Quantum Master Equation for QED in Exact Renormalization Group

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Recently, one of us (H. S.) gave an explicit form of the Ward-Takahashi identity for the Wilson action of QED. We first rederive the identity using a functional method. The identity makes it possible to realize the gauge symmetry even in the presence of a momentum cutoff. In the cutoff dependent realization, the nilpotency of the BRS transformation is lost. Using the Batalin-Vilkovisky formalism, we extend the Wilson action by including the antifield contributions. Then, the Ward-Takahashi identity for the Wilson action is lifted to a quantum master equation, and the modified BRS transformation regains nilpotency. We also obtain a flow equation for the extended Wilson action.

§1. Introduction

One of the most important subjects in the exact renormalization group (ERG)1–3 is to find a method to treat gauge symmetry, which is naively incompatible with a momentum cutoff Λ introduced as a regularization (see Ref. 4) and references therein for a manifestly gauge invariant formalism using ERG). Constraints on the induced symmetry breaking terms are described by some identities for Green functions, so-called “the broken Ward-Takahashi (WT) identities” or “the modified Slavnov-Taylor (ST) identities”.5,6) These identities are written either for the Wilson action $S$ or its Legendre transformed effective action $Γ$. They have additional regulator dependent terms which are absent in the standard WT or ST identities for the cutoff removed theories. This is why they are called “broken” or “modified” identities.

Even if the standard realization of the symmetry no longer works, the symmetry can be realized in a regularization dependent way. This is most conveniently done using the Batalin-Vilkovisky (BV) antifield formalism7,8)(see also Ref. 9)), which has been recognized as the most general and powerful method for dealing with symmetries. Any local as well as global symmetries are described by the quantum master equation (QME) in the BV formalism. It was shown10) that if the QME holds for a cutoff-removed action $S[φ]$, this should also be the case for the Wilson action $S[Φ]$ with a finite value of momentum cutoff. Hence, at least conceptually, the presence of the symmetry along ERG flow is established. This result may suggest that, for consideration of symmetries, the use of the Wilson action $S$ is preferable to that of the Legendre action $Γ$.

The above argument remains, however, at a formal level, and only little has been known concerning how actions satisfying the QME look like for concrete cases,
especially for gauge theories.\textsuperscript{*}) So far, only for the lattice chiral symmetry, which is a prototype of the regularization dependent symmetry, it has been shown that the QME contains the Ginsparg-Wilson relation and was solved to give an action for self-interacting fermions.\textsuperscript{14)}

Recently, an important result was obtained in formulating gauge symmetry in terms of ERG: one of the present authors (H. S.) has derived the WT identity\textsuperscript{**}) $\Sigma_\Phi = 0$ for the Wilsonian QED action $S[\Phi]$ and its flow equation.\textsuperscript{15, 16)}

From the WT identity, we obtain the BRS transformation $\delta$ that has the following properties: 1) it depends on the Wilson action, and therefore, is non-linear; 2) a non-trivial Jacobian factor associated with $\delta$ is generated to cancel the change of the action; 3) it is not nilpotent, $\delta^2 \neq 0$. We interpret the last property as a signal that field variables should be enlarged to include antifields.

The main aim of the present paper is to describe how these results fit in the formal argument in treating the gauge theory. As pointed out above, any symmetry, if it exists, can be described by the QME in the BV antifield formalism. Actually we show that the WT identity $\Sigma_\Phi = 0$ for QED can be lifted to the QME $\Sigma_{\Phi, \Phi^*} = 0$.

In Ref. 15), the action $S[\Phi]$ that satisfies the WT identity is obtained perturbatively. Here we show how to construct the action $S[\Phi, \Phi^*]$ that fulfills the QME, assuming that we have the action satisfying the WT identity. We call the action $S[\Phi, \Phi^*]$ as the master action.

It is found that our master action, a formal solution to the QME, has non-trivial antifield dependence: the infinite power series expansion w.r.t. antifields takes the form of a Taylor expansion of the Dirac fields, which corresponds to a shift of field variables in the Wilson action. As a byproduct of introducing antifields, we employ the “quantum BRS transformation”,\textsuperscript{19)} $\delta_Q$, which is assured to be nilpotent, $\delta_Q^2 = 0$, thanks to the QME. Using this, we show the BRS invariance of the Polchinski equation for our master action. We emphasize that the nilpotency is recovered in the BV formalism even though the BRS transformation read off from the WT identity is not nilpotent.

The rest of this paper is organized as follows. In the next section, we give a general method for deriving the WT identity $\Sigma_\Phi = 0$ for a regularized theory, and apply it to QED to obtain the WT identity derived in Ref. 15). From the WT identity, we read off the BRS transformation. In §3, after a brief explanation of the antifield formalism, we construct a master action for QED that satisfies the QME $\Sigma_{\Phi, \Phi^*} = 0$ starting from an action satisfying the WT identity $\Sigma_\Phi = 0$. In the final section, the Polchinski flow equation is given for our master action, and its BRS invariance is shown.

\textsuperscript{*}) As for global symmetries, see Ref. 11). The RG flow equation for QED was studied in Ref. 12) with some approximations. Reference 13) discusses roles of the modified WT identity in relation to numerical studies.

\textsuperscript{**}) This WT identity is expected to carry the same information as “the broken WT identity” for QED given in Refs. 17) and 18) for the Legendre effective action $\Gamma$. 

