ANALYTIC CAPACITY AND PROJECTIONS

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Abstract. In this paper we study the connection between the analytic capacity of a set and the size of its orthogonal projections. More precisely, we prove that if $E \subset \mathbb{C}$ is compact and $\mu$ is a Borel measure supported on $E$, then the analytic capacity of $E$ satisfies

$$\gamma(E) \geq c \frac{\mu(E)^2}{\int_I \|P_\theta \mu\|_2^2 \, d\theta},$$

where $c$ is some positive constant, $I \subset [0, \pi)$ is an arbitrary interval, and $P_\theta \mu$ is the image measure of $\mu$ by $P_\theta$, the orthogonal projection onto the line $\{re^{i\theta} : r \in \mathbb{R}\}$. This result is related to an old conjecture of Vitushkin about the relationship between the Favard length and analytic capacity. We also prove a generalization of the above inequality to higher dimensions which involves related capacities associated with signed Riesz kernels.

1. Introduction

The objective of this paper is to study the connection between the analytic capacity of a set and the size of its projections onto lines. First we introduce some notation and definitions. A compact set $E \subset \mathbb{C}$ is said to be removable for bounded analytic functions if for any open set $\Omega$ containing $E$, every bounded function analytic on $\Omega \setminus E$ has an analytic extension to $\Omega$. In order to study removability, in the 1940s, Ahlfors [Ah] introduced the notion of analytic capacity. The analytic capacity of a compact set $E \subset \mathbb{C}$ is

$$\gamma(E) = \sup \{|f'|(\infty)|,$$

where the supremum is taken over all analytic functions $f : \mathbb{C} \setminus E \to \mathbb{C}$ with $|f| \leq 1$ on $\mathbb{C} \setminus E$, and $f'(\infty) = \lim_{z \to \infty} z(f(z) - f(\infty))$. In [Ah], Ahlfors showed that $E$ is removable for bounded analytic functions if and only if $\gamma(E) = 0$.

In the 1960s, Vitushkin conjectured that a compact set in the plane is non-removable for bounded analytic functions (or equivalently, has positive analytic capacity) if and only if its orthogonal projections have positive length in a set of directions of positive measure, or in other words, if and only if it has positive Favard length. The Favard length of a Borel set $E \subset \mathbb{C}$ is

$$\text{Fav}(E) = \int_0^\pi \mathcal{H}^1(P_\theta(E)) \, d\theta,$$

where, for $\theta \in [0, \pi)$, $P_\theta$ denotes the orthogonal projection onto the line $L_\theta := \{re^{i\theta} : r \in \mathbb{R}\}$, and $\mathcal{H}^1$ stands for the 1-dimensional Hausdorff measure. In 1986, Mattila [Mal] showed that Vitushkin’s conjecture is false. Indeed, he proved that the property of having zero...
Favard length is not invariant under conformal mappings while removability for bounded analytic functions remains invariant. Mattila’s result didn’t tell which implication in the above conjecture is false. This question was partially solved in 1988, when Jones and Murai [JM] constructed a set with zero Favard length and positive analytic capacity. Later on, Joyce and Mörters [JoyM] later obtained an easier example using curvature of measures.

Although Vitushkin’s conjecture is not true in full generality, it turns out that it holds in the particular case of sets with finite length. This was proved by G. David [Da] in 1998. Indeed, he showed that such sets are removable if and only if they are purely unrectifiable, which is equivalent to having zero Favard length, by the Besicovitch projection theorem. For sets with arbitrary length, removability can be characterized in terms of curvature of measures, by [To1] (see Theorem 2.1 below for more details).

As mentioned above, one of the directions of Vitushkin’s conjecture is false. However, it is not known yet if the other implication holds. Namely, does positive Favard length imply positive analytic capacity? In a sense, the main result of this paper asserts that if one strengthens the assumption of positive Favard length in a suitable way, then the answer is positive. See the survey [EV] for more information on this and other related questions.

Given a Borel measure $\mu$ in $\mathbb{R}^2$, we denote by $P_\theta \mu$ the image measure of $\mu$ by the orthogonal projection $P_\theta$ from $\mathbb{R}^2$ onto the line $L_\theta := \{re^{i\theta} : r \in \mathbb{R}\}$. Our main result is the following:

**Theorem 1.1.** Let $I \subset [0, \pi)$ be an interval. For any compact set $E \subset \mathbb{C}$ and any Borel measure $\mu$ on $\mathbb{C}$, we have

$$\gamma(E) \geq c \frac{\mu(E)^2}{\int_I \|P_\theta \mu\|_2^2 d\theta},$$

where $c$ is some positive constant depending only on $\mathcal{H}^1(I)$.

In the theorem above, $\|P_\theta \mu\|_2$ stands for the $L^2$ norm of the density of $P_\theta \mu$ with respect to length in $L_\theta$, and the $L^2$ norm is computed with respect to length in $L_\theta$ too.

It is worth mentioning that the Favard length of $E$ satisfies an estimate very similar to (1.3):

$$\text{Fav}(E) \geq c_I \frac{\mu(E)^2}{\int_I \|P_\theta \mu\|_2^2 d\theta},$$

for some constant $c_I > 0$. This follows easily from the Cauchy-Schwarz inequality. Indeed, for any Borel measure $\mu$ supported on $E$ and $\theta \in [0, \pi)$,

$$\mu(E) = \|P_\theta \mu\|_1 \leq \|P_\theta \mu\|_2 \mathcal{H}^1(P_\theta E)^{1/2}.$$

Thus,

$$\mu(E) \mathcal{H}^1(I) \leq \int_{\mathbb{R}} \|P_\theta \mu\|_2 \mathcal{H}^1(P_\theta E)^{1/2} d\theta \leq \left( \int_I \|P_\theta \mu\|_2^2 d\theta \right)^{1/2} \text{Fav}(E)^{1/2},$$

which yields (1.4) with $c_I = \mathcal{H}^1(I)^2$. So a natural question is the following: does there exist some absolute constant $c > 0$ such that

$$\gamma(E) \geq c \text{Fav}(E)?$$

If the answer were positive, then Theorem 1.1 would be an immediate consequence of this and (1.4).
The result stated in Theorem 1.1 extends to higher dimensions. In \(\mathbb{R}^d\) the role of the analytic capacity \(\gamma\) is played by the capacities \(\Gamma_{d,n}\) associated to the vector-valued Riesz kernels \(x/|x|^{n+1}\). Given an integer \(0 < n < d\) and a compact \(E \subset \mathbb{R}^d\), one sets
\[
\Gamma_{d,n}(E) = \sup \langle |T|, 1 \rangle,
\]
where the supremum is take over all real distributions \(T\) supported in \(E\) such that
\[
\frac{x}{|x|^{n+1}} * T \in L^\infty(\mathbb{R}^d) \quad \text{and} \quad \left\| \frac{x}{|x|^{n+1}} * T \right\|_{L^\infty(\mathbb{R}^d)} \leq 1.
\]
In the case \(n = 1, d = 2\), \(\Gamma_{2,1}\) is the real version of the analytic capacity \(\gamma\), and from [To1] it holds that \(\gamma \approx \Gamma_{2,1}\).

In the codimension 1 case (i.e., \(n = d - 1\), \(\Gamma_{d,d-1}\) is the so-called Lipschitz harmonic capacity introduced by Paramonov [Pa]. The analogue of Vitushkin’s conjecture also holds for sets with finite \(\mathcal{H}^n\)-measure. That is, \(E\) removable for Lipschitz harmonic functions in \(\mathbb{R}^{n+1}\) if and only if \(E\) is purely \(n\)-unrectifiable, or equivalently, the orthogonal projections of \(E\) on almost all hyperplanes have \(\mathcal{H}^n\)-measure zero. See [NT01] and [NT02]. The analogous result for \(1 < n < d - 1\) is still an open problem.

The higher dimensional extension of Theorem 1.1 is the following:

**Theorem 1.2.** Let \(V_0 \subset \mathbb{R}^d\) be an \(n\)-plane through the origin and let \(s > 0\). For any compact set \(E \subset \mathbb{R}^d\) and any Borel measure \(\mu\) on \(\mathbb{R}^d\), we have
\[
\Gamma_{d,n}(E) \geq c \int_{B(V_0,s)} \|P_V \mu\|_2^2 \, d\gamma_{d,n}(V),
\]
where \(c\) is some positive constant depending only on \(s, n,\) and \(d\).

In this theorem, \(P_V\) is the orthogonal projection onto the \(n\)-dimensional subspace \(V\), and \(\|P_V \mu\|_2\) is the \(L^2\) norm (with respect to \(n\)-dimensional Lebesgue measure) in \(V\) of \(P_V \mu\) (we identify \(P_V \mu\) with its density with respect to \(n\)-dimensional Lebesgue measure in \(V\)). Also, \(\gamma_{d,n}\) is the natural probability measure on the Grassmanian \(G(d,n)\) (see [Ma2, Chapter 3]), and \(B(V_0,s)\) is a ball of radius \(s\) in \(G(d,n)\). See Section 3.2 below for the definition of the metric in \(G(d,n)\).

The first fundamental step towards the proof of Theorems 1.1 and 1.2 is a Fourier calculation which shows that there exist constants \(c, \lambda > 1\) (depending only on \(s, n, d\)) such that
\[
(1.5) \quad \int \int_{x-y \in K(V_0^\perp, \lambda^{-1}s)} \frac{d\mu(x) \, d\mu(y)}{|x-y|^n} \leq c \int_{B(V_0,s)} \|P_V \mu\|_2^2 \, d\gamma_{d,n}(V),
\]
where, given \(U \in G(d,m)\) and \(t > 0\), \(K(U,t)\) is the cone
\[
K(U,t) = \{ x \in \mathbb{R}^d : \text{dist}(x,U) < t \, |x| \},
\]
and \(V_0^\perp\) is the subspace orthogonal to \(V_0\). In the planar case \(n = 1, d = 2\), the calculation is particularly clean and we obtain a more precise result. See Corollaries 3.3 and 3.11 for more details. Our inspiration for proving the estimate (1.5) comes from the work of Martikainen and Orponen [MO]. In that paper, the authors characterize the “big pieces of Lipschitz graph” condition on \(n\)-AD-regular sets in terms of an integral of the form
\[
\int_{B(V_0,s)} \|P_V \mu\|_2^2 \, d\gamma_{d,n}(V),
\]
with \(\mu\) equal to \(n\)-dimensional Hausdorff measure \(\mathcal{H}^n\) restricted
to a suitable subset $E$. Roughly speaking, in the proof of the main lemma (Lemma 1.10) of [MO], the authors obtain a variant of the estimate (1.5), but with the left-hand side replaced by a discretized version of the integral. They do not use the Fourier transform, but instead use more geometric arguments.

The second step in the proof of Theorems 1.1 and 1.2 is the construction of a corona type decomposition which will allow us to bound the $L^2(\mu)$ norm of the Riesz transform

$$\mathcal{R}^n \mu(x) = \int \frac{x-y}{|x-y|^{n+1}} d\mu(y),$$

assuming that $\mu$ satisfies the growth condition

$$\mu(B(x,r)) \leq c_0 r^n \quad \text{for all } x \in \mathbb{R}^d, \ r > 0.$$ 

Using this corona decomposition we will deduce that

$$\|\mathcal{R}^n \mu\|_{L^2(\mu)}^2 \lesssim \mu(\mathbb{R}^d) + \iint_{x-y \in K(U_0,s)} \frac{d\mu(x) d\mu(y)}{|x-y|^n}$$

for any $U_0 \in G(d,d-n)$ and $s > 0$. In the case $n = 1$, we will get the following estimate involving the curvature of $\mu$:

$$\iiint \frac{1}{R(x,y,z)^2} d\mu(x) d\mu(y) d\mu(z) \lesssim \mu(\mathbb{C}) + \iint_{x-y \in K(U_0,s)} \frac{d\mu(x) d\mu(y)}{|x-y|},$$

where $R(x,y,z)$ stands for the radius of the circumference passing through $x, y, z$. In both (1.6) and (1.7), the implicit constant depends only on $s$, $c_0$, $n$ and $d$. See Theorem 10.2 for more details. We remark that other related corona decompositions have already appeared in [To2] and [AT], for example. However, the use of the Riesz-type energy in (1.5) in connection with corona decompositions is totally new, as far as we know.

Using (1.5), (1.6), (1.7), and the characterization of analytic capacity in terms of curvature of measures from [To1] and the characterization of the capacities $\Gamma_{d,n}$ in terms of $L^2$ estimates of Riesz transforms from [Vo] and [Pr], we will obtain Theorem 1.1 and Theorem 1.2 respectively.

2. Notation and preliminaries

2.1. Generalities. We write $a \lesssim b$ if there is a $C > 0$ such that $a \leq Cb$, and we write $a \lesssim_t b$ if the constant $C$ depends on the parameter $t$. We write $a \approx b$ to mean $a \lesssim b \lesssim a$ and define $a \approx_t b$ similarly.

We denote the open ball of radius $r$ centered at $x$ by $B(x,r)$. For a ball $B = B(x,r)$ and $\delta > 0$ we write $r(B)$ for its radius and denote $\delta B = B(x,\delta r)$.

