A short proof of the zero-two law for cosine functions

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Abstract: Let \((C(t))_{t \in \mathbb{R}}\) be a cosine function in a unital Banach algebra. We give a simple proof of the fact that if \(\lim sup_{t \to 0} \|C(t) - 1_A\| < 2\), then \(\lim sup_{t \to 0} \|C(t) - 1_A\| = 0\).

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1 Introduction

Recall that a cosine function taking values in a unital normed algebra \(A\) with unit element \(1_A\) is a family \(C = (C(t))_{t \in \mathbb{R}}\) of elements of \(A\) satisfying the so-called d'Alembert equation

\[
C(0) = 1_A, \quad C(s + t) + C(s - t) = 2C(s)C(t) \quad (s, t \in \mathbb{R}).
\]

One can define in a similar way cosine sequences \((C(n))_{n \in \mathbb{Z}}\). A cosine sequence depends only on the value of \(C(1)\), since we have, for \(n \geq 1\),

\[
C(-n) = C(n) = T_n(C(1)),
\]

where \(T_n(x) = \sum_{k=0}^{[n/2]} C_n^k x^{n-2k} (x^2 - 1)^k\) is the \(n\)-th Tchebychev polynomial.

Strongly continuous operator valued cosine functions play an important role in the study of abstract nonlinear second order differential equations, see for example [11]. In a paper to appear in the Journal of Evolution Equations [9], Schwenninger and Zwart showed that if a strongly continuous cosine family \((C(t))_{t \in \mathbb{R}}\) of bounded operators on a Banach space \(X\) satisfies \(\lim sup_{t \to 0} \|C(t) - I_X\| < 2\), then the generator \(a\) of this cosine function is a bounded operator, so that \(\lim sup \|C(t) - I_X\| = \lim sup_{t \to 0} \|\cos(ta)I_X - I_X\| = 0\), and they asked whether a similar zero-two law holds for general cosine functions \((C(t))_{t \in \mathbb{R}}\). This
question was answered positively by Chojnacki in [5]. Using a sophisticated argument based on ultrapowers, Chojnacki deduced this zero-two law from the fact that if a cosine sequence \( C(t) \) satisfies \( \sup_{t \in \mathbb{R}} \| C(t) - 1_A \| < 2 \), then \( C(t) = 1_A \) for \( t \in \mathbb{R} \). This second result, which was obtained independently by the author in [7], is proved by Chojnacki in [5] by adapting methods used by Bobrowski, Chojnacki and Gregosiewicz in [3] to show that if a cosine sequence \( (C(t))_{t \in \mathbb{R}} \) satisfies \( \sup_{t \in \mathbb{R}} \| C(t) - \cos(at)1_A \| \leq \frac{3}{2\sqrt{3}} \) for some \( a \in \mathbb{R} \), then \( C(t) = \cos(at)1_A \) for \( t \in \mathbb{R} \), a result also obtained independently by the author in [7], which improves previous results of [2], [4] and [10].

The purpose of this paper is to give a short direct proof of the zero-two law. The zero-two law for complex-valued cosine functions is a folklore result, which easily implies that if \( \limsup_{t \to 0} \rho(C(t) - 1_A) < 2 \) then \( \lim sup_{t \to 0} \rho(C(t) - 1_A) = 0 \), where \( \rho(x) \) denotes the spectral radius of an element \( x \) of a Banach algebra, see section 2. Our proof of the zero-two law is then based on the fact that if \( \| C(t) - 1_A \| \leq 2 \), and if \( \rho \left( \frac{1}{2} - 1_A \right) < 1 \), then we have

\[
C \left( \frac{t}{2} \right) = \sqrt{1_A - \frac{C(t) - 1_A}{2}},
\]

where \( \sqrt{1_A - u} \) is defined by the usual series for \( \| u \| \leq 1 \). It follows from this identity and from the fact that the coefficients of the Taylor series at the origin of the function \( t \to 1 - \sqrt{1 - t} \) are positive that in this situation we have

\[
\left\| C \left( \frac{t}{2} \right) - 1_A \right\| \leq 1 - \sqrt{1 - \frac{C(t) - 1_A}{2}},
\]

and the zero-two law follows.

