Image transmission through a stable paraxial cavity

Sylvain Gigan, Laurent Lopez, Nicolas Treps, Agnès Maître, Claude Fabre
Laboratoire Kastler-Brossel, Université Pierre et Marie Curie, Case 74, 75252 PARIS cedex 05.

We study the transmission of a monochromatic "image" through a paraxial cavity. Using the formalism of self-transform functions, we show that a transverse degenerate cavity transmits the self-transform part of the image, with respect to the field transformation over one round-trip of the cavity. This formalism gives a new insight on the understanding of the behavior of a transverse degenerate cavity, complementary to the transverse mode picture. An experiment of image transmission through a hemiconfocal cavity show the interest of this approach.

I. INTRODUCTION

Image transmission and propagation in a paraxial system, using optical devices such as lenses and mirrors is a well-known and extensively studied problem [1]. The free-propagation of a field changes its transverse distribution, but in some planes, such as conjugate planes, or Fourier plane, one get simple transformations of the image. On the other hand, transmission through cavities has also been studied extensively for a long time, starting from the Fabry-Perot resonator, then to the laser[2], and their are commonly used as temporal frequency filters. Less known are their spatial frequency characteristics and the resonances of the cavities. Optical cavities have also been studied extensively for a long time, starting from the Fabry-Perot resonator, then to the cavity which is only resonant for one transverse mode is gated. Whereas the transmission of an image through a cavity, complementary to the transverse mode picture. An experiment of image transmission through a simple degenerate cavity : the hemi-confocal cavity.

II. "ABCD" CAVITY ROUND-TRIP MATRIX TRANSFORMS

All the theory developed in this paper will be performed within the paraxial approximation. We consider a monochromatic electromagnetic field \( E(\vec{r},t) \) at frequency \( \omega \), linearly polarized along a vector \( \vec{u} \) and propagating along a direction \( z \) of space. The position in the transverse plane will be represented by the vector \( \vec{r} = x\vec{i} + y\vec{j} \). The electric field is supposed stationary and can be written in a given transverse plane as :

\[
\vec{E}(\vec{r}, t) = \text{Re}[E(\vec{r}) e^{-i \omega t} \vec{u}] \tag{1}
\]

where \( \vec{u} \) is the polarization unit vector. The local intensity in this plane is then :

\[
I(\vec{r}) = 2\epsilon_0 c E(\vec{r}) E^*(\vec{r}) \tag{2}
\]

The input image considered all along this paper is defined by a given transverse repartition of the complex transverse field \( E_{in}(\vec{r}) \) in an input plane \( z_{in} \). We suppose that its extension is finite around the optical axis, and that its transverse variations are such that this image propagates within the paraxial approximation. We will consider both intensity images and "field" images, i.e. not only the transverse intensity distribution of the field, but also the amplitude distribution itself.

A. Image propagation in a paraxial system

The field \( E(\vec{r}) \) is propagating through an optical system along the \( z \) axis. An input-output relation of the
form can be written:

$$E_{\text{out}}(\vec{r}) = \mathcal{T}[E_{\text{in}}(\vec{r})]$$  \hspace{1cm} (3)

where $E_{\text{in}}(\vec{r})$ and $E_{\text{in}}(\vec{r})$ are the fields just before and just after the optical system and $\mathcal{T}$ is the transformation of the field associated to the optical system. If the system is only a linear paraxial system (made of lenses, or curved mirrors, but without diaphragms), the propagation properties of the system are described by its Gauss matrix $T$ (often called ABCD matrix) which writes:

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$  \hspace{1cm} (4)

All the properties of the system can be inferred from the values of the coefficients $A$, $B$, $C$ and $D$, and of the total optical length $L$ of the system (responsible for a phase factor which is not included in the ABCD coefficients). We will assume that the index of refraction is the same at the input and at the output of the system. As a consequence, we have $\det(T) = AD - BC = 1$. In particular, the transformation $\mathcal{T}$ of the field can be derived from the Huygens-Fresnel equation in free space in the case $B \neq 0$ [2]:

$$\mathcal{T} : E(\vec{r}_1) \rightarrow E(\vec{r}_2) = -e^{ikL}\frac{i}{B\lambda} \iint d^2\vec{r}_1 E(\vec{r}_1) \exp \left[ -i\frac{\pi}{B\lambda} (A\vec{r}_1^2 - 2\vec{r}_1\vec{r}_2 + D\vec{r}_2^2) \right]$$  \hspace{1cm} (5)