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§2. Path integral derivation of WT identity for cutoff QED

We will derive the WT identity for the QED with a momentum cutoff using the path integral formalism. Our derivation is based on the method\(^*\) which has already been discussed by several authors.\(^5)\,\,20),\,\,21)\) The formalism given here may be applicable to any theory with symmetry. We will first discuss a generic theory with a momentum cutoff, and then apply the results to QED.

2.1. WT identity for a cutoff theory

The fields are denoted collectively by \(\phi^A\). The index \(A\) represents the Lorentz indices of tensor fields, the spinor indices of the fermions, and/or indices distinguishing different types of fields. The Grassmann parity for \(\phi^A\) is expressed as \(\epsilon(\phi^A) = \epsilon_A\), so that \(\epsilon_A = 0\) if the field \(\phi^A\) is Grassmann even (bosonic) and \(\epsilon_A = 1\) if it is Grassmann odd (fermionic). The generating functional for this theory in the presence of sources \(J_A\) is given by

\[
Z_{\phi}[J] = \int D\phi \exp \left( -S[\phi] + J \cdot \phi \right),
\]

where the action \(S\) is decomposed into the kinetic and interaction terms

\[
S[\phi] = \frac{1}{2} \phi \cdot D \cdot \phi + S_I[\phi].
\]

In this paper we use the matrix notation in momentum space:

\[
J \cdot \phi = \int \frac{d^d p}{(2\pi)^d} J_A(-p)\phi^A(p),
\]

\[
\phi \cdot D \cdot \phi = \int \frac{d^d p}{(2\pi)^d} \phi^A(-p)D_{AB}(p)\phi^B(p).
\]

We now introduce an IR momentum cutoff \(A\) through a positive function that behaves as

\[
K\left(\frac{p}{A}\right) \rightarrow \begin{cases} 
1, & (p^2 < A^2) \\
0, & (p^2 \rightarrow \infty)
\end{cases}
\]

where the function goes to 0 sufficiently rapidly as \(p^2 \rightarrow \infty\). For simplicity, we write the function as \(K(p)\) in the rest of the paper. Using this function, we decompose the original fields \(\phi^A\) with the propagator \((D_{AB}(p))^{-1}\) into two classes of fields: the IR fields \(\Phi^A\) with the propagator \(K(p)(D_{AB}(p))^{-1}\), and the UV fields \(\chi^A\) with \((1-K(p))(D_{AB}(p))^{-1}\). To this end, we substitute a gaussian integral over new fields \(\theta^A\)

\[
\int D\theta \exp \left( -\frac{1}{2} \left( \theta - J(1-K)D^{-1} \right) \cdot \frac{D}{K(1-K)} \cdot \left( \theta - (-)^{\epsilon(J)} D^{-1}(1-K)J \right) \right) = \text{const}
\]

\(^\ast\) In Ref. 5), it is called “the quantum action principle”.
into the path-integral (2.1), and introduce new variables $\Phi$ and $\chi$ by

$$
\phi^A = \Phi^A + \chi^A, \quad \theta^A = (1 - K)\Phi^A - K\chi^A.
$$

(2.6)

Then, we obtain

$$
Z_{\phi}[J] = N_J \int D\Phi D\chi \exp \left( -\frac{1}{2} \Phi \cdot K^{-1} D \cdot \Phi - J \cdot K^{-1} \Phi 
+ \frac{1}{2} \chi \cdot (1 - K)^{-1} D \cdot \chi + S_I[\Phi + \chi] \right),
$$

(2.7)

where

$$
N_J \equiv \exp \frac{1}{2} (-)^{\epsilon} (1 - K)^{-1} D \cdot J_B.
$$

(2.8)

The Wilson action is given by

$$
S[\Phi] \equiv \Phi \cdot K^{-1} D \cdot \Phi/2 + S_I[\Phi],
$$

(2.9)

where $S_I[\Phi]$ is defined by

$$
\exp -S_I[\Phi] \equiv \int D\chi \exp \left( -\frac{1}{2} \chi \cdot (1 - K)^{-1} D \cdot \chi + S_I[\Phi + \chi] \right).
$$

(2.10)

Note that the gaussian integral (2.5) is chosen in such a way that the UV fields $\chi^A$ do not couple to source terms, and hence the Wilson action $S_I$ depends only on the IR fields $\Phi^A$. The partition function for $\Phi^A$

$$
Z_{\phi}[J] = \int D\Phi \exp \left( -S[\Phi] + K^{-1} J \cdot \Phi \right)
$$

(2.11)

is related to that for $\phi$ by

$$
Z_{\phi}[J] = N_J Z_{\phi}[J].
$$

(2.12)

This implies that the full generating functional $Z_{\phi}$ can be constructed from the Wilson action $S[\Phi]$. In Eq. (2.11), note that the source to the IR field $\Phi^A$ is multiplied by $K^{-1}$. Therefore, the correlation functions in two theories are related as

$$
\langle \phi^{A_1} \ldots \phi^{A_N} \rangle_{\phi} |_{J=0} = \langle (K^{-1} \phi^{A_1}) \ldots (K^{-1} \phi^{A_N}) \rangle_{\Phi} |_{J=0}. \quad (N \geq 3)
$$

(2.13)

As for the two-point functions, there are extra contributions from the factor $N_J$.

