New rigid string instantons in $R^4$*

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Abstract

New rigid string instanton equations are derived. Contrary to standard case, the equations split into three families. Their solutions in $R^4$ are discussed and explicitly presented in some cases.

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0 Introduction

It is known that perturbatively rigid string [1, 2] is trivial in the sense that at low energies it is equivalent to the Nambu-Goto string. There is a hope that non-perturbative effects may change this behaviour. In fact, lattice simulations indicate appearance of a non-perturbative IR fixed point for the 3d rigid string [3]. For higher dimensional target spaces the situation is unclear. One of the non-perturbative signature of field theory models are instantons. Certain instanton equations of rigid string appeared for the first time in [1] and then their solutions and properties were discussed in [4, 5]. These instantons appeared to be non-compact surfaces in $\mathbb{R}^4$ and as we shall show they are somehow exceptional examples of more general instantons. One can also find some remarks about rigid string instantons in [6]. Despite these works not much have been established toward classifications of instantons and their relevance for string dynamics.

In this note we are going to investigate rigid string instantons in $\mathbb{R}^4$ more thoroughly. In particular we show that equations of [1] give instantons of very limited type - our construction yield much bigger family. The instantons are classified by two topological invariants: the Euler characteristic $\chi$ of the immersed (closed) Riemann surface $\Sigma$ and the self-intersection number $I$ of the immersion. The constructed set of instantons is rich enough to cover all possible values of $\chi$, $I$. It is interesting to note that, contrary to ordinary instantons, the rigid string instantons split into three families. The intersection of these families is non-trivial and, except one case, is equivalent to the instantons of [1].

Let us recall some basic facts about the rigidity (sometimes called extrinsic curvature). The action is given by (1) and it is known to be plagued with plethora of identities which allow to exhibit its different aspects.

$$\int_{\Sigma} \sqrt{g} g^{ab} \partial_a t^{\mu\nu} \partial_b t^{\mu\nu} = 2 \int_{\Sigma} \sqrt{g} K^b_a K^a_b = 2 \int_{\Sigma} \sqrt{g} (\Delta \vec{X})^2 - 8\pi \chi.$$ (1)

In the above $t^{\mu\nu} \equiv e^{ab} \partial_a X^\mu \partial_b X^\nu / \sqrt{g}$ is the element of the Grassman manifold $G_{4,2}$. Throughout the paper we shall exclusively use the induced metric $g_{ab} \equiv \partial_a \vec{X} \partial_b \vec{X}$. The tension tensor $K^i_{ab}$ is defined by the relation: $\partial_a \partial_b \vec{X} = \Gamma^c_{ab} \partial_c \vec{X} + K^i_{ab} \bar{N}_i$, where $\bar{N}_i$ ($i=1,2$) are two vectors normal to the immersed surface. The Euler characteristic of the Riemann surface $\Sigma$ is given by Gauss-Bonnet formula $\chi = \frac{1}{4\pi} \int_{\Sigma} \sqrt{g} R$. In the course of the paper we
shall heavily use identities expressing $\chi$ and the self-intersection number $I$ of an immersion in terms of $t^{\mu\nu}$.

\begin{align}
\chi &= \frac{1}{4\pi} \int_{\Sigma} \epsilon^{ab} \partial_a t^{\mu\nu} \partial_b t^{\mu\rho} t^{\nu\rho} \quad (2) \\
I &= -\frac{1}{16\pi} \int_{\Sigma} \sqrt{g} g^{ab} \partial_a t^{\mu\nu} \partial_b \tilde{t}^{\mu\nu} = \frac{1}{8\pi} \int_{\Sigma} \epsilon^{ab} \partial_a t^{\mu\nu} \partial_b t^{\mu\rho} \tilde{t}^{\nu\rho} \quad (3)
\end{align}

where $\tilde{t}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} t_{\rho\sigma}$.

