On the integral cohomology of quotients of manifolds by cyclic groups
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February 28, 2017

Abstract
We propose new tools based on basic lattice theory to calculate the integral cohomology of the quotient of a manifold by an automorphism group of prime order. As examples of applications, we provide the Beauville-Bogomolov forms of some irreducible symplectic V-manifolds; we also show a new expression for a basis of the integral cohomology of a Hilbert scheme of two points on a surface.

1 Introduction
Consider a compact connected orientable manifold (without boundary) \(X\) and a finite automorphism group \(G\), of prime order \(p\), acting on \(X\). In this article, we study the integral cohomology of the quotient \(X/G\). The group \(G\) acts naturally on the cohomology of \(X\). In the case of cohomology with rational coefficients, the problem is much simpler, \(H^*(X/G, \mathbb{Q})\) is isomorphic to the invariant space \(H^*(X, \mathbb{Q})^G\). Nevertheless, for integral cohomology this property does not hold anymore.

A fundamental tool for studying this question is given by the work of Smith in [28]. Let \(\pi : X \to X/G\) be the quotient map. Smith has constructed a push-forward map \(\pi_* : H^*(X, \mathbb{Z}) \to H^*(X/G, \mathbb{Z})\) such that:

\[
\pi_* \circ \pi^* = d \text{id}_{H^*(X/G, \mathbb{Z})}, \quad \pi^* \circ \pi_* = \sum_{g \in G} g^*.
\]

Let denote by \(H^*_{\text{tors}}(Y, \mathbb{Z}) := H^*_{\text{tors}}(Y, \mathbb{Z})\) the torsion-free part of the cohomology of a topological space \(Y\). Then, (1) provides the exact sequence

\[
0 \longrightarrow \pi_* (H^*_{\text{tors}}(X, \mathbb{Z})) \longrightarrow H^*_{\text{tors}}(X/G, \mathbb{Z}) \longrightarrow \mathbb{Z}/p \mathbb{Z} \alpha_k(X) \longrightarrow 0,
\]

where \(\alpha_k(X)\) is a non-negative integer which we call the \(k\)-th coefficient of surjectivity of \(X\) (See Section 3.1).

The objective of this paper is to provide tools for calculating \(\alpha_k(X)\). Using basic lattice theory, we provide an upper bound for the coefficients of surjectivity. For a finer result, one has to study the local action of \(G\) around the fixed points. A simple fixed point is a fixed point such that the local action of \(G\) around is given by a matrix of the form \(\text{diag}(1, \ldots, 1, \xi_d^p, \ldots, \xi_d^p)\), where \(\xi_d^p\) is a \(p\)-th root of unity with \(d \in \{1, \ldots, p-1\}\). These fixed points have the interest to provide singularities on \(X/G\) that can be solved with only one blow-up. For instance, all fixed points are simple when \(p = 2\); we will see other examples of such fixed points in Section 7. Our main theorem is:

**Theorem 1.1.** Let \(X\) be a compact Kähler manifold of complex dimension \(n\) and \(G\) an automorphism group of prime order \(p \leq 19\). Let \(c := \text{Codim Fix } G\). Assume that:

(i) \(H^*(X, \mathbb{Z})\) and \(H^*(\text{Fix } G, \mathbb{Z})\) are \(p\)-torsion-free,

(ii) the spectral sequence of equivariant cohomology with coefficients in \(\mathbb{F}_p\) degenerates at the second page,

(iii) all fixed points of \(G\) are simple,
(iv) $c \geq \frac{n}{2} + 1$.

Then, for all $2n - 2c + 1 \leq k \leq 2c - 1$, the $k$-th coefficient of surjectivity $\alpha_k(X)$ of $X$ vanishes.

Condition (ii) is natural in the sense that there is already a theory to show the degeneration of spectral sequences at the second page (see for instance [8]). Sarti, Boisier and Nipper-Wisskirchen, in [5], also developed theorems of degeneration of the equivariant spectral sequence that we will use in our applications. We will also see (Theorem 6.1) that condition (ii) can be replaced by a numerical condition involving invariants related to the action of the group and the cohomology of the fixed locus.

This result can be applied to compute the Beauville-Bogomolov form of some irreducible symplectic $V$-manifolds. A $V$-manifold is a compact analytic complex space with at worst finite quotient singularities. A $V$-manifold is called symplectic if its nonsingular locus is endowed with an everywhere non-degenerate holomorphic 2-form which extends to a resolution of singularities. A symplectic $V$-manifold is called irreducible if it is complete, simply connected, and the holomorphic 2-form is unique up to scaling. Such varieties are good candidates to generalize the short list of known irreducible symplectic manifolds. Indeed, some aspects of the theory were already generalized in [22] and [16], for instance, the Beauville-Bogomolov form, the local Torelli theorem, and the Fujiki formula. We recall that the Fujiki formula (45) establishes a proportionality between the cup-product and the Beauville-Bogomolov form; the coefficient of proportionality is called the Fujiki constant.

In this paper we compute the Beauville-Bogomolov form of a manifold of $K3^{[2]}$-type quotiented by symplectic automorphisms of order 3 and 11. There are two different examples of symplectic automorphisms $\sigma$ of order 11 on a manifold of $K3^{[2]}$-type $X$ (Example 4.5.1 and Example 4.5.2 in [19]). As explained in [19, Section 7.4.4], the lattice $H^2(X, \mathbb{Z})^G$ will be either $\begin{pmatrix} 6 & 2 & 2 \\ 2 & 8 & -3 \\ 2 & -3 & 8 \end{pmatrix}$ or $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 22 \end{pmatrix}$. We denote the quotients $X/\sigma$, respectively, by $M_{11}^1$ and $M_{11}^2$.

**Theorem 1.2.** Let $X$ be a manifold of $K3^{[2]}$-type. Let $G$ be a symplectic automorphism group of order 11 acting on $X$. Then, the Fujiki constant of both $M_{11}^1$ and $M_{11}^2$ is 33 and the Beauville-Bogomolov lattices are:

$$H^2(M_{11}^1, \mathbb{Z}) \simeq \begin{pmatrix} 2 & -1 & 3 \\ -1 & 8 & -1 \\ 3 & -1 & 6 \end{pmatrix}, \quad H^2(M_{11}^2, \mathbb{Z}) \simeq \begin{pmatrix} 2 & 1 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In [19, Theorem 7.2.7], Mongardi distinguishes two kinds of symplectic automorphism of order 3 on a $K3^{[2]}$-type manifold one with 27 isolated fixed points and another with a fixed abelian surface.

**Theorem 1.3.** Let $X$ be a manifold of $K3^{[2]}$-type. Let $G$ be a symplectic automorphisms group of order 3 of $X$ with 27 isolated fixed points. We denote $M_3 = X/G$. Then, the Beauville-Bogomolov lattice $H^2(M_3, \mathbb{Z})$ is isomorphic to $U(3) \oplus U^2 \oplus A_2^3 \oplus (-6)$, and the Fujiki constant of $M_3$ is 9.

These examples provide new small-dimensional moduli spaces of singular irreducible symplectic varieties. In particular, $M_{11}^1$ and $M_{11}^2$ provide two positive definite Beauville-Bogomolov lattices of rank 3, whose period domains are cones in $\mathbb{P}^2$. The varieties $M_{11}^1$ and $M_{11}^2$ look very different from the known smooth irreducible symplectic varieties. For example, none of their deformations admits a Lagrangian fibration since their Beauville-Bogomolov form is positive definite (see for instance [12, Remark 2.17]). Hence, $M_{11}^1$ and $M_{11}^2$ could be interesting examples to study in order to develop the theory of singular irreducible symplectic varieties.

As it is well known, the Hilbert scheme of two points $A^{[2]}$ on a surface $A$ can be obtained as a quotient of $\mathrm{Bl}_A(A \times A)$ the blow-up of $A \times A$ in the diagonal $\Delta$. Therefore, studying the integral cohomology of $A^{[2]}$ is a possible application of our theory. As another application of our main theorem, we provide a new expression of a basis of the cohomology of $A^{[2]}$, already studied in [25]
and [27]. This basis has the advantage to only require that the surface is Kähler. Moreover, the
basis is expressed in terms of pull-backs and push-forward of classes of the surface, allowing to
calculate easily the ring structure of $H^\ast(A[[2]],\mathbb{Z})$ (see Lemma 7.9).

This paper generalizes results of the author’s Ph.D. thesis [18], and it was already applied in
[13] for finding the Beauville-Bogomolov form of a partial resolution of the quotient by a symplectic
involution of the generalized Kummer of dimension 4.

The paper is structured as follows. Since the integral cohomology has a lattice structure, in
Section 2, we study a general lattice $T$ endowed with an isometry group $G$ of prime order. In
particular, using the $\mathbb{Z}[G]$-module structure of $T$, we compute the discriminant of the invariant
lattice $T^G$ (Proposition 2.15). In Section 3 we apply these results to find the discriminant of
$\pi_\ast(H^\ast(X, \mathbb{Z}))$ (Corollary 3.8), and, using basic lattice theory, we provide an upper bound for
the coefficients of surjectivity $\alpha_k(X)$ (Proposition 3.10). Thanks to Section 2, we can calculate
the cohomology of $G$ in Section 4 (Corollary 4.2); this allows us to use equivariant cohomology to
calculate the cohomology of non-ramified quotients. In Section 5 we study resolution of singularities,
which leads to the proof of the main Theorem 1.1 in Section 6. Section 7 is dedicated to some
applications, namely the computation of the Beauville-Bogomolov forms of certain V-manifolds
and the description of the integral cohomology of the Hilbert scheme of two points.

Acknowledgements. I am very grateful to Samuel Boissière for helpful discussions and to
Simon Kapfer for inspiring in the problem of Section 7.3. I also want to thank Dimitri Marku-
shевич, Emilio Franco and Henrique Sá Earp for many helpful comments on some preliminary
versions of this work. This work was supported by Fapesp grant 2014/05733-9.

2 The $\mathbb{Z}[G]$-module structure of a lattice

2.1 Basic properties of lattices

We recall the basic facts on lattices which are used in this paper (see for example [9, Chapter
8.2.1]). A lattice $T$ is a free $\mathbb{Z}$-module endowed with a non-degenerate bilinear form. We denote by
discr $T$ the discriminant of $T$, which is the absolute value of the determinant of the bilinear form
of $T$. We say that $T$ is unimodular if its discriminant is 1. A sublattice $N$ of $T$ is said primitive if
$T/N$ is torsion-free.

If $N$ is a sublattice of a lattice $T$ of the same rank, we have the basic formula:

$$\# \frac{T}{N} = \sqrt{\frac{\text{discr } N}{\text{discr } T}}. \quad (3)$$

If $T$ is an unimodular lattice and $N$ is a primitive sublattice, then

$$\text{discr } N = \text{discr } N^\perp. \quad (4)$$

More precisely, the projection

$$\frac{T}{N \oplus N^\perp} \to A_N \quad (5)$$
is an isomorphism, where $A_N := N^\perp/N$ is the discriminant group. Moreover, this provide an
isometry:

$$\nu : A_N \to A_N^\perp, \quad (6)$$
between the discriminant groups.

Let $X$ be a compact connected orientable manifold of dimension $n$. We endowed $H^\ast(X, \mathbb{Z})$ with
the natural bilinear form $B$ defined as follows. Let $x \in H^k(X, \mathbb{Z})$ and $y \in H^q(X, \mathbb{Z})$,

$$B(x, y) = 0 \text{ if } k + q < n,$$

$$B(x, y) = x \cdot y \text{ if } k + q = n, \text{ where } \cdot \text{ refers to the cup-product}. \quad (7)$$
By Poincaré duality, it turns \( H^*(X, \mathbb{Z}) \) into an unimodular lattice.

In our case, the lattice \( T \) will be endowed with an isometry group \( G \) of prime order \( p \). We denote by \( T^G \) the invariant lattice of \( T \). We also denote by \( (T^G)^\vee(p) \) the dual of \( T^G \) with its bilinear form multiplied by \( p \). Let \( G \) be an automorphism group of prime order \( p \) on \( X \) and \( T \) an unimodular sublattice of \( H^*(X, \mathbb{Z}) \); in Section 3.2, we will see that:

\[
(T^G)^\vee(p) \simeq \pi_*(T),
\]

where \( \pi: X \to X/G \) is the quotient map. For this reason, we will call \( (T^G)^\vee(p) \) the push-forward lattice of \( (T, G) \); an key step for our purpose, which will be accomplished in Section 2.3, is the calculation of its discriminant. To do so, we will use the \( \mathbb{Z}[G] \)-module structure of \( T \) given by \( g \cdot x = g(x) \) for all \( g \in G \) and all \( x \in T \).

2.2 The Boissière–Nieper-Wisskirchen–Sarti invariants

Here we review some important notions introduced in [5]. Let \( p \) be a prime number, \( T \) a \( p \)-torsion-free \( \mathbb{Z} \)-module of finite rank and \( G = \langle g \rangle \) a group of prime order \( p \) acting linearly on \( T \). Then, \( T \otimes \mathbb{F}_p \) is a \( \mathbb{F}_p \)-vector space equipped with a linear action of \( G \). The minimal polynomial of \( g \), as an endomorphism of \( T \otimes \mathbb{F}_p \), divides \( X^p - 1 = (X - 1)^p \in \mathbb{F}_p[X] \), hence \( g \) admits a Jordan normal form over \( \mathbb{F}_p \). We can decompose \( T \otimes \mathbb{F}_p \) as a direct sum of some \( \mathbb{F}_p[G] \)-modules \( N_q \) of dimension \( q \) for \( 1 \leq q \leq p \), where \( g \) acts on \( N_q \) in a suitable basis by a matrix of the following form:

\[
\begin{pmatrix}
1 & 1 & \cdots & 0 \\
0 & & & 1 \\
\end{pmatrix}
\]

**Definition 2.1.** We define the integer \( \ell_q(T) \) as the number of blocks of size \( q \) in the Jordan decomposition of the \( \mathbb{F}_p[G] \)-module \( T \otimes \mathbb{F}_p \), so that \( T \otimes \mathbb{F}_p \simeq \bigoplus_{q=1}^p N_q^{\ell_q(T)} \).

When \( p \leq 19 \), we can define the \( \ell_q(T) \) from another point of view. Let \( \xi_p \) be a primitive \( p \)-th root of unity, \( K := \mathbb{Q}(\xi_p) \) and \( \mathcal{O}_K := \mathbb{Z}[\xi_p] \) the ring of algebraic integers of \( K \). By a classical theorem of Masley-Montgomery [15], \( \mathcal{O}_K \) is a PID if and only if \( p \leq 19 \). The \( \mathbb{Z}[G] \)-module structure of \( \mathcal{O}_K \) is defined by \( g \cdot x = \xi_p^r x \) for \( x \in \mathcal{O}_K \). For any \( a \in \mathcal{O}_K \), we denote by \( (\mathcal{O}_K, a) \) the module \( \mathcal{O}_K + \mathbb{Z} \) whose \( \mathbb{Z}[G] \)-module structure is defined by \( g \cdot (x, k) = (\xi_p^r x + ka, k) \). In the proof of [5, Proposition 1.1], using [7, Theorem 74.3] of Diederichsen and Reiner, Boissière, Nieper-Wisskirchen and Sarti show the following.

**Proposition 2.2.** Let \( T \) be a free \( \mathbb{Z} \)-module of finite rank and let \( G = \langle g \rangle \) be a group of prime order \( p \leq 19 \) acting on \( T \). Then, we have an isomorphism of \( \mathbb{Z}[G] \)-module:

(i) \[
T \simeq \bigoplus_{i=1}^r (\mathcal{O}_K, a_i) \oplus \mathcal{O}_K^{\oplus s} \oplus \mathbb{Z}^{\oplus t},
\]

where \( r, s, t \) are integers and \( a_i \notin (\xi_p^r - 1)\mathcal{O}_K \). Moreover, the integers \( r, s, t \) are uniquely determined by the \( \mathbb{Z}[G] \)-module structure of \( T \).

(ii) When \( p \geq 3 \), \( (\mathcal{O}_K, a_i) \otimes \mathbb{F}_p = \mathbb{N}_p, \mathcal{O}_K \otimes \mathbb{F}_p = \mathbb{N}_p - 1 \) and \( \mathbb{Z} \otimes \mathbb{F}_p = \mathbb{N}_1 \), so \( r = \ell_p(T), s = \ell_{p-1}(T) \) and \( t = \ell_1(T) \).

(iii) We have:

\[
T^G \simeq \bigoplus_{i=1}^r (\mathcal{O}_K, a_i)^G \oplus \mathbb{Z}^{\oplus t},
\]

with \( \text{rk}(\mathcal{O}_K, a_i)^G = 1 \).

**Remark 2.3.** In the particular case \( p = 2 \), we have \( \ell_1(T) = t + s \) and \( \ell_2(T) = r \).
Hence, to distinguish the integers \( t \) and \( s \) in the case \( p = 2 \), we choose the following notation.

**Definition 2.4.** When \( p = 2 \), we define \( \ell_+(T) := t \), \( \ell_-(T) := s \).

**Remark 2.5.** When \( p = 2 \), it follows: \( \ell_1(T) = \ell_+(T) + \ell_-(T) \).

Moreover, to avoid distinguishing the cases \( p = 2 \) and \( p \geq 3 \) all over, we adopt the following notation.

**Notation 2.6.** When \( p \geq 3 \), we also denote \( \ell_+(T) := \ell_1(T) = t \) and \( \ell_-(T) := \ell_{p-1}(T) = s \).

We can reformulate Proposition 2.2 in those terms:

**Corollary 2.7.** Let \( T \) be a free \( \mathbb{Z} \)-module and let \( G = \langle g \rangle \) be a group of prime order \( p \leq 19 \) acting on \( T \). We have:

(i) \( \text{rk} T = p \ell_p(T) + (p - 1) \ell_-(T) + \ell_+(T) \),

(ii) \( \text{rk} T^G = \ell_p(T) + \ell_+(T) \).

As in [5], in the case of cohomology, we choose the following notation for our invariants:

**Notation 2.8.** For all \( 0 \leq k \leq \dim X \), we denote:

\( \ell_k^G(X) = \ell_+(H^k_f(X, \mathbb{Z})) \), \( \ell_k^l(X) = \ell_-(H^k_f(X, \mathbb{Z})) \) and for all \( q \in \{1, \ldots, p\} \), \( \ell_q^k(X) = \ell_q(H^k_f(X, \mathbb{Z})) \).

### 2.3 Discriminant of the push-forward lattice

**Proposition 2.9.** Let \( T \) be an unimodular lattice and let \( G = \langle g \rangle \) be a group of prime order \( p \) acting on \( T \). Then,

\[ \text{discr}(T^G)^\vee (p) = p^{\ell^G(T)} - \ell_p(T). \]

If moreover \( p \leq 19 \), we have:

\[ \text{discr}(T^G)^\vee (p) = p^{\ell^+ (T)}. \]

This section is dedicated to proving this proposition. We need several lemmas.

**Lemma 2.10.** Let \( T \) be a lattice and \( G = \langle g \rangle \) a group of prime order \( p \) acting on \( T \). Let \( \sigma = \text{id} + g + \ldots + g^{p-1} \) then:

\( \ker \sigma = (T^G)^\perp \).

**Proof.** We have \( \ker \sigma \subset (T^G)^\perp \). Indeed, let \( x \in \ker \sigma \) and \( y \in T^G \). For all \( 0 \leq k \leq p - 1 \), we have: \( g^k(x) \cdot y = x \cdot y \). Hence, \( 0 = \sigma(x) \cdot y = px \cdot y \). So, \( x \cdot y = 0 \). Conversely, let \( x \in (T^G)^\perp \), let \( y = x + g(x) + \ldots + g^{p-1}(x) \). Then, we have \( y \in (T^G)^\perp \cap T^G \). Since the bilinear form on \( T \) is non-degenerate, we have \( y = 0 \).

**Lemma 2.11.** Let \( T \) be an unimodular lattice and \( G = \langle g \rangle \) a group of prime order \( p \) acting on \( T \).

\[ (T^G)^\vee = \left\{ \frac{y + g(y) + \ldots + g^{p-1}(y)}{p} \mid y \in T \right\}. \]

**Proof.** For all \( x \in T^G \) and all \( y \in T \),

\[ \frac{y + g(y) + \ldots + g^{p-1}(y)}{p}, x = y \cdot x. \]

Hence:

\[ (T^G)^\vee \supset \left\{ \frac{y + g(y) + \ldots + g^{p-1}(y)}{p} \mid y \in T \right\}. \]

Now, we prove the converse inclusion. Let \( x \) be a primitive element of \( T^G \) and \( q \in \mathbb{N}^* \) such that \( \frac{x}{q} \in (T^G)^\vee \). Then, it provides \( \frac{x}{q} \in A_T^G \). By (5), there is \( z \in T \) and \( y \in (T^G)^\perp \) such that \( z = \frac{z}{q} + \frac{y + g(y) + \ldots + g^{p-1}(y)}{q} \). But by Lemma 2.10,
\[ y + g(y) + \ldots + g^{p-1}(y) = 0. \] Hence, \( z + g(z) + \ldots + g^{p-1}(z) = \frac{p}{q} x. \) Since \( x \) is primitive in \( T^G \), \( q \) divides \( p \). Hence, \( q = 1 \) or \( q = p \). If \( q = p \), we get \( z + g(z) + \ldots + g^{p-1}(z) = x. \) If \( q = 1 \), we can write \( x = \overbrace{x + \ldots + x}^{p \text{ times}} \).