Now we consider how the IR cutoff affects the realization of symmetry. Suppose the original gauge-fixed action $S[\phi]$ is invariant under the BRS transformation

$$
\phi^A \rightarrow \phi^{A'} = \phi^A + \delta \phi^A, \quad \delta \phi^A = R^A[\phi] \lambda,
$$

(2.14)

* The construction of the Wilson action via similar techniques can be found in Refs. 20), 22) and 23). The idea of the decomposition of fields was also discussed in a non-local regularization scheme.24)
where $\lambda$ is an anticommuting constant. Hence,

$$
\delta S = \frac{\partial^r S}{\partial \phi^A} \delta \phi^A \equiv \Sigma_\phi \lambda = 0.
$$

(2.15)

Assuming the invariance of the functional measure $D\phi$, we obtain the standard WT identity for $Z_\phi$:

$$
\langle \Sigma_\phi \rangle_{\phi, J} = Z_\phi^{-1}[J] \int D\phi \ J \cdot R[\phi] \ \exp (-S[\phi] + J \cdot \phi)
= Z_\phi^{-1}[J] \left( J \cdot R[J] \right) Z_\phi[J] = 0.
$$

(2.16)

Equation (2.16) may be rewritten as the WT identity for the cutoff theory by using Eq. (2.12),

$$
\langle \Sigma_\Phi \rangle_{\Phi, K} - 1 J = Z_\Phi^{-1} J \cdot R[J] Z_\Phi[J] = Z_\Phi^{-1} (N^{-1}_J J \cdot R[J] N J Z_\Phi[J]).
$$

(2.17)

We expect that the last expression in the above can be written as the expectation value of an operator in the cutoff theory: its vanishing is a consequence of the symmetry of the original theory. Therefore, the operator is appropriately regarded as the “WT operator”. We denote it by $\Sigma_\Phi$ so that

$$
\langle \Sigma_\Phi \rangle_{\Phi, K} = 0.
$$

(2.18)

In the next subsection, we obtain the WT operator explicitly for the QED with a momentum cutoff.

2.2. WT identity for the cutoff QED

In addition to the gauge and Dirac fields $\{A_\mu, \psi, \bar{\psi}\}$, we consider, for the BRS symmetry, the (non-interacting) ghost and anti-ghost $\{c, \bar{c}\}$ as well as the auxiliary field $B$. Thus we have $\phi^A = \{A_\mu, B, c, \bar{c}, \psi, \bar{\psi}\}$ and the corresponding sources $J_A = \{J_\mu, J_B, J_c, J_{\bar{c}}, J_\psi, J_{\bar{\psi}}\}$.

The action is given as

$$
S[\phi] = \frac{1}{2} \phi \cdot D \cdot \phi + S_I[\phi].
$$

(2.19)

The free part is

$$
\frac{1}{2} \phi \cdot D \cdot \phi = \int_k \left[ \frac{1}{2} A_\mu(-k)(k^2 \delta_{\mu\nu} - k_\mu k_\nu)A_\nu(k) + \bar{c}(-k)ik^2c(k)
-B(-k)(ik_\mu A_\mu(k) + \frac{\xi}{2} B(k)) \right] + \int_p \bar{\psi}(-p)(\slashed{p} + im)\psi(p),
$$

(2.20)

* We assume the presence of BRS invariant regularization scheme such as the dimensional regularization in order for the $Z_\phi$ theory to be well-defined. However, the knowledge of the $Z_\phi$ theory is only used as the boundary condition for the $Z_\Phi$ theory at $\Lambda \to \infty$. 
where \( \xi \) is the gauge parameter, and \( S_I[\phi] \) gives the interaction part. We assume that the above action is invariant under the standard BRS transformation

\[
\delta A_\mu(k) = -ik_\mu c(k), \quad \delta \bar{c}(k) = iB(k), \quad \delta c(k) = \delta B(k) = 0 ,
\]
\[
\delta \psi(p) = -ie \int_k \psi(p-k) c(k), \quad \delta \bar{\psi}(-p) = ie \int_k \bar{\psi}(-p-k) c(k). \quad (2.21)
\]

The source dependent normalization factor \( N_J \) in Eq. (2.12) can be calculated explicitly as

\[
\ln N_J = \left( \frac{-\epsilon^A}{2} \right) J_A \left( \frac{1-K}{K} \right) (D^{-1})^{AB} J_B
\]
\[
= \int_k \left( \frac{1-K}{K} \right)(k) \left\{ J_\varepsilon(-k) - \frac{i}{k^2} J_\bar{c}(k) - J_B(-k) - \frac{i k_\mu}{k^2} J_\mu(k) \right\}
\]
\[
- \frac{1}{2} J_\mu(-k) \frac{1}{k^2} \left( \delta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right) J_\nu(k) \}
\]
\[
+ \int_p \left( \frac{1-K}{K} \right)(p) J\psi(-p) \frac{1}{\not{p} + im} J\bar{\psi}(p). \quad (2.22)
\]

