POLYAK-LOJASIEWICZ INEQUALITY ON THE SPACE OF MEASURES AND CONVERGENCE OF MEAN-FIELD BIRTH-DEATH PROCESSES

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Abstract. The Polyak-Lojasiewicz inequality (PLI) in \( \mathbb{R}^d \) is a natural condition for proving convergence of gradient descent algorithms [13]. In the present paper, we study an analogue of PLI on the space of probability measures \( \mathcal{P}(\mathbb{R}^d) \) and show that it is a natural condition for showing exponential convergence of a class of birth-death processes related to certain mean-field optimization problems. We verify PLI for a broad class of such problems for energy functions regularised by the KL-divergence.

1. Introduction

Consider a classical optimization problem, where one is interested in finding a global minimum of a differentiable function \( f : \mathbb{R}^d \to \mathbb{R} \). A natural condition on \( f \), under which the gradient descent algorithm has a geometric convergence rate to \( \min_{y \in \mathbb{R}^d} f(y) \), is the Polyak-Lojasiewicz inequality (PLI)

\[
\frac{1}{\kappa} \| \nabla f(x) \|^2 \geq f(x) - \min_{y \in \mathbb{R}^d} f(y),
\]

required to hold with a positive constant \( \kappa > 0 \), for all \( x \in \mathbb{R}^d \) (see [13] and the references therein, or [5][4] for other variants of Lojasiewicz inequalities). It is easy to see that when \( f \) is strictly convex, (1.1) holds, but the converse is not necessarily true.

In the present paper we are concerned with an optimization problem on the space of probability measures \( \mathcal{P}(\mathbb{R}^d) \). We consider a function \( V : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \), and we want to find a minimizing measure \( m^* \in \mathcal{P}(\mathbb{R}^d) \). Such optimization problems have attracted considerable attention in recent years, see e.g. [9][16][12][18][7]. In this setting, there exist multiple different choices of flows of probability measures \( (m_t)_{t \geq 0} \) that can serve as analogues of the gradient descent algorithm in \( \mathbb{R}^d \), as well as multiple different choices of conditions on \( V \) analogous to (1.1) that can be used to prove convergence of such flows.

The main example of \( V \) considered in this paper is an energy function regularised by the KL-divergence. Consider \( F : \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \) (which can be non-linear) and a probability measure \( \pi(dx) \propto e^{-U(x)}dx \) with a potential \( U : \mathbb{R}^d \to \mathbb{R} \). For any \( \sigma \geq 0 \), we put

\[
V^\sigma(m) = F(m) + \frac{\sigma^2}{2} \text{KL}(m|\pi), \quad m \in \mathcal{P}(\mathbb{R}^d),
\]

where for any \( m \in \mathcal{P}(\mathbb{R}^d) \),

\[
\text{KL}(m|\pi) = \begin{cases} 
\int_{\mathbb{R}^d} \log \left( \frac{m(x)}{\pi(x)} \right) m(x) dx & m \text{ absolutely continuous with respect to } \pi, \\
\infty & \text{otherwise}.
\end{cases}
\]

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It is known (see e.g. Proposition 2.5 in [12]) that $V^\sigma$ is minimized by a measure $m^{\sigma,*} \in \mathcal{P}(\mathbb{R}^d)$ satisfying
\begin{equation}
(1.3) 
  m^{\sigma,*}(x) = \frac{1}{Z} \exp \left( -\frac{2}{\sigma^2} \left( \frac{\delta F}{\delta m}(m^{\sigma,*}, x) + U(x) \right) \right),
\end{equation}
where $Z$ is the normalising constant, and for any $m \in \mathcal{P}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, by $\frac{\delta F}{\delta m}(m, x)$ we denote the flat derivative of $F$ with respect to $m$, in the direction of $x \in \mathbb{R}^d$, evaluated at $m$. For any $m, m' \in \mathcal{P}(\mathbb{R}^d)$, the function $\frac{\delta F}{\delta m} : \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$ satisfies
\begin{equation}
F(m') - F(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta F}{\delta m}(m + \lambda(m' - m), x) (m' - m) \, (dx) \, d\lambda.
\end{equation}

See Appendix 5 for more details on flat derivatives. This notion of derivative appears in the literature under several different names, including the linear functional derivative (see Section 5.4.1 in [6]) or the first variation [2]. It is important to note that $\frac{\delta F}{\delta m}$ is defined only up to a constant, i.e., for any $C \in \mathbb{R}$, the function $\frac{\delta F}{\delta m} + C$ is also a flat derivative of $F$. Everywhere in this paper we will adopt a normalizing convention requiring $\int_{\mathbb{R}^d} \frac{\delta F}{\delta m}(m, x) m(dx) = 0$, which then makes the choice of the constant unique.

The objective of this work is to identify a flow of measures $(m_t)_{t \geq 0}$ such that $V^\sigma(m_t) \to V^\sigma(m^{\sigma,*})$ as $t \to \infty$, as well as conditions that ensure that this convergence is exponential. To this end, we equip the space $\mathcal{P}(\mathbb{R}^d)$ with a suitable distance function $d : \mathcal{P}(\mathbb{R}^d) \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ and consider a corresponding gradient flow, where the form of the flow is dictated by the choice of $d$. Our main focus is on the Fisher-Rao metric.

**Fisher-Rao Gradient Flow.** Let $\mathcal{P}_{ac}(\mathbb{R}^d)$ be the space of probability measures on $\mathbb{R}^d$ that are absolutely continuous with respect to the Lebesgue measure. Then the Fisher-Rao distance between $\mu_0, \mu_1 \in \mathcal{P}_{ac}(\mathbb{R}^d)$ is defined by
\begin{equation}
\text{FR}(\mu_0, \mu_1) = \int_{\mathbb{R}^d} \left| \sqrt{\mu_0(x)} - \sqrt{\mu_1(x)} \right|^2 \, dx.
\end{equation}

One can also consider a dynamic representation of the Fisher-Rao metric (see e.g. Section 2.2 in [11] and the references therein), which, for any $\mu_0, \mu_1 \in \mathcal{P}_{ac}(\mathbb{R}^d)$ states that
\begin{equation}
\text{FR}(\mu_0, \mu_1) = \inf_{\nu \in L^2([0,1] \times \mathcal{P}_{ac}(\mathbb{R}^d))} \left\{ \int_0^1 \int_{\mathbb{R}^d} |\nu_s|^2 \mu_s(dx) \, ds : \text{s.t. } \partial_s \mu_s = \nu_s \mu_s \right\}.
\end{equation}

This result tells us that measures in the space $(\mathcal{P}_{ac}(\mathbb{R}^d), \text{FR})$ are transported along curves prescribed by a birth-death (or reaction) equation. The main focus of this work is to identify a corresponding Polyak-Lojasiewicz inequality from which we can deduce the exponential convergence to $m^{\sigma,*}$ of the flow $(m_t)_{t \geq 0}$ described by the birth-death equation
\begin{equation}
(1.4) \quad \partial_t m_t(x) = -a(m_t, x)m_t(x), \quad a(m, x) := \frac{\delta F}{\delta m}(m, x) + \log \left( \frac{m(x)}{\pi(x)} \right) - \text{KL}(m|\pi).
\end{equation}

