STABILITY AND SLICING INEQUALITIES FOR INTERSECTION BODIES

ALEXANDER KOLDOBSKY AND DAN MA

Abstract. We prove a generalization of the hyperplane inequality for intersection bodies, where volume is replaced by an arbitrary measure $\mu$ with even continuous density and sections are of arbitrary dimension $n-k$, $1 \leq k < n$. If $K$ is a generalized $k$-intersection body, then

$$\mu(K) \leq \frac{n}{n-k} c_{n,k} \max_H \mu(K \cap H) \text{Vol}_n(K)^{k/n}.$$

Here $c_{n,k} = |B_n^2|^{(n-k)/n}/|B_{n-k}^2| < 1$, $|B_n^2|$ is the volume of the unit Euclidean ball, and maximum is taken over all $(n-k)$-dimensional subspaces of $\mathbb{R}^n$. The constant is optimal, and for each intersection body the inequality holds for every $k$. We also prove a stronger “difference” inequality. The proof is based on stability in the lower dimensional Busemann-Petty problem for arbitrary measures in the following sense. Let $\varepsilon > 0$, $1 \leq k < n$. Suppose that $K$ and $L$ are origin-symmetric star bodies in $\mathbb{R}^n$, and $K$ is a generalized $k$-intersection body. If for every $(n-k)$-dimensional subspace $H$ of $\mathbb{R}^n$

$$\mu(K \cap H) \leq \mu(L \cap H) + \varepsilon,$$

then

$$\mu(K) \leq \mu(L) + \frac{n}{n-k} c_{n,k} \text{Vol}_n(K)^{k/n} \varepsilon.$$

1. Introduction

The Busemann-Petty problem, posed in 1956 in [11], asks the following question. Suppose that $K$ and $L$ are origin-symmetric convex bodies in $\mathbb{R}^n$ so that

$$\text{Vol}_{n-1}(K \cap \xi) \leq \text{Vol}_{n-1}(L \cap \xi), \quad \forall \xi \in S^{n-1},$$

where $\xi$ is the central hyperplane perpendicular to $\xi$. Does it follow that

$$\text{Vol}_n(K) \leq \text{Vol}_n(L)?$$

The answer is affirmative if $n \leq 4$ and negative if $n \geq 5$. The solution was completed at the end of the 90’s as the result of a sequence of
papers \cite{26, 11, 14, 5, 27, 31, 10, 11, 36, 37, 18, 19, 38, 13}; see \cite{20, p. 3} or \cite{12, p. 343} for details.

It is natural to ask what happens if hyperplane sections are replaced by sections of lower dimensions. Suppose that for every \((n - k)\)-dimensional subspace \(H \in \mathbb{R}^n\),

\[
\text{Vol}_{n-k}(K \cap H) \leq \text{Vol}_{n-k}(L \cap H).
\]

Does it follow that

\[
\text{Vol}_n(K) \leq \text{Vol}_n(L)?
\]

Zhang \cite{39} proved that the answer is affirmative if and only if all origin-symmetric convex bodies in \(\mathbb{R}^n\) are generalized \(k\)-intersection bodies (see definition in Section 2; this is similar to the connection between the original Busemann-Petty problem and intersection bodies established by Lutwak in \cite{27}). Using this connection, Bourgain and Zhang \cite{6} proved that the answer is negative if the dimension of sections \(n - k > 3\) (see also \cite{33} and different later proof in \cite{21}). However, the cases of two- and three-dimensional sections remain open. Other results on the lower dimensional Busemann-Petty problem can be found in \cite{28–30, 32–35}.

In this paper, we establish stability in the affirmative part of the lower dimensional Busemann-Petty problem. Stability problems in convex geometry have been considered for a long time; see \cite{16} for numerous results and references. Stability in volume comparison problems was first studied in \cite{22}, where such results were proved for the Busemann-Petty and Shephard problems. We extend the result of \cite{22, Theorem 1} to sections of lower dimensions in the following way.

