Double canard cycles in singularly perturbed planar systems

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Abstract We study the bifurcations of slow-fast cycles with two canard points in singularly perturbed planar systems. After analyzing the local dynamics of two canard points lying on the $S$-shaped critical manifolds, we give a sufficient condition under which there exist three hyperbolic limit cycles bifurcating from some slow-fast cycles. The proof is based on the geometric singular perturbation theory. Then, we apply the results to cubic Liénard equations with quadratic damping, and prove the coexistence of three large limit cycles enclosing three equilibria. This is a new dynamical configuration and has never been previously found in the existing references.

Keywords Limit cycle · Canard cycle · Normal form · Melnikov theory · Liénard equation

1 Introduction

Singular perturbation problems in multi-scale dynamical systems widely appear in science and engineering, such as cellular physiology, fluid mechanics, population dynamics and so on [25, 28, 35]. These singularly perturbed systems usually admit a clear separation in one slow time scale and one fast time scale. So they are also referred to as slow-fast systems. Based on the normally hyperbolic invariant manifolds theory (see, for instance, [19, 41]), Fenichel [20] in 1979 laid the foundation of geometric singular perturbation theory to investigate multiple time scales dynamics. Since then, geometric singular perturbation theory has become a hotspot research subject in the field of dynamical systems. There is an enormous literature on this topic. We refer the readers to [8–10, 18, 22, 24, 29] and the references therein.

Our aim is to study a singularly perturbed planar system

$$\varepsilon \frac{dx}{d\tau} = \varepsilon \dot{x} = f(x, y, \mu, \varepsilon),$$
$$\frac{dy}{d\tau} = \dot{y} = g(x, y, \mu, \varepsilon),$$

(1.1)

where $(x, y) \in \mathbb{R}^2, \mu \in \mathbb{R}^{1+m}$ with $m \geq 1$, the parameters $\varepsilon$ satisfies $0 < \varepsilon \leq \varepsilon_0 \ll 1$ for some small $\varepsilon_0$, and the functions $f$ and $g$ are $C^k$ with $k \geq 3$. By a time rescaling $\tau = \varepsilon t$, system (1.1) is changed to
\[
\frac{dx}{dt} = x' = f(x, y, \mu, \varepsilon), \\
\frac{dy}{dt} = y' = \varepsilon g(x, y, \mu, \varepsilon).
\] (1.2)

For simplification, let \(X_{\varepsilon, \mu}\) denote the vector field of system (1.2). Clearly, systems (1.1) and (1.2) are equivalent for \(\varepsilon \neq 0\). To obtain the dynamics of system (1.1) or (1.2) for sufficiently small \(\varepsilon\), we consider the limiting case \(\varepsilon = 0\). Then system (1.1) becomes the reduced equation

\[
0 = f(x, y, \mu, 0), \\
\dot{y} = g(x, y, \mu, 0),
\] (1.3)

and system (1.2) becomes the layer equation

\[
x' = f(x, y, \mu, 0), \\
y' = 0,
\] (1.4)

For each fixed \(\mu\), the phase state of system (1.3) is defined on the set of equilibria of system (1.4), that is,

\[
C_{\mu,0} := \left\{ (x, y) \in \mathbb{R}^2 : f(x, y, \mu, 0) = 0 \right\}.
\]

This set is called the critical set. If it is a submanifold of \(\mathbb{R}^2\), then it is called the critical manifold. The branches of the set \(C_{\mu,0}\) are called the slow curves. By the Fenichel theory [20], a normally hyperbolic submanifold \(M_{\mu,0}\) (with or without boundary) of the critical manifold \(C_{\mu,0}\) is perturbed to a slow manifold \(M_{\mu,\varepsilon}\) near \(M_{\mu,0}\) for sufficiently small \(\varepsilon\). We call \(M_{\mu,0}\) the normally hyperbolic manifold if \(\partial f / \partial x \neq 0\) along \(M_{\mu,0}\). The points in \(C_{\mu,0}\) with \(\partial f / \partial x = 0\) are called the contact points, where the normal hyperbolicity breaks down. The most generic contact points are jump points, for which the reduced flow (1.3) directs toward the contact points. More degenerate contact points are canard points, a simple zero of the function \(g\) in system (1.2), which leads to a possibility of periodic orbits in its neighborhood. Canard points are also called the turning points in some references. Geometric analysis of the contact points was initiated in [10], where Dumortier and Roussarie applied the blow-up technique to study the singularly perturbed van der Pol equation. Following the pioneering work [10] of Dumortier and Roussarie, many efforts have been devoted to expand the capabilities of this technique. For example, Krupa and Szmolyan used the technique provided in [10] to extend the slow manifolds of singularly perturbed planar systems near jump points and canard points, and more results on the blow-up technique and its generalizations are referred to [5, 14, 28].

Jump points and canard points in singularly perturbed planar systems with one-parameter layer equations can lead to relaxation oscillations and canard solutions, respectively. Relaxation oscillation is a periodic orbit which spends a long time along the slow manifold toward a jump point, jumps from this contact point, spends a short time parallel to the fast orbits toward another stable slow manifold, follows the slow manifold again until another jump point is reached, and finally returns to its starting point via several similarly successive motions [27]. Canards are the orbits contained in the intersection of an attracting slow manifold and a repelling slow manifold. Canards are subject to a generic Hopf breaking mechanism, that is, the flow of the layer equation (1.4) has the same direction on an attracting slow curve and a repelling slow curve which are connected by a generic canard point. Periodic orbits containing canards are referred to as canard cycles. Relaxation oscillations and canard cycles can be both seen as the perturbations of slow-fast cycles, which are closed loops formed by a connected succession of critical manifolds, fast orbits of the layer equations and contact points. More precisely, relaxation oscillations arise from common cycles which only contain repelling critical manifolds or attracting critical manifolds (see Fig. 1a), canard cycles from canard slow-fast cycles which contain at least one attracting and one repelling critical manifolds (see Figs. 1b, 1c and 1d).

Besides these slow-fast cycles with at most one canard points, there are numerous models of the form (1.1) whose S-shaped critical manifolds have two canard points, such as the Tyson–Hong–Thron–Novak circadian oscillator model [3, 37], cubic Liénard equations with quadratic damping [12], predator-prey models of generalized Holling type III [23, 38] and so on. In these models, it is interesting to study the limit cycles bifurcating from a singular double canard cycle (also called two-layer canard slow-fast cycles [16]), that is, a slow-fast canard cycle passes through two layers of fast orbits and contains two generic Hopf breaking mechanisms at two non-degenerate canard points. See Fig. 2. Limit cycles bifurcating from a singular double canard cycles have two canards. For this reason, we refer to these canard cycles under consideration as double canard cycles.

The study of slow-fast cycles with two breaking mechanisms was initiated in [15], where Dumortier and Roussarie considered the limit cycles bifurcating from slow-fast cycles containing two breaking mech-
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Fig. 1 1a Common cycle. 1b Canard slow-fast cycle without head. 1c Canard slow-fast cycle with head. 1d Transitory canard

Fig. 2  Singular double canard cycle

anisms, one is the Hopf breaking mechanism and the other one is the jump breaking mechanism induced by a fast orbit connecting two jump points. Each of these two mechanisms is controlled by one parameter, so these cycles are also called canard slow-fast cycles with two breaking parameters. Later, the study made in [15] was generalized to canard slow-fast cycles with multiple breaking mechanisms by Dumortier and Roussarie [16], and canard slow-fast cycles of finite codimension with two breaking mechanisms by Mamouhdi and Roussarie [32]. By the results in [15, 16, 32], we can obtained the maximal number of limit cycles bifurcating from canard slow-fast cycles with two or multiple breaking mechanisms by analyzing the properties of the associated slow divergence integrals. However, the analysis in [15, 16, 32] did not give the explicit expressions of the associated breaking parameters and the parameter conditions for the existence of perturbed canard cycles in terms of system parameters.

Note that singular double canard cycles are special canard slow-fast cycles with two breaking parameters. As a complementary work of [15, 16, 32], this paper presents a systematic approach to the explicit determination of two breaking parameters and gives a sufficient condition under which three hyperbolic limit cycles bifurcate from a singular double canard cycle. The proofs for these results are based on the geometric singular perturbation theory. More specifically, we first give the detailed study of the local dynamics for canard points by the normal form theory, the blow-up technique and the extended Melnikov theory obtained by Wechselberger in [40]. And then we establish the existence of double canard cycles by the results in [15, 32]. As an application, we use the results to prove the existence of the configuration of three large limit cycle enclosing three equilibria in a cubic Liénard equation with quadratic damping. This configuration has never been observed previously in the existing results. See Sect. 6 for more details.

This paper is organized as follows. In Sect. 2, we make some hypotheses and state the results on the bifurcation of singular double canard cycles. In Sect. 3, we give the normal forms of system (1.2) near canard points. Section 4 is contributed to investigating the existence of canards near canard points by the Melnikov theory. We give the proof for the results on the bifurcation of singular double canard cycles in Sect. 5, and then, we apply the obtained results to cubic Liénard equations with quadratic damping in Sect. 6. We make some remarks in the final section.

