A HÖRMANDER-MIKHLIN MULTIPLIER THEORY FOR FREE GROUPS
AND AMALGAMATED FREE PRODUCTS OF VON NEUMANN ALGEBRAS

TAO MEI, ÉRIC RICARD, AND QUANHUA XU

Abstract. We establish a platform to transfer $L_p$-completely bounded maps on tensor products of von Neumann algebras to $L_p$-completely bounded maps on the corresponding amalgamated free products. As a consequence, we obtain a Hörmander-Mikhlin multiplier theory for free products of groups. Let $\mathbb{F}_\infty$ be a free group on infinite generators $(g_1, g_2, \cdots)$. Given $d \geq 1$ and a bounded symbol $m$ on $\mathbb{T}^d$ satisfying the classical Hörmander-Mikhlin condition, the linear map $M_m : \mathbb{C}[\mathbb{F}_\infty] \to \mathbb{C}[\mathbb{F}_\infty]$ defined by $\lambda(g) \mapsto m(k_1, \cdots, k_d)\lambda(g)$ for $g = g_{k_1} \cdots g_{k_n} \in \mathbb{F}_\infty$ in reduced form (with $k_l = 0$ in $m(k_1, \cdots, k_d)$ for $l > n$), extends to a complete bounded map on $L_p(\mathbb{F}_\infty)$ for all $1 < p < \infty$, where $\mathbb{F}_\infty$ is the group von Neumann algebra of $\mathbb{F}_\infty$. A similar result holds for any free product of discrete groups.

1. Introduction

Fourier multipliers are by far the most important operators in analysis. One can say that classical harmonic analysis has been developed around Fourier multipliers. The main task there is to find criteria for the boundedness of Fourier multipliers on various function spaces, notably classical harmonic analysis has been developed around Fourier multipliers. The main task there is to find criteria for the boundedness of Fourier multipliers on various function spaces, notably $L^p$-spaces. However, it is in general impossible to characterize their $L_p$-boundedness for finite $p \neq 2$. In this regard, the most fundamental result is the celebrated Hörmander-Mikhlin multiplier theorem which asserts that if $m$ is a $C^{[\frac{d}{2}]+1}$ function on $\mathbb{R}^d \setminus \{0\}$ such that

$$\sup_{\alpha \leq |\alpha| \leq |\xi|+1} \sup_{\xi \in \mathbb{R}^d \setminus \{0\}} |\xi|^{|\alpha|} |\partial^\alpha m(\xi)| < \infty,$$

where $\alpha = (\alpha_1, \cdots, \alpha_d)$ denotes a multi-index of nonnegative integers and $|\alpha| = \alpha_1 + \cdots + \alpha_d$, then the associated Fourier multiplier $T_m$, defined via Fourier transform by $T_m(f) = \hat{m} \hat{f}$, is a bounded map on $L_p(\mathbb{R}^d)$ for all $1 < p < \infty$. A similar statement holds for the periodic case, i.e., for $\mathbb{T}^d$ instead of $\mathbb{R}^d$. More precisely, if $m : \mathbb{Z}^d \to \mathbb{C}$ satisfies

$$(1) \quad \|m\|_{HM} = \sup_{0 \leq |\alpha| \leq |\xi|+1} \sup_{k \in \mathbb{Z}^d} |k|^{|\alpha|} |\partial^\alpha m(k)| < \infty,$$

where $\partial^\alpha = \partial_{k_1}^{\alpha_1} \cdots \partial_{k_d}^{\alpha_d}$ denotes the partial discrete difference operator of order $\alpha = (\alpha_1, \cdots, \alpha_d)$, then the associated Fourier multiplier $T_m$ is a bounded map on $L_p(\mathbb{T}^d)$ for all $1 < p < \infty$.

In noncommutative harmonic analysis and operator algebras, the study Fourier multipliers on general groups is of paramount importance. These multipliers are also called Herz-Schur multipliers in the literature. The underlying $L_p$-spaces are the noncommutative $L_p$-spaces on the group von Neumann algebras. This line of investigation finds one of its origins in the study of approximation properties of operator algebras inaugurated by Haagerup’s pioneering work [6] on Fourier multipliers on free groups in which he solved the longstanding problem on Grothendieck’s approximation property for the reduced $C^*$-algebra of a free group: this algebra has the completely bounded approximation property. Later, together with De Cannière [2] and Cowling [5], Haagerup studied completely bounded multipliers on the Fourier algebras of locally compact groups.

One of the remarkable recent achievements in the interplay between multipliers and noncommutative $L_p$-spaces is the result of Lafforgue and de la Salle [19] asserting that the noncommutative $L_p$-space with $p > 4$ associated to the von Neumann algebra of $SL_3(\mathbb{Z})$ fails the completely bounded approximation property. Haagerup and coauthors [9, 10, 8] extended their result to connected simple Lie groups; later, de Laat and de la Salle [18] discovered that $L_p$ multipliers are very tightly

2000 Mathematics Subject Classification: Primary: 46L07, 46L50. Secondary: 46L52, 46L54.

Key words: Free groups, free products of groups, amalgamated free products of von Neumann algebras, noncommutative $L_p$-spaces, Hörmander-Mikhlin multipliers, free Fourier multipliers, completely bounded maps.

1
connected with Banach space geometry and group representations. Other striking illustrations of this interplay are Junge and Parcet’s work [16] on the Maurey-Rosenthal factorization for non-commutative $L_p$-spaces which relies on Schur multiplier estimates. Multipliers are also intimately related to functional calculus in von Neumann algebras as shown by the resolution of Krein’s celebrated Lipschitz continuity problem in noncommutative $L_p$-spaces by Potapov and Sukochev [32], see also [3] where the crucial use of Fourier multipliers is more apparent.

In a recent series of articles [7, 11], Haagerup and co-authors obtained a beautiful handy characterization of the completely bounded radial multipliers on the von Neumann algebras (i.e. $L_\infty$) of free groups and p-adic groups, and, more generally, on homogeneous trees and free products; see also Wysoczański’s previous work [35]. All this has motivated many other publications of the same nature, in particular, building on Ozawa’s work [26], Mei and de la Salle [22] obtained a similar characterization for hyperbolic groups. Nevertheless, little has been understood for non-radial multipliers\(^1\), nor for $L_p$-spaces with $2 < p < \infty$. Here, the most challenging problem is to find a Hörmander-Mikhlin type criterion for Fourier multipliers on $L_p$-spaces to be completely bounded.

The recent papers [14, 15, 27] provide a first advance towards this direction. The work [23] by the first and second named authors is more related to the present article, it introduced the free Hilbert transforms on free groups and free products of von Neumann algebras and showed their $L_p$-boundedness for all $1 < p < \infty$. We pursue this line of investigation by going considerably beyond, and will exhibit a large family of $L_p$ multipliers in the free setting.

To proceed further, we need some notation and refer to the next section for all unexplained notions. Let $\Gamma$ be a discrete group and $\lambda$ its left regular representation on $L_2(\Gamma)$. The group von Neumann algebra $vN(\Gamma)$ is the von Neumann algebra generated by $\lambda(\Gamma)$, it is also equal to the weak* closure of the group algebra $\mathbb{C}[\Gamma]$ that consists of all polynomials in $\lambda$:

$$\mathbb{C}[\Gamma] = \left\{ \sum_{g \in \Gamma} \alpha(g) \lambda(g) : \alpha(g) \in \mathbb{C} \right\}.$$ 

Following notation in quantum groups, we will denote $vN(\Gamma)$ by $\widehat{\Gamma}$ in this article. $\widehat{\Gamma}$ is equipped with its canonical tracial state $\tau$ defined by $\tau(x) = \langle \delta_x, x \delta_e \rangle$ for any $x \in \widehat{\Gamma}$, where $\delta_e$ denotes the Dirac mass at the identity $e$ of $\Gamma$. $L_p(\widehat{\Gamma})$ is the noncommutative $L_p$-space based on $(\widehat{\Gamma}, \tau)$, it is equipped with the natural operator space structure introduced by Pisier.

For a complex function $m$ on $\Gamma$, the associated Fourier multiplier $T_m$ is a linear map on $\mathbb{C}[\Gamma]$ determined by $\lambda(g) \mapsto m(g)\lambda(g)$. We call $m$ a (completely) bounded $L_p$-multiplier on $\Gamma$ if $T_m$ extends to a (completely) bounded map on $L_p(\widehat{\Gamma})$. Our main aim is to find sufficient conditions for $m$ to be such a multiplier. Note that the periodic Hörmander-Mikhlin theorem recalled at the beginning concerns the case $\Gamma = \mathbb{Z}^d$ in which $\widehat{\Gamma} = L_\infty(\mathbb{T}^d)$, $\mathbb{T}$ being the unit circle equipped with normalized Haar measure. In this case, every function $m$ on $\mathbb{Z}^d$ satisfying (1) is a completely bounded $L_p$-multiplier on $\mathbb{Z}^d$.

Throughout the article, $F_\infty$ will denote a free group on infinite generators $\{g_1, g_2, \ldots\}$. An element $g \in F_\infty$ different from the identity $e$ is written as a reduced word on $\{g_1, g_2, \ldots\}$:

$$g = g_{i_1}^{k_1} g_{i_2}^{k_2} \cdots g_{i_n}^{k_n} \quad \text{with} \quad i_1 \neq i_2 \neq \cdots \neq i_n, \quad k_j \in \mathbb{Z} \setminus \{0\}, \quad 1 \leq j \leq n. \quad (2)$$

Given a complex function $m$ on $\mathbb{Z}^d$ and a fixed $d \in \mathbb{N}$, we define $M_m : \mathbb{C}[F_\infty] \to \mathbb{C}[F_\infty]$ by

$$M_m(\lambda(g)) = \begin{cases} m(k_1, \cdots, k_d) \lambda(g) & \text{if } d \leq n, \\ m(k_1, \cdots, k_n, 0, \cdots, 0) \lambda(g) & \text{if } d > n \end{cases}$$

for every $g$ as in (2). Below is our first main theorem. It is contained in [24] for $d = 1$.

**Theorem 1.1.** Let $d \in \mathbb{N}$ and $1 < p < \infty$. If $m$ is a completely bounded $L_p$-multiplier on $\mathbb{Z}^d$, then $M_m$ extends to a completely bounded map on $L_p(\widehat{F}_\infty)$. In particular, the conclusion holds if $m$ satisfies (1).

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\(^1\)A better understanding of non-radial multipliers seem to be in order to answer some fundamental open questions, such as the construction of Schauder basis, for the free group reduced $C^*$-algebras and the corresponding noncommutative $L_p$-spaces.
To achieve Theorem 1.1, we establish a new platform to transfer the $L_p$-complete boundedness of Fourier multipliers on tori to Fourier multipliers on free groups. The key tool is a group of unitary actions on $L_2(\mathbb{R}^\infty)$ and its complete boundedness on $L_p(\mathbb{R}^\infty)$ for all $1 < p < \infty$. Given $z = (z_{j,i})_{1 \leq j \leq d, 1 \leq i < \infty} \in \mathbb{T}^d \times \mathbb{T}^\infty$, let
$$a^L_d(\lambda(g)) = z_{1,1}^{k_1} z_{2,2}^{k_2} \cdots z_{d, d}^{k_d} \lambda(g)$$
for $g$ as in (2) (with $k_\ell = 0$ if $\ell > n$). It is obvious that $a^L_d$ extends to a unitary action of the group $\mathbb{T}^d \times \mathbb{T}^\infty$ on $L_2(\mathbb{R}^\infty)$.

**Theorem 1.2.** The above action $a^L_d$ of $\mathbb{T}^d \times \mathbb{T}^\infty$ on $L_2(\mathbb{R}^\infty)$ extends to a uniformly completely bounded action on $L_p(\mathbb{R}^\infty)$ for every $1 < p < \infty$ with constant depending only on $d$ and $p$.

Theorem 1.1 will follow from Theorem 1.2 and a standard transference argument. Our initial proof of Theorem 1.2 is very long and technical. We have then luckily found a new and shorter proof of Theorem 1.2.

**Theorem 1.3.** Let $T_{j,i}$ be a family of finite von Neumann algebras equipped with normal faithful tracial states. Let $B$ be a common von Neumann subalgebra of the $A_i$’s and $E_i : A_i \to B$ be the corresponding trace preserving conditional expectation. We denote by $(A, \tau) = \tau_{i \in I} B(A_i, \tau_i)$ the amalgamated free product of the $(A_i, \tau_i)$’s over $B$. Let $E$ be the conditional expectation from $A$ to $B$. As usual, for $X \subseteq L_p(A)$ we write $X$ for the set $\{x - E(x) : x \in X\}$.

