Instability of a Square Sheet of Rubberlike Material under Symmetric Biaxial Stretching

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Abstract. The purpose of this work is to analyse the stability of the solution of the biaxial stretching test, in the case of Ishihara-Zahorski's constitutive model of rubber-like material. Two solutions were obtained, the symmetric and asymmetric one. It was shown that the asymmetric solution is possible for significant forces and this solution is always stable. However, the symmetric solution is stable only for relatively small principal stretches. It is worth emphasizing that the stored energy function of the considered model is polyconvex.

1. Introduction

In one of its pioneering experiments on rubber membranes, Treloar in 1948 [1] stretched a square membrane with equal forces in two directions and reported observing a stable deformation of two unequal principal extensions corresponding to the equal forces. After 30 years, the existence and the possibility of asymmetric solution was investigated by Kearsley [2] and MacSithig [3] within the framework of non-linear elasticity. They investigated the existence of this solution on the example of the Mooney-Rivlin material model (MR). They showed more than could be seen in the Treolar's experiment. They proved that bifurcation occurs for sufficiently large forces, i.e. the symmetrical solution ceases to be instable, and two equivalent asymmetrical solutions are stable.

In the paper [4], there is another interpretation of Treolar's experiments [1], i.e. according to the authors of the work [4], unequal extensions of a uniformly stretched sample result from the anisotropy of the material being tested. However, it should be emphasized, that in the theory of hyperelasticity there is generally no single solution to problems of compression or stretching of a sample realized by the stress boundary conditions, which is obvious when the sample is squeezing. Instabilities, bifurcations of solutions are observed, for example, in the issue of balloon inflation [6,14,15]. This is also foreseen in analytical solutions of this boundary value problem, cf. [9].

The purpose of this work is to analyse the instability of the solution of the biaxial stretching test, in the case of Ishihara-Zahorski's constitutive model of rubber-like material [8, 10].

2. Basic assumptions and relationships of hyperelasticity

In the case of incompressible materials, the function of stored elastic energy is not a potential of elasticity, because the spherical part of the Cauchy stress tensor is not determined [3,9,11,15]. The incompressible material may be subjected only to isochoric deformations, i.e. deformations without changing the volume of the body. Each deformation of the incompressible body is defined by the incompressibility condition
where \( J = \text{det} \mathbf{F} \) is determination of the tensor gradient \( \mathbf{F} \) \([9, 11, 15]\). Consequently, the elastic potential must incorporate the constraint (3.1) with the Lagrange multiplier \( p \), being the hydrostatic pressure of the spherical part of the Cauchy stress \( \sigma \).

The tensor \( \mathbf{F} \) is decomposed into the isochoric and volumetric part as follows:

\[
\mathbf{F} = J^{\frac{1}{3}} \mathbf{F}, \quad \text{det} \mathbf{F} = 1.
\]

Further, we introduce

\[
\mathbf{B} = \mathbf{F}^T \mathbf{F}, \quad \mathbf{C} = \mathbf{F}^T \mathbf{F},
\]

where \( \mathbf{B} \) and \( \mathbf{C} \) denote the left and the right Cauchy-Green strain tensors of isochoric deformations, cf. \([9, 11, 12]\) and the references therein.

In the case of incompressible isotropic materials, the stored energy function \( W \) is a function of only two independent invariants of isochoric deformations \([9, 11]\)

\[
W = W \left( I_1, I_2 \right) = \frac{1}{2} \mu \left( I_1 - 3 \right) + \frac{1}{2} \lambda \left( I_2 - 3 \right) + \frac{1}{4} \zeta (I_2 - 3)^2.
\]

The elastic energy function \( W = U \left( T_1, T_2 \right) \) is a function of two invariants of isochoric deformations. Basic invariants \( T_1 \) and \( T_2 \), due to the constraints of incompressibility (1), can be considered as functions of only two independent eigenvalues \( \lambda_1 \) and \( \lambda_2 \) of the stretch tensors \( \mathbf{U} = \sqrt{\mathbf{C}} \) and \( \mathbf{V} = \sqrt{\mathbf{B}} \) \([9, 11, 15]\), i.e.: \( T_1 = \lambda_1^2 + \lambda_2^2 + \left( \lambda_1 \lambda_2 \right)^2 \), \( T_2 = \lambda_1^{-2} + \lambda_2^{-2} + \left( \lambda_1 \lambda_2 \right)^2 \).

