Mass relations in noncommutative geometry revisited
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Abstract: We generalize the notion of the ‘noncommutative coupling constant’ given by Kastler and Schücker by dropping the constraint that it commute with the Dirac-operator. This leads essentially to the vanishing of the lower bound for the Higgs mass and of the upper bound for the W mass. Thus it can be concluded that these bounds stem from the equal weighting of right- and left-handed fermions.
1 Introdution

One of the interesting features of Connes’ version of the standard model [1] is the appearance, at the classical level, of mass relations between gauge boson, scalar and fermion masses. In particular one obtains a prediction for the Higgs mass as a function of the top mass with only a small conceptual uncertainty (‘fuzzyness’) [2]. However, these results depend on a particular definition of the action, in contrast to other predictions (e.g. the occurrence of the Higgs field or the absence of massive neutrinos), which follow directly from the axioms for spectral triples. For instance the spectral action defined in [4] leads to values for the Higgs mass which differ from the prediction cited above.

The latter result was obtained from the noncommutative Yang-Mills-action $\mathcal{L}_{YM} = (F,F)$, where $(\cdot,\cdot)$ denotes a gauge invariant scalar product on the space of two-forms. In this article we shall generalize the definition of $(\cdot,\cdot)$ as compared to [2]. Using this generalized scalar product one only finds an upper bound for the Higgs mass instead of a prediction. We hope that this (more general) point of view leads to a better, physical understanding of the mass relations, which arise in noncommutative Yang-Mills theories.

2 Noncommutative Yang-Mills theories

The formulation of Yang-Mills-theories within noncommutative geometry is based on a spectral triple [1] $\{A,\mathcal{H},D,\Gamma,J\}$ that is constructed as the tensor product of a discrete spectral triple [4,8] $\{A_f,\mathcal{H}_f,D_f,\gamma_f,J_f\}$ and the spectral triple describing space-time $\{C^\infty(\mathcal{M}),L^2(\mathcal{M},S),\varphi_M,\gamma_5,C\}$. Here $L^2(\mathcal{M},S)$ denotes the Hilbert-space of square-integrable sections of the spin-bundle $S$ and $C = i\gamma_2\gamma_0$ is the operator corresponding to charge conjugation. Thus we have

\[
\begin{align*}
\mathcal{A} &= C^\infty(\mathcal{M}) \otimes A_f, \\
\mathcal{H} &= L^2(\mathcal{M},S) \otimes \mathcal{H}_f, \\
J &= C \otimes J_f, \\
\Gamma &= \gamma_5 \otimes \gamma_f, \\
D &= \varphi_M \otimes 1 + \gamma_5 \otimes D_f.
\end{align*}
\]
The representation $\pi(\mathcal{A})$ on $\mathcal{H}$ induces a representation of the universal differential algebra $\Omega(\mathcal{A})$ as bounded operators on $\mathcal{H}$:

$$
\pi : \Omega(\mathcal{A}) \rightarrow \mathcal{B}(\mathcal{H})
$$

$$
a_0 da_1 \cdots da_n \mapsto \pi(a_0) [\mathcal{D}, \pi(a_1)] \cdots [\mathcal{D}, \pi(a_n)] .
$$

The differential algebra belonging to the spectral triple, $\Omega_D(\mathcal{A})$, is then constructed from $\Omega(\mathcal{A})$ by dividing out the ideal $\mathcal{K}$ generated by $\ker \pi \cup d(\ker \pi)$. This raises a technical difficulty since $\Omega_D(\mathcal{A})$ consists of equivalence classes and therefore it is quite nontrivial to compute its algebraic structure or, at least, its structure as an $\mathcal{A}$-bimodule. Following [2] this problem is resolved as follows.

