Representation theory for the Jordanian quantum algebra $U_h(sl(2))$ is developed using a nonlinear relation between its generators and those of $sl(2)$. Closed form expressions are given for the action of the generators of $U_h(sl(2))$ on the basis vectors of finite dimensional irreducible representations. In the tensor product of two such representations, a new basis is constructed on which the generators of $U_h(sl(2))$ have a simple action. Using this basis, a general formula is obtained for the Clebsch-Gordan coefficients of $U_h(sl(2))$. Some remarkable properties of these Clebsch-Gordan coefficients are derived.

1 Introduction

The group $GL(2)$ admits, up to isomorphism, only two quantum group deformations with central determinant: $GL_q(2)$ and $GL_h(2)$, see [1]. The quantum group $GL_q(2)$ has been well studied, being the prototype example for many works on quantum groups. Investigations of the Jordanian quantum group $GL_h(2)$, or $SL_h(2)$, and its dual quantum algebra $U_h(sl(2))$ started more recently. Its defining relations were given in [2,3], and a construction of the dual Hopf algebra in [4]. Recently, also for the 2-parameter Jordanian quantum group $GL_{g,h}(2)$ its dual was constructed [5].

For a development of its differential calculus or differential geometry we refer to [6] and [7]. A construction of the universal $R$-matrix was given in [8,9,10].

In this paper we are primarily interested in the irreducible finite dimensional representations of $U_h(sl(2))$. Also here, there has been progress in recent years. In [11], a direct construction of these representations was given by factorising the Verma module. An important development was given by Abdesselam et al [12]: they gave a nonlinear relation between the generators of $U_h(sl(2))$ and the classical generators of $sl(2)$. As a consequence they obtained expressions for the action of the generators of $U_h(sl(2))$ on basis vectors of the finite dimensional irreducible representations. These expressions were not always in closed form, and this was solved in [13]. In [14], finite and infinite dimensional representations of $U_h(sl(2))$ are constructed, and for the first time the tensor product of two representations is
considered, yielding some examples of Clebsch-Gordan coefficients. The problem of determining Clebsch-Gordan coefficients was then completely solved in [13].

In the present paper we shall discuss a number of interesting properties of the Clebsch-Gordan coefficients of $\mathcal{U}_h(\text{sl}(2))$, after recalling some of the main results of [13].

2 SL$_h(2)$ and $\mathcal{U}_h(\text{sl}(2))$

Consider the bialgebra $A_h(2)$ with parameter $h$ and four generators $a, b, c, d$ subject to the relations:

\[
ba = ab - ha^2 + hD \quad ca = ac + hc^2 \\
da = ad + hdc - hac \quad bd = db - hd^2 + hD \\
cd = dc + hc^2 \quad cb = bc + hdc + h^2c^2
\]

where $D = ad - bc - hac$. It is easy to verify the $D$ is central. With $t = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, there is a comultiplication given by $\Delta(t) = t \otimes t$, and a co-unit $\epsilon(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, turning $A_h(2)$ into a coalgebra. The element $D$ is group-like, so one can extend $A_h(2)$ by $D^{-1}$, and then an antipode $S$ can be defined leading to the Hopf algebra $GL_h(2)$. Putting $D = 1$ gives rise to the matrix quantum group $SL_h(2)$, see [11].

The dual Hopf algebra of SL$_h(2)$ is denoted by $\mathcal{U}_h(\text{sl}(2))$. It is an associative algebra generated by $H, Y, T$ and $T^{-1}$ satisfying quadratic relations [4]. For us it is more convenient to work with $X = (\log T)/h$, i.e. $T = e^{hX}$ and $T^{-1} = e^{-hX}$. Then the relations read:

\[
[H, X] = 2 \sinh \frac{hX}{h}, \quad [X, Y] = H, \\
[H, Y] = -Y(\cosh hX) - (\cosh hX)Y.
\]

The comultiplication is given by:

\[
\Delta(H) = H \otimes e^{hX} + e^{-hX} \otimes H, \\
\Delta(X) = X \otimes 1 + 1 \otimes X, \\
\Delta(Y) = Y \otimes e^{hX} + e^{-hX} \otimes Y.
\]

The other ingredients (co-unit, antipode) are also defined, but not needed here.

