Phase structure of black branes in grand canonical ensemble

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Abstract

This is a companion paper of our previous work [1] where we studied the thermodynamics and phase structure of asymptotically flat black \( p \)-branes in a cavity in arbitrary dimensions \( D \) in a canonical ensemble. In this work we study the thermodynamics and phase structure of the same in a grand canonical ensemble. Since the boundary data in two cases are different (for the grand canonical ensemble boundary potential is fixed instead of the charge as in canonical ensemble) the stability analysis and the phase structure in the two cases are quite different. In particular, we find that there exists an analog of one-variable analysis as in canonical ensemble, which gives the same stability condition as the rather complicated known (but generalized from black holes to the present case) two-variable analysis. When certain condition for the fixed potential is satisfied, the phase structure of charged black \( p \)-branes is in some sense similar to that of the zero charge black \( p \)-branes in canonical ensemble up to a certain temperature. The new feature in the present case is that above this temperature, unlike the zero-charge case, the stable brane phase no longer exists and ‘hot flat space’ is the stable phase here. In the grand canonical ensemble there is an analog of Hawking-Page transition, even for the charged black \( p \)-brane, as opposed to the canonical ensemble. Our study applies to non-dilatonic as well as dilatonic black \( p \)-branes in \( D \) space-time dimensions.
1 Introduction

Black holes in asymptotically AdS space have attracted a lot of attention in recent years. The reason is partly due to the AdS/CFT correspondence proposed by Maldacena [2] and its consequences [3, 4, 5]. The large black holes in AdS space are thermodynamically stable\(^4\) [7] and so, the equilibrium states and the phase structure of such space-times can be easily studied. By AdS/CFT correspondence this in turn will provide us with the information about the similar states and phase structure on the CFT or gauge theory side [8, 9]. Black hole space-time is naturally associated with a Hawking temperature [10], so the gauge theory will also be at a finite temperature [11]. In particular, AdS black holes are well-known to undergo a Hawking-Page transition [7] and by AdS/CFT, this corresponds to the confinement-deconfinement phase transition [11] in SU(N) gauge theory at large N.

However, it is well-known that the phase structure just mentioned is not unique to the AdS black holes, but similar structure also arises in suitably stabilized flat as well as dS black holes [12, 13]. Higher dimensional theories like string or M-theory are known to admit higher dimensional black objects in the form of black \(p\)-branes [14, 15, 16] which are asymptotically flat and so, it would be interesting to see what kind of equilibria and phase structure they give rise to. With this motivation we obtained the equilibrium states and the phase structure of the black \(p\)-branes in the canonical ensemble in the previous work [1]. In this paper we study the same for the black \(p\)-branes in grand canonical ensemble. Since the boundary data in these two ensembles are quite different, their phase structures are also expected to be different and it is of interest to see how they differ with the change of the boundary data.

We will consider the black \(p\)-branes (in \(D\) dimensions) which are the solutions of \(D\)-dimensional gravity coupled to a dilaton (a scalar field) and a \((p+1)\)-form gauge field [15]. These are asymptotically flat and so are thermodynamically unstable\(^5\) as an isolated black \(p\)-brane would radiate energy in the form of Hawking radiation [10]. In order to restore thermodynamic stability so that equilibrium thermodynamics and phase structure can be studied we must consider ensembles that include not only the branes but also their environment. As self-gravitating systems are spatially inhomogeneous, any specification of such ensembles requires not just thermodynamic quantities of interest but the place at which they take the specified values. In other words, we place the black brane in a cavity \textit{a la} York [19] (see also [20, 21, 22, 23, 24]) and its extension in the charged case. The wall

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\(^4\)This is not the case for small black holes, for example, see [6].

\(^5\)A detailed analysis of this kind of instability for neutral black holes/branes/rings is given in [17, 18].
of the cavity is fixed at a radial distance $\rho_B$, and we will keep the temperature and the gauge potential at the wall of the cavity (at $\rho_B$) fixed. This will define a grand canonical ensemble (Note that for the canonical ensemble, the charge enclosed in the cavity is fixed instead.). So, for the grand canonical ensemble charge can vary. We will study the phase structure of black $p$-branes in this ensemble. Charged black holes in the grand canonical ensemble have been studied in [25, 26].

We will employ the Euclidean action formalism [27, 19, 28] to the dilatonic black $p$-brane geometry in $D$ space-time dimensions. We first write the Euclidean action containing the gravitational part including a Gibbons-Hawking boundary term [27], the dilatonic part and the form-field part. By using the equation of motion (i.e., on-shell), we can express the usual Einstein-Hilbert gravitational action plus the dilatonic part in terms of the form-field, therefore simplify the action and then evaluate it for the given black $p$-brane geometry placed in a cavity with the potential and the temperature fixed on its wall. To the leading order this action is related to the grand potential or Gibbs free energy of the system and is an essential entity for the stability analysis. It can explicitly be expressed in terms of parameters of the branes and the cavity. The stationary point of this action with respect to the relevant variables such as the horizon size will determine the relevant thermal states with the temperature and potential fixed at the wall of the cavity. The second derivative or derivatives of the action at the stationary point can be used to give us the information about the stability of the black brane at that point. We find the condition which determines at least the local stability of the black brane phase. In the case when there is a stable black brane phase, we find that there exists a minimum temperature below which there is only ‘hot flat space’ phase and no black brane phase. But above this temperature there exist two black brane phases with two different radii. The smaller one is unstable which corresponds to the maximum of the free energy and the larger one is locally stable and corresponds to the local minimum of the free energy. This local minimum becomes a global one only after the temperature rises above a certain value. Below this value and above minimum temperature, the locally stable black brane eventually makes a transition to the ‘hot flat space’ by a phase transition. But above this value, the large black brane is globally stable. Upto this point the picture is very similar to the chargeless black brane case in the canonical ensemble [1]. But now for the grand canonical ensemble as the temperature rises more, the globally stable black brane phase disappears after a certain value and at this point there is only one unstable black brane phase and the stable phase here would be the ‘hot flat space’. Also note that unlike in the case of canonical ensemble [8, 9, 12, 13, 1], here we do not have a situation where we

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6For the case of canonical ensemble with fixed non-zero charge, the phase structure of the asymptoti-
have two locally stable black brane phases and so there is no such phase transition similar to the van der Waals-Maxwell liquid-gas phase transition or even a transition from one black brane phase to the other. We would like to remark that in this grand canonical ensemble, unlike in the case of AdS black holes [8, 9], we do not have a critical value for the potential in the sense defined there since the existence of this critical value in AdS case is largely due to the presence of the cosmological constant (as an additional parameter). One can check easily that if we set the cosmological constant vanish (i.e., setting the parameter $l \to \infty$ there), the extremal black holes becomes supersymmetric BPS ones, which is in analog of the present discussion, and the potential will not have the critical value anymore but has only an upper bound. The existence of the upper bound of potential for the present case is explained below and in section 3.

We now discuss the case of extremal black $p$-branes. Mathematically, the extremal brane configuration can be obtained simply by taking the ‘mass-equal-charge’ limit of the corresponding non-extremal one. However, this does not mean that the extremal branes and non-extremal branes can actually be related to each other by a physical process which can also be understood from the semi-classical discussion given by Gibbons and Kallosh [29], Hawking et al [30, 31] and Teitelboim [32], for the case of black holes including the dilatonic ones (the latter resembles the present discussion). So, for example, the entropy changes discontinuously from the non-extremal case to the extremal one. The non-extremal case has a non-vanishing entropy, given always by one quarter of the event horizon area while the extremal one always has vanishing entropy\(^7\). This very fact, just

\[^7\text{The discrepancy in the entropy between the semi-classical calculation and the microstate counting in string theory in the AdS/CFT context has been addressed recently in [33]. It was pointed out there that the non-vanishing of the entropy is due to a different reason and the semi-classical result of vanishing entropy for the extremal black hole still holds.}\]
like the case of black holes, implies that the non-extremal branes and extremal branes are qualitatively different objects and a non-extremal black brane cannot turn into an extremal brane. Actually, the nearer the mass of the non-extremal brane gets to the charge the lower the temperature of the brane and so the lower the rate of radiation of mass. Thus the mass will never be exactly equal to the charge. This, in turn, implies, as it should, that there does not exist any real physical process which can bring down the temperature of a thermal system to absolute zero since otherwise it would be against the third law of thermodynamics that the zero temperature can never be reached in reality. Moreover, also like the case of black holes, an extremal brane cannot become a non-extremal brane even when matter or radiation are thrown into the extremal one, which seems at first sight contrary to the common sense [30]. This is due to the fact that one can identify extremal branes with any period and therefore they can be in equilibrium with thermal radiation at any temperature. Thus they must be able to radiate at any rate, unlike non-extremal branes, which can radiate only at the rate corresponding to their temperature. So extremal objects always radiate in such a way as to keep themselves extreme when matter or radiation is sent into them. This discussion applies to the asymptotically flat branes in canonical ensemble for which the charge inside the cavity is fixed.

In this paper, we are considering the asymptotically flat branes in grand canonical ensemble for which the potential at the wall of the cavity is fixed instead. Before we discuss the thermodynamical stability, we first need to address the question: given the boundary data in this ensemble mentioned earlier, what are the possible stable configurations allowed in the cavity which can be used to discuss this stability? Unlike in the canonical ensemble, the charge inside the cavity in the grand canonical ensemble is not fixed and can fluctuate or exchange with the environment. So one expects first that the ‘hot flat space’ can be a stable phase under certain conditions. The second candidate is the non-extremal branes, charged or uncharged, and the third possible one is the extremal branes. However, the probability for the existence of non-extremal branes is much higher than that of the extremal branes for the following reasons. In the case of black holes, the pair creation rate of the non-extremal black holes is enhanced by a factor of \( e^S \) over that of the extremal branes where \( S \) is the entropy of the non-extremal black holes [30, 29]. We expect this to apply to the case of branes as well. Further, unlike a non-extremal brane, an extremal brane with a large charge can fission (split) into extremal branes with smaller charges without violating the second law of thermodynamics since this process is suppressed by a factor of \( e^{\Delta S} \) with \( \Delta S \) the change of total entropy of the system of extremal branes and for extremal branes \( \Delta S = 0 \). Note that an extremal brane with a macroscopic charge in the present context (for the gravity description to remain valid) is actually a BPS
brane, preserving one half of the underlying spacetime supersymmetries and there is no binding energy among BPS branes with smaller charges, even the individual BPS branes with unit charge (which constitutes the extremal branes with macroscopic charge). So a large extremal branes can easily split into smaller extremal ones. This provides another means to understand the above fission process. In other words, the extremal branes are unstable in the first place in this ensemble and should be excluded from consideration in our following thermodynamical stability analysis.

This paper is organized as follows. In section 2, we discuss the dilatonic black \( p \)-brane solution and evaluate the action in Euclidean signature. Then we discuss the stability of various equilibrium states from this action in section 3. The phase structure of the black \( p \)-brane is discussed in section 4. Then we conclude in section 5. A more general two-variable stability analysis (as opposed to one-variable stability analysis performed in section 3) is presented in the Appendix.

