STRONG STABILITY OF SAMPLED-DATA RIESZ-SPECTRAL SYSTEMS

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Abstract. Suppose that a continuous-time linear infinite-dimensional system with a static state-feedback controller is strongly stable. We address the following question: If we convert the continuous-time controller to a sampled-data controller by applying an idealized sampler and a zero-order hold, will the resulting sampled-data system be strongly stable for all sufficiently small sampling periods? In this paper, we restrict our attention to the situation where the generator of the open-loop system is a Riesz-spectral operator and its point spectrum has a limit point at the origin. We present conditions under which the answer to the above question is affirmative. In the robustness analysis, we show that the sufficient condition for strong stability obtained in the Arendt-Batty-Lyubich-Vu theorem is preserved between the original continuous-time system and the sampled-data system under fast sampling.

Key words. infinite-dimensional systems, sampled-data control, stabilization, strong stability, robustness

AMS subject classifications. 47A55, 47D06, 93C25, 93C57, 93D15

1. Introduction.

We consider systems with state space $X$ and input space $C$ of the form

\begin{equation}
\dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0; \quad x(0) = x^0 \in X,
\end{equation}

where $X$ is a Hilbert space, $A$ is the generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on $X$, and $B$ is a bounded linear operator from $C$ to $X$. Suppose that a continuous-time feedback control $u(t) = Fx(t)$, where $F$ is a bounded linear operator from $X$ to $C$, achieves the strong stability of the closed-loop system in the sense that $A + BF$ generates a strongly stable semigroup $(T_{BF}(t))_{t \geq 0}$ on $X$, i.e.,

$$\lim_{t \to \infty} \|T_{BF}(t)x^0\| = 0 \quad \forall x^0 \in X.$$ 

Instead of this continuous-time controller, we use the following digital controller with an idealized sampler and a zero-order hold:

\begin{equation}
\begin{align*}
    u(t) &= Fx(k\tau), \quad k\tau \leq t < (k+1)\tau, \\
    \lim_{t \to \infty} \|x(t)\| &= 0 \quad \forall x^0 \in X.
\end{align*}
\end{equation}

Our objective is to show that a certain class of infinite-dimensional systems possess this robustness property with respect to sampling.

In the finite-dimensional case, stability is preserved for all sufficiently small sampling periods. This result has been extended to the exponential stability of some

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classes of infinite-dimensional systems in [18, 31], but even exponential stability is much more delicate in the infinite-dimensional case [30]. Sampled-data systems are ubiquitous in computer-based control systems, and various sampled-data control problems have been studied for infinite-dimensional systems; for example, stabilization [11, 12, 16, 17, 19, 29, 34, 37] and output regulation [13–15, 20, 36]. Robustness of strong stability with respect to sampling has been posed as an open problem in [32], and it has not been solved yet.

Strong stability of strongly continuous semigroups is rather weak compared with exponential stability. In fact, exponential stability is preserved under all sufficiently small bounded perturbations, whereas it is easy to find a strongly stable semigroup and an arbitrarily small perturbation such that the perturbed semigroup is unstable; see, e.g., Section 1 of [22]. The difficulty of robustness analysis of strong stability arises from the high level of generality of strong stability. For example, define the bounded linear operator \( A_1 \) on \( \ell^2(\mathbb{C}) \) by

\[
A_1 x := \sum_{n=1}^{\infty} -\frac{1}{n} \langle x, \phi_n \rangle \phi_n
\]

and the operator \( A_2 \) on \( \ell^2(\mathbb{C}) \) by

\[
A_2 x := \sum_{n=1}^{\infty} \left( -\frac{1}{n} + in \right) \langle x, \phi_n \rangle \phi_n
\]

with domain

\[
D(A_2) := \left\{ x \in \ell^2(\mathbb{C}) : \sum_{n=1}^{\infty} n^2 |\langle x, \phi_n \rangle|^2 < \infty \right\},
\]

where \( \{ \phi_n : n \in \mathbb{N} \} \) is the standard basis of \( \ell^2(\mathbb{C}) \). Both operators \( A_1 \) and \( A_2 \) generate strongly stable semigroups. However, the behaviors of the semigroups and the spectral properties of \( A_1 \) and \( A_2 \) are quite different. Focusing on important subclasses of strongly stable semigroups, the author of [22–27] has studied robustness of strong stability. To study strong stability of delay semigroups, perturbation results for strongly stable semigroups have been developed in [28]. The preservation of strong stability under discretization via the Cayley transformation has been investigated in [2, 9]. We can regard discretization by sampling as a perturbation, but this structured perturbation has not been investigated in the above previous studies.

In this paper, we concentrate on the situation where the system (1.1) is a Riesz-spectral system, i.e., the generator \( A \) is a Riesz-spectral operator; see Definition 2.5 below for the definition of Riesz-spectral operators. We further assume that \( A \) has no eigenvalues on the imaginary axis and only finitely many eigenvalues in \( \{ \lambda \in \mathbb{C} \setminus \{0\} : \text{Re} \lambda > -\alpha, |\arg \lambda| < \pi/2 + \delta \} \) (the gray area in Fig. 1) for some \( \alpha > 0 \) and \( 0 < \delta \leq \pi/2 \) but that there exists a sequence of the eigenvalues of \( A \) such that it is contained in the sector \( \{ \lambda \in \mathbb{C} \setminus \{0\} : \pi/2 + \delta \leq |\arg \lambda| \leq \pi \} \) and converges to 0. Consequently, 0 belongs to the continuous spectrum of \( A \). For example, the operator \( A_1 \) given in (1.3) has this spectral property, but \( A_2 \) in (1.4) does not. The sectorial constraint on the eigenvalues avoids any losses of high-frequency information caused by sampling. Such a sectorial constraint on the spectrum of the generator \( A \) has been also placed in the previous study [18] in order to prove that exponential stability is preserved under fast sampling in the case of boundary or pointwise control.

Another important assumption of this study is that \( A + BF \) satisfies the sufficient condition for strong stability obtained in the well-known Arendt-Batty-Lyubich-Vũ...
theorem [1,21], that is, sup_{t \geq 0} \| TBF(t) \| < \infty, \sigma_p(A+BF) \cap i\mathbb{R} = \emptyset, and \sigma(A+BF) \cap i\mathbb{R} = \{0\}, where \sigma_p(A+BF) and \sigma(A+BF) denote the point spectrum and the spectrum of A + BF, respectively. It is straightforward to show that the sampled-data system (1.1) and (1.2) is strongly stable if and only if the discrete semigroup \((\Delta(\tau)^k)_{k \in \mathbb{N}}\) on X, where

\[
\Delta(\tau) := T(\tau) + \int_0^\tau T(s)BFds,
\]
is strongly stable, i.e.,

\[
\lim_{k \to \infty} \| \Delta(\tau)^k x^0 \| = 0 \quad \forall x^0 \in X;
\]

see Section 2. Then the robustness analysis of strong stability with respect to sampling becomes the problem of determining whether or not the discrete semigroup \((\Delta(\tau)^k)_{k \in \mathbb{N}}\) is strongly stable for all sufficiently small \(\tau > 0\). To check the strong stability of \((\Delta(\tau)^k)_{k \in \mathbb{N}}\), we use the discrete version of the Arendt-Batty-Lyubich-Vu theorem. More precisely, we prove that sup_{k \in \mathbb{N}} \| \Delta(\tau)^k \| < \infty, \sigma_p(\Delta(\tau)) \cap \mathbb{T} = \emptyset, and \sigma(\Delta(\tau)) \cap \mathbb{T} = \{1\}. In summary, we here show that if the continuous-time closed-loop operator \(A + BF\) satisfies the sufficient condition for strong stability in the continuous case of the Arendt-Batty-Lyubich-Vu theorem, then the discretized closed-loop operator \(\Delta(\tau)\) also satisfies the sufficient condition of the discrete counterpart for all sufficiently small sampling periods \(\tau > 0\). This means that the sufficient condition for strong stability obtained in the Arendt-Batty-Lyubich-Vu theorem is preserved between the original continuous-time system and the sampled-data system under fast sampling.

This paper is organized as follows. In Section 2, we first review useful results on strong stability and Riesz-spectral operators and then state our main result on robustness of strong stability with respect to sampling. To prove this result, we study the spectrum of \(\Delta(\tau)\) in Section 3. In Section 4, we investigate the boundedness of the discrete semigroup \((\Delta(\tau)^k)_{k \in \mathbb{N}}\) in order to complete the proof of the main result. Concluding remarks are made in Section 5.

Notation and terminology. For \(\alpha \in \mathbb{R}\) and \(r > 0\), we define

\[
\mathbb{C}_\alpha := \{ \lambda \in \mathbb{C} : \text{Re} \lambda > \alpha \}, \quad \mathbb{D}_r := \{ \lambda \in \mathbb{C} : |\lambda| < r \}, \quad \mathbb{E}_r := \{ \lambda \in \mathbb{C} : |\lambda| > r \}.
\]
We denote the unit circle by $\mathbb{T} := \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$ and the imaginary axis by $i\mathbb{R} := \{ i\omega : \omega \in \mathbb{R} \}$. For $\delta \in (0, \pi]$, we define the sector $\Sigma_\delta := \{ \lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \delta \}$. Let $X$ and $Y$ be Banach spaces. For a linear operator $A : X \to Y$, we denote by $D(A)$, $\text{ran}(A)$, and $\ker(A)$ the domain, the range, and the kernel of $A$, respectively. The space of all bounded linear operators from $X$ to $Y$ is denoted by $\mathcal{L}(X,Y)$, and we define $\mathcal{L}(X) := \mathcal{L}(X,X)$. For a linear operator $A : D(A) \subset X \to X$, we denote by $\sigma(A)$, $\sigma_p(A)$, and $\rho(A)$ the spectrum, the point spectrum, and the resolvent set of $A$, respectively. The resolvent operator is denoted by $R(\lambda,A) = (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$. For a set $S \subset X$ and a linear operator $A : D(A) \subset X \to Y$, we write for $A|_S$ the restriction of $A$ to $S$, i.e., $A|_S x = Ax$ with domain $D(A|_S) := D(A) \cap S$. If $X$ is a Hilbert space, then we denote the inner product by $\langle x,y \rangle$ for $x,y \in X$ and the Hilbert space adjoint by $A^*$ for a linear operator $A$ with dense domain in $X$. Sequences $\{\phi_n : n \in \mathbb{N}\}$ and $\{\psi_n : n \in \mathbb{N}\}$ on a Hilbert space are called biorthogonal if

$$\langle \phi_n, \psi_m \rangle = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise}. \end{cases}$$

Let $X$, $U$, and $Y$ be Banach spaces, $A$ generate a strongly continuous semigroup on $X$, $B \in \mathcal{L}(U,X)$, $C \in \mathcal{L}(X,Y)$, and $\beta \in \mathbb{R}$. The control system $(A,B,\cdot)$ is called $\beta$-exponentially stabilizable if there exists $F \in \mathcal{L}(X,U)$ such that the growth bound of the semigroup generated by $A+BF$ is less than $\beta$. If $(A,B,\cdot)$ is $0$-exponentially stabilizable, then it is called exponentially stabilizable. The control system $(\cdot,\cdot,C)$ is called $\beta$-exponentially detectable if there exists $L \in \mathcal{L}(Y,X)$ such that the growth bound of the semigroup generated by $A+LC$ is less than $\beta$. If $(\cdot,\cdot,C)$ is $0$-exponentially detectable, then it is called exponentially detectable. A strongly continuous semigroup $(T(t))_{t \geq 0}$ on $X$ is called uniformly bounded if $\sup_{t \geq 0} \|T(t)\| < \infty$ and strongly stable if $\lim_{t \to \infty} T(t)x = 0$ for every $x \in X$. By a discrete semigroup on $X$, we mean a family $(\Delta^k)_{k \in \mathbb{N}}$ of operators, where $\Delta \in \mathcal{L}(X)$. A discrete semigroup $(\Delta^k)_{k \in \mathbb{N}}$ on $X$ is called power bounded if $\sup_{k \in \mathbb{N}} \|\Delta^k\| < \infty$ and strongly stable if $\lim_{k \to \infty} \|\Delta^k x\| = 0$ for every $x \in X$.