Being a subspace of $\mathcal{B}(\mathcal{H})$, $\pi(\Omega(\mathcal{A}))$ has a natural class of bilinear forms:

$$(A, B)_z := \text{Tr}_\omega(z A^* B |\mathcal{D}|^{-4}),$$

where, a priori, $z$ can be any bounded operator. To avoid technical difficulties we restrict $z$ to be of the form $1 \otimes z_f$. (In this case one will recover the classical Yang-Mills-Higgs-action from the action functional that will be defined below. We have not examined whether a more general ansatz for $z$ could also lead to this action.) $(\cdot, \cdot)_z$ will then be a nondegenerate scalar product on $\pi(\Omega(\mathcal{A}))$ if and only if the matrix $z_f$ is selfadjoint and strictly positive (i.e. $\ker z_f = 0$). Taking the representative which is orthogonal to $\mathcal{K}$, it becomes possible to identify $\Omega_D(\mathcal{A})$ as a subspace of $\pi(\Omega(\mathcal{A}))$. This provides a representation $\tilde{\pi}$ of $\Omega_D(\mathcal{A})$ as an $\mathcal{A}$-bimodule as well as a scalar product $(\cdot, \cdot)_z$ on it.

Using $\tilde{\pi}$ one can now define the Yang-Mills-action

$$\mathcal{L}_{YM} = (F, F)_z$$

where the curvature $F \in \tilde{\pi}(\Omega^2_D(\mathcal{A}))$ is defined as usual $F = dA + A^2$, and the gauge potential $A$ is a selfadjoint element of $\tilde{\pi}(\Omega^1_D(\mathcal{A})) = \pi(\Omega^1(\mathcal{A}))$. $\mathcal{L}_{YM}$ will be invariant under the gauge transformations

$$A \mapsto \pi(u) A \pi(u^*) + \pi(u) \pi(du^*) \quad u \in \mathcal{U} = \{ u \in \mathcal{A} : uu^* = u^* u = 1 \}$$

A sufficient condition for this construction to be well-defined is that $\pi(\mathcal{K})$ is closed in the closure of $\pi(\Omega(\mathcal{A}))$. It is easy to check, however, that there are no serious problems in the case at hand.

We should mention that, in general, this representation will fail to be a representation of the algebra $\Omega_D(\mathcal{A})$ and even more so of the differential algebra.
if the matrix $z_f$ commutes with $\pi(\mathcal{A})$. We shall also require that it commutes with $\pi^o(\mathcal{A}) := J\pi(\mathcal{A})^*J^{-1}$. This makes the parametrization of the matrix $z_f$ easier, but it does not restrict $\mathcal{L}_{YM}$: according to the general classification of finite spectral triples $\mathfrak{H}_f$ is given as the direct sum of representation spaces for $\mathcal{A} \otimes \mathcal{A}^o$. Each of these spaces is a tensor product, say $V \otimes V^o$, such that $\pi(\mathcal{A})$ and $\pi(\mathcal{O}_2(\mathcal{A}))$ act as $1$ on $V^o$. $[z_f, \pi^o(\mathcal{A})] = 0$ means then that the restriction of $z_f$ to one of these spaces is of the form $\zeta \otimes 1$.

Recall that the gauge potential $A$ describes both the vector bosons and the scalar particles of the theory and that $\mathcal{L}_{YM}$ gives the complete bosonic part of the action, possibly including symmetry breaking terms. To define the fermionic action one uses the so-called Majorana-spinors, i.e. the elements of the subspace:

$$\mathcal{H}_{real} = \{ \psi \in \mathcal{H} : J\psi = \psi \}.$$  

With the notation $\langle \cdot, \cdot \rangle$ for the scalar product in $\mathcal{H}$ one sets

$$\mathcal{L}_{fermions} := \langle \psi, (\mathcal{D} + A + JAJ^{-1})\psi \rangle \quad \psi \in \mathcal{H}_{real}.$$  

$\mathcal{D}_f$ can then be interpreted in terms of fermion masses and mixing angles which are completely arbitrary unless the choice of $\mathcal{D}_f$ is restricted by an additional principle. With our ansatz $z = 1 \otimes z_f$ the coupling constants, gauge boson masses and scalar masses are determined by the selfadjoint, strictly positive matrix $z_f$, which has to commute with $\pi(\mathcal{A}_f)$ and the opposite representation of the algebra $\pi^o(\mathcal{A}_f)$. This also implies $[z_f, \gamma_f] = 0$ as $\gamma_f$ is an element of $\pi(\mathcal{A}) \otimes \pi^o(\mathcal{A})$.