Assume that we are given maps $T_{j,i} : L_p(A_i) \to L_p(A_i)$ with $1 \leq j \leq d$ and $i \in I$ satisfying the following conditions

- $T_{j,i}$ is $B$-bimodular, that is, $T_{k,i}(axb) = aT_{j,i}(x)b$ for $a, b \in B$ and $x \in L_p(A_i)$;
- $T_{j,i}(L_p(A_i)) \subseteq L_p(A_i)$.

We can define a map $T^L_d$ as follows. $T^L_d(b) = b$ for $b \in B$ and
$$T^L_d(a) = \begin{cases} T_{i_1,i_1}(a_{i_1}) \otimes \cdots \otimes T_{d,i_d}(a_{i_d}) \otimes a_{d+1} \otimes \cdots \otimes a_n & \text{if } d < n, \\ T_{i_1,i_1}(a_{i_1}) \otimes \cdots \otimes T_{n,i_n}(a_{i_n}) & \text{if } d \geq n \end{cases}$$
for $a = a_1 \otimes \cdots \otimes a_n \in \bigotimes_{i_1} B_i \otimes \cdots \otimes \bigotimes_{i_n} A_{i_n}$ with $i_1 \neq i_2 \neq \cdots \neq i_n$.

The following theorem and its proof contain the main novelty of the article, it gives a surprisingly simple condition to ensure the complete boundedness of the map $T^L_d$ on $L_p(A)$.

**Theorem 1.3.** Let $1 < p < \infty$. Assume that the $T_{j,i}$’s extend to completely bounded maps on $L_p(A_i)$ and on $L_2(A_i)$ uniformly in $i, j$. Then $T^L_d$ extends to a completely bounded map on $L_p(A)$ with
$$\|T^L_d\|_{cb(L_p(A))} \lesssim_{p,d} \prod_{j=1}^d \sup_{i \in I} \left( \|T_{j,i}\|_{cb(L_p(A_i))} + \|T_{j,i}\|_{cb(L_2(A_i))} \right).$$

It is clear that the assumption above is necessary for the validity of the conclusion. Thus the theorem gives a characterization of the complete boundedness of $T^L_d$ on $L_p(A)$. Another aspect to be emphasized is the fact that $p$ is a single fixed index. One of the main ingredients in our argument is a length reduction formula for the $L_p$-norms associated with amalgamated free products that we state as Theorem 4.6. This formula is interesting for its own right, it was previously proved in [17, Theorem B] for homogeneous free polynomials.

The article is organized as follows. After a brief introduction to necessary preliminaries, we give in section 3 some norms on Hilbert $B$-modules that are the modular extensions of the usual column and row noncommutative $L_p$-norms; we show that bounded modular maps extend to these modular noncommutative $L_p$-spaces. The results in this section provide crucial tools for section 4 that is the core of the article and contains our major ideas. Section 4 gives the proof of Theorem 1.3 that is done through a reduction formula for polynomials in a free product of von Neumann algebras, the latter is proved in its turn with the help of an intermediate result (Theorem 4.5) that is a special case of Theorem 1.3 where $d = 1$, $p > 2$ and every $T_i$ is a $*$-representation on $A_i$, leaving $B$ invariant and commuting with $E_i$ for all $i \in I$. Theorems 1.1 and 1.2 easily follow from the results.
in section 4; moreover, the arguments can be extended to the free products of general (non abelian) discrete groups. We do all this in section 5. This section also contains some more paraproducts which might be interesting in free analysis. We end the article with an appendix on the endpoint \( p = \infty \).

We will use the following convention through the article: \( A \subseteq B \) (resp. \( A \preceq_p B \)) means that \( A \leq cB \) (resp. \( A \leq c_pB \)) for some absolute positive constant \( c \) (resp. a positive constant \( c_p \) depending only on \( p \)). \( A \sim B \) or \( A \simeq_p B \) means that these inequalities as well as their inverses hold.

2. Preliminaries

This section presents some preliminaries on noncommutative \( L_p \)-spaces and amalgamated free products.

Murray and von Neumann’s work [25] demonstrates von Neumann algebras as a natural framework to do noncommutative analysis. The elements in a von Neumann algebra \( \mathcal{M} \) can be integrated over the equipped trace \( \tau \) and measured by the associated \( L_p \)-norms. In the sequel, \( \mathcal{M} \) will denote a von Neumann algebra with a normal faithful semifinite trace. For each \( 1 \leq p \leq \infty \), let \( L_p(\mathcal{M}) \) be the noncommutative \( L_p \)-space associated with the pair \( (\mathcal{M}, \tau) \). Note that \( L_\infty(\mathcal{M}) = \mathcal{M} \) with the operator norm. We refer to [31] for details and more historical references.

All \( L_p \)-spaces in this article will be equipped with their natural operator space structure introduced by Pisier in [29, 30]. We also refer to [30] for operator space theory. In fact, what we will need from the latter theory is only [29, Lemma 1.7] that asserts a linear map \( T : L_p(\mathcal{M}) \to L_p(\mathcal{M}) \) is completely bounded iff \( \text{Id}_{\mathcal{S}_{\mathcal{E}}} \otimes T : L_p(B(\ell_2^\infty)\overline{\otimes}\mathcal{M}) \to L_p(B(\ell_2^\infty)\overline{\otimes}\mathcal{M}) \) is bounded; in this case, the completely bounded norm of \( T \) is equal to the usual norm of \( \text{Id}_{\mathcal{S}_{\mathcal{E}}} \otimes T \) and denoted by \( ||T||_{cb(\mathcal{E})} \) or simply \( ||T||_{cb} \) when no ambiguity is possible. Note that \( L_2(\mathcal{M}) \) is an operator Hilbert space and every bounded map on it is automatically completely bounded with the same norm.

We will often be concerned with group von Neumann algebras. Let \( \Gamma \) be a discrete group. Recall the definition of its left regular representation \( \lambda \); for any \( g \in \Gamma \), \( \lambda(g) \) is the left translation by \( g \) on \( \ell_2(\Gamma) \), i.e., \( \lambda(g)(f)(h) = f(g^{-1}h) \) for any \( h \in \Gamma \) and \( f \in \ell_2(\Gamma) \). Thus \( \lambda(g)(\delta_h) = \delta_{gh} \), where \( \delta_h \) is the Dirac mass at \( h \). \( \mathbb{C}[\Gamma] \) denotes the linear span of \( \lambda(\Gamma) \). Then the group von Neumann algebra \( \hat{\mathcal{G}} \) is the weak* closure of \( \mathbb{C}[\Gamma] \) in \( B(\ell_2(\Gamma)) \). The canonical trace of \( \hat{\mathcal{G}} \) is the vector state induced by \( \delta_e \) with \( e \) the identity of \( \Gamma \). When \( \Gamma \) is abelian (e.g. \( \Gamma = \mathbb{Z}^d \)), \( L_p(\hat{\mathcal{G}}) \) coincides with the usual \( L_p \)-space on the dual group of \( \Gamma \).

We turn to the second part of the article on amalgamated free products. Let \( (\mathcal{A}_i, \tau_i)_{i \in I} \) be a family of finite von Neumann algebras equipped with normal faithful normalized traces. Let \( \mathcal{B} \) be a common von Neumann subalgebra of \( \mathcal{A}_i \) for all \( i \in I \), and \( E_i : \mathcal{A}_i \to \mathcal{B} \) the trace preserving conditional expectation. Denote \( (\mathcal{A}, \tau) = *_{i \in I}(\mathcal{A}_i, \tau_i) \), the amalgamated free product of \( (\mathcal{A}_i, \tau_i)_{i \in I} \) over \( \mathcal{B} \). We will briefly recall the construction to fix notation.

For any \( x \in \mathcal{A}_i \), we denote by \( \hat{x} = x - E_0 x \) and \( \hat{\mathcal{A}}_i = \{ \hat{x} : x \in \mathcal{A}_i \} \), which yields a natural decomposition \( \mathcal{A}_i = \mathcal{B} \oplus \hat{\mathcal{A}}_i \). We use the multi-index notation: \( \hat{\mathcal{A}}_i = \bigoplus_{n \geq 1} \bigoplus_{i_1 \neq \ldots \neq i_n} \mathcal{A}_{i_1} \otimes \cdots \otimes \mathcal{A}_{i_n} = \bigoplus_{n \geq 0} \bigoplus_{i_1 \neq \ldots \neq i_n} \mathcal{B} \otimes \cdots \otimes \mathcal{B} \hat{\mathcal{A}}_i = \bigoplus_{n \geq 0} \bigoplus_{(i_1, \ldots, i_n) \in I^n} \mathcal{W}_n \) is a \( * \)-algebra by using concatenation and centering with respect to \( \mathcal{B} \). The natural projection \( E \) from \( \mathcal{W} \) onto \( \mathcal{B} \) is a conditional expectation. \( \tau \circ E \) is a trace on \( \mathcal{W} \) that is independent of \( \tau \) and will be denoted by \( \tau \). Then \( (\mathcal{A}, \tau) \) is the finite von Neumann algebra obtained by the GNS construction from \( (\mathcal{W}, \tau) \). Thus \( \mathcal{W} \) is weak* dense in \( \mathcal{A} \) and dense in \( L_p(\mathcal{A}) \) for \( p < \infty \). Moreover, \( E \) extends to a trace preserving conditional expectation from \( \mathcal{A} \) onto \( \mathcal{B} \). As \( E \) restricts to \( E_i \) on \( \mathcal{A}_i \), from now on we skip the index \( i \) for the conditional expectations.

For \( n \geq 0 \) we denote by \( P_n \) the natural projection

\[
P_n : \mathcal{W} \to \mathcal{W}_n = \bigoplus_{(i_1, \ldots, i_n) \in I^n} \mathcal{A}_{i_1} \otimes \cdots \otimes \mathcal{A}_{i_n}
\]
and $P_{≥ n} = \Id - (P_1 + \cdots + P_{n-1})$. Note that $P_n$ extends to a completely bounded projection on $L_p(A)$ with cb-norm $≤ 2n + 1$ for all $1 ≤ p ≤ ∞$ (see [34]).

For $k ∈ I$ let

$$L_k = \bigoplus_{\varepsilon_i = k} W_{\varepsilon_i} \quad \text{and} \quad R_k = L_k^*.$$ 

We denote the associated orthogonal projections on $W$ by $L_k$ and $R_k$. Given a family $ε = (ε_i)_{i ∈ \{0, 1\}}$ of elements in $B$, and $x ∈ W$, we define

$$(3) \quad H_ε(x) = ε_0 E(x) + \sum_{k ∈ I} ε_k L_k(x) \quad \text{and} \quad H_ε^{\mathsf{op}}(x) = E(x) ε_0^* + \sum_{k ∈ I} R_k(x) ε_k^*.$$ 

The following is the heart of [23], we state it as a lemma for later reference.

**Lemma 2.1.** Let $1 < p < ∞$ and $ε = (ε_i)_{i ∈ \{0, 1\}}$ be a family of unitaries in the center $Z(B)$ of $B$. Then $H_ε$ and $H_ε^{\mathsf{op}}$ extend to completely bounded maps on $L_p(A)$. Moreover,

$$\|H_ε(x)\|_p ≍_p \|x\|_p ≍ \|H_ε^{\mathsf{op}}(x)\|_p, \quad x ∈ L_p(A).$$

3. **NORMS ON MODULES**

This section is a preparatory part for the next one. Here we will introduce the modular versions of the usual row and column p-operator spaces and show the corresponding boundedness results. This is quite tedious but we have to cope it since they will be the key tools for the proofs of our main results on amalgamated free products in the next section. Our references for Hilbert $C^*$-modules are [20, 28].

Let $B$ be a von Neumann subalgebra of a semifinite von Neumann algebra $(M, τ)$ such that the restriction of $τ$ to $B$ is semifinite too. We need to introduce several norms on tensor products related to $L_p$-modules as in [17].

