Convergence of the dispersion Camassa-Holm $N$-Soliton

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Abstract

In this paper, we show that the peakon (peaked soliton) solutions can be recovered from the smooth soliton solutions, in the sense that there exists a sequence of smooth $N$-soliton solutions of the dispersion Camassa-Holm equation converging to the $N$-peakon of the dispersionless Camassa-Holm equation uniformly with respect to the spatial variable $x$ when the dispersion parameter tends to zero. The main tools are asymptotic analysis and determinant identities.

Keywords Camassa-Holm equation, peakon, soliton, asymptotic analysis, determinant technique

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1 Introduction

Consider the Camassa-Holm (CH) equation [1]

$$u_t + 2\omega u_x - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx},$$  \hspace{1cm} (1.1)

where $u$ is the fluid velocity in the $x$ direction, $\omega$ is a constant related to the critical shallow water speed, and the subscripts denote partial derivatives. The CH equation models the unidirectional propagation of shallow water waves over a flat bottom [1,4].

When the dispersion coefficient $\omega > 0$, we refer to (1.1) as the dispersion CH equation in this paper, which possesses solitary wave solutions whose limiting form as $\omega \to 0$ has peaks where the first derivative are discontinuous [1,2]. There are abundant literature on the soliton solutions, involving several techniques in the soliton theory; see [5-15] for details.

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When $\omega = 0$, the equation (1.1), i.e.

$$u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx},$$

(1.2)

compared with the KdV equation, has the advantage of admitting peakons (peaked solitons) of the form $ce^{-|x-ct|}$, which capture the main feature of the exact traveling wave solutions of greatest height of the governing equations for water waves in irrotational flow [16][18], and modeling wave breaking [2][19][20]. Equation (1.2) admits multi-soliton solution of the form

$$u(x,t) = \sum_{i=1}^{N} m_i(t)e^{-|x-x_i(t)|},$$

(1.3)

which is called multipeakons. The system of evolution equations for amplitudes $m_i(t)$ and positions $x_i(t)$ of peaks is a completely integrable finite dimensional Hamiltonian system, explicit formulas of which have been obtained by Beals e.l. [19][21] via the inverse scattering transform method using the Stieltjes theorem on continued fractions. Blow-up results for certain initial data and global existence theorems for for a large class of initial data have been given by [1][2] and [22], respectively.

Note that (1.2) is a formal limit of (1.1), a natural question is to analyze the relation between the parameter solitons of (1.1) and the peakons of (1.2). Some theoretical and numerical results have been obtained on this question. Li and Olver [23][24] have showed how the single peakon of (1.2) can be recovered as limits of classical solitary wave solutions via the theory of dynamical systems. Johnson [7] and Parker [11] have claimed that the profile of the analytic one, two-soliton solutions may become peaked for some singular limit involving the wave parameters and dispersion parameter in their settings. These answers obtained are not so satisfactory.

In this paper, we prove rigorously that the non-analytic $N$-peakon (amplitudes $m_i$ in (1.3) are all positive) solutions can be recovered from the analytic $N$-soliton solutions given by [8][9]. Our work is motivated by [23][25], and establishes the relation between solutions of (1.1) and (1.2). We believe that there is certain physics behind this mathematical fact, while we still cannot explain it completely.

The layout of this paper is as follows. In Section 2, we simply review some results on explicit solutions of (1.1) and (1.2), and prove some determinant identities that are important to the proof of convergence. In Section 3, results on the convergence of the $N$-soliton are proposed. In Section 4, we prove the convergence with the aid of asymptotic analysis and determinant technique.

## 2 Explicit solutions to Camassa-Holm equation and determinant identities

Let $\omega = \kappa^2$, and without lose of generality, we assume that $\kappa > 0$. The $N$-soliton solutions of (1.1) given by Li and Zhang [8][9] reads:

$$u(y,t) = \left(\ln \left| \frac{f_1}{f_2} \right| \right)_t, \quad x(y,t) = \ln \left| \frac{f_1}{f_2} \right|,$$
where
\[
\begin{align*}
&f_1 = \frac{W(\Phi_1, \ldots, \Phi_N, e^{\frac{\pi}{N}})}{W(\Phi_1, \ldots, \Phi_N)}, & f_2 = \frac{W(\Phi_1, \ldots, \Phi_N, e^{-\frac{\pi}{N}})}{W(\Phi_1, \ldots, \Phi_N)}, \\
&\Phi_i(y,t) = \begin{cases} 
\cosh \xi_i, & i = 2l-1, \\
\sinh \xi_i, & i = 2l,
\end{cases} \quad 1 \leq i \leq N,
\end{align*}
\]

and
\[
\begin{align*}
&\xi_i = k_i(y - \kappa \epsilon i t), & k_i = \frac{1}{2\kappa} \left( 1 - \frac{2\kappa^2}{c_i} \right)^{\frac{1}{2}}, \quad k_1 < \cdots < k_N.
\end{align*}
\]

**Remark 2.1.** According to the examples in \[26\] and our proof in section 5, we suppose that \(0 < 2\kappa k_i < 1\) (\(1 \leq i \leq N\)), then the formulas above really give smooth \(N\)-soliton and

\[
\begin{align*}
&c_i = \frac{2\kappa^2}{1 - 4\kappa^2 k_i^2} > 2\kappa^2 (1 \leq i \leq N). \\
&\text{Let } I = \{i_1 < i_2 \cdots < i_n\} \text{ be an } n\text{-element subset of the integer interval } [1,N] = \{1,2,\cdots,N\}, \text{ and } J = \{1,2,\cdots,N\} \setminus I. \text{ Define } a_i \text{ and } b_i \text{ as follows:} \\
&a_i = \frac{1+2\kappa k_i}{1-2\kappa k_i} \quad \text{and} \quad b_i = \frac{1-2\kappa k_i}{1+2\kappa k_i},
\end{align*}
\]

and set
\[
\begin{align*}
a_I = \prod_{i \in I} a_i, & \quad b_I = \prod_{i \in I} b_i, & \quad \xi_I = \sum_{i \in I} \xi_i, & \quad \Gamma_I = \prod_{i,j \in I, i < j} (k_j - k_i),
\end{align*}
\]

with the proviso that \(\Gamma_{\{i\}} = 1\).

