HAUSDORFF DIMENSION OF CLOSED SUBSETS IN PROFINITE GROUPS

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Abstract. A countably based profinite group can be naturally seen as a metric space with respect to a given filtration, and thus, it has a well defined Hausdorff dimension function. Barnea and Shalev found a group theoretical expression for the Hausdorff dimension of the closed subgroups of a profinite group $G$, opening, in this way, a bunch of possibilities to explore. In this paper we generalize Barnea’s and Shalev’s result to arbitrary closed subsets of $G$.

1. Introduction

Hausdorff dimension was introduced in 1918 as a generalization of the usual concept of topological dimension. In the last decades, however, based on the pioneering work of Barnea and Shalev [3], the concept of Hausdorff dimension has led to interesting results in the context of countably based profinite groups. Barnea’s and Shalev’s results are based, in turn, on Abercrombie’s results in [2], where the Billingsley dimension in profinite groups and profinite rings is studied. For countably based profinite groups, the notion of Billingsley dimension turns out to be a more general concept than that of Hausdorff dimension.

Let $G$ be a countably based profinite group and let $S : G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \ldots$ be a filtration of $G$, that is, a chain of open normal subgroups of $G$ such that $\cap_{n \geq 0} G_n = 1$. Then, the family $\mathcal{B}$ consisting of all cosets of the subgroups of $S$ forms an open basis of $G$. Suppose, in addition, that $G$ is also endowed with a (not necessarily translation-invariant) Borel probability measure $\mu$ in $G$. Then, we can define the Billingsley dimension function $\text{bdim}^S G, \mu$ in $G$ with respect to the measure $\mu$ and the filtration $S$ in the following way.

Let $X$ be a subset of $G$. We say that $C$ is a $\rho$-covering of $X$, where $\rho \in \mathbb{R}_{\geq 0}$, if $C$ is a covering of $X$ such that for every $B \in C$ we have $\mu(B) \leq \rho$. For each $\delta, \rho \in \mathbb{R}_{\geq 0}$ we define

$$l^\delta_{\mu, \rho}(X) = \inf \left\{ \sum_{B \in C} \mu(B)^{\delta} \mid C \text{ is a } \rho\text{-covering of } X \text{ such that } C \subseteq B \right\},$$

and we write

$$l^\delta_{\mu}(X) = \lim_{\rho \to 0} l^\delta_{\mu, \rho}(X).$$

As shown in [4, pp. 140–141], there exists a real number $\Delta$ such that $l^\delta_{\mu}(X) = \infty$ if $\delta < \Delta$ and $l^\delta_{\mu}(X) = 0$ if $\delta > \Delta$. This number $\Delta$ is called the Billingsley dimension.

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of $X$ with respect to $\mu$ and $S$ and we denote it by $\text{bdim}^{S,\mu}_G(X)$. This notion will be essential in the proof of Theorem 1.2 below.

On the other hand, the filtration $S$ induces a translation-invariant metric $d^S$ on $G$ defined as

$$d^S(x, y) = \inf \{|G : G_n|^{-1} | xy^{-1} \in G_n\},$$

where $x, y \in G$. This metric, in turns, defines the Hausdorff dimension function $\text{hdim}^S_G(X)$ for any subset $X \subseteq G$ with respect to the filtration series $S$ (just follow the same procedure as for the Billingsley dimension but replacing $\mu(B)$ by $\text{diam}(B)$ in the definition of $\rho$-covering and in (1)).

These two dimension functions, namely, the Billingsley dimension and the Hausdorff dimension functions, are closely related. Indeed, if $\eta$ is the usual Haar measure of $G$, then $\eta(gG_i) = \text{diam}(gG_i)$ for every $g \in G$ and $i \in \mathbb{N}$, so $\text{bdim}^{S,\eta}_G$ coincides with the Hausdorff dimension function $\text{hdim}^S_G$.

Finally, for $\rho > 0$, let $N_\rho(X)$ be the minimal number of sets of diameter at most $\rho$ needed to cover $X$. Then, the lower box dimension of $X$ with respect to the filtration $S$ is

$$\text{dim}^S_0(X) = \liminf_{\rho \to 0} \frac{\log N_\rho(X)}{-\log \rho}.$$ 

The Hausdorff dimension and the box dimension can be defined exactly in the same way for general metric spaces. The relation between these dimension functions is given by the following lemma.