If $B = 0$, the Gauss matrix can be written $T = \begin{pmatrix} M & 0 \\ C & \frac{1}{M} \end{pmatrix}$. In this case the field in the output plane is given by:

$$\mathcal{T} : E(\vec{r}_1) \rightarrow E(\vec{r}_2) = -e^{ikL}Mu_1(M\vec{r}_1)e^{i\frac{kCMR^2}{2}}$$  \hspace{1cm} (6)

In terms of imaging, a conjugate plane corresponds to a transformation for which one retrieves the input image within a magnification factor $M$. From equations (5) and (6), it can be inferred that:

- if $B = 0$ one retrieves the intensity image but not the amplitude (there is a phase curvature coming from the term $e^{i\frac{kCMR^2}{2}}$ of equation (6)). We will call such a transform an "Intensity-Conjugate Transform", or ICT.

- if $B = 0$ and $C = 0$ one retrieves the amplitude image (and the intensity image of course). We will call such a transform an "Amplitude-Conjugate Transform", or ACT. This transform is sometimes also called a Near-Field (NF).

Another interesting transformation is the one for which one obtains the spatial Fourier transform of the image. Still from equations (5) and (6), one sees that:

- if $A = 0$ one obtains the Fourier transform for the field, within a curvature phase term corresponding to the factor $e^{-\frac{i\pi DR^2}{2}}$ of equation (5)). This factor does not affect the intensity distribution. We will call this transformation a "Intensity Fourier Transform", or IFT.

- if $A = 0$ and $D = 0$ one obtains a perfect spatial Fourier transform for the amplitude field. We will call this transformation an "Amplitude Fourier Transform", or AFT. It is sometimes called a far-field (FF).

It is straightforward to see that a 2f-2f system (a lens of focal distance $f$ placed before and after a distance $2f$) performs an ACT, and that a f-f system performs an AFT. Whereas AFT and ACT can be simply and directly juxtaposed side-by-side, this is not the case for IFT and ICT transformations because of the phase factors.

Let us remind a few obvious facts, which will be nonetheless useful to understand the rest of the discussion. Two length scales have to be considered for the optical system length $L$. The "rough length", important to understand propagation (diffraction) effects, and the "exact length", which must be known on the scale of $\lambda$, necessary to determine the exact phase of the field.

**B. Transverse degeneracy of a resonator**

For simplicity purposes, all our discussion about cavities will be restricted to the case of linear optical cavities with two spherical mirrors. Its extension to more complex cases (ring cavity, cylindrical mirrors, etc...) is straightforward. We also assume that the transverse extension of the field is not limited by the size of the mirrors. In this simple case the cavity is fully described by its round-trip Gauss matrix $T_{\text{cav}}$, starting from a given reference plane.

We consider here only geometrically stable cavities ($|A + D| > 2$). In this case, the eigenmodes of the device are the Hermite-Gauss modes (HG) adapted to the cavity, i.e. having wavefront coinciding with the mirror surfaces. The normalized transverse electric field in the $T_{E_{mn}}$ mode basis is given by:

$$A_{mn}(\vec{r}, z) = C_{mn} \frac{1}{w(z)} H_m \left( \frac{\sqrt{2}x}{w(z)} \right) H_n \left( \frac{\sqrt{2}y}{w(z)} \right) e^{i\left( \frac{q^2}{\pi^2} \right)} e^{-i(n+m+1)\arctan \left( \frac{\sqrt{2}y}{w(z)} \right)} e^{ikz}$$  \hspace{1cm} (7)
where:
\[
C_{mn} = \frac{1}{\sqrt{\pi}2^{m+n-1}m!n!}
\]
\[
z_R = \frac{\pi w_0}{\lambda}
\]
\[
q(z) = z - iz_R
\]
\[
w(z) = w_0 \left[ 1 + \left( \frac{z}{z_R} \right)^2 \right]
\]
\[
\Psi(z) = (n + m + 1) \arctan \left( \frac{z}{z_R} \right)
\]

\(w_0\) is the waist of the \(TEM_{00}\) mode of the cavity taken in its focal plane, of coordinate \(z = 0\), and \(q\) is the complex radius of curvature. It is important to note that \(q\) is independent from the mirror positions and size of the waist. Finally, \(\Psi(z)\), the Gouy phase-shift, will play a major role in this discussion.