2 Bifurcation of singular double canard cycles

In this section, we first introduce some essential hypotheses and then state the results on the bifurcation of singular double canard cycles. In order to define the nondegeneracy conditions for canard points, we first set \( \mu = (\lambda, \eta) = (\lambda, \eta_1, \ldots, \eta_m) \in \mathbb{R} \times \mathbb{R}^m. \) Now we assume that for a fixed \( \mu = \mu_0 = (\lambda_0, \eta_1, \ldots, \eta_m, 0) \), the function \( f \) satisfies the following hypotheses:

(H1) For a fixed \( \mu = \mu_0 \), there exists a smooth function \( \phi \) having precisely two different extreme points \( \alpha_1 \) and \( \alpha_2 \) with \( \alpha_1 < \alpha_2 \) such that the critical manifold \( C_0 := C_{\mu_0,0} \) is represented by

\[
C_0 := \left\{ (x, y) \in \mathbb{R}^2 : y = \phi(x) \right\}.
\]
(H2) At the points \((\alpha_j, \omega_j) := (\alpha_j, \phi(\alpha_j)), j = 1, 2\), the function \(f\) satisfies:

\[
\begin{align*}
& f(\alpha_j, \omega_j, \mu, 0) = 0, \quad \frac{\partial f}{\partial x}(\alpha_j, \omega_j, \mu, 0) = 0, \\
& \quad \frac{\partial^2 f}{\partial x^2}(\alpha_j, \omega_j, \mu, 0) \neq 0, \quad \frac{\partial f}{\partial y}(\alpha_j, \omega_j, \mu, 0) \neq 0.
\end{align*}
\]

(2.1)

and the following non-degenerate conditions:

\[
\frac{\partial^2 f}{\partial x^2}(\alpha_j, \omega_j, \mu, 0) \neq 0, \quad \frac{\partial f}{\partial y}(\alpha_j, \omega_j, \mu, 0) \neq 0.
\]

(H3) Along the critical manifold \(C_0\), the function \(f\) satisfies that

\[
\begin{align*}
& \frac{\partial f}{\partial x}(x, \phi(x), \mu, 0, 0) < 0 \text{ for } x \in (-\infty, \alpha_1) \cup (\alpha_2, +\infty), \\
& \quad \frac{\partial f}{\partial x}(x, \phi(x), \mu, 0, 0) > 0 \text{ for } x \in (\alpha_1, \alpha_2).
\end{align*}
\]

We remark that by (H2) and the Implicit Function Theorem, there is an open neighborhood \(U_\mu(\mu_0) \subset \mathbb{R}^{1+m}\) of \(\mu = \mu_0\) and exactly four \(C^0\) functions

\[
\begin{align*}
\tilde{x}_j(\mu) &= \alpha_j + \frac{f_x f_y - f_y f_{\alpha_j}}{f_y f_{xx}}(\alpha_j, \omega_j, \mu_0, 0) (\lambda - \lambda_0) \\
&\quad + \sum_{k=1}^{m} \frac{f_{\omega_k} f_x - f_x f_{\omega_k}}{f_y f_{xx}}(\alpha_j, \omega_j, \mu_0, 0) (\eta_k - \eta_0) \\
&\quad + O\left(\|\mu - \mu_0\|^2\right), \\
\tilde{y}_j(\mu) &= \omega_j + \frac{f_y f_{\alpha_j} - f_y f_{xx}}{f_y f_{xx}}(\alpha_j, \omega_j, \mu_0, 0) (\lambda - \lambda_0) \\
&\quad + \sum_{k=1}^{m} \frac{f_{\omega_k} f_x - f_x f_{\omega_k}}{f_y f_{xx}}(\alpha_j, \omega_j, \mu_0, 0) (\eta_k - \eta_0) \\
&\quad + O\left(\|\mu - \mu_0\|^2\right),
\end{align*}
\]

(2.2)

such that

\[
\begin{align*}
f(\tilde{x}_j(\mu), \tilde{y}_j(\mu), \mu, 0) &= 0, \quad \frac{\partial f}{\partial x}(\tilde{x}_j(\mu), \tilde{y}_j(\mu), \mu, 0) = 0, \quad \mu \in U_\mu(\mu_0), \quad j = 1, 2.
\end{align*}
\]

(2.4)

We can choose an appropriate set \(U_\mu(\mu_0)\) such that \((\tilde{x}_j(\mu), \tilde{y}_j(\mu))\) satisfy the non-degenerate conditions:

\[
\begin{align*}
& \frac{\partial^2 f}{\partial x^2}(\tilde{x}_j(\mu), \tilde{y}_j(\mu), \mu, 0) \neq 0, \quad \frac{\partial f}{\partial y}(\tilde{x}_j(\mu), \tilde{y}_j(\mu), \mu, 0) \neq 0, \\
& \quad \mu \in U_\mu(\mu_0), \quad j = 1, 2.
\end{align*}
\]

(2.5)

Then, we get two manifolds \(\mathcal{L}_\mu^0\) and \(\mathcal{R}_\mu^0\), which are parameterized by \(\mu\) and given by

\[
\begin{align*}
\mathcal{L}_\mu^0 &= \left\{ (\tilde{x}_1(\mu), \tilde{y}_1(\mu)) \in \mathbb{R}^2 : \mu \in U_\mu(\mu_0) \right\}, \\
\mathcal{R}_\mu^0 &= \left\{ (\tilde{x}_2(\mu), \tilde{y}_2(\mu)) \in \mathbb{R}^2 : \mu \in U_\mu(\mu_0) \right\}.
\end{align*}
\]

We refer to \(\mathcal{L}_\mu^0\) and \(\mathcal{R}_\mu^0\) as the contact point manifolds. Without loss of generality, assume that \(U_\mu(\mu_0) = \mathbb{R}^{1+m}\).

Under the above hypotheses, we see that \((\alpha_j, \omega_j)\) are both contact points. Our interest is to study the case that they are both canard points. Then, we further make the following hypotheses on the function \(g\).

(H4) For \(\mu = \mu_0\), the function \(g\) satisfies that

\[
\begin{align*}
& g(\alpha_j, \omega_j, \mu_0, 0) = 0, \\
& \quad \frac{\partial g}{\partial x}(\alpha_j, \omega_j, \mu_0, 0) \neq 0, \quad j = 1, 2,
\end{align*}
\]

and in the extended space \(\{(x, y, \lambda) \in \mathbb{R}^3\}\), the curves \(\mathcal{L}_\mu^0\) and \(\mathcal{R}_\mu^0\) transversely intersect the manifold given by \(g(x, y, \lambda, \eta_0, 0) = 0\) at \((\alpha_j, \omega_j, 0)\), that is,

\[
\begin{align*}
& G_j := \frac{\partial g}{\partial x}(\alpha_j, \omega_j, \lambda_0, \eta_0, 0) \cdot \frac{\partial \tilde{x}_j}{\partial \lambda}(\mu_0) \\
&\quad + \frac{\partial g}{\partial y}(\alpha_j, \omega_j, \lambda_0, \eta_0, 0) \cdot \frac{\partial \tilde{y}_j}{\partial \lambda}(\mu_0) \\
&\quad + \frac{\partial g}{\partial \lambda}(\alpha_j, \omega_j, \lambda_0, \eta_0, 0) \neq 0.
\end{align*}
\]

(H5) For \(\mu = \mu_0\), the function \(\phi\) reaches its minimum and maximum values at \(x = \alpha_1\) and \(x = \alpha_2\), respectively, system (1.1) has precisely one equilibrium \(E_0 := (x_m, y_m)\) on the section \(M := \{(x, y) : y = \phi(x), \alpha_1 < x < \alpha_2\}\), and \(E_0\) is a saddle. The slow motions governed by

\[
\phi'(x)\dot{x} = \phi'(x)\frac{dx}{d\tau} = g(x, \phi(x), \mu_0, 0),
\]

satisfy \(\dot{x} > 0\) for \(x < \alpha_1\) and \(\dot{x} < 0\) for \(x > \alpha_2\).

Under the hypotheses (H1)–(H5), for \(\mu = \mu_0\) both contact points are canard points. The dynamics of the limiting systems (1.3) and (1.4) are shown in Fig. 3a.

We next construct singular double canard cycles in the following way. For each \(s \in (0, \omega_2 - \omega_1)\), let the functions \(\alpha_L(s), \alpha_M(s)\) and \(\alpha_R(s)\) satisfy \(\alpha_L(s) < \alpha_1 < \alpha_M(s) < \alpha_2 < \alpha_R(s)\) and \(\phi(\alpha_L(s)) = \phi(\alpha_M(s)) = \phi(\alpha_R(s)) = \phi(\omega_1 + s)\). For each
\((s_1, s_2) \in \Omega := \{0 < s_2 < y_m - \omega_1 < s_1 < \omega_2 - \omega_1\}\), define a slow-fast cycle \(\Gamma(s_1, s_2)\) by (see Fig. 3b)
\[
\Gamma(s_1, s_2) = \begin{cases} 
(x, y) \in \mathbb{R}^2 : y = \phi(x), \ \alpha_L(s_1) \leq x \\
\cup \{x, \omega_1 + s_2 \in \mathbb{R}^2 : \alpha_M(x) \leq x \} \\
\cup \{x, y \in \mathbb{R}^2 : y = \phi(x), \ \alpha_M(s_1) \leq x \} \\
\cup \{x, \omega_1 + s_1 \in \mathbb{R}^2 : \alpha_L(s_1) \leq x \} \\
\end{cases} \leq \alpha_M(s_1) \}. 
\] (2.6)

The limit cycles bifurcating from \(\Gamma(s_1, s_2)\) is closely related to slow divergence integrals. Here for \(x_1\) and \(x_2\) in \(\mathbb{R}\), the slow divergence integrals \(I(x_1, x_2)\) along the critical manifold are defined by (see, for instance, \([7, 17, 27, 30, 31]\))
\[
I(x_1, x_2) = \int_{s_1}^{s_2} \frac{\partial f}{\partial x} (x, \phi(x), \mu_0, 0) \cdot \frac{\phi'(x)}{g(x, \phi(x), \mu_0, 0)} \, dx. 
\]

Four slow divergence integrals associated with \(\Gamma(s_1, s_2)\) are given by
\[
\begin{align*}
\mathcal{I}_1(s_1) &= I(\alpha_L(s_1), \alpha_1), \\
\mathcal{I}_2(s_2) &= I(\alpha_1, \alpha_M(s_2)), \\
\mathcal{I}_3(s_2) &= I(\alpha_R(s_2), \alpha_2), \\
\mathcal{I}_4(s_1) &= I(\alpha_2, \alpha_M(s_1)).
\end{align*}
\]

Before stating the main results, we define some important constants which are useful in the subsequent proof. Let the constants \(d_{j,r_2}, d_{j,\lambda_2}\) and \(d_{j,\eta_1}\) be defined by
\[
d_{j,r_2} = -\sqrt{\frac{\pi}{8}} (4a_{j,1} - a_{j,2} + 3a_{j,3} - 2a_{j,4} + 2a_{j,5}), \\
d_{j,\lambda_2} = -\sqrt{\frac{\pi}{8}}, \ d_{j,\eta_1} = \sqrt{\pi}a_{j,5+i}, \ i = 1, ..., m, \quad \text{(2.7)}
\]
where \(a_{j,i}\) are given by
\[
a_{j,1} = \frac{\partial \phi_{j,3}}{\partial x}(0), \\
a_{j,2} = \frac{\partial \phi_{j,1}}{\partial x}(0), \quad a_{j,3} = \frac{\partial \phi_{j,2}}{\partial x}(0), \\
a_{j,4} = \frac{\partial \phi_{j,4}}{\partial x}(0), \quad a_{j,5} = \phi_{j,6}(0), \\
a_{j,5+i} = \phi_{j,6+i}(0), \quad j = 1, 2, \quad i = 1, ..., m,
\]
and \(\phi_{j,i} = \phi_i\) whose lengthy expressions are given as in Lemma 3.1 for \((\alpha, \omega) = (\alpha_j, \omega_j)\). The results on the bifurcation of singular double canard cycles are summarized in the following.