3 Relation between $\mathcal{U}_h(\text{sl}(2))$ and $\text{sl}(2)$, and representations

With the following definition [12]

\[
Z_+ = \frac{2}{h} \tanh \frac{hX}{2}, \\
Z_- = (\cosh \frac{hX}{2})Y(\cosh \frac{hX}{2}),
\]

The matrix $Z_+$ is a generator of the quantum group $SU_q(2)$.
it follows that the elements \( \{ H, Z_+, Z_- \} \) satisfy the commutation relations of a classical \( sl(2) \) basis:

\[
[H, Z_\pm] = \pm 2Z_\pm, \quad [Z_+, Z_-] = H.
\]

These relations can be inverted, e.g.

\[
eh X = (1 + \frac{h}{2} Z_+)(1 - \frac{h}{2} Z_+)^{-1}.
\]

These relations can also be used to give explicit matrix elements for the finite dimensional representations of \( U_h(sl(2)) \).

Recall that finite dimensional irreducible representations of \( sl(2) \) are labeled by a number \( j \), with \( 2j \) a non-negative integer. The representation space can be denoted by \( V^{(j)} \) with basis \( e^{j}_m (m = -j, -j + 1, \ldots, j) \), and the action is

\[
He^{j}_m = 2m e^{j}_m, \\
Z_+ e^{j}_m = \sqrt{(j + m)(j + m + 1)} e^{j}_{m+1}.
\]

For us, a more convenient basis for computations is the following \( v \)-basis related to the above \( e \)-basis by:

\[
v^{j}_m = \alpha^{j,m} e^{j}_m, \quad \text{with } \alpha^{j,m} = \sqrt{(j + m)!/(j - m)!}.
\]

The \( sl(2) \) matrix elements in this basis are:

\[
Hv^{j}_m = 2m v^{j}_m, \\
Xv^{j}_m = \frac{\lfloor (j - m + 1)/2 \rfloor}{2k + 1} v^{j}_{m+1+2k}, \\
Yv^{j}_m = (j + m)(j - m + 1)v^{j}_{m+1} - (j - m)(j + m + 1) \left( \frac{h}{2} \right)^2 v^{j}_{m+1} \\
+ \sum_{s=1}^{\lfloor (j - m + 1)/2 \rfloor} \left( \frac{h}{2} \right)^{2s} v^{j}_{m+1+2s}.
\]

**Proposition 1** The action of the generators of \( U_h(sl(2)) \) on the representation space \( V^{(j)} \) is given by

\[
Hv^{j}_m = 2m v^{j}_m, \\
Xv^{j}_m = \frac{\lfloor (j - m + 1)/2 \rfloor}{2k + 1} v^{j}_{m+1+2k}, \\
Yv^{j}_m = (j + m)(j - m + 1)v^{j}_{m+1} - (j - m)(j + m + 1) \left( \frac{h}{2} \right)^2 v^{j}_{m+1} \\
+ \sum_{s=1}^{\lfloor (j - m + 1)/2 \rfloor} \left( \frac{h}{2} \right)^{2s} v^{j}_{m+1+2s},
\]
It should be noted that the matrix elements of $X$ were already obtained in [12]. Those of $Y$ were also determined in [12], however not in closed form but as a complicated sum. In [13] we showed how such sums can be reduced to a simple form, using recently developed algorithms [15]. Proposition 1 is easy to apply and gives immediately all matrix elements of the $U_h(\text{sl}(2))$ generators. For example, the representatives for $X$ and $Y$, respectively, in the $v$-basis for $j = 2$ are given by:

$$\begin{pmatrix}
0 & 1 & 0 & h^2/12 & 0 \\
0 & 0 & 1 & 0 & h^2/12 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad \begin{pmatrix}
0 & -3h^2/4 & 0 & h^4/16 & 0 \\
4 & 0 & -5h^2/4 & 0 & h^4/16 \\
0 & 6 & 0 & -5h^2/4 & 0 \\
0 & 0 & 6 & 0 & -3h^2/4 \\
0 & 0 & 0 & 4 & 0
\end{pmatrix},$$

with $H$ given by the usual matrix $\text{diag}(4, 2, 0, -2, -4)$. Note that the sl(2) representatives in the $v$-basis are recovered simply by putting $h = 0$.

### 4 Tensor product of $U_h(\text{sl}(2))$ representations

Consider $V^{(j_1)} \otimes V^{(j_2)}$ with basis $v^j_{m_1} \otimes v^j_{m_2}$. Our purpose is to show that this decomposes into the direct sum of representations $V^{(j)}$, $j = |j_1 - j_2|, \ldots, j_1 + j_2$. Note that the vectors $v^j_{m_1} \otimes v^j_{m_2}$ are in general no eigenvectors of $\Delta(H)$, since the comultiplication is given by:

$$\Delta(H) = H \otimes e^{hX} + e^{-hX} \otimes H = H \otimes 1 + 1 \otimes H + 2H \otimes \sum_{n=1}^{\infty} \left( \frac{hZ_+}{2} \right)^n + \sum_{n=1}^{\infty} \left( \frac{-hZ_+}{2} \right)^n \otimes 2H. \quad (8)$$

The eigenvectors of $\Delta(H)$ are linear combinations of the vectors $v^j_{m_1} \otimes v^j_{m_2}$, and the coefficients play a crucial role in this work. To define these coefficients, recall the definition of the Pochhammer symbol:

$$(a)_n = \begin{cases} 
a(a + 1) \cdots (a + n - 1) & \text{if } n = 1, 2, \ldots; \\
1 & \text{if } n = 0. \end{cases} \quad (9)$$

Next we define

$$b_{k,l}^{m_1,m_2} = \begin{cases} 
\frac{(-2m_1 - k)(-2m_2 - l)_k}{k!!} & \text{if } k \geq 0 \text{ and } l \geq 0; \\
0 & \text{otherwise,} \end{cases} \quad (10)$$

and finally the essential coefficients:

$$a_{k,l}^{m_1,m_2} = (-1)^k(h/2)^k(l) b_{k,l}^{m_1,m_2} - b_{k-1,l-1}^{m_1,m_2}. \quad (11)$$

Then we have the following important result:
Proposition 2 In $V^{(j_1)} \otimes V^{(j_2)}$, the vectors

$$w_{m_1, m_2}^{j_1, j_2} = \sum_{k=0}^{j_2-m_2} \sum_{l=0}^{j_1-m_1} a_{k,l}^{m_1,m_2} v_{m_1+k}^{j_1} \otimes v_{m_2+l}^{j_2}$$  \hspace{1cm} (12)$$

form a basis consisting of eigenvectors of $\Delta(H)$. The explicit action of $\Delta(H)$, $\Delta(X)$ and $\Delta(Y)$ is given by

$$\Delta(H) w_{m_1, m_2}^{j_1, j_2} = 2(m_1 + m_2) w_{m_1, m_2}^{j_1, j_2},$$

$$\Delta(Z_+) w_{m_1, m_2}^{j_1, j_2} = w_{m_1+1, m_2}^{j_1, j_2} + w_{m_1, m_2+1}^{j_1, j_2},$$

$$\Delta(Z_-) w_{m_1, m_2}^{j_1, j_2} = (j_1 + m_1)(j_1 - m_1 + 1) w_{m_1-1, m_2}^{j_1, j_2} + (j_2 + m_2)(j_2 - m_2 + 1) w_{m_1, m_2-1}^{j_1, j_2}.$$  \hspace{1cm} (13)$$