Let us note \(z_1\) and \(z_2\) the mirror positions and \(L = z_2 - z_1\) the total length of the cavity. The resonant cavity eigenmodes will be the HG modes \(A_{mn}\), having a total round-trip phase-shift equal to \(2p'\pi\), with \(p'\) integer. If the input field has a fixed wavelength \(\lambda\), this will occur only for a comb of cavity length values \(L_{mnp'}\) given by:
\[
L_{mnp'} = \frac{\lambda}{2} \left( p' + (n + m + 1) \frac{\alpha}{2\pi} \right)
\]
where
\[
\alpha = 2 \left( \arctan \left( \frac{z_2}{z_R} \right) - \arctan \left( \frac{z_1}{z_R} \right) \right)
\]
is the Gouy phase shift accumulated by the \(TEM_{00}\) mode along one cavity round-trip. It is related to the cavity Gauss matrix \(T_{\text{cav}}\) eigenvalues \(\mu_{12}\) by the relation:
\[
\mu_{1,2} = e^{\pm i\alpha}
\]
This simple relation has been shown in [8] for a linear cavity with two curved mirrors. We give in the appendix a demonstration of this result valid for any stable paraxial cavity.

A cavity is called "transverse degenerate" when for a given frequency and cavity length, several transverse modes are simultaneously resonant. From equation (9), we can see that:

- there is a natural degeneracy for HG modes giving the same value to \(s = m + n\), related to the cylindrical symmetry of the device. We will not consider this natural degeneracy any longer, and call \(s\) the transverse mode order and \(p\) the longitudinal mode order;
- the cavity is transverse degenerate when \(\alpha\) is a rational fraction of \(2\pi\). Let us write \(\alpha/2\pi\) as an irreducible fraction:
\[
\alpha = 2\pi \frac{K}{N} \{2\pi\},
\]
with \(K, N\) integers and \(0 < \frac{K}{N} < 1\). \(K/N\) is called the degeneracy order of the cavity[8].

As the degeneracy order is the remainder part of the total Gouy phase-shift over one turn of the cavity, we conclude that there exists an infinite number of cavity configurations with the same degeneracy order. Furthermore, the rational fraction ensemble being dense in \(\mathbb{R}\), transverse degenerate cavities is a dense ensemble among all the possible cavities.

Let us first consider the comb of cavity resonant lengths (see figure (1)). Rewriting equation (9) as:
\[
L_{sp} = \frac{\lambda}{2N} \left( Np + K(s + 1) \right).
\]
where \(p\) is an integer. One sees that whereas the free spectral range of the cavity for longitudinal modes \(p\) periodicity) remains equal to the usual value \(\lambda/2\), the natural unit to describe the comb is \(\frac{\lambda}{2N}\). \(N\) and \(K\) appear as the steps, in this natural unit, for the longitudinal comb (when fixing \(s\)) and for the transverse comb (when fixing \(p\)). Within a free spectral range, there exist \(N\) lengths for which the teeth of the comb coincide, allowing us to define \(N\) families of modes.

Let us now consider the cavity in terms of rays optics. (11) implies that a paraxial cavity with a degeneracy order \(K/N\) verifies[8, 9]:
\[
(T_{\text{cav}})^N = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.
\]
where \(I_2\) is the identity matrix of size \(2 \times 2\). This relation means that any incoming ray will retrace back onto itself after \(N\) round-trips, forming a closed trajectory, or orbit. The total phase accumulated on such an orbit is \(2K\pi\) (as can be seen on equation (11)).

Up to now, only perfect Fabry-Perot resonators have been considered. If one consider a cavity with a given finesse \(F \simeq \frac{\lambda}{2\gamma}\), where \(\gamma\) is the energy loss coefficient over one round-trip, supposed small, then \(\frac{1}{F}\) is the mean number of round-trip of the energy in the cavity before it escapes. As a consequence, for a given finesse, and a cavity with a degeneracy order of \(K/N\), we have to compare \(F\) to \(N\). If the finesse is low (i.e. \(F \ll N\)) then light will escape before retracing its path and the previous discussion is not relevant. In the rest of the discussion we will then stay in the high finesse limit (\(F \gg N\)).

We now have all the tools necessary to study the propagation of a paraxial image in a stable resonant cavity.

