On the fundamental 3-classes of knot group representations

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Abstract
We discuss the fundamental (relative) 3-classes of knots (or hyperbolic links), and provide diagrammatic descriptions of the push-forwards with respect to every link-group representation. The point is an observation of a bridge between the relative group homology and quandle homology from the viewpoints of Inoue–Kabaya map [IK]. Furthermore, we give an algorithm to algebraically describe the fundamental 3-class of any hyperbolic knot.

Keywords
Knot, relative group homology, hyperbolicity, Malnormal subgroups, quandle,

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1 Introduction.
In the study of an oriented compact 3-manifold $M$ with torus boundary, the (relative) fundamental homology 3-class $[M, \partial M]$ in $H_3(M, \partial M; \mathbb{Z}) \cong \mathbb{Z}$ essentially has basic information. To further analyze it quantitatively, relative viewpoints are usually practice: To describe this, we suppose a pair of groups $K \subset G$ and a homomorphism $f : \pi_1(M) \rightarrow G$, which sends every boundary elements in $\pi_1(M)$ to some elements in $K$. Then, for any relative group 3-cocycle
θ ∈ H^3(G, K; A) with local coefficients (see [10] for the explicit definition), we can consider the following pairing valued in the coinvariant \( M_G = H_0(G; A) \):

\[
\langle \theta, f_\ast [M, \partial M] \rangle \in A_G = A/\{a - g \cdot a \}_{a \in A, \; g \in G}.
\]

These settings appear in many topics of low dimensional topology. For example, the volumes and the Chern-Simons invariants of hyperbolic manifolds can be described as this paring, where \( G = SL_N(\mathbb{C}) \) (see [Neu, Zic, GTZ]). Furthermore, this paring includes the triple cup products of the form \( \theta = a \sim b \sim c \); see [MS, No4]. In addition, if \( G \) is of finite order, the pairing is called the Dijkgraaf-Witten invariant [DW], as a toy model of TQFT.

However, the paring in the general situation is often considered to be far from being computable. Actually, we come up against difficulties: First, it is troublesome to explicitly describe a (truncated) triangulation in \( M \), which represents the 3-class \( f_\ast [M, \partial M] \). Moreover, the 3-class \( f_\ast [M, \partial M] \) is not always unique, but depends on the choices of \( 2^{2\pi_0(\partial M)} \) decorations, as mentioned in [Zic, §5]; Next, the boundary condition is important; when dealing with the condition, we mostly need long and verbose explanations, as in [DW, Neu, Zic, Mor, Kab] (cf. Homotopy quantum field theory [Tur]). In addition, since the relative homology is defined from some projective resolution (see §3), it is essentially a critical problem to choose an appreciate resolution and to find a presentation of the 3-cocycle \( \theta \). Further, even if we can succeed in doing so on \( \theta \), such presentations are mostly quite intricate.

Nevertheless, this paper develops a diagrammatic computation of the pairing, according to geometric structures of links. Precisely, let \( L \subset S^3 \) be an oriented link in the 3-sphere, and \( M = E_L \) be the 3-manifold which is obtained from \( S^3 \) by removing an open tubular neighborhood of \( L \), i.e., \( E_L = S^3 \setminus \nu L \). When \( L \) has “malnormal property” as a broad class (i.e., hyperbolic links), we succeed in giving a computation of the pairing; see Theorem 2.1 (however, in cable cases we need some conditions; see §7, cf. cabling formula [Ishi]). Here, we stress that the computation is constructed from only a choice of a link diagram of \( L \), while the construction needs no triangulation of \( E_L \). More precisely, the result is summarized to that, if we know the presentation of \( \theta \) and the JSJ decomposition of \( L \), we can compute the pairing from a diagram; see §5.

Let us roughly explain our approach to the theorem. As seen in [CKS, IK, No3, No1], quandle theory [Joy] and homology [CKS] have advantages to some diagrammatic computation in knot theory. Thus, inspired by the works [IK, NM], we will construct a bridge between the quandle and group homologies using chain maps, in order to reduce the 3-class \( [M, \partial M] \) to a quandle 3-class. However, as a technical reason appearing in [IK] or the scissors congruence in [Dup, Neu], the bridge factors through Hochschild relative homology [Hoc], and is formulated as a zigzag sequence; see §5. As a solution, this paper points out (Theorem 2.3) that “malnormal” property of groups is a suitable condition for obtaining a quasi inverse in the zigzag; see §5 (Here, we are based on [Sim, HW, NM] which studied malnormal property of knots). To summarize, composing the chain maps give the required computation of the pairing.

In applications, we obtain four advantages from the approaches as follows. First, the previous composite gives an algorithm to algebraically describe the fundamental 3-class \([M, \partial M]\)
for hyperbolic links; see \[\text{5.2}\]. Next, our results emphasize topological advantages of the quandle cocycle invariant [CKS]. Especially, for malnormal pairs \((G, K)\), we will give a method to produce many quandle cocycles, and obtain an easier formulation of computing the pairing (see Theorem \[\text{2.3}\]). The third is some results of determining the third homology of the link quandle \(Q_L\), where \(L\) is a knot or a hyperbolic link. This quandle \(Q_L\) is analogous to the fundamental groups (defined in \[\text{Joy}\]), and play a key role in the proof of the main theorems; see Appendix \[A\] for details. The forth one is to show a generalization and applications of the work [IK]. To be precise, although the paper [IK] showed the same theorem in only the case \(G = SL_2(\mathbb{C})\), we point out the generalization applicable for malnormality as mentioned above.

This paper is organized as follows. Section 2 states the theorems. Section 3 introduces relative group homology, and Section 4 reviews the quandle homology [CKS] and Inoue–Kabaya chain map [IK]. Section 5 explains the algorithm to describe \([M, \partial M]\), and gives an example from the 4_1-knot; Section 6 proves the main theorem, and Section 7 discusses for cable knots. Appendix A computes the third homology of the link quandle \(Q_L\) for some links.

## 2 Statements; the main results.

This section states the main results. For this, we first fix terminology throughout this paper.

**Notation and assumption throughout this paper.**

- By a link we mean a \(C^\infty\)-embedding of solid tori into the 3-sphere \(S^3\) or into the solid torus \(D^2 \times S^1\). We often suppose orientation, and denote \(\pi_0(L)\) by \(\#L \in \mathbb{Z}\).

- In short, the fundamentals group \(\pi_1(S^3 \setminus L)\) is abbreviated to \(\pi_L\), and the complement space \(S^3 \setminus L\) is often done to \(E_L\).

- Furthermore, fix a pherihedral group, \(\mathcal{P}_\ell\), with respect to \(\ell \leq \#L\), which is generated by a meridian-longitude pair \((m_\ell, l_\ell)\).

- Moreover, we fix a group \(G\) and subgroups \(K_\ell\) with \(\ell \leq \#L\), and suppose a homomorphism \(f : \pi_1(S^3 \setminus L) \to G\) such that \(f(\mathcal{P}_\ell) \subset K_\ell\).

In this situation, although it seems easy to define a pushforward of the 3-class \(f_*([E_L, \partial E_L])\) in the relative group homology \(H_3(G, K_1, \ldots, K_{\#L}; \mathbb{Z})\), it is known (see [Zic, §5] or [NM, §10]) that a canonical definition of such pushforwards depends on the choice of “\(f\)-decorations”. However, if \(E_L\) is decomposed as a union of complete hyperbolic 3-manifolds, the 3-class \(f_*(|[E_L, \partial E_L]|)\) is known to be well-defined (see [Zic, §5]). Therefore, similarly to (1), we can consider the pairing between this 3-class and a 3-cocycle of \(G\) relative to \((K_\ell)_t \leq \#L\).

### 2.1 The first statement.

We will set up some diagrammatic terminology, and state Theorem \[\text{2.1}\]. Take the set of the arcs of \(D\) and the set of the complementary regions of \(D\), denoted by \(\text{Arc}_D\) and \(\text{Reg}_D\), respectively.
Given a map $\phi : \text{Reg}_D \times \text{Arc}_D \times \text{Arc}_D \to A$, let us consider a weight sum of the form

$$\Psi_\phi(D) := \sum_{\tau} \epsilon_{\tau} \cdot \phi(x_{\tau}, y_{\tau}, z_{\tau}) \in A$$

by running over all the crossings $\tau$ of $D$, where $x_{\tau}$ and $y_{\tau}, z_{\tau}$ are the region and the arcs shown in Figure 1 and $\epsilon_{\tau} \in \{\pm 1\}$ is the sign of $\tau$. Then, the main statement is as follows:

![Figure 1: Positive and negative crossings with labeled regions and labeled arc.](image)

**Theorem 2.1** (See §6 for the proof.). Assume that $L$ is either a prime non-cable knot\(^1\) or a hyperbolic link. Then, for any relative group 3-cocycle $\theta \in H^3(G; K_1, \ldots, K_{\#L}; A)$, there is a map $\phi_\theta : \text{Reg}_D \times \text{Arc}_D \times \text{Arc}_D \to A$ for which the following equality holds in the coinvariant $A_G$:

$$\langle \theta, f_*[E_L, \partial E_L] \rangle = \Psi_{\phi_\theta}(D) \in A_G.$$  \hspace{1cm} (3)

In conclusion, the right hand side ensures a computation of the pairing, with describing no triangulation. Thus, it is important to present $\phi_\theta$ concretely; in Sections 5–6, we give a concrete presentation of $\phi_\theta$ for such links. However, the map $\phi_\theta$ is abstractly obtained from homology algebra without functoriality on $f$, and essentially depends on the link type of $L$. Actually, even for the figure eight knot $L$, the map $\phi_\theta$ is a sum of many terms; see Section 5.2.

Incidentally, Section 7 similarly discusses the cable cases and concludes a similar statement (Theorem 7.1). Here, we see that the statement essentially turns out to be modulo some integers, and to have no more information than the homology of cyclic groups.

### 2.2 The second statement from malnormality and transfer.

In contrast, we will consider some conditions to get the map $\phi_\theta$ and the diagrammatic description in a concrete way. Here, the subgroups $K_1, \ldots, K_{\#L} \subset G$ are said to be malnormal (in $G$), if

(\#) For any $(i, j) \in I^2$ and any $g \in G$ with $g \notin K_j$, the intersection $g^{-1}K_i g \cap K_j$ equals $\{1_G\}$.

The papers [HWO, HW] give such examples, as in Gromov hyperbolic groups.