**Theorem 1.** Let \(K\) and \(L\) be origin-symmetric star bodies in \(\mathbb{R}^n\), and \(1 \leq k < n\). Suppose \(K\) is a generalized \(k\)-intersection body and \(\varepsilon > 0\). If for every \((n - k)\)-dimensional subspace \(H\) of \(\mathbb{R}^n\)

(1) \[
\text{Vol}_{n-k}(K \cap H) \leq \text{Vol}_{n-k}(L \cap H) + \varepsilon,
\]

then

(2) \[
\text{Vol}_n(K)^{\frac{n-k}{n}} \leq \text{Vol}_n(L)^{\frac{n-k}{n}} + c_{n,k} \varepsilon,
\]

where \(c_{n,k} = \frac{|B_n^2|^{(n-k)/n}}{|B_{n-k}^2|} \) and \(|B_n^2|\) is the volume of the unit Euclidean ball.

Note that \(c_{n,k} < 1\), which immediately follows from the log-convexity of the \(\Gamma\)-function (see for example \cite{24, Lemma 2.1}). Also, in the formulation of Theorem 1 in \cite{22} the constant \(c_{n,1}\) was replaced by 1, though the proof there gives the result with \(c_{n,1}\).
Zvavitch [40] found a remarkable generalization of the Busemann-Petty problem to arbitrary measures. It appears that one can replace volume by any measure with even continuous density in \( \mathbb{R}^n \). Let \( f \) be an even continuous non-negative function on \( \mathbb{R}^n \), and denote by \( \mu \) the measure on \( \mathbb{R}^n \) with density \( f \). For every closed bounded set \( B \subset \mathbb{R}^n \) define

\[
\mu(B) = \int_B f(x) \, dx.
\]

It was proved in [40] that, for \( n \leq 4 \) and any origin-symmetric convex bodies \( K \) and \( L \) in \( \mathbb{R}^n \), the inequalities

\[
\mu(K \cap \xi^\perp) \leq \mu(L \cap \xi^\perp), \quad \forall \xi \in S^{n-1}
\]

imply

\[
\mu(K) \leq \mu(L).
\]

Zvavitch also proved that this is generally not true if \( n \geq 5 \), namely, for any \( \mu \) with strictly positive even continuous density there exist \( K \) and \( L \) providing a counterexample.

Stability in Zvavitch’s result was established in [23, Theorem 2]. Here we extend this result to sections of lower dimensions, as follows.

**Theorem 2.** Let \( K \) and \( L \) be origin-symmetric star bodies in \( \mathbb{R}^n \), and \( 1 < k < n \). Suppose \( K \) is a generalized \( k \)-intersection body and \( \varepsilon > 0 \). If for every \((n-k)\)-dimensional subspace \( H \) of \( \mathbb{R}^n \)

\[
\mu(K \cap H) \leq \mu(L \cap H) + \varepsilon,
\]

then

\[
\mu(K) \leq \mu(L) + \frac{n}{n-k} c_{n,k} \text{Vol}_n(K)^{k/n} \varepsilon.
\]

In the case \( f \equiv 1 \), we get another stability result for volume which is weaker than what is provided by Theorem 1. This is the reason why we state Theorem 1 separately. However, for arbitrary measures the constant in Theorem 2 is the best possible, as follows from the example after Corollary 5.

The stability results mentioned above were applied in [22,23] to the hyperplane (or slicing) problem of Bourgain [2,3] that can be formulated as follows. Does there exist an absolute constant \( C \) so that for any origin-symmetric convex body \( K \) in \( \mathbb{R}^n \)

\[
\text{Vol}_n(K)^{n-1} \leq C \max_{\xi \in S^{n-1}} \text{Vol}_{n-1}(K \cap \xi^\perp)?
\]

The best-to-date estimate \( C \sim n^{1/4} \) is due to Klartag [17], who removed the logarithmic term from the previous estimate of Bourgain [4]. We
refer the reader to recent papers [8,9] for the history and current state of the hyperplane problem.

In the case where $K$ is an intersection body (see Section 2 for definitions and properties), the inequality (4) is known for sections of arbitrary dimension with the best possible constant. For any $1 \leq k < n$,

$$\text{Vol}_n(K) \frac{n-k}{n} \leq c_{n,k} \max_{H \in G(n,n-k)} \text{Vol}_{n-k}(K \cap H), \tag{5}$$

where $G(n,n-k)$ is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$, and the equality is attained when $K = B_{2}^{n}$. In particular, if the dimension $n \leq 4$, then (5) is true for any origin-symmetric convex body $K$. The proof is an immediate consequence of Zhang’s connection between generalized intersection bodies and the lower dimensional Busemann-Petty problem; apply this connection to any generalized $k$-intersection body $K$ and $L = B_{2}^{n}$. Then use the fact that every intersection body is a generalized $k$-intersection body for every $k$ (see [15] or [28]). For every fixed $k$, the inequality (5) holds for any generalized $k$-intersection body $K$.

