ON THE BRIDGE NUMBER OF KNOT DIAGRAMS WITH MINIMAL CROSSINGS

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Abstract. Given a diagram $D$ of a knot $K$, we consider the number $c(D)$ of crossings and the number $b(D)$ of overpasses of $D$. We show that, if $D$ is a diagram of a nontrivial knot $K$ whose number $c(D)$ of crossings is minimal, then $1 + \sqrt{1 + c(D)} \leq b(D) \leq c(D)$. These inequalities are shape in the sense that the upper bound of $b(D)$ is achieved by alternating knots and the lower bound of $b(D)$ is achieved by torus knots. The second inequality becomes an equality only when the knot is an alternating knot. We prove that the first inequality becomes an equality only when the knot is a torus knot.

1. Introduction

A diagram $D$ of a knot $K$ is a nice representative of the knot type $[K]$ which is the isotopy class of $K$. It can be obtained from a regular planar projection (or simply, regular projection) $P$ of $K$ as follows. Let us take a sufficiently small neighborhood of each double point of $P$ so that the intersection of the neighborhood and $P$ is of the ‘shape X’ on the plane. Then modify the interior of each neighborhood so that we get a knot $D$ which is isotopic to $K$ and regularly projected to $P$. In this sense, a knot diagram can be ‘almost planar’, i.e., it lies in the plane except for a sufficiently small neighborhood of each double point of the regular planar projection.

Tabulation of knots is usually done by listing knots using their diagrams of minimal numbers of crossings. In general, such a minimal knot diagram is not unique within its isotopy class. Can this nonuniqueness be fixed by considering some other quantities associated with knot diagrams? In this paper, we study the number of overpasses of a minimal knot diagram. An immediate concern in this investigation is the relationship between the number of crossings and the number of overpasses of a knot diagram. As it turns out, the number of crossings can be estimated from below and above by that of overpasses if the number of crossings is minimal among all diagrams of the same knot type (Theorem 2.9).

As an interesting consequence of this estimation of the minimal crossing number by the number of overpasses, we may put two nice and relatively well understood classes of knots, that of alternating knots and that of $(k - 1, \pm k)$-torus knots, on the opposite
extremes of a measurement of nonalternatingness of knot diagrams (Theorem 2.10). Under this measurement, we prove that the ‘most nonalternating’ knot diagrams are exactly the standard minimal \((k - 1, \pm k)\)-torus knot diagrams (Theorem 3.1).

Notice that the minimal number of overpasses of a knot was the classical bridge number of a knot, first studied by Schubert in [5], where the effect of various operations on knots (satellite, cabling, connected sum) on this number was investigated.

Throughout this paper, all knots are oriented and lie in the 3-dimensional sphere \(S^3\). Also, all knots are tame, i.e., they are isotopic to polygonal or smooth knots. Hence, every knot has a diagram whose number of crossings is finite. For convenience, we distinguish knot diagrams and knots. From now on, a knot means an isotopy knot type \([D]\) for some knot diagram \(D\). We will refer the reader to textbooks of knot theory (such as [1] or [2]) for standard terminologies. Also, we assume the well ordering property of the set \(\mathbb{N} \cup \{0\}\) of nonnegative integers to guarantee existence of the smallest element of any nonempty subset of \(\mathbb{N} \cup \{0\}\).

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2. Minimal crossings and bridge number of knot diagrams

The number of crossings of a knot diagram \(D\) is denoted by \(c(D)\). For each knot \(K\), we denote \(\min \{c(D) \mid D \text{ is a diagram of } K\}\) by \(c(K)\). Note that we may assume a knot diagram \(D\) lies in the plane by indicating ‘overcrossings’ and ‘undercrossings’. A ‘crossing’, in normal sense, of a knot diagram \(D\) means a ‘double point’ of the regular projection of \(D\). Hence, \(c(D)\) is the number of all double points of the regular projection of \(D\). If \(x\) is a crossing of a knot diagram \(D\), we denote the overcrossing and the undercrossing of \(D\) projected to \(x\) by \(x_+\) and \(x_-\), respectively, where \(x_+\) is above \(x\) and \(x_-\) is below \(x\). Alternatively, we may regard a crossing of \(D\) as a pair of two points, overcrossing and undercrossing, in \(D\) which are projected to the same double point. In this case, a crossing is considered as the preimage of a double point under the projection map.

**Proposition 2.1.** Let \(D\) be a knot diagram with \(c(D) \geq 1\). Then there is a unique positive integer \(k\) such that there is a finite sequence

\[s_1, f_1, s_2, f_2, \ldots, s_k, f_k\]

of \(2k\) points of \(D\), each of which is neither an overcrossing nor an undercrossing of \(D\), such that

\[[s_1, f_1], [s_2, f_2], \ldots, [s_{k-1}, f_{k-1}], [s_k, f_k]\]

and

\[[f_1, s_2], [f_2, s_3], \ldots, [f_{k-1}, s_k], [f_k, s_1]\]
are the overpasses and the underpasses of $D$ with respect to $s_1, f_1, s_2, f_2, \ldots, s_k, f_k$, respectively, where $[s_i, f_i]$, for each $i \in \{1, ..., k\}$, is the closed arc of $D$ from $s_i$ to $f_i$ which contains at least one overcrossing but has no undercrossing; similarly, $[f_i, s_{i+1}]$, for each $i \in \{1, ..., k\}$, with the subscript counted modulo $k$, is the closed arc of $D$ from $f_i$ to $s_{i+1}$ which contains at least one undercrossing but has no overcrossing.

Proof. We may assume $c(D) \geq 2$. If $c(D) = 1$, then $D$ is of the ‘shape 8’, and hence, $k = 1$.

Existence: We will construct a finite sequence as stated above. The following procedure is one way to get it. Let us start at a point $*$ on $D$ slightly before an undercrossing and go along with $D$ until arriving at the first overcrossing $a_1$ from $*$. Let $a_2$ be the undercrossing just before $a_1$. Take a point $s_1$ of $D$ between $a_2$ and $a_1$. Then go along with $D$ from $s_1$ until arriving at the first undercrossing $a_3$ from $s_1$, and let $a_4$ be the overcrossing just before $a_3$. Take a point $f_1$ of $D$ between $a_4$ and $a_3$. We may now repeat this procedure to take the other points $s_2, f_2, s_3, f_3, \ldots, s_k, f_k$ until there is no overcrossing between $f_k$ and $*$. Since the number $c(D)$ of crossings of $D$ is finite, $k$ must be finite. By its construction, the sequence of points $s_1, f_1, s_2, f_2, \ldots, s_k, f_k$ of $D$ is a one as stated in the proposition.