We will give a description of the pairing, by using quandle theory. For this, we now review a quandle and colorings. A quandle [Joy] is a set, $X$, with a binary operation $\triangleright : X \times X \to X$ such that

---

\(^1\)A knot $K$ is cable, if there is a solid torus $V$ embedded in $S^3$ such that $V$ contains $K$ as the $(p, q)$-torus knot for some $p, q \in \mathbb{Z}$. Furthermore, a knot $L$ is prime, if it cannot be written as the knot sum of two non-trivial knots. Incidentally, the assumption in the theorems relies on Theorem 3.9 of [HW] and the JSJ decomposition; see [5].
(I) The identity $a \triangleleft a = a$ holds for any $a \in X$.

(II) The map $(\bullet \triangleleft a) : X \to X$ that sends $x$ to $x \triangleleft a$ is bijective, for any $a \in X$.

(III) The distributive identity $(a \triangleleft b) \triangleleft c = (a \triangleleft c) \triangleleft (b \triangleleft c)$ holds for any $a, b, c \in X$.

Let $X$ be a quandle. An $X$-coloring of $D$ is a map $C : \text{Arc}_D \to X$ such that $C(\alpha_\tau) \triangleleft C(\beta_\tau) = C(\gamma_\tau)$ at each crossings $\tau$ of $D$ illustrated as Figure 2. Further, for $x_0 \in X$, a shadow coloring is a pair of an $X$-coloring $C$ and a map $\lambda : \text{Reg}_D \to X$ such that the unbounded exterior region is assigned by $x_0$ and that if two regions $R$ and $R'$ are separated by an arc $\delta$ as shown in the right of Figure 2, then $\lambda(R) \triangleleft C(\delta) = \lambda(R')$. It is worth keeping in mind that, from arbitrary $x_0 \in X$ and $X$-coloring, we obtain uniquely a shadow coloring such that the unbounded region is labeled by $x_0$.

This paper mainly deal with the following class of quandles.

Example 2.2 ([Joy]). This example is due to Joyce [Joy]. Under the above settings $(f, G, K_\ell)$, let $X$ be the union of the left quotients $(K_\ell \backslash G)$, that is,

$$X = \sqcup_{\ell=1}^{\#L}(K_\ell \backslash G).$$

Let $k_\ell$ be $f(m_\ell) \in K_\ell$. Assume that $k_\ell$ is commutative with any elements of $K_\ell$. Then, the union $X$ is made into a quandle under the operation

$$[K_\ell x] \triangleleft [K_\ell y] = [K_\ell^{-1}xy^{-1}k_\ell y],$$

for any $x, y \in G$. In what follows, we will write the triple $(G, K, C)$ for this quandle.

Furthermore, as is known (see [No3, Appendix] for the details), the homomorphism $f$ admits uniquely an $X$-coloring $C$ with $C(m_\ell) = k_\ell$ such that $f$ is recovered from the assignment

$$\text{Arc}_D \to G; \quad \gamma \mapsto C(\gamma)^{-1}k_\ell C(\gamma)$$

via Wirtinger presentation, where $\gamma$ lies in the $\ell$-the component. Hence, by applying $k_1$ to $x_0$, we have the associated shadow coloring $S$.

Then, the main theorem in this subsection is stated as follows:

Theorem 2.3. Suppose that $k_\ell$ commutates with any elements of $K_\ell$. Furthermore, assume one of the two followings: (i) $(G, K_1, \ldots, K_{\#L})$ is malnormal. (ii) $K_1, \ldots, K_{\#L}$ are of finite order and all the order $|K_1|$ is invertible in the coefficient group $A$.

Then, any relative group 3-cocycle of $(G, K_1, \ldots, K_{\#L})$ is represented as a map $\theta : X^4 \to A$ such that, for any hyperbolic link and any prime non-cable knot, the fundamental 3-class $\langle \theta, f_\ell[E_L, \partial E_L] \rangle$ is equal to the sum

$$\sum_{\tau} \epsilon_\tau \left( \theta(k_1, a_\tau, b_\tau, c_\tau) - \theta(k_1, a_\tau \triangleleft b_\tau, b_\tau, c_\tau) - \theta(k_1, a_\tau \triangleleft c_\tau, b_\tau \triangleleft c_\tau, c_\tau) + \theta(k_1, (a_\tau \triangleleft b_\tau) \triangleleft c_\tau, b_\tau \triangleleft c_\tau, c_\tau) \right).$$
running over the all crossings $\tau$. Here, for the assignment $(x_\tau, y_\tau, z_\tau)$ around $\tau$ as in Figure 7, we define $(a_\tau, b_\tau, c_\tau) \in X^3$ by setting $S(x_\tau), S(y_\tau), S(z_\tau)$.

3 Relative group homology.

The construction of $\phi_0$ in Theorem 2.1 relies on relative group homology.

We give an outline of the proof. The first step is to introduce three chain groups, and two chain maps (see §3–4). These chain maps will be summarized to

$$C^\ast_R(X) \overset{\varphi_*}{\longrightarrow} C^\Delta(X; \mathbb{Z}) \otimes_{\mathbb{Z}[\text{As}(X)]} \mathbb{Z} \xleftarrow{\alpha} C^\ast_{gr}(G, K; \mathbb{Z}).$$

(6)

Roughly speaking, the right hand side denominates relative groups cocycles (see §3.1), and the left one can be diagrammatically described (see §4). Thus, if we construct a bridge from the left side hand to the right one, we can obtain diagrammatic computations as in the theorems. However, as seen in §4, the existence of such a bridge depends on some properties of knot type. Thus, in the proof, we need careful verifications to deal with the chain maps.

From now on, we will accomplish the outline in details. To begin with, in §3.1, we introduce relative group homology in the family version. After that, we will give a key proposition 3.6.

Throughout this section, we fix a group $G$ and subgroups $K_1, \ldots, K_m \subset G$ as above. Furthermore, we denote the index set $\{1, \ldots, m\}$ by $I$, and denote $(K_1, \ldots, K_m)$ by $K$, in short.

3.1 Preliminaries; Two versions of group relative homology.

The relative group homology is usually defined from a group pair $K \subset G$; see, e.g., [NM, §3] or [Zic]. However, this paper generalizes it in the family version so as to deal with link cases.

Consider the union of the left quotients, $\sqcup_{i \in I}(K_i \setminus G)$, and define the subgroup of the form

$$C^\ast_{0\text{red}}(G, K, I) := \{ (a_1, \ldots, a_m) \in \mathbb{Z}[G]^m \mid \sum_{i \in I} a_i = 0 \}.$$  

(7)

Then, we canonically have a right $G$-module homomorphism

$$P_1 : C^\ast_{0\text{red}}(G, K, I) \longrightarrow \mathbb{Z}[\sqcup_{i \in I}(K_i \setminus G)].$$

We define the relative group homology of $(G, K)$ to be the torsion $\text{Tor}^\ast_{\mathbb{Z}[G]}(\text{Coker}(P_1), M)$, where $M$ is a left $\mathbb{Z}[G]$-module. Precisely, taking the augmentation map $\varepsilon : \mathbb{Z}[\text{Coker}(P_1)] \to \mathbb{Z}$ with a choice of a projective resolution

$$\mathcal{P}_* : \cdots \overset{\partial_{n+1}}{\longrightarrow} \mathcal{P}_n \overset{\partial_n}{\longrightarrow} \cdots \overset{\partial_2}{\longrightarrow} \mathcal{P}_1 \overset{\partial_1}{\longrightarrow} \text{Coker}(P_1) \overset{\varepsilon}{\longrightarrow} \mathbb{Z} \quad \text{(exact)},$$

as right $\mathbb{Z}[G]$-modules, the relative homology is defined to be

$$H_n(G, K; M) := H_n(\mathcal{P}_* \otimes_{\mathbb{Z}[G^n]} M, \partial_*).$$

Dually, we can define the cohomology as $\text{Ext}^\ast_{\mathbb{Z}[G]}(\text{Ker}(\varepsilon), M)$. For enough projectivity, we now cite an example of such a $\mathcal{P}_*$.  

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Example 3.1 (Mapping cone). For any set $B$, we define the map $\partial_n^A : \mathbb{Z}[B^{n+1}] \to \mathbb{Z}[B^n]$ by setting
\[
\partial_n^A(x_0, \ldots, x_n) = \sum_{i \leq n} (-1)^i (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{Z}[B^n].
\] Furthermore, for $n > 1$, consider the following free $G$-module:
\[
C_n^{gr}(G, K) := \left( (\mathbb{Z}[G^{n+1}] \otimes_{\mathbb{Z}[G]} \mathbb{Z}) \oplus \left( \bigoplus_{i \leq m} \mathbb{Z}[K_i^n] \otimes_{\mathbb{Z}[K_i]} \mathbb{Z} \right) \right) \otimes_{\mathbb{Z}} \mathbb{Z}[G].
\]
Furthermore, when $n = 1$, we define $C_n^{gr}(G, K) = C_1(G) = \mathbb{Z}[G^2]$. Then, we can easily see that the following assignments define a differential on these modules: $\partial_1 := \partial_1^A$ and for $n > 1$,
\[
\partial_n(\tilde{g}, \tilde{k}_1, \ldots, \tilde{k}_m) := (\partial_n^A(\tilde{g}) + (-1)^n \sum_{i \in I} \tilde{k}_i, \partial_n^A(\tilde{k}_1), \ldots, \partial_n^A(\tilde{k}_m))
\]
for any $(\tilde{g}, \tilde{k}_1, \ldots, \tilde{k}_m) \in G^{n+1} \times K_1^n \times \cdots \times K_m^n$. Thus, we can define $H_n^{gr}(G, K; M)$ to be the homology of the complex $(C_n^{gr}(G, K) \otimes M, \partial_n)$. Then, a standard discussion of mapping cones deduces the long exact sequence with $n \geq 2$:
\[
\cdots \to H_{n+1}^{gr}(G, K; M) \xrightarrow{\delta} \bigoplus_j H_n^{gr}(K_j; M) \xrightarrow{\oplus (i_j)_*} H_n^{gr}(G; M) \to H_n^{gr}(G, K; M) \to \cdots. \tag{9}
\]

Proposition 3.2. Then, the pair $(C_n^{gr}(G, K), \partial_n)$ gives a free resolution of $\text{Coker}(P_1)$.

Proof. As is known (see [Zic, Theorem 2.1]), if $|I| = 1$ or $|I| = 0$, the statement is true. Then, we may assume $|I| > 1$. For any $i \in I$, consider the inclusion $C_n^{gr}(G)^{m-1} \to \bigoplus C_n^{gr}(G, K_i)$. Then, we have two sequences with commutativity:
\[
\cdots \to C_n^{gr}(G)^{m-1} \to C_{n-1}^{gr}(G)^{m-1} \to \cdots \to C_0^{red}(G, \emptyset, I) \to \varepsilon \mathbb{Z} \tag{exact}
\]
\[
\cdots \to \bigoplus C_n^{gr}(G, K_i) \to \bigoplus C_{n-1}^{gr}(G, K_i) \to \cdots \to \bigoplus \text{Coker}(P_{(i)}) \to \varepsilon \mathbb{Z} \tag{exact}
\]
Here, the exactness is ensured by the cases with $|I| \leq 1$. The cokernel of the vertical map is exactly the complex $(C_n^{gr}(G, K), \partial_n)$ from definitions. Hence, by the five lemma, the cokernel gives a free resolution of $\text{Coker}(P_1)$.