We prove several generalizations of (5) using the stability results formulated above. First, interchanging $K$ and $L$ in Theorem 1, we get the following “difference” inequality, previously established in [22, Corollary 1] in the hyperplane case.

**Corollary 3.** Let $K$ and $L$ be origin-symmetric star bodies in $\mathbb{R}^n$, and $1 \leq k < n$. Suppose $K$ and $L$ are generalized $k$-intersection bodies, then

$$\left| \text{Vol}_n(K) \frac{n-k}{n} - \text{Vol}_n(L) \frac{n-k}{n} \right| \leq c_{n,k} \max_{H \in G(n,n-k)} \left| \text{Vol}_{n-k}(K \cap H) - \text{Vol}_{n-k}(L \cap H) \right|.$$

Putting $L = \emptyset$ in the latter inequality, we get (5) for any generalized $k$-intersection body $K$.

Interchanging $K$ and $L$ in Theorem 2, we get the following inequality, which was earlier proved for $k = 1$ in [23, Corollary 1].

**Corollary 4.** Let $K$ and $L$ be origin-symmetric star bodies in $\mathbb{R}^n$, and $1 \leq k < n$. Suppose that $K$ and $L$ are generalized $k$-intersection bodies. Then

$$|\mu(K) - \mu(L)| \leq \frac{n}{n-k} c_{n,k} \max_H |\mu(K \cap H) - \mu(L \cap H)| \max \left\{ \text{Vol}_n(K)^{k/n}, \text{Vol}_n(L)^{k/n} \right\},$$

where maximum is taken over all $(n-k)$-dimensional subspaces $H$ of $\mathbb{R}^n$. 
Putting $L = \emptyset$, we generalize to lower dimensions the hyperplane inequality for arbitrary measures from [23, Theorem 1].

**Corollary 5.** Let $1 \leq k < n$, and suppose that $K$ is a generalized $k$-intersection body in $\mathbb{R}^n$. Then

$$
\mu(K) \leq \frac{n - k}{n} c_{n,k} \max_{H \in G(n, n - k)} \mu(K \cap H) \text{Vol}_n(K)^{k/n}.
$$

The constant in the right-hand side is the best possible. In fact, let $K = B_2^n$ and, for every $j \in \mathbb{N}$, let $f_j$ be a non-negative continuous function on $[0, 1]$ supported in $(1 - \frac{1}{j}, 1)$ and such that $\int_0^1 f_j(t)dt = 1$. Let $\mu_j$ be the measure on $\mathbb{R}^n$ with density $f_j(|x|_2)$, where $|x|_2$ is the Euclidean norm. We have

$$
\mu_j(B_2^n) = |S^{n-1}| \int_0^1 r^{n-1} f_j(r)dr,
$$

where $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere in $\mathbb{R}^n$. For every $H \in G(n, n - k)$,

$$
\mu_j(B_2^n \cap H) = |S^{n-k-1}| \int_0^1 r^{n-k-1} f_j(r)dr.
$$

Clearly,

$$
\lim_{j \to \infty} \frac{\int_0^1 r^{n-1} f_j(r)dr}{\int_0^1 r^{n-k-1} f_j(r)dr} = 1.
$$

Using $|S^{n-1}| = n|B_2^n|$, we get

$$
\lim_{j \to \infty} \frac{\mu_j(B_2^n)}{\max_H \mu_j(B_2^n \cap H) \text{Vol}_n(B_2^n)^{k/n}} = \frac{|S^{n-1}|}{|S^{n-k-1}| |B_2^n|^{k/n}} = \frac{n}{n - k} c_{n,k},
$$

which shows that the constant is asymptotically optimal.

2. **Stability**

We say that a closed bounded set $K$ in $\mathbb{R}^n$ is a *star body* if every straight line passing through the origin crosses the boundary of $K$ at exactly two points different from the origin, the origin is an interior point of $K$, and the *Minkowski functional* of $K$ defined by

$$
\|x\|_K = \min\{a \geq 0 : x \in aK\}
$$

is a continuous function on $\mathbb{R}^n$.

The *radial function* of a star body $K$ is defined by

$$
\rho_K(x) = \|x\|_K^{-1}, \quad x \in \mathbb{R}^n.
$$

If $x \in S^{n-1}$ then $\rho_K(x)$ is the radius of $K$ in the direction of $x$. 

