SIMPLE MODULES OVER QUANTUM TORUS AND QUANTUM GROUP

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Abstract. In this paper, we classify all simple modules over the quantum torus \( \mathbb{C}_\nu[x^{\pm 1}, y^{\pm 1}] \) and the quantum group \( U_q(\mathfrak{sl}_2) \) for generic case.

Keywords: quantum torus, quantum group, simple module

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1. Introduction

Quantum group \( U_q(\mathfrak{sl}_2) \) is the \( q \)-deformation of the universal enveloping algebra \( U(\mathfrak{sl}_2) \) of the 3-dimensional simple Lie algebra \( \mathfrak{sl}_2 \). In some sense, \( \mathfrak{sl}_2 \) and \( U_q(\mathfrak{sl}_2) \) perhaps are the most fundamental objects in the theory of Lie algebras and quantum groups. The classification of simple modules over \( \mathfrak{sl}_2 \) or \( U_q(\mathfrak{sl}_2) \) is a very important problem in their representation theory.

Let \( \mathfrak{z} \) be the Casimir element of \( U(\mathfrak{sl}_2) \). The quotient algebra \( U(\mathfrak{sl}_2)/\langle \mathfrak{z} - c \rangle \) is isomorphic to a subalgebra of Weyl algebra \( A \) for any \( c \in \mathbb{C} \). In 1981, Block completely classified all simple modules over the Weyl algebra \( A \) and the Lie algebra \( \mathfrak{sl}_2 \) (see [B]).

Let \( Z(U_q(\mathfrak{sl}_2)) \) be the center of quantum group \( U_q(\mathfrak{sl}_2) \). If \( q \) is a root of unity, then \( K^n, E^n, F^n \in Z(U_q(\mathfrak{sl}_2)) \) for some positive integer \( n \) and the quotient \( U_q(\mathfrak{sl}_2)/\langle K^n - c_1, E^n - c_2, F^n - c_3 \rangle \) is a finite-dimensional algebra for all \( c_1 \in \mathbb{C}^* \) and \( c_2, c_3 \in \mathbb{C} \). In this case, all simple modules have been determined (See [CK]). However, it is still open to classify all simple modules over quantum group \( U_q(\mathfrak{sl}_2) \) for generic \( q \in \mathbb{C}^* \).

The quantum torus \( \mathbb{C}_\nu[x^{\pm 1}, y^{\pm 1}] \) is the quantum analogue of the Weyl algebra \( A \), which arises as a localization of some group algebra (see [M]) and plays an important role in noncommutative geometry (see [M]). If \( \nu \) is a root of unity of order \( n \), the center algebra of the quantum torus is generated by \( x^n, y^n \), and the quotient algebra \( \mathbb{C}_\nu[x^{\pm 1}, y^{\pm 1}]/\langle x^n - c_1, y^n - c_2 \rangle \) is isomorphic to \( \mathfrak{gl}_n(\mathbb{C}) \) for all \( c_1, c_2 \in \mathbb{C}^* \). It is also an open problem to classify all simple modules over the quantum torus for generic \( \nu \in \mathbb{C}^* \).

Similar to the classical case, let \( Z_q \) be the Casimir element of \( U_q(\mathfrak{sl}_2) \), then the quotient algebra \( U_q(\mathfrak{sl}_2)/\langle Z_q - c \rangle \) is isomorphic to a subalgebra of \( \mathbb{C}_\nu[x^{\pm 1}, y^{\pm 1}] \) for any \( c \in \mathbb{C} \).

In this paper, we classify all simple modules over the quantum torus \( \mathbb{C}_\nu[x^{\pm 1}, y^{\pm 1}] \) and the quantum group \( U_q(\mathfrak{sl}_2) \) for generic \( \nu = q^2 \).

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2. Localizations

In this section, we first recall some definitions and facts about localizations of noncommutative rings, then we list some known results about simple modules over noncommutative rings and their localizations (see [B] and [BC]).

Let \( R \) be a ring with 1 and \( S \) a multiplicative subset of \( R \) containing 1. We say that \( S \) satisfies the left Ore condition if

\[
Rs \cap Sa \neq \emptyset, \ \forall (s, a) \in S \times R.
\]

A localization of \( R \) with respect to \( S \) is a ring \( B = S^{-1}R \) containing \( R \) as a subring such that every \( s \in S \) is invertible and \( B = \{ s^{-1}a \mid s \in S, a \in R \} \). The localization \( B = S^{-1}R \) exists if and only if \( S \) has no zero divisor and satisfies the left Ore condition.

