Homological mirror symmetry for open Riemann surfaces from pair-of-pants decompositions

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Punctured surfaces

Follow G. Mikhalkin:

\[ C \in \text{a degenerating family of hypersurfaces} \]

\[
C_t = \left\{ f_t(z) = \sum_{\alpha=(\alpha_1, \alpha_2) \in A \subset \mathbb{Z}^2} c_{\alpha} t^{-\nu(\alpha)} z_1^{\alpha_1} z_2^{\alpha_2} = 0 \right\} \subset (\mathbb{C}^*)^2
\]

\[ \nu : \text{Conv}(A) \to \mathbb{R} \text{ convex piecewise linear} \]

\[
\text{Log}_t(C_t) \xrightarrow{t\to\infty} \Gamma = \text{singular locus of } \\
L_\nu(\xi) = \max\{ \langle \alpha, \xi \rangle - \nu(\alpha) | \alpha \in A \}
\]

\[
\mathbb{R}^2 \setminus \Gamma = \bigcup_{\alpha \in A} C_\alpha
\]
SYZ mirror symmetry for $C$ \cite{Abouzaid-Auroux-Katzarkov 1205.0053}

$(Y, W)$

\[
\Delta_Y = \{(\xi_1, \xi_2, \eta) \in \mathbb{R}^3 | \eta \geq L_\nu(\xi)\}
\]

\[
W = -z^{(0,0,1)} \in \mathcal{O}(Y)
\]

\[
W^{-1}(0) = D = \coprod_{\alpha \in A} D_\alpha
\]

Example:

\[
C = \{1 + x_1 + x_2 = 0\}
\]

$(\mathbb{C}^3, xyz)$
Example:

\[ C = \left\{ 1 + x_1 + x_2 + \frac{t}{x_1 x_2} = 0 \right\} \]

\[ \mathcal{O}(-3) \quad s \quad \text{, } \quad sxyz \]

\[ \mathbb{P}^2 \quad (x : y : z) \]

\( Y \) is glued together from affine toric pieces according to the tropical hypersurface,

\[ MF(Y, W) = \lim MF(\text{pieces}) \]
Homological Mirror Symmetry

\[ \mathcal{W}(C) \cong D_{\text{sing}}^b(\mathcal{D} = W^{-1}(0)) \]

whenever \( C \) is noncompact

[AAEKO 1103.4322
Bocklandt 1111.3392
\ldots]
Matrix factorization

\[ MF(Y, W) \quad \cong \quad D_{\text{sing}}^b(D) = D^b(\text{Coh}(D))/\text{Perf} \]

\[ T_\alpha(k) := \mathcal{O}(-D_\alpha)(k) \xrightarrow{t_{\alpha;1}} \mathcal{O}(k) \xleftarrow{t_{\alpha;0}} \mathcal{O}(k) \quad \leftrightarrow \quad \text{coker}(t_{\alpha;1}) = \mathcal{O}_{D_\alpha}(k) \]

Example: mirror of a pair of pants = \((\mathbb{C}^3, xyz), \hat{R} = \mathbb{C}[x, y, z], R = \hat{R}/\langle xyz \rangle\)

\[ \hat{R} \xrightarrow{x} \hat{R} \quad \xleftrightarrow{yz} \quad \hat{R} \quad \leftrightarrow \quad \{ \cdots \xrightarrow{x} R \xrightarrow{yz} R \xrightarrow{x} R \} \to \hat{R}/\langle x \rangle \quad (2\text{-periodic}) \]
Objects of $\mathcal{W}(C)$

Abouzaid’s generation

$\mathcal{W}(C)$ is split-generated by $L_\alpha(k)$, $\alpha \in A$, $k \in \mathbb{Z}$.

$L_\alpha(k)$ winds around each edge $k_{\alpha \gamma} = n_{\alpha \gamma} k + \delta_{\alpha, \gamma}$ times

$\delta_{\gamma, \alpha} - \delta_{\alpha, \gamma} = 1 + d_{\alpha, \gamma}$, where $d_{\alpha, \gamma} = \deg O(D_\alpha)|_{D_{\alpha} \cap D_{\gamma}}$

e.g. $\delta_{\gamma, \alpha} = \delta_{\gamma, \beta} = 2$

$\delta_{\alpha, \gamma} = \delta_{\beta, \gamma} = 0$
Wrapped Fukaya category  [Abouzaid-Seidel 0712.3177]

Example: Cylinder $\mathbb{R} \times S^1$, $\omega = d(rd\theta)$

Model: choice of Hamiltonian $H = 2\pi |r|$ when $|r| \gg 1$, $H_n = nH$

Floer complexes: $CF^*(L_1, L_2; H_n) = \langle \phi^1_{H_n} (L_1) \cap L_2 \rangle$.

