Network coding for multicasting over Rayleigh fading multi access channels

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Abstract—This paper examines the problem of rate allocation for multicasting over slow Rayleigh fading channels using network coding. In the proposed model, the network is treated as a collection of Rayleigh fading multiple access channels. In this model, rate allocation schemes that are based solely on the statistics of the channels are presented. The schemes are aimed to minimize the outage probability. An upper bound is presented for the probability of outage in the fading MAC. A suboptimal solution based on this bound is given for the MAC network model. We also consider outage probability minimization for TDMA schemes for the multiple access channels.

Index Terms—Network coding, wireless networks, multicast, outage capacity, Rayleigh fading, multiple access channels.

I. INTRODUCTION

Network coding extends the functionality of intermediate nodes from storing/forwarding packets to performing algebraic operations on received data. If network coding is permitted, the multicast capacity has been shown to be equal to the minimal min-cut between the source and each of its destinations [1]. In the past decade, the concept of combining data by network coding has been extensively extended and it is well known that in order to achieve the multicast rate, a linear combination over a finite field suffices if the field size is larger than the number of destinations. Moreover, centralized linear network coding can be designed in polynomial time. Decentralized linear network coding can be found using a random code approach (see [2], §15.3 and the references therein).

We briefly introduce related work on topology management and rate allocation for network coding in multicast over wireless networks. The problem of finding a minimum-cost scheme (while maintaining a certain multicast rate) in coded networks was studied by Lun et al. [3, 4]. They showed that there is no loss of optimality when the problem is decoupled into: finding the optimal coding rate allocation vector (also known as subgraph) and designing the code that is applied over the optimal subgraph. When addressing the problem of rate allocation for multicast with network coding in wireless networks, Lun et al., [4, 5] tackled the problem through the so-called wireless multicast advantage phenomenon. This phenomenon simply comes down to the fact that when interference is avoided in the network (e.g., by avoiding simultaneous transmissions), communication between any two nodes is overheard by their nearby nodes due to the broadcast nature of the wireless medium. In [5], the wireless multicast advantage was used to reduce the transmission energy of the multicast scheme (since when two nodes communicate, some of their nearby nodes get the packet for “free”). Therefore, their wireline minimum-cost optimization problem was updated accordingly [see 5, eq,(1) and (40)]. In [6] interference is allowed but is assumed to be limited. Joint optimal power control, network coding and congestion control is presented for the case of very high SINR (signal to noise plus interference ratio). This interference assumption implies that there are some limitations on simultaneous transmissions and this is taken into account in the optimization problem.

The demand for interference free channels at all nodes means that some level of orthogonality is required between different transmissions in the network. However, avoiding interference between all nodes comes at the cost of loss of expensive bandwidth, or alternatively leads to rate degradation in band limited systems. The same argument can be applied to the limited interference model since some orthogonality at a certain radius is required. The multiple access channel (MAC) network model was introduced in [7]. In the MAC network model, the superposition property of the wireless medium is exploited and the network is treated as a collection of MACs, such that each receiver simultaneously receives data from all its in-neighbors. Due to the convexity of the capacity region of the MAC [2], [7] presented a joint power control and rate allocation solution for a (convex) network utility.

There are certain other considerations that must be taken into account in the search for a rate allocation vector in wireless networks. The wireless medium varies over time and suffers from fading channels due to multipath or shadowing for example. In [8] the block fading model was introduced. In this model the channel gain is assumed to be constant over each coherence time interval. Typically, fading models are classified as fast fading or slow fading. In fast fading, the coherence time of the channel is small relative to a code block length and as a consequence the channel is ergodic with a well-defined Shannon capacity (also known as ergodic capacity [9]). In slow fading the code block length and the coherence time of the channel are of the same order. As a consequence the channel is not ergodic and usually the Shannon capacity is not a good
The notion of outage capacity was introduced in [8] for transmitting over fading channels when the channel gain is available only at the receiver. In this approach, we transmit at a certain rate and tolerate some information loss when an outage event occurs. An outage event occurs whenever the transmitted rate is not supported by the instantaneous channel gain; i.e., when the channel gain is too low for successful decoding of the transmitted message. It is assumed that the outage event occurs with low probability and therefore a reliable communication is available most of the time. A different strategy to deal with slow fading is the broadcast channel approach [10]. In this approach different states of the channel are treated as channels toward different receivers (a receiver for each state). Hence, the same strategy used for sending common and private messages to different users on the Gaussian broadcast channel can be applied here. When the channel gain is also available also at the encoder, the encoder can adapt the power and the transmission rate according to the instantaneous state of the channel and thus it achieves a higher rate on average. Moreover, as regards the outage capacity, the transmitter can use power control to conserve power by not transmitting at all during designated outage periods.