Let $E$ be a right Hilbert $B$-module with $B$-inner product $(\cdot, \cdot)$ (or $(\cdot, \cdot)_E$ if we deal with several modules). $E$ is equipped with the norm induced by its inner product. However, we do not assume that $E$ is complete. A typical example is, given an index set $I$, the module

$$C_I(B) = \{(x_α)_{α ∈ I} ∈ B : \sup_{S ⊆ I \text{ finite}} \| ∑_{α ∈ S} x_α^* x_α \|_B < ∞\}$$

with inner product $(x, y) = \lim_S \sum_{α ∈ S} x_α^* y_α$, which is well defined as a weak* limit over finite subsets of $I$ for inclusion. By [20, 28], a general module $E$ can be embedded into a self-dual Hilbert module and there exist an index set $I$ and a right $B$-module map $u = (u_α)_{α ∈ I} : E → C_I(B)$ so that $(x, y) = ⟨u(x), u(y)⟩$.

Given $2 ≤ p < ∞$, we introduce a norm on the amalgamated tensor product $E ⊗_B L_p(M)$ as follows:

$$\|x\|_{E^c ⊗_p L_p(M)} = \left\| \left( ∑_{i,j=1}^n a_i^* (x_i, x_j) a_j \right)^{1/2} \right\|_{L_p(M)}$$

for $x = ∑_{i=1}^n x_i ⊗ a_i ∈ E ⊗_B L_p(M)$. Equipped with this norm, $E ⊗_B L_p(M)$ is denoted by $E^c ⊗_p L_p(M)$ (the superscript $c$ refers to column). In [17], this norm is denoted by $\| ∑_{i} |x_i| a_i \|_p$.

As explained in section 3 there, via the above concrete embedding of $E$ into $C_I(B)$, the map

$$u ⊗ \Id_{L_p(M)} : E^c ⊗_B L_p(M) → C_I(B) ⊗_B L_p(M) ⊂ L_p(B(ℓ_2(I)) ⊗ M)$$

is then an isometry that allows us to view $E^c ⊗_B L_p(M)$ as a subspace of the column subspace of the last space. This also fully justifies that $\| · \|_{E^c ⊗_p L_p(M)}$ is indeed a norm.

Similarly, given $F$ a left Hilbert $B$-module, we define a norm on $L_p(M) ⊗_p F$ by

$$\|y\|_{L_p(M) ⊗_p F^c} = \left\| \left( ∑_{i,j=1}^n b_i^*(y_i, y_j) b_j^* \right)^{1/2} \right\|_{L_p(M)}$$

for $y = ∑_{i=1}^n b_i ⊗ y_i$. This norm is denoted by $\| ∑_{i} |b_i| y_i \|_p$ in [17]. As above, $L_p(M) ⊗_p F^c$ can be identified with a subspace of the row subspace of $L_p(B(ℓ_2(J)) ⊗ M)$ for some index set $J$.
Thus we deduce $E$ immediately implies the following:

$$\sum_{i,j} a_{i,j}^* (T(x_i), T(x_j)) a_{i,j} \leq \|T\|^2 \sum_{i,j} a_{i,j}^* (x_i, x_j) a_{i,j},$$

so we can reinterpret this inequality in the matrix algebra $\mathbb{M}_n(\mathcal{B})$ as

$$0 \leq \langle T(x_i), T(x_j) \rangle_{i,j} \leq \|T\|^2 \langle x_i, x_j \rangle_{i,j}.$$

The latter immediately implies

$$\left( \sum_{i,j} a_{i,j}^* (T(x_i), T(x_j)) a_{i,j} \right)^{1/2} \leq \|T\| \left( \sum_{i,j} a_{i,j}^* (x_i, x_j) a_{i,j} \right)^{1/2}$$

for any $a_{i,j} \in L_p(\mathcal{M})$. This yields the first assertion.

The second follows from the same argument by noticing that $\text{Id}_{E'} \otimes T$ is bounded on the right Hilbert $\mathcal{B}$-module $E' \otimes_{\mathcal{B}} E$ with norm $\|T\|$. Indeed, for finite families $x_i \in E$ and $y_i \in E'$ with $1 \leq i \leq n$

$$\langle y_i \otimes T(x_i), \sum_i y_i \otimes T(x_i) \rangle_{E' \otimes_{\mathcal{B}} E} = \sum_{i,j} \langle T(x_i), \langle y_i, y_j \rangle_{E'} T(x_j) \rangle_E.$$

As a positive element in $\mathbb{M}_n(\mathcal{B})$, the matrix $\langle y_i, y_j \rangle_{E'}_{i,j}$ can be written as

$$\langle y_i, y_j \rangle_{E'}_{i,j} = \sum_k (b_{k,i}^* b_{k,j})_{i,j} \text{ with } b_{k,j} \in \mathcal{B}.$$

Thus we deduce

$$\langle \sum_i y_i \otimes T(x_i), \sum_i y_i \otimes T(x_i) \rangle_{E' \otimes_{\mathcal{B}} E} \leq \|T\|^2 \sum_k \langle \sum_i b_{k,i} x_i, \sum_i b_{k,i} x_i \rangle_E$$

$$= \|T\|^2 \sum_i \langle y_i \otimes x_i, \sum_i y_i \otimes x_i \rangle_{E' \otimes_{\mathcal{B}} E}.$$

This is the desired boundedness of $\text{Id}_{E'} \otimes T$ on $E' \otimes_{\mathcal{B}} E$. Thus the proposition is proved. \qed

Thanks to the identifications recalled at the beginning of this section, the previous proposition immediately implies the following...
Remark 3.3. If $T : E \to E$ is a bounded right $B$-module map, then for any left $B$-module $F$
\[
\|T \otimes \text{Id}_{E_p(M)} \otimes \text{Id}_F : E^c \otimes_p L_p(M) \otimes_p F^r \to E^c \otimes_p L_p(M) \otimes_p F^r\| \leq \|T\|.
\]
A similar statement holds for bimodules, namely, $\text{Id}_{E^r} \otimes T \otimes \text{Id}_{E_p(M)} \otimes \text{Id}_F$ is bounded on $(E' \otimes_B E^c) \otimes_p L_p(M) \otimes_p F^r$ with norm less than or equal to $\|T\|$ if additionally $E$ is equipped with a left $B$-action and $T$ is $B$-bimodular.

Remark 3.4. Proposition 3.2 is the modular version of the fact that the row and column $p$-operator
spaces are homogeneous (i.e., when $B = \mathbb{C}$ and $E$ and $F$ are just Hilbert spaces).

A typical situation to which we will apply the previous results is the case where $E = F = A$ with $A$
a finite von Neumann algebra containing $B$. The right inner product on $A$ is $\langle x, y \rangle = \mathbb{E}(x^* y)$
and the left $\langle x, y \rangle = \mathbb{E}(x y^*)$, $\mathbb{E}$ being the trace preserving conditional expectation from $A$
on $B$. In this case, we have $A^c \otimes_2 L_2(B) = L_2(A)$ isometrically. Thus if $T : A \to A$ is right $B$-modular
and bounded for $\langle , \rangle$, it automatically extends to a map $\tilde{T} : L_2(A) \to L_2(A)$ with
$\|\tilde{T}\|_{B(L_2(A))} \leq \|T\|$. Since $\|\langle x, y \rangle\|_B = \sup_{b \in L_2(B)} \|b\|_2 \|bx\|_2$ for $x \in A$, we actually have $\|\tilde{T}\|_{B(L_2(A))} = \|T\|$. We
may still write $T$ instead of $\tilde{T}$.

We state this fact as a lemma for later use.

Lemma 3.5. Let $(A, \tau)$ be a finite von Neumann and $B \subset A$ a von Neumann subalgebra with
the associated conditional expectation $\mathbb{E}$. Assume that $A$ is equipped with the right $B$-module inner
product $(x, y) = \mathbb{E}(x^* y)$. Then any bounded right $B$-module map $T : A \to A$ extends to a bounded
map on $L_2(A)$ with $\|T\|_{B(L_2(A))} = \|T\|$.

If $T : A \to A$ is completely positive and leaves $B$ invariant, then it satisfies the above conditions. If
additionally $\mathbb{E} \circ T \leq \mathbb{E}$, then $T$ also extends to a completely bounded map on $L_p(A)$.

We will need the following

Proposition 3.6. Let $T : L_p(M) \to L_p(M)$ be a completely bounded map. Then for any right
Hilbert $B$-module $E$ and left Hilbert $B$-module $F$ we have
\[
\|\text{Id}_E \otimes T \otimes \text{Id}_F\|_{cb(E^c \otimes_p L_p(M) \otimes_p F^r)} \leq \|T\|_{cb}.
\]

Proof. This is a direct consequence of the fact that identifying $E^c \otimes_p L_p(M) \otimes_p F^r$ with a subspace
of $L_p(B_{\ell_2(\mathbb{C}^2)} \otimes M)$, then $\text{Id}_E \otimes T \otimes \text{Id}_F$ acts like $\text{Id} \otimes T$. $\square$

4. Multipliers on amalgamated free products

This section is the core of the article and contains our major novelty. The principal result
is Theorem 4.9 that will imply Theorem 1.3 by iterations. Throughout the section, $(A_i, \tau_i)_{i \in I}$
will denote a family of finite von Neumann algebras containing $B$ as a common subalgebra. We
will use notation introduced in section 2 on amalgamated free products, in particular, $(A, \tau) = 
\ast_{i \in I} (A_i, \tau_i)$.

4.1. An intermediate result. Given a family $\pi = (\pi_i)_{i \in I}$ of *-representations $\pi_i : A_i \to A_i$ such that
$\pi_i(b) = b$ for all $b \in B$ and $\mathbb{E} \circ \pi_i = \mathbb{E}$, we introduce a linear map $T_\pi$ on $W$ by $T_\pi(b) = b$
for $b \in B$ and
\[
T_\pi(a_1 \otimes \cdots \otimes a_n) = \pi_i(a_1) \otimes a_2 \otimes \cdots \otimes a_n
\]
for $n \geq 1$ and $a_1 \otimes \cdots \otimes a_n \in W_\pi$ where $\pi = (\pi_1, \cdots, \pi_i)$ and $i_1 \neq \cdots \neq i_n$. Define $T_\pi^p$ on $W$ as
$T_\pi^p(x) = T_\pi(x^*)^*$. Note that both $T_\pi$ and $T_\pi^p$ commute with the projections $P_i$.

We aim to show that $T_\pi$ extends to a bounded map on $L_p(A)$ for $1 < p < \infty$. To this end, we
introduce some paraproducts on $W \times W$ in the manner of [23]: for $x, y \in W$
\[
\begin{align*}
x \uparrow y &= xy - \mathbb{E}_x[H_x^p(xH_y^p(y))], \\
0 \uparrow 1 &= xy - \mathbb{E}_x[H_x^p(H_y(x)y)], \\
1 \uparrow 1 &= 1, \quad 0 \uparrow 0, \quad 1 \uparrow 0, \quad 0 \uparrow 1,
\end{align*}
\]
where $H_x$ and $H_x^p$ are the free Hilbert transform defined in (3) and $\mathbb{E}_x$ is the conditional expectation
over all possible choices of symmetric independent signs $\varepsilon = (\varepsilon_i)$. Note that when $x$ and $y$ are


elementary tensors, then $x \not\uparrow y$ collects in $xy$ the parts that do not end in the same algebra as $y$, that is, all letters in $y$ must have been simplified. Similarly, $x \not\uparrow y$ collects in $xy$ the parts that do not start in the same algebra as $x$.

We will need some elementary free algebraic facts.

**Lemma 4.1.** Let $g \in W_l$, $h \in W_n$ with $l, n \geq 0$.

i) If $l > n$, then $T_\pi(g h) = T_\pi(g) h$.

ii) If $n > l$, then $T_\pi^\text{op}(g h) = g T_\pi^\text{op}(h)$.

iii) If $l = n$, then $P_{\geq 2}[T_\pi(g h)] = P_{\geq 2}[T_\pi^\text{op}(g) h]$ and $P_{\geq 2}[T_\pi^\text{op}(g h)] = P_{\geq 2}[g T_\pi^\text{op}(h)]$.

**Proof.** One can assume that $g$ and $h$ are elementary tensors. The first item is then clear as the first letter of $g$ cannot be cancelled if $l > n$. The second is obtained by passing to adjoints. Similarly, if $n = l$, $P_{\geq 2}(g h)$ is a sum of elementary tensors that all start with the first letter of $g$ and end with the last letter of $h$, up to a multiplication by an element of $E$.

The following Cotlar type formula immediately follows from the previous lemma.

