ON THE RELATIONSHIP BETWEEN RANK-$(n-1)$ CONVEXITY AND $S$-QUASICONVEXITY

MARIAPIA PALOMBARO

SISSA
Via Beirut 2-4, 34014 Trieste, Italy
Email: palombar@sissa.it

Abstract. We prove that rank-$(n-1)$ convexity does not imply $S$-quasiconvexity (i.e., quasiconvexity with respect to divergence free fields) in $M^{m \times n}$ for $m > n$, by adapting the well-known Šverák’s counterexample [5] to the solenoidal setting. On the other hand, we also remark that rank-$(n-1)$ convexity and $S$-quasiconvexity turn out to be equivalent in the space of $n \times n$ diagonal matrices. This follows by a generalization of Müller’s work [4].

Key words: $A$-quasiconvexity, lower semicontinuity, solenoidal fields.

2000 Mathematics Subject Classification: 49J45.

1. Introduction

The purpose of this note is to generalize some known results about the relationship between rank-one convexity and quasiconvexity to the context of divergence free fields. This is motivated by the lower semicontinuity results provided by Fonseca and Müller [2, Theorems 3.6-3.7]. Let us recall the relevant definitions. A function $f : M^{m \times n} \to \mathbb{R}$ on the $m \times n$ matrices is called rank-one convex if it is convex on each rank-one line, i.e., for every $A, Y \in M^{m \times n}$ with $\text{rank}(Y) = 1$, the function $t \to f(A + tY)$ is convex. It is quasiconvex if

$$\int_{T^n} f(A + \nabla \varphi) \, dx \geq f(A),$$

for all $A \in M^{m \times n}$ and for all $T^n$-periodic functions $\varphi \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$, where $T^n := (0,1)^n$. Quasiconvexity implies rank-one convexity. Whether the converse is true for $m = 2$ and $n \geq 2$ is an outstanding open problem. In the higher dimensional case $m \geq 3$, Šverák’s counterexample [5] shows that rank-one convexity is not the same as quasiconvexity. On the other hand, Müller [4] proved that the two notions are equivalent for $2 \times 2$ diagonal matrices. See also [1, 3] for further generalizations.

In the spirit of $A$-quasiconvexity (see, e.g., [2]), we provide in this note the counterpart of these results in the context of divergence free fields. The corresponding notion of quasiconvexity for solenoidal fields, that we call $S$-quasiconvexity, is defined as follows.
Definition 1.1. A function \( f : \mathbb{M}^{m \times n} \to \mathbb{R} \) is said to be \( S \)-quasiconvex if for each smooth \( \mathbb{T}^n \)-periodic matrix field \( B : \mathbb{R}^n \to \mathbb{M}^{m \times n} \) such that \( \text{Div}B = 0 \), the following inequality holds:

\[
\int_{\mathbb{T}^n} f(B) \, dx \geq f\left( \int_{\mathbb{T}^n} B \, dx \right).
\]

The symbol \( \text{Div} \) in the Definition 1.1 denotes the operator which acts as the divergence on each row of the matrix field \( B \). While quasiconvexity implies convexity along rank-one lines, it is easily checked that \( S \)-quasiconvexity implies convexity along rank-(\( n-1 \)) lines. Indeed, if a function \( f \) is \( S \)-quasiconvex, then \( t \to f(A + tV) \) is convex for every \( A, V \in \mathbb{M}^{m \times n} \) with rank\( (V) \leq n-1 \). Our aim in this note is to show that rank-(\( n-1 \)) convexity implies \( S \)-quasiconvexity in \( \mathbb{M}^{m \times n} \), for \( m \geq n+1 \geq 4 \). More precisely we prove the following result.

Theorem 1.2. For all \( n \geq 3 \) and \( m \geq n+1 \), there exists \( F : \mathbb{M}^{m \times n} \to \mathbb{R} \) such that \( F \) is rank-(\( n-1 \)) convex but not \( S \)-quasiconvex.

The proof of Theorem 1.2 is essentially based on the Šverák’s counterexample adapted to the solenoidal setting and is reminiscent of an example given by Tartar in connection with a theorem in compensated compactness (see [6], pp. 185–6).

We do not know whether \( S \)-quasiconvexity and rank-(\( n-1 \)) convexity are equivalent in the case when \( m = n \). Nevertheless Müller’s result on quasiconvexity on diagonal matrices extends as well to the divergence free fields. If we identify the space \( D(n) \) of diagonal \( n \times n \) matrices with \( \mathbb{R}^n \) via \( y \to \text{diag}(y_1, \ldots, y_n) \), then a rank-(\( n-1 \)) convex function on \( D(n) \) may be regarded as a function on \( \mathbb{R}^n \) which is convex on each hyperplane \( \{y_i = \text{const}\} \), \( i = 1, \ldots, n \). Then, a straightforward generalization of Theorem 1 in [4] (see also Theorem 1.5 in [3]) asserting that rank-one convexity implies quasiconvexity on diagonal matrices, leads to the following statement assuring that rank-(\( n-1 \)) convexity implies \( S \)-quasiconvexity on diagonal matrices.