The operator that appears in Eq. (2.17) takes the following form for QED:

\[
J \cdot R[\partial^j] = \left( J \cdot R[\partial^j] \right)_{\text{gauge}} + \left( J \cdot R[\partial^j] \right)_{\text{matter}}, \quad (2.23)
\]

where

\[
\left( J \cdot R[\partial^j] \right)_{\text{gauge}} = i \int_k \left\{ -k \cdot J(-k) \frac{\partial^l}{\partial J\varepsilon(-k)} + J_\bar{c}(-k) \frac{\partial^l}{\partial J_B(-k)} \right\}, \quad (2.24)
\]
\[
\left( J \cdot R[\partial^j] \right)_{\text{matter}} = -ie \int_{p,k} \left\{ J\psi(-p) \frac{\partial^l}{\partial J\bar{\psi}(p+k)} - J_\bar{\psi}(p) \frac{\partial^l}{\partial J\varepsilon(p+k)} \right\}. \quad (2.25)
\]

Let us now derive the WT-identity for the Wilson action (2.9) for QED. In the following, we use the same notation for the IR fields as for the original fields: \( \Phi^A = \{ A_\mu, B, c, \bar{c}, \psi, \bar{\psi} \} \). The kinetic term is given by

\[
\frac{1}{2} \Phi \cdot K^{-1} D \cdot \Phi = \int_k K^{-1}(k) \left[ \frac{1}{2} A_\mu(-k)(k^2 \delta_{\mu\nu} - k_\mu k_\nu) A_\nu(k) \right.
\]
\[\left. + \bar{c}(-k) ik^2 c(k) - B(-k)\left( ik \cdot A(k) + \frac{\xi}{2} B(k) \right) \right]
\]
\[+ \int_p K^{-1}(p) \bar{\psi}(-p)(\not{p} + im) \psi(p). \quad (2.26)
\]

It follows from Eq. (2.17) that our central task for finding \( \Sigma_\Phi \) is to compute \( Z_\Phi^{-1} N_J^{-1} J \cdot RN_J Z_\Phi \). It is easy to realize that the non-trivial deformation from the standard WT identity has two origins: 1) the normalization factor \( N_J \), and 2) the
scale factor $K^{-1}$ in the source terms $K^{-1}J \cdot \Phi$ and in the kinetic terms $\Phi \cdot K^{-1}D \cdot \Phi/2$. Now, from Eq. (2.17), we have

$$0 = Z_\Phi^{-1} \left[ N_J^{-1}(J \cdot R) N_J \right] Z_\Phi$$

$$= Z_\Phi^{-1} \left[ (J \cdot R)_{\text{gauge}} + N_J^{-1}(J \cdot R)_{\text{matter}} N_J \right] Z_\Phi. \quad (2.27)$$

The second line is a result of the fact, $(J \cdot R)_{\text{gauge}} N[J] = 0$. The matter sector which contains non-trivial contributions may be written as follows:

$$Z_\Phi^{-1}(J \cdot R)_{\text{matter}} Z_\Phi = -ie \left( \int_{p, k} \left\{ \frac{J_\psi(-p)}{K(p)} U(-p, p-k) \frac{J_\bar{\psi}(p-k)}{K(p-k)} \right. 
+ \left. J_\psi(-p) \frac{\partial^l}{\partial J_\psi(-p + k)} - J_\bar{\psi}(p) \frac{\partial^l}{\partial J_\bar{\psi}(p + k)} \right\} c(k) \right)_{\Phi, K^{-1}J}, \quad (2.28)$$

where

$$U(-p, p-k) \equiv \frac{1 - K(p-k)}{\bar{p} - k + im} K(p) - \frac{1 - K(p)}{\bar{p} + im} K(p-k). \quad (2.29)$$

Using

$$Z_\Phi^{-1} J_A Z_\Phi = \left\langle K \frac{\partial^r S}{\partial \Phi^A} \right\rangle_{\Phi, K^{-1}J}, \quad Z_\Phi^{-1} \frac{\partial^l}{\partial J_A} Z_\Phi = \left\langle K^{-1} \Phi^A \right\rangle_{\Phi, K^{-1}J}, \quad (2.30)$$

we obtain

$$\langle \Sigma_\Phi \rangle_{\Phi, K^{-1}J} = 0 \quad (2.31)$$

with

$$\Sigma_\Phi = \int_k \left\{ \frac{\partial S}{\partial A_\mu(k)} (-ik_\mu) c(k) + \frac{\partial^r S}{\partial \bar{c}(k)} i B(k) \right\}$$

$$- ie \int_{p, k} \left\{ \frac{\partial^r S}{\partial \psi(p)} \frac{K(p)}{K(p-k)} \psi(p-k) - \frac{K(p)}{K(p+k)} \bar{\psi}(-p-k) \frac{\partial^l S}{\partial \bar{\psi}(-p)} \right\} c(k)$$

$$- ie \int_{p, k} \text{tr} \left\{ \left( \frac{\partial^l S}{\partial \bar{\psi}(-p + k)} \frac{\partial^r S}{\partial \psi(p)} - \frac{\partial^l S}{\partial \bar{\psi}(-p + k)} \frac{\partial^r S}{\partial \psi(p)} \right) U(-p, p-k) \right\} c(k). \quad (2.32)$$

From the identity (2.31), we note that any correlation function with a $\Sigma_\Phi$ insertion vanishes,

$$\langle \Sigma_\Phi \Phi A_1 \Phi A_2 \ldots \Phi A_N \rangle_{\Phi, J=0} = 0. \quad (2.33)$$

Therefore, we obtain the operator identity $\Sigma_\Phi = 0$, which is the WT identity derived in Refs. 15) and 16) for the Wilson action of QED.

