Higher-order photon rings can be expected to be detected in a more detailed image of the black hole found in future observations. This rings are lensed images of the luminous matter surrounding the black hole and are formed by photons that loop around it. In this paper we have succeeded to derive an analytical expression for the shape of the higher-order rings in the form that is most convenient for application: the explicit equation of the curve in polar coordinates. The formula describes the apparent shape of the higher-order image of the circular orbit with the given radius around Schwarzschild black hole as viewed by distant observer with an arbitrary inclination. For the derivation, the strong deflection limit of the gravitational deflection is used. Our formula is a simple and efficient alternative to the numerical calculation of ray trajectories, with the main application to studying the shape of $n = 2$ and $n = 3$ photon rings.

## I. INTRODUCTION

One consequence of the gravitational deflection of light is the formation of multiple images of the same source. In the case when gravitational lensing occurs on a black hole, so-called higher-order images can appear. These images are formed by photons that have experienced one or more turns around the black hole before reaching the observer.

Higher-order images have been extensively studied, see, e.g., reviews [1, 3]; see also Introduction in our previous article [4]. Among early papers on the subject, one can refer to [5, 10]. More active research started about two decades ago. Numerical studies of higher-order images were presented in the paper by Virbhadra and Ellis [11] (the authors introduced the term ‘relativistic images’), see also [12, 13]. In analytical studies, the strong deflection limit of gravitational deflection was used: an analytical formula describing the logarithmically diverging deflection angle of photons making one or more revolutions around the black hole [1, 14, 15]. Analytical calculations of relativistic images were developed in the works of Bozza et al. [14–22]. The strong deflection limit (also known as strong field limit) was also used in the series of paper by Eiroa et al. [23–26]. Gravitational lensing beyond the weak deflection approximation was studied also in Frittelli et al. [27] and Perlick [1, 28]. Further, higher-order images have been also investigated both numerically and analytically by many groups [29–52]. The strong deflection limit for massive particles was first found by Tsupko [53], see also [54, 55].

Most of the articles considered the case when the source is compact and is at a large distance from the black hole, where the gravity of the black hole can be neglected. The important generalization has been made in the work of Bozza and Scarpetta [57] who have developed the strong deflection limit for sources at arbitrary distance from the black hole. It was further used in the papers of Bozza et al. [2, 51, 58]. In particular, this approach was used for analytical investigations of higher-order images of the accretion disk in the work of Aldi and Bozza [58].

The recent emergence of great interest in the subject of higher-order images is associated with an observational breakthrough in black hole imaging. In 2019, the Event Horizon Telescope has been presented the shadow of supermassive black hole in the M87 galaxy [59–67] (see also recent results about Sgr A* [68–73]). After that, attention has been attracted to the study of higher-order photon rings which can be expected in a more detailed image, being concentrated near the edge of the black hole shadow [74–82]. These rings are lensed images of the luminous matter surrounding the black hole. It is now quite common to denote these images by the number of half-orbits $n$, e.g., [74, 75, 83, 84]. The direct image is denoted as $n = 0$, secondary image has $n = 1$, and we refer all images $n \geq 2$ as higher-order photon rings (Fig. 1).

Numerical simulations are used to obtain a detailed image of the black hole that can be compared with observations, see Event Horizon Telescope papers [60, 62, 65, 72] and, e.g., [59, 63, 103]. At the same time, a great number of works are devoted to analytical studies of the black hole shadow, in which the size and shape of the shadow boundary (also known as the critical curve) are studied. E.g., the deformed shape of the shadow of the Kerr black hole has known analytical representation, e.g., [104, 107]: for an observer at an arbitrary distance from the black hole, see [107, 109]. Analytical investigations of shape of Kerr black hole shadow can be found, e.g., in [20, 104–124]. For another examples of theoretical consideration of black hole shadow see, e.g., [123, 150]. We refer to the recent review by Perlick and Tsupko [107] as an overview of analytical studies of the shadow, see also the review by Cunha and Herdeiro [106].

