ASYMPTOTIC BEHAVIOUR OF GRADED LOCAL COHOMOLOGY MODULES VIA LINKAGE

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Abstract. Assume that \( R = \bigoplus_{n \in \mathbb{N}_0} R_n \) is a standard graded algebra over the local ring \((R_0, m_0)\), \( a \) is a homogeneous ideal of \( R \), \( M \) is a finitely generated graded \( R \)-module and \( R_+ := \bigoplus_{n \in \mathbb{N}} R_n \) denotes the irrelevant ideal of \( R \). In this paper, we study the asymptotic behaviour of the set \( \{ \text{grade}(a \cap R_0, H^{\text{grade}(R_+)}_{R_+}(M)_n) \}_{n \in \mathbb{Z}} \) as \( n \to -\infty \), in the case where \( a \) and \( R_+ \) are homogenously linked over \( M \).

1. Introduction

Let \( R = \bigoplus_{n \in \mathbb{N}_0} R_n \) be a positively graded commutative Noetherian ring, \( R_+ = \bigoplus_{n \in \mathbb{N}} R_n \) be the irrelevant ideal of \( R \) and \( M \) denotes a graded \( R \)-module. Also, assume that \( a \) is a homogeneous ideal of \( R \). Then for all \( i \in \mathbb{N}_0 \), the set of non-negative integers, the \( i \)-th local cohomology module \( H^i_a(M) \) of \( M \) with respect to \( a \) has a natural grading (our terminology on local cohomology comes from [3]). Also, in the case where \( M \) is finitely generated, it is well-known that the graded components of local cohomology modules \( H^i_{R_+}(M)_n \) of \( M \) with respect to the irrelevant ideal are trivial for sufficiently large values of \( n \) and they are finitely generated as \( R_0 \)-module for all \( n \in \mathbb{Z} \), the set of integers ([3, 16.1.5]). The asymptotic behaviour of the components \( H^i_{R_+}(M)_n \) when \( n \to -\infty \) is more complicated and has been studied by many authors, too. See for example [1], [2], [4] and [8].

Assume that \( R \) is standard graded, i.e. \( R_0 \) is a commutative Noetherian ring and \( R \) is generated, as an \( R_0 \)-algebra, by finitely many elements of degree one, and \( M \) is finitely generated. In [4] the authors considered the problem of asymptotic behaviour of the set \( \{ \text{grade}(a_0, H^i_{R_+}(M)_n) \}_{n \in \mathbb{Z}} \) when \( n \to -\infty \), in the case where \((R_0, m_0)\) is local and \( a_0 \) is a proper ideal of \( R_0 \). They showed that this set is asymptotically stable in some special cases. In this paper, we consider this problem, via linkage.

Following [5], two homogeneous ideals \( a \) and \( b \) are said to be homogenously linked (or h-linked) by \( I \) over \( M \), denoted by \( a \sim_{I(M)} b \), if \( I \) is generated by a homogeneous \( M \)-regular sequence and \( aM = IM :_M b \) and \( bM = IM :_M a \). This is a generalization of the classical concept of linkage of ideals introduced by Peskine and Szpiro [10].

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Assume that the homogeneous ideal \( a \) and \( R_+ \) are h-linked over \( M \). We show that the set 
\[ \{ \text{grade}(a \cap R_0, H^\text{grade}(R_+,M)(M)_n), n \in \mathbb{Z} \} \]
is asymptotically stable when \( n \to -\infty \), and also we determine its stable value, in some cases, see Proposition 2.2. Also, in general case, we find a lower bound for this set. More precisely, let 
\[ f^R_+(M) := \inf \{ i \mid R_+ \nsubseteq \sqrt{0 : H^i_a(M)} \} \]
denotes the \( R_+ \)-finiteness dimension of \( M \) relative to \( a \) and \( f_R^+(M) := f^R_+(M) \) be the finiteness dimension of \( M \) relative to \( R_+ \). Then, we show that if \( a \sim^h_{(I,M)} R_+ \) and \( \text{grade}(R_+, M) = f^R_+(M) \) then \( \text{grade}(a \cap R_0, H^\text{grade}(R_+,M)(M)_n) \geq f^R_+(M) - f_R(a) \) for all \( n \ll 0 \) (Theorem 2.4). As a corollary, we show that the graded components \( H^R_+(M)_n \) are relative Cohen-Macaulay with respect to \( a \cap R_0 \) of degree \( f^R_+(M) - f_R(a) \), in some certain cases (Corollary 2.5). Recall that a non-zero finitely generated \( R \)-module \( X \) is said to be relative Cohen-Macaulay with respect to the ideal \( c \) of degree \( n \) if \( H^i_c(X) = 0 \) for all \( i \neq n \).

