DISTRIBUTION OF MOMENTS OF TRACE OF FROBENIUS IN ARITHMETIC PROGRESSIONS AND HOLOMORPHIC PROJECTION

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Abstract. In this paper, we study moments of the trace of Frobenius of elliptic curves if the trace is restricted to a fixed arithmetic progression. In particular, we fix the arithmetic progression and consider the ratio of the $2k$-th moment to the zeroeth moment as one varies the size of the finite field $\mathbb{F}_{p^r}$. We obtain asymptotic formulas for these ratios for cases for which the prime $p$ goes to infinity for fixed $r$ and cases where the power $r$ goes to infinity with fixed $p$. These results follow from similar asymptotic formulas relating sums and moments of Hurwitz class numbers where the sums are restricted to certain arithmetic progressions which we investigate in this paper.

1. Introduction and statement of results

For an elliptic curve $E$ defined over the finite field $\mathbb{F}_{p^r}$ with $p^r$ elements ($p$ prime, $r \in \mathbb{N}$), the trace of Frobenius is given by

$$\text{tr}(E) = \text{tr}_{p^r}(E) := p^r + 1 - \# E(\mathbb{F}_{p^r}).$$

Here $E(\mathbb{F}_{p^r})$ is the set of points on the elliptic curve over the finite field $\mathbb{F}_{p^r}$. If we take an elliptic curve over $\mathbb{Q}$ and apply reduction to the finite fields with $p^r$ elements, then by the Modularity Theorem [4, 32, 33] we obtain the $p^r$-th Fourier coefficients of a weight two newform (one obtains the other coefficients by multiplicativity). For $m \in \mathbb{Z}$ and $M \in \mathbb{N}$, we restrict to the set

$$\mathcal{E}_{m,M,p^r} := \{ E/\mathbb{F}_{p^r} : \text{tr}(E) \equiv m \pmod{M} \}.$$

Understanding the distribution of the numbers $\text{tr}(E)$ in this arithmetic progression is closely related to investigating the weighted $\kappa$-th moment with respect to $\text{tr}(E)$ (for $\kappa \in \mathbb{N}_0$)

$$S_{\kappa,m,M}(p^r) := \sum_{E/\mathbb{F}_{p^r} : \text{tr}(E) \equiv m \pmod{M}} \frac{\text{tr}(E)^\kappa}{\# \text{Aut}_{\mathbb{F}_{p^r}}(E)} = \sum_{E \in \mathcal{E}_{m,M,p^r}} \frac{\text{tr}(E)^\kappa}{\# \text{Aut}_{\mathbb{F}_{p^r}}(E)}. \tag{1.1}$$

Before stating our first result, we discuss how $S_{\kappa,m,M}(p^r)$ is expected to grow as $p^r \to \infty$. For simplicity, we consider the case $n = p$ prime in this heuristic argument. By the Hasse bound [16], conjectured by Artin [1] in his thesis, we have that $|\text{tr}(E)| \leq 2\sqrt{p}$. Taking $-1 \leq a \leq b \leq 1$ and a fixed elliptic curve $E$ over $\mathbb{Q}$, it was independently conjectured by Sato and Tate (see e.g. [14]) that if $E$ does not have complex multiplication, then

$$\lim_{N \to \infty} \frac{\# \{ p \leq N : 2a\sqrt{p} \leq \text{tr}(E) \leq 2b\sqrt{p} \}}{\# \{ p \leq N \}} = \frac{2}{\pi} \int_a^b \frac{1}{\sqrt{1-x^2}} dx.$$
This was later proven in a series of collaborations between Barnet-Lamb, Clozel, Geraghty, Harris, Shepherd-Barron, and Taylor [2, 7, 15]. Similarly, one can fix a prime \( p \) and vary the elliptic curve in some family, which is a version of the vertical Sato–Tate conjecture that was first considered by Sarnak [26] and Serre [28]. An analogous question asks about the distribution of the average value over all elliptic curves \( E/F_p^r \) over the finite field \( F_p^r \) with \( \sigma d(p^r) p^{2r} \leq \text{tr}(E) \leq \text{bd}(p^r) p^5 \) as one varies \( p^r \). Here \( d(n) \) denotes the number of divisors of \( n \). If there is no cancellation (this holds automatically for \( \kappa \) even because every term in (1.1) is non-negative), then we expect that for some constant \( D_{\kappa} \)

\[
S_{\kappa, m, M}(p^r) \sim D_{\kappa} \sum_{\substack{E/F_p^r \text{ (mod } M)}} \frac{p^{2\kappa r}}{\# \text{ Aut}(E)} = D_{\kappa, p^r} S_{\kappa, m, M}(p^r),
\]

where here and throughout we omit the first subscript if \( \kappa = 0 \). In this paper, we show that for \( \kappa = 2k \in 2\mathbb{N} \) the constant does indeed exist and equals the \( k \)-th Catalan number \( C_k \) if \( p \) is fixed and \( r \to \infty \) or if \( r \leq 2 \) is fixed and \( p \to \infty \).

**Theorem 1.1.** Let \( m \in \mathbb{Z}, M \in \mathbb{N} \) and \( \varepsilon > 0 \) be given.

1. For primes \( p \to \infty \), we have

\[
\frac{S_{2k, m, M}(p)}{p^k S_{m, M}(p)} = C_k + O_{k, M, \varepsilon}\left(p^{-\frac{r}{2}+\varepsilon}\right), \quad \frac{S_{2k, m, M}(p^r)}{p^{rk} S_{m, M}(p^r)} = C_k + O_{k, M, r, \varepsilon}\left(p^{-1+\varepsilon}\right) \quad (r \geq 2).
\]

2. Let \( p > 3 \) be a prime for which \( p \not| \gcd(m, M) \) and \( k \in \mathbb{N} \). As \( r \to \infty \), we have

\[
\frac{S_{2k, m, M}(p^r)}{p^{rk} S_{m, M}(p^r)} = C_k + O_{k, p, M, \varepsilon}\left(p^{\left(-\frac{1}{2}+\varepsilon\right)r}\right).
\]

**Remarks.**

1. For \( M = 1 \), these sums were studied by Birch [3] and implicitly appear in the work of Ihara [18] (see also [19, Theorem 1, Theorem 2]). They obtained a formula for these sums in terms of the trace of Hecke operators that yields the asymptotic obtained in Theorem 1.1. For \( M = 2 \), formulas for \( S_{2k, m, 2} \) were obtained by Kaplan and Petrow (see for example [20, Theorem 8]).

2. The implied constant in the error term is ineffective due to an ineffective lower bound in Lemma 3.7 that uses Siegel’s ineffective lower bound [29] for the class numbers of imaginary quadratic fields (see Lemma 2.4 below). Using Littlewood’s conditional effective bound for the class numbers [22], it can be made effective under the Generalized Riemann Hypothesis. Moreover, for fixed \( M \) one should in principle be able to obtain an effective version of Lemma 3.7 by computing the Eisenstein series components of certain modular forms; this was carried out for primes \( M \leq 7 \) in [5, Section 4] and [23, Corollary 7.3].

