Exact and linearized refractive index stress-dependence in anisotropic photoelastic crystals

Fabrizio Daví

DICEA and ICRYS, Università Politecnica delle Marche, via Brecce Bianche, 60131 Ancona, Italy

For the permittivity tensor of photoelastic anisotropic crystals, we obtain the exact nonlinear dependence on the Cauchy stress tensor. We obtain the same result for its square root, whose principal components, the crystal principal refractive index, are the starting point for any photoelastic analysis of transparent crystals. From these exact results we then obtain, in a totally general manner, the linearized expressions to within higher-order terms in the stress tensor for both the permittivity tensor and its square root. We finish by showing some relevant examples of both nonlinear and linearized relations for optically isotropic, uniaxial and biaxial crystals.

1. Introduction

In crystal photoelasticity, the evaluation of the principal refractive index and their dependence on the stress, either applied or residual, is a mandatory step for any theoretical and experimental analysis of the optical properties of transparent crystals (vid. e.g. [1–3]).

For \( \mathbf{B}_0 \) the dielectric permeability tensor in the unstressed state and \( \mathbf{M} \) the fourth-order piezo-optic tensor, the dielectric permeability of a stressed crystal is described by the Maxwell linear relation\(^1\):

\[
\mathbf{B}(\mathbf{T}) = \mathbf{B}_0 + \mathbf{M}[\mathbf{T}],
\]

where \( \mathbf{T} \) is the symmetric Cauchy stress tensor. The relation between the principal values \((B_1, B_2, B_3)\) of \( \mathbf{B}(\mathbf{T}) \) and the principal refractive index \((n_1, n_2, n_3)\) is:

\[
B_k = n_k^{-2}, \quad k = 1, 2, 3;
\]

\(^1\)To denote the relevant tensor fields we are dealing with, we follow the notation that can be found, e.g. in [4].
the difference between principal refractive index, the birefringence:

\[ \Delta n = n_i - n_j, \quad i, j = 1, 2, 3, i \neq j, \quad (1.3) \]

is one of the most important measurable quantities in photoelastic experiments (cf. e.g. [5–7]).

Clearly, any analytical evaluation of the principal refractive index \( n_k \) can be done provided that we are first able to write the inverse of \( \mathbf{B}(\mathbf{T}) \), the dielectric permittivity

\[ \mathbf{K}(\mathbf{T}) = \mathbf{B}^{-1}(\mathbf{T}) = (\mathbf{B}_0 + \mathbf{M}[\mathbf{T}])^{-1}, \quad (1.4) \]

and then to obtain its square root:

\[ \mathbf{N}(\mathbf{T}) = \mathbf{K}^{1/2}(\mathbf{T}), \quad (1.5) \]

the principal values \( (n_1, n_2, n_3) \) of \( \mathbf{N}(\mathbf{T}) \) being the principal refractive index.

The typical solution of this problem is to first find the eigencouples \( (\mathbf{B}_k, \mathbf{u}_k) \) of \( \mathbf{K}(\mathbf{T}) \), then take the square root of the inverse of (1.2) and, finally, if we need linearized relations, linearize the result about the unstressed state \( \mathbf{T} = 0 \), like we did for instance in [8,9]. Such an approach has many limitations, since the possibility to find an explicit expression for the eigencouples \( (\mathbf{B}_k, \mathbf{u}_k) \) (here \( \mathbf{u}_k \) is the eigenvector associated with \( \mathbf{B}_k \)) depends heavily on the crystal symmetry through \( \mathbf{M} \) and on the stress tensor \( \mathbf{T} \): indeed, in [8,9] we considered a special state of stress. Moreover, for optically uniaxial materials the linearization about the unstressed state may not be well defined since the derivative of \( n_k \) with respect to \( \mathbf{T} \) may blow up for \( \mathbf{T} \to 0 \).

Recently, searching for an easy way to represent the rotation in the polar decomposition of the deformation gradient, in a serendipitous way I found an old paper by Hoger & Carlson [10] dealing with the inversion of a tensor like (1.1) and with the square-root extraction like in (1.5). The most interesting thing, which is the novelty of the present approach, is that the exact results obtained there did not require an \textit{a priori} solution of an eigenvalue problem but rather they were obtained by a repeated application of the Cayley–Hamilton theorem: indeed we arrive at an exact nonlinear relationship between the refractive index tensors \( \mathbf{N} \) and \( \mathbf{T} \), which can be the starting point for either a linearized relation, an exact or approximate representation for the principal refractive index, or both.

In our paper, we apply the results presented in [10] to obtain explicit, exact and nonlinear relations for the permittivity tensor (1.4) and for its square root (1.5), in terms of \( \mathbf{B}_0 \) and \( \mathbf{M}[\mathbf{T}] \). Then, by starting from these exact results, we give a general linearization procedure, which leads to within higher-order terms into \( \mathbf{T} \), to two relations which are equivalent to (1.1), namely:

\[
\begin{align*}
\mathbf{K}(\mathbf{T}) &= \mathbf{K}_0 + \mathbb{K}[\mathbf{T}] \\
\mathbf{N}(\mathbf{T}) &= \mathbf{N}_0 + \mathbb{N}[\mathbf{T}],
\end{align*}
\]

(1.6)

with the two fourth-order piezo-optic tensors \( \mathbb{K} \) and \( \mathbb{N} \) expressed solely in terms of the components of the eigencouples of \( \mathbf{B}_0 \) and \( \mathbf{M} \).

As a matter of fact, however, in order to obtain the principal refractive index from \( \mathbf{N}(\mathbf{T}) \) we still need to solve an eigenvalues problem: besides special cases of stress in which one eigenvector of \( \mathbf{N}(\mathbf{T}) \) is known, we have to solve the problem by the means of an approximate method like that proposed in [2]; on the other hand no further approximations besides the linearization, and no special hypothesis on \( \mathbf{T} \), are necessary to obtain (1.6)_2.

(a) Notation

Let \( \mathcal{V} \) be the three-dimensional vector space whose elements we denote, \( \mathbf{v} \in \mathcal{V} \) and \( \text{Lin} \) the space of second-order tensors \( \mathbf{A} \in \text{Lin} \). For \( \{\mathbf{e}_k\}, k = 1, 2, 3 \) an orthonormal base in \( \mathcal{V} \), the components of \( \mathbf{v} \) and \( \mathbf{A} \) are given by \( v_i = \mathbf{v} \cdot \mathbf{e}_i \) and \( A_{ij} = \mathbf{A} \cdot \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{A} \mathbf{e}_j \cdot \mathbf{e}_i, \; i, j = 1, 2, 3 \). We denote Sym and
Sym^1 as the subspaces of Lin of symmetric and positive-definite symmetric tensors, respectively; in Sym, we find it useful to use the orthogonal base \{W_k\}, \(h = 1, \ldots 6\):^2

\[
\begin{align*}
W_1 &= e_1 \otimes e_1, & W_2 &= e_2 \otimes e_2, & W_3 &= e_3 \otimes e_3, \\
W_4 &= \text{sym}(e_2 \otimes e_3), & W_5 &= \text{sym}(e_1 \otimes e_3), & W_6 &= \text{sym}(e_1 \otimes e_2),
\end{align*}
\]

(1.7)

with \(I = W_1 + W_2 + W_3\). We define the spherical and deviatorical parts of \(T \in \text{Sym}\) as:

\[
\begin{align*}
\text{sph} \; T &= \sigma_m I, & \text{dev} \; T &= T - \text{sph} \; T, & \sigma_m &= \frac{1}{3} \text{tr} \; T,
\end{align*}
\]

(1.8)

the underlying associated subspaces of Sym being Sph and Dev, Sym = Sph \oplus Dev; in the base (1.7) we have:

\[
\text{dev} \; T = \hat{T}_{11} W_1 + \hat{T}_{22} W_2 + \hat{T}_{33} W_3 + T_{23} W_4 + T_{13} W_5 + T_{12} W_6,
\]

(1.9)

where

\[
\begin{align*}
\hat{T}_{11} &= \frac{2T_{11} - T_{22} - T_{33}}{3}, & \hat{T}_{22} &= \frac{2T_{22} - T_{11} - T_{33}}{3}, & \hat{T}_{33} &= \frac{2T_{33} - T_{11} - T_{22}}{3}.
\end{align*}
\]

(1.10)

If \((\sigma_k, e_k), k = 1, 2, 3\) are the eigencouples of \(T\), then by the decomposition (1.8) we have

\[
\sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3), \quad \text{dev} \; T = \hat{\sigma}_k W_k, \quad k = 1, 2, 3,
\]

(1.11)

where \(\hat{\sigma}_k\) are the eigenvalues of dev \(T\).