When dealing with outage capacity for fading MAC, the common outage has a similar definition to the outage event in the point to point case. A common outage event is declared whenever we transmit with rates that are not supported by the instantaneous channel gains. If the channel gains are available at both the decoder and the encoders, there are additional notions of capacities for the fading MAC to be taken into account. The throughput capacity region for the Gaussian fading MAC was introduced in [11]. In a nutshell, this is the Shannon capacity region when the codewords can be chosen as a function of the realization of the fading with arbitrarily long coding delays. However, as for the point to point case, this approach is not realistic in slow fading cases since it requires a very long delay to average out the fading effect. [12] derived the delay limited capacity for the Gaussian fading MAC (also known as the zero outage capacity). In the delay limited capacity, unlike the throughput capacity, the chosen coding delay has to work uniformly for all fading processes with a given stationary distribution. However, the delay limited capacity is somewhat pessimistic due to the demand of maintaining a constant rate under any fading condition. The outage capacity region and the optimal power allocation for a fading MAC were described in [13]. As was pointed out in [13], in a slow fading environment, the decoding delay depends solely on the code-length employed and not on the time variation of the channel.

In this paper we study the problem of rate allocation for multicasting over slow Rayleigh fading channels using network coding. The problem is examined in the MAC network model, where the network is treated as a collection of Rayleigh fading MACs. In our network model, we assume that links on the network vary faster than the entire network can respond to the variations. Therefore, our goal is to find a rate allocation scheme that is based solely on the statistics of the channels which minimizes the outage probability. The communication model is described in details in section II. In section III the optimal rate allocation scheme is given for a fading time-division multiple access (TDMA) based network. In section IV a suboptimal solution for the rate allocation problem is presented for the MAC network model. The solution is based on an upper bound on the probability of outage in the fading MAC. In section V we report some simulation results. We end with concluding remarks.

II. COMMUNICATION MODEL

Let \( G = (V, E) \) be a directed graph with the set of nodes \( V \) and directed edges \( E \subset V \times V \), where transceivers are nodes and channels are edges representing a wireless communication network. The cardinality of any set \( A \) is denoted by \(|A|\). Random variables are denoted by a capital letter. Inequalities between vectors are defined element-wise; i.e., \( v \leq u \) implies \( v_i \leq u_i \) for all \( i \). For any node \( j \in V \), we denote the in-neighborhood and out-neighborhood of \( j \) by \( I(j) \) and \( O(j) \) respectively, i.e., \( I(j) = \{ i : (i, j) \in E \} \) and \( O(j) = \{ i : (j, i) \in E \} \). The rate transmitted on a link \((i, j)\) is denoted by \( r_{i,j} \), the rate allocation vector is denoted by \( r = [r_{i,j} : (i, j) \in E] \) and the local rate allocation vector is denoted by \( r_j = [r_{i,j} : i \in I(j)] \).

