HEAT KERNEL ESTIMATES FOR NON-SYMMETRIC STABLE-LIKE PROCESSES

PENG JIN

Abstract. Let $d \geq 1$ and $0 < \alpha < 2$. Consider the integro-differential operator

$$L f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x+h) - f(x) - \chi_{\alpha}(h) \nabla f(x) \cdot h \right] \frac{n(x,h)}{|h|^{d+\alpha}} \, dh + 1_{\alpha>1} b(x) \cdot \nabla f(x),$$

where $\chi_{\alpha}(h) := 1_{\alpha>1} + 1_{\alpha=1} 1_{|h| \leq 1}$, $b : \mathbb{R}^d \to \mathbb{R}^d$ is bounded measurable, and $n : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is measurable and bounded above and below respectively by two positive constants. Further, we assume that $n(x,h)$ is Hölder continuous in $x$, uniformly with respect to $h \in \mathbb{R}^d$. In the case $\alpha = 1$, we assume additionally $\int_{\partial B_r} n(x,h) \, dS_r(h) = 0$, $\forall r \in (0,\infty)$, where $dS_r$ is the surface measure on $\partial B_r$, the boundary of the ball with radius $r$ and center $0$. In this paper, we establish two-sided estimates for the heat kernel of the Markov process associated with the operator $L$. This extends a recent result of Z.-Q. Chen and X. Zhang.

1. Introduction

In probability theory, stable distributions play a very important role. They appear naturally when one studies the limits of the sum of suitably rescaled independent and identically distributed random variables. A stable distribution is firstly characterized by an index $\alpha \in (0,2]$, which is called the index of stability. Stable distributions with index $\alpha = 2$ are nothing but the Gaussian ones, while those with index $\alpha \in (0,2)$ have heavy tails and are particularly interesting for applications, see, e.g., [20]. One feature of stable distributions is their analytical tractability, which is due to the simple form of their characteristic functions. In particular, density estimates for stable distributions with index $\alpha \in (0,2)$ were done in [12] for the one-dimensional case, and the higher dimensional analogues were obtained in [6, 16, 24].

A Lévy process whose distribution is $\alpha$-stable is called an $\alpha$-stable process. Due to [16, 24], density estimates of $\alpha$-stable processes with $\alpha \in (0,2)$ have been well-understood. Moreover, as shown in [23, 14, 15], many other Lévy processes, whose Lévy measure resembles that of an $\alpha$-stable processes, possess similar or slightly different density estimates.

2010 Mathematics Subject Classification. primary 60J35, 47G20, 60J75.

Key words and phrases. Stable-like process, heat kernel, integro-differential operator, martingale problem, Levi's method.
Stable-like processes are extensions of stable processes and refer to Markov processes that behave, at each point of the state space, like a single stable process. In the literature there are different definitions of these processes, see, e.g., [1, 16, 8, 3, 5]. Symmetric stable-like processes can be defined through the corresponding symmetric Dirichlet forms, as done in [8]. Note that sharp heat kernel estimates for symmetric stable-like processes have been obtained in [8]. Compared to the symmetric case, non-symmetric stable-like processes are usually given as solutions of the martingale problem for stable-like operators. Following [5], a stable-like symmetric case, non-symmetric stable-like processes are usually given as solutions of symmetric stable-like processes. We will consider an integro-differential operator that is more general given in (1.1). Let $d \geq 1$ and $0 < \alpha < 2$. Consider the operator

$$
\mathcal{L} f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x + h) - f(x) - \chi_\alpha(h) \nabla f(x) \cdot h \right] \frac{n(x, h)}{|h|^{d+\alpha}} \mathrm{d}h + \mathbf{1}_{\{\alpha > 1\}} b(x) \cdot \nabla f(x),
$$

(1.2)

where $\chi_\alpha(h) := \mathbf{1}_{\alpha > 1} + \mathbf{1}_{\alpha = 1} \mathbf{1}_{\{|h| \leq 1\}}$, the vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ and the function $n : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ are measurable. Throughout this paper, we assume the following assumptions:

**Assumption 1.1.** The function $n$ satisfies $0 < \kappa_0 \leq n(x, h) \leq \kappa_1$ for all $x, h \in \mathbb{R}^d$, where $\kappa_0$ and $\kappa_1$ are constants. Further, there exist constants $\theta \in (0, 1)$ and $\kappa_2 > 0$ such that

$$
|n(x, h) - n(y, h)| \leq \kappa_2 |x - y|^\theta, \quad \forall x, y, h \in \mathbb{R}^d.
$$

(1.3)

In the case $\alpha = 1$, we assume additionally

$$
\int_{\partial B_r} n(x, h) h \mathrm{d}S_r(h) = 0, \quad \forall r \in (0, \infty),
$$

(1.4)

where $dS_r$ is the surface measure on $\partial B_r$, the boundary of the ball with center 0 and radius $r$.

**Remark 1.2.** Note that we don’t assume the symmetry of $n(x, h)$ in $h$, i.e., it is possible that $n(x, h) \neq n(x, -h)$ for some $x, h \in \mathbb{R}^d$.

**Assumption 1.3.** There exists a constant $\kappa_3 > 0$ such that $|b(x)| \leq \kappa_3$ for all $x \in \mathbb{R}^d$. 

According to [19, Proposition 3], the martingale problem for $\mathcal{L}$ is well-posed under Assumptions 1.1 and 1.3. In spite of the presence of the drift term $b \cdot \nabla$ in $\mathcal{L}$, we will call the Markov process associated with $\mathcal{L}$ a stable-like process. The main result of this paper is as follows:

**Theorem 1.4.** Suppose that the operator $\mathcal{L}$ defined in (1.2) satisfies Assumptions 1.1 and 1.3. Let $(X,(L^x))$ be the Markov process associated with $\mathcal{L}$, i.e., $L^x$ is the unique solution to the martingale problem for $\mathcal{L}$ starting from $x \in \mathbb{R}^d$ and $X = (X_t)$ is the canonical process on $D([0,\infty);\mathbb{R}^d)$. Then $(X,(L^x))$ has a jointly continuous transition density $l(t,x,y)$ such that $L^x (X_t \in E) = \int_E l(t,x,y) dy$ for all $t > 0$, $x \in \mathbb{R}^d$ and $E \in \mathcal{B}(\mathbb{R}^d)$. Moreover, for each $T > 0$, there exists a constant $C_1 = C_1(d,\alpha,\kappa_0,\kappa_1,\kappa_2,\theta,\kappa_3,T) \in (1,\infty)$ such that

$$C_1^{-1} \left( \frac{t}{|x-y|^{d+\alpha}} \wedge t^{-d/\alpha} \right) \leq l(t,x,y) \leq C_1 \left( \frac{t}{|x-y|^{d+\alpha}} \wedge t^{-d/\alpha} \right)$$

for all $x,y \in \mathbb{R}^d$ and $0 < t \leq T$. For the case $1 < \alpha < 2$, there exists also a constant $C_2 = C_2(d,\alpha,\kappa_0,\kappa_1,\kappa_2,\theta,\kappa_3,T) > 1$ such that

$$|\nabla_x l(t,x,y)| \leq C_2 t^{-1/\alpha} \left( \frac{t}{|x-y|^{d+\alpha}} \wedge t^{-d/\alpha} \right), \quad \forall x,y \in \mathbb{R}^d, \quad t \in (0,T].$$

To prove Theorem 1.4, we will use the same approach as in [9], namely, we will apply the parametrix method of Levi. However, we have to overcome two main difficulties. The first one is, surprisingly, that sharp two-sided density estimates for a jump-type Lévy process with Lévy measure $K(h)|h|^{\frac{d-\alpha}{2}} dh$, where $K(\cdot)$ is bounded from above and below by two positive constants, are not completely known. To solve this problem, we will start with the upper bounds derived in [23], then use the rescaling argument in [4, Proposition 2.2] and some ideas from [5]. The second difficulty is due to the fact that $n(x,h)$ is not symmetric in $h$, which makes some rescaling arguments in [9] fail to work. As a result, in the case $\alpha = 1$, we obtain some estimates that are weaker than those in [9] (see, e.g., Lemma 3.6 below and [9, Theorem 2.4]). However, these weaker forms of estimates don’t essentially effect the proof of Theorem 1.4.

The rest of the paper is organized as follows. After a short section on preliminaries, in Section 3 we derive the two-sided density estimates for jump-type Lévy processes, whose Lévy measure is comparable to that of a rotationally symmetric $\alpha$-stable process. In Section 4 we construct the transition density of $(X,(L^x))$, with the additional assumption that the drift $b$ in $\mathcal{L}$ is identically 0. In Section 5 we treat the case where $1 < \alpha < 2$ and the drift term $b \cdot \nabla$ in $\mathcal{L}$ is present. Section 6 is devoted to the proof of Theorem 1.4.

Finally, we give a few remarks on the notation for the constants appearing in the statements or proofs of the results. The letter $c$ with subscripts will only appear in proofs and denote positive constants whose exact value is unimportant. The labeling of the constants $c_1, c_2, ...$ starts anew in the proof of each result. We write $C(d,\alpha,...)$ for a positive constant $C$ that depends only on the parameters $d, \alpha, ...$.
2. Preliminaries

2.1. Notation. The inner product of \(x\) and \(y\) in \(\mathbb{R}^d\) is written as \(x \cdot y\). We use \(|v|\) to denote the Euclidean norm of a vector \(v \in \mathbb{R}^m, m \in \mathbb{N}\). We use \(B_r(x)\) for the open ball of radius \(r\) with center \(x\) and simply write \(B_r\) for \(B_r(0)\). The boundary of \(B_r(x)\) is denoted by \(\partial B_r(x)\).

For a bounded function \(g\) on \(\mathbb{R}^d\) we write \(|g| := \sup_{x \in \mathbb{R}^d} |g(x)|\). Let \(C^2_0(\mathbb{R}^d)\) denote the class of \(C^2\) functions such that the function and its first and second order partial derivatives are bounded.

Let \(D = D([0, \infty); \mathbb{R}^d)\), the set of paths in \(\mathbb{R}^d\) that are right continuous with left limits, be endowed with the Skorokhod topology. Set \(X_t(\omega) = \omega(t)\) for \(\omega \in D\) and let \(\mathcal{D} = \sigma(X_t : 0 \leq t < \infty)\) and \(\mathcal{F}_t := \sigma(X_r : 0 \leq r \leq t)\). A probability measure \(\mathbb{P}\) on \((D, \mathcal{D})\) is called a solution to the martingale problem for \(\mathcal{L}\) starting from \(x \in \mathbb{R}^d\), if \(\mathbb{P}(X_0 = x) = 1\) and under the measure \(\mathbb{P}\), \(f(X_t) - \int_0^t \mathcal{L}f(X_u)du, \ t \geq 0\), is an \(\mathcal{F}_t\)-martingale for all \(f \in C^2_0(\mathbb{R}^d)\).

2.2. Rescaling. Instead of \(\mathcal{L}\), we first consider the operator

\[
\mathcal{A}f(x) := \int_{\mathbb{R}^d \setminus \{0\}} [f(x + h) - f(x) - \chi_\alpha(h)h \cdot \nabla f(x)] \frac{n(x, h)}{|h|^{d+\alpha}} dh.
\]

(2.1)

It turns out that the the Markov process associated with \(\mathcal{A}\) has the following rescaling property, which is analogous to [4, Proposition 2.2].

**Lemma 2.1.** Consider the operator \(\mathcal{A}\) defined in (2.1) with \(n(\cdot, \cdot)\) satisfying Assumption 1.1. Let \((X, (\mathbb{P}^x))\) be the Markov process associated with the operator \(\mathcal{A}\), i.e., \(\mathbb{P}^x\) is the unique solution to the martingale problem for \(\mathcal{A}\) starting from \(x \in \mathbb{R}^d\) and \(X = (X_t)\) is the canonical process on \(D([0, \infty); \mathbb{R}^d)\). Let \(a > 0\). Define \(\tilde{\mathbb{P}}^x = \mathbb{P}^{x/a}\) and \(Y_t := aX_{a^{-t}}\), \(t \geq 0\). Then \(\tilde{\mathbb{P}}^x(Y_0 = x) = 1\) and \(f(Y_t) - \int_0^t \tilde{\mathcal{A}}f(Y_u)du, \ t \geq 0\), is a \(\tilde{\mathbb{P}}^x\)-martingale for all \(f \in C^2_0(\mathbb{R}^d)\), where

\[
\tilde{\mathcal{A}}f(x) := \int_{\mathbb{R}^d \setminus \{0\}} [f(x + h) - f(x) - \chi_\alpha(h)h \cdot \nabla f(x)] \frac{\tilde{n}(x, h)}{|h|^{d+\alpha}} dh
\]

with \(\tilde{n}(x, h) := n(x/a, h/a)\).

**Proof.** In view of (1.4), the proof of [4, Proposition 2.2] works also here without any changes. \(\square\)

**Remark 2.2.** In Lemma 2.1, after the transformation \(\tilde{n}(x, h) = n(x/a, h/a)\), we have

\[
|\tilde{n}(x, h) - \tilde{n}(y, h)| = \left| n\left(\frac{x}{a}, \frac{h}{a}\right) - n\left(\frac{y}{a}, \frac{h}{a}\right)\right| \leq \kappa_2 \left|\frac{x}{a} - \frac{y}{a}\right|^{\theta} = \kappa_2 a^{-\theta}|x - y|^\theta
\]

for all \(x, y\) and \(h \in \mathbb{R}^d\).

2.3. Estimate of the first exit time from a ball.

**Lemma 2.3.** Let \(\mathcal{A}\) and \((X, (\mathbb{P}^x))\) be as in Lemma 2.1. Then there exists a constant \(C_3 > 0\) not depending on \(x\) such that for all \(r > 0\) and \(t > 0\),

\[
\mathbb{P}^x(\tau_{B_r(x)} \leq t) \leq C_3 tr^{-\alpha},
\]

where \(\tau_{B_r(x)} := \inf \{t \geq 0 : X_t \notin B_r(x)\}\).
Proof. The proof is essentially identical to that of [3, Proposition 3.1]. Let $f \in C^2_0(\mathbb{R}^d)$ be a non-negative function that is equal to $|x|^2$ for $|x| \leq 1/2$, which equals 1 for $|x| \geq 1$. Let $r > 0$ and $x_0 \in \mathbb{R}^d$ be arbitrary. Define $u(x) := r^2 f\left(r^{-1}(x - x_0)\right)$, $x \in \mathbb{R}^d$. Then $u \in C^2_0(\mathbb{R}^d)$, and $\|u\| \leq c_1 r^2$, $\|\nabla u\| \leq c_1 r$ and $\|D^2 u\| \leq c_1$ for some positive constant $c_1$. As shown in the proof of [3, Proposition 3.1], there exists a constant $c_2 > 0$ such that

$$\left| \int_{|h|\leq r} \left[ u(x + h) - u(x) - h \cdot \nabla u(x) \right] \frac{n(x,h)}{|h|^{d+\alpha}} dh \right| \leq c_2 r^{2-\alpha} \tag{2.2}$$

and

$$\left| \int_{|h|> r} \left[ u(x + h) - u(x) \right] \frac{n(x,h)}{|h|^{d+\alpha}} dh \right| \leq c_2 r^{2-\alpha}. \tag{2.3}$$

We now distinguish between the following three cases:

(i) $1 < \alpha < 2$. Since

$$\left| \int_{|h|> r} h \cdot \nabla u(x) \frac{n(x,h)}{|h|^{d+\alpha}} dh \right| \leq c_1 r \left| \int_{|h|> r} \frac{n(x,h)}{|h|^{d+\alpha-1}} dh \right| \leq c_3 r^{2-\alpha},$$

we get from (2.2) and (2.3) that $\|Au\| \leq c_4 r^{2-\alpha}$. (ii) $\alpha = 1$. In view of (1.4), it follows directly from (2.2) and (2.3) that $\|Au\| \leq c_2 r^{2-\alpha}$. (iii) $0 < \alpha < 1$. We have

$$\left| \int_{|h|\leq r} \left[ u(x + h) - u(x) \right] \frac{n(x,h)}{|h|^{d+\alpha}} dh \right| \leq \|\nabla u\| \left| \int_{|h|\leq r} \frac{n(x,h)}{|h|^{d+\alpha-1}} dh \right| \leq c_5 r^{2-\alpha},$$

which together with (2.3) implies $\|Au\| \leq c_6 r^{2-\alpha}$.

