EFFECTIVE DYNAMICS OF TRANSLATIONALLY INVARIANT
MAGNETIC SCHRODINGER EQUATIONS IN THE HIGH FIELD LIMIT

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ABSTRACT. We study the large field limit in Schrödinger equations with magnetic vector potentials describing translationally invariant $B$-fields with respect to the $z$-axis. In a first step, using regular perturbation theory, we derive an approximate description of the solution, provided the initial data is compactly supported in the Fourier-variable dual to $z \in \mathbb{R}$. The effective dynamics is thereby seen to produce high-frequency oscillations and large magnetic drifts. In a second step we show, by using the theory of almost invariant subspaces, that this asymptotic description is stable under polynomially bounded perturbations that vanish in the vicinity of the origin.

1. INTRODUCTION

In this work we study a class of linear Schrödinger equations which describe the behavior of charged, spinless particles under the influence of translationally invariant magnetic fields $B(x) = \nabla \times A(x)$ in $\mathbb{R}^3$. More specifically, we denote the spatial variables by $(x_1, x_2, z) \in \mathbb{R}^3$, with $x \equiv (x_1, x_2)$, and consider magnetic vector potentials of the form

$$A(x) = (0, 0, A(x)).$$

Note that this implies that $A(x)$ is automatically divergence free, i.e. $\nabla \cdot A(x) = 0$. Prototypical examples include the field around an infinite current-carrying wire,

$$A = (0, 0, b \ln(|x|)), \quad b \in \mathbb{R},$$

and azimuthal fields of constant magnitude,

$$A = (0, 0, b|x|), \quad b \in \mathbb{R}. \tag{1.1}$$

The dynamics of quantum particles in such translationally invariant fields is governed by

$$i \partial_t \Psi = H \Psi, \quad \Psi_{t=0} = \Psi_0 \in L^2(\mathbb{R}^3), \tag{1.2}$$

with (purely) magnetic Hamiltonian

$$H = (-i \nabla + A(x))^2.$$

As previously observed in [8, 9], the operator $H$ can be treated as a family of two-dimensional operators under the partial Fourier transform $F : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, given by

$$F \varphi(x, p) \equiv \hat{\varphi}(x, p) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i z p} \varphi(x, z) \, dz, \quad \varphi \in L^2(\mathbb{R}^3).$$

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The translation invariance of the magnetic vector potential then implies that $\mathcal{H}$ is unitarily equivalent to the direct integral in $L^2(\mathbb{R}^2; L^2(\mathbb{R}^2))$ of the family of fiber Hamiltonians $\mathcal{H}(p)$, i.e.,

$$\mathcal{F} \mathcal{H} \mathcal{F}^{-1} = \int_\mathbb{R} \mathcal{H}(p) \, dp.$$ 

Within each fiber, the dynamics is then governed by

$$(1.3) \quad i \partial_t \hat{\Psi} = \mathcal{H}(p) \hat{\Psi}, \quad \hat{\Psi}_{|t=0} = \hat{\Psi}_0(\cdot, p).$$

where, for fixed $p \in \mathbb{R}$, $\mathcal{H}(p)$ is an operator acting on $L^2(\mathbb{R}^2)$ only. We note that the hereby obtained fiber Hamiltonian

$$\mathcal{H}(p) = -\Delta_x + (p + A(x))^2$$

is symmetric and positive on $C_0^\infty(\mathbb{R}^2)$, where throughout this paper we shall always impose the condition $A \in L^\infty_{\text{loc}}(\mathbb{R}^2)$. A classical result in the mathematical theory of Schrödinger operators (see, e.g., [5]) then implies that $\mathcal{H}(p)$ is essentially self-adjoint on $L^2(\mathbb{R}^2)$ and we shall denote its unique self-adjoint extension by the same letter. By Stone’s theorem, $\mathcal{H}(p)$ is seen to be the generator of a group of unitary operators

$$S_t(p) = e^{-it\mathcal{H}(p)} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2).$$

which yields the existence of a unique solution to (1.3). In the following, we will be interested in an effective description of this type of quantum dynamics in the high magnetic field limit, i.e., the regime in which the field strength $|B| \to +\infty$.

1.1. Asymptotic regime and rescaling. We first consider the case of homogenous vector potentials, which can be seen as a generalization of (1.1). More specifically, we consider

$$(1.4) \quad A^\varepsilon(x) = \frac{1}{\varepsilon^{\alpha+1}} \left( 0, 0, \frac{|x|}{\varepsilon} \Theta(\vartheta) \right), \quad \text{with } \alpha > 0,$$

where $|x| \geq 0$ and $\vartheta \in [0, 2\pi)$ denote polar coordinates in $\mathbb{R}^2$. The function $\Theta \in C^\infty_{\text{per}}([0, 2\pi); \mathbb{R})$ describes a possible azimuthal dependence of the magnetic field. Throughout this work, we shall impose:

Assumption 1.1. The function $\Theta \in C^\infty_{\text{per}}([0, 2\pi); \mathbb{R})$ satisfies

$$0 < c_1 \leq \Theta(\vartheta) \leq c_2 < \infty, \quad \forall \vartheta \in [0, 2\pi).$$

In addition, the small dimensionless parameter $0 < \varepsilon \ll 1$ describes the inverse of the magnetic field strength $|B^\varepsilon|$, and we are consequently interested in deriving the effective dynamics of solutions to (1.3) and (1.4) as $\varepsilon \to 0_+$. The power $\alpha + 1$ appearing in (1.4) might seem ad hoc at first glance, but it is chosen in a way that will allow us to expand various quantities in integer powers of the perturbation parameter ($\varepsilon p$), see below. In principle, other powers can be considered as well, but the formulas for the expansion become much more cumbersome. In view of the fact that $A^\varepsilon$ can be rewritten as

$$A^\varepsilon(x) = \frac{1}{\varepsilon^\alpha} \left( 0, 0, \frac{|x|}{\varepsilon} \Theta(\vartheta) \right) = \frac{1}{\varepsilon^\alpha} \left( 0, 0, \frac{A(x)}{\varepsilon} \right)$$

it is natural to introduce rescaled spatial variables via

$$x' = \frac{x}{\varepsilon}, \quad z' = z.$$

The new unknown is then given by

$$\psi^\varepsilon(t, x', z') = \varepsilon \Psi(t, \varepsilon x', z'),$$
where the prefactor ensures that the $L^2$-norm remains invariant. From now on, we shall also assume that the initial data $\Psi_0$ is $\varepsilon$-independent in the rescaled variables $(x', z')$ and normalized s.t. $\|\psi_0\|_{L^2} = 1$.

**Remark 1.2.** In terms of the original variables, this means that

$$\Psi_0(x, z) = \frac{1}{\varepsilon} \psi_0 \left( \frac{x}{\varepsilon}, z \right).$$

In other words, $\Psi_0$ is concentrated on the scale $\varepsilon$ w.r.t. to the $x$-directions, an assumption consistent with the asymptotic regime considered.

Rescaling the initial value problem (1.2) accordingly, we obtain (after dropping all the primes $'$, for simplicity)

$$i\varepsilon^2 \partial_t \psi^\varepsilon = \mathcal{H}^\varepsilon \psi^\varepsilon, \quad \psi^\varepsilon_{|t=0} = \psi_0(x, z),$$

where the rescaled Hamiltonian reads

$$\mathcal{H}^\varepsilon = -\Delta_x - \varepsilon^2 \partial^2_z + |x|^{2\alpha} \Theta(\vartheta)^2 - 2i\varepsilon |x|^{\alpha} \Theta(\vartheta) \partial_z.$$

We seek an asymptotic description as $\varepsilon \to 0_+$ of

$$\psi^\varepsilon(t, x, z) = e^{-it\mathcal{H}^\varepsilon/\varepsilon^2} \psi_0(x, z),$$

solution to (1.5). To this end, we again have the fiber decomposition

$$\mathcal{F} \mathcal{H}^\varepsilon \mathcal{F}^{-1} = \int_\mathbb{R} \mathcal{H}^\varepsilon(p) \, dp,$$

where $\mathcal{H}^\varepsilon(p)$ can be written as a perturbed Hamiltonian of the form

$$\mathcal{H}^\varepsilon(p) = H_0 + \varepsilon p \hat{V}^\varepsilon p(x).$$

Here, and in the following, $H_0$ denotes the $\varepsilon$-independent part of the rescaled Hamiltonian, i.e.,

$$H_0 := -\Delta + |x|^{2\alpha} \Theta(\vartheta)^2, \quad x \in \mathbb{R}^2,$$

and, for each fixed $p \in \mathbb{R}$, we have an effective potential given by

$$\hat{V}^\varepsilon p(x) := \varepsilon p + 2|x|^{\alpha} \Theta(\vartheta).$$

Note that the unperturbed operator $H_0$ describes the magnetic confinement of quantum particles in the $(x_1, x_2)$-plane, since for all $\alpha > 0$, the potential

$$V_0(x) = |x|^{2\alpha} \Theta(\vartheta)^2 \to +\infty \quad \text{as } |x| \to \infty,$$

in view of Assumption 1.1. In particular, the spectrum $\sigma(H_0)$ is purely discrete, see Lemma 2.1.

### 1.2. Effective dynamics.

Assume, for simplicity, that $\lambda \in \sigma(H_0)$ is a given, non-degenerate eigenvalue of $H_0$ with associated eigenfunction $\chi \in L^2(\mathbb{R}^2)$. Then, for each fixed $p \in \mathbb{R}$ and $\varepsilon \in (0, 1]$ sufficiently small, we may, in view of (1.7), regard $(\varepsilon p) \ll 1$ as an effective perturbation parameter within the fiber decomposition of $\mathcal{H}^\varepsilon$. Using classical techniques from analytic perturbation theory allows us to derive (for $\varepsilon$ small enough) a convergent series for the perturbed eigenvalues $\lambda^\varepsilon$ in the form

$$\lambda^\varepsilon = \lambda + \varepsilon p \lambda_1 + (\varepsilon p)^2 \lambda_2 + \ldots$$

The coefficients $\lambda_j \in \mathbb{R}$ are thereby computed by an iterative procedure starting from the unperturbed eigenvalue $\lambda \in \sigma(H_0)$, see Section 2.2. for more details. In turn, this yields an effective description of $\psi^\varepsilon(t)$ as $\varepsilon \to 0_+$:

Indeed, let $\rho_0 > 0$ denote some (large) cut-off parameter in the momentum variable $p \in \mathbb{R}$ dual to $z$, and assume that $\psi_0$ is of the form

$$\psi_0(x, z) = a(z) \chi(x),$$
where the modulating amplitude $a \in L^2(\mathbb{R})$ is such that $\hat{a}(p) = 0$ for $|p| > p_0$. We shall prove (cf. Theorem 2.5) that in this case the solution to (1.5) satisfies
\begin{equation}
\psi^\varepsilon(t, x, z) = \phi^\varepsilon(t, x) + O(\varepsilon(1 + |t|)),
\end{equation}
where the $O$-notation should be understood w.r.t. the $L^2$-norm, and
\begin{equation}
\phi^\varepsilon(t, x) = e^{-it\lambda_1/\varepsilon^2} \chi(x) e^{it\lambda_2 a}\left(z - \frac{t\lambda_1}{\varepsilon}\right).
\end{equation}
We observe from (1.10) and (1.11) that the solution $\psi^\varepsilon$ is highly oscillatory in time, with a constant but singular phase $\propto \varepsilon^{-2}$. More interestingly, along the $z$-axis the solution exhibits the dispersive behavior of a free particle with effective mass $M = \lambda_2^{-1}$, while also being subject to a strong drift with velocity $v^\varepsilon = \lambda_1/\varepsilon$. We shall show in Section 2.2. that $\lambda_1 > 0$ and given by
\begin{equation}
\lambda_1 = 2 \langle \chi, | \cdot |^2 \Theta \rangle_{L^2(\mathbb{R}^2)},
\end{equation}
while the formula for $\lambda_2$ is slightly more involved. Such strong drifts along $z$ can also be observed in numerical simulations of the corresponding classical particle dynamics, see [1].