**Lemma 1.1** ([6], p. 46). Let $(M, d)$ be a metric space. Then, for every $X \subseteq M$, we have

$$\text{hdim}(X) \leq \text{dim}^d_0(X),$$

where $\text{hdim}$ and $\text{dim}^d_0$ stand for the Hausdorff dimension and the lower box dimension functions in $M$ with respect to $d$, respectively.

In the aforementioned work [3], Barnea and Shalev proved that if $H$ is a closed subgroup of a countably based profinite group $G$, then its Hausdorff dimension is given by the formula

$$\text{hdim}^S_G(H) = \liminf_{n \to \infty} \frac{\log |HG_n : G_n|}{\log |G : G_n|}.$$ 

In this paper we will see that this formula can be generalized to general uncountable closed subsets. In this way, we obtain an expression for the Hausdorff dimension of all closed subsets of $G$.

**Theorem 1.2.** Let $G$ be a countably based profinite group and let $S : G = G_0 \geq G_1 \geq G_2 \geq \ldots$ be a filtration of $G$. Let $X$ be a closed subset of $G$. Then:

(i) If $X$ is countable, then $\text{hdim}^S_G(X) = 0$.

(ii) If $X$ is uncountable, then

$$\text{hdim}^S_G(X) = \text{dim}^S_0(X) = \liminf_{n \to \infty} \frac{\log |XG_n : G_n|}{\log |G : G_n|},$$

where $|XG_n : G_n|$ stands for the number of cosets of $G_n$ of the form $xG_n$ with $x \in X$.

Let $w$ be a word in $k$ variables, that is, an element of the free group $F_k$ with $k$ generators. For any group $G$, this word can be identified with a map, which is also denoted by $w$, from the Cartesian product of $k$ copies of $G$ to the group $G$ itself.
by substituting group elements for the variables. Suppose $G$ is a profinite groups. In the last years, there has been growing interest in the set $\text{Im}(w) = G_w$ of word values of $w$ in $G$ (see for example [7], [8], [9], [10] and [5]). This set is known to be closed in $G$, as it is the continuous image of a compact set in a Hausdorff space. Thus, if the subgroup $w(G) = \langle G_w \rangle$ is countably based and if $W$ is a filtration of $w(G)$, then one may apply Theorem 1.2 to the set $G_w$ and, in this way, obtain the Hausdorff dimension of $G_w$ in $w(G)$ with respect to the filtration $W$. It would be interesting if this notion could be related with the width of the word $w$ (taking the work in [7], [8] and [9] one step further) or with the conciseness of $w$ in the class of profinite groups (in line with [5] or [10]).

2. Proof of the theorem

In the reminder, $G$ stands for a countably based profinite group and $S : G = G_0 \geq G_1 \geq \ldots$ for a filtration of $G$. Before we begin, we recall a couple of definitions.

**Definition 2.1.** The **coset tree of $G$** with respect to the filtration $S$ is a rooted tree with set of vertices 

$$B = \{gG_n \mid g \in G, n \in \mathbb{N}_0\},$$

root $G$, and an edge joining two vertices $B_1$ and $B_2$ if there exists $g \in G$ such that $B_1 = gG_n$ and $B_2 = gG_{n+1}$ for some $n \in \mathbb{N}_0$.

For every $n \in \mathbb{N}_0$, the set $B_n$ of vertices at level $n$ is the set of all vertices of the form $gG_n$, with $g \in G$.

For every $B \in B_n$, we define the set $T_B$ of direct descendants of $B$ as the set of all vertices in $B_{n+1}$ that are adjacent to $B$.

Note that if $B \in B_n$, then $|T_B| = |G_n : G_{n+1}|$. Moreover, for every $g \in G$, there is a unique infinite path $\{B_0, B_1, \ldots\}$ such that $B_n \in B_n$ for every $n \in \mathbb{N}_0$ and $\bigcap_{n \geq 0} B_n = \{g\}$ (in fact, such a path must be the path $\{gG_0, gG_1, \ldots\}$, that is, it must satisfy $B_n = gG_n$ for every $n$).

**Definition 2.2.** A measure $\mu$ on a measure space $M$ is said to be **non-atomic** if for any measurable set $E$ of $M$ with $\mu(E) > 0$ there exists a measurable subset $F$ of $E$ such that $\mu(E) > \mu(F) > 0$.

Since $G$ is a profinite group, this property is equivalent to $\mu(g) = 0$ for every $g \in G$ (for simplicity, we just write $\mu(g)$ instead of $\mu(\{g\})$).

