Slow-roll Inflation for Generalized Two-Field Lagrangians

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Abstract

We study the slow-roll regime of two field inflation, in which the two fields are also coupled through their kinetic terms. Such Lagrangians are motivated by particle physics and by scalar-tensor theories studied in the Einstein frame. We compute the power spectra of adiabatic and isocurvature perturbations on large scales to first order in the slow-roll parameters. We discuss the relevance of the extra coupling terms for the amplitude and indexes of the power spectra. Beyond the consistency condition which involves the amplitude of gravitational waves, additional relations may be found in particular models based on such Lagrangians: as an example, we find an additional general consistency condition in implicit form for Brans-Dicke theory in the Einstein frame.

1 Introduction

A period of inflation in the early Universe explains the origin of the large-scale structure by the evolution of initial, quantum vacuum fluctuations of matter (see [1] for a textbook review). In the simplest inflationary model the dynamics is driven by a single scalar field, whose quantum fluctuations produce a primordial, scale invariant spectrum for curvature perturbations. The amplitude of curvature perturbation remains constant on super-horizon

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scales until the time when the perturbation re-enters into the Hubble scale, when the universe is dominated by radiation or matter. As soon as more than one matter field is considered during inflation, isocurvature perturbations may arise among different component and may also affect the curvature perturbations on large scales. Isocurvature fluctuations arise naturally when two or more scalar fields slow-roll during inflation [2,3,4].

A useful formalism which splits the original two-field dynamics in a tangential and orthogonal (to the trajectory in phase space) basis has been developed [5]: this method results in a straightforward identification of curvature and isocurvature fluctuations at first order in perturbation theory, whose evolution equations are regular during inflation. This formalism has been further developed for general theories with two scalar fields [6,7,8,9].

In this paper we study the two-field slow-roll regime for theories described by the following action

\[
S = \int d^4x \sqrt{-g} \left[ \frac{R}{2} - \frac{g^{\mu\nu}}{2} \partial_{\mu}\varphi \partial_{\nu}\varphi - \frac{e^{2b\phi}}{2} g^{\mu\nu} \partial_{\mu}\chi \partial_{\nu}\chi - V(\varphi, \chi) \right]
\]  

where \(M_{\text{pl}} = 1/\sqrt{8\pi G}\) is the reduced Planck mass. The non-standard kinetic term for \(\chi\) appears in \(\sigma\)-model theories or in scalar-theories for gravity after a transformation to the conformal Einstein frame [12,13]. In a previous paper [3] the splitting of adiabatic and isocurvature perturbations for the action (1) was studied. It was found that the extra term generated by the coupling in kinetic term for \(\chi\), couples adiabatic and isocurvature modes for scales larger than Hubble radius also for scaling solutions [8]. A stronger correlation of adiabatic and isocurvature modes is therefore expected for \(b_\varphi \neq 0\).

It is interesting to investigate if there are generic predictions for inflationary models in which the dynamics is not driven by a single field [10]. For \(b_\varphi = 0 \ast\) in Eq. (1), the only model-independent prediction is a consistency relation among the tensor to scalar ratio, the gravity waves spectral index and the cross-correlation between curvature and isocurvature perturbations [11], which modifies the single field consistency relation. The present paper is devoted to the prediction of inflationary models based on the action (1) for \(b_\varphi \neq 0\).

The outline of the paper is as follows. In section II we review the formalism and the main equations obtained in [8] for the theory in Eq. (1). In section III we study the slow-roll approximation. In section IV we study the dynamics of curvature and isocurvature perturbations during inflation and in section V we give the final power spectra. In section VI we focus on some model dependent relations and in section VII we apply our results to

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\*The case with \(b \neq 0\) and \(b_\varphi = 0\) is trivial since this constant can be included in a redefinition of \(\chi\).
scalar-tensor theories studied in the Einstein frame. We conclude in section VIII.

2 Basic Equations

In this first section we shall review the equations of motion deriving from (1). Such equations can be also found in [8], but we feel to rewrite them here in order to make our paper self-contained. The equations of motion for the two homogeneous field are:

\[ \ddot{\phi} + 3H \dot{\phi} + V_\phi = b_\phi e^{2b} \dot{\chi}^2, \]
\[ \ddot{\chi} + (3H + 2b_\phi \dot{\phi}) \dot{\chi} + e^{-2b} V_\chi = 0, \]

and the Einstein equations are:

\[ H^2 = \frac{8\pi G}{3} \left[ \frac{1}{2} \dot{\phi}^2 + \frac{e^{2b}}{2} \dot{\chi}^2 + V \right], \]
\[ \dot{H} = -4\pi G \left[ \dot{\phi}^2 + e^{2b} \dot{\chi}^2 \right] \equiv -4\pi G \dot{\sigma}^2, \]

where the last equation is not independent from the others. The average and orthogonal fields are [3]:

\[ d\sigma = \cos \theta \, d\phi + \sin \theta \, e^b \, d\chi, \]
\[ ds = e^b \cos \theta \, d\chi - \sin \theta \, d\phi, \]

with:

\[ \cos \theta = \frac{\dot{\phi}}{\sqrt{\dot{\phi}^2 + e^{2b} \dot{\chi}^2}}, \]
\[ \sin \theta = \frac{e^b \dot{\chi}}{\sqrt{\dot{\phi}^2 + e^{2b} \dot{\chi}^2}}. \]