Uniqueness: Suppose that $l$ is a positive integer and $s'_1, f'_1, s'_2, f'_2, \ldots, s'_l, f'_l$ is a sequence of $D$ as stated in the proposition. Let

$$a_2, a_1, a_4, a_3, \ldots, a_{2(2k-1)}, a_{2(2k-1)-1}, a_{2(2k)}, a_{2(2k)-1}$$

be the sequence of points on $D$ as described above. Then $s'_1$ must be contained in an arc of $D$ which contains $s_i$ for some $i \in \{1, ..., k\}$ but no overcrossing and no undercrossing of $D$. In other words, $s'_1$ must be between $a_{2(2i-1)}$ and $a_{2(2i-1)-1}$ for some $i \in \{1, ..., k\}$. Hence, $f'_1$ must be between $a_{2(2i)}$ and $a_{2(2i)-1}$. Keep going on like this. If $i = 1$, then $f'_1$ must be between $a_{2(2k)}$ and $a_{2(2k)-1}$, and if $1 < i \leq k$, then $f'_i$ must be between $a_{2(2(i-1))}$ and $a_{2(2(i-1))-1}$. Hence, we have $l = k$. This proves the uniqueness of $k$. 

Such a sequence $s_1, f_1, s_2, f_2, \ldots, s_k, f_k$ as in Proposition 2.1 is said to be an over-underpass sequence of $D$. Since any over-underpass sequence of $D$ consists of $2k$ points, the number of overpasses (or underpasses) with respect to any over-underpass sequence of $D$ is $k$. Hence, we define the number of overpasses (or underpasses) of the knot diagram $D$ as $k$, and denote it by $b(D)$, sometimes called the length of over-underpass sequence. If a knot diagram $D$ has no crossing, we define $b(D)$ as 0. Also, for each knot $K$, we denote $\min\{b(D) \mid D \text{ is a diagram of } K\}$ by $b(K)$. This number $b(K)$ is called the bridge number of $K$.

Notice that (1) $c(D)$ and $b(D)$ are plane isotopy invariants of knot diagrams, i.e., if $D_1$ and $D_2$ are plane isotopic knot diagrams, then $c(D_1) = c(D_2)$ and $b(D_1) = b(D_2)$; (2) $c(K)$ and $b(K)$ are isotopy invariants of knots, i.e., if $K_1$ and $K_2$ are isotopic knots, then $c(K_1) = c(K_2)$ and $b(K_1) = b(K_2)$.
**Corollary 2.2.** If \( b(D) \leq 1 \), then \( D \) is a diagram of a trivial knot. Therefore, a knot \( K \) is trivial if and only if \( K \) has a diagram \( D \) with \( b(D) \leq 1 \).

Obviously, for any positive integer \( k \), there is a diagram of a trivial knot whose number of overpasses is greater than \( k \). On the other hand, given a knot diagram \( D \) with at least one crossing, we can add crossings to \( D \) as many as we want without changing the knot type and the number of overpasses of \( D \). For example, take a sufficiently small arc of \( D \) from \( s_1 \) to a point between \( s_1 \) and the first overcrossing of \( D \) from \( s_1 \), and twist it alternatively so that the number of overpasses of \( D \) is not changed. Or, we may modify the interior of a sufficiently small neighborhood of a crossing of \( D \) to achieve the goal of increasing \( c(D) \) arbitrarily while keeping \( b(D) \) and the knot type of \( D \) fixed. Hence, we have the following corollary.

**Corollary 2.3.** If \( D \) is a diagram of a knot \( K \) such that \( c(D) \geq 1 \), then for every positive integer \( n \), there is a diagram \( D' \) of \( K \) such that \( b(D') = b(D) \) and \( c(D') \geq c(D) + n \).

Remark that the number of crossings of a knot diagram with a minimal number of overpasses can be arbitrarily large.

**Lemma 2.4.** \( b(D) \leq c(D) \) for any knot diagram \( D \). The equality holds if and only if \( D \) is an alternating knot diagram. Furthermore, \( b(K) \leq c(K) \) for any knot \( K \).

*Proof.* Suppose that \( b(D) = k \geq 1 \) and \( s_1, f_1, s_2, f_2, ..., s_k, f_k \) is an over-underpass sequence of \( D \). Let \( m_i \) be the number of overcrossings of \( D \) on \([s_i, f_i]\) for each \( i \in \{1, ..., k\} \), \( n_i \) the number of undercrossings of \( D \) on \([f_i, s_{i+1}]\) for each \( i \in \{1, ..., k-1\} \), and \( n_k \) the number of undercrossings of \( D \) on \([f_k, s_1]\). Then

\[
2c(D) = \sum_{i=1}^{k} m_i + \sum_{i=1}^{k} n_i \geq \sum_{i=1}^{k} 1 + \sum_{i=1}^{k} 1 = 2b(D).
\]

\( D \) is an alternating knot diagram if and only if every overpass and underpass has exactly one overcrossing and undercrossing, respectively, i.e., \( m_i = n_i = 1 \) for each \( i \in \{1, ..., k\} \).

Also, it follows immediately that \( b(K) \leq c(K) \) for any knot \( K \). \( \Box \)

The following lemma is the first key to prove the first main theorem of this paper (Theorem 2.9). By this lemma, if an overpass of a knot diagram \( D \) with respect to an over-underpass sequence of \( D \) crosses an underpass more than once, then the number of crossings of \( D \) is not minimal any more.

**Lemma 2.5.** If \( D \) is a minimal diagram of a knot \( K \) with respect to crossings, i.e., \( c(D) = c(K) \), \( b(D) \) is a nonnegative integer \( k \), and \( s_1, f_1, s_2, f_2, ..., s_k, f_k \) is an over-underpass sequence of \( D \), then every overpass of \( D \) with respect to \( s_1, f_1, s_2, f_2, ..., s_k, f_k \) crosses each underpass at most once.
Proof. We may assume $K$ is nontrivial. If $K$ is trivial, then $c(D) = c(K) = 0$, hence, $b(D) = 0$. Let $D$ be a diagram of a knot $K$ such that $c(D) = c(K)$. Suppose that $b(D) = k \geq 1$ and $s_1, f_1, s_2, f_2, ..., s_k, f_k$ is an over-underpass sequence of $D$. Then, by Lemma 2.4, $c(D) \geq k \geq 1$. Let $o_i = [s_i, f_i]$ and $u_i = [f_i, s_{i+1}]$ for each $i \in \{1, ..., k-1\}$, and let $o_k = [s_k, f_k]$ and $u_k = [f_k, s_1]$. Suppose that the number of crossings between $o_i$ and $u_j$ is at least 2 for some $i, j \in \{1, ..., k\}$. Let $x_1$ and $x_2$ be two crossings of $D$ such that $x_1^+$ and $x_2^+$ are the first and the second overcrossings from $s_i$ among all overcrossings between $o_i$ and $u_j$, respectively, and let $y_1$ and $y_2$ be two crossings of $D$ such that $y_1^-$ and $y_2^-$ are the first and the second undercrossings from $f_j$ in $\{x_1^-, x_2^-, \}$, respectively. Then either $y_1 = x_1$, $y_2 = x_2$ or $y_1 = x_2$, $y_2 = x_1$. There are two different diagrams for each case. One of them for the case of $y_1 = x_1$, $y_2 = x_2$ is shown in Figure 1. It should be not hard for the reader to figure out the other diagrams.

