 1-forms $\sigma^i$ we use the right-invariant 1-forms of Bianchi Type $VII_0$

$$
\begin{align*}
    r^1 &= dy - z \, d\theta, \\
    r^2 &= dz + y \, d\theta, \\
    r^3 &= d\theta
\end{align*}
$$

(22)

and it can be directly verified that the metric

$$
ds^2 = \frac{a^2}{2} \sinh 2\chi \left[ d\chi^2 + (r^3)^2 \right] + \tanh \chi (r^1)^2 + \coth \chi (r^2)^2
$$

(23)

is Ricci-flat and anti-self-dual as well. Even though the explicit expression for the metric changes drastically under this exchange of left and right-invariant 1-forms, there will be no change in the principal results we shall present below.

In order to investigate the geometrical properties of the metric (18) we shall use the Newman-Penrose formalism for Euclidean signature. For this metric the natural complex isotropic dyad is given by the co-frame

$$
\begin{align*}
    l &= \frac{a}{2} (\sinh 2\chi)^{1/2} \left( d\chi + i \sigma^3 \right) \\
    m &= \frac{1}{\sqrt{2}} \left[ (\tanh \chi)^{1/2} \sigma^1 + i (\coth \chi)^{1/2} \sigma^2 \right]
\end{align*}
$$

and this choice results in an anti-self-dual gauge which can be verified by calculating the spin coefficients through eqs.(1.21). We find

$$
\epsilon = \frac{1}{a} \cosh 2\chi (\sinh 2\chi)^{-3/2}, \quad \sigma = \frac{2}{a} (\sinh 2\chi)^{-3/2}
$$

(24)

while all other spin coefficients vanish. Now from eq.(5) it is readily seen that the vector field $l$ determines an isotropic geodesic congruence up to affine reparametrization. Using the spin coefficients in eqs.(1.36) we find that the connection 1-forms are given by

$$
\begin{align*}
    \Gamma_0^0 &= -\Gamma_1^1 = \frac{1}{a} \cosh 2\chi (\sinh 2\chi)^{-3/2} \left( \bar{l} - l \right)
    \\
    \Gamma_0^1 &= -\bar{\Gamma}_1^0 = -\frac{2}{a} \left( \sinh 2\chi \right)^{-3/2} \bar{m}
\end{align*}
$$

(25)
and in particular all $\bar{\Gamma}_{x'y'}$ vanish so that this is an anti-self-dual gauge. The curvature tetrad scalars are obtained through the Ricci identities by substituting these spin coefficients in eqs. (I.93, 95) and we find that the non-vanishing Weyl scalars are given by

$$
\Psi_2 = \frac{4}{a^2} (\sinh 2\chi)^{-3},
$$

$$
\Psi_0 = \bar{\Psi}_4 = -\frac{12}{a^2} \cosh 2\chi (\sinh 2\chi)^{-3},
$$

thus the curvature is anti-self-dual, the metric is algebraically general, or Petrov Type I [23] and there is a curvature singularity $r = 0$, or $\chi = 0$. We conclude that the instanton metric that corresponds to the catenoid minimal surface, where $a^2 \rightarrow -a^2$ in eq. (13), is Bianchi Type $VI_0$. Both of these metrics can be obtained from the Belinski-Gibbons-Page-Pope Bianchi Type $IX$ and $VIII$ solutions [25] by group contraction [26] which requires that the modulus of the elliptic functions must be set equal to unity.

## 4 Symmetries

Well-known gravitational instanton metrics admit hidden symmetries [27] and we shall now show that the instanton metric derived from helicoid minimal surface provides another such example. First of all the metric (18) admits a three parameter group of motions. The Killing vectors are given by

$$
\xi_1 = \frac{\partial}{\partial y}, \quad \xi_2 = \frac{\partial}{\partial z}, \quad \xi_3 = \frac{\partial}{\partial \theta} - z \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}
$$

and they satisfy the Lie algebra

$$
[\xi_1, \xi_2] = 0, \quad [\xi_1, \xi_3] = \xi_2, \quad [\xi_3, \xi_2] = \xi_1
$$

of Bianchi Type $VII_0$ with the structure constants (21). The Killing vectors (27) are the right-invariant vector fields satisfying

