Poincaré and Logarithmic Sobolev Inequalities for Nearly Radial Measures

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Abstract Poincaré inequality has been studied by Bobkov for radial measures, but few are known about the logarithmic Sobolev inequality in the radial case. We try to fill this gap here using different methods: Bobkov’s argument and super-Poincaré inequalities, direct approach via $L_1$-logarithmic Sobolev inequalities. We also give various examples where the obtained bounds are quite sharp. Recent bounds obtained by Lee–Vempala in the log-concave bounded case are refined for radial measures.

Keywords Radial measure, log-concave measure, Poincaré inequality, logarithmic Sobolev inequality, super-Poincaré inequality

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1 Introduction

Let $\mu(dx) = Z^{-1}e^{-V(x)}dx$ be a probability measure defined on $\mathbb{R}^n$ ($n \geq 2$). We do not require regularity for $V$ and allow it to take values in $\mathbb{R} \cup \{-\infty, +\infty\}$. We only require that $\int e^{-V}dx = 1$, or more generally that the previous integral is finite. We denote by $\mu(f)$ the integral of $f$ w.r.t. $\mu$.

In this note we will be interested in functional inequalities verified by the measure $\mu$. Recall that $\mu$ satisfies a Poincaré inequality if for all smooth $f$,

$$\text{Var}_\mu(f) := \mu(f^2) - \mu^2(f) \leq C_P(\mu)\mu(\nabla f^2), \quad (1.1)$$

and that it satisfies a log-Sobolev inequality if for all smooth $f$

$$\text{Ent}_\mu(f^2) := \mu(f^2 \ln(f^2)) - \mu(f^2) \ln(\mu(f^2)) \leq C_{LS}(\mu)\mu(\nabla f^2). \quad (1.2)$$

$C_P$ and $C_{LS}$ are understood as the best constants for the previous inequalities to hold. We refer to [1, 2, 29] among many others, for a comprehensive introduction to some of the useful
consequences of these inequalities and their most important properties, such as convergence to
equilibrium (in $L^2$ or in entropy) or concentration of measure.

If $\mu$ is not normalized as a probability measure, (1.1) reads as

$$\mu(f^2) - (1/\mu(\mathbb{R}^n))\mu^2(f) \leq C_P(\mu)\mu(\nabla f^2).$$  \hfill (1.3)

One key feature of these inequalities is their tensorization property namely

$$C_P(\mu \otimes \nu) = \max(C_P(\mu), C_P(\nu))$$

(the same for $C_{LS}$) giving a natural way to control these constants for product measures.

Another particular family of measures is the set of nearly radial defined as follows:

**Definition 1.1**  A measure $\mu$ is called nearly radial if it admits a decomposition

$$\mu(dx) = \mu_r(dp)\mu_a(d\theta)$$  \hfill (1.4)

with $x = r\theta$, $r \in \mathbb{R}^+$ and $\theta \in S^{n-1}$. This amounts to $V(x) = V_r(r) + V_a(\theta)$ and

$$\mu(dx) = n_\omega_n r^{n-1} e^{-V_r(r)} e^{-V_a(\theta)} dp \sigma_n(d\theta),$$

where $\sigma_n$ denotes the uniform distribution on $S^{n-1}$ and $\omega_n$ denotes the volume of the unit euclidean ball.

If in addition $V_a$ is bounded below and above, we shall call these measures almost radial.

When $\mu_a = \sigma_n$, $\mu$ is said to be radial (or spherically symmetric) as in the usual literature.

Note also that as our main purpose is functional inequalities such as Poincaré inequality or logarithmic Sobolev inequality which are translation invariant, one may always consider probability measures which are nearly radial measures up to a preliminary translation.

It is natural to ask how to control $C_P(\mu)$ and $C_{LS}(\mu)$ in terms of constants related to $\mu_r$ and $\mu_a$. Since $\mu_a$ is supported by the sphere we will use the natural riemanian gradient, in other words, for $\theta \in S^{n-1}$, we will decompose

$$\nabla f = (\nabla_\theta f, \nabla_{\theta \perp} f) := (\langle \nabla f, \theta \rangle, \Pi_{\theta \perp} \nabla f),$$

where $\Pi_{\theta \perp}$ denotes the orthogonal projection onto $\theta \perp$.

Though natural it seems that the previous question was not often addressed in the literature with the notable exception of radial log-concave measures for which Bobkov (see [8]) studied the Poincaré constant (his result was improved in [13]) and for which Huet (see [22]) studied isoperimetric property.

Our main results in the radial (or almost radial) case say that both the Poincaré and the log-Sobolev constant are controlled up to universal constants, by the corresponding constants for the radial part $\mu_r$ and $\mu(\rho^2)/(n-1)$ for Poincaré and some slightly more intricate combination for log-Sobolev, i.e.:

**Theorem 1.2**  Let $\mu$ be a radial measure.

1. (Bobkov’s result)

$$C_P(\mu) \leq \max \left( C_P(\mu_r), \frac{\mu(\rho^2)}{n-1} \right).$$

2. (Theorem 4.5) there exists a universal constant $c$ such that

$$C_{LS}(\mu) \leq c \left( C_{LS}(\mu_r) + \mu(\rho) \max \left( C_P(\mu_r), \frac{\mu(\rho^2)}{n-1} \right)^{1/2} \right).$$
Other results and some consequences are also described. More precisely, the “tensorization” part of Bobkov’s proof is elementary and will be explained in Section 2, where we will also show how it applies to other types of Poincaré inequalities (weak or super). As a byproduct we will obtain a first (bad) bound for the log-Sobolev constant.

In Section 4, we propose a direct approach of the logarithmic Sobolev inequality for (almost) radial measures. This approach uses in particular $L^q$ ($1 \leq q \leq 2$) log-Sobolev inequalities for the uniform measure on the sphere we establish in Section 3, based on the study made in [10] and results in [26]. In the framework of general log-concave measures, similar ideas already appear in [7].

All these results are applied in Section 5 to some examples. In particular, in the radial case, we improve upon the bound recently obtained by Lee and Vempala ([25]) for compactly supported (isotropic) log-concave measures. It reads

**Theorem 1.3** (Theorem 5.9) For any radial log-concave probability measure $\mu$ with support contained in the ball centered at 0 of radius $R$, then

$$C_{LS}(\mu) \leq C \frac{R^2}{n-1}$$

for some universal constant $C$.

It is not difficult to see that, according to Hahn–Banach theorem, for a compactly supported and centered log-concave measure such an $R$ always exists and is less than the diameter of the support. The previous result can thus be immediately compared with the one in [25], in the radial case.

## 2 Poincaré Inequality and Variants for Nearly Radial Measures

### 2.1 Bobkov’s Tensorization Method Revisited

Let us begin by explaining Bobkov’s tensorization method.

For $\mu(dx) = \mu_r(dp)\mu_a(d\theta)$ and a smooth bounded function $f$ one has firstly

$$\int f^2(\rho\theta)\mu_r(dp) \leq C_P(\mu_r) \int (\nabla f(\rho\theta), \theta)^2\mu_r(dp) + \left( \int f(\rho\theta)\mu_r(dp) \right)^2$$

$$= C_P(\mu_r) \int |\nabla_\theta f(\rho\theta)|^2\mu_r(dp) + \left( \int f(\rho\theta)\mu_r(dp) \right)^2. \quad (2.1)$$

*(Be careful: therefore $\nabla_\theta$ is NOT the gradient w.r.t. $\theta$, but the derivation in direction $\theta$, used throughout the paper.)*

Integrating with respect to $\mu_a$, we obtain

$$\mu(f^2) \leq C_P(\mu_r)\mu(|\nabla_\theta f|^2) + \int \left( \int f(\rho\theta)\mu_r(dp) \right)^2 \mu_a(d\theta). \quad (2.2)$$

But if we define $w(\theta) = \int f(\rho\theta)\mu_r(dp)$, it holds

$$\int \left( \int f(\rho\theta)\mu_r(dp) \right)^2 \mu_a(d\theta) \leq C_P(\mu_a) \int |\nabla w(\theta)|^2\mu_a(d\theta) + \left( \int w(\theta)\mu_a(d\theta) \right)^2$$

$$\leq C_P(\mu_a) \int \left| \int \rho\nabla_\theta f(\rho\theta)\mu_r(dp) \right|^2\mu_a(d\theta) + \mu^2(f)$$

$$\leq C_P(\mu_a)\mu_r(\rho^2)\mu(|\nabla_\theta f|^2) + \mu^2(f), \quad (2.3)$$
where we have used the Cauchy–Schwarz inequality in the last inequality. Here we assume that
the Poincaré constant of $\mu_a$ on the sphere $S^{n-1}$ is w.r.t. the usual gradient and not the gradient
on the sphere. Using that
\[ |\nabla_\theta f|^2 + |\nabla_{\theta \perp} f|^2 = |\nabla f|^2, \]
we have thus obtained

**Theorem 2.1** (Essentially due to Bobkov) For a nearly radial probability measure $\mu(dx) = \mu_r(dp)\mu_a(d\theta)$ then
\[ C_P(\mu) \leq \max(C_P(\mu_r), \mu_r(\rho^2)C_P(\mu_a)). \]

Recall that if $C_P(\mu_r) < +\infty$ then $\mu_r(e^{\lambda t}) < +\infty$ for $\lambda < 2/\sqrt{C_P(\mu_r)}$ (see e.g. [1]) so that $\mu_r(\rho^2)$ is finite too. See the sharp estimate (5.1) due to Bonnefont–Joulin–Ma [13] for log-concave radial measures.

