A quantum model of space-time-matter

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We study a quantum mechanics with the usual postulates but in which the Heisenberg algebra of canonical commutation relations and the Poincare algebra are replaced by the Lie algebra of the homogeneous Lorentz group SO(5,1). It arises from the hypothesis that the above group is the fundamental group of invariance for the laws of physics. The observables of the theory like position, time, momentum, energy, angular momentum and others are the generators of the algebra of the group. Neither position and time observables commute between them, nor momentum and energy observables. The algebra of Poincare quantum mechanics is recovered in the limit in which two parameters, that we physically interpret as the Hubble constant and the Planck mass, are taken to zero and infinite respectively. We consider the equations that are satisfied by the spinor representation of the group.

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A satisfactory quantum theory of space-time is lacking. One may trace the difficulty to obtain such a theory to the early history of the theory of quantum mechanics as it was developed by Heisenberg, Schroedinger, Dirac and others. In the study of a physical system, say a particle in a box, one assumes a classical physical space-time as its framework and the correspondence principle is then used to obtain the canonical commutation rules of the theory and to write the equation of evolution. But what to assume in the case that the physical system is the space-time itself? Moreover, is it possible to separate clearly space-time from matter in a consistent quantum treatment of both? If we turn our attention to space-time, it is worth to remember how Einstein found a road to that beautiful place in which the abysm between space and time that existed in Newton’s mechanical view vanished, and space and time became fused into the Minkowski space-time continuum. One may then, dream of a quantum path, close to that of Einstein, that lead us to a place where the conceptual gap between space-time and matter is also absent.

To try to convert such a hope in reality we need, first, to pay attention, at what is required to define a quantum theory: We have to select the observables, give the commutation relations between them, i.e., its algebra, and finally, we have to specify its dynamics. A set of observables appropriate for the description of space-time-matter should include the position and time observables as well as the observables commonly associated with matter, as the energy, momentum and angular momentum.

Next, we expect that if space, time, and matter are conceptually similar they would be reflected in the properties of the algebra. Knowing of the connection between symmetry transformations and observables, we have, then, to face the apparent existence of two distinct types of motion, to wit, rotations and translations. One might think that the essential characteristic of rotations, the periodicity, is absent in translations, but in crystallography we know how to make compatible translations and periodicity. Nevertheless, if we view translations as rotations we are led to accept their non-commutativity. Being the generators of space-time translations the momentum observables in quantum mechanics, we have, then, that the different components of the momentum and the energy don’t commute between them. We would be departing from the Poincare algebra in which these objects commute. But we may think that the Poincare algebra is only an approximation to reality. Similarly, if we regard the position and time observables as rotations we are confronted with the non-commutativity of the different position components and the time. Besides, if position observables and momentum observables are now rotations, the canonical commutation rules between position and momentum observables will be valid only approximately. Finally, the generators of space-time rotations are the angular momentum observables and with the boosts constitute the algebra of the homogeneous Lorentz group SO(3,1).

The question that emerges is, whether there is a rotation algebra that contains all these fourteen generators and that reproduces at least at some limit the known properties of these observables. We found that the smallest appropriate algebra is the algebra of the group SO(5,1). The assignment of the remaining generator and the boosts to its corresponding observables will be done later.

The path we were looking for, salvo a mirage, is now visible. Minkowski space-time is a necessary consequence of the Principle of Special Relativity, that states that the laws of physics are invariant under the group of inhomogeneous Lorentz transformations SO(3,1), i.e., the Poincare group. Lorentz transformations leave invariant
the differential form
\[ ds^2 = -dt^2 + \frac{1}{c^2}(dx^2 + dy^2 + dz^2), \]
for the interval \( ds \) between two events. The corresponding form for the group \( SO(5,1) \) is a global form that we shall write in a convenient way and its invariance will constitute the starting point of the model.

It is the purpose of this note to study a quantum mechanics with the usual postulates but in which the Heisenberg algebra of canonical commutation relations and the Poincare algebra are replaced by the Lie algebra of the homogeneous Lorentz group \( SO(5,1) \). It arises from the hypothesis that the above group is the fundamental group of invariance for the laws of physics. The observables of the theory like position, time, momentum, energy, angular momentum and others are the generators of the algebra of the group. Neither position and time observables commute between them, nor momentum and energy observables. The algebra of Poincare quantum mechanics is recovered in the limit in which two parameters, that we physically interpret as the Hubble constant and the Planck mass are taken to zero and infinite respectively. We consider the equations that are satisfied by the spinor representation of the group.

