UNIQUENESS OF ANCIENT COMPACT NON-COLLAPSED
SOLUTIONS TO THE 3-DIMENSIONAL RICCI FLOW

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Abstract. In this paper we study the classification of compact $\kappa$-noncollapsed ancient solutions to the 3-dimensional Ricci flow which are rotationally and reflection symmetric. We prove that any such solution is isometric to the sphere or the type II ancient solution constructed by G. Perelman in [29].

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1. Introduction

Consider an ancient compact 3-dimensional solution to the Ricci flow

\[
\frac{\partial}{\partial t} g_{ij} = -2R_{ij}
\]  

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existing for $t \in (−\infty, 0)$ so that it shrinks to a round point at $T$. The goal in this work is to provide the classification of such solutions under natural geometric assumptions.

Ancient compact solutions to the 2-dimensional Ricci flow were classified by Daskalopoulos, Hamilton and Sesum in [18]. It turns out that in this case, the complete list contains (up to conformal invariance) only the shrinking sphere solitons and the King solution. The latter is a well known example of ancient collapsed Ricci flow solution and can be written in closed form. It was first discovered by J. King [25] in the context of the logarithmic fast-diffusion equation on $\mathbb{R}^2$ and later independently by Rosenau [30] in the same context. It also appears as the sausage model in the context of quantum field theory, in the independent work of Fateev-Onofri-Zamolodchikov [20]. Although the King ancient solution is not a soliton, it may be visualized as two steady solitons, called “cigars”, coming from opposite spatial infinities glued together. Let us remark that the classification work in [18] classifies both collapsed and non-collapsed solutions.

In [28], Lei Ni showed that any $\kappa$-noncollapsed ancient solution to the Ricci flow which is of Type I and has positive curvature operator has constant sectional curvature. In [16], Brendle, Huisken and Sinestrari proved that any ancient solution to the Ricci flow in dimension $n \geq 3$ which satisfies a suitable curvature pinching condition must have constant sectional curvature. Fateev’s examples (in [19]) which are collapsed show that the pinching curvature condition in [16] can not be removed. They also show the classification of closed ancient solutions even in dimension three, if we do not assume noncollapsedness may be very difficult, if not impossible. In [8], Bakas, Kong and Ni construct several higher dimensional examples of type I ancient closed solutions to the Ricci flow which are non-collapsed and with positive sectional curvature. Observe that Perelman’s solution is of positive curvature operator, non-collapsed, but of type II.

Regarding the classification of ancient solutions to other geometric flows, let us mention related work in the mean curvature flow setting. In [2] the authors showed that every closed, uniformly 2-convex and non-collapsed ancient solution to the mean curvature flow must be either the family of contracting spheres or the unique, up to isometries, ancient oval constructed by White in [31] and later by Haslhofer and Hershkovits in [24]. On the other hand, ancient noncompact non-collapsed uniformly 2-convex solutions were considered by Brendle and Choi in [14] and [15], where the authors showed that any noncollapsed, uniformly 2-convex noncompact ancient solution to the mean curvature flow must be the rotationally symmetric translating soliton, and hence the Bowl soliton, up to scaling and isometries. Ancient compact collapsed mean curvature flow solutions were studied in a recent interesting work by Bourni, Langford and Tignalia in [10].

Let us now turn our attention to the 3-dimensional Ricci flow. In [29], G. Perelman established the existence of a rotationally symmetric ancient $\kappa$-noncollapsed solution on $S^3$ which is not a soliton. This is a type II ancient solution backward in time, namely its scalar curvature satisfies $\sup_{M \times (−\infty, 0)} |t||R(x, t)| = \infty$ and forms a type I singularity forward in time, since it shrinks to a round point. Perelman’s ancient solution has backward in time limits which are the Bryant soliton and the round cylinder $S^2 \times \mathbb{R}$, depending on how the sequence of points and times about which one rescales are chosen. These are the only backward in time limits
of the Perelman ancient solution. Let us remark that in contrast to the collapsed King ancient solution of the 2-dimensional Ricci flow, the Perelman ancient solution is noncollapsed. In fact there exist other ancient compact solutions to the 3-dimensional Ricci flow which are collapsed and are the analogues of the King solution (see in [19], [8]).

In [29], Perelman introduced the following notion of $\kappa$-noncollapsed metrics.

**Definition 1.1** ($\kappa$-noncollapsed property). The metric $g$ is called $\kappa$-noncollapsed on the scale $\rho$, if every metric ball $B_r$ of radius $r < \rho$ which satisfies $|Rm| \leq r^{-2}$ on $B_r$ has volume at least $\kappa r^n$. An ancient Ricci flow solution is called $\kappa$-noncollapsed, if it is $\kappa$-noncollapsed on all scales $\rho$, for some $\kappa > 0$.

It turns out that this is an important notion in the context of ancient solutions and singularities. In fact, in [29] Perelman proved that every ancient solution arising as a blow-up limit at a singularity of the Ricci flow on compact manifolds is $\kappa$-noncollapsed on all scales for some $\kappa > 0$. We have the following conjecture made by Perelman.

**Conjecture 1.2** (Perelman). Let $(S^3, g(t))$ be a compact, ancient $\kappa$-noncollapsed solution to the Ricci flow (1.1) on $S^3$. Then $g(t)$ is either a family of contracting spheres or Perelman’s solution.

The well known Hamilton-Ivey pinching estimate tells us that any two or three dimensional Ricci flow ancient solution, with bounded curvature at each time slice, has nonnegative sectional curvature. Since our solution $(S^3, g(t))$ is closed, the strong maximum principle implies that the sectional curvatures, and hence the entire curvature operator, are strictly positive. It follows by Hamilton’s Harnack estimate (see in [23]) that $R(t) \geq 0$, yielding the existence of a uniform constant $C > 0$ so that $R(t, t) \leq C$, for all $t \in (-\infty, t_0]$. Since the curvature is positive, one concludes that

**eq-curv-bound**

$$\|\text{Rm}\|_{g(t)} \leq C,$$

for all $-\infty < t \leq t_0$, for a uniform constant $C$. The above discussion yields that any closed 3-dimensional $\kappa$-noncollapsed ancient solution is actually a $\kappa$-solution, in the sense that was defined by Perelman in [29].

In a recent important paper by S. Brendle ([11]), the author proved that a 3-dimensional non-compact ancient $\kappa$-solution is isometric to either a family of shrinking cylinders or their quotients, or to the Bryant soliton. The author first shows that all 3-dimensional ancient $\kappa$-solutions which are non-compact have to be rotationally symmetric. After that he shows that such a rotationally symmetric solution, if not a cylinder or its quotient, must be a steady Ricci soliton and hence the Bryant soliton by one of his earlier works ([12]) about classification of steady Ricci solitons.

The techniques of Brendle in [11] can be also applied to show the rotation symmetry of ancient compact and $\kappa$-noncollapsed solution to the Ricci flow (1.1) on $S^3$. Brendle has recently shown this in [13]. Bamler and Kleiner in [9], obtained the same result as Brendle in the compact case, using different methods. However, since the rotationally symmetric solutions discovered by Perelman are not solitons, the classification of rotationally symmetric ancient compact and $\kappa$-noncollapsed solutions is a difficult problem. Our goal in this work is to establish this classification
under the additional assumption of reflection symmetry. In an upcoming work we plan to remove this technical assumption. Our main result states as follows.

**Theorem 1.3.** Let \((S^3, g(t))\) be a compact, \(\kappa\)-noncollased ancient solution to the Ricci flow on \(S^3\) which is symmetric with respect to rotation and reflection. Then \(g(t)\) is either a family of contracting spheres or Perelman’s solution.

Combining Theorem 1.3 and recent results in [13] immediately yield the following result.

**Theorem 1.4.** Let \((S^3, g(t))\) be a compact, \(\kappa\)-noncollased ancient solution to the Ricci flow on \(S^3\), which is symmetric with respect to reflection. Then \(g(t)\) is either a family of contracting spheres or Perelman’s solution.

Assume from now on that \((S^3, g(t))\) is a Ricci flow solution which satisfies the assumptions of Theorem 1.3. We will next see how one can express the Ricci flow under rotational symmetry as a single equation. Let us first remark that by the work of Perelman we know that the asymptotic soliton of \((S^3, g(t))\) is either a round cylinder or a sphere. We can understand this that every \(\kappa\)-solution has a gradient shrinking soliton buried inside of it, in an asymptotic sense as time approaches \(-\infty\) (for more details on asymptotic solitons see [29]). In our recent work [3] we show that if the asymptotic soliton is the sphere, then the solution \((S^3, g(t))\) must be the round sphere itself. Hence, from now on we may assume that the asymptotic soliton of our closed \(\kappa\)-noncollapsed solution is the round cylinder \(S^2 \times \mathbb{R}\).

Since at each time slice, the metric is \(SO(3)\)-invariant, it can be written as

\[
g = \phi^2 \, dx^2 + \psi^2 \, g_{can},
\]

where \((-1,1) \times S^2\) may be naturally identified with the sphere \(S^3\) with its north and south poles removed. The function \(\psi(x,t) > 0\) may be regarded as the radius of the hypersurface \(\{x\} \times S^2\) at time \(t\). The distance function from the equator is given by

\[
s(x,t) = \int_0^x \phi(x',t) \, dx'.
\]

and abbreviating \(ds = \phi(x,t) \, dx\), we write our metric as \(g = ds^2 + \psi^2 \, g_{can}\). As it was remarked in [4], for our metric (2.2) to define a smooth metric on \(S^3\) we need to have \(\psi(s_- (t)) = 1, \psi(s_+ (t)) = -1\) holding at the two tips of our solution. Under the Ricci flow, the profile function \(\psi : (s_-(t), s_+(t)) \times (-\infty, 0) \to \mathbb{R}\) evolves by

\[
\psi_t = \psi_{ss} - \frac{1 - \psi^2}{\psi}.
\]

Consider next a type I scaling of our metric, which leads to the rescaled profile \(u(\sigma, \tau)\) defined by

\[
u(\sigma, \tau) := \psi(s, t), \quad \sigma := \frac{s}{\sqrt{-t}}, \quad \tau = \log(-t).
\]

A direct calculation shows that \(u : (\sigma_- (\tau), \sigma_+ (\tau)) \times (-\infty, 0) \to \mathbb{R}\) satisfies the equation

\[
u_\tau = u_{\sigma\sigma} + \frac{u^2}{u} - \frac{1}{u} + \frac{u}{2}
\]

with boundary conditions at the tips \(u_\sigma(\sigma_- (\tau), \tau) = 1, u_\sigma(\sigma_+ (\tau), \tau) = -1\).
It follows from the discussion above, since we know our solution is rotationally symmetric ([9], [13]), our main result, Theorem 1.3, is equivalent to the following uniqueness result.

**Theorem 1.5.** Let \((S^3, g_1(t))\) and \((S^3, g_2(t)), -\infty < t < T\), be two compact nonspherical rotationally and reflection symmetric, \(\kappa\)-noncollapsed ancient solutions to the 3-dimensional Ricci flow which have the same axis of symmetry and whose profile functions \(\psi_1(s,t)\) and \(\psi_2(s,t)\) satisfy equation (1.3). Then, they are the same up to translations in time and parabolic rescaling. In particular, they coincide with the Perelman solution.

A crucial first step in showing Theorem 1.5 is to establish the (unique up to scaling) asymptotic behavior of any compact rotationally and reflection symmetric \(\kappa\)-noncollapsed ancient solution to the Ricci flow on \(S^3\) which is not isometric to a sphere. This was recently established by the authors in [3] and is summarized, for the reader’s convenience, in the next theorem.

**Theorem 1.6** (Angenent, Daskalopoulos, Sesum in [3]). Let \((S^3, g(t))\) be any reflection and rotationally symmetric compact \(\kappa\)-noncollapsed ancient solution to the Ricci flow on \(S^3\) which is not isometric to a round sphere. Then the rescaled profile \(u(\sigma,\tau)\) solution to (1.5) has the following asymptotic expansions:

(i) For every \(L > 0\),

\[
u(\sigma,\tau) = \sqrt{2} \left(1 - \frac{\sigma^2 - 2}{8|\tau|}\right) + o(|\tau|^{-1}), \quad \text{on } |\sigma| \leq L\]

as \(\tau \to -\infty\).

(ii) Define \(z := \sigma/\sqrt{|\tau|}\) and \(\bar{u}(\sigma,\tau) := u(z\sqrt{|\tau|},\tau)\). Then,

\[
\lim_{\tau \to -\infty} \bar{u}(z,\tau) = \sqrt{2 - \frac{z^2}{2}}
\]

uniformly on compact subsets of \(|z| < 2\).

(iii) Let \(k(t) := R(p_t, t)\) be the maximal scalar curvature which is attained at each one of the two tips \(p_t\), for \(t \ll -1\). Then the rescaled Ricci flow solutions \((S^3, \bar{g}_t(s), p_t)\), with \(\bar{g}_t(s) = k(t) g(.t + k(t)^{-1} s)\), converge to the unique Bryant translating soliton with maximal scalar curvature one. Furthermore, \(k(t)\) and the diameter \(d(t)\) satisfy the asymptotics

\[
k(t) = \frac{\log |t|}{|t|} (1 + o(1)) \quad \text{and} \quad d(t) = 4\sqrt{|t| \log |t|} (1 + o(1))
\]

as \(t \to -\infty\).

The outline of the paper is as follows. In section 2 we give a detailed outline of our proof, including all equations and norms that we consider in the two different regions, which are the cylindrical and the tip regions. In section 3 we study the linearized equation around the cylinder. We show that the norm of the projections of the difference of our solutions onto unstable and stable modes of the linearized operator around the cylinder is controlled by a tiny multiple of the norm of the projection of a difference of our solutions onto a neutral mode and a tiny multiple of the norm of a difference of our solutions outside the cylindrical region. Section 4 is devoted to the analysis of the tip region. We show how to define a suitable weighted norm in the tip so that the norm of the difference of our solutions in the tip
region is controlled by a tiny multiple of the norm of the difference of our solutions inside the cylindrical region. In Section 5 we give the proof of Theorem 1.5 using the results from previous two sections and carefully analyzing the error terms that appear when we approximate the Ricci flow equation by its linearization at the cylinder (in the cylindrical region) or the Bryant soliton (in the soliton region). In our final Section appendix A, we include needed a priori estimates whose proofs are somewhat similar to the proofs of analogous a priori estimates in [2] but still different enough to have their proofs included for the sake of completeness of our arguments.

2. Outline of the proof of Theorem 1.5

Since the proof of Theorem 1.5 is quite involved, in this preliminary section we give an outline of its main steps. Our method is based on a priori estimates for the “distance” between two given ancient solutions which is measured in appropriate weighted $L^2$-norms. We need to consider two different regions: the cylindrical region and the tip region. In each of these regions, our distance norms are dictated by the behavior of our solutions. In what follows, we define these regions, review the equations in each region and define the weighted $L^2$-norms with respect to which we will prove coercive type estimates in the subsequent sections. At the end of the section we will give an outline of the proof of Theorem 1.5.

2.1. Equations under rotational symmetry. We have seen in the introduction that a solution $g$ of (1.1) on $S^3$ which is $SO(3)$-invariant, can be written as

$$g = \phi^2 \, dx^2 + \psi^2 \, g_{can}, \quad \text{on } (-1, 1) \times S^2$$

where $(-1, 1) \times S^n$ may be naturally identified with the sphere $S^3$ with its North and South poles removed. The function $\psi(x, t) > 0$ may be regarded as the radius of the hypersurface $\{x\} \times S^2$ at time $t$. By the reflection symmetry assumption we may assume that $\phi(x, t) = \phi(-x, t)$ and $\psi(x, t) = \psi(-x, t)$, for all $x \in (-1, 1)$, that is $x = 0$ is the point of reflection symmetry for our profile functions. Then the distance function to the center of symmetry (we will refer to it as to the equator) is given by

$$s(x, t) = \int_0^x \phi(x', t) \, dx'. \quad \text{(2.1)}$$

We will write

$$s_{\pm}(t) := \lim_{x \to \pm 1} s(x, t),$$

or shortly $s_{\pm}$, for the distance from the equator to the South and the North poles, respectively, which depend on time, along the Ricci flow. If we abbreviate

$$ds = \phi(x, t) \, dx \quad \text{and} \quad \frac{\partial}{\partial s} = \frac{1}{\phi(x, t)} \, \frac{\partial}{\partial x},$$

then we can write our metric as

$$g = ds^2 + \psi^2 \, g_{can}. \quad \text{(2.2)}$$

Let us next review how you derive the evolution equation of the profile function $\psi(s, t)$ from the Ricci flow equation. The time derivative does not commute with
the \( s \)-derivative, and in general we must use
\[
\frac{\partial}{\partial t} ds = \frac{\phi}{\phi} \cdot ds \quad \text{and} \quad \left[ \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right] = -\frac{\phi}{\phi} \frac{\partial}{\partial s}.
\]

The Ricci tensor is given by
\[
Rc = 2K_0 ds^2 + [K_0 + K_1] \psi^2 g_{\text{can}}
\]
\[
= -2\psi_{ss} ds^2 + \{-\psi \psi_{ss} - 1\} \psi^2 + 1 \} g_{\text{can}}
\]
where \( K_0 \) and \( K_1 \) are the two distinguished sectional curvatures that any metric of the form (2.2) has. They are the curvature of a plane tangent to \( \{s\} \times S^n \), given by

\[\text{eq-K1} \quad K_1 := \frac{1 - \psi^2}{\psi^2}, \]

and the curvature of an orthogonal plane given by

\[\text{eq-K0} \quad K_0 := -\frac{\psi_{ss}}{\psi}.\]

Moreover, the scalar curvature is given by
\[
R = g^{jk} R_{jk} = 4K_0 + 2K_1.
\]

The time derivative of the metric is
\[
\frac{\partial g}{\partial t} = 2 \frac{\phi_t}{\phi} ds^2 + 2 \psi \psi_t g_{\text{can}}.
\]

Therefore, if the metrics \( g(t) \) evolve by Ricci flow \( \partial_t g = -2Rc \), then
\[
\frac{\phi_t}{\phi} = 2 \frac{\psi_{ss}}{\psi},
\]

so that
\[
\frac{\partial}{\partial t} ds = 2 \frac{\psi_{ss}}{\psi} ds \quad \text{and} \quad \left[ \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right] = -2 \frac{\psi_{ss}}{\psi} \frac{\partial}{\partial s}.
\]

Under Ricci flow the radius \( \psi(s, t) \) satisfies the equation

\[\text{eq-RF} \quad \psi_t = \psi_{ss} - \frac{1 - \psi^2}{\psi}.\]

As in [4], for our metric (2.2) to define a smooth metric on \( S^3 \) we need to have

\[\text{closing-psi} \quad (2.6) \quad \psi(s_-) = 1, \quad \psi^{(2k)}(s_-) = 0 \quad \text{and} \quad \psi(s_+) = -1, \quad \psi^{(2k)}(s_+) = 0, \]

for \( k \in \mathbb{N} \cup \{0\} \).

### 2.2. Ancient Ricci ovals and their invariances.

In [3] we have discussed results that lead to the conclusion that the asymptotic soliton of any \( SO(3) \)-invariant closed \( \kappa \)-solution is either a sphere or a cylinder. In the case it is a sphere, we have proved in [3] that our solution has to be the family of shrinking round spheres itself. Hence, for the rest of the paper we may assume that any closed \( \kappa \)-solution in consideration below has a round cylinder as its asymptotic cylinder.

**Definition 2.1.** We define an Ancient Ricci Oval to be any rotationally symmetric closed \( \kappa \)-solution which has the round cylinder as its asymptotic soliton.
Let \( g_1 = ds^2 + \psi_1^2 g_{an} \), for \( i \in \{1, 2\} \) be two reflection symmetric Ancient Ricci Oval solutions satisfying the assumptions of Theorem 1.5. In particular we have fixed the axis of symmetry and the center of reflection symmetry \( s = 0 \) for both solutions. Under the Ricci flow each profile \( \psi_i(s, t) \) satisfies (2.5). In the statement of our main Theorem 1.5 we claim the uniqueness of \( g_1 \) and \( g_2 \) up to parabolic scaling in space-time and translations in time. Since each solution \( g_i(t) \) gives rise to a two parameter family of solutions

\[
\begin{equation}
(2.7) \quad g_i^{\beta \gamma}(\cdot, t) = e^{\gamma/2} g_i(\cdot, e^{-\gamma}(t - \beta))
\end{equation}
\]


the theorem claims the following: given two ancient oval solutions we can find \( \beta, \gamma \) and \( t_0 \in \mathbb{R} \) such that

\[
\begin{equation}
(2.8) \quad g_1(\cdot, t) = g_2^{\beta \gamma}(\cdot, t), \quad \text{for } t \leq t_0.
\end{equation}
\]

The profile function \( \psi_2^{\beta \gamma} \) corresponding to the modified solution \( g_2^{\beta \gamma}(\cdot, t) \) is given by

\[
\begin{equation}
(2.9) \quad g_i(\cdot, t) = \sqrt{-t} \bar{g}_i(\cdot, \tau), \quad \tau := -\log(-t).
\end{equation}
\]

These are again rotationally and reflection symmetric with profile function \( u \), which is related to \( \psi \) by

\[
\begin{equation}
(2.10) \quad \psi(s, t) = \sqrt{-t} u(s, \tau), \quad \sigma = \frac{s}{\sqrt{-t}}, \quad \tau = -\log(-t).
\end{equation}
\]

If the \( \psi_i \) satisfy the Ricci flow equation (1.3), then the rescaled profiles \( u_i \) satisfy

\[
\begin{equation}
(2.11) \quad u_\tau = u_{\sigma \sigma} + \frac{u_\sigma^2}{u} - \frac{1}{u} + \frac{u}{2}
\end{equation}
\]

with boundary conditions at the tips \( u_\sigma(\sigma_-) = 1, u_\sigma(\sigma_+) = -1 \).

The vector fields \( \partial_\sigma \) and \( \partial_\tau \) do not commute. However, we can make them commute by adding a non-local term in equation (2.11) (see [3], Section 2 for details). In fact in commuting variables the equation for \( u \) becomes

\[
\begin{equation}
(2.12) \quad u_\tau = u_{\sigma \sigma} - \frac{\sigma}{2} u_\sigma - J(\sigma, \tau) u_\sigma + \frac{u_\sigma^2}{u} - \frac{1}{u} + \frac{u}{2}
\end{equation}
\]

where

\[
\begin{equation}
(2.13) \quad J(\sigma, \tau) = 2 \int_0^\sigma \frac{u_\sigma}{u} \, ds'.
\end{equation}
\]

Changing \( g_i(\cdot, t) \) to \( g_i^{\beta \gamma}(\cdot, t) \) has the following effect on \( u_i(\sigma, \tau) \):

\[
\begin{equation}
(2.14) \quad u_i^{\beta \gamma}(\sigma, \tau) = \sqrt{1 + \beta e^\tau} u_i \left( \frac{\sigma}{\sqrt{1 + \beta e^\tau}}, \tau + \gamma - \log(1 + \beta e^\tau) \right).
\end{equation}
\]

To prove the uniqueness theorem we will look at the difference \( \psi_1 - \psi_2^{\beta \gamma} \), or equivalently at \( u_1 - u_2^{\beta \gamma} \). The parameters \( \beta, \gamma \) will be chosen so that the projections of \( u_1 - u_2^{\beta \gamma} \) onto positive eigenspace (that is spanned by two independent eigenvectors) and zero eigenspace of the linearized operator \( \mathcal{L} \) at the cylinder are equal to zero at
time $\tau_0$, which will be chosen sufficiently close to $-\infty$. Correspondingly, we denote the difference $\psi_1 - \psi_2^\beta$ by $\psi_1 - \psi_2$ and $u_1 - u_2^\beta$ by $u_1 - u_2$. We will actually observe that the parameters $\beta, \gamma$ can be chosen to lie in a certain range, which allows our main estimates to hold independently of the choice of these parameters during the proof, as long as they stay in the specified range. We will show in Section 5 that for a given small $\epsilon > 0$ there exists $\tau_0 \ll -1$ sufficiently negative for which we have

$$\beta \leq \epsilon e^{\tau_0} / |\tau_0| \quad \text{and} \quad \gamma \leq \epsilon |\tau_0|$$

and our estimates hold for $(u_1 - u_2^\beta)(., \tau)$, for all $\tau \leq \tau_0$, as long as $\beta$ and $\gamma$ satisfy (2.15). This inspires the following definition.

**Definition 2.2** (Admissible triple of parameters $(\beta, \gamma)$). We say that the pair of parameters $(\beta, \gamma)$ is admissible with respect to time $\tau_0$ if they satisfy (2.15).

Throughout the proof of Theorem 1.5 we will make sure that our choice of parameters $(\beta, \gamma)$ satisfies the admissibility condition given by this definition.

2.3. The two regions: equations, norms and crucial estimates. The proof of Theorem 1.5 relies on sharp coercive estimates in appropriate norms for the difference of our two solutions $w := u_1 - u_2^\beta$. Since the behavior of our solutions changes from being a cylinder near the equator to being the Bryant soliton at the two tips (see Theorem 1.6) we will need to consider these two regions separately. Namely for a given small positive constant $\theta$, we define the cylindrical region by

$$C_\theta = \{ (\sigma, \tau) : u_1(\sigma, \tau) \geq \theta / 4 \}$$

and the tip region

$$T_\theta = \{ (u, \tau) : u_1 \leq 2\theta, \tau \leq \tau_0 \}.$$

We will next outline how we treat each region separately and obtain a coercive estimate for the difference of the two solutions in appropriate weighted norms. At the end of the outline we will show how these estimates imply uniqueness.