## 2 Black \( p \)-brane solution and the action

The black \( p \)-brane solution was originally constructed [14] as a solution to the ten-dimensional supergravity containing a metric, a dilaton and a \((p + 1)\)-form gauge field and was generalized to arbitrary dimensions in [15]. These solutions are given in Lorentzian signature, but for the purpose of studying thermodynamics, we write the black \( p \)-brane solution in Euclidean signature as (see for example [34]),

\[
\begin{align*}
\mathbf{d} s^2 &= \Delta_+ \Delta_-^{d/2} \mathbf{d} t^2 + \Delta_-^{d/2} \sum_{i=1}^{d-1} (\mathbf{d} x^i)^2 + \Delta_+^{-1} \Delta_-^{d/2} \mathbf{d} \rho^2 + \rho^2 \Delta_+^{-1} \mathbf{d} \Omega_{d+1}^2, \\
A_{[p+1]} &= -ie^{a\phi_0/2} \left[ \left( \frac{r_-}{r_+} \right)^{d/2} - \left( \frac{r_- - r_+}{\rho^2} \right)^{d/2} \right] \mathbf{d} t \wedge \mathbf{d} x^1 \wedge \ldots \wedge \mathbf{d} x^p, \\
F_{[p+2]} &= dA_{[p+1]} = -ie^{a\phi_0/2} \tilde{\mathbf{d}} \left( \frac{r_- - r_+}{\rho^d} \right)^{d/2} \mathbf{d} \rho \wedge \mathbf{d} t \wedge \mathbf{d} x^1 \wedge \ldots \wedge \mathbf{d} x^p, \\
e^{2(\phi - \phi_0)} &= \Delta_a^a, 
\end{align*}
\]

Here we have defined

\[
\Delta_\pm = 1 - \left( \frac{r_\pm}{\rho} \right)^d
\]
where, \( r_\pm \) are the two parameters characterizing the solution and are related to the mass and the charge of the black brane. In the metric (1) the Euclidean time is periodic and so, the metric has an isometry \( S^1 \times SO(d-1) \times SO(\tilde{d} + 2) \) indicating that it represents a \((d-1) \equiv p\)-brane in Euclidean signature. The total space-time dimension is \( D = d + \tilde{d} + 2 \), where the space transverse to the \( p \)-brane has the dimensionality \( \tilde{d} + 2 \). \( \phi \) is the dilaton and \( \phi_0 \) is its asymptotic value and related to the string coupling as \( g_s = e^{\phi_0} \). \( a \) is the dilaton coupling and is given for the supergravity theory with maximal supersymmetry by,

\[
a^2 = 4 - \frac{2d\tilde{d}}{D - 2}.
\]

(3)

It is clear from the Lorentzian form of the above metric that when \( r_- = 0 \), and \( a = 0 \), it reduces to the \( D \)-dimensional Schwarzschild solution which has an event horizon at \( \rho = r_+ \), whereas, at \( \rho = r_- \), there is a curvature singularity. So, the metric in (1) represents a black \( p \)-brane only for \( r_+ > r_- \), with \( r_+ = r_- \), being its extremal limit [14]. A \( p \)-brane naturally couples to the \((p+1)\)-form gauge field whose form and its field strength are given in (1). Note that we have defined the gauge potential with a constant shift, following [26], in such a way that it vanishes on the horizon so that it is well-defined on the local inertial frame. The black \( p \)-brane will be placed in a cavity with its wall at \( \rho = \rho_B \). It is clear from the metric in (1) that the physical radius of the cavity is

\[
\bar{\rho}_B = \Delta_\pm \rho_B
\]

(4)

while \( \rho_B \) is merely the coordinate radius. So, it is \( \bar{\rho}_B \) which we should fix in the following discussion and not \( \rho_B \). Note that when the black brane is non-dilatonic \( a = 0 \) and in that case \( \rho_B = \bar{\rho}_B \). Also we fix the dilaton \(^9\) at \( \bar{\rho}_B \), which indicates that the asymptotic value of the dilaton is not fixed for the present consideration and this is crucial for our discussion in expressing relevant quantities in ‘barred’ parameters. By this argument we also have

\[
\bar{r}_\pm = \Delta_\pm \bar{r}_\pm
\]

(5)

and \( \bar{r}_\pm \) are the proper parameters which we should use in the present context. In terms of the ‘barred’ parameters \( \Delta_\pm \) remain the same as before,

\[
\Delta_\pm = 1 - \frac{r_\pm^d}{\rho_B^d} = 1 - \frac{\bar{r}_\pm^d}{\bar{\rho}_B^d}.
\]

(6)

\(^9\)This enables us to obtain its correct equation of motion from the corresponding action in the presence of a boundary.
Since the Euclidean time coordinate in (1) is periodic so, for the metric to be well-defined without a conical singularity at $\rho = r_+$, the Euclidean time must have a periodicity,

$$\beta^* = \frac{4\pi r_+}{d} \left( 1 - \frac{r_d^2}{r_+^d} \right)^{\frac{1}{2}-\frac{1}{2}},$$

(7)

which is the inverse temperature at $\rho = \infty$. The local $\beta(\bar{\rho}_B)$ is given\(^\text{10}\) as

$$\beta = \beta(\bar{\rho}_B) = \Delta_+^{\frac{1}{d}} \Delta_-^{-\frac{d}{2(d+d)}} \beta^*$$

(8)

which is the inverse of local temperature at $\bar{\rho}_B$ when in thermal equilibrium with the environment (the wall of the cavity) at the inverse temperature $\beta$. Note that in terms of the ‘barred’ parameters the inverse of local temperature $\beta(\bar{\rho}_B)$ can be expressed from (8) and (4) as,

$$\beta(\bar{\rho}_B) = \frac{4\pi r_+}{d} \Delta_+^{\frac{1}{d}} \Delta_-^{-\frac{d}{2(d+d)}} \left( 1 - \frac{r_d^2}{r_+^d} \right)^{\frac{1}{2}-\frac{1}{2}} = \frac{4\pi \bar{r}_+}{d} \Delta_+^{\frac{1}{d}} \Delta_-^{-\frac{d}{2}} \left( 1 - \frac{\bar{r}_d^2}{\bar{r}_+^d} \right)^{\frac{1}{2}-\frac{1}{2}},$$

(9)

where in the second equality $\Delta_\pm$ are also expressed in terms of ‘barred’ parameters. Also the charge is defined as,

$$Q_d = \frac{i}{\sqrt{2\kappa}} \int e^{-a(d)\phi} \ast F_{[p+2]} = \frac{\Omega_{d+1} e^{-a\phi_0/2} \bar{d}(r_+ r_-)^{d/2}}{\sqrt{2\kappa}} = \frac{\Omega_{d+1} \bar{d}}{\sqrt{2\kappa}} e^{-a\phi/2} (\bar{r}_+ \bar{r}_-)^{d/2}.$$  

(10)

In (10), $\kappa$ is a constant with $1/(2\kappa^2)$ appearing in front of the Hilbert-Einstein action in canonical frame but containing no string coupling $g_s$, $\ast F_{[p+2]}$ denotes the Hodge dual of the $(p + 2)$-form field given in (1). Also, $\Omega_n$ denotes the volume of a unit $n$-sphere. In the last line of (10) we have expressed the asymptotic value of the dilaton in terms of the fixed dilaton via the boundary condition $\phi(\bar{\rho}_B) = \bar{\phi}$ at the wall of the cavity from the relation (1) and then expressed $r_\pm$ by $\bar{r}_\pm$ from (5).

In the grand canonical ensemble, the fixed quantities are the physical radius of the wall of the cavity $\bar{\rho}_B$, the temperature $T$, the brane volume $V_p$ (The local volume at $\bar{\rho}_B$, $V_p(\bar{\rho}_B) = \Delta_+^{\frac{d(d-1)}{2}} V_p^*$, with $V_p^* = \int d^{d}x$, is set equal to the fixed value $V_p$) and the potential $\bar{\Phi}$, all at the wall of the cavity. They are independent parameters and the corresponding local quantities for the branes have to be equal to these pre-selected fixed values on-shell

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\(^{10}\)The corresponding local temperature is related to the temperature at spatial infinity by the so-called Tolman relation [35].
at the wall of the cavity, respectively. The potential in the local inertial frame at the wall of the cavity can be obtained from $A_{[p+1]}$ and the metric in (1) as,

$$A_{[p+1]} = -ie^{\phi/2} (\Delta - \Delta_+)^{-\frac{1}{2}} \left( \frac{r_-}{r_+} \right)^{\frac{d}{2}} \left( 1 - \frac{r_+}{\rho_B} \right) d\bar{t} \wedge d\bar{x}^1 \ldots d\bar{x}^p$$

$$\equiv -i\sqrt{2}\kappa\Phi d\bar{t} \wedge d\bar{x}^1 \ldots d\bar{x}^p$$

(11)

where $(\bar{t}, \bar{x}^1, \ldots, \bar{x}^p)$ are the coordinates in the local inertial frame and is related to the original coordinates as $\bar{t} = \Delta_+^{\frac{1}{2}} \Delta_-^{-\frac{d}{2d-2}} t$ and $\bar{x}^i = \Delta_-^{-\frac{d}{2d-2}} x^i$ for $i = 1, 2, \ldots, p$ as can be seen from the metric in (1). So, $\Phi$ is the potential conjugate to the charge and is set equal to the fixed value $\bar{\Phi}$ at the wall. We will now evaluate the action with these boundary data.

The Euclidean action for the dilatonic black branes in the canonical ensemble has already been evaluated in the Appendix of our previous work [1]. We will use that result to evaluate the action for the grand canonical ensemble. The relevant action for the gravity coupled to the dilaton and a $(p+1)$-form gauge field in the canonical ensemble is,

$$I_{CE}^C = I_{CE}^C(g) + I_{CE}^C(\phi) + I_{CE}^C(F)$$

(12)

where, $I_{CE}^C(g)$ is the gravitational part of the action which has a Hilbert-Einstein term and a Gibbons-Hawking boundary term [27], $I_{CE}^C(\phi)$ is the dilatonic part and $I_{CE}^C(F)$ is the form-field part. The first two terms remain exactly the same in the grand canonical ensemble, however, the form-field part which in the canonical ensemble has the form,

$$I_{CE}^C(F) = \frac{1}{2\kappa^2} \frac{1}{2(d+1)!} \int_M d^Dx \sqrt{g} e^{-a(d)\phi} F^2_{d+1}$$

$$- \frac{1}{2\kappa^2} \frac{1}{d!} \int_{\partial M} d^{D-1}x \sqrt{\gamma} n_\mu e^{-a(d)\phi} F^{\mu\mu_1\mu_2\ldots\mu_d} A_{\mu_1\mu_2\ldots\mu_d},$$

(13)

will differ in the grand canonical ensemble since the potential at the wall of the cavity is fixed and so, the last term in (13) will be absent in the grand canonical ensemble. Therefore, the action for the grand canonical ensemble can be obtained from that of the canonical ensemble by the relation,

$$I_{CE}^{GC} = I_{CE}^C + \frac{1}{2\kappa^2} \frac{1}{d!} \int_{\partial M} d^{D-1}x \sqrt{\gamma} n_\mu e^{-a(d)\phi} F^{\mu\mu_1\mu_2\ldots\mu_d} A_{\mu_1\mu_2\ldots\mu_d},$$

(14)