Let $r$ be the number of overcrossings on the arc $\overrightarrow{x_1^+ x_2^+}$ of $o_i$ from $x_1^+$ to $x_2^+$, and let $s$ be the number of undercrossings on the arc $\overrightarrow{y_1^- y_2^-}$ of $u_j$ from $y_1^-$ to $y_2^-$. To prove this lemma, for each of the cases $r \leq s$ and $r > s$, we will construct a diagram $D'$ of $K$ by modifying an arc of $o_i$ or $u_j$ such that $c(D') < c(D)$. Temporarily, we regard the knot diagram $D$ as its regular projection. Hence, $D$ lies in the plane, and all overcrossings and all undercrossings of $D$ are the double points of $D$ as the regular projection. Consider the $r$ double points $d_1, d_2, ..., d_r$ on the arc $\overrightarrow{x_1^+ x_2^+}$ of $o_i$ from $x_1^+$ to $x_2^+$ such that $d_1 < d_2 < ... < d_r$ with respect to the order from $s_i$ and the $s$ double points $e_1, e_2, ..., e_s$ on the arc $\overrightarrow{y_1^- y_2^-}$ of $u_j$ from $y_1^-$ to $y_2^-$ such that $e_1 < e_2 < ... < e_s$ with respect to the order from $f_j$. Then $x_1 = d_1$, $x_2 = d_r$ and $y_2 = e_1$, $y_1 = e_s$. Notice that we can take a sufficiently small positive real number $\varepsilon$, an $\varepsilon$-neighborhood $U_{o_i, \varepsilon}$ of $\overrightarrow{x_1^+ x_2^+}$, and an $\varepsilon$-neighborhood $V_{u_j, \varepsilon}$ of $\overrightarrow{y_1^- y_2^-}$ such that $U_{o_i, \varepsilon} \cap \{s_1, f_1, s_2, f_2, ..., s_k, f_k\} = \emptyset$, $V_{u_j, \varepsilon} \cap \{s_1, f_1, s_2, f_2, ..., s_k, f_k\} = \emptyset$, the set of all double points of $D$ contained in $U_{o_i, \varepsilon}$ is $\{d_1, d_2, ..., d_r\}$, the set of all double points of $D$ contained in $V_{u_j, \varepsilon}$ is $\{e_1, e_2, ..., e_s\}$, and, for every positive real number $\varepsilon' \leq \varepsilon$, $|Bd(U_{o_i, \varepsilon'}) \cap D| = 2(r + 1)$ and $|Bd(V_{u_j, \varepsilon'}) \cap D| = 2(s + 1)$, where $Bd(U_{o_i, \varepsilon'})$ and $Bd(V_{u_j, \varepsilon'})$ are the boundaries of $U_{o_i, \varepsilon'}$ and $V_{u_j, \varepsilon'}$, respectively. Hence, we may assume that $U_{o_i, \varepsilon} \cap D = \overrightarrow{a_1 a_2} \cup (l_1 \cup l_2 \cup ... \cup l_r)$ and $V_{u_j, \varepsilon} \cap D = \overrightarrow{b_1 b_2} \cup (l' \cup l' \cup ... \cup l')$, where $a_1$ and $a_2$ are the first and the second points from $s_i$ in $Bd(U_{o_i, \varepsilon}) \cap o_i$, respectively, $b_1$ and $b_2$ are the first and the second points from $f_j$ in $Bd(V_{u_j, \varepsilon}) \cap u_j$, respectively, $l_t$ is an arc of the underpass passing through $d_t$ whose endpoints are on $Bd(U_{o_i, \varepsilon})$ for each $t \in \{1, ..., r\}$, and $l'$ is an arc of the overpass passing through $e_t$ whose endpoints are on $Bd(V_{u_j, \varepsilon})$ for each $t \in \{1, ..., s\}$. Remark that $\{l_1, ..., l_r\}$ is pairwise disjoint, $l_t \cap o_i = \{d_t\}$ and $|l_t \cap Bd(U_{o_i, \varepsilon})| = 2$ for each $t \in \{1, ..., r\}$, and $|Bd(U_{o_i, \varepsilon}) \cap o_i| = 2$; similarly, $\{l', ..., l'\}$ is pairwise disjoint, $l' \cap u_j = \{e_t\}$ and $|l' \cap Bd(V_{u_j, \varepsilon})| = 2$ for each $t \in \{1, ..., s\}$, and $|Bd(V_{u_j, \varepsilon}) \cap u_j| = 2$. 
Let \( p_1, p_2, p_3, p_4 \) be the first, the second, the third, the fourth points from \( f_j \) in \( \text{Bd}(U_{o_i, \epsilon}) \cap (l_1 \cup l_r) \), respectively, and let \( \alpha \) be a point on \( p_1p_2u_j \) between \( p_1 \) and \( y_1 \). Draw an arc \( A \) in \( U_{o_i, \epsilon} \) starting at \( \alpha \) so that \( A \) intersects \( l_t \) transversely only once for each \( t \in \{1, \ldots, r\} \) but \( A \) does not intersect \( o_i \). Then we take \( \beta \) as the intersecting point of \( A \) and \( p_3p_4u_j \) and denote the arc of \( A \) from \( \alpha \) to \( \beta \) by \( \alpha\beta \). Similarly, let \( q_1, q_2, q_3, q_4 \) be the first, the second, the third, the fourth points from \( s_i \) in \( \text{Bd}(V_{u_j, \epsilon}) \cap (l^1 \cup l^s) \), respectively, and let \( \gamma \) be a point on \( q_1q_2o_i \) between \( q_1 \) and \( x_1 \). Draw an arc \( B \) in \( V_{u_i, \epsilon} \) starting at \( \gamma \) so that \( B \) intersects \( l^t \) transversely only once for each \( t \in \{1, \ldots, s\} \) but \( B \) does not intersect \( u_j \). Then we take \( \delta \) as the intersecting point of \( B \) and \( q_3q_4o_i \) and denote the arc of \( B \) from \( \gamma \) to \( \delta \) by \( \gamma\delta \).