In the MAC network model, the network is treated as a collection of multiple access channels such that each receiver simultaneously receives data from all its in-neighbors. It is assumed that there is no interference between transmissions toward different receivers (see Fig. 1). This can be achieved by orthogonal transmissions (e.g., by using a different frequency band, time sharing or using directional antennas). Clearly, this is an improvement over a model where all transmissions are orthogonal. The MAC network model is extensively described for the deterministic case in [7]. This model can easily be adapted to deal with fast fading channels in the case

![Fig. 1. The MAC network model: An illustration of the wireless network as a directed graph \( G \). In the MAC network model each receiver receives data from all its in-neighborhood nodes. For example the nodes in \( I(5) = \{3, 4\} \) transmit toward node 5 and the nodes in \( I(7) = \{2, 3, 6, 8\} \) transmit toward 7. However, it is assumed that (for example), there is no interference between the transmissions toward node 5 and the transmissions toward node 7.](Image)
of a constant power allocation vector by using the ergodic MAC capacity region instead of the MAC capacity region. This ergodic capacity region is easily obtained by taking the expectations of the capacity constraints [2]. Here, we examine the MAC network model in the case of slow fading channels.

The channel gain of link \((i, j)\) is denoted by \(H_{i,j}\). \(H_{i,j}\) is a circular complex normal random variable \(H_{i,j} \sim \mathcal{CN}(0, \sigma_{i,j}^2)\). It is assumed that all \(H_{i,j}\) are independent of each other. The transmission on link \((i, j)\) is denoted by \(x_{i,j}\) and it is transmitted with an average power \(p_{i,j}\). We assume that \(\sigma_j^2\) is the variance of \(N_j\) - the zero mean Gaussian noise at node \(j\). Hence, the received signal at node \(j\) is given by:

\[
Y_j = \sum_{i \in I(j)} H_{i,j} x_{i,j} + N_j.
\]

Fig. 2 illustrates the MAC of node 7 in the network of Fig. 1. When the instantaneous channel gains \(H_{i,j}\) are deterministic and known, this is the well-known Gaussian multi access channel [2]. Hence, the instantaneous MAC capacity region is given by:

\[
\gamma_j^{\text{ins}}(H) := \left\{ r_{i,j} : \sum_{i \in M(j)} r_{i,j} \leq \log_2 \left( 1 + \frac{P_{M(j),j}}{\sigma_j^2} \right) \right\},
\]

where \(P_{M(j),j} = \sum_{i \in M(j)} p_{i,j}|H_{i,j}|^2\). However, when dealing with Rayleigh channels this capacity region may not be a good measure of performance and the outage capacity is a better and more practical alternative. A common outage event is jointly declared for all links whenever we transmit toward a certain node with rates that are not supported by the instantaneous MAC capacity region.

**Definition 1:** For a rate vector \(r_j = [r_{i,j} : i \in I(j)]\) and for the MAC associated with node \(j\), the common outage event is

\[
r_j \notin \gamma_j^{\text{ins}}(H),
\]

where the capacity region \(\gamma_j^{\text{ins}}(H)\) is defined in (2).

**Definition 2:** The probability of outage in the fading MAC of node \(j\) is given by

\[
P_j^{\text{out}} = \Pr\left( r_j \notin \gamma_j^{\text{ins}}(H) \right).
\]

Similar to these definitions, we define an outage event and outage probability for the MAC network model.

**Definition 3:** The outage event for the MAC network model is the event for which there exists node \(j \in V\) such that \(r_j \notin \gamma_j^{\text{ins}}(H)\).

Hence, the probability of outage in the MAC network model is given by

\[
P_{\text{out}}^{\text{MAC}} = \Pr \left( \bigcup_{j \in V} \{ r_j \notin \gamma_j^{\text{ins}}(H) \} \right).
\]

To complete the description of the local communication model, we associate the codebooks, the encoders \((F_{i,j} : i \in I(j))\) and the decoder \(g_j\) that establish the connection between \(I(j)\) and \(j\) at rates \(r_{i,j} : i \in I(j)\) to any node in the network. Obviously, node \(j\) shares the appropriate codebooks and encoders with its in-neighbors.

We now briefly define the underlying network coding model. The source node is denoted by \(s \in V\) and it is assumed that \(I(s) = \emptyset\). The set of all destinations (sinks) is denoted by \(D_s \subseteq V \setminus \{s\}\).

