Asymptotic analysis for a second-order curved thin film

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Abstract
We consider a second-order thin curved film whose behavior is governed by an energy made up of a first-order nonlinear part depending on the gradient of the deformation augmented by a quadratic second-order part depending on the tensor of second derivatives of the deformation. We carry out a 3D–2D analysis through an asymptotic expansion in powers of the thickness of the film as it tends to zero.

Keywords
Formal asymptotic expansion, curved thin films, variational methods

1. Introduction
We are interested in this study in some second-order functionals that model the mechanics of thin curved films. We study the behavior of such functionals as the thickness of the thin film goes to zero by performing a 3D–2D analysis. The energies considered contain a first-order part depending on the gradient of the deformation, as in the case of hyperelastic materials, through a nonlinear density. The second part of the energy contains a quadratic term depending on the tensor of the second derivatives of the deformation. When the quadratic form is the square of the Euclidean norm, we recover the example of the interfacial energy term of Van der Waals type for martensitic materials (see [1–6]).

Many recent works were interested in the study of second gradient thin films. Most of them in the context of fluid mechanics and fluid–structure interactions [7–9]. The second-order term might represent some higher-order perturbation that induce weak compactness in the Sobolev space \( W^{2,p} \) and thus strong compactness in \( W^{1,p} \). Adding such a higher-order term was also necessary in order to induce the existence of some curve energy terms (see [10]). Second-order terms appear also in some Phase Field Crystal models where they control the variations represented by the first-order term (see [11]). We also cite some works involving higher-order terms related to phase transitions [12–17].

The 3D–2D dimension reduction for thin films is an efficient way to obtain a two-dimensional (2D) model that approximates a three-dimensional (3D) realistic one when the thickness is small enough. 2D approximations are easier to study and simpler in terms of numerical simulations with less cost in data and time solving (see [18, 19]).

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 Basically, there are two main ways in proceeding with the dimension reduction. The first method, that we consider in this work, is the formal asymptotic expansion method. It is based on an Ansatz assuming that the deformation of the 3D thin film admits an asymptotic expansion in powers of the thickness in order to characterize the limit behavior. The method of formal asymptotic expansion for plates in a functional framework was introduced by Ciarlet and Destuynder [20] and Ciarlet [21], after the pioneering works of Friedricks and Dressler [22] and Goldenveizer [23]. Many authors used this method in performing dimension reduction for different models such as Sanchez-Palencia [24–26], Ciarlet and Lods [27], Ciarlet [28], Miara [29], Castineira and Rodriguez-Aros [30, 31], and Chen et al. [32]. Another approach of this method using variational formulation only in terms of displacements was derived by Raoult [33]. A general description of this method and of works where it was used can be found in Ciarlet [34] (see [29, 35–37]). Later on, this method was refined and improved in terms of assumptions by Fox et al. [38]. Their method is based on the resolution of a sequence of Euler–Lagrange equations.

The second approach for dimension reduction is variational. It needs no ansatz on the deformations and uses arguments of Γ-convergence, a kind of convergence that was introduced by De Giorgi [39], De Giorgi and Franzoni [40]. The main properties of the Γ-convergence are as follows: first, working on a separable metric space and up to a subsequence, the Γ-limit always exist (see Proposition 1.42 in Braides [41]), and second, if a sequence of almost minimizers stays in a compact then the limits of the converging subsequences minimize the Γ-limit. Thus, the limit model obtained, provides a description of the behavior of the minimizers of the sequence of energies depending on the thickness, as it goes to zero. Also, the limit obtained being lower semi-continuous with respect to the considered topology, the existence of minimizers is thus guaranteed (see [42]). This method was used in many works in the context of dimension reduction such as Ignat and Zorgati [11], Acerbi et al. [43], Babadjian et al. [44], Braides et al. [45], Fonseca and Francfort [46], Le Dret and Raoult [47, 48], and Zorgati [49–51].

In this work, we use the formal asymptotic expansion method introduced by Pantz [52], which leads to the resolution of a succession of minimization problems (see [53, 54]). Indeed, it appeared that the formal expansion method yields an incorrect result compared to the Γ-convergence method in some cases. Pantz’s method makes it possible to obtain formal expansions that agree with Γ-convergence limits as much as is possible as it was the case for the nonlinear membrane model obtained using Γ-convergence techniques by Le Dret and Raoult [47], where Pantz’ limit model [52, 55] contained the same energy density before relaxation, in comparison with the one obtained precendently by Fox et al. [38]. Since Pantz’ method is based on minimization problems (see Section 4), it capts some phenomena that are not seen by the classical algebraic approach based on the resolution of a series of partial differential equations.

The equilibrium state of the thin curved film occupying the domain $\tilde{\Omega}_h \subset \mathbb{R}^3$ of thickness $h$ with midsurface, a bounded 2D submanifold $\tilde{S}$, and undergoing a deformation $\tilde{\varphi}$ is thus described by the minimizers of the energy

$$\tilde{J}^h(\tilde{\varphi}) = \int_{\tilde{\Omega}_h} \left[ Q(\nabla^2 \tilde{\varphi}) + W(\nabla \tilde{\varphi}) \right] dx$$

(1)

where $\nabla^2 \tilde{\varphi}$ is the $3 \times 3 \times 3$ tensor of second derivatives, $Q$ is a positive quadratic form and $W(\nabla \tilde{\varphi})$ represents the elastic energy by unit volume of the material depending on the gradient of the deformation $\tilde{\varphi}$. We are interested in the behavior of this energy and its minimizers when the thickness $h$ of the film goes to zero.

We begin our study by introducing the notation and some geometrical preliminaries, then we rescale the energy in order to work on a domain independent of the thickness $h$. Next, we carry out the computations of the terms appearing in the asymptotic expansion of the energy, which will bring us to the resolution of the minimization problems. In order to be able to perform such an analysis, we will consider a first-order nonlinear term given by the Saint Venant–Kirchhoff material stored energy function.

The term of order zero obtained by solving the first four minimization problems will provide the 2D scaled limit energy that we illustrate with an example in the case of plates and also in some simple examples of curved films. Notice that we recover, as a particular case supposing the quadratic form to be the square of the norm, the 2D second-order energy part, obtained by convergence arguments in Bhattacharya and James [1] for the planar films and Le Dret and Zorgati [2, 3] for the curved films. The first-order energy part generalizes to the case of curved films the results obtained in Pantz [52]. Moreover, our results prove the efficiency of Pantz’ method for formal asymptotic expansion in the case of second-order energies and also in the case of curved thin films. The limit model obtained generalizes preceding works containing second-order terms with a general quadratic form. It provides explicit limit energy functionals that can be used for numerical simulations. The asymptotic
expansions obtained can also lead, for given curved midsurface, to go along with the minimization problems in order to obtain terms of higher order.

Finally, in order to make the paper more concise, we will postpone some technical proofs in an appendix section.

2. Notations and geometrical preliminaries

Let \((e_1, e_2, e_3)\) be the orthonormal canonical basis of the Euclidean space \(\mathbb{R}^3\). The norm of a vector \(v\) of \(\mathbb{R}^3\) is denoted by \(|v|\), the scalar product of two vectors \(u\) and \(v\) by \(u \cdot v\), their vector product by \(u \wedge v\), and their tensor product \(u \otimes v\). Let \(M_{33}\) be the space of \(3 \times 3\) matrices with real coefficients equipped with the usual norm \(|F| = \sqrt{\text{tr}(F^T F)}\). We denote by \(A = (a_1|a_2|a_3)\) the matrix of \(M_{33}\) whose first column is \(a_1\), the second is \(a_2\), and the third is \(a_3\). We denote by \(\nabla\) and \(\nabla^2\) the \(3 \times 3\) and \(3 \times 3 \times 3\) tensor of first and second derivatives of a deformation \(\varphi: \Omega \to \mathbb{R}^3\), where \(\Omega\) is an open subset of \(\mathbb{R}^3\).

Let \(M_{333}\) be the space of \(3 \times 3 \times 3\) tensors. When \(P = (p_{ijk})\) is a tensor of order 3, the tensor with components \(r_{ijk} = p_{ijk}\) is the transpose of \(P\) with respect to the second and third index. The tensor \(P\) is said to be symmetric when it is equal to its transpose with respect to the second and third indices. Let \(P\) be a tensor of order \(p \geq 1\) and \(R\) a tensor of order \(r \geq 1\). We define the contracted tensor product of the tensors \(P\) and \(R\) and we denote by \(P \otimes R\), the tensor of order \(p + r - 2\) whose components are obtained by summation of the products of the components of \(P\) and \(R\), on the last index of the components of \(P\) and the first index of those of \(R\). For example, using the summation convention, if \(p = 3\) and \(r = 2\) we have

\[
P \otimes R = P_{ijk} R_{km}(e_i \otimes e_j \otimes e_m).
\]

Let \(Q: M_{333} \to \mathbb{R}^+\) be a positive quadratic form whose corresponding bilinear symmetric form is \(B: M_{333} \times M_{333} \to \mathbb{R}\). In order to simplify our computations, we will make the following assumptions on the quadratic form \(Q\): we suppose that the basis \((e_i \otimes e_j \otimes e_k)_{1 \leq i,j,k \leq 3}\) of \(M_{333}\) is orthogonal with respect to the quadratic form \(Q\), that is, the matrix of the quadratic form in this basis is diagonal so that for every \(A \in M_{333}\), we have

\[
Q(A) = \sum_{1 \leq i,j,k \leq 3} A_{ijk}^2 Q_{ijk},
\]

where \(A_{ijk}\) denote the components of the tensor \(A\) in the orthogonal basis and \(Q_{ijk} = Q(e_i \otimes e_j \otimes e_k)\). Notice that some components \(Q_{ijk}\) might vanish since we do not suppose \(Q\) to be definite. Nevertheless, in order to solve the minimization problems, we suppose that at least one of the \(Q_{ijk}\) is strictly positive.

Throughout this work, we will use the convention of summation. The Greek indices take their values in the set \(\{1, 2, 3\}\) and the Latin indices take their values in \(\{1, 2\}\).

