Two non existence results for the self-similar equation in Euclidean 3-space

Henri Anciaux*
email: henri.anciaux@staff.ittralee.ie

Abstract

We prove that the only self-similar surfaces of Euclidean 3-space which are foliated by circles are the self-similar surfaces of revolution discovered by S. Angenent and that the only ruled, self-similar surfaces are the cylinders over planar self-similar curves.

2000 MSC: 53C44

Introduction

The Mean Curvature Flow (denoted by MCF in the following) is the gradient flow of the area functional on the space of $n$-submanifolds of some Riemannian manifold. From the viewpoint of analysis, this flow is governed by a non-linear parabolic equation. Although classical results of analysis show short-time existence of the MCF, understanding its long-time behaviour is a hard problem which requires to control the possible singularities that may appear along the flow.

Self-similar flows arise as special solutions of the MCF that preserve the shape of the evolving submanifold. Analytically speaking, this amounts to making a particular Ansatz in the parabolic PDE describing the flow in order to eliminate the time variable and reduce the equation to an elliptic one.

The simplest and most important example of a self-similar flow is when the evolution is a homothety. Such a self-similar submanifold $X$ with mean curvature vector $\vec{H}$ satisfies the following non-linear, elliptic system:

$$\vec{H} + \lambda X^\perp = 0,$$
where $X^\perp$ stands for the projection of the position vector $X$ onto the normal space. If $\lambda$ is any strictly positive constant, the submanifold shrinks in finite time to a single point under the action of the MCF, its shape remaining unchanged. If $\lambda$ is strictly negative, the submanifold will expand, its shape again remaining the same; in this case the submanifold is necessarily non-compact. The case of vanishing $\lambda$ is the well-known case of a minimal submanifold, which of course is stationary under the action of the flow. The first case is of particular importance because at certain types of singularity the MCF is asymptotically self-shrinking.

Before stating our own results, we mention some work that has been done on the subject: in [AbLa], all self-shrinking planar curves where classified; in particular, the only simple self-shrinking curves are the round circles. In [Ang], the existence of non-spherical self-similar hypersurfaces of revolution in $\mathbb{R}^n$ were shown; in [An], we described rotationally symmetric Lagrangian self-shrinkers and self-expanders in $\mathbb{R}^{2n}$. Very recently, a wider class of self-similar Lagrangian submanifolds has been derived in [JLT]. On the other hand, few classification results have been obtained so far. It was shown in [ACR] and [AR] that the only Lagrangian self-similar submanifolds of $\mathbb{R}^{2n}$ which are foliated by $(n-1)$-dimensional spheres are the examples found in [An]; in another direction spherical self-shrinkers have been characterized in [Sm].

In this note we give a characterization of the only self-similar surfaces of $\mathbb{R}^3$ known until now: we first prove that the self-similar surfaces of revolution discovered by S. Angenent in [Ang] are the only cyclic self-similar surfaces (Theorem 1), and next that the cylinders over planar self-similar curves are the only ruled self-similar surfaces (Theorem 2).

1 The self-similar equation in coordinates

Let $X : U \to \mathbb{R}^3$ a local parametrization of some surface $\Sigma$. We denote by $E, F$ and $G$ the coefficients of the first fundamental form of $\Sigma$:

$$E = |X_s|^2 \quad F = \langle X_s, X_t \rangle \quad G = |X_t|^2.$$

Let $N$ be the unit normal vector given by $N = \frac{X \times X_t}{|X \times X_t|}$. Here and in the remainder of the section, $\times$ denotes the canonical vectorial product of $\mathbb{R}^3$. The coefficients of the second fundamental form are defined to be:

$$e = \langle X_{ss}, N \rangle \quad f = \langle X_{st}, N \rangle \quad g = \langle X_{tt}, N \rangle.$$
In order to simplify further calculations, we introduce the following coefficients, which are proportional to the previous ones:

\[
\bar{e} = \langle X_{ss}, X_s \times X_t \rangle \quad \bar{f} = \langle X_{st}, X_s \times X_t \rangle \quad \bar{g} = \langle X_{tt}, X_s \times X_t \rangle.
\]

