Quantum two-photon algebra from non-standard $U_z(sl(2, \mathbb{R}))$ and a discrete time Schrödinger equation

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Abstract

The non-standard quantum deformation of the (trivially) extended $sl(2, \mathbb{R})$ algebra is used to construct a new quantum deformation of the two-photon algebra $h_6$ and its associated quantum universal $R$-matrix. A deformed one-boson representation for this algebra is deduced and applied to construct a first order deformation of the differential equation that generates the two-photon algebra eigenstates in Quantum Optics. On the other hand, the isomorphism between $h_6$ and the (1+1) Schrödinger algebra leads to a new quantum deformation for the latter for which a differential-difference realization is presented. From it, a time discretization of the heat-Schrödinger equation is obtained and the quantum Schrödinger generators are shown to be symmetry operators.
1 Introduction

The two-photon Lie algebra $h_6$ is generated by the operators \{$N, A_+, A_-, B_+, B_-$, $M$\} endowed with the following commutation rules

\[
\begin{align*}
[N, A_+] &= A_+ & [N, A_-] &= -A_- & [A_-, A_+] &= M \\
[N, B_+] &= 2B_+ & [N, B_-] &= -2B_- & [B_-, B_+] &= 4N + 2M \\
[A_+, B_-] &= -2A_- & [A_+, B_+] &= 0 & [M, \cdot] &= 0 \\
[A_-, B_+] &= 2A_+ & [A_-, B_-] &= 0.
\end{align*}
\]

(1.1)

In this basis, two remarkable Lie subalgebras of $h_6$ can be easily detected: the harmonic oscillator algebra $h_4$ with generators \{$N, A_+, A_-, M$\}, and the (trivially) extended $\mathfrak{sl}(2, \mathbb{R})$ algebra generated by \{$N, B_+, B_-, M$\} and denoted as $\mathfrak{sl}(2, \mathbb{R})$ (the central extension $M$ is trivial for the latter since it can be absorbed by redefining $N \to N + M/2$).

The two-photon algebra can be used to generate a large zoo of squeezed and coherent states for (single mode) one and two-photon processes which have been recently analysed in [2] (in this respect, see also [1]). More explicitly, the following one-boson representation of $h_6$

\[
\begin{align*}
N &= a_+ a_- \\
M &= 1 \\
A_+ &= a_+ \\
B_+ &= a_+^2 \\
A_- &= a_- \\
B_- &= a_-^2
\end{align*}
\]

(1.2)

where

\[
[a_-, a_+] = 1,
\]

(1.3)

shows that one-photon processes are algebraically encoded within the subalgebra $h_4$ and that $\mathfrak{sl}(2, \mathbb{R})$ contains the information concerning two-photon dynamics. The realization (1.3) can be translated into a Fock–Bargmann representation where the $h_6$ generators act in the Hilbert space of entire analytic functions $f(\alpha)$ as linear differential operators:

\[
\begin{align*}
N &= \alpha \frac{d}{d\alpha} \\
M &= 1 \\
A_+ &= \alpha \\
B_+ &= \alpha^2 \\
A_- &= \frac{d}{d\alpha} \\
B_- &= \frac{d^2}{d\alpha^2}.
\end{align*}
\]

(1.4)

The two-photon algebra eigenstates are given by the analytic eigenfunctions that fulfill

\[
(\beta_1 N + \beta_2 B_- + \beta_3 B_+ + \beta_4 A_- + \beta_5 A_+) f(\alpha) = \lambda f(\alpha).
\]

(1.5)

In the Fock–Bargmann representation (1.4), the following differential equation is deduced from (1.5):

\[
\beta_2 \frac{d^2 f}{d\alpha^2} + (\beta_1 \alpha + \beta_4) \frac{df}{d\alpha} + (\beta_3 \alpha^2 + \beta_5 \alpha - \lambda) f = 0
\]

(1.6)

where $\beta_i$ are arbitrary complex coefficients and $\lambda$ is a complex eigenvalue. The solutions of this equation (provided a suitable normalization is imposed) give rise
to the two-photon coherent/squeezed states \[2\]. One and two-photon coherent and squeezed states corresponding to the subalgebras \( h_4 \) and \( sl(2, \mathbb{R}) \) can be derived from equation (1.6) by setting \( \beta_2 = \beta_3 = 0 \) and \( \beta_4 = \beta_5 = 0 \), respectively.