We consider an open domain \(\bar{\Omega}_h\) occupied by a thin curved film of thickness \(h\) whose midsurface is \(\bar{S}\), a 2D bounded submanifold of class \(C^2\) of \(\mathbb{R}^3\) that admits an atlas consisting in one chart. Let \(\psi\) be this chart. It is thus a \(C^2\)-diffeomorphism from an open bounded domain \(\omega \subset \mathbb{R}^2\) into \(\bar{S}\). We suppose that \(\omega\) has a Lipschitz boundary \(\partial \omega\) and that \(\psi\) is extendable into a function of \(C^2(\bar{\omega}, \mathbb{R}^3)\). Let \(a_{\alpha}(x) = \psi_{,\alpha}(x)\) be the vectors of the covariant basis of the tangent plane \(T_{\psi(x)}\bar{S}\) associated to the chart \(\psi\), where \(\psi_{,\alpha}\) denote the derivatives of \(\psi\) with respect to the variable \(\alpha\). We suppose that there exist \(\delta > 0\) such that

\[
|a_1(x) \wedge a_2(x)| \geq \delta \quad \text{on } \bar{\omega},
\]

and we define the vector \(a_{3}(x) = \frac{a_1(x) \wedge a_2(x)}{|a_1(x) \wedge a_2(x)|}\) belonging to \(C^1(\bar{\omega}, \mathbb{R}^3)\). This is the normal unit vector to the tangent plane. The vectors \(a_1(x), a_2(x),\) and \(a_3(x)\) constitute the covariant basis. We define the vectors of the contravariant basis by the relations

\[
a^{\alpha}(x) \in T_{\psi(x)}\bar{S}, \quad a^{\alpha}(x) \cdot a_{\beta}(x) = \delta_{\alpha\beta} \quad \text{and} \quad a^{3}(x) = a_3(x).
\]

We set \(A(x) = (a_1(x)|a_2(x)|a_3(x))\). We notice that \(A(x)\) is an invertible matrix on \(\bar{\omega}\) and its inverse is given by

\[
A^{-1}(x) = (a^{\alpha}(x)|a^{3}(x)|a^{3}(x))^T.
\]

We also notice that

\[
\det A(x) = |\text{cof} A(x) \cdot e_3| = |a_1(x) \wedge a_2(x)| > \delta > 0 \quad \text{on } \bar{\omega}.
\]
The domain \( \tilde{\Omega}_h \) is defined as
\[
\tilde{\Omega}_h = \left\{ x \in \mathbb{R}^3, \exists \tilde{x} \in \tilde{\Omega}, x = \tilde{x} + \eta a_3(\psi^{-1}(\tilde{x})) \text{ with } -\frac{h}{2} < \eta < \frac{h}{2} \right\},
\]
and for \( \tilde{x} \in \tilde{\Omega}_h \), we have
\[
\tilde{x} = \tilde{\pi}(\tilde{x}) + \left[ (\tilde{x} - \tilde{\pi}(\tilde{x})) \cdot a_3(\psi^{-1}(\tilde{\pi}(\tilde{x}))) \right] \cdot a_3(\psi^{-1}(\tilde{\pi}(\tilde{x}))),
\]
where \( \tilde{\pi} \) is the orthogonal projection which sends \( \tilde{\Omega}_h \) onto \( \tilde{S} \). The map \( \tilde{\pi} \) is well defined and is of class \( C^1 \) for \( h \) small enough compared to the radius of curvature of \( \tilde{S} \). We can thus define the curvilinear coordinates of a point of the domain \( \Omega_h \) associated to the chart \( \psi \) by
\[
(x_1, x_2) = \psi^{-1}(\tilde{\pi}(\tilde{x})) \text{ and } x_3 = (\tilde{x} - \tilde{\pi}(\tilde{x})) \cdot a_3(\psi^{-1}(\tilde{\pi}(\tilde{x}))).
\]

We now define the domain \( \Omega_h \) by
\[
\Omega_h = \left\{ x \in \mathbb{R}^3, x = (x_1, x_2, x_3), (x_1, x_2) \in \omega, -\frac{h}{2} < x_3 < \frac{h}{2} \right\}.
\]

We choose a small enough \( h^* \) and define a \( C^2 \)-diffeomorphism \( \Psi: \Omega^*_h \to \tilde{\Omega}^*_h \) by
\[
\Psi(x_1, x_2, x_3) = \psi(x_1, x_2) + x_3 a_3(x_1, x_2). \tag{2}
\]
This diffeomorphism is the inverse of the rescaling that we will use in order to work on a flat domain, and its gradient verifies:
\[
\nabla \Psi(x_1, x_2, x_3) = A(x_1, x_2) + x_3(a_{3,1}(x_1, x_2)|a_{3,2}(x_1, x_2)|0).
\]

The matrix \( \nabla \Psi(x_1, x_2, x_3) \) is everywhere invertible and its determinant is the Jacobian of the rescaling. In the following, \( h \) will denote some sequence of real numbers less than \( h^* \) and decreasing into zero.

3. The 3D and rescaled problems

The thin film occupies, in its reference configuration, the domain \( \tilde{\Omega}_h \). We associate with each deformation \( \tilde{\varphi}: \tilde{\Omega}_h \to \mathbb{R}^3 \) the energy
\[
\tilde{\mathcal{J}}^h(\tilde{\varphi}) = \tilde{K}^h(\tilde{\varphi}) + \tilde{\mathcal{I}}^h(\tilde{\varphi}),
\]
where
\[
\tilde{K}^h(\tilde{\varphi}) = \int_{\tilde{\Omega}_h} Q(\nabla^2 \tilde{\varphi}) \, dx,
\]
represents the second-order energy and
\[
\tilde{\mathcal{I}}^h(\tilde{\varphi}) = \int_{\tilde{\Omega}_h} W(\nabla \tilde{\varphi}) \, dx,
\]
is the elastic energy, with \( W: \mathbb{M}^3_{33} \to [0, +\infty[ \) regular and verifying the following growth and coercivity assumptions: There exists \( c_1, c_2 > 0 \) such that
\[
\forall A \in \mathbb{M}^3_{33}, \ c_1(|A|^2 - 1) \leq W(A) \leq c_2(|A|^q + 1) \quad \text{with } 2 \leq q < 6.
\]
Notice that we do not consider in this work any constraints of determinant type related to loss of orientation or matter compenetration, see Anza-Hafsa and Mandallena [56]. For simplicity, we will only consider homogeneous boundary conditions of placement imposed on the lateral boundary of the film. We thus introduce the space of admissible deformations
\[
\bar{V}_h = \{ \tilde{\varphi} \in H^2(\tilde{\Omega}_h; \mathbb{R}^3); \tilde{\varphi}(\tilde{x}) = \tilde{\pi}x \text{ on } \tilde{\Gamma}_h \},
\]
where $\widetilde{A} = (\tilde{a}_1|\tilde{a}_2|\tilde{a}_3) \in M_{33}$ is a given constant matrix and $\tilde{\Gamma}_h = \Psi(\partial \omega \times [-h/2, h/2])$ is the lateral surface of $\tilde{\Omega}_h$. Note that due to the growth condition satisfied by $W$ and the Sobolev embedding theorem, the energy functional $\tilde{J}_h$ is well defined and takes its values in $\mathbb{R}$ for $\phi \in \tilde{V}_h$.

The minimization problem consists in finding $\tilde{\phi}_h \in \tilde{V}_h$ such that

$$
\tilde{J}_h(\tilde{\phi}_h) = \min_{\phi \in \tilde{V}_h} \tilde{J}_h(\phi).
$$

It is fairly obvious that such a minimizer exists if we suppose the quadratic form to be definite, using weak convergence in $H^2$ that entails strong convergence in $W^{1,q}$ for $q < 6$. This is not the case for general positive quadratic form.

To study the behavior of this energy and its eventual minimizers, we begin by flattening and rescaling the minimizing problem in order to work on a fixed cylindrical domain. For clarity, we proceed in two steps, flatten first and then rescale.

If $\tilde{\phi}_h$ is a deformation of the film in its reference configuration, we define $\phi_h : \Omega_h \rightarrow \mathbb{R}^3$ by setting $\phi_h(x) = \tilde{\phi}_h(\Psi(x))$ for all $x \in \Omega_h$. Since $\Psi$ is a $C^2$-diffeomorphism, $\phi_h$ is in $H^2$ whenever $\tilde{\phi}_h$ is in $H^2$. For every such deformation $\phi_h$, we thus set $J_h(\phi_h) = J_h(\phi_h \circ \Psi^{-1})$ and we obtain that

$$
J_h(\phi_h) = \int_{\Omega_h} \left\{ \mathcal{Q} \left( \left( \nabla^2 \phi_h \otimes (\nabla \Psi^{-1} \circ \Psi) \right)^T \otimes (\nabla \Psi^{-1} \circ \Psi) \right) + W(\nabla \phi_h (\nabla \Psi^{-1} \circ \Psi)) \right\} \det \nabla \Psi \, dx.
$$

This completes the flattening step.

Let us now turn to the rescaling step. Since the total energy in the limit membrane deformation regime is of order $h$, we are actually interested in the asymptotic behavior of the energy per unit thickness $\frac{1}{h}J_h$.

We define $z_h : \Omega_1 \rightarrow \Omega_h$ by

$$
z_h(x_1, x_2, x_3) = (x_1, x_2, hx_3).
$$

To each deformation $\phi_h$ on $\Omega_h$, we associate the deformation $\phi(h)$ on $\Omega_1$ defined by $\phi(h)(x) = \phi_h(z_h(x))$ and its rescaled deformation gradient:

$$
\nabla_h \phi(h) = \begin{pmatrix} \phi(h),1 & \phi(h),2 & \frac{1}{h} \phi(h),3 \end{pmatrix} = \nabla \phi(h) + \frac{1}{h} \phi(h),3 \otimes e_3,
$$

and rescaled tensor of second derivatives:

$$
\nabla^2_h \phi(h) = \nabla^2 \phi(h) + \frac{1}{h} (\nabla \phi(h),3 \otimes e_3 + (\nabla \phi(h),3 \otimes e_3)^T) + \frac{1}{h^2} \phi(h),33 - \phi(h),33 \otimes e_3 \otimes e_3,
$$

where the notations $\nabla \phi = \phi,\alpha \otimes e_\alpha$ and $\nabla^2 \phi = \phi,\alpha\beta \otimes e_\alpha \otimes e_\beta$ stand for in-plane first and second gradients, respectively. Note that $\nabla \phi(h),3 = (\nabla \phi),3$, hence the unambiguous notation $\nabla \phi,3$. Note also that $\nabla \phi$ is $M_{33}$-valued and $\nabla^2 \phi$ is $\mathbb{M}_{3333}$-valued.

Finally, we set $J(h)(\phi(h)) = \frac{1}{h} J_h(\phi_h)$. Thus, we obtain a rescaled energy of the form:

$$
J(h)(\phi(h)) = K(h)(\phi(h)) + I(h)(\phi(h)),
$$

with

$$
I(h)(\phi) = \int_{\Omega_1} W(\nabla_h \phi A_h) d_x, dx,
$$

and

$$
K(h)(\phi) = \int_{\Omega_1} \mathcal{Q} \left( (\nabla^2_h \phi \otimes A_h)^T \otimes A_h + \nabla_h \phi \otimes B_h \right) d_x, dx,
$$

where $\tilde{\phi}_h \in \tilde{V}_h$, $\phi_h \in \tilde{V}_h$, and $\mathcal{Q}$ is the quadratic form as defined above.
where
\[ d_h(x) = \det \nabla \Psi(z_h(x)), \quad A_h(x) = \nabla \Psi^{-1}(\Psi(z_h(x))) \text{ and } B_h(x) = \nabla^2 \Psi^{-1}(\Psi(z_h(x))). \]

Problem (3) becomes: Find \( \varphi(h) \in V_h \) such that
\[ J(h)(\varphi(h)) = \min_{\varphi \in V_h} J(h)(\varphi), \]
with
\[ V_h = \{ \varphi \in H^2(\Omega_1; \mathbb{R}^3); \varphi(x) = \tilde{A}\Psi(z_h(x)) \text{ on } \partial \omega \times [-\frac{1}{2}, \frac{1}{2}] \}. \]

Since problems (3) and (4) are equivalent, and that the existence of solutions to problem (3) is obvious, there exists a solution \( \varphi(h) \in V_h \) for problem (4) for every \( 0 < h < h^* \).

We can now compute the formal asymptotic expansion on \( \Omega_1 \).