From now on, \( D \) is the diagram of \( K \) again, i.e., \( D \) is not a regular projection of a knot but a knot diagram. Let \( \alpha\beta \) be a regularly projecting arc in \( \mathbb{R}^3 \) whose regular projection is \( \overrightarrow{\alpha\beta} \) such that \( \overrightarrow{\alpha\beta} \) has no overcrossing, and let \( \overrightarrow{\gamma\delta} \) be a regularly projecting arc in \( \mathbb{R}^3 \) whose regular projection is \( \overrightarrow{\gamma\delta} \) such that \( \overrightarrow{\gamma\delta} \) has no undercrossing. Then

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Proof of Lemma 2.5: Case 1.(1) and Case 2.(2) when \( y_1 = x_1, y_2 = x_2 \).}
\end{figure}
Case 1. $r \leq s$: Let $D' = (D - \alpha \beta_{u_j}) \cup \alpha \beta$. Then $D'$ is a diagram of $K$.

(1) If $p_1p_{2u_j}$ and $p_3p_{4u_j}$ make the same sign with $o_i$, then
\[ c(D') = c(D) - (s - 1) + (r - 2), \text{ hence, } c(D) = c(D') + (s - r) + 1 > c(D'); \]

(2) If $p_1p_{2u_j}$ and $p_3p_{4u_j}$ make opposite signs with $o_i$, then
\[ c(D') = c(D) - s + (r - 2), \text{ hence, } c(D) = c(D') + (s - r) + 2 > c(D'). \]

Case 2. $r > s$: Let $D' = (D - \gamma \delta^{\alpha}) \cup \gamma \delta$. Then $D'$ is a diagram of $K$.

(1) If $\frac{q_1}{q_2} \gamma^{\alpha}$ and $\frac{q_3}{q_4} \delta^{\alpha}$ make the same sign with $u_j$, then
\[ c(D') = c(D) - (r - 1) + (s - 2), \text{ hence, } c(D) = c(D') + (r - s) + 1 > c(D'); \]

(2) If $\frac{q_1}{q_2} \gamma^{\alpha}$ and $\frac{q_3}{q_4} \delta^{\alpha}$ make opposite signs with $u_j$, then
\[ c(D') = c(D) - r + (s - 2), \text{ hence, } c(D) = c(D') + (r - s) + 2 > c(D'). \]

Therefore, we have a diagram $D'$ of $K$ such that $c(D') < c(D)$. This is a contradiction to $c(D) = c(K)$.

Now, we prove the second key lemma for Theorem 2.9 by a similar idea as the one used in the proof of Lemma 2.5. By this lemma, if an overpass of a knot diagram $D$ with respect to an over-underpass sequence of $D$ crosses an adjacent underpass or an underpass crosses an adjacent overpass, then the number of crossings of $D$ is not minimal.

**Lemma 2.6.** If $D$ is a minimal diagram of a knot $K$ with respect to crossings, $b(D) = k \geq 2$, and $s_1, f_1, s_2, f_2, ..., s_k, f_k$ is an over-underpass sequence of $D$, then no overpass of $D$ with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ crosses its adjacent underpasses and no underpass crosses its adjacent overpasses, where the adjacent underpasses of the $i$-th overpass $o_i$ are the $(i - 1)$-th and the $i$-th underpasses $u_{i-1}$ and $u_i$ for each $i \in \{2, ..., k\}$; the adjacent underpasses of $o_1$ are $u_k$ and $u_1$; the adjacent overpasses of $u_i$ are $o_i$ and $o_{i+1}$ for each $i \in \{1, ..., k - 1\}$; the adjacent overpasses of $u_k$ are $o_k$ and $o_1$.

**Proof.** We may assume that each overpass of $D$ with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ crosses each underpass at most once by Lemma 2.5.

Suppose that there is $i \in \{1, ..., k\}$ such that the $i$-th overpass $o_i = [s_i, f_i]$ of $D$ with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ crosses the $i$-th underpass $u_i = [f_i, s_{i+1}]$, with the subscript counted modulo $k$. Let $x$ be the crossing of $D$ between $o_i$ and $u_i$. Let $y$ be the crossing of $D$ such that $y_+$ is the overcrossing of $o_i$ just before $f_i$, and let $z$ be the crossing of $D$ such that $z_-$ is the undercrossing of $u_i$ just after $f_i$. Note that $x$, $y$, $z$ need not be distinct. However, if some of them are identical, then, by Reidemeister moves, we can reduce the crossing easily. Let $r$ be the number of overcrossings on the arc $\overrightarrow{x+y}$ of $o_i$ from $x_+$ to $y_+$, and let $s$ be the number of undercrossings on the arc $\overrightarrow{z-x}$ of $u_i$ from $z_-$ to $x_-$. Temporarily, we regard the knot diagram $D$ as its regular
projection as we did in the proof of Lemma 2.5. Let us take an \( \epsilon \)-neighborhood \( U_{a_i, \epsilon} \) of \( \overrightarrow{xy} \) and an \( \epsilon \)-neighborhood \( V_{a_i, \epsilon} \) of \( \overrightarrow{zy} \) for a sufficiently small positive real number \( \epsilon \) as described in the proof of Lemma 2.5. Let \( a_1 \) and \( a_2 \) be the first and the second points from \( s_i \) in \( Bd(U_{a_i, \epsilon/2}) \cap \partial_i \), respectively, and let \( a_3 \) and \( a_4 \) be the first and the second points from \( s_i \) in \( Bd(U_{a_i, \epsilon/2}) \cap \partial_i \), respectively. Similarly, let \( b_1 \) and \( b_2 \) be the first and the second points from \( s_i \) in \( Bd(V_{a_i, \epsilon/2}) \cap \partial_i \), respectively, and let \( b_3 \) and \( b_4 \) be the first and the second points from \( s_i \) in \( Bd(V_{a_i, \epsilon/2}) \cap \partial_i \), respectively. Let \( \overrightarrow{a_2a_4} \) be the arc of \( Bd(U_{a_i, \epsilon/2}) \) from \( a_2 \) to \( a_4 \) such that \( |\overrightarrow{a_2a_4} \cap D| = (r - 1) + 2 \), and let \( \overrightarrow{b_1b_3} \) be the arc of \( Bd(V_{a_i, \epsilon/2}) \) from \( b_1 \) to \( b_3 \) such that \( |\overrightarrow{b_1b_3} \cap D| = (s - 1) + 2 \).

From now on, \( D \) is the diagram of \( K \) again. Let \( \overrightarrow{a_2a_4} \) be a regularly projecting arc in \( R^3 \) whose regular projection is \( \overrightarrow{a_2a_4} \) such that \( \overrightarrow{a_2a_4} \) has no overcrossing, and let \( \overrightarrow{b_1b_3} \) be a regularly projecting arc in \( R^3 \) whose regular projection is \( \overrightarrow{b_1b_3} \) such that \( \overrightarrow{b_1b_3} \) has no undercrossing. Then

Case 1. \( r \leq s \): Let \( D' = (D - (\overrightarrow{a_2a_4} \cup \overrightarrow{b_1b_3})) \cup \overrightarrow{a_2a_4} \). Then \( D' \) is a diagram of \( K \) and \( c(D') = c(D) - s + (r - 1) \), hence, \( c(D) = c(D') + (s - r) + 1 > c(D') \).