2.3.1. The cylindrical region. For a given $\tau \leq \tau_0$ and constant $\theta$ positive and small, consider the cylindrical region $C_\theta = \{ (\sigma, \tau) : u_1(\sigma, \tau) \geq \theta / 4 \}$. Let $\varphi_C(\sigma, \tau)$ denote a standard cut-off function with the following properties:

1. $\text{supp } \varphi_C \subset C_\theta$
2. $0 \leq \varphi_C \leq 1$
3. $\varphi_C \equiv 1$ on $C_{2\theta}$.

The solutions $u_i$, $i = 1, 2$, satisfy equation (2.11). Setting

$$w := u_1 - u_2^\beta$$

we see that $w_C$ satisfies the equation

$$\frac{\partial}{\partial \tau} w_C = \mathcal{L}[w_C] + \mathcal{E}[w, \varphi_C]$$

where the operator $\mathcal{L}$ is given by

$$\mathcal{L} = \partial^2_{\sigma} - \frac{\sigma}{2} \partial_\sigma + 1$$

and where the error term $\mathcal{E}$ is described in detail in Section 3. We will see that

$$\mathcal{E}[w, \varphi_C] = \mathcal{E}(w_C) + \bar{\mathcal{E}}[w, \varphi_C] + \mathcal{E}_{nl}$$
where $\mathcal{E}(w_C)$ is the error introduced due to the nonlinearity of our equation and is given by (3.6), $\mathcal{E}[w, \varphi_C]$ is the error introduced due to the cut off function $\varphi_C$ and is given by (3.7) (to simplify the notation we have set $u_2 := u_2^{\beta\gamma}$) and $\mathcal{E}_{nl}$ is a nonlocal error term and is given by (3.8).

The differential operator $L$ is a well studied self-adjoint operator on the Hilbert space $\mathcal{H} := L^2(\mathbb{R}, e^{-\sigma^2/4} d\sigma)$ with respect to the norm and inner product

\begin{equation}
\|f\|_D^2 = \int_{\mathbb{R}} f(\sigma)^2 e^{-\sigma^2/4} d\sigma, \quad \langle f, g \rangle = \int_{\mathbb{R}} f(\sigma)g(\sigma) e^{-\sigma^2/4} d\sigma.
\end{equation}

We split $\mathcal{H}$ into the unstable, neutral, and stable subspaces $\mathcal{H}^+, \mathcal{H}_0$, and $\mathcal{H}^-$, respectively. The unstable subspace $\mathcal{H}^+$ is spanned by the eigenfunction $\psi_0 \equiv 1$ corresponding to the only positive eigenvalue 1 (that is, $\mathcal{H}^+$ is one dimensional, due to our assumption on reflection symmetry). The neutral subspace $\mathcal{H}_0$ is the kernel of $L$, and is one dimensional space spanned by $\psi_2 = \sigma^2 - 2$. The stable subspace $\mathcal{H}^-$ is spanned by all other eigenfunctions. Let $P^\pm$ and $P_0$ be the orthogonal projections on $\mathcal{H}^\pm$ and $\mathcal{H}_0$.

For any function $f : \mathbb{R} \times (-\infty, \tau_0]$, we define the cylindrical norm

\begin{equation}
\|f\|_{\mathcal{H}, \infty}(\tau) = \sup_{\sigma \leq \tau} \left( \int_{\sigma-1}^{\tau'} \|f(\cdot, s)\|_D^2 \, ds \right)^{1/2}, \quad \tau \leq \tau_0
\end{equation}

and we will often simply set

\begin{equation}
\|f\|_{\mathcal{H}, \infty} := \|f\|_{\mathcal{H}, \infty}(\tau_0).
\end{equation}

In the course of proving necessary estimates in the cylindrical region we define yet another Hilbert space $\mathcal{D}$ by

$$
\mathcal{D} = \{f \in \mathcal{H} : f, f_\sigma \in \mathcal{H}\},
$$
equation-norm

equipped with a norm

\begin{equation}
\|f\|_D^2 = \int_{\mathbb{R}} \{f(\sigma)^2 + f'(\sigma)^2\} e^{-\sigma^2/4} d\sigma.
\end{equation}
equation-norm

We will write

$$
(f, g)_\mathcal{D} = \int_{\mathbb{R}} \{f'(\sigma)g'(\sigma) + f(\sigma)g(\sigma)\} e^{-\sigma^2/4} d\sigma,
$$
equation-norm

for the inner product in $\mathcal{D}$.

Since we have a dense inclusion $\mathcal{D} \subset \mathcal{H}$ we also get a dense inclusion $\mathcal{H} \subset \mathcal{D}^*$ where every $f \in \mathcal{H}$ is interpreted as a functional on $\mathcal{D}$ via

$$
g \in \mathcal{D} \mapsto (f, g).
$$
equation-norm

Because of this we will also denote the duality between $\mathcal{D}$ and $\mathcal{D}^*$ by

$$
(f, g) \in \mathcal{D} \times \mathcal{D}^* \mapsto (f, g).
$$
equation-norm

Similarly as above define the cylindrical norm

\begin{equation}
\|f\|_{\mathcal{D}, \infty}(\tau) = \sup_{\sigma \leq \tau} \left( \int_{\sigma-1}^{\tau'} \|f(\cdot, s)\|_D^2 \, ds \right)^{1/2},
\end{equation}
equation-cyl-norm

and analogously we define the cylindrical norm $\|f\|_{\mathcal{D}^*, \infty}(\tau)$ and set for simplicity $\|f\|_{\mathcal{D}^*, \infty} := \|f\|_{\mathcal{D}^*, \infty}(\tau_0)$. 
In Section 3 we will show a coercive estimate for \( w_C \) in terms of the error \( E[w, \varphi_C] \). However, as expected, this can only be achieved by removing the projection \( P_0 w_C \) onto the kernel of \( L \), generated by \( \psi_2 \). More precisely, setting

\[
\hat{w}_C := P_+ w_C + P_- w_C = w_C - P_0 w_C
\]

we will prove that for any given \( \theta \in (0, \sqrt{2}) \) and \( \epsilon > 0 \) there exist \( \tau_0 \ll -1 \) (depending on \( \theta \) and \( \epsilon \)) such that the following bound holds

\[
\|\hat{w}_C\|_{\mathcal{D}, \infty} \leq C \|E[w, \varphi_C]\|_{\mathcal{D}, \infty}
\]

provided that \( P_+ w_C(\tau_0) = 0 \). In fact, we will show in Proposition 5.1 that the parameters \( \beta \) and \( \gamma \) can be adjusted so that for \( w^{\beta\gamma} := u_1 - u_2^{\beta\gamma} \), we have

\[
\|\hat{w}_C\|_{\mathcal{D}, \infty} = 0 \quad \text{and} \quad \|\hat{w}_C\|_{\mathcal{D}, \infty} = 0.
\]

Thus (2.20) will hold for such a choice of \( \beta, \gamma \) and \( \tau_0 \ll -1 \). The condition \( P_0 w_C(\tau_0) = 0 \) is essential and will be used in Section 5 to give us that \( w^{\beta\gamma} \equiv 0 \). In addition, we will show in Proposition 5.1 that \( \beta \) and \( \gamma \) can be chosen to be admissible according to our Definition 2.2.

The norm of the error term \( \|E[w, \varphi_C]\|_{\mathcal{D}, \infty} \) on the right hand side of (2.20) will be estimated in Section 3. We will show that given \( \epsilon > 0 \) small, there exists a \( \tau_0 \ll -1 \) such that

\[
\|E[w, \varphi_C]\|_{\mathcal{D}, \infty} \leq \epsilon \left( \|w_C\|_{\mathcal{D}, \infty} + \|w \chi_{D_\theta}\|_{\mathcal{D}, \infty} \right).
\]

where \( D_\theta := \{ (\sigma, \tau) : \theta/4 \leq u_1(\sigma, \tau) \leq \theta/2 \} \) denotes the support of the derivative of \( \varphi_C \). Combining (2.20) and (2.22) yields the bound

\[
\|\hat{w}_C\|_{\mathcal{D}, \infty} \leq \epsilon \left( \|w_C\|_{\mathcal{D}, \infty} + \|w \chi_{D_\theta}\|_{\mathcal{D}, \infty} \right),
\]

holding for all \( \epsilon > 0 \) and \( \tau_0 := \tau_0(\epsilon) \ll -1 \).

To close the argument we need to estimate \( \|w \chi_{D_\theta}\|_{\mathcal{D}, \infty} \) in terms of \( \|w_C\|_{\mathcal{D}, \infty} \). This will be done by considering the tip region and establishing an appropriate \textit{a priori} bound for the difference of our two solutions there.

2.3.2. The tip region. The tip region is defined by \( T_\theta = \{ (u, \tau) : u_1 \leq 2\theta, \tau \leq \tau_0 \} \). Since equation (2.11) becomes singular at the tips \( \sigma_\pm(\tau) \), in the tip region we introduce the change of variables \( Y := u_2^\beta \) and we view \( Y \) as a function of \( (u, \tau) \) (see in [3], Section 2 for details). A simple calculation shows that \( Y(u, \tau) \) satisfies the equation

\[
Y + \frac{u}{2} Y_u = Y Y_{uu} - \frac{1}{2} (Y_u)^2 + (1 - Y) \frac{Y_u}{u} + 2(1 - Y) \frac{Y}{u^2}.
\]

Under this change of variables, our solutions \( u_1(\sigma, \tau) \) and \( u_2^{\beta\gamma}(\sigma, \tau) \) become \( Y_1(u, \tau) \) and \( Y_2^{\beta\gamma}(u, \tau) \). In this region we consider a cut-off function \( \varphi_T(u) \) with the following properties:

\[
(\text{i}) \quad \text{supp} \varphi_T \subset T_\theta \quad (\text{ii}) \quad 0 \leq \varphi_T \leq 1 \quad (\text{iii}) \quad \varphi_T \equiv 1, \text{ on } T_{\theta/2}.
\]

Let \( \Psi_1(u, \tau) = \sqrt{Y_1(u, \tau)} \) and \( \Psi_2^{\beta\gamma}(u, \tau) = \sqrt{Y_2^{\beta\gamma}(u, \tau)} \). We will see in Section 4.3 that the difference \( W(u, \tau) := \Psi_1(u, \tau) - \Psi_2^{\beta\gamma}(u, \tau) \) satisfies equation (4.29). Our next goal is to define an appropriate weighted norm in the tip region \( T_\theta \), by defining the weight \( \mu(u, \tau) \). To this end we need to further distinguish between two
regions in $\mathcal{T}_0$: for $L > 0$ sufficiently large to be determined in Section 4, we define the collar region to be the set

$$\mathcal{K}_{\theta,L} := \left\{ y \mid \frac{L}{\sqrt{|\tau|}} \leq u_1(\sigma, \tau) \leq 2\theta \right\}$$

and the soliton region to be the set

$$\mathcal{S}_L := \left\{ y \mid 0 \leq u_1(\sigma, \tau) \leq \frac{L}{\sqrt{|\tau|}} \right\}.$$

This is necessary, as the behavior of our solutions changes from being cylindrical in the collar region to resembling the Bryant soliton in the soliton region.

In the soliton region we further rescale our solutions by setting for each solutions

$$Z(\rho, \tau) = Y(u, \tau), \quad \rho := u \sqrt{|\tau|}.$$

A direct calculation using (2.24) shows that each rescaled solution $Z$ satisfies the equation

$$\frac{1}{|\tau|} \left( Z_\tau - \frac{1}{2|\tau|} \rho Z_\rho + \frac{\rho Z_\mu}{2} \right) = ZZ_{\rho \rho} - \frac{1}{2} Z_\rho^2 + (1 - Z)\frac{Z_\rho}{\rho} + \frac{2(1 - Z)Z}{\rho^2}.$$  

The collar region can be viewed as the transition region between the cylindrical and soliton regions. Furthermore, the result in Theorem 1.6 implies that each solution $Z(\rho, \tau)$ converges smoothly on compact subsets of $\rho \leq L$ to the translating Bowl soliton $Z_0(\rho)$ with maximal curvature one (see in [17] for more details about the Bryant soliton). It was shown in [17] that $Z_0(\rho)$ satisfies the following asymptotics

$$Z_0(\rho) = \begin{cases} 1 - \rho^2/6 + O(\rho^4), & \text{as } \rho \to 0 \\ \rho^{-2} + O(\rho^{-4}), & \text{as } \rho \to \infty. \end{cases}$$

Let us next define our weight $\mu(u, \tau)$ in the tip region as a function of $(u, \tau)$. Let $\zeta(u)$ be a nonnegative smooth decreasing function defined on $u \in (0, \infty)$ such that

$$\zeta(u) = 1, \quad \text{for } u \geq \theta/2 \quad \text{and } \quad \zeta(u) = 0, \quad \text{for } u \leq \theta/4.$$

Such a function can be chosen to satisfy the derivative estimate $|\zeta'(u)| \leq 5\theta^{-1}$.

For our given solution $u(\sigma, \tau)$ which after the coordinate change gives rise to $\sigma(u, \tau)$ and $Y(\sigma, \tau) := u_\sigma^2(u, \tau)$ (recall that we have dropped the index and denote $\sigma, Y_1$ by $\sigma, Y$ respectively) we define our weight $\mu(u, \tau)$ in the tip region to be

$$\mu(u, \tau) = -\frac{\sigma^2(\theta, \tau)}{4} + \int_0^u \mu_u(u', \tau) \, du'$$

where

$$\mu_u := \zeta(u) \left( -\frac{\sigma^2(u, \tau)}{4} \right)_u + (1 - \zeta(u)) \frac{1 - Y(u, \tau)}{u Y(u, \tau)}.$$

Note that since $\zeta \equiv 1$ for $u \geq \theta/2$, we have $\mu(u, \tau) = -\frac{\sigma^2(u, \tau)}{4}$ in this region, hence it coincides with our weight in the cylindrical region. This is important as our norms in the intersection of the cylindrical and tip regions need to coincide.

Now that we have defined the weight $\mu(u, \tau)$, let us define the norm in the tip region. For a function $W : [0, 2\theta] \times (-\infty, \tau_0] \to \mathbb{R}$ and any $\tau \leq \tau_0$,
we define the weighted $L^2$ norm with respect to the weight $\Psi^{-2} e^{\mu(\cdot, \tau)}$ by

\begin{equation}
\|W(\cdot, \tau)\|^2 := \int_{\theta}^{\theta} W^2(u, \tau) \Psi^{-2} e^{\mu(u, \tau)} \, du, \quad \tau \leq \tau_0
\end{equation}

where $\Psi := \Psi_1 := \sqrt{1}$ denotes one of our solutions in the tip coordinates. Furthermore, we define the tip norm to be

\begin{equation}
\|W\|_{2, \infty}(\tau) = \sup_{\tau' \leq \tau} |\tau'|^{-1/4} \left( \int_{\tau'-1}^{\tau} \|W(\cdot, s)\|^2 \, ds \right)^{1/2}
\end{equation}

for any $\tau \leq \tau_0$. We include the weight in time $|\tau|^{-1/4}$ to make the norms equivalent in the transition region, between the cylindrical and the tip regions, as will become apparent in Corollary 5.4. We will also abbreviate

\begin{equation}
\|W\|_{2, \infty}(\tau_0).
\end{equation}

For a cutoff function $\varphi_T$ as in (2.25), we set $W_T(u, \tau) := W(u, \tau) \varphi_T$, where $W := \Psi_1 - \Psi_2^{\beta \gamma}$. In Section 4 we will show the following crucial estimate which roughly states that the norm of the difference $W$ of our two solutions in the tip region can be estimated by the norm of $W$ in the region $\theta \leq u \leq 2 \theta$ which is included in our cylindrical region $u \geq \theta/4$. More precisely, we will show that there exists a small $\theta > 0$ and $\tau_0 \ll -1$ depending on $\theta$ for which

\begin{equation}
\|W_T\|_{2, \infty} \leq \frac{C}{\sqrt{\tau_0}} \|W \chi_{[\theta, 2\theta]}\|_{2, \infty}
\end{equation}

for $\tau \leq \tau_0$. Here $\chi_{[\theta, 2\theta]}$ is the characteristic function of the interval $[\theta, 2\theta]$. We will next outline how this estimate is combined with that in the cylindrical region to close the argument and conclude uniqueness.

2.4. The conclusion. The statement of Theorem 1.5 is equivalent to showing there exist parameters $\beta$ and $\gamma$ so that $u_1(\sigma, \tau) = u_2^{\beta \gamma}(\sigma, \tau)$, where $u_2^{\beta \gamma}(\sigma, \tau)$ is defined by (2.14) and both functions, $u_1(\sigma, \tau)$ and $u_2^{\beta \gamma}(\sigma, \tau)$, satisfy equation (2.11). We set $w := u_1 - u_2^{\beta \gamma}$, and $W := \Psi_1 - \Psi_2^{\beta \gamma}$, where $(\beta, \gamma)$ is an admissible pair of parameters with respect to $\tau_0$, such that (2.21) holds for a $\tau_0 \ll -1$.

Fix $\theta > 0$ and small such that (2.35) holds for $\tau_0 \ll -1$ depending on $\theta$. For that fixed $\theta$ and any $\epsilon > 0$ we also have that (2.23) holds, for $\tau_0 \ll -1$ depending on $\theta, \epsilon$. Combining (2.23) and (2.35) with the estimates in Corollary 5.4 which compare our norms in the intersection of cylindrical and tip regions, we finally show that $\|P_0 w_C\|_{P, \infty}$ dominates over the norms of $P_+ w_C$ and $P_- w_C$ (this happens by applying (2.23) for $\epsilon$ sufficiently small depending only on $\theta$). We will use this fact in Section 5 to conclude that $w(\sigma, \tau) := w^{\beta \gamma}(\sigma, \tau) \equiv 0$ for our choice of parameters $\beta$ and $\gamma$. To do so we will look at the projection $a(\tau) := P_0 w_C$ and define its norm

$$\|a\|_{\beta, \infty}(\tau) = \sup_{\tau \leq \tau_0} \left( \int_{\sigma = \tau}^{\tau} \|a(s)\|^2 \, ds \right)^{1/2},$$

with $\|a\|_{\beta, \infty}(\tau_0)$. By projecting equation (2.16) onto the zero eigenspace spanned by $\psi_2$ and estimating error terms by $\|a\|_{\beta, \infty}$ itself, we will conclude in Section 5 that $a(\tau)$ satisfies a certain differential inequality which combined with our assumption that $a(\tau_0) = 0$ (that follows from the choice of parameters $\beta$, $\gamma$ so that (2.21) hold) will yield that
Lemma 3.3
and let
we show
Remark
we write simply
Proposition 3.2. For every \( \epsilon > 0 \) and \( \theta > 0 \) small, there exists a \( \tau_0 \ll -1 \) so that if
\[
\| \tilde{w}_C \|_{\mathcal{D}, \infty} \leq \epsilon (\| w_C \|_{\mathcal{D}, \infty} + \| w \chi_{D_0} \|_{\mathcal{B}, \infty}),
\]
where \( D_0 := \{ \sigma \mid \theta/2 \leq u_1(\sigma, \theta) \leq 0 \} \) and \( \tilde{w}_C = \mathcal{P}_- w_C + \mathcal{P}_+ w_C \).

Our linear operator \( \mathcal{L}(f) = f_{\sigma\sigma} - \frac{\sigma}{\tau} + f \) is the same as in [2], and hence, the linear theory we derived in [2] carries over to the Ricci flow case as well. In order for this article to be self-contained, we will state the results from [2] that we will use later, but for the proofs of the same we refer reader to look at [2]. More precisely, in [2] we obtained energy type estimate for ancient solutions \( f : (-\infty, \tau_0] \to \mathcal{D} \) of the linear cylindrical equation
\[
(3.1) \quad \frac{\partial f}{\partial \tau} - \mathcal{L} f(\tau) = g(\tau).
\]

Lemma 3.3 (Lemma 5.8 in [2]). Let \( f : (-\infty, \tau_0] \to \mathcal{D} \) be a bounded solution of equation (3.1). If \( T > 0 \) is sufficiently large, then there is a constant \( C_* \), such that
\[
\sup_{\tau \leq \tau_0} \| \tilde{f}(\tau) \|_{\mathcal{B}}^2 + C_*^{-1} \sup_{n \geq 0} \int_{I_n} \| \tilde{f}(\tau) \|_{\mathcal{B}}^2 d\tau 
\leq \| f_+(\tau_0) \|_{\mathcal{B}}^2 + C_* \sup_{n \geq 0} \int_{I_n} \| \tilde{g}(\tau) \|_{\mathcal{B}}^2 d\tau,
\]
where $I_n$ is the interval $I_n = [\tau_0 - (n + 1)T, \tau_0 - nT]$ and where $f_+ = \mathcal{P}_+ f$ and \( \dot{f} = \mathcal{P}_+ f + \mathcal{P}_- f \).

We also summarize in the next lemma the following bounds on various operators between the Hilbert spaces $\mathcal{H}$ and $\mathcal{D}$ with norms (2.17) and (3.4) respectively (see in Section 2.3.1 for their definitions). All these bounds were shown in [2].

**Lemma 3.4.** The following hold:

i. $f \to \sigma f$ is bounded from $\mathcal{D}$ to $\mathcal{H}$.

ii. $f \to \sigma f$, $f \to \partial_{\sigma} f$, $f \to \partial_{\sigma}^2 f = (-\partial_{\sigma} f + \sigma f)$ are bounded from $\mathcal{H}$ to $\mathcal{D}^*$.

iii. $f \to \sigma^2 f$, $f \to \sigma \partial_{\sigma} f$, $f \to \partial_{\sigma}^2 f$ are bounded from $\mathcal{D}$ to $\mathcal{D}^*$.

iv. $f \to f$ is bounded from $\mathcal{D}$ to $\mathcal{H}$ and hence from $\mathcal{H}$ to $\mathcal{D}^*$.

The rest of this section will be devoted to the proof of Proposition 3.2. To simplify the notation we will simply denote $u_2^{\beta\gamma}$ by $u_2$ and set $w := u_1 - u_2$. The difference $w$ satisfies the equation

\[
\frac{\partial}{\partial \tau} w = \mathcal{L}[w] + \frac{w_2}{u_1} + \frac{w^2}{2u_1} + \frac{2w_\sigma u_2\sigma}{u_1} - \frac{w_2^2 w}{u_1 u_2} + \frac{w}{2u_1 u_2} (2 - u_2^2) - J_1 w_\sigma + u_2 (J_2 - J_1)
\]

(3.3)

Note that

\[
J_1 - J_2 = -2 \int_0^\sigma \frac{u_1 u_2}{u_1} d\sigma' + 2 \int_0^\sigma \frac{u_2 u_2\sigma}{u_2} d\sigma' = -2 \int_0^\sigma \left(\frac{2u_2\sigma - u_2^2}{u_1^2} + \frac{u_2^2 - 2}{2u_1 u_2}\right) w d\sigma'.
\]

(3.4)

Let $\varphi_C$ be a cut off function as in section 2.3.1 and let $w_C = w \varphi_C$. An easy computation shows that

\[
\frac{\partial}{\partial \tau} w_C = \mathcal{L}[w_C] + \mathcal{E}(w_C) + \mathcal{E}[w, \varphi_C] + \mathcal{E}_{nl}
\]

(3.5)

where

\[
\mathcal{E}(w_C) := \left(\frac{w_2}{u_1} + \frac{2u_2\sigma}{u_1} - J_1\right)(w_C)_\sigma - \left(\frac{w}{2u_1} + \frac{w_2^2 + u_2^2 - 2}{2u_1 u_2}\right) w_C
\]

(3.6)

and

\[
\mathcal{E}[w, \varphi_C] := \varphi_C \partial_{\tau} w - (\varphi_C)_\sigma w - 2(\varphi_C)_\sigma w_\sigma + \frac{\sigma}{2}(\varphi_C)_\sigma w
\]

(3.7)

and

\[
\mathcal{E}_{nl} := u_2 \varphi_C (J_2 - J_1).
\]

(3.8)

The proof of Proposition 3.2 will follow easily by combining Lemma 3.3 with the next estimate on the error terms in equation (3.5).

**Proposition 3.5.** For every $\epsilon > 0$ and $\theta > 0$ small, there exists a $\tau_0 \ll -1$, so that the error term $\mathcal{E} := \mathcal{E}(w_C) + \mathcal{E}[w, \varphi_C] + \mathcal{E}_{nl}$ satisfies the estimate

\[
\|\mathcal{E}\|_{\mathcal{D}^*} \leq \epsilon \left(\|w_C\|_{\mathcal{D}, \infty} + \|w \chi D_\theta\|_{\mathcal{D}, \infty} \right)
\]

(3.9)

where $D_\theta := \{\sigma \mid \theta/2 \leq u_1(\sigma, \tau) \leq \theta\}$. 