Here $\partial M$ denotes the boundary of the space-time $M$ and $n_\mu$ is a space-like vector normal to the boundary. $\gamma$ is the determinant of the boundary metric. Note that the spacetime $M$ here is the bulk region defined by $\bar{\rho}_B \geq \bar{\rho} \geq \bar{r}_+$ while the boundary $\partial M$ is defined as
the surface at $\bar{\rho} = \bar{\rho}_B$ plus the one at $\bar{\rho} = \bar{r}_+$. The surface integration in (14) at $\bar{\rho} = \bar{r}_+$ vanishes due to the vanishing form-potential there. Evaluating the last term in (14) for the black $p$-brane configuration given in (1) we get,

$$I_{CE}^{GC} = I_{CE} - \frac{\beta V_p^*}{2\kappa^2} \Omega_{d+1} \bar{d}_{\bar{r}} \frac{\Delta_+}{\Delta_-} = I_{CE} - \beta V_p Q_d \bar{\Phi}, \quad (15)$$

where we have used the charge expression and the potential from (10) and (11) as,

$$Q_d = \frac{\Omega_{d+1}}{\sqrt{2\kappa}} e^{-a\bar{\phi}/2} (\bar{r}_+ - \bar{r}_-)^{\bar{d}} / 2,$$

$$\bar{\Phi} = \Phi(\bar{\rho}_B) = \frac{1}{\sqrt{2\kappa}} e^{a\bar{\phi}/2} \left( \frac{\Delta_+}{\Delta_-} \right)^{\frac{1}{2}}.$$

(16)

Now using the explicit form\(^{11}\) of the Euclidean action for the canonical ensemble given in [1] we write (15) as\(^{12}\)

$$I_E(\beta, \bar{\Phi}, \bar{\rho}_B, V_p; Q_d, \bar{r}_+) = \beta \Omega - \beta E(\bar{\rho}_B, V_p; Q_d, \bar{r}_+) - S(\bar{\rho}_B, V_p; Q_d, \bar{r}_+) - \beta V_p Q_d \bar{\Phi}$$

$$= -\frac{\beta V_p \Omega_{d+1} \bar{d}_{\bar{r}_B}}{2\kappa^2} \left[ (\bar{d} + 2) \left( \frac{\Delta_+}{\Delta_-} \right)^{\frac{1}{2}} + \bar{d}(\Delta_+ \Delta_-)^{\frac{1}{2}} - 2(\bar{d} + 1) \right]$$

$$- \frac{4\pi V_p \Omega_{d+1} \bar{r}_+^{\bar{d}+1} \Delta_-^{\frac{1}{2} - \frac{d}{2} - \frac{1}{2} + \frac{1}{d}}}{2\kappa^2} \left( 1 - \frac{\bar{r}_-^{\bar{d}}}{\bar{r}_+^{\bar{d}}} \right)^{\frac{1}{2} + \frac{1}{d}} - \beta V_p Q_d \bar{\Phi}, \quad (18)$$

where $\Omega = E - TS - Q\bar{\Phi}$ is the Gibbs free energy. In the above, we have the total charge enclosed in the cavity as $Q = V_p Q_d$, the entropy

$$S = \frac{4\pi V_p \Omega_{d+1} \bar{r}_+^{\bar{d}+1} \Delta_-^{\frac{1}{2} - \frac{1}{2} + \frac{1}{d}}}{2\kappa^2} \left( 1 - \frac{\bar{r}_-^{\bar{d}}}{\bar{r}_+^{\bar{d}}} \right)^{\frac{1}{2} + \frac{1}{d}},$$

$$= \frac{4\pi V_p^* \Omega_{d+1} \bar{r}_+^{\bar{d}+1} \Delta_-^{\frac{1}{2} - \frac{1}{2} + \frac{1}{d}}}{2\kappa^2} \left( 1 - \frac{\bar{r}_-^{\bar{d}}}{\bar{r}_+^{\bar{d}}} \right)^{\frac{1}{2} + \frac{1}{d}}.$$

(19)

\(^{11}\)In obtaining this, we have made use of the on-shell equation of motion

$$R - \frac{1}{2} (\partial \phi)^2 = \frac{\bar{d} - d}{2(D - 2)(d + 1)!} e^{-\alpha(d) \phi} F^2, \quad (17)$$

to express $I_E(g) + I_E(\phi)$ in terms of the form-field $F$ to simplify the computation.

\(^{12}\)In obtaining the explicit action, we have used the background subtraction approach. The natural background here is taken as the ‘hot flat space’ with a constant dilaton $\bar{\phi}$, a fixed potential $\bar{\Phi}$ and other boundary data. The equations of motion are satisfied trivially inside the cavity for the ‘hot flat space’ with the given boundary data.
where \( V^*_p = \int d^p x \) as defined before and is independent of the location of the cavity as it should be, and the energy for the cavity as

\[
E = -\frac{V_p \Omega_{d+1} \tilde{\rho}_B}{2\kappa^2} \left[ (\tilde{d} + 2) \left( \frac{\Delta_+}{\Delta_-} \right)^{1/2} + \tilde{d}(\Delta_+\Delta_-)^{1/2} - 2(\tilde{d} + 1) \right],
\]

(20)

which gives the ADM mass when \( \rho_B \to \infty \). Since this is the Euclidean action for the black \( p \)-brane in the grand canonical ensemble we will be working with, for brevity, we have removed the superscript ‘GC’ from \( I_E \) in writing (18). Note that we have expressed everything on the r.h.s. in terms of the ‘barred’ parameters showing that the formalism works for both the non-dilatonic as well as dilatonic branes. In (18) \( \beta, \bar{\rho}_B, \Phi \) are the inverse of temperature, the physical radius and the potential of the cavity, therefore are all fixed, and \( Q_d, \bar{r}_+ \) are variables. Note that \( \bar{r}_- \) is not independent and can be expressed in terms of \( Q_d \) and \( \bar{r}_+ \) from (16) as,

\[
\bar{r}_-^d = \left( \frac{\sqrt{2\kappa Q_d} e^{\phi/2}}{\Omega_{d+1} \tilde{d}} \right)^2 \frac{1}{\bar{r}_+^d}. \tag{21}
\]

Substituting (21) in (18) we can express the action in terms of two variables \( Q_d \) and \( \bar{r}_+ \). Then varying the action with respect to \( Q_d \) and putting that to zero, i.e., at the stationary point we obtain (after some simplification),

\[
\frac{\partial I_E}{\partial Q_d} = 0 \quad \Rightarrow \quad \Phi = \Phi(\bar{\rho}_B) = \frac{Q_d e^{\alpha \phi}}{d\Omega_{d+1} \tilde{r}_+^d} \left( \frac{\Delta_+}{\Delta_-} \right)^{1/2} \left[ 1 + \frac{\tilde{d} + 2}{d} \Delta_+^{-1} \left( \frac{4\pi \bar{r}_+^{1/2} \Delta_+^{1/2} \Delta_-^{-1/2}}{\beta \tilde{d} \left( 1 - \frac{\bar{r}_+^d}{\bar{r}_-^d} \right)^{1/2}} - 1 \right) \right], \tag{22}
\]

Similarly, varying the action with respect to the other variable \( \bar{r}_+ \) and setting that to zero, i.e., at the stationary point we obtain,

\[
\frac{\partial I_E}{\partial \bar{r}_+} = 0 \quad \Rightarrow \quad \beta = \beta(\bar{\rho}_B) = \frac{4\pi \bar{r}_+^{1/2} \Delta_+^{1/2} \Delta_-^{-1/2}}{d} \left( 1 - \frac{\bar{r}_+^d}{\bar{r}_-^d} \right)^{1/2 - 1/2}. \tag{23}
\]

This is the correct form of \( \beta \) we had given earlier in (9). Using this (23) in (22) and substituting the form of \( Q_d \) given in the first equation of (16) we recover the correct form of the potential given in the second equation of (16). This is a verification that the action \( I_E \) given in (18) is indeed correct. We will use this form of the action in the next section to
study the stability of the equilibrium states of the black $p$-branes in the grand canonical ensemble.

Before we close this section, we would like to discuss briefly the validity of the use of the effective action in describing the phase structure of black $p$-branes throughout the parameter space in the present ensemble. Since we are considering a situation where the black $p$-branes are placed in a cavity, this implies that the size of the cavity $\bar{\rho}_B$ must be larger than the horizon size $\bar{r}_+$ of the branes. In obtaining the explicit Euclidean action (18), we have performed the integration of the bulk action in the range of $\bar{\rho}_B > \bar{\rho} > \bar{r}_+$. To have a valid description in terms of the effective action, we need to keep the curvature of black brane spacetime uniformly weak throughout this range and if the fundamental theory is string theory, we need in addition to keep the effective string coupling uniformly weak. One can check that the requirement(s) can be satisfied with the expected condition

$$\bar{r}_+ = r_+ \Delta^{a^2/4\bar{d}} \gg l,$$

where $l$ is the fundamental length scale of the underlying theory, for example, it is the Planck scale $l_p$ in eleven dimensions or the string scale $l_s$ in ten dimensions.

There appears no further subtlety in the grand canonical ensemble considered in this paper since the extremal branes are excluded from consideration as discussed in the Introduction. The relevant configurations here are the ‘hot flat space’ and the non-extremal branes. Note also that the location of the cavity wall along the radial direction satisfies $\bar{\rho}_B > \bar{\rho} > \bar{r}_+ > \bar{r}_-$. However, there appears to have a potential issue regarding the fixed dilaton value at the wall of the cavity in the canonical ensemble, considered in our previous work [1], for certain extremal branes such as D3 branes for which the so-called attractor mechanism is at work\textsuperscript{13}. Once this mechanism is at work, the dilaton value at the horizon is completely determined by the relevant charge(s), independent of its asymptotic value. The extremal $p$-brane configuration from the supergravities with maximal SUSY in spacetime dimension $D \geq 4$ can be obtained from the non-extremal one given in (1) by taking the extremal limit, i.e., $r_+ = r_-$. These branes preserve one half of the space-time supersymmetries and the dilaton at a given $\rho$ is related to its asymptotic value via the relation given in (1). The (arbitrary) fixed value at the wall of the cavity is therefore related to the (arbitrary) asymptotic value of the dilaton with the fixed cavity size. So in the present context, one can re-phrase the attractor mechanism as: the value of the dilaton at the horizon is independent of the fixed value of the dilaton at the wall of the cavity. So if the attractor mechanism is at work, the wall of the cavity cannot be too close to the horizon since otherwise the approach will be inconsistent with this

\textsuperscript{13}We thank the anonymous referee for raising this point to us.
mechanism or it will give rise to a jump of the value of the dilaton in the bulk which
is impossible. Fortunately, the situation is not so restrictive as indicated above. Once
the attractor mechanism is at work, any given point outside the horizon has actually an
infinite physical distance away from the horizon (see, for example, [36, 37, 38]) and so the
condition $\rho_B > r_+ = r_-$ with fixed $\rho_B$, the radial location of the wall of the cavity, will
be enough.