The shape of the higher-order photon rings is very close to the boundary of the black hole shadow (critical curve),...
However, it differs from it and is of considerable interest from the observational point of view, e.g. [73, 74]. Even in the simplest case of Schwarzschild black hole, for which the shadow boundary is a circle, the photon ring is a circle only for an observer on the axis of symmetry. For observer with inclination, this is a deformed curve, the shape of which would be very useful to know in an analytical form.

In this paper, we derive the analytical expression for the shape of higher-order rings in the form that is most convenient for application: the explicit equation of the curve in polar coordinates. The formula describes the apparent shape of higher-order image of circular orbit with given radius around the Schwarzschild black hole for distant observer with an arbitrary inclination. A simple analytical formula for the curve allows us to find a number of properties of the ring images.  

In our derivation, we use the strong deflection limit of light deflection for arbitrary source position mentioned above [2, 57]. Since photon rings are formed by light sources near the black hole, this method is very convenient for analytical calculation of their properties (strong deflection approximation is appropriate for \( n \geq 2 \) rings). E.g., in our previous work [4], we have considered the thin accretion disk around Schwarzschild black hole and the observer located on the symmetry axis (polar view). For that configuration, we calculated analytically the angular radii, thicknesses, and solid angles of higher-order rings in the form of compact analytical expressions; we also made estimates for fluxes. In our present article, we consider the observer with arbitrary inclination angle and focus only on the deformed shape of photon ring images of higher orders (\( n \geq 2 \)).

The paper is organized as follows. In the next Section we derive an explicit analytic formula for the shape of the higher-order rings. In the Section III we present graphs, and discuss properties of higher-order rings that are derived analytically using our results. The Section IV is our Conclusions. In the Appendix, we present the alternative derivation of our formula based on results of Aldi and Bozza [58], finding complete agreement.

II. POLAR CURVE FOR THE SHAPE OF HIGHER-ORDER PHOTON RINGS

In this Section we derive the explicit analytical expression for the shape of the higher-order photon rings, using the strong deflection limit of gravitational deflection.

---

1 Throughout the article, by 'circular orbit' or 'circular ring' we mean thin circular radiating ring of given constant radius orbiting the black hole in the equatorial plane (e.g., Luminet [157]). This ring should be not confused with the 'photon ring' (sometimes referred to as 'ring of emission'), which are lensed image of circular orbit on the observer’s sky. See also the discussion on p.7 of [107].
FIG. 2. Geometry of the problem and the variables used. The left panel shows the circular orbit of radius \( r_S \) and the light ray from an element of circular orbit making one complete revolution around the black hole and arriving at the observer with inclination \( \vartheta_O \). Such ray gives the tertiary (\( n = 2 \)) image of the source. The angle \( \gamma \) is measured in the plane of the ray trajectory. The picture on the right shows the observer’s reference frame. The angle \( \gamma \) in the plane of the ray is related to the angle \( \alpha \) in the observer’s sky by the formula (8), see article [157] and Fig. 3 there. The angle \( \alpha \) is then converted to the more familiar angle \( \phi \), measured counterclockwise from the horizontal axis, see eq. (10). The final shape \( b_2(\varphi) \) of the \( n = 2 \) lensed image of the circular orbit is given by the formula (11).

Note that the variable \( \tilde{\phi} \) is defined in the ray plane and therefore should not be confused with the variable \( \phi \) in Eq. (1).

We start from Eq. (2) and find

\[
b = b_{cr} \left[ 1 + f(r_S) e^{-\Delta \tilde{\phi}} \right].
\]

Using Eq. (6) in Eq. (5), we find:

\[
b(\gamma) = 3\sqrt{3}m \left[ 1 + f(r_S) e^{-2\pi-\gamma} \right].
\]