Proof. Note that our unique asymptotics result in [3], together with more refined asymptotics in the collar region (see section A for precise statements and their proofs) we have that for every $\epsilon > 0$ there exists a $\tau_0 \ll -1$ so that for $\tau \leq \tau_0$ we have $|w(\sigma, \tau)| \leq \epsilon$ in $C_\theta$. Combining this, estimate (3.6) and (A.4), since $u \geq \theta/2$ in $C_\theta$, yield

\begin{equation}
|\mathcal{E}(w_C)| \leq \frac{\epsilon}{6} \left(|w_C| + |(w_C)_\sigma|\right) + \frac{|u_2^2 - 2|}{2u_1u_2} |w_C| \tag{3.10}
\end{equation}

and

\begin{equation}
|\mathcal{E}[w, \varphi_C]| \leq \frac{C(\theta)}{\sqrt{|\tau_0|}} |w|_D + C(\theta)|(|\varphi_C|)_\sigma w_{\sigma}\tag{3.11}
\end{equation}

in $C_\theta$, for $\tau \leq \tau_0$. Then, using the estimates in Lemma 3.4 and (3.10) we have

\[\|\mathcal{E}(w_C)\|_D \leq \frac{\epsilon}{6} \|w_C\|_D + \frac{|u_2^2 - 2|}{2u_1u_2} \|w_C\|_D\]

\[\leq \frac{\epsilon}{6} \|w_C\|_D + C(\theta) \left( \int_{0}^{K} (u_2 - \sqrt{2})^2 w_{\sigma}^2 \, d\mu \right)^{\frac{1}{2}} + \frac{C(\theta)}{\sqrt{|\tau_0|}} \left( \int_{0}^{\infty} w_{\sigma}^2 \, d\mu \right)^{\frac{1}{2}}\]

where $C_0$ is a uniform constant, that is, an upper bound of the norm of the operator $f \rightarrow \sigma f$ from $H^2$ to $D^\ast$. For a given $\epsilon > 0$, choose $K$ large so that $C(\theta)/K \leq \epsilon/12$ and then for this $K$ choose $\tau_0 \ll -1$ so that for $\tau \leq \tau_0$ and for $\sigma \in [0, K]$ we have $|u_2 - \sqrt{2}| \leq \epsilon/12$. The latter follows from the fact that the $\lim_{\tau \rightarrow -\infty} u_2(\sigma, \tau) = \sqrt{2}$, uniformly on compact sets. This finally yields

\begin{equation}
\|\mathcal{E}(w_C)\|_{D^\ast, \infty} \leq \frac{\epsilon}{3} \|w_C\|_{D^\ast, \infty}. \tag{3.12}
\end{equation}

Furthermore, using (3.11), Lemma 3.4 and the definition of the cut off function $\varphi_C$, for all $\tau \leq \tau_0$ we have

\[\|\mathcal{E}[w, \varphi_C]\|_D \leq \frac{C(\theta)}{\sqrt{|\tau_0|}} \|w|_{D^\ast} + C(\theta) \|(|\varphi_C|)_\sigma w_{\sigma}\|_D\]

\[\leq \frac{C(\theta)}{\sqrt{|\tau_0|}} \|w|_{D^\ast} + C(\theta) \left( \|(|\varphi_C|)_\sigma w\|_D + \|(|\varphi_C|)_\sigma w\|_{\mathcal{B}} \right)\]

\[\leq \frac{C(\theta)}{\sqrt{|\tau_0|}} \|w|_{D^\ast} + C(\theta) \|w|_{D^\ast} \| \|\varphi_C\|_{\mathcal{B}},\]

where the constant $C(\theta)$ may vary from line to line, but is uniform in time. This leads to

\begin{equation}
\|\mathcal{E}[w, \varphi_C]\|_{D^\ast, \infty} \leq \frac{C(\theta)}{\sqrt{|\tau_0|}} \|w|_{D^\ast} \| \|\varphi_C\|_{\mathcal{B}}. \tag{3.13}
\end{equation}

It remains to deal with the more delicate bound of the non-local term $\mathcal{E}_{nl}$. Using (3.4) we have

\[\mathcal{E}_{nl} = -2 \left( u_{2\sigma}^2 \varphi C \int_{0}^{\sigma} \frac{w_{\sigma\sigma}}{u_1} \, d\sigma' + u_{2\sigma}^2 \varphi C \int_{0}^{\sigma} \frac{w_{\sigma\sigma}}{u_1 u_2} \, d\sigma' \right) =: -2(I_1 + I_2)\]

and hence,

\[\|\mathcal{E}_{nl}\|_{D^\ast} \leq 2 \left( \|I_1\|_{D^\ast} + \|I_2\|_{D^\ast} \right).\]
Using (A.4) we have
\[
\|I_2\|_{B^*}^2 \leq C_0 \|I_2\|_{\beta}^2 \leq \frac{C(\theta)}{|\tau|^2} \int_{\sigma \geq 0} \varphi_C^2 \left( \int_0^\sigma |w| \, d\sigma' \right)^2 e^{-\frac{\tau}{\sigma}} \, d\sigma
\]
\[+ \frac{C(\theta)}{|\tau|^2} \int_{\sigma \leq 0} \varphi_C^2 \left( \int_0^\sigma |w| \, d\sigma' \right)^2 e^{-\frac{\tau}{\sigma}} \, d\sigma.
\tag{3.14}
\]

It is enough to show how we deal with one of the two terms since the other one is handled similarly. Note that by definition \(\varphi_C\) is nonincreasing for \(\sigma \geq 0\) and is nondecreasing for \(\sigma \leq 0\). Using this and the Fubini theorem we get
\[
\int_{\sigma \geq 0} \varphi_C^2 \left( \int_0^\sigma |w| \, d\sigma' \right)^2 e^{-\frac{\tau}{\sigma}} \, d\sigma \leq \int_{\sigma \geq 0} \varphi_C^2 \left( \int_0^\sigma w^2 \, d\sigma' \right)e^{-\frac{\sigma^2}{4}} \, d\sigma
\]
\[
\leq \int_{\sigma \geq 0} \varphi_C^2 (\sigma^4)w^2(\sigma) \, d\sigma e^{-\frac{\sigma^2}{4}} \, d\sigma
\]
\[
= \int_0^{\tau^{-1}(\theta, \sigma)} w_C^2 \left( \int_{\sigma'}^{\tau^{-1}(\theta, \sigma)} e^{-\frac{\sigma'^2}{4}} \, d\sigma' \right) \, d\sigma
\leq C_0 \int_{\sigma \geq 0} w_C^2 \, d\mu \leq C_0 \|w_C\|_{\beta}^2.
\]

This yields the bound
\[
\|I_2\|_{B^*, \infty} \leq \frac{C(\theta)}{|\tau_0|} \|w_C\|_{\beta, \infty}.
\tag{3.15}
\]

We deal with the term \(I_1\), similarly as with term \(I_2\) above. Using (A.4), integration by parts and Fubini’s theorem, gives that for all \(\tau \leq \tau_0\) we have
\[
\|I_1\|_{B^*} \leq C_0 \|I_1\|_{\beta} \leq \frac{C(\theta)}{|\tau|^2} \left( \|\varphi_C \frac{\int_0^\sigma w_\sigma \, d\sigma'}{u_1^2} \|_{\beta}^2 + \|\varphi_C \frac{w_\sigma}{u_1}(\sigma, \tau) \|_{\beta}^2 \right)
\]
\[\leq \frac{C(\theta)}{|\tau|} \left( \|\varphi_C w_\sigma\|_{\beta}^2 \right) \leq \frac{C(\theta)}{|\tau_0|} \left( \|w_C\|_{\beta}^2 + \|w \chi_{D_\alpha}\|_{\beta}^2 \right).
\tag{3.16}
\]

It follows that
\[
\|I_1\|_{B^*, \infty} \leq \frac{C(\theta)}{|\tau_0|} \left( \|w_C\|_{B^*, \infty}^2 + \|w \chi_{D_\alpha}\|_{B^*, \infty}^2 \right).
\tag{3.17}
\]

Finally, (3.12), (3.13), (3.15) and (3.17) imply that for every \(\epsilon > 0\), there exists a \(\tau_0 \ll -1\) so that (3.9) holds, hence concluding the proof of Proposition.

We can finally finish the proof of Proposition 3.2.

**Proof of Proposition 3.2.** Apply Lemma 3.3 to \(w_C\) solving (3.5), to conclude that there exist \(\tau_0 \ll -1\) and constant \(C_0 > 0\), so that if the parameters \((\beta, \gamma)\) are chosen to ensure that \(P^+ w_C(\tau_0) = 0\), then \(\hat{w}_C := P^+ w_C + P^- w_C\) satisfies the estimate
\[
\|\hat{w}_C\|_{B^*, \infty} \leq C_0 \|E\|_{B^*, \infty}, \quad \text{for all } \tau \leq \tau_0
\]
where \(E := E(w_C) + \tilde{E}(w, \varphi_C) + \mathcal{E}_{\text{el}}\). Combining this together with Proposition 3.5 concludes the proof of Proposition 3.2. □
4. The tip region

Let \( u_1(\sigma, \tau) \) and \( u_2(\sigma, \tau) \) be the two solutions to equation (2.11) as in the statement of Theorem 1.5 and let \( u_2^{\beta \gamma} \) be defined by (2.14). We will now estimate the difference of these solutions in the tip region

\[
\mathcal{T}_0 = \{ (\sigma, \tau) \mid u_1(\sigma, \tau) \leq 2\theta \}
\]

for \( \theta > 0 \) sufficiently small, and \( \tau \leq \tau_0 \ll -1 \), where \( \tau_0 \) is to be chosen later. Recall from Section 2.3.2 that the tip region is further divided into the collar region \( K_L,\theta \) defined by (2.26) and the soliton region \( S_L \) defined by (2.27).

As we have seen in Section 2.3.2, in the tip region we exchange the variables \( \sigma \) and \( u \) and consider \( Y(u, \tau) := u_2^{\beta \gamma}(\sigma, \tau) \) as a function of \( u \). This means that for our given solutions \( u_1(\sigma, \tau), u_2^{\beta \gamma}(\sigma, \tau) \) of (2.12), we consider

\[
Y_1(u, \tau) := u_1^2(\sigma, \tau), \quad \text{where} \quad \sigma = \sigma_1(u, \tau) \iff u = u_1(\sigma, \tau)
\]

and similarly

\[
Y_2^{\beta \gamma}(u, \tau) := (u_2^{\beta \gamma})^2(\sigma, \tau), \quad \text{where} \quad \sigma = \sigma_2^{\beta \gamma}(u, \tau) \iff u = u_2^{\beta \gamma}(\sigma, \tau).
\]

Note that by the definition of \( u_2^{\beta \gamma}(\sigma, \tau) \) (see (2.14)) we have that

\[
Y_2^{\beta \gamma}(u, \tau) = Y_2 \left( \frac{u}{\sqrt{1 + \beta e^\tau}}, \tau + \gamma - \log(1 + \beta e^\tau) \right).
\]

We have seen in Section 2.3.2 that in the soliton region we need further rescale of our solutions, namely define

\[
Z_1(\rho, \tau) := Y_1(u, \tau) \quad \text{and} \quad Z_2(\rho, \tau) := Y_2(u, \tau), \quad \rho := \sqrt{|\tau|} u.
\]

Both rescaled solutions satisfy equation (2.28). Also, using (4.1) we see that

\[
Z_2^{\beta \gamma}(\rho, \tau) = Z_2 \left( \frac{\rho \sqrt{\tau + \gamma - \log(1 + \beta e^\tau)}}{\sqrt{|\tau|} \sqrt{1 + \beta e^\tau}}, \tau + \gamma - \log(1 + \beta e^\tau) \right).
\]

The following simple consequence of Theorem 1.6 will be used in the sequel.

**Proposition 4.1.** If \((\beta, \gamma)\) are \( \tau_0 \) admissible in the sense of Definition 2.2, then

\[
Z_1(\rho, \tau) \to Z_0(\rho) \quad \text{and} \quad Z_2^{\beta \gamma}(\rho, \tau) \to Z_0(\rho) \quad \text{as} \quad \tau \to -\infty,
\]

uniformly on compact sets and smoothly, where \( Z_0(\rho) \) is the unique rotationally symmetric Bryant soliton with maximal scalar curvature equal to one.

**Proof.** Let’s first show that each of the rescaled solutions \( Z_i(\rho, \tau) \) according to (4.2) converges, as \( \tau \to -\infty \), uniformly smoothly to the unique rotationally symmetric Bryant soliton \( Z_0(\rho) \) whose maximum curvature is equal to one. Let’s drop the subscript \( i \) from the solutions setting for simplicity, that is set \( Z := Z_i \). Denote by \((S^3, g(\cdot, t))\) the unrescaled solution of (1.1).

By Theorem 1.6 we know that the maximal scalar curvature \( k(t) \) of the solution \((S^3, g(\cdot, t))\) satisfies \( k(t) = \frac{\log(-t)}{-t} (1 + o(1)) \), as \( t \to -\infty \). Moreover, according to Theorem 1.6, the rescaled solution \((S^3, \hat{g}(\cdot, \tau))\), defined by

\[
\hat{g}(\cdot, \tau) = k(t) g(\cdot, t), \quad \tau = -\log(-t),
\]

...
whose maximal scalar curvature is equal to one, converges, as \( \tau \to -\infty \), to the unique rotationally symmetric Bryant soliton whose maximal scalar curvature is equal to one. Since,

\[
\hat{g}(\cdot, \tau) = |\tau| \left( \frac{du^2}{Y(u, \tau)} + u^2 g_{S^2} \right) = \frac{d\rho^2}{Z(\rho, \tau)} + \rho^2 g_{S^2},
\]
we conclude from the above discussion that \( Z(\rho, \tau) \) converges, as \( \tau \to -\infty \), in \( C^\infty_{loc} \) to \( Z_0(\rho) \), where \( \frac{d\rho^2}{Z_0(\rho)} + \rho^2 g_{S^2} \) is the Bryant soliton metric whose maximal scalar curvature is one.

This in particular shows that \( Z_1(\rho, \tau) \to Z_0(\rho) \) and \( Z_2^{\beta\gamma}(\rho, \tau) \to Z_0(\rho) \) in \( C^\infty_{loc} \).

For a cutoff function \( \varphi_T(u) \) supported in the tip region (see (2.25) for its definitions), we set

\[
W_T := \varphi_T W \quad \text{where} \quad W := \Psi_1 - \Psi_2^{\beta\gamma},
\]
and \( \Psi_1 := \sqrt{Y_1} \) and \( \Psi_2^{\beta\gamma} := \sqrt{Y_2^{\beta\gamma}} \). The reason for passing from \( Y \) to \( \Psi := \sqrt{V} \) is that it allows us to easier recognize the divergence structure of the equation for \( W \) which is essential in establishing our energy estimate in the soliton region (see in Section 4.3).

Recall the definition of the norm \( \| \cdot \|_{2, \infty} \) in the tip region is given by (2.33)-(2.34) in Section 2.3.2. The main goal in this section is to establish the following estimate.

**Proposition 4.2.** There exist \( \theta \) with \( 0 < \theta \ll 1 \), \( \tau_0 \ll -1 \) and \( C < +\infty \) such that

\[
\|W_T\|_{2, \infty} \leq \frac{C}{\sqrt{|\tau_0|}} \|W \chi_{[\theta, 2\theta]}\|_{2, \infty}
\]
holds. Constant \( C \) is a uniform constant, independent of \( \tau_0 \), as long as \( \tau_0 \ll -1 \).

To simplify the notation throughout this section we will denote \( Y_1 \) by \( Y \) and \( \Psi_1 \) by \( \Psi \). Also, we will denote \( Y_2^{\beta\gamma} \) by \( \tilde{Y} \) and \( \Psi_2^{\beta\gamma} \) by \( \tilde{\Psi} \). The proof of this Proposition will be based on a Poincaré inequality for the function \( W_T \) which is supported in the tip region. These estimates will be shown to hold with respect to an appropriately chosen weight \( e^{\phi(u, \tau)} du \), where \( \mu(u, \tau) \) is given by (2.30)-(2.31). We will begin by establishing various properties on the weight \( \mu(u, \tau) \), especially estimates which we later use in this section. We will continue with the Poincaré inequality and the energy estimate. Note that the energy estimate will require careful integration by parts which is based on the divergence structure of the equation for \( W \) with respect to our appropriately defined weight \( \mu(u, \tau) \). This estimate is quite more delicate than its analogue in [2]. The proof of Proposition 4.2 will be finished in Section 4.4.

### 4.1. Properties of the weight \( \mu(u, \tau) \)

Let \( \zeta(u) \) be a nonnegative smooth decreasing function defined on \( u \in (0, \infty) \) such that

\[
\zeta(u) = 1, \quad \text{for } u \geq \theta/2 \quad \text{and} \quad \zeta(u) = 0, \quad \text{for } u \leq \theta/4.
\]
Such a function can be chosen to satisfy the derivative estimate \( |\zeta'(u)| \leq 5\theta^{-1} \).

For our given solution \( u(\sigma, \tau) \) which after the coordinate change gives rise to \( \sigma(u, \tau) \) and \( Y(\sigma, \tau) := u_\sigma^2(u, \tau) \) (recall that we have dropped the index and denote
σ, Y₁ by σ, Y respectively) we define our weight μ(u, τ) in the tip region as in (2.30) where we recall that

\[ \mu_u := \zeta(u) \left( -\frac{\sigma^2(u, \tau)}{4} \right)_u + \left( 1 - \zeta(u) \right) \frac{1 - Y(u, \tau)}{u Y(u, \tau)}. \]

Note that since \( \zeta \equiv 1 \) for \( u \geq \theta/2 \), we have \( \mu(u, \tau) = -\frac{\sigma^2(u, \tau)}{4} \) in this region, hence it coincides with our weight in the cylindrical region. This is important as our norms in the intersection of the cylindrical and tip regions need to coincide.

In this section we will prove sharp estimates on our weight \( \mu \) and its derivatives which will be used in the following two sub-sections. Crucial role will play the convexity estimate which is shown in the Appendix, Proposition A.2, namely that

\[ (u^2)_{\sigma \sigma} \leq 0, \quad \text{on } u \geq \frac{L}{\sqrt{|\tau|}} \]

holds for \( L \gg 1 \) and \( \tau \leq \tau_0 \ll -1 \). We will also use its consequence (A.15). To facilitate future references, let us remark that (4.6) expressed in terms of \( Y := u_\sigma^2 \) implies that

\[ u Y_u + 2 Y \leq 0, \quad \text{on } K_{L, \theta} \]

holds for \( L \gg 1 \) and \( \tau \leq \tau_0 \ll -1 \). Also, throughout this whole section we will use the bound

\[ 1 - \frac{\eta}{10} \leq \frac{\sigma^2}{4|\tau|} \leq 1 + \frac{\eta}{10}, \quad \text{on } T_\theta \]

which holds for \( \tau \leq \tau_0 \ll -1 \) and \( \theta = \theta(\eta) \) sufficiently small. This bound is an immediate consequence of Theorem 1.6.

**Lemma 4.3.** Fix \( \eta > 0 \). There exist \( \theta > 0 \) small, \( L \gg 1 \) and \( \tau_0 \ll -1 \) such that

\[ (1 - \eta) \mu_u \leq \frac{1 - Y}{u Y} \leq (1 + \eta) \mu_u \]

and

\[ (1 - \eta) |\tau| \leq \frac{1}{u^2 Y} \leq (1 + \eta) |\tau| \]

hold on \( K_{L, \theta} \) and \( \tau \leq \tau_0 \).

**Proof.** Both bounds follow from our crucial estimate (A.15), which says that

\[ 1 - \frac{\eta}{10} \leq -\frac{\sigma u u_\sigma}{2} \leq 1 + \frac{\eta}{10} \]

holds on \( K_{L, \theta} \) for \( \theta > 0 \) small, \( L \gg 1 \) and \( \tau \leq \tau_0 \ll -1 \).

The first estimate simply follows from the definition of \( \mu_u \), and the fact that

\[ \left( -\frac{\sigma^2}{4} \right)_u = -\frac{\sigma u u_\sigma}{2} = -\frac{\sigma}{u} \frac{1}{u_\sigma^2} = -\frac{\sigma u u_\sigma}{2} \frac{1}{u Y} \]

which with the aid of (4.11) gives

\[ \left| \mu_u - \frac{1 - Y}{u Y} \right| \leq \left| \left( -\frac{\sigma^2}{4} \right)_u - \frac{1 - Y}{u Y} \right| \leq \frac{\eta}{10} \frac{1}{u Y} + \frac{1}{u} \leq \eta \frac{1 - Y}{Y u} \]

where in the last inequality we used that \( Y \ll \eta \) in the considered region.
For the second inequality (4.10), we observe that
\[ |\sigma| u^2 Y = \frac{4|\sigma| u^2 u^2}{4} \]
hence the bound readily follows by (4.11) and (4.8).

**Corollary 4.4.** There exists an absolute \( \theta \ll 1 \) and \( \tau_0 \ll -1 \) such that

\[ \mu(u, \tau) \leq -\frac{|\tau|}{4} \]
holds on \( T_\theta \) for \( \tau \leq \tau_0 \ll -1 \).

**Proof.** We claim that \( \mu u \geq 0 \) in \( T_\theta \). Indeed, in \( K_{\theta,L} \) this is true by (4.9). In \( S_L \), \( \zeta = 0 \) and thus \( \mu u = 1 - Y u \sigma \). Hence, in \( T_\theta \) we have
\[
\mu(u, \tau) = -\frac{\sigma^2(\theta, \tau)}{4} - \int_\theta^u \mu(u', \tau) du' \leq -\frac{\sigma^2(\theta, \tau)}{4}.
\]
The claim (4.12) now immediately follows by (4.8), by taking \( \eta \) there sufficiently small.

We will next estimate \( u_\tau \) in the region \( \theta/4 \leq u \leq 2\theta \). This will be used later to estimate the time derivative \( \mu_\tau (u, \tau) \) of our weight.

**Lemma 4.5.** Fix \( \eta > 0 \). There exist \( \theta > 0 \) small, \( L \gg 1 \) and \( \tau_0 \ll -1 \) such that the bounds

\[ |u_\tau(\sigma, \tau)| \leq \frac{\eta}{u} \quad \text{and} \quad \left| \left( -\frac{\sigma^2(u, \tau)}{4} \right) \right| \leq \frac{\eta}{u^2 Y} \]
hold in the collar region \( K_{\theta,L} \) and \( \tau \leq \tau_0 \).

**Proof.** We recall equation (2.12), namely that
\[
u_\tau = u_{\sigma\sigma} - \frac{\sigma}{2} u_\sigma - J(\sigma, \tau) u_\sigma + \frac{u_\sigma}{u} - \frac{1}{u} + \frac{u}{2},
\]
Using the bounds
\[
0 \leq -J \leq 2 \frac{|u_\sigma|}{u}, \quad |u_\sigma| \leq \frac{\eta}{10 u}, \quad \left| 1 + \frac{\sigma u u_\sigma}{2u} \right| \leq \frac{\eta}{10}, \quad |u_\sigma| \leq \frac{\eta}{20}
\]
which hold for \( \tau_0 \ll -1 \), we conclude that
\[
|u_\tau| \leq \frac{\eta}{2u} + \frac{1}{u} \left| \frac{\sigma u u_\sigma}{2u} + 1 \right| \leq \frac{\eta}{2u} + \frac{2\theta^2}{u} \leq \frac{\eta}{u},
\]
if we take \( \theta \) so that \( 4\theta^2 \leq \eta \). Now, using this bound, we compute
\[
\left| \left( -\frac{\sigma^2}{4} \right) \right| = \left| \frac{\sigma \sigma_\tau}{2} \right| = \left| \frac{\sigma u_\tau}{2u_\sigma} \right| = \frac{1}{u} \left| \frac{\sigma u u_\sigma}{2u} \right| |u_\tau| \leq \frac{\eta}{u^2 Y}.
\]

We will also need an estimate for the time \( \mu_\tau (u, \tau) \) derivative of our weight in the whole tip region \( T_\theta \) where \( u \leq 2\theta \). But before we do so, we will estimate \( Y u_\tau, Y u_{uu} \) as well as \( Y_\tau \) in this region. Recall that \( Y \) satisfies equation (2.24). Our estimates will be based on the bounds in Proposition A.4, namely that given an \( \eta > 0 \) there exist \( \theta \) and \( L \gg 1 \) and \( \tau_0 \ll -1 \) such that for \( \tau \leq \tau_0 \),

\[ 0 \leq -u u_{\sigma\sigma} \leq \eta \quad \text{and} \quad u^2 |u_{\sigma\sigma}| \leq \eta \]
Lemma 4.6. Fix $\eta > 0$. There exist $\theta > 0$ small, $L \gg 1$ and $\tau_0 \ll -1$ such that $Y$ satisfies derivative bounds

\begin{equation}
|Y_u| \leq \frac{\eta}{u}, \quad |Y_{uu}| \leq \frac{\eta}{u^2\sqrt{Y}}, \quad Y_\tau \leq \frac{\eta\sqrt{Y}}{u^2} + \frac{\eta}{4}, \quad |Y_\tau| \leq \frac{\eta}{u^2}
\end{equation}

hold on $K_{\theta,L}$ and for $\tau \leq \tau_0$. It follows that $\Psi := \sqrt{Y}$ satisfies the derivative bounds

\begin{equation}
|\Psi_u| \leq \frac{\eta}{u\Psi}, \quad |\Psi_{uu}| \leq \frac{\eta}{u^2\Psi^2}, \quad \Psi_\tau \leq \frac{\eta}{u^2} + \frac{\eta}{8\Psi}, \quad |\Psi_\tau| \leq \frac{\eta}{u^2}
\end{equation}

in the same region.

Proof. Recall that $Y_u = 2u_\sigma u_{\sigma u} = 2u_{\sigma u}$ and $Y_{uu} = (u_{\sigma \sigma})_u = u_{\sigma \sigma \sigma} u_u$. Hence, the first two bounds in (4.15) readily follow from the two bounds in (4.14). For the third bound we combine (2.24) with (4.7) and the two bounds we just obtained (with $\eta/2$ instead of $\eta$) to conclude

\[ Y_\tau \leq Y Y_{uu} - \frac{u}{2} Y_u \leq \frac{\eta\sqrt{Y}}{2u^2} + \frac{\eta}{4}. \]

The last bound in (4.15) follows the same way, using the bounds we just obtained (with $\eta/10$ instead of $\eta$) and $Y \ll 1$. The bounds (4.16) readily follow from (4.15).

We will next combine the estimates above to obtain the bounds for $\mu_{uu}(u, \tau)$ and $\mu_\tau(u, \tau)$ which will be used in the the next two subsections.

Lemma 4.7. Fix $\eta > 0$. There exist $\theta > 0$, $L \gg 1$ and $\tau_0 \ll -1$ such that the bound

\begin{equation}
\mu_{uu} \leq \eta \mu_u^2
\end{equation}

holds in the collar region $K_{\theta,L}$ and for all $\tau \leq \tau_0 \ll -1$.

