Simultaneous controllability of wave sound propagations

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Abstract
In this work we study one problem of mathematical interest for their applications in several topics in Applied Science. We study simultaneous controllability of a pair of systems which model the evolution of sound in a compressible flow considered as a transmission problem. We show the well posed of the problem. Furthermore provided appropriate conditions in the geometry of the domain are valid and suitable assumptions on the fluid, is possible to conduct the pair of systems to the equilibrium in a simultaneous way using only one control.

Key words: Controllability, wave sound, H.U.M

1 Introduction

In this work, we considered an equations system to describe an evolution of the wave sound or compressible fluids. A linear model well know is given by a system \[12\]

\[
\begin{aligned}
\frac{\partial u}{\partial t} + \alpha \nabla p &= 0, \quad \text{in } \Omega \times (0,T) \\
\frac{\partial p}{\partial t} + \beta \text{div}(u) &= 0, \quad \text{in } \Omega \times (0,T) \\
u \cdot \eta &= Q, \quad \text{in } S_0 \times (0,T) \\
p &= 0, \quad \text{in } S_1 \times (0,T) \\
u(x,0) &= u_0(x), p(x,0) = p_0(x)
\end{aligned}
\]

where \(p = p(x,t)\) is acoustic precision, \(u = (u_1, u_2, u_3)\) and \(u_j = u_j(x,t)\) are fluid velocity field, \(\alpha > 0\) is the density of equilibrium and \(\beta > 0\) is the compressibility factor of fluid. Here \(\Omega\) is an open subset of \(\mathbb{R}^3\) with regularity boundary conditions \(S_0 \cup S_1 = \partial \Omega\) and \(S_0 \cap S_1 = \emptyset\).

To solve the simultaneous controllability we considered a system given by

\[
\begin{aligned}
\frac{\partial v}{\partial t} + \gamma \nabla q &= 0, \quad \text{in } \Omega \times (0,T) \\
\frac{\partial q}{\partial t} + \tau \text{div}(v) &= 0, \quad \text{in } \Omega \times (0,T) \\
q &= P, \quad \text{in } S_0 \times (0,T) \\
q &= 0, \quad \text{in } S_1 \times (0,T) \\
v(x,0) &= v_0(x), q(x,0) = q_0(x)
\end{aligned}
\]

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where $\gamma > 0$ and $\tau > 0$. $Q$ and $P$ in (1) and (2), respectively; these are control functions. In 1986, D.L. Russell[20] and J.L.Lions [13] proposed to solve a exact controllability problem for an evolution model, using only one control function. They called that problem as simultaneous controllability. The absences of dissipative effects as in (1) and (2), the problem present difficulties for the solution, see the examples [7], [8], [10] and [13], where they perturbed the multipliers used for the controllability.

The problem of simultaneous controllability for the systems (1) and (2) is to take a control for both of system using only one control function, i.e., given $T > 0$ any initial condition, $(u_0, \bar{p}_0, \bar{v}_0, q_0)$ and final $(\tilde{u}_0, \tilde{p}_0, \tilde{v}_0, \tilde{q}_0)$ in appropriate functional space, find $P(x, t)$ and $Q(x, t)$ such that

a) A solution $\{u, p, v, q\}$ of (1) and (2) satisfied in $T$

$$\begin{align*}
(u(., T), p(., T), v(., T), q(., T)) &= (\tilde{u}_0, \tilde{p}_0, \tilde{v}_0, \tilde{q}_0)
\end{align*}$$

b) The control function, $P(x, t)$, for (2) was given in terms of $Q(x, t)$.

A method to solve the controllability problem is Hilbert Uniqueness Method (H.U.M) proposed by J.L.Lions, it is a construction of an appropriate structure for the Hilbert space in the initial conditions space.

These structure are connected by uniqueness properties. An important contribution to the controllability problems (1) and (2) were made by Kapitonov et. G. Perla Menzala [8], [10]. In [10] and [8] the author answered positively for these structure are connected by uniqueness properties. An important contribution to the controllability problems (1) and (2) were made by Kapitonov et. G. Perla Menzala [8], [10]. In [10] and [8] the author answered positively for a simultaneous control and They showed that the control $P = -\frac{2}{\gamma}Q$ could be use to solve a problem. In this work we study a controllability problem of these systems with a perspective for applications as a problem of transmission; this is described below.

Given $\sigma_0$ and $\sigma_1$ open limited subset and conexo in $\mathbb{R}^3$, with $\bar{\sigma}_1 \subseteq \sigma_0$. Also $\omega = \sigma_0 \setminus \bar{\sigma}_1$, we denoted $\partial \sigma_0 = s_0$, $\partial \sigma_1 = s_1$. And fixed an integer $m > 1$ and $k = 1, 2, \ldots, m$. For each $k$, $b_k$ is an open subset and conexo, with regularity in the boundary such that, $\bar{\sigma}_1 \subseteq b_k \subseteq \sigma_0$, $\bar{b}_k \subseteq b_{k+1}$. We put $\omega_0 = b_1 \setminus \bar{\sigma}_1$, $\omega_k = b_{k+1} \setminus \bar{b}_k$, $k = 1, 2, \ldots, m - 1$ and $\omega_m = \sigma_0 \setminus \bar{b}_m$. and, $\omega = \bigcup_{j=0}^{m} \omega_j$, for $i \neq j$, we take $\omega_i \cap \omega_j = \emptyset$ and $\partial \omega = s_0 \cup s_1$. Examples for this decomposition is showed in Figure 1.

![Figure 1. Case $m = 0$ and $m = 3$](image)

We need a solution defined by part on each sub domain; for that, we considered the systems (1) and (2) rewrite on sub domains $\Omega_k$, and

$$\begin{align*}
\frac{\partial u^k}{\partial t} + \alpha^k \nabla p^k &= 0, \quad \text{in } \Omega_k \times (0, T) \\
\frac{\partial p^k}{\partial t} + \beta^k \text{div}(u^k) &= 0, \quad \text{in } \Omega_k \times (0, T) \\
u^k(x, 0) &= \tilde{u}_0^k(x), \quad p^k(x, 0) = \tilde{p}_0^k(x) \\
\end{align*}$$

$$\begin{align*}
\frac{\partial q^k}{\partial t} + \gamma^k \nabla q^k &= 0, \quad \text{in } \Omega_k \times (0, T) \\
\frac{\partial v^k}{\partial t} + \tau^k \text{div}(v^k) &= 0, \quad \text{in } \Omega_k \times (0, T) \\
v^k(x, 0) &= \tilde{v}_0^k(x), \quad q^k(x, 0) = \tilde{q}_0^k(x) \\
\end{align*}$$

where $\alpha^k$, $\beta^k$, $\gamma^k$, $\tau^k$ are the coefficients of the systems (1) and (2), and $\tilde{u}_0^k$, $\tilde{p}_0^k$, $\tilde{v}_0^k$, $\tilde{q}_0^k$ are the initial conditions.
$k = 0, 1, 2, \ldots, m$. 

with boundary conditions (1) and (2). The interfaces of transmission conditions $\Gamma_k = \partial \Omega_k$, given by 

\begin{align}
\begin{cases}
\alpha^{k-1} p^{k-1} = \alpha^k p^k \\
\beta^{k-1}(u^{k-1}, \eta) = \beta^k(u^k, \eta) \\
k = 2, \ldots, m, (x, t) \in \Gamma_k \times (0, T)
\end{cases}
\end{align}

(5) 

\begin{align}
\begin{cases}
\gamma^{k-1} q^{k-1} = \gamma^k q^k \\
\tau^{k-1}(v^{k-1}, \eta) = \tau^k(v^k, \eta) \\
k = 2, \ldots, m, (x, t) \in \Gamma_k \times (0, T)
\end{cases}
\end{align}

(6) 

for the systems (3) and (4), respectively.

The functions $\alpha^k, \beta^k, \gamma^k$ and $\tau^k$ are the restriction for the functions $\alpha, \beta, \gamma, \tau$ on the systems (1) and (2), we assumed that those functions were constant by parts, strictly positive and we lost the continuity only in $\Gamma_k$, $k = 1, 2, \ldots, m$.

The objective in this section is to get the estimation of 

\[(T - T_0) \sum_{k=0}^m \int_{\Omega_k} [\beta^k |u^k|^2 + \alpha^k(p^k)^2 + \tau^k |v^k|^2 + \gamma^k(q^k)^2] \, dx \leq C \int_0^T \int_{S_0} [\alpha p - \tau(v, \eta)]^2 \frac{\partial h}{\partial \eta} \, dS_0 dt.\]  

(7) 

For some $T_0 > 0$, $C > 0$ and $T > T_0$. The inequality (7) is named from an inequality of observation which is in the theorem 3.12 assuming geometrical properties on domain $\Omega$ and in the interfaces $\Gamma_k$. Such that, to prove (7) we assumed monotonicity conditions in the coefficients of the systems (3) and (4). The requirement necessary were found by Lions[13] in his study of transmission problem. Lagnese[7] used the same hypothesis to prove the result of controllability for a hyperbolic problem. Then this will be done.

(1) We showed that (3)-(6) is well posed problem, we used the semigroup theory [17]

(2) We obtained an inequality of simultaneous observability, for both systems (3) and (4), these we solved using a multipliers technique[6]

(3) We applied the H.U.M (Hilbert Uniqueness Method) to obtain the simultaneous controllability [13].