A special case of Theorem 1.1 yields a result about elliptic curves with \( M \)-torsion points \( (M \in \mathbb{N}) \)

\[
E[M] := \{P \in E : \text{ord}(P) \mid M\}.
\]

Here \( \text{ord}(P) \) means the order of the point under the group law defined on elliptic curves. We denote the subset of torsion points of precise order \( M \) by

\[
E^*[M] := \{P \in E : \text{ord}(P) = M\}
\]
and define

$$S^*_k,M(p^r) := \sum_{E/F_{p^r}} \frac{\text{tr}(E)^\kappa}{\# \text{Aut}_{F_{p^r}}(E)}.$$ 

**Corollary 1.2.** Let $M$ be a squarefree integer.

1. As $p \to \infty$, we have

$$\frac{S^*_{2k,M}(p)}{p^k S^*_M(p)} = C_k + O_{k,M,\varepsilon} \left( p^{-\frac{1}{2} + \varepsilon} \right),$$

$$\frac{S^*_{2k,M}(p^r)}{p^{rk} S^*_M(p^r)} = C_k + O_{k,M,r,\varepsilon} \left( p^{-\frac{1}{2} + \varepsilon} \right) \quad (r \geq 2).$$

2. If $p > 3$ is a prime we have, as $r \to \infty$

$$\frac{S^*_{2k,M}(p^r)}{p^{rk} S^*_M(p^r)} = C_k + O_{k,p,M,\varepsilon} \left( p^{-(\frac{1}{2} + \varepsilon)} \right) \quad (r \geq 2).$$

Theorem 1.1 is a consequence of a more general theorem about moments of sums of Hurwitz class numbers that are of independent interest. To describe these, let $Q_D$ denote the set of integral binary quadratic forms of discriminant $D < 0$. The $|D|$-th Hurwitz class number is defined by

$$H(|D|) := \sum_{Q \in Q_D/SL_2(\mathbb{Z})} \frac{1}{\omega_Q},$$

where $\omega_Q$ is half the size of the stabilizer group $\Gamma_Q$ of $Q$ in $SL_2(\mathbb{Z})$. By convention, we set $H(0) := -\frac{1}{12}$ and $H(r) := 0$ for $r \notin \mathbb{N}_0$ or $r \equiv 1,2 \pmod{4}$. Sums of moments of these Hurwitz class numbers analogous to $S^*_{k,m,M}$ are given by

$$H_{k,m,M}(n) := \sum_{t \equiv m \pmod{M}} t^\kappa H\left( 4n - t^2 \right). \quad (1.2)$$

In the following we drop the condition $t \in \mathbb{Z}$ in the summation. Sums of this type have occurred throughout the literature and satisfy many nice identities. For example, for $M = 1$, $\kappa = 0$, and $n = p$ prime we have the famous identity (see [11, p. 154])

$$H_{0,1}(p) = 2p.$$

Similar identities such as

$$H_{1,5}(p) = \begin{cases} 
\frac{1}{3}(p+1) & \text{if } p \equiv 1, 2 \pmod{5}, \\
\frac{1}{2}(p-1) & \text{if } p \equiv 3 \pmod{5}, \\
\frac{5}{12}(p+1) & \text{if } p \equiv 4 \pmod{5}
\end{cases}$$

were proven in [5] and [6]. Theorem 1.1 follows from the following theorem.

**Theorem 1.3.** Let $m, M, k \in \mathbb{N}$ be given. As $n \to \infty$, we have

$$\frac{H_{2k,m,M}(n)}{n^k H_{m,M}(n)} = C_k + O_{k,M,\varepsilon} \left( n^{-\frac{1}{2} + \varepsilon} \right).$$

After some preliminary setup in Section 2, we begin by investigating Hurwitz class numbers and then the moments in Section 3, proving Theorem 1.3. We then return to the application of these moments to elliptic curves in Section 4, proving Theorem 1.1 and Corollary 1.2.
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2. Preliminaries

2.1. Holomorphic and non-holomorphic modular forms. We give a brief overview of the theory of modular forms here; for details, see [21, 25]. For \( d \) odd, we set

\[
\varepsilon_d := \begin{cases} 
1 & \text{if } d \equiv 1 \pmod{4}, \\
i & \text{if } d \equiv 3 \pmod{4}.
\end{cases}
\]

Let \( \Gamma \) be a congruence subgroup containing \( T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) and if \( \kappa \in \frac{1}{2} \mathbb{Z} \), then we also require that \( \Gamma \subseteq \Gamma_0(4) \). A function \( F : \mathbb{H} \to \mathbb{C} \) satisfies modularity of weight \( \kappa \in \frac{1}{2} \mathbb{Z} \) on \( \Gamma \subseteq \text{SL}_2(\mathbb{Z}) \) with character \( \chi \) if for every \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \) we have

\[
F|_{\kappa \gamma} = \chi(d) F.
\]

Here the weight \( \kappa \) slash operator is defined by

\[
F|_{\kappa \gamma}(\tau) := \left( \frac{c}{d} \right)^{2\kappa} \varepsilon_d^{2\kappa} (c\tau + d)^{-\kappa} F(\gamma \tau),
\]

where \( (\cdot) \) denotes the extended Legendre symbol. We call \( F \) a (holomorphic) modular form if \( F \) is holomorphic on \( \mathbb{H} \) and \( F(\tau) \) grows at most polynomially in \( v \) as \( \tau = u + iv \to \mathbb{Q} \cup \{i\infty\} \).

To define certain non-holomorphic modular forms, let \( \Delta_\kappa := -v^2 \left( \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + i\kappa v \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right) \) be the weight \( \kappa \) hyperbolic Laplace operator. A smooth function \( F \) transforming modular of weight \( \kappa \) is a harmonic Maass form of weight \( \kappa \) if \( \Delta_\kappa(F) = 0 \) and there exists \( a \in \mathbb{R} \) such that

\[
F(\tau) = O(e^{av}) \text{ as } v \to \infty \quad \text{and} \quad F(u + iv) = O(e^{\frac{\pi}{u}}) \text{ for } u \in \mathbb{Q} \text{ as } v \to 0^+.
\]

If \( F \) is moreover holomorphic on \( \mathbb{H} \), then we call \( F \) a weakly holomorphic modular form. For a harmonic Maass form \( F \) of weight \( \kappa \), \( \xi_\kappa(F) \) with \( \xi_\kappa := 2iv^\kappa \frac{\partial}{\partial \tau} \) is a weakly holomorphic modular form of weight \( 2 - \kappa \).

Suppose that \( \kappa \neq 1 \). Letting \( \Gamma(\alpha, x) \) denote the incomplete gamma function, a harmonic Maass form of weight \( \kappa \) on \( \Gamma \) has a Fourier expansion of the form

\[
F(\tau) = F^+(\tau) + F^-(\tau)
\]

with (for \( q := e^{2\pi i \tau} \))

\[
F^+(\tau) = \sum_{n \gg -\infty} c_F^+(n) q^n
\]

\[
F^-(\tau) = c_F^-(0) v^{1-\kappa} + \sum_{0 \neq n < \infty} c_F^-(n) \Gamma(1 - \kappa, -4\pi nv) q^n.
\]

Another type of non-holomorphic modular form that naturally occurs is an almost holomorphic modular form, which is a function \( F : \mathbb{H} \to \mathbb{C} \) satisfying weight \( \kappa \) modularity on \( \Gamma \) for which there exist holomorphic functions \( F_j \) \((0 \leq j \leq \ell)\) such that \( F(\tau) = \sum_{j=0}^{\ell} F_j(\tau)v^{-j} \).

We call \( F_0 \) a quasimodular form.
There are natural operators that preserve modularity. In particular, suppose that 
\[ F(\tau) = \sum_{n \geq n_0} c_{F,v}(n)q^n \] satisfies weight \( \kappa \) modularity with Nebentypus character \( \chi \) (of modulus \( N \)) on \( \Gamma_0(N) \cap \Gamma_1(M) \) with \( M \mid N \). We have that
\[ F\big| V_\delta(\tau) := F(\delta \tau) \]
satisfies weight \( \kappa \) modularity on \( \Gamma_0(\text{lcm}(4,\delta N)) \cap \Gamma_1(M) \) with Nebentypus \( \chi \cdot (\delta)^{2k} \), and
\[ F\big| U_\delta(\tau) := \sum_{n \geq n_0} c_{F,v}(\delta n)q^n \]
satisfies weight \( \kappa \) modularity on \( \Gamma_0(\text{lcm}(4,N,\delta)) \cap \Gamma_1(M) \) with Nebentypus \( \chi \cdot (\delta)^{2k} \).