In the series of papers the authors assumed also

$$[z_f, \mathcal{D}_f] = 0. \quad (1)$$

This assumption leads to the occurrence of lower and upper bounds for the $W$ as well as for the Higgs mass. In particular, the two bounds for the Higgs mass differ only by $34$ MeV and one has (for $m_t = 180 \pm 12$ GeV) the prediction:

$$m_H = 288 \pm 22\text{GeV}.$$  

However, there might be reasons to drop this condition. For instance, a symmetry requirement can lead to constraints for $z_f$ which are not compatible with (1).

In the next section we will show that this prediction disappears if one uses the generalized scalar product $(\cdot, \cdot)_z$, although there is still an upper bound for $m_H$ of about $380$ GeV.
3 The calculation of the standard model parameters

The standard model of elementary particle physics is obtained from the following spectral data. The real matrix algebra is chosen as

$$\mathcal{A}_f = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

and is represented on the space $$\mathcal{H}_f = \mathbb{C}^{90}$$ with the basis

$$u_R \ d_R \ u_L \ d_L \ e_R \ e_L \ \nu_L \ \nu^c_L$$

$$u^c_R \ d^c_R \ u^c_L \ d^c_L \ e^c_R \ e^c_L \ \nu^c_L .$$

Here $$u_R$$, for instance, represents the nine right-handed up-type quarks and $$u^c_R = J u_R$$ will be identified with their charge conjugate by the Majorana-condition $$J \psi = \psi$$. An element $$(\lambda, q, m) = a \in \mathcal{A}_f$$ acts on these basis elements as

$$\pi(a) :$$

$$u_R \mapsto \lambda u_R$$

$$d_R \mapsto \bar{\lambda} d_R$$

$$\left( \begin{array}{c} u_L \\ d_L \end{array} \right) \mapsto q \otimes 1_3 \otimes 1_{N_f} \left( \begin{array}{c} u_L \\ d_L \end{array} \right)$$

$$e_R \mapsto \bar{\lambda} e_R$$

$$\left( \begin{array}{c} \nu_L \\ e_L \end{array} \right) \mapsto q \otimes 1_{N_f} \left( \begin{array}{c} \nu_L \\ e_L \end{array} \right)$$

from the left and as

$$\pi^o(a) :$$

$$Q \mapsto (m^T \otimes 1_3 \otimes 1_{N_f}) Q$$

$$\ell \mapsto \bar{\lambda} \ell$$

from the right, where we have used $$Q$$ for quarks and $$\ell$$ for leptons. We refer the reader to [1], [2] or [4] for further details of this spectral triple. The most general matrix $$z_f$$ leading to a gauge invariant nondegenerate scalar product is parametrized by ten selfadjoint and strictly positive $$3 \times 3$$-matrices.
ζ_i, which act on the different families. Thus z_f acts on \( u_R \) by \( 1_3 \otimes \zeta_1 \), on \( d_R \) by \( 1_3 \otimes \zeta_2 \), on \( (u_L)^{aL} \) by \( 1_2 \otimes 1_3 \otimes \zeta_3 \) and similarly for the other particles.

Following exactly the lines of [2] but using the general scalar product \((\cdot, \cdot)_z\), one obtains the following expressions for the coupling constants:

\[
\begin{align*}
g_{\text{strong}} &= \left[ \text{tr}(\zeta_6 + \zeta_7 + 2\zeta_9) \right]^{-\frac{1}{2}} \\
g_{\text{weak}} &= \left[ \text{tr}(3\zeta_4 + \zeta_5) \right]^{-\frac{1}{2}} \\
g_{\text{hyper}} &= \left[ \frac{1}{2} \text{tr}(3\zeta_1 + 3\zeta_2 + \zeta_3 + \zeta_8 + 2\zeta_{10}) + \frac{1}{6}g_{\text{strong}}^{-2} \right]^{-\frac{1}{2}}.
\end{align*}
\]

(2)

From (3) and the usual definition \( \sin^2 \theta_w = \frac{g_{\text{weak}}^{-2}}{g_{\text{hyper}} + g_{\text{weak}}} \), it is immediately clear that we have the bound

\[
\sin^2 \theta_w \leq \left[ 1 + \frac{1}{6} \left( \frac{g_{\text{weak}}}{g_{\text{strong}}} \right)^2 \right]^{-1}.
\]