Writing the volume of $K$ in polar coordinates, one gets

$$\text{Vol}_n(K) = \frac{1}{n} \int_{S^{n-1}} \rho^n_K(\theta) d\theta = \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^n d\theta.$$  

(7)

The spherical Radon transform $R : C(S^{n-1}) \mapsto C(S^{n-1})$ is a linear operator defined by

$$Rf(\xi) = \int_{S^{n-1} \cap \xi^\perp} f(x) \ dx, \quad \xi \in S^{n-1}$$

for every function $f \in C(S^{n-1})$.

The polar formula (7) for the volume of a hyperplane section expresses this volume in terms of the spherical Radon transform (see for example [20, p.15]):

$$S_K(\xi) = \text{Vol}_{n-1}(K \cap \xi^\perp) = \frac{1}{n-1} R(\| \cdot \|_{-1}^{-1})(\xi).$$

(8)

The spherical Radon transform is self-dual (see [16, Lemma 1.3.3]): for any functions $f, g \in C(S^{n-1})$

$$\int_{S^{n-1}} Rf(\xi) \ g(\xi) \ d\xi = \int_{S^{n-1}} f(\xi) \ Rg(\xi) \ d\xi.$$  

(9)

Using self-duality, one can extend the spherical Radon transform to measures. Let $\mu$ be a finite Borel measure on $S^{n-1}$. We define the spherical Radon transform of $\mu$ as a functional $R\mu$ on the space $C(S^{n-1})$ acting by

$$(R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x) d\mu(x).$$

By Riesz’s characterization of continuous linear functionals on the space $C(S^{n-1})$, $R\mu$ is also a finite Borel measure on $S^{n-1}$. If $\mu$ has continuous density $g$, then by [9] the Radon transform of $\mu$ has density $Rg$.

The class of intersection bodies was introduced by Lutwak [27]. Let $K, L$ be origin-symmetric star bodies in $\mathbb{R}^n$. We say that $K$ is the intersection body of $L$ if the radius of $K$ in every direction is equal to the $(n - 1)$-dimensional volume of the section of $L$ by the central hyperplane orthogonal to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\rho_K(\xi) = \|\xi\|_K^{-1} = \text{Vol}_{n-1}(L \cap \xi^\perp).$$

(10)

All the bodies $K$ that appear as intersection bodies of different star bodies form the class of intersection bodies of star bodies.

Note that the right-hand side of (10) can be written in terms of the spherical Radon transform using (8):

$$\|\xi\|_K^{-1} = \frac{1}{n-1} \int_{S^{n-1} \cap \xi^\perp} \|\theta\|_{L^{-1}}^{n+1} d\theta = \frac{1}{n-1} R(\| \cdot \|_{L^{-1}}^{n+1})(\xi).$$

(10)
It means that a star body $K$ is the intersection body of a star body if and only if the function $\| \cdot \|_K^{-1}$ is the spherical Radon transform of a continuous positive function on $S^{n-1}$. This allows to introduce a more general class of bodies. A star body $K$ in $\mathbb{R}^n$ is called an intersection body if there exists a finite Borel measure $\mu$ on the sphere $S^{n-1}$ so that $\| \cdot \|_K^{-1} = R\mu$ as functionals on $C(S^{n-1})$, i.e. for every continuous function $f$ on $S^{n-1}$,

$$\int_{S^{n-1}} \|x\|_K^{-1} f(x) \, dx = \int_{S^{n-1}} Rf(x) \, d\mu(x). \tag{11}$$

Intersection bodies played the crucial role in the solution of the original Busemann-Petty problem due to the following connection found by Lutwak [27]. If $K$ is an origin-symmetric intersection body in $\mathbb{R}^n$ and $L$ is any origin-symmetric star body in $\mathbb{R}^n$, then the inequalities $S_K(\xi) \leq S_L(\xi)$ for all $\xi \in S^{n-1}$ imply that $\text{Vol}_n(K) \leq \text{Vol}_n(L)$, i.e. the answer to the Busemann-Petty problem in this situation is affirmative. For more information about intersection bodies, see [20, Chapter 4], [25], [12, Chapter 8] and references there. In particular, every origin-symmetric convex body in $\mathbb{R}^n$, $n \leq 4$ is an intersection body; see [11,13,38]. Also the unit ball of any finite dimensional subspace of $L_p$, $0 < p \leq 2$ is an intersection body; see [18].