Remark 3.3. Here, we mention a topological meaning of the relative homology $H_n(G, K; M)$. Let $K(K_j, 1)$ and $K(G, 1)$ be the Eilenberg-MacLane spaces of $K_j$ and of $G$, respectively. Let $(i_j)_* : K(K_j, 1) \to K(G, 1)$ be the map induced from the inclusions. Then, similar to [Zic, §2 and §5], we can see that $H_n(G, K; M)$ is isomorphic to the homology of the mapping cone of $\sqcup i_j : \sqcup K(K_j, 1) \to K(G, 1)$ with local coefficients $M$.

Example 3.4. As an example, let us describe the non-homogenous cochain. Specifically, a 3-cocycle of $G$ relative to $K$ is represented by maps $\theta : G^3 \to A$ and $\eta_k : (K_k)^2 \to A$ satisfying the two equations
\[
g_1 \cdot \theta(g_2, g_3, g_4) - \theta(g_1g_2, g_3, g_4) + \theta(g_1, g_2g_3, g_4) - \theta(g_1, g_2, g_3g_4) + \theta(g_1, g_2, g_3) = 0, \tag{10}
\]
\[ \theta(k_1, k_2, k_3) = k_1 \cdot \eta(k_2, k_3) - \eta(k_1 k_2, k_3) + \eta(k_1, k_2 k_3) - \eta(k_1, k_2), \]  
for any \( g_i \in G \) and \( k_i \in K_i. \)

In another way, we will introduce the relative homology of \( \sqcup_{i \in I} (K_i \setminus G) \), which is originally defined by Hochschild [Hoc]. Let \( Y \) be the union \( \sqcup_{i \in I} (K_i \setminus G) \), and let \( C_n^{\text{pre}}(Y) \) be the free \( \mathbb{Z} \)-module generated by \((n + 1)\)-tuples \((y_0, y_1, \ldots, y_n) \in Y^{n+1}\). Consider the differential homomorphism defined by \( \partial^\Delta \) as above. As is basically known (see [NM, Bro, Zic]), the chain complex \((C^\Delta(Y), \partial^\Delta)\) is acyclic. Noting the natural action \( Y \curvearrowright G \), equip \( C_n^{\text{pre}}(Y) \) with the diagonal action. Furthermore, as a parallel to (7), for any \( \sigma \in Y \), we similarly have a \( G \)-module defined by \( \text{Coker}(P_n) \). The proof is essentially due to [NM]. Consider the submodule

\[ C_n^{\text{red}}(G, I) := \{ (a_i)_{i \in I} \in \bigoplus_{i \in I} \mathbb{Z}[G^{n+1}] \mid \sum_{i \in I} a_i = 0 \}. \]

Then, we similarly have a \( G \)-homomorphism \( P_n : C_n^{\text{red}}(G, I) \to C_n^{\text{pre}}(Y) \).

**Definition 3.5.** We define the chain complex \((C^\Delta_s(Y), \partial^\Delta_s)\) to be the cokernel \( \text{Coker}(P_n) \), which is diagonally acted on by \( G \).

Furthermore, \( H^s_*(Y; M) \) denotes the homology of the quotient complex \( \text{Coker}(P_n) \otimes_{\mathbb{Z}[G]} M \). Namely, \( H^s_*(Y; M) = H_s(\text{Coker}(P_n) \otimes_{\mathbb{Z}[G]} M) \).

This chain complex \((\text{Coker}(P_s), \partial^\Delta_s)\) is acyclic and is not always projective, even if \( |I| = 1 \). However, the projectivity of \( P_s \) admits, uniquely up to homotopy, a chain \( \mathbb{Z}[G] \)-map

\[ \alpha : (P_s, \partial_s) \to (C^\Delta_s(Y), \partial^\Delta_s). \]

**3.2 A key proposition from malnormality, and some examples.**

Whereas this \( \alpha \) is not always a quasi-isomorphism (see [NM, §3.2] for counter-examples), we give a criterion which is a key in this paper.

**Proposition 3.6** (A modification of [NM Proposition 3.23]). The set \( Y = \sqcup_{i \in \#L} (K_i \setminus G) \) is assumed to be of infinite order. Furthermore, the subgroups \( K_1, \ldots, K_{\#L} \subset G \) are malnormal.

Then, the chain map \( \alpha \) induces an isomorphism \( H^s_*(Y; M) \cong H_*(G, K; M) \) for any coefficient \( M \).

**Proof.** The proof is essentially due to [NM]. Consider the submodule

\[ C_n^\#(Y) := \mathbb{Z}\{ ([x_0, \ldots, x_n]) \in \text{Coker}(P_n) \mid \text{If } s \neq t, \text{ then } x_s \neq x_t. \}. \]

Since \( Y \) is of infinite order, this \( C_n^\#(Y) \) is an acyclic subcomplex of \( \text{Coker}(P_n) \), and the injection is quasi-isomorphic; see [NM Proposition 3.20] for the details. Furthermore, we can easily check that, if \( \sigma \cdot g = \sigma \) with \( g \in G \) and \( \sigma := (x_0, \ldots, x_n) \in C_n^\#(Y) \), the malnormal assumption implies \( g = 1_G \in G \). That is, the action is free; therefore, \( C_n^\#(Y) \) is the free \( \mathbb{Z}[G] \)-module. Since the above \( \alpha \) factors through \( C_n^\#(Y) \), we have the conclusion.

We end this section by giving three examples satisfying the assumptions.
Example 3.7 (hyperbolic 3-manifolds). Let \( N \) be a compact hyperbolic 3-manifold with torus boundary \( \partial N = T_1 \sqcup \cdots \sqcup T_m \). Apply \( G \) to \( \pi_1(N) \) and \( K_i \) to \( \pi_1(T_i) \) with a choice of base point. As is well-known as “algebraic atoroidality” in hyperbolic geometry (see [AFW]), the boundary group \( \pi_1(T_i) \) injects \( \pi_1(N) \), and the malnormal condition holds. Since \( N \) is a \( K(G,1) \)-space by hyperbolicity, we thus have the isomorphisms

\[
H^\Delta_*(Y; \mathbb{Z}) \cong H^\Theta_*(G, \mathcal{K}; \mathbb{Z}) \cong H_*(N, \partial N; \mathbb{Z}).
\]

Example 3.8 (Knots). Furthermore, given a non-trivial knot \( L \) in the 3-sphere \( S^3 \), we replace \( G \) by \( \pi_1(S^3 \setminus L) \) and \( K_1 \) by a peripheral subgroup \( \pi_1(\partial(S^3 \setminus L)) \cong \mathbb{Z}^2 \), which is generated by a meridian-longitude pair \((m, l)\). By the loop theorem of 3-manifolds, \( K_1 \) injects \( G \). Furthermore, \( S^3 \setminus L \) is basically known to be a \( K(G,1) \)-space.

Moreover, we mention a theorem to detect the malnormality of the pair.

Theorem 3.9 ([Sim][HW]). Let \( K_1 \subset G \) be as above. The pair \((K_1, G)\) is malnormal if and only if the knot \( L \) is neither coprime nor cable.

In particular, in the case, the isomorphism \( \alpha: H^\Delta_*(Y; \mathbb{Z}) \cong H_*(E_L, \partial(E_L); \mathbb{Z}) \) holds.

Example 3.10 (Link quandles). More generally, let us consider a link \( L \subset S^3 \) and the link group \( G = \pi_L = \pi_1(S^3 \setminus L) \). Let \( J \) with \( 1 \leq j \leq \pi_0(L) \) be the abelian subgroup generated by a meridian-longitude pair \((m_j, l_j)\) with respect to the \( j \)-th link component, that is, \( K_j \) is a peripheral group generated by \((m_j, l_j)\). We denote \( \sqcup J K_j \) by \( \partial \pi_L \) hereafter.

However, there are many non-malnormal pairs \((\pi_L, \partial \pi_L)\). A typical example is the Hopf link. More generally, malnormality for splittable links is completely characterized in [HW Corollary 4].

Incidentally, the union \( \sqcup J (K_j \setminus \pi_L) \) with the binary operation \([4]\) is called the link quandle \([Joy]\). We denote the link quandle by \( Q_L \), since we later use it in many times.

4 Review; quandle homology and Inoue–Kabaya map.

Next, regarding the middle term in the zigzag sequence \([6]\), this section reviews the quandle homology \([CJLS]\) and Inoue–Kabaya chain map \([IK]\). As seen in \([CKS]\) \([IK]\) \([No3]\) \([No1]\), quandle theory is useful for reducing some 3-dimensional discussions to diagrammatic objects.

We briefly explain the rack and quandle (co)homology groups \([CJLS]\) \([CKS]\). Let \( X \) be a quandle, and \( C_n^R(X) \) be the free right \( \mathbb{Z} \)-module generated by \( X^n \). Namely, \( C_n^R(X) := \mathbb{Z}[X^n] \).