Intermediate nodes are allowed to send out packets that are a combination of their received information and as a result they break the flow conservation by increasing/decreasing the outside rate. However, the main theorem of network coding for multicast is stated in terms of the max-flow (min-cut) between each source and its destinations. Therefore, we distinguish between the flow at an edge \((i, j)\) and the actual rate at that link. Let \(f_{i,j}\) be the flow at edge \((i, j)\) destined for destination \(d \in D_s\), and let \(r_{i,j}\) be the actual rate at edge \((i, j)\).

The communication parameters are summarized in Table I.

**Table I: A summary of communication model parameters**

| Parameter   | Description                                                                 |
|-------------|-----------------------------------------------------------------------------|
| \(x_{i,j}\) | Transmission on link \((i, j)\)                                             |
| \(H_{i,j}\) | Channel gain of link \((i, j)\), \(H_{i,j} \sim \mathcal{CN}(0, \sigma_{i,j}^2)\) |
| \(p_{i,j}\) | Average power of the transmission on link \((i, j)\)                        |
| \(N_j\)    | Noise at node \(j\), \(N_j \sim \mathcal{CN}(0, \sigma_j^2)\)             |
| \(Y_j\)    | Received signal at node \(j\)                                              |
| \(r_{i,j}\) | Rate at link \((i, j)\)                                                     |
| \(r_j\)    | Local rate allocation vector \(r_j = [r_{i,j} : i \in I(j)]\)              |
| \(r\)      | Rate allocation vector \(r = [r_{i,j} : (i, j) \in E]\)                    |
| \(\gamma_j^{\text{ins}}(H)\) | Instantaneous MAC capacity region of node \(j\)                           |
| \(P_j^{\text{out}}\) | Probability of outage in the fading MAC of node \(j\)                  |
| \(P_{\text{out}}^{\text{MAC}}\) | Probability of outage in the MAC network model                         |
| \(s\)      | Source node                                                                  |
| \(D_s\)    | Set of all destinations \(D_s \subseteq V \setminus \{s\}\)              |
| \(f_{i,j}\) | Flow at edge \((i, j)\) destined for destination \(d \in D_s\)            |

As mentioned in section I, there is no loss of optimality by first finding the optimal rate allocation solution and then designing the coding scheme that realizes the connection. In the following section the rate allocation vector for the MAC network model is given as the solution to an optimization problem and the coding scheme that realizes the connection is assumed to be given. For large scale networks, where global network information is not available, the random network coding shown in [14, 15] can be employed. In general, in random network coding, intermediate nodes store all their received packets in their memory and when a packet injection occurs on an outgoing link, the node forms a packet that is a random linear combination of the packets in its memory.
In order to enable decoding at the destinations, the random coefficients of the linear combinations are included in the header of the packet as side information. These coefficients are called the global encoding vector of the packet. Decoding is possible if all destinations collected enough packets with linearly-independent global encoding vectors.

We use the algorithm shown in [15] and adjust it to the MAC network model. At initiation, the source node $s$ stores a block of $K$ words $w_{1s}, w_{2s}, \ldots, w_{Ks}$ in its memory, where $w_{ks}$ is a vector of length $\lambda_s$ over a finite field $\mathbb{F}_q$. The source creates output words for link $(s,j)$ at rate $r_{sj}$. The output words are formed by taking linear combinations of the words in the source’s memory; i.e., the $k$’th output word of source $s$ is obtained by $x_k = \sum_{n=1}^{K} \gamma_k(n)x_n$. The coefficients of the combinations are drawn uniformly from $\mathbb{F}_q$ and they are included in the header of the packet which is denoted by $\gamma \in \mathbb{F}_q^K$. The header is called the global encoding vector.