Zhang in [39] introduced a generalization of intersection bodies. For $1 \leq k \leq n-1$, the $(n-k)$-dimensional spherical Radon transform is an operator $\mathcal{R}_{n-k} : C(S^{n-1}) \mapsto C(G(n,n-k))$ defined by

$$\mathcal{R}_{n-k}(f)(H) = \int_{S^{n-1} \cap H} f(x) \, dx, \quad H \in G(n,n-k).$$

Denote the image of the operator $\mathcal{R}_{n-k}$ by $X$:

$$\mathcal{R}_{n-k} \left( C(S^{n-1}) \right) = X \subset C(G(n,n-k)).$$

Let $M^+(X)$ be the space of linear positive continuous functionals on $X$, i.e. for every $\nu \in M^+(X)$ and non-negative function $f \in X$, we have $\nu(f) \geq 0$.

An origin-symmetric star body $K$ in $\mathbb{R}^n$ is called a generalized $k$-intersection body if there exists a functional $\nu \in M^+(X)$, so that for every $f \in C(S^{n-1})$,

$$\int_{S^{n-1}} \|x\|_K^{-k} f(x) \, dx = \nu(\mathcal{R}_{n-k}(f)).$$

When $k = 1$ we get the class of intersection bodies. It was proved by Grinberg and Zhang [15, Lemma 6.1] that every intersection body in $\mathbb{R}^n$ is a generalized $k$-intersection body for every $k < n$. More generally, as proved later by Milman [28], if $m$ divides $k$, then every generalized
$m$-intersection body is a generalized $k$-intersection body. Zhang \cite{39} showed that the answer to the lower dimensional Busemann-Petty problem is affirmative if and only if every origin-symmetric convex body in $\mathbb{R}^n$ is a generalized $k$-intersection body.

Denote by $1_{S^n} \equiv 1$ and $1_{G} \equiv 1$ the functions which are equal to 1 everywhere on the unit sphere $S^{n-1}$ and the Grassmanian $G(n, n-k)$, correspondingly. Then, $\mathcal{R}_{n-k}(1_S) = |S^{n-k-1}| \cdot 1_G$.

We are now ready to prove the stability in the lower dimensional Busemann-Petty problem.

\textbf{Proof of Theorem 1.} By the polar formula for volume (7), for each $H \in G(n, n-k)$ we have
\begin{equation}
\text{Vol}_{n-k}(K \cap H) = \frac{1}{n-k} \mathcal{R}_{n-k} \left( \| \cdot \|_{K}^{-n+k} \right)(H),
\end{equation}
Then the inequality (1) can be written as
\begin{equation}
\mathcal{R}_{n-k} \left( \| \cdot \|_{K}^{-n+k} \right)(H) \leq \mathcal{R}_{n-k} \left( \| \cdot \|_{L}^{-n+k} \right)(H) + (n-k) \varepsilon.
\end{equation}
Since $K$ is a generalized $k$-intersection body, there exists $\mu_0 \in M^+$, such that for each $\psi \in C(S^{n-1})$,
\begin{equation}
\int_{S^{n-1}} \|x\|_{K}^{-k} \psi(x) dx = \mu_0(\mathcal{R}_{n-k}(\psi)).
\end{equation}
Since $\mu_0$ is a positive functional, by (13) and (14), we have
\begin{equation}
n \text{Vol}_n(K) = \int_{S^{n-1}} \|x\|_{K}^{-k} \|x\|_{K}^{-n+k} dx
\leq \mu_0 \left( \mathcal{R}_{n-k} \left( \| \cdot \|_{K}^{-n+k} \right) \right)
\leq \mu_0 \left( \mathcal{R}_{n-k} \left( \| \cdot \|_{L}^{-n+k} \right) \right) + (n-k) \varepsilon \mu_0(1_G)
\end{equation}
Using (14), Hölder’s inequality and polar formula for the volume, we get
\begin{equation}
\int_{S^{n-1}} \|x\|_{K}^{-k} \|x\|_{L}^{-n+k} dx
\leq \left( \int_{S^{n-1}} \|x\|_{K}^{-n} dx \right)^{k/n} \left( \int_{S^{n-1}} \|x\|_{L}^{-n} dx \right)^{(n-k)/n}
= n \text{Vol}_n(K)^{k/n} \text{Vol}_n(L)^{(n-k)/n}.
\end{equation}
Now, by (14), the well-known formula $|S^{n-1}| = n|B^n_2|$ (see [20, p. 33]) and Hölder’s inequality,