Define a boundary \( \partial_n^R : C_n^R(X) \rightarrow C_{n-1}^R(X) \) by

\[
\partial_n^R(x_1, \ldots, x_n) = \sum_{1 \leq i \leq n} (-1)^i \left( (x_1 \lhd x_i, \ldots, x_{i-1} \lhd x_i, x_{i+1}, \ldots, x_n) - (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \right).
\]

Since \( \partial_n^R \circ \partial_n^R = 0 \) as usual, we can define the homology \( H_n^R(X) \) and call it the rack homology. Furthermore, let \( C_n^D(X) \) be the submodule of \( C_n^R(X) \) generated by \( n \)-tuples \((x_1, \ldots, x_n)\) with \( x_i = x_{i+1} \) for some \( i \in \{1, \ldots, n-1\} \). One can easily see that this \( C_n^D(X) \) is a subcomplex of \( C_n^R(X) \). Then, the quandle homology, \( H_n^Q(X) \), is defined to be the homology of the quotient complex \( C_n^R(X)/C_n^D(X) \). In general, it is difficult to compute these homology groups.
In addition, we will review the Inoue–Kabaya map whose codomain is the Hochschild complex in Definition 3.5. For this, we need some notation. A map \( f : X \to X' \) between quandles is a quandle homomorphism, if \( f(a \triangleleft b) = f(a) \triangleleft f(b) \) for any \( a, b \in X \). Furthermore, given a quandle \( X \), we set up the abstract group, \( \text{As}(X) \), with presentation

\[
\text{As}(X) := \langle e_x \ (x \in X) \mid e_x^{-1} \cdot e_y^{-1} \cdot e_x \cdot e_y \ (x, y \in X) \rangle.
\]

We call \( \text{As}(X) \) the associated group. Further, \( \text{As}(X) \) has a right action on \( X \) defined to be \( x \cdot e_y := x \triangleleft y \), where \( x, y \in X \). Let \( O(X) \) be the orbit set of \( X \triangleleft \text{As}(X) \). With respect to \( i \in O(X) \), we fix \( x_i \in X \) in the orbit. As in Example 2.2, denoting \( \text{Stab}(x_i) \subset \text{As}(X) \) by \( K_i \), we can consider the setting

\[
X = Y = \bigsqcup_{i \in O(X)} (K_i \setminus G) \quad \text{with} \quad G = \text{As}(X).
\]

Furthermore, we set up the following set consisting of some maps:

\[
I_n := \{ \ i : \{2, 3, \ldots, n\} \to \{0, 1\} \},
\]

which is of order \( 2^{n-1} \). Moreover, given a tuple \( (x_1, \ldots, x_n) \in X^n \) and each \( i \in I_n \), we define \( x(i, i) \in X \) by the formula

\[
x(i, i) := x_i \cdot (e_{x_{i+1}}^{(i+1)} \cdots e_{x_n}^{(n)}).
\]

Then, with a choice of an element \( p \in X \), we define a homomorphism

\[
\varphi_n : C_n^R(X) \to C_n^\Delta(X) \otimes \mathbb{Z}[\text{As}(X)] \mathbb{Z}
\]

by setting

\[
\varphi_n(x_1, \ldots, x_n) := \sum_{i \in I_n} (-1)^{e(2) + e(3) + \cdots + e(n)} (p, x(i, 1), \ldots, x(i, n)).
\]

Here are the descriptions of \( \varphi_* \) of lower degree:

\[
\begin{align*}
\varphi_2(a, b) &= (p, a, b) - (p, a \triangleleft b, b), \\
\varphi_3(a, b, c) &= (p, a, b, c) - (p, a \triangleleft b, b, c) - (p, a \triangleleft c, b \triangleleft c, c) + (p, (a \triangleleft b) \triangleleft c, b \triangleleft c, c).
\end{align*}
\]

Then, it is shown [IK §4] that this \( \varphi_n \) is a chain map, i.e., \( \partial_n^\Delta \circ \varphi_n = \varphi_{n-1} \circ \partial_n^R \), and that if \( n \leq 3 \), the image of the subcomplex \( C_n^D(X) \) is nullhomotopic. Hence, the map \( \varphi_3 \) with \( n = 3 \) induces a homomorphism

\[
(\varphi_3)_* : H_3^Q(X) \to H_3^\Delta(X; \mathbb{Z}).
\]

We refer the reader to several studies on the chain map; see [IK, Kab, No1, No2, No3].

Next, we review the quandle cocycle invariant. Given a shadow coloring \( S \) on a link diagram \( D \), the fundamental 3-class of \( S \) denoted by \( [S] \) is defined to be the sum

\[
[S] := \sum_{\tau} \varepsilon_\tau (\lambda(x_\tau), C(y_\tau), C(z_\tau)) \in C_3^Q(X)
\]

by running over all the crossings \( \tau \) illustrated in Figure 11, where the triple \( (x_\tau, y_\tau, z_\tau) \) consists of the three arcs around \( \tau \), and \( \varepsilon_\tau \in \{\pm 1\} \) is the sign of \( \tau \). Then, we can easily see that \( [S] \)
is a quandle 3-cycle in $C^Q_3(X)$; see [CKS]. If we have a quandle 3-cocycle $\phi : X^3 \to A$, the pairing $\langle \phi, [S] \rangle \in A$ is called the quandle cocycle invariant of $S$. Here, a map $\phi : X^3 \to A$ is a quandle 3-cocycle, if the followings hold by definition:

$$
\phi(x, z, w) - \phi(x \triangleleft y, z, w) - \phi(x, y, w) + \phi(x \triangleleft w, y \triangleleft w, z \triangleleft w) - \phi(x, y, z),
$$

$$
\phi(x, x, y) = \phi(y, z, z) = 1,
$$

for any $x, y, z, w \in X$.

For calculating the invariant $\langle \phi, [S] \rangle$, it is important to find explicit formulas of quandle 3-cocycles, although it is difficult in general.

**Example 4.1.** Let $X$ be the link quandle $Q_L$ of a link; see Example 3.10. Consider the identity $\pi_L \to \pi_L$. Then, by (5), we have the $Q_L$-coloring $S_{\text{id}_{Q_L}}$, and have the associated 3-class $[S_{\text{id}_{Q_L}}]$. This homology 3-class plays a key role later.

**Example 4.2.** In hyperbolic case, Inoue and Kabaya obtain a cocycle from the chain map $\varphi_*$, with a relation to the Chern-Simons invariant. Let $X$ be the quotient set $\mathbb{C}^2 \setminus \{(0, 0)\}/\sim$ subject to the relation $(a, b) \sim (-a, -b)$. Equip $X$ with a quandle operation

$$
(a, b) \triangleleft (c, d) = (a, b) \left( \begin{array}{cc} 1 + cd & d^2 \\ -c^2 & 1 - cd \end{array} \right).
$$

One can easily verify that $X$ is isomorphic to the triple $(G, K, z_0)$ as in Example 2.2, where $G$ is $PSL_2(\mathbb{C})$ and $K$ is the unipotent subgroup of the form $\left\{ \left( \begin{array}{cc} 1 & a \\ 0 & 1 \end{array} \right) \mid a \in \mathbb{C} \right\}$, and $z_0 = \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$.

Moreover, this $(G, K)$ is widely known to be malnormal; see [HWO, HW]. Therefore, the chain map $\alpha$ is a quasi-isomorphism, which ensures a quasi inverse $\beta$. Furthermore, Neumann [Neu] and Zickert [Zic] described the Chern-Simons 3-class as a relative group 3-cocycle

$$
CS \in C^3_{gr}(PSL_2(\mathbb{C}), K; \mathbb{C}/\pi^2\mathbb{Z}),
$$

(13)

together with a cocycle presentation (see [Zic] for the detail). As a consequence, we concretely get a quandle 3-cocycle $\varphi^* \circ \beta^*(CS)$.

Further, we consider a hyperbolic link $L$. From the viewpoint of Example 2.2, the associated holonomy representation $\rho : \pi_L \to PSL_2(\mathbb{C})$ is regarded as a shadow $X$-coloring $S_\rho$. Then, Inoue and Kabaya [IK] Theorem 7.3 showed the equality

$$
\langle \rho^*(CS), [E_L, \partial E_L] \rangle = \langle (\beta \circ \varphi_3)^*(CS) , [S_\rho] \rangle \in \mathbb{C}/\pi^2\mathbb{Z}.
$$

(14)

As a result, we can compute the Chern-Simons invariant without triangulation, as a quandle cocycle invariant; see [IK] for examples.

### 5 Algebraic presentation of the fundamental homology 3-class.

In this section, we give a method to algebraically present the fundamental 3-class of $(E_L, \partial E_L)$, where $L$ is either a hyperbolic link or a prime non-cable knot.

To describe this, the following plays a key viewpoint (see [6] for the proof).
Theorem 5.1. Assume that the knot $L$ is neither coprime nor cable, as in Theorem \[A\]. Then, the Inoue–Kabaya chain map $\varphi_3$ induces an isomorphism $H^3_\Delta(Q_L) \rightarrow H^3_\Delta(Q_L; \mathbb{Z})$ as $\mathbb{Z}$.

Theorem [A.1] implies that $H^3_\Delta(Q_L)$ is generated by some fundamental 3-class $[S_{idQ_L}]$. Hence, if we can explicitly formulate a quasi-inverse $\beta : C^\Delta_*(Q_L)_{\pi_L} \rightarrow C^gr_*(\pi_L, \partial\pi_L; \mathbb{Z})$ in \[16\], then we obtain an algebraic presentation of the fundamental 3-class $[E_L, \partial E_L]$.

We will explain the reason why we focus on only hyperbolic links in \[5,2\]. The key is based on the JSJ decomposition of knots. Precisely, the geometrization theorem and the JSJ decomposition on a prime knot in the 3-sphere (see, e.g., [Bud, Theorem 4.18] or [AFW, HW]) say that there exist open sets $V_1 \subset V_2 \subset \cdots V_n \subset S^3$ satisfying the followings:

1. The set $V_i$ for any $i$ is an open solid torus in $S^3$, and $V_i$ contains the prime knot $L$.
2. The difference $V_i - \overline{V}_{i-1}$ for any $i \in \mathbb{Z}_{\geq 0}$ is either a hyperbolic knot or an $(n_i, m_i)$-torus knot in the solid torus for some $(n_i, m_i) \in \mathbb{Z}^2$. Here we denote the knot $L$ by $V_0$.

As is known, the decomposition is unique in some sense. Here, remark (see [Bud, Corollary 4.19]) that $L$ is a cable knot if and only if there is a difference $V_j - \overline{V}_{j-1}$ for some $j$ is an $(n_j, m_j)$-torus knot in the solid torus; see Figure 5.

Following the JSJ-decomposition, let us further examine the pairing \[1\]. Denote the inclusion $V_i - V_{i-1} \subset S^3 - L$ by $\iota_i$, and the torus-boundary $\overline{V}_{i-1} \cap V_i$ by $B_i$. Then, given $f : \pi_L \rightarrow G$ and $\theta$ as before, we have $f \circ (\iota_i)_* : \pi_1(V_i - V_{i-1}) \rightarrow G$. Let $K_i \subset G$ be the image of $\pi_1(B_i)$ via $f \circ (\iota_i)_*$, where we fix an appreciate base point. Then, we can regard the pullback $\iota_i^* \circ f^*(\theta)$ as a relative group 3-cocycle of $(G, K_i, K_{i+1})$.

In summary, the excision axiom on $\iota_i$’s ensures the equality

$$
\langle f^*(\theta), [E_L, \partial E_L] \rangle = \sum_{i:1 \leq i \leq m} \langle \iota_i^* \circ f^*(\theta), [V_i - V_{i-1}, \partial(V_i - V_{i-1})] \rangle.
$$

To conclude, for prime non-cable knots, it is enough to deal with the fundamental 3-classes of hyperbolic links.