The average field \( \sigma \) and the angle \( \theta \) satisfy, respectively:

\[ \ddot{\sigma} + 3H \dot{\sigma} + V_\sigma = 0, \]
\[ \dot{\theta} = -\frac{V_s}{\dot{\sigma}} - b_\phi \dot{\sigma} \sin \theta, \]

where:

\[ V_\sigma = V_\phi \cos \theta + e^{-b} V_\chi \sin \theta, \]
\[ V_s = -V_\phi \sin \theta + e^{-b} V_\chi \cos \theta. \]

We then pass to equations for the fluctuations [3]. By using the longitudinal gauge for the metric fluctuations:

\[ ds^2 = -(1 + 2\Phi) dt^2 + a^2(1 - 2\Phi) dx^2, \]
the equation for the average Mukhanov variable $Q_\sigma = \delta \sigma + \dot{\Phi} \Phi$ is:

$$\ddot{Q}_\sigma + 3H \dot{Q}_\sigma + \left[ \frac{k^2}{a^2} + V_{\sigma \sigma} + \theta \frac{V_s}{\sigma} - \frac{1}{M_{pl}^2 a^3} \left( \frac{a^3 \dot{\phi}^2}{H} \right) - b_\phi \frac{V_\chi}{\sigma} e^{-b} \sin \theta \right] Q_\sigma = -2 \left( \frac{V_s}{\sigma} \delta s \right) + 2 \left( \frac{V_s}{\sigma} + \frac{\dot{H}}{H} \right) \frac{V_s}{\sigma} \delta s, \tag{14}$$

and for $\delta s$ we have:

$$\ddot{\delta s} + 3H \dot{\delta s} + \left[ \frac{k^2}{a^2} + V_{ss} + 3\dot{\phi}^2 - b_\phi \dot{\phi}^2 + b_\phi^2 g(t) + b_\phi f(t) \right] \delta s = -\frac{k^2 \Phi}{a^2 2\pi G \sigma^2}, \tag{15}$$

where:

$$g(t) = -\dot{\phi}^2 (1 + 3 \sin^2 \theta), \quad f(t) = V_\phi (1 + \sin^2 \theta) - 4V_s \sin \theta. \tag{16}$$

We note again that Eq. (14) has the correct single field limit and all the equations of this paragraph agree with those in [5] when $b_\phi = 0$.

3 Slow-Roll Expansion

Under the assumption of the slow-roll for both fields $\phi$ and $\chi$ the equations of motions at first-order are:

$$\dot{\phi} = \dot{\sigma} \cos \theta \simeq -\frac{V_\phi}{3H}, \quad \dot{\chi} = \dot{\sigma} \sin \theta e^{-b} \simeq -\frac{V_\chi}{3H} e^{-2b}, \tag{17}$$

$$H^2 (\varphi, \chi) \simeq \frac{8\pi G}{3} V(\varphi, \chi). \tag{18}$$

We note that we do not keep the viscous term $2b_\phi \dot{\phi} \dot{\chi}$ in Eq. (14) to lowest order in a slow-roll expansion. By defining the slow-roll parameters as:

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \simeq \frac{M_{pl}^2}{2} \left( \frac{V_\sigma}{V} \right)^2, \tag{19}$$

$$\epsilon_\varphi = \frac{M_{pl}^2}{2} \left( \frac{V_\varphi}{V} \right)^2 \simeq \epsilon \cos^2 \theta, \quad \epsilon_\chi = \frac{M_{pl}^2}{2} \left( \frac{V_\chi}{V} \right)^2 e^{-2b} \simeq \epsilon \sin^2 \theta, \tag{20}$$

$$\eta_{\varphi \varphi} = M_{pl}^2 \frac{V_{\varphi \varphi}}{V}, \quad \eta_{\varphi \chi} = M_{pl}^2 \frac{V_{\varphi \chi}}{V} e^{-b}, \quad \eta_{\chi \chi} = M_{pl}^2 \frac{V_{\chi \chi}}{V} e^{-2b}, \tag{21}$$

$$\epsilon_b = 8M_{pl}^2 b_\phi^2, \tag{22}$$
neglecting the other terms in the equations (2) and (3):

\[
\left\{ |\ddot{\varphi}|, |b_\varphi e^{2b}\chi^2| \right\} \ll \{ |V_\varphi|, 3|H\dot{\varphi}| \}, \\
\left\{ |\ddot{\chi}| e^{2b}, |b_\varphi \dot{\varphi} \chi| e^{2b} \right\} \ll \{ |V_\chi|, 3|H\dot{\varphi}| \},
\]

we have the conditions for the slow-roll: \( \epsilon_i \ll 1, \quad |\eta_{ij}| \ll 1 \forall i, j = \varphi, \chi \) and \( \epsilon_b \ll 1 \). These conditions for the parameters (20) and (21) are the generalization of the slow-roll conditions, and the one for \( \epsilon_b \) arises directly by requiring that \( \varphi \) and \( \chi \) slow-roll. Now we can extend the formalism of average and entropy field to the slow-roll parameters and we define:

\[
\eta_{\sigma\sigma} \equiv \eta_{\varphi\varphi} \cos^2 \theta + \eta_{\varphi\chi} \sin 2\theta + \eta_{\chi\chi} \sin^2 \theta = \frac{V_{\sigma\sigma}}{3H^2},
\]

\[
\eta_{\sigma s} \equiv (\eta_{\chi\chi} - \eta_{\varphi\varphi}) \sin \theta \cos \theta + \eta_{\varphi\chi} (\cos^2 \theta - \sin^2 \theta) = \frac{V_{\sigma s}}{3H^2},
\]

\[
\eta_{ss} \equiv \eta_{\varphi\varphi} \sin^2 \theta - \eta_{\varphi\chi} \sin 2\theta + \eta_{\chi\chi} \cos^2 \theta = \frac{V_{ss}}{3H^2}.
\]

