ELLIPTIC Diffeomorphisms
of Symplectic 4-Manifolds.

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Abstract. We show that symplectically embedded \((-1)\)-tori give rise to certain elements in the symplectic mapping class group of 4-manifolds. An example is given where such elements are proved to be of infinite order.

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0. Introduction

Let \((X,\omega)\) be a closed symplectic 4-manifold. Denote by \(\pi_0\mathcal{I}mp(X,\omega)\) the group of symplectic diffeomorphisms of \(X\) modulo symplectic isotopy. Let us consider a forgetful homomorphism

\[ \pi_0\mathcal{I}mp(X,\omega) \to \pi_0\mathcal{D}iff(X). \]

Here \(\pi_0\mathcal{D}iff(X)\) denotes the smooth mapping class group for \(X\). It is known this homomorphism is not necessary injective. If \(\Sigma\) is a smooth Lagrangian sphere in \(X\), then there
exists a symplectomorphism $T_\Sigma^2 : X \to X$, called **symplectic Dehn twist along** $\Sigma$ such that $T_\Sigma^2$ is smoothly isotopic to the identity. In his thesis [SeiT], Seidel proved that in many cases $T_\Sigma^2$ is not symplectically isotopic to the identity. He then proved that for certain $K3$ surfaces containing two Lagrangian spheres $\Sigma_1$ and $\Sigma_2$, the element $T_{\Sigma_1}^2$ has an infinite order, and hence the forgetful homomorphism has an infinite kernel. The reader is invited to look at [Se2] for a detailed description of symplectic Dehn twists.

In spite of this deep result of Seidel, and other results that followed it [Ab-McD, Bu, AG, Ev, LLW, AL, TI], we still have no general way to construct non-isotopic symplectomorphisms for 4-manifolds. For instance, there are 4-manifolds which do not contain Lagrangian spheres, yet it is believed that they have a non-trivial symplectic mapping class group.

In this paper we introduce and study a new type of symplectomorphisms for 4-manifolds. In short, our construction is as follows. Let $(X, \omega)$ be a symplectic 4-manifold which contains a symplectically embedded torus $C \subset X$ of self-intersection number $(-1)$. In particular, $\mu := \int_C \omega > 0$. Then we construct a family of symplectic forms $\omega_t$ on $X$ in the cohomology class $[\omega]$ such that $\int_C \omega_t = \mu - t$. We show that such a family exists for $t$ large enough to have $\int_C \omega_t < 0$, and that the construction is unique up to isotopy.

For each $t \geq \mu$ we construct an $\omega_t$-symplectomorphism $E_{C} : X \to X$, called **elliptic twist along** $C$. As smooth maps, these symplectomorphisms $E_{C}$ are isotopic, so we can think of $E_{C}$ as of one diffeomorphism defined up to isotopy (as the family $\omega_t$ above).

We use the notation $\omega$ for any of the forms $\omega_t$ in the family above, and denote by $E_{C}$ any $\omega_t$-symplectic elliptic twist. Due to importance for our paper, we use the notation \((-1)\text{-torus}\) for any smoothly embedded torus of self-intersection number $(-1)$.

This note is devoted to answering the following question: given an elliptic twist $E_{C}$, when is it symplectically trivial, i.e. is it symplectically isotopic to the identity? It appears that it is so in the case when $\int_C \omega > 0$ the elliptic twist $E_{C}$ is symplectically trivial. In particular, $E_{C}$ is always smoothly isotopic to identity. As we shall see below, it is not so in the case $\int_C \omega \leq 0$, and $E_{C}$ could be non-trivial.

Our first result is an example of a 4-manifold $X$ and a $(-1)$-torus $C$ in it where the elliptic twist $E_{C}$ turns out to be always symplectically trivial.

**Theorem 0.1.** Let $(X, \omega)$ be a symplectic ruled 4-manifold diffeomorphic to the total space of the non-trivial $S^2$-bundles over $T^2$; we denote it by $S^2 \times T^2$ for short. Then

i) there is a symplectic form $\omega_0$ on $X$ which admits an $\omega_0$-symplectic $(-1)$-torus $C \subset X$.

In particular, the elliptic twist $E_{C}$ is well-defined.

ii) the forgetful homomorphism $\pi_0 \mathcal{Symp}(X, \omega) \to \pi_0 \mathcal{Diff}(X)$ is injective, no matter the symplectic form $\omega$ is. In particular, the elliptic twist $E_{C}$ is always symplectically isotopic to the identity.

The injectivity property claimed in part ii) previously was proved by McDuff for every irrational ruled 4-manifold except $S^2 \times T^2$, see [McD-B]. We thus cover the remaining case. Note however that our proof is slightly different. In contrast to the other geometrically ruled surfaces, $S^2 \times T^2$ contains an embedded symplectic $(-1)$-torus and hence an elliptic twist does occur.

The main result of this note shows that it is possible for an elliptic twist to contribute into a symplectic mapping class group.