Theorem 1.3. Let \( 1 < p < \infty \). Assume that \( f : \mathbb{R}^n \to \mathbb{R} \) is convex on each hyperplane \( \{y_i = \text{const}\} \), \( i = 1, \ldots, n \), and satisfy \( 0 \leq f(y) \leq C(1 + |y|^p) \). Suppose that

\[
\begin{align*}
u_h^i & \to u^i_\infty, & & \text{in } L^p_{\text{loc}}(\mathbb{R}^n) \text{ as } h \to \infty, & & i = 1, \ldots, n, \\
\partial_i u^i_h & \to \partial_i u^i_\infty, & & \text{in } W^{-1,p}_{\text{loc}}(\mathbb{R}^n) \text{ as } h \to \infty, & & i = 1, \ldots, n.
\end{align*}
\]

Then for every open set \( V \subset \mathbb{R}^n \)

\[
\int_V f(u^1_\infty, \ldots, u^n_\infty) \, dx \leq \liminf_{h \to \infty} \int_V f(u^1_h, \ldots, u^n_h) \, dx.
\]

We remark that, for \( n = 2 \), Theorem 1.3 reduces itself to Theorem 1 in [4] (in [4] the case \( n = p = 2 \) is considered; see Theorem 1.5 in [3] for generalization to any \( n \geq 2 \) and \( 1 < p < \infty \)). Indeed, in dimension two, the notion of \( S \)-quasiconvexity coincides with that of quasiconvexity since any divergence free field defines a gradient field upon left multiplication by the rotation \(
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\). Therefore Šverák’s example [5] shows that the conclusion of Theorem 1.2 already holds for \( n = 2 \).

Finally we recall that another result in the direction of Theorem 1.3 is that if \( f \) is a quadratic form and is rank-(\( n-1 \)) convex, then \( f \) is also \( S \)-quasiconvex (see [7]).

2. Proof of Theorem 1.2

Theorem 1.2 is a consequence of Lemma 2.2 and Corollary 2.3 below. We will basically follow Šverák’s strategy. The key idea is to find three rank-(\( n-1 \)) directions such that these directions
are the only rank-(\(n - 1\)) directions in the vector space spanned by them, which we call \(L\). Then one defines a rank-(\(n - 1\)) convex function on \(L\) and seeks a divergence free field that takes values only in \(L\) and for which the inequality (1.1) is violated. The desired function \(F\) is then obtained by suitably extending the rank-(\(n - 1\)) convex function defined on \(L\) to the whole space. We first construct an example in \(\mathbb{M}^{4 \times 3}\) and then we will extend it to \(\mathbb{M}^{m \times n}\). Let \(V_1, V_2, V_3 \in \mathbb{M}^{4 \times 3}\) be given by

\[
(2.1) \quad V_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.
\]

We consider the three-dimensional subspace of \(\mathbb{M}^{4 \times 3}\) generated by \(V_1, V_2, V_3\):

\[
(2.2) \quad L := \text{span}\{V_1, V_2, V_3\} = \{\eta_1 V_1 + \eta_2 V_2 + \eta_3 V_3, \eta_1, \eta_2, \eta_3 \in \mathbb{R}\},
\]

and we define the function \(f : L \to \mathbb{R}\) in the following way

\[
(2.3) \quad \forall \eta_1, \eta_2, \eta_3 \in \mathbb{R} \quad f(\eta_1 V_1 + \eta_2 V_2 + \eta_3 V_3) = -\eta_1 \eta_2 \eta_3.
\]

It can be checked that the only rank-two directions in \(L\) are given by \(V_1, V_2, V_3\) and therefore the function \(f\) is convex (in fact linear) on each rank-two line contained in \(L\).

**Lemma 2.1.** Let \(L\) and \(f\) be defined by (2.2) and (2.3), respectively and let \(P : \mathbb{M}^{4 \times 3} \to L\) be the orthogonal projection onto \(L\). Then for each \(\varepsilon > 0\) there exists \(k = k(\varepsilon) > 0\) such that the function \(F : \mathbb{M}^{4 \times 3} \to \mathbb{R}\) given by

\[
(2.4) \quad F(X) = f(PX) + \varepsilon|X|^2 + \varepsilon|X|^4 + k|X - PX|^2
\]

is rank-two convex on \(\mathbb{M}^{4 \times 3}\).

