ON REPULSIVE GRAVITATIONAL ACTIONS

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Abstract. In particular instances and in particular regions the Ein-steinian gravity exerts a repulsive action – and without any cosmolo-
gical term. This fact could have an interest for the explanation of the
dark energy, and for the gravitational collapses.

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1. – In a recent paper (see sects. 5 and 5bis of [1]) we have emphasized,
reconsidering some lucid and straightforward computations by Hilbert [2],
that in particular instances and in particular regions the Einsteinian gra-
vity – without any cosmological term – becomes of a repulsive kind. We
give now an impressive illustration of this fact by exhibiting six diagrams
(five concerning test-particles and one concerning light-rays, moving in a
Schwarzschild field) which are specially enlightening.

2. – Of course, Hilbert [2] uses the standard form of solution to Schwarz-
schild problem, which was discovered by him, by Droste, and by Weyl, quite
independently:

\[
g_{00} = 1 - \frac{2m}{r}; \quad g_{01}^{-1} = g_{\theta\theta}; \quad g_{\phi\phi} = r^2 \sin^2 \vartheta; \quad g_{\nu\nu} = 0 \quad \text{for} \quad j \neq k.
\]

Remark that for \( r > 2m \) the standard form is diffeomorphic to the original
Schwarzschild’s form and to Brillouin’s form, that hold for \( r > 0 \) (see, e.g.,
[1], and the Appendix).

Hilbert gives a full treatment of the geodesic lines of test-particles and
light-rays in a Schwarzschild field. We summarize here (see also [1]) the
Hilbertian results regarding the circular paths and the radial ones. For a
simple reason of physical reality – and in accordance with ideas of Einstein
and Hilbert – we assume that the gravitating centre is an extended spherical
body with the minimal radius \((9/8)2m\) [3].

The circular geodesics of the test-particles are restricted by the following
inequalities (Hilbert puts \( c = 1 \), and his \( \alpha \) coincides with \( 2m \)):

\[
(1) \quad r > \frac{3}{2}2m \quad \left[ > \frac{9}{8}(2m) \right],
\]

\[
(2) \quad \frac{v}{c} < \frac{1}{\sqrt{3}},
\]
where \( v = c(m/r)^{1/2} \) is the linear velocity. These relations tell us that here the Einsteinian gravity acts repulsively for small values of \( r \).

For the circular trajectories of the light-rays the coordinate-radius \( r \) is equal to \((3/2)2m\), with a velocity \( v = c/\sqrt{3} \).

In 1939 Einstein [4], employing a system of isotropic coordinates, rediscovered relation (1); he found that (his \( \mu \) is equal to our \( m \)):

\[
\frac{m}{2} \left( \frac{2 + \sqrt{3}}{2} \right)
\]

where \( r \) is now the radial isotropic coordinate. The concordance of (1) with (3) is a straightforward consequence of the passage from standard \( r \) to isotropic \( r \):

\[
\frac{r}{r'} = \left(1 + \frac{m}{2r} \right)^2 r
\]

In the standard coordinates the differential equation of the radial motions is:

\[
\frac{1}{c^2} \frac{d^2r}{dt^2} - \frac{3}{2} \frac{2m}{r(r-2m)} \left( \frac{dr}{c dt} \right)^2 + \frac{m(r-2m)}{r^3} = 0
\]

with the integral:

\[
\left( \frac{dr}{c dt} \right)^2 = \left( \frac{r-2m}{r} \right)^2 + A \left( \frac{r-2m}{r} \right)^3
\]

the constant \( A \) is zero for the light-rays, negative for the material particles, and such that \((2/3) \leq |A| \leq 1 \).

