CONSISTENCY OF \( \neg AC^3 \) + ‘\( \chi(E_{G_1}) = 3 \), \( \chi(E_{G_2}) \geq \omega \) \( \implies \chi(E_{G_1 \times G_2}) = 3 \)' AND RELATIVE CONSISTENCY VIA STRONGLY COMPACTNESS

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Abstract. We prove András Hajnal’s Theorem 2 of [Haj85] in different ways and observe a permutation model where the axiom of choice for 3 element sets fails but the statement in Theorem 2 of [Haj85] still holds for \( k = 3 \). We also observe that the Dilworth’s decomposition theorem for infinite p.o.sets of finite width holds and a weaker form of Loé’s lemma (p. 253 of [HoRa08]) fails in the permutation model of Theorem 7 of [HT18] due to Lorenz Halbeisen and Eleftherios Tachtsis. Secondly, we weaken the large cardinal assumption of the results from [AC13] due to Arthur Apter and Brent Cody, from a supercompact cardinal to a strongly compact cardinal. Further, applying the appropriate automorphism technique from [AH91] we remove the additional assumption that ‘every strongly compact cardinal is a limit of measurable cardinals’ from corollary 2.32 of section 4, chapter 2 of [Di11] by Ioanna Dimitriou.

1. Introduction

§. In Theorem 2 of [Haj85], András Hajnal proved that if the chromatic number of a graph \( G_1 \) is finite (say \( k < \omega \)), and the chromatic number of another graph \( G_2 \) is infinite, then the chromatic number of \( G_1 \times G_2 \) is \( k \). Hajnal mentioned the usage of Gödel’s compactness theorem for propositional logic in his proof in one line. The second author helped to observe Lemma 2.1 which helped us to figure out a different proof using the compactness theorem, explicitly in this note. Using that proof we approach the problem in several other ways. In the solution of problem 12, chapter 23 of [KT06], Péter Komjáth and Vilmos Totik gave a second argument using the ultrafilter lemma which is an equivalent formulation of the Gödel’s compactness theorem in ZF. We observe twelve different proofs categorized as follows.

(1) In [Cow77], Robert Cowen obtained a generalized version of König’s lemma, which is equivalent to the ultrafilter lemma in ZF. We provide an argument incorporating the methods of Cowen from [Cow77].

(2) We observe a straightforward argument using Rado’s selection lemma from [Rado49] when \( AC^{fin} \) is assumed, following the well-known applications of Rado’s selection lemma in [EB51], [Mir71] and [Rado65].

(3) We observe a straightforward argument using Cowen–Engeler lemma, which is another equivalent formulation of the ultrafilter lemma in ZF.

(4) Following the proof of Marshall Hall’s infinite Hall’s theorem from [Hal66], we observe an argument using Tychonoff’s theorem for finite discrete spaces.

(5) Following the proof of Rado’s selection lemma from [Lux62], we observe an argument which involves the usage of both ultrafilters and ultrapowers.

(6) Péter Komjáth communicated to us two different ways of proving the De Bruijn–Erdős theorem using the ultrafilter lemma. We incorporate the arguments to provide two more proofs using the ultrafilter lemma.

Key words and phrases. Chromatic number of product of graphs, ultrafilter lemma, permutation models, strongly compactness, symmetric extension.

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1 Rado’s selection lemma was used to prove the De Bruijn-Erdős theorem from [EB51], Dilworth’s decomposition theorem for infinite p.o.sets with finite width from [Mir71] and Marshall Hall’s infinite Hall’s theorem from [Hal48] in the presence of \( AC^{fin} \).
(7) We also observe an argument using the methods concerning compactness via prime semi lattices from [Cow83] due to Robert Cowen.

(8) We observe an argument using nonstandard analysis from [HLS5] due to Albert.E. Hurd and Peter.A. Loeb.

(9) Péter Komjáth communicated to us three different ways of proving De Bruijn–Erdős theorem using Zorn’s lemma. We incorporate two of those arguments, to provide two more proofs using Zorn’s lemma, by constructing two different partially ordered sets.

(10) We observe one argument using transfinite recursion following De Bruijn’s original proof of the De Bruijn–Erdős theorem.

In particular, we can prove the following theorem in ZFC in thirteen different ways in section 2, each of which has distinguished character and each of which is different from the proof in [KT06].

Theorem 1.1. (Theorem 2 of [Haj85]), (ZFC). If the chromatic number of a graph $G_1$ is finite (say $k < \omega$), and the chromatic number of another graph $G_2$ is infinite, then the chromatic number of $G_1 \times G_2$ is $k$.

Clearly, the methods of section 2 can be applied to give several other proofs to problems related to the Gödel’s Compactness theorem. We list a few of those well-known results.

1. $\chi(E_{G_1}) \leq \mu$ if $G = (V_G, E_G)$ is a $K_\mu$-chordal graph [Kom15].
2. Dilworth’s decomposition theorem for infinite p.o.sets with finite width [Dil50], [Tac19], [Mir71].
3. Uniqueness of every field has an algebraic closure [Bana92].
4. For every finite field $F$, for every nontrivial vector space $V$ over $F$, there exists a non-zero linear functional $f : V \to F$, (c.f. Theorem 18 of [HT13]).
5. The infinite Hall’s theorem [Haj88], [Haj96].
6. a restricted version of Tukey-Teichmüller Theorem [Hod05].
7. Given an infinite graph $X$ and a finite graph $H$, if every finite subgraph of $X$ has a homomorphism into $H$, then so has $X$.

§. In section 3, we observe that although the ultrafilter lemma or some choice principle stronger than it, is needed in all the proofs of Theorem 1.1, the ultrafilter lemma can consistently fail even if the statement in Theorem 1.1 holds for $k = 3$. In particular, the first author observe that there is a permutation model where the axiom of choice for 3 element sets fails\footnote{axiom of choice for 3 element sets is strictly weaker than the ultrafilter lemma.} but if $\chi(E_{G_1}) = 3$ and $\chi(E_{G_2}) \geq \omega$ then $\chi(E_{G_1 \times G_2}) = 3$ holds, if we denote $\chi(E_{G})$ as the chromatic number of the graph $G = (V_G, E_G)$. We denote $\mathfrak{P}_k$ as the following statement for some natural number $k < \omega$,

$\chi(E_{G_1}) = k < \omega$ and $\chi(E_{G_2}) \geq \omega$ implies $\chi(E_{G_1 \times G_2}) = k$.’

and prove the following theorem.

Theorem 1.2. ($\mathfrak{P}_3$ does not imply $AC_3$ in ZFA). There exists a permutation model where the axiom of choice for 3 element sets fails, but $\mathfrak{P}_3$ is true. Moreover, in the permutation model there is a denumerable family $A$ of 3 element sets, which has no partial Kinoa–Wegner selection function\footnote{i.e., there is no infinite subfamily $A'$ of $A$ with a function $f$ such that $dom(f) = A'$ and for all $U \in A'$, $\emptyset \neq f(U) \subseteq U$ in the permutation model.}.

§. In section 4, adopting the methods from the proof of Theorem 3.1 (i) of [Tac19] due to Eleftherios Tachtsis with some minor modifications, the first author observe a new proof of Theorem 1.1 in ZF, if $G_1$ is a graph on some well-orderable set of vertices. Similarly, we figure out a new proof of the De Bruijn–Erdős theorem for graphs on a well-orderable set of vertices.
Observation 1.3. (ZF). If the chromatic number of a graph $G_1$ on some well-orderable set of vertices is finite (say $k < \omega$), and the chromatic number of another graph $G_2$ is infinite, then the chromatic number of $G_1 \times G_2$ is $k$. Consequently, we can also prove the following.

1. If $\mathcal{P}_k'$ denotes the following statement,
   
   ‘If the chromatic number of a comparability graph $G_1$ (of some p.o.set) whose independent sets are well-orderable, is finite (say $k < \omega$), and the chromatic number of another graph $G_2$ is infinite, then the chromatic number of $G_1 \times G_2$ is $k.’$

   then, $\mathcal{P}_k'$ does not imply ‘There are no amorphous sets’ in ZFA. In particular, we observe two different permutation models where $\mathcal{P}_k'$ holds but ‘There are no amorphous sets’ fails.

2. If $\mathcal{P}_k''$ denotes the following statement,
   
   ‘Given any natural number $n < \omega$, if the chromatic number of a comparability graph $G_1$ (of some p.o.set) whose independent sets have size at most $n$, is finite (say $k < \omega$), and the chromatic number of another graph $G_2$ is infinite, then the chromatic number of $G_1 \times G_2$ is $k.’$

   then, $\mathcal{P}_k''$ does not imply AC$^{\text{fin}}$ in ZFA.

Observation 1.4. (ZF). A graph $G$ on some well-orderable set of vertices is $n$-colorable if and only if each of its subgraphs is $n$-colorable.

In [Bab69] Laszlo Babai proved that a graph on some well-orderable set of vertices is finitely chromatic if and only if every subset of order-type $\omega$ is finitely chromatic. In the proof (c.f. solution of Problem 10, chapter 23 of [Kan80]), full strength of dependent choice (DC or DC$^\omega$) and the De Bruijn–Erdős theorem for $\geq 3$ colorings was assumed. In [Lau71], Läuchli proved that the De Bruijn–Erdős theorem for $\geq 3$ colorings is equivalent to the ultrafilter lemma (UL) in ZF. Since De Bruijn–Erdős theorem for graphs on a well-orderable set of vertices is used here, using Observation 1.4, we may obtain the result in ZF+DC$^\omega$ only.

Corollary 1.5. (of Observation 1.4). It is possible to prove the following well-known result in ZF+DC$^\omega$ only.

‘Let $X$ be a graph on some well-orderable set $V$. Then $X$ is finitely chromatic if and only if every subset of order-type $\omega$ is finitely chromatic.’

§. In Theorem 7 of [HTT18], Lorenz Halbeisen and Eleftherios Tachtsis constructed a permutation model $N$ where for arbitrary $n \geq 2$, $C_n^{-}$ fails. In section 5, the first author observe that not only the Dilworth’s decomposition theorem for infinite p.o.sets with finite width (DT) holds but also $\mathcal{P}_k'$ for all finite integer $k < \omega$ holds in $N$. Consequently, DT does not imply $C_n^{-}$, for each $n \geq 2$ in ZFA. Moreover, LT (c.f. Appendix C) fails in $N$ following [Tac19].

Theorem 1.6. (DT does not imply $C_n^{-}$ in ZFA for any $n \geq 2$). For every natural number $n \geq 2$, there is a model $N$ of ZFA where the Dilworth’s decomposition theorem for infinite p.o.sets with finite width holds but there is an infinite family of $n$-element sets which have no partial choice function. Moreover, we can see the following in $N$.

- $\mathcal{P}_k'$ holds for all finite integer $k < \omega$.
- If LT denote the following statement,

   ‘If $\mathcal{A} = (A, R^A)$ is a non-trivial relational $L$-structure over some language $L$, and $\mathcal{U}$ be an ultrafilter on a non-empty set $I$, then the ultrapower $\mathcal{A}/\mathcal{U}$ and $\mathcal{A}$ are elementarily equivalent.’

   then LT fails in $N$.

4 Following the terminologies of [HTT18], this means that there is an infinite family of $n$-element sets which has no partial choice function.

5 Following the proof of Consequence (1) of Observation 1.3.
Secondly, we reduce the large cardinal assumptions of two previously well-known results by working on symmetric extensions on strongly compact Prikry forcing.

§. In Theorem 1 of [AC13], Arthur Apter and Brent Cody obtained a symmetric extension where \( \kappa \) and \( \kappa^+ \) are both singular, and there is a sequence of distinct subsets of \( \kappa \) of length equal to any predefined ordinal, assuming a supercompact cardinal \( \kappa \). They used the fact that it is possible to obtain a forcing extension where a supercompact cardinal \( \kappa \) can become indestructible under \( \kappa \)-directed closed forcing notions\(^6\) and worked on symmetric extension based on supercompact Prikry forcing to obtain the result. In section 6 the first author observe that applying a recent result of Toshimichi Usuba, which is Theorem 3.1 of [ADU19], followed by working on symmetric extensions based on strongly compact Prikry forcing (c.f. [AH91]), it is possible to weaken the assumption of a supercompact cardinal \( \kappa \) to a strongly compact cardinal \( \kappa \).

Theorem 1.7. (Reducing the assumption of Theorem 1 of [AC13]). Suppose \( \kappa \) is a strongly compact cardinal, GCH holds, and \( \theta \) is an ordinal in a ground model \( V \) of ZFC. There is then a symmetric extension \( V(G) \) where AC fails, \( \kappa \) and \( \kappa^+ \) are both singular with \( (\text{cf}(\kappa))^{V(G)} = \omega \) and \( (\text{cf}(\kappa^+))^{V(G)} < \kappa \). Moreover, \( \kappa \) is a strong limit cardinal that is a limit of inaccessible cardinals and there is a sequence of distinct subsets of \( \kappa \) of length \( \theta \) in the symmetric extension \( V(G) \).

§. In [ADK16], Arthur Apter, Ioanna Dimitriou and Peter Koepke proved that in Gitik’s model \( G \), every singular cardinal is a Rowbottom cardinal with a Rowbottom filter. Further, they conjectured about the possibility of removing the additional assumption that ‘every strongly compact cardinal is a limit of measurable cardinals’. Arthur Apter communicated to us that the methods of [AH91] can be applied to prove the conjecture. The conjecture is still open, but inspired from the appropriate automorphism technique used in [AH91], in section 7 the first author observe a symmetric extension with a sequence of successive singular Rowbottom cardinals that has order type larger than \( \omega \), and smaller than or equal to \( (\omega_1)^V \), if \( V \) is the ground model. This may remove the additional assumption that ‘every strongly compact cardinal is a limit of measurable cardinals’ from corollary 2.32 of [Dim11] by Ioanna Dimitriou.

Theorem 1.8. (Reducing the assumption of corollary 2.32 of [Dim11]). Suppose for some ordinal \( \rho \in (\omega, \omega_1] \), there is a \( \rho \)-long sequence \( \langle \kappa_\alpha : 0 < \alpha < \rho \rangle \) of strongly compact cardinals, which sequence has limit \( \eta \) in a ground model \( V \) of ZFC. There is then a symmetric extension \( \langle \kappa_\alpha : 0 < \alpha < \rho \rangle \rangle^V \) of a symmetric system \( \langle \mathcal{P}, \mathcal{G}, \mathcal{F} \rangle \) where all cardinals in the interval \( (\omega, \eta) \) are uncountable, singular and almost Ramsey. Moreover, all uncountable singular cardinals in \( (\omega, \eta) \) carry a Rowbottom filter in \( V(G) \).

Finally, we remark that if we work with strongly compact Prikry forcing instead of injective tree Prikry forcing as done in the proof of Theorem 2.12 of [Dim11], then we can extend Theorem 2.12 from [Dim11] as follows.

Observation 1.9. (Extending Theorem 2.12 of [Dim11]). Suppose there is an increasing sequence \( \langle \kappa_n : 0 < n < \omega \rangle \) of strongly compact cardinals in a ground model \( V \) of ZFC, which sequence has limit \( \eta \). For any function \( f : \omega \to 2 \) in the ground model, there is then a symmetric extension \( V(G) \) in terms of a symmetric system \( \langle \mathcal{P}, \mathcal{G}, \mathcal{F} \rangle \) where \( g_{n+1} \) is regular if \( f(n) = 1 \) and singular if \( f(n) = 0 \). Moreover, each singular cardinal in the obtained pattern of regular and singular cardinals, carry a Rowbottom filter.

In Appendix A, we observe that if \( G_1 \) has countably many vertices, then we can obtain a proof of Theorem 1.1 using König’s lemma, a well-known equivalent of AC\(_\omega^\infty\) and another proof using sequential compactness of topological spaces. We prove a weakly compact variant of Theorem 1.1 in Appendix B. In Appendix C we provide a list of forms from [HorRe98] which we use in this note.

\(^6\)Using Laver’s indestructibility of supercompactness.
\(^7\)In [Ban] the first author observed that the first supercompact cardinal could be the first uncountable regular cardinal at any successor height in Gitik’s model.
In this section, we prove Theorem 1.1 in thirteen different ways. We define the cartesian product of two graphs \( G_1 = (V_{G_1}, E_{G_1}) \) and \( G_2 = (V_{G_2}, E_{G_2}) \) as the graph \( G_1 \times G_2 = (V_{G_1 \times G_2}, E_{G_1 \times G_2}) = (V_{G_1} \times V_{G_2}, \{(x_0, y_0), (y_0, y_1)\} : \{x_0, y_0\} \in E_{G_1}, \{x_1, y_1\} \in E_{G_2}\) where \( V_{G_1} \times V_{G_2} \) is the cartesian product of the vertex sets \( V_{G_1} \) and \( V_{G_2} \). If the chromatic number of \( G \) is \( \chi(G) \), then \( \chi(G_1 \times G_2) \leq \min(\chi(G_1), \chi(G_2)) \) if \( \chi(G) \) is denoted as the chromatic number of the graph \( G = (V_{G_1} \times V_{G_2}) \). It can be seen that \( \chi(G_1 \times G_2) \leq \min(\chi(G_1), \chi(G_2)) \) if \( \chi(G) \) is denoted as the chromatic number of the graph \( G = (V_{G_1} \times V_{G_2}) \).

We prove the following theorem.

**Theorem 1.1**. There exists a map \( i_F : F \rightarrow \{1, \ldots, k-1\} \) such that for any \( x, y \in V_{G_1} \), \( \{x, i_F(y)\} \) is independent.

**Lemma 2.1**. For all finite \( F \subset V_{G_1} \), there exists a mapping \( i_F : F \rightarrow \{1, \ldots, k-1\} \) such that for any \( x, x' \in F, A_{x,i_F(x)} \cap A_{x',i_F(x')} \) is not independent.

Proof. Since any superset of non-independent set is non-independent, it is enough to show that for all finite \( F \subset V_{G_1} \), there exists an \( i_F : F \rightarrow \{1, \ldots, k-1\} \) such that \( \cap_{x \in F} A_{x,i_F(x)} \) is not independent. For the sake of contradiction suppose that there exist a finite \( F \subset V_{G_1} \) such that for all \( i_F : F \rightarrow \{1, \ldots, k-1\}, \cap_{x \in F} A_{x,i_F(x)} \) is independent. Assume that \( V_{G_2} = \cup_{x \in F} A_{x,i_F(x)} \). Thus \( V_{G_2} \) can be written as a finite union of independent sets contradicting the fact that \( \chi(G_2) \) is infinite. Thus for all finite \( F \subset V_{G_1} \), we can obtain a mapping \( i_F : F \rightarrow \{1, \ldots, k-1\} \) such that \( \cap_{x \in F} A_{x,i_F(x)} \) is not independent.

In all our subsequent solutions, which are either dependent on some equivalent formulations of the axiom of choice or some weaker choice principles, we prove the following either using Lemma 2.1 or without it.

**Theorem 2.2.** (Compactness theorem for propositional logic). A set of propositional logic formulae is satisfiable if and only if every finite subset of it is satisfiable.

Proof. (1st proof of Theorem 1.1). By well-ordering principle, we enumerate \( V_{G_1} \) as \( V_{G_1} = \{x_1, x_2, \ldots\} \). We work with propositional language with the following sentence symbols.
A proof applying Cowen’s generalized form of König’s lemma. We prove Theorem 1.1 using Cowen’s generalization of König’s lemma from [Cow77] (c.f. Theorem 1 of [Cow77]).

We consider $\Sigma$ to be the collection of the following well-founded formulae.

1. $A'_{x_i,m} \land A'_{x_j,l}$ if $A_{x_i,m} \cap A_{x_j,l}$ is not an independent set where $l, m \in \{1, 2, \ldots, k-1\}$ and $x_i, x_j \in V_{G_1}$, such that $x_i \neq x_j$.
2. $\neg(A'_{x_i,j} \land A'_{x_j,l})$ for any $l, j \in \{1, 2, \ldots, k-1\}$ such that $l \neq j$ and each $x_i \in V_{G_1}$.
3. $A'_{x_1,1} \lor A'_{x_2,2} \lor \ldots \lor A'_{x_k,k-1}$ for each $x_i \in V_{G_1}$.

**Claim 2.3.** If $v$ is a truth assignment which satisfies $\Sigma$, then we can define a mapping $i : V_{G_1} \to \{1, 2, \ldots, k-1\}$ such that the intersection of any two elements in $\{A_{x_i(x)} : x \in V_{G_1}\}$ is not independent by

$$i(x_i) = A'_{x_i,j} \text{ if and only if } v(A'_{x_i,j}) = T.$$ 

**Proof.** By (2) and (3) for each $x_i \in V_{G_1}$, each collection $S_{x_i} = \{A_{x_i,1}, \ldots, A_{x_i,k-1}\}$ gets assigned a unique representative. By (1), for any $x_i, x_j \in V_{G_1}$ such that $x_i \neq x_j$, the representatives of $S_{x_i}$ and $S_{x_j}$ are such that the intersection of them is not independent. □

**Claim 2.4.** Any finite subset $\Sigma' \subseteq \Sigma$, is satisfiable.

**Proof.** Given any finite subset $\Sigma' \subseteq \Sigma$, let $F = \{x_{i_1}, \ldots, x_{i_l}\}$ be the vertices that are mentioned in $\Sigma'$. By Lemma 2.1 there is a mapping $i_F : F \to \{1, 2, \ldots, k-1\}$ such that for any $x, x' \in F$ such that $x \neq x'$, $A_{x,i_F(x)} \cap A_{x',i_F(x')} \text{ is not independent.}$ Let $v_0$ be a truth assignment such that for all $1 \leq r \leq l$ and $x \in S_{x_{i_r}} = \{A_{x_{i_r},1}, \ldots, A_{x_{i_r},k-1}\}$,

$$v_0(A'_{x_{i_r},r}) = T \text{ if and only if } x = i_F(x_{i_r}).$$

Clearly, $v_0$ satisfies $\Sigma'$.