The second step of derivation is to relate the angle \( \gamma \) characterizing the trajectory with some angle in the observer’s frame. A convenient way to do this is to follow Luminet’s paper [157] who used the polar angle \( \alpha \) and got the following simple relation:

\[
\cos \gamma = \frac{\cos \alpha}{\sqrt{\cos^2 \alpha + \cot^2 \vartheta_O}},
\]

where \( \epsilon = \frac{b - b_{cr}}{b_{cr}} \ll 1, \quad b_{cr} = 3\sqrt{3}m \),

\[
f(r_S) = \frac{65 \left( 1 - \frac{3m}{r_S} \right)^2}{(3 + \sqrt{3})^2} \left( 3 + \sqrt{3 + \frac{18m}{r_S}} \right)^{-2}.
\]
Using eq. (8) in (7), we find the apparent shape of image of circular orbit as the function $b(\alpha)$:

$$b(\alpha) = 3\sqrt{3}m \left\{ 1 + f(r_S) \exp \left[ -2\pi - \arccos \left( \frac{\cos \alpha}{\sqrt{\cos^2 \alpha + \cot^2 \vartheta_O}} \right) \right] \right\} .$$  \hspace{1cm} (9)

Our third step, for purposes of convenience mainly, is to rewrite the formula (9) through the polar angle $\varphi$, measured in the ‘usual’ way: counterclockwise from the horizontal axis (Fig 2). We write:

$$\cos \alpha = \cos(3\pi/2 - \varphi) = -\sin \varphi .$$  \hspace{1cm} (10)

Using the relation $\arccos(-x) = \pi - \arccos x$, we obtain finally the shape of tertiary image ($n = 2$ photon ring) of circular orbit with radius $r_S$ as the polar curve $b_2(\varphi)$:

$$b_2(\varphi) = 3\sqrt{3}m \left\{ 1 + f(r_S) \exp \left[ -3\pi + \arccos \left( \frac{\sin \varphi}{\sqrt{\sin^2 \varphi + \cot^2 \vartheta_O}} \right) \right] \right\} .$$  \hspace{1cm} (11)

To obtain a complete curve, the angle $\varphi$ must vary from 0 to $2\pi$.

The next image ($n = 3$ photon ring) is formed by photons that have passed behind the black hole and then turned around once more before reaching the observer. Such ray experiences the following change of azimuthal coordinate:

$$\Delta \tilde{\varphi} = 3\pi + (\pi - \gamma) .$$  \hspace{1cm} (12)

Compared to $n = 2$ image, the light ray forming $n = 3$ image reaches the point in opposite part of the observer’s sky. Therefore we have to replace $\alpha$ with $\pi + \alpha$ in the Eq.(8). We get:

$$\cos \gamma = \frac{-\cos \alpha}{\sqrt{\cos^2 \alpha + \cot^2 \vartheta_O}} .$$  \hspace{1cm} (13)

Correspondingly, we find the dependence $b(\alpha)$ for $n = 3$ image as

$$b(\alpha) = 3\sqrt{3}m \left\{ 1 + f(r_S) \exp \left[ -4\pi - \arccos \left( \frac{-\cos \alpha}{\sqrt{\cos^2 \alpha + \cot^2 \vartheta_O}} \right) \right] \right\} .$$  \hspace{1cm} (14)

Transforming from $\alpha$ to $\varphi$ with eq.(10), we obtain finally the shape of $n = 3$ photon ring as the polar curve $b_3(\varphi)$:

$$b_3(\varphi) = 3\sqrt{3}m \left\{ 1 + f(r_S) \exp \left[ -4\pi + \arccos \left( \frac{\sin \varphi}{\sqrt{\sin^2 \varphi + \cot^2 \vartheta_O}} \right) \right] \right\} .$$  \hspace{1cm} (15)

To obtain the formula for general $n$, we notice that for rings with $n = 2, 4, 6, \ldots$ we have:

$$\Delta \tilde{\varphi} = n\pi + \gamma ,$$  \hspace{1cm} (16)

and eq.(8) should be used, whereas for $n = 3, 5, 7, \ldots$ we have:

$$\Delta \tilde{\varphi} = n\pi + (\pi - \gamma) = (n + 1)\pi - \gamma ,$$  \hspace{1cm} (17)

and eq.(13) should be used.