The paper is organized as follows: in the first section we show that there is an infinite energy barrier between instantons belonging to different topological sectors of rigid string. This indicates the existence of instantons in each topological sector of the model. In Sec. 2 we derive basic equations, while in the next section we discuss their solutions. Finally we comment on other works devoted to the subject and state conclusions.

1 Energy barrier between different instantons

Before we go to the discussion of the instanton equations we shall show that any action containing the rigidity has a minimum in each topological sector given by the Euler characteristic $\chi$ of $\Sigma$ and the self-intersection number $I$ of the immersion $X$. The considerations are valid for compact surfaces only.

It is known that generic maps of a Riemann surface of genus $h$ to $R^4$ ($X: \Sigma \rightarrow R^4$) are immersion. Immersions of given $\Sigma$ are classified, up to regular homotopies, by the self-intersection number $I$ [9] (see also [10] for a brief review and some definitions). Hereafter we shall identify both topological numbers $\chi, I$ with analytical expressions (2) and (3), respectively. If so the genus $h$ of $\Sigma$ is not really an invariant of continuous deformations of $X$ but can acquire arbitrary values from metric singularities. Similar behaviour characterizes $I$ what can be inferred from the similarity of the expressions (2) and (3). In the following we shall discuss the latter case more thoroughly. We shall construct a continuous family of maps $X_\alpha$ which will connect two immersions with $I$ different by one. Thus the family will not be a regular homotopy. $X_\alpha$ must go through a singularity i.e. a point where the induced metric will vanish. We shall show that at this point the rigidity is infinite. An action with rigidity will separate different topological sector of field configurations. Hence there must exist a minimum of the rigidity for each $I$ and also for each $\chi$. 

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Any map \(X\) with given \((\chi, I)\) can be locally deformed, by a homotopy which is not regular, in such a way that \(I\) will change by one. For a certain value of the deformation parameter, say \(\alpha = 0\), the map \(X_{\alpha=0}\) ceases to be an immersion. The problem is to characterize singularities of \(X_\alpha\) under such deformations. We shall parameterize family \(X_\alpha\) by \(\alpha\) from neighborhood of zero \(\alpha \in D^1\). Because deformations are local instead of considering the whole Riemann surface \(\Sigma\) we take a 2d disc \(D^2 \subset \Sigma\). Thus the family of discs is a 3d manifold \(D^1 \times D^2\). Maps \(X_\alpha\) from \(D^2 \subset \Sigma\) to \(R^4\) will be constructed as a composition of two maps: \(X_\alpha = g \circ f_\alpha\), where \(f_\alpha: D^2 \to D^1 \times D^2\) and \(g: D^1 \times D^2 \to R^4\). The first map \(f\) must be non-singular i.e. it must be embedding because \(X_\alpha\) must be immersion for all \(\alpha \neq 0\).

In order to analyze singularities of \(X_\alpha\) we must consider maps \(\tilde{g}\) of \(D^2\) together with parameter space \(\alpha \in D^1\) into the 5-manifold \(D^1 \times R^4\). The requirement is that the parameter space is embedded into \(D^1\) of \(D^1 \times R^4\). Hence \(\partial_\alpha \tilde{g}\) is never zero. The generic singularities of such maps are well known [11] to be cross-caps which in suitable coordinate system have the form:

\[
\tilde{g}: (t_1, t_2, x) \to (t_1, t_2, t_1x, t_2x, x^2).
\] (4)

The map has the line of self-intersections \(\tilde{g}(0, 0, x) = \tilde{g}(0, 0, -x)\) which terminates at the singular point \(x = 0\). We must immerse family \((f_\alpha)\) of discs \(D^1 \times D^2 \to D^1 \times R^4\) in such a way that it intersects (in 2 points) the line \(t_1 = t_2 = 0\) for \(\alpha < 0\) and ceases to do it otherwise. Hence, for \(a < 0\), \(X_\alpha = g \circ f_\alpha\) is an immersion of \(D^2 \subset \Sigma\) in \(R^4\) with one self-intersection point. \(X_0 = g \circ f_0\) ceases to be an immersion because it goes through singularity point \((0, 0, 0)\) of \(\tilde{g}\).