By Lemma 1.1 in order to prove that $\text{hdim}^S_G(X) = \text{dim}^S_B(X)$, it only remains to show that

$$\text{hdim}^S_G(X) \geq \text{dim}^S_B(X).$$

For that purpose, we will first need the following lemmas.

**Lemma 2.3 ([3], pp. 136–141).** Let $G$ be as above and let $\mu$ be a non-atomic Borel probability measure on $G$. Then:

(i) The function $\text{hdim}^S_G(\mu)$ is a non-negative increasing set function.

(ii) If $S$ is a Borel subset of $G$, then $\mu(S) > 0$ implies $\text{hdim}^S_G(\mu)(S) = 1$.

**Lemma 2.4 ([3], pp. 144–145).** Let $G$ be as above and let $\mu_1$ and $\mu_2$ be non-atomic Borel probability measures on $G$. Let $\delta \in \mathbb{R}_{>0}$ and consider a subset $S$ of the set

$$\left\{ g \in G \mid \liminf_{n \to \infty} \frac{\log \mu_1(gG_n)}{\log \mu_2(gG_n)} \geq \delta \right\}.$$
Then, we have
\[ \text{bdim}_{\mathcal{S}}^{S, \mu_2}(S) \geq \delta \text{bdim}_{\mathcal{S}}^{S, \mu_1}(S). \]

**Lemma 2.5** ([1], Lemma 5.2). Let \( G \) be as above and let \( \mu' : \mathcal{B} \to [0, 1] \) be a function such that \( \mu'(G) = 1 \) and
\[ \sum_{C \in \mathcal{T}_B} \mu'(C) = 1 \]
for all \( B \in \mathcal{B} \). Then there exists a unique Borel probability measure \( \mu \) of \( G \) such that if \( B \in \mathcal{B}_n \), then
\[ \mu(B) = \prod_{i=0}^{n} \mu'(B_i), \]
where \( \{B_0, B_1, \ldots, B_n\} \) is the unique path from \( G \) to \( B \).

We are now in a position to prove Theorem 1.2. Even if our proof follows the basic ideas of Abercrombie in [2], it requires us to introduce a new measure which is tailored to the specific closed subset \( X \).

**Proof of Theorem 1.2.** It is known that, in general, the Hausdorff dimension of countable subsets is 0 ([4, p. 138]), so let us prove part (ii).

First, it is clear that if \( \rho = |G : G|^{-1} \), then \( N_\rho(X) = |XG_n : G_n|. \) Thus, as the only possible values for the distance \( d_S \) in \( G \) are precisely the \( |G : G|^{-1} \), we have
\[ \dim_S(X) = \liminf_{n \to \infty} \frac{\log |XG_n : G_n|}{\log |G : G_n|}, \]
so the second equality follows (even for countable sets).

For the first equality, using Lemma 2.5, we will construct a suitable Borel probability measure on \( G \) (unlike the probability measure defined in [2], our construction will deeply depend on the subset \( X \)). In order to do so, we define a function \( \nu' : \mathcal{B} \to [0, 1] \) recursively as follows. First set \( \nu'(G) = 1 \) and suppose that for some \( n \geq 0 \), we have defined \( \nu' \) for all \( B \in \mathcal{B}_m \) with \( \nu(B) \neq 0 \) we have \( |B \cap X| > \aleph_0 \) and
\[ \nu(B) \geq |XG_m : G_m|^{-1}. \]
Hence, we also assume that this holds for every \( m \leq n \).

Fix a vertex \( B \in \mathcal{B}_n \). Let us define \( \nu' \) on the set \( T_B \) of direct descendants of \( B \). To do this, suppose first \( \nu(B) = 0 \). In this case, we set
\[ \nu'(C) = |G_n : G_{n+1}|^{-1} \]
for all \( C \in T_B \) and clearly equality (2) holds. In this way we extend the definition of \( \nu \) so that \( \nu(C) = \nu(B) \nu'(C) = 0 \) for all \( C \in T_B \).