With these slow-roll conditions:

\[
\epsilon_i \ll 1, \quad |\eta_{ij}| \ll 1 \forall i, j = \sigma, s, \quad \epsilon_b \ll 1,
\]

the background slow-roll solution is:

\[
\dot{\sigma}^2 \simeq \frac{2}{3} \epsilon V, \quad \frac{\dot{\theta}}{H} \simeq -\eta_{ss} + \frac{1}{2} \text{sign}(b_\varphi) \text{sign} \left( \frac{V_\chi}{V} \right) \sqrt{\epsilon_b \epsilon_\chi} \cos^2 \theta,
\]

\[
\frac{\ddot{\sigma}}{H \dot{\sigma}} \simeq \epsilon - \eta_{\sigma\sigma} + \frac{1}{2} \text{sign}(b_\varphi) \text{sign} \left( \frac{V_\chi}{V} \right) \sqrt{\epsilon_b \epsilon_\chi} \sin \theta \cos \theta,
\]

and the equations of motion for \( Q_\sigma \) and \( \delta s \) on large scales become:

\[
\dot{Q}_\sigma = AHQ_\sigma + BH\delta s,
\]

\[
\dot{\delta s} = HS\delta s,
\]

where:

\[
A(\epsilon_i, \eta_{ij}) = -\eta_{ss} + 2\epsilon + \frac{1}{2} \text{sign}(b_\varphi) \text{sign} \left( \frac{V_\chi}{V} \right) \sqrt{\epsilon_b \epsilon_\chi} \sin \theta \cos \theta,
\]

\[
B(\epsilon_i, \eta_{ij}) = -2\eta_{ss} - \text{sign}(b_\varphi) \text{sign} \left( \frac{V_\chi}{V} \right) \sqrt{\epsilon_b \epsilon_\chi} \sin^2 \theta
\]

\[
= 2 \frac{\dot{\theta}}{H} - \text{sign}(b_\varphi) \text{sign} \left( \frac{V_\chi}{V} \right) \sqrt{\epsilon_b \epsilon_\chi},
\]

\[
S(\epsilon_i, \eta_{ij}) = -\eta_{ss} - \frac{1}{2} \text{sign}(b_\varphi) \text{sign} \left( \frac{V_\varphi}{V} \right) \sqrt{\epsilon_b \epsilon_\varphi} (1 + \sin^2 \theta).
\]
On large scales the entropy field perturbations evolve independently of the adiabatic field, but in contrast to the $b_\varphi = 0$ case isocurvature perturbations do affect curvature perturbations also when $\eta_{\varphi s} = 0$ (or $\dot{\theta} = 0$). From Eqs. (34) it is clear that we need to know two more parameters, $\theta$ and $\epsilon_b$, with respect to the four needed in the case with $b_\varphi = 0$: this means that the original asymmetry between $\varphi$ and $\chi$ cannot be completely hidden by the diagonalization in $\sigma$ and $s$.

4 Evolution of Fluctuations During Inflation

Following [5] we write at Hubble crossing the amplitude of perturbations as:

$$Q_\sigma|_{(k=a_s H_* )} = \frac{H_s}{\sqrt{2k^3}} e_\sigma(k), \quad \delta s|_{(k=a_s H_*)} = \frac{H_s}{\sqrt{2k^3}} e_s(k),$$

(35)

where $H_s$ is the Hubble parameter evaluated at horizon crossing, and the random variables $e_\sigma(k)$ and $e_s(k)$ satisfy:

$$\langle e_I(k) \rangle = 0 \quad \text{and} \quad \langle e_I(k) \bar{e}_J(k') \rangle = \delta_{IJ} \delta(k-k') \{ I,J \} = \{ \sigma,s \}. \quad (36)$$

It is important to stress that the above Eqs. (35,36) imply that adiabatic and isocurvature fluctuations have same spectrum and amplitude and vanishing correlation at horizon crossing. At the end of this section we shall elaborate more on this assumption.

Integrating the (31) and supposing that the change of $S$ is negligible during inflation, after the substituing the second formula of (35) we find that the isocurvature perturbations evolve as:

$$\delta s(t) = \frac{H_s}{2k^3} e^{S(N_*-N(t))} e_s(k) \quad (37)$$

where $N_* = \int_{t_*}^{tF} H dt$ corresponds to the number of the e-folds between the horizon crossing and the end of inflation. The slow-roll solution for $Q_\sigma$ is:

$$Q_\sigma(t) = \frac{H_s}{2k^3} e^{A(N_*-N(t))} e_\sigma(k) + \frac{H_s}{2k^3} e^{S(N_*-N(t))} e_s(k) . \quad (38)$$