The compact support condition on $\hat{a}$ is needed in our approach to justify the use of $\varepsilon = \varepsilon \rho$ as a small parameter. To this end, one should note that the solution to (1.5) is to be understood via
\begin{equation}
\psi^\varepsilon(t, x, z) = \mathcal{F}^{-1}(S_1(p) \hat{\psi}_0(x, p))(z),
\end{equation}
where $\mathcal{H}(p)$ is the fiber Hamiltonian defined in (1.7) and $S_1(p)$ the associated Schrödinger group acting on $L^2(\mathbb{R}^2)$. In particular, if $\hat{\psi}_0$ is compactly supported in $p \in \mathbb{R}$, then
\begin{equation}
\psi^\varepsilon(t, x, p) = \hat{\psi}^\varepsilon(t, x, p) 1_{\{|p| \leq p_0\}} \quad \text{for all } t \in \mathbb{R},
\end{equation}
since the characteristic function $1_{\{|p| \leq p_0\}}$ clearly commutes with $S_1(p)$.

1.3. Comparison with existing results. When comparing the effective dynamics (1.10) to earlier results in the literature, we note that in the case of a purely radial vector potential $A(x) = (0, 0, A(|x|))$, Yafaev gave a detailed investigation of the spectral properties of $\mathcal{H}$ and the associated long-time behavior of solutions in [8, 9]. The fact that $A(|x|)$ is purely radial thereby allows for a description of the spectrum $\sigma(\mathcal{H}(p))$ in terms of spectral band-functions. In turn this allows for a representation of the long-time behavior of solutions as $t \to \pm \infty$, which is governed by group velocities that play the same role as $v^\varepsilon = \lambda_1/\varepsilon$ does in formula (1.11). We also note that an analogous study for the case of unitary $B$-fields (induced by vector potentials (1.1) in general spatial dimensions) has later been done in [1].

In contrast to all of these works we can allow for an azimuthal dependence of the vector potential described by the function $\Theta$. While we do not study the asymptotic regime as $t \to \pm \infty$, we instead consider the time-evolution on large time-scales up to $t \sim O(\varepsilon^{-1})$ in the large field limit as $\varepsilon \to 0_+$. Our results can therefore be seen as complementary to [8, 9].

1.4. Stability under more general perturbations. The effective dynamics (1.10) is obtained for the particular class of vector potentials given in (1.4). It is a natural question whether a similar effective description holds true for more general, translation invariant magnetic fields. To answer this question, the main observation is that any $\psi^\varepsilon$ which is (approximately) given by (1.10) remains strongly localized near the origin of the $(x_1, x_2)$-plane. Thus we expect that only the behavior of $A^\varepsilon$ in a small neighborhood near this origin plays a significant role.
In the second part of this work, we shall rigorously show that (1.10) indeed remains valid for a much more general class of vector potentials, given by

\[
A_\alpha^\varepsilon(x) = \frac{1}{\varepsilon^{\alpha+1}} (0, 0, (|x|^{\alpha} \Theta(\vartheta) + a(x))).
\]

Here, \(a \in L^\infty_{\text{loc}}(\mathbb{R}^2)\) is assumed to be vanishing in a small neighborhood around the origin of the \((x_1, x_2)\)-plane and polynomially bounded at infinity (see Section 3 for a precise formulation). In particular, \(a\) is not necessarily homogenous and can become arbitrarily large as \(|x| \to \infty\). The class of vector potentials (1.12) is therefore by no means a small perturbation of the previous case (1.4).

Nevertheless, the heuristic picture described above can be made mathematically rigorous by using the fact that all eigenfunctions \(\chi\) of the unperturbed Hamiltonian \(H_0\), defined in (1.8), admit a strong, i.e. exponential, decay. This implies that the contribution of \(A_\alpha^\varepsilon(x)\) for large \(|x|\) are strongly suppressed in our perturbative analysis. Indeed, if \(P_\lambda\) denotes the spectral projection onto a simple eigenvalue \(\lambda \in \sigma(H_0)\) and \(H_\alpha^\varepsilon\) the magnetic Hamiltonian with an additional perturbation given by \(a\), we shall prove that (cf. Theorem 3.3):

\[
\left( e^{-itH_\alpha^\varepsilon/\varepsilon^2} - e^{-itH_\varepsilon/\varepsilon^2} \right) \left( e^{itH_\alpha^\varepsilon/\varepsilon^2} - e^{-itH_\varepsilon/\varepsilon^2} \right) \|P_\lambda\|_{L^2(\mathbb{R}^3)} = O(\varepsilon(1 + |t|)).
\]

In particular, this implies that for initial data

\[
\psi_0(x, z) = P_\lambda \psi_0(x, z) = a(x) \chi(z),
\]

with compactly supported amplitude \(\tilde{a}\), we have, by triangle inequality:

\[
\|e^{-itH_\alpha^\varepsilon/\varepsilon^2} \psi_0 - \phi^\varepsilon(t)\|_{L^2(\mathbb{R}^3)} \leq \left\| \left( e^{-itH_\alpha^\varepsilon/\varepsilon^2} - e^{-itH_\varepsilon/\varepsilon^2} \right) P_\lambda \psi_0 \right\|_{L^2(\mathbb{R}^3)} + \|e^{-itH_\varepsilon/\varepsilon^2} \psi_0 - \phi^\varepsilon(t)\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon(1 + |t|).
\]

In view of (1.10) and (1.13), both terms in the middle are seen to be of the same order, resulting in the last inequality. Thus, the effective dynamics \(\phi^\varepsilon\) given by (1.11) is also an approximate solution in the case with an additional inhomogeneous perturbation \(a\).

Let us emphasize, however, that a rigorous mathematical justification of this approach is not straightforward. In particular it cannot be obtained within the framework of regular perturbation theory, since \(a\) is allowed to grow polynomially as \(|x| \to \infty\) and thus is not an \(H_0\)-bounded perturbation. Indeed, the mere assumption of a polynomial bound on \(a\), combined with the fact that \(a\) vanishes in a vicinity of the origin, even allows for \(\sigma(H_\alpha^\varepsilon)\) to be purely continuous, see Remark 3.4. To overcome this obstacle one has to use the more sophisticated framework of asymptotic perturbation theory, cf. [3, 4, 7] for a general overview. More precisely, we shall employ the theory of almost invariant subspaces developed in [6].

The latter relies strongly on exponential decay bounds of the eigenfunctions of \(H_0\). While these bounds can, in principle, be obtained from the general theory of Agmon-Combes-Thomas (see e.g. [2]), for the reader’s convenience we shall give an elementary proof of this fact in the Appendix.

This paper is now organized as follows: In Section 2 we shall give a rigorous proof of the effective dynamics described in (1.10). We shall also show how to generalize this result to arbitrary initial data in \(L^2(\mathbb{R}^3)\), if one is willing to give up on an explicit \(\varepsilon\)-dependent approximation error. The case of polynomially bounded perturbations of \(A^\varepsilon\) in the form (1.12) is then studied in Section 3. In it we shall first recall some general aspects of the theory of almost invariant subspaces and then apply these results to our setting.
Notation: Throughout this work, we shall write $a \lesssim b$ if there exists a constant $C > 0$, independent of $\varepsilon \in (0, 1]$ and $t \in \mathbb{R}$, such that $a \leq Cb$.

2. THE HOMOGENEOUS CASE: REGULAR PERTURBATION THEORY

Assume that $\mathcal{A}^\varepsilon$ is given by (1.4), and recall that, after applying the partial Fourier-transformation to (1.5), we are lead to consider the following perturbed initial value problem:

$$i\varepsilon^2 \partial_t \tilde{\psi}^\varepsilon = (H_0 + \varepsilon p\hat{V}^\varepsilon p(x))\tilde{\psi}^\varepsilon, \quad \tilde{\psi}^\varepsilon_{|t=0} = \tilde{\psi}_0(\cdot, p),$$

where $H_0$ is a confining Hamiltonian of the form

$$H_0 = -\Delta_x + |x|^{2\alpha} \Theta(\vartheta)^2, \quad \alpha > 0.$$  

Classical results from spectral theory (see, e.g., [5]), describe the main properties of the unperturbed operator $H_0$:

Lemma 2.1. Assume that $\Theta$ satisfies Assumption (1.1). Then, for all $\alpha > 0$, the operator $H_0$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$. Moreover, we have:

(i) $H_0 \geq 0$ and $(H_0 + 1)^{-1}$ is compact.

(ii) The spectrum $\sigma(H_0)$ is an increasing sequence $(\lambda_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ of positive eigenvalues with finite multiplicity and $\lambda_n \to +\infty$ as $n \to \infty$. In particular,

$$\sigma(H_0) = \sigma_{\text{disc}}(H_0) \text{ and } \sigma_{\text{cova}}(H_0) = \emptyset.$$  

(iii) The associated eigenfunctions $(\chi_n)_{n \in \mathbb{N}}$, counted with multiplicity, form an orthonormal basis of $L^2(\mathbb{R}^2)$.