We now state some results on \(N\)-soliton solutions of (1.1).

**Theorem 2.2.** The \(N\)-soliton given by Li and Zhang can be expressed as follows,

\[
\begin{align*}
u(y,t) &= \left( \ln \frac{g_1}{g_2} \right), \\
x(y,t) &= \frac{y}{\kappa} + \ln \frac{g_1}{g_2} + \alpha,
\end{align*}
\]

where
\[
\begin{align*}
g_1 &= 1 + \sum_{n=1}^{N} \left( \sum_{l} b_{l} e^{2\xi_l} \prod_{j \in J, l \in I} \text{sgn}(j-l) \frac{k_j + k_l}{k_j - k_l} \right), \\
g_2 &= 1 + \sum_{n=1}^{N} \left( \sum_{l} a_{l} e^{2\xi_l} \prod_{j \in J, l \in I} \text{sgn}(j-l) \frac{k_j + k_l}{k_j - k_l} \right).
\end{align*}
\]

We have left the proof of Theorem 2.2 to section 5 the one who is not interested in can skip it without affecting the comprehension of the main results.
Theorem 2.3. For the smooth N-soliton of (1.1), the result of Li and Zhang obtained via Darboux transformation is equivalent to Parker’s proposed by Hirota bilinear method.

Proof. Let

\[ \phi_i = \ln \left( \prod_{j=1, j \neq i}^{N} \frac{\text{sgn}(j-i)k_j + k_i}{k_j - k_i} \right), \quad (2.4) \]

and \( \sum_{\mu=0,1} \) be the summation over all possible combination of \( \mu_1 = 0, 1, \ldots, \mu_N = 0, 1 \).

Under the transformation: \( \bar{\xi}_i \to \xi_i - \frac{\alpha_i}{2}, \) \( k_i \to \frac{k_i}{2}, \) the N-soliton solutions of (1.1) with arbitrary initial phase becomes

\[ u(y,t; \kappa) = \left( \ln \frac{g_1}{g_2} \right)_{,y}, \quad x(y,t; \kappa) = \frac{y}{\kappa} + \ln \frac{g_1}{g_2} + \alpha, \quad (2.5) \]

where

\[ g_1 = \sum_{\mu=0,1} \exp \left( \sum_{i=1}^{N} \mu_i (\bar{\xi}_i - \phi_i) + \sum_{i,j}^{N} \mu_i \mu_j \gamma_{ij} \right), \]

\[ g_2 = \sum_{\mu=0,1} \exp \left( \sum_{i=1}^{N} \mu_i (\bar{\xi}_i + \phi_i) + \sum_{i,j}^{N} \mu_i \mu_j \gamma_{ij} \right), \quad (2.6) \]

with

\[ \bar{\xi}_i = k_i(y - \kappa c_i t - y_0), \quad c_i = \frac{2\kappa^2}{1 - \kappa^2 k_i^2}, \quad 0 < \kappa k_i < 1 \quad (1 \leq i \leq N) \quad (2.7) \]

\[ \phi_i = \ln \frac{1 + \kappa k_i}{1 - \kappa k_i}, \quad e^{\gamma_{ij}} = \left( \frac{k_i - k_j}{k_i + k_j} \right)^2, \quad 0 < k_1 < \cdots < k_N, \]

and \( \alpha, y_0, (1 \leq i \leq N) \) are arbitrary constants. \( \square \)

We now review the N-peakon of the dispersionless CH equation (1.2).

Consider the spectral problem associated to (1.2):

\[ \psi_{xx} = \left( \frac{1}{4} + \hat{\lambda} m \right) \psi, \quad e^{\pm \frac{i}{2} \psi}(x) \to 0 \quad \text{as} \quad x \to \mp \infty \quad (2.8) \]

with \( m = 2 \sum_{i=1}^{N} m_i \delta_i \). Denote the eigenvalues of (2.8) by \( \hat{\lambda}_i \) \((i = 1, \ldots, N)\), let \( \Delta_{m}^n \) be the determinant of the \( n \times n \) submatrix of an infinite Hankel matrix, whose (1, 1) entry is \( \hat{\Delta}_m \), i.e. \( \Delta_{m}^n = \det(\hat{\Delta}_{m+i+j})_{i,j=0}^{n-1} \), the moments \( \hat{\Delta}_m \) are restricted by

\[ \hat{\Delta}_m = \sum_{i=0}^{N} (-\hat{\lambda}_i)^m a_i, \quad a_i = a_i(0)e^{-\frac{\tilde{\lambda}_0}{2}} \quad (i \geq 0), \quad \tilde{\lambda}_0 = 0, \quad a_0 = \frac{1}{2}. \]

Let \( \hat{\Delta}_{m}^n = \det(\hat{\Delta}_{m+i+j})_{i,j=0}^{n-1} \) with

\[ \hat{\Delta}_m = \sum_{i=1}^{N} (-\hat{\lambda}_i)^m a_i, \quad a_i = a_i(0)e^{-\frac{\tilde{\lambda}_0}{2}}. \]
Lemma 2.4 (N-peakon, [19]). The dispersionless CH equation (1.2) admits N-peakon solutions

\[ u(x,t) = \sum_{i=1}^{N} m_i(t) e^{-|x-x_i(t)|}, \]

where

\[ m_i = \frac{2\tilde{\lambda}_0\Delta_{N-i+1}^2 - \Delta_{N-i+1}^4}{\Delta_{N-i+1}^4 - \Delta_{N-i}^4}, \quad x_i = \ln \left( \frac{2\tilde{\lambda}_0\Delta_{N-i+1}^2}{\Delta_{N-i}^4} \right). \]  

(2.9)

Last, we present some determinant identities necessary to the proofs of the convergence of smooth N-soliton.