5.1 The fundamental relative 3-class of hyperbolic links.

This subsection gives an explicit algorithm for describing the fundamental relative 3-class of hyperbolic links. Here, the description is done in truncated terms; see Theorem \[5.3\].

We begin by reviewing the truncated complex, which is defined by Zickert [Zic, §3]. Fix a group $G$, and subgroups $K_1, \ldots, K_m$. For $n \geq 1$, consider the free abelian group $\mathbb{Z}[G^{n^2+n}]$, and denote the $(ij)$-th generator $g \in G$ by $g_{ij}$ with $i \neq j$. Define $\overline{C}_n(G, \mathcal{K})$ by the submodule of $\mathbb{Z}[G^{n^2+n}]$ which is generated by $g_{ij}$ satisfying

- for any $i \in \{0, \ldots, n\}$, there exists $m_i$ such that the $n$ elements $g_{i0}, \ldots, g_{ii}, \ldots, g_{in}$ subject to $K_{m_i}$ are equal in the coset $K_{m_i} \backslash G$.

Then, right multiplication endows $\overline{C}_n(G, \mathcal{K})$ with a $G$-module structure, and the usual simplicial boundary map gives rise to a boundary map $\partial_*$ on $\overline{C}_n(G, \mathcal{K})$. The complex $(\overline{C}_*(G, \mathcal{K}), \partial_*)$
is called the truncated complex of \((G, K)\). As was analogously shown \cite{Zic} Remark 3.2 and Proposition 3.7, we can easily verify that this complex is a free resolution of \(\text{Coker}(P_I)\).

In addition, let us examine the case where \(G_C\) is \(PSL_2(\mathbb{C})\) and every \(K_{C,t}\) is conjugate to the unipotent subgroup such that \(K_{C,s} \cap K_{C,t} = \{1_{G_C}\}\) for \(s \neq t\). Then, we have the quandle \(X'\), from Example \[1.2\] as the union of the parabolic quandles \(\sqcup^{#_L}K_{C,t}\backslash G_C = \sqcup^{#_L}(\mathbb{C}^2 \setminus \{0,0\})/\sim\). We will describe a quasi-inverse \(\beta\) mentioned in Proposition \[3.6\]. For this, consider the following subcomplex of \(C^A_n(X_C; \mathbb{Z})\):

\[
C_{n}^{h\#}(X_C) = \langle \{ (a_0, b_0), \ldots, (a_n, b_n) \} \in C^A_n(X_C; \mathbb{Z}) \mid a_i b_j \neq a_j b_i \text{ for any } i, j \text{ with } i \neq j \rangle.
\]

Then, this complex is known to be an acyclic \(\mathbb{Z}[G_C]\)-free complex. Consider the correspondence

\[
(X_C)^{n+1} \to \mathbb{G}; \quad ((a_0, b_0), (a_1, b_1), \ldots, (a_n, b_n)) \mapsto g_{ij} := \left( \begin{array}{cc} a_i & b_i \\ a_j/(a_i b_j - a_j b_i) & b_j/(a_i b_j - a_j b_i) \end{array} \right),
\]

This gives rise to a homomorphism

\[
\beta : C_{n}^{h\#}(X_C) \to \overline{C}_n(G_C, K_C).
\]

Then, Zickert \cite{Zic} §3 (see also \cite{NM} Corollary 9.6) showed that this map is a \(\mathbb{Z}[G]\)-module complex homomorphism. To summarize, this \(\beta\) gives a quasi-inverse of the chain map \(\alpha : C^\text{gr}_n(G_C, K_C) \to C_n(X_C)\).

We return to the discussion of a hyperbolic link \(L\), and state Theorem 5.3 below. Fix a diagram \(D\) of \(L\). Then, we have the holonomy representation \(\rho : \pi_L \to PSL_2(\mathbb{C})\). As is well-known, \(\rho\) is injective. Thus, it is more sensible to use matrices in \(PSL_2(\mathbb{C})\), than to use (Wirtinger) group presentations of \(\pi_L\). Here, we should mention the following lemma obtained from hyperbolicity.

**Lemma 5.2** (see \cite{Zic} §5 or \cite{IK} Lemma 7.2). Let \(\sigma \in C_3(X_C)\) be a 3-cycle which represent the fundamental 3-class \(\rho_*(E_L, \partial E_L)\) of a hyperbolic link. Then, this 3-cycle \(\sigma\) lies in the subcomplex \(C^\#_3(X_C)\).

To sum up, we obtain the conclusion:

**Theorem 5.3.** Let \(L\) be a hyperbolic link with the holonomy representation \(\rho : \pi_L \to PSL_2(\mathbb{C})\). Let \(G\) be the image \(\rho(\pi_L)\), and \(K\) be the subgroups \(\rho(\partial \pi_L)\). Fix a diagram \(D\), and take the quandle 3-class \([S]\); see \[4\] Then, the following 3-cycle represents the fundamental 3-class in \(H_3(E_L, \partial E_L; \mathbb{Z}) \cong H^\text{gr}_3(\pi_L, \partial \pi_L; \mathbb{Z}) \cong \mathbb{Z}\).

\[
\text{res}(\beta) \circ \varphi_3([S]) \in \overline{C}_3(G, K; \mathbb{Z}).
\]
5.2 Example on the figure eight knot.

As the simplest case, we let $L$ be the figure eight knot as in Figure 4. By the Wirtinger presentation, we have

$$\pi_1(S^2 \setminus L) \cong \langle g, h \mid h^{-1}gh = g^{-1}h^{-1}ghg^{-1}h \rangle,$$

where $g$ and $h$ are meridians derived from the arcs $\alpha_1$ and $\alpha_2$, respectively. We denote the two classes in $Q_L$ of $g$ and $h \in \pi_L$ by $a$ and $b$, respectively. Then, by definition, the fundamental 3-class $[S_{id_{Q_L}}]$ is given by

$$-(b \triangleleft a, a, b) - (b \triangleleft a, b, a) + ((b \triangleleft a) \triangleleft b, a, a \triangleleft b) + (b, b, b \triangleleft a) \in C^Q_3(Q_L; \mathbb{Z}).$$

Notice that the first term is sent to zero, by $\varphi_*$. Then, $\varphi_*[S_{id_{Q_L}}]$ is computed as

$$-(p, b \triangleleft a, a, b) + (p, (b \triangleleft a) \triangleleft a, a, b) + (p, (b \triangleleft a) \triangleleft b, a \triangleleft b, b) - (p, ((b \triangleleft a) \triangleleft a) \triangleleft b, a \triangleleft b, b)
- (p, b \triangleleft a, b, a) + (p, (b \triangleleft a) \triangleleft b, b, a) + (p, (b \triangleleft a) \triangleleft a, b \triangleleft a, a) - (p, ((b \triangleleft a) \triangleleft b) \triangleleft a, b \triangleleft a, a)
+ (p, (b \triangleleft a) \triangleleft b, a \triangleleft b) - (p, ((b \triangleleft a) \triangleleft b) \triangleleft a, a \triangleleft b) - (p, ((b \triangleleft a) \triangleleft b) \triangleleft a, a \triangleleft b, a \triangleleft b)
+ (p, ((b \triangleleft a) \triangleleft b) \triangleleft a) \triangleleft (a \triangleleft a, a \triangleleft (a \triangleleft b), a \triangleleft b)).$$

As matters now stand, they seem complicated.

Thus, following Theorem 5.3 we consider the well-known holonomy representation $\rho : \pi_L \to PSL_2(\mathbb{Z}[\omega])$, where $\omega = (-1 + \sqrt{-3})/2$. This is represented by the $X_\omega$-coloring $C$ in Figure 4. Accordingly, replacing $a$ by $(1, 0)$ and $b$ by $(0, \omega)$. The 3-cycle above is reduced to

$$-(p, (-\omega, \omega), (1, 0), (0, \omega)) + (p, (-2\omega, \omega), (1, 0), (0, \omega)) + (p, (-\omega, \omega + 1), (1, \omega - 1), (0, \omega))
- (p, (-2\omega, \omega + 2), (1, \omega - 1), (0, \omega))
- (p, (-\omega, \omega), (0, \omega), (1, 0)) + (p, (-2\omega, \omega), (0, \omega), (1, 0)) + (p, (-\omega, \omega), (-\omega, \omega), (1, 0))
- (p, (-2\omega - 1, \omega + 1), (-\omega, \omega), (1, 0))
+ (p, (-\omega - 1 + \omega), (0, \omega), (1, \omega - 1)) - (p, (-\omega, 2 + \omega), (0, \omega), (1, \omega - 1))
- (p, (-2\omega - 1, 4), (-\omega, 1 + \omega), (1, \omega - 1)) + (p, (-2\omega - 2, 6 - \omega), (-\omega, 1 + \omega), (1, \omega - 1)).$$

Hence, for example, if $p = (0, 1)$ and we apply the composite $\beta$ to this cycle, we can describe explicitly the fundamental 3-class. However, the description forms long; we omit writing it.

Figure 4: The holonomy representation of $4_1$ as an $X$-coloring. Here $\omega = (-1 + \sqrt{-3})/2$.

6 Proofs of the main theorems.

We will prove the theorems in §2. If $L$ is the trivial knot, the theorems are obvious. Thus, we may assume that $L$ is non-trivial in what follows. We begin by proving Theorem 2.1. Since the discussion is not functorial, the proof may seem intricate.
6.1 Proof of Theorem 2.1

Proof. Recall the complexes in Sections 3, 4. Since they are functorial by construction, we obtain a commutative diagram:

\[
\begin{array}{cccc}
C^R_*(Q_L; \mathbb{Z}) & \xrightarrow{\varphi_*} & C^A_*(Q_L; \mathbb{Z}) & \xrightarrow{\alpha} C^B_*(\pi_L, \partial \pi_L; \mathbb{Z}) \\
\downarrow{f_*} & & \downarrow{f_*} & \downarrow{f_*} \\
C^R_*(X; \mathbb{Z}) & \xrightarrow{\varphi_*} & C^A_*(X; \mathbb{Z}) & \xrightarrow{\alpha} C^B_*(G, K; \mathbb{Z})
\end{array}
\]

(16)

Since \( L \) is either a prime non-cable knot or a hyperbolic link by assumption, the right \( \beta \) comes from the quasi-isomorphism \( \alpha \) in Examples 3.7 and 3.8.