At the end, we show that if \((R_0, \mathfrak{m}_0)\) is local then the local cohomology modules \( H^R_{\mathfrak{m}_0}H^R_{\Gamma_R(M)} \) are Artinian, for all \( i, j \in \mathbb{N}_0 \), when \( R_+ \) is linked by an \( \mathfrak{m}_0 \)-primary ideal (Proposition 2.6).

Throughout the paper, \( R = \bigoplus_{n \in \mathbb{N}_0} R_n \) is a standard graded ring over the local base ring \((R_0, \mathfrak{m}_0)\), \( M \) is a finitely generated graded \( R \)-module with \( M \neq \Gamma_R(M) \). Also, \( a \) denotes a proper homogeneous ideal of \( R \) and we set \( a_0 := a \cap R_0 \).

2. The results

First, we recall the concept of homogeneously linked ideals from [5], as well as some basic definitions which will be used later in the paper.

Definitions and Remark 2.1. Let \( b \) be a second homogeneous ideal of \( R \).

(i) Assume that \( aM \neq M \neq bM \) and \( I \subseteq a \cap b \) be an ideal generated by a homogeneous \( M \)-regular sequence. Then we say that the ideals \( a \) and \( b \) are homogeneously linked (or h-linked) by \( I \) over \( M \), denoted \( a \sim^h_{(I,M)} b \), if \( bM = IM :_M a \) and \( aM = IM :_M b \). The ideals \( a \) and \( b \) are said to be geometrically h-linked by \( I \) over \( M \) if, in addition, \( aM \cap bM = IM \). Also, we say that the ideal \( a \) is h-linked over \( M \) if there exist homogeneous ideals \( b \) and \( I \) of \( R \) such that \( a \sim^h_{(I,M)} b \). \( a \) is \( h \)-M-selflinked by \( I \) if \( a \sim^h_{(I,M)} a \).

(ii) Following [3, 9.1.5], the \( b \)-finiteness dimension of \( M \) relative to \( a \) is defined to be
\[ f^h_a(M) := \inf \{ i \in \mathbb{N}_0 \mid b \nsubseteq \sqrt{0 : H^i_a(M)} \} \]

In the case where \( b = a \), in view of [3, 9.1.3], this invariant equals the finiteness dimension of \( M \) relative to \( a \). That is
\[ f_a(M) := \inf \{ i \in \mathbb{N} \mid H^i_a(M) \ \text{is not finitely generated} \} \].
In the next proposition, we want to study the stability of the set \( \{ \text{grade}(a_0, H^{f_{R_+}(M)}(M)_n) \}_{n \in \mathbb{Z}} \) of integers where \( n \to -\infty \). This problem has already considered in [4]. Note that, by [3, 6.2.7] and [5, 3.16(ii)], \( \text{grade}(R_+, M) \leq f_{R_+}(M) \leq f_{a+R_+}(M) \).

**Proposition 2.2.** Let \( a \overset{h}{\sim}_{(I, M)} R_+ \) and set \( t := \text{grade}(R_+, M) \). If \( f_{R_+}(M) \neq t \) or \( f_{a+R_+}(M) = t \), then \( \text{grade}(a_0, H^{f_{R_+}(M)}(M)_n) = 0 \) for all \( n \ll 0 \).

**Proof.** By the homogeneous Mayer-Vietoris sequence and [6, 2.2], we have the following homogeneous exact sequence of \( R \)-modules

\[
\ldots \rightarrow H^{i-1}_I(M) \rightarrow H^i_{a+R_+}(M) \rightarrow H^i_a(M) \oplus H^i_{R_+}(M) \rightarrow H^i_I(M) \rightarrow \ldots.
\]

It yields

\[
H^i_{a+R_+}(M) \cong H^i_a(M) \oplus H^i_{R_+}(M) \quad \text{for all } i \geq t + 1
\]

and, by [3, 6.2.7], the exact sequence

\[
0 \rightarrow H^t_{a+R_+}(M) \rightarrow H^t_a(M) \oplus H^t_{R_+}(M) \rightarrow H^t_I(M) \rightarrow H^{t+1}_{a+R_+}(M) \rightarrow H^{t+1}_a(M) \oplus H^{t+1}_{R_+}(M) \rightarrow 0.
\]