2 Functional spaces

Given the Hilbert space $X_1 = [L^2(\Omega)]^3 \times [L^2(\Omega)]$, associate to (3). We define an scalar product in $X_1$, given by $(\tilde{u}, \tilde{p}), (u, p) \in X_1$, then:

\[\langle (u, p), (\tilde{u}, \tilde{p}) \rangle_{X_1} = \sum_{k=0}^m \int_{\Omega_k} \{ \beta^k u^k \cdot \tilde{u}^k + \alpha^k p^k \tilde{p}^k \} \, dx.\]  

(8) 

Consequently, we considered $X_2 = [L^2(\Omega)]^3 \times [L^2(\Omega)]$ associate to (4). We define a scalar product in $X_2$, as $(\tilde{v}, \tilde{q}), (v, q) \in X_2$, then:

\[\langle (v, q), (\tilde{v}, \tilde{q}) \rangle_{X_2} = \sum_{k=0}^m \int_{\Omega_k} \{ \tau^k v^k \cdot \tilde{v}^k + \gamma^k q^k \tilde{q}^k \} \, dx.\]  

(9)
We have considered a total energy to the problem (3), (4), (5), (6) and the boundary conditions in (1), (2), as

\[ E(t) = \frac{1}{2} \sum_{k=0}^{m} \int_{\Omega_k} \left\{ \beta_k |u^k|^2 + \alpha^k(p^k)^2 + \tau_k |v^k|^2 + \gamma^k(q^k)^2 \right\} \, dx \]  

(10)

Making a rigorously way for the interfaces conditions, we can see a lemma 2.1; for more details see Perla et al.\[10\].

**Lemma 2.1** Given \( \Omega \) bounded region in \( \mathbb{R}^3 \), with regularity in the boundary \( \partial \Omega \). The application

\[ [C^1(\Omega)]^3 \rightarrow C^1(\partial \Omega) \]

\( u = (u_1, u_2, u_3) \rightarrow u \cdot \eta \)

where \( \eta = \eta(x) \) is as exterior unit normal vector in \( x \in \partial \Omega \). We can extend by continuity application

\[ \tilde{H} \rightarrow H^{-1/2}(\partial \Omega) \]

where \( \tilde{H} = \left\{ u \in [L^2(\Omega)]^3, \text{ such that, } \text{div}(u) \in L^2(\Omega) \right\} \) and \( H^{-1/2}(\partial \Omega) \) is dual space of \( H^{1/2}(\partial \Omega) \)

To simplify the notation we write \( u^k \) as \( u, \beta^k \) as \( \beta \), the same way for all symbols in the region \( \Omega_k \), by the lemma 2.1 is clearly that the spaces

\[
\begin{align*}
H_1 &= \{(u,p) \in X_1, \text{ such that, } (-\alpha \nabla p, \beta \text{div}(u)) \in X_1\} \subseteq X_1 \\
H_2 &= \{(v,q) \in X_2, \text{ such that, } (-\gamma \nabla q, -\tau \text{div}(v)) \in X_2\} \subseteq X_2
\end{align*}
\]

we can define the sub spaces:

\[
Z_1 = \left\{ (u,p) \in H_1, \text{ such that } \beta^{k-1}(u^{k-1} \cdot \eta) = \beta^k(u \cdot \eta), \text{ in } \Gamma_k, k = 2, \ldots, m \right\}
\]

\[ u \cdot \eta = 0 \in S_0, \quad p = 0 \in S_1 \]

and,

\[
Z_2 = \left\{ (v,q) \in H_2, \text{ such that } \tau^{k-1}(v^{k-1} \cdot \eta) = \tau^k(v \cdot \eta), \text{ in } \Gamma_k, \quad k = 2, \ldots, m \right\}
\]

\[ q = 0, \quad \text{in } \partial \Omega = S_0 \cup S_1 \]

Observe that \( [C^1(\Omega)] \subset Z_j, \quad j = 1, 2 \). Also \( Z_1 \) and \( Z_2 \) are dense in \( X_1 \) and \( X_2 \), respectively. Considering the bounded operator

\[ A_j : Z_j = D(A_j) \subseteq X_j \rightarrow X_j \]

defined as

(1) Given \((u,p) \in D(A_1)\), then, \( A_1(u,p) = (-\alpha \nabla p, \beta \text{div}(u)) \)

(2) Given \((v,q) \in D(A_2)\), then, \( A_2(v,q) = (-\gamma \nabla q, -\tau \text{div}(v)) \)

The adjoint operator of \( A_1 \) is denoted by \( A_1^* \); It is calculated and given as :

\[ A_1^*(\tilde{u},\tilde{p}) = (\alpha \nabla \tilde{p}, \beta \text{div}(\tilde{u})) \]

and
Perla et al. [10] showed that operator $A_1$ is skew-adjoint, i.e., $A_1^* = -A_1$, the same result was proved for $A_2$. Using the Stone’s theorem, we have proved that $A_1$ and $A_2$ generate infinitesimally a group of strongly continuous unit operators $\{U_j(t)\}_{t \in \mathbb{R}}$, in $X_1$ and $X_2$, respectively. Moreover, $U_j(t)w_j$ is strongly differentiable in relation to $t$ and for any $w_j \in D(A_j)$, 

$$\frac{d}{dt} U_j(t)w_j = A_j U(t)w_j$$

Now, we study some properties of the solutions (3), these are used in

**Lemma 2.2** Given $V_j = [\text{Ker}(A_j^*)]^{1}$, considering the orthogonality in relation with the scalar product defined in $X_1$ and $X_2$, respectively. Then, the following results are valid:

1. $U_j(t) (V_j \cap D(A_j)) \subset V_j \cap D(A_j)$
2. Fixing $t \in \mathbb{R}$, and $(u, p) \in V_1 \cap D(A_1)$, then, in the distributions manner
   
   - $(a)$ $\text{curl}(u^k) \equiv 0$, in $\Omega_k$, $k = 0, 1, \ldots, m$
   - $(b)$ $u \times \eta \equiv 0$ in $S_1$
   - $(c)$ $u^{k-1} \times \eta = u^k \times \eta$ in $\Gamma_k$, $k = 2, \ldots, m$

   where $\times$ denoted a vectorial product of $\mathbb{R}^3$ and $\eta(x)$ is an exterior normal vector $\Gamma_k$

3. Fixing $t \in \mathbb{R}$, $(v, q) \in V_2 \cap D(A_2)$, then, in the distribution manner
   
   - $(a)$ $\text{curl}(v^k) \equiv 0$, in $\Omega_k$, $k = 0, 1, \ldots, m$
   - $(b)$ $v \times \eta \equiv 0$ in $\Gamma$
   - $(c)$ $v^{k-1} \times \eta = v^k \times \eta$ in $\Gamma_k$, $k = 2, \ldots, m$

**Proof.** The proof of $A_1$, $A_2$ are similar. We make the proof for the first one. Given $(u, p) \in V_1 \cap D(A_1)$, then $(u, p) \in V_1$ and $(u, p) \in D(A_1)$. As consequence of semigroup theory, we know that $U(t)(u, p) \in D(A_1)$, $\forall t \in \mathbb{R}$.

Now we need to prove that $U(t)(u, p) \in V_1$. See that, $\text{Ker}(A_1^*) \neq \emptyset$ has elements of the form $(\beta^{-1} \text{Curl}(v), 0)$ where $v \in [H^2(\Omega)]^3$, $v = 0$ in $S_0$. Given $w = (w_1, w_2) \in \text{Ker}(A_1^*)$, then $A_1^*(w_1, w_2) = 0$. And,

$$\frac{d}{dt} (U_1(t)(u, p), (w_1, w_2)) = (A_1 U_1(t)(v, p), (w_1, w_2))_{X_1} = (U_1(t)(u, p), A_1^*(w_1, w_2)) = 0$$

we have that,

$$U_1(t)(u, p), (w_1, w_2) = C, \quad C=\text{constant} \quad \forall t \in \mathbb{R}.$$

In particular, for $t = 0$, $(u, p), (w_1, w_2))_{X_1} = C$, and, $(u, p) \in [\text{Ker}(A_1^*)]^{1}$ and $(w_1, w_2) \in \text{Ker}(A_1^*)$, this implied that $C = 0$. In the same form, $(U_1(t)(u, p), (w_1, w_2)) = 0, \quad \forall t \in \mathbb{R}$. Given $U_1(t)(u, p) \in [\text{Ker}(A_1^*)]^{1} = V_1$ and the item 1) was proved. Now, we prove the first item 2). Given $v \in [H^2(\Omega)]^3$ with support in $\Omega_k$ and considering the elements $(\beta^{-1} \text{Curl}(v), 0) \in \text{Ker}(A_1^*)$. Then, for all $(u, p) \in V_1 \cap D(A_1)$, we have;

$$0 = \langle (u, p), (\beta^{-1} \text{Curl}(v), 0) \rangle_{X_1} = \sum_{k=0}^{m} \int_{\Omega_k} u \cdot \text{Curl}(v) \, dx$$

because the support of $v$ is in $\Omega_k$. The same form, $\text{Curl}(v) = 0$ in $\Omega_k$, $k = 0, 1, \ldots, m$ in the distributions way. For proving the item b) of 2) We use the identities such as,

$$\int_{\Omega} \text{Curl}(u) \cdot v \, dx = \int_{\Omega} u \cdot \text{Curl}(v) \, dx - \int_{\partial \Omega} v \cdot (u \times \eta) \, d\Gamma$$ (11)
Given \( v \in [H^2(\Omega)]^3 \) and \( (\beta^{-1}\text{Curl}(v), 0) \in \text{Ker}(A_1^*) \), as, \( v = 0 \) in \( \bigcup_{j=1}^m \bar{\Omega}_j \). Using (11) we have

\[ 0 = \langle (u, p), (\beta^{-1}\text{Curl}(v), 0) \rangle_{X_1} = \sum_{k=0}^m \int_{\Omega_k} u^k \cdot \text{Curl}(v^k) \, dx \]

\[ = \sum_{k=0}^m \int_{\Omega_k} \text{Curl}(u^k) \cdot v^k \, dx + \sum_{k=0}^m \int_{\partial\Omega_k} v^k \cdot (u^k \times \eta) \, d\Gamma_k \]

\[ = \sum_{k=0}^m \int_{\partial\Omega_k} v^k \cdot (u^k \times \eta) \, d\Gamma_k = \int_{S_1} v \cdot (u \times \eta) \, dS_1 \] (12)

0 = \sum_{k=0}^m \int_{\partial\Omega_k} v^k \cdot (u^k \times \eta) \, d\Gamma_k 

\[ = \sum_{k=0}^m \int_{\partial\Omega_k} v^k \cdot (u^k \times \eta) \, d\Gamma_k \] (13)

Now, we chose \( v \) such that \( v = 0 \) in \( S_0 \) and \( v = 0 \) in \( \bigcup_{j=1}^m \bar{\Omega}_j \), by (13) we have that

\[ 0 = \int_{\Gamma_k} v \cdot \{u^{k-1} \times \eta - u^k \times \eta\} \, d\Gamma_k \]

that is the expected result.