Further, it was shown in [3, Proposition 3.1] that

$$r^2 P_{x_0}^T \left( \tau_{B_r(x_0)} \leq t \right) \leq E_{x_0} \left[ u \left( X_{t \wedge \tau_{B_r(x_0)}} \right) \right]$$

$$= E_{x_0} \left[ \int_0^{t \wedge \tau_{B_r(x_0)}} A u(X_s) ds \right] \leq c_7 t r^{2-\alpha}, \tag{2.4}$$

which implies the assertion. \hfill $\Box$

Lemma 2.4. Assume $1 < \alpha < 2$. Let $\mathcal{L}$ and $(X, (\mathcal{L}^x))$ be as in Theorem 1.4. Define $\tau_{B_r(x)}$ as in Lemma 2.3. Then for each $T > 0$, there exists a constant $C_4 > 0$ not depending on $x$ such that for all $0 < r < T$ and $t > 0$,

$$\mathcal{L}^x \left( \tau_{B_r(x)} \leq t \right) \leq C_4 t r^{-\alpha} \tag{2.5}$$

Proof. Let the function $u$ be as in the proof of Lemma 2.3. Note that $\mathcal{L} u = A u + b \cdot \nabla u$ and $\|Au\| \leq c_1 r^{2-\alpha}$, $r > 0$, which was already proved in proof of Lemma 2.3. Then we obtain from $\|b \cdot \nabla u\| \leq c_2 \kappa_2 r$ that $\|Lu\| \leq c_3 (r^{2-\alpha} + r)$, $r > 0$. Similarly to (2.4), we get

$$r^2 P_{x_0}^T \left( \tau_{B_r(x_0)} \leq t \right) \leq c_4 t (r^{2-\alpha} + r) \leq c_5 r^{2-\alpha}, \quad 0 < r < T.$$

So (2.5) follows. \hfill $\Box$
2.4. Some inequalities and estimates. Let $\gamma > 0$ be a constant. It follows from [9, p.277, (2.9)] that for $|z| \leq (2t^{1/\alpha}) \lor (|x|/2)$,

$$
\left( t^{1/\alpha} + |x| + z \right)^{-\gamma} \leq 4^\gamma \left( t^{1/\alpha} + |x| \right)^{-\gamma}.
$$

(2.6)

Following the notation in [9], we write

\[
\hat{g}_i^\alpha(t, x) := t^{\gamma/\alpha}(|x|^\beta \land 1) \left( t^{1/\alpha} + |x| \right)^{-d-\alpha}, \quad (t, x) \in (0, \infty) \times \mathbb{R}^d.
\]

As shown in [9], the following convolution inequalities hold.

Lemma 2.5. ([9, Lemma 2.1]) (i) For all $\beta \in [0, \alpha/2]$ and $\gamma \in \mathbb{R}$, there exists some constant $C_5 = C_5(d, \alpha) > 0$ such that

\[
\int_{\mathbb{R}^d} \hat{g}_i^\alpha(t, x) dx \leq C_5 t^{\frac{\gamma + \beta}{\alpha}}, \quad (t, x) \in (0, 1) \times \mathbb{R}^d.
\]

(2.7)

(ii) For all $\beta_1, \beta_2 \in [0, \alpha/4]$, and $\gamma_1, \gamma_2 \in \mathbb{R}$, there exists some constant $C_6 = C_6(d, \alpha) > 0$ such that for all $0 < s < t \leq 1$ and $x \in \mathbb{R}^d$,

\[
\begin{align*}
\int_{\mathbb{R}^d} \hat{g}_1^\beta(t-s, x-z) \hat{g}_2^\beta(s, z) dz & \leq C_6 \left( (t-s)^{\frac{\gamma_1 + \beta_1 + \beta_2 - \alpha}{\alpha}} + (t-s)^{\frac{\gamma_2 + \beta_2 - \alpha}{\alpha}} \right) \hat{g}_0^\beta(t, x) \\
& + C_6(t-s)^{\frac{\gamma_1 + \beta_1 + \beta_2 - \alpha}{\alpha}} \hat{g}_1^\beta(t, x) + C_6(t-s)^{\frac{\gamma_2 + \beta_2 - \alpha}{\alpha}} \hat{g}_1^\beta(t, x).
\end{align*}
\]

(2.8)

(iii) For all $\beta_1, \beta_2 \in [0, \alpha/4]$, $\gamma_1 + \beta_1 > 0$ and $\gamma_2 + \beta_2 > 0$, there exists some constant $C_7 = C_7(d, \alpha) > 0$ such that for all $0 < s < t \leq 1$ and $x \in \mathbb{R}^d$,

\[
\int_0^1 \int_{\mathbb{R}^d} \hat{g}_1^\beta(t-s, x-z) \hat{g}_2^\beta(s, z) dz ds \\
\leq C_7 B \left( \frac{\gamma_1 + \beta_1}{\alpha}, \frac{\gamma_2 + \beta_2}{\alpha} \right) \left( \hat{g}_{\gamma_1+\gamma_2+\beta_1+\beta_2}^0 + \hat{g}_{\gamma_1+\gamma_2+\beta_2}^0 + \hat{g}_{\gamma_1+\gamma_2+\beta_1}^0 \right)(t, x),
\]

(2.9)

where $B(\gamma, \beta)$ is the Beta function with parameters $\gamma, \beta > 0$.

For $\lambda > 0$, define $u_\lambda(x) := \int_0^\infty e^{-\lambda t} \hat{g}_0^\alpha(s, x) ds$, $x \in \mathbb{R}^d$. According to [7, Lemma 3, Lemma 7 and Theorem 8], there exist constants $C_8 = C_8(d, \alpha) > 1$ and $C_9 = C_9(d, \alpha) > 1$ such that for all $\lambda > 0$ and $x, y, z \in \mathbb{R}^d$,

\[
C_8^{-1} \left( \lambda^{(d-\alpha)/\alpha} \lor |x|^{\alpha-d} \right) \land \left( \lambda^{-2} |x|^{-d-\alpha} \right) \\
\leq u_\lambda(x) \leq C_8 \left( \lambda^{(d-\alpha)/\alpha} \lor |x|^{\alpha-d} \right) \land \left( \lambda^{-2} |x|^{-d-\alpha} \right)
\]

(2.10)

and

\[
u_\lambda(x-z) \land u_\lambda(z-y) \leq C_9 u_\lambda(x-y).
\]

(2.11)

Lemma 2.6. Assume $1 < \alpha < 2$. Define $k_\lambda(x) := \int_0^\infty e^{-\lambda t} \hat{g}_{\alpha-1}^0(s, x) ds$, $x \in \mathbb{R}^d$. Then there exist constants $C_{10} = C_{10}(d, \alpha) > 0$ and $C_{11} = C_{11}(d, \alpha) > 0$ such that

\[
k_\lambda(x) \leq C_{10} \left( |x|^{\alpha-d-1} \lor \left( \lambda^{-2+1/\alpha} |x|^{-d-\alpha} \right) \right), \quad \lambda > 0, x \in \mathbb{R}^d.
\]

(2.12)
and
\[ \int_{\mathbb{R}^d} u_\lambda(x-z)k_\lambda(z-y)dz \leq C_{11}(1 + 1/\alpha)u_\lambda(x-y), \quad \lambda > 0, \ x, y \in \mathbb{R}^d. \quad (2.13) \]

**Proof.** It is easy to see that \( k_\lambda(x) = \lambda^{(d+1)/\alpha}k_1(\lambda^1/\alpha)x \). So it suffices to show (2.12) for \( \lambda = 1 \). For \( x \in \mathbb{R}^d \), we have
\[ k_1(x) \leq \int_0^{1/\alpha} e^{-t} t^{1-1/\alpha} dt + \int_{1/\alpha}^{\infty} e^{-t} t^{(d+1)/\alpha} dt. \]
Therefore, for \( |x| > 1 \),
\[ k_1(x) \leq c_1 |x|^{-d-\alpha} + |x|^{-d-1} \int_{1/\alpha}^{\infty} e^{-t} dt \leq c_1 |x|^{-d-\alpha} + |x|^{-d-1} \leq c_2 |x|^{-d-\alpha}; \]
for \( |x| \leq 1 \),
\[ k_1(x) \leq |x|^{-d-\alpha} \int_0^{1/\alpha} t^{1-1/\alpha} dt + \int_{1/\alpha}^{1} t^{(d+1)/\alpha} dt \leq c_3 |x|^{-d+\alpha-1}. \]
So (2.12) is true. To show (2.13), we proceed in the same way as in the proof of [7, Lemma 17]. Set \( w_\lambda(x) := (\lambda^{-(d-\alpha)/\alpha} |x|^{\alpha-1}) \). It follows from (2.10) and (2.12) that \( k_\lambda(x) \leq c_4 w_\lambda(x)u_\lambda(x) \) for all \( \lambda > 0 \) and \( x \in \mathbb{R}^d \). So
\[ \int_{\mathbb{R}^d} u_\lambda(x-z)k_\lambda(z-y)dz \]
\[ \leq \int_{\mathbb{R}^d} u_\lambda(x-y) \int_{\mathbb{R}^d} w_\lambda(z-y)u_\lambda(z-y)u_\lambda(x-y)dz \]
\[ \leq c_4 u_\lambda(x-y) \int_{\mathbb{R}^d} w_\lambda(z-y)(u_\lambda(x-z) \vee u_\lambda(z-y))dz \]
\[ \leq c_4 u_\lambda(x-y) \int_{\mathbb{R}^d} [(w_\lambda(x-z)u_\lambda(x-z)) \vee (w_\lambda(z-y)u_\lambda(z-y))]dz \quad (2.14) \]
\[ \leq c_4 u_\lambda(x-y) \int_{\mathbb{R}^d} [(w_\lambda(x-z)u_\lambda(x-z)) + (w_\lambda(x-z)u_\lambda(z-y))]dz \]
\[ \leq 2c_4 u_\lambda(x-y) \int_{\mathbb{R}^d} w_\lambda(z)u_\lambda(z)dz, \quad (2.15) \]
where in (2.14) we used the fact that \( w(z-y) \) and \( u_\lambda(z-y) \) are decreasing in \( |z-y| \). By (2.10) and the definition of \( w_\lambda \), we have
\[ w_\lambda(z)u_\lambda(z) \leq c_5 (|z|^{\alpha-\alpha} + \lambda^{-2+1/\alpha}|z|^{-\alpha}), \quad \lambda > 0, \ z \in \mathbb{R}^d. \]
Thus
\[ \int_{\mathbb{R}^d} w_\lambda(z)u_\lambda(z)dz \leq c_5 \int_{|z| \leq 1/\alpha} |z|^{\alpha-\alpha} dz + c_5 \int_{|z| \leq 1/\alpha} \lambda^{-2+1/\alpha}|z|^{-\alpha} dz \]
\[ \leq c_5 \lambda^{-1+1/\alpha}. \quad (2.16) \]
So (2.13) follows by (2.15) and (2.16). \( \square \)
3. Stable-like Lévy processes and their density estimates

Consider a Lévy process $Z = (Z_t)_{t \geq 0}$ with $Z_0 = 0$ a.s., which is defined on some probability space $(\Omega, \mathcal{A}, P)$ and whose characteristic function is given by

$$E[e^{iZ_t u}] = e^{-t\psi(u)}, \quad u \in \mathbb{R}^d,$$

$$\psi(u) = -\int_{\mathbb{R}^d \setminus \{0\}} (e^{iu \cdot h} - 1 - \chi_\alpha(h)iu \cdot h)K(h)dh.$$

Throughout this section we assume that the function $K: \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\frac{\kappa_0}{|h|^{d+\alpha}} \leq K(h) \leq \frac{\kappa_1}{|h|^{d+\alpha}}, \quad h \in \mathbb{R}^d,$$  \hspace{1cm} (3.1)

where $\kappa_1 > \kappa_0 > 0$ are the constants appearing in Assumption 1.1. In the case $\alpha = 1$, we assume in addition to (3.1) that

$$\int_{\partial B_r} K(h)z dS_r(h) = 0, \quad \forall r \in (0, \infty).$$  \hspace{1cm} (3.2)

In view of (3.1), we call $Z$ a stable-like Lévy process. The aim of this section is to establish some estimates for the density functions of $Z$. To this end, we follow the same idea as in [9]. Define $\tilde{K}: \mathbb{R}^d \to \mathbb{R}$ by $\tilde{K}(h) := K(h) - \kappa_0/(2|h|^{d+\alpha})$, $z \in \mathbb{R}^d$. So

$$\frac{2^{-1}\kappa_0}{|h|^{d+\alpha}} \leq \tilde{K}(h) \leq \frac{\kappa_1 - 2^{-1}\kappa_0}{|h|^{d+\alpha}}, \quad h \in \mathbb{R}^d.$$  \hspace{1cm} (3.3)

Note that if $\alpha = 1$, then

$$\int_{\partial B_r} \tilde{K}(h)z dS_r(h) = 0, \quad \forall r \in (0, \infty).$$  \hspace{1cm} (3.4)

Let

$$\tilde{\psi}(u) := -\int_{\mathbb{R}^d \setminus \{0\}} \left(e^{iu \cdot h} - 1 - \chi_\alpha(h)iu \cdot h\right)\tilde{K}(h)dh, \quad u \in \mathbb{R}^d,$$  \hspace{1cm} (3.5)

and $\tilde{Z} = (\tilde{Z}_t)_{t \geq 0}$ be a stable-like Lévy process with the characteristic exponent $\tilde{\psi}$. Without loss of generality, we assume that the process $(\tilde{Z}_t)$ is also defined on $(\Omega, \mathcal{A}, P)$.

We can write

$$\psi(u) = -\int_{\mathbb{R}^d \setminus \{0\}} \left(e^{iu \cdot h} - 1 - \chi_\alpha(h)iu \cdot h\right)\left(\frac{\kappa_0}{2|h|^{d+\alpha}} + \tilde{K}(h)\right)dh$$

$$= C_{12}|u|^\alpha + \tilde{\psi}(u),$$

where $C_{12} = C_{12}(d, \alpha, \kappa_0) > 0$ is a constant. It holds

$$e^{-t\Re(\psi(u))} = |e^{-t\psi(u)}| = |e^{-t(C_{12}|u|^\alpha + \tilde{\psi}(u))}| = e^{-tC_{12}|u|^\alpha}e^{-t\tilde{\psi}(u)} \leq e^{-tC_{12}|u|^\alpha},$$  \hspace{1cm} (3.6)

where $\Re(x)$ denotes the real part of $x \in \mathbb{C}$. Therefore, we get

$$\Re(\psi(u)) \geq C_{12}|u|^\alpha, \quad u \in \mathbb{R}^d, \quad t \geq 0.$$  \hspace{1cm} (3.7)
By (3.6) and the inversion formula of Fourier transform, the law of \( Z_t \) has a density (with respect to the Lebesgue measure) \( f_t \in L^1(\mathbb{R}^d) \cap C_b(\mathbb{R}^d) \) that is given by

\[
f_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iu \cdot x} e^{-t\psi(u)} du, \quad x \in \mathbb{R}^d, \quad t > 0.
\]  

(3.8)

Similarly, we define

\[
g_t(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iu \cdot x} e^{-tC_{12}|u|^\alpha} du
\]  

(3.9)

and

\[
\tilde{f}_t(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iu \cdot x} e^{-t\tilde{\psi}(u)} du
\]  

for \( x \in \mathbb{R}^d, \quad t > 0 \). Then \( g_t \) and \( h_t \) are densities of some rotationally symmetric \( \alpha \)-stable process \( (S_t) \) and the stable-like Lévy process \( (\tilde{Z}_t) \), respectively. It is clear that \( f_t = g_t \ast \tilde{f}_t \). Since \( g_t \) is the density of a rotationally symmetric \( \alpha \)-stable process, we have the following scaling property of \( g_t \): for all \( x \in \mathbb{R}^d \) and \( t > 0 \),

\[
g_t(x) = t^{-d/\alpha} g_1(t^{-1/\alpha}x).
\]  

(3.10)

It is well-known that the following estimates for \( g_t \) hold: there exists some constant \( C_{13} = C_{13}(d, \alpha, \kappa_0) > 0 \) such that

\[
C_{13}^{-1} t \left( t^{1/\alpha} + |x| \right)^{-d-\alpha} \leq g_t(x) \leq C_{13} t \left( t^{1/\alpha} + |x| \right)^{-d-\alpha},
\]  

(3.11)

for all \( x \in \mathbb{R}^d \) and \( t > 0 \). Moreover, for each \( k \in \mathbb{N} \), we can find a constant \( C_{14} = C_{14}(d, \alpha, \kappa_0, k) > 0 \) such that

\[
|\nabla^k g_t(x)| \leq C_{14} t \left( t^{1/\alpha} + |x| \right)^{-d-\alpha-k}
\]  

(3.12)

for all \( x \in \mathbb{R}^d \) and \( t > 0 \), see [9, Lemma 2.2].

We next show that the same estimate as in (3.11) is also true for the density \( f_t \). For \( |\nabla f_t| \) we shall derive an estimate that is slightly worse than the estimate on \( |\nabla g_t| \) given in (3.12). As the first step, we have the following upper estimate that is actually a special case of [23, Theorem 1].

**Lemma 3.1.** ([23]) Let \( f_t \) be as in (3.8). Then there exists some constant \( C_{15} = C_{15}(d, \alpha, \kappa_0, \kappa_1) > 0 \) such that

\[
f_t(x) \leq C_{15} \left( 1 \wedge |x|^{-d-\alpha} \right), \quad x \in \mathbb{R}^d.
\]  

(3.13)

**Proof.** Note that (3.7) is true. The assertion thus follows by [23, Theorem 1]. Indeed, to apply [23, Theorem 1], we only need to take \( \mu \) as the surface measure \( dS_1 \) on \( \partial B_1 \), \( q(\cdot) \equiv \kappa_1, \phi(\cdot) \equiv 1, \beta = \alpha, \gamma = d, \) and \( k_1 = k_2 = 1 \) there. Then we obtain

\[
f_1(x + v) \leq c_1 \left( 1 \wedge |x|^{-d-\alpha} \right), \quad \forall x \in \mathbb{R}^d,
\]  

(3.14)
Proof. Let \( x \) be as in Lemma 3.1. Then we have

\[
f_t(x) \leq C_{15} t \left( t^{1/\alpha} + |x| \right)^{-d-\alpha}, \quad x \in \mathbb{R}^d, \ t > 0. \tag{3.15}
\]

Moreover, there exists some constant \( C_{16} = C_{16}(d, \alpha, \kappa_0, \kappa_1) > 0 \) such that

\[
|\nabla f_t(x)| \leq C_{16} t^{-1/\alpha} \left( t^{1/\alpha} + |x| \right)^{-d-\alpha} \tag{3.16}
\]

for all \( x \in \mathbb{R}^d \) and \( t > 0 \).