### 2.2 Weak and Super Poincaré Inequality

A weak version of the Poincaré inequality has been introduced in [28] (also see the related papers [3, 9, 16]). It has, as the usual Poincaré inequality, interesting properties such as polynomial (or stretched exponential) convergence to equilibrium or concentration property. A weak Poincaré inequality is satisfied) and $\text{Osc}(f) := \sup_{x,y} |f(x) - f(y)|$ denotes the oscillation of $f$.

The integrability of $\rho^2$ is ensured as soon as $\beta_{WP}^\mu$ does not explode too quickly at the origin (see e.g. [3]).

Similarly one can reinforce the Poincaré inequality introducing super Poincaré inequalities ([2, 4, 5, 18, 29, 30]): for any $t \geq 1$ and all smooth $f$
\[ \mu(f^2) \leq \beta_{WP}^\mu(t)\mu(|\nabla f|^2 + t\text{Osc}^2(f)) \]
(2.4)

where $\beta_{WP}^\mu$ is a non-increasing function that can explode at $t = 0$ (otherwise the classical Poincaré inequality is satisfied) and $\text{Osc}(f)$ denotes the oscillation of $f$.

The previous proof yields (the detail is left to the reader)

**Theorem 2.2** If $\mu(dx) = \mu_r(dp)\mu_a(d\theta)$ then
\[ \beta_{WP}^\mu(t) \leq \max(\beta_{WP}^\mu(t/2), \mu_r(\rho^2)\beta_{WP}^\mu(t/2)). \]

The integrability of $\rho^2$ is ensured as soon as $\beta_{WP}^\mu$ does not explode too quickly at the origin (see e.g. [3]).
for all $t > 0$ and nice function $f$. Applying this inequality to constant functions shows that $eta(t) \geq 1$ for all $t$. But we may replace $\beta(t)$ by 1 as soon as $t \geq C_P(\mu)$.

It is known that if $\beta^\mu_{SP}(t)$ behaves like $Ce^{c/t}$ as $t \to 0$, (2.6) together with a Poincaré inequality is equivalent to the logarithmic Sobolev inequality (we will provide precise estimates below).

Following the same route we immediately get that for any positive $t$ and $s$,

$$
\mu(f^2) \leq (s + \beta^\mu_{SP}(s)t\mu_r(\rho^2))\mu(|\nabla f|^2) + \beta^\mu_{SP}(s)\beta^\mu_{SP}(t)\mu^2(|f|). 
$$

(2.7)

Of course this is a new super Poincaré inequality or more precisely a new family of super Poincaré inequalities. The most natural choice (not necessarily the best one) is

$$
t = \frac{s}{\mu(\rho^2)\beta^\mu_{SP}(s)}
$$

yielding the following

**Theorem 2.3** If $\mu(dx) = \mu_r(dp)\mu_a(d\theta)$, then

$$
\beta^\mu_{SP}(t) \leq \beta^\mu_{SP}(t/2)\beta^\mu_{SP} \left(\frac{t}{2\mu(\rho^2)\beta^\mu_{SP}(t/2)}\right).
$$

If, for example, we suppose that a logarithmic Sobolev inequality holds for $\mu_r$, it is not hard to remark that the angular part has to satisfy a much stronger inequality than a logarithmic Sobolev inequality for such an inequality to hold for $\mu$.

### 2.3 The Radial Case

If $\mu$ is radial i.e. $\mu_a = \sigma_n$ the uniform measure on $S^{n-1}$, we deduce from the famous Lichnerowitz’s estimate $C_P(\sigma_n) = \frac{1}{n-1}$ and Theorem 2.1, that

$$
C_P(\mu) \leq \max \left(C_P(\mu_r), \frac{\mu_r(\rho^2)}{n-1}\right).
$$

(2.8)

Actually $\sigma_n$ satisfies the much stronger Sobolev inequality (for $n \geq 4$ see [2, p.308] written for spherical gradient but recall the introduction)

$$
\|g\|^2_{\frac{2n-2}{n-3}} \leq \sigma_n(g^2) + \frac{4}{(n-1)(n-3)}\sigma_n(|\nabla g|^2).
$$

(2.9)

We deduce from this and the Poincaré inequality, the following inequality

$$
\|g\|^2_{\frac{2n-2}{n-3}} \leq \sigma_n^2(g) + \frac{1}{n-3}\sigma_n(|\nabla g|^2).
$$

(2.10)

One can thus derive the corresponding $\beta^\sigma_{n}$. If one wants to see the dimension dependence one has to be a little bit careful.

First we apply Hölder’s inequality for $p > 2$,

$$
\sigma_n(g^2) \leq \sigma_n^{\frac{1}{n-2}}(|g|^p)\sigma_n^{\frac{n-2}{n-3}}(|g|),
$$

then $p = \frac{2n-2}{n-3}$, yielding according to what precedes

\[
\sigma_n(g^2) \leq \left(\sigma_n^2(g) + \frac{1}{n-3}\sigma_n(|\nabla g|^2)\right)^{\frac{n-1}{n-2}} \sigma_n^{\frac{1}{n-2}}(|g|) \\
\leq \sigma_n^2(|g|) + \left(\frac{1}{n-3}\sigma_n(|\nabla g|^2)\right)^{\frac{n-1}{n-2}} \sigma_n^{\frac{1}{n-2}}(|g|).
\]
Recall Young’s inequality: for all $p > 1$ and $a, b, t > 0$,
\[ ab \leq t \frac{a^p}{p} + t^{-\left(\frac{q-1}{q}\right)} \frac{b^q}{q} \]
with $1/p + 1/q = 1$. We deduce from what precedes, this time with $p = (n + 1)/(n - 1)$,
\[ \sigma_n(g^2) \leq t \sigma_n(|\nabla g|^2) + \left(1 + \frac{1}{4(n - 3)(n - 1)/2} t^{-\left(\frac{n-1}{2}\right)}\right) \sigma_n^2(|g|) \]
and finally that
\[ \beta_{\sigma_n}^{SP}(t) \leq 1 + \frac{1}{4(n - 3)(n - 1)/2} t^{-\left(\frac{n-1}{2}\right)}. \tag{2.11} \]

We can thus plug (2.11) in Theorem 2.3 and get

**Corollary 2.4** If $\mu(dx) = \mu_r(d\rho) \sigma_n(d\theta)$ (radial) and $n \geq 4$, then
\[ \beta_{\mu}^{SP}(t) \leq \beta_{\mu_r}^{SP}(t/2) \left(1 + \frac{\mu(\rho^2)^{\beta_{\mu}^{SP}(t/2)}}{(n - 3)} \right)^{(n - 1)/2}. \]

The cases $n = 2$ and $n = 3$ can be studied separately.

### 2.4 Application to the Log-Sobolev Inequality

The equivalence between a log-Sobolev inequality and a super-Poincaré inequality is well known (see e.g. [29, Theorems 3.3.1 and 3.3.3], despite some points we do not understand in the proofs). But here we need precise estimates on the constants. One way to get these estimates is to use the capacity-measure description of these inequalities following the ideas in [5] (also see [31] for some additional comments).

**Lemma 2.5** (see [15, Proposition 3.4]) If $\mu$ satisfies a logarithmic-Sobolev inequality with constant $C_{LS}(\mu)$ then $\beta_{\mu}^{SP}(t) \leq 2 e^{2C_{LS}(\mu)/t}$.

A simplified version of the results of [4, 5] is contained in [2], see in particular Proposition 8.3.2 and Proposition 8.4.1, from which one can deduce the (slightly worse bound) $\beta_{\mu}^{SP}(t) \leq e^{96C_{LS}(\mu)/t}$.