In order to ensure that $\hat{V}^\varepsilon p(x) = \varepsilon p + 2|x|^\alpha \Theta(\vartheta)$ can indeed be considered as a regular perturbation of $H_0$, we have the following simple lemma:

Lemma 2.2. For all $u(\cdot, p) \in \mathcal{D}(H_0) \subset L^2(\mathbb{R}^2)$,

$$\|\hat{V}^\varepsilon p u(\cdot, p)\|_{L^2(\mathbb{R}^2)} \leq (\sqrt{2} + \varepsilon |p|)\|u(\cdot, p)\|_{L^2(\mathbb{R}^2)} + \sqrt{2}\|H_0 u(\cdot, p)\|_{L^2(\mathbb{R}^2)}.$$  

Thus, the potential $\hat{V}^\varepsilon p$ is $H_0$-bounded.

Proof. First, let $u \in C_0^\infty(\mathbb{R}^2)$ and denote $v(x) = |x|^{\alpha} \Theta(\vartheta)$. Since $-\Delta$ is positive, i.e., $\langle u, (-\Delta) u \rangle_{L^2} \geq 0$, we have, by Cauchy-Schwarz:

$$\|uv\|_{L^2}^2 = \langle u, v^2 u \rangle_{L^2} \leq \langle u, H_0 u \rangle_{L^2} \leq \|u\|_{L^2} \|H_0 u\|_{L^2} \leq \frac{1}{2} (\|u\|_{L^2}^2 + \|H_0 u\|_{L^2}^2).$$  

Hence by triangle inequality,

$$\|\hat{V}^\varepsilon p u\|_{L^2} \leq \varepsilon |p|\|u\|_{L^2} + 2\|vu\|_{L^2} \leq (\sqrt{2} + \varepsilon |p|)\|u\|_{L^2} + \sqrt{2}\|H_0 u\|_{L^2}.$$  

Since $H_0$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$, this estimate extends to $u \in \mathcal{D}(H_0)$. \qed

Lemma 2.2 implies that $\mathcal{H}^\varepsilon(p)$ is an analytic family of operators in the sense of Kato, and hence we may apply the following result from regular perturbation theory, see [4, 7]:

Proposition 2.3. Let $p \in \mathbb{R}$ be fixed and $\lambda \in \sigma(H_0)$ be an $m$-degenerate eigenvalue of $H_0$. Then, there exists an $\varepsilon_0 > 0$ such that for $|\varepsilon p| < \varepsilon_0|p|$, and $k = 1, \ldots, m$ there exist:

(i) $\lambda_k(\varepsilon p)$, not necessarily distinct, real functions;

(ii) $\chi_k(\cdot, \varepsilon p) \in L^2(\mathbb{R}^2)$, normalized via $\|\chi_k(\cdot, \varepsilon p)\|_{L^2} = 1$;

(iii) one-dimensional orthogonal projection $P_k(\varepsilon p) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, with

$$P_k(\varepsilon p)P_k(\varepsilon p) = \delta_k, P_k(\varepsilon p).$$  

where we assume that \( \text{supp} \phi \) solution (2.2) \( \lambda \) in the form that shall write the asymptotic expansions of the perturbed eigenvalues, as same as for \( H \).

Theorem 2.5. Let \( \Theta \) satisfy Assumption 1.1 and let the initial data \( \psi_0 \in L^2(\mathbb{R}^3) \) be such that
\[
F \psi_0(x, p) \equiv \hat{\psi}_0(x, p) = \hat{a}(p) \chi_k(x),
\]
where we assume that \( \text{supp} \hat{a} \subset [-p_0, p_0] \), for some \( p_0 > 0 \). Define an approximate solution \( \psi^\varepsilon \) to (1.5) by
\[
\phi^\varepsilon_k(t, x, z) = e^{-it\lambda/\varepsilon^2} \chi_k(x)e^{it\lambda_k z \Delta_\varepsilon} a \left( z - \frac{t\lambda_k}{\varepsilon^2} \right).
\]
Then, there exists an \( \varepsilon_0 \in (0, 1] \), such that for all \( 0 < \varepsilon < \varepsilon_0 \) and all \( t \in \mathbb{R} \):
\[
\left\| e^{-itH_{\varepsilon^2}} \psi_0 - \phi^\varepsilon_k(t) \right\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon (1 + |t|).
\]

Remark 2.4. One observes from (1.9) that the term \( \varepsilon p \) within \( \hat{V}^{\varepsilon, p} \) just shifts the energy by a constant factor. In particular, the eigenfunctions for \( H^{\varepsilon} \) are the same as for \( H - (\varepsilon p)^2 \).

Now, let \( \lambda \in \sigma(H_0) \) be an \( m \)-degenerate eigenvalue of \( H_0 \). In what follows, we shall write the asymptotic expansions of the perturbed eigenvalues, as \( \varepsilon \to 0 \), in the form
\[
\lambda_k(\varepsilon p) = \lambda + \sum_{j=1}^{\infty} (\varepsilon p)^j \lambda_{k,j}, \quad k = 1, \ldots, m,
\]
where \( \lambda_{k,j} \in \mathbb{R} \). Analogously, we shall write for the associated spectral projections
\[
P_k(\varepsilon p) = P_k(0) + \sum_{j=1}^{\infty} (\varepsilon p)^j Q_{k,j},
\]
and denote
\[
\chi_k \equiv \chi_k(\cdot, 0).
\]

We can now state the first main result describing the effective dynamics of solutions to (1.5).

Theorem 2.5. Let \( \Theta \) satisfy Assumption 1.1 and let the initial data \( \psi_0 \in L^2(\mathbb{R}^3) \) be such that
\[
F \psi_0(x, p) \equiv \hat{\psi}_0(x, p) = \hat{a}(p) \chi_k(x),
\]
where we assume that \( \text{supp} \hat{a} \subset [-p_0, p_0] \), for some \( p_0 > 0 \). Define an approximate solution \( \psi^\varepsilon \) to (1.5) by
\[
\phi^\varepsilon_k(t, x, z) = e^{-it\lambda/\varepsilon^2} \chi_k(x)e^{it\lambda_k z \Delta_\varepsilon} a \left( z - \frac{t\lambda_k}{\varepsilon^2} \right).
\]
Then, there exists an \( \varepsilon_0 \in (0, 1] \), such that for all \( 0 < \varepsilon < \varepsilon_0 \) and all \( t \in \mathbb{R} \):
\[
\left\| e^{-itH_{\varepsilon^2}} \psi_0 - \phi^\varepsilon_k(t) \right\|_{L^2(\mathbb{R}^3)} \lesssim \varepsilon (1 + |t|).
\]

Proof. First, let \( p_0 > 0 \) be a momentum cut-off, such that \( \hat{a}(p) = 0 \) for all \( |p| > p_0 \). Then, there exists an \( \varepsilon_0 > 0 \), such that both \( \lambda_k(\varepsilon p) \) and \( \chi_k(\cdot, \varepsilon p) \in L^2(\mathbb{R}^2) \) are analytic for all \( \varepsilon < \varepsilon_0 \) and \( |p| < p_0 \) on
\[
I_0 = ( -\varepsilon_0 p_0, \varepsilon_0 p_0 ) \subset \mathbb{R}.
\]
This then implies that there exists a constant \( c_0 = c_0(p_0) > 0 \), such that
\[
\left\| \chi_k - \chi_k(\cdot, \varepsilon p) \right\|_{L^2(\mathbb{R}^2)} \lesssim c_0 \varepsilon |p|.
\]
Keeping in mind that \( \tilde{\psi}_0(x, p) = \tilde{a}(p) \chi_k(x) \), we can thus estimate
\[
\left\| \tilde{\psi}_0(\cdot, p) - \tilde{a}(p) \chi_k(\cdot, \epsilon p) \right\|_{L^2(\mathbb{R}^2)} \leq c_0 \epsilon |p| |\tilde{a}(p)|.
\]
Next, we observe that
\[
e^{-it\epsilon^2 (H_0 + \epsilon p \hat{V}^{x,p})} \chi_k(x, \epsilon p) = e^{-it\epsilon^2 \lambda_k(\epsilon p)} \chi_k(x, \epsilon p).
\]
In view of (2.2) and using Taylor expansion of the exponential function, we find
\[
e^{-it\epsilon^2 \lambda_k(\epsilon p)} - e^{-it\epsilon^2 \lambda_k(\epsilon p)} e^{-it\epsilon^2 \lambda_k z} \lesssim \epsilon |p|^3 |t|,
\]
uniformly on \( I_0 \). Combining this expansion with (2.5), and using a triangle inequality, we obtain
\[
\left\| e^{-it\epsilon^2 \lambda_k(\epsilon p)} \chi_k(\cdot, \epsilon p) - e^{-it\epsilon^2 \lambda_k(\epsilon p)} \chi_k \right\|_{L^2(\mathbb{R}^2)}
\leq \left\| e^{-it\epsilon^2 \lambda_k(\epsilon p)} (\chi(\cdot, \epsilon p) - \chi_k) \right\|_{L^2(\mathbb{R}^2)} + 
\left\| e^{-it\epsilon^2 \lambda_k(\epsilon p)} - e^{-it\epsilon^2 \lambda_k(\epsilon p)} e^{-it\epsilon^2 \lambda_k z} \right\|_{L^2(\mathbb{R}^2)}
\lesssim \epsilon |p|(1 + p^2 |t|).
\]
Recall that the exact solution to (2.1) is given by
\[
\tilde{\psi}^c(t, x, p) = e^{-it\epsilon^2 (H_0 + \epsilon p \hat{V}^{x,p})} \tilde{\psi}_0(x, p),
\]
where \( \tilde{\psi}_0(x, p) = \tilde{a}(p) \chi_k(x) \). Denoting
\[
\tilde{\phi}_k(t, x, p) = e^{-it\epsilon^2 \lambda_k e^{-it\epsilon^2 \lambda_k z} \tilde{a}(p) \chi_k(x),
\]
we obtain, in view of (2.6) and (2.8), that
\[
\left\| \tilde{\psi}^c(t, \cdot, p) - \tilde{\phi}_k(t, \cdot, p) \right\|_{L^2(\mathbb{R}^2)} \lesssim \epsilon |p|(1 + p^2 |t|) |\tilde{a}(p)|.
\]
This holds uniformly for \( p \in I_0 \), and so by the compact support of \( \tilde{a}(p) \) we may further estimate
\[
\left\| \tilde{\psi}^c(t) - \tilde{\phi}_k(t) \right\|_{L^2(\mathbb{R}^2)} \leq C \epsilon (1 + |t|),
\]
for some constant \( C = C(p_0, a) > 0 \). Next, we consider the approximate solution under the (partial) inverse Fourier transform,
\[
\phi_k^c(t, x, z) = \left( \mathcal{F}^{-1} \tilde{\phi}_k^c \right)(t, x, z)
= e^{-it\epsilon^2 \lambda_k} \chi_k(x) \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} e^{ip(z-t\epsilon^{-1} \lambda_k z)} (e^{-it\epsilon^2 \lambda_k z} \tilde{a}(p)) dp,
\]
and note that the term \( e^{-it\epsilon^2 \lambda_k z} \) is just the Fourier transform of the free evolution \( e^{it\lambda_k z \Delta_z} \) in \( z \)-direction. Thus, we explicitly obtain
\[
\frac{1}{\sqrt{2\pi}} \int_\mathbb{R} e^{ip(z-t\epsilon^{-1} \lambda_k z)} (e^{-it\epsilon^2 \lambda_k z} \tilde{a}(p)) dp = e^{it\lambda_k z \Delta_z} a \left( z - t\epsilon^{-1} \lambda_k z \right),
\]
where \( a \) is the Fourier-inverse of \( \tilde{a} \). Finally, applying this to (2.9), along with Plancherel’s theorem, gives us
\[
\left\| \psi^c(t) - \phi_k^c(t) \right\|_{L^2(\mathbb{R}^2)} = \left\| \mathcal{F}^{-1} (\tilde{\psi}^c(t) - \tilde{\phi}_k(t)) \right\|_{L^2(\mathbb{R}^2)} \lesssim \epsilon (1 + |t|),
\]
for all \( \epsilon < \epsilon_0 \). This completes the proof. \( \square \)
2.1. More general initial data. The previous result requires us to impose rather severe restrictions on the initial data. In this subsection we shall show how to generalize Theorem 2.5 to arbitrary initial data in $L^2$. The price one pays, however, is the loss of an explicit, purely $\varepsilon$-dependent convergence rate:

Indeed, by Lemma 2.1 we know that $\sigma(H_0)$ is an increasing sequence of eigenvalues $\lambda^n \in \mathbb{R}$, $n \in \mathbb{N}$, with finite-multiplicity $m_n \in \mathbb{N}$. Note that here we use the superscript $n$ to index the eigenvalues, not to indicate a power. For each $n \in \mathbb{N}$ we denote the perturbed eigenvalues of $H^\varepsilon(p)$ by $\lambda^n_1(\varepsilon p), \ldots, \lambda^n_{m_n}(\varepsilon p)$, where $m_n \in \mathbb{N}$ is the multiplicity of a given $\lambda^n \in \sigma(H_0)$. Similarly to (2.2), we can express $\lambda^n_k(\varepsilon p)$ as a power series:

$$\lambda^n_k(\varepsilon p) = \lambda^n + \sum_{j=1}^{\infty} (\varepsilon p)^j \lambda^n_{k,j}, \quad n \in \mathbb{N}.$$ 

We recall that the set of unperturbed eigenfunctions associated to $\lambda^n$, i.e.

$$\{ \chi^n_k : 1 \leq n < \infty, 1 \leq k \leq m_n \}$$

forms an orthonormal basis of $L^2(\mathbb{R}^2)$. Hence, any $\psi_0 \in L^2(\mathbb{R}^3) \simeq L^2(\mathbb{R}^2) \otimes L^2(\mathbb{R})$ can be written as

$$\psi_0(x, z) = \sum_{n=1}^{\infty} \sum_{k=1}^{m_n} a^n_k(z) \chi^n_k(x),$$

with amplitudes $a^n_k \in L^2(\mathbb{R}^2)$. Under the partial Fourier transform this becomes

$$\tilde{\psi}_0(x, p) = \sum_{n=1}^{\infty} \sum_{k=1}^{m_n} \hat{a}^n_k(p) \chi^n_k(x),$$

where the coefficients $\hat{a}^n_k \in L^2(\mathbb{R})$ are not necessarily compactly supported. For such general initial data, we have the following result:

**Proposition 2.6.** Assume $\Theta$ satisfies assumption (1.1) and let $\psi_0 \in L^2(\mathbb{R}^3)$. Then, for any $\delta > 0$ there exists $N_\delta \in \mathbb{N}$ and $\varepsilon_\delta > 0$ such that for all $0 < \varepsilon < \varepsilon_\delta$, an approximate solution to (1.5) is given by

$$\phi^\varepsilon(t, x, z) = \sum_{n=1}^{N_\delta} \sum_{k=1}^{m_n} e^{-it\lambda^n_kz^2/\varepsilon} \chi^n_k(x) e^{it \lambda^n_k z^2 \Delta_t + \lambda^n_k (z - \frac{t\lambda^n_k}{\varepsilon})},$$

where all $\hat{a}^n_k \in C_0(\mathbb{R})$ are compactly supported $L^2$-approximations of the coefficients $\hat{a}^n_k$ appearing in (2.12), and we have the estimate

$$\left\| e^{-it \mathcal{H}^\varepsilon/\varepsilon^2} \psi_0 - \phi^\varepsilon(t) \right\|_{L^2(\mathbb{R}^3)} \lesssim \delta (1 + |t|),$$

**Proof.** By linearity of (1.5), we will be able to reduce our initial data to a finite sum and apply the same analysis from Theorem 2.5: First, for any $\delta > 0$ we can choose $N_\delta \in \mathbb{N}$ large enough, such that

$$\tilde{\varphi}_1(x, p) = \sum_{n=1}^{N_\delta} \sum_{k=1}^{m_n} \hat{a}^n_k(p) \chi^n_k(x)$$

satisfies

$$\left\| \tilde{\psi}_0 - \tilde{\varphi}_1 \right\|_{L^2(\mathbb{R}^2)} \lesssim \delta.$$

Then, for any $\delta_1 > 0$ we may approximate each $\hat{a}^n_k \in L^2(\mathbb{R})$, where $1 \leq n \leq N_\delta$, $1 \leq k \leq m_n$, by a compactly supported $\tilde{\delta}^n_k \in C_0(\mathbb{R})$, such that

$$\left\| \hat{a}^n_k - \tilde{\delta}^n_k \right\|_{L^2(\mathbb{R})} \lesssim \delta_1.$$
Defining
\[ \hat{\varphi}_2(x, p) = \sum_{n=1}^{N_\delta} \sum_{k=1}^{m_n} \hat{b}_k^n(p) \chi_n^k(x), \]
we thus have
\[ \| \hat{\psi}_0 - \hat{\varphi}_2 \|_{L^2(\mathbb{R}^2)} \lesssim \sum_{n=1}^{N_\delta} m_n \delta_1 + \delta. \]
Finally, let
\[ \hat{\varphi}_3(x, p) = \sum_{n=1}^{N_\delta} \sum_{k=1}^{m_n} \hat{b}_k^n(p) \chi_n^k(x, \varepsilon p), \]
then, by (2.5) we can estimate
\[ \| \hat{\psi}_0 - \hat{\varphi}_3 \|_{L^2(\mathbb{R}^2)} \lesssim \sum_{n=1}^{N_\delta} m_n (\varepsilon + \delta_1) + \delta. \]
Having reduced our initial data to a finite sum of terms compactly supported in \( p \in \mathbb{R} \), we may now directly apply Theorem 2.5 to each term in the approximate initial data given by (2.16). Hence, we obtain
\[ \| \hat{\psi}^{\varepsilon}(t) - \hat{\varphi}^{\varepsilon}(t) \|_{L^2(\mathbb{R}^2)} \lesssim \sum_{n=1}^{N_\delta} m_n (\varepsilon(1 + |t|) + \delta_1) + \delta, \]
where \( \hat{\varphi}^{\varepsilon} \) is given by (2.13) and \( \hat{b}_k^n \) denotes the Fourier-inverse of \( \hat{b}_k^n \). To finally bound the error terms, we first choose \( \delta > 0 \) arbitrarily small. This choice will fix the values of \( N_\delta \) and \( \sum_{n=1}^{N_\delta} m_n \). We can then choose \( \varepsilon < 1 \) and \( \delta_1 < 1 \) small enough, such that for all \( 0 < \varepsilon < \varepsilon_\delta \),
\[ \sum_{n=1}^{N_\delta} m_n (\varepsilon(1 + |t|) + \delta_1) \lesssim \delta(1 + |t|). \]
This yields the claim.

2.2. Computation of the main perturbation coefficients. We finally turn to the computation of the perturbation coefficients appearing in our leading order approximation \( \hat{\varphi}^{\varepsilon} \). Recall that for the \( n \)-th eigenspace of \( H_0 \), the corresponding perturbed eigenvalue of \( H^{\varepsilon}(p) \) can be written as
\[ \lambda_n^p(\varepsilon p) = \lambda_n + \varepsilon p \lambda_{n,1}^p + (\varepsilon p)^2 \lambda_{n,2}^p + O((\varepsilon p)^3). \]
Since only \( \lambda_{n,1}^p \) and \( \lambda_{n,2}^p \) enter into the definition of \( \hat{\varphi}^{\varepsilon} \), we shall in the following only focus on the computation of these two coefficients.
Indeed, using well-known formulas from perturbation theory (see, e.g., [7]), we have:
\[ \tilde{\lambda}_{k,1}^n = \langle \lambda_k^n, \hat{V}^{\varepsilon,p} \chi_n^k \rangle_{L^2(\mathbb{R}^2)}, \]
\[ \tilde{\lambda}_{k,2}^n = - \sum_{\varepsilon, \ell \neq n}^{\infty} \sum_{j=1}^{m_\ell} (\lambda_\ell - \lambda_n)^{-1} |\langle \lambda_k^n, \hat{V}^{\varepsilon,p} \chi_j^\ell \rangle_{L^2(\mathbb{R}^2)}|^2. \]
Here, we use the tilde notation to indicate that these are not yet the final values for \( \lambda_{k,1}^p \) and \( \lambda_{k,2}^p \), since \( \hat{V}^{\varepsilon,p} \) will introduce some additional factors of \( \varepsilon p \). To obtain an
expansion of the form (2.17), we thus need to recombine terms according to their respective order in $\varepsilon p$. To this end, we calculate

$$
\langle \chi_k^n \cdot , 0 \rangle, \tilde{V}^\varepsilon \cdot \chi_j^\ell \rangle_{L^2(\mathbb{R}^2)} = \langle \chi_k^n \cdot , (\varepsilon p + 2) |^n | \chi_j^\ell \rangle_{L^2(\mathbb{R}^2)}
$$

$$
= 2 \langle \chi_k^n \cdot , |^n | \chi_j^\ell \rangle_{L^2(\mathbb{R}^2)} + \varepsilon p \delta_{k,j} \delta_{n,\ell},
$$

where $\delta_{k,j}$ is the Kronecker delta. Denoting

$$
e_k^n := 2 \langle \chi_k^n \cdot , |^n | \chi_j^\ell \rangle_{L^2(\mathbb{R}^2)} \in \mathbb{R}^+,
$$

we find $\tilde{\lambda}_{k,1}^n = v_{k,k} + \varepsilon p$, and

$$
\tilde{\lambda}_{k,2}^n = - \sum_{\ell = 1, \ell \neq n}^\infty m_{\ell} (\lambda^\ell - \lambda^n)^{-1} |v_{k,j}^n + \varepsilon p \delta_{k,j} \delta_{n,\ell}|^2
$$

$$
= - \sum_{\ell = 1, \ell \neq n}^\infty m_{\ell} (\lambda^\ell - \lambda^n)^{-1} |v_{k,j}^n|^2.
$$

Hence, in accordance with the power series (2.17) we have:

$$
\lambda_{k,1}^n = v_{k,k}^n, \quad \lambda_{k,2}^n = 1 - \sum_{\ell = 1, \ell \neq n}^\infty m_{\ell} (\lambda^\ell - \lambda^n)^{-1} |v_{k,j}^n|^2.
$$

Unfortunately, it seems that for general $\alpha > 0$ there is no explicit expression available for the $L^2$-inner products which define the coefficients $v_{k,j}^n$, even in the case where $\Theta \equiv 1$.

