Thermodynamics of Black Holes in Hořava-Lifshitz Gravity

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Abstract

By using the canonical Hamiltonian method, we obtain the mass and entropy of the black holes with general dynamical coupling constant $\lambda$ in Hořava-Lifshitz Gravity. Regardless of whether the horizon is sphere, plane or hyperboloid, we find these black holes are thermodynamically stable in some parameter space and unstable phase also exists in other parameter space. The relation between the entropy and horizon area of the black holes has an additional coefficient depending on the coupling constant $\lambda$, compared to the $\lambda = 1$ case. For $\lambda = 1$, the well-known coefficient of one quarter is recovered in the infrared region.

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1 Introduction

Recently a field theory model for a UV complete theory of gravity was proposed by Hořava [1], which is a non-relativistic renormalisable theory of gravity and reduces to Einstein’s general relativity at large scales for the dynamical coupling constant $\lambda = 1$. Much attention has been paid to this gravity theory [2]–[28]. The authors of [11] found some static spherically symmetric black hole solutions in Hořava-Lifshitz theory and [16] presented topological black hole solutions and discussed the associated thermodynamic properties with those black hole solutions.

In the (3 + 1)-dimensional ADM formalism, where the metric can be written as

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt),$$

and for a spacelike hypersurface with a fixed time, its extrinsic curvature $K_{ij}$ is

$$K_{ij} = \frac{1}{2N}(g_{ij} - \nabla_i N_j - \nabla_j N_i),$$

where a dot denotes a derivative with respect to $t$ and covariant derivatives defined with respect to the spatial metric $g_{ij}$. The action of Hořava-Lifshitz theory is [1]

$$I = \int dtd^3x (\mathcal{L}_0 + \mathcal{L}_1),$$

$$\mathcal{L}_0 = \sqrt{g}N \left\{ \frac{2}{\kappa^2} (K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2 (\Lambda R - 3\Lambda^2)}{8(1 - 3\lambda)} \right\},$$

$$\mathcal{L}_1 = \sqrt{g}N \left\{ \frac{\kappa^2 \mu^2 (1 - 4\lambda)}{32(1 - 3\lambda)} R^2 - \frac{\kappa^2}{2\omega^2} Z_{ij}Z^{ij} \right\},$$

where

$$Z_{ij} = C_{ij} - \frac{\mu \omega^2}{2} R_{ij}.$$  

and $\kappa^2$, $\lambda$, $\mu$, $\omega$ and $\Lambda$ are constant parameters and the Cotton tensor, $C_{ij}$, is defined by

$$C^{ij} = \epsilon^{ikl} \nabla_k \left( R^l_j - \frac{1}{4} R^l \delta^j_i \right) = \epsilon^{ikl} \nabla_k R^l_j - \frac{1}{4} \epsilon^{ikj} \partial_k R.$$

In (3), the first two terms are the kinetic terms, while the others give the potential of the theory in the so-called “detailed-balance” form.

Comparing the action to that of general relativity, one can see that the speed of light, Newton’s constant and the cosmological constant are

$$c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda}{1 - 3\lambda}}, \quad G = \frac{\kappa^2 c}{32\pi}, \quad \tilde{\Lambda} = \frac{3}{2} \Lambda,$$

respectively. Let us notice that when $\lambda = 1$, the first three terms in (3) could be reduced to the usual ones of Einstein’s general relativity. However, in Hořava-Lifshitz theory, $\lambda$ is a dynamical coupling constant, susceptible to quantum correction [1]. In addition, we see from (3) that when $\lambda > 1/3$, the cosmological constant $\Lambda$ must be negative.
However, the cosmological constant can be positive if we make an analytic continuation 
\( \mu \rightarrow i\mu, \ w^2 \rightarrow -iw^2 \) [11]. In this paper, we consider the former case with a negative 
cosmological constant.