Proof. Fix $\eta > 0$, and assume that $\sigma > 0$ and $u_\sigma < 0$, as the case $\sigma < 0$ and $u_\sigma > 0$ is similar. Recall that $\mu(u, \tau)$ is defined so that it satisfies (2.31). We know that the bounds (A.15), (4.9) and (4.15) hold on $K_{\theta,L}$, for $\bar{\eta} := \eta/10$ and $\theta$ small, $L$ large and $\tau \leq \tau_0 \ll -1$. Using these bounds, we find that in the region where $\mu_u = (\frac{-\sigma_u^2}{2})_u$, we have

\[ \mu_{uu} = \left(\frac{-\sigma_u}{2}\right)_u = \frac{-\sigma_{uu} + \sigma_u^2}{2} \leq \frac{\sigma_{uu} - \sigma_u^2}{2u_{\sigma u}^2} \leq \frac{1}{2} \left| \frac{\sigma_{uu}}{u_{\sigma u}^2} \right| \frac{\bar{\eta}}{u^2 Y^2} \leq \bar{\eta} \mu_u^2 \]

while in the region where $\mu_u = (\frac{1 - Y^{-1}}{uY})_u$, we have

\[ \mu_{uu} = \left(\frac{Y^{-1} - 1}{u} \right)_u = \frac{1 - Y^{-1}}{u^2 Y} - \frac{Y_u}{u Y^2} \leq \frac{|Y_u|}{u Y^2} \leq \frac{\bar{\eta}}{u^2 Y^2} \leq 2\bar{\eta} \mu_u^2. \]

We conclude, with the aid of (4.9), (A.15), applied with $\eta/10$ instead of $\eta$, (using also the bound $\zeta'(u) \leq 5\theta^{-1} \leq 20u^{-1}$ and $Y \ll 1$) that

\[ \mu_{uu} \leq (1 - \zeta) \frac{\eta}{u^2 Y^2} + \zeta'(u) \left| \left(\frac{-\sigma_u}{2}\right) - \frac{Y^{-1} - 1}{u} \right| \leq 4\bar{\eta} \mu_u^2 \leq \eta \mu_u^2 \]

holds on $K_{\theta,L}$ and $\tau \leq \tau_0 \ll -1$. 

\[ \square \]
Lemma 4.8. Fix \( \eta > 0 \). There exist \( \theta > 0 \) and \( \tau_0 \ll -1 \) such that the bound

\[
\mu_{\tau}(u, \tau) \leq \eta |\tau| \left( 1 + \rho^{-1} \chi_{[0, 1]}(\rho) \right) \leq C_0 \frac{\eta}{u^2 Y}, \quad \rho := u \sqrt{|\tau|}
\]

holds in the whole tip region \( T_{\theta} \), for all \( \tau \leq \tau_0 \ll -1 \).

**Proof.** We use the definition of \( \mu(u, \tau) \) in (2.30)-(2.31) and that \( \zeta \equiv 1 \) for \( u \geq \theta/2 \). Integration by parts gives

\[
\mu_{\tau} = \left( -\frac{\sigma^2(\theta, \tau)}{4} \right)_{\tau} + \int_\theta^u \zeta \left( -\frac{\sigma^2(u, \tau)}{4} \right)_{u\tau} + (1 - \zeta) \left( \frac{Y^{-1}(u, \tau) - 1}{u} \right)_{\tau} \, du
\]

\[
= \zeta \left( -\frac{\sigma^2(u, \tau)}{4} \right)_{\tau} + \int_\theta^u \zeta' \left( -\frac{\sigma^2}{4} \right)_{\tau} \, du + \int_\theta^u (1 - \zeta) \frac{Y}{uY^2} \, du
\]

where, to simplify the notation, we will denote the variable of integration by \( u \) (instead of \( u' \)) when there is no danger of confusion.

Fix \( \eta > 0 \) and small. We will treat separately the two cases of the collar and soliton regions, \( L/\sqrt{|\tau|} \leq u \leq \theta \) and \( u \leq L/\sqrt{|\tau|} \), respectively.

**Case 1: Given \( \eta > 0 \), there exists \( 0 < \theta \ll 1 \) and \( L \gg 1 \) such that (4.18) holds on \( L/\sqrt{|\tau|} \leq u \leq \theta \).

Observe first that on the region \( u \geq \theta \), we have \( \zeta(u) = 1, \zeta'(u) = 0 \), hence the desired bound simply follows from (4.13) and (4.10) for \( \theta \) small and \( \tau \leq \tau_0(\theta) \ll -1 \).

Assume now that \( u \leq \theta \). Then the second bound in (4.13) (with \( \eta \) replaced by \( \eta/10 \) and (4.10) readily give that

\[
\left| \zeta \left( -\frac{\sigma^2(u, \tau)}{4} \right)_{\tau} + \int_\theta^u \zeta' \left( -\frac{\sigma^2}{4} \right)_{\tau} \, du \right| \leq \frac{\eta}{4} |\tau|
\]

holds on the whole tip region \( T_{\theta} \), for \( \theta \) small and \( \tau \leq \tau_0 \ll -1 \) (recall that both \( \zeta \) and \( \zeta' \) are zero for \( u \leq \theta/4 \)).

Let’s now look at the last integral in (4.19). Using equation (2.24) to substitute for \( Y_{\tau} \) and integrating by parts we obtain

\[
\int_\theta^u (1 - \zeta) \frac{Y}{uY^2} \, du = \int_\theta^u (1 - \zeta) \left( \frac{Y_{uu}}{uY} - \frac{Y_u^2}{2uY^2} + \frac{1 - Y}{u^4Y^2} (uY_u + 2Y) - \frac{1}{2} \frac{Y_u}{Y^2} \right) \, du
\]

\[
= \int_\theta^u (1 - \zeta) \left( \frac{1}{2} \frac{Y_u^2}{uY^2} + \frac{1 - Y}{u^4Y^2} (uY_u + 2Y) - \frac{1}{2} \frac{Y_u}{Y^2} \right) \, du
\]

\[
+ \int_\theta^u \zeta_u \frac{Y}{uY} \, du + (1 - \zeta(u)) \frac{Y_u}{uY} (u, \tau)
\]

To obtain the desired bound, we cannot just use the estimates (4.15), we need to use careful integration by parts together with (4.7). In fact we will use the negative term \( uY_u + 2Y \leq 0 \) in our favor to bound the first term on the right hand side of
the last formula. To this end, we write

\[
\int_{u}^{\theta} (1 - \zeta) \frac{Y_u}{u Y^2} du = \int_{u}^{\theta} (1 - \zeta) \left( \frac{1}{2} \frac{Y_u}{u^2 Y^2} (u Y_u + 2Y) + \frac{1 - Y}{u^3 Y^2} (u Y_u + 2Y) \right) du \\
- \int_{u}^{\theta} (1 - \zeta) \left( \frac{Y_u}{u^2 Y^2} + \frac{1}{2} \frac{Y_u}{Y^2} \right) du + \int_{u}^{\theta} \zeta \frac{Y_u}{u Y} du \\
+ (1 - \zeta(u)) \frac{Y_u}{u Y}(u, \tau) \\
= \int_{u}^{\theta} (1 - \zeta) \frac{1}{u^3 Y^2} \left( \frac{1}{2} u^2 Y_u + 1 - Y \right) (u Y_u + 2Y) du \\
- \int_{u}^{\theta} (1 - \zeta) \left( \frac{Y_u}{u^2 Y} + \frac{1}{2} \frac{Y_u}{Y^2} \right) du + \int_{u}^{\theta} \zeta \frac{Y_u}{u Y} du \\
+ (1 - \zeta(u)) \frac{Y_u}{u Y}(u, \tau) \\
\leq \int_{u}^{\theta} \frac{(-Y_u)}{u^2 Y} + \int_{u}^{\theta} \frac{(-Y_u)}{2Y^2} du + \int_{u}^{\theta} \zeta \frac{Y_u}{u Y} du \\
+ (1 - \zeta(u)) \frac{Y_u}{u Y}(u, \tau), \text{ since } 0 \leq \zeta \leq 1,
\]

where in the last inequality we used \(1/2 u^2 Y_u + 1 - Y \geq -\eta u + 1 - Y > 0\) and \(u Y_u + 2Y \leq 0\). By \((4.10)\) (with \(\eta = 1\)), we know that throughout collar region the values of \(1/u^2 Y\) are comparable. Hence, we can pull out the \(1/u^2 Y(u, \tau)\) from the first integral in the last line above, evaluated at the end point \(u\), and use \(Y_u \leq 0\), \(u \leq \theta < 1\) to obtain that

\[
\int_{u}^{\theta} \left( \frac{(-Y_u)}{u^2 Y} + \frac{1}{2} \frac{(-Y_u)}{Y^2} \right) du \leq \frac{2}{u^2 Y} \int_{u}^{\theta} (-Y_u) du' + \frac{1}{2} \int_{u}^{\theta} \left( \frac{1}{Y} \right) u' du' \\
\leq \frac{2}{u^2} + \frac{1}{2Y(\theta, \tau)} \leq |\tau| \left( \frac{2}{L^2 + \theta^2} \right) \leq \frac{\eta}{20} |\tau|
\]

holds, for \(L \gg 1\) and \(0 < \theta \ll 1\) both depending on \(\eta\). Furthermore, by \((4.15)\) we have

\[
\int_{u}^{\theta} \zeta \frac{Y_u}{u Y} du + (1 - \zeta(u)) \frac{Y_u}{u Y}(u, \tau) < \frac{\eta}{20} |\tau|.
\]

We conclude that

\[
\text{eqn-zeta5} \quad \int_{u}^{\theta} (1 - \zeta) \frac{Y_u}{u Y^2} du \leq \frac{\eta}{10} |\tau|
\]

Finally combining the bounds \((4.20)\) and \((4.21)\) we conclude that the first bound in \((4.18)\) holds, in \(K_{\theta, L}\), provided \(L \gg 1\), \(0 < \theta \ll 1\) and \(\tau \leq \tau_0 \ll -1\).

**Case 2:** For the given \(\eta > 0\), let \(L \gg 1\) be such that \((4.18)\) holds in \(K_{L, \theta}\) and for \(\tau \leq \tau_0 \ll -1\). Then, we may choose \(\tau_0 \ll -1\) such that \((4.18)\) also holds on \(S_L\) and \(\tau \leq \tau_0\).
Since (4.20) holds in the whole tip region, using also (4.21) at \( u = L/\sqrt{|\tau|} \) and that \( \zeta = 0 \) on \( u < \theta/4 \), we obtain
\[
\mu_{\tau} \leq \frac{\eta}{4}|\tau| + \int_{L/\sqrt{|\tau|}}^{\theta} (1 - \zeta) \frac{Y_{\tau}}{uY^2} du + \int_{L/\sqrt{|\tau|}}^{u} (1 - \zeta) \frac{Y_{\tau}}{uY^2} du
\]  
(4.22)
\[
\leq \frac{\eta}{2}|\tau| + \int_{L/\sqrt{|\tau|}}^{u} \frac{Y_{\tau}}{uY^2} du.
\]
Recall that in the soliton region \( S_L \), the rescaled solution \( Z(\rho, \tau) := Y(u, \tau) \), with \( \rho := u \sqrt{|\tau|} \) converges to the Bowl soliton \( Z_0(\rho) \), which implies (using (2.28)) that
\[
\frac{1}{|\tau|} Y_{\tau} = \frac{1}{|\tau|} (Z_{\tau} - \frac{\rho}{2|\tau|} Z_{\rho}) \rightarrow 0, \quad \text{as } \tau \rightarrow -\infty.
\]  
(4.23)
uniformly on \( \rho \in [0, L] \). We conclude that for our given constant \( L > 0 \) and any \( \eta' > 0 \), there exists \( \tau_0 < -1 \), such that for all \( \rho \leq L \), we have
\[
\int_{L/\sqrt{|\tau|}}^{u} \frac{Y_{\tau}}{uY^2} du \leq \eta' |\tau| \int_{\rho}^{L} \frac{1}{\rho Z_0} d\rho \leq \eta' |\tau| \frac{C(L)}{\rho} \leq \frac{\eta}{10} \frac{|\tau|}{\rho}
\]
by taking \( \eta' = \eta/(10 \cdot C(L)) \), for our given \( \eta > 0 \). Plugging this last estimate in (4.22) concludes the first bound in (4.18) holds in the soliton region.

Let's now check that the second bound in (4.18) holds for \( \tau \ll \tau_0 < -1 \). To this end, we fix \( L_0 \) universal constant so that \( |\tau| \leq \frac{2}{\rho^2 Y^2} \) holds on \( u \geq L_0/\sqrt{|\tau|} \) (we use again (4.10)). On the other region where \( \rho \leq L_0 \), we use \( Z(\rho, \tau) = Y(u, \tau) \leq 1 \) to get \( \rho^2 Z (1 + \rho^{-1} \chi_{[0,1]}(\rho)) \leq C_0 = C(L_0) \), which readily gives the desired bound.

\[\Box\]

4.2. Poincaré inequality in the tip region. Our goal in this section is to derive the following Poincaré inequality:

\[\textbf{Proposition 4.9} \text{ (Poincaré inequality). There exists an absolute constant } C_0 > 0, \text{ a small absolute constant } \theta_0, \text{ and } \tau_0 \ll -1, \text{ such that the inequality}\]
\[\int \mu_u^2 f^2 e^{-\mu(u, \tau)} du \leq C_0 \left( \int f_u^2 e^{-\mu(u, \tau)} du + \int \frac{f^2}{u^2} e^{-\mu(u, \tau)} du \right)\]
holds, for any smooth compactly supported function \( f \) in \( \mathcal{T}_{\theta_0} \) and for all \( \tau \leq \tau_0 \).

\[\textbf{Proof.} \quad \text{First we show the following weighted estimate}\]
\[\int \frac{1}{2} \mu_u^2 f^2 e^{-\mu} du - \int \mu_{uu} f^2 e^{-\mu} \leq 2 \int f_u^2 e^{-\mu} du\]
which simply follows by completing the square and integrating by parts. To this end, we write
\[
0 \leq \left( f_{u} - \frac{\mu_u}{2} f \right)^2 = f_u^2 + \frac{\mu_u^2}{4} f^2 - \mu_u f f_u = f_u^2 + \frac{\mu_u^2}{4} f^2 - \frac{1}{2} \mu_u (f^2)_u
\]
and integrate by parts to obtain
\[
\int \left( f_u^2 + \frac{\mu_u^2}{4} f^2 \right) e^{-\mu} du \geq \frac{1}{2} \int \mu_u (f^2)_u e^{-\mu} du
\]
\[
= \frac{1}{2} \int \left( - \mu_{uu} + \frac{\mu_u^2}{2} \right) f^2 e^{-\mu} du.
\]
Rearranging terms leads to
\[ \frac{1}{4} \int \mu_u^2 f^2 e^{-\mu} du - \frac{1}{2} \int \mu_{uu} f^2 e^{-\mu} du \leq \int f_u^2 e^{-\mu} du. \]
which readily implies (4.25).

We will now apply (4.25) to our special case where the weight \( \mu(u, \tau) \) is given by (2.30)-(2.31). Let \( \theta_0, L_0 \) be universal constants such that Corollary 4.7 holds with \( \eta := 1/8 \), namely that \( \mu_{uu} \leq \frac{1}{8} \mu_u^2 \) holds on \( L_0/\sqrt{|\tau|} \leq u \leq 2\theta_0 \), for \( \tau \leq \tau_0 \ll -1 \).

To deal with the region \( u \leq L_0/\sqrt{|\tau|} \) we will next consider the change of variable \( \rho := u/\sqrt{|\tau|} \) and we will use the \( C^\infty \) convergence of \( Z(\rho, \tau) := Y(u, \tau) \) to the Bryant soliton \( Z_0(\rho) \). Since,
\[ 2|\Psi_u| |\Psi| \leq |\tau| \frac{|Z_\rho|}{\rho^2 Z^2} \]
the convergence of \( Z(\rho, \tau) \rightarrow Z_0 \) implies that
\[ |Z_\rho| \leq \frac{C(L_0)}{\rho} \leq \frac{C(L_0)}{\rho^2} \]
where \( C(L_0) = L_0 C(L_0) \). Hence,
\[ (4.26) \]
\[ \mu_{uu} \leq 2 \frac{|\Psi_u|}{u \Psi^3} \leq |\tau| \frac{C(L_0)}{u^2} = \frac{\bar{C}(L_0)}{u^2} \]
holds for \( u \leq L_0/\sqrt{|\tau|} \). Combining the two estimates (4.17) with (4.26) finally gives the bound
\[ \mu_{uu} \leq \frac{1}{8} \mu_u^2 + \frac{\bar{C}(L_0)}{u^2} \]
which holds on the whole tip region \( T_{\bar{\theta}_0} \). The last estimate combined with (4.25) readily gives (4.24).

\[ \square \]

### 4.3. Energy estimate for \( W_T \)

We will next derive an energy estimate for difference \( W := \Psi_1 - \Psi_2 \) in the weighted \( L^2 \)-space with respect to our weight \( e^{\mu} du \), as defined in Section 2.3.2 (see (2.30)-(2.31) for the definition of \( \mu := \mu(u, \tau) \) and (2.32)-(2.34) for the definition of the \( L^2 \)-norm). Recall that we denote \( \Psi_1, Y_1 \) by \( \Psi, Y \) and \( \bar{Y}, \bar{\Psi} \).

A direct calculation shows that both \( \Psi(u, \tau), \Psi_2(u, \tau) \) satisfy the equation
\[ (4.27) \]
\[ \Psi^{-2} \left( \Psi_u + \frac{u}{2} \Psi_u \right) = \Psi_{uu} + \Psi_u - 1 + \Psi_u + \Psi^{-1} - \Psi \]
since \( Y(u, \tau), \bar{Y}(u, \tau) \) satisfy (2.24). In fact the reason for considering \( \Psi := \sqrt{Y} \) instead of \( Y \) is that the equation (4.27) for \( \sqrt{Y} \) is simpler than that of \( Y \) and, in particular, it has a nice divergence structure which will help us derive a sharp energy estimate, suitable for our purposes.

Let \( W_T = W \varphi_T \), where \( \varphi_T(u) \) is the cut-off function defined in (2.25). In this subsection we will derive a weighted energy estimate for \( W_T \) and combine it with our Poincaré inequality to obtain following differential inequality.
**Proposition 4.10** (Integral Differential inequality). There exist absolute constants $\theta > 0$ small, $\lambda > 0$ and $\tau_0 \ll -1$ and a constant $C(\theta)$ such that

\[
\frac{d}{d\tau} \int W_{\tau}^2 \Psi^{-2} e^\mu du \leq -2\lambda|\tau| \int W_{\tau}^2 \Psi^{-2} e^\mu du + C(\theta) \int_0^{2\theta} W^2 \Psi^{-2} e^\mu du.
\]

holds for all $\tau \leq \tau_0$.

**Proof.** We will begin by computing the equation that the difference $W := \Psi - \Psi_2$ satisfies. Subtracting the equation for $\Psi_2$ from the equation for $\Psi$, we find

\[
\frac{1}{\Psi^2} \left( W_{\tau} + \frac{u}{2} W_u \right) = W_{uu} + \frac{\Psi^{-2} - 1}{u} W_u + \frac{\Psi^{-2} - \Psi_2^{-2}}{\mu} \Psi_{2u} - \left( \frac{1}{u^2 \Psi_2^2} + \frac{1}{u^2} \right) W - (\Psi^{-2} - \Psi_2^{-2}) \left( \Psi_{2\tau} + \frac{u}{2} \Psi_{2u} \right).
\]

We can further express

\[
\Psi^{-2} - \Psi_2^{-2} = \frac{\Psi_2^{-2} - \Psi_2^{-2}}{\Psi^2 \Psi_2^2} = -\frac{\Psi + \Psi_2}{\Psi^2} W = -\frac{2\Psi_2}{\Psi^2} W
\]

where, to simplify the notation, we have denoted by $\Psi_{12} := \frac{1}{2} \Psi + \Psi_2$. Under this notation, the above equation for $W$ becomes

\[
\frac{1}{\Psi^2} \left( W_{\tau} + \frac{u}{2} W_u \right) = W_{uu} + \frac{\Psi^{-2} - 1}{u} W_u - \left( \frac{2\Psi_{2u}}{u \Psi^2} + \frac{1}{u^2 \Psi_2^2} + \frac{1}{u^2} \right) W + \frac{2\Psi_{12}}{\Psi^2} \left( \Psi_{2\tau} + \frac{u}{2} \Psi_{2u} \right) W
\]

where we have arranged terms in such a way that the terms in the second line will be considered error terms.

In an attempt to recognize a divergence structure in the above equation for $W$, we next observe that since $\Psi \approx \Psi_2 \approx \Psi_{12}^{-1}$ we have

\[
\frac{2\Psi_{2u}}{u \Psi_{12}^2 \Psi^2} + \frac{1}{u^2 \Psi_2^2} + \frac{1}{u^2} \approx \frac{2\Psi_{u}}{u \Psi_2^2} + \frac{\Psi^{-2} - 1}{u^2} + \frac{2}{u^2} = -\left( \frac{\Psi^{-2} - 1}{u} \right) W + \frac{2}{u^2} W
\]

It follows that the equation above becomes

\[
\frac{1}{\Psi^2} \left( W_{\tau} + \frac{u}{2} W_u \right) = W_{uu} + \frac{\Psi^{-2} - 1}{u} W_u + \frac{\partial}{\partial u} \left( \frac{\Psi^{-2} - 1}{u} \right) W - \frac{2}{u^2} W + \frac{1}{u^2} B W
\]

where

\[
B := \left( \frac{2\Psi_{u}}{u \Psi} - \frac{2\Psi_{2u}}{u \Psi_{12}^2 \Psi} + \frac{\Psi - \Psi_2}{u^2 \Psi_2^2} \right) + 2\Psi_{12} \left( \Psi_{2\tau} + \frac{u}{2} \Psi_{2u} \right).
\]

The second and third term on the right-hand side of the above equation can be combined together as one term in divergence form. This finally gives us the equation

\[
\frac{1}{\Psi^2} W_{\tau} = W_{uu} + \frac{\partial}{\partial u} \left( \frac{\Psi^{-2} - 1}{u} W \right) - \frac{u}{2 \Psi^2} W_u - \frac{2}{u^2} W + \frac{1}{u \Psi^2} B W.
\]

Also, using the bounds in (4.16) (with $\eta/10$) and the bounds $\Psi/2 \leq \Psi_2 \leq 2\Psi$ (which readily follow from (4.10)) we may estimate $B$ as

\[
B \leq \frac{\eta}{u^2 \Psi^2}.
\]

Let $W_T = W\varphi_T$, where $\varphi_T(u)$ is the cut-off function defined in (2.25). To simplify the notation, we drop the subscript $T$ from $\varphi_T$, and simply set $\varphi := \varphi_T$.  