Case 2. \( r > s \): Let \( D' = (D - (\overrightarrow{b_1b_3} \cup \overrightarrow{a_2a_4})) \cup \overrightarrow{b_1b_3} \). Then \( D' \) is a diagram of \( K \) and \( c(D') = c(D) - r + (s - 1) \), hence, \( c(D) = c(D') + (r - s) + 1 > c(D') \).

Therefore, we have a diagram \( D' \) of \( K \) such that \( c(D') < c(D) \). This is a contradiction to \( c(D) = c(K) \).

Figure 2. Proof of Lemma 2.6.
As the other case, if there is $i \in \{1,...,k\}$ such that the $i$-th underpass $u_i = [f_i, s_{i+1}]$ of $D$ with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ crosses the $(i + 1)$-th underpass $o_{i+1} = [s_{i+1}, f_{i+1}]$, with the subscript counted modulo $k$, then we can construct a diagram $D'$ of $K$ such that $c(D') < c(D)$ by the same argument above. This proves the lemma. □

The following two lemmas are immediate consequences of Lemma 2.6 and the other keys for Theorem 2.9.

**Lemma 2.7.** If $D$ is a minimal diagram of a knot $K$ with respect to crossings, $b(D)$ is a positive integer $k$, and $s_1, f_1, s_2, f_2, ..., s_k, f_k$ is an over-underpass sequence of $D$, then there is no overpass of $D$ with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ which crosses all underpasses.

*Proof.* Suppose that an overpass of $D$ with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ crosses all underpasses. Then the overpass crosses its adjacent underpasses. This is a contradiction to $c(D) = c(K)$ by Lemma 2.6. □

**Lemma 2.8.** If $D$ is a minimal diagram of a knot $K$ with respect to crossings, $b(D) = k \geq 2$, and $s_1, f_1, s_2, f_2, ..., s_k, f_k$ is an over-underpass sequence of $D$, then every overpass of $D$ with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ crosses at most $k - 2$ underpasses.

*Proof.* Suppose that an overpass $o$ of $D$ with respect to $s_1, f_1, s_2, f_2, ..., s_k, f_k$ crosses $k - 1$ underpasses. Then, by Lemma 2.5, $o$ crosses each of the $k - 1$ underpasses exactly once. Since every overpass has two adjacent underpasses and $o$ crosses $k - 1$ underpasses, $o$ must cross at least one of its adjacent underpasses. This is a contradiction to $c(D) = c(K)$ by Lemma 2.6. □

**Theorem 2.9.** If $D$ is a minimal diagram of a knot $K$ with respect to crossings, then $b(D) \leq c(D) \leq b(D)(b(D) - 2)$.

*Proof.* Let $D$ be a diagram of a knot $K$ such that $c(D) = c(K)$. If $K$ is a trivial knot, then $D$ is a simple closed curve on the plane, hence, $c(D) = b(D) = 0$. Remark that $b(D)$ can not be 1. If $b(D) = 1$, then, by Corollary 2.2, $D$ represents a trivial knot, i.e., $K$ is a trivial knot, hence, $c(D) = c(K) = 0$. This contradicts to Lemma 2.4. Next, we claim that $b(D)$ can not be 2 if $D$ is a minimal diagram of a nontrivial knot $K$ with respect to crossings. This follows from the same argument in the proof of Lemma 2.8. Suppose that $K$ is a nontrivial knot, $c(D) = c(K)$, $b(D) = 2$, and $s_1, f_1, s_2, f_2$ is an over-underpass sequence of $D$. Then each overpass of $D$ with respect to $s_1, f_1, s_2, f_2$ must cross one of the underpasses. However, all underpasses are adjacent to each overpass. Suppose that $b(D) = k \geq 3$ and $s_1, f_1, s_2, f_2, ..., s_k, f_k$ is an over-underpass sequence of $D$. We may assume that the underpasses $[s_1, f_1], [s_2, f_2], ..., [s_{k-1}, f_{k-1}], [s_k, f_k]$ are disjoint closed arcs on the projection plane of $D$ and the underpasses $[f_1, s_2], [f_2, s_3], ..., [f_{k-1}, s_k], [f_k, s_1]$ are also...
disjoint closed arcs on it. Remark that the number of crossings of $D$ is the number of intersections of projections of overpasses and underpasses of $D$. Since we have $k$ overpasses and $k$ underpasses, by Lemma 2.5 and Lemma 2.8, there are at most $k(k - 2)$ crossings among overpasses and underpasses. □

Notice that $c(K) \leq b(K)(b(K) - 2)$ does not hold in general. An example can be given with $K$ being the $(2,p)$-torus knot. We have $b(K) = 2$ in this case.

The following theorem is an immediate consequence of Theorem 2.9 if $K$ is nontrivial. It means that the number of overpasses of a minimal knot diagram with respect to crossings can also be estimated by the number of crossings. Once more, in this case, the number of overpasses needs not be minimal and is at least 3 as shown in the proof of Theorem 2.9.

**Theorem 2.10.** If $K$ is a nontrivial knot and $D$ is a minimal diagram of $K$ with respect to crossings, then $1 + \sqrt{1 + c(D)} \leq b(D) \leq c(D)$.

### 3. The Most Nonalternating Knots are $(k - 1, \pm k)$-Torus Knots

Theorem 2.10 provides us the most optimal lower bound for the number of overpasses of $D$ when $D$ is a minimal knot diagram with respect to crossings. Here, an interesting problem occurs. In Lemma 2.4, we showed that a knot diagram $D$ is alternating if and only if $c(D) = b(D)$. Also, by Kauffman [3], an alternating knot has a minimal knot diagram with respect to crossings which is alternating. Hence, the second inequality of Theorem 2.10 becomes an equality if and only if the knot is an alternating knot. On the other hand, as shown by Murasugi [4], the standard diagram $D$ of a $(k - 1, \pm k)$-torus knot is a minimal knot diagram with respect to crossings and $c(D) = k(k - 2)$. So the first inequality of Theorem 2.10 becomes an equality for the $(k - 1, \pm k)$-torus knot with $b(D) = k$. Is the converse true under the condition that $D$ is a minimal knot diagram with respect to crossings? We prove this as the last theorem of this paper.

**Theorem 3.1.** If $D$ is a minimal diagram of a nontrivial knot $K$ with respect to crossings and $c(D) = b(D)(b(D) - 2)$, then $D$ is the standard diagram of either the $(b(D) - 1, b(D))$-torus knot or the $(b(D) - 1, -b(D))$-torus knot. Hence, $K$ is the $(b(D) - 1, \pm b(D))$-torus knot.