The equations of motion for the action (3) are given as [9, 11]

\[
\frac{2}{\kappa^2} \left( K_{ij} K^{ij} - \lambda K^2 \right) - \frac{\kappa^2 \mu^2 (\Lambda R - 3\Lambda^2)}{8(1 - 3\lambda)} - \frac{\kappa^2 \mu^2 (1 - 4\Lambda)}{32(1 - 3\lambda)} R^2 + \frac{\kappa^2}{2\omega^4} Z_{ij} Z^{ij} = 0, \quad (7)
\]

and

\[
\nabla_k \left( K^{kl} - \lambda K g^{kl} \right) = 0, \quad (8)
\]

where the tensors \( E_{ij}^{(1)}, E_{ij}^{(2)}, E_{ij}^{(3)}, E_{ij}^{(4)}, E_{ij}^{(5)} \) and \( E_{ij}^{(6)} \) are composed by \( K_{ij}, g_{ij}, N, N_i \) and their covariant derivatives with respect to the three dimensional metric. The explicit 
forms of these tensors can be found, for example, in Ref. [11].

The static, spherically symmetric solutions have been found in [11]. The solutions for 
\( \lambda = 1 \) are asymptotically AdS\(_4\) and may be of some interest in AdS/CFT correspondence.
The solutions have been extended to general topological black holes, in which the two 
dimensional sphere as black hole horizon has been generalized to two dimensional constant 
curvature spaces, and their thermodynamic properties including the definition of the mass 
and entropy are discussed in Ref. [16] for the case of \( \lambda = 1 \). Another remarkable point 
is that the solution of general relativity is found not always to be recovered at large 
distance [11]. For large distance, the Einstein theory could only arise for the case with 
\( \lambda = 1 \). This indicates that the infrared region of Hořava-Lifshitz theory can deviate from 
Einstein’s general relativity.

It is extremely interesting to study the properties of this kind of black holes for general 
\( \lambda \). This is because all of the important properties of the Hořava-Lifshitz theory may not 
be revealed by just studying some special cases like \( \lambda = 1 \). It is possible that some 
important properties of the theory will emerge in the case with general \( \lambda \). Black hole 
thermodynamics can give some lights on some aspects of the quantum effects of gravity. 
Compared with other UV complete theories, such as string theory, Hořava-Lifshitz theory 
has quite different UV behavior. So it is an urgent problem to examine the effect of 
quantum gravity by studying the thermodynamics of black holes in this theory. Even 
at the semi-classical level, it is also interesting to study the thermodynamical stability 
of these black holes. In this paper, we extend our previous study [16] of black hole 
thermodynamics of topological black holes for \( \lambda = 1 \) to general values of \( \lambda \). In particular 
we discuss how to define the mass and entropy in this general situation by using the 
canonical Hamilton formulation [29, 30, 31, 32]. In [16] we have used the first law of black 
hole thermodynamics to find the entropy expression of the topological black hole solutions 
in Hořava-Lifshitz theory. Both methods give the same result.

In the course of writing this paper, a paper appeared [23] which discusses related 
subject, but the proper definitions of the mass and entropy are not discussed.
2 Topological Black Holes with General $\lambda$

Here we briefly review the topological black hole solutions with general $\lambda$ which have been presented in [16]. Assume that the metric of the black hole is given by

$$ds^2 = -\tilde{N}^2 f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_k^2.$$

(10)

In terms of the new function $F$ defined by

$$F(r) = k - \Lambda r^2 - f(r),$$

(11)

the action takes the form

$$I = \frac{\kappa^2 \mu^2 \Omega_k}{8(1 - 3\lambda)} \int dt dr \tilde{N} \left\{ \frac{\lambda - 1}{2} F' r^2 - \frac{2\lambda}{r} FF' + \frac{(2\lambda - 1)}{r^2} F^2 \right\}.$$

(12)

After variation this reduced action, we get the equations of motion

$$0 = \left( \frac{2\lambda}{r} F - (\lambda - 1) F' \right) \tilde{N}' + (\lambda - 1) \left( \frac{2}{r^2} FF' - \frac{2\lambda - 1}{r^2} F^2 \right) \tilde{N},$$

(13)

$$0 = (\lambda - 1) r^2 F' r^2 - 4\lambda r F F' + 2(2\lambda - 1) F^2.$$

(14)