$$II = \left( n - k \right) \varepsilon \mu_0(1_G) = \frac{(n-k)\varepsilon}{|S^{n-k-1}|} \int_{S^{n-1}} \|x\|_{K}^{n-k} 1_{S}(x) \, dx$$

$$\leq \frac{(n-k)\varepsilon}{|S^{n-k-1}|} \left( \int_{S^{n-1}} \|x\|_{K}^{n} \, dx \right)^{k/n} |S^{n-1}|^{\frac{n-k}{n}}$$

$$= \frac{n^{k/n}(n-k)|S^{n-1}|^{\frac{n-k}{n}}}{|S^{n-k-1}|} \text{Vol}_n(K)^{k/n} \varepsilon$$

$$= \frac{n|B^n_2|^{\frac{n-k}{n}}}{|B^{n-k-1}_2|} \text{Vol}_n(K)^{k/n} \varepsilon.$$

Combining this with (15) and (16), we get the result.

We now pass to stability for arbitrary measures. Let $\mu$ be a measure on $\mathbb{R}^n$ with even continuous density $f$. Let $\chi$ be the indicator function of the interval $[0, 1]$. The measure $\mu$ of a star body $K$ can be expressed in polar coordinates as follows:

$$\mu(K) = \int_{K} f(x) \, dx = \int_{\mathbb{R}^n} \chi(\|x\|_{K}) f(x) \, dx$$

$$= \int_{S^{n-1}} \left( \int_{0}^{r} t^{n-1} f(t\theta) \, dt \right) \, d\theta. \tag{17}$$

Similarly, we can express the volume of a section of $K$ by an $(n-k)$-dimensional subspace $H$ of $\mathbb{R}^n$ as

$$\mu(K \cap H) = \int_{H} \chi(\|x\|_{K}) f(x) \, dx$$

$$= \int_{S^{n-1} \cap H} \left( \int_{0}^{t^{n-k-1} f(t\theta)} dt \right) \, d\theta$$

$$= \mathcal{R}_{n-k} \left( \int_{0}^{r^{n-k-1} f(r\theta)} dr \right) (H), \tag{18}$$

where the Radon transform is applied to a function of the variable $\theta \in S^{n-1}$.

We need the following lemma, which was also used by Zvavitch in his proof.
**Lemma 6.** Let $a, b, k \in \mathbb{R}^+$, and $\alpha$ be a non-negative function on $(0, \max\{a, b\})$, such that the integral below converges. Then

$$
\int_0^a r^{n-1} \alpha(r) \, dr - a^k \int_0^a r^{n-k-1} \alpha(r) \, dr \\
\leq \int_0^b r^{n-1} \alpha(r) \, dr - a^k \int_0^b r^{n-k-1} \alpha(r) \, dr
$$

**Proof.** The result follows from

$$a^k \int_a^b r^{n-k-1} \alpha(r) \, dr \leq \int_a^b r^{n-1} \alpha(r) \, dr.
$$

\[ \Box \]

**Proof of Theorem 2.** Using (18), inequality (3) can be written as

\begin{equation}
\mathcal{R}_{n-k} \left( \int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta) \, dr \right) (H) \\
\leq \mathcal{R}_{n-k} \left( \int_0^{\|\theta\|_L^{-1}} r^{n-k-1} f(r\theta) \, dr \right) (H) + \varepsilon, \quad \forall H \in G(n, n - k).
\end{equation}

As in the proof of Theorem 1, let $\mu_0$ be the positive functional associated with the generalized $k$-intersection body $K$. Applying $\mu_0$ to both sides of (19) and then using (14), we get

\begin{equation}
\int_{S^{n-1}} \|\theta\|_K^{-k} \left( \int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta) \, dr \right) d\theta \\
\leq \int_{S^{n-1}} \|\theta\|_K^{-k} \left( \int_0^{\|\theta\|_L^{-1}} r^{n-k-1} f(r\theta) \, dr \right) d\theta + \varepsilon \mu_0(1_G).
\end{equation}