Rather than the classical formula for the mean curvature,

\[
2H = \frac{eG + gE - 2fF}{EG - F^2},
\]

it will be more convenient to use the following one:

\[
(1) \quad 2H = \frac{\bar{e}G + \bar{g}E - 2\bar{f}F}{(EG - F^2)^{3/2}}.
\]

In codimension one, the self-similar equation \( \vec{H} + \lambda X^\perp = 0 \) becomes scalar, namely: \( H + \lambda \langle X, N \rangle = 0 \). Moreover, in \( \mathbb{R}^3 \) have:

\[
(2) \quad \langle X, N \rangle = \frac{1}{\sqrt{EG - F^2}} \langle X, X_s \times X_t \rangle = \frac{1}{\sqrt{EG - F^2}} \det(X, X_s, X_t).
\]

Finally, from Equations (1) and (2) we deduce:

**Lemma 1** A surface of \( \mathbb{R}^3 \) is self-similar if and only if, for any local parametrization \( X : U \to \mathbb{R}^3 \) of \( \Sigma \), the following formula holds:

\[
(3) \quad \bar{e}G + \bar{g}E - 2\bar{f}F + 2\lambda(EG - F^2) \det(X, X_s, X_t) = 0.
\]

## 2 Cyclic surfaces in \( \mathbb{R}^3 \)

**Theorem 1** Let \( \Sigma \) be a self-similar (non minimal) cyclic surface in \( \mathbb{R}^3 \). Then either \( \Sigma \) is a round sphere or a surface of revolution described by S. Angenent (cf \( \text{[Ang]} \)).

**Lemma 2** Let \( \Sigma \) be a self-similar cyclic surface in \( \mathbb{R}^3 \). Then the circles of the foliation are parallel or is a piece of a round sphere.

**Proof of Lemma 2.** The proof is by contradiction and is based on a method due to J. Nitsche (cf \( \text{[Ni1],[Ni2],[Ta]} \)). Let \( C(s) \) be a one-parameter family of circles, \( R(s) \) its radius and \( \vec{t}(s) \) the unit normal vector to \( C(s) \). There exists some space curve \( \gamma(s) \) whose unit tangent vector is \( \vec{t}(s) \). Moreover, if the circles are not parallel, the curve \( \gamma \) is not a straight line, so its curvature \( k(s) \) does not vanish, except in a discrete set of points. Away from those
points, let \((\vec{t}(s), \vec{n}(s), \vec{b}(s))\) be the Frénet frame related to \(\gamma(s)\). Finally, let \(z(s)\) be the center of the circle \(C(s)\). Hence, the corresponding cyclic surface is locally parametrized by

\[
X : \quad I \times \mathbb{S}^1 \to \mathbb{R}^3 \\
(s, t) \mapsto R(\vec{n} \cos t + \vec{b} \sin t) + z.
\]

Following the notation of [Ni1], we define \((\alpha, \beta, \gamma)\) to be the coordinates of \(z'(s)\) in the Frénet frame \((\vec{t}, \vec{n}, \vec{b})\). A long calculation (cf [Ni1],[Ta]) shows that \(\ddot{e}G + \dot{g}E - 2\dot{f}F\) is a trigonometric polynomial in the variable \(t\) whose linearization takes the form:

\[
\ddot{e}G + \dot{g}E - 2\dot{f}F = \sum_{j=0}^{3} a_j \cos(jt) + b_j \sin(jt).
\]

For later convenience, we write the explicit expression of certain of its coefficients:

\[
\begin{align*}
a_3 &= -\frac{R^3 k}{2}(k^2 R^2 + \beta^2 - \gamma^2), \\
b_3 &= -k R^3 \beta \gamma, \\
a_2 &= \frac{R^3}{2}(5 \alpha k^2 R + \beta' k R - \beta k' R - 6 \beta k R'), \\
b_2 &= \frac{R^3}{2}((\gamma' k R - \gamma k' R - 6 \gamma k R').
\end{align*}
\]