2. Infinite-dimensional sample-data system. Let $X$ be a Hilbert space, and consider the following sampled-data system with state space $X$:

$$\begin{align*}
(2.1a) & \quad \dot{x}(t) = Ax(t) + Bu(t), \quad t \geq 0; \quad x(0) = x^0 \in X \\
(2.1b) & \quad u(t) = Fx(k\tau), \quad k\tau \leq t < (k+1)\tau,
\end{align*}$$

where $x(t) \in X$ is the state, $u(t) \in \mathbb{C}$ is the control input, $\tau > 0$ is the sampling period, $A : D(A) \subset X \to X$ generates a strongly continuous semigroup $(T(t))_{t \geq 0}$ on $X$, $B \in \mathcal{L}(\mathbb{C},X)$ is the control operator, and $F \in \mathcal{L}(X,\mathbb{C})$ is the feedback operator.

**Definition 2.1.** The sampled-data system (2.1) is called strongly stable if

$$\lim_{t \to \infty} \|x(t)\| = 0$$

for every initial state $x^0 \in X$.

The objective of this paper is to show that if the strongly continuous semigroup $(T_{BF}(t))_{t \geq 0}$ generated by $A + BF$ is strongly stable, then the sampled-data system (2.1) is also strongly stable for all sufficiently small sampling periods $\tau > 0$.

For $t \geq 0$, define $S(t) \in \mathcal{L}(\mathbb{C},X)$ and $\Delta(t) \in \mathcal{L}(X)$ by

$$S(t) := \int_0^t T(s)Bds, \quad \Delta(t) := T(t) + S(t)F,$$
respectively. Then the state \( x \) of the sampled-data system (2.1) satisfies

\[(2.3) \quad x((k+1)\tau) = \Delta(\tau)x(k\tau) \quad \forall k \in \mathbb{N} \cup \{0\}.
\]

By the following proposition, it suffices to investigate the strong stability of the discrete semigroup \( (\Delta(\tau)^k)_{k \in \mathbb{N}} \) in order to study the strong stability of the sampled-data system (2.1).

**Proposition 2.2.** The sampled-data system (2.1) is strongly stable if and only if the discrete semigroup \( (\Delta(\tau)^k)_{k \in \mathbb{N}} \) is strongly stable.

**Proof.** Since \( (\Rightarrow) \) immediately follows from (2.3), we here show only \( (\Leftarrow) \). Suppose that \( (\Delta(\tau)^k)_{k \in \mathbb{N}} \) is strongly stable. Let \( x^0 \in X \) be given. We obtain

\[x(k\tau + t) = \Delta(t)x(k\tau) = \Delta(t)\Delta(\tau)^kx^0 \quad \forall t \in [0, \tau), \forall k \in \mathbb{N} \cup \{0\}.
\]

By the strong continuity of \( (T(t))_{t \geq 0} \), there exists \( c \geq 1 \) such that

\[\|\Delta(t)\| \leq c \quad \forall t \in [0, \tau).
\]

It follows that

\[\|x(k\tau + t)\| \leq c\|\Delta(\tau)^kx^0\| \quad \forall t \in [0, \tau), \forall k \in \mathbb{N} \cup \{0\}.
\]

By assumption, \( \|\Delta(\tau)^kx^0\| \to 0 \) as \( k \to \infty \). Thus, we obtain \( x(t) \to 0 \) as \( t \to \infty \). \( \Box \)

Instead of dealing with strong stability directly, we employ the following sufficient conditions obtained in the Arendt-Batty-Lyubich-Vu theorem [1, 21].

**Theorem 2.3** (Continuous case). Let \( (T(t))_{t \geq 0} \) be a uniformly bounded semigroup generated by \( A \) on a Hilbert space. If \( \sigma_p(A) \cap i\mathbb{R} = \emptyset \) and if \( \sigma(A) \cap i\mathbb{R} \) is countable, then \( (T(t))_{t \geq 0} \) is strongly stable.

**Theorem 2.4** (Discrete case). Let \( (\Delta^k)_{k \in \mathbb{N}} \) be a power bounded discrete semigroup on a Hilbert space. If \( \sigma_p(\Delta) \cap \mathbb{T} = \emptyset \) and if \( \sigma(\Delta) \cap \mathbb{T} \) is countable, then \( (\Delta^k)_{k \in \mathbb{N}} \) is strongly stable.

### 2.1. Basic facts on Riesz-spectral operators.

In the sampled-data system (2.1), we assume that \( A \) is a Riesz-spectral operator, which is defined as follows:

**Definition 2.5** (Definition 3.2.6 of [3]). Let \( A \) be a closed linear operator on a Hilbert space \( X \) with simple eigenvalues \( \{\lambda_n : n \in \mathbb{N}\} \) and corresponding eigenvectors \( \{\phi_n : n \in \mathbb{N}\} \). We say that \( A \) is a Riesz-spectral operator if the following two conditions are satisfied:

- a) \( \{\phi_n : n \in \mathbb{N}\} \) is a Riesz basis for \( X \), that is,
  - (i) the closed linear span of \( \{\phi_n : n \in \mathbb{N}\} \) is \( X \); and
  - (ii) there exist constants \( M_a, M_b > 0 \) such that for all \( N \in \mathbb{N} \) and all \( a_n \in \mathbb{C} \), \( 1 \leq n \leq N \),

\[(2.4) \quad M_a \sum_{n=1}^{N} |a_n|^2 \leq \left\| \sum_{n=1}^{N} a_n\phi_n \right\|^2 \leq M_b \sum_{n=1}^{N} |a_n|^2;
\]

- b) the set of eigenvalues \( \{\lambda_n : n \in \mathbb{N}\} \) has at most finitely many accumulation points.

Before stating the main result, we recall some basic facts on Riesz bases and Riesz-spectral operators; see [3, 8, 35] for more details.
LEMMA 2.6 (Lemma 3.2.4 of [3]). Let \( \{ \phi_n : n \in \mathbb{N} \} \) form a Riesz basis on a Hilbert space \( X \). Then the following hold:

a) There exists a unique biorthogonal sequence \( \{ \psi_n : n \in \mathbb{N} \} \), and \( \{ \psi_n : n \in \mathbb{N} \} \) is also a Riesz basis for \( X \).

b) Every \( x \in X \) can be represented uniquely by

\[
x = \sum_{n=1}^{\infty} \langle x, \psi_n \rangle \phi_n.
\]

Moreover, using constants \( M_a, M_b > 0 \) satisfying (2.4), one has

\[
M_a \sum_{n=1}^{\infty} |\langle x, \psi_n \rangle|^2 \leq ||x||^2 \leq M_b \sum_{n=1}^{\infty} |\langle x, \psi_n \rangle|^2 \quad \forall x \in X.
\]

LEMMA 2.7 (Lemma 3.2.5 of [3]). Suppose that a closed linear operator \( A \) on a Hilbert space \( X \) has simple eigenvalues \( \{ \lambda_n : n \in \mathbb{N} \} \) and that their corresponding eigenvectors \( \{ \phi_n : n \in \mathbb{N} \} \) form a Riesz basis in \( X \). If \( \{ \psi_n : n \in \mathbb{N} \} \) are the eigenvectors of the adjoint \( A^* \) of \( A \) corresponding to the eigenvalues \( \{ \lambda_n : n \in \mathbb{N} \} \), then \( \{ \psi_n : n \in \mathbb{N} \} \) can be suitably scaled so that \( \{ \phi_n : n \in \mathbb{N} \} \) and \( \{ \psi_n : n \in \mathbb{N} \} \) are biorthogonal.

THEOREM 2.8 (Theorem 3.2.8 of [3]). Suppose that \( A \) is a Riesz-spectral operator on a Hilbert space \( X \) with simple eigenvalues \( \{ \lambda_n : n \in \mathbb{N} \} \) and corresponding eigenvectors \( \{ \phi_n : n \in \mathbb{N} \} \). Let \( \{ \psi_n : n \in \mathbb{N} \} \) be the eigenvectors of \( A^* \) such that \( \{ \phi_n : n \in \mathbb{N} \} \) and \( \{ \psi_n : n \in \mathbb{N} \} \) are biorthogonal. Then \( A \) has the following properties:

a) \( A \) satisfies \( \rho(A) = \{ \lambda \in \mathbb{C} : \inf_{n \in \mathbb{N}} |\lambda - \lambda_n| > 0 \} \), \( \sigma(A) = \{ \lambda_n : n \in \mathbb{N} \} \), and

\[
(\lambda I - A)^{-1}x = \sum_{n=1}^{\infty} \frac{1}{\lambda - \lambda_n} \langle x, \psi_n \rangle \phi_n \quad \forall x \in X, \forall \lambda \in \rho(A).
\]

b) \( A \) has the representation

\[
Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \psi_n \rangle \phi_n \quad \forall x \in D(A),
\]

and \( D(A) \) can be written as

\[
D(A) = \left\{ x \in X : \sum_{n=1}^{\infty} |\lambda_n|^2 \cdot |\langle x, \psi_n \rangle|^2 < \infty \right\}.
\]

c) \( A \) is the generator of a strongly continuous semigroup \( (T(t))_{t \geq 0} \) if and only if \( \sup_{n \in \mathbb{N}} \Re \lambda_n < \infty \). The semigroup \( (T(t))_{t \geq 0} \) satisfies

\[
T(t)x = \sum_{n=1}^{\infty} e^{t \lambda_n} \langle x, \psi_n \rangle \phi_n \quad \forall x \in X, \forall t \geq 0,
\]

and the growth bound of \( (T(t))_{t \geq 0} \) is given by \( \sup_{n \in \mathbb{N}} \Re \lambda_n \).

2.2. Main result. We place the following assumption on the sampled-data system.
Assumption 2.9. Let $A$ be a Riesz-spectral operator on a Hilbert space $X$ with simple eigenvalues $\{\lambda_n : n \in \mathbb{N}\}$ and corresponding eigenvectors $\{\phi_n : n \in \mathbb{N}\}$. Let $\{\psi_n : n \in \mathbb{N}\}$ be the eigenvectors of $A^*$ that are biorthogonal with $\{\phi_n : n \in \mathbb{N}\}$. Let the control operator $B \in \mathcal{L}(\mathbb{C}, X)$ and the feedback operator $F \in \mathcal{L}(X, \mathbb{C})$ be represented as

$$(2.7) \quad Bu = bu, \quad u \in \mathbb{C}; \quad Fx = \langle x, f \rangle, \quad x \in X$$

for some $b, f \in X$. Assume that the operators $A$, $B$, and $F$ satisfy the following conditions:

(A1) there exist $\alpha > 0$ and $0 < \delta \leq \pi/2$ such that the set $\mathbb{C}_{-\alpha} \cap \Sigma_{\pi/2+\delta}$ has only finite elements of $\{\lambda_n : n \in \mathbb{N}\}$ (see also Fig. 1 for the set $\mathbb{C}_{-\alpha} \cap \Sigma_{\pi/2+\delta}$);

(A2) $\{\lambda_n : n \in \mathbb{N}\} \cap i\mathbb{R} = \emptyset$;

(A3) $0 \in \{\lambda_n : n \in \mathbb{N}\}$;

(A4) $A + BF$ generates a uniformly bounded semigroup on $X$ and satisfies $\sigma_p(A + BF) \cap i\mathbb{R} = \emptyset$, $\sigma(A + BF) \cap i\mathbb{R} = \{0\}$, and

$$(2.8) \quad \sup_{\omega \in \mathbb{R}} \|R(i\omega, A + BF)\| < \infty;$$

(A5) $b$ satisfies

$$\sum_{n=1}^{\infty} \left| \frac{(b, \psi_n)}{\lambda_n} \right|^2 < \infty;$$

(A6) $b$ and $f$ satisfy

$$\sum_{n=1}^{\infty} \frac{(b, \psi_n)(\phi_n, f)}{\lambda_n} \neq -1.$$

By (A1), $\sup_{n \in \mathbb{N}} \Re \lambda_n < \infty$. Therefore, Theorem 2.8 c) shows that $A$ generates a strongly continuous semigroup. Since $\sigma(A) = \{\lambda_n : n \in \mathbb{N}\}$ by Theorem 2.8 a), it follows from (A2) and (A3) that $0 \in \sigma(A) \setminus \sigma_p(A)$. Applying the mean ergodic theorem (see, e.g., Theorem I.2.25 of [4]) to the stable part of $A$, we find that $0$ belongs to the continuous spectrum of $A$; see Remark 4.4 for details. Note that the control system $(A, B, -)$ is not exponentially stabilizable by Theorem 8.2.3 of [3]. By (A4) and the Arendt-Batty-Lyubich-Vu theorem, the semigroup $(T_{BF}(t))_{t \geq 0}$ generated by $A + BF$ is strongly stable. Using the mean ergodic theorem again, we see that $0$ is still in the continuous spectrum of $A + BF$.