The item 3) is proved in the same way.

The theorem 2.3 has a summary of the results.

**Theorem 2.3** Given \( V_j \) an orthogonal complement of the subset \( \text{Ker}(A_j^*) \), \( j = 1, 2 \), in \( X_j \). Consider the problems (3), (4), (1), (2) and the initial conditions \((u_0, p_0) \in V_1 \cap D(A_1) \) and \((v_0, q_0) \in V_2 \cap D(A_2) \). Then \((u, p) = U_1(t)(u_0, p_0)\) and \((v, q) = U_2(t)(v_0, q_0)\) are the uniqueness solutions, respectively. That is,

\((u, p) \in C(\mathbb{R}; V_1 \cap D(A_1)) \cap C(\mathbb{R}, X_1)\)

\((v, q) \in C(\mathbb{R}; V_2 \cap D(A_2)) \cap C(\mathbb{R}, X_2)\)

In addition, these solutions satisfies the properties in the lemma 2.2.

Before to show the inequality of observability, we prove some important properties.

The energy associate to the systems (3) and (4), with null boundary conditions, are given by:

\[ E_1(t) = \frac{1}{2} \sum_{k=0}^m \int_{\Omega_k} \{ \beta_k \cdot |u^k|^2 + \alpha^k (p^k)^2 \} \, dx \]

and

\[ E_2(t) = \frac{1}{2} \sum_{k=0}^m \int_{\Omega_k} \{ \gamma_k \cdot |v^k|^2 + \gamma^k (q^k)^2 \} \, dx \]
respectively. To prove that these are not dependent of the time $t$, In fact, we multiplied the first equation of (3) by $\beta^k u^k$ and integrating in $\Omega_k$ and adding in $k = 0, 1, \ldots, m$, we have

$$\frac{1}{2} \frac{d}{dt} \sum_{k=0}^{m} \int_{\Omega_k} \beta^k |u^k|^2 \, dx - \sum_{k=0}^{m} \int_{\Omega_k} \beta^k \alpha^k p^k \text{div}(u^k) \, dx + \sum_{k=0}^{m} \int_{\partial \Omega_k} \beta^k \alpha^k p^k (u^k \cdot \eta) \, dx. \quad (14)$$

Multiplying the second equation of (3) by $\alpha^k p^k$, after integrating in $\Omega_k$ and adding in $k$, we have that

$$\frac{1}{2} \frac{d}{dt} \sum_{k=0}^{m} \int_{\Omega_k} \alpha^k (p^k)^2 \, dx + \sum_{k=0}^{m} \int_{\Omega_k} \alpha^k \beta^k p^k \text{div}(u^k) \, dx = 0 \quad (15)$$

adding (14) and (15), we have that

$$\frac{1}{2} \frac{d}{dt} \sum_{k=0}^{m} \int_{\Omega_k} \{ \beta^k |u^k|^2 + \alpha^k (p^k)^2 \} \, dx + \sum_{k=0}^{m} \int_{\partial \Omega_k} \beta^k \alpha^k p^k (u^k \cdot \eta) \, dx = 0$$

moreover,

$$\sum_{k=0}^{m} \int_{\partial \Omega_k} \beta^k \alpha^k p^k (u^k \cdot \eta) \, dx = \int_{S_1} \alpha \beta p(u \cdot \eta) \, dS_1 + \sum_{k=1}^{m} \int_{\Gamma_k} \{ \alpha^{k-1} \beta^{k-1} p^{k-1} (u^{k-1} \cdot \eta) - \}

\alpha^k \beta^k p^k (u^k \cdot \eta) \, d\Gamma_k + \int_{S_0} \alpha \beta p(u \cdot \eta) \, dS_0$$

Using the contour and interface conditions (5), we have that $\sum_{k=0}^{m} \int_{\partial \Omega_k} \alpha^k \beta^k p^k (u^k \cdot \eta) \, d\Gamma_k = 0$. Follows the affirmation. The case $E_2(t)$ is similar.

### 3 Inequality of observability

In this section we show the inequality of observability. This inequality satisfies the systems (3) and (4) simultaneously. Using the multiplier’s theory (see Komornik[6]), we make the proof. The multiplier was modified to get a good estimates in the boundary. These multiplier were used in several works. The invariant of the systems (1) and (2), in relations to dilatations groups in all variables, see [8] and [10]. Given $h : C(\Omega) \cap C^1(\Omega) \longrightarrow \mathbb{R}$ an auxiliary function, it will be chosen in the next steps; and, given $(u, p) \in V_1 \cap \mathcal{D}(A_1)$ a solution of the system (1). Considering the multiplier given by:

$$\begin{cases}
M_1 = 2 (\alpha p - u \cdot \nabla h + \alpha \int_0^t p(x, s) \, ds) \\
M_2 = 2 (\beta tu - p \nabla h) \\
M_3 = 2 \beta u
\end{cases}$$

and $(u, p)$ solution of (1), we have the identities

$$0 = M_1 \{ p_t + \beta \text{div} u \} + M_2 \{ u_t + \alpha \nabla p \} + M_3 \{ \int_0^t (u_s + \alpha \nabla p) \, ds \}$$

The expression above, we make rewrite as

$$0 = \frac{\partial A}{\partial t} - \text{div}(\vec{B}) - J \quad (16)$$
where

\[ A = t \left[ \beta \left| u \right|^2 + \alpha p^2 \right] - 2p \left( u \cdot \nabla h \right) + 2\alpha p \int_0^t p(x, s) ds - 2\beta u(x, 0) \cdot \int_0^t u(x, s) ds \]

\[ \bar{B} = -2\alpha \beta p u + \alpha p^2 \nabla h - \beta \left| u \right|^2 \nabla h + 2\beta (u \cdot \nabla h) u - 2\alpha \beta \left( \int_0^t p(x, s) ds \right) u \]

\[ J = \beta (\Delta - 1) \left| u \right|^2 - 2\beta \sum_{i,j=1}^3 \frac{\partial^2 h}{\partial x_i \partial x_j} u_i u_j - \alpha (\Delta h - 3)p^2 \]

**Remark 3.1** We have considered \( h(x) = \frac{1}{2} \left| x - x_0 \right|^2 \) for some \( x_0 \in \mathbb{R} \) fixed, then \( J = 0 \). In this case (16) represents a conservation law. Integrating (16) and \( \Omega_k \), we observe that: in the expression \( B \) need to fix the \( \frac{\partial h}{\partial n} \) to get good estimations; then, we chose \( h(x) \) as a small perturbation of \( \frac{1}{2} \left| x - x_0 \right|^2 \) for some \( x_0 \in \mathbb{R}^3 \).

Integrating the identity (16) in \( \Omega_k \times (0, s) \) and adding in \( k \), we have

\[ 0 = \sum_{k=0}^m \int_{\Omega_k} \left[ A^k(x, s) - A^k(x, 0) \right] dx - \sum_{k=0}^m \int_0^s \int_{\partial \Omega_k} \bar{B}^k \cdot \eta d\Gamma - \sum_{k=0}^m \int_0^s J^k dt dx \]  

(17)

replacing the expression \( A \) in (17), we have that

\[ 0 = \sum_{k=0}^m \int_{\Omega_k} \left\{ s \left[ \beta \left| u \right|^2 + \alpha k (p^k)^2 \right] - 2p \int_0^s (u \cdot \nabla h) dx - 2\alpha k \int_0^s p^k(x, \tau) d\tau \right. \]

\[ - 2\beta u_0^k(x) \cdot \int_0^s u^k(x, r) dr + 2p_0^k (u_0^k \cdot \nabla h) \left. \right\} dx - \sum_{k=0}^m \int_0^s \int_{\partial \Omega_k} \bar{B}^k \cdot \eta d\Gamma_k - \sum_{k=0}^m \int_0^s J^k dt dx \]

thus,

\[ s \sum_{k=0}^m \int_{\Omega_k} \left[ \beta k \left| u \right|^2 + \alpha k (p^k)^2 \right] = 2 \sum_{k=0}^m \int_{\Omega_k} 2p^k(x, s) (u \cdot \nabla h) dx - 2 \sum_{k=0}^m \int_{\Omega_k} \alpha k p^k \int_0^s p^k(x, \tau) d\tau \]

\[ + 2 \sum_{k=0}^m \int_{\Omega_k} \beta u_0^k(x) \cdot \int_0^s u^k(x, r) dr dx - 2 \sum_{k=0}^m \int_{\Omega_k} p_0^k (u_0^k \cdot \nabla h) dx \]

\[ + \sum_{k=0}^m \int_0^s \int_{\partial \Omega_k} \bar{B}^k \cdot \eta d\Gamma_k + \sum_{k=0}^m \int_0^s J^k dt dx \]  

