A local hidden-variable model violating Bell’s inequalities: a reply to Matzkin.

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Recently, Matzkin claimed the construction of a hidden variable (HV) model [1] which is both local and equivalent with the quantum-mechanical predictions. In this paper we will briefly present this HV model and argue, by identifying an extra non-local “hidden” HV , why this model is not local.

I. INTRODUCTION

With the derivation of his well-known inequalities, Bell [2] proved that any local model based on hidden variables (HV) can not reproduce the empirical predictions of quantum-mechanics. Recently however, Matzkin claimed that he has constructed a HV model [1] which is both local and equivalent with the quantum-mechanical predictions.

Such a claim is remarkable, especially when we take into account that stochastic local HV models (like the HV model of Matzkin) also have to obey the Bell inequalities [3]. In this paper we will briefly present the HV model of Matzkin and argue why this model is not local.

II. EPRB

In the EPRB-experiment, studied by both Bell and Matzkin, a pair of spin-\(\frac{1}{2}\) particles are formed in the singlet spin state \(|\Psi_0\rangle\):

\[
|\Psi_0\rangle = \frac{1}{\sqrt{2}}(|z\uparrow\rangle \otimes |z\downarrow\rangle - |z\downarrow\rangle \otimes |z\uparrow\rangle).
\]

(1)

Both particles move freely in opposite directions towards two measuring devices. The spin component \(a\) of particle 1 is measured in direction \(\vec{a}\) and the spin component \(b\) of particle 2 is measured in direction \(\vec{b}\). If we measure the spin component for both particles in the same direction \((\vec{a} = \vec{b})\), anti-correlation of the particles leads to the measurement outcome \(a = -b\). More generally, for arbitrary directions \(\vec{a},\vec{b}\), the quantum-mechanical expectation value of the product of the two spin components \(< S_{\vec{a}} \otimes S_{\vec{b}} >\) is:

\[
E_{QM}(\vec{a},\vec{b}) \equiv < \Psi_0 | S_{\vec{a}} \otimes S_{\vec{b}} | \Psi_0 > = -\frac{\vec{a} \cdot \vec{b}}{4},
\]

(2)

where \(S_{\vec{a}} = \frac{1}{2}(\vec{\sigma} \cdot \vec{a})\) and \(S_{\vec{b}} = \frac{1}{2}(\vec{\sigma} \cdot \vec{b})\).

III. MATZKINS HV MODEL

Matzkins HV model makes use of the following elements:

1. HV. A single particle is specified by a hidden variable \(\vec{\lambda}\); a normalized vector in \(\mathbb{R}^3\). For the two particles specified by \((\vec{\lambda}_1, \vec{\lambda}_2)\) in the singlet spin state, anti-correlation is described by the relation:

\[
\vec{\lambda}_1 = -\vec{\lambda}_2.
\]

(3)

2. HV distribution. As we will see, the measurement outcome of a measurement on a single particle does not only depend on the HV \(\vec{\lambda}\) describing the particle, but it does also depend on a HV distribution \(R\). Matzkin states

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that a one-particle system in an eigenstate of the spin-operator in (the normalized) direction $\vec{u}$ with a spin component $u = \pm \frac{1}{2}$, can be described with a HV distribution:

$$R_{\pm \frac{1}{2}\vec{u}}(\vec{\lambda}) = \frac{1}{\sqrt{3\pi}}\delta(\vec{\lambda} \cdot \vec{u} \pm \frac{1}{2}).$$

(4)

In his paper, Matzkin does not make clear how we have to interpret this HV distribution $R$. The fact that two particles in the singlet state, are anti-correlated by relation (3) however, suggest that a single particle is characterized by only one HV $\vec{\lambda}$. It is therefore plausible that the distribution $R$ is a probability distribution which describes an ensemble of all different $\vec{\lambda}$ corresponding to the same quantum state of the system. Each element of the ensemble corresponds thereby with a single particle described by one HV $\vec{\lambda}$.

3. Measurement outcome. A measurement induces a perturbation of the initial HV distribution. If we measure the spin component of a particle in direction $\vec{u}$, the perturbation of the initial HV distribution $(R_{t0})$ depends on the measurement outcome $M(R_{t0}, \vec{u}, \vec{\lambda}) = \pm \frac{1}{2}$. Matzkin states that for such a spin measurement in direction $\vec{u}$, the post-measurement HV distribution $(R_{t1})$ will be equal to the distributions given in (4), depending on the measurement outcomes $\pm \frac{1}{2}$.