So by Theorem 2.2 and claim 2.4, $\Sigma$ is satisfiable. By claim 2.3 we can obtain an $i : V_{G_1} \to \{1, 2, \ldots, k-1\}$ such that intersection of any two elements in $\{A_{x_i(x)} : x \in V_{G_1}\}$ is not independent. □

2.2. A proof applying Cowen’s generalized form of König’s lemma. We prove Theorem 1.1 using Cowen’s generalization of König’s lemma from [Cow77] (c.f. Theorem 1 of [Cow77]).

\begin{figure}[ht]
\centering
\includegraphics[scale=0.5]{figure1.png}
\caption{A map $i : V_{G_1} \to \{1, \ldots, k-1\}$ such that intersection of any two elements in $\{A_{x_i(x)} : x \in V_{G_1}\}$ is not independent.}
\end{figure}
2.3. A proof applying Rado’s selection lemma in presence of the axiom of choice for finite sets. In [EB51], De Bruijn and Erdős mentioned that the De Bruijn–Erdős theorem follows from Rado’s selection lemma [Rado9]. In [Mir71], Mirsky gave a proof of Dilworth’s decomposition theorem using Rado’s selection lemma. In [Rado65], Rado himself gave another proof of the infinite Hall’s theorem using the Rado’s selection lemma. In all those proofs, \( AC \) was assumed. Trivially Theorem 1.1 also follows from Rado’s selection lemma if we assume \( AC^{fin} \).

Lemma 2.5. (Rado’s selection lemma). Let \( \{A_i : i \in I\} \) be a family of finite sets with arbitrary index and suppose for each finite \( F \subset I \), there is a \( f_F \in \Pi_{i \in F} A_i \), then there is a \( f \in \Pi_{i \in I} A_i \) such that for all finite \( W \subset I \), there is a finite \( V \subset I \) such that \( W \subset V \) and \( f \restriction W = f_V \restriction W \).

Proof. (3rd proof of Theorem 1.1). Let \( I = V_{G_1} \) and \( A_i = \{1, \ldots, k-1\} \) for each \( i \in V_{G_1} \). By \( AC^{fin} \), \( \Pi_{i \in V_{G_1}} A_i \) is non-empty. By Lemma 2.1, for all finite \( F \subset V_{G_1} \), there exists a mapping \( i_F : F \to \{1, \ldots, k-1\} \) such that for any \( x, x' \in F \), \( A_{x,i_F(x)} \cap A_{x',i_F(x')} \) is not independent. The mapping obtained by applying Lemma 2.5 is our desired mapping \( i \).

Remark 1. In ZF, the ultrafilter lemma is equivalent to ‘Rado’s selection lemma + \( AC^{fin} \)’ (Fact 1 of [HT14]). On the other hand, Rado’s selection lemma don’t imply \( AC^{fin} \) in ZF. So, we are mentioning the usage of \( AC^{fin} \) explicitly in the above proof. In particular, in Fraenkel’s 2nd-model (Fraenkel-Mostowski model \( N_2 \) from [How84a]), the Rado’s selection lemma holds [How84a], but the ultrafilter lemma fails. Consequently in \( N_2 \), \( AC^{fin} \) fails too. We cant say whether we can prove Theorem 1.1 in \( N_2 \) or not. So, we ask the following question.

Question 2.6. Can we prove Theorem 1.1 using only Rado’s selection lemma in ZF?

Remark 2. We note that in all other proofs in section 2, either some equivalent of the ultrafilter lemma is assumed or some equivalent of the axiom of choice like well-ordering principle or Zorn’s lemma is assumed. Since the Zorn’s lemma implies the ultrafilter lemma, which implies \( AC^{fin} \) in ZF, we are not mentioning the usage of \( AC^{fin} \) explicitly in other proofs.
(2) for each finite subset \( F \) of \( X \), there exists an element \( f \in M \) such that the domain of \( f \) is \( F \).

(3) \( M \) has finite character.

**Proof.** (4th proof of Theorem 1.1). Let \( M \) be the set of all mappings \( g : X' \to \{1, 2, \ldots, k - 1\} \) where \( X' \subseteq V_{G_1} \) such that for any \( x, y \in X' \) and \( x \neq y \) we have \( A_{x,g(x)} \cap A_{y,g(y)} \) is not independent. We can see the following.

1. For every \( x \in V_{G_1} \), \( \phi(x) = \{g(x) : g \in M \land x \in \text{dom}(g)\} \) is finite.
2. By Lemma 2.1, for each finite subset \( X' \) of \( V_{G_1} \), there is a \( g : X' \to \{1, 2, \ldots, k - 1\} \) such that for any \( x, y \in X' \) and \( x \neq y \) we have \( A_{x,g(x)} \cap A_{y,g(y)} \) is not independent. Thus \( \text{dom}(g) = X' \).
3. \( M \) has finite character. Let \( g \in M \) and \( F \) be a finite subset of \( V_{G_1} \), then \( g \upharpoonright F \in M \). In the other direction, suppose \( g : X' \to \{1, 2, \ldots, k - 1\} \) such that for any \( x, y \in X' \subseteq V_{G_1} \) and \( x \neq y \) we have \( A_{x,g(x)} \cap A_{y,g(y)} \) is not independent and the restriction of \( g \) to each finite subset of \( \text{dom}(g) \) be in \( M \). Let \( p, q \in \text{dom}(g) \) be such that \( p \neq q \). Now, \( \{p, q\} \) is a finite subset of \( \text{dom}(g) \) and so \( g \upharpoonright \{p, q\} \in M \). So, \( A_{p,g(p)} \cap A_{q,g(q)} \) is not independent. Consequently, \( g \in M \). So, \( M \) has a finite character. Thus by the Lemma 2.7, \( M \) has an element \( i \) such that \( \text{dom}(i) = V_{G_1} \). Thus \( i : V_{G_1} \to \{1, 2, \ldots, k - 1\} \) is a mapping such that for any \( x, y \in V_{G_1} \) and \( x \neq y \) we have that \( A_{x,i(x)} \cap A_{y,i(y)} \) is not independent. \( \square \)

2.5. A proof applying Tychonoff’s theorem for finite discrete spaces. Tychonoff’s theorem, in general, is equivalent to the axiom of choice \((AC)\) and Tychonoff’s theorem for compact Hausdorff spaces as well as finite discrete spaces are equivalent to the Boolean prime ideal theorem which is strictly weaker than \( AC \).

**Theorem 2.8.** (Tychonoff’s theorem for finite discrete spaces). Arbitrary product of finite discrete spaces is compact.

In [Hal66], James Halpern proved the infinite Hall’s theorem using Theorem 2.8. We incorporate the methods from [Hal66] to give a proof of Theorem 1.1.

**Proof.** (5th proof of Theorem 1.1). Endow \( \{1, 2, \ldots, k - 1\} \) with the discrete topology. The product space \( \{1, 2, \ldots, k - 1\} V_{G_1} \) with product topology is compact by Theorem 2.8. For \( s \in \{V_{G_1}\}^{<\omega} \), define \( F_s = \{f \in \{1, 2, \ldots, k - 1\} V_{G_1} : x, y \in s, x \neq y \to A_{x,f(x)} \cap A_{y,f(y)} \) is not independent\}. We can see the following.

- Following Lemma 2.1, for each \( s \in \{V_{G_1}\}^{<\omega} \) we have that \( F_s \) is non-empty.
- Since we are considering the product topology of \( \{1, 2, \ldots, k - 1\} V_{G_1} \), for each \( s \in \{V_{G_1}\}^{<\omega} \), complement of \( F_s \) is open, and so \( F_s \) is closed.
- We can see that \( \{F_s : s \in \{V_{G_1}\}^{<\omega}\} \) has finite intersection property as \( F_{s_1} \cap \ldots \cap F_{s_k} \subseteq F_{s_1 \cup \ldots \cup s_k} \).

Thus by compactness of \( \{1, 2, \ldots, k - 1\} V_{G_1} \), there is an \( i \in \cap \{F_s : s \in \{V_{G_1}\}^{<\omega}\} \). We can observe that \( i : V_{G_1} \to \{1, \ldots, k - 1\} \) is a function such that the intersection of any two elements in \( \{A_{x,i(x)} : x \in V_{G_1}\} \) is not independent. Pick arbitrary \( x, y \in V_{G_1} \). There is then \( \{x, y\} \in \{V_{G_1}\}^{<\omega} \) such that \( i \in F_{\{x, y\}} \). Consequently, \( A_{x,i(x)} \cap A_{y,i(y)} \) is not independent. \( \square \)

2.6. Different proofs using ultrafilter lemma. In problem 12, chapter 23 of [KT06], using the ultrafilter lemma, Péter Komjáth and Vilmos Totik proved Theorem 1.1. We give a brief description of the ultrafilter lemma and provide three different proofs using it. Let \( X \) be a set. \( F \subseteq P(X) \) is a filter on \( X \), if it is closed under upward inclusion, finite intersections and doesn’t contain the empty set. We say \( F \) is an ultrafilter if it is a filter and for all \( A \subseteq X \), either

\( 8 \)In Howard Rubin’s first model \((\Lambda'38 \text{ in } [HoRu08])\), Howard Rubin’s second model \((\Lambda'40 \text{ in } [HoRu08])\) and Cohen’s first model \((M1 \text{ in } [HoRu98])\), the Boolean prime ideal theorem holds, but \( AC \) fails.
Proof. \( \) of the graphs two elements \( f, g \) for each \( f \) the set Definition 2.12. Consequently, \( \) as the De Bruijn–Erdős theorem combining Proof. \( A \) as the De Bruijn–Erdős theorem combining induction and ultrafilters on infinite cardinal \( \kappa \) which extends the filter of end segments. We incorporate the arguments to prove Theorem 1.1.

Proof. \( \) of Theorem 1.1. We show by transfinite induction on the infinite cardinal \( \kappa \), that if \( |V_{G_1}| = \kappa \), then there exists an \( i : V_{G_1} \to \{1, \ldots, k-1 \} \) such that intersection of any two elements in \( \{A_{x,(i)} : x \in V_{G_1} \} \) is not independent.

First we enumerate \( V_{G_1} \) as \( V_{G_1} = \{v_\alpha : \alpha < \kappa \} \) and assume that the sentence holds for all cardinals less than \( \kappa \). Let \( \mathcal{U} \) be an ultrafilter on \( \kappa \) which extends the filter of end segments by Lemma 2.9. Formally, \( \mathcal{U} \) be an ultrafilter on \( \kappa \) such that,

\[ \{\beta : \alpha < \beta < \kappa \} \in \mathcal{U} \text{ for all } \alpha < \kappa. \]

By assumption, for each \( \alpha < \kappa \) there is a \( f_\alpha : \{v_\beta : \beta < \alpha \} \to \{1, \ldots, k-1 \} \) such that the intersection of any two elements in \( \{A_{x,f_\alpha(x)} : x \in \{v_\beta : \beta < \alpha \} \} \) is not independent. By Lemma 2.10, for each \( \beta < \kappa \), there is a unique \( 1 \leq i(\beta) \leq k-1 \) such that \( A_\beta = \{\alpha : \beta < \alpha \text{ and } f_\alpha(v_\beta) = i(\beta)\} \in \mathcal{U}. \)

Claim 2.11. \( v_\beta \mapsto i(\beta) \) is the required mapping on \( V_{G_1}. \)

Proof. Let us pick arbitrary \( v_\gamma \) and \( v_\delta \) in \( V_{G_1}. \) Since \( A_\gamma \cap A_\delta \in \mathcal{U} \), we can pick an \( \alpha \in A_\gamma \cap A_\delta. \) Consequently, \( i(\gamma) = f_\alpha(v_\gamma), \) \( i(\delta) = f_\alpha(v_\delta) \) and \( A_{\gamma,f_\alpha(v_\gamma)} \cap A_{\delta,f_\alpha(v_\delta)} \) is not independent. So, \( A_{\gamma,i(\gamma)} \cap A_{\delta,i(\delta)} \) is not independent. \( \square \)

In [Lux62], Luxemburg gave a unique and interesting proof of the Rado’s selection lemma as well as the De Bruijn–Erdős theorem combining ultraproducts and Lemma 2.9. We incorporate the method from [Lux62] to give another proof of Theorem 1.1. We briefly describe the concept of ultraproducts before sketching the proof.

Definition 2.12. (Ultraproducts) Let \( I \) be an indexing set and \( \mathcal{U} \) be an ultrafilter over \( I \) and for each \( i \in I, \) let \( \mathfrak{A}_i = (A_i, f^{A_i}, R^{A_i}, \ldots) \) be a non-trivial \( \mathcal{L} \) structure for some language \( \mathcal{L}. \) For two elements \( f, g \in \Pi_{i \in I} A_i, \) we say \( f \sim_\mathcal{U} g \) if and only if \( \{i \in I : f(i) = g(i)\} \in \mathcal{U}. \) We define the set \( f/\mathcal{U} = \{g : f \sim_\mathcal{U} g\}. \) The ultraproduct \( \Pi_{i \in I} \mathfrak{A}_i/\mathcal{U} = (\Pi_{i \in I} A_i/\mathcal{U}, f^{\Pi_{i \in I} A_i/\mathcal{U}}, R^{\Pi_{i \in I} A_i/\mathcal{U}}) \) is an \( \mathcal{L} \)-structure which is defined as follows.

- The domain of the ultraproduct \( \Pi_{i \in I} \mathfrak{A}_i/\mathcal{U} \) is defined as \( \Pi_{i \in I} A_i/\mathcal{U} = \{f/\mathcal{U} : f \in \Pi_{i \in I} A_i\}. \)
- \( f^{\Pi_{i \in I} A_i/\mathcal{U}}(a_1/\mathcal{U}, \ldots, a_n/\mathcal{U}) = (f^{A_i}(a_1(i), \ldots, a_n(i))) : i \in I/\mathcal{U}. \)
- \( \{a_1/\mathcal{U}, \ldots, a_n/\mathcal{U}\} \in R^{\Pi_{i \in I} A_i/\mathcal{U}} \) if and only if \( \{i \in I : \{a_1(i), \ldots, a_n(i)\} \in R^{A_i}\} \in \mathcal{U}. \)

Proof. \( \) of Theorem 1.1. Applying Lemma 2.9, let \( \mathcal{U} \) be an ultrafilter on \( |V_{G_1}|^{<\omega} \) such that for each \( x \in V_{G_2} \) we have \( \{s \in |V_{G_1}|^{<\omega} : x \in s\} \in \mathcal{U}. \) Let \( (W, Y) \) be the ultraproduct of the graphs \( G_1 \mid s \) for all \( s \in |V_{G_1}|^{<\omega} \) with respect to \( \mathcal{U}. \) Formally,

\[ (W, Y) = \Pi_{s \in |V_{G_1}|^{<\omega}} \{s, G_1 \mid s\}/\mathcal{U}. \]

\(^9\)Consequently, any filter on \( X \) can be extended to an ultrafilter on \( X. \)
By Lemma 2.1, for each \( s \in [V_{G_1}]^{<\omega} \) there is a \( F_s : s \to \{1, \ldots, k-1\} \) such that for any \( x_i, x_j \in s \), \( A_{x_i, F_s(x_i)} \cap A_{x_j, F_s(x_j)} \) is not independent. Then \( F : W \to \{1, \ldots, k-1\} \) with \( F([f]) = i \iff \{s \in [V_{G_1}]^{<\omega} : F_s(f(s)) = i\} \in U \) is a mapping in the ultraproduct such that the intersection of any two elements in \( \{A_{x, f(x)} : [f] \in W\} \) is not independent. Now, the graph \( G_1 \) can be embedded into the ultraproduct \((W, Y)\). The embedding is given by \( v \to [f_v] \) where \( f_v(s) = v \) for \( v \in s \) if \( s \in [V_{G_1}]^{<\omega} \). Thus we can obtain a desired mapping \( i : X \to \{1, \ldots, k-1\} \) such that intersection of any two elements in \( \{A_{x, i(x)} : x \in V_{G_1}\} \) is not independent.

Péter Komjáth communicated to us a proof of De Bruijn–Erdős theorem using a straightforward application of Lemma 2.9. We incorporate the arguments to give another proof of Theorem 1.1.

**Proof. (8th proof of Theorem 1.1).** As in the previous solution, applying Lemma 2.9, let \( U \) be an ultrafilter on \([V_{G_1}]^{<\omega}\) such that for each \( x \in V_{G_1} \) we have \( \{s \in [V_{G_1}]^{<\omega} : x \in s\} \in U \). By Lemma 2.1, for each \( s \in [V_{G_1}]^{<\omega} \) there is a \( f_s : s \to \{1, \ldots, k-1\} \) such that for any \( x_i, x_j \in s \), \( A_{x_i, f_s(x_i)} \cap A_{x_j, f_s(x_j)} \) is not independent. By Lemma 2.10, for each \( x \in V_{G_1} \) there is a unique \( i(x) \in \{1, \ldots, k-1\} \) such that

\[
A_x = \{s \in [V_{G_1}]^{<\omega} : x \in s, f_s(x) = i(x)\} \in U.
\]

**Claim 2.13.** \( i \) is the desired mapping on \( V_{G_1} \).

**Proof.** Assume two vertices \( x, y \in V_{G_1} \). Now, \( B = \{s : x, y \in s\} \in U \), and for each element \( s \in B \), \( A_{x, f_s(x)} \cap A_{y, f_s(y)} \) is not independent. Now if \( s \in B \cap A_x \cap A_y \), then \( i(x) = f_s(x) \) and \( i(y) = f_s(y) \). Thus we obtain that \( A_{x, i(x)} \cap A_{y, i(y)} \) is not independent.

**2.7. A proof applying compactness via prime semi lattices.** In this section we give an easy proof of Theorem 1.1 applying the methods used in [Cow83]. We first recall the following basic terminologies from [Cow83] if \((s, S)\) is a semilattice.

1. The semilattice \((s, S)\) is prime if whenever \( s \in S \) and \( s_1 \lor s_2 \lor \cdots s_n \in S \) then \( (s \land s_1) \lor \cdots (s \land s_n) \in S \) and \( s \land (s_1 \lor \cdots s_n) = (s \land s_1) \lor \cdots (s \land s_n) \).
2. \( I \subseteq S \) is an ideal of the semi lattice \((s, S)\) if \( s \in I \) and \( t \leq s \) implies \( t \in I \).
3. An ideal \( I \) is regular if for every \( s, t \in I \) and \( s \lor t \in S \) we have \( s \land t \in I \).
4. If \( I \) is an ideal of the semilattice \((s, \leq)\), a subset \( W \subseteq S \) avoids \( I \) if \( \forall W \not\subseteq I \) and \( W \) finitely avoids \( I \) if for every finite \( W_0 \subseteq W \) we have \( \forall W_0 \not\subseteq I \).
5. A subset \( C \subseteq S \) is \( \kappa \)-compact with respect to an ideal \( I \) if for every \( K \subseteq C \) with \( |K| < \kappa \), \( K \) finitely avoids \( I \) implies \( K \) avoids \( I \).
6. A subset \( C \subseteq S \) is compact with respect to an ideal \( I \) if for every \( K \subseteq C \), \( K \) finitely avoids \( I \) implies \( K \) avoids \( I \).

We recall a few more terminologies from [Cow83]. If \( Q \subseteq S \), we let \( \Pi(Q) = \{\land T : T \subseteq Q \} \) and \( \Sigma(Q) = \{\lor T : T \subseteq Q \} \) and \( \Sigma(Q) = \{\forall T : T \subseteq Q \} \). If \( T \) is only allowed to range over subsets of \( Q \) of cardinality \( \leq \kappa \) we write \( \Pi_\kappa(Q) \) and \( \Sigma_\kappa(Q) \) respectively and if \( T \) only ranges over finite subsets of \( Q \), we write \( \Pi_F(Q) \) and \( \Sigma_F(Q) \) respectively. We recall Theorem 2 and Theorem 3 from [Cow83].

**Theorem 2.14. (Theorem 2 of [Cow83]).** Let \((S, \leq)\) be a prime semilattice with ideal \( I \). If \( Q \subseteq S \) is \( \kappa \)-compact, then \( \Pi_\kappa(Q) \) is \( \kappa \)-compact. If \( Q \) is compact, then \( \Pi(Q) \) is compact.

**Theorem 2.15. (Theorem 3 of [Cow83]).** Let \((S, \leq)\) be a prime semilattice and \( I \) be a regular ideal of \((S, \leq)\). If \( Q \subseteq S \) is compact \( \Sigma_F(Q) \) is compact.

A partially ordered set \((W, <)\) is said to be directed if for any \( w_1, \ldots, w_n \in W \) there is a \( w \in W \) such that \( w_i \leq w \) for all \( 1 \leq i \leq n \). In [Cow83], applying Theorem 2.14 and Theorem 2.15,
Robert Cowen proved the following theorem which we will use to give another proof of Theorem 1.1.

**Theorem 2.16. (Theorem 5 of [Cow83]).** Let $(W, <)$ be a directed partially ordered set and for each $w \in W$, let $f_w$ be a finite nonempty set of functions with domain $D_w$. Suppose $w_1 \leq w_2$ and $f \in F_{w_2}$ implies $f \upharpoonright D_{w_1} \in F_{w_1}$. There is then a function $f$ such that $f \upharpoonright D_w \in F_w$ for all $w \in W$.

