Intertwining of simple characters in $\mathrm{GL}(n)$

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Abstract. Let $F$ be a non-Archimedean local field and let $G$ be the general linear group $G = \mathrm{GL}_n(F)$. Let $\theta_1, \theta_2$ be simple characters in $G$. We show that $\theta_1$ intertwines with $\theta_2$ if and only if $\theta_1$ is endo-equivalent to $\theta_2$. We also show that any simple character in $G$ is a $G$-type.

Let $F$ be a non-Archimedean local field and let $G = \mathrm{GL}_n(F)$, for some $n \geq 1$. Following [1], the category $\mathrm{Rep}_G$ of smooth complex representations of $G$ decomposes as a direct sum of indecomposable blocks,

$$\mathrm{Rep}_G = \bigoplus_{s \in \mathcal{B}(G)} \mathrm{Rep}_s G,$$

indexed by a certain set $\mathcal{B}(G)$. Let $\mathcal{S}$ be a finite subset of $\mathcal{B}(G)$. As in [7], an $\mathcal{S}$-type in $G$ is an irreducible smooth representation $\rho$, of some compact open subgroup of $G$, with the property that an irreducible smooth representation of $G$ contains $\rho$ if and only if it lies in $\mathrm{Rep}_s G$, for some $s \in \mathcal{S}$.

One knows [8] how to construct an $\{s\}$-type in $G$, for any $s \in \mathcal{B}(G)$. Those types are all built from simple characters in groups $\mathrm{GL}_m(F)$, in the sense of [6], for various integers $m \leq n$. Here, we return to the simple characters themselves and prove:

**Type Theorem.** Let $G = \mathrm{GL}_n(F)$, for some $n \geq 1$, and let $\theta$ be a simple character in $G$. The character $\theta$ is then an $\mathcal{S}_\theta$-type in $G$, for some finite subset $\mathcal{S}_\theta$ of $\mathcal{B}(G)$.

The proof, and a description of $\mathcal{S}_\theta$, are given in §4 below.

We use the Type Theorem to prove a powerful result concerning the intertwining properties of simple characters. As part of the definition, a simple character in $G = \mathrm{GL}_n(F)$ is attached, in an invariant manner, to a hereditary order in the matrix algebra $A = \mathrm{M}_n(F)$. A cornerstone of the theory is the fact ((3.5.11) of [6]) that two simple characters in $G$, attached to the same order and which intertwine in $G$, are actually conjugate. This result is taken one further level in [4]. There is a canonical procedure for transferring simple characters between hereditary orders, in possibly different matrix algebras. Given two simple characters $\theta_i$ in $\mathrm{GL}_n(F)$, attached to hereditary orders $\mathfrak{a}_i$, one can find an integer $n$ and a hereditary order $\mathfrak{a}$ in $\mathrm{M}_n(F)$ to which both characters may be transferred. If the transferred characters are conjugate in $\mathrm{GL}_n(F)$, one says they are endo-equivalent. One knows that endo-equivalence is an equivalence relation on

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the class of simple characters in all general linear groups. Moreover, endo-equivalent characters attached to the same order are necessarily conjugate. Here, we consider a general pair of simple characters in one group. We prove:

**Intertwining Theorem.** Let \( \theta_1, \theta_2 \) be simple characters in \( G = \text{GL}_n(F) \). The characters \( \theta_1, \theta_2 \) intertwine in \( G \) if and only if they are endo-equivalent.

To the specialist in the area, these results provide clear and satisfying conclusions to several lines of development, but the non-specialist may wish for more motivation. This is first provided by our examination [5] of the congruence properties of the local Langlands correspondence, where the Intertwining Theorem provides a crucial step in the argument: see [5] 4.3 Lemma.

The results also give a framework in which to investigate representations in more general settings. There is a fully functional theory of simple characters and endo-equivalence spanning all inner forms \( \text{GL}_n(D) \) of \( \text{GL}_n(F) \), where \( D \) is a finite-dimensional central \( F \)-division algebra [2], [9], [10]. However, certain new structures come into play, and one is led to ask how these are reflected or clarified in analogues of our results. In a different direction, one may consider smooth representations of \( \text{GL}_n(F) \) over fields of positive characteristic \( \ell \). Provided \( \ell \) is not the residual characteristic of \( F \), one has an identical general theory of simple characters. One deduces readily that the Intertwining Theorem holds unchanged. However, as Vincent Sécherre reminds us, a simple character need not be a type in this situation.