The orthogonal invariants \(i_{kA}, k = 1, 2, 3\) of \(A \in \text{Lin}\) are defined by:

\[
\begin{align*}
i_{1A} &= I \cdot A = \text{tr} \; A, \\
i_{2A} &= I \cdot A^* = \frac{1}{2}((\text{tr} \; A)^2 - \|A\|^2), \\
i_{3A} &= \text{det} \; A,
\end{align*}
\]

(1.12)

and

\[
i_{3A} = (\det A)A^{-T}
\]

here \(A^* = (\det A)A^{-T}\) denotes the cofactor of \(A\). For \(\alpha \in \mathbb{R}\), the following identity holds:

\[
\det(A + \alpha I) = \alpha^3 + \alpha^2 i_{1A} + \alpha i_{2A} + i_{3A},
\]

(1.13)

moreover, for \(C = A + \alpha e \otimes e + \beta g \otimes g\) with \(\alpha, \beta \in \mathbb{R}\) and \(\|e\| = \|g\| = 1, e \cdot g = 0\), it is:

\[
\begin{align*}
i_{1C} &= i_{1A} + \alpha + \beta, \\
i_{2C} &= i_{2A} + \alpha \beta + i_{1A}(\alpha + \beta) - A \cdot (\alpha e \otimes e + \beta g \otimes g), \\
i_{3C} &= i_{3A} + A^* \cdot (\alpha e \otimes e + \beta g \otimes g).
\end{align*}
\]

(1.14)

Let \(M : \text{Sym} \rightarrow \text{Sym}\) be the piezo-optic fourth-order tensor, then its components are defined as

\[
M_{ijkl} = M[e_i \otimes e_k] \cdot e_j \otimes e_l, \quad M_{ijkl} = M_{ikjl} = M_{jikl},
\]

(1.15)

or in Voigt’s two-index notation:

\[
M_{AB} = M[W_B] \cdot W_A, \quad A, B = 1, \ldots 6.
\]

(1.16)

We denote \(I\) as the fourth-order identity and with \(M^T\) as the transpose of \(M\):

\[
M[A] \cdot B = M^T[B] \cdot A, \quad \forall A, B \in \text{Lin}.
\]

(1.17)

For a given \(B, C \in \text{Lin}\) we shall make use of the two fourth-order tensors \(B \boxtimes C\) and \(B \otimes C\) defined by:

\[
(B \boxtimes C)M[A] = BM[A]C, \quad (B \otimes C)M[A] = (M[A] \cdot C)B, \quad \forall A.
\]

(1.18)

^2We make use here of Voigt’s two-index notation for second- and fourth-order tensors, provided the identification 11 = 1, 22 = 2, 33 = 3, 23 = 4, 13 = 5, 12 = 6.
2. The dielectric permittivity tensor

In [10], an analytical exact expression of the inverse was obtained:

$$ (cI + M)^{-1}, \quad (2.1) $$

provided $c > 0$ and $M \in \text{Sym}^+$; the result was obtained by the means of a repeated application of the Cayley–Hamilton theorem. Further, by the same tool, an explicit and exact analytical expression for the square root of a symmetric positive definite tensor was also given.

In the following subsections, we shall show how the result given in [10] allows for an explicit, exact and nonlinear expression of the permittivity tensor $K(T)$ defined by (1.4) and of its square root $N(T)$ defined by (1.5).

We shall treat separately the three cases of optically isotropic, optically uniaxial and optically biaxial crystals, which differ for the different multiplicities of the eigenvalues of $B_0$.

(a) Optically isotropic crystals

For optically isotropic crystals (which are comprised of isotropic materials and cubic crystals), the tensor $B_0 \in \text{Sph}$, with

$$ B_0 = n_o^{-2}I. \quad (2.2) $$

Hence, the results of [10] can be used directly provided in (2.1) we identify:

$$ c = n_o^{-2}, \quad M = \mathbb{M}[T]. \quad (2.3) $$

We notice that, whereas both $B(T) \in \text{Sym}^+$ and $B_0 \in \text{Sym}^+$, nothing can be said about the difference $B(T) - B_0 = \mathbb{M}[T]$; however, in [10], the positive-definiteness of $M$ is an invertibility requirement; accordingly, we simply made the weaker assumption that $M = \mathbb{M}[T]$ is invertible for all $T$.

Granted such an assumption, from eqn (2.2) of [10], we obtain the explicit representation for $K(T)$:

$$ K(T) = \frac{1}{\alpha_3} \left( \alpha_1 I - \alpha_2 \mathbb{M}[T] + \mathbb{M}[T]^2 \right), \quad (2.4) $$

where the three functions $\alpha_j, j = 1, 2, 3$ are given by

$$ \begin{align*}
\alpha_1(n_o, \mathbb{M}[T]) &= n_o^{-4} + n_o^{-2} \iota_{1M} + \iota_{2M}, \\
\alpha_2(n_o, \mathbb{M}[T]) &= n_o^{-2} + \iota_{1M} \\
\alpha_3(n_o, \mathbb{M}[T]) &= n_o^{-6} + n_o^{-4} \iota_{1M} + n_o^{-2} \iota_{2M} + \iota_{3M}
\end{align*} \quad (2.5) $$

and the orthogonal invariants of $\mathbb{M}[T]$ by:

$$ \begin{align*}
\iota_{1M} &= \text{tr} \mathbb{M}[T], \\
\iota_{2M} &= \frac{1}{2}((\text{tr} \mathbb{M}[T])^2 - \|\mathbb{M}[T]\|^2) \\
\iota_{3M} &= \det \mathbb{M}[T] \neq 0.
\end{align*} \quad (2.6) $$

We remark that relation (2.4) can also be arrived at directly by the representation theorem for isotropic functions (vid. e.g. [11]):

$$ K(M) = \mathcal{I}_0(\iota_{km})I + \mathcal{I}_1(\iota_{km})M + \mathcal{I}_2(\iota_{km})M^2; \quad (2.7) $$

in our treatment the dependence of the three functions $\mathcal{I}_j(\iota_{km}), j = 0, 1, 2$ on the invariants of $M$ is made explicit by (2.5) as a consequence of the procedure followed in [10].
To obtain the square root of $K(T)$, we then make use of formula (3.7) of [10], which gives $N(T)$ in terms of functions of the invariants of both $K(T)$ and $N(T)$:

$$N(T) = \frac{1}{v_4} \left( v_1 I + v_2 K(T) - v_3 K(T)^2 \right),$$  

(2.8)

where

$$v_1 = \iota_{1N}^{13}K + \iota_{3N}(\iota_{2N}(\iota_{1N} + \iota_{1K}) + \iota_{2K}),$$

$$v_2 = (\iota_{1N}^{12N} - \iota_{3N})(\iota_{2N} + \iota_{1K}),$$

$$v_3 = \iota_{1N}^{12N} - \iota_{3N},$$

$$v_4 = \iota_{2N}(\iota_{2N} + \iota_{1K}) + \iota_{2K} + \iota_{3K};$$  

(2.9)

and the invariants $\iota_{iN}$ in (2.9) can be represented explicitly in terms of the invariants $\iota_{ik}$ (vid. [10], §5 and eqns (D2) and (D5) in the Dataset [DS] for this paper).