Proof. Let \( a > 0 \) and define \( Y_t := aZ_{\alpha^{-1}t}, \ t \geq 0 \). Then \( (Y_t) \) is a Lévy process and for \( u \in \mathbb{R}^d \),

\[
\mathbb{E}[e^{iy_t \cdot u}] = \mathbb{E}[e^{ia_{\alpha^{-1}} t u \cdot y_t}] = \exp \left( ta^{-\alpha} \int_{d\{0\}} \left( e^{iu \cdot h} - 1 - \chi_\alpha(h) iu \cdot h \right) K(h)dh \right).
\]

By (3.2) and a change of variables, we obtain

\[
\mathbb{E}[e^{iy_t \cdot u}] = \exp \left( \int_{d\{0\}} \left( e^{iu \cdot h} - 1 - \chi_\alpha(h) iu \cdot h \right) a^{-d-\alpha} K(a^{-1}h)dh \right), \quad u \in \mathbb{R}^d.
\]

Set \( M(h) := a^{-d-\alpha} K(a^{-1}h), \ h \in \mathbb{R}^d \). Then the function \( M \) satisfies

\[
\frac{\kappa_0}{|h|^{d+\alpha}} \leq M(h) \leq \frac{\kappa_1}{|h|^{d+\alpha}}, \quad h \in \mathbb{R}^d, \tag{3.17}
\]

where the positive constants \( \kappa_0 \) and \( \kappa_1 \) are the same as in (3.1). Therefore, \( (Y_t) \) is also a stable-like Lévy process. Let \( \rho(x), \ x \in \mathbb{R}^d \), be the probability density of \( Y_1 \). By choosing \( a \) such that \( a^{-\alpha} = t \), we obtain \( Y_t = t^{-1/\alpha} Z_t \), which implies \( \rho(x) = td^{1/\alpha} f_t(t^{1/\alpha} x), \ x \in \mathbb{R}^d \). It follows from Lemma 3.1 that \( td^{1/\alpha} f_t(t^{1/\alpha} x) \leq C_{15} (1 \wedge |x|^{-d-\alpha}), \ x \in \mathbb{R}^d \). So (3.15) is true.

Next, we will use the fact that \( f_t = g_t \ast \hat{f}_t \) to show (3.16). Since \( \hat{f}_t \) is the density of \( \tilde{L}_t \) and \( \tilde{L}_t \) is a stable-like Lévy process with the jump kernel \( \hat{K} \) that satisfies (3.3) and (3.4), we obtain, using (3.15), the existence of a constant \( \hat{C}_{15} = \hat{C}_{15}(d, \alpha, \kappa_0, \kappa_1) > 0 \) such that

\[
\hat{f}_t(x) \leq \hat{C}_{15} t \left( t^{1/\alpha} + |x| \right)^{-d-\alpha}, \quad x \in \mathbb{R}^d, \ t > 0. \tag{3.18}
\]
Note that $\nabla f_t = (\nabla g_t) \ast \tilde{f}_t$. By (3.12), we get that for all $x \in \mathbb{R}^d$ and $t > 0$,

$$|\nabla f_t(x)| \leq \int_{\mathbb{R}^d} |\nabla g_t(x - h)| \tilde{f}_t(h) dh$$

$$\leq C_{14} C_{15} \int_{\mathbb{R}^d} t^{(1/\alpha + |x - h|)^{-d-\alpha-1}} \left( t^{1/\alpha} + |h| \right)^{-d-\alpha} dh$$

$$\leq C_{16} t^{1-\alpha} \left( t^{1/\alpha} + |x| \right)^{-d-\alpha}.$$

This completes the proof. \qed

By (3.15) and the same argument as in [9, Lemma 2.3], we easily obtain the following corollary.

**Corollary 3.3.** There exists a constant $C_{17} = C_{17}(d, \alpha, \kappa_0, \kappa_1) > 0$ such that

$$|f_t(x) - f_t(x')| \leq C_{17} \left( (t^{-1/\alpha}|x - x'|) \wedge 1 \right) \{ g_0^0(t, x) + g_0^0(t, x') \}$$

(3.19)

for all $x, x' \in \mathbb{R}^d$ and $t > 0$.

**Lemma 3.4.** There exists some constant $C_{18} = C_{18}(d, \alpha, \kappa_0, \kappa_1) > 0$ such that

$$f_t(x) \geq C_{18} t \left( t^{1/\alpha} + |x| \right)^{-d-\alpha}, \quad \forall x \in \mathbb{R}^d, \ t > 0.$$

**Proof.** We will use the fact that $f_t = g_t \ast \tilde{f}_t$ to show this lemma. According to Lemma 2.3, there exists some constant $c_1 = c_1(d, \alpha, \kappa_0, \kappa_1) > 0$ such that

$$\mathbb{P}(\tilde{\tau}_{B_r} \leq t) \leq c_1 \, t r^{-\alpha}, \quad \forall r > 0,$$

(3.20)

where $\tilde{\tau}_{B_r} := \inf \left\{ t \geq 0 : \tilde{Z}_t \notin B_r \right\}$. Choose $c_2 > 0$ such that

$$(2^{-1} c_2)^\alpha = 2 c_1.$$  

(3.21)

If $|x| \leq c_2 t^{1/\alpha}$, then

$$f_t(x) \geq \int_{B_{2c^2 t^{1/\alpha}}(x)} g_t(x - y) \tilde{f}_t(y) dy \geq c_3 t^{-d/\alpha} \int_{B_{2c^2 t^{1/\alpha}}(x)} \tilde{f}_t(y) dy$$

$$\geq c_3 t^{-d/\alpha} \int_{B_{c^2 t^{1/\alpha}}} \tilde{f}_t(y) dy = c_3 t^{-d/\alpha} \mathbb{P}(\tilde{Z}_t \in B_{c^2 t^{1/\alpha}})$$

$$\geq c_3 t^{-d/\alpha} \mathbb{P} \left( \sup_{0 \leq s \leq t} |\tilde{Z}_s| < c_2 t^{1/\alpha} \right) = c_3 t^{-d/\alpha} \left( 1 - \mathbb{P} \left( \sup_{0 \leq s \leq t} |\tilde{Z}_s| \geq c_2 t^{1/\alpha} \right) \right)$$

$$= c_3 t^{-d/\alpha} \left( 1 - \mathbb{P}(\tilde{\tau}_{B_{c^2 t^{1/\alpha}}} \leq t) \right)^{(3.20)} \geq c_3 t^{-d/\alpha} \left( 1 - c_1 t \left( c_2 t^{1/\alpha} \right)^{-\alpha} \right)$$

$$\overset{(3.21)}{=} c_4 t^{-d/\alpha}. $$
If $|x| > c_2 t^{1/\alpha}$, then
\[
f_t(x) \geq \int_{\mathbb{R}^d \setminus B_{2-1/\alpha}(x)} g_t(x - y) \tilde{f}_t(y) dy \geq c_5 \int_{\mathbb{R}^d \setminus B_{2-1/\alpha}(x)} \frac{t}{|x - y|^{d + \alpha}} \tilde{f}_t(y) dy
\]
\[
\geq c_5 \int_{\mathbb{R}^d \setminus B_{2-1/\alpha}(x)} \frac{t}{|x - y|^{d + \alpha}} \tilde{f}_t(y) dy \geq c_5 \int_{B_{2-1/\alpha}(x)} \frac{t}{|x - y|^{d + \alpha}} \tilde{f}_t(y) dy
\]
\[
\geq c_6 \frac{t}{|x|^{d + \alpha}} \int_{B_{2-1/\alpha}(x)} \tilde{f}_t(y) dy = c_6 \frac{t}{|x|^{d + \alpha}} \left( 1 - c_1 t \left( 2^{-1} c_2 t^{1/\alpha} \right) - \alpha \right). \tag{3.21}
\]
This completes the proof. \qed

Next, we derive some useful estimates for $f_t$. In the subsequent proofs we will use very often the following identities: for $t > 0$ and $x, h \in \mathbb{R}^d$,
\[
g_t(x + h) - g_t(x) = \int_0^1 \nabla g_t(x + rh) \cdot h dr, \tag{3.22}
\]
\[
g_t(x + h) - g_t(x) - h \cdot \nabla g_t(x) = \int_0^1 \left( \int_0^1 \nabla^2 g_t(x + rr' h) \cdot r dr' \right) \cdot h dr. \tag{3.23}
\]

For each $\alpha \in (0, 2)$, it was proved in [9, p. 282] that there exists some constant $C_{19} = C_{19}(d, \alpha) > 0$ such that for all $0 < t \leq 1$ and $x \in \mathbb{R}^d$,
\[
\int_{\mathbb{R}^d} \left( \left( t^{-2/\alpha} |h|^2 \right) \wedge 1 \right) \left( \rho_0^0(t, x + h) + \rho_0^0(t, t, x) \right) \cdot |h|^{-d - \alpha} dh \leq C_{19} \rho_0^0(t, x). \tag{3.24}
\]

**Lemma 3.5.** Assume $\alpha \neq 1$. Then there exists constant $C_{20} = C_{20}(d, \alpha, \kappa_0, \kappa_1) > 0$ such that for all $0 < t \leq 1$ and $x \in \mathbb{R}^d$,
\[
\int_{\mathbb{R}^d} |f_t(x + h) - f_t(x) - \chi_\alpha(h) h \cdot \nabla f_t(x) \cdot |h|^{-d - \alpha} dh \leq C_{20} \rho_0^0(t, x). \tag{3.25}
\]

**Proof.** The idea of proof is borrowed from [9, Theorem 2.4]. If we can find a constant $\tilde{C}_{20} = \tilde{C}_{20}(d, \alpha, \kappa_0, \kappa_1) > 0$ such that for all $0 < t \leq 1$ and $x \in \mathbb{R}^d$,
\[
\int_{\mathbb{R}^d} |g_t(x + h) - g_t(x) - \chi_\alpha(h) h \cdot \nabla g_t(x) | \cdot |h|^{-d - \alpha} dh \leq \tilde{C}_{20} \rho_0^0(t, x), \tag{3.26}
\]
then the assertion follows from $f_t = g_t * \tilde{f}_t$ and
\[
\int_{\mathbb{R}^d} \rho_0^0(t, x - y) \tilde{f}_t(y) dy \leq c_1 t^{-1} \int_{\mathbb{R}^d} \tilde{f}_t(x - y) \hat{f}_t(y) dy
\]
\[
= c_1 t^{-1} \hat{f}_2(t) \leq c_2 \rho_0^0(t, x).
\]

Next, we proceed to prove (3.26).
(i) We first consider the case 0 < α < 1. If |h| ≤ 1, then
\[
|g_1(x + h) - g_1(x)| \leq c_5 (|h| \wedge 1) \left( \frac{t^{1/\alpha}}{|h|^{\alpha}} + 1 \right) \left( g_0^0(t, x + h) + g_0^0(t, x) \right).
\] (3.27)

By (3.10), we get
\[
|g_t(x + h) - g_t(x)| \leq c_5 \left( (1 + |x|^{1/\alpha}) \wedge 1 \right) \left( g_0^0(t, x + h) + g_0^0(t, x) \right).
\]

Therefore,
\[
\int_{\mathbb{R}^d} |g_t(x + h) - g_t(x)| \cdot |h|^{-d-\alpha} \, dh
\leq c_5 \int_{\mathbb{R}^d} \left( (1 + |x|^{1/\alpha}) \wedge 1 \right) g_0^0(t, x + h) \cdot |h|^{-d-\alpha} \, dh
\]
\[
+ c_5 \int_{\mathbb{R}^d} \left( (1 + |x|^{1/\alpha}) \wedge 1 \right) g_0^0(t, x) \cdot |h|^{-d-\alpha} \, dh =: I_1 + I_2.
\]

We have
\[
I_1 \leq c_5 t^{1-1/\alpha} \int_{|h| \leq t^{1/\alpha}} g_0^0(t, x + h) \cdot |h|^{-d-\alpha} \, dh
\]
\[
+ c_5 \int_{|h| > t^{1/\alpha}} g_0^0(t, x + h) \cdot |h|^{-d-\alpha} \, dh =: I_{11} + I_{12}.
\]

Further,
\[
I_{11} \leq c_6 t^{1-1/\alpha} \int_{|h| \leq t^{1/\alpha}} \left( 1 \wedge |x|^{1/\alpha} \right) \left( \frac{t^{1/\alpha}}{|h|^{\alpha}} + |x| \right)^{-d-\alpha} \cdot |h|^{-d-\alpha} \, dh
\]
\[
\leq c_6 t^{1-1/\alpha} \int_{|h| \leq t^{1/\alpha}} |h|^{-d-\alpha} \, dh \leq c_7 g_0^0(t, x).
\] (2.6)

If |x| ≤ 2t^{1/\alpha}, then
\[
I_{12} \leq c_9 t \int_{|h| > t^{1/\alpha}} \left( 1 \wedge |x|^{1/\alpha} \right) \left( \frac{t^{1/\alpha}}{|h|^{\alpha}} + |x| \right)^{-d-\alpha} \cdot |h|^{-d-\alpha} \, dh
\]
\[
\leq c_9 t^{-d/\alpha} \int_{|h| > t^{1/\alpha}} |h|^{-d-\alpha} \, dh \leq c_8 t^{-1 - d/\alpha} \leq c_9 g_0^0(t, x);\]
if \(|x| > 2t^{1/\alpha}\), then

\[
I_{12} \leq c_5 \left( \int_{|t^{1/\alpha} < |h| \leq \frac{|x|}{2}} \frac{\theta^0_\alpha(t, x + h) \cdot |h|^{-d-\alpha}}{h} \, dh + \int_{|h| > \frac{|x|}{2}} \frac{\theta^0_\alpha(t, x + h) \cdot |h|^{-d-\alpha}}{h} \, dh \right)
\]

\[
\leq c_5 t \int_{|t^{1/\alpha} < |h| \leq \frac{|x|}{2}} \left( t^{1/\alpha} + |x + h| \right)^{-d-\alpha} \cdot |h|^{-d-\alpha} \, dh + c_{10} |x|^{-d-\alpha} \int_{|h| > \frac{|x|}{2}} \frac{\theta^0_\alpha(t, x + h) \cdot |h|^{-d-\alpha}}{h} \, dh
\]

\[
\leq c_{11} t \left( t^{1/\alpha} + |x| \right)^{-d-\alpha} \int_{|t^{1/\alpha} < |h| \leq \frac{|x|}{2}} |h|^{-d-\alpha} \, dh + c_{12} |x|^{-d-\alpha} \leq c_{14} \theta^0_\alpha(t, x).
\]

For \(I_2\), by setting \(\tilde{h} := t^{-1/\alpha} h\), we have

\[
I_2 = c_5 \theta^0_\alpha(t, x) \int_{\mathbb{R}^d} \left( |\tilde{h}| \wedge 1 \right) \cdot |t^{1/\alpha} \tilde{h}|^{-d-\alpha} \tilde{t}^{d/\alpha} \, d\tilde{h}
\]

\[
= c_5 t^{-1} \theta^0_\alpha(t, x) \int_{\mathbb{R}^d} \left( |\tilde{h}| \wedge 1 \right) \cdot |\tilde{h}|^{-d-\alpha} \, d\tilde{h} \leq c_{15} \theta^0_\alpha(t, x).
\]

Summarizing the above estimates for \(I_{11}, I_{12}\) and \(I_2\), we obtain (3.26).

(ii) Let \(1 < \alpha < 2\). For \(|h| > 1\), we have

\[
|g_1(x + h) - g_1(x) - \chi_\alpha(h) h \cdot \nabla g_1(x)| \leq g_1(x + h) + |h| \cdot |\nabla g_1(x)| \leq c_{16} \left( \theta^0_\alpha(1, x + h) + \theta^0_\alpha(1, x) \right) + c_{17} |h| \theta^0_{\alpha - 1}(1, x).
\]

For \(|h| \leq 1\), we have

\[
|g_1(x + h) - g_1(x) - \chi_\alpha(h) h \cdot \nabla g_1(x)| \leq |h|^2 \int_0^1 \int_0^1 |\nabla^2 g_1(x + rr'h)| \, dr \, dr
\]

\[
\leq c_{18} |h|^2 \int_0^1 \int_0^1 (1 + |x + rr'h|)^{-d-\alpha - 2} \, dr \, dr
\]

\[
\leq c_{19} |h|^2 (1 + |x|)^{-d-\alpha - 2} \leq c_{19} |h|^2 (1 + |x|)^{-d-\alpha}.
\]

So

\[
|g_1(x + h) - g_1(x) - \chi_\alpha(h) h \cdot \nabla g_1(x)| \leq c_{20} \left( |h|^2 \wedge 1 \right) \left( \theta^0_\alpha(1, x + h) + \theta^0_\alpha(1, x) \right) + c_{21} 1_{(|h| > 1)} |h| \theta^0_{\alpha - 1}(1, x).
\]
By (3.10), we get
\[ |g_t(x + h) - g_t(x) - \chi_\alpha(h) h \cdot \nabla g_t(x)| \]
\[ = t^{-d/\alpha} \left| g_t(t^{-1/\alpha} x + t^{-1/\alpha} h) - g_t(t^{-1/\alpha} x) - t^{-1/\alpha} h \cdot \nabla g_t(t^{-1/\alpha} x) \right| \]
\[ \leq c_{20} \left( \left( t^{-2/\alpha} |h|^2 \right) \wedge 1 \right) (\mathcal{g}^0_\alpha(t, x + h) + \mathcal{g}^0_\alpha(t, x)) + c_{21} 1_{\{|h| > t^{1/\alpha}\}} |h| \mathcal{g}^0_{\alpha-1}(t, x). \] (3.30)

Since
\[ \int_{|h| > t^{1/\alpha}} |h| \mathcal{g}^0_{\alpha-1}(t, x)|h|^{-d-\alpha} \, dh \leq c_{22} \mathcal{g}^0_\alpha(t, x), \]
the assertion now follows from (3.24) and (3.30).