From a basic calculation (see proof of \([5, \text{Lemma 3.1}]\)), we obtain:

**Lemma 2.12.** Let \( T \) be a free \( \mathbb{Z} \)-module and \( G = \langle g \rangle \) a group of prime order \( p \) acting on \( T \). We denote \( \tau = g - \text{id} \) and \( \sigma = \text{id} + g + g^2 + \ldots + g^{p-1} \). Moreover, we denote by \( \tau, \sigma \) \( \in F_p[G] \) respectively the reduction of \( \tau \) and \( \sigma \) modulo \( p \). Let \( (v_1, \ldots, v_q) \) be the canonical basis of the \( F_p[G] \)-module \( N_q \) defined in Section 2.2; that is \( g(v_1) = v_1 \) and \( g(v_i) = v_i + v_{i-1} \) for all \( i \geq 2 \). Then the following hold:

1. \( \ker(\tau) = \langle v_1 \rangle \) and \( \text{Im}(\tau) = \langle v_1, \ldots, v_{q-1} \rangle \), for all \( q \leq p \).
2. \( \ker(\sigma) = N_q \) and \( \text{Im}(\sigma) = 0 \), for all \( q < p \). Moreover, \( \ker(\tau) = \langle v_1, \ldots, v_{q-1} \rangle \) and \( \text{Im}(\sigma) = \langle v_1 \rangle \), if \( q = p \).

**Lemma 2.13.** Let \( T \) be a free \( \mathbb{Z} \)-module and \( G = \langle g \rangle \) a group of prime order \( p \) acting on \( T \). Following notation of Section 2.2:

\[
T \otimes \mathbb{F}_p \simeq \bigoplus_{q=1}^{p} N_q^{\oplus \ell_q(T)}.
\]

For all \( z \in T \), we denote by \( \overline{z} \in T \otimes \mathbb{F}_p \) its reduction modulo \( p \).

Let \( x \in T^G \), there exists \( y \in T \) such that \( x = y + g(y) + \ldots + g^{p-1}(y) \) if and only if \( \overline{x} \in N_p^{\oplus \ell_p(T)} \).

**Proof.** Assume that \( x = y + g(y) + \ldots + g^{p-1}(y) \), with \( y \in T \). If \( \overline{x} = 0 \), then \( \overline{x} \in N_p^{\oplus \ell_p(T)}. \) Now we assume that \( \overline{x} \neq 0 \). Then, \( y \notin \ker(\tau) \), so by Lemma 2.12, (ii), \( y \in N_p^{\oplus \ell_p(T)}. \) Hence, \( \overline{x} \in N_p^{\oplus \ell_p(T)}. \)

Conversely, assume that \( \overline{x} \in N_p^{\oplus \ell_p(T)} \), we can write \( \overline{x} = \sum_i a_i v_{1,i} \), where \( v_{1,i} \) are invariant elements of the direct summands of \( N_p^{\oplus \ell_p(T)} \) (see Lemma 2.12, (i)). But, we have \( v_{1,i} = v_{p,i} + g(v_{p,i}) + \ldots + g^{p-1}(v_{p,i}) \) by Lemma 2.12, (iii). The result follows.

**Proposition 2.14.** Let \( T \) be an unimodular lattice and let \( G = \langle g \rangle \) be a group of prime order \( p \) acting on \( T \). Then:

\[
\frac{T}{T^G \oplus (T^G)^\perp} \simeq (\mathbb{Z}/p\mathbb{Z})^{\ell_p(T)}.
\]

**Proof.** Let \( T \otimes \mathbb{F}_p \simeq \bigoplus_{q=1}^{p} N_q^{\oplus \ell_q(T)} \) be the isomorphism of \( \mathbb{F}_p[G] \)-module provided in Section 2.2. By Lemma 2.11 and 2.13, we can define a surjective morphism:

\[
\begin{array}{ccc}
(T^G)^\vee & \longrightarrow & (N_p^{G})^{\oplus \ell_p(T)} \\
\overline{x} & \longmapsto & \overline{x},
\end{array}
\]

where \( \overline{x} \) denotes the reduction modulo \( p \) of \( x \). Moreover, \( \overline{x} \in T^G \iff p|x \iff \overline{x} = 0. \) Hence, the previous morphism provides an isomorphism

\[
A_{T^G} \simeq (N_p^{G})^{\oplus \ell_p(T)}.
\]

Since \( N_p^{G} \simeq (\mathbb{Z}/p\mathbb{Z}) \), the result follows by (5).

Now, by Proposition 2.14 and (3), we have

\[
\text{discr} (T^G \oplus (T^G)^\perp) = p^{2\ell_p(T)}.
\]

So, from (4) and (7), we obtain the discriminant of \( T^G \):

**Proposition 2.15.** Let \( T \) be an unimodular lattice and let \( G = \langle g \rangle \) be a group of prime order \( p \) acting on \( T \). Then:

\[
\text{discr} T^G = p^{\ell_p(T)}.
\]
Now, we conclude the proof of Proposition 2.9. Since \( T \) is unimodular, we know by Proposition 2.14 that the discriminant group \( \mathsf{A}_{p^G} := \frac{(\mathbb{Z}/p\mathbb{Z})^G}{\mathfrak{f}_{p^G}} \) of \( T^G \) is isomorphic to \( \mathfrak{f}_{p^G}/\mathfrak{fr}_{p^G} = (\mathbb{Z}/p\mathbb{Z})^\ell_p(T) \). Hence, by Proposition 2.15 and (3):

\[
\text{discr}(T^G) = p^{-\ell_p(T)}.
\]

It follows:

\[
\text{discr}(T^G)(p) = p^k \tau_{p^G - \ell_p(T)}.
\]

So, by Corollary 2.7, in the case \( p \leq 19 \), we obtain:

\[
\text{discr}(T^G)(p) = p^{\ell_p(T)}.
\]

3 The \( \mathbb{Z}[G] \)-module structure of the cohomology lattice

In all Section 3, \( X \) be a compact connected orientable manifold of dimension \( n \) and \( G = \langle g \rangle \) an automorphism group of prime order \( p \). We denote by \( \pi : X \to X/G \) the quotient map.

3.1 Coefficients of surjectivity

The following proposition is straightforward from [1, Theorem 5.4 and Corollary 5.8] (it is a generalization to cohomology of original results of Smith [28]).

**Proposition 3.1.** We can define a push-forward map \( \pi_* : H^*(X, \mathbb{Z}) \to H^*(X/G, \mathbb{Z}) \) such that:

\[
\pi_* \circ \pi^* = \text{id}_{H^*(X/G, \mathbb{Z})}, \quad \pi^* \circ \pi_* = \sum_{g \in G} g^*.
\]

From this proposition, let us examine the torsion of \( H^*(X/G, \mathbb{Z}) \). Notice that \( H^*(X, \mathbb{Z}) \) can be decomposed into three parts: the torsion-free part, the \( p \)-torsion part and the other torsion part.

**Remark 3.2.** Denote by \( H^*_p(X, \mathbb{Z}) \) the torsion part of \( H^*(X, \mathbb{Z}) \) without the \( p \)-torsion part. By using Proposition 3.1

\[
H^*_p(X, \mathbb{Z})^G \simeq H^*_p(X/G, \mathbb{Z}).
\]

Hence, the torsion which is not \( p \)-torsion does not provide any difficulties. On the other hand, the technique we propose does not say anything about the \( p \)-torsion part of \( H^*(X/G, \mathbb{Z}) \). So, in this paper, we will study only the torsion-free part of \( H^*(X/G, \mathbb{Z}) \).

**Notation 3.3.** Let \( Y \) be a topological space and \( p \) a prime number. Set:

- \( H^*_f(Y, \mathbb{Z}) := H^*(Y, \mathbb{Z})/\text{torsion} \),
- \( H^*_p(Y, \mathbb{Z}) := H^*(Y, \mathbb{Z})/(p \text{-torsion}) \).

We want to calculate the integer \( \alpha_k(X) \) provided by the exact sequence (2):

\[
\begin{array}{cccccc}
0 & \longrightarrow & \pi_*(H^*_f(X, \mathbb{Z})) & \longrightarrow & H^*_f(X/G, \mathbb{Z}) & \longrightarrow & (\mathbb{Z}/p\mathbb{Z})^{\alpha_k(X)} & \longrightarrow & 0.
\end{array}
\]

**Definition 3.4.** The integer \( \alpha_k(X) \) is called the \( k \)-th coefficient of surjectivity of \((X, G)\).

Hence, with our notation, \( \alpha_k(X) \) is expressed as follows:

\[
\frac{H^*_p(X/G, \mathbb{Z})}{\pi_*(H^*_p(X, \mathbb{Z}))} = \frac{H^*_f(X/G, \mathbb{Z})}{\pi_*(H^*_f(X, \mathbb{Z}))} = (\mathbb{Z}/p\mathbb{Z})^{\alpha_k(X)}.
\]

Moreover, from (2), we obtain the following formula:

\[
\sum_{k=0}^{2n} \alpha_k(X) = \frac{\log_p \text{disc} \pi_*(H^*_f(X, \mathbb{Z})) - \log_p \text{disc} H^*_f(X/G, \mathbb{Z})}{2}.
\]
This formula is available under many different versions considering natural sublattices of $H^*(X, \mathbb{Z})$ as $H^p(X, \mathbb{Z}) \oplus H_2^{-n-k}(X, \mathbb{Z})$:

$$\alpha_n(X) + \alpha_{-n}(X) = \log_p \left( \frac{\pi_* \left( H^p(X, \mathbb{Z}) \oplus H_2^{-n-k}(X, \mathbb{Z}) \right) - \log \det \left( H^p(X/G, \mathbb{Z}) \oplus H_2^{-n-k}(X/G, \mathbb{Z}) \right)}{2} \right).$$

(9)

In the rest of this section, we will see how this formula leads to an upper bound on the coefficient of surjectivity.

### 3.2 The push-forward lattice

Now in order to apply Proposition 2.9, in this section we prove the following proposition.

**Proposition 3.5.** Let $T$ be a unimodular sublattice of $H^*(X, \mathbb{Z})$ stable under the action of $G$. Then, we have an isometry:

$$\pi_*(T) \simeq (T^G)^\vee(p).$$

Before proving the proposition, the following lemma allows us to understand the cup-product form of the lattice $\pi_*(H^*(X, \mathbb{Z})^G)$.

**Lemma 3.6.** (i) Let $0 \leq k \leq n, q$ an integer such that $kq \leq n$ and $x \in H^k(X, \mathbb{Z})^G$. Then:

$$\pi_*(x)^q = p^{q-1} \pi_*(x^q) + p - \text{torsion}.$$  

If moreover $H^{kq}(X, \mathbb{Z})$ is $p$-torsion-free, then the property that $\pi_*(x)$ is divisible by $p$ implies that $\pi_*(x^q)$ is divisible by $p$.

(ii) Let $(x_i)_{1 \leq i \leq q}$ be elements of $H^{k_i}(X, \mathbb{Z})^G$ then:

$$\pi_*(x_1) \cdot \ldots \cdot \pi_*(x_q) = p^{q-1} \pi_*(x_1 \cdot \ldots \cdot x_q) + p - \text{torsion}.$$  

**Proof.** (i) By Proposition 3.1:

$$\pi^*(\pi_*(x)^q) = p^q x^q = \pi^*(p^{q-1} \pi_*(x^q)).$$

The map $\pi^*$ is injective on the $p$-torsion-free part, which implies the claimed equality. If moreover $\pi_*(x)$ is divisible by $p$, we can write $\pi_*(x) = py$ with $y \in H^{kq}(X/G, \mathbb{Z})$. This gives:

$$p^q y^q = p^{q-1} \pi_*(x^q) + p - \text{torsion}. \quad (10)$$

We cannot divide by $p^{q-1}$ because of the possible torsion of $H^{kq}(X/G, \mathbb{Z})$. We will use the fact that $H^{kq}(X, \mathbb{Z})$ is $p$-torsion-free to get around the problem. Applying $\pi^*$ to this equality, we obtain:

$$p^q \pi^*(y^q) = p^q x^q + \pi^*(p - \text{torsion}).$$

Since $H^{kq}(X, \mathbb{Z})$ is $p$-torsion-free, $\pi^*(p - \text{torsion}) = 0$, and we have

$$\pi^*(y^q) = x^q.$$  

Pushing down by $\pi_*$, we obtain:

$$p y^q = \pi_*(x^q).$$

(ii) Same proof as (10). \qed

**Remark 3.7.** When $k_1 + \ldots + k_q = n$, there is no problem of torsion and we have:

$$\pi_*(x_1) \cdot \ldots \cdot \pi_*(x_q) = p^{q-1} \pi_*(x_1 \cdot \ldots \cdot x_q).$$

Indeed by Poincaré duality $H^n(X, \mathbb{Z}) = \mathbb{Z}$ is torsion-free and identifying $H^n(X, \mathbb{Z})$ with $\mathbb{Z}$, we can write $\pi_*(x_1 \cdot \ldots \cdot x_q) = x_1 \cdot \ldots \cdot x_q$.  

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Now, we prove Proposition 3.5. By Lemma 2.11, we have:

\[ \pi_*(T) \supset \pi_*(T^G)^\vee. \]

Indeed, let \( z \in (T^G)^\vee \), by Lemma 2.11 there exist \( y \in T \) such that \( z = \frac{y + g(y) + \ldots + g^{p-1}(y)}{p} \). So, \( \pi_*(y) = \pi_*(z) \). Now let us prove the other inclusion:

\[ \pi_*(T) \subset \pi_*(T^G)^\vee. \]

Let \( z \in T \), by Proposition 2.14, we can write \( z = \frac{x + y}{p} \) with \( x \in T^G \) and \( y \in (T^G)^\perp \). Hence, by Lemma 2.10, we obtain \( x = z + g(z) + \ldots + g^{p-1}(z) \). Hence, \( \pi_*(z) = \pi_*(\frac{x}{p}) \in \pi_*(T^G)^\vee \). It follows

\[ \pi_*(T) = \pi_*(T^G)^\vee. \]

Then, by Lemma 3.6 (ii), we have an isometry:

\[ \pi_*(T) \simeq (T^G)^\vee(p). \]

That proves Proposition 3.5.

Moreover, combining Proposition 2.9 and 3.5, we obtain the following corollary.

**Corollary 3.8.** Let \( T \) be a unimodular sublattice of \( H^*(X,\mathbb{Z}) \) stable under the action of \( G \); then

\[ \text{disc} \; \pi_*(T) = p^{\ell^+(T)}. \]

### 3.3 Upper bound for the coefficients of surjectivity

If we apply Corollary 3.8, to the lattice \( H^k_f(X,\mathbb{Z}) \oplus H^{n-k}_f(X,\mathbb{Z}) \), we obtain:

\[ \text{disc} \; \pi_*(H^k_f(X,\mathbb{Z}) \oplus H^{n-k}_f(X,\mathbb{Z})) = p^{\ell^k_f(X,\mathbb{Z})G + H^{n-k}_f(X,\mathbb{Z})G - \ell^1_p(\pi_*) - \ell^1_p(\pi_*)}. \]

Using (3), it follows that

\[ \alpha_k(X) + \alpha_{n-k}(X) \leq \frac{\text{rk} (H^k_f(X,\mathbb{Z})G + H^{n-k}_f(X,\mathbb{Z})G) - \ell^1_p(\pi_*) - \ell^1_p(\pi_*)}{2}. \]

**Proposition 3.9.** For all \( 0 \leq k \leq n \):

(i) \( \text{rk} H^k_f(X,\mathbb{Z})^G = \text{rk} H^{n-k}_f(X,\mathbb{Z})^G \),

(ii) \( \ell^k_f(X) = \ell^{n-k}_f(X) \), where \( * \) means \(+,-,0\) or \( q \in \{1,...,p\} \).

**Proof.** By Poincaré duality, we have:

\[ H^k_f(X,\mathbb{Z}) \longrightarrow H^{n-k}_f(X,\mathbb{Z})^\vee \]

\[ x \longmapsto f_x := (y \mapsto x \cdot y), \]

which is an isomorphism. Moreover, we can endow \( H^{n-k}_f(X,\mathbb{Z})^\vee \) with a structure of \( \mathbb{Z}[G] \)-module by setting \( g \cdot f = f(g^{-1}) \) for all \( f \in H^{n-k}_f(X,\mathbb{Z})^\vee \) and \( g \in G \). With this structure, the Poincaré isomorphism becomes an isomorphism of \( \mathbb{Z}[G] \)-modules. So, it shows that:

\[ \ell^k_f(X) = \ell_*(H^k_f(X,\mathbb{Z})) = \ell_*(H^{n-k}_f(X,\mathbb{Z})^\vee). \]

Now, it remains to show that

\[ \ell_*(H^{n-k}_f(X,\mathbb{Z})^\vee) = \ell_*(H^{n-k}_f(X,\mathbb{Z})) = \ell^{n-k}_f(X). \]

At this stage of the proof, the case \( p = 2 \) differs from the case \( p \geq 3 \).
Let us first assume that \( p \geq 3 \). We have to prove that, for \( 1 \leq q \leq p \), the \( \mathbb{F}_p[G] \)-modules \( N_q \) and \( \mathbb{J}_q^{\ast} \) are isomorphic (where \( N_q \) was defined in Section 2.2). The matrix of the action of \( g \) on \( N_q \) is given by the inverse of the transpose matrix of the Jordan block of size \( q \):

\[
\begin{pmatrix}
1 & 1 & 0 \\
0 & \ddots & 1 \\
0 & \cdots & 1
\end{pmatrix}^{-1} t.
\]

Denote by \( J_q \) the matrix of the Jordan block of size \( q \). The minimal polynomial of \( J_q \) is \((X - 1)^q\). However, \(( (J_q^{-1})^t)^t = (J_q^{-1})^t = I_q \), with \( I_q \) the identity matrix of size \( q \). So, the minimal polynomial of \((J_q^{-1})^t\) divides \((X - 1)^q\). Let \( 0 \leq d \leq q \) minimal integer such that \(( (J_q^{-1})^t)^d = I_q \). However, this implies \( J_q^d = I_q \). So, \( d = q \). It follows that the Jordan form of \((J_q^{-1})^t\) is given by \( J_q \).

Hence, \( N_q \cong \mathbb{J}_q^{\ast} \) as \( \mathbb{F}_p[G] \)-module. This proves the result for \( p \geq 3 \).

Now assume that \( p = 2 \). It is clear that the dual of \( \mathbb{Z} \) endowed with the trivial \( \mathbb{Z}[G] \)-module structure is again \( \mathbb{Z} \) endowed with the trivial \( \mathbb{Z}[G] \)-module structure. The \( \mathbb{Z}[G] \)-module \( \mathcal{O}_K \) defined in Section 2.2, is only \( \mathbb{Z} \) endowed with the \( \mathbb{Z}[G] \)-module structure given by \( g \cdot t = -t \) for all \( t \in \mathbb{Z} \).

Hence, we also have an isomorphism of \( \mathbb{Z}[G] \)-module between \( \mathcal{O}_K \) and \( \mathcal{O}_K^{\ast} \), endowed with the \( \mathbb{Z}[G] \)-module structure given by \( g \cdot f = f(g^{-1} \cdot ) \) for all \( f \in \mathcal{O}_K^{\ast} \). Let us consider now \(( \mathcal{O}_K, a)^\vee \) endowed with the \( \mathbb{Z}[G] \)-module structure given by \( g \cdot f = f(g^{-1} \cdot ) \) for all \( f \in ( \mathcal{O}_K, a)^\vee \). We can apply Proposition 2.2, (i) to \( ( \mathcal{O}_K, a)^\vee \). If \( \ell_\ast(( \mathcal{O}_K, a)^\vee) \) and \( \ell_\ast(( \mathcal{O}_K, a)^\vee) \) are not both zero, we obtain a contradiction by considering the double dual which is the dual of \( ( \mathcal{O}_K, a)^\vee \) and is canonically isomorphic to \( ( \mathcal{O}_K, a)^\vee \). Hence, there exists \( b \in \mathcal{O}_K \) such that \( ( \mathcal{O}_K, a)^\vee \simeq ( \mathcal{O}_K, b) \) as \( \mathbb{Z}[G] \)-module. This concludes the proof of (ii).

