Algebraic treatment of non-Hermitian quadratic Hamiltonians

Francisco M. Fernández

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Abstract
We generalize a recently proposed algebraic method in order to treat non-Hermitian Hamiltonians. The approach is applied to several quadratic Hamiltonians studied earlier by other authors. Instead of solving the Schrödinger equation we simply obtain the eigenvalues of a suitable matrix representation of the operator. We take into account the existence of unitary and antiunitary symmetries in the quantum-mechanical problem.

Keywords Quadratic Hamiltonian · Algebraic method · Adjoint matrix · Unitary symmetry · Antiunitary symmetry · Exceptional point

1 Introduction

Hamiltonian operators that are quadratic functions of the coordinates and momenta proved to be useful for the study of interesting physical phenomena [1–4]. The eigenvalues of such Hamiltonians may be real or complex. The occurrence of real or complex spectrum depends on the values of the model parameters that determine the experimental setting. The transition from one regime to the other is commonly interpreted as the breaking of PT symmetry. In some cases those PT symmetric Hamiltonians are also Hermitian [5, 6].

In addition to those Hamiltonians directly related to experiment there are other quadratic oscillators that have been used to illustrate physical concepts in a more theoretical setting. They may be Hermitian [7–9] or non-Hermitian [10–14].

The eigenvalue equation for a quadratic Hamiltonian can be solved exactly in several different ways [3, 8, 15]. The algebraic method [5, 6], based on well known properties of Lie algebras [16, 17], is particularly simple and straightforward. It focusses on the natural frequencies of the quantum-mechanical problem and reveals the transition from real to complex spectrum without solving the eigenvalue equation explicitly or
writing the Hamiltonian in diagonal form. The whole problem reduces to diagonalizing a $2N \times 2N$ matrix, where $N$ is the number of coordinates.

Those earlier applications of the algebraic method focused on Hermitian Hamiltonians [5, 6] but the approach can also be applied to non-Hermitian quadratic ones. The purpose of this paper is to generalize those results and take into account possible unitary and antiunitary symmetries of the quadratic Hamiltonians.

In Sect. 2 we briefly address the concepts of unitary and antiunitary symmetries. In Sect. 3 we outline the main ideas of the algebraic method and derive the regular or adjoint matrix representation for non-Hermitian quadratic Hamiltonians. The approach is similar to that in the previous papers [5, 6] but the results are slightly more general and convenient for present aims. In this section we also consider Hermitian quadratic Hamiltonians and illustrate the main results by means of a simple one-dimensional example. In Sect. 4 we consider three two-dimensional quadratic Hamiltonians already studied earlier by other authors. They prove useful for illustrating the transition from real to complex spectrum, when the model is either Hermitian or non-Hermitian, which depends on a suitable choice of the model parameters. Finally, in Sect. 5 we summarize the main results and draw conclusions.

2 Unitary and antiunitary symmetry

In this section we outline some well known concepts and definitions that appear in most textbooks on quantum mechanics [18] with the purpose of facilitating the discussion in subsequent sections. Given a linear operator $H$ its adjoint $H^\dagger$ satisfies

$$\langle f|H^\dagger|f\rangle = \langle f|H|f\rangle^\ast,$$

(1)

for any vector $|f\rangle$ in the complex Hilbert space where $H$ is defined. If $H = H^\dagger$ we say that the operator is Hermitian [18].

If $|\psi\rangle$ is an eigenvector of the Hermitian operator $H$ with eigenvalue $E$

$$H|\psi\rangle = E|\psi\rangle,$$

(2)

then $\langle \psi|H|\psi\rangle = \langle \psi|H|\psi\rangle^\ast$ leads to $(E - E^\ast)\langle \psi|\psi\rangle = 0$. Therefore, if $|\psi\rangle$ belongs to the Hilbert space where $H$ is defined ($0 < \langle \psi|\psi\rangle < \infty$) then $E$ is real.