We shall now compute the term \((EG - F^2) \det(X, X_s, X_t)\). Firstly we define \((p, q, r)\) to be the coordinates of \(z(s)\) in the Frénet frame \((\vec{t}, \vec{n}, \vec{b})\). By deriving the relation \(z = p\vec{t} + q\vec{n} + r\vec{b}\), we get

\[
\begin{cases}
\alpha = p' - kq, \\
\beta = q' + pk - \tau r, \\
\gamma = r' + \tau q.
\end{cases}
\]

It follows that the coordinates of \(X\) in the frame \((\vec{t}, \vec{n}, \vec{b})\) are \((p, q + R \cos t, r + R \sin t)\). We also have the following expressions for the first derivatives of the immersion:

\[
X_s = (\alpha - kR \cos t, \beta + R' \cos t + \tau R \sin t, \gamma - \tau R \cos t + R' \sin t),
\]

\[
X_t = (0, -R \sin t, R \cos t).
\]

Next we calculate:

\[
\det(X, X_s, X_t) = \begin{vmatrix}
p & \alpha - kR \cos t & 0 \\
q + R \cos t & \beta + R' \cos t + \tau R \sin t & -R \sin t \\
r + R \sin t & \gamma + R' \sin t - \tau R \cos t & R \cos t
\end{vmatrix}
\]
R = q as a linear system in the variables $p$ and $r$.

Thus

$$R = R \left( \begin{array}{c|cc}
 p & \alpha - kR \cos t & 0 \\
 q + R \cos t & \beta + R' \cos t & -R \sin t \\
 r + R \sin t & \gamma + R' \sin t & R \cos t \\
\end{array} \right) + (kR \cos t - \alpha) \left( \begin{array}{c|c}
 q + R \cos t & -R \sin t \\
 r + R \sin t & \cos t \\
\end{array} \right)$$

The next step is the computation of the first fundamental form:

$$E = \alpha^2 + \beta^2 + \gamma^2 + (R')^2 + R^2 \tau^2 + (2R' \beta - 2R\alpha k - 2R\gamma \tau) \cos t,$$

$$F = (2R\alpha \tau + 2R' \gamma) \sin t + R^2 k^2 \cos^2 t,$$

$$G = R(-R\tau - \beta \sin t + \gamma \cos t).$$

Thus

$$\frac{EG - F^2}{R^2} = (R^2 k^2 - \gamma^2) \cos^2 t - \beta^2 \sin^2 t + 2\beta \gamma \cos t \sin t$$

$$+ 2(R' \beta - R\alpha k) \cos t + 2R' \gamma \sin t + (\alpha^2 + \beta^2 + \gamma^2 + (R')^2)$$

$$= \frac{1}{2} \left( R^2 k^2 - \gamma^2 + \beta^2 \right) \cos 2t + \beta \gamma \sin 2t + 2(R' \beta - R\alpha k) \cos t$$

$$+ 2R' \gamma \sin t + \left( \frac{1}{2} R^2 k^2 + \alpha^2 + \frac{3}{2} \beta^2 + \frac{1}{2} \gamma^2 + (R')^2 \right).$$

We deduce that $R^{-3}(EG - F^2) \det(X_s, X_t)$ is also a trigonometric polynomial of order 4, whose linearization take the form:

$$R^{-3}(EG - F^2) \det(X, X_s, X_t) = \sum_{j=0}^{4} a'_j \cos(jt) + b'_j \sin(jt).$$