If we use (2.4) in (2.8) we arrive at the explicit expression for the square root of the permittivity tensor:

$$N(T) = a_0 I + a_1 M[I^2] + a_2 M[I^2]^2 + a_3 M[I^3] + a_4 M[I^4],$$  

(2.10)

where the five functions $a_j = a_j(n, M[I]) = a_j(\alpha, \nu)$, $j = 0, 1, \ldots, 4$ are given explicitly by:

$$a_0 = \frac{v_1 \alpha^2_3 + v_2 \alpha_1 \alpha_3 - v_3 \alpha_1^2}{v_4 \alpha_3^2},$$

$$a_1 = -\frac{v_2 \alpha_2 \alpha_3 - 2v_3 \alpha_2 \alpha_1}{v_4 \alpha_3^2},$$

$$a_2 = \frac{v_2 \alpha_2 + 2v_3 \alpha_1 - v_3 \alpha_2}{v_4 \alpha_3^2},$$

$$a_3 = \frac{2v_3 \alpha_2}{v_4 \alpha_3^2},$$

$$a_4 = -\frac{v_3}{v_4 \alpha_3^2},$$  

(2.11)

and

The principal values of (2.10) then give the explicit and nonlinear formula for the principal refraction index.

(b) Optically uniaxial crystals

Optically uniaxial crystals belong to trigonal, tetragonal and hexagonal symmetry groups. In all these cases, the tensor $B_o$ for uniaxial crystals has representation, provided we identify the optic axis direction with $e_3$:

$$B_o = n_o^{-2}(I - W_3) + n_e^{-2}W_3 = n_o^{-2}I + (n_e^{-2} - n_o^{-2})W_3,$$  

(2.12)

where $n_o$ is the ordinary and $n_e$ the extraordinary refractive index; accordingly

$$B(T) = n_o^{-2}I + (n_e^{-2} - n_o^{-2})W_3 + M[I],$$  

(2.13)

and the results obtained for optically isotropic materials still hold provided we set the tensor $M$ in place of $M$:

$$M = M + DW_3, \quad M = M[I], \quad D = n_e^{-2} - n_o^{-2}.$$  

(2.14)
Equations (2.4), (2.5), (2.9), (2.10) and (2.11) still hold provided we replace $\mathbf{M}$ with $\mathbf{\tilde{M}}$, the relation between their invariants being obtained by the means of identities (1.14):

$$
\begin{align*}
\dot{t}_{1\mathbf{\tilde{M}}} &= t_{1\mathbf{M}} + D, \\
\dot{t}_{2\mathbf{\tilde{M}}} &= t_{2\mathbf{M}} + D\mathbf{M} \cdot (\mathbf{I} - W_3), \\
\dot{t}_{3\mathbf{\tilde{M}}} &= t_{3\mathbf{M}} + D\mathbf{M}^* \cdot W_3.
\end{align*}
$$

(2.15)

and

The dielectric permittivity tensor for uniaxial crystals is then given by:

$$
\mathbf{K}(T) = \frac{1}{\beta_3} \left( \beta_1 (\mathbf{I} - W_3) + (\beta_1 - \beta_2 D + D^2)W_3 - \beta_2 \mathbf{M}[T] + D(\mathbf{\bar{M}}[T]W_3 + W_3 \mathbf{M}[T]) + \mathbf{M}[T]^2 \right),
$$

(2.16)

where the relations between the functions $\beta_k, k = 1, 2, 3$ are given by:

$$
\begin{align*}
\beta_1(n_o, n_e, \mathbf{M}[T]) &= \alpha_1 + Dn_o^{-2} + D\mathbf{M} \cdot (\mathbf{I} - W_3), \\
\beta_2(n_o, n_e, \mathbf{M}[T]) &= \alpha_2 + D \\
\beta_3(n_o, n_e, \mathbf{M}[T]) &= \alpha_3 + n_o^{-4}D + n_o^{-2}D\mathbf{M} \cdot (\mathbf{I} - W_3) + D\mathbf{M}^* \cdot W_3.
\end{align*}
$$

(2.17)

By setting (2.16) into (2.8) we then get the nonlinear relation for $\mathbf{N}(T)$ in uniaxial crystals:

$$
\mathbf{N}(T) = b_0 \mathbf{I} + b_1 W_3 + b_2 \mathbf{M} + b_3 (\mathbf{MW}_3 + W_3 \mathbf{M}) + b_4 W_3 \mathbf{MW}_3 + b_5 \mathbf{M}^2 + b_6 (\mathbf{MW}_3 W_3 + W_3 \mathbf{M}^2 + \mathbf{M}^2 W_3) + b_7 ((\mathbf{MW}_3)^2 + (\mathbf{W}_3 \mathbf{M})^2 + \mathbf{W}_3 \mathbf{M}^2 \mathbf{W}_3) + b_8 \mathbf{M}^3 + b_9 (\mathbf{M}^3 W_3 + \mathbf{W}_3 \mathbf{M}^3 + \mathbf{M} \mathbf{W}_3 \mathbf{M}^2 + \mathbf{M}^2 \mathbf{W}_3 \mathbf{M}) + b_{10} \mathbf{M}^4,
$$

(2.18)

where the 11 functions $b_k(n_o, n_e, \mathbf{M}[T]) = b_k(\beta_j, v_l), k = 0, 1, \ldots, 10$ are given explicitly by eqn (D8) of the dataset [DS].

(c) Optically biaxial crystals

Optically biaxial crystals are of the triclinic, monoclinic and orthorhombic symmetry groups and have three different principal refractive indexes: if we assume that the orthonormal frame $\{\mathbf{e}_k\}, k = 1, 2, 3$ is also the principal frame for $\mathbf{B}_0$, then:

$$
\mathbf{B}_0 = B_1 \mathbf{W}_1 + B_2 \mathbf{W}_2 + B_3 \mathbf{W}_3, \quad B_k = n_k^{-2}, k = 1, 2, 3;
$$

(2.19)

let $B_1 > B_2 > B_3$, then we can rewrite (2.19) as:

$$
\mathbf{B}_0 = B_3 \mathbf{I} + D_1 \mathbf{W}_1 + D_2 \mathbf{W}_2, \quad D_\alpha = B_\alpha - B_3, \alpha = 1, 2,
$$

(2.20)

and by replacing again into (2.4), (2.5), (2.8) and (2.9) the tensor $\mathbf{M}$ with $\mathbf{\tilde{M}}$

$$
\mathbf{\tilde{M}} = \mathbf{M} + D_1 \mathbf{W}_1 + D_2 \mathbf{W}_2,
$$

(2.21)

we arrive at the following relation for the permittivity tensor:

$$
\mathbf{K}(T) = \frac{1}{\gamma_3} \left( \gamma_1 \mathbf{I} + \sum_{\alpha=1}^{2} (D_\alpha^2 - \gamma_2 D_\alpha) \mathbf{W}_\alpha - \gamma_2 \mathbf{\bar{M}}[T] + \sum_{\alpha=1}^{2} D_\alpha (\mathbf{\bar{M}}[T] \mathbf{W}_\alpha + \mathbf{W}_\alpha \mathbf{\bar{M}}[T]) + \mathbf{\bar{M}}[T]^2 \right),
$$

(2.22)