A unitary operator $S$ satisfies

$$\langle Sf|Sg\rangle = \langle f|g\rangle,$$

(3)

for any pair of vectors $f$ and $g$ in the Hilbert space where $S$ is defined. If

$$SH = HS,$$

(4)

we say that the linear operator $H$ exhibits a unitary symmetry (we assume that both $H$ and $S$ are defined on the same Hilbert space). We can also write this expression as $SHS^\dagger = H$ because $S^\dagger = S^{-1}$. It follows from equations (2) and (4) that

$$HS|\psi\rangle = SH|\psi\rangle = ES|\psi\rangle.$$

(5)
An antiunitary operator $A$ satisfies [19]

$$\langle Af | Ag \rangle = \langle f | g \rangle^*, \quad (6)$$

for any pair of vectors $f$ and $g$ in the Hilbert space where $A$ is defined. It follows from this expression that

$$A(af + bg) = a^*Af + b^*Ag, \quad (7)$$

for any pair of complex numbers $a$ and $b$ and we say that $A$ is antilinear [19].

If

$$HA = AH, \quad (8)$$

we say that the linear operator $H$ exhibits an antiunitary symmetry (we assume that $H$ and $A$ are defined on the same Hilbert space). Since $A$ is invertible this last expression can be rewritten as $AHA^{-1} = H$. An important consequence of this equation is that

$$HA |\psi\rangle = AH |\psi\rangle = AE |\psi\rangle = E^*A |\psi\rangle. \quad (9)$$

Therefore, if $A |\psi\rangle = a |\psi\rangle$ (that is to say, the antiunitary symmetry is exact) the eigenvalue $E$ is real (even if $H$ is non-Hermitian).

3 The algebraic method

In two earlier papers we applied the algebraic method to a class of Hermitian Hamiltonians that includes those that are quadratic functions of the coordinates and their conjugate momenta [5, 6]. In what follows we apply the approach to non-Hermitian Hamiltonians that are also of remarkable physical interest. Although most of the expressions are similar to those derived in the earlier articles we repeat the treatment here in order to make this paper sufficiently self contained. The main difference is that we do not assume that $H$ is Hermitian and the main results will be more general. In addition to it, present algebraic approach will take into account the possibility that $H$ exhibits unitary or antiunitary symmetries.

The algebraic method enables us to solve the eigenvalue equation for a linear operator $H$ in the particular case that there exists a set of Hermitian operators $S_N = \{O_1, O_2, \ldots, O_N\}$ that satisfy the commutation relations

$$[H, O_i] = \sum_{j=1}^{N} H_{ji} O_j. \quad (10)$$

Without loss of generality we assume that the operators in $S_N$ are linearly independent so that the only solution to
\[
\sum_{j=1}^{N} d_j O_j = 0,
\]

is \( d_i = 0 \) for all \( i = 1, 2, \ldots, N \).

Because of Eq. (10) it is possible to find an operator of the form

\[
Z = \sum_{i=1}^{N} c_i O_i,
\]

such that

\[
[H, Z] = \lambda Z.
\]

The operator \( Z \) is important for our purposes because \( Z|\psi\rangle \) is eigenvector of \( H \) with eigenvalue \( E + \lambda \):

\[
HZ|\psi\rangle = ZH|\psi\rangle + \lambda Z|\psi\rangle = (E + \lambda)Z|\psi\rangle,
\]

provided that \( Z|\psi\rangle \) is nonzero.

It follows from Eqs. (10), (12) and (13), and from the fact that the set \( S_N \) is linearly independent, that the coefficients \( c_i \) are solutions to

\[
(H - \lambda I)C = 0,
\]

where \( H \) is an \( N \times N \) matrix with elements \( H_{ij} \), \( I \) is the \( N \times N \) identity matrix, and \( C \) is an \( N \times 1 \) column matrix with elements \( c_i \). \( H \) is called the adjoint or regular matrix representation of \( H \) in the operator basis set \( S_N \) [16, 17]. Equation (15) admits non-trivial solutions for those values of \( \lambda \) that are roots of the characteristic polynomial

\[
P(\lambda) = \det(H - \lambda I) = 0.
\]

In some cases the matrix \( H \) may not be diagonalizable because it is not normal

\[
HH^\dagger \neq H^\dagger H.
\]

If there exists a linear operator \( W \) that commutes with all the basis operators \( O_i \) ([\( W, O_i \] = 0, \( i = 1, 2, \ldots, N \)) then \( H \) and \( H + W \) share the same adjoint matrix \( H \). Such an occurrence is not a serious difficulty in the cases studied here because any operator \( W \) with such a property is proportional to the identity operator and its effect on the spectrum is trivial.