A novel (to our knowledge) and interesting feature of \( h_6 \) has been recently pointed out in \[4\]: if we define

\[
D = -N - \frac{1}{2} M \quad P = A_+ \quad K = A_- \quad H = \frac{1}{2} B_+ \quad C = \frac{1}{2} B_- ,
\]

(1.7)

and compute the commutation rules in this new basis we obtain

\[
\begin{align*}
[D, P] &= -P & [D, K] &= K & [K, P] &= M \\
[D, H] &= -2H & [D, C] &= 2C & [H, C] &= D \\
[K, H] &= P & [K, C] &= 0 & [M, \cdot] &= 0 \\
[P, C] &= -K & [P, H] &= 0 ,
\end{align*}
\]

(1.8)

which are just the defining relations of the (1+1) dimensional Schrödinger algebra \( S(1 + 1) \) \[5, 6\] generated by the time translation operator \( H \), space translation \( P \), Galilean boost \( K \), dilation \( D \), conformal transformation \( C \) and mass \( M \). Therefore, the isomorphism (1.7) relates in a simple way two physically different frameworks and enhances the role played by the underlying abstract Lie symmetry. The kinematical nature of \( S(1 + 1) \) can be explicitly put forward by recalling the usual differential realization of the Schrödinger generators in terms of the time and space coordinates \((t, x)\):

\[
\begin{align*}
H &= \partial_t \quad P = \partial_x \quad M = m \\
K &= -t \partial_x - mx \quad D = 2t \partial_t + x \partial_x - a \\
C &= t^2 \partial_t + tx \partial_x - at + mx^2 / 2
\end{align*}
\]

(1.9)

where \( a, m \) are the constants that label the representation. This algebra appears as the symmetry algebra for the (1+1) heat/Schrödinger equation (SE), that can be straightforwardly obtained from the representation (1.9) as the Casimir operator \( P^2 - 2 \, M \, H \) of the (1+1) Galilei subalgebra of \( S(1 + 1) \).

The direct connection of \( S(1 + 1) \) with the SE has recently motivated the search for \( q \)-deformations of this algebra connected with discretizations (in both space and time variables) of this equation on geometric lattices of the type \( y_n = q^n y_0 \). In \[7, 8\] \( q \)-deformations of the vector field realization (1.9) arise as symmetry algebras of different discretized versions of the SE. From a constructive point of view, another \( q \)-Schrödinger algebra has been directly introduced in \[9\] in order to get a generalized \( q \)-SE from its deformed representation theory. However, none of these \( q \)-algebras has been found to be consistent with a Hopf algebra structure \[10\] yet. On the other hand, a discretization of the SE on a regular space-time lattice \( y_n = y_0 + n \, z \) has been also studied from a symmetry approach in \[11\]. In this case, the resultant symmetry operators close a non-deformed Schrödinger algebra. This classical nature of the discrete symmetries on a regular grid has been also proven for the wave equation in \[12\].
The first Hopf algebra deformation of $h_6$ and the isomorphism (1.7) used to derive the corresponding quantum Schrödinger algebra relations have been recently introduced in [4]. This quantum algebra was obtained by starting from the (non-standard) quantum deformation of its one-photon subalgebra $h_4$ [13]. This paper is devoted to the investigation of a new quantum deformation of $h_6 \equiv S(1 + 1)$ coming from the (non-standard) quantum deformation of the other relevant subalgebra $\mathfrak{sl}(2, \mathbb{R})$ given in [14]. As a result, the corresponding deformed analogues of (1.4) and (1.9) are shown to provide two systematic applications of this deformed symmetry: among them, we shall insist in the construction of a differential-difference analogue of the SE with quantum algebra symmetry.

In the next section the explicit form of the quantum two-photon algebra is presented starting from the classical $r$-matrix $r = zN \wedge B_4$ that underlies the deformation. Some mathematical properties are stated: among them, the embedding $U_z(\mathfrak{sl}(2, \mathbb{R})) \subset U_z(h_6)$ and the proof that the universal $R$-matrix of $U_z(\mathfrak{sl}(2, \mathbb{R}))$ is also valid for $U_z(h_6)$. A deformed one-boson representation is afterwards presented and translated into a Fock–Bargmann realization. This representation suggests the construction of deformed states of light through a differential equation (whose classical version is (1.6)) for their corresponding quantum optical analytic functions. Since the general equation so obtained is rather involved, a truncated deformation (valid for small values of $z$) is explicitly presented.