The converse part is a consequence of [2].

**Lemma 2.6** (see [2, Proposition 8.3.2 and Proposition 8.4.1]) Conversely, if $\beta_{\mu}^{SP}(t) \leq C_{1} e^{C_{2}/t}$ and $\mu$ satisfies a Poincaré inequality with constant $C_{P}(\mu)$, then
\[ C_{LS}(\mu) \leq 64 \left(C_{2} + \ln \left(1 + \frac{(1 + 2e^{2}) C_{1}}{4}\right) C_{P}(\mu) \right). \]

**Proof** Recall that it is enough to look at $t \leq C_{P}(\mu)$ and then take $\beta_{\mu}^{SP}(t) = 1$ for $t \geq C_{P}(\mu)$. So we may replace $C_{1} e^{C_{2}/t}$ by the larger $\frac{1}{1+2e^{2}} e^{C_{2}/t}$ with
\[ C_{2}' = C_{2} + C_{P}(\mu) \ln \left(1 + \frac{(1 + 2e^{2}) C_{1}}{4}\right). \]

In [2] terminology we thus have $\delta(s) = C_{2}'/\ln(s(1 + 2e^{2})/4)$, that satisfies the assumptions of [2, Proposition 8.4.1] with $q = 4$. We thus get for $\mu(A) \leq \frac{1}{4}$,
\[ \text{Cap}_{\mu}(A) \geq \frac{\mu(A) \ln((1 + 2e^{2})/2 \mu(A))}{8C_{2}'}. \]
But for $\mu(A) \leq 1/2$, 

$$\ln((1 + 2e^2)/2\mu(A)) \geq \ln \left(1 + \frac{e^2}{\mu(A)}\right).$$

We may thus apply [2, Proposition 8.3.2] yielding $C_{LS}(\mu) \leq 64C^2$. \(\square\)

Now assume that $\mu$ is radial. If $\mu_r$ satisfies a log-Sobolev inequality, $\beta^{SP}_{\mu_r}(t) \leq 2e^{2C_{LS}(\mu_r)/t}$ so that using Corollary 2.4,

$$\beta^{SP}_{\mu}(t) \leq 2e^{4C_{LS}(\mu_r)/t} + \frac{1}{2 \left(\frac{2\mu(\rho^2)}{n-3}\right)^{(n-1)/2}} e^{2(\mu_r)^2 C_{LS}(\mu_r)/t}. \quad (2.12)$$

Once again since we only have to look at $t < C_F(\mu)$, and using (2.8), we obtain the following worse bound

$$\beta^{SP}_{\mu}(t) \leq 2 \left(e^{-(n-1)C_{LS}(\mu_r)/\max(C_F(\mu_r), \frac{\mu(\rho^2)}{n-1})} + \frac{1}{4} \left(\frac{2\mu(\rho^2)}{n-3}\right)^{(n-1)/2}\right) e^{2(\mu_r)^2 C_{LS}(\mu_r)/t}. \quad (2.13)$$

But we can use the following homogeneity property of the log-Sobolev and the Poincaré inequalities: defining $\mu_\lambda$ for $\lambda > 0$, by

$$\int f(z)\mu_\lambda(dz) = \int f(\lambda z)\mu(dz),$$

it holds $C_{LS}(\mu_\lambda) = \lambda^2 C_{LS}(\mu)$ (the same for $C_F(\mu)$).

Looking at the pre-factor in (2.13), we see that making $\lambda$ go to 0, the second term goes to 0 while the first one is unchanged. Using the homogeneity property for both $\mu$ and $\mu_r$, and using Lemma 2.6 again, we have thus obtained

$$C_{LS}(\mu) \leq 64 \left(n + 1 \right) C_{LS}(\mu_r) + C \max \left(C_F(\mu_r), \frac{\mu(\rho^2)}{n-1}\right), \quad (2.14)$$

where

$$C = \ln \left(1 + \frac{1 + 2e^2}{2} e^{-(n-1)C_{LS}(\mu_r)/\max(C_F(\mu_r), \frac{\mu(\rho^2)}{n-1})}\right).$$

Of course in many (almost all) situations $C = 0$. Using $C_F(\mu_r) \leq C_{LS}(\mu_r)/2$ it is not very difficult to show that $C \neq 0$ if and only if

$$\mu(\rho^2) \geq \frac{2(n-1)^2 C_{LS}(\mu_r)}{1 + e^2},$$

in which case $C \leq \ln(1 + e^2)$.

These results extend to the “almost” radial situation, using the standard perturbation result for a (super)-Poincaré inequality or a log-Sobolev inequality, as explained in the next corollary.

**Corollary 2.7** Assume that $\mu(dx) = \mu_r(dp)\mu_\sigma(d\theta)$ is almost radial (recall Definition 1.1) with

$$m \leq \frac{\|dp_\mu\|}{\|d\sigma_n\|} \leq M,$$

where $\sigma_n$ is the uniform probability measure on $\mathbb{S}^{n-1}$. Then

$$C_F(\mu) \leq C_F(\mu_r) + \frac{M}{m} \frac{\mu_r(\rho^2)}{n-1}.$$ 

If $n \geq 4$ and $\mu_r$ satisfies a log-Sobolev inequality then so does $\mu$ and

$$C_{LS}(\mu) \leq 64 \frac{M}{m}(n + 1) C_{LS}(\mu_r),$$
except if $\mu(\rho^2) \geq \frac{2(n-1)^2 C_{LS}(\mu_r)}{1+e^2}$ in which case

$$C_{LS}(\mu) \leq 64 \frac{M}{m} \left( (n+1) C_{LS}(\mu_r) + \ln(1 + e^2) \frac{\mu(\rho^2)}{n-1} \right).$$

The previous two bounds amount to the existence of a universal constant $C$ such that

$$C_{LS}(\mu) \leq C \frac{M}{m} \max \left( n C_{LS}(\mu_r), \frac{\mu(\rho^2)}{n-1} \right).$$

We may of course adapt the above proof to characterize the logarithmic Sobolev inequality starting from Theorem 2.3, i.e.:

**Theorem 2.8** Assume that $n \geq 4$, $\mu(dx) = \mu_r(d\rho) \mu(a(d\theta))$ and that $\mu_a$ satisfies a super Poincaré inequality with $\beta_{SP}^\mu(t) = C_a t^{-r}$. Then

$$C_{LS}(\mu) \leq 64 \left( c(\kappa) C_{LS}(\mu_r) + c \mu_r(\rho^2) C_P(\mu_a) \right).$$

### 3 \( L^q \) Log-Sobolev Inequality for the Uniform Measure on the Sphere

In this section we shall recall some isoperimetric property of the uniform measure $\sigma_n$ on the unit sphere $S^{n-1}$, in the purpose of obtaining the $L^q$-log-Sobolev inequality of $\sigma_n$. This will be crucial for our results in the next section.

#### 3.1 Recalls

Let us recall the $L^1$ inequalities we are interested in, namely Cheeger (or $L^1$ Poincaré) inequality

$$\mu(|f - m_\mu f|) \leq C \int |\nabla f| d\mu$$

(3.1)

where $m_\mu f$ denotes a $\mu$-median of $f$ and similarly the $L^q$ log-Sobolev inequality ($1 \leq q \leq 2$)

$$\mu(|f|^q \ln(|f|^q)) - \mu(|f|^q) \ln(\mu(|f|^q)) \leq C \int |\nabla f|^q d\mu.$$  (3.2)

As usual we denote by $C_C(\mu)$ and $C_{LSq}(\mu)$ the optimal constants in the previous inequalities. It is known (see e.g. [7]) that $C_P(\mu) \leq 4C_{LS}^2(\mu)$. One can also show that there exists a universal constant $D$ such that $C_{LS}(\mu) \leq DC_{LS1}^2(\mu)$ (see below).

These inequalities are strongly related to the isoperimetric profile of $\mu$. Recall that the isoperimetric profile $I_\mu$ of $\mu$ is defined for $p \in [0, 1]$ as

$$I_\mu(p) = \inf_{A \text{ s.t. } \mu(A) = p} \mu^+_n(\partial A),$$

where

$$\mu^+_n(\partial A) = \lim_{h \to 0} \inf \frac{\mu(A^h) - \mu(A)}{h},$$

$A^h$ being the geodesic enlargement of $A$ of size $h$. Of course in “smooth” situations, as absolutely continuous measures w.r.t. the natural Riemannian measure on a Riemannian manifold, $I_\mu(p) = I_\mu(1 - p)$ so that it is enough to consider $p \in [0, \frac{1}{2}]$.