Note that the map $(m, x) \mapsto a(m, x)$ formally corresponds to $\frac{\delta V^\sigma}{\delta m}(m_t, \cdot)$ which may not exist since the KL-divergence is only lower semi-continuous. The map $(m, x) \mapsto a(m, x)$ is a well-defined function under the assumption of flat-differentiability of $F$ (note that $\text{KL}(m|\pi)$ in [1.4]) corresponds to the normalizing constant needed in our normalizing convention mentioned above).
To see why the particular form of \((m, x) \mapsto a(m, x)\) in (1.4) is a good choice one needs to show that \(t \mapsto V^\sigma(m_t)\) is differentiable and that
\[
\partial_t (V^\sigma(m_t) - V^\sigma(m^{\ast\ast})) = \int_{\mathbb{R}^d} \left( \frac{\delta F}{\delta m}(m_t, x) + \log \left( \frac{m_t(x)}{\pi(x)} \right) - \text{KL}(m_t | \pi) \right) m_t(x) dx,
\]
(1.5)
\[
= -\int_{\mathbb{R}^d} |a(m_t, x)|^2 m_t(x) dx.
\]
One can then deduce that \(V^\sigma(m_t)\) converges to a local minimum. The Polyak-Lojasiewicz condition that implies the exponential convergence of \(V^\sigma(m_t)\) to \(V^\sigma(m^{\ast\ast})\), requires that there exists a constant \(\kappa > 0\) such that for any \(m^\ast \in \arg\min_m V^\sigma(m)\) and any \(m \in \mathcal{P}(\mathbb{R}^d)\),
\[
\frac{1}{\kappa} \|a(m, \cdot)\|^2_{L^2(m)} \geq V^\sigma(m) - V^\sigma(m^\ast).
\]
(1.6)
We call (1.6) the flat Polyak-Lojasiewicz condition, since the function \(a(m, x)\) formally corresponds to the flat derivative of \(V^\sigma\), as explained above. With such an inequality at hand, one immediately sees that
\[
\partial_t (V^\sigma(m_t) - V(m^{\ast\ast})) \leq -\kappa (V^\sigma(m_t) - V(m^{\ast\ast})).
\]

The main contributions of this work are:
\begin{itemize}
  \item We establish the existence and uniqueness of the non-linear infinite dimensional birth-death flow (1.4).
  \item We demonstrate that \(t \mapsto V^\sigma(m_t)\) is differentiable, which implies that the energy dissipation equality (1.5) holds.
  \item We show that for a large class of energy functions \(V^\sigma\), the Polyak-Lojasiewicz condition (1.6) can be verified under relatively mild assumptions.
\end{itemize}

The remaining part of the paper is organised as follows. In Section 2 we formulate our main results and the assumptions we work with. In Section 2.1 we present a result on the verification of the flat Polyak-Lojasiewicz inequality (1.6) for general energy functions (not necessarily of the form (1.2)) under certain quadratic growth conditions. This section is of independent interest and can be seen as a counterpart of the results that were proved in \textit{13} in \(\mathbb{R}^d\), or the results that were proved on the space of measures in \textit{4} for a quadratic growth condition with respect to the \(L^2\)-Wasserstein distance (while we work with the KL-divergence and the \(\chi^2\)-divergence). In Section 2.2 we review the literature and we present a more in-depth discussion on the motivation for studying the gradient flow (1.4). In Section 3 we prove our main results on the existence of the gradient flow and the differentiability of the energy function. Finally, the Appendix includes some general auxiliary results on comparing different \(f\)-divergences, adapted from [10] and a brief overview of the notion of the flat derivative.

2. Main Results

We work with the energy function \(V^\sigma: \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}\) given by (1.2), for some possibly non-linear \(F: \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}\) and \(\sigma \geq 0\). We have the following assumptions on \(F\).

\textbf{Assumption 1.} Suppose \(F\) has the first and the second order flat derivatives \((\delta F/\delta m): \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}\) and \((\delta^2 F/\delta m^2): \mathcal{P}(\mathbb{R}^d) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}\), respectively. Furthermore, suppose that
(1) $F$ is convex, i.e., for any $m, m' \in \mathcal{P}(\mathbb{R}^d)$ we have

$$F(m) - F(m') \leq \int_{\mathbb{R}^d} \frac{\delta F}{\delta m}(m, x) (m - m)(dx).$$

(2) There exists a constant $C > 0$ such that for all $m \in \mathcal{P}(\mathbb{R}^d)$ and for all $x \in \mathbb{R}^d$ we have

$$\left| \frac{\delta F}{\delta m}(m, x) \right| \leq C.$$

(3) There exists a constant $C_2 > 0$ such that for all $m \in \mathcal{P}(\mathbb{R}^d)$ and for all $x, y \in \mathbb{R}^d$ we have

$$\left| \frac{\delta^2 F}{\delta m^2}(m, x, y) \right| \leq C_2.$$

Furthermore, suppose we have absolutely continuous probability measures $\pi, m_0 \in \mathcal{P}(\mathbb{R}^d)$ such that $\pi(dx) \propto e^{-U(x)}dx$ for a potential $U : \mathbb{R}^d \to \mathbb{R}$ and the following conditions are satisfied.

Assumption 2. Suppose $m_0 \in \mathcal{P}(\mathbb{R}^d)$ is absolutely continuous and comparable with $\pi$ in the following sense.

(1) There exists a constant $r > 0$ such that

$$\inf_{x \in \mathbb{R}^d} \frac{m_0(x)}{\pi(x)} \geq r.$$

(2) There exists a constant $R > 1$ such that

$$\sup_{x \in \mathbb{R}^d} \frac{m_0(x)}{\pi(x)} \leq R.$$

Note that here $\pi$ is just a reference measure, and the actual measure of interest (the minimizer of $V^\sigma$) is given implicitly by the following equation

$$m^{\sigma,*}(x) = \frac{1}{Z} \exp \left( -\frac{2}{\sigma^2} \left( \frac{\delta F}{\delta m}(m^{\sigma,*}, x) + U(x) \right) \right),$$

where $Z$ is the normalizing constant. We immediately observe that, under condition (2.2), conditions (2.4) and (2.5) together are equivalent to assuming that there exist constants $\tilde{r} > 0, \tilde{R} > 1$ such that for all $x \in \mathbb{R}^d$,

$$\tilde{r} \leq \frac{m_0(x)}{m^{\sigma,*}(x)} \leq \tilde{R}.$$

As we will explain in more detail in Subsection 2.2, Assumption 2 is a kind of ”warm start” condition that says that once we fix the reference measure $\pi$ in (1.2), the initial measure $m_0$ of our gradient flow should be comparable to $\pi$. We have the following result.

Theorem 2.1. Under Assumption 1 and condition (2.5) from Assumption 2, equation (1.4) has a unique solution. Moreover, for $t \geq 0$,

$$\text{KL}(m_t|\pi) \leq 2 \ln \tilde{R} + \frac{4C}{\sigma^2}.$$

As we explained in the discussion in Section 1, the crucial property needed for showing the exponential convergence of $(m_t)_{t \geq 0}$ is the differentiability of the energy function along the gradient flow.
Theorem 2.2. Under Assumption 7 and condition (2.5) from Assumption 2, the function \( t \mapsto V^\sigma(m_t) \) is differentiable and

\[
(2.8) \quad \partial_t V^\sigma(m_t) = \int_{\mathbb{R}^d} \left[ \frac{\delta F}{\delta m}(m_t, x) + \log \left( \frac{m_t(x)}{\pi(x)} \right) - \text{KL}(m_t|m) \right] \partial_t m_t(x) dx.
\]

Finally, we have the following Polyak-Łojasiewicz inequality.

Theorem 2.3. Under Assumptions 7 and 2, the flow \((m_t)_{t \geq 0}\) solving (1.4) satisfies

\[
(2.9) \quad V^\sigma(m_t) - V^\sigma(m^{*, \sigma}) \leq \frac{4\bar{R}}{\sigma^2 F} \|a(m_t, \cdot)\|_{L^2(m_t)}^2
\]

for all \( t > 0 \), where the constants \( \bar{r} \) and \( \bar{R} \) are determined by (2.6).

Hence, based on the discussion in Section 1, we have the following result.

Corollary 2.4. Under Assumptions 7 and 2, the flow \((m_t)_{t \geq 0}\) solving (1.4) satisfies

\[
V^\sigma(m_t) - V^\sigma(m^{*, \sigma}) \leq (V^\sigma(m_0) - V^\sigma(m^{*, \sigma})) e^{-\kappa t},
\]

where \( \kappa = \sigma^2 \bar{r}/4\bar{R} \).

The proofs of all the results formulated above are postponed to Section 3.

In Subsection 2.1 we will explain how to deduce the Polyak-Łojasiewicz inequality (2.9) for a general class of energy functions that satisfy a certain growth condition with respect to the KL-divergence. We will now formulate a lemma where we verify that growth condition for the energy function \( V^\sigma \) given by (1.2).

Lemma 2.5. For \( V^\sigma \) given by (1.2), if \( F \) is convex, then \( V^\sigma \) satisfies the quadratic growth condition

\[
V^\sigma(m) - V^\sigma(m^{*, \sigma}) \geq \frac{\sigma^2}{2} \text{KL}(m|m^{*, \sigma})
\]

for any \( m \in \mathcal{P}(\mathbb{R}^d) \).