Given an $m$-plane $V \subset \mathbb{R}^d$, $z \in \mathbb{R}^d$, and $s > 0$, we consider the (open) cone

$$K(z,V,s) = \{x \in \mathbb{R}^d : \text{dist}(x-z,V) < s |x-z|\}.$$ 

In the case $z = 0$, we also write $K(V,s) = K(0,V,s).$
2.2. **Measures and rectifiability.** The Lebesgue measure of a set $A \subset \mathbb{R}^d$ is denoted by $\mathcal{L}^d(A)$. Given $0 < \delta \leq \infty$, we set

$$\mathcal{H}^n_\delta(A) = \inf \left\{ \sum_i \text{diam}(A_i)^n : A_i \subset \mathbb{R}^d, \text{diam}(A_i) \leq \delta, A \subset \bigcup_i A_i \right\}.$$ 

We define the $n$-dimensional Hausdorff measure as

$$\mathcal{H}^n(A) = \lim_{\delta \to 0} \mathcal{H}^n_\delta(A).$$

A set $E \subset \mathbb{R}^d$ is called $n$-rectifiable if there are Lipschitz maps $f_i : \mathbb{R}^n \to \mathbb{R}^d$, $i = 1, 2, \ldots$, such that

$$(2.1) \quad \mathcal{H}^n \left( E \setminus \bigcup_i f_i(\mathbb{R}^n) \right) = 0.$$ 

On the other hand, $E$ is called purely $n$-unrectifiable if any $n$-rectifiable subset $F \subset E$ has zero $\mathcal{H}^n$-measure.

Also, one says that a Radon measure $\mu$ on $\mathbb{R}^d$ is $n$-rectifiable if $\mu$ vanishes out of an $n$-rectifiable set $E \subset \mathbb{R}^d$ and moreover $\mu$ is absolutely continuous with respect to $\mathcal{H}^n|_E$.

A measure $\mu$ is called $n$-AD-regular (or just AD-regular or Ahlfors-David regular) if there exists some constant $c > 0$ such that

$$c^{-1} r^n \leq \mu(B(x, r)) \leq c r^n \quad \text{for all } x \in \text{supp}(\mu) \text{ and } 0 < r \leq \text{diam}(\text{supp}(\mu)).$$

2.3. **Cauchy and Riesz transform, and capacities.** Given a signed Radon measure $\nu$ in $\mathbb{C}$, the Cauchy transform is defined by

$$\mathcal{C}\nu(x) = \int \frac{1}{z-w} \, d\nu(w),$$

whenever the integral makes sense. The $\epsilon$-truncated version is

$$\mathcal{C}_\epsilon \nu(x) = \int_{|z-w| > \epsilon} \frac{1}{z-w} \, d\nu(w).$$

For a signed Radon measure $\nu$ in $\mathbb{R}^d$, we consider the $n$-dimensional Riesz transform

$$\mathcal{R}^n\nu(x) = \int \frac{x-y}{|x-y|^{n+1}} \, d\nu(y),$$

whenever the integral makes sense. For $\epsilon > 0$, its $\epsilon$-truncated version is given by

$$\mathcal{R}_\epsilon^n\nu(x) = \int_{|x-y| > \epsilon} \frac{x-y}{|x-y|^{n+1}} \, d\nu(y).$$

The curvature of a non-negative Borel measure $\mu$ in $\mathbb{C}$ is defined by

$$c^2(\mu) = \iint \frac{1}{R(x, y, z)} \, d\mu(x) \, d\mu(y) \, d\mu(z),$$

where $R(x, y, z)$ stands for the radius of the circumference passing through $x, y, z$. For $\epsilon > 0$, its $\epsilon$-truncated version is

$$c^2_\epsilon(\mu) = \iint_{|x-y| > \epsilon} \frac{1}{R(x, y, z)} \, d\mu(x) \, d\mu(y) \, d\mu(z).$$
As shown in [MeV], if $\mu$ is a finite Borel measure in $\mathbb{C}$ satisfying the linear growth condition

$$\mu(B(z,r)) \leq c_0 r \quad \text{for all } z \in \mathbb{C}, \ r > 0,$$

then

$$\|C_z \mu\|_{L^2(\mu)}^2 = \frac{1}{6} c_0^2(\mu) + O(\mu(\mathbb{C})), $$

where $|O(\mu(\mathbb{C}))| \leq \mu(\mathbb{C})$, with the implicit constant depending only on $c_0$. The connection between the Cauchy kernel and curvature of measures was first observed by Melnikov while studying analytic capacity [Me].

We denote by $L_n(E)$ the set of positive Borel measures $\mu$ supported on $E$ satisfying

$$\mu(B(x,r)) \leq r^n \quad \text{for all } x \in E, \ r > 0.$$

The following theorem characterizes analytic capacity in terms of measures from $L_1(E)$ with finite curvature.

**Theorem 2.1.** Let $E \subset \mathbb{C}$ be compact. Then we have:

$$(2.2) \quad \gamma(E) \approx \sup \{\mu(E) : \mu \in L_1(E), \ c^2(\mu) \leq \mu(E)\}.$$

The fact that $\gamma(E)$ is bigger than a constant multiple of the supremum is due to Melnikov [Me], and the more difficult converse estimate to Tolsa [To1].

The extension of the preceding result to the capacities $\Gamma_{d,n}$ is the following:

**Theorem 2.2.** Let $E \subset \mathbb{R}^d$ be compact. Then we have:

$$(2.3) \quad \Gamma_{d,n}(E) \approx \sup \{\mu(E) : \mu \in L_n(E), \ \sup_{\epsilon > 0} \|R^\epsilon \mu\|_{L^2(\mu)}^2 \leq \mu(E)\}.$$ 

Theorem 2.2 was proved by Volberg in the case $n = d - 1$, and it was later extended to all the values $0 < n < d$ by Prat [Pt].

### 3. The Fourier calculation

#### 3.1. The planar case

We think it is worth first studying the planar case because it is simpler the higher dimensional case, and the result is more precise.

Recall that for any Schwartz function $\phi : \mathbb{R}^2 \to \mathbb{C}$, we have

$$(3.1) \quad \hat{P_\theta \phi}(x) = \hat{\phi}(x) \quad \text{for all } x \in L_\theta,$$

where $(P_\theta \phi)(x) = \int_{x+L_\theta^+} \phi d\mathcal{L}^1 = \int_{L_\theta^+} \phi(x+y) d\mathcal{L}^1y$ and $\hat{P_\theta \phi}$ denotes the 1-dimensional Fourier transform on $L_\theta$. (Note that $P_\theta \phi$ is the density of $P_\theta \nu$ where $\nu = \phi(x) dx$.) To see (3.1), observe that for $\xi \in L_\theta$,

$$\hat{P_\theta \phi}(\xi) = \int_{L_\theta} P_\theta \phi(x) e^{-2\pi i x \cdot \xi} dx = \int_{L_\theta} \int_{L_\theta^+} \phi(x+y) e^{-2\pi i y \cdot \xi} dy dx = \hat{\phi}(\xi),$$

where we use the fact that $y \cdot \xi = 0$ for $y \in L_\theta^\perp$.

**Lemma 3.1.** Let $K_1$ be the cone $K_1 = \{re^\theta : r \in \mathbb{R}, \ \theta \in [0, \pi]\}$. Let $I^\perp = I + \frac{\pi}{2}$ (mod $\pi$) and define the cone $K_{I^\perp}$ similarly. Then the (distributional) Fourier transform of $\chi_{K_1}(x) |x|^{-1}$ is $\chi_{K_{I^\perp}}(x) |x|^{-1}$. 

Proof. Let \( \phi : \mathbb{R}^2 \to \mathbb{C} \) be a Schwartz function. By applying the identity [3.1] to \( \phi \) and using the Fourier inversion formula, we have \( \int_{L_\vartheta} \hat{\phi} d\mathcal{L}^1 = (P_\vartheta \phi)(0) = \int_{L_\vartheta} \phi d\mathcal{L}^1 \). Thus, by polar coordinates,

\[
\int_{K_I} |x|^{-1} \hat{\phi}(x) dx = \frac{1}{2} \int_I \int_{L_\vartheta} \hat{\phi} d\mathcal{L}^1 d\vartheta = \frac{1}{2} \int_I \int_{L_\vartheta} \phi d\mathcal{L}^1 d\vartheta = \int_{K_{I_\perp}} |x|^{-1} \phi(x) dx,
\]

which completes the proof. \( \square \)

Proposition 3.2. Let \( \psi : \mathbb{R}^2 \to \mathbb{R} \) be a Schwartz function. Then, for any set \( I \subset [0, \pi] \), we have

\[
\int_{I_+} \|P_\vartheta \psi\|_2^2 d\vartheta = 2 \int_{x-y \in K_I} \frac{\psi(x) \psi(y)}{|x-y|} dx dy.
\]

Proof. Let \( k(x) = \chi_{K_I}(x) |x|^{-1} \), so that \( \hat{k}(x) = \chi_{K_I}(x) |x|^{-1} \) and \( \int_{x-y \in K_I} \psi(x) \psi(y) dx dy = \int (k * \psi)(x) dx \). Since \( \psi \) is a real-valued Schwartz function, we have \( \int (k * \psi)(x) dx = \int \hat{k}(\psi)^2 dx \), so it follows that

\[
\int_{x-y \in K_I} \frac{\psi(x) \psi(y)}{|x-y|} dx dy = \int (k * \psi)(x) dx = \int \hat{k} |\hat{\psi}|^2 dx = \int_{K_{I_\perp}} |x|^{-1} |\hat{\psi}(x)|^2 dx.
\]

Finally, by Plancherel and the identity [3.1] applied to \( \psi \), we have

\[
\int_{I_+} \|P_\vartheta \psi\|_2^2 d\vartheta = \int_{I_+} \int_{L_\vartheta} |\hat{P_\vartheta \psi}|^2 d\mathcal{L}^1 d\vartheta = \int_{I_+} \int_{-\infty}^{\infty} |\hat{\psi}(r e^{i\theta})|^2 dr d\theta = 2 \int_{K_{I_\perp}} |x|^{-1} |\hat{\psi}(x)|^2 dx,
\]

which completes the proof. \( \square \)

Corollary 3.3. Let \( \mu \) be a finite Borel measure in \( \mathbb{C} \) with compact support and \( I \subset [0, \pi] \) an arbitrary open set. Then we have

\[
2 \int_{x-y \in K_I \setminus \{0\}} \frac{1}{|x-y|} d\mu(x) d\mu(y) \leq \int_{I_+} \|P_\vartheta \mu\|_2^2 d\vartheta,
\]

where \( K_I \) is the cone \( K_I = \{re^{i\theta} : r \in \mathbb{R}, \theta \in I \} \) and \( I_\perp = I + \frac{\pi}{2} \) (mod \( \pi \)).

Proof. We assume that the integral on the right hand side above is finite. Let \( \{\phi_\varepsilon\}_{\varepsilon > 0} \) be an approximation of the identity, so that \( \phi_\varepsilon \) has compact support and is radial and \( C^\infty \). Denote \( \mu_\varepsilon = \mu \star \phi_\varepsilon \). It is straightforward to check that the identity [3.2] holds both for \( \mu \) and \( \mu_\varepsilon \). Then, by the dominated convergence theorem we deduce that

\[
\lim_{\varepsilon \to 0} \int_{I_+} \|P_\vartheta \mu_\varepsilon\|_2^2 d\vartheta = 2 \lim_{\varepsilon \to 0} \int_{K_{I_\perp}} |x|^{-1} |\hat{\mu}(\varepsilon x)|^2 dx \leq 2 \int_{K_{I_\perp}} |x|^{-1} |\hat{\mu}(x)|^2 dx = \int_{I_+} \|P_\vartheta \mu\|_2^2 d\vartheta.
\]

Hence, by Proposition 3.2 we have

\[
\limsup_{\varepsilon \to 0} \int_{x-y \in K_I} \frac{1}{|x-y|} d\mu_\varepsilon(x) d\mu_\varepsilon(y) \leq \int_{I_+} \|P_\vartheta \mu\|_2^2 d\vartheta.
\]
So it suffices to show that
\[
\int \int_{x-y \in K_I \setminus \{0\}} \frac{1}{|x-y|} d\mu(x) d\mu(y) \leq \limsup_{\varepsilon \to 0} \int \int_{x-y \in K_I} \frac{1}{|x-y|} d\mu_\varepsilon(x) d\mu_\varepsilon(y).
\]
To this end, consider an arbitrary non-negative continuous, compactly supported, function \( f(x) \leq \chi_{K_I}(x) |x|^{-1} \). We have that
\[
\int f \ast \mu_\varepsilon d\mu_\varepsilon = \int (f \ast \mu \ast \phi_\varepsilon) d\mu.
\]
Since \( f \ast \mu \) is compactly supported and continuous, \( f \ast \mu \ast \phi_\varepsilon \ast \phi_\varepsilon \) converges uniformly to \( f \ast \mu \) as \( \varepsilon \to 0 \), and thus
\[
\int f \ast \mu d\mu = \lim_{\varepsilon \to 0} \int f \ast \mu_\varepsilon d\mu_\varepsilon \leq \limsup_{\varepsilon \to 0} \int \int_{x-y \in K_I} \frac{1}{|x-y|} d\mu_\varepsilon(x) d\mu_\varepsilon(y).
\]

3.2. The higher dimensional case. Let \( G(d,n) \) denote the Grassmanian space of \( n \)-dimensional linear subspaces of \( \mathbb{R}^d \). Let \( \gamma_{d,n} \) denote the natural probability measure on \( G(d,n) \). We define a metric on \( G(d,n) \) by \( d(V,W) = \| P_V - P_W \| \), where \( \| \cdot \| \) denotes the operator norm.

Recall that for any Schwartz function \( \phi: \mathbb{R}^d \to \mathbb{C} \) and any \( V \in G(d,n) \), we have
\[
\widehat{P_V \phi}(x) = \hat{\phi}(x) \quad \text{for all } x \in V,
\]
where \( (P_V \phi)(x) = \int_{x+V} \phi d\mathcal{L}^{d-n} \) and \( \widehat{P_V \phi} \) denotes the \( n \)-dimensional Fourier transform on \( V \). The proof is identical to the proof of \((3.1)\).

The following is the higher dimensional analogue of Lemma \((3.1)\).