1. A review of simple characters

Let \( \mathfrak{o}_F \) be the discrete valuation ring in \( F \) and \( \mathfrak{p}_F \) the maximal ideal of \( \mathfrak{o}_F \). We choose a smooth character \( \psi \) of \( F \) which is of level one, in the sense that \( \ker \psi \) contains \( \mathfrak{p}_F \) but not \( \mathfrak{o}_F \).

Let \( V \) be an \( F \)-vector space of finite dimension and set \( A = \text{End}_F(V), G = \text{Aut}_F(V) \). Let \( \mathfrak{a} \) be a hereditary \( \mathfrak{o}_F \)-order in \( A \), with Jacobson radical \( \mathfrak{p}_A \). A **simple character in \( G \)**, attached to \( \mathfrak{a} \), is one of the following objects. The **trivial** simple character attached to \( \mathfrak{a} \) is the trivial character of the group \( U^1_{\mathfrak{a}} = 1 + \mathfrak{p}_A^{-1} \): we denote this \( 1^1_{\mathfrak{a}} \).

To define a **non-trivial** simple character attached to \( \mathfrak{a} \), we recall briefly the definition [6] (1.5.5) of a simple stratum \( [\mathfrak{a}, l, 0, \beta] \) in \( A \). First, \( \beta \in G \) and the algebra \( E = F[\beta] \) is a field. The hereditary order \( \mathfrak{a} \) is \( E \)-pure, in that \( x^{-1}ax = a \) for \( x \in E^\times \). The integer \( l \) is positive and given by \( \beta^{-1}a = p_A^l \). The quadruple \( [\mathfrak{a}, l, 0, \beta] \) is then a simple stratum in \( A \) if \( \beta \) satisfies a technical condition \( \kappa_0(\beta, \mathfrak{a}) < 0 \) loc. cit. Since we will not use this directly, we say no more of it.

Following the recipes of [6] 3.1, the simple stratum \( [\mathfrak{a}, \beta] = [\mathfrak{a}, l, 0, \beta] \) defines open subgroups \( H^1(\beta, \mathfrak{a}) \subset J^1(\beta, \mathfrak{a}) \subset J^0(\beta, \mathfrak{a}) \) of the unit group \( U_A = \mathfrak{a}^\times \), such that \( J^1(\beta, \mathfrak{a}) = J^0(\beta, \mathfrak{a}) \cap U^1_{\mathfrak{a}} \). The choice of \( \psi \) then gives rise to a finite set \( \mathcal{C}(\mathfrak{a}, \beta, \psi) \) of smooth characters of \( H^1(\beta, \mathfrak{a}) \), called **simple characters**: see [6] 3.2 for the full definition. The choice of \( \psi \) is essentially irrelevant, so we treat it as fixed and henceforth omit it from the notation.

We recall a fundamental property of simple characters attached to a fixed hereditary order [6] (3.5.11).

**Intertwining implies conjugacy.** For \( i = 1, 2 \), let \( [\mathfrak{a}, \beta_i] \) be a simple stratum in \( A \) and let \( \theta_i \in \mathcal{C}(\mathfrak{a}, \beta_i) \). If the characters \( \theta_1, \theta_2 \) intertwine in \( G \) then they are conjugate by an element of \( U_A \).

We shall also need systems of **transfer maps**. Let \( [\mathfrak{a}, l, 0, \beta] \) be a simple stratum in \( A \), as before. Suppose we have another \( F \)-vector space \( V' \) of finite dimension, an \( F \)-embedding \( \iota : F[\beta] \to A' = \text{End}_F(V') \), and an \( F[\iota(\beta)] \)-pure hereditary order \( \mathfrak{a}' \) in \( A' \): any two such embeddings
\(\ell\) are \(U_a\)-conjugate, so we are justified in omitting \(\ell\) from the notation. There is a unique integer \(l'\) such that \([a',l',0,\beta]\) is a simple stratum in \(A'\). There is a canonical bijection

\[(1.1) \quad \tau_{\alpha,a}^\beta : \mathcal{C}(\alpha,\beta) \xrightarrow{\sim} \mathcal{C}(a',\beta).\]

This family of maps is transitive with respect to the orders: in the obvious notation, we have

\[\tau_{\alpha,a}^\beta = \tau_{\alpha,a'}^\beta \circ \tau_{\alpha,a'}^\beta.\]

Full details may be found in [6] section 3.6 and [4] section 8.