As a result, we can write the general formula valid for all higher-order images of $n$-th order:

$$b_n(\varphi) = 3\sqrt{3}m \left\{ 1 + f(r_S) \exp \left[ -(n + 1)\pi + \arccos \left( \frac{\sin \varphi}{\sqrt{\sin^2 \varphi + \cot^2 \vartheta_O}} \right) \right] \right\} , \hspace{1cm} n \geq 2 .$$  \hspace{1cm} (18)

For the reader’s convenience, we remind our notations here. Eq. (18) describes the apparent shape of higher-order image (photon ring of $n$-th order) of circular orbit with constant radius $r_S$ in the equatorial plane of Schwarzschild black hole. Angle $\vartheta_O$ defines the observer’s inclination angle measured from the axis of symmetry. Variable $b_n$ is the impact parameter in the observer’s sky; polar angle $\varphi$ is set in the usual way: counterclockwise from the horizontal axis (Fig 2). Function $f(r_S)$ is given by (4). Mass parameter $m$ is defined in (1), and the shadow boundary has the
radius \( b_{cr} = 3\sqrt{3} m \).

If one considers an accretion disk with given inner \( r^i_S \) and outer \( r^o_S \) radii, then the photon ring will also have the inner \( b^i_n(\varphi) \) and outer \( b^o_n(\varphi) \) boundaries, determined by the values of these radii. The thickness of \( n \)-photon ring as the function of \( \varphi \) will be:

\[
\Delta b_n(\varphi) \equiv b^o_n(\varphi) - b^i_n(\varphi) = 3\sqrt{3} m \left[ f_S^{\text{out}}(\varphi) - f_S^{\text{in}}(\varphi) \right] \exp \left[ -(n+1)\pi + \arccos \left( \frac{\sin \varphi}{\sqrt{\sin^2 \varphi + \cot^2 \vartheta}} \right) \right],
\]  \hspace{1cm} (19)

and the area of \( n \)-photon ring will be:

\[
\Delta S_n = \frac{1}{2} \int_0^{2\pi} (b^o_n(\varphi) - b^i_n(\varphi)) \, d\varphi \simeq 3\sqrt{3} m \int_0^{2\pi} \Delta b_n(\varphi) \, d\varphi,
\]  \hspace{1cm} (20)

or \( \Delta S_n = 6\sqrt{3} m \pi \langle \Delta b_n(\varphi) \rangle \), where \( \langle \Delta b_n(\varphi) \rangle = \frac{1}{2\pi} \int_0^{2\pi} \Delta b_n(\varphi) \, d\varphi \).  \hspace{1cm} (21)

\[\]

III. DISCUSSION

In this Section, we present graphs for the higher-order photon rings based on our formula, and derive some analytical properties of the rings.

A. Shape of photon rings

First of all, our analytical formula allows us to plot graphs showing the shape of photon rings. The Fig. 3 shows graphs for different radii of the emitting circle and different angles of inclination of the observer. It should be kept in mind that the difference between the shape of the higher-order rings and the shadow boundary is difficult to notice by looking at the ring itself, because the higher-order images are exponentially close to the shadow edge. In particular, for the Schwarzschild black hole considered here, the shape of the higher-order rings differs very little from the circle.

To demonstrate the shape of the \( n = 2 \) ring, in Fig. 3 we plot the impact parameter \( b_2 \) as the function of \( \varphi \), where \( \varphi \) is presented on the horizontal axis (left panels).

In addition, for a better understanding of the shape, we use the following technique: instead of the plotting the impact parameter itself, we make the polar plot of the ‘excess’

\[
\tilde{b}_2(\varphi) = b_2(\varphi) - 3\sqrt{3} m.
\]  \hspace{1cm} (22)

Such a graph shows the radial offset of the image point from the shadow boundary in each polar direction, see right panels in Fig. 3.

In the graph for the angle \( \vartheta_O = 17^\circ \) in Fig. 3, one can notice a sinusoidal oscillation of the impact parameter; see also Fig. 7 in [4] for Kerr case. Having an analytical formula for the curve allows us to derive this property analytically. Expanding eq. (18) in small angle \( \vartheta_O \), we find:

\[
b_n(\varphi) \simeq 3\sqrt{3} m \left\{ 1 + f(r_S) e^{- (n+1)\pi/2} \times \left(1 - \vartheta_O \sin \varphi\right) \right\},
\]  \hspace{1cm} (23)

As can be seen, the linear term of the expansion is proportional to \( \sin \varphi \). We conclude that for nearly polar observer, \( \vartheta_O \ll 1 \), the ring size oscillates sinusoidally.