Of particular interest for the present paper is a basis set of operators that satisfy

\[
[O_i, O_j] = U_{ij} \hat{1},
\]

where \( U_{ij} \) is a complex number and \( \hat{1} \) is the identity operator that we will omit from now on. It follows from \([O_j, O_i] = -[O_i, O_j]\) and \([O_i, O_j]^\dagger = -[O_i, O_j]\) that

\[
U_{ij} = -U_{ji}^* = -U_{ji},
\]

and
\[ \mathbf{U}^\dagger = \mathbf{U}, \]  

where \( \mathbf{U} \) is the \( N \times N \) matrix with elements \( U_{ij} \). The well known Jacobi identity

\[ [O_k, [H, O_i]] + [O_i, [O_k, H]] + [H, [O_i, O_k]] = 0, \]

leads to

\[ [O_k, [H, O_i]] = [O_i, [H, O_k]]. \]

Therefore, Eqs. (10), (18), (19) and (22) lead to

\[ (\mathbf{U} \mathbf{H})^\dagger = \mathbf{H}^\dagger \mathbf{U} = -\mathbf{H}^\dagger \mathbf{U} = \mathbf{U} \mathbf{H}, \]

where \( \mathbf{U} \) is invertible because the set of operators \( \mathbf{S}_N \) is linearly independent. Taking into account that \( \mathbf{U}^{-1} \mathbf{H} \mathbf{U} = -\mathbf{H} \) we conclude that \( P(-\lambda) = \det(\mathbf{H} + \lambda \mathbf{I}) = 0 \); that is to say: both \( \lambda \) and \( -\lambda \) are eigenvalues.

It follows from (15) and (23) that

\[ \mathbf{H}^\dagger \mathbf{U} \mathbf{C} = -\lambda \mathbf{U} \mathbf{C} \]

\[ \mathbf{H}^\dagger \mathbf{U} \mathbf{C}^* = -\lambda^* \mathbf{U} \mathbf{C}^*. \]

If \( \mathbf{H} \) has a unitary symmetry given by the unitary operator \( \mathbf{S} \) such that

\[ \mathbf{S} \mathbf{O}_i \mathbf{S}^\dagger = \sum_{j=1}^{N} s_{ji} \mathbf{O}_j, \]

then it follows from

\[ \mathbf{S} [\mathbf{H}, \mathbf{O}_i] \mathbf{S}^\dagger = [\mathbf{H}, \mathbf{S} \mathbf{O}_i \mathbf{S}^\dagger], \]

that

\[ \mathbf{S} \mathbf{H} = \mathbf{H} \mathbf{S}, \]

where \( \mathbf{S} \) is the matrix with elements \( s_{ij} \). Under these conditions we have

\[ \mathbf{H} \mathbf{S} \mathbf{C} = \mathbf{S} \mathbf{H} \mathbf{C} = \lambda \mathbf{S} \mathbf{C}. \]

Therefore, if \( \lambda_i \neq \lambda_j \) for all \( i \neq j \) then \( \mathbf{S} \mathbf{C}_i = s_i \mathbf{C}_i \).