**Theorem 0.2.** Let $Z$ be $S^2 \times T^2 \# \mathbb{CP}^2$, there exist a symplectic form $\omega$ on $Z$ and three $(-1)$-tori $C_1, C_2$, and $C_3$ in $Z$ such that such that the elliptic twists $E_{C_i}$ are well-defined
and no of them is symplectically isotopic to the identity. Moreover, each symplectomorphism $E_C$ has infinite order in the symplectic mapping class group.

The methods we use to prove both theorems were introduced by Abreu-McDuff [Ab-McD], and then extended further by McDuff, see [McD-B].

The main technique we use in the proof is Gromov’s theory of pseudoholomorphic curves. This theory involve various Banach manifolds and constructions with them. Dealing with them we often pretend being in the finite-dimensional case. We refer the reader to the book [Iv-Sh-1] and articles [Iv-Sh-2, Iv-Sh-3] for a comprehensive analytic setup to Gromov’s theory of pseudoholomorphic curves. Of course, reader is free to address to any of numerous alternative sources and expositions of the theory such as [McD-Sa-3] or the seminal paper [Gro].

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0.1. Symplectic economics. Here we give a brief description of the inflation technique developed by McDuff, see e.g. [McD-B], and a generalization of this procedure given by Buş, see [Bu].

**Theorem 0.3** (Inflation). Let $J$ be an $\omega_0$-tamed almost complex structure on a symplectic 4-manifold $(X, \omega_0)$ that admits an embedded $J$-holomorphic curve $C$ with $[C] \cdot [C] \geq 0$. Then there is a family $\omega_s$, $s \geq 0$, of symplectic forms that all tame $J$ and have cohomology class $[\omega_s] = [\omega_0] + s \text{PD}([C])$, where $\text{PD}([C])$ is Poincaré dual to $[C]$.

For negative curves a somewhat reverse procedure exists, called negative inflation or deflation.

**Theorem 0.4** (Deflation). Let $J$ be an $\omega_0$-tamed almost complex structure on a symplectic 4-manifold $(X, \omega_0)$ that admits an embedded $J$-holomorphic curve $C$ with $[C] \cdot [C] = -m$. Then there is a family $\omega_s$ of symplectic forms that all tame $J$ and have cohomology class $[\omega_s] = [\omega_0] + s \text{PD}([C])$

for all $0 \leq s < \frac{\omega_0([C])}{m}$.

0.2. The Abreu-McDuff framework. One can attack the problem of computing a symplectic mapping class group $\mathcal{Symp}(X, \omega)$ by using the fibration first introduced by Kronheimer [Kh], which is as follows:

$$\mathcal{Symp}(X, \omega) \cap \text{Diff}_0(X) \to \text{Diff}_0(X) \to \Omega(X, \omega),$$
where $\text{Diff}_0(X)$ is the identity component of the diffeomorphism group of $X$ and $\Omega(X, \omega)$ is the space of all symplectic forms on $X$ that are isotopic to $\omega$. Here the last arrow stands for the map

$$\varphi : \text{Diff}_0(X) \to \Omega(X, \omega) \quad \text{with} \quad \varphi : f \mapsto f_* \omega.$$ 

To shorten notation, we write $\text{Symp}^\ast(X, \omega)$ instead of $\text{Symp}(X, \omega) \cap \text{Diff}_0(X)$. Clearly, we have this homotopy fibration sequence

$$\ldots \to \pi_1(\text{Diff}_0(X)) \xrightarrow{\varphi_*} \pi_1(\Omega(X, \omega)) \xrightarrow{\partial} \pi_0(\text{Symp}^\ast(X, \omega)) \to 0.$$ 

Let $\mathcal{J}(X, \omega)$ be the space of almost-complex structures for which there exists a taming symplectic form in $\Omega(X, \omega)$. The space $\mathcal{J}(X, \omega)$ is canonically homotopy equivalent to $\Omega(X, \omega)$, see \cite{McD-B}. Let $\psi : \Omega(X, \omega) \to \mathcal{J}(X, \omega)$ be the homotopy equivalence. There is a commutative homotopy diagram

$$\begin{array}{ccc}
\text{Diff}_0(X) & \xrightarrow{\varphi} & \Omega(X, \omega) \\
\downarrow{\nu} & & \downarrow{\psi} \\
\mathcal{J}(X, \omega), & & \end{array}$$

where the diagonal arrow is for the map

$$\nu : \text{Diff}_0(X) \to \mathcal{J}(X, \omega) \quad \text{with} \quad \nu : f \mapsto f_* J,$$

for an arbitrarily chosen $\omega$-tamed almost-complex structure $J$. Thus, it becomes more convenient to center the arguments on the space $\mathcal{J}(X, \omega)$ rather than on $\Omega(X, \omega)$.

We see form the following diagram

$$\begin{array}{ccc}
\ldots \longrightarrow & \pi_1(\text{Diff}_0(X)) & \xrightarrow{\varphi_*} \pi_1(\Omega(X, \omega)) \xrightarrow{\partial} \pi_0(\text{Symp}^\ast(X, \omega)) \longrightarrow 0 \\
\downarrow{id} & & \downarrow{\psi_*} \\
\ldots \longrightarrow & \pi_1(\text{Diff}_0(X)) & \xrightarrow{\nu_*} \pi_1(\mathcal{J}(X, \omega)), \quad (0.1)
\end{array}$$

that each loop in $\mathcal{J}(X, \omega)$ contributes to the symplectic mapping class group of $X$, provided this loop does not come from $\text{Diff}_0(X)$.