The following results are then well known (see e.g. [7] for the first one and [10, Theorem 1.1] for the second one)

**Proposition 3.1** There is an equivalence between the following two statements:

1. $I_\mu(p) \geq C \min(p, 1 - p)$,
We turn to the uniform distribution

3.2 The Uniform Measure on the Sphere

some new universal constant

so that one can find universal constants

Using the extension of Stirling’s formula to the Γ function, one sees that

\[
F^{-1} = \text{inverse function of } F
\]

\(F^{-1}\) are optimal.

Proposition 4.5.5 iii]). For the logarithmic Sobolev inequality see [21]. Actually these bounds

These bounds are also true for

that appear in several places or immediate consequences of these results. In what follows \(s_n\)
denotes the area of \(S^{n-1}\) which is equal to \(\frac{2\pi^{n/2}}{\Gamma(n/2)}\).

Many properties rely on the fact that the sphere \(S^{n-1}\) satisfies the curvature-dimension condition \(CD(n-2, n-1)\) (see [2, p. 87]). It follows from [2, Proposition 4.8.4 and Theorem 5.7.4] that for \(n \geq 3\),

\[
C_P(\sigma_n) \leq \frac{1}{n-1} \quad \text{and} \quad C_{LS}(\sigma_n) \leq \frac{2}{n-1}.
\]

These bounds are also true for \(n = 2\). It is easy to check for the Poincaré constant using e.g. [2, Proposition 4.5.5 iii]). For the logarithmic Sobolev inequality see [21]. Actually these bounds are optimal.

The isoperimetric profile \(I_{\sigma_n}\) is described by Bobkov and Houdré in [10, Lemma 9.1 and Lemma 9.2].

**Proposition 3.2** Let \(n \geq 3\). Define

\[
f_n(t) = \frac{s_{n-1}}{s_n}(1 - t^2)^{\frac{n-2}{2}}, \quad -1 \leq t \leq 1.
\]

Let \(F_n\) be the distribution function on \([-1, 1]\) whose probability density is \(f_n\), \(G_n = F_n^{-1}\) be the inverse function of \(F_n\). Then

\[
I_{\sigma_n}(p) = \frac{s_{n-1}}{s_n}(1 - G_n^2(p))^{\frac{n-2}{2}}.
\]

Notice in particular that

\[
I_{\sigma_n}(1/2) = \frac{s_{n-1}}{s_n} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n/2)}{\Gamma((n-1)/2)}.
\]

Using the extension of Stirling’s formula to the Γ function, one sees that

\[
\lim_{n \to +\infty} \sqrt{2\pi/(n-1)} \frac{s_{n-1}}{s_n} = 1,
\]

so that one can find universal constants \(c\) and \(C\) such that

\[
c\sqrt{n-1} \leq \frac{s_{n-1}}{s_n} \leq C\sqrt{n-1}.
\]

Using their Lemma 8.2, Bobkov and Houdré ([10]) also show in their Section 9 that the minimum of \(I_{\sigma_n}(p)/p(1-p)\) is attained for \(p = 1/2\) so that for all \(p \in [0, 1/2]\)

\[
I_{\sigma_n}(p) \geq \frac{2s_{n-1}}{s_n}p
\]
and

\[ C_C(\sigma_n) = \frac{1}{2} \frac{s_n}{s_{n-1}} \leq \frac{C}{\sqrt{n-1}} \quad (3.5) \]

for some universal constant \(C\). Actually, the application of [10, Lemma 8.2] to \(I_{\alpha_n}^2\) for \(\alpha \in [1, n/(n-1)]\) furnishes some Sobolev inequality (see [10, Proposition 8.1]).

**Remark 3.3** Since the curvature dimension condition \(CD(n-2, n-1)\) implies \(CD(0, +\infty)\) for \(n \geq 2\) (i.e., \(\sigma_n\) is log-concave), it is known (see e.g. [27] for numerous references) that \(I_{\alpha_n}\) is concave on \([0, 1/2]\). This furnishes another proof of (3.4).

Actually there is a control of the isoperimetric profile related to the Poincaré constant.

Another remarkable property of log-concave measures identified by Ledoux ([23, Theorem 5.3]) for the usual \((L^2)\) log-Sobolev inequality and generalized in [26, Theorem 1.2] by Milman, is that the \(L^q\) log-Sobolev inequality also furnishes such a control for the isoperimetric profile, more precisely for any log-concave probability measure \(\mu\) and \(p \in [0, 1/2]\),

\[ I_\mu(p) \geq \frac{\sqrt{2}}{34C_{LS}(\mu)} p \ln^\frac{1}{q}(1/p), \]

in the case \(q = 2\), and more generally there exists a universal constant \(c\) such that for all \(1 \leq q \leq 2\),

\[ I_\mu(p) \geq \frac{c}{C_{LS}^{1/q}(\mu)} p \ln^\frac{1}{q}(1/p). \]

The converse statement \(I_\mu(p) \geq c_\mu p \ln^\frac{1}{q}(1/p)\) implies \(C_{LS}(\mu) \leq C_{\mu}^{\frac{1}{q}}\) for some universal constant \(C\) does not require log-concavity and was shown by Bobkov–Zegarlinski [12].

**A Precise Dimension Dependence in the Isoperimetric Profile**

Our goal is now to determine the best possible constant \(C_n(q)\) (best in terms of the dimension) such that

\[ I_{\sigma_n}(p) \geq C_n(q) p \ln^{1/q}(1/p), \quad (3.6) \]

and then to apply the equivalence we explained before to derive the best possible \(L^q\) log-Sobolev inequality for \(\sigma_n\).

We shall consider \(p = F_n(x)\) in order to have a tractable expression

\[ I_{\sigma_n}(F_n(x)) = \frac{s_{n-1}}{s_n} (1 - x^2)^{\frac{n-2}{2}}. \]

To this end we will use the following elementary lemma.

**Lemma 3.4** For all \(x \in [-1, 0)\) define \(A_n(x) = \frac{1}{n+1}(1 - x^2)^{1/2}I_{\sigma_n}(F(x))\). It holds

\[ A_n(x) \leq F_n(x) \leq \frac{A_n(x)}{-x}. \]

**Proof** For all \(x \in [-1, 0)\), we have on one hand,

\[ F_n(x) = \int_{-1}^x \frac{s_{n-1}}{s_n} (1 - u^2)^{(n-3)/2} du \]
\[ \int_{-1}^{x} \frac{s_{n-1} - u}{s_n} \frac{1}{s_n - x} (1 - u^2)^{(n-3)/2} du \]

\[ = \frac{s_{n-1}}{(-x)(n-1)s_n} (1 - x^2)^{(n-1)/2} = \frac{1}{(-x)(n-1)} (1 - x^2)^{1/2} I_{\sigma_n}(F(x)). \]

On the other hand,

\[ F_n(x) = \int_{-1}^{x} \frac{s_{n-1}}{s_n} (1 - u^2)^{(n-3)/2} du \]

\[ \geq \int_{-1}^{x} \frac{s_{n-1}}{s_n} (-u)(1 - u^2)^{(n-3)/2} du \]

\[ = \frac{s_{n-1}}{(n-1)s_n} (1 - x^2)^{(n-1)/2} = \frac{1}{(n-1)} (1 - x^2)^{1/2} I_{\sigma_n}(F(x)). \]

It follows for \( F_n(x) \leq 1/2 \) and provided \((-x)/A_n(x) > 1 \) (for its logarithm to be positive),

\[ F_n(x) \ln^{1/q}(1/F_n(x)) \geq F_n(x) \ln^{1/q}((-x)/A_n(x)) \geq A_n(x) \ln^{1/q}((-x)/A_n(x)). \]

But, remarking that \( \ln((n-1)s_n/s_{n-1}) \geq 0 \),

\[ \ln((-x)/A_n(x)) \geq \ln(-x) + \frac{n-1}{2} \ln(1/(1-x^2)) \geq \ln(-x) + \frac{1}{4} \ln(1/(1-x^2)) \geq 0 \]

for all \( x \in (-1, -a) \) for some \( 0 < a < 1 \) using continuity, so that for \( x \leq -a \),

\[ \ln(-x) + \frac{n-1}{2} \ln(1/(1-x^2)) \geq \frac{2n-3}{4} \ln(1/(1-x^2)). \]

This yields for such an \( x \),

\[ F_n(x) \ln^{1/q}(1/F_n(x)) \]

\[ \geq \frac{1}{(n-1)} (1 - x^2)^{1/2} I_{\sigma_n}(F(x)) \left( \frac{2n-3}{4} \ln(1/(1-x^2)) + \ln \left( \frac{(n-1)s_n}{s_{n-1}} \right) \right)^{1/q} \]

and finally that there exists a constant \( c(q) \) such that for such an \( x \),

\[ F_n(x) \ln^{1/q}(1/F_n(x)) \geq c(q)(n-1)^{\frac{1-q}{q}} I_{\sigma_n}(F_n(x)). \]

(3.7)

In particular the constant \( C_n(q) \) in (3.6) cannot be bigger than a constant times \( (n-1)^{\frac{1-q}{q}} \).