3 Stability analysis of the black $p$-branes

For the purpose of the stability analysis we will define some new parameters following
refs.[26, 13, 12] and rewrite the action, the potential and the inverse temperature in terms
of those parameters. Let us first define the reduced charge as,

$$Q_d^* = \left( \frac{\sqrt{2} A Q_d e^{a\phi/2}}{\Omega_{d+1} d} \right)^{\frac{1}{d}}$$

Then from (21) we have

$$\bar{r}_- = \left( \frac{Q_d^*}{r_+} \right)^2$$

Next we define the following parameters,

$$x = \left( \frac{r_+}{\rho_B} \right)^d, \quad \bar{b} = \frac{\beta}{4\pi \rho_B}, \quad q = \left( \frac{Q_d^*}{\rho_B} \right)^{\frac{1}{d}}$$

where the dimensionless parameter $\bar{b}$ is fixed, but the other two parameters $x$ and $q$ vary.
Note that the parameter $\bar{b}$ is related to the inverse of temperature of the environment, $q$
is related to the charge and $x$ is related to the horizon size. In terms of these parameters we have

$$\Delta_+ = 1 - \frac{\bar{r}_d^{\frac{1}{d}}}{\rho_B^{\frac{1}{d}}} = 1 - x$$

$$\Delta_- = 1 - \frac{\bar{r}_d^{\frac{1}{d}}}{\rho_B^{\frac{1}{d}}} = 1 - \frac{(Q_d^*)^{2\frac{1}{d}}}{\bar{r}^{d} + \rho_B^{\frac{1}{d}}} = 1 - \frac{q^2}{x}$$

$$1 - \frac{\bar{r}_d^{\frac{1}{d}}}{\bar{r}^{\frac{1}{d}}} = 1 - \frac{q^2}{x^2}$$

Using (28) the two equations at the equilibrium (at the stationary point) giving the
constant temperature and the potential at the wall of the cavity given in (23) and (22)
can be written as

$$\bar{b} = b(x, q), \quad \bar{\varphi} = \varphi(x, q)$$
where we have defined $\varphi = \sqrt{2\kappa e^{-a\bar{\phi}/2}} \Phi$ and

$$b(x, q) = \frac{1}{d} \left( 1 - \frac{q^2}{x^2} \right)^{-\frac{d+1}{2}} \left( 1 - \frac{q^2}{x^2} \right)^{-\frac{d}{2}}, \quad \varphi(x, q) = \frac{q}{x} \left( 1 - \frac{x - q^2}{x} \right)^{\frac{1}{2}}. \quad (30)$$

The two equations in (29) can also be derived from the following reduced action in a similar fashion as (23) and (22) but now with respect to variables $x$ and $q$, respectively,

$$\tilde{I}_E(x, q) \equiv \frac{2\kappa^2 I_E}{4\pi^2 \beta_{\bar{b}} V_p \Omega \tilde{d} + 1}$$

$$= -\tilde{b} \left[ (\tilde{d} + 2) \left( 1 - \frac{x - q^2}{x} \right)^{\frac{1}{2}} + \tilde{d} (1 - x)^{\frac{1}{2}} \left( 1 - \frac{q^2}{x} \right)^{\frac{1}{2}} - 2(\tilde{d} + 1) + \tilde{d} \tilde{q} \varphi \right]$$

$$-x^{1+\frac{1}{d}} \left( 1 - \frac{q^2}{x} \right)^{\frac{1}{2} + \frac{1}{d}}, \quad (31)$$

where $I_E$ is as given in (18). These two equations, for given $\tilde{b}$ and $\varphi$, determine $x$ and $q$ completely. However, in the presence of two variables, the analysis of stability at the extremal points determined by these two equations is a bit more complicated than that given in [1], in the canonical ensemble. There the relation at the charge equilibrium for fixed charge has been employed to reduce the two variables to only one\(^\text{14}\). While the stability analysis with two variables, which we will perform in the Appendix, is more complete, there actually exists an analogous simpler analysis as in the canonical case. But for this we need to employ the potential relation at the equilibrium for fixed potential, as given in the second equation of (29), to reduce the two variables to one variable. In what follows, we will first perform the simpler one-variable analysis and find the stability condition. We will perform the more complete yet more complicated two-variable analysis in the Appendix and show that these two approaches give the same stability condition.

Note that since $x/q = (\bar{r}_+ + Q^*)^d = (\bar{r}_+ + \bar{r}_-)^{d/2} \geq 1$, so, $x \geq q$ and since $x = (\bar{\rho}_B + \bar{r}_+)^d \leq 1$, so, $x$ has the range $q \leq x \leq 1$. The condition (24) may put a more strict lower bound for $x$ if $q$ is too small. The end point $x = 1$ corresponds to taking $\tilde{\rho}_B = \bar{r}_+$, i.e., the cavity is placed on the horizon where the ensemble temperature is infinity. In practice, we can keep

\(^{14}\)While the one-variable stability analysis given in [1] appears natural in the canonical ensemble, there exists also a more general two-variable analysis which is more complete yet more complicated and gives the same stability condition. For the present case, the two-variable analysis seems more natural but as we will show in the text that when the second equation in (29) for the fixed potential is used to reduce the two variables to only one, the same stability condition can be obtained.
an ensemble at any given high temperature and the infinitely high temperature should be understood as a limiting process. So we should take $x < 1$. The other end point $x = q$ corresponds to the extremal case which should be excluded from the consideration given what has been discussed in the Introduction. For this reason, we take $x > q$. So, following [26], we limit ourselves to the physical region of $0 \leq q < x < 1$ in what follows, keeping in mind the condition (24)\textsuperscript{15}. Also note from the expression in (30) that since $1 - q^2/x = 1 - (q^2/x^2)x > 1 - x$, so we have $0 \leq \varphi < 1$, with $\varphi = 0$ corresponding to the chargeless ($q = 0$) case. As discussed in Introduction, we should exclude the $\varphi = 1$ ($x = q$) since it corresponds to the extremal case. $\varphi > 1$ cannot be in a black brane phase and it actually corresponds to the ‘hot flat space’ phase.

Let us now consider the one-variable case first, i.e., when the second equation in (29) $0 \leq \bar{\varphi} = \varphi(x, q) < 1$ is satisfied. We can now solve this equation to get,

$$
\frac{q^2}{x^2} = \frac{\varphi^2}{1 - (1 - \varphi^2)x}. \tag{32}
$$

Now substituting (32) in $b(x, q)$ equation in (30) we have for fixed potential $\bar{\varphi}$,

$$
b(x) \equiv b(x, q) = \frac{x^{\frac{1}{d}}[1 - (1 - \bar{\varphi}^2)x]^{\frac{1}{d}}}{\bar{\varphi}^{1 + 1/d}(1 - \bar{\varphi}^2)^{1/2 + 1/d}}, \tag{33}
$$

and in the reduced action in (31), we have

$$
\tilde{I}_E(x) \equiv \tilde{I}_E(x, q) = -2\bar{b} \left(\tilde{d} + 1\right) \left[\left(1 - (1 - \bar{\varphi}^2)x\right)^{1/2} - 1\right] - x^{1+1/d} \left(1 - \bar{\varphi}^2\right)^{1/2+1/d}. \tag{34}
$$

One can easily check that

$$
\frac{d\tilde{I}_E(x)}{dx} = \frac{(1 + \tilde{d})(1 - \bar{\varphi}^2)}{[1 - (1 - \bar{\varphi}^2)x]^{1/2}} \left[\tilde{b} - b(x)\right], \tag{35}
$$

where $b(x)$ is defined in (33). This vanishes at the stationary point of the action, i.e.,

$$
\tilde{b} = b(\bar{x}), \tag{36}
$$

which is the first equation in (29) in the present consideration. The local stability at the stationary point is determined by requiring it to be a local minimum of the action, i.e.,

$$
\frac{d^2I_E(x)}{dx^2} \bigg|_{x=\bar{x}} = -\frac{(1 + \tilde{d})(1 - \bar{\varphi}^2)}{[1 - (1 - \bar{\varphi}^2)x]^{1/2}} \frac{db(\bar{x})}{d\bar{x}} > 0. \tag{37}
$$

\textsuperscript{15}As will become clear, the local stability condition given in (41) does satisfy the condition (24).
In other words, we need to have (since the other factors are all positive)

\[ \frac{db(\bar{x})}{d\bar{x}} = \frac{\bar{x}^{1/\tilde{d} - 1} \left[ 2 - (2 + \tilde{d})\bar{x}(1 - \tilde{\varphi}^2) \right]}{2\tilde{d}^2(1 - \tilde{\varphi}^2)^{1/2 - 1/\tilde{d}} [1 - (1 - \tilde{\varphi}^2)\bar{x}]^{1/2}} < 0, \tag{38} \]

which gives the local stability condition as

\[ 2 - (2 + \tilde{d})\bar{x}(1 - \tilde{\varphi}^2) < 0. \tag{39} \]

In the above, \( \bar{x} \) is a solution of the present equation of state as given in (36).

As we will show in detail in the Appendix that the above local stability condition (39) continues to hold when we consider the more general yet more complicated two-variable situation and this justifies the simplified one-variable analysis of stability performed above.

We therefore conclude that for the local stability of the system we must have (39) to be satisfied. To understand the meaning of the stability condition (39) we rewrite it as

\[ \bar{x}(1 - \tilde{\varphi}^2) > \frac{2}{d + 2} \tag{40} \]

which further implies,

\[ \frac{2}{d + 2} < \bar{x} < 1, \quad \text{and} \quad 0 < \tilde{\varphi} < \sqrt{\frac{d}{d + 2}}. \tag{41} \]

In other words, for given \( \tilde{\varphi} \) satisfying the above constraint, only \( \bar{x} \) which is in the range specified in (41), gives at least a locally stable system. Otherwise the system is not stable. Note that the lower bound \( 2/(\tilde{d} + 2) \) for \( \bar{x} \) is of the order of unity for allowed \( \tilde{d} \leq 7 \) and the condition (24) in the present context is \( l/\tilde{\rho}_B \ll 1 \). So we can easily have \( 1 > \bar{x} > 2/(\tilde{d} + 2) \gg l/\tilde{\rho}_B \) to satisfy the condition (24).

This concludes our discussion on the stability of asymptotically flat black \( p \)-branes in the grand canonical ensemble. Using this information we will construct the phase structure of the equilibrium states of the black \( p \)-branes in this ensemble and compare with that in the canonical ensemble.