**Lemma 2.1** (Lemma 2.4.2 of [B]). Suppose that \( B \) is a localization of \( R \) with respect to \( S \) and a principal left ideal domain which is not a division ring, \( M \) is a simple \( S \)-torsion-free \( R \)-module, \( \alpha \in R \) is irreducible in \( B \) and annihilates some nonzero element of \( M \). Then \( M \cong \text{Soc}_R B / B \alpha \).

**Remark 2.2.** Suppose that \( \alpha \in R \) is irreducible in \( B \), then \( B / B \alpha \) is a simple \( B \)-module. However, the \( R \)-module \( \text{Soc}_R B / B \alpha = (R + B \alpha) / B \alpha \) may be not simple. The Lemma 5.2 gives examples for this case.

3. Quantum torus \( \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \)

Let \( \nu \in \mathbb{C}^* \) be generic and \( R = \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \) with the defining relation \( \nu xy = yx \). In this section, we classify all simple \( R \)-modules.

Let \( S = \mathbb{C}[x, x^{-1}] \setminus \{0\} \), which is multiplicative, contains 1 and has no zero divisor.

**Lemma 3.1.** The subset \( S \) satisfies the left Ore condition.

**Proof.** If \( a = 0 \), we have \( 0 \in Sa \cap Rs \) for all \( s \in S \).

For all \( s = s(x) \in \mathbb{C}[x, x^{-1}] \) and \( a = \sum f_i(x) y^i \in R \setminus \{0\} \), choose

\[
h(x) = \prod_{f_i \neq 0} s(\nu^i x), \quad b_i(x) = \frac{h(x) f_i(x)}{s(\nu^i x)} \in \mathbb{C}[x, x^{-1}].
\]

Then we have

\[
R \cdot s(x) \ni \left( \sum b_i(x)y^i \right) s(x) = \sum h(x) f_i(x) y^i = h(x) a \in S \cdot a.
\]

So \( S \) satisfies the left Ore condition. \( \square \)

The localization of \( R \) with respect to \( S \) is \( B = S^{-1}R = \mathbb{C}[x, y, y^{-1}] \). In particular, \( yf(x) = f(\nu x)y \), for all \( f(x) \in \mathbb{C}[x] \).

**Lemma 3.2.** The ring \( B = \mathbb{C}[x, y, y^{-1}] \) is a principal left ideal domain and is not a division ring.

**Proof.** In fact, \( \alpha \in B \) is invertible if and only if it is a non-zero monomial in variable \( y \). So \( B \) is not a division ring. Moreover, \( B \) is Euclidean, and then it is a principal left ideal domain. \( \square \)

**Lemma 3.3.** For any \( \lambda \in \mathbb{C}^* \), let \( M_{\lambda} \) be the vector space \( \mathbb{C}[y, y^{-1}] 1_{\lambda} \). Then \( M_{\lambda} \) is a simple module over \( R \) with \( x \cdot 1_{\lambda} = \lambda 1_{\lambda} \) and \( y \) acts by multiplying.
Theorem 3.4. Let $U$.

Proof. Similar to Lemma 3.1, Lemma 4.2.

The localization of $U$.

Proof. Straightforward.

Simple modules over $U$.

Proof. For all $v = (\sum_{i=k}^l c_i y^i)1_\lambda \neq 0$, we have

$$\lambda^{-n} x^n \cdot v = (\sum_{i=k}^l \nu^{-in} c_i y^i)1_\lambda.$$ 

Since $\nu$ is generic, the matrix $(q^{-in})_{k \leq i \leq l, 0 \leq n \leq l-k}$ is invertible, this forces $c_i y^i 1_\lambda \in M$. Because $v \neq 0$, we have $c_i y^i 1_\lambda \neq 0$ for some $i$ and $1_\lambda = \frac{1}{y^{-i}} c_y y^i 1_\lambda \in M$. So $M$ is simple. \qed

Theorem 3.4. Let $M$ be a simple module over $R$. Then one of the following holds:

(i) There exists a $\lambda \in \mathbb{C}^*$ such that $M \cong M_\lambda$;

(ii) There exists an $\alpha \in R$ such that $\alpha$ is irreducible in $B$ and $M \cong (R + B\alpha)/B\alpha$.

Proof. If $M$ is $S$-torsion, then there exists $\lambda \in \mathbb{C}^*$ and $v \in M$ such that $xv = \lambda v$ and $M = \mathbb{C}[y, y^{-1}]v$, which is isomorphic to $M_\lambda$.

