CONFORMAL EXTENSIONS OF FUNCTIONS DEFINED ON
ARBITRARY SUBSETS OF RIEMANN SURFACES

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Abstract. For a function defined on an arbitrary subset of a Riemann surface, we give conditions which allow the function to be extended conformally. One folkloric consequence is that two common definitions of an analytic arc in $\mathbb{C}$ are equivalent.

The purpose of this note is to extend conformally a function given on an arbitrary subset of a Riemann surface. Our original motivation was to prove that a construction of Nestoridis and Zadik in [3] holds more generally, so there is no need to use the specific form of the extended function used in [3, Prop. 2.3, iii]. A corollary of our result is that two common definitions of analytic arcs in $\mathbb{C}$ are equivalent.

Definition 1. A (topological or Jordan) open arc $J$ in a topological space $X$ is a subset $J \subset X$, which is a homeomorphic image of the open unit interval

$$I = \{ t \in \mathbb{R} : 0 < t < 1 \},$$

equivalently, of the real line. Thus, if $J$ is an open arc, there is a parametrization $\varphi : I \to J$, which is a homeomorphism.

If we are considering an open arc in a Riemann surface $X$, then topological notions (closure, boundary, etc.) will be with respect to $X$. In particular, if we are considering an open arc in the Riemann sphere $\mathbb{C}$, then topological notions will be with respect to $\mathbb{C}$. For a holomorphic mapping $f : X \to Y$ between two Riemann surfaces and a point $p \in X$, we write $f'(p) = 0$ to signify that, for some (hence any) local coordinate mappings $\varphi$ and $\psi$ at $p$ and $f(p)$ respectively, with $\varphi(p) = 0$, we have $(\psi \circ f \circ \varphi^{-1})'(0) = 0$. In particular, either $X$ or $Y$ may be the Riemann sphere $\mathbb{C}$.

For an open arc $J$ in $X$ with parametrization $\varphi$, we define the initial end $J(0)$ and the terminal end $J(1)$ of $J$ as

$$J(0) = \bigcap_{0 < t < 1} \varphi(0, t], \quad J(1) = \bigcap_{0 < t < 1} \varphi(t, 1).$$

Since $\varphi$ is a homeomorphism onto $J$ and $J$ has the relative topology induced by $X$, it follows that the open arc $J$ is disjoint from both of its ends. Each end is a closed connected set. If the initial end is a point, we call this the initial point of the arc (even though it is not on the arc). Similarly, if the terminal end is a point, we call it the terminal point.

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If $\varphi : I \to J$ is an open arc and $f : J \to \mathbb{C}$, we define the initial and terminal cluster sets of $f$ on $J$:

$$C_0(f, J) = \bigcap_{t \in I} \{f(\varphi(s)) : 0 < s < t\}, \quad C_1(f, J) = \bigcap_{t \in I} \{f(\varphi(s)) : t < s < 1\}.$$ 

The cluster set $C(f, J)$ is the union of the initial cluster set of $f$ and the terminal cluster set of $f$. In other words,

$$C(f, J) = C_0(f, J) \cup C_1(f, J) = \bigcap_{\varepsilon > 0} \{f \circ \varphi|[I \setminus [\varepsilon, 1 - \varepsilon]]\}.$$ 

**Theorem 1.** Suppose $J$ is an open arc in $\mathbb{C}$ and $f : U \to \mathbb{C}$ is a holomorphic mapping on an open neighborhood $U$ of $J$ in $\mathbb{C}$. Suppose $f|_J$ is injective, $f'(z) \neq 0$, for $z \in J$, and the sets $f(J)$ and $C(f, J)$ are disjoint. Then, $f$ is injective (one-to-one conformal) on some neighborhood of $J$.

**Proof.** Assume, first of all, that the initial and terminal cluster sets $C_0(f, J)$ and $C_1(f, J)$ are disjoint. Consider the mapping $\psi : I \to f(J)$, given as $\psi = f \circ \varphi$. As a composition of continuous injective mappings, $\psi$ is also continuous and injective. We claim that $\psi^{-1}$, (which is well-defined) is also continuous. Suppose for the sake of contradiction, that there is a sequence $\psi(t_j), t_j \in I$, which converges to a point $\psi(\alpha), \alpha \in I$, but $t_j \not\to \alpha$. We may assume that $t_j$ converges to point $\beta \in [0, 1]$. If $\beta \in I$, then $\psi(t_j) \to \psi(\beta) \neq \psi(\alpha)$, which is a contradiction. If $\beta = 0$, then $t_j \to 0$, so $\psi(\alpha) = \lim \psi(t_j) \in C_0(f, J)$, since $t_j \to 0$, which again is a contradiction, since $f(J)$ is disjoint from $C_0(f, J)$. The same argument shows that $t_j$ cannot converge to $1$. Thus, $\psi^{-1}$ is continuous and so $\psi$ is a homeomorphism. This shows that $f(J)$ is also an open arc.