### 4 Phase structure of black \( p \)-branes

We have seen in the previous section that the stability condition for the black \( p \)-branes is given by (39). When we use the potential condition at the equilibrium, \( \tilde{\varphi} = \varphi(x, q) \), to eliminate \( q \) and obtain (33), i.e,

\[ b(x) = \frac{x^{1/\tilde{d}} [1 - (1 - \tilde{\varphi}^2) x]^{1/2}}{\tilde{d} (1 - \tilde{\varphi}^2)^{1/2 - 1/\tilde{d}}} \tag{42} \]
we have the range of \( x \) as \( 0 < x < 1 \). This is different from the fixed charge case of the canonical ensemble where the range is \( q < x < 1 \), since here we are considering the grand canonical ensemble where the potential \( \bar{\varphi} \) is fixed and not the charge \( q \). Even for the zero charge case of the canonical ensemble, where the range of \( x \) in the two cases are the same, the phase structure in these two cases, as we will see, will be quite different due to the different boundary data in the two ensembles. Note that as \( x \to 0 \), \( b(x \to 0) = 0 \), exactly as in the canonical case, but as \( x \to 1 \), \( b(x \to 1) \neq 0 \), where it has the value,

\[
b(1) = \frac{\bar{\varphi}}{d(1 - \bar{\varphi}^2)^{\frac{1}{2} - \frac{1}{d}}} > 0
\]

and this is different from the fixed charge canonical case including the zero charge case. The local temperature at this point has the value

\[
T(1) = \frac{\bar{d}(1 - \bar{\varphi}^2)^{\frac{1}{2} - \frac{1}{d}}}{4\pi \bar{\rho}_B \bar{\varphi}}.
\]

Also note from (42) that in the range \( 0 < x < 1 \), \( b(x) > 0 \) and so, if we plot \( b(x) \) vs. \( x \) curve it will start from zero when \( x \to 0 \) and then it rises and ends at \( b(1) = 0 \) given above when \( x \to 1 \). In between there can be extrema for \( b(x) \) and in fact, it is not difficult to check from

\[
\frac{db(x)}{dx} = \frac{b(x) \left[ 2 - (d + 2)x(1 - \bar{\varphi}^2) \right]}{2d x \left[ 1 - x(1 - \bar{\varphi}^2) \right]},
\]

that there can be only one extremum and that is a maximum if it exists at all. This maximum is not always guaranteed to exist unlike the chargeless case of canonical ensemble. However, when it exists indeed, the characteristic behavior of \( b(x) \) vs. \( x \) is given in Figure 1. The plot of \( T(x) \) vs. \( x \) curve will have opposite behavior. It will start from infinity when \( x \to 0 \) then the curve will drop and end at \( T(1) \) as \( x \to 1 \). In between there can be an extremum and in this case it is a minimum. Let us now determine the condition for the existence of the maximum in \( b(x) \) or minimum of \( T(x) \). Equating (45) to zero gives us the maximum of \( x \) as,

\[
x_{\text{max}} = \frac{2}{(2 + \bar{d})(1 - \bar{\varphi}^2)}.
\]

Now substituting this value in (42) we determine the maximum value of \( b(x) \) and from there the minimum value of \( T(x) \) as,

\[
b_{\text{max}} = \left( \frac{2}{2 + \bar{d}} \right)^{\frac{1}{d}} \left[ \bar{d}(\bar{d} + 2)(1 - \bar{\varphi}^2) \right]^{-\frac{1}{d}} \Rightarrow \\
T_{\text{min}} = \left( \frac{2 + \bar{d}}{2} \right)^{\frac{1}{d}} \left[ \frac{\bar{d}(\bar{d} + 2)(1 - \bar{\varphi}^2)}{4\pi \bar{\rho}_B \bar{\varphi}} \right]^{\frac{1}{d}}.
\]
Figure 1: The typical behavior of the (reduced) inverse temperature \( b(x) \) vs. the (reduced) horizon radius \( x \) at a fixed potential \( \bar{\varphi} \).

The condition for the existence of maximum is \( x_{\text{max}} < 1 \), which gives \( \bar{\varphi} < (\tilde{d}/(\tilde{d} + 2))^{1/2} \), i.e., the same constraint we obtained earlier in (41), which is expected since the non-existence of maximum implies no local stability. From this we have the following constraint on \( b_{\text{max}} \) or \( T_{\text{min}} \):

\[
b_{\text{max}} < \frac{1}{\sqrt{2d}} \left( \frac{2}{2 + d} \right)^{\frac{1}{2}} \quad \Rightarrow \quad T_{\text{min}} > \frac{\sqrt{2d}}{4\pi \rho_B} \left( \frac{2 + d}{2} \right)^{\frac{1}{2}}
\]

where we have used the constraint on \( \bar{\varphi} \) given in (41). So, when \( b_{\text{max}} \) (\( T_{\text{min}} \)) given in eq.(47) satisfy the constraint (48) we will have the maximum (minimum). From Figure 1, we can see that when \( x_{\text{max}} < x < 1 \), \( db/dx < 0 \) and we have a locally stable system and when \( 0 < x < x_{\text{max}} \), \( db/dx > 0 \) and we have an unstable system consistent with what we discussed earlier.

For any given \( \bar{\varphi} < 1 \) and \( \bar{b} \), satisfying \( b(1) < \bar{b} < b_{\text{max}} \) with \( b(1) \) as given in (43), one expects two black brane solutions with radii \( x_1 \) and \( x_2 \), where \( x_1 < x_2 \) from \( \bar{b} = b(x) \) as shown in Figure 1 and only the large one (\( x_2 \)) will give a locally stable phase. The small one (\( x_1 \)) will be unstable. Since here for each given \( \bar{\varphi} \) and \( \bar{b} \), we have only one locally stable phase, we don’t have a thermodynamical phase transition like in the canonical ensemble\(^\text{16}\), even though \( x_1 = x_2 = x_{\text{max}} \) seems to appear as a critical point. When \( \bar{\varphi} \geq (\tilde{d}/(\tilde{d} + 2))^{1/2} \) and/or \( \bar{b} > b_{\text{max}} \) (\( T < T_{\text{min}} \)) no black brane phase is possible in the

\(^{16}\)In the canonical ensemble there is a phase transition between two stable black brane phases, one of which is locally stable and the other is globally stable. Which one is locally stable and which one is globally stable depend on the temperature of the environment in contact with the system [1].
grand canonical ensemble and the only possible thermally stable phase here is the ‘hot flat space’. Also for \(0 = b(0) < \bar{b} < b(1)\), assuming \(\bar{\varphi} < (\bar{d}/(\bar{d} + 2))^{1/2}\) from now on, no stable black brane phase is possible and the ‘hot flat space’ is again the thermally stable phase. This clearly shows that the thermodynamics and the phase structure are quite different for the grand canonical ensemble than those in the canonical ensemble [1] and this is expected since the boundary data are different in the two cases.

To the leading order, the Euclidean action is directly related to the grand potential or Gibbs free energy as \(\beta \Omega = I_E\). So, for a given temperature, smaller the value of \(I_E\), the more stable the system is. We will consider the reduced action \(\tilde{I}_E\). For a given \(\bar{b}\) and \(\bar{\varphi}\) satisfying their respective constraints such that the system is locally stable, if the reduced action is positive at the stationary point, it is metastable since the reduced action has smaller value, i.e., zero, for the ‘hot flat space’ with the same boundary data and will make a transition to this phase. In other words, if the system is initially at the locally stable black brane phase, after some time, a phase transition would occur so the black brane phase will become the ‘hot flat space’ phase with a different topology via a phase transition. So, for making sure that we have a stable black brane phase, we need to look for condition such that the minimum of the reduced action is actually a global minimum. This can be simply realized by conditions such that the stationary action at the minimum is negative.

The reduced action at the stationary point can be expressed from the action (34) as,

\[
\tilde{I}_E = \frac{\bar{b}}{y}(\bar{d} + 2)(1 - y)\left(y - \frac{\bar{d}}{\bar{d} + 2}\right)
\]

where we have used \(b(\bar{x}) = \bar{b}\) with \(\bar{x}\) the stationary point and \(y = \sqrt{1 - \bar{x}(1 - \bar{\varphi}^2)}\). Given the fact that \(y < 1\), so a negative reduced action requires,

\[
y < \frac{\bar{d}}{\bar{d} + 2}
\]

which in turn gives,

\[
\bar{x}(1 - \bar{\varphi}^2) > \frac{4(\bar{d} + 1)}{(\bar{d} + 2)^2},
\]

where we have used the expression of \(y\) given above. Note that this condition is consistent with the local stability condition given in (40) since \(4(\bar{d} + 1)/(\bar{d} + 2)^2 > 2/(\bar{d} + 2)\). Also note that \(4(\bar{d} + 1)/(\bar{d} + 2)^2 < 1\) for all \(\bar{d}\) and therefore, there is no contradiction. Actually, the constraint (51) gives the following new constraints

\[
\frac{4(\bar{d} + 1)}{(\bar{d} + 2)^2} < \bar{x} < 1, \quad 0 < \bar{\varphi} < \frac{\bar{d}}{\bar{d} + 2}.
\]
So (51) or (52) is the condition for global stability while (40) or (41) is merely the local stability condition. If we denote

$$\bar{x}_g = \frac{4(\tilde{d} + 1)}{(d + 2)^2 (1 - \varphi^2)} (> x_{\text{max}})$$

then for $0 < \varphi < \tilde{d}/(\tilde{d} + 2)$ and $b(1) < \bar{b} < b_g$ ($T_g < \bar{T} < T(1)$), where,

$$b_g = \frac{\left(4(\tilde{d} + 1)\right)^{\frac{1}{4}}}{(\tilde{d} + 2)^{\frac{1}{4}} \sqrt{1 - \varphi^2}} < b_{\text{max}}$$

and

$$T_g = \frac{4\pi \rho_B \left(4(\tilde{d} + 1)\right)^{\frac{1}{4}}}{4(\tilde{d} + 2)^{\frac{1}{4}} \sqrt{1 - \varphi^2}} > T_{\text{min}},$$

the reduced action has a global minimum at $\bar{x}_g < x_2 < 1$. In other words, this ensemble can give a sensible description of black brane thermodynamics only when the above conditions are satisfied. We would like to point out that when the maximum of $b(x)$ exists (or minimum of $T(x)$ exists), the grand canonical ensemble is in some sense similar to the chargeless case of canonical ensemble except around $x = 1$.

While the above gives a clear picture of phase structure for black $p$-branes in the grand canonical ensemble, we can also consider the dependence of the free energy on the temperature at fixed potential, as in the usual approach, to understand the same structure. For this, we note the known results from the above: 1) When $\bar{\varphi} > 1$, the only stable phase is the ‘hot flat space’ and we have the vanishing free energy ($\tilde{I}_E = 0$). 2) When $(\tilde{d}/(\tilde{d} + 2))^{1/2} < \bar{\varphi} < 1$, we have the (reduced) on-shell free energy for the branes

$$\tilde{\Omega}(\bar{x}) \equiv \frac{\tilde{I}_E(\bar{x})}{\bar{b}} = (\tilde{d} + 2) \frac{1 - y}{y} \left( y - \frac{\tilde{d}}{\tilde{d} + 2} \right) > 0,$$

where $y = \sqrt{1 - \bar{x}(1 - \bar{\varphi}^2)}$ and $(\tilde{d}/(\tilde{d} + 2))^{1/2} < y < 1$. When this happens, $b(x)$ does not have a maximum and there exists only one solution $\bar{x}$ from the equation $\bar{b} = b(\bar{x})$ for $0 = b(0) < \bar{b} < b(1)$. The reduced temperature defined as $t \equiv 1/\bar{b} = 4\pi \rho_B T$ is now in the range of $t(1) < t < \infty$ with