**Proof. (9th proof of Theorem 1.1).** If $H \in [V_{G_1}]^{\leq \omega}$, let $F_H$ be the set of mappings $f : H \to \{1, 2, ..., k - 1\}$ such that for any $x, y \in H$ such that $x \neq y$, $A_{x, f(x)} \cap A_{y, f(y)}$ is not independent. By Lemma 2.1, $F_H$ is non empty. For $H_1, H_2 \in [V_{G_1}]^{\leq \omega}$, we define the ordering $< \text{ as } H_1 < H_2$ if $H_1$ is a subset of $H_2$. Clearly, if $H_1 < H_2$ and $f \in F_{H_1}$ then $f \upharpoonright V_{H_1} \in F_{H_1}$. By Theorem 2.16 we obtain a function $i$ such that $i \upharpoonright V_H \in F_H$ for all $H \leq G_1$. Consequently, we obtain the desired mapping $i$. 

**2.8. A proof applying Nonstandard analysis.** A feature of the nonstandard analysis is that one can extend any mathematical object $A$ to an object $^*A$ which inherits all the elementary properties of the initial object. We follow Theorem 5.14 from [HLS85] directly to observe another argument.

**Proof. (10th proof of Theorem 1.1).** Let $S$ denote the set of all finite exhausting subsets of $V_{G_1}$. By Lemma 2.1 for each $F \in S$, there is a mapping $f_F : F \to \{1, 2, ..., k - 1\}$ such that for any $x, y \in F$ and $x \neq y$, we have $A_{x, f_F(x)} \cap A_{y, f_F(y)}$ is not independent. Thus in $V(V_{G_1} \cup N)$ the following formula $\phi$ is true.

$$\phi = \forall F \in S(\exists f_F : F \to \{1, 2, ..., k - 1\})(\forall x, y \in F)[x \neq y \to P_{x, f_F(x), y, f_F(y)}].$$

if $P_{x, f_F(x), y, f_F(y)}$ is a first order logic formula symbolising 'A_{x, f_F(x)} \cap A_{y, f_F(y)}$ is not independent'.

By the definition of enlargement, there exists a $B \in ^*S$ so that $V_{G_1} \subseteq B$ and thus we can embed $V_{G_1}$ into a hyperfinite element $B$ of $^*S$. By transfer of $\phi$ we can see that,

there is a mapping $f_B : B \to ^*\{1, 2, ..., k - 1\}$ so that if $x, y \in B$ and $x \neq y$, then $P_{x, f_B(x), y, f_B(y)}$ is true.

We restrict $f_B$ to $A$ to get a mapping $f_A : A \to \{1, 2, ..., k - 1\}$. We can see that if $x, y \in A$ and $x \neq y$ then $P_{x, f_A(x), y, f_A(y)}$ holds, i.e., $A_{x, f_A(x)} \cap A_{y, f_A(y)}$ is not independent.

**2.9. Proofs using Zorn’s lemma.** Péter Komjáth communicated to us three different proofs of De Bruijn–Erdős theorem using Zorn’s lemma or Zorn’s maximal theorem, a well-known equivalent of the axiom of choice. We incorporate two of those proofs to give two new proofs of Theorem 1.1.

**Proof. (11th proof of Theorem 1.1).** By well-ordering theorem, let $V_{G_1} = \{v_1, ..., v_n\}$ be an enumeration of $V_{G_1}$. Define a partially ordered set $(\mathbb{P}, \leq)$ as follows.

- $(p(v_i) : v_i \in V_{G_1}) \in \mathbb{P}$ if $\text{rng}(p(v_i)) \subseteq \{A_{i, c} : c \in \{1, 2, ..., k - 1\}\}$ such that for all $s \in [V_{G_1}]^{\leq \omega}$, there is a choice function $f_s$ satisfying the following properties.
  - For every $v_i \in s$, $f_s(v_i) \in p(v_i)$.
  - For every $x_1, x_2 \in s$, $A_{x_1, f_s(x_1)} \cap A_{x_2, f_s(x_2)}$ is not independent.
- $p(v_i) : v_i \in V_{G_1}) \leq (q(v_i) : v_i \in V_{G_1})$ if $\text{rng}(q(v_i)) \subseteq \text{dom}(p(v_i))$ for all $v_i \in V_{G_1}$.

**Claim 2.17.** If $L \subseteq \mathbb{P}$ is a chain, then there is an upper bound for $L$. 

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**10**We may recall the definition of $^*\text{transform of } \phi$ which is Definition 5.3 of chapter I of [HLS85]. Transfer principle which is Theorem 5.4 of chapter I of [HLS85], definition of enlargement which is Definition 5.6 of chapter II of [HLS85] and the definition of exhausting sets which is Definition 5.12 of chapter II of [HLS85].
Proof. For every \( v_i \in V_{G_1} \), set \( q(v_i) = \cap \{ p(v_i) : v_j \in V_{G_1} \in L \} \). It is enough to show that \( \langle q(v_i) : v_i \in V_{G_1} \rangle \in \mathbb{P} \), in the following steps, which guarantees that it is an upper bound for \( L \).

- Let \( s \in \{ V_{G_1} \}^{<\omega} \). For every \( v_i \in s \), there is \( p''(v_i) : v_i \in V_{G_1} \in L \) such that \( q(v_i) = p''(v_i) \). Since \( L \) is a chain, there is a \( p'(v_i) : v_i \in V_{G_1} \in L \) such that \( p''(v_i) : v_i \in V_{G_1} \leq p'(v_i) : v_i \in V_{G_1} \). As \( p'(v_i) : v_i \in V_{G_1} \in \mathbb{P} \), there is a choice function \( f \) with the following properties for all \( s' \in \{ V_{G_1} \}^{<\omega} \):
  - for every \( v_i \in s \), \( f_v(v_i) \in p'(v_i) \).
  - For every \( v_i, v_j \in s', A_{v_i, f_v(v_i)} \cap A_{v_j, f_v(v_j)} \) is not independent.

- We have seen that for all \( s \in \{ V_{G_1} \}^{<\omega} \) and for every \( v_i \in s \), \( p'(v_i) \subseteq q(v_i) \), and so for every \( v_i \in s \), \( f_v(v_i) \in p'(v_i) \subseteq q(v_i) \), and consequently, \( \langle q(v_i) : v_i \in V_{G_1} \rangle \in \mathbb{P} \).

By Zorn’s lemma there is a maximal element \( \langle p(v_i) : v_i \in V_{G_1} \rangle \in \mathbb{P} \). We prove that for every \( v_i \in V_{G_1} \), \( \| p(v_i) \| = 1 \). Consequently, \( f : V_{G_1} \rightarrow \{ 1, \ldots, k - 1 \} \) is a choice function where for every \( v_i \in V_{G_1} \), \( f(v_i) = p(v_i) \) with the property that for every \( v_i, v_j \in V_{G_1} \), \( A_{v_i, f(v_i)} \cap A_{v_j, f(v_j)} \) is not independent.

claim 2.18. For every \( v_i \in V_{G_1} \), \( \| p(v_i) \| = 1 \).

Proof. Otherwise we may assume that for some \( v_k \in V_{G_1} \), we have \( a_0 \neq a_1 \) and \( a_0, a_1 \in p(v_k) \). For \( j \in \{ 0, 1 \} \), we define \( p^j(v_i) : v_i \in V_{G_1} \) as follows.

\[
\text{for } v_i \in V_{G_1}, \quad p^j(v_i) = p(v_i) \text{ if } v_i \in A_{v_i, f(v_i)} \text{ and } p^j(v_i) = p(v_k) - \{ a_j \} \text{ if } v_i = v_k.
\]

Fix \( j \in \{ 0, 1 \} \). Since \( \langle p^j(v_i) : v_i \in V_{G_1} \rangle \in \mathbb{P} \) would imply \( \langle p^j(v_i) : v_i \in V_{G_1} \rangle > \langle p(v_i) : v_i \in V_{G_1} \rangle \) which contradicts the maximality of \( \langle p(v_i) : v_i \in V_{G_1} \rangle \), we have \( \langle p^j(v_i) : v_i \in V_{G_1} \rangle \not\in \mathbb{P} \). Following the definition of \( \mathbb{P} \), then there exists \( s_j \in \{ V_{G_1} \}^{<\omega} \) for which there is no choice function \( f \) with the following properties.

- For every \( v_i \in s_j \), \( f(v_i) \in p^j(v_i) \).
- For every \( v_i, v_j \in s_j \), \( A_{v_i, f(v_i)} \cap A_{v_j, f(v_j)} \) is not independent.

Let \( s = s_0 \cup s_1 \). Since \( \langle p(v_i) : v_i \in V_{G_1} \rangle \not\in \mathbb{P} \) and \( s \in \{ V_{G_1} \}^{<\omega} \), let \( f \) be a choice function of \( V_{G_1} \mid s \) with \( f(v_i) \in p(v_i) \) for \( v_i \in s \) and for each \( v_0, v_1 \in s \), \( A_{v_0, f(v_0)} \cap A_{v_1, f(v_1)} \) is not independent.

Let \( j \in \{ 0, 1 \} \) be such that \( f(v_k) \neq a_j \). Consequently, \( f \upharpoonright s_j \) is a choice function such that \( (f \upharpoonright s_j)(v_i) \in p^j(v_i) \) for all \( v_i \in s_j \), and for every \( v_r, v_s \in s_j \), \( A_{v_r, f|s_j(v_r)} \cap A_{v_s, f|s_j(v_s)} \) is not independent, which is a contradiction.

\( \square \)

Proof. (12th proof of Theorem 1.1). Define a partially ordered set \( (\mathbb{P}, \leq) \) as follows.

- \( f \in \mathbb{P} \) if \( f \) is a function from some \( \text{dom}(f) \subseteq V_{G_1} \) into \( \{ 1, \ldots, k - 1 \} \) such that for every finite \( s \subseteq \text{Dom}(f) \) and finite \( t \subseteq V_{G_1} - \text{dom}(f) \), there is a mapping \( g : s \cup t \rightarrow \{ 1, \ldots, k - 1 \} \) which extends \( f \upharpoonright s \) such that the following holds.
  - For every \( v_i, v_j \in s \cup t \), \( A_{v_i, g(v_i)} \cap A_{v_j, g(v_j)} \) is not independent.
- \( f \leq f' \) if \( f \subseteq f' \).

claim 2.19. If \( L \subseteq \mathbb{P} \) is a chain, then there is an upper bound for \( L \).

Proof. If \( L = \{ f_i : i \in I \} \), then we can see that \( \cup L = \cup \{ f_i : i \in I \} \) is the upper bound for \( L \). We prove that \( \cup L \in \mathbb{P} \). Let \( s \in \{ \text{dom}(f) \}^{<\omega} \) and \( t \in [V_{G_1} - \text{dom}(f)]^{<\omega} \). Then there is an \( i \in I \) such that \( s \subseteq \text{dom}(f_i) \). Consequently, there is a mapping \( g : s \cup t \rightarrow \{ 1, \ldots, k - 1 \} \) which extends \( f_i \upharpoonright s \) such that for every \( v_i, v_j \in s \cup t \), \( A_{v_i, g(v_i)} \cap A_{v_j, g(v_j)} \) is not independent.
By Zorn’s lemma, there is a maximal element $f \in \mathbb{P}$. We prove that $\text{dom}(f) = V_{G_1}$. Consequently, $f$ is the choice function such that for every $x_1, x_2 \in V_{G_1}$ such that $x_1 \neq x_2$, $A_{x_1, f(x_1)} \cap A_{x_2, f(x_2)}$ is not independent.

**Claim 2.20.** $\text{dom}(f) = V_{G_1}$.

**Proof.** For the sake of contradiction, we assume that $\text{dom}(f) \neq V_{G_1}$. Pick a $v \in V_{G_1} - \text{dom}(f)$. For each $i$, let $f_i$ be the function which extends $f$ such that the following holds.

$$\text{dom}(f_i) = \text{dom}(f) \cup \{v\}, \quad f_i(v) = i.$$  

Since $f$ is a maximal element of $\mathbb{P}$, $f_i \notin \mathbb{P}$ for each $i$. Thus for each $i$, there are $s_i \in |\text{dom}(f)|^{<\omega}$ and $t_i \in |V_{G_1} \setminus (\text{dom}(f) \cup \{v\})|^{<\omega}$ such that there is no mapping $g : s_i \cup \{v\} \cup t_i \to k$ which extends $f_i \cup s_i \cup \{v\}$ such that for every $v_i, v_j \in s_i \cup \{v\} \cup t_i$, $A_{v_i, g} \cap A_{v_j, g}$ is not independent. As $f \notin \mathbb{P}$, there is a mapping $g' : s \cup t \to \{1, 2, \ldots, k-1\}$ which extends $f \cup s$, such that for every $v_i, v_j \in s \cup t$, $A_{v_i, g'}(v_j)$ is not independent where $s = \cup \{s_i : 1 \leq i \leq k-1\}$ and $t = \cup \{t_i : 1 \leq i \leq k-1\}$. If $g'(v) = i$, then we obtain a contradiction because of the choice of $s_i$ and $t_i$. □

2.10. **A proof applying transfinite recursion.** A proof of De Bruijn–Erdős theorem was due to De Bruijn using well-ordering the vertex set and transfinite recursion. We incorporate the methods from De Bruijn’s proof of the De Bruijn–Erdős theorem to give another possible solution of Theorem 1.1 using transfinite recursion.

**Proof.** (13th proof of Theorem 1.1). Using well-ordering principle we first enumerate $V_{G_1}$ as $V_{G_1} = \{v_\alpha : \alpha < \phi\}$ for some ordinal $\phi$. Using transfinite recursion we will obtain a mapping $f_\gamma : \{v_\alpha : \alpha < \gamma\} \to \{1, 2, \ldots, k-1\}$ which extends $f_\beta$ for each $\beta < \gamma$ with the following properties.

1. For each $x, y \in \{v_\alpha : \alpha < \gamma\}$, $A_{x, f_\gamma(x)} \cap A_{y, f_\gamma(y)}$ is not independent.
2. For each $s \in \{v_\alpha : \alpha < \gamma\}$, $f_\gamma$ can be extended to a mapping $g : \{v_\alpha : \alpha < \gamma\} \cup s \to \{1, 2, \ldots, k-1\}$ such that for each $x, y \in \{v_\alpha : \alpha < \gamma\} \cup s$, $A_{x, g} \cap A_{y, g}$ is not independent.

**Successor cardinals.** We assume that we have $f_\gamma$. We define $f_{\gamma+1}$. We will define $f_\gamma$ for each $1 \leq i \leq k-1$ as follows.

- $f_\gamma^i$ is an extension of $f_\gamma$.
- $f_\gamma^i(v_{\gamma}) = i$.

**Claim 2.21.** There is an $1 \leq i \leq k-1$ for which $f_\gamma^i$ fits for the definition of $f_{\gamma+1}$.

**Proof.** Otherwise, for every $1 \leq i \leq k-1$, there is $s_i \in \{v_\alpha : \alpha < \gamma\}^{<\omega}$ such that $f_\gamma^i$ cannot be extended to any mapping from $\{v_\alpha : \alpha < \gamma\} \cup v_i \cup s_i$ to $\{1, 2, \ldots, k-1\}$ such that for each $x, y \in \{v_\alpha : \alpha < \gamma\} \cup v_i \cup s_i$, $A_{x, g} \cap A_{y, g}$ is not independent. Consequently, $f_\gamma$ cannot be extended to any mapping from $\{v_\alpha : \alpha < \gamma\} \cup v_i \cup \bigcup \{s_i : 1 \leq i \leq k-1\}$ to $\{1, 2, \ldots, k-1\}$. But then since $\{v_\alpha : \alpha < \gamma\} \cup v_\gamma \cup \bigcup \{s_i : 1 \leq i \leq k-1\}$ is finite, we arrive at a contradiction because of the property of $f_\gamma$. □

**Limit cardinals.** Let $\gamma$ be a limit cardinal. We prove that $\cup \{\alpha : \alpha < \gamma\}$ fits for the definition of $f_\gamma$ in the following steps.

**Verifying property (1).** Pick any $x, y \in \cup \{v_\alpha : \alpha < \gamma\}$ such that $x \neq y$. Since $\cup \{\alpha : \alpha < \gamma\}$ is an increasing union of $f_\alpha$’s, there exists a $\beta < \gamma$ such that $A_{x, f_\alpha(x)} \cap A_{y, f_\beta(y)}$ is not independent. Thus, $A_{x, \cup \{f_\alpha : \alpha < \gamma\}(x)} \cap A_{y, \cup \{f_\alpha : \alpha < \gamma\}(y)}$ is not independent since $\cup \{\alpha : \alpha < \gamma\}$ extends $f_\beta$. □
Verifying property $(2)$. Assume that there is some finite subset $s$ of $\{v_\beta : \gamma \leq \beta < \phi\}$ such that $f_\alpha$ cannot be extended to a mapping $g : \{v_\alpha : \alpha < \gamma\} \cup s \to k$ such that for each $x, y \in \{v_\alpha : \alpha < \gamma\} \cup s, A_{x,g(x)} \cap A_{y,g(y)}$ is not independent. Let $g_0, \ldots, g_{m-1}$ be the collection of mappings $g_\iota : s \to \{1, 2, \ldots, k-1\}$ such that for each $x, y \in s, A_{x,g_\iota(x)} \cap A_{y,g_\iota(y)}$ is not independent. Thus none of $f_\alpha \cup g_\iota$ has the property that for each $x, y \in \{v_\alpha : \alpha < \gamma\} \cup s, A_{x,g(x)} \cap A_{y,g(y)}$ is not independent. Thus there must be some $\alpha_i < \gamma$ and $\omega_i \in s$ such that $A_{v_{\alpha_i}, f_\alpha(v_{\alpha_i})} \cap A_{v_{\omega_i}, g_\iota(v_{\omega_i})}$ is independent. Since $\gamma$ is the limit, there is some $\alpha < \gamma$ such that $\alpha_i < \alpha$ for some $i < m$. Consequently, $f_\alpha$ cannot be extended to $s$ which is a contradiction. \hfill $\blacksquare$

**Remark.** We can see that $\mathfrak{P}_k$ (c.f. section 1) for any $k < \omega$, does not imply the Antichain principle (every p.o.set has a maximal antichain) as well as Urysohn’s lemma (Form 78 in [HoRu98]) in ZFA. In Mostowski’s linearly ordered model ($\mathcal{N}_3$ in [HoRu98]), the ultrafilter lemma is true and hence $\mathfrak{P}_k$ is true for any $k < \omega$ in $\mathcal{N}_3$. Since the Antichain principle implies ‘every linearly ordered set can be well-ordered’ (c.f. Theorem 9.1(a) of [Jec73]) and the set $A$ of atoms is linearly ordered but not well-ordered in $\mathcal{N}_3$, the Antichain principle fails in $\mathcal{N}_3$. Moreover, in [Bru83], Brunner proved that the Urysohn’s lemma fails in Mostowski’s linearly ordered model ($\mathcal{N}_3$ in [HoRu98]).

**A historical note and a few applications.** There are several results that appeal to the effectiveness of compactness arguments to bridge the gap between the finite and the infinite. Even if they follow via Gödel’s compactness theorem, a few of them also appeal to several other non-trivial combinatorial arguments from the 1950s. For instance, the proof of De Bruijn–Erdős theorem by Lajos Pósa and Gabriel Andrew Dirac appeal to some non-trivial usage of Zorn’s lemma. The purpose of this section is to approach such problems in several different ways, apart from giving a solution using Gödel’s compactness theorem, using the methods from [Cow77], [Hal66], [Lux62], [Cow83], [HL85], [EB51] and some more. We can observe that the methods adopted in this section can be applied to prove the following already well-known results, each of which appeals to some compactness arguments, in several different ways.

1. $\chi(E_G) \leq \mu$ if $G = (V_G, E_G)$ is a $K_\mu$-chordal graph [Kem15].
2. Dilworth’s decomposition theorem for infinite p.o.sets with finite width [Dil50], [Tac19], [Mir71].
3. Uniqueness of every field has an algebraic closure [Bana92].
4. For every finite field $\mathcal{F}$, for every non-trivial vector space $V$ over $\mathcal{F}$, there exists a non-zero linear functional $f : V \to F$. (c.f. Theorem 18 of [HT13]).
5. the infinite Hall’s theorem [Hal48], [Hal66].
6. a restricted version of Tukey-Teichmüller Theorem [Hod05].
7. Given an infinite graph $X$ and a finite graph $H$, if every finite subgraph of $X$ has a homomorphism into $H$, then so has $X$.

3. **Proving Theorem 1.2**

In section 2, we observe that all the proofs of Theorem 1.1 appeal to either some equivalent formulation of the axiom of choice, like Zorn’s lemma or some equivalent formulation of the ultrafilter lemma, like compactness theorem for propositional logic, Tyconoff theorem for finite discrete spaces, Cowen–Engeler lemma, etc. In particular, if we recall the statement of $\mathfrak{P}_k$ for some natural number $k < \omega$ from section 1, as follows,

'\(\chi(E_{G_1}) = k < \omega \land \chi(E_{G_2}) \geq \omega \implies \chi(E_{G_1 \times G_2}) = k\).'

then there are a few proofs from section 2 where the ultrafilter lemma implies $\mathfrak{P}_k$ in the context where the axiom of choice fails. It is thus natural to wonder whether the ultrafilter lemma follows from $\mathfrak{P}_k$. In this section, we use the technique of permutation models and observe a model of ZFA where not only the ultrafilter lemma but also the axiom of choice for 3 element sets fails but $\mathfrak{P}_3$ holds. We give a brief description of permutation models and prove Theorem 1.2.
3.1. Permutation models. Let $V$ be a model of $ZFA + AC$ where $A$ is a set of atoms or ur-elements. Each permutation $\pi : A \to A$ extends uniquely to a permutation of $\pi' : V \to V$ by $\epsilon$-induction. Let $G$ be a group of permutations of $A$ and $F$ be a normal filter of subgroups of $G$. We follow the steps of Theorem 3.2 of \[How84\].