2.1. Verification of the flat Polyak-Łojasiewicz condition. In this subsection we adapt the proof of Theorem 2 in [13] to the setting of the space of measures. In [13] it was shown how the classical Polyak-Łojasiewicz inequality (1.1) for functions on \( \mathbb{R}^d \) can be inferred from a certain type of a quadratic growth condition. Here we will work with functions on \( \mathcal{P}(\mathbb{R}^d) \) and we will carry out a similar argument, based on certain quadratic growth conditions expressed in terms of either the KL-divergence of the \( \chi^2 \)-divergence. This result can be interpreted as an analogue of Theorem 1 in [4], which showed that a certain type of the Łojasiewicz inequality can be inferred from a quadratic growth condition with respect to the \( L^2 \)-Wasserstein distance. We will present our reasoning in a series of lemmas.

Lemma 2.6. Suppose that \( G : \mathcal{P}(\mathbb{R}^d) \rightarrow \mathbb{R} \) has the first order flat derivative and that \( G \) is convex (cf. (2.1)). Then for any absolutely continuous probability measures \( m, m' \in \mathcal{P}(\mathbb{R}^d) \),

\[
G(m) - G(m') \leq \left( \int_{\mathbb{R}^d} \left| \frac{\delta G}{\delta m}(m, x) \right|^2 m(x) dx \right)^{1/2} \left( \int_{\mathbb{R}^d} \left( \frac{m'(x)}{m(x)} - 1 \right)^2 m(x) dx \right)^{1/2}
\]

\[
= \left\| \frac{\delta G}{\delta m}(m, \cdot) \right\|_{L^2(m)} \cdot \chi^2(m'|m)^{1/2}.
\]
Proof. Since \( \int_{\mathbb{R}^d} \frac{\partial G}{\partial m}(m, x)m(x)dx = 0 \) by convention, from the convexity condition \((2.1)\) we get
\[
G(m) - G(m') \leq - \int_{\mathbb{R}^d} \frac{\partial G}{\partial m}(m, x)m'(x)dx = - \int_{\mathbb{R}^d} \frac{\partial G}{\partial m}(m, x) \left( \frac{m'(x)}{m(x)} - 1 \right) m(x)dx.
\]
A simple application of the Cauchy-Schwarz inequality in \( L^2(m) \) proves the desired assertion. \( \square \)

Next we need a lemma that allows us to compare the \( \chi^2 \)-divergence and the KL-divergence, between two absolutely continuous measures, such that the ratio of their densities is bounded from above and below.

**Lemma 2.7.** Suppose we have absolutely continuous \( m, m' \in \mathcal{P}(\mathbb{R}^d) \) such that there exist constants \( r, R > 0 \) such that for any \( x \in \mathbb{R}^d \) we have
\[
r \leq \frac{m(x)}{m'(x)} \leq R.
\]
Then we have
\[
(2.10) \quad \text{KL}(m'|m) \leq \frac{1}{r} \text{KL}(m|m') \quad \text{and} \quad \chi^2(m|m') \leq 2R \text{KL}(m|m').
\]

**Proof.** The proof can be adapted from the proofs of Proposition 1 and Proposition 2 in \([10]\), which covered the case of discrete probability measures. For completeness, we include the proof in the appendix. \( \square \)

Based on the above lemmas, we can show the following result.

**Theorem 2.8.** Suppose that \( G: \mathcal{P}(\mathbb{R}^d) \to \mathbb{R} \) has the first order flat derivative and that \( G \) is convex. Suppose further that \( G \) is minimized by an absolutely continuous measure \( m^* \) and that there exists a constant \( \lambda > 0 \) such that for any \( m' \in \mathcal{P}(\mathbb{R}^d) \),
\[
(2.11) \quad G(m') - G(m^*) \geq \lambda \text{KL}(m'|m^*).
\]
Moreover, suppose that we have an absolutely continuous measure \( m \in \mathcal{P}(\mathbb{R}^d) \) such that there exist constants \( r, R > 0 \) such that for any \( x \in \mathbb{R}^d \) we have
\[
r \leq \frac{m(x)}{m^*(x)} \leq R.
\]
Then
\[
(2.12) \quad G(m) - G(m^*) \leq \frac{2R}{\lambda r} \left\| \frac{\partial G}{\partial m}(m, \cdot) \right\|_{L^2(m)}^2.
\]

**Proof.** We follow the argument from the proof of Theorem 1 in \([4]\). Since \( V \) is assumed to be convex, from Lemma 2.6 we get
\[
(2.13) \quad V(m) - V(m^*) \leq \left\| \frac{\delta}{\delta m} V(m, \cdot) \right\|_{L^2(m)} \cdot \chi^2(m^*|m)^{1/2}.
\]
However, due to Lemma 2.7 we have
\[
\chi^2(m^*|m) \leq 2R \text{KL}(m^*|m) \leq \frac{2R}{r} \text{KL}(m|m^*),
\]
which, together with \((2.13)\) and \( V(m) - V(m^*) \geq \lambda \text{KL}(m|m^*) \) leads to
\[
\text{KL}(m|m^*)^{1/2} \leq \frac{1}{\lambda} \left( \frac{2R}{r} \right)^{1/2} \left\| \frac{\partial G}{\partial m}(m, \cdot) \right\|_{L^2(m)}.
\]
In particular,
\begin{equation}
(2.14) 
\chi^2(m^*|m)^{1/2} \leq \frac{2R}{\lambda r} \left\| \frac{\delta G}{\delta m}(m, \cdot) \right\|_{L^2(m)}.
\end{equation}
Plugging (2.14) into the right hand side of (2.13), we obtain
\begin{equation}
G(m) - G(m^*) \leq \frac{2R}{\lambda r} \left\| \frac{\delta G}{\delta m}(m, \cdot) \right\|_{L^2(m)}^2.
\end{equation}
\[\square\]

Remark 2.9. Under the assumptions of Theorem 2.8 we obtain the flat Polyak-Łojasiewicz condition of the type (1.6) with the constant
\[\kappa = \left( \frac{2R}{\lambda r} \right)^{-1}.\]

In what follows, we will prove that the flow \((m_t)_{t \geq 0}\) given by (1.4) is such that \(\bar{r} \leq m_t(x) \leq \bar{R}\) for all \(t > 0\) and \(x \in \mathbb{R}^d\), which will allow us to show (2.12) with \(G\) on the left hand side replaced by \(V_\sigma\) and \(\delta G/\delta m(m, x)\) on the right hand side replaced by \(a(m, x)\) given by (1.4). This will be the basis of the proof of our main results in Section 3 and will provide us with an exponential convergence rate \(\kappa\) of \(V_\sigma(m_t)\) to \(V_\sigma(m_\sigma, \cdot^*)\). We can easily observe that the convergence rate \(\kappa\) degenerates to zero when \(\lambda \to 0\) or \(r \to 0\) or \(R \to \infty\).

Condition (2.11) corresponds to the classical quadratic growth condition for functions \(f : \mathbb{R}^d \to \mathbb{R}\) that can be used (see Theorem 2 in [13]) to prove the classical Polyak-Łojasiewicz inequality (1.1) under the additional assumption of convexity of \(f\) (but not necessarily strong convexity). More precisely, the quadratic growth condition in \(\mathbb{R}^d\) states that
\[f(x) - f^* \geq \frac{\mu}{2} \|x - x_p\|^2,
\]
where \(f^*\) is the minimum of \(f\) and \(x_p \in \arg \min_{x \in \mathbb{R}^d} f(x)\). Specifying an analogous condition for functions on the space of measures is non-straightforward, as there are multiple choices of the notion of the distance. Blanchet and Bolte in [4] proved that a certain type of a Łojasiewicz inequality can be implied by a condition such as (2.11) but with the \(L^2\)-Wasserstein distance instead of the KL-divergence, see formula (2) and Theorem 1 in [4]. Based on the proof of our Theorem 2.8 it is clear that we can also consider a quadratic growth condition with respect to the \(\chi^2\)-divergence with reversed arguments, i.e.,
\begin{equation}
(2.15) 
G(m) - G(m^*) \geq \lambda \chi^2(m^*|m).
\end{equation}
Using (2.13) and (2.15), one immediately obtains
\begin{equation}
\chi^2(m^*|m)^{1/2} \leq \frac{1}{\lambda} \left\| \frac{\delta G}{\delta m}(m, \cdot) \right\|_{L^2(m)},
\end{equation}
which can be plugged back into (2.13) to obtain
\begin{equation}
G(m) - G(m^*) \leq \frac{1}{\lambda} \left\| \frac{\delta G}{\delta m}(m, \cdot) \right\|_{L^2(m)}^2.
\end{equation}
It is clear based on Lemma 2.7 that (2.11) implies (2.15), but we are presently unaware of any examples of energy functions that would satisfy (2.15) but not (2.11).
2.2. Literature review. In order to present our results in a broader context, let us first discuss a different type of gradient flows and associated Lojasiewicz-type inequalities. We will also provide two heuristic examples in order to build a better intuition for the type of problems discussed in our paper.