We will use diagram (0.1) to prove both Theorem 0.1 and Theorem 0.2. The reader is referred to \cite{McD-B} for more extensive discussion of the topic.

### 0.3. Elliptic twists

We start with a symplectic 4-manifold $(X, \omega)$ which contains an embedded symplectic torus $C$ of self-intersection number $(-1)$, $[C]^2 = -1$.

We choose an appropriate $\omega$-tamed almost-complex structure $J_*$ for which the torus $C$ is pseudoholomorphic. One can think of $J_*$ as point of the subspace $D_{[C]} \subset \mathcal{J}(X, \omega)$ of those almost-complex structures which admit a pseudoholomorphic curve in class $[C]$. In what follows, we refer to $D_{[C]}$ as the elliptic divisorial locus for the class $[C]$. The term divisorial locus is taken from the fact that the subspace $D_{[C]}$ in some neighbourhood of $J_*$ locally behaves as a submanifold of real codimension 2 of $\mathcal{J}(X, \omega)$ provided the $J_*$-holomorphic curve $C$ is smooth, see e.g. \cite{Iv-Sh-1}.

Further, we shall say that an almost-complex structure $J_*$ satisfies the wall-crossing property if

- there exists a small 2-disc $\Delta \subset \mathcal{J}(X, \omega)$ which intersects $D_{[C]}$ transversally at $J_*$,
- there exists a cohomology class $\xi \in H^2(X; \mathbb{R})$, such that $\int_{[C]} \xi \leq 0$ and such that every $J \in \Delta \setminus \{J_*\}$ is tamed by some symplectic form $\theta_J$ with the cohomology class $[\theta_J] = \xi$. 

One observes that all the forms $\theta_J$, $J \in \Delta \setminus \{J_s\}$, are deformation equivalent to each other through symplectic forms of cohomology class $\xi$. With this understood, one applies Moser’s theorem to show that the forms $\theta_J$, $J \in \Delta \setminus \{J_s\}$, are isotopic to each other.

Let us use the parameter $t$ for the points on the boundary $\partial \Delta$ of the disc $\Delta$ and $J(t)$ for the structure parameterized by $t \in \partial \Delta$. We thus have $\psi(\theta_{J(t)}) = J(t)$. Set $\theta := \theta_{J(0)}$. Obviously, the space $\mathcal{J}(X, \theta)$ does not contain $J_s$, as well as no other points of $\mathcal{D}_{[C]}$, but it does contain $\Delta \setminus \{J_s\}$. Therefore the loop $J(t)$, which is a boundary of the disc $\Delta$, is locally non-contractible in $\mathcal{J}(X, \theta)$ and potentially gives certain element in $\pi_1(J(X, \theta))$ which does not lie in the image of the homomorphism $\nu_s$ from the diagram (0.1).

Therefore, the symplectomorphism $E_C := \partial(\theta_{J(t)})$, which is well-defined up to symplectic isotopy, may give a nontrivial element in $\pi_0(\mathcal{J}^*(X, \omega))$. We call $E_C$ the elliptic twist along $C$.

0.4. When do elliptic twists occur? In order to study elliptic twists, we better have examples of 4-manifolds where the wall-crossing property can be easily verified. Here is a series of such 4-manifolds.

Let $X$ be a symplectic 4-manifold containing an embedded symplectic torus $C$ of self-intersection number 0, i.e. $[C]^2 = 0$. Choose a tamed almost-complex structure for which $C$ is pseudoholomorphic, so $C$ becomes a smooth elliptic curve. We then deform this structure slightly to make it integrable in some tubular neighbourhood of $C$, and assume that a sufficiently small neighbourhood of $C$ has an elliptic fibering with $C$ being a multiple fiber of multiplicity $m > 1$.

Let $g : \Delta \to X$ be a holomorphic embedding of a small complex 2-disc $\Delta$ into $X$ such that $g(\Delta)$ intersects $C$ transversally at a single point $P$. Choose a complex coordinate $t$ on $\Delta$ such that $g(0) = P$. Consider the product $\Delta \times X$ and the embedding $h : \Delta \to \Delta \times X$ for which $h(\Delta)$ is the diagonal of $\Delta \times g(\Delta)$.

We now define a 3-fold $Z$ to be the blow-up of $\Delta \times X$ along $h(\Delta)$. There is a natural mapping $Z \to \Delta \times X \to \Delta$, where the first arrow is the contraction map, while the second one is the projection map. Because of this, our 3-fold $Z$ forms a complex-analytic family $Z_t$, $t \in \Delta$, of small deformations of $Z_0$, where $Z_0$ is $X$ blown-up at the point $P$.