**Lemma 1.** For \(j = 1, 2\), let \([a_j,l_j,0,\beta_j]\) be a simple stratum in \(A_j = \text{End}_F(V_j)\).

1. There exists a finite-dimensional \(F\)-vector space \(V\), a hereditary order \(a\) in \(A = \text{End}_F(V)\) and a pair of \(F\)-embeddings \(\iota_j : F[\beta_j] \rightarrow A\), such that \(a\) is \(F[\iota_j\beta_j]\)-pure, for \(j = 1, 2\).

2. Let \(\theta_j \in \mathcal{C}(a_j,\beta_j)\). The following are equivalent:
   
   (a) There exists a system \((V,a,\iota_j)\), as in (1), such that \(\tau_{a_1,a}(\theta_1)\) intertwines with \(\tau_{a_2,a}(\theta_2)\) in \(G = \text{Aut}_F(V)\).
   
   (b) For any system \((V,a,\iota_j)\), as in (1), the character \(\tau_{a_1,a}(\theta_1)\) intertwines with \(\tau_{a_2,a}(\theta_2)\) in \(G = \text{Aut}_F(V)\).

**Proof.** Part (1) is elementary. If (2)(a) holds, then \([F[\beta_1]:F] = [F[\beta_2]:F]\) by [6] (3.5.1) and the intertwining implies conjugacy property. Part (2) is then given by Theorem 8.7 of [4]. \(\square\)

Developing this theme, if the (non-trivial) simple characters \(\theta_j\) of Lemma 1(2) satisfy condition (a), we say they are endo-equivalent. In particular, in the context of (1.1), \(\theta\) is endo-equivalent to \(\tau_{\alpha,a}(\theta)\). Further, two endo-equivalent simple characters attached to the same order intertwine, and so are conjugate. It follows that endo-equivalence is indeed an equivalence relation on the class of non-trivial simple characters cf. [4] 8.10.

It is convenient to extend this framework to include the trivial simple characters. We set \(\tau_{\alpha,a}(1^1_a) = 1^1_{a'}\) and deem that any two trivial simple characters are endo-equivalent. Any two such characters in the same group intertwine, so the approach is consistent with the main case. Moreover, a trivial simple character can never intertwine with a non-trivial one: this follows from [3] Theorem 1 and [6] (2.6.2).

2. **Heisenberg extensions**

Let \(\theta\) be a simple character in \(G = \text{Aut}_F(V)\), attached to a hereditary order \(a\) in \(A = \text{End}_F(V)\). Thus \(\theta\) is a character of an open subgroup \(H^1_\theta\) of \(U^1_a\). Let \(J^0_\theta\) denote the \(U_a\)-normalizer of \(\theta\) and put \(J^1_\theta = J^0_\theta \cap U^1_a\). If \(\theta\) is non-trivial, we choose a simple stratum \([a,\beta]\) in \(A\) such that \(\theta \in \mathcal{C}(a,\beta)\). We then get \(J^1_\theta = J^k(\beta,a)\) and \(H^1_\theta = H^1(\beta,a)\), in the notation of \(\S 1\). If \(\theta\) is the trivial simple character \(1^1_a\) attached to \(a\), we have \(H^1_\theta = J^1_\theta = U^1_a\) and \(J^0_\theta = U^1_a\).

With \(E = F[\beta]\) (if \(\theta\) is non-trivial) or \(F\) (otherwise), let \(B = \text{End}_E(V)\) be the \(A\)-centralizer of \(E\) and set \(b = a \cap B\). Thus \(b\) is a hereditary \(a_E\)-order in \(B\) with radical \(q = p_a \cap B\). We then have \(J^0_\theta = J^0_b U_b^1\) and \(J^1_\theta \cap U_b^1 = U^1_b\).