### III. IMAGE TRANSMISSION THROUGH A PARAXIAL STABLE CAVITY

We will consider for simplicity sake an impedance matched cavity, where the input and output mirror have the same reflectivity and no other losses exist in the cavity, so that at exact resonance a mode is transmitted with
as a complex transmission pending on the length and geometry of the cavity. The fine the input image as the transverse field configuration an efficiency equal to unity. As shown on figure 2, we de-
gresented on (iii) the peaks corresponding to other combs on grey dashed line. (i) low K/N cavity (ii) cavity with K/N=1/2, for instance confocal (iii) cavity with K/N=2/3. K and N are integers. for (ii) and (iii) we indicated besides each peak the first possible modes \((p, s)\). For simplicity sake, we represented on (iii) the peaks corresponding to other combs on grey dashed line.

an efficiency equal to unity. As shown on figure 2, we define the input image as the transverse field configuration \(E_{in}(\vec{r})\) at a chosen plane before the cavity. We want to image it on a detection plane after the cavity. After propagation along a first paraxial system corresponding to an ACT of magnification equal to 1, \(E_{in}(\vec{r})\) is transformed into its near field at a given reference plane inside the cavity (position \(z_{ref}\)). After propagation along a second identical paraxial system, a new near field (output image) is obtained with unity magnification after the cavity on a detection plane (a CCD for instance). As the three planes are perfectly imaged on each other, we will use the same notation \(\vec{r}\) for these three transverse coordinates, and we will omit the z coordinate.

Let \(a_{m,n}\) be the projection of the image on the mode \(A_{m,n}\) of the cavity:

\[
a_{m,n} = \int E_{in}(\vec{r}) A_{m,n}^*(\vec{r}) d^2\vec{r} \tag{15}
\]

We can write \(E_{in}\) as:

\[
E_{in}(\vec{r}) = \sum_{m,n} a_{m,n} A_{m,n}(\vec{r}). \tag{16}
\]

The effect of the cavity on the image can be understood as a complex transmission \(t_{m,n}\) on each mode \(A_{m,n}\), depending on the length and geometry of the cavity. The output image will then be written as:

\[
E_{out}(\vec{r}) = \sum_{m,n} t_{m,n} a_{m,n} A_{m,n}(\vec{r}) \tag{17}
\]

A. Single mode cavity

Let us consider a single mode cavity having a length \(L\) chosen so that only the \(TEM_{00}\) resonates. The transmission function of the cavity is:

\[
t_{m,n} = \delta_{m0} \delta_{n0} \tag{18}
\]

and the output image is:

\[
E_{out}(\vec{r}, t) = \sum_{m,n} t_{m,n} a_{m,n} A_{m,n}(\vec{r}) = a_{0,0} A_{0,0}(\vec{r}) \tag{19}
\]

All the transverse information about the input image \(E_{in}\) is then lost when passing through the cavity. In such a single mode-cavity, the Gouy phase-shift \(\alpha/2\pi\) is not a rational fraction, so that whatever \(N, T_{cav}^N \neq I_2\). In terms of geometrical optics, this means that no ray (except the ray on the optical axis) ever retraces its path on itself. This is the usual understanding of the effect of a cavity on an image, where the image is completely destroyed.

In general the precise length of the resonator is controlled through a piezo-electric transducer. If the single-mode cavity length is scanned over one free spectral range, every Laguerre-Gauss cavity eigenmode will be transmitted one after the other. The intensity field averaged over time on a CCD will be, at a given transverse position:

\[
< I_{out}(\vec{r}) > \propto \sum_{m,n} |a_{m,n}|^2 |u_{m,n}(\vec{r})|^2. \tag{20}
\]

Each mode is transmitted at a different moment and does not interfere with the others. As a consequence we obtain the sum of the intensity into each \(TEM_{pq}\) mode of the image, and not the image since

\[
\sum_{m,n} |a_{m,n}|^2 |u_{m,n}(\vec{r})|^2 \neq \left| \sum_{m,n} a_{m,n} u_{pq}(\vec{r}) \right|^2. \tag{21}
\]

This means that, even scanned, a single-mode cavity does not transmit correctly an image.

FIG. 1: partial transverse and longitudinal comb in 3 configurations. (i) low K/N cavity (ii) cavity with K/N=1/2, for instance confocal (iii) cavity with K/N=2/3. K and N are integers. for (ii) and (iii) we indicated besides each peak the first possible modes \((p, s)\). For simplicity sake, we represented on (iii) the peaks corresponding to other combs on grey dashed line.