Now assume that $M$ is $S$-torsion-free. By Lemma 2.1 there exists $\alpha \in R$ such that $\alpha$ is irreducible in $B$ and

$$M \cong \text{Soc}_R B/B\alpha = (R + B\alpha)/B\alpha.$$ 

\qed

4. Quantum group $U_q(\mathfrak{sl}_2)$

Let $q \in \mathbb{C}^*$ be generic. The quantum group $U_q(\mathfrak{sl}_2)$ is the complex unital algebra generated by elements $E, F, K, K^{-1}$ with relations

$$KK^{-1} = 1, KEk^{-1} = q^2 E, KFK^{-1} = q^{-2} F,$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$ 

It is well known that the center of $U_q(\mathfrak{sl}_2)$ is the polynomial algebra $\mathbb{C}[Z_q]$ in variable (see [3])

$$Z_q = EF + \frac{1}{(q - q^{-1})^2} (q^{-1} K + qK^{-1}).$$

For any simple $U_q(\mathfrak{sl}_2)$-module $V$, there exists a $c \in \mathbb{C}$ such that $Z_q$ acts as $c \cdot \text{id}_V$. Thus the classification problem of simple $U_q(\mathfrak{sl}_2)$-modules is equivalent to the classification problem of simple modules over $U_c = U_q(\mathfrak{sl}_2)/(Z_q - c)U_q(\mathfrak{sl}_2)$ for all $c \in \mathbb{C}$.

Lemma 4.1. The map $F \mapsto y, K^{\pm 1} \mapsto x^{\pm 1}, E \mapsto cy^{-1} - \frac{1}{(q - q^{-1} c - (q^{-1} K + qK^{-1})y^{-1} = -\psi(y^{-1} K + qK^{-1}y^{-1})}$ defines an algebra injection $\phi : U_c \rightarrow R = \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ with $\nu = q^2$.

Proof. Straightforward. \qed

Let $R_c = \phi(U_c)$ and $S = \mathbb{C}[x, x^{-1}] \setminus \{0\}$ as in Section 3.

Lemma 4.2. The localization of $R_c$ with respect to $S$ exists and $S^{-1} R_c = S^{-1} R$.

Proof. Similar to Lemma 3.1, $S$ satisfies the left Ore condition in $R_c$. Then $S^{-1} R_c$ exists. Since

$$y^{-1} = \frac{(q - q^{-1})^2}{(q - q^{-1})^2 c - (q^{-1} K + qK^{-1})\psi(E)} \in S^{-1} R_c,$$

it is easy to see that $S^{-1} R_c = \mathbb{C}[x][y, y^{-1}] = S^{-1} R$. \qed
Theorem 4.3. Let $V$ be a simple module over $U_c$. Then one of the following holds:

(i) $V$ is a lowest weight module;
(ii) $V$ is a highest weight module;
(iii) $V$ is a simple module of intermediate series;
(iv) There exists an $\alpha \in R_c$ such that $\alpha$ is irreducible in $B$ and $V \cong (R_c + B\alpha)/B\alpha$.

Proof. If $V$ is $S$-torsion, then there exits a $\lambda \in \mathbb{C}^*$ and $v \in V$ such that $K^{\pm 1}v = \lambda \pm 1v$. So $V$ is a simple weight module, and it is known that $V$ is a lowest weight module, a highest weight module or a simple module of intermediate series.

Next assume that $V$ is $S$-torsion-free. By Lemma 5.1, there exists an $\alpha \in R_c$ such that $\alpha$ is irreducible in $B$ and

$$V \cong \text{Soc}_{R_c}B/B\alpha = (R_c + B\alpha)/B\alpha.$$ 

5. Examples of new simple $U_q(\mathfrak{sl}_2)$-modules

For all $f(x), g(x) \in S$, the polynomial $\alpha = f(x)y - g(x)$ is irreducible in $B$. Let $V = (R_c + B\alpha)/B\alpha$. Then there exists $v \in V$ such that

$$Fv = \frac{g(K)}{f(K)}v,$$

$$Ev = \left(c - 2^{-1}K + qK^{-1}\right) \frac{f(q^{-2}K)}{g(q^{-2}K)}v.$$ 

In particular, if $\frac{g(K)}{f(K)} \in \mathbb{C}$ or $\left(c - 2^{-1}K + qK^{-1}\right) \frac{f(q^{-2}K)}{g(q^{-2}K)} \in \mathbb{C}$, $V$ is called a Whittaker module, which is the $q$-analogue of Whittaker module of Lie algebra $\mathfrak{sl}_2$.