It follows from

\[ \mathbf{S} [\mathbf{O}_i, \mathbf{O}_j] \mathbf{S}^\dagger = [\mathbf{S} \mathbf{O}_i \mathbf{S}^\dagger, \mathbf{S} \mathbf{O}_j \mathbf{S}^\dagger] = \mathbf{U}_{ij}, \]

that

\[ \mathbf{S}^\dagger \mathbf{U} \mathbf{S} = \mathbf{U}. \]

Suppose that \( \mathbf{H} \) has an antiunitary symmetry given by the antiunitary operator \( \mathbf{A} \) and that
Then, it follows from
\[ AO_i A^{-1} = \sum_{j=1}^{N} a_{ij} O_j. \] (31)
that
\[ A[H, O_i]A^{-1} = [H, AO_i A^{-1}], \] (32)
where \( A \) is the matrix with elements \( a_{ij} \). Therefore, if \( C \) is an eigenvector of \( H \) with eigenvalue \( \lambda \), we have \( HAC^* = AH^* C^* = A(HC)^* \) that leads to
\[ HAC^* = \lambda^* AC^* . \] (34)
We conclude that both \( \lambda \) and \( \lambda^* \) are eigenvalues of \( H \) and roots of the characteristic polynomial \( P(\lambda) \). If \( AC^* = bC \), where \( b \) is a scalar, then the antiunitary symmetry is exact and \( \lambda = \lambda^* \).

Since the set of operators \( S_N \) is assumed to be linearly independent then \( A \) is invertible and
\[ H^* = A^{-1} HA. \] (35)
Besides, it follows from
\[ A[O_i, O_j]A^{-1} = [AO_i A^{-1}, AO_j A^{-1}] = U^*_i , \] (36)
that
\[ A'UA = U^* = U' = - U. \] (37)

### 3.1 Hermitian Hamiltonians

The case \( H^\dagger = H \) is of particular interest for several physical applications \([1–4]\) and was studied in detail in our earlier papers \([5, 6]\). In what follows we summarize the main results.

When \( H \) is Hermitian we have some additional useful relationships. For example, the commutator relation \( [H, O_i]^\dagger = -[H, O_i] \) leads to
\[ H^\dagger_{ij} = - H_{ij}. \] (38)
It follows from Eqs. (23) and (20) that \( H \) is \( U \)-pseudo-Hermitian \([6]\):
\[ H^\dagger = UHU^{-1}, \] (39)
(see \([20–24]\) for a more general and detailed discussion of pseudo-Hermiticity or quasi-Hermiticity). Note that \( H \) and \( H^\dagger \) share eigenvalues.
\[ H^\dagger U C = \lambda U C. \quad (40) \]

Another important relationship follows from the fact that \([H, Z]^\dagger = -[H, Z^\dagger];\]

\[ [H, Z^\dagger] = -\lambda^* Z^\dagger. \quad (41) \]

According to what was argued above if \( |\psi\rangle \) and \( Z|\psi\rangle \) belong to the Hilbert space, then both \( E \) and \( \lambda \) are real. In the language of quantum mechanics we often say that \( Z \) and \( Z^\dagger \) are a pair of annihilation-creation or ladder operators because, in addition to (14), we also have

\[ HZ^\dagger |\psi\rangle = (E - \lambda)Z^\dagger |\psi\rangle. \quad (42) \]

### 3.2 Quadratic Hamiltonians

The simplest Hamiltonians that can be treated by the algebraic method are those that are quadratic functions of the coordinates and their conjugate momenta:

\[ H = \sum_{i=1}^{2K} \sum_{j=1}^{2K} \gamma_{ij} O_i O_j, \quad (43) \]

where \( \{O_1, O_2, \ldots, O_{2K}\} = \{x_1, x_2, \ldots, x_K, p_1, p_2, \ldots, p_K\} \), \([x_m, p_n] = i\delta_{mn}\), and \([x_m, x_n] = [p_m, p_n] = 0\). In this case the matrix \( U \) has the form

\[ U = i\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \quad (44) \]

where \( 0 \) and \( I \) are the \( K \times K \) zero and identity matrices, respectively, which already satisfies \( U^\dagger = U^{-1} = U \). The matrices \( H, \gamma \) and \( U \) are related by

\[ H = (\gamma + \gamma') U, \quad (45) \]

where \( \gamma \) is the matrix with elements \( \gamma_{ij}. \)