Notice that if we replace the constant 2 by \( \frac{3}{2} \) a "three line argument" due to Arendt [11] shows that if \( \limsup_{t \to 0} \| C(t) - 1_A \| < \frac{3}{2} \) then \( \limsup_{t \to 0} \| C(t) - 1_A \| = 0 \). The proof presented here has some analogy with Arendt's proof, and the difficulty to estimate \( \| (1_A + C(\frac{t}{2}))^{-1} \| \) is circumvented in the present paper by using the formula above.

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2 The zero-two law for the spectral radius

The zero-two law for scalar cosine functions pertains to folklore, but we could not find a reference in the literature for the following certainly well-known lemma, which is a variant of proposition 3.1 of [7].
Lemma 2.1. Let \( c = (c(t))_{t \in \mathbb{R}} \) be a complex-valued cosine function. Then \( c \) satisfies one of the following conditions

(i) \( \limsup_{t \to 0} |c(t) - 1| = +\infty \),

(ii) \( \limsup_{t \to 0} |c(t) - 1| = 2 \),

(iii) \( \limsup_{t \to 0} |c(t) - 1| = 0 \).

First assume that \( M := \limsup_{t \to 0} |c(t)| < +\infty \), and denote by \( S \) the set of all complex numbers \( \alpha \) for which there exists a sequence \( (t_m)_{m \geq 1} \) of positive real numbers such that \( \lim_{m \to +\infty} t_m = 0 \) and \( \lim_{m \to +\infty} c(t_m) = \alpha \). Then \( |\alpha| \leq M \) for every \( \alpha \in S \). Notice that if \( \alpha \in S \), and if a sequence \( (t_m)_{m \in \mathbb{Z}} \) satisfies the above conditions with respect to \( \alpha \), then \( T_n(\alpha) = \lim_{m \to +\infty} T_n(C(t_m)) = \lim_{m \to +\infty} C(nt_m) \), and so \( T_n(\alpha) \in S \), and \( |T_n(\alpha)| \leq M \) for \( n \geq 1 \). Now write \( \alpha = \cos(z) = \sum_{k=0}^{+\infty} \frac{z^k}{2k!} \), and set \( u = \Re(z), v = \Im(z) \). We have, for \( n \geq 1 \),

\[
T_n(\alpha) = \cos(nz) = \frac{e^{inu}e^{-nv} + e^{-inu}e^{nv}}{2}.
\]

Since \( \sup_{n \geq 1} |T_n(\alpha)| \leq M \), we have \( v = 0 \), \( S \subset [-1,1] \), and \( \limsup_{t \to 0} |c(t) - 1| \leq 2 \).

Now assume that \( S \neq \{1\} \), and let \( \alpha \in S \setminus \{1\} \). We have \( \alpha = \cos(u) \) for some \( u \in \mathbb{R} \). We see as above that \( \cos(nu) \in S \) for every \( n \geq 1 \). If \( u/\pi \) is irrational, then the set \( \{e^{inu}\}_{n \geq 1} \) is dense in the unit circle \( T \), and so \( S = [-1,1] \) since \( S \) is closed, and in this situation \( \limsup_{t \to 0} |c(t) - 1| = 2 \).