2.2.1. Wasserstein Gradient Flow. The dynamic representation of the $L^2$-Wasserstein metric $\mathcal{W}_2$ due to Benamou and Brenier [3, 21] states that

\begin{equation}
\mathcal{W}_2(\mu_0, \mu_1) = \inf_{\nu \in L^2([0,1] \times \mathcal{P}_2(\mathbb{R}^d))} \left\{ \int_0^1 \int_{\mathbb{R}^d} |\nu_s|^2 \mu_s(dx) ds : \text{s.t } \partial_s \mu_s + \text{div}(\nu_s \mu_s) = 0 \right\} .
\end{equation}

This result tells us that measures in the space $(\mathcal{P}_2(\mathbb{R}^d), \mathcal{W}_2)$ of probability measures with finite second moments are transported along curves described by the forward-Kolmogorov PDE.

Following the notation from [12] we have

\begin{equation}
\nabla a(m, x) = \left( \nabla \frac{\delta F}{\delta m} \right)(m, x) + \frac{\sigma^2}{2} \nabla \left( \frac{\log m(x)}{\pi(x)} \right) .
\end{equation}

One can show [12] that $V^\sigma(m_t)$ is decreasing along the gradient flow $(m_t)_{t \geq 0}$ satisfying

\begin{equation}
\partial_t m_t = \text{div} (\nabla a(m_t, \cdot) m_t) .
\end{equation}

Note that this flow corresponds to the mean-field Langevin equation (see e.g. (1.4) and (1.5) in [12]), and in particular becomes the classical overdamped Langevin equation when $F = 0$. Indeed, if we can show that $t \mapsto V^\sigma(m_t)$ is differentiable, we obtain

\begin{equation}
\begin{aligned}
\partial_t V^\sigma(m_t) &= \int_{\mathbb{R}^d} a(m_t, x) \partial_t m_t(x) dx = \int_{\mathbb{R}^d} a(m_t, x) \text{div}(\nabla a(m_t, x) m_t(x)) dx \\
&= - \int_{\mathbb{R}^d} |\nabla a(m_t, x)|^2 m_t(dx) .
\end{aligned}
\end{equation}

From here, one can deduce that $V^\sigma(m_t)$ converges to a local minimum as $t \to \infty$. In the case when $F$ is convex, and hence $V^\sigma$ is strictly convex, $V^\sigma(m_t) \to V^\sigma(m^{\sigma,*})$, see [12]. More recently, [17] and [8] under additional structural assumptions proved that this convergence is exponential.

In this setting, the Polyak-Lojasiewicz condition that implies the exponential convergence $V^\sigma(m_t) \to V^\sigma(m^{\sigma,*})$, requires that there exists a constant $\kappa > 0$ such that for any $m^* \in \arg \min_{m \in \mathcal{P}([0,1])} V^\sigma(m)$ and any $m \in \mathcal{P}([0,1])$,

\begin{equation}
\frac{1}{\kappa} \|\nabla a(m, \cdot)\|_{L^2(m)}^2 \geq V^\sigma(m) - V^\sigma(m^{\sigma,*}) .
\end{equation}

With such an inequality at hand, one immediately sees that

\begin{equation}
\partial_t (V^\sigma(m_t) - V^\sigma(m^{\sigma,*})) = - \int_{\mathbb{R}^d} |\nabla a(m_t, x)|^2 m_t(dx) \leq -\kappa (V^\sigma(m_t) - V^\sigma(m^{\sigma,*})) ,
\end{equation}

and the exponential convergence follows due to the Gronwall lemma.

Example 2.10. Let $F = 0$ in [12]. In this case the optimal measure $m^{\sigma,*} = \arg \min_{m} V^\sigma(m) = \pi$. Then, assuming that we can show that $t \mapsto \text{KL}(m_t|\pi)$ is differentiable, we have

\begin{equation}
\partial_t (V^\sigma(m_t) - V^\sigma(m^{\sigma,*})) = \frac{\sigma^2}{2} \partial_t \text{KL}(m_t|\pi) = -\frac{\sigma^2}{4} \int_{\mathbb{R}^d} \left| \nabla \log \frac{m_t(x)}{\pi(x)} \right|^2 m_t(dx) .
\end{equation}
In this case the Polyak-Lojasiewicz inequality is just the well-known log-Sobolev inequality
\[
\frac{1}{\kappa} \int_{\mathbb{R}^d} \left| \nabla \log \frac{m_t(x)}{\pi(x)} \right|^2 m_t(dx) \geq \text{KL}(m_t|\pi).
\]

**Example 2.11.** Let us consider a heuristic example with a different type of energy function. Consider \( V^\sigma(m) := \chi^2(m|\pi) = \int_{\mathbb{R}^d} \left( \frac{m(x)}{\pi(x)} - 1 \right)^2 \pi(x)dx \) for probability measures \( m \in \mathcal{P}(\mathbb{R}^d) \) absolutely continuous with respect to \( \pi \), and denote
\[
\nabla \tilde{a}(m, x) := 2\nabla \left( \frac{m(x)}{\pi(x)} \right),
\]
which formally corresponds to the flat derivative of the \( \chi^2 \)-divergence. Then \( V^\sigma(m_t) \) is decreasing along the gradient flow \((m_t)_{t \geq 0}\) satisfying
\[
\partial_t m_t = \text{div} \left( \nabla \tilde{a}(m_t, \cdot) \pi \right),
\]
i.e., similarly as in (2.19), assuming \( t \mapsto \chi^2(m_t|\pi) \) is differentiable, we have
\[
\partial_t V^\sigma(m_t) = - \int_{\mathbb{R}^d} \left| \nabla \tilde{a}(m_t, x) \right|^2 \pi(dx).
\]
Here the Polyak-Lojasiewicz inequality becomes the Poincaré inequality
\[
\frac{1}{\kappa} \int_{\mathbb{R}^d} \left| \nabla \left( \frac{m_t(x)}{\pi(x)} \right) \right|^2 \pi(dx) \geq \chi^2(m_t|\pi).
\]
Note that this corresponds to (2.20) with the \( L^2(\pi) \) norm instead of \( L^2(m_t) \), since we used a different gradient flow (compare (2.22) to (2.18)).

2.2.2. Wasserstein-Fisher-Rao Gradient Flow. A natural idea is to combine the Wasserstein (2.18) and the Fisher-Rao (1.4) gradient flows which in our setting leads to
\[
\partial_t m_t = \text{div} \left( \nabla a(m_t, \cdot)m_t - a(m_t, x)m_t \right).
\]
Flows of this type have been the subject of intensive research over the last few years \cite{14,11,15,19}. If the differentiability of \( t \mapsto V^\sigma(m_t) \) can be verified, one can then check that
\[
\partial_t (V^\sigma(m_t) - V^\sigma(m^\sigma *)) = - \left| \nabla a(m_t, \cdot) \right|_{L^2(m_t)}^2 - \left| a(m_t, \cdot) \right|_{L^2(m_t)}^2.
\]
This shows that both the Langevin part and the birth-death part can independently contribute to the convergence of \( V^\sigma(m_t) \), if the right corresponding conditions (2.20) or (1.6) are satisfied.

Even though \cite{19} studied the convergence of flows similar to (2.23), their definition used the energy function \( V^\sigma \) instead of the function \( a \) corresponding to the flat derivative of \( V^\sigma \), and hence their flow does not seem to converge to the right minimizer \( m^\sigma * \).