4. Asymptotic expansion

Performing the formal asymptotic expansion method with a general elastic energy term is obviously impossible. We need an explicit expression of the internal elastic energy density \( W \). This conducts us to consider an example of an elastic energy term that is relatively simple in order to permit the needed computations. We thus consider an elastic energy part that corresponds to a Saint Venant–Kirchhoff material. This choice is also widely considered for fluid–structure interaction problems (see, for instance, Boulakia and Guerrero [9]). Analogous energies associating the Saint Venant–Kirchhoff elastic part and second-order part were obtained as limit models in the context of fiber reinforced composite [57]. Thus we suppose that
\[ W(A) = \frac{\lambda}{8}(\text{tr}(A^T A - I)^2) + \frac{\mu}{4} \text{tr}((A^T A - I)^2) \text{ for every } A \in \mathbb{M}_{33}, \]
where \( \lambda > 0 \) and \( \mu > 0 \) are the Lamé constants of the material. Notice that we suppose \( \lambda > 0 \) rather then the usual condition \( 2\mu + 3\lambda > 0 \), which is a more restrictive condition but necessary for the resolution of the following minimizing problems.

We denote by \( P(h) \) the problem of finding \( \varphi(h) \) verifying
\[ J(h)(\varphi(h)) = \inf_{\varphi \in V_h} J(h)(\varphi), \]
with
\[ V_h = \{ \varphi \in H^2(\Omega_1; \mathbb{R}^3); \varphi(x) = \tilde{A} \circ \Psi(z_h(x)) \text{ on } \partial \omega \times \left(-\frac{1}{2}, \frac{1}{2}\right) \}, \]
where \( \tilde{A} = (\tilde{a}_i | \tilde{a}_2 | \tilde{a}_3) \) is a constant matrix of \( \mathbb{M}_{33} \).

**Ansatz 1.** We suppose that the solution \( \varphi(h) \) of the problem \( P(h) \) admits a formal expansion in powers of \( h \), that is,
\[ \varphi(h) = \sum_{n \geq 0} h^n \varphi^n. \]

**Remark 1.** The first and second derivatives of \( \Psi^{-1} \) taken at the point \( \Psi(z_h(x)) \) admit each one an asymptotic expansion in powers of \( h \). This can be obtained using Taylor–Lagrange formula and supposing that \( \Psi^{-1} \) is sufficiently regular.

Using Ansatz 1 and Remark 1, we will show that the energy \( J(h) \) admits an asymptotic expansion in powers of \( h \). We begin by proving the following proposition.

**Proposition 1.** The energy \( J(h) \) admits a first formal asymptotic expansion in powers of \( h \):
\[ J(h)(\varphi) = \sum_{n=-4}^{\infty} h^n J^n_h(\varphi), \]
with
\[ J^{-4}_h(\varphi) = \int_{\Omega_1} \left( Q(d_h^{-2} \otimes e_i \otimes e_j) + C^{ijkl} E_h^{-2} E_h^{-2} \right) c_0 \, dx, \]
$$J_h^{-3}(\phi) = \sum_{p=-4}^{-3} \int_{\Omega_1} \left( B(d_{ij}^{p+2} \otimes e_i \otimes e_j, d_{ij}^{p-5} \otimes e_i \otimes e_j) + C^{ijkl} E_{ij}^{p+2} E_{kl}^{p-5} \right) c_0$$

$$+ \left( Q(d_{ij}^{-2} \otimes e_i \otimes e_j) + C^{ijkl} E_{ij}^{-2} E_{kl}^{-2} \right) c_1 \, dx$$

and for every $n \geq -2$

$$J_h^n(\phi) = \sum_{p=-4}^{n} \int_{\Omega_1} \left( B(d_{ij}^{p+2} \otimes e_i \otimes e_j, d_{ij}^{n-p-2} \otimes e_i \otimes e_j) + C^{ijkl} E_{ij}^{p+2} E_{kl}^{n-p-2} \right) c_0 \, dx$$

$$+ \sum_{p=-4}^{n-1} \int_{\Omega_1} \left( B(d_{ij}^{p+2} \otimes e_i \otimes e_j, d_{ij}^{n-p-3} \otimes e_i \otimes e_j) + C^{ijkl} E_{ij}^{p+2} E_{kl}^{n-p-3} \right) c_1 \, dx$$

$$+ \sum_{p=-4}^{n-2} \int_{\Omega_1} \left( B(d_{ij}^{p+2} \otimes e_i \otimes e_j, d_{ij}^{n-p-4} \otimes e_i \otimes e_j) + C^{ijkl} E_{ij}^{p+2} E_{kl}^{n-p-4} \right) c_2 \, dx,$$

where $d_{ij}, E_{ij}$ will be defined in equations (15)–(17), $c_0, c_1, c_2$ in equation (9), $C^{ijkl}$ are the components of the elasticity tensor that will be defined in equation (7), and $B$ is the symmetric bilinear form associated with the quadratic form $Q$.

**Remark 2.** We notice that the terms $J_h^n$ depend on $h$. It is not an asymptotic expansion in the usual sense of the term. Nevertheless, Remark 1 will enable us to deduce (see Proposition 3) that $J(h)$ admits a second asymptotic expansion in powers of $h$ containing terms independent of $h$ of the form

$$J(h)(\phi) = \sum_{n \geq -4} J^n(\phi) h^n.$$

Nonetheless, the useful expansion is actually the one given by Proposition 1.

Next, we use Pantz’s method for the asymptotic expansion consisting in solving a succession of minimization problems and based on the following Proposition 2 [52]. We provide the proof in the appendix section for the convenience of the reader. In the following, we will indicate by $\phi$ the sequence $(\phi_i)_{i \in \mathbb{N}}$ and we will write

$$J(h)(\phi) = J(h)(\phi(h)).$$

**Proposition 2.** The solution $\phi = (\phi^0, \phi^1, \phi^2, ...)$ of the sequence of problems $P(h)$ verifies

$$\phi \in \bigcap_{n=-4}^{\infty} Q_n,$$

with

$$Q_{-4} = \left\{ \phi \in C(\overline{\Omega_1}; \mathbb{R}^3) : \sum_n \phi^n h^n \in V_h \right\},$$

and

$$Q_{n+1} = \left\{ \phi \in Q_n, J^n(\phi) = \inf_{\psi \in \Omega_1} J^n(\psi) \right\}.$$

We denote by $P_n$ the problem consisting in finding the minimizers of $J^n$ on $Q_n$.

Proposition 2 enables us to obtain the solution $\phi(h)$ of $P(h)$ solving the sequence of problems $P_n$. 
4.1. Proof of Proposition 1 and computation of the asymptotic expansion terms

Since the first-order energy term is of a Saint Venant–Kirchhoff material type, we have

$$\tilde{T}^h(\tilde{\varphi}) = \int_{\Omega^h} W(\nabla \tilde{\varphi}),$$

with

$$W(\nabla \tilde{\varphi}) = C^{ijkl} \tilde{E}^h_{ij}(\tilde{\varphi}) \tilde{E}^h_{kl}(\tilde{\varphi}),$$

where

$$C^{ijkl} = \lambda \delta^{ij} \delta^{kl} + \mu (\delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk}),$$

(7)

is the elasticity tensor and

$$\tilde{E}^h(\tilde{\varphi}) = \frac{1}{2} [\tilde{\varphi}^T \nabla \tilde{\varphi} - I]$$

is the Green–Lagrange strain tensor.

**Lemma 1.** The Jacobian of the change of variables process \( \det \nabla \Psi(z_h(x)) \) is polynomial with respect to \( h \) with

$$\det \nabla \Psi(z_h(x)) = c_0 + hc_1 + h^2c_2.$$  

(8)

**Proof.** Expanding the determinant with respect to the first and second columns of the gradient, we obtain

$$c_0 = \det(a_1|a_2|a_3), \quad c_1 = x_3 \left( \det(a_1|a_{3,2}|a_3) + \det(a_{1,1}|a_2|a_3) \right) \quad \text{and} \quad c_2 = x_3^2 \det(a_{3,1}|a_{3,2}|a_3).$$  

(9)

**Lemma 2.** The components \( \tilde{E}^h_{ij}(\tilde{\varphi}) \) of the strain tensor \( \tilde{E}^h(\tilde{\varphi}) \) admit an asymptotic expansion in powers of \( h \),

$$\tilde{E}^h_{ij}(\tilde{\varphi}) = \sum_{n \geq -2} \tilde{E}^h_{ij} h^n,$$

(10)

where

$$\tilde{E}^h_{ij} = \tilde{E}^0_{ij} = \frac{1}{2} (\tilde{A}^0_{ij} + \tilde{B}^0_{ij} + \tilde{C}^0_{ij}),$$

$$\tilde{E}^h_{ij} = \frac{1}{2} (\tilde{A}^0_{ij} + \tilde{B}^0_{ij} + \tilde{C}^0_{ij} - \delta_{ij}),$$

and

$$\tilde{E}^n_{ij} = \frac{1}{2} (\tilde{A}^n_{ij} + \tilde{B}^{n+1}_{ij} + \tilde{C}^{n+2}_{ij}) \text{ for } n > 0,$$

(11)

(12)

(13)

with

$$\tilde{A}^n_{ij} = \Psi_{a,i}^{-1} \Psi_{,a,j}^{-1} \sum_{p=0}^{n} \varphi^{p}_{a} \cdot \varphi^{n-p}_{,a},$$

$$\tilde{B}^n_{ij} = \left( \Psi_{a,d}^{-1} \Psi_{a,j}^{-1} + \Psi_{3,i}^{-1} \Psi_{3,j}^{-1} \right) \left( \sum_{p=0}^{n} \varphi^{p}_{a} \cdot \varphi^{n-p}_{a} \right),$$

(14)

and

$$\tilde{C}^n_{ij} = \Psi_{a,d}^{-1} \Psi_{3,j}^{-1} \sum_{p=0}^{n} \varphi^{p}_{,3} \cdot \varphi^{n-p}_{,3},$$

where the derivatives of \( \varphi^p \) are taken at point \( z_h^{-1} \circ \Psi^{-1}(\tilde{\varphi}) \).
Notice that the coefficients \( \tilde{E}_i^n \) depend on \( h \).