This is true for all biaxial crystals but monoclinic and triclinic. For monoclinic crystals, however, the monoclinic $b$-axis is a principal direction and hence we can obtain an explicit representation for the eigencouples of $\mathbf{B}_\alpha$. 
where the functions $\gamma_k = \gamma_k(n_j, M[T]), j, k = 1, 2, 3$ are defined by

$$
\begin{align*}
\gamma_1 &= a_1 + n_3^2(D_1 + D_2) + D_1M \cdot (I - W_1) + D_2M \cdot (I - W_2), \\
\gamma_2 &= a_2 + D_1 + D_2
\end{align*}
$$

and

$$
\begin{align*}
\gamma_3 &= a_3 + n_3^{-4}(D_1 + D_2) + n_3^{-2}D_1D_2 \\
&+ \sum_{\alpha=1}^{2} (n_3^{-2}D_\alpha M \cdot (I - W_\alpha) + D_\alpha M^\star \cdot W_\alpha),
\end{align*}
$$

and the orthogonal invariants of $\tilde{M}$ are:

$$
\begin{align*}
t_{1\tilde{M}} &= t_{1M} + D_1 + D_2, \\
t_{2\tilde{M}} &= t_{2M} + D_1D_2 + D_1M \cdot (I - W_1) + D_2M \cdot (I - W_2), \\
t_{3\tilde{M}} &= t_{3M} + M^\star \cdot (D_1W_1 + D_2W_2).
\end{align*}
$$

As in the previous cases, by (2.8) and (2.22) we get the relation for the square root of the permittivity tensor for biaxial crystals:

$$
N(T) = c_0I + c_1W_1 + c_2W_2 + c_3M + c_4(MW_1 + W_1M) \\
+ c_5(MW_2 + W_2M) + c_6W_1MW_1 \\
+ c_7W_2MW_2 + c_8(W_1MW_2 + W_2MW_1) \\
+ c_9M^2 + c_{10}MW_1M + c_{11}MW_2M \\
+ c_{12}((MW_1)^2 + (W_1M)^2 + W_1M^2W_1) \\
+ c_{13}((MW_2)^2 + W_2M^2 + W_2MW_2) \\
+ c_{14}(W_1M^2W_2 + MW_1MW_2 + W_1MW_2M) + c_{15}M^3 \\
+ c_{16}(MW_1M^2 + W_1M^2 + M^3W_1 + M^2W_1M) \\
+ c_{17}(MW_2M^2 + W_2M^2 + M^3W_2 + M^2W_2M) + c_{18}M^4,
$$

with the 19 functions $c_k = c_k(n_j, M[T]) = c_k(\gamma_j, \nu_k), k = 0, 1, \ldots, 18, j = 1, 2, 3$ are given explicitly by eqn. (D10) of the dataset [DS].

3. Linearized relations

Relation (1.1) is a linear relation in the stress $T$; on the other hand, the exact relations for $K(T)$ and $N(T)$ we obtained in §2 are nonlinear, involving the inversion of (1.1) and the extraction of its square root.

Crystals however are brittle materials, with a limited elastic range and a low brittle fracture tensile strength: accordingly it makes sense to consider an expression for the principal refractive index which is linearized in the stress, to arrive at a relation which is equivalent to (1.1).

In previous papers dedicated to the same problem [8,9], we obtained linear relations for $n_k(T)$ by a linearization procedure which involved the eigenvalues $B_k$ of $B(T)$:

$$
n_k(T) = \frac{1}{\sqrt{B_k}} \bigg|_{T=0} - \frac{1}{2\sqrt{B_k}} \frac{\partial B_k}{\partial T} \bigg|_{T=0} \cdot T + o(||T||^2);
$$

this procedure however was far from general (we must have explicit relations for $B_k$, which is not possible for all crystallographic classes and all stress) and moreover the derivative of $B_k$ blows up to infinity for $T = 0$ for uniaxial crystals and some special cases of stress.
We give here, for both the permittivity tensor and for its square root, a general linearization scheme which holds for any stress $T$ and leads to (1.6)

$$K(T) = K_0 + K(T) + o(||T||^2)$$

and

$$N(T) = N_0 + N(T) + o(||T||^2),$$

with $K_0 = K(0)$, $N(0) = N_0$ and the two fourth-order tensors

$$K = \frac{\partial K}{\partial T} \bigg|_{T=0}, \quad N = \frac{\partial N}{\partial T} \bigg|_{T=0};$$

as in the previous section we shall treat in order the three cases of optically isotropic, uniaxial and biaxial crystals.

(a) Optically isotropic

In this case, since $B_0 = n_o^{-2}I$, it is trivial to evaluate its inverse and the associated square root.

$$K(0) = n_o^2I, \quad N(0) = n_oI.$$  \hfill{(3.4)}

As far as the fourth-order tensor $K$ is concerned, from (2.4) we have

$$K = I \otimes \frac{\partial}{\partial T} \left( \frac{\alpha_1}{\alpha_3} \right) \bigg|_{T=0} - \frac{\alpha_{20}}{\alpha_{30}} M,$$  \hfill{(3.5)}

where the terms $\alpha_{j0} = \alpha_j(n_o, 0)$, $j = 1, 2, 3$ are

$$\alpha_{10} = n_o^{-4}, \quad \alpha_{20} = n_o^{-2}, \quad \alpha_{30} = n_o^{-6};$$  \hfill{(3.6)}

then, since

$$\frac{\partial}{\partial T} \left( \frac{\alpha_1}{\alpha_3} \right) \bigg|_{T=0} = \frac{1}{\alpha_{30}^2} \left( \frac{\alpha_3 \alpha_1}{\alpha_3} \right) \bigg|_{T=0} - \frac{\alpha_{10}}{\alpha_{30}} \frac{\partial \alpha_3}{\partial T} \bigg|_{T=0}$$

$$= \frac{1}{\alpha_{30}^2} \left( \alpha_3 n_o^{-2} - \alpha_{10} n_o^{-4} \right) M^T I = 0,$$  \hfill{(3.7)}

we have, to within higher-order terms:

$$K(T) = n_o^2I - n_o^4 M[T].$$  \hfill{(3.8)}

When we turn our attention to (3.2)$_2$, by (2.10) we have that:

$$N = I \otimes \frac{\partial a_o}{\partial T} \bigg|_{T=0} + a_1(n_o, 0) M,$$  \hfill{(3.9)}

and since by eqns (D12) and (D16) from the dataset [DS] it is

$$\frac{\partial a_o}{\partial T} \bigg|_{T=0} = 0, \quad a_1(n_o, 0) = -\frac{n_o^3}{2},$$  \hfill{(3.10)}

then we are led, to within higher-order terms in the stress tensor, to

$$N(T) = n_o I - \frac{n_o^3}{2} M[T].$$  \hfill{(3.11)}

The piezo-optic tensors $K$ and $N$ for an optically isotropic material accordingly admit the following simple representation:

$$K = -n_o^4 M, \quad N = -\frac{n_o^3}{2} M,$$  \hfill{(3.12)}

both being proportional to the Maxwell piezo-optic tensor $M$. 
(b) Optically uniaxial

For optically uniaxial crystals, from (2.12), trivially:

\[ K(0) = n_o^2(I - W_3) + n_e^2 W_3, \quad N(0) = n_o(I - W_3) + n_e W_3, \]

(3.13)

whereas from (2.22) we get:

\[
\begin{align*}
\frac{\partial K}{\partial T} \bigg|_{T=0} &= I \otimes \frac{\partial}{\partial T} \left( \frac{\beta_1}{\beta_3} \right) \bigg|_{T=0} + W_3 \otimes \frac{\partial}{\partial T} \left( \frac{D(D - \beta_2)}{\beta_3} \right) \bigg|_{T=0} \\
&- \frac{\beta_2}{\beta_{20}} M + \frac{D}{\beta_{30}} (I \otimes W_3 + W_3 \otimes I) M, \\
\end{align*}
\]

(3.14)

where

\[
\begin{align*}
\beta_{10} &= \beta_1(n_o, n_e, 0) = \alpha_{10} + Dn_o^{-2}, \\
\beta_{20} &= \beta_2(n_o, n_e, 0) = \alpha_{20} + D, \\
\beta_{30} &= \beta_3(n_o, n_e, 0) = \alpha_{30} + Dn_o^{-4},
\end{align*}
\]