On the other hand, \cite{15} studied (2.23) corresponding to the linear case \( (F = 0) \) of our Example 2.10 and obtained an exponential rate of convergence to \( \pi \), measured in the KL-divergence (see Theorem 3.3 therein). Interestingly, even though the authors of \cite{15} did not explicitly make a connection to the Polyak-Lojasiewicz inequalities, their proof is in fact based on showing a special case of condition (1.7) as specified above (see their inequality \( (2 - 2\delta)H_1(f) \leq H_2(f) \) in the proof of Theorem 3.3, integrate it with respect to \( \rho_t \) and note that our \( m_t \) corresponds to their \( \rho_t \). This Polyak-Lojasiewicz inequality is verified in \cite{15} under a positive lower bound on the ratio of densities \( \inf_{x \in \mathbb{R}^d} \frac{\rho(x)}{\pi(x)} \) that is required to hold for all \( t > 0 \), see (B.1) in \cite{15}. Then they use an argument
based on the maximum principle (which is possible due to the Langevin component of their dynamics) to show that this condition in fact only has to hold for $t = 0$. As a consequence, they conclude that compared to the classical result on the exponential convergence of the Langevin dynamics to $\pi$ under the log-Sobolev inequality, by adding the birth-death component to the dynamics they can get rid of the log-Sobolev assumption and replace it by a ”warm start” condition $\inf_{x \in \mathbb{R}^d} \frac{\rho_0(x)}{\pi(x)} \geq c$ for some $c > 0$. However, in [15] the Langevin part of the dynamics is only applied to make the use of the maximum principle possible, and does not directly contribute to the convergence rate. Moreover, some important technical issues such as the question of the existence of the gradient flow and the differentiability of $t \mapsto V^\sigma(m_t)$ were not addressed in [15].

In this paper we study a more general setting than [15], including non-linear functions $F$ in the energy function $V^\sigma$ in (1.2), and we rigorously prove the existence of the corresponding birth-death gradient flow $(m_t)_{t \geq 0}$, as well as the differentiability of $t \mapsto V^\sigma(m_t)$. We also verify the flat Polyak-Lojasiewicz inequality (1.6) and thus establish the exponential rate of convergence of $V^\sigma(m_t)$ to $V^\sigma(m^{\sigma,*})$. Our condition guaranteeing that (1.6) holds (Assumption 2) resembles the warm start condition from [15], however, in order to show that it propagates from $t = 0$ to all $t > 0$, we do not need to use the Langevin component of the dynamics and hence we work with a ”pure” birth-death dynamics (the Fisher-Rao gradient flow).

Other recent papers studying the mean-field optimization problem specified by (1.2), such as [17] and [8], focused on the Wasserstein gradient flow (2.18). Both [17] and [8] proved the exponential convergence rate of $V^\sigma(m_t)$ to $V^\sigma(m^{\sigma,*})$ under the assumption of the log-Sobolev inequality for a class of proximal Gibbs measures related to $m^{\sigma,*}$. Compared to [17, 8], working with the Fisher-Rao gradient flow allows us to get rid of that assumption, at the cost of introducing the additional ”warm start” conditions in Assumption 2.

With all that said, we would like to point out that from the point of view of practical algorithms (that will be the subject of our future work), combining the birth-death dynamics with the Langevin dynamics seems advisable. The Wasserstein-Fisher-Rao gradient flow (2.23) can be seen as the mean-field limit of an interacting particle system that can be used as a basis of practically implementable algorithms (as studied in Section 6 in [15]). The support of the birth-death flow does not change in time and hence, intuitively, if we do not include the diffusion component in our dynamics and we initialize it with the empirical measure of a set of particles, the dynamics will just keep re-arranging the mass between the particles but will not change their positions. Hence the convergence of such dynamics should be expected to be worse than the convergence of a particle system utilizing both the Langevin and the birth-death components. This issue is not apparent in the analysis of the mean-field limit process in the present paper (as our results use a ”warm start” assumption on the initial condition), but we will investigate it in detail in our future work on the particle system approximations and the corresponding algorithms. From the practical point of view, the main message of this paper is that the birth-death component of such algorithms can be defined in terms of the function $a$ given by (1.4), which corresponds to the flat derivative of the energy function $V^\sigma$, but the focus here is on the theoretical analysis of the gradient flow rather than applications.
3. Existence of the gradient flow and other proofs

In order to prove the existence of a solution \((m_t)_{t \geq 0}\) to

\[
\partial_t m_t(x) = -\left( \frac{\delta F}{\delta m}(m_t, x) + \log \left( \frac{m_t(x)}{\pi(x)} \right) - \text{KL}(m_t|\pi) \right) m_t(x),
\]

we first notice that (3.1) is equivalent to

\[
\partial_t \ln m_t(x) = -\left( \frac{\delta F}{\delta m}(m_t, x) + \frac{\sigma^2}{2} \ln \frac{m_t(x)}{\pi(x)} - \frac{\sigma^2}{2} \text{KL}(m_t|\pi) \right).
\]

By Duhamel’s formula, (3.2) is equivalent to

\[
\ln m_t(x) = e^{-\frac{\sigma^2}{2} t} \ln m_0(x) - \int_0^t \frac{\sigma^2}{2} e^{-\frac{\sigma^2}{2} (t-s)} \left( \frac{2}{\sigma^2} \frac{\delta F}{\delta m}(m_s, x) - \ln \pi(x) - \text{KL}(m_s|\pi) \right) ds.
\]

Based on this formula, we will define a Picard iteration scheme. To this end, let us first fix \(T > 0\) and choose a flow of probability measures \((m_t^{(0)})_{t \in [0,T]}\) such that

\[
\int_0^T \text{KL}(m_s^{(0)}|\pi) ds < \infty.
\]

For each \(n \geq 1\), we want to fix \(m_0^{(n)} = m_0^{(0)} = m_0\) (with \(m_0^{(0)}\) satisfying condition (2.3) from Assumption 2) and define \((m_t^{(n)})_{t \in [0,T]}\) by

\[
\ln m_t^{(n)}(x) = e^{-\frac{\sigma^2}{2} t} \ln m_0(x) - \int_0^t \frac{\sigma^2}{2} e^{-\frac{\sigma^2}{2} (t-s)} \left( \frac{2}{\sigma^2} \frac{\delta F}{\delta m}(m_s^{(n-1)}, x) - \ln \pi(x) - \text{KL}(m_s^{(n-1)}|\pi) \right) ds.
\]

We have the following result.

**Lemma 3.1.** The sequence of flows \((m_t^{(n)})_{t \in [0,T]}\) given by (3.4) is well-defined and such that for all \(n \geq 1\) and all \(t \in [0,T]\) we have

\[
\text{KL}(m_t^{(n)}|\pi) \leq 2 \ln R + \frac{4}{\sigma^2} C.
\]

**Proof.** Consider \(n = 1\). By (2.2) and (3.3), the integral on the right hand side of (3.4) is finite, and hence \((m_t^{(1)})_{t \in [0,T]}\) is well-defined. Note that due to (2.2), the only potential issue with the definition of \((m_t^{(n)})_{t \in [0,T]}\) is due to the KL-divergence term under the integral, since a priori we do not know whether it is integrable. We will now prove by induction how to bound that term. Suppose that \(\int_0^T \text{KL}(m_s^{(n-1)}|\pi) ds < \infty\) and, based on (3.4), write

\[
\ln \frac{m_t^{(n)}(x)}{\pi(x)} = e^{-\frac{\sigma^2}{2} t} \ln \frac{m_0(x)}{\pi(x)} - \int_0^t \frac{\sigma^2}{2} e^{-\frac{\sigma^2}{2} (t-s)} \left( \frac{2}{\sigma^2} \frac{\delta F}{\delta m}(m_s^{(n-1)}, x) - \text{KL}(m_s^{(n-1)}|\pi) \right) ds.
\]