**Proof.** We have

\[
\tilde{E}_i(h) = \frac{1}{2}(\tilde{\varphi}_i \cdot \tilde{\varphi}_j - \delta_{ij}).
\]

Writing the derivatives of \( \tilde{\varphi} \) in term of those of \( \varphi \), we get

\[
\tilde{\varphi}_{ij}(\tilde{x}) = (\varphi_{i,k} \circ z_h^{-1} \circ \Psi^{-1}(\tilde{x}))(z_h^{-1})_{i,j} \circ \Psi^{-1}(\tilde{x})\Psi_i^{-1}(\tilde{x}),
\]

with

\[
(z_h^{-1})_{\alpha,\beta} = \delta_{\alpha\beta} \text{ et } (z_h^{-1})_{i,3} = (z_h^{-1})_{3,i} = \frac{1}{h}\delta_{i3}.
\]

Since \( \varphi(x) = \sum_{p \geq 0} h^p \varphi^p(x) \), we obtain

\[
\tilde{\varphi}_i(\tilde{x}) = \sum_{p \geq 0} h^p \left( (\varphi^p_{i,a} \circ z_h^{-1} \circ \Psi^{-1}(\tilde{x}))\Psi_i^{-1}(\tilde{x}) + \frac{1}{h} (\varphi^p_{3,a} \circ z_h^{-1} \circ \Psi^{-1}(\tilde{x}))\Psi_i^{-1}(\tilde{x}) \right).
\]

Taking the scalar product, we obtain

\[
\tilde{\varphi}_i(\tilde{x}) \cdot \tilde{\varphi}_j(\tilde{x}) = \sum_{n \geq 0} h^n \left( (\varphi^p_{i,a} \circ z_h^{-1} \circ \Psi^{-1}(\tilde{x}))\Psi_i^{-1}(\tilde{x}) + \frac{1}{h} (\varphi^p_{3,a} \circ z_h^{-1} \circ \Psi^{-1}(\tilde{x}))\Psi_i^{-1}(\tilde{x}) \right)
\cdot \left( (\varphi^{n-p}_{j,b} \circ z_h^{-1} \circ \Psi^{-1}(\tilde{x}))\Psi_{j,i}^{-1}(\tilde{x}) + \frac{1}{h} (\varphi^{n-p}_{3,b} \circ z_h^{-1} \circ \Psi^{-1}(\tilde{x}))\Psi_{j,i}^{-1}(\tilde{x}) \right),
\]

and thus

\[
\tilde{\varphi}_i(\tilde{x}) \cdot \tilde{\varphi}_j(\tilde{x}) = \sum_{n \geq 0} h^n \left( (A^n_{ij} + \frac{1}{h} B^n_{ij} + \frac{1}{h^2} C^n_{ij}) \right),
\]

with \( A^n_{ij}, B^n_{ij}, \) and \( C^n_{ij} \) defined as above, which gives the result. \( \square \)

**Lemma 3.** The elastic energy term \( I(h)(\varphi) \) admits an asymptotic expansion in powers of \( h \)

\[
I(h)(\varphi) = \sum_{n=-4}^{+\infty} I_h^n(\varphi)h^n,
\]

where

\[
I_h^{-4}(\varphi) = \int_{\Omega_1} C^{ijkl} E_{ij}^{-2} E_{kl}^{-2} c_0 \, dx,
\]

\[
I_h^{-3}(\varphi) = \sum_{p=-4}^{-3} \int_{\Omega_1} C^{ijkl} E_{ij}^{p+2} E_{kl}^{-p-5} c_0 \, dx + \int_{\Omega_1} C^{ijkl} E_{ij}^{-2} E_{kl}^{-2} c_1 \, dx,
\]

and for every \( n \geq -2 \)

\[
I_h^n(\varphi) = \sum_{p=-4}^{n} \int_{\Omega_1} C^{ijkl} E_{ij}^{p+2} E_{kl}^{n-p-2} c_0 \, dx + \sum_{p=-4}^{n-1} \int_{\Omega_1} C^{ijkl} E_{ij}^{p+2} E_{kl}^{-p-3} c_1 \, dx
\]

\[
+ \sum_{p=-4}^{n-2} \int_{\Omega_1} C^{ijkl} E_{ij}^{p+2} E_{kl}^{-p-4} c_2 \, dx,
\]
where
\[ E_{ij}^{-2} = \frac{1}{2} C_{ij}^{0}, \quad E_{ij}^{-1} = \frac{1}{2}(B_{ij}^{0} + C_{ij}^{1}), \quad E_{ij}^{0} = \frac{1}{2}(A_{ij}^{0} + B_{ij}^{1} + C_{ij}^{2} - \delta_{ij}), \]  
and for every \( n > 0 \)
\[ E_{ij}^{n} = \frac{1}{2}(A_{ij}^{n} + B_{ij}^{n+1} + C_{ij}^{n+2}), \]  
with \( A_{ij}^{n}, B_{ij}^{n}, \) and \( C_{ij}^{n} \) analogous to \( A_{ij}^{0}, B_{ij}^{0}, \) and \( C_{ij}^{0} \) defined in equations (12)–(14) with \( \Psi_{i,j}^{-1} \) being taken at point \( \Psi(z_{0}(x)) \), and where \( c_{0}, c_{1}, c_{2} \) are defined in Lemma 1.

**Proof.** We have that
\[ T_{h}(\tilde{\varphi}) = \int_{\Omega_{h}} C^{ijkl}\tilde{E}_{ij}^{h}(\tilde{\varphi}^{b})\tilde{E}_{kl}^{h}(\tilde{\varphi}^{b})dx. \]

Replacing \( \tilde{E}_{ij}^{h}(\tilde{\varphi}^{b}) \) and \( \tilde{E}_{kl}^{h}(\tilde{\varphi}^{b}) \) by their asymptotic expansion in powers of \( h \) and applying the product of the two series, we obtain
\[ T_{h}(\tilde{\varphi}) = \sum_{n \geq -4} \left( \left( \sum_{p=-4}^{n} \int_{\Omega_{h}} C^{ijkl}\tilde{E}_{ij}^{p+2} E_{kl}^{n-p-2}(\tilde{\varphi}^{b})dx \right) \right) h^{n}. \]

Next, by changing variables, we obtain
\[ I(\varphi) = \sum_{n \geq -4} \left( \left( \sum_{p=-4}^{n} \int_{\Omega_{1}} C^{ijkl}\tilde{E}_{ij}^{p+2} E_{kl}^{n-p-2}(c_{0} + hc_{1} + h^{2}c_{2})dx \right) \right) h^{n}, \]

Then, replacing the Jacobian by its asymptotic expansion in powers of \( h \), we obtain
\[ I(\varphi) = \sum_{n \geq -4} \left( \left( \sum_{p=-4}^{n} \int_{\Omega_{1}} C^{ijkl}\tilde{E}_{ij}^{p+2} E_{kl}^{n-p-2}(c_{0} + hc_{1} + h^{2}c_{2})dx \right) \right) h^{n}, \]

which gives the result. \( \square \)

**Lemma 4.** The second-order energy term \( K(h)(\varphi) \) admits an asymptotic expansion in powers of \( h \)
\[ K(h)(\varphi) = \sum_{n=-4}^{+\infty} K_{h}^{n}(\varphi)h^{n}, \]

with
\[ K_{h}^{-4}(\varphi) = \int_{\Omega_{1}} c_{0} Q(d_{ij}^{-2} \otimes e_{i} \otimes e_{j})dx, \]
\[ K_{h}^{-3}(\varphi) = \int_{\Omega_{1}} \left( c_{1} Q(d_{ij}^{-2} \otimes e_{i} \otimes e_{j}) + \sum_{p=-4}^{3} c_{0} B\left( d_{ij}^{p+2} \otimes e_{i} \otimes e_{j}, d_{ij}^{n-p-2} \otimes e_{i} \otimes e_{j} \right) \right)dx, \]

and for every \( n \geq -2 \)
\[ K_{h}^{n}(\varphi) = \int_{\Omega_{1}} \left( c_{2} \sum_{p=-4}^{n-4} B\left( d_{ij}^{p+2} \otimes e_{i} \otimes e_{j}, d_{ij}^{n-p-4} \otimes e_{i} \otimes e_{j} \right) + \sum_{p=-4}^{n-1} c_{1} B\left( d_{ij}^{p+2} \otimes e_{i} \otimes e_{j}, d_{ij}^{n-p-3} \otimes e_{i} \otimes e_{j} \right) + c_{0} \sum_{p=-4}^{n} B\left( d_{ij}^{p+2} \otimes e_{i} \otimes e_{j}, d_{ij}^{n-p-2} \otimes e_{i} \otimes e_{j} \right) \right)dx, \]
where
\[ d_{ij}^2 = \xi_{ij}, \quad d_{ij}^{-1} = \gamma_{ij} + \zeta_{ij} \quad \text{and for every} \quad n \geq 0, \quad d_{ij}^n = \xi_{ij}^n + \gamma_{ij}^{n+1} + \zeta_{ij}^{n+2}, \]
(17)
with
\[ \xi_{ij}^p = \varphi_{ij}^p \Psi_{ij}^{-1}, \]
(18)
\[ \gamma_{ij}^p = \varphi_{ij}^p \Psi_{ij}^{-1}, \]
(19)
and
\[ \zeta_{ij}^p = \varphi_{ij}^p \Psi_{ij}^{-1}. \]
(20)

In the previous formulas, the derivatives \( \Psi_{ij}^{-1} \) and \( \varphi_{ij}^{-1} \) are taken at \( \Psi(z_h(x)) \).

**Proof.** We have
\[ \varphi(x) = \varphi \circ z_h^{-1} \circ \Psi^{-1}(x). \]

Differentiating twice, we obtain
\[
\varphi(x) = \varphi \circ z_h^{-1} \circ \Psi^{-1}(x).
\]

Next, we replace \( \varphi \) by its asymptotic expansion in powers of \( h \), we obtain
\[ \varphi_{ij} = \sum_{p \geq -2} d_{ij}^p h^p, \]
with
\[ d_{ij}^{-2} = \tilde{\xi}_{ij}, \quad d_{ij}^{-1} = \tilde{\gamma}_{ij} + \tilde{\zeta}_{ij} \quad \text{and for every} \quad n \geq 0, \quad d_{ij}^n = \tilde{\xi}_{ij}^n + \tilde{\gamma}_{ij}^{n+1} + \tilde{\zeta}_{ij}^{n+2}, \]
where \( \tilde{\xi}_{ij}^p, \tilde{\gamma}_{ij}^p, \) and \( \tilde{\zeta}_{ij}^p \) have the same algebraic expressions respectively as \( \xi_{ij}^p, \gamma_{ij}^p, \) and \( \zeta_{ij}^p \) defined in equations (18)–(20) noticing that \( \varphi_{ij}, \varphi_{ij}^{-1} \) and \( \varphi_{ij}^{-1}, \varphi_{ij}^{-1} \) are taken at \( z_h^{-1} \circ \Psi^{-1}(x) \) and \( \Psi^{-1}_{ij}, \varphi_{ij}^{-1} \) are taken at \( x \). Next, since we can write
\[ \nabla^2 \varphi = \tilde{\varphi}_{ij} \otimes e_i \otimes e_j, \]
we obtain that
\[ Q(\nabla^2 \varphi) = \sum_{n \geq -4} \left( \sum_{p=-4}^{n} B(d_{ij}^{p+2} \otimes e_i \otimes e_j, d_{ij}^{p-n-2} \otimes e_i \otimes e_j) \right) h^n. \]

Finally, by changing the variables, we obtain that
\[
K(h(\varphi)) = \frac{1}{h} \int_{\tilde{\Omega}_h} \tilde{K}(\tilde{\varphi}) = \frac{1}{h} \int_{\tilde{\Omega}_1} Q(\nabla^2 \tilde{\varphi}) dx
\]
\[
= \int_{\tilde{\Omega}_1} \sum_{n \geq -4} \left( \left( \sum_{p=-4}^{n} B(d_{ij}^{p+2} \otimes e_i \otimes e_j, d_{ij}^{p-n-2} \otimes e_i \otimes e_j) \right)(c_0 + hc_1 + h^2 c_2) \right) h^n dx,
\]
which gives the result.

**Proof of Proposition 1.** It suffices to write that \( J(h(\varphi)) = J(h(\varphi)) + K(h(\varphi)) \) and to apply Lemmas 3.3 and 3.4, to obtain the result.
Remark 3. The terms of the asymptotic expansion of \( J(h) \) are not independent on the thickness \( h \) since the first and second derivatives of \( \Psi^{-1} \) are taken at the point \( \Psi(x_1, x_2, hx_3) \). However, thanks to Remark 1 we have the following proposition.

**Proposition 3.** The energy \( J(h) \) admits an asymptotic expansion of the form

\[
J(h)(\varphi) = \sum_{n \geq -4} J^n(\varphi)h^n,
\]

where the \( J^n \) are independent on \( h \).