Let $E_t$ be the exceptional ($-1$)-curve in $Z_t$, and let $\sigma_t : Z_t \to X$ be a map that contracts $E_t$ to a point $Q_t \in X$, $g(t) = Q_t$. We denote by $E$ the homology class of the exceptional lines $E_t$, which is the same for each $t \in \Delta$. Since $C$ is a multiple fiber, there are no curves in homology class $[C]$ passing through $Q_t$, provided $t$ is nonzero. However, since $C$ is of the multiplicity $m$, for each $t \neq 0$ there exists a unique elliptic curve in homology class $m[C]$ such that it does pass through $Q_t$.

We thus claim that:

i) $Z_0$ contains a unique smooth elliptic curve in homology class $[C] - E$, which is the strict transform of the curve $C$ in $X$, and

ii) $Z_t$, $t \neq 0$, contains no elliptic curves in homology class $[C] - E$. However, what $Z_t$ contains is a smooth elliptic curve $C_m$ in homology class $m[C] - E$, which is the strict transform of a certain elliptic curve in $X$.

Now let $\mathcal{D}_{[C] - E}$ be the elliptic divisorial locus for $[C] - E$. The manifold $Z_0$ corresponds to some point $J_0 \in \mathcal{D}_{[C] - E}$, as well as the family $Z_t$ corresponds to some 2-disc $J_t$, $t \in \Delta$, which intersects $\mathcal{D}_{[C] - E}$ transversally at the single point $J_0$. 
To see $J_0$ satisfies the wall-crossing property it is needed to construct a family $\theta_t$, $t \in \Delta - J_0$, of cohomologous symplectic forms on $Z$ such that $J_t$ is $\theta_t$-tamed and $\int_{[C]-E} \theta_t \leq 0$.

We construct them as follows: let $\omega$ be some symplectic form on $Z_0$ which is taming the almost-complex structure $J_0$. Clearly, the almost-complex structures $J_t$ are $\omega$-tamed for $|t|$ small enough. We have $\int_{[C]} \omega > 0$, but $\int_{[C]-E} \omega > 0$, and so the form $\omega$ should be changed.

Every $Z_t$, $t \neq 0$, contains the smooth elliptic curve $C_m$, which is in class $m[C] - E$. Thus the negative inflation technique can be used to deform $\omega$ to be a form $\theta_t$ such that

$$\int_{m[C]-E} \theta_t = \varepsilon \quad \text{for } \varepsilon > 0 \text{ arbitrary small.}$$

Recall that the negative inflation deformation does not violate the taming condition for $Z_t$, and it is being performed in a small neighbourhood of the curve $C_m$, so it does not affect the symplectic area of the curve $C$, see [Bu]. We thus have

$$\int_{[C]-E} \theta_t = \varepsilon - (m-1) \int_{[C]} \omega$$

and hence if we take $m$ sufficiently large, we can make the area of the class $[C] - E$ as negative as desired.

1. **Elliptic geometrically ruled surfaces**

1.1. **General remarks.** A complex surface $X$ is called ruled if there exists a holomorphic map $\pi : X \to Y$ to a Riemann surface $Y$ such that each fiber $\pi^{-1}(y)$ is a rational curve; if, in addition, each fiber is irreducible, then $X$ is called geometrically ruled. A ruled surface is obtained by blowing up a geometrically ruled surface. Note however that a geometrically ruled surface need not be minimal (the blow up of $\mathbb{CP}^2$, denoted by $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, is de facto a unique example of a geometrically ruled surface that is not a minimal one). Unless otherwise noted, all ruled surfaces are assumed to be geometrically ruled. One can speak of the genus of the ruled surface $X$, meaning thereby the genus of $Y$. We thus have rational ruled surfaces, elliptic ruled surfaces and so on.

Up to diffeomorphism, there are two total spaces of orientable $S^2$-bundles over a Riemann surface: the product $S^2 \times Y$ and the non-trivial bundle $S^2 \tilde{\times} Y$. The product bundle admits sections $Y_{2k}$ of even self-intersection number $[Y_{2k}]^2 = 2k$, and the non-trivial bundle admits sections $Y_{2k+1}$ of odd self-intersection number $[Y_{2k+1}]^2 = 2k+1$. We will choose the basis $Y = [Y_0]$, $S = [pt \times S^2]$ for $H_2(S^2 \times Y; \mathbb{Z})$, and use the basis $Y_+ = [Y_1]$, $Y_- = [Y_{-1}]$ for $H_2(S^2 \tilde{\times} Y; \mathbb{Z})$. To simplify notations, we denote both the classes $S$ and $Y_+ - Y_-$, which are the fiber classes of the ruling, by $F$. Further, the class $Y_+ + Y_-$, which is a class for a bisection of $X$, will be of particular interest for us, and will be widely used in forthcoming computations; we denote this class by $B$. Throughout this paper we will freely identify homology and cohomology by Poincaré duality.

Clearly, we have $[Y_{2k}] = Y + kF$ and $[Y_{2k+1}] = Y_+ + k(Y_+ + Y_-)$. This can be seen by evaluating the intersection forms for these 4-manifolds on the given basis:

$$Q_{S^2 \times Y} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Q_{S^2 \tilde{\times} Y} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Observe that these forms are non-isomorphic. That is why the manifolds $S^2 \times Y$ and $S^2 \tilde{\times} Y$ are non-diffeomorphic. One more way to express the difference between them is to note that the product $S^2 \times Y$ is a spin 4-manifold, but $S^2 \tilde{\times} Y$ is not of that kind.
This divergence, however, is sort of fragile: after one blow-up they become diffeomorphic, $S^2 \times Y \# \mathbb{CP}^2 \simeq S^2 \times Y \# \mathbb{CP}^2$.