It remains to prove (i). By Poincaré duality isomorphism (13), we only have to show that \( \text{rk} H^q_{\ast}(X, \mathbb{Z})^G = \text{rk}(H^{q-k}_{\ast}(X, \mathbb{Z})^{\vee})^G \), where the action of \( G \) on \( H^{q-k}_{\ast}(X, \mathbb{Z})^{\vee} \) is given by \( g \cdot f = f(g^{-1} \cdot ) \) for all \( f \in H^{q-k}_{\ast}(X, \mathbb{Z})^{\vee} \). This is equivalent to show that \( \dim H^{q-k}(X, \mathbb{Q})^G = \dim(H^{q-k}(X, \mathbb{Q})^{\vee})^G \), which is readily verified considering a basis \( \mathcal{B} = \{v_1, \ldots, v_d, v_{d+1}, \ldots, v_m\} \) of \( H^{q-k}(X, \mathbb{Q}) \) where \( \{v_1, \ldots, v_d\} \) is a basis of \( H^{q-k}(X, \mathbb{Q})^{\vee} \) and taking the dual basis of \( \mathcal{B} \). 

Then, it follows from (12) (and from Corollary 2.7 in the case \( p \leq 19 \)):

**Proposition 3.10.** For all \( 0 \leq k \leq n \):

\[
\alpha_k(X) + \alpha_{n-k}(X) \leq \text{rk} H^q_{\ast}(X, \mathbb{Z})^G - \ell^k_p(X);
\]

if moreover \( p \leq 19 \), then

\[
\alpha_k(X) + \alpha_{n-k}(X) \leq \ell^k(X).
\]

Furthermore, these inequalities become equalities when \( X/G \) is smooth.

**Proof.** The first inequality follows from (12) and Proposition 3.9. Then, we deduce the second one from Corollary 2.7. When \( X/G \) is smooth, the lattice \( H^q_{\ast}(X/G, \mathbb{Z}) \oplus H^{n-k}_{\ast}(X/G, \mathbb{Z}) \) is unimodular. Therefore, (9), (11) and Proposition 3.9 provide the equality. 

**Remark 3.11.** When \( n = 2m \) is even and \( k = m \), we calculated the discriminant of two copies of the lattice \( \pi_\ast(H^m_{\ast}(X, \mathbb{Z})) \) and the inequalities of Proposition 3.10 are:

\[
\alpha_m(X) \leq \frac{\text{rk} H^m_{\ast}(X, \mathbb{Z})^G - \ell^m_p(X)}{2};
\]

and if \( p \leq 19 \):

\[
\alpha_m(X) \leq \frac{\ell^m(X)}{2}.
\]
3.4 The $H^k$-normality

A particular case of interest is when a coefficient of surjectivity $\alpha_k(X)$ vanishes, for some $k$. In such a case, the push-forward map $\pi_* : H^k_{p-f}(X, \mathbb{Z}) \to H^k_{p-f}(X/G, \mathbb{Z})$ is surjective.

Definition 3.12. Let $0 \leq k \leq n$. When $\alpha_k(X) = 0$, we will say that $(X, G)$ is $H^k$-normal.

Proposition 3.13. Let $0 \leq k \leq n$. The following statements are equivalent:

(i) $(X, G)$ is $H^k$-normal.

(ii) For $x \in H^k_{p-f}(X/G, \mathbb{Z})$, $\pi_*(x)$ is divisible by $p$ if and only if there exists $y \in H^k_{p-f}(X, \mathbb{Z})$ such that $x = y + g(y) + \ldots + g^{p-1}(y)$.

Proof. We assume (ii). Let $z \in H^k_{p-f}(X/G, \mathbb{Z})$, by (2), we can find $x \in H^k_{p-f}(X, \mathbb{Z})$ such that $z = \pi_*(x)$. So by (ii), there exists $y \in H^k_{p-f}(X, \mathbb{Z})$ such that $x = y + g(y) + \ldots + g^{p-1}(y)$. Hence $z = \pi_*(y)$.

Now, assume (i). Let $x \in H^k_{p-f}(X, \mathbb{Z})$ such that $p$ divides $\pi_*(x)$. Since $\pi_* : H^k_{p-f}(X, \mathbb{Z}) \to H^k_{p-f}(X/G, \mathbb{Z})$ is surjective, there is $y \in H^k_{p-f}(X, \mathbb{Z})$ such that $p\pi_*(y) = \pi_*(x)$. We apply $\pi^*$ and Proposition 3.1 to get $p(y + g(y) + \ldots + g^{p-1}(y)) = px$.

We provide two important properties of $H^k$-normality. First, in certain circumstances, it is possible to deduce the $H^k$-normality from $H^k$-normality.

Proposition 3.14. We assume $p \leq 19$ and $H^*(X, \mathbb{Z})$ $p$-torsion-free (so $H^*(X, \mathbb{Z}) \otimes \mathbb{F}_p = H^*(X, \mathbb{F}_p)$).

Let $0 \leq k \leq n$, $t$ an integer such that $kt \leq n$. Assume that $(X, G)$ is $H^{kt}$-normal. For all $x \in H^*(X, \mathbb{Z})$, we denote by $\overline{x}$ in $H^*(X, \mathbb{F}_p)$ its reduction modulo $p$. If

$$\mathcal{J} : \text{Sym}^t H^k(X, \mathbb{F}_p) \to H^{kt}(X, \mathbb{F}_p)$$

$$\overline{x_1} \otimes \ldots \otimes \overline{x_t} \mapsto \overline{x_1 + \ldots + x_t}$$

is injective and $\mathcal{J}(\text{Sym}^t H^k(X, \mathbb{F}_p))$ admits a complementary vector space, stable by the action of $G$, then $(X, G)$ is $H^k$-normal.

Proof. We use the same notation for both Jordan decompositions of $H^k(X, \mathbb{Z})$ and of $H^{kt}(X, \mathbb{Z})$:

$$H^k(X, \mathbb{F}_p) = \sum_{q=1}^p N_q^{\otimes t}(X) := \sum_{q=1}^p N_q^t,$$

Moreover, since $p \leq 19$:

$$H^k(X, \mathbb{F}_p) = N_1 \oplus N_{p-1} \oplus N_p.$$

Hence, by Proposition 2.2:

$$H^k(X, \mathbb{F}_p)^G = N_1 \oplus N_{p-1}^G. \quad (15)$$

Let $x \in H^k_{p-f}(X, \mathbb{Z})$. We assume that there is no $y \in H^k_{p-f}(X, \mathbb{Z})$ such that $x = y + \varphi^*(y) + \ldots + (\varphi^*)^{p-1}(y)$ and we show that $\pi_*(x)$ is not divisible by $p$.

By Lemma 2.13, $\overline{\emptyset} \notin N_p$. Since $\mathcal{J}$ is injective and $N_q^{\otimes t} = N_1$, by (15), we have $\overline{\emptyset} \notin N_p$. Again by Lemma 2.13, there is no $z \in H^k_{p-f}(X, \mathbb{Z})$ such that $x = z + \varphi^*(z) + \ldots + (\varphi^*)^{p-1}(z)$. Since $(X, G)$ is $H^{kt}$-normal, $\pi_*(x)$ is not divisible by $p$. Now, since $H^{kt}(X, \mathbb{Z})$ is torsion-free, by Lemma 3.6 (i), $\pi_*(x)$ is not divisible by $p$.

We will apply this proposition to an example in Section 7.2. Now, we introduce the notion of the pullback of a pair $(X, G)$.

Definition 3.15. Let $s : \tilde{X} \to X$ be a surjective morphism such that $\tilde{X}$ is a compact connected orientable manifold of dimension $n$ and suppose that there is an automorphism $\tilde{g}$ of order $p$ of $\tilde{X}$ which verifies $s \circ \tilde{g} = g \circ s$. Moreover, we assume that the degree of $s$ is not divisible by $p$. We denote $\langle \tilde{g} \rangle$ by $\tilde{G}$.

The triple $(\tilde{X}, \tilde{G}, s)$ will be called a pullback of $(X, G)$. 
Lemma 3.16. Let \((\tilde{X}, \tilde{G}, s)\) be a pullback of \((X, G)\). We denote by \(\tilde{s} : \tilde{X} \to X/G\) the quotient map and by \(r : \tilde{X}/\tilde{G} \to X/G\) the induced map. Let \(0 \leq k \leq n\) and \(x \in H^k(X, \mathbb{Z})^G\). Then:

\[ \tilde{s}^*(x) = r^*(\pi_*^*(x)) + p - \text{torsion}. \]

If moreover \(H^k(\tilde{X}, \mathbb{Z})\) is \(p\)-torsion-free, then the property that \(r^*(\pi_*^*(x))\) is divisible by \(p\) implies that \(\tilde{s}^*(x)\) is divisible by \(p\).

**Proof.** The proof follows the same idea as the proof of Lemma 3.6.

We have a Cartesian diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{s} & X \\
\downarrow{\tilde{\pi}} & & \downarrow{\pi} \\
\tilde{X}/\tilde{G} & \xrightarrow{r} & X/G.
\end{array}
\]

It induces a commutative diagram in cohomology:

\[
\begin{array}{ccc}
H^k(X/G, \mathbb{Z}) & \xrightarrow{\tilde{s}^*} & H^k(X, \mathbb{Z}) \\
\downarrow{\tilde{\pi}^*} & & \downarrow{\pi^*} \\
H^k(\tilde{X}/\tilde{G}, \mathbb{Z}) & \xrightarrow{s^*} & H^k(\tilde{X}, \mathbb{Z}).
\end{array}
\]

It follows from Proposition 3.1 that

\[ \tilde{s}^*(\pi_*^*(x)) = s^*(\pi_*^*(x)) + p \cdot s^*(x) = \tilde{s}^*(s^*(x)). \]

The map \(\tilde{s}^*\) is injective on the torsion-free part, so we get the equality. If moreover \(r^*(\pi_*^*(x))\) is divisible by \(p\), we can write \(r^*(\pi_*^*(x)) = py\) with \(y \in H^k(M, \mathbb{Z})\). Thus gives:

\[ \tilde{s}^*(x) + p - \text{torsion} = py. \]

Applying \(\tilde{s}^*\), we get:

\[ ps^*(x) = p\tilde{s}^*(y). \]

Since \(H^k(\tilde{X}, \mathbb{Z})\) is \(p\)-torsion-free, this is also the case for the group \(s^*(H^k(X, \mathbb{Z}))\), hence:

\[ \tilde{s}^*(y) = s^*(x). \]

Hence, by applying \(\tilde{s}^*\), we get \(\tilde{s}^*(s^*(x)) = py. \)

**Proposition 3.17.** Let \((\tilde{X}, \tilde{G}, s)\) be a pullback of \((X, G)\). Assume that \(H^*(X, \mathbb{Z})\) and \(H^*(\tilde{X}, \mathbb{Z})\) are \(p\)-torsion-free. Assume that \(s^*(H^k(X, \mathbb{F}_p))\) admits a complementary vector space, stable under the action of \(\tilde{G}\). Then, if \((\tilde{X}, \tilde{G})\) is \(H^k\)-normal then \((X, G)\) is \(H^k\)-normal.

**Proof.** We use the notion for the Jordan decomposition as in (14). First note that since \(s\) is surjective, \(s^* : H^k(X, \mathbb{Z}) \to H^k(\tilde{X}, \mathbb{Z})\) is injective too. Then, we can prove that \(s^* : H^k(X, \mathbb{Z}) \to H^k(\tilde{X}, \mathbb{Z})\) is injective. Since \(H^*(X, \mathbb{Z})\) and \(H^*(\tilde{X}, \mathbb{Z})\) are \(p\)-torsion-free, we only have to verify that \(p|s^*(x)\) implies \(p|x\) for all \(x \in H^k(\tilde{X}, \mathbb{Z})\). However, if \(s^*(x)\) is divisible by \(p\) then \(s^*(x) \cdot s^*(y) = s^*(x \cdot y)\) if divisible by \(p\) for all \(y \in H^k(\tilde{X}, \mathbb{Z})\). Since the degree of \(s\) is not divisible by \(p\), it follows that \(p\) divides \(x \cdot y\) for all \(y \in H^k(\tilde{X}, \mathbb{Z})\). So, by Poincaré duality, it means that \(x\) is divisible by \(p\).

Now, let \(x \in H^k(X, \mathbb{Z})^G\). We assume that there is no \(y \in H^k(X, \mathbb{Z})\) such that \(x = y + \varphi^*(y) + \ldots + (\varphi^*)^{p-1}(y)\) and we show that \(\pi_*^*\) is not divisible by \(p\). By Lemma 2.13 \(\pi \notin \mathcal{N}_p\).

Since \(s^* : H^k(X, \mathbb{F}_p) \to H^k(\tilde{X}, \mathbb{F}_p)\) is injective and \(s^*(H^k(X, \mathbb{F}_p))\) admits a complementary vector space, stable under the action of \(\tilde{G}\), \(s^*(x) \notin \mathcal{N}_p\). Hence, by Lemma 2.13, there is no \(z \in H^k(X, \mathbb{Z})\) such that \(s^*(x) = z + \varphi(z) + \ldots + \varphi^{p-1}(z)\). Since \((\tilde{X}, \tilde{G})\) is \(H^k\)-normal, \(\tilde{s}^*(s^*(x))\) is not divisible by \(p\). Hence, by Lemma 3.16, \(r^*(\pi_*^*(x))\) is not divisible by \(p\). It follows that \(\pi_*^*(x)\) is not divisible by \(p\).

We will see an example of the use of this proposition in Section 7.6.
4 Application to equivariant cohomology

Let $X$ be a topological space and $G$ a group acting on $X$. Let $EG \rightarrow BG$ be a universal $G$-bundle in the category of CW-complexes. Denote by $X_G = EG \times_G X$ the orbit space for the diagonal action of $G$ on the product $EG \times X$ and $f : X_G \rightarrow BG$ the map induced by the projection onto the first factor. The map $f$ is a locally trivial fiber bundle with typical fiber $X$ and structure group $G$. We define the $G$-equivariant cohomology of $X$ by $H^*_G(X, Z) := H^*(EG \times_G X, Z)$. Then, the Leray–Serre spectral sequence associated to the map $f$ gives a spectral sequence converging to the equivariant cohomology:

$$E_2^{p,q} := H^p(G; H^q(X, Z)) \Rightarrow H^{p+q}_G(X, Z).$$

Hence, we are interested in calculating the cohomology of our prime order group $G = (g)$, with coefficient in a $Z$-module. We have the following projective resolution of $Z$ considered as a $G$-module:

$$\cdots \xrightarrow{\epsilon} Z[G] \xrightarrow{\sigma} Z[G] \xrightarrow{i} Z[G] \xrightarrow{\tau} Z,$$

where $\tau = g - \text{id}$, $\sigma = \text{id} + g + \ldots + g^{p-1}$ and $\epsilon$ is the summation map: $\epsilon(\sum_{j=0}^{p-1} \alpha_j g^j) = \sum_{j=0}^{p-1} \alpha_j$. If $H$ is a $Z[G]$-module of finite rank over $Z$, the cohomology of $G$ with coefficients in $H$ is computed as the cohomology of the complex:

$$0 \rightarrow H \xrightarrow{\tau} H \xrightarrow{\sigma} H \xrightarrow{i} \cdots.$$

**Proposition 4.1.** Assume that $H$ is a $p$-torsion-free $Z$-module and a $Z[G]$-module with $G = (g)$ group of prime order $p \leq 19$. Then:

(i) $H^0(G, H) = H^G$,

(ii) $H^{2i-1}(G, H) = (\mathbb{Z}/p\mathbb{Z})^{l_i(H)}$,

(iii) $H^{2i}(G, H) = (\mathbb{Z}/p\mathbb{Z})^{s_i(H)}$.

**Proof.** (i) By definition, $H^0(G, H) = \text{Ker} \tau$.

(ii) First, we remark that the cohomology of the torsion part is trivial. Since the torsion part $H_{\text{torsion}}$ is $p$-torsion-free, $p$ is always invertible in $H_{\text{torsion}}$. Hence, all $x \in H_{\text{torsion}} \cap \text{Ker} \tau$ can be written $x = \frac{1}{p} \left( x + \ldots + x \right)$ in $\text{Im} \sigma$; and for $x \in H_{\text{torsion}}$ such that $x + \ldots + g^{p-1}(x) = 0$, we can write $x = g(y) - y$ with $y = -\frac{1}{p} \left( (p-1)x + (p-2)g(x) + \ldots + 2g^{p-3}(x) + g^{p-1}(x) \right)$.

So, henceforth, we assume that $H$ is torsion-free. In odd degrees, $H^{2i-1}(G, H) = \text{Ker} \sigma / \text{Im} \tau$.

Using the notation of Section 2.2, in the proof of [7, Theorem 74.3], it is shown that:

$$\text{Ker} \sigma = O_K b_1 \oplus \ldots \oplus O_K b_{s-1} \oplus A_{r+s},$$

$$\text{Im} \tau = E_1 b_1 \oplus \ldots \oplus E_n b_{s-1} \oplus E_{r+s} A_{r+s},$$

with $b_1, \ldots, b_n$ $O_K$-free elements in $\text{Ker} \sigma$, $A$ an $O_K$-ideal of $K$ and

$$E_i = \ldots = E_r = O_K,$$

$$E_{r+1} = \ldots = E_{r+s} = (\xi - 1)O_K.$$

The numbers $s$ and $s$ in the last equalities are the same as in Proposition 2.2. Moreover, we find in the proof of [7, Theorem 74.3] that $O_K / (\xi - 1)O_K = A / (\xi - 1)A = \mathbb{Z}/p\mathbb{Z}$. Hence, we get $\text{Ker} \sigma / \text{Im} \tau = (\mathbb{Z}/p\mathbb{Z})^{l_i(X)}$.

(iii) For $i \geq 1$, $H^{2i}(G, H) = \text{Ker} \tau / \text{Im} \sigma$. By (iii) of Proposition 2.2:

$$H^G \simeq \bigoplus_{i=1}^{\ell_1} (O_K, a_i) \oplus \mathbb{Z}^{B_G}.$$
Remark for all the following inclusion:

Let us consider the following diagram:

\[
\begin{array}{c}
H^k(X, Z) \\
\downarrow \pi_* \\
H^k(G, X) \to H^k(G, X/G)
\end{array}
\]

Remark first that since \( r \) is surjective, \( r^* \) is an injective map.

Definition 5.1. The sublattice

\[ N_{k,r} := r^* \left[ \pi_* \left( H^k_f(X, Z) \oplus H^{n-k}_f(X, Z) \right) \right] \]

of \( H^k_f(X/G, Z) \oplus H^{n-k}_f(X/G, Z) \) will be called the \( k \)-th exceptional lattice of \( r \).

It follows that \( N_{k,r} \) is a primitive over-lattice of \( r^* \left[ \pi_* \left( H^k_f(X, Z) \oplus H^{n-k}_f(X, Z) \right) \right] \). We have the following inclusion:

\[ r^* \left[ \pi_* \left( H^k_f(X, Z) \oplus H^{n-k}_f(X, Z) \right) \right] \subset r^* \left[ H^k_f(X/G, Z) \oplus H^{n-k}_f(X/G, Z) \right] \subset N_{k,r}^+ \]

Now, since \( r^* \) is injective, it induces the following injective map of \( \mathbb{Z}/p\mathbb{Z} \)-vector space:

\[
\frac{H^k_f(X/G, Z) \oplus H^{n-k}_f(X/G, Z)}{\pi_* \left( H^k_f(X, Z) \oplus H^{n-k}_f(X, Z) \right)} \hookrightarrow r^* \left[ \pi_* \left( H^k_f(X, Z) \oplus H^{n-k}_f(X, Z) \right) \right].
\]
Then, using (3) and taking the logarithm in base $p$, it follows that:

$$
\log_p \text{disc} \pi_* \left( H^k_f(X, Z) \oplus H^{n-k}_f(X, Z) \right) - \log_p \text{disc} H^k_f(X/G, Z) \oplus H^{n-k}_f(X/G, Z)
$$

$$
\leq \log_p r^* \left[ \pi_* \left( H^k_f(X, Z) \oplus H^{n-k}_f(X, Z) \right) \right] - \log_p \text{disc} N^k_f.
$$

By (9):

$$
\alpha_k(X) + \alpha_{n-k}(X) \leq \frac{\log_p \text{disc} r^* \left[ \pi_* \left( H^k_f(X, Z) \oplus H^{n-k}_f(X, Z) \right) \right] - \log_p \text{disc} N^k_f}{2}.
$$

Moreover, since $r^*$ is injective:

$$
\text{disc} r^* \left[ \pi_* \left( H^k_f(X, Z) \oplus H^{n-k}_f(X, Z) \right) \right] = \text{disc} \pi_* \left( H^k_f(X, Z) \oplus H^{n-k}_f(X, Z) \right). \quad (16)
$$

Furthermore, $X/G$ is smooth, so $H^k_f(X/G, Z) \oplus H^{n-k}(X/G, Z)$ is unimodular. Hence, by (4),

$$
\text{disc} N_k = \text{disc} N^k_f. \quad \text{It follows that}
$$

$$
\alpha_k + \alpha_{n-k} \leq \frac{\log_p \text{disc} \pi_* \left( H^k_f(X, Z) \oplus H^{n-k}_f(X, Z) \right) - \log_p \text{disc} N_k}{2}.
$$

Finally, by (11) and Proposition 3.9, we obtain:

**Proposition 5.2.** Let $X$ be a compact connected orientable manifold of dimension $n$ and $G$ an automorphism group of prime order $p$. Let $r : X/G \to X/G$ be a resolution of singularities. Let $N_{k,r}$ be the $k$-th exceptional lattice of $r$. Then:

$$
\alpha_k(X) + \alpha_{n-k}(X) \leq \text{rk} H^k_f(X, Z)^G - \ell^k_p(X) - \frac{1}{2} \log_p \text{disc} N_{k,r}.
$$

If moreover, $p \leq 19$:

$$
\alpha_k(X) + \alpha_{n-k}(X) \leq \ell^k_p(X) - \frac{1}{2} \log_p \text{disc} N_{k,r}.
$$

**Remark 5.3.** When $n = 2m$ is even and $k = m$, the discriminant of $N_m$ is given by the discriminant of two copies of $N_m^{\otimes \frac{1}{2}} := r^* \left[ \pi_* \left( H^m_f(X, Z) \right)^\perp \right]$ and the inequalities of Proposition 5.2 are

$$
\alpha_m \leq \frac{\text{rk} H^m_f(X, Z)^G - \ell^m_p(X) - \log_p \text{disc} N^m_m}{2};
$$

and, when $p \leq 19$,

$$
\alpha_m \leq \frac{\ell^m_p(X) - \log_p \text{disc} N^m_m}{2}.
$$

Now, the objective will be to express $N_k$ in terms of exceptional divisors and calculate its discriminant.