We will explain (17) below. Recall the quandle 3-class \( [S_{idQL}] \) from Example 4.1. Then, denoting by \([\pi_L, \partial \pi_L] \) a generator of \( H^3_3(\pi_L, \partial \pi_L) \cong \mathbb{Z} \), the diagram (16) admits \( N_L \in \mathbb{Z} \) such that

\[
\beta \circ \varphi_3([S_{idQL}]) = N_L[\pi_L, \partial \pi_L] \in H^3_3(Q_L; \mathbb{Z}) \cong \mathbb{Z}.
\]

Then, for every group relative 3-cocycle \( \theta \), setting \( \phi_\theta = (\beta \circ \varphi_3)^* \circ f^* (\theta) \) yields the equalities

\[
\langle \phi_\theta, [S_{idQL}] \rangle = \langle f^*(\theta), \beta \circ \varphi_3([S_{idQL}]) \rangle = N_L\langle f^*(\theta), [\pi_L, \partial \pi_L] \rangle \in A_G.
\]

(17)

Hence, it is enough to show \( N_L = \pm 1 \). Actually, if \( N_L = \pm 1 \), we canonically obtain a map \( \phi_\theta : Reg_D \times (Arc_D)^2 \rightarrow A \) from the definition of \([S_{idQL}]\), which justifies the desired equality (3).

In hyperbolic case, letting \( f \) be the associated holonomy \( \pi_L \rightarrow PSL_2(\mathbb{C}) \) with \( \theta = CS \), we immediately have \( N_L = \pm 1 \) by (14).

It remains to work in the prime case where \( L \) is neither hyperbolic nor cable. Then, thanks to the JSJ-decomposition (15) above, there is a solid torus \( V_1 \subset S^3 \), which contains the link \( L \) and \( V_1 \setminus L \) is hyperbolic.

We will give a diagram (18) below. Regarding \( V_2 \) as a link component, we have another hyperbolic link \( L' := L \cup (S^3 \setminus V_2) \) in the 3-sphere. Denote by \( K_2 \) another subgroup of \( \pi_1(S^3 \setminus L') \) arising from \( \partial(S^3 \setminus L') \). Then, the link quandle \( Q_{L'} \) is bijective to \( K_1 \setminus \pi_L' \cup K_2 \setminus \pi_L' \) by definition. Furthermore, letting \( W = V_1 \setminus L \), we consider the homogenous quandle of the form \( K_1 \setminus \pi_L \cup K_2 \setminus \pi_L \), and denote it by \( Q_W \). Then, the inclusion \( j : S^3 \setminus L' \rightarrow S^3 \setminus L \) defines a quandle map \( j_* : Q_{L'} \rightarrow Q_W \), and we have a canonical injection \( Q_L \rightarrow Q_W \). In summary, we have the commutative diagram on the third homology groups:

\[
\begin{array}{cccc}
H^3_3(Q_L; \mathbb{Z}) & \xrightarrow{i^Q_*} & H^3_3(Q_W; \mathbb{Z}) & \xrightarrow{j^Q_*} H^3_3(Q_{L'}; \mathbb{Z}) \\
\downarrow{\varphi_*} & & \downarrow{\varphi_*} & \downarrow{\varphi_*} \\
H^3_3(Q_L; \mathbb{Z}) & \xrightarrow{i^Q_*} & H^3_3(Q_W; \mathbb{Z}) & \xrightarrow{j^Q_*} H^3_3(Q_{L'}; \mathbb{Z})
\end{array}
\]

(18)

In the appendix (Lemma A.3), we later show the isomorphisms \( H^3_3(Q_W; \mathbb{Z}) \cong H^3_3(Q_L; \mathbb{Z}) \cong \mathbb{Z}^2 \), and the matrixes of \( i^Q_* \) and \( j^Q_* \) are given by \((1, 0)\) and \(\left(\begin{array}{cc}1 & 1 \\ 1 & 1\end{array}\right)\), respectively. Hence, since the right \( \varphi_* \) is surjective by the former discussion, the commutativity with (13) implies \( N_L = \pm 1 \) as desired. \( \square \)
Proof of Theorem 6.1. Since $L$ is neither coprime nor cable, the above discussion readily implies the bijectivity of $\phi_\ast$. \qed

Remark 6.1. We mention the assumption of hyperbolicity. As a counter example, consider the Hopf link $L$. Since $\pi_1(S^3 \setminus L) \cong \mathbb{Z}^2$ and the boundary inclusions induce isomorphisms on $\pi_1$, the link quandle $Q_L$ consists of two points. Hence, the chain map $\varphi_\ast$ is zero by definition. However, the pairing $\langle \theta, f_*[E_L, \partial E_L] \rangle$ is not always trivial. In summary, it is seemingly hard to generalize the theorem 2.1 in every link case.

6.2 Proof of Theorem 2.3. Malnormality and Transfer.

Next, tuning to malnormality and transfer, we will complete the proof of Theorem 2.3. For this, we shall mention a key proposition obtained from transfer.

Proposition 6.2 (cf. Transfer; see [Bro §III.10]). Let $K_1, \ldots, K_{\#L}$ be finite subgroups of $G$, and let $Y$ be $\sqcup_i(K_i \setminus G)$ with action. Assume that all the order $|K_i|$ is invertible in the coefficient group $A$. Then, the chain map

$$\alpha : (\mathcal{P}_*, \otimes \mathbb{Z} A, \partial_*) \longrightarrow (C_i^\Delta(Y) \otimes \mathbb{Z} A, \partial_*)$$

with coefficients $A$ is a quasi-isomorphism.

Proof. According to the same discussion on the transfer; see [Bro §III.9–10] \qed

Proof of Theorem 2.3. Since $K \subset G$ is malnormal by assumption, Proposition 3.6 again ensures a quasi-inverse $\beta' : C_i^\Delta(X)_G \rightarrow C_i^\beta(G, K)$, where $X = \sqcup_i K_i / G$ as in Example 2.2. Then, for any group 3-cocycle $\theta$, we set $\phi_\theta = (\beta' \circ f_\ast \circ \varphi_3)^\ast(\theta) \in A$ as a quandle 3-cocycle.

We will show the equality below. First, assume that $L$ is either a hyperbolic link or a prime knot. Compute the pairing $\langle \phi_\theta, [S_f] \rangle$ as

$$\langle \phi_\theta, [S_f] \rangle = \langle f^* \circ (\beta')^\ast(\theta), (\varphi_3)^\ast[S_f] \rangle = \langle f^* \circ (\beta')^\ast(\theta), (\alpha)^\ast[\pi_L, \partial \pi_L] \rangle$$

$$= \langle \alpha^* \circ f^* \circ (\beta')^\ast(\theta), [\pi_L, \partial \pi_L] \rangle = \langle f^*(\theta), [\pi_L, \partial \pi_L] \rangle.$$ (19)

Here, the second equality is obtained by (17) and Proposition 3.6, and the others are done by functoriality.

Since this assumption is the same as that in Theorem 2.3, we can a quasi inverse $\beta$ of $\alpha$ in the coefficients $M$. Hence, the required equality is done by the same discussion as (19). \qed

Finally, we will point out that, in some cases, such group 3-cocycles $\theta$ have much simpler expressions in the terms of quandle cocycles. We end this section by giving two examples.

Example 6.3 (Cyclic group). The first is an analogous discussion to [No1 §5]. Let us consider a dihedral group $G = \mathbb{F}_p \rtimes \mathbb{Z}/2$ with the subgroup $K = \mathbb{Z}/2$, where $p \in \mathbb{N}$ is a prime. Let $X = G / K = \mathbb{F}_p$. Then, $H^3(G; \mathbb{F}_p) \cong \mathbb{F}_p$ is known. Furthermore, we can easily check that the following map is a 3-cocycle of $C_3^\Delta(X; \mathbb{F}_p)$:

$$\theta : X^4 \longrightarrow \mathbb{F}_p; \quad (x, y, z, w) \longmapsto (x - y)(y - z)^p + (z - w)^p - (y - w)^p.$$
Then, we can compute the pullback as a quandle 3-cocycle
\[ \varphi^*(\theta)(x, y, z) = (x - y)^2y^p - (2y - z)^p + z^p. \]

This formula is first found by Mochizuki [Moc], and is shown to be non-trivial in \( H_3^3(X; \mathbb{F}_p) \cong \mathbb{F}_p \). Some papers in quandle theory compute the quandle cocycle invariants.

Furthermore, we will see that some triple Massey product are much simpler in the viewpoint of quandle cohomology. This example is essentially due to [No1, §4.2].

**Example 6.4** (cf. [No1]). Regard the finite field \( \mathbb{F}_q \) as the abelian group \((\mathbb{Z}/p)^m\), where \( q = p^m \) and \( p \neq 2 \). Consider the (nilpotent) group on the set
\[ G := \mathbb{Z}/2 \times \mathbb{F}_q \times (\mathbb{F}_q \wedge \mathbb{F}_q), \]
with operation
\[ (n, a, \kappa) \cdot (m, b, \nu) = (n + m, (1)^m a + b, \kappa + \nu + [(-1)^m a \otimes b]). \]  

Letting the subgroup \( K = \mathbb{Z}/2 \times \{0\} \times \{0\} \subset G \) and let \( x_0 \in K \) be \((1, 0, 0)\), we have the quandle of the form \( X = \mathbb{F}_q \times (\mathbb{F}_q \wedge \mathbb{F}_q) \).

The cohomology \( H^3(G; \mathbb{F}_q) \) is more complicated. In fact, the cohomology has 3-cocycles \( \theta_T \), which are derived from triple Massey product and are complicated (see [No1, Proposition 4.8]). However, the author [No1, Lemma 4.7] showed that the pullback \( \varphi^* \theta_T : X^3 \to \mathbb{F}_q \) is formulated as
\[ (\varphi^* \theta_T)((x, \alpha), (y, \beta), (z, \gamma)) = (x - y)^{q_1}(y - z)^{q_2+q_3}z^{q_4}, \]
with some prime powers \( q_1, q_2, q_3, q_4 \in \mathbb{Z} \). Since this formula is quite simple, we can compute the relative fundamental 3-class in an easier way than the group theoretic method; see [No1, §5] for a computation.

### 7 Theorem for cable knots.

While Theorems 2.1 and 2.3 assumed non-cabling knots, this section considers cable knots. For this consider a solid torus \( V \subset S^3 \) such that \( V \setminus L \) is the \((m, n)\)-torus knot. By the formula \( (15) \) from the JSJ-decomposition, it is sensible to consider either the torus knot \( T_{m, n} \) in \( S^3 \) or the solid one in \( V \). We denote the latter by \( S_{m, n} \); Regarding \( S^3 \setminus (L \sqcup (S^3 \setminus V)) \) as a link complement, the knot \( S_{m, n} \) admits a link-diagram in \( \mathbb{R}^2 \); see Figure 5.