This section is mainly about the non-spin elliptic ruled surface $S^2 \times T^2$. When study this manifold we sometimes use the notations $T_+$ and $T_-$ instead of $Y_+$ and $Y_-$ for the standard homology basis in $H^2(S^2 \times T^2; \mathbb{Z})$.

From the algebro-geometric viewpoint every such $X$ is a holomorphic $\mathbb{CP}^1$-bundle over a Riemann surface $Y$ whose structure group is $\text{PGL}(2, \mathbb{C})$. Biholomorphic classification of ruled surfaces is well understood, at least for low values of the genus. Below we recall part of the classification of elliptic ruled surfaces given by Atiyah in [At-2]; this being the first step towards understanding almost-complex geometry of these surfaces. We also provided a short summary of Suwa’s results: i) an explicit construction of a complex analytic family of ruled surfaces, where one can see the jump phenomenon of complex structures, see subsection 1.3, ii) an examination of those complex surfaces which are both ruled and admit an elliptic pencil, see Theorem 1.3.

It maybe should be mentioned that no matter what complex structure we are dealing with, the formula describing the first Chern class of a geometrically ruled surface is as follows:

$$c_1(S^2 \times Y) = 2Y + \chi(Y) S, \quad c_1(S^2 \times Y) = (1 + \chi(Y)) Y_+ + (1 - \chi(Y)) Y_-.$$  \hspace{1cm} (1.1)

The symplectic geometry of ruled surfaces has been extensively studied by many authors [Li-Li, Li-Liu-1, Li-Liu-2, AGK, Sh-1, H-Hy]. Ruled surfaces are of great interest from the symplectic point of view mainly because of the following significant result due to Lalonde and McDuff, see [La-McD].

**Theorem 1.1** (The classification of ruled 4-manifolds). Let $X$ be oriented diffeomorphic to a minimal rational or ruled surface, and let $\xi \in H^2(X)$. Then there is a symplectic form (even a Kähler one) on $X$ in the class $\xi$ iff $\xi^2 > 0$. Moreover, any two symplectic forms in class $\xi$ are diffeomorphic.

Thus all symplectic properties of ruled surfaces depend only on the cohomology class of a symplectic form.

Our main interest is to study symplectic $(-1)$-tori in $X$ and the corresponding elliptic twists. It is easy to prove that, except of $S^2 \times T^2$, there are no symplectic $(-1)$-tori in ruled surfaces. The homology class $T_- \in H_2(S^2 \times T^2; \mathbb{Z})$ can be represented by a symplectic $(-1)$-torus, but none of the other classes of $H_2(S^2 \times T^2; \mathbb{Z})$ can.

Let $(X, \omega_\mu)$ be a symplectic ruled 4-manifold $(S^2 \times T^2, \omega_\mu)$, where $\omega_\mu$ is a symplectic structure of the cohomology class $[\omega_\mu] = T_+ - \mu T_-$, $\mu \in (-1, 1)$. By Theorem 1.1 $(X, \omega_\mu)$ is well-defined up to symplectomorphism. As promised in the introduction, we will prove that $\mathcal{Symp}^*(X, \omega_\mu)$ is trivial.

Given $\mu > 0$, the elliptic divisorial locus in the space $\mathcal{J}(X, \omega_\mu)$ is empty. Following McDuff [McD-B], we will show that the group $\mathcal{Symp}^*(X, \omega_\mu)$ coincides with a group of certain diffeomorphisms, see Lemma 1.15 the latter group can be computed to be trivial by standard topological technique, see Proposition 1.11.

When $\mu \leq 0$, the elliptic divisorial locus $\mathcal{D}_{T_-}$ enters. The geometry of this divisorial locus is studied below in subsection 1.7, and particularly it is proved that: i) every point of $\mathcal{D}_{T_-}$ satisfies the wall-crossing property, and hence $(X, \omega_\mu)$ admits certain elliptic twists, see Lemma 1.12, ii) the symplectic mapping class group $\mathcal{Symp}^*(X, \omega_\mu)$ is generated by elliptic twists of the divisorial locus, iii) each of them is symplectically isotopic to the identity, see Lemma 1.17.
1.2. Classification of complex surfaces ruled over elliptic curves. Here we very briefly describe possible complex structures on elliptic ruled surfaces and study some of their properties.

Let $X$ be diffeomorphic to either $S^2 \times Y^2$ or $S^2 \times \mathbb{R}^2$. The Enriques-Kodaira classification of complex surfaces (see e.g. [BHPV]) ensures the following:

1) Every complex surface $X$ of this diffeomorphism type is algebraic and hence Kähler.