Eqs. (3) and (6) tell us that the acceleration is negative (attractive gravity) or positive (repulsive gravity) where, respectively:

\[
\left| \frac{dr}{c dt} \right| < \frac{1}{\sqrt{3}} \frac{r-2m}{r}
\]

\[
\left| \frac{dr}{c dt} \right| > \frac{1}{\sqrt{3}} \frac{r-2m}{r}
\]

For the radial motions of the light-rays we have from eq. (6) with \( A = 0 \):

\[
\left| \frac{dr}{c dt} \right| = \frac{r-2m}{r}
\]

and therefore the light is everywhere repulsed by the gravitating body. (see inequality (5)). If, in particular, a light-ray starts from \( r = \infty \) with a velocity \( c \), it arrives at \( r = (9/8)2m \) with velocity \((1/9)c \).

Putting \( x := r/(2m) \) and \( y := (dr/c dt)^2 \), eq. (6) can be rewritten as follows:

\[
y(x) = \left( \frac{x-1}{x} \right) \left( 1 - |A| \frac{x-1}{x} \right)
\]

\((1 < x < \infty) \).
Figs. 1÷5 represent five diagrams of eq. (11) for the following values of $|A|$: 1; 0.9; 0.8; 0.7; 2/3. Fig. 6 gives the diagram of eq. (10) for $A = 0$. In these diagrams the variable $x$ goes from $(9/8)$ to infinite. The regions in which the gravitation acts repulsively are quite evident. Remark that in the instances of Fig. 5 (test-particles) and Fig. 6 (light-rays) the gravitation is everywhere repulsive.
Figure 1. Diagram of $y(x) = [(x - 1)/x] \cdot [1 - (x - 1)/x]$ for some values of $x$: $(9/8) \leq x < +\infty$; max(3.0; 4/27); $[y(9/8)]^{1/2} = 2\sqrt{2}/27$.

Figure 2. Diagram of $y(x) = [(x - 1)/x]^2 [1 - 0.9 \cdot (x - 1)/x]$ for some values of $x$: $(9/8) \leq x < +\infty$; max(3.75; 0.182844); $[y(9/8)]^{1/2} = 0.105409$. 
Figure 3. Diagram of $y(x) = [(x - 1)/x]^2[1 - 0.8 \cdot (x - 1)/x]$ for some values of $x$: $(9/8) \leq x < +\infty$; max$(6.0; 0.231481)$; $[y(9/8)]^{1/2} = 0.106058$.

Figure 4. Diagram of $y(x) = [(x - 1)/x]^2[1 - 0.7 \cdot (x - 1)/x]$ for some values of $x$: $(9/8) \leq x < +\infty$; max$(21.0; 0.302343)$; $[y(9/8)]^{1/2} = 0.106703$. 
Figure 5. Diagram of $y(x) = [(x - 1)/x]^2[1 - (2/3) * (x - 1)/x]$ for some values of $x$; $(9/8) \leq x < +\infty$; $\max(+\infty, 1/3)$; $[y(9/8)]^{1/2} = 0.106917$.

Figure 6. Diagram of $y(x) = [(x - 1)/x]^2$ for some values of $x$; $(9/8) \leq x < +\infty$; $\max(+\infty, 1.0)$; $[y(9/8)]^{1/2} = 1/9$. 
3. – It is clear that a cosmological model incorporating the existence of the above repulsive actions could explain the so-called dark energy, i.e. the energy that is responsible for the accelerated expansion of the universe [5].

Further, the repulsive gravity could play a role in the dynamics of gravitational collapses.

4. – Till now people have tried to explain the dark energy by means of Friedmann model with a cosmological term, or with a “quintessence” [5].

However, Friedmann model is rather poor, and is isomorphic to a corresponding Newtonian model [6]. Accordingly, it cannot give a physically sensible explanation of the dark energy.

**APPENDIX**

The Hilbertian conception about the singularities of the metric tensor of GR is illustrated by these considerations concerning the Schwarzschild problem [2]:

“Für $\alpha \equiv 2m \neq 0$ erweisen sich $r = 0$ und bei positivem $\alpha$ auch $r = \alpha$ als solche Stellen, an denen die Maßbestimmung nicht regulär ist. Dabei nenne ich eine Maßbestimmung oder ein Gravitationsfeld $g_{\mu\nu}$ an einer Stelle regulär, wenn es möglich ist, durch umkehrbar eindeutige Transformation ein solches Koordinatensystem einzuführen, daß für dieses die entsprechenden Funktionen $g'_{\mu\nu}$ an jener Stelle regulär, i.d.h. in ihr und in ihrer Umgebung stetig und beliebig oft differenzierbar sind und eine von Null verschiedene Determinante $g'_{\mu\nu}$ haben.