While the statements analogous to the theorems 2.1 and 2.3 do not hold, we get similar results modulo some integer. Precisely,

**Theorem 7.1** (cf. Theorems 2.1 and 2.3). Assume that \( L \) is either the torus knot \( T_{m, n} \subset S^3 \) or that \( S_{m, n} \subset V \). Let \( N = n \) if \( L \) is \( S_{m, n} \), and let \( N = mn \) if \( L \) is \( T_{m, n} \).

Then, for any relative group 3-cocycle \( \theta \in H^3(G, K; A) \), there is a map \( \phi_\theta : \text{Reg}_D \times \text{Arc}_D \times \text{Arc}_D \to A \) for which the following equality holds in the coinvariant \( A_G \):
\[ \langle \theta, f_*(E_L, \partial E_L) \rangle = \Psi_{\phi_\theta}(D) \in A_G, \quad \text{modulo the integer } N. \]
Furthermore, if the pair \((G, K)\) is malnormal, then there are a quandle 3-cocycle \(\phi_\theta\) and an \(X\)-coloring \(S_f\) such that \(\langle \theta, f_*[E_L, \partial E_L]\rangle = \langle \phi_\theta, [S_f]\rangle\) modulo the integer \(N\).

In conclusion, we have a diagrammatic computation for cable knots, although the statement is modulo some integer. In addition, we later see that this discussion is reduced to the homology of cyclic groups; hence, the pairing does not have more information than cyclic groups.

### 7.1 Observations on the torus knots.

Before going to the proof, we show two lemmas, and observe an essential reason why the statement is modulo \(N\).

Let \(L\) be the \((n, m)\)-torus knot \(T_{n,m}\) in the 3-sphere \(S^3\), and let \(G = \pi_1(S^3 \setminus L)\) and \(K\) be the pherihedral subgroup. Fix \((a, b, n, m) \in \mathbb{Z}^4\) with \(an + bm = 1\). According to [HWO, §2], we mention these group presentations

\[
\pi_1(S^3 \setminus T_{n,m}) \cong \langle x, y \mid x^n = y^m \rangle \supset \langle x^ay^b, (x^ay^b)^{-nm}x^n \rangle = K.
\]

Then, Theorem 3.9 says that the pair \((G, K)\) is not malnormal. Considering the center \(Z = \langle x^n \rangle \subset G\), the quotients \(G/Z\) and \(K/Z\) are shown to be isomorphic to the free product \(\mathbb{Z}/n \ast \mathbb{Z}/m\) and \(\mathbb{Z}\), respectively. Denote the quotients \(G/Z\) and \(K/Z\) by \(G\) and \(K\), respectively. Then, it is known [HWO, §2] that the pair \((G, K)\) is malnormal. We will show the following:

**Lemma 7.2.** There are isomorphisms

\[
H^\Delta_e(K \setminus G; \mathbb{Z}) \cong H^\Delta_e(K \setminus G; \mathbb{Z}) \cong H^\Delta_e(G, K; \mathbb{Z}) \cong \begin{cases} 
\mathbb{Z}, & \text{if } * = 1, 2, \\
\mathbb{Z}/nm, & \text{if } * \text{ is odd, and } * \geq 3, \\
0, & \text{otherwise.}
\end{cases}
\]

**Proof.** The first one is obtained by the settheoretic equality \(K \setminus G = K \setminus G\), the second is done by the malnormality. We explain the last one: by a Mayer-Vietoris argument, the homology of \(\mathbb{Z}/n \ast \mathbb{Z}/m\) is that of the pointed sum \(L^\infty_\mathbb{Z} \vee L^\infty_\mathbb{Z}\), where \(L^\infty_\mathbb{Z}\) is the infinite dimensional lens space with fundamental group \(\mathbb{Z}/m\). Hence, the sequence (9) can lead to the conclusion.

In summary, since the proofs in this paper often employ the simplicial homology \(H^\Delta_e(K \setminus G)\), it is sensible to consider the pairing modulo \(nm\).

In addition, we similarly observe another knot \(S_{n,m}\) in the solid torus, in details. Fix four integers \((n, m, a, b) \in \mathbb{Z}^4\) with \(an + bm = 1\). Consider the following subspace

\[
\{ (z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1, \quad |z^n + w^m| < \frac{1}{nm^m}, \quad |z| < \frac{1}{3} \}.
\]
We can easily check that the space is homeomorphic to the \((n, m)\)-torus knot \(V \setminus T_{n,m}\). Since the space is regarded as a restriction of a Milnor fibration over \(S^1\), it is an Eilenberg-MacLane space. Furthermore, as in the usual computation of \(\pi_1(S^3 \setminus T_{n,m})\), set up the two subsets

\[
U_1 := \{ (z, w) \in V \setminus T_{n,m} \mid |z|^2 \leq 1/2 \}, \quad U_2 := \{ (z, w) \in V \setminus T_{n,m} \mid |z|^2 \geq 1/2 \}.
\]

Since \(U_1 \cong S^1\) and \(U_2 \cong S^1 \times S^1\), a van-Kampen argument (see [BZ, §15]) can conclude

\[
\pi_1(V \setminus T_{n,m}) \cong \langle x, y, z \mid x^n = y^m, \ yz = zy \rangle.
\]

Here, the meridian \(m\) and the longitude \(l\) are represented by \(x^ay^b\), and \(m^{-nm}x^n\), respectively, and the generator \(z\) arises from the loop of \(\{x_0\} \times S^1\). Geometrically, this \(z\) is the singular fiber in the Seifert 3-manifold. Furthermore, replacing \((y, z)\) by \((y', z') = (y^m, z)\), we have

\[
\pi_1(V \setminus T_{n,m}) \cong \langle x, z \mid x^mz' = z'x^m \rangle.
\]

In summary, we can easily obtain the following:

**Lemma 7.3.**

(i) The center of \(\pi_1(V \setminus T_{n,m})\) is generated by \(x^p\) and is isomorphic to \(\mathbb{Z}\).

(ii) The quotient group of \(\pi_1(V \setminus T_{n,m})\) subject to the center is isomorphic to \(\mathbb{Z}/m \ast \mathbb{Z}\).

(iii) The subgroup generated by the meridian \([m]\) is malnormal, and is isomorphic to \(\mathbb{Z}\).

Since \(S_{n,m}\) is embedded in \(S^3\), we have the link quandle \(Q_V\). As a result, we similarly have

\[
H^\Delta_k(Q_V; \mathbb{Z}) \cong H^\Delta_k(K\setminus G; \mathbb{Z}) \cong H^\text{gr}_k(\mathbb{Z}/m \ast \mathbb{Z}, \mathbb{Z}, \mathbb{Z}; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^2, & \text{if } k = 1,2, \\ \mathbb{Z}/m, & \text{if } * \text{ is odd, and } * \geq 3. \\ 0, & \text{otherwise.} \end{cases}
\]

Finally, we give a comment on the homomorphism on \(\pi_1\) induced from the inclusion \(j : V \setminus T_{n,m} \rightarrow S^3 \setminus T_{n,m}\). Note that this homomorphism can be regarded as the quotient subject to \(z\) in the group presentation.

### 7.2 Proof of Theorem 7.1

We will give the proof of Theorem 7.1 based on the discussion in §7.1.

**Proof.** In this proof, we alway deal with homology in torsion coefficients \(\mathbb{Z}/N\). Since the latter part from malnormality can be proven in a similar way to Theorem 2.1, we will only show the former.

By the above computations of homologies, the map \(\alpha\) yields quasi isomorphism on \(H_\ast\), which gives a quasi inverse \(\beta : C^\text{gr}_\ast(Q_V) \rightarrow C^\text{gr}_\ast(\pi_V, \partial \pi_V)\). Furthermore, we suppose a quandle homomorphism \(f : Q_{T_{m,n}} \rightarrow X\).

Similarly to (16), we have the following commutative diagram by functoriality.

\[
\begin{array}{cccccc}
H^\text{gr}_3(Q_V; \mathbb{Z}/N) & \xrightarrow{\varphi^*} & H^\Delta_3(Q_V; \mathbb{Z}/N) & \xrightarrow{\alpha} & H^\text{gr}_3(\pi_V, \partial \pi_V; \mathbb{Z}/N) \\
\downarrow j_* & & \downarrow j_* & & \downarrow j_* \\
H^\text{gr}_3(Q_{T_{m,n}}; \mathbb{Z}/N) & \xrightarrow{\varphi^*} & H^\Delta_3(Q_{T_{m,n}}; \mathbb{Z}/N) & \xrightarrow{\alpha} & H^\text{gr}_3(\pi_{T_{m,n}}, \partial \pi_{T_{m,n}}; \mathbb{Z}/N) \\
\downarrow f_* & & \downarrow f_* & & \downarrow f_* \\
H^\text{gr}_3(X; \mathbb{Z}/N) & \xrightarrow{\varphi^*} & H^\Delta_3(X; \mathbb{Z}/N) & \xrightarrow{\alpha} & H^\text{gr}_3(G, K; \mathbb{Z}/N).
\end{array}
\]
Here, by the above discussion, the middle $j_*$ is reduced to the injection $\mathbb{Z}/m \to \mathbb{Z}/nm$. Hence, if we show that all the left $\varphi_i$’s are surjective, then the rest of the proof runs as in the proof of Theorems 2.1 and 2.3.

To prove the surjectivity, it is enough to set up appreciate $X$ and $f$. Although there are many choices of $X$ and $f$ for the proof, this paper relies on some results in [No1] as follows: Take arbitrary prime $p$ which divides $m$. Further, choose minimal $k \in \mathbb{N}$ such that $n$ is not relatively prime to $1+p^k$. We set up the semiprodudct $G = (\mathbb{Z}/p)^k \rtimes \mathbb{Z}/(1+p^k)$ and the subgroup $K = \mathbb{Z}/(1+p^k)$. Then, we can easily construct a group homomorphism $f : \mathbb{Z}/m*\mathbb{Z}/n \to G$ that sends the subgroup $\mathbb{Z}$ to $K$, and induces the injection $f_* : H_3(\mathbb{Z}/m*\mathbb{Z}/n; \mathbb{Z}/p) \to H_3(G; \mathbb{Z}/p)$ on homology. Thus, it is enough for the surjectivity of the left $\varphi_i$’s to show that the map $\varphi_* : H_3^R(X; \mathbb{Z}/N) \to H_3^R(X; \mathbb{Z}/N)_G$ is injective. However, noticing that $X = (\mathbb{Z}/p)^k = \mathbb{F}_{p^k}$ forms an Alexander quandle, the injectivity of $X$ is already shown in the previous paper [No1] Lemmas 4.5–4.6], which studies the chain map $\varphi_*$ for Alexander quandles.