2.2. Rankin-Cohen brackets. For \( F_1, F_2 \) transforming like modular forms of weight \( \kappa_1, \kappa_2 \in \frac{1}{2}\mathbb{Z} \), respectively, define for \( k \in \mathbb{N}_0 \) the \( k \)-th Rankin-Cohen bracket
\[ [F_1, F_2]_k := \frac{1}{(2\pi i)^k} \sum_{j=0}^{k} (-1)^j \left( \binom{\kappa_1+k-1}{j} \binom{\kappa_2+k-1}{k-j} \right) F_1^{(j)} F_2^{(k-j)} \]
with \( \binom{\alpha}{j} := \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-j+1)} \). Then \( [F_1, F_2]_k \) transforms modular form of weight \( \kappa_1 + \kappa_2 + 2k \).

2.3. Elliptic curves and trace of Frobenius. A good introduction to elliptic curves is [30]. For an elliptic curve \( E \) defined over \( \mathbb{F}_p \), we define the Frobenius endomorphism \( \text{Fr} \) from \( E \) to itself via the \( p \)-th power map. Namely, for a point \( P = (X,Y) \in E \), we set
\[ \text{Fr}(P) := (X^p, Y^p). \]
The trace of Frobenius is given by \( \text{tr}(E) \). For \( r = 1 \), Hasse [16] showed that
\[ |\text{tr}(E)| \leq 2\sqrt{p}. \]
The distribution of \( \frac{1}{2\sqrt{p}} \text{tr}(E) \) has been well-studied and it is natural to group those elliptic curves whose trace of Frobenius agree. For \( t \in \mathbb{Z} \), we hence define
\[ \mathcal{E}_{p^r,t} := \{ E/\mathbb{F}_{p^r} : \text{tr}(E) = t \}. \]
As is well-known, there is a group law defined on elliptic curves and the automorphisms of the group we denote by \( \text{Aut}_{\mathbb{F}_{p^r}}(E) \). The automorphism group gives a natural weighting on the elliptic curves in \( \mathcal{E}_{p^r,t} \), leading to the definition
\[ N_A(p^r; t) := \sum_{E \in \mathcal{E}_{p^r,t}} \frac{1}{\# \text{Aut}_{\mathbb{F}_{p^r}}(E)}. \]
These sums naturally occur when investigating \( S_{\kappa,m,M}(p^r) \) because
\[ S_{\kappa,m,M}(p^r) = \sum_{t \equiv m \pmod{M}} t^\kappa N_A(p^r; t). \]
2.4. The class number generating function. Let

\[ \mathcal{H}(\tau) := \sum_{n \in \mathbb{Z}} H(n) q^n \]

be the generating function for the Hurwitz class numbers. Its modular properties follow by [17, Theorem 2].

**Theorem 2.1.** The function

\[ \hat{H}(\tau) := \mathcal{H}(\tau) + \frac{1}{8\pi \sqrt{v}} + \frac{1}{4\sqrt{\pi}} \sum_{n=1}^{\infty} n \Gamma \left( -\frac{1}{2}, 4\pi n^2 v \right) q^{-n^2} \]

is a harmonic Maass form of weight \( \frac{3}{2} \) on \( \Gamma_0(4) \).

2.5. Elliptic curves and class numbers. For \( m \in \mathbb{Z} \), \( n, M \in \mathbb{N} \), and \( \kappa \in \mathbb{N}_0 \), we next relate \( S_{m,M,\kappa}(n) \) to certain sums of Hurwitz class numbers given by

\[ \mathcal{H}_{\kappa,m,M}(p; n) := \sum_{\substack{t \equiv m \pmod{M} \atop p \nmid t}} t^\kappa H \left( 4n - t^2 \right). \]

We are mostly interested in the case \( n = p^r \) with \( r \in \mathbb{N} \), which we abbreviate by \( \mathcal{H}_{\kappa,m,M}(p^r) := \mathcal{H}_{\kappa,m,M}(p; p^r) \). To state the result, we set

\[ E_{\kappa,m,M}(p^r) := \delta_M | m \delta_{\kappa=0} \delta_{2|r} H(4p) + \delta_M | m \delta_{\kappa=0} \delta_{2|r} \frac{1}{2} \left( 1 - \left( -\frac{1}{p} \right) \right) \]

\[ + \frac{1}{3} \left( 1 - \left( -\frac{3}{p} \right) \right) p^{\kappa \frac{r}{2}} \eta_{\kappa,m,M}(p^r) + \frac{1}{3} (p - 1) 2^{\kappa - 2} p^{\kappa \frac{r}{2}} \sigma_{\kappa,m,M}(p^r) \]

with

\[ \eta_{\kappa,m,M}(p^r) := \sum_{\substack{t \equiv m \pmod{M} \atop t^2 = p^r}} \text{sgn}(t)^\kappa, \quad \sigma_{\kappa,m,M}(p^r) := \sum_{\substack{t \equiv m \pmod{M} \atop t^2 = 4p^r}} \text{sgn}(t)^\kappa. \]

Here and throughout \( \delta_S := 1 \) if a statement \( S \) is true and \( \delta_S := 0 \) otherwise. We note that if \( \kappa \in 2\mathbb{N}_0 \), then \( \eta_{\kappa,m,M}(p^r) = \eta_{m,M}(p^r) \), where

\[ \eta_{m,M}(p^r) := \# \left\{ t = \pm 2p^\frac{r}{2} \in \mathbb{Z} : t \equiv m \pmod{M} \right\}, \]

\[ \sigma_{m,M}(p^r) := \# \left\{ t = \pm p^\frac{r}{2} \in \mathbb{Z} : t \equiv m \pmod{M} \right\}. \]

**Lemma 2.2.** For a prime \( p > 3 \), \( \kappa \in \mathbb{N}_0 \), and \( r \in \mathbb{N} \) we have

\[ 2S_{\kappa,m,M}(p^r) = \mathcal{H}_{\kappa,m,M}(p^r) + E_{\kappa,m,M}(p^r). \]
Proof. The claim easily follows, using that by [20, Theorem 3], for a prime \( p > 3 \) and \( r \in \mathbb{N} \) we have

\[
2N_A(p^r; t) = \begin{cases} 
H(4p^r - t^2) & \text{if } t^2 < 4p^r, \ p \nmid t, \\
H(4p^r) & \text{if } t = 0 \text{ and } r \text{ is odd}, \\
\frac{1}{2} \left( 1 - \left( \frac{-1}{p} \right) \right) & \text{if } t = 0 \text{ and } r \text{ is even}, \\
\frac{1}{3} \left( 1 - \left( \frac{-3}{p} \right) \right) & \text{if } t^2 = p^r, \\
\frac{1}{12} (p - 1) & \text{if } t^2 = 4p^r, \\
0 & \text{otherwise}.
\end{cases}
\]

The sums \( \mathcal{H}_{\kappa,m,M}(p;n) \) are related to \( H_{\kappa,m,M}(n) \), as a direct calculation shows.