The third section is devoted to the analysis of the quantum Schrödinger algebra originated from the previous deformation through the isomorphism (1.7). A systematic construction of a deformed SE is presented from the quantum deformation of the representation (1.9), that is obtained in terms of differential-difference operators. The fact that the Galilei generators \{K, H, P, M\} close a (deformed) subalgebra allows us to consider the corresponding Casimir operator, that leads to a time discretization of the SE on a uniform time lattice. Finally, the full quantum algebra invariance of the equation is proven by checking that the remaining generators $D$ and $C$ also transforms solutions into solutions. This result can be interpreted as a first example of uniform-lattice discretization induced from quantum algebras. From a physical point of view, an approach to the discrete time dynamics on such a time lattice has been recently developed in [15].

2 A new quantum two-photon algebra

2.1 The quantum algebra $U_z(h_6)$

A general property of non-standard classical $r$-matrices linked to a given Lie algebra $g$ is their automatic compatibility with any other Lie algebra $g'$ that contains $g$ as a subalgebra. Therefore, some Lie bialgebra structures on $g'$ can be obtained by using (known) classical $r$-matrices of $g$. If we recall that Lie bialgebras are just the first order of quantum deformations [10], it turns out that the construction of (non-standard) quantum algebra deformations of $g'$ can be guided by known results
for their subalgebras.

This was the case in [4], where the non-standard quantum deformation of $h_4$ introduced in [13] was used to obtain a so-called Jordanian quantum two-photon algebra such that $U_z(h_4) \subset U_z(h_6)$; the classical $r$-matrix linked to that deformation was $r = zN \wedge A_+$, and the primitive generators in $h_6$ turned out to be $M$ and $A_+$.

The same kind of construction can be obtained by means of the classical $r$-matrix of $sl(2, \mathbb{R})$, the subalgebra of $h_6$ linked to “pure” two-photon processes:

$$r = zN \wedge B_+.$$ (2.1)

By construction, (2.1) is a solution of the classical Yang–Baxter equation for $h_6$ and generates the Lie bialgebra with cocommutators, $\delta(X) = [1 \otimes X + X \otimes 1, r]$, given by:

$$\begin{align*}
\delta(B_+) &= 0 & \delta(M) &= 0 \\
\delta(N) &= 2zN \wedge B_+ & \delta(A_+) &= -zA_+ \wedge B_+ \\
\delta(A_-) &= z(A_- \wedge B_+ + 2N \wedge A_+) \\
\delta(B_-) &= 2z(B_- \wedge B_+ + N \wedge M).
\end{align*}$$ (2.2)

The four generators $\{N, B_+, B_-, M\}$ close a sub-bialgebra which is isomorphic to the one described for $\mathfrak{sl}(2, \mathbb{R})$ in [14] with the aid of the identification:

$$\begin{align*}
J_+ &= \frac{B_+}{2} & J_- &= -\frac{B_-}{2} & J_3 &= N & I &= -\frac{M}{2}
\end{align*}$$ (2.3)

and the replacement $z \to 2z$. Thus we can profit from this fact getting a new quantum two-photon algebra with different properties with respect to the Jordanian one introduced in [4].

The steps involved in the construction of the quantum deformation of the above two-photon bialgebra are as follows: i) Take the Hopf structure of $U_z(\mathfrak{sl}(2, \mathbb{R}))$ written in the two-photon basis. ii) Find a coassociative coproduct $\Delta$ of the remaining two generators $A_+$ and $A_-$ by taking into account that (2.2) gives its first order deformation. iii) Compute the remaining deformed commutation rules by imposing the coproduct to be a Lie algebra homomorphism. iv) Finally, counit $\epsilon$ and antipode $\gamma$ are deduced from the previous results.