The operation of an intermediate node is as follows. It is assumed that each node in the network has a memory of size $(\lambda + 1)K$. Whenever a node receives a word, it stores the word in its memory if the word’s global encoding vector is linear-independent (over the finite field $\mathbb{F}_q$) of the words that have already been stored. This ensures that there are at most $K$ words in the memory of each node. In the outgoing links, the operation of an intermediate node is very similar to the operation of a source node. Each node $i$ creates output words for link $(i,j)$ at rate $r_{ij}$. Each word is formed by taking linear combinations of the words that are in the node’s memory. The coefficients of the combinations are drawn uniformly from $\mathbb{F}_q$. Assume that the node has $L$ words in its memory $\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_L$ with headers $\alpha_1, \alpha_2, \ldots, \alpha_L$, respectively.

The $k$’th output word is formed by $\hat{x}_k = \sum_{\ell=1}^{L} \tilde{\gamma}_k(\ell)\hat{x}_\ell$, where the coefficients $\tilde{\gamma}_k(\ell) \in \mathbb{F}_q$ for $1 \leq \ell \leq L$ are chosen according to a uniform distribution over the elements of $\mathbb{F}_q$. Since all operations on the network are linear (and over a finite field), the $k$’th output word can be written as $\tilde{x}_k = \sum_{n=1}^{K} \gamma_k(n)x_n$, where $\gamma_k(n) = \sum_{\ell=1}^{L} \tilde{\gamma}_k(\ell)\alpha_1(n)$. Hence, the header of the $k$’th output word is set to $\gamma_k := \{\gamma_k(1), \gamma_k(2), \ldots, \gamma_k(K)\}$. In other words, the same linear combination applied to the word is applied to the global encoding vector of the words. This will help us keep track of which linear combinations are applied to the original source words when they flow through the network.

In the lower level protocol, the words are encoded at node $i$ by the encoder $F_{i,j}$. In particular, node $i$ injects packets at link $(i,j)$ at rate $r_{ij}$, where each packet is the encoding of the output words that were created at node $i$ towards $j$ since the transmission of the last packet at link $(i,j)$.

Whenever a sink node $d \in D_s$ collects $K$ words with linearly-independent global encoding vectors, it is able to decode the $K$ words of source $s$ and it informs all nodes to flush the words from their memory. The random network coding scheme for the MAC network model is discussed in more details in [7].

### III. Rate Allocation for TDMA Based Network Model

Before we discuss the rate allocation for the fading MAC network model, we present a simpler model. Consider a TDMA based network where each link is an interference-free point to point channel. It is assumed that the bandwidth is equally divided between the in-neighbors of each receiver (for example, in the network shown in Fig. 1, we can transmit at link $(4, 9)$ only half of the time, and during that time link $(6, 9)$ is silenced and vice versa). In this model, the data on each link $(i,j)$ is transmitted at rate $r_{ij}$ only $\frac{1}{2|U_j|}$ of the time. Hence, the effective rate on each link is given by $r_{ij}^{\text{eff}} = \frac{r_{ij}}{2|U_j|}$ and the effective rate allocation vector is denoted by $\mathbf{r}^{\text{eff}} = \{r_{ij}^{\text{eff}} : (i,j) \in E\}$. The transmission on link $(i,j)$ is denoted by $x_{ij}$ and it is transmitted with an average power of $\delta_j p_{i,j}$, where $\delta_j = 1$ for the naive TDMA and $\delta_j = |T(j)|$ for the non-naive TDMA. Note that in cases where the power constraints must be strictly satisfied we are constrained to the naive TDMA (e.g., under power mask constraints). Denote by

$$e_{\text{ins}}^{\text{ins}}(h) = \log_2 \left( 1 + \frac{\delta_j p_{i,j} |h|^2}{\sigma_j^2} \right)$$

the instantaneous capacity of link $(i,j)$. Obviously, since all $H_{i,j}$ are independent random variables so are the instantaneous capacities $C_{\text{ins}}^{\text{ins}}(H_{i,j})$. The probability of outage in this model for a given rate allocation vector $\mathbf{r}$ is the probability that there exists a link with rate $r_{ij}$ that is not supported by the instantaneous capacity $C_{\text{ins}}^{\text{ins}}(H_{i,j})$. Hence, the probability of outage is given by

$$P_{\text{out}^{\text{TDMA}}} = 1 - \prod_{(i,j) \in E} \Pr \{ C_{\text{ins}}^{\text{ins}}(H) \geq r_{ij} \}. \quad (7)$$