Since, $\varphi$ is independent of $\tau$, integration with respect to our measure $e^\mu du$ and differentiation in time $\tau$ yields
\[
\frac{1}{2} \frac{d}{d\tau} \int \frac{W^2}{\Psi^2} e^\mu du = \int \frac{1}{\Psi^2} WW_\tau \varphi^2 e^\mu du + \int \left( \frac{1}{2} \mu_\tau - \frac{\Psi_\tau}{\Psi} \right) \frac{W^2}{\Psi^2} e^\mu du.
\]

Using equation (4.29) while integrating by parts the first two terms we obtain
\[
\frac{1}{2} \frac{d}{d\tau} \int \frac{W^2}{\Psi^2} e^\mu du = - \int \frac{W^2 \varphi^2 e^\mu du}{u} - \int \left( \mu_u + \frac{\Psi^2 - 1}{u} + \frac{u}{2\Psi^2} \right) WW_u \varphi^2 e^\mu du - \frac{2}{u^2} W^2 e^\mu du - \frac{u}{\Psi^2} \int \frac{2}{u^2} W^2 e^\mu du + \int \frac{1}{2} \mu_\tau - \frac{\Psi_\tau}{\Psi} \frac{W^2}{\Psi^2} e^\mu du - 2 \int W W_u \varphi_u e^\mu du - 2 \int \frac{\Psi^2 - 1}{u} W^2 \varphi \varphi_u e^\mu du.
\]

Set
\[
G_1 := \mu_u + \frac{\Psi^2 - 1}{u} + \frac{u}{2\Psi^2} \quad \text{and} \quad G_2 := \frac{1}{\Psi^2} \left( \beta + \frac{1}{2} \mu_\tau - \frac{\Psi_\tau}{\Psi} \right).
\]

Furthermore, use $(W_T)_u = W_u \varphi + \varphi_u$ to write
\[
\int G_1 \int W W_u \varphi^2 e^\mu du = \int G_1 \int W_T (W_T)_u e^\mu du - \int G_1 \int W^2 \varphi \varphi_u e^\mu du
\]
and $(W_T)_u^2 = W_u^2 \varphi^2 + 2 W W_u \varphi \varphi_u + W^2 \varphi^2$ to write
\[
- \int W_u^2 \varphi^2 e^\mu du = - \int (W_T)_u^2 e^\mu du + \int W_u^2 \varphi_u e^\mu du + 2 \int W W_u \varphi \varphi_u e^\mu du.
\]

Inserting this in (4.31) we obtain (after cancelling terms)
\[
\frac{1}{2} \frac{d}{d\tau} \int \frac{W^2}{\Psi^2} e^\mu du = - \int (W_T)_u^2 e^\mu du - \int G_1 \int W_T (W_T)_u e^\mu du - \int \mu_u \frac{\Psi^2 - 1}{u} W^2 e^\mu du - \frac{2}{u^2} W^2 e^\mu du + \int G_2 \int W_T^2 \varphi^2 e^\mu du - \int \left( - G_1 + 2 \frac{\Psi^2 - 1}{u} \right) W^2 \varphi \varphi_u e^\mu du + \int W^2 \varphi_u e^\mu du.
\]

We will see in the sequel that our weight $e^{\mu(u, \tau)}$ is chosen so that $|\mu_u - \frac{\Psi^2 - 1}{u}|$ is small compared to $\mu_u$ in the whole tip region. Moreover the third term in $G_1$ is small compared to the other two terms. This inspires us to combine the first three terms on the right hand side of the above formula to complete a square
\[
- \int (W_T)_u + \mu_u W_T)^2 e^\mu du = - \int \left( e^\mu W_T \right)_u^2 e^{-\mu} du
\]

plus the remaining terms
\[
\int \left( \mu_u - \frac{\Psi^2 - 1}{u} - \frac{u}{2\Psi^2} \right) W_T (W_T)_u e^\mu du + \int \left( \mu_u - \frac{\Psi^2 - 1}{u} \right) \mu_u W^2_T e^\mu du.
\]

Setting
\[
G_0 := \mu_u - \frac{\Psi^2 - 1}{u} = \mu_u - \frac{1 - Y}{uY} \quad \text{(recall that } Y = \Psi^2)\]
we finally obtain the following energy inequality which holds on the whole tip region:

\[
\frac{1}{2} \frac{d}{dt} \int \frac{W_T^2}{\Psi^2} e^\mu du = - \int \left( (\mu W_T)_u^2 e^{-\mu} du + \int \left( G_0 - \frac{u}{2\Psi^2} \right) W_T(W_T)_u e^\mu du \right.
\]

\[
- \int \frac{2}{u^2} W_T^2 e^\mu du + \int \left( G_0 \mu_u + G_2 \right) W_T^2 e^\mu du
\]

\[
- \int \left( \left( -G_1 + 2 \frac{\Psi^{-2} - 1}{u} \right) \varphi \varphi_u + \varphi_u^2 \right) W^2 e^\mu du.
\]

To absorb the cross term with \( W_T(W_T)_u \), we set \( G_3 := G_0 - \frac{u}{2\Psi^2} \) and write

\[
\int G_3 W_T(W_T)_u e^\mu du = \int G_3 W_T(e^\mu W_T)_u du - \int G_3 \mu_u W_T^2 e^\mu du
\]

\[
\leq \frac{1}{2} \int (e^\mu W_T)_u^2 e^{-\mu} du + \int \left( \frac{1}{2} G_3^2 - G_3 \mu_u \right) W_T^2 e^\mu du.
\]

Combining the last two estimates, we finally obtain that

\[
\frac{1}{2} \frac{d}{dt} \int \frac{W_T^2}{\Psi^2} e^\mu du \leq \int \left( \frac{1}{2} (G_0 - \frac{u}{2\Psi^2})^2 + \frac{u}{2\Psi^2} \mu_u + G_2 \right) W_T^2 e^\mu du
\]

\[
- \int \left( \left( -G_1 + 2 \frac{\Psi^{-2} - 1}{u} \right) \varphi \varphi_u + \varphi_u^2 \right) W^2 e^\mu du.
\]

Let us now combine our energy estimate (4.33) with the Poincaré inequality (4.24) applied to \( f := e^\mu W_T \). The latter can be written as

\[
\int (e^\mu W_T)_u^2 e^{-\mu} du \geq C_0^{-1} \mu_u W_T^2 e^\mu du - \frac{1}{u^2} \int W_T^2 e^\mu du.
\]

Combining (4.33) with (4.34) yields the differential inequality

\[
\frac{1}{2} \frac{d}{dt} \int \frac{W_T^2}{\Psi^2} e^\mu du \leq -c_0 \int \mu_u^2 W_T^2 e^\mu du - \frac{3}{2} \int \frac{1}{u^2} W_T^2 e^\mu du
\]

\[
+ \int \left( \frac{1}{2} \left( G_0 - \frac{u}{2\Psi^2} \right)^2 + \frac{u}{2\Psi^2} \mu_u + G_2 \right) W_T^2 e^\mu du
\]

\[
- \int \left( \left( -G_1 + 2 \frac{\Psi^{-2} - 1}{u} \right) \varphi \varphi_u + \varphi_u^2 \right) W^2 e^\mu du
\]

where \( c_0 := C_0^{-1/2} \) is an absolute constant.

We will see below that the terms in the first line of the above formula are our main order terms while the terms in the second and third lines are small. The negative term \(-\frac{3}{2} \int \frac{1}{u} W_T^2 e^\mu du\) is a low order term in the collar region and will not help us estimating the error terms, however near the tip \( u = 0 \) it becomes large and will help us deal with errors in the soliton region. These estimates will be done next and we will do them separately in the collar and soliton regions.

**Claim 4.11** (Estimate of error terms in \( K_{\theta,L} \)). Fix \( \eta > 0 \). There exists \( \theta > 0 \) small, \( L \gg 1 \) and \( \tau_0 \ll -1 \) (all three depending on \( \eta \)) and an absolute constant \( C_1 \), such that

\[
\frac{1}{2} \left( G_0 - \frac{u}{2\Psi^2} \right)^2 + \frac{u}{2\Psi^2} \mu_u + G_2 \leq C_1 \eta \mu_u^2
\]
holds on $\mathcal{K}_{\theta,L}$ for $\tau \leq \tau_0$. Furthermore, the bound

$$
\text{eqn-Gcollarc0} \quad (4.37) \quad \left| (- \frac{G_1}{G} + 2 \frac{\Psi^2 - 1}{u}) \varphi \varphi_u + \varphi_u^2 \right| \leq C(\theta) \Psi^{-2} \chi_{[\rho,2\rho]}^u
$$

holds on the support of $\varphi_u$ and for $\tau \leq \tau_0 \ll -1$.

**Proof of Claim 4.11.** First, lets use the bounds in (4.16), (4.30), (4.18) and the lower bound $\mu_u^2 \geq |\tau|/(2Y)$ (which follows from (4.9) and (4.10)) to obtain the bound

$$
G_2 \leq C_0 \frac{\eta}{u^2 \Psi^4} \leq 2C_0 \eta \mu_u^2
$$

for an absolute constant $C_0$. Moreover, the bound $|G_0| \leq \eta \mu_u$ (which readily follows from (4.9)) implies that

$$
\frac{1}{2} (G_0 - \frac{u}{2\Psi^2})^2 + \frac{u}{2\Psi^2} \mu_u \leq \eta \mu_u^2
$$

for $u \leq 2\theta$, if $\theta$ is sufficiently small depending on $\eta$ and $\eta^2 \ll \eta$. Combining the two estimates yields (4.36) for a different absolute constant $C_1$.

The second estimate (4.37) easily follows from the definition of $G_1$ and (4.9).

We will now fix $\theta > 0$ small and $L \gg 1$ so that

$$
\text{eqn-collarc0} \quad (4.38) \quad \left| \frac{1}{2} (G_0 - \frac{u}{2\Psi^2})^2 + \frac{u}{2\Psi^2} \mu_u + G_2 \right| \leq C_1 \eta \mu_u^2 \leq \frac{c_0}{4} \mu_u^2
$$

holds on $\mathcal{K}_{L,\theta}$ for $\tau \leq \tau_0 \ll -1$, where $c_0$ is the absolute constant from our Poincaré inequality (4.34). For this choice of $L$, we will consider the soliton region $S_L$ and we will use the $C^\infty$ convergence of $Z(\rho, \tau), Z_2(\rho, \tau)$ to the Bryant $Z_0(\rho)$ (see Proposition 4.1) to absorb the error terms in (4.33) by the good negative terms.

We will next show the analogous estimate on $S_L$, where we notice that $G_0 = 0$ (since $\zeta \equiv 0$ there). Hence we claim the following:

**Claim 4.12** (Estimate of error terms in $S_L$). Let $c_0$ be the constant from (4.34) and assume that $c_0 < 1$. For the given choice of $L$ so that (4.38) holds on $\mathcal{K}_{L,\theta}$, there exists $\tau_0 \ll -1$ such that

$$
\text{eqn-Gsoliton} \quad (4.39) \quad \frac{u^2}{8\Psi^4} + \frac{u}{2\Psi^2} \mu_u + G_2 \leq \frac{c_0}{4} \frac{1}{u^2 \Psi^4}
$$

holds on $S_L$, for all $\tau \leq \tau_0$.

**Proof of Claim.** First, observe that $\frac{u^2}{8\Psi^4} + \frac{u}{2\Psi^2} \mu_u \ll \frac{c_0}{8} \frac{1}{u^2 \Psi^4}$ on $S_L$ hence, it is sufficient to show that $G_2 \leq \frac{c_0}{8} \frac{1}{u^2 \Psi^4}$ for $\tau \leq \tau_0 \ll -1$, where $G_2$ is defined in (4.32). Transforming to soliton region variables, this is equivalent to showing that

$$
|B| + \frac{1}{2} \mu_\tau + \frac{1}{2} |Z_\tau - \frac{\rho}{2|\tau|} Z_\rho| \leq \frac{c_0}{8} \frac{|\tau|}{\rho^2 Z}
$$

and it is sufficient to show that each of these three terms is bounded by $\frac{\eta |\tau|}{\rho^2 Z}$ for $\tau \leq \tau_0 \ll -1$, with $\eta$ sufficiently small. The desired bound for the second term follows by (4.18) and for the third term by (4.23) by taking $\tau \leq \tau_0 \ll -1$.
first term, using the convergence $Z(\rho, \tau) \rightarrow Z_0$, $Z_2(\rho, \tau) \rightarrow Z_0$ and (4.23), we find that for any given $\eta > 0$ we can find $\tau_0 \ll -1$, such that

$$|B| \leq \frac{\eta|\tau|}{\rho^2 \rho} \leq \frac{\eta|\tau|}{\rho^2 Z} \quad \text{(since } Z \leq 1)$$

holds on $S_L$ and for $\tau \leq \tau_0 \ll -1$. Combining these three bounds for $\eta = c_0/40$, we finally conclude (4.39). \qed

We will combine (4.35) with (4.36), (4.37) and (4.39). In fact, using all our bounds, it is easy to see that (4.35) implies the following differential inequality

$$\frac{d}{d\tau} \int \frac{W_T}{\Psi^2} e^\mu du \leq - \int (c_0 \mu^2 + \frac{2}{u^2}) W^2 e^\mu du + C(\theta) \int \frac{W^2 \chi[\theta, 2\theta]}{\Psi^2} e^\mu du.$$ 

We may assume that $c_0 < 1$. Then, using (4.9) once more

$$c_0 \mu^2 + \frac{2}{u^2} \geq c_0 \frac{(1 - \Psi^2)^2}{2u^2 \Psi^4} + \frac{2}{u^2} \geq c_0 \frac{1}{100 u^2 \Psi^4},$$

By (4.9) and the soliton asymptotics we have that there exists a constant $\lambda > 0$ (depending on $c_0$) such that on the whole tip region

$$c_0 \frac{1}{100 u^2 \Psi^2} \geq c_0 \frac{1}{100 u^2 Y} \geq 2\lambda |\tau|.$$ 

Hence, we finally conclude the desired integral differential inequality (4.28). \qed

4.4. Proof of Proposition 4.2. The proof of Proposition 4.2 easily follows from the integral differential inequality (4.28). Setting

$$f(\tau) := \int W_T^2 \Psi^{-2} e^\mu du, \quad g(\tau) := \int W^2 \Psi^{-2} \chi[\theta, 2\theta] e^\mu du$$

we may express (4.28) as

$$\frac{d}{d\tau} f(\tau) \leq -2\lambda |\tau| f(\tau) + C(\theta) g(\tau).$$

Furthermore, setting $F(\tau) := \int_{\tau-1}^\tau f(s) ds$ and $G(\tau) := \int_{\tau-1}^\tau g(s) ds$, we have

$$\frac{d}{d\tau} F(\tau) = f(\tau) - f(\tau - 1) = \int_{\tau-1}^\tau \frac{d}{ds} f(s) ds \leq -2\lambda \int_{\tau-1}^\tau |s| f(s) ds + C(\theta) \int_{\tau-1}^\tau g(s) ds

implying

$$\frac{d}{d\tau} F \leq -2\lambda |\tau| F + C(\theta) G.$$ 

This is equivalent to

$$\frac{d}{d\tau} (e^{-\lambda \tau^2} F(\tau)) \leq \frac{C(\theta)}{|\tau|} e^{-\lambda \tau^2} G(\tau).$$

Furthermore, since $W^2 := (\Psi - \Psi^2)^2 \leq 1$ and $\Psi^{-2} e^\mu du \leq 1$ on $S_L$, for $\tau \leq \tau_0 \ll -1$ and $\theta \ll 1$ (which follows from (4.12)), the functions $F(\tau)$ and $G(\tau)$ are uniformly
bounded functions for \( \tau \leq \tau_0 \). Hence, \( \lim_{\tau \to -\infty} e^{-\lambda|\tau|^2} F(\tau) = 0 \), so that from the last differential inequality we get

\[
e^{-\lambda|\tau|^2} F(\tau) \leq C \int_{-\infty}^{\tau} \frac{G(s)}{|s|} \left( |s| e^{-\lambda s^2} \right) ds
\]

\[
\leq \frac{C}{|\tau|} \sup_{s \leq \tau} \frac{G(s)}{\sqrt{|s|}} \int_{-\infty}^{\tau} |s| e^{-\lambda s^2} ds
\]

where \( C = C(\theta) \). It follows that for all \( \tau \leq \tau_0 \ll -1 \) we have

\[
|\tau|^{-1/2} F(\tau) \leq \frac{C}{|\tau_0|} \sup_{\tau \leq \tau_0} |\tau|^{-1/2} G(\tau)
\]

and hence,

\[
\sup_{\tau \leq \tau_0} |\tau|^{-1/2} F(\tau) \leq \frac{C}{|\tau_0|} \sup_{\tau \leq \tau_0} |\tau|^{-1/2} G(\tau)
\]

or equivalently,

\[
\|W_T\|_{2,\infty} \leq \frac{C(\theta)}{\sqrt{|\tau_0|}} \|W\chi_{[\theta,2\theta]}\|_{2,\infty}
\]

therefore concluding the proof of Proposition 4.2.

5. PROOF OF THEOREM 1.5

We will now combine Propositions 3.2 and 4.2 to conclude the proof of our main result Theorem 1.5. Throughout this section we will fix the constant \( \theta > 0 \) given in Proposition 4.2. For this constant \( \theta \), let us recall our notation of the various regions

\[
C_\theta = \{ u_1 \geq \theta/4 \}, \quad D_\theta = \{ \theta/4 \leq u_1 \leq \theta/2 \}, \quad T_\theta = \{ u_1 \leq 2\theta \}
\]

and that \( \varphi_C \), \( \varphi_T \) are supported on \( C_\theta \), \( T_\theta \) respectively and \( \varphi_C \equiv 1 \) on \( C_{2\theta} \) and \( \varphi_T \equiv 1 \) on \( T_{\theta/2} \).

We have seen at the beginning of Section 2 that translating and dilating the original solution has an effect on the rescaled rotationally symmetric solution \( u(\sigma, \tau) \), as given in formula (2.14). Let \( u_1(\sigma, \tau) \) and \( u_2(\sigma, \tau) \) be any two solutions to equation (2.11) as in the statement of Theorem 1.5 and let \( u_{\beta\gamma} \) be defined by (2.14). Our goal is to find parameters \( (\beta, \gamma) \) so that the difference

\[
w_{\beta\gamma} := u_1 - u_2 \equiv 0.
\]

Proposition 4.2 says that the weighted \( L^2 \)-norm \( \|W_{\beta\gamma}\|_{2,\infty} \) of the difference of our solutions \( W_{\beta\gamma}(u, \tau) := \Psi_1(u, \tau) - \Psi_2(u, \tau) \) (after we change the variables) in the whole tip region \( T_\theta := \{ (\sigma, \tau) : u_1(\sigma, \tau) \leq 2\theta \} \) is controlled by \( \|W_{\beta\gamma} \chi_{[\theta,2\theta]}\|_{2,\infty} \), where \( \chi_{[\theta,2\theta]}(u) \) is supported in the transition region between the cylindrical and tip regions. By (5.7) which will be shown below, \( \|W_{\beta\gamma} \chi_{[\theta,2\theta]}\|_{2,\infty} \) can be estimated in terms of \( \|u_{\beta\gamma}\|_{L^2} \|\delta\|_{2,\infty} \), where \( D_\theta = \{ (\sigma, \tau) : \theta/4 \leq u_1(\sigma, \tau) \leq \theta/2 \} \). Therefore combining Propositions 3.2 and 4.2 gives the crucial estimate (5.8) which will be shown in detail in Proposition 5.5 below. This estimate says that the norm of the difference \( u_{\beta\gamma} \) of our solutions when restricted in the cylindrical region is dominated by the norm of its projection onto the zero eigenspace of the operator \( \mathcal{L} \) (the
linearization of our equation on the limiting cylinder). Note that Proposition 3.2 holds under the assumption that the projection of \( \omega_C^{\beta\gamma} \) onto the positive eigenspace of \( \mathcal{L} \) is zero, that is \( \mathcal{P}_+ \omega_C^{\beta\gamma}(\tau_0) = 0 \).

After having established that the projection onto the zero eigenspace \( a(\tau) := \langle \omega_C^{\beta\gamma}, \psi_2 \rangle \) dominates in the \( \| \omega_C^{\beta\gamma} \|_{\beta, \infty} \), the conclusion of Theorem 1.5 will follow by establishing an appropriate differential inequality for \( a(\tau) \), for \( \tau \leq \tau_0 \ll -1 \) and also having that \( a(\tau_0) = \mathcal{P}_0 \omega_C^{\beta\gamma}(\tau_0) = 0 \). It follows from this discussion that it is essential to have

\[
\mathcal{P}_+ \omega_C^{\beta\gamma}(\tau_0) = \mathcal{P}_0 \omega_C^{\beta\gamma}(\tau_0) = 0
\]

(5.1)

holding for the same \( \tau_0 \) when \( \tau_0 \ll -1 \). This will be done by appropriately choosing the parameters \( \beta \) and \( \gamma \). In fact, we will next show that for every \( \tau_0 \ll -1 \) we can find parameters \( \beta = \beta(\tau_0) \) and \( \gamma = \gamma(\tau_0) \) such that (5.1) holds and we will also give their asymptotics relative to \( \tau_0 \). Let us emphasize that we need to be able for every \( \tau_0 \ll -1 \) to find parameters \( \beta, \gamma \) so that (5.1) holds, since up to the final step of our proof we have to keep adjusting \( \tau_0 \) by taking it even more negative so that our estimates hold (see Remark 5.2 below).

We will need the following result whose proof is identical to the analogous result in [2].

**Proposition 5.1.** There is a number \( \tau_* \ll -1 \) such that for all \( \tau \leq \tau_* \) there exist \( b \) and \( \Gamma \) such that the difference \( w^{\beta\gamma} := u_1 - u_2^{\beta\gamma} \) satisfies

\[
\langle \varphi_C \omega_C^{\beta\gamma}, \psi_0 \rangle = 0 \quad \text{and} \quad \langle \varphi_C \omega_C^{\beta\gamma}, \psi_2 \rangle = 0
\]

where \( \psi_0(\sigma) \equiv 1 \) and \( \psi_2(\sigma) = \sigma^2 - 2 \) are the positive and null eigenvectors of the operator \( \mathcal{L} \). In addition, the parameters \( \beta \) and \( \gamma \) can be chosen so that

\[
b := \sqrt{1 + \beta e^\tau} - 1 \quad \text{and} \quad \Gamma := \frac{\gamma - \log(1 + \beta e^\tau)}{\tau}
\]

satisfy

\[
|b| = o(|\tau|^{-1}) \quad \text{and} \quad |\Gamma| = o(1), \quad \text{as} \ \tau \to -\infty.
\]

Equivalently, this means that \((\beta, \gamma)\) is admissible with respect to \( \tau \), according to our Definition 2.2.

**Proof.** The proof relies only on the asymptotics of our solution in the cylindrical region (see Theorem 1.6 and its proof in [3]). Since the asymptotics of our Ricci flow rotationally symmetric solution in the cylindrical region are very similar to the cylindrical region asymptotics of ancient mean curvature flow solution with rotational symmetry (see [1]) and they differ just by a constant, the proof of this Proposition is identical to the proof of corresponding Proposition 7.1 in [2].

**Remark 5.2** (The choice of parameters \((\beta, \gamma)\)). We can choose \( \tau_0 \ll -1 \) to be any small number so that \( \tau_0 \leq \tau_* \), where \( \tau_* \) is as in Proposition 5.1 and so that all our uniform estimates in previous sections hold for \( \tau \leq \tau_0 \). Note also that having Proposition 5.1 we can decrease \( \tau_0 \) if necessary and choose parameters \( \beta \) and \( \gamma \) again so that we still have \( \mathcal{P}_+ \omega_C(\tau_0) = \mathcal{P}_0 \omega_C(\tau_0) = 0 \), without effecting our estimates. Hence, from now on we will be assuming that we have fixed parameters \( \beta \) and \( \gamma \) at some time \( \tau_0 \ll -1 \), to have both projections zero at time \( \tau_0 \). As a consequence of Proposition 5.1 which shows that the parameters \((\beta, \gamma)\) are admissible with respect
to \( \tau_0 \) and Remark 3.1, all the estimates for \( w = u_1 - u_2^{\beta \gamma} \) will then hold for all \( \tau \leq \tau_0 \), independently of our choice of \((\beta, \gamma)\).

As we pointed out above, we need to show next that the norms of the difference of our two solutions with respect to the weights defined in the cylindrical and the tip regions satisfy comparison inequalities in the intersection between the regions, so called transition region. We need those inequalities to conclude the proof of Theorem 1.5. We have the following.

**Lemma 5.3.** Given \( 0 < \theta \ll 1 \), there exists \( C(\theta) > 0 \) such that

\[
|\tau| \int (w \chi_{D_n})^2 e^{-\sigma^2/4} \, d\sigma \leq C(\theta) \left( \int \chi_{\tau/2} w_\sigma^2 e^{-\sigma^2/4} \, d\sigma + o(1) \int w_\sigma^2 e^{-\sigma^2/4} \, d\sigma \right).
\]

*Proof.* Fix a \( \tau \leq \tau_0 \ll -1 \) and consider a smooth function \( \eta(\sigma, \tau) \) defined on \( \sigma \geq 0 \), satisfying \( 0 \leq \eta \leq 1 \), and

\[
\eta \equiv 1 \text{ on } \{ \sigma : u_1(\sigma, \tau) \leq \theta/2 \} \quad \text{and} \quad \eta \equiv 0 \text{ on } \{ \sigma : u_1(\sigma, \tau) \geq \theta \}.
\]

Denote by \( \ell_1 := \ell_1(\theta, \tau) > 0 \), \( \ell_2 := \ell_2(\theta, \tau) > 0 \) the points at which \( u_1(\ell_1, \tau) = \theta \) and \( u_1(\ell_2, \tau) = \theta/4 \). Clearly, \( \ell_2 > \ell_1 \) by the monotonicity of \( u_1(\cdot, \tau) \) on the set \( u_1(\sigma, \tau) \leq \theta \), for \( \tau \ll -1 \).

Next set, \( v := \eta w \) and use the inequality

\[
\int_0^{\ell_2} v_\sigma^2 e^{-\sigma^2/4} \, d\sigma + \frac{1}{4} \int_0^{\ell_1} v_\sigma^2 e^{-\sigma^2/4} \, d\sigma
\geq \frac{1}{4} \ell_2 e^{-\ell_2^2/4} v(\ell_2)^2 + \frac{1}{16} \int_0^{\ell_2} \sigma^2 v_\sigma^2 e^{-\sigma^2/4} \, d\sigma
\]

where \( \ell_2 = \ell_2(\theta, \tau) \) is the number defined above. This standard weighted Poincaré type inequality was shown in Lemma 4.12 in [1]. Since \( v \equiv 0 \) on \( 0 \leq \sigma \leq \ell_1 \) (corresponding to \( u_1(\sigma, \tau) \geq \theta \)) we obtain (after dropping the positive boundary term) the bound

\[
\frac{1}{16} \int_{\ell_1}^{\ell_2} \sigma^2 v_\sigma^2 e^{-\sigma^2/4} \, d\sigma \leq \int_{\ell_1}^{\ell_2} v_\sigma^2 e^{-\sigma^2/4} \, d\sigma + \frac{1}{4} \int_{\ell_1}^{\ell_2} v_\sigma^2 e^{-\sigma^2/4} \, d\sigma.
\]

Using that \( \ell_1 \gg 1000 \) for \( \tau \ll -1 \), we get that

\[
\frac{1}{32} \int_{\ell_1}^{\ell_2} \sigma^2 v_\sigma^2 e^{-\sigma^2/4} \, d\sigma \leq \int_{\ell_1}^{\ell_2} v_\sigma^2 e^{-\sigma^2/4} \, d\sigma.
\]

Since, \( v := \eta w \) and \( v_\sigma^2 \leq 2 \left( \eta^2 w_\sigma^2 + \eta_\sigma^2 w^2 \right) \), we conclude the bound

\[
\int_{\ell_1}^{\ell_2} \sigma^2 \eta^2 w_\sigma^2 e^{-\sigma^2/4} \, d\sigma \leq 64 \int_{\ell_1}^{\ell_2} \eta^2 w_\sigma^2 e^{-\sigma^2/4} \, d\sigma + 64 \int_{\ell_1}^{\ell_2} \eta^2 w^2 e^{-\sigma^2/4} \, d\sigma.
\]

However, by definition \( \eta_\sigma \neq 0 \) only on the set \( C_{2\theta} := \{ u_1 \geq \theta/2 \} \) which is contained in the set where \( \varphi_C \equiv 1 \), i.e. on the set where \( w = w_C \). In addition, \( \eta_\sigma \ll o(1) \), as \( \tau \to -\infty \). Using also that \( \eta \equiv 1 \) on \( D_\theta := \{ \sigma : \theta/4 \leq u_1(\sigma, \tau) \leq \theta/2 \} \) and that \( \eta \) is supported on \( T_{\theta/2} := \{ \sigma : u_1(\sigma, \tau) \leq \theta \} \), we finally conclude the bound

\[
\int \sigma^2 (w \chi_{D_0})^2 e^{-\sigma^2/4} \, d\sigma \leq 64 \int \chi_{\tau/2} w_\sigma^2 e^{-\sigma^2/4} \, d\sigma + o(1) \int w_\sigma^2 e^{-\sigma^2/4} \, d\sigma.
\]

Furthermore we have shown that on the region \( D_\theta \) we have \( \sigma^2 \geq c_9 |\tau| \). Combining the above readily implies (5.4). \( \square \)
We recall the definitions of the norms $\|w\|_{D,\infty}$ and $\|W\|_{2,\infty}$ given in (2.19) and (2.32)-(2.34) respectively. As a corollary of the previous Lemma we have the following relations between our norms.