Obwohl nach meiner Auffassung nur reguläre Lösungen der physikalischen Grundgleichungen die Wirklichkeit unmittelbar darstellen, so sind doch gerade die Lösungen mit nicht regulären Stellen ein wichtiges mathematisches Mittel zur Annäherung [approximation] an charakteristische reguläre Lösungen – und in diesem Sinne ist nach dem Vorgange von Einstein [Berl. Ber., (1915) 831] und Schwarzschild [Berl. Ber., (1916) 189; an English version in arXiv:physics/9905030 (May 12th, 1999), and in Gen. Rel. Grav., 35 (2003) 951] die für $r = 0$ und $r = \alpha$ nicht reguläre Maßbestimmung (35) $:$ $G(dr, d\vartheta, d\varphi, dt) = \{r/(r-\alpha)\}dr^2 + r^2d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2 - \{(r-\alpha)/r\}dt^2$ als Ausdruck der Gravitation einer in der Umgebung des Nullpunktes zentratisch-symmetrisch verteilten [and therefore extended] Masse anzusehen1. The footnote1 says: “Die Stellen $r = \alpha$ nach dem Nullpunkt zu transformieren, wie es Schwarzschild [loc. cit.] tut, ist meiner Meinung nach nicht zu empfehlen [not to be recommended]; die Schwarzschildsche Transformation ist überdies nicht die einfachste, die diesen Zweck erreicht.”

**A comment. - i)** The above definition of regularity of the metric tensor is chiefly significant so far as physical reality is concerned. In particular, it tells us that the well-known coordinate system by Kruskal and Szekeres is destitute of any physical value: indeed, the derivatives $\partial u/\partial r$ and $\partial v/\partial r$ of Kruskal’s coordinates $u$ and $v$ are singular at $r = 2m$: this means that the singularity at $r = 2m$ of the standard interval has been “incorporated” in the new coordinates. Consequently, the Kruskal-Szekeres form of solution does not give a proper transformation of standard form. –
ii) Hilbert thinks that the singularities at $r = 0$ and at $r = 2m$ are only an approximate description of a small extended gravitating mass having its centre at $r = 0$; the first statement of footnote\(^1\) reveals its true meaning if we take into account this consideration; it does not represent a disowning of Schwarzschild’s procedure. On the other hand, the above quoted Schwarzschild’s memoir concerns “das Gravitationsfeld eines Massenpunktes”, whereas Hilbert’s treatment intends to find the Einsteinian field outside a generic spherically symmetrical distribution of matter.

The second statement of footnote\(^1\), according to which there exist choices of the coordinate system characterized by a unique singularity of metric tensor at $r = 0$, and that are simpler than Schwarzschild’s choice, finds a well-known example in Brillouin’s choice [Journ. Phys. Radium, 23 (1923) 43; an English version in arXiv:physics/0002009 (February 3rd, 2000)], which can be obtained from standard $r$ with the substitution $r \rightarrow r + 2m$. (The corresponding substitution for Schwarzschild’s choice is: $r \rightarrow [r^3 + (2m)^3]^{1/3}$. Brillouin’s and Schwarzschild’s forms of solution are maximally extended).

A last remark. After 1960 innumerable papers on the singularities of the metric tensor have been written, and various definitions of regularity of the solutions to Einstein equations have been proposed. In particular, the distinction between “hard” and “soft” singularities has been repeatedly emphasized, and many contributions of a worthy value from the geometric standpoint have been given. We think, however, that Hilbert’s simple definition of the regularity of a metric tensor is chiefly preferable for physical reasons.

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