Now assume that \( u/\pi \) is rational, and let \( s \geq 2 \) be the smallest positive integer such that \( e^{isu} = 1 \). Then \( e^{2isu} = e^{ipu} \) for some positive integer \( p \), and so \( \cos \left( \frac{2isu}{s} \right) \) \( \in S \). Let \( (t_m)_{m \geq 1} \) be a sequence of positive reals such that \( \lim_{m \to +\infty} t_m = 0 \) and \( \lim_{m \to +\infty} c(t_m) = \cos \left( \frac{2isu}{s} \right) \), let \( q \geq 2 \), and let \( \beta \) be a limit point of the sequence \( c \left( \frac{2isu}{s} \right) \), where \( n \geq 1 \). There exists \( y \in \mathbb{R} \) such that \( \cos(y) = \beta \) and such that \( s^{q-1}y = \frac{2isu}{s} + 2k\pi \), with \( k \in \mathbb{Z} \). Then \( y = (1 + ks)\frac{2isu}{s} \). Since \( \gcd(1 + ks, s^{q-1}) = 1 \), there exists a positive integer \( r \) such that \( r\frac{2isu}{s} \in 2\pi\mathbb{Z} \), so that \( \cos \left( \frac{2isu}{s} \right) \) \( \in S \). This implies that \( \cos \left( \frac{2isu}{s} \right) \) \( \in S \) for \( p \geq 1, q \geq 1 \), and \( S \) is dense in \([-1,1]\). Since \( S \) is closed, we obtain again \( S = [-1,1] \), which implies that \( \limsup_{t \to 0} |c(t) - 1| = 2 \). So if neither (i) nor (ii) holds, we have \( S = \{1\} \), which implies (iii). □

Notice that if a cosine function \( C = (C(t))_{t \in \mathbb{R}} \) in a Banach algebra \( A \) satisfies \( \sup_{|t| \leq \eta} \|C(t)\| \leq M < +\infty \) for some \( \eta > 0 \), then \( \sup_{|t| \leq L} \|C(t)\| < +\infty \) for every \( L > 0 \), since \( \sup_{|t| \leq M} \|C(t)\| \leq \sup_{|y| \leq M} \|T_n(y)\| \) for every \( n \geq 1 \), where \( T_n \) denotes the \( n \)-th Tchebychev polynomial. In particular if a complex-valued cosine function \( c = (c(t))_{t \in \mathbb{R}} \) satisfies (iii), then the identity

\[
(1 - c(s - t))(1 - c(s + t)) = (c(s) - c(t))^2
\]
shows as is well-known that the cosine function \(c\) is continuous on \(\mathbb{R}\), which implies that \(c(t) = \cos(ta)\) for some \(a \in \mathbb{C}\).

If \(A\) is commutative and unital, we will denote \(1_A\) the unit element of \(A\), and we will denote by \(\hat{A}\) the space of all characters on \(A\), equipped with the Gelfand topology, i.e. the compact topology induced by the weak* topology on the unit ball of the dual space of \(A\).

**Proposition 2.2.** Let \(C = (C(t))\) be a cosine function in a unital Banach algebra \(A\). If \(\limsup_{t \to 0} \rho(C(t) - 1_A) < 2\), then \(\limsup_{t \to 0} \rho(C(t) - 1_A) = 0\).

**Proof:** We may assume that unital Banach algebra \(A\) is generated by \((C(t))_{t \in \mathbb{R}}\). Let \(\chi \in \hat{A}\). Then the cosine complex-valued function \((\chi(C(t)))_{t \in \mathbb{R}}\) satisfies condition (iii) of the lemma, and so there exists \(a_\chi \in \mathbb{C}\) such that we have

\[
\chi(C(t)) = \cos(ta_\chi) \quad (t \in \mathbb{R}).
\]

Set \(u_\chi = \text{Re}(a_\chi), v_\chi = \text{Im}(a_\chi).\) We have

\[
\rho(C(t) - 1) \geq |1 - \cos(tu_\chi) \cosh(tv_\chi)|.
\]