If \( \gamma' = \gamma \) the quadratic Hamiltonian (43) is Hermitian and \( H \) is \( U \)-pseudo Hermitian: \( H^\dagger = U H U \). In this case \( \gamma + \gamma' = \gamma + \gamma^* = 2\Re \gamma. \)

The Schrödinger equation for a quadratic Hamiltonian is exactly solvable and its eigenvalues and eigenvectors can be obtained by several approaches [3, 8, 15]. In what follows we apply the algebraic method and just obtain the eigenvalues \( \lambda_i \) of the adjoint or regular matrix representation which is sufficient for present purposes. In this case the adjoint or regular matrix is closely related to the fundamental matrix [25].

### 3.3 One-dimensional example

We first consider the simplest quadratic Hamiltonian
\[ H = p^2 + x^2 + b(xp + px), \]  
which is Hermitian when \( b \) is real. On choosing the set of operators \( S_2 = \{ x, p \} \) we obtain the matrix representation

\[
H = \begin{pmatrix} -2ib & 2i \\ -2i & 2ib \end{pmatrix}.
\]  

Its eigenvalues

\[
\lambda_2 = -\lambda_1 = 2\sqrt{1 - b^2},
\]

are real when \( b^2 < 1 \).

The eigenvalues of the Hermitian case are real when \(-1 < b < 1\) and this operator does not have eigenvectors in the Hilbert space when \( b^2 \geq 1 \). When \( b = 1 \) the matrix \( H \) exhibits only one eigenvalue \( \lambda = 0 \) and one eigenvector

\[
C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]

and is defective or not diagonalizable. When \( b = -1 \) we obtain the same eigenvalue but in this case the only eigenvector is

\[
C = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix},
\]

and again the matrix is defective. These particular values of \( b \) are commonly called exceptional points [26–29].

In the coordinate representation, the ground-state eigenfunction and eigenvalue are

\[
\psi_0(x) = Ne^{-\alpha x^2}, \quad \alpha = \frac{1}{2} \left( \sqrt{1 - b^2} + ib \right),
\]

\[
E = \sqrt{1 - b^2},
\]

respectively. We clearly appreciate that \( \psi_0(x) \) is square-integrable only when \( b^2 < 1 \) in agreement with the result of the algebraic method.

When \( b = i\beta \), the eigenvalues of \( H \) are real for all real values of \( \beta \). In this case there is an antiunitary symmetry \( A = PT \) commonly called PT symmetry given by \( AH(x, p)A = TH(-x, -p)T = H(-x, p)^* \). Its matrix representation

\[
A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},
\]

satisfies \( AA = I \) and \( AHA = H^* \) as argued in Sect. 3. Note that since \( PH(x, p)P = H(-x, -p) = H(x, p) \) we also have \( AH(x, p)A = H(x, -p)^* \). The matrix representation in this case is \(-A\) and satisfies exactly the same two conditions.
At any of the exceptional points the adjoint matrix representation can be written in Jordan canonical form by means of a suitable similarity transformation. For example, when \( b = 1 \) we have

\[
H = \begin{pmatrix}
-2i & 2i \\
-2i & 2i
\end{pmatrix}.
\]

(53)

By means of the matrix

\[
P = \begin{pmatrix}
-2i & 1 \\
-2i & 0
\end{pmatrix},
\]

(54)

we obtain

\[
P^{-1}HP = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}.
\]

(55)

### 4 Two-dimensional examples

In this section we consider three two-dimensional quadratic Hamiltonians studied earlier by other authors. The first example

\[
H = p_x^2 + p_y^2 + x^2 + ay^2 + bxy, \quad a > 0,
\]

(56)

is closely related to the one discussed by Cannata et al. [10], Calliceti et al. [25], Fernández and García [12] and Beygi et al. [13] and is Hermitian when \( b \) is real. On choosing the basis set of operators \( S_4 = \{x, y, p_x, p_y\} \) we obtain the regular matrix representation

\[
H = \begin{pmatrix}
0 & 0 & 2i & bi \\
0 & 0 & bi & 2ai \\
-2i & 0 & 0 & 0 \\
0 & -2i & 0 & 0
\end{pmatrix}.
\]