**Proof.** Let $D$ be a minimal diagram of a nontrivial knot $K$ with respect to crossings such that $c(D) = b(D)(b(D) - 2)$. Suppose that $b(D) = k$. Then $k \geq 3$ (See the proof of Theorem 2.9). In order to prove this theorem, we will consider all possible knot diagrams satisfying the hypothesis and show that they can only be the standard diagrams of either the $(k - 1, k)$-torus knot or the $(k - 1, -k)$-torus knot depending on signs of crossings. For convenience, we regard the knot diagram $D$ as its regular projection here. Also, to visualize overpasses and underpasses clearly, we imagine blue
and red colors for overpasses and underpasses, respectively. In this sense, crossings are only the intersections of blue arcs and red arcs on the plane, i.e., crossings are purple!

Now, let us start drawing all possible knot diagrams satisfying the hypothesis. Notice that such knot diagrams must satisfy the following 3 rules:

Rule 1: Every overpass and every underpass intersect each underpass and each overpass at most once, respectively (By Lemma 2.5);
Rule 2: No overpass and no underpass intersect its adjacent underpasses and its adjacent overpasses, respectively (By Lemma 2.6);
Rule 3: Every overpass and every underpass intersect $k - 2$ underpasses and $k - 2$ overpasses, respectively (By Lemma 2.8).

If some overpass has less than $k - 2$ overcrossings, then at least one overpass has more than $k - 2$ overcrossings; similarly, if some underpass has less than $k - 2$ undercrossings, then at least one underpass has more than $k - 2$ undercrossings. This is a contradiction to Lemma 2.8. From now on, we describe drawing the knot diagrams. Let us take a point $s_1$ on the plane and draw the first overpass $o_1 = [s_1, f_1]$. Draw the first underpass $u_1 = [f_1, s_2]$ and the second overpass $o_2 = [s_2, f_2]$. Until now, we can not have any crossing by Rule 2. When we draw the second underpass $u_2$ from $f_2$, we must make $u_2$ intersect $o_1$ first by Rule 2,3. Notice that we have only 2 cases that $u_2$ intersects $o_1$ first as follows.

Case 1. The sign $\text{sign}(u_2, o_1)$ of crossing between $u_2$ and $o_1$ is $-1$,

where $\text{sign}(u_2, o_1) = +1 = \text{sign}(o_1, u_2)$ if $o_1$ intersects $u_2$ from left to right and $\text{sign}(u_2, o_1) = -1 = \text{sign}(o_1, u_2)$ if $o_1$ intersects $u_2$ from right to left when we look $u_2$ as a line segment passing through the crossing upward. Notice that, by Rule 1, we can define the sign of each crossing by this way.

After $u_2$ intersects $o_1$ first with $\text{sign}(u_2, o_1) = -1$, we must change color from red to blue by Rule 1,2. Take a point $s_3$ so that $u_2 = [f_2, s_3]$ has only one undercrossing. Then the third overpass $o_3$ must intersect $u_1$ first with $\text{sign}(o_3, u_1) = -1$ by Rule 2,3. After $o_3$ intersects $u_1$ first, we must change color from blue to red by Rule 1,2. Take a point $f_3$ so that $o_3 = [s_3, f_3]$ has only one overcrossing. Then the third underpass $u_3$ can intersect only either $o_1$ first with $\text{sign}(u_3, o_1) = -1$ or $o_2$ first with $\text{sign}(u_3, o_2) = -1$. We claim that $u_3$ must intersect $o_2$ first with $\text{sign}(u_3, o_2) = -1$. Suppose that $u_3$ intersects $o_1$ first with $\text{sign}(u_3, o_1) = -1$. Then, after $u_3$ intersects $o_1$ first, we must change color from red to blue by Rule 1,2. Take a point $s_4$ so that $u_3 = [f_3, s_4]$ has only one undercrossing. In this case, we can not draw any knot diagram such that $u_3$ intersects $o_2$. This is a contradiction to Rule 2,3. Hence, $u_3$ intersects $o_2$ first with $\text{sign}(u_3, o_2) = -1$. After $u_3$ intersects $o_2$ first, take a point $x_3$ which is neither $s_1$ nor a crossing so that the arc $[f_3, x_3]$ of $u_3$ has only one undercrossing. Notice that we can connect $x_3$ and $s_1$ by an arc without crossing.
This diagram is impossible.

Figure 3. The first three overpasses and underpasses, $\text{sign}(u_2, o_1) = -1$.

If we have only 3 overpasses and 3 underpasses, we must connect $x_3$ and $s_1$ by an arc without crossing. This is the only way to draw the knot diagrams satisfying Rule 1,2,3 and gives us the standard diagram of the $(2, -3)$-torus knot.

See Figure 3, where solid arcs are supposed to be of the blue color (overpasses) and dashed arcs of the red color (underpasses).

Suppose that we have more than 3 overpasses, i.e., $k > 3$. Then the third underpass $u_3$ must intersect $o_1$ second by Rule 1,2,3 and we have only 2 cases that $u_3$ intersects $o_1$ second.

**Claim:** $u_3$ must intersect $o_1$ second with $\text{sign}(u_3, o_1) = -1$. See Figure 4(i).

**Proof of the Claim:** Suppose that $u_3$ intersects $o_1$ second with $\text{sign}(u_3, o_1) = +1$. See Figure 4(ii). Then, after $u_3$ intersects $o_1$ second, we must change color from red to blue by Rule 1. Take a point $s_4$ so that $u_3 = [f_3, s_4]$ has only two undercrossings. Then $o_4$ must intersect $u_2$ first with $\text{sign}(o_4, u_2) = -1$ and $u_1$ second with $\text{sign}(o_4, u_1) = +1$ by Rule 1,2,3. See Figure 4(iii).

After $o_4$ intersects $u_1$ second, we must change color from blue to red by Rule 1. Take a point $f_4$ so that $o_4 = [s_4, f_4]$ has only two overcrossings. Then $u_4$ can intersect only either $o_1$ first with $\text{sign}(u_4, o_1) = +1$ or $o_3$ first with $\text{sign}(u_4, o_3) = -1$. However, $u_4$ can not intersect $o_1$ first. If $u_4$ intersects $o_1$ first with $\text{sign}(u_4, o_1) = +1$, we must change color from red to blue by Rule 1,2. By a similar argument as the one used before, in this case, we can not draw any knot diagram such that $u_4$ intersects $o_2$. This is a contradiction to Rule 2,3. Hence, $u_4$ must intersect $o_3$ first with $\text{sign}(u_4, o_3) = -1$.