3. Stability under Perturbations Vanishing in the Vicinity of the Origin

In this section we show that the previous results are stable against a large class of singular perturbations. To this end, we consider magnetic vector potentials of the form:

$$
\mathcal{A}_Q^\varepsilon(x) = \frac{1}{\varepsilon^{\alpha+1}}(0, 0, |x|^\alpha \Theta(y) + a(x)), \quad \alpha > 0,
$$

where $a \in L^\infty_{loc}(\mathbb{R}^2)$ is polynomially bounded at infinity and vanishes in an arbitrarily small neighborhood of the origin. More specifically, we impose:

**Assumption 3.1.** There exist constants $x_0 > 0, M < \infty$, and $\beta < \infty$, such that

$$
a(x) = 1_{\{|x| \geq x_0\}} a(x) \quad and \quad |a(x)| \leq M |x|^{\beta}.
$$

Note that, since for any $s > 0$:

$$
\sup_{|x| \geq x_0} |x|^{-s} = \frac{1}{x_0^s},
$$

Assumption 3.1 implies that

$$
|a(x)| \leq \frac{M}{x_0^s} |x|^{\beta + s}.
$$

Thus, one can take w.l.o.g. $\beta > 0$ sufficiently large, to ensure

$$
\beta \geq 3 + \alpha.
$$

Using the same rescaling of the spatial variables as in Section 1, a straightforward computation shows that instead of (1.7), one obtains

$$
\mathcal{H}^\varepsilon_a(p) = H_0 + \varepsilon p \tilde{V}^\varepsilon \cdot (x) + \varepsilon^{3-\alpha} \tilde{W}^\varepsilon \cdot (x)
$$

$$
= \mathcal{H}^\varepsilon(p) + \varepsilon^{3-\alpha} \tilde{W}^\varepsilon \cdot (x),
$$

where
where $H_0$ and $\hat{V}^{\varepsilon,p}$ are the same as in (1.8) and (1.9), respectively, and
\begin{equation}
\hat{W}_a^{\varepsilon,p}(x) = 2\varepsilon^{-\beta} a(\varepsilon x) + 2\varepsilon^{-\beta} |x|^\alpha \Theta(\theta) a(\varepsilon x) + \varepsilon^{-\beta-\alpha} (a(\varepsilon x))^2.
\end{equation}

**Lemma 3.2.** Assumption 3.1 implies that $\hat{W}_a^{\varepsilon}$ is polynomially bounded, uniformly for $0 < \varepsilon \leq 1$, i.e.
\begin{equation}
|\hat{W}_a^{\varepsilon}(x)| \lesssim (1 + |x|)^{2\beta}.
\end{equation}

**Proof.** We first estimate, using Assumption 3.1 and the fact that $\Theta$ is bounded:
\begin{align*}
|\hat{W}_a^{\varepsilon,p}(x)| \lesssim |p| \varepsilon^{1-\beta} |a(\varepsilon x)| + \varepsilon^{-\beta} |x|^{\alpha} |a(\varepsilon x)| + \varepsilon^{-\beta-\alpha} |a(\varepsilon x)|^2 \\
\lesssim (1 + |p|)(\varepsilon |x|^\beta + |x|^{\alpha+\beta} + \varepsilon^{\beta-\alpha}|x|^{2\beta})
\end{align*}

We now notice that $\beta - \alpha > 3$, in view of (3.2), and thus all the powers of $\varepsilon$ are greater than zero, which yields
\begin{equation}
|\hat{W}_a^{\varepsilon,p}(x)| \lesssim (1 + |p|)(1 + |x|)^{2\beta}.
\end{equation}

This setting will allow us to prove that the result of Section 2 remains valid. More precisely we shall show:

**Theorem 3.3.** Suppose $A_0^\varepsilon$ is given by (3.1), where $a$ satisfies Assumption 3.1. Let $\lambda \in \sigma(H_0)$ and $P_\lambda$ the corresponding spectral projection. Then there exists $\varepsilon_0 \in (0,1]$, such that for all $0 < \varepsilon < \varepsilon_0$:
\begin{equation}
\left\| \left(e^{-\iota t H_a^\varepsilon} - e^{-\iota t \hat{H}^\varepsilon} \right) P_\lambda \right\| \lesssim \varepsilon (1 + |t|).
\end{equation}

Here, $H_a^\varepsilon$ and $\hat{H}^\varepsilon$ are defined via their fiber representations given in (3.3).

For the proof of this theorem, we shall rely on the asymptotic perturbation theory developed in [6] (see also [3, 4] and the references therein). Sending the reader to [6] for more technical details, we shall give in the next subsection the main ideas behind this perturbation theory and the construction of almost invariant subspaces. Using these, the proof of Theorem 3.3 will then be given in Subsection 3.2, where we end with some concluding remarks.

### 3.1. Construction of almost invariant subspaces

The fact that $a$ is in general not homogenous implies that, in contrast to Section 2, the perturbation parameter is no longer given by $\varepsilon p$, but instead $\varepsilon$ and $p$ have to be seen as independent from each other. However, it is clear from the previous discussions that, once a momentum cut-off $p_0 > 0$ has been chosen, the precise value of $|p| \leq p_0$ no longer matters. We shall therefore stop tracking the dependence on $p$ and instead only focus on the dependence on $\varepsilon \ll 1$. To emphasize this fact, we introduce the following shorthand notation: Let
\begin{equation}
\mathcal{V}_a^\varepsilon(x) := p\hat{V}^{\varepsilon,p}(x) + \varepsilon^{\beta-\alpha-1}\hat{W}_a^{\varepsilon,p}(x),
\end{equation}

so that for $a \equiv 0$ we get back $\mathcal{V}_0^\varepsilon \equiv p\hat{V}^{\varepsilon,p}$. Using this, we consider the perturbed Hamiltonian $H_a^\varepsilon$ in the form
\begin{equation}
H_a^\varepsilon = H_0 + \varepsilon \mathcal{V}_a^\varepsilon(x).
\end{equation}

From (3.5) and the explicit form of $\hat{V}^{\varepsilon,p}(x)$, we know that $\mathcal{V}_a^\varepsilon(x)$ is polynomially bounded uniformly for $0 < \varepsilon \leq 1$:
\begin{equation}
|\mathcal{V}_a^\varepsilon(x)| \lesssim (1 + p_0^2)(1 + |x|)^{2\beta}.
\end{equation}
Now, let $\lambda \in \sigma(H_0)$, $P_\lambda$ be the corresponding spectral projection, and recall that if $a \equiv 0$, the potential $\mathcal{V}_a^\varepsilon$ is $H_0$-bounded (see Section 2). Thus, for $\varepsilon \in (0, 1]$ sufficiently small, we already know from regular perturbation theory that

$$\tag{3.7} P_\lambda(\varepsilon) = \sum_{j=0}^{\infty} \varepsilon^j Q_j(\varepsilon) \equiv P_\lambda + \sum_{j=1}^{\infty} \varepsilon^j Q_j(\varepsilon).$$

The $Q_j$ can thereby be expressed via (cf. [4]):

$$\tag{3.8} Q_j(\varepsilon) = (-1)^j \frac{i}{2\pi} \oint_{\Gamma} (H_0 - z)^{-1} (\mathcal{V}_0^\varepsilon (H_0 - z)^{-1})^j dz,$$

where

$$\Gamma = \left\{ z \in \mathbb{C} : |z - \lambda| = \frac{d}{2} \right\}, \quad d = \text{dist} (\lambda, \sigma(H_0) \setminus \lambda).$$

From (3.7) and $P_\lambda = P_\lambda^2$, we can compute

$$\sum_{j=0}^{\infty} \varepsilon^j Q_j(\varepsilon) = \left( \sum_{i=0}^{\infty} \varepsilon^i Q_i(\varepsilon) \right) \left( \sum_{k=0}^{\infty} \varepsilon^k Q_k(\varepsilon) \right),$$

and matching terms with equal powers of $\varepsilon$, we find

$$Q_j(\varepsilon) = \sum_{i=0}^{j} Q_i(\varepsilon) Q_{j-i}(\varepsilon).$$

Similarly, one can use (3.7) to expand $P_\lambda(\varepsilon) \mathcal{H}^\varepsilon = \mathcal{H}^\varepsilon P_\lambda(\varepsilon)$, and combining terms by their power in $\varepsilon$, one finds

$$[H_0, Q_{j+1}(\varepsilon)] = -[\mathcal{V}_0^\varepsilon, Q_j(\varepsilon)].$$

In the case $a \neq 0$, however, we cannot directly follow the same approach, since (3.6) does not imply that $\mathcal{V}_a^\varepsilon$ is $H_0$-bounded. Thus, (3.8) is a-priori not well defined if $\mathcal{V}_0^\varepsilon$ is replaced by $\mathcal{V}_a^\varepsilon$.

**Remark 3.4.** In the particular case where $a \in C^1(\mathbb{R}^2)$ and such that

$$a(x) = \begin{cases} 0 & \text{for } |x| \leq 1, \\ -|x|^a \Theta(\theta) & \text{for } |x| \geq 2, \end{cases}$$

we have

$$\mathcal{V}_a^\varepsilon(x) = \begin{cases} \mathcal{V}_0^\varepsilon(x) & \text{for } |x| \leq \frac{1}{\varepsilon}, \\ \varepsilon p^2 & \text{for } |x| \geq \frac{2}{\varepsilon}. \end{cases}$$

This implies that the spectrum of $\mathcal{H}_a^\varepsilon$ includes the interval $[(\varepsilon p)^2, \infty) \subset \mathbb{R}$ and thus $\sigma(H_a^\varepsilon)$ is no longer discrete, in contrast to $\sigma(H_0^\varepsilon)$.