(3.15)

and

By (2.17) and (3.15), it is:

\[
\begin{align*}
\frac{\partial}{\partial T} \left( \frac{\beta_1}{\beta_3} \right) \bigg|_{T=0} &= \frac{1}{\beta_{20}^2} \left( \beta_{30} \frac{\partial \beta_1}{\partial T} \bigg|_{T=0} - \beta_{10} \frac{\partial \beta_3}{\partial T} \bigg|_{T=0} \right); \\
\end{align*}
\]

(3.16)

then, since:

\[
\begin{align*}
\frac{\partial \beta_1}{\partial T} \bigg|_{T=0} &= \frac{\partial \alpha_1}{\partial T} \bigg|_{T=0} + D M^T [I - W_3] \\
\frac{\partial \beta_3}{\partial T} \bigg|_{T=0} &= \frac{\partial \alpha_3}{\partial T} \bigg|_{T=0} + Dn_o^{-2} M^T [I - W_3],
\end{align*}
\]

(3.17)

by (3.15)_{1,3}, (3.17) and (3.7), then from (3.16) we arrive at

\[
\frac{\partial}{\partial T} \left( \frac{\beta_1}{\beta_3} \right) \bigg|_{T=0} = 0.
\]

(3.18)

In a similar manner:

\[
\begin{align*}
\frac{\partial}{\partial T} \left( \frac{D(D - \beta_2)}{\beta_3} \right) \bigg|_{T=0} &= -\frac{D}{\beta_{20}^2} \left( \beta_{30} \frac{\partial \beta_2}{\partial T} \bigg|_{T=0} + (D - \beta_{20}) \frac{\partial \beta_3}{\partial T} \bigg|_{T=0} \right); \\
\end{align*}
\]

(3.19)

since by (3.15)_2

\[
\frac{\partial \beta_2}{\partial T} \bigg|_{T=0} = \frac{\partial \alpha_2}{\partial T} \bigg|_{T=0} = M^T [I],
\]

(3.20)

then by (3.19), (3.15)_3 and (3.7) we arrive at

\[
\frac{\partial}{\partial T} \left( \frac{D(D - \beta_2)}{\beta_3} \right) \bigg|_{T=0} = n_o^2(n_o^2 - n_e^2)^2 M^T [W_3].
\]

(3.21)

From (3.18) and (3.21) then we get, to within higher-order terms, the linearized relation for the permittivity:

\[
K(T) = n_o^2(I - W_3) + n_e^2 W_3 - n_o^4 M^T [T] \\
+ n_o^2(n_o^2 - n_e^2)(M^T [T] \cdot W_3)W_3 \\
+ n_o^2(n_e^2 - n_o^2)(M^T [W_3 + W_3 M^T [T]].
\]

(3.22)
As far as the tensor \( \mathbf{N}(T) \) is concerned, from (2.18) we have
\[
\frac{\partial \mathbf{N}}{\partial T}_{T=0} = \mathbf{I} \otimes \frac{\partial b_0}{\partial T}_{T=0} + \mathbf{W}_3 \otimes \frac{\partial b_1}{\partial T}_{T=0} + b_2(n_o, n_e, 0)\mathbf{M} + b_3(n_o, n_e, 0)(\mathbf{I} \otimes \mathbf{W}_3 + \mathbf{W}_3 \otimes \mathbf{I})\mathbf{M} + b_4(n_o, n_e, 0)(\mathbf{W}_3 \otimes \mathbf{W}_3)\mathbf{M},
\]
and by the means of eqn (D24) of [DS], we obtain the linearized expression for the square root of the permittivity tensor:
\[
\mathbf{N}(T) = n_o(\mathbf{I} - \mathbf{W}_3) + n_e\mathbf{W}_3 - \frac{n_o^2}{2} \left( \mathbf{M}[\mathbf{T}] + F_1(\xi)(\mathbf{M}[\mathbf{T}]\mathbf{W}_3 + \mathbf{W}_3\mathbf{M}[\mathbf{T}]) \right) + F_2(\xi)\mathbf{W}_3\mathbf{M}[\mathbf{T}]\mathbf{W}_3),
\]
where the functions \( F_k = F_k(\xi), \xi = n_e/n_o, \) \( k = 1, 2, \) are defined by (vid. [DS]):
\[
F_1(\xi) = \frac{(1 - \xi^2)(\xi^2 + 2\xi + 2)}{2(1 + \xi)^2},
\]
and
\[
F_2(\xi) = \frac{(1 - \xi)^2}{(1 + \xi)^2},
\]
with \( F_k(1) = 0. \)

The piezo-optic tensors \( \mathbb{K} \) and \( \mathbb{N} \) for optically uniaxial crystals have the following representation:
\[
\mathbb{K} = -n_o^4 \left[ \mathbb{I} - (\xi^2 - 1)(\mathbf{W}_3 \otimes \mathbf{W}_3 + \mathbf{I} \otimes \mathbf{W}_3 + \mathbf{W}_3 \otimes \mathbf{I}) \right] \mathbf{M}
\]
and
\[
\mathbb{N} = -\frac{n_o^3}{2} \left[ \mathbb{I} + F_1(\xi)(\mathbf{I} \otimes \mathbf{W}_3 + \mathbf{W}_3 \otimes \mathbf{I}) + F_2(\xi)\mathbf{W}_3 \otimes \mathbf{W}_3 \right] \mathbf{M}.
\]

(c) Optically Biaxial

Since \( \mathbf{B}_o \) for optically biaxial crystals is given by (2.19), then \( \mathbf{K}(0) \) and \( \mathbf{N}(0) \) admit the explicit representations:
\[
\mathbf{K}(0) = n_1^2\mathbf{W}_1 + n_2^2\mathbf{W}_2 + n_3^2\mathbf{W}_3, \quad \mathbf{N}(0) = n_1\mathbf{W}_1 + n_2\mathbf{W}_2 + n_3\mathbf{W}_3;
\]
the derivative of permittivity with respect to \( T \) is
\[
\frac{\partial \mathbf{K}}{\partial T} \bigg|_{T=0} = \mathbf{I} \otimes \frac{\partial \gamma_1}{\partial T} \bigg|_{T=0} + \sum_{\alpha=1}^{2} \mathbf{W}_\alpha \otimes \frac{\partial \gamma_3}{\partial T} \bigg|_{T=0} \left( \frac{D_\alpha(D_\alpha - \gamma_2)}{\gamma_3} \right) \bigg|_{T=0} \mathbf{M} + \sum_{\alpha=1}^{2} \frac{D_\alpha}{\gamma_3} (\mathbf{I} \otimes \mathbf{W}_\alpha + \mathbf{W}_\alpha \otimes \mathbf{I}) \mathbf{M}.
\]
Since, by a repeated application of the same procedure we used for uniaxial materials, we have:
\[
\frac{\partial}{\partial T} \left( \frac{\gamma_1}{\gamma_3} \right) \bigg|_{T=0} = 0
\]
and
\[
\frac{\partial}{\partial T} \left( \frac{D_\alpha(D_\alpha - \gamma_2)}{\gamma_3} \right) \bigg|_{T=0} = n_o^2(n_o^2 - n_e^2)\mathbf{M}^T[k\mathbf{W}_\alpha], \quad \alpha = 1, 2,
\]
then we obtain, to within higher-order terms, the linearized relation for the permittivity:

\[
K(T) = n_1^2 W_1 + n_2^2 W_2 + n_3^2 W_3 - n_3^3 M[T] \\
+ \sum_{\alpha=1}^{2} n_2^3 (n_\alpha^2 - n_3^2) (M[T] \cdot W_\alpha) W_\alpha \\
+ \sum_{\alpha=1}^{2} n_3^3 (n_\alpha^2 - n_3^2) (M[T] W_\alpha + W_\alpha M[T]).
\]  