We also mention that for polar observer (zero inclination, \( \vartheta_O = 0 \)), the higher-order images have the circular shape with radius (in terms of impact parameter):

\[
b_n = 3\sqrt{3} m \left\{ 1 + f(r_S) e^{- (n+1)\pi/2} \right\},
\]  \hspace{1cm} (24)

which agrees with the results of our previous paper [4].

B. Thickness of photon rings

In order to discuss the thickness (19) of the photon rings, consider a thin luminous accretion disk with given inner and outer radii, \( r^i_S \) and \( r^o_S \).

Let us discuss the tertiary image of such a disk, as viewed at different inclinations. If the observer looks at the disk with zero inclination (polar view, \( \vartheta_O = 0 \)), all parts of the ring will have the same thickness. All of them will be formed by photons for which the change of angular coordinate during the motion to observer is equal to \( \Delta \hat{\phi} = 5\pi/2 \), see [24] with \( n = 2 \). In this case, the thickness is so small that it is rather difficult to show in the picture, see Fig. 7 in [4].

To make the thickness of the ring on the graph more noticeable, it is better to consider the observer at a high
FIG. 3. Apparent shape of tertiary image (n = 2 photon ring) of concentric rings of different radii $r_S$ for different inclinations $\vartheta _{\text{O}}$ of the observer. The curves are plotted according to the analytical formula (11). Each left panel shows the impact parameter $b_2$ as the function of angle $\varphi$ in the observer’s sky, for the following values of ring radii $r_S$: 3.3m, 4m, 6m, 10m, 20m. Each right panel shows the corresponding excess $\tilde{b}_2(\varphi)$ plotted in polar coordinates, see eq. (22), which can provide better understanding of the deformed shape of the image.

Inclination (close to the equatorial view, $\vartheta _{\text{O}} \lesssim \pi / 2$). For such observer, different parts of the photon ring will have quite different thicknesses. The lower parts of the image will have the biggest thickness because they are formed by photons with a smaller total bending. Namely, for the tertiary ring, the lowest points of the image are formed by photons with $\Delta \tilde{\varphi} \sim 2\pi$ and the upper points of the image will have $\Delta \tilde{\varphi} \sim 3\pi$.

Based on the discussion above, in Fig 4a, we try to show the thickness of the ring on a real scale, for the case when observer is close to equatorial plane and, correspondingly, the lower part of the ring is relatively thick. The dependence of thickness on the polar direction can be understood from Fig 4b,c.
FIG. 4. Thickness and shape of tertiary ($n = 2$) ring image of luminous accretion disk with inner and outer boundaries $r^\text{in}_S = 6\text{m}$ and $r^\text{out}_S = 30\text{m}$, as seen by nearly equatorial observer, $\vartheta_O = 85^\circ$. All graphs are based on the analytical formula (11). Panel 'a' shows the photon ring in real scale. The shape of the curve is almost indistinguishable from a circle (we remind that the shadow of Schwarzschild black hole is a circle with the radius $b_{cr} = 3\sqrt{3} \approx 5.196\text{m}$). However, it is still possible to notice the thickness of the ring at the lower half of the image. Panel 'b' shows the impact parameter $b_2$ in the observer’s sky as the function of $\varphi$. Panel 'c' shows the polar plot of the excess $\tilde{b}_2(\varphi)$, defined by eq.(22).