The way out of this problem is to use an alternative formula for $Q_j(\varepsilon)$, obtained from (3.8) via the residue theorem (see e.g. [7], or [4, Chapter II.2.1] in the self-adjoint case):

$$\tag{3.9} Q_j(\varepsilon) = (-1)^{j+1} \sum_{\nu_1, \ldots, \nu_{j+1} \geq 0 : \sum_{l=1}^{j+1} \nu_l = j} S^{\nu_1} \mathcal{V}_0^\varepsilon S^{\nu_2} \mathcal{V}_0^\varepsilon \cdots \mathcal{V}_0^\varepsilon S^{\nu_{j+1}}.$$ 

Here, $S$ is the reduced resolvent of $H_0$ at $\lambda$, i.e.

$$\tag{3.10} S = \frac{i}{2\pi} \oint_{\Gamma} (H_0 - \zeta)^{-1} (\lambda - \zeta)^{-1} d\zeta,$$

and, by convention, for $\nu_j = 0$:

$$\tag{3.11} S^0 = -P_\lambda.$$
The fact that (3.9) remains well-defined if $V_0^\epsilon$ is replaced by $V_0^\epsilon$ is the main ingredient in the development of an asymptotic perturbation theory. Indeed, we have the following lemma, originally proved in [6]:

**Lemma 3.5.** For $j \in \mathbb{N}$, let

\[ Q_j^\epsilon(e) = (-1)^{j+1} \sum_{\nu_1, \ldots, \nu_{j+1} \geq 0; \Sigma \nu_k = j} S^{\nu_1} V_{a_1} S^{\nu_2} V_{a_2} \ldots V_{a_j} S^{\nu_{j+1}}. \]

Then, all $Q_j^\epsilon(e)$ are well defined and bounded on $C^\infty_0(\mathbb{R}^2)$, and their extensions by continuity satisfy $Q_j^\epsilon(e) L^2(\mathbb{R}^2) \subset D(H_0) \cap D(V_{a_0}^\epsilon)$, as well as

\[ Q_j^\epsilon(e) = \sum_{l=0}^j Q_l^\epsilon(e) Q_{j-l}^\epsilon(e). \]

Denoting by $[\cdot, \cdot]$ the usual commutator bracket, we also have

\[ [H_0, Q_j^\epsilon(e)] f = -[V_{a_0}^\epsilon, Q_{j-1}^\epsilon(e)] f, \text{ for } f \in D(H_0) \cap D(V_{a_0}^\epsilon), \]

and

\[ [V_{a_0}^\epsilon, Q_j^\epsilon(e)] \text{ is bounded on } D(V_{a_0}^\epsilon), \text{ for all } j \in \mathbb{N}. \]

The main point of this lemma concerns the boundedness of $Q_j^\epsilon(e)$, which in itself relies on the fact that the reduced resolvent $S$ preserves the exponential decay of the eigenfunctions of $H_0$, see the appendix.

**Proof (Sketch).** First, by taking $K = \Gamma$ in Proposition A.1 and applying it to (3.10), there exists $\omega_\Gamma > 0$ and $M_\Gamma < \infty$ such that for all $\omega \in [0, \omega_\Gamma]$:

\[ \| e^{i\omega \cdot x} S e^{-\omega \cdot x} \| \leq M_\Gamma, \]

where $(x) = (1 + |x|^2)^{1/2}$. Similarly, as a consequence of Proposition A.2, there exists $M_{\omega_T} < \infty$ such that

\[ \| e^{\omega_T \cdot x} P_\lambda e^{-\omega_T \cdot x} \| \leq M_{\omega_T}. \]

Notice that each term in the sum of (3.12) contains at least one $S^0 = -P_\lambda$ by (3.11) and the restrictions on $\nu_i$. In particular,

\[ Q_0(e) = Q_0^\epsilon(e) = P_\lambda. \]

We may therefore insert appropriate $e^{i\omega_T \cdot x} e^{-i\omega_T \cdot x}$ (or its opposite) between terms in (3.12) in order to group the appearing factors $V_{a_j}^\epsilon$ (which are polynomially bounded in view of (3.6)) with exponential decay terms $e^{-i\omega_T \cdot x}$, while grouping the growth terms $e^{i\omega_T \cdot x}$ with $S$ and $P_\lambda$ in such a way that these terms are bounded by (3.16) and (3.17). Hence, we obtain that (3.12) is indeed well defined and bounded. The algebraic properties (3.13)–(3.15) can be derived along the same lines as in the case of regular perturbation theory. \(\square\)

With Lemma 3.5 in hand, the construction of almost invariant subspaces of $H_0^\epsilon$ follows closely Section III of [6]: to begin with, we consider the truncated perturbation series

\[ T_N^\epsilon(e) = \sum_{j=0}^N e^j Q_j^\epsilon(e), \]

where $Q_j^\epsilon$ is given by (3.12). The operator $T_N^\epsilon$ defines an almost projection, since from (3.18) and (3.13), it follows

\[ \| T_N^\epsilon(e)^2 - T_N^\epsilon(e) \| \lesssim e^{N+1}. \]
Since $Q^p_0(\varepsilon) = P_\lambda$, we also have
\[
\lim_{\varepsilon \to 0} \| T^\varepsilon_N(\varepsilon) - P_\lambda \| = 0.
\]

The next step is to construct a \textit{bona fide} orthogonal projection close to $T_N(\varepsilon)$. To this end, we recall the following abstract result on almost projections:

**Proposition 3.6.** Let $T$ be a bounded self-adjoint operator satisfying
\[
\| T^2 - T \| \leq c < \frac{1}{8}
\]
Then, $\{ z \in \mathbb{C} : |z - 1| = \frac{1}{2} \} \subset \rho(T)$ and
\[
\| P - T \| \lesssim \varepsilon,
\]
where
\[
P = \frac{i}{2\pi} \oint_{|z - 1| = \frac{1}{2}} (T - z)^{-1} \, dz.
\]

**Proof.** See Section 3 of [6].

In view of (3.19), $T^\varepsilon_N(\varepsilon)$ satisfies the condition for this proposition, provided $\varepsilon$ is small enough. We may then define the orthogonal projection
\[
P^\varepsilon_N(\varepsilon) := \frac{i}{2\pi} \oint_{|z - 1| = \frac{1}{2}} (T^\varepsilon_N(\varepsilon) - z)^{-1} \, dz,
\]
which, in view of [6, identity (6.3)], satisfies for $\varepsilon$ sufficiently small:
\[
P^\varepsilon_N(\varepsilon) - T^\varepsilon_N(\varepsilon) = (T^\varepsilon_N(\varepsilon)^2 - T^\varepsilon_N(\varepsilon))
\]
\[
\times \frac{i}{2\pi} \oint_{|z - 1| = \frac{1}{2}} \frac{1}{z(1 - z)} \left(1 + \frac{T^\varepsilon_N(\varepsilon)}{z(1 - z)}\right)^{-1} \left(1 + \frac{T^\varepsilon_N(\varepsilon)^2 - T^\varepsilon_N(\varepsilon)}{z(1 - z)}\right)^{-1} \, dz.
\]
Proposition 3.6 therefore implies
\[
\| P^\varepsilon_N(\varepsilon) - T^\varepsilon_N(\varepsilon) \| \lesssim \varepsilon^{N+1}.
\]
Moreover, (3.14) implies
\[
[H^\varepsilon_a, T^\varepsilon_N(\varepsilon)] = \varepsilon^{N+1}[V^\varepsilon_a, Q^\varepsilon_N(\varepsilon)],
\]
and hence, from (3.20) we find
\[
[H^\varepsilon_a, P^\varepsilon_N(\varepsilon)] = -\varepsilon^{N+1} \frac{i}{2\pi} \oint_{|z - 1| = \frac{1}{2}} (T_N(\varepsilon) - z)^{-1} [V^\varepsilon_a, Q^\varepsilon_N(\varepsilon)](T^\varepsilon_N(\varepsilon) - z)^{-1} \, dz.
\]
This together with (3.15) gives the crucial estimate
\[
\| [H^\varepsilon_a, P^\varepsilon_N(\varepsilon)] \| \lesssim \varepsilon^{N+1},
\]
i.e., in the language of [6, Section II]:

$P^\varepsilon_N(\varepsilon) L^2(\mathbb{R}^2)$ are almost invariant subspaces of order $\varepsilon^{N+1}$ associated to $H^\varepsilon_a$.

For our purposes, $P^\varepsilon_N(\varepsilon)$ will serve to replace the perturbed spectral projections in the analytic case.

Next, we recall that $V^\varepsilon_a = p\hat{V}^{\varepsilon,p} + \varepsilon^{\beta - \alpha - 1} \hat{W}^{\varepsilon,p}_a$. Inserting this expression into (3.12) and collecting all the terms which do not contain $\hat{W}^{\varepsilon,p}_a$ allows us to rewrite
\[
Q^\varepsilon_j(\varepsilon) = Q_j(\varepsilon) + \varepsilon^{\beta - \alpha - 1} R^\varepsilon_j(\varepsilon),
\]
and since $R^\varepsilon_j(\varepsilon)$ inherits the boundedness from $Q^\varepsilon_j(\varepsilon)$, we have
\[
\| Q^\varepsilon_j(\varepsilon) - Q_j(\varepsilon) \| \lesssim \varepsilon^{\beta - \alpha - 1}.
\]
With estimates (3.22), (3.23), and (3.24) at hand, we can now turn to the proof of Theorem 3.3.