**Lemma 2.3.** For \( m \in \mathbb{Z}, M \in \mathbb{N}, p \) prime, and \( \kappa \in \mathbb{N}_0 \), we have

\[
\mathcal{H}_{\kappa,m,M}(p;n) = \sum_{\ell \equiv m \pmod{p}} \sum_{\ell \equiv M \ell \pmod{p}} H_{\kappa,m+M\ell,Mp}(n).
\]

2.6. Generating functions for sums of moments of class numbers. Taking the generating function of (1.2), for \( m \in \mathbb{Z}, M \in \mathbb{N}, \) and \( \kappa \in \mathbb{N}_0 \), we study sums of the type

\[
\mathcal{H}_{\kappa,m,M}(\tau) := \sum_{\tau} H_{\kappa,m,M}(\tau) q^n = \sum_{\tau} \sum_{n=0}^{\infty} t^n H(4n - t^2) q^n.
\]

We directly see that

\[
\mathcal{H}_{\kappa,m,M} = (\mathcal{H} \theta_{\kappa,m,M}) |_{U_4},
\]

where

\[
\theta_{\kappa,m,M}(\tau) := \sum_{n=0}^{\infty} n^\kappa q^{n^2}.
\]

2.7. Properties of class numbers. We use the following bounds that follow from results of Siegel [29].

**Lemma 2.4.** For a discriminant \( D < 0 \) and \( \varepsilon > 0 \), we have

\[
|D|^{\frac{3}{2} - \varepsilon} \ll \varepsilon H(|D|) \ll \varepsilon |D|^{\frac{3}{4} + \varepsilon}.
\]

If \( D = \Delta f^2 \) with \( \Delta < 0 \) a fundamental discriminant and \( f \in \mathbb{N} \), then (see [8, p. 273])

\[
H(|\Delta f^2|) = H(|\Delta|) \sum_{d|f} \mu(d) \left( \frac{\Delta}{d} \right) \sigma \left( \frac{f}{d} \right),
\]

where \( \sigma(n) := \sum_{d|n} d \). Setting \( D_p := \frac{D_p}{p^{2\alpha}} \) for an odd prime \( p \) such that \( 2\alpha \leq \text{ord}_p(D) \leq 2\alpha + 1 \), this leads to the following useful relation.

**Lemma 2.5.** If \( D < 0 \) is a discriminant and \( p \) is an odd prime, then

\[
H(|D|p^2) = pH(|D|) + \left( 1 - \left( \frac{D_p}{p} \right) \right) H(D_p).
\]
Proof. Let $D = \Delta f^2$ with $\Delta < 0$ a fundamental discriminant and $f \in \mathbb{N}$. Since the sum in (2.1) is multiplicative, if $f = p^\alpha g$ with $\alpha \in \mathbb{N}_0$ and $p \nmid g$, then we can write

\begin{align}
H (|D|) &= H (|\Delta|g^2) \left( \sigma (p^\alpha) - \delta_{\alpha \geq 1} \left( \frac{\Delta}{p} \right) \sigma (p^{\alpha - 1}) \right), \quad (2.2) \\
H (|D|p^2) &= H (|\Delta|g^2) \left( \sigma (p^{\alpha + 1}) - \left( \frac{\Delta}{p} \right) \sigma (p^\alpha) \right). \quad (2.3)
\end{align}

Note that for $\ell \geq 0$ we have

$$
\sigma (p^\ell) = 1 + \delta_{\ell \geq 1} p \sigma (p^{\ell - 1}).
$$

Comparing (2.3) with (2.2) yields

$$
H (|D|p^2) = pH (|D|) + \left( 1 - \left( \frac{\Delta}{p} \right) \right) H (|\Delta|g^2).
$$

Since $p \nmid g$, we have $\left( \frac{\Delta^2}{p^2} \right) = 1$ and hence $\left( \frac{\Delta}{p} \right) = \left( \frac{\Delta^2}{p^2} \right)$. The claim follows from the fact that $\Delta g^2 = D_p$. \qed

2.8. Holomorphic projection. Let $F (\tau) = \sum_{n \in \mathbb{Z}} c_{F,v}(n)q^n$ be a (not necessarily holomorphic) function satisfying weight $\kappa \geq 2$ modularity. Suppose furthermore that $F (\tau) - P_{\infty} (q^{-1})$ has moderate growth, where $P_{\infty} \in \mathbb{C}[x]$ and that a similar condition holds as $\tau \to \mathbb{Q}$. We define (see [13, Proposition 5.1, p. 288] for the general statement and [24] for it written in this generality) the holomorphic projection of $F$

$$
\pi^{\text{reg}}_{\text{hol}} (F) (\tau) := P_{\infty} (q^{-1}) + \sum_{n=1}^{\infty} c_F(n)q^n.
$$

Here for $n \in \mathbb{N}$

$$
c_F(n) := \frac{(4\pi n)^{-1}}{\Gamma (\kappa - 1)} \lim_{s \to 0^+} \int_{0}^{\infty} c_{F,v}(n) u^{\kappa - 2 - s} e^{-4\pi uv} dv.
$$

Note that for a non-holomorphic modular form $F$, there exists a unique cusp form $f$ satisfying $\langle F, g \rangle = \langle f, g \rangle$ for every cusp form $g$, where $\langle \cdot, \cdot \rangle$ denotes the Petersson inner product. The function $f$ is Sturm’s [31] original definition for the holomorphic projection of $F$ and Gross and Zagier showed in [13, Proposition 5.1, p. 288] that Sturm’s definition matches the definition given here if one additionally assumes that $F$ decays polynomially towards all cusps. We have the following properties of $\pi^{\text{reg}}_{\text{hol}} (F)$ (see [13, Proposition 5.1 and Proposition 6.2] as well as [23, (4.6)])

**Lemma 2.6.** Suppose that $F$ is continuous and transforms modular of weight $\kappa \geq 2$ on $\Gamma_1 (N)$. Then the following hold.

1. If $F$ is holomorphic, then $\pi^{\text{reg}}_{\text{hol}} (F) = F$.
2. If $\kappa > 2$ and $F$ is bounded towards all cusps, then $\pi^{\text{reg}}_{\text{hol}} (F)$ is a holomorphic modular form. If $\kappa = 2$, then $\pi^{\text{reg}}_{\text{hol}} (F)$ is a quasimodular form of weight two.
3. If $F_1$ is a weight $\kappa_1 \in \frac{1}{2} \mathbb{N}$ harmonic Maass form and $f_2$ is a weight $\kappa_2 \in \frac{1}{2} \mathbb{N}$ holomorphic modular form, and $k \in \mathbb{N}_0$ with $\kappa := \kappa_1 + \kappa_2 + 2k \geq 2$, then

$$
\pi^{\text{reg}}_{\text{hol}} ([F_1, f_2]_k) = \left[ F^{+}_1, f_2 \right]_k + \pi^{\text{reg}}_{\text{hol}} ([F^{-}_1, f_2]_k).
$$
Remark. Lemma 2.6 (3) appears in a slightly different form in [23, (4.6)] because Mertens did not include the constant term in the definition of $F^-$ in [23, (3.1)].

Rearranging Lemma 2.6 (3) yields a formula for $[F_1^+, F_2]_k$. In [23, Theorem 1.2], Mertens considered the special case that $\kappa_1 = \frac{3}{2}$, $\kappa_2 = \frac{1}{2}$, and $\xi_\frac{3}{2}(F_1)$ and $F_2$ are both weight $\frac{1}{2}$ holomorphic modular forms on $\Gamma_1(N)$. Serre and Stark showed in [27, Theorem A] that this space is spanned by unary theta functions. For a character $\chi$ and $a \in \mathbb{N}$, these are given by $\theta_{\chi} | V_a$, where

$$\theta_{\chi}(\tau) := \sum_{n \in \mathbb{Z}} \chi(n) q^n.$$

Using Serre and Stark’s classification, one may assume without loss of generality that $F_2 = \theta_{\chi} | V_a$ and $\xi_\frac{3}{2}(F_1) = \theta_{\psi} | V_b$ for some characters $\chi, \psi$ and $a, b \in \mathbb{N}$. In the proof of [23, Theorem 1.2], Mertens related the second term on the right-hand side of Lemma 2.6 (3) to

$$\Lambda_{\ell,a,b}^{\chi,\psi}(\tau) := 2 \sum_{n=1}^{\infty} \lambda_{\ell,a,b}^{\chi,\psi}(n) q^n,$$

where here and throughout $\sum^*$ means that the terms in the sum with $s = 0$ are weighted by $\frac{1}{2}$. Using the fact that $\pi_{\text{reg}}^\tau([F_1, F_2]_k)$ is a quasimodular form by Lemma 2.6 (2), one then obtains the following.