From Eq. (2.32), it is easy to realize that the WT identity is nothing but the BRS invariance of the action under the standard BRS transformation (2.21) as far as the gauge sector is concerned. Since the gauge sector is free, this is quite natural.
Though the matter contribution to $\Sigma_\Phi$ is slightly complicated, we will see presently that it also allows an interpretation as a change of the action under some symmetry transformation. We may rewrite the matter contributions in $\Sigma_\Phi$ as

$$
\begin{align*}
&ie \int_{p, k} \frac{\partial^r S}{\partial \psi(p)} c(k) \left\{ \frac{K(p)}{K(p-k)} \psi(p-k) - U(-p, p-k) \frac{\partial^l S}{\partial \psi(-p + k)} \right\} \\
&-ie \int_{p, k} \left\{ \frac{K(p)}{K(p+k)} \psi(-p - k) \right\} c(k) \frac{\partial^l S}{\partial \psi(-p)} \\
&+ie \int_{p, k} tr \frac{\partial^l \partial^r S}{\partial \psi(-p + k) \partial \psi(p)} U(-p, p-k) c(k).
\end{align*}
$$

(2.34)

From the first two lines of Eq. (2.34), we read off the BRS transformation of the fermion. Including the transformation for the gauge sector, we find

$$
\delta A_\mu(k) = -ik_\mu c(k), \quad \delta c(k) = iB(k), \quad \delta \bar{c}(k) = \delta B(k) = 0
$$

$$
\delta \psi(p) = ie \int_k c(k) \left\{ \frac{K(p)}{K(p-k)} \psi(p-k) - U(-p, p-k) \frac{\partial^l S}{\partial \psi(-p + k)} \right\},
$$

$$
\delta \bar{\psi}(-p) = ie \int_k \left\{ \frac{K(p)}{K(p+k)} \bar{\psi}(-p - k) \right\} c(k).
$$

(2.35)

With this transformation (2.35), $\Sigma_\Phi$ is now written as

$$
\Sigma_\Phi = \frac{\partial^r S}{\partial \Phi^A} \delta \Phi^A + ie \text{tr} \left( \frac{\partial^l \partial^r S}{\partial \psi \partial \bar{\psi}} U \right) c .
$$

(2.36)

The second term can be interpreted as the Jacobian factor associated with the BRS transformation (2.35).

We have three remarks on the BRS transformation (2.35). (i) It depends on the Wilson action $S[\Phi]$, and therefore it is non-linear. (ii) It is not unique: the non-linear contribution could appear both in $\delta \psi$ and $\delta \bar{\psi}$. (iii) The nilpotency is lost on $\psi$, though it holds for other fields.

Obviously, the nilpotency is the most important property of the BRS symmetry. It is desirable to elevate Eq. (2.35) to the one with nilpotency. In order to achieve this, we need to find out a way to take care of the Jacobian factor appearing in Eq. (2.36). This can be realized with the BV anti-field formalism.

**§3. Antifield formalism and QME**

Let us first explain the antifield formalism briefly. For each IR field $\Phi^A$, we introduce its antifield $\Phi_A^*$ with the opposite Grassmann parity, $\epsilon(\Phi_A^*) = \epsilon(\Phi^A) + 1$,

$$
\Phi_A^* = \{ A_\mu^*, B^*, c^*, \bar{c}^*, \psi^*, \bar{\psi}^* \}.
$$

(3.1)

The canonical structure of fields and their anti-fields is specified by the anti-bracket. For any pair of operators, $X$ and $Y$, it is defined as

$$
(X, Y) \equiv \frac{\partial^r X}{\partial \Phi_A} \frac{\partial^l Y}{\partial \Phi_A^*} - \frac{\partial^r X}{\partial \Phi_A^*} \frac{\partial^l Y}{\partial \Phi_A}.
$$

(3.2)
Consider a gauge theory with an action $S[\Phi, \Phi^*]$ and calculate the operator defined as

$$\Sigma[\Phi, \Phi^*] \equiv \frac{1}{2} (S, S) - \Delta S,$$

(3.3)

where

$$\Delta \equiv (-)^{\epsilon A+1} \frac{\partial^r}{\partial \Phi^A} \frac{\partial^r}{\partial \Phi^*_A}.$$

(3.4)

The equation $\Sigma[\Phi, \Phi^*] = 0$ is the quantum master equation of the BV formalism. The action satisfying $\Sigma[\Phi, \Phi^*] = 0$ is called a quantum master action, or simply a master action. Later, we denote a master action as $S_M[\Phi, \Phi^*].$

Our aim in this section is to construct a master action, by using the WT identity (2.32) for our Wilson action $S[\Phi].$

In a standard gauge theory with a gauge-fixed action $S[\phi]$ and a nilpotent BRS transformation $\delta \phi^A,$ the master action is $S[\phi] + \phi^*_A \delta \phi^A,$ linear in anti-fields. To start with, let us try an extended action linear in the anti-fields $\Phi^*$:

$$S_{\text{lin}}[\Phi, \Phi^*] = S[\Phi] + \phi^*_A \delta \Phi^A.$$  

This action, however, does not satisfy the QME: $\Sigma[\Phi, \Phi^*] \propto c \bar{c} \psi U U.$ To cancel this contribution, one should add suitable terms $S_{\text{quad}}[\Phi, \Phi^*]$ quadratic in the anti-fields, and so on. After several trials, we realize that this expansion w.r.t. antifields is the Taylor expansion of the action, where $\bar{\psi}$ is replaced by $\bar{\psi} - i e \bar{c} \psi^* c U.$