As \(f_\alpha\) we consider a family of quadrics: \(f_\alpha(s, t) = \{s^2 + t^2 + \alpha, s, t\}\) in \(D^1 \times D^2\). It respects all requirement just imposed on the family of embedded surfaces. As \(g\) we take the last four components of the map (4) dropping the coordinate which corresponds to an embedding of the deformation parameter \((\alpha \sim t_1)\) in \(D^1\). Thus \(X_\alpha\) is:

\[
X_\alpha(s, t) = g \circ f_\alpha(s, t) = \{s, (s^2 + t^2 + \alpha)t, st, t^2\}
\] (5)

As we expected, at \(\alpha = 0\) the image of \(D^2\) under \(X\) is singular i.e. \(\partial X_0/\partial t = 0\) at \((s = t = 0)\). For \(\alpha \neq 0\) the map (5) is an immersion. For \(\alpha > 0\) it does not have self-intersection points \((I = 0)\). For \(\alpha < 0\) it has one self-intersection point: \(X_\alpha(s = 0, t = 0, t = \ldots)\).
\[ \sqrt{\alpha} = X_\alpha(s = 0, t = -\sqrt{\alpha}) \ (I = 1) \]. The above arguments show that \(5\) is the generic form of maps with the desired properties.

Now we calculate the rigidity for such a family of maps. The relevant formulae are \(\text{[8]}\). At \(\alpha = 0\) the density \(\sqrt{g} g^{ab} g_{cd} K_a^{ib} K_b^{ia}\) diverges as: \(4\pi/r^3 + O(1/r)\), where \(r\) is the polar coordinate on \(D^2 \subset \Sigma\). Existence of the singularity means that the rigidity tends to infinity at \(\alpha = 0\) i.e. when \(X\) ceases to be an immersion. Thus the rigidity separates configurations with different self-intersection number by an infinite barrier. Hence, we can expect a minimum of an action with the rigidity for each topological sector of the theory. This is the main conclusion of this part of the paper. Let us stress the local aspect of the considerations, what implies its validity for an arbitrary target space-time.

In the rest of the paper we shall be looking for these minima in terms of instantons. It will appear that in some cases the minima do not exist if we bound considerations to compact surfaces in \(R^4\).

2 Basic equations

In this section we shall derive instanton equations. We recall that the tensor \(t^{\mu\nu}\) is the (Gauss) map \(\Sigma \rightarrow G_{4,2}\), where \(\Sigma\) is the Riemann surface and \(G_{4,2} \equiv O(4)/(O(2) \times O(2)) = S^2 \times S^2\) is the Grassman manifold of planes in \(R^4\) \((\mu = 0, 1, 2, 3)\). The product structure of \(G_{4,2}\) is related to the fact that \(t^{\mu\nu}\) splits into self-dual (+) and anti-self-dual (−) parts: \(t^{\mu\nu}_\pm \equiv t^{\mu\nu} \pm \tilde{t}^{\mu\nu}\). Both tensors assume values in \(S^2\) due to \(t^{\mu\nu}_\pm t^{\mu\nu}_\pm = 4\). In order to simplify notation we introduce two vectors: \(n^i_\pm = t^{0i}_\pm (i = 1\ldots3)\) which parameterize all components of \(t^{\mu\nu}_\pm\) and respect \(\vec{n}^2_+ = \vec{n}^2_- = 1\). There are associated topological invariants \(I_\pm\) which classify homotopy classes of maps \(\Sigma \rightarrow G_{4,2}\). These are the degrees (winding numbers) of maps \(t^{\mu\nu}_\pm : \Sigma \rightarrow S^2\). Both topological invariants \(\text{[3]}\) can be expressed in terms of \(I_\pm\).