The higher-order ring image of accretion disk of given size has the maximum thickness at $\varphi = 3\pi/2$, see Eq.(19). The maximum thickness of the $n$-ring ($\Delta b_n$)$_{\text{max}}$ depends on the inclination $\vartheta_O$ as

$$(\Delta b_n)_{\text{max}} \propto \exp \left[ \arccos \left( -\frac{1}{\sqrt{1 + \cot^2 \vartheta_O}} \right) \right],$$

or, using $\arccos(-x) = \pi - \arccos(x)$, as

$$(\Delta b_n)_{\text{max}} \propto \exp \left[ -\arccos \left( \frac{1}{\sqrt{1 + \cot^2 \vartheta_O}} \right) \right].$$

In Fig.4 we present the graph of the right-hand side of the Eq.(20). We can also conclude that the observation of the tertiary ring might be most promising for the observer located slightly above the equatorial plane.

C. Area occupied by the ring

Let us calculate the area of $n$-th photon ring, as viewed at the small inclination, $\vartheta_O \ll 1$. Up to the linear order in angle $\vartheta_O$, we find:

$$\Delta S_n \simeq 54\pi m^2 [f(r^\text{out}_S) - f(r^\text{in}_S)] e^{-(n+1)\pi/2}.$$  

The term, linear in $\vartheta_O$ in (23), gives zero contribution when integrated over $\varphi$.

We can conclude that, for nearly polar observer, the area occupied by the image does not depend on the inclination of observer (up to the first order in $\vartheta_O$). This means that some of the results obtained in the paper [4] for zero inclination can be used to make estimations for the case of nonzero inclination as well.

IV. CONCLUSIONS

(i) Studies of higher-order photon rings, especially tertiary (number of half-orbits $n = 2$) and quaternary ($n = 3$), are of great interest in the perspective of future observations. In this regard, it would be very useful to have an analytical description of the shape of these rings, even for simple case.

(ii) In this paper, we derive the equation describing the shape of higher-order ring as the polar curve: the impact parameter as the function of the angle on the observer’s sky. Our formula describes the apparent shape of lensed image of circular ring with given radius around the Schwarzschild black hole, for an arbitrary observer’s inclination (Fig.2). The analytical expression for any higher-order ring $n \geq 2$ is given by Eq.(18). The summary of all variables used is given after Eq.(18).
(iii) Our formula is a simple and efficient alternative to the numerical calculation of ray trajectories. First, it allows one to easily plot the higher-order ring curves. As an example of the use of our formula, we have presented graphs illustrating the shape of the tertiary ring (Figs. III). Second, the analytical expression allows one to derive a number of universal properties without the need for numerical simulations. As an example, we derived analytically some properties of rings, see Section III.

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APPENDIX A. DERIVATION FROM RESULTS OF ALDI AND BOZZA

In this Appendix we show how to obtain our formula from the results of Aldi and Bozza [58]. In their work, the authors have found an analytical relationship between the higher-order image parameters in the observer’s sky and the position of the emitting source in the equatorial plane. For every image order \( n \), the two halves of the accretion disk are considered separately, with the corresponding choice of the parameter \( m \) (should not be confused with our mass parameter) and the additional parameter \( \sigma \) (see below). We have succeeded to show that for our problem these two ‘branches’ of solution can be combined into one formula containing only the image order \( n \). For simplicity, we compare only \( n = 2 \) and \( n = 3 \) rings. The full agreement is found.

According to [58], the variables \((\epsilon, \xi)\) specifying the position of the image element on the observer’s sky are associated with variables \((r_e, \phi_e)\) specifying the position of the source element as

\[
\xi = -\sigma \frac{\tan \phi_e}{\sqrt{\mu_O^2 + \tan^2 \phi_e}}, \tag{28}
\]

and

\[
\epsilon = \tilde{e}(r_e) \tilde{\epsilon}(\phi_e), \tag{29}
\]

where the function \( \tilde{e}(r_e) \) agrees with our function \( f(r_S) \) (we will show it later), and the function \( \tilde{\epsilon}(\phi_e) \) is given by

\[
\tilde{\epsilon}(\phi_e) = \exp \left[ -m \pi - \sigma \arcsin \sqrt{\mu_O^2 \cos^2 \phi_e + \sin^2 \phi_e} \right]. \tag{30}
\]