These, result that \( H^i_a(M) \) and \( H^i_{R_+}(M) \) are respectively \( R_+ \)-torsion and \( a \)-torsion for all \( i \neq t \). So, if \( f_{R_+}(M) \neq t \), then \( \Gamma_a(H^{f_{R_+}(M)}(M)) = H^{f_{R_+}(M)}(M) \). Therefore, by [3, 14.1.12] and [1, 4.8],

\[
\Gamma_a(H^{f_{R_+}(M)}(M)_n) = H^{f_{R_+}(M)}(M)_n \neq 0 \quad \text{for all } n \ll 0.
\]

This, in view of [3, 6.2.7], prove the claim. Now, assume that \( f_{a+R_+}(M) = t \). By [9, 3.4],

\[
\Gamma_a(H^t_{R_+}(M)) \cong \Gamma_a(H^0_I(M)) = \Gamma_{a+R_+}(H^t_I(M)) \cong H^t_{a+R_+}(M).
\]

So, using [4, 2.2] and [3, 14.1.12], \( \Gamma_a(H^{f_{R_+}(M)}(M)_n) \neq 0 \) for all \( n \ll 0 \). This implies that \( \text{grade}(a_0, H^{f_{R_+}(M)}(M)_n) = 0 \) for all \( n \ll 0 \), as desired. \( \square \)

Let \( N = \bigoplus_{n \in \mathbb{Z}} N_n \) be a graded \( R \)-module. Then following [8], \( N \) is called finitely graded if \( N_n = 0 \) for all but finitely many \( n \in \mathbb{Z} \). Also, we set

\[
g_a(N) := \sup \{ k \in \mathbb{N}_0 | H^i_a(N) \text{ is finitely graded for all } i < k \}.
\]

**Remark 2.3.** Note that, by [8, 2.3], if \( N \) is finitely generated, then \( g_a(N) = f_{a+R_+}^R(N) \).

In view of [5, 3.17], if \( a \overset{h}{\sim}_{(I, M)} R_+ \) and one of the conditions of proposition 2.2 holds, then \( \text{grade}(a_0, H^{f_{R_+}(M)}(M)_n) = f_{a+R_+}(M) - f_{R_+}(M) \) for all \( n \ll 0 \). In the next theorem and corollary, we consider a case where \( a \overset{h}{\sim}_{(I, M)} R_+ \) and none of the conditions of 2.2 establish.
We show that \( \text{grade}(a_0, H^\text{f}_{R_+}(M)(n)) \) is bounded below for all \( n \ll 0 \) and also, in a special case, it is asymptotically stable, as \( n \to -\infty \).

**Theorem 2.4.** Let \( a \stackrel{\sim}{\longrightarrow} (I; M) R_+ \) and assume that \( \text{grade}(R_+, M) = f_{R_+}(M) < f_{R_+}^a(M) \). Then \( \text{grade}(a_0, H^\text{f}_{R_+}(M)(n)) \geq f_{a+R_+}^a(M) - f_{R_+}(M) \) for all \( n \ll 0 \).

**Proof.** Set \( t := \text{grade}(R_+, M) \). Note that, in view of \([5, 3.16(ii)]\), \( f_{a+R_+}^a(M) \geq t + 1 \). Now, consider the following convergence of spectral sequences \([11, 11.38]\),

\[
E_2^{ij} = H^{j}_{a_0R}(H^j_{R_+}(M)) \Rightarrow H^{i+j}_{a+R_+}(M).
\]

By \((2.2)\) and \((2.3)\), \( H^j_{R_+}(M) \) is \( a \)-torsion for all \( j \neq t \). As a result,

\[
E_2^{ij} = 0 \quad \text{for all } i \geq 1 \text{ and all } j \neq t,
\]

and the above convergence of spectral sequences yields

\[
\Gamma_{a_0}(H^{i+t}_{R_+}(M))_n = (E^{0,t+i}_2)_n = \ldots = (E^{0,t+i}_{i+1})_n \supset \ker(d^{0,t+i}_{i+1})_n = (E^{0,t+i}_{i+2})_n = (E^{0,t+i}_\infty)_n,
\]

for all \( i \in \mathbb{N} \) and all \( n \in \mathbb{Z} \). Therefore, if \( 1 \leq i \leq f_{a+R_+}^a(M) - t \) then, by \((2.2)\), \((2.3)\) and \((2.3)\), \( H^{i+t}_{R_+}(M) \) is finitely graded and \((2.5)\) implies that

\[
(E^{0,t+i}_2)_n = (E^{0,t+i}_\infty)_n = 0 \quad \text{for all } 1 \leq i \leq f_{a+R_+}^a(M) - t \text{ and all } n \ll 0.
\]