The latter is easily solved to give

$$F(r) = \alpha r^s,$$

(15)

where

$$s = \frac{2\lambda \pm \sqrt{2(3\lambda - 1)}}{\lambda - 1},$$

(16)

and then the first gives

$$\tilde{N} = \gamma r^{1-2s},$$

(17)

where $\alpha$ and $\gamma$ are both integration constants. If we use the usual units in gravity theory, $\gamma$ has an inverse dimension of $r^{1-2s}$. For $\lambda = 1$ or $s = 1/2$ case, in which it is dimensionless, one can set $\gamma = 1$ by rescaling the time coordinate $t$ [16]. When $\alpha = 0$ or $F = 0$, Eq. (13) does not restrict $\tilde{N}$. For the case $k = 1$, our solution reduces to the one given in Ref. [11]. Substituting this metric into the equations (7), (8) and (9), we find that this metric with $f$ and $\tilde{N}$ above indeed satisfies the equations of motion.

It is interesting to note that there are two branches in (16). It is easy to find that the range of $s$ is $(-1, 2)$ for the negative branch in the case of $\lambda > 1/3$, which will be assumed in the present paper. Note that the exponent $s$ of Eq. (15) for the negative branch is always less than 2 for positive $\lambda$, and thus the $r^2$ term in the metric function (11) dominates in large distance. This suggests that negative branch solution has some asymptotic behavior of AdS spacetime. On the other hand, the positive branch $s$ gives a power larger than 2 for $\lambda > 1$. In that case, the $F$ term will dominate at large distances and the solution will have a cosmological horizon-like if $\alpha > 0$. In this case, the physical meaning of the solution is not very clear since the solution is not asymptotic to the vacuum solution with $\alpha = 0$ at infinity. In the case of $1 > \lambda > 1/3$, the positive branch gives a negative power. In this case, the solution seemingly makes sense. However, some physical quantities are not well defined in this case. Therefore in the present paper we limit ourselves to the negative branch with $s$ in the range $s \in [-1, 2)$. 

3
3 Black Hole Thermodynamics

In this section, we discuss black hole thermodynamics by using the canonical Hamilton formulation [29, 30, 31, 32, 33, 34]. The partition function for a thermodynamical ensemble is identified with the Euclidean path integral in the saddle point approximation around Euclidean continuation of the classical solution.

For our solutions, their asymptotically behaviors are complicated. Those solutions are neither asymptotically flat nor asymptotically AdS. As a result, the definitions of ADM mass or conformal mass [35, 36] are not applicable here (in fact, one will obtain a divergent result if naively uses these definitions). On the other hand, we find that the canonical Hamiltonian method works well in our case and enables us to define finite mass associated with those solutions.

Consider the Euclidean continuation of the action of the topological black holes for general $\lambda$ in Hamiltonian form ($I \rightarrow -I_E$)

$$I_E = \int d^3x dt \left[ \pi^{ij} \dot{g}_{ij} - N \mathcal{H} - N^i \mathcal{H}_i \right] + B,$$

where the $B$ is a boundary term. $N$ and $N_i$ are lapse function and shift function respectively. In our case, the lapse function is given by $N^2 = \tilde{N}^2 f$. Since we are considering static black hole case, we need not give the explicit form of the momentum $H_i$ and conjugate $\pi^{ij}$ of $\dot{g}_{ij}$. For the black hole metric (10), the Euclidean action is reduced to

$$I_E = -\beta \Omega_k \int_{r_+}^{\infty} \tilde{N}(r) \mathcal{H}(r) dr + B,$$

where $\mathcal{H}(r)$ is given by

$$\mathcal{H}(r) = \frac{\kappa^2 \mu^2 \Omega_k}{8(1 - 3\lambda)} \left\{ \frac{(\lambda - 1)}{2} r^2 - \frac{2\lambda}{r} F F' + \frac{(2\lambda - 1)}{r^2} F^2 \right\},$$

$\beta$ is the period of Euclidean time and $r_+$ is the radius of the black hole horizon defined by the largest root of $f(r) = 0$ or $F(r) = k - \Lambda r^2$ from (11). The Euclidean black holes are static and satisfy the constraint $\mathcal{H} = 0$. So the Euclidean action is just the boundary term $B$. The existence of this boundary term ensures that we can get correct equations of motion from variation of the Euclidean action.