$$
i_{\xi_k} r^j = \delta^j_k
$$

upon contraction with the right-invariant 1-forms (22).
Hidden symmetries play an important role in the study of first integrals of motion. Starting from the Lagrangian for geodesics

\[ L = \frac{1}{2} \left( \frac{ds}{d\lambda} \right)^2 \]

where \( \lambda \) is an affine parameter, we have the conserved canonical momenta

\[ p_y = \frac{2}{\sinh 2\chi} \left( (\sinh^2 \chi + \sin^2 \theta) \dot{y} - \sin \theta \cos \theta \dot{z} \right) \]
\[ p_z = \frac{2}{\sinh 2\chi} \left( (\sinh^2 \chi + \cos^2 \theta) \dot{z} - \sin \theta \cos \theta \dot{y} \right) \]

conjugate to \( y \) and \( z \) respectively, where dot denotes differentiation with respect to \( \lambda \). Since the metric (18) admits two evident Killing vectors \( \partial_y \) and \( \partial_z \), these two first integrals of motion are manifest. Another immediate constant of motion is the square of the 4-momentum

\[ g^{\mu\nu} p_\mu p_\nu = \mu^2, \]

however, these three integrals of motion are not sufficient for a complete integration of the geodesic equations. Thus, following Walker and Penrose [28] we shall consider possible quadratic integrals of motion

\[ K_i = K_i^{\mu\nu} p_\mu p_\nu \]

where \( K_i^{\mu\nu} \) are symmetric Killing tensors which satisfy the Walker-Penrose equations

\[ K_i^{(\mu\nu; \lambda)} = 0 \]

and round parentheses denote symmetrization. It is evident that the existence of a non-trivial Killing tensor in the instanton metric (18) should give rise to a fourth integral of motion. In order to integrate eqs. (33) we shall use the Newman-Penrose formalism for Euclidean signature. We begin with the resolution of the Killing tensor along the legs of the complex isotropic dyad (24)

\[ K^{\mu\nu} = \chi_{11} \bar{l}^\mu \bar{l}^\nu + \chi_{33} \bar{m}^\mu \bar{m}^\nu + \chi_{12} \bar{l}^\mu \bar{m}^\nu + \chi_{34} \bar{m}^\mu \bar{m}^\nu + 2\chi_{13} \bar{l}^{(\mu} \bar{m}^{\nu)} + 2\chi_{14} \bar{l}^{(\mu} \bar{m}^{\nu)} + cc \]
where \( cc \) denotes the complex conjugate of the foregoing terms and the Killing tetrad scalars will be defined as

\[
\begin{align*}
\chi_{11} & = \bar{\chi}_{22} = K_{\mu\nu} l^\mu l^\nu \\
\chi_{33} & = \bar{\chi}_{44} = K_{\mu\nu} m^\mu m^\nu \\
\chi_{13} & = \bar{\chi}_{24} = K_{\mu\nu} l^\mu m^\nu \\
\chi_{14} & = \bar{\chi}_{23} = K_{\mu\nu} l^\mu \bar{m}^\nu \\
\chi_{12} & = \bar{\chi}_{21} = K_{\mu\nu} l^\mu \bar{l}^\nu \\
\chi_{34} & = \bar{\chi}_{43} = K_{\mu\nu} m^\mu \bar{m}^\nu .
\end{align*}
\] (35)

From eqs. (33) immediately it follows that the dyad scalars \( \chi_{13} \) and \( \chi_{14} \) must vanish while the remaining satisfy a set of coupled equations

\[
\begin{align*}
D\chi_{11} - 2\epsilon \chi_{11} & = 0, \\
(D + 2\epsilon)\chi_{11} + 2D\chi_{12} & = 0, \\
(D + 2\epsilon)\chi_{44} + 2\sigma \chi_{11} & = 0, \\
(D - 2\epsilon)\chi_{33} + 2\sigma(\chi_{12} - \chi_{34}) & = 0, \\
D\chi_{34} - \sigma \chi_{44} & = 0.
\end{align*}
\] (36)