Let \(\eta = \eta_\theta\) be the unique irreducible representation of \(J^0_\theta\) which contains \(\theta\) [6] (5.1.1). Thus \(\eta|_{H^1_\theta}\) is a multiple of \(\theta\). Let \(\mathfrak{X}^0(\theta)\) be the set of equivalence classes of irreducible representations of \(J^0_\theta\) which contain \(\theta\). Let \(\mathfrak{X}^0(\theta)\) be the set of \(\kappa \in \mathfrak{X}^0(\theta)\) with the following two properties. First, \(\kappa|_{J^0_b} \cong \eta_\theta\). Second, \(\kappa\) is intertwined by every element of \(G\) which intertwines \(\theta\). In the language of [6], \(\mathfrak{X}^0(\theta)\) consists of the “\(\beta\)-extensions” of \(\eta_\theta\) and is non-empty [6] (5.2.2).
In particular, $\mathcal{R}^0(1^1_b)$ is the set of equivalence classes of irreducible representations of $U_b$ trivial on $U_b^1$. For $\sigma \in \mathcal{R}^0(1^1_b)$, there is a unique irreducible representation $\sigma_\theta$ of $J^0_b$ which agrees with $\sigma$ on $U_b$ and is trivial on $J^1_b$.

**Lemma 2.** Let $\kappa \in \mathcal{R}^0(\theta)$, $\sigma \in \mathcal{R}^0(1^1_b)$. The representation $\kappa \otimes \sigma_\theta$ of $J^0_b$ is irreducible, and lies in $\mathcal{R}^0(\theta)$. For any $\kappa \in \mathcal{R}^0(\theta)$, the map

$$\mathcal{R}^0(1^1_b) \rightarrow \mathcal{R}^0(\theta), \quad \sigma \mapsto \kappa \otimes \sigma_\theta,$$

is a bijection.

**Proof.** The restriction of $\kappa \otimes \sigma_\theta$ to $H^1_\theta$ is surely a multiple of $\theta$. The other assertions are given by [5] 1.5 Proposition. \(\square\)

3. **Residually cuspidal representations**

We continue in the same situation. Let $k_E$ denote the residue field of $E$. The group $J^0_b/J^1_b$ takes the form

$$J^0_b/J^1_b \cong U_b/U_b^1 \cong \prod_{i=1}^r \text{GL}_{m_i}(k_E),$$

for integers $r, m_i \geq 1$ such that $\sum_{i \leq i \leq r} = n/[E:F]$. In particular, $U_b/U_b^1$ is the group of rational points of a connected reductive $k_E$-group. We fix $\kappa \in \mathcal{R}^0(\theta)$. If $\lambda \in \mathcal{R}^0(\theta)$ then, by Lemma 2, $\lambda \cong \kappa \otimes \sigma_\theta$ where $\sigma$ is the inflation of a uniquely determined irreducible representation $\tilde{\sigma}$ of $U_b/U_b^1$. We say that $\lambda$ is residually cuspidal if the representation $\tilde{\sigma}$ is cuspidal. The representation $\kappa$ is uniquely determined, up to tensoring with a character of the form $(\phi \circ \det_{U_b})_\theta$, where $\phi$ is a character of $U_E$ trivial on $U_b^1$ [6] (5.2.2), so this property of $\lambda$ does not depend on the choice of $\kappa$. We denote by $\mathcal{R}^0_\kappa(\theta)$ the subset of residually cuspidal elements of $\mathcal{R}^0(\theta)$.

**Proposition 1.** Let $\theta$ be a simple character in $G$, and let $[\alpha, \beta]$ be a simple stratum in $A$ such that $\theta \in \mathcal{C}([\alpha, \beta])$. Let $E$ denote the field $F[\beta]$.

1. Let $a'$ be an $E$-pure hereditary $\alpha_E$-order in $A$, containing $\alpha$. Let $\theta' = \tau_{\alpha,a'}^\beta$, and let $\lambda \in \mathcal{R}^0_\kappa(\theta)$. An irreducible representation $\pi$ of $G$ containing $\lambda$ then contains some element of $\mathcal{R}^0(\theta')$.

2. Suppose that $\lambda \in \mathcal{R}^0(\theta)$ is not residually cuspidal. There exists an $E$-pure hereditary $\alpha_E$-order $a''$ in $A$, with $a'' \subset \alpha$, and an element $\lambda'' \in \mathcal{R}^0_\kappa(\theta'')$, where $\theta'' = \tau_{\alpha,a''}^\beta \theta$, with the following property: any irreducible representation of $G$ containing $\lambda''$ also contains $\lambda''$.

**Proof.** All assertions follow from (8.3.5) Proposition of [6]. \(\square\)

**Remark 1.** Proposition 1 applies equally when $\theta$ is a trivial simple character, as noted in [6], following (8.3.5).