Let

\[ A_m = \sum_{i=1}^{n} \lambda_i^m E_i, \quad E_i = e^{x_i t}, \quad x_0 = \frac{y_0}{\kappa}, \quad \lambda_i = \frac{2}{\epsilon_i}, \]  

(2.10)

where \( m, n \in \mathbb{Z} \) and \( 0 \leq n \leq N \). Let \( D_n^m = \det(A_{m+j})_{i,j=0}^{n-1} \), and adopt the convention that \( D_0^m = 1 \).

Lemma 2.5. For \( n = 1, \ldots, N \),

\[ D_n^m = \sum_{1 \leq i_1 < \cdots < i_m \leq n} \Delta_n(i_1, \ldots, i_m)(\lambda_{i_1} \cdots \lambda_{i_m})^m E_{i_1} \cdots E_{i_m}, \]  

(2.11)

where

\[ \Delta_n(i_1, \ldots, i_m) = \prod_{1 \leq l < m \leq n} (\lambda_{i_l} - \lambda_{i_m})^2. \]

We adopt the convention that \( D_0^m = 1 \) and \( D_n^0 = 0 \) \( (n < 0) \). Since \( \lambda_i \)’s are distinct, we have \( D_n^m > 0 \) \( (0 \leq n \leq N) \), and \( D_n^0 = 0 \) \( (n > N) \) by [19] Theorem 6.1.

Lemma 2.6 (Jacobi identity, [27]). For any determinant \( D \), let \( D(i,j;k,l) \) be the determinant obtained from \( D \) by deleting the rows \( i,j \) and the columns \( k,l \), respectively, then

\[ DD(i,j;k,l) = D(i,k)D(j;l) - D(j,k)D(i;l), \quad i < j, \quad k < l, \]

Lemma 2.7. For any \( m, n \in \mathbb{Z} \) and \( n \geq 0 \),

\[ D_{n+2}^m D_n^{m+2} = D_{n+1}^{m+2} D_{n+1}^m - (D_{n+1}^{m+1})^2, \]  

(2.12)

\[ D_{n+1}^m D_n^m - D_{n+1}^2 D_n^2 = 2D_{n+1} D_n^3, \]  

(2.13a)

\[ D_{n+2}^0 D_n^0 - D_{n+1}^0 D_n^1 = 2D_{n+1} D_n^1, \]  

(2.13b)

\[ D_{n+1}^2 D_{n+2}^0 - D_{n+2}^2 D_{n+1}^0 = -2D_{n+2}^0(1;2)D_{n+1}^1, \]  

(2.14)

\[ D_{n+1}^0 D_{n+2}^0 - D_{n+2}^0 D_{n+1}^0 = 2D_{n+2}^0(1;2)D_{n+1}^1. \]  

(2.15)

Corollary 2.8.

\[ \sum_{i=0}^{n} \frac{(D_i^2)^2}{D_{i+1} D_i} = \frac{D_n^3}{D_{n+1}^4}, \quad \sum_{i=n+1}^{N-1} \frac{(D_i^0)^2}{D_{i+1} D_i} = \frac{D_{n+2}^{-1}}{D_{n+1}^4}. \]  

(2.16)
Proof of Lemma 2.7  The identity (2.12) follows from Lemma 2.6 with \( D = D_n^\mu \) and \( i = k = 1, j = l = n + 2 \). By the characteristics of the determinant \( D_n^\mu \), (2.13b) is equivalent to (2.13a). Note that \( A_m \) is given by (2.10), we have \( A_{n+1} = 2A_{n+2} \), therefore,
\[
m2D_n^{m-2}(1; 2) = D_n^m = 2D_{n+1}^m(n + 1; 2).
\]
Using the relation above, (2.13a) and (2.14)–(2.15) can be rewritten as follows,
\[
\begin{align*}
D_1^1 & = D_1^0 = D_{n+1}^1 - D_{n+2}^1(n + 3; 2)D_n^1 - D_{n+3}^1(n + 3; 2)D_n^1, \\
D_2^1 & = D_2^0 = -D_{n+1}^1(1; 2)D_{n+2}^0 - D_{n+2}^1(n + 1; 2)D_{n+1}^0, \\
D_3^1 & = D_3^0 = -D_{n+1}^1(1; 2)D_{n+2}^0 - D_{n+1}^1(n + 1; 2)D_{n+2}^0.
\end{align*}
\]
Consider the following determinants of order \( n + 2 \),
\[
D_1 = \begin{vmatrix}
0 & A_1 & A_2 & \cdots & A_{n+1} \\
1 & A_2 & A_3 & \cdots & A_{n+2} \\
0 & A_3 & A_4 & \cdots & A_{n+3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & A_{n+2} & A_{n+3} & \cdots & A_{2n+2}
\end{vmatrix}, \quad D_3 = D_{n+2}^0, \quad D_4 = D_{n+2}^1.
\]
The identities (2.13a) and (2.14)–(2.15) follow readily from Lemma 2.6
- For \( D_1 \) and \( D_4 \), set \( i = k = 1, j = l = n + 2 \), we have (2.13a) and (2.15).
- For \( D_3 \), set \( i = 1, k = 2, j = l = n + 2 \), we have (2.14).

By (2.12), we have
\[
(D_1^2)^2 = D_1^1D_1^3 - D_1^1D_1^3, \quad (D_3^0)^2 = D_{n+1}^1D_{n+1}^1 - D_{n+2}^1D_{n+2}^1.
\]
Dividing the equations above by \( D_{n+1}^1D_1^1 \) and summing up, we obtain Corollary 2.8 by taking account of the fact that \( D_n^m = 0 \) for any \( n > N \) or \( n < 0 \).

3 Results on convergence

Motivated by Li and Olver [23, 24], we consider a sequence of \( N \)-soliton solutions with the velocity of the \( i \)-th soliton in \((x,t)\)-space independent of the dispersion parameter \( \kappa \).