For \( q = 1 \), it is known (see [24, Theorem 2]) that

\[ I_{\sigma_n}(p) \geq \frac{1}{2\pi} p \ln(1/p) \]

for \( p \in [0, 1/2] \), so that if not optimal, this result is optimal up to a constant factor.

Since we cannot hope for a better result, our goal will be now to prove that

\[ C_n(q) \geq C(q)(n-1)^{\frac{1-q}{q}}. \]

To this end we will take advantage of Ledoux’s result applied to \( \sigma_n \), i.e.,

\[ I_{\sigma_n}(p) \geq C(2) \sqrt{n - 1} p \ln^{1/2}(1/p). \]

(3.8)

Indeed, it is easy to check that

\[ \sqrt{n - 1} p \ln^{1/2}(1/p) \geq (n-1)^{\frac{1-q}{q}} p \ln^{1/q}(1/p) \]
as soon as \( p \geq e^{-(n-1)} \). It is thus enough to consider the remaining \( p = F_n(x) \leq e^{-(n-1)} \). Using Lemma 3.4 we thus have
\[
\frac{s_{n-1}}{(n-1)s_n}(1 - x^2)^{(n-1)/2} \leq e^{-(n-1)}
\]
so that
\[
\ln(1 - x^2) \leq 2 \left( -1 + \frac{n(n-1)}{n-1} \right) \leq -2 \frac{e-1}{e},
\]
which implies
\[
x \leq -\left( 1 - e^{-2(e-1)/e} \right)^{1/2} = y. \tag{3.9}
\]
We can thus deduce, for such an \( x \),
\[
F_n(x) \ln^{1/q}(1/F_n(x)) \leq \frac{1}{(-x)(n-1)}(1 - x^2)^{1/2} I_{\sigma_n}(F(x))
\]
\[
\left( \ln((n-1)s_n/s_{n-1}) + \frac{n-1}{2} \ln(1/(1 - x^2)) \right)^{1/q}
\]
\[
\leq \frac{1}{(-y)(n-1)}(1 - x^2)^{1/2} I_{\sigma_n}(F(x))
\]
\[
\left( \ln^{1/q}((n-1)s_n/s_{n-1}) + \left( \frac{n-1}{2} \right)^{1/q} \ln^{1/q}(1/(1 - x^2)) \right)
\]
\[
\leq D(n-1)^{\frac{1}{q-1}} I_{\sigma_n}(F(x)) \tag{3.10}
\]
for some constant \( D \). We may thus state

**Proposition 3.5** Let \( n \geq 3 \). For any \( 1 \leq q \leq 2 \) there exist constants \( C \) and \( C(q) \) such that for all \( p \in [0,1/2] \) one has
\[
I_{\sigma_n}(p) \geq C(n-1)^{1-\frac{q}{p}} p \ln^{1/q}(1/p),
\]
yielding
\[
C_{LSq}(\sigma_n) \leq C(q) \left( \frac{1}{n-1} \right)^{q-1}.
\]
Actually all constants can be chosen independently of \( q \in [1,2] \). The bound is of optimal order with respect to the dimension.

4 The Logarithmic-Sobolev Inequality in the Nearly Radial Case

4.1 An Alternate Direct Approach

Instead of using super Poincaré inequalities, let us try to directly mimic Bobkov’s tensorization in the case of log-Sobolev. First
\[
\int (f^2 \ln(f^2))(\rho \theta)\mu_\rho(d\rho) \leq C_{LS}(\mu_\rho) \int \left| \nabla_x f(\rho \theta) \right|^2 \mu_\rho(d\rho)
\]
\[
+ \left( \int f^2(\rho \theta)\mu_\rho(d\rho) \right) \ln \left( \int f^2(\rho \theta)\mu_\rho(d\rho) \right), \tag{4.1}
\]
so that integrating with respect to $\mu_a$ we get

$$\text{Ent}_\mu(f^2) \leq C_{LS}(\mu_r)\mu(|\nabla f|^2) + \text{Ent}_{\mu_a}(w^2)$$

(4.2)

with

$$w(\theta) = \left( \int f^2(\rho\theta)\mu_r(d\rho) \right)^{\frac{1}{2}} ,$$

after having remarked that

$$\mu(f^2) \ln(\mu(f^2)) = \mu_a(w^2) \ln(\mu_a(w^2)).$$

We are thus facing a difficulty. Indeed

$$\nabla w(\theta) = \frac{\int f(\rho\theta)\rho\nabla_{\theta} f(\rho\theta)\mu_r(d\rho)}{\left( \int f^2(\rho\theta)\mu_r(d\rho) \right)^{\frac{1}{2}}} .$$

Hence if we use the classical log-Sobolev inequality

$$\text{Ent}_{\mu_a}(w^2) \leq C_{LS}(\mu_a)\mu_a(|\nabla w|^2)$$

$$\leq C_{LS}(\mu_a) \int \frac{\left( \int f(\rho\theta)\rho\nabla_{\theta} f(\rho\theta)\mu_r(d\rho) \right)^2}{\int f^2(\rho\theta)\mu_r(d\rho)} \mu_a(d\theta).$$

(4.3)

Using Cauchy-Schwarz inequality in two different ways we obtain

**Proposition 4.1** If $\mu(dx) = \mu_r(d\rho)\mu_a(d\theta)$, then

1. \[\text{Ent}_\mu(f^2) \leq \max(C_{LS}(\mu_r), \|\rho\|_{L^\infty(\mu_r)}C_{LS}(\mu_a))\mu(|\nabla f|^2).\]

2. \[\text{Ent}_\mu(f^2) \leq \max(C_{LS}(\mu_r), C_{LS}(\mu_a))\mu((1 \lor \rho)^2|\nabla f|^2).\]

This is observed also by Joulin and Ma (in a private communication). The first inequality in Proposition 4.1 implies

**Corollary 4.2** If $\mu(dx) = \mu_r(d\rho)\mu_a(d\theta)$ and $\mu$ is supported by a bounded ball $B(0, R)$ centered at the origin of radius $R$, then

$$C_{LS}(\mu) \leq \max\left(C_{LS}(\mu_r), R^2C_{LS}(\mu_a)\right).$$

The second inequality in Proposition 4.1 is a *weighted* log-Sobolev inequality, with weight $(1 \lor \rho)^2$, which is much weaker than the log-Sobolev inequality. These inequalities have been studied for instance in [20, 30]. Consequences in terms of concentration, rate of convergence or transport are in particular discussed in [20, Section 3]. The weight $(1 \lor \rho)^2$ is however too big for being really interesting. In particular this weighted log-Sobolev inequality does not imply a Poincaré inequality in whole generality.

Now consider the almost radial situation. According to Section 3, $C_{LS}(\sigma_n) \leq \frac{2}{n-1}$ for $n \geq 2$. It thus follows from Corollary 4.2 and perturbation arguments

**Corollary 4.3** For all $\mu(dx) = \mu_r(d\rho)\mu_a(d\theta)$ supported by some bounded ball $B(0, R)$ and satisfying

$$m \leq \left\| \frac{d\mu_a}{d\sigma_n} \right\|_{\infty} \leq M,$$
it holds
\[ C_{LS}(\mu) \leq \max \left( C_{LS}(\mu_r), \frac{M}{m} \frac{2R^2}{n-1} \right). \] (4.4)

### 4.2 The (Almost) Radial Case

Our main new results for the log-Sobolev inequality of almost radial measure are

**Theorem 4.4** (Bounded support case) There exists a universal constant \( c \) such that, for every almost radial measure \( \mu(dx) = \mu_r(d\rho)\mu_a(d\theta) \) with support in \( B(0,R) \) and satisfying
\[
m \leq \left\| \frac{d\mu_a}{d\sigma_n} \right\|_{\infty} \leq M,
\]
it holds
\[
C_{LS}(\mu) \leq 2C_P(\mu) + \max \left( C_{LS}(\mu_r), c \frac{M}{m} \min \left( \frac{R\sqrt{C_P(\mu)}}{R^2}, \frac{R^2}{n-1} \right) \right).
\] (4.5)

**Theorem 4.5** (Unbounded support case) There exists a universal constant \( c \) such that, for every almost radial measure \( \mu(dx) = \mu_r(d\rho)\mu_a(d\theta) \) satisfying
\[
m \leq \left\| \frac{d\mu_a}{d\sigma_n} \right\|_{\infty} \leq M,
\]
it holds
\[
C_{LS}(\mu) \leq \frac{M}{m} \left( C_{LS}(\mu_r) + \mu(\rho) \max \left( C_P(\mu_r), \frac{\mu(\rho^2)}{n-1} \right) \right)^{1/2}.
\] (4.6)

Notice that if \( \mu \) is supported by a bounded ball \( B(0,R) \), we recover only partially the conclusion of Corollary 4.3.

