UNIVALENCY OF CERTAIN TRANSFORM OF UNIVALENT FUNCTIONS

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Abstract

We consider univalency problem in the unit disc \( \mathbb{D} \) of the function
\[
g(z) = \frac{(z/f(z)) - 1}{-a_2},
\]
where \( f \) belongs to some classes of univalent functions in \( \mathbb{D} \) and \( a_2 = \frac{f''(0)}{2} \neq 0 \).

Key words: analytic, univalent, transform

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1. Introduction. Let \( \mathcal{A} \) denote the family of all analytic functions \( f \) in the unit disk \( \mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \} \) satisfying the normalization \( f(0) = 0 = f'(0) - 1 \), i.e., \( f \) has the form
\[
f(z) = z + a_2z^2 + a_3z^3 + \ldots
\]

Let \( \mathcal{S}, \mathcal{S} \subset \mathcal{A} \), denote the class of univalent functions in \( \mathbb{D} \), let \( \mathcal{S}^* \) be the subclass of \( \mathcal{A} \), and \( \mathcal{S} \) which are starlike in \( \mathbb{D} \), and let \( \mathcal{U} \) denote the set of all \( f \in \mathcal{A} \) satisfying the condition
\[
|U_f(z)| < 1 \quad (z \in \mathbb{D}),
\]

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where

$$U_f(z) := \left( \frac{z}{f(z)} \right)^2 f'(z) - 1.$$  

In [1], Theorem 4 the authors consider the problem of univalency for the function

$$g(z) = \frac{(z/f(z)) - 1}{-a_2},$$

where \( f \in \mathcal{U} \) has the form (1) with \( a_2 \neq 0 \). They proved the following 

**Theorem A.** Let \( f \in \mathcal{U} \). Then, for the function \( g \) defined by expression (4) we have

(a) \(|g'(z) - 1| < 1\) for \(|z| < |a_2|/2\);

(b) \( g \in \mathcal{S}^* \) in the disk \(|z| < |a_2|/2\), and even more

\[
\left| \frac{zg'(z)}{g(z)} - 1 \right| < 1
\]

in the same disk;

(c) \( g \in \mathcal{U} \) in the disk \(|z| < |a_2|/2\) if \( 0 < |a_2| \leq 1 \).

These results are the best possible.

For the proof of the previous theorem the authors used the next representation for the class \( \mathcal{U} \) (see [2] and [3]). Namely, if \( f \in \mathcal{U} \), then

$$z = \frac{1}{f(z)} - a_2 z - z \omega(z),$$

where the function \( \omega \) is analytic in \( \mathbb{D} \) with \(|\omega(z)| \leq |z| < 1\) for all \( z \in \mathbb{D} \). The appropriate function \( g \) from (4) has the form

$$g(z) = z + \frac{1}{a_2} z \omega(z).$$

2. Results. In this paper we consider other cases of Theorem A(c) and certain related results.

**Theorem 1.** Let \( f \in \mathcal{U} \). Then the function \( g \) defined by equation (4) belongs to \( \mathcal{U} \) in the disc

$$|z| < \sqrt{\frac{1 - |a_2| + \sqrt{|a_2|^2 + 2|a_2| - 3}}{2}},$$

i.e., satisfies (2) on this disc, if \( \frac{5}{3} \leq |a_2| \leq 2 \).
Proof. For the first part of the proof we use the same method as in [1]. By the definition of the class $U$, i.e., inequality (2), and using the next estimation for the function $\omega$

$$|z\omega'(z) - \omega(z)| \leq \frac{r^2 - |\omega(z)|^2}{1 - r^2},$$

where $|z| = r$ and $|\omega(z)| \leq r$, after some calculations we obtain

$$|U_g(z)| = \frac{1}{1 - \frac{1}{a_2^2}} \frac{|z\omega'(z) - \omega(z)|}{1 + \frac{1}{a_2^2} \omega_1(z)} \leq \frac{|a_2| \cdot |z\omega'(z) - \omega(z)| + |\omega(z)|^2}{(|a_2| - |\omega(z)|)^2} \leq \frac{1}{1 - r^2} \cdot \varphi(t).$$