(18)

where \( p_0^k = p^k(x, 0), u_0^k = u^k(x, 0) \). The proof of the main result is to get the inequality of the observability; we obtain this estimation using the right side of (18). Making good presentation of the proof, we show many lemmas

**Lemma 3.2** Given \( \{u, p\} \) regular solution for the problem (3)-(5), this was given by the theorem 2.3. Then

\[ \sum_{k=0}^m \int_{\Omega_k} 2p^k(x, s) (u \cdot \nabla h) dx \leq C_1 \sum_{k=0}^m \left\{ \beta k \left| u \right|^2 + \alpha k (p^k)^2 \right\} dx \]

where \( C_1 = 3 \max_{k=0,1,\ldots,m} \left\{ (\alpha k)^{-1}, (\beta k)^{-1} \right\} \max_{x \in \Omega} \left| \nabla h \right| \)
Proof. Using the Holder inequality in the first term in the right-hand side of (18), we have that

$$2 \sum_{k=0}^{m} \int_{\Omega_k} p_k(x,u_k,\nabla h) dx = 2 \sum_{k=0}^{m} \sum_{i=1}^{3} \int_{\Omega_k} p_k u_k \frac{\partial h}{\partial x_i} dx \leq 2 \max_{x \in \Omega} |\nabla h| \sum_{k=0}^{m} \sum_{i=1}^{3} \int_{\Omega_k} p_k u_i dx$$

$$= 2 \max_{x \in \Omega} |\nabla h| \left[ \sum_{k=0}^{m} \left( \int_{\Omega_k} (p_k)^2 \right)^{1/2} \left( \int_{\Omega_k} (u_k)^2 \right)^{1/2} \right]$$

$$\leq \max_{x \in \Omega} |\nabla h| \left\{ 2 \int_{\Omega_k} (p_k)^2 dx + \int_{\Omega_k} |u_k|^2 dx \right\}$$

$$\leq 3 \max_{k=0,1,\ldots,m} \left\{ (\alpha_k)^{-1}, (\beta_k)^{-1} \right\} \max_{x \in \Omega} |\nabla h| \int_{\Omega_k} \left\{ \beta_k |u_k|^2 + \alpha_k (p_k)^2 \right\} dx$$

$$= C_1 \sum_{k=0}^{m} \left\{ \beta_k |u_k|^2 + \alpha_k (p_k)^2 \right\} dx$$

where,

$$C_1 = 3 \max_{k=0,1,\ldots,m} \left\{ (\alpha_k)^{-1}, (\beta_k)^{-1} \right\} \max_{x \in \Omega} |\nabla h|$$

The second term in the right-hand side of (18), $2 \sum_{k=0}^{m} \int_{\Omega_k} \alpha_k p_k \int_{0}^{s} p_k(x,\tau) d\tau$, we may write as

$$2 \sum_{k=0}^{m} \int_{\Omega_k} \alpha_k p_k \int_{0}^{s} p_k(x,\tau) d\tau = -\frac{\partial}{\partial s} \sum_{k=0}^{m} \int_{\Omega_k} \alpha_k \left[ \int_{0}^{s} p_k(x,r) dr \right] dx. \tag{20}$$

To estimate the fourth term in the right-hand side of (18), $-2 \sum_{k=0}^{m} \int_{\Omega_k} p_k(u_0 \nabla h) dx$, we used the assumption the independence of the energy with the time the estimation is (19), Thus:

$$-2 \sum_{k=0}^{m} \int_{\Omega_k} p_k(u_0 \nabla h) dx \leq C_1 \sum_{k=0}^{m} \int_{\Omega_k} \left[ \beta_k |u_k|^2 + \alpha_k (p_k)^2 \right] dx. \tag{21}$$

The fifth term of right-hand side of (18), $\sum_{k=0}^{m} \int_{\partial \Omega_k} \vec{B}^{k-1} \eta d\Gamma_k$, we need to make an analysis more carefully.

Lemma 3.3 Given $\{u,p\}$ a regular solution of the problem (1)-(5) by theorem 2.3. For $k = 1, 2, \ldots, m$, we have the following identity

$$\vec{B}^{k-1} \eta - \vec{B}^k \eta = -\frac{\partial h}{\partial \eta} \left\{ \frac{\alpha_k^{k-1} - \alpha_k}{\alpha_k-1} \alpha_k p_k^2 + (\beta_k^{k-1} - \beta_k) \frac{\beta_k}{\beta_k-1} |u_k|^2 + (\beta_k^{k-1} - \beta_k) |u^k \times \eta|^2 \right\}$$

is validated in $\Gamma_k$, $k = 1, 2, \ldots, m$.

Proof. Using the boundary conditions (5), we have that $x \in S_1$, then

$$\vec{B} \eta = -\beta |u|^2 \frac{\partial h}{\partial \eta} + 2\beta (u \cdot \nabla h)(u \cdot \eta)$$

$$= -\beta |u|^2 \frac{\partial h}{\partial \eta} + 2\beta \left\{ |u|^2 \frac{\partial h}{\partial \eta} + (u \cdot \eta)(\nabla h \times u) \right\}$$

$$= -\beta |u|^2 \frac{\partial h}{\partial \eta} + 2\beta |u|^2 \frac{\partial h}{\partial \eta}$$

$$= \beta |u|^2 \frac{\partial h}{\partial \eta}. \tag{22}$$
and \( x \in S_0 \), then

\[
\vec{B} \cdot \eta = \alpha p^2 \frac{\partial h}{\partial \eta} - \beta \mid u \mid^2 \frac{\partial h}{\partial \eta} + \beta k - \beta^k \mid \eta \mid^2 \frac{\partial h}{\partial \eta}.
\]

Using the interface conditions (5), for \( x \in \Gamma_k \) we have the following identity

\[
\vec{B}^{k-1} \cdot \eta - \vec{B}^k \cdot \eta = -2\alpha^{k-1} \beta^{k-1} + \beta^k \left( \mid u^{k-1} \cdot \eta \mid - \beta^{k-1} \mid \eta \mid^2 \right) - \beta^{k-1} \left( \int_0^t p^{k-1}(x, s) ds \right) \eta - 2\beta^{k-1} \left( u^{k-1} \cdot \eta \right) + 2\beta^k \left( \int_0^t \left( u^k \cdot \eta \right) ds \right) \eta.
\]

(23)

else,

\[
\alpha^{k-1} \frac{p^{k-1}}{\partial \eta} = \alpha^k \frac{p^k}{\partial \eta} = \frac{\alpha^{k-1} p^{k-1}}{\alpha^k} \frac{\partial h}{\partial \eta} = -\alpha^k \frac{\partial h}{\partial \eta}.
\]

(24)

For \( \mid \eta \mid = 1 \) is validated

\[
\mid u \mid^2 = \mid (u \cdot \eta) \mid^2 + \mid u \times \eta \mid^2.
\]

(25)

Substituting (25) in (23), we have that

\[
\beta^k \mid u^k \mid^2 \frac{\partial h}{\partial \eta} - \beta^{k-1} \mid u^{k-1} \mid^2 \frac{\partial h}{\partial \eta} = \beta^k \left( \mid (u^k \cdot \eta) \mid^2 + \mid (u^k \times \eta) \mid^2 \right) \frac{\partial h}{\partial \eta} - \beta^{k-1} \left( \mid (u^{k-1} \cdot \eta) \mid^2 + \mid (u^{k-1} \times \eta) \mid^2 \right) \frac{\partial h}{\partial \eta} + \beta^{k-1} \mid u^{k-1} \mid^2 \frac{\partial h}{\partial \eta} + \beta^k \mid u^k \times \eta \mid^2 - \beta^{k-1} \mid u^{k-1} \times \eta \mid^2 \frac{\partial h}{\partial \eta}.
\]

(26)
Finally,

\[ 2\beta^{k-1}(u^{k-1} \cdot \nabla h)(u^{k-1} \cdot \eta) - 2\beta^k(u^k \cdot \nabla h)(u^k \cdot \eta) = 2\beta^{k-1} |u^{k-1} \cdot \eta|^2 - 2\beta^k |u^k \cdot \eta|^2 \frac{\partial h}{\partial \eta} \]

\[ = 2 \left( \frac{\beta^{k-1}}{\beta^k} - 1 \right) \frac{\beta^k}{\beta^{k-1}} |u^k \cdot \eta|^2 \frac{\partial h}{\partial \eta} \]

\[ = 2 \left( \frac{1}{\beta^k} - 1 \right) (\beta^k |u^k \cdot \eta|)^2 \frac{\partial h}{\partial \eta} \]

Substituting (24), (25) and (27) in (23), we have that

\[ \bar{B}^{k-1} \cdot \eta - \bar{B}^k \cdot \eta = - (\alpha^{k-1} - \alpha^k) \frac{\alpha^k}{\alpha^{k-1}} (p^k)^2 \frac{\partial h}{\partial \eta} - (\beta^{k-1} - \beta^k) \frac{\beta^k}{\beta^{k-1}} |u^k \cdot \eta|^2 \frac{\partial h}{\partial \eta} \]

\[ = - \frac{\partial h}{\partial \eta} \left\{ \frac{(\alpha^{k-1} - \alpha^k)}{\alpha^{k-1}} (p^k)^2 + (\beta^{k-1} - \beta^k) \frac{\beta^k}{\beta^{k-1}} |u^k \cdot \eta|^2 + (\beta^{k-1} - \beta^k) |u^k \times \eta|^2 \right\} \]