These formulae allow to calculate the power spectra:

$$\langle Q_\sigma(k) \bar{Q}_\sigma(k') \rangle = \frac{2\pi^2}{k^3} P_{Q_\sigma} \delta(k-k') , \quad \langle \delta s(k) \bar{\delta s}(k') \rangle = \frac{2\pi^2}{k^3} P_s \delta(k-k') , \quad (39)$$

and the equation:

$$\langle Q_\sigma \bar{\delta s} \rangle = \frac{2\pi^2}{k^3} C_{Q_\sigma s} \delta(k-k') \quad (40)$$

defines the correlation between the variables $Q_\sigma$ and $\delta s$. 
Now we would like to express the results in terms of curvature and isocurvature fluctuations, defined as:

\[ \zeta = H \frac{Q_\sigma}{\sigma}, \quad S = H \frac{\delta s}{\sigma}, \]

which are related by [8]:

\[ \dot{\zeta} = H \frac{k^2}{a^2} \Phi + \frac{2H}{\sigma} \dot{\delta s} + 2b_\varphi H \sin \theta \delta s \]
\[ - \frac{H k^2}{a^2} \Phi - \frac{2V_\varphi}{\sigma}, \]

and whose power spectra at Hubble crossing are given by:

\[ \mathcal{P}_\zeta |_* \simeq \mathcal{P}_S |_* \simeq \frac{1}{(2\pi)^2 \frac{H_*^4}{\sigma_*^2}}. \]

In terms of these quantities the relevant power spectra at the end of inflation are:

\[ \mathcal{P}_\zeta = \frac{H_*^2}{(2\pi)^2} \frac{1}{2M_{pl}^2 \epsilon_*} \left[ 1 + \left( \frac{B}{\gamma} \right)^2 (1 - e^{-\gamma N_*})^2 \right], \]
\[ \mathcal{P}_S = \frac{H_*^2}{(2\pi)^2} \frac{1}{2M_{pl}^2 \epsilon_*} e^{-2\gamma N_*}, \]
\[ \mathcal{P}_C = C_{\zeta S} = \frac{H_*^2}{(2\pi)^2} \frac{1}{2M_{pl}^2 \epsilon_*} \frac{B}{\gamma} e^{-2\gamma N_*} (e^{\gamma N_*} - 1) \]

where:

\[ \gamma = A - S. \]

We note that despite appearance the limit for \( \gamma \to 0 \) (in which isocurvature perturbations are not damped) is well defined. The power spectrum of gravitational waves is:

\[ \mathcal{P}_T = \mathcal{P}_T |_* = \frac{8}{M_{pl}^2 (2\pi)^2} \frac{H_*^2}{\epsilon_*} \]

and remains unchanged after horizon crossing and through the radiation era. The spectral indexes are defined as:

\[ n_m - 1 \equiv \frac{d \ln \mathcal{P}_m}{d \ln k}, \quad m = \zeta, S, C, \quad n_T \equiv \frac{d \ln \mathcal{P}_T}{d \ln k}. \]

Once evaluated at horizon crossing these spectral indexes are:

\[ n_\zeta - 1|_* = n_S - 1|_* = -6\epsilon_* + 2\eta_{\sigma\sigma*} - \text{sign}(b_\varphi) \text{sign} \left( \frac{V_\chi}{V} \right) \sqrt{\epsilon_{b*} \epsilon_{\chi*} \sin \theta_* \cos \theta_*}, \]
\[ n_T|_* = -2\epsilon_*, \]
and $P_C|_* = 0$, i.e. adiabatic and isocurvature fluctuations are considered uncorrelated at Hubble crossing as from Eqs. (35) (36).

Eqs. (35) (36) mean that the variables to quantize (or randomize in terms of classical numbers) are $Q_\sigma, \delta s$ and their spectra and amplitude are the same at horizon crossing, although their evolution equations are different and coupled, as is clear from Eqs. (14) (15). It is conceivable that this assumption may be violated in certain models [15] for $b_\phi = 0$. We therefore expect that a non-vanishing correlation at horizon crossing may be present also for $b_\phi \neq 0$. It is also conceivable to quantize the Mukhanov variables $Q_\varphi, Q_\chi$ (associated to $\varphi, \chi$, respectively) instead of $Q_\sigma, \delta s$. Since

$$Q_\sigma = \cos \theta Q_\varphi + \sin \theta e^{b_\chi} Q_\chi, \quad \delta s = - \sin \theta Q_\varphi + \cos \theta e^{b_\chi} Q_\chi,$$

by imposing Eq. (36) for $I, J = Q_\varphi, Q_\sigma$ one has

$$P_C|_* = \frac{H^2}{\sigma_*^2} \sin \theta_* \cos \theta_* \left( e^{2b_\chi} \langle Q^2_\chi \rangle - \langle Q^2_\varphi \rangle \right).$$

It is then clear that adiabatic and isocurvature modes are really uncorrelated at horizon crossing when one of the two field dominates also for $b_\phi \neq 0$. If Eq. (35) is used for $Q_\varphi$ and $\langle Q^2_{\chi, k} \rangle = H^2 e^{-2b_\chi}/(2k^3)$ in order to take into account the extra damping term in the equation of motion for $\chi$, the correlation is again zero also for the $b_\phi \neq 0$ case. When both fields are important at horizon crossing, a correlation may be present if the two Mukhanov variables do not have the same amplitude at horizon crossing up to rescaling.

5 After Inflation

The end of inflation depends strongly on the form of the potential $V$. As an example, let us focus on the case $V(\varphi, \chi) = e^{-\beta \varphi/M_{pl}} \tilde{V}(\chi)$, where $\beta > 0$.