**Lemma 2.7.** Suppose that $\chi$ and $\psi$ are characters of conductors $N_\chi$ and $N_\psi$, respectively and $N, a, b \in \mathbb{N}$ with $bN_\psi^2 | N$. If $F$ is a harmonic Maass form of weight $\frac{3}{2}$ on $\Gamma_1(4N)$ that grows at most polynomially towards all cusps and satisfies $\xi_\frac{3}{2}(F) = \theta_{\psi} | V_b$, then

$$\left( [F^+, \theta_{\chi} | V_a]_k - 2^{3-k} \pi \right) \Lambda_{2k+1,a,b}^{\chi,\psi} \right) | U_4$$

is a holomorphic cusp form of weight $2k + 2$ on $\Gamma_1(\text{lcm}(4N, 4aN_\chi^2))$ if $k > 0$ and a quasimodular form of weight two if $k = 0$.

**Remark.** The statement of Lemma 2.7 corrects an error in [23, (5.2)] where the constant in front of the second term differs by a factor of $-2\sqrt{\pi}$.

3. Holomorphic projection and the proof of Theorem 1.3

In this section, we prove Theorem 1.3.

3.1. Fourier coefficients of certain Rankin–Cohen brackets. Define (compare with [23, (7.7)], although the notation is different there)

$$G_{k,m,M}(n) := \sum_{t \equiv \pm m \pmod{M}} p_{2k}(t, n) H \left( 4n - t^2 \right),$$

where $p_{2k}(t, n)$ denotes the $(2k)$-th coefficients in the Taylor expansion of $(1 - tX + nX^2)^{-1}$. These appear in the Fourier expansion of $[H, \theta_{m,M}]_k | U_4$ as a direct calculation using the results of Cohen [8] shows.

**Lemma 3.1.** The $n$-th Fourier coefficient of $[H, \theta_{m,M}]_k | U_4$ equals $\frac{(2k)!}{2^k} G_{k,m,M}(n)$. 

Using the explicit evaluation ([12, (3), page 29])

\[ p_{2k}(t, n) = \frac{(2k)!}{k!} \sum_{\mu=0}^{k} (-1)^\mu \binom{2k-\mu}{\mu} t^{2k-2\mu} n^\mu \]

we directly obtain the following lemma.

**Lemma 3.2.** For \( m \in \mathbb{Z} \) and \( k, M \in \mathbb{N} \), we have

\[ H_{2k,m,M}(n) = \frac{k!}{(2k)!} G_{k,m,M}(n) - \frac{1}{\mu!/(2k-2\mu)!} \sum_{\mu=1}^{k} (-1)^\mu (2k-\mu)! n^\mu H_{2k-2\mu,m,M}(n). \]

In order to investigate \( H_{2k,m,M}(n) \), it suffices by Lemma 3.1 and Lemma 3.2 to study the coefficients of \([\mathcal{H}, \theta_{m,M}]_k|U_4\). In [23, Theorem 1.2] Mertens applied holomorphic projection on functions related to \([\hat{\mathcal{H}}, \theta_{m,M}]_k|U_4\). To state his result (noted two paragraphs before [23, Proposition 7.2]), we set

\[ \Lambda_{\ell,m,M}^\chi(\tau) := \sum_{n=1}^{\infty} \lambda_{\ell,m,M}(n) q^n, \quad \text{where} \quad \lambda_{\ell,m,M}(n) := 2 \sum_{t+s \geq n} \sum_{t \equiv \pm \ell \pmod{M}} (t-s)^\ell. \]

**Lemma 3.3.** For \( k \in \mathbb{N}_0 \), \( m \in \mathbb{Z} \), and \( M \in \mathbb{N} \), the function

\[ \left( [\mathcal{H}, \theta_{m,M}]_k + 2^{-1-2k} \binom{2k}{k} \Lambda_{2k+1,m,M} \right) |U_4 \]

is a holomorphic cusp form of weight \( 2 + 2k \) on \( \Gamma_0(4M^2) \cap \Gamma_1(M) \) (resp. \( \Gamma_0(4M^2) \)) if \( M \nmid m \) (resp. \( M \mid m \)) if \( k > 0 \) and quasimodular on that group if \( k = 0 \).

**Remark.** Lemma 3.3 was stated in a different form in [23]. Specifically, Mertens assumed that \( M \) is prime and an error appears in the constant in front of \( \Lambda_{2k+1,m,M} \) in [23]. The proof in [23] goes through without the assumption that \( M \) is prime, however.

**Proof of Lemma 3.3.** Using Lemma 2.7 and relating the functions \( \Lambda_{\ell,a,b}^\chi \) (resp. \([\mathcal{H}, \theta_{a,b}]_k\)) to \( \Lambda_{\ell,m,M} \) (resp. \([\mathcal{H}, \theta_{m,M}]_k\)) via orthogonality of characters yields the claim. \( \square \)

In light of Lemma 3.2, it is natural to recursively define \( C_0 := 1 \) and

\[ C_k := -\sum_{\mu=1}^{k} (-1)^\mu \binom{2k-\mu}{\mu} C_{k-\mu}. \]

As we show in the next lemma, \( C_k \) are the Catalan numbers

\[ C_k := \frac{1}{k+1} \binom{2k}{k}. \]

**Lemma 3.4.** We have \( C_k = C_k \).

**Proof.** For \( k = 0 \) the claim holds directly by definition. Assume inductively that the claim holds for all \( \ell < k \). Then we have

\[ C_k = -\sum_{\mu=1}^{k} (-1)^\mu \binom{2k-\mu}{\mu} C_{k-\mu}. \]
It remains to show that
\[- \sum_{\mu=1}^{k} (-1)^\mu \binom{2k - \mu}{\mu} C_{k-\mu} = \begin{cases} C_k & \text{if } k \geq 1, \\ 0 & \text{if } k = 0. \end{cases} \tag{3.1} \]

To show the claim, we take the generating function of the left-hand side of (3.1):
\[L(X) := - \sum_{k=1}^{\infty} \sum_{\mu=1}^{k} (-1)^\mu \binom{2k - \mu}{\mu} C_{k-\mu} X^k \tag{3.2} \]

using the binomial series expansion. We then recall the evaluation of the generating function (see [10, 26.5.2])
\[F(Z) := \sum_{k=0}^{\infty} C_k Z^k = 1 - \sqrt{1 - 4Z} \]
valid for \(|Z| < \frac{1}{4}\). Therefore, for \(0 < X < \frac{1}{4}\), we obtain
\[F \left( \frac{X}{(1+X)^2} \right) = 1 + X. \]

Plugging this into the first sum in (3.2) yields \(L(X) = \sum_{k=1}^{\infty} C_k X^k\). This gives the claim. \(\square\)

### 3.2. Asymptotic growth of the moments.

In order to obtain the asymptotic growth of \(G_{k,m,M}\) and \(H_{2k,m,M}\), we require the following straightforward estimates.

**Lemma 3.6.** We have
\[\lambda_{\ell,m,M}(n) \leq n^{\frac{\ell}{2}} \lambda_{m,M}(n) \ll_{\varepsilon} n^{\frac{\ell}{2} + \varepsilon}. \]

We are now ready to prove an asymptotic formula for \(H_{2k,m,M}(n)\).