(57)

Its four eigenvalues are the square roots of

\[
\xi_{\pm} = 2 \left[ a + 1 \pm \sqrt{b^2 + (a - 1)^2} \right],
\]

(58)

and are real provided that \( \xi_{\pm} > 0 \). More precisely, they are real if \(- (a - 1)^2 < b^2 < 4a \) that reveals four exceptional points, two at each endpoint. The right exceptional points \( b = \pm 2 \sqrt{a} \) appear in the Hermitian case. On the other hand, when \( b = i \beta \), \( \beta \) real, the eigenvalues are real if \( |\beta| < |a - 1| \). In this case the non-Hermitian operator is similar to a self-adjoint one [25]. The most symmetric case \( a = 1 \) exhibits real eigenvalues only in the Hermitian region \( 0 < b^2 < 4 \).

When \( b^* = -b \) the operator exhibits two antiunitary symmetries \( A_x = S_xT \) and \( A_y = S_yT \), where \( S_xH(x, y, p_x, p_y)S_x = H(-x, y, -p_x, p_y) \).
and $S_y H(x, y, p_x, p_y) S_y = H(-x, -y, p_x, -p_y)$, that lead to $A_y H(x, y, p_x, p_y) A_y = H(-x, -y, -p_x, p_y)$ and $A_x H(x, y, p_x, p_y) A_x = H(x, -y, -p_x, p_y)^*$ with matrix representations $A_x$ and $A_y$ that satisfy $A_q A_q = I$ and $A_q H A_q = H^*$, $q = x, y$. These antiunitary symmetries have been named partial parity-time symmetry [13, 30] and are particular cases of other antiunitary symmetries that one can find in quadratic Hamiltonians [31].

The second example was discussed earlier by Li and Miao [11] and more recently by Miao and Xu [14] as a three-parameter model but for present purposes we can rewrite it as a two-parameter one:

$$H = p_x^2 + p_y^2 + x^2 + ay^2 + bpxpy, \quad a > 0.$$  \hfill (59)

In those articles $b$ was chosen to be imaginary but here we allow it to be also real, in which case the Hamiltonian is Hermitian.

The four eigenvalues of its regular matrix representation

$$H = \begin{pmatrix}
0 & 0 & 2i & 0 \\
0 & 0 & 0 & 2ai \\
-2i & -bi & 0 & 0 \\
-bi & -2i & 0 & 0
\end{pmatrix},$$  \hfill (60)

are the square roots of

$$\xi_{\pm} = 2\left[a + 1 \pm \sqrt{ab^2 + (a-1)^2}\right].$$  \hfill (61)

These eigenvalues are real provided that

$$\frac{(a-1)^2}{a} < b^2 < 4,$$  \hfill (62)

which reveals that there are four exceptional points as in the preceding example. The right ones $b = \pm 2$ appear in the Hermitian case. On the other hand, when $b = \pm i$, $\beta$ real, the eigenvalues are real if $|\beta| < |a-1|/\sqrt{a}$. The most symmetric case $a = 1$ exhibits real eigenvalues only in the Hermitian region $0 < b^2 < 4$.

Clearly, most of the mathematical features of the models (56) and (59) are similar. To what has just been said we add that when $b^* = -b$ both operators exhibit exactly the same two antiunitary symmetries $A_x$ and $A_y$ already discussed above.

According to Li and Miao [11] and Miao and Xu [14] when $b^* = -b$ the operator (59) is neither Hermitian nor PT symmetric but it is PT-pseudo Hermitian because $PTHPT = H^* = H^\dagger$. However, an operator $H$ is $\eta$-pseudo Hermitian if there is an invertible Hermitian operator $\eta$ such that $H^\dagger = \eta H \eta^{-1}$ [20–24]. Since the operator $PT$ is not Hermitian (it is in fact antilinear and antiunitary) it is not correct to speak of PT-pseudo Hermiticity. The operator (59) exhibits two antiunitary symmetries given by $A_x$ and $A_y$ as well as the true pseudo-Hermiticity provided by an operator $\tilde{\eta} = \tilde{\Omega} \tilde{\Omega}$, where $\tilde{\Omega}$ is a suitable exponential operator [14].