**Lemma 3.6.** Assume \( \alpha = 1 \). Then there exists a constant \( C_{21} = C_{21}(d, \alpha, \kappa_0, \kappa_1) > 0 \) such that for all \( 0 < t \leq 1 \) and \( x \in \mathbb{R}^d \),
\[ \int_{\mathbb{R}^d} \left| f_t(x + h) - f_t(x) - \chi_\alpha(h) h \cdot \nabla f_t(x) \right| \cdot \frac{1}{|h|^{d+\alpha}} \, dh \leq C_{21} (1 + \ln (t^{-1})) \mathcal{g}^0_\alpha(t, x). \] (3.31)

**Proof.** Note that \( \chi_\alpha(h) = 1_{\{|h| \leq 1\}} \) when \( \alpha = 1 \). Similarly to (3.28), we have that for \( |h| \leq 1 \),
\[ |g_t(x + h) - g_t(x) - \chi_\alpha(h) h \cdot \nabla g_t(x)| \leq c_1 |h|^2 (1 + |x|)^{-d-\alpha}. \]

For \( |h| > 1 \), we have
\[ |g_t(x + h) - g_t(x) - \chi_\alpha(h) h \cdot \nabla g_t(x)| \leq c_2 \left( \mathcal{g}^0_\alpha(1, x + h) + \mathcal{g}^0_\alpha(1, x) \right). \]

So
\[ |g_t(x + h) - g_t(x) - \chi_\alpha(h) h \cdot \nabla g_t(x)| \leq c_3 \left( |h|^2 \wedge 1 \right) \left( \mathcal{g}^0_\alpha(1, x + h) + \mathcal{g}^0_\alpha(1, x) \right). \] (3.32)

By the scaling property \( g_t(x) = t^{-d/\alpha} g_1(t^{-1/\alpha} x) \), we obtain
\[ |g_t(x + h) - g_t(x) - \chi_\alpha(h) h \cdot \nabla g_t(x)| \]
\[ = t^{-d} \left| g_1(t^{-1} x + t^{-1} h) - g_1(t^{-1} x) - t^{-1} \chi_1(h) h \cdot \nabla g_1(t^{-1} x) \right| \]
\[ = t^{-d} \left| g_1(t^{-1} x + t^{-1} h) - g_1(t^{-1} x) - \chi_1(t^{-1} h) t^{-1} h \cdot \nabla g_1(t^{-1} x) \right| \]
\[ - 1_{\{t < |h| \leq 1\}} (h) t^{-1} h \cdot \nabla g_1(t^{-1} x) \]
\[ \leq c_{34} \left( |t^{-1} h|^2 \wedge 1 \right) \left( \mathcal{g}^0_\alpha(t, x + h) + \mathcal{g}^0_\alpha(t, x) \right) \]
\[ + c_4 1_{\{t < |h| \leq 1\}} (h) t^{-1} (1 + |t^{-1} x|)^{-d-2} |h| \]
\[ \leq c_3 \left( |t^{-1} h|^2 \wedge 1 \right) \left( \mathcal{g}^0_\alpha(t, x + h) + \mathcal{g}^0_\alpha(t, x) \right) + c_4 1_{\{t < |h| \leq 1\}} (h) \mathcal{g}^0_\alpha(t, x)|h|. \] (3.33)

Note that
\[ \int_{\mathbb{R}^d} 1_{\{t < |h| \leq 1\}} (h) |h| \cdot \frac{1}{|h|^{d+1}} \, dh = \int_{\{t < |h| \leq 1\}} \frac{1}{|h|^d} \, dh = c_5 \ln (t^{-1}). \] (3.34)

Combining (3.24), (3.33) and (3.34), we obtain (3.31).
For a function $f$ on $\mathbb{R}^d$ we define the function $\delta_f$ on $\mathbb{R}^{2d}$ by

$$\delta_f(x, x') := f(x) - f(x'), \quad x, x' \in \mathbb{R}^d.$$ 

**Lemma 3.7.** Assume $\alpha \neq 1$. Then there exists a constant $C_{22} = C_{22}(d, \alpha, \kappa_0, \kappa_1) > 0$ such that for all $0 < t \leq 1$ and $x, x' \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |\delta_{f_1}(x + h, x' + h) - \delta_{f_1}(x, x')| - \chi_\alpha(h)h \cdot \delta_{\nabla f_1}(x, x')| \cdot |h|^{-d-\alpha}dh$$

$$\leq C_{22} \left( \left( t^{-1/\alpha} |x - x'| \right)^{\wedge} 1 \right) \{(\delta_t(t, x) + \delta_t(t, x')) \}.$$  

(3.35)

**Proof.** As in Lemma 3.5, we only need to prove that for all $0 < t \leq 1$ and $x, x' \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |\delta_{g_1}(x + h, x' + h) - \delta_{g_1}(x, x') - \chi_\alpha(h)h \cdot \delta_{\nabla g_1}(x, x')| \cdot |h|^{-d-\alpha}dh$$

$$\leq \tilde{C}_{22} \left( \left( t^{-1/\alpha} |x - x'| \right)^{\wedge} 1 \right) \{(\delta_t(t, x) + \delta_t(t, x')) \},$$

where $\tilde{C}_{22} = \tilde{C}_{22}(d, \alpha, \kappa_0, \kappa_1) > 0$ is a constant.

(i) We first consider the case $\alpha > 1$. If $|h| \leq 1$ and $|x - x'| \leq 1$, then

$$|\delta_{g_1}(x + h, x' + h) - \delta_{g_1}(x, x') - \chi_\alpha(h)h \cdot \delta_{\nabla g_1}(x, x')|$$

$$\leq c_1 |h|^2 |x - x'| \int_0^1 \int_0^1 \int_0^1 \left| \nabla^3 g_1(x + rr'h + rr''(x - x')) | \cdot \chi_\alpha(h)h \cdot \delta_{\nabla g_1}(x, x') | \cdot |h|^{-d-\alpha}dh$$

$$\leq c_2 |h|^2 |x - x'| |1 + |x + rr'h + rr''(x - x')|)^{-d-\alpha-3} dh$$

$$\leq c_3 |h|^2 |x - x'| (1 + |x|)^{-d-\alpha-3} \leq c_3 |h|^2 |x - x'| \delta_\alpha^0(1, x). \hspace{1cm} (3.36)$$

If $|h| > 1$ and $|x - x'| \leq 1$, then

$$|\delta_{g_1}(x + h, x' + h) - \delta_{g_1}(x, x') - \chi_\alpha(h)h \cdot \delta_{\nabla g_1}(x, x')|$$

$$\leq |x - x'| \int_0^1 |\nabla^2 g_1(x + r(x' - x))| dr$$

$$+ |x - x'| \int_0^1 |\nabla g_1(x + r(x' - x))| dr$$

$$+ |h| \cdot |x - x'| \int_0^1 |\nabla^2 g_1(x + r(x' - x))| dr$$

$$\leq c_4 |x - x'| (1 + |x + h|)^{-d-\alpha-1} + c_4 |x - x'| (1 + |x|)^{-d-\alpha-1}$$

$$+ c_4 |h| \cdot |x - x'| (1 + |x|)^{-d-\alpha-2}. \hspace{1cm} (3.37)$$
In view of (3.29), we thus get
\[
|\delta_{g_1}(x+h, x'+h) - \delta_{g_1}(x, x') - \chi_\alpha(h)h \cdot \delta_{\nabla g_1}(x, x')| \\
\leq c_5 \left(|x-x'| \land 1 \right) \left(|h|^2 \land 1 \right) \left((\mathcal{Q}_\alpha(1, x+h) + \mathcal{Q}_\alpha(1, x) + \mathcal{Q}_\alpha(1, x'+h) + \mathcal{Q}_\alpha(1, x')) \right) \\
+ c_6 1_{\{|h|>1\}} |h| \left(|x-x'| \land 1 \right) \left((\mathcal{Q}_\alpha(1, x) + \mathcal{Q}_\alpha(1, x')) \right). 
\]

Then we can proceed in the same way as in the proof of Lemma 3.5 to obtain (3.35).

(ii) Let $0 < \alpha < 1$. Similarly to (3.36), we have that for $|h| \leq 1$ and $|x-x'| \leq 1$,
\[
|\delta_{g_1}(x+h, x'+h) - \delta_{g_1}(x, x')| \leq c_7 |h| \cdot |x-x'| \delta_0(1, x);
\]
Similarly to (3.37), for $|h| > 1$ and $|x-x'| \leq 1$, we obtain
\[
|\delta_{g_1}(x+h, x'+h) - \delta_{g_1}(x, x')| \leq c_8 |x-x'| \left((\mathcal{Q}_\alpha(1, x) + \mathcal{Q}_\alpha(1, x+h) \right). 
\] (3.38)

Noting (3.27), we thus get
\[
|\delta_{g_1}(x+h, x'+h) - \delta_{g_1}(x, x')| \\
\leq c_9 \left(|x-x'| \land 1 \right) \left(|h| \land 1 \right) \left((\mathcal{Q}_\alpha(1, x+h) + \mathcal{Q}_\alpha(1, x) + \mathcal{Q}_\alpha(1, x'+h) + \mathcal{Q}_\alpha(1, x') \right). 
\]

The rest of the proof is completely similar to Lemma 3.5. We omit the details. □

**Lemma 3.8.** Assume $\alpha = 1$. Then there exists a constant $C_{23} = C_{23}(d, \alpha, \kappa_0, \kappa_1) > 0$ such that for all $0 < t \leq 1$ and $x, x' \in \mathbb{R}^d$,
\[
\int_{\mathbb{R}^d} |f_t(x+h, x'+h) - f_t(x, x') - \chi_\alpha(h)h \cdot \nabla f_t(x, x')| \cdot \frac{1}{|h|^{d+\alpha}} dh \\
\leq C_{23} \left(1 + \ln(t^{-1}) \right) \left((t^{-1/\alpha}|x-x'|) \land 1 \right) \left((\mathcal{Q}_\alpha(1, x) + \mathcal{Q}_\alpha(t, x') \right). 
\] (3.39)

**Proof.** By (3.36), (3.38) and (3.32), we have
\[
|\delta_{g_1}(x+h, x'+h) - \delta_{g_1}(x, x') - \chi_\alpha(h)h \cdot \nabla g_1(x, x')| \\
\leq c_1 \left(|x-x'| \land 1 \right) \left(|h|^2 \land 1 \right) \left((\mathcal{Q}_\alpha(1, x+h) + \mathcal{Q}_\alpha(1, x) + \mathcal{Q}_\alpha(1, x'+h) + \mathcal{Q}_\alpha(1, x') \right). 
\]

Similarly to (3.36), if $t^{-1}|x-x'| \leq 1$, then
\[
|\nabla g_1(t^{-1}x) - \nabla g_1(t^{-1}x')| \leq c_2 t^{-1}|x-x'| (1 + |t^{-1}x|)^{-d-3}. 
\]

Noting (3.12), we thus get
\[
|\nabla g_1(t^{-1}x) - \nabla g_1(t^{-1}x')| \\
\leq c_3 \left((t^{-1}|x-x'|) \land 1 \right) \left((1 + |t^{-1}x|)^{-d-2} + (1 + |t^{-1}x'|)^{-d-2} \right). 
\]

The rest of the proof goes in the same way as in Lemma 3.6. □
4. Transition density of the Markov process associated with \( \mathcal{A} \)

In this section we will use Levi’s method (parametrix) to construct the transition density of the Markov processes that corresponds to the generator \( \mathcal{A} \), where

\[
\mathcal{A}f(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x + h) - f(x) - \nabla f(x) \cdot \chi_\alpha(h)h \right] \frac{n(x, h)}{|h|^{d+\alpha}} \, dh. \tag{4.1}
\]

Throughout this section, we assume that \( n(\cdot, \cdot) \) satisfies Assumption 1.1.

Levi’s method has been applied in [9] and [16] to construct transition densities of stable-like processes that are similar to what we consider here. In the sequel we will follow closely the approach of [9].

According to Assumption 1.1, for each \( y \in \mathbb{R}^d \), \( h \mapsto n(y, h)|h|^{-d-\alpha} \) is a function that satisfies (3.1) and (3.2). Let \( f_t^y(\cdot) \), \( t > 0 \), be the density functions of the stable-like Lévy process with the jump kernel \( n(y, h)|h|^{-d-\alpha} \), namely,

\[
f^y_t(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iux} e^{-i\psi^y(u)} \, du, \quad x \in \mathbb{R}^d, \quad t > 0, \tag{4.2}
\]

where

\[
\psi^y(u) = -\int_{\mathbb{R}^d \setminus \{0\}} (e^{iu \cdot h} - 1 - \chi_\alpha(h)iu \cdot h) \frac{n(y, h)}{|h|^{d+\alpha}} \, dh. \tag{4.3}
\]

Define the operator \( \mathcal{A}^y \) by

\[
\mathcal{A}^y f(x) := \int_{\mathbb{R}^d \setminus \{0\}} \left[ f(x + h) - f(x) - \nabla f(x) \cdot \chi_\alpha(h)h \right] \frac{n(y, h)}{|h|^{d+\alpha}} \, dh. \tag{4.4}
\]

**Remark 4.1.** In view of Assumption 1.1, all the estimates that we established in Lemmas 3.2 – 3.8 are also true for \( f^y_t \) (in place of \( f_t \)).

The following Lemma is analogous to [9, Theorem 2.5].

**Lemma 4.2.** Suppose \( \gamma \in (0, \alpha/4) \). Then there exists some constant \( C_{24} = C_{24}(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \gamma) > 0 \) such that for all \( 0 < t \leq 1 \) and \( x, x' \in \mathbb{R}^d \),

\[
|f^y_t(x) - f^y_t(x')| \leq C_{24} \left( |y - y'|^\gamma + 1 \right) \left( \hat{\eta}_{\alpha - \gamma} + \check{\eta}_{\alpha - \gamma} \right) (t, x), \tag{4.5}
\]

\[
|\nabla_x f^y_t(x) - \nabla_x f^y_t(x')| \leq C_{24} \left( |y - y'|^\gamma + 1 \right) \left( \hat{\eta}_{\alpha - 1} + \check{\eta}_{\alpha - 1} \right) (t, x), \tag{4.6}
\]

and

\[
\int_{\mathbb{R}^d} \left| \left( f^y_t - f^y_t \right)(x + h) - \left( f^y_t - f^y_t \right)(x) - \chi_\alpha(h)h \cdot \nabla \left( f^y_t - f^y_t \right)(x) \right| \cdot |h|^{-d-\alpha} \, dh \leq C_{24} \left( |y - y'|^\gamma + 1 \right) \left( \hat{\eta}_{\alpha} + \check{\eta}_{\alpha} \right) (t, x). \tag{4.7}
\]

**Proof.** The proof is almost the same as that of [9, Theorem 2.5], and we only need to verify that for \( t > 0 \), \( x, y, y' \in \mathbb{R}^d \),

\[
f^y_t(x) - f^y_t(x) = \int_0^t \int_{\mathbb{R}^d} \left( f^y_{t-s}(z) - f^y_{t-s}(x) \right) \left( \mathcal{A}^y - \mathcal{A}^y \right) (f^y_t(y - \cdot))(z) \, dz \, ds. \tag{4.8}
\]

By (4.2) and (4.4), we have

\[
(\mathcal{A}^y - \mathcal{A}^y)(f^y_t(y - \cdot))(z) = -\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left( \psi^y(u) - \psi^y(u) \right) e^{-iu\cdot(y-z)} \, du.
\]
Note that \( \int_{\mathbb{R}^d} (A^v - A^{v'}) (f^v_{\varepsilon}(x - \cdot)) (z) \, dz = 0 \). By the Fubini’s theorem, we have that for \( 0 < \varepsilon < t \),

\[
\int_{\varepsilon}^{t} \int_{\mathbb{R}^d} (f^v_{t-s}(z) - f^v_{t-s}(x)) (A^v - A^{v'}) (f^v_{\varepsilon}(x - \cdot)) (z) \, dz \, ds
\]

\[
= \int_{\varepsilon}^{t} \int_{\mathbb{R}^d} f^v_{t-s}(z) (A^v - A^{v'}) (f^v_{\varepsilon}(x - \cdot)) (z) \, dz \, ds
\]

\[
= -\frac{1}{(2\pi)^d} \int_{\varepsilon}^{t} \int_{\mathbb{R}^d} f^v_{t-s}(z) \left( \int_{\mathbb{R}^d} (\psi^v(u) - \psi^v(u)) e^{-s\psi^v(u)} e^{-iuv_z} \right) \, du \, dz \, ds
\]

\[
= -\frac{1}{(2\pi)^d} \int_{\varepsilon}^{t} \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} (\psi^v(u) - \psi^v(u)) e^{-s\psi^v(u)} e^{-iuv_z} \right) e^{-iu\psi^v(w)} \, du \, dz \, ds
\]

\[
= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iu\psi^v(w)} \left( e^{-iu\psi^v(w)} - e^{-\epsilon\psi^v(u)} e^{\epsilon\psi^v(u)} \right) \, du
\]

\[
= f^v_{t}(x) - \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-iu\psi^v(w)} \, du. \tag{4.9}
\]

By (2.8), (3.6), (3.19) and the dominated convergence theorem, we can let \( \varepsilon \to 0 \) in (4.9) to obtain (4.8). \( \square \)

For \( t \in (0, 1] \) and \( x, y \in \mathbb{R}^d \), define

\[
q(t, x, y) := f^v_{t}(y - x) \tag{4.10}
\]

and

\[
F(t, x, y) := (A - A^v) q(t, \cdot, y)(x)
\]

\[
= \int_{\mathbb{R}^d \setminus \{0\}} \left[ q(t, x + h, y) - q(t, x, y) \right. \\
- \chi_\alpha(h) h \cdot \nabla_x q(t, x, y) \left. \right] \frac{(n(x, h) - n(y, h))}{|h|^{d+\alpha}} \, dh.
\]

For functions \( \varphi_1, \varphi_2 \) on \( (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \), we introduce the notation \( \varphi_1 \otimes \varphi_2 \) by

\[
\varphi_1 \otimes \varphi_2(t, x, y) := \int_{0}^{t} \int_{\mathbb{R}^d} \varphi_1(t - s, x, z) \varphi_2(s, z, y) \, dz \, ds, \quad t \in (0, 1], \, x, y \in \mathbb{R}^d.
\]

Next, we study the convergence of the series \( \sum_{n=1}^{\infty} F^{\otimes n} \), where \( F^{\otimes 1} := F \) and \( F^{\otimes n} := F \otimes (F^{\otimes (n-1)}) \). Recall that the constant \( \theta \) is given in (1.3). In the rest of this paper, let \( \tilde{\theta} := \theta \wedge (\alpha/4) \).