Since $|H_{i,j}| \sim \text{Rayleigh}(\sigma_j)$, it is easy to verify that $C_{\text{ins}}^{\text{ins}}(H)$ has a cumulative distribution function (CDF)

$$F_{C_{\text{ins}}^{\text{ins}}(r_{ij})} = 1 - e^{-\lambda_{i,j}(2^{r_{ij}} - 1)}$$

where $\lambda_{i,j} = \frac{1}{2\sigma_j^2} \frac{n a}{\delta_j p_{i,j}}$. Substituting (8) into (7) yields

$$P_{\text{out}^{\text{TDMA}}} = 1 - \prod_{(i,j) \in E} e^{-\lambda_{i,j}(2^{r_{ij}} - 1)}.$$ \quad (9)

Therefore, it can be verified that the optimal rate allocation that minimizes the outage probability for the TDMA model while maintaining a multicast-rate demand $R_s$ is given by the
following convex optimization problem

\[
\min_{f, x} \sum_{(i, j) \in E} \lambda_{i, j} 2^{x_{i,j}} \quad \text{(10)}
\]
subject to

\[
0 \leq f_{i,j}^d \quad \forall (i, j) \in E, d \in D_s,
\]

\[
0 \leq f_{i,j}^d \leq r_{i,j}^\text{eff} = \frac{r_{i,j}}{[2(n)]} \quad \forall (i, j) \in E, d \in D_s,
\]

\[
\sum_{i \in \mathcal{I}(j)} f_{i,j}^d = \sum_{i \in \mathcal{O}(j)} f_{j,i}^d = \begin{cases} \lambda_{s,d} & j = d \in D_s \\ 0 & j \notin \{s, d\} \end{cases} \quad \forall j \in V, d \in D_s,
\]

where (10a) and (10b) are the flow positivity and the flow conservation constraints, respectively. Consider the effective-rate graph \( \tilde{G} = (V, E, r^\text{eff}) \), the constraints (10a)-(10c) guarantee that any feasible solution of (10) provides a minimum min-cut of at least \( R_s \) between the source and each destination. Therefore, a multicast rate of \( R_s \) is achievable by network coding, see Theorem 1 in [4].

It is worth mentioning that when we have common signal to noise ratio (SNR) for all links; i.e., \( \text{SNR} = \frac{P_i}{\sigma_i^2} \) \( \forall (i, j) \in E \), (10) degenerates into

\[
\min_{f, x} \sum_{(i, j) \in E} \frac{1}{\delta_{i, j}} 2^{x_{i,j}} \quad \text{(11)}
\]
subject to

\[
0 \leq f_{i,j}^d \leq r_{i,j}^\text{eff} = \frac{r_{i,j}}{[2(n)]} \quad \forall (i, j) \in E, d \in D_s
\]

\[
\sum_{i \in \mathcal{I}(j)} f_{i,j}^d = \sum_{i \in \mathcal{O}(j)} f_{j,i}^d = \begin{cases} \lambda_{s,d} & j = d \in D_s \\ 0 & j \notin \{s, d\} \end{cases} \quad \forall j \in V, d \in D_s
\]

and the optimal solution of (11) does not depend on the SNR but solely on the topology of the network and the multicast rate demand. The probability of outage in the i.i.d case is then given by

\[
\Pr_{\text{TDMA}}^{\text{out}} = 1 - \exp\left(-D \cdot \text{SNR}^{-1}\right),
\]
where \( D = \sum_{(i, j) \in E} \frac{1}{2\delta_{i, j}} (2^{x_{i,j}} - 1) \) is a constant that depends solely on the network topology (and the multicast rate demand). Note that asymptotically - in high SNR; i.e., \( \text{SNR} = \frac{P_i}{\sigma_i^2} \gg 1 \), the minimal probability of outage in the i.i.d network case is approximately

\[
\Pr_{\text{TDMA}}^{\text{out}} \approx D \cdot \text{SNR}^{-1}.
\]