For $x \in V$, we denote symmetric group with respect to $G$ by $sym^G x = \{g \in G \mid g(x) = x\}$. We say $x$ is $F$-symmetric if $sym^G (x) \in F$ and $x$ is hereditarily $F$-symmetric if $x$ is symmetric and each element of transitive closure of $x$ is symmetric. We define the permutation model $\mathcal{N}$ with respect to $G$ and $F$, to be the class of all hereditarily $F$-symmetric sets. It is well-known that $\mathcal{N}$ is a model of $ZFA$.

3.2. Proof of Theorem 1.2. We consider the permutation model $\mathcal{N}$ from section 3 of \[How84\] where the the axiom of choice for 3 element sets fails, following Theorem 3.1 of \[How84\]. We show that in $\mathcal{N}$, $\mathfrak{P}_3$ holds. For the sake of convenience, we recall the definition of the permutation model $\mathcal{N}$ from \[How84\].

(1) Defining the ground model $M$. Let $M$ be a model of $ZF U + AC$ where $U$ is a countable set of atoms. Assume $U$ be the disjoint union $\bigcup_{i \in \omega} U_i$ where for each $i \in \omega$, $U_i = \{a_i, b_i, c_i\}$.

(2) Defining the group $G$ of permutations and the filter $F$ of subgroups of $G$.

- Defining $\mathcal{G}$. For each $i \in \omega$, define $\eta_i : U_i \to U_i$ by $\eta_i(a_i) = b_i$, $\eta_i(b_i) = c_i$, and $\eta_i(c_i) = a_i$. We define the group of permutations $G$ as $G = \{\phi : \phi$ is a bijection from $U$ to $U$ such that for all $i \in \omega$, $\phi|_{U_i} = \eta_i$ or $\phi|_{U_i} = \eta_i^2$ or $\phi|_{U_i} = 1_{U_i}\}$ where $1_{U_i}$ is the identity permutation on $U_i$.

- Defining $F$. If $S \in [\omega]^{< \omega}$, we define the subgroup $G_S$ of $G$ by $G_S = \{\phi \in G : (\forall i \in S) \phi \text{ fixes } U_i \text{ pointwise} \}$ (c.f. figure 2). Let $F = \{G_S : S \in [\omega]^{< \omega}\}$ be the filter of subgroups of $G$.

(3) Defining the permutation model $\mathcal{N}$. We define the permutation model $\mathcal{N} \subset M$ with respect to $M$, $G$ and $F$, to be the class of all hereditarily $F$-symmetric sets.

\[CON(\neg \text{AC}^3 + \chi(E_{G_1}) = 3, \chi(E_{G_2}) \geq \omega \implies \chi(E_{G_1 \times G_2}) = 3)\] AND STRONGLY COMPACTNESS 15

\[\mathfrak{P}_3\] holds. For the sake of contradiction assume that $F : V_{G_1} \times V_{G_2} \to \{1, 2\}$ is a good coloring of $G_1 \times G_2$ in $\mathcal{N}$. For each color $c \in \{1, 2\}$ and each vertex $x \in V_{G_1}$ we let $A_{x,c} = \{y \in V_{G_2} : F(x, y) = c\}$. Define a relation $R$ on $\{1, 2\}$ as $(v_1, i)R(v_2, j)$ if and only if $v_1 \neq v_2$.
implies $A_{V_1,i} \cap A_{G_2,j}$ is not independent’ for $v_1, v_2 \in V_{G_1}$. Following Lemma 2.1, we have that for all finite $F \subset V_{G_1}$, there exists a mapping $i_F : F \rightarrow \{1, 2\}$ such that for any $x, x' \in F$, $A_{x,i_F(x)} \cap A_{x',i_F(x')}$ is not independent. Let $A = \{\{1, 2\} : i \in V_{G_1}\}$ be an indexed set of pairs $\{1, 2\}$. Since the axiom of choice is assumed in $V$, apply Gödel’s Compactness theorem or some equivalent formulation of ultrafilter lemma in $V$, to obtain a choice function $g$ for $A$ in $V$ such that for any $x, x' \in V_{G_1}$, $A_{x,g(x)} \cap A_{x',g(x')}$ is not independent using the methods of section 2. We modify $g$ to obtain a choice function $f$ for $A$ in $N$ such that for each $x \neq x' \in V_{G_1}$, $A_{x,f(x)} \cap A_{x',f(x')}$ is not independent, in the following steps.

(1) Constructing the choice function $f$. Let $S_0 \subseteq \omega$ be the support of $(A, R)$. For each $t \in A$, let $Orb_{S_0}(t) = \{\phi(t) : \phi \in G_{S_0}\}$ be the orbit of $t$ under group $G_{S_0}$ and $Orb_A = \{Orb_{S_0}(t) : t \in A\}$. Then $A = \cup\{Orb_{S_0}(t) : t \in A\}$ and $Orb_A$ is well-orderable since each element in this family is supported by $S_0$. We construct a choice function $f$ as follows.

(a) Defining $perm(S)$ for each $S \subseteq \omega$ such that $S \cap S_0 = \emptyset$. Define $\eta_i'$ such that $\eta_i' \mid U_i = \eta_i$ and $\eta_i' \mid U_j = 1_{U_j}$ for all $i \neq j$. Following the definition of $G$, $\eta_i' \in G$. Define $perm(S) = \{\Pi \in S(\eta_i')^S : \delta_i \in \{0, 1, 2\}, \text{for each } i \in S\}$ to be the collection of all choice functions $\psi : S \rightarrow \cup_{i \in S} \eta_i'^0, \eta_i'^1, \eta_i'^2$ for each $S \subseteq \omega$ such that $S \cap S_0 = \emptyset$. Consequently, $|perm(S)| = 3^3$.

(b) Defining $perm(t, 1)$ and $perm(t, 2)$. For each $t \in A$, let $sup(t) = S - S_0$ where $S$ is the support of $t$. We denote $perm(t)$ by $perm(sup(t))$ for each $t \in A$. Clearly, $|perm(t)| = 3^{sup(t)}$ by arguments from (a). For each $t = \{1, 2\} \in A$, we define $perm(t, 1) = \{\psi \in perm(t) : g(\psi(t)) = \psi(1)\}$ to be the collection of all choice functions $\psi : S - S_0 \rightarrow \cup_{i \in S - S_0} \eta_i'^0, \eta_i'^1, \eta_i'^2$ such that $g(\psi(t)) = \psi(1)$ and $perm(t, 2) = \{\psi \in perm(t) : g(\psi(t)) = \psi(2)\}$ to be the collection of all choice functions $\psi : S - S_0 \rightarrow \cup_{i \in S - S_0} \eta_i'^0, \eta_i'^1, \eta_i'^2$ such that $g(\psi(t)) = \psi(2)$.

(c) Defining $f$. Clearly, $|perm(t)| = |perm(t, 1)| + |perm(t, 2)|$. So, $|perm(t, 1)| \neq |perm(t, 2)|$ since $|perm(t)|$ is odd. For each $t \in A$, define $f(t) = 1$ if $|perm(t, 1)| > |perm(t, 2)|$ and 2 otherwise.

(2) Verifying that $f$ is in $N$. Let $\psi'$ be any element of $G_{S_0}$. Define $\psi(x) \in \psi'(x)$ if $x \in U_i$, for some $i \in sup(t)$ and $\psi(x) \in x$ otherwise. So $\psi \in perm(t)$ and since for all $x$ such that $x \in U_i$ for some $i \in supp(t)$, $\psi^{-1}(\psi(x)) = x$ we have $\psi^{-1}(\psi(t)) = t$. So, $\psi(t) = \psi'(t) = t$. Also, $\psi(1) = \psi'(1)$ and $\psi(2) = \psi'(2)$. Otherwise for the sake of contradiction, if $\psi(2) = \psi'(1)$ then $\psi^{-1}(\psi'(1) = 2$ but $\psi^{-1}(\psi'(\{1, 2\}) = \{1, 2\}$. Consequently, $(\psi^{-1}(\psi')^3 \neq 1_U$, which contradicts the definition of $G$ and the fact that $(\psi^{-1}(\psi')^3 \in G$.

Thus, $|perm(t, 1)| = |\{\eta \in perm(t) : g(\eta(t)) = \eta(1)\}|$

$= |\{\psi \in perm(t) : g(\eta(t)) = \eta(1)\}|$

$= |\{\eta \in perm(t) : g(\eta(t)) = \eta(1)\}|$

$= |\{\eta \in perm(t, 1) : \psi(1)\}|$

$= |\{\eta \in perm(t, 2) : \psi(1)\}|$

Similarly, $|perm(t, 2)| = |\{\psi \in perm(t, 2) : \psi(2)\}|$. By definition of $f$, $f(t) = 1$ if and only if $f(\psi'(t)) = \psi'(1)$. Thus $\psi'(\psi'(t)) = \psi'(1)$ if and only if $f(\psi'(t)) = \psi'(1)$. Thus $\psi'$ fixes $f$. Consequently, $G_{S_0}$ fixes $f$ and $f \in N$.

(3) Verifying that for each $x \neq x' \in V_{G_1}$, $A_{x,f(x)} \cap A_{x',f(x')}$ is not independent. For the sake of contradiction assume that there exists $x, x' \in V_{G_1}$ such that $x \neq x'$ holds and $(x, f(x)) R (x', f(x'))$ is false, i.e. $A_{x,f(x)} \cap A_{x',f(x')}$ is independent. Let $t_x$ and $t_{x'}$ be 2 sets in $A$. Following Lemma 6 from [Hows], for every $\eta \in sup(t_x) \cap sup(t_{x'})$ either $(\forall \psi \in perm(sup(t_x) - sup(t_{x'}))) \psi(\eta) \in perm(t_x, t_{x'} - f(x))$ or $(\forall \psi \in perm(sup(t_{x'}) - sup(t_x))) \psi(\eta) \in perm(t_{x'}, t_x - f(x'))$. Therefore either,

$2. |\{\eta \in perm(sup(t_x) \cap sup(t_{x'})) : (\forall \psi \in perm(sup(t_x) - sup(t_{x'}))) \psi(\eta) \in perm(t_x, t_{x'} - f(x))\}| \geq |perm(sup(t_x) \cap sup(t_{x'}))|$. So,

$|perm(t_x, t_{x'} - f(x))| > |perm(t_{x'}, f(x))|$ which contradicts the choice of $f$ at $x$ and the definition of $f$. 
Remark 1. Following the methods used in the proof of Lemma 3.1, we can see that in the above permutation model \( N \) the De Bruijn–Erdős theorem holds for 2 colorings. Consequently, \( AC^2 \) and hence \( AC^4 \) also holds in \( N \).

Remark 2. Following the proof of Theorem 20 in [HT13], we may say that in \( N \), the axiom of choice for \( n \) element sets fails for any \( n \geq 3, \neq 4 \). For \( n \geq 3, \neq 4 \), \( A_n = \{X : X \text{ is an } n \text{ element subset of } \cup_{i \in \omega} U_i \text{ for some } i \in \omega \} \) belongs to \( N \), but admits no choice function in \( N \). Since \( AC^3 \) fails in \( N \), the ultrafilter lemma fails too and so the De Bruijn–Erdős theorem for \( n \) colorings fails for each \( n \geq 3 \) in \( N \).

Remark 3. Since \( \{U_i : i \in \omega\} \) has no partial choice function in \( N \), \( C^\omega \) fails in \( N \) (c.f. proof of Theorem 3 (3) of [HT18]). Following Theorem 1 (7) of [HT18], \( RC_3 \) fails too. Consequently, \( \Psi_k \) do not imply \( RC_3 \) as well as \( C^\omega \) in ZFA.

Remark 4. We may observe that in \( N \), \( A = \{U_i : i \in \omega\} \) has no partial Kinna–Wagner selection function\(^{11}\) also. For the sake of contradiction, assume the existence of an infinite subfamily \( A' \) of \( A \), with a Kinna–Wagner selection function \( f \). Following the terminologies in [How84] and Theorem 3.1 of [How84], suppose \( A' \) and \( f \) are supported by some \( K \in [\omega]^{<\omega} \). We can see that there exists a \( j \in \omega \), such that \( U_j \in A' \) but \( j \in \omega \) and \( K \) is finite and \( K \). Let \( B_j = f(U_j) \subseteq U_j \). Consider the permutation \( \pi_j \) of \( A \) where \( \pi_j \) is the 3-cycle \( a_j \rightarrow b_j \rightarrow c_j \rightarrow a_j \) and \( \pi_j \upharpoonright A' \cup U_j \) is the identity mapping \( 1_{A' \cup U_j} \) on \( A' \cup U_j \). Clearly, \( \pi_j \in \fix_G(E) \), \( \pi_j(U_j) = U_j \) and \( \pi_j(B_j) \neq B_j \). Consequently, we have the following:

\[
(U_j, B_j) \in f \implies (U_j, B_j) \in f(f) \implies (U_j, \pi_j(B_j)) \in f.
\]

Since \( \pi_j(B_j) \neq B_j \), \( (U_j, B_j) \in f \), and \( (U_j, \pi_j(B_j)) \in f \), we obtain a contradiction to the fact that \( f \) is a function.

Question 3.2. If \( k > 3 \), does the ultrafilter lemma follows from \( \Psi_k \)? Otherwise is there any symmetric extension where \( \Psi_k \) holds for \( k > 3 \) but the ultrafilter lemma fails?

4. Graphs on well-orderable set of vertices and more consistency results

We give a new proof of the fact that \( 2^X \) is compact in ZF for a well-orderable set \( X \) and use this fact to prove Observation 1.3 and Observation 1.4 in ZF.

Lemma 4.1. (ZF). If \( X \) is well-orderable, then \( 2^X \) is compact.

Proof. We first show that \([0,1]^X \) is compact in ZF for a well-orderable set \( X = \{ \alpha : \alpha < \lambda \} \). Since compact spaces and filter-compact spaces are same in ZF, it is enough to show that every filter \( F \) in \([0,1]^X \) has a cluster point. We denote the \( \alpha \)-th projection by \( \pi_\alpha : [0,1]^X \rightarrow [0,1] \) and

\[\text{CON}(\neg AC^3 + \chi(E_{G_1}) = 3, \chi(E_{G_2}) \geq \omega \implies \chi(E_{G_1} \times G_2) = 3') \quad \text{AND STRONGLY COMPACTNESS} \]

\[17 \quad \text{or,} \]

\[2, \left| \{ \eta \in \operatorname{perm}(\sup(t_x) \cap \sup(t_x')) : (\forall \psi \in \operatorname{perm}(\sup(t_x) - \sup(t_x'))) \psi(\eta) \in \operatorname{perm}(t_x, t_x' - f(x')) \} \right| \geq |\operatorname{perm}(\sup(t_x) \cap \sup(t_x'))| \].

So, \( |\operatorname{perm}(t_x, t_x' - f(x'))| > |\operatorname{perm}(t_x, f(x'))| \) which contradicts the definition of \( f \) and the choice of \( f \) at \( x \).

Thus there exist a choice function \( f \in N \) for \( A \) such that for any \( x, x' \in G_1 \), \( A_{x, f(x)} \cap A_{x', f(x')} \) is not independent. Since \( x \rightarrow f(x) \) is not a good coloring in \( G_1 \) as \( \chi(E_{G_1}) = 3 \), there are \( x, x' \in G_1 \) with \( f(x) = f(x') = j \) and \( \{x, x'\} \in E_{G_1} \). Consequently, \( A' = A_{x, f(x)} \cap A_{x', f(x')} \) is not independent. Pick \( y, y' \in A' \) joined by an edge in \( G_2 \). Then \( (x, y) \) and \( (x', y') \) are joined in \( G_1 \times G_2 \) and get the same color \( j \) which is a contradiction to the fact that \( F \) is a good coloring of \( G_1 \times G_2 \). On the other hand if \( f : V_{G_1} \rightarrow \{1, 2, 3\} \) is a good 3-coloring of \( G_1 \), then \( F(x, y) = f(x) \) is a good 3-coloring of \( G_1 \times G_2 \). Consequently, \( \Psi_3 \) holds in \( N \). \( \Box \)

The terminology of \( RC_3 \) and \( C^\omega \) is from [HT18].

\[\text{i.e., there is no infinite subfamily } A' \text{ of } A \text{ with a function } f \text{ such that } \text{dom}(f) = A' \text{ and for all } U \in A', \emptyset \neq f(U) \subseteq U.\]
the neighbourhood filter of a point \( x \in [0,1] \) by \( \mathcal{U}(x) \). By recursion on \( \alpha \), we define a family of filters \( \{ \mathcal{F}_\alpha : \alpha < \lambda \} \) on \([0,1]^X\) and a sequence \( \{ x_\alpha : \alpha < \lambda \} \) as follows.

- Let \( \mathcal{F}_0 = \mathcal{F} \). Consider the push-forward filter \( \pi_0^{-1}(\mathcal{F}_0) = \{ G \subseteq [0,1] : \pi_0^{-1}(G) \in \mathcal{F}_0 \} \) in \([0,1] \). The set of cluster points of \( \pi_0^{-1}(\mathcal{F}_0) \) is a non-empty closed subset of \([0,1] \) and hence contains a smallest member. Let \( x_0 \) be the smallest cluster point of \( \pi_0^{-1}(\mathcal{F}_0) \).

- Let, \( \mathcal{F}_{\alpha+1} \) be the filter on \([0,1]^X\) generated by the set \( \mathcal{F}_\alpha \cup \{ \pi_\alpha^{-1}(U) : U \in \mathcal{U}(x_\alpha) \} \). Similarly, consider the push-forward filter \( \pi_\alpha^{-1}(\mathcal{F}_{\alpha+1}) = \{ G \subseteq [0,1] : \pi_\alpha^{-1}(G) \in \mathcal{F}_{\alpha+1} \} \) in \([0,1] \). The set of cluster points of \( \pi_\alpha^{-1}(\mathcal{F}_{\alpha+1}) \) is a non-empty closed subset of \([0,1] \) and hence contains a smallest member. Let \( x_{\alpha+1} \) be the smallest cluster point of \( \pi_\alpha^{-1}(\mathcal{F}_{\alpha+1}) \).

- If \( \alpha \) is a limit cardinal, then we define \( \mathcal{F}_\alpha \) to be the filter on \([0,1]^X\) generated by \( \cup \{ \mathcal{F}_\beta : \beta < \alpha \} \) and \( x_\alpha \) to be the smallest cluster point of \( \pi_\alpha^{-1}(\mathcal{F}_\alpha) \).\(^\text{13}\)

Consequently, \( x = \{ x_\alpha : \alpha < \lambda \} \) is the cluster point of a filter \( \mathcal{G} \) generated by \( \cup \{ \mathcal{F}_\alpha : \alpha < \lambda \} \) and hence for \( \mathcal{F} \) in \([0,1] X \). Now \( 2^X \) is a closed subspace of the compact space \([0,1]^X \). Let \( \mathcal{B} \) be an open cover of \( 2^X \). Let \( \mathcal{A} = \{ A : A \text{ is open in } [0,1]^X \text{ and } (2^X \cap A) \in \mathcal{B} \} \). Since, \( \mathcal{A} \cup \{ \{0,1\}^X \setminus 2^X \} \) is an open cover of \([0,1]^X \), it contains a finite cover \( \mathcal{F} \) as \([0,1]^X \) is compact. Clearly, \( \mathcal{G} = \{ 2^X \cap F : F \in \mathcal{F} \cap \mathcal{A} \} \) is a finite cover of \( 2^X \). So, \( 2^X \) is compact. \( \square \)

Remark. We can also prove Lemma 4.1 applying Theorem 1 of [Loeb65].

4.1. Proving Observation 1.3. We follow the methods used by Tachtsis from Theorem 3.1(i) [Taci19] or the methods used in Theorem 18 of [HT13]. We work with propositional language \( \mathcal{L} \) with the following sentence symbols.

\[
\mathcal{A}'_{x_i,j} \text{ where } j \in \{1, 2, ..., k-1\} \text{ and } x_i \in V_{G_1}.
\]

Let \( \mathcal{F} \) be the set of all formulae of \( \mathcal{L} \) and \( \Sigma \subset \mathcal{F} \) be the collection of the following well-founded formulas.

1. \( \mathcal{A}'_{x_i,m} \land \mathcal{A}'_{x_j,j} \) if \( \mathcal{A}'_{x_i,m} \cap \mathcal{A}'_{x_j,j} \) is not an independent set where \( l, m \in \{1, 2, ..., k-1\} \) and \( x_i, x_j \in V_{G_1} \), such that \( x_i \neq x_j \).
2. \( \neg(\mathcal{A}'_{x_i,j} \land \mathcal{A}'_{x_j,l}) \) for any \( l, j \in \{1, 2, ..., k-1\} \) such that \( l \neq j \) and \( x_i \in V_{G_1} \).
3. \( \mathcal{A}'_{x_i,1} \lor \mathcal{A}'_{x_i,2} \lor ... \lor \mathcal{A}'_{x_i,k-1} \) for each \( x_i \in V_{G_1} \).

We enumerate \( \mathcal{Var} = \{ \mathcal{A}'_{x_i,i} : x \in V_{G_1}, i \in \{1, 2, ..., k-1\} \} \) since \( V_{G_1} \times \{1, 2, ..., k-1\} \) is well-orderable. For every \( W \in [V_{G_1}]^{\omega \setminus \{0\}} \), we set \( \Sigma_W \) be the subset of \( \mathcal{F} \), which is defined as \( \Sigma \) except that the subscripts in the formulae are from the set \( W \cup \{1, 2, ..., k-1\} \). Endow the discrete 2-element space \([0,1]\) with the discrete topology and consider the product space \( 2^\mathcal{Var} \) with the product topology. Let \( \mathcal{F}_W = \{ f \in 2^\mathcal{Var} : \forall \phi \in \Sigma_W(f'(\phi) = 1) \} \) where for \( f \in 2^\mathcal{Var} \), the element \( f' \) of \( 2^\mathcal{F} \) denotes the valuation mapping determined by \( f \). By Lemma 2.1 the family \( \mathcal{X} = \{ F_W : W \in [V_{G_1}]^{\omega \setminus \{0\}} \} \) has the finite intersection property. Also for each \( W \in [V_{G_1}]^{\omega \setminus \{0\}} \), \( F_W \) is closed in the topological space \( 2^\mathcal{Var} \). By Lemma 4.1 since \( 2^\mathcal{Var} \) is compact in \( ZF, \cap \mathcal{X} \) is non-empty. Pick an \( f \in \cap \mathcal{X} \) and let \( f' \in 2^\mathcal{F} \) be the unique valuation mapping that extends \( f \). Clearly, \( f'(\phi) = 1 \) for all \( \phi \in \Sigma \). Thus, by claim 2.3 we can obtain an \( i : V_{G_1} \to \{1, 2, ..., k-1\} \) such that the intersection of any two elements in \( \{ \mathcal{A}'_{x_i(i)} : x \in V_{G_1} \} \) is not independent.