The assumption (2.8) will be used to guarantee that $|1 - FR(\lambda, A)B|$ is bounded from below by a positive constant on $\overline{\mathbb{C}_0} \setminus \mathbb{D}_\eta$ for every $\eta > 0$. This assumption (2.8) appears also in the robustness analysis of strong stability developed in [25], and the assumption in the form

$$\sup_{0 < |\omega| \leq 1} |\omega| \cdot \|R(i\omega, A + BF)\| < \infty$$

is additionally placed in [25]. Instead of this assumption, we place (A5) and (A6) in order to obtain a positive lower bound of $|1 - FR(\lambda, A)B|$ on $(\overline{\mathbb{C}_0} \cap \mathbb{D}_\eta) \setminus \{0\}$ for a sufficiently small $\eta > 0$. The sectorial condition on the eigenvalues in (A1) is also used for this purpose. Similarly, when the robustness of exponential stability with respect to sampling is analyzed for systems with unbounded control operators in [18], the semigroup $(T(t))_{t \geq 0}$ is assumed to be holomorphic, which guarantees that $\sigma(A)$ is contained in a certain sector.
We easily see that $b$ belongs to the domain of the algebraic inverse of $A$ under (A5). To deal with the high sensitivity of strong stability to perturbations, we assume by (A5) that the control operator $B$ has the boundedness property related to the continuous spectrum of $A$ in addition to the standard boundedness property $B \in \mathcal{L}(\mathbb{C}, X)$. It is assumed also in [25] that perturbations have such stronger boundedness properties.

The following theorem, which presents robustness of strong stability with respect to sampling, is the main result of this paper.

**Theorem 2.10.** If Assumption 2.9 is satisfied, then there exists $\tau^* > 0$ such that for every $\tau \in (0, \tau^*)$, the sampled-data system (2.1) is strongly stable.

The idea of the proof is to show that the discrete semigroup $\Delta(\tau^k)_{k \in \mathbb{N}}$ satisfies the sufficient condition for strong stability in the Arendt-Batty-Lyubich-Vă theorem. The sufficient condition given in this theorem consists of the spectral property and the boundedness property of a discrete semigroup. These properties of $\Delta(\tau^k)_{k \in \mathbb{N}}$ are investigated in Sections 3 and 4, respectively.

**3. Spectrum and sampling.** Our first goal is to show that the spectral properties in the Arendt-Batty-Lyubich-Vă theorem is satisfied for $\Delta(\tau)$ with sufficiently small $\tau > 0$.

**Theorem 3.1.** If Assumption 2.9 is satisfied, then there exists $\tau^* > 0$ such that $\sigma_p(\Delta(\tau)) \cap \tau = \emptyset$ and $\sigma(\Delta(\tau)) \cap \tau = \{1\}$ for every $\tau \in (0, \tau^*)$.

With a slight modification of Theorem 2.8 a), we easily obtain the following properties of the spectrum and the resolvent of $T(t)$ represented by (2.6).

**Lemma 3.2.** Let $\{\lambda_n : n \in \mathbb{N}\} \subset \mathbb{C}$ satisfy $\sup_{n \in \mathbb{N}} \text{Re} \lambda_n < \infty$. Suppose that $\{\phi_n : n \in \mathbb{N}\}$ is a Riesz basis for a Hilbert space $X$, and let $\{\psi_n : n \in \mathbb{N}\}$ be the biorthogonal sequence to $\{\phi_n : n \in \mathbb{N}\}$. If we define $T(t) \in \mathcal{L}(X)$ by (2.6) for $t \geq 0$, then $\sigma_p(T(t)) = \{e^{i\lambda_n} : n \in \mathbb{N}\}$ and $\sigma(T(t)) = \{e^{i\lambda_n} : n \in \mathbb{N}\}$ for every $t \geq 0$. Moreover, for all $z \in \rho(T(t))$ and $t \geq 0$, the resolvent $R(z, T(t))$ is given by

$$
R(z, T(t))x = \sum_{n=1}^{\infty} \frac{1}{z - e^{i\lambda_n}} (x, \psi_n)\phi_n \quad \forall x \in X.
$$

(3.1)

We also immediately obtain a representation of the algebraic inverse of a Riesz-spectral operator $A$ with $0 \notin \sigma_p(A)$.

**Lemma 3.3.** Let $A$ be a Riesz-spectral operator on a Hilbert space $X$ as in Theorem 2.8, and assume that the eigenvalues $\{\lambda_n : n \in \mathbb{N}\}$ satisfy $\sup_{n \in \mathbb{N}} \text{Re} \lambda_n < \infty$ and $\lambda_n \neq 0$ for all $n \in \mathbb{N}$. The operator $A_0$ defined by

$$
A_0x := \sum_{n=1}^{\infty} \frac{1}{\lambda_n} (x, \psi_n)\phi_n
$$

with domain

$$
D(A_0) := \left\{ x \in X : \sum_{n=1}^{\infty} \left| \frac{(x, \psi_n)}{\lambda_n} \right|^2 < \infty \right\}
$$

is the algebraic inverse $A^{-1}$ of $A$, i.e., satisfies $D(A_0) = \text{ran}(A)$, $A_0Ax = x$ for every $x \in D(A)$, and $AA_0x = x$ for every $x \in D(A_0)$. Moreover, for all $x \in D(A^{-1})$
and $t \geq 0$, the semigroup $(T(t))_{t \geq 0}$ generated by $A$ satisfies $T(t)x \in D(A^{-1})$ and $A^{-1}T(t)x = T(t)A^{-1}x$.

Lemma 3.3 shows that if (A5) also holds, i.e., $b \in D(A^{-1})$, then $S(t)$ defined by (2.2) is written as

\[(3.2) \quad S(t) = A^{-1}(T(t) - I)B = \sum_{n=1}^{\infty} \frac{e^{t\lambda_n} - 1}{\lambda_n} \langle b, \psi_n \rangle \phi_n \quad \forall t \geq 0.
\]

This and (3.1) yield

\[(3.3) \quad (zI - T(t))^{-1}S(t) = \sum_{n=1}^{\infty} \frac{e^{t\lambda_n} - 1}{z - e^{t\lambda_n}} \frac{\langle b, \psi_n \rangle}{\lambda_n} \phi_n \quad \forall z \in \rho(T(t)), \forall t \geq 0.
\]

Using the expression (3.2), we obtain the following simple result on the spectrum of $\Delta(t)$.

**Lemma 3.4.** Let $A$ be a Riesz-spectral operator on a Hilbert space $X$ whose eigenvalues $\{\lambda_n : n \in \mathbb{N}\}$ satisfy $\sup_{n \in \mathbb{N}} \Re \lambda_n < \infty$, (A2), and (A3). Assume that $B \in \mathcal{L}(X, \mathbb{C})$ and $F \in \mathcal{L}(\mathbb{C}, X)$ in the form of (2.7) satisfy (A5) and (A6). Then $1 \in \sigma(\Delta(t)) \setminus \sigma_p(\Delta(t))$ for every $t > 0$.

**Proof.** Let $t > 0$. The essential spectrum of $T(t)$ contains 1 under (A3), and it follows from the compactness of $S(t)F$ that $T(t)$ and $\Delta(t) = T(t) + S(t)F$ have the same essential spectrum; see, e.g., Section IV.1.20 of [5] and Section XVII.2 of [6] for an essential spectrum. Hence $1 \in \sigma(\Delta(t))$.

To show that $1 \notin \sigma_p(\Delta(t))$, let $v \in X$ satisfy $\Delta(t)v = v$. By (3.2), we obtain

\[ (T(t) - I)(I + A^{-1}BF)v = 0. \]

Lemma 3.2 and (A2) give $1 \notin \sigma_p(T(t))$. Therefore, $(I + A^{-1}BF)v = 0$. If $Fv = 0$, then $v = 0$. Otherwise,

\[(3.4) \quad - FA^{-1}BFv = Fv \neq 0.
\]

By (A6),

\[ FA^{-1}B = \langle A^{-1}b, f \rangle = \sum_{n=1}^{\infty} \frac{\langle b, \psi_n \rangle \langle \phi_n, f \rangle}{\lambda_n} \neq -1,
\]

which contradicts (3.4). Thus, $v = 0$; i.e., $1 \notin \sigma_p(\Delta(t))$. \(\square\)

The next lemma provides a useful property of the spectrum of the product of bounded operators; see, e.g., (3) in Section III.2 of [6].

**Lemma 3.5.** Let $X$ and $Y$ be Banach spaces. For all $T \in \mathcal{L}(X, Y)$ and $S \in \mathcal{L}(Y, X)$, $\sigma(TS) \setminus \{0\} = \sigma(ST) \setminus \{0\}$ holds.

We obtain the estimate of $|1 - F(\lambda I - A)^{-1}B|$ for $\lambda \in \rho(A) \cap \mathbb{C}_0$.

**Lemma 3.6.** Let $A$ be a Riesz-spectral operator on a Hilbert space $X$ whose eigenvalues $\{\lambda_n : n \in \mathbb{N}\}$ satisfy (A1) and $0 \notin \{\lambda_n : n \in \mathbb{N}\} \setminus \{\lambda_n : n \in \mathbb{N}\}$. Assume that $B \in \mathcal{L}(X, \mathbb{C})$ and $F \in \mathcal{L}(\mathbb{C}, X)$ in the form of (2.7) satisfy (A4)–(A6). Then there exists $\epsilon > 0$ such that $|1 - F(\lambda I - A)^{-1}B| > \epsilon$ for every $\lambda \in \rho(A) \cap \mathbb{C}_0$.

**Proof.** We divide the proof into two cases. First, we consider the case where $\lambda \in \rho(A)$ belongs to $\overline{\mathbb{C}_0 \setminus B_\eta}$ for $\eta > 0$. To this end, we employ the condition (2.8) in
(A4). Second, we study the case \( \lambda \in (\mathbb{C}_0 \cap \mathbb{D}_\eta) \setminus \{0\} \) for a sufficiently small \( \eta > 0 \), using (A1), (A5), and (A6).

Let \( \lambda \in \rho(A) \) be given. Since
\[
\lambda I - A - BF = (\lambda I - A)(I - (\lambda I - A)^{-1}BF)
\]
and since \( \sigma((\lambda I - A)^{-1}BF) \setminus \{0\} = \sigma(F(\lambda I - A)^{-1}B) \setminus \{0\} \) by Lemma 3.5, it follows that
\[
\lambda \in \rho(A + BF) \iff 1 \in \rho((\lambda I - A)^{-1}BF) \iff 1 \in \rho(F(\lambda I - A)^{-1}B).
\]
Define \( G(\lambda) := F(\lambda I - A)^{-1}B \). A straightforward calculation shows that
\[
(3.5) \quad \frac{1}{1 - G(\lambda)} = F(\lambda I - A - BF)^{-1}B + 1 \quad \forall \lambda \in \rho(A) \cap \rho(A + BF).
\]
Using this equation, we can extend \( 1/(1 - G) \), which is defined only on \( \rho(A) \cap \rho(A + BF) \), to a holomorphic function on \( \rho(A + BF) \supset \mathbb{C}_0 \setminus \{0\} \).