To avoid conical singularity at horizon of the Euclidean black hole solution, we have to set the time period $\beta$ to

$$\beta(\tilde{N}(r) F'(r))|_{r=r_+} = 4\pi,$$

which gives the temperature of the black hole

$$T = \frac{1}{\beta} = \frac{\gamma}{4\pi r_+^2} \left[ -\Lambda r_+^2 (2 - s) - sk \right].$$

When $\lambda \rightarrow 1$, from L’Hospital rule, we have $s = 1/2$. In this case, we obtain

$$T = \frac{\gamma}{8\pi r_+} \left[ -3\Lambda r_+^2 - k \right],$$
which is just the temperature given in Ref. [16] for $\gamma = 1$.

In the canonical ensemble, this temperature should be kept fixed under the variation of the action. From the variation of the Euclidean action, we find that the variation of the boundary term is given by

$$
\delta B = \delta B|_{\infty} - \delta B|_{r_+} = -\beta \frac{\kappa^2 \mu^2 \Omega_k}{8(1 - 3\lambda)} \left[ \lambda \left( \frac{2}{r} \tilde{N} F \delta F \right) - (\lambda - 1)(\tilde{N} F' \delta F) \right]_{r_+}^{\infty}.
$$

(24)

To get equations of motion, we need not know the explicit form of $\delta F$ or $\delta \tilde{N}$, but here we need them, which can be obtained from our solutions (15). Another point that should be noted is that the coordinate $r$ is invariant under the variation; this is the same as in the process of variation to get the equations of motion (13) and (14). Near infinity, from the expression of $F$ in (15), we find

$$
\delta F = \delta (\alpha r^s) = r^s \delta \alpha,
$$

(25)

so $\alpha$ is the only thermodynamic parameter. From the expression $\tilde{N}$ in (17), we have

$$
\lambda \left( \frac{2}{r} \tilde{N} F \delta F \right) \bigg|_{\infty} = \frac{2\lambda}{r} \left( \gamma r^{1-2s} (\alpha r^s) (r^s \delta \alpha) \right) = 2\lambda \gamma \alpha \delta \alpha.
$$

(26)

Although $\delta F$, $F$, or $\tilde{N}$ diverge at infinity, the combination $(2/r)\tilde{N} F \delta F$ is finite. In fact, even for the case $\lambda = 1$ or $s = 1/2$, the $\delta F$ is divergent as $\sqrt{r}$, but $(2/r)F \delta F$ is finite. The only special point of this case is that $\tilde{N}$ is constant. Certainly, for the case $\lambda = 1/3$ or $s = -1$, this combination is also finite, and $\delta F$ has good behavior like $1/r$ (but $\tilde{N}$ rapidly increases asymptotically as $r^3$). For the case $1/2 < s < 2$, $\delta F$ increases faster than $s = 1/2$ case. However, since $(2/r)\tilde{N}$ decreases so as to cancel this divergence, we can always get finite result. Similarly, we find

$$
(\lambda - 1) \left( \tilde{N} F' \delta F \right) \bigg|_{\infty} = \left( 2\lambda - \sqrt{6\lambda - 2} \right) \gamma \alpha \delta \alpha.
$$

(27)

Combining equations (24), (26) and (27), we get the variation of the boundary term at infinity

$$
\delta B|_{\infty} = -\beta \frac{\kappa^2 \mu^2 \Omega_k}{8(1 - 3\lambda)} \sqrt{6\lambda - 2} \gamma \alpha \delta \alpha.
$$

(28)

This suggests that this boundary term is given by

$$
B|_{\infty} = \beta \frac{\kappa^2 \mu^2 \Omega_k}{16 \sqrt{3\lambda - 1}} \gamma \alpha^2.
$$

(29)

For the boundary at the horizon, the variation of $F$ is given by [29, 32]

$$
\delta F|_{r_+} = \left( \frac{\partial F}{\partial f} \right)_{r_+} [\delta f]_{r_+}.
$$