The resulting quantum Hopf algebra $U_z(h_6)$ reads:

$$\begin{align*}
\Delta(B_+) &= 1 \otimes B_+ + B_+ \otimes 1 & \Delta(M) &= 1 \otimes M + M \otimes 1 \\
\Delta(N) &= 1 \otimes N + N \otimes e^{2zB_+} & \Delta(A_+) &= 1 \otimes A_+ + A_+ \otimes e^{-zB_+} \\
\Delta(A_-) &= 1 \otimes A_- + A_- \otimes e^{2zB_+} + 2zN \otimes e^{2zB_+} A_+ \\
\Delta(B_-) &= 1 \otimes B_- + B_- \otimes e^{2zB_+} + 2zN \otimes e^{2zB_+} M
\end{align*}$$ (2.4)

$$\epsilon(X) = 0 \quad \text{for} \quad X \in \{N, A_+, A_-, B_+, B_-, M\}$$ (2.5)

$$\begin{align*}
\gamma(A_+) &= -A_+ e^{zB_+} & \gamma(M) &= -M \\
\gamma(N) &= -Ne^{-2zB_+} & \gamma(B_+) &= -B_+ \\
\gamma(A_-) &= -(A_- - 2zNA_+) e^{-zB_+} \\
\gamma(B_-) &= -(B_- - 2zNM) e^{-2zB_+}.
\end{align*}$$ (2.6)
\[ [N, A_+] = A_+ \quad [N, A_-] = -A_- \quad [A_-, A_+] = M \]
\[ [N, B_+] = e^{2zB_+} - 1 \quad [N, B_-] = -2B_- - 4zN^2 \]
\[ [B_-, B_+] = 4N + 2Me^{2zB_+} \quad [M, \cdot] = 0 \quad (2.7) \]
\[ [A_+, B_-] = -2A_- + 2z(NA_+ + A_+ N) \quad [A_+, B_+] = 0 \]
\[ [A_-, B_+] = 2e^{2zB_+}A_+ \quad [A_-, B_-] = -2z(NA_+ + A_- N). \]

Clearly by construction we have \( U_z(\mathfrak{sl}(2, \mathbb{R})) \subset U_z(\hbar_6) \). Note that the oscillator algebra \( \hbar_4 \) is preserved as an undeformed subalgebra only at the level of commutation relations.

### 2.2 Universal \( R \)-matrix

The quantum universal \( R \)-matrix for \( U_z(\hbar_6) \) is given by

\[ R = \exp\{-zB_+ \otimes N\} \exp\{zN \otimes B_+\}, \quad (2.8) \]

which is a solution of the quantum Yang–Baxter equation and verifies the property

\[ R\Delta(X)R^{-1} = \sigma \circ \Delta(X) \quad \text{for} \quad X \in \hbar_6 \quad (2.9) \]

where \( \sigma(a \otimes b) = b \otimes a \).

In order to prove this statement we have to take into account that the element \((2.8)\) is a universal \( R \)-matrix for \( U_z(\mathfrak{sl}(2, \mathbb{R})) \) \([13]\). Therefore, as \( U_z(\mathfrak{sl}(2, \mathbb{R})) \subset U_z(\hbar_6) \) we only need to show that \( R \) also satisfies \((2.9)\) for the remaining two generators \( A_+ \) and \( A_- \).

We consider the formula

\[ e^f \Delta(X) e^{-f} = \Delta(X) + \sum_{n=1}^{\infty} \frac{1}{n!} [f, \ldots [f, \Delta(X)]^n] \ldots \quad (2.10) \]

and set \( X \equiv A_+, f \equiv zN \otimes B_+ \). A direct computation shows that

\[ [zN \otimes B_+, \ldots [zN \otimes B_+, \Delta(A_+)]^n] \ldots = A_+ \otimes e^{-zB_+}(zB_+)^n \quad n \geq 1 \quad (2.11) \]

so that

\[ e^{zN \otimes B_+} \Delta(A_+) e^{-zN \otimes B_+} = \Delta(A_+) + \sum_{n=1}^{\infty} A_+ \otimes e^{-zB_+}(zB_+)^n \frac{1}{n!} \]
\[ = 1 \otimes A_+ + A_+ \otimes e^{-zB_+} + A_+ \otimes e^{-zB_+}(e^{zB_+} - 1) \]
\[ = 1 \otimes A_+ + A_+ \otimes 1 \equiv \Delta_0(A_+). \quad (2.12) \]