**Lemma 4.3.** (i) Define

\[
\Phi(t, x, y) := \sum_{n=1}^{\infty} F^{\otimes n}(t, x, y), \quad (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d. \tag{4.11}
\]

Then the series on the right-hand side of (4.11) converges locally uniformly on \( (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \). Moreover, \( \Phi \) is continuous on \( (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \), and there exists a constant \( C_{26} := C_{26}(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \theta) > 0 \) such that for all \( (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \),

\[
|\Phi(t, x, y)| \leq C_{26} \left( g^0_{\tilde{\theta}}(t, x - y) + g_0^{\tilde{\theta}}(t, x - y) \right). \tag{4.12}
\]
(ii) Given $\gamma \in (0, \hat{\theta})$, there exists a constant $C_{27} = C_{27}(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \gamma) > 0$ such that for all $t \in (0, 1]$ and $x, x', y \in \mathbb{R}^d$,
\[|\Phi(t, x, y) - \Phi(t, x', y)| \leq C_{27} \left(|x - x'|^{\beta-\gamma} \wedge 1\right) \left\{ \left(\gamma + \gamma_{\gamma-\theta}^0\right)(t, x - y) + \left(\gamma + \gamma_{\gamma-\theta}^0\right)(t, x' - y) \right\}.\]

**Proof.** In view of Lemma 3.2 – Lemma 4.2 and Remark 4.1, the proof is essentially the same as in [9, Theorem 4.1]. We omit the details. \[\Box\]

By (2.9), (3.15) and (4.12), there exists a constant $C_{28} = C_{28}(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \theta) > 0$ such that
\[q \otimes \Phi(t, x, y) \leq C_{28} \left(q_{\alpha + \theta}^0 + q_{\alpha}^0\right)(t, x - y), \quad t \in (0, 1], \ x, y \in \mathbb{R}^d. \quad (4.13)\]

It follows that
\[p(t, x, y) := q(t, x, y) + q \otimes \Phi(t, x, y), \quad (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d, \quad (4.14)\]
is well-defined.

**Proposition 4.4.** There exists a constant $C_{29} = C_{29}(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \theta) > 0$ such that
\[|p(t, x, y)| \leq C_{29}q_{\alpha}^0(t, x - y), \quad (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d. \quad (4.15)\]
Moreover, the function $(t, x, y) \mapsto p(t, x, y)$ is continuous on $(0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$.

**Proof.** The estimate (4.15) is a simple consequence of (3.15) and (4.13). By (3.6) and Assumption 1.1, there exists a constant $c_1 = c_1(d, \alpha, \kappa_0) > 0$ with
\[|\exp(-iu \cdot x - t\psi^\nu(u))| \leq \exp(-c_1t|u|^\alpha), \quad \forall t > 0, \ x, y, u \in \mathbb{R}^d,
\]
where $\psi^\nu$ is given in (4.3). The continuity of $(t, x, y) \mapsto q(t, x, y)$ now follows from (4.10), (4.2) and the dominated convergence. Since $q(t, x, y)$ and $\Phi(t, x, y)$ are both continuous, again by dominated convergence, the function $(0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \ni (t, x, y) \mapsto p(t, x, y)$ is also continuous. \[\Box\]

In the remaining part of this section we will show that $p(t, x, y)$ is the transition density of the Markov process associated with $\mathcal{A}$. The ideas for the proof of the next two propositions come from [13, Chap. 1, Theorems 4 - 5].

**Proposition 4.5.** Suppose that the function $\varphi : (0, 1] \times \mathbb{R}^d \to \mathbb{R}$ is continuous and such that for all $x, x' \in \mathbb{R}^d$ and $t \in (0, 1]$,
\[|\varphi(t, x)| \leq c_{\varphi}t^{-1+\theta/\alpha} \quad (4.16)\]
and
\[|\varphi(t, x) - \varphi(t, x')| \leq c_{\varphi}t^{-1+\gamma/\alpha} \left(|x - x'|^{\theta-\gamma} \wedge 1\right), \quad (4.17)\]
where $c_{\varphi} > 0$ and $\gamma \in (0, \hat{\theta})$ are constants. Consider the function $V$ defined by
\[V(t, x) := \int_0^t \int_{\mathbb{R}^d} q(t - s, x, z)\varphi(s, z)dzds, \quad (t, x) \in (0, 1] \times \mathbb{R}^d. \quad (4.18)\]
Then for each \( t \in (0, 1] \), \( \mathcal{A}V(t, \cdot) \) is well-defined and

\[
\mathcal{A}V(t, \cdot)(x) = \int_0^t \int_{\mathbb{R}^d} \mathcal{A}q(t-s, \cdot, z)(x)\varphi(s, z)dz\,ds, \quad x \in \mathbb{R}^d. \tag{4.19}
\]

We also have the estimate

\[
|\mathcal{A}V(t, \cdot)(x)| \leq C_{30} \left( 1 + \ln \left( \frac{t}{s} \right) \right) t^{-1}, \quad \forall x \in \mathbb{R}^d, \ t \in (0, 1], \tag{4.20}
\]

where \( C_{30} = C_{30}(d, \alpha, \kappa_0, \kappa_1, \theta, \gamma, c_\varphi) > 0 \) is a constant.

**Proof.** Let \( 0 < s < t \leq 1 \) and \( x \in \mathbb{R}^d \) be arbitrary. By (3.15) and (4.16), we have

\[
\int_{\mathbb{R}^d} |q(t-s, x, z)| \varphi(s, z)|dz| \leq c_1 \int_{\mathbb{R}^d} \tilde{g}_0^0(t-s, x-z)s^{-1+\theta/\alpha}dz \tag{2.7} \leq c_2 s^{-1+\theta/\alpha}. \tag{4.21}
\]

So the function \( V \) in (4.18) is well-defined. Let

\[
J(t, s, x) := \int_{\mathbb{R}^d} q(t-s, x, z)\varphi(s, z)dz. \tag{4.22}
\]

By (3.16), (3.25) and (3.31), we obtain that for \( |x-x_0| \leq (t-s)^{1/\alpha} \),

\[
|\nabla_x q(t-s, x, z)| \leq c_3 \tilde{g}^0_{\alpha-1}(t-s, x-z) \leq c_4 \tilde{g}^0_{\alpha-1}(t-s, x_0-z). \tag{2.6}
\]

So it is easy to see that for \( 0 < s < t \leq 1 \) and \( x \in \mathbb{R}^d \),

\[
\nabla_x J(t, s, x) = \int_{\mathbb{R}^d} \nabla_x q(t-s, x, z)\varphi(s, z)dz. \tag{4.23}
\]

Similarly, we have

\[
|\mathcal{A}q(t-s, \cdot, z)(x)| \leq c_5 \left( 1 + \ln \left( \frac{t}{s} \right) \right) \tilde{g}^0_0(t-s, x-z) \tag{4.24}
\]

and

\[
\mathcal{A}J(t, s, \cdot)(x) = \int_{\mathbb{R}^d} \mathcal{A}q(t-s, \cdot, z)(x)\varphi(s, z)dz. \tag{4.25}
\]

Let \( y \in \mathbb{R}^d \) be arbitrary. We now write

\[
J(t, s, x) = \int_{\mathbb{R}^d} q(t-s, x, z) (\varphi(s, z) - \varphi(s, y))dz + \varphi(s, y) \int_{\mathbb{R}^d} \left( q(t-s, x, z) - f^p_{t-s}(z-x) \right)dz + \varphi(s, y). \tag{4.26}
\]

We will complete the proof in two steps.

"Step 1": We show that if \( \alpha \geq 1 \), then

\[
\nabla_x V(t, x) = \int_0^t \nabla_x J(t, s, x)ds, \quad (t, x) \in (0, 1] \times \mathbb{R}^d. \tag{4.27}
\]
By (4.10), (4.16), (4.17) and (4.26), we have
\[ |\nabla_x J(t, s, x)| \leq \left| \int_{\mathbb{R}^d} \nabla_x \left( f^x_{t-s}(z-x) \right) \left( \varphi(s, z) - \varphi(s, y) \right) dz \right| + |\varphi(s, y)| \cdot \left| \int_{\mathbb{R}^d} \left( \nabla_x \left( f^y_{t-s}(z-x) \right) - \nabla_x \left( f^y_{t-s}(z-x) \right) \right) dz \right| \]
\[ \leq c_5 \int_{\mathbb{R}^d} s^{-1+\gamma/\alpha} \left( |y-z|^{\theta-\gamma} \wedge 1 \right) \theta_0^0(t-s, s-x) dz + c_6 \int_{\mathbb{R}^d} \left( |y-z|^{\theta} \wedge 1 \right) \left( \theta_0^0 + \theta_0^{\gamma-\gamma} \right) (t-s, s-x, z) dz, \]
(4.28)
where the constants $c_5$ and $c_6$ are independent of $y$. Choosing $y = x$ in (4.28), we get
\[ |\nabla_x J(t, s, x)| \leq c_5 \int_{\mathbb{R}^d} s^{-1+\gamma/\alpha} (t-s)^{\theta-\gamma} + c_6 s^{-1+\theta/\alpha} (t-s)^{\theta-1}. \]
(4.29)
If $\alpha \geq 1$, then the right-hand side of (4.29), as a function with the variable $s$, is integrable on $[0, \ell]$. The equation (4.27) now follows by the dominated convergence theorem.

"Step 2": We consider a general $\alpha \in (0, 2)$ and show that $AV(t, x)J(t, s, x)$ is well-defined and (4.19) holds. For $h \in \mathbb{R}^d$ and $h \neq 0$, it follows from (4.27) that
\[ V(t, x + h) - V(t, x) - \chi_0(h)h \cdot \nabla_x V(t, x) \]
\[ = \int_0^1 [J(t, s, x + h) - J(t, s, x) - \chi_0(h)h \cdot \nabla_x J(t, s, x)] ds. \]
(4.30)
By (4.10), (4.16), (4.17) and (4.26), we get
\[ |J(t, s, x + h) - J(t, s, x) - \chi_0(h)h \cdot \nabla_x J(t, s, x)| \]
\[ \leq c_{10} \int_{\mathbb{R}^d} |f^x_{t-s}(z-x-h) - f^x_{t-s}(z-x) - \chi_0(h)h \cdot \nabla_x \left( f^{y}_{t-s}(z-x) \right)| s^{-1+\gamma/\alpha} \]
\[ \times \left( |y-z|^{\theta-\gamma} \wedge 1 \right) dz + c_{11} s^{-1+\theta/\alpha} \int_{\mathbb{R}^d} \left| f^y_{t-s}(z-x-h) - f^{y}_{t-s}(z-x-h) \right| \]
\[ - f^z_{t-s}(z-x) + f^{R}_{t-s}(z-x) - \chi_0(h)h \cdot \nabla_x \left( f^y_{t-s}(z-x) \right) \]
\[ + \chi_0(h)h \cdot \nabla_x \left( f^y_{t-s}(z-x) \right) dz. \]
(4.31)
It follows from (3.25), (3.31), (4.7), (4.31) and the Fubini’s theorem that
\[ I(t, s, x) := \int_{\mathbb{R}^d \setminus \{0\}^d} \left| J(t, s, x + h) - J(t, s, x) - \chi_0(h)h \cdot \nabla_x J(t, s, x) \right| \frac{n(x, h)}{|h|^{d+\alpha}} dh \]
\[ \leq c_{12} \left( 1 + \ln \left( (t-s)^{-1} \right) \right) \int_{\mathbb{R}^d} \theta_0^0(t-s, s-x) s^{-1+\gamma/\alpha} \left( |y-z|^{\theta-\gamma} \wedge 1 \right) dz \]
\[ + c_{13} s^{-1+\theta/\alpha} \left( 1 + \ln \left( (t-s)^{-1} \right) \right) \int_{\mathbb{R}^d} \left( |y-z|^{\theta} \wedge 1 \right) \left( \theta_0^0 + \theta_0^{\gamma-\gamma} \right) (t-s, s-x, z) dz. \]
(4.32)
Choosing $y = x$ in (4.32) and applying (2.7), we get
\[
I(t, s, x) \leq c_{14} \left( 1 + \ln \left( (t-s)^{-1} \right) \right) s^{-1+\gamma/\alpha} (t-s)^{(\delta-\gamma-\alpha)/\alpha}
+ c_{15} \left( 1 + \ln \left( (t-s)^{-1} \right) \right) s^{-1+\theta/\alpha} (t-s)^{(\delta-\alpha)/\alpha}
\]
\[
\gamma \leq c_{16} \left( 1 + \ln \left( (t-s)^{-1} \right) \right) s^{-1+\gamma/\alpha} (t-s)^{(\delta-\gamma-\alpha)/\alpha},
\]
which implies
\[
\int_0^t I(t, s, x) ds \leq c_{16} \int_0^t \left( 1 + \ln \left( (t-s)^{-1} \right) \right) s^{-1+\gamma/\alpha} (t-s)^{(\delta-\gamma-\alpha)/\alpha} ds
\leq c_{17} \left( 1 + \ln (t^{-1}) \right) t^{(\delta-\alpha)/\alpha} + c_{17} t^{-1+\gamma/\alpha} (t-\gamma)/(2\alpha)
\leq c_{18} \left( 1 + \ln (t^{-1}) \right) t^{-1}.
\]
So $A^t V(t, q)$ is well-defined and (4.20) is true. By (4.25), (4.30) and the Fubini's theorem, we obtain
\[
A^t V(t, q) = \int_0^t A J(t, s, q)(x) ds = \int_0^t \int_{\mathbb{R}^d} A q(t-s, \cdot, z)(x) \varphi(s, z) dz ds.
\]
This completes the proof. \(\square\)

**Proposition 4.6.** Let $\varphi$ and $V$ be as in Proposition 4.5. Then for all $t \in (0, 1]$ and $x \in \mathbb{R}^d$, $\partial_t V(t, x)$ exists and satisfies
\[
\partial_t V(t, x) = \varphi(t, x) + \int_0^t \int_{\mathbb{R}^d} A^t q(t-s, \cdot, z)(x) \varphi(s, z) dz ds.
\]
Moreover, for each $x \in \mathbb{R}^d$, $t \mapsto \partial_t V(t, x)$ is continuous on $(0, 1]$.

**Proof.** Let $J$ be the same as in (4.22). It is easy to verify that $\partial_t J(t, s, x)$ exists for $0 < s < t \leq 1$ and $x \in \mathbb{R}^d$.