If the family \((u_\chi)_{\chi \in \hat{A}}\) were unbounded, there would exist a sequence \((t_n)_{n \geq 1}\) of real numbers converging to zero and a sequence \((\chi_n)_{n \geq 1}\) of characters of \(A\) such that \(\cos(t_n u_{\chi_n}) = -1\), and we would have \(\rho(C(t_n) - 1) \geq 2\) for \(n \geq 1\). So the family \((u_\chi)_{\chi \in \hat{A}}\) is bounded. If the family \((v_\chi)_{\chi \in \hat{A}}\) were unbounded, there would exist a sequence \((t'_n)_{n \geq 1}\) of real numbers converging to zero and a sequence \((\chi'_n)_{n \geq 1}\) of characters of \(A\) such that \(\lim_{n \to +\infty} \cosh(t'_n v_{\chi'_n}) = +\infty\). But this would imply that \(\limsup_{t \to 0} \rho(C(t)) = +\infty\). Hence the family \((a_\chi)_{\chi \in \hat{A}}\) is bounded, and we have

\[
\limsup_{t \to 0} \rho(C(t) - 1_A) = \lim_{t \to 0} \sup_{\chi \in \hat{A}} |\cos(ta_\chi) - 1| = 0.
\]

\(\square\)

### 3 The zero-two law for cosine functions

Set \(a_n = \frac{1}{n!^2} \left(\frac{1}{2} - 1\right) \ldots \left(\frac{1}{2} - n + 1\right)\) for \(n \geq 1\), with the convention \(a_0 = 0\), and for \(|z| < 1\), set

\[
\sqrt{1-z} = \sum_{n=0}^{+\infty} (-1)^n a_n z^n,
\]

so that \((\sqrt{1-z})^2 = 1 - z\), and \(\sqrt{1-t}\) is the positive square root of \(1 - t\) for \(t \in (-1, 1)\). Also \(\text{Re}(\sqrt{1-z}) > 0\) for \(|z| < 1\).
We have, for $t \in [0, 1)$,
\[
\sum_{n=1}^{+\infty} (-1)^{n-1} \alpha_n t^n = 1 - \sqrt{1 - t}.
\]

Since $(-1)^{n-1} \alpha_n \geq 0$ for $n \geq 1$, the series $\sum_{n=1}^{+\infty} |\alpha_n| = \sum_{n=1}^{+\infty} (-1)^{n-1} \alpha_n$ is convergent, and we have
\[
\sum_{n=1}^{+\infty} |\alpha_n| t^n = 1 - \sqrt{1 - t} \quad (0 \leq t \leq 1).
\]

Now let $A$ be a commutative unital Banach algebra, and let $x \in A$ such that $\|x\| \leq 1$. Set
\[
\sqrt{1_A - x} = \sum_{n=0}^{+\infty} (-1)^n \alpha_n x^n.
\]

Then $(\sqrt{1_A - x})^2 = 1_A - x$, and we have
\[
\left\|1_A - \sqrt{1_A - x}\right\| = \left\|\sum_{n=1}^{+\infty} (-1)^n \alpha_n x^n\right\| \leq \sum_{n=1}^{+\infty} |\alpha_n| \|x\|^n = 1 - \sqrt{1 - \|x\|} \quad (2)
\]

Notice also that if $A$ is commutative, then we have
\[
\text{Re} \left(\sqrt{1_A - x} \right) = \text{Re} \left(\sqrt{1 - \chi(x)} \right) \geq 0 \quad (\chi \in \hat{A}). \quad (3)
\]

We obtain the following formula

**Lemma 3.1.** Let $(C(t))_{t \in \mathbb{R}}$ be a cosine function in a unital Banach algebra $A$. Assume that $\|C(t) - 1_A\| \leq 2$ and that $p \left(\left(\frac{t}{2}\right) - 1\right) < 1$. Then we have
\[
C \left(\frac{t}{2}\right) = \sqrt{1_A - \frac{1_A - C(t)}{2}}.
\]