The simple characters $\theta'$, $\theta''$ of Proposition 1 are both endo-equivalent to $\theta$.

We say that a simple stratum $[\alpha, \beta]$ in $A$ is $m$-simple if $\alpha$ is maximal among $F[\beta]$-pure hereditary $\alpha_E$-orders in $A$. We say that a simple character $\theta$ is $m$-simple if $\theta \in \mathcal{C}([\alpha, \beta])$, where $[\alpha, \beta]$ is $m$-simple. (This depends on $\theta$, not the choice of $[\alpha, \beta]$.) Similarly for trivial characters.

**Proposition 2.** Let $\lambda \in \mathcal{R}^0(\theta)$. The following are equivalent:

1. $\theta$ is $m$-simple and $\lambda$ is residually cuspidal;
2. $\lambda$ is contained in some irreducible cuspidal representation of $G$;
3. any irreducible representation of $G$ containing $\lambda$ is cuspidal.
Proof. The equivalence of (2) and (3) is [6] (6.2.1, 6.2.2). The implication (1) \(\Rightarrow\) (2) is [5] (6.2.3). For the converse, suppose that either \(\theta\) is not \(m\)-simple or that \(\lambda\) is not residually cuspidal. Let \(\pi\) be an irreducible representation of \(G\) containing \(\lambda\). In either case, part (2) of Proposition 1 implies the existence of the following objects:

1. a non-maximal \(E\)-pure hereditary order \(A'\) in \(A\),
2. a simple character \(\theta'\) attached to \(A'\) and endo-equivalent to \(\theta\),
3. a representation \(\lambda' \in R^0_e(\theta')\) occurring in \(\pi\).

The representation \(\pi\) is then not cuspidal, by [6] (8.3.3 or 7.3.16).

\[\Box\]

Corollary 1. An irreducible cuspidal representation \(\pi\) of \(G\) contains exactly one conjugacy class of simple characters \(\theta\), and all of those characters are \(m\)-simple.

Proof. This follows from Proposition 2 and [6] (6.2.4).

\[\Box\]

So, if \(\pi\) is an irreducible cuspidal representation of \(G\), all simple characters contained in \(\pi\) belong to the same endo-equivalence class, which we denote \(\vartheta(\pi)\).

4. The Type Theorem

Let \(a\) be a hereditary \(o_F\)-order in \(A = \text{End}_F(V)\), with Jacobson radical \(p_a\). Thus

\[a/p_a \cong \prod_{i=1}^{r} M_{n_i}(k_F),\]

for positive integers \(n_i\) with sum \(n\). Let \(M_a\) be an \(F\)-Levi subgroup of \(G\) such that \(M_a \cong \prod_{1 \leq i \leq r} \text{GL}_{n_i}(F)\). The group \(M_a\) is determined uniquely, up to conjugation in \(G\). If \(M\) is an \(F\)-Levi subgroup of \(G\), we say that \(M\) is subordinate to \(a\) if \(M\) is \(G\)-conjugate to a Levi subgroup of \(M_a\).

We recall some further definitions. A cuspidal datum in \(G\) is a pair \((M,\sigma)\), where \(M\) is a Levi subgroup of \(G\) and \(\sigma\) is an irreducible cuspidal representation of \(M\). The set of such data carries the equivalence relation “\(G\)-inertial equivalence”, as in [7] §1. The set of equivalence classes for this relation will be denoted \(B(G)\).

If \(\pi\) is an irreducible smooth representation of \(G\), there is a cuspidal datum \((M,\sigma)\) in \(G\) and a parabolic subgroup \(P\) of \(G\), with Levi component \(M\), such that \(\pi\) is equivalent to a subquotient of the induced representation \(\text{Ind}_P^G \sigma\). The inertial equivalence class of \((M,\sigma)\) is thereby uniquely determined: we call it the inertial support of \(\pi\) and denote it \(\mathcal{I}(\pi)\). If \(\mathcal{S}\) is a finite subset of \(B(G)\), an \(\mathcal{S}\)-type in \(G\) is a pair \((K,\rho)\), where \(K\) is a compact open subgroup of \(G\) and \(\rho\) is an irreducible smooth representation of \(K\) such that, if \(\pi\) is an irreducible smooth representation of \(G\), then \(\pi\) contains \(\rho\) if and only if \(\mathcal{I}(\pi) \in \mathcal{S}\) [7] 4.1, 4.2.