FIG. 2: Scheme of the transmission of an image through a cavity.
B. Planar cavity

It is important to study the planar cavity, since it is both widely used experimentally and often taken as a model cavity in theoretical works. Let us consider a planar cavity of length L. The Gauss matrix is:

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix} = \begin{pmatrix}
1 & 2L \\
0 & 1
\end{pmatrix}
\]  

(21)

It does not fulfill condition (14) for any N value, and is therefore not degenerate. Strictly speaking, the planar cavity is not a paraxial cavity, even for rays making a small angle \( \beta \) with the cavity axis, which escape from the axis after a great number of reflections. As a consequence, there is no gaussian basis adapted to this cavity. The planar cavity eigenmodes are the tilted plane waves \( e^{ik(\beta x + \beta y)} \), which are not degenerate since they resonate for different lengths: \( L = p \frac{\lambda}{2} (1 + \beta_1^2/2 + \beta_2^2/2) \). For a given length the cavity selects a cone of plane waves with a given value of \( \beta_1^2 + \beta_2^2 \). The planar cavity is therefore not an imaging cavity. However, given a detail size, if the finesse is low enough and the cavity short enough for the diffraction to be negligible, then the image can be roughly transmitted. This study is again outside the scope of this paper.

C. Totally degenerate cavity

Let us now consider a completely degenerate paraxial cavity, in which all the transverse modes are resonant for the same cavity length. As a consequence the transmission function of this cavity brought to resonance is:

\[ t_{m,n} = 1 \]  

(22)

and the output field will be:

\[ E_{\text{out}}(\vec{r}) = \sum_{m,n} t_{m,n} a_{m,n} u_{m,n}(\vec{r}) = E_{\text{in}}(\vec{r}). \]  

(23)

Its Gauss matrix is \( T_{\text{cav}} = I_2 \), its degeneracy order is 1: every input ray will retrace its path after a single round-trip. A completely degenerate cavity can be called self-imaging. Examples of self imaging cavities have been described in [9].

D. cavity of degeneracy order \( K/N \)

Let us now study the propagation of an image through a transverse degenerate cavity with degeneracy order \( K/N \). We will use a formalism of self-transform function, that we introduce in the next subsection.

1. Cyclic transforms

Some functions are there own Fourier transform. They verify:

\[ \hat{f}(u) = f(u) \]  

(24)

where the Fourier transform \( \hat{f} \) is defined by:

\[ \hat{f}(u) = \int_{-\infty}^{+\infty} f(x) e^{2\pi iux} \, dx. \]  

(25)

Two well known examples are the gaussian functions \( f(x) = ae^{-\pi x^2} \) and the infinite dirac comb \( f(x) = \sum_n \delta(x-n) \). These functions are called Self-Fourier functions or SFFs. Caola[10] showed that for any function \( g(x) \), then \( f \) defined as:

\[ f(x) = g(x) + g(-x) + \bar{g}(x) + \bar{g}(-x) \]  

(26)

is a SFF. Lohmann and Mendlovic[12] showed later that this construction method for a SFF (equation (26)) is not only sufficient but necessary. Any SFF \( f(x) \) can be generated through equation (26) from another function \( g(x) \). Lipson [11] remarked that such distributions should exist in the middle of a confocal resonator. Lohmann [13] also studied how such states could be used to enhance the resolution in imaging.

It is straightforward to generalize this approach to a N-cyclic transform. A transform \( T_C \) is said to be N-cyclic if applied N times to any function \( F \) one gets the initial function:

\[ T_C^N[F(x)] = F(x) \]

Let \( T \) be any transform. A function \( F_S \) will be a self-transform function of \( T \) if:

\[ T[F_S(x)] = F_S(x) \]

Given \( T_C \) a N-cyclic transform, and \( g(x) \) a function, it has been shown in [12] that \( F_S(x) \) defined as:

\[ F_S(x) = g(x) + T_C[g(x)] + T_C^2[g(x)] + \ldots + T_C^{N-1}[g(x)] \]  

(27)

is a self-transform function of \( T_C \) and that any self-transform function \( F_S \) of \( T_C \) can be generated in this manner (take \( g = F_S/N \) for instance). The Fourier transform is 4-cyclic. Other cyclic transforms and associated self-transform functions are studied in [11, 13, 14].

We will show here that degenerate cavities produce such self-transform functions from an input image through a transformation similar to equation (27).