Let us assume this form for the master action and prove that it indeed satisfies the QME. Our master action is

$$S_M[\Phi, \Phi^*] = S[\Phi'] + \int_k \left( \Phi^*_A (-k)(-i)k_\mu c(k) + i c^*(-k)B(k) \right)$$

$$+ i e \int_{p,k} \left( \psi^*(-p) \frac{K(p)}{K(p-k)} c(k) \psi(p-k) + \bar{\psi}(-p-k)c(k) \frac{K(p)}{K(p+k)} \bar{\psi}^*(p) \right).$$

(3.5)

Here we have introduced the shifted field $\bar{\psi}'$ and $\Phi'^A$

$$\bar{\psi}'(-p) \equiv \bar{\psi}(-p) - i e \int_k \psi^*(-p-k)c(k) U(-p-k, p),$$

$$\Phi'^A = \{ A_\mu, B, c, \bar{c}, \psi, \bar{\psi}' \}.$$  

(3.6)

In Eq. (3.5), note that the second term in $\delta \psi$ of Eq. (2.35) is absorbed into $S[\Phi']$ due to the shift.

In proving the QME for $S_M$, it is important that the action $S[\Phi]$ satisfies the WT identity. For convenience, we rewrite the identity for $S[\Phi']$ with the shifted fields (3.6). We obtain

$$\int_k \left\{ \frac{\partial S[\Phi']}{\partial A_\mu(k)} (-i k_\mu) c(k) + \frac{\partial^r S[\Phi']}{\partial \bar{c}(k)} i B(k) \right\}$$

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Combining all the terms in Eqs. (3.8) – (3.10), we find
\[ +ie \int_{p,k} \frac{\partial^r S[\Phi']}{\partial \psi(p)} c(k) \left\{ \frac{K(p)}{K(p - k)} \psi(p - k) - U(-p - k, p + k) \frac{\partial^l S[\Phi']}{\partial \psi(-p + k)} \right\} \]
\[ +ie \int_{p,k} \frac{\partial^r S[\Phi']}{\partial \psi(p + k)} K(p) \]
\[ \times \left\{ \bar{\psi}(-p - k) - ie \psi^*(-p - k - l)c(l)U(-p - k - l, p + k) \right\} c(k) \]
\[ +ie \int_{p} \text{tr} \left( \frac{\partial^l \partial^r S[\Phi']}{\partial \psi(-p) \partial \psi(p + k)} U(-p - k, p) \right) c(k) = 0. \] (3.7)

Now it is straightforward to verify that the action (3.5) satisfies the QME. Here we calculate the contributions to \((S_M, S_M)/2\) from the matter sector.

\[ \int_{p} \frac{\partial^r S_M}{\partial \psi(p)} \frac{\partial^l S_M}{\partial \psi^*(p)} = ie \int_{p,k} \frac{\partial^r S[\Phi']}{\partial \psi(-p)} \frac{K(p)}{K(p + k)} \bar{\psi}(-p - k) c(k), \] (3.8)
\[ \int_{p} \frac{\partial^r S_M}{\partial \psi(p)} \frac{\partial^l S_M}{\partial \psi^*(p)} = ie \int_{p,l} \frac{\partial^r S[\Phi']}{\partial \psi(p)} c(l) \left( \frac{K(p)}{K(p - l)} \psi(p - l) \right) \]
\[ -U(-p, p - l) \frac{\partial^l S[\Phi']}{\partial \psi(-p + l)} \right) + e^2 \int_{p,k,l} \frac{K(p + k + l)}{K(p + l)} c(k)c(l)\psi^*(-p - k - l)U(-p - l, p) \frac{\partial^l S[\Phi']}{\partial \psi(-p)} \] . (3.9)

The quantum term may be calculated as
\[ \Delta \Phi S_M = -ie \int_{p,k} \text{tr} \left( \frac{\partial^l \partial^r S_M}{\partial \psi(-p) \partial \psi(p + k)} U(-p - k, p) \right) c(k). \] (3.10)

Combining all the terms in Eqs. (3.8) – (3.10), we find
\[ \Sigma[\Phi, \Phi^*] \equiv \frac{1}{2} (S_M, S_M)_{\Phi} - \Delta \Phi S_M = 0 , \] (3.11)

thanks to the identity (3.7). Therefore the action \(S_M\) defined by Eq. (3.5) is indeed a master action. Note that the same \(e^2\) term appears in both Eqs. (3.9) and (3.7).

In summary, we have observed that the \((\partial S/\partial \bar{\psi})(\partial S/\partial \psi)\) term of \(\Sigma_{\Phi}\) is absorbed into the classical part \((\partial S_M/\partial \psi)(\partial S_M/\partial \psi^*)\) of the QME, corresponding to the shift of \(\bar{\psi}\). Likewise, the \(\partial \delta S/\partial \psi \partial \bar{\psi}\) term of \(\Sigma_{\Phi}\) turns into the jacobian associated with the BRS transformation. The shift of \(\bar{\psi}\) needed for constructing \(S_M\) from \(S\) now appears quite natural.