On the other hand we find

\[ [-zB_+ \otimes N, \ldots [-zB_+ \otimes N, \Delta_0(A_+)]^n] \ldots = (-zB_+)^n \otimes A_+ \quad n \geq 1 \quad (2.13) \]
and the proof for $A_+$ follows

$$e^{-zB_+ \otimes N} \Delta_0(A_+) e^{zB_+ \otimes N} = \Delta_0(A_+) + \sum_{n=1}^{\infty} \frac{(-zB_+)^n}{n!} \otimes A_+$$

$$= 1 \otimes A_+ + A_+ \otimes 1 + (e^{-zB_+} - 1) \otimes A_+ = \sigma \circ \Delta(A_+). \quad (2.14)$$

Now let $X \equiv A_-$ and $f \equiv zN \otimes B_+$:

$$[zN \otimes B_+, \Delta(A_-)] = -2zN \otimes e^{2zB_+} A_+ - zA_- \otimes B_+ e^{zB_+} \quad (2.15)$$

$$[zN \otimes B_+, \ldots [zN \otimes B_+, \Delta(A_-)]^n \ldots ] = A_- \otimes e^{zB_+} (-zB_+)^n \quad n \geq 2. \quad (2.16)$$

Then

$$e^{zN \otimes B_+} \Delta(A_-) e^{-zN \otimes B_+} = \Delta(A_-) - 2zN \otimes e^{2zB_+} A_+ + \sum_{n=1}^{\infty} A_- \otimes e^{zB_+} \frac{(-zB_+)^n}{n!}$$

$$= 1 \otimes A_- + A_- \otimes 1 \equiv \Delta_0(A_-). \quad (2.17)$$

Similarly we find that

$$[-zB_+ \otimes N, \Delta(A_-)] = 2ze^{2zB_+} A_+ \otimes N + zB_+ \otimes A_- \quad (2.18)$$

$$[-zB_+ \otimes N, \ldots [-zB_+ \otimes N, \Delta_0(A_+)]^n \ldots ] = (zB_+)^n \otimes A_- \quad n \geq 2 \quad (2.19)$$

and, consequently,

$$e^{-zB_+ \otimes N} \Delta_0(A_-) e^{zB_+ \otimes N} = \Delta_0(A_-) + 2ze^{2zB_+} A_+ \otimes N + \sum_{n=1}^{\infty} \frac{(zB_+)^n}{n!} \otimes A_-$$

$$= A_- \otimes 1 + e^{zB_+} \otimes A_- + 2ze^{2zB_+} A_+ \otimes N = \sigma \circ \Delta(A_-). \quad (2.20)$$

### 2.3 Deformed Fock-Bargmann realization

Let $a_-, a_+$ be the boson generators fulfilling (1.3). Then a one-boson representation of $U_z(h_6)$ (2.7) is given by:

$$B_+ = a_+^2 \quad M = 1 \quad N = \frac{e^{2za_+^2} - 1}{2za_+} a_-$$

$$A_+ = \left( \frac{1 - e^{-2za_+^2}}{2z} \right)^{1/2} \quad A_- = \frac{e^{2za_+^2} \left( 1 - e^{-2za_+^2} \right) ^{1/2} a_-}{a_+}$$

$$B_- = \left( \frac{e^{2za_+^2} - 1}{2za_+^2} \right) a_- + \left( \frac{e^{2za_+^2}}{a_+} + \frac{1 - e^{2za_+^2}}{2za_+^2} \right) a_- \quad (2.21)$$

Note that in the limit $z \to 0$ we recover the classical representation (1.2). Let us also remark that, in spite of the fact that (2.7) presents a non-deformed oscillator subalgebra in terms of the abstract generators $\{N, A_+, A_-, M\}$, their representation (2.21) includes strong deformations in terms of usual boson operators.
above deformed one-boson representation provides the Fock–Bargmann one by setting \( a_+ \equiv \alpha \) and \( a_- \equiv \frac{d}{d\alpha} \). However, due to its rather complicated form we will restrict to the first order in \( z \). In such approximation, the two-photon differential operators acting on the space of entire analytic functions \( f(\alpha) \) turn out to be:

\[
\begin{align*}
B_+ &= \alpha^2 M + 1 \quad N = (\alpha + z\alpha^3) \frac{d}{d\alpha} \\
A_+ &= \alpha - z\alpha^3/2 \quad A_- = (1 + 3z\alpha^2/2) \frac{d}{d\alpha} \\
B_- &= (1 + z\alpha^2) \frac{d^2}{d\alpha^2} + z\alpha \frac{d}{d\alpha}.
\end{align*}
\]