The exact scaling solution found in [16] (and discussed in more detail in [8]) for $\tilde{V}$ independent on $\chi$ is a threshold regime between the domination of $\varphi$ and $\chi$. The field $\chi$ may end up not oscillating even when $\tilde{V}(\chi)$ is convex depending on $b(\varphi)$. In such a case inflation may be ended by instant preheating [17] or by symmetry breaking in a third field, like in hybrid models (but in this last case our formalism is not sufficient).

In the case in which inflation ends by $\chi$ oscillations ($\varphi$ may oscillate as well) one should study if parametric amplification of scalar perturbations occurs during the preheating phase [18] [19]. Indeed, the fact that both $\varphi$ and $\chi$ slow-roll during inflation guarantees that a mixture of adiabatic and isocurvature perturbations with similar infrared spectra is present at the end of inflation: this is one of the necessary conditions for having an amplification of curvature perturbations during preheating [19]. In the worst case (as for a quartic potential potential for the inflaton coupled to a second
field by a dimensionless parameter $g^2$ fluctuations may grow until the non-linear stage even on large scales and the scenario would not be compatible with our universe. This fine tuning for the inflaton coupling parameters leads to the conclusion that a simple quartic potential is under theoretical, and not only observational, pressure.

In a scenario compatible with observations fluctuations should remain small, although they may change drastically after inflation is ended. On large scales, adiabatic and isocurvature fluctuations evolve according to:

$$\dot{\zeta} = \alpha(t)H(t)S,$$

$$\dot{S} = \delta(t)H(t)S. \tag{54}$$

By integrating over time we can apply the formalism of the transfer matrix in order to study how the correlation between adiabatic and isocurvature perturbations builds up:

$$\begin{pmatrix} \zeta(t) \\ S(t) \end{pmatrix} = \begin{pmatrix} 1 & T_{\zeta S} \\ 0 & T_{SS} \end{pmatrix} \begin{pmatrix} \zeta(t_*) \\ S(t_*) \end{pmatrix}, \tag{55}$$

where:

$$T_{SS}(t_*,t) = \exp \left( \int_{t_*}^t \delta(t')H(t')dt' \right),$$

$$T_{\zeta S}(t_*,t) = \int_{t_*}^t \alpha(t')H(t')T_{SS}(t_*,t')dt'. \tag{56}$$

We note that we are implicitly assuming that $T_{\zeta S}, T_{SS}$ depend on $k$ only through $t_*$, the instant at which fluctuations leave the Hubble radius. Such assumption is useful also for models compatible with observations, in which fluctuations are amplified during (p)reheating, but in a $k$-independent way in the region $k \sim 0$ (where the fluctuations relevant for observations are located during (p)reheating). The power spectra are therefore:

$$P_\zeta = (1 + T_{\zeta S}^2)P_\zeta|_* = (1 + \cot^2 \Delta), \tag{57}$$

$$P_S = T_{SS}^2P_\zeta|_* \tag{58}$$

$$C_{\zeta S} = T_{\zeta S}T_{SS}P_\zeta|_* \tag{59}$$

where the measure of the correlation is introduced as the cross-correlation angle $\Delta$:

$$\cos \Delta = \frac{P_C}{\sqrt{P_\zeta P_S}} = \frac{T_{\zeta S}}{\sqrt{P_\zeta P_S}}; \tag{60}$$

$\Delta$ allows to reconstruct curvature perturbation spectrum at horizon crossing:

$$P_\zeta|_* \simeq P_\zeta \sin^2 \Delta. \tag{61}$$
We note that with the relation \( \tan \Delta = \gamma e^{\gamma N^*} / (B(e^{\gamma N^*} - 1)) \) Eqs. agree with Eqs. [15, 16, 17], the formalism of transfer functions may be also used during inflation leading to the correct results.

The spectral indexes defined as in Eqs. (49) are:

\[
\begin{align*}
n_{\zeta} - 1 & = -6\epsilon + 4\epsilon (\cos \Delta)^2 + 2\eta_{ss} (\sin \Delta)^2 + 4\eta_{s\sigma} \sin \Delta \cos \Delta \\
& + 2\eta_{ss} (\cos \Delta)^2 + 2 \text{sign}(b_{\varphi}) \text{sign} \left( \frac{V_{\chi}}{V} \right) \sqrt{\epsilon_b \epsilon_{\chi}} (\sin \theta)^2 \sin \Delta \cos \Delta \\
& + \text{sign}(b_{\varphi}) \text{sign} \left( \frac{V_{\varphi}}{V} \right) \sqrt{\epsilon_b \epsilon_{\varphi}} (1 + \sin^2 \theta) \cos^2 \Delta \\
& - \text{sign}(b_{\varphi}) \text{sign} \left( \frac{V_{\chi}}{V} \right) \sqrt{\epsilon_b \epsilon_{\chi}} \sin \theta \sin \theta \sin^2 \Delta , \\
\end{align*}
\]

\[
\begin{align*}
n_{S} - 1 & = -2\epsilon - 2S = -2\epsilon + 2\eta_{ss} + \text{sign}(b_{\varphi}) \text{sign} \left( \frac{V_{\varphi}}{V} \right) \sqrt{\epsilon_b \epsilon_{\varphi}} (1 + (\sin \theta)^2) , \\
\end{align*}
\]