2. Image propagation through a \( K/N \) degenerate cavity

Let us consider a resonator cavity with order of degeneracy \( K/N \). Let \( \gamma \) be the (low) intensity losses over one round trip on the mirrors. For an impedance-matched cavity without internal losses, the losses are identical on
the two mirrors, meaning that the amplitude transmission of one mirror is \( t = \sqrt{\frac{R}{2}} \). For a cavity at a resonance, we have:

\[
T_N^{cav}[E_{in}(\vec{r})] = E_{in}(\vec{r})
\]

(28)
since after \( N \) turns the field comes back onto itself. It means we can view \( T_{cav} \) as a \( N \)-cyclic transform on the intensity.

The output field at resonance will be:

\[
E_{out}(\vec{r}) = \frac{t^2}{1 - (1 - t^2)^N} \sum_{n=0}^{N-1} (1 - t^2)^n T_n^{cav}[E_{in}(\vec{r})]
\]

(29)

the factor \((1 - t^2)\) taking into account the double reflection of the field at each round-trip. Using the fact that \( T_n^{cav}[E_{in}(\vec{r})] = E_{in}(\vec{r}) \), we can finally rewrite the output field as:

\[
E_{out}(\vec{r}) = \frac{t^2}{1 - (1 - t^2)^N} \sum_{n=0}^{N-1} (1 - t^2)^n T_n^{cav}[E_{in}(\vec{r})]
\]

(30)

In the high finesse limit \((\mathcal{F} \gg N)\) \((1 - t^2)^n \simeq 1 - nt^2 \simeq 1\) for \( n \leq N \), so that:

\[
E_{out}(\vec{r}) \simeq \frac{1}{N} \sum_{n=0}^{N-1} T_n^{cav}[E_{in}(\vec{r})]
\]

(31)

The output image is thus the self-transform field for \( N \)-cyclic transform \( T_{cav} \), constructed from the input image through the method of equation (27).

Let us finally note that most of this discussion can be extended to more complex cavities, provided \( T_{cav} \) is a cyclic transform, and that the present formalism holds for single-mode or totally degenerate cavities: in the former case it means that a self-transform function for a non-cyclic transform is just a cavity mode; in the latter case the transform is just the identity, and of course any field is a self-transform for identity.

### IV. HEMI-CONFOCAL CAUSITY

We will now illustrate this formalism by considering in more detail a particular cavity, the hemi-confocal cavity, which is made of a plane mirror and a curved mirror \( R \) of radius of curvature \( R \) separated by a distance \( L = R/2 \) (see figure 3), which has already been studied in terms of beam trajectories[16]. We have studied this kind of cavity both theoretically and experimentally in the framework of our experimental investigations on cw amplification of images in a cavity[17].

#### A. Theoretical study

It is straightforward to show that the round-trip Gouy phase-shift \( \alpha \) is equal to \( \pi/2 \) for a hemi-confocal cavity, so that its degeneracy order is \( 1/4 \): there are four distinct families of transverse modes, depending on the value \( p + q \) modulo 4. The round-trip Gauss matrix, starting from the plane mirror, is:

\[
T_{cav} = \begin{pmatrix} 0 & \frac{R}{2} \\ -\frac{2}{\pi} & 0 \end{pmatrix}
\]

(32)

so that:

\[
T_{cav}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T_{cav}^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]

(33)

So two round-trips give the opposite of the identity (symmetry with respect to the cavity axis), which is the Gauss matrix of the confocal cavity, and four round-trips give the identity, as expected for a cavity with degeneracy order 1/4. \( T_{cav} \) is the transformation of a f-f system, and is an exact AFT transform:

\[
T_{cav} : u(\vec{r}_1) \rightarrow -e^{ikt} \frac{2i}{R\lambda} \int d^2\vec{r}_2 u(\vec{r}_2) \exp \left[ -i \frac{4\pi}{\lambda R} \vec{r}_1 \cdot \vec{r}_2 \right]
\]

(34)

It is equal to the 2-D spatial Fourier transform, of the form:

\[
\tilde{u}(\vec{y}) = \frac{2}{\lambda R} \int u(\vec{r}) e^{-i \frac{4\pi}{\lambda R} \vec{r} \cdot \vec{y}} d^2\vec{r}
\]

(35)

multiplied by a phase factor \( a = ie^{ikL} \), which depends on the exact length of the cavity. It must verify \( a^4 = 1 \) at resonance, so that \( a = 1, i, -1 \) or \( -i \).