**Lemma 4.2.** For $g, h \in W$ we have

\[
\begin{align*}
P_{\geq 2}[T_\pi(g T_\pi^\text{op}(h))] = P_{\geq 2}[T_\pi(g T_\pi^\text{op}(h))] + T_\pi^\text{op}(T_\pi(g) h) - T_\pi T_\pi^\text{op}(g h).
\end{align*}
\]

**Proof.** By linearity, it suffices to show the formula for $g \in W_l$, $h \in W_n$. Then it remains to check it by using Lemma 4.1 according to the different cases. We omit the details.

**Lemma 4.3.** Let $g = g_1 \otimes \cdots \otimes g_l \in W_l$, $h = h_n \otimes \cdots \otimes h_1 \in W_n$ with $l, n \geq 0$, and let $g' = g_2 \otimes \cdots \otimes g_l$ and $h' = h_n \otimes \cdots \otimes h_2$.

i) If $n > l$, then $P_1(g h) = \delta_{n+l,1} E(g h') h_1$.

ii) If $n > l$, then $P_1(g h) = g_{n+1} g_1 E(g' h)$.

iii) If $l = n$, then $P_1(g h) = P_1(g_1 E(g' h') h_1)$.

**Proof.** This proof is easy. Let us verify only i). If $n > l + 1$, then both $P_1(g h)$ and $E(g h') h_1$ vanish. The case $n = l + 1$ is checked by induction on $l$ and $n$ thanks to the following simplification formula:

\[
g h = x \otimes g_{n+1} \otimes y + x E(g h_n) y,
\]

where $x = g_1 \otimes \cdots \otimes g_{l-1}$ and $y = h_{n-1} \otimes \cdots \otimes h_1$. It then follows that $P_1(g h) = P_1(x E(g h_n) y)$. 

The following is a complement to the Cotlar formula (4).

**Lemma 4.4.** For $g, h \in W$ we have

\[
\begin{align*}
P_1(T_\pi(g T_\pi^\text{op}(h))) &= T_\pi\bigl(P_1(g 1 T_\pi^\text{op}(h))\bigr) + T_\pi^\text{op}\bigl(P_1(T_\pi(g) 0 h)\bigr) + T_\pi\bigl(P_1(g 0 h)\bigr).
\end{align*}
\]

**Proof.** We will check the formula according to the three cases in Lemma 4.3 and use notation there.

For i), we can assume $n = l + 1$, otherwise all terms are $0$. Then

\[
P_1(T_\pi(g T_\pi^\text{op}(h))) = E(T_\pi(g h') \pi_{j_1}(h_1)).
\]

On the other hand, for elementary tensors $x \in W_{i_1 \cdots i_l}$ and $y \in W_{j_n \cdots j_1}$, we have $x \not\uparrow y = 0$ as the last letter of $y$ in $xy$ is not cancelled. To deal with $x \not\uparrow y$, note that

\[
H_x[P_1(H_x(x) y)] = \varepsilon_{i_1 j_1} P_1(x y).
\]

It thus follows that $P_1(x \not\uparrow y) = P_1(x y)$ since $E(xy') = 0$ unless $i_1 = j_2$ but then $j_1 \neq j_2$ and $E_{l}(\varepsilon_{i_1 j_1}) = 0$. Hence the right hand side of (5) is exactly $T_\pi^\text{op}\bigl[P_1(T_\pi(g) h)\bigr] = E(T_\pi(g h') \pi_{j_1}(h_1))$.

The case ii) is obtained from i) by passing to adjoints.

For iii), let $x \in W_{i_1 \cdots i_l}$ and $y \in W_{j_n \cdots j_1}$ be elementary tensors. Then $P_1(x y) = 0$ unless $i_k = j_k$ for all $1 \leq k \leq n$. Hence we can assume that $i_1 = j_1$. Then noting that

\[
H_x[P_1(H_x(x) y)] = \varepsilon_{i_1 j_1}^2 P_1(x y) = P_1(x y),
\]
we deduce $P_1(x \mapsto y) = 0$; by symmetry, $P_1(x \mapsto y) = 0$ too. Thus the right hand side of (5) becomes

$$T_\pi \left[ P_1 \left( g \mapsto h \right) \right] = T_\pi \left[ P_1 \left( g_1 E(g'h')h_1 \right) \right] = \pi_i \left[ P_1 \left( g_1 E(g'h')h_1 \right) \right].$$

However, the left hand side is

$$P_1 \left[ \pi_i \left( g_1 E(g'h') \right) \pi_i (h_1) \right] = P_1 \left[ \pi_i \left( g_1 E(g'h') \right) \pi_i (h_1) \right] = \pi_i \left[ P_1 \left( g_1 E(g'h')h_1 \right) \right]$$

for $\pi_i$ is a $*$-representation, leaves the elements of $B$ invariant and $P_1 \pi_i = \pi_i P_1$. (5) is thus proved in the case $l = n$ too.

The following is an intermediate result to Theorem 4.9, it will be the key for the reduction formula in Theorem 4.6 below.

**Theorem 4.5.** The map $T_\pi$ extends to a completely bounded map on $L_p(A)$ for all $2 \leq p < \infty$ with cb-norm majorized by a constant depending only on $p$.

**Proof.** As the $\pi_i$'s are trace preserving and leave $B$ invariant, $T_\pi$ is an isometry on $W$ for the $L_2(A)$-norm, thus it extends to an isometry on $L_2(A)$. We need only to prove the boundedness of $T_\pi$ since the complete boundedness will then be automatic thanks to the usual trick of replacing $B$ by the matrix algebra $M_n(B)$.

As in [23], we show that the $L_p$-boundedness of $T_\pi$ implies its $L_{2p}$-boundedness for $2 \leq p < \infty$. Starting with $p = 2$, using iteration and interpolation, we will deduce the assertion for the full range $2 < p < \infty$. In the following, we will denote by $\gamma_{2p}$ the norm of $T_\pi$ and $T_\pi^{op}$ on $W$ equipped with the $L_{\mu}$-norm.

For $x \in W$ we write

$$T_\pi(x)T_\pi(x)^* = T_\pi(x)T_\pi^{op}(x^*)$$

$$= P_{\geq 2} [T_\pi(x)T_\pi^{op}(x^*)] + P_1 [T_\pi(x)T_\pi^{op}(x^*)] + E[T_\pi(x)T_\pi^{op}(x^*)].$$

By (4) and the fact that $P_{\geq 2}$ has norm less than 5 on $L_p(A)$, we have

$$\|P_{\geq 2} [T_\pi(x)T_\pi^{op}(x^*)]\|_p \leq 5 [2\gamma_p \|x\|_{2p}\|T_\pi(x)\|_{2p} + \gamma_{2p}^2 \|x\|_{2p}^2].$$

On the other hand, by [23, Proposition 3.14] or Lemma 2.1, the paraproductions $i,j$ are bounded from $L_{2p}(A) \times L_{2p}(A)$ to $L_p(A)$ with norm less than $\eta_p$. Thus using (5) and the fact that $P_1$ has norm less than 3 on $L_p(A)$, we get

$$|P_1 [T_\pi(x)T_\pi^{op}(x^*)]|_p \leq 3\eta_p \gamma_p \|x\|_{2p}\|T_\pi(x)\|_{2p} + \|x\|_{2p}^2.$$ 

Clearly,

$$\|E[T_\pi(x)T_\pi^{op}(x^*)]\|_p = \|E(x^*x)\|_p \leq \|x\|_{2p}.$$ 

So combining all the estimates yields

$$\|T_\pi(x)\|_{2p}^2 \leq \gamma_{2p} \left( 10 + 6\eta_p \right) \|x\|_{2p}\|T_\pi(x)\|_{2p} + \left( 1 + 3\eta_p \gamma_{2p} + 5\gamma_{2p}^2 \right) \|x\|_{2p}^2.$$ 

It thus follows that

$$\|T_\pi(x)\|_{2p} \leq C\gamma_{2p} \|x\|_{2p}$$

for some absolute constant $C$, whence $\gamma_{2p} \leq C\gamma_{2p} \eta_p$. This finishes the proof. \qed

**4.2. A length reduction formula.** We show here how to recursively estimate the $L_p$-norm of an element in $W$ in the spirit of [17]. The space $W$ is naturally a right Hilbert $B$-module with inner product $\langle x, y \rangle = E(x^*y)$ and also a left Hilbert $B$-module with $\langle \langle x, y \rangle \rangle = E(xy^*)$. The same holds for $W_1$ and $\tilde{W}$ as submodules. A typical element in $W$ can be written as a finite sum

$$x = x_0 + x_1 + \sum_{i, \alpha} a_i(\alpha) \otimes b_i(\alpha) = x_0 + x_1 + z,$$

where $x_0 \in B$, $x_1 \in W_1$ and $a_i(\alpha) \in \mathcal{A}$, and $b_i(\alpha) \in \tilde{W}$ with $L_i(b_i(\alpha)) = 0$.

The following result extends the main result of [17] on homogeneous polynomials to any polynomials, which is the key tool for the argument in the next subsection.

**Theorem 4.6.** With the notation above, for $2 \leq p < \infty$, we have (considering $z \in W_1 \otimes \tilde{W}$)

$$\|x\|_p \approx_p \|x_0\|_p + \|x_1\|_p + \|z\|_{W_1 \otimes_p L_p(A)} + \|z\|_{L_p(A) \otimes_p \tilde{W}}.$$
Proof. By the boundedness of the projections $P_0$ and $P_1$, it suffices to prove the estimate for $z$. To this end, we consider two copies of $\mathcal{A}$, and put a superscript to distinguish them. We use associativity of the free product to write
\[ \mathcal{A}^{(1)} \ast_B \mathcal{A}^{(2)} = (\ast_{i \in I, B} \mathcal{A}_i^{(1)}) \ast_B (\ast_{i \in I, B} \mathcal{A}_i^{(2)}) = \ast_{i \in I, B} (\mathcal{A}_i^{(1)} \ast_B \mathcal{A}_i^{(2)}) \text{ def} \ast_{i \in I, B} \mathcal{A}_i. \]
Since the traces are compatible, $\|x\|_{L_p(\mathcal{A})} = \|x^{(1)}\|_{L_p(\mathcal{A}^{(1)} \ast_B \mathcal{A}^{(2)})}$ for every $x \in L_p(\mathcal{A})$.

For each $i$, we define the swap map $\pi_i$ on $\mathcal{A}_i$ by
\[ a_i^{(j)} \otimes a_i^{(k)} \mapsto a_i^{(j)} \otimes a_i^{(k)} \]
for $\mathcal{B}$-centered elements $a_i^{(j)}$ and by $\pi_i(b) = b$ for $b \in \mathcal{B}$. It is clear that $\pi_i$ is a $*$-representation, $E$-preserving and leaves the elements of $\mathcal{B}$ invariant. Thus, noticing that $\pi_i^2 = \text{Id}_{\mathcal{A}_i}$, we use Theorem 4.5 to get
\[ \left\| \sum_{\alpha} a_i(\alpha) \otimes b_i(\alpha) \right\|_{L_p(\mathcal{A})} = \sum_{\alpha} \left\| a_i(\alpha) \otimes b_i(\alpha) \right\|_{L_p(\mathcal{A}^{(1)} \ast_B \mathcal{A}^{(2)})}. \]

The element $\sum_{\alpha} a_i(\alpha) \otimes b_i(\alpha)$ is homogeneous of degree 2 with respect to the length of $\mathcal{A}^{(1)} \ast_B \mathcal{A}^{(2)}$. Thus by [17, Theorem B]
\[ \left\| \sum_{\alpha} a_i(\alpha) \otimes b_i(\alpha) \right\|_{L_p(\mathcal{A}^{(1)} \ast_B \mathcal{A}^{(2)})} \leq \left\| \sum_{\alpha} |a_i(\alpha)(2)\otimes b_i(\alpha)(1)|_{p} \right\| + \left\| \sum_{\alpha} |a_i(\alpha)(2)\otimes b_i(\alpha)(1)|_{p} \right\|. \]

Since taking copies clearly does not change norms, we have
\[ \left\| \sum_{\alpha} |a_i(\alpha)(2)\otimes b_i(\alpha)(1)|_{p} \right\| = \left\| \sum_{\alpha} a_i(\alpha)(2) \otimes b_i(\alpha)(1) \right\|_{(\mathcal{W}^{(2)})^* \otimes \mathcal{L_p}(\mathcal{A}^{(1)} \ast_B \mathcal{A}^{(2)})} \]
\[ = \left\| \sum_{\alpha} a_i(\alpha) \otimes b_i(\alpha) \right\|_{\mathcal{W}^* \otimes \mathcal{L_p}(\mathcal{A})}. \]

Similarly,
\[ \left\| \sum_{\alpha} a_i(\alpha)(2) \otimes b_i(\alpha)(1) \right\|_{p} = \left\| \sum_{\alpha} a_i(\alpha) \otimes b_i(\alpha) \right\|_{\mathcal{L_p}(\mathcal{A}) \otimes \mathcal{W}^*}. \]

This concludes the proof of the theorem. \qed

Remark 4.7. We could also have used [17, Theorem C] to make some terms more explicit. Namely,
\[ \|x\|_{p} \simeq \| (\mathbb{E}(x_1^* x_1))^{1/2} \|_{p} + \left( \sum_{t} \| L_t(x_1) \|_{p}^{1/p} \right)^{1/p} + \| (\mathbb{E}(x_1 x_1^*))^{1/2} \|_{p}, \]
and
\[ \|z\|_{L_p(\mathcal{A}) \otimes \mathcal{W}^*} \simeq \|z\|_{\mathcal{W}^* \otimes \mathcal{W}^*} + \| (\mathbb{E}(zz^*))^{1/2} \|_{p} + \left( \sum_{\alpha, \beta} \| \mathbb{E}(b_i(\alpha) b_i(\beta^*) a_i(\beta)) \|_{p} \right)^{1/p}. \]

Here $\mathcal{W}^* \otimes \mathcal{W}^*$ has to be understood as $\mathcal{W}^* \otimes_{\mathcal{B}_p} \mathcal{L_p}(\mathcal{A}) \otimes \mathcal{W}^*$.