Remark 3 This proposition tells us that the action of $\Delta(H)$, $\Delta(X)$ and $\Delta(Y)$ on the $w$-vectors is the same as the action of the $su(2)$ generators (under the trivial Lie algebra comultiplication) on the uncoupled vectors $v_{m_1}^{j_1} \otimes v_{m_2}^{j_2}$. This observation implies the results on the tensor product decomposition and Clebsch-Gordan coefficients for $U_h(sl(2))$. In particular, the Clebsch-Gordan coefficients for $U_h(sl(2))$ are essentially given by linear combinations of $su(2)$ Clebsch-Gordan coefficients, with $a_{k,l}^{m_1,m_2}$ the coefficients of this linear combination.

Let us first consider an example, say $V^{(1)} \otimes V^{(1/2)}$. Using the formulas (10)-(12), the $w$-vectors are explicitly given by

$$\begin{pmatrix}
 w_{-1,-1/2}^{1/2} \\
 w_{-1,1/2}^{1/2} \\
 w_{0,-1/2}^{1/2} \\
 w_{0,1/2}^{1/2} \\
 w_{1,-1/2}^{1/2} \\
 w_{1,1/2}^{1/2} \\
 w_{1,1/2}^{1/2} \\
 w_{1,1/2}^{1/2}
\end{pmatrix} = \begin{pmatrix}
 1 & h & -h/2 & h^2/4 & h^2/4 & -h^3/8 \\
 0 & 1 & 0 & h/2 & 0 & 0 \\
 0 & 0 & 1 & 0 & -h/2 & h^2/4 \\
 0 & 0 & 0 & 1 & 0 & h/2 \\
 0 & 0 & 0 & 0 & 1 & -h \\
 0 & 0 & 0 & 0 & 0 & h \\
 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
 v_{-1}^{1} \otimes v_{-1/2}^{1/2} \\
 v_{-1}^{1} \otimes v_{1/2}^{1/2} \\
 v_{0}^{1/2} \otimes v_{-1/2}^{1/2} \\
 v_{0}^{1/2} \otimes v_{1/2}^{1/2} \\
 v_{1}^{1} \otimes v_{-1/2}^{1/2} \\
 v_{1}^{1} \otimes v_{1/2}^{1/2}
\end{pmatrix}. $$

It is easy to verify that the inverse of the above upper-triangular matrix is given by reflecting the matrix along its second diagonal, i.e. by its skew-transpose:

$$\begin{pmatrix}
 1 & -h & h/2 & h^2/4 & 0 & -h^3/8 \\
 0 & 1 & 0 & -h/2 & 0 & h^2/4 \\
 0 & 0 & 1 & 0 & h/2 & h^2/4 \\
 0 & 0 & 0 & 1 & 0 & -h/2 \\
 0 & 0 & 0 & 0 & 1 & h \\
 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}. $$

This turns out to be a general property of these matrices of $a_{k,l}^{m_1,m_2}$ coefficients. In other words, we have
Proposition 4 The coefficients $a_{k,l}^{m_1,m_2}$ satisfy

$$
\sum_{n_1,n_2} a_{n_1-m_1,n_2-m_2}^{m_1,m_2} a_{-M_1,-M_2}^{-M_1,-M_2} = \delta_{m_1,M_1} \delta_{m_2,M_2}.
$$

(14)

Note that the above formula is nontrivial only for $M_1 \geq m_1$ and $M_2 \geq m_2$, otherwise the indices of the $a$-coefficients are negative and thus automatically zero. The above property follows from the following remarkable identity holding for arbitrary parameters $x$ and $y$:

$$
\sum_{k=0}^{K} \sum_{l=0}^{L} \frac{(-x-k)(-y-l)_{k!} (x+K+k)(y+L+l)_{K-k}}{(K-k)! (L-l)!} \frac{(x+k)(y+l)}{(x+k)(y+l)} \delta_{K,0} \delta_{L,0},
$$

(15)

by putting $x = 2m_1$, $y = 2m_2$, $K = M_1 - m_1$ and $L = M_2 - m_2$. The proof of (15) falls beyond the scope of the present paper.