2) Every such complex surface $X$ is ruled, i.e. there exists a holomorphic map $\pi : X \to \mathbb{Y}$ such that $\mathbb{Y}$ is a complex curve, and each fiber $\pi^{-1}(y)$ is an irreducible rational curve. Note that, with the single exception of $\mathbb{C}P^1 \times \mathbb{C}P^1$, a ruled surface admits at most one ruling.

It was shown by Atiyah [At-2] that every holomorphic $\mathbb{C}P^1$-bundle over a curve $\mathbb{Y}$ with structure group the projective group $\text{PGL}(2, \mathbb{C})$ admits a holomorphic section, and hence the structure group of such bundle can be reduced to the affine group $\text{Aff}(1, \mathbb{C}) \subset \text{PGL}(2, \mathbb{C})$.

All of what is said works perfectly for any ruled surface, no matter the genus. Keep in mind, however, that everything below is for genus one surfaces. It was Atiyah who gave a classification of ruled surfaces with the base elliptic curve. The description presented here is taken from [Sw].

**Theorem 1.2** (Atiyah). Every $\text{PGL}(2, \mathbb{C})$-bundle over an elliptic curve can expressed uniquely as one of the following:

1) a $\mathbb{C}^*$-bundle of nonpositive degree,

2) $\mathcal{A}^{\text{Spin}}$,

3) $\mathcal{A}$,

where $\mathcal{A}^{\text{Spin}}$ and $\mathcal{A}$ are affine bundles.

We shall proceed with a little discussion of these bundles:

1 With a bit luck the structure group of a $\mathbb{C}P^1$-bundle can be reduced further to $\mathbb{C}^* \subset \text{Aff}(1, \mathbb{C}) \subset \text{PGL}(2, \mathbb{C})$. We now give an explicit description of such bundles.

Let $y \in \mathbb{Y}$ be a point on the curve $\mathbb{Y}$, and let $\{V_0, V_1\}$ be an open cover of $\mathbb{Y}$ such that $V_0 = \mathbb{Y} \setminus \{y\}$ and $V_1$ is a small neighbourhood of $y$, so the domain $V_0 \cap V_1 =: \hat{\mathbb{Y}}$ is a punctured disc. We choose a multivalued coordinate $u$ on $\mathbb{Y}$ centered at $y$.

A surface $X_k$ associated to the line bundle $\mathcal{O}(ky)$ (or if desired, a $\mathbb{C}^*$-bundle) can be described as follows:

$$X := (V_0 \cup \mathbb{C}P^1) \cup (V_1 \cup \mathbb{C}P^1) / \sim,$$

where $(u, z_0) \in V_1 \cup \mathbb{C}P^1$ and $(u, z_1) \in V_2 \cup \mathbb{C}P^1$ are identified iff $u \in \hat{\mathbb{Y}}$, $z_1 = z_0 u^k$. Here $z_0, z_1$ are inhomogeneous coordinates on $\mathbb{C}P^1$s.

Clearly, the biholomorphism $(z_0, u) \to (z_0^{-1}, u), (z_1, u) \to (z_1^{-1}, u)$ maps $X_k$ to $X_{-k}$. Thus it is sufficient to consider only values of $k$ that are nonpositive.

There is a natural $\mathbb{C}^*$-action on $X_k$ via $g \cdot (z_0, u) := (gz_0, u), g \cdot (z_1, u) := (gz_1, u)$ for each $g \in \mathbb{C}^*$. The fixed point set of this action consists of two mutually disjoint sections $Y_k$ and $Y_{-k}$ defined respectively by $z_0 = z_1 = 0$ and $z_0 = z_1 = \infty$. We have $[Y_k]^2 = k$ and $[Y_{-k}]^2 = -k$.

It is very well known that any line bundle $L$ of degree $\text{deg}(L) = k \not= 0$ is isomorphic to $\mathcal{O}(ku)$ for some $u \in \mathbb{Y}$. Thus all the ruled surfaces associated with line bundles of non-zero degree $k$ are biholomorphic to one and the same surface $X_k$. 
On the other hand, the parity of the degree of the underlying line bundle is a topological invariant of a ruled surface. More precisely, a ruled surface \( X \) associated with a line bundle \( L \) is diffeomorphic to \( Y \times S^2 \) for \( \deg(L) \) even, and to \( Y \tilde{\times} S^2 \) for \( \deg(L) \) odd.

ii) Again, we start with an explicit description of the ruled surface \( X_A \) associated with the affine bundle \( A \). Let \( \{V_0, V_1, \hat{V}\} \) be the open cover of \( Y \) as before, \( u \) be a coordinate on \( Y \) centered at \( y \), and \( z_0, z_1 \) be fiber coordinates. Define

\[
X_A := (V_0 \cup \mathbb{CP}^1) \cup (V_1 \cup \mathbb{CP}^1) \sim,
\]

where \((z_0, u) \sim (z_0, u)\) for \( u \in \hat{V} \) and \( z_0 = z_1 u + u^{-1} \).

There is an obvious section \( Y_1 \) defined by the equation \( z_0 = z_1 = \infty \), but in contrast to \( \mathbb{C}^* \)-bundles, the surface \( X_A \) contains no section disjoint from that one. This can be shown by means of direct computation in local coordinates, but one easily deduce this from Theorem 1.3 below.