4.2. Consequence 1 of Observation 1.3. We prove that \( \Psi_k^f \) does not imply ‘There are no amorphous sets’ in \( ZFA \). We recall the model constructed in the proof of Theorem 2.1 of [Tac16] as follows.

1. Defining the ground model \( M \). We start with a ground model \( M \) of \( ZFA + AC \) where \( A \) is a countably infinite set of atoms written as a disjoint union \( \cup \{ A_i : i \in \omega \} \) where for each \( i \in \omega, A_i = \{ a_i, b_i \} \).
2. Defining the group \( \mathcal{G} \) of permutations and the filter \( \mathcal{F} \) of subgroups of \( \mathcal{G} \).

\(^{13}\)The second author helped to observe this.
• Defining $\mathcal{G}$. $\mathcal{G}$ be the group of all permutations $\phi$ of $A$ such that $\phi$ moves only finitely many atoms

• Defining $\mathcal{F}$. $\mathcal{F}$ be the filter of subgroups of $\mathcal{G}$ generated by $\{\text{fix}_E(E) : E \in [A]^{<\omega}\}$.

(3) Defining the permutation model. Consider the permutation model $\mathcal{N}$ determined by $M$, $\mathcal{G}$ and $\mathcal{F}$.

In $\mathcal{N}$, the Ramsey’s theorem (RT) fails since $A = \{A_i : i \in \omega\}$ has no (partial) choice function (c.f. claim 1 of Theorem 2.1 and Theorem 1.2 in [Tac16]). Also both $A$ and the set $A$ of atoms are amorphous.

Lemma 4.2. In $\mathcal{N}$, $AC(LO,LO)$ holds.

Proof. Following Lemma 2 of [Tac16], every linearly orderable set is well-orderable (LW) holds in $\mathcal{N}$. Following Lemma 3 of [Tac16], the union of a well-orderable family of well-orderable sets is well-orderable ($UT(WO,WO,WO)$) holds in $\mathcal{N}$. Thus $AC(LO,LO)$ holds in $\mathcal{N}$.

Lemma 4.3. $\mathcal{N}_k$ holds in $\mathcal{N}_1$ for all $k \in \omega$.

Proof. Let $E \subset A$ be a finite support of the comparability graph $G_1$. Then $V_{G_1} = P = \bigcup \{\text{Orb}_B(E) : p \in V_{G_1}\}$ where $\text{Orb}_B(E) = \{\phi(p) : \phi \in \text{fix}_E(E)\}$ for $p \in P$. Since, every permutation $\psi \in \mathcal{G}$ moves only finitely many atoms, following the arguments in Claim 3 of [Tac16] or Claim 3.5 of [Tac19], $\text{Orb}_B(E)$ is an antichain in $P$ and so an independent set in the comparability graph $G_1$. By assumption, $\text{Orb}_B(E)$ is well-orderable and $\mathcal{O} = \{\text{Orb}_B(E) : p \in V_{G_1}\}$ is well-orderable since every element of this family $\mathcal{O}$ is supported by $E$. Following Lemma 3 of [Tac16], well-orderable union of well-orderable sets is well-orderable ($UT(WO,WO,WO)$) holds in $\mathcal{N}$, and so $V_{G_1}$ is well-orderable in $\mathcal{N}$. Applying Observation 1.3, $\mathcal{N}_k$ holds in $\mathcal{N}$ for all $k \in \omega$.

An alternative argument. We can also observe consequence (1) in the basic Fraenkel model. We recall the basic Fraenkel model (in [HoRu98]) as follows.

(1) Defining the ground model $M$. We start with a ground model $M$ of ZFA + AC, where $A$ is a countably infinite set of atoms.

(2) Defining the group $\mathcal{G}$ of permutations and the filter $\mathcal{F}$ of subgroups of $\mathcal{G}$.

• Defining $\mathcal{G}$. $\mathcal{G}$ be the group of all permutations of $A$.

• Defining $\mathcal{F}$. $\mathcal{F}$ be the filter of subgroups of $\mathcal{G}$ generated by $\{\text{fix}_E(E) : E \in [A]^{<\omega}\}$.

(3) Defining the permutation model. Consider the permutation model $\mathcal{N}_1$ determined by $M$, $\mathcal{G}$ and $\mathcal{F}$.

In $\mathcal{N}_1$, the Ramsey’s theorem (RT) holds (c.f. Theorem 2 of [Blu77]) and the set $A$ of atoms is amorphous [HoRu98].

Lemma 4.4. $\mathcal{N}_k$ holds in $\mathcal{N}_1$ for all $k \in \omega$.

Proof. Let $E \subset A$ be a finite support of the comparability graph $G_1$. Then $V_{G_1} = P = \bigcup \{\text{Orb}_B(E) : p \in V_{G_1}\}$. Following the proof of Lemma 9.2(ii) and Lemma 9.3 in [Jec73], $\text{Orb}_B(E)$ is an antichain in $P$ for all $p \in P$ and so an independent set in the comparability graph $G_1$. By assumption, $\text{Orb}_B(E)$ is well-orderable and $\mathcal{O} = \{\text{Orb}_B(E) : p \in V_{G_1}\}$ is well-orderable in $\mathcal{N}$ since every element of this family $\mathcal{O}$ is supported by $E$. Since the union of a well-orderable

\footnote{We follow the terminology from Definition 1 of [HT18] and denote $AC(LO,LO)$ by every linearly orderable family of linearly orderable sets has a choice function.}

\footnote{We observe another argument following the proof in claim 4.10 of [Tac19a]. Let $(X, \leq)$ be a linearly ordered set in $\mathcal{N}$ supported by $E$. We show $\text{fix}_X(E) \subseteq \text{fix}_X X$ which implies that $X$ is well-orderable in $\mathcal{N}$. For the sake of contrary assume $\text{fix}_X(E) \not\subseteq \text{fix}_X X$. So there is an element $y \in X$ which is not supported by $E$ and there is a $\phi \in \text{fix}_X E$ such that $\phi(y) \neq y$. Since $\phi(y) \neq y$ and $\leq$ is a linear order on $X$, we obtain either $\phi(y) < y$ or $y < \phi(y)$. Let $\phi(y) < y$. Since every permutation $\phi \in \mathcal{G}$ moves only finitely many atoms there exists some $k < \omega$ such that $\phi^k = 1_A$. Thus, $p = \phi^k(p) < \phi^{k-1}(p) < \ldots < \phi(p) < p$. By transitivity of $<, p < p$, which is a contradiction. Similarly we can arrive at a contradiction if we assume $y < \phi(y)$.}
family of well-orderable sets is well-orderable (UT(\(WO,WO,WO\))) holds in \(N_1\) (c.f. [HorRu98]), \(V_{G_1}\) is well-orderable in \(N_1\). Applying Observation 1.3, \(\Psi^\omega_k\) holds in \(N_1\) for all \(k \in \omega\). \(\square\)

4.3. Consequence 2 of Observation 1.3. We prove that \(\Psi^\omega_k\) doesn't imply \(AC_{\omega}^{\mathbb{P}}\) in ZFA. We recall the Levy’s permutation model \((N_6, [\text{HorRu98}])\).

(1) Defining the ground model \(M\). We start with a ground model \(M\) of \(ZFA + AC\) where \(A\) is a countably infinite set of atoms written as a disjoint union \(\bigcup \{P_n : n \in \omega\}\), where \(P_n = \{a^n_0 ... a^n_{\omega} \} \) such that \(a^n_0\) is the \(n^{th}\)-prime number.

(2) Defining the group \(G\) of permutations and the filter \(F\) of subgroups of \(G\).

- Defining \(G\). \(G\) be the group generated by the following permutations \(\pi_n\) of \(A\).

\[ \pi_n: a^n_0 \mapsto a^n_2 \mapsto ... a^n_{\omega} \mapsto a^n_0 \text{ and } \pi_n(x) = x \text{ for all } x \in A \setminus P_n. \]

- Defining \(F\). \(F\) be the filter of subgroups of \(G\) generated by \(\{\text{fix}_G(E) : E \in [A]^{< \omega}\}\).

(3) Defining the permutation model. Consider the permutation model \(N_6\) determined by \(M, G, F\).

It is well-known that in \(N_6\), \(AC_{\omega}^{\mathbb{P}}\) fails since \(\{P_i : i \in \omega\}\) has no (partial) choice function. Consequently, following Lemma 4.4 of [Tuc19a], ‘every infinite p.o.set has either an infinite chain or an infinite antichain (CA\(C\))’ is false in \(N_6\). Following Remark 4 in section 3, \(\{P_i : i \in \omega\}\) has no (partial) Kinna–Wegner selection function also.

**Lemma 4.5.** \(\Psi^\omega_k\) holds in \(N_6\) for all \(k \in \omega\).

**Proof.** Let \(E \subset A\) be a finite support of the comparability graph \(G_1\). Then \(V_{G_1} = \bigcup \{\text{Orb}_E(p) : p \in V_{G_1}\}\). Following Claim 3.5 of [Tuc19a], \(\text{Orb}_E(p)\) is an antichain in \(P\) and so an independent set in the comparability graph \(G_1\). By assumption, \(\text{Orb}_E(p)\) has cardinality at most \(n\) and \(\mathcal{O} = \{\text{Orb}_E(p) : p \in V_{G_1}\}\) is well-orderable since every element of this family \(\mathcal{O}\) is supported by \(E\). Since \(AC_n\) holds in \(N_6\) for all integers \(n \geq 2\) (c.f. [Jec73], Theorem 7.11), \(V_{G_1}\) is well-orderable in \(N_6\). Applying Observation 1.3, \(\Psi^\omega_k\) holds in \(N_6\) for all \(k \in \omega\). \(\square\)

4.4. De Bruijn–Erdős theorem for graphs on well-orderable set of vertices. We prove Observation 1.4. We work with propositional language \(L\) with the following sentence symbols.

\[ A'_{x,j} \text{ where } j \in \{1, 2, ..., k\} \text{ and } x_i \in V_{G}. \]

Let \(\mathcal{F}\) be the set of all formulae of \(L\) and \(\Sigma \subset \mathcal{F}\) be the collection of the following well-founded formulae.

1. \(\neg (A'_{x,m} \land A'_{x,m})\) if \(\{x_i, x_j\} \in E_{G}\) and \(m \in \{1, 2, ..., k\}\).
2. \(\neg (A'_{x,i} \land A'_{x,i})\) for any \(l, j \in \{1, 2, ..., k\}\) such that \(l \neq j\) and each \(x_i \in V_{G}\).
3. \(A'_{x,i} \lor A'_{x,2} \lor ... \lor A'_{x,k}\) for each \(x_i \in V_{G}\).

Following the methods used in the proof of Observation 1.3, we may obtain a \(f' \in 2^\mathcal{F}\) such that \(f'(\phi) = 1\) for all \(\phi \in \Sigma\). Consequently, we can obtain a \(n\)-coloring of \(G\).

**Remark.** We can also apply the methods of this section to prove the following.

- the infinite Hall’s theorem for a well-orderable system of sets in ZF.
- for every finite field \(\mathcal{F}\), for every nontrivial well-orderable vector space \(V\) over \(\mathcal{F}\), there exists a non-zero linear functional \(f : V \to F\) in ZF. (Modifying the proof of Theorem 18 from [HT13].)

4.5. Babai’s result holds under ZF+DC only. We prove Corollary 1.5 of Observation 1.4. Let \(X\) be an infinitely chromatic graph on some well-ordered set \(V\). Without loss of generality, we can assume that for every \(a \in V\), the graph on vertices \(V_a = \{x \in V : x < a\}\) is finitely chromatic. Consequently, \(V^a = \{x \in V : a < x\}\) is infinitely chromatic. Let \(a_0\) be the initial vertex in the well-ordering. We obtain an increasing sequence \(a_0 < a_1, ...\) of vertices from \(V\) and the finite subgraphs \(F_n\) between \(a_n\) and \(a_{n+1}\) so that \(X\) restricted to \(F_n\) is at least...
n-chromatic. Suppose we already have an increasing sequence of elements \(a_0 < a_1 \ldots < a_{n-1}\) and a sequence of graphs \(\{F_i\}_{i<n}\), we choose \(a_n\) and \(F_n\) so that \(F_n\) is the finite subgraph with elements between \(a_n\) and \(a_{n+1}\) and \(X\) restricted to \(F_n\) is at least \(n\)-chromatic.

- Since \(V\) is well-ordered, by Observation 1.4 we can find a finite subgraph \(H\) of \(X\), so that \(H\) dont have a \((n-1)\)-coloring in ZF, which can determine \(a_n\) and \(F_n\).

Applying \(DC_\omega\), we can obtain an increasing sequence \(a_0 < a_1 \ldots\) of vertices from \(V\) and the finite subgraphs \(F_n\) between \(a_n\) and \(a_{n+1}\) so that \(X\) restricted to \(F_n\) is at least \(n\)-chromatic. The union of \(\{F_i\}_{i<\omega}\) gives an \(\omega\)-type subset which is infinitely chromatic.

5. Dilworth’s theorem don’t imply that every infinite family of \(n\) (greater than or equal to 2) element sets has a partial choice function

5.1. Dilworth’s decomposition theorem. Dilworth’s decomposition theorem states that if \(\mathbb{P}\) is an arbitrary p.o.set and \(k\) a natural number, then if \(\mathbb{P}\) has no antichains of size \(k+1\), while at least one \(k\)-element subset of \(\mathbb{P}\) is an antichain, then \(\mathbb{P}\) can be partitioned into \(k\)-chains. We refer to [Tac19] for details concerning the Dilworth’s decomposition theorem in ZF. We denote the Dilworth’s decomposition theorem for infinite p.o.sets of finite width by DT and recall Theorem 3.1(i) from [Tac19].

Lemma 5.1. (Theorem 3.1(i), [Tac19]). Dilworth’s decomposition theorem for well-orderable infinite p.o.sets of finite width is provable in ZF.

5.2. A weaker form of Löś’s lemma. We recall a weaker form of Löś’s lemma (which is Form 253 of [HoRu98] and denote it by LT. Paul E. Howard proved that in ZF, ‘LT + BPI is equivalent to AC’ (c.f. Theorem 2.2 of [Tac19a]). Modifying the proof of Howard, Tachtsis proved the following.

Lemma 5.2. (Theorem 4.1(i), [Tac19a]). LT implies ‘Every amorphous set of non-empty sets has a choice function’ in ZFA.

5.3. Proving Theorem 1.6. We fix an arbitrary integer \(n \geq 2\) and recall the model constructed in the proof of Theorem 7 of [HT18] as follows.

1. Defining the ground model \(M\). We start with a ground model \(M\) of ZFA + AC where \(A\) is a countably infinite set of atoms written as a disjoint union \(\bigcup \{A_i : i < \omega\}\) where for each \(i \in \omega\), \(A_i = \{a_{i_1}, a_{i_2}, \ldots, a_{i_n}\}\).

2. Defining the group \(G\) of permutations and the filter \(\mathcal{F}\) of subgroups of \(G\).

- Defining \(G\). \(G\) is defined in [HT18] in a way so that if \(\eta \in G\), then \(\eta\) only moves finitely many atoms and for all \(i \in \omega\), \(\eta(A_i) = A_k\) for some \(k \in \omega\). We recall the details from [HT18] as follows. For all \(i \in \omega\), let \(\tau_i\) be the \(i\)-cycle \(a_{i_1} \rightarrow a_{i_2} \rightarrow \ldots \rightarrow a_{i_n} \rightarrow a_{i_1}\). For every permutation \(\psi\) of \(\omega\), which moves only finitely many natural numbers, let \(\phi_{\psi}\) be the permutation of \(A\) defined by \(\phi_{\psi}(a_i) = a_{\psi(i)}\) for all \(i \in \omega\) and \(j = 1, 2, \ldots, n\). Let \(\eta \in G\) if and only if \(\eta = \rho \phi_{\psi}\), where \(\psi\) is a permutation of \(\omega\) which moves only finitely many natural numbers and \(\rho\) is a permutation of \(A\) for which there is a finite \(F \subseteq \omega\) such that for every \(k \in F\), \(\rho^{-1} A_k = \tau_j^\eta\) for some \(j < n\), and \(\rho\) fixes \(A_m\) pointwise for every \(m \in \omega\setminus F\).

- Defining \(\mathcal{F}\). \(\mathcal{F}\) be the filter of subgroups of \(G\) generated by \(\{\text{fix} \mathcal{E} \mathcal{E} (E) : E \in [A]^{<\omega}\}\).

3. Defining the permutation model. Consider the permutation model \(\mathcal{N}\) determined by \(M, G, F\) and \(\mathcal{F}\).

Following point 1 in the proof of Theorem 7 of [HT18], both \(A\) and \(A = \{A_i : i < \omega\}\) are amorphous in \(\mathcal{N}\). Consequently, the principle that ‘every infinite family of \(n\)-element sets has a (partial) choice function’ fails, and so the Ramsey’s theorem (RT) fails (following [Tac16]).

Theorem 1.2. In point 1 in the proof of Theorem 7 of [HT18], it was stated that in \(\mathcal{N}\), \(A = \{A_i : i < \omega\}\) has no infinite subfamily \(B\) with a Kinna–Wegner selection function. For reader’s convenience we sketch an argument.
Lemma 5.3. In \( \mathcal{N} \), \( \mathcal{A} = \{ A_i : i \in \omega \} \) has no partial Kinna–Wegner selection function.

Proof. For the sake of contradiction, assume the existence of an infinite subfamily \( \mathcal{A}' \) of \( \mathcal{A} \), with a Kinna–Wagner selection function \( f \). Suppose \( \mathcal{A}' \) and \( f \) are supported by some \( E \). Without loss of generality, we may assume that \( E = \cup \{ A_i : i \in K \} \) for some \( K \subseteq \omega < \omega \). We can see that there exists a \( j \in \omega \), such that \( A_j \in \mathcal{A}' \) but \( j \in \omega \setminus K \) since \( \mathcal{A}' \) is infinite and \( K \) is finite. Let \( B_j = f(A_j) \).

Consider the permutation \( \pi_j \) of \( A \) where \( \pi_j \mid A_j \) is the \( n \)-cycle \( a_{j_1} \mapsto a_{j_2} \mapsto \ldots \mapsto a_{j_n} \mapsto a_{j_1} \), and \( \pi_j \mid A \setminus A_j \) is the identity mapping \( 1_{A \setminus A_j} \) on \( A \setminus A_j \). Clearly, \( \pi_j \in \text{fix}_G(E) \), \( \pi_j(A_j) = A_j \) and \( \pi_j(B_j) \neq B_j \). Consequently, we have the following.

\[
(A_j, B_j) \in f \implies \pi_j(A_j, B_j) \in \pi_j(f) \implies (A_j, \pi_j(B_j)) \in f.
\]

Since \( \pi_j(B_j) \neq B_j, (A_j, B_j) \in f \), and \( (A_j, \pi_j(B_j)) \in f \), we obtain a contradiction to the fact that \( f \) is a function. \( \square \)

Lemma 5.4. \( LT \) fails in \( \mathcal{N} \).

Proof. Since \( \mathcal{A} \) is an amorphous set of non-empty sets which has no choice function in \( \mathcal{N} \), following Lemma 5.2, \( LT \) fails in \( \mathcal{N} \).

Applying Observation 1.3 and following the proof of Consequence 1 of Observation 1.3, \( \mathcal{P}'_k \) holds in \( \mathcal{N} \) for all \( k \in \omega \). Following [Tac19], we observe that DT holds in \( \mathcal{N} \).

Lemma 5.5. In \( \mathcal{N} \), DT holds.

Proof. Let \( E \subseteq A \) be a finite support of an infinite p.o.set \( \mathbb{P} = (P, \prec) \) with finite width. Then \( P = \bigcup \{ \text{Orb}_E(p) : p \in P \} \). Since if \( \eta \in G \), then \( \eta \) only moves finitely many atoms, \( \text{Orb}_E(p) \) is an antichain in \( \mathbb{P} \) for each \( p \in P \) following Claim 3 of [Tac16]. For reader’s convenience we write the argument explicitly.

Claim 5.6. For each \( p \in P \), \( \text{Orb}_E(p) \) is an antichain in \( \mathbb{P} \).

Proof. Otherwise there is a \( p \in P \), such that \( \text{Orb}_E(p) \) is not an antichain in \( \mathbb{P} \). Thus, for some \( \phi, \psi \in \text{fix}_G(E) \), \( \phi(p) \) and \( \psi(p) \) are comparable. Without loss of generality we may assume \( \phi(p) \prec \psi(p) \). Since if \( \eta \in G \), then \( \eta \) only moves finitely many atoms, there exists some \( k < \omega \) such that \( \phi^k = 1_A \). Let \( \pi = \psi^{-1} \phi \). Consequently, \( \pi(p) \prec p \) and \( \pi^k = 1_A \) for some \( k \in \omega \). Thus, \( p = \pi^k(p) \prec \pi^{k-1}(p) \prec \ldots \prec \pi(p) \prec p \). By transitivity of \( \prec \), \( p < p \), which is a contradiction. \( \square \)

By Claim 5.6, \( \text{Orb}_E(p) \) is finite for each \( p \in P \) since the width of \( \mathbb{P} \) is finite. Now, \( \{ \text{Orb}_E(p) : p \in P \} \) is well-orderable in \( \mathcal{N} \) since every element of this family is supported by \( E \). Following point 4 in the proof of Theorem 7 of [HT18] and Lemma 3 of [Tac16], well-orderable union of well-orderable sets is well-orderable (\( UT(WO, WO, WO) \)) holds in \( \mathcal{N} \), and \( P \) is well-orderable in \( \mathcal{N} \). Applying Lemma 5.1, DT holds in \( \mathcal{N} \).