We may realize the group \( SO(5,1) \) as maps onto itself of a hypersurface in \( R^{5,1} \)
\[- (ct)^2 + x^2 + y^2 + z^2 + \left( \frac{cu}{H} \right)^2 = \frac{\hbar}{H m_{pl}}, \]
with \( c \) and \( \hbar \) being the speed of light and Planck’s constant on one hand, and \( m \) and \( H \) being the Planck mass and the Hubble constant. The coordinates \( u \) and \( v \) are dimensionless. Even tough the coordinates \( x, y, z \) and \( t \) have dimensions of space and time respectively, they should be taken as the physical space-time variables.

The generators \( L_{ij} \) of the algebra satisfy the commutation relations
\[ L_{ij} = i(x_i \partial_j - x_j \partial_i), \]
\[ [L_{ij}, L_{kl}] = i(\eta_{jk} L_{il} - \eta_{ik} L_{jl} - \eta_{jl} L_{ik} + \eta_{il} L_{jk}), \]
with \( i, j, k, l = 0, 1, 2, 3, 5, 6 \). The nonzero elements of the metric \( \eta_{ij} \) are \( \eta_{00} = -1 \) and \( \eta_{11} = \eta_{22} = \eta_{33} = \eta_{55} = \eta_{66} = 1 \). Here \( x^i \equiv (x^0, x^1, x^2, x^3, x^5, x^6) \equiv (ct, x, y, z, \frac{cu}{H}, \frac{\hbar u}{m_{pl} c}). \)

The invariance of the form \( \mathbb{H} \) may be resumed in the following two postulates:
1. **The laws of physics are the same for all observers.**
2. **The speed of light \( c \), the planck constant \( \hbar \), the Hubble constant \( H \) and the Planck mass \( m_{pl} \) are the same for all observers.**

We associate with a given observer a set of observables. These physical observables are identified with the set of hermitian operators \( L_{ij} \) of the algebra of the Lorentz group \( SO(5,1) \) as follows:

1) The physical position three-vector \( \mathbf{X} = \{X^1, X^2, X^3\} \) and the physical time \( T = X^0 \) are represented by the three-vector operator \( \frac{\hbar}{m_{pl} c} \{L^{01}, L^{02}, L^{03}\} \) and \( \frac{\hbar}{m_{pl} c} L^{00} \) respectively.
2) The momentum three-vector \( \{P^1, P^2, P^3\} \) and the energy \( P^0 \) are represented by the three-vector operator \( \mathbf{P} = \frac{\hbar}{c} \{L^{11}, L^{22}, L^{33}\} \) and \( \hbar H L^{50} \) respectively.
3) The angular momentum three-vector \( \mathbf{J} = J^{23}, J^{31}, J^{12} \) is represented by the three-vector operator \( \hbar \{L^{23}, L^{31}, L^{12}\} \) and the three-vector operator \( \frac{\hbar}{c} \{L^{01}, L^{02}, L^{03}\} \) represents a three-vector observable \( \{J^{01}, J^{02}, J^{03}\} \) that we suggest is the acceleration three-vector.
4) Finally, the mass scale \( S \) is represented by \( \frac{\hbar}{m_{pl}} J^{56} \).

In natural units, where the speed of light \( c \), and the Planck constant \( \hbar \) are set to 1, the commutation relations \( \mathbb{H} \) read
\[ [P_\mu, P_\nu] = i H^2 J_{\mu\nu}, \]
\[ [J_{\mu\nu}, P_\rho] = -i \eta_{\rho\nu} P_\mu + i \eta_{\rho\mu} P_\nu, \]
\[ [X_\mu, X_\nu] = i \frac{m_{pl}}{\hbar} J_{\mu\nu}, \]
\[ [J_{\mu\nu}, X_\rho] = -i \eta_{\rho\nu} X_\mu + i \eta_{\rho\mu} X_\nu, \]

\[ [J_{\mu\nu}, J_{\rho\sigma}] = i \eta_{\rho\nu} J_{\mu\sigma} - i \eta_{\rho\mu} J_{\nu\sigma} - i \eta_{\sigma\nu} J_{\mu\rho} + i \eta_{\sigma\mu} J_{\nu\rho}, \]
\[ [X_\mu, P_\nu] = i \eta_{\nu\mu} S, \]
\[ [S, X_\mu] = \frac{i m_{pl}}{\hbar} P_\mu, \]
\[ [S, P_\mu] = -i H^2 X_\mu, \]
\[ [S, J_{\mu\nu}] = 0, \]

with \( \mu, \nu = 0, 1, 2, 3 \).