Also, using the concept of convergence of spectral sequences and \((2.3)\), we have

\[
0 = H^{i+t}_{a+R_+}(M)_n \cong (E^{i+t}_\infty)_n = (E^{i,t}_2)_n = H^{i}_{a_0}(H^t_{R_+}(M)_n),
\]

for all \( 1 \leq i \leq f_{a+R_+}^a(M) - t \) and all \( n \ll 0 \). In addition, in view of \((2.4)\) and the assumption on \( t \),

\[
0 = H^t_{a+R_+}(M)_n \cong (E^{0,t}_\infty)_n = (E^{0,t}_2)_n = \Gamma_{a_0}(H^t_{R_+}(M)_n) \quad \text{for all } n \ll 0.
\]

Therefore, by \([3, 6.2.7]\),

\[
\text{grade}(a_0, H^t_{R_+}(M)_n) \geq f_{a+R_+}^a(M) - t \quad \text{for all } n \ll 0.
\]

\( \square \)

Let \( N \) be an \( R \)-module. The cohomological dimension of \( N \) with respect to \( a \) is defined to be

\[
\text{cd}(a, N) := \sup \{ i \in \mathbb{Z} | H^i_a(N) \neq 0 \}.
\]

A finitely generated \( R \)-module \( X \) is called relative Cohen-Macaulay with respect to the ideal \( b \) of degree \( n \) if \( H^i_b(X) = 0 \) for all \( i \neq n \).

In the next item, using the above theorem, we show that the graded components \( H^\text{grade}(R_+; M)(n) \) are relative Cohen-Macaulay \( R_0 \)-modules with respect to \( a_0 \), in some special cases.
Corollary 2.5. Let the situations be as in theorem 2.4 and set $t := \text{grade}(R_+, M)$. Then in each of the following cases, $H^t_{R_+}(M)_n$ is a relative Cohen-Macaulay $R_0$-module with respect to $a_0$ of degree $f^R_{a+R_+}(M) - t$ for all $n \ll 0$, i.e. $\text{grade}(a_0, H^t_{R_+}(M)_n) = \text{cd}(a_0, H^t_{R_+}(M)_n) = f^R_{a+R_+}(M) - t$ for all $n \ll 0$.

(i) $M$ is relative Cohen-Macaulay with respect to $a + R_+$.  

(ii) $a$ is generated by homogeneous elements of degree 0.  

(iii) $\text{cd}(a, M) \in \{t, t + 1\}$.

Proof.  

(i) By hypothesis and [5, 3.15(ii)], $\text{cd}(a + R_+, M) = f^R_{a+R_+}(M)$. Since $H^i_t(M) = 0$ for all $i \neq t$, using a homogeneous Mayer-Vietoris sequence and [6, 2.2] we have the following homogeneous exact sequences of $R$-modules

$$0 \longrightarrow H^t_{a+R_+}(M) \longrightarrow H^t_a(M) \oplus H^t_{R_+}(M) \longrightarrow X \longrightarrow 0,$$

and

$$0 \longrightarrow X \longrightarrow H^t_a(M) \longrightarrow Y \longrightarrow 0,$$

where $X$ is a submodule of $H^t_a(M)$ and $Y$ is a submodule of $H^t_{a+R_+}(M)$.

Now, by affecting $\Gamma_a(\cdot)$ on exact sequences (2.6) and (2.7) and using [9, 3.4], the following isomorphisms have been obtained

$$H^i_a(H^t_{R_+}(M)) \cong H^i_a(H^t_a(M)) \cong H^i_a(M), \quad \text{for all } i > 1.$$

Also, using [6, 2.3], $\text{cd}(a, M) \leq \text{cd}(a + R_+, M) = f^R_{a+R_+}(M)$. Therefore,

$$H^i_a(H^t_{R_+}(M)) = 0 \quad \text{for all } i \geq f^R_{a+R_+}(M) - t.$$

On the other hand, by [3, 14.1.12] and the fact that $H^t_{R_+}(M)$ is $R_+$-torsion, we have

$$H^i_a(H^t_{R_+}(M))_n \cong H^i_{a_0}(H^t_{a+R_+}(M))_n \cong H^i_{a_0}(H^t_{R_+}(M))_n \cong H^i_{a_0}(H^t_{R_+}(M)_n)$$

for all $i$ and all $n$. Therefore, by (2.9),

$$\text{grade}(a_0, H^t_{R_+}(M)_n) \leq \text{cd}(a_0, H^t_{R_+}(M)_n) \leq f^R_{a+R_+}(M) - t, \quad \text{for all } n \in \mathbb{Z}.$$

Now, using 2.4, the result follows.