The incompressibility constraints show that \( \lambda_3 = \left( \lambda_1 \lambda_2 \right)^{-1} \). Principal stretches \( \lambda_1 \) and \( \lambda_2 \) are independent but they are unordered eigenvalues of stretch tensors. It is easy to check that the function \( T_1 \) is a convex function with respect to \( \lambda_1 \) and \( \lambda_2 \), and the function \( T_2 \), for sufficiently large elongations, is not a convex function with respect to \( \lambda_1 \) and \( \lambda_2 \).

3. Ishihara-Zahorski’s model of incompressible rubber-like materials

The stored energy function of the so-called Ishihara-Zahorski’s model (MIZ) \([8, 10]\) of incompressible rubber-like materials we can write in the following form:

\[
W = \frac{1}{2} \mu \left[ f \left( T_1 - 3 \right) + (1 - f) (T_2 - 3) \right] + \frac{1}{4} \zeta (T_2 - 3)^2.
\]
where $\mu_0 > 0$ is an initial shear modulus and $f \in (0,1)$, and $c > 0$ are material parameters. If $c \geq 0$ the function (7) is polyconvex [1]. For $c = 0$, the stored energy function (7) reduces to the commonly known energy function of MR model [9, 11, 12, 15].

The stress tensor in the current configuration is obtained from (5) and (7):

$$\sigma = -\mu_0 I + \left(\mu_0 f + c(\bar{T}_1 - 3)\right)\bar{B} - \mu_0 (1 - f)\bar{B}^{-1},$$

i.e. we get Eq. (8) after substituting the functions

$$\bar{p}_1 = \mu_0 f + c(\bar{T}_1 - 3), \quad \bar{p}_{-1} = -\mu_0 (1 - f).$$

to the constitutive relationship (5).

It should be noted that if $f = 1$ and $c = 0$, then from the MIZ model, we obtain the Neo-Hookean model (NH), which is the simplest model of the hyperelastic incompressible material [9, 11, 15].

4. Universal relationships for biaxial tension tests

In the case of rubber-like materials, typical experimental tests of homogeneous deformations from which material parameters are determined are uniaxial and biaxial stretching tests (interpreted as plane stress state tests (PSC)) and uniaxial compression / tension tests, assuming a plane deformation state (PDS) [9, 11]. In the above-mentioned tests, assuming material incompressibility, principal stretches and nominal stresses are measured.

In order to interpret experimental tests, it is necessary to know the relationship between the components of the first Piola-Kirchhoff strain tensor and stretch tensors. We remind you that the following relationship applies [9, 11, 15]:

$$S = J\sigma \left(F^{-1}\right)^T = \sigma \text{Cof } F.$$  \hspace{1cm} (10)

It should be clearly emphasized that the ES function is not in general a stress potential for incompressible materials, cf. (5) and (8). In the relation (8) there is a Lagrange multiplier, because the material has incompressibility constraints $J - 1 = 0$. However, in the case of a plane stress case (PSC) the Lagrange multiplier can be calculated from the condition $\sigma_3 = 0$.

After some calculations we obtain the following dependences on the nominal main stresses for PSC:

$$S_1 = \bar{p}_1 (\bar{\lambda}_1 - \bar{\lambda}_2^2 \bar{\lambda}_3^3) + \bar{p}_{-1} (\bar{\lambda}_3^3 - \bar{\lambda}_2 \bar{\lambda}_1^3) = (\bar{\lambda}_1 - \bar{\lambda}_2^2 \bar{\lambda}_3^3)(\bar{p}_1 - \bar{p}_{-1} \bar{\lambda}_1^2),$$

$$S_2 = (\bar{\lambda}_2 - \bar{\lambda}_2 \bar{\lambda}_3)(\bar{p}_1 - \bar{p}_{-1} \bar{\lambda}_1^2).$$

where the functions $\bar{p}_1$ and $\bar{p}_{-1}$ are defined by Eq. (9).