**Theorem 2.1** Assume that system (1.1) satisfies (H1)–(H5), the curves \(\{(s_1, s_2) \in \Omega : \mathcal{I}_1(s_1) + \mathcal{I}_2(s_2) = 0\}\) and \(\{(s_1, s_2) \in \Omega : \mathcal{I}_3(s_2) + \mathcal{I}_4(s_1) = 0\}\) transversally intersects at the point \(\{(s_1^*, s_2^*) \in \Omega, \text{ and for a certain } i \in \{1, ..., m\}\}
\[
\text{Rank} \begin{pmatrix} -f_{xx} (P_1) (2g_x (P_1))^{-1} g_1 d_{1,1,2} d_{1,2,1} \\ -f_{xx} (P_2) (2g_x (P_2))^{-1} g_2 d_{2,1,2} d_{2,2,1} \end{pmatrix} = 2, \quad \text{(2.8)}
\]
where \(P_j = (\alpha_j, \omega_j, \mu_0, 0)\) and \(\beta_j(\mu) = g(\tilde{x}_j(\mu), \tilde{y}_j(\mu), \mu, 0)\) for \(\mu \in \mathbb{R}^{1-m}\) and \(j = 1, 2\). Then, there exists a sufficiently small \(\varepsilon_0\) and two continuous functions \(\lambda(\varepsilon)\) and \(\eta(\varepsilon)\) in the form
\[
\lambda(\varepsilon) = \lambda_0 + (A_1B_2 - A_2B_1)^{-1} \times (B_2C_1 - B_1C_2)\varepsilon + O(\varepsilon^{3/2}),
\]
\[
\eta_j(\varepsilon) = \eta_{j,0} + (A_1B_2 - A_2B_1)^{-1} \times (A_1C_2 - A_2C_1)\varepsilon + O(\varepsilon^{3/2}),
\]
for \(0 < \varepsilon < \varepsilon_0\), where \(A_j, B_j, C_j\) are given by
\[
A_j = -\frac{f_{xx}(P_j)g_{x}(P_j)}{2g_{s}(P_j)}d_{j,r_2}, \quad B_j = d_{j,n_2},
\]
\[
C_j = \frac{f_{y}(P_j)g_{x}(P_j)}{2g_{s}(P_j)}d_{j,r_2} + \frac{f_{xx}(P_j)g_{s}(P_j)}{2g_{s}(P_j)}d_{j,s_2}, \quad j = 1, 2,
\]
such that system (1.1) with \((\varepsilon, \mu) = (\varepsilon, \lambda(\varepsilon), \eta_1, 0, \ldots, \eta_{i-1,0}, \eta_0(\varepsilon), \eta_{i+1,0}, \ldots, \eta_{m,0})\) has a limit cycle \(\Gamma(\varepsilon, s^1_1, s^2_2)\) in a small neighborhood of \(\Gamma(s^1_1, s^2_2)\), and \(\Gamma(\varepsilon, s^1_1, s^2_2) \to \Gamma(s^1_1, s^2_2)\) as \(\varepsilon \to 0\) in the sense of Hausdorff distance. Moreover, for each fixed \(\varepsilon \in (0, \varepsilon_0)\), there exists a pair of \((\hat{\lambda}(\varepsilon), \hat{\eta}(\varepsilon))\) satisfying
\[
|\hat{\lambda}(\varepsilon) - \lambda(\varepsilon), \hat{\eta}(\varepsilon) - \eta(\varepsilon)| = O(e^{-K/\varepsilon}) \quad (2.10)
\]
for some positive constant \(K\) such that system (1.1) with \((\varepsilon, \mu) = (\varepsilon, \hat{\lambda}(\varepsilon), \eta_1, 0, \ldots, \eta_{i-1,0}, \hat{\eta}(\varepsilon), \eta_{i+1,0}, \ldots, \eta_{m,0})\) has three hyperbolic limit cycles bifurcating from \(\Gamma(\varepsilon, s^1_1, s^2_2)\).

After analyzing the local dynamics of canard points, we give the length proof for Theorem 2.1 in sect. 5. Note that \((\alpha_1, \alpha_{01})\) (resp. \((\alpha_2, \alpha_{02})\)) is a canard point when \(\beta_1(\mu) = 0\) (resp. \(\beta_2(\mu) = 0\)). Because canard points are closely related to Hopf bifurcations, the new parameters \(\beta_1\) and \(\beta_2\) are called the Hopf breaking parameters [15]. If (2.8) holds, then by the Implicit Function Theorem, we can also define the parameters \(\lambda\) and \(\eta_j\) as the Hopf breaking parameters. In the end of this section, we also remark that the preceding discussion also present a systematic approach to the determination of the Hopf breaking parameters for a general system with two canard points lying on the \(S\)-shaped critical manifolds.

3 Normal forms near canard points

Let the functions \(\tilde{g}_j\) associated with \((\alpha_j, \omega_j)\) be defined by
\[
\tilde{g}_j(x, y, \mu, \varepsilon) = g \left( x + \tilde{x}_j (\mu + \mu_0), y + \tilde{y}_j (\mu + \mu_0), \mu + \mu_0, \varepsilon \right), \quad j = 1, 2.
\]
By (H4), we obtain that the functions \(\tilde{g}_j\) satisfy the following:
\[
\tilde{g}_j(0, 0, \tilde{\lambda}_j(0), 0, \varepsilon) = 0, \quad \frac{\partial \tilde{g}_j}{\partial x}(0) \neq 0, \quad \frac{\partial \tilde{g}_j}{\partial \lambda}(0) \neq 0, \quad j = 1, 2.
\]
(3.11)

Whenever there is no confusion, the zero vector is denoted by \(0\). Then by the Implicit Function Theorem, there exists an open neighborhood \(U_\varepsilon(0) \subset \mathbb{R}\) of \(\varepsilon = 0\) and exactly two \(C^k\) functions
\[
\tilde{\lambda}_j(\varepsilon) = -\frac{\partial \tilde{g}_j}{\partial \varepsilon}(0) \left( \frac{\partial \tilde{g}_j}{\partial \lambda}(0) \right)^{-1} \varepsilon + O(\varepsilon^2), \quad (3.12)
\]
such that
\[
\tilde{g}_j(0, 0, \tilde{\lambda}_j(\varepsilon), 0, \varepsilon) = 0, \quad \varepsilon \in U_\varepsilon(0), \quad j = 1, 2.
\]
(3.13)

Without loss of generality, we assume that \(U_\varepsilon(0) = \mathbb{R}\). The normal forms of system (1.2) near the canard points are given in the next lemma.