Let $x \in \mathbb{R}^d$ be fixed. We only consider the case with $0 < t < 1$, $h > 0$ and $t + h \leq 1$, since the argument we will use works similarly when $0 < t - h < t \leq 1$. We have
\[
h^{-1} (V(t+h, x) - V(t, x))
= h^{-1} \int_0^{t+h} J(t+h, s, x) ds - h^{-1} \int_0^t J(t, s, x) ds
= h^{-1} \int_t^{t+h} J(t+h, s, x) ds + \int_0^t h^{-1} [J(t+h, s, x) - J(t, s, x)] ds
= h^{-1} \int_t^{t+h} [J(t+h, s, x) - \varphi(t, x)] ds + \varphi(t, x) + \int_0^t J_1(t^*, s, x) ds,
\]
where $J_1(t, s, x) := \partial_t J(t, s, x)$ and $t^* \in [t, t+h]$. 
We will complete the proof in several steps. "Step 1": We show that
\[ \lim_{h \to 0} h^{-1} \int_{t}^{t+h} |J(t + h, s, x) - \varphi(t, x)| \, ds = 0. \] (4.37)

For \( s \in (t, t + h) \), we have
\[
|J(t + h, s, x) - \varphi(t, x)| \\
= \left| \int_{\mathbb{R}^d} \left[ q(t + h - s, x, z) - f^c_{t+h-s}(z - x) \right] \varphi(s, z) \, dz + \sum_{\beta = 1}^{m} f^c_{t+h-s}(z - x) \right| \, ds \]
\[
\leq \int_{\mathbb{R}^d} \left| f^c_{t+h-s}(z - x) - f^c_{t+h-s}(z - x) \right| \, ds \]
\[
= I_1 + I_2. \] (4.38)

For \( I_1 \), by (4.5), (4.16) and noting that \( s \in (t, t + h) \), we have
\[
I_1 \leq c_1 s^{-1+\hat{\gamma}/\alpha} \int_{\mathbb{R}^d} \left( |z - x|^{\hat{\gamma}} \wedge 1 \right) \left( g_0^0 + g_{\alpha}^\gamma \right) (t + h - s, z - x) \, dz \]
\[
\leq c_2 t^{-1+\hat{\gamma}/\alpha} (t + h - s)^{\hat{\gamma}/\alpha} \leq c_2 t^{-1+\hat{\gamma}/\alpha} h^{\hat{\gamma}/\alpha}. \] (4.39)

For \( I_2 \) and \( n \in \mathbb{N} \), by (3.15), (4.16) and noting that \( s \in (t, t + h) \), we have
\[
I_2 \leq c_3 \int_{|z - x| \geq 1/n} g_0^0 (t + h - s, z - x) \cdot |\varphi(s, z) - \varphi(t, x)| \, dz \]
\[
+ c_3 \int_{|z - x| \leq 1/n} g_0^0 (t + h - s, z - x) \cdot |\varphi(s, z) - \varphi(t, x)| \, dz \]
\[
\leq c_4 t^{-1+\hat{\gamma}/\alpha} \int_{|z - x| \geq 1/n} g_0^0 (t + h - s, z - x) \, dz \]
\[
+ c_3 \int_{|z - x| \leq 1/n} g_0^0 (t + h - s, z - x) \cdot |\varphi(s, z) - \varphi(t, x)| \, dz. \] (4.40)

For any given \( \varepsilon > 0 \), by the continuity of \( \varphi \), we can find \( n_0 \in \mathbb{N} \) and \( h_0 > 0 \) such that
\[
|\varphi(s, z) - \varphi(t, x)| < \varepsilon, \quad \forall s \in (t, t + h_0), \, |z - x| \leq \frac{1}{n_0}. \] (4.41)
By (4.40) and (4.41), we get that for $t < s < t + h < t + h_0$,

$$
I_2 \leq c t^{-1+\hat{\beta}/\alpha} \int_{\{|z-x| \geq 1/n_0\}} \varphi_0^h(t+h-s, z-x)dz + c_5 \varepsilon
$$

$$
= c t^{-1+\hat{\beta}/\alpha} \int_{\{|z| \geq 1/n_0\}} \varphi_0^h(t+h-s, z)dz + c_5 \varepsilon
$$

$$
= c t^{-1+\hat{\beta}/\alpha} \int_{\{|z'| \geq (t+s-h)^{-1/\alpha}/n_0\}} \varphi_0^h(1, z')dz' + c_5 \varepsilon
$$

$$
\leq c t^{-1+\hat{\beta}/\alpha} \int_{\{|z'| \geq h^{-1/\alpha}/n_0\}} \varphi_0^h(1, z')dz' + c_5 \varepsilon. \quad (4.42)
$$

Combining (4.38), (4.39), and (4.42) yields

$$
\lim_{h \downarrow 0} h^{-1} \int_t^{t+h} |J(t+h, s, x) - \phi(t, x)| ds \leq c_5 \varepsilon.
$$

Since $\varepsilon > 0$ is arbitrary, the convergence in (4.37) follows.

"Step 2": We evaluate the integral $\int_0^t \partial_t J(t^*, s, x)ds$. If $t > s$, then

$$
\partial_t J(t, s, x) = \int_{\mathbb{R}^d} \partial_t q(t-s, x, z) \psi(s, z) dz
$$

$$
= \int_{\mathbb{R}^d} A^c q(t-s, z, x) \psi(s, z) dz
$$

$$
= \int_{\mathbb{R}^d} (A^c - A) q(t-s, z, x) \psi(s, z) dz
$$

$$
+ \int_{\mathbb{R}^d} A q(t-s, z, x) \psi(s, z) dz
$$

$$
=: I_3 + I_4. \quad (4.44)
$$

For $III$, by (3.25), (3.31) and (4.16), we have

$$
|I_3| \leq c_6 s^{-1+\beta/\alpha} \left(1 + \ln \left((t-s)^{-1}\right)\right) \int_{\mathbb{R}^d} \varphi_0^h(t-s, z-x) dz
$$

$$
\leq c_7 s^{-1+\beta/\alpha} \left(1 + \ln \left((t-s)^{-1}\right)\right) (t-s)^{-1+\hat{\beta}/\alpha}. \quad (4.45)
$$

The term $I_4$ has already been treated in Proposition 4.5, see (4.25) and (4.33). Altogether we obtain

$$
|\partial_t J(t, s, x)| \leq c_8 s^{-1+\gamma/\alpha} \left(1 + \ln \left((t-s)^{-1}\right)\right) (t-s)^{-1+(\hat{\beta}-\gamma)/\alpha}. \quad (4.46)
$$

Consider

$$
H := \int_0^t J_1(t^*, s, x)ds - \int_0^t J_1(t, s, x)ds.
$$

Note that for $0 < s < t$ and $t^* \in [t, t+h]$, it holds that

$$
|J_1(t^*, s, x) - J_1(t, s, x)| \leq 2c_8 s^{-1+\gamma/\alpha} \left(1 + \ln \left((t-s)^{-1}\right)\right) (t-s)^{-1+(\hat{\beta}-\gamma)/\alpha}. \quad (4.47)
$$
Since for \( s < t \leq t^* \leq t + h \), \( \lim_{h \to 0} J_1(t^*, s, x) = J_1(t, s, x) \), by (4.47) and dominated convergence, we obtain
\[
\lim_{h \to 0} \int_0^t |J_1(t^*, s, x) ds - J_1(t, s, x)| ds = 0.
\]
So we get \( \lim_{h \to 0} |H| = 0 \). By (4.36), (4.37) and (4.43), we obtain (4.35).

“Step 3”: To see that the function \( t \mapsto \partial_t V(t, x) \) is continuous, we can argue as above, namely, for \( h \in (0, \delta) \),
\[
\int_0^{t+h} \int_{\mathbb{R}^d} \mathcal{A}^2 q(t + h - s, \cdot, z)(x) \varphi(s, z) dz ds
\]
\[
= \int_0^t \int_{\mathbb{R}^d} \mathcal{A}^2 q(t + h - s, \cdot, z)(x) \varphi(s, z) dz ds
\]
\[
+ \int_t^{t+h} \int_{\mathbb{R}^d} \mathcal{A}^2 q(t + h - s, \cdot, z)(x) \varphi(s, z) dz ds,
\]
where the second term on the right-hand side goes to 0 as \( h \to 0 \), since by (4.46),
\[
\lim_{h \to 0} \int_0^{t+h} |J_1(t + h, s, x)| ds
\]
\[
\leq \lim_{h \to 0} \int_0^{t+h} s^{-1+\gamma/\alpha} \left( 1 + \ln \left( (t + h - s)^{-1} \right) \right) (t + h - s)^{-1+(\theta-\gamma)/\alpha} ds = 0,
\]
while the first term converges to \( \int_0^t \int_{\mathbb{R}^d} \mathcal{A}^2 q(t - s, \cdot, z)(x) \varphi(s, z) dz ds \) by (4.43), (4.46) and dominated convergence.

**Corollary 4.7.** Let \( \varphi \) and \( V \) be as in Proposition 4.5. Then the function \( (t, x) \mapsto V(t, x) \) is bounded continuous on \( [0, 1] \times \mathbb{R}^d \).

**Proof.** According to (4.21), the function \( V \) is obviously bounded on \( (0, 1] \times \mathbb{R}^d \).
Let \((t_0, x_0) \in (0, 1] \times \mathbb{R}^d \) be fixed. Choose \( \epsilon > 0 \) such that \( \epsilon < t_0 \). In view of (4.43) and (4.46), we obtain for \( s < t \) and \( x \in \mathbb{R}^d \),
\[
\left| \int_{\mathbb{R}^d} \mathcal{A}^2 q(t - s, \cdot, z)(x) \varphi(z) dz \right|
\]
\[
\leq c_1 s^{-1+\gamma/\alpha} \left( 1 + \ln \left( (t - s)^{-1} \right) \right) (t - s)^{-1+(\theta-\gamma)/\alpha}.
\]
Arguing as in (4.34), we get
\[
\left| \int_0^t \int_{\mathbb{R}^d} \mathcal{A}^2 q(t - s, \cdot, z)(x) \varphi(z) dz ds \right| \leq c_2 \left( 1 + \ln \left( t^{-1} \right) \right) t^{-1}, \quad t \in (0, 1], x \in \mathbb{R}^d.
\]
By (4.35), we see that \( \partial_t V(t, x) \) is bounded on \([\epsilon, 1] \times \mathbb{R}^d \). Therefore, for \((t, x) \in [\epsilon, 1] \times \mathbb{R}^d \),
\[
|V(t, x) - V(t_0, x_0)| \leq |V(t, x) - V(t_0, x)| + |V(t_0, x) - V(t_0, x_0)| \leq c_3 |t - t_0| + |V(t_0, x) - V(t_0, x_0)|.
\]
By (4.23), \( J(t, s, x) \) is continuous in \( x \). Since \( V(t, x) = \int_0^t J(t, s, x) ds \), it follows from (4.21) and dominated convergence that for each \( t \in (0, 1] \), the function \( x \mapsto \)
$V(t, x)$ is continuous. In view of (4.48), we get $\lim_{(t, x) \to (t_0, x_0)} V(t, x) = V(t_0, x_0)$. □

Next, we show that $p(t, x, y)$ defined in (4.14) is the fundamental solution to the Cauchy problem of the equation $\partial_t u = Au$.

**Proposition 4.8.** Let $\phi \in C^\infty_0(\mathbb{R}^d)$. Define $u(t, x) := \int_{\mathbb{R}^d} p(t, x, y) \phi(y) dy$, $t \in (0, 1]$, and $u(0, x) := \phi(x)$, where $x \in \mathbb{R}^d$. Then $u \in C_0([0, 1] \times \mathbb{R}^d)$ and

$$\partial_t u(t, x) = Au(t, \cdot)(x), \quad t \in (0, 1], \ x \in \mathbb{R}^d. \quad (4.49)$$

Moreover, for each $x \in \mathbb{R}^d$, $t \mapsto \partial_t u(t, x)$ is continuous on $(0, 1]$; for each $t \in (0, 1]$, $x \mapsto \partial_t u(t, x)$ is continuous on $\mathbb{R}^d$.

**Proof.** Set

$$I_1(t, x) := \int_{\mathbb{R}^d} q(t, x, y) \phi(y) dy$$

and

$$I_2(t, x) := \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} q(t - s, x, z) \Phi(s, z, y) \phi(y) dy dz ds$$

$$= \int_0^t \int_{\mathbb{R}^d} q(t - s, x, z) \varphi(s, z) dz ds,$$

where $\varphi(s, z) := \int_{\mathbb{R}^d} \Phi(s, z, y) \phi(y) dy$. Then $\varphi$ satisfies (4.16) and (4.17).

By Proposition 4.5, $AI_2(t, \cdot)(x)$ is well-defined for all $t \in (0, 1]$ and $x \in \mathbb{R}^d$, and it holds that

$$Au(t, \cdot)(x) = \int_{\mathbb{R}^d} A q(t, \cdot, y)(x) \phi(y) dy$$

$$+ \int_0^t \int_{\mathbb{R}^d} \partial_z A q(t - s, \cdot, z)(x) \varphi(s, z) dz ds. \quad (4.50)$$

For $t \in (0, 1]$ and $x \in \mathbb{R}^d$, we have

$$\partial_t I_1(t, x) = \int_{\mathbb{R}^d} A^v q(t, \cdot, y)(x) \phi(y) dy,$$  \quad (4.51)

and, by Proposition 4.6,

$$\partial_t I_2(t, x) = \varphi(t, x) + \int_0^t \int_{\mathbb{R}^d} A^z q(t - s, \cdot, z)(x) \varphi(s, z) dz ds. \quad (4.52)$$
So for \( t \in (0,1] \) and \( x \in \mathbb{R}^d \),
\[
\varphi(t,x) = \int_{\mathbb{R}^d} \Phi(t,x,y)\phi(y)dy \\
\quad = \int_{\mathbb{R}^d} F(t,x,y)\phi(y)dy \\
\quad + \int_{\mathbb{R}^d} \left( \int_0^t \int_{\mathbb{R}^d} F(t-s,x,z)\Phi(s,z,y)dzds \right) \phi(y)dy
\]
\[
= \int_{\mathbb{R}^d} (A - A^q) q(t,\cdot,y)(x)\phi(y)dy \\
\quad + \int_0^t \int_{\mathbb{R}^d} (A - A^q) q(t-s,\cdot,z)(x)\varphi(s,z)dzds. \tag{4.53}
\]
Combining (4.50), (4.51), (4.52) and (4.53), we arrive at (4.49).

By Corollary 4.7, we see that \( u(0,t,x) = u(t,x) \) at \( (t,x) = (0,x_0) \), where \( x_0 \in \mathbb{R}^d \). We have
\[
\|u(t,x) - u(0,x_0)\| \leq \|u(t,x) - u(0,x)\| + \|\phi(x) - \phi(x_0)\|.
\]
So it suffices to show that \( \lim_{t \to 0} u(t,x) = u(0,x) \), and the convergence is uniform with respect to \( x \in \mathbb{R}^d \). Noting that \( \|\phi(y) - \phi(x)\| \leq c_2 (1 \wedge |x - y|^{\alpha/2}) \), we obtain
\[
|I_1(t,x) - \phi(x)| \leq \left| \int_{\mathbb{R}^d} q(t,x,y) [\phi(y) - \phi(x)] dy \right|
\]
\[
\quad + \left| \int_{\mathbb{R}^d} q(t,x,y)\phi(y)dy - \phi(x) \right|
\]
\[
\quad \leq c_2 \int_{\mathbb{R}^d} \phi_\alpha^{\alpha/2}(t,y-x)dy + \left| \phi(x) \int_{\mathbb{R}^d} [f_1^n(y-x) - f_1^n(y-x)] dy \right|
\]
\[
\quad \leq c_4 t^{1/2} + c_4 t^{\theta/\alpha}, \tag{4.5}
\]
which shows that \( \lim_{t \to 0} \sup_{x \in \mathbb{R}^d} |I_1(t,x) - \phi(x)| = 0 \). Finally, it follows from (4.21) that \( \lim_{t \to 0} \sup_{x \in \mathbb{R}^d} |I_2(t,x)| = 0 \). So \( u(t,x) \to u(0,x) \) uniformly in \( x \in \mathbb{R}^d \) as \( t \to 0 \).