After $u_4$ intersects $o_3$ first, $u_4$ can intersect only either $o_1$ second with $\text{sign}(u_4, o_1) = +1$ or $o_2$ second with $\text{sign}(u_4, o_2) = +1$. However, $u_4$ can not intersect $o_1$ second. If $u_4$ intersects $o_1$ second with $\text{sign}(u_4, o_1) = +1$, after $u_4$ intersects $o_1$ second, we must change color from red to blue by Rule 1,2. In this case, we can not draw any knot diagram such that $u_4$ intersects $o_2$. This is a contradiction to Rule 2,3. Hence, the only way to draw $u_4$ is that $u_4$ intersects $o_3$ first with $\text{sign}(u_4, o_3) = -1$ and $o_2$ second with $\text{sign}(u_4, o_2) = +1$. See Figure 4(iv).
After $u_4$ intersects $o_2$ second, we must change color from red to blue by Rule 1. Take a point $s_5$ so that $u_4 = [f_1, s_5]$ has only two undercrossings. Then $o_5$ can intersect only either $u_1$ first with $\text{sign}(o_5, u_1) = +1$ or $u_3$ first with $\text{sign}(o_5, u_3) = -1$. However, $o_5$ can not intersect $u_1$ first. If $o_5$ intersects $u_1$ first with $\text{sign}(o_5, u_1) = +1$, then we must change color from blue to red by Rule 1,2. In this case, we can not draw any knot diagram such that $o_5$ intersects $u_2$. This is a contradiction to Rule 2,3. Hence, $o_5$ must intersect $u_3$ first with $\text{sign}(o_5, u_3) = -1$. Again, see Figure 4(iv).

After $o_5$ intersects $u_3$ first, $o_5$ can intersect only either $u_1$ second with $\text{sign}(o_5, u_1) = +1$ or $u_2$ second with $\text{sign}(o_5, u_2) = +1$. However, $o_5$ can not intersect $u_1$ second. If $o_5$ intersects $u_1$ second with $\text{sign}(o_5, u_1) = +1$, after $o_5$ intersects $u_1$ second, we must change color from blue to red by Rule 1,2. In this case, we can not draw any knot diagram such that $o_5$ intersects $u_2$. This is a contradiction to Rule 2,3. Hence, $o_5$ must intersect $u_2$ second with $\text{sign}(o_5, u_2) = +1$. Once more, see Figure 4(iv).
After $o_5$ intersects $u_2$ second, we must change color from blue to red by Rule 1. In this case, we can not draw any knot diagram such that $o_5$ intersects $u_1$. This is a contradiction to Rule 2,3. Hence, when $\text{sign}(u_3, o_1) = +1$, we can not draw any knot diagram satisfying Rule 1,2,3. Therefore, $\text{sign}(u_3, o_1)$ must be $-1$. This finishes the proof of the Claim.

Now, we use an induction on the order of the over-underpass sequence. The argument used here is just a generalization of the previous one. Hence, our argument will be slightly sketchy here.

Suppose, without loss of generality, that $3 \leq n < k$ and $s_1, f_1, s_2, f_2, ..., s_n, f_n$ is an over-underpass sequence of the standard diagram of the $(n-1, -n)$-torus knot.

![Figure 5](image-url)  
Figure 5. It is impossible for $u_n$ to intersect $o_1$ with $\text{sign}(u_n, o_1) = -1$. 
Let $o_i = [s_i, f_i]$ and $u_i = [f_i, s_{i+1}]$ for each $i \in \{1, \ldots, n-1\}$, and let $o_n = [s_n, f_n]$ and $u_n^* = [f_n, s_1]$. Take a point $x_n$ of $u_n^*$ between the $(n-2)$-th undercrossing of $u_n^*$ and $s_1$. Then the arc $[f_n, x_n]$ of $u_n^*$ has also $n - 2$ undercrossings.

Notice that $o_1, o_2, o_3, \ldots, o_{n-1}, o_n$ are overpasses and underpasses of a knot diagram $D$ satisfying Rule 1,2,3 and $[f_n, x_n]$ is an arc of the $n$-th underpass $u_n$ of $D$. Suppose that $D$ has more than $n$ overpasses. Then there is an $((n-2) + 1)$-th undercrossing on $u_n$ by $o_1$ according to Rule 1,2,3. We claim that at this undercrossing, $u_n$ must intersect $o_1$ with $\text{sign}(u_n, o_1) = -1$.

To prove this claim, suppose that $u_n$ intersects $o_1$ with $\text{sign}(u_n, o_1) = +1$. See Figure 5. Then, after $u_n$ intersects $o_1$ with $\text{sign}(u_n, o_1) = +1$, we must change color from red to blue by Rule 1. Take a point $s_{n+1}$ so that $u_n = [f_n, s_{n+1}]$ has $(n-2) + 1$ undercrossings. Then the only way to draw $o_{n+1}$ is that $o_{n+1}$ intersects $u_{n-1}$ first, $u_{n-2}$ second, ..., $u_2$ $(n-2)$-th, and $u_1$ $(\text{sign}(o_{n+1}, u_{n-1}) = +1)$ so that $\text{sign}(o_{n+1}, u_{n-2}) = \ldots = \text{sign}(o_{n+1}, u_2) = -1$, and $\text{sign}(o_{n+1}, u_1) = +1$ by Rule 1,2,3. After $o_{n+1}$ intersects $u_1$ with $\text{sign}(o_{n+1}, u_1) = +1$, we must change color from blue to red by Rule 1. Take a point $f_{n+1}$ so that $o_{n+1}$ has $(n - 2) + 1$ overcrossings. Then the only way to draw $u_{n+1}$ is that $u_{n+1}$ intersects $o_n$ first, $o_{n-1}$ second, ..., $o_3$ $(n-2)$-th, and $o_2$ $(\text{sign}(o_{n+1}, u_{n-1}) = +1)$ so that $\text{sign}(o_{n+1}, o_3) = \text{sign}(u_{n+1}, o_{n-1}) = \ldots = \text{sign}(u_{n+1}, o_3) = -1$, and $\text{sign}(u_{n+1}, o_2) = +1$ by Rule 1,2,3. After $u_{n+1}$ intersects $o_2$ with $\text{sign}(u_{n+1}, o_2) = +1$, we must change color from red to blue by Rule 1. Take a point $s_{n+2}$ so that $u_{n+1}$ has $(n-2) + 1$ undercrossings. Then the only way to draw $o_{n+2}$ is that $o_{n+2}$ intersects $u_n$ first, $u_{n-1}$ second, ..., $u_3$ $(n-2)$-th, and $u_2$ $(\text{sign}(o_{n+1}, u_{n-1}) = +1)$ so that $\text{sign}(o_{n+2}, u_n) = \text{sign}(o_{n+2}, u_{n-1}) = \ldots = \text{sign}(o_{n+2}, u_3) = -1$, and $\text{sign}(o_{n+2}, u_2) = +1$ by Rule 1,2,3. After $o_{n+2}$ intersects $u_2$ with $\text{sign}(o_{n+2}, u_2) = +1$, we must change color from blue to red by Rule 1. However, in this case, we can not draw any knot diagram such that $o_{n+2}$ intersects $u_1$. This is a contradiction to Rule 2,3. Therefore, $\text{sign}(u_n, o_1)$ cannot be $+1$.