Now, we estimate the sixth term, \( \sum_{m=0}^{k-1} \int_{\Omega_k} \int_0^s f^k dt dx \). Remember that,

\[ \sum_{m=0}^{k-1} \int_{\Omega_k} \int_0^s f^k dt dx = \sum_{m=0}^{k-1} \int_{\Omega_k} \int_0^s \left\{ \beta^k(\Delta h - 1) |u^k|^2 - 2\beta^k \sum_{i,j=1}^{m} \frac{\partial^2 h}{\partial x_i \partial x_j} u_i^k u_j^k - \alpha^k(\Delta h - 3)(p^k)^2 \right\} \]

To estimate (29), we choose a function \( h(x) = \frac{1}{2} |x - x_0|^2 + \delta_0 \Phi(x) \)

where \( x_0 \in \sigma_1 \) and \( \Phi \) satisfy

\[
\begin{align*}
\Delta \Phi &= 1 \text{ en } \Omega \\
\frac{\partial \Phi}{\partial \eta} &= \frac{2 \text{ Vol}(\Omega)}{\text{ Vol}(S_0)}, \text{ en } S_0 \\
\frac{\partial \Phi}{\partial \eta} &= \frac{\text{ area}(S_1)}{\text{ area}(S_1)}, \text{ en } S_1
\end{align*}
\]

Remark 3.4 Given \( \mu = \mu(\Omega) \), by

\[ \mu(\Omega) = \inf_{x \in \Omega} \sum_{i=1}^{3} \frac{\partial^2 \Phi(x)}{\partial x_i \partial x_j} \xi_i \xi_j \]

we may observe that, considering \( \xi = (1, 0, 0), \xi = (0, 1, 0), \xi = (0, 0, 1) \), have that

\[ \mu(\Omega) \leq 2 \frac{\partial^2 \Phi(x)}{\partial x_1^2}, \quad \mu(\Omega) \leq 2 \frac{\partial^2 \Phi(x)}{\partial x_2^2}, \quad \mu(\Omega) \leq 2 \frac{\partial^2 \Phi(x)}{\partial x_3^2} \]

that, adding the last expressions we have that

\[ 3\mu(\Omega) \leq 2\Delta \Phi \implies \mu(\Omega) \leq \frac{2}{3} \]
the expression of $h(x)$, then
\[
\frac{\partial^2 h(x)}{\partial x_i \partial x_j} = \delta_{ij} + \delta_0 \frac{\partial^2 \Phi(x)}{\partial x_i \partial x_j}
\]
and $\Delta h = 3 + \delta_0$.

**Lemma 3.5** Given \( \{u, p\} \) a regular solution of the problem (1)-(5) by theorem 2.3. Choosing $h$ as (30), we have that
\[
\sum_{k=0}^{m} \int_{\Omega_k} \int_{0}^{s} J^k dt dx \leq \delta_0 (1 - \mu(\Omega)) \sum_{k=0}^{m} \int_{\Omega_k} \int_{0}^{s} \sum_{i,j=1}^{2} \left( \delta_{ij} + \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \right) u_i^k u_j^k - \alpha^k (p^k)^2 \right\} dt dx
\]
for any $\delta_0 > 0$

**Proof.** Using (30), (31), (32) and the observation (3.4), we have that
\[
\sum_{k=0}^{m} \int_{\Omega_k} \int_{0}^{s} J^k dt dx = \sum_{k=0}^{m} \int_{\Omega_k} \int_{0}^{s} \left\{ \beta^k (2 + \delta_0) | u^k |^2 - 2\beta^k \sum_{i,j=1}^{2} \left( \delta_{ij} + \frac{\partial^2 \Phi}{\partial x_i \partial x_j} \right) u_i^k u_j^k - \alpha^k (p^k)^2 \right\}
\]
\[
\leq \sum_{k=0}^{m} \int_{\Omega_k} \int_{0}^{s} \left\{ \beta^k (2 + \delta_0) | u^k |^2 - 2\beta^k | u^k |^2 - \beta^k \mu(\Omega) \delta_0 | u^k |^2 - \alpha^k \delta_0 (p^k)^2 \right\}
\]
\[
\leq \sum_{k=0}^{m} \int_{\Omega_k} \int_{0}^{s} \left\{ \delta_0 \beta^k | u^k |^2 - \mu(\Omega) \delta_0 \beta^k | u^k |^2 - \alpha^k \delta_0 (p^k)^2 \right\} dt dx
\]
\[
\leq \delta_0 (1 - \mu(\Omega)) \sum_{k=0}^{m} \int_{\Omega_k} \int_{0}^{s} \left\{ \beta^k | u^k |^2 + \alpha^k (p^k)^2 \right\} dt dx
\]

To estimate the rest of the terms of (18), $2 \sum_{k=0}^{m} \int_{\Omega_k} \int_{0}^{s} \beta^k u_0^k(x) \cdot \int_{0}^{s} u^k(x, r) dr dx$, considering a hypothesis about initial condition $u_0(x)$, we assumed that, it satisfied the following system:
\[
\begin{cases}
  u_0^{-1} \eta = u_0 \cdot \eta \quad \text{in} \quad \Gamma_k \\
u_0^{-1} \times \eta = u_0 \times \eta \quad \text{in} \quad \Gamma_k \\
u_0 = \nabla l^k(x) \quad \text{in} \quad \Gamma_k \\
l^k \in H^2(\Omega_k) \quad \text{and} \quad h = 0 \quad \text{in} \quad S_1
\end{cases}
\]

**Remark 3.6** The hypothesis have been made in (34), inclusive the solution of the problem satisfy the properties of the lemma (2.2), these are necessaries because the domain is conexo.

**Lemma 3.7** Given \( \{u, p\} \) regular solution of the problem (3)-(5) by theorem 2.3, and the initial condition $u_0$ that satisfied (34). Then
\[
2 \sum_{k=0}^{m} \int_{\Omega_k} \beta^k u_0^k(x) \int_{0}^{s} u^k(x, r) dr dx = 2 \sum_{k=1}^{m} \int_{\Gamma_k} l^k(p^k(x, s) - p_0^k(x)) dx
\]

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Proof. Using the hypothesis made in the initial condition (34), we have that

\[
2 \sum_{k=0}^{m} \int_{\Omega_k} \beta^k u_0^k(x) \int_0^s u^k(x,r) dr dx = 2 \sum_{k=0}^{m} \int_{\Omega_k} \int_0^s \beta^k \nabla l^k \cdot u^k
\]

\[
= 2 \sum_{k=0}^{m} \int_0^s \left\{ - \int_{\Omega_k} \beta^k \nu l^k \operatorname{div}(u^k) - \int_{\partial \Omega_k} \beta^k l^k (u^k \cdot \eta) \right\}
\]

\[
= 2 \sum_{k=0}^{m} \int_0^s \int_{\Omega_k} l^k \frac{\partial p^k}{\partial t} + 2 \sum_{k=0}^{m} \int_s^0 \int_{\partial \Omega_k} \beta^k l^k (u^k \cdot \eta)
\]

\[
= 2 \sum_{k=0}^{m} \int_0^s \int_{\Omega_k} l^k \frac{\partial p^k}{\partial t} + 2 \int_0^s \int_{\partial S_1} \beta l (u \cdot \eta) + \int_0^s \int_{\partial S_0} \beta^k l^k (u^k \cdot \eta)
\]

by the hypothesis made in the initial condition (34), we have that

\[
\begin{align*}
\text{De, } u_0^{k-1} \cdot \eta &= u_0^k \cdot \eta \quad \text{in } \Gamma_k \implies \nabla l^{k-1} \cdot \eta &= \nabla l^k \cdot \eta \quad \text{in } \Gamma_k \\
&\implies \nabla (l^{k-1} - l^k) \cdot \eta = 0 \quad \text{in } \Gamma_k \\
&\implies \nabla (l^{k-1} - l^k) \cdot \eta \perp \eta \quad \text{in } \Gamma_k.
\end{align*}
\]

\[
\begin{align*}
\text{De, } u_0^{k-1} \times \eta &= u_0^k \times \eta \quad \text{in } \Gamma_k \implies \nabla l^{k-1} \times \eta = \nabla l^k \times \eta \quad \text{in } \Gamma_k \\
&\implies \nabla (l^{k-1} - l^k) \times \eta = 0 \quad \text{in } \Gamma_k \\
&\implies \nabla (l^{k-1} - l^k) \times \eta \parallel \eta \quad \text{in } \Gamma_k.
\end{align*}
\]

thereby, \(\nabla (l^{k-1} - l^k) = 0\) in \(\Gamma_k\), it implies that \(l^{k-1} - l^k = C\) in \(\Gamma_k\), \(C = \text{constant}\). Substituting (36) in the second term on the right-hand side of (35), then

\[
2 \sum_{k=1}^{m} \int_0^s \int_{\Gamma_k} \left[ \beta^{k-1} l^{k-1} (u^{k-1} \cdot \eta) - \beta^k l^k (u^k \cdot \eta) \right] = 2 \sum_{k=1}^{m} \int_0^s \int_{\Gamma_k} (l^{k-1} - l^k) \beta^k (u^k \cdot \eta)
\]

\[
= 2C \sum_{k=1}^{m} \int_0^s \int_{\Gamma_k} \beta^k (u^k \cdot \eta)
\]

\[
= 2 \int_0^s \int_{S_0} \beta (u \cdot \eta) = 0
\]

and following substituting (37) in (35).
Substituting the obtained estimations, (19), (20), (21), (22), (29), (37) in (18), then,

\[
\sum_{k=0}^{m} \int_{\Omega_k} [\beta^k | u^k |^2 + \alpha^k (p^k)^2] \, dx = 2C_1 s \sum_{k=0}^{m} \int_{\Omega_k} [\beta^k | u^k |^2 + \alpha^k (p^k)^2] \, dx - \\
\frac{\partial}{\partial s} \sum_{k=0}^{m} \alpha^k [p^k(x, r) dr]^2 \, dx + \int_{0}^{s} \beta | u|^2 \frac{\partial h}{\partial \eta} + \\
\sum_{k=0}^{m} \int_{0}^{s} \int_{\Gamma_k} [\beta_k^{k-1} \eta - \beta_k^{k} \eta] + \int_{0}^{s} \int_{S_0} \left( \alpha \rho \frac{\partial h}{\partial \eta} - \beta | u|^2 \frac{\partial h}{\partial \eta} \right) + \\
\delta_0 (1 - \mu(\Omega)) \sum_{k=0}^{m} \int_{0}^{s} \int_{\Gamma_k} [\beta^k | u^k |^2 + \alpha^k (p^k)^2] + \\
2 \sum_{k=0}^{m} \int_{\Omega_k} l_k (p^k(x, s) - p_0^k(x)) \, dx
\]