It follows from Equation 3 that $a'_4$ and $b'_4$ must vanish, which can be viewed as a linear system in the variables $q$ and $r$:

$$\begin{cases}
a'_4 = k \left( \frac{1}{2} (R^2 k^2 - \gamma^2 + \beta^2) q + \beta \gamma \right) = 0 \\
b'_4 = k \left( -\beta q + \frac{1}{2} (R^2 k^2 - \gamma^2 + \beta^2) r \right) = 0.
\end{cases}$$
We deduce that either (i) \( q \) and \( r \) vanish, (which in turn implies the vanishing of \( \gamma \)), or (ii) the determinant \( \frac{1}{4}(R^2k^2-\gamma^2+\beta^2)^2+\beta^2\gamma^2 \) of the system vanishes, and then both \( R^2k^2-\gamma^2+\beta^2 \) and \( \beta\gamma \) vanish, so in particular \( \beta \) or \( \gamma \) must vanish. But the vanishing of \( \gamma \) would imply the vanishing of \( R^2k^2+\beta^2 \), a contradiction (since \( R > 0 \) and \( k > 0 \)). So in the second case, \( \beta \) vanishes.

**First case:** \( q = r = \gamma = 0 \).

We first observe that here \( X(s, t) = R(\vec{n}\cos t + \vec{b}\sin t) + p\vec{t} \) so that \( |X(s, t)|^2 = R^2 + p^2 \). Next, using the fact that \( \beta = pk \) and \( \alpha = p' \), we get simpler expressions for the following:

\[
\det(X, X_s, X_t) \frac{R}{R^2} = (p\beta + kR^2) \cos t + pR' - \alpha R = k(p^2 + R^2) \cos t + pR' - p'R.
\]

\[
\frac{EG - F^2}{R^2} = \frac{1}{2}(R^2k^2+\beta^2) \cos 2t + 2(R'\beta - R\alpha k) \cos t + \left( \frac{1}{2}R^2k^2 + \alpha^2 + \frac{3}{2}\beta^2 + (R')^2 \right).
\]

\[
= \frac{1}{2}k^2(R^2+p^2) \cos 2t + 2k(R'p - Rp') \cos t + \left( \frac{1}{2}R^2k^2 + (p')^2 + \frac{3}{2}p^2k^2 + (R')^2 \right).
\]

It follows that:

\[
a'_3 = \frac{1}{4}(R^2k^2 + \beta^2)(p\beta + kR^2) = \frac{1}{4}k^3(R^2 + p^2)^2.
\]

On the other hand

\[
a_3 = -\frac{R^3k^3}{2}(k^2R^2 + (kp)^2) = -\frac{R^3k^3}{2}(R^2 + p^2).
\]

From Lemma 1, we have

\[
\sum_{j=0}^{4} a_j \cos(jt) + b_j \sin(jt) + 2\lambda R^3 \left( \sum_{j=0}^{4} a'_j \cos(jt) + b'_j \sin(jt) \right) = 0,
\]

so that \( \lambda = -\frac{a_3}{2R^3a'_4} = \frac{2}{R^2+p'} = \frac{2}{|X|^2} \). It implies that \( |X| \) is constant, thus the surface is a piece of a sphere.

**Second case:** \( \beta = 0 \) and \( R^2k^2 = \gamma^2 \).

In this case \( \gamma = \pm Rk \) does not vanish and we have:

\[
\det(X, X_s, X_t) \frac{R}{R} = \frac{1}{2}kRq \cos 2t + \frac{1}{2}kRr \sin 2t + (-\alpha q + kR^2) \cos t + (p\gamma - \alpha' r) \sin t + \frac{1}{2}kRq + pR' - \alpha R,
\]
\[
\frac{EG - F^2}{R^2} = -2Rak \cos t + 2R' \gamma \sin t + (\alpha^2 + \gamma^2 + (R')^2).
\]

As \(a_3\) and \(b_3\) vanish,
\[
\begin{cases}
    a'_3 = Rk(-qRok - R'\gamma) &= 0, \\
    b'_3 = Rk(qR'\gamma - rRok) &= 0.
\end{cases}
\]

we deduce that either both \(Rak\) and \(R'\gamma\) vanish, or \(q\) and \(r\) vanish. We can discard the second case because \(\gamma = r' + \tau q = 0\), a contradiction. Thus, using the fact that \(\alpha\) and \(R'\) vanish, we have
\[
\det(X, X_s, X_t) \cdot R = \frac{1}{2} kRq \cos 2t + \frac{1}{2} kRr \sin 2t + kR^2 \cos t + p\gamma \sin t + \frac{1}{2} kRq,
\]
\[
\frac{EG - F^2}{R^2} = \gamma^2,
\]
so that \(a'_2 = \frac{1}{2}kRq\gamma^2\) and \(b'_2 = \frac{1}{2}kRr\gamma^2\). On the other hand, \(a_2\) and \(b_2\) vanish, so by Lemma 1 \(a'_2\) and \(b'_2\) must vanish as well; again we get a contradiction since it implies the vanishing of \(q\) and \(r\).