If \( E_{in}(\vec{r}) \) is the input image, then the output field is (see figure 4) :

\[
E(\vec{r}) = \frac{1}{4} \left( E_{in}(\vec{r}) + a^2 E_{in}(-\vec{r}) + a \tilde{E}_{in}(\vec{r}) + a^2 \tilde{E}_{in}(-\vec{r}) \right)
\]

(36)

In terms of imaging, \( a \) is the phase between then even/odd parts of the field and its spatial Fourier transform, and \( a^2 \) gives the parity of the output image. Each value of \( a \) corresponds to a given family of modes, more precisely:

\[
\begin{align*}
  a = 1 & \rightarrow \text{modes } m + n = 0 \ (4) \\
  a = i & \rightarrow \text{modes } m + n = 1 \ (4) \\
  a = -1 & \rightarrow \text{modes } m + n = 2 \ (4) \\
  a = -i & \rightarrow \text{modes } m + n = 3 \ (4)
\end{align*}
\]

(37)
For example, the hemi-confocal cavity tuned on the $m + n = 0[4]$ family will transmit the sum of the even part of the image and of the even part of its Fourier transform.

**B. Numerical simulation**

We will now give results of a numerical simulation in a simple experimental configuration: in order to create the input image $E_{in}$, a large Gaussian $TEM_{00}$ mode is intercepted by a single infinite slit of width $w_0$, shifted from the optical axis by $1.5w_0$, which is imaged (near field) onto the reference plane ($z_{ref}$) of the cavity. Without the slit, the input $TEM_{00}$ mode has in the reference plane a size equal to three times the waist of the $TEM_{00}$ cavity eigenmode. We study the transmission of this input image through the cavity at the near field detection plane. We represented on figure (5) the input image, its decomposition over the first 400 $TEM_{pq}$ modes, (with $0 < p, q < 20$), and its spatial Fourier transform. Limiting the decomposition of the image to only $\sim 400$ modes is equivalent to cutting high-order spatial frequencies, and therefore takes into account the limited transverse size of the optics.

Figure 6 gives the expected transmission peak as a function of cavity length, and displays the four families of modes in a given free spectral range. The height of each peak is proportional to the intensity of the image projection on the family of mode. For instance a symmetric field will have no intensity on the $1[4]$ and $3[4]$ families.

Figure 7 gives the amplitude of the transmitted field for each family of modes, calculated from the transmission of the 400 first $TEM_{mn}$ modes. For each family, one easily recognizes the even or odd part of the image (two slits) and the Fourier transform along the axis perpendicular to the slits.

The expected intensity image is represented on figure 8 for each family. One observes that the Fourier transform is much more intense than the image, which is the reason why the local intensity is higher. As the parity information on the field disappears when looking at the intensity, it is difficult to infer from it which resonance is involved. An indication can come from the intensity on the optical axis, which is always zero for an antisymmetric output. One can note that if we add the amplitude fields corresponding to the resonances $m + n = j[4]$ and $m + n = j + 2[4]$, the
two terms corresponding to the Fourier transform vanish. One only gets the even or odd part of the image, which corresponds to the action on the image of a confocal cavity. It is interesting to note that no combination of modes transmit only the Fourier transform of the image.

An interesting question is to know which information is lost in the transmission through the cavity, since one only transmits a quarter of the input intensity. By looking at the transmitted image, it seems that no information is really lost, except on areas where the image and its Fourier transform overlap. But here we have some a priori information on the image that we have sent and we know which part of the output is the image, and which part is the Fourier transform. In more general cases, the information we lose is the knowledge about whether what we observe is the image or the Fourier transform, as well as half the image (since the parity is fixed by the geometry of the cavity, only half the output image is relevant to reconstruct it). Therefore for a given resonance, this cavity not only cuts 75% of the modes, it also destroys 75% of the information.

As a conclusion, the transmission through the cavity transforms the input image into its self-transform function corresponding to the round-trip transform of the hemi-confocal cavity. One may notice that for the resonance \( m + n = i[4] \), a self-Fourier image, i.e a SFF field, is obtained.

C. Experimental demonstration

FIG. 9: Schematic representation of the experimental setup.