Combining (2.8) in (A4) and the Neumann series of resolvents (see, e.g., Proposition IV.1.3 of [5]), we have that \( \|R(\lambda, A + BF)\| \leq 2M \) for every \( \lambda \in \Lambda_1 := \{ \lambda \in \mathbb{C} : 0 \leq \text{Re} \lambda \leq c, |\text{Im} \lambda| > 1 \} \), where
\[
M := \sup_{\omega \in \mathbb{R}, |\omega| > 1} \|R(i\omega, A + BF)\|, \quad c := \frac{1}{2M}.
\]
By the uniform boundedness of \( (T_{BF}(t))_{t \geq 0} \), the Hille-Yosida theorem (see, e.g., Theorem II.3.8 of [5]) shows that
\[
\|R(\lambda, A + BF)\| \leq \frac{\sup_{t \geq 0} \|T_{BF}(t)\|}{c}
\]
for every \( \lambda \in \Lambda_2 := \mathbb{C}_c \). For \( 0 < \eta < \min\{c, 1\} \), the resolvent \( R(\lambda, A + BF) \) is holomorphic on the compact set
\[
\Lambda_3 := \{ \lambda \in \mathbb{C} : |\lambda| \geq \eta, 0 \leq \text{Re} \lambda \leq c, |\text{Im} \lambda| \leq 1 \}.
\]
Theorem 2.8 a) and Lemma 3.3 that

\[ \epsilon \]

for all \( r \) (see Fig. 2), we conclude from (3.5) that

\[ |\eta| < \epsilon \]

for every \( r \). Note that

\[ (3.6) \]

Hence \( (\overline{D}_{\eta_1} \cap \Sigma_{\pi/2+\delta} \cap \{\lambda_n : n \in \mathbb{N}\}) = \emptyset \).

Note that \( b \in D(A^{-1}) \) by (A5). It is enough to show that \( (re^{i\theta} - A)^{-1}b \) converges to \( -A^{-1}b \) uniformly on \( \theta \in [-\pi/2, \pi/2] \) as \( r \to 0 \). More precisely, we will show that for every \( \epsilon_2 > 0 \), there exists \( \eta \in (0, \eta_1) \) such that

\[ (re^{i\theta} - A)^{-1}b + A^{-1}b < \epsilon_2 \]

for all \( r \in (0, \eta) \) and all \( \theta \in [-\pi/2, \pi/2] \). Indeed, using this fact and \( \epsilon_3 := |1 + FA^{-1}b| > 0 \) by (A6), we see that

\[ |1 - G(re^{i\theta})| \geq |1 + FA^{-1}b| - |F(re^{i\theta} - A)^{-1}b + FA^{-1}b| > \epsilon_3 - \|F\|\epsilon_2 \]

for all \( r \in (0, \eta) \) and all \( \theta \in [-\pi/2, \pi/2] \). Hence if \( \|F\|\epsilon_2 < \epsilon_3 \), then (3.7) holds with \( \epsilon_1 := \epsilon_3 - \|F\|\epsilon_2 > 0 \).

Since \( re^{i\theta} \in \rho(A) \) for every \( r \in (0, \eta_1) \) and \( \theta \in [-\pi/2, \pi/2] \), it follows from Theorem 2.8 a) and Lemma 3.3 that

\[ (re^{i\theta} - A)^{-1}b + A^{-1}b = \sum_{n=1}^{\infty} \left( \frac{1}{re^{i\theta} - \lambda_n} + \frac{1}{\lambda_n} \right) \langle b, \psi_n \rangle \phi_n \]

for every \( r \in (0, \eta_1) \) and \( \theta \in [-\pi/2, \pi/2] \). Using (2.4), we obtain

\[ \|(re^{i\theta} - A)^{-1}b + A^{-1}b\|^2 \leq M_0 \sum_{n=1}^{\infty} \left| \left( \frac{1}{re^{i\theta} - \lambda_n} + \frac{1}{\lambda_n} \right) \langle b, \psi_n \rangle \right|^2 \]

\[ \leq M_0 \sum_{n=1}^{\infty} g_n(r) \quad \forall r \in (0, \eta_1), \quad \forall \theta \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] , \]

where

\[ g_n(r) := \sup_{-\pi/2 \leq \theta \leq \pi/2} \left| \left( \frac{1}{re^{i\theta} - \lambda_n} + \frac{1}{\lambda_n} \right) \langle b, \psi_n \rangle \right|^2 . \]

Let \( r \in (0, \eta_1/2) \) and \( \theta \in [-\pi/2, \pi/2] \) be given. Suppose that \( \lambda_n \in \Sigma_{\pi/2+\delta} \). Then \( |\lambda_n| \geq \eta_1 \) by (3.6), and hence

\[ |re^{i\theta} - \lambda_n| \geq \frac{\eta_1}{2} > r. \]

Since

\[ (3.8) \]

\[ \frac{1}{re^{i\theta} - \lambda_n} + \frac{1}{\lambda_n} = \frac{re^{i\theta} - \lambda_n}{re^{i\theta} - \lambda_n} \cdot \frac{1}{\lambda_n}, \]
it follows that

\[(3.9) \quad g_n(r) \leq \left| \frac{\langle b, \psi_n \rangle}{\lambda_n} \right|^2 \quad \text{if } \lambda_n \in \Sigma_{\pi/2+\delta}.\]

Suppose next that \(\lambda_n \in \mathbb{C} \setminus \Sigma_{\pi/2+\delta}\). Using the estimates

|\lambda_n| \leq \frac{|\text{Re}\lambda_n|}{\sin \delta}, \quad |r e^{i\theta} - \lambda_n| \geq |\text{Re}\lambda_n|,

we obtain

| \frac{r e^{i\theta}}{r e^{i\theta} - \lambda_n} | \leq 1 + \left| \frac{\lambda_n}{r e^{i\theta} - \lambda_n} \right| \leq 1 + \frac{1}{\sin \delta}.

By (3.8),

\[(3.10) \quad g_n(r) \leq \left( 1 + \frac{1}{\sin \delta} \right)^2 \left| \frac{\langle b, \psi_n \rangle}{\lambda_n} \right|^2 \quad \text{if } \lambda_n \in \mathbb{C} \setminus \Sigma_{\pi/2+\delta}.\]

The estimates (3.9) and (3.10) yield

\[g_n(r) \leq \left( 1 + \frac{1}{\sin \delta} \right)^2 \left| \frac{\langle b, \psi_n \rangle}{\lambda_n} \right|^2 \quad \forall r \in \left( 0, \frac{\eta}{2} \right), \quad \forall n \in \mathbb{N}.\]

Combining this with (A5), we obtain

\[\lim_{r \to 0} \sum_{n=1}^{\infty} g_n(r) = \sum_{n=1}^{\infty} \lim_{r \to 0} g_n(r) = 0.\]

Thus, \((r e^{i\theta} - A)^{-1} b\) converges to \(-A^{-1} b\) uniformly on \(\theta \in [-\pi/2, \pi/2]\) as \(r \to 0\). This completes the proof.

Remark 3.7. Combining (3.5) and Lemma 3.6, we see that the transfer function \(G_{BF}(\lambda) := F(\lambda I - A - BF)^{-1} B\) is holomorphic and bounded on \(C_0\). This property is equivalent to the exponential stability of \((T_{BF}(t))_{t \geq 0}\) if \((A, B, F)\) is exponentially stabilizable and exponentially detectable; see, e.g., Theorem VI. 8.35 of [5]. However, since \(0\) is an accumulation point of \(\sigma_p(A)\) in our problem setting, \((A, B, F)\) is not exponentially stabilizable or exponentially detectable by Theorem 8.2.3 of [3].

As in the robustness analysis of exponential stability with respect to sampling given in Theorem 2.1 of [31], we connect the estimates of the continuous-time system and the discrete-time system.

Lemma 3.8. Let \(A\) be a Riesz-spectral operator on a Hilbert space \(X\) whose eigenvalues \(\{\lambda_n : n \in \mathbb{N}\}\) satisfy (A1) and \(0 \in \{\lambda_n : n \in \mathbb{N}\}\} \setminus \{\lambda_n : n \in \mathbb{N}\}.\ Let \(B \in L(X, \mathbb{C})\) and \(F \in L(\mathbb{C}, X)\) be given by (2.7), and assume that (A5) holds. If there exists \(\epsilon_c \in (0, 1)\) such that

\[(3.11) \quad |1 - F(\lambda I - A)^{-1} B| > \epsilon_c \quad \forall \lambda \in \rho(A) \cap \overline{C_0},\]

then, for every \(\epsilon_d \in (0, \epsilon_c),\) there exists \(\tau^* > 0\) such that for every \(\tau \in (0, \tau^*),\)

\[(3.12) \quad |1 - F(zI - T(\tau))^{-1} S(\tau)| > \epsilon_d \quad \forall z \in \rho(T(\tau)) \cap \overline{E_1}.\]
Proof. The proof is divided into four steps. In Steps 1 and 2, we study the infinite-dimensional tail of the series of \((\lambda I - A)^{-1}B\) and \((z I - T(\tau))^{-1}S(\tau)\), respectively. These infinite-dimensional tails can be regarded as the errors of finite-dimensional approximations. The objective of Steps 1 and 2 is to show that such an error becomes arbitrarily small as the approximation order increases. In Step 3, we look at the finite-dimensional approximation of the transfer function \(F(z I - T(\tau))^{-1}S(\tau)\). We do not encounter any difficulties arising from strong stability in this step, and so the result on exponential stability obtained in the proof of Theorem 2.1 of [31] can be applied without modifications. In Step 4, we investigate the set \(\rho(T(\tau)) \cap E_1\) in order to complete the proof.

Step 1: We show that for every \(\epsilon > 0\), there exists \(N_0^\epsilon \in \mathbb{N}\) such that

\[
\sup_{\lambda \in \mathbb{C} \setminus \{0\}} \left\| \sum_{n=N}^{\infty} \frac{\langle b, \psi_n \rangle}{\lambda - \lambda_n} \phi_n \right\| \leq \epsilon \quad \forall N \geq N_0^\epsilon.
\]

Let \(\lambda \in \mathbb{C} \setminus \{0\}\). By (A1), there exists \(N_1 \in \mathbb{N}\) such that

\[
\lambda_n \in \mathbb{C} \setminus \mathbb{C}_{-\alpha} \quad \text{or} \quad \lambda_n \in \mathbb{C} \setminus \Sigma_{\pi/2+\delta} \quad \forall n \geq N_1.
\]

If \(\lambda_n \in \mathbb{C} \setminus \mathbb{C}_{-\alpha}\), then

\[
\left| \frac{1}{\lambda - \lambda_n} + \frac{1}{\lambda_n} \right| \leq \frac{2}{\alpha} := \Gamma_1.
\]

Suppose that \(\lambda_n \in \mathbb{C} \setminus \Sigma_{\pi/2+\delta}\). We obtain

\[
\frac{1}{\lambda - \lambda_n} + \frac{1}{\lambda_n} = \left(1 + \frac{\lambda_n}{\lambda - \lambda_n}\right) \frac{1}{\lambda_n}.
\]

Since

\[
|\lambda_n| \leq \frac{|\text{Re}\lambda_n|}{\sin \delta}, \quad |\lambda - \lambda_n| \geq |\text{Re}\lambda_n|,
\]

it follows that

\[
\left| \frac{1}{\lambda - \lambda_n} + \frac{1}{\lambda_n} \right| \leq \left(1 + \frac{1}{\sin \delta}\right) \frac{1}{|\lambda_n|} = \Gamma_2.
\]

Let \(\epsilon > 0\) be given. By Lemma 2.6 b) and (A5),

\[
\sum_{N=1}^{\infty} |\langle b, \psi_n \rangle|^2 < \infty, \quad \sum_{N=1}^{\infty} \left| \frac{\langle b, \psi_n \rangle}{\lambda_n} \right|^2 < \infty.
\]

Therefore, there exists \(N_2 \geq N_1\) such that

\[
\sum_{n=N}^{\infty} |\langle b, \psi_n \rangle|^2 < \frac{\epsilon^2}{8M_b \Gamma_1^2}, \quad \sum_{n=N}^{\infty} \left| \frac{\langle b, \psi_n \rangle}{\lambda_n} \right|^2 < \frac{\epsilon^2}{8M_b \Gamma_2^2} \quad \forall N \geq N_2.
\]