As a next step, we claim that the only way to draw $o_{n+1}$ and $u_{n+1}$ gives us the standard diagram of the $((n+1)-1, -(n+1))$-torus knot (see Figure 6). This will finish the induction.

After $u_n$ intersects $o_1$ $(\text{sign}(u_n, o_1) = -1)$, we must change color from red to blue by Rule 1,2. Take a point $s_{n+1}$ so that $u_n = [f_n, s_{n+1}]$ has only $(n-2)+1$ undercrossings. Then the only way to draw $o_{n+1}$ is that $o_{n+1}$ intersects $u_{n-1}$ first, $u_{n-2}$ second, ..., $u_1$ $(\text{sign}(o_{n+1}, u_{n-1}) = -1)$ so that $\text{sign}(o_{n+1}, u_{n-2}) = \ldots = \text{sign}(o_{n+1}, u_1) = -1$ by Rule 1,2,3. Suppose that $o_{n+1}$ intersects $u_i$ first for some $i < n-1$. Then $\text{sign}(o_{n+1}, u_i)$ must be $-1$ and $o_{n+1}$ must intersect $u_i$ first, $u_{i-1}$ second, ..., $u_1$ $i$-th so that $\text{sign}(o_{n+1}, u_i) = \text{sign}(o_{n+1}, u_{i-1}) = \ldots = \text{sign}(o_{n+1}, u_1) = -1$ by Rule 1,2,3. However, after $o_{n+1}$ intersects $u_i$ $i$-th with $\text{sign}(o_{n+1}, u_i) = -1$, $o_{n+1}$ must stop before intersecting $u_n$ and we must change color from blue to red by Rule 1. In this case, we can not draw any knot diagram such that $o_{n+1}$ intersects $u_{n-1}$. This is
a contradiction to Rule 2,3. Hence, $o_{n+1}$ must intersect $u_{n-1}$ first, $u_{n-2}$ second, ..., $u_1$ 
($(n - 2) + 1$)-th so that $\text{sign}(o_{n+1}, u_{n-1}) = \text{sign}(o_{n+1}, u_{n-2}) = ... = \text{sign}(o_{n+1}, u_1) = -1$. Take a point $f_{n+1}$ so that $o_{n+1} = [s_{n+1}, f_{n+1}]$ has only $(n - 2) + 1$ overcrossings. Then, by a similar argument as before, we can show that the only way to draw $u_{n+1}$ is that $u_{n+1}$ intersects $o_n$ first, $o_{n-1}$ second, ..., $o_2$ ($(n - 2) + 1$)-th so that $\text{sign}(u_{n+1}, o_n) = \text{sign}(u_{n+1}, o_{n-1}) = ... = \text{sign}(u_{n+1}, o_2) = -1$ by Rule 1,2,3.

![Figure 6](image_url)  
Figure 6. The only way to draw $o_{n+1}$ and $u_{n+1}$. 
Suppose that \( u_{n+1} \) intersects \( o_i \) first for some \( i < n \). Then \( \text{sign}(u_{n+1}, o_i) \) must be \(-1\) and \( u_{n+1} \) must intersect \( o_i \) first, \( o_{i-1} \) second, ..., \( o_1 \) \( i \)-th so that \( \text{sign}(u_{n+1}, o_i) = \text{sign}(u_{n+1}, o_{i-1}) = ... = \text{sign}(u_{n+1}, o_1) = -1 \) by Rule 1,2,3. However, after \( u_{n+1} \) intersects \( o_1 \) \( i \)-th with \( \text{sign}(u_{n+1}, o_1) = -1 \), \( u_{n+1} \) must stop before intersecting \( o_{n+1} \) and we must change color from red to blue by Rule 1,2. In this case, we can not draw any knot diagram such that \( u_{n+1} \) intersects \( o_n \). This is a contradiction to Rule 2,3. Hence, \( u_{n+1} \) must intersect \( o_n \) first, \( o_{n-1} \) second, ..., \( o_2 \) \((n - 2) + 1\)-th so that \( \text{sign}(u_{n+1}, o_n) = \text{sign}(u_{n+1}, o_{n-1}) = ... = \text{sign}(u_{n+1}, o_2) = -1 \). Take a point \( x_{n+1} \) so that the arc \([f_{n+1}, x_{n+1}]\) of \( u_{n+1} \) has only \((n - 2) + 1\) undercrossings. Notice that, in this case, we can connect \( x_{n+1} \) and \( s_1 \) by an arc without crossing so that we complete drawing the knot diagram.

This is the only way to draw a knot diagram with \( n + 1 \) overpasses satisfying the Rules 1,2,3.

To complete the argument, when \( n = k - 1 \), we can get the standard diagram \( D \) of the \((k - 1, -k)\)-torus knot and this is the only way to draw the knot diagrams satisfying Rule 1,2,3, and hence, the theorem is proved in Case 1.

Case 2. The sign \( \text{sign}(u_2, o_1) \) of crossing between \( u_2 \) and \( o_1 \) is \( +1 \).

By the same argument as used in Case 1, we can get only standard diagrams of the \((b(D) - 1, b(D))\)-torus knot. This proves the theorem. \( \square \)
4. A final remark

One may view Lemmas 2.5 and 2.6 from another perspective. Suppose \( D \) is a knot diagram, not necessarily being minimal with respect to crossings. If either an overpass intersects an underpass more than once or an overpass intersects its adjacent underpass, then Lemmas 2.5 and 2.6 give a specific way to change the knot diagram \( D \) by isotopy with the number of crossings reduced.

Consider a knot diagram \( D \) satisfying the following two conditions:

1. each overpass intersects each underpass at most once; and
2. each overpass does not intersect its adjacent underpasses.

For such a knot diagram \( D \), we have

\[
b(D) \leq c(D) \leq b(D)(b(D) - 2).
\]

When \( c(D) = b(D) \), \( D \) is an alternating knot diagram. By Theorem 3.1, when \( c(D) = b(D)(b(D) - 2) \), \( D \) is the standard diagram of a \((b(D) - 1, \pm b(D))\) torus knot. Thus, it is natural to wonder whether conditions (1) and (2) above are sufficient for a knot diagram of a prime knot being minimal with respect to crossings. Goeritz’s unknot diagrams show that the answer to this question is negative. See Figure 7.

Are there any other necessary conditions on the overpasses and underpasses of a knot diagram for it being minimal with respect to crossings? This will be the topic of our further investigation.

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