### 5.2 Using blow-ups

In this section $X$ is a complex manifold and $G$ an automorphism group of prime order $p$. The simplest case that we can consider for applying Proposition 5.2 is the following.

Let $s : \tilde{X} \to X$ be the blow-up of $X$ in the fixed locus $Z$ of $G$, let $\tilde{G}$ be the automorphism group induced by $G$ on $\tilde{X}$. When $X$ is Kähler, the cohomology of $\tilde{X}$ can be calculated using [30, Theorem 7.31].

**Theorem 5.4.** ([30]) Let $X$ be a Kähler manifold and let $Z \subset X$ be a submanifold of codimension $d$. Let $s : \tilde{X} \to X$ be the blow-up of $X$ in $Z$. Let $E = s^{-1}(Z)$ be the exceptional divisor and
$j : E \hookrightarrow \tilde{X}$ the embedding. Let $h = c_1(\mathcal{O}_E(1)) \in H^2(E, \mathbb{Z})$. Then, we have an isomorphism of Hodge structures:

$$H^k(X, \mathbb{Z}) \oplus \left( \bigoplus_{i=1}^{d-2} H^{k-2i-2}(Z, \mathbb{Z}) \right) \xrightarrow{\tau^* + \sum, j, \psi_j \circ \eta_{|E}} H^k(\tilde{X}, \mathbb{Z}).$$

Here, $h^i$ is the morphism by the cup-product with $h^i \in H^{2i}(E, \mathbb{Z})$.

Now if $\tilde{X}/\tilde{G}$ is smooth, $\tilde{X}/\tilde{G}$ provides a resolution of singularities of $X/G$:

$$\begin{align*}
\tilde{X} & \xrightarrow{s} X, \\
\tilde{X}/\tilde{G} & \xrightarrow{r} X/G
\end{align*}$$

where the exceptional lattice of $r$ could be expressed in terms of the cohomology of the fixed locus $Z$ via Theorem 5.4. So, in this section, we will study when such $\tilde{X}/\tilde{G}$ is smooth.

**Remark 5.5.** The triple $(\tilde{X}, \tilde{G}, r)$ is a pullback of $(X, G)$ (cf. Definition 3.15).

At each fixed point of $G$, by Cartan [6, Lemma 1] we can locally linearize the action of $G$. Thus, at a fixed point $x \in X$, the action of $G$ on $X$ is locally equivalent to the action of $G = \langle g \rangle$ on $\mathbb{C}^n$ via

$$g = \text{diag}(\xi_1, ..., \xi_n),$$

where $\xi_i$ is a $p$-th root of unity. Without loss of generality, we can assume that $k_1 \leq ... \leq k_n \leq p-1$.

**Lemma 5.6.** Let $X$ be a complex manifold of dimension $n$ and $Z$ a connected component of $\text{Fix} G$, the fixed locus of an automorphism group $G$ of prime order $p$. Let $m = \dim Z$ and $E \rightarrow Z$ be the exceptional divisor of the blow-up $\tilde{X} \rightarrow X$ of $X$ in $Z$. Let $G$ be the automorphism group induced by $G$ on $\tilde{X}$. Assume that the local action of $G$ around a point of $Z$ is given by

$$g = \text{diag}(1, ..., 1, \xi_{k_1+1}^{\pm 1}, ..., \xi_{k_n}^{\pm 1}),$$

then the different local actions of $\tilde{G}$ around its fixed points contained in $E$ are given by the diagonal matrices:

$$\text{diag}(1, ..., 1, \xi_{k_i+1}^{\pm 1}, ..., \xi_{k_i-1}^{\pm 1}, \xi_{k_i}^{\pm 1}, ..., \xi_{k_n}^{\pm 1})$$

for all $m + 1 \leq i \leq n$.

**Proof.** Let $G = \langle \text{diag}(\xi_1, ..., \xi_n) \rangle$ acting on $\mathbb{C}^n$. Since $k_1 = ... = k_m = 0$, we have to consider the blow-up of $\mathbb{C}^n \times \mathbb{C}^{n-m}$ in $\mathbb{C}^n \times \{0\}$, which is given by $\mathbb{C}^m \times \mathbb{C}^{n-m}$ where $\mathbb{C}^{n-m}$ is the blow-up of $\mathbb{C}^{n-m}$ in $0$. So, without loss of generality, we can assume that $m = 0$.

Let $\mathbb{C}^{n}$ be the blow-up of $\mathbb{C}^n$ in 0 and $\tilde{G}$ the automorphism group of $\mathbb{C}^{n}$ induced by $G$. We will describe the action of $\tilde{G}$ on

$$\tilde{C}^n = \left\{ ((x_1, ..., x_n), (a_1 : ... : a_n)) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid \text{rk}((x_1, ..., x_n), (a_1, ..., a_n)) = 1 \right\}.$$ We denote by

$$\Theta_i = \left\{ ((x_1, ..., x_n), (a_1 : ... : a_n)) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid a_i \neq 0 \right\}$$

the chart $a_i \neq 0$. We have

$$\tilde{C}^n \cap \Theta_i = \left\{ ((x_1, ..., x_n), (a_1, ..., a_{i-1}, a_{i+1}, ..., a_n)) \in \mathbb{C}^n \times \mathbb{C}^{n-1} \mid x_j = x_ia_j, \ j \in \{1, ..., n\} \right\}.$$ Hence, we have an isomorphism:

$$f : \tilde{C}^n \cap \Theta_i \rightarrow C^n$$

$$((x_1, ..., x_n), (a_1 : ... : a_n)) \rightarrow (a_1, ..., a_{i-1}, x_i, a_{i+1}, ..., a_n)$$
Thus, the action of $\tilde{G}$ on $\mathbb{C}^n \cap \mathcal{O}_1$ provides an action on $\mathbb{C}^n$ given by the diagonal matrix

\[
\text{diag}(\xi_p^{k_1-k}, \ldots, \xi_p^{k_{i-1}-k_i}, \xi_p^{k_i}, \xi_p^{k_{i+1}-k_i}, \ldots, \xi_p^{k_n-k}).
\]

We recall the following property from the proof of [24, Proposition 6]:

**Proposition 5.7.** ([24]) The quotient $\mathbb{C}^n/G$ is smooth if and only if $\text{rk} g - \text{id} = 1$; that is $k_1 = \ldots = k_{n-1} = 0$.

Hence, we deduce from Lemma 5.6 the following.

**Corollary 5.8.** Let $X$ be a complex manifold of dimension $n$ and $G$ an automorphism group of prime order $p$. Let $x \in \text{Fix} G$.

(i) The variety $X/G$ is smooth in $\pi(x)$ if and only if the local action around $x$ is given by $\text{diag}(0, \ldots, 0, \xi_p^k)$.

(ii) Let $\tilde{X}$ be the blow-up of $X$ in the connected components of $\text{Fix} G$ of codimension $\geq 2$ and $\tilde{G}$ the automorphism group induced by $G$ on $\tilde{X}$.

The quotient $\tilde{X}/\tilde{G}$ is smooth if and only if the local action of $G$ around all $x \in \text{Fix} G$, is given by a matrix of the form: $\text{diag}(0, \ldots, 0, \xi_p^k, \ldots, \xi_p^k)$.

So, the local actions which are given by a diagonal matrix of the form $\text{diag}(0, \ldots, 0, \xi_p^k, \ldots, \xi_p^k)$ are of particular interest for our purpose of using Proposition 5.2.

**Definition 5.9.** A fixed point is simple if there is $i \in \{1, \ldots, n\}$ such that $k_1 = \ldots = k_i = 0$ and $k_{i+1} = \ldots = k_n$.

Let $X$ be a complex manifold and $G$ an automorphism group of prime order acting on $X$. We have studied the case of a single blow-up, we now consider a sequence of blow-ups:

\[
\begin{array}{c}
\xymatrix{ & X_k \ar[rr]^{s_k} & & \cdots \ar[r]^{s_2} & X_1 \ar[r]^{s_1} & X_i \ar[d]^{\pi_i} & \cdots \ar[r]^{r_2} & X_1/G \ar[r]^{r_1} & X/G \ar[d]^{\pi} }
\end{array}
\]

where each $s_{i+1}$ is the blow-up of $X_i$ in $\text{Fix} G_i$ ($X = X_0$ and $G = G_0$). However, in most cases, it is not possible to find $k$ such that $X_k/G_k$ is smooth.

**Proposition 5.10.** There exists $k \in \mathbb{N}$ such that $X_k/G_k$ is smooth if and only if either all the fixed points of $G$ are simple or $p = 3$. Moreover, in the case where $p = 3$, $X_2/G_2$ is smooth.

**Proof.** By [6, Lemma 1], we can assume that $X = \mathbb{C}^n$ and

\[
G = \left\langle \text{diag}(\xi_p^{k_1}, \ldots, \xi_p^{k_n}) \right\rangle.
\]

First, if 0 is a simple fixed point, by Corollary 5.8, $M_1$ is smooth. If $p = 3$, we will show that $M_2$ is smooth. Let $G = \left\langle \text{diag}(\xi_3^{k_1}, \ldots, \xi_3^{k_n}) \right\rangle$ acting on $\mathbb{C}^n$. Without loss of generality, we can assume that all $k_i$ are different from 0. Let $\mathbb{C}^n$ be the blow-up of $\mathbb{C}^n$ in 0. By Lemma 5.6, in the chart $a_i \neq 0$ the action of $G_1$ is given by the diagonal matrix

\[
\text{diag}(\xi_3^{k_1-k}, \ldots, \xi_3^{k_{i-1}-k_i}, \xi_3^{k_i}, \xi_3^{k_{i+1}-k_i}, \ldots, \xi_3^{k_n-k}).
\]

As $p = 3$, there is a $j$ such that $k_1 = \ldots = k_j = 1$ and $k_{j+1} = \ldots = k_n = 2$. So, if $i \leq j$, by permuting the $i$-th and the $j$-th coordinates of the chart $a_i \neq 0$, we reduce the action to the form

\[
\text{diag}(1, \ldots, 1, \xi_3, \ldots, \xi_3).
\]
If \( i > j \), by a permutation of coordinates we obtain
\[
\text{diag}(\xi_2^2, \ldots, \xi_3^2, 1, \ldots, 1).
\]

In both cases, these are simple fixed points. Hence, all the points of \( \text{Fix}G_1 \) are simple fixed points. Moreover, as there are no fixed points with both eigenvalues \( \xi_3, \xi_3^2 \) present in the diagonal matrix of the action, we can conclude that the components of \( \text{Fix}G_1 \) with spectra \( (1, \ldots, 1, \xi_3, \ldots, \xi_3) \) and \( (\xi_3^2, \ldots, \xi_3^2, 1, \ldots, 1) \) are disjoint and can be blown up independently. Hence, by Corollary 5.8, \( \hat{M}_2 \) is smooth.

Secondly, we will show that in the other cases, \( X_k/G_k \) will never be smooth. We start with \( \dim X = 2 \). By [6, Lemma 1], we can assume that \( X = \mathbb{C}^2 \) and \( G = \langle \text{diag}(\xi_\alpha^\beta, \xi_\alpha^\beta) \rangle \), \( p > 3 \) and \( \alpha \) not equal to 0 or 1.

Let \( \xi_\alpha^\beta \) be a fixed non-trivial \( p \)-th root of the unity, for \( x \in \text{Fix}G_i \), we can write:
\[
(X_i, G_i, x) \sim (\mathbb{C}^2, \langle \text{diag}(\xi_\alpha^\beta), 0 \rangle),
\]
where \( \sim \) means that the local actions are the same. Hence, we can define a sequence as follows:
\[
u_i = \{ \beta \in \mathbb{Z}/p\mathbb{Z} | \exists x \in X_i : (X_i, G_i, x) \sim (\mathbb{C}^2, \langle \text{diag}(\xi_\alpha^\beta) \rangle, 0) \}.
\]

For instance, \( u_0 = \{ \alpha, \beta \} \), \( u_1 = \{ \alpha - 1, \frac{1}{\alpha - 1}, -1,\frac{1}{1- \alpha} \} \). Now assume that there is \( i \in \mathbb{N} \) such that \( X_i/G_i \) is smooth. Let \( i \) be the smallest integer such that \( X_i/G_i \) is smooth. By Corollary 5.8, \( i \), we have \( u_i = \{ 0 \} \) and we can write \( u_{i-1} = \{ \alpha_1, \ldots, \alpha_k \} \). Let \( x \in \text{Fix}G_{i-1} \) such that
\[
(X_{i-1}, G_{i-1}, x) \sim (\mathbb{C}^2, \langle \text{diag}(\xi_\alpha^\beta) \rangle, 0),
\]
with \( \alpha_j \in u_i \). Hence, necessarily, \( \alpha_j = 1 \) and so \( u_{i-1} = \{ 1 \} \).

We do the same calculation with \( u_{i-2} = \{ \alpha'_1, \ldots, \alpha'_k \} \). Let \( x \in \text{Fix}G_{i-2} \) be such that:
\[
(X_{i-2}, G_{i-2}, x) \sim (\mathbb{C}^2, \langle \text{diag}(\xi_\alpha^\beta) \rangle, 0),
\]
with \( \alpha'_j \in u_{i-2} \setminus \{ 0, 1 \} \). We remark that \( u_{i-2} \setminus \{ 0, 1 \} \) is not empty because \( X_{i-1}/G_{i-1} \) is not smooth by definition of \( i \). Let \( \hat{C}^2 \) be the blow-up of \( \mathbb{C}^2 \) in 0. The action of \( \langle \text{diag}(\xi_\alpha^\beta) \rangle \) on \( \hat{C}^2 \) has two fixed points \( a_1 \) and \( a_2 \) with (see Lemma 5.6):
\[
(\hat{C}^2, \langle \text{diag}(\xi_\alpha^\beta) \rangle, a_1) \sim (\mathbb{C}^2, \text{diag}(\xi_\alpha^\beta, \xi_\alpha^\beta^{-1}), 0)
\]
and
\[
(\hat{C}^2, \langle \text{diag}(\xi_\alpha^\beta) \rangle, a_2) \sim (\mathbb{C}^2, \text{diag}(\xi_\alpha^\beta, \xi_\alpha^\beta^{1-\alpha_j}), 0).
\]

Hence, necessarily, \( \alpha'_j = 2 \) and \( \alpha'_j = \frac{1}{2} \). In \( \mathbb{Z}/p\mathbb{Z}, \frac{1}{2} \) is a solution, if and only if, \( p = 3 \). Hence, \( p = 3 \) and we are done.

Now, we assume \( n > 2 \). By [6, Lemma 1], we can assume that \( X = \mathbb{C}^n \) and \( G = \langle \text{diag}(\xi_{k_1}^{\beta_1}, \ldots, \xi_{k_n}^{\beta_n}) \rangle \). Without loss of generality, we can assume that all the \( k_i \) are different from 0. Moreover, our assumption, \( p > 3 \) and the \( k_i \) are not all equal. Without loss of generality, we can assume that
Since not all the $k_i$ are equal, there is $j \in \{1, \ldots, n\}$ such that $k_j \neq 1$. We also denote $\alpha = k_j$. We denote $X' = \mathbb{C}^n$ and $G' = \langle \text{diag}(\xi_p, \xi_p^\alpha) \rangle$. And we define as before the sequence:

$$u'_i = \{ \beta \in \mathbb{Z}/p\mathbb{Z} \mid \exists x \in X'_i : (X'_i, G'_i, x) \sim (\mathbb{C}^n, \langle \text{diag}(\xi_p, \xi_p^\alpha) \rangle, 0) \}. $$

We also define the following sequence:

$$U_i = \{ \beta \in \mathbb{Z}/p\mathbb{Z} \mid \exists x \in X_i : (X_i, G_i, x) \sim (\mathbb{C}^n, \langle \text{diag}(\xi_p, \xi_p^\alpha, \xi_p^t, \ldots, \xi_p^t_{n-1}) \rangle, 0) \}. $$

We have to show that $U_i \neq \{0\}$ for all $i \in \mathbb{N}$. Indeed, $U_i \supset u'_i$, and we have seen that $u'_i \neq \{0\}$ for all $i \in \mathbb{N}$. \hfill \qed

\section{Proof of Theorem 1.1 and outcomes}

Theorem 1.1 is a direct consequence of the following result which will be proven in this section.

**Theorem 6.1.** Let $X$ be a compact Kähler manifold of complex dimension $n$ and $G = \langle g \rangle$ an automorphism group of prime order $p \leq 19$. Let $c := \text{Codim Fix} G$. Assume that

(i) $H^\ast(X, \mathbb{Z})$ and $H^\ast(\text{Fix} G, \mathbb{Z})$ are $p$-torsion-free,

(ii) $h^\ast(\text{Fix} G, \mathbb{Q}) \geq \sum_{1 \leq q < p} \ell_q^\ast(X),$

(iii) all fixed points of $G$ are simple,

(iv) $c \geq \frac{n}{2} + 1.$

Then, for all $2n - 2c + 1 \leq k \leq 2c - 1$, $(X, G)$ is $H^k$-normal.

We first explain why Theorem 1.1 follows from Theorem 6.1. Boissière, Nieper-Wisskirchen and Sarti in [5, Corollary 4.3] have proven the following formula:

**Proposition 6.2.** Let $X$ be a compact connected orientable manifold of dimension $n$ and $G$ an automorphism group of prime order $p$. If the spectral sequence of equivariant cohomology with coefficients in $\mathbb{F}_p$ degenerates at the second page, then:

$$h^\ast(\text{Fix} G, \mathbb{F}_p) = \sum_{1 \leq q < p} \ell_q^\ast(X).$$

Hence, since we assume that $H^\ast(\text{Fix} G, \mathbb{Z})$ is $p$-torsion-free, we obtain Theorem 1.1 from Theorem 6.1 and Proposition 6.2.

Before proving Theorem 6.1 in Section 6.1, we discuss its hypotheses in few words.

**Remark 6.3.** The condition $p \leq 19$ is necessary to work with the spectral sequence of equivariant cohomology with coefficients in $\mathbb{Z}$ and use Corollary 4.2 (see Lemma 6.10). However, it is possible to avoid the condition $p \leq 19$ working, under some additional assumptions, with the spectral sequence of equivariant cohomology with coefficients in $\mathbb{F}_p$. I made the choice not to provide such a theorem, since in practice it is very rare to have to work with $p > 19$.

**Remark 6.4.** The condition on the codimension of Fix $G$ in Theorem 6.1 could be improved. However, hypothesis (ii) would have to be replaced by a more elaborate condition.

**Remark 6.5.** As explained in Section 5.2, to avoid the condition that the fixed points of $G$ are simple, we need to find another practical way of resolving the singularities of $X/G$. We could also use two blow-ups in the case $p = 3$. We provide such an example in Section 7.6.
6.1 Proof of Theorem 6.1

The idea of the proof is to calculate the discriminant of the exceptional lattice $N_{k,r}$, where $r$ is obtained from a blow-up, and apply Proposition 5.2.