Since \( \partial_t u(t,x) = \partial_t I_1(t,x) + \partial_t I_2(t,x) \), the continuity of \( t \mapsto \partial_t I_1(t,x) \) follows easily by (4.51), (4.52) and Proposition 4.6. Noting that \( x \mapsto A^q(t,\cdot,y)(x) \) is continuous and for \( |x - x_0| \leq t^{1/\alpha} \),
\[
|A^q(t,\cdot,\cdot)(x)| \leq c_5 \hat{g}_0(t,x - z) \tag{2.6}
\]
the continuity of \( x \mapsto \partial_t I_1(t,x) \) follows by (4.51) and dominated convergence. Similarly, \( x \mapsto \partial_t I_2(t,x) \) is also continuous. So the continuity of \( x \mapsto \partial_t u(t,x) \) follows. This completes the proof. \( \square \)

**Proposition 4.9.** Let \((X, (P^x))\) be the Markov process associated with the operator \( A \) defined in (4.1). Then the function \( p(t,x,y) \), \( (t,x,y) \in (0,1] \times \mathbb{R}^d \), is the transition density of \((X, (P^x))\), namely, for each \( 0 < t \leq 1 \) and \( x \in \mathbb{R}^d \),
\[
P^x (X_t \in E) = \int_E p(t,x,y)dy, \quad \forall E \in \mathcal{B}(\mathbb{R}^d),
\]
Proof. Let $0 < t \leq 1$ be fixed. Consider $\phi \in C_0^\infty(\mathbb{R}^d)$ that is arbitrary. Define $u(s, x) := \int_{\mathbb{R}^d} p(s, x, y) \phi(y) \, dy$, $s > 0, x \in \mathbb{R}^d$, and $u(0, \cdot) = \phi$. Let
\[
\tilde{u}(s, x) := u(t - s, x), \quad 0 \leq s \leq t, x \in \mathbb{R}^d.
\]
By Theorem 4.8, $\tilde{u} \in C_b([0, t] \times \mathbb{R}^d)$ and
\[
\partial_\nu \tilde{u}(s, x) + A\tilde{u}(s, x) = 0, \quad 0 \leq s < t, x \in \mathbb{R}^d, \quad \tilde{u}(t, x) = \phi(x). \tag{4.54}
\]
Let $(\rho_n)_{n \in \mathbb{N}}$ be a mollifying sequence in $\mathbb{R}^d$. Set
\[
\tilde{u}_n(s, \cdot) := \tilde{u}(s, \cdot) * \rho_n.
\]
Then for $0 < \varepsilon < t$, we have $\tilde{u}_n \in C^{1,2}_b([0, t - \varepsilon] \times \mathbb{R}^d)$. Indeed, for $(s, x) \in [0, t - \varepsilon] \times \mathbb{R}^d$,
\[
\partial_s \tilde{u}_n(s, x) = \int_{\mathbb{R}^d} \partial_s \tilde{u}(s, x - y) \rho_n(y) \, dy.
\]
Note that for each $x \in \mathbb{R}^d$, $s \mapsto \partial_s \tilde{u}(s, x)$ is continuous, which implies that for each $x \in \mathbb{R}^d$, $s \mapsto \partial_s \tilde{u}_n(s, x)$ is continuous. Since, by (4.20), (4.50) and (4.54), $\partial_s \tilde{u}(s, x)$ is bounded on $[0, t - \varepsilon] \times \mathbb{R}^d$, it follows that $\partial_s \tilde{u}_n(s, x)$ is Lipschitz in $x$, uniformly with respect to $s \in [0, t - \varepsilon]$. Similarly to Corollary 4.7, we conclude that $\partial_s \tilde{u}_n \in C_b([0, t - \varepsilon] \times \mathbb{R}^d)$. It is obvious that
\[
\partial_t \tilde{u}_n(s, x) = \int_{\mathbb{R}^d} \partial_t \rho_n(x - y) \tilde{u}(s, y) \, dy
\]
\[
= \int_{\mathbb{R}^d} \tilde{u}(s, x - y) \partial_t \rho_n(y) \, dy \in C_b([0, t - \varepsilon] \times \mathbb{R}^d).
\]
The cases for second order derivatives are similar. So $\tilde{u}_n \in C^{1,2}_b([0, t - \varepsilon] \times \mathbb{R}^d)$.

According to [22, Theorem (1.1)], the process
\[
\tilde{u}_n(s, X_s) - \int_0^s (\partial_r + A)\tilde{u}_n(r, X_r) \, dr, \quad s \in [0, t - \varepsilon],
\]
is a $\mathbb{P}^x$-martingale. So
\[
\mathbb{E}^x[\tilde{u}_n(t - \varepsilon, X_{t-\varepsilon})] - \mathbb{E}^x[\tilde{u}_n(0, X_0)] = \mathbb{E}^x \left[ \int_0^{t-\varepsilon} (\partial_r + A)\tilde{u}_n(r, X_r) \, dr \right].
\]
As $n \to \infty$, it is clear that $\partial_s \tilde{u}_n(s, x) \to \partial_s \tilde{u}(s, x)$, since for each $s \in [0, t - \varepsilon]$, $x \mapsto \partial_s \tilde{u}(s, x)$ is continuous; moreover, according to (4.34),
\[
A\tilde{u}_n(s, x) = A \left( \int_{\mathbb{R}^d} \tilde{u}(s, x - y) \rho_n(y) \, dy \right)
\]
\[
= \int_{\mathbb{R}^d} A\tilde{u}(s, \cdot - y)(x) \rho_n(y) \, dy \to A\tilde{u}(s, x),
\]
where we used the fact that for each $s \in [0, t - \varepsilon]$, $x \mapsto A\tilde{u}(s, \cdot)(x)$ is continuous. So $(\partial_r + A)\tilde{u}_n(r, X_r)$ converges boundedly and pointwise to $(\partial_r + A)\tilde{u}(r, X_r)$. By dominated convergence, we obtain
\[
\mathbb{E}^x[\tilde{u}(t - \varepsilon, X_{t-\varepsilon})] - \mathbb{E}^x[\tilde{u}(0, X_0)] = \mathbb{E}^x \left[ \int_0^{t-\varepsilon} (\partial_r + A)\tilde{u}(r, X_r) \, dr \right] = 0.
\]
So

\[ \mathbb{E}^x[u(\varepsilon, X_{t-\varepsilon})] = \tilde{u}(0, x) = u(t, x). \]

Letting \( \varepsilon \to 0 \), we get

\[ u(t, x) = \mathbb{E}^x[u(0, X_{t-})] = \mathbb{E}^x[u(0, X_t)] = \mathbb{E}^x[\phi(X_t)], \]

at least for \( t \in I := \{ t \in (0, 1] : X_{t-} = X_t, \text{ P}_x\text{-a.s.} \} \). By [11, Chap. 3, Lemma 7.7], the set \((0, 1] \setminus I \) is at most countable. Then by the right continuity of \( t \mapsto X_t \) and the continuity of \( t \mapsto u(t, x) \), we obtain for all \( t \in (0, 1] \),

\[ \mathbb{E}^x[\phi(X_t)] = u(t, x) = \int_{\mathbb{R}^d} p(t, x, y)\phi(y)dy, \quad \forall \phi \in C^\infty_0(\mathbb{R}^d). \]

This means that \( p(t, x, \cdot) \) is the density function of the distribution of \( X_t \) under \( \text{P}^x \).

□

The next proposition is about a gradient estimate on \( p(t, x, y) \) for the case \( 1 < \alpha < 2 \).

**Proposition 4.10.** Suppose that \( 1 < \alpha < 2 \). Then there exists a constant \( C_{31} = C_{31}(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \theta) > 0 \) such that for all \( (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \),

\[ |\nabla_x p(t, x, y)| \leq C_{31} t^{1-1/\alpha} \left( t^{1/\alpha} + |x| \right)^{-d-\alpha}. \]

**Proof.** Recall that \( p = q + q \otimes \Phi \). By (3.16) and Remark 4.1, we obtain

\[ |\nabla_x q(t, x, y)| \leq c_1 \delta_{0-1}^0(t, x - y), \quad (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d. \] (4.55)

Since

\[ \nabla_x (q \otimes \Phi(t, x, y)) = \int_0^t \int_{\mathbb{R}^d} \nabla_x q(t-s, x, z)\Phi(s, z, y)dzds, \]

we get that for \( (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \),

\[ |\nabla_x (q \otimes \Phi(t, x, y))| \]

\[ \leq c_2 \int_0^t \int_{\mathbb{R}^d} \delta_{0-1}^0(t-s, x-z) \left\{ \delta_{0-1}^0(s, z-y) + \delta_{0-1}^0(s, z-y) \right\} dzds \]

\[ \leq c_3 \delta_{0+\alpha-1}^0(t, x, y) + c_4 \delta_{0-1}^0(t, x, y) \leq c_5 \delta_{0-1}^0(t, x, y). \] (4.56)

Now, the assertion follows by (4.55) and (4.56). □

We conclude this section with the following theorem.

**Proposition 4.11.** Consider the operator \( A \) given in (4.1) and assume that \( n(\cdot, \cdot) \) satisfies Assumption 1.1. Then for the Markov process \((X, (\text{P}^x))\) associated with \( A \), there exists a jointly continuous transition density \( p(t, x, y) \) such that for all \( t > 0, x \in \mathbb{R}^d \) and \( E \in \mathcal{B}(\mathbb{R}^d) \),

\[ \text{P}^x(X_t \in E) = \int_E p(t, x, y)dy. \]

Moreover, for each \( T > 0 \), there exists a constant \( C_{32} = C_{32}(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \theta, T) > 0 \) such that

\[ p(t, x, y) \leq C_{32} t^{(t^{1/\alpha} + |x-y|)^{-d-\alpha}}, \quad x, y \in \mathbb{R}^d, \quad 0 < t \leq T. \] (4.57)
For the case $1 < \alpha < 2$, there exists also a constant $C_{33} = C_{33}(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \theta, T) > 0$ such that
\[
|\nabla_x p(t, x, y)| \leq C_{33} t^{1-1/\alpha} \left( t^{1/\alpha} + |x - y| \right)^{-d-\alpha}, \quad x, y \in \mathbb{R}^d, \quad 0 < t \leq T. \tag{4.58}
\]

Proof. Let $T > 0$ be fixed and set $a := T^{-1/\alpha}$. Define $\tilde{P}_x = P_{x/a}$ and $Y_t := aX_{a^{-t}}$, $t \geq 0$. By Lemma 2.1, Remark 2.2, and Propositions 4.4 and 4.9, the Markov process $\left(Y, \left(\tilde{P}_x\right)\right)$ has a jointly continuous transition density $\tilde{p}(t, x, y)$, $(t, x, y) \in (0, 1] \times \mathbb{R}^{2d}$. Moreover, there exists a constant $c_1 = c_1(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \theta) > 0$ such that
\[
\tilde{p}(t, x, y) \leq c_1 t \left( t^{1/\alpha} + |x - y| \right)^{-d-\alpha}, \quad t \in (0, 1], \ x, y \in \mathbb{R}^d. \tag{4.59}
\]

It follows that for each $t \in (0, T]$ and $x \in \mathbb{R}^d$, the law of $X_t$ under $P_x$ is absolutely continuous with respect to the Lebesgue measure and thus has a density function $p(t, x, \cdot)$. Since
\[
\tilde{p}(t, x, y) \, dy = \tilde{P}_x(Y_t \in dy) = P_{x/a}(aX_{a^{-t}} \in dy) = a^{-d} p\left(a^{-t} x/a, a^{-t} y/a\right) \, dy,
\]
we obtain
\[
p(t, x, y) = a^d \tilde{p}(a^\alpha t, ax, ay) \tag{4.59} \leq c_1 a^d a^\alpha t \left( (a^\alpha t)^{1/\alpha} + |ax - ay| \right)^{-d-\alpha} \\
= c_1 t \left( t^{1/\alpha} + |x - y| \right)^{-d-\alpha}, \quad \forall x, y \in \mathbb{R}^d, \quad 0 < t \leq T.
\]
Moreover, by the continuity of $\tilde{p}(t, x, y)$, the function $(t, x, y) \mapsto p(t, x, y)$ is continuous on $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$. In view of Proposition 4.10, the estimate (4.58) can be similarly proved. This completes the proof. \(\square\)

Remark 4.12. Let $p(t, x, y)$ be as in Proposition 4.11. It follows from (4.14), (4.13) and Lemma 3.4 that there exist $t_0 = t_0(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \theta) \in (0, 1)$ and $C_{34} = C_{34}(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \theta) > 0$ such that
\[
p(t, x, y) \geq C_{34} t^{-d/\alpha}, \quad \forall t \in (0, t_0], \ |x - y| \leq t^{1/\alpha}. \tag{4.60}
\]

5. Transition density of the Markov process associated with $\mathcal{L}$

In this section we assume $1 < \alpha < 2$. In this case, we still need to handle the extra term $b(x) \cdot \nabla f(x)$ in the definition of $\mathcal{L} f$. Throughout this section we assume Assumptions 1.1 and 1.3 are true.

Let $p(t, x, y)$ be as in Proposition 4.11. It follows from the continuity of $p(t, x, y)$ and the Markov property that
\[
\int_{\mathbb{R}^d} p(s, x, z)p(t, z, y)dz = p(t+s, x, y), \quad t, s > 0, \ x, y \in \mathbb{R}^d. \tag{5.1}
\]

By (5.1) and Theorem 4.11, there exists a constant $C_{35} = C_{35}(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \theta) > 0$ such that for all $t > 0$ and $x, y \in \mathbb{R}^d$,
\[
p(t, x, y) \leq C_{35} e^{C_{35} t} \left( t^{1/\alpha} + |x - y| \right)^{-d-\alpha} \tag{5.2}
\]
and
\[|\nabla_x p(t, x, y)| \leq C_{35} e^{C_{35} t} t^{1-1/\alpha} \left(t^{1/\alpha} + |x - y|\right)^{-d-\alpha}. \tag{5.3}\]

For \( t > 0 \) and \( x, y \in \mathbb{R}^d \), let \( l_0(t, x, y) := p(t, x, y) \). Then
\[
\int_0^t \int_{\mathbb{R}^d} |l_0(t-s, x, z)b(z) \cdot \nabla_z p(s, z, y)| dz \, ds \\
\leq \kappa_3 C_{35}^2 e^{C_{35} t} \int_0^t \int_{\mathbb{R}^d} \phi_0^0(t-s, x-z) \phi_{\alpha-1}^0(s, z-y) dz \, ds \\
\leq \kappa_3 C_{35}^2 e^{C_{35} t} B\left(1, 1-\alpha^{-1}\right) \phi_{\alpha-1}^0(t, x, y).
\]

So
\[
l_1(t, x, y) := \int_0^t \int_{\mathbb{R}^d} l_0(t-s, x, z)b(z) \cdot \nabla_z p(s, z, y) dz \, ds, \quad t > 0, \ x, y \in \mathbb{R}^d, \tag{5.4}\]
is well-defined. Similarly, we can define recursively
\[
l_n(t, x, y) := \int_0^t \int_{\mathbb{R}^d} l_{n-1}(t-s, x, z)b(z) \cdot \nabla_z p(s, z, y) dz \, ds, \quad t > 0, \ x, y \in \mathbb{R}^d. \tag{5.5}\]

By induction, we easily get that for \( t > 0 \) and \( x, y \in \mathbb{R}^d \),
\[
|l_n(t, x, y)| \\
\leq C_{35} (\kappa_3 C_{35} C_{35})^n e^{C_{35} t} \prod_{i=1}^n \mathcal{B}\left(\frac{\alpha + (i-1)(\alpha - 1)}{\alpha}, \frac{\alpha - 1}{\alpha}\right) \phi_{\alpha+n(\alpha-1)}^0(t, x, y) \\
= C_{35} (\kappa_3 C_{35} C_{35} \Gamma(1-\alpha^{-1}))^n e^{C_{35} t} \frac{\prod_{i=1}^n \mathcal{B}\left(\frac{i(\alpha - 1)}{\alpha}, \frac{\alpha - 1}{\alpha}\right) \phi_{\alpha+n(\alpha-1)}^0(t, x, y)}{\Gamma(1+n(1-\alpha^{-1}))} \\
\tag{5.6}
\]
and
\[
|\nabla_x l_n(t, x, y)| \\
\leq C_{35} (\kappa_3 C_{35} C_{35})^n e^{C_{35} t} \prod_{i=1}^n \mathcal{B}\left(\frac{i(\alpha - 1)}{\alpha}, \frac{\alpha - 1}{\alpha}\right) \phi_{\alpha+n(\alpha-1)}^0(t, x, y) \\
= C_{35} (\kappa_3 C_{35} C_{35})^n \left(\frac{\Gamma(1-\alpha^{-1})}{\Gamma((1+n)(1-\alpha^{-1}))}\right)^{n+1} e^{C_{35} t} \frac{\prod_{i=1}^n \mathcal{B}\left(\frac{i(\alpha - 1)}{\alpha}, \frac{\alpha - 1}{\alpha}\right) \phi_{\alpha+n(\alpha-1)}^0(t, x, y)}{\Gamma((1+n)(1-\alpha^{-1}))}. \tag{5.7}\]

Remark 5.1. Similarly as above, for \((t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d\), define \(|l|_0(t, x, y) := p(t, x, y)\) and then recursively
\[
|l|_n(t, x, y) := \int_0^t \int_{\mathbb{R}^d} |l|_{n-1}(t-s, x, z)b(z) |\nabla_z p(s, z, y)| dz \, ds.
\]

In view of Lemma 2.6, we can follow the same argument as in \([7, \text{p. 191, (40)}]\) to obtain the existence of \(\lambda_0 > 0\) and \(C_{36} = C_{36}(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \theta, \kappa_3) > 0\) such that
\[
\sum_{n=0}^{\infty} \int_0^\infty e^{-\lambda t} |l|_n(t, x, y) dt \leq C_{36} u_\lambda(x - y), \quad \forall \lambda > \lambda_0, \ x, y \in \mathbb{R}^d, \tag{5.8}\]
where \(u_\lambda\) is defined in Sect. 2.4.
Proposition 5.2. Assume $1 < \alpha < 2$. Let $\mathcal{L}$ and $(X, (\mathcal{L}_t^x))$ be as in Theorem 1.4, and $l_n$ be as in (5.5). Then $(X, (\mathcal{L}_t^x))$ has a jointly continuous transition density $l(t, x, y)$ given by

$$l(t, x, y) := \sum_{n=0}^{\infty} l_n(t, x, y), \quad (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d,$$  

(5.9)

where the series on the right-hand side of (5.9) converges locally uniformly on $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$. Moreover, it holds that for all $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$l(t, x, y) = p(t, x, y) + \int_0^t \int_{\mathbb{R}^d} l(\tau, x, z)b(z) \cdot \nabla_z p(t - \tau, z, y) dz d\tau.$$  

(5.10)

Proof. Let $T > 1$ be fixed. By (5.6), we get for $t \in (0, T]$ and $x, y \in \mathbb{R}^d$,

$$|l_n(t, x, y)| \leq \frac{C_{35} \left( \kappa_3 C_7 C_{35} T(1 - \alpha^{-1}) \Gamma \left( 1 - \alpha^{-1} \right) \right)^n e^{C_{35} T}}{\Gamma(1 + n(1 - \alpha^{-1}))} \delta_{a}^0(t, x, y).$$  

(5.11)

The local uniform convergence of $\sum_{n=0}^{\infty} l_n(t, x, y)$ follows from (5.11). It is also easy to see that (5.10) is true. By induction and a similar argument as in [7, Lemma 14], we see that $l_n(t, x, y)$ is jointly continuous in $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$, which, together with the local uniform convergence, implies the joint continuity of $l(t, x, y)$.