The components of the energy-momentum relations \( \mathbb{H} \) as well as the components of the position and time \( \mathbb{H} \) do not commute. They represent a departure of the Poincare algebra, but when the Planck mass \( m_{pl} \to \infty \) and the Hubble constant \( H \to 0 \) their commutation is restored and the set of commutation relations \( \mathbb{H} \) gives the Poincare algebra. This is the basic reason for the physical interpretation of the parameters \( m_{pl} \) and \( H \), because it is known that the Poincare algebra should be just an approximation at large scales, i.e., at the scale of the Hubble constant \( H \), and at short scales, i.e., at the scale of the Planck mass \( m_{pl} \).

The mass scale \( S \) in the commutation relations \( \mathbb{H} \) replaces the identity 1 in the Heisenberg algebra. The non-commutativity of the mass scale \( S \) as reflected in the relations \( \mathbb{H} \) means a departure of that algebra, only to be restored in the same limit \( m_{pl} \to \infty \) and \( H \to 0 \). In that case, the observable \( S \) commutes with all the generators of the algebra. If we consider an
irreducible representation of the group, it is a multiple to the identity.

On the other hand, the quadratic Casimir operator associated to the Lie algebra of the Lorentz group $SO(5,1)$ is

$$\frac{1}{m_{pl}^2} P^2 + \frac{H^2}{m_{pl}^2} X^2 + \frac{H^2}{m_{pl}^2} \eta_{\mu\nu} J^{\mu\nu} + S^2 = \frac{H^2}{m_{pl}^2} C_2, \quad (14)$$

with $P^2 = \eta_{\mu\nu} P^\mu P^\nu$ and $X^2 = \eta_{\mu\nu} X^\mu X^\nu$. $C_2$ is a multiple of the identity that depends on the representation of the group.

If we take the limit $H \to 0$ and $m_{pl} \to \infty$ but in such a way that $\frac{H}{m_{pl}} \to 0$, then the Eq. (14) becomes $S^2 = 0$. All the observables commute now, and then we may conclude saying that it reached the classical Poincare geometry.

If we take $H \to 0$ and $m_{pl} \to \infty$ but such that $\frac{H}{m_{pl}} \to s$, with $s$ finite, we get instead

$$\eta_{\mu\nu} J^{\mu\nu} + \frac{1}{s^2} S^2 = C_2, \quad (15)$$

It appears possible to choose $s$ such that the mass scale $S$ becomes the identity $I$. We may conclude now that we reached the algebra of Poincare quantum mechanics.

If we only take the Hubble constant $H \to 0$, leaving the Planck mass $m_{pl}$ finite, the Eq. (14) becomes

$$P^2 + \frac{m_{pl}^2}{4} S^2 = 0. \quad (16)$$

We may find the correspondence rules in the same limit. It is convenient to be at the point with coordinates $t = x = y = z = v = 0$, $u=1$. In such a limit, the observable $P_\mu$ becomes

$$P_\mu = H L_{65} = i H \left( \frac{u}{H} \partial_\mu - H x_\mu \partial_u \right) \to i \partial_\mu, \quad (17)$$

when the Hubble constant $H \to 0$. Similarly, the mass observable $S$ becomes,

$$S = H L_{65} = i \left( \frac{H v}{m_{pl}} \partial_u - \frac{m_{pl}}{H} x_\mu \partial_\mu \right) \to -i \partial_v, \quad (18)$$

If one assumes that the wave-function in the $v$-direction is a plane wave $\Phi(v) \sim \exp(iv)$, one obtains $S = I$ and the Eq. (16) becomes Einstein’s energy-momentum mass relation, where $m_{pl}$ takes the role of invariant mass of the representation. We may be confident that the physical interpretation of the observable $S$ as a mass scale makes sense.

On the other hand in the limit in which the Planck mass $m_{pl} \to \infty$, the observable $X_\mu$ becomes

$$X_\mu = \frac{L_{6\mu}}{m_{pl}} = i \left( \frac{v}{m_{pl}} \partial_\mu - m_{pl} x_\mu \partial_u \right) \to -ix_\mu \partial_u. \quad (19)$$

With the same behavior of the wave-function in the $v$-direction one obtains, $X_\mu = x_\mu$.

The relations (15)-(18) are the commutations relations for the set observables associated with a given observer. Another observer’s set satisfies the same commutations relations, pair of sets being related by an element of the group $SO(5,1)$. This means that going from one observer to another, the observables mix up. Given that, by construction, the sets includes space, time, and matter observables, we can not disentangle space-time from matter.