Let \(\theta\) be a (possibly trivial) simple character in \(G\), attached to the hereditary order \(a\). Let \(\Theta\) denote the endo-equivalence class of \(\theta\).

**Definition.** Let \(s \in B(G)\) be the \(G\)-inertial equivalence class of \((M,\sigma)\), where

\[M \cong \prod_{j=1}^{s} \text{GL}_{m_j}(F), \quad \sigma = \bigotimes_{j=1}^{s} \sigma_j,\]

and \(\sigma_j\) is an irreducible cuspidal representation of \(\text{GL}_{m_j}(F)\). We say that \(s\) is subordinate to \(\theta\) if \(M\) is subordinate to \(a\) and \(\vartheta(\sigma_j) = \Theta\), for all \(j\).

We prove the following version of the Type Theorem.
**Theorem 3.** Let $V$ be a finite-dimensional $F$-vector space, and let $G$ denote the group $\text{Aut}_F(V)$. Let $\theta$ be a simple character in $G$. Let $\mathcal{S}_\theta$ be the set of $s \in \mathcal{B}(G)$ that are subordinate to $\theta$. The character $\theta$ is then an $\mathcal{S}_\theta$-type in $G$.

**Proof.** We have to show that an irreducible representation $\pi$ of $G$ contains $\theta$ if and only if the inertial support of $\pi$ is an element of $\mathcal{S}_\theta$. We assume that $\theta$ is non-trivial: the proof for trivial simple characters is parallel but easier, so we omit it. We choose a simple stratum $[\alpha, \beta]$ such that $\theta \in \mathcal{C}(\alpha, \beta)$ and set $E = F[\beta]$.

We start in a slightly more general situation, with a cuspidal datum $s$ of the form $(M, \sigma)$ such that

\[ (4.1) \quad M \cong \prod_{k=1}^{s} \text{GL}_{n_k}(F), \quad \sigma = \bigotimes_{k=1}^{s} \sigma_k, \]

for various integers $n_k \geq 1$, and $\theta(\sigma_k) = \Theta$ for all $k$. Replacing $M$ by a $G$-conjugate and each $\sigma_k$ by an equivalent representation, we can assume we are in the following situation. First, $M$ is the $G$-stabilizer of a decomposition $V = \bigoplus_{1 \leq k \leq s} V_k$, in which the $V_k$ are non-zero $E$-subspaces of $V$. Second, each $\sigma_k$ contains a simple character $\theta_k \in \mathcal{C}(\alpha_k, \beta)$, endo-equivalent to $\theta$, for some simple stratum $[\alpha_k, \beta]$ in $\text{End}_F(V_k)$. Observe that, by Corollary 1, each $\theta_k$ is $m$-simple, so $J^0_{\theta_k}/J^0_{\theta_k} \cong M_{nk}/[E:F](\mathcal{X}_E)$.

We may impose a further normalization. We suppose given an $E$-pure hereditary order $\mathfrak{A}$ in $A = \text{End}_F(V)$ such that $\mathfrak{A}/p_\mathfrak{A} \cong \prod_{k=1}^s \text{GL}_{n_k}(k_F)$, the integers $n_k$ being as in (4.1). There is then an integer $N > 0$ such that $[\mathfrak{A}, N, 0, \beta]$ is a simple stratum in $A$. Let $\theta_{\mathfrak{A}} = \tau_{a_0, \mathfrak{A}}^\beta \theta \in \mathcal{C}(\mathfrak{A}, \beta)$. In particular, $\theta_{\mathfrak{A}}$ is endo-equivalent to $\theta$. Theorem 7.2 of [8] gives an $s$-type in $G$ of the form $(J^0(\beta, \mathfrak{A}), \lambda)$, where $\lambda_{\mathfrak{A}} \in \mathcal{X}^0(\theta_{\mathfrak{A}})$.

**Remark 2.** To be more precise, the construction in [8] 7.2 yields an $s$-type $(K, \tau)$ where, in our notation, $H^1(\beta, \mathfrak{A}) \subset K \subset J^0(\beta, \mathfrak{A})$. The representation of $J^0(\beta, \mathfrak{A})$ induced by $\tau$ is our $\lambda_{\mathfrak{A}}$.