All the fixed points of $G$ are simple, so we recall from Section 5.2 that we are in the configuration of the following commutative diagram:

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{s} & X \\
\downarrow{\tilde{\pi}} & & \downarrow{\pi} \\
X/G & \xrightarrow{r} & X/G,
\end{array}
\]

where $\tilde{G}$ be the automorphism group induced by $G$ and $\tilde{X}/\tilde{G}$ is smooth. We also denote $M = X/G$, $M = \tilde{X}/\tilde{G}$, $V = X \setminus \text{Fix} G$, $U = \pi(V)$, $F = s^{-1}(\text{Fix} G)$, we use the same symbol $F$ for its image by $\tilde{\pi}$ and we adopt the notation introduced in Section 2.2 and Section 3.1. Diagram (17) induces a commutative diagram on cohomology:

\[
\begin{array}{ccc}
H^k(M, \mathbb{Z}) & \xrightarrow{\pi_*} & H^k(X, \mathbb{Z}) \\
\downarrow{r^*} & & \downarrow{s^*} \\
H^k(M/G, \mathbb{Z}) & \xrightarrow{\tilde{\pi}_*} & H^k(\tilde{X}, \mathbb{Z}).
\end{array}
\]

Lemma 6.6. Let $0 \leq k \leq 2n$ and $x \in H^k(X, \mathbb{Z})$. Then:

(i) $\tilde{\pi}_*(s^*(x)) = r^*(\pi_*(x)) + p - \text{torsion}$,

(ii) $N_k = \tilde{\pi}_*(s^*(H^k_f(X, \mathbb{Z}) \oplus H^{2n-k}_f(X, \mathbb{Z})))^\perp$.

Proof. We remark that $(\tilde{X}, \tilde{G}, s)$ is a pull-back of $(X, G)$, so (i) follows from Lemma 3.16. applying (i) to $H^k_f(M, \mathbb{Z})$, we obtain:

\[
\tilde{\pi}_*(s^*(H^k_f(X, \mathbb{Z}))) = r^*(\pi_*(H^k_f(X, \mathbb{Z}))).
\]

Then, it follows (ii) by definition of $N_k$. \qed

Lemma 6.7. (i) For all $i \leq 2c - 2$, $H^i(X, \mathbb{Z}) = H^i(V, \mathbb{Z})$,

(ii) $H^{2c-1}(V, \mathbb{Z})$ is $p$-torsion-free.

Proof. We have the following exact sequence:

\[
H^i(X, \mathbb{Z}) \longrightarrow H^i(X, \mathbb{Z}) \longrightarrow H^i(V, \mathbb{Z}) \longrightarrow H^{i+1}(X, V, \mathbb{Z}).
\]

Moreover, by Thom’s isomorphism for each connected component $S_m$ of $\text{Fix} G$: $H^i(X, X \setminus S_m, \mathbb{Z}) = H^{i-2} \text{Codim} S_m(S_m, \mathbb{Z})$. Hence:

\[
H^i(X, V, \mathbb{Z}) = \sum_{S_m \subseteq \text{Fix} G} H^i(X, X \setminus S_m, \mathbb{Z}) = 0,
\]

for all $i \leq 2c - 1$. So, we get (i). Moreover, we obtain the following exact sequence:

\[
0 \longrightarrow H^{2c-1}(X, \mathbb{Z}) \longrightarrow H^{2c-1}(V, \mathbb{Z}) \longrightarrow H^{2c}(X, V, \mathbb{Z}).
\]

From the hypotheses, we know that $H^{2c-1}(X, \mathbb{Z})$ and $H^*(\text{Fix} G, \mathbb{Z})$ are $p$-torsion-free; so, by Thom’s isomorphism, $H^{2c}(X, V, \mathbb{Z})$ is $p$-torsion-free. Then, (ii) follows from (19). \qed
We consider the following commutative diagram:

\[
\begin{array}{ccc}
H^k(\mathcal{M}_F, \mathcal{M}/F - 0, \mathbb{Z}) = H^k(\widetilde{M}, U, \mathbb{Z}) & \xrightarrow{\pi^*} & H^k(\widetilde{M}, \mathbb{Z}) \\
\downarrow d\pi^* & & \downarrow \pi^* \\
H^k(\mathcal{M}_F, \mathcal{M}/F - 0, \mathbb{Z}) = H^k(\widetilde{X}, V, \mathbb{Z}) & \xrightarrow{\beta} & H^k(\widetilde{X}, \mathbb{Z}),
\end{array}
\]

where \(\mathcal{M}_F - 0\) and \(\mathcal{M}/F - 0\) are vector bundles minus the zero section. We denote \(T_k := h(H^k(\widetilde{X}, V, \mathbb{Z}))\).

**Lemma 6.8.** When \(2n - 2c + 1 \leq k \leq 2c - 1\), we have:

\[
H^k(\widetilde{X}, \mathbb{Z}) = s^*(H^k(X, \mathbb{Z})) \oplus T_k.
\]

Moreover:

\[
T_k \perp s^*(H^{2n-k}(X, \mathbb{Z})).
\]

**Proof.** By Thom’s isomorphism, \(H^k(\widetilde{X}, V, \mathbb{Z}) = H^{k-2}(F, \mathbb{Z})\), and the map \(h\) can be identified with the morphism \(j : H^{k-2}(F, \mathbb{Z}) \to H^k(\widetilde{X}, \mathbb{Z})\), where \(j\) is the inclusion in \(\widetilde{X}\). As in the proof of [30, Theorem 7.31], the map

\[
(s^*, j_k) : H^k(X, \mathbb{Z}) \oplus H^{k-2}(F, \mathbb{Z}) \to H^k(\widetilde{X}, \mathbb{Z})
\]

is surjective and its kernel is the image of the map

\[
\bigoplus_{S_d \subset \text{Fix } G} H^{k-2r_d}(S_d, \mathbb{Z}) \to H^k(X, \mathbb{Z}) \oplus H^{k-2}(F, \mathbb{Z}),
\]

where \(r_k\) is the codimension of the component \(S_d\) of \(\text{Fix } G\). But in our case \(\bigoplus_{S_d \subset \text{Fix } G} H^{k-2r_d}(S_d, \mathbb{Z}) = 0\).

The orthogonality of this sum follows from the projection formula. \(\square\)

Moreover, Theorem 5.4 also shows that:

\[
\text{rk } T_k = \text{rk } \bigoplus_{S_d \subset \text{Fix } G} H^{k-2r_d-2}(S_d, \mathbb{Z}),
\]

where \(r_d\) is the codimension of the component \(S_d\) of \(\text{Fix } G\). Hence, when \(2n - 2c + 1 \leq k \leq 2c - 1\), we obtain:

\[
\text{rk } T_k = h^{2+\epsilon}(\text{Fix } G, \mathbb{Z}),
\]

where \(\epsilon = 1\) if \(k\) is odd and \(\epsilon = 0\) if \(k\) is even.

**Lemma 6.9.** We have:

(i) When \(2n - 2c + 1 \leq k \leq 2c - 1\):

\[
\text{discr } \tilde{\pi}_*(T_k \oplus T_{2n-k}) = p^{2h^{2+\epsilon}(\text{Fix } G, \mathbb{Z})},
\]

where \(\epsilon = 1\) if \(k\) is odd and \(\epsilon = 0\) if \(k\) is even.

(ii) \(N_{k,r}\) is the primitive over-lattice of \(\tilde{\pi}_*(T_k \oplus T_{2n-k})\).

**Proof.** By Lemma 6.8, we have:

\[
H^k(\widetilde{X}, \mathbb{Z}) \oplus H^{2n-k}(\widetilde{X}, \mathbb{Z}) = s^*(H^k(X, \mathbb{Z}) \oplus H^{2n-k}(X, \mathbb{Z})) \oplus (T_k \oplus T_{2n-k}).
\]

Since \(\widetilde{X}\) and \(X\) are smooth, \(H^k(\widetilde{X}, \mathbb{Z}) \oplus H^{2n-k}(\widetilde{X}, \mathbb{Z})\) and \(H^k(X, \mathbb{Z}) \oplus H^{2n-k}(X, \mathbb{Z})\) are unimodular. It follows that \(T_k \oplus T_{2n-k}\) is unimodular. Moreover, by Thom’s isomorphism \(H^k(\widetilde{X}, V, \mathbb{Z}) =\)
From (21), where \( j \) is the inclusion in \( \tilde{X} \). It follows that \( T_k \subset H^k(\tilde{X}, \mathbb{Z}) \). Hence, by Lemma 3.6 (ii), we obtain:

\[
\text{discr } \tilde{\pi}_* (T_k \oplus T_{2n-k}) = p^{\text{rk} T_k \oplus T_{2n-k}}.
\]

From (21), when \( 2n - 2c + 1 \leq k \leq 2c - 1 \),

\[
\text{rk} T_k = h^{2+c} (\text{Fix } G, \mathbb{Z}),
\]

where \( c = 1 \) if \( k \) is odd and \( c = 0 \) if \( k \) is even. Remark that if \( 2n - 2c + 1 \leq k \leq 2c - 1 \) then \( 2n - 2c + 1 \leq 2n - k \leq 2c - 1 \). Hence, \( \text{rk} T_k = \text{rk} T_{2n-k} = h^{2+c}(\text{Fix } G, \mathbb{Z}) \). So, it follows (i).

Now taking the image of (22) by \( \tilde{\pi}_* \), we obtain \( \tilde{\pi}_*(s^*(H^k(X, \mathbb{Z}) \oplus H^{2n-k}(X, \mathbb{Z}))) \oplus \tilde{\pi}_*((T_k \oplus T_{2n-k})) \) as a sublattice of full rank of \( H^k(\tilde{M}, \mathbb{Z}) \oplus H^{2n-k}(\tilde{M}, \mathbb{Z}) \). Moreover, applying Proposition 3.1, the direct sum remains orthogonal. So, (ii) follows from Lemma 6.6 (ii).

By the property of the Thom isomorphism,

\[
d\tilde{\pi}^*(H^k(\mathcal{A}_{\tilde{M}/F}, \mathcal{A}_{\tilde{M}/F} - 0, \mathbb{Z})) = pH^k(\mathcal{A}_{\tilde{X}/F}, \mathcal{A}_{\tilde{X}/F} - 0, \mathbb{Z}).
\]

Then, by commutativity of diagram (20) and Proposition 3.1, we have \( g(H^k(\tilde{M}, U, \mathbb{Z})) = \tilde{\pi}_*(T_k) \). It follows the exact sequence:

\[
\begin{array}{c}
0 \longrightarrow \tilde{\pi}_*(T_k) \longrightarrow H^k(\tilde{M}, \mathbb{Z}) \longrightarrow H^k(U, \mathbb{Z}).
\end{array}
\]

(23)

In order to calculate the discriminant of \( N_k, r \) which is by Lemma 6.9 (ii), the primitive over lattice of \( \tilde{\pi}_*(T_k) \oplus \tilde{\pi}_*(T_{2n-k}) \), we need to know the \( p \)-torsion of \( H^k(U, \mathbb{Z}) \oplus H^{2n-k}(U, \mathbb{Z}) \). Let \( H^k_p(U, \mathbb{Z}) \) be the \( F_p \)-vector spaces given by the \( p \)-torsion part of \( H^k(U, \mathbb{Z}) \).

To lighten a bit the notation, in the following calculations, we denote \( \ell^k_a \) for \( \ell^k(X) \) and \( N_k \) for \( N_k, r \).

**Lemma 6.10.** Let \( k \leq 2c - 1 \), we have if \( k = 2a \):

\[
\dim_{F_p} H^k_p(U, \mathbb{Z}) \leq \sum_{j=0}^{a-1} \ell^{2j} + \sum_{j=0}^{a-1} \ell^{2j+1}.
\]

If \( k = 2a + 1 \):

\[
\dim_{F_p} H^k_p(U, \mathbb{Z}) \leq \sum_{j=0}^{a-1} \ell^{2j+1} + \sum_{j=0}^{a} \ell^{2j}.
\]

**Proof.** Since \( G \) acts freely on \( V \), we recall from Section 4 that

\[
H^*(U, \mathbb{Z}) \cong H^*_G(V, \mathbb{Z}).
\]

Using the second page of the spectral sequence of equivariant cohomology, it follows that

\[
\dim_{F_p} H^k_p(U, \mathbb{Z}) \leq \sum_{i=1}^k \dim_{F_p} H^i(G, H^{k-i}(V, \mathbb{Z})).
\]

(24)

The group \( H^0(G, H^k(V, \mathbb{Z})) \) does not provide any contribution to \( H^k_p(U, \mathbb{Z}) \) because by Corollary 4.2, \( H^0(G, H^k(V, \mathbb{Z})) = H^k(V, \mathbb{Z}) \) and by Lemma 6.7, \( H^k(V, \mathbb{Z}) \) is \( p \)-torsion-free.

Moreover, by Lemma 6.7 (i), (24) becomes:

\[
\dim_{F_p} H^k_p(U, \mathbb{Z}) \leq \sum_{i=1}^k \dim_{F_p} H^i(G, H^{k-i}(X, \mathbb{Z})).
\]

Hence, by Corollary 4.2, if \( k = 2a \):

\[
\dim_{F_p} H^k_p(U, \mathbb{Z}) \leq \sum_{j=0}^{a-1} \ell^{2j} + \sum_{j=0}^{a-1} \ell^{2j+1}.
\]
If \( k = 2n + 1 \):

\[
\dim_{\mathbb{F}_p} H^k_p(U, \mathbb{Z}) \leq \sum_{j=0}^{a-1} \ell^j_+ + \sum_{j=0}^{a} \ell^j_-. \]

From (23) and Lemma 6.9 (ii):

\[
\dim_{\mathbb{F}_p} \left( \frac{N_k}{\pi_*(T_k \oplus T_{2n-k})} \right) \leq \dim_{\mathbb{F}_p} H^k_p(U, \mathbb{Z}) \oplus H^{2n-k}_p(U, \mathbb{Z}).
\]

When \( 2n-2c+1 \leq k \leq 2c-1 \), we can apply Lemma 6.10 to \( H^k_p(U, \mathbb{Z}) \) and to \( H^{2n-k}_p(U, \mathbb{Z}) \). When \( k = 2n \),

\[
\dim_{\mathbb{F}_p} \left( \frac{N_k}{\pi_*(T_k \oplus T_{2n-k})} \right) \leq \sum_{j=0}^{a-1} \ell^j_+ + \sum_{j=0}^{a-1} \ell^j_- + \sum_{j=0}^{n-a-1} \ell^j_+ + \sum_{j=0}^{n-a-1} \ell^j_-.
\]

Hence, by Lemma 6.9 (i) and (3):

\[
\log_p \text{disc} N_k \geq 2h^*(\text{Fix } G, \mathbb{Z}) - 2 \left[ \sum_{j=0}^{a-1} \ell^j_+ + \sum_{j=0}^{a-1} \ell^j_- + \sum_{j=0}^{n-a-1} \ell^j_+ + \sum_{j=0}^{n-a-1} \ell^j_- \right].
\]

Similarly, when \( k = 2a + 1 \), we obtain:

\[
\log_p \text{disc} N_k \geq 2h^{2a+1}(\text{Fix } G, \mathbb{Z}) - 2 \left[ \sum_{j=0}^{a-1} \ell^j_+ + \sum_{j=0}^{a} \ell^j_- + \sum_{j=0}^{n-a-2} \ell^j_+ + \sum_{j=0}^{n-a-1} \ell^j_- \right].
\]

Now the strategy to finish the proof is the following. Equations (25) and (26) alone are not practical but, as we will see, their sum can be simplified. Hence, the objective is to use (25)+(26) to prove our theorem. Notice first that since \( c \geq \frac{2}{a} + 1 \), the interval \( \{2n - 2c + 1, \ldots, 2c - 1\} \) always has cardinal at least 3. Hence, we can always find \( k \) such that \( k, k+1 \in \{2n - 2c + 1, \ldots, 2c - 1\} \). Using (25)+(26) we will provide an upper bound for \( \log_p \text{disc} N_k + \log_p \text{disc} N_{k+1} \) which will provide the \( H^k \oplus H^{k+1} \)-normality. Then, the claim follows covering \( \{2n - 2c + 1, \ldots, 2c - 1\} \) by pairs of consecutive numbers \( \{k, k+1\} \).

So, let \( k, k+1 \in \{2n - 2c + 1, \ldots, 2c - 1\} \), we can assume that \( k = 2a \), if \( k \) is odd the proof is identical. Hence, summing (25)+(26) and applying Proposition 3.9 (ii), we have:

\[
\log_p \text{disc} N_k + \log_p \text{disc} N_{k+1} - 2h^*(\text{Fix } G, \mathbb{Z}) \geq -2 \left[ \sum_{j=0}^{k-1} \ell^j_+ + \sum_{j=0}^{k} \ell^j_+ + \sum_{j=0}^{2n-k-2} \ell^j_+ + \sum_{j=0}^{2n-k-1} \ell^j_- \right] - 2 \left[ \sum_{j=0}^{k-1} \ell^j_+ + \sum_{j=0}^{k} \ell^j_- + \sum_{j=0}^{2n-k-2} \ell^j_- + \sum_{j=0}^{2n-k-1} \ell^j_- \right] - 2 \left[ \sum_{j=0}^{k-1} \ell^j_+ + \sum_{j=0}^{k} \ell^j_- + \sum_{j=k+2}^{2n} \ell^j_+ + \sum_{j=k+1}^{2n} \ell^j_- \right] - 2 \left[ \sum_{1 \leq q < p} \ell^q_+ \right].
\]

Hence, by hypothesis,

\[
\log_p \text{disc} N_k + \log_p \text{disc} N_{k+1} \geq 2 \left[ \ell^k_+ + \ell^{k+1}_+ \right],
\]

and, by Proposition 5.2,

\[
\alpha_k = \alpha_{2n-k} = \alpha_{k+1} = \alpha_{2n-k-1} = 0.
\]
Remark 6.11. We have proved a bit more than what is claimed in Theorem 6.1. Under its hypotheses, we have seen that \( \tilde{\pi}_*(s^*(H^k(X, \mathbb{Z}) \oplus H^{2n-k}(X, \mathbb{Z}))) \) is primitive, in \( H^*(M, \mathbb{Z}) \), for all \( 2n-2e+1 \leq k \leq 2e-1 \).

Indeed, from (27) and Proposition 5.2, \( \text{discr } N_k = p^{k^2} \). Moreover, from (16) and Corollary 3.8, \( \text{discr } r^*(\pi_*(H^k_f(X, \mathbb{Z}) \oplus H^{2n-k}_f(X, \mathbb{Z}))) = p^{k^2} \). So, by (18), we have:

\[
\log_p \text{discr } N_k = \text{discr } \tilde{\pi}_*(s^*(H^k_f(X, \mathbb{Z}) \oplus H^{2n-k}_f(X, \mathbb{Z}))).
\]

Since \( \tilde{M} \) is smooth, by (4), we have \( \text{discr } N_k = \text{discr } N_k^\perp \). However, by Lemma 6.6 (ii), \( \tilde{\pi}_*(s^*(H^k(X, \mathbb{Z}) \oplus H^{2n-k}(X, \mathbb{Z}))) \subset N_k^\perp \). So by (28), \( \tilde{\pi}_*(s^*(H^k(X, \mathbb{Z}) \oplus H^{2n-k}(X, \mathbb{Z}))) = N_k^\perp \), which is primitive in \( H^*(\tilde{M}, \mathbb{Z}) \).

6.2 Refinement on the codimension of the fixed locus

We provide an extension of Theorem 1.1 when \( \text{Fix } G \) is a bit larger. As mentioned in Remark 6.4, we will need more technical conditions. However, when \( \text{Codim } \text{Fix } G = \left\lceil \frac{n}{2} \right\rceil \), it is still possible to provide a result which is interesting in practice (see for instance the application in [13, Section 15]). We will see another example in Section 7.3.

Theorem 6.12. Let \( X \) be a compact Kähler manifold of complex dimension \( n \) and \( G = \langle g \rangle \) an automorphism group of prime order \( p \leq 19. \) Assume that:

(i) \( H^*(X, \mathbb{Z}) \) and \( H^*(\text{Fix } G, \mathbb{Z}) \) are \( p \)-torsion-free.

(ii) \( h^{2*}(X^G, \mathbb{Z}) \geq \ell^{2*}_+(X) + \ell^{2*+1}_+(X) \) if \( n \) is even and \( h^{2*+1}(X^G, \mathbb{Z}) \geq \ell^{2*}_+(X) + \ell^{2*}_+(X) \) if \( n \) is odd.

(iii) All fixed points of \( G \) are simple.

(iv) \( \text{Codim } \text{Fix } G = \left\lceil \frac{n}{2} \right\rceil \).

(v) When \( n \) is even, let \( \Sigma \) be the component of \( \text{Fix } G \) of dimension \( \frac{n}{2} \). We assume that \( \Sigma \) is connected. Let \( j : \text{Fix } G \hookrightarrow X \) be the inclusion. We assume that \( j_* : H^n(\Sigma, \mathbb{Z}) \to H^n(X, \mathbb{Z}) \) is injective and its image is primitive.

Then, \( (X, G) \) is \( H^n \)-normal.

This section is dedicated to the proof of this theorem. The proof is different when \( n \) is even or odd. When \( n \) is odd, it is a straightforward consequence of the results in the previous section. When \( n \) is even we have to do more work. Let us first prove the theorem when \( n \) is odd.