**Proof.** We obtain this result using Proposition 1 and Remark 1.

Remark 4. The expression of \( J^n \) derives from that of \( J^n_0 \) using the asymptotic expansion of \( \Psi^{-1}(\Psi(x_1, x_2, hx_3)) \). First, it is obvious that \( J^{-4} = J_0^{-4} \), where \( J_0^n = J^n_0 \) when \( h = 0 \). Next, minimizing \( J^{-4} \), we obtain that \( J^{-4} = J_0^{-4} = 0 \). This implies that \( J^{-3} = J_0^{-3} \). Similarly, we obtain that \( J^{-2} = J_0^{-2} \) minimizing \( J^{-3} \), also that \( J^{-1} = J_0^{-1} \) and finally, that \( J^0 = J_0^0 \) minimizing \( J^{-1} \).

5. **Solving the minimization problems**

We will use Proposition 2 to obtain the expression of \( J^0 \) following the minimization of the energies \( J^{-4}, J^{-3}, J^{-2}, \) and \( J^{-1} \). We have the following result.

**Proposition 4.** Minimizing the energies \( J^{-4}, J^{-3}, J^{-2}, \) and \( J^{-1} \), we obtain the following \( J^0 \) energy

\[
J^0(\varphi) = \int_{\Omega_1} \left( \sum_{i,j=1}^{3} Q(d_{ij}^0 \otimes e_i \otimes e_j) + \sum_{i,j,k,l=1}^{3} C_{ijkl} E_{ij}^0 E_{kl}^0 \right) \det(A(x)) \, dx,
\]

where \( \varphi \in V_h \) verifies

\[
\psi^0_{3} = 0 \quad \text{and} \quad \psi^1_{33} = 0,
\]

with

\[
d_{ij}^0 = \psi^0_{\alpha \beta} \psi^{-1}_{\alpha \beta} \psi_{ij}^{-1} + \psi^1_{\alpha \beta} \left( \psi^{-1}_{\alpha \beta} \psi_{ij}^{-1} + \psi^{-1}_{\alpha \beta} \psi_{ij}^{-1} \psi_{ij}^{-1} \right) + \psi^3_{\alpha \beta} \psi_{ij}^{-1} + \psi^4_{\alpha \beta} \psi_{ij}^{-1} + \psi^2_{\alpha \beta} \psi_{ij}^{-1} \psi_{ij}^{-1},
\]

\[
E_{ij}^0 = \frac{1}{2} \left[ (\psi^0_{\alpha \beta} \psi^0_{\beta \gamma}) \psi_{ij}^{-1} + \psi^1_{\alpha \beta} \psi_{ij}^{-1} + (\psi^0_{\alpha \beta} \psi^0_{\beta \gamma}) \psi_{ij}^{-1} \right] - \delta_{ij},
\]

and the functions \( \Psi^{-1}_{ij}, \Psi^{-1}_{ik} \) are taken at point \( \Psi(x_1, x_2, 0) \).

The proof of the Proposition is a consequence of the following lemmas and propositions. The proofs of Lemma 5 and Lemma 6 are postponed into the appendix section for the convenience of the reader.

**Lemma 5.** The energy \( J^{-4} \) is of the form

\[
J^{-4}(\varphi) = \int_{\Omega_1} \left( Q(\psi^0_{33} \otimes e_i \otimes e_j) \psi_{ij}^{-1} + 1 \frac{1}{4} C_{ijkl} \psi_{ij}^{-1} \psi_{ij}^{-1} \psi_{ij}^{-1} \psi_{ij}^{-1} \right) c_0 \, dx,
\]

where the functions \( \Psi_{ij}^{-1} \) are taken at point \( \Psi(x_1, x_2, 0) \).

Now that we have the expression of the energy \( J^{-4} \), we can minimize it on \( V_h \) following Pantz’s method.

**Proposition 5.** Solving the first minimization problem, that is, minimizing \( J^{-4} \) on \( V_h \), we obtain

\[
Q_{-3} = \{ \varphi \in V_h, \psi^0_{3} = 0 \}.
\]

Moreover, \( J^{-4}(\varphi) = 0 \) on \( Q_{-3} \).
**Proof.** Using Proposition 2, we have

\[ Q_{-3} = \{ \varphi \in V_h, J^{-4}(\varphi) = \inf_{\varphi' \in V_h} J^{-4}(\varphi') \} . \]

Next, using Lemma 5, we notice that to minimize the energy \( J^{-4} \) on \( V_h \), it suffices to consider deformations \( \varphi \) verifying

\[ \varphi_{,3}^0 = 0. \tag{26} \]

For these deformations, we have

\[ J^{-4}(\varphi) = 0, \tag{27} \]

which gives the result since \( J^{-4}(\varphi) \geq 0 \) and \( J^{-4}(\varphi) > 0 \) if \( \varphi_{,3}^0 \neq 0 \). \( \square \)

**Corollary 1.** If \( \varphi \in Q_{-3} \), then

\[ J^{-4}_h(\varphi) = 0. \tag{28} \]

**Proof.** The proof is obvious using equation (65). \( \square \)

Next, we consider the second minimization problem. We have the following result.

**Proposition 6.** The second minimization problem gives

\[ Q_{-2} = Q_{-3}. \]

**Proof.** Using Proposition 2, we have

\[ Q_{-2} = \{ \varphi \in Q_{-3}, J^{-3}(\varphi) = \inf_{\varphi' \in Q_{-3}} J^{-3}(\varphi') \}, \]

and using Proposition 1, we have

\[
J^{-3}_h(\varphi) = \sum_{p=-4}^{-3} \int_{\Omega_1} \left( B(d_{ij}^{p+2} \otimes e_i \otimes e_j, d_{ij}^{-p-5} \otimes e_i \otimes e_j) + C_{ijkl} E_{ij}^{p+2} E_{kl}^{-p-5} \right) c_0 \\
+ \left( Q(d_{ij}^{-2} \otimes e_i \otimes e_j) + C_{ijkl} E_{ij}^{-2} E_{kl}^{-2} \right) c_1 \, dx. \tag{29} \]

Then, for \( \varphi \in Q_{-3} \), Proposition 5 gives that \( d_{ij}^{-2} = 0 \) and \( E_{ij}^{-2} = 0 \) which implies using equation (29) that

\[ J^{-3}_h(\varphi) = 0. \tag{30} \]

Since we already proved that

\[ J^{-4}_h(\varphi) = J^{-4}(\varphi) = 0, \tag{31} \]

we obtain

\[ h^3 J(h)(\varphi) = \sum_{n \geq 0} J^{n-3}_h(\varphi) h^n = \sum_{n \geq 0} J^{n-3}(\varphi) h^n. \tag{32} \]

Thus, for every \( \varphi \in Q_{-3} \) we have

\[ J^{-3}(\varphi) = J^{-3}_0(\varphi) = 0, \tag{33} \]

which gives the result. \( \square \)

Next, we solve the following minimization problem that will provide a condition on the second derivative of \( \varphi^1 \).

**Lemma 6.** For every \( \varphi \in Q_{-2} \), we have

\[ J^{-2}(\varphi) = \int_{\Omega_1} Q(\varphi,_{33}^1 \otimes e_i \otimes e_j) \Psi_{3,j}^{-12} \Psi_{3,j}^{-12} \det(a_1 | a_2 | a_3) \, dx, \tag{34} \]

where the functions \( \Psi_{ij}^{-1} \) are taken at point \( \Psi(x_1, x_2, 0) \).
As a consequence of Lemma 6, we obtain the following Proposition.

**Proposition 7.** The third minimization problem gives

\[ Q_{-1} = \{ \varphi \in V_h, \varphi_{0,3}^0 = 0 \text{ and } \varphi_{33}^1 = 0 \}. \]  

**Proof.** Using Proposition 2 we have

\[ Q_{-1} = \{ \varphi \in Q_{-2}, J_{-2}(\varphi) = \inf_{\varphi' \in Q_{-2}} J_{-2}(\varphi') \}. \]

Next, using Lemma 6 and since the quadratic form is positive, we know that in order to minimize the energy \( J^{-2} \), it is necessary and sufficient to consider \( \varphi \) such that \( \varphi_{1,33}^1 = 0 \), which gives the result. \( \square \)

**Corollary 2.** For every \( \varphi \in Q_{-1} \), we have

\[ J_{-2}(\varphi) = 0. \]  

**Proof.** We obtain the result using equation (72). \( \square \)

Next, we pass to the following minimization problem. We have the following Proposition.

**Proposition 8.** The fourth minimization problem is trivial and gives

\[ Q_0 = Q_{-1}. \]

**Proof.** Using Proposition 2, we have

\[ Q_0 = \{ \varphi \in Q_{-1}, J_{-1}(\varphi) = \inf_{\varphi' \in Q_{-1}} J_{-1}(\varphi') \}. \]

Moreover, using Proposition 1, we have

\[
J_{-1}(\varphi) = \sum_{p=-4}^{-1} \int_{\Omega_1} \left( B(d_{ij}^{p+2} \otimes e_i \otimes e_j, d_{ij}^{p-3} \otimes e_i \otimes e_j) + C_{ijkl} E_{ij}^{p+2} E_{kl}^{p-3} \right) c_0 \, dx \\
+ \sum_{p=-4}^{-2} \int_{\Omega_1} \left( B(d_{ij}^{p+2} \otimes e_i \otimes e_j, d_{ij}^{p-4} \otimes e_i \otimes e_j) + C_{ijkl} E_{ij}^{p+2} E_{kl}^{p-4} \right) c_1 \, dx \\
+ \sum_{p=-4}^{-3} \int_{\Omega_1} \left( B(d_{ij}^{p+2} \otimes e_i \otimes e_j, d_{ij}^{p-5} \otimes e_i \otimes e_j) + C_{ijkl} E_{ij}^{p+2} E_{kl}^{p-5} \right) c_2 \, dx. \tag{37}
\]

Next, using equations (17) and (15), we have that for every \( \varphi \in Q_{-1} \),

\[ d_{ij}^{-2} = d_{ij}^{-1} = 0 \quad \text{and} \quad E_{ij}^{-2} = E_{ij}^{-1} = 0, \]  

which implies that for every \( \varphi \in Q_{-1} \), we have

\[ J_{-1}(\varphi) = 0. \]  

Since, for every \( \varphi \in Q_{-1} \), we have

\[ J^{-2}(\varphi) = J_{-2}(\varphi) = 0, \]  

it implies that for every \( \varphi \in Q_{-1} \), we have

\[ J^{-1}(\varphi) = J_{0}^{-1}(\varphi) = 0, \]  

which gives the result. \( \square \)
Finally, the following minimization problem will provide the proof of Proposition 4.