(30)

Since on the horizon, we have

$$
[\delta f]_{r_+} + \left( \frac{df}{dr} \right)_{r_+} \delta r_+ = 0,
$$

(31)
\[ \delta F |_{r+} = - \left( \frac{\partial F}{\partial f} \right)_{r+} \left( \frac{df}{dr} \right)_{r+} \delta r_+ = \left( \frac{df}{dr} \right)_{r+} \delta r_+. \]  

(32)

As a result, by using the relation (21), we arrive at

\[ \delta B |_{r+} = - \frac{\pi \kappa^2 \mu^2 \Omega_k}{2(1 - 3\lambda)} \left[ \frac{2\lambda}{r_+} F(r_+) - (\lambda - 1) F'(r_+) \right] \delta r_+. \]  

(33)

This way we obtain the black hole entropy

\[ B |_{r+} = S, \]  

(34)

where

\[ S = \frac{\pi \kappa^2 \mu^2 \Omega_k}{\sqrt{2(3\lambda - 1)}} \int G(r_+) dr_+ + S_0, \]  

(35)

with integration constant \( S_0 \). The integration constant \( S_0 \) should be fixed as discussed in [16]. The integrand \( G(r_+) \) is given by

\[ G(r_+) = \frac{1}{r_+} F(r_+) = \frac{1}{r_+} (k - \Lambda r_+^2). \]  

(36)

For the on-shell solution, the Euclidean action is just the boundary term. Namely, we have

\[ I_E = B = B |_{\infty} - B |_{r+}, \]  

(37)

which gives

\[ I_E = \beta \frac{\sqrt{2\kappa^2 \mu^2 \Omega_k}}{16\sqrt{3\lambda - 1}} \gamma \alpha^2 - S. \]  

(38)

Since the Euclidean action has relation to free energy \( F_e \) by

\[ I_E = \beta F_e = \beta M - S, \]  

(39)

where \( S \) is the entropy and \( M \) is the mass. Thus we get the mass, temperature and the entropy of the black holes as follows.

\[ M = \frac{\sqrt{2\kappa^2 \mu^2 \Omega_k}}{16\sqrt{3\lambda - 1}} \gamma \alpha^2, \]  

(40)

\[ T = \frac{\gamma}{4\pi r_+^2} \left[ -\Lambda r_+^2 (2 - s) - sk \right], \]  

(41)

\[ S = \frac{\pi \kappa^2 \mu^2 \Omega_k}{\sqrt{2(3\lambda - 1)}} \left[ k \ln(\sqrt{-\Lambda} r_+) + \frac{1}{2} (\sqrt{-\Lambda} r_+)^2 \right] + S_0. \]  

(42)

We can also express the mass by radius of the horizon

\[ M = \frac{\sqrt{2\kappa^2 \mu^2 \gamma \Omega_k}}{16\sqrt{3\lambda - 1}} \left( k - \Lambda r_+^2 \right)^2 \frac{r_+^2}{r_+^{2s}}. \]  

(43)
Defining $\ell^2 = -1/\Lambda$ and using (6) and (16), we have

$$M = \frac{c^3}{16\pi G} \left( \frac{1+s}{2-s} \right) (\gamma \Omega_k \ell^{2-2s}) \left[ \frac{k + (r_+/\ell)^2}{(r_+/\ell)^s} \right]^2,$$

where $c$ is the light velocity defined in Eq. (6) and can be re-expressed in terms of $s$ and $\ell$ instead of $\lambda$ and $\Lambda$:

$$c = \left( \frac{2-s}{1+s} \right) \left( \frac{\kappa^2 \mu}{4 \sqrt{2} \ell} \right).$$

The temperature is given by

$$T = \frac{\gamma}{4 \pi r_+^2} \left[ (r_+/\ell)^2 (2-s) - ks \right].$$

The entropy can also be expressed as

$$S = \frac{c^3}{4G} \left( \frac{1+s}{2-s} \right) (\Omega_k \ell^2) \left[ k \ln \left( \frac{r_+}{\ell} \right)^2 + \left( \frac{r_+}{\ell} \right)^2 \right] + S_0.$$

When $s = 1/2$ or $\lambda = 1$, it goes to the one obtained in [16]. It is easy to confirm that these thermodynamical quantities satisfy the first law of thermodynamics

$$dM = TdS.$$

Note that in Ref. [16], we have derived the entropy using the first law, but here we have shown that the canonical Hamiltonian formalism allows us to define the entropy which satisfies the first law.