Lemma 3.1 Suppose that the functions \(f\) and \(g\) in system (1.2), respectively, satisfy the conditions of (H2) and (H4) at \((\alpha_j, \omega_j), j = 1, 2\). For each \(j = 1, 2\), let the functions \(\tilde{f}\) and \(\tilde{g}\) be in the form
\[
\tilde{f}(x, y, \lambda, \eta, \varepsilon) = f \left( x + \tilde{x} \left( \left( \lambda + \tilde{\lambda}(\varepsilon), \eta \right) + \mu_0 \right), y + \tilde{y} \left( \left( \lambda + \tilde{\lambda}(\varepsilon), \eta \right) + \mu_0, \varepsilon \right) + \mu_0 \right),
\]
\[
\tilde{g}(x, y, \lambda, \eta, \varepsilon) = g \left( x + \tilde{x} \left( \left( \lambda + \tilde{\lambda}(\varepsilon), \eta \right) + \mu_0 \right), y + \tilde{y} \left( \left( \lambda + \tilde{\lambda}(\varepsilon), \eta \right) + \mu_0, \varepsilon \right) + \mu_0, \varepsilon \right),
\]
where the \(C^k\) functions \(\tilde{x} = \tilde{x}_j, \tilde{y} = \tilde{y}_j\) and \(\tilde{\lambda} = \tilde{\lambda}_j\) respectively, have the expansions (2.2), (2.3) and (3.12). Then near the point \((\alpha, \omega) = (\alpha_j, \omega_j), system (1.2) can be changed into the form
\[
x' = -y \left( 1 + \phi_1(x, y, \lambda, \eta, \varepsilon) \right) + x^2 \left( 1 + \phi_2(x, y, \lambda, \eta, \varepsilon) \right) + \varepsilon \phi_3(x, y, \lambda, \eta, \varepsilon),
\]
\[
y' = \varepsilon \left( \xi x \left( 1 + \phi_4(x, y, \lambda, \eta, \varepsilon) \right) - \lambda \left( 1 + \phi_5(x, y, \lambda, \eta, \varepsilon) \right) + y \phi_6(x, y, \lambda, \varepsilon) \right) + \sum_{i=1}^{m} \eta_i \phi_{6+i}(x, y, \lambda, \eta, \varepsilon),
\]
where \(\xi = \pm 1\), the \(C^k\) functions \(\phi_i\) are defined by
\[
\phi_1(x, y, \mu, \varepsilon) = \left( \tilde{f}_j(0) \right)^{-1} \phi_1 \circ T(x, y, \mu, \varepsilon),
\]
\[
\phi_2(x, y, \mu, \varepsilon) = \left( \tilde{f}_j(0) \right)^{-1} \phi_2 \circ T(x, y, \mu, \varepsilon),
\]
\[
\phi_3(x, y, \mu, \varepsilon) = \left( \tilde{f}_j(0) \right)^{-1} \phi_3 \circ T(x, y, \mu, \varepsilon),
\]
\[
\phi_4(x, y, \mu, \varepsilon) = \left( \tilde{f}_j(0) \right)^{-1} \phi_4 \circ T(x, y, \mu, \varepsilon),
\]
\[
\phi_5(x, y, \mu, \varepsilon) = \left( \tilde{f}_j(0) \right)^{-1} \phi_5 \circ T(x, y, \mu, \varepsilon),
\]
\[
\phi_6(x, y, \mu, \varepsilon) = \left( \tilde{f}_j(0) \right)^{-1} \phi_6 \circ T(x, y, \mu, \varepsilon),
\]
\[
\phi_{6+i}(x, y, \mu, \varepsilon) = \left( \tilde{f}_j(0) \right)^{-1} \phi_{6+i} \circ T(x, y, \mu, \varepsilon), \quad i = 1, 2, \ldots, m.
\]
\[
\phi_2(x, y, \mu, \varepsilon) = 2(f_0(x, 0))^{-1}\hat{\phi}_2 \circ \mathcal{T}(x, y, \mu, \varepsilon),
\]
\[
\phi_3(x, y, \mu, \varepsilon) = -\zeta \hat{f}_x(x, 0)(2\hat{f}_y(x, 0))^{-1}
\hat{\phi}_3 \circ \mathcal{T}(x, y, \mu, \varepsilon),
\]
\[
\phi_4(x, y, \mu, \varepsilon) = (\hat{g}_x(x, 0))^{-1}\hat{\phi}_4 \circ \mathcal{T}(x, y, \mu, \varepsilon),
\]
\[
\phi_5(x, y, \mu, \varepsilon) = (\hat{g}_x(x, 0))^{-1}\hat{\phi}_5 \circ \mathcal{T}(x, y, \mu, \varepsilon),
\]
\[
\phi_6(x, y, \mu, \varepsilon) = -\zeta (\hat{f}_x(x, 0))^{-1}
\hat{\phi}_6 \circ \mathcal{T}(x, y, \mu, \varepsilon),
\]
\[
\phi_{6+j}(x, y, \mu, \varepsilon) = \zeta \hat{f}_x(x, 0)(2\hat{g}_x(x, 0))^{-1}
\left(\hat{g}_{6+j}(x, 0) + \hat{\phi}_{6+j} \circ \mathcal{T}(x, y, \mu, \varepsilon)\right),
\]
\]
\[
\text{the function } \hat{\phi}_j \text{ have the form}
\]
\[
\hat{\phi}_1(x, y, \mu, \varepsilon) = -\hat{f}_y(x, 0) + \int_0^1 D_2 \ddot{f}(ux, uy, \mu, ue) du
\]
\[
+ x \int_0^1 \int_0^1 uD_{12} \dddot{f}(ux, uy, \mu, ue) dudv,
\]
\[
\hat{\phi}_2(x, y, \mu, \varepsilon) = -\frac{1}{2} \hat{f}_x(x, 0)
\]
\[
+ \int_0^1 \int_0^1 uD_{11} \ddot{f}(ux, uy, \mu, ue) dudv,
\]
\[
\hat{\phi}_3(x, y, \mu, \varepsilon) = -\hat{f}_y(x, 0) + \int_0^1 D_{4+m} \ddot{f}(ux, uy, \mu, ue) du
\]
\[
+ x \int_0^1 \int_0^1 uD_{4+m} \dddot{f}(ux, uy, \mu, ue) dudv,
\]
\[
\hat{\phi}_4(x, y, \mu, \varepsilon) = -\hat{g}_y(x, 0) + \int_0^1 D_1 \ddot{g}(ux, uy, \mu, ue) du,
\]
\[
\hat{\phi}_5(x, y, \mu, \varepsilon) = -\hat{g}_y(x, 0) + \int_0^1 D_2 \dddot{g}(ux, uy, \mu, ue) dudv,
\]
\[
\hat{\phi}_6(x, y, \mu, \varepsilon) = -\hat{g}_y(x, 0) + \int_0^1 D_{3+j} \dddot{g}(ux, uy, \mu, \eta, \varepsilon) dudv,
\]
\]
\[
\text{the transformation } \tilde{T} : \mathbb{R}^{4+m} \to \mathbb{R}^{4+m} \text{ is in the form}
\]
\[
\tilde{T}(x, y, \lambda, \eta, \varepsilon) = \left(\frac{2}{\hat{f}_x(x, 0)}, x - \frac{2}{\hat{f}_x(x, 0)} \hat{f}_y(x, 0),
\right.
\]
\[
\left. - \frac{2\zeta \hat{g}_x(x, 0)}{\hat{f}_x(x, 0)} \lambda, \eta, - \frac{\zeta}{\hat{f}_x(x, 0)} \varepsilon\right),
\]
\]
\[
\text{and } D_j \text{ denotes the } j-th \text{ partial derivative with respect to the } j-th \text{ variable, and } D_{1j} \text{ with } i = 1, j = 1, 2, 4+m, \text{ are in the form } D_{1j} = D_j \circ D_i.
\]
\[
\text{Proof } \text{ It suffices to study the following system}
\]
\[
\frac{dx}{dt} = x' = \bar{f}(x, y, \mu, \varepsilon),
\]
\[
\frac{dy}{dt} = y' = \varepsilon \bar{g}(x, y, \mu, \varepsilon).
\]
Similarly, note that $g(0,0,0,0) = 0$ for $\varepsilon \in U_\varepsilon(0)$, then $g$ can be written as the form
\[ g(x, y, \lambda, \eta, \varepsilon) = g(x, y, \lambda, \eta, \varepsilon) - g(0, 0, 0, 0, \varepsilon) = x \left( \hat{g}_x(0) + \hat{\phi}_1(x, y, \lambda, \eta, \varepsilon) \right) + \lambda \left( \hat{g}_y(0) + \hat{\phi}_2(x, y, \lambda, \eta, \varepsilon) \right) + y \left( \hat{g}_y(0) + \hat{\phi}_2(x, y, \lambda, \eta, \varepsilon) \right) + \sum_{j=1}^{m} \eta_j \left( \hat{g}_{\eta_j}(0) + \hat{\phi}_{\eta_j}(x, y, \lambda, \eta, \varepsilon) \right), \]
(3.19)
where the functions $\hat{\phi}_i, i = 4, \ldots, 6 + m$, are defined as in this lemma. Clearly, the functions $\hat{\phi}_1$ and $\hat{\psi}_j, i = 1, 2, 3, j = 1, \ldots, 3 + m$, are $C^k$ and satisfy $\hat{\phi}_1(0) = 0$ and $\hat{\psi}_j(0) = 0$. Then by taking the transformation $\hat{T}$, we obtain the normal form (3.14). Therefore, the proof is now complete.

4 Canard solutions

In order to study two canards near canard points $(x_j, \omega_j)$, we consider the dynamics of system (3.14) near the origin for sufficiently small $\| (\lambda, \eta, \varepsilon) \|$. We first take a quasi-homogeneous blow-up transformation $\Pi$ of the form
\[ x = r \tilde{x}, \quad y = r^2 \tilde{y}, \quad \varepsilon = r^2 \tilde{\varepsilon}, \quad \lambda = \tilde{\lambda}, \quad \eta_i = r \tilde{\eta}_i, \quad i = 1, \ldots, m. \]
In the chart $K_2$, this blow-up transformation is reduced to $\Pi_2$ of the form
\[ x = r_2 x_2, \quad y = r_2^2 y_2, \quad \varepsilon = r_2^2 \tilde{\varepsilon}, \quad \lambda = r_2 \tilde{\lambda}, \quad \eta_i = r_2 \tilde{\eta}_i \quad i = 1, \ldots, m. \]
(4.1)
By substituting (4.1) into (3.14) and taking a time rescaling, system (3.14) is changed into
\[ \dot{x}_2(x_2, y_2, r_2, \lambda_2, \eta_2) : \]
\[ \dot{x}_2' = -y_2 + x_2^2 + r_2 \left( a_1 x_2 - a_2 x_2 y_2 + a_3 x_2^3 \right) + O \left( r_2 (r_2 + \lambda_2 + \sum_{i=1}^{m} \eta_2, i) \right), \]
(4.2)
\[ \dot{y}_2 = x_2 - \lambda_2 + r_2 \left( a_4 x_2^2 + a_5 y_2 \right) + \sum_{i=1}^{m} a_{5+i} \eta_2, i + O \left( r_2 \left( r_2 + \lambda_2 + \sum_{i=1}^{m} \eta_2, i \right) \right), \]
where the constants $a_i$ are given by
\[ a_1 = \frac{\partial \phi_1}{\partial x}(0), \quad a_2 = \frac{\partial \phi_1}{\partial x^2}(0), \quad a_3 = \frac{\partial \phi_2}{\partial x^3}(0), \]
\[ a_4 = \frac{\partial \phi_1}{\partial x^4}(0), \quad a_5 = \phi_0(0), \quad a_{5+i} = \phi_{0+i}(0), \quad i = 1, \ldots, m. \]
When $r_2 = 0, \lambda_2 = 0, \eta_2 = 0$, system (4.2) is reduced to an integral system
\[ \dot{X}_2(x_2, y_2) : \quad \dot{x}_2' = -y_2 + x_2^2, \quad \dot{y}_2 = x_2. \]
(4.3)
The solutions of this integral system are determined by the level curves of the function $H$, which is in the form
\[ H(x_2, y_2) := \frac{1}{2} e^{-2y_2} (y_2 - x_2^2 + \frac{1}{2}), \quad (x_2, y_2) \in \mathbb{R}^2. \]
System (4.3) has a solution $\gamma(t), t \in \mathbb{R}$, of the form
\[ \gamma(t) = \left( \frac{1}{2} t, \frac{1}{4} t^2 - \frac{1}{2} \right), \quad t \in \mathbb{R}. \]
(4.4)

By applying the Poincaré compactification (see, for instance, [42, section V.1, p.321]), we can obtain the phase portrait of this integral system in the Poincaré disk, which is shown in Fig. 4a. We see that $\gamma$ is a heteroclinic orbit connecting two infinite equilibria. Let $\gamma_a = \{ \gamma(t) : t \leq 0 \}$ and $\gamma_r = \{ \gamma(t) : t \geq 0 \}$. Consider the perturbed system (4.2) of the integral system (4.3). Assume that the heteroclinic orbit $\gamma$ is broken into $\gamma_{a,p}$ and $\gamma_{r,p}$. By the continuous dependency on parameters, the orbits $\gamma_{a,p}$ and $\gamma_{r,p}$ transversely intersect $y_2$-axis at $(0, y_{2,a})$ and $(0, y_{2,r})$, which are in a small neighborhood of $(0, -1/2)$. See Fig. 4b.