Integrating (38) in \((0, T)\) and using the independence of energy of the model with the time, we have that

\[
\frac{T}{2} [1 - \delta_0 (1 - \mu(\Omega))] \sum_{k=0}^{m} \int_{\Omega_k} [\beta^k | u^k |^2 + \alpha^k (p^k)^2] \, dx \leq 2C_1 T \sum_{k=0}^{m} \int_{\Omega_k} [\beta^k | u^k |^2 + \alpha^k (p^k)^2] \, dx - \\
\sum_{k=0}^{m} \int_{\Omega_k} \alpha^k \left[ \int_{0}^{T} p^k(x, r) \right]^2 \, dx + \int_{0}^{T} \int_{0}^{s} \int_{S_1} \beta | u|^2 \frac{\partial h}{\partial \eta} + \sum_{k=0}^{m} \int_{0}^{T} \int_{\Gamma_k} [\beta_k^{k-1} \eta - \beta_k^{k} \eta] + \\
\int_{0}^{T} \int_{0}^{s} \int_{S_0} \left( \alpha \rho \frac{\partial h}{\partial \eta} - \beta | u|^2 \frac{\partial h}{\partial \eta} \right) + 2 \sum_{k=0}^{m} \int_{0}^{T} \int_{\Omega_k} l_k (p^k(x, s) - p_0^k(x))
\]

Now, we need the hypothesis in the domain \(\Omega\). Given \(\delta_0 > 0\) such that, some \(x_0 \in \sigma_1\), we have

\[
\begin{align*}
\delta_0 (1 - \mu(\Omega)) &< 1, \\
(x - x_0).\eta &\geq -2\delta_0 \frac{\text{Vol}(\Omega)}{\text{area}(S_0)}, \quad \text{for } x \in S_0 \\
(x - x_0).\eta &\leq \delta_0 \frac{\text{Vol}(\Omega)}{\text{area}(S_1)}, \quad \text{for } x \in S_1 \\
(x - x_0).\eta + \delta_0 \frac{\partial \Phi}{\partial \eta} &\geq 0, \quad \forall x \in \Gamma_k, \quad k = 1, 2, \ldots, m
\end{align*}
\]

**Remark 3.8** The hypothesis made in (40), These are true when \(\delta_0 = 0\) satisfied the surf of kind "star-shaped".

Using (40) in \(S_1\), we have

\[
\frac{\partial h}{\partial \eta} = \nabla h.\eta = (x - x_0).\eta + \delta_0 \frac{\partial \Phi}{\partial \eta} = (x - x_0).\eta - \delta_0 \frac{\text{Vol}(\Omega)}{\text{area}(S_1)} \leq 0
\]

and, for \(x \in S_0\),

\[
\frac{\partial h}{\partial \eta} = \nabla h.\eta = (x - x_0).\eta + \delta_0 \frac{\partial \Phi}{\partial \eta} = (x - x_0).\eta + 2\delta_0 \frac{\text{Vol}(\Omega)}{\text{area}(S_0)} \geq 0
\]

Substituting (42) and (41) in (39), then

\[
\int_{0}^{T} \int_{0}^{s} \int_{S_1} \beta | u|^2 \frac{\partial h}{\partial \eta} \leq 0
\]

and

\[
- \int_{0}^{T} \int_{0}^{s} \int_{S_0} \beta | u|^2 \frac{\partial h}{\partial \eta} \leq 0
\]
moreover, if the coefficients $\alpha^k, \beta^k$ satisfied

\[
\begin{align*}
\alpha^{k-1} &\leq \alpha^k \\
\beta^{k-1} &\leq \beta^k
\end{align*}
\tag{45}
\]

then, the fourth condition in (40), (45) and the lemma (3.3) we have that

\[
\sum_{k=1}^{m} \int_{0}^{T} \int_{0}^{s} \int_{\Gamma_k} \left[ \tilde{B}^{k-1} \cdot \eta - \tilde{B}^{k} \cdot \eta \right] \leq 0
\tag{46}
\]

Using that the associate energy to the system (1)-(3); with not dependency of the time, we prove the next lemma

**Lemma 3.9** Given a regular solution $\{u, p\}$ of the problem (1) by theorem 2.3, and the initial condition $u_0$ that satisfied (34). Then

\[
2 \sum_{k=0}^{m} \int_{0}^{T} \int_{\Omega_k} t^k (p^k(x, s) - p_0^k(x)) dx ds \leq \sum_{k=0}^{m} \int_{\Omega_k} \left[ \int_{0}^{T} p^k(x, s) \right]^2 dx + \\
(C_3 + C_4 T) \sum_{k=0}^{m} \int_{\Omega_k} [\beta^k | u^k |^2 + \alpha^k (p^k)^2] dx
\]

where $C_3 = C_2 \max_k \{(\alpha^k \beta^k)^{-1}\}$ and $C_4 = \max_k \{C_2(\beta^k)^{-1}, (\alpha^k)^{-1}\}$
Proof.

\[ 2 \sum_{k=0}^{m} \int_{0}^{T} \int_{\Omega_{k}} |t^{k}(p^{k}(x, s) - p_{0}^{k}(x))| = 2 \sum_{k=0}^{m} \int_{0}^{T} \int_{\Omega_{k}} |t^{k}p^{k}(x, s)| + 2 \sum_{k=0}^{m} \int_{0}^{T} \int_{\Omega_{k}} |t^{k}p_{0}^{k}(x)| \]

\[ 2 \leq \sum_{k=0}^{m} \int_{0}^{T} |t^{k}(x)| \int_{0}^{T} |t^{k}p^{k}(x, t)| + 2 \sum_{k=0}^{m} \int_{0}^{T} \left( \int_{0}^{T} |t^{k}(x)|^{2} \right)^{1/2} \left( \int_{0}^{T} |p^{k}(x, s)|^{2} \right)^{1/2} \]

\[ \leq 2 \sum_{k=0}^{m} \left[ \int_{0}^{T} (|t^{k}(x)|^{2})^{1/2} \left( \int_{\Omega_{k}} [p^{k}(x, s)]^{2} \right)^{1/2} \right] + \sum_{k=0}^{m} \int_{0}^{T} \int_{\Omega_{k}} \{(|t^{k}|^{2} + (p^{k})^{2}) \right\} \]

\[ \leq \sum_{k=0}^{m} \left\{ (\alpha^{-1})^{-1} \int_{0}^{T} (|t^{k}(x)|^{2}) + \alpha^{k} \int_{0}^{T} |p^{k}(x, s)|^{2} \right\} + \sum_{k=0}^{m} \int_{0}^{T} \int_{\Omega_{k}} \{(|t^{k}|^{2} + (p^{k})^{2}) \}

\[ \leq C_{2} \sum_{k=0}^{m} \int_{0}^{T} (|t^{k}|^{2}) + \sum_{k=0}^{m} \alpha^{k} \int_{0}^{T} \int_{\Omega_{k}} |p^{k}(x, s)|^{2} \right\} + \sum_{k=0}^{m} \int_{0}^{T} \int_{\Omega_{k}} \{(|t^{k}|^{2} + (p^{k})^{2}) \}

\[ \leq C_{2} \max_{k} \{(\alpha^{k} \beta^{k})^{-1}\} \sum_{k=0}^{m} \int_{0}^{T} \int_{\Omega_{k}} \beta^{k} |u^{k}|^{2} + \alpha^{k}(p^{k})^{2} \} + \sum_{k=0}^{m} \alpha^{k} \int_{0}^{T} \int_{\Omega_{k}} |p^{k}(x, s)|^{2} \]

\[ \leq C_{3} = \max_{k} \{(\alpha^{k} \beta^{k})^{-1}\} \text{ and } C_{4} = \max_{k} \left\{ C_{2}(\beta^{k})^{-1}, (\alpha^{k})^{-1} \right\} \text{ where } C_{3} = C_{2} \max_{k} \{(\alpha^{k} \beta^{k})^{-1}\} \text{ and } C_{4} = \max_{k} \left\{ C_{2}(\beta^{k})^{-1}, (\alpha^{k})^{-1} \right\} \]