Let $W_o$ be the component of $\mathbb{C} \setminus C_0(f, J)$ which contains the connected set $f(J) \cup C_1(f, J)$ and let $W_1$ be the component of $\mathbb{C} \setminus C_1(f, J)$ which contains the connected set $f(J) \cup C_0(f, J)$. Let $\hat{C}_0(f, J)$ be the union of $C_0(f, J)$ with all of its complementary components in $\mathbb{C}$ which do not meet the connected set $f(J) \cup C_1(f, J)$. We define $\hat{C}_1(f, J)$ similarly. We may also say that $\hat{C}_0(f, J) = \mathbb{C} \setminus W_o$ and $\hat{C}_1(f, J) = \mathbb{C} \setminus W_1$.

The sets $f(J), \hat{C}_0(f, J)$ and $\hat{C}_1(f, J)$ are pairwise disjoint. Both compact connected sets $\hat{C}_0(f, J)$ and $\hat{C}_1(f, J)$ have only one complementary component $W_o$ and $W_1$ respectively in $\mathbb{C}$ which both contain $f(J)$. There exists a homeomorphism

$$h : \mathbb{C} \setminus [\hat{C}_0(f, J) \cup \hat{C}_1(f, J)] \to \mathbb{C} \setminus \{p_o, p_1\},$$

mapping the topological annulus $\mathbb{C} \setminus [\hat{C}_0(f, J) \cup \hat{C}_1(f, J)]$ onto the twice punctured sphere $\mathbb{C} \setminus \{p_o, p_1\}$, where $p_o$ and $p_1$ are distinct finite points. The homeomorphism $h$ maps the open arc $f(J)$ to the open arc $h(f(J))$, whose initial and terminal points are respectively $p_o$ and $p_1$. By composing with a Möbius transformation, we may assume that the open arc $h(f(J))$ does not pass through $\infty$. Let $\alpha_o$ and $\alpha_1$ be disjoint open arcs in $\mathbb{C}$, where $\alpha_o$ joins $\infty$ to $p_o$ and $\alpha_1$ joins $p_1$ to infinity. Set

$$\alpha = \alpha_o \cup \{p_o\} \cup h(f(J)) \cup \{p_1\} \cup \alpha_1.$$ 

The open arc $h(f(J))$ is the homeomorphic image of the open unit interval $I$ under the parametrization $h \circ f \circ \varphi$ with initial point $p_o$ and terminal point $p_1$. We may extend this to a parametrization $\eta : (-\infty, +\infty) \to \alpha$ of the open arc $\alpha$. By the
Schoenflies Theorem [2, page 81], we may further extend \( \eta \) to a homeomorphism \( \eta : \mathbb{C} \to \mathbb{C} \). Let us denote the homeomorphism

\[
h^{-1} \circ \eta : \mathbb{C} \setminus \{0, 1\} \to \mathbb{C} \setminus \widehat{\mathcal{C}}_0(f, J) \cup \widehat{\mathcal{C}}_1(f, J)
\]

by \( H \).