Remark 3.1. There exists a sequence of analytic \( N \)-soliton solutions, the velocity of the \( i \)-th soliton in \((x,t)\)-space given by positive constant independent of \( \kappa \). In fact, since \( k_i \)'s are arbitrary and satisfy \( 0 < \kappa k_i < 1(1 \leq i \leq N) \), \( 0 < k_1 < \cdots < k_N \), let \( \alpha_i (1 \leq i \leq N) \) be \( N \) distinct positive constants independent of \( \kappa \), and
\[
k_i = \left( \frac{1}{\kappa} \left( 1 - \frac{2\kappa^2}{\alpha_i} \right)^{\frac{1}{2}} \right), \quad \text{ (3.1)}
\]
then \( \tilde{c}_i \) given by (2.7) with \( k_i \) chosen by (3.1) equals to \( \alpha_i \). Thus the sequence of \( N \)-soliton solutions given by (2.5)–(2.7) and (3.1) is required.
In the following, we will still use $c_i$ to denote the constant $\tilde{c}_i$, and give the results on the convergence of this sequence of smooth $N$-soliton solutions. With the preparations above, we have the convergence of the $N$-soliton solutions of the dispersion CH equation.

**Theorem 3.2.** Let

$$\bar{x}_n = \ln \left( \frac{2D^0_{N-n+1}}{D^2_{N-n}} \right), \quad 1 \leq n \leq N. \quad (3.2)$$

Under appropriate translation and scaling transformations, for any given $t$, when $\kappa \to 0$, the $N$-soliton given by (2.5) of (2.7) and (3.1) has the limit

$$\bar{u}(x; t) = \begin{cases} u_n, & \bar{x}_{n-1} < x \leq \bar{x}_n (n = 1, 2, \ldots, N) \\ u_{N+1}, & x > \bar{x}_N \end{cases} \quad (3.3)$$

with

$$u_n = e^{\lambda_i(D^0_{N-n} + 4e^{-\lambda_i}D^1_{N-n+2})}, \quad u_{N+1} = 2 \sum_{i=1}^{N} c_i e^{-\lambda_i}. \quad (3.4)$$

**Remark 3.3.** If we set $\lambda_i = -4\tilde{\lambda}_i$, i.e. $\tilde{c}_i = -\frac{1}{2\lambda_i}$, which is the asymptotic velocity of the position $x_i$ at large positive time for the peakon solutions given by Lemma 1.3 and $c_i$’s are distinct positive constants by [19, Theorems 4.1 and 6.4]. Then $A_m = 4^mA_m$ and $D^0_m = 4^{n(m+n-1)}\Delta^0_m$ for any $m, n \in \mathbb{Z}$ and $0 \leq n \leq N$, hence,

$$D^0_{N-i+1} = 4^{(N-i+1)(N-i)}\Delta^0_{N-i+1}, \quad D^2_{N-i} = 4^{(N-i)(N-i+1)}\Delta^2_{N-i}. \quad (3.5)$$

Since $\Delta^m_n = \Delta^m_n (m \geq 1)$, we obtained that the second expression in (2.9) is fixed under the transformation:

$$\Delta^2 \to D^2, \quad \Delta^0 \to D^0.$$

Therefore, when $\lambda_i = -4\tilde{\lambda}_i$, we have $\bar{x}_n = x_n (1 \leq n \leq N)$.

**Theorem 3.4.** Under appropriate translation and scaling transformations, the $N$-soliton solutions of (1.1) given by (2.5) of (2.7) and (3.1) with $\alpha_i = -\frac{1}{2\lambda_i}$ converges to the $N$-peakon of (1.2) uniformly with respect to $x$ as $\kappa \to 0$.

### 4 Proof of convergence

This section is devote to the proof of Theorems 3.2 and 3.4, the asymptotic analysis is the main tool.

Let $g = g_1 / g_2$, then under the inverse of the reciprocal transformation, i.e. $dx = \frac{1}{r}dy + ut\, dt$, we have $u(x(y,t), t; \kappa) = u(y(t), t; \kappa) = \left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) \ln g$, and

$$u(x, t; \kappa) = \frac{g_t}{g} - g_s. \quad (4.1)$$

Therefore, for Theorem 3.2, we only need to show: for any given $t$, under appropriate translation and scaling transformations, the limit on $\kappa \to 0$ of the right hand side of
identities listed in Section 2. 

Theorem 3.2 and 3.4 follow from some determinant (4.1) exists and is given by (3.3). We present some lemmas and propositions to show the existence of this limit, then Theorems 3.2 and 3.4 follow from some determinant identities listed in Section 2.

**Lemma 4.1.** Let $z_i = \exp(x - c_i t - x_0) = e^t E_i^{-1}$, $\varepsilon = \frac{x_0}{t}$, when $\varepsilon \to 0,$

\[
\kappa k_i = 1 - \frac{\kappa^2}{c_i} + o(\varepsilon), \quad v_i = e^{-\phi_i} = \lambda_i \varepsilon + o(\varepsilon),
\]

\[e^\xi = \frac{z_i}{g} + o(1), \quad \gamma_i = \varepsilon^2 (\lambda_i - \lambda_j)^2 + o(\varepsilon^2).
\]

Under the phase shift: $\xi_i \to \xi_i - \phi_i,$ when $\kappa \to 0$ i.e. $\varepsilon \to 0,$ we have

\[
g_1 = 1 + \sum_{n=1}^{N} \varepsilon^{n(n+1)} g^{-n} \sum_{1 \leq i_1 < \cdots < i_n \leq N} (\lambda_{i_1} \cdots \lambda_{i_n})^2 \Delta_n(i_1, \cdots, i_n)z_{i_1} \cdots z_{i_n} + o(\varepsilon^{N(N+1)}),
\]

\[
g_2 = 1 + \sum_{n=1}^{N} \varepsilon^{n(n-1)} g^{-n} \sum_{1 \leq i_1 < \cdots < i_n \leq N} \Delta_n(i_1, \cdots, i_n)z_{i_1} \cdots z_{i_n} + o(\varepsilon^{N(N-1)}).
\]