\[ \chi = I_+ - I_- \quad I = \frac{1}{2}(I_+ + I_-) \]  \hspace{1cm} (6)

where \(I_\pm = \frac{1}{8\pi} \int_\Sigma \epsilon^{ab} \partial_a n_\pm (\partial_b n_\pm \times \vec{n}_\pm)\). We also note another useful identity:

\[ I = -\frac{1}{16\pi} \int_\Sigma \sqrt{g} \epsilon^{ab} (\partial_a n_+ \partial_b n_+ - \partial_a n_- \partial_b n_-) \]  \hspace{1cm} (7)

\[ \text{[In the case under consideration, homotopy classes of maps } \Sigma \rightarrow G_{4,2} \text{ are classified by their degrees. This follows from the Pontryagin-Thom construction [3].]} \]
which stems from (3). Using (6) and (7) we get

\[ \int \Sigma \sqrt{g^a b} \partial_t \eta^\mu \partial_b t^{\mu \nu} = \int \Sigma \sqrt{g^a b} (\partial_a \eta_+ \partial_b \eta_+ + \partial_a \eta_- \partial_b \eta_-) \]
\[ = 2 \int \Sigma \sqrt{g^a b} \partial_a \eta_+ \partial_b \eta_+ + 16\pi I \]
\[ = 2 \int \Sigma \sqrt{g^a b} \partial_a \eta_- \partial_b \eta_- - 16\pi I \]

In order to derive instanton equations we follow the standard route. Let us write the inequalities:

\[ \int \Sigma \sqrt{g^a b} (\partial_a \eta_+ \pm \epsilon_a^c \partial_c \eta_+ \times \eta_+) (\partial_b \eta_+ \pm \epsilon_b^d \partial_d \eta_+ \times \eta_+) \geq 0 \]

which imply \( \int \Sigma \sqrt{g^a b} \partial_a \eta_+ \partial_b \eta_+ \geq 8\pi |I_+| \). They are saturated by the following instanton equations for the self-dual part of \( t^{\mu \nu} \) i.e. for \( \eta_+ \):

\[ (+, \pm) \equiv \partial_a \eta_+ \pm \frac{\epsilon_a^c}{\sqrt{g}} \partial_c \eta_+ \times \eta_+ = 0 \]

There is a twin set of instanton equations for \( \eta_- \) i.e. for the anti-self-dual part of \( t^{\mu \nu} \).

\[ (-, \pm) \equiv \partial_a \eta_- \pm \frac{\epsilon_a^c}{\sqrt{g}} \partial_c \eta_- \times \eta_- = 0 \]

As we shall see below, (12) and (13) are not independent equations. This is obvious if one notices that \( \eta_+ \) and \( \eta_- \) carry altogether the same degrees of freedom as \( X \). It follows that if one threatens \( \eta_+ \) and \( \eta_- \) as would be independent the so-called integrability conditions appears. These will be discussed in the end of the paper. Moreover one must realizes that metric also depends only on the same degrees of freedom.

Hereafter we shall discuss relations between Eqs. (12, 13). If (12) holds then \( I_+ \geq 0 \) for \((+, -) = 0 \) and \( I_+ \leq 0 \) for \((+, +) = 0 \). On the other hand if (13) holds then \( I_- \geq 0 \) for \((-,-) = 0 \) and \( I_- \leq 0 \) for \((-,+)= 0 \). Let us check when two instanton equations can be respected simultaneously. From (7) we get \( I = \frac{1}{2}(-|I_+| + |I_-|) \). Confronting with (8) we conclude that in this case \(|I_-| - I_- = I_+ + |I_-| \) must be respected. All possible solutions to this condition are listed below.