Fix \( t \in J \). We may choose a closed disc \( \overline{D}_t \) with center \( t \) so small that \( 0, 1 \notin \overline{D}_t \) and \( H \) maps \( D_1 \) homeomorphically onto a Jordan domain \( W_t \) containing \( f(\varphi(t)) \). Since \( f'(\varphi(t)) \neq 0 \), we may choose a branch \( g_t \) of \( f^{-1} \) in a neighborhood of \( f(\varphi(t)) \), such that \( f^{-1} \circ f \) is the identity in a neighborhood of \( \varphi(t) \). We may choose \( D_t \) so small that the Jordan domain \( H(D_t) \) is contained in the domain of definition of this inverse branch \( g_t \). We claim that these inverse branches, for various \( t \) are compatible. Indeed, since \( H \) is a homeomorphism, two Jordan domains \( H(D_s) \) and \( H(D_t) \) intersect if and only if the discs \( D_s \) and \( D_t \) intersect and in this case the intersection \( H(D_s) \cap H(D_t) \) has only one component, which is \( H(D_s \cap D_t) \). Since \( f \) is injective on \( J \), the branches \( g_s \) and \( g_t \) agree on the (non-empty) arc of \( f(J) \) in \( H(D_s) \cap H(D_t) \). By the uniqueness principle, \( g_s = g_t \) on \( H(D_s) \cap H(D_t) \). We have verified that the inverse branches \( g_t, t \in I \), are compatible. Thus, we may define a branch \( g \) of \( f^{-1} \) on the neighborhood \( W = \bigcup_{t \in I} H(D_t) \) of \( f(J) \). We have that \( f \) maps the open neighborhood \( g(W) \) of \( J \) biholomorphically onto the neighborhood \( W \) of \( f(J) \). This completes the proof, in case the initial and terminal cluster sets \( C_0(f, J) \) and \( C_1(f, J) \) are disjoint.

Suppose the initial and terminal cluster sets \( C_0(f, J) \) and \( C_1(f, J) \) are not disjoint. The cluster set \( C(f, J) \) is then a continuum or a point, so the open set \( \mathbb{C} \setminus C(f, J) \) is simply connected. In particular, the component \( \Omega \) of \( \mathbb{C} \setminus C(f, J) \) which contains the connected set \( f(J) \) is simply connected. Set \( E = \mathbb{C} \setminus \Omega \). If \( E \) is a singleton, we may map \( \Omega \) to \( \mathbb{C} \) by a Möbius transformation so that \( E \) goes to \( \infty \). Suppose \( E \) is a continuum. By the Riemann mapping theorem, we may assume that \( \Omega \) is the unit disc and \( f(J) \) is an open arc in \( \Omega \). There is a homeomorphism \( h : \Omega \to \mathbb{C} \) so that \( h(w) \to \infty \), as \( |w| \to 1 \). Thus, whether \( E \) is a singleton or not, there is a homeomorphism from \( \Omega \) to \( \mathbb{C} \). After this mapping, the image of \( f(J) \) is an open arc both ends of which are \( \infty \). We may parametrize \( f(J) \) by the real line and use the Schoenflies theorem as above.

This result extends to arcs on Riemann surfaces. For that purpose, the following lemma is useful and of independent interest.

**Lemma 1.** Let \( J \) be an open arc in a Riemann surface \( X \). Then, \( J \) has a fundamental system of simply connected neighborhoods.

**Proof.** For simplicity of notation, it will be convenient to consider \( J \) as being parametrized by \( \mathbb{R} \) rather than \( (0, 1) \). Thus, \( \varphi : (-\infty, +\infty) \to J \). We may choose an increasing sequence \( t_j, j \in \mathbb{Z} \), such that \( \lim_{j \to \pm \infty} = \pm \infty \) and each \( \varphi(t_j, t_{j+1}) = J_j \) is contained in a chart \( U_j \). We may construct Jordan domains \( V_j \), such that \( J_j \subset V_j \subset U_j \), the end points of \( J_j \) are on \( \partial V_j \) and the \( V_j \) are disjoint except possibly for end points. By construction, the set \( V = \bigcup_j V_j \) covers \( J \setminus \bigcup \{ \varphi(t_j) \} \). By introducing suitable small neighborhoods of the end points \( \varphi(t_j) \), we may enlarge \( V \) to a “strip” \( S \) which is homeomorphic to \( \{ z = x + iy : -\infty < x < +\infty, |y| < 1 \} \) and hence simply connected. If \( W \) is a neighborhood of \( J \), we can replace \( X \) by the component of \( W \) containing \( J \). We have thus shown that \( J \) has a fundamental system of simply connected neighborhoods.
Theorem 2. Let $J$ be an open arc in a Riemann surface $X$ and $f : U \to \mathbb{C}$ a holomorphic mapping on an open neighborhood $U$ of $J$ in $X$. Suppose $f |_J$ is injective, $f'(p) \neq 0$, for $p \in J$, and the sets $f(J)$ and $C(f, J)$ are disjoint. Then, $f$ is injective (one-to-one conformal) on some neighborhood of $J$.

Proof. We may assume that $U$ is simply connected and so it is conformally equivalent to $\mathbb{C}$, to $\mathbb{C}$ or to the unit disc. The case $\mathbb{C}$ is excluded, since $f$ is not constant. The conclusion now follows from the previous theorem. \hfill \Box

We shall now extend our results on arcs to arbitrary sets. Of course, a necessary condition that a function be extendable conformally is that it be extendable holomorphically. To this end we have the following.