Lemma 2.1 is an obvious extension of Lemma 2 in [5] and therefore we refer the reader to [5] for its proof. We remark that an extension of the form (2.4) is always possible if \(V_1, V_2, V_3\) are any three rank-(\(n - 1\)) directions in \(\mathbb{M}^{m \times n}\) such that they are the only rank-(\(n - 1\)) directions in the subspace spanned by them and \(f\) is defined as in (2.3).

**Lemma 2.2.** There exist \(\varepsilon > 0\) and \(k > 0\) such that the function \(F\) given by (2.4) is rank-two convex but not \(S\)-quasiconvex.

**Proof.** Let \(B : \mathbb{T}^3 \to \mathbb{M}^{4 \times 3}\) be defined by

\[
B(x) = \begin{pmatrix}
\cos 2\pi x_3 & \cos 2\pi x_1 & 0 \\
0 & \cos 2\pi x_3 & 0 \\
0 & 0 & \cos 2\pi (x_1 - x_3)
\end{pmatrix}.
\]

It is readily seen that the matrix field \(B\) defined above is divergence-free and it satisfies

\[
\begin{cases}
B(x) \in L & \forall x \in \mathbb{R}^3, \\
\int_{\mathbb{T}^3} B \, dx = 0, \\
\int_{\mathbb{T}^3} f(B) \, dx = -\int_{\mathbb{T}^3} (\cos 2\pi x_1)^2 (\cos 2\pi x_3)^2 \, dx < 0.
\end{cases}
\]
Since $B$ is bounded, we can choose $\varepsilon > 0$ such that

\begin{equation}
\int_{\mathbb{T}^3} \left( f(B) + \varepsilon |B|^2 + \varepsilon |B|^4 \right) \, dx < 0.
\end{equation}

By Lemma 2.2 there exists $k = k(\varepsilon)$ such that the function

$$F(X) = f(PX) + \varepsilon |X|^2 + \varepsilon |X|^4 + k |X - PX|^2$$

is rank-two convex. Since $|B(x) - PB(x)| = 0$ for all $x$ in $\mathbb{R}^3$, we have from (2.5)

$$\int_{\mathbb{T}^3} F(B(x)) \, dx < 0,$$

which concludes the proof. \qed

**Corollary 2.3.** For all $n > 3$ and $m \geq n + 1$, there exists $F^{(n)} : M^{m \times n} \rightarrow \mathbb{R}$ such that $F$ is rank-$(n - 1)$ convex but not $S$-quasiconvex in $L^p(\mathbb{R}^n, M^{m \times n})$, for every $p \geq 1$.

**Proof.** We show how to adapt the counterexample constructed in Lemma 2.2 to an arbitrary dimension $n$. Since one can always increase the number of rows by adding some zeros while preserving the rank of the matrices, it is enough to consider the case when $m = n + 1$. In this situation we will exhibit three matrices $V_1^{(n)}, V_2^{(n)}, V_3^{(n)}$ which satisfy the following properties

\begin{equation}
\text{rank}(V_i^{(n)}) \leq n - 1 \quad \forall i = 1, 2, 3,
\end{equation}

\begin{equation}
\text{rank}(\alpha V_1^{(n)} + \alpha_2 V_2^{(n)} + \alpha_3 V_3^{(n)}) = n \quad \forall \alpha \in S^2 \setminus \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}.
\end{equation}

For each $\alpha$ in $S^2$ we set

$$M^{(n)}(\alpha) := \alpha V_1^{(n)} + \alpha_2 V_2^{(n)} + \alpha_3 V_3^{(n)}.$$ 

We first consider the case when $n = 4$. We define $V_1^{(4)}, V_2^{(4)}, V_3^{(4)}$ as follows

\begin{equation}
V_1^{(4)} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad V_2^{(4)} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad V_3^{(4)} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix}.
\end{equation}

We have that $\text{rank}(V_1^{(4)}) = \text{rank}(V_3^{(4)}) = 3$ and $\text{rank}(V_2^{(4)}) = 2$. In order to see that the condition (2.7) is satisfied it is convenient to write the explicit formula for $M^{(4)}(\alpha)$:

$$M^{(4)}(\alpha) = \begin{pmatrix}
\alpha_1 & \alpha_2 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 \\
0 & \alpha_3 & \alpha_2 & 0 \\
\alpha_3 & \alpha_3 & \alpha_3 & \alpha_1 \\
0 & 0 & 0 & \alpha_3
\end{pmatrix}.$$ 