1. \( I_+ > 0 \) implies \( I_- < 0 \). Instanton equations: \((-,-) = 0 \), \((+,+) = 0 \). Below we shall show that, in fact \((-,-) = 0 \Leftrightarrow (+,+) = 0 \Leftrightarrow \Delta X^\mu = 0 \).
2. \( I_+ = 0 \) implies \( I_- \geq 0 \). Instanton equations are \((+, -) = (+, +) = 0\) and \((-,-) = 0\). Due to the first point we get \( \partial_a t^{\mu\nu}_+ = 0 \).

3. \( I_- = 0 \) implies \( I_+ \leq 0 \). Instanton equations \((-+, -) = (-, -) = 0\) and \((+, +) = 0\). Due to the first point we get \( \partial_a t^{\mu\nu}_- = 0 \).

4. \( I_+ < 0 \) implies \( I_- > 0 \). Instanton equations: \((+, -) = 0\), \((-+, +) = 0\). Due to (6) and \( \chi \leq 2 \), both equations can be respected simultaneously only for \( I_+ = 1, I_- = -1 \) (non self-intersecting sphere).

We rewrite \((+, +) = 0\) in terms of components of the stress tensor \( K^{ij}_{ab} \).

\[
K^{ij}_{ef} \left[ \delta_{ij} \left( -\frac{\epsilon^e_a \epsilon^f_b}{\sqrt{g}} + \frac{\epsilon^f_a \epsilon^e_b}{\sqrt{g}} \right) + \epsilon_{ij} \left( \delta^e_f \delta^f_a + \frac{\epsilon^e_a \epsilon^f_b}{g} \right) \right] = 0
\] (14)

The l.h.s. of (14) is equivalent to \( K^{i\alpha}_{a} = 0 \) i.e. to \( \Delta X^\mu = 0 \). Analogously one can show that \((-,-) = 0 \iff \Delta X^\mu = 0 \). Instantons respecting \( \Delta X^\mu = 0 \) are called minimal. Because the l.h.s. of (14) is non-negative, minimal instantons can not exist for Riemann surfaces of genus smaller than 2. There is one exception to this: torus with \( \chi = 0 \) - from (1) we get \( \partial_a t^{\mu\nu}_+ = 0 \) i.e. the “torus” is in fact degenerate to \( R^2 \).

We summarize this discussion noting that we obtained three families of instanton equations: \((+, -) = 0\), \((-+, +) = 0\), \((+, +) \equiv (-, -) = 0\). Instantons considered in \([4, 5]\) lies in the intersection of these families and corresponds to the equations \( \partial_a t^{\mu\nu}_+ = 0 \).

It is useful to construct a map of all possible instantons on the \((I, h)\) plane (here \( h \) is the genus of the Riemann surface \( \Sigma_h \)). Minimal instantons respect \( I_+ < 0 \) and \( I_- > 0 \) thus leads to inequality \( |I| \leq h - 1 \); \((+, -) = 0\) instantons respect \( I_+ \geq 0 \) and from (8) we get \( I + I_+ = 2I_+ + h - 1 \geq 0 \); \((-+, +) = 0\) instantons respect \( I_- \leq 0 \) and from (9) \( I + I_- = 2I_+ + 1 - h \leq 0 \). We also notice that instantons may exist for all possible \( \chi \) and \( I \). Fig.1 summarizes the relation between different type of instanton equations.

### 3 Solutions of instanton equations

Apparently the problem of solving Eqs. (12, 13) is very complicated. Despite this some results are known. On of the tools is the Gauss map of an immersion \([4]\). Let us recall some basic facts. The Gauss map of an immersion \( X : \Sigma \to R^4 \) is defined to be the map
Figure 1: Rigid string instantons on the \((I, h)\) plane. Minimal instantons, are denoted by empty circles, \((+, -) = 0\) instantons by '+'s, \((-,-) = 0\) instantons by '-'s, respectively. Instantons \(\partial_a \vec{n}_\pm = 0\) are denoted by full circles. The solution found in this paper corresponds to the ± point.