(3)

The matrices \( \zeta_6, \ldots, \zeta_{10} \) do not appear in the expressions for the W and Higgs mass so that \( g_{\text{strong}} \) and \( g_{\text{hyper}} \) can be chosen independently of these two parameters. Before we present these expressions it is convenient to introduce the following shorthand notations: let \( m_i, i = 0, \ldots, 8 \) denote the fermion masses in decreasing order (i.e. \( m_0 = m_t \)), we set

\[
\begin{align*}
a_i &:= \frac{m_i^2}{m_W^2}, \quad i = 0, \ldots, 8 \\
b_1 &:= \frac{m_t^2 + m_b^2}{m_W^2}, \quad b_2 := \frac{m_c^2 + m_s^2}{m_W^2}, \quad \ldots \quad b_6 := \frac{m_e^2}{m_W^2}.
\end{align*}
\]

(4)

(5)

For reasons that will become clear later on, we choose the parametrization of the matrices \( \zeta_1, \ldots, \zeta_5 \) as follows: the diagonal elements of \( 3\zeta_4 \) and \( \zeta_5 \) are denoted \( \nu_k, \quad k = 1, \ldots, 6 \), while those of \( \zeta_3, 3\zeta_1 \) and \( 3V_{CKM}^*\zeta_2 V_{CKM} \) are
denoted by \( \mu_i, \quad i = 0, \ldots, 8 \). The boson masses are then found to be

\[ g_{\text{weak}}^{-2} = \sum_k \nu_k \]  
\[ m^2_W = \frac{1}{2} \sum_k \nu_k \left[ \sum_i \mu_i (a_i m^2_W) + \sum_k \nu_k (b_k m^2_W) \right] \]  
\[ m^2_H = \frac{\sum_{i,j} \mu_i \mu_j (a_i - a_j)^2}{\sum_{i,k} \mu_i \nu_k} + \frac{\sum_{k,l} \nu_k \nu_l (b_k - b_l)^2}{\sum_{l,k} \nu_l \nu_k} \]  

Because all the parameters of the right hand side in (7) are positive one gets

\[ m^2_W < \infty. \]

Now, the problem is to find bounds for the Higgs mass (8) under the constraint

\[ 2 \sum_k \nu_k = \sum_i a_i \mu_i + \sum_k b_k \nu_k \]  

which is just a rephrasing of (4). Choosing \( \nu_0 = 1 \), \( \mu_0 = \frac{2 - b_6}{a_0} \) and all other parameters \( a_i \), \( 2 - b_k \) of the order \( \epsilon \) one sees from (8) that the Higgs mass comes out to be of the order \( \epsilon \). Therefore there exists no nontrivial lower bound. To find the lowest upper bound under the constraint (9) is more difficult, but it is straightforward to prove the following estimate. Taking into account the experimental values for the fermion masses one has the relations \( (b_2 - b_6)^2 = \min_{k,l>2} (b_k - b_l)^2 \) and \( \frac{(a_0 - a_8)^2}{a_0 + a_8} = \max_{i,j} \frac{(a_i - a_j)^2}{a_i + a_j} \), from which one obtains

\[ \frac{\sum_{i,j} \mu_i \mu_j (a_i - a_j)^2}{\sum_{i,k} \nu_k \mu_i} \leq \frac{(2 - b_6) 16 (a_0 + a_1) (a_0 - a_8)^2}{(8 (a_0 + a_1) + a_8) (a_0 + a_8)} \]
\[ \frac{\sum_{k,l} \nu_k \nu_l (b_k - b_l)^2}{\sum_{l,k} \nu_l \nu_k} \leq \frac{(b_1 - b_6)^2}{2 - \eta^2}, \quad \eta = \frac{(b_2 - b_6) ^2}{b_1 - b_6}. \]

\( ^4 \) with some suitable ordering of the indices \( k, i \).
As stated above these estimates do not provide inf\(_{\mu_i\nu_k}\)(m^2_H). A suitable choice of the parameters \(\mu_i, \nu_k\) leads, however, to values of the Higgs mass which are very close to the upper bound obtained so far. In a more transparent form our result can thus be stated as

\[
0 < m^2_H < \frac{17}{4} m_t^2 + O(m_b^2),
\]

where we have only retained the top mass contribution.