Clearly, the constants $y_{2,a}$ and $y_{2,r}$ depend on the parameters $r_2, \lambda_2$ and $\eta_2$. Here, we write $y_{2,a}$ and $y_{2,r}$ for simplicity. To investigate the persistence of the heteroclinic orbit $\gamma$, we define the so-called distance function $D(r_2, \lambda_2, \eta_{2,1}, \ldots, \eta_{2,m})$ by
\[ D(r_2, \lambda_2, \eta_{2,1}, \ldots, \eta_{2,m}) = y_{2,a} - y_{2,r}. \]
If $D(r_2, \lambda_2, \eta_{2,1}, \ldots, \eta_{2,m}) = 0$ for some suitable parameters, then the heteroclinic orbit $\gamma$ is persistent. Note that the heteroclinic orbit $\gamma$ in the integral system (4.3) is unbounded. Then, the major obstacle is to make sure whether the distance function $D(r_2, \lambda_2, \eta_{2,1}, \ldots, \eta_{2,m})$ can be given by the classical Melnikov computation (see, for instance, [4, 21, 33]). This problem can be solved by the method obtained by Wechselberger in [40]. Roughly speaking, for the
extended system of (4.2) in the form
\[
\begin{align*}
x_2' &= -y_2 + x_2^2 + r_2 \left( a_1 x_2 - a_2 x_2 y_2 + a_3 x_2^3 \right) + O \left( r_2 \left( r_2 + \lambda_2 + \sum_{i=1}^{m} \eta_{2,i} \right) \right), \\
y_2' &= x_2 - \lambda_2 + r_2 \left( a_4 x_2^2 + a_5 y_2 \right) + \sum_{i=1}^{m} a_{5+i} \eta_{2,i} + O \left( r_2 \left( r_2 + \lambda_2 + \sum_{i=1}^{m} \eta_{2,i} \right) \right), \\
r_2' &= 0, \quad \lambda_2' = 0, \quad \eta_{2,i}' = 0, \quad i = 1, \ldots, m,
\end{align*}
\]
(4.5)

if all solutions of the extended system (4.5) near the heteroclinic orbit \( \gamma \) are of at most algebraic growth for \( t \to \pm \infty \). Then, the distance function \( D(r_2, \lambda_2, \eta_{2,1}, \ldots, \eta_{2,m}) \) can be similarly obtained as in the classical case. More precisely, we have the following lemma.

**Lemma 4.1** The distance function \( D(r_2, \lambda_2, \eta_{2,1}, \ldots, \eta_{2,m}) \) has the expansion
\[
\begin{align*}
D(r_2, \lambda_2, \eta_{2,1}, \ldots, \eta_{2,m}) &= d_{r_2} r_2 + d_{\lambda_2} \lambda_2 + \sum_{i=1}^{m} d_{\eta_{2,i}} \eta_{2,i} + O \left( |(r_2, \lambda_2, \eta_{2,1}, \ldots, \eta_{2,m})|^2 \right),
\end{align*}
\]
where
\[
d_{r_2} = -\frac{\sqrt{2\pi}}{8} \left( 4a_1 - a_2 + 3a_3 - 2a_4 + 2a_5 \right),
\]
\[
d_{\lambda_2} = -\sqrt{2\pi}, \quad d_{\eta_{2,i}} = \sqrt{2\pi} a_{5+i}, \quad i = 1, \ldots, m.
\]
Furthermore, the distance function \( D \) is \( C^k \) smooth.

**Proof** By a direct computation, the linearization of system (4.3) about the heteroclinic orbit \( \gamma \) is in the form
\[
\begin{align*}
u' &= tu - v \quad \text{and} \quad v' = u,
\end{align*}
\]
and the vector \( \tilde{\gamma}(0) = (0, 1) \). Then by applying Theorem 1 and Proposition 1 in [40], the distance function \( D(r_2, \lambda_2, \eta_{2,1}, \ldots, \eta_{2,m}) \) is \( C^k \) and has the expansion
\[
D(r_2, \lambda_2, \eta_{2,1}, \ldots, \eta_{2,m}) = d_{r_2} r_2 + d_{\lambda_2} \lambda_2 + \sum_{i=1}^{m} d_{\eta_{2,i}} \eta_{2,i} + O \left( |(r_2, \lambda_2, \eta_{2,1}, \ldots, \eta_{2,m})|^2 \right),
\]
where the coefficients \( d_{r_2}, d_{\lambda_2} \) and \( d_{\eta_{2,i}} \) are given by
\[
d_{r_2} = \frac{\partial D}{\partial r_2}(0)
\]
\[
= \int_{-\infty}^{+\infty} \langle \tilde{\gamma}(t), \frac{\partial X_2}{\partial r_2}(\gamma(t), 0) \rangle \, dt
\]
\[
= -\frac{\sqrt{2\pi}}{8} \left( 4a_1 - a_2 + 3a_3 - 2a_4 + 2a_5 \right).
\]
\[ d_{h_2} = \frac{\partial D}{\partial \lambda_2}(0) \]
\[ = \int_{-\infty}^{+\infty} \langle \dot{y}(t), \frac{\partial X_2}{\partial \lambda_2}(y(t), 0) \rangle \, dt = -\sqrt{2}\pi, \]
\[ d_{\eta_2,j} = \frac{\partial D}{\partial \eta_{2,i}}(0) \]
\[ = \int_{-\infty}^{+\infty} \langle \dot{y}(t), \frac{\partial X_2}{\partial \eta_{2,i}}(y(t), 0) \rangle \, dt \]
\[ = \sqrt{2}\pi d_{5+i}, \quad i = 1, ..., m, \]
the symbol \( \langle \cdot, \cdot \rangle \) denotes the inner product of two vectors. Therefore, the proof is now complete. \( \square \)

By Lemma 4.1, we can obtain the simultaneous occurrence of two canard solutions near \((\alpha_j, \omega_j), j = 1, 2.\)

**Lemma 4.2** Suppose that system (1.1) satisfies the hypotheses (H1)–(H5), and

\[ \text{Rank } \begin{pmatrix} -f_{xx}(P_1)(2g_x(P_1))^{-1}G_{d_1j, \lambda_2, \eta_{2,1}, ..., \eta_{2,m}} \\ -f_{xx}(P_2)(2g_x(P_2))^{-1}G_{d_2j, \lambda_2, \eta_{2,1}, ..., \eta_{2,m}} \end{pmatrix} = 2. \]

(4.6)

Then, there exists a \( C^k \) function \( \mu(\epsilon) \) defined on \((0, \epsilon_1)\) for a small \( \epsilon_1 > 0 \) such that two canard solutions occur concurrently near the canard points \((\alpha_j, \omega_j), j = 1, 2.\)

**Proof** By Lemma 3.1, we obtain that by the translations \( T_j \) of the form

\[ T_j(x, y, \lambda, \eta, \epsilon) = \left( x + \tilde{x}_j \left( (\lambda + \tilde{\lambda}_j(\epsilon), \eta) + \mu_0 \right), y + \tilde{y}_j \left( (\lambda + \tilde{\lambda}_j(\epsilon), \eta) + \mu_0, \epsilon \right), \right. \]

and then by the rescaling \( S_j \) of the form

\[ S_j(x, y, \lambda, \eta, \epsilon) = \left( \frac{2}{f_{xx}(P_j)} x, - \frac{2}{f_{xx}(P_j) f_y(P_j)} y, \right. \]

\[ \left. - \frac{2g_x(P_j)}{f_{xx}(P_j) g_x(P_j)} \lambda, \eta, \right) \]

\[ - \left( \frac{1}{f_y(P_j) g_x(P_j)} \right) \epsilon; \]

where \( P_j = (\alpha_j, \omega_j, \mu_0, 0) \), and \( \tilde{x}_j, \tilde{y}_j \) and \( \tilde{\lambda}_j \) are, respectively, given by (2.2), (2.3) and (3.12), system (1.2) near the canard points \((\alpha_j, \omega_j)\) can be changed into (3.14). Similarly to Lemma 4.1, the distance functions \( D_j \) associated with the canard points \((\alpha_j, \omega_j)\) have the expansions

\[ D_j \left( r_2, \lambda_2, \eta_{2,1}, ..., \eta_{2,m} \right) = d_{j,r_2} + d_{j,\lambda_2} \]

\[ + \sum_{i=1}^{m} d_{j,\eta_i, \eta_{2,i}} + O \left( \left( r_2, \lambda_2, \eta_{2,1}, ..., \eta_{2,m} \right)^2 \right), \]