Combining this with (2.4), we obtain

\[
\left\| \sum_{n=N}^{\infty} \frac{\langle b, \psi_n \rangle}{\lambda_n} \phi_n \right\|^2 \leq M_b \sum_{n=N}^{\infty} \left| \frac{\langle b, \psi_n \rangle}{\lambda_n} \right|^2 < \frac{\epsilon^2}{4} \quad \forall N \geq N_2.
\]
Moreover, since (3.15) and (3.16) yield
\[
\left| \frac{1}{\lambda - \lambda_n} \right|^2 \leq \frac{\Gamma_1^2 + \Gamma_2^2}{|\lambda_n|^2} \quad \forall N \geq N_1,
\]
it follows from (2.4) that, for every \( N \geq N_2 \),
\[
\left\| \sum_{n=N}^{\infty} \frac{\langle b, \psi_n \rangle}{\lambda - \lambda_n} \phi_n + \sum_{n=N}^{\infty} \frac{\langle b, \psi_n \rangle}{\lambda_n} \phi_n \right\|^2 \leq M_b \sum_{n=N}^{\infty} \left| \frac{1}{\lambda - \lambda_n} \right|^2 |\langle b, \psi_n \rangle|^2
\]
\[
\leq M_b \left( \Gamma_1^2 \sum_{n=N}^{\infty} |\langle b, \psi_n \rangle|^2 + \Gamma_2^2 \sum_{n=N}^{\infty} |\lambda_n|^2 \right)< \epsilon^2.
\]
Therefore,
\[
\left\| \sum_{n=N}^{\infty} \frac{\langle b, \psi_n \rangle}{\lambda - \lambda_n} \phi_n \right\| \leq \left\| \sum_{n=N}^{\infty} \frac{\langle b, \psi_n \rangle}{\lambda - \lambda_n} \phi_n + \sum_{n=N}^{\infty} \frac{\langle b, \psi_n \rangle}{\lambda_n} \phi_n \right\| = \left\| \sum_{n=N}^{\infty} \frac{\langle b, \psi_n \rangle}{\lambda_n} \phi_n \right\| < \epsilon
\]
for every \( N \geq N_2 \). Thus, (3.13) holds with \( N_0^d := N_2 \).

**Step 2:** Recall that \((zI - T(t))^{-1}S(t)\) can be represented in the form (3.3). We shall show that for every \( \epsilon > 0 \), there exists \( N_0^d \in \mathbb{N} \) such that
\[
(3.18) \quad \sup_{z \in \mathbb{E}_1 \setminus \{1\}} \left\| \sum_{n=1}^{\infty} \frac{1-e^{\tau \lambda_n}}{z-e^{\tau \lambda_n}} \cdot \frac{\langle b, \psi \rangle}{\lambda_n} \phi_n \right\| \leq \epsilon \quad \forall \tau > 0, \forall N \geq N_0^d.
\]

Let \( z \in \mathbb{E}_1 \setminus \{1\} \) and \( \tau > 0 \). As in Step 1, we choose \( N_1 \in \mathbb{N} \) so that (3.14) holds. The following inequality is useful to obtain the estimate (3.18):
\[
(3.19) \quad \left| \frac{1-e^{\tau \lambda_n}}{z-e^{\tau \lambda_n}} \right| \leq \left| \frac{1-e^{\tau \lambda_n}}{1-e^{\tau \Re \lambda_n}} \right| = \frac{|1-e^{\tau \lambda_n}|}{|1-e^{\tau \Re \lambda_n}|} \cdot \frac{|\lambda_n|}{|\Re \lambda_n|} \quad \forall n \geq N_1.
\]

Let \( n \geq N_1 \). Recalling that (3.14) holds, we first consider the case \( \lambda_n \in \mathbb{C} \setminus \mathbb{C}_{\alpha} \), i.e., \( \Re \lambda_n \leq -\alpha \). Suppose that \(-1 \leq \tau \Re \lambda_n \leq 0\). The function
\[
g(\lambda) := \begin{cases} \frac{1-e^\lambda}{\lambda} & \text{if } \lambda \neq 0 \\ -1 & \text{if } \lambda = 0 \end{cases}
\]
is holomorphic on \( \mathbb{C} \). Hence, on a compact set \( \{ \lambda \in \mathbb{C} : -1 \leq \Re \lambda \leq 0, \ |\Im \lambda| \leq \pi \} \), there exists \( M_1 > 0 \) such that \( |g(\lambda)| \leq M_1 \). For every \( \lambda \in \mathbb{C} \) with \( |\Im \lambda| \leq \pi \),
\[
|g(\lambda + 2\ell \pi i)| = \left| \frac{1-e^{\lambda + 2\ell \pi i}}{\Re \lambda + i(\Re \lambda + 2\ell \pi)} \right| \leq |g(\lambda)| \quad \forall \ell \in \mathbb{N}.
\]
Therefore, \( |g(\lambda)| \leq M_1 \) if \(-1 \leq \Re \lambda \leq 0\). This estimate on \( g \) shows that
\[
|g(\lambda + 2\ell \pi i)| = \left| \frac{1-e^{\ell \pi \lambda}}{\Re \lambda + 2\ell \pi} \right| \leq M_1.
\]

(3.20)
Moreover, by the mean value theorem,
\begin{equation}
\frac{1 - e^{\tau \text{Re} \lambda_n}}{\tau |\text{Re} \lambda_n|} \geq e^{-1}.
\end{equation}

It follows from (3.19)–(3.21) that
\begin{equation}
\left| \frac{1 - e^{\tau \lambda_n}}{z - e^{\tau \lambda_n}} \right| \cdot \frac{1}{\lambda_n} \leq \frac{eM_1}{|\text{Re} \lambda_n|} \leq \frac{eM_1}{\alpha}.
\end{equation}

Suppose that $\tau \text{Re} \lambda_n < -1$. Then (3.19) leads to
\begin{equation}
\left| \frac{1 - e^{\tau \lambda_n}}{z - e^{\tau \lambda_n}} \right| \leq \frac{e}{1 - e^{-1}}.
\end{equation}

Thus, if $\text{Re} \lambda_n \leq -\alpha$, then
\begin{equation}
\left| \frac{1 - e^{\tau \lambda_n}}{z - e^{\tau \lambda_n}} \right| \cdot \frac{1}{\lambda_n} \leq \max \left\{ \frac{eM_1}{\alpha}, \frac{2}{1 - e^{-1}} \right\} =: \Upsilon_1.
\end{equation}

Next we consider the case $\lambda_n \in \mathbb{C} \setminus \Sigma_{\pi/2+\delta}$, i.e., $\text{Re} \lambda_n \leq 0$ and
\begin{equation}
|\lambda_n| \leq \frac{|\text{Re} \lambda_n|}{\sin \delta}.
\end{equation}

If $-1 \leq \tau \text{Re} \lambda_n \leq 0$, then (3.19)–(3.21) yield
\begin{equation}
\left| \frac{1 - e^{\tau \lambda_n}}{z - e^{\tau \lambda_n}} \right| \leq \frac{eM_1}{\sin \delta}.
\end{equation}

If $\tau \text{Re} \lambda_n < -1$, then (3.22) holds. Thus, in the case $\lambda_n \in \mathbb{C} \setminus \Sigma_{\pi/2+\delta}$, we obtain
\begin{equation}
\left| \frac{1 - e^{\tau \lambda_n}}{z - e^{\tau \lambda_n}} \right| \leq \max \left\{ \frac{eM_1}{\sin \delta}, \frac{2}{1 - e^{-1}} \right\} =: \Upsilon_2.
\end{equation}

By the estimates (3.23) and (3.24), for every $N \geq N_1$,
\begin{equation}
\left\| \sum_{n=N}^{\infty} \frac{1 - e^{\tau \lambda_n}}{z - e^{\tau \lambda_n}} \cdot \frac{\langle b, \psi_n \rangle}{\lambda_n} \phi_n \right\|^2 \leq M_0 \sum_{n=N}^{\infty} \left| \frac{1 - e^{\tau \lambda_n}}{z - e^{\tau \lambda_n}} \cdot \frac{\langle b, \psi_n \rangle}{\lambda_n} \right|^2 \leq M_0 \left( \Upsilon_1 \sum_{n=N}^{\infty} |\langle b, \psi_n \rangle|^2 + \Upsilon_2 \sum_{n=N}^{\infty} \left| \frac{\langle b, \psi_n \rangle}{\lambda_n} \right|^2 \right).
\end{equation}

Similarly to Step 1, it follows from (3.17) that for every $\epsilon > 0$, there exists $N_0^\epsilon \geq N_1$ such that (3.18) holds.

**Step 3:** By (3.3),
\begin{equation}
1 - F(zI - T(\tau))^{-1}S(\tau) = 1 + \sum_{n=1}^{\infty} \frac{1 - e^{\tau \lambda_n}}{z - e^{\tau \lambda_n}} \cdot \frac{\langle b, \psi_n \rangle}{\lambda_n} \langle \phi_n, f \rangle
\end{equation}

for all $z \in \rho(T(\tau))$ and $\tau > 0$. Assume that $\epsilon_c \in (0, 1)$ satisfies (3.11), and choose $\epsilon \in (0, \epsilon_c/3)$ arbitrarily. By Steps 1 and 2, there exists $N_0 \in \mathbb{N}$ such that for every
\[ N \geq N_0 \text{ and } \tau > 0, \]
\[
\text{(3.26a)} \quad \sup_{\lambda \in \mathbb{T}_0 \setminus \{0\}} \left| \sum_{n=N}^{\infty} \frac{\langle b, \psi_n \rangle \langle \phi_n, f \rangle}{\lambda - \lambda_n} \right| \leq \epsilon \]
\[
\text{(3.26b)} \quad \sup_{\zeta \in \mathbb{B}_1 \setminus \{1\}} \left| \sum_{n=N}^{\infty} \frac{1 - e^{\tau \lambda_n}}{z - e^{\tau \lambda_n}} \cdot \frac{\langle b, \psi_n \rangle \langle \phi_n, f \rangle}{\lambda_n} \right| \leq \epsilon. \]

Let \( N_1 \in \mathbb{N} \) satisfy (3.14), and take \( N \geq \max\{N_0, N_1\} \). We investigate the finite-dimensional truncation:
\[
\sum_{n=1}^{N-1} \frac{1 - e^{\tau \lambda_n}}{z - e^{\tau \lambda_n}} \cdot \frac{\langle b, \psi_n \rangle \langle \phi_n, f \rangle}{\lambda_n}. \]

This finite sum has no difficulty arising from strong stability, i.e., \( 0 \in \sigma(A) \setminus \sigma_p(A) \). Hence we can apply the result on exponential stability developed in the proof of Theorem 2.1 of [31].

For \( \tau, \eta, a > 0 \), define the sets \( \Omega_0, \Omega_1, \Omega_2, \) and \( \Omega_3 \) by:
\[
\Omega_0 := \{ z = e^{\tau \lambda} : \text{Re} \lambda \geq 0, \ |\tau \lambda| < \eta \} \]
\[
\Omega_1 := \{ z = e^{\tau \lambda} : |\lambda - \lambda_n| \geq a \text{ for all } 1 \leq n \leq N - 1 \}
\]
\[
\cup \{ z = e^{\tau \lambda} : 0 < |\lambda - \lambda_n| < a, \ \langle b, \psi_n \rangle \langle \phi_n, f \rangle = 0 \text{ for some } 1 \leq n \leq N - 1 \}
\]
\[
\Omega_2 := \{ z = e^{\tau \lambda} : 0 < |\lambda - \lambda_n| < a, \ \langle b, \psi_n \rangle \langle \phi_n, f \rangle \neq 0 \text{ for some } 1 \leq n \leq N - 1 \}
\]
\[
\Omega_3 := \mathbb{B}_1 \setminus \Omega_0. \]

If \( 0 < \eta < \pi \), then for every \( z \in \Omega_0 \), there uniquely exists \( \lambda \in \mathbb{T}_0 \) such that \( z = e^{\tau \lambda} \) and \( |\tau \lambda| < \eta \). This \( \lambda \) is the complex variable in the continuous-time setting corresponding to the complex variable \( z \) in the discrete-time setting.