IV. RATE ALLOCATION FOR THE FADING MAC NETWORK MODEL

In this section we study the problem of finding the rate allocation vector for the fading MAC network model discussed in the previous sections. In our wireless model we assume a slow fading model with independent Rayleigh fading channels; i.e., the absolute value of the channel gain of all links are independent and follows the Rayleigh distribution. While it is assumed that the instantaneous channels gain may be available at both the encoders and the decoders, we assume that the rate allocation vector is determined a-priori, based solely on the statistic. The reason for this assumption is that the rate allocation vector is determined based on network considerations, whereas the instantaneous state of each component of the network varies faster than the entire network can respond to the variations. Note that this assumption is practical as well when the power constraints must be satisfied in each encoding block (e.g. when we are under Federal Communications Commission (FCC) regulations).

The probability of outage in a Rayleigh fading MAC is related to the joint distribution of the linear combinations of exponential random variables. However, the computation of the probability of a common outage becomes extremely complicated in a MAC with more than 2 links. Therefore, instead of using the exact expression of the probability of common outage, we relax the problem and minimized an upper bound on the outage probability of a multiple access channel. For simplicity, we only consider a MAC with i.i.d links; i.e., \( \lambda_{i,j} = \frac{1}{2^{n_i} \sigma_i^2} \lambda_j \). Note that in [16] it was pointed out that the outage probability is bounded from below by

\[
\Pr_{\text{MAC}}^{\text{out}} \geq 1 - e^{-\lambda S_n} \Gamma[n, \lambda (2R_n - 1 - S_n)] \frac{1}{[n-1]!} \]

where \( \Gamma(n, x) \) is the incomplete gamma function, \( S_n = \sum_{k=1}^{n} (2^{n_k} - 1) \) and \( R_n = \sum_{i=1}^{n} r_i \). The bound may also be expressed as

\[
\Pr_{\text{MAC}}^{\text{out}} \geq 1 - e^{-\lambda (2R_n - 1 - S_n)} G \left( \lambda (2R_n - 1 - S_n) \right),
\]

where, \( G(x) = \sum_{k=1}^{n} \frac{1}{k!} x^k \).

The upper bound for a MAC with \( n \) links is given in the following theorem:

\textbf{Theorem 1:} The probability of common outage of a MAC with \( n \geq 3 \) i.i.d Rayleigh channels is bounded by

\[
\Pr_{\text{MAC}}^{\text{out}} \leq 1 - e^{-\lambda (2R_n - 1)} \tilde{G}(\lambda R_n) \]

where \( \tilde{G}(x) = \frac{1}{2} x^2 + x + 1 \) and \( \alpha_n = \prod_{i=1}^{n} (2^{i^2} - 1) \).

\textbf{Proof:} The proof will be given in [17]. \qed

\textbf{Remark 1:} The probability of common outage of MAC with \( n \) i.i.d Rayleigh channels is bounded by \( \Pr_{\text{MAC}}^{\text{out}} \leq 1 - e^{-\lambda (2R_n - 1)} \).

\textbf{Proof:} For \( n = 1 \), we have a Rayleigh fading Gaussian channel with outage probability given by (8):

\[
\Pr_{\text{MAC}}^{\text{out}} = \Pr_{\text{out}} = 1 - e^{-\lambda (2^{r_1} - 1)}.
\]

For \( n = 2 \), a simple computation yields that

\[
1 - \Pr_{\text{MAC}}^{\text{out}} = e^{-\lambda (2^{r_1+1}r_2 - 1)} (1 + \lambda (2^{r_1 - 1} (2^{r_2} - 1))) \geq e^{-\lambda (2^{r_1+1}r_2 - 1)}.
\]
For \( n \geq 3 \), the claim follows from Theorem 1 and the fact that for any non-negative \( x \), we have \( G(x) \geq 1 \).