For practical reasons, we had to use a hemi-confocal cavity in our experimental set-up designed to study parametric image amplification in optical cavities (see figure 9). We placed a USAF (US Air Force) resolution target on the path of a \( TEM_{00} \) mode produced by a Nd:YAG laser, and imaged it onto the plane mirror of a hemi-confocal cavity, of length 50 mm, servo-locked on a resonance peak. The size of the \( TEM_{00} \) mode inside the cavity was three times larger than the eigenmode waist of the cavity. The finesse of the cavity was about 600. The plane mirror of the cavity was then imaged on a CCD camera. The experimental transmitted images, together with the corresponding objects, are represented on figure (10). The size of the \( TEM_{00} \) cavity mode is roughly equal to the width of the transmitted slit in the second line. One notices that each output image is symmetric, the center of symmetry being the axis of the cavity. It is possible to recognize on the transmitted images the symmetrized input image, and the patterns at the center corresponding to the Fourier transform of the input. For a slit it is well known that its Fourier transform is the sinc-squared diffraction pattern, perpendicular to the slit. This kind of pattern can be recognized on the upper two images of the figure. On the last image the symmetrized "2" is somewhat truncated by the limited field of view.
imposed by the size of the illuminating $TEM_{00}$ mode, whereas the diffraction pattern has the general shape of the Fourier transform of the slit formed by the main bar of the "2", tilted at 45°, plus a more complex shape corresponding to the Fourier transform of the remaining part of the image.

FIG. 10: Image on the resolution target (left) and their transmission through the hemi-confocal cavity (right).

V. CONCLUSION

In summary, this paper has studied in a general way the problem of image transmission through a paraxial cavity, characterized by its round-trip (ABCD) matrix, the eigenvalues of which give the round-trip Gouy phase shift, and therefore the order of transverse degeneracy of the cavity. We have shown that the formalism of self-transform functions, already applied in optics but never to cavities, was very useful to understand how an image is transmitted through a degenerate cavity: at resonance the cavity transmits the self-transform part of the input field. We have then focused our attention on the hemi-confocal cavity, which performs a spatial Fourier transform over one round trip, and shown that it transmits the self-Fourier part of the image. This property was demonstrated experimentally on various shapes of input images. Furthermore we have shown that a transverse degenerate cavity is a very convenient way to produce a self-transform field from any input field, for instance in the case of the hemi-confocal cavity a field which is its own Fourier transform, i.e. its own far-field. Such states are interesting for optics[15] and in physics in general[11].

From a more practical point of view, transverse degenerate cavities can be useful for imaging purposes. For example they are necessary for intracavity c.w. parametric amplification of images. The observation of c.w. image amplification with low pump powers will be reported in a forthcoming publication [17]. These experimental results open the way to the observation of specific quantum aspects of imaging which have been predicted to occur in such devices, such as noiseless phase-sensitive amplification, local squeezing or spatial entanglement.

APPENDIX: EIGENVECTORS AND GOUY PHASE OF A CAVITY

Let A,B,C and D be the coefficients of the cavity round-trip Gauss matrix $T_{cav}$, starting from any plane. Given that $AD - BC = 1$, the eigenvalues of this Gauss are:

$$\mu_{1,2} = e^{\pm i \arccos \frac{A^2 + D^2}{2}} \quad (A.1)$$

They are simply related to the matrix trace $A + D$, and as expected independent of the reference plane one choses in the cavity to calculate the Gauss matrix. Let us now consider the fundamental gaussian mode of the cavity, $E(r) = E(0)e^{-ikr^2/2q}$, where $q$ is the complex radius of curvature. Using the propagation relation (5), one easily computes that, on axis, it becomes after one round trip:

$$E'(0) = E(0)e^{ikL} \frac{1}{B/q + A} \quad (A.2)$$

The round-trip Gouy phase shift $\alpha$ for this mode is therefore:

$$\alpha = \arccos \left( \frac{1}{B/q + A} \right)^{-1} \quad (A.3)$$

On the other hand, after one round trip in the cavity, the complex radius of curvature becomes $\frac{Aq + B}{Cq + d}$, but since it is an eigenmode of the cavity, $q$ must verify:

$$q = \frac{Aq + B}{Cq + d} \quad (A.4)$$

from which one deduces:

$$A + \frac{B}{q} = \frac{D + A}{2} + i\sqrt{1 - (A + D)^2/4} \quad (A.5)$$

From equation(A.3), one then find that $\alpha = \arccos \left( \frac{A + D}{2} \right)$, and therefore using Eq(A.1) one retrieves relation(11).
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