Define \( a^* := \min \{|\lambda_n - \lambda_m|/2 : 1 \leq n < m \leq N - 1\} \). If \( |\lambda - \lambda_n| < a^* \) for some \( 1 \leq n \leq N - 1 \), then \( |\lambda - \lambda_m| \geq a^* \) for \( 1 \leq m \leq N - 1 \) with \( m \neq n \). By Steps 3) and 4) of the proof of Theorem 2.1 in [31], there exist \( \tau^* > 0, \eta \in (0, \pi) \), and \( a \in (0, a^*) \) such that the following three statements hold for every \( \tau \in (0, \tau^*) \):

(i) for all \( z \in \Omega_4 := \Omega_0 \cap \Omega_1 \) and the corresponding \( \lambda \),
\[
\text{(3.27)} \quad \left| \sum_{n=1}^{N-1} \frac{\langle b, \psi_n \rangle \langle \phi_n, f \rangle}{\lambda - \lambda_n} + \sum_{n=1}^{N-1} \frac{1 - e^{\tau \lambda_n}}{z - e^{\tau \lambda_n}} \cdot \frac{\langle b, \psi_n \rangle \langle \phi_n, f \rangle}{\lambda_n} \right| < \epsilon; \]

(ii) \( e^{\tau \lambda_n} \in \mathbb{C} \setminus \Omega_3 \) for all \( 1 \leq n \leq N - 1 \); and

(iii) for all \( z \in \Omega_5 := (\Omega_0 \cap \Omega_2) \cup \Omega_3 \),
\[
\text{(3.28)} \quad \left| 1 + \sum_{n=1}^{N-1} \frac{1 - e^{\tau \lambda_n}}{z - e^{\tau \lambda_n}} \cdot \frac{\langle b, \psi_n \rangle \langle \phi_n, f \rangle}{\lambda_n} \right| > \epsilon_c. \]

In what follows, we set \( \tau, \eta, a > 0 \) so that the above statements (i)–(iii) hold.

Suppose that \( z \in \Omega_4 \setminus \{1\} \), and let \( \lambda \in \mathbb{T}_0 \setminus \{0\} \) be the corresponding complex variable in the continuous-time setting. Since
\[
|1 - F(\lambda I - A)^{-1}B| = \left| 1 - \sum_{n=1}^{\infty} \frac{\langle b, \psi_n \rangle \langle \phi_n, f \rangle}{\lambda - \lambda_n} \right| > \epsilon_c, \]
it follows from the estimates (3.26a), (3.26b), and (3.27) that

$$1 + \sum_{n=1}^{\infty} \frac{1 - e^{\tau \lambda_n}}{z - e^{\tau \lambda_n}} \cdot \frac{\langle b, \psi_n \rangle \langle \phi_n, f \rangle}{\lambda_n} > \epsilon_c - 3\epsilon.$$  

On the other hand, if $z \in \Omega_5 \setminus \{1\}$, then (3.26b) and (3.28) yield

$$1 + \sum_{n=1}^{\infty} \frac{1 - e^{\tau \lambda_n}}{z - e^{\tau \lambda_n}} \cdot \frac{\langle b, \psi_n \rangle \langle \phi_n, f \rangle}{\lambda_n} > \epsilon_c - \epsilon.$$  

Step 4: It remains to show that

(3.29) $$\Omega \setminus \{1\} \cup (\Omega_5 \setminus \{1\}) = \rho(T(\tau)) \cap E_1.$$  

By definition,

$$(\Omega_0 \cap \Omega_1) \cup (\Omega_0 \cap \Omega_2) = \Omega_0 \cap (\Omega_1 \cup \Omega_2) = \Omega_0 \setminus \{e^{\tau \lambda_n} : 1 \leq n \leq N - 1\}.$$  

Moreover, the statement (ii) above yields

$$\Omega_3 \cap \{e^{\tau \lambda_n} : 1 \leq n \leq N - 1\} = \emptyset.$$  

Since $N \geq N_1$, it follows from (3.14) that $E_1 \cap \{e^{\tau \lambda_n} : n \geq N\} = \emptyset$. Hence

$$(\Omega_4 \setminus \{1\}) \cup (\Omega_5 \setminus \{1\}) = (\Omega_4 \cup \Omega_3) \setminus \{1\}$$

$$= (\Omega_0 \setminus \{e^{\tau \lambda_n} : 1 \leq n \leq N - 1\}) \cup \Omega_3 \setminus \{1\}$$

$$= (\Omega_0 \cup \Omega_3) \setminus \{1\} \cup \{e^{\tau \lambda_n} : 1 \leq n \leq N - 1\}$$

$$= E_1 \setminus \{1\} \cup \{e^{\tau \lambda_n} : n \in \mathbb{N}\}.$$  

Since $\sigma(T(\tau)) = \{e^{\tau \lambda_n} : n \in \mathbb{N}\}$ by Lemma 3.2, we obtain

$$E_1 \setminus \{1\} \cup \{e^{\tau \lambda_n} : n \in \mathbb{N}\} = E_1 \setminus \sigma(T(\tau)).$$  

Thus, (3.29) holds. This completes the proof.  

We are now in the position to prove Theorem 3.1.

Proof of Theorem 3.1. By Lemmas 3.6 and 3.8, there exists $\tau^* > 0$ such that

(3.30) $$1 \in \rho(FR(z, T(\tau))S(\tau)) \quad \forall z \in \rho(T(\tau)) \cap E_1, \quad \forall \tau \in (0, \tau^*).$$  

Let $\tau \in (0, \tau^*)$. By (A1)–(A3) and Lemma 3.2, $\sigma(T(\tau)) \cap \mathbb{T} = \{1\}$. This and (3.30) imply that

(3.31) $$\mathbb{T} \setminus \{1\} \subset \rho(T(\tau)) \cap E_1, \quad 1 \in \rho(FR(z, T(\tau))S(\tau)) \quad \forall z \in \mathbb{T} \setminus \{1\}.$$  

On the other hand,

$$zI - \Delta(\tau) = (zI - T(\tau))(I - (zI - T(\tau))^{-1}S(\tau)F) \quad \forall z \in \rho(T(\tau)).$$  

Since $\sigma(R(z, T(\tau))S(\tau)F) \setminus \{0\} = \sigma(FR(z, T(\tau))S(\tau)) \setminus \{0\}$ by Lemma 3.5, it follows that for every $z \in \rho(T(\tau))$,

(3.32) $$1 \in \rho(FR(z, T(\tau))S(\tau)) \iff z \in \rho(\Delta(\tau)).$$  

By (3.31) and (3.32), we obtain $\mathbb{T} \setminus \{1\} \subset \rho(\Delta(\tau))$. Since $1 \notin \sigma_p(\Delta(\tau))$ and $1 \in \sigma(\Delta(\tau))$ by Lemma 3.4, it follows that $\sigma_p(\Delta(\tau)) \cap \mathbb{T} = \emptyset$ and $\sigma(\Delta(\tau)) \cap \mathbb{T} = \{1\}$.  

\[\Box\]
4. Preservation of boundedness. In this section, we prove the power boundedness of \((\Delta(k)^k)_{k\in\mathbb{N}}\) in order to finish the proof of the main theorem.

**Theorem 4.1.** If Assumption 2.9 is satisfied, then there exists \(\tau^* > 0\) such that the discrete semigroup \((\Delta(k)^k)_{k\in\mathbb{N}}\) is power bounded for every \(\tau \in (0, \tau^*)\).

**Remark 4.2.** Combining Theorems 3.1 and 4.1 with the discrete version of the mean ergodic theorem (see, e.g., Theorem I.2.9 and Corollary I.2.11 in [4]), we obtain

\[
X = \ker(\Delta(\tau) - 1) \oplus \text{ran}(\Delta(\tau) - 1) = \overline{\text{ran}(\Delta(\tau) - 1)}
\]

for all sufficiently small \(\tau > 0\). Therefore, 1 belongs to the continuous spectrum of \(\Delta(\tau)\).

Before proving Theorem 4.1, we apply a spectral decomposition for \(A\); see, e.g., Lemma 2.4.7 of [3] or Proposition IV.1.16 in [5]. Assume that (A1) is satisfied. Then only finite elements of \(\{\lambda_n : n \in \mathbb{N}\}\) are in \(\mathbb{C}_{-a} \cap \Sigma_{\pi/2+\delta}\). For every \(\beta > 0\), there exists a smooth, positively oriented, and simple closed curve \(\Phi\) in \(\mathbb{R}^n\) containing \(\sigma(A) \cap \overline{\mathbb{C}_0}\) in its interior and \(\sigma(A) \cap (\mathbb{C} \setminus \overline{\mathbb{C}_\beta})\) in its exterior. Here we choose \(\beta > 0\) so that \(\sigma(A) \cap \overline{\mathbb{C}_\beta} = \{\lambda_n : n \in \mathbb{N}\} \cap \mathbb{C}_0\). The operator

\[
(4.1) \quad \Pi := \frac{1}{2\pi i} \int_{\Phi} (\lambda I - A)^{-1} d\lambda
\]

is a projection on \(X\). We have

\[
X = X^+ \oplus X^-,
\]

where \(X^+ := \Pi X\) and \(X^- := (I - \Pi)X\). Then \(\dim X^+ < \infty\), \(X^+ \) and \(X^-\) are \(T(t)\)-invariant for all \(t \geq 0\), and

\[
\sigma(A^+) = \sigma(A) \cap \overline{\mathbb{C}_\beta}, \quad \sigma(A^-) = \sigma(A) \cap (\mathbb{C} \setminus \overline{\mathbb{C}_\beta}),
\]

where \(A^+ := A|_{X^+}\) and \(A^- := A|_{D(A) \cap X^-}\). For \(t \geq 0\), we define

\[
T^+(t) := T(t)|_{X^+}, \quad T^-(t) := T(t)|_{X^-}.
\]

Then \((T^+(t))_{t \geq 0}\) and \((T^-(t))_{t \geq 0}\) are strongly continuous semigroups on \(X^+\) and \(X^-\), respectively, and their generators are given by \(A^+\) and \(A^-\), respectively. Let

\[
(4.2) \quad \{\lambda_n : 1 \leq n \leq N_s - 1\} := \{\lambda_n : n \in \mathbb{N}\} \cap \mathbb{C}_0 = \sigma(A^+)
\]

by changing the order of \(\{\lambda_n : n \in \mathbb{N}\}\) if necessary.

By construction, \((T^-(t))_{t \geq 0}\) is uniformly bounded. Moreover, Lemma 4.2.7 of [3] shows that \((T^-(t))_{t \geq 0}\) is strongly stable under (A2). Hence we easily obtain the following properties of the discrete semigroup \((T^-(\tau)^k)_{k \in \mathbb{N}}\).

**Lemma 4.3.** Let \(\tau > 0\). The discrete semigroup \((T^- \tau)^k)_{k \in \mathbb{N}}\) constructed as above is power bounded. If (A2) additionally holds, then \((T^- \tau)^k)_{k \in \mathbb{N}}\) is strongly stable.

**Remark 4.4.** Applying the mean ergodic theorem (see, e.g., Theorem I.2.25 of [4]) to the uniformly bounded semigroup \((T^- \tau)^k)_{k \in \mathbb{N}}\), we obtain

\[
X^- = \ker(A^-) \oplus \text{ran}(A^-) = \overline{\text{ran}(A^-)}.
\]

Since the finite-dimensional unstable part \(A^+\) is invertible, it follows that \(X^+ = \text{ran}(A^+)\). Thus, \(X = \text{ran}(A)\), which implies that 0 belongs to the continuous spectrum of \(A\).
A characterization of uniformly bounded semigroups on Hilbert spaces has been obtained in [7, 33]. The following theorem is its discrete analogue and provides a necessary and sufficient condition for a discrete semigroup on a Hilbert space to be power bounded.

**Theorem 4.5 (Theorem II.1.12 of [4]).** Let \( X \) be a Hilbert space and \( \Delta \in \mathcal{L}(X) \) satisfy \( \mathbb{E}_1 \subset \rho(\Delta) \). The discrete semigroup \( (\Delta^k)_{k \in \mathbb{N}} \) is power bounded if and only if for every \( x, y \in X \),

\[
\limsup_{r \to 1^+} \left( r - 1 \right) \int_0^{2\pi} \left( \| R(re^{i\theta}, \Delta)x \|^2 + \| R(re^{i\theta}, \Delta)^*y \|^2 \right) d\theta < \infty.
\]

To use Theorem 4.5, we show that \( \mathbb{E}_1 \subset \rho(\Delta(\tau)) \) holds for all sufficiently small \( \tau > 0 \).