Replace $z_i (i = 1, \ldots, N)$ by

\[
\frac{\prod_{j=1}^{n} \lambda_j}{2 \prod_{j=1, j \neq i}^{n} (\lambda_i - \lambda_j)^2} \frac{z_i}{\lambda_i^2},
\]

i.e. under scaling transformations, we obtain the following estimate involving $D^m_n$ for $g_1, g_2.$

**Lemma 4.2.** When $\kappa \to 0$ i.e. $\varepsilon \to 0,$

\[
g_1 = 1 + 2 \sum_{n=1}^{N} \varepsilon^{n(n+1)} g^{-n} d_n e^{nD^0_{N-n}} + o(\varepsilon^{N(N+1)}),
\]

\[
g_2 = 1 + \sum_{n=1}^{N} \varepsilon^{n(n-1)} g^{-n} d_n e^{nD^2_{N-n}} + o(\varepsilon^{N(N-1)}),
\]

with $d_n > 0$ defined by

\[
d_n(t) = \frac{\prod_{j=1}^{n} \lambda_j}{2^n \Delta N \prod_{i=1}^{N} E_i}, \quad (n = 1, 2, \ldots, N + 1).
\]

**Proof.** For any $1 \leq n \leq N,$ let $I_n = \{i_1 < i_2 < \cdots < i_n\}$ and $J_n = \{1, \ldots, N\} \setminus I_n$ be two ordered subsets of the integer interval $[1, N] = \{1, 2, \cdots, N\}.$ Denote by $Q$ the following product

\[
\prod_{j=1, j \neq i_1}^{n} (\lambda_{i_1} - \lambda_j)^2 \prod_{j=1, j \neq i_n}^{n} (\lambda_{i_n} - \lambda_j)^2,
\]

then

\[
Q = \Delta_n^2 \prod_{l \in I_n, m \in J_n} (\lambda_l - \lambda_m)^2,
\]
Proof. Next, taking account of the characteristics of (4.2), to calculate\[ \lim_{\varepsilon \to 0} \frac{g}{\varepsilon}, \]
we have\[ Q = \frac{\Delta N}{\Delta_{N_n}}. \]
Under the phase shift
\[ z_i \to \frac{\prod_{j=1, j \neq i}^{N} \lambda_j^2}{2 \prod_{j=1}^{N} (\lambda_i - \lambda_j)^2} \frac{z_i}{\lambda_i^2}, \quad i \in I_n, \]
and the equality
\[ \Delta_n \prod_{j=1, j \neq i}^{N} \lambda_j^2 e^{\alpha x E_{i_1} \cdots E_{i_n}} = \frac{\prod_{j=1}^{N} \lambda_j^2 \prod_{j=1}^{N} \lambda_j^2 (n-1) e^{\alpha x \Delta_{N_n} e^x E_{i_1} \cdots E_{i_n}}}{2^{n-1} \Delta_{N_n} E_{i_1} \cdots E_{i_n}}, \]
which completes the proof by taking account of (2.11).

Proposition 4.3.\] The function $g$ satisfies
\[ \sum_{n=0}^{N+1} \epsilon e^{n-1} h_0 g^{N-n+1} = o(e^{N(N+1)}), \] where
\[ h_0 = 1, \quad h_{N+1} = -\frac{\prod_{i=1}^{N} \lambda_i^2 e^{N x}}{2^{N} \Delta_{N} \prod_{i=1}^{N} E_i}, \]
\[ h_n = d_n e^{(n-1)x}(e^{x}D_{N-n}^2 - 2D_{N-n}^0), \quad n = 1, \ldots, N. \]

Proof. By Lemma 4.2,
\[ g^{N+1} + \sum_{n=1}^{N} \epsilon e^{n-1} g^{N-n+1} d_n e^{\alpha x} D_{N-n}^2 = g^{N} + \sum_{n=1}^{N} \epsilon e^{n-1} g^{N-n+1} d_n e^{\alpha x} D_{N-n}^0 + o(e^{N(N+1)}). \]
combining the equation in terms of $e^{n-1} g^{N-n+1}$ ($n = 0, 1, \ldots, N+1$) leads to the conclusion.

Taking derivatives of (4.2) with respect to $x$ and $t$, we have the following equation:
\[ \frac{g_t}{g - g_x} = \frac{\sum_{i=1}^{N} \epsilon_i g^{N+1} + \sum_{n=1}^{N} \epsilon e^{(n-1)x} g^{N-n+1} d_n e^{(n-1)x}(e^{x}D_{N-n}^2 - 2D_{N-n}^0) + o(e^{N(N+1)})}{g^{N+1} + \sum_{n=1}^{N} \epsilon e^{(n-1)x} g^{N-n+1} e^{\alpha x} D_{N-n}^2 + o(e^{N(N+1)})}. \]
Next, taking account of the characteristics of (4.2), to calculate $\lim_{\varepsilon \to 0} \frac{g}{\varepsilon - g_x}$, we only need to seek positive series solution
\[ g = g^{(0)} + g^{(2)} x^2 + g^{(4)} x^4 + \cdots, \quad g^{(m)} \geq 0 (m = 0, 2, \ldots). \]

In the rest of this section, we will use $x_n$ to denote $\tilde{x}_n$ given by (3.2) (see Remark 3.3).
Proposition 4.4. Suppose that \( \{ h_n \} \) satisfies
\[
h_1 > 0, h_2 > 0, \ldots, h_{n-1} > 0, h_n \leq 0, h_{n+1} < 0, \ldots, h_N < 0, \tag{4.6}
\]
and the series (4.5) satisfies (4.2), then
\[
g \sim g^{(2n-2)} e^{2n-2} = -\frac{h_n}{h_{n-1}} e^{2n-2}, \tag{4.7}
\]
and \( x_{n-1} < x \leq x_n \) \((n = 1, 2, \ldots, N)\).