Figure below: generators of $CF^*(L, L; H_2)$
Wrapped Fukaya category

Morphism:

\[ \text{hom}(L_1, L_2) = CW^*(L_1, L_2) \]
\[ = \lim_{\rightarrow n} CF^*(L_1, L_2; H_n) = \bigsqcup_{n=1}^{\infty} CF^*(L_1, L_2; nH) / \sim. \]

When all higher order continuation maps are trivial.

In the example: \( \text{hom}(L, L) = \langle \bigcup_{i \in \mathbb{Z}} x^i \rangle. \)
Wrapped Fukaya category

Product (and $A_\infty$-products are similar with more terms)

$$\mu^2_H : CF^*(L_1, L_2; nH) \otimes CF^*(L_0, L_1; nH) \to CF^*(L_0, L_2; 2nH)$$

$$\mu^2 : CW^*(L_1, L_2) \otimes CW^*(L_0, L_1) \to CW^*(L_0, L_2)$$

$\mathcal{W}(\mathbb{R} \times S^1)$ generated by $L$

$Coh(\mathbb{C}^*) = \text{finite modules over } \mathbb{C}[x, x^{-1}]$ generated by $\mathcal{O}$

$CW^*(L, L) \cong \mathbb{C}[x, x^{-1}] \cong \text{End}(\mathcal{O})$
Can we compute $\mathcal{W}(C)$ from $\mathcal{W}(P_{\alpha\beta\gamma})$’s?

Main theorem [L.]:

- $\mathcal{W}(C)$ is split-generated by the objects $L_\alpha(k), \alpha \in A, k \in \mathbb{Z}$.

- In a suitable model for $\mathcal{W}(C)$, the morphism complex between any two objects, $L_\alpha(k)$ and $L_\beta(l)$, is generated by

$$\mathcal{X}(L_\alpha(k), L_\beta(l)) = \left( \bigcup \mathcal{X}_{P_{\alpha\beta\gamma}}(L_\alpha(k), L_\beta(l)) \right) / \sim$$

with $x \in \mathcal{X}_{P_{\alpha\beta\gamma}} \cap \mathcal{C}_{\alpha\beta} \sim y \in \mathcal{X}_{P_{\alpha\beta\eta}} \cap \mathcal{C}_{\alpha\beta}$ whenever $\rho^{\gamma}_{\alpha\beta}(x) = \rho^{\eta}_{\alpha\beta}(y)$.

- In this model, the $A_\infty$-products in $\mathcal{W}(C)$ are given by those in the pairs of pants.
Restriction maps

\[ \bigoplus_{L_i, L_j} CW_{P_{\alpha\beta\gamma}}^*(L_i, L_j) \xrightarrow{\rho_{\alpha\beta}^\gamma} \bigoplus_{L_i, L_j} CW_{C_{\alpha\beta}}^*(L_i, L_j) \]
Ingredient: Hamiltonian perturbation
Equivalence relations

\[ \rho^\gamma_{\alpha\beta} \left( x^i_{\alpha;\beta} \right) = \rho^n_{\alpha\beta} \left( \tilde{x}^{n_{\alpha\beta}(l-k)-i}_{\alpha;\beta} \right) \quad \Rightarrow \quad x^i_{\alpha;\beta} \sim \tilde{x}^{n_{\alpha\beta}(l-k)-i}_{\alpha;\beta} \in \mathcal{X}(L_{\alpha}(k), L_{\alpha}(l)) \]

\[ \rho^\gamma_{\alpha\beta} \left( x^i_{\alpha;\beta} \right) = \rho^n_{\alpha\beta} \left( \tilde{x}^{n_{\alpha\beta}(l-k)+d_{\alpha;\beta}-i}_{\alpha;\beta} \right) \quad \Rightarrow \quad x^i_{\alpha;\beta} \sim \tilde{x}^{n_{\alpha\beta}(l-k)+d_{\alpha;\beta}-i}_{\alpha;\beta} \in \mathcal{X}(L_{\alpha}(k), L_{\beta}(l)) \]
Homological Mirror Symmetry

\[ \text{Knörrer periodicity theorem} \]
Pick a split-generating set of Lagrangians so that for each $L$:

- $L \cap e = \bigcup$ (disjoint arcs), each arc intersects each circle fiber of $e$ just once.
- For any two arcs, the portion of $L$ in the complement of $e$ connected by these two arcs cannot be homotopically trivial.