which admit the general solution

\[
\begin{align*}
\chi_{11} & = \chi_{22} = A \sinh \chi \cosh \chi, \quad \chi_{12} = -\chi_{11} + B, \\
\chi_{33} & = \chi_{44} = a^2 \tanh \chi \left( -A + \frac{C}{a^4 \sinh^2 \chi} \right), \\
\chi_{34} & = -\frac{a^2}{\sinh \chi \cosh \chi} \left( A \sinh^2 \chi + \frac{C}{a^4} \cosh 2\chi \right) + B
\end{align*}
\] (37)

with arbitrary constants \( A, B \) and \( C \). Substituting these expressions into eq. (34) we find that the additive constants of integration \( B \) and \( C \) do not give rise to new Killing tensors. They lead to constants of motion derived from the metric tensor and the symmetric tensor product of the two commuting Killing vectors respectively. Therefore without loss of generality we can discard them and we are left with the independent integral of motion

\[
K = -p_\theta^2 - a^2 (\cos \theta p_y + \sin \theta p_z)^2
\] (38)
which is not expressible in terms of symmetric bilinear combinations of the Killing vectors (27). Thus the instanton metric that corresponds to the helicoid minimal surface admits the quadratic Killing tensor (38) which provides a fourth integral of motion for geodesics.

The existence of this Killing tensor leads to a separation of variables in the Hamilton-Jacobi equation for geodesics. The separability ansatz

$$S = p_y y + p_z z + S_\chi(\chi) + S_\theta(\theta)$$  

leads to decoupled ordinary differential equations in the Hamilton-Jacobi equation

$$
\left(\frac{dS_\chi}{d\chi}\right)^2 + a^2 \left(p_y^2 + p_z^2\right) \sinh^2 \chi - \frac{\mu^2 a^2}{2} \sinh 2\chi = K \\
\left(\frac{dS_\theta}{d\theta}\right)^2 + a^2 \left(\cos \theta p_y + \sin \theta p_z\right)^2 = -K
$$

where the separation constant $K$ coincides with (38). As it was noted in [27] some well-known instanton metrics with two commuting Killing vectors which, however, are not of Petrov type $D$ may admit a Killing tensor. We have seen that the metric (18) which is an anti-self-dual Petrov type I instanton provides another example of such metrics.

## 5 Scalar Green’s function

The Green function for a massless scalar field obeys the equation

$$\Delta G(x, x') = -\delta^4(x, x')$$

where $\Delta = \nabla_\mu g^{\mu\nu} \nabla_\nu$ is the Laplacian with $\nabla$ denoting the covariant derivative. For the metric (18) it has the following explicit form

$$
\Delta = \partial_\chi \partial_\chi + \partial_\theta \partial_\theta + a^2 \sinh^2 \chi \left(\partial_{yy} + \partial_{zz}\right) + a^2 \left(\cos \theta \partial_y + \sin \theta \partial_z\right)^2
$$

and we shall now show that the existence of a Killing tensor leads to the solution of Laplace’s equation by separation of variables which enables us to
construct the scalar Green function in closed form. Following Carter[29] we begin with the second order operator which is constructed from the Killing tensor (38)

$$\hat{K} = \nabla_\mu K^{\mu\nu} \nabla_\nu = -\partial_{\theta\theta} - a^2 (\cos \theta \partial_y + \sin \theta \partial_z)^2$$  \hspace{1cm} (43)

which commutes with the Laplacian (42) and the vector fields $\partial_y$ and $\partial_z$. Thus the three commuting operators

$$\begin{align*}
\partial_y \Phi &= k_y \Phi \\
\partial_z \Phi &= k_z \Phi \\
\hat{K} \Phi &= \lambda \Phi
\end{align*}$$  \hspace{1cm} (44)

with eigenvalues $k_y$, $k_z$, $\lambda$ have common eigenfunctions. Hence eigenfunctions of the Laplacian (42) admit separation of variables of the form

$$\Phi = R(\chi) S(\theta) e^{i(k_y y + k_z z)}$$  \hspace{1cm} (45)

where the angular functions satisfy the last one of eqs.(44). This can be reduced to a pair of Mathieu equations by passing to polar coordinates in the constants of separation $k_y = k \cos \phi$ and $k_z = k \sin \phi$. Then we have