In the antifield formalism, the “quantum” BRS transformation\(^{10}\) is defined by
\[ \delta_Q X \equiv (X, S_M) - \Delta X \] (3.12)

for any operator \(X\). For the fields in QED, it takes the following form:
\[ \delta_Q A_\mu(k) = -ik_\mu c(k), \quad \delta_Q \bar{c}(k) = iB(k), \quad \delta_Q c(k) = \delta_Q B(k) = 0 , \]
\[ \delta_Q \psi(p) = ie \int_k c(k) \left\{ \frac{K(p)}{K(p - k)} \psi(p - k) - U(-p, p - k) \frac{\partial S_M}{\partial \psi(-p + k)} \right\}, \]
\[ \delta_Q \bar{\psi}(-p) = ie \int_k \left\{ \frac{K(p)}{K(p + k)} \bar{\psi}(-p - k) \right\} c(k). \]  
\hspace{1cm} (3.13)

This transformation has the same form as Eq. (2.35). However, the action on the r.h.s. of \( \delta_Q \psi \) is now \( S_M[\Phi, \Phi^*] \), and the BRS transformation has a non-trivial antifield dependence. The BRS transformation in the gauge sector is quite simple, while that of the matter sector is rather complicated.

The quantum BRS transformation is nilpotent if and only if the QME holds:
\[ \delta^2_Q X = (X, \Sigma[\Phi, \Phi^*]) = 0. \]  
\hspace{1cm} (3.14)

In other words, the QME enables us to define the nilpotent BRS transformation. This should be compared with the classical counterpart, \( \delta X \equiv (X, S_M) \) which does not vanish due to the lack of the Jacobian factor,
\[ \delta^2 X = \frac{1}{2} (X, (S_M, S_M)) \neq 0. \]  
\hspace{1cm} (3.15)

§4. Polchinski flow equation for the master action and its BRS invariance

In this section, we derive the Polchinski flow equation for our master action and show its BRS invariance.

Let us begin with the well-known generic result on the Polchinski flow equation for the Wilson action \( S[\Phi] \) without antifields. It is given by
\[ \partial_t S[\Phi] = - \int_p \Phi^A(p) (K^{-1} \dot{K})(p) \frac{\partial S}{\partial \Phi^A(p)} \]
\[ + \frac{1}{2} \int_p \left[ \frac{\partial S}{\partial \Phi^A(p)} (\dot{K}D^{-1}(p))^{AB} \frac{\partial S}{\partial \Phi^B(-p)} - (-)^{\epsilon A} (\dot{K}D^{-1}(p))^{AB} \frac{\partial S}{\partial \Phi^B(-p)} \right] \]  
\hspace{1cm} (4.1)

up to terms independent of fields. Here, we use a dimensionless parameter \( t = \log(\Lambda/\mu) \) and \( \dot{K} = \partial_t K \).

The flow equation for our master action \( S_M[\Phi, \Phi^*] \) of QED can be obtained through a straightforward calculation. From the definition (3.5), we have
\[ \partial_t S_M[\Phi, \Phi^*] = \partial_t S[\Phi] |_{\Phi = \Phi^*} - ie \int_{p,k} \psi^*(-p - k)c(k) \partial_t U(-p - k, p) \frac{\partial S[\Phi]}{\partial \psi(-p)} \]
\[ + \int_{p,k} c(k) \left[ \psi^*(-p)c(k) U(-p - k, p) \right] . \]  
\hspace{1cm} (4.2)

In replacing \( S[\Phi] \) by \( S_M[\Phi, \Phi^*] \), one takes account of the following points specific to the abelian nature of QED: (1) the ghost is a free field, and the BRS transformation
for the gauge and ghost sector \( \{ A_\mu, B, c, \bar{c} \} \) is cutoff independent; (2) the shift of the fermionic field \( \bar{\psi} \rightarrow \bar{\psi} - i e c \psi^* U \) generates a non-trivial antifield dependence in the flow equation. We also note the following identity for the matrix \( U \):

\[
\partial_t U(-p - k, p) = \left( \frac{\dot{K}(p + k)}{K(p + k)} + \frac{\dot{K}(p)}{K(p)} \right) U(-p - k, p) + \frac{\dot{K}(p + k)}{K(p + k)} \frac{1}{\dot{p} + k + im} - \frac{\dot{K}(p)}{K(p)} K(p + k) \frac{1}{\dot{p} + im}.
\]

(4.3)