In a biaxial symmetric stretching: $\bar{\lambda}_1 = \bar{\lambda}_2$, and from (11) we get
\[ S_1 = S_2 = \left( \bar{\lambda}_1 - \bar{\lambda}_1^{-3} \right) \left( \bar{\lambda}_1 - \bar{\lambda}_2^{-3} \right). \]  \hspace{1cm} (12)

In this test the basic invariants are defined by:

\[ T_1 = 2\bar{\lambda}_1^{-2} + 2\bar{\lambda}_2^{-2}, \quad T_2 = 2\bar{\lambda}_1^{-2} + \bar{\lambda}_2^{-4}. \]  \hspace{1cm} (13)

It is worth noticing that functions \( T_1(\bar{\lambda}) \) and \( T_2(\bar{\lambda}) \), defined by formulas (13), are convex functions for \( \bar{\lambda} > 0 \). In general the invariant \( T_2 \) is not a convex function of \( \bar{\lambda}_1 \) and \( \bar{\lambda}_2 \).

5. Energy function

After substituting invariants (6) in the ES function (7), we get the elastic energy function of the MIZ model in a form dependent on the principal stretches \( \bar{\lambda}_1 \) and \( \bar{\lambda}_2 \):

\[
W = W(\bar{\lambda}_1, \bar{\lambda}_2) = \frac{1}{2} \mu_0 \left[ f \left( \bar{\lambda}_1^2 + \bar{\lambda}_2^2 + (\bar{\lambda}_1 \bar{\lambda}_2)^2 - 3 \right) + (1 - f) \left( \bar{\lambda}_1^{-2} + \bar{\lambda}_2^{-2} + (\bar{\lambda}_1 \bar{\lambda}_2)^2 - 3 \right) \right] + \frac{1}{4} c \left( \bar{\lambda}_1^2 + \bar{\lambda}_2^2 + (\bar{\lambda}_1 \bar{\lambda}_2)^2 - 3 \right)^2. \]  \hspace{1cm} (14)

Figures 1 and 2 contain contour plots of the ES functions of MR and MIZ models.

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**Figure 1.** Contour plots of the ES function \( W/\mu_0 \) of the MR model for different values of the parameter \( f : f = 1 \), i.e. the NH model (black solid lines), \( f = 0.5 \) (black dashed lines), \( f = 0.8 \) (continuous blue lines). Red lines indicate the relationship between elongation in uniaxial and biaxial stretching tests.
The principal values of the nominal stress are determined by differentiation the function (14) with respect to principal stretches. For example, we get the following dependencies in biaxial symmetrical stretching:

\[ S_i = S_2 = \frac{\partial W}{\partial \lambda_i} \bigg|_{\lambda_i=\lambda} = (\lambda_i - \lambda_i^{-1}) \left[ f\mu_0 + (1-f)\mu_0 \lambda_i^{-1} + c \left( 2\lambda_i^2 + \lambda_i^{-4} - 3 \right) \right], \tag{15} \]

Substituting \( c = 0 \) in equations (14) - (15) we get relationships for the MR model, and additionally assuming \( f = 1 \) we have equations for the NH model.

6. Stability of a square sheet under symmetric biaxial stretching

In case of square membrane (with dimensions \( L \times L \times H \)) the loads \( F_1 \) and \( F_2 \), and the principal stretches are related by:

\[ F_i = S_i HL = (\lambda_i - \lambda_i^{-3}) (\bar{\beta}_i - \bar{\beta}_i \lambda_i^2), \]

\[ F_2 = S_2 HL = (\lambda_2 - \lambda_2^{-3}) (\bar{\beta}_2 - \bar{\beta}_2 \lambda_2^{-2}), \tag{16} \]

according to (11). In in equations (16), we substitute material functions (9) with regard to invariant formulas (6) in the case of the MIZ model.