**Remark 4.6** Of course the previous results are much better, in terms of the dimension dependence, than Corollary 2.7 since the pre-factor of \( C_{LS}(\mu_r) \) is “dimension free” (more precisely can be bounded from above by a universal constant), while the dimension appears in front of \( C_{LS}(\mu_r) \) in Corollary 2.7. Let us remark also that the constant appearing in the second formulation is close from the one obtained by Bobkov in dimension one [7]. It will appear again in the next Section. Remark also that contrary to the corollary, the proof cannot be extended to more general cases, except if the angular part \( \mu_a \) satisfies a similar \( L^1 \) log-Sobolev inequality.

**Proof of Theorem 4.4** Rewrite (4.2),
\[
\mu(f^2 \ln(f^2)) \leq C_{LS}(\mu_r)\mu(|\nabla_{\theta} f|^2) + \int \frac{g^q}{q} \ln(g) d\mu_a
\] (4.7)
with \( g(\theta) = \left( \int f^2(\rho \theta)\mu_r(d\rho) \right)^{1/q} \) and \( 1 \leq q \leq 2 \).

Instead of the usual log-Sobolev inequality, we now use the \( L^q \) log-Sobolev inequality for \( \sigma_n \) we have obtained in Section 3. By the standard perturbation result, it thus holds
\[
\int g^q \ln(g^q) d\mu_a \leq \frac{M}{m} (n-1)^{1-q} \int |\nabla g|^q d\mu_a + \left( \int g^q d\mu_a \right) \ln \left( \int g^q d\mu_a \right),
\] (4.8)
so that
\[
\text{Ent}_\mu(f^2) \leq C_{LS}(\mu_r)\mu(|\nabla_{\theta} f|^2) + \frac{M}{m} (n-1)^{1-q} \int |\nabla g|^q d\mu_a.
\]
Now
\[
\int |\nabla g|^q d\mu_a = \frac{2q}{q} \int \left( \int f^2(\rho \theta)\mu_r(d\rho) \right)^{1-q} \left( \int f \nabla_{\theta} f \rho \mu_r(d\rho) \right)^q d\mu_a
\]
Sobolev constant. We first use the same approach will provide us with an easy to apprehend formulation of the logarithmic Sobolev constant. We may of course stop here to get a first control of the logarithmic Sobolev constant of

\[ \int |\nabla g|^q d\mu_\alpha \leq \frac{2q}{q} \|\rho\|_\infty^q \left( \int |\nabla \theta f|^2 d\mu_\tau \right)^{\frac{q}{2}} \]  

where we have used Hölder’s inequality in the last line.

Recall Rothaus lemma

\[ \text{Ent}_\mu (f^2) \leq \text{Ent}_\mu ((f - \mu(f))^2) + 2\text{Var}_\mu(f). \]

We may thus replace \( f \) by \( f - \mu(f) \) (that changes \( g \)) and use Poincaré’s inequality in order to get

\[ \int |\nabla g|^q d\mu_\alpha \leq \frac{2q}{q} \|\rho\|_\infty^q C_P(\mu)^{1-\frac{q}{2}} \left( \int |\nabla \theta f|^2 d\mu \right). \]

Gathering all these results, using again Poincaré inequality for bounding the variance, we thus have if \( \mu \) is supported by \( B(0, R) \),

\[ \text{Ent}_\mu (f^2) \leq C_{LS}(\mu_\tau) \int |\nabla \theta f(\rho\theta)|^2 d\mu + 2C_P(\mu) \left( \int |\nabla f|^2 d\mu \right) 
\]

\[ + c \frac{M}{m}(n - 1)^{1-q} \frac{2q}{q} R^q C_P(\mu)^{1-\frac{q}{2}} \left( \int |\nabla \theta f|^2 d\mu \right). \]

Taking \( q = 2 \) if \( R \leq (n - 1) \) and \( q = 1 \) if \( R \geq (n - 1) \), we obtain the desired estimate (4.5) by Theorem 2.1.

**Proof of Theorem 4.5** We will use the preceding argument with \( q = 1 \) (i.e., the \( L^1 \) log-Sobolev inequality for \( \sigma_n \)). Instead of (4.9) we have

\[ \int |\nabla g| d\sigma_n \leq 2 \left( \int f^2 \rho^2 d\mu_\tau \right)^{1/2} \left( \int |\nabla \theta f|^2 d\mu_\tau \right)^{\frac{1}{2}} d\sigma_n \]

obtained from the first equality in (4.9) and the Cauchy–Schwarz inequality.

Let us consider \( \int f^2 \rho^2 d\mu \) integrating first w.r.t. \( \mu_\tau \). By using the variational description of the relative entropy we have for any \( t > 0 \),

\[ \int f^2(\rho\theta)\rho^2 \mu_\tau(d\rho) \leq \text{Ent}_{\mu_\tau}(f^2(\rho\theta)) + \frac{1}{t} \ln \left( \int e^{t\rho^2} \mu_\tau(d\rho) \right) \left( \int f^2(\rho\theta)\mu_\tau(d\rho) \right) \]

We may of course stop here to get a first control of the logarithmic Sobolev constant of \( \mu \) by using recentering and Rothaus lemma (see the end of the argument) but let us see how using the same approach will provide us with an easy to apprehend formulation of the logarithmic Sobolev constant. We first use

\[ \int f^2 \rho^2 d\mu_\tau \leq 2 \int f^2(\rho - \mu(\rho))^2 d\mu + 2\mu^2(\rho) \int f^2 d\mu_\tau, \]

and using again the variational description of the relative entropy we have for any \( t > 0 \),

\[ \int f^2(\rho\theta)(\rho - \mu(\rho))^2 \mu_\tau(d\rho) \leq \text{Ent}_{\mu_\tau}(f^2(\rho\theta)) + \frac{1}{t} \ln \left( \int e^{t(\rho - \mu(\rho))^2} \mu_\tau(d\rho) \right) \left( \int f^2(\rho\theta)\mu_\tau(d\rho) \right). \]

(4.14)
But since $\rho \mapsto \rho - \mu_r(\rho)$ is 1-Lipschitz and of $\mu_r$ mean equal to 0, it is known (see e.g. [7, Formula (4.9)]) that for $t < 1/C_{LS}(\mu_r)$
\[
\int e^{t(\rho - \mu(\rho))^2} \mu(d\rho) \leq \frac{1}{\sqrt{1-tC_{LS}(\mu_r)}}.
\]
For $t = 1/2C_{LS}(\mu_r)$ we thus deduce
\[
\int f^2(\rho \theta)(\rho - \mu(\rho))^2 \mu_r(d\rho) \leq C_{LS}(\mu_r) \int |\nabla f(\rho \theta)|^2 \mu_r(d\rho) + \ln(2)C_{LS}(\mu_r) \int f^2(\rho \theta) \mu_r(d\rho).
\]
Finally
\[
\int f^2 \rho^2 d\mu \leq 2C_{LS}(\mu_r) \int |\nabla f|^2 d\mu + 2(\ln(2)C_{LS}(\mu_r) + \mu_r^2(\rho)) \int f^2 d\mu.
\] (4.15)
Replacing $f$ by $f - \mu(f)$ and using the Poincaré inequality, we thus deduce
\[
\int (f - \mu(f))^2 \rho^2 d\mu \leq 2C_{LS}(\mu_r) \int |\nabla f|^2 d\mu + 2(\ln(2)C_{LS}(\mu_r) + \mu_r^2(\rho))C_P(\mu) \int |\nabla f|^2 d\mu.
\]
\[
\text{Gathering all what precedes we get}
\]
\[
\text{Ent}_\mu((f - \mu(f))^2) \leq A\mu(|\nabla f|^2)
\]
with, for some universal constant $c$,
\[
A = C_{LS}(\mu_r) + c \left( C_{LS}(\mu_r) + (C_{LS}(\mu_r) + \mu_r^2(\rho))C_P(\mu) \right)^{\frac{1}{2}}.
\] (4.16)
To conclude it remains to use Rothaus lemma again. We obtain (4.6) after some simple manipulations using in particular $2C_P \leq C_{LS}$, the concavity of the square root, and the homogeneity of the inequalities w.r.t. dilations as we did in order to get (2.14) but this time with $\lambda \to +\infty$, and finally (2.8) (replacing for simplicity the max by the sum).