The proof of Proposition 4.4 need the following two lemmas.

Lemma 4.5. For \( h_n \) given by (4.3), we have
\[
x = x_n \Leftrightarrow h_n = 0, \quad x > x_n \Leftrightarrow h_n > 0, \quad x < x_n \Leftrightarrow h_n < 0. \tag{4.8}
\]

Lemma 4.6. For \( n = 1, 2, \ldots, N+1, \)
\[
x_{n-1} < x_n,
\tag{4.9}
\]
with the convention that \( x_0 = -\infty, x_{N+1} = +\infty. \)

Lemma 4.5 follows from the definition of \( h_n. \) For Lemma 4.6 we recall that \( D^1_{N-n+1} > 0 \) by Lemma 2.12 and then (4.9) follows.

The proof of Proposition 4.4. By Lemmas 4.5 and 4.6 the condition (4.6) holds for some \( n, \) therefore, \( x_{n-1} < x \leq x_n. \) Substituting
\[
g = g^{(2m)} e^{2m} + O(e^{2m+2})
\]
into (4.2), the lowest order of \( \epsilon \) on left hand side of (4.7) is \( m(2N-m+1). \) Note that
\[
m(2N-m+1) \leq N(N+1), \quad 1 \leq m \leq N,
\]
which leads to the coefficient of \( e^{m(2N-m+1)} \) being zero, that is,
\[
h_m(g^{(2m)})^{N-m+1} + h_{m+1}(g^{2m})^{N-m} = 0.
\]
By (4.6) and \( g^{(2m)} > 0, \) we have \( m = n - 1 \) and (4.7), which completes the proof.

According to Lemmas 4.6 and 4.5, for any given \( t, \) if \( x_{n-1} < x \leq x_n \) \((n = 1, 2, \ldots, N)\) and the series (4.5) satisfies (4.2), then \( g \) admits the estimate (4.7) on the order of \( \epsilon. \)

Obviously,
\[
m(m-1) + (2n-2)(N-m+1) \geq (n-1)(2N-n+2), \quad 1 \leq m \leq N,
\]
and the equality holds for \( m = n \) or \( m = n - 1. \) Therefore, for any given \( t, \) the numerator and denominator of the right-hand side of (4.4) share the same lowest power \( e^{(n-1)(2N-n+2)} \) on the interval \( x_{n-1} < x \leq x_n, \) which come from the \( (n-1)\)-th and the \( n\)-th term, respectively, while they share the same lowest power \( e^{N(N+1)} \) given by the \( N\)-th term on \( x > x_N. \) Thus, we have the following proposition.
By Theorem 3.2, we need to show that

\[
\frac{g_t}{g - g_s} = \frac{d_ne^{(N-1)x}(e^x D_n^2 - 2D_n^0)g^{(2N-2)}e^{N(N+1)} + o(e^{N(N+1)})}{-d_ne^{x(N+1)} - 2D_n^0g^{(2N-2)}e^{N(N+1)} + o(e^{N(N+1)})}, \quad x > x_N,
\]

and

\[
\frac{g_t}{g - g_s} = -\frac{G_n}{F_n}, \quad x_{n-1} < x \leq x_n (n = 1, \ldots, N),
\]

with

\[
G_n = \left( d_ne^{(n-1)x}(e^x D_n^2 - 2D_n^0)g^{(2N-2)(N-n+1)} + d_ne^{x(N+1)} \right),
\]

\[
F_n = \left( d_ne^{(n-1)x}D_n^2 - 2D_n^0 \right)g^{(2N-2)(N-n+1)} + d_ne^{x(N+1)}g^{(2N-2)(N-n+1)} + o(e^{(n-1)(2N-n+2)}),
\]

With all the preparations above, we can complete the proof of theorems on convergence.

**Proof of Theorem 3.2** According to Proposition 4.7, \( \lim_{\varepsilon \to 0} \frac{g_t}{g - g_s} \) exists. Using (4.7), for \( n = 1, \ldots, N \), we obtain

\[
\lim_{\varepsilon \to 0} \frac{g_t}{g - g_s} = \frac{e^{-x}d_n - (e^x D_n^2 - 2D_n^0)h_n - d_n(e^x D_n^2 - 2D_n^0)h_n}{-h_nD_n^2 + h_nD_n^2 + \varepsilon}, \quad (4.10)
\]

on \( x_{n-1} < x \leq x_n \). Substituting \( h_n \) defined by (4.3) into (4.10), by Lemma 2.7 (replace \( n \) in (2.13b) with \( N - n \)), we have

\[
\lim_{\varepsilon \to 0} \frac{g_t}{g - g_s} = \frac{e^x D_n^2 + 4e^{-x}D_n^2}{D_n^2}, \quad x_{n-1} < x \leq x_n (n = 1, \ldots, N).
\]

For \( n = N + 1 \), taking account of \( D_0^2 = 1, D_1^2 = A_0 \), we have

\[
\lim_{\varepsilon \to 0} \frac{g_t}{g - g_s} = 2e^{-x}D_1^2 = 2 \sum_{i=1}^{N} c_i^2, \quad x > x_N.
\]

Now we conclude that under appropriate scaling transformations, \( \frac{g_t}{g - g_s} \) converges uniformly to \( \tilde{u}(x; t) \) when \( \varepsilon \to 0 \) i.e. \( \kappa \to 0 \), which completes the proof.