If the membrane is symmetrically loaded by \( F_1 = F_2 \) on the two sides, the right-hand sides of equations (16) must be equal. The condition \( F_1 = F_2 \) can be written as

\[ \lambda_1 = \lambda_2. \]
\[
F_1 - F_2 = \frac{(\lambda_1 - \lambda_2)}{\lambda_1 \lambda_2} \left( \bar{\beta}_1 + \bar{\beta}_2 (\lambda_1^4 - \lambda_2^4) - \bar{\beta}_4 \left( \lambda_1^4 \lambda_2^4 - \lambda_1 \lambda_2 - \lambda_1^2 - \lambda_2^2 \right) \right) = 0, \tag{17}
\]

which immediately gives the symmetric solution, \( \lambda_1 = \lambda_2 \). The asymmetric solution may exist and can be found from the equation

\[
\bar{\beta}_1 + \bar{\beta}_2 \lambda_1^4 - \bar{\beta}_4 \left( \lambda_1^4 \lambda_2^4 - \lambda_1 \lambda_2 - \lambda_1^2 - \lambda_2^2 \right) = 0,
\]

\[
\mu_0, f + c (\lambda_1^4 + \lambda_2^4 + (\lambda_1, \lambda_2)^4 - 3) (1 + \lambda_1^4 \lambda_2^4) + \mu_0 (1 - f) (\lambda_1^4 \lambda_2^4 - \lambda_1 \lambda_2 - \lambda_1^2 - \lambda_2^2) = 0. \tag{18}
\]

For such solution, in general, \( \lambda_1 \) and \( \lambda_2 \) are different, and the square sheet becomes rectangular after stretching. In the figures 3 and 4 we have plotted \( \lambda_1 \) against \( \lambda_2 \) for typical values of parameters \( f \) and \( c \) for MR and MIZ models, respectively. In each figure the straight red line represents the symmetric solution \( \lambda_1 = \lambda_2 \), and the red curve corresponds to the asymmetric solution \( \lambda_1 \neq \lambda_2 \). The intersection of red lines is the bifurcation point. We can see that for large enough forces, there are two possible solutions with different stresses, corresponding to the one with \( \lambda_1 = \lambda_2 \) and the other with \( \lambda_1 \neq \lambda_2 \), to the traction boundary value problem of a square membrane with biaxial symmetric loading. One may ask which one is a stable solution?

**Figure 3.** Contour plots of the ES function \( W/\mu_0 \) of the MR model for \( f = 0.8 \), black solid lines. Red lines indicate the relationship between extensions in biaxial stretching with the same forces, the intersection of red lines is the point of bifurcation \( \lambda_n = \lambda_1 = \lambda_2 = 2.055 \). Green curves correspond to solutions of equation (19) for \( \varepsilon = 0.02 \).
Figure 4. Contour plots of the ES function $W/\mu_0$ of the MIZ model for $f = 0.8$ and $c = 0.01\mu_c$, black solid lines. Red lines indicate the relationship between extensions in biaxial stretching with the same forces, the intersection of red lines is the point of bifurcation $\lambda_y = \lambda_1 = \lambda_2 = 2.6$. Green curves correspond to solutions of equation (19) for $\varepsilon = 0.02$.