For a system consisting of a single particle in a quantum state with a positive spin along the $\vec{z}$-axis (having an initial HV distribution $R_{\pm \frac{1}{2}\vec{z}}$) the probabilities of $M(R = R_{\pm \frac{1}{2}\vec{z}}, \vec{u}, \vec{\lambda}) = \pm \frac{1}{2}$ by measuring the spin component in direction $\vec{u}$ are given by and equal to the quantum-mechanical probabilities:

$$P(M(R = R_{\pm \frac{1}{2}\vec{z}}, \vec{u}, \vec{\lambda}) = \frac{1}{2}) = \cos^2 \theta_{\vec{z},\vec{u}}$$

$$P(M(R = R_{\pm \frac{1}{2}\vec{z}}, \vec{u}, \vec{\lambda}) = -\frac{1}{2}) = \sin^2 \theta_{\vec{z},\vec{u}}$$

(5) (6)

with $\theta_{\vec{z},\vec{u}}$ the angle between $\vec{z}$ and $\vec{u}$.

Note that this probabilities not only depend on $\vec{\lambda}$ and $\vec{u}$, but are also dependent on the initial HV distribution $R$. Because there is no correlation between $\vec{\lambda}$ and the HV distribution of a system, we make this dependency clear by adding $R$ as an extra (hidden) variable:

$$M(R, \vec{u}, \vec{\lambda}) = \pm \frac{1}{2}.$$  

(7)

Because it is important for our argument, we stress this point again: in general, the outcome of a measurement in de HV model of Matzkin does not only depend on a HV $\vec{\lambda}$, but it does also depend on a HV distribution $R$. We therefore recognize this HV distribution $R$ as an extra hidden variable of the model.

4. Locality. The only statement Matzkin makes about locality is that “…the HV distribution is dynamically and locally affected by the measurement …”. To be more precise we assume that, following Bell, for a HV model the measurement outcome of particle 2 has to be a function of information locally available at particle 2. In particular, it may therefore not depend on the measurement direction and measurement outcome of a measurement on particle 1.

IV. A HV MODEL FOR TWO PARTICLES

With these elements a HV-model for the EPRB experiment is constructed. The system consists of two particles, described with the HV $(\vec{\lambda}_1, \vec{\lambda}_2)$ respecting correlation (3). The initial distribution of HV for each particle in the singlet state is (as given by Matzkin) a uniform distribution $R_{\Sigma}$ of HV on a unit sphere. As noted, the measurement outcome of a measurement on one of the two particles depends on the measurement direction $\vec{u}$, the HV $\vec{\lambda}$ and the initial HV distribution. For a distribution $R_{\Sigma}$, the measurement outcome $M$ of a measurement on one of the particles (described by $\vec{\lambda}$), is defined by:

$$M(R = R_{\Sigma}, \vec{u}, \vec{\lambda}) = \frac{1}{2} \frac{\vec{u} \cdot \vec{\lambda}}{||\vec{u}||} \in \{ -\frac{1}{2}, \frac{1}{2} \}$$

(8)

Simply stated: the sign of the measurement outcome depends on the sign of the inner product of the HV $\vec{\lambda}$ and the measurement direction $\vec{u}$. 
Knowing the initial HV distributions \( R_\Sigma \) for the two particles in the singlet state, formula (3) describing the relation between \( (\vec{\lambda}_1, \vec{\lambda}_2) \) and formula (5), we are able to calculate the expectation value \( E_{HV}(\vec{a}, \vec{b}) \) for the HV model of Matzkin. Integrating over the initial HV distribution and using relation (3) gives:

\[
E_{HV}(\vec{a}, \vec{b}) = \langle \mathcal{M}_1(R_\Sigma, \vec{a}), \mathcal{M}_2(R_\Sigma, \vec{b}) \rangle = \int \mathcal{M}_1(R_\Sigma, \vec{a}, \vec{\lambda}_1) \mathcal{M}_2(R_\Sigma, \vec{b}, -\vec{\lambda}_1) \frac{1}{4\pi} d\Omega_{\vec{\lambda}_1} = -\frac{1}{4} + \frac{1}{2\pi} \theta_{\vec{a},\vec{b}}.
\]

(9)

Unfortunately, this result is not equal to the quantum-mechanical prediction (2). In fact, the result is exactly the result of a naive HV model proposed by Bell [2], which does not violate the Bell inequalities at all.