**Lemma 3.4.** Let \( B \subset G(d,n) \). Let \( \sigma, \nu \) be measures on \( \mathbb{R}^d \) given by
\[
\int f \, d\sigma = \int_B \int_V f \, d\mathcal{L}^n \, d\gamma_{d,n}(V), \tag{3.5}
\]
\[
\int f \, d\nu = \int_B \int_{V^\perp} f \, d\mathcal{L}^{d-n} \, d\gamma_{d,n}(V). \tag{3.6}
\]
Then the (distributional) Fourier transform of \( \sigma \) is \( \nu \).

**Proof.** Let \( \phi: \mathbb{R}^d \to \mathbb{C} \) be a Schwartz function. Then, as in the proof of Lemma \((3.1)\)
\[
\int \phi \, d\sigma = \int_B \int_V \hat{\phi} \, d\mathcal{L}^n \, d\gamma_{d,n}(V) = \int_B \int_V \widehat{P_V \phi} \, d\mathcal{L}^n \, d\gamma_{d,n}(V)
= \int_B \phi_V(0) \, d\gamma_{d,n}(V) = \int_B \int_{V^\perp} \phi \, d\mathcal{L}^{d-n} \, d\gamma_{d,n}(V) = \int \phi \, d\nu. \quad \square
\]

**Lemma 3.5.** Let \( B \subset G(d,n) \). Let \( \sigma \) be given by \((3.5)\). Then \( \text{supp} \sigma \subset \bigcup_{V \in B} V, \sigma \ll \mathcal{L}^d \), and \( \frac{d\sigma}{dx}(x) \leq \frac{c(d,n)}{|x|^{d-n}} \).
Proof. From the definition of $\sigma$, it is immediate that $\text{supp} \sigma \subset \bigcup_{V \in \mathcal{B}} V$. The next two properties follow from the following identity:

$$
\int_{G(d,n)} \int_{V} f \, d\mathcal{L}^{n} \, d\gamma_{d,n}(V) = c(d,n) \int_{\mathbb{R}^{n}} \frac{f(x)}{|x|^{d-n}} \, d\mathcal{L}^{n}.
$$

For a proof of this identity, see (24.2) in [Ma3].

**Lemma 3.6.** Let $\psi : \mathbb{R}^{2} \to \mathbb{R}$ be a Schwartz function. Then for any set $B \subset G(d,n)$, we have

$$
\int_{B} \|P_{V} \psi\|_{2}^{2} \, d\gamma_{d,n}(V) = \int \int \frac{d\nu}{d\mathcal{L}^{n}}(x-y) \psi(x) \psi(y) \, dx \, dy.
$$

**Proof.** The proof is identical to the proof of Lemma 3.2. □

To make Lemma 3.6 more useful, we obtain a lower bound on $\frac{d\nu}{d\mathcal{L}^{n}}$ via the following three lemmas.

**Lemma 3.7.** For all $V \in G(d,n)$ and for $\delta \leq d,n 1$,

$$
\gamma_{d,n}(B(V, \delta)) \approx_{d,n} \delta^{n(d-n)}
$$

**Proof.** This is Proposition 4.1 of [FO]. □

**Lemma 3.8.** Let $x \in \mathbb{R}^{d} \setminus \{0\}$. Then $G_{x} := \{ V \in G(d,n) : x \in V \}$ and $G(d-1,n-1)$ are isomorphic as metric spaces.

**Proof.** Let $A : \mathbb{R}^{d-1} \to \mathbb{R}^{d}$ be a linear map satisfying $A^{T}A = \text{id}$ and whose image is the orthogonal complement of $x$. Consider the map $\Psi : G(d-1,n-1) \to G_{x}$ given by $V \mapsto \text{span}(AV,x)$. We will show $\Psi$ is an isometry. First we make two observations.

1. For any $V \in G_{x}$, we have $P_{V}x = x$.
2. For any $z \in \mathbb{R}^{d-1}$ and $V \in G(d-1,n-1)$, we have $P_{\Psi V}Az = AP_{V}z$.

Let $V,W \in G(d-1,n-1)$. We need to show

$$
\|P_{\Psi V} - P_{\Psi W}\|_{\mathbb{R}^{d} \to \mathbb{R}^{d}} = \|P_{V} - P_{W}\|_{\mathbb{R}^{d-1} \to \mathbb{R}^{d-1}}.
$$

Let $y \in \mathbb{R}^{d}$ and write $y = \lambda x + Az$, where $\lambda \in \mathbb{R}$ and $z \in \mathbb{R}^{d-1}$. Note that $z = A^{T}y$. Using the two observations above, we have

$$
(P_{\Psi V} - P_{\Psi W})y = (P_{\Psi V} - P_{\Psi W})Az = A(P_{V} - P_{W})z = A(P_{V} - P_{W})A^{T}y.
$$

Hence $P_{\Psi V} - P_{\Psi W} = A(P_{V} - P_{W})A^{T}$. Since $\|A\|_{\mathbb{R}^{d-1} \to \mathbb{R}^{d}} = \|A^{T}\|_{\mathbb{R}^{d} \to \mathbb{R}^{d-1}} = 1$, it follows that $\|P_{\Psi V} - P_{\Psi W}\| \leq \|P_{V} - P_{W}\|$. To show the reverse inequality, note that for any $z \in \mathbb{R}^{d-1}$, we have

$$
\|(P_{V} - P_{W})z\| = \|A(P_{V} - P_{W})z\| \\
= \|(P_{\Psi V} - P_{\Psi W})Az\| \\
\leq \|P_{\Psi V} - P_{\Psi W}\| \|Az\| \\
= \|P_{\Psi V} - P_{\Psi W}\| |z|,
$$

which implies $\|P_{V} - P_{W}\| \leq \|P_{\Psi V} - P_{\Psi W}\|$. □
Lemma 3.9. Let \( B = B(V_0, r) \subset G(d, n) \). Let \( \sigma \) be given by \((3.5)\). Then there is a \( c = c(d, n, r) > 0 \) such that

\[
\frac{d\sigma}{dx}(x) \geq \frac{c}{|x|^{d-n}} \quad \text{on the cone} \quad \bigcup_{V \in B(V_0, \frac{1}{2}r)} V.
\]

Proof. Note that \( \sigma(\lambda A) = \lambda^n \sigma(A) \), which implies \( \frac{d\sigma}{dx}(\lambda x) = \lambda^{n-d} \frac{d\sigma}{dx}(x) \). Hence, it suffices to show \( \frac{d\sigma}{dx}(x) \gtrsim_{d,n,r} 1 \) for all \( x \in \bigcup_{V \in B(V_0, \frac{1}{2}r)} V \) with \( |x| = 1 \).

Fix \( x \in \bigcup_{V \in B(V_0, \frac{1}{2}r)} V \) with \( |x| = 1 \). Let \( G_x = \{ V \in G(d, n) : x \in V \} \), and let \( (G_x)^\delta \subset G(d, n) \) denote the \( \delta \)-neighborhood of \( G_x \).

We claim that

\[
(3.7) \quad \sigma(B(x, s)) \gtrsim s^n \gamma_{d,n}(\langle G_x \rangle^{s/2} \cap B(V_0, r)) \quad \text{for all} \quad s > 0.
\]

To see this, note that if \( V \in \langle G_x \rangle^{s/2} \), then there is some \( W \in G_x \) such that \( d(V, W) < \frac{s}{2} \).

Then \( |x - P_V x| = |P_W x - P_V x| \leq \|P_W - P_V\| \leq d(W, V) < \frac{s}{2} \), so \( \mathcal{L}^n(V \cap B(x, s)) \gtrsim_{d,n} s^n \).

Hence,

\[
\sigma(B(x, s)) = \int_{B(V_0, r)} \mathcal{L}^n(V \cap B(x, s)) \, d\gamma_{d,n}(V) \gtrsim_{d,n} s^n \gamma_{d,n}(\langle G_x \rangle^{s/2} \cap B(V_0, r)),
\]

which proves \((3.7)\).

Next, we bound \( \gamma_{d,n}(\langle G_x \rangle^{s/2} \cap B(V_0, r)) \) from below. Fix a \( V_1 \in G_x \cap B(V_0, \frac{1}{2}r) \). (Since \( x \in K \), \( V_1 \) exists.)

Suppose \( s < r \). Let \( F_s \) be a maximal \( s \)-separated subset of \( G_x \cap B(V_1, \frac{1}{2}r) \). It follows from the maximality of \( F_s \) that

\[
(3.8) \quad G_x \cap B(V_1, \frac{1}{2}r) \subset \bigcup_{W \in F_s} (G_x \cap B(W, s)).
\]

By \((3.8)\), Lemma 3.8 and Lemma 3.7 applied to \( G(d-1, n-1) \), it follows that for \( s \lesssim_{d,n,r} 1 \),

\[
(3.9) \quad \#F_s \gtrsim_{d,n,r} s^{-(n-1)(d-n)}.
\]

Next, observe that the balls \( \{ B(W, \frac{s}{2}) \}_{W \in F_s} \) are pairwise disjoint and contained in \( \langle G_x \rangle^{s/2} \cap B(V_0, r) \), so

\[
(3.10) \quad \gamma_{d,n}(\langle G_x \rangle^{s/2} \cap B(V_0, r)) \geq \sum_{W \in F_s} \gamma_{d,n}(B(W, \frac{s}{2})) \gtrsim_{d,n,r} s^{d-n} \quad \text{for} \quad s \lesssim_{d,n,r} 1,
\]

where we used \((3.9)\) and \((3.7)\) in the last inequality. Finally, \((3.7)\) and \((3.10)\) imply \( \frac{d\sigma}{dx}(x) \gtrsim_{d,n,r} 1 \), as desired. \( \square \)
Corollary 3.10. Let $V_0 \in G(d,n)$ and $s > 0$. Then there exist constants $\lambda, c > 1$ such that for any Schwartz function $\psi : \mathbb{R}^d \to \mathbb{R}$,
\[
c^{-1} \iint_{x - y \in K(V_0^+, \lambda^{-1}s)} \frac{\psi(x) \psi(y)}{|x - y|^n} \, dx \, dy \leq \int_{B(V_0, s)} ||P_V \mu||_2^2 \, d\gamma_{d,n}(V) \leq c \iint_{x - y \in K(V_0^+, \lambda s)} \frac{\psi(x) \psi(y)}{|x - y|^n} \, dx \, dy.
\]

Proof. By Lemma 3.3 and Lemma 3.9 (applied to $G(d, d - n)$),
\[
c_1 \frac{\chi_{K_1}(x)}{|x|^n} \leq \frac{d\nu}{dx}(x) \leq c_2 \frac{\chi_{K_2}(x)}{|x|^n},
\]
where $K_1 = \bigcup_{V \in B(V_0, \frac{s}{2})} V^\perp$, $K_2 = \bigcup_{V \in B(V_0, s)} V^\perp$.

For $\lambda$ sufficiently large, we have $K(V_0^+, \lambda^{-1}s) \subset K_1 \subset K_2 \subset K(V_0^+, \lambda s)$. Thus,
\[
c_1 \frac{\chi_{K(V_0^+, \lambda^{-1}s)}(x)}{|x|^n} \leq \frac{d\nu}{dx}(x) \leq c_2 \frac{\chi_{K(V_0^+, \lambda s)}(x)}{|x|^n},
\]
so this corollary follows easily from Lemma 3.6.

\[\square\]

Corollary 3.11. Let $V_0 \in G(d,n)$ and $s > 0$. Then there exist constants $\lambda, c > 1$ such that for any finite Borel measure $\mu$ in $\mathbb{R}^d$,
\[
\iint_{x - y \in K(V_0^+, \lambda^{-1}s)} \frac{d\mu(x) \, d\mu(y)}{|x - y|^n} \leq c \int_{B(V_0, \lambda s)} ||P_V \mu||_2^2 \, d\gamma_{d,n}(V).
\]

The proof of this corollary follows from Corollary 3.10 along the same lines as the one of Corollary 3.3, and so we skip it.

Remark 3.12. The converse inequality
\[
\int_{B(V_0, s)} ||P_V \mu||_2^2 \, d\gamma_{d,n}(V) \leq c \iint_{x - y \in K(V_0^+, \lambda s)} \frac{d\mu(x) \, d\mu(y)}{|x - y|^n}
\]
do not hold for arbitrary measures. Indeed, in the case $d = 2$, $n = 1$, consider a segment $L$ through the origin, and let $K(V_0^+, \lambda s)$ be a cone such that $L$ is not contained in the closure of the cone. Then with $\mu = \mathcal{H}^1|_L$, the integral on the left hand side is positive (and finite) while the one on the right hand side is zero.

Instead, one can show that the following holds:
\[
(3.11) \quad \int_{B(V_0, s)} ||P_V \mu||_2^2 \, d\gamma_{d,n}(V) \leq c \iint_{x - y \in K(V_0^+, \lambda s)} \frac{d\mu(x) \, d\mu(y)}{|x - y|^n} + C(\lambda) \int M_n \mu(x) \, d\mu(x),
\]
where
\[
M_n \mu(x) = \sup_{r > 0} \frac{\mu(B(x, r))}{r^n}.
\]
However, the estimate (3.11) will not be necessary in this paper and so we skip the proof.
4. The dyadic lattice of David and Mattila

Now we will introduce the dyadic lattice of cubes with small boundaries of David-Mattila associated with a Radon measure $\mu$. This lattice has been constructed in [DM, Theorem 3.2]. Its properties are summarized in the next lemma.

**Lemma 4.1** (David, Mattila). Let $\mu$ be a compactly supported Radon measure in $\mathbb{R}^d$. Consider two constants $C_0 > 1$ and $A_0 > 5000C_0$ and denote $W = \text{supp} \mu$. Then there exists a sequence of partitions of $W$ into Borel subsets $Q$, $Q \in \mathcal{D}_{\mu,k}$, with the following properties:

- For each integer $k \geq 0$, $W$ is the disjoint union of the “cubes” $Q$, $Q \in \mathcal{D}_{\mu,k}$, and if $k < l$, $Q \in \mathcal{D}_{\mu,l}$, and $R \in \mathcal{D}_{\mu,k}$, then either $Q \cap R = \emptyset$ or else $Q \subset R$.