Then, putting altogether, we obtain

\[
\partial_t S_M[\Phi, \Phi^*] = \frac{1}{2} \int \frac{K(p)}{p^2} \left( \delta_{\mu\nu} - (1 - \xi) \frac{p_\mu p_\nu}{p^2} \right) \left( \frac{\partial S_M}{\partial A_\mu(p)} \cdot \frac{\partial S_M}{\partial A_\nu(-p)} - \frac{\partial^2 S_M}{\partial A_\mu(p) \partial A_\nu(-p)} \right)
\]

\[
- \int \frac{K(p)}{K(p)} \left[ A_\mu(p) \left( \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) \frac{\partial S_M}{\partial A_\nu(p)} + i p_\nu \xi B(p) \frac{\partial S_M}{\partial A_\nu(p)} \right]
\]

\[
+ \int \frac{K(p)}{K^2(p)} \left[ B(-p) (i p \cdot A(p) + \xi B(p)) - i p^2 \bar{c}(-p) c(p) \right]
\]

\[
+ \int \frac{K(p)}{K(p)} \left[ \frac{\partial^r S_M}{\partial \psi(p)} \frac{1}{\dot{p} + im} \frac{\partial S_M}{\partial \psi(-p)} + \text{tr} \left( \frac{1}{\dot{p} + im} \cdot \frac{\partial \partial^r S_M}{\partial \psi(p)} \right) \right]
\]

\[
- \int \frac{K(p)}{K(p)} \left[ \bar{\psi}(-p) \frac{\partial S_M}{\partial \psi(p)} + \frac{\partial^r S_M}{\partial \psi(p)} \psi(p) - \psi^*(-p) \frac{\partial^r S_M}{\partial \psi^*(p)} + \frac{\partial S_M}{\partial \psi^*(-p)} \right]
\]

\[
- ie \int \frac{K(p)}{K(p)} K(p - k) c(k) \times \left( \psi^*(-p) \frac{1}{\dot{p} + im} \frac{\partial S_M}{\partial \psi(-p + k)} - \frac{\partial^r S_M}{\partial \psi(p)} \frac{1}{\dot{p} + im} \bar{\psi}^*(p - k) \right) \right) \]

(4.4)

Thanks to the abelian nature of the theory, no antifields appear in the gauge and ghost sector. The fermionic sector has explicit antifield dependence.

Let us discuss the BRS invariance of the flow equation (4.4). For the RG flow of the WT operator, we obtain the relation,

\[
\partial_t \Sigma[\Phi, \Phi^*] = (\partial_t S_M, S_M) - \Delta \partial_t S_M = \delta_Q \partial_t S_M = 0,
\]

(4.5)

which implies that the flow itself should be written as a quantum BRS transform of something. Actually, up to the QME, we find\(^{10}\)

\[
\partial_t S_M = -\delta_Q G,
\]

(4.6)

where \( G \) is the generator of a canonical transformation

\[
G = G_1 + G_2 + G_3
\]

(4.7)

that has three parts:

\[
G_1 \equiv \int_k A^*_\mu(-k) \left[ \frac{1}{2} \frac{\dot{K}(k)}{k^2} \left( \delta_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right) \frac{\partial S_M}{\partial A_\nu(-k)} \right]
\]
Quantum Master Equation for QED in Exact Renormalization Group

\[ \dot{K}(k) \frac{i k \mu}{K(k)} \frac{k^2}{k^2} (ik \cdot A(k) - \xi B(k)) \]  
\[ G_2 \equiv - \int \frac{\dot{K}(k)}{K(k)} \left[ A^*_\mu(-k) A_\mu(k) + B^*(-k) B(k) \right. 
+ c^*(-k) \bar{c}(k) + \psi^*(-k) \psi(k) + \bar{\psi}(k) \bar{\psi}^*(-k) \left. \right], \]  
\[ G_3 \equiv \int_p \psi^*(-p) \frac{\dot{K}(p)}{p + im} \left[ \partial_t S_M + \frac{ie}{K(p)} \int_k \bar{c}(k) K(p - k) \bar{\psi}^*(p - k) \right]. \]

§5. Summary and discussion

In this paper, we have rederived the WT identity for the Wilson action of QED using a functional method and shown that it can be lifted to a QME in the BV antifield formalism. The master action, our formal solution to the QME, generically has non-linear but simple anti-field dependence which appears merely as a shift of field variables. We have also found that the master action is not unique, and that it can be deformed by canonical transformations in the space of fields and antifields. No deformation can remove the non-linear anti-field dependence in the master action. We believe that the non-linear anti-field dependence is an inherent feature of any local symmetries in cutoff field theories.

We have also derived an extended flow equation for the master action. Since the master action is determined up to canonical transformations, the flow equation is not unique, and can be modified by canonical transformations.

A pair of fundamental equations, the WT identity \( \Sigma \Phi = 0 \) and the Polchinski equation \( \partial_t S[\Phi] - F[\Phi] = 0 \), can be interpreted as a gauge fixed version of the QME and extended flow equation:

\[ \Sigma \Phi = \Sigma[\Phi, \Phi^*]|_{\Phi^* \rightarrow 0} = 0, \]
\[ \partial_t S[\Phi] - F[\Phi] = (\partial_t S_M[\Phi, \Phi^*] - F[\Phi, \Phi^*])|_{\Phi^* \rightarrow 0} = 0. \]

It should be emphasized that the QME plays a crucial role not only in constructing a nilpotent BRS transformation, but also in showing the BRS invariance of the extended flow equation. These properties imply that the exact gauge symmetry does exist in the Wilson action of QED despite the presence of a finite momentum cutoff.

A perturbative solution to the WT identity \( \Sigma \Phi = 0 \) and the Polchinski equation \( \partial_t S[\Phi] - F[\Phi] = 0 \) has been obtained in Refs. 15) and 16). It is straightforward to find the corresponding perturbative solution to the QME: \( \Sigma[\Phi, \Phi^*] = 0 \) and the extended flow equation: \( \partial_t S_M[\Phi, \Phi^*] - F[\Phi, \Phi^*] = 0 \).

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