**Proof of Theorem 3.4** By Theorem 3.2, we need to show that

\[
u_n = \sum_{i=1}^{n-1} m_i e^{-(x-x_i)} + \sum_{i=n}^{N} m_i e^{-(x-x_i)}, \quad n = 1, \ldots, N + 1 \quad (4.11)
\]

with \( x_i, m_i \) given by (2.9).
For $n = N + 1$, by $D_{1,N}^0 = 2D_{1}^{-1}$, (4.11) is equivalent to
\[ \sum_{i=0}^{N-1} \frac{(D_{i+1}^0)^2}{D_{i+1}^1} = D_{1}^{-1}. \]
For $n = 1, \ldots, N$, (4.11) is equivalent to
\[ \sum_{i=0}^{N-n} \frac{(D_{i+1}^2)^2}{D_{i+1}^1} = \sum_{i=N-n+1}^{N-1} \frac{(D_{i+1}^0)^2}{D_{i+1}^1} = D_{N-n+2}^{-1} - D_{1}^{-1}. \]
The three identities follow readily from Corollary 2.8 by replacing $n$ in (2.16) with $-1$ and $N-n$, respectively.

5 Proof of Theorem 2.2

Consider the following determinant
\[
A = \begin{vmatrix}
1 & e^{2\xi_1} & \cdots & e^{2\xi_N} & (-1)^{N-1} \cdot 2\kappa

k_1 e^{2\xi_1} - k_1 & k_2 e^{2\xi_2} + k_2 & \cdots & k_N e^{2\xi_N} + (-1)^{N-1} (-k_N) & (2\kappa)^{N-1}

k_1^2 e^{2\xi_1} + k_1^2 & k_2^2 e^{2\xi_2} - k_2^2 & \cdots & k_N^2 e^{2\xi_N} + (-1)^{N-1} (-k_N)^2 & (2\kappa)^{N-2}

\vdots & \vdots & \ddots & \vdots & \vdots

k_1^N e^{2\xi_1} + (-k_1)^N & k_2^N e^{2\xi_2} - (-k_2)^N & \cdots & k_N^N e^{2\xi_N} + (-1)^{N-1} (-k_N)^N & 1
\end{vmatrix}
\]
(5.1)

Easy to find that the determinant $A$ equals to the sum of all the terms $e^{2\xi_{i_1} + \cdots + 2\xi_{i_n}}$ ($n = 1, 2, \ldots, N$) and the term without exponential functions.

Lemma 5.1. For any $N \geq 1$, the determinant $A$ can be expressed as follows:
\[
A = \prod_{i<j} (k_j - k_i) \prod_{i=1}^{N} (1 + 2\kappa k_i) + \sum_{n=1}^{N} e^{2\xi_n} \Gamma_j \Gamma_J \prod_{i<j, i \in I, j \in J} (1 - 2\kappa k_i)(1 + 2\kappa k_j)(k_i + k_j).
\]
(5.2)

Proof. We only need to show that the term without exponential functions equals to
\[
\prod_{i<j} (k_j - k_i) \prod_{i=1}^{N} (1 + 2\kappa k_i),
\]
(5.3)
and for any $1 \leq n \leq N$, $I = \{i_1 < i_2 \ldots < i_n\}$, the coefficient of the term $e^{2\xi_{i_1} + \cdots + 2\xi_{i_n}}$ can be given by
\[
\Gamma_j \Gamma_J \prod_{i<j, i \in I, j \in J} (1 - 2\kappa k_i)(1 + 2\kappa k_j)(k_i + k_j).
\]
(5.4)

Let
\[
\alpha_i = e^{2\xi_i}, \quad \beta_j = \begin{bmatrix} (-1)^{j-1} \\ (-1)^{j-1} (-k_j) \\ \vdots \\ (-1)^{j-1} (-k_j)^N \end{bmatrix}, \quad \gamma = \begin{bmatrix} (2\kappa)^{N} \\ (2\kappa)^{N-1} \\ \vdots \\ 1 \end{bmatrix}
\]
then
\[ A = |\alpha_1 + \beta_1 \quad \alpha_2 + \beta_2 \quad \cdots \quad \alpha_N + \beta_N \quad \gamma|, \]
and the term without exponential function equals to \(|\beta_1 \quad \beta_2 \quad \cdots \quad \beta_N \quad \gamma| \triangleq \tilde{A}. \]
Computing the determinant \(\tilde{A}\), we have
\[
\tilde{A} = (-1)^{\frac{N(N-1)}{2}} \prod_{i<j} (k_i - k_j) \prod_{i=1}^N (1 + 2\kappa k_i) = \prod_{i<j} (k_j - k_i) \prod_{i=1}^N (1 + 2\kappa k_i), \tag{5.5}
\]
we have used the convention: \(k_1 < k_2 < \cdots < k_N\) in the last step.

Replacing the columns of \(\tilde{A}\) with indices \(i_1, \ldots, i_n\) by \(\alpha_{i_1}, \ldots, \alpha_{i_n}\), respectively, we obtain a determinant \(B\), then the coefficient of the term \(e^{2\sum_{i=1}^n \beta_i k_i}\) in determinant \(A\) can be given by \(B\). Let \(\beta'_j = (-1)^{j-1} \beta_j\) \((j \in J)\), \(\gamma' = (2\kappa)^{-N}\gamma\), replacing the columns of \(B\) with indices \(j (j \in J)\), \(N + 1\) by \(\beta'_j (j \in J)\) and \(\gamma'\), respectively, we obtain a Vandermonde determinant \(C\). Besides, according to the operations above, we have
\[
B = (-1)^{\frac{N(N-1)}{2}-(i_1+\cdots+i_n-n)(2\kappa)^N} C,
\]
with
\[
C = (-1)^{\frac{(N-n)(N-2)}{2}} \Gamma_1 \Gamma_J \prod_{i=1}^n \left( \prod_{j \in J, j < i} \prod_{j, j > i} (k_i + k_j) \prod_{i \in J} (\frac{1}{2\kappa} - k_i) (\frac{1}{2\kappa} + k_j) \right) \prod_{i \in J, j \in J} \prod_{i \in J} \prod_{j \in J} (1 + 2\kappa k_i)(1 + 2\kappa k_j)(k_i + k_j), \tag{5.5}
\]
we have applied the convention: \(i_1 < i_2 < \cdots < i_n\) and \(k_1 < k_2 < \cdots < k_N\) multiple times to adjust the factors involving -1. Therefore
\[
B = \Gamma_1 \Gamma_J \prod_{i \in I, j \in J} (1 - 2\kappa k_i)(1 + 2\kappa k_j)(k_i + k_j),
\]
which together with \eqref{5.5} completes the proof of Lemma\ref{5.1} \hfill \Box