**Corollary 5.4.** Given $0 < \theta \ll 1$, there exists $C(\theta)$ and $\tau_0 \ll -1$ such that for all $\tau \leq \tau_0$ we have,

**eqn-help1**

\[
\|w \chi_{D_{\theta}}\|_{\delta,\infty} \leq \frac{C(\theta)}{\sqrt{|\tau_0|}} \left( \|W_T\|_{2,\infty} + o(1) \|w_C\|_{\delta,\infty} \right)
\]

and also

**eqn-help2**

\[
\|W \chi_{(\theta,2\theta]}\|_{2,\infty} \leq C(\theta) \|w_{\sigma} \chi_{D_{\theta}}\|_{\delta,\infty}.
\]

**Proof.** To prove (5.6) we recall that $W_T^2 = w_\sigma^2$ and $\Psi := |u_{1\sigma}| \leq C(\theta) |\tau|^{-1/2}$ on $T_{\theta/2}$ (since this bound holds in $D_{2\theta}$ and $u_1$ are concave). Hence, also using the change of variables $du = u_{1\sigma} d\sigma$, we deduce from (5.4) that for any $\tau \leq \tau_0 \ll -1$ we have

\[
\int (w \chi_{D_{\theta}})^2 e^{-\sigma^2/4} d\sigma \leq \frac{C(\theta)}{|\tau|} \left( \int W_T^2 \Psi^{-1} e^\mu du + o(1) \int w_{\sigma}^2 e^{-\sigma^2/4} d\sigma \right)
\]

\[
\leq \frac{C(\theta)}{|\tau|} \left( |\tau|^{-1/2} \int W_T^2 \Psi^{-2} e^\mu du + o(1) \int w_{\sigma}^2 e^{-\sigma^2/4} d\sigma \right)
\]

and (5.6) readily follows using the definitions of our norms.

On the other hand, using $du = u_{1\sigma} d\sigma$ and the bound $\Psi := |u_{1\sigma}| \geq C(\theta) |\tau|^{-1/2}$ on $D_{\theta} := \{ \sigma: \theta \leq u_{1\sigma}(\sigma, \tau) \leq 2\theta \}$ we also have

\[
\int (W \chi_{(\theta,2\theta]})^2 \Psi^{-2} e^\mu du = \int (w_{\sigma} \chi_{D_{\theta}})^2 \Psi^{-1} e^{-\sigma^2/2} d\sigma
\]

\[
\leq C(\theta) |\tau|^{1/2} \int (w_{\sigma} \chi_{D_{\theta}})^2 e^{-\sigma^2/2} d\sigma
\]

which combined with he definition of our norms proves (5.7). \qed

We next combine our main results in the previous two sections, Propositions 3.2 and 4.2, with Corollary 5.4 to establish our crucial estimate which says that what actually dominates in the norm $\|w_C\|_{D,\infty}$ is $\|P_0 w_C\|_{D,\infty}$.

**Proposition 5.5.** For any $\epsilon > 0$ there exists a $\tau_0 \ll -1$ so that we have

**eqn-w1230**

\[
\|\tilde{w}_C\|_{D,\infty} \leq \epsilon \|P_0 w_C\|_{D,\infty}.
\]

**Proof.** By Proposition 5.1 we know that for every $\tau_0 \ll -1$ sufficiently close to negative infinity, we can choose parameters $(\beta, \gamma)$ which are admissible with respect to $\tau_0$ and such that $P_+ w_C(\tau_0) = P_0 w_C(\tau_0) = 0$. From now on we will always consider $w(\sigma, \tau) = u_1(\sigma, \tau) - u_{\sigma}^{\beta,\gamma}(\sigma, \tau)$, for these chosen parameters $\beta$ and $\gamma$.

Let $0 < \theta \ll 1$ be a sufficiently small number so that Proposition 4.2 holds, namely there exists $\tau_0 \ll -1$ so that

\[
\|W_T\|_{2,\infty} \leq \frac{C(\theta)}{\sqrt{|\tau_0|}} \|W \chi_{(\theta,2\theta]}\|_{2,\infty}.
\]

In addition, we may choose $\tau_0 \ll -1$ so that Corollary 5.4 holds, that is

\[
\|W \chi_{(\theta,2\theta]}\|_{2,\infty} \leq C(\theta) \|w_{\sigma} \chi_{D_{\theta}}\|_{\delta,\infty}.
\]
implies for

\[ 4.2 \]

\[ 3.8 \]

\[ 3.7 \]

\[ 3.2 \]

\[ 3.6 \]

\[ 5.6 \]

\[ 5.10 \]

\[ 1.5 \]

\[ 5.10 \]

\[ 36 \]

\[ 36 \]

\[ 36 \]

\[ 36 \]

\[ 36 \]

\[ 36 \]

\[ 36 \]

\[ 36 \]

\[ 36 \]

Since \( D_{d\theta} := \{ \sigma : \theta \leq u_1(\sigma, \tau) \leq 2\theta \} \) is contained in the set where \( \varphi_C = 1 \), we have

\[ \| w_\sigma \chi_{D_{d\theta}} \|_{\dot{B}_\infty^0} \leq \| w_C \|_{\dot{B}_\infty^0}. \]

Hence, the last two estimates yield

\[ \| W_T \|_{2, \infty} \leq \frac{C(\theta)}{\sqrt{|\tau_0|}} \| w_C \|_{\dot{B}_\infty^0}. \]  

(5.9)

Now given any \( \epsilon > 0 \) and the \( \theta > 0 \) as above, Proposition 3.2 implies for \( \tau_0 \) sufficiently negative we have

\[ \| \dot{w}_C \|_{\dot{B}_\infty^0} \leq \frac{\epsilon}{3} \left( \| w_C \|_{\dot{B}_\infty^0} + \| w_\chi \chi_{D_{d\theta}} \|_{\dot{B}_\infty^0} \right) \]

for all \( \tau \leq \tau_0 \), where as before \( D_{d\theta} = \{ \sigma : \theta/4 \leq u_1(\sigma, \tau) \leq \theta/2 \} \). Using (5.6) to estimate \( \| w_\chi \chi_{D_{d\theta}} \|_{\dot{B}_\infty^0} \), we get the bound

\[ \| \dot{w}_C \|_{\dot{B}_\infty^0} \leq \frac{\epsilon}{2} \left( \| w_C \|_{\dot{B}_\infty^0} + C(\theta) \| W_T \|_{2, \infty} \right). \]  

(5.10)

Combining (5.9) with (5.10) yields

\[ \| \dot{w}_C \|_{\dot{B}_\infty^0} \leq \frac{\epsilon}{2} \left( \| w_C \|_{\dot{B}_\infty^0} + \frac{C(\theta)}{\sqrt{|\tau_0|}} \| w_C \|_{\dot{B}_\infty^0} \right) \leq \epsilon \| w_C \|_{\dot{B}_\infty^0} \]

by choosing \( |\tau_0| \) sufficiently large relative to \( C(\theta) \). The last estimate readily yields (5.8) finishing the proof of the Proposition.

\[ \square \]

**Proof of the Main Theorem 1.5.** Fix the constant \( \theta > 0 \) given in Proposition 4.2 and for this constant consider the cylindrical and tip regions as defined before. Recall that \( w^{\beta_2}(\sigma, \tau) = u_1(\sigma, \tau) - u_2^{\beta_2}(\sigma, \tau) \), which we shortly denote by \( w(\sigma, \tau) = u_1(\sigma, \tau) - u_2(\sigma, \tau) \), where \( u_2^{\beta_2}(\sigma, \tau) \) is given by (2.14).

Following the notation from previous sections we have

\[ \frac{\partial}{\partial \tau} w_C = \mathcal{L}[w_C] + \mathcal{E}[w_C] + \mathcal{E}[w, \varphi_C] + \mathcal{E}_{nl}, \]

with \( w_C = \dot{w}_C + a(\tau) \psi_2 \), where \( a(\tau) = \langle w_C, \psi_2 \rangle \). The error terms \( \mathcal{E}[w_C], \mathcal{E}[w, \varphi_C] \) and \( \mathcal{E}_{nl} \) are given by formulas (3.6), (3.7) and (3.8), respectively. Projecting the above equation on the null eigenspace generated by \( \psi_2 \) while using that \( \langle \mathcal{L}[w_C], \psi_2 \rangle = 0 \) we obtain

\[ \frac{d}{d\tau} a(\tau) = \langle \mathcal{E}[w_C] + \mathcal{E}[w, \varphi_C] + \mathcal{E}_{nl}, \psi_2 \rangle. \]

Since \( \frac{\langle \psi_2, \psi_2 \rangle}{\| \psi_2 \|^2} = 8 \) we can write the above equation as

\[ \frac{d}{d\tau} a(\tau) = 2 \frac{a(\tau)}{|\tau|} + F(\tau) \]

where

\[ F(\tau) := \frac{\langle \mathcal{E}(w_C) - \frac{a(\tau)}{8|\tau|} \psi_2^2 + \mathcal{E}[w, \varphi_C] + (\mathcal{E}_{nl} - \frac{a(\tau)}{8|\tau|} \psi_2^2), \psi_2 \rangle}{\| \psi_2 \|^2} \]  

(5.11)

Solving the above ordinary differential equation for \( a(\tau) \) in terms of \( F(\tau) \) yields

\[ a(\tau) = C_0 \tau^{-2} - \tau^{-2} \int_\tau^{\tau_0} F(s) s^2 \, ds. \]
By Remark 5.2 we may assume $\alpha(\tau_0) = 0$. This means that $C_0 = 0$ which implies
\begin{equation}
|a(\tau)| = |\tau|^{-2} \int_{\tau}^{\tau_0} F(s) s^2 \, ds.
\end{equation}
Define
\begin{equation*}
\|a\|_{\mathcal{H}, \infty}(\tau) = \sup_{\tau' \leq \tau} \left( \int_{\tau'}^{\tau'} |a(\zeta)|^2 \, d\zeta \right)^{\frac{1}{2}}
\end{equation*}
and
\begin{equation*}
\|a\|_{\mathcal{H}, \infty} := \|a\|_{\mathcal{H}, \infty}(\tau_0).
\end{equation*}
Since $P_{wC}(. \tau) = a(\tau) \psi_2(.)$, we have
\begin{equation*}
\|P_{wC}\|_{\mathcal{H}, \infty}(\tau) = \|a\|_{\mathcal{H}, \infty}(\tau) \|\psi_2\|_{\mathcal{H}}.
\end{equation*}
Next we need the following claim.

**Claim 5.6.** For every $\epsilon > 0$ there exists a $\tau_0$ depending only on $\epsilon$, $\theta$ and dimension $n$ so that
\begin{equation*}
\left| \int_{\tau-1}^{\tau} F(s) \, ds \right| \leq \frac{\epsilon}{|\tau|} \|a\|_{\mathcal{H}, \infty}
\end{equation*}
for $\tau \leq \tau_0$.

Assume for the moment that the Claim holds. We will finish the proof of the theorem. Set $\epsilon := 1/2$ and choose a sufficiently negative number $\tau_0 < \min(\tau^*, -100)$ (where $\tau^*$ is as in Proposition 5.1) and such that Claim 5.6 holds. Such a number $\tau_0$ depends only on the constant $\theta$ and on dimension $n$. Proposition 5.1 tells us that for the chosen $\tau_0$ we can choose parameters $\beta$ and $\gamma$ and such that $P_{wC}(\tau_0) = P_0wC(\tau_0) = 0$. We will next see that for that choice of parameters $w(\sigma, \tau) \equiv 0$. To this end, observe that for $\tau \leq \tau_0$ we have
\begin{align*}
\left| \int_{\tau}^{\tau_0} F(s) s^2 \, ds \right| &\leq \sum_{j=|\tau|-1}^{|\tau|} \int_{j}^{j+1} s^2 F(s) \, ds \\
&\leq \epsilon \|a\|_{\mathcal{H}, \infty} \sum_{j=|\tau|-1}^{\tau_0} |j| \\
&\leq \frac{1}{2} |\tau|^2 \|a\|_{\mathcal{H}, \infty}.
\end{align*}
Combining the last inequality with (5.12) while choosing $\epsilon = 1/2$ yields
\begin{equation*}
|a(\tau)| \leq \frac{1}{2} \|a\|_{\mathcal{H}, \infty},
\end{equation*}
for all $\tau \leq \tau_0$.

This implies
\begin{equation*}
\|a\|_{\mathcal{H}, \infty} \leq \frac{1}{2} \|a\|_{2, \infty}
\end{equation*}
and hence $\|a\|_{\mathcal{H}, \infty} = 0$, which further gives
\begin{equation*}
\|P_{wC}\|_{\mathcal{H}, \infty} = 0.
\end{equation*}
Finally, (5.8) implies $\hat{w}_C \equiv 0$ and hence, $w_C \equiv 0$ for $\tau \leq \tau_0$. By (5.7) and the fact that $\varphi_C \equiv 1$ on $D_{\theta}$ we have $W_C(\varphi_C, 2\theta) \equiv 0$ for $\tau \leq \tau_0$. Proposition 4.2 then yields that $W_T \equiv 0$ for $\tau \leq \tau_0$. All these imply
\begin{equation*}
u_1(y, \tau) = \nu_2^{\gamma}(y, \tau),
\end{equation*}
for $\tau \leq \tau_0$. By forward uniqueness of solutions to the mean curvature flow (or equivalently to cylindrical equation (2.11)), we have
\begin{equation*}
u_1 \equiv \nu_2^{\gamma},
\end{equation*}
and hence $g_1 \equiv g_2^{\gamma}$. This concludes the proof of the main Theorem 1.5.
To complete the proof of Theorem 1.5 we still need to prove Claim 5.6, and we do it next.

**Proof of Claim 5.6.** Throughout the proof we will use the estimates

\[ \| \dot{\omega} \|_{D, \infty} \leq \varepsilon \| a \|_{D, \infty} \quad \text{and} \quad \| \omega \|_{D, \infty} \leq C \| a \|_{D, \infty} \]

which hold for all \( \tau_0 \ll -1 \). These estimates follow from Proposition 5.5 and we can achieve the first estimate to hold for any \( \varepsilon > 0 \), for \( \tau \leq \tau_0 \), by choosing \( \tau_0 = \tau_0(\varepsilon) \ll -1 \) sufficiently small. By (5.6) and (5.9), we also have

\[ \| w \chi_{D_0} \|_{D, \infty} \leq \frac{C(\theta)}{\sqrt{\tau_0}} \| w \|_{D, \infty} \]

for \( \tau_0 \ll -1 \). The last bound and (5.14) give us the bound

\[ \| w \|_{D, \infty} \leq \varepsilon \| w \|_{D, \infty} \]

From the definition of \( \hat{E}[w, \varphi_c] \) given in (3.7) and the definition of cut off function \( \varphi_c \), we see that the support of \( \hat{E}[w, \varphi_c] \) is contained in

\[ \left( \sqrt{2 - \theta^2} - \epsilon_1 \right) \sqrt{2|\tau|} \leq |\sigma| \leq \left( \sqrt{2 - \frac{\theta^2}{4}} + \epsilon_1 \right) \sqrt{2|\tau|} \]

where \( \epsilon_1 \) is so tiny that \( \sqrt{2 - \frac{\theta^2}{4}} + \epsilon_1 < \sqrt{2} \). Also by the a priori estimates in Appendix A we have

\[ |u_\sigma| + |u_{\sigma\sigma}| \leq \frac{C(\theta)}{\sqrt{|\tau|}}, \quad \text{for } |\sigma| \leq \left( \sqrt{2 - \frac{\theta^2}{4}} + \epsilon_1 \right) \sqrt{2|\tau|}. \]

Furthermore, in [3] we have showed that our ancient solutions \( u_i, i \in \{1, 2\} \) satisfy

\[ \sup_{\tau \leq \tau} \left\| u_i - \sqrt{2} + \frac{\sqrt{2}}{8|\tau|} \psi_2 \right\| = o(|\tau|^{-1}), \]

\[ \sup_{\tau \leq \tau} \left\| u_i + \frac{\sqrt{2}}{8|\tau|} \psi_2 \right\| \rho \sup_{\tau \leq \tau} \left\| u_{i\sigma} \right\| \sup_{\tau \leq \tau} \left\| u_{i\sigma\sigma} \right\| = O(|\tau|^{-1}). \]

In particular, this implies

\[ \sup_{\tau' \leq \tau} \left\| u_i - \sqrt{2} + \frac{\sqrt{2}}{8|\tau|} \psi_2 \right\| = o(|\tau|^{-1}), \]

Moreover, by standard regularity parabolic estimates, for every \( L > 0 \), there exists a \( \tau_0 \ll -1 \) so that for \( \tau \leq \tau_0 \),

\[ \sup_{|\sigma| \leq L} \left( |u_{i\sigma}| + |u_i - \sqrt{2}| \right) = O(|\tau|^{-1}). \]
We will now apply the estimates above to achieve the desired bound. We need to estimate all projections of error terms on the right hand side of (5.11), and we will treat each of the three terms separately in three different steps. We start with the simplest bound, which is the projection of the error $\bar{E}[w, \varphi_C]$ due to the cut-off function.

**Step 1.** For every $\epsilon > 0$ there exists a $\tau_0 \ll -1$ so that for $\tau \leq \tau_0$ we have

\[
\|\psi_2\|^{-2} \left| \int_{\tau-1}^{\tau} \langle \bar{E}[w, \varphi_C], \psi_2 \rangle \, ds \right| \leq \epsilon \frac{\|a\|_{\mathcal{D}, \infty}}{|\tau|}.
\]  

(5.20)

We have

\[
\int_{\tau-1}^{\tau} |\langle \bar{E}[w, \varphi_C], \psi_2 \rangle| \, d\tau' \leq \int_{\tau-1}^{\tau} \|\bar{E}[w, \varphi_C]\|_{\mathcal{D}'} \|\psi_2\|_{\mathcal{D}} \, d\tau' \\
\leq e^{-|\tau|/4} \|\bar{E}[w, \varphi_C]\|_{\mathcal{D}' \infty},
\]

where $\tilde{\chi}$ denotes a smooth function with a support in $|\sigma| \geq (\sqrt{2-\theta^2/4-2\epsilon_1}) \sqrt{2|\tau|}$, being equal to one for $|\sigma| \geq (\sqrt{2-\theta^2/4} - \epsilon_1) \sqrt{2|\tau|}$. Combining the last estimate with Proposition 3.5, Proposition 5.5 and (5.14), also using (5.13), implies that for every $\epsilon > 0$ we can find a $\tau_0 \ll -1$ so that for $\tau \leq \tau_0$ the desired bound (5.20) holds.

We will next estimate the main error term.

**Step 2.** For every $\epsilon > 0$ there exists a $\tau_0 \ll -1$ so that for $\tau \leq \tau_0$ we have

\[
\|\psi_2\|^{-2} \left| \int_{\tau-1}^{\tau} \langle \mathcal{E}(w_C) - \frac{a(\tau)}{8|\tau|}, \psi_2 \rangle \, d\tau' \right| \leq \epsilon \frac{\|a\|_{\mathcal{D}, \infty}}{|\tau|}.
\]  

(5.21)

We have

\[
\mathcal{E}(w_C) - \frac{a(\tau)}{8|\tau|} \psi_2 = \left( \frac{w_\sigma}{u_1} + \frac{2u_{2\sigma}}{u_1} - J_1 \right) (w_C)_\sigma - \left( \frac{w}{2u_1} + \frac{u_{2\sigma}}{u_1u_2} \right) w_C \\
- \left( \frac{u_{2}^2 - 2}{2u_1u_2} w_C + \frac{a(\tau)}{8|\tau|} \psi^2 \right),
\]

where (since $u_{1\sigma} = 0$) we have

\[
J_1 := 2 \int_0^\sigma \frac{u_{1\sigma}}{u_1} \, d\sigma' = 2 \int_0^\sigma \frac{u_{2\sigma}}{u_1} \, d\sigma' + 2 \frac{u_{1\sigma}}{u_1} (\sigma, \tau).
\]

These imply

\[
\mathcal{E}(w_C) - \frac{a(\tau)}{8|\tau|} \psi_2 = \left( - \frac{w_\sigma}{u_1} + 2 \int_0^\sigma \frac{u_{2\sigma}}{u_1} \, d\sigma' \right) (w_C)_\sigma \\
- \left( \frac{w}{2u_1} + \frac{u_{2\sigma}}{u_1u_2} \right) w_C - \left( \frac{u_{2}^2 - 2}{2u_1u_2} w_C + \frac{a(\tau)}{8|\tau|} \psi^2 \right). \]

\[
\text{eq-E-recall}
\]

(5.22)

We use the same symbol $\epsilon$ to denote possibly different, but uniformly small constants from line to line. Also, we will often use the bound $|\psi_2| \leq 2(\sigma^2 + 1)$ and the fact that the operator $f \to \sigma f$ is bounded from $\mathcal{D}$ to $\mathcal{H}$. We start with the last
term in (5.22). Using the decomposition $w_C = \hat{w}_C + a(\tau) \psi_2$ we write this term as

$$\int \left( \frac{u_2^2 - 2}{2u_1u_2} w_C + \frac{a(\tau)}{8|\tau|} \psi_2 \right) \psi_2 \, d\mu = \int \frac{u_2^2 - 2}{2u_1u_2} \hat{w}_C \psi_2 \, d\mu$$

(5.23)

$$+ a(\tau) \int \left( \frac{u_2 - \sqrt{2}}{2u_1u_2} \right) \left( (u_2 - \sqrt{2})(1 - \sqrt{2}u_1) + 2(\sqrt{2} - u_1) \right) \psi_2^2 \, d\mu$$

$$+ \frac{a(\tau)}{\sqrt{2}} \int \left( u_2 - \sqrt{2} + \frac{\sqrt{2}}{8|\tau|} \right) \psi_2^2 \, d\mu.$$

Since the operator $f \rightarrow \sigma f$ is bounded from $\mathcal{D}$ to $\mathcal{H}$ we have

$$\left| \int \frac{u_2^2 - 2}{2u_1u_2} \hat{w}_C \psi_2 \, d\mu \right| \leq C(\theta) \left( \int (u_2 - \sqrt{2})^2 (\sigma^2 + 1) \, d\mu \right)^{1/2} \left( \int u_2^2 (\sigma^2 + 1) \, d\mu \right)^{1/2}$$

$$\leq C(\theta) (\|u_2 - \sqrt{2}\| + \|u_2\|) \|\hat{w}_C\|_{\mathcal{D}}.$$

Hence, Proposition 5.5 and (5.18) imply that for $\tau \leq \tau_0 \ll -1$ we have

$$\|\psi_2\|^{-2} \int_{\tau - 1}^\tau \int \frac{u_2^2 - 2}{2u_1u_2} \hat{w}_C \psi_2 \, d\mu \, d\tau' \leq \frac{\epsilon}{|\tau|} \|a\|_{B,\infty}.$$

To estimate the second term on the right hand side in (5.23), it is enough to estimate

$$\int_{\tau - 1}^\tau |a(\tau')| \int (u_2 - \sqrt{2})^2 \psi_2^2 \, d\mu \, d\tau'$$

$$\leq \|a\|_{B,\infty} \left( \int_{\tau - 1}^\tau (u_2 - \sqrt{2})^2 \, d\mu \int (u_2 - \sqrt{2})^2 \psi_2^4 \, d\mu \, d\tau' \right)^{1/2}$$

$$\leq C \|a\|_{B,\infty} \sup_{\tau \leq \tau_0} \left( \int (u_2 - \sqrt{2})^2 \psi_2^4 \, d\mu \right)^{1/2}$$

$$\leq C \|a\|_{B,\infty} \sup_{\tau \leq \tau_0} \left( \int_{|\sigma| \leq L} (u_2 - \sqrt{2})^2 \psi_2^4 \, d\mu + \int_{\{|\sigma| \geq L\} \cap C_\theta} (u_2 - \sqrt{2})^2 \psi_2^4 \, d\mu \right)^{1/2}$$

$$\leq \frac{\epsilon}{|\tau|} \|a\|_{B,\infty},$$

where in the second inequality in the previous estimate we have used (5.18) and in the last inequality, for $|\sigma| \leq L$ we have used (5.19) and for $|\sigma| \geq L$, we have found $L$ sufficiently big so that we can make $\int_{|\sigma| \geq L} (u_2 - \sqrt{2})^2 \psi_2^4 \, d\mu$ as small as we want.

For the third term by the Cauchy-Schwarz inequality and (5.17) we have

(5.24) \quad \left| \int_{\tau - 1}^\tau a(\tau') \int (u_2 - \sqrt{2} + \frac{\sqrt{2}}{8|\tau|} \psi_2) \psi_2^2 \, d\mu \, d\tau' \right| \leq \frac{\epsilon}{|\tau|} \|a\|_{B,\infty}.

Inserting these three last estimates into (5.23) we conclude the following bound for the last term in (5.22)

(5.25) \quad \left| \int_{\tau - 1}^\tau \int \left( \frac{u_2^2 - 2}{2u_1u_2} w_C + \frac{a(\tau)}{8|\tau|} \psi_2^2 \right) \psi_2 \, d\mu \, d\tau' \right| \leq \frac{\epsilon}{|\tau|} \|a\|_{B,\infty}.