Here \( m \) is the number of polar inversions of the photon, which depends on the image order \( n \) and the considered half of the disk, see the explanations after Eq. (3.28) in [58]. In turn, the parameter \( \sigma \) is

\[
\sigma = \pm (-1)^m. \tag{31}
\]

The choice of sign in \( \sigma \) is explained after Eq. (3.21) in [58]. One needs to choose a positive sign for photons emitted upwards from the disk and negative for photons emitted downwards [58]. Variable \( \mu_O \) defines the observer inclination: \( \mu_O \equiv \cos \phi_O \). Also, the position angle in the observer sky \( \varphi \) (note that authors [58] use another letter here) is given by

\[
\varphi = -\sigma \arccos(-\xi). \tag{32}
\]

We start from \( n = 2 \) photon ring. First, we rewrite our eq. (11) as

\[
\epsilon = f(r_S) \exp \left[ -3\pi + \arccos \left( \frac{\sin \varphi}{\sqrt{\sin^2 \varphi + \cot^2 \theta_O}} \right) \right]. \tag{33}
\]

For \( n = 2 \) ring, with our geometry, we need to choose ‘plus’ sign in \( \sigma \) for this image.

Authors [58] put the observer to the position \( \phi_O = \pi \) and consider two halves of the accretion disk separately. First, consider the half of the disk closer to the observer, \( \pi/2 < \phi_e < 3\pi/2 \). According to discussion after eq.(3.28) in [58], for image of \( n = 2 \) order, one needs to take \( m = 2 \). Therefore, we find \( \sigma = +1 \).

With these values of \( m \) and \( \sigma \), Eqs. (32) and (28) lead to:

\[
\varphi = -\arccos \frac{\tan \phi_e}{\sqrt{\mu_O^2 + \tan^2 \phi_e}}. \tag{34}
\]

Using the relation \( \sin(\arccos x) = \sqrt{1 - x^2} \), one can get:

\[
\sin \varphi = -\frac{\mu_O}{\sqrt{\mu_O^2 + \tan^2 \phi_e}} < 0. \tag{35}
\]
Now we transform arcsin to arccos in the formula (30), using the relation \( \arcsin x = \arccos \sqrt{1 - x^2} \):
\[
\arcsin \sqrt{\mu_O^2 \cos^2 \phi_e + \sin^2 \phi_e} = \arccos \sqrt{1 - \mu_O^2 \cos^2 \phi_e}.
\]

We need to express \( \cos^2 \phi_e \) via \( \varphi \). Squaring the equation (30), we get
\[
\sin^2 \varphi = \frac{\mu_O^2}{\mu_O^2 + \tan^2 \phi_e}.
\]

Rearranging the terms, we find:
\[
\tan^2 \phi_e = \frac{\mu_O^2 \cot^2 \varphi}{1 + \mu_O^2 \cot^2 \varphi}.
\]

Expressing \( \cos^2 \phi_e \) via \( \tan^2 \phi_e \), with help of (38), we find:
\[
\cos^2 \phi_e = \frac{1}{1 + \mu_O^2 \cot^2 \varphi}.
\]

Substituting expression (39) into Eq.(36), after some transformations, we find:
\[
\arcsin \sqrt{\mu_O^2 \cos^2 \phi_e + \sin^2 \phi_e} = \arccos \sqrt{\frac{\sin^2 \varphi}{\sin^2 \varphi + \cot^2 \theta_O}}.
\]

According to Eq. (35), for \( \pi/2 < \phi_e < 3\pi/2 \), we have \( \sin \varphi < 0 \). Therefore,
\[
\sqrt{\sin^2 \varphi} = -\sin \varphi,
\]
and finally we find:
\[
\arcsin \sqrt{\mu_O^2 \cos^2 \phi_e + \sin^2 \phi_e} = \arccos \frac{-\sin \varphi}{\sqrt{\sin^2 \varphi + \cot^2 \theta_O}}.
\]

Using (12) in (30), together with \( m = 2 \) and \( \sigma = +1 \), we find:
\[
\hat{e}(\phi_e) = \exp \left[ -2\pi - \arccos \frac{-\sin \varphi}{\sqrt{\sin^2 \varphi + \cot^2 \theta_O}} \right].
\]

Using the relation \( \arccos(-x) = \pi - \arccos x \), we find the final expression for the function \( \hat{e}(\phi_e) \):
\[
\hat{e}(\phi_e) = \exp \left[ -3\pi + \arccos \frac{\sin \varphi}{\sqrt{\sin^2 \varphi + \cot^2 \theta_O}} \right],
\]
which agrees with our Eq. (33).