**Proof of Theorem 1.**

By Lemma 2 we know that the circles of a (non spherical) self-similar cyclic surface must be parallel. Without loss of generality, we may assume that they are horizontal. Thus the surface may be locally parametrized by an immersion of the form:
\[
X : I \times S^1 \to \mathbb{R}^3, \quad (s, t) \mapsto (a(s) + R(s) \cos t, b(s) + R(s) \sin t, s).
\]

We compute
\[
X_s = (a' + R' \cos t, b' + R' \sin t, 1),
\]
\[
X_t = (-R \sin t, R \cos t, 0),
\]
from which we deduce the coefficients of the first fundamental form:
\[
E = (a')^2 + (b')^2 + (R')^2 + 1 + 2R'(a' \cos t + b' \sin t),
\]
\[
F = R(b' \cos t - a' \sin t),
\]
\[
G = R^2.
\]

We now compute the second derivatives of the immersion:
\[
X_{ss} = (a'' + R'' \cos t, b'' + R'' \sin t, 0),
\]
\[ X_{st} = (-R' \sin t, R' \cos t, 0), \]
\[ X_{tt} = (-R \cos t, R \sin t, 0), \]
from which we deduce
\[ \bar{e} = \det(X_{ss}X_s, X_t) = R(-R'' - a'' \cos t + b'' \sin t), \]
\[ \bar{f} = \det(X_{st}, X_s, X_t) = 0, \]
\[ \bar{g} = \det(X_{tt}, X_s, X_t) = R^2. \]

Finally, we compute
\[ \det(X, X_s, X_t) = RR' s - R^2 + R(a's - a) \cos t + R(b's - b) \sin t. \]

We are now in position to write Equation (3) as a trigonometric polynomial. There are no terms of order 3 in \( \bar{e}G + \bar{g}E - 2\bar{f}F \), and a straightforward computation shows that the coefficients in \( \cos 3t \) and \( \sin 3t \) of \( (EG - F^2) \det(X, X_s, X_t) \) are respectively \( (a's - a)((a')^2 + (b')^2) \) and \( (b's - b)((a')^2 + (b')^2) \), up to a multiplicative constant. It follows that either \( a' \) and \( b' \) vanish, or \( a's - a \) and \( b's - b \) vanish. The first case is the case of the surfaces of revolution, which has been treated by S. Angenent in [Ang]. If \( a's - a \) and \( b's - b \) vanish, we deduce that \( a(s) = a_0 s \) and \( b(s) = b_0 s \), for some constants \( a_0 \) and \( b_0 \). It implies the vanishing of \( a'' \) and \( b'' \) and thus the expression \( \bar{e}G + \bar{g}E - 2\bar{f}F \) becomes a polynomial of degree 1. Moreover, the coefficients in \( \cos 2t \) and \( \sin 2t \) of \( (EG - F^2) \det(X, X_s, X_t) \) are respectively \( (R's - sR)((a_0)^2 - (b_0)^2) \) and \( (R's - sR)a_0b_0 \). Again there are two cases: either \( R's - sR \) vanishes, or \( a_0 \) and \( b_0 \) vanish. If both \( a_0 \) and \( b_0 \) vanish, we fall back again on the case of surfaces of revolution. On the another hand, if \( R's - sR \) vanishes, so does \( \det(X, X_s, X_t) \), thus \( \bar{e}G + \bar{g}E - 2\bar{f}F \) must vanish as well, which means that the immersion is minimal. Therefore a self-similar cyclic surface must be of revolution and the proof is complete.