Remark 4.8. It is now rather easy to get an analogue of [17, Theorem C]. The norm of the last term $\|z\|_{\mathcal{W}^* \otimes \mathcal{L_p}(\mathcal{A})}$ corresponds to that of an element in $S_p \otimes \mathcal{L_p}(\mathcal{A})$ and we can formally iterate the argument. Thus, we can write the norm of $x \in \mathcal{P}_{\geq k}(\mathcal{L_p}(\mathcal{A}))$ as a sum of $2k+1$ norms. For simplicity assume $x \in \mathcal{P}_{\geq k}(\mathcal{W})$, they are given by
\[ \|x\|_{\mathcal{W}^* \otimes \mathcal{W}^*}, \quad 0 \leq l \leq k, \]
\[ \|x\|_{\mathcal{W}^* \otimes \mathcal{L_p}(\mathcal{A})^* \otimes \mathcal{W}^*}, \quad 0 \leq l \leq k - 2 \]
\[ \|x\|_{\mathcal{W}^* \otimes \mathcal{L_p}(\mathcal{A})}. \]

The last one being recursive. We leave the details to the interested reader.
4.3. Maps of the $d$-th letters and the proof of Theorem 1.3. This subsection contains our principal result that is the key step of the proof of Theorem 1.3. Fix $1 < p < \infty$. Given a family of maps $T_i : L_p(\mathcal{A}_i) \to L_p(\mathcal{A}_i)$ we will define an associated map of the $d$-th letters of reduced words in $\mathcal{W}$ for $d \geq 1$. The minimal assumption required for the $T_i$'s is the following

(H1) $T_i$ is $\mathcal{B}$-bimodular and $T_i(\mathcal{L}_q(\mathcal{A}_i)) \subset \mathcal{L}_q(\mathcal{A}_i)$ for $q = 2$ and $q = p$.

(H2) $T_i : L_q(\mathcal{A}_i) \to L_q(\mathcal{A}_i)$ is completely bounded and

$$ cb_q = \sup_{i \in I} \| T_i \|_{cb(\mathcal{L}_q(\mathcal{A}_i))} < \infty \quad \text{for} \quad q = 2 \quad \text{and} \quad q = p. $$

Note that the cb-norm of $T_i$ on $L_2(\mathcal{A}_i)$ coincides with its usual norm for $L_2(\mathcal{A}_i)$ is a homogeneous operator space. On the other hand, it is obvious that

$$ cb_q = \| \oplus_i T_i : L_q(\oplus_i \mathcal{A}_i) \to L_q(\oplus_i \mathcal{A}_i) \|_{cb}. $$

Now we define a linear map $T^{(d)}$ on $\mathcal{W}$ by $T^{(d)}(b) = b$ for $b \in \mathcal{B}$ and

$$ T^{(d)}(a_1 \otimes \cdots \otimes a_n) = \left\{ \begin{array}{ll}
    a_1 \otimes \cdots \otimes a_{d-1} \otimes T_d(a_d) \otimes a_{d+1} \otimes \cdots \otimes a_n & \text{if } d \leq n, \\
    a_1 \otimes \cdots \otimes a_n & \text{if } d > n
\end{array} \right. $$

for $n \geq 1$ and $a_1 \otimes \cdots \otimes a_n \in \mathcal{W}_d$ with $\mathbf{i} = (i_1, \cdots, i_n)$ and $i_1 \neq \cdots \neq i_n$. Note that the range of $T^{(d)}$ is not inside $\mathcal{W}$ but clearly in $L_p(\mathcal{A})$.

**Theorem 4.9.** Under the hypotheses (H1) and (H2), $T^{(d)}$ extends to a completely bounded map on $L_p(\mathcal{A})$ with

$$ \| T^{(d)} \|_{cb(L_p(\mathcal{A}))} \lesssim_{p,d} cb_2 + cb_p. $$

**Proof.** We start the proof by a crucial observation related to Lemma 3.5. We view $\mathcal{A}_i$ as a right Hilbert $\mathcal{B}$-module with the inner product $\langle x, y \rangle = \mathbb{E}(x^* y)$. We claim that $\mathbb{E}(T_i(x)^* T_i(y))$ belongs to $\mathcal{B}$ for any $x, y \in \mathcal{A}$. By polarization, we can assume $x = y$. Then for $b \in L_2(\mathcal{A}_i)$, by the modularity of $T_i$,

$$ \tau[b \mathbb{E}(T_i(x)^* T_i(x))] = \tau[\mathbb{E}(T_i(xb)^* T_i(xb))] = \| T_i(xb) \|_2 \leq \| T_i \| \| xb \|_2. $$

This implies the claim, as well as $\mathbb{E}(T_i(x)^* T_i(x)) \leq \| T_i \| \langle x, x \rangle$. Thus $T_i(\mathcal{A}_i)$ is a $\mathcal{B}$-bimodule. A similar statement holds when $\mathcal{A}_i$ is viewed as a left $\mathcal{B}$-module with the inner product $\langle x, y \rangle = \mathbb{E}(xy^*)$.

We have stated the results in section 3 for maps $T : E \to E$ but they clearly remain true if $T : E \to E'$. We can thus apply them to the restriction of $T_i$ to $\mathcal{A}_i$ with range $T_i(\mathcal{A}_i)$. The same holds for direct sums.

An immediate consequence is the boundedness of $T^{(d)}$ on $\mathcal{W}_d$ equipped with the $L_2$-norm, this follows from the orthogonality of the $\mathcal{W}_d$'s and Proposition 3.2 combined with the above observation. So the theorem holds for $p = 2$. On the other hand, by duality, we need only to consider the case $2 < p < \infty$ that will be assumed in the remainder of the proof.

Before going into the core of the proof, we point out that the result of Theorem 4.6 can be applied to $T^{(d)}(x)$ when $x \in \mathcal{W}$ using an obvious approximation argument. One just needs to adapt correctly the modules, replacing $\mathcal{W}_d^\mathcal{B}$ by $T^{(1)}(\mathcal{W}_d^\mathcal{B})$ if $d = 1$ and $\mathcal{W}^\mathcal{B}$ by $T^{(d-1)}(\mathcal{W})^\mathcal{B}$ if $d \geq 1$.

We continue the proof by induction on $d$. The main part is the initial step: $d = 1$. By the usual argument of tensoring with the matrix algebras $\mathcal{M}_{an}$, it suffices to prove the boundedness of $T^{(1)}$. We will apply Theorem 4.6. Let $x = x_0 + x_1 + z \in \mathcal{W}$ as (6).

To deal with $\| T^{(1)}(x_1) \|_p$, we use the Khintchine inequality from Remark 4.7. We have that $T^{(1)}$ is bounded on $L_p(\oplus_i \mathcal{A}_i)$ with norm majorized by $cb_p$, that is,

$$ \left( \sum_i \| L_i(T^{(1)}(x_1)) \|_p^p \right)^{1/p} = \left( \sum_i \| T_i(L_i(x_1)) \|_p^p \right)^{1/p} \leq cb_p \left( \sum_i \| x_i \|_p^p \right)^{1/p}. $$

On the other hand, thanks to the previous observation, $\oplus_i T_i$ can be viewed as a bounded modular map on $\mathcal{W}_d$ with respect to both inner products $\langle \cdot, \cdot \rangle$ and $\langle \cdot', \cdot' \rangle$ with norm bounded by $cb_2$. Thus by Proposition 3.2,

$$ \| (\mathbb{E}[T^{(1)}(x_1)^* T^{(1)}(x_1)]) \|^{1/2}_p = \| (\mathbb{E}[(\oplus_i T_i(x_1))^* (\oplus_i T_i(x_1))]) \|^{1/2}_p \leq cb_2 \| (\mathbb{E}(x_1 x_1))^1/2 \|^{1/2}_p. $$
and similarly for the second inner product. Hence, 
\[ \|T^{(1)}(x_1)\|_p \lesssim (cb_2 + cb_p)\|x_1\|_p. \]

For the remaining part \( z \), note that 
\[ T^{(1)}(z) = \sum_{i,\alpha}[\oplus_i T_i]\{a_i(\alpha)\} \otimes b_i(\alpha). \]

Thus by the observation that \( \oplus_i T_i : \mathcal{W}_1 \to T^{(1)}(\mathcal{W}_1) \) is \( \mathcal{B} \)-bimodular and Proposition 3.2, we again have
\[ \left\| \sum_{i,\alpha} T_i(a_i(\alpha)) \otimes b_i(\alpha) \right\|_{T^{(1)}(\mathcal{W}_1) \otimes_p L_p(\mathcal{A})} \leq cb_2 \left\| \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \right\|_{L^*_p(\mathcal{A}) \otimes_p L_p(\mathcal{A})}. \]

For the other norm of \( z \), we use Proposition 3.6 to obtain
\[ \left\| \sum_{i,\alpha} T_i(a_i(\alpha)) \otimes b_i(\alpha) \right\|_{L_p(\mathcal{A}) \otimes_p \mathcal{W}^v} \leq cb_p \left\| \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \right\|_{L_p(\mathcal{A}) \otimes_p \mathcal{W}^v}. \]

Thus \( T^{(1)} \) extends to a (completely) bounded map on \( L_p(\mathcal{A}) \).

Now assume that \( d \geq 2 \) and \( T^{(d-1)} \) is completely bounded on \( \mathcal{W} \) for the \( L_p \)-norm. For \( x = x_0 + x_1 + z \) as above, we have
\[ T^{(d)}(x_0 + x_1) = x_0 + x_1 \quad \text{and} \quad T^{(d)}(z) = \sum_{i,\alpha} a_i(\alpha) \otimes T^{(d-1)}(b_i(\alpha)). \]

Using the boundedness of \( T^{(d-1)} \) on \( L_2(\mathcal{A}) \) and Proposition 3.2, we have
\[ \left\| T^{(d)} \left( \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \right) \right\|_{L_p(\mathcal{A}) \otimes_p T^{(d-1)}(\mathcal{W})^v} \leq cb_2 \left\| \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \right\|_{L_p(\mathcal{A}) \otimes_p \mathcal{W}^v}. \]

Similarly, the complete boundedness of \( T^{(d-1)} \) on \( L_p(\mathcal{A}) \) and Proposition 3.6 imply
\[ \left\| T^{(d)} \left( \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \right) \right\|_{L^*_p(\mathcal{A}) \otimes_p L_p(\mathcal{A})} \lesssim cb_p \left\| \sum_{i,\alpha} a_i(\alpha) \otimes b_i(\alpha) \right\|_{L^*_p(\mathcal{A}) \otimes_p L_p(\mathcal{A})}. \]

This concludes the induction, and the proof of the theorem too. \( \square \)

Theorem 1.3 immediately follows from Theorem 4.9.