Theorem 3. Let $X$ be a Riemann surface, $E$ an arbitrary subset of $X$ and $f : E \to Y$ a mapping from $E$ to a complex manifold $Y$, such that, for each $p \in E$, there is an open neighborhood $U_p \subset X$ of $p$ and a holomorphic mapping $\Phi_p : U_p \to Y$, such that $\Phi_p(q) = f(q)$, for all $q \in U_p \cap E$. Then, $f$ extends to a holomorphic mapping $\Phi : U \to Y$ on some neighborhood $U$ of $E$, such that $\Phi$ locally coincides with some $\Phi_p$.

A form of this theorem was proved in [H] Th. 5] for the special case of meromorphic functions, that is, when $Y = \mathbb{C}$. The theorem in [H] is also weaker in the sense that it is not claimed that, $\Phi$ locally coincides with some $\Phi_p$, but merely that $\Phi(p) = f(p)$.

Proof. The proof is a modification of an argument taken from [H]. Let $E'$ be the set of accumulation points of $E$ which are in $E$. Choose a distance function on $X$ (see [H]). For $p \in E$, denote by $r_p$ the distance of $p$ from $\partial U_p$. For each $p \in E'$, we choose a parametric disc $D_p$ for $X$ at $p$, such that $\text{diam} D_p < r_p/2$.

Claim: for every two such discs $D_p, D_q$, with $p, q \in E'$, we have

$$\Phi_p(z) = \Phi_q(z), \quad \text{for all} \quad z \in D_p \cap D_q.$$ 

We may suppose that $D_p \cap D_q \neq \emptyset$ and $\text{diam} D_q \leq \text{diam} D_p$. Then, $D_q \subset U_p$. Since $q$ is a limit point of $E$, and both $\Phi_p$ and $\Phi_q$ equal $f$ on $E \cap D_q$, it follows that $\Phi_p = \Phi_q$ on the component of $U_p \cap U_q$ containing $D_q$. Obviously, this component contains $D_p \cap D_q$ so the claim follows.

We have

$$E' \subset U' \overset{def}{=} \bigcup_{p \in E'} D_p.$$ 

By the claim, we may define a holomorphic function $\Phi$ on the open neighborhood $U'$ of $E'$, by setting $\Phi = \Phi_p$ on each $D_p, p \in E'$. Moreover, $\Phi = f$ on $U' \cap E$.

Arrange the points of $E \setminus U'$ in a sequence $p_n$. Denote by $U_p$ the neighborhood of $p$. For each $p = p_n$, choose a disc $D_n = D_p$ centered at $p$ and contained in $U_p$, so small that the radius is less than $1/n$, and such that $E \cap D_p = \{p\}$. We can also arrange that the discs $D_n$ are pairwise disjoint. For instance, it suffices that the radius of $D_n$ be smaller than $1/2$ the distance of $p_n$ to the rest of $E$. Set $\Phi = \Phi_{p_n}$ on $D_n$. Let $U$ be defined as

$$U = [U' \setminus \bigcup_{p \in E' \setminus U'} \overline{D_p}] \cup \bigcup_{p \in E' \setminus U'} D_p.$$
Lemma 2.\hspace{1em} The following lemma \cite[Lemma 3.6]{4} is helpful.

In order to check that the set $U$ is open one can use the fact that the radii of the $D_n$ converge to zero. The mapping $\Phi$ is well defined on the neighborhood $U$ of $E$ and has the desired properties. \hfill $\square$

Remark 1. In the proof of Theorem 3 the function $\Phi$ locally coincides with some $\Phi_p$. Therefore, if we assume that the derivative of $\Phi_p$ at $p$ is non zero, then we can consider smaller open sets $U_p$ so that the derivative of $\Phi_p$ is everywhere non zero. It follows that the derivative of $\Phi$ is non zero everywhere on $U$ and for every $z$ in $U$ the mapping $\Phi$ is locally a homeomorphism between two open sets containing $z$ and $\Phi(z)$, respectively.

A holomorphic curve in a complex manifold $Y$ is a nonconstant holomorphic mapping $\Phi : X \to Y$ from a Riemann surface $X$ into $Y$.