Here,

$$(7) \quad \varphi(t) = \frac{|a_2| r^2 - (|a_2| - 1 + r^2)t^2}{(|a_2| - t)^2}$$

and $|\omega(z)| = t$, $0 \leq t \leq r$. From here we have that

$$\varphi'(t) = \frac{2|a_2|}{(|a_2| - t)^3} \cdot \left[ r^2 - (|a_2| - 1 + r^2)t \right],$$

(where $|a_2| - t > 0$ since $|a_2| \geq \frac{5}{4} > 1 > t$). Next, $\varphi'(t) = 0$ for

$$t_0 = \frac{r^2}{|a_2| - 1 + r^2}$$

and $0 \leq t_0 \leq r$ if

$$\frac{r^2}{|a_2| - 1 + r^2} \leq r,$$

which is equivalent to

$$r^2 - r + |a_2| - 1 \geq 0.$$

The last relation is valid for $\frac{5}{4} \leq |a_2| \leq 2$ and every $0 \leq t < 1$. It means that the maximal value of the function $\varphi$ on $[0, r]$ is

$$\varphi(t_0) = \frac{(|a_2| - 1 + r^2)r^2}{(|a_2| - 1)(|a_2| + r^2)}.$$

Finally,

$$|U_g(z)| \leq \frac{1}{1 - r^2} \cdot \varphi(t_0) = \frac{(|a_2| - 1 + r^2)r^2}{(1 - r^2)(|a_2| - 1)(|a_2| + r^2)} < 1$$
if
\[ r^4 - (1 - |a_2|)r^2 + (1 - |a_2|) < 0, \]
or if
\[ r < \sqrt{\frac{1 - |a_2| + \sqrt{|a_2|^2 + 2|a_2| - 3}}{2}}. \]
This completes the proof.

For our next consideration we need the following lemma.

**Lemma 1.** Let \( f \in A \) be of the form (1). If
\[ \sum_{n=2}^{\infty} n|a_n| \leq 1, \]
then
\[ |f'(z) - 1| < 1 \quad (z \in \mathbb{D}), \]
\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 \quad (z \in \mathbb{D}) \]
(i.e. \( f \in S^* \)), and \( f \in \mathcal{U} \).

For the proof of \( f \in \mathcal{U} \) in the lemma see [3], while the rest easily follows.

Further, let \( \mathcal{S}^+ \) denote the class of univalent functions in the unit disc with the representation
\[ \frac{z}{f(z)} = 1 + b_1z + b_2z^2 + \cdots, \quad b_n \geq 0, \quad n = 1, 2, 3, \ldots. \]
For example, the Silverman class (the class with negative coefficients) is included in the class \( \mathcal{S}^+ \), as well as the Koebe function \( k(z) = \frac{z}{(1+z)^2} \in \mathcal{S}^+ \). The next characterization is valid for the class \( \mathcal{S}^+ \) (for details see [4])
\[ f \in \mathcal{S}^+ \iff \sum_{n=2}^{\infty} (n-1)b_n \leq 1. \]

**Theorem 2.** Let \( f \in \mathcal{S}^+ \). Then the function \( g \) defined by (4) belongs to the class \( \mathcal{U} \) in the disc \(|z| < |a_2|/2\) and the result is the best possible.

**Proof.** Using the representation (9), the corresponding function \( g \) has the form
\[ g(z) = \frac{\frac{z}{f(z)} - 1}{-a_2} = \frac{\frac{z}{f(z)} - 1}{b_1} = z + \sum_{n=2}^{\infty} \frac{b_n}{b_1}z^n \quad (b_1 \neq 0), \]
and from here
\[ \frac{1}{r}g(rz) = z + \sum_{n=2}^{\infty} \frac{b_n}{b_1}r^{n-1}z^n \quad (0 < r \leq 1). \]
Then, after applying Lemma 1, we have
\[ \sum_{n=2}^{\infty} n|a_n| = \sum_{n=2}^{\infty} \frac{b_n}{b_1} r_n = 1 + \sum_{n=2}^{\infty} (n-1) \frac{b_n}{b_1} r_n \]
\[ \leq \frac{2r}{b_1} \sum_{n=2}^{\infty} (n-1) b_n \leq \frac{2r}{b_1} \leq 1 \]
if \( r \leq b_1/2 = |a_2|/2 \). It means, by the same lemma, that \( g \in \mathcal{U} \) in the disc \(|z| < |a_2|/2\).