Observe that the $4 \times 3$ minor of $M^{(4)}(\alpha)$ which is obtained eliminating the fifth row and the fourth column is a linear combination of the matrices $V_1, V_2, V_3$ defined by (2.1). Then using the fact that $V_1, V_2, V_3$ satisfy (2.1) for $n = 3$, one easily checks that $\text{rank}(M^{(4)}(\alpha)) = 4$. Remark that replacing the entry $M_{4,4}^{(4)}(\alpha) = \alpha_1$ by $M_{4,4}^{(4)}(\alpha) = \alpha_2$ would give another possible choice of $V_1^{(4)}, V_2^{(4)}, V_3^{(4)}$.
For $n = 5$ we choose $V_1^{(5)}, V_2^{(5)}, V_3^{(5)}$ such that $M^{(5)}(\alpha)$ is given by

$$M^{(5)}(\alpha) = \begin{pmatrix}
\alpha_1 & \alpha_2 & 0 & 0 & 0 \\
0 & \alpha_1 & 0 & 0 & 0 \\
0 & \alpha_3 & \alpha_2 & 0 & 0 \\
\alpha_3 & \alpha_3 & \alpha_3 & a_{44} & 0 \\
0 & 0 & 0 & \alpha_3 & a_{55} \\
0 & 0 & 0 & 0 & \alpha_3
\end{pmatrix}$$

where $a_{ii}$ can be chosen in the set $\{\alpha_1, \alpha_2\}$. Proceeding in a similar way, for every $n \geq 4$ we define the matrix $M^{(n)}(\alpha)$ such that

$$M^{(n)}(\alpha) \in M^{(n+1)\times n},$$

$$M_{i,j}^{(n)}(\alpha) = M_{i,j}^{(n-1)}(\alpha) \quad \text{for } i \leq n, j \leq n - 1,$$

$$M_{n,n}^{(n)}(\alpha) = a_{nn} \quad \text{where } a_{nn} \in \{\alpha_1, \alpha_2\},$$

$$M_{n+1,n}^{(n)}(\alpha) = \alpha_3,$$

$$M_{i,j}^{(n)}(\alpha) = 0 \quad \text{otherwise}.$$

By construction we have that rank$(M^{(n)}(\alpha)) = n$ for all $\alpha \in \mathbb{S}^2 \setminus \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}$. For each $n$ we set $L^{(n)} := \text{span}\{V_1^{(n)}, V_2^{(n)}, V_3^{(n)}\}$ and we define the function $f^{(n)} : L^{(n)} \to \mathbb{R}$ as in (2.4), i.e., $f^{(n)}(\eta_1 V_1^{(n)} + \eta_2 V_2^{(n)} + \eta_3 V_3^{(n)}) := -\eta_1 \eta_2 \eta_3$. The sought function $F^{(n)}$ is then defined as in (2.4) with $f^{(n)}$ in the place of $f$ and with $P$ the orthogonal projection onto $L^{(n)}$. Considering the divergence free field $B^{(n)}(x) := \cos(2\pi x_3) V_1^{(n)} + \cos(2\pi x_1) V_2^{(n)} + \cos 2\pi(x_1 - x_3) V_3^{(n)}$, we see that $F^{(n)}$ is not $S$-quasiconvex.

\[ \square \]

Acknowledgements

I am grateful to Enzo Nesi for drawing my attention to this problem and for fruitful discussions. This work was done during my post-doc at the Max Planck Institute for Mathematics in the Sciences in Leipzig.

References

[1] Chaudhuri N., Müller S.: Rank-one convexity implies quasi-convexity on certain hypersurfaces. Proc. Roy. Soc. Edinburgh Sect. A 133 (2003), no.6, 1263–1272.

[2] Fonseca I., Müller S.: A-quasiconvexity, lower semicontinuity, and Young measures. SIAM J. Math. Anal. 30 (1999), no. 6, 1355–1390 (electronic).

[3] Lee J., Müller P.F.X., Müller S.: Compensated compactness, separately convex functions and interpolatory estimates between Riesz transforms and Haar projections. Comm. PDE, accepted (2008). Preprint 7 (2008), Max Planck Institute for Mathematics in the Sciences, Leipzig.

[4] Müller S.: Rank-one convexity implies quasiconvexity on diagonal matrices. Int. Math. Research Not. 1999, 1087–1095.

[5] Šverák V.: Rank-one convexity does not imply quasiconvexity. Proc. Roy. Soc. Edinburgh Sect. A 120 (1992), no. 1-2, 185–189.

[6] Tartar L.: Compensated compactness and applications to partial differential equations, in Nonlinear analysis and mechanics: Heriot-Watt Symposium IV, Pitman Research Notes in Mathematics 39, 136–212. (London: Pitman, 1979.)
[7] Tartar L.: Estimations fines des coefficients homogénéisés, in Ennio De Giorgi’s Colloquium (Paris 1983), ed. P. Kree, 168–187. (Boston: Pitman, 1985.)