6. Weakening the assumption of supercompactness by strong compactness

6.1. Strongly compact and supercompact cardinals. We recall the definition of a supercompact cardinal and a strongly compact cardinal from ‘The Higher Infinite’ [Kanamori] of Akihiro Kanamori. For a set \( A \) we say \( \mathcal{U} \) a fine measure on \( \mathcal{P}_\kappa(A) \) if \( \mathcal{U} \) is a \( \kappa \)-complete ultrafilter and for any \( i \in A \), \( \{ x \in \mathcal{P}_\kappa(A) : i \in x \} \in \mathcal{U} \). We say \( \mathcal{U} \) is a normal measure on \( \mathcal{P}_\kappa(A) \), if \( \mathcal{U} \) is a fine measure and if \( f : \mathcal{P}_\kappa(A) \rightarrow A \) is such that \( f(X) \in X \) for a set in \( \mathcal{U} \), then \( f \) is constant on a set in \( \mathcal{U} \). Let \( \kappa \) be an uncountable regular cardinal. We say the following.

1. \( \kappa \) is \( \lambda \)-strongly compact if there is a fine measure on \( \mathcal{P}_\kappa(\lambda) \), it is strongly compact if it is \( \lambda \)-strongly compact for all \( \kappa \leq \lambda \).
(2) \( \kappa \) is \( \lambda \)-supercompact if there is a normal measure on \( \mathcal{P}_\kappa(\lambda) \), it is supercompact if it is \( \lambda \)-supercompact for all \( \kappa \leq \lambda \).

6.2. Homogeneity of forcing notions. We recall the definition of weakly homogeneous and cone homogeneous forcing notions from [DD08].

**Definition 6.1. (Definition 2 of [DD08].)** Let \( \mathbb{P} \) be a set forcing notion.

- We say \( \mathbb{P} \) is weakly homogeneous if for any \( p, q \in \mathbb{P} \), there is an automorphism \( a : \mathbb{P} \to \mathbb{P} \) such that \( a(p) \) and \( q \) are compatible.\(^{16}\)
- For \( p \in \mathbb{P} \), let \( \text{Cone}(p) \) denote \( \{ r \in \mathbb{P} : r \leq p \} \). We say \( \mathbb{P} \) is cone homogeneous if and only if for any \( p, q \in \mathbb{P} \), there exist \( p' \leq p, q' \leq q \), and an isomorphism \( \pi : \text{Cone}(p') \to \text{Cone}(q') \).

Following Fact 1 of [DD08], if \( \mathbb{P} \) is a weakly homogeneous forcing notion, then it is cone homogeneous too. Also, the finite support products of weakly (cone) homogeneous forcing notions are weakly (cone) homogeneous.

6.3. Strongly compact Prikry forcing. Suppose \( \lambda > \kappa \) and \( \kappa \) be a \( \lambda \)-strongly compact cardinal in the ground model \( V \). Let \( \mathcal{U} \) be a fine measure on \( \mathcal{P}_\kappa(\lambda) \) and \( \mathcal{F} = \{ f : f \) is a function from \( [\mathcal{P}_\kappa(\lambda)]^{\leq \omega} \) to \( \mathcal{U} \} \). We recall the definition of a strongly compact Prikry forcing \( \mathbb{P}_\mathcal{U} \) and details concerning it from [AH91]. In particular, \( \mathbb{P}_\mathcal{U} \) is the set of all finite sequences of the form \( \langle p_1, ..., p_n, f \rangle \) satisfying the following properties.

- \( \langle p_1, ..., p_n \rangle \in [\mathcal{P}_\kappa(\lambda)]^{\leq \omega} \).
- For \( 0 \leq i < j \leq n \), \( p_i \cap \kappa \neq p_j \cap \kappa \).
- \( f \in \mathcal{F} \).

The ordering on \( \mathbb{P}_\mathcal{U} \) is given by \( \langle q_1, ..., q_m, g \rangle \preceq \langle p_1, ..., p_n, f \rangle \) if and only if we have the following.

- \( n \leq m \).
- \( \langle p_1, ..., p_n \rangle \) is the initial segment of \( \langle q_1, ..., q_m \rangle \).
- For \( i = n + 1, ..., m \), \( q_i \in f(\langle p_1, ..., p_n, q_{n+1}, ..., q_{i-1} \rangle) \).
- For \( \langle q, \bar{r} \rangle \in [\mathcal{P}_\kappa(\lambda)]^{\leq \omega} \), \( g(\langle q, \bar{r} \rangle) \preceq f(\langle q, \bar{r} \rangle) \).

Let \( G \) be \( V \)-generic over \( \mathbb{P}_\mathcal{U} \). A density argument tells us that for any regular \( \delta \in [\kappa, \lambda] \), \( r \upharpoonright \delta = \{ < p_0 \cap \delta, ..., p_n \cap \delta > : f \in \mathcal{F}[\{ p_0, ..., p_n \} \in G] \} \) codes a cofinal \( \omega \)-sequence through \( \delta \). In \( V[r \upharpoonright \delta] \subseteq V[G] \), \( \kappa \) is a singular cardinal having cofinality \( \omega \). Also, \( (\lambda)^V \) is collapsed to \( \kappa \) in \( V[G] \). Since any two conditions having the same stems are compatible, \( \mathbb{P}_\mathcal{U} \) is \( (\lambda^{< \kappa})^+ \)-c.c. We recall the following Prikry like lemma which implies that \( \mathbb{P}_\mathcal{U} \) does not add bounded subsets to \( \kappa \).

**Lemma 6.2. (Lemma 1.1 of [AH01]).** If \( \phi \) is a formula in the forcing language with respect to \( \mathbb{P}_\mathcal{U} \), then for every forcing condition \( \langle p_1, ..., p_n, f \rangle \), there is some \( g \subseteq f, g \in \mathcal{F} \) so that \( \langle p_1, ..., p_n, g \rangle \) decides \( \phi \).

Let \( \delta \in [\kappa, \lambda] \) be an inaccessible cardinal. If \( x \subseteq \mathcal{P}_\kappa(\lambda) \), let \( x \upharpoonright \delta = \{ Z \cap \delta : Z \in x \} \) and \( \mathcal{U} \upharpoonright \delta = \{ x \upharpoonright \delta : x \in \mathcal{U} \} \). Since, \( \mathcal{U} \) is a \( \kappa \)-complete, fine ultrafilter on \( \mathcal{P}_\kappa(\delta) \), \( \mathcal{U} \upharpoonright \delta \) is a \( \kappa \)-complete, fine ultrafilter on \( \mathcal{P}_\kappa(\delta) \). We refer to [AH01] for further details concerning the strongly compact Prikry forcing \( \mathbb{P}_\mathcal{U} \) and \( \mathbb{P}_\mathcal{U} \upharpoonright \delta \).

6.4. Symmetric extension. Symmetric extensions in terms of symmetric system \( \langle \mathbb{P}, \mathbb{G}, \mathcal{F} \rangle \) are intermediate models of the form \( \text{HOD}(V \cup a)^{V[G]} \) as \( a \) varies over \( V[G] \) according to Serge Grigorieff [GT01]. We recall the basics of symmetric extensions in terms of symmetric system \( \langle \mathbb{P}, \mathbb{G}, \mathcal{F} \rangle \) in this section. Let \( \mathbb{P} \) be a forcing notion that is a partially ordered set with a maximum

\(^{16}\)Given an infinite cardinal \( \kappa \) and a regular cardinal \( \lambda \), the Levy collapse \( \text{Col}(\kappa, \lambda) \) is weakly homogeneous.

\(^{17}\)i.e. any two conditions of the form \( \langle p_1, ..., p_n, f \rangle \) and \( \langle p_1, ..., p_n, g \rangle \) are compatible.
element 1, \(\mathcal{G}\) be a group of automorphisms of \(\mathbb{P}\) and \(\mathcal{F}\) be a normal filter of subgroups over \(\mathcal{G}\). We recall the following symmetry lemma from \cite{Jec03}.

**Theorem 6.3. (Symmetry Lemma, Lemma 14.37 of \cite{Jec03}).** Let \(\mathbb{P}\) be a forcing notion, \(\varphi\) be a formula of the forcing language with \(n\) variables and let \(\sigma_1, \sigma_2, \ldots, \sigma_n \in V^\mathbb{P}\) be \(\mathbb{P}\)-names. If \(a \in \text{Aut}(\mathbb{P})\), then \(p \Vdash \varphi(\sigma_1, \sigma_2, \ldots, \sigma_n) \iff a(p) \Vdash \varphi(a(\sigma_1), a(\sigma_2), \ldots, a(\sigma_n))\).

For \(\tau \in V^\mathbb{P}\), we denote the symmetric group with respect to \(\mathcal{G}\) by \(\text{sym}^\mathcal{G}\tau = \{g \in \mathcal{G} : g\tau = \tau\}\) and say \(\tau\) is symmetric with respect to \(\mathcal{F}\) if \(\text{sym}^\mathcal{G}\tau \in \mathcal{F}\). Let \(HS^\mathcal{F}\) be the class of all hereditary symmetric names. We define symmetric extension of \(V\) or symmetric submodel of \(V[G]\) with respect to \(\mathcal{F}\) as \(V(G)^{\mathcal{F}} = \{\tau^G : \tau \in HS^\mathcal{F}\}\). For the sake of our convenience we omit the subscript \(\mathcal{F}\) sometimes and call \(V(G)^{\mathcal{F}}\) as \(V(G)\).

**Definition 6.4. (Symmetric System).** We say \(\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle\) is a symmetric system if \(\mathbb{P}\) is a forcing notion, \(\mathcal{G}\) the automorphism group of \(\mathbb{P}\) and \(\mathcal{F}\) a normal filter of subgroups over \(\mathcal{G}\).

**Theorem 6.5. (Lemma 15.51 of \cite{Jec03}).** If \(\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle\) is a symmetric system and \(G\) is a \(V\)-generic filter, then \(V(G)\) is a transitive model of \(ZF\) and \(V \subseteq V(G) \subseteq V[G]\).

For \(E \subseteq \mathbb{P}\), let us define the pointwise stabilizer group to be \(\text{fix}_E = \{g \in \mathcal{G} : \forall p \in E, g(p) = p\}\). i.e. it is the set of automorphisms which fix \(E\) pointwise. Again we denote \(\text{fix}_E\) by \(\text{fix}^E\) for the sake of convenience. A subset \(I \subseteq \mathcal{P}(\mathbb{P})\) is called \(\mathcal{G}\)-symmetry generator if it is closed under unions and if for all \(g \in \mathcal{G}\) and \(E \in I\), there is an \(E' \in I\) s.t. \(g(\text{fix}^E) g^{-1} \supseteq \text{fix}^E'\). It is possible to see that if \(I\) is a \(\mathcal{G}\)-symmetry generator, then the set \(\{\text{fix}^E : E \in I\}\) generates a normal filter over \(\mathcal{G}\) (Proposition 1.23 of Chapter 1 in \cite{Dim11}). Let \(I\) be the \(\mathcal{G}\)-symmetry generator generating a normal filter \(\mathcal{F}_I\) over \(\mathcal{G}\). We say \(E \in I\) supports a name \(\sigma \in HS\) if \(\text{fix}E \subseteq \text{sym}(\sigma)\). Since \(\mathbb{P}, \mathcal{G}\) and \(I\) are enough to define a symmetric extension, we define a symmetric extension \(V(G)^{\mathcal{F}_I}\) using \(\langle \mathbb{P}, \mathcal{G}, I \rangle\) at times and work with it.

### 6.5. Proving Theorem 1.7.

We weaken the assumption of supercompactness by strongly compactness from Theorem 1 of \cite{AC13}.

1. **Defining ground model** \((V)\): We start with a model \(V_0\) of \(ZFC\) where \(\kappa\) is a strongly compact cardinal, \(\theta\) an ordinal and \(GCH\) holds. By Theorem 3.1 of \cite{ADU19} we can obtain a forcing extension \(V\) where \(2^\kappa = \theta\) and strong compactness of \(\kappa\) is preserved. We assume \(\lambda > \kappa\) in \(V\) such that \((cf(\lambda))^V < \kappa\).

2. **Defining symmetric system** \(\langle \mathbb{P}, \mathcal{G}, \mathcal{F}_I \rangle\):
   - Let \(\mathbb{U}\) be a fine measure on \(\mathcal{P}_\kappa(\lambda)\) and \(\mathcal{P} = \mathcal{P}_\mathbb{U}\) be the strongly compact Prikry forcing.
   - Let \(\mathcal{G}\) be full permutation group of \(\lambda\) and extend it to the partial order by permuting the range of the conditions.
   - For each inaccessible \(\alpha \in [\kappa, \lambda)\) we define \(E_\alpha = \mathcal{P}_{\mathbb{U}|\alpha}\) where \(\mathbb{U} \upharpoonright \alpha\) is the fine measure on \(\mathcal{P}_\alpha(\lambda)\) induced by some fine measure \(\mathbb{U}\) on \(\mathcal{P}_\kappa(\lambda)\) and \(\mathcal{P}_{\mathbb{U}|\alpha}\) is the strongly compact Prikry forcing with respect to the fine measure \(\mathbb{U} \upharpoonright \alpha\). Let \(I = \{E_\alpha : \alpha \text{ is an inaccessible cardinal in } [\kappa, \lambda]\}\). Let \(\mathcal{F}_I\) be the normal filter over \(\mathcal{G}\) generated by \(\{E_\alpha : E_\alpha \in I\}\).

3. **Defining symmetric extension of** \(V\): Let \(G\) be a \(\mathbb{P}\)-generic filter. We consider the symmetric extension \(V(G)^{\mathcal{F}_I}\) by the symmetric system \(\langle \mathbb{P}, \mathcal{G}, \mathcal{F}_I \rangle\) to be our desired model.

Intuitively, \(V(G)\) is the model constructed in \cite{AH91} which is the least model of \(ZF\) extending \(V\) and containing \(r \upharpoonright \delta\) for each inaccessible \(\delta \in [\kappa, \lambda)\) where \(r \upharpoonright \delta = \{<p_0 \cap \delta, \ldots, p_n \cap \delta > : \exists f \in \mathcal{F}[\{p_0, ..., p_n, f\} \in G]\}\) but not the \(\lambda\)-sequence of \(r \upharpoonright \delta\)'s. We follow the homogeneity of strongly compact Prikry forcing mentioned in Lemma 2.1 of \cite{AH91} to observe the following lemma.

**Lemma 6.6.** If \(A \subseteq V(G)\) is a set of ordinals, then \(A \in V[G \upharpoonright E_\delta]\) for some inaccessible \(\delta \in [\kappa, \lambda)\).

**Lemma 6.7.** In \(V(G), \kappa\) is a strong limit cardinal.
Proof. Since, $V \subseteq V(G) \subseteq V[G]$ and $\mathbb{P}$ does not add bounded subsets to $\kappa$ following Lemma 6.2, $V$ and $V(G)$ have same bounded subsets of $\kappa$\(^1\). Consequently, in $V(G)$, $\kappa$ is a limit of inaccessible cardinals and thus a strong limit cardinal as well. □

Lemma 6.8. If $\gamma \geq \lambda$ is a cardinal in $V$, then $\gamma$ remains a cardinal in $V(G)$.

Proof. For the sake of contradiction, let $\gamma$ is not a cardinal in $V(G)$. There is then a bijection $f : \alpha \rightarrow \gamma$ for some $\alpha < \gamma$ in $V(G)$. Since $f$ can be coded by a set of ordinals, by Lemma 6.6 $f \in V[G \upharpoonright E_4]$ for some inaccessible $\delta \in [\kappa, \lambda)$. Since GCH is assumed in $V_0$ we have $(\delta^{<\kappa})^{V_0} = \delta$, and since $\text{Add}(\kappa, \theta)$ preserves cardinals and adds no sequences of ordinals of length less than $\kappa$, we conclude that $(\delta^{<\kappa})^V = (\delta^{<\kappa})^{V_0} = \delta$. Now $\mathbb{P}_{\delta^+}$ is $(\delta^{<\kappa})^+\text{-c.c.}$ in $V$ and hence $\delta^+\text{-c.c.}$ in $V$. Consequently, $\gamma$ is a cardinal in $V$ which is a contradiction. □

Lemma 6.9. In $V(G)$, $\text{cf}(\kappa) = \omega$. Moreover, $(\kappa^+)^{V(G)} = \lambda$ and $\text{cf}(\lambda)^{(G)} = \text{cf}(\lambda)^V$.

Proof. For each $\delta \in [\kappa, \lambda)$, we have $V[\mathbb{G} \upharpoonright E_4] \subseteq V(G)$. Consequently, $\text{cf}(\kappa)^{(G)} = \omega$ since $\text{cf}(\kappa)^{(G)} = \omega$. Following Lemma 2.4 of [AH91], every ordinal in $(\kappa, \lambda)$ which is a cardinal in $V$ in collapses to have size $\kappa$ in $V(G)$, and so $(\kappa^+)^{V(G)} = \lambda$. Since $V$ and $V(G)$ have same bounded subsets of $\kappa$, we see that $\text{cf}(\lambda)^{(G)} = \text{cf}(\lambda)^V < \kappa$. □

We see that since, $V \subseteq V(G)$ and $(2^\kappa = \theta)^V$, there is a $\theta$-sequence of distinct subsets of $\kappa$ in $V(G)$. Since $\text{cf}(\kappa^+)^{V(G)} < \kappa$ we can also see that $\mathcal{A}C_\kappa$ fails in $V(G)$.

Corollary 6.10. (Reducing the assumption of Theorem 2 of [AC13]). Suppose $\kappa$ is a strongly compact cardinal, GCH holds, $\theta$ is an ordinal in a ground model $V$ of ZFC. There is then a model of $\mathcal{Z}F + \neg \mathcal{A}C_\omega$ in which $\text{cf}(\kappa_1) = \text{cf}(\kappa_2) = \omega$, and there is a sequence of distinct subsets of $\mathcal{R}_\omega$ of length $\theta$.

Corollary 6.11. (Reducing the assumption of Theorem 3 of [AC13]). Suppose $\kappa$ is a strongly compact cardinal, GCH holds, $\theta$ is an ordinal in a ground model $V$ of ZFC. There is then a model of $\mathcal{Z}F + \neg \mathcal{A}C_\omega$ in which $\mathcal{R}_\omega$ and $\mathcal{R}_{\omega+1}$ are both singular with $\omega \leq \text{cf}(\mathcal{R}_{\omega+1}) < \mathcal{R}_\omega$, and there is a sequence of distinct subsets of $\mathcal{R}_\omega$ of length $\theta$.

7. REMOVING THE ASSUMPTION THAT ALL STRONGLY COMPACT CARDINALS ARE LIMITS OF MEASURABLE CARDINALS

7.1. Rowbottom cardinals. We recall the definition of Rowbottom cardinals from [Kan09]. An uncountable regular cardinal $\kappa$ is $\mu$-Rowbottom if for all $\alpha < \kappa$ and $f : [\kappa]^{<\omega} \rightarrow \alpha$, there is a homogeneous set $X \subseteq \kappa$ for $f$ of order type $\kappa$ such that $\|f''[X]^{<\omega}\| < \mu$. An uncountable regular cardinal $\kappa$ is Rowbottom if it is $\omega_1$-Rowbottom. Filter $\mathcal{F}$ on $\kappa$ is a Rowbottom filter on $\kappa$ if for any $f : [\kappa]^{<\omega} \rightarrow \lambda$, where $\lambda < \kappa$ there is a set $X \in \mathcal{F}$ such that $\|f''[X]^{<\omega}\| \leq \omega$.

7.2. Proving Theorem 1.8. We recall the basics of symmetric extension and strongly compact Prikry forcing from the previous section. Given $\rho \in (\omega, \omega^1]$, we start with a $\rho$-sequence of strongly compact cardinals as assumed in Chapter 2, Section 4 of [Dim11] and construct our desired symmetric extension.

1. Defining ground model($V$): Let $V$ be a model of ZFC where for some $\rho \in (\omega, \omega^1]$, there is a $\rho$-sequence $\langle \kappa_\alpha : 0 < \alpha < \rho \rangle$ of strongly compact cardinals and $\eta$ be the limit of this sequence. Let $\text{Reg}^\rho$ be the set of infinite regular cardinals $\alpha \in (\omega, \eta)$. We classify each $\alpha \in \text{Reg}^\rho$ in three types as follows.

- (type 0). If $\alpha \in (\omega, \kappa_1)$.
- (type 1). If $\alpha \geq \kappa_1$ and there is a largest $\kappa_\epsilon \leq \alpha$, i.e., $\alpha \in [\kappa_\epsilon, \kappa_{\epsilon+1})$.

\(^{18}\)We can observe another argument from Lemma 2.2 of [AH91].
• (type 2). If $\alpha \geq \kappa_1$ and there is no largest strongly compact $\leq \alpha$, then let $\beta_\alpha = \cup\{\kappa_i : \kappa_i < \alpha\}$. We ditto Gitik’s treatment for type 2 cardinals from chapter 2, subsection 4 of [Dim11].