\[ G : \Sigma \rightarrow G_{2,4} = S^2 \times S^2 \text{ i.e. } G(z) \text{ gives tangent plane to the immersion at the point } X(z). \] For the conformal metric \(g_{ab} \propto \delta_{ab}\) one can identify \(G_{2,4}\) with a quadric in \(CP^3\):

\[ \sum_{\mu=1}^4 Z_\mu^2 = 0, \text{ where } Z_\mu \text{ are coordinates on } CP^3 \subset C^4. \] 

\(Z_\mu\) is \(\partial X\) up to a \(C\)-number function \(\Psi: \partial X^\mu = \Psi Z^\mu\). Unfortunately not every map \(G: \Sigma \rightarrow G_{2,4} = S^2 \times S^2\) can be a Gauss map of an immersion. The so-called integrability conditions have to be respected \([8]\). They originate from the fact that \(\bar{\partial}\partial X^\mu\) must be orthogonal to \(\partial X^\mu\) and real. Both conditions reads:

\[ \bar{\partial}\ln(\Psi) = -\frac{\bar{\partial}Z^\mu Z^\mu}{|Z|^2}, \quad \text{Im} \left[ \Psi \left( \bar{\partial}Z^\mu (\delta^{\mu\nu} - \frac{Z^\mu Z^\nu}{|Z|^2}) \right) \right] = 0 \] \hspace{1cm} (15)

There is a nice parameterization of \(Z\)

\[ Z = \{1 + f_+ f_-, i(1 - f_+ f_-), f_+ - f_-, -i(f_+ + f_-)\}, \] \hspace{1cm} (16)

where \(f_i: \Sigma \rightarrow S^2\). From Eq. (13) one can derive the following integrability conditions \([8]\):

\[ \frac{|\bar{\partial}f_+|}{1 + |f_+|^2} = \frac{|\bar{\partial}f_-|}{1 + |f_-|^2}, \] \hspace{1cm} (17)
\[
\text{Im} \left[ \partial \left( \frac{\partial \bar{f} + 2 \frac{\partial f + \bar{f}}{1 + |f|^2}}{1 + |f|^2} + \frac{\partial \bar{f} - 2 \frac{\partial f - \bar{f}}{1 + |f|^2}}{1 + |f|^2} \right) \right] = 0
\]  

Both conditions take relatively simple form when expressed in terms of \((+,+),(-,-)\): \(|(+,+)|= |(-,-)|\), \(dA = 0\) where \(A = [(+,+)\partial(+,+) + (-,-)\partial(-,-)]/(+)^2 \, dz + \text{c.c.}\). We see that the first integrability condition guarantee the equality (9)=(10). For minimal instantons the first condition is a tautology while the second one looks singular. There is a theorem [8] which says that while the integrability conditions (17,18) are solved for appropriately regular maps we can reconstruct the surface \(X\) up to a 4d shift and a scale.

It is worth to notice that the integrability conditions posses symmetry groups. First of all it is the rotation group \(SO(4) \sim SO_{+}(3) \times SO_{-}(3)\).

\[
f_{\pm} \to \frac{\alpha_\pm f_\pm + \beta_\pm}{-\beta_\pm f_\pm + \alpha_\pm}, \quad |\alpha_\pm|^2 + |\beta_\pm|^2 = 1, \quad \alpha_\pm, \beta_\pm \in C
\]  

Both conditions are also invariant under (restricted) conformal transformations performed on \(f_+\) and \(f_-\) simultaneously: \(f_{\pm}(z, \bar{z}) \to f_{\pm}(g(z), \bar{g}(z))\). This symmetry is the remnant of the reparameterization invariance of the original theory.