Applying Lemma 6 with $a = \|\theta\|_K^{-1}$, $b = \|\theta\|_L^{-1}$ and $\alpha(r) = f(r\theta)$ and then integrating over the sphere, we get

$$\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta) \, dr - \|\theta\|_K^{-k} \int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta) \, dr \\
\leq \int_0^{\|\theta\|_L^{-1}} r^{n-1} f(r\theta) \, dr - \|\theta\|_K^{-k} \int_0^{\|\theta\|_L^{-1}} r^{n-k-1} f(r\theta) \, dr,
$$

and
\begin{align}
(21) \quad & \int_{S^{n-1}} \left( \int_0^1 r^{n-1} f(r\theta) \, dr \right) d\theta \\
& - \int_{S^{n-1}} \|\theta\|^{-k}_K \left( \int_0^1 r^{n-k-1} f(r\theta) \, dr \right) d\theta \\
& \leq \int_{S^{n-1}} \left( \int_0^1 r^{n-1} f(r\theta) \, dr \right) d\theta \\
& - \int_{S^{n-1}} \|\theta\|^{-k}_L \left( \int_0^1 r^{n-k-1} f(r\theta) \, dr \right) d\theta.
\end{align}

Adding (20) and (21) and using (17) we get
\[
\mu(K) \leq \mu(L) + \varepsilon \mu_0(1_G).
\]

As shown in the proof of Theorem 1
\[
\mu_0(1_G) \leq \frac{n}{n-k} c_{n,k} \operatorname{Vol}_n(K)^{k/n},
\]
which completes the proof.

As mentioned earlier, every intersection body is a generalized $k$-intersection body for every $k$, so if $K$ is an intersection body, the results of Theorems 1 and 2 hold for all $k$ at the same time, as well as the results of Corollaries 3, 4, 5.

Acknowledgement. The first named author wishes to thank the US National Science Foundation for support through grant DMS-1001234.

References

[1] K. Ball, Some remarks on the geometry of convex sets, Geometric aspects of functional analysis (1986/87), Lecture Notes in Math. 1317, Springer-Verlag, Berlin-Heidelberg-New York, 1988, 224–231.
[2] J. Bourgain, On high-dimensional maximal functions associated to convex bodies, Amer. J. Math. 108 (1986), 1467–1476.
[3] J. Bourgain, Geometry of Banach spaces and harmonic analysis, Proceedings of the International Congress of Mathematicians (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, 871–878.
[4] J. Bourgain, On the distribution of polynomials on high-dimensional convex sets, Geometric aspects of functional analysis, Israel seminar (1989-90), Lecture Notes in Math. 1469, Springer, Berlin, 1991, 127–137.
[5] J. Bourgain, On the Busemann-Petty problem for perturbations of the ball, Geom. Funct. Anal. 1 (1991), 1–13.
[6] J. Bourgain and Gaoyong Zhang, *On a generalization of the Busemann-Petty problem*. Convex Geometric Analysis, Math. Sci. Res. Inst. Publ. **34**, Cambridge Univ. Press, Cambridge, 1999, 53–58.

[7] H. Busemann and C. M. Petty, *Problem on convex bodies*. Math. Scand., **4** (1956), 88–94.

[8] N. Dafnis and G. Paouris, *Small ball probability estimates, $\psi_2$-behavior and the hyperplane conjecture*, J. Funct. Anal. **258** (2010), 1933–1964.

[9] R. Eldan and B. Klartag, *Approximately gaussian marginals and the hyperplane conjecture*, preprint. [arXiv:1001.0875](https://arxiv.org/abs/1001.0875).

[10] R. J. Gardner, *Intersection bodies and the Busemann-Petty problem*, Trans. Amer. Math. Soc. **342** (1994), 435–445.

[11] R. J. Gardner, *A positive answer to the Busemann-Petty problem in three dimensions*, Ann. of Math. (2) **140** (1994), 435–447.

[12] R. J. Gardner, *Geometric tomography*. Cambridge Univ. Press, New York, 2nd edition, 2006.

[13] R. J. Gardner, A. Koldobsky and Th. Schlumprecht, *An analytic solution to the Busemann-Petty problem on sections of convex bodies*, Ann. of Math. (2) **149** (1999), 691–703.

[14] A. Giannopoulos, *A note on a problem of H. Busemann and C. M. Petty concerning sections of symmetric convex bodies*, Mathematika **37** (1990), 239–244.