**Proposition 3.7.** We have
\[H_{2k,m,M}(n) = C_k n^k H_{m,M}(n) + O_{k,M,\varepsilon} \left( n^{k+\frac{1}{2}+\varepsilon} \right). \]

**Proof.** We argue by induction using Lemma 3.2. Since \(C_0 = 1\), the claim holds trivially for \(k = 0\). For \(k \geq 1\), Lemma 3.1 and Lemma 3.3 imply that
\[G_{k,m,M}(n) + \frac{1}{2^{2k} k!} \lambda_{2k+1,m,M}(4n) \]
is the \(n\)-th coefficient of a weight \(2k + 2\) cusp form. By Deligne’s bound [9] it thus may be bound against \(O_{k,M,\varepsilon}(n^{k+\frac{1}{2}+\varepsilon})\). The implied constant in the error term a priori depends on \(m\) as well, but by taking the maximum over all of the choices of \(m \pmod{M}\), we may drop the dependence on \(m\) throughout. Using Lemma 3.5, we obtain
\[G_{k,m,M}(n) \ll_{k,M,\varepsilon} n^{k+\frac{1}{2}+\varepsilon}. \]

Plugging this into Lemma 3.2, using the inductive hypothesis and the fact that \(C_k\) satisfies the recurrence defining \(C_k\) by Lemma 3.4, we obtain the claim. \(\square\)
3.3. The main term. In this subsection we investigate the growth of \( H_{m,M}(n) \).

**Lemma 3.7.** For \( m, M \in \mathbb{N} \) fixed, we have, as \( n \to \infty \),
\[
n^{1-\varepsilon} \ll_{\varepsilon,M} H_{m,M}(n) \ll_{\varepsilon} n^{1+\varepsilon}.
\]

**Proof.** We begin with the lower bound. Since for \( 4n - t^2 \not= 0 \), \( H(4n - t^2) \geq 0 \) and since \( H(0) = -\frac{1}{12} \), we obtain
\[
H_{m,M}(n) \geq \sum_{\substack{t \equiv m \pmod{M} \\ |t| \leq \sqrt{n}}} H(4n - t^2) - \frac{1}{6}.
\]
Using the lower bound in Lemma 2.4, we obtain
\[
\sum_{\substack{t \equiv m \pmod{M} \\ t \leq \sqrt{n}}} H(4n - t^2) \gg_{\varepsilon} M \sum_{\substack{t \equiv m \pmod{M} \\ t \leq \sqrt{n}}} (4n - t^2)^{\frac{1}{2} - \varepsilon} \gg_{\varepsilon,M} n^{1-\varepsilon}.
\]
Plugging this into (3.3) yields the lower bound.

For the upper bound, we again use the fact that every term is non-negative except \( 4n - t^2 = 0 \) to bound
\[
H_{m,M}(n) \leq \sum_{|t| < 2\sqrt{n}} H(4n - t^2).
\]
The upper bound in Lemma 2.4 then yields
\[
\sum_{|t| < 2\sqrt{n}} H(4n - t^2) \ll_{\varepsilon} \sum_{|t| < 2\sqrt{n}} (4n - t^2)^{\frac{1}{2} + \varepsilon} \ll_{\varepsilon} \sum_{|t| < 2\sqrt{n}} n^{\frac{3}{2} + \varepsilon} \ll_{\varepsilon} n^{1+\varepsilon}.
\]

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** By Proposition 3.6, we have
\[
\frac{H_{2k,m,M}(n)}{n^k H_{m,M}(n)} = C_k + O_{k,M,\varepsilon} \left( \frac{n^{\frac{3}{2} + \varepsilon}}{H_{m,M}(n)} \right).
\]
The claim now follows by using the lower bound in Lemma 3.7 to bound the error term. \( \square \)

4. PROOF OF THEOREM 1.1 AND COROLLARY 1.2

The next lemma relates \( \mathcal{K}_{2k,m,M}(p^r) \) to linear combinations of \( H_{2k,m,M}(p^j) \) with \( 0 \leq j \leq r \).

**Lemma 4.1.** Suppose that \( p > 3 \), \( m \in \mathbb{Z} \), \( M \in \mathbb{N} \), and \( k \in \mathbb{N}_0 \).

1. If \( r \leq 1 \) or both \( p \mid M \) and \( p \nmid m \), then
\[
\mathcal{K}_{2k,m,M}(p^r) = H_{2k,m,M}(p^r) - \delta_{M|m}\delta_{k=0} H(4p).
\]
2. If \( p \not\mid M \) and \( r \geq 2 \), then
\[
\mathcal{K}_{2k,m,M}(p^r) = H_{2k,m,M}(p^r) - p^{2k+1} H_{2k,m,M}(p^{r-2}) - \delta_{M|m}\delta_{k=0}\delta_{2|\sigma_{2,p}} \left( 1 - \left( -\frac{1}{p} \right) \right) - \delta_{M|m}\delta_{k=0}\delta_{2|\sigma_{2,p}} H(4p) - \frac{4^k}{12} p^{rk+1} \left( 1 - \frac{1}{p} \right) \varphi_{m,M}(p^r) - \frac{1}{3} \left( 1 - \left( -\frac{3}{p} \right) \right) p^{rk} \sigma_{m,M}(p^r).
\]
Proof. (1) By definition, we have

\[ \mathcal{H}_{2k,m,M}(p^r) = H_{2k,m,M}(p^r) - \sum_{t \equiv m \pmod{M}} t^{2k} H \left( 4p^r - t^2 \right). \] (4.1)

It is not hard to see that under the assumptions of (1) the only possible term in the second sum is \( t = 0 \), which only occurs if \( M \mid m \) and \( k = 0 \), giving the claim.

(2) Letting \( t \mapsto pt \), the second summand in (4.1) equals (we assume that \( r \geq 2 \))

\[ p^{2k} \sum_{pt \equiv m \pmod{M}} t^{2k} H \left( \left( 4p^{r-2} - t^2 \right) p^2 \right) = p^{2k} \sum_{t \equiv m \pmod{M}} t^{2k} H \left( \left( 4p^{r-2} - t^2 \right) p^2 \right). \] (4.2)

We next use Lemma 2.5 with \( D = t^2 - 4p^{r-2} \) to rewrite the terms in (4.2) with \( |t| < 2p^{5-1} \), where this condition is required to assure that \( D < 0 \). If \( 0 < |t| < 2p^{5-1} \) and \( p > 3 \), then \( \text{ord}_p(t^2 - 4p^{r-2}) = \text{ord}_p(t^2) \) and hence \( \alpha = \text{ord}_p(t) \) in Lemma 2.5, yielding

\[ H \left( \left( 4p^{r-2} - t^2 \right) p^2 \right) = pH \left( 4p^{r-2} - t^2 \right) \]

\[ + \left( 1 - \left( \frac{4}{p^2} \right)^2 - 4p^{r-2-2\alpha} \right) \left( \frac{t}{p^\alpha} \right)^2. \] (4.3)

For \( t = 0 \), the choice of \( \alpha \) in Lemma 2.5 is \( \alpha = \frac{r}{2} - 1 \) if \( r \) is even and \( \alpha = \frac{r-3}{2} \) if \( r \) is odd, so in this case Lemma 2.5 gives

\[ H \left( 4p^r \right) = pH \left( 4p^{r-2} \right) + \delta_{2|r} \left( 1 - \left( \frac{-1}{p} \right) \right) H(4) + \delta_{2|p} H(4p). \]

Noting that the Legendre symbol in (4.3) equals 1 unless \( \text{ord}_p(t) = \frac{r}{2} - 1 \), plugging back into (4.2) for the \( |t| < 2p^{5-1} \) terms and using the evaluations \( H(0) = -\frac{1}{12} \), \( H(3) = \frac{1}{3} \), and \( H(4) = \frac{1}{2} \) easily yields the claim. \( \square \)

We next bound \( |E_{2k,m,M}(p^r)| \).