3 Ruled surfaces in \( \mathbb{R}^3 \)

**Theorem 2** Let \( \Sigma \) be a self-similar ruled surface in \( \mathbb{R}^3 \). Then \( \Sigma \) is a cylinder over a self-similar planar curve.

**Proof.** A ruled surface of \( \mathbb{R}^3 \) may be locally parametrized by an immersion of the form
\[
X : I \times \mathbb{R} \rightarrow \mathbb{R}^3
\]
\[
(s, t) \mapsto \gamma(s)t + p(s),
\]
where \( |\gamma(s)| = 1 \) and \( \langle p(s), \gamma(s) \rangle = 0 \).

Our discussion being local, we divide the problem in two cases: either the rulings are parallel, in which case \( \gamma(s) \) is constant, or they are not, and then \( \gamma(s) \) is a regular curve in \( \mathbb{S}^2 \). The easy task of checking that if the rulings are parallel, then the ruled surface is a cylinder over a self-similar planar curve is left to the Reader. Such curves have been classified by Abresch and Langer (cf [AbLa]). Hence, we assume from now on that \( \gamma(s) \) is a regular spherical curve and that \( s \) is its arclength parameter. It follows that \((e_1, e_2, e_3) := (\gamma, \gamma', \gamma \times \gamma')\) is an orthonormal frame. Denoting by \( k(s) = \langle \gamma''(s), \gamma(s) \times \gamma'(s) \rangle \) the curvature of \( \gamma \) in \( \mathbb{S}^2 \), we can write the Frénet equations as follows:

\[
\begin{align*}
e_1' &= e_2, \\
e_2' &= ke_3 - e_1, \\
e_3' &= -ke_2.
\end{align*}
\]

Introducing the coordinates of \( p \) in the frame \((e_1, e_2, e_3)\), i.e. \( p = ae_2 + be_3 \), we get

\[
p' = -ae_1 + (a' - kb)e_2 + (b' + ka)e_3.
\]

We now compute the first derivatives of the immersion:

\[
X_s = \gamma' t + p' \quad X_t = \gamma,
\]
from which we deduce the coefficients of the first fundamental form:

\[
E = t^2 + 2t \langle \gamma', p' \rangle + |p'|^2 = t^2 + 2(a' - kb)t + a^2 + (a' - kb)^2 + (b' + ka)^2
\]

\[
F = \langle \gamma, p' \rangle = a \quad G = 1
\]

\[
EG - F^2 = t^2 + 2(a' - kb)t + (a' - kb)^2 + (b' + ka)^2.
\]

From the second derivatives of the immersion,

\[
X_{ss} = \gamma'' t + p'' \quad X_{st} = \gamma' \quad X_{tt} = 0,
\]
we deduce:

\[
\bar{e} = \det(\gamma'' t + p'', \gamma' t + p', \gamma) = t^2 \det(\gamma'', \gamma', \gamma) + t \left( \det(\gamma'', p', \gamma) + \det(p'', \gamma', \gamma) \right) + \det(p'', p', \gamma)
\]

\[
= kt^2 + t \left( -k(a' - kb) + \det(p'', \gamma', \gamma) \right) + \det(p'', p', \gamma),
\]

\[
\bar{f} = \det(\gamma', \gamma' t + p', \gamma) = \det(\gamma', p', \gamma) = -(b' + ka),
\]

\[
\bar{g} = 0.
\]

Finally, we calculate:

\[
\det(X, X_s, X_t) = \det(\gamma t + p, \gamma' t + p', \gamma)
\]

9
\[
= \det(p, \gamma t + p', \gamma) \\
= t \det(p, \gamma', \gamma) + \det(p, p', \gamma) \\
= bt + ab' - ba' + k(a^2 + b^2).
\]