Let $s \in \mathcal{S}_\theta$. Thus $s$ is subordinate to $\theta$ and we may therefore choose $\mathfrak{A} \subset a$. Let $\pi$ be an irreducible representation of $G$ of inertial support $s$. By definition, $\pi$ contains $\lambda_{\mathfrak{A}}$. By Proposition 1, $\pi$ contains a simple character $\theta' \in \mathcal{C}(a, \beta)$ which is endo-equivalent to $\theta_{\mathfrak{A}}$. It follows that $\theta'$ is endo-equivalent to $\theta$, and hence $G$-conjugate to $\theta$. In particular, $\pi$ contains $\theta$.

Conversely, let $\pi$ be an irreducible representation of $G$ which contains $\theta$. Proposition 1(2) gives an $E$-pure hereditary order $a' \subset a$, a simple character $\theta' \in \mathcal{C}(a', \beta)$ and a representation $\lambda \in \mathcal{X}^0(\theta')$ which occurs in $\pi$. Comparing with Theorem 7.2 of [8] again, we see that $\lambda$ is an $s$-type in $G$, for some $s \in \mathcal{S}_\theta$. Consequently, the inertial support of $\pi$ is an element of $\mathcal{S}_\theta$, as required. □

## 5. The Intertwining Theorem

We prove the Intertwining Theorem. Let $G = \text{GL}_n(F)$, $A = M_n(F)$. Let $\theta_1, \theta_2$ be simple characters in $G$, with endo-classes $\Theta_1, \Theta_2$ respectively.

Suppose first that $\Theta_1 = \Theta_2$. We have to show that $\theta_1$ intertwines with $\theta_2$ in $G$. If the $\theta_i$ are trivial, this is clear, so suppose otherwise. Choose simple strata $[a_i, \beta_i]$ in $A$ such that $\theta_i \in \mathcal{C}(a_i, \beta_i)$ and put $E_i = F[\beta_i]$. The relation $\Theta_1 = \Theta_2$ implies that the field extensions $E_i/F$ have the same ramification indices and the same residue class degrees [4] (8.11). So, there exists an $E_2$-pure hereditary order $a$ in $A$ which is isomorphic to $a_1$. Set $\theta = \tau_{a_2, a}^\beta \theta_2$. According to Lemma 1, the simple characters $\theta_1, \theta$ must intertwine in $G$ and hence be $G$-conjugate, say
\( \theta = \theta_1^g \), for some \( g \in G \). By [6] (3.6.1), the characters \( \theta_1, \theta_2 \) agree on \( H^1(\beta_2, a_2) \cap H^1(\beta_2, a) \). Therefore \( g \) intertwines \( \theta_1 \) with \( \theta_2 \), as required.

For the converse, suppose that \( \theta_1 \) intertwines with \( \theta_2 \). Abbreviating \( H_i = H^1(\beta_i, a_i) \), this hypothesis implies the existence of a non-trivial \( G \) homomorphism

\[
(5.1) \quad c\text{-Ind}_{H_1}^G \theta_1 \rightarrow c\text{-Ind}_{H_2}^G \theta_2.
\]

Frobenius Reciprocity, for compact induction from open subgroups, implies that the space \( \Pi_i = c\text{-Ind}_{H_i}^G \theta_i \) is generated over \( G \) by its \( \theta_i \)-vectors. Since \( \theta_i \) is a type in \( G \) (Theorem 3), every irreducible \( G \)-subquotient of \( \Pi_i \) contains \( \theta_i \) [7] 4.1. The existence of the non-trivial map (5.1) implies there exists an irreducible representation \( \pi \) of \( G \) containing both \( \theta_i \). If the inertial support of \( \pi \) is of the form \( (\prod_k \text{GL}_{n_k}(F), \otimes_k \sigma_k) \) then, by Theorem 3 again, \( \vartheta(\sigma_k) = \Theta_1 = \Theta_2 \), for all \( k \).

Since endo-equivalence is an equivalence relation, the Intertwining Theorem implies that simple characters, in a fixed group, exhibit the following surprising property.

**Corollary 2.** Let \( \theta_1, \theta_2, \theta_3 \) be simple characters in \( G = \text{GL}_n(F) \). If \( \theta_1 \) intertwines with \( \theta_2 \) and \( \theta_2 \) intertwines with \( \theta_3 \), then \( \theta_1 \) intertwines with \( \theta_3 \).

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