\textbf{Proof of Proposition 4.} Using Proposition 1, we have

\begin{equation}
J_h^0(\phi) = \sum_{p=-4}^{0} \int_{\Omega_1} \left( B(d_{ij}^{p+2} \otimes e_i \otimes e_j, d_{ij}^{p-2} \otimes e_i \otimes e_j) + C_{ijkl} E_{ij}^{p+2} E_{kl}^{p-2} \right) c_0 \, dx
\end{equation}

\begin{equation}
+ \sum_{p=-4}^{-1} \int_{\Omega_1} \left( B(d_{ij}^{p+2} \otimes e_i \otimes e_j, d_{ij}^{p-3} \otimes e_i \otimes e_j) + C_{ijkl} E_{ij}^{p+2} E_{kl}^{p-3} \right) c_1 \, dx
\end{equation}

\begin{equation}
+ \sum_{p=-4}^{-2} \int_{\Omega_1} \left( B(d_{ij}^{p+2} \otimes e_i \otimes e_j, d_{ij}^{p-4} \otimes e_i \otimes e_j) + C_{ijkl} E_{ij}^{p+2} E_{kl}^{p-4} \right) c_2 \, dx. \tag{42}
\end{equation}

Canceling the terms \(d_{ij}^{-2}, d_{ij}^{-1}, E_{ij}^{-2}, \) and \(E_{ij}^{-1},\) we obtain that

\begin{equation}
J_h^0(\phi) = \int_{\Omega_1} \left( Q(d_{ij}^0 \otimes e_i \otimes e_j) + C_{ijkl} E_{ij}^{0} E_{kl}^{0} \right) c_0 \, dx. \tag{43}
\end{equation}

Next, using Lemma 2, we have

\begin{equation}
E_{ij}^0 = \frac{1}{2}(d_{ij}^0 + B_{ij}^0 + C_{ij}^2 - \delta_{ij})
\end{equation}

\begin{equation}
= \frac{1}{2} \left( \Psi_{x,i}^{-1} \Psi_{x,j}^{-1} \varphi_i^0 \cdot \varphi_j^0 + \left( \Psi_{x,j}^{-1} \Psi_{x,i}^{-1} + \Psi_{x,i}^{-1} \Psi_{x,j}^{-1} \right) \left( \sum_{p=0}^{1} \varphi_i^p \cdot \varphi_j^{1-p} \right) + \Psi_{x,i}^{-1} \Psi_{x,j}^{-1} \left( \sum_{p=0}^{2} \varphi_i^p \cdot \varphi_j^{2-p} - \delta_{ij} \right) \right), \tag{44}
\end{equation}

which gives for \(\varphi \in Q_0\)

\begin{equation}
E_{ij}^0 = \frac{1}{2} \left( \Psi_{x,i}^{-1} \Psi_{x,j}^{-1} \varphi_i^0 \cdot \varphi_j^0 + \left( \Psi_{x,j}^{-1} \Psi_{x,i}^{-1} + \Psi_{x,i}^{-1} \Psi_{x,j}^{-1} \right) \varphi_i^1 \cdot \varphi_j^0 + \Psi_{x,j}^{-1} \Psi_{x,j}^{-1} |\varphi_j^1|^2 - \delta_{ij} \right). \tag{45}
\end{equation}

Then, using Lemma 4, we have

\begin{equation}
d_{ij}^0 = \delta_{ij} + \gamma_{ij}^1 + \gamma_{ij}^2 = \varphi_i^0 \Psi_{x,i}^{-1} \Psi_{x,j}^{-1} + \varphi_j^0 \Psi_{x,j}^{-1} + \varphi_i^1 \Psi_{x,i}^{-1} \Psi_{x,j}^{-1} + \varphi_j^1 \Psi_{x,j}^{-1} + \varphi_i^2 \Psi_{x,i}^{-1} \Psi_{x,j}^{-1}, \tag{46}
\end{equation}

where the functions \(\Psi_{x,j}^{-1} \) and \(\Psi_{x,j}^{-1} \) are taken at point \(\Psi(x_1, x_2, h x_3).\) Since for \(\varphi \in Q_0,\) we have

\begin{equation}
J_h^{-1}(\varphi) = J^{-1}(\varphi) = 0, \tag{47}
\end{equation}

it implies that

\begin{equation}
J(h)(\varphi) = \sum_{n \geq 0} J_h^n(\varphi) h^n = \sum_{n \geq 0} J^n(\varphi) h^n. \tag{48}
\end{equation}

Then, we obtain that

\begin{equation}
J_0^0(\varphi) = J_0^0(\varphi), \tag{49}
\end{equation}

which completes the proof of Proposition 4. 

\textbf{Remark 5.} The limit model obtained is in agreement with the one obtained in Le Dret and Zorgati [2, 3] using \(\Gamma\)-convergence techniques for the case of curved martensitic thin films where the quadratic form is the square of the norm.

\textbf{Remark 6.} Applying our results for the case of planar films, that is, for \(\Psi(x) = x\) we recover the formal asymptotic expansion of Pantz for the elastic energy of a Saint Venant–Kirchhoff material and the limit interfacial energy part obtained by Bhattacharya and James for the martensitic thin films again when the quadratic form is the square of the norm.
6. Application to the planar, cylindrical, and spherical cases

In this section we will illustrate our results for films that are planar, cylindrical, or spherical. In the cylindrical and spherical cases, we will suppose for simplicity that the quadratic form is the square of the norm in $M_{333}$.

6.1. Case of planar films

In this section, we will apply our results for the case of planar films, that is, $\Psi(x) = x$. The expression of the energy $e^0$ becomes

$$J^0(\varphi) = \int_{\Omega_1} \left\{ \frac{\lambda}{4} \left( |\varphi_{,1}^0|^2 + |\varphi_{,2}^0|^2 + |\varphi_{,3}^0|^2 - 3 \right)^2 \right\} \, dx.$$  

Adding $\mu/2 \left( (|\varphi_{,1}^0|^2 - 1)^2 + (|\varphi_{,2}^0|^2 - 1)^2 + (|\varphi_{,3}^0|^2 - 1)^2 + 2(\varphi_{,1}^0 \cdot \varphi_{,2}^0)^2 + 2(\varphi_{,1}^0 \cdot \varphi_{,3}^0)^2 + 2(\varphi_{,2}^0 \cdot \varphi_{,3}^0)^2 \right)$ and $Q(\varphi_{\alpha \beta}^0 \otimes e_\alpha \otimes e_\beta) + Q(\varphi_{, \alpha \beta}^1 \otimes e_\alpha \otimes e_\beta) + Q(\varphi_{, \alpha \alpha}^1 \otimes e_\alpha \otimes e_\alpha) + Q(\varphi_{, \alpha \alpha}^1 \otimes e_3 \otimes e_3)$

Remark 7. In the case of planar films, the derivatives of $\Psi$ are constants, and thus, the dependence on $h$ that appears in the curved case following the first five minimization problems does not exist. Consequently, we can go further in solving the minimization problems.

We have the following result.

Theorem 1. Suppose that $Q_{i33} > 0$ for $i = 1, 2, 3$. Thus, minimizing $J^0$ on $Q_0$, we obtain that

$$Q_1 = \left\{ \varphi \in V_h \text{ verifying } \varphi_{,3}^0 = 0, \varphi_{,33}^1 = 0, J^0_1(\varphi^0, \varphi^1) = \inf_{(\xi, \eta) \in Q_0^1} J^0_1(\xi, \eta), \varphi_{,33}^2 = 0 \right\},$$

and $\varphi^0(x) = (\tilde{a}_1 | \tilde{a}_2 | 0)x$, $\varphi_{,3}^1(x) = x_3 \tilde{a}_3$ on $\partial \omega \times \left( -\frac{1}{2}, \frac{1}{2} \right)$,

with $\tilde{A} = (\tilde{a}_1 | \tilde{a}_2 | \tilde{a}_3)$ being the constant matrix appearing in the boundary condition,

$$Q_2^0 = \left\{ (z_0, z_1) \in H^2(\Omega_1; \mathbb{R}^3) \times H^2(\Omega_1; \mathbb{R}^3) \text{ such that there exist } (\varphi^2, \varphi^3, ...) \text{ verifying} \right\},$$

and

$$J^0_1(\varphi^0, \varphi^1) = \int_{\Omega_1} \left\{ \left( Q(\varphi_{, \alpha \beta}^0 \otimes e_\alpha \otimes e_\beta) + Q(\varphi_{, \alpha \alpha}^1 \otimes e_\alpha \otimes e_\alpha) + Q(\varphi_{, \alpha \alpha}^1 \otimes e_3 \otimes e_3) \right) + \frac{\lambda}{4} \left( |\varphi_{,1}^0|^2 + |\varphi_{,2}^0|^2 + |\varphi_{,3}^0|^2 - 3 \right)^2 \right\} \, dx.$$  

Proof. Recall that

$$Q^0 = \left\{ \varphi = \sum_{i \geq 0} h^i \varphi^i \text{ with } \varphi^0 \in H^2(\Omega_1; \mathbb{R}^3), \varphi_{,3}^0 = 0, \varphi_{,33}^0 = 0 \right\},$$

and $\varphi(x) = (\tilde{a}_1 | \tilde{a}_2 | h \tilde{a}_3)x$ on $\partial \omega \times \left( -\frac{1}{2}, \frac{1}{2} \right)$.
where \( \tilde{A} = (\tilde{a}_1 | \tilde{a}_2 | \tilde{a}_3) \) is a constant matrix in \( \mathbb{M}_{33} \). The boundary condition on \( \partial \omega \times \left( -\frac{1}{2}, \frac{1}{2} \right) \) writes
\[
\varphi^0(x) = (\tilde{a}_1 | \tilde{a}_2 | 0)x, \quad \varphi^1(x) = (0 | 0 | \tilde{a}_3)x = x_3 \tilde{a}_3 \text{ and } \varphi^n(x) = 0 \text{ for } n \geq 2.
\]

Let
\[
J_0^1(\varphi^0, \varphi^1) = \inf_{z^3 \in Q^1_0} J(\varphi^0, \varphi^1, z^2),
\]

with
\[
Q^1_0 = \left\{ z \in H^2(\Omega_1; \mathbb{R}^3) \text{ such that there exist } (\varphi^0, \varphi^1, \varphi^3, ...) \text{ verifying } \varphi^0 + h\varphi^1 + h^2z + h^3 \varphi^3 + ... \in Q^0 \right\}.
\]

We have that for every \( \varphi^2 \) verifying \( \varphi^2_{33} = 0 \), we obtain equation (53).

**Remark 8.** Notice that the result is in accordance with the limit model obtained by Bhattacharya and James (see [1]) for a martensitic plate obtained by \( \Gamma \) convergence techniques.

**Remark 9** The hypothesis \( q_{i33} > 0 \) for \( i = 1, 2, 3 \) is not necessary to obtain equation (53). Indeed, if one of the \( q_{i33} \) is equal to zero, then the corresponding term do not appear in the energy from the beginning.