**Lemma 4.6.** Let \( A \) be a Riesz-spectral operator on a Hilbert space \( X \) whose eigenvalues \( \{\lambda_n : n \in \mathbb{N} \} \) satisfy (A1) and \( 0 \in \{\lambda_n : n \in \mathbb{N} \} \setminus \{\lambda_n : n \in \mathbb{N} \} \). Assume that \( B \in \mathcal{L}(X, \mathbb{C}) \) and \( F \in \mathcal{L}(\mathbb{C}, X) \) in the form of (2.7) satisfy (A4)–(A6). Then there exists \( \tau^* > 0 \) such that \( \mathbb{E}_1 \subset \rho(\Delta(\tau)) \) for every \( \tau \in (0, \tau^*) \).

**Proof.** Lemmas 3.6 and 3.8 show that there exist \( \epsilon > 0 \) and \( \tau^* > 0 \) such that for every \( \tau \in (0, \tau^*) \),

\[
\tau(\lambda_n - \lambda_m) \neq 2\ell\pi i \quad \forall \ell \in \mathbb{Z} \setminus \{0\}, \ 1 \leq n, m \leq N_n - 1
\]

(3.3)

\[
|1 - FR(z, T(\tau))S(\tau)| > \epsilon \quad \forall z \in \rho(T(\tau)) \cap \mathbb{E}_1,
\]

where \( N_n \in \mathbb{N} \) is as given in (4.2). Let \( \tau \in (0, \tau^*) \). Since

\[
zI - \Delta(\tau) = (zI - T(\tau))(I - (zI - T(\tau))^{-1}S(\tau)F) \quad \forall z \in \rho(T(\tau)),
\]

it follows from Lemma 3.5 that for every \( z \in \rho(T(\tau)) \),

\[
(4.5) \quad 1 \in \rho(FR(z, T(\tau))S(\tau)) \iff z \in \rho(\Delta(\tau)).
\]

Combining this with (4.4), we obtain

\[
(4.6) \quad \rho(T(\tau)) \cap \mathbb{E}_1 \subset \rho(\Delta(\tau)).
\]

We see from (4.6) that if \( \sigma(T(\tau)) \cap \mathbb{E}_1 \subset \rho(\Delta(\tau)) \), then the desired conclusion \( \mathbb{E}_1 \subset \rho(\Delta(\tau)) \) holds. Assume, to get a contradiction, that \( \sigma(T(\tau)) \cap \mathbb{E}_1 \cap \sigma(\Delta(\tau)) \neq \emptyset \), and let \( z_0 \in \sigma(T(\tau)) \cap \mathbb{E}_1 \cap \sigma(\Delta(\tau)) \). By Lemma 3.2,

\[
\sigma(T(\tau)) \cap \mathbb{E}_1 = \{e^{r\lambda_n} : 1 \leq n \leq N_n - 1\},
\]

and (3.3) yields \( e^{r\lambda_n} \neq e^{r\lambda_m} \) for \( 1 \leq n, m \leq N_n - 1 \) with \( n \neq m \). Therefore, there uniquely exists \( 1 \leq n_0 \leq N_n - 1 \) such that \( z_0 = e^{r\lambda_n} \). Since \( \Delta(\tau) - T(\tau) = S(\tau)F \) is compact, it follows that \( z_0 \) is an eigenvalue of \( \Delta(\tau) \).

Let \( v \in X \) be an eigenvector of \( \Delta(\tau) \) corresponding to \( z_0 \). By (3.2),

\[
(4.7) \quad T(\tau)v + A^{-1}(T(\tau) - I)BFv = z_0v.
\]

Since

\[
\langle T(\tau)v, \psi_{n_0} \rangle = e^{r\lambda_{n_0}}\langle v, \psi_{n_0} \rangle
\]

\[
\langle A^{-1}(T(\tau) - I)BFv, \psi_{n_0} \rangle = \frac{e^{r\lambda_{n_0}} - 1}{\lambda_{n_0}}\langle BFv, \psi_{n_0} \rangle,
\]

we have

\[
\langle T(\tau)v, BFv \rangle = z_0\langle v, BFv \rangle.
\]

Hence, for every \( x, y \in X \),

\[
\limsup_{r \to 1^+} \left( r - 1 \right) \int_0^{2\pi} \left( \| R(re^{i\theta}, \Delta)x \|^2 + \| R(re^{i\theta}, \Delta)^*y \|^2 \right) d\theta < \infty.
\]
it follows from (4.7) that
\[ \frac{e^{r\lambda_0} - 1}{\lambda_0} \langle BFv, \psi_{n_0} \rangle = 0. \]
By \( \lambda_{n_0} \in \mathbb{C}_0 \), we obtain \( \langle BFv, \psi_{n_0} \rangle = 0 \). This occurs if and only if \( \langle b, \psi_{n_0} \rangle = 0 \) or \( Fv = 0 \). Since \( A + BF \) generates a uniformly bounded semigroup by (A4), it follows that \( (A, B, -) \) is \( \beta \)-exponentially stabilizable for every \( \beta > 0 \). Hence \( \langle b, \psi_{n_0} \rangle \neq 0 \) by Theorem 8.2.3 of [3], and so \( Fv = 0 \). Substituting it into (4.7), we obtain \( T\tau v = e^{r\lambda_0}v \). There exists a nonzero constant \( \gamma \in \mathbb{C} \) such that \( v = \gamma \phi_{n_0} \). Therefore,
\[ (A + BF)v = Av = \lambda_{n_0}v, \]
which implies that \( \lambda_{n_0} \in \sigma_p(A + BF) \) and contradicts the assumption that \( A + BF \) generates a uniformly bounded semigroup. This completes the proof.

To study power boundedness based on Theorem 4.5, we use the well-known Sherman-Morrison-Woodbury formula given in the next lemma, which can be obtained from a straightforward calculation.

**Proposition 4.7.** Let \( X, U \) be Banach spaces, \( A : D(A) \subset X \rightarrow X \) be a closed operator, \( B \in \mathcal{L}(U, X) \), \( F \in \mathcal{L}(X, U) \), and \( \lambda \in \rho(A) \). If \( 1 \in \rho(FR(\lambda, A)B) \), then \( \lambda \in \rho(A + BF) \) and
\[ R(\lambda, A + BF) = R(\lambda, A) + R(\lambda, A)B(I - FR(\lambda, A)B)^{-1}FR(\lambda, A). \]

After these preparations, we are now ready to prove that the discrete semigroup \((\Delta(\tau^k))_{k \in \mathbb{N}}\) is power bounded for all sufficiently small \( \tau > 0 \). The proof is inspired by Paunonen’s proof of Theorem 4 in [25].

**Proof of Theorem 4.1.** By Lemmas 3.6, 3.8, and 4.6, there exist \( \tau^* > 0 \) and \( M_0 > 0 \) such that for every \( \tau \in (0, \tau^*) \), we obtain \( E_1 \subset \rho(\Delta(\tau)) \) and
\[ \left| \frac{1}{1 - FR(z, \tau)/S(\tau)} \right| \leq M_0 \quad \forall z \in \rho(\tau) \cap E_1. \]
Let \( \tau \in (0, \tau^*) \) be given. By Theorem 4.5, it suffices to show that
\[ \lim_{r \downarrow 1} (r - 1) \int_0^{2\pi} \left( \| R(re^{i\theta}, \Delta(\tau))x \|^2 + \| R(re^{i\theta}, \Delta(\tau)^*y \|^2 \right) d\theta < \infty \]
for every \( x, y \in X \). Since \( \sigma(T(\tau)) = \{e^{r\lambda_n} : n \in \mathbb{N}\} \) by Lemma 3.2, it follows from (A1) that there exists \( r_1 > 1 \) such that \( re^{i\theta} \in \rho(T(\tau)) \) for every \( r \in (1, r_1) \) and every \( \theta \in [0, 2\pi] \). Since the Sherman-Morrison-Woodbury formula given in Proposition 4.7 yields
\[ R(re^{i\theta}, T(\tau) + S(\tau)F)x = R(re^{i\theta}, T(\tau))x + \frac{R(re^{i\theta}, T(\tau))S(\tau)FR(re^{i\theta}, T(\tau))x}{1 - FR(re^{i\theta}, T(\tau))S(\tau)} \]
for all \( x \in X \) and all \( r \in (1, r_1) \), we can estimate
\[ \int_0^{2\pi} \| R(re^{i\theta}, T(\tau) + S(\tau)F)x \|^2 d\theta \]
\[ \leq 2 \int_0^{2\pi} \| R(re^{i\theta}, T(\tau))x \|^2 d\theta \]
\[ + 2M_0^2 \| x \|^2 \int_0^{2\pi} \| R(re^{i\theta}, T(\tau))S(\tau) \|^2 \cdot \| FR(re^{i\theta}, T(\tau)) \|^2 d\theta \]
for all \( x \in X \) and all \( r \in (1, r_1) \).

To estimate the first term of the right-hand side of (4.9), we apply the spectral decomposition by the projection \( \Pi \) given in (4.1). Take \( x \in X \). Then \( x = x^+ + x^- \) with \( x^+ = \Pi x \in X^+ \) and \( x^- = (I - \Pi)x \in X^- \). There exists \( c_1 > 0 \) such that \( |re^{i\theta} - e^{\tau \lambda_n}| \geq c_1 \) for all \( r \in (1, r_1), \theta \in [0, 2\pi) \), and \( 1 \leq n \leq N - 1 \). Therefore, (2.4) and Lemma 3.2 yield

\[
\int_0^{2\pi} \| R(re^{i\theta}, T(\tau)) x^+ \|^2 d\theta \leq M_b \sum_{n=1}^{N-1} |\langle x^+, \psi_n \rangle|^2 \int_0^{2\pi} \frac{1}{|re^{i\theta} - e^{\tau \lambda_n}|^2} d\theta \leq \frac{2\pi M_b}{c_1^2} \sum_{n=1}^{N-1} |\langle x^+, \psi_n \rangle|^2.
\]

Therefore,

\[
\limsup_{r \downarrow 1} (r - 1) \int_0^{2\pi} \| R(re^{i\theta}, T(\tau)) x^+ \|^2 d\theta = 0.
\]

Since the discrete semigroup \((T^- (\tau^k))_{k \in \mathbb{N}}\) is power bounded by Lemma 4.3, we see from Theorem 4.5 that

\[
\limsup_{r \downarrow 1} (r - 1) \int_0^{2\pi} \| R(re^{i\theta}, T(\tau)) x^- \|^2 d\theta < \infty.
\]

Consequently,

(4.10) \[
\limsup_{r \downarrow 1} (r - 1) \int_0^{2\pi} \| R(re^{i\theta}, T(\tau)) x \|^2 d\theta < \infty.
\]

We next investigate the second term of the right-hand side of (4.9). Using (2.4) and (3.3), we have that for every \( re^{i\theta} \in \rho(T(\tau)) \),

\[
\| R(re^{i\theta}, T(\tau)) S(\tau) \|^2 \leq M_b \sum_{n=1}^{\infty} \frac{1}{|re^{i\theta} - e^{\tau \lambda_n}|^2} \cdot \left| \frac{\langle b, \psi_n \rangle}{\lambda_n} \right|^2.
\]

Under (A1), there is \( N_1 \in \mathbb{N} \) such that \( \lambda_n \in \mathbb{C} \setminus \mathbb{C}_{-\alpha} \) or \( \lambda_n \in \mathbb{C} \setminus \Sigma_{\pi/2+\delta} \) for every \( N \geq N_1 \). As shown in (3.25) in Step 2 of the proof of Lemma 3.8, there exists \( M_1 > 0 \) such that for every \( z = \mathbb{E}_1 \setminus \{1\} \),

\[
\sum_{n=N_1}^{\infty} \left| \frac{1 - e^{\tau \lambda_n}}{z - e^{\tau \lambda_n}} \right|^2 \cdot \left| \frac{\langle b, \psi_n \rangle}{\lambda_n} \right|^2 \leq M_1.
\]

It follows from (A2) that there exist \( c_2 > 0 \) and \( M_2 > 0 \) such that for all \( 1 \leq n \leq N_1 - 1 \),

\[
|re^{i\theta} - e^{\tau \lambda_n}| \geq c_2 \quad \forall r \in (1, r_1), \forall \theta \in [0, 2\pi)
\]

\[
|1 - e^{\tau \lambda_n}| \leq 1 + |e^{\tau \lambda_n}| \leq M_2.
\]

By these inequalities,

\[
\sum_{n=1}^{N_1-1} \left| \frac{1 - e^{\tau \lambda_n}}{re^{i\theta} - e^{\tau \lambda_n}} \right|^2 \cdot \left| \frac{\langle b, \psi_n \rangle}{\lambda_n} \right|^2 \leq \left( \frac{M_2}{c_2} \right)^2 \sum_{n=1}^{N_1-1} \left| \frac{\langle b, \psi_n \rangle}{\lambda_n} \right|^2 \quad \forall r \in (1, r_1), \forall \theta \in [0, 2\pi).
\]
Hence we obtain
\[(4.11) \quad \|R(re^{i\theta}, T(\tau))S(\tau)\|^2 \leq M_3 \quad \forall r \in (1, r_1), \quad \forall \theta \in [0, 2\pi)\]
for some $M_3 > 0$.