In order to show that the result is the best possible, let us consider the function \( f_1 \) defined by
\[ \left( \frac{z}{f_1(z)} \right)^2 = 1 + bz + z^2, \quad 0 < b \leq 2. \]
Then, \( f_1 \in \mathcal{S}^+ \) is of type \( f_1(z) = z - bz^2 + \cdots \), so the function
\[ g_1(z) = \frac{z}{f_1(z)} - 1 = z + \frac{1}{b} z^2 \]
is such that
\[ \left| \left( \frac{z}{g_1(z)} \right)^2 g_1'(z) - 1 \right| \leq \frac{1}{\left( 1 - \frac{1}{b} \right)^2} < 1 \]
when \(|z| < b/2\). This implies that \( g_1 \) belongs to the class \( \mathcal{U} \) in the disk \(|z| < b/2\). On the other hand, since \( g_1'(-b/2) = 0 \), the function \( g_1 \) is not univalent in a bigger disc, implying that the result is the best possible. \( \square \)

**Theorem 3.** Let \( f \in \mathcal{S} \). Then the function \( g \) defined by (4) belongs to the class \( \mathcal{U} \) in the disk \(|z| < r_0\), where \( r_0 \) is the unique real root of equation
\[ \frac{3r^2 - 2r^4}{(1 - r^2)^2} - \ln(1 - r^2) = |a_2|^2 \]
on the interval \((0, 1)\).

**Proof.** We apply the same method as in the proof of the previous theorem. Namely, if \( f \in \mathcal{S} \) has the representation (9), then
\[ \sum_{n=2}^{\infty} (n-1)|b_n|^2 \leq 1 \]
(see [5], Theorem 11, p. 193, Vol. 2). Also, using (4), (9) and (13), we have \( a_2 = -b_1 \), and
\[ \frac{1}{r} g(rz) = z + \sum_{n=2}^{\infty} \frac{b_n}{b_1} r^{n-1} z^n, \quad 0 < r \leq 1. \]
So,
\[
\sum_{n=2}^{\infty} n|a_n| = \sum_{n=2}^{\infty} n \frac{|b_n|}{|b_1|} r^{n-1} \\
= \frac{1}{|b_1|} \sum_{n=2}^{\infty} \sqrt{n-1} \cdot |b_n| \cdot \frac{n}{\sqrt{n-1}} \cdot r^{n-1} \\
\leq \frac{1}{|b_1|} \left( \sum_{n=2}^{\infty} (n-1)|b_n|^2 \right)^{1/2} \cdot \left( \sum_{n=2}^{\infty} \frac{n^2}{n-1} r^{2(n-1)} \right)^{1/2} \\
\leq \frac{1}{|b_1|} \left( r^2 \sum_{n=2}^{\infty} (n-1)(r^2)^{n-2} + 2r^2 \sum_{n=2}^{\infty} (r^2)^{n-2} + \sum_{n=2}^{\infty} \frac{1}{n-1}(r^2)^{n-1} \right)^{1/2} \\
= \frac{1}{|b_1|} \left[ \frac{3r^2 - 2r^4}{(1-r^2)^2} - \ln(1-r^2) \right]^{1/2} \leq 1
\]

if \(|z| < r_0\), where \(r_0\) is the root of the equation
\[
\frac{3r^2 - 2r^4}{(1-r^2)^2} - \ln(1-r^2) = |b_1|^2 (= |a_2|^2).
\]

We note that the function on the left side of this equation is an increasing one on the interval \((0, 1)\), so the equation has a unique root when \(0 < |a_2| \leq 2\). \(\Box\)

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