Next, let’s bound the middle term in (5.22). First, we have

$$\left| \int \frac{u_2^2}{u_1u_2} w_C \psi_2 \, d\sigma \right| \leq C(\theta) \int_{|\sigma| \leq L} u_2^2 |w_C| |\psi_2| \, d\sigma + C(\theta) \int_{|\sigma| \geq L} u_2^2 |w_C||\psi_2| \, d\sigma.$$
For any $\epsilon > 0$, choose $L$ big enough so that $\int_{|\sigma| \geq L} |\psi_2| \, d\mu \leq \epsilon$. For that $L$ we can choose a $\tau_0 \ll -1$ so that (5.19) holds for $\tau \leq \tau_0$. By Hölder’s inequality and by (5.16) (where the latter is used in the second integral above) we get

$$\left| \int \frac{u_{2\sigma}^2}{u_1 u_2} w_C \psi_2 \, d\mu \right| \leq C(\theta) \|w_C\|_B \sup_{|\sigma| \leq L} u_{2\sigma}^2 + \frac{\epsilon}{|\tau|} \|w_C\|_B.$$

If we choose $\tau_0 \ll -1$ sufficiently small, the last inequality, (5.13) and (5.19) yield

$$\|\psi_2\|^2 \left| \int_{\tau-1}^\tau \int \frac{u_{2\sigma}^2}{u_1 u_2} w_C \psi_2 \, d\mu \, d\tau' \right| \leq \frac{\epsilon}{|\tau|} \|a\|_{B, \infty}.$$

Furthermore, using that $f \to \sigma f$ is a bounded operator from $\mathcal{D}$ to $\mathcal{H}$, we can bound

$$\left| \int \frac{w w_C}{2u_1} \psi_2 \, d\mu \right| \leq C(\theta) \left( \int w_C^2 |\psi_2| + C(\theta) \int |w_C||w| \chi_{D_0} |\psi_2| \, d\mu \right),$$

$$\leq C(\theta) \|w_C\|_D^2 + C(\theta) \|w_C\|_B \left( \int w_2^2 \chi_{D_0} \, d\mu \right)^{\frac{1}{2}}.$$

This together with (5.18) yield

$$\|\psi_2\|^2 \left| \int_{\tau-1}^\tau \int \frac{w w_C}{2u_1} \psi_2 \, d\mu \, d\tau' \right| \leq \frac{\epsilon}{|\tau|} \|a\|_{B, \infty} e^{-\alpha_0|\tau|},$$

for $\tau \leq \tau_0 \ll -1$ sufficiently small. These last two bounds yield to the estimate for the middle term.

To deal with the first term, we first see that similarly to the last estimate above we have

$$\left| \int \frac{w_2 (w_C)_{\sigma}}{u_1} \psi_2 \, d\mu \right| \leq C(\theta) \left( \int (w_C)^2 \psi_2 \, d\mu + C(\theta) \int |w_\sigma| (w_C)_{\sigma} |\psi_2| \chi_{D_0} \, d\mu \right),$$

$$\leq C(\theta) \|w_C\|_D^2 + C(\theta) \int (w_C)^2_{\sigma} \, d\mu + C(\theta) \|w_C\|_{D, \infty} e^{-\epsilon_0 |\tau|}.$$

This implies

$$\left| \int_{\tau-1}^\tau \int \frac{w_2 (w_C)_{\sigma}}{u_1} \psi_2 \, d\mu \, d\tau' \right| \leq \frac{\epsilon}{|\tau|} \|a\|_{B, \infty} + \int_{\tau-1}^\tau \int (w_C)^2_{\sigma} \, d\mu \, d\tau'.$$

By the proof of Lemma A.7 we have for $\tau \leq \tau_0$,

$$\int_{\tau-1}^\tau \int (w_C)^2_{\sigma} \, d\mu \, d\tau' \leq C \int_{\tau-2}^\tau \int w_2^2 \, d\mu \, d\tau' + \int_{\tau-2}^\tau \int \mathcal{E}^2 \, d\mu \, d\tau'.$$

where $\mathcal{E} = \mathcal{E}(w_C) + \mathcal{E}[w, \varphi_C] + \mathcal{E}_{nl}$ is the error term given by (3.6), (3.7) and (3.8). Similarly as in Proposition 3.5 where we have estimated $\|\mathcal{E}\|_{D^{\star, \infty}}$, we could estimate
\[ \| \mathcal{E} \|_{\mathcal{B}, \infty}(\tau) \text{ for } \tau \leq \tau_0. \]

By carefully analyzing all the terms we estimated in the proof of Proposition 3.5 we conclude that for \( \tau \leq \tau_0 \ll -1 \),

\[
\sup_{\tau \leq \tau} \int_{\tau-1}^{\tau} \mathcal{E}^2 \, d\mu \, dt' \leq \epsilon \| wC \|_{D, \infty}^2(\tau) + \epsilon \| wX_{D_0} \|_{\mathcal{B}, \infty}^2(\tau) + \epsilon \| wX_{\tilde{D}_0} \|_{\mathcal{B}, \infty}^2(\tau) + \epsilon \| (u_2^2 - 2) wC \|_{\mathcal{B}, \infty}(\tau). 
\]

Combining this and (5.15) yields

\[ (5.30) \quad \sup_{\tau \leq \tau} \int_{\tau-1}^{\tau} \mathcal{E}^2 \, d\mu \, dt' \leq C(\theta) \epsilon \| wC \|_{D, \infty}^2(\tau) \leq C(\theta) \frac{\epsilon}{|\tau|} \| a \|_{\mathcal{B}, \infty}, \]

where in the last inequality we used similar arguments that we used to obtain (5.27).

This estimate, (5.29) and similar arguments to the ones we used to derive (5.27) yield

\[ (5.31) \quad \int_{\tau-1}^{\tau} (wC)_{\sigma}^2 \, d\mu \, dt' \leq C \| wC \|_{D, \infty}^2(\tau) \leq \frac{\epsilon}{|\tau|} \| a \|_{\mathcal{B}, \infty}. \]

Finally, (5.28) and (5.31) imply the bound

\[ (5.32) \quad \| \psi_2 \|^{-2} \left( \int_{\tau-1}^{\tau} \int_{\tau-1}^{\tau} \frac{w_\sigma(wC)_\sigma}{u_1} \psi_2 \, d\mu \, dt' \right) \leq \frac{\epsilon}{|\tau|} \| a \|_{\mathcal{B}, \infty} \]

for a different but still arbitrarily and uniformly small constant \( \epsilon \). It only remains to bound

\[ (5.33) \quad \int (wC)_\sigma \left( \int_0^\sigma \frac{u_2^2}{u_1^2} \, d\sigma' \right) \psi_2 \, d\mu = 2 \int_{A_+} (wC)_\sigma \left( \int_0^\sigma \frac{u_2^2}{u_1^2} \, d\sigma' \right) \psi_2 \, d\mu \]

where \( A_+ := \text{supp} \, \varphi_C \cap \{ \sigma \geq 0 \} \). Applying Cauchy-Schwarz inequality and Fubini’s theorem imply

\[
\left| \int_{A_+} (wC)_\sigma \left( \int_0^\sigma \frac{u_2^2}{u_1^2} \, d\sigma' \right) \psi_2 \, d\mu \right| \leq C(\theta) \int (wC)_\sigma \psi_2 \left( \int_0^\sigma \frac{u_2^2}{u_1^2} \, d\sigma' \right) \, d\mu \\
= C(\theta) \int_0^{u_1^{-1}(\theta, \tau)} u_1^2 \sigma \, \sigma' \psi_2 \left( \int_0^\sigma \frac{u_2^2}{u_1^2} \, d\sigma' \right) \, d\mu \\
\leq C(\theta) \| wcC \|_D \int_0^{u_1^{-1}(\theta, \tau)} u_1^2 \sigma \left( \int_0^{\infty} \frac{\psi_2}{u_1^2} \, d\sigma' \right)^{\frac{3}{2}} \, d\sigma' \\
\leq C(\theta) \| wcC \|_D \int_0^{u_1^{-1}(\theta, \tau)} u_1^2 \sigma \, d\sigma' \\
\leq C(\theta) \| wcC \|_D \left( \int_{|\sigma| \leq L} u_1^2 \sigma e^{-\frac{\sigma^2}{T_0^2}} \, d\sigma + \int_{|\sigma| \geq L} u_1^2 \sigma e^{-\frac{\sigma^2}{T_0^2}} \, d\sigma \right) 
\]

To estimate the second integral in the last line above we use (A.4) and then find \( L \gg 1 \) big enough so that \( \int_{|\sigma| \geq L} e^{-\frac{\sigma^2}{T_0^2}} \, d\sigma \) can be made as small as we want. Then we find \( \tau_0 \ll -1 \) sufficiently small so that (5.19) holds on \( |\sigma| \leq L \) for \( \tau \leq \tau_0 \) and this we apply to the first integral in the last line in the previous estimate. Finally, for \( \tau \leq \tau_0 \) we have

\[ (5.34) \quad \| \psi_2 \|^{-2} \left( \int_{\tau-1}^{\tau} \int (wC)_\sigma \left( \int_0^\sigma \frac{u_2^2}{u_1^2} \, d\sigma' \right) \psi_2 \, d\mu \, dt' \right) \leq \frac{\epsilon}{|\tau|} \| a \|_{\mathcal{B}, \infty}. \]
Estimates (5.25), (5.26), (5.27), (5.32) and (5.34) lead to (5.21) as desired. \(\square\)

**Step 3.** For every \(c > 0\) there exists a \(\tau_0 \ll -1\) so that for \(\tau \leq \tau_0\) we have

\[
\|\psi_2\|^{-2} \left| \int_{\tau-1}^{\tau} \langle \mathcal{E}_{nl} - \frac{a(\tau)}{8|\tau|} \psi_2^2, \psi_2 \rangle \, d\tau' \right| \leq \frac{c}{|\tau|} \|a\|_{\mathcal{B}_1}. \tag{5.35}
\]

We recall that \(\langle \mathcal{E}_{nl}, \psi_2 \rangle = \langle u_{2\sigma} \varphi_C (J_2 - J_1), \psi_2 \rangle\). By (3.4) this can be written as

\[
\langle \mathcal{E}_{nl}, \psi_2 \rangle = -2 \int u_{2\sigma} \varphi_C \psi_2 \left( \int_0^{\sigma} \frac{w_{\sigma\sigma}}{u_1} \, d\sigma' \right) \, d\mu \tag{5.36}
\]

Furthermore, we can write

\[
I := \int u_{2\sigma} \varphi_C \psi_2 \int_0^{\sigma} \frac{u_{2\sigma\sigma}}{u_1 u_2} \, w \, d\sigma' \, d\mu =: I_1 + I_2
\]

where

\[
I_1 := \int u_{2\sigma} \varphi_C \psi_2 \int_0^{\sigma} \frac{u_{2\sigma\sigma}}{u_1 u_2} \, w \, d\sigma' \, d\mu
\]

and

\[
I_2 := \int u_{2\sigma} \varphi_C \psi_2 \int_0^{\sigma} \frac{u_{2\sigma\sigma}}{u_1 u_2} \, (1 - \varphi_C) w \, d\sigma' \, d\mu.
\]

To estimate \(I_1\), we write \(I_1 = 2I_1|_{A_+}\) where \(I_1|_{A_+}\) denotes the same integral \(I_1\) restricted on \(\text{supp} \varphi_C \cap \{ \sigma \geq 0 \}\). To estimate \(I_1|_{A_+}\), we note that by Fubini’s Theorem and (A.4) for a small \(\eta > 0\), we have

\[
\left| I_1|_{A_+} \right| \leq C(\theta) \int_0^{u_{1^{-1}}(\theta, \tau)} |u_{2\sigma\sigma}| |w_C| \int_{\sigma'}^{u_{1^{-1}}(\theta, \tau)} |u_{2\sigma}| \varphi_C |\psi_2| e^{-\frac{\eta^2}{4}} \, d\sigma \, d\sigma'. \tag{5.37}
\]

Integration by parts readily gives the bound

\[
\int_{\sigma'}^{u_{1^{-1}}(\theta, \tau)} |\psi_2| e^{-\eta^2 \sigma^2 / 4} \, d\sigma \leq \int_{\sigma'}^{\infty} (\sigma^2 + 2) e^{-\eta^2 \sigma^2 / 4} \, d\sigma \leq C \left( 1 + \sigma' \right) e^{-\eta^2 \sigma^2 / 4}
\]

which inserting in the above estimate yields

\[
\left| I_1|_{A_+} \right| \leq \frac{C(\theta)}{\sqrt{|\tau|}} \int_0^{u_{1^{-1}}(\theta, \tau)} |u_{2\sigma\sigma}| |w_C| (\sigma + 1) e^{-\frac{\eta^2}{4}} \, d\sigma
\]

\[
\leq \frac{C(\theta)}{\sqrt{|\tau|}} \left( \int_{\mathcal{C}_0} u_{2\sigma\sigma}^2 \, d\mu \right)^{\frac{1}{2}} \left( \left( \int_{\mathcal{C}_0} w_C^2 \sigma^2 \, d\mu \right)^{\frac{1}{2}} + \|w_C\| \right)
\]

\[
\leq \frac{C(\theta)}{\sqrt{|\tau|}} \left( \int_{\mathcal{C}_0} u_{2\sigma\sigma}^2 \, d\mu \right)^{\frac{1}{2}} \|w_C\|_\mathcal{D}.
\]

Using (5.17) we conclude the bound

\[
\int_{\tau-1}^{\tau} |I_1| \, d\tau' = 2 \int_{\tau-1}^{\tau} |I_1|_{A_+} \, d\tau' \leq \frac{C(\theta)}{|\tau|^{\frac{1}{2}}} \|a\|_{\mathcal{B}_1} \leq \frac{c}{|\tau|} \|a\|_{\mathcal{B}_1}.
\]
To estimate $I_2$ we simply use
\[ |I_2| \leq C(\theta) \left( \int w^2 \chi_{D_\theta} \, d\mu \right)^{\frac{1}{2}} e^{-c_0(\theta) |\tau|} \]
which together (5.13) and (5.14) yields
\[ \int_{\tau-1}^{\tau} |I_2| \, d\tau' \leq \frac{\epsilon}{|\tau|} \|a\|_{B_\infty}. \]

Combining these two estimates for $I_1$ and $I_2$ finally gives us the bound
\[ \int_{\tau-1}^{\tau} |I| \, d\tau' \leq \frac{\epsilon}{|\tau|} \|a\|_{B_\infty}. \]  

It remains to estimate $\bar{I}$. For that we first integrate by parts using that $w_\sigma(0, \tau) = 0$, to obtain
\[ \bar{I} := \int u_{2\sigma} \varphi_C \psi_2 \int_0^\sigma \frac{w_{\sigma \psi_2}}{u_1} \, d\sigma' \, d\mu \]
\[ = \int u_{2\sigma} \varphi_C \psi_2 \int_0^\sigma \frac{w_{\sigma u_1}}{u_1} \, d\sigma' \, d\mu + \int u_{2\sigma} \varphi_C \psi_2 \frac{w_{\sigma}}{u_1} \, d\mu \]
\[ = \int u_{2\sigma} \varphi_C \psi_2 \int_0^\sigma \frac{w_{\sigma u_1}}{u_1} \, d\sigma' \, d\mu - \int \frac{u_{2\sigma}}{u_1} (\varphi_C)_\sigma w \, d\mu + \int \frac{u_{2\sigma}}{u_1} (w_C)_\sigma \, d\mu. \]

Note that similarly to estimating $I_1$ term above (see (5.37)) we have
\[ \left| \int u_{2\sigma} \varphi_C \psi_2 \int_0^\sigma \frac{w_{\sigma u_1}}{u_1} \, d\sigma' \, d\mu \right| \leq C(\theta) \left( \int u_{2\sigma}^2 \, d\mu \right)^{\frac{1}{2}} \left( \|w_C\|_D + \|(w_C)_\sigma\|_{B_\infty} \right) \]

hence, using (5.13), (5.17), (5.29), (5.30) and (5.31) we obtain
\[ \left| \int_{\tau-1}^{\tau} \int u_{2\sigma} \varphi_C \psi_2 \int_0^\sigma \frac{w_{\sigma u_1}}{u_1} \, d\sigma' \, d\mu \, d\tau' \right| \leq \frac{\epsilon}{|\tau|} \|a\|_{B_\infty}. \]  

Similarly, we obtain
\[ \left| \int_{\tau-1}^{\tau} \int \frac{u_{2\sigma}}{u_1} (\varphi_C)_\sigma w \, d\mu \, d\tau' \right| \leq \frac{\epsilon}{|\tau|} \|a\|_{B_\infty}. \]  

For the last term that gives $\bar{I}$ we need to estimate
\[ \int \frac{u_{2\sigma}}{u_1} (w_C)_\sigma \, d\mu = \frac{1}{\sqrt{2}} \int \frac{u_{2\sigma}}{u_1} (\sqrt{2} - u_1) (w_C)_\sigma \, d\mu + \frac{1}{\sqrt{2}} \int u_{2\sigma} \psi_2 (w_C)_\sigma \, d\mu. \]

For first term in (5.41), after applying Cauchy-Schwarz inequality, (A.4) and using Lemma 4.12 from [1], (5.29), (5.30), (5.31) and (5.18) we obtain for $\tau \leq \tau_0 \ll -1$
the bound
\[
\left| \int u_{2\sigma} \psi_2 (\sqrt{2} - u_1) (w_C)_\sigma \, d\mu \right| \leq \frac{C(\theta)}{\sqrt{|\tau|}} \left( \| (w_C)_{\sigma} \|_{\delta} + \| \sigma (w_C)_{\sigma} \|_{\delta} \right) \left( \| u_1 - \sqrt{2} \|_{\delta} + \| \sigma (u_1 - \sqrt{2}) \|_{\delta} \right)
\]
\leq \frac{C(\theta)}{\sqrt{|\tau|}} \left( \| w_C \|_{\delta} + \| (w_C)_{\sigma} \|_{\delta} \right) \left( \| u_1 - \sqrt{2} \|_{\delta} \right)
\leq \frac{C(\theta)}{|\tau|} \| w_C \|_{\delta} \| u_1 - \sqrt{2} \|_{\delta} \leq \frac{\epsilon}{|\tau|} \| a \|_{\delta, \infty}.
\]

For the second term on the right hand side in (5.41) we have
\[
\frac{1}{\sqrt{2}} \int u_{2\sigma} (w_C)_{\sigma} \psi_2 \, d\mu
= \frac{1}{\sqrt{2}} \int \left( u_2 - \sqrt{2} + \frac{\sqrt{2} \psi_2}{8|\tau|} \right)_{\sigma} (w_C)_{\sigma} \psi_2 \, d\mu - \int \frac{\psi_{2\sigma}}{8|\tau|} \psi_2 (w_C)_{\sigma} \, d\mu
\]
\[
= \frac{1}{\sqrt{2}} \int \left( u_2 - \sqrt{2} + \frac{\sqrt{2} \psi_2}{8|\tau|} \right)_{\sigma} (w_C)_{\sigma} \psi_2 \, d\mu - \frac{1}{8|\tau|} \int \psi_{2\sigma} \psi_2 \psi_2 (w_C)_{\sigma} \, d\mu - \frac{a(\tau)}{8|\tau|} \int \psi_{2\sigma}^2 \psi_2 \, d\mu.
\]

Note that by Cauchy-Schwarz inequality, Lemma 4.12 in [1], (5.13), (5.17) and (5.31) we have
\[
\| \psi_2 \|_{-1} \left| \frac{1}{\sqrt{2}} \int_{\tau-1}^\tau \int \left( u_2 - \sqrt{2} + \frac{\sqrt{2} \psi_2}{8|\tau|} \right)_{\sigma} (w_C)_{\sigma} \psi_2 \, d\mu \, d\tau' \right| \leq \frac{\epsilon}{|\tau|} \| a \|_{\delta, \infty}.
\]

Also, by Cauchy-Schwarz inequality and (5.13) we get
\[
\left| \int_{\tau-1}^\tau \frac{1}{8|\tau'|} \int \psi_{2\sigma} \psi_2 \hat{w}_{C, \sigma} \, d\mu \, d\tau' \right| \leq \frac{\epsilon}{|\tau|} \| a \|_{\delta, \infty}
\]
and using \( \int \psi_{2\sigma}^2 \psi_2 \, d\mu = 4 \| \psi_2 \|_{\delta}^2 \) and that \( \langle \psi_2, \psi_2 \rangle = 8 \| \psi_2 \|_{\delta}^2 \) we have
\[
\int \psi_{2\sigma}^2 \psi_2 \, d\mu = \frac{1}{2} \langle \psi_2^2, \psi_2 \rangle.
\]

The last identity and estimates (5.39), (5.40), (5.41), (5.42), (5.43), (5.44) and (5.45) yield
\[
\left| \int_{\tau-1}^\tau \left( - 2 \bar{I} - \frac{a(\tau)}{8|\tau|} \langle \psi_2^2, \psi_2 \rangle \right) \, d\tau' \right| \leq \frac{\epsilon}{|\tau|} \| a \|_{\delta, \infty}.
\]
Combining this with (5.36), (5.38) and (5.46) leads to (5.35) as desired.

Finally, inserting in (5.11) the main estimates from the three steps, (5.20), (5.21) and (5.35) concludes the proof of Claim 5.6.

The proof of Theorem 1.5 is now complete.
A. A priori bounds for rotationally symmetric data

In this section we will establish some preliminary a’priori bounds for rotationally symmetric solutions. We assume throughout this section that \( u(\cdot, \tau) \) is a solution of (2.11). We recall that we have denoted by \( \sigma_{\pm}(\tau) \) the points of maximal scalar curvature of our rescaled solution \( \bar{M}_{\tau} \). For any \( \theta \in (\sqrt{2}, 0) \), let us recall the definition of the cylindrical region (in un-rescaled coordinates), that is
\[
C_\theta := \{ (\sigma, \tau) : u(\sigma, \tau) \geq \frac{\theta}{4} \}.
\]

A.1. Derivative bounds in the cylindrical region. Results in [3] and monotonicity of \( u_{\sigma} \) immediately imply that for every \( \theta \in (0, \sqrt{2}) \) there exist \( C(\theta), c(\theta) \) and \( \tau_0 \ll -1 \) such that for all \( \tau \leq \tau_0 \) we have
\[
|u_{\sigma}(\sigma, \tau)| \leq C(\theta) \frac{\sqrt{|\tau|}}{}.
\]
It also follows that
\[
|\sigma - \sigma_{\pm}(\tau)| \geq c(\theta)|\sigma_{\pm}(\tau)|, \quad \text{for } \tau \leq \tau_0 \ll -1.
\]
Moreover, using convexity of \( u(\sigma, \tau) \) as in [1], we have
\[
|u_{\sigma}(\sigma, \tau)| \leq \frac{C}{\sigma_{\pm}(\tau) - \sigma},
\]
if \( u_{\sigma} \leq 0 \). If \( u_{\sigma} \geq 0 \), similar estimate holds if we replace \( \sigma_{\pm}(\tau) \) by \( |\sigma_{-}(\tau)| \).

We will next derive higher order derivative estimates which hold away from the tips of our surface.

**Lemma A.1.** For any \( \theta < \sqrt{2} \) there exist constants \( C(\theta) > 0 \) and \( \tau_0 = \tau_0(\theta) \ll -1 \) so that the bounds
\[
|u_{\sigma}| + |u_{\sigma\sigma}| + |u_{\sigma\sigma\sigma}| \leq \frac{C(\theta)}{\sqrt{|\tau|}}
\]
hold on \( C_\theta \), for all \( \tau \leq \tau_0 \).

**Proof.** We first notice that by (A.2) we have that for any \( \theta < \sqrt{2} \), there exists an \( \alpha = \alpha(\theta) < 1 \) such that
\[
(\sigma, \tau) \in C_\theta \implies \sigma \in [\alpha \sigma_{-}(\tau), \alpha \sigma_{+}(\tau)] \quad \text{and} \quad |\sigma_{\pm} - \sigma| \geq (1 - \alpha) |\sigma_{\pm}(\tau)|.
\]
The bound on \( |u_{\sigma}| \) readily follows by (A.1). To obtain higher order derivative estimates on \( u \) we first differentiate the evolution equation (2.11) with respect to \( \sigma \). If we write \( z := u_{\sigma} \), then we obtain
\[
\frac{\partial z}{\partial \tau} = z_{\sigma\sigma} + \frac{z(1 - z^2)}{u^2}.
\]
The vector fields \( \partial_{\sigma} \) and \( \partial_{\tau} \) do not commute. To overcome this we introduce commuting variables as in Section 2. Then the equation becomes
\[
\frac{\partial z}{\partial \tau} = z_{\sigma\sigma} - \frac{\sigma}{2} z_{\sigma} - J(\sigma, \tau) z_{\sigma} + \frac{z(1 - z^2)}{u^2}
\]
where \( J(\sigma, \tau) \) is given by (2.13). We will localize the proof of our desired estimate (A.4) by introducing the following change of variables. Assume with no loss of
generality that $\sigma_0 > 0$. Given a point $(\sigma_0, \tau_0)$ in space-time with $\sigma_0 \leq \alpha \sigma_+ (\tau_0)$, we get

$$\bar{\epsilon}(\eta, \bar{\tau}) := z(\sigma_0 \bar{e}^{\bar{\tau}/2} + \eta, \tau_0 + \bar{\tau}).$$

If we choose $-\tau_0$ large enough, depending on $\alpha \in (0, 1)$, then this function is defined on the rectangle

$$Q := \{(\eta, \bar{\tau}) \mid |\eta| \leq 1, -1 \leq \bar{\tau} \leq 0\}.$$

To see this, recall that $s_1(t)$ for the unrescaled flow is monotonically decreasing. Hence $e^{-\tau/2} \sigma_+ (\tau) = s(-e^{-\tau})$, is a decreasing function of $\tau$, and thus

$$\sigma_+ (\tau_0) \leq e^{-\bar{\tau}/2} \sigma_+ (\tau_0 + \bar{\tau}) \quad \text{for} \quad \bar{\tau} \in [-1, 0].$$

For any $\alpha < 1$, we choose $\alpha' = (1 + \alpha)/2$. We also choose $\tau_0(\alpha)$ so that

$$\alpha \sigma_+ (\tau') + 1 \leq \alpha' \sigma_+ (\tau') \quad \text{for all} \quad \tau' < \tau_0.$$

For any $(\eta, \bar{\tau}) \in Q$ we then get

$$\sigma_0 \bar{e}^{\bar{\tau}/2} + \eta \leq \alpha \sigma_+ (\tau_0) e^{\bar{\tau}/2} + 1 \leq \alpha \sigma_+ (\tau_0 + \bar{\tau}) + 1 \leq \alpha' \sigma_+ (\tau_0 + \bar{\tau}).$$

It follows that $\bar{\epsilon}(\eta, \bar{\tau}) = z(\sigma_0 \bar{e}^{\bar{\tau}/2} + \eta, \tau_0 + \bar{\tau})$ is indeed defined on $Q$.