Below we shall shortly discuss solutions to the instantons equations and the above integrability conditions. In the parameterization (16):

\[
\vec{n}_+ = \frac{f_+ \bar{f}_+ - 1}{1 + |f_+|^2}, \quad i \frac{f_+ - \bar{f}_+}{1 + |f_+|^2}, \quad \frac{f_+ + \bar{f}_+}{1 + |f_+|^2}
\]

\[
\vec{n}_- = \frac{f_- \bar{f}_- - 1}{1 + |f_-|^2}, \quad i \frac{f_- - \bar{f}_-}{1 + |f_-|^2}, \quad \frac{f_- + \bar{f}_-}{1 + |f_-|^2}
\]  

so that we get

\((+, -) = 0 \Leftrightarrow \partial f_+ = 0, \quad (-,+) = 0 \Leftrightarrow \partial f_- = 0\)  

respectively. Hereafter we shall concentrate on the first of Eqs (21). It can be easily solved:

\[
f_+ = \eta_+ \prod_{j=1}^{I_+} \frac{\bar{z} - \bar{a}_j}{\bar{z} - \bar{b}_j}
\]  

We solve the integrability conditions for the \(I_+ = 1\) case. Using conformal invariance we can put \(f_+ = \bar{z}\). Next we choose the ansatz for \(f_-\): \(f_- = \frac{a_+ + b_+}{\bar{a}_+ + \bar{b}_+}, \quad ad - bc = 1, \quad a, b, c, d \in C\). This is equivalent to an assumption that both Eqs. (21) are respected. The second integrability
condition holds identically. The first integrability condition gives \( d = \bar{a}, \ c = -\bar{b} \) thus setting the solution on the \( SO_-(3) \) manifold. Hence the whole moduli space of solutions for \( f_\pm \) consists of one point (up to irrelevant rotations of space-time and reparameterizations of the world-sheet). From (15) we can determine \( \Psi : \Psi = i\lambda/|Z|^2, \ \lambda \in \mathbb{R} \). Integrating \( \partial X = \Psi Z \) we get the immersion \( X \)

\[
X - X_0 = \frac{\lambda}{1 + |z|^2} \{y, x, 1\} \tag{23}
\]

The above is the sphere \( (X^0 - X^0_0)^2 + (X^1 - X^1_0)^2 + (X^3 - X^3_0 - \lambda/2)^2 = \lambda^2/4, \ X^2 - X^2_0 = 0. \) The formula (23) gives 5-dimensional family of instantons. In the forthcoming paper [13] we show that this is really the most general instanton family with \( \chi = 2, I = 0. \)

Unfortunately for \( I_+ > 1 \) the situation is much more complicated. The simple ansatz \( f_− = \eta_− \prod_{j=1}^m \frac{z - z_0}{z - \bar{z}_j} \prod_{k=1}^{m'} \frac{z - \bar{z}'_k}{z - b_k} \) appeared to be too restrictive and we were not able to find any solutions to the integrability conditions. Definitely, different methods are required [13].

## 4 Final comments

Let us finally comment on other works concerning the rigid string instantons and state conclusions.

Certain instanton equations for rigid string were proposed in [1] and farther elaborated in [4, 5]. The considered equations were

\[
\partial_a \vec{n}_\pm = 0 \tag{24}
\]

One can see that they belong to the set Eqs. (12,13) restricted be the condition \( I_\pm = 0. \) Eq. (9) implies that for (24) instantons \( I = \pm \frac{1}{2} \chi \) holds, so e.g. for the torus the equations can describe only the standard \( (I = 0) \) immersion in \( \mathbb{R}^3. \) Moreover (24) implies also \( \Delta X^\mu = 0. \) Hence no compact surface can be immersed in \( \mathbb{R}^4 \) while (24) is respected.

In the present paper we have shown that, contrary to (24) general instanton equations (12,13) can have a representant for each value of \( \chi \) and \( I. \) Not all of them can have compact representant in non-compact space-time \( \mathbb{R}^4. \) Non-compactness of the space-time makes

\[\text{Surfaces constructed in [3] must be singular i.e. they are not immersions. By a direct computations one can find that their Euler characteristic is (except one case) greater then 2.}\]
some of the immersions to “run away” to infinity i.e. instantons become non-compact and hard to control. It is known that minimal instantons \((+, +) = 0\) can not exist in \(R^4\) \([12, 14]\). For \((+, -)\) and the twin \((-+, +)\) family we have found explicitly one compact instanton with topological numbers \(\chi = 2, I = 0\). In the forthcoming paper we shall show that, in fact, all these instantons are compact \([13]\).