\[
\begin{align*}
n_{C} - 1 & = -2\epsilon + 2\eta_{ss} + 2\eta_{s\sigma} \tan \Delta + \text{sign}(b_{\varphi}) \text{sign} \left( \frac{V_{\varphi}}{V} \right) \sqrt{\epsilon_b \epsilon_{\varphi}} [1 + (\sin \theta)^2] \\
& + \text{sign}(b_{\varphi}) \text{sign} \left( \frac{V_{\chi}}{V} \right) \sqrt{\epsilon_b \epsilon_{\chi}} (\sin \theta)^2 \tan \Delta \\
& = n_{S} - 1 + \left( 2\eta_{s\sigma} + \text{sign}(b_{\varphi}) \text{sign} \left( \frac{V_{\varphi}}{V} \right) \sqrt{\epsilon_b \epsilon_{\varphi}} (\sin \theta)^2 \right) \tan \Delta , \\
n_{T} & = -2\epsilon ,
\end{align*}
\]

(62)

where we have omitted to indicate that the slow-roll parameters are evaluated at Hubble crossing (as thereafter in the paper). The crucial assumption of explicit \( k \)-independence of the transfer functions has allowed to derive the final spectral indexes just in terms of the slow-roll parameters at Hubble crossing and the correlation angle \( \Delta \), since \( \alpha^* = B \) and \( \delta^* = -\gamma \).

As already said, the power spectrum of gravitational waves remains unchanged after horizon crossing and we find the consistency condition:

\[
\frac{P_T}{P_{\zeta}} = -8n_T \left( 1 - \frac{c_{T S}^2}{P_{\zeta} P_S} \right)
\]

(63)
as for the case of double inflation with \( b_{\varphi} = 0 \) [10, 11]. As expected this relation does not change for \( b_{\varphi} \neq 0 \), but it becomes an upper bound in presence of additional fields [11].

6 Model-Dependent Relations

The class of inflationary models studied here contains two more parameters than usual double inflation with \( b_{\varphi} = 0 \) [10, 11]. The amount of parameters is therefore 9: 6 inflationary parameters plus the Hubble scale during inflation plus two transfer functions. The number of input parameters for
observations always remain 8 to first order in slow-roll expansion: 4 spectra plus 4 spectral indexes. In a situation where the parameters are more than the "observables", the relation for gravitational waves which persists with respect to the \( b_\phi = 0 \) case - although expected - is a benefit. Therefore we need to look for model dependent relations.

We have already noticed in the introduction that isocurvature and adiabatic perturbations are coupled also for scaling solution, i. e. when \( \dot{\theta} = 0 \) (since \( V_\chi \) is not simply proportional to \( \dot{\theta} \)). This coupling is also evident in the definition of the \( B \) parameter in Eq. (33), which is not vanishing also for \( \eta_{\sigma s} \sim 0 \). Therefore, perturbations can be effectively decoupled when \( \varphi \) dominates, but never when \( \chi \) dominates. As a consequence, \( n_C \neq n_S \) despite \( \eta_{\sigma s} \sim 0 \).

We conclude this section observing that, for the curvaton case [23], where

\[
\sin \Delta \sim 0
\]

and the adiabatic perturbation at horizon crossing is negligible, we find that tensor perturbations are negligible (\( P_T \approx 0 \)) and a relation among the scalar indices:

\[
n_C \simeq n_C \simeq n_S = 1 - 2\epsilon + 2\eta_{\sigma s} + \text{sign}(b_\varphi)\text{sign} \left( \frac{V_\varphi}{V} \right) \sqrt{\epsilon_b \epsilon_\varphi} \left[ 1 + (\sin \theta)^2 \right]
\]

\[
= 1 - 2\epsilon + 2\eta_{\sigma s} + 2b_\varphi M_{\text{pl}} \sqrt{2\epsilon} \cos \theta \left[ 1 + (\sin \theta)^2 \right]
\]

which is qualitatively similar to the case with \( \epsilon_b = 0 \) - the three spectral indexes are the same - but quantitatively different. All the curvaton phenomenology is therefore changed if \( \epsilon_b \neq 0 \) and \( \varphi \) was not negligible during inflation.

7 Application to Scalar-Tensor Theories in the Einstein Frame

We now apply our study to the particular case of scalar-tensor theories studied in the Einstein frame. The analysis of the consistency conditions in the Jordan frame is in progress. For the case of a massless dilaton (\( \varphi \)) the potential in the action [11] is:

\[
V(\varphi, \chi) = e^{4b(\varphi)} U(\chi).
\]

The case of Brans-Dicke cosmology is obtained for \( b(\varphi) \propto \varphi \) [14] and will be discussed in Sec. (7.3). Starting from a scalar-tensor theory in the Jordan frame

\[
S = \int d^4x \sqrt{-g} \left[ \frac{\bar{R}}{16\pi} F(\phi) - G(\phi) \frac{\tilde{g}^{\mu \nu}}{16\pi} \partial_\mu \phi \partial_\nu \phi - \frac{\tilde{g}^{\mu \nu}}{2} \partial_\mu \chi \partial_\nu \chi - U(\chi) \right],
\]

(66)
the action \( \mathcal{L} \) with the potential \( V(\phi) \) is recovered by rewriting \( V(\phi) \) in the conformal frame with metric \( g_{\mu\nu} = \tilde{g}_{\mu\nu}(G\phi(\phi)) \) (the function \( b(\phi) \) which parametrizes the relation between the original dilaton \( \phi \) and its conformal one \( \varphi \)).