As a parallel discussion, when we choose any prime $p$ which divides $n$, the similar injectivity can be shown. To summarize, since such a $p$ is arbitrary, we have shown the surjectivity of the left $\varphi_i$’s. Hence, we complete the proof.

Finally, we give a corollary:

**Corollary 7.4.** Assume a solid torus $V \subset S^3$ such that $V \setminus L$ is the $(m, n)$-torus knot. Let $K$ be a malnormal subgroup of $G$. Then, for any relative 3-cocycle $\theta$, the $\ell$-torsion part of the pairing $\langle \theta, f_*(E_L, \partial E_L) \rangle$ is zero. Here $\ell$ is either the prime number coprime to $nm$ or $\ell = 0$.

Actually, as seen in the previous proof, the fundamental 3-class of the torus knot must factor through $H_3^\Delta(Q_V; \mathbb{Z}) \cong \mathbb{Z}/mn$. For example, if $\theta$ is the Chern-Simon 3-class as in (13), the free (volume) part of the pairing turns out to be zero.

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## A Appendix; the third quandle homology of some link quandles.

The purpose is to determine the third quandle homology of some link quandles (Theorems A.1 and A.2). In what follows, we assume the terminology in [14] and we deal with only integral homology (so we often omit writing $\mathbb{Z}$).

For the purpose, we use an approach analogous to [No3, §8]. Thus, we shall review rack spaces from a quandle $X$. Consider the orbit decomposition $X = \bigsqcup_{i \in O(X)} X_i$ from the action of $\text{As}(X)$ on $X$. For each orbit $i \in O_X$, we fix $x_i \in X_i$. Let $Y$ be either $X_i$ or the single point with their discrete topology. Then, we start with a disjoint union $\bigsqcup_{n \geq 0}(Y \times ([0, 1] \times X)^n)$, and consider the following two relations:

$$(y, t_1, x_1, \ldots, x_{j-1}, 0, x_j, t_{j+1}, \ldots, t_n, x_n) \sim (y, t_1, x_1, \ldots, x_{j-1}, t_{j-1}, x_{j-1}, t_{j+1}, x_j, t_{j+1}, \ldots, t_n, x_n),$$
(y, t₁, x₁, . . . , t_{j−1}, x_{j−1}, 1, x_j, t_{j+1}, x_{j+1}, . . . , t_n, x_n) \sim (y, e_{x_j}, t₁, x₁ \preceq x_j, . . . , t_{j−1}, x_{j−1} \preceq x_j, t_{j+1}, x_{j+1}, . . . , t_n, x_n).

Then, the rack space \( B(X, Y) \) is defined to be the quotient space, which is path connected. When \( Y \) is a single point, we denote it by \( BX \) in short.

We will list some properties on the space from [FRS]. By observing the cellular complexes, the following isomorphisms are known:

\[
H_n^R(X) \cong H_n(BX; \mathbb{Z}), \quad H_{n+1}^R(X) \cong \bigoplus_{i \in \mathcal{O}_X} H_n^R(B(X, X_i); \mathbb{Z}).
\] (23)

Furthermore, concerning fundamental groups, we mention the following isomorphisms [FRS]:

\[
\pi_1(BX) \cong As(X), \quad \pi_1(B(X, X_i)) \cong \text{Stab}(x_i) \subset As(X).
\] (24)

It is shown [FRS Proposition 5.2] that the action of \( \pi_1(BX) \) on \( \pi_*(BX) \) is trivial, and the projection \( p : B(X, X_i) \to BX \) is a covering. Therefore, we have functorially the Postnikov tower written in

\[
\begin{array}{cccccc}
H_3(\text{Stab}(x_i)) & \longrightarrow & \pi_2(B(X, X_i)) & \longrightarrow & H_2(B(X, X_i)) & \longrightarrow & H^R_2(\text{Stab}(x_i)) & \longrightarrow 0 \\
\downarrow p_* & & \downarrow \cong & & \downarrow p_* & & \downarrow p_* & \\
H_3(\pi_1(BX)) & \longrightarrow & \pi_2(BX) & \longrightarrow & H_2(BX) & \longrightarrow & H^R_2(\pi_1(BX)) & \longrightarrow 0
\end{array} \quad (\text{exact})
\] (25)

Furthermore, we mention [LN Theorem 7], which claims the isomorphisms

\[
H^R_3(X) \cong H^Q_2(X) \oplus \mathbb{Z}^{\mathcal{O}(X)}, \quad H^R_3(X) \cong H^Q_3(X) \oplus H^Q_2(X) \oplus \mathbb{Z}^{\mathcal{O}(X) \times \mathcal{O}(X)}. \quad (26)
\]

Using the above results, we will show the following theorem:

**Theorem A.1.** Let \( Q_L \) be the link quandle of a non-trivial knot \( L \). Then,

\[
H^R_3(Q_L) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, \quad H^Q_3(Q_L) \cong \mathbb{Z}.
\]

Furthermore, the quandle homology \( H^Q_3(Q_L) \cong \mathbb{Z} \) is generated by the fundamental 3-class \([S_{id, Q_L}].\)

**Proof.** Note \( O(X) = 1 \). Then, \( \pi_1(B(X, X)) \cong \text{Stab}(x_0) \) is the peripheral group \( \cong \mathbb{Z}^2 \). Hence, \( H^R_1(\text{Stab}(x_i)) \cong H_2(S^1 \times S^1; \mathbb{Z}) \). Furthermore, \( \pi_2(BX) \cong \mathbb{Z}^2 \) is known [FRS]. Hence, the above sequence reduces \( H^R_3(Q_L) \cong \mathbb{Z}^3 \) as desired. Furthermore, since \( H^R_2(Q_L) \cong H_1(B(Q_L, Q_L)) \cong H^R_1(\text{Stab}(x_i)) \cong \mathbb{Z}^2 \) (which is already known [RS, FRS]), we have \( H^Q_2(Q_L) \cong \mathbb{Z} \) by (21). Hence, we obviously obtain \( H^Q_3(Q_L) \cong \mathbb{Z} \) from (26).

We complete the remaining proof on the generator. Consider a chain map \( q : C^n_3(X) \to C^n_{q−1}(X) \) induced by \((x_1, \ldots, x_n) \mapsto (x_2, \ldots, x_n)\). We can easily check that the map on the homology level coincides with the above \( p_* \). Here note the fact [FRS, RS] that \( H^Q_2(Q_L) \cong \mathbb{Z} \) is generated by the 2-class \( q_*[S_{id, Q_L}]. \) Hence, \( H^Q_3(Q_L) \) must be generated by the 3-class \([S_{id, Q_L}]. \) \( \Box \)

Next, we will deal with the link case. As seen in Remark 6.1 or Seifert pieces, it is complicated to deal with \( H^Q_3(Q_L) \) of every links. Thus, we assume a property:

The centrerizer subgroup of each perihedra group \( \mathcal{P}_i \subset \pi_L \) is equal to \( \mathcal{P}_i. \) \( \dagger. \)

For example, if \( L \) is hyperbolic or the subgroups \( \mathcal{P}_i \) are malnormal, the condition hold.
**Theorem A.2.** Let $L$ be a non-splitting link with the property $(†)$. Then,

$$H_3^R(Q_L) \cong \mathbb{Z}^\#L(\#L+3), \quad H_3^Q(Q_L) \cong \mathbb{Z}^\#L.$$  

**Proof.** First, we show $H_2^R(Q_L) \cong \mathbb{Z}^2$. By $(†)$, we can easily verify $\text{Stab}(x_i) \cong \mathbb{Z}^2$ and $|O(Q_L)| = \#L$. Hence, it follows from (23) that $H_2^R(Q_L) \cong \oplus_{x \in O(X)} H_1^R(\text{Stab}(x_i)) \cong \mathbb{Z}^2$ as desired.

Next, we show $\pi_2(BX) \cong \mathbb{Z}^\#L + 1$. Since $L$ is non-splitting, $S^3 \setminus L$ is an Eilenberg-MacLane space of $\pi_L \cong \text{As}(Q_L)$; see [AFW]. Accordingly, $H^g_*(\text{As}(Q_L)) \cong H_*(S^3 \setminus L)$. Thus, $H^g_3(\text{As}(Q_L)) \cong 0$ and $H^g_2(\text{As}(Q_L)) \cong \mathbb{Z}^\#L - 1$. By (23) and $H_2(BQ_L) \cong \mathbb{Z}^2$, we obviously have $\pi_2(BQ_L) \cong \mathbb{Z}^\#L + 1$.

Next, we will complete the proof. Since $H_2(BQ_L) \cong \mathbb{Z}^2$ and $\text{Stab}(x_i) \cong \mathbb{Z}^2$, we have $H_2(B(X, X_i)) \cong \mathbb{Z}^3$ by (23). Hence, we have $H_3^R(Q_L) \cong \mathbb{Z}^\#L(\#L+3)$ by (23) with $n = 2$. Furthermore, regarding $H_3^Q(Q_L)$, the proof is readily due to (26).

Finally, we prove Lemma A.3 which is used in (6). In what follows, we use the notation in [6]. More precisely, we recall the associated quandles $Q_L$ and $Q_{L'}$ of links $L$ and $L'$, respectively, and employ the homogenous quandle of the form $Q_W = (K_1 \setminus \pi_L) \sqcup (K_2 \setminus \pi_L)$, together with the injections $i_* : Q_L \to Q_W$ and $j_* : Q_{L'} \to Q_W$; see [6] for the details.

**Lemma A.3.** Assume that $L'$ is a hyperbolic link. Then, $H_3^Q(Q_W) \cong \mathbb{Z}^2$. Furthermore, the induced maps on the third homology level

$$i_*^Q : H_3^Q(Q_L) \to H_3^Q(Q_W), \quad \text{and} \quad j_*^Q : H_3^Q(Q_{L'}) \to H_3^Q(Q_W)$$

are given by $(1, 0)$ and $\left( \begin{array}{c}1 \\ 1 \\ 1 \end{array} \right)$, respectively.

**Proof.** (Sketch) The proof is done by similar discussions. By the form of $Q_W$, we can easily show $\text{As}(Q_W) \cong \pi_1(S^3 \setminus L)$. Then, the isomorphism (23) readily implies $H_2^R(Q_W) \cong \mathbb{Z}^4$. Since $S^3 \setminus L'$ is hyperbolic, it follows from $(†)$ that $\text{Stab}(x_i) \cong \mathbb{Z}^2$ and $|O(X)| = 2$. Then, by the same discussion of Theorem A.2, we can show $H_3^Q(Q_W) \cong \mathbb{Z}^2$.

Furthermore, by the above proofs, we already know the basis of quandle homologies. Thus, by diagram chasing, we can check the matrix presentations, although we omit describing the details. \(\square\)

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