In general we cannot determine whether the black holes are thermodynamic stable or not since one cannot fix the integration constant $S_0$ here. As argued in [16], to fix the integration constant $S_0$, one has to invoke the quantum theory of the gravity. For the Ricci flat black holes with $k = 0$, the logarithmic term is absent and the entropy is proportional to the horizon area. In this case we can set $S_0 = 0$ by the assumption that black hole entropy vanishes when horizon goes to zero. Thus we have the free energy of the black hole

$$F_e \sim \gamma (s-1)M.$$

This result is also valid for large black holes. We see that the large black holes and those with $k = 0$ are always thermodynamically stable globally when $s \leq 1$. When $s > 1$, the free energy turns out to be positive, which means that the black hole is thermodynamically unstable globally. This is quite different from the situation in Einstein’s general relativity, where large AdS black holes are always thermodynamically stable regardless of the horizon topology.

### 4 Some Special Cases

#### 4.1 Einstein gravity: $\lambda \to 1$

For the case $\lambda = 1$, we have $s = 1/2$. Those thermodynamic quantities become

$$M = \frac{c^3}{16\pi G} (\Omega_k \ell) \left[ \frac{k + (r_+/\ell)^2}{(r_+/\ell)^{1/2}} \right]^2,$$

$$7$$
\[ T = \frac{1}{8\pi r_+} \left[ 3(r_+ / \ell)^2 - k \right], \]  
(51)

and

\[ S = \frac{c^3}{4G} \left[ k \ln \left( \frac{r_+}{\ell} \right)^2 + \left( \frac{r_+}{\ell} \right)^2 \right] + S_0. \]  
(52)

Here we have set \( \gamma = 1 \) by the rescaling of the time. These are just what we have found in \([16]\). Since this case has been discussed in some details in \([16]\), we will not repeat the discussions here.

### 4.2 Black holes with flat horizon

In this case, the thermodynamic quantities have the forms

\[ M = \frac{c^3}{16\pi G} \left( \frac{1 + s}{2} \right) (\gamma \Omega_k \ell^2) (r_+ / \ell)^{2(2-s)}, \]  
(53)

\[ T = \frac{\gamma}{4\pi} \ell^{-2s}(2-s)(r_+ / \ell)^{2-2s}, \]  
(54)

\[ S = \frac{c^3}{4G} \left( \frac{1 + s}{2} \right) (\Omega_k \ell^2) \left( \frac{r_+}{\ell} \right)^2 + S_0, \]  
(55)

where \( S_0 \) can be set to zero as argued above. So, for the Ricci flat horizon case, the entropy is proportional to the horizon area and the log term disappears. The difference from the well-known area formula in Einstein’s general relativity is the additional factor \((1 + s)/(2 - s)\) or \(\sqrt{(3\lambda - 1)/2}\) in the black hole entropy. The free energy is given by

\[ F_e = \gamma(s - 1)M. \]  
(56)

The global thermodynamic stability is discussed in Sec. 3.

### 4.3 Non-Einstein Case: \( \lambda \to 1/3 \)

In the case, notice the definition of the speed of light in \([3]\), we see that the temperature of the black hole is still well defined. However, the mass and entropy of the black hole diverge. To have a finite result in this case, one could take a rescaling of the speed of light so that \((c^3/G)\sqrt{3\lambda - 1}\) goes to a finite constant in the limit \(\lambda \to 1/3\). But the physical meaning (if any) is not clear at the moment for this rescaling.