Proposition 5.1. If $V$ is of rank one as free $\mathbb{C}[K, K^{-1}]$-module, then one of the following holds:

(i) $F1 = \mu K^n1$ and $E1 = \frac{1}{\mu} \left(c - 2^{-1}K + qK^{-1}\right) K^{-n}1$;
(ii) $E1 = \mu K^n1$ and $F1 = \frac{1}{\mu} \left(c - 2K^{-1}K^{-1}\right) q^{-2n}K^{-1}1$;
(iii) $E1 = \mu K^n(q^{-1}K - x_1)1$ and $F1 = \frac{1}{\mu(q-x_1)^2} (1 - q^{-1}x_2K^{-1}) q^{-2n}K^{-1}1$.

Where $\mu \in \mathbb{C}^*$, $n \in \mathbb{Z}$ and $x_1, x_2$ are solutions of $q^{-1}x + qx^{-1} = q^{-1}c = 0$.

Proof. Suppose that $E1 = f(K)1, F1 = g(K)1$, then $f(K)g(q^{-2}K) = c - \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}$. By direct computations, this proposition holds.

Lemma 5.2. Suppose $V = \mathbb{C}[K, K^{-1}]1$ such that

$$E1 = \mu K^n(q^{-1}K - x_1)1, \quad F1 = \frac{1}{\mu(q-x_1)^2} (1 - q^{-1}x_2K^{-1}) q^{-2n}K^{-1}1.$$ 

(1) If $x_1 = q^{-s+1}, x_2 = q^{s+1}$ for some positive integer $s$, then $V$ is not a simple module.
(2) If $x_1 = -q^{-s+1}, x_2 = -q^{s+1}$ for some positive integer $s$, then $V$ is not a simple module.

In particular, in these cases we have $c = \pm \frac{q^s + q^{-s}}{(q-q^{-1})^2}$. 

Proof. We only prove (1), the proof for (2) is similar.

Let \( f(K) = \sum_{i=0}^{s} a_{i}K^{i} \) such that \( a_{0} \neq 0 \) and \( a_{j} = -q^{-2}a_{j-1}q^{s-j} \) for all \( j \geq 1 \). Then we have

\[
\frac{1}{\mu}K^{-n}q^{2s+1}Ef(K)1 = (K - q^{s+2})f(K)1, \\
\mu(q - q^{-1})^{2}K^{n+1}q^{-2s}Ff(K)1 = (K - q^{-s})f(K)1.
\]

Thus the subspace \( \mathbb{C}[K, K^{-1}]f(K) \) is a proper submodule over \( U_{q}(\mathfrak{sl}_{2}) \).

By the following lemma, the proper submodule \( \mathbb{C}[K, K^{-1}]f(K) \) is simple.

**Lemma 5.3.** Let \( V = \mathbb{C}[K, K^{-1}]1 \) the polynomial module over \( U_{q}(\mathfrak{sl}_{2}) \) such that

\[
E1 = \mu K^{n}(q^{-1}K - x_{1})1, \\
F1 = \frac{1}{\mu(q - q^{-1})^{2}}(1 - q^{-1}x_{2}K^{-1})q^{-2n}K^{-n}1.
\]

If \( x_{1} \neq \pm q^{-s+1} \) for any positive integer \( s \), then \( V \) is a simple module.

Proof. We may assume that \( n = 0 \) and \( \mu = 1 \). The proof for general case is very similar. Then \( E1 = (q^{-1}K - x_{1})1 \) and \( F1 = \frac{1}{(q - q^{-1})^{2}}(1 - q^{-1}x_{2}K^{-1})1 \). By Proposition [5.1] \( x_{1} \cdot x_{2} = q^{2} \).