We substitute these operators in (1.5) getting the differential equation:

\[
\beta_2(1 + z\alpha^2) \frac{d^2f}{d\alpha^2} + \left( \beta_1\alpha + \beta_4 + z(\beta_1\alpha^3 + \beta_2\alpha + 3\beta_4\alpha^2/2) \right) \frac{df}{d\alpha} + (\beta_3\alpha^2 + \beta_5\alpha - z\beta_5\alpha^3/2 - \lambda)f = 0.
\]

The particular equation with \( \beta_4 = \beta_5 = 0 \) is associated to the Hopf subalgebra \( U_z(sl(2, \mathbb{R})) \) while the case \( \beta_2 = \beta_3 = 0 \) corresponds to the harmonic oscillator sector (recall that the latter is not a Hopf subalgebra). Besides the problem of finding particular solutions of the equation (2.23), there are two other important questions to be considered: it is necessary to define a deformed ‘invariant’ measure and the possible solutions must be analytic.

### 3 Quantum Schrödinger algebra and time discretization

As it was mentioned in the Introduction, a precise Lie subalgebra of \( S(1 + 1) \) can be considered in order to explain the significance of this algebra as the Lie symmetry algebra of the (1+1) SE: the extended (1+1) Galilei algebra generated by \( \{K, H, P, M\} \). In fact, the (free) heat-Schrödinger equation

\[
(\partial_x^2 - 2m\partial_t)\phi(x, t) = 0,
\]

is just the Casimir operator \( P^2 - 2MH \) of the Galilei subalgebra in the representation (1.9). If we denote the equation operator as

\[
E = P^2 - 2MH,
\]

we can say that an operator \( S \) is a symmetry of \( E \) if \( S \) transforms solutions into solutions:

\[
(E S)\phi(x, t) = (\Lambda E)\phi(x, t),
\]

where \( \phi(x, t) \) is a solution of (3.1) and \( \Lambda \) is another operator. In particular, since \( E \) is given by the Casimir operator of the extended Galilei algebra, all its generators are
symmetries $S$ of the SE fulfilling (3.3) with $\Lambda = S$. Moreover, the dilation operator $D$ is also a symmetry, since (1.8) implies that $[E, D] = 2E$. Finally, we can compute

$$[E, C] = -(KP + PK + 2MD).$$

(3.4)

Now, if we take the particular representation with $a = -1/2$, we obtain that the right hand side of (3.4) is just the operator $2tE$, so that the conformal operator $C$ also transforms solutions of $E$ into solutions. As we shall see in the sequel, this procedure can be generalized to the quantum case.

### 3.1 Quantum Schrödinger algebra

Let us now use the isomorphism (1.7) in order to obtain another non-standard quantum deformation of $S(1 + 1)$. The new classical $r$-matrix and cocommutators read

$$r = 2zH \wedge D + zH \wedge M$$

(3.5)

$$\delta(H) = 0 \quad \delta(M) = 0 \quad \delta(P) = -2zP \wedge H$$

$$\delta(K) = z(2K \wedge H + 2P \wedge D + P \wedge M)$$

$$\delta(D) = z(4D \wedge H + 2M \wedge H)$$

$$\delta(C) = z(4C \wedge H - D \wedge M).$$

(3.6)

The coproduct and the commutation rules of the Hopf algebra $U_z(S(1 + 1))$ are

$$\Delta(H) = 1 \otimes H + H \otimes 1$$

$$\Delta(M) = 1 \otimes M + M \otimes 1$$

$$\Delta(P) = 1 \otimes P + P \otimes e^{-2zH}$$

$$\Delta(K) = 1 \otimes K + K \otimes e^{2zH} - 2z(D + M/2) \otimes e^{4zH}P$$

$$\Delta(D) = 1 \otimes D + D \otimes e^{4zH} + M \otimes (e^{4zH} - 1)/2$$

$$\Delta(C) = 1 \otimes C + C \otimes e^{4zH} - z(D + M/2) \otimes e^{4zH}M,$$

(3.7)

$$[D, P] = -P$$

$$[D, K] = K$$

$$[K, P] = M$$

$$[D, H] = \frac{1 - e^{4zH}}{2z}$$

$$[D, C] = 2C + 2z(D + M/2)^2$$

$$[H, C] = D + M(1 - e^{4zH})/2$$

$$[K, H] = e^{4zH}P$$

$$[K, C] = z(DK + KD + KM)$$

$$[P, H] = 0$$

$$[P, C] = -K - z(DP + PD + PM)$$

$$[M, \cdot] = 0.$$

(3.8)