$$t(1) = \frac{\tilde{d} (1 - \bar{\varphi}^2)^{\frac{1}{4}}}{\bar{\varphi}},$$

from (43), and the plot of the (reduced) free energy vs the (reduced) temperature gives only the unstable branch as shown in Figure 2(a). For this case, we do not have stable
brane phase and the only stable phase is the ‘hot flat space’. Here we have \( \tilde{\Omega}(t \to \infty) \to 0 \) and
\[
\tilde{\Omega}(t \to t(1)) \to (\tilde{d} + 2) \frac{1 - \bar{\varphi}}{\bar{\varphi}} \left( \bar{\varphi} - \frac{\tilde{d}}{\bar{\varphi} + 2} \right),
\]
which is positive for the allowed range \((\bar{d}/(\tilde{d} + 2))^{1/2} \leq \bar{\varphi} < 1\). Let us examine the slope of \( \tilde{\Omega}(t) \) with respect to \( t \) in general. We have
\[
\frac{d\tilde{\Omega}(t)}{dt} = -\frac{1 - \bar{\varphi}^2}{2y^3} \left[ \bar{x}(1 - \bar{\varphi}^2) - \frac{2}{\bar{\varphi} + 2} \right] \frac{d\bar{x}(t)}{dt}
\]
\[
= -\frac{\tilde{d}}{\bar{\varphi} + 2} \frac{\bar{x}(t)(1 - \bar{\varphi}^2)}{t y(t)} < 0,
\]
where in the second line we have used
\[
b(\bar{x}) t = \tilde{b} t = 1, \quad \frac{db(\bar{x})}{d\bar{x}} \frac{d\bar{x}(t)}{dt} = -\frac{1}{t^2} < 0,
\]
and (45). In other words, the slope is always negative. 3) When \( 0 < \bar{\varphi} < (\bar{d}/(\bar{d} + 2))^{1/2} \), \( b(x) \) has a maximum and occurs at \( x_{\text{max}} \) given in (46). This gives a minimum (reduced) temperature as
\[
t_{\text{min}} = 1/b(x_{\text{max}}) = \left( \frac{2 + \tilde{d}}{2} \right)^{\frac{1}{2}} \left[ \tilde{d}(\tilde{d} + 2)(1 - \bar{\varphi}^2) \right]^{\frac{1}{2}}.
\]
There exist two solutions \( x_1(t) \) and \( x_2(t) \) \((x_1 < x_2)\) from \( b(\bar{x}) = 1/t \) for each given \( t \) satisfying \( t(1) > t > t_{\text{min}} \) as discussed earlier and shown in Figure 1. The small \( x_1(t) \) corresponds to the unstable brane and the large \( x_2(t) \) corresponds to the locally stable brane. This implies that there always exist two branches in the plot of the (reduced) free energy vs the (reduced) temperature. The unstable branch is obtained by substituting the solution \( x_1(t) \) in the reduced free energy (56) with \( t_{\text{min}} < t < \infty \) while the locally stable branch is obtained similarly but now by substituting the solution \( x_2(t) \) instead with \( t_{\text{min}} < t < t(1) \), where \( t(1) \) is again given by (57) but \( \bar{\varphi} \) has its present range as given above. The unstable branch always has positive free energy which is larger than the free energy in the locally stable branch except at \( t = t_{\text{min}} \) where the two become equal and it is
\[
\tilde{\Omega}(t = t_{\text{min}}) = (\tilde{d} + 2) \left( 1 - \sqrt{\frac{\tilde{d}}{\bar{\varphi} + 2}} \right)^2 > 0,
\]
which is explicitly independent of \( \bar{\varphi} \) in the present case. The (reduced) free energy at the other end in this branch is again \( \tilde{\Omega}(t \to \infty) \to 0 \), also independent of \( \bar{\varphi} \). In this branch, we
Figure 2: The typical behavior of the (reduced) free energy $\tilde{\Omega}(t)$ vs. the (reduced) temperature $t$ at a fixed potential $\bar{\varphi}$. Plot (a) considers the case $\sqrt{\tilde{d}/(\tilde{d} + 2)} \leq \bar{\varphi} < 1$ which has only one unstable branch. Plot (b) considers the case $\tilde{d}/(\tilde{d} + 2) \leq \bar{\varphi} < \sqrt{\tilde{d}/(\tilde{d} + 2)}$ which has two branches, one unstable and the other only locally stable branch. Plot (c) considers the case $0 < \bar{\varphi} < \tilde{d}/(\tilde{d} + 2)$ which also has two branches, one unstable and the other (locally) stable branch. The unstable branch has always positive free energy for $t_{\text{min}} \leq t < \infty$ while the stable branch has negative free energy in the range $t_g < t < t(1)$, where it becomes globally stable.
always have $x_1(t)(1 - \bar{\varphi}^2) < 2/(\tilde{d} + 2)$ which can be obtained from $x_1(t) < x_{\text{max}}$ with $x_{\text{max}}$ given in (46). This branch gives no stable brane phase. For the locally stable branch, when $\tilde{d}/(\tilde{d} + 2) \leq \bar{\varphi} < (\tilde{d}/(\tilde{d} + 2))^{1/2}$, $\tilde{\Omega}(t) > 0$ for $t_{\text{min}} < t < t(1)$ with the positive $\tilde{\Omega}(t = t_{\text{min}})$ given by (62) and the non-negative $\tilde{\Omega}(t \to t(1))$ still given by (58) but now for the present range of $\bar{\varphi}$ given above. The plot of the free energy vs. the temperature is given in Figure 2(b) for this subcase. This locally stable branch gives only meta-stable brane phase and will eventually decay to the ‘hot flat space’ after their formation. If we want the existence of globally stable branes, the free energy needs to be negative at certain temperature. This can be realized when $0 < \bar{\varphi} < \tilde{d}/(\tilde{d} + 2)$. With this, the free energy vanishes at $t = t_g$ and becomes negative when $t_g < t < t(1)$ with

$$t_g \equiv 1/b_g = \frac{(\tilde{d} + 2)^{1+\frac{2}{d}}}{4(\tilde{d} + 1)}$$

from (54), therefore this branch becomes globally stable in this range. One can check that the (reduced) free energy $\tilde{\Omega}(t \to t(1)) < 0$ as expected in the stable branch using its expression given again by (58). Note that the value of this limit increases as $\bar{\varphi}$ increases. One can also check that $t_{\text{min}} < t_g < t(1)$ in the present subcase. The plot of the (reduced) free energy vs the (reduced) temperature for fixed $\bar{\varphi}$ is now given in Figure 2(c). For each given $t$ satisfying $t_{\text{min}} \leq t < t(1)$ in the present two subcases, the corresponding plot just reflects what has been described. In the range $0 < \bar{\varphi} < \tilde{d}/(\tilde{d} + 2)$, $t_{\text{min}}, t_g$ and $t(1)$ all decrease when $\bar{\varphi}$ increases which can be seen from their respective expressions. Note that $\tilde{\Omega}(t = t_{\text{min}})$ in both branches and $\tilde{\Omega}(t \to \infty)$ in the unstable branch are both independent of $\bar{\varphi}$, and $\tilde{\Omega}(t \to t(1))$ increases as $\bar{\varphi}$ increases. As shown above, the slope of $\tilde{\Omega}(t)$ is negative in either branch. These features give the general character of the free energy vs the temperature plot in both branches for different fixed potentials. One may take Figure 2(b) as the consequence of increasing $\bar{\varphi}$ from Figure 2(c).

5 Conclusion

To conclude, in this paper we have studied the equilibria and the phase structure of the asymptotically flat dilatonic black $p$-branes in a fixed cavity in arbitrary dimensions $D$ in a grand canonical ensemble. For this, we have considered both the temperature and the potential fixed at the wall of the cavity and compared with our previous study of the same system in a canonical ensemble [1] for which the charge enclosed in the cavity, rather than the potential at the wall of the cavity, is fixed. We computed the Euclidean action
corresponding to the dilatonic black $p$-brane solution of $D$-dimensional supergravity with maximal supersymmetry with proper consideration of the above fixed quantities at the wall of cavity in this ensemble. To the leading (or zeroth-loop) order this action is related to the grand potential or Gibbs free energy of the system and is an essential entity for the stability analysis. This action has two variables, namely, the horizon size $r_+$ (related to the variable $x$) and the charge $Q_d$ (related to the variable $q$). At the stationary point of this action, we derived the expected equations for which $x$ and $q$ have to satisfy with both the temperature and the potential fixed at the wall of cavity. The second derivatives of the action with respect to $x$ and $q$ at the stationary point can be used to analyze the local thermal stability of the system in this ensemble.

Unlike in the canonical case, the directly obtained action here is a function of two variables and this makes the stability analysis a bit involved. However, there exists also an analog of one-variable analysis as in the canonical case and we have shown in the text that the condition so obtained for the stability of the equilibrium states as given in (39) is the same as that of the rather complicated two-variable case. For the two-variable analysis given in the Appendix, we have made use of a trick to use the $b, \varphi$ variables as defined in (30) instead of original $x, q$ variables. This makes the analysis a bit simpler and straightforward. The stability condition can also be expressed as those given in (41). So, for a given potential satisfying the constraint given in (41), only the horizon size lying in the range also given in (41) gave locally stable system.

We then obtained the phase structure of the black $p$-branes in this ensemble. We found that locally stable black brane phase exists only when the minimum temperature of the system given in (47) satisfies the constraint (48). In this case, below the minimum temperature, there is no stable black brane phase and only stable phase is the ‘hot flat space’. Above this temperature and below the value $T(1)$ given in (44), there are two black brane phases, the larger one is locally stable and the smaller one is unstable. The locally stable black brane phase becomes globally stable only above the temperature $T_g$ given in (55) and below $T(1)$. Below $T_g$ and above $T_{\text{min}}$ (given in (47)), the stable black brane is only locally stable and will eventually make a transition to the ‘hot flat space’. Finally, above $T(1)$ there is no stable brane phase and the only stable phase is again the ‘hot flat space’ phase. We also commented on the similarity of this phase structure with that of the zero charge canonical case except at one end of the $x$ variable, namely, near $x = 1$. This structure is reminiscent of the Hawking-Page phase transition of the AdS, dS and flat black holes discussed in [8, 12, 13]. However, unlike in the canonical case, here we found that at a given temperature when the stable phase exists it does not come in pairs – only a single stable phase appears. So, there is no thermodynamical phase transition
between two stable black brane phases analogous to the van der Waals-Maxwell liquid-gas phase transition like that appeared in the canonical case.

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Appendix: The two-variable stability analysis

In this appendix we perform the more general yet more complicated two-variable analysis of the stability of black p-branes and show that the same local stability condition derived from the simple one-variable consideration as given in (39) continues to hold.

It is clear from (30) that both $b(x, q)$ and $\varphi(x, q)$ are smooth functions of $x$ and $q$ and so, in principle they can be inverted to give $x$ and $q$ as functions of $b$ and $\varphi$. Although we will not need their explicit form, however, we will need the form of $b$ as a function of $x$ and $\varphi$. For that we will eliminate $q$ from $b(x, q)$-equation given in (30) and express it as a function of $x$ and $\varphi$ instead. From the $\varphi$ equation in (30) we obtain $q^2/x^2$ as,

$$\frac{q^2}{x^2} = \frac{\varphi^2}{1 - (1 - \varphi^2)x}$$

(64)

Now substituting (64) in $b$ equation in (30) we obtain,

$$b = \frac{x^{\frac{d}{2}} [1 - (1 - \varphi^2)x]^{\frac{d}{2}}}{\tilde{d}(1 - \varphi^2)^{\frac{d}{2} - \frac{d}{2}}}.$$  

(65)

The reduced action (31) can now be re-expressed as,

$$\tilde{I}_E(x, \varphi) = -\tilde{b} \left[ (\tilde{d} + 2)y + \tilde{d} (1 - x(1 - \varphi\varphi)) y^{-1} - 2(\tilde{d} + 1) \right] + b\tilde{d}(y^2 - 1)y^{-1},$$

(66)

where we have used (64) and (65). In the above, we have also defined

$$y = \sqrt{1 - x(1 - \varphi^2)},$$

(67)
with \( x(b, \varphi) \) determined by (65). We will use both of (65) and (66) later.