$$\frac{d^2 S}{d\Theta^2} + \left( \lambda - k^2 a^2 \cos^2 \Theta \right) S = 0$$  \hspace{1cm} (46)

where $\Theta = \theta - \phi$. We are interested only in periodic solutions of this equation with period $2\pi$. These solutions exist only for discrete values of the separation constant $\lambda$ and they are given by even and odd periodic Mathieu functions $Se_n(ka, \cos \Theta)$ and $So_n(ka, \cos \Theta)$ respectively[30]. When the parameter $ka$ tends to zero, these solutions reduce to the trigonometric functions

$$Se_n(ka, \cos \Theta) \to \cos(n\Theta), \quad So_n(ka, \cos \Theta) \to \sin(n\Theta)$$

while the separation constant

$$\lambda e_n \to \lambda o_n \to n^2$$

goes over into the square of an integer.
Similarly, the equation for radial modes can also be transformed into Mathieu’s equation

\[
\frac{d^2 R}{d \tilde{\chi}^2} + (k^2 a^2 \cosh^2 \tilde{\chi} - \lambda) R = 0
\]  

(47)

where

\[
\tilde{\chi} = \chi + \frac{i \pi}{2}
\]

is a new complex coordinate. The solutions of this equation which satisfy the regularity conditions at the origin and at infinity are expressed in terms of Bessel-like Mathieu functions \(J_n(ka, \cosh \tilde{\chi})\), \(J_0(ka, \cosh \tilde{\chi})\) and Hankel-like Mathieu functions \(H_n(ka, \cosh \tilde{\chi})\), \(H_0(ka, \cosh \tilde{\chi})\) respectively. With these solutions of eqs. (46)-(47) we use the general procedure \[3\] to obtain the Green function

\[
G(x, x') = \frac{1}{(2\pi)^2} \int_0^\infty \int_0^{2\pi} k \, dk \, d\phi \, e^{ikY\cos \phi} \, g_h(\tilde{\chi}, \tilde{\chi}', \Theta, \Theta')
\]  

(48)

with the factorized function

\[
g_h(\tilde{\chi}, \tilde{\chi}', \Theta, \Theta') = i\pi H_0^{(1)}(k\tilde{Z}) = 4\pi i \left\{ \sum_{n=0}^{\infty} \left( \frac{S_n(h, \cos \Theta')}{M_n(h)} \right) S_n(h, \cos \Theta) \right. \\
\times [\theta(\tilde{\chi} - \tilde{\chi}') J_n(h, \cosh \tilde{\chi}) H_n(h, \cosh \tilde{\chi}) \\
+ \theta(\tilde{\chi}' - \tilde{\chi}) J_n(h, \cosh \tilde{\chi}) H_n(h, \cosh \tilde{\chi})] \\
+ \sum_{n=0}^{\infty} \left( \frac{S_n(h, \cos \Theta')}{M_n(h)} \right) S_n(h, \cos \Theta) \\
\times [\theta(\tilde{\chi} - \tilde{\chi}') J_n(h, \cosh \tilde{\chi}) H_n(h, \cosh \tilde{\chi}) \\
+ \theta(\tilde{\chi}' - \tilde{\chi}) J_n(h, \cosh \tilde{\chi}) H_n(h, \cosh \tilde{\chi})] \right\}
\]  

(49)

where \(h = ka\),

\[
M_n = \int_0^{2\pi} |S_n|^2 d\Theta, \quad M_0 = \int_0^{2\pi} |S_0|^2 d\Theta
\]

are the normalization constants. The Heaviside unit step function is denoted by \(\theta\) and \(\tilde{Z}\) is the distance between two points. Returning back to the original set of independent variables we note that \(Z^2(\tilde{\chi}, \Theta) \equiv \tilde{Z}^2 = -Z^2(\chi, \Theta)\) and,
accordingly, in eq.(49) the function \( H_0 \) goes over into \( K_0 \), the modified Hankel function. Then the expression (48) takes the form

\[
G(x, x') = \frac{1}{(2\pi)^3} \int_0^{2\pi} d\phi \int_0^\infty k \, dk \, e^{ikY \cos \phi} K_0(kZ)
\]  