Substituting in (39), and using the \( \frac{\partial h}{\partial \eta} \geq 0 \) in \( S_{0} \), we have that

\[ \frac{T}{2} \int_{0}^{T} \int_{\Omega_{k}} \beta^{k} |u^{k}|^{2} + \alpha^{k}(p^{k})^{2} \right\} \]

\[ \leq \left\{ (2C_{1} + C_{4})T + C_{3} \right\} \sum_{k=0}^{m} \int_{0}^{T} \int_{\Omega_{k}} \beta^{k} |u^{k}|^{2} + \alpha^{k}(p^{k})^{2} \right\} + \int_{0}^{T} \int_{S_{0}} \alpha p^{2} \frac{\partial h}{\partial \eta} \]

thus,

\[ \frac{T}{2} \int_{0}^{T} \int_{\Omega_{k}} \beta^{k} |u^{k}|^{2} + \alpha^{k}(p^{k})^{2} \right\} \]

\[ \leq 2 \int_{0}^{T} \int_{S_{0}} \alpha p^{2} \frac{\partial h}{\partial \eta} dS_{0} dt \]

where \( C_{5} = C_{3} + (2C_{1} + C_{4})T \) and considered \( T > \max 1, (2C_{1} + C_{4}) \). We have proved the
\textbf{Theorem 3.10} Taking $\Phi$ as in (31), the geometry properties (40), and the hypothesis of monotony of coefficients (45) and the hypothesis (34); these were made for the initial condition. Then, $\exists C_5 > 0$, with independence of $t, u, v, p_0$, such that

$$\frac{T}{2} [1 - \delta_0(1 - \mu(\Omega)) - 2C_5] \sum_{k=0}^{m} \int_{\Omega_k} [\beta^k | u^k |^2 + \alpha^k (p^k)^2] \, dx \leq 2 \int_0^T \int_{S_0} \alpha p^2 \frac{\partial h}{\partial \eta} \, dS_0 \, dt.$$ 

The same manner, we obtain the inequality of observability for the system (2)-(4) with their interface conditions and the monotonicity of the coefficients, given by:

$$\gamma^{k-1} \leq \gamma^k$$

$$\tau^{k-1} \leq \tau^k$$

\textbf{Theorem 3.11} Assuming $\Phi$ as in (31), the monotonicity for the coefficients (50) and the hypothesis of the theorem 3.10 with $h(x) = \frac{1}{2} |x - x_0|^2 + \delta_0 \Phi(x)$ and $(v_0, q_0) \in V_2 \cap D(A_2)$, $v_0^k = \nabla m^k$, with $m^k \in H^2(\Omega_k)$, $m = 0$ in $S_1$. Then, there is a constant $C_6 > 0$, with independence of $t, v_0, q_0$ such that

$$T [1 - \delta_0(1 - \mu(\Omega))] \sum_{k=0}^{m} \int_{\Omega_k} [\tau^k | v^k |^2 + \gamma^k (q^k)^2] \, dx - 2C_6 \sum_{k=0}^{m} \int_{\Omega_k} [\tau^k | v^k |^2 + \gamma^k (q^k)^2] \, dx$$

$$\leq 2 \int_0^T \int_{S_0} \tau | v.\eta |^2 \frac{\partial h}{\partial \eta} \, dS_0 \, dt.$$ 

\textbf{Proof.} The proof was obtained using the results in [10] and the estimations that were made in the proof of the theorem 3.10

Assuming the hypothesis of the theorems 3.10 and 3.11, we obtain the inequalities of the observability:

$$(T - T_0) \sum_{k=0}^{m} \int_{\Omega_k} [\beta^k | u^k |^2 + \alpha^k (p^k)^2 + \tau^k | v^k |^2 + \gamma^k (q^k)^2] \, dx$$

\begin{equation}
\leq C_7 \int_0^T \int_{S_0} [\alpha p^2 + \tau | v.\eta |^2] \frac{\partial h}{\partial \eta} \, dS_0 \, dt.
\end{equation}

For any $T > T_0 = \max \left\{1, \frac{C_5 + C_6}{\delta_0(1 - \mu(\Omega))} \right\}$. That was made in [10] and [8], (51). This is an inequality of observability, moreover, it is not convenient to use the H.U.M technique. Starting of (51), we obtain an appropriate inequality.

\textbf{Theorem 3.12} Assuming the hypothesis of the theorem 3.10, 3.11. Moreover, we suppose that:

$$\alpha^k \beta^k = \gamma^k \tau^k$$

$$\beta^{k-1} \tau^k = \beta^k \tau^{k-1}.$$ 

\textbf{Then, There is a positive constant $C > 0$ such that}

$$(T - T_0) \sum_{k=0}^{m} \int_{\Omega_k} [\beta^k | u^k |^2 + \alpha^k (p^k)^2 + \tau^k | v^k |^2 + \gamma^k (q^k)^2] \, dx$$

\begin{equation}
\leq C \int_0^T \int_{S_0} [\alpha p - \tau (v.\eta)]^2 \frac{\partial h}{\partial \eta} \, dS_0 \, dt.
\end{equation}

$\forall T > \max \left\{1, \frac{C_5 + C_6}{\delta_0(1 - \mu(\Omega))} \right\}$. 

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Proof. Using the theorems (4) and (52), we have that

\[
\frac{d}{dt} \int_{\Omega} (\tau u.v + \alpha p q) dx = \frac{d}{dt} \sum_{k=0}^{m} \int_{\Gamma_k} (\tau^k u^k v^k + \alpha^p q^k) dx = \sum_{k=0}^{m} \int_{\Omega_k} \left\{ \tau^k \frac{\partial u^k}{\partial t} + \alpha^p q^k \frac{\partial q^k}{\partial t} \right\} dx
\]

\[
= \sum_{k=0}^{m} \int_{\Omega_k} \left\{ -\alpha^k \tau^k \nabla p^k \cdot v^k - \tau^k \gamma^k u^k \nabla q^k - \alpha^k \beta^k \text{div}(u^k) q^k - \alpha^k \tau^k p^k \text{div}(v^k) \right\}
\]

\[
= -\sum_{k=0}^{m} \int_{\partial \Omega_k} \alpha^k \tau^k p^k (v^k, \eta) + \sum_{k=0}^{m} \int_{\Omega_k} \left\{ \alpha^k \tau^k p^k \text{div}(v^k) - \tau^k \gamma^k u^k \nabla q^k - \alpha^k \beta^k \text{div}(u^k) q^k - \alpha^k \tau^k p^k \text{div}(v^k) \right\} dx
\]

\[
= -\sum_{k=0}^{m} \int_{\partial \Omega_k} \alpha^k \tau^k p^k (v^k, \eta) - \sum_{k=0}^{m} \int_{\partial \Omega_k} \tau^k \gamma^k q^k (u^k, \eta) + \sum_{k=0}^{m} \int_{\partial \Omega_k} \alpha^k \beta^k \text{div}(u^k) q^k - \alpha^k \tau^k p^k \text{div}(u^k) q^k \]

\[
= \sum_{k=0}^{m} \int_{\partial \Omega_k} \left\{ \gamma^k \tau^k \text{div}(u^k) q^k - \alpha^k \beta^k \text{div}(u^k) q^k \right\}
\]

\[
= -\sum_{k=0}^{m} \int_{\partial \Omega_k} \alpha^k \tau^k p^k (v^k, \eta) - \sum_{k=0}^{m} \int_{\partial \Omega_k} \tau^k \gamma^k q^k (u^k, \eta)
\]

\[
= -\int_{S_1} \alpha \tau p (v, \eta) - \sum_{k=0}^{m} \int_{\Gamma_k} \left[ \alpha^k - \tau^k \gamma^k (u^k, \eta) \right] - \int_{S_0} \alpha \tau q (u, \eta) - \int_{S_1} \tau \gamma (u, \eta) - \sum_{k=0}^{m} \int_{\Gamma_k} \left[ \tau^k - \gamma^k q^k (u^k, \eta) \right] - \int_{S_0} \tau \gamma q (u, \eta)
\]

and, using the boundary condition and (52) in the interfaces, we have that

\[
\frac{d}{dt} \sum_{k=0}^{m} \int_{\Omega_k} (\tau^k u^k v^k + \alpha^k p^k q^k) dx = -\int_{S_0} \alpha \tau p (v, \eta) dS_0
\]

(54)

This is clearly that the first, second, fourth and sixth term of (53) are null only substituting directly the interface conditions. The fifth term is made using (52)

\[
\sum_{k=0}^{m} \int_{\Gamma_k} \left[ \tau^k - \gamma^k q^k (u^k, \eta) \right] \]

\[
= \sum_{k=0}^{m} \int_{\Gamma_k} \left[ \frac{\tau^k}{\beta_k^k} \gamma^k q^k (u^k, \eta) \right]
\]

\[
= \sum_{k=0}^{m} \int_{\Gamma_k} \left[ \frac{\tau^k}{\beta_k^k} \gamma^k q^k (u^k, \eta) \right]
\]

\[
= \sum_{k=0}^{m} \int_{\Gamma_k} \left[ \frac{\gamma^k}{\beta_k^k} \gamma^k q^k (u^k, \eta) \right]
\]

\[
= 0
\]

and ,

\[
\int_{S_0}^T \int_{S_0} \left[ \alpha p - \tau (v, \eta) \right]^2 = \int_{S_0}^T \int_{S_0} \alpha^2 p^2 + \int_{S_0}^T \int_{S_0} \tau^2 (v, \eta)^2 - 2 \tau \alpha \int_{S_0}^T \int_{S_0} p (v, \eta)
\]