(4.7)

where \( d_{j,r_2}, d_{j,\lambda_2} \) and \( d_{j,\eta_{2,i}} \) are given by (2.7). Let a projection \( P_2 \) be defined by \( P_2(x_2, y_2, \lambda_2, \eta_2) = (r_2, \lambda_2, \eta_2) \). Then to finish the proof, it suffices to solve the equations \( D_j \circ P_2 \circ \Pi_2 \circ S_j \circ T_j(x, y, \lambda, \eta, \epsilon) = 0 \), which are equivalent to the existence of solutions for equations

\[ f_y(P_j, \mu_0, 0, g_x(P_j)) \left( d_{j,\lambda_2} \right( \lambda - \lambda_0 = \tilde{\lambda}_j(\epsilon) \right) \]

\[ + \frac{G_j}{2g_x(P_j)} \sum_{i=1}^{m} d_{j,\eta_i, \eta_{2,i}} + O \left( \left( \epsilon, \mu - \mu_0 \right) \right) = 0, \quad j = 1, 2. \]

(4.8)

where \( G_j \) are defined as in (H4). Since (4.6) holds, there exists a certain \( i \in \{1, ..., m\} \) such that

\[ \det \left( \begin{pmatrix} -f_{xx}(P_1)(2g_x(P_1))^{-1}G_{d_1j, \lambda_2, \eta_{2,1}, ..., \eta_{2,m}} \\ -f_{xx}(P_2)(2g_x(P_2))^{-1}G_{d_2j, \lambda_2, \eta_{2,1}, ..., \eta_{2,m}} \end{pmatrix} \right) \neq 0. \]

This together with the Implicit Function Theorem yields that there is an open interval \((0, \epsilon_2)\) for a small \( \epsilon_2 > 0 \) and exactly two \( C^k \) functions \( \tilde{\lambda}(\epsilon) \) and \( \tilde{\eta}_i(\epsilon) \) having the same expansions as in (2.9), such that for each \( \epsilon \in (0, \epsilon_2) \) equations (4.8) have a solution \((\epsilon, \lambda, \eta_1, ..., \eta_m) = (\epsilon, \tilde{\lambda}(\epsilon), \eta_{1,0}, ..., \eta_{m-1,0}, \tilde{\eta}_i(\epsilon), \eta_{i+1,0}, ..., \eta_{m,0}) \). Thus, the proof is now complete. \( \square \)

**Remark 4.1** A standard method to deal with the existence of canard solutions near a canard point is the blow-up technique. See, for instance, [10, 15–17, 26, 32]. We also adopt this technique to prove the coexistence of two canard solutions. After observing that the canard solutions correspond to the persistent heteroclinic orbits in the Poincaré disk, we only consider system (3.14) in the chart \( K_2 \) instead of introducing different charts as previously done in [10, 26], and give the conditions under which two canard solutions appear simultaneously by the extended Melnikov method [40].

5 Proof of Theorem 2.1

In this section, we give the proof for Theorem 2.1 by Lemma 4.1 and [32, Theorem 12]. We will see that there exists a codimension 2 limit cycle bifurcating from a
singular double canard cycle. We refer the readers to [16, 32] for the precise definition of the codimension of a limit cycle.

Proof of Theorem 2.1. We first prove that under some suitable conditions, there exists a codimension 2 limit cycle bifurcating from \( \Gamma(s^*_1, s^*_2) \), from which three hyperbolic limit cycles can arise. Since (4.6) holds, there exists a local diffeomorphism \( \mathcal{F} \) transforming \( \partial := (\lambda, \eta) \) near \( (\lambda_0, \eta_0, 0) \) to \( (\beta_1, \beta_2) = (\beta_1(\partial), \beta_2(\partial)) \) such that the canard point \((\alpha_1, \omega_1)\) appears when \( \beta_1 = 0 \), and the other canard point \((\alpha_2, \omega_2)\) appears when \( \beta_2 = 0 \). No confusion would arise, here we still use \((\beta_1, \beta_2)\) to denote the two breaking parameters for convenience. Note that the curves \( \{(s_1, s_2) \in \Omega : I_1(s_1) + I_2(s_2) = 0\} \) and \( \{(s_1, s_2) \in \Omega : I_3(s_2) + I_4(s_1) = 0\} \) transversally intersect at the point \((s^*_1, s^*_2) \in \Omega \). Then \( \Gamma(s^*_1, s^*_2) \) is codimension 2 (see [32, Definition 7]). By [32, Theorem 12] there exists a sufficiently small \( \varepsilon_0 \) and two continuous functions \( \beta_1(\varepsilon) \) and \( \beta_2(\varepsilon) \) for \( 0 < \varepsilon < \varepsilon_0 \) such that system (1.1) with \( (\lambda, \eta) = \mathcal{F}^{-1}(\beta_1(\varepsilon), \beta_2(\varepsilon)) \) and \( \eta = \eta_{j,0} \) for \( j \neq i \) has a codimension 2 limit cycle \( \Gamma_\varepsilon(s^*_1, s^*_2) \). Furthermore, this perturbed limit cycle \( \Gamma_\varepsilon(s^*_1, s^*_2) \) can produce three hyperbolic limit cycles by slightly perturbing \((\lambda, \eta)\) again.

Secondly, we give the explicit expansion of \( (\lambda(\varepsilon), \eta(\varepsilon)) \) with

\[
(\lambda(\varepsilon), \eta(\varepsilon)) = \mathcal{F}^{-1}(\beta_1(\varepsilon), \beta_2(\varepsilon)), \quad 0 < \varepsilon < \varepsilon_0.
\]

Fix \( \eta_{j,0} \) for \( j \neq i \) and set \( P_j = (\alpha_j, \omega_j + s^*_j), j = 1, 2 \). Let the forward orbits and the backward orbits of \( P_j \) under the flow of system (1.2) be, respectively, denoted by \( \gamma_j, f(t), t \geq 0 \), and \( \gamma_j, b(t), t \leq 0 \), which satisfy \( \gamma_j, f(0) = \gamma_j, b(0) = P_j \). For each \( j = 1, 2 \), we take a small open neighborhood \( V_j \) of the canard points \((\alpha_j, \omega_j)\) such that near the point \((\alpha_j, \omega_j)\), system (1.2) can be changed into (3.14). By the Fenichel Theorem [20, Theorem 9.1], there exist two open sets \( I_{i, f} := (a_{i, f}^+, b_{i, f}^+) \subset \mathbb{R}^+ \) and \( I_{i, b} := (a_{i, b}^-, b_{i, b}^-) \subset \mathbb{R}^- \) such that for \( t \in I_{i, f} \) and \( \gamma_{i, f}(t) \notin V_j \), for \( t \in I_{i, b} \) and \( \gamma_{i, b}(t) \notin V_j \), for \( \gamma_{1, f}(t) \in V_j \), there exists \( \gamma_{1, f}(t) \notin V_j \), for \( \gamma_{1, b}(t) \in V_j \), and there exists \( \gamma_{1, b}(t) \notin V_j \), for \( \gamma_{2, f}(t) \in V_j \), and \( \gamma_{2, f}(t) \notin V_j \), for \( \gamma_{2, b}(t) \leq t \leq 0 \), and \( \gamma_{2, b}(t) \leq t \leq 0 \), and \( \gamma_{2, b}(t) \notin V_j \), for \( \gamma_{2, b}(t) \notin V_j \), for \( \gamma_{2, b}(t) \leq t \leq 0 \). See Fig. 5.

Consider system (1.2) in the sets \( V_j \). Following the discussion in Sect. 4, system (1.2) can be transformed into the similar systems as (4.5). Let these two transformed systems be, respectively, denoted by \( X_{2, j}(x_1, x_2, \gamma, \eta) \), which are the perturbations of the integral system \( X_j(x_1, x_2, \gamma, \eta) \). To distinguish two canard points \((\alpha_j, \omega_j), j = 1, 2 \), let the heteroclinic orbit \( \gamma = \gamma_i \cup \gamma_r \), which is the perturbation of system (1.2) with \( \gamma \rightarrow \gamma_i \), \( \gamma \rightarrow \gamma_r \). Define the perturbations of \( \gamma_i \) and \( \gamma_r \) be \( \gamma_i, p \) and \( \gamma_r, p \), which intersect \( x_j = 0 \) at \( (0, y_{j, a}) \) and \( (0, y_{j, r}) \), respectively. Let the intersection point of the transformed orbits of \( \gamma_{i, f} \) (resp. \( \gamma_{r, b} \)) and \( x_j = 0 \) be denoted by \( (0, y_{j, f}) \) (resp. \( (0, y_{j, r}) \)). By [20, Theorem 9.1], the slow manifolds are exponentially close to normally hyperbolic branches of the critical manifold \( C_0 \). By the similar argument as in [26, Lemma 5.1], we can obtain that there exists a positive constant \( k \) such that

\[
|y_{j, a} - y_{j, f}| = O(e^{-k/r_j}),
\]

\[
|y_{j, r} - y_{j, b}| = O(e^{-k/r_j}),
\]

\[
|y_{j, a} - y_{j, b}| = O(e^{-k/r_j})
\]

and the partial derivatives of \( y_{j, a} - y_{j, f}, y_{j, r} - y_{j, b}, y_{j, a} - y_{j, r}, y_{j, b} - y_{j, r} \) with respect to \( r_j, \lambda_j \) and \( \eta_{j,i}, i = 1, ..., m \), have the similar estimates as above.

Note that there exists a double canard cycle bifurcating from \( \Gamma(s^*_1, s^*_2) \) if and only if

\[
\tilde{D}_1(r_2, \lambda_2, \eta_{2,i}) := y_{2, a} - y_{2, r} = 0,
\]

\[
\tilde{D}_2(r_2, \lambda_2, \eta_{2,i}) := y_{2, r} - y_{2, b} = 0.
\]

Set \( \tilde{D}_j(r_2, \lambda_2, \eta_{2,i}) := y_{j, a} - y_{j, r} \). Note that

\[
\tilde{D}_1(r_2, \lambda_2, \eta_{2,i}) = D_1(r_2, \lambda_2, \eta_{2,i})
\]
Then, there exists a double canard cycle bifurcating from $\Gamma(x_i^*, s_j^2)$ if and only if two equations with the expansions (4.8) has solutions. Let $\eta_j = \eta_{j,0}$ for $j \neq i$.