3.2. Proof of Theorem 3.3. We start by collecting the following facts, all of which will be used without further notice in the proof:

Since \( P_\lambda(\varepsilon) \) is a finite dimensional projection, the results in the appendix applied to \( \mathcal{H}_\varepsilon^p(p) \) imply that \( \mathcal{W}'^\varepsilon P_\lambda(\varepsilon) \) and thus also \( \mathcal{H}_\varepsilon^p P_\lambda(\varepsilon) \), is bounded. Moreover, the arguments showing boundedness of \( Q_\varepsilon^p(\varepsilon) \) also yield boundedness of \( \mathcal{H}_\varepsilon^p Q_\varepsilon^p(\varepsilon) \), which in itself implies that \( \mathcal{H}_\varepsilon^p T_\varepsilon^p(\varepsilon) \) is bounded. In turn this yields boundedness of \( \mathcal{H}_\varepsilon^p P_\lambda(\varepsilon) \) by invoking formula (3.21). Furthermore, since \( \mathcal{H}_\varepsilon^p \), \( P_N^p(\varepsilon) \), and \( P_\lambda(\varepsilon) \) are all self-adjoint, we have that

\[
\|\mathcal{H}_\varepsilon^p P_\lambda(\varepsilon)\| = \|P_\lambda(\varepsilon)\mathcal{H}_\varepsilon^p\| = \|P_N^p(\varepsilon)\mathcal{H}_\varepsilon^p\|.
\]

With this in mind, our goal is to derive an asymptotic estimate for the operator norm of

\[
\Delta(\varepsilon, t) := \left( e^{-i\varepsilon^{-2}t\mathcal{H}_\varepsilon^p} - e^{-i\varepsilon^{-2}t\mathcal{H}_\varepsilon^p} \right) P_\lambda.
\]

To do so, we start by using the triangle inequality, together with the fact that both of the Schrödinger groups are unitary, to obtain

\[
\left\| \Delta(\varepsilon, t) - \left( e^{-i\varepsilon^{-2}t\mathcal{H}_\varepsilon^p} P_N^p(\varepsilon) - e^{-i\varepsilon^{-2}t\mathcal{H}_\varepsilon^p} P_\lambda(\varepsilon) \right) \right\| \leq \|P_N^p(\varepsilon) - P_\lambda\| + \|P_\lambda(\varepsilon) - P_\lambda\|.
\]

By (3.7), we have

\[
\|P_\lambda(\varepsilon) - P_\lambda\| \preceq \varepsilon,
\]

while (3.18) and (3.22) imply

\[
\|P_N^p(\varepsilon) - P_\lambda\| \preceq \|P_N^p(\varepsilon) - T_N(\varepsilon)\| + \|T_N(\varepsilon) - P_\lambda\| \preceq \varepsilon^{N+1} + \varepsilon \preceq \varepsilon.
\]

Thus

\[
\left\| \Delta(\varepsilon, t) - \left( e^{-i\varepsilon^{-2}t\mathcal{H}_\varepsilon^p} P_N^p(\varepsilon) - e^{-i\varepsilon^{-2}t\mathcal{H}_\varepsilon^p} P_\lambda(\varepsilon) \right) \right\| \preceq \varepsilon,
\]

and since \([\mathcal{H}_\varepsilon^p, P_\lambda(\varepsilon)] = 0\), we can rewrite

\[
e^{-i\varepsilon^{-2}t\mathcal{H}_\varepsilon^p} P_\lambda(\varepsilon) = e^{-i\varepsilon^{-2}tP_\lambda(\varepsilon)\mathcal{H}_\varepsilon^p P_\lambda(\varepsilon)} P_\lambda(\varepsilon),
\]

which yields

\[
\left\| \Delta(\varepsilon, t) - \left( e^{-i\varepsilon^{-2}t\mathcal{H}_\varepsilon^p} P_N^p(\varepsilon) - e^{-i\varepsilon^{-2}tP_\lambda(\varepsilon)\mathcal{H}_\varepsilon^p P_\lambda(\varepsilon)} P_\lambda(\varepsilon) \right) \right\| \preceq \varepsilon.
\]

To further estimate the difference of the two Schrödinger groups involved, we recall the following general fact: Let \( A \) and \( B \) be two self-adjoint operators, such that \( A - B \) is bounded, and denote

\[
W_s := e^{isA}e^{-isB}, \quad \text{for } s \in \mathbb{R}.
\]

Then \( W_0 = 1 \) and

\[
W_s = 1 + \int_0^s W_\tau \, d\tau.
\]

Writing out the time-derivative \( W_\tau \), and multiplying by \( e^{-isA} \), yields Duhamel’s formula (or, equivalently, Dyson’s formula in the interaction picture) for the difference of two Schrödinger groups, i.e.

\[
e^{-isA} - e^{-isB} = -i \int_0^s e^{-i(s-\tau)A} (A - B) e^{-isB} e^{-isA} \, d\tau.
\]

Next, we decompose \( \mathcal{H}_\varepsilon^p \) into its diagonal and off-diagonal elements, i.e.

\[
\mathcal{H}_\varepsilon^{p} = \mathcal{H}_\varepsilon^{p,\text{diag}} + \mathcal{H}_\varepsilon^{p,\text{off}},
\]

where

\[
\mathcal{H}_\varepsilon^{p,\text{diag}} = \mathcal{H}_\varepsilon^{p}(\varepsilon) \mathcal{H}_\varepsilon^{p}(\varepsilon) \mathcal{H}_\varepsilon^{p}(\varepsilon) (1 - P_N^p(\varepsilon)) + (1 - P_N^p(\varepsilon)) \mathcal{H}_\varepsilon^{p}(\varepsilon) P_N^p(\varepsilon) + P_N^p(\varepsilon) \mathcal{H}_\varepsilon^{p}(\varepsilon) (1 - P_N^p(\varepsilon)).
\]
In view of (3.23), we have
\[ \|P^a_{\text{off}}\| = \|(1 - 2P^a_N(e))[\mathcal{H}^e, P^a_N(e)]\| \lesssim \varepsilon^{N+1}. \]

Duhamel’s formula (3.27) for \( s = \frac{1}{2\tau} \), \( A = \mathcal{H}^e \) and \( B = \mathcal{H}^{e, \text{diag}} \), consequently implies
\[
\left\| e^{-i\varepsilon^{-2}\mathcal{H}^e} - e^{-i\varepsilon^{-2}\mathcal{H}^{e, \text{diag}}(t)} \right\| P^a_N(e) \| \leq \int_0^{\tau \varepsilon^{-2}} \| \mathcal{H}^{e, \text{diag}} \| d\tau \lesssim \varepsilon^{N-1}|\tau|. \]

To proceed further, we note that
\[ \mathcal{H}^{e, \text{diag}} \big|_{\text{ran}P^a_N(e)} = P^a_N(e)\mathcal{H}^e P^a_N(e) \big|_{\text{ran}P^a_N(e)}. \]

The functional calculus of self-adjoint operators therefore yields
\[
e^{-i\varepsilon^{-2}\mathcal{H}^{e, \text{diag}}} P^a_N(e) = e^{-i\varepsilon^{-2}P^a_N(e)} \mathcal{H}^e P^a_N(e), \]
and hence
\[
(3.28) \quad \left\| \left( e^{-i\varepsilon^{-2}\mathcal{H}^e} - e^{-i\varepsilon^{-2}P^a_N(e)} \right) P^a_N(e) \right\| \lesssim \varepsilon^{N-1}|\tau|. \]

Another application of the triangle inequality allows us to combine this estimate with (3.26) to infer
\[
\left\| \Delta(e, t) - \left( e^{-i\varepsilon^{-2}P^a_N(e)} \mathcal{H}^e P^a_N(e) - e^{-i\varepsilon^{-2}P_N(e)} \mathcal{H}^e P_N(e) \right) \right\| \lesssim \varepsilon + \varepsilon^{N-1}|\tau|. \]

In here, we can further rewrite
\[ P_N(e) = P_N(e) - P^a_N(e) + P^a_N(e), \]
and note that
\[ P^a_N(e) - P_N(e) = P^a_N(e) - T^a_N(e) + \sum_{j=1}^N \varepsilon^j (Q^a(e) - Q_j)) - \sum_{j=1}^N \varepsilon^j Q_j(e). \]

Recalling (3.22) and (3.24), we see that
\[ (3.29) \quad \left\| P^a_N(e) - P_N(e) \right\| \lesssim \varepsilon^{N+1} + \varepsilon^{3-\alpha}, \]
which consequently yields
\[
\left\| \Delta(e, t) - \left( e^{-i\varepsilon^{-2}P^a_N(e)} \mathcal{H}^e P^a_N(e) - e^{-i\varepsilon^{-2}P_N(e)} \mathcal{H}^e P_N(e) \right) \right\| \lesssim \varepsilon + \varepsilon^{3-\alpha} + \varepsilon^{N-1}|\tau|. \]

Finally, we can use formula (3.27) once more to estimate
\[
\left\| e^{-i\varepsilon^{-2}P^a_N(e)} \mathcal{H}^e P^a_N(e) - e^{-i\varepsilon^{-2}P_N(e)} \mathcal{H}^e P_N(e) \right\| \lesssim \int_0^{\tau \varepsilon^{-2}} \| \mathcal{H}^e P_N(e) \| d\tau.
\]

Here, we can express the difference of operators appearing within the integral via
\[
P^a_N(e)\mathcal{H}^e P^a_N(e) - P_N(e)\mathcal{H}^e P_N(e)
= P^a_N(e)\mathcal{H}^e P^a_N(e) - P_N(e)\mathcal{H}^e P_N(e) + e^{3-\alpha} P_N(e)\tilde{W}^{e,p}_A P_N(e)
= (P^a_N(e) - P_N(e))\mathcal{H}^e P^a_N(e) + P_N(e)\mathcal{H}^e (P^a_N(e) - P_N(e)) + e^{3-\alpha} P_N(e)\tilde{W}^{e,p}_A P_N(e).
\]

Estimate (3.29), together with our previous discussion on the boundedness of all the appearing operators then yields
\[
\left\| P^a_N(e)\mathcal{H}^e P^a_N(e) - P_N(e)\mathcal{H}^e P_N(e) \right\| \lesssim \varepsilon^{N+1} + \varepsilon^{3-\alpha},
\]
and thus
\[
\int_0^{\tau \varepsilon^{-2}} \| P^a_N(e)\mathcal{H}^e P^a_N(e) - P_N(e)\mathcal{H}^e P_N(e) \| d\tau \lesssim \varepsilon^{N-1}|\tau| + \varepsilon^{3-\alpha-2}|\tau|. \]
Recalling that $\beta - \alpha \geq 3$ we infer that 
\[
\left\| e^{-it\varepsilon^{-2}P_N^\omega(x)\mathcal{H}_w P_N^\omega(x)} - e^{-it\varepsilon^{-2}P_N^\omega(x)\mathcal{H}^\varepsilon P_N^\omega(x)} \right\| P_N^\omega(x) \lesssim \varepsilon|t|, \quad \text{for } N \geq 2.
\]

Hence, by choosing some fixed $N \geq 2$:
\[
\| \Delta(\varepsilon, t) \| \leq \left\| \Delta(\varepsilon, t) - \left( e^{-it\varepsilon^{-2}P_N^\omega(x)\mathcal{H}_w P_N^\omega(x)} - e^{-it\varepsilon^{-2}P_N^\omega(x)\mathcal{H}^\varepsilon P_N^\omega(x)} \right) P_N^\omega(x) \right\| 
+ \left\| \left( e^{-it\varepsilon^{-2}P_N^\omega(x)\mathcal{H}_w P_N^\omega(x)} - e^{-it\varepsilon^{-2}P_N^\omega(x)\mathcal{H}^\varepsilon P_N^\omega(x)} \right) P_N^\omega(x) \right\| \lesssim \varepsilon(1 + |t|),
\]

which is the desired result.

\[\square\]

**Remark 3.7.** It is clear from our proof that the condition that $a$ vanishes in a neighborhood of the origin can be relaxed to a being sufficiently small there. More precisely, Theorem 3.3 remains valid if
\[
a(x) \lesssim \begin{cases} |x|^{3+\alpha} & \text{for } |x| < |x_0|, \\ |x|^\beta & \text{for } |x| \geq |x_0|. \end{cases}
\]
The details are left to the interested reader.