To finish this section, let us remark that one could also get another way to control (4.12) by using Lyapunov conditions rather than using the Logarithmic Sobolev inequality for the radial part. Of course, by [17], one also knows that in our setting a logarithmic Sobolev inequality is equivalent to some Lyapunov type conditions. However we will see that in order to control (4.12) one needs a slightly weaker inequality. Indeed let us suppose here that there exist some function $W(\rho) \geq 1$ and constants $a, b > 0$ such that
\[
\rho^2 \leq -a \frac{L_{\rho}W}{W} + b, \quad \rho > 0
\] (4.17)
where $L_{\rho}f = f'' - 2(\frac{1}{\rho} - \frac{n-1}{\rho^2})f'$ is the generator corresponding to the radial part of $\mu$. Recall now that we need to control
\[
\int f^2 \rho^2 \mu_r(d\rho).
\]
Using (4.17) we get
\[
\int f^2 \rho^2 \mu_r(d\rho) \leq a \int f^2 \frac{-L_{\rho}W}{W} \mu_r(d\rho) + b \int f^2 \mu_r(d\rho).
\]
The first term is easily dealt with, using integration by parts or a large deviations argument as in [18]:
\[
\int f^2 \frac{-L_{\rho}W}{W} \mu_r(d\rho) \leq \int |\nabla f(\rho \theta)|^2 \mu_r(d\rho).
\]
We then use the same trick as before, i.e., centering and Rothaus lemma to get
Proposition 4.7 Assume (4.17). There exists a universal constant \( c \) such that, for all \( \mu(dx) = \mu_r(dp)\mu_a(d\theta) \) satisfying

\[
m \leq \left\| \frac{d\mu_a}{d\sigma_n} \right\|_\infty \leq M,
\]

it holds

\[
C_{LS}(\mu) \leq c \frac{M}{m} \left( C_{LS}(\mu_r) + \sqrt{a} + \sqrt{b} \max \left( C_P(\mu_r), \frac{\mu(\rho^2)}{m-1} \right)^{1/2} \right).
\]

Remark 4.8 Let us make a few comments about the Lyapunov condition (4.17). It has been shown in [19] that it is a sufficient condition for Talagrand inequality, and that there exist examples satisfying this condition and not a logarithmic Sobolev inequality. Nevertheless, we need for the first part of the proof that the radial part satisfies a logarithmic Sobolev inequality. More crucial are the values of the constants \( a \) and \( b \) with respect to the dimension. If \( a \) can be chosen dimension free in usual cases, say \( V_r(\rho) = \rho^k \), \( b \) is then of order \( n \) and we then get an additional \( \sqrt{n} \) factor for the logarithmic Sobolev constant in this case. Of course, it is surely better than the \( n \) factor by using super-Poincaré inequality.

5 Some Applications in the (Almost) Radial Case

The main interest of the previous results is that they reduce the study of functional inequalities for \( \mu \) to the one of its radial part \( \mu_r \) which is supported by the half line. For such one dimensional measures explicit criteria of Muckenhoupt (or Hardy) type are well known [1, 2, 4–6]). Let us recall the case of the Poincaré inequality (see e.g. [2, Theorem 4.5.1]) and of the log-Sobolev inequality (see [5, Theorem 7] with \( T(u) = 2u \), or [15, Proposition 2.4] for a slightly different version).

Proposition 5.1 Assume that \( \mu_r \) is absolutely continuous w.r.t. Lebesgue measure with density \( \rho_r \). Assume in addition that the support of \( \mu_r \) is an interval \( I \). Let \( m \) be a median of \( \mu_r \), then

\[
\frac{1}{2} \max(b_-, b_+) \leq C_P(\mu_r) \leq 4 \max(b_-, b_+)
\]

and

\[
\frac{1}{12} \max(B_-, B_+) \leq C_{LS}(\mu_r) \leq 10 \max(B_-, B_+)
\]

where

\[
b_- = \sup_{x \in I, 0 \leq x < m} \mu_r([0, x]) \int_x^m \frac{1}{\rho_r(\rho)} d\rho,
\]

\[
b_+ = \sup_{x \in I, x > m} \mu_r([x, +\infty)) \int_x^m \frac{1}{\rho_r(\rho)} d\rho
\]

while

\[
B_- = \sup_{x \in I, 0 \leq x < m} \mu_r([0, x]) \ln \left( 1 + \frac{1}{\mu_r([0, x])} \right) \int_x^m \frac{1}{\rho_r(\rho)} d\rho,
\]

\[
B_+ = \sup_{x \in I, x > m} \mu_r([x, +\infty)) \ln \left( 1 + \frac{1}{\mu_r([x, +\infty))} \right) \int_x^m \frac{1}{\rho_r(\rho)} d\rho.
\]

Several methods are known to furnish estimates for quantities like \( \mu_r([a, +\infty]) \) or \( \int (1/\rho_r) \) (see e.g. [1, Chapter 6.4]).
Remark 5.2 We will not study in more details the $L^1$ inequalities on the real (half) line. However, because it is immediate, let us only give an upper bound for $C_C$.

Proposition 5.3 Let $\nu(dx) = e^{-W(x)}dx$ be a probability measure on $\mathbb{R}$.

Then $C_C(\nu) \leq \max(b_+, b_-)$ where

$$b_- = \sup_{t \in \mathbb{R}} e^{W(t)} \nu([-\infty, t]) \quad \text{and} \quad b_+ = \sup_{t \in \mathbb{R}} e^{W(t)} \nu([t, +\infty]).$$

Proof We simply write for any $a$,

$$\nu(|f - m_\nu(f)|) \leq \nu(|f - f(a)|) = \int_a^a \int_{-\infty}^x f'(t) dt \nu(dx) \leq \int_{-\infty}^a \int_{-\infty}^x |f'(t)| dt \nu(dx) + \int_a^x |f'(t)| dt \nu(dx) \leq \int_{-\infty}^a |f'(t)| \nu([-\infty, t]) dt + \int_a^{+\infty} |f'(t)| \nu([t, +\infty]) dt,$$

and then write $dt = e^{W(t)} \nu(dt)$ to get the result. \qed

It is possible to get a lower bound, and bounds for $C_{LS1}$ using ad-hoc Orlicz spaces as in [10] and the previous trick in one dimension.

Remark 5.4 Before to study some simple examples, let us say a word about “optimality” in the radial case.

Notice first that if $f(x) = g(|x|)$, $\nabla f(x) = g'(|x|) \frac{x}{|x|}$ so that $|\nabla f|^2(x) = (g'(|x|))^2$. It follows that $C_P(\mu) \geq C_P(\mu_r)$ and similarly for the log-Sobolev constant.

If we take $f(x) = \sum_{i=1}^n x_i/|x|$, $\mu_\mu(f) = 1$ while $\mu(|\nabla f|^2) = (n-1)\mu(1/\rho^2)$ so that $C_P(\mu) \geq 1/(n-1)\mu(1/\rho^2)$. The latter is smaller than $\mu(\rho^2)/(n-1)$ but both quantities are comparable in many situations. We shall see it on the examples below.

In all the examples below, we will pay a particular attention to dimension dependence, so that any sentence like “the good order” or “good dependence” has to be understood “with respect to the dimension $n$.

Example 5.5 The Uniform Measure on a Euclidean Ball

Consider the simplest example for radial $\mu$, i.e., the uniform measure on the Euclidean ball of radius $R$. The good order for the log-Sobolev constant has been derived in [11, Proposition 5.3] by pushing forward the Gaussian distribution onto the uniform one on the ball (also see Proposition 5.4 therein for the $L^{\infty}$ inequality).