For $\lambda > C_{35} \lor \lambda_0$ and $f \in B_0(\mathbb{R}^d)$, define

$$R^\lambda f(x) := \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} p(t, x, y)f(y) dy dt, \quad x \in \mathbb{R}^d,$$

and

$$S^\lambda f(x) := \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} l(t, x, y)f(y) dy dt, \quad x \in \mathbb{R}^d.$$  

(5.12)

Note that $S^\lambda$ in (5.12) is well-defined by (5.8). If $f$ is bounded measurable, then

$$S^\lambda f(x) - R^\lambda f(x) = \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} f(y) \left( \int_0^t \int_{\mathbb{R}^d} l(\tau, x, z)b(z) \cdot \nabla_z p(t - \tau, z, y) dz d\tau \right) dy dt.$$  

(5.13)

Since (5.2), (5.3) and (5.8) hold, we can apply Fubini’s theorem to get

$$S^\lambda f(x) - R^\lambda f(x) = \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} l(\tau, x, z) \left[ b(z) \cdot \nabla z \left( \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda(t - \tau)} p(t - \tau, z, y)f(y) dy dt \right) \right] dz d\tau,$$

namely,

$$S^\lambda f(x) - R^\lambda f(x) = S^\lambda B R^\lambda f,$$  

(5.14)

where $B R^\lambda f := b \cdot \nabla R^\lambda f$. Applying (5.13) $i$ times, we get

$$S^\lambda g = \sum_{k=0}^{i} R^k(B R^\lambda)^k g + S^\lambda(B R^\lambda)^{i+1} g, \quad \forall \lambda > C_{35} \lor \lambda_0, \ g \in B_0(\mathbb{R}^d).$$  

(5.15)

It follows from (5.3) that

$$\| B R^\lambda g \| \leq N_\lambda \| b \| \cdot \| g \| \leq \kappa_3 N_\lambda \| g \|, \quad g \in B_0(\mathbb{R}^d),$$
where $N_\alpha > 0$ is a constant with $N_\alpha \downarrow 0$ as $\alpha \uparrow \infty$. So we can find $\lambda_1 > C_{35} \vee \lambda_0$ such that $N_\alpha < 1/\kappa_3$ for all $\alpha > \lambda_1$. It follows from (5.8) and (5.15) that
\[
\lim_{\alpha \to \infty} \|S^{\lambda}(BR^{\lambda})^{i+1}g\| = 0 \text{ for all } \alpha > \lambda_1.
\] Therefore,
\[
S^{\lambda}g = \sum_{k=0}^{\infty} R^{\lambda}(BR^{\lambda})^{k}g, \quad \forall \lambda > \lambda_1, \ g \in B_b(\mathbb{R}^d). \tag{5.16}
\]

Next, we show that $l(t, x, y)$ is the transition density of $(X, (L^x))$. Let $x \in \mathbb{R}^d$ be fixed. For $\lambda > 0$ and $f \in B_b(\mathbb{R}^d)$, define
\[
V^{\lambda}f := E_{L^x}\left[ \int_0^\infty e^{-\lambda t} f(X_t) dt \right].
\]
For $f \in C_b^2(\mathbb{R}^d)$, we know that
\[
f(X_t) - f(X_0) - \int_0^t Lf(X_u) du, \quad t \geq 0,
\]
is a $L^x$-martingale. So
\[
E_{L^x}\left[ f(X_t) \right] = f(x) = E_{L^x}\left[ \int_0^t Lf(X_u) du \right]. \tag{5.17}
\]
Multiplying both sides of (5.17) by $e^{-\lambda t}$, integrating with respect to $t$ from 0 to $\infty$ and then applying Fubini’s theorem, we get for $f \in C_b^2(\mathbb{R}^d)$,
\[
E_{L^x}\left[ \int_0^\infty e^{-\lambda t} f(X_t) dt \right] = \frac{1}{\lambda} f(x) + \frac{1}{\lambda} E_{L^x}\left[ \int_0^\infty e^{-\lambda u} Lf(X_u) du \right]. \tag{5.18}
\]
We now claim
\[
\lambda V^{\lambda}f = f(x) + V^{\lambda}(L^x f), \quad \forall f \in C^{\alpha+\beta}(\mathbb{R}^d), \tag{5.19}
\]
where $0 < \beta < 2 - \alpha$ (see [17, Sect. 3.1] for the definition of the Hölder space $C^{\alpha+\beta}(\mathbb{R}^d)$). Indeed, if $f \in C^{\alpha+\beta}(\mathbb{R}^d)$, by convolution with mollifiers, we can find a sequence $(f_n) \subset C^\infty_c(\mathbb{R}^d)$ such that $f_n \to f$ in $C^{\alpha+\beta}(K)$, for any compact set $K \subset \mathbb{R}^d$ and $0 < \beta' < \beta$. Moreover, $\|f_n\|_{C^{\alpha+\beta}(\mathbb{R}^d)} \leq \|f\|_{C^{\alpha+\beta}(\mathbb{R}^d)}$. For details the reader is referred to [21, p. 438]. Noting that for $|h| \leq 1$,
\[
|f_n(x+h) - f_n(x) - \nabla f_n(x) \cdot h| = \left| \int_0^1 (\nabla f_n(x+rh) - \nabla f_n(x)) \cdot h dr \right|
\leq c_1 \|f_n\|_{C^{\alpha+\beta}(\mathbb{R}^d)} |h|^\alpha |h|^{\beta-1} \leq c_1 \|f\|_{C^{\alpha+\beta}(\mathbb{R}^d)} |h|^\alpha |h|^{\beta-1},
\]
by dominated convergence, we see that $Lf_n \to Lf$ boundedly and pointwise as $n \to \infty$. Since (5.19) is true for $f_n$ by (5.18), the passage to the limit gives (5.19).

Given $g \in C^2(\mathbb{R}^d)$, it follows from [2, Proposition 7.4] and [2, Theorem 7.2] that there exists $f \in C^{\alpha+\beta}(\mathbb{R}^d)$ such that $(\lambda - A)f = g$, where $\lambda > 0$. For this $f$, as in (5.19) we have $\lambda R^{\lambda}f = f + R^{\lambda}(Af)$, which implies $f = R^{\lambda}g$. Substituting this $f$ in (5.19), we obtain
\[
V^{\lambda}g = R^{\lambda}g(x) + V^{\lambda}(BR^{\lambda}g), \quad g \in C^\beta(\mathbb{R}^d). \tag{5.20}
\]
After a standard approximation procedure, the equality (5.20) holds for any $g \in C_b(\mathbb{R}^d)$. Then we can use a monotone class argument to extend (5.20) to all $g \in B_b(\mathbb{R}^d)$.
Similarly to (5.14), we obtain from (5.20) that
\[ V^\lambda g = \sum_{i=0}^{k} R^\lambda(B R^\lambda)^i g(x) + V^\lambda(B R^\lambda)^{k+1} g, \quad g \in \mathcal{B}_0(\mathbb{R}^d). \] (5.21)

For \( \lambda > \lambda_1 \), by (5.15) and the definition of \( \lambda_1 \), we obtain
\[ V^\lambda g = \sum_{i=0}^{\infty} R^\lambda(B R^\lambda)^i g(x), \quad \forall \lambda > \lambda_1, \quad g \in \mathcal{B}_0(\mathbb{R}^d). \] (5.22)

It follows from (5.16) and (5.22) that for all \( \lambda > \lambda_1 \) and \( g \in \mathcal{B}_0(\mathbb{R}^d) \),
\[ E_{L^\infty} \left[ \int_0^\infty e^{-\lambda t} g(X_t) dt \right] = \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda t} l(t, x, y) g(y) dy dt. \]

Note that for \( g \in \mathcal{B}_b(\mathbb{R}^d) \), the function \( t \mapsto \int_{\mathbb{R}^d} l(t, x, y) g(y) dy \) is bounded continuous on \((0, T)\) for any \( T > 0 \). By the uniqueness of the Laplace transform, we obtain
\[ E_{L^\infty} [g(X_t)] = \int_{\mathbb{R}^d} l(t, x, y) g(y) dy, \quad \forall g \in \mathcal{B}_b(\mathbb{R}^d), \quad t > 0. \]

This implies that \( l(t, x, \cdot) \) is the density function of the law of \( X_t \) under the measure \( L^x \). □

**Remark 5.3.** Let \( l(t, x, y) \) be as in Proposition 5.2. By (4.60), (5.9) and (5.6), there exist \( t_0 = t_0(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \theta, \kappa_3) \in (0, 1) \) and \( C_{37} = C_{37}(d, \alpha, \kappa_0, \kappa_1, \kappa_2, \theta, \kappa_3) > 0 \) such that
\[ l(t, x, y) \geq C_{37} t^{-d/\alpha}, \quad \forall t \in (0, t_0], \quad |x-y| \leq t^{1/\alpha}. \]

6. Proof of Theorem 1.4

Finally, we give the proof of our main result.

**Proof of Theorem 1.4.** By Propositions 4.11 and 5.2, we get the existence of a jointly continuous transition density \( l(t, x, y) \) for \((X, (L^x))\). The claimed upper bounds of \( l(t, x, y) \) and \(|\nabla_x l(t, x, y)|\) follow from (4.57), (5.6), (5.7) and Proposition 5.2.

We now prove the lower bound of \( l(t, x, y) \) by following [9, Sect. 4.4]. Arguing in the same way as in [9, p. 306-307] (see also the proof of [4, Prop. 2.3]), we conclude that if \( A \) and \( B \) are bounded Borel subsets of \( \mathbb{R}^d \) with \( B \) being closed and having a positive distance from \( A \), then
\[ \sum_{s \leq t} 1_A(X_{s-}) 1_B(X_s) \left( \int_0^t 1_A(X_s) \left( \int_B \frac{n(X_s, y - X_s)}{|y - X_s|^{d+\alpha}} dy \right) ds \right. \]

is a \( L^x \)-martingale.

Let \( T > 0 \) be fixed. By Remarks 4.12 and 5.3, there exist constants \( t_0 \in (0, 1) \) and \( c_1 > 0 \) such that
\[ l(t, x, y) \geq c_1 t^{-d/\alpha}, \quad \forall t \in (0, t_0], \quad |x-y| \leq t^{1/\alpha}. \]

As in (5.1), \( l(t, x, y) \) satisfies also the Chapman-Kolmogorov’s equation. Iterating \([T/t_0] + 1\) times, we obtain
\[ l(t, x, y) \geq c_2 t^{-d/\alpha}, \quad \forall t \in (0, T], \quad |x-y| \leq 3c_3 t^{1/\alpha}, \]
where \(c_2, c_3 > 0\) are constants. By Lemmas 2.3 and 2.4, there is a constant \(\lambda \in (0, 1/2)\) such that for all \(t \in (0, T)\) and \(x \in \mathbb{R}^d\),

\[
L^x (\tau_{B_{c_3 t^{1/\alpha}}(x)} < \lambda t) \leq \frac{1}{2}.
\]

Below, assume \(0 < t \leq T\) and \(|x - y| > 3c_3 t^{1/\alpha}\). Set \(A_1 := B_{c_3 t^{1/\alpha}}(x)\) and \(A_2 := B_{c_3 t^{1/\alpha}}(y)\). Let \(\overline{A}_i\) the closure of \(A_i\), \(i = 1, 2\). Similarly to [9, p. 309, (4.36)], we have

\[
L^x (X_{\lambda t} \in B_{c_3 t^{1/\alpha}}(y)) \geq \frac{1}{2} L^x (X_{\lambda t} \wedge \tau_{A_1} \in \overline{A}_2),
\]

where \(\tau_{A_1} := \inf\{t \geq 0 : X_t \notin A_1\}\). Since

\[
1_{X_{\lambda t} \wedge \tau_{A_1} \in \overline{A}_2} = \sum_{s \leq \lambda t \wedge \tau_{A_1}} 1_{\tau_{A_1}}(X_s) 1_{\overline{A}_2}(X_s),
\]

by (6.1) and optional sampling, we have

\[
L^x (X_{\lambda t} \wedge \tau_{A_1} \in \overline{A}_2) = E_{L^x} \left[ \int_0^{\lambda t \wedge \tau_{A_1}} 1_{\tau_{A_1}}(X_s) \left( \int_{\overline{A}_2} \frac{n(X_s, y - X_s)}{|y - X_s|^{d+\alpha}} \, dy \right) \, ds \right]
\]

\[
= E_{L^x} \left[ \int_0^{\lambda t \wedge \tau_{A_1}} \int_{\overline{A}_2} \frac{n(X_s, y - X_s)}{|y - X_s|^{d+\alpha}} \, dy \, ds \right].
\]

The rest of the proof is then the same as in [9, p. 310]. So we get

\[
l(t, x, y) \geq c_4 t |x - y|^{-d-\alpha}, \quad \forall t \in (0, T], \ |x - y| > 3c_3 t^{1/\alpha}.
\]

The theorem is proved. \(\square\)

References

[1] Bass, R.F.: Uniqueness in law for pure jump Markov processes. Probab. Theory Related Fields 79(2), 271–287 (1988).
[2] Bass, R.F.: Regularity results for stable-like operators. J. Funct. Anal. 257(8), 2693–2722 (2009).
[3] Bass, R.F., Kassmann, M.: Harnack inequalities for non-local operators of variable order. Trans. Amer. Math. Soc. 357(2), 837–850 (2005).
[4] Bass, R.F., Levin, D.A.: Harnack inequalities for jump processes. Potential Anal. 17(4), 375–388 (2002).
[5] Bass, R.F., Tang, H.: The martingale problem for a class of stable-like processes. Stochastic Process. Appl. 119(4), 1144–1167 (2009).
[6] Blumenthal, R.M., Getoor, R.K.: Some theorems on stable processes. Trans. Amer. Math. Soc. 95, 263–273 (1960).
[7] Bogdan, K., Jakubowski, T.: Estimates of heat kernel of fractional Laplacian perturbed by gradient operators. Comm. Math. Phys. 271(1), 179–198 (2007)
[8] Chen, Z.Q., Kumagai, T.: Heat kernel estimates for stable-like processes on \(d\)-sets. Stochastic Process. Appl. 108(1), 27–62 (2003).
[9] Chen, Z.Q., Zhang, X.: Heat kernels and analyticity of non-symmetric jump diffusion semigroups. Probab. Theory Related Fields 165(1-2), 267–312 (2016).
[10] Chen, Z.Q., Zhang, X.: Uniqueness of stable-like processes. arXiv preprint arXiv:1604.02681 (2016)
[11] Ethier, S.N., Kurtz, T.G.: Markov processes: Characterization and convergence. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York (1986).
[12] Feller, W.: An introduction to probability theory and its applications. Vol. II. Second edition. John Wiley & Sons, Inc., New York-London-Sydney (1971)
[13] Friedman, A.: Partial differential equations of parabolic type. Prentice-Hall Inc., Englewood Cliffs, N.J. (1964)
[14] Kaleta, K., Sztonyk, P.: Upper estimates of transition densities for stable-dominated semigroups. J. Evol. Equ. 13(3), 633–650 (2013).
[15] Kaleta, K., Sztonyk, P.: Estimates of transition densities and their derivatives for jump Lévy processes. J. Math. Anal. Appl. 431(1), 260–282 (2015).
[16] Kolokoltsov, V.: Symmetric stable laws and stable-like jump-diffusions. Proc. London Math. Soc. (3) 80(3), 725–768 (2000).
[17] Krylov, N.V.: Lectures on elliptic and parabolic equations in Hölder spaces, Graduate Studies in Mathematics, vol. 12. American Mathematical Society, Providence, RI (1996).
[18] Mikulevicius, R., Pragarauskas, H.: On the Cauchy problem for integro-differential operators in Hölder classes and the uniqueness of the martingale problem. Potential Anal. 40(4), 539–563 (2014).
[19] Mikulevičius, R., Pragarauskas, H.: On the Cauchy problem for integro-differential operators in Sobolev classes and the martingale problem. J. Differential Equations 256(4), 1581–1626 (2014).
[20] Nolan, J.: Stable Distributions: Models for Heavy-Tailed Data. Springer New York (2016).
[21] Priola, E.: Pathwise uniqueness for singular SDEs driven by stable processes. Osaka J. Math. 49(2), 421–447 (2012).
[22] Stroock, D.W.: Diffusion processes associated with Lévy generators. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete 32(3), 209–244 (1975).
[23] Sztonyk, P.: Transition density estimates for jump Lévy processes. Stochastic Process. Appl. 121(6), 1245–1265 (2011).
[24] Watanabe, T.: Asymptotic estimates of multi-dimensional stable densities and their applications. Trans. Amer. Math. Soc. 359(6), 2851–2879 (electronic) (2007).

Peng Jin: Fakultät für Mathematik und Naturwissenschaften, Bergische Universität Wuppertal, 42119 Wuppertal, Germany
E-mail address: jin@uni-wuppertal.de