Case: \( n \) is odd

We have \( n = 2m + 1 \) and \( \text{Codim } \text{Fix } G = m + 1, \) so \( n \leq 2m + 1. \) It follows that Lemmas 6.6, 6.7, 6.8, 6.9 remain true. The problem is that the interval \( \{2n - 2e + 1, \ldots, 2e - 1\} \) has only cardinality one; hence it is no longer possible to sum (25) and (26). However, (26) remains true and becomes

\[
\log_p \text{discr } N_n \geq 2h^{2*+1}(\text{Fix } G, \mathbb{Z}) - 2 \left[ \sum_{j=0}^{m-1} \ell^{2j+1}_+ + \sum_{j=0}^{m} \ell^{2j}_+ + \sum_{j=0}^{m-1} \ell^{2j+1}_- + \sum_{j=0}^{m} \ell^{2j}_- \right].
\]

Applying Proposition 3.9, it follows:

\[
\log_p \text{discr } N_n \geq 2h^{2*+1}(\text{Fix } G, \mathbb{Z}) - 2 \left[ \sum_{j=0}^{m-1} \ell^{2j+1}_+ + \sum_{j=0}^{m} \ell^{2j}_+ + \sum_{j=0}^{m-1} \ell^{2n-2j-1}_+ + \sum_{j=0}^{m} \ell^{2n-2j}_- \right].
\]

So:

\[
\log_p \text{discr } N_n \geq 2h^{2*+1}(\text{Fix } G, \mathbb{Z}) - 2 \left[ \ell^{2*+1}_+ + \ell^{2*}_- - \ell^{m+1}_+ \right].
\]

Then, we get the result using hypothesis (ii) and Proposition 5.2.
Lemma 6.13. 

(i) \( H^{2m-1}(X, \mathbb{Z}) = H^{2m-1}(V, \mathbb{Z}) \),

(ii) \( H^{2m}(V, \mathbb{Z}) \) is \( p \)-torsion-free.

Proof. We have the following exact sequence:

\[
\begin{align*}
H^{2m-1}(X, V, \mathbb{Z}) &\longrightarrow H^{2m-1}(X, \mathbb{Z}) \oplus H^{2m}(V, \mathbb{Z}) \longrightarrow H^{2m}(X, V, \mathbb{Z}) \oplus H^{2m}(X, \mathbb{Z}) \\
&\quad \quad \quad \quad \text{by Thom’s isomorphism}
\end{align*}
\]

By Thom’s isomorphism

\[
H^{2m-1}(X, V, \mathbb{Z}) = 0
\]

and \( \rho \) can be identified with \( j_*: H^0(\Sigma, \mathbb{Z}) \hookrightarrow H^{2m}(X, \mathbb{Z}) \) that we assumed injective. So,

\[
H^{2m-1}(X, \mathbb{Z}) = H^{2m-1}(V, \mathbb{Z}).
\]

Moreover, we obtain the following exact sequence:

\[
\begin{align*}
0 &\longrightarrow H^0(\Sigma, \mathbb{Z}) \xrightarrow{j_*} H^{2m}(X, \mathbb{Z}) \xrightarrow{\rho} H^{2m}(V, \mathbb{Z}) \longrightarrow H^{2m+1}(X, V, \mathbb{Z}) \\
&\text{From the hypotheses, we know that } H^{2m}(X, \mathbb{Z}) \text{ and } H^*(\text{Fix } G, \mathbb{Z}) \text{ are } \rho \text{-torsion-free; so by Thom’s isomorphism } H^{2m+1}(X, V, \mathbb{Z}) \text{ is } \rho \text{-torsion-free. Moreover, we assumed that the image of } j_* \text{ is primitive. Hence, it follows (ii).}
\end{align*}
\]

We consider the following commutative diagram:

\[
\begin{align*}
H^{2m}(\mathcal{N}_{\tilde{M}/F}, \mathcal{N}_{\tilde{M}/F} - 0, \mathbb{Z}) &\xrightarrow{\delta} H^2m(\tilde{M}, U, \mathbb{Z}) \\
&\downarrow \pi^* \\
H^{2m}(\mathcal{N}_{\tilde{X}/F}, \mathcal{N}_{\tilde{X}/F} - 0, \mathbb{Z}) &\xrightarrow{\delta} H^2m(\tilde{X}, V, \mathbb{Z})
\end{align*}
\]

where \( \mathcal{N}_{\tilde{X}/F} - 0 \) and \( \mathcal{N}_{\tilde{M}/F} - 0 \) are vector bundles minus the zero section. We denote \( R := h(H^{2m}(\tilde{X}, V, \mathbb{Z})) \). We prove the following adaptation of Lemma 6.8:

Lemma 6.14. We can write:

\[
H^{2m}(\tilde{X}, \mathbb{Z}) = s^*(H^{2m}(X, \mathbb{Z})) \oplus T,
\]

with \( R = T \oplus j_*(H^0(\Sigma, \mathbb{Z})) \), where \( j \) is the inclusion in \( \tilde{X} \).

Proof. As in proof of Lemma 6.8, the map \( h \) can be identified with the morphism \( j_*: H^{2m-2}(F, \mathbb{Z}) \to H^{2m}(\tilde{X}, \mathbb{Z}) \); moreover, the map

\[
(s^*, j_*): H^{2m}(X, \mathbb{Z}) \oplus H^{2m-2}(F, \mathbb{Z}) \to H^{2m}(\tilde{X}, \mathbb{Z})
\]

is surjective and its kernel is the image of the map

\[
\bigoplus_{S_d \subset \text{Fix } G} H^{2m-2r_d}(S_d, \mathbb{Z}) \to H^{2m}(X, \mathbb{Z}) \oplus H^{2m-2}(F, \mathbb{Z}),
\]

where \( r_d \) is the codimension of the component \( S_d \) of \( \text{Fix } G \). However, in our case

\[
\bigoplus_{S_d \subset \text{Fix } G} H^{2m-2r_d}(S_d, \mathbb{Z}) = H^0(\Sigma, \mathbb{Z}).
\]

So, we obtain \( H^{2m}(\tilde{X}, \mathbb{Z}) = s^*(H^{2m}(X, \mathbb{Z})) \oplus T \), and the orthogonality of the sum can also be deduced using the projection formula. \( \square \)
Let $\tilde{R}$ be the minimal primitive over-lattice of $\tilde{\pi}_*(R)$ in $H^{2m}(\tilde{M}, \mathbb{Z})$ and $\tilde{T}$ the minimal primitive over-lattice of $\tilde{\pi}_*(T)$ in $H^{2m}(\tilde{M}, \mathbb{Z})$. From Lemmas 6.14 and 6.6 (ii), by the same proof as Lemma 6.9 (ii):

$$ N_{2m} = \tilde{T}^{2\oplus}. \quad (30) $$

As in the previous proof, we need to calculate the discriminant of $\tilde{T}$. By the property of the Thom isomorphism,

$$ d\tilde{\pi}^*(H^{2m}(\mathcal{N}_{\tilde{M}/F}, \mathcal{N}_{\tilde{M}/F} - 0, \mathbb{Z})) = pH^{2m}(\mathcal{N}_{\tilde{X}/F}, \mathcal{N}_{\tilde{X}/F} - 0, \mathbb{Z}). $$

Then, by commutativity of diagram (20) and Proposition 3.1, we have $g(H^{2m}(\tilde{M}, U, \mathbb{Z})) = \tilde{\pi}_*(R)$. It follows the exact sequence:

$$ 0 \longrightarrow \tilde{\pi}_*(R) \longrightarrow H^{2m}(\tilde{M}, \mathbb{Z}) \longrightarrow H^{2m}(U, \mathbb{Z}). \quad (31) $$

With the same proof as Lemma 6.10, we obtain from Lemma 6.13:

$$ \dim_{\mathbb{P}} H^{2m}_p(U, \mathbb{Z}) \leq \sum_{j=0}^{m-1} \ell_j + \sum_{j=0}^{m-1} \ell_{j+1}. \quad (32) $$

However, (31) will provide only an upper bound for $\dim_{\mathbb{P}} \frac{\tilde{R}}{\tilde{\pi}_*(R)}$. We need some lemmas to compare $\frac{\tilde{R}}{\tilde{\pi}_*(R)}$ and $\frac{\tilde{T}}{\pi_*(T)}$.

**Lemma 6.15.** There exists $x \in \tilde{\pi}_*(T)$ such that $\frac{x + (-1)^{n-1}\pi_*(a^*(\Sigma))}{p} \in H^{2m}(\tilde{M}, \mathbb{Z})$.

**Proof.** Let $s_1 : Y \rightarrow X$ be the blow-up of $X$ in $\Sigma$ and $\Sigma_1$ the exceptional divisor, and $s_2 : \tilde{X} \rightarrow Y$ the blow-up in the other components of Fix $G$ such that $s = s_2 \circ s_1$. We denote $\Sigma_2 = s_2^*(\Sigma_1)$ and $\tilde{\Sigma} = \tilde{\pi}_*(\Sigma_2)$. Consider the following diagram:

$$ \begin{array}{ccc}
\Sigma_2 & \xrightarrow{l_2} & \tilde{X} \\
\downarrow{g_2} & & \downarrow{s_2} \\
\Sigma_1 & \xrightarrow{l_1} & Y \\
\downarrow{g_1} & & \downarrow{s_1} \\
\Sigma & \xrightarrow{l_0} & X,
\end{array} $$

where $l_0$, $l_1$ and $l_2$ are the inclusions and $g_i := s_i|_{\Sigma_i}$, $i \in \{1, 2\}$.

We have $\tilde{\pi}_*(O_{\tilde{X}}) = O_{\tilde{M}} \oplus \mathcal{L}$, with $\mathcal{L}^p = O_{\tilde{M}} \left( -\left( \sum_{S_k \in \text{Fix} G \setminus \Sigma} \tilde{S}_k + \tilde{\Sigma} \right) \right)$, where each $\tilde{S}_k$ is the exceptional divisor associated to the connected component $S_k \neq \Sigma$ of Fix $G$. Thus,

$$ \frac{\sum_{S_k \in \text{Fix} G \setminus \Sigma} \tilde{S}_k + \tilde{\Sigma}}{p} \in H^2(\tilde{M}, \mathbb{Z}). $$

It follows that

$$ \left( \frac{\sum_{S_k \in \text{Fix} G \setminus \Sigma} \tilde{S}_k + \tilde{\Sigma}}{p} \right)^m \in H^{2m}(\tilde{M}, \mathbb{Z}). $$

By Lemma 3.6 (i), we get:

$$ \frac{x + \tilde{\pi}_*(\Sigma_2^m)}{p} \in H^{2m}(\tilde{M}, \mathbb{Z}), \quad (33) $$

with $x \in \tilde{\pi}_*(T)$.

Now, it remains to calculate $\Sigma_2^m$. By [10, Proposition 6.7], we have

$$ s_1^*l_{0*}(\Sigma) = l_{1*}(c_{m-1}(E)). $$
where \( E := g_! (\mathcal{M}_\Sigma/Y) ) / \mathcal{M}_{\Sigma_1/Y} \). So
\[
s_1^* l_1^*(\Sigma) = l_1^* \left( \sum_{i=0}^{m-1} (-1)^{m-1-i} c_i (g_! (\mathcal{M}_\Sigma/X)) \cdot c_i (\mathcal{M}_{\Sigma_1/Y})^{m-1-i} \right)
\]
\[
= l_1^* \left( \sum_{i=1}^{m-1} (-1)^{m-1-i} c_i (g_! (\mathcal{M}_\Sigma/X)) \cdot c_i (\mathcal{M}_{\Sigma_1/Y})^{m-1-i} \right)
+ (-1)^{m-1} l_1^* (c_1 (\mathcal{M}_{\Sigma_1/Y})^{m-1})
\]
\[
= l_1^* \left( \sum_{i=1}^{m-1} (-1)^{m-1-i} c_i (g_! (\mathcal{M}_\Sigma/X)) \cdot c_i (\mathcal{M}_{\Sigma_1/Y})^{m-1-i} \right)
+ (-1)^{m-1} \Sigma^m.
\]

By applying \( s_2^* \), we get:
\[
\Sigma^m_2 = (-1)^{m-1} (s^*(\Sigma) - s_2^* l_1^*(a)),
\]
where \( a = \sum_{i=1}^{m-1} (-1)^{m-1-i} c_i (g_! (\mathcal{M}_\Sigma/X)) \cdot c_i (\mathcal{M}_{\Sigma_1/Y})^{m-1-i} \in T \). And pushing forward via \( \tilde{\pi}_* \), we get:
\[
\tilde{\pi}_* (\Sigma^m_2) = (-1)^{m-1} (\tilde{\pi}_* (s^*(\Sigma)) - \tilde{\pi}_* (s_2^* l_1^*(a))).
\]
The result follows from (33). \(\square\)

**Lemma 6.16.** We have:
\[
\dim_p \frac{\tilde{T}}{\tilde{\pi}_* (T)} \leq \dim_p H^{2m}_p (U, Z) - 1,
\]
where \( H^{2m}_p (U, Z) \) is the \( p \)-torsion part of \( H^{2m} (U, Z) \).

**Proof.** From (31):
\[
\dim_p \frac{\tilde{R}}{\tilde{\pi}_* (R)} \leq \dim_p H^{2m}_p (U, Z).
\]
But by Lemma 6.15, we already know that there exists \( x \in \tilde{\pi}_* (T) \) such that \( \frac{2(-1)^{m-1} \tilde{\pi}_* (s^*(\Sigma))}{p} \in H^{2m}(\tilde{M}, Z) \).

We are going to deduce \( \dim_p \frac{\tilde{T}}{\tilde{\pi}_* (T)} \) from \( \dim_p \frac{\tilde{R}}{\tilde{\pi}_* (R)} \). If \( \tilde{\pi}_* (s^*(\Sigma)) \) is divisible by \( p \) in \( H^{2m}(\tilde{M}, Z) \), then \( \frac{\tilde{\pi}_* (s^*(\Sigma))}{p} \in (\tilde{R}/\tilde{\pi}_* (R)) \setminus (\tilde{T}/\tilde{\pi}_* (T)) \), if not \( \frac{2(-1)^{m-1} \tilde{\pi}_* (s^*(\Sigma))}{p} \in (\tilde{R}/\tilde{\pi}_* (R)) \setminus (\tilde{T}/\tilde{\pi}_* (T)) \). In both cases:
\[
\dim_p \frac{\tilde{T}}{\tilde{\pi}_* (T)} = \dim_p \frac{\tilde{R}}{\tilde{\pi}_* (R)} - 1.
\]
\(\square\)

So, from Lemma 6.16 and (32):
\[
\dim_p \frac{\tilde{T}}{\tilde{\pi}_* (T)} \leq \sum_{j=0}^{m-1} \ell_+^{(2 j)} + \sum_{j=0}^{m-1} \ell_-^{(2 j+1)} - 1.
\]
(34)

From Theorem 5.4 and Lemma 6.14:
\[
\text{rk} T = \text{rk} \bigoplus_{S_d \subseteq \text{Fix } G} \bigoplus_{i=0}^{r_d-2} H^{2m-2i-2}(S_d, Z),
\]
where \( r_d \) is the codimension of the component \( S_d \) of \( \text{Fix } G \). Hence, we obtain:
\[
\text{rk} T = h^2(\text{Fix } G, Z) - 2,
\]
(35)
With the same proof as Lemma 6.9, it follows:
\[
\text{discr } \tilde{\pi}_* (T) = p^{h_2(\text{Fix } G, Z) - 2}.
\]
(36)
Combining (36), (34) and (3), then applying Proposition 3.9, we have
\[
\log_p \text{discr} \tilde{T} - h^{2*}(\text{Fix} G, \mathbb{Z}) \geq -2 \left[ \sum_{j=0}^{m-1} \ell_+^{2j} + \sum_{j=0}^{m-1} \ell_-^{2j+1} \right]
\geq - \left[ \sum_{j=0}^{m-1} \ell_+^{2j} + \sum_{j=0}^{m-1} \ell_-^{2j+1} + \sum_{j=0}^{m-1} \ell_+^{2n-2j} + \sum_{j=0}^{m-1} \ell_-^{2n-2j-1} \right]
\geq -\ell_+^{2m} - \ell_-^{2m+1} + \ell_+^{2m}.
\]

Hence, using hypothesis (ii):
\[
\log_p \text{discr} \tilde{T} \geq \ell_+^{2m}.
\]

Finally, we conclude with (30) and Remark 5.3.

Remark 6.17. Same remark as Remark 6.11, we have also proved that \(\tilde{\pi}^*(s^*(H^n(X, \mathbb{Z})))\) is primitive in \(H^*(\tilde{M}, \mathbb{Z})\).

7 Examples of applications

7.1 Simply connected surfaces

Boissière, Sarti and Nieper-Wisskirchen have proved the following ([5, Proposition 4.5]):

**Proposition 7.1.** Let \(X\) be a simply connected compact surface and \(G\) an automorphism group with at least a fixed point. Then, the spectral sequence of equivariant cohomology with coefficients in \(\mathbb{F}_p\) is degenerate at the second page.

Then, from this proposition and Theorem 1.1, we obtain the following corollary.

**Corollary 7.2.** Let \(X\) be a simply connected compact Kähler surface and \(G\) an automorphism group of prime order \(p \leq 19\). Assume that \(\text{Fix} G\) is finite but not empty and contains simple points. Then, \((X,G)\) is \(H^2\)-normal.

This result can be applied for example to a K3 surface endowed with a symplectic involution.

7.2 The case of the \(K3^{[2]}\)-type manifolds

Boissière, Sarti and Nieper-Wisskirchen also proved in [5, Theorem 1.1] that for \(X\) a \(K3^{[2]}\)-type manifolds and \(G\) an automorphism group of prime order \(p \notin \{2, 5, 23\}\), the spectral sequence of equivariant cohomology with coefficients in \(\mathbb{F}_p\) is degenerate at the second page.

Together with Theorem 1.1, this implies:

**Corollary 7.3.** Let \(X\) be a \(K3^{[2]}\)-type manifolds and \(G\) an automorphism group of prime order \(p \notin \{2, 5, 23\}\). Let \(c := \text{Codim} \text{Fix} G\). Assume that:

(i) all fixed points of \(G\) are simple;

(ii) \(c \geq 3\).

Then, \((X,G)\) is \(H^2\) and \(H^4\)-normal.

We remark that \(H^*(X, \mathbb{Z})\) is torsion-free by [14] and \(H^*(\text{Fix} G, \mathbb{Z})\) is torsion-free because \(\text{Fix} G\) contains only isolated points or smooth complex curves. So, \(H^4\)-normality follows from Theorem 1.1. It remains to show the \(H^2\)-normality. To do so, we will use Proposition 3.14.

**Lemma 7.4.** Let \(X\) be a topological space. Let \(t\) and \(k\) be integers and \(p\) a prime number. Assume that \(H^*(X, \mathbb{Z})\) is \(p\)-torsion-free. We denote by \(\tau \in H^*(X, \mathbb{F}_p)\) the reduction modulo \(p\) of an element \(x \in H^*(X, \mathbb{Z})\).
If the cup product map \( \text{Sym}^t H^k(X, \mathbb{Q}) \to H^k(X, \mathbb{Q}) \) is an isomorphism and \( \frac{H^k(X, \mathbb{Z})}{\text{Sym}^t H^k(X, \mathbb{Z})} \) is \( p \)-torsion-free, then:

\[
\mathcal{F} : \text{Sym}^t H^k(X, \mathbb{F}_p) \to H^k(X, \mathbb{F}_p)
\]

\[
\mathcal{F} : \bigotimes_{1}^{t} \to \bigotimes_{1}^{t-1} \to \bigotimes_{1}^{t}
\]

is an isomorphism.

**Proof.** We prove the injectivity.

Let \( \otimes_{1}^{t} \in \text{Sym}^t H^k(X, \mathbb{F}_p) \) such that \( \otimes_{1}^{t} \cdot x_t = 0 \). Then, there exists \( y \in H^k(X, \mathbb{Z}) \) such that \( x_1 \cdot \ldots \cdot x_t = py \). Hence, \( \hat{y} \in H^k(X, \mathbb{Z})/\text{Sym}^t H^k(X, \mathbb{Z}) \) is a \( p \)-torsion element (here \( \hat{y} \) is the class of \( y \) modulo \( \text{Sym}^t H^k(X, \mathbb{Z}) \)). Hence, by the hypothesis \( \hat{y} = 0 \). It follows that \( y \in \text{Sym}^t H^k(X, \mathbb{Z}) \), so \( y = y_1 \cdot \ldots \cdot y_t \) with \( y_i \in H^k(X, \mathbb{Z}) \). Since \( \text{Sym}^t H^k(X, \mathbb{Q}) \to H^k(X, \mathbb{Q}) \) is injective, \( x_1 \otimes \ldots \otimes x_t = py_1 \otimes \ldots \otimes y_t \). So, \( \otimes_{1}^{t} \otimes \ldots \otimes \otimes_{1}^{t} = 0 \).

We prove the surjectivity.