(ii) Since $M$ is finitely generated, by (2.8), [3, 14.1.12] and [7, Theorem 1],

$$H^i_{a_0}(H^t_{R_+}(M)_n) \cong H^t_{a_0+R_+}(M)_n = 0 \quad \text{for all } i > 1 \text{ and all } n \ll 0.$$

So, $\text{grade}(a_0, H^t_{R_+}(M)_n) \leq \text{cd}(a_0, H^t_{R_+}(M)_n) \leq 1$ for all $n \ll 0$. Also, by 2.4,

$$\text{grade}(a_0, H^t_{R_+}(M)_n) \geq f^R_{a+R_+}(M) - t \geq 1, \quad \text{for all } n \ll 0.$$ 

This proves the claim.

(iii) By hypothesis and (2.8), $H^i_a(H^t_{R_+}(M)) = 0$ for all $i > 1$ and the statement holds as part (ii). 

$\square$
The cohomological finite length dimension of $M$ is defined to be

$$g(M) := \inf\{i \in \mathbb{N} \mid l_{R_0}(H_{R_0}^i(M)_n) = \infty \text{ for infinitely many } n\},$$

where $l_{R_0}(H_{R_0}^i(M)_n)$ is the length of the $R_0$-module $H_{R_0}^i(M)_n$.

The following proposition considers Artinianness of some local cohomology modules with respect to linked ideals as well as the stability of a set of integers.

**Proposition 2.6.** Assume that $q_0$ is an $m_0$-primary ideal of $R_0$ such that $q_0 R \not\subseteq (I; M)$ $R_+$ and set $t := \text{grade}(R_+, M)$. Then the following statements hold.

(i) The set $\{\dim_{R_0} M_n\}_{n \in \mathbb{Z}}$ of integers is asymptotically stable for $n \to +\infty$ and it equals $t$, i.e. $\dim_{R_0} M_n = t$ for all $n \gg 0$.

(ii) $H_{R_+}^j(M)$ and $H_{m_0 R}^j(M)$ are Artinian $R$-modules for all $j \neq t$. Also, if $g(M) < \infty$, then $H_{m_0 R}^i(H_{R_+}^j(M))$ is Artinian for all $i$ and all $j$ and that $g(M) = t$.

**Proof.**

(i) Since $M$ is finitely generated, by [7, Theorem 1], $R_1 M_n = M_{n+1}$ for all $n \gg 0$. It shows that $\dim_{R_0} M_n \geq \dim_{R_0} M_{n+1}$ for all $n \gg 0$. Thus, there exists $l \in \mathbb{N}$ such that

$$\dim_{R_0} M_n = l$$

for all $n \gg 0$.

On the other hand, by [3, 14.1.12] and [5, 3.9],

$$H_{m_0}^i(M_n) \cong H_{q_0}^i(M_n) \cong H_{q_0 R}^i(M_n) = 0 \quad \text{for all } i \neq t \text{ and all } n \gg 0.$$

So, using [3, 6.1.4], $l = t$.

(ii) The first statement holds by [3, 7.1.3], (2.2) and (2.3). (2.2) and (2.3), also, ensure that $H_{R_+}^j(M)$ is $m_0 R$-torsion for all $j \neq t$, so $H_{m_0 R}^i(H_{R_+}^j(M))$ is Artinian for all $i$ and all $j \neq t$.

For the case $j = t$, according to the definition of $g(M)$, $t \leq g(M)$. So, by [4, 2.4], $H_{m_0 R}^i(H_{R_+}^j(M))$ is Artinian for $i = 0, 1$ and also it does by [3, 1.2.3] and (2.8) for all $i > 1$.

Moreover, since $H_{R_+}^j(M)$ is Artinian for all $j \neq t$, by [7, Theorem 1], $H_{R_+}^j(M)_n$ is an Artinian $R_0$-module for all $j \neq t$ and all $n \in \mathbb{Z}$. Therefore, using [3, 16.1.5], $l_{R_0}(H_{R_+}^j(M)_n) < \infty$ for all $j \neq t$. As a result $g(M) = t$.

□

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