Then by (2.8) and the Implicit Function Theorem, there exists a sufficiently small $\varepsilon_0$ and a unique continuous curve $\mu(\varepsilon) := (\varepsilon, \tilde{\lambda}(\varepsilon), \eta_{i,0}, \ldots, \eta_{i-1,0}, \tilde{\eta}_{i}(\varepsilon), \eta_{i+1,0}, \ldots, \eta_{m,0})$ for $0 < \varepsilon < \varepsilon_0$ such that (4.8) has the solution $\mu = \mu(\varepsilon)$. Then by the uniqueness we have that $(\tilde{\lambda}(\varepsilon), \eta_i(\varepsilon)) = F^{-1}(\beta_1(\varepsilon), \beta_2(\varepsilon)) = (\tilde{\lambda}(\varepsilon), \tilde{\eta}_i(\varepsilon))$ for $0 < \varepsilon < \varepsilon_0$. Finally, by the second statement in [32,Theorem 12] and the preceding discussion, we obtain (2.10). This finishes the proof. □

6 Applications to Liénard equations

In this section, we apply Theorem 2.1 to a singularly perturbed Liénard equation

$$
\frac{dx}{dt} = x' = \tilde{F}(x) - y, \\
\frac{dy}{dt} = y' = \varepsilon \left( \tilde{\eta} + \tilde{\lambda} x - \tilde{F}(x) \right),
$$

(6.1)

where $(x, y) \in \mathbb{R}^2$, $\tilde{F}(x)$ is a cubic polynomial with the first-order derivative $F'(x)$, the parameters $\tilde{\eta}$, $\tilde{\lambda}$ and $\varepsilon$ are real and $\varepsilon > 0$ is sufficiently small. Liénard equation (6.1) can turn to a scalar equation of the form

$$
x'' + \tilde{F}'(x)x' + P(x) = 0,
$$

where $\tilde{F}(x)$ and $P(x) = -\varepsilon(\tilde{\eta} + \tilde{\lambda} x - \tilde{F}(x))$ are polynomials of degree two and three, respectively. Then system (6.1) is called a cubic Liénard equation with quadratic damping (also called a Liénard equation of type (3, 2)). We refer to [1, 2, 11, 12, 39] and the references therein for more details on this class of Liénard equations.

Note that for each $(x, y) \in \mathbb{R}^2$,

$$
\frac{\partial}{\partial x} \left( \tilde{F}(x) - y \right) + \frac{\partial}{\partial y} \left( \varepsilon \left( \tilde{\eta} + \tilde{\lambda} x - \tilde{F}(x) \right) \right) = \tilde{F}'(x),
$$

(6.2)

then by Bendixson’s Theorem [13,Theorem 7.10, p.188], system (6.1) has no limit cycles in $\mathbb{R}^2$ if either $\tilde{F}'(x) \geq 0$ or $\tilde{F}'(x) \leq 0$ holds for all $x \in \mathbb{R}$. Thus, an essential assumption for the existence of limit cycles is that $\tilde{F}$ has precisely two different extreme points. This implies that the critical manifold of (6.1) is $S$-shaped.

By some changes, we can check that system (6.1) with this essential assumption is equivalent to

$$
x' = F(x) - y =: f(x, y), \\
y' = \varepsilon \left( \eta + \left( \lambda + \frac{1}{6} \right) x - F(x) \right) =: \varepsilon g(x, \eta, \lambda).
$$

(6.3)

where the parameters $\eta$, $\lambda$ and $\varepsilon$ are real, $\varepsilon > 0$ is sufficiently small, and $F(x)$ is in the form

$$
F(x) = -\frac{1}{3}x^3 + \frac{1}{2}x, \quad x \in \mathbb{R}.
$$

Then, the first-order derivative $F'$ of the cubic polynomial $F$ is given by

$$
F'(x) = -x^2 + \frac{1}{4}, \quad x \in \mathbb{R}.
$$

Letting $s = \varepsilon t$, system (6.3) is changed into the form

$$
\varepsilon \frac{dx}{ds} = \varepsilon \dot{x} = F(x) - y = f(x, y), \\
\frac{dy}{ds} = \dot{y} = \left( \eta + \left( \lambda + \frac{1}{6} \right) x - F(x) \right) = \varepsilon g(x, \eta, \lambda).
$$

(6.4)

The critical manifold $C_0$ corresponds to $X_{\eta, \lambda}$ is in the form

$$
C_0 = \left\{ (x, y) \in \mathbb{R} : y = F(x) = -\frac{1}{3}x^3 + \frac{1}{2}x \right\}.
$$

Note that $F'(\pm 1/2) = 0$. Then $(\pm 1/2, \pm 1/12)$ are both contact points, at which the nondegeneracy conditions hold:

$$
\frac{\partial^2 f}{\partial x^2}(-1/2, -1/12) = 1, \\
\frac{\partial^2 f}{\partial x^2}(1/2, 1/12) = \frac{\partial f}{\partial y}(\pm 1/2, \pm 1/12) = -1.
$$

If $g(\pm 1/2, \eta, \lambda) \neq 0$, then $(\pm 1/2, \pm 1/12)$ are jump points, for which the reduced flow directs toward the jump points $(\pm 1/2, \pm 1/12)$. If $g(\pm 1/2, \eta, \lambda) = 0$, then $(\pm 1/2, \pm 1/12)$ are canard points, for which the reduced passes through the canard points $(\pm 1/2, \pm 1/12)$. One the other hand, if $(1/2, 1/12)$ is a canard point of system (6.3), then $g(1/2, \eta, \lambda) = 0$ yields

$$
\alpha := \eta + \frac{1}{2} \lambda = 0.
$$

(6.5)

Similarly, if $(-1/2, -1/12)$ is a canard point of system (6.3), then

$$
\beta := \eta - \frac{1}{2} \lambda = 0.
$$

(6.6)
Then $\alpha$ and $\beta$ now independently control the appearances of canard points. More specifically, $(1/2, 1/12)$ (resp. $(-1/2, -1/12)$) becomes a canard point when $\alpha = 0$ (resp. $\beta = 0$). So $\alpha$ and $\beta$ are the Hopf breaking parameters. Note that (6.5) and (6.6) give an inverse transformation from the parameters plane $(\eta, \lambda)$ to the Hopf breaking parameters plane $(\alpha, \beta)$. Thus, no confusion should arise, we also use $X_{\varepsilon, \alpha, \beta}$ replacing $X_{\varepsilon, \eta, \lambda}$ in the following.

By changes
$$T_\pm: (x, y, \varepsilon, \eta, \lambda) \rightarrow \left(\pm \left(\frac{1}{2} - 2\tilde{x}\right), \pm \left(\frac{1}{12} - 2\tilde{y}\right), 6\tilde{\varepsilon}, \eta, \lambda\right),$$

system (6.3) is transformed into
$$\tilde{x}' = -\tilde{y} + \tilde{x}^2 \left(1 - \frac{4}{3}\tilde{x}\right),$$
$$\tilde{y}' = \tilde{\varepsilon} \left(\tilde{x} \left(1 + 6\lambda - 6\tilde{x} + 8\tilde{x}^2\right) \mp 3\eta - \frac{3}{2}\lambda\right).$$

By the quasi-homogeneous blow-up transformation $\Pi$ of the form
$$\Pi: \quad \tilde{x} = \tilde{r}\tilde{x}, \quad \tilde{y} = \tilde{r}^2\tilde{y}, \quad \tilde{\varepsilon} = \tilde{r}^2, \quad \eta = \tilde{r}\eta, \quad \lambda = \tilde{r}\lambda, \quad \tilde{r} > 0,$$

and then by a time rescaling $t \rightarrow t/\tilde{r}$, systems in (6.7) are changed to
$$\tilde{\tilde{x}}': = -\tilde{y} + \tilde{x}^2 - \frac{4}{3}\tilde{r}\tilde{x}^3,$$
$$\tilde{y}' = \tilde{\varepsilon} \left(\tilde{x} \left(6\tilde{x}^2 + 6\lambda\tilde{x}\right) + 8\tilde{x}^2\tilde{x}^3 \mp 3\eta - \frac{3}{2}\lambda\right).$$

Then by Lemma 4.1, we have the following results.

**Lemma 6.1** Assume that $(\pm 1/2, \pm 1/12)$ are both canard points of system (6.3). Then, there exists a sufficiently small $\varepsilon_0$ and two $C^\infty$ functions $\eta_c$ and $\lambda_c$ in the form
$$\eta_c(\varepsilon) = O(\varepsilon^{3/2}), \quad \lambda_c(\varepsilon) = -\varepsilon/9 + O\left(\varepsilon^{3/2}\right),$$
$$\varepsilon \in (0, \varepsilon_0],$$

such that two canard solutions arise near two canard points simultaneously.