Let \( \overline{y} \in H^k(X, \mathbb{F}_p) \), with \( y \in H^k(X, \mathbb{Z}) \). Since \( \text{Sym}^t H^k(X, \mathbb{Q}) \to H^k(X, \mathbb{Q}) \) is an isomorphism, there is \( q \in \mathbb{N} \) and \( x_1 \otimes \ldots \otimes x_t \in \text{Sym}^t H^k(X, \mathbb{Z}) \) such that \( \frac{1}{q}x_1 \cdot \ldots \cdot x_t = y \). Hence, \( \hat{y} \in H^k(X, \mathbb{Z})/\text{Sym}^t H^k(X, \mathbb{Z}) \) is a \( q \)-torsion element. But since \( H^k(X, \mathbb{Z})/\text{Sym}^t H^k(X, \mathbb{Z}) \) is \( p \)-torsion-free, \( p \) does not divide \( q \). And \( \mathcal{F}(\otimes_{1}^{t} \otimes \ldots \otimes \otimes_{1}^{t}) = \overline{y} \).

When \( X \) is a \( K3^{[2]} \)-type manifold, we know from [29] that \( \text{Sym}^2 H^2(X, \mathbb{Q}) \to H^4(X, \mathbb{Q}) \) is an isomorphism. Moreover, from [5, Proposition 6.6], we know that:

\[
H^4(X, \mathbb{Z})/\text{Sym}^2 H^2(X, \mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^3 \oplus (\mathbb{Z}/5\mathbb{Z}).
\]

It follows from the previous lemma that for \( p \notin \{2, 5\}, \)

\[
\text{Sym}^2 H^2(X, \mathbb{F}_p) \simeq H^4(X, \mathbb{F}_p).
\]

So, by Proposition 3.14, we obtain the following corollary which finishes the proof of Corollary 7.3.

**Corollary 7.5.** Let \( X \) be a \( K3^{[2]} \)-type manifold and \( G \) an automorphism group of prime order \( p \notin \{2, 5\} \). If \( (X, G) \) is \( H^2 \)-normal, \( (X, G) \) is \( H^2 \)-normal.

We propose the following example as an application of Corollary 7.3. First, we recall that an automorphism on a Hilbert scheme of points \( S^{[n]} \) on a K3 surface \( S \) is called natural if it is the automorphism induced by an automorphism on the K3 surface \( S \).

**Example 7.6.** Let \( X \) be a Hilbert scheme of 2 points on a K3 surface and \( G \) a natural non-symplectic automorphism group of order 3 on \( X \). Then, \( (X, G) \) is \( H^4 \) and \( H^2 \)-normal.

**Proof.** Let \( G \) be such an isomorphism group on \( S^{[2]} \). Since the action on the K3 surface is non-symplectic, the fixed points will be simple. Moreover, from [4, Section 6.1], Fix \( G \) consists in 3 isolated points and 3 rational curves.

**Remark 7.7.** An analogous of Corollary 7.3 could also be stated when \( p = 2 \) or \( 5 \) and \( X \) is a Hilbert scheme of 2 points on a K3 surface and \( G \) a natural automorphism group using statement (2) of [5, Theorem 1.1].

7.3 Integral basis of the Hilbert scheme of two points on a surface

In this section, we complete the work started in [13, Section 7]. Let \( A \) be a smooth compact Kähler surface with \( H^*(A, \mathbb{Z}) \) 2-torsion-free and \( A^{[2]} \) the Hilbert scheme of two points. It can be constructed as follows: Consider the direct product \( A \times A \). Denote

\[
b : \text{Bl}_A(A \times A) \to A \times A
\]
the blow-up along the diagonal $h : \Delta \simeq A$ with exceptional divisor $E$ (for simplicity we denote by $x$ the pull-back $h^*(x)$ for all $x \in H^*(A, \mathbb{Z})$). Let $f$ be a fiber of the map $E \to \Delta$. Let $j : E \hookrightarrow \text{Bl}_\Delta(A \times A)$, $g : \Delta \to A \times A$ be the embeddings and $p_1 : A \times A \to A$, $p_2 : A \times A \to A$ the projections. We also denote by $A$ the class which generates $H^0(A, \mathbb{Z})$ and by $pt$ the class which generates $H^0(A, \mathbb{Z})$. We have the following commutative diagram:

$$
\begin{array}{ccc}
E & \xrightarrow{j} & \text{Bl}_\Delta(A \times A) \\
| & & | \\
\Delta & \xrightarrow{g} & A \times A \\
| & & | \\
A & \xrightarrow{p_1} & A \\
| & & | \\
& \xrightarrow{p_2} & A
\end{array}
$$

The action of $S_2$ extends to an action on $\text{Bl}_\Delta(A \times A)$. Then, $A^{[2]} = \text{Bl}_\Delta(A \times A)/S_2$ and we denote by $\pi : \text{Bl}_\Delta(A \times A) \to A^{[2]}$ the quotient map. Moreover, from [27, Theorem 2.2], we know that $H^*(A^{[2]}, \mathbb{Z})$ is 2-torsion-free. We want to prove the following proposition:

**Proposition 7.8.**

(i) $H^1(A^{[2]}, Z) = \pi_*(b^*(H^1(A \times A, Z)))$,

(ii) $H^2(A^{[2]}, Z) = \pi_*(b^*(H^2(A \times A, Z))) \oplus \mathbb{Z} \pi_*(E)$,

(iii) $H^3(A^{[2]}, Z) = \pi_*(b^*(H^3(A \times A, Z))) \oplus \pi_*(j)_*b^*_E(H^1(\Delta, Z))$,

(iv) The lattices $\pi_*(b^*(H^4(A \times A, Z)))$ and $\pi_*(j)_*b^*_E(H^2(\Delta, Z))$ are primitive in $H^4(A^{[2]}, \mathbb{Z})$. Moreover:

$$Q := \frac{H^4(A^{[2]}, \mathbb{Z})}{\pi_*(b^*(H^4(A \times A, Z))) \oplus \pi_*(j)_*b^*_E(H^2(\Delta, Z))} = (\mathbb{Z}/2\mathbb{Z})^{\pi_*(A)}$$

and there exists an isometry $\theta$ of $H^2(A, \mathbb{Z})$ such that the classes \(\frac{\pi_*(j)_*b^*_E(\theta(x)) + \pi_*(b^*(x \otimes x))}{2}\) generate $Q$.

(v) $H^5(A^{[2]}, Z) = \pi_*(b^*(H^5(A \times A, Z))) \oplus \frac{1}{2}\pi_*(j)_*b^*_E(H^3(\Delta, Z))$,

(vi) $H^6(A^{[2]}, Z) = \pi_*(b^*(H^6(A \times A, Z))) \oplus \pi_*(f)$,

(vii) $H^7(A^{[2]}, Z) = \pi_*(b^*(H^7(A \times A, Z)))$.

This section is dedicated to the proof of this proposition. Statements (iii) and (v) were proven in [13, Proposition 7.1]. Before proving the remaining statements, we will see that they are also interesting because they provide the ring structure of $H^*(A^{[2]}, \mathbb{Z})$.

**Lemma 7.9.**

(i) For all $x_1, x_2, y_1, y_2$ in $H^*(A, \mathbb{Z})$:

$$\pi_*b^*(x_1 \otimes y_1) \cdot (\pi_*b^*(x_2 \otimes y_2)) = \pi_*b^*(x_1 \cdot x_2) \otimes (y_1 \cdot y_2) + (x_1 \cdot y_2) \otimes (y_1 \cdot x_2) .$$

(ii) For all $x, y, z$ in $H^*(A, \mathbb{Z})$:

$$\pi_*b^*(x \otimes y) \cdot (\pi_*(j)_*b^*_E(z)) = 2\pi_*(j)_*b^*_E(x \cdot y \cdot z) .$$

(iii) $\left(\frac{1}{2}\pi_*(j)_*(E)\right)^2 = \frac{1}{2} \left(\pi_*(j)_*b^*_E(c_1(A)) - \pi_*b^*(\Delta)\right)$.

(iv) For all $x, y$ in $H^*(A, \mathbb{Z})$:

$$\left(\pi_*(j)_*b^*_E(x)\right) \cdot (\pi_*(j)_*b^*_E(y)) = 2 \left(\pi_*(j)_*b^*_E(x \cdot y \cdot c_1(A)) - \pi_*b^*g_*c_1(x \cdot y)\right) .$$
Proof. (i) Using Lemma 3.6 (ii), we have:

\[
(\pi_* b^*(x_1 \otimes y_1)) \cdot (\pi_* b^*(x_2 \otimes y_2)) = \frac{1}{4} (\pi_* b^*(x_1 \otimes y_1 + y_1 \otimes x_1)) \cdot (\pi_* b^*(x_2 \otimes y_2 + y_2 \otimes x_2)) \\
= \frac{1}{2} \pi_* ((b^*(x_1 \otimes y_1 + y_1 \otimes x_1)) \cdot (b^*(x_2 \otimes y_2 + y_2 \otimes x_2))) \\
= \frac{1}{2} \pi_* b^* ((x_1 \otimes y_1 + y_1 \otimes x_1) \cdot (x_2 \otimes y_2 + y_2 \otimes x_2)) \\
= \pi_* b^* ((x_1 \cdot x_2) \otimes (y_1 \cdot y_2) + (x_1 \cdot y_2) \otimes (y_1 \cdot x_2)).
\]

(ii) Using Lemma 3.6 (ii), then the projection formula and finally the commutativity of digram (38), we have:

\[
(\pi_* b^*(x \otimes y)) \cdot (\pi_* j_* b^*_{E}(z)) = \frac{1}{2} (\pi_* b^*(x \otimes y \otimes x)) \cdot (\pi_* j_* b^*_{E}(z)) \\
= \pi_* ((b^*(x \otimes y \otimes x)) \cdot (j_* b^*_{E}(z))) \\
= \pi_* j_* (j^* b^*(x \otimes y \otimes x)) \cdot (b^*_{E}(z)) \\
= \pi_* j_* b^*_{E} (g^* (x \otimes y \otimes x) \cdot z).
\]

By definition: \(x \otimes y = p_1^*(x) \cdot p_2^*(y)\). It follows: \(g^*(x \otimes y) = h^*(x \cdot y)\) and by convention, we do not write the \(h^*\). So:

\[
(\pi_* b^*(x \otimes y)) \cdot (\pi_* j_* b^*_{E}(z)) = 2 \pi_* j_* b^*_{E} (x \cdot y \cdot z).
\]

(iii) By [10, Proposition 6.7], we have:

\[
b^*(\Delta) = j_*(c_1(\mathcal{F})),
\]

where \(\mathcal{F} := b^*_{E}(\mathcal{M}_{\Delta/A \otimes A})/\mathcal{M}_{E/BL(\Delta/A \otimes A)}\). Moreover, \(j_*(\mathcal{M}_{E/BL(\Delta/A \otimes A)}) = \pi_* (E)^2\); it follows:

\[
b^*(\Delta) = j_*(b^*_{E}(c_1(\mathcal{M}_{\Delta/A \otimes A}))) - j_*(E)^2.
\]

Furthermore, we remark that

\[
c_1(\mathcal{M}_{\Delta/A \otimes A}) = c_1 \left( \frac{T_{A \otimes A | \Delta}}{T_{\Delta}} \right) = c_1(T_{\Delta}) = h^* c_1(T_A),
\]

So:

\[
j_*(E)^2 = j_*(b^*_{E}(c_1(A))) - b^*(\Delta).
\]

Then, the result follows from Lemma 3.6 (ii).

(iv) By (ii) and then by (iii), (i):

\[
\left( \pi_* j_* b^*_{E}(x) \right) \cdot \left( \pi_* j_* b^*_{E}(y) \right) = \frac{1}{4} (\pi_* b^*(x \otimes 1)) \cdot (\pi_* j_* (E))^2 \cdot (\pi_* b^*(y \otimes 1)) \\
= \frac{1}{2} (\pi_* b^*(x \otimes y \otimes 1) + x \otimes y) \cdot \left( \pi_* j_* (b^*_{E}(c_1(A))) - \pi_* b^*(\Delta) \right).
\]

Then, again by (ii):

\[
\left( \pi_* j_* b^*_{E}(x) \right) \cdot \left( \pi_* j_* b^*_{E}(y) \right) = \frac{1}{2} \left( 4 \pi_* j_* b^*_{E}(x \cdot y \cdot c_1(A)) - \pi_* b^*(x \otimes 1 + x \otimes y) \cdot \pi_* b^*(\Delta) \right).
\]

Now, we calculate \(\pi_* b^*(x \cdot y \otimes 1 + x \otimes y) \cdot \pi_* b^*(\Delta)\) separately:

\[
\pi_* b^*(x \cdot y \otimes 1 + x \otimes y) \cdot \pi_* b^*(\Delta) = \frac{1}{2} \pi_* b^*(x \cdot y \otimes 1 + 1 \otimes (x \cdot y) + x \otimes y + y \otimes x) \cdot \pi_* b^*(\Delta).
\]
By Lemma 3.6 (ii), then by projection formula and finally by the definition $x \otimes y = p_1^*(x) \cdot p_2^*(y)$, we have:

$$\pi_* b^*((x \otimes y) \otimes 1 + x \otimes y) \cdot \pi_* b^*(\Delta) = \pi_* b^*([(x \otimes y) \otimes 1 + 1 \otimes (x \otimes y) + x \otimes y + y \otimes x] \cdot \Delta)$$

$$= \pi_* b^* g_*(x \otimes y) \cdot \pi_* b^*(x \otimes y) + \pi_* b^* g_*(x \otimes y) \cdot \pi_* b^*(x \otimes y) + \pi_* b^* g_*(x \otimes y) \cdot \pi_* b^*(x \otimes y)$$

$$= 4\pi_* b^* g_*(x \otimes y) \cdot \pi_* b^*(x \otimes y).$$

So, by (39):

$$\left(\pi_* j_* b_{1E}(x)\right) \cdot \left(\pi_* j_* b_{1E}(y)\right) = 1 \left(4\pi_* j_* b_{1E}(x \cdot y \cdot c_1(A)) - 4\pi_* b^* g_*(x \cdot y)\right).$$

Now we start the proof of Proposition 7.8, we first prove (i) and (vii). By Theorem 5.4, we have:

$$H^k(B\Delta(A \times A), \mathbb{Z}) = b^*(H^k(A \times A, \mathbb{Z})),$$

for $k = 1$ and $k = 7$. It follows from Künneth formula that $\ell^1_+(B\Delta(A \times A)) = 0$. Hence, from Proposition 3.10, the coefficient of surjectivity $\alpha_1(B\Delta(A \times A)) = \alpha_7(B\Delta(A \times A)) = 0$. It follows (i) and (vii).

Now let us prove (ii) and (vi). By Theorem 5.4, we have:

$$H^2(B\Delta(A \times A), \mathbb{Z}) = b^*(H^2(A \times A, \mathbb{Z})) \oplus j_* b_{1E}(H^2(\Delta, \mathbb{Z})).$$

We have $j_* b_{1E}(H^2(\Delta, \mathbb{Z})) = Z E$. From Künneth formula, we remark that $\ell^1_+(B\Delta(A \times A)) = 1$. Hence, from Proposition 3.10:

$$\alpha_2(B\Delta(A \times A)) + \alpha_6(B\Delta(A \times A)) = 1.$$

Moreover, we have $\pi_* (\mathcal{O}_{B\Delta(A \times A)}) = \mathcal{O}_{A[\Delta]} \oplus \mathcal{L}$, with $\mathcal{L}^2 = \mathcal{O}_{A[\Delta]}(\pi_*(E))$. So, as it is well known $\pi_*(E)$ is divisible by 2. Hence, $\alpha_2(B\Delta(A \times A)) = 1$ and $\alpha_6(B\Delta(A \times A)) = 0$. So, (ii) follows from (40). Similarly, by Theorem 5.4, we have:

$$H^6(B\Delta(A \times A), \mathbb{Z}) = b^*(H^6(A \times A, \mathbb{Z})) \oplus j_* b_{1E}(H^6(\Delta, \mathbb{Z})),$$

with $j_* b_{1E}(H^6(\Delta, \mathbb{Z})) = Z f$. So, we obtain (vi) because $\alpha_6(B\Delta(A \times A)) = 0$.

It remains to study the most interesting part which is $H^4(A[2], \mathbb{Z})$. Again by Theorem 5.4, we have:

$$H^4(B\Delta(A \times A), \mathbb{Z}) = b^*(H^4(A \times A, \mathbb{Z})) \oplus j_* b_{1E}(H^2(\Delta, \mathbb{Z})).$$

(41)

First, we are going to use Remark 6.17 to show that $\pi_* (b^*(H^4(A \times A, \mathbb{Z})))$ is primitive in $H^4(A[2], \mathbb{Z})$. To do so, we verify the hypothesis of Theorem 6.12 for the couple $(A \times A, \mathcal{L}_2)$. By Künneth formula, we have $\ell^k_+(A \times A) = 0$ for all $k \in \{0, 1, 2, 3\}$, $\ell^k_+(A \times A) = b_2(A)$ and $\ell^k_+(A \times A) = 0$. It follows condition (ii) of Theorem 6.12. Moreover, $\Delta \in H^4(A \times A, \mathbb{Z})$ is primitive because $\Delta \cdot \mathcal{A}[\Delta] = 1$. So, by Theorem 6.12, $(A \times A, \mathcal{L}_2)$ is $H^*$-normal and by Remark 6.17 the lattice $\pi_* (b^*(H^4(A \times A, \mathbb{Z})))$ is primitive in $H^4(A[2], \mathbb{Z})$. Moreover, by Corollary 3.8, $\log_2 \text{disc} \pi_* (b^*(H^4(A \times A, \mathbb{Z}))) = b_2(A)$ and by Lemma 3.6 (ii), $\log_2 \text{disc}(\pi_* j_* b_{1E}(H^2(\Delta, \mathbb{Z})))$ = $b_2(A)$. So:

$$\text{disc} \pi_* (b^*(H^4(A \times A, \mathbb{Z}))) = \text{disc} \pi_* j_* b_{1E}(H^2(\Delta, \mathbb{Z})).$$

(42)

Since $A[2]$ is smooth, we know by (4) that $\text{disc} \pi_* (b^*(H^4(A \times A, \mathbb{Z}))) = \text{disc} \pi_* (b^*(H^4(A \times A, \mathbb{Z})))^\perp$.

Since $\pi_* j_* b_{1E}(H^2(\Delta, \mathbb{Z})) \subset \pi_* (b^*(H^4(A \times A, \mathbb{Z})))^\perp$, by (42), $\pi_* j_* b_{1E}(H^2(\Delta, \mathbb{Z})) = \pi_* (b^*(H^4(A \times A, \mathbb{Z})))^\perp$. So, $\pi_* j_* b_{1E}(H^2(\Delta, \mathbb{Z}))$ is also primitive in $H^4(A[2], \mathbb{Z})$.

Now, we are going to calculate

$$H^4(A[2], \mathbb{Z}) \oplus \pi_* j_* b_{1E}(H^2(\Delta, \mathbb{Z})).$$

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For a lattice $L$, we denote by $A_L := L^\perp/L$ its discriminant group and for $x \in L^\perp$, we denote by $\overline{x}$ its reduction in $A_L$. Let denote $N := \pi_* (b^*(H^4(A \times A, \mathbb{Z})))$ and $N^\perp := \pi_* j_* b_{E}^*(H^2(\Delta, \mathbb{Z}))$. By Lemma 3.6 (ii), $\frac{1}{2} N^\perp = (N^\perp)^\perp$. It follows that:

$$A_{N^\perp} \simeq \frac{1}{2} N^\perp = (\mathbb{Z}/2\mathbb{Z})^{b_{N}(A)}.$$ (43)

Since $H^4(A^{[2]}, \mathbb{Z})$ is unimodular, by (6), we have an isometry between the discriminant groups:

$$\nu : A_N \cong A_{N^\perp}.$$  

Again by Lemma 3.6 (ii), $\frac{1}{2} \pi_* b^*(x \otimes x) \in N^\perp$, for all $x \in H^2(A, \mathbb{Z})$. So, $A_N$ is generated by the classes $\frac{1}{2} \pi_* b^*(x \otimes x)$, where $x \in H^2(A, \mathbb{Z})$. Let $\{ z(x) \in N^\perp \mid x \in H^2(A, \mathbb{Z}) \}$ be a basis of $N^\perp$ such that $\nu \left( \frac{\pi_* b^*(x \otimes x)}{2} \right) = \frac{z(x)}{2}$. Since $\nu$ is an isometry, we have:

$$\frac{\pi_* b^*(x \otimes x)}{2} \cdot \frac{\pi_* b^*(y \otimes y)}{2} = \frac{z(x)}{2} \cdot \frac{z(y)}{2} \mod \mathbb{Z}.$$ (44)

However, by Lemma 7.9 (i):

$$\pi_* b^*(x \otimes x) \cdot \pi_* b^*(y \otimes y) = 2(x \cdot y)^2.$$  

And by Lemma 7.9 (iv):

$$\pi_* j_* b_{E}^*(x) \cdot \pi_* j_* b_{E}^*(y) = 2x \cdot y.$$  

Since $(x \cdot y) = (x \cdot y)^2 \mod 2$, it follows from (44) that:

$$\frac{z(x)}{2} \cdot \frac{z(y)}{2} = \frac{\pi_* j_* b_{E}^*(x)}{2} \cdot \frac{\pi_* j_* b_{E}^*(y)}{2} \mod \mathbb{Z}.$$  

So, we have constructed an isometry of $A_{N^\perp}$ sending $\frac{z(x)}{2}$ to $\frac{\pi_* j_* b_{E}(x)}{2}$. However, [23, Theorem 3.6.3] says that the natural map $O(N^\perp) \to O(A_{N^\perp})$ is surjective when $N^\perp$ is an indefinite lattice which is always the case apart when $A$ is the projective space; but in this cases, $\text{rk} \: N^\perp = 1$ and the natural map $O(N^\perp) \to O(A_{N^\perp})$ is obviously surjective.