Of course (9) is not the result of Matzkin. So what went wrong? The difference is that Matzkin makes an extra assumption before calculating the expectation value \( E_{HV} \). Matzkin states that if we measure the spin component of particle 1 along axis \( \vec{a} \) \((S_{\vec{a}})\) and obtain, for example, result \( a = +\frac{1}{2} \), we know (using (3)) that \( \vec{\lambda}_1 \cdot \vec{a} \geq 0 \). With the anti-correlation (3) in mind, we therefore also know that \( \vec{\lambda}_2 \cdot \vec{a} \leq 0 \). After a measurement of the spin of particle 1, we will therefore know the HV \( \vec{\lambda}_2 \) (describing particle 2) more precisely.

By knowing \( \vec{\lambda}_2 \) more precisely, Matzkin continues, we are also able to limit the initial HV distribution \( R \) for particle 2. Given the original uniform distribution \( R_\Sigma \) and the relation \( \vec{\lambda}_2 \cdot \vec{a} \leq 0 \), the HV \( \vec{\lambda}_2 \) is, for example, also certainly an element of the uniform distribution on a half-sphere \( R_{\Sigma \leq \frac{1}{2} \vec{a}} \). We can therefore, Matzkin argues, use \( R_{\Sigma \leq \frac{1}{2} \vec{a}} \) as the HV distribution of particle 2 after measuring \( S_{1,\vec{a}} = +\frac{1}{2} \).

By postulating that a particle described with a HV distribution \( R_{\Sigma \leq \frac{1}{2} \vec{a}} \) behaves like it is in an eigenstate of \( S_{\vec{a}} \) with spin component \( a = -\frac{1}{2} \), Matzkin argues that the measurement-outcomes \( \mathcal{M} \) for such a particle will obey the (quantum-mechanical) probabilities given in (5). If so, the calculated expectation value \( E_{HV} \) will correspond to the quantum-mechanical prediction (2) [1].

\[
E_{HV}(\vec{a}, \vec{b}) = \langle \mathcal{M}_1(\vec{a}), \mathcal{M}_2(\vec{b}) \rangle = \int \mathcal{M}_1(R_\Sigma, \vec{a}, \vec{\lambda}_1) \mathcal{M}_2(R_{\Sigma a\vec{a}}, \vec{b}, -\vec{\lambda}_1) \frac{1}{4\pi} d\Omega_{\vec{\lambda}_1} = -\frac{\vec{a} \cdot \vec{b}}{4}.
\]

(10)

Using (10), it is easily seen that the HV distribution \( R_{\Sigma a\vec{a}} \) of particle 2 depends both on the direction \( \vec{a} \) and on the measurement outcome \( a \) of the spin-measurement of particle 1. Both of these properties are in conflict with the definition of “locality” as given by Bell.

Matzkin does not give us any local mechanism (based on information locally available) to explain the perturbation of the HV distribution of particle 2 after measuring particle 1. So, we have to conclude that the initial HV distribution \( R_\Sigma \) of particle 2 is modified in a non-local way when a measurement on particle 1 is made. Ironically, the only statement Matzkin made about locality is hereby not obeyed.

V. CONCLUSION

We recognize that measurement outcomes in the HV model of Matzkin not only depend on the HV \( \vec{\lambda} \) and the the measurement direction \( \vec{a} \), but also on an initial HV distribution \( R \). In the HV model for the two particles in the EPRB experiment, the initial HV distribution of particle 2 is modified, based on both the measurement direction and the measurement outcome of a measurement on particle 1. Both of these properties of the HV model of Matzkin conflict with the criteria of a local theory, as defined by Bell. Therefore, we conclude that the given HV model is not local.

[1] A. Matzkin, quant-ph/0703271 (2007).
[2] J.S. Bell, Physics 1, 195 (1964).
[3] J. S. Bell, Bertlmanns socks and the nature of reality, in Speakable and Unspeakable in Quantum Mechanics, Cambridge University Press, (1987).
[4] Matzkin uses a different notation in his paper, making the dependency between the measurement direction and outcome of the measurement of the spin of particle 1 and the measurement outcome of a measurement on particle 2 less clear. He writes the distributions as a subscript and directly substitutes the probabilities of (3) in the integral.
[5] In fact, modifying a local variable (the HV-distribution) of particle 2 based on the knowledge we obtain by measuring particle 1, seems to us no more than some sophisticated way of formalizing the sentence: “by knowing the spin of particle
1 is $|\vec{a}\rangle$, measuring the spin of particle 2 in direction $\vec{b}$ should respect the quantum-mechanical predictions as if particle 2 is in a quantum state $|\vec{a}\rangle^\ast$. This may be true, but can not be called a local model.