Notice that in this deformation the primitive generator (besides the mass) is the time translation $H$, while in $[4]$ this role was played by the space translation $P$. The quantum $\mathfrak{sl}(2, \mathbb{R})$ algebra corresponds now to the Hopf subalgebra spanned by $\{D, H, C, M\}$, and the extended Galilei generators together with the dilation, $\{K, H, P, M, D\}$, also close a Hopf subalgebra.

On the other hand, the universal $R$-matrix for $U_z(S(1 + 1))$ is provided by (2.8):

$$R = \exp\{2zH \otimes D\} \exp\{zH \otimes M\} \exp\{-zM \otimes H\} \exp\{-2zD \otimes H\}.$$  

(3.9)
3.2 A time discretization of the Schrödinger equation

A realization of $U_z(S(1 + 1))$ (with classical limit (1.9)) reads

$$
H = \partial_t \quad P = \partial_x \quad M = m
$$

$$
K = -(t + 4z) e^{4z \partial_t} \partial_x - mx
$$

$$
D = 2(t + 4z) \frac{e^{4z \partial_t} - 1}{4z} + x \partial_x - a
$$

$$
C = (t^2 - 4zbt) \frac{e^{4z \partial_t} - 1}{4z} + tx \partial_x - at + mx^2/2 - 4z(b + 1)e^{4z \partial_t}
$$

$$
- zx^2 \partial_x^2 - 2z(b - a + 1/2)x \partial_x - z(b - a)^2
$$

where $b = m/2 - 2$. If we introduce a discrete time derivative as

$$
\mathcal{D}_t f(t, x) := \left( \frac{e^{4z \partial_t} - 1}{4z} \right) f(t, x) = \frac{f(t + 4z, x) - f(t, x)}{4z},
$$

(3.11)

then (3.10) adopts the form of a differential-difference realization:

$$
H = \partial_t \quad P = \partial_x \quad M = m
$$

$$
K = -(t + 4z)(1 + 4z \mathcal{D}_t) \partial_x - mx
$$

$$
D = 2(t + 4z) \mathcal{D}_t + x \partial_x - a
$$

$$
C = (t^2 - 4zbt) \mathcal{D}_t + tx \partial_x - at + mx^2/2 - 4z(b + 1)(1 + 4z \mathcal{D}_t)
$$

$$
- zx^2 \partial_x^2 - 2z(b - a + 1/2)x \partial_x - z(b - a)^2
$$

(3.12)

We can now try to reproduce the symmetry procedure before outlined for the non-deformed Schrödinger algebra. The first important feature of the deformation just introduced is that, at the level of commutation rules, the Galilei structure remains as a (deformed) subalgebra. It is not difficult to check that the deformed Casimir operator for that subalgebra is

$$
E_z = P^2 - 2M \frac{1 - e^{-4zH}}{4z}.
$$

(3.13)

It seems quite natural to define the corresponding SE as the action of (3.13) on $\phi(x, t)$ through (3.10). It reads

$$
\left( \partial_x^2 - 2m \frac{1 - e^{-4zH}}{4z} \right) \phi(x, t) = 0 \equiv \left( \partial_x^2 - 2m \mathcal{D}_t \right) \phi(x, t) = 0,
$$

(3.14)

which is just a time discretization of the free SE. The discrete time derivative $\mathcal{D}_t$ is related to (3.11) by changing $z \rightarrow -z$.