The formalism of quantum mechanics tell us how to proceed. We must construct the Hilbert space on which the observables act. A basis of the Hilbert space is obtained selecting from that set, a complete set of commuting observables. The algebra of the Lorentz group $SO(5,1)$ is of rank three and this means that the simultaneous measurement of only three observables completely determines a particular state of the physical system. This is in marked contrast with Poincare quantum mechanics where we need the four components of the energy-momentum to label the physical states. Moreover, here we may choose only one of the components to label states, e.g., the energy $P^0$. We are then forced to choose, one and only one of the components of the position three-vector, let’s say the component $Z$ and together with it, the corresponding component $J_z$ of the angular momentum.

Before turning to the consideration of the representation of the homogeneous Lorentz group analogous to the one introduced by Dirac, we may note in passing that the algebra of observables (15)-(18) is invariant under the automorphism defined by

$$X_\mu \to P_\mu, \quad J_{\mu\nu} \to J_{\mu\nu}, \quad P_\mu \to -X_\mu$$

$$H \leftrightarrow \frac{1}{m_{pl}}. \quad (20)$$

To write a linear equation invariant under the group $SO(5,1)$, we follow Dirac’s footsteps. We construct a set of six irreducible matrices $\Gamma_i$ that satisfy the anticommutation relations

$$\{ \Gamma_i, \Gamma_j \} = 2 \eta_{ij}. \quad (21)$$

The matrices $\sigma_{ij}$

$$\sigma_{ij} = -\frac{i}{2} [\Gamma_i, \Gamma_j] \quad (22)$$

satisfy the commutation relations (21) and hence provide an irreducible spinor representation [5].

We have at our disposal the differential operators $L_{ij}$ [6] and the matrices $\sigma_{ij}$ to work out a linearized Casimir expression. An equation that satisfies the requirements is

$$H \sigma^{ij} L_{ij} \Psi(x) = \lambda \Psi(x), \quad (23)$$

where $\lambda$ is a parameter with dimensions of energy.
This equation may be derived from the following Lagrangian density

\[ L(x) = -\bar{\Psi}(x)(H\sigma^{ij}L_{ij} - \lambda)\Psi(x) \]  

(24)

with \( \bar{\Psi}(x) = \Psi^+(x)s^6 \).

The lagrangian density (24) is invariant global gauge transformations of the field \( \Psi(x) \), \( \Psi(x) \rightarrow \exp(-i\alpha)\Psi(x) \). One can make it local, by introducing a gauge field \( A_{ij}(x) \) that transforms \( A_{ij}(x) \rightarrow A_{ij}(x) + iL_{ij}\alpha(x) \). The action \( S \) is then

\[ S = -\int d^6x \bar{\Psi}(x)(\sigma^{ij}D_{ij} - \lambda)\Psi(x) + \frac{1}{4g}F^2 \]  

(25)

with \( F_{ij\ kl} = [D_{ij}, D_{kl}] \) and \( D_{ij} = HL_{ij} + A_{ij}(x) \). Note that the coupling constant \( g \) has dimension of length square. The Euler-Lagrange equations are

\[ (H\sigma^{ij}L_{ij} - \lambda)\Psi(x) = \sigma^{ij}A_{ij}(x)\Psi(x), \]  

(26)

\[ HL_{ij}F_{ij\ kl}(x) = g \bar{\Psi}\sigma_{kl}\Psi(x). \]  

(27)

The action (25) has in-built both an ultraviolet cutoff, the Planck mass \( m_{pl} \), as well as, an infrared one, the Hubble constant \( H \), that opens the door to the speculation that the resulting quantum field theory is finite.

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[1] Space-Time-Matter is the title of Weyl’s classic book. H. Weyl, Space-Time-Matter, (New York, Dover Publications, 1950)

[2] We use the word rotations for pseudo-rotations also.

[3] We avoid taking the track that maintains the commutativity and leads to the conformal group, but at the cost of requiring a nonlinear realization for its physical interpretation.

[4] It has naturally the correct dimensions.

[5] These matrices are in principle \( 2^3 \)-dimensional.

[6] It is straightforward to prove that if the lagrangian density \( L \) is a functional of \( \phi(x) \) and \( L_{ij}\phi(x) \) only, then the Euler-Lagrange field equations are \( L_{ij}\frac{\partial L}{\partial L_{ij}\phi(x)} - \frac{\partial L}{\partial \phi(x)} = 0 \).