On the other hand, a computation shows that $\bar{\epsilon}$ satisfies

$$\frac{\partial \bar{\epsilon}}{\partial \bar{\tau}} = \frac{\bar{\epsilon}_\eta - \bar{\eta}}{2} - J(\sigma, \tau_0 + \bar{\tau}) \frac{\bar{\epsilon}_\eta}{2} + \frac{\bar{\epsilon}(1 - \bar{\epsilon}^2)}{u^2}$$

which can be written as

$$\frac{\partial \bar{\epsilon}}{\partial \bar{\tau}} = a(\eta, \bar{\tau}, \bar{\epsilon}, \bar{\epsilon}_\eta) \bar{\epsilon}_\eta + b(\eta, \bar{\tau}, \bar{\epsilon}, \bar{\epsilon}_\eta),$$

where

$$a(\eta, \bar{\tau}, \bar{\epsilon}, \bar{\epsilon}_\eta) = 1, \quad b(\eta, \bar{\tau}, \bar{\epsilon}, \bar{\epsilon}_\eta) = \frac{\bar{\epsilon}(1 - \bar{\epsilon}^2)}{u^2} - \frac{\bar{\eta}}{2} p - J(\sigma, \tau_0 + \bar{\tau}) p,$$

and

$$J(\sigma, \tau_0 + \bar{\tau}) = 2 \frac{\bar{\epsilon}}{u(\sigma, \tau_0 + \bar{\tau})} + 2 \int_0^\sigma \frac{u^2}{u^2} \, d\sigma', \quad \text{for} \quad \sigma := \sigma_0 \bar{e}^{\bar{\tau}/2} + \eta.$$

Estimate (A.3) combined with $u(0, \tau) = \sqrt{2(\nu - 1)} + \delta(\tau)$ tell us that on the rectangle $Q$ we can bound $\bar{\epsilon} = u_\sigma (\sigma_0 \bar{e}^{\bar{\tau}/2} + \eta, \tau_0 + \bar{\tau})$ by

$$|\bar{\epsilon}| \leq \frac{C}{\sigma_+(\tau_0 + \bar{\tau}) - \sigma_0 \bar{e}^{\bar{\tau}/2} - \eta} \leq \frac{C}{\sigma_+(\tau_0 + \bar{\tau}) - \sigma_0 \bar{e}^{\bar{\tau}/2} - 1}.$$

By (A.5) we have $e^{\bar{\tau}/2} \sigma_+ (\tau_0) \leq \sigma_+ (\tau_0 + \bar{\tau})$, which then implies

$$|\bar{\epsilon}| \leq \frac{C}{e^{\bar{\tau}/2} \sigma_+ (\tau_0) - \sigma_0 \bar{e}^{\bar{\tau}/2} - 1} \leq \frac{C}{\sigma_+(\tau_0) - \sigma_0 - e^{-\bar{\tau}/2}},$$

for $\bar{\tau} \in [-1, 0]$. Since $\sigma_0 \leq \alpha \sigma_+(\tau_0)$, we have

$$\sigma_0 - e^{-\bar{\tau}/2} \leq \sigma_0 + e^{1/2} \leq \alpha \sigma_+(\tau_0),$$

with $\alpha' = (1 + \alpha)/2$, assuming again that $-\tau_0$ is sufficiently large. In the end we get the following estimate for $\bar{\epsilon}$ on the rectangle $Q$

$$|\bar{\epsilon}| \leq \frac{C(\alpha)}{\sigma_+(\tau_0)}.$$
We next apply this bound to the coefficients $a$ and $b$ in the equation \((A.6)\) for $\bar{z}$. We have that $a(\eta, \tau, \bar{z}, p) = 1$. Using \((A.7), (A.8)\) and asymptotics proved in [3], we get
\[
|J(\bar{\sigma}, \tau_0 + \bar{\tau})| \leq \frac{C(\alpha)}{\sigma_+(\tau_0 + \bar{\tau}) - \sigma_0 e^{\bar{\tau}^2/2} - 1} + C(\alpha) \int_0^{\bar{\sigma}} \frac{d\sigma'}{(\sigma_+(\tau_0 + \bar{\tau}) - \sigma')^2}.
\]
Using that $\bar{\sigma} = \sigma_0 e^{\bar{\tau}^2/2} + \eta$, $\sigma_+(\tau_0 + \bar{\tau}) \geq e^{\bar{\tau}^2/2} \sigma_+(\tau_0)$, $\sigma_0 \leq \alpha \sigma_+(\tau_0)$ and $\sigma_+(\tau_0) \sim \sqrt{|\tau_0|}$, for $\tau_0 \ll -1$ (the latter follows from our results in [3]) we have
\[
|J(\bar{\sigma}, \tau_0 + \bar{\tau})| \leq \frac{C(\alpha)}{(e^{\bar{\tau}^2/2} \sigma_+(\tau_0) - \sigma_0) - 1} + C(\alpha) \int_0^{\bar{\sigma}} \frac{d\sigma'}{(e^{\bar{\tau}^2/2} \sigma_+(\tau_0) - \sigma')^2}
\leq \frac{C(\alpha)}{(e^{\bar{\tau}^2/2} \sigma_+(\tau_0) - \sigma_0) - 1} + \frac{C(\alpha)}{(e^{\bar{\tau}^2/2} \sigma_+(\tau_0)(1 - \alpha) - 1)^2}
\leq \frac{C(\alpha)}{\sqrt{|\tau_0|}}.
\]
The above bound on $J(\bar{\sigma}, \tau_0 + \bar{\tau})$, the lower bound $u \geq \theta$ and \((A.8)\) imply
\[
|b(\eta, \tau, \bar{z}, p)| \leq C(1 + p^2) = C a(\eta, \tau, \bar{z}, p) (1 + p^2).
\]
As a consequence of these bounds on the coefficients $a$ and $b$ and classical interior estimates for equation \((A.6)\) (see [27]), we obtain
\[
|\bar{z}_\eta(0, 0)| + |\bar{z}_\eta(0, 0)| \leq C_0 \sup_{\bar{Q}} |\bar{z}(\eta, \tau)| \leq \frac{C(\alpha)}{\sqrt{|\tau_0|}}.
\]
Finally, since $z_\eta(0, 0) = u_{\sigma\sigma}(\sigma_0, \tau_0)$, $\bar{z}_\eta(0, 0) = u_{\sigma\sigma\sigma}(\sigma_0, \tau_0)$ and $\alpha = \alpha(\theta)$ this completes the proof of Lemma A.1.

**A.2. The concavity of $u^2$.** Our aim in this section is to prove the following result which will allow us to better estimate $u(\cdot, \tau)$ and its derivatives in the collar region.

**Claim-2**

**Proposition A.2.** Let $(S^3, g(t))$ be a compact, ancient solution to the Ricci flow on $S^3$. Then there exist $\tau_0 \ll -1$ and $L \gg 1$ so that the function $u^2$ is concave on $u \geq L/\sqrt{|\tau|}$, for $\tau \leq \tau_0 \ll -1$.

The proof of this Proposition combines a contradiction argument based on scaling and the following maximum principle Lemma.

**Lemma A.3.** Under the assumptions of Proposition A.2, there exist a $\tau_0 \ll -1$ such that if $\left((u^2)_{\sigma\sigma}\right)_{\text{max}} > 0$ is attained in $\left\{u \geq L/\sqrt{|\tau|}\right\}$, for a sufficiently large number $L$, then
\[
\frac{d}{d\tau} \max(u^2)_{\sigma\sigma} \leq 0.
\]

**Proof.** Since $(u^2)_{\sigma\sigma}$ is scaling invariant quantity, we will work in the original variables $(s, t, \psi(s, t))$. Define $Q(s, t) := \psi^2(s, t)$. Note that $(\psi^2)_{ss} = (u^2)_{\sigma\sigma}$. Hence, it is sufficient to show that $\frac{d}{d\tau} \max(\psi^2)_{ss} \leq 0$.

It is easy to compute that in commuting variables $s$ and $t$ we have
\[
Q_t = Q_{ss} - 2Q_s \int_0^s \frac{\psi_{ss}}{\psi} ds - 2
\]
implying,
\[
(Q_{ss})_t = (Q_{ss})_{ss} - \left(J + \frac{Q_s}{\psi^3}\right) (Q_{ss})_s + \frac{4\psi_{ss}}{\psi^3} (Q_s^2 - QQ_{ss}),
\]

**proof**
where \( J = 2 \int_0^3 \frac{\psi}{\sqrt{\psi}} ds' \). We also have that \( Q_{ss}^2 - QQ_{ss} = 2\psi^2 \psi_s^2 - 2\psi^3 \psi_{ss} \geq 0 \), yielding that at the maximum of \( Q_{ss} \) we have
\[
\frac{d}{d\tau} \max Q_{ss} \leq 0.
\]
If the maximum of \( Q_{ss} \) is attained in the set \( \{ u > L/\sqrt{|\tau|} \} \), since \( (Q_{ss})_s = 0 \) at the maximum point and \( \psi > 0 \) there, we conclude that \( \frac{d}{d\tau} \max(u^2)_{ss} \leq 0 \). \( \square \)

**Proof of Proposition A.2.** Denote by \( q(\sigma, \tau) = u^2(\sigma, \tau) \) and recall that \( q_{ss} = Q_{ss} \).
We claim that at the boundary of the set \( \{ u > L/\sqrt{|\tau|} \} \) we have \( q_{ss} < 0 \). To see that, let's write
\[
q_{ss} = uY_u + Y = \rho Z_{\rho} + 2 Z
\]
where \( Z(\rho, \tau) := Y(u, \tau), \rho := u \sqrt{|\tau|} \) so that the boundary \( u = L/\sqrt{|\tau|} \) corresponds to \( \rho = L \). We know that \( Z(L, \tau) \) converges, as \( \tau \to -\infty \) to the Bryant soliton \( Z_0(L) \), whose maximal scalar curvature is equal to one. On the other hand, the asymptotics (2.29) of \( Z_0(L) \), imply that
\[
LZ_{\rho}(L) + 2Z_0(L) = -4L^{-4} + o(0^{-4}) < 0, \quad \text{for } L \gg 1.
\]
This means that for \( L \gg 1 \) and \( \tau \leq \tau_0 \ll -1 \) we have that \( q_{ss} \) is negative at the boundary, namely
\[
q_{ss}|_{u=L/\sqrt{|\tau|}} = L Z_{\rho}(L, \tau) + 2Z(L, \tau) < 0.
\]
Next, we claim there exist \( \tau_0 \ll -1 \) and \( L \gg 1 \) so that \( \max_{u \geq \frac{L}{\sqrt{|\tau|}}} q_{ss} \leq 0 \). Assume the statement were not true. Since, \( q_{ss} < 0 \) at the boundary, this means that there exist sequences \( \tau_j \to -\infty, \sigma_j \) and \( L_j \to \infty \) so that
\[
q_{ss}(\sigma_j, \tau_j) = \max_{u \geq \frac{L_j}{\sqrt{|\tau_j|}}} q_{ss} > 0 \quad \text{and} \quad u(\sigma_j, \tau_j) > \frac{L_j}{\sqrt{|\tau_j|}}
\]
Lemma A.3 then implies that \( q_{ss}(\sigma_j, \tau_j) \geq c > 0 \), for some uniform constant \( c > 0 \). Since \( q_{ss} = 2(u_{ss} + u^2) \) and \( u_{ss} \leq 0 \), we conclude that \( u_{ss}^2(\sigma_j, \tau_j) \geq c/2 \), or expressed in the tip region variables, that \( Y(u_j, \tau_j) \geq c/2 \), where \( u_j > \frac{L_j}{\sqrt{|\tau_j|}} \). Since \( Y_u = 2u_{ss} \leq 0 \) we conclude that
\[
Z(L_j, \tau_j) = Y\left( \frac{L_j}{\sqrt{|\tau_j|}}, \tau_j \right) \geq Y(u_j, \tau_j) \geq \frac{c}{2}.
\]
Since for any \( L \) large we have \( L_j \geq L \), for \( j \) sufficiently large, and since \( Z(\rho, \tau) \) is decreasing in \( \rho \), we have that \( Z(L, \tau_j) \geq Z(L_j, \tau_j) \geq \frac{c}{2} \). On the other hand, we have that the limit \( \lim_{j \to \infty} Z(L, \tau_j) = Z_0(L) \approx 1/L^2 \), provided \( L \gg 1 \). All these lead to a contradiction, if we choose \( L \) sufficiently large. \( \square \)

**A.3. Estimates in the Collar region.** In this section we will prove that any of our solutions satisfies the sharp estimate (A.15) in the collar region \( K_{L, \theta} \), provided that \( L \gg 1 \) and \( \tau \leq \tau_0 \ll -1 \). This estimate played a crucial role in estimating error terms in the entire Section 4 dealing with the tip region. We first show, in the next Proposition, that our solutions behave geometrically as cylinders in the region \( u \geq L/\sqrt{|\tau|} \), for \( L \gg 1 \) and \( \tau \leq \tau_0 \ll -1 \).
Proposition A.4. Given an $\eta > 0$ there exist $L \gg 1$ and $\tau_0 \ll -1$ such that for $\tau \leq \tau_0$,
\[
\frac{K_1}{K_0} = \frac{u u_{\sigma\sigma}}{1 - u_{\sigma}^2} \leq \eta, \quad \text{on} \quad u \geq \frac{L}{L}.
\]
and moreover,
\[
\frac{u}{\sqrt{\tau}} \leq \eta.
\]
Proof. We will use similar arguments to the ones we used to prove the analogous statement in [2], based on the following claim:

Claim A.5. For every $\tilde{L} > 0$ there exist an $L \gg 1$ and $\tau_0 \ll -1$ so that
\[
\frac{L}{L} \geq \frac{L}{L} \quad \Rightarrow \quad \text{dist}_{g(\tau)}(p, p^k_j) \leq \frac{\tilde{L}}{\sqrt{R(p, \tau)}}
\]
where $p$ is any point in our manifold, corresponding to $\sigma$, and $p^k_j$, $k = 1, 2$ is any of the two tip points where the scalar curvature becomes maximal, corresponding to $\sigma_{\pm}$. 

Proof of Claim. To show above claim we argue by contradiction. Assume the claim is not correct, meaning there exist an $\tilde{L} > 0$ and sequences $L_j \rightarrow \infty, \tau_j \rightarrow -\infty$ and $\sigma_j$, so that

\[
u(\sigma_j, \tau_j) \geq \frac{L_j}{L_j} \quad \text{but} \quad \text{dist}_{g(\tau_j)}(p_j, p^k_j) \leq \frac{\tilde{L}}{\sqrt{R(p_j, \tau_j)}}
\]
for say $k = 1$, where $p_j \in M$ corresponds to $\sigma_j$ and $p^k_j$ is a brief notation for $p^k_{\tau_j}$. Note that the distance between points $p_j$ and $p^k_j$ is measured with respect to metric $g(\sigma_j, \tau_j)$. Rescale the flow around $(p_j, \tau_j)$ by $\lambda_j := R(p_j, \tau_j)$, that is set $\tilde{g}_j(\sigma_j, \tau_j) = \lambda_j g(\sigma_j, \tau_j + \lambda_j^{-1} \tau_j)$. Then $R_{\tilde{g}_j}(p_j, 0) = 1$ and by Perelman’s compactness theorem for $\kappa$-solutions (see section 11 in [29]) we can extract a convergent subsequence $(M, g_j(\sigma_j, \tau_j), \sigma_j)$ that converges to a complete $\kappa$-solution. Since the limit is complete, noncompact, by [11] we know it is either a shrinking round cylinder or a Bryant soliton. By (A.12) we have that the rescaled distance from $p_j$ to $p^k_j$ at time zero is $\text{dist}_{\tilde{g}_j}(p_j, p^k_j) \leq \tilde{L}$. Moreover at the tip point the scaling invariant quantity $(K_1/K_0)(p^1_j, \tau_j) = 1$, yielding that the limiting metric is actually the Bryant soliton.

By results in [3] we have $R(p^1_j, \tau_j) \sim |\tau_j|$. By (A.12) and Perelman’s compactness theorem we have that $1 \leq R_{\tilde{g}_j}(p^1_j, 0) = \frac{R(p^1_j, \tau_j)}{R(p^k_j, \tau_j)} \leq C$, for all $j \geq j_0$ and a uniform constant $C$. Hence, for $j \geq j_0$ we have that $R(p_j, \tau_j) \sim |\tau_j|$ as well. Furthermore, if $\sigma_j$ and $\sigma^1_j$ are corresponding to points $p_j$ and $p^1_j$, respectively, we have
\[
|\sigma_j - \sigma^1_j| \leq \text{dist}_{\tilde{g}(\tau_j)}(p_j, p^1_j) \leq \frac{\tilde{L}}{\sqrt{R(p_j, \tau_j)}} \leq \frac{C \tilde{L}}{\sqrt{|\tau_j|}}.
\]
On the other hand, using $|u_{\sigma}| \leq 1$ we get
\[
|\sigma_j - \sigma^1_j| = \int_0^{u(\sigma_j, \tau_j)} \frac{du}{|u_{\sigma}|} \geq u(\sigma_j, \tau_j) \geq \frac{L_j}{L_j}.
\]
Combining (A.13) and (A.14), if we let $j \rightarrow \infty$, yield contradiction. This concludes the proof of the claim. \qed
To prove (A.9), due to (A.11) it is enough to prove the following statement: for every $\eta > 0$ there exist $L \gg 1$ and $\tau_0 \ll -1$ such that for $\tau \leq \tau_0$ we have

$$\min (\text{dist}_{g(\tau)}(p, p_1^1), \text{dist}_{g(\tau)}(p, p_2^2)) \geq \frac{L}{\sqrt{R(p, \tau)}} \Rightarrow \frac{K_1}{K_0}(p, \tau) \leq \eta.$$ 

To prove this we argue again by contradiction. Assume there exist $\eta > 0$ and sequences $L_j \to \infty$, $\tau_j \to -\infty$ and $p_j$ so that $\frac{K_1}{K_0}(p_j, \tau_j) \geq \eta$ but $\min (\text{dist}_{\tau_j}(p_j, p_1^1), \text{dist}_{\tau_j}(p_j, p_2^2)) \geq L_j \sqrt{R(p_j, \tau_j)}$.

Rescale the metric around $(p_j, \tau_j)$ by $\lambda_j := R(p_j, \tau_j)$, that is consider a sequence of rescaled metrics $\tilde{g}_j(\cdot, \tau_j) = \lambda_j g(\cdot, \tau_j + \tau \lambda_j^{-1})$. Then the rescaled distance satisfies $\tilde{\text{dist}}_j(p_j, p_k^k) \geq L_j$, for both $k = 1$ and $k = 2$. Using Perelman’s compactness theorem for $\kappa$-solutions, the fact that $K_1/K_0$ is scaling invariant quantity and Brendle’s classification result of complete noncompact $\kappa$-solutions (see [11]), after passing to a subsequence we conclude that the sequence of rescaled solutions $(M, \tilde{g}_j(\cdot, \tau_j), p_j)$ subconverges to a Bryant soliton. Since the $\lim_{j \to \infty} \tilde{\text{dist}}_j(p_j, p_k^k) = \infty$ for both $k = 1$ and $k = 2$ and since $K_0, K_1 \geq 0$ for our rotationally symmetric solution, the splitting theorem implies that the limit has to split off a line, implying that the limit has to be a cylinder, which contradicts the above fact the limit has to be the Bryant soliton at the same time.

Finally, the bound (A.10) follows by similar arguments using again that $u^2 u, u, u, u$ is a scaling invariant quantity. This finishes the proof of the proposition.

Based on Proposition A.2 we next show the following crucial for our purposes sharp bound. This bound is extensively used in Section 4.

**Lemma A.6.** Fix $\eta > 0$ small. There exists $\theta, L$ and $\tau_0 \ll -1$ such that

$$|1 + \frac{\sigma u u}{2}| < \eta$$

holds on $K_{L, \theta}$, for $\tau \leq \tau_0 \ll -1$.

**Proof.** Having Proposition A.2 and unique asymptotics that we proved in [3], the proof of the first estimate is identical to the proof of Corollary 4.7 in [2].

\[\square\]

**A.4. Energy estimate for the linear equation.** In this final section we prove the following standard energy estimate adopted to ancient solutions.

**Lemma A.7.** Let $w$ be a compactly supported, ancient solution to

$$w_{\tau} = L[w] + g, \quad \text{on } \mathbb{R} \times (-\infty, \tau_0].$$

Then, there exists a uniform constant $C$ so that

$$\sup_{\tau \leq \tau_0} \int w^2_{\tau} \, d\mu + \sup_{\tau \leq \tau_0} \int_{\tau-1}^{\tau} \int w^2_{\tau\sigma} \, d\mu \, d\tau' \leq C \left( \sup_{\tau \leq \tau_0} \int_{\tau-1}^{\tau} \int g^2 \, d\mu \, d\tau' + \sup_{\tau \leq \tau_0} \int_{\tau-1}^{\tau} \int g^2 \, d\mu \, d\tau' \right).$$
Proof. If we multiply the linear equation by \(w\) and integrate it by parts, we obtain

\[
\frac{1}{2} \frac{d}{d\tau} \int w^2 \, d\mu = - \int w_\sigma^2 \, d\mu + \int w^2 \, d\mu + \int gw \, d\mu.
\]

For any number \(\tau \in (-\infty, \tau_0)\), set \(\eta(\tau') = \tau' - \tau + 2\) so that \(0 \leq \eta(\tau') \leq 2\) for \(\tau' \in [\tau - 2, \tau]\). Then, for any \(\tau' \in [\tau - 2, \tau]\), we have

\[
\frac{1}{2} \frac{d}{d\tau'} \left( \eta(\tau') \int w^2 \, d\mu \right) = - \eta(\tau') \int w_\sigma^2 \, d\mu + (\eta(\tau') + \frac{1}{2}) \int w^2 \, d\mu + \eta(\tau') \int gw \, d\mu.
\]

If we integrate it from \(\tau - 2\) to \(\tau' \in [\tau - 1, \tau]\), for every \(\tau \leq \tau_0\) and Cauchy-Schwarz inequality for the last term on the right hand side, we get

\[
\sup_{\tau \leq \tau_0} \int \eta(\tau') \int w^2 \, d\mu \, d\tau' 
\leq C \left( \sup_{\tau \leq \tau_0} \int \tau - 1 \int w^2 \, d\mu \, d\tau' + \sup_{\tau \leq \tau_0} \int \tau - 1 \int g^2 \, d\mu \, d\tau' \right)
\]

for a uniform constant \(C\).

Next, let's multiply the equation for \(w\) by \(w_\sigma\) and integrate by parts to get

\[
\frac{1}{2} \frac{d}{d\tau} \int w_\sigma^2 \, d\mu + \int (w_\sigma^2 - \sigma w_\sigma w_\sigma + \frac{1}{4} \sigma^2 w_\sigma^2) \, d\mu = \int (\frac{1}{2} \sigma w_\sigma + \frac{1}{2} \sigma w_\sigma g - w_\sigma g - w_\sigma g) \, d\mu.
\]

If we multiply the previous equation by the function \(0 \leq \eta(\tau) \leq 2\) which is defined as above, while using

\[
\int \sigma w_\sigma w_\sigma \, d\mu = - \frac{1}{2} \int w_\sigma^2 \, d\mu + \frac{1}{4} \int \sigma^2 w_\sigma^2 \, d\mu
\]

and Cauchy-Schwarz, we obtain for \(\epsilon > 0\)

\[
\frac{d}{d\tau'} \left( \eta(\tau') \int w_\sigma^2 \, d\mu \right) + \eta(\tau') \int w_\sigma^2 \, d\mu + \frac{1}{2} \eta(\tau') \int w_\sigma^2 \, d\mu 
\leq \epsilon \int \sigma^2 w_\sigma^2 \, d\mu + C \epsilon \int w^2 \, d\mu + C \epsilon \int g^2 \, d\mu + C \int w_\sigma^2 \, d\mu.
\]

Choose \(\epsilon > 0\) sufficiently small. Using that the operator \(f \to \sigma f\) is bounded from \(D\) to \(\mathcal{H}\), by choosing small enough, if we integrate the previous estimate for \(\tau' \in [\tau - 2, \tau]\), for every \(\tau \leq \tau_0\), similarly as above we get

\[
\sup_{\tau \leq \tau_0} \int w_\sigma^2 \, d\mu + \sup_{\tau \leq \tau_0} \int \tau - 1 \int w_\sigma^2 \, d\mu
\leq C \sup_{\tau \leq \tau_0} \int \tau - 1 \left( \int w^2 + w_\sigma^2 \right) \, d\mu + \sup_{\tau \leq \tau_0} \int \tau - 1 \int g^2 \, d\mu.
\]

Combining this estimate with (A.16) concludes the proof of the Lemma. \(\square\)
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