We also have

\[
\ln \frac{\pi(x)}{m_t^{(n)}(x)} = -e^{-\frac{\sigma^2}{2} t} \ln \frac{m_0(x)}{\pi(x)} - \int_0^t \frac{\sigma^2}{2} e^{-\frac{\sigma^2}{2} (t-s)} \left( -\frac{2}{\sigma^2} \frac{\delta F}{\delta m}(m_s^{(n-1)}, x) + \text{KL}(m_s^{(n-1)}|\pi) \right) ds.
\]
Due to (2.2) and (2.5), we can multiply both sides of (3.5) by \( m_t^{(n)}(x) \) and integrate with respect to \( x \) in order to obtain

\[
\text{KL}(m_t^{(n)}|\pi) \leq \ln R + \frac{2}{\sigma^2} C + \int_0^t \frac{\sigma^2}{2} e^{-\frac{\sigma^2}{2}(t-s)} \text{KL}(m_s^{(n-1)}|\pi) ds.
\]

Similarly, by multiplying both sides of (3.6) by \( \pi(x) \) and integrating with respect to \( x \), we obtain

\[
\text{KL}(\pi|m_t^{(n)}) \leq \ln R + \frac{2}{\sigma^2} C - \int_0^t \frac{\sigma^2}{2} e^{-\frac{\sigma^2}{2}(t-s)} \text{KL}(m_s^{(n-1)}|\pi) ds.
\]

Consequently, we obtain

\[
\text{KL}(m_t^{(n)}|\pi) \leq \text{KL}(m_t^{(n)}|\pi) + \text{KL}(\pi|m_t^{(n)}) \leq 2 \ln R + \frac{4}{\sigma^2} C,
\]

which finishes the proof by induction. \( \square \)

We will now consider the sequence of flows \( (m_t^{(n)})_{t\in[0,T]} \) in \( \mathcal{P}([0,T]) \) equipped with the distance \( TV_T \), defined for any \( (\mu_t)_{t\in[0,T]}, (\nu_t)_{t\in[0,T]} \in \mathcal{P}([0,T]) \) by

\[
TV_T ((\mu_t)_{t\in[0,T]}, (\nu_t)_{t\in[0,T]}) := \int_0^T TV(\mu_t, \nu_t) dt.
\]

Since \( \mathcal{P}(\mathbb{R}^d) \) equipped with the total variation distance \( TV \) is complete, we can apply the argument from Lemma A.5 in [22] with \( p = 1 \) to conclude that \( \mathcal{P}([0,T]) \) equipped with \( TV_T \) is also complete. We will now consider the Picard iteration mapping \( \Psi((m_t^{(n-1)})_{t\in[0,T]}):= (m_t^{(n)})_{t\in[0,T]} \) defined via (3.4), and show that \( \Psi \) is contractive in \( (\mathcal{P}([0,T]), TV_T) \). Then the Banach fixed point theorem will give us the existence of a solution to (3.1).

**Theorem 3.2.** The mapping \( \Psi((m_t^{(n-1)})_{t\in[0,T]}):= (m_t^{(n)})_{t\in[0,T]} \) defined via (3.4) is contractive in \( (\mathcal{P}([0,T]), TV_T) \).

**Proof.** From (3.4) we have

\[
\ln m_t^{(n)}(x) - \ln m_t^{(n-1)}(x) = - \int_0^t \frac{\sigma^2}{2} e^{-\frac{\sigma^2}{2}(t-s)} \times
\]

\[
\times \left[ \frac{2}{\sigma^2} \left( \frac{\delta F}{\delta m_s^{(n-1)}}(m_s^{(n-1)}, x) - \frac{\delta F}{\delta m_s^{(n-2)}}(m_s^{(n-2)}, x) \right) - \text{KL}(m_s^{(n-1)}|\pi) + \text{KL}(m_s^{(n-2)}|\pi) \right] ds.
\]

Multiplying both sides by \( m_t^{(n)}(x) \) and integrating with respect to \( x \), we obtain

\[
\text{KL}(m_t^{(n)}|m_t^{(n-1)}) = - \int_0^t \frac{\sigma^2}{2} e^{-\frac{\sigma^2}{2}(t-s)} \left[ \frac{2}{\sigma^2} \int_{\mathbb{R}^d} \left( \frac{\delta F}{\delta m_s^{(n-1)}}(m_s^{(n-1)}, x) - \frac{\delta F}{\delta m_s^{(n-2)}}(m_s^{(n-2)}, x) \right) \right.
\]

\[
\times m_t^{(n)}(dx) - \text{KL}(m_s^{(n-1)}|\pi) + \text{KL}(m_s^{(n-2)}|\pi) \left] ds.
\]

(3.7)
Moreover, note that
\[
\int_{\mathbb{R}^d} \left( \frac{\delta F}{\delta m}(m_s^{(n-1)}, x) - \frac{\delta F}{\delta m}(m_s^{(n-2)}, x) \right) m_t^{(n)}(dx) = \int_{\mathbb{R}^d} \int_{0}^{1} \frac{2}{\delta m^{2}} (m_s^{(n-2)} + \lambda (m_s^{(n-1)} - m_s^{(n-2)}), x, y) d\lambda \\
\times (m_s^{(n-1)} - m_s^{(n-2)}) (dy)m_t^{(n)}(dx).
\]

Similarly, again from (3.7), we have
\[
\ln m_t^{(n-1)}(x) - \ln m_t^{(n)}(x) = - \int_{0}^{t} \frac{\sigma^2}{2} e^{-\frac{s^2}{2}(t-s)} \times \\
\times \left[ \frac{2}{\sigma^2} \left( \frac{\delta F}{\delta m}(m_s^{(n-2)}, x) - \frac{\delta F}{\delta m}(m_s^{(n-1)}, x) \right) - \text{KL}(m_s^{(n-2)}|\pi) + \text{KL}(m_s^{(n-1)}|\pi) \right] ds.
\]

Multiplying both sides by \(m_t^{(n-1)}(x)\) and integrating with respect to \(x\), we obtain
\[
\text{KL}(m_t^{(n-1)}|m_t^{(n)}) = - \int_{0}^{t} \frac{\sigma^2}{2} e^{-\frac{s^2}{2}(t-s)} \times \\
\times \left[ \frac{2}{\sigma^2} \left( \frac{\delta F}{\delta m}(m_s^{(n-2)}, x) - \frac{\delta F}{\delta m}(m_s^{(n-1)}, x) \right) - \text{KL}(m_s^{(n-2)}|\pi) + \text{KL}(m_s^{(n-1)}|\pi) \right] ds.
\]

Similarly as before, we note that
\[
\int_{\mathbb{R}^d} \left( \frac{\delta F}{\delta m}(m_s^{(n-2)}, x) - \frac{\delta F}{\delta m}(m_s^{(n-1)}, x) \right) m_t^{(n-1)}(dx) = - \int_{\mathbb{R}^d} \int_{0}^{1} \frac{2}{\delta m^{2}} (m_s^{(n-2)} + \lambda (m_s^{(n-1)} - m_s^{(n-2)}), x, y) d\lambda \\
\times (m_s^{(n-1)} - m_s^{(n-2)}) (dy)m_t^{(n-1)}(dx).
\]

Combining (3.7) and (3.8), we obtain
\[
\text{KL}(m_t^{(n)}|m_t^{(n-1)}) + \text{KL}(m_t^{(n-1)}|m_t^{(n)}) = - \int_{0}^{t} e^{-\frac{s^2}{2}(t-s)} \times \\
\times \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{0}^{1} \frac{2}{\delta m^{2}} (m_s^{(n-2)} + \lambda (m_s^{(n-1)} - m_s^{(n-2)}), x, y) d\lambda (m_s^{(n-1)} - m_s^{(n-2)}) (dy) \times \\
\times (m_t^{(n)} - m_t^{(n-1)}) (dx) ds.
\]

Hence, due to (2.3), we get
\[
\text{KL}(m_t^{(n)}|m_t^{(n-1)}) + \text{KL}(m_t^{(n-1)}|m_t^{(n)}) \\
\leq \int_{0}^{t} e^{-\frac{s^2}{2}(t-s)} C_2 TV(m_s^{(n-1)}, m_s^{(n-2)}) TV(m_t^{(n)}, m_t^{(n-1)}) ds
\]

By the Pinsker-Csiszar inequality, \(TV^2(m_t^{(n)}, m_t^{(n-1)}) \leq \frac{1}{2} \text{KL}(m_t^{(n)}|m_t^{(n-1)})\) and hence
\[
4TV^2(m_t^{(n)}, m_t^{(n-1)}) \leq C_2 TV(m_t^{(n)}, m_t^{(n-1)}) \int_{0}^{t} e^{-\frac{s^2}{2}(t-s)} TV(m_s^{(n-1)}, m_s^{(n-2)}) ds,
\]

\[13\]
where the second inequality follows from Lemma 3.1.