### 5 Local Stability of Black Hole Thermodynamics

By studying Euclidean action, or free energy, we can find information on the global stability of black hole thermodynamics. However, to discuss the local stability, we have to calculate the heat capacity of black holes. From the expressions for the mass and temperature, we get heat capacity as

\[ C = \frac{\partial M}{\partial T} = \frac{c^3}{4G} (\Omega_k \ell^2) \cdot \frac{(1 + s)}{2 - s} \cdot \left\{ \frac{(k + r_+^2 / \ell^2) [(2 - s)r_+^2 / \ell^2 - ks]}{ks^2 + (s - 1)(s - 2)r_+^2 / \ell^2} \right\}, \]  
(57)
For the case \( \lambda = 1 \) or \( s = 1/2 \), this result goes back to the one in [16]. Here, we give some discussions on thermodynamical stability of the black holes.

1. \( k = 0 \): It is easy to find, for \( k = 0 \), the dominator will change sign at \( s = 1 \). So for \( s > 1 \) case, the Ricci flat black holes are thermodynamically unstable. Note that in this case, the black hole is also globally unstable according to its free energy. This behavior of the heat capacity for this case can be directly found from Fig. 1.

![Figure 1: Heat capacity for \( k = 0 \), where \( x = r_+ / \ell \).](image)

2. \( k = -1 \): In this case, the two factors in the numerator of the heat capacity are both positive: the first one is positive because the minimal horizon is at \( r_+ = \ell \) for massless black hole; the second comes from the requirement of the positive definiteness of Hawking temperature. Thus the sign of the heat capacity is completely determined by the denominator. Therefore when \( s \geq 1 \), the heat capacity is always negative. When \(-1 < s < 1 \), the heat capacity is positive for \( r_+^2 / \ell^2 > s^2 / (1 - s)(2 - s) \). Otherwise, it is negative and it diverges when \( r_+^2 / \ell^2 = s^2 / (1 - s)(2 - s) \). Note that requiring \( r_+^2 / \ell^2 \geq 1 \) leads to \( s \geq 2/3 \), which means that for \( 2/3 < s < 1 \), the black hole is thermodynamically stable if \( r_+^2 / \ell^2 > s^2 / (1 - s)(2 - s) \). Fig. 2 depicts the heat capacity for \( r_+ > \ell \).

3. \( k = 1 \): In this case, the positive definiteness of temperature demands \( r_+^2 / \ell^2 \geq s / (2 - s) \). When the equality holds, it corresponds to an extremal black hole with vanishing temperature. Then we can see that when \(-1 < s \leq 1 \), the heat capacity is always positive. When \( 1 < s < 2 \), it is positive for

\[
\sqrt{\frac{s}{2 - s}} < \frac{r_+}{\ell} < \sqrt{\frac{s^2}{(s - 1)(2 - s)}},
\]

(58)
and it becomes negative for \( r_+ / \ell > \sqrt{\frac{s^2}{(s-1)(2-s)}} \). One can find this behavior in Fig. 3. It is interesting to note that in this case even for a large black hole, it is not thermodynamically stable.

In summary we have found that in three cases with different horizon topologies, there always exist locally thermodynamically stable phases and unstable phases in suitable parameter regimes.
6 Conclusion and Discussions

In this paper, using the canonical Hamiltonian method we have generalized the discussion of thermodynamics of topological black holes for $\lambda = 1$ case [16] to the general $\lambda \geq 1/3$ case. All the thermodynamical quantities we have got reduce to those in [16] when $\lambda = 1$, although we have used different approaches. We have also studied the global and local thermodynamical stability of these black holes, and found that there exist rich phase structures compared to the case of AdS Schwarzschild black holes in Einstein’s general relativity. In all three different horizon topologies, locally stable or unstable phases exist in the proper parameter space. It is quite different from the case in Einstein’s general relativity.

For general $\lambda$, up to a constant, the entropy of black hole not only receives a logarithm correction, but also gets a multiplicative factor which is a function of $\lambda$ or $s$. This function reduces to one when $\lambda = 1$. In this case, the one quarter in the relation between the entropy and the horizon area is recovered, the well-known entropy formula in units of $c = G = 1$. This means that the area formula of the entropy is not restored in the infrared region of the theory unless the dynamical coupling constant is one.

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