Using the estimate (4.11), we have that for every $r \in (1, r_1)$,
\[(4.12) \quad \int_0^{2\pi} \|R(re^{i\theta}, T(\tau))S(\tau)\|^2 \cdot \|FR(re^{i\theta}, T(\tau))\|^2 d\theta \leq M_3 \int_0^{2\pi} \|R(re^{i\theta}, T^*(\tau))F^*\|^2 d\theta.
\]
The adjoint semigroup $(T^*(t))_{t \geq 0}$ is given by
\[T^*(t)x = \sum_{n=1}^{\infty} e^{i\lambda_n t} \langle x, \phi_n \rangle \psi_n \quad \forall x \in X, \quad \forall t \geq 0,
\]
and its generator is $A^*$; see, e.g., Theorem 2.3.6 of [3]. Define the operator $A_1 : D(A_1) \subset X \to X$ by
\[A_1 x := \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle \psi_n
\]
with domain
\[D(A_1) := \left\{ x \in X : \sum_{n=1}^{\infty} |\lambda_n|^2 \cdot |\langle x, \phi_n \rangle|^2 < \infty \right\}.
\]
By Corollary 3.2.10 of [3], $A_1$ is a Riesz-spectral operator and generates the semigroup $(T^*(t))_{t \geq 0}$. Therefore, $A_1 = A^*$. Similarly to (4.10), we obtain
\[(4.13) \quad \limsup_{r \downarrow 1} (r - 1) \int_0^{2\pi} \|R(re^{i\theta}, T^*(\tau))y\|^2 d\theta < \infty \quad \forall y \in X.
\]
Since $F^*u = fu$ for every $u \in \mathbb{C}$, it follows from (4.12) and (4.13) that
\[(4.14) \quad \limsup_{r \downarrow 1} (r - 1) \int_0^{2\pi} \|R(re^{i\theta}, T(\tau))S(\tau)\|^2 \cdot \|FR(re^{i\theta}, T(\tau))\|^2 d\theta < \infty.
\]
Applying the estimates (4.10) and (4.14) to (4.9), we obtain
\[
\limsup_{r \downarrow 1} (r - 1) \int_0^{2\pi} \|R(re^{i\theta}, \Delta(\tau))x\|^2 d\theta < \infty.
\]
We have from a similar calculation that
\[
\limsup_{r \downarrow 1} (r - 1) \int_0^{2\pi} \|R(re^{i\theta}, \Delta(\tau))^* y\|^2 d\theta < \infty \quad \forall y \in X,
\]
using the following estimate:
\[
\int_0^{2\pi} \|R(re^{i\theta}, T(\tau) + S(\tau)F)^* y\|^2 d\theta
\]
\[
= \int_0^{2\pi} \left\| R(re^{i\theta}, T(\tau))^* y + \frac{R(re^{i\theta}, T(\tau))S(\tau)FR(re^{i\theta}, T(\tau))}{1 - FR(re^{i\theta}, T(\tau))S(\tau)F} \right\|^2 d\theta
\]
\[
\leq 2 \int_0^{2\pi} \|R(re^{i\theta}, T^*(\tau))y\|^2 d\theta
\]
\[
+ 2M_2 \|y\|^2 \int_0^{2\pi} \|R(re^{i\theta}, T(\tau))S(\tau)\|^2 \cdot \|FR(re^{i\theta}, T(\tau))\|^2 d\theta.
\]
for all \( y \in X \) and all \( r \in (1, r_1) \). Thus, the desired estimate (4.8) is obtained for every \( x, y \in X \).

We see from Theorems 3.1 and 4.1 that the sufficient condition for strong stability in the Arendt-Batty-Lyubich-Vũ theorem is satisfied. We finally prove the main theorem of this article, Theorem 2.10.

**Proof of Theorem 2.10.** There exists \( \tau^* > 0 \) such that for every \( \tau \in (0, \tau^*) \),

(i) \( \sigma_p(\Delta(\tau)) \cap \mathbb{T} = \emptyset \) and \( \sigma(\Delta(\tau)) \cap \mathbb{T} = \{1\} \) by Theorem 3.1; and

(ii) the discrete semigroup \( (\Delta(\tau))^k \) is power bounded by Theorem 4.1.

By the Arendt-Batty-Lyubich-Vũ theorem, Theorem 2.4, \( (\Delta(\tau))^k \) is strongly stable. This and Proposition 2.2 show that the sampled-data system (2.1) is strongly stable.

We conclude this section by applying Theorem 2.10 to an infinite-dimensional system whose generator is a simple diagonal operator.

**Example 4.8.** Let \( X = \ell^2(\mathbb{C}) \) with standard basis \( \{\phi_n : n \in \mathbb{N}\} \), \( N_\alpha \in \mathbb{N} \), and \( \{\lambda_n \in \mathbb{C} : 1 \leq n \leq N_\alpha \} \) be distinct. Define \( A \in \mathcal{L}(X) \) by

\[
Ax := \sum_{n=1}^{N_\alpha-1} \lambda_n \langle x, \phi_n \rangle \phi_n + \sum_{n=N_\alpha}^{\infty} -\frac{1}{n} \langle x, \phi_n \rangle \phi_n.
\]

The operator \( A \) satisfies (A1)–(A3). Let \( b \in X \) and the control operator \( B \in \mathcal{L}(\mathbb{C}, X) \) be represented as \( Bu = bu \) for \( u \in \mathbb{C} \). We apply the spectral decomposition by the projection \( \Pi \) given in (4.1), and we define \( B^+ := \Pi B \) and \( B^- := (I - \Pi)B \). Suppose that \( b \) satisfies

\[
(4.15) \quad \langle b, \phi_n \rangle \neq 0, \quad 1 \leq n \leq N_\alpha - 1; \quad \sum_{n=N_\alpha}^{\infty} n^2 |\langle b, \phi_n \rangle|^2 < \infty.
\]

These conditions are equivalent to the controllability of the unstable part \( (A^+, B^+) \) and (A5), respectively.

Since \( (A^+, B^+) \) is controllable, there exists \( f_1 \in X \) such that the matrix

\[
\begin{bmatrix}
\lambda_1 & 0 \\
\vdots & \ddots \\
0 & \lambda_{N_\alpha-1}
\end{bmatrix}
+ \begin{bmatrix}
\langle b, \phi_1 \rangle \\
\vdots \\
\langle b, \phi_{N_\alpha-1} \rangle
\end{bmatrix} = \begin{bmatrix}
\langle \phi_1, f_1 \rangle & \cdots & \langle \phi_{N_\alpha-1}, f_1 \rangle
\end{bmatrix}
\]

is Hurwitz and

\[
\langle \phi_n, f_1 \rangle = 0 \quad \forall n \geq N_\alpha.
\]

Let \( F_1 \in \mathcal{L}(X, \mathbb{C}) \) be represented as \( F_1x = \langle x, f_1 \rangle \) for \( x \in X \), and define \( F_1^+ := F_1|_{X^+} \). Then \( \rho(A^+ + B^+ F_1^+) \supset \mathbb{C}_\alpha \). For every \( \lambda \in \rho(A^+ + B^+ F_1^+) \cap \rho(A^-) \), we obtain \( \lambda \in \rho(A + BF_1) \) and write

\[
(4.16) \quad R(\lambda, A + BF_1) = \begin{bmatrix}
R(\lambda, A^+ + B^+ F_1^+) & 0 \\
R(\lambda, A^-)B^- F_1^+ R(\lambda, A^+ + B^+ F_1^+) & R(\lambda, A^-)
\end{bmatrix}
\]

under the decomposition \( X = X^+ \oplus X^- \). Moreover, we see from Theorem 2.8 a) that

\[
(4.17) \quad \|R(i\omega, A^-)\| = \frac{1}{|\omega|}.
\]
By (4.16) and (4.17), the generator $\tilde{A} := A + BF_1$ satisfies the condition (2.8) in (A4). It is not difficult to see that the other conditions in (A4) are also satisfied. We find that the feedback operator $F_1$ satisfies (A6), by adding a small perturbation if needed. Therefore, we can apply Theorem 2.10 to the sampled-data system with the feedback operator $F_1$. However, the discrete-time counterpart of Lemma 20 in [10] immediately shows that the structured feedback operator $F$ achieves the strong stability of the sampled-data system. To illustrate the effectiveness of Theorem 2.10, we here consider feedback operators that affect the stable part $(A^-, B^-)$.

By (4.16) and (4.17), we obtain

$$\sup_{0<|\omega| \leq 1} |\omega| \cdot \|R(i\omega, \tilde{A})\| < \infty.$$ 

Furthermore, $b \in D(\tilde{A}^{-1})$ holds. Indeed, since $b^- := (I - \Pi)b \in D((A^-)^{-1})$ by the latter condition on $b$ given in (4.15), there exists $x^-_b \in X^-$ such that $b^- = A^-x^-_b$. We obtain

$$\tilde{A} \begin{bmatrix} x^+ \\ x^- \end{bmatrix} = \begin{bmatrix} (A^+ + B^+F_1)x^+ \\ B^-F_1x^+ + A^-x^- \end{bmatrix} \quad \forall x^+ \in X^+, \; \forall x^- \in X^-.$$ 

Since $A^+ + B^+F_1$ is invertible, there exists $x^+_0 \in X^+$ such that $\Pi b = (A^+ + B^+F_1)x^+_0$. Moreover, if we set $x^-_0 := (1 - F_1x^+_0)x^-_b$, then

$$B^-F_1x^+_0 + A^-x^-_0 = (F_1x^+_0)b^- + (1 - F_1x^+_0)b^- = b^-.$$ 

Hence $b \in \text{ran}(\tilde{A}) = D(\tilde{A}^{-1})$.

Theorem 4 of [25] shows that there exists $\kappa > 0$ such that $\tilde{A} + BF_2 = A + B(F_1 + F_2)$ satisfies the conditions in (A4) for every $F_2 \in \mathcal{L}(X, \mathbb{C})$ with $\|F_2\| < \kappa$. As in the case of the structured feedback operator $F_1$, we see that $F := F_1 + F_2$ also satisfies (A6), by adding a small perturbation if necessary. Thus Theorem 2.10 can be applied to the sampled-data system with the nonstructured feedback operator $F$.

5. Concluding remarks. In this paper, we have analyzed robustness of strong stability with respect to sampling. We have limited our attention to the situation where the generator $A$ is a Riesz-spectral operator and $0 \in \sigma(A) \setminus \sigma_p(A)$. We have presented conditions under which the sufficient condition for strong stability in the Arendt-Batty-Lyubich-Vũ theorem is preserved between the original continuous-time system and the sampled-data system under fast sampling. Our future work is to analyze robustness of polynomial stability with respect to sampling.

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