From Lemma 1 we deduce that the immersion is self-similar if and only if the following vanishes:

\[
\bar{e} G + \bar{g} E - 2f F + 2\lambda(EG - F^2) \det(X, X_s, X_t) = 0
\]

\[
\iff k t^2 + t[-k(a' - kb) + \det(p'', \gamma', \gamma)] + \det(p'', p', \gamma) - 2\bar{f} F + 2\lambda(t^2 + 2(a' - kb)t + (a' - kb)^2 + (b' + ka)^2)(bt + ab' - ba' + k(a^2 + b^2)) = 0.
\]

This is a polynomial in \( t \) whose coefficient in \( t^3 \) is \( 2\lambda b \). So \( b \) must vanish and we get

\[
k t^2 + \det(p'', \gamma', \gamma) + \det(p'', p', \gamma) + 2\lambda[t^2 + 2a't + (a')^2 + (ka)^2]ka^2 = 0.
\]

Now the coefficient in \( t^2 \) is \( k + 2\lambda ka^2 \), so either \( k \) vanishes, or \( \lambda < 0 \) and \( a \) is a non-vanishing constant. If the curvature vanishes, \( \gamma \) is a great circle of \( S^2 \). As \( p = a\gamma' \), it follows that the \( X(s, t) = \gamma(s) + a(s)\gamma'(s) \) so the image of \( X \) lies in the span of \( \gamma \) and \( \gamma' \) and therefore is a piece of a plane. If \( a \) is a constant, using the fact that \( p'' = -a(1 + k^2)e_2 - ak'e_3 \), we deduce that

\[
\det(p'', \gamma', \gamma) = ak',
\]

\[
\det(p'', p', \gamma) = -a^2k(1 + k^2).
\]

Hence the self-similar equation is reduced to

\[
k t^2 + ak't - a^2k(1 + k^2) + 2\lambda(t^2 + (ka)^2)ak = 0.
\]

The coefficient in \( t \) is \( ak' \), therefore \( k \) is constant. Finally the constant term in the above expression is \( a^2k(1 + k^2 + 2\lambda ka^2) = a^2k(1 + k^2 + 2\lambda a^2) = a^2k \). Again, we get the vanishing of \( k \), hence the surface is a piece of a plane.

References

[AbLa] U. Abresch, J. Langer, The normalized curve shortening flow and homothetic solutions, J. of Diff. Geom. 23 (1986), 175–196

[ACR] H. Anciaux, I. Castro, P. Romon, Lagrangian submanifolds of \( \mathbb{R}^{2n} \) which are foliated by spheres, Acta Math. Sinica (English Series), 22 (2006) no. 4, 1197–1214.
[An] H. Anciaux, *Construction of equivariant self-similar solutions to the mean curvature flow in $\mathbb{C}^n$*, Geom. Dedicata, 120 (2006), no. 1, 37–48

[AR] H. Anciaux, P. Romon, *Cyclic and ruled Lagrangian surfaces in complex Euclidean space* math.DG/0703645

[Ang] S. Angenent, *Shrinking donuts*, in *Nonlinear diffusion reaction equations & their equilibrium, States 3*, editor N.G. Lloyd, Birkhäuser, Boston, 1992

[JLT] D. Joyce, Y.-I. Lee, M.-P. Tsui, *Self-similar solutions and translating solitons for Lagrangian mean curvature flow*, arXiv:0801.3721

[Ni1] J. Nitsche, *Lectures on minimal surfaces. Vol. 1* Cambridge University Press, Cambridge, 1989

[Ni2] J. Nitsche, *Cyclic surfaces of constant mean curvature*, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 1989, 1–5

[Lo] R. López, *Cyclic surfaces of constant Gauss curvature*, Houston J. Math. 27 (2001), no. 4, 799–805

[Sm] K. Smoczyk, *Self-shrinkers of the mean curvature flow in arbitrary codimension*, IMRN 48 (2005), 2983-3004

[Ta] L. Tavares, Master dissertation, PUC-Rio, 2007

Henri Anciaux
Departement of Mathematics and Computing
Institute of Technology, Tralee
Co. Kerry, Ireland
henri.anciaux@staff.ittralee.ie