- The general position of the cubes $Q$ can be described as follows. For each $k \geq 0$ and each cube $Q \in \mathcal{D}_{\mu,k}$, there is a ball $B(Q) = B(x_Q, r(Q))$ such that

$$x_Q \in W, \quad A_0^{-k} \leq r(Q) \leq C_0 A_0^{-k},$$

$$W \cap B(Q) \subset Q \subset W \cap 28 B(Q) = W \cap B(x_Q, 28r(Q)),$$

and

the balls $5B(Q)$, $Q \in \mathcal{D}_{\mu,k}$, are disjoint.

- The cubes $Q \in \mathcal{D}_{\mu,k}$ have small boundaries. That is, for each $Q \in \mathcal{D}_{\mu,k}$ and each integer $l \geq 0$, set

$$N^\text{ext}_l(Q) = \{x \in W \setminus Q : \text{dist}(x, Q) < A_0^{-k-l}\},$$

$$N^\text{int}_l(Q) = \{x \in Q : \text{dist}(x, W \setminus Q) < A_0^{-k-l}\},$$

and

$$N_l(Q) = N^\text{ext}_l(Q) \cup N^\text{int}_l(Q).$$

Then

$$\mu(N_l(Q)) \leq (C^{-1}C_0^{-3d-1}A_0)^{-l} \mu(90B(Q)).$$

(4.1)

- Denote by $\mathcal{D}^{db}_{\mu,k}$ the family of cubes $Q \in \mathcal{D}_{\mu,k}$ for which

$$\mu(100B(Q)) \leq C_0 \mu(B(Q)).$$ \hspace{1cm} (4.2)

We have that $r(Q) = A_0^{-k}$ when $Q \in \mathcal{D}_{\mu,k} \setminus \mathcal{D}^{db}_{\mu,k}$ and

$$\mu(100B(Q)) \leq C_0^{-l} \mu(100^{l+1}B(Q)) \quad \text{for all } l \geq 1 \text{ with } 100^l \leq C_0 \text{ and } Q \in \mathcal{D}_{\mu,k} \setminus \mathcal{D}^{db}_{\mu,k}. \hspace{1cm} (4.3)$$

We use the notation $\mathcal{D}_{\mu} = \bigcup_{k \geq 0} \mathcal{D}_{\mu,k}$. Observe that the families $\mathcal{D}_{\mu,k}$ are only defined for $k \geq 0$. So the diameter of the cubes from $\mathcal{D}_{\mu}$ are uniformly bounded from above. For $Q \in \mathcal{D}_{\mu}$, we set $J(Q) = \{P \in \mathcal{D}_{\mu} : P \subset Q\}$. Given $Q \in \mathcal{D}_{\mu,k}$, we denote $J(Q) = k$, and we set $\ell(Q) = 56 C_0 A_0^{-k}$ and we call it the side length of $Q$. Notice that

$$\frac{1}{28} C_0^{-1} \ell(Q) \leq \text{diam}(28B(Q)) \leq \ell(Q).$$
Observe that \( r(Q) \approx \text{diam}(Q) \approx \ell(Q) \). Also we call \( x_Q \) the center of \( Q \), and the cube \( Q' \in \mathcal{D}_{\mu,k-1} \) such that \( Q' \supset Q \) the parent of \( Q \). We set \( B_Q = 28B(Q) = B(x_Q, 28r(Q)) \), so that

\[
W \cap \frac{1}{28}B_Q \subset Q \subset B_Q.
\]

We assume \( A_0 \) big enough so that the constant \( C^{-1}C_0^{-3d-1}A_0 \) in (4.1) satisfies

\[
C^{-1}C_0^{-3d-1}A_0 > A_1/2 > 10.
\]

Then we deduce that, for all \( 0 < \lambda \leq 1 \),

\[
\mu(\{ x \in Q : \text{dist}(x, W \setminus Q) \leq \lambda \ell(Q) \}) + \mu(\{ x \in 3.5B_Q : \text{dist}(x, Q) \leq \lambda \ell(Q) \}) \leq c\lambda^{1/2} \mu(3.5B_Q).
\]

(4.4)

We denote \( \mathcal{D}_{\mu}^{db} = \bigcup_{k \geq 0} \mathcal{D}_{\mu,k}^{db} \). Note that, in particular, from (4.2) it follows that

\[
\mu(3B_Q) \leq \mu(100B(Q)) \leq C_0 \mu(Q) \quad \text{if} \quad Q \in \mathcal{D}_{\mu}^{db}.
\]

(4.5)

For this reason we will call the cubes from \( \mathcal{D}_{\mu}^{db} \) doubling. Given \( Q \in \mathcal{D}_\mu \), we denote by \( \mathcal{D}_\mu(Q) \) the family of cubes from \( \mathcal{D}_\mu \) which are contained in \( Q \). Analogously, we write \( \mathcal{D}_\mu^{db}(Q) = \mathcal{D}_\mu^{db} \cap \mathcal{D}_\mu(Q) \).

As shown in \([DM, \text{Lemma 5.28}]\), every cube \( R \in \mathcal{D}_\mu \) can be covered \( \mu \)-a.e. by a family of doubling cubes:

**Lemma 4.2.** Let \( R \in \mathcal{D}_\mu \). Suppose that the constants \( A_0 \) and \( C_0 \) in Lemma 4.1 are chosen suitably. Then there exists a family of doubling cubes \( \{ Q_i \}_{i \in I} \subset \mathcal{D}_\mu^{db} \), with \( Q_i \subset R \) for all \( i \), such that their union covers \( \mu \)-almost all \( R \).

The following result is proved in \([DM, \text{Lemma 5.31}]\).

**Lemma 4.3.** Let \( R \in \mathcal{D}_\mu \) and let \( Q \subset R \) be a cube such that all the intermediate cubes \( S \), \( Q \subset S \subset R \) are non-doubling (i.e. belong to \( \mathcal{D}_\mu \setminus \mathcal{D}_\mu^{db} \)). Then

\[
\mu(100B(Q)) \leq A_0^{-10d(J(Q) - J(R) - 1)} \mu(100B(R)).
\]

(4.6)

Given a ball (or an arbitrary set) \( B \subset \mathbb{R}^d \) and a fixed \( n \geq 1 \), we consider its \( n \)-dimensional density:

\[
\Theta_\mu(B) = \frac{\mu(B)}{\text{diam}(B)^n}.
\]

From the preceding lemma we deduce:

**Lemma 4.4.** Let \( Q,R \in \mathcal{D}_\mu \) be as in Lemma 4.3. Then

\[
\Theta_\mu(100B(Q)) \leq C_0 A_0^{-9d(J(Q) - J(R) - 1)} \Theta_\mu(100B(R))
\]

and

\[
\sum_{S \in \mathcal{D}_\mu, Q \subset S \subset R} \Theta_\mu(100B(S)) \leq c \Theta_\mu(100B(R)),
\]

with \( c \) depending on \( C_0 \) and \( A_0 \).

For the easy proof, see \([Co, \text{Lemma 4.4}]\), for example.

We will also need the following technical result.
Lemma 4.5. Let $R \in \mathcal{D}_\mu$ such that $\mu(2B_R) \leq C_1 \mu(R)$. Then there exists another cube $Q \subset R$ from $\mathcal{D}_\mu^{db}$ such that

$$\mu(Q) \approx \mu(R) \quad \text{and} \quad \ell(Q) \approx \ell(R),$$

with the implicit constants depending on $C_1$.

Proof. Suppose that $R \in \mathcal{D}_{\mu,k}$. For some $N > 1$ to be fixed later, denote by $I_N$ the family of cubes from $\mathcal{D}_{\mu,k+N}$ which are contained in $R$. Recall that the balls $B(Q)$, $Q \in I_N$, are disjoint and that their radii satisfy

$$A_0^{-k-N} \leq r(Q) \leq C_0 A_0^{-k-N}.$$

All the balls from $I_N$ intersect $R$ and are contained in $2B_R$ for $N$ big enough. Thus we have

$$\# I_N \cdot C_d A_0^{(k-N)d} \leq L^d \left( \bigcup_{Q \in I_N} B(Q) \right) \leq C_d (2C_0 A_0^{-k})^d,$$

and so

$$\# I_N \leq 2^d C_0^d A_0^{Nd}.$$

Therefore, the cube $Q' \in I_N$ with maximal measure satisfies

$$\mu(Q') \geq 2^{-d} C_0^{-d} A_0^{-Nd} \mu(R).$$

We claim now that if $N$ is big enough then there exists some cube $Q \in \mathcal{D}_\mu^{db}$ such that $Q' \subset Q \subset R$. Indeed, if such cube $Q$ does not exist, then denoting by $R'$ the son of $R$ that contains $Q'$, we deduce

$$\mu(Q') \leq \mu(100B(Q')) \leq A_0^{-10d(N-2)} \mu(100B(R')) \leq A_0^{-10d(N-2)} \mu(2B_R) \leq C_1 A_0^{-10d(N-2)} \mu(R),$$

taking into account that $100B(R') \subset 2B_R$. For $N$ big enough, this estimate contradicts \[4.7\], and thus the cube $Q$ mentioned above exists. It is clear that this satisfies the estimates $\mu(Q) \approx \mu(R)$ and $\ell(Q) \approx \ell(R)$, as wished. \hfill \square

5. The corona decomposition

Let $\mu$ a Borel measure in $\mathbb{R}^d$ satisfying the growth condition

$$\mu(B(x,r)) \leq c_0 r^\alpha \quad \text{for all} \ x \in \mathbb{R}^d, \ r > 0.$$ \hfill (5.1)

We consider the dyadic lattice $\mathcal{D}_\mu$ of David-Mattila associated to $\mu$, and we assume that $\text{supp} \mu \in \mathcal{D}_\mu$ is the biggest cube in this lattice. (To this end, we assume $\mathcal{D}_{\mu,k}$ to be defined for $k \geq k_0$, with an appropriate $k_0$.) Sometimes we will also denote by $R_0$ the initial cube $\text{supp} \mu$. We allow all constants $c$, $C$, and other implicit constants to depend on $n$, $d$, and the parameters in the definition of the David-Mattila cubes.

Let $\text{Top}$ be a family of cubes from $\mathcal{D}_\mu^{db}$ to be fixed below, with $R_0 \in \text{Top}$. For every $R \in \text{Top}$, denote by $\text{Next}(R)$ the family of maximal cubes $Q \in \text{Top}$ that are contained in $R$, and by $\text{Tr}(R)$ the family of cubes $Q \in \mathcal{D}_\mu$ that are contained in $R$ and not contained in any $Q' \in \text{Next}(R)$. Then, define

$$\text{Good}(R) = R \setminus \bigcup_{Q \in \text{Next}(R)} Q,$$
and for \( Q, S \in \mathcal{D}_\mu \) with \( Q \subset S \),
\[
\delta_\mu(Q,S) = \int_{2B_S \setminus 2B_Q} \frac{1}{|y-x|^n} \, d\mu(y).
\]

The next lemma is the main tool which will allow us to connect the energy
\[
\int_\mathbb{R}^d \frac{1}{|x-y|^n} \, d\mu(x) \, d\mu(y)
\]
to the curvature of \( \mu \).

**Lemma 5.1** (Corona decomposition). Given \( V_0 \in G(d,d-n) \) and \( s > 0 \), consider the cone \( K := K(V_0, s) \subset \mathbb{R}^d \). Let \( \mu \) be a Borel measure in \( \mathbb{R}^d \) satisfying the growth condition \( (5.1) \). There exists a family \( \text{Top} \subset \mathcal{D}_\mu \) as above such that, for all \( R \in \text{Top} \), there exists an \( n \)-dimensional Lipschitz graph \( \Gamma_R \) with the slope depending only on \( s \) such that:

(a) \( \mu \)-almost all \( \text{Good}(R) \) is contained in \( \Gamma_R \).

(b) For all \( Q \in \text{Next}(R) \) there exists another cube \( \tilde{Q} \in \mathcal{D}_\mu \) such that \( \delta_\mu(Q,\tilde{Q}) \leq c \Theta_\mu(2B_R) \) and \( 2B_{\tilde{Q}} \cap \Gamma_R \neq \emptyset \).

(c) For all \( Q \in \text{Tr}(R) \), \( \Theta_\mu(2B_Q) \leq c \Theta_\mu(2B_R) \).

Furthermore, the following packing condition holds:
\[
\sum_{R \in \text{Top}} \Theta_\mu(2B_R) \mu(R) \lesssim \mu(R_0) + \int_\mathbb{R}^d \frac{1}{|x-y|^n} \, d\mu(x) \, d\mu(y),
\]
with the implicit constant depending only on \( c_0 \) and the aperture \( s \) of the cone \( K \).

The next Sections 6-9 are devoted to the proof of this lemma. In these sections we assume that \( \mu \) is a measure in \( \mathbb{R}^d \) that satisfies \( (5.1) \) and that \( K = K(V_0, s) \) is a cone, with \( V_0 \in G(d,d-n) \), \( s > 0 \), such that
\[
\int_\mathbb{R}^d \frac{1}{|x-y|^n} \, d\mu(x) \, d\mu(y) < \infty.
\]

### 6. The Construction of an Approximate Lipschitz Graph

**6.1. The stopping cubes.** We consider constants \( A \gg 1 \) and \( 0 < \tau, \eta, \alpha \ll 1 \) to be fixed below. For \( Q \in \mathcal{D}_\mu \), we denote
\[
\mathcal{E}_\mu(Q) = \frac{1}{\mu(Q)} \int_{\mathbb{R}^d} \frac{1}{|x-y|^n} \, d\mu(x) \, d\mu(y).
\]

Observe that \( \mathcal{E}_\mu(Q) \) scales like \( \Theta_\mu(2B_Q) \).