Proof of Theorem 1.3. For \( 1 \leq k \leq d \) let \( T^{(k)} \) be the map \( T^{(k)} \) in Theorem 4.9 associated to the family \( (T_{k,i})_{i \in I} \). Then
\[ T^{Ld} = T^{(1)}_{1} T^{(2)}_{2} \cdots T^{(d)}_{d}. \]

This yields the assertion. \( \square \)

We extend Lemma 2.1 to the Hilbert transform of the \( d \)-th letters in the spirit of Theorem 4.9 and Theorem 1.3. Let \( \varepsilon = (\varepsilon_i)_{i \in I} \) be a family of elements in the unit ball of \( \mathcal{Z}(\mathcal{B}) \). Let \( T_{i} \) be the left multiplication map on \( L_q(\mathcal{A}) \) by \( \varepsilon_i \). Clearly, the \( T_{i} \)'s satisfy the hypotheses \( (H_1) \) and \( (H_2) \). Denote the corresponding \( T^{(d)} \) by \( \mathcal{H}_{\varepsilon}^{(d)} \). If \( d = 1 \), this coincides with the free Hilbert transform in (3). More generally, given \( d \) let \( \varepsilon = (\varepsilon_{j,i})_{1 \leq j \leq d, i \in I} \) be a family in the unit ball of \( \mathcal{Z}(\mathcal{B}) \). The corresponding map \( T^{L,\varepsilon} \) as in Theorem 1.3 is denoted by \( \mathcal{H}_{\varepsilon}^{L,\varepsilon} \).

The following is a particular case of Theorem 1.3, it extends [23, Theorem 4.7] to the amalgamated free product case.

Corollary 4.10. Both \( \mathcal{H}_{\varepsilon}^{(d)} \) and \( \mathcal{H}_{\varepsilon}^{L,\varepsilon} \) extend to completely bounded maps on \( L_p(\mathcal{A}) \) for all \( 1 < p < \infty \) with \( cb \)-norms controlled by constants depending only on \( p \) and \( d \).

We conclude this section with the boundedness of some paraproducts that generalize those introduced at the beginning of subsection 4.1. These paraproducts are of independent interest in free analysis.
Let $\varepsilon = (\varepsilon_{j,i})_{j \geq 1, i \in J}$ be an independent family of symmetric random variables with values $\pm 1$. Let $\mathcal{H}_{\varepsilon}^{L_d}$ be the map associated to $(\varepsilon_{j,i})_{1 \leq j \leq d, i \in J}$ as in the previous corollary, and let $\mathcal{H}_{\varepsilon}^{L_d, \text{op}}(x) = [\mathcal{H}_{\varepsilon}^{L_d}(x^*)]^*$. We use the convention that $\mathcal{H}_{\varepsilon}^{L_d, \text{op}} = \text{Id}$. For $j, k \geq 0$ and $x, y \in \mathcal{W}$, define

$$x \ast y = \mathbb{E}_x \mathbb{E}_y \mathcal{H}_{\varepsilon}^{L_j} \mathcal{H}_{\varepsilon}^{L_k, \text{op}}[\mathcal{H}_{\varepsilon}^{L_j}(x) \mathcal{H}_{\varepsilon}^{L_k, \text{op}}(y)],$$

where $\mathbb{E}_x$ denotes the underlying expectation and $\varepsilon'$ is an independent copy of $\varepsilon$. This paraproduct is easily understood for elementary tensors $x$ and $y$: $x \ast y$ then collects all those terms in the development of $xy$ into elementary tensors whose first $j$ letters come from the same algebras of the first $j$ letters of $x$, and whose last $k$ letters from the same algebras of the last $k$ letters of $y$.

The previous corollary implies the following

**Corollary 4.11.** The paraproduct $\mathcal{H}_{\varepsilon}^{L_d} \ast$ extends to a bounded bilinear map from $L_{2p}(A) \times L_{2p}(A)$ to $L_p(A)$ for all $1 < p < \infty$ with norm majorized by a constant depending only on $p, j$ and $k$.

**Remark 4.12.** The reader familiar with Haagerup noncommutative $L_p$-spaces can extend, with necessary modifications, all results of this section to the type III case, that is, to amalgamated free products of von Neumann algebras equipped with faithful normal states instead of traces.

## 5. Multipliers on Free Products of Groups

In this section we will first prove Theorems 1.1 and 1.2, then consider Fourier multipliers on free products of general discrete groups. Recall that $\hat{\Gamma}$ denotes the group von Neumann algebra of a discrete group $\Gamma$ generated by the left regular representation $\lambda$.

### 5.1. Proofs of Theorems 1.1 and 1.2

**Proof of Theorem 1.2.** We will apply Theorem 1.3 to the special case where $\mathcal{A}_i = \hat{\mathbb{Z}} = L_\infty(\mathbb{T})$ for all $i \in \mathbb{N}$. Fix a family $z = (z_{j,i})_{1 \leq j \leq d, 1 \leq i \leq \infty}$ of complex numbers with modulus $1$. Define $T_{j,i}$ to be the measure preserving $*$-representation on $\mathcal{A}_i$ given by $T_{j,i}(\lambda(n)) = z_{j,i}\lambda(n)$ for any $n \in \mathbb{Z}$, or equivalently in terms of the generator $\zeta \in L_\infty(\mathbb{T})$, $T_{j,i}(\zeta^n) = z_{j,i}^n\zeta^n$. $T_{j,i}$ extends to a complete isometry on $L_p(\mathbb{T})$ for $1 \leq p < \infty$. The corresponding map $T^{L_d}$ is exactly the map $\alpha^{L_d}_z$ in Theorem 1.2. Thus Theorem 1.3 implies that $\alpha^{L_d}_z$ is completely bounded on $L_p(\hat{\mathbb{F}}_\infty)$ for $1 < p < \infty$, whence Theorem 1.2. \hfill $\Box$

**Proof of Theorem 1.1.** We will use $\alpha^{L_d}_z$ in the previous proof for the special case where $z_{j,i} = z_j$ for all $i$, and write $\alpha_z = \alpha^{L_d}_z$ for $z \in \mathbb{T}^d$. Thus $\alpha$ is a uniformly completely bounded action of $\mathbb{T}^d$ on $L_p(\hat{\mathbb{F}}_\infty)$. We then easily deduce Theorem 1.1 by the standard transference argument as presented in [4]. Let us give the details.

Let $m$ be a Hörmander-Mikhlin multiplier on $\mathbb{Z}^d$, that is, $m$ satisfies (1). Then the associated Fourier multiplier $T_m$ on $L_p(\mathbb{T}^d)$ is completely bounded with cb-norm majorized by $C_{p,d}\|m\|_{HM}$. This follows from [1] for $d = 1$ and [21, 37] for $d \geq 2$ since the Schatten $p$-class $S_p$ is a UMD space. Note that valid for general UMD spaces, the results in these papers require more regularity on $m$ than the condition (1), that is, the partial discrete derivations should run to all orders up to $d$ instead of $\left[\frac{d}{2}\right] + 1$ in (1). However, using the arguments of [36, section 4.1], we can show that when the UMD space in consideration is a noncommutative $L_p(\mathcal{M})$, we can go down again to the classical order $\left[\frac{d}{2}\right] + 1$. Hence

$$\|T_m \otimes \text{Id}_{S_p}: L_p(\mathbb{T}^d; S_p) \to L_p(\mathbb{T}^d; S_p)\|_{cb} \lesssim_{p,d} \|m\|_{HM}.$$  

The Schatten $p$-class $S_p$ here can be replaced by $L_p(\mathcal{M})$ for any QWEP $\mathcal{M}$, in particular, by $L_p(\hat{\mathbb{F}}_\infty)$. Thus

$$\|T_m \otimes \text{Id}_{L_p(\hat{\mathbb{F}}_\infty)}: L_p(\hat{\mathbb{T}}^d; L_p(\hat{\mathbb{F}}_\infty)) \to L_p(\hat{\mathbb{T}}^d; L_p(\hat{\mathbb{F}}_\infty))\|_{cb} \lesssim_{p,d} \|m\|_{HM}. $$

Now given $x \in L_p(\hat{\mathbb{F}}_\infty)$ define $f \in L_p(\mathbb{T}^d)$ by $f(z) = \alpha_z(x)$ for $z \in \mathbb{T}^d$. Then by Theorem 1.2

$$\|f(z)\|_{L_p(\hat{\mathbb{F}}_\infty)} \lesssim_{p,d} \|x\|_{L_p(\hat{\mathbb{F}}_\infty)}, \quad z \in \mathbb{T}^d.$$
Clearly, we have the intertwining identity:

\[ [T_m \otimes \text{Id}_{L_p(\hat{\mathbb{F}}_\infty)}](f)(z) = \alpha_z(M_m(x)), \quad z \in \mathbb{T}^d. \]

Thus we deduce

\[
\|M_m(x)\|_{L_p(\hat{\mathbb{F}}_\infty)}^p \lesssim_{p,d} \int_{\mathbb{T}^d} \|\alpha_z(M_m(x))\|_{L_p(\hat{\mathbb{F}}_\infty)}^p \, dz
\]

\[ = \|[T_m \otimes \text{Id}_{L_p(\hat{\mathbb{F}}_\infty)}](f)\|_{L_p(\hat{\mathbb{F}}_\infty)}^p
\]

\[ \lesssim_{p,d} \|m\|_{\text{HM}}^p \|f\|_{L_p(\mathbb{T}^d,L_p(\hat{\mathbb{F}}_\infty))}^p
\]

\[ \lesssim_{p,d} \|m\|_{\text{HM}}^p \|x\|_{L_p(\hat{\mathbb{F}}_\infty)}. \]

Therefore, \( M_m \) is bounded on \( L_p(\hat{\mathbb{F}}_\infty) \) with norm controlled by \( C_{p,d}\|m\|_{\text{HM}} \). The complete boundedness follows from the usual argument of tensoring with \( S_p \).

We end this subsection with some examples of Fourier multipliers on the free group. The free Hilbert transform of \([23]\) is a typical example of Fourier multipliers studied in this article. Theorem 1.1 allows us to exhibit plenty of examples of Fourier multipliers on the free group. We give here just some typical ones.

**Example 5.1.** Let \( \mathcal{A}_i = L_\infty(\mathbb{T}) \) for all \( i \in \mathbb{N} \) and \( z = (z_{j,i})_{1 \leq j \leq d, i \in I} \) be a family of complex numbers of modulus 1. Then the corresponding transforms \( \mathcal{H}_z^{(d)} \) and \( \mathcal{H}_z^{L_d} \) in Corollary 4.10 are completely bounded on \( L_p(\hat{\mathbb{F}}_\infty) \) for all \( 1 < p < \infty \).

**Example 5.2.** We give two more examples of similar nature: For \( 1 \leq j \leq d \) let \( T_{j,i} \) be the map on \( \mathcal{A}_i = L_\infty(\mathbb{T}) \) defined by \( T_{j,i}(\zeta^n) = z_{j,i}^{n\alpha(n)} \zeta^n \) for \( \zeta \in \mathbb{T} \) and \( n \in \mathbb{Z} \). Clearly, \( T_{j,i} \) is completely bounded on \( L_p(\mathbb{T}) \) for all \( 1 < p < \infty \). Let \( \mathcal{H}_z^{L_d} \) be the corresponding map \( T^{L_d} \) in Theorem 1.3 and \( \mathcal{H}_z^{(d)} \) the map \( T^{(d)} \) in Theorem 4.9 associated to \( (T_{d,i})_{i \in \mathbb{N}} \). Again, \( \mathcal{H}_z^{L_d} \) and \( \mathcal{H}_z^{(d)} \) are completely bounded on \( L_p(\hat{\mathbb{F}}_\infty) \) for all \( 1 < p < \infty \).

**Example 5.3.** The Riesz transforms \( R_j, 1 \leq j \leq d \), on \( L_p(\mathbb{T}^d) \) are the Fourier multipliers of symbols

\[ m_j(k) = \frac{k_j}{\|k\|} \quad \text{for} \quad k = (k_1, \cdots, k_d) \in \mathbb{Z}^d, \quad \|k\| = \sqrt{k_1^2 + \cdots + k_d^2}. \]

It is classical that \( R_j \) is completely bounded on \( L_p(\mathbb{T}^d) \) for \( 1 < p < \infty \). The corresponding multipliers \( m_{R_{j,d}} \) in Theorem 1.1 are denoted by \( R_{j,d} \) and may be called the free Riesz transforms of the first \( d \) letters on \( \mathbb{F}_\infty \). \( R_{j,d} \) is just the free Hilbert transform of \([23]\). \( R_{j,d} \), \( 1 \leq j \leq d \), are completely bounded on \( L_p(\hat{\mathbb{F}}_\infty) \) for \( 1 < p < \infty \).