Finally, in the case when the quadratic form represents the square of the norm in \( \mathbb{M}_{33} \), let us define the 2D energy \( J \) by
\[
J(u) = \int_\omega \left[ 2|\nabla u|^2 + \frac{\lambda}{4} \left( |\varphi_1^0|^2 + |\varphi_2^0|^2 + |u|^2 - 3 \right)^2 \right. \\
\left. + \frac{\mu}{2} \left( (|\varphi_1^0|^2 - 1)^2 + (|\varphi_2^0|^2 - 1)^2 + (|u|^2 - 1)^2 + 2(\varphi_1^0 \cdot \varphi_2^0)^2 + 2(\varphi_1^0 \cdot u)^2 + 2(\varphi_2^0 \cdot u)^2 \right) \right] dx.
\]

Notice that the minimizer of the energy \( J \) is the \( \varphi^1_3 \) that minimizes the energy \( J^0 \). Writing the corresponding Euler–Lagrange equation, we obtain that the minimizers are solutions of the following partial differential equation:
\[
\begin{cases}
\lambda (|\varphi_1^0|^2 + |\varphi_2^0|^2 + |u|^2 - 3) + 2\mu (|u|^2 - 1)u + 2\mu [(u \cdot \varphi_1^0)\varphi_1^0 + (\varphi_2^0 \cdot u)\varphi_2^0] - 4\Delta u = 0 & \text{in } \omega \\
\quad u = Aa_3 & \text{on } \partial \omega.
\end{cases}
\]

### 6.2. Case of a cylindrical film

In order to illustrate our results on an explicit example of curved film, we apply them in the case of a thin film of cylindrical form, length \( l \) and of radius \( r \) when the quadratic form represents the square of the norm. We recall that in the case of a portion of a cylinder, the diffeomorphism \( \psi \) is of the form:
\[
\psi: \omega := ]0; 1[ \times ]0; \pi/2[ \rightarrow \mathbb{R}^3 \\
(x_1, x_2) \mapsto (x_1, r \cos x_2, r \sin x_2),
\]

which implies that
\[
a_1(x_1, x_2) = (1, 0, 0), \quad a_2(x_1, x_2) = (0, -r \sin x_2, r \cos x_2) \text{ and } a_3(x_1, x_2) = (0, -\cos x_2, -\sin x_2).
\]

Thus, \( c_0 = r \) and \( \Psi \) writes
\[
\Psi: \Omega_h := ]0; 1[ \times ]0; \pi/2[ \rightarrow \widehat{\Omega}_h \\
(x_1, x_2, x_3) \mapsto (x_1, (r - x_3) \cos x_2, (r - x_3) \sin x_2),
\]
which gives that
\[ \Psi^{-1} : \tilde{\Omega}_h \rightarrow \Omega_h \]
\[ (z_1, z_2, z_3) \mapsto \left( z_1, \arctan \frac{z_3}{z_2}, r - \sqrt{z_2^2 + z_3^2} \right). \]

Next, we compute \( \Psi^{-1} \) and its derivatives at the point
\[ \Psi(x_1, x_2, 0) = (x_1, r \cos x_2, r \sin x_2). \]

We obtain the values of \( d^0_\nu \) and \( E^0_\nu \), which give the following energy \( J^0 \)
\[ J^0(\varphi^0, \varphi^1, \varphi^2) = \int_{\tilde{\Omega}_1} \left\{ \frac{\lambda}{4} \left( |\varphi^1|^2 + \frac{1}{r} |\varphi^0_2|^2 + |\varphi^1_3|^2 - 3 \right)^2 \right. \]
\[ + \frac{\mu}{2} \left[ \left( |\varphi^0_1|^2 - 1 \right)^2 + \left( \frac{1}{r} |\varphi^0_2|^2 - 1 \right)^2 + \left( |\varphi^1_3|^2 - 1 \right)^2 + 2(\varphi^0_1 \cdot \varphi^0_2)^2 + 2(\varphi^0_1 \cdot \varphi^1_3)^2 + 2(\varphi^1_2 \cdot \varphi^1_3)^2 \right] \]
\[ + \left( |\varphi^1_1|^2 + \frac{2}{r^2} |\varphi^0_1|^2 + 2 |\varphi^1_3|^2 + \left( \frac{1}{r^2} \varphi^0_{22} - \frac{1}{r} \varphi^1_3 \right)^2 + 2 \left( \frac{1}{r} \varphi^1_{23} + \frac{1}{r^2} \varphi^1_2 \right)^2 \right) \}
\[ \left. \right\} \, r \, dx. \quad (59) \]

Notice that we can minimize \( J^0 \) with respect to \( \varphi^2 \), taking
\[ \varphi^2_{33} = 0. \quad (60) \]

The energy becomes
\[ J^0(\varphi^0, \varphi^1) = \int_{\tilde{\Omega}_1} \left\{ \frac{\lambda}{4} \left( |\varphi^0_1|^2 + \frac{1}{r} |\varphi^0_2|^2 + |\varphi^1_3|^2 - 3 \right)^2 \right. \]
\[ + \frac{\mu}{2} \left[ \left( |\varphi^0_1|^2 - 1 \right)^2 + \left( \frac{1}{r} |\varphi^0_2|^2 - 1 \right)^2 + \left( |\varphi^1_3|^2 - 1 \right)^2 + 2(\varphi^0_1 \cdot \varphi^0_2)^2 + 2(\varphi^0_1 \cdot \varphi^1_3)^2 + 2(\varphi^1_2 \cdot \varphi^1_3)^2 \right] \]
\[ + \left( |\varphi^1_1|^2 + \frac{2}{r^2} |\varphi^0_1|^2 + 2 |\varphi^1_3|^2 + \left( \frac{1}{r^2} \varphi^0_{22} - \frac{1}{r} \varphi^1_3 \right)^2 + 2 \left( \frac{1}{r} \varphi^1_{23} + \frac{1}{r^2} \varphi^1_2 \right)^2 \right) \}
\[ \left. \right\} \, r \, dx. \quad (61) \]

The main difference, at this stage, with the case of plates, appears in the existence of terms of order 1 in the second-order energy part, namely, \( \frac{1}{r^2} \varphi^2 \) and \( \frac{1}{r} \varphi^1 \). This makes the minimization of \( J^0 \) with respect to \( \varphi^1 \) clearly different from the case of planar films. Indeed, in order to minimize our energy with respect to \( \varphi^1 \), we consider the energy defined on \( H^1(\omega; \mathbb{R}^3) \) with \( \varphi^1_3, \varphi^0_2, \) and \( \varphi^0_{22} \) constant, writing
\[ J(u) = \int_{\omega} \left\{ \frac{\lambda}{4} \left( |\varphi^0_1|^2 + \frac{1}{r} |\varphi^0_2|^2 + |u|^2 - 3 \right)^2 \right. \]
\[ + \frac{\mu}{2} \left[ \left( |u|^2 - 1 \right)^2 + 2(\varphi^0_1 \cdot u)^2 + 2(\varphi^0_2 \cdot u)^2 \right] \]
\[ + \left( 2 |u_{11}|^2 + \left| \frac{1}{r} u - \frac{1}{r^2} \varphi^0_{22} \right|^2 + 2 \left| \frac{1}{r} u_{23} + \frac{1}{r^2} \varphi^0_2 \right|^2 \right) \}
\[ \left. \right\} \, r \, dx. \quad (62) \]

Notice that the minimizer of the last energy is the same \( \varphi^1 \) that minimizes \( J^0 \). Considering the corresponding Euler–Lagrange equation, we obtain that the minimizers verify the following partial differential equations
\[ \left\{ \begin{aligned}
\left[ \lambda (|\varphi^0_1|^2 + \frac{1}{r} |\varphi^0_2|^2 + |u|^2 - 3) + 2\mu (|u|^2 - 1) \right] u + 2\mu [(u \cdot \varphi^0_1) \varphi^0_3 + (u \cdot \varphi^0_2) \varphi^0_2] \\
+ 2\frac{\lambda}{r^2} \left( u - \frac{1}{r^2} \varphi^0_{22} \right) - 4\Delta u = 0 \quad \text{in } \omega \\
u = \bar{A} a_3 \text{ on } \partial \omega, \\
\end{aligned} \right. \]

where
\[ \bar{\Delta} u = u_{,11} + \frac{1}{r^2} u_{,22}. \quad (63) \]
6.3. Case of a portion of a sphere

We consider the case of a thin film of spherical form, which is rather different from that of the cylinder and further away from the case of planar films. Supposing that the quadratic form is the square of the norm, let us consider a spherical midsurface for which the diffeomorphism $\psi$ reads:

$$\psi : \omega := I_1 \times I_2 \subset \mathbb{R}^2 \rightarrow \tilde{S} \subset \mathbb{R}^3$$

$$(x_1, x_2) \mapsto (\sin x_1 \cos x_2, \sin x_1 \sin x_2, \cos x_1),$$

where $x_1$ and $x_2$ represents the latitudinal and longitudinal coordinates, $I_1$ and $I_2$ are open intervals of $\mathbb{R}$ such that $\tilde{S}$ do not contain the poles of the sphere in order to avoid singularities. We obtain that

$$a_1(x_1, x_2) = (\cos x_1 \cos x_2, \cos x_1 \sin x_2, -\sin x_1),$$
$$a_2(x_1, x_2) = (-\sin x_1 \sin x_2, \sin x_1 \cos x_2, 0),$$
and $a_3(x_1, x_2) = (\sin x_1 \cos x_2, \sin x_1 \sin x_2, \cos x_1)$.

Thus, $c_0 = \sin x_1$ and $\Psi$ reads

$$\Psi : \Omega_h := I_1 \times I_2 \times \mathbb{R} \rightarrow \tilde{\Omega}_h$$

$$(x_1, x_2, x_3) \mapsto ((1 + x_3) \sin x_1 \cos x_2, (1 + x_3) \sin x_1 \sin x_2, (1 + x_3) \cos x_1),$$

which implies that

$$\Psi^{-1} : \tilde{\Omega}_h \rightarrow \Omega_h$$

$$(z_1, z_2, z_3) \mapsto (\arctan \sqrt{\frac{z_1^2 + z_2^2}{z_3^2}}, \arctan \frac{z_2}{z_1}, -1 + \sqrt{\frac{z_1^2 + z_2^2 + z_3^2}}).$$

Then, we compute $\Psi^{-1}$ and its derivatives at point $\Psi(x_1, x_2, 0)$ to obtain the values of $d_y^0$ and $E_y^0$, which gives the following energy $J^0$,

$$J^0(\varphi) = \int_{\Omega_1} \left[ \left| \varphi_{,11}' \right|^2 + \frac{1}{\sin^2 x_1} \left| \varphi_{,22}' \right|^2 + \frac{2}{\sin^2 x_1} \left| \varphi_{,12}' \right|^2 + 2 \left| \varphi_{,13}' \right|^2 + 2 \left| \varphi_{,23}' \right|^2 + \left| \varphi_{,33}' \right|^2 ight.$$
$$+ \frac{2 \cos^2 x_1}{\sin^2 x_1} \varphi_{,12}' \cdot \varphi_{,1}' - \frac{4 \cos x_1}{\sin^2 x_1} \left( \varphi_{,12}' \cdot \varphi_{,2}' - 4 \varphi_{,13}' \cdot \varphi_{,3}' - 4 \frac{1}{\sin^2 x_1} \varphi_{,23}' \cdot \varphi_{,2}' \right)$$
$$+ \frac{2}{\sin^2 x_1} \varphi_{,22}' \cdot \varphi_{,1}' + \left| \varphi_{,1}' \right|^2 + \left| \varphi_{,2}' \right|^2 \left( 4 \cos^2 x_1 \sin^2 x_2 + \cot^2 x_1 + 2 \right) + \left| \varphi_{,2}' \right|^2 
$$
$$+ \frac{\mu}{2} \left[ \left( \left| \varphi_{,1}' \right|^2 - 1 \right)^2 + \left( \frac{1}{\sin^2 x_1} \left| \varphi_{,2}' \right|^2 - 1 \right)^2 + \left( \left| \varphi_{,3}' \right|^2 - 1 \right)^2 + \frac{2}{\sin^2 x_1} \left( \varphi_{,1}' \cdot \varphi_{,2}' \right)^2 + 2 \left( \varphi_{,1}' \cdot \varphi_{,3}' \right)^2 
$$
$$+ \frac{2}{\sin^2 x_1} \left( \varphi_{,22}' \cdot \varphi_{,3}' \right)^2 \right] \sin x_1 dx.$$