(2) Defining symmetric system $(\mathcal{P}, \mathcal{G}, \mathcal{F})$: Let $\text{Reg}_0^\eta$ be the set of all regular type 0 cardinals in $(\omega, \eta)$, $\text{Reg}_1^\eta$ be the set of all regular type 1 cardinals in $(\omega, \eta)$ and $\text{Reg}_2^\eta$ be the set of all regular type 2 cardinals in $(\omega, \eta)$.

• Defining the partially ordered set.
  - Let $\mathbb{P}_\alpha = \{p : \omega \rightarrow \alpha : |p| < \omega\}$ for every $\alpha \in \text{Reg}_0^\eta$ and $\mathbb{P}_0 = \prod_{\alpha \in \text{Reg}_0^\eta} \mathbb{P}_\alpha$.
  - Let $\mathcal{U}$ be the fine measure on $\mathcal{P}_{\kappa_\kappa}(\kappa+1)$, then we let $\mathcal{P}_{\kappa_\kappa}$ to be the strongly compact Prikry forcing $\mathbb{P}_\mathcal{U}$. Let $\mathbb{P}_1 = \prod_{\kappa \in \rho} \mathbb{P}_{\kappa_\kappa}$ be the finite support product of $\mathcal{P}_{\kappa_\kappa}$.
  - For each $\alpha \in \text{Reg}_2^\eta$, let $\mathbb{P}_\alpha$ be the forcing notion as described in section 4, chapter 2 of [Dim11] for type 2 cardinals. Let $\mathbb{P}_2 = \prod_{\alpha \in \text{Reg}_2^\eta} \mathbb{P}_\alpha$.

We define the desired forcing notion $\mathcal{P}$ as the product of $\mathbb{P}_0$, $\mathbb{P}_1$ and $\mathbb{P}_2$.

• Defining the group of automorphisms $\mathcal{G}$.
  - For each $\alpha \in \text{Reg}_0^\eta$ let $\mathcal{G}_\alpha$ be the group of permutations of $\alpha$ that only moves finitely many elements of $\alpha$ and $\mathcal{G}_0^\eta$ be the finite support product of all these $\mathcal{G}_\alpha$’s.
  - Let $\mathcal{G}_\kappa$ be the permutation group of $\kappa$ that only moves finitely many elements of $\kappa$ and extend it to the partial order by permuting the range of the conditions. Let $\mathcal{G}_1$ be the finite support product of all these $\mathcal{G}_\kappa$’s.
  - For each $\alpha \in \text{Reg}_2^\eta$, let $\mathcal{G}_\alpha$ be the group of permutations of $\alpha$ that only moves finitely many elements of $\alpha$ and $\mathcal{G}_2$ be the finite support product of all these $\mathcal{G}_\alpha$’s.

We desire the group of automorphisms $\mathcal{G}$ as the product of $\mathcal{G}_0$, $\mathcal{G}_1$ and $\mathcal{G}_2$.

• Defining the normal filter $\mathcal{F}_\mathcal{I}$ of subgroups over $\mathcal{G}$.
  - For every finite $e_0 \subseteq \text{Reg}_0^\eta$, we define $\mathcal{E}_e_0 = \{p \upharpoonright e_0 : p \in \mathcal{P}_0\}$.
  - For $m < \omega$ and $e_1 = \{\alpha_1, \ldots, \alpha_m\} \subseteq \text{Reg}_1^\eta$ a sequence of inaccessible cardinals in $\text{Reg}_0^\eta$ such that for each $\alpha_1 \in e_1$, there is a distinct $e_\alpha_i \in \text{Ord}$ such that $\alpha_i \in [\kappa_{\alpha_i}, \kappa_{\alpha_i} + 1]$ we define $\mathcal{E}_e_1 = \prod_{\alpha \in \{1, 2, \ldots, m\}} \mathcal{P}_{\mathcal{U}_{\alpha_i}}[\alpha]$ where $\mathcal{U}_{\alpha_i}[\alpha]$ is the fine measure on $\mathcal{P}_{\mathcal{U}_{\alpha_i}}(\alpha)$ induced by some fine measure $\mathcal{U}_{\alpha_i}$ on $\mathcal{P}_{\mathcal{U}_{\alpha_i}}(\kappa_{\alpha_i} + 1)$ and $\mathcal{P}_{\mathcal{U}_{\alpha_i}}[\alpha]$ is the strongly compact Prikry forcing with respect to the fine measure $\mathcal{U}_{\alpha_i}[\alpha]$.
  - For every finite $e_2 \subseteq \text{Reg}_2^\eta$, we define $\mathcal{E}_e_2 = \{T \upharpoonright e_2 : T \upharpoonright e_2 \in \mathcal{P}_2\}$.

Let $\mathcal{I} = \{\mathcal{E}_e_1 \cup \mathcal{E}_e_2 : e_0 \text{ is any finite subset of } \text{Reg}_0^\eta, e_1 \text{ is any finite subset of } \text{Reg}_1^\eta \}\cup \{e_1 : \text{each } \alpha_1 \in e_1 \text{, there is a distinct } e_\alpha_i \in \text{Ord} \text{ such that } \alpha_i \in [\kappa_{\alpha_i}, \kappa_{\alpha_i} + 1]\}$.

(3) Defining symmetric extension of $V$: Let $G$ be a $\mathbb{P}$-generic filter. Consider the symmetric model $V(G)^{\mathcal{I}}$.

Following the homogeneity of strongly compact Prikry forcings from Lemma 2.1 of [AH91], homogeneity of injective tree Prikry forcings from Lemma 2.15 of [Dim11] and Lemma 2.23 of [Dim11], and the fact that finite support product of weakly (cone) homogeneous forcings are weakly (cone) homogeneous, we can obtain the desired homogeneity of $\mathbb{P}$. Consequently, we can have the following lemma.

Lemma 7.1. If $X'$ is a set of ordinals in $V(G)$, then for some $X \subseteq X$, $X' \subseteq V[G \upharpoonright X]$.

19We quote the following from chapter 2, subsection 4 of [Dim11]. “Since $\mathcal{U}$ is a countable limit ordinal, we can obtain an increasing cofinal function $g : \omega \rightarrow \chi$ in $V$. Let $\chi$ be such a function. We obtain $\beta_\alpha = \cup\{\kappa_{\alpha_i}(n) : n < \omega\}$ and an ascending sequence $\{\kappa_{\alpha_i}(n) : n < \omega\}$. For each $n < \omega$, let $\mathcal{U}_{\alpha_i}$ be a fine ultrafilter over $\mathcal{P}_{\kappa_{\alpha_i}}(\alpha)$ and $h_{\alpha_i, n} : \mathcal{P}_{\kappa_{\alpha_i}}(\alpha) \rightarrow \alpha$ be a surjection. If $\alpha$ is inaccessible then we let $h_{\alpha, n}$ to be a bijection. Define $\phi_{\alpha, n} = \{X \subseteq \alpha : h_{\alpha, n}(X) \in \mathcal{U}_{\alpha_i, n}\}$ which is a $\kappa_{\alpha_i}(n)$-complete uniform ultrafilter over $\alpha$.”

20i.e., if $\alpha_i \neq \alpha_j \in e_1$, $\alpha_i \in [\kappa_{\alpha_i}, \kappa_{\alpha_i} + 1]$ and $\alpha_j \in [\kappa_{\alpha_j}, \kappa_{\alpha_j} + 1]$ then $\epsilon_\alpha \neq \epsilon_\alpha_j$. 

We recall the Prikry like lemma for the injective tree Prikry forcing which is Lemma 2.24 of [Dim11] and the Prikry like lemma for the strongly compact Prikry forcing which is Lemma 1.1 of [AH91] or Lemma 6.2. We apply this to show that all $\kappa_\alpha$ for $0 < \alpha < \rho$, and their limits are still cardinals in $V(G)$.

**Lemma 7.2.** For every $0 < \epsilon < \rho$, $\kappa_\epsilon$ is a cardinal in $V(G)$. Consequently, their limits are also preserved.

**Proof.** For the sake of contradiction we assume that for some $0 < \epsilon < \rho$, there is some $\beta < \kappa_\epsilon$ and a bijection $f : \beta \to \kappa_\epsilon$ in $V(G)$. By Lemma 7.1, for some $X \subseteq \mathcal{I}$, $f \in V[G \upharpoonright X]$. Let $X$ be $E_0 \cup E_1 \cup E_2$ such that $e_0$ is some finite subset of $Reg_{\kappa_\epsilon}$, $e_2$ is some finite subset of $Reg_{\kappa_\epsilon}$ and $e_1$ is a finite collection of inaccessible cardinals in $Reg_{\kappa_\epsilon}$ such that for each $\alpha_i \in e_1$, there is a distinct $e_{\alpha_i} \in \text{Ord}$ such that $\alpha_i \in [\kappa_{\alpha_i}, \kappa_{\alpha_i+1})$.

We may imagine $V[G \upharpoonright X]$ as $V[G \upharpoonright E_0][G \upharpoonright E_1][G \upharpoonright E_2]$ and show that $f$ is not added in $V[G \upharpoonright E_0][G \upharpoonright E_1][G \upharpoonright E_2]$ to obtain a contradiction.

**Step 1.** $f$ is not added in $V[G \upharpoonright E_0]$. Clearly, $E_0$ is $\kappa_\epsilon$-c.c. Thus, $f$ cannot exist in $V[G \upharpoonright E_0]$.

**Step 2.** $f$ is not added in $V[G \upharpoonright E_0][G \upharpoonright E_1]$. Let $\{\alpha_1, \ldots, \alpha_m\}$ be an increasing enumeration of $e_1$, and let for each $1 \leq i \leq m$ there is a distinct $e_{\alpha_i}$ such that $\alpha_i \in [\kappa_{\alpha_i}, \kappa_{\alpha_i+1})$. Let $1 \leq j \leq m$ be the greatest such that $\kappa_\epsilon > \alpha_j$. We can write $E_{\alpha_j}$ as $\Pi_{i=1}^{j} P_{\mathcal{U}_{\alpha_i} | \alpha_i} \times \Pi_{i=j+1}^{m} P_{\mathcal{U}_{\alpha_i} | \alpha_i}$, where for each $1 \leq i \leq m$, $\mathcal{U}_{\alpha_i} | \alpha_i$ is the fine measure on $\mathcal{P}_{\kappa_{\alpha_i}}(\alpha_i)$ and $P_{\mathcal{U}_{\alpha_i} | \alpha_i}$ is the strongly compact Prikry forcing with respect to the fine measure $\mathcal{U}_{\alpha_i} | \alpha_i$. Clearly, $\Pi_{i=j+1}^{m} P_{\mathcal{U}_{\alpha_i} | \alpha_i}$ do not add any bounded subset of $\kappa_\epsilon$ following Lemma 6.2 (the Prikry like lemma for the strongly compact Prikry forcing). Moreover, $\Pi_{i=1}^{j} P_{\mathcal{U}_{\alpha_i} | \alpha_i} < \kappa_\epsilon$. Thus, $f$ is not added in $V[G \upharpoonright E_0][G \upharpoonright E_1]$ either.

**Step 3.** $f$ is not added in $V[G \upharpoonright E_0][G \upharpoonright E_1][G \upharpoonright E_2]$. Clearly, $E_2 = E_2 \cap \kappa_\epsilon \times E_2 | \kappa_\epsilon$ where $E_2 | \kappa_\epsilon$ is $\kappa_\epsilon$-c.c. and $E_2 | \kappa_\epsilon$ does not add bounded subsets to $\kappa_\epsilon$ following the Prikry like lemma for the injective tree Prikry forcing from Lemma 2.24 of [Dim11]. Thus, no such $f$ can exist in $V[G \upharpoonright E_0][G \upharpoonright E_1][G \upharpoonright E_2]$ also.

**Lemma 7.3.** In $V(G)$, the regular cardinals of type 2 have collapsed to their singular limits of strongly compact cardinals below them and if $\alpha \in (\kappa_\epsilon, \kappa_{\epsilon+1})$ is a regular cardinal of type 1 where $0 < \epsilon < \rho$, then $(|\alpha| = \kappa_\epsilon)^{V(G)}$.

**Proof.** Following Lemma 2.28 from [Dim11], the regular cardinals of type 2 have collapsed to their singular limits of strongly compact cardinals below them. Following Lemma 2.4 of [AH91], if $\alpha \in (\kappa_\epsilon, \kappa_{\epsilon+1})$ is a regular cardinal of type 1, then $(|\alpha| = \kappa_\epsilon)^{V(G)}$.

Consequently, we can have the following corollary similar to Corollary 2.29 of [Dim11].

**Corollary 7.4.** In $V(G)$, a cardinal in $(\omega, \eta)$ is a successor cardinal if and only if it is in $\{\kappa_\epsilon : \epsilon < \rho\}$ and a cardinal in $(\omega, \eta)$ is a limit cardinal if and only if it is a limit in the sequence $\{\kappa_\epsilon : \epsilon < \rho\}$ in $V$.

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[21] The argument goes as follows. Let $\alpha \in (\kappa_\epsilon, \kappa_{\epsilon+1})$ is a type 1 regular cardinal and $\beta \in (\alpha, \kappa_{\epsilon+1})$ be an inaccessible cardinal in $V$. We first show that $\alpha$ is no longer a cardinal in $V[G \upharpoonright E_{\beta}]$. More specifically, we show that there are no cardinals in the interval $(\kappa_\epsilon, \beta)$ in $V[G \upharpoonright E_{\beta}]$. For the sake of contrary, let $\alpha \in (\kappa_\epsilon, \beta)$ be the least cardinal in $V$ which remains a cardinal in $V[G \upharpoonright E_{\beta}]$. We observe contradiction in each of the following two cases.

**Case (i).** If $\alpha_1$ is a regular cardinal in $V$. We can see that $cf(\alpha_1) = \omega$ in $V[G \upharpoonright E_{\beta}]$. By the least-ness of the cardinality of $\alpha_1$, $\alpha_1 = \kappa_\epsilon^+$. But, $cf(\kappa_\epsilon^+) = \omega$ in $V[G \upharpoonright E_{\beta}]$, which is impossible since $V[G \upharpoonright E_{\beta}]$ is a model of $\text{AC}$.  

**Case (ii).** If $\alpha_1$ is a singular cardinal in $V$. Once more $\alpha_1 = \kappa_\epsilon^+$ in $V[G \upharpoonright E_{\beta}]$ which is impossible since $V[G \upharpoonright E_{\beta}]$ is a model of $\text{AC}$, and so the successor cardinal cannot be a singular cardinal. 

Thus, there are no cardinals in the interval $(\kappa_\epsilon, \beta)$ in $V[G \upharpoonright E_{\beta}]$. As $V[G \upharpoonright E_{\beta}] \subseteq V(G)$, the collapsing function for $\alpha$ is in $V(G)$ as well. Consequently, $\alpha$ is not a cardinal in $V(G)$ and so $(|\alpha| = \kappa_\epsilon)^{V(G)}$. 

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[Dim11], [AH91]
Lemma 7.5. In \( V(G) \), every ordinal in \( \text{Reg}^9 \) is singular of cofinality \( \omega \). Consequently, the interval \((\kappa, \eta)\) only contains singular cardinals.

Proof. Let \( \alpha \) is in \( \text{Reg}^9 \) is either of type 1 or type 2. There is a \( \omega \)-Prikry sequence cofinal in \( \alpha \) supported by \( \{\alpha\} \) following Lemma 2.3 of [AH91] and Lemma 2.26 of [Dim11]. \( \square \)

Similar arguments as in Lemma 2.20 and Lemma 2.30 of [Dim11] guarantees that all cardinals in the interval are almost Ramsey. Adopting the appropriate automorphism technique from Lemma 3.1 of [AH91], we observe that in \( V(G) \), all the successor cardinals in \((\omega, \eta)\) carry Rowbottom filter as well.

Lemma 7.6. In \( V(G) \), all cardinals in \((\omega, \eta)\) carry Rowbottom filter.

Proof. Following Lemma 2.31 of [Dim11] and Theorem 8.7 of [Kan09], we can see that all limit cardinals in \((\omega, \eta)\) are Rowbottom cardinals carrying a Rowbottom filter. In \( V(G) \), if a cardinal \( \kappa \) in \((\omega, \eta)\) is a successor cardinal, then there is an \( \epsilon < \rho \) such that \( \kappa = \kappa_\epsilon \). We show that \( \kappa_\epsilon \) carries a Rowbottom filter in \( V[G | E_{\{\kappa_{\epsilon+1}\}}] \subset V(G) \).

Firstly, we see that \( \kappa_\epsilon \) carries a Rowbottom filter in \( V[G'] \) where \( G' \) is a \( \gamma \)-generic filter over \( \mathbb{P}_{\kappa_\epsilon} \). Suppose for the sake of contradiction \( p = (p_0, ..., p_\gamma, u) \) \( G' \) forces that \( F_\gamma : [\mathbb{P}_{\kappa_\epsilon}(\kappa_{\epsilon+1})]^{\omega} \rightarrow \gamma < \kappa_\epsilon \) is a counter example to the Rowbottomness of \( \kappa_\epsilon \). Let \( U \) be the fine measure on \( \mathbb{P}_{\kappa_\epsilon}(\kappa_{\epsilon+1}) \) such that \( \mathbb{P}_{\kappa_\epsilon} = \mathcal{P}_U \).

- **Defining \( \mathcal{U}_{\kappa_\epsilon} \) and \( \mathcal{F}_{\kappa_\epsilon} \).** Let \( k : \mathbb{P}_{\kappa_\epsilon}(\kappa_{\epsilon+1}) \rightarrow \kappa_\epsilon \) be a map. We define \( \mathcal{U}_{\kappa_\epsilon} \) to be the push-forward ultrafilter \( \kappa_\epsilon(U) \). We may assume that \( \mathcal{U}_{\kappa_\epsilon} \) is a normal measure on \( \kappa_\epsilon \). Otherwise, we can change \( \mathcal{U}_{\kappa_\epsilon} \), so that it becomes normal, as follows.
  - Define a map \( l : \mathbb{P}_{\kappa_\epsilon}(\kappa_{\epsilon+1}) \rightarrow \mathbb{P}_{\kappa_\epsilon}(\kappa_{\epsilon+1}) \) as follows.
    \[ l(p) = (p \setminus \kappa_\epsilon) \cup (p \cap r(p)). \]

    We can see that \( l(U) \) is a fine measure on \( \mathbb{P}_{\kappa_\epsilon}(\kappa_{\epsilon+1}) \) and \( k_\epsilon(l(U)) \) is a normal measure on \( \kappa_\epsilon \). Let \( \mathcal{F}_{\kappa_\epsilon} = \{f : f : [\mathbb{P}_{\kappa_\epsilon}(\kappa_{\epsilon+1})]^{<\omega} \rightarrow \mathcal{U}_{\kappa_\epsilon}\} \).

- **Defining a subset \( X_{H,f} \) of \( \kappa_\epsilon \) and \( \mathcal{U}_H \).** For any \( f \in \mathcal{F}_{\kappa_\epsilon} \) and any \( H \) which is \( V \)-generic over \( \mathbb{P}_H \) we define the following subset \( X_{H,f} \) of \( \kappa_\epsilon \) in \( V[H] \).

  \[ X_{H,f} = \{f(\emptyset) \cap (p_0 \cap \kappa_\epsilon) \cup f(p_0) \cap (p_1 \cap \kappa_\epsilon) \cup (p_0 \cap \kappa_\epsilon) \cup f(p_0) \cap (p_1 \cap \kappa_\epsilon) \cup ... \} \]

  We define, \( \mathcal{U}_H = \{X_{H,f} : f \in \mathcal{F}_{\kappa_\epsilon}\} \).

We can clearly observe that \( \mathcal{U}_H \) is a filter on \( \kappa_\epsilon \). We recall \( F = \{f : f \) is a function from \( [\mathbb{P}_{\kappa_\epsilon}(\kappa_{\epsilon+1})]^{<\omega} \) to \( U \} \) from the definition of \( \mathcal{P}_U \). Let \( \mathcal{T} \) be the collection of finite sequences of 0’s and 1’s.

- **Defining an appropriate pair.** For \( \pi \in \mathcal{T}, g \in \mathcal{F}, h \in \mathcal{F}_{\kappa_\epsilon}, \sigma = (s_0, ..., s_k) \in [\mathbb{P}_{\kappa_\epsilon}(\kappa_{\epsilon+1})]^{<\omega} \) with each \( s_l \cap \kappa_\epsilon \) a cardinal for \( 0 \leq l \leq k, \tau = (t_0, ..., t_n) \in [\kappa_\epsilon]^{<\omega} \), we say \( \langle \sigma, \tau \rangle \) is appropriate for \( \pi, g, h \) if and only if the following holds.

  \[ \begin{align*}
  &- s_0 \cap \kappa_\epsilon < s_1 \cap \kappa_\epsilon < ..., \\
  &- t_0 < t_1, ..., \\
  &- t_j \neq s_j \cap \kappa_\epsilon \text{ for all } i \text{ and } j. \\
  &- \text{In case } \{t_i\}_{i<k}, \{s_j \cap \kappa_\epsilon\}_{j<k} \text{ are arranged in order, we have a sequence } \rho \text{ with the following.} \]

  \[ \begin{align*}
  &\ast \text{ len}(\rho) = \text{len}(\pi). \\
  &\ast \text{ if } \pi(i) = 0 \text{ then } \rho(i) = t_j \text{ for some } j \text{ and } \rho(i) \in h(t_0, ..., t_{j-1}). \]

  \[ \ast \text{ if } \pi(i) = 1 \text{ then } \rho(i) \notin \tau \text{ and } \rho(i) \in g(t_0, ..., t_j) \text{ where } t_j \text{ is the greatest member of } \tau \text{ below } \rho(i). \]

Similar to the claim in the proof of Lemma 3.1 of [AH91] we can observe the following.
claim 7.7. for all \( \pi \in \mathcal{T} \) and for all \( \sigma \in [\mathcal{P}_\kappa, (\kappa_{\kappa+1})]^{<\omega} \) extending \( \langle p_0, \ldots, p_n \rangle \) there are \( g \in F \), \( h \in \mathcal{F}_{\kappa_\pi}, \alpha \prec \kappa_\pi \) such that for all \( (\sigma', \tau) \) appropriate for \( \pi, g, h \), \( (\sigma \prec \sigma', g) \models \neg \text{"}F(\tau) = \alpha\" \).