(3.30)

We turn our attention to (3.3)2 and then, by (2.25), (3.27)2 and (3.2)2 we get

\[
N(T) = n_1 W_1 + n_2 W_2 + n_3 W_3 + I \otimes \frac{\partial c_\alpha}{\partial T} \bigg|_{T=0} + \sum_{\alpha=1}^{2} W_\alpha \otimes \frac{\partial c_\alpha}{\partial T} \bigg|_{T=0} \\
+ c_3(n_\alpha, 0) M[T] + c_4(n_\alpha, 0) (M[T] W_1 + W_1 M[T]) \\
+ c_5(n_\alpha, 0) (M[T] W_2 + W_2 M[T]) + c_6(n_\alpha, 0) W_1 M[T] W_1 \\
+ c_7(n_\alpha, 0) W_2 M[T] W_2 + c_8(n_\alpha, 0) (W_1 M[T] W_2 + W_2 M[T] W_1);
\]  

(3.31)

since \((\text{vid. DS})\):

\[
\frac{\partial c_\alpha}{\partial T} \bigg|_{T=0} = 0, \quad \frac{\partial c_\alpha}{\partial T} \bigg|_{T=0} = 0, \quad \alpha = 1, 2,
\]  

(3.32)

and

\[
c_k(n_\alpha, 0) = n_3^3 G_k(\xi_1, \xi_2), \quad k = 3, 4, \ldots, 8,
\]  

(3.33)

where the six functions \(G_k, k = 3, \ldots, 8\) of the two parameters \(\xi_\alpha = n_\alpha / n_3, \alpha = 1, 2\), given explicitly in \([\text{DS}]\), are such that \(G_3(1, 1) = 1/2\) and \(G_j(1, 1) = 0, j \neq 3\), then from (3.31) we get

\[
N(T) = n_1 W_1 + n_2 W_2 + n_3 W_3 - n_3^3 \left( G_3(\xi_1, \xi_2) M[T] \\
+ G_4(\xi_1, \xi_2) (W_1 M[T] + M[T] W_1) + G_5(\xi_1, \xi_2) (W_2 M[T] + M[T] W_2) \\
+ G_6(\xi_1, \xi_2) W_1 M[T] W_1 + G_7(\xi_1, \xi_2) W_2 M[T] W_2 \\
+ G_8(\xi_1, \xi_2) (W_1 M[T] W_2 + W_2 M[T] W_1) \right);
\]  

(3.34)

By (3.30) and (3.34) then the tensors \(K\) and \(N\) for a biaxial crystal have the explicit form:

\[
K = -\left[ n_3^2 \mathbb{I} - \sum_{\alpha=1}^{2} n_2^3 (n_\alpha^2 - n_3^2) (W_\alpha \otimes W_\alpha + I \otimes W_\alpha + W_\alpha \otimes I) \right] M,
\]

and

\[
N = -n_3^3 \left[ G_3(\xi_1, \xi_2) \mathbb{I} + G_4(\xi_1, \xi_2) (I \otimes W_1 + W_1 \otimes I) \\
+ G_5(\xi_1, \xi_2) (I \otimes W_2 + W_2 \otimes I) \\
+ G_6(\xi_1, \xi_2) W_1 \otimes W_1 + G_7(\xi_1, \xi_2) W_2 \otimes W_2 \\
+ G_8(\xi_1, \xi_2) (W_1 \otimes W_2 + W_2 \otimes W_1) \right] M.
\]  

(3.35)
4. Examples

(a) Optically isotropic crystals

(i) Isotropic materials

For isotropic materials, the piezo-optic tensor has the representation [12]

\[ \mathbb{M} = M_1 \frac{1}{3} \mathbf{I} \otimes \mathbf{I} + M_2 (\mathbf{I} - \frac{1}{3} \mathbf{I}) \],

(4.1)

where the two piezo-optic moduli \( M_1 \) and \( M_2 \) are expressed in terms of the components \( \mathbb{M}_{ijhk} = \mathbb{M}_{jikh} \) of the piezo-optic tensor by

\[ M_1 = \mathbb{M}_{1111} + 2\mathbb{M}_{1122}, \quad M_2 = \mathbb{M}_{1111} - \mathbb{M}_{1122}. \]

(4.2)

Accordingly, by (4.1) and (1.8) we have

\[ \mathbb{M}[\mathsf{T}] = M_1 \sigma_m \mathbf{I} + M_2 \, \text{dev} \, \mathsf{T}. \]

(4.3)

Since \( \mathsf{B}_o \) is given by (2.2), then the permittivity tensor is then given by relation (2.4) together with (4.1):

\[ \mathbb{K}(\mathsf{T}) = \frac{1}{\alpha_3} \left( (\alpha_1 - \alpha_2 \sigma_m + M_1^2 \sigma_m^2) \mathbf{I} + M_2 (2 \sigma_m - \alpha_2) \, \text{dev} \, \mathsf{T} + M_2^2 (\text{dev} \, \mathsf{T})^2 \right), \]

(4.4)

where, explicitly

\[ \begin{align*}
\alpha_1 &= n_o^{-2} \alpha_2 + 3 M_2 \sigma_m^2 + M_2 \, \text{dev} \, \mathsf{T}^2, \\
\alpha_2 &= n_o^{-2} + 3 M_1 \sigma_m \\
\alpha_3 &= n_o^{-2} \alpha_1 + (M_1 \sigma_m)^3 + M_1^2 M_2^2 \sigma_m i_{2T} + M_2^2 i_{3T},
\end{align*} \]

(4.5)

and with \( i_{2T} \) and \( i_{3T} \) the invariants of \( \text{dev} \, \mathsf{T} \). We write (4.4) in terms of (1.11) together with (4.5) and since (4.4) is still in spectral form, then we get directly the principal refractive index as the square root of the principal values of (4.4):

\[ n_k = \sqrt{A_k / B}, \quad k = 1, 2, 3, \]

(4.6)

with

\[ \begin{align*}
A_k &= n_o^{-4} + 2 n_o^{-2} M_1 \sigma_m + M_1^2 \sigma_m^2 + M_2^2 (\hat{\sigma}_1^2 + \hat{\sigma}_2^2 + \hat{\sigma}_3^2) \\
&\quad - \hat{\sigma}_k (M_1 M_2 \sigma_m - n_o^{-2} M_2) + M_2^2 \hat{\sigma}_k^2, \\
B &= n_o^{-2} (n_o^{-2} + 3 M_1 \sigma_m) + M_1^2 \sigma_m^2 + M_2^2 (\hat{\sigma}_1^2 + \hat{\sigma}_2^2 + \hat{\sigma}_3^2) + M_1^2 \sigma_m^3 + M_1 M_2 \sigma_m (\hat{\sigma}_1 \hat{\sigma}_2 + \hat{\sigma}_1 \hat{\sigma}_3 + \hat{\sigma}_2 \hat{\sigma}_3) + M_2^2 \hat{\sigma}_1 \hat{\sigma}_2 \hat{\sigma}_3,
\end{align*} \]

(4.7)

When we consider the linearized relation (3.11) then we arrive directly at the very well-known formula:

\[ n_k = n_o - \frac{n_o^3}{2} (M_1 \sigma_m + M_2 \hat{\sigma}_k), \quad k = 1, 2, 3, \]

(4.8)

which shows how, in order to have birifringence, an isotropic material must be loaded by a deviatoric stress. From (1.3) and (4.8) then it follows trivially the well-known Brewster’s Law [13,14]:

\[ \Delta n = - \frac{n_o^3}{2} M_2 (\hat{\sigma}_i - \hat{\sigma}_k) = - \frac{n_o^3}{2} M_2 (\sigma_i - \sigma_k), \quad i, k = 1, 2, 3; \ i \neq k. \]

(4.9)

In order to compare the linearized relation (4.8) with the nonlinear relation (4.6) we consider separately a spherical stress \( \sigma_m \) and a generic deviatorical stress \( \hat{\sigma}_k \) for a LG-812 Nd-doped glass
Figure 1. Linear and nonlinear relation: spherical and deviatorical stress. (Online version in colour.)