In order to answer this question, we can consider an imperfect problem for which the square membrane is stretched with unequal forces

$F_i = (1 + \varepsilon)F_2 \Rightarrow \left( \lambda_1 - \lambda^{-3}_2 \right)(\bar{\beta}_1 - \bar{\beta}_2 \lambda^2_2) = (1 + \varepsilon)\left( \lambda_1 - \lambda^{-3}_2 \right)(\bar{\beta}_1 - \bar{\beta}_2 \lambda^2_2)$. \hfill (19)

The equation (19) can be solved numerically for a given value of $\varepsilon$. It is worth to note that there are two solution curves (see the green curves and blue curves in figures 3 and 4). They do not intercept each other. They approach the two intercepting curves for $F_1 = F_2$ when $\varepsilon \rightarrow 0$. Beyond the bifurcation point, any small imbalance of forces causes the membrane to change into a rectangular shape. The symmetric solution is unstable.

The stability of a square membrane under symmetric biaxial stretching may also be analyzed via a thermodynamic criterion following the procedure discussed in [13]. Assuming that the membrane undergoes a homogeneous deformation, its potential energy, equals

$\Pi = W(\lambda_1, \lambda_2) - S_1\lambda_1 - S_2\lambda_2$, \hfill (20)

where elastic energy is given by (14). The equilibrium solutions can then be found by minimizing $\Pi$ with respect to $\lambda_1$ and $\lambda_2$,

$\frac{\partial \Pi}{\partial \lambda_1} = \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_1} - S_1 = 0, \quad \frac{\partial \Pi}{\partial \lambda_2} = \frac{\partial W(\lambda_1, \lambda_2)}{\partial \lambda_2} - S_2 = 0$. \hfill (21)
The equations (21) are equivalent to previously obtained relationships (16). The solutions are stable when the Hessian of the potential energy

\[
\begin{bmatrix}
\frac{\partial^2 \Pi}{\partial \lambda_i^2} & \frac{\partial^2 \Pi}{\partial \lambda_i \lambda_j} \\
\frac{\partial^2 \Pi}{\partial \lambda_j \lambda_i} & \frac{\partial^2 \Pi}{\partial \lambda_j^2}
\end{bmatrix},
\]

is positive semi-definite or, equivalently,

\[
\frac{\partial^2 W}{\partial \lambda_i^2} \geq 0, \quad \frac{\partial^2 W}{\partial \lambda_j^2} - \left( \frac{\partial^2 W}{\partial \lambda_i \lambda_j} \right)^2 \geq 0,
\]

because the function \( \Pi \) has the form (20).

In the case of a membrane made of MIZ material, we substitute the energy function (14) in the above relationships. We obtain the symmetric solution (15) from equations (21) when \( S_1 = S_2 \). For the considered material parameters of the MIZ model, the conditions (13) show that this solution is not stable for elongations greater than \( \lambda_B \). For an asymmetric solution \( \lambda_1 \neq \lambda_2 \), from the conditions (13), doing this numerically, we conclude that the asymmetric solution is always stable.

7. Final remarks

In this chapter the MIZ model of an incompressible, isotropic hyprelastic material, called in the literature Ishihara-Zahorski model [8, 10], whose elastic energy function depends on two invariants of isochoric deformation, were analysed. From the point of view of phenomenological mechanics, the MIZ model is a generalization of the MR model [9, 11, 13, 15, 17]. There are three constants in the ES function: the initial shear modulus \( \mu_0 > 0 \) (identical to the linear Hooke’s relationship), the parameter \( f \) (setting the linear combination between invariants and, i.e. the combination between the first and second fundamental invariants of isochoric deformation) and the material parameter \( c > 0 \), which scales the square effect of the first invariant in the ES function. The MIZ model is desirable from the point of view of the mathematical theory of hypereasticity, i.e. if \( f \in (0,1) \) and \( c \geq 0 \), the function (7) is a polyconvex function, which assures the existence of solutions of a wide class of boundary value problems [1].

In this paper, analysing the problem of equal stretching of the square membrane from the MR or MIZ material (with polyconvex stored energy functions) two solutions were obtained, the symmetric and asymmetric one. It was shown that the asymmetric solution (depending on parameters \( f \) and \( c \)) is possible for significant forces and this solution is always stable. It is worth emphasizing that the symmetric solution is stable only for principal stretches smaller than \( \lambda_B \).

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