[15] E. Grinberg and Gaoyong Zhang, *Convolutions, transforms and convex bodies*, Proc. London Math. Soc. **78** (1999), 77–115.

[16] H. Groemer, *Geometric Applications of Fourier Series and Spherical Harmonics*. Cambridge University Press, New York, 1996.

[17] B. Klartag, *On convex perturbations with a bounded isotropic constant*, Geom. Funct. Anal. **16** (2006), 1274–1290.

[18] A. Koldobsky, *Intersection bodies, positive definite distributions and the Busemann-Petty problem*, Amer. J. Math. **120** (1998), 827–840.

[19] A. Koldobsky, *Intersection bodies in $\mathbb{R}^4$*, Adv. Math. **136** (1998), 1–14.

[20] A. Koldobsky, *Fourier analysis in convex geometry*, Amer. Math. Soc., Providence RI, 2005.

[21] A. Koldobsky, *A functional analytic approach to intersection bodies*. Geom. Funct. Anal., **10** (2000), 1507–1526.

[22] A Koldobsky, *Stability in the Busemann-Petty and Shephard problems*. Adv. Math., doi:10.1016/j.aim.2011.06.031

[23] A. Koldobsky, *A hyperplane inequality for measures of convex bodies in $\mathbb{R}^n$, $n \leq 4$*, Discrete Comput. Geom., doi:10.1007/s00454-011-9362-8

[24] A. Koldobsky and M. Lifshits, *Average volume of sections of star bodies*, Geometric Aspects of Functional Analysis, V. Milman and G. Schechtman, eds., Lecture Notes in Mathematics **1745** (2000), 119–146.

[25] A. Koldobsky and V. Yaskin, *The interface between convex geometry and harmonic analysis*, CBMS Regional Conference Series in Mathematics, 108, American Mathematical Society, Providence, RI, 2008.

[26] D. G. Larman and C. A. Rogers, *The existence of a centrally symmetric convex body with central sections that are unexpectedly small*, Mathematika **22** (1975), 164–175.
[27] E. Lutwak, *Intersection bodies and dual mixed volumes*, Adv. Math. *71* (1988), 232–261.

[28] E. Milman, *Generalized intersection bodies*. J. Funct. Anal., *240* (2) (2006), 530–567.

[29] E. Milman, *A comment on the low-dimensional Busemann-Petty problem*. GAFA Seminar Notes 2004-5, Lecture Notes in Math. *1910*, 2007, 245–253.

[30] E. Milman, *Generalized intersection bodies are not equivalent*. Adv. Math., *217* (6) (2008), 2822–2840.

[31] M. Papadimitrakis, *On the Busemann-Petty problem about convex, centrally symmetric bodies in $\mathbb{R}^n$*, Mathematika *39* (1992), 258–266.

[32] B. Rubin, *The lower dimensional Busemann-Petty problem for bodies with the generalized axial symmetry*. Israel J. of Math., *173* (2009), 213–233.

[33] B. Rubin and Gaoyong Zhang, *Generalizations of the Busemann-Petty Problem for sections of convex bodies*. J. Funct. Anal., *213* (2) (2004), 473–501.

[34] V. Yaskin, *The Busemann-Petty problem in hyperbolic and spherical spaces*. Adv. Math., *203* (2) (2006), 537–553.

[35] V. Yaskin, *A solution to the lower dimensional Busemann-Petty problem in the hyperbolic space*. J. Geom. Anal., *16* (4) (2006), 735–745.

[36] Gaoyong Zhang, *Centered bodies and dual mixed volumes*, Trans. Amer. Math. Soc. *345* (1994), 777–801.

[37] Gaoyong Zhang, *Intersection bodies and Busemann-Petty inequalities in $\mathbb{R}^4$*, Ann. of Math. (2) *140* (1994), 331–346.

[38] Gaoyong Zhang, *A positive answer to the Busemann-Petty problem in four dimensions*, Ann. of Math. (2) *149* (1999), 535–543.

[39] Gaoyong Zhang, *Section of convex bodies*. Amer. J. Math., *118* (1996), 319–340.

[40] A. Zvavitch, *The Busemann-Petty problem for arbitrary measures*. Math. Ann., *331* (2005), 267–887.

Department of Mathematics, University of Missouri, Columbia, MO 65211

E-mail address: koldobskiya@missouri.edu

Department of Mathematics, University of Missouri, Columbia, MO 65211

E-mail address: madan516@gmail.com