4 Conclusions

In [2] the authors have obtained upper and lower bounds for the W-mass as well as for the Higgs mass by using a scalar product on \(\Omega_D(A)\) that was restricted to fulfill the condition \([z, D] = 0\). We have shown that the upper bound for the W-mass and the lower bound for the Higgs mass disappear if one uses a more general scalar product. Now, with our notation, the above condition leads to the equations

\[
\begin{align*}
\zeta_1 &= \zeta_2 = \zeta_4, & \zeta_3 &= \zeta_5 \\
\zeta_6 &= \zeta_7 = \zeta_9, & \zeta_8 &= \zeta_{10}
\end{align*}
\]

which are due to the fact that \(D_f\) maps the right-handed fermions to their left-handed partners. In addition all the matrices \(\zeta_i\) have to be chosen diagonal and the matrices \(\zeta_2, \zeta_7\) have also to be proportional to \(1_3\) (if one assumes that the CKM-matrix is nondegenerate). It is clearly the requirement (10) which leads to the appearence of a lower bound for the Higgs mass as compared to our result. We can state that the additional bounds found in [2] are a consequence of the equal weighting of particles of different chirality. The other restrictions on the matrices \(\zeta_i\) have only a numerical effect for the bounds we have obtained, but do not lead to a qualitatively different result.

Our last remark concerns the relation for \(\sin \theta_w\) coming from (8). It is possible to generalize the Dirac-operator by taking e.g.

\[
D_\theta := (\sigma_M \otimes 1 + \gamma_5 \otimes D_f) \cdot (1 \otimes \theta),
\]

if the matrix \(\theta\) fulfills certain conditions which come from the axioms for spectral triples. A possible choice of \(\theta\) would be the diagonal matrix that multiplies right-handed fermions by \(\sin \theta\) and left-handed fermions by \(\cos \theta\).
with $\theta \in \mathbb{R}$. Obviously the additional parameter $\theta$ would be sufficient to fix $\sin \theta_w$ arbitrary. A similar result ( in a different model) was also noted in [7]. One should mention that there are other deriviations of the classical standard model Lagrangean from noncommutative geometry, which do not start from a spectral triple [8][1][9], see also [10] for a short review. In particular, Wulkenhaar obtains results which are quite similar to those of [2].

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References

[1] A.Connes, Gravity coupled with matter and the foundation of noncommutative geometry, Comm.Math.Phys,182 (1996), 155

[2] D.Kastler, T.Schücker, The standard model à la Connes-Lott, CPT-94/P.3091, hep-th/9412187
D.Kastler, T.Schücker, A detailed account of Alain Connes’ version of the standard model in non-commutative differential geometry, Rev.Math.Phys.8(1996)205-228
B.Iochum, D.Kastler, T.Schücker, Fuzzy mass relations in the standard model CPT-95/P.3235, hep-th/9507150
L.Carminati, B.Iochum, T.Schücker, The noncommutative constraints on the standard model à la Connes CPT-96/P.3307, hep-th/9604169

[3] M.Paschke, A.Sitarz, Discrete spectral triples and their symmetries, q-alg/9612029

[4] M.Paschke, A.Sitarz, Standard Model parameters and interaction from noncommutative geometry, in preparation

[5] A.Connes, A.Chamseddine, The Spectral Action Principle, hep-th/9606001

[6] T.Krajewski, Classification of finite spectral triples, CPT-96/P.3409, hep-th/9701081
[7] A.Sitarz, *Higgs mass and noncommutative geometry*, Phys. Lett. B **308** (1993) 311-314

[8] R.Wulkenhaar, *The standard model within nonassociative geometry*, hep-th/9607096
R.Wulkenhaar, *The mathematical footing of nonassociative geometry*, hep-th/9607094

[9] M.Dubois-Violette,R.Kerner, J.Madore, J.Math.Phys **31**(2) (1990) 323

[10] F.Scheck, *The standard model within noncommutative geometry: a comparison of models MZ 97-01*, hep-th/9701073

[11] N.A.Papadopoulos, J.Plass, *Natural extensions of the Connes-Lott model and comparison with the Mainz-Marseille model* hep-th/9605072