We want to stress that despite this, the general arguments of Sec.1 shows that solutions to the instanton equations should exist for all possible topological sectors for compact space-times. This subject goes beyond the scope of this paper.

We finish with few remarks concerning possible applications of the rigid string instantons described in this paper. It is conceivable that they may play prominent role in string description of gauge fields. For example, it is known that YM2 in 1/N expansion is localized on surface-to-surface holomorphic and anti-holomorphic maps \([13]\) (see also \([16]\)). Four dimensional version of this construction was proposed in \([17, 10]\). Unfortunately, in this case no definite set of maps was given. One may speculate that the rigid string instantons should be the appropriate maps. Work in this direction is in progress.

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**References**

[1] A.M.Polyakov, Nucl. Phys. B268 (1986) 406.

[2] L. Peliti and S. Leibler, Phys. Rev. Lett. 54 (1985) 1690; D. Foerster, Phys. Lett. 114A (1986) 115; H. Kleinert, Phys. Lett. 114A (1986) 263, Phys. Lett. 174B (1987) 335.

[3] S.M.Catterall, Phys. Lett. 243B (1990) 121

J.Ambjørn, Z.Burda, J.Jurkiewicz and B.Petersson, Nucl. Phys. B393 (1993) 517.

[4] J.F.Wheater, Phys. Lett. 208B (1988) 388.

[5] G.D.Robertson, Phys. Lett. 226B (1989) 244.
[6] P.O.Mazur and V.P.Nair, Nucl. Phys. B 284 (1986) 146. K.S. Viswanathan, R.Parthasarathy and D.Kay, Ann. Phys. (N.Y.) 101 (1991) 237.

[7] G.E.Bredon, Topology and Geometry, Springer, 1993. B.A.Dubrovin, A.T.Fomenko and S.P.Novikov, Modern Geometry - Methods and Applications, vol.III, Springer-Verlag.

[8] D.A.Hoffman and R.Osserman, J.Diff.Geom. 18 (1983) 733; Proc. London. Math. Soc. (3) 50 (1985) 21.

[9] H.Whitney, “The self-intersections of a smooth n-manifold in 2n-space”, Ann.Math. 45 (1944) 220; R.Lashof and S.Smale, “On the immersion of manifolds in Euclidean space”, Ann.Math. 68 (1958) 562.

[10] J. Pawełczyk, ”Immersions and folds in string theories of gauge fields”, preprint UW-IFT-24/94, December 1994, hep-th/9604053, to be published in Int.J.Mod.Phys. A.

[11] B.Morin, Comptes Rendus Acad.Sci.,Paris 260 (1965) 5662; M.Golubitsky and V.Guillemin, Stable Mappings and Their Singularities, Springer-Verlag 1973.

[12] R.Osserman, A Survey of Minimal Surfaces, Van Nostrand Reinhold Math.Studies. No.25. 1969.

[13] work in preparation.

[14] Harmonic maps, Selected papers of J.Eells and collaborators, World-Scientific 1992.

[15] S.Corder, G.Moore and S.Ramgoolam, Yale preprint YCTP-P23-93, hep-th/9402101.

[16] P. Horava, EFI-93-66, hep-th/9311156. To appear in Proc. of The Cargese Workshop, 1993; “Topological Rigid String Theory and Two Dimensional QCD”, PUPT-1547, June 1995, hep-th/9507061. A.M.Polyakov, Lectures given at the CRM-CAP Summer School Particle and Fields ’94, August 16-24 1994, Banff, Alberta,Canada.

[17] J. Pawełczyk, Phys. Rev. Lett. 74, 3924 (1995), hep-th/9403175.