It follows that there is an isometry $\tilde{\theta}$ of $N^\perp$ such that:

$$\tilde{\theta}(\pi_* j_* b_{E}^*(x)) = z(x) \mod 2N^\perp.$$  

Then, $\tilde{\theta}$ provides an isometry $\theta$ on $H^2(A, \mathbb{Z})$ such that:

$$\pi_* j_* b_{E}^*(\theta(x)) = z(x) \mod 2N^\perp.$$  

Then, by definition of $\nu$, the elements

$$\pi_* j_* b_{E}(\theta(x)) + \pi_* b^*(x \otimes x)$$

are divisible by 2 for all $x \in H^2(A, \mathbb{Z})$.

### 7.4 Symplectic groups and Beauville–Bogomolov forms

We will study the Beauville–Bogomolov form of some quotients of an irreducible symplectic $n$-fold $X$ by a symplectic group $G$ of prime order. So, $M := X/G$ will be a singular irreducible symplectic $V$-manifold. One of our most important ingredients will be the Fujiki formula generalized by Matsushita in [16] (see [18, Section 1.2.2] for an overview). Assume that $\text{Codim} \: \text{Sing} \: M \geq 4$, then if $Y = M$ or $X$, we have for all $\alpha \in H^2(Y, \mathbb{Z})$:

$$\alpha^{2n} = c_Y B_Y (\alpha, \alpha)^n,$$ (45)
where $c_Y$ is the Fujiki constant. Moreover, if $\omega \neq 0$ is the holomorphic 2-form of $Y$, the convention is:

$$B_Y(\omega + \overline{\omega}, \omega + \overline{\omega}) > 0.$$  \hfill (46)

Furthermore, the convention is to choose the Beauville–Bogomolov form to be integral and indivisible.

We know the Beauville–Bogomolov form $B_X$ of $X$. Then, we will deduce the Beauville–Bogomolov form $B_M$ of $M$ from $B_X$ and $H^2(M, \mathbb{Z})$. The link between $B_X$ and $B_M$ is the matter of the following proposition.

**Proposition 7.10.** Let $X$ be an irreducible symplectic manifold of dimension $2n$, $G$ a symplectic group of prime order $p$ with $\text{Codim Fix} G \geq 4$. Then,

$$B_M(\pi_*(\alpha), \pi_*(\beta)) = \sqrt{\frac{c_X p^{2n-1}}{c_M}} B_X(\alpha, \beta),$$

where $c_M$ is the Fujiki constant of $M$ and $\alpha, \beta$ are in $H^2(X, \mathbb{Z})^G$.

**Proof.** By (45), we have:

$$(\pi_*(\alpha))^{2n} = c_M B_M(\pi_*(\alpha), \pi_*(\alpha))^n,$$ \hfill (47)

and:

$$\alpha^{2n} = c_X B_X(\alpha, \alpha)^n.$$ \hfill (48)

Moreover, by Lemma 3.6 (ii):

$$(\pi_*(\alpha))^{2n} = p^{2n-1} \alpha^{2n}. $$ \hfill (49)

So, by (47), (48), (49), we have:

$$B_M(\pi_*(\alpha), \pi_*(\beta))^n = \frac{c_X p^{2n-1} B_X(\alpha, \alpha)^n}{c_M}. $$

By (46), we get the result. \hfill $\square$

When we assume that $X$ is a manifold of $K3^{[2]}$-type, from [3, Section 9], we know that the Fujiki constant of $X$ is $3$. So for $\alpha, \beta$ in $H^2(X, \mathbb{Z})^G$:

$$B_M(\pi_*(\alpha), \pi_*(\beta)) = \sqrt{\frac{3 p^3}{c_M}} B_X(\alpha, \beta). $$ \hfill (50)

We need an adaptation of Lemma 2.11 in the case of the lattice of $H^2(X, \mathbb{Z})$ endowed with the Beauville–Bogomolov form.

**Lemma 7.11.** Let $X$ be an irreducible symplectic manifold of $K3^{[2]}$-type and $G = \langle g \rangle$ a symplectic automorphism group of order $3 \leq p \leq 19$. We have:

(i) $A_{H^2(X, \mathbb{Z})^G} = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/p\mathbb{Z})^{G}(X),$

(ii) $\text{discr } H^2(X, \mathbb{Z})^G = 2 p^{G}(X).$

(iii) We denote $A_{H^2(X, \mathbb{Z})^{G,p}} := (\mathbb{Z}/p\mathbb{Z})^{G}(X)$. Then, the projection

$$\frac{H^2(X, \mathbb{Z})^G \oplus \ker \sigma}{H^2(X, \mathbb{Z})^{G,p}} \rightarrow A_{H^2(X, \mathbb{Z})^{G,p}}$$

is an isomorphism, where $\sigma = \text{id} + g + \ldots + g^{p-1}$.

(iv) Moreover, let $x \in H^2(X, \mathbb{Z})^G$. We have $\frac{x}{p} \in (H^2(X, \mathbb{Z})^G)^\vee$ if and only if there is $z \in H^2(X, \mathbb{Z})$ such that $x = z + g(z) + \ldots + g^{p-1}(z)$.

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(v) Also:

\[
\frac{\pi_*(H^2(X,\mathbb{Z}))}{\pi_*(H^2(X,\mathbb{Z})^G)} = (\mathbb{Z}/p\mathbb{Z})^{\ell^2_p(X)}.
\]

**Proof.** The first three assumptions follow from [5, Lemma 6.5] and its proof (the number \(\ell^2_p(X)\) in [5, Lemma 6.5] equal \(\ell^2_p(X)\) from [5, Corollary 5.8]). Then, the proof of (iv) is identical to the proof of Lemma 2.11.

It remains to prove (v). Let

\[
A := \left\{ \frac{z}{p} \in (H^2(X,\mathbb{Z})^G) \mid z \in H^2(X,\mathbb{Z}) \right\}.
\]

By (iv):

\[
A = \left\{ \frac{x}{p} \in (H^2(X,\mathbb{Z})^G) \mid x \in H^2(X,\mathbb{Z}) \right\}.
\]

So:

\[
\frac{H^2(X,\mathbb{Z})^G}{\ell^2_p(X)} = A_{H^2(X,\mathbb{Z})^G,p}.
\]

The following morphism

\[
f : A \rightarrow \frac{\pi_*(H^2(X,\mathbb{Z}))}{\pi_*(H^2(X,\mathbb{Z})^G)}
\]

is surjective by definition. Quotiented \(A\) by \((H^2(X,\mathbb{Z})^G)\), \(f\) provides an isomorphism between \(A_{H^2(X,\mathbb{Z})^G,p}\) and \(\pi_*(H^2(X,\mathbb{Z})) / \pi_*(H^2(X,\mathbb{Z})^G)\). Then, the result follows (iii). \(\square\)

### 7.5 Quotient of a \(K^{[3]}\)-type manifold by an automorphism of order 11

As mentioned in the introduction, there are two different examples of symplectic automorphisms of order 11 on a manifold of \(K^{[3]}\)-type (Example 4.5.1 and Example 4.5.2 in [19]). In both examples, the fixed locus is a set of 5 isolated points. But as it is explained in [19, Section 7.4.4], the lattice \(H^2(X,\mathbb{Z})^G\) will be different in these cases. In the first case the lattice is \(\left[\begin{array}{ccc} 6 & 2 & 2 \\ 2 & 8 & -3 \\ 2 & -3 & 8 \end{array}\right]\), and in the second \(\left[\begin{array}{ccc} 2 & 1 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 22 \end{array}\right]\). Moreover, it is proceed in [19, Section 7.4.4] that these are the only possible invariant lattices for \(p = 11\). So, let \(X\) be a manifold of \(K^{[3]}\)-type and \(G\) a symplectic automorphism group of \(X\) of order 11. If \(H^2(X,\mathbb{Z})^G\) is isomorphic to \(\left[\begin{array}{ccc} 6 & 2 & 2 \\ 2 & 8 & -3 \\ 2 & -3 & 8 \end{array}\right]\), we will denote the quotient \(X/G\) by \(M^1_{11}\), and if \(H^2(X,\mathbb{Z})^G\) is isomorphic to \(\left[\begin{array}{ccc} 2 & 1 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 22 \end{array}\right]\), we will denote the quotient \(X/G\) by \(M^2_{11}\). Hence, we obtain the following corollary.

**Corollary 7.12.** Let \(X\) be a manifold of \(K^{[3]}\)-type. Let \(G\) be a symplectic automorphism group of \(X\) of order 11. Then, \((X,G)\) is \(H^2\)-normal and \(H^4\)-normal.

**Proof.** In the both cases, we have \(\text{discr} H^2(X,\mathbb{Z})^G = 2 \times 11^2\). Hence, by Lemma 7.11 (ii):

\[
\ell^2_{11}(X) = 2.
\]

It follows from Corollary 2.7 that

\[
\ell^2_x(X) = 1 \text{ and } \ell^2_x(X) = 0.
\]

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We recall from Section 2.2, that when \( p \geq 3 \), we have \( \ell^k_p(X) = \ell^k_1(X) \) and \( \ell^k_p(X) = \ell^{k-1}_p(X) \) for all \( k \). Hence, by (37) and [5, Lemma 6.14]:

\[
\ell^4_1(X) = 1.
\]

So, it follows from Remark 3.11 that:

\[
\alpha_4(X) \leq \frac{1}{2}.
\]

Since \( \alpha_4(X) \) is an integer, we have \( \alpha_4(X) = 0 \). So, \((X,G)\) is \( H^4 \)-normal. Therefore, the \( H^2 \)-normality follows from Corollary 7.5.

The end of the section is dedicated to the proof of Theorem 1.2.

Assume

\[
H^2(X,\mathbb{Z})^G \cong \begin{pmatrix} 6 & 2 & 2 \\ 2 & 8 & -3 \\ 2 & -3 & 8 \end{pmatrix}.
\]

We denote by \( \{a,b,c\} \) an integral basis of \( H^2(X,\mathbb{Z})^G \) with \( B_X(a,a) = 6, B_X(b,b) = 8, B_X(c,c) = 8, B_X(a,b) = B_X(a,c) = 2 \) and \( B_X(b,c) = -3 \). Let

\[
B = \left\{ \frac{\pi_*(b) - \pi_*(c)}{11}, \frac{4\pi_*(a) - \pi_*(b)}{11}, \frac{\pi_*(a) - 3\pi_*(c)}{11} \right\}.
\]

We show that \( B \) is a basis of \( H^2(M_{11}^1,\mathbb{Z}) \). By Lemma 7.11 (iv), \( B \subseteq \pi_*(H^2(X,\mathbb{Z})) \). Moreover, we have

\[
\langle B \rangle = \pi_*(H^2(X,\mathbb{Z})) = (\mathbb{Z}/11\mathbb{Z})^2.
\]

Hence, by Lemma 7.11 (v) and (51):

\[
\langle B \rangle = \pi_*(H^2(X,\mathbb{Z})).
\]

It follows from Corollary 7.12 that \( B \) is a basis of \( H^2(M_{11}^1,\mathbb{Z}) \).

The matrix of the sublattice of \( H^2(X,\mathbb{Z})^G \) generated by \( b-c, 4a-b \) and \( a-3c \) is

\[
\begin{pmatrix}
2 \times 11 & -11 & 3 \times 11 \\
-11 & 8 \times 11 & -11 \\
3 \times 11 & -11 & 6 \times 11
\end{pmatrix}.
\]

Then, by (50), the Beauville–Bogomolov form on \( H^2(M_{11}^1,\mathbb{Z}) \) gives the lattice

\[
\frac{1}{111} \sqrt{3 - 11^3 c_{M_{11}^1}} \begin{pmatrix} 2 & -1 & 3 \\ -1 & 8 & -1 \\ 3 & -1 & 6 \end{pmatrix}.
\]

Since the Beauville–Bogomolov form is integral and indivisible, it follows that \( c_{M_{11}^1} = 33 \), and we get the lattice

\[
\begin{pmatrix} 2 & -1 & 3 \\ -1 & 8 & -1 \\ 3 & -1 & 6 \end{pmatrix}.
\]

Now assume that

\[
H^2(X,\mathbb{Z})^G \cong \begin{pmatrix} 2 & 1 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 22 \end{pmatrix}.
\]

The proof is identical taking the basis

\[
B = \left\{ \frac{\pi_*(a) - 2\pi_*(b)}{11}, \frac{6\pi_*(a) - \pi_*(b)}{11}, \frac{\pi_*(c)}{11} \right\},
\]

where \( \{a,b,c\} \) is an integral basis of \( H^2(X,\mathbb{Z})^G \) with \( B_X(a,a) = 2, B_X(b,b) = 6, B_X(c,c) = 22, B_X(b,c) = B_X(a,c) = 0 \) and \( B_X(a,b) = 1 \).
7.6 Quotient of a $K3^{[2]}$-type manifold by a symplectic automorphisms of order 3

As mentioned in the introduction, in [19, Theorem 7.2.7], Mongardi distinguishes two kinds of symplectic automorphisms of order 3 on a $K3^{[2]}$-type manifold one with 27 isolated fixed points and another with a fixed abelian surface. We recall from [19, Example 4.2.1]:

Remark 7.13. Let $S$ be a $K3$ surface. A natural symplectic automorphism of order 3 on $S^{[2]}$ has 27 isolated fixed points.

Moreover, Mongardi in [20, Theorem 2.5] shows the following result:

Proposition 7.14. Let $X$ be a $K3^{[2]}$-type manifold and $\varphi$ a symplectic automorphism of order 3 with 27 isolated fixed points. Then, $(X, \varphi)$ can be deformed to a couple $(S^{[2]}, \phi^{[2]})$, where $S$ is a $K3$ surface and $\phi^{[2]}$ is the automorphism induced on $S^{[2]}$ by a symplectic automorphism of order 3 on $S$.

In this section, we are considering these symplectic automorphisms $\varphi$ of order 3, with 27 isolated points, on a $K3^{[2]}$-type manifold $X$, and we denote the quotient $X/\varphi$ by $M_3$.

Corollary 7.15. Let $X$ be a manifold of $K3^{[2]}$-type. Let $G$ be a symplectic automorphisms group of order 3 of $X$ with 27 isolated fixed points. Then, $(X, G)$ is $H^2$-normal and $H^4$-normal.

Proof. By Corollary 7.5, if $(X, G)$ is $H^4$-normal then $(X, G)$ is $H^2$-normal. Hence, we have only to show the $H^2$-normality.

The action of $G$ is symplectic on $X$, in this case, the only possible local action around the fixed points is given by:

$$\text{diag}(\eta_3, \eta_3^2, \eta_3^3),$$

where $\eta_3$ is a 3-root of the unity. So, unfortunately, the fixed points of $G$ are not simple. However, since $G$ is of order 3, we can get around this difficulty using Proposition 5.10.

Let $X_1$ be the blow-up of $X$ in $\text{Fix} G$ and $G_1$ the automorphisms group induced from $G$. From Lemma 5.6 the fixed points of $G_1$ are simple. Hence, it will be possible to apply Theorem 6.1 to $(X_1, G_1)$.

First, using the same method as in the proof of Lemma 5.6, we remark that for each isolated fixed point of $G$, we get 2 rational fixed lines of $G_1$. So, $\text{Fix} G_1$ consists in 54 rational lines. Moreover, from [14], $H^*(X, \mathbb{Z})$ is torsion-free, then from Theorem 5.4, $H^*(X_1, \mathbb{Z})$ is also torsion-free. Hence, it only remains to verify the condition (ii) of Theorem 6.1.

From Theorem 5.4:

$$H^i(X_1, \mathbb{Z}) \cong H^i(X, \mathbb{Z}) \oplus H^0(pt, \mathbb{Z})^{27} \oplus H^j(X_1, \mathbb{Z}) \cong H^j(X, \mathbb{Z}),$$

for $i \in \{2, 4, 6\}$ and $j \in \{0, 1, 3, 5, 7, 8\}$. Since the exceptional divisors are fixed by the action of $G_1$, it follows that:

$$\ell_q^i(X_1) = \ell_q^i(X) + 27, \quad (52)$$

for all $i \in \{2, 4, 6\}$. The others $\ell_q^i$ are the same for $X$ and $X_1$. Moreover, from [5, Theorem 1.1] and Proposition 6.2:

$$h^*(\text{Fix} G, \mathbb{Q}) = \sum_{1 \leq q < p} \ell_q^i(X).$$

Since $\text{Fix} G$ consists in 27 isolated points, we get:

$$27 = \sum_{1 \leq q < p} \ell_q^i(X). \quad (53)$$

By (52):

$$\sum_{1 \leq q < p} \ell_q^i(X_1) = \sum_{1 \leq q < p} \ell_q^i(X) + 3 \times 27. \quad (54)$$

Since $\text{Fix} G_1$ consists in 54 rational curves, we have:

$$h^*(\text{Fix} G_1, \mathbb{Q}) = 2 \times 54 = 4 \times 27. \quad (55)$$
Finally, from (53), (54) and (55), we get:

\[ h^*(\text{Fix} G_1, \mathbb{Q}) = \sum_{1 \leq q < p} \ell_q^*(X_1). \]

It follows from Theorem 6.1 that \((X_1, G_1)\) is \(H^4\)-normal. We conclude with Proposition 3.17.

The end of the section is dedicated to the proof of Theorem 1.3.

From [11, Theorem 4.1] and Proposition 7.14, there is an isometry of lattices \(H^2(X, \mathbb{Z})^G \cong U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2 \oplus (-2)\). In the rest of the proof, we identify \(H^2(X, \mathbb{Z})^G\) with the lattice \(U \oplus U(3) \oplus U(3) \oplus A_2 \oplus A_2 \oplus (-2)\).

By Lemma 7.11 (iv), we have:

\[ \frac{1}{3} \pi_*(U(3)) \subset H^2(M_3, \mathbb{Z}). \] (56)

Let \((a, b)\) an integral basis of \(A_2\), with \(B_X(a, a) = B_X(b, b) = -2\) and \(B_X(a, b) = 1\). Idem, by Lemma 7.11 (iv), we have:

\[ \frac{1}{3} \pi_*(a) - \frac{1}{3} \pi_*(b) \in H^2(M_3, \mathbb{Z}). \]

The lattice generated by \(a - b\) and \(a + 2b\) is isomorphic to \(A_2(3) \overset{\sim}{=} \begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}\). So, we denote \(A_2(3) := \langle a - b, a + 2b \rangle\). We have:

\[ \frac{1}{3} \pi_*(A_2(3)) \in H^2(M_3, \mathbb{Z}). \] (57)

Then, by (56) and (57):

\[ \pi_*(H^2(X, \mathbb{Z})) \supset \pi_* \left( U \oplus \frac{1}{3} U(3)^2 \oplus \frac{1}{3} A_2(3)^2 \oplus (-2) \right). \]

Therefore, by (ii) and (v) of Lemma 7.11, we obtain:

\[ \pi_*(H^2(X, \mathbb{Z})) = \pi_* \left( U \oplus \frac{1}{3} U(3)^2 \oplus \frac{1}{3} A_2(3)^2 \oplus (-2) \right). \]

So by Corollary 7.15,

\[ H^2(M_3, \mathbb{Z}) = \pi_* \left( U \oplus \frac{1}{3} U(3)^2 \oplus \frac{1}{3} A_2(3)^2 \oplus (-2) \right). \]

Then, by (50), the Beauville-Bogomolov form of \(H^2(M_3, \mathbb{Z})\) gives the lattice

\[
U \left( \sqrt{\frac{81}{c_{M_3}}} \oplus \frac{1}{3} U^2 \left( \sqrt{\frac{81}{c_{M_3}}} \oplus \frac{1}{3} A_2^2 \left( \sqrt{\frac{81}{c_{M_3}}} \oplus (-2) \sqrt{\frac{81}{c_{M_3}}} \right) \right) \right)
\]

\[ = U \left( 3 \sqrt{\frac{9}{c_{M_3}}} \oplus U^2 \left( \sqrt{\frac{9}{c_{M_3}}} \oplus A_2^2 \left( \sqrt{\frac{9}{c_{M_3}}} \oplus (-6) \sqrt{\frac{9}{c_{M_3}}} \right) \right) \right). \]

Then knowing that the Beauville-Bogomolov form is integral and indivisible, we have \(c_{M_3} = 9\) and we get the lattice

\[ U(3) \oplus U^2 \oplus A_2^2 \oplus (-6). \]
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