With respect to other multi-field inflationary theories, scalar-tensor cosmologies may follow a simple evolution: the inflaton \( \chi \) decays in matter fields after inflation and \( \varphi \) evolves coupled to the matter trace until the present time, determining the coupling constants. For the background Eqs. (2,3) are simply rewritten as:

\[
\ddot{\varphi} + 3H\dot{\varphi} = b_T \chi,
\]

\[
\dot{\rho}_\chi + 3H(\rho_\chi + p_\chi) = -b_T \dot{\varphi} T_\chi,
\]

where the \( \chi \) energy density is \( \rho_\chi = e^{2b} \chi^2/2 + e^{4b} U(\chi) \) and \( T_\chi = -\rho_\chi + 3p_\chi \) is the \( \chi \) trace. Adiabatic perturbations were studied for the case in Eq. (65) neglecting the correlation between adiabatic and isocurvature modes [24].

The first consequence of the choice (65) is the reduction from six to four independent slow-roll parameters:

\[
\epsilon_\varphi = 8M_{Pl}^2 b_\varphi^2, \tag{68}
\]

\[
\epsilon_\chi = \frac{M_{Pl}^2}{2} \left( \frac{U_\chi}{U} \right)^2 e^{-2b}, \tag{69}
\]

\[
\eta_{\varphi\phi} = 4M_{Pl}^2 (b_{\varphi\phi} + 4b_\phi^2) = 4M_{Pl}^2 b_{\varphi\phi} + 2\epsilon_\varphi, \tag{70}
\]

\[
\eta_{\chi\chi} = M_{Pl}^2 \left( \frac{U_{\chi\chi}}{U} \right) e^{-2b}, \tag{71}
\]

and two which are not independent:

\[
\epsilon_b = \epsilon_\varphi, \quad \eta_{\varphi\chi} = 2\text{sign}(b_\varphi)\text{sign}\left( \frac{U_\chi}{U} \right) \sqrt{\epsilon_\varphi\epsilon_\chi}. \tag{72}
\]

### 7.1 Extended inflation: \( \sin \theta \sim 0 \)

If we suppose that the evolution of the massless dilaton dominates, \( \sin \theta \ll \cos \theta \) (\( \epsilon_\chi \ll \epsilon_\varphi \)), we have:

\[
A = -4M_{Pl}^2 b_{\varphi\phi} + 2\epsilon_\chi, \quad B = -4\text{sign}(b_\varphi)\text{sign}\left( \frac{U_\chi}{U} \right) \sqrt{\epsilon_\varphi\epsilon_\chi}, \tag{73}
\]

\[
S = -\eta_{\chi\chi} - \frac{1}{2}\epsilon_\varphi. \tag{74}
\]

The isocurvature spectral index is:

\[
n_S - 1 = -\epsilon_\varphi + 2\eta_{\chi\chi}. \tag{75}
\]

When \( \cos \Delta = 0 \) the adiabatic spectrum coincides with Eq. (4.18) of [24], and the spectral index of the adiabatic perturbation

\[
n_\zeta - 1 = -2\epsilon_\varphi + 8M_{Pl}^2 b_{\varphi\phi} \tag{76}
\]

agrees with [24].
7.2 Chaotic inflation: $\cos \theta \sim 0$

If the inflaton $\chi$ dominates, $\cos \theta \ll \sin \theta$ ($\epsilon_\varphi \ll \epsilon_\chi$) and we have:

$$A = -\eta_{\chi\chi} + 2(\epsilon_\chi + \epsilon_\varphi), \quad B = 3\text{sign}(b_\varphi)\text{sign}\left(\frac{U_\chi}{U}\right)\sqrt{\epsilon_\varphi \epsilon_\chi}, \quad (77)$$

$$S = -3\epsilon_\varphi - 4M^2_\text{Pl}b_{\varphi\varphi}, \quad (78)$$

and the isocurvature index is:

$$n_S - 1 = -2\epsilon_\chi + 2\eta_{\phi\varphi} = -2\epsilon_\chi + 4\epsilon_\varphi + 8M^2_\text{Pl}b_{\varphi\varphi}. \quad (79)$$

In the absence of correlation ($B = 0$ and $\cos \Delta = 0$) the spectral index of adiabatic perturbation is:

$$n_\zeta - 1 = -6\epsilon_\chi + 2\eta_{\chi\chi} \quad (80)$$

which coincides with the Eq. (5.7) of [24] if one poses $\alpha_s = 0$ and corrects the factor $e^{2\alpha_s}$. This implies that the growth of the factor $b$ reduces the variation of the tilt.

7.3 Brans-Dicke Theory

The Brans-Dicke theory [14] was thoroughly investigated in the past [25, 24]: it can be obtained by $F(\phi) = \phi$ and $G(\phi) = \omega/\phi$, with $\omega$ as a $\phi$ independent parameter. In such way one obtains $2b(\varphi) = -\varphi/(\sqrt{\omega^2 + 3/2}M^2_\text{Pl})$. For $b(\varphi) \propto \varphi$ we have three slow-roll parameters since $\eta_{\varphi\varphi} = 2\epsilon_\varphi$. Therefore it is possible to make one prediction in addition to Eq. (63). We have checked that in the simplest case of a quadratic and quartic potential for $\chi$ there is no amplification during preheating †. It is interesting to note that for a quartic potential, $\varphi$ displays oscillations to leading order, too. Indeed, the $\chi$ background time average energy density redshift like radiation, but the oscillations in the $\chi$ trace drive $\varphi$. Any field coupled to the dilaton may then be excited by parametric resonance. However, we take $\Delta$ in order to take our results completely general.