From (2.2), (2.7) and Lemma 3.3 we obtain for any $t$

which gives

$$TV(m_t^{(n)}, m_t^{(n-1)}) \leq \frac{C_2}{4} \int_0^t e^{-\frac{e^2}{t}(t-s)} TV(m_s^{(n-1)}, m_s^{(n-2)}) ds$$

$$\leq \left( \frac{C_2}{4} \right)^{n-1} e^{\frac{e^2}{t}} \int_0^t \int_0^{t_i} \ldots \int_0^{t_{n-2}} e^{\frac{e^2}{t}} TV(m_{t_{n-1}}, m_{t_{n-1}}) dt_{n-1} \ldots dt_2 dt_1$$

$$\leq \left( \frac{C_2}{4} \right)^{n-1} e^{\frac{e^2}{t}} \frac{t^{n-2}}{(n-2)!} \int_0^t e^{\frac{e^2}{t}} TV(m_{t_{n-1}}, m_{t_{n-1}}) dt_{n-1}$$

$$\leq \left( \frac{C_2}{4} \right)^{n-1} \frac{t^{n-2}}{(n-2)!} \int_0^t TV(m_{t_{n-1}}, m_{t_{n-1}}) dt_{n-1},$$

where in the third inequality we bounded $\int_0^{t_{n-2}} dt_{n-1} \leq \int_0^t dt_{n-1}$ and in the fourth inequality we bounded $e^{\frac{e^2}{t}} \leq e^{\frac{e^2}{t}}$. Hence we obtain

$$\int_0^T TV(m_t^{(n)}, m_t^{(n-1)}) dt \leq \left( \frac{C_2}{4} \right)^{n-1} \frac{T^{n-1}}{(n-2)!} \int_0^T TV(m_{t_{n-1}}, m_{t_{n-1}}) dt_{n-1}.$$

For sufficiently large $n$, the constant on the right hand side becomes less than 1 and the proof is complete. \hfill \Box

We can now finalize the proof of Theorem 2.1.

**Proof of Theorem 2.1.** By Theorem 3.2 for any $T > 0$ we obtain the existence of a flow $(m_t)_{t \in [0, T]}$ satisfying (3.1). Moreover, for Lebesgue-almost all $t \in [0, T]$ we have

$$TV(m_t^{(n)}, m_t) \to 0 \quad \text{as } n \to \infty,$$

which implies

$$m_t^{(n)} \to m_t \quad \text{weakly, as } n \to \infty.$$ 

Hence, using the lower semi-continuity of the KL-divergence (see e.g. Theorem 2.34 in [1]) we obtain

$$(3.9) \quad KL(m_t | \pi) \leq \liminf_{n \to \infty} KL(m_t^{(n)} | \pi) \leq 2 \ln R + \frac{4C}{\sigma^2},$$

where the second inequality follows from Lemma 3.4.

Note that the unique solution $(m_t)_{t \in [0, T]}$ to (3.1) can also be expressed as

$$m_t(x) = m_0(x) \exp \left( - \int_0^t \left( \frac{\delta F}{\delta m}(m_s, x) + \log \left( \frac{m_s(x)}{\pi(x)} \right) - KL(m_s | \pi) \right) ds \right).$$

From (2.2), (2.7) and Lemma 3.3 we obtain for any $t \in [0, T]$,

$$\left| \frac{\delta F}{\delta m}(m_t, x) + \log \frac{m_t(x)}{\pi(x)} - KL(m_t | \pi) \right| \leq 2C + 3 \ln R + \frac{\sigma^2}{2} \left( 2 \ln R + \frac{4C}{\sigma^2} \right) + \frac{4C}{\sigma^2} =: C_V.$$

This gives $\|m_t\|_{TV} \leq \|m_0\|_{TV} e^{C_V t}$, and shows that $m_t$ does not explode in any finite time, hence we obtain a global solution $(m_t)_{t \in [0, \infty)}$. \hfill \Box

Before we can prove the differentiability of $t \mapsto V^\pi(m_t)$, we need an auxiliary result that shows that the bounds on the ratio $m_0(x)/\pi(x)$ from Assumption 2 carry over to the ratio $m_t(x)/\pi(x)$ for all $t > 0$. 

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Lemma 3.3. Under conditions (2.4) and (2.5), there exist constants \( r_1, R_1 > 0 \) such that for all \( t > 0 \) and for all \( x \in \mathbb{R}^d \) we have

\[
r_1 \leq \frac{m_t(x)}{\pi(x)} \leq R_1.
\]

Proof. Following the discussion from the beginning of Section 3, we see that for \( (m_t)_{t \geq 0} \) we have

\[
\ln \frac{m_t(x)}{\pi(x)} = e^{-\frac{2}{\sigma^2} \ln \frac{m_0(x)}{\pi(x)}} - \frac{1}{2} \int_0^t e^{-\frac{2}{\sigma^2}(t-s)} \left( \frac{2}{\sigma^2} \frac{\delta F}{\delta m}(m_s, x) - \text{KL}(m_s|\pi) \right) ds.
\]

Using (2.2), (2.5) and (2.7) we obtain

\[
\ln \frac{m_t(x)}{\pi(x)} \leq \ln R + C + \frac{\sigma^2}{2} \left( 2 \ln R + \frac{4C}{\sigma^2} \right).
\]

Obtaining a lower bound on \( \frac{m_t(x)}{\pi(x)} \) follows similarly, by using (2.4) instead of (2.5). \( \square \)

We are now ready to complete the proof of differentiability.

Proof of Theorem 2.2. We have the differentiability of \( F(m_t) \) as a consequence of Assumption 1. In order to show the differentiability of \( \text{KL}(m_t|\pi) \), we need to prove that \( \frac{\partial}{\partial t} \frac{m_t(x)}{\pi(x)} m_t(x) \leq g(x) \) for some integrable function \( g \), which is sufficient by a standard result in measure theory (see e.g. Theorem 11.5 in [20]). Indeed, by (2.2), (2.5) and (2.7), we get

\[
\left| \frac{\partial}{\partial t} \frac{m_t(x)}{\pi(x)} m_t(x) \right| = \left| \frac{\pi(x)}{m_t(x)} \frac{\partial m_t(x) m_t(x)}{\pi(x)} + \frac{\partial}{\partial t} \frac{m_t(x)}{\pi(x)} m_t(x) \right|
\]

\[
= \left| \left( \frac{\pi(x) + \ln \frac{m_t(x)}{\pi(x)}}{\pi(x)} \right) \left( \frac{\delta F}{\delta m}(m_t, x) + \ln \frac{m_t(x)}{\pi(x)} - \text{KL}(m_t|\pi) \right) \right|
\]

\[
\leq \left( \frac{\pi(x) + \ln R}{\pi(x)} \right) \left( C + 3 \log R + \frac{4C}{\sigma^2} \right).
\]

We can now write

\[
\frac{\partial}{\partial t} V^\sigma(m_t) = \int_{\mathbb{R}^d} \frac{\delta F}{\delta m}(m_t, x) \partial_t m_t(x) dx + \int_{\mathbb{R}^d} \partial_t \left( \frac{m_t(x)}{\pi(x)} m_t(x) \right) dx
\]

\[
= \int_{\mathbb{R}^d} \left( \frac{\delta F}{\delta m}(m_t, x) + \left( 1 + \ln \frac{m_t(x)}{\pi(x)} \right) \right) \partial_t m_t(x) dx,
\]

which is equivalent to (2.8) and hence finishes the proof. \( \square \)