Given a cube \( R \in \mathcal{D}_\mu \), we consider the following families of cubes:

- The high density family \( \text{HD}_0(R) \), which is made up of the cubes \( Q \in \mathcal{D}_\mu \) which satisfy \( \Theta_\mu(2B_Q) \geq A \Theta_\mu(2B_R) \).
- The low density family \( \text{LD}_0(R) \), which is made up of the cubes \( Q \in \mathcal{D}_\mu \) which satisfy \( \Theta_\mu(2B_Q) \leq \tau \Theta_\mu(2B_R) \).
In the case $\lambda > 0$ we denote $K^\lambda_x = K(x, V_0, \lambda s)$.

Given $Q \in D_\mu$, we also set $K^\lambda_Q = \bigcup_{x \in 2B_Q} K^\lambda_x$.

In the case $\lambda = 1$, we write $K_x$ and $K_Q$ instead of $K^1_x$ and $K^1_Q$.
Lemma 6.3. There exists some constant $M > 1$ depending only on $s$ such that the following are true.

(a) Suppose $Q \in \Tree(R)$ and $Q' \in \mathcal{D}_\mu(R)$ satisfy $Q' \cap (K_Q^{1/2} \setminus MB_Q) \neq \emptyset$ and $\ell(Q') \leq \frac{1}{\eta} \dist(Q', Q)$. Then $Q' \not\in \Tree(R)$.

(b) Let $J \subset \Tree(R)$ be a family of pairwise disjoint cubes. Then

\begin{equation}
\mu\left( \bigcup_{Q \in J} K_Q^{1/2} \cap (R \setminus M B_Q) \right) \leq C(A, \tau, M) \varepsilon \mu(R).
\end{equation}

Proof of (a). Let $P \in \mathcal{D}_\mu(R)$ be such that $Q' \subset P \subset R$, $P \subset K_Q^{3/4}$ and $\ell(P) \approx \dist(P, Q)$. Let $S \in \mathcal{D}_\mu(R)$ be such that $Q \subset S \subset R$, $\ell(S) \approx \frac{1}{\eta} \ell(P)$, and $\dist(P, S) \approx \ell(P)$. For $M$ big enough and an appropriate choice of the implicit constants, we have

$$2B_P \subset K_x \quad \text{for all } x \in 2B_S.$$ 

Therefore,

\begin{equation}
\mu(2B_P) \mu(2B_S) \frac{1}{\ell(P)^n} \leq \int_{x \in 2B_S} \int_{x-y \in K} \frac{1}{|x-y|^n} \, d\mu(x) \, d\mu(y) = \mu(S) \mathcal{E}_\mu(S),
\end{equation}

assuming $\eta$ small enough (depending on $M$).

Since $Q \in \Tree(R)$ and $Q \subset S$, we have $S \not\in \Stop(R)$ and thus

$$\Theta_\mu(2B_P) = \frac{\mu(2B_P)}{\ell(P)^n} \leq \frac{\mu(S)}{\mu(2B_S)} \mathcal{E}_\mu(S) \leq \mathcal{E}_\mu(S) \leq \varepsilon \Theta_\mu(2B_R).$$

Thus if $\varepsilon$ is small enough, then $P \in LD_0(R)$. Since $Q' \subset P$, it follows that $Q' \not\in \Tree(R)$. \qed

Proof of (b). We can assume that the cubes of the family $J$ cover $R$. Otherwise we replace $J$ by a suitable enlarged family $J'$.

For a fixed $Q \in J$, if $M$ is big enough, we can cover $K_Q^{1/2} \cap (R \setminus MB_Q)$ by a family of cubes $P \in \mathcal{D}_\mu(R)$ such that $P \subset K_Q^{3/4}$, $P \cap (R \setminus MB_Q) \neq \emptyset$, and $\ell(P) \approx \dist(P, Q)$. We denote by $I_Q$ this family. We assign a cube $S_{P,Q} \in \mathcal{D}_\mu(R)$ to each $P \in I_Q$ such that $Q \subset S_{P,Q} \subset R$, $\ell(S_{P,Q}) \approx \frac{1}{\eta} \ell(P)$, and $\dist(P, S_{P,Q}) \approx \ell(P)$. As in the proof of (a), for $M$ big enough, $\eta$ small enough, and an appropriate choice of the implicit constants, we have

$$\mu(2B_P) \mu(2B_{S_{P,Q}}) \frac{1}{\ell(P)^n} \leq \mu(S_{P,Q}) \mathcal{E}_\mu(S_{P,Q}).$$

Since $Q \subset S_{P,Q}$, we have $S_{P,Q} \not\in \Stop(R)$ and thus

$$\mu(2B_{S_{P,Q}}) \frac{1}{\ell(P)^n} \approx_M \mu(2B_{S_{P,Q}}) \frac{1}{\ell(S_{P,Q})^n} \approx_{A, \tau, M} \Theta_\mu(2B_R),$$

so

$$\Theta_\mu(2B_R) \mu(P) \lesssim_{A, \tau, M} \mu(S_{P,Q}) \mathcal{E}_\mu(S_{P,Q}).$$

Consider now a maximal subfamily $\mathcal{A} \subset \bigcup_{Q \in J} I_Q$, so that the cubes from $\mathcal{A}$ are pairwise disjoint and

$$\bigcup_{Q \in J} K_Q^{1/2} \cap (R \setminus M B_Q) \subset \bigcup_{P \in \mathcal{A}} P.$$
For each $P \in A$ we choose a cube $S(P) := S_{P,Q}$, where $Q$ is such that $P \in I_Q$. The precise choice of $Q$ does not matter as long as $P \in I_Q$. Observe that for each cube $S \in \mathcal{D}_\mu$ there is at most a bounded number (depending on $M$) of cubes $P \in A$ such that $S = S(P)$, taking into account that $\ell(S(P)) \approx \frac{1}{M} \ell(P)$ and $\text{dist}(P, S(P)) \approx \ell(P)$. Further, all the cubes $\{S(P)\}_{P \in A}$ belong to $\text{Tree}(R) \setminus \text{Stop}(R)$. As a consequence,

\[(6.4) \quad \mu\left(\bigcup_{Q \in J} K_{Q}^{1/2} \cap (R \setminus MB_Q)\right) \leq \sum_{P \in A} \mu(P) \]

\[\lesssim_{A,\tau,M} \frac{1}{\Theta_\mu(2B_R)} \sum_{P \in A} \mu(S(P)) \mathcal{E}_\mu(S(P)) \]

\[\lesssim_{A,\tau,M} \frac{1}{\Theta_\mu(2B_R)} \sum_{S \in \text{Tree}(R) \setminus \text{Stop}(R)} \mu(S) \mathcal{E}_\mu(S).\]

By Fubini, we get

\[
\sum_{S \in \text{Tree}(R) \setminus \text{Stop}(R)} \sum_{Q \in J, Q \subset S} \mu(S) \mathcal{E}_\mu(S) = \sum_{S \in \text{Tree}(R) \setminus \text{Stop}(R)} \sum_{Q \in J, Q \subset S} \mu(Q) \mathcal{E}_\mu(S) \]

\[= \sum_{Q \in J} \mu(Q) \sum_{S \in \text{Tree}(R) \setminus \text{Stop}(R)} \mathcal{E}_\mu(S).\]

where we used the fact that the cubes from $J$ cover $R$ in the first identity. Note now that for any $Q \in J$,

\[
\sum_{S \in \text{Tree}(R) \setminus \text{Stop}(R)} \mathcal{E}_\mu(S) \leq \varepsilon \Theta_\mu(2B_R),
\]

because of the stopping condition involving the cubes from BRE($R$). Therefore,

\[
\sum_{S \in \text{Tree}(R) \setminus \text{Stop}(R)} \mu(S) \mathcal{E}_\mu(S) \leq \varepsilon \Theta_\mu(2B_R) \sum_{Q \in J} \mu(Q) \leq \varepsilon \Theta_\mu(2B_R) \mu(R).
\]

From (6.4) and the preceding estimate, the lemma follows. \qed

Denote

\[G_R = R \setminus \bigcup_{Q \in \text{Stop}(R)} Q \quad \text{and} \quad \widetilde{G}_R = \bigcap_{k=1}^{\infty} \bigcup_{Q \in \text{Tree}(R)} 2MB_Q \]

and observe that $G_R \subset \widetilde{G}_R$. As an immediate consequence of the preceding lemma, we can show that $\widetilde{G}_R$ is contained in a Lipschitz graph.

**Lemma 6.4.** For all $x, y \in \widetilde{G}_R$, we have $x - y \notin K^{1/2}$. Hence, $\widetilde{G}_R$ is contained in an $n$-dimensional Lipschitz graph with Lipschitz constant depending only on $s$.

**Proof.** Suppose for contradiction that $x, y \in \widetilde{G}_R$ and $x - y \notin K^{1/2}$. Let $Q, Q' \in \text{Tree}(R)$ be such that $x \in 2MB_Q$ and $y \in 2MB_{Q'}$, with $\ell(Q), \ell(Q')$ so small that $Q' \cap (K_{Q}^{1/2} \setminus MB_Q) \neq \emptyset$ and $\ell(Q') \leq \frac{1}{M} \text{dist}(Q', Q)$. By Lemma 6.3(a), it follows that $Q' \notin \text{Tree}(R)$, which is a contradiction. \qed
6.4. An algorithm to construct a Lipschitz graph close to the stopping cubes.

Given \( t > 1 \), we say that two cubes \( Q, Q' \subset \mathcal{D}_t \) are \( t \)-neighbors if

\[
(6.5) \quad t^{-1} \ell(Q') \leq \ell(Q) \leq t \ell(Q')
\]

and

\[
(6.6) \quad \text{dist}(Q, Q') \leq t(\ell(Q) + \ell(Q')).
\]

We say that a family of cubes is \( t \)-separated if there is not any pair of cubes in this family which are \( t \)-neighbors.

Given a big constant \( t > M \) to be fixed below, we denote by \( \text{Sep}(R) \) a maximal \( t \)-separated subfamily of \( \text{Stop}(R) \). It is easy to check that such subfamily exists. Next, we define

\[
\widetilde{\text{Sep}}(R) = \{ Q \in \text{Sep}(R) : 2MB_Q \cap \tilde{G}_R = \emptyset \text{ and } Q' \in \text{Sep}(R) \text{ such that } 2MB_{Q'} \subset 2MB_Q \}.
\]

**Lemma 6.5.** Suppose \( t \) is sufficiently large (depending on \( M \)). Then for all \( Q, Q' \in \widetilde{\text{Sep}}(R) \), we have \( Q' \not\in 2MB_Q \).

**Proof.** Suppose \( Q' \in \widetilde{\text{Sep}}(R) \) and \( Q' \subset 2MB_Q \). We will show \( Q \not\in \widetilde{\text{Sep}}(R) \). If \( \ell(Q') > t^{-1} \ell(Q) \), then \( Q', Q' \subset 2MB_Q \) implies that \( Q, Q' \) are \( t \)-neighbors if \( t \) is sufficiently large. On the other hand, if \( \ell(Q') \leq t^{-1} \ell(Q) \), then \( Q' \subset 2MB_Q \) implies that \( 2MB_{Q'} \subset 2MB_Q \) if \( t \) is sufficiently large. Hence, in both cases, we have \( Q \not\in \widetilde{\text{Sep}}(R) \). \( \square \)

**Lemma 6.6.** The following holds:

(a) For each \( Q \in \text{Stop}(R) \) there exists some cube \( Q' \in \text{Sep}(R) \) which is \( t \)-neighbor of \( Q \).

(b) For each \( Q \in \text{Sep}(R) \), at least one of the following is true:

- \( 2MB_Q \cap \tilde{G}_R \neq \emptyset \).

- There exists some \( P \in \widetilde{\text{Sep}}(R) \) such that \( P \subset 2MB_Q \).

**Proof.** The first statement is obvious from the maximality of the separated family \( \text{Sep}(R) \).

For the second one, note first that the statement is clearly true if \( Q \in \text{Sep}(R) \). If \( Q \in \text{Sep}(R) \setminus \widetilde{\text{Sep}}(R) \) and \( 2MB_Q \cap \tilde{G}_R = \emptyset \), then there exists another cube \( Q_1 \in \text{Sep}(R) \) such that \( \ell(Q_1) \leq t^{-1} \ell(Q) \) and \( 2MB_{Q_1} \subset 2MB_Q \).

If \( Q_1 \in \text{Sep}(R) \), then we take \( P = Q_1 \). Otherwise, since \( 2MB_{Q_1} \cap \tilde{G}_R \subset 2MB_{Q_1} \cap 2MB_Q = \emptyset \), there exists some cube \( Q_2 \in \text{Sep}(R) \) such that \( \ell(Q_2) \leq t^{-1} \ell(Q_1) \) and \( 2MB_{Q_2} \subset 2MB_{Q_1} \). Iterating this process, we will get a sequence of cubes \( Q \equiv Q_0, Q_1, \ldots Q_m \) such that \( \ell(Q_j) \leq t^{-1} \ell(Q_{j-1}) \) and \( 2MB_{Q_j} \subset 2MB_{Q_{j-1}} \), for \( j = 1, \ldots, m \).