**Corollary 1.** Let $X$ be a Riemann surface, $E$ a connected subset of $X$ and $f : E \to Y$ a mapping from $E$ to a complex manifold $Y$, such that, for each $p \in E$, there is an open neighborhood $U_p \subset X$ of $p$ and a holomorphic mapping $\Phi_p : U_p \to Y$, such that $\Phi_p(q) = f(q)$, for all $q \in U_p \cap E$. Then, $f$ extends to a holomorphic curve $\Phi : V \to Y$ mapping some connected neighborhood $V$ of $E$ into $Y$, such that $\Phi$ locally coincides with some $\Phi_p$.

**Proof.** In Theorem 3 let $V$ be the component of $U$ containing $E$. Then, $V$ is a Riemann surface and so $\Phi : V \to Y$ is, by definition, a holomorphic curve. \hfill $\square$

The Corollary applies, in particular, to the case that $E$ is an open arc.

In order to extend a function, not only holomorphically, but even biholomorphically, the following lemma \cite[Lemma 3.6]{4} is helpful.

**Lemma 2.** Let $U, Y$ be Hausdorff spaces with countable bases and $U$ be locally compact. If $\Phi : U \to Y$ is a local homeomorphism and the restriction of $\Phi$ to a closed subset $E$ is a homeomorphism, then $\Phi$ is a homeomorphism on some neighborhood $V$ of $E$.

Let $E$ be a subset of a Riemann surface $X$, and let $f : E \to Y$. For a point $p \in X$, we define the cluster set $C(f,p)$ of $f$ at $p$ as the set of all values $w \in Y$, such that there is a sequence $z_n \in E, z_n \to p$, for which $f(z_n) \to w$.

**Theorem 4.** Let $X$ and $Y$ be Riemann surfaces and $E$ be an arbitrary subset of $X$. Suppose, for a function $f : E \to Y$, that the cluster sets $C(f,p), p \in E$, are pairwise disjoint and, for each $p \in E$, there is an open neighborhood $U_p \subset X$ of $p$ and a holomorphic mapping $\Phi_p : U_p \to Y$, such that $\Phi_p(q) = f(q)$, for all $q \in U_p \cap E$ and $\Phi'_p(p) \neq 0$. Then $f$ extends to a one-to-one conformal mapping of some open neighborhood $V$ of $E$ onto an open subset of $Y$.

**Proof.** By Theorem 3, $f$ extends to a holomorphic mapping into $Y$. According to Remark 1, for every $z$ in $U$, the mapping $\Phi$ is locally a homeomorphism between two open sets containing $z$ and $\Phi(z)$, respectively. Set $F = f(E)$. We claim that $f : E \to F$ is a homeomorphism. First of all, $f$ is continuous, because it is the restriction of the continuous function $\Phi$. The continuity of $f$ implies that $C(f,z) = f(z)$, for all $z \in E$. The hypothesis on cluster sets therefore implies that $f$ is injective and hence has an inverse function $\psi : F \to E$. We claim that $\psi$ is continuous. To see this, let $b = f(a)$ be a point of $F$ and suppose $w_n = f(z_n)$ converges to $b$. Let $a'$ be any limit point of $\psi(w_n) = z_n$. Then, from the definition of $C(f,a')$, it follows that $b \in C(f,a')$. Since $b$ is also in $C(f,a)$, the hypothesis on cluster sets implies
that $a' = a$. We have shown that $\psi(w_n) \to a = \psi(b)$. This confirms the claim that $\psi$ is continuous and also that $f$ is a homeomorphism.

If $E$ is closed, the theorem now follows from Lemma 2.

In general, fix some distance function on $Y$. For each $z \in E$, we may choose an open neighborhood $\tilde{U}_z$ such that $\Phi(\tilde{U}_z)$ is a disc $D_w$ centered at $w = f(z)$ and we may choose a branch $\Psi_w$ of $\Phi^{-1}$ in $D_w$ such that $\Psi_w \circ \Phi$ is the identity on $\tilde{U}_z$. We claim that $\Psi_w = \psi$ on $F \cap D_w$. To verify the last claim, suppose $b \in F \cap D_w$. From the definition of $\Psi_w$, there is a point $a \in E \cap \tilde{U}_z$, such that $f(a) = b$ and $a = \Psi_w(b)$. Since $f$ is injective, $a = f^{-1}(b) = \psi(b)$. We have shown that $\Psi_w(b) = a = \psi(b)$, which establishes the claim. By Theorem 3 there is an open neighborhood $W$ of $F$ and a holomorphic mapping $\Psi : W \to X$, such that $\Psi = \Psi_w$ on $D_w$, for each $w \in F$. The mapping $\Psi$ is holomorphic on $W$ and maps $W$ to an open neighborhood $V$ of $E$, contained in the domain of definition of $\Phi$. Moreover, $\Psi \circ \Phi$ is the identity on $V$. Thus, $\Phi$ is biholomorphic from $V$ to $W$ and $\Phi$ restricted to $E$ is $f$. This finishes the proof.