We will make repeated use of the following geometric characterization of \( X_A \), whose proof is given in [Sw].

**Theorem 1.3.** The surface \( X_A \) associated with the affine bundle \( A \) has fibering of smooth elliptic curves. The elliptic structure is uniquely specified by \( X_A \): the base curve is the rational one, a general fiber is a torus in class \( 2Y_+ + 2Y_- \), there are but 3 double fibers, and there are no other multiple fibers.

The following corollary will be used later. The reader is invited to look at [McD-D] for the definition of the Gromov invariants and some examples of their computation.

**Corollary 1.4.** \( \text{Gr}(Y_+ + Y_-) = 3 \).

**Proof.** There are no smooth curves in class \( Y_+ + Y_- \) apart from those three curves which are the double fibers of the elliptic fibration described above. Since each of these curves is a double fiber of an elliptic fibering, its normal bundle is isomorphic to the square root of the trivial bundle and hence the \( J \)-regularity property holds for it, see [McD-D].

Based on this theorem, Suwa then gives another construction of \( X_A \). We mention this construction here because it appears to have interest for the sequel.

Let us identify \( Y \) with the quotient \( \mathbb{C}/\Lambda \), where \( \Lambda \) is a lattice in \( \mathbb{C} \), a discrete additive subgroup \( \mathbb{Z} \oplus \mathbb{Z} \subset \mathbb{C} \). Consider the representation

\[
\mathbb{Z} \oplus \mathbb{Z} \to \text{PGL}(2, \mathbb{C}) : (n, m) \rightarrow f^n g^m,
\]

where \( f, g \in \text{PGL}(2, \mathbb{C}) \) form a pair of nonidentity distinct commuting involutions, say

\[
f : z \rightarrow -z, \quad g : z \rightarrow \frac{1}{z}.
\]

The product \( \mathbb{C} \times \mathbb{CP}^1 \) can be equipped with a free \( \mathbb{Z}^2 \) action in such a way that \( \mathbb{Z}^2 \) acts as a translation on the the first factor, and by automorphisms \( f, g \) on the second one.

It is clear that the quotient \( X_A := \mathbb{C} \times_{\mathbb{Z}^2} \mathbb{CP}^1 \) is an elliptic ruled surface equipped with the ruling \( X_A \to \mathbb{C}/\Lambda \). This surface is non-spin, for an explanation, see [Sw], where this is proved by constructing a section for \( X_A \) of odd self-intersection number, see also [McD-Sa-1] for a different explanation.

Further, because \( \mathbb{Z}^2 \) acts by elements of order 2, this gives rise to an effective action of \( T = \mathbb{C}/2\Lambda \), which is a complex torus, on \( X_A \). The desired fibering is constructed by
means of this action. It consists of regular fibers, where the action is free, and of three multiple fibers, whose isotropy groups correspond to the three pairwise different order two subgroups of $\mathcal{T}$.

\[ \mathcal{A}^{Spin} \] The ruled surface associated to $\mathcal{A}^{Spin}$ is diffeomorphic to $S^2 \times T^2$, thus it will not discussed here in this note, but see [Sw].

Summarizing our above observations we see that $X \cong S^2 \times T^2$ admits countably many complex structures. These structures are as follows:

- the structures $J \in \mathcal{J}_{1-2k}$, $k > 0$, such that the ruled surface $(X,J)$, which is biholomorphic to $X_{1-2k}$, contains a section of self-intersection number $1 - 2k$, and
- the affine structures $J \in \mathcal{J}_A$ such that the ruled surface $J \in (X,J)$, which is biholomorphic to $X_A$, contains no sections of negative self-intersection number but does contain a triple of smooth bisections.

1.3. One family of ruled surfaces over elliptic base. Here is a construction of a one-parametric complex-analytic family $p : X \rightarrow \mathbb{C}$ of non-spin elliptic ruled surfaces, such that the surfaces $p^{-1}(t), t \neq 0$, are biholomorphic to $X_A$ and $p^{-1}(0) \cong X_{-1}$.

As before, we take a point $y$ on $Y$, let $u$ be a coordinate of the center $y$, and put $\{V_0, V_1, \hat{V}\}$ to be an open cover for $Y$ such that $V_0 := Y \setminus \{y\}$, $V_1$ is a small neighbourhood of $y$, and $\hat{V} := V_0 \cap V_1$. Further, let $\Delta$ be a complex plane, and let $t$ be a coordinate on it.

We construct the complex 3-manifold $X$ by patching $\Delta \times V_0 \times \mathbb{CP}^1$ and $\Delta \times V_1 \times \mathbb{CP}^1$ in such a way that $(t, z_0, u) \sim (t, z_0, u)$ for $u \in \hat{V}$ and $z_0 = z_1 u + tu^{-1}$.

The preimage of 0 and 1 under the natural projection $p : X \rightarrow \Delta$ are biholomorphic respectively to $X_{-1}$ and $X_A$. In fact, it is not hard to see that for each $t \neq 0$, the surface $p^{-1}(t)$ is biholomorphic to $X_A$ as well. One way to prove this is to use the $\mathbb{C}^*$-action on $X$

$$g \cdot (t, z_0, u) := (tg^{-1}, gz_0, u), \quad g \cdot (t, u, z_1, u) := (tg^{-1}, g z_1, u)$$

for each $g \in \mathbb{C}^*$.