For each $(u, v) \in \mathcal{U} := \{(u, v) \in \mathbb{R}^2 : 0 < u, v < 1/12\}$, let the constants $x_{1j}(u)$ and $x_{2j}(v)$, $j = l, m, r$, satisfy $x_{1l}(u) < x_{1m}(u) < x_{1r}(u)$, $x_{2l}(v) < x_{2m}(v) < x_{2r}(v)$ and
$$F(x_{1j}(u)) = \frac{1}{12} - u, \quad F(x_{2j}(v)) = -\frac{1}{12} + v,$$
$$j = l, m, r.$$

A singular double canard cycle $\Gamma(u, v)$ for each $(u, v) \in \mathcal{U}$ is defined by:
$$\Gamma(u, v) = \left\{(x, y) \in \mathbb{R}^2 : y = F(x), \ x_{1l}(u) \leq x \leq x_{2m}(v)\right\} \cup \left\{(x, y) \in \mathbb{R}^2 : x_{2m}(v) \leq x \leq x_{2r}(v)\right\} \cup \left\{(x, y) \in \mathbb{R}^2 : y = F(x), \ x_{1m}(u) \leq x \leq x_{2r}(v)\right\} \cup \left\{(x, y) \in \mathbb{R}^2 : x_{1l}(u) \leq x \leq x_{1m}(u)\right\}.$$

The slow divergence integrals of the slow curve between $x_1$ and $x_2$ for $x_j \in \mathbb{R}$ are in the form
$$I_j(x_1, x_2) = \int_{x_1}^{x_2} \text{div} X(0,0,0) \cdot \frac{F'(x)}{g(x,0,0)} \, dx = \int_{x_1}^{x_2} \frac{3(2x + 1)(2x - 1)}{4x} \, dx.$$

For each $(u, v) \in \mathcal{U}$, we define the integrals
$$\tilde{I}_1(u) = I_j(x_{1l}(u), -1/2), \quad \tilde{I}_2(u) = I_j(-1/2, x_{2m}(v)), \quad \tilde{I}_3(u) = I_j(x_{2r}(v), 1/2), \quad \tilde{I}_4(u) = I_j(1/2, x_{1m}(u)), \quad (u, v) \in \mathcal{U}.$$

By making the change of variable, the integrals $\tilde{I}_i$ can be written as the form
$$\tilde{I}_1(u) = \int_{x_{1l}(u)}^{x_{1l}(u)} -3x^{-1} \cdot F'(x) \, dx,$$
$$= -3 \int_{-1/2}^{1/2} \left(F_{m}^{-1}(y)^{-1}\right) \, dy,$$
$$\tilde{I}_2(v) = \int_{-1/2}^{x_{2m}(v)} -3x^{-1} \cdot F'(x) \, dx,$$
$$= -3 \int_{-1/2}^{x_{2m}(v)} \left(F_{m}^{-1}(y)^{-1}\right) \, dy,$$
$$\tilde{I}_3(v) = \int_{x_{2r}(v)}^{1/2} -3x^{-1} \cdot F'(x) \, dx,$$
$$= -3 \int_{-1/2}^{x_{2m}(v)} \left(F_{m}^{-1}(y)^{-1}\right) \, dy,$$
$$\tilde{I}_4(u) = \int_{1/2}^{x_{1m}(u)} -3x^{-1} \cdot F'(x) \, dx,$$
$$= -3 \int_{-1/2}^{x_{1m}(u)} \left(F_{m}^{-1}(y)^{-1}\right) \, dy.$$
where \( x = F_l^{-1}(y), \) \( x = F_m^{-1}(y) \) and \( x = F_r^{-1}(y) \) are the single-value inverse functions of \( y = F(x) \) for \( x \) in the intervals \((-\infty, -1/2], [-1/2, 1/2] \) and \([1/2, +\infty)\), respectively. The properties of the above integrals are summarized in next two lemmas.

**Lemma 6.2** The slow divergence integrals \( I_j \) satisfy the following conditions:

(i) \( I_1(u) = I_3(v) \) and \( I_2(v) = I_4(u) \) for \( u = v \), and
\[
I_2(0) = I_4(0) = 0, \quad I_1(0) = I_3(0) = -\frac{9}{8} + \frac{3}{4} \ln 2 < 0,
\]
\[
I_1 \left( \frac{1}{12} \right) = I_3 \left( \frac{1}{12} \right) = -\frac{3}{4} + \frac{3}{8} \ln 3 < 0,
\]
\[
\lim_{v \to \left( \frac{1}{12} \right)^-} I_2(v) = \lim_{u \to \left( \frac{1}{12} \right)^-} I_4(u) = +\infty.
\]

(ii) For each \((u, v) \in U\),
\[
I_1'(u) = -3 \left( F_l^{-1} \left( \frac{1}{12} - u \right) \right)^{-1} > 0,
\]
\[
I_2'(v) = -3 \left( F_m^{-1} \left( v - \frac{1}{12} \right) \right)^{-1} > 0,
\]
\[
I_3'(v) = 3 \left( F_r^{-1} \left( v - \frac{1}{12} \right) \right)^{-1} > 0,
\]
\[
I_4'(u) = 3 \left( F_m^{-1} \left( \frac{1}{12} - u \right) \right)^{-1} > 0.
\]

**Proof** Since \( F \) is an odd function, the assertion (i) holds by a direct computation. By the explicit expressions of \( I_i \) as above, we can obtain (ii). This finishes the proof. \( \square \)

**Lemma 6.3** Let the functions \( F_1 \) and \( F_2 \) be in the form
\[
F_1(u, v) = I_1(u) + I_2(v), \quad F_2(u, v) = I_3(v) + I_4(u), \quad (u, v) \in U.
\] (6.11)

Then, there exist precisely two smooth functions \( \tilde{v}(u) \) for \( 0 < u < 1/12 \) and \( \tilde{u}(v) \) for \( 0 < v < 1/12 \) such that \( F_1(u, \tilde{v}(u)) = 0 \) for \( 0 < u < 1/12 \) and \( F_2(\tilde{u}(v), v) = 0 \) for \( 0 < v < 1/12 \). Furthermore, the curves \( \{(u, v) \in U : F_1(u, v) = 0\} \) and \( \{(u, v) \in U : F_2(u, v) = 0\} \) have precisely one intersection at a point \((u_0, v_0) \in U \) and are transverse at \((u_0, v_0) \).

**Proof** By Lemma 6.2, we have that
\[
F_1(0, 0) = F_2(0, 0) = -\frac{9}{8} + \frac{3}{4} \ln 2 < 0,
\]
\[
F_1(1/12, 0) = F_2(0, 1/12) = -\frac{3}{4} + \frac{3}{8} \ln 3 < 0,
\]
\[
F_1(0, (1/12)^-) = F_2((1/12)^-), 0) = +\infty,
\]
\[
F_1((1/12)^-, (1/12)^-) = F_2((1/12)^-, (1/12)^-) = +\infty,
\]
then by (ii) in Lemma 6.2 and the Implicit Function Theorem, there exist exactly two smooth functions \( \tilde{v}(u) \) for \( 0 < u < 1/12 \) and \( \tilde{u}(v) \) for \( 0 < v < 1/12 \) such that \( F_1(u, \tilde{v}(u)) = 0 \) and \( F_2(\tilde{u}(v), v) = 0 \).

By a direct computation, we have that \( \tilde{v} \) and \( \tilde{u} \) satisfy that
\[
\tilde{v}(0^+) > 0, \quad \tilde{v}'(u) = -\frac{I_1'(u)}{I_2'(v)} < 0,
\]
\[
\tilde{u}(0^+) > 0, \quad \tilde{u}'(v) = -\frac{I_3'(v)}{I_4'(u)} < 0.
\] (6.12)

Then the curves \( \{(u, v) \in U : F_1(u, v) = 0\} \) and \( \{(u, v) \in U : F_2(u, v) = 0\} \) have at least one intersection.

Further, by the properties of \( F \) we have that
\[
-1 < \left( F_l^{-1} \left( \frac{1}{12} - u \right) \right)^{-1} < -\frac{\sqrt{3}}{2} < -\frac{1}{2},
\]
\[
< \left( F_m^{-1} \left( v - \frac{1}{12} \right) \right)^{-1} < 0,
\]
\[
0 < \left( F_m^{-1} \left( \frac{1}{12} - u \right) \right)^{-1} < \frac{\sqrt{3}}{2} < 1,
\]
\[
< \left( F_r^{-1} \left( v - \frac{1}{12} \right) \right)^{-1} < 1.
\]

Then at each intersection \((u_0, v_0)\),
\[
\tilde{v}'(u_0) \cdot \tilde{u}'(v_0) = \frac{I_1'(u_0)}{I_2'(v_0)} \cdot \frac{I_3'(v_0)}{I_4'(u_0)} > 1.
\]

This finishes the proof. \( \square \)

By applying Theorem 2.1 and Lemma 6.3, we have the following results.

**Theorem 6.1** Suppose that the slow-fast Liénard Eq. (6.3) possesses two canard points at \((\pm 1/2, \pm 1/12)\). Then, there exists a point \((u_0, v_0) \in U \) such that the cyclicity of \( \Gamma(u_0, v_0) \) is at most three. More specifically, there is a sufficiently small \( \varepsilon_0 \) and two continuous functions \( \eta(\varepsilon) = O(\varepsilon^{3/2}) \) and \( \lambda(\varepsilon) = -\varepsilon /9 + O(\varepsilon^{3/2}) \) for \( \varepsilon \in (0, \varepsilon_0) \) such that a double canard cycle \( \gamma_\varepsilon \) bifurcates from \( \Gamma(u_0, v_0) \) in \( X_{\varepsilon, \eta(\varepsilon), \lambda(\varepsilon)} \) and \( \gamma_\varepsilon \) produces three hyperbolic limit cycles for fixed \( \varepsilon \in (0, \varepsilon_0) \) and some \( (\eta, \lambda) \) in the exponentially small neighborhood of \( (\eta(\varepsilon), \lambda(\varepsilon)) \).

**Remark 6.1** Note that three large limit cycles appear in exponentially small parameter regions. Although these
canard cycles are hyperbolic, it is not easy to find them numerically. Along this direction, some efforts were made in [6] to find perturbed canard cycles in classical Liénard equations by using AUTO. However, less limit cycles were found in [6] than those obtained by the geometric singular perturbation theory.

7 Concluding remarks

We have studied the limit cycles bifurcating from slow-fast cycles in singularly perturbed planar systems with two canard points. By applying the obtained results, we further investigate the limit cycles in Liénard equations of type (3, 2). Wang and Jing [39] once gave the sufficient conditions under which Liénard equations of type (3, 2) have at most three limit cycles and numerically found that there exists the configuration of one large limit cycles enclosing two small ones in a certain Liénard equation of type (3, 2). Here by the slow divergence integral, we prove the existence of the configuration of three large limit cycles in Liénard equations of type (3, 2). It is also possible to obtain more limit cycles in general Liénard equations of type \( (m, n) \) with \( m + n \geq 5 \) by the technique of geometric singular perturbation theory. This topic on planar systems with more than two contact points was considered in [15, 16, 32, 36].

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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