[15] with (at $\lambda = 632$ nm):

$$n_0 = 1.49, \quad M_1 = 2,01 \cdot 10^{-6} \text{ mm}^2/\text{N}, \quad M_2 = -0.63 \cdot 10^{-6} \text{ mm}^2/\text{N};$$

(4.10)

from figure 1 we see that the two relations diverge for a stress which is about $10^2$ times the brittle fracture stress $\sigma_f = 100 \text{ N mm}^{-2}$: for a stress which is below this brittle fracture stress (for $\log(\sigma/\sigma_f) < 100$) the nonlinear and the linear relations give the same results.

(ii) Cubic crystals

For cubic crystals of the classes $432, \bar{4}3m$ and $m3m$ (the higher-symmetry classes), the piezo-optic tensor can be represented in terms of three moduli:

$$\mathbb{M} = M_1 \frac{1}{3} \mathbb{I} \otimes \mathbb{I} + M_2 \sum_{k=1}^{3} (\mathbb{W}_k \otimes \mathbb{W}_k - \frac{1}{3} \mathbb{I} \otimes \mathbb{I}) + M_3 \sum_{j=4}^{6} \mathbb{W}_j \otimes \mathbb{W}_j,$$

(4.11)

where

$$M_1 = M_{1111} + 2M_{1122}, \quad M_2 = M_{1111} - M_{1122}, \quad M_3 = M_{1212};$$

(4.12)

from (4.11) it follows that

$$\mathbb{M}[\mathbb{T}] = M_1 \sigma_m \mathbb{I} + M_2 (\hat{T}_{11}\mathbb{W}_1 + \hat{T}_{22}\mathbb{W}_1 + \hat{T}_{33}\mathbb{W}_3) + M_3 (T_{23}\mathbb{W}_4 + T_{13}\mathbb{W}_5 + T_{12}\mathbb{W}_6).$$

(4.13)

For the lower-symmetry cubic classes $23, m3$ in place of (4.13) we have instead

$$\mathbb{M}[\mathbb{T}] = M_1 (\sigma_m + \hat{T}_{11})\mathbb{W}_1 + M_2 (\sigma_m + \hat{T}_{22})\mathbb{W}_1 + M_3 (\sigma_m + \hat{T}_{33})\mathbb{W}_3 + M_{1212} (T_{23}\mathbb{W}_4 + T_{13}\mathbb{W}_5 + T_{12}\mathbb{W}_6),$$

(4.14)

where

$$M_1 = M_{1111} + 2M_{1122} + M_{1133}, \quad M_2 = M_{1111} + M_{2211} + M_{1122}, \quad M_3 = M_{1111} + M_{3311} + M_{2211}.$$
are still able to write in an explicit form the principal refractive index, since we know one of the principal directions.

(b) Optically uniaxial crystals

For the optically uniaxial crystals, in force of the considerations we did about isotropic materials, we shall deal only with the linearized relation (3.26)\textsubscript{2}; by using the representation (D31) from the Dataset [DS], which gives the tensor \( \mathbb{M}[T] \) in terms of the six components \( N_k = N_k(\mathbb{M}, \sigma_m, \text{dev } T) \) in (3.26)\textsubscript{2} we arrive at:

\[
\mathbb{N}[T] = -\frac{n_0^3}{2} \left( N_1 \mathbf{W}_1 + N_2 \mathbf{W}_2 + N_6 \mathbf{W}_6 + (1 + F_1(\xi) + F_2(\xi))N_3 \mathbf{W}_3 \right)
+ (1 + F_1(\xi))(N_4 \mathbf{W}_4 + N_5 \mathbf{W}_5),
\]

(4.16)
a relation which holds true for any optically uniaxial crystal. Clearly \( \mathbb{N}[T] \) has not, for a generic \( T \), the same eigenvectors of \( \mathbb{B}_0 \) and we are left with the problem to solve the eigenvalues problem for the tensor

\[
\mathbb{N}(T) = n_o(\mathbf{I} - \mathbf{W}_3) + n_e \mathbf{W}_3 + \mathbb{N}[T];
\]

(4.17)
however from (4.16) we can obtain the restrictions on the stress in order that \( \mathbb{N}[T] \) and \( \mathbb{B}_0 \) have at least a common eigenvector. Trivially, this can be obtained if two between the components \( N_4, N_5 \) and \( N_6 \) vanish.

For the Trigonal classes 3, 3\(^\prime\) the request that two of these components be zero requires \( \hat{T}_{11} = \hat{T}_{22} \) and \( T_{ij} = 0, \ i \neq j \) which implies that also the third constant vanishes: accordingly \( \{\mathbf{e}_k\} \) is a base of eigenvectors for \( \mathbb{N}(T) \) and hence for \( \mathbb{B}(T) \). For the classes 32, 3\(^m\), 3\(^{m'}\), when \( T_{12} = T_{23} = 0 \) we have \( N_5 = N_6 = 0 \) and \( \mathbf{e}_1 \) is an eigenvector for \( \mathbb{B}(T) \).

For the Tetragonal and Hexagonal lower-symmetry classes, the condition \( T_{13} = T_{23} = 0 \) makes both \( N_4 = N_5 = 0 \) and the symmetry direction \( \mathbf{e}_3 \) is an eigenvector for \( \mathbb{B}(T) \), whereas for the high-symmetry classes it is sufficient that two shear stress vanish.

If we consider for instance the Tetragonal 4/\( m \) lead-tungstate PbWO\(_4\) (PWO) with \( n_o = 2.270 \) and \( n_e = 2.186 \) at \( \lambda = 525 \text{ nm} \) [16], then \( \xi = 0.962 \) with

\[
1 + F_1 + F_2 \approx 1 + F_1 = 0.925;
\]

(4.18)
the six components of the tensor \( \mathbb{N}(T) \) then are given by:

\[
\begin{align*}
N_{11} & = n_o - \frac{n_0^3}{2} (\sigma_m M_1 + M_{11} \hat{T}_{11} + M_{12} \hat{T}_{22} + M_{13} \hat{T}_{33} + M_{16} T_{12}) \\
N_{22} & = n_o - \frac{n_0^3}{2} (\sigma_m M_1 + M_{12} \hat{T}_{11} + M_{11} \hat{T}_{22} + M_{13} \hat{T}_{33} - M_{16} T_{12}) \\
N_{33} & = n_e - 0.925 \frac{n_0^3}{2} (\sigma_m M_3 + (M_{33} - M_{31}) \hat{T}_{33}) \\
N_{12} & = M_{66} T_{12} \\
N_{13} & = 0.925 M_{44} T_{13} \\
N_{23} & = 0.925 M_{44} T_{23}.
\end{align*}
\]

(4.19)
The eigenvalues of \( \mathbb{N}(T) \) can be obtained either in exact form whenever at least two shear stress vanish or, provided \( \|\mathbb{B}_0\| \gg \|\mathbb{M}[T]\| \), with approximated methods like the one proposed, e.g. in [2].
(c) Optically biaxial crystals