If the process does not terminate, then \( \bigcap_{j=0}^{\infty} 2MB_{Q_j} \) is nonempty. By definition of \( \tilde{G}_R \), we have \( \bigcap_{j=0}^{\infty} 2MB_{Q_j} \subset \tilde{G}_R \), which contradicts our assumption that \( 2MB_Q \cap \tilde{G}_R = \emptyset \). Hence, this process terminates at some \( Q_m \) (i.e., \( Q_m \in \widetilde{\text{Sep}}(R) \)). We take \( P = Q_m \), and obtain \( P \subset 2MB_P \subset 2MB_Q \). \( \square \)

**Lemma 6.7.** Assume that \( t \) is chosen big enough (depending on \( M \), but not on \( A, \tau, \text{ or } \varepsilon \)). Then:

(a) For all \( Q, Q' \in \widetilde{\text{Sep}}(R) \), we have

\[
(6.7) \quad Q \cap K^{1/2}_Q = Q' \cap K^{1/2}_Q = \emptyset.
\]
(b) For all \( x \in \tilde{G}_R \) and for all \( Q \in \tilde{\text{Sep}}(R) \), we have

\[
x \notin K_{x_2}^{1/2} \text{ and } Q \cap K_{x_2}^{1/2} = \emptyset.
\]

**Proof of (a).** Suppose that (6.7) fails. Observe that this implies that both \( Q \cap K_{Q_2}^{1/2} \) and \( Q' \cap K_{Q_2}^{1/2} \) are nonempty. Suppose that \( \ell(Q) \leq \ell(Q') \). Since \( Q \) and \( Q' \) are not \( t \)-neighbors, then we must have \( \ell(Q') \leq t^{-1} \ell(Q) \).

We claim that \( Q' \subset MB_Q \). Suppose not. Then \( Q' \cap (K_{Q_2}^{1/2} \setminus MB_Q) \neq \emptyset \), and \( \ell(Q') \leq t^{-1} \ell(Q) \leq \frac{1}{t} \text{dist}(Q, Q') \). Hence it follows that \( Q' \notin \text{Tree}(R) \), a contradiction. This shows that \( Q' \subset MB_Q \). But that contradicts Lemma 6.5.

**Proof of (b).** Suppose for contradiction that \( x \in K_{x_2}^{1/2} \). We claim that \( x \in MB_Q \). Suppose not, so that \( x \in K_{Q_2}^{1/2} \setminus MB_Q \). Let \( Q' \in \text{Tree}(R) \) be such that \( x \in 2MB_{Q'} \) with \( \ell(Q') \) so small that \( \ell(Q') \leq \frac{1}{t} \text{dist}(Q', Q) \). By this inequality and the fact that \( Q' \cap (K_{Q_2}^{1/2} \setminus MB_Q) \neq \emptyset \), it follows from Lemma 6.3(a) that \( Q' \notin \text{Tree}(R) \), which is a contradiction. Hence \( x \in MB_Q \), so \( 2MB_Q \cap \tilde{G}_R \neq \emptyset \). But this contradicts \( Q \in \tilde{\text{Sep}}(R) \).

**Lemma 6.8.** Let \( \Lambda_0 > 0 \) be big enough, depending on \( M \) and \( t \). There is a Lipschitz graph \( \Gamma_R \) with slope depending only on \( s \) such that \( \tilde{G}_R \subset \Gamma_R \) and \( \Lambda_0 B_Q \cap \Gamma_R \neq \emptyset \) for every \( Q \in \text{Tree}(R) \).

**Proof.** For each \( Q \in \tilde{\text{Sep}}(R) \), pick a point \( z_Q \in Q \). Let \( F = \{z_Q : Q \in \tilde{\text{Sep}}(R)\} \cup \tilde{G}_R \). By Lemma 6.4 and Lemma 6.7 it follows that

\[
x - y \notin K_{x_2}^{1/2} \text{ for all } x, y \in F.
\]

Hence there is a Lipschitz graph \( \Gamma_R \) containing \( F \).

By Lemma 6.6 any cube \( P \in \tilde{\text{Stop}}(R) \) is \( t \)-neighbor of some cube \( P' \in \tilde{\text{Sep}}(R) \) and \( 2MB_{P'} \cap F \neq \emptyset \). So if \( Q \in \text{Tree}(R) \), then either \( Q \) intersects \( \tilde{G}_R \) or \( Q \) contains some cube \( P \in \tilde{\text{Stop}}(R) \) and thus

\[
\text{dist}(Q, \Gamma_R) \leq \text{dist}(P, \Gamma_R) \leq C(t) \ell(P) \leq C(t) \ell(Q).
\]

which implies \( \Lambda_0 B_Q \cap \Gamma_R \neq \emptyset \).

**6.5. The small measure of the low density set.** The goal in this section is to estimate the total measure of the cubes in the low density set.

**Lemma 6.9.** We have

\[
\sum_{Q \in \text{LD}(R)} \mu(Q) \leq c(\tau + C(A, \tau, M) \varepsilon) \mu(R).
\]

To prove Lemma 6.9 we will construct an auxiliary \( n \)-dimensional Lipschitz graph, by arguments quite similar to the ones for \( \Gamma_R \). We denote by \( \text{Stop}(R) \) the subfamily of cubes \( Q \in \text{Stop}(R) \) such that

\[
Q \notin \bigcup_{P \in \text{Stop}(R)} K_{P_2}^{1/2} \cap (R \setminus MB_P),
\]
so that, by Lemma 6.3(b),

\[
\mu \left( \bigcup_{Q \in \text{Stop}(R) \setminus \widetilde{\text{Stop}}(R)} Q \right) \leq C(A, \tau, M) \varepsilon \mu(R).
\]

We claim that we can choose a subfamily \( \text{LD}_{\text{Sep}}(R) \subset \text{LD}(R) \cap \widetilde{\text{Stop}}(R) \) which is \( t \)-separated and such that

\[
\sum_{Q \in \text{LD}(R) \cap \widetilde{\text{Stop}}(R)} \mu(Q) \leq C(t) \sum_{Q \in \text{LD}_{\text{Sep}}(R)} \mu(Q).
\]

To this end we argue as follows: let \( \text{LD}_1(R) \) be a maximal \( t \)-separated subfamily of \( \text{LD}(R) \cap \text{Stop}(R) \). Let \( \text{LD}_2(R) \) be a maximal \( t \)-separated subfamily of \( \text{LD}(R) \cap \widetilde{\text{Stop}}(R) \setminus \text{LD}_1(R) \). By induction, let \( \text{LD}_j(R) \) be a maximal \( t \)-separated subfamily of \( \text{LD}(R) \cap \widetilde{\text{Stop}}(R) \setminus \left( \text{LD}_1(R) \cup \ldots \cup \text{LD}_{j-1}(R) \right) \). It turns out that there is bounded number \( N_0 \) of non-empty families \( \text{LD}_j(R) \), with \( N_0 \) depending on \( t \). Indeed, if \( Q \in \text{LD}_j(R) \), then \( Q \) is a \( t \)-neighbor of some cubes \( Q_1 \in \text{LD}_1(R) \), \( Q_2 \in \text{LD}_2(R) \), \ldots, \( Q_{j-1} \in \text{LD}_{j-1}(R) \), by the maximality of \( \text{LD}_k(R) \) for \( k = 1, \ldots, j-1 \). Since the number of \( t \)-neighbors of any cube has some bound depending on \( t \), we get \( j \leq N_0 \). Now we just let \( \text{LD}_{\text{Sep}}(R) \) be the family \( \text{LD}_j(R) \) for which \( \sum_{Q \in \text{LD}_j(R)} \mu(Q) \) is maximal, and then we have

\[
\sum_{Q \in \text{LD}(R) \cap \widetilde{\text{Stop}}(R)} \mu(Q) \leq N_0 \sum_{Q \in \text{LD}_{\text{Sep}}(R)} \mu(Q),
\]

which proves our claim.

Next, we modify the family \( \text{LD}_{\text{Sep}}(R) \) as follows: if there are two cubes \( Q, Q' \in \text{LD}_{\text{Sep}}(R) \) such that

\[
1.1B_Q \cap 1.1B_{Q'} \neq \emptyset \quad \text{and} \quad \ell(Q) < \ell(Q'),
\]

then we eliminate \( Q' \). We denote by \( \widetilde{\text{LD}}(R) \) the resulting family after eliminating all cubes \( Q' \) of this type in \( \text{LD}_{\text{Sep}}(R) \). We have the following variant of Lemma 6.6(b).

**Lemma 6.10.** For each \( Q \in \text{LD}_{\text{Sep}}(R) \), at least one of the following is true:

- \( 1.2B_Q \cap \widetilde{G}_R \neq \emptyset \)
- There exists some \( P \in \text{LD}(R) \) such that \( P \subset 1.2B_Q \).

**Proof.** If \( Q, Q' \in \text{LD}_{\text{Sep}}(R) \) satisfy (6.11), then (6.6) holds, and thus (6.5) must fail. Therefore, \( Q \) must be much smaller than \( Q' \), and so

\[
\ell(Q) \leq t^{-1}\ell(Q') \quad \text{and} \quad 1.2B_Q \subset 1.2B_{Q'} \quad \text{if} \quad \ell(Q) < \ell(Q'),
\]

assuming \( t \) big enough to guarantee the last inclusion. Now we can copy the proof of Lemma 6.6(b) with \( 2M \) replaced everywhere by \( 1.2 \). \( \square \)

For each \( Q \in \text{LD}(R) \) we choose a point

\[
w_Q \in Q \setminus \bigcup_{P \in \text{Stop}(R)} K_{P}^{1/2} \cap (R \setminus MB_P).
\]

**Lemma 6.11.** There exists an \( n \)-dimensional Lipschitz graph \( \Gamma_0 \) which passes through every point \( w_P \), \( P \in \text{LD}(R) \).
Proof. It is enough to show that for any pair $w_Q, w_{Q'}$, with $Q, Q' \in \overline{\dom}(R)$, $Q \neq Q'$, we have
\begin{equation}
    w_Q - w_{Q'} \notin K^{1/2}.
\end{equation}
To show this, suppose that $\ell(Q) \leq \ell(Q')$. By the construction of the points $w_P$, $P \in \overline{\dom}(R)$, it follows that
\begin{equation}
    w_{Q'} \notin K^{1/2} \cap (R \setminus M B_Q) \quad \text{and} \quad w_Q \notin K^{1/2} \setminus (R \setminus M B_{Q'})
\end{equation}
which implies that
\begin{equation}
    w_{Q'} \notin K^{1/2}_{w_Q} \setminus B(w_Q, c_1 M \ell(Q)) \quad \text{and} \quad w_Q \notin K^{1/2}_{w_{Q'}} \setminus B(w_{Q'}, c_1 M \ell(Q'))
\end{equation}
for some $c_1 \approx 1$.

So to conclude the proof of (6.12) it suffices to show that
\begin{equation}
    w_{Q'} \notin B(w_Q, c_1 M \ell(Q)).
\end{equation}
To this end, notice that if $w_{Q'} \in B(w_Q, c_1 M \ell(Q))$, then
\begin{equation*}
    \text{dist}(Q, Q') \leq |w_Q - w_{Q'}| \leq c_1 M \ell(Q) \leq t(\ell(Q) + \ell(Q')),
\end{equation*}
assuming $t = t(M)$ big enough. Since $Q$ and $Q'$ are not $t$-neighbors, we must have
\begin{equation*}
    \ell(Q) \leq t^{-1} \ell(Q').
\end{equation*}
Together with the fact that $1.1 B_Q \cap 1.1 B_{Q'} = \emptyset$, and recalling that $w_Q \in Q \subset B_Q$ and $w_{Q'} \in Q' \subset B_{Q'}$, this implies that
\begin{equation*}
    |w_Q - w_{Q'}| \geq \frac{1}{10} r(B_{Q'}) \geq \frac{c t}{10} r(B_Q) > c_1 M \ell(Q)
\end{equation*}
if $t(M)$ is big enough again. So (6.13) holds, and the lemma follows. \hfill \square

Proof of Lemma 6.7. Consider the family of balls $\{1.5 B_Q\}_{Q \in \overline{\dom}_{\text{sep}}(R)}$. By the covering Theorem 9.31 from [To4], there exists a subfamily $\mathcal{F} \subset \overline{\dom}_{\text{sep}}(R)$ such that:
\begin{enumerate}
    \item[(i)] $\bigcup_{Q \in \overline{\dom}_{\text{sep}}(R)} 1.5 B_Q \subset \bigcup_{Q \in \mathcal{F}} 2 B_Q$,
    \item[(ii)] $\sum_{Q \in \mathcal{F}} \chi_{1.5 B_Q} \leq C$.
\end{enumerate}
Then
\begin{equation*}
    \sum_{Q \in \overline{\dom}_{\text{sep}}(R)} \mu(Q) \leq \sum_{Q \in \mathcal{F}} \mu(2 B_Q) \leq c \tau \Theta_{\mu}(2 B_R) \sum_{Q \in \mathcal{F}} r(B_Q)^n.
\end{equation*}
Recall now that for each $B_Q$, with $Q \in \mathcal{F} \subset \overline{\dom}_{\text{sep}}(R)$, there exists some point $w_P \in \Gamma_0 \cap 1.2 B_Q$ or some point $x \in \check{G}_R \cap 1.2 B_Q \subset \Gamma_R \cap 1.2 B_Q$. Then we have
\begin{equation*}
    \mathcal{H}^n(1.5 B_Q \cap (\Gamma_0 \cup \Gamma_R)) \approx r(B_Q)^n.
\end{equation*}
So using the property (ii) of the covering, we obtain
\begin{equation*}
    \sum_{Q \in \mathcal{F}} r(B_Q)^n \lesssim \sum_{Q \in \mathcal{F}} \mathcal{H}^n(1.5 B_Q \cap (\Gamma_0 \cup \Gamma_R)) \leq C \mathcal{H}^n(2 B_R \cap (\Gamma_0 \cup \Gamma_R)) \leq C' \ell(R)^n.
\end{equation*}
Thus,
\begin{equation*}
    \sum_{Q \in \overline{\dom}_{\text{sep}}(R)} \mu(Q) \leq c \tau \Theta_{\mu}(2 B_R) \ell(R)^n \leq c \tau \mu(R).
\end{equation*}
Together with (6.10), this yields

\[ \sum_{Q \in \text{LD}(R) \setminus \text{Stop}(R)} \mu(Q) \leq C \tau \mu(R), \]

with \( C \) depending on \( t \). Finally to conclude the lemma, we just take into account that, by (6.9), we have

\[ \sum_{Q \in \text{LD}(R) \setminus \text{Stop}(R)} \mu(Q) \leq C(A, \tau, M) \varepsilon \mu(R). \]

\[ \square \]