We now expand the reduced action (31) at the stationary point determined by the equation (29) with \( x = \bar{x} \) and \( q = \bar{q} \) to quadratic order as,

\[
\tilde{I}_E(x, q) = \tilde{I}_E(\bar{x}, \bar{q}) + \tilde{I}_{ij} \bigg|_{z_i = \bar{z}_i, z_j = \bar{z}_j} (z_i - \bar{z}_i)(z_j - \bar{z}_j) + \cdots
\]  

(68)

where \( z_1 = x, z_2 = q \) with \( i, j = 1, 2 \) and we have used the stationary conditions for the first order terms. In the above

\[
\tilde{I}_{ij} \equiv \frac{\partial^2 \tilde{I}_E(x, q)}{\partial z_i \partial z_j}
\]

(69)

The local stability of the system is determined by whether the quadratic terms in the expansion are always positive definite. This can be easily understood if we diagonalize the matrix \( \tilde{I}_{ij} \) and demand that each of the two eigenvalues \( \lambda_1 \) and \( \lambda_2 \) is positive definite. Now since,

\[
\lambda_1 \lambda_2 = \det \tilde{I}_{ij}, \quad \text{and} \quad \lambda_1 + \lambda_2 = \tilde{I}_{xx} + \tilde{I}_{qq}
\]

(70)

\( \lambda_1, \lambda_2 > 0 \) implies that both \( \tilde{I}_{xx} + \tilde{I}_{qq} > 0 \) and \( \det \tilde{I}_{ij} > 0 \). Now we will rewrite these two conditions such that our analysis becomes simpler as\(^{17}\),

\[
\frac{\tilde{I}_{qq}}{\det \tilde{I}_{ij}} > 0, \quad \text{(71)}
\]

\[
\frac{\tilde{I}_{qq}}{\det \tilde{I}_{ij}} > 0 \quad \text{(72)}
\]

In order to understand the meaning of the condition (71) we have to evaluate \( \tilde{I}_{qq} \) and then set it to greater than zero. From the form of \( \tilde{I}_E \) in (31) we obtain the first condition (71) as,

\[
\tilde{I}_{qq} \equiv \frac{\partial^2 \tilde{I}_E}{\partial q^2} \bigg|_{x = \bar{x}, q = \bar{q}} = \frac{\bar{b}d\Phi}{\bar{q}(1 - \frac{\bar{x}^2}{2})} \left[ 1 + \frac{(\bar{d} + 2)\Phi^2 [1 - \frac{2}{d} + \frac{2}{d}\bar{x}(1 - \Phi^2)]}{\bar{d}(1 - \Phi^2)(1 - \bar{x})} \right] > 0.
\]

(73)

Once this is satisfied the thermodynamic stability is completely determined by the condition (72). A direct evaluation of \( \tilde{I}_{qq}/\det \tilde{I}_{ij} \) from the expression (in \( (x, q) \) variables) of

\(^{17}\)It is not difficult to see that the conditions given in (71) and (72) automatically implies \( \det \tilde{I}_{ij} > 0 \) and \( \tilde{I}_{xx} + \tilde{I}_{qq} > 0 \). To see this note that we first have to have \( \tilde{I}_{qq} > 0 \) (if \( I_{qq} < 0 \), the above two conditions can never be satisfied), then (72) implies \( \det \tilde{I}_{ij} > 0 \) and so, \( \det \tilde{I}_{ij} = \tilde{I}_{qq} \tilde{I}_{xx} - \tilde{I}_{qq}^2 = \tilde{I}_{qq}(\tilde{I}_{xx} - \tilde{I}_{qq}/\tilde{I}_{qq}) > 0 \Rightarrow \tilde{I}_{xx} - \tilde{I}_{qq}/\tilde{I}_{qq} > 0 \Rightarrow \tilde{I}_{xx} > \tilde{I}_{qq}/\tilde{I}_{qq} > 0 \) and this in turn implies \( \tilde{I}_{xx} + \tilde{I}_{qq} > 0 \).
\( \bar{I}_E \) in (31) is complicated and tedious and we will make use of a trick in evaluating it. This will make the analysis of stability much more elegant and simpler. For this, we will instead evaluate the l.h.s. of (72) by first going to the \((b, \varphi)\) variable and evaluate \( \bar{I}_{ab} \), where

\[
\bar{I}_{ab} \equiv \frac{\partial^2 \bar{I}_E(b, \varphi)}{\partial \xi_a \partial \xi_b}
\]

with \( a, b = 1, 2 \) and \( \xi_1 = \varphi, \xi_2 = b \) and then relate this matrix to the original matrix \( \bar{I}_{ij} \) in \((x, q)\) variables by chain rules as follows,

\[
\bar{I}_{ab} = \begin{pmatrix}
\bar{I}_{\varphi \varphi} & \bar{I}_{\varphi b} \\
\bar{I}_{b \varphi} & \bar{I}_{bb}
\end{pmatrix}
= \begin{pmatrix}
\frac{\partial}{\partial \varphi} & \frac{\partial}{\partial q} \\
\frac{\partial}{\partial q} & \frac{\partial}{\partial b}
\end{pmatrix} \begin{pmatrix}
\bar{I}_{ij}
\end{pmatrix} \begin{pmatrix}
\frac{\partial}{\partial \varphi} & \frac{\partial}{\partial q} \\
\frac{\partial}{\partial q} & \frac{\partial}{\partial b}
\end{pmatrix}^T,
\]

where ‘\(^T\)’ denotes the transpose of a matrix. After that we invert this matrix relation to obtain,

\[
\bar{I}_{ij}^{-1} = \begin{pmatrix}
\frac{\partial}{\partial \varphi} & \frac{\partial}{\partial q} \\
\frac{\partial}{\partial q} & \frac{\partial}{\partial b}
\end{pmatrix}^T \bar{I}_{ab}^{-1} \begin{pmatrix}
\frac{\partial}{\partial \varphi} & \frac{\partial}{\partial q} \\
\frac{\partial}{\partial q} & \frac{\partial}{\partial b}
\end{pmatrix},
\]

Again inverting (76) we will evaluate \( \bar{I}_{ij}/\det \bar{I}_{ij} \) in terms of \( \bar{I}_{ab} \) and other known functions given in (76). Now to evaluate \( \bar{I}_{ab} \), we first calculate

\[
\left( \frac{\partial \bar{I}_E}{\partial \varphi} \right)_b = \left( \frac{\partial \bar{I}_E}{\partial x} \right)_q \left( \frac{\partial x}{\partial \varphi} \right)_b + \left( \frac{\partial \bar{I}_E}{\partial q} \right)_x \left( \frac{\partial q}{\partial \varphi} \right)_b
= \left[ \bar{b} - b(x, q) \right] f(x, q) \left( \frac{\partial x}{\partial \varphi} \right)_b
- \left[ \bar{d} \bar{d}(\bar{\varphi} - \varphi(x, q)) + \frac{\bar{d} + 2}{1 - \frac{q^2}{x}}(\bar{b} - b(x, q)) \right] \left( \frac{\partial q}{\partial \varphi} \right)_b,
\]

and

\[
\left( \frac{\partial \bar{I}_E}{\partial b} \right)_\varphi = \left( \frac{\partial \bar{I}_E}{\partial x} \right)_q \left( \frac{\partial x}{\partial b} \right)_\varphi + \left( \frac{\partial \bar{I}_E}{\partial q} \right)_x \left( \frac{\partial q}{\partial b} \right)_\varphi
= \left[ \bar{b} - b(x, q) \right] f(x, q) \left( \frac{\partial x}{\partial b} \right)_\varphi
- \left[ \bar{d} \bar{d}(\bar{\varphi} - \varphi(x, q)) + \frac{\bar{d} + 2}{1 - \frac{q^2}{x}}(\bar{b} - b(x, q)) \right] \left( \frac{\partial q}{\partial b} \right)_\varphi.
\]
where we have used the form of $\tilde{I}_E$ given in (31) and the function $f(x, q)$ is defined as,

$$f(x, q) = (1 - x)^{-1} \left( 1 - \frac{q^2}{x} \right)^{-1} \left[ \tilde{d} + 2 - \frac{\tilde{d} + 2}{2} \left( \frac{1 - \tilde{q}^2}{1 - \frac{q^2}{x}} \right) + \frac{\tilde{d}}{2} \left( 1 - \frac{q^2}{x} \right) \right] > 0, \quad (79)$$

Thus we have from eqs. (77) and (78)

$$\begin{align*}
\frac{\partial^2 \tilde{I}_E}{\partial \varphi^2} &= \tilde{b} \tilde{d} \frac{\partial q}{\partial \varphi}, \\
\frac{\partial^2 \tilde{I}_E}{\partial b \partial \varphi} &= -f(\bar{x}, \bar{q}) \frac{\partial x}{\partial \varphi} + \frac{(\tilde{d} + 2)\bar{\varphi}}{1 - \frac{q^2}{x}} \frac{\partial q}{\partial \varphi}, \\
\frac{\partial^2 \tilde{I}_E}{\partial b^2} &= \bar{b} \tilde{d} \frac{\partial q}{\partial b}, \\
\frac{\partial^2 \tilde{I}_E}{\partial b \partial \varphi} &= -f(\bar{x}, \bar{q}) \frac{\partial x}{\partial b} + \frac{(\tilde{d} + 2)\bar{\varphi}}{1 - \frac{q^2}{x}} \frac{\partial q}{\partial b},
\end{align*} \quad (80)$$

where the second derivatives are evaluated at $b = \bar{b}$ and $\varphi = \bar{\varphi}$. So, from the above equations (80) we obtain $\tilde{I}_{ab}$ as,

$$\tilde{I}_{ab} = \begin{pmatrix}
\tilde{b} \tilde{d} \frac{\partial q}{\partial \varphi} \\
\tilde{b} \tilde{d} \frac{\partial q}{\partial b} \\
-f(\bar{x}, \bar{q}) \frac{\partial x}{\partial \varphi} + \frac{(\tilde{d} + 2)\bar{\varphi}}{1 - \frac{q^2}{x}} \frac{\partial q}{\partial \varphi}
\end{pmatrix}, \quad (81)$$

where we have used

$$\tilde{b} \tilde{d} \frac{\partial q}{\partial b} = -f(\bar{x}, \bar{q}) \frac{\partial x}{\partial \varphi} + \frac{(\tilde{d} + 2)\bar{\varphi}}{1 - \frac{q^2}{x}} \frac{\partial q}{\partial \varphi}. \quad (82)$$