**Example 5.4.** Our final example is given by the classical Littlewood-Paley multiplier. It is well known that the following Littlewood-Paley multiplier

\[ m = \sum_{i=0}^{\infty} \varepsilon_i 1_{\{|k| \in \mathbb{Z}^d : 2^i-1 \leq |k| < 2^{i+1}\}} \quad \text{with} \quad \varepsilon_i = \pm 1, \quad |k| = |k_1| + \cdots + |k_d| \]

is a completely bounded Fourier \( L_p \)-multiplier on \( \mathbb{Z}^d \) for all \( 1 < p < \infty \). The corresponding multiplier \( M_m \), denoted by \( LP_{c,L_d} \), is completely bounded on \( L_p(\hat{\mathbb{F}}_\infty) \). One can equally consider the more commonly used Littlewood-Paley multiplier:

\[ m = \sum_{(i_1, \cdots, i_d) \in \mathbb{Z}^d} \varepsilon_i 1_{R_{(i_1, \cdots, i_d)}} \]

where \( R_{(i_1, \cdots, i_d)} = I_{i_1} \times \cdots \times I_{i_d} \) and \( I_j = \{ k \in \mathbb{Z} : 2^j - 1 \leq |k| < 2^{j+1} \} \). It gives rise to a completely bounded multiplier on \( L_p(\hat{\mathbb{F}}_\infty) \) too.
5.2. More paraproducts. We make the paraproducts in Corollary 4.11 more precise in the case of free groups. Let \( z = (z_{j,i})_{j,i \in \mathbb{N}} \in \mathbb{T}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \). Let \( \alpha_{zd} \) be the map associated to \((z_{j,i})_{1 \leq j \leq d, i \in \mathbb{N}}\) in Theorem 1.2. Note that

\[
\alpha_{zd} = T_{zd}^{(1)} \cdots T_{zd}^{(d)},
\]

where \( T_{zd}^{(j)}(\lambda(g)) = \alpha_{pd}^{j} \alpha_{qj}^{j} \lambda(g) \) (with \( k_{j} = 0 \) for \( j > n \)) for \( g = g_{1}^{k_{1}} \cdots g_{n}^{k_{n}} \in F_{\infty} \) in reduced form. Put

\[
\alpha_{zd,op}(x) = [\alpha_{zd}(x^{*})]^{*} \quad \text{and} \quad T_{zd}^{(j),op}(x) = [T_{zd}^{(j)}(x^{*})]^{*}.
\]

Let \( \varepsilon = (\varepsilon_{j,i})_{j,i \in \mathbb{N}, i \in \mathbb{N}} \) be an independent family of symmetric signs. Recall that we use \( H_{\varepsilon}^{(j)} \) and \( H_{\varepsilon}^{(j),op} \) to denote the free Hilbert transforms on the (last) \( j \)-th letters considered in Corollary 4.10,

\[
H_{\varepsilon}^{(j)}(\lambda_{y}) = \varepsilon_{j,i} \lambda_{y} \quad \text{and} \quad H_{\varepsilon}^{(j),op}(\lambda_{y}) = \varepsilon_{j,i,m+1-j} \lambda_{y}.
\]

We use again the convention that \( T_{zd}^{0} = H_{\varepsilon}^{(0)} \). For \( j, k \geq 0 \) and \( x, y \in \mathbb{C}[F_{\infty}] \), define

\[
\begin{align*}
&x^{j,k} + y = E_{x} E_{y} \alpha_{jd}^{j} \alpha_{qj}^{k} \left[ \alpha_{pd}^{j} \alpha_{qj}^{k} \right] \left( x \alpha_{jd}^{j} \alpha_{qj}^{k} \right) \left( y \right), \\
&x^{j,k+1} + y = E_{x} E_{y} H_{\varepsilon}^{(j+1)} H_{\varepsilon}^{(j+1)} (x \alpha_{jd}^{j} \alpha_{qj}^{k+1} \alpha_{pd}^{j+1} \alpha_{qj}^{k+1} \alpha_{pd}^{j+1} \alpha_{qj}^{k}) \left( y \right),
\end{align*}
\]

where \( E_{x} \) and \( E_{y} \) denote the expectations on \( z \) and \( \varepsilon \), respectively.

As in the setting of free products of von Neumann algebras, we can easily interpret these paraproducts for \( x = \lambda(g) \) and \( y = \lambda(h) \). To this end, we say that the first \( j \) blocks of \( g \) survive in \( gh \) if the first \( j \) blocks of \( gh \) and \( g \) are exactly the same, and that the \( j \)-th block of \( g \) marks in \( gh \) if the \( j \)-th blocks of \( gh \) and \( g \) are powers of a same generator. Replacing \"first\" by \"last\" (i.e., counting the letters of a reduced word in the reverse order), we get similar notions.

Thus for \( g, h \in F_{\infty} \)

- \( \lambda(g)^{j,k} \lambda(h) = \lambda(gh) \) if the first \( j \) blocks of \( g \) and the last \( k \) blocks of \( h \) survive in \( gh \), \( \lambda(g)^{j,k} \lambda(h) = 0 \) otherwise;
- \( \lambda(g)^{j+1,k} \lambda(h) = \lambda(gh) \) if the first \( j \) blocks of \( g \) and the last \( k \) blocks of \( h \) survive in \( gh \), and in addition, the \( (j+1) \)-th block of \( g \) marks in \( gh \), \( \lambda(g)^{j+1,k} \lambda(h) = 0 \) otherwise.

Our first approach to Theorem 1.2 heavily relies on the boundedness of the above paraproducts and several variants of them. Now this boundedness immediately follows from Theorem 1.2.

**Proposition 5.5.** All the above paraproducts extend to bounded bilinear maps from \( L_{2p}(\hat{\mathbb{F}}_{\infty}) \times L_{2p}(\hat{\mathbb{F}}_{\infty}) \) to \( L_{p}(\hat{\mathbb{F}}_{\infty}) \) for all \( 1 < p < \infty \) with norm majorized by constants depending only on \( p, j \) and \( k \).

5.3. Extension to free products of groups. In the proof of Theorem 1.1 one can easily replace \( \mathbb{Z} \) by any abelian discrete group \( \Gamma \). However, to go beyond the abelian case, one needs extra efforts. In this subsection, \( \Gamma \) will denote a general discrete group. Let \( \Gamma_{\infty} = \Gamma^{-\infty} \) be the infinite free power of \( \Gamma \). Each \( g \in \Gamma_{\infty} \setminus \{e\} \) is written as a reduced word:

\[ g = g_{1}g_{2} \cdots g_{n} \]

with \( g_{j} \neq e \) belonging to the \( i_{j} \)-th copy of \( \Gamma \) in \( \Gamma_{\infty} \) and \( i_{1} \neq i_{2} = \cdots = i_{n} \).

We begin by extending Theorem 1.2 to this general setting. Define a linear map \( \alpha_{zd} : \mathbb{C}[\Gamma_{\infty}] \to \mathbb{C}[\Gamma^{d} \times \Gamma_{\infty}] \) as follows: for \( g = g_{1} \cdots g_{n} \in \Gamma_{\infty} \) as above in reduced form,

\[ \alpha_{zd}(\lambda(g)) = \lambda^{(\ell)}(g_{1}, \cdots, g_{d} \otimes \lambda(g)) \]

with \( g_{\ell} \in \lambda^{(\ell)}(g_{1}, \cdots, g_{d}) \) if \( \ell > n \). Here we have denoted by \( \lambda^{(\ell)} \) the left regular representation of \( \Gamma^{d} \) to avoid ambiguity (\( \lambda \) being that of \( \Gamma_{\infty} \)).

**Theorem 5.6.** Let \( d \in \mathbb{N} \) and \( 1 < p < \infty \). Then the map \( \alpha_{zd} \) extends to a completely isomorphic embedding of \( L_{p}(\hat{\Gamma}_{\infty}) \) into \( L_{p}(\hat{\Gamma}^{d} \otimes \hat{\Gamma}_{\infty}) \).
Proof. We use an argument similar to that of Theorem 4.6. Let \( G = \ast_{i=1} \cdots d+1 \Gamma \) and \( G_\infty = \ast_{i \geq 1} G \). To avoid confusion we use \( \Gamma_{i,l} \) to denote the \( l \)-th copy of \( \Gamma \) in the \( i \)-th copy of \( G \) in \( G_\infty \).

First, let \( \pi : G \to \hat{G} \) be the \( * \)-representation given by the cyclic permutation of the copies that sends the \( l \)-th copy of \( \Gamma \) to the \( l + (d + 1) \)-th copy. Consider the map \( T^{Ld} \) on \( L_p(\hat{G}_\infty) \) given by Theorem 1.3 associated to \( T_k = \pi_k, 1 \leq k \leq d \), as well as its inverse associated to \( T_k^{-1} = \pi_k^{-1}, 1 \leq k \leq d \). By Theorem 1.3, \( T^{Ld} \) is a complete isomorphism on \( L_p(\hat{G}_\infty) \). We will need the restriction of \( T^{Ld} \) to a copy of \( L_p(\hat{\Gamma}_{i,l}) \) in \( L_p(\hat{G}_\infty) \) and will make its presentation more precise.

Let us identify \( \Gamma_{i,l} \) with \( \ast_{i \geq 1} \Gamma_{i,l} \). For an element in \( g \in \Gamma_{i,l} \), we denote its copy in \( \Gamma_{i,l} \) by \( g^{(i)} \). Thus, for \( g = g_1 \cdots g_n \in \Gamma_{i,l} \), we have

\[
T^{Ld}(\lambda(g)) = \lambda(g_1^{(2)} \cdots g_d^{(d+1)} g_{d+1}^{(1)} \cdots g_n^{(1)}).
\]

As we have explained, (8) defines a complete isomorphic embedding of \( L_p(\hat{\Gamma}_{i,l}) \) into \( L_p(\hat{G}_\infty) \).

Next, we consider the group morphism \( \phi \) from \( G_\infty \) onto \( \Gamma^d \) such that for all \( i \) and \( g \in \Gamma_{i,k} \subset G_\infty \)

\[
\phi(g) = e \quad \text{if} \quad k = 1 \quad \text{and} \quad \phi(g) = \underbrace{e, \cdots, e, g, \cdots, e}_{k-2} \quad \text{if} \quad 2 \leq k \leq d + 1.
\]

Let \( U = \lambda_{\Gamma^d} \circ \phi \). Then \( U \) is a unitary representation of \( G_\infty \) on \( \ell_2(\Gamma^d) \). Applying the Fell absorption principle to \( U \), we get a completely isometric embedding of \( L_p(\hat{G}_\infty) \) into \( L_p(\Gamma^d \otimes \hat{G}_\infty) \):

\[
\Pi_{\phi} : \sum_{g \in G_\infty} c(g)\lambda(g) \mapsto \sum_{g \in G_\infty} c(g)\lambda_{\Gamma^d}(\phi(g)) \otimes \lambda(g).
\]

By (8) and (9), we see that

\[
\alpha^{Ld}(x) = \Pi_{\phi} \cdot T^{Ld}(x), \quad x \in C[\Gamma_{i,l}].
\]

Thus \( \alpha^{Ld} \) extends to a completely isomorphic embedding on \( L_p(\hat{G}_\infty) \). \hfill \Box

Remark 5.7. There is an alternate proof to Theorem 5.6. First, one may apply Theorem 4.6 and Remark 4.7 for \( A_i = \Gamma^d \otimes \hat{\Gamma} \) for all \( i \in \mathbb{N} \), and \( B = \Gamma^d \otimes \mathbb{C} \), and check that

\[
\|\lambda^{L}(x)\|_{L_p(\hat{\Gamma} \otimes \hat{\Gamma}_\infty)} \simeq_p \|x\|_{L_p(\hat{\Gamma}_\infty)}
\]

for any \( x \in C[\Gamma_{\infty}] \). Next, one may apply Theorem 4.6 repeatedly to get

\[
\|\alpha^{L_d}(x)\|_{L_p(\hat{\Gamma} \otimes \hat{\Gamma}_\infty)} \simeq_p \|\lambda^{L(d-1)}(x)\|_{L_p(\hat{\Gamma} \otimes \hat{\Gamma}_\infty)} \simeq_p \|\cdots \simeq_p \|x\|_{L_p(\hat{\Gamma}_{\infty})}.
\]

This alternate proof gives a better constant.