Suppose now that \( \nu(B) \neq 0 \) and let \( T_B \) be the set of all direct descendants \( C \) of \( B \) such that \( |C \cap X| > \aleph_0 \) (notice that \( T_B \) is non-empty as \( |B \cap X| > \aleph_0 \).
First, we put \( \nu'(C) = 0 \) for every \( C \in T_B \setminus T_B^* \) and, as before, \( \nu(C) = 0 \). Now, if \( T_B^* \) consists only of one vertex \( C \), then we are forced, in order for (2) to be satisfied, to set \( \nu'(C) = 1 \) and \( \nu(C) = \nu(B) \). If, on the contrary, \( |T_B^*| \geq 2 \), then we consider two cases in turn, namely, \( \nu(B)/2 < |XG_{n+1} : G_{n+1}|^{-1} \) and \( \nu(B)/2 \geq |XG_{n+1} : G_{n+1}|^{-1} \).

If \( \nu(B)/2 < |XG_{n+1} : G_{n+1}|^{-1} \), then we choose a vertex \( C \in T_B^* \) and set \( \nu'(C) = 1 \). Hence, we put \( \nu(C) = \nu(B) \) and \( \nu'(D) = 0 \) for all \( D \in T_B \setminus \{C\} \).

If, however, \( \nu(B)/2 \geq |XG_{n+1} : G_{n+1}|^{-1} \), then we choose two vertices \( C_1, C_2 \in T_B^* \) and set \( \nu'(C_1) = \nu'(C_2) = 1/2 \). In this way, we obtain \( \nu(C_1) = \nu(C_2) = \nu(B)/2 \) and \( \nu'(D) = 0 \) for all \( D \in T_B \setminus \{C_1, C_2\} \). It is clear, in addition, that in both cases (2) is satisfied.

Finally, if \( \nu(C) \neq 0 \), then it clearly follows that \( |C \cap X| > \aleph_0 \). Let us see that we also have \( \nu(C) \geq |XG_{n+1} : G_{n+1}|^{-1} \). Observe first that \( \nu(C) \neq 0 \) implies \( \nu(B) \neq 0 \), so that \( \nu(B) \geq |XG_n : G_n|^{-1} \). Thus, since

\[
\nu(B) \geq |XG_n : G_n|^{-1} \geq |XG_{n+1} : G_{n+1}|^{-1},
\]

and as \( \nu'(C) = 1/2 \) only if \( \nu(B)/2 \geq |XG_{n+1} : G_{n+1}|^{-1} \) and otherwise \( \nu'(C) = 0 \), the assertion follows. In this way we finish the definition of \( \nu' \) for the vertices in \( T_B \), and we follow in this way until we define it on the whole tree.

Now, by Lemma 2.5 there exists a unique Borel probability measure in \( G \), that, abusing notation, as it takes the same values as \( \nu \) in \( B \), we still denote by \( \nu \).

We claim that \( \nu \) is a non-atomic Borel probability measure of \( G \). Take \( g \in G \) and suppose first that there exists \( B \in B \) with \( g \in B \) such that \( \nu'(B) = 0 \). In particular we have \( \nu(B) = 0 \), so

\[
\nu(g) \leq \nu(B) = 0.
\]

Suppose then \( \nu'(B) \neq 0 \) for every \( B \in B \) with \( g \in B \), and let \( \{B_0, B_1, \ldots\} \) be the unique infinite chain of vertices such that \( B_n \in B_n \) for every \( n \in \mathbb{N}_0 \) and \( \cap_{i \geq 0} B_i = \{g\} \). Let us see that for every \( j \geq 0 \) there exists \( k > j \) such that

\[
\nu(B_k) \leq \frac{1}{2j} \nu(B_j).
\]

Suppose by way of contradiction that there exists \( j \geq 0 \) such that \( \nu(B_k) > \frac{1}{2j} \nu(B_j) > 0 \) for every \( k \geq j \). From the construction of \( \nu \) we can deduce that this only occurs if \( \nu(B_k) = \nu(B_j) \) and, in particular, \( \nu'(B_k) = 1 \) for every \( k \geq j + 1 \).