Now we will consider the photons emitted from the far side of the disk, where \( -\pi/2 < \phi_e < \pi/2 \). According to explanations after eq.(3.28) in [58], for image of \( n = 2 \) order, one needs to take \( m = 3 \). Therefore, we find \( \sigma = -1 \). With these values of \( m \) and \( \sigma \), Eqs. (32) and (28) lead to:
\[
\varphi = + \arccos \frac{-\tan \phi_e}{\sqrt{\mu_O^2 + \tan^2 \phi_e}}.
\]
From this, one can get
\[
\sin \varphi = + \frac{\mu_O}{\sqrt{\mu_O^2 + \tan^2 \phi_e}} > 0.
\]

Part of transformations of arcsin with the square root in Eq. (34) will be exactly the same, as for previous range of \( \phi_e \), and we reach the formula (41). But now, according to (44), we have \( \sin \varphi > 0 \). Keeping in mind that the values of \( m \) and \( \sigma \) have also changed, we finally come to the same Eq. (41).

Thus, we have shown that with the help of our transformations, formulas for two ranges of angle \( \phi_e \) can be combined into one expression (41), which is in agreement with our Eq. (33).

Now we consider \( n = 3 \) photon ring. With our geometry, we need to choose 'minus' sign in (41) for this image.

First, consider the half of the disk closer to the observer, \( \pi/2 < \phi_e < 3\pi/2 \). According to discussion after eq.(3.28) in [58], for image of \( n = 3 \) order, one needs to take \( m = 4 \). Therefore, we find \( \sigma = -1 \). We obtain:
\[
\varphi = + \arccos \frac{-\tan \phi_e}{\sqrt{\mu_O^2 + \tan^2 \phi_e}}, \quad \sin \varphi > 0.
\]

Analogous to the transformations done for the \( n = 2 \) ring, we find:
\[
\hat{e}(\phi_e) = \exp \left[ -4\pi + \arccos \frac{\sin \varphi}{\sqrt{\sin^2 \varphi + \cot^2 \theta_O}} \right].
\]

For the photons emitted from the far side of the disk, where \( -\pi/2 < \phi_e < \pi/2 \), we have \( m = 3 \) and \( \sigma = +1 \). Correspondingly, we obtain:
\[
\varphi = - \arccos \frac{-\tan \phi_e}{\sqrt{\mu_O^2 + \tan^2 \phi_e}}, \quad \sin \varphi < 0.
\]

Analogous to the transformations done for the \( n = 2 \) ring, we find the same Eq. (41) for these range of \( \phi_e \). Eq. (48) agrees with our Eq. (11).

Now we will show that our function \( f(r_S) \) is equivalent to \( \hat{e}(r_e) \) used in eq.(29) and defined in eq.(3.23) of [58].

We rewrite eq.(4) as:
\[
f(r_S) = \frac{6^5(r_S - 3m)}{(3 + \sqrt{3})^2(3\sqrt{r_S} + \sqrt{3r_S + 18m})^2}.
\]
Using the following relations,

\[ (1 + \sqrt{3})^2 = 2 \left( 2 + \sqrt{3} \right), \]  
(51)

\[ \frac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}, \]  
(52)

we find

\[ f(r_S) = \frac{216 \cdot (2 - \sqrt{3})(\sqrt{3} r_S - \sqrt{r_S + 6m})}{\sqrt{3} r_S + \sqrt{r_S + 6m}}. \]  
(54)

This expression agrees with eq. (3.23) of [58] if one change \( r_S \rightarrow r_e \). Note that authors [58] use the units of the Schwarzschild radius, \( 2MG/c^2 = 1 \).

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