In this case we also deal with the linearized relation (3.35) only, and by a simple calculation with the representation (D31) from the Dataset [DS] we arrive at:

\[
\mathbf{N}[\mathbf{T}] = n_3^2 \left( (G_3(\xi_1, \xi_2) + G_4(\xi_1, \xi_2) + G_6(\xi_1, \xi_2))N_1 \mathbf{W}_1 + (G_3(\xi_1, \xi_2) + G_5(\xi_1, \xi_2) + G_7(\xi_1, \xi_2))N_2 \mathbf{W}_2 + G_3(\xi_1, \xi_2)N_3 \mathbf{W}_3 + (G_3(\xi_1, \xi_2) + G_5(\xi_1, \xi_2))N_4 \mathbf{W}_4 + (G_3(\xi_1, \xi_2) + G_4(\xi_1, \xi_2))N_5 \mathbf{W}_5 + (G_3(\xi_1, \xi_2) + G_4(\xi_1, \xi_2) + G_5(\xi_1, \xi_2) + G_6(\xi_1, \xi_2))N_6 \mathbf{W}_6 \right).
\]

(4.20)

As in the case of uniaxial crystals, in order that \(\mathbf{B}(\mathbf{T})\) and \(\mathbf{B}_1\) have at least a common eigenvector we require that two components between \(N_4, N_5\) and \(N_6\) must vanish. In monoclinic crystals, from the relations for \(N_K\) provided in [DM], this means that we may have \(T_{13} = T_{23} = 0\) with the monoclinic \(b\)-axis \(e_2\) as the common eigenvector. For orthorhombic crystals instead, it suffices that two of the shear stress must vanish. There is no such possibility for triclinic crystals instead.

We consider as an example the cerium-doped Lu\(_x\)Y\(_{2-x}\)SiO\(_5\) (LYSO) which is monoclinic, class 2/m with \(n_1 = 1.8313, n_2 = 1.8524\) and \(n_3 = 1.8277\) at \(\lambda = 409\) nm [17]: in this case \(\xi_1 = 1.002\) and \(\xi_2 = 1.013\), with:

\[
G_3 = -0.2304, \quad G_4 = -0.0009, \quad G_5 = -0.0056
\]

and

\[
G_6 = -0.0074, \quad G_7 = -0.0010, \quad G_8 = 3 \times 10^{-5},
\]

and the six components of \(\mathbf{N}(\mathbf{T})\):

\[
\begin{align*}
N_{11} &= n_1 - 0.238 \frac{n_3^2}{2} (\sigma_{m} M_1 + M_{11} \hat{T}_{11} + M_{12} \hat{T}_{22} + M_{13} \hat{T}_{33} + M_{16} T_{12}) \\
N_{22} &= n_2 - 0.237 \frac{n_3}{2} (\sigma_{m} M_2 + M_{21} \hat{T}_{11} + M_{22} \hat{T}_{22} + M_{23} \hat{T}_{33} + M_{26} T_{12}) \\
N_{33} &= n_3 - 0.230 \frac{n_3^2}{2} (\sigma_{m} M_3 + M_{31} \hat{T}_{11} + M_{32} \hat{T}_{22} + M_{33} \hat{T}_{33} + M_{36} T_{12}) \\
N_{12} &= -0.236 \frac{n_3^2}{2} (M_{14} T_{23} + M_{15} T_{13}) \\
N_{13} &= -0.231 \frac{n_3^2}{2} (M_{54} T_{23} + M_{55} T_{13}) \\
N_{23} &= -0.236 \frac{n_3^2}{2} (M_{16} \hat{T}_{11} + M_{26} \hat{T}_{22} + M_{36} \hat{T}_{33} + M_{66} T_{12})
\end{align*}
\]

(4.21)

here for \(T_{13} = T_{23} = 0\) the eigenvalues can be obtained in explicit form, whereas in the other cases we need an approximate method to find the eigenvalues like the one proposed in [2].

5. Conclusion

We first obtained the exact expression for the permittivity tensor and its square root for optically isotropic, uniaxial and biaxial crystals, by applying a result obtained in [10]: the principal components of the permittivity square root are the principal refractive index.

Then we get the linearized relations for both the permittivity tensor and its square root, to within higher-order terms in the stress tensor: these relations hold for any crystallographic symmetry and any stress tensor. By the means of an example concerning glass, which is optically
anisotropic, we show that the linearized and the exact relations coincide for stress which are two orders larger than the brittle fracture stress.

We finish by writing the components of the square root of the permittivity tensor for optically uniaxial and biaxial crystals and by showing the restriction on the stress, which allows for an explicit evaluation of the principal refraction index, the other cases being dealt with by one of the approximate methods which can be found in the literature [2].

We think that these relations generalize and simplify those presented elsewhere for special cases of stress and crystallographic symmetries [8,9].

Data accessibility. In order to make the paper more concise and readable, many calculations and explicit expressions are presented and collected in Dataset [DS], which is available on Mendeley. (doi:10.17632/3yz353c8ms.1)

Competing interests. I declare I have no competing interests.

Funding. This work was completely supported by the Università Politecnica delle Marche, Progetto Strategico di Ateneo 2017: ‘Scintillating crystals: an interdisciplinary, applications-oriented approach aimed to the scientific knowledge and process control for application concerning life-quality improvement’.

Acknowledgements. The research leading to these results is within the scope of CERN R&D Experiment 18 ‘Crystal Clear Collaboration’ and the PANDA Collaboration at GSI-Darmstadt.

References

1. Born M, Wolf E. 1999 Principles of optics, VIIth edn. Cambridge, UK: Cambridge University Press.
2. Perelomova MV, Tagieva MM. 1983 Problems in crystal physics with solutions. Moscow, Russia: MIR Publishers, English trans.
3. Sirotin Y, Shaskolskaya MP. 1982 Fundamentals of crystal physics. Moscow, Russia: MIR Publishers.
4. Wang CC. 1979 Mathematical principles of mechanics and electromagnetism. Part B: electromagnetism and gravitation. New York, NY: Springer Science.
5. Kuske A, Robertson GS. 1974 Photoelastic stress analysis. New York, NY: Wiley.
6. Aben H, Guillemet C. 1993 Photoelasticity of glass. Berlin, Germany: Springer.
7. Bain AK. 2019 Crystal optics: properties and applications. New York, NY: Wiley.
8. Rinaldi D, Daví F, Montalto L. 2018 On the photoelastic constants and the Brewster law for stressed tetragonal crystals. Math. Models Appl. Sci. 41, 3103–3116. (doi:10.1002/mma.4804)
9. Rinaldi D, Daví F, Montalto L. 2019 On the photoelastic constants for stressed anisotropic crystals. Nuclear Inst. Methods Phys. Res. Sect. A: Accel. Spectrom. Detect. Assoc. Equip. 947, 162782.
10. Hoger A, Carlson D. 1984 Determination of the stretch and and rotation in the polar decomposition of the deformation gradient. Q. Appl. Math. 42, 113–117. (doi:10.1090/qam/736511)
11. Gurtin ME. 1981 An introduction to continuum mechanics. New York, NY: Academic Press.
12. Authier A. 2003 International tables for crystallography. Volume D: physical properties of crystals. Dordrecht, The Netherlands: Kluwer Academic Publisher.
13. Brewster D. 1830 Papers on optics. Royal Society of London.
14. Daví F. 2020 On the generalization of the Brewster law. Math. Mech. Complex Syst. 8, 29–46. (doi:10.2140/memocs.2020.8.29)
15. Waxier RM, Feldman A. 1980 Piezooptic coefficients of four neodymium-doped laser glasses. Appl. Opt. 19, 2481–2482. (doi:10.1364/AO.19.002481)
16. Borgia B et al. 1997 Ordinary and extraordinary complex refractive index of the lead tungstate PbWO₄ crystal. Nuclear Instrum. Methods Phys. Res. A 385, 209–214. (doi:10.1016/S0168-9002(96)01016-9)
17. Erdei G, Berze N, Péter Á, Játékos B, Lőrincz E. 2012 Refractive index measurement of cerium-doped LuₓY₂₋ₓSiO₅ single crystal. Opt. Mater. 34, 781–785. (doi:10.1016/j.optmat.2011.11.006)