Since \( X \) is infinite and \( \{G_n\}_{n \in \mathbb{N}_0} \) is a base of neighbourhoods of the identity of \( G \), we have

\[
\lim_{n \to \infty} |XG_n : G_n|^{-1} = 0,
\]

so there exists \( d \geq j \) such that

\[
\frac{1}{2} \nu(B_n) = \frac{1}{2} \nu(B_j) \geq |XG_n : G_n|^{-1}
\]

for every \( n \geq d \). Now, for each \( n \geq d + 1 \), define

\[
U_n = \bigcup \{C \in T_{B_{n-1}} \mid C \neq B_n\}.
\]

Since \( \nu'(B_n) = 1 \) for all \( n \geq d + 1 \), it follows by (2) that \( \nu'(C) = 0 \) for all \( C \in U_n \) and \( n \geq d + 1 \). Therefore, by the definition of \( \nu' \), it follows for all \( n \geq d + 1 \) that \( |B_n \cap X| > \aleph_0 \) and, since \( \nu(B_n)/2 \geq |XG_n : G_n|^{-1} \), that \( |C \cap X| \leq \aleph_0 \) for all \( C \in U_n \). Hence, \( |U_n \cap X| \leq \aleph_0 \) for every \( n \geq d + 1 \). Let \( U = \bigcup_{n=d+1}^{\infty} (U_n \cap X) \) and observe that \( U \) is countable, being a countable union of countable sets. However, it
is clear that \((B_d \cap X) \setminus U = \{g\}\), which is a contradiction since \(B_d \cap X\) is uncountable. Therefore, for every \(j \geq 1\), there exists \(k > j\) such that
\[
\nu(B_k) \leq \frac{1}{2^i} \nu(B_j).
\]
Thus, since \(g \in B_n\) for all \(n \geq 1\), it follows that \(\nu(g) \leq 1/2^i\) for every \(i \in \mathbb{N}\). This yields \(\nu(g) = 0\) and \(\nu\) is a non-atomic Borel probability measure.

Now, from the construction of \(\nu'\), we deduce that if \(gG_n \cap X = \emptyset\) with \(g \in G\) and \(n \in \mathbb{N}_0\), then \(\nu(gG_n) = 0\). Hence we have \(\nu(G \setminus XG_n) = 0\), and in particular, since \(\nu(G) = 1\), we obtain \(\nu(XG_n) = 1\) for every \(n \in \mathbb{N}_0\). Since \(X\) is closed in \(G\), we thus get
\[
\nu(X) = \nu(X) = \nu\left( \bigcap_{n \in \mathbb{N}_0} XG_n \right) = \lim_{n \to \infty} \nu(XG_n) = 1.
\]
On the other hand, consider the set
\[
S = \{h \in X \mid \nu'(B) \neq 0 \text{ for all } B \in \mathcal{B} \text{ with } h \in B\}.
\]
For all \(n \in \mathbb{N}\), write also
\[
R_n = \bigcup\{B \in B_n \mid \nu'(B) = 0\}
\]
and notice that each \(R_n\) is open, being a union of open sets. Moreover, it is obvious that
\[
S = X \setminus \bigcup_{n=1}^{\infty} R_n,
\]
and hence, since \(X\) is closed in \(G\), it follows that \(S\) is also a closed subset of \(G\). In particular, both \(S\) and \(X \setminus S\) are Borel subsets of \(G\).

Clearly, for every \(g \in X \setminus S\) there exists \(B \in \mathcal{B}\) with \(g \in B\) such that \(\nu(B) = 0\), and so, since \(G\) is countably based, \(X \setminus S\) is contained in a countable union of open spheres with probability measure 0. This gives \(\nu(X \setminus S) = 0\). Thus
\[
\nu(S) = \nu(X) - \nu(X \setminus S) = 1,
\]
and so Lemma 2.3 yields \(\text{bdim}^{S,\nu}_G(S) = 1\).

Recall from (4) that for every \(s \in S\) and every \(n \in \mathbb{N}_0\) we have \(\nu(sG_n) \geq |XG_n : G_n|^{-1}\). Thus, by (3), we deduce that
\[
\liminf_{n \to \infty} \frac{\log \nu(sG_n)}{\log \eta(sG_n)} \geq \liminf_{n \to \infty} \frac{\log |XG_n : G_n|^{-1}}{\log |G : G_n|^{-1}} = \dim^S_B(X),
\]
where \(\eta\) is the usual Haar measure of \(G\). Hence, \(S\) is a subset of
\[
\left\{ g \in G \mid \liminf_{n \to \infty} \frac{\log \nu(gG_n)}{\log \eta(gG_n)} \geq \dim^S_B(X) \right\},
\]
and therefore, by Lemma 2.3 and Lemma 2.4 we have
\[
\text{hdim}^{S}_G(X) \geq \text{hdim}^{S}_{G}(S) = \text{bdim}^{S,\eta}_G(S) \geq \dim^S_B(X) \text{bdim}^{S,\nu}_G(S) = \dim^S_B(X),
\]
as desired. □
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