6.6. The approximate Lipschitz graph. In the next lemma we gather some of the previous results and estimates.

**Lemma 6.12.** Let \( R \in \mathcal{D}_\mu^{db} \), and suppose that \( \tau, \eta, \varepsilon \) are small enough and \( \varepsilon \ll \tau \). Then there exists an \( n \)-dimensional Lipschitz graph \( \Gamma_R \) whose slope is bounded above by some constant depending only on \( s \) such that the following holds:

- (a) \( R \setminus \bigcup_{Q \in \text{Stop}(R)} Q \subset \Gamma_R \).
- (b) There exists some constant \( \Lambda_0 > 1 \) such that for all \( Q \in \text{Tree}(R) \),
  \[ \Lambda_0 B_Q \cap \Gamma_R \neq \emptyset. \]
- (c) We have:
  \[ (6.14) \quad \sum_{Q \in \text{LD}(R)} \mu(Q) \leq \tau^{1/2} \mu(R), \]

  and also
  \[ (6.15) \quad \sum_{Q \in \text{BRE}(R)} \mu(Q) \leq \frac{1}{\varepsilon \Theta_\mu(2B_R)} \sum_{S \in \text{Tree}(R)} \mathcal{E}_\mu(S) \mu(S). \]

**Proof.** The statement (a) follows from Lemma 6.4 and (b) from Lemma 6.8. On the other hand, the estimate (6.14) follows from the analogous one proved in Lemma 6.9 choosing \( \varepsilon \) and \( \tau \) suitably small. Finally, concerning (6.15), recall that if \( Q \in \text{BRE}(R) \), then

\[ \sum_{S \in \mathcal{D}_\mu : Q \subset S \subset R} \mathcal{E}_\mu(S) \geq \varepsilon \Theta_\mu(2B_R). \]

Therefore,

\[ \Theta_\mu(2B_R) \sum_{Q \in \text{BRE}(R)} \mu(Q) \leq \frac{1}{\varepsilon} \sum_{Q \in \text{BRE}(R)} \mu(Q) \sum_{S \in \mathcal{D}_\mu : Q \subset S \subset R} \mathcal{E}_\mu(S) \]

\[ = \frac{1}{\varepsilon} \sum_{S \in \text{Tree}(R)} \mathcal{E}_\mu(S) \sum_{Q \in \text{BRE}(R) : Q \subset S} \mu(Q) \]

\[ \leq \frac{1}{\varepsilon} \sum_{S \in \text{Tree}(R)} \mathcal{E}_\mu(S) \mu(S). \]

\[ \square \]
7. THE FAMILY OF Top CUBES

7.1. The family Top. We are going to construct a family of cubes \( \text{Top} \subset \mathcal{D}_{\mu}^{db} \) inductively. To this end, we need to introduce some additional notation. Given a cube \( Q \in \mathcal{D}_{\mu} \), we denote by \( \mathcal{M}D(Q) \) the family of maximal cubes (with respect to inclusion) from \( \mathcal{D}_{\mu}^{db}(Q) \setminus \{Q\} \). Recall that, by Lemma 4.2, this family covers \( \mu \)-almost all of \( Q \). Moreover, by Lemma 4.4 it follows that if \( P \in \mathcal{M}D(Q) \), then \( \Theta_{\mu}(2B_P) \leq c \Theta_{\mu}(2B_Q) \). Given \( R \in \mathcal{D}_{\mu}^{db} \), we denote

\[
\text{Next}(R) = \bigcup_{Q \in \text{Stop}(R)} \mathcal{M}D(Q).
\]

By the construction above, it is clear that the cubes in \( \text{Next}(R) \) are different from \( R \).

For the record, notice that if \( P \in \text{Next}(R) \), then

\[
(7.1) \quad \Theta_{\mu}(2B_S) \leq c(A) \Theta_{\mu}(2B_R) \quad \text{for all} \quad S \in \mathcal{D}_{\mu} \text{ such that} \quad P \subset S \subset R.
\]

We are now ready to construct the aforementioned family \( \text{Top} \). We will have \( \text{Top} = \bigcup_{k \geq 0} \text{Top}_k \). First we set

\[
\text{Top}_0 = \{R_0\}.
\]

(Recall that \( R_0 \equiv \text{supp} \mu \).) Assuming \( \text{Top}_k \) has been defined, we set

\[
\text{Top}_{k+1} = \bigcup_{R \in \text{Top}_k} \text{Next}(R).
\]

Note that the families \( \text{Next}(R) \), with \( R \in \text{Top}_k \), are pairwise disjoint.

7.2. The family of cubes ID. We distinguish a special type of cubes from \( \text{Top} \). For \( R \in \text{Top} \), we write \( R \in \text{ID}(\text{increasing density}) \) if

\[
\mu \left( \bigcup_{Q \in \text{HD}(R)} Q \right) \geq \frac{1}{2} \mu(R).
\]

Lemma 7.1. Suppose that \( A \) is big enough. If \( R \in \text{ID} \), then

\[
(7.2) \quad \Theta_{\mu}(2B_R) \mu(R) \leq \frac{1}{2} \sum_{Q \in \text{Next}(R)} \Theta_{\mu}(2B_Q) \mu(Q).
\]

Proof. Recalling that \( \Theta_{\mu}(2B_Q) \geq A \Theta_{\mu}(2B_R) \) for every \( Q \in \text{HD}(R) \), we deduce that

\[
\Theta_{\mu}(2B_R) \mu(R) \leq 2 \sum_{Q \in \text{HD}(R)} \Theta_{\mu}(2B_Q) \mu(Q) \leq 2A^{-1} \sum_{Q \in \text{HD}(R)} \Theta_{\mu}(2B_Q) \mu(Q).
\]

Since the cubes from \( \text{HD}(R) \) belong to \( \mathcal{D}_{\mu}^{db} \), it follows immediately from Lemma 4.3 that for any \( Q \in \text{HD}(R) \),

\[
\Theta_{\mu}(2B_Q) \mu(Q) \lesssim \sum_{P \in \text{Next}(R): P \subset Q} \Theta_{\mu}(2B_P) \mu(P),
\]

and then it is clear that (7.2) holds if \( A \) is taken big enough. \( \square \)
8. The Packing Condition

Lemma 8.1. Suppose that

\[ (8.1) \quad \mu(B(x, r)) \leq c_0 r^n \quad \text{for all } x \in \text{supp} \mu, \ r > 0. \]

For all \( S \in \mathcal{D}_\mu \) we have

\[ (8.2) \quad \sum_{R \in \text{Top}_{k+1}(S), \ R \in S} \Theta_\mu(2B_R) \mu(R) \leq \frac{1}{2} \sum_{Q \in \text{Next}(R)} \Theta_\mu(2B_Q) \mu(Q) \]

assuming that the constants \( A, \tau, \varepsilon, \) and \( \eta \) have been chosen suitably.

Proof. Observe first that it is enough to prove the lemma assuming that \( S \in \text{Top} \). So we assume \( S \in \text{Top} \), and we denote \( \text{Top}(S) = \text{Top} \cap \mathcal{D}_\mu(S) \) and \( \text{Top}_j(S) = \text{Top}_j \cap \mathcal{D}_\mu(S) \). For a given \( k \geq 0 \), we also write

\[ \text{Top}_0^k(S) = \bigcup_{0 \leq j \leq k} \text{Top}_j(S), \]

and also \( \mathcal{I}_D^k = \mathcal{I} \cap \text{Top}_0^k(S) \).

To prove (8.2), first we deal with the cubes from the \( \mathcal{I} \) family. By Lemma 7.1 for every \( R \in \mathcal{I} \) we have

\[ \Theta_\mu(2B_R) \mu(R) \leq \frac{1}{2} \sum_{Q \in \text{Next}(R)} \Theta_\mu(2B_Q) \mu(Q) \]

and hence we obtain

\[ \sum_{R \in \mathcal{I}_D^k} \Theta_\mu(2B_R) \mu(R) \leq \frac{1}{2} \sum_{R \in \mathcal{I}_D^k} \sum_{Q \in \text{Next}(R)} \Theta_\mu(2B_Q) \mu(Q) \leq \frac{1}{2} \sum_{Q \in \text{Top}^k(S)} \Theta_\mu(2B_Q) \mu(Q), \]

because the cubes from \( \text{Next}(R) \) with \( R \in \text{Top}_0^k(S) \) belong to \( \text{Top}^k(S) \). Thus,

\[ \sum_{R \in \text{Top}_0^k(S)} \Theta_\mu(2B_R) \mu(R) = \sum_{R \in \text{Top}_0^k(S) \setminus \mathcal{I}_D^k} \Theta_\mu(2B_R) \mu(R) + \sum_{R \in \mathcal{I}_D^k} \Theta_\mu(2B_R) \mu(R) \]

\[ \leq \sum_{R \in \text{Top}_0^k(S) \setminus \mathcal{I}_D^k} \Theta_\mu(2B_R) \mu(R) + \frac{1}{2} \sum_{R \in \text{Top}_0^k(S)} \Theta_\mu(2B_R) \mu(R) + C c_0 \mu(S), \]

where, for the last inequality, we took into account that \( \Theta_\mu(2B_R) \leq C c_0 \) for every \( R \in \text{Top}_{k+1}(S) \) because of the assumption (8.1). Using that

\[ \sum_{R \in \text{Top}_0^k(S)} \Theta_\mu(2B_R) \mu(R) \leq (k + 1) c_0 \mu(S) < \infty, \]

we deduce

\[ \sum_{R \in \text{Top}_0^k(S)} \Theta_\mu(2B_R) \mu(R) \leq 2 \sum_{R \in \text{Top}_0^k(S) \setminus \mathcal{I}_D^k} \Theta_\mu(2B_R) \mu(R) + C c_0 \mu(S). \]

Letting \( k \to \infty \), we derive

\[ (8.3) \quad \sum_{R \in \text{Top}(S)} \Theta_\mu(2B_R) \mu(R) \leq 2 \sum_{R \in \text{Top}(S) \setminus \mathcal{I}} \Theta_\mu(2B_R) \mu(R) + C c_0 \mu(S). \]
To estimate the first term on the right hand side of (8.3) we use the fact that, for \( R \in \text{Top}(S) \setminus ID \), we have
\[
\mu\left( R \setminus \bigcup_{Q \in \text{HD}(R)} Q \right) \geq \frac{1}{2} \mu(R),
\]
and then we get
\[
\mu(R) \leq 2 \mu\left( R \setminus \bigcup_{Q \in \text{Stop}(R)} Q \right) + 2 \mu\left( \bigcup_{Q \in \text{HD}(R)} Q \right)
= 2 \mu\left( R \setminus \bigcup_{Q \in \text{Next}(R)} Q \right) + 2 \sum_{Q \in \text{LD}(R)} \mu(Q) + 2 \sum_{Q \in \text{BRE}(R)} \mu(Q).
\]
(8.4)

Recall now that, by (6.14),
\[
\sum_{Q \in \text{LD}(R)} \mu(Q) \leq \frac{\tau}{2} \mu(R).
\]
Choosing \( \tau \leq 1/16 \), say, from (8.4) we infer that
\[
\mu(R) \leq 4 \mu\left( R \setminus \bigcup_{Q \in \text{Next}(R)} Q \right) + 4 \sum_{Q \in \text{BRE}(R)} \mu(Q).
\]

So we deduce that
\[
\sum_{R \in \text{Top}(S) \setminus ID} \Theta_{\mu}(2B_R) \mu(R) \leq 4 \sum_{R \in \text{Top}(S)} \Theta_{\mu}(2B_R) \mu\left( R \setminus \bigcup_{Q \in \text{Next}(R)} Q \right) + 4 \sum_{R \in \text{Top}(S)} \Theta_{\mu}(2B_R) \sum_{Q \in \text{BRE}(R)} \mu(Q).
\]
(8.5)

To deal with the first sum on the right hand side above, we take into account that the sets \( R \setminus \bigcup_{Q \in \text{Next}(R)} Q \), with \( R \in \text{Top}(S) \), are pairwise disjoint, and also that \( \Theta_{\mu}(2B_R) \leq C_0 \), by the condition (8.1). Then we get
\[
\sum_{R \in \text{Top}(S)} \Theta_{\mu}(2B_R) \mu\left( R \setminus \bigcup_{Q \in \text{Next}(R)} Q \right) \leq C_0 \mu(S).
\]
(8.6)

To deal with the second sum in (8.5), we use (6.15) to obtain
\[
\sum_{R \in \text{Top}(S)} \sum_{Q \in \text{BRE}(R)} \mu(Q) \leq \frac{1}{\varepsilon} \sum_{R \in \text{Top}(S)} \sum_{P \in \text{Tree}(R)} \sum_{E \in \mathcal{E}_\mu(P)} \mu(P) \leq \frac{1}{\varepsilon} \sum_{P \in \mathcal{D}_{\mu,k}(S)} \mathcal{E}_\mu(P) \mu(P).
\]

Denote by \( \ell_k \) the side length of the cubes from \( \mathcal{D}_{\mu,k} \). By the definition of \( \mathcal{E}_\mu(P) \) (see (6.1)) and the finite overlapping of the balls \( 2B_P \) among the cubes \( P \) of the same generation, we
get

\[
\sum_{P \in \mathcal{D}_\mu(S)} \mathcal{E}_\mu(P) \mu(P) = \sum_k \sum_{P \in \mathcal{D}_{\mu,k}(S)} \int_{x-y \in \mathbb{R}^n} \frac{1}{|x-y|^n} \, d\mu(x) \, d\mu(y) \\
\lesssim \sum_k \int_{x-y \in \mathbb{R}^n} \frac{1}{|x-y|^n} \, d\mu(x) \, d\mu(y) \\
\lesssim \eta \int_{x-y \in \mathbb{R}^n} \frac{1}{|x-y|^n} \, d\mu(x) \, d\mu(y).
\]

Therefore,

\[
\sum_{R \in \text{Top}(S)} \Theta_\mu(2B_R) \sum_{Q \in \text{BRE}(R)} \mu(Q) \lesssim \eta \int_{x-y \in \mathbb{R}^n} \frac{1}{|x-y|^n} \, d\mu(x) \, d\mu(y).
\]

Together with (8.3), (8.5), and (8.6), this yields (8.2).