Proof: The abstract version of the formula $\sin^2 \left(\frac{u}{2}\right) = \frac{1 - \cos(u)}{2}$ gives
\[
1_A - C \left(\frac{t}{2}\right)^2 = 1_A - C(t), C \left(\frac{t}{2}\right)^2 = 1_A - 1_A - C(t) = \left(\sqrt{1_A - \frac{1_A - C(t)}{2}}\right)^2,
\]

\[
\left( C \left(\frac{t}{2}\right) - \sqrt{1_A - \frac{1_A - C(t)}{2}} \right) \left( C \left(\frac{t}{2}\right) + \sqrt{1_A - \frac{1_A - C(t)}{2}} \right) = 0.
\]
We may assume that $A$ is commutative. Let $\chi \in \hat{A}$. Since $\rho \left( C \left( \frac{t}{2} \right) - 1 \right) < 1$, we have $\text{Re} \left( \chi \left( C \left( \frac{t}{2} \right) \right) \right) > 0$. Since $\text{Re} \left( \chi \left( \sqrt{1 - \frac{1}{2} C(t)} \right) \right) \geq 0$, $C \left( \frac{t}{2} \right) + \sqrt{1 - \frac{1}{2} C(t)}$ is invertible in $A$, and $C \left( \frac{t}{2} \right) - \sqrt{1 - \frac{1}{2} C(t)} = 0$. □

**Theorem 3.2.** Let $(C(t))_{t \in \mathbb{R}}$ be a cosine sequence in a Banach algebra. If $\limsup_{t \to 0} \|C(t) - 1_A\| < 2$, then $\limsup_{t \to 0} \|C(t) - 1_A\| = 0$.

Proof: It follows from proposition 2.2 and lemma 3.1 that there exists $\eta > 0$ such that we have, for $|t| \leq \eta$,

$$\|C(t) - 1_A\| < 2, C \left( \frac{t}{2} \right) = \sqrt{1 - \frac{1}{2} C(t)}.$$ 

Using (1), we obtain, for $|t| \leq \eta$,

$$\left\| C \left( \frac{t}{2} \right) - 1_A \right\| \leq 1 - \sqrt{1 - \frac{C(t) - 1_A}{2}}.$$ 

Set $l = \limsup_{t \to 0} \|C(t) - 1_A\|$. We obtain

$$l \leq 1 - \sqrt{1 - \frac{l}{2}} \leq 1,$$

and so $l = 0$. □

Notice that the proof above gives a little bit more than the zero-two law: if $\|1 - C(t)\| \leq 2$ and $\rho \left( 1 - C \left( \frac{t}{2} \right) \right) < 1$ for $|t| \leq \eta$, then we have, for $n \geq 1$,

$$\sup_{|t| \leq 2^{-n} \eta} \|C(t) - 1_A\| \leq u_n,$$

where the sequence $u_n$ satisfies $u_0 = 2, u_{n+1} = 1 - \sqrt{1 - \frac{u_n}{2}}$ for $n \geq 1$, and $\lim_{n \to +\infty} u_n = 0$, which gives an explicit control on the convergence to 0 of $\|C(t) - 1_A\|$ as $t \to 0$.

Notice also that the fact that the coefficients of the Taylor expansion at the origin of the function $t \to 1 - \sqrt{1 - t}^2$ are positive was used in [6] to show that $\|x^2 - x\| \geq 1/4$ for every quasinilpotent element $x$ of a Banach algebra such that $|x| \geq 1/2$. Similar argument were used in [8] to show that if a semigroup $(T(t))_{t \geq 0}$ in a Banach algebra $A$ satisfies $\limsup_{t \to 0^+} \|T(t) - T((n+1)t)\| < \frac{n}{(n+1)^{1+\frac{1}{2}}}$ for some $n \geq 1$, then there exists an idempotent $J$ of $A$ such that $\lim_{t \to 0} \|T(t) - J\| = 0$, so that $\limsup_{t \to 0^+} \|T(t) - T((n+1)t)\| = 0$.  

6
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