5 Clebsch-Gordan coefficients and properties

From Remark 3 it is easy to deduce that the decomposition of the tensor product is given by

$$
V^{(j_1)} \otimes V^{(j_2)} = \bigoplus_{j=|j_1-j_2|} V^{(j)},
$$

and we have

Proposition 5 The Clebsch-Gordan coefficients for $\mathcal{U}_h(\text{sl}(2))$, in

$$
e^{(j_1,j_2)}_m = \sum_{n_1,n_2} C_{n_1,n_2,m}^{j_1,j_2,j} e^{j_1}_{n_1} \otimes e^{j_2}_{n_2},
$$

are given by

$$
C_{n_1,n_2,m}^{j_1,j_2,j} (h) = \sum_{m_1+m_2=m} C_{m_1,m_2,m}^{j_1,j_2,j} A_{m_1,m_2}^{m_1,m_2},
$$

with $C_{m_1,m_2,m}^{j_1,j_2,j}$ the usual $\text{su}(2)$ Clebsch-Gordan coefficients, and $A_{m_1,m_2}^{m_1,m_2}$ determined by

$$
A_{k,l}^{m_1,m_2} = a_{k,l}^{m_1,m_2} \frac{\alpha_{j_1+m_1+k} \alpha_{j_2,m_2+l}}{\alpha_{j_1,m_1} \alpha_{j_2,m_2}}.
$$

So apart from the $\alpha$-factors (which appear here because we have formulated the proposition in the $e$-basis rather than in the $v$-basis), the Clebsch-Gordan matrix is essentially the product of the corresponding $\text{su}(2)$ Clebsch-Gordan matrix with the upper triangular matrix of $a$-coefficients considered in the previous section.

From the explicit form of the $a$-coefficients, and Proposition 5, it follows that
Proposition 6 The Clebsch-Gordan coefficients of $U_h(sl(2))$ satisfy

- if $m = n_1 + n_2$ then $C_{j_1,j_2,j}^{j_1,j_2,j}(h) = C_{n_1,n_2,m}^{j_1,j_2,j}(h)$;
- if $m > n_1 + n_2$ then $C_{j_1,j_2,j}^{j_1,j_2,j}(h) = 0$;
- if $m < n_1 + n_2$ then $C_{j_1,j_2,j}^{j_1,j_2,j}(h)$ is a monomial in $h^{n_1+n_2-m}$.

The most interesting property follows from Proposition 4:

Proposition 7 The Clebsch-Gordan coefficients of $U_h(sl(2))$ satisfy the skew-orthogonality relations

$$
\sum_{n_1,n_2} (-1)^{j_1+j_2-j} C_{n_1,n_2,m}^{j_1,j_2,j}(h) C_{-n_1,-n_2,-m}^{j_1,j_2,j'}(h) = \delta_{j,j'}\delta_{m,m'},
$$

$$
\sum_{j,m} (-1)^{j_1+j_2-j} C_{j_1,j_2,j}^{j_1,j_2,j}(h) C_{n_1,n_2,m}^{j_1,j_2,j}(h) = \delta_{n_1,n_1'}\delta_{n_2,n_2'}.
$$

This property gives in fact the inverse matrix of a general Clebsch-Gordan matrix of $U_h(sl(2))$. The proof is as follows: recall that the Clebsch-Gordan matrix of $U_h(sl(2))$ is essentially the product of an upper-triangular matrix of $a$-coefficients with an $su(2)$ Clebsch-Gordan matrix. But the upper-triangular matrix has an easy inverse, namely its skew-transpose; and also the $su(2)$ Clebsch-Gordan matrix has an easy inverse, namely its transpose (since it is orthogonal). This, and some symmetry properties of $su(2)$ Clebsch-Gordan coefficients, leads to Proposition 7.

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