We remark that if $E$ has no isolated points, then the extension in Theorem 4 is unique in the sense that any two extensions agree on some neighborhood of $E$. In particular, this applies to the case that $E$ is a curve.

**Definition 2.** [analytic arc] An analytic open arc $J$ in a Riemann surface $X$ is an arc in $X$, whose parametrization $\varphi$ is analytic. Thus, for every $t_0 \in I = (0, 1)$ and local coordinate $z$ in a neighborhood of $\varphi(t_0)$, the function $z \circ \varphi$ has a representation as a power series near $t_0 : (z \circ \varphi)(t) = \sum a_j(t - t_0)^j$. We shall say that $J$ is a regular open analytic arc if $\varphi'(t) \neq 0$, for all $t \in (0, 1)$.

**Definition 3.** [conformal arc] A conformal open arc $J$ in a Riemann surface is an arc in $X$, with a parametrization which extends to a one-to-one conformal mapping from a neighborhood of $I \subset \mathbb{C}$ into $X$.

**Corollary 2.** An open arc $J$ in a Riemann surface $X$ is a regular analytic arc if and only if it is a conformal arc.

**Proof.** Clearly, if $J$ is a conformal arc, then it is a regular analytic arc.

Suppose, conversely, that $J$ is a regular analytic arc. Consider firstly the case that $X = \mathbb{C}$. Thus, there is a parametrization $\varphi : I \to J$ which is analytic and such that $\varphi'(t) \neq 0$, for $t \in I$. For each $t \in I$, we may extend $\varphi$ to a holomorphic function $\varphi_t$ in a disc $D_t$ centered at $t$ and disjoint from $\{0, 1\}$. If $D_s \cap D_t \neq \emptyset$, then $\varphi_s = \varphi = \varphi_t$ on $I \cap D_s \cap D_t$. Thus, $\varphi_s = \varphi_t$ on $D_s \cap D_t$. By setting $\varphi = \varphi_t$ on $D_t$, for each $t \in I$, we obtain a well-defined holomorphic extension of $\varphi$ to the open neighborhood $U = \cup_{t \in I} D_t$ of $I$.

We note that the initial and terminal ends of $J$ are the same as the initial and terminal cluster sets $C_\alpha(\varphi, I)$ and $C_1(\varphi, I)$. Since $\varphi : I \to J$ is a homeomorphism, these cluster sets are disjoint from $I$. It follows directly from Theorem 4 that $J$ is a conformal arc. This completes the proof of the converse, in case $X = \mathbb{C}$.

In the general case, it follows from Lemma 1, that there is a simply connected neighborhood $U$ of $J$ in $X$. By the Riemann mapping theorem, $U$ is conformally equivalent to an open subset of $\mathbb{C}$, and so it follows from the case just treated that $J$ is a conformal arc.

This equivalence, stated in the corollary, is probably folkloric, at least in case the analytic arc is the image of the closed rather than open unit interval or in case
we have a Jordan curve. Osserman [5] says that an analytic Jordan curve on a Riemann surface $X$ is defined by an analytic mapping of the unit circle into $X$ and this mapping extends to a conformal mapping of an annulus into $X$. In these cases, one can use compactness to give a simpler proof than the one we have given.

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References

[1] Gauthier, P. M. Complex extension-interpolation and approximation on arbitrary sets in one dimension (submitted).
[2] Greenberg, M. J. Lectures on algebraic topology. W. A. Benjamin, Inc., New York-Amsterdam 1967.
[3] Nestoridis, V.; Zadik, I. Padé Approximants, density of rational functions in $A^\infty(\Omega)$ and smoothness of the integration operator, arXiv:1212.4394.
[4] Milnor, J.; Differential Topology. Lectures by John Milnor, Princeton University, Fall term 1958. Notes by James Munkres.
[5] Osserman, R. A lemma on analytic curves. Pacific J. Math. 9 1959 165-167.
[6] Pfluger, A. Theorie der Riemannschen Flächen. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1957.

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