As in the free group case, the previous theorem, together with transference, easily implies a multiplier result on \( \Gamma_{\infty} \). Given a bounded function \( m \) on \( \Gamma^d \), define a linear map \( M_m \) on \( L_2(\hat{\Gamma}_{\infty}) \) by

\[
M_m(\lambda(g)) = m(g_1, g_2, \cdots, g_d)\lambda(g)
\]

with \( g_\ell = e \) in \( m(g_1, g_2, \cdots, g_d) \) if \( \ell > n \) for every \( g \in \Gamma_{\infty} \) in reduced form. Note that \( \hat{\Gamma} \) is QWEP iff \( \Gamma \) is hyperlinear (cf. [33]).

Theorem 5.8. Assume that \( \hat{\Gamma} \) is QWEP. Let \( d \in \mathbb{N} \) and \( 1 < p < \infty \). If the Fourier multiplier \( T_m \) is completely bounded on \( L_p(\hat{\Gamma}^d) \), then \( M_m \) extends to a completely bounded map on \( L_p(\hat{\Gamma}_{\infty}) \).

Proof. This proof is similar to that of Theorem 1.1. As \( \hat{\Gamma} \) is QWEP, so is \( \hat{\Gamma}_{\infty} \). Thus \( T_m \otimes \text{Id}_{L_p(\hat{\Gamma}_{\infty})} \) is completely bounded on \( L_p(\hat{\Gamma}^d \otimes \hat{\Gamma}_{\infty}) \) by Junge’s noncommutative Fubini theorem [12]. Let \( x \in C[\Gamma_{\infty}] \). Using the action \( \alpha^{L_d} \) in Theorem 5.6, we have

\[
\|x\|_{L_p(\hat{\Gamma}_{\infty})} \simeq_{d,p} \|\alpha^{L_d}(x)\|_{L_p(\hat{\Gamma}^d \otimes \hat{\Gamma}_{\infty})}.
\]

It remains to use the interviewing formula

\[
T_m \otimes \text{Id}[\alpha^{L_d}(x)] = a^{L_d}[M_m(x)]
\]

to conclude as in the proof of Theorem 1.1. \hfill \Box
In the same line, we conclude this subsection by stating another application. Consider a family of discrete groups $\Gamma_i$, $i \in I$, and its free product $\Gamma_\infty = \ast_{i \in I} \Gamma_i$; consider also a family of finite von Neumann algebras $(M_i, \tau_i), i \in I$, and its von Neumann tensor product $(M, \tau) = \otimes_{i \in I} (M_i, \tau_i)$. Let $\{m_{i,g}\}_{g \in \Gamma_i} \subset M_i$ for all $i \in I$, and let $M_i$ be the operator-valued Fourier multiplier on $\Gamma_i$:

$$M_i(\lambda(g)) = m_{i,g} \otimes \lambda(g), \quad g \in \Gamma_i.$$ 

We construct a map $M^{Ld}$ similar to Theorem 5.8. Given $g = g_1 \cdots g_n \in \Gamma_\infty$ in reduced form, define

$$M^{Ld}(\lambda(g)) = \begin{cases} m_{i_1,g_1} \otimes \cdots \otimes m_{i_d,g_d} \otimes \lambda(g) & \text{if } n \geq d, \\
_{i_1,g_1} \otimes \cdots \otimes m_{i_n,g_n} \otimes \lambda(g) & \text{if } n < d. \end{cases}$$

**Theorem 5.9.** Let $1 < p < \infty$. Then $M^{Ld}$ extends to a completely bounded Fourier multiplier from $L_p(\hat{\Gamma}_\infty)$ to $L_p(M \otimes \hat{\Gamma}_\infty)$ iff the family $\{\|M_i\|_{cb}(L_p(\hat{\Gamma}_i), L_p(M_i \otimes \hat{\Gamma}_i))\}_{i \in I}$ is bounded. In this case, we have

$$\|M^{Ld}\|_{cb}(L_p(\hat{\Gamma}_\infty), L_p(M \otimes \hat{\Gamma}_\infty)) \lesssim_{p,d} \sup_{i \in I} \|M_i\|_{cb}(L_p(\hat{\Gamma}_i), L_p(M_i \otimes \hat{\Gamma}_i)).$$

**Proof.** We can easily adapt the proofs of Theorem 5.6 to the present setting, so we omit the details. \hfill \Box

### Appendix A. Endpoint boundedness of free Hilbert transforms

Our arguments rely on the $L_p$-boundedness ($1 < p < \infty$) of the free Hilbert transforms $H_z$ in Lemma 2.1 and their variants $H_z^{(j)}$ in Corollary 4.10. We will discuss their bounds on homogeneous polynomials when $p = \infty$, since they cannot be bounded in full generality at the end point. It is also natural to ask if one can get an $L_\infty$-BMO boundedness for the BMO spaces studied in [13].

#### A.1. Bounds on homogeneous polynomials

We work in the setting of amalgamated free products of von Neumann algebras as in section 4. Let $H_z^{(j)}$ be the maps introduced in Corollary 4.10 with $\varepsilon$ a family of signs.

The case $j = 1$ of the following theorem follows from [17, Proposition 2.8].

**Theorem A.1.** Let $d \geq 1$ and $1 \leq j \leq d$. Then for any $x \in \mathcal{W}_d$ we have

$$\|H_z^{(j)}(x)\|_\infty \lesssim \min \{ \log(j + 2), \log(d - j + 2) \} \|x\|_\infty.$$

**Proof.** For $z \in T$, let $U_z$ be the unitary on $L_2(\mathcal{A})$ sending $w \in \mathcal{W}_n$ to $z^n w$ for $n \geq 0$. Given $x \in \mathcal{W}_d, y \in L_2(\mathcal{A})$ and $0 \leq k \leq 2d$, let

$$x_{\perp}^k = E_z[z^{d-k}U^*_z(xU_z(y))].$$

It is easy to verify that

$$xy = \sum_{k=0}^{2d} x_{\perp}^k y \quad \text{and} \quad \|x_{\perp}^k y\|_2 \leq \|x\|_\infty \|y\|_2.$$ 

Moreover, we easily show

$$H_z^{(j)}(x)_{\perp}^k y = \begin{cases} H_z^{(j)}(x_{\perp}^k y) & \text{if } k \leq 2(d-j), \\
x_{\perp}^k H_z^{(d+1-j)}(y) & \text{if } k > 2(d-j). \end{cases}$$

Thus

$$H_z^{(j)}(x)y = \sum_{0 \leq k \leq 2(d-j)} H_z^{(j)}(x_{\perp}^k y) + \sum_{2(d-j) < k \leq 2d} x_{\perp}^k H_z^{(d+1-j)}(y).$$

On the other hand, using the elementary estimate

$$\|z^k\|_{L_1(T)} \simeq \log(n+2),$$

we get

$$\|\sum_{k=0}^n x_{\perp}^k y\|_2 + \|\sum_{k=n}^{2d} x_{\perp}^k y\|_2 \lesssim \min \{ \log(n+2), \log(2d - n + 2) \} \|x\|_\infty \|y\|_2.$$
Therefore, since $H_z^{(j)}$ is isometric on $L_2(A)$, we deduce
\[
\|H_z^{(j)}(x)y\|_2 \leq \|\sum_{k=0}^{2(d-j)} x^\perp k y\|_2 + \|\sum_{k=2(d-j)+1}^{2d} x^\perp k H_z^{(d+1-j)}(y)\|_2 \\
\lesssim \min \{ \log(j + 2), \log(d - j + 2) \}\|x\|_\infty \|y\|_2,
\]
whence the desired estimate on $\|H_z^{(j)}(x)\|_\infty$.

\[\square\]

A.2. Failure of the $L_\infty$-BMO boundedness. Let us restrict ourselves to the free group case. Recall that the Poisson semigroup $(S_t)_{t \geq 0}$ on $F_\infty$ is the normal unital completely positive semigroup given by
\[
S_t(\lambda(g)) = e^{-t|\lambda(g)|}, \quad g \in F_\infty.
\]
We also recall the definitions of various BMO-spaces according to [13]. As usual we denote by $L^p_0(\hat{F}_\infty)$ the subspace of centered elements (i.e., elements with vanishing trace) in $L^p(\hat{F}_\infty)$. Define
\[
\text{BMO}^\alpha(S) = \{ x \in L^2_0(\hat{F}_\infty) : \|x\|_{\text{BMO}^\alpha} < \infty \},
\]
\[
\text{bmo}^\alpha(S) = \{ x \in L^2_0(\hat{F}_\infty) : \|x\|_{\text{bmo}^\alpha} < \infty \},
\]
where
\[
\|x\|_{\text{BMO}^\alpha(S)} = \sup_{t > 0} \|S_t([x - S_t(x)]^2)\|_\infty^{1/2},
\]
\[
\|x\|_{\text{bmo}^\alpha(S)} = \sup_{t > 0} \|S_t([x]^2) - |S_t(x)|^2\|_\infty^{1/2}.
\]
Similarly, we define the row versions $\text{BMO}^\alpha(S)$ and $\text{bmo}^\alpha(S)$ by passing to adjoints. One of the main results of [13] states that the intersection space $\text{BMO}^\alpha(S) \cap \text{BMO}^\alpha(S)$ behaves well with complex interpolation, i.e., it replaces $L_\infty$ as an endpoint space in the complex interpolation scale $\{L^p(\hat{F}_\infty)\}_{p > 1}$.

Let $H_z$ denote the free Hilbert transform of the first letters associated to a sequence of signs, see Lemma 2.1. It is easy to see that $\|H_z(x)\|_{\text{bmo}^\alpha(S)} = \|x\|_{\text{bmo}^\alpha(S)}$ for $x \in \mathbb{C}[F_\infty]$. We will explain why one cannot hope the boundedness of $H_z$ from $L_\infty(\hat{F}_\infty) = \hat{F}_\infty$ to any of $\text{BMO}^\alpha(S)$, $\text{BMO}^\alpha(S)$ or $\text{bmo}^\alpha(S)$.

**Lemma A.2.** Let $a, b \in F_\infty$ be two free elements. Let $z$ be a finite sum $z = \sum_{k \geq 1} c_k \lambda_{a^k}$ and $z_n = z \lambda(b^n)$. Then

i) $\lim_{n \to \infty} \|z_n\|_{\text{bmo}^\alpha(S)} = \|z\|_\infty$,

ii) $(e^{-1} - e^{-2})\|z\|_\infty \leq \limsup_{n \to \infty} \|x_n\|_{\text{BMO}^\alpha(S)} \leq 2\|z\|_\infty$, $\alpha \in \{c, r\}$.

**Proof.**

i) It is clear that
\[
\|z_n\|_{\text{bmo}^\alpha(S)} \leq \|z_n\|_\infty = \|z\|_\infty.
\]
For a fixed $t > 0$, we have $\lim_{n \to \infty} \|S_t(z_n)\|_\infty = 0$. Thus,
\[
\limsup_{n \to \infty} \|S_t(z_n)\|_{\text{bmo}^\alpha(S)} \leq \sup_t \|S_t(z_n^*)\|_\infty = \|z\|^2_\infty.
\]

ii) The upper bound is clear. For the lower, we use the Kadison inequality to ensure
\[
\|S_t(z_n^* - S_t(z_n^*))\|_\infty, \quad \|S_t(z_n^* - S_t(z_n))^2\|_\infty \geq \|(S_t - S_{2t})z_n\|^2_\infty.
\]
But in $\hat{F}_\infty$,
\[
\lim_{n \to \infty} \|(S_t - S_{2t})z_n\|_\infty = (e^{-1} - e^{-2})z.
\]
This finishes the proof.

As $H_z(z_n) = H_z(z)\lambda(b^n)$, we get

**Corollary A.3.** The transform $H_z$ is unbounded from $L_\infty(\hat{F}_\infty)$ to any of $\text{BMO}^\alpha(S)$, $\text{BMO}^\alpha(S)$ and $\text{bmo}^\alpha(S)$. 

Acknowledgements. The first author is partially supported by NSF award DMS-1700171. The second and third authors are partially supported by the French ANR project No. ANR-19-CE40-0002, and the third author is also partially supported by the Natural Science Foundation of China (No.12031004).

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Department of Mathematics, Baylor University, Waco, TX USA
Email address: tao.mei@baylor.edu

Normandie Univ, UNICAEN, CNRS, LMNO, 14000 Caen, France
Email address: eric.ricard@unicaen.fr

Laboratoire de Mathématiques, Université de Bourgogne Franche-Comté, 25030 Besançon Cedex, France
Email address: qxu@univ-fcomte.fr