**Lemma 4.2.** For \( k \in \mathbb{N}_0 \) and \( \varepsilon > 0 \) we have

\[ E_{2k,m,M}(p^r) = \frac{1}{3} 2^{2k-2} p^{r+1} \varrho_{m,M}(p^r) + O_{k,\varepsilon} \left( p^{r+\frac{4k-a}{2}+\varepsilon} \right). \]

Proof. The first term in the definition of \( E_{2k,m,M}(p^r) \) only occurs if \( k = 0 \) and we use the upper bound in Lemma 2.4 to conclude that

\[ H(4p) \ll \varepsilon p^{\frac{1}{2}+\varepsilon} = p^{r+k+\frac{4k-a}{2}+\varepsilon}. \]

The second term is clearly \( O(1) \) and the third term is \( O(p^k) \) because \( 0 \leq \varrho_{m,M}(p^r) \leq 2 \). We finally use \( 0 \leq \sigma_{m,M}(p^r) \leq 2 \) to split the last term in the definition of \( E_{2k,m,M}(p^r) \) as

\[ \frac{1}{3} (p-1) 2^{2k-2} p^{r-k} \sigma_{m,M}(p^r) = \frac{1}{3} 2^{2k-2} p^{r-k} \sigma_{m,M}(p^r) + O_k \left( p^k \right), \]

yielding the claim. \( \square \)

We are now ready to prove Theorem 1.1.
Proof of Theorem 1.1. (1) By Lemma 2.2, we have
\[
\frac{S_{2k,m,M}(p^r)}{p^{rk}S_{m,M}(p^r)} = \frac{\mathcal{H}_{2k,m,M}(p^r) + E_{2k,m,M}(p^r)}{p^{rk}(\mathcal{H}_{m,M}(p^r) + E_{m,M}(p^r))}.
\] (4.4)
We first consider \( r = 1 \). By Lemma 4.1 (1) the right-hand side of (4.4) equals
\[
\frac{H_{2k,m,M}(p) + E_{2k,m,M}(p)}{p^k(\mathcal{H}_{m,M}(p) - \delta_{M|m}\delta_{k=0}(4p) + E_{m,M}(p))} = \frac{\frac{H_{2k,m,M}(p)}{p^{rk}\mathcal{H}_{m,M}(p)} + \frac{E_{2k,m,M}(p)}{p^{rk}E_{m,M}(p)}}{1 - \delta_{M|m}\delta_{k=0}\frac{H(4p)}{\mathcal{H}_{m,M}(p)} + \frac{E_{m,M}(p)}{E_{m,M}(p)}}.
\]
Using Lemma 4.2 and Lemma 3.7 together with Lemma 2.4, we obtain (note that \( e_{m,M}(p^r) = 0 \) for \( r \) odd)
\[
\frac{H(4p)}{H_{m,M}(p)} \ll_{M,\varepsilon} p^{-1+\varepsilon}, \quad \left| \frac{E_{2k,m,M}(p)}{p^kH_{m,M}(p)} \right| \ll_{k,M,\varepsilon} p^{\frac{k-1}{2}+1+\varepsilon} \ll p^{-\frac{1}{2}+\varepsilon}.
\]
Using Theorem 1.3 yields the case \( r = 1 \).
For \( r \geq 2 \), we use Lemma 4.1 (2) to rewrite
\[
\mathcal{H}_{2k,m,M}(p^r) = H_{2k,m,M}(p^r) - p^{2k+1}H_{2k,m,M}(p^{r-2}) + O_{k,M}(p^{rk+1}).
\] (4.5)
By Proposition 3.6 and the upper bound in Lemma 3.7, we have
\[
p^{2k+1}H_{2k,m,M}(p^{r-2}) = C_k p^{rk+1}H_{m,M}(p^{r-2}) + O_{k,M,\varepsilon} \left( p^{2k+1+(r-2)(k+\frac{1}{2})+1+\varepsilon} \right)
\ll_{k,M,\varepsilon} p^{r(k+1+\varepsilon)-1} + p^{r(k+\frac{1}{2}+\varepsilon)} \ll_{k,M,\varepsilon} p^{r(k+1+\varepsilon)-1},
\]
where in the last bound we use the fact that \( r - 1 \geq \frac{r}{2} \) for \( r \geq 2 \). Plugging this back into (4.5) yields
\[
\mathcal{H}_{2k,m,M}(p^r) = H_{2k,m,M}(p^r) + O_{k,M,\varepsilon} \left( p^{r(k+1+\varepsilon)-1} \right).
\] (4.6)
By Lemma 4.2, we have
\[
E_{2k,m,M}(p^r) = O_{k} \left( p^{rk+1} \right) = O_{k} \left( p^{r(k+1+\varepsilon)-1} \right).
\] (4.7)
Plugging (4.6) and (4.7) into (4.4) yields
\[
\frac{S_{2k,m,M}(p^r)}{p^{rk}S_{m,M}(p^r)} = \frac{H_{2k,m,M}(p^r) + O_{k,M,\varepsilon} \left( p^{r(k+1+\varepsilon)-1} \right)}{p^{rk}H_{m,M}(p^r) + O_{k,M,\varepsilon} \left( p^{r(k+1+\varepsilon)-1} \right)} = \frac{\frac{H_{2k,m,M}(p^r)}{p^{rk}H_{m,M}(p^r)} + \frac{O_{k,M,\varepsilon} \left( p^{r(k+1+\varepsilon)-1} \right)}{p^{rk}H_{m,M}(p^r)}}{1 + \frac{O_{k,M,\varepsilon} \left( p^{r(k+1+\varepsilon)-1} \right)}{H_{m,M}(p^r)}}.
\]
The lower bound in Lemma 3.7 now yields the claim.
(2) First assume that \( p \nmid M \). Since \( p \nmid m \) in this case by assumption, Lemma 4.1 (1) yields that \( \mathcal{H}_{2k,m,M}(p^r) = H_{2k,m,M}(p^r) \) and by (4.4) we have
\[
\frac{S_{2k,m,M}(p^r)}{p^{rk}S_{m,M}(p^r)} = \frac{H_{2k,m,M}(p^r) + E_{2k,m,M}(p^r)}{p^{rk}(H_{m,M}(p^r) + E_{m,M}(p^r))} = \frac{\frac{H_{2k,m,M}(p^r)}{p^{rk}H_{m,M}(p^r)} + \frac{E_{2k,m,M}(p^r)}{p^{rk}E_{m,M}(p^r)}}{1 + \frac{E_{m,M}(p^r)}{H_{m,M}(p^r)}}
\]
By Lemma 4.2 and Lemma 3.7 we obtain
\[
\frac{E_{2k,m,M}(p^r)}{p^{rk}H_{m,M}(p^r)} \ll_{p,M,\varepsilon} p^{r(-\frac{1}{2}+\varepsilon)}
\]
and the proof follows as in the case \( r = 1 \).
Next suppose that $p \nmid M$. We first rewrite the numerator of (4.4). Plugging Theorem 1.3 into the right-hand side of Lemma 2.3 yields

$$\mathcal{H}_{2k,m,M}(p^r) = \sum_{\ell \pmod{p}} p^k H_{m + M\ell, Mp}(p^r) \left( C_k + O_{k,p,M,\varepsilon}\left(p^r\left(-\frac{1}{2} + \varepsilon\right)\right)\right).$$

(4.8)

We then insert Lemma 2.3 into the right-hand side of (4.8) to obtain

$$\mathcal{H}_{2k,m,M}(p^r) = p^k \mathcal{H}_{m,M}(p^r) \left( C_k + O_{k,p,M,\varepsilon}(p^r)\right).$$

Plugging back into (4.4) yields

$$S_{2k,m,M}(p^r) = \frac{C_k + O_{k,p,M,\varepsilon}\left(p^r\left(-\frac{1}{2} + \varepsilon\right)\right) + O\left(E_{2k,m,M}(p^r)\right)}{1 + O\left(E_{m,M}(p^r) / \mathcal{H}_{m,M}(p^r)\right)}.$$

(4.9)

By Lemma 2.3 and Lemma 3.7, for any choice $\lambda \pmod{p}$ such that $p \nmid (m + M\lambda)$

$$\mathcal{H}_{m,M}(p^r) = \sum_{\ell \pmod{p}} H_{m + M\ell, Mp}(p^r) \geq H_{m + M\lambda, Mp}(p^r) \gg_{p, M, \varepsilon} p^r (1 - \varepsilon).$$

Hence by Lemma 4.2 we have

$$\frac{E_{2k,m,M}(p^r)}{p^k \mathcal{H}_{m,M}(p^r)} \ll_{k,p,M,\varepsilon} \frac{p^r \left(p^{-1} + \varepsilon\right)}{p^r (k + 1 - \varepsilon)} \ll_{p, \varepsilon} p^r (-1 + \varepsilon).$$

The claim now follows from (4.9).