(56)
Integrating (54) in \((0, T)\) and substituting in (56), we have that
\[
\int_0^T \int_{S_0} [\alpha p + \tau(v, \eta)]^2 = \int_0^T \int_{S_0} \alpha^2 p^2 + \int_0^T \int_{S_0} \tau^2(v, \eta)^2 + \sum_{k=0}^m \int_{\Omega_k} (\tau^k u^k, v^k + \alpha^k p^k q^k) |^T_0
\]
Now, we have that
\[
\int_0^T \int_{S_0} (\alpha^2 p^2 + \tau^2(v, \eta)^2) = \int_0^T \int_{S_0} [\alpha p - \tau(v, \eta)]^2 - \sum_{k=0}^m \int_{\Omega_k} (\tau^k u^k, v^k + \alpha^k p^k q^k) |^T_0
\]
using (57) to estimate the term of right-hand side of the inequality (51)
\[
\int_0^T \int_{S_0} [\alpha p - \tau(v, \eta)]^2 - C_8 \sum_{k=0}^m \int_{\Omega_k} (\tau^k u^k, v^k + \alpha^k p^k q^k) |^T_0
\]
Substituting in (51), we have that
\[
(T - T_0) \sum_{k=0}^m \int_{\Gamma_k} [\beta_k | u^k |^2 + \alpha^k (p^k)^2 + \tau^k | v^k |^2 + \gamma^k (q^k)^2] \, dx + C_8 \sum_{k=0}^m \int_{\Omega_k} (\tau^k u^k, v^k + \alpha^k p^k q^k) |^T_0
\]
\[
\leq C_7 C_8 \int_0^T \int_{S_0} [\alpha p - \tau(v, \eta)]^2
\]
and the proof of the theorem follows the inequalities:
\[
\sum_{k=0}^m \int_{\Omega_k} (\tau^k u^k, v^k + \alpha^k p^k q^k) |^T_0 \leq C_9 \int_0^T \int_{\Gamma_k} [\beta_k | u^k |^2 + \alpha^k (p^k)^2 + \tau^k | v^k |^2 + \gamma^k (q^k)^2] \, dx
\]
where \(C_9 = \max \{\tau^k(\beta^k)^{-1}, \alpha^k(\tau^k)^{-1}\}\)

As a corollary , of uniqueness of the theorem 3.12, we have that:

**Corollary 3.13** with the hypothesis of the theorem 3.12, given \(\{u, p\}\) and \(\{v, q\}\) solutions of the problem (3) and (2), respectively. Making that
\[
\alpha p(x, t) = \tau v(x, t), \eta \quad \forall (x, t) \in \Gamma_0 \times (0, T)
\]
then, in \(T > T_0, u = v = 0\) and \(p = q = 0\) for all \((x, t) \in \Omega \times (0, T)\)

**4 Exact controllability**

As a consequence of the corollary 3.13 , we have that: for \(T > T_0\), the expression
\[
\left[ \int_0^T \int_{\Gamma_0} [\alpha p - \tau(v, \eta)] \right]^{1/2}
\]
define a norm in a space of initial data \((u_0, p_0)\) and \((v_0, q_0)\) the problems (3) and (2). We denote by \(Y\) the Hilbert space defined as closure of \(V_1 \cap D(A_1) \times V_1 \cap D(A_1)\) in \(X = X_1 \times X_2\) with the norm (61). The obtained number in (61) is denoted by \(\| (u_0, p_0, v_0, q_0) \|_Y\). Now, \(Y \subset X\) and
\[
\| (u_0, p_0, v_0, q_0) \|_X = \| (u_0, p_0) \|_{X_1}^2 + \| (v_0, q_0) \|_{X_2}^2
\]
\[
\leq C \| (u_0, p_0, v_0, q_0) \|_Y^2
\]
for some positive constant $C$. The dual space of $Y$ respect to $X$ is denoted by $Y'$. In $\Omega \times (0, T)$ consider the systems (3) and (4) with initial condition $(u_0, p_0, v_0, q_0) \in Y'$. Using the transportation method, the solution to the problems (3) and (4) with no homogeneous contour conditions.

**Definition 4.1** Given $(u(x,t), p(x,t), v(x,t), q(x,t)) \in C(O,T; Y')$, is a solution of (3) and (4), if

$$\langle (u, p, v, q), (\tilde{u}, \tilde{p}, \tilde{v}, \tilde{q}) \rangle_X = \langle (u_0, p_0, v_0, q_0), (\tilde{u}_0, \tilde{p}_0, \tilde{v}_0, \tilde{q}_0) \rangle_X - \int_0^T \int_{\Gamma_0} [\alpha \beta Q\tilde{p} + \gamma \tau P(\tilde{u}, \eta)] d\Gamma_0 ds$$

for all $(\tilde{u}_0, \tilde{p}_0, \tilde{v}_0, \tilde{q}_0) \in Y$ and $0 < t < T$. Here $(\tilde{u}_0, \tilde{p}_0)$ and $(\tilde{v}_0, \tilde{q}_0)$ are the solutions of (3) and (4), respectively, for the functions $P, Q \in C(0, T; L^2(\Gamma_0))$. In (63) is given by

$$\langle (u,p,v,q),(\tilde{u},\tilde{p},\tilde{v},\tilde{q})\rangle_X = \langle (u,p),(\tilde{u},\tilde{p})\rangle_{X_1} + \langle (v,q),(\tilde{v},\tilde{q})\rangle_{X_2}$$

**Definition 4.2** A solution of (3) and (4); This is null in the time $t = T$. The function $(u(x,t), p(x,t), v(x,t), q(x,t))$ of (3) and (4) in $C(0,T; Y')$ such that

$$\langle (u, p, v, q), (\tilde{u}, \tilde{p}, \tilde{v}, \tilde{q}) \rangle_X = \int_0^T \int_{\Gamma_0} [\alpha \beta Q\tilde{p} + \gamma \tau P(\tilde{u}, \eta)] d\Gamma_0 ds$$

for all $(\tilde{u}, \tilde{p}, \tilde{v}, \tilde{q}) \in Y$ and $0 < t < T$.

Given the lineal and reversible systems (3) and (4) in the time; it is clearly to solve the problem of exact controllability, it is sufficient to prove that, for all initial condition in $Y'$, and their solutions, it can be take in the equilibrium of time $T$. Given $G_1 = (w_0, k_0)$ and $G_2 = (m_0, l_0)$ arbitrary elements of $Y$. We denote by

$$\begin{align*}
(w(x,t), k(x,t)) &= U_1(t)(w_0, k_0) \\
(m(x,t), l(x,t)) &= U_2(t)(m_0, l_0)
\end{align*}$$

consider the following functions

$$Q = \beta (ak(x,t) - km(x,t)\eta)$$

$$P = -\frac{\beta}{\gamma} Q$$

and given $(u, p)$ and $(v, q)$ the solution of (3) and (4), these are null in the instant $T$, $(T > T_0)$ and the contour conditions (65).

Considering the map

$$\wedge : Y \rightarrow Y'$$

$$(G_1, G_2) \rightarrow \wedge(G_1, G_2) = (u, p, v, q) |_{t=0}$$

Using (64) in $t = 0$ and substituting $P$ and $Q$ given by (65), we have

$$\langle \wedge(G_1, G_2), (\tilde{u}_0, \tilde{p}_0, \tilde{v}_0, \tilde{q}_0) \rangle_X = \int_0^T \int_{\Gamma_0} [\alpha \beta Q\tilde{p} + \gamma \tau P(\tilde{u}, \eta)] d\Gamma_0 ds$$

$$= \int_0^T \int_{\Gamma_0} (\gamma k - \tau m(\eta))(\alpha \tilde{p} - \tau \tilde{v}(\eta)) d\Gamma_0 ds$$

$$= \langle (G_1, G_2), (\tilde{u}_0, \tilde{p}_0, \tilde{v}_0, \tilde{q}_0) \rangle_Y$$

of (66), we can conclude that $\wedge$ is an isomorphism of $Y$ in $Y'$. Putting

$$\begin{align*}
(G_1, G_2) &= \wedge^{-1}((u_0, p_0), (v_0, q_0)) \\
Q &= \beta^{-1} (ak(x,t) - km(x,t)\eta) \\
P &= -\frac{\beta}{\gamma} Q
\end{align*}$$

(67)
Using (63) with $t = T > T_0$, we have that

$$\langle (u(x,T), p(x,T), v(x,T), q(x,T)), (U_1(t)(\tilde{u}_0, \tilde{p}_0), U_2(t)(\tilde{v}, \tilde{q})) \rangle_X =$$

$$= \langle \langle G_1, G_2 \rangle, (\tilde{u}_0, \tilde{p}_0, \tilde{v}, \tilde{q}) \rangle_X - \langle \langle G_1, G_2 \rangle, (\tilde{u}_0, \tilde{p}_0, \tilde{v}, \tilde{q}) \rangle_Y$$

for all $(\tilde{u}_0, \tilde{p}_0, \tilde{v}, \tilde{q}) \in Y$. Using (66), we have that $(u(x,T), p(x,T), v(x,T), q(x,T))$ is a functional null on $Y$. In conclusion we prove the following theorem

**Theorem 4.3** Assuming the hypothesis of theorem 3.12. If given $T > T_0$ and initial condition $(u_0, p_0, v_0, q_0) \in Y'$ the problem (1), (2), (5), (5). Then, there is exist an control $Q(x,t) \in C(0,T; L^2(\Gamma_0))$ such that the corresponding solution $(u, p, v, q)$ with the boundary condition

$$\begin{cases}
  u.\eta = Q, & \text{in } S_0 \times (0,T) \\
  p = 0, & \text{in } S_1 \times (0,T) \\
  u(x,0) = u_0(x), p(x,0) = p_0(x)
\end{cases}$$

(68)

and

$$\begin{cases}
  q = P, & \text{in } S_0 \times (0,T) \\
  q = 0, & \text{in } S_1 \times (0,T) \\
  v(x,0) = v_0(x), q(x,0) = q_0(x)
\end{cases}$$

(69)

with $P = -\beta \gamma^{-1} Q$, satisfy , for $x \in \Omega$

$$\begin{cases}
  u(x,T) = 0 \\
  p(x,T) = 0 \\
  v(x,T) = 0 \\
  q(x,T) = 0
\end{cases}$$

(70)

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