9. PROOF OF LEMMA 5.1

We have to show that the family \text{Top} satisfies the properties stated in Lemma 5.1. By the definition of the family \text{Next}(R), we have

\[
\bigcup_{Q \in \text{Stop}(R)} Q = \bigcup_{Q \in \text{Next}(R)} Q \quad \text{up to a set of } \mu\text{-measure 0.}
\]

Thus, by Lemma 6.12(a), \( \mu\)–almost all of \( R \setminus \bigcup_{Q \in \text{Next}(R)} Q \) is contained in \( \Gamma_R \), and we have verified property (a) of Lemma 5.1.

Next we deal with the property (b). Given \( Q \in \text{Next}(R) \) we have to check that there exists some \( \bar{Q} \in \mathcal{D}_\mu \) such that \( \delta_\mu(Q, \bar{Q}) \leq \varepsilon \Theta_\mu(2B_R) \) and \( 2B_{\bar{Q}} \cap \Gamma_R \neq \emptyset \). Let \( Q' \in \text{Stop}(R) \) such that \( Q \subset Q' \). By Lemma 6.12(b), there exists some constant \( \Lambda_0 > 1 \) such that \( \Lambda_0 B_{Q'} \cap \Gamma_R \neq \emptyset \). This implies that there exists one cube \( \bar{Q} \in \text{Tree}(R) \) such that \( Q' \subset \bar{Q} \), \( \ell(\bar{Q}) \approx \ell(Q') \), and \( 2B_{\bar{Q}} \cap \Gamma_R \neq \emptyset \). We split

\[
\delta_\mu(Q, \bar{Q}) = \int_{2B_{\bar{Q}} \setminus 2B_{Q'}} \frac{1}{|y-x|^n} \, d\mu(y) + \int_{2B_{Q'} \setminus 2B_{\bar{Q}}} \frac{1}{|y-x|^n} \, d\mu(y).
\]

To estimate the first integral we use the fact that \( |y-x| \approx \ell(Q') \approx \ell(\bar{Q}) \) in the domain of integration, and we derive

\[
\int_{2B_{\bar{Q}} \setminus 2B_{Q'}} \frac{1}{|y-x|^n} \, d\mu(y) \lesssim \Theta_\mu(2B_{\bar{Q}}) \lesssim A \Theta_\mu(2B_R).
\]

To estimate the second one we take into account that there are no doubling cubes strictly between \( Q \) and \( Q' \). Then from Lemma 4.4 and standard estimates, it easily follows that

\[
\int_{2B_{Q'} \setminus 2B_{\bar{Q}}} \frac{1}{|y-x|^n} \, d\mu(y) \lesssim \Theta_\mu(100B(Q')).
\]
If $Q'$ is doubling, then $\Theta_\mu(100B(Q')) \lesssim \Theta_\mu(2BQ') \lesssim A \Theta_\mu(2BR)$. Otherwise $Q' \neq R$ and the parent of $Q'$, which we denote by $\overline{Q}'$, belongs to $\text{Tree}(R) \setminus \text{Stop}(R)$. Thus, by Lemma 6.1

$$\Theta_\mu(100B(Q')) \lesssim \Theta_\mu(2B\overline{Q}') \lesssim A \Theta_\mu(2BR),$$

taking into account that $100B(Q') \subset 2B\overline{Q}'$ for the first inequality (since $A_0 \gg 1$ in the David-Mattila lattice). Hence in any case, $\delta_\mu(Q,\overline{Q}) \lesssim_A \Theta_\mu(2BR)$ and (b) in Lemma 5.1 holds.

Next, we observe that (c) in Lemma 5.1 follows from Lemma 6.1 in case that $Q \in \text{Tree}(R)$, and from (7.1) otherwise.

Finally, the packing condition (7.2) has been proved in Lemma 8.1.

10. Application to curvature, Riesz transforms, and capacities

10.1. Curvature of measures and Riesz transforms. To estimate the curvature of $\mu$ we will use the following result:

Lemma 10.1. Let $\mu$ be a measure satisfying the growth condition (5.1). Suppose that there exists a family $\text{Top} \subset \mathcal{D}^d_\mu$ as in Section 5 such that, for all $R \in \text{Top}$, there exists an $n$-dimensional Lipschitz graph $\Gamma_R$ whose slope is uniformly bounded above by some constant independent of $R$ such that:

(a) $\mu$-almost all $\text{Good}(R)$ is contained in $\Gamma_R$.

(b) For all $Q \in \text{Next}(R)$ there exists another cube $\tilde{Q} \in \mathcal{D}_\mu$ such that $\delta_\mu(Q,\tilde{Q}) \leq c \Theta_\mu(2BR)$ and $2B\tilde{Q} \cap \Gamma_R \neq \emptyset$.

(c) For all $Q \in \text{Tr}(R)$, $\Theta_\mu(2BQ) \leq c \Theta_\mu(2BR)$.

In the case $n = 1$, we have:

$$c^2(\mu) \lesssim \|\mu\| + \sum_{R \in \text{Top}} \Theta_\mu(2BR)^2 \mu(R),$$

and for any integer $n \in (0, d)$,

$$\sup_{\varepsilon > 0} \|\mathcal{R}_\varepsilon^2 \mu\|_{L^2(\mu)} \lesssim \|\mu\| + \sum_{R \in \text{Top}} \Theta_\mu(2BR)^2 \mu(R),$$

with the implicit constant depending only on $c_0$ in both estimates.

For $n = 1, d = 2$, a version of this result which uses the usual dyadic lattice of $\mathbb{R}^2$ instead of the David-Mattila lattice is proven in [102]. For arbitrary $n, d$, another version in terms of the David-Mattila lattice is shown in [34], which in fact is valid for other singular integral operators with odd kernel, besides the Cauchy and Riesz transforms.

By combining Lemmas 5.1 and 10.1 we obtain the following result.

Theorem 10.2. Given $V_0 \in G(d, d - n)$ and $s > 0$, consider the cone $K := K(V_0, s) \subset \mathbb{R}^d$. Let $\mu$ be a Borel measure in $\mathbb{R}^d$ satisfying the growth condition

$$\mu(B(x, r)) \leq c_0 r^n$$

for all $x \in \mathbb{R}^d$, $r > 0$.

In the case $n = 1$ we have

$$c^2(\mu) \lesssim \|\mu\| + \int_{x \in K} \int_{y \in K} \frac{1}{|x - y|} \, d\mu(x) \, d\mu(y),$$
and for any integer \( n \in (0, d) \),

\[
\sup_{\epsilon > 0} \| \mathcal{R}_\epsilon^n \mu \|_{L^2(\mu)}^2 \lesssim \| \mu \| + \int \int_{x-y \in K} \frac{1}{|x-y|^n} \, d\mu(x) \, d\mu(y).
\]

The implicit constants in both inequalities depend only on \( d, n, c_0, \) and \( s \).

**Proof.** From Lemmas 5.1 and 10.1 in the case \( n = 1 \) we deduce the following:

\[
c^2(\mu) \lesssim \| \mu \| + \sum_{R \in \text{Top}} \Theta_\mu(2B_R)^2 \mu(R) \lesssim \| \mu \| + \sum_{R \in \text{Top}} \Theta_\mu(2B_R) \mu(R)
\]

\[
\lesssim \| \mu \| + \int \int_{x-y \in K} \frac{1}{|x-y|^2} \, d\mu(x) \, d\mu(y),
\]

and analogously, for any integer \( n \in (0, d) \), with \( \sup_{\epsilon > 0} \| \mathcal{R}_\epsilon^n \mu \|_{L^2(\mu)}^2 \) instead of \( c^2(\mu) \). \( \square \)

10.2. **Analytic capacity.** To prove Theorem 1.1, recall that by Theorem 2.1 for any compact set \( E \subset \mathbb{C} \) we have

\[
\gamma(E) \approx \sup \{ \sigma(E) : \sigma \in L_1(E), c^2(\sigma) \leq \sigma(E) \},
\]

where \( L_1(E) \) stands for the set of positive Borel measures supported on \( E \) satisfying \( \sigma(B(x,r)) \leq r \) for all \( x \in E, \, r > 0 \).

Let \( \mu \) be a measure supported on \( E \) such that \( \int_I \| P_{\theta} \mu \|_2^2 \, d\theta < \infty \), and denote

\[
\lambda = \frac{1}{\mu(E)} \int_I \| P_{\theta} \mu \|_2^2 \, d\theta.
\]

We intend to construct a suitable measure \( \sigma \) with linear growth from \( \mu \), and then we will apply (10.3) to \( \sigma \).

Let \( \theta_0 \in I \) be such that

\[
\| P_{\theta_0} \mu \|_2^2 \leq \frac{1}{\mathcal{H}^1(I)} \int_I \| P_{\theta} \mu \|_2^2 \, d\theta = \frac{\lambda \mu(E)}{\mathcal{H}^1(I)}.
\]

Denote \( \eta = P_{\theta_0} \mu \) and let \( L_{\theta_0} = \{ r e^{i\theta_0} : r \in \mathbb{R} \} \). Observe that the preceding estimate is equivalent to

\[
\int_{L_{\theta_0}} \left| \frac{d\eta}{dr} \right|^2 \, dr = \int_{P_{\theta_0}(E)} \frac{d\eta}{dr} \, d\eta(r) \leq \frac{\lambda \eta(P_{\theta_0}(E))}{\mathcal{H}^1(I)}.
\]

So by Chebyshev’s inequality we have

\[
\eta \left\{ r \in L_{\theta_0} : \frac{d\eta}{dr}(r) > \frac{2 \lambda}{\mathcal{H}^1(I)} \right\} \leq \frac{1}{2} \eta(P_{\theta_0}(E)).
\]

Hence there exists a compact set \( F_0 \) contained in

\[
\left\{ r \in L_{\theta_0} : \frac{d\eta}{dr}(r) \leq \frac{2 \lambda}{\mathcal{H}^1(I)} \right\}
\]

such that \( \eta(F_0) \geq \frac{1}{4} \eta(P_{\theta_0}(E)) = \frac{1}{4} \mu(E) \). Clearly,

\[
\eta(F_0 \cap B(x,s)) \leq \frac{4 \lambda}{\mathcal{H}^1(I)} \, s \quad \text{for all} \quad x \in \mathbb{C}, \, s > 0.
\]
Next we consider the closed set $F = P_{\theta_0}^{-1}(F_0) \cap \text{supp } \mu$, and the measure 
\[
\sigma = \frac{\mathcal{H}^1(I)}{4 \lambda} \mu|_F.
\]
Note that 
\[
(10.4) \quad \sigma(F) = \frac{\mathcal{H}^1(I)}{4 \lambda} \mu(F) = \frac{\mathcal{H}^1(I)}{4 \lambda} \eta(F_0) \geq \frac{\mathcal{H}^1(I)}{16 \lambda} \mu(E).
\]
Further, for any $x \in \text{supp } \sigma$ and $s > 0$, 
\[
\sigma(B(x,s)) \leq \sigma(P_{\theta_0}^{-1}(P_{\theta_0}(B(x,s)))) = \frac{\mathcal{H}^1(I)}{4 \lambda} \eta(P_{\theta_0}(F \cap B(x,s))) \leq s,
\]
and so $\sigma$ has linear growth with constant 1. Also, by the definition of $\lambda$ and (10.4), 
\[
\int \|P_{\theta_0} \sigma\|^2 d\theta = \left(\frac{\mathcal{H}^1(I)}{4 \lambda}\right)^2 \int \|P_{\theta_0} \mu|_F\|^2 d\theta \leq \left(\frac{\mathcal{H}^1(I)}{4 \lambda}\right)^2 \int \|P_{\theta_0} \mu\|^2 d\theta = \frac{\mathcal{H}^1(I)^2}{16 \lambda} \mu(E) \leq \mathcal{H}^1(I) \sigma(F).
\]
Hence, by Theorem 10.2 and Corollary 3.3, we deduce that 
\[
c^2(\sigma) \lesssim \sigma(F) + \int \int_{x-y \in K_{\perp}} \frac{1}{|x-y|} d\sigma(x) d\sigma(y) \lesssim \sigma(F) + \int \|P_{\theta_0} \sigma\|^2 d\theta \leq C_1 \sigma(F),
\]
where the constant $C_1$ depends only on $\mathcal{H}^1(I)$. Then, from (10.3) and (10.4), we deduce that 
\[
\gamma(E) \geq \gamma(F) \gtrsim \sigma(F) \gtrsim \frac{\mu(E)}{\lambda} = \frac{\mu(E)^2}{\int \|P_{\theta_0} \mu\|^2 d\theta},
\]
with the implicit constants depending on $\mathcal{H}^1(I)$. This concludes the proof of Theorem 1.1.

10.3. The capacities $\Gamma_{d,n}$. The proof of Theorem 1.2 is analogous to the one of Theorem 1.1. The only difference is that we have to replace the curvature $c^2(\mu)$ by $\sup_{\epsilon > 0} \|\mathcal{R}_\epsilon^a \mu\|^2_{L^2(\mu)}$, and use Theorem 2.2 and Corollary 3.11 instead of Theorem 2.1 and Corollary 3.3 respectively. We skip the details.

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