By construction, the quantum algebra generators $\{K, H, P, M\}$ are symmetries of $E_z$. Furthermore, $D$ and $C$ are also deformed Schrödinger symmetries. This fact follows straightforwardly for the dilation since from (3.8)

$$
[E_z, D] = 2E_z.
$$

(3.15)
For the conformal transformation, we have
\[
[E_z, C] = -(KP + PK + 2MDe^{-4zH}) + M(M + 2)(1 - e^{-4zH})
- z(DP^2 + 2PDP + P^2D + 2P^2M).
\] (3.16)

By introducing the realization (3.10) we find that
\[
[E_z, C] = 2(t + 4z - 2zx\partial_x)E_z - 2z\{m + 2(1 - a)\} \partial_x^2
+ m(1 + 2ae^{-4z\partial_t}) + m(m + 2)(1 - e^{-4z\partial_t}).
\] (3.17)

Now, as in the non-deformed case, we have to set \(a = -1/2\) in order to obtain \(C\) as a symmetry:
\[
[E_z, C] = 2\{t + z(1 - m - 2x\partial_x)\} E_z.
\] (3.18)

## 4 Concluding remarks

The connection between \(q\)-algebras and geometric \(q\)-lattices seems to be a general fact [8], although a general theory of this kind of systems from the point of view of symmetries is still lacking. The results presented in this paper and the structural properties of non-standard deformations suggest a natural link between non-standard quantum algebras and uniform lattices whose generality and physical implications should be better understood.

In this sense, it is remarkable that the quantum Schrödinger algebra here presented induces in a very simple way a natural definition of a discrete time SE that is invariant under the quantum algebra generators. It becomes apparent that a similar construction for the non-standard deformation given in [3] would lead to a (uniform) discretization of the SE in the spatial coordinate. This different discretization would be, in the kinematical context, the algebraic consequence of the use of either the \(sl(2, \mathbb{R})\) or the \(h_4\) subalgebra as generating structures for the construction of the deformation.

Other discretizations of the SE linked with other quantum algebras can be found in [10, 17, 18]. In particular, the use in [17] of the \(sl(2, \mathbb{R})\) symmetry in order to include some potentials was also analysed in [8] in a \(q\)-lattice context and would deserve further attention from the point of view of our deformation. Another open problem arising from the results here presented is the study of the special functions that can be defined from them. Work in all these directions is in progress.

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References

[1] W.-M. Zhang, D. H. Feng and R. Gilmore, Rev. Mod. Phys. 62, 867 (1990).

[2] C. Brif, Ann. Phys. 251, 180 (1996).

[3] V. A. Fock, Z. Phys. 49, 339 (1928); V. Bargmann, Comm. Pure Appl. Math 14, 187 (1961).

[4] A. Ballesteros, F. J. Herranz and P. Parashar, “A Jordanian quantum two-photon/Schrödinger algebra”, q-alg/9706014.

[5] C. R. Hagen, Phys. Rev. D5, 377 (1972).

[6] U. Niederer, Helv. Phy. Acta 45, 802 (1972).

[7] R. Floreanini and L. Vinet, Lett. Math. Phys. 32, 37 (1994).

[8] R. Floreanini and L. Vinet, J. Math. Phys. 36, 3134 (1995).

[9] V. K. Dobrev, H.-D. Doebner and C. Mrugalla, J. Phys. A29, 5909 (1996).

[10] Chari V and Pressley A, A Guide to Quantum Groups, Cambridge University Press (1995).

[11] R. Floreanini, J. Negro, L. M. Nieto and L. Vinet, Lett. Math. Phys. 36, 351 (1996).

[12] J. Negro and L. M. Nieto, J. Phys. A29, 1107 (1996).

[13] A. Ballesteros and F. J. Herranz, J. Phys. A29, 4307 (1996).

[14] A. Ballesteros, F. J. Herranz and J. Negro, “Boson representations, non-standard quantum algebras and contractions”, J. Phys. A30 to appear.

[15] G. Jaroskiewicz and K. Norton, J. Phys. A30, 3145 (1997).

[16] F. Bonechi, N. Ciccoli, R. Giachetti, E. Sorace and M. Tarlini, Commun. Math. Phys. 175, 161 (1996).

[17] Ö. F. Dayi and I. H. Duru, “q-Schrödinger Equations for V = u² + 1/u² and Morse Potentials in terms of the q-Canonical Transformation”, Int. J. Mod. Phys. A to appear.

[18] M. Micu, “q-Deformed Schrödinger Equation”, q-alg/9703026.