We shall obtain here similar bounds by using our results. First

$$\mu_r(d\rho) = n\rho^{n-1} 1_{0 \leq \rho \leq R}d\rho$$

so that the mean, the (unique) median, the second moment, the variance of $\mu_r$ are respectively:

$$\mu_n = \frac{n}{n + 1} R, \quad m_n = (1/2)^{1/n} R, \quad \mu_r(\rho^2) = \frac{n}{n + 2} R^2, \quad v_n = \frac{n}{(n+2)(n+1)^2} R^2.$$

It is then easily seen that $b_-$ and $b_+$ are both less than $\frac{R^2}{(n-2)}$ provided $n > 2$. The case $n = 2$ can be handled separately. It follows that $C_P(\mu_r) \leq 4(R^2/n^2)$. We also have $C_P(\mu_r) \geq v_n$ so that $R^2/n^2$ is the good order. Finally

$$C_P(\mu) \leq \max \left( \frac{4}{n^2}, \frac{n}{(n+2)(n-1)} \right) R^2.$$
This bound is not sharp, but asymptotically sharp. Indeed \( \mu(1/\rho^2) = \frac{n}{n-2} R^{-2} \), so that according to the discussion in remark, \( C_P(\mu) \geq \frac{n-2}{n(n-1)} R^2 \).

Notice that the upper bound is exactly the one obtained in [13, Theorem 1.2], while the lower bound is better than the one in [13]. [13, Theorem 1.2] deals with general radial log-concave measures by using dedicated tools for log-concave one dimensional measures, we shall come back to this later.

We turn now to the log-Sobolev constant assuming first that \( n \geq 3 \).

First

\[
B_- = \sup_{0 \leq x < m_n} \frac{x^2}{n(n-2)} \left( 1 - \frac{(x/m_n)^{n-2}}{(1+(R/x)^n)} \right) \ln \left( 1 + \frac{1}{(1+(R/x)^n)} \right),
\]

so that using \( \ln(1+u^n) \leq \ln 2 + n \ln(u) \) for \( u \geq 1 \), we deduce

\[
B_- \leq \frac{R^2}{n-2} \sup_{u \geq 1} \left( \frac{1}{u^2} \left( \frac{\ln 2}{n} + \ln u \right) \right) \leq c_1 \frac{R^2}{n-2},
\]

with \( c_1 = \frac{1}{2e} + \frac{\ln 2}{n} \leq 1 \). Similarly, using \( \ln(1+u^n) \geq n \ln(u) \) for \( u \geq 1 \) we have

\[
B_- \geq \frac{R^2}{n-2} \sup_{2^{1/n} \leq u} \left( \frac{\ln u}{u^2} \left( 1 - \frac{2^{(n-2)/n} u^{-(n-2)}}{1+(R/x)^n} \right) \right) \geq c_2 \frac{R^2}{n-2},
\]

with for instance \( c_2 = \ln 2/16 \) obtained by choosing \( u = 4 \).

Next

\[
B_+ = \sup_{m_n, x \leq R} \left( 1 - \frac{(x/R)^n}{(n-2)} \right) \frac{R^2}{n(n-2)} \left( 2^{(n-2)/n} - (R/x)^n \right) \ln \left( 1 + \frac{1}{1 - (x/R)^n} \right),
\]

so that

\[
B_+ \leq \frac{R^2}{n(n-2)} \sup_{0 < v \leq 1} (v \ln(1+(1/v))) \leq \frac{R^2}{n(n-2)}.
\]

Gathering all this we obtain that

\[
\frac{\ln 2}{192} \frac{R^2}{n-2} \leq C_{LS}(\mu_r) \leq 10 \frac{R^2}{n-2}.
\]

Note that choosing \( f(x) = x \) one gets the better lower bound

\[
C_{LS}(\mu_r) \geq \frac{R^2}{n(n-2)} \frac{n^2}{n+2} \left( \ln \left( 1 + \frac{2}{n} \right) - \frac{2}{n+2} \right) \geq 2R^2 \frac{n-2}{(n+2)^2}.
\]

Using Corollary 4.3 we have thus shown, since \( C_{LS}(\mu) \geq C_{LS}(\mu_r) \), and after simple manipulations

**Proposition 5.6** For all \( n \geq 3 \) the uniform measure \( \mu \) on the Euclidean ball of radius \( R \) satisfies

\[
\frac{2^{(n-2)/n} R^2}{(n+2)^2} \leq C_{LS}(\mu) \leq 10 \frac{R^2}{n-2}.
\]

**Example 5.7 Spherically Symmetric Log-concave Measures**

The previous example is a particular example of a radial log-concave measure, i.e., \( \mu(dx) = e^{-V(|x|)}dx \) where \( V \) is convex and non-decreasing. Actually for what follows (in the radial situation) we do not need \( V \) to be non-decreasing, but still convex.

For such measures the Poincaré constant was first studied by Bobkov in [8]. Bobkov’s result was improved by Bonnefont–Joulin–Ma in [13, Theorem 1.2] who states that for such measures,

\[
\frac{\mu(\rho^2)}{n} \leq C_P(\mu) \leq \frac{\mu(\rho^2)}{n-1}.
\]
The case of the log-Sobolev constant was not really addressed in the specific radial situation, but rather for general log concave distributions. Define

\[ \text{Orl}(\mu) = \inf \left\{ t > 0; \int \exp(|x - \mu(x)|^2/t^2)\mu(dx) \leq 2 \right\}. \]

Then, according to Bobkov’s [7, Theorem 1.3], if \( \mu \) is log-concave

\[ C_{LS}(\mu) \leq C\text{Orl}^2(\mu) \]  

the right hand side being finite or infinite. When \( \mu \) is supported by a bounded set \( K \), this yields the rough bound \( C_{LS}(\mu) \leq C\text{diam}^2(K) \). The latter has been improved in [25, Theorem 8], where it is shown that \( C_{LS}(\mu) \leq C\text{diam}(K) \) provided \( \mu \) is isotropic (i.e., its covariance matrix equals identity). Notice that Bobkov’s [7, Corollary 2.3] tells that

\[ C_{LS}(\mu) \leq 2(C_P(\mu) + \text{diam}(K)\sqrt{C_P(\mu)}), \]  

so that, it also furnishes the diameter bound if the K-L-S conjecture is true.

Actually the specific radial case was only addressed in [22] where the author studies the isoperimetric profile of radial log concave distributions (see Theorem 4 and Theorem 5 therein). Connections between the log-Sobolev constant and the isoperimetric profile are strong in the log-concave situation (see [26, Theorem 1.2]) but the results of [22] are not easy to handle with. We will thus use our previous results to derive explicit bounds.

Writing \( \mu_r(d\rho) = n\omega_n\rho^{n-1}e^{-V(\rho)}d\rho \) with \( \omega_n \) the volume of the unit Euclidean ball, we see that \( \mu_r \) is also log-concave. One can thus apply Bobkov’s results in [7], starting with Proposition 4.4 therein

\[ \frac{3}{4}\text{Orl}^2(\mu_r) \leq C_{LS}(\mu_r) \leq 48\text{Orl}^2(\mu_r) \]  

where

\[ \text{Orl}(\mu_r) = \inf \left\{ t > 0; \int \exp((\rho - \mu_r(\rho))^2/t^2)\mu_r(d\rho) \leq 2 \right\}. \]

As a byproduct we get thanks to Theorem 4.5 and (5.1)

**Proposition 5.8** If \( \mu \) is a radial log-concave distribution,

\[ C_{LS}(\mu) \leq C \left( \text{Orl}^2(\mu_r) + \frac{\mu(\rho^2)}{\sqrt{n-1}} \right). \]

Using (5.4) and Remark 5.5, one can also derive some lower bound.

If we assume in addition that \( \mu \) is supported by a bounded ball \( B(0, R) \), we may obtain other bounds using our Corollary 4.3 (the same with Theorem 4.4). We remark that

\[ \mu_r(d\rho) = Z_n^{-1}e^{(n-1)\ln \rho - V(\rho)}d\rho = Z_n^{-1}e^{-W(\rho)}d\rho \]

with

\[ W''(\rho) = V''(\rho) + \frac{n-1}{\rho^2} \geq \frac{(n-1)}{R^2}. \]

According to Bakry–Émery criterion (see e.g. [2, Proposition 5.7.1]) we deduce

\[ C_{LS}(\mu_r) \leq \frac{R^2}{n-1}. \]  

Applying Corollary 4.3 we have thus obtained
Theorem 5.9 For any radial log-concave probability measure $\mu$ with support contained in $B(0,R)$,
\[
C_{LS}(\mu) \leq C \frac{R^2}{n-1}.
\]
This bound is sharp in the sense that a similar lower bound is true for the uniform measure on a ball (recall Proposition 5.6).

When the radius is less than $n/2$ this result is better than [25]. Actually we do not need $\mu$ to be isotropic contrary to [25].

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