For an arbitrary \( f(K) = \sum_{i=r}^{s} a_{i}K^{i} \in \mathbb{C}[K, K^{-1}] \) such that \( a_{r} \cdot a_{s} \neq 0 \), we may assume \( r = 0 \) with replacing \( f(K) \) by \( K^{-r}f(K) \). So \( s \geq 0 \). If \( s = 0 \), then we obtain \( 1 \) by multiplying \( a_{s}^{-1} \). Next assume \( s > 0 \). We have

\[
(q^{2s+1}E - K)f(K) \\
= (a_{s-1}(q^{2} - 1) - a_{s}q x_{1})K^{s} \\
+ (a_{s-2}(q^{4} - 1) - a_{s-1}q^{3} x_{1})K^{s-1} \\
\vdots \\
+ (a_{0}(q^{2s} - 1) - a_{1}q^{2s-1} x_{1})K \\
- a_{0}q^{2s+1} x_{1},
\]

and

\[
(q^{-2s}(q - q^{-1})^{2}KF - K)f(K) \\
= (a_{s-1}(q^{-2} - 1) - a_{s}q^{-1} x_{2})K^{s} \\
+ (a_{s-2}(q^{-4} - 1) - a_{s-1}q^{-3} x_{2})K^{s-1} \\
\vdots \\
+ (a_{0}(q^{-2s} - 1) - a_{1}q^{-2s+1} x_{2})K \\
- a_{0}q^{-2s+1} x_{2}.
\]

**Case 1.** \( (q^{2s+1}E - K)f(K) \) is not a scalar of \( f(K) \). Then \( g(K) := (a_{s-1}(q^{2} - 1) - a_{s}q x_{1})f(K) - a_{s}(q^{2s+1}E - K)f(K) \neq 0 \)

and \( g(K) = \sum_{i=0}^{s-1} b_{i}K^{i} \) for some constants \( b_{i} \in \mathbb{C} \).

**Case 2.** \( (q^{-2s}(q - q^{-1})^{2}KF - K)f(K) \) is not a scalar of \( f(K) \). Similar to Case 1.

**Case 3.** Both \( (q^{2s+1}E - K)f(K) \) and \( (q^{-2s}(q - q^{-1})^{2}KF - K)f(K) \) are scalars of \( f(K) \).
Note that \( a_0 q^{2s+1} x_1 \neq 0, a_0 q^{-2s-1} x_2 \neq 0 \). By
\[
\begin{align*}
  a_0(q^{2s} - 1) - a_1 q^{2s-1} x_1 &= -a_1 q^{2s+1} x_1, \\
  a_0(q^{-2s} - 1) - a_1 q^{-2s+1} x_2 &= -a_1 q^{-2s-1} x_2,
\end{align*}
\]
we have \( a_1 \neq 0 \) and \( x_2 = x_1 q^{2s} \). Hence \( x_2 = \pm q^{s+1} \) and \( x_1 = \pm q^{-s+1} \). Contradiction to assumption.

Induction on \( s \), we may obtain \( s = 0 \) and then \( V \) is a simple module. \( \square \)

**Proposition 5.4.** Suppose that \( V \) is a simple \( U_q(\mathfrak{sl}_2) \)-module and it is of rank one as free \( \mathbb{C}[K,K^{-1}] \)-module. Then one of the following holds:

\( (i) \) \( F \mathbf{1} = \mu K^n \mathbf{1} \) and \( E \mathbf{1} = \frac{1}{\mu} \left( c - \frac{q^{-1} K + qK^{-1}}{(q^{-q} - 1)^2} \right) K^{-n} \mathbf{1}; \)

\( (ii) \) \( E \mathbf{1} = \mu K^n \mathbf{1} \) and \( F \mathbf{1} = \frac{1}{\mu} \left( c - \frac{qK + q^{-1} K^{-1}}{(q^{-q} - 1)^2} \right) q^{-2n} K^{-n} \mathbf{1}; \)

\( (iii) \) \( E \mathbf{1} = \mu K^n(q^{-1} K - x_1) \mathbf{1} \) and \( F \mathbf{1} = \frac{1}{\mu(q^{-q} - 1)^2} (1 - q^{-1} x_2 K^{-1}) q^{-2n} K^{-n} \mathbf{1}. \)

Where \( \mu \in \mathbb{C}^*, n \in \mathbb{Z}, x_1, x_2 \) are solutions of \( q^{-1} x + qx - (q - q^{-1})^2 c = 0 \) and \( x_1 \neq \pm q^{-s+1} \) for all positive integer \( s \).

**Proof.** For case (i), we have \( \frac{1}{\mu} K^{-n} F \cdot \phi(K) = \phi(q^2 K) \), then it is easy to know the module is simple.

For case (ii), the proof is similar.

Case (iii) can be obtained by Lemmas 5.2-5.3. \( \square \)

Note that the Whittaker modules are those modules (i) and (ii) in Proposition 5.4 with \( n = 0 \). Consequently, we have the following corollary.

**Corollary 5.5.** The Whittaker modules over \( U_q(\mathfrak{sl}_2) \) are simple.

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