(50)

and the distance functions \( Z \) and \( Y \) are given by

\[
\begin{align*}
Z^2 & = r^2 + a^2 \cos^2(\theta - \phi) + r'^2 + a^2 \cos^2(\theta' - \phi) \\
& - 2 \left[ (r^2 + a^2)^{1/2} (r'^2 + a^2)^{1/2} \cos(\theta - \phi) \cos(\theta' - \phi) \right. \\
& \left. + rr' \sin(\theta - \phi) \sin(\theta' - \phi) \right] \\
Y^2 & = (y - y')^2 + (z - z')^2.
\end{align*}
\]  

(51)

We shall first perform the \( k \) integration using the standard integral

\[
\int_0^\infty k \, dk \, e^{ik\alpha} K_0(kZ) = -\frac{\alpha}{(\alpha^2 + Z^2)^{3/2}} \left[ \ln \left( \frac{\alpha}{Z} + \sqrt{\frac{\alpha^2}{Z^2} + 1} \right) + i\pi 2 \right] \\
+ \frac{1}{\alpha^2 + Z^2}
\]  

(52)

with \( \alpha = Z \cos \phi \). Substituting this expression into eq.(50) and carrying out the definite integral over an angular variable \( \phi \) we find the final closed expression for the Green function

\[
G(x, x') = \frac{1}{64\pi^2} \frac{F}{F^2 + Y^2 E},
\]  

(53)

with

\[
F^2 = 2 \left\{ (\cos(\theta' - \theta) - 1)^2 [X^4 + a^2X^2 + X^2g] + 6x^2X^2 \\
+ (\cos(\theta' - \theta) + 1)^2 [x^4 + a^2x^2 - x^2g] - 2x^2X^2 \cos^2(\theta' - \theta) \right\},
\]

\[
E = \cos(\theta' - \theta)(-X^2 + x^2 - g) + 2X^2 + 2x^2 + a^2 \\
- [a^2 \cos(\theta' - \theta) - g + X^2 - x^2] \cos(\theta + \theta')
\]  

(54)

where we have used the notations

\[
X = \frac{r + r'}{2}, \quad x = \frac{r' - r}{2}, \quad g = [(a^2 + X^2)^2 + 2x^2(a^2 - X^2) + x^4]^{1/2}.
\]
This Green function can be used to evaluate the vacuum expectation value of the stress-energy tensor for a massless scalar field. Using the point-splitting procedure we find a finite expression for renormalized Green function by substraction from (53) its singular part [32]. This gives us the vacuum expectation value of the stress-energy tensor

\[
\langle T^\nu_\mu (x) \rangle = \frac{1}{1536\pi^2} \frac{a^2}{16r^2} \frac{28r^2 - 5a^2}{(r^2 + a^2)^3} \begin{pmatrix}
3 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

at the coincidence points.

6 Conclusion

The gravitational instanton metrics generated by classical minimal surfaces, the catenoid and the helicoid, are Euclidean Bianchi Type \( VI_0 \) and \( VII_0 \) metrics respectively. In this paper we have discussed some properties of the gravitational instanton that corresponds to the helicoid. We have shown that it is an anti-self-dual solution of Petrov Type \( I \) and admits a Killing tensor which leads to the solution of the Hamilton-Jacobi and Laplace equations by separation of variables. The interpretation of the self-dual metric generated by the helicoid as a gravitational instanton is problematic due to its incompleteness and the curvature singularity at \( r = 0 \). The final resolution of this issue must await an analysis of the global structure of this exact solution. However, hyper-Kähler instanton metrics sometimes admit an \( M \)-theory resolution [4] and are therefore of interest in supergravity even though they are singular and not complete. The remarkable symmetry properties of this metric enables us to obtain the scalar Green’s function by separation of variables and calculate the vacuum expectation value of the stress-energy tensor.

7 Acknowledgments

We thank G. W. Gibbons for many interesting and helpful conversations. We thank also U. Camci for his interest in this work.
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The unit disk at the cross-section represents $S^2$ and its geodesics consist of great circles that intersect the boundary at right angles. The helicoid surface is generated by one such geodesic, that is the surface is *ruled* by this geodesic of $S^2$.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9812007v2