Let \(A'\) be the determinant given by replacing the \(N + 1\) column of \(A\) by
\[
\delta = \left( 1 \quad \frac{1}{2\kappa} \quad (-\frac{1}{2\kappa})^2 \quad (\frac{1}{2\kappa})^N \right)^T.
\]
Similar to the proof of Lemma\ref{5.1} we have
\[
A' = \prod_{i<j} (k_j - k_i) \prod_{i=1}^N (1 - 2\kappa k_i) + \sum_{n=1}^N e^{2\sum_{i=1}^n \beta_i k_i} \Gamma_1 \Gamma_J \prod_{i \in J, j \in J} (1 + 2\kappa k_i)(1 - 2\kappa k_j)(k_i + k_j). \tag{5.6}
\]
Furthermore, it is easy to check that \eqref{5.2} and \eqref{5.6} can be rewritten as
\[
A = \prod_{i<j} (k_j - k_i) \prod_{i=1}^N (1 + 2\kappa k_i) g_1, \quad A' = \prod_{i<j} (k_j - k_i) \prod_{i=1}^N (1 - 2\kappa k_i) g_2,
\]
where
with $g_1$ and $g_2$ given by (2.3a) and (2.3b), respectively. Taking account of the following identities

$$(2\kappa)^N 2^N \exp \left( \sum_{i=1}^{N} \xi_i \right) W(\Phi_1, \ldots, \Phi_N, e^{\frac{\kappa}{2}}) = e^{\frac{\kappa}{2} A},$$

$$(−2\kappa)^N 2^N \exp \left( \sum_{i=1}^{N} \xi_i \right) W(\Phi_1, \ldots, \Phi_N, e^{-\frac{\kappa}{2}}) = e^{-\frac{\kappa}{2} A'},$$

We obtain

$$(-1)^N \frac{f_1}{f_2} = e^{\frac{\kappa}{2} g_1} \frac{g_1}{g_2} \prod_{i=1}^{N} a_i, \quad (5.7)$$

Note that $g_1 > 0$, $g_2 > 0$, hence,

$$\left| \frac{f_1}{f_2} \right| = e^{\frac{\kappa}{2} g_1} \frac{g_1}{g_2} \prod_{i=1}^{N} a_i,$$

For any given $N$,

$$r = \frac{\kappa f_1 f_2}{\prod_{i=1}^{N} (k_i^2 - 4\kappa^2)} \quad (5.8)$$

(see [8,9]) is positive by the equation (5.7). Thus, the reciprocal transformation: $dy = rdx - urdt$ has the inverse: $dx = \frac{1}{r} dy + u dt$, and

$$\frac{\partial x}{\partial y} = \frac{1}{r(y,t)}, \quad \frac{\partial x}{\partial t} = u(y,t). \quad (5.9)$$

is integrable, therefore,

$$\left( \frac{1}{r} - \frac{1}{\kappa} \right)_t = \left( \frac{1}{r} \right)_t = u_y \left( \ln \frac{g_1}{g_2} \right)_y.$$

Since $u, u_x, u_{xx} \to 0$, $r \to \kappa$ when $|x| \to \infty$, we have $u, u_y \to 0$, $r \to \kappa$ when $|y| \to \infty$, which leads to

$$\left( \ln \frac{g_1}{g_2} \right)_y + \beta(y) = \frac{1}{r} - \frac{1}{\kappa},$$

where $\beta(y)$ is an arbitrary function of $y$, and $\beta(y) \to 0$ as $|y| \to \infty$, particularly, we can choose $\beta(y) = 0$. Thus, by the first equation in (5.9), we have (2.2) with $\alpha$ an arbitrary integral constant.

We now complete the proof of Theorem 2.2.

**Remark 5.2.** In [12], Parker provided the N-soliton solutions via Hirota bilinear method up to $N = 4$ by means of Mathematica, these solutions take the same form as Theorem 2.2, but the proof for the general case $N > 4$ has not been given.
6 Concluding remarks

In this paper, we provide a new representation for the $N$-soliton solutions of the dispersion CH equation, obtained via Darboux transformation by Li. The new representation also provides a rigorous proof for bilinear solution of dispersion CH equation (see Theorems 2.2 and 2.3), which completes the work in [11, 12]. We also show that the $N$-peakon of the dispersionless CH equation can be recovered from a sequence of smooth $N$-soliton solutions of (1.1), which establishes the relation between $N$-soliton solutions of (1.1) and $N$-peakon solutions of (1.2) for any $N \geq 1$. We believe that there is certain physics behind the mathematics we proved, while we still cannot explain it completely.

In [15], the authors also obtained smooth soliton solutions of (1.1) via Darboux transformation, while for the occurrence of the first and second derivatives of Wronskian, the multi-soliton they construct is very complicate, and it is difficult to investigate the convergence. Whether the multi-soliton solutions given by them are equivalent to the one studied in this paper is still unclear. If so, we may obtain the transformation between peakon solution of (1.2), which is attractive.

At present, the existence of soliton solutions with both positive and negative asymptotic speeds is still not clear, though the peakon-antipeakon (amplitudes $m_i$ in (1.3) have different signs) solutions of the dispersionless CH equation (1.2) have been given by the same formulas (1.3) in Lemma 1.3. In [26], the authors investigated multi-soliton solutions of dispersive CH equation, the formulas in [8, 9] may not give smooth soliton solutions, the existence of soliton solutions with both positive and negative asymptotic speeds and the relation to the peakon-antipeakon solutions both are interesting topics for further study.

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