From the form of $\tilde{I}_{ab}$ given in (81) we can calculate its inverse as,

$$\tilde{I}_{ab}^{-1} = \frac{1}{\det \tilde{I}_{ab}} \begin{pmatrix}
-f(\bar{x}, \bar{q}) \frac{\partial x}{\partial \varphi} + \frac{(\tilde{d} + 2)\bar{\varphi}}{1 - \frac{q^2}{x}} \frac{\partial q}{\partial \varphi} \\
-\tilde{b} \tilde{d} \frac{\partial q}{\partial b} \\
\tilde{b} \tilde{d} \frac{\partial q}{\partial \varphi}
\end{pmatrix}, \quad (83)$$

with $\det \tilde{I}_{ab}$ having the from (as can be seen from (81))

$$\det \tilde{I}_{ab} = \tilde{f} \bar{b} \tilde{d} \left( \frac{\partial q}{\partial \varphi} \frac{\partial x}{\partial b} - \frac{\partial q}{\partial b} \frac{\partial x}{\partial \varphi} \right) \quad (84)$$

where we have made use of (82). Now substituting (83) in (76) we obtain,

$$\tilde{I}_{ij}^{-1} = \begin{pmatrix}
\frac{1}{\tilde{b} \tilde{d}} \frac{\partial q}{\partial \varphi} \\
\frac{1}{\tilde{b} \tilde{d}} \frac{\partial x}{\partial \varphi} \\
\frac{1}{\tilde{b} \tilde{d}} (\frac{\tilde{d} + 2}{1 - \frac{q^2}{x}}) \frac{\partial x}{\partial \varphi} - \frac{1}{\tilde{f} \bar{b} \tilde{d}} \frac{\partial x}{\partial \varphi}
\end{pmatrix}, \quad (85)$$

where we also made use of (82). Inverting (85) we get

$$\tilde{I}_{ij} = \det \tilde{I}_{ij} \begin{pmatrix}
\frac{(\tilde{d} + 2)}{\tilde{f} \bar{b} \tilde{d} (1 - \frac{q^2}{x})} \frac{\partial x}{\partial \varphi} - \frac{1}{\tilde{f} \bar{b} \tilde{d}} \frac{\partial x}{\partial \varphi} - \frac{1}{\tilde{b} \tilde{d} \bar{x} \bar{\varphi}} \\
-\frac{1}{\tilde{b} \tilde{d} \bar{x} \bar{\varphi}} \\
\frac{1}{\tilde{b} \tilde{d} \bar{x} \bar{\varphi}}
\end{pmatrix}. \quad (86)$$

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From (85) we also obtain, \( \Delta I^{-1}_{ij} = (\Delta I_{ij})^{-1} = \Delta I_{ab}/(f\tilde{b})^2 \). Now from (86) we finally get,

\[
\frac{\Delta I_{qq}}{\det \Delta I_{ij}} = \frac{(\tilde{d} + 2)\bar{\varphi}}{f\tilde{b} (1 - \bar{x}^2)} \frac{\partial x}{\partial \varphi} - \frac{1}{f} \frac{\partial x}{\partial \tilde{b}},
\]

(87)

We can compute \( \partial x/\partial b \) and \( \partial x/\partial \varphi \) at the stationary point from (65) and obtain,

\[
\frac{\partial x}{\partial b} = \frac{2\tilde{d}\bar{x} [1 - \bar{x}(1 - \varphi^2)]}{b \left[ 2 - (\tilde{d} + 2)\bar{x}(1 - \varphi^2) \right]}, \quad \frac{\partial x}{\partial \varphi} = -\frac{2\tilde{d}\bar{x} \varphi [1 - \frac{2}{\tilde{d}} + \frac{2}{\tilde{d}} \bar{x}(1 - \varphi^2)]}{(1 - \varphi^2) \left[ 2 - (\tilde{d} + 2)\bar{x}(1 - \varphi^2) \right]},
\]

(88)

Substituting (88) in (87) we have from the condition (72),

\[
\frac{\Delta I_{qq}}{\det \Delta I_{ij}} = -\frac{2\tilde{d}\bar{x} [1 - \bar{x}(1 - \varphi^2)]}{f\tilde{b} \left[ 2 - (\tilde{d} + 2)\bar{x}(1 - \varphi^2) \right]} \left[ 1 + \frac{(\tilde{d} + 2)\varphi^2 \left[ 1 - \frac{2}{\tilde{d}} + \frac{2}{\tilde{d}} \bar{x}(1 - \varphi^2) \right]}{\tilde{d}(1 - \varphi^2)(1 - \bar{x})} \right] > 0.
\]

(89)

Thus for the stability of the system the above condition has to be satisfied. Now since the second factor in the square bracket is positive definite by the first condition written in (73) it is clear that for (89) to hold the denominator of the first factor has to be negative definite. This gives, as promised, precisely the same condition as given in (39).

Now it can be checked that once (39) is satisfied the first condition written explicitly in (73) is automatically satisfied. To see this we look at the numerator of the second term in the square bracket in the expression of \( \Delta I_{qq} \) given in (73),

\[
1 - \frac{2}{\tilde{d}} + \frac{2}{\tilde{d}} \bar{x}(1 - \varphi^2) = \frac{1}{\tilde{d}} \left[ \tilde{d} - 2(1 - \bar{x}(1 - \varphi^2)) \right] = [1 - \bar{x}(1 - \varphi^2)] \left[ 1 + \frac{(\tilde{d} + 2)\bar{x}(1 - \varphi^2) - 2}{\tilde{d}} \right]
\]

(90)

and this is positive definite since the first term is positive definite and the second term is positive definite by (39). Therefore, \( \Delta I_{qq} \) is positive definite. This shows that (39) is equivalent to both the stability conditions given in (71) and (72).

References

[1] J. X. Lu, S. Roy and Z. Xiao, “Phase transitions and critical behavior of black branes in canonical ensemble,” JHEP 1101, 133 (2011), arXiv:1010.2068 [hep-th].

[2] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1999)] [arXiv:hep-th/9711200].
[3] E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

[4] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory,” Phys. Lett. B 428, 105 (1998) [arXiv:hep-th/9802109].

[5] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323, 183 (2000) [arXiv:hep-th/9905111].

[6] G. T. Horowitz and V. E. Hubeny, “Quasinormal modes of AdS black holes and the approach to thermal equilibrium,” Phys. Rev. D 62, 024027 (2000) [arXiv:hep-th/9909056].

[7] S. W. Hawking and D. N. Page, “Thermodynamics Of Black Holes In Anti-De Sitter Space,” Commun. Math. Phys. 87, 577 (1983).

[8] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, “Charged AdS black holes and catastrophic holography,” Phys. Rev. D 60, 064018 (1999) [arXiv:hep-th/9902170].

[9] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, “Holography, thermodynamics and fluctuations of charged AdS black holes,” Phys. Rev. D 60, 104026 (1999) [arXiv:hep-th/9904197].

[10] S. W. Hawking, “Particle Creation By Black Holes,” Commun. Math. Phys. 43, 199 (1975) [Erratum-ibid. 46, 206 (1976)].

[11] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” Adv. Theor. Math. Phys. 2, 505 (1998) [arXiv:hep-th/9803131].

[12] S. Carlip and S. Vaidya, “Phase transitions and critical behavior for charged black holes,” Class. Quant. Grav. 20, 3827 (2003) [arXiv:gr-qc/0306054].

[13] A. P. Lundgren, “Charged black hole in a canonical ensemble,” Phys. Rev. D 77, 044014 (2008) [arXiv:gr-qc/0612119].

[14] G. T. Horowitz and A. Strominger, “Black strings and P-branes,” Nucl. Phys. B 360, 197 (1991).

[15] M. J. Duff and J. X. Lu, “Black and super p-branes in diverse dimensions,” Nucl. Phys. B 416, 301 (1994) [arXiv:hep-th/9306052].
[16] M. J. Duff, H. Lu and C. N. Pope, “The black branes of M-theory,” Phys. Lett. B 382, 73 (1996) [arXiv:hep-th/9601052].

[17] O. J. C. Dias, P. Figueras, R. Monteiro, H. S. Reall and J. E. Santis, “An instability of higher-dimensional rotating black holes,” JHEP 1005, 076 (2010) [arXiv:1001.4527 [hep-th]].

[18] D. Astefanesei, M. J. Rodriguez and S. Theisen, “Thermodynamic instability of doubly spinning black objects,” JHEP 1008, 046 (2010) [arXiv:1003.2421 [hep-th]].

[19] J. W. York, “Black hole thermodynamics and the Euclidean Einstein action,” Phys. Rev. D 33, 2092 (1986).

[20] J. D. Brown, “Black hole thermodynamics in a box,” arXiv:gr-qc/9404006.

[21] R. Parentani, J. Katz and I. Okamoto, “Thermodynamics of a black hole in a cavity,” Class. Quant. Grav. 12, 1663 (1995) [arXiv:gr-qc/94010015].

[22] C. S. Peca and J. P. S. Lemos, “Thermodynamics of toroidal black holes,” J. Math. Phys. 41, 4783 (2000) [arXiv:gr-qc/9809029].

[23] J. P. Gregory and S. F. Ross, “Stability and the negative mode for Schwarzschild in a finite cavity,” Phys. Rev. D 64, 124006 (2001) [arXiv:hep-th/0106220].

[24] O. B. Zaslavskii, “Boulware state and semiclassical thermodynamics of black holes in a cavity,” Phys. Rev. D 68, 127502 (2003) [arXiv:gr-qc/0310090].

[25] B. F. Whiting and J. W. York, “Action Principle and Partition Function for the Gravitational Field in Black Hole Topologies,” Phys. Rev. Lett. 61, 1336 (1988).

[26] H. W. Braden, J. D. Brown, B. F. Whiting and J. W. York, “Charged black hole in a grand canonical ensemble,” Phys. Rev. D 42, 3376 (1990).

[27] G. W. Gibbons and S. W. Hawking, “Action Integrals And Partition Functions In Quantum Gravity,” Phys. Rev. D 15, 2752 (1977).

[28] J. D. Brown and J. W. York, “The path integral formulation of gravitational thermodynamics,” arXiv:gr-qc/9405024.

[29] G. W. Gibbons and R. E. Kallosh, “Topology, entropy and Witten index of dilaton black holes,” Phys. Rev. D 51, 2839 (1995) [arXiv:hep-th/9407118].
[30] S. W. Hawking, G. T. Horowitz and S. F. Ross, “Entropy, Area, and black hole pairs,” Phys. Rev. D 51, 4302 (1995) [arXiv:gr-qc/9409013].

[31] S. W. Hawking and G. T. Horowitz, Class. Quant. Grav. 13, 1487 (1996) [arXiv:gr-qc/9501014].

[32] C. Teitelboim, Phys. Rev. D 51, 4315 (1995) [Erratum-ibid. D 52, 6201 (1995)] [Phys. Rev. D 52, 6201 (1995)] [arXiv:hep-th/9410103].

[33] S. M. Carroll, M. C. Johnson and L. Randall, “Extremal limits and black hole entropy,” JHEP 0911, 109 (2009) [arXiv:0901.0931 [hep-th]].

[34] M. J. Duff, R. R. Khuri and J. X. Lu, “String solitons,” Phys. Rept. 259, 213 (1995) [arXiv:hep-th/9412184].

[35] R. C. Tolman, “On the Weight of Heat and Thermal Equilibrium in General Relativity,” Phys. Rev. 35, 904 (1930).

[36] R. Kallosh, N. Sivanandam and M. Soroush, “The Non-BPS black hole attractor equation,” JHEP 0603, 060 (2006) [arXiv:hep-th/0602005].

[37] K. Goldstein, N. Iizuka, R. P. Jena and S. P. Trivedi, “Non-supersymmetric attractors,” Phys. Rev. D 72, 124021 (2005) [arXiv:hep-th/0507096].

[38] M. R. Garousi and A. Ghodsi, “On Attractor Mechanism and Entropy Function for Non-extremal Black Holes/Branes,” JHEP 0705, 043 (2007) [arXiv:hep-th/0703260].