The next example is a pair of harmonic oscillators coupled by an angular momentum and was proposed for the study of “a particle in a rotating anisotropic
harmonic trap or a charged particle in a fixed harmonic potential in a magnetic field” [9]. For simplicity we rewrite this Hamiltonian in the following form

$$H = p_x^2 + p_y^2 + x^2 + ay^2 + b(xp_y - yp_x), \quad a > 0.$$  \hspace{1cm} (63)

It is Hermitian when $b$ is real which is exactly the case considered by Rebón et al. [9] but here we also allow it to be complex.

The regular matrix representation for this operator is

$$H = \begin{pmatrix}
0 & -bi & 2i & 0 \\
bi & 0 & 0 & 2ai \\
-2i & 0 & 0 & -bi \\
0 & -2i & bi & 0
\end{pmatrix},$$  \hspace{1cm} (64)

and its eigenvalues are the square roots of

$$\xi_{\pm} = 2a + b^2 + 2 \pm \sqrt{(a - 1)^2 + 2(a + 1)b^2}. \hspace{1cm} (65)$$

These eigenvalues are real provided that

$$b^2 > -\frac{(a - 1)^2}{2(a + 1)},$$

$$0 > (b^2 - 4)(4a - b^2). \hspace{1cm} (66)$$

Once again we appreciate that when $a = 1$ the eigenvalues are real only when the Hamiltonian is Hermitian ($-2 < b < 2$).

The Hamiltonian (63) is parity invariant and when $b^* = -b$ it is also PT symmetric: $PTHPT = TH(x, y, p_x, p_y)T = H(x, y, -p_x, -p_y)^* = H(x, y, p_x, p_y)$. The matrix representation $A$ of this antiunitary symmetry satisfies $A \cdot A = I$ and $A \cdot H \cdot A = H^*$ as argued above.

The characteristic polynomial for the three examples discussed in this section is

$$\lambda^4 - (\xi_+ + \xi_-) \lambda^2 + \xi_+ \xi_- = 0,$$  \hspace{1cm} (67)

therefore, the dynamical variables satisfy the differential equation of fourth order

$$\frac{d^4}{dt^4} q + (\xi_+ + \xi_-) \frac{d^2}{dt^2} q + \xi_+ \xi_- = 0,$$  \hspace{1cm} (68)

as shown in the “Appendix”.

The equations above show that $\xi_{\pm} = \omega^2 \pm \Delta$, so that the case of equal frequencies takes place at the exceptional points given by the condition $\Delta = 0$. In this case the corresponding adjoint matrix representation $H_\omega$ can be written in Jordan canonical form.
where the form of the matrix $\mathbf{P}$ depends on the model.

### 5 Further comments and conclusions

Quadratic Hamiltonians are suitable models for the study of several physical phenomena [1, 3, 4] as well as theoretical investigations [7–9]. The algebraic method is a suitable tool for the analysis of the spectra of such oscillators [5, 6]. In this paper we extended the treatment from Hermitian quadratic Hamiltonians to non-Hermitian ones taking into account possible unitary and antiunitary symmetries.

A common feature of the two-dimensional oscillators discussed in Sect. 4 is that they do not exhibit real eigenvalues when $a = 1$ and $b^2 < 0$. According to the algebraic method an eigenvalue $\lambda$ of the regular matrix representation $\mathbf{H}$ is the difference between two energy levels of the Hamiltonian $H$. Therefore this frequency reveals the dependence of the energy levels on the model parameter $b$. For this reason, when $a \neq 1$ the energy levels can be expanded in a Taylor series of the form

$$ E_{n_1n_2} = \sum_{j=0}^{\infty} E_{n_1n_2}^{(j)} b^j, \quad E_{n_1n_2}^{(2j+1)} = 0, $$