We therefore obtain in implicit form the following consistency condition among the four spectral indexes and the correlation angle $\Delta$. An intermediate step is to give the relation for $\eta_{\chi\chi}$:

$$2\eta_{\chi\chi} = \frac{n_\zeta - 1}{\sin^2 \frac{\Delta}{2}} + n_S - 1 - \cot^2 \Delta(2n_C - n_S - 1) - \frac{n_T}{2}(8 - 5\cos^2 \theta), \quad (81)$$

where we have used $n_T = -2\epsilon$. Plugging this relation in the relation for the spectral indexes we get:

†During preheating the splitting used here in $Q_\varphi$ and $\delta$ is not so useful because the evolution equations become singular. It is more useful to use the Mukhanov variables associated to $\delta\varphi$ and $\delta\chi$ as in [19].
Figure 1: Evolution of $\chi(t)$ (top) and $\varphi(t)$ (bottom) for $U(\chi) = m^2\chi^2/2$ (left) and $U(\chi) = \lambda\chi^4/4$ (right). The fields are in $1/\sqrt{G}$ units and the $x$-axis is $mt$ for the massive case and $\sqrt{\lambda/G}t$ for the self-interacting case. After a slow-roll regime, $\varphi$ oscillates as well in the case of a self-interacting potential (for $\chi$ bottom panel, to the right).

\[
\begin{align*}
(n_s - 1)(\sin \Delta)^2(\sin \theta)^2 &= n_T(\sin \Delta)^2(1 - \cos^2 \theta \sin^2 \theta - 2 \cos^2 \theta) \\
+ (n_\zeta - 1) \cos^2 \theta - (\cos \Delta)^2(2n_c - n_s - 1) \cos^2 \theta \\
(n_c - n_s) \sin \Delta \cos \Delta &= [(n_\zeta + n_s - 1) - (\cos \Delta)^2(2n_c - 1)] \cos \theta \sin \theta \\
- (\sin \Delta)^2 \cos \theta \sin \theta [1 + \frac{n_T}{2}(3 + 2 \sin^2 \theta)].
\end{align*}
\]

(82)

As expected on the number of parameters, the latter system of equations in terms of $\theta$ is the additional consistency condition in implicit form. We note that for $\sin \Delta \sim 0$ the above relation is a subcase of Eq. (64).

We stress that Brans-Dicke theory - studied in the Einstein frame - is just an example of the possibility to have a consistency condition in addition to Eq. (64). Other physical models based on Einstein gravity may display the same interesting feature.

8 Conclusions

We have studied double field inflationary models based on the action (1). As already pointed out previously [8] the correlation of isocurvature and curvature perturbations is strengthened by the non-standard kinetic term for $\chi$. 
We have computed the power spectra for adiabatic and isocurvature perturbations to first order in the slow-roll parameters, taking into account their correlation which builds up after fluctuations leave the Hubble radius. Our approach was limited by three assumptions. First, we consider slow-roll parameters to lowest order, although there are several efforts to go beyond this approximation [26]: on the other hand, this approximation allows us to take into account the correlation between adiabatic and isocurvature perturbations to lowest order. Second, we consider adiabatic and isocurvature fluctuations uncorrelated at horizon crossing. Third, we have considered the transfer functions independent on $k$, apart from the dependence on the instant in which fluctuations leave the Hubble radius. Within these usual assumptions, our results are completely general.

The search for inflationary consistency conditions to first order in slow-roll parameters for theories based on Eq. (11) seems desperate. Indeed, the parameters are in general more than the ”observables”, and the consistency condition in Eq. (63) remains only because tensor and scalar modes are decoupled. However, for some physical models in which there are just three independent slow-roll parameters an additional prediction should be present: Brans-Dicke theories - studied in the Einstein frame - is just an example in which the consistency condition in Eq. (82) among the four spectral indexes and the correlation angle holds. On the basis of numbers of parameters, a consistency condition in addition to Eq. (82) is also expected in other class of scalar-tensor theories and may become an important theoretical predictions. Therefore, in some inflationary models based on string theory, spatial variation of the coupling constants (given by fluctuations in the dilaton) are correlated with density fluctuations in a way which is predictable.

It is important to note that the observational relevance of isocurvature perturbations of massless moduli is strictly connected to scalar-tensor theories to first order in perturbation theory. Indeed, for $b_{\varphi} = 0$ the background density of an uncoupled massless scalar is completely washed out after inflation: because of the $\dot{\varphi}$ fast redshift dependence ($\propto a^{-3}$), the observational relevance of such perturbation is washed out in the later evolution. On the opposite, with $b_{\varphi} \neq 0$, $\varphi$ is coupled both to the inflaton in the early universe and to non-relativistic matter, becoming a non-negligible ingredient of the primordial soup at late times.

We conclude by adding that dilaton isocurvature perturbations may be relevant for CMB and LSS observations, both in the massless or effectively massive case. Comparing to other nearly massless mode, such as radiation-quintessence [27, 28], dilaton isocurvature perturbations may be more relevant for observations because of the dilaton coupling to non-relativistic matter.
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