We finally prove Corollary 1.2.

**Proof of Corollary 1.2.** Note that if $\text{tr}(E) \equiv p^r + 1 \pmod{M}$, then we have

$$\#E(\mathbb{F}_{p^r}) = p^r + 1 - \text{tr}(E) \equiv 0 \pmod{M}.$$ 

Therefore $\text{tr}(E) \equiv p^r + 1 \pmod{M}$ if and only if $M$ divides the order $E(\mathbb{F}_{p^r})$ of the group of points on the elliptic curve. Letting $\text{rad}(M) := \prod_{\ell \mid M} \ell$ be the *radical of $M$, we see by Lagrange’s Theorem and the fact that $E$ is abelian that for $E$ with $\text{tr}(E) \equiv p^r + 1 \pmod{M}$ there exists an element of $E$ of order $\text{rad}(M)$. In particular, for $M$ squarefree we obtain an element of order $M$ if and only if $\text{tr}(E) \equiv p^r + 1 \pmod{M}$. Thus, plugging in the definition (1.1), we see that for every $\kappa \in \mathbb{N}_0$

$$S_{k,p^r+1,M}(p^r) = \sum_{E \in E(\mathbb{F}_{p^r})} \frac{\text{tr}(E)^\kappa}{\#\text{Aut}_{\mathbb{F}_{p^r}}(E)} = S_{k,M}^*(p^r).$$

The proof now follows immediately from Theorem 1.1 by plugging in $m \equiv p^r + 1 \pmod{M}$. Here we group together those prime powers $p^r$ in the same congruence class $\pmod{M}$ and apply the result in each case, using the fact that the error term is uniform in $m$. □
References

[1] E. Artin, *Quadratische Körper im Gebiete der höheren Kongruenzen. II. Analytischer Teil*, Math. Z. **19** (1924), 207–246.

[2] T. Barnet-Lamb, D. Geraghty, M. Harris, and R. Taylor, *A family of Calabi–Yau varieties and potential automorphy II*, Publ. Res. Inst. Math. Sci. **47** (2011), 29–98.

[3] B. Birch, *How the number of points of an elliptic curve over a fixed prime field varies*, J. London Math. Soc. **43** (1968), 57–60.

[4] C. Breuil, B. Conrad, F. Diamond, and R. Taylor, *On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercises*, J. Amer. Math. Soc. **14** (2001), 843–939.

[5] K. Bringmann and B. Kane, *Sums of class numbers and mixed mock modular forms*, Math. Proc. Cambridge Phil. Soc. **167** (2019), 321–333.

[6] B. Brown, N. Calkin, T. Flowers, K. James, E. Smith, and A. Stout, *Elliptic curves, modular forms, and sums of Hurwitz class numbers*, J. Number Theory **128** (2008), 1847–1863.

[7] L. Clozel, M. Harris, and R. Taylor, *Automorphy for some $\ell$-adic lifts of automorphic mod $\ell$ Galois representations*, Publ. Math. Inst. Hautes Études Sci. **108** (2008), 1–181.

[8] H. Cohen, *Sums involving the values at negative integers of $L$-functions of quadratic characters*, Math. Ann. **217** (1975), 217–285.

[9] P. Deligne, *La conjecture de Weil I*, Inst. Hautes Études Sci. Publ. Math. **43** (1974), 273–307.

[10] Digital Library of Mathematical Functions, National Institute of Standards and Technology, http://dlmf.nist.gov/.

[11] M. Eichler, *On the class of imaginary quadratic fields and sums of divisors of natural numbers*, J. Indian Math. Soc. **19** (1956), 153–180.

[12] M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progr. Math. **55**, Birkhäuser, 1985.

[13] B. Gross, D. Zagier, *Heegner points and derivatives of $L$-series*, Invent. Math. **84** (1986), 225–320.

[14] M. Harris, *Galois representations, automorphic forms, and the Sato-Tate conjecture*, Indian J. Pure Appl. Math. **45** (2014), no. 5, 707–746.

[15] M. Harris, N. Shepherd-Barron, and R. Taylor, *A family of Calabi–Yau varieties and potential automorphy*, Ann. Math. **171** (2010), 779–813.

[16] H. Hasse, *Zur Theorie der abstrakten elliptischen Funktionenkörper. I, II, and III*, J. reine angew. Math. **175** (1936), 55–62, 69–88, 193–208.

[17] F. Hirzebruch and D. Zagier, *Intersection numbers of curves on Hilbert modular surfaces and modular forms of Nebentypus*, Invent. Math. **36** (1976), 57–113.

[18] Y. Ihara, *Hecke polynomials as congruence $\zeta$ functions in elliptic modular case*, Ann. Math. **85** (1967), 267–295.

[19] N. Kaplan and I. Petrow, *Elliptic curves over a finite field and the trace formula*, Proc. London Math. Soc. **115** (2017), 1317–1372.

[20] N. Kaplan and I. Petrow, *Traces of Hecke operators and refined weight enumerators of Reed–Solomon codes*, Trans. Amer. Math. Soc. **370** (2018), 2537–2561.

[21] N. Koblitz, *Introduction to elliptic curves and modular forms*, Graduate texts in Math. **97**, Springer-Verlag, 1993.

[22] J. Littlewood, *On the class number of the corpus $P(\sqrt{-k})$*, Proc. London Math. Soc. **27** (1928), 358–372.

[23] M. Mertens, *Eichler-Selberg type identities for mixed mock modular forms*, Adv. Math. **301** (2016), 359–382.

[24] M. Mertens, K. Ono, and L. Rolen, *Mock Modular Eisenstein series with Nebentypus*, preprint.

[25] K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and $q$-series*, CMBS Regional Conference Series in Mathematics **102** (2004), American Mathematical Society, Providence, RI, USA.

[26] P. Sarnak, *Statistical properties of eigenvalues of the Hecke operators*, Analytic Number Theory and Diophantine Problems, Progr. Math. **70**, Stillwater, 1984, Birkhäuser, Basel (1987), 321–331.

[27] J.-P. Serre and H. Stark, *Modular forms of weight $\frac{1}{2}$*, in Modular functions of one variable VI, Lecture notes in Math. **627** (1977), Springer, Berlin, 27–67.
[28] J.-P. Serre, Répartition asymptotique des valeurs propres de l’opérateur de Hecke $T_p$, J. Amer. Math. Soc. 10 (1997), 75–102.
[29] C. Siegel, Über die Classenzahl quadratischer Zahlkörper, Acta. Arith. 1 (1935), 83–86.
[30] J. Silverman, The arithmetic of elliptic curves, Graduate texts in Math. 106, Springer-Verlag, 2009.
[31] J. Sturm, Projections of $C^\infty$ automorphic forms, Bull. Amer. Math. Soc. 2 (1980), 435–439.
[32] R. Taylor and A. Wiles, Ring theoretic properties of certain Hecke algebras, Ann. Math. 141 (1995), 553–572.
[33] A. Wiles, Modular elliptic curves and Fermat’s Last Theorem, Ann. Math. 141 (1995), 443–551.

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