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### Appendix A. Spectral Estimates

For the sake of the reader we shall present in here an elementary approach to the proof of exponential decay bounds for the eigenfunctions of a wide class of confinement Hamiltonians. Suppose
\[
H = -\Delta + U(x), \quad \mathcal{D}(H) = C_0^\infty(\mathbb{R}^d),
\]
such that $U : \mathbb{R}^d \to \mathbb{R}$ satisfies
\[
\begin{align*}
U(x) & \geq 0, \quad U \in L_\text{loc}^\infty(\mathbb{R}^d), \\
\lim_{R \to \infty} \left( \text{ess inf}_{x \in \mathbb{R}^d} \left( 1 - \mathbbm{1}_{B_R(0)} \right) U(x) \right) & = \infty,
\end{align*}
\]
where $\mathbbm{1}_{B_R(x)}$ is the indicator function on the ball of radius $R$. As in Lemma 2.1, we have that $H$ is essentially self-adjoint with positive, discrete spectrum. Finally, let
\[
\langle x \rangle := (1 + |x|^2)^{1/2}
\]
and note that
\[
|\nabla \langle x \rangle| \leq 1, \quad |\Delta \langle x \rangle| \leq d.
\]
Then, we have the following:

**Proposition A.1.** Let $K \subset \rho(H)$ be compact, where $\rho(H)$ is the resolvent set of $H$. Then, there exists $\omega_K > 0$ and $M_K < \infty$ such that for all $z \in K$ and $0 \leq \omega \leq \omega_K$, the resolvent satisfies
\[
\left\| e^{\omega \langle \cdot \rangle} (H - z)^{-1} e^{-\omega \langle \cdot \rangle} \right\| \leq M_K.
\]
Finally, let $b \in (0, \infty)$. By Cauchy-Schwarz,

$$4d\langle f, Hf \rangle_{L^2} \leq 4d\|bf\|_{L^2} \|\frac{1}{b}Hf\|_{L^2} \leq 2d\left(b\|f\|_{L^2} + \frac{1}{b}\|Hf\|_{L^2}\right)^2,$$

and so we have shown

$$\|2\nabla \cdot \nabla f\|_{L^2} \leq \sqrt{2d}\left(b\|f\|_{L^2} + \frac{1}{b}\|Hf\|_{L^2}\right).$$

The second and third term in $D_\omega$ are easily bounded via (A.2), and so with (A.7) we find

$$\|D_\omega f\|_{L^2} \leq \sqrt{2d}\left(b\|f\|_{L^2} + \frac{1}{b}\|Hf\|_{L^2}\right) + (d + \omega)\|f\|_{L^2} \leq \frac{1}{b}\sqrt{2d}\|Hf\|_{L^2} + \left(b\sqrt{2d} + d + \omega\right)\|f\|_{L^2}.$$
This inequality holds in general whenever \( f \in \mathcal{D}(H) \), where we now denote \( \mathcal{D}(H) \) to be the domain of the self-adjoint extension of \( H \). Hence, we may take \( g \in L^2(\mathbb{R}^d) \) and substitute \( f = (H - z)^{-1}g \) into (A.8):

\[
\|D_\omega(H - z)^{-1}g\|_{L^2} \leq \frac{1}{b} \sqrt{2d} \|H(H - z)^{-1}g\|_{L^2} + \left( b\sqrt{2d} + d + \omega \right) \|H - z\|^{-1}g\|_{L^2}.
\]

(A.9)

Next, recalling that \( K \subset \rho(H) \) is compact, we denote

\[
\delta_K := \inf_{z \in K} \text{dist} \left( z, \sigma(H) \right) > 0, \quad z_K := \sup_{z \in K} |z| < \infty.
\]

By functional calculus, this gives

\[
\sup_{z \in K} \|(H - z)^{-1}\| \leq \frac{1}{\delta_K}, \quad \sup_{z \in K} \|H(H - z)^{-1}\| \leq 1 + \frac{z_K}{\delta_K}.
\]

(A.10)

Returning to (A.9), we choose \( b = \sqrt{\delta_K} \) to find

\[
\|D_\omega(H - z)^{-1}g\|_{L^2} \leq \left( \sqrt{\frac{2d}{\delta_K}} \left( 1 + \frac{z_K}{\delta_K} \right) + \left( \sqrt{2d\delta_K} + d + \omega \right) \frac{1}{\delta_K} \right) \|g\|_{L^2}.
\]

(A.11)

Therefore, we can always find an \( \omega_K > 0 \), such that for all \( 0 \leq \omega \leq \omega_K \):

\[
\omega \|D_\omega(H - z)^{-1}g\|_{L^2} \leq \frac{1}{2} \|g\|_{L^2},
\]

proving (A.5), and in turn (A.6) with \( c = \frac{1}{2} \).

Finally, we apply the estimates (A.6) and (A.10) within identity (A.4) to finish the proof:

\[
\left\| e^{\omega(\cdot)}(H - z)^{-1}e^{-\omega(\cdot)} \right\| = \left\| \left( e^{\omega(\cdot)}(H - z)e^{-\omega(\cdot)} \right)^{-1} \right\| \leq \frac{2}{\delta_K} := M_K.
\]

□

Next, we show that the eigenfunctions and spectral projections of \( H \) decay at worst exponentially:

**Proposition A.2.** Let \( \lambda \in \mathbb{R} \) be an \( m \)-degenerate eigenvalue of \( H = -\Delta + U \), with associated orthonormalized eigenfunction \( \chi_k \), \( k = 1, \ldots, m \). Then, for any \( \omega_0 > 0 \) there exists \( M_{\lambda,\omega_0} < \infty \), such that

\[
\|e^{\omega_0(\cdot)}\chi_k\|_{L^2(\mathbb{R}^d)} \leq M_{\lambda,\omega_0}.
\]

(A.12)

Additionally, for the associated \( m \)-dimensional spectral projection \( P_\lambda \) it holds:

\[
\|P_\lambda e^{\omega_0(\cdot)}f\|_{L^2(\mathbb{R}^d)} \leq m M_{\lambda,\omega_0} \|f\|_{L^2(\mathbb{R}^d)}.
\]

(A.13)

**Proof.** Suppose \( \lambda \in \sigma(H) \) is an eigenvalue of finite multiplicity \( m \in \mathbb{N} \) and \( \chi_k \) is an associated (normalized) eigenfunction. For \( \nu > 0 \), let

\[
\Omega_\nu := \{ x \in \mathbb{R}^d : U(x) < \lambda + \nu \},
\]

and define

\[
H_\nu := -\Delta_x + (1 - \mathbbm{1}_{\Omega_\nu})U + (\lambda + \nu) \mathbbm{1}_{\Omega_\nu},
\]

where \( \mathbbm{1}_{\Omega_\nu} \) is the indicator function of \( \Omega_\nu \). Then, we can rewrite \( H\chi_k = \lambda \chi_k \) via

\[
(H_\nu - \lambda)\chi_k = (\lambda + \nu - U) \mathbbm{1}_{\Omega_\nu} \chi_k.
\]

By construction, \( H_\nu \geq \lambda + \nu > \lambda \), and so \( \lambda \in \rho(H_\nu) \), the resolvent set of \( H_\nu \). Hence,

\[
\chi_k = (H_\nu - \lambda)^{-1}(\lambda + \nu - U) \mathbbm{1}_{\Omega_\nu} \chi_k
\]
and we can estimate
\[ \|e^{\omega_0(t)} \chi_k\|_{L^2} \leq \|e^{\omega_0(t)}(H_\nu - \lambda)^{-1}e^{-\omega_0(t)}\| \times \|e^{\omega_0(t)}(\lambda + \nu - W)\Omega_\nu\|_{L^\infty} \|\chi_k\|_{L^2}. \]
(A.14)

For the term involving \((H_\nu - \lambda)^{-1}\), we seek to invoke Proposition A.1. Note that the latter \textit{a-priori} only holds for \(\omega_0 \leq \omega_K\). However from (A.11) it is clear that we may take \(\omega_K\) to be arbitrarily large by increasing \(\delta_K = \inf_{z \in K} \text{dist}(z, \sigma(H_\nu))\), where \(K \subset \rho(H_\nu)\) is compact. This can be done by setting \(K = \{\lambda\}\) and noting that \(\delta_K \geq \nu\). Hence we may choose \(\nu\) large enough such that \(\omega_0 \leq \omega_K\), and Proposition A.1 yields
\[ \|e^{\omega_0(t)}(H_\nu - \lambda)^{-1}e^{-\omega_0(t)}\| \leq M_K < \infty. \]

Then, since \(\Omega_\nu\) is bounded we have
\[ \|e^{\omega_0(t)}(\lambda + \nu - W)\Omega_\nu\|_{L^\infty} = \sup_{x \in \Omega_\nu} |e^{\omega_0(x)}(\lambda + \nu - W(x))| \leq C_{\lambda,\omega_0} < \infty. \]

In view of (A.14), this yields
\[ \|e^{\omega_0(t)} \chi_k\|_{L^2} \leq M_K C_{\lambda,\omega_0} \equiv M_{\lambda,\omega_0}. \]

With (A.12) established, we can now obtain an analogous bound for \(P_\lambda\):
\[ \|e^{\omega_0(t)} P_\lambda e^{\omega_0(t)} f\|_{L^2} \leq \sum_{k=1}^m \|e^{\omega_0(t)}(e^{\omega_0(t)} f, \chi_k)_{L^2}\|\chi_k\|_{L^2} \leq \sum_{k=1}^m M_{\lambda,\omega_0} \|f\|_{L^2} \|e^{\omega_0(t)} \chi_k\|_{L^2} \leq m M_{\lambda,\omega_0}^2 \|f\|_{L^2}. \]

\[ \square \]

**Remark A.3.** In general, one expects the eigenfunctions \(\chi_k\) to decay even stronger, as can be seen from the well-known case of the harmonic oscillator where \(\Theta \equiv 1\) and \(\alpha = 1\). However, the proof of such stronger decay properties is usually much more involved, while (A.12) is sufficient for our purposes.

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