Proof of Lemma 2.5. The proof is a straightforward extension of the proof of Proposition 1 in [17], where this was shown for \( V = F + H \), where \( H \) is the negative entropy.
By convexity of $F$, for any probability measures $m, m′ ∈ \mathcal{P}(\mathbb{R}^d)$ we get

\[
V^\sigma(m′) = F(m′) + \frac{σ^2}{2} \text{KL}(m′ | π)
\]

\[
\geq F(m) + \int_{\mathbb{R}^d} \frac{δF}{δm}(m, x)(m′ - m)(dx) + \frac{σ^2}{2} \text{KL}(m′ | π)
\]

\[
= F(m) + \int_{\mathbb{R}^d} \left( \frac{δF}{δm}(m, x) + \frac{σ^2}{2} \log \frac{m(x)}{π(x)} - \frac{σ^2}{2} \log \frac{m(x)}{π(x)} \right) (m′ - m)(dx) + \frac{σ^2}{2} \text{KL}(m′ | π)
\]

\[
= F(m) + \int_{\mathbb{R}^d} a(m, x)(m′ - m)(dx) - \int_{\mathbb{R}^d} \frac{σ^2}{2} \log \frac{m(x)}{π(x)} (m′ - m)(dx) +\frac{σ^2}{2} \text{KL}(m′ | π)
\]

\[
= F(m) + \int_{\mathbb{R}^d} a(m, x)(m′ - m)(dx) + \frac{σ^2}{2} \text{KL}(m′ | m) + \frac{σ^2}{2} \text{KL}(m | π)
\]

\[
\geq V^\sigma(m) + \int_{\mathbb{R}^d} a(m, x)(m′ - m)(dx) + \frac{σ^2}{2} \text{KL}(m′ | m).
\]

Taking $m = m^\sigma*$ in the above calculation finishes the proof, since $a(m^\sigma*, ·)$ is constant by Proposition 2.5 in \[12\].

**Proof of Theorem 2.3.** By Lemma 2.5 the quadratic growth condition \(2.11\) required in Theorem 2.8 is satisfied for $m = m_t$ for all $t > 0$, with $λ = σ^2/2$. Moreover, by Lemma 3.3 the ratio condition is also satisfied for $m = m_t$ for all $t > 0$ (note that by the discussion below Assumption 2, the ratio condition involving $π$ with constants $r$ and $R$ is equivalent to the ratio condition involving $m^\sigma*$ with corresponding constants $\bar{r}$ and $\bar{R}$). Note that by the proof of Lemma 2.5 in the case of $G = V^\sigma$, the convexity condition needed in the proof of Lemma 2.6 (and thus in Theorem 2.8) can be applied with $a(m, x)$ instead of $\frac{δG}{δm}$. Hence, by Theorem 2.8 the flat Polyak-Łojasiewicz condition \(2.9\) is satisfied for all $m_t$, with the constant $κ = (4\bar{R}/(σ^2\bar{r}))^{-1}$. □

4. **Appendix: Relations between different f-divergences**

Suppose we have absolutely continuous probability measures $m, m′ ∈ \mathcal{P}(\mathbb{R}^d)$ and a convex function $f : [0, \infty) → \mathbb{R}$. Then the f-divergence of $m$ with respect to $m′$ is defined by

\[
I_f(m|m′) := \int_{\mathbb{R}^d} f \left( \frac{m(x)}{m′(x)} \right) m′(x)dx.
\]

For instance, choosing $f(t) = t \log t$ leads to the KL-divergence and $f(t) = (t - 1)^2$ leads to the $χ^2$-divergence. We have the following result adapted from Theorem 6 in \[10\].

**Lemma 4.1.** Let $f : [0, \infty) → \mathbb{R}$ be convex and such that $f(1) = 0$. Let us consider an interval $(r, R) ⊂ (0, \infty)$ such that

(i) $f$ is twice differentiable on $(r, R)$

(ii) there exist real constants $a, A$ such that

\[
a ≤ tf''(t) ≤ A \quad \text{for all } t ∈ (r, R).
\]

Then for any absolutely continuous probability measures $μ$ and $ν$, we have the inequality

\[
a \text{KL}(μ | ν) ≤ I_f(μ | ν) ≤ A \text{KL}(μ | ν).
\]

**Proof.** Let us define a mapping $F_a : [0, \infty) → \mathbb{R}$ given by $F_a(t) := f(t) - at \log t$. Then $F_a$ is such that $F_a(1) = 0$, and is twice differentiable and convex on $(r, R)$ since $F_a''(t) ≥ 0$
on $(r, R)$. Note that the $f$-divergence associated to a convex $F_a$ with $F_a(1) = 0$ is always non-negative due to Jensen’s inequality, and hence we have

$$0 \leq I_{F_a}(\mu|\nu) = I_f(\mu|\nu) - a \text{KL}(\mu|\nu).$$

We now define a mapping $F_A : (0, \infty) \to \mathbb{R}$ by setting $F_A(t) := At \log t - f(t)$. Then $F_A$ is such that $F_A(1) = 0$, and is twice differentiable and convex on $(r, R)$ since $F''_A(t) \geq 0$ on $(r, R)$. We again use the fact that the corresponding $f$-divergence is non-negative, and we obtain

$$0 \leq I_{F_A}(\mu|\nu) = A \text{KL}(\mu|\nu) - I_f(\mu|\nu),$$

which finishes the proof. \hfill \Box

**Proof of Lemma 2.7.** We consider the mapping $f_1 : (0, \infty) \to \mathbb{R}$ given by $f_1(t) = -\log(t)$. Note that the $f$-divergence corresponding to this $f_1$ is the KL-divergence with swapped arguments, i.e., for any absolutely continuous probability measures $\mu$ and $\nu$, we have

$$I_{f_1}(\mu|\nu) = \int_{\mathbb{R}} \log \frac{\mu(x)}{\nu(x)} \nu(x) dx = \int_{\mathbb{R}} \log \frac{\nu(x)}{\mu(x)} \nu(x) dx = \text{KL}(\nu|\mu).$$

We remark that $f_1(1) = 0$ and that $f_1$ is twice differentiable on any interval $(r, R) \subset (0, \infty)$. We also have

$$\frac{1}{R} \leq tf''_1(t) \leq \frac{1}{r} \quad \text{for all } t \in (r, R)$$

since $f''(t) = 1/t^2$. Applying Lemma 4.1 with $a = 1/R$ and $A = 1/r$, we have

$$\frac{1}{R} \text{KL}(\mu|\nu) \leq \text{KL}(\nu|\mu) \leq \frac{1}{r} \text{KL}(\mu|\nu).$$

This shows the first inequality in (2.10). We now consider the mapping $f_2 : (0, \infty) \to \mathbb{R}$ defined by $f_2(t) := (t - 1)^2$, i.e., $I_{f_2}$ is the $\chi^2$-divergence. Again, $f_2(1) = 0$ and $f_2$ is twice differentiable on any interval $(r, R) \subset (0, \infty)$. Moreover, we have

$$2r \leq tf''_2(t) \leq 2R \quad \text{for all } t \in (r, R)$$

since $f''_2 = 2$. Applying Lemma 4.1 with $a = 2r$ and $A = 2R$, we have

$$2r \text{KL}(\mu|\nu) \leq \chi^2(\mu|\nu) \leq 2R \text{KL}(\mu|\nu).$$

This shows the second inequality in (2.10) and concludes the proof. \hfill \Box

5. APPENDIX: FLAT DERIVATIVE

**Definition 5.1.** Fix $q \geq 0$ and let $\mathcal{P}_q(\mathbb{R}^d)$ be the space of probability measures on $\mathbb{R}^d$ with finite $q$-th moments. A functional $F : \mathcal{P}_q(\mathbb{R}^d) \to \mathbb{R}$, is said to admit a first order linear derivative (or a flat derivative), if there exists a functional $\frac{\delta F}{\delta m} : \mathcal{P}_q(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$, such that

1. For all $a \in \mathbb{R}^d$, $\mathcal{P}_q(\mathbb{R}^d) \ni m \mapsto \frac{\delta F}{\delta m}(m, a)$ is continuous.
2. For any $\nu \in \mathcal{P}_q(\mathbb{R}^d)$, there exists $C > 0$ such that for all $a \in \mathbb{R}^d$ we have

$$\left| \frac{\delta F}{\delta m}(\nu, a) \right| \leq C(1 + |a|^q).$$

3. For all $m, m' \in \mathcal{P}_q(\mathbb{R}^d)$,

$$F(m') - F(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta F}{\delta m}(m + \lambda(m' - m), a) (m' - m) (da) d\lambda.$$

The functional $\frac{\delta F}{\delta m}$ is then called the linear (functional) derivative of $F$ on $\mathcal{P}_q(\mathbb{R}^d)$. 17
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