The probability of outage in the MAC network model is given in equation (5). By assumption, all \( H_{i,j} \) are independent of each other. Therefore, the probability of outage in the MAC network model can be rewritten as

\[
P_{\text{out}}^{\text{MAC}} = 1 - \prod_{j \in V \setminus \{s\}} (1 - P_{j}^{\text{out}}).
\]

(19)

Obviously,

\[
P_{\text{out}}^{\text{MAC}} \leq 1 - \prod_{j \in V \setminus \{s\}} \left(1 - \tilde{P}_{j}^{\text{out}}\right)
\]

(20)

if \( P_{\text{out}}^{j} \) is bounded above by \( \tilde{P}_{j}^{\text{out}} \). Although the bound in Remark 1 is weaker than the one we get from Theorem 1, we used the weaker bound for finding a rate allocation vector for the outage MAC model. Hence we have

\[
P_{\text{out}}^{\text{MAC}} \leq 1 - \prod_{j \in V \setminus \{s\}} e^{-\lambda_j \left(2^{R_j} - 1\right)},
\]

(21)

where \( \tilde{R}_j = \sum_{i \in I(j)} r_{i,j} \). Finally, since \( e^{-\lambda \left(2^{R_j} - 1\right)} \) is log-concave the problem becomes computationally tractable:

\[
\min_{f, r} \sum_{j \in V \setminus \{s\}} \lambda_j 2^{\tilde{R}_j}
\]

subject to

\[
0 \leq f_{i,j}^{d} \leq r_{i,j} \quad \forall (i,j) \in E, d \in D \\
\sum_{i \in I(j)} f_{i,j}^{d} = \sum_{i \in O(j)} f_{j,i}^{d} = \begin{cases} 
R_s & j \notin \{s,d\} \\
R_s & j = d \end{cases} \\
\quad \forall j \in V, d \in D_s \\
\tilde{R}_j = \sum_{i \in I(j)} r_{i,j} \quad \forall j \in V \setminus \{s\}.
\]

(22)

V. SIMULATION RESULTS

In this section the probability of outage of the suboptimal algorithm for the MAC network model is presented. In the simulation we consider the network shown in Fig. 3 where it was assumed that all links are i.i.d Rayleigh(\( \nu \)) channels, with \( \nu = 1 \). We solved (22) for various values of SNR = \( \frac{P}{\sigma^2} \) and the results are shown in Fig. 4. The lower and upper bounds for the outage probability were obtained by calculating (14) and (16) (respectively) for each MAC associated with node \( j \) if \( I(j) \geq 3 \) and the exact expression of the outage probability for each receiver \( j \) with \( I(j) \leq 2 \). As can be seen, up to 6 bits/sec/Hz the bounds are quite tight.

In addition we compared the performance of the fading MAC network model to the performance of the TDMA model. It is shown in Fig. 5 (for the naive TDMA model) and in Fig. 6 (for the non-naive TDMA model) that the MAC network model outperforms both TDMA based models.
VI. CONCLUSIONS

In this paper we studied the rate allocation problem for multicasting over slow Rayleigh fading channels using network coding. Rate allocation schemes based solely on the statistics of the channels were presented. In the TDMA based model, where the links are interference-free slow Rayleigh fading channels, the optimal rate allocation that minimizes the outage probability is found by solving a convex optimization problem. In the MAC network model, where the network is treated as a collection of slowly Rayleigh fading multiple access channels, we proposed a suboptimal scheme that is given as the solution to a convex optimization problem. This suboptimal solution is based on an upper bound on the probability of outage of a fading multiple access channel. In the simulation results, it is shown that the MAC network model outperforms the TDMA based model.

Acknowledgements: This research was supported by the Israel Ministry of Labor, Trade and Commerce, as part of the RESCUE consortium.

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