Now let \( \sigma \) be \( \langle p_0, \ldots, p_n \rangle \), and choose \( g_\pi, h_\pi, \alpha_\pi \) for each \( \pi \in \mathcal{T} \). Consider the following.

- \( g \) be the intersection \((\cap g_\pi) \cap f\),
- \( h \) be \( \cap h_\pi \),
- \( Z = \{ \alpha_\pi \}_{\pi \in \mathcal{T}}. \)

Let \( H \) be a \( V \)-generic filter over \( \mathbb{P}_{U^*} \) such that \( \langle \sigma, g \rangle \in H \). For any \( \tau \in [X_{H, f}]^{<\omega} \), we can find \( \sigma' \) and a \( \pi \) such that \( (\sigma', \tau) \) is appropriate for \( g_\pi, h_\pi \) and \( \pi \). Thus \( (\sigma \prec \sigma', g_\pi) \models \text{"}F(\tau) = \alpha_\pi\" \) and so \( (\sigma \prec \sigma', g) \models \text{"}F(\tau) \in Z'\" \) and \( (\sigma, g) \models \text{"}F(\tau) \in X_{H, f}]^{<\omega} \subseteq Z' \" \). Now \( |Z| \leq \omega \) contradicts the assumption that \( (\sigma, f) \) forces that \( F \) is a counterexample to Rowbottomness of \( \kappa_\pi \). Consequently, we can observe that \( \mathcal{U}_{G'} \) is a Rowbottom filter on \( \kappa_\pi \) in \( V[G'] \).

Now, the definition of \( X_{H, f} \) above for a \( V \)-generic filter \( H \) over \( \mathbb{P}_{\kappa_\pi} \), depends only on \( H \restriction \kappa_\pi \). Consequently, \( \mathcal{U}_{G'} \) is in \( V[G \restriction E_{\kappa_\pi}] \). \( \square \)

Remark. In section 3, chapter 2 of \([\text{Dim11}]\), Ioanna Dimitriou worked on symmetric extension based on injective tree-Prikry forcing and Lévy Collapse to construct a countable sequence of cardinals, in any desired pattern of regular and singular cardinals. If we replace the injective tree-Prikry forcing with strongly compact Prikry forcing as done in this section, then using the methods of \([\text{AH91}]\) (specifically the methods applied in \text{Lemma 7.6} all the singular cardinals in the \( \omega \)-long pattern of singular and regular cardinals can carry a Rowbottom filter. Consequently, we can prove \text{Observation 1.9}.

\text{Question 7.8. (asked in \[\text{ADK16}\]). Is it possible to remove the additional assumption that ‘every strongly compact cardinals are the limit of measurable cardinals’ from \text{Theorem 1.1} of \[\text{ADK16}\]?}

8. (Appendix A): Different proofs if the vertex set of one graph is countable

In ZFC, we prove \text{Theorem 1.1} when the graph \( G_1 \) has countably many vertices, in two different ways.

\text{Theorem 8.1. If the chromatic number of a graph } G_1 = (V_{G_1}, E_{G_1}) \text{ with countably many vertices, is finite (say } k < \omega \text{), and the chromatic number of another graph } G_2 \text{ is infinite, then the chromatic number of } G_1 \times G_2 \text{ is } k. \)

8.1. First proof of \text{Theorem 8.1}. In \([\text{Kon27}]\), Denis König introduced a classic result in infinite graphs known as König’s lemma which is an equivalent of \( AC_{\omega}^{fin} \) in ZF\(^{22}\). It is well-known that the De Bruijn Erdős theorem for graphs on a countable set of vertices is provable using König’s lemma. We incorporate the arguments and give the first proof of \text{Theorem 8.1}, which has a different character than the methods used in \text{section 2}.

\text{Lemma 8.2. (König’s lemma). Suppose that } (\mathcal{T}, <) \text{ is an infinite finitely branching tree. There is then an infinite branch } B \text{ through } (\mathcal{T}, <). \)

We first enumerate \( V_{G_1} \) as \( V_{G_1} = \{ v_1, ... \} \) and for each \( n \geq 1 \), let \( V_{G_1n} = \{ v_1, ... v_n \} \). Let \( C_n \) be the collection of functions \( f: V_{G_1n} \rightarrow \{ 1, 2, ..., k-1 \} \) such that for all \( x, y \in V_{G_1n}, A_{x,f(x)} \cap A_{y,f(y)} \)

\(^{22}\)There are several equivalent versions of \( AC_{\omega}^{fin} \) in ZF, namely Cowen–Engeler lemma for partial valuations on countable sets, Prime ideal theorem for boolean algebras or distributive lattices with countably many generators, compactness theorem for propositional logic for countable collection of formulae, infinitary Ramsey theorem for \( 2 \) colorings, Hall’s theorem for countable collection of sets, Tychonoff theorem for countable collection of finite discrete spaces etc. Following the methods from \text{section 2}, we can observe several arguments to prove \text{Theorem 8.1} using Cowen–Engeler lemma for partial valuations on countable sets or Prime ideal theorem for boolean algebras with countably many generators or compactness theorem for propositional logic for countable collection of formulae.
is not independent. By Lemma 2.1, \( C_n \) is nonempty. Clearly, \( C_n \) is finite as well. Let \( (\mathcal{T}, <) \) be the tree with levels and partial orderings defined as follows.

- \( \text{Lev}_0 = \emptyset \). \( \text{Lev}_n = C_n \) for \( n \geq 0 \).
- Suppose \( \chi_1 \in \text{Lev}_n \) and \( \chi_2 \in \text{Lev}_m \) where \( 1 \leq n \leq m \). Then we define \( \chi_1 < \chi_2 \) if and only if \( \chi_1 = \chi_2 \cap \{v_1, \ldots, v_k\} \).

Clearly \( (\mathcal{T}, <) \) is the infinite finitely branching tree. By Lemma 8.2, there is an infinite branch \( B = \{x_n : n \in \omega\} \) through \( \mathcal{T} \) where \( \chi_n \in \text{Lev}_n \).

**Claim 8.3.** \( \chi = \cup_n \chi_n \) is a mapping from \( V_{G_1} \) to \( \{1, 2, \ldots, k - 1\} \) such that for all \( x, y \in V_{G_1} \), \( A_{x, f(x)} \cap A_{y, f(y)} \) is not independent.

**Proof.** Pick \( x, y \in V_{G_1} \) such that \( x \neq y \). There is then a \( n \in \omega \) such that \( x, y \in V_{G_{1n}} \). Since \( \chi_n \in C_n \), we have that \( A_{x, \chi_n(x)} \cap A_{y, \chi_n(y)} \) is not independent. Consequently, \( A_{x, \chi(y)} \cap A_{y, \chi(y)} \) is not independent by the definition of \( \chi \).

### 8.2. Second proof of Theorem 8.1.

It is well-known that the De Bruijn–Erdős theorem for graphs on a countable set of vertices is provable using sequential compactness of topological spaces. We incorporate the arguments to give another proof of **Theorem 8.1**. We enumerate \( V_{G_1} \) as \( V_{G_1} = \{v_1, \ldots\} \) and for each \( n \geq 1 \), let \( V_{G_{1n}} = \{v_1, \ldots, v_n\} \). Let \( X = \{1, 2, \ldots, k - 1\}^{V_{G_1}} \).

We can assume \( X \neq \emptyset \) since we are working in ZFC. For each \( n \in \omega \), by Lemma 2.1 we can pick \( f_n \in X \) such that \( f_n \upharpoonright \{v_1, \ldots, v_n\} \) is a mapping from \( V_{G_{1n}} \) to \( \{1, 2, \ldots, k - 1\} \) where for each \( x, y \in V_{G_{1n}}, A_{x, f_n(x)} \cap A_{y, f_n(y)} \) is not independent. \( X \) is sequentially compact since \( G_1 \) is countable and so, \( \{f_n\} \) has a convergent sub-sequence converging to some point \( f \).

**Claim 8.4.** \( f \) is a mapping from \( V_{G_1} \) to \( \{1, 2, \ldots, k - 1\} \) such that for all \( x, y \in V_{G_1}, A_{x, f(x)} \cap A_{y, f(y)} \) is not independent.

**Proof.** For the sake of a contradiction, let us assume that there exists \( v_i \neq v_j \), such that \( A_{v_i, f(v_i)} \cap A_{v_j, f(v_j)} \) is independent. Without loss of generality, we may assume \( i < j \). Let \( A_j \) be the collection of mappings \( g : V_{G_1} \to \{1, \ldots, k - 1\} \) such that \( g \upharpoonright \{v_0, \ldots, v_j\} = f \upharpoonright \{v_0, \ldots, v_j\} \). Since \( \{f_n\} \) has a convergent subsequence converging to \( f \), for infinitely many \( n \) we have \( f_n \in A_j \). Pick an \( m > j \) such that \( f_m \in A_j \). So, \( f_m \upharpoonright \{v_0, \ldots, v_j\} = f \upharpoonright \{v_0, \ldots, v_j\} \). Consequently, \( A_{v_j, f_m(v_j)} \cap A_{v_j, f_m(v_j)} \) is independent which contradicts the fact that \( f_m \upharpoonright \{v_1, \ldots, v_m\} \) is a mapping from \( V_{G_{1m}} \) to \( \{1, 2, \ldots, k - 1\} \) such that for each \( x, y \in V_{G_{1m}}, A_{x, f_m(x)} \cap A_{y, f_m(y)} \) is not independent.

### 9. (Appendix B): A weakly compact variant in ZFC

In [Haj85], András Hajnal proved that for every infinite cardinal \( \kappa \) there are two graphs \( G_1 \) and \( G_2 \) with \( \chi(E_{G_1}) = \chi(E_{G_2}) = \kappa^+ \) and \( \chi(E_{G_1} \times G_2) = \kappa \). Thus the property of \( \mathfrak{P}_\kappa \) changes, when \( E_{G_1} \) is an infinite successor cardinal and no longer a finite cardinal. In [Haj85], Hajnal mentioned a strongly compact variant of the problem. We observe the following variant of **Theorem 1.1**, if \( \kappa \) is a weakly compact cardinal.

**Theorem 9.1.** Let \( \kappa \) be a weakly compact cardinal such that \( \chi(E_{G_2}) = \kappa \) and \( |V_{G_1}| \leq \kappa \), and \( \mu < \kappa \) be an arbitrary cardinal such that \( \chi(E_{G_1}) = \mu \). Then \( \chi(E_{G_1} \times G_2) = \mu \).

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23We sketch a well-known argument. Recall that a topological space \( X \) is sequentially compact if every sequence of points in \( X \) has a convergent subsequence converging to a point in \( X \). Let \( \{x_n : n \in \omega\} \) be an infinite sequence of points in \( X \) and \( \{x_n : n \in \omega\} \) be the infinite sequence in the first coordinate induced by \( \{x_n : n \in \omega\} \), which must have a convergent subsequence say \( \{x_{n^1} : n \in \omega\} \). Since \( \{1, \ldots, k\} \) has a discrete topology, \( \{x_{n^1} : n \in \omega\} \) must be constant for all \( n^1 \geq m_1 \) for some \( m_1 \in \omega \). Let \( y_1 = x_{m_1} \). After constructing \( \{y_1, \ldots, y_{n^1-1}\} \), we construct \( y_{n^1} \) as follows. Let \( \{x_{n^1} : n \in \omega\} \) be the infinite sequence in the \( n^1 \)-coordinate induced by \( \{x_{n^1} : n \in \omega\} \), which must have a convergent subsequence \( \{x_{n^1}^i : n \in \omega\} \). Since \( \{1, \ldots, k\} \) has a discrete topology, \( \{x_{n^1}^i : n \in \omega\} \) must be constant for all \( n^1 \geq m_i \) for some \( m_i \in \omega \). Let \( y_i = x_{m_i} \). Clearly, \( \{y_n : n \in \omega\} \) is a subsequence of \( \{x_n : n \in \omega\} \) which converges to \( (y_{11}, y_{22}, \ldots) \).
9.1. Weak compactness theorem. Given an infinite cardinal \( \kappa \), the language \( \mathcal{L}_{\kappa, \omega} \) consists of the following.

1. \( \kappa \) variables,
2. various relation and function symbols,
3. logical connectives and infinitary connectives \( \land_{\xi < \alpha} \phi_{\xi}, \lor_{\xi < \alpha} \phi_{\xi} \) for \( \alpha < \kappa \).
4. quantifiers \( \exists v, \forall v \).

We say that the language \( \mathcal{L}_{\kappa, \omega} \) satisfies the weak compactness theorem if whenever \( \Sigma \) is a set of sentences of \( \mathcal{L}_{\kappa, \omega} \) such that \( |\Sigma| \leq \kappa \) and every \( S \subset \Sigma \) with \( |S| < \kappa \) is satisfiable, then \( \Sigma \) is satisfiable. An uncountable cardinal \( \kappa \) is a weakly compact cardinal, if and only if \( \mathcal{L}_{\kappa, \omega} \) satisfies the weak compactness theorem.

9.2. Proof of Theorem 9.1. Fix some \( \lambda < \mu \). For the sake of contradiction we assume that \( F : V_{G_1} \times V_{G_2} \to \{1, 2, ..., \lambda\} \) is a good coloring of \( G_1 \times G_2 \). For each color \( c \in \{1, 2, ..., \lambda\} \) and each vertex \( x \in V_{G_1} \) we let \( A_{x,c} = \{ y \in V_{G_2} : F(x, y) = c \} \). By well-ordering principle, we enumerate \( V_{G_1} \), as \( V_{G_1} = \{x_1, x_2, ...\} \).

Claim 9.2. For all \( F \subset V_{G_1} \) such that \( |F| < \kappa \), there is a mapping \( i_F : F \to \{1, 2, ..., \lambda\} \) such that for any \( x, x' \in F \) where \( x \neq x' \), \( A_{x, i_F(x)} \cap A_{x', i_F(x')} \) is not independent.

Proof. Since any superset of non-independent set is non-independent, it is enough to show that for all \( F \subset V_{G_1} \) such that \( |F| < \kappa \), there exists an \( i_F : F \to \{1, 2, ..., \lambda\} \) such that \( \cap_{x \in F} A_{x, i_F(x)} \) is not independent. For the sake of contradiction we assume that there exists a \( F \subset V_{G_1} \) where \( |F| < \kappa \) such that for all \( i_F : F \to \{1, 2, ..., \lambda\} \), \( \cap_{x \in F} A_{x, i_F(x)} \) is independent. Thus \( V_{G_2} \) can be written as a \( \lambda^{<\kappa} \) union of independent sets which contradicts the assumption that \( \chi(E_{G_2}) = \kappa \) since \( \kappa > \lambda^{<\kappa} \).

We work with the language \( \mathcal{L}_{\kappa, \omega} \) with the following sentence symbols.

\[ A'_{x,i,j} \text{ where } j \in \{1, 2, ..., \lambda\} \text{ and } x_i \in V_{G_1}. \]

We consider \( \Sigma \) to be the collection of the following well founded formulae of \( \mathcal{L}_{\kappa, \omega} \).

1. \( A'_{x,i,m} \land A'_{x,j,l} \) if \( A_{x,i,m} \cap A_{x,j,l} \) is not an independent set where \( x_i \neq x_j, x_i, x_j \in V_{G_1} \) and \( i, m \in \{1, 2, ..., \lambda\} \).
2. \( \neg (A'_{x,i,j} \land A'_{x',j,l}) \) for any \( i, j \in \{1, 2, ..., \lambda\} \) such that \( i \neq j \) and each \( x_i \in V_{G_1} \).
3. \( A'_{x_1} \lor A'_{x_2} \lor A'_{x_3} \land A'_{x_4} \land A'_{x_5} \) for each \( x_i \in V_{G_1} \).

Claim 9.3. If \( v \) is a truth assignment which satisfies \( \Sigma \), then we can define a mapping \( i : V_{G_1} \to \{1, 2, ..., \lambda\} \) such that the intersection of any two elements in \( \{A_{x,i(x)} : x \in V_{G_1}\} \) is not independent by

\[ i(x_i) = A'_{x,i,j} \text{ if and only if } v(A'_{x,i,j}) = T. \]

Proof. By (2) and (3) for each \( x_i \in V_{G_1} \), each collection \( S_{x_i} = \{A_{x_1,1}, ..., A_{x_1,\lambda}\} \) gets assigned a unique representative. By (1), for any \( x_i, x_j \in V_{G_1} \) such that \( x_i \neq x_j \), the representatives of \( S_{x_i} \) and \( S_{x_j} \) are such that the intersection of them is not independent.

Claim 9.4. Any subset \( \Sigma' \subseteq \Sigma \) such that \( |\Sigma'| < \kappa \) is satisfiable.

Proof. Given any subset \( \Sigma' \subseteq \Sigma \) such that \( |\Sigma'| < \kappa \), let \( F = \{x_{i_1}, ..., x_{i_\eta}\} \) be the vertices that are mentioned in \( \Sigma' \). By claim 9.2, there is a mapping \( i_F : F \to \{1, 2, ..., \lambda\} \) such that for any \( x, x' \in F \) where \( x \neq x' \), \( A_{x,i_F(x)} \cap A_{x', i_F(x')} \) is not independent. Let \( v_0 \) be a truth assignment such that for all \( 1 \leq r \leq \eta \) and \( x \in S_{x_{i_r}} = \{A_{x_{i_1},1}, ..., A_{x_{i_\eta},\lambda}\} \),

\[ v_0(A'_{x_{i_r},r}) = T \text{ if and only if } x = i_F(x_{i_r}). \]

Clearly, \( v_0 \) satisfies \( \Sigma' \).
Since \( \kappa \) is a weakly compact cardinal and \( |\Sigma| \leq \kappa \), \( L_{\kappa, \omega} \) satisfies the weak-compactness property. Thus, by claim 9.4, \( \Sigma \) is satisfiable. Thus, by claim 9.3 we can obtain an \( i : V_{G_1} \to \{1, \ldots, \lambda\} \) such that intersection of any two elements in \( \{A_{x,i(x)} : x \in V_{G_1}\} \) is not independent. Since \( \chi(E_{G_1}) = \mu > \lambda \), \( x \mapsto i(x) \) is not a good coloring in \( G_1 \). Thus there are \( x, x' \in V_{G_1} \) with \( i(x) = i(x') = j \) for some \( j \in \{1, \ldots, \lambda\} \) and \( \{x, x'\} \in E_{G_1} \). Consequently, \( A = A_{x,i(x)} \cap A_{x',i(x')} \) is not independent. Pick \( y, y' \in A \) joined by an edge in \( E_{G_2} \). Then \( (x, y) \) and \( (x', y') \) are joined in \( E_{G_1} \times E_{G_2} \) and get the same color \( j \) which is a contradiction to the fact that \( F \) is a good coloring of \( G_1 \times G_2 \).

10. (Appendix C): A List of forms

- (AC, Form 1 of \([HoRu98]\)). Every family of nonempty sets has a choice function.
- (UL, Form 90 of \([HoRu98]\)). Every linearly ordered set can be well-ordered.
- (UT(WO,WO,WO), Form 231 of \([HoRu98]\)). The union of a well-ordered collection of well-orderable sets is well-orderable.
- (Boolean Prime Ideal theorem (BPI), Form 14 of \([HoRu98]\)). Every non-trivial boolean algebra has a prime ideal.
- (Form 14M of \([HoRu98]\)). Robert Cowen’s generalization of König’s lemma \([Cow77]\).
- (Form 14AW of \([HoRu98]\)). The Compactness theorem for propositional logic.
- (Form 14Z of \([HoRu98]\)). Tychonoff’s Compactness theorem for families of finite spaces.
- (Ultrafilter lemma (UL), Form 14A of \([HoRu98]\)). Every proper filter over a set \( S \) can be extended to an ultrafilter.
- (Cowen-Engeler lemma, Form 14X of \([HoRu98]\)).
- (Form 14G of \([HoRu98]\)). De Bruijn–Erdős theorem for \( n \geq 3 \) colorings.
- (Rado’s selection lemma (RSL), Form 99 of \([HoRu98]\)).
- (\( AC_{fin} \), Form 62 of \([HoRu98]\)). Every set of non-empty finite sets has a choice function.
- (\( AC_{\omega} \), Form 10 of \([HoRu98]\)). Every denumerable set of non-empty finite sets has a choice function.
- (Form 10F of \([HoRu98]\)). König’s lemma.
- (Dependent choice (DC/\( DC_\omega \)), Form 43 of \([HoRu98]\)).
- (Ramsey’s theorem (RT), Form 17 of \([HoRu98]\)). For every infinite set \( X \) and for every partition of \( [X]^2 \), into two sets \( A \) and \( B \), there is an infinite subset \( Y \) of \( X \) such that either \( |Y|^2 \subseteq A \) or \( |Y|^2 \subseteq B \).
- (Chain/Antichain Principle (CAC), Form 212 of \([HoRu98]\)). Every infinite p.o.set has an infinite chain or an infinite antichain.
- (Form 64 of \([HoRu98]\)). There are no amorphous sets.
- (LT, Form 253 of \([HoRu98]\)). If \( \mathfrak{A} = (A, \mathcal{R}) \) is a non-trivial relational \( \mathcal{L} \)-structure over some language \( \mathcal{L} \), and \( \mathcal{U} \) be an ultrapower on a non-empty set \( I \), then the ultrapower \( \mathfrak{A}^I/\mathcal{U} \) and \( \mathfrak{A} \) are elementarily equivalent.

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