We notice that here also we can minimize our energy with respect to $\varphi_{,33}'$ only by canceling it, which means that $\varphi^2$ is linear with respect to the third variable. Consequently, all the remaining terms are independent of the
third variable. This enables us to integrate with respect to the third variable obtaining the following expression

\[
J^0(\varphi) = \int_{\omega} \left[ \left| \varphi_{,11}^0 \right|^2 + \frac{2}{\sin^2 x_1} \left| \varphi_{,12}^0 \right|^2 + 2 |\varphi_{,13}^1 - \varphi_{,1}^0| |\varphi_{,23}^1 - \varphi_{,2}^0| + \frac{2}{\sin^2 x_1} |\varphi_{,23}^1 - \varphi_{,2}^0|^2 \\
+ \frac{2 \cos x_1}{\sin^3 x_1} \varphi_{,22}^0 \cdot \varphi_{,1}^0 - \frac{4 \cos x_1}{\sin^3 x_1} \varphi_{,12}^0 \cdot \varphi_{,2}^0 + \frac{1}{\sin^2 x_1} \varphi_{,22}^0 + \varphi_{,3}^1 \right)^2
\]

where

\[
\|u\|^2 + \frac{2}{\sin^2 x_1} |\varphi_{,2}^0|^2 + \left| \frac{1}{\sin^2 x_1} \varphi_{,22}^0 + u \right|^2
\]

Proceeding as in the cylindrical case, we consider the energy \( \tilde{J} \) defined by

\[
\tilde{J}(u) = \int_{\omega} \left[ \left| 2 u_{,1} - \varphi_{,1}^0 \right|^2 + \frac{2}{\sin^2 x_1} \left| u_{,2} - \varphi_{,2}^0 \right|^2 + \left| \frac{1}{\sin^2 x_1} \varphi_{,22}^0 + u \right|^2 \right]
\]

We notice that the \( \varphi_{,3}^1 \) minimizing \( J^0 \) is the minimizer of \( \tilde{J} \). Next, considering the corresponding Euler–Lagrange equations, we obtain that the minimizer is a solution of the following partial differential equations

\[
\begin{aligned}
\left\{ \begin{array}{l}
\lambda \left( \varphi_{,11}^0 - \frac{1}{\sin x_1} \varphi_{,22}^0 + |u|^2 - 3 \right) + 2 \mu (|u|^2 - 1) + 2 |u - \varphi_{,1}^0| (u \cdot \varphi_{,1}^0) + \frac{1}{\sin^2 x_1} (u \cdot \varphi_{,2}^0) \varphi_{,2}^0 = 0 \text{ in } \omega \\
-4 \Delta \tilde{u} + 4 \cot x_1 u_{,1} + 2 u + \frac{6}{\sin^2 x_1} \varphi_{,22} + 4 \varphi_{,11} = 0 \text{ in } \omega \\
u = \partial_3 a \text{ on } \partial \omega,
\end{array} \right.
\end{aligned}
\]

where

\[
\tilde{u} = u_{,11} + \frac{1}{\sin^2 x_1} u_{,22} + \cot x_1 u_{,1},
\]

with \( x_1 \) being the latitudinal coordinate.

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Appendix 1

In this section, we provide the proofs of some results stated in Sections 4 and 5.

Proof of Proposition 2. Let \( \psi (h) = (\psi^n)_{n \in \mathbb{N}} \) verifying:

\[
J(h)(\psi(h)) = \inf_{\psi \in V_h} J(h)(\psi),
\]

with

\[
J(h)(\psi) = \sum_{n \geq -4} J^n(\psi)h^n. \]

We will prove by induction that \( \psi(h) \in \bigcap_{n=4}^{\infty} Q_n \). We have already seen that \( \psi(h) \in Q_{-4} = V_h \). Suppose that \( \psi(h) \in \bigcap_{\bar{n}=4}^{\bar{n}} Q_{\bar{n}} \) and let us prove that \( \psi(h) \in Q_{\bar{n}+1} \). By definition of \( Q_n \), we have that for every \( \tilde{\psi} \in Q_n \):

\[
J(h)(\tilde{\psi}) = \sum_{k=-4}^{\bar{n}-1} h^k \inf_{\psi \in Q_k} J^k(\psi) + h^\bar{n} J^\bar{n}(\tilde{\psi}) + \sum_{k=\bar{n}+1}^{\infty} h^k J^k(\tilde{\psi}) = C_\bar{n}(h) + h^{\bar{n}} \left[ J^{\bar{n}}(\tilde{\psi}) + \sum_{k=\bar{n}+1}^{\infty} h^{k-\bar{n}} J^k(\tilde{\psi}) \right].
\]

Since \( \psi(h) \) is a minimizer of \( J(h) \) over \( V_h \supseteq Q_n \), we have

\[
J(h)(\psi(h)) = \inf_{\psi \in Q_n} J(h)(\psi).
\]

Thus, for every \( \psi \in Q_n \), we have

\[
J^{\bar{n}}(\psi(h)) + \sum_{k=\bar{n}+1}^{\infty} h^{k-\bar{n}} J^k(\psi(h)) \leq J^{\bar{n}}(\psi) + \sum_{k=\bar{n}+1}^{\infty} h^{k-\bar{n}} J^k(\psi).
\]
Next, for every $\alpha > 0$, there exists $\psi_\alpha \in Q_n$ such that
\[
J_n^\alpha(\psi_\alpha) \leq \inf_{\psi \in \mathcal{L}_h} J_n^\alpha(\psi) + \alpha.
\]

Thus, we obtain that
\[
J_n^\alpha(\varphi(h)) + h \sum_{k=0}^\infty h^k J^{n+k+1}(\varphi(h)) \leq J_n^\alpha(\varphi) + h \sum_{k=0}^\infty h^k J^{n+k+1}(\psi_\alpha)
\]
\[
\leq \inf_{\psi \in \mathcal{L}_h} J_n^\alpha(\psi) + \alpha + h \sum_{k=0}^\infty h^k J^{n+k+1}(\psi_\alpha).
\]

Letting $h \to 0$, we obtain
\[
J_n^\alpha(\varphi(h)) \leq \inf_{\psi \in \mathcal{L}_h} J_n^\alpha(\psi) + \alpha,
\]
and thus the result by letting $\alpha \to 0$. \hfill \Box

**Proof of Lemma 5.** We use Proposition 1, which implies that
\[
J_{h^{-4}}^n(\varphi) = \int_{\Omega_1} \left( Q(d_{ij}^{-2} \otimes e_i \otimes e_j) + C_{ijkl} E_{ij}^{-2} E_{kl}^{-2} \right) c_0 \, dx,
\]
(65)

where, using Lemma 4, we have
\[
d_{ij}^{-2} = s_{ij}^0 = \varphi_{ij,3}^0 \Psi_{ij,3}^{-1} \Psi_{ij}^{-1},
\]
(66)

and using Lemma 2, we have
\[
E_{ij}^{-2} = \frac{1}{2} C_{ij}^0 = \Psi_{ij,3}^{-1} |\varphi_{ij,3}|^2,
\]
(67)

noticing that the functions $\Psi_{ij}^{-1}$ are taken at point $\Psi(x_1, x_2, h_3)$. Next, since we have
\[
h^k J(h)(\varphi) = \sum_{n=0}^\infty J_{h^n}(\varphi) h^n = \sum_{n=0}^\infty J_{h^{-4}}^n(\varphi) h^n,
\]
(68)

this implies that for every $\varphi \in Q_{-4}$, we have
\[
J_{h^{-4}}(\varphi) = J_{0^{-4}}(\varphi),
\]
(69)

where $J_{h^n} = J_{h^n}^n$ for $h = 0$, and thus the result. \hfill \Box

**Proof of Lemma 6.** Using Proposition 1, we have
\[
J_{h^{-2}}^0(\varphi) = \sum_{p=-4}^{-2} \int_{\Omega_1} \left( B(d_{ij}^{p+2} \otimes e_i \otimes e_j, d_{ij}^{-p-4} \otimes e_i \otimes e_j) + C_{ijkl} E_{ij}^{p+2} E_{kl}^{-p-4} \right) c_0 \, dx
\]
\[
+ \sum_{p=-4}^{-3} \int_{\Omega_1} \left( B(d_{ij}^{p+2} \otimes e_i \otimes e_j, d_{ij}^{-p-5} \otimes e_i \otimes e_j) + C_{ijkl} E_{ij}^{p+2} E_{kl}^{-p-5} \right) c_1 \, dx
\]
\[
+ \int_{\Omega_1} \left( Q(d_{ij}^{-2} \otimes e_i \otimes e_j) + C_{ijkl} |E_{ij}^{-2}|^2 \right) c_2 \, dx.
\]
(70)

When $\varphi \in Q_{-2}$, that is $\varphi_{ij,3}^0 = 0$, we know that $d_{ij}^{-2} = 0$ and $E_{ij}^{-2} = 0$. Thus, we have
\[
J_{h^{-2}}^0(\varphi) = \int_{\Omega_1} \left( Q(d_{ij}^{-1} \otimes e_i \otimes e_j) + C_{ijkl} E_{ij}^{-1} E_{kl}^{-1} \right) \det(a_1 |a_2 |a_3) \, dx.
\]
(71)
Moreover, Lemma 2 implies that
\[
E^{-1}_{ij} = \frac{1}{2}(B^0_{ij} + C^1_{ij}) = \frac{1}{2} \left( (\Psi_{3,3}^{-1} \Phi_{a,j} + \Psi_{a,j} \Psi_{3,3}^{-1}) \varphi_{3}^{0} \cdot \varphi_{a}^{0} + \Psi_{3,3}^{-1} \Psi_{3,3}^{-1} \sum_{p=0}^{1} \varphi_{3}^{p} \cdot \varphi_{3}^{p+1} \right)
\]
and Lemma 4 implies that
\[
d^{-1}_{ij} = \gamma^{0}_{ij} + \zeta^{1}_{ij} = \varphi_{a,3}^{0} (\Psi_{3,3}^{-1} \Phi_{a,j} + \Psi_{a,j} \Psi_{3,3}^{-1}) + \varphi_{a,3}^{0} \Psi_{3,3}^{-1} \Psi_{3,3}^{-1} \Psi_{3,3}^{-1} \Psi_{3,3}^{-1}.
\]
Thus, when \( \varphi \in Q_{-2} \), we have
\[
E^{-1}_{ij} = 0 \quad \text{and} \quad d^{-1}_{ij} = \varphi_{3,3}^{1} \Psi_{3,3}^{-1} \Psi_{3,3}^{-1},
\]
where the functions \( \Psi_{3,3}^{-1} \) are taken at point \( \Psi(x_1, x_2, hx_3) \). Replacing these expressions into (71), we obtain that
\[
J^{-2}_{h} (\varphi) = \int_{\Omega_{1}} Q_{3,3}^{1} \otimes e_{i} \otimes e_{j} \Psi_{3,3}^{-1}^{2} \Psi_{3,3}^{-1} \Psi_{3,3}^{-1} \Psi_{3,3}^{-1} \det(a_{1} | a_{2} | a_{3}) \, dx,
\]
where the functions \( \Psi_{3,3}^{-1} \) are taken at point \( \Psi(x_1, x_2, hx_3) \). Next, since \( \varphi \in Q_{-2} \), we have
\[
h^{2} J(h)(\varphi) = \sum_{n \geq 0} J_{h}^{n-2}(\varphi) h^{n} = \sum_{n \geq 0} J_{h}^{n-2}(\varphi) h^{n},
\]
and thus, for every \( \varphi \in Q_{-2} \), we have
\[
J^{-2}(\varphi) = J_{0}^{-2}(\varphi),
\]
which gives the result. \( \square \)