and the eigenvalues are real for $b^2 < 0$ within its radius of convergence. If, on the other hand, $a = 1$ the perturbation corrections of odd order do not vanish and the eigenvalues can only be real for real $b$. In a series of papers we proposed to calculate the perturbation correction of first order to determine whether the eigenvalues of a given non-Hermitian operator are real or complex [12, 32–34]. If the perturbation correction of first order is nonzero, then the eigenvalues are complex. This simple and straightforward argument applies even if the perturbation series is divergent as long as it is asymptotic to the actual eigenvalue. Whether the perturbation correction of first order vanishes or not depends on the symmetry of $H_0$; for this reason the application of group theory proved to be quite successful [12, 32–34]. The quadratic Hamiltonians studied in Sect. 4 are exactly solvable problems that confirm the argument put forward in those earlier papers. Note that the greater symmetry of $H_0$ takes place when $a = 1$.

### Appendix

In this “Appendix” we derive some additional results that may be useful for future applications of the algebraic method.

For every eigenvalue $\lambda_i$ we construct the operator

$$ \mathbf{P} \mathbf{H}_\omega \mathbf{P} = \begin{pmatrix} -\omega & 1 & 0 & 0 \\ 0 & -\omega & 0 & 0 \\ 0 & 0 & \omega & 1 \\ 0 & 0 & 0 & \omega \end{pmatrix}, $$

(69)
For convenience we label the eigenvalues in such a way that \( \lambda_j = -\lambda_{2K-j+1} \), \( j = 1, 2, \ldots, K \), and when they are real we organize them in the following way:

\[
\lambda_1 < \lambda_2 < \cdots < \lambda_K < 0 < \lambda_{K+1} < \cdots < \lambda_{2K}.
\]

(72)

If we take into account that \([H, Z_i Z_j] = (\lambda_i + \lambda_j) Z_i Z_j\) then we conclude that

\[
[H, [Z_i, Z_j]] = (\lambda_i + \lambda_j) [Z_i, Z_j] = 0,
\]

(73)

which tells us that \( Z_i \) and \( Z_j \) commute when \( \lambda_i + \lambda_j \neq 0 \). If \([Z_j, Z_{2K-j+1}] = \sigma_j \neq 0\) for all \( j = 1, 2, \ldots, K \) then we can write \( H \) in the following way

\[
H = -\sum_{j=1}^{K} \frac{\lambda_j}{\sigma_j} Z_{2K-j+1} Z_j + E_0.
\]

(74)

If \( \psi_0 \) is a vector in the Hilbert space where \( H \) is defined that satisfies

\[
Z_j \psi_0 = 0, \ j = 1, 2, \ldots, K,
\]

(75)

then \( H \psi_0 = E_0 \psi_0 \).

Consider the time-evolution of the dynamical variables

\[
O_j(t) = e^{itH} O_j e^{-itH},
\]

(76)

so that

\[
\dot{O}_j(t) = i e^{itH} [H, O_j] e^{-itH} = i \sum_{k=1}^{2K} H_{kj} O_k(t).
\]

(77)

If we define the row vector \( \mathbf{O}(t) = (O_1(t) \ O_2(t) \ \ldots \ O_{2K}(t)) \) then we have the matrix differential equation \( \dot{\mathbf{O}}(t) = i \mathbf{O}(t) \mathbf{H} \) with the following solution:

\[
\mathbf{O}(t) = \mathbf{O} e^{it\mathbf{H}}, \ \mathbf{O} = \mathbf{O}(0).
\]

(78)

Since \( P(\mathbf{H}) = 0 \) then

\[
P \left( -i \frac{d}{dt} \right) \mathbf{O}(t) = \mathbf{O} P(\mathbf{H}) e^{it\mathbf{H}} = 0,
\]

(79)

gives us a differential equation of order \( 2K \) for the dynamical variables. Obviously, \( Z_j(t) = e^{it\lambda_j} Z_j, \ j = 1, 2, \ldots, 2K, \) satisfies this equation.

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