This proves even more than we desired, namely, that there exists a $\mathbb{C}^*$-action on $X$ such that for each $g \in \mathbb{C}^*$ we get a commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{g} & X \\
p & & p \\
\mathbb{C} & \xrightarrow{g} & \mathbb{C},
\end{array}$$

where $X \xrightarrow{g} X$ denotes the biholomorphism induced by $g \in \mathbb{C}^*$.

The construction of the complex-analytic family $X$ is due to Suwa, see [Sw], though the existence of the $\mathbb{C}^*$-action was not mentioned in Suwa’s paper.

1.4. Embedded curves and almost-complex structures. In subsection 1.2 the classification for non-spin elliptic ruled surfaces was given. It turns out that this classification can be extended to the almost-complex geometry of $S^2 \times T^2$.

Let $X$ be diffeomorphic to $S^2 \times T^2$, and let $\mathcal{J}(X)$ be the space of almost-complex structures on $X$ that are tamed by some symplectic form; the symplectic forms need not be the same. Here we use the short notation $\mathcal{J}$ for $\mathcal{J}(X)$.

Given $k > 0$, let $\mathcal{J}_{1-2k}(X)$ (for shorten, we will refer to it by $\mathcal{J}_{1-2k}$) be the subset of $J \in \mathcal{J}$ consisting of elements that admit a smooth irreducible $J$-holomorphic elliptic
curve in the class $T_+ - kF$. It is well known that $J_{1-2k}$ forms a subvariety of $\mathcal{J}$ of real codimension $2(2k-1)$.

Further, define $\mathcal{J}(X)$ (or $\mathcal{J}_A$, for short) be the subset $J \in \mathcal{J}$ of those element for which there exists a smooth irreducible $J$-holomorphic elliptic curve in the class $B$.

By pretty straightforward computation one can show that the sets $\mathcal{J}_{1-2k}$ are mutually disjoint, and each $\mathcal{J}_{1-2k}$ is disjoint from $\mathcal{J}_A$. Further, it is not hard to see that $\mathcal{J}_1 \subset \mathcal{J}_A$ and $\mathcal{J}_{1-2(k+1)} \subset \mathcal{J}_{1-2k}$, where $\mathcal{J}_{1-2k}$ is for the closure of $\mathcal{J}_{1-2k}$. A less trivial fact is that

$$\mathcal{J} = \mathcal{J}_A + \bigcup_{k=1}^{\infty} \mathcal{J}_{1-2k},$$

it can be also stated as follows.

**Proposition 1.5** (cf. Lemma 4.2 in [McD-B]). Let $(X,\omega)$ be a symplectic ruled 4-manifold diffeomorphic to $S^2 \times T^2$. Then every $\omega$-tamed almost-complex structure $J$ admits a smooth irreducible $J$-holomorphic representative in either $B$ or $T_+ - kF$ for some $k > 0$.

**Proof.** The proof is analogous to the one of Lemma 4.2 in [McD-B]. Observe that the expected codimension for the class $B$ is zero. By Lemma 4.4 we have $\text{Gr}(T_+ + T_-) > 0$. Hence, $\mathcal{J}_A$ is an open dense subset of $\mathcal{J}$, and, thanks to the Gromov compactness theorem, for each $J \in \mathcal{J}$ the class $B$ has at least one $J$-holomorphic representative, possibly singular, reducible or having multiple components.

By virtue of Theorem 1.7 no matter what $J$ was chosen, our manifold $X$ admits the smooth $J$-holomorphic ruling $\pi$ by rational curves in class $F$.

Since $B \cdot F > 0$, it follows from positivity of intersections that any $J$-holomorphic representative $B$ of the class $B$ must either intersect a $J$-holomorphic fiber of $\pi$ or must contain this fiber completely.

a) First assume that $B$ is irreducible. Then it is of genus not greater than 1 because of the adjunction formula. This curve is of genus 1 because every spherical homology class of $X$ is proportional to $F$. We now can apply the adjunction formula one more time to conclude that $B$ is smooth, i.e. $J \in \mathcal{J}_A$.

b) The curve $B$ is reducible but contains no irreducible components which are the fibers of $\pi$. Then it contains precisely two components $B_1$ and $B_2$, since $B \cdot F = 2$. Both the curves $B_1$ and $B_2$ are smooth sections of $\pi$, and hence $[B_i] = T_+ + k_iF$, $i = 1,2$. Since $[B_1] + [B_2] = B$, it follows that $k_1 + k_2 = -1$, and hence either $k_1$ or $k_2$ is negative. Thus we have that either $B_1$ or $B_2$ is a smooth $J$-holomorphic section of negative self-intersection number.

c) If some of the irreducible components of $B$ are in the fibers class $F$, then one can apply arguments similar to that used in a) and b) to prove that the part $B'$ of $B$ which contains no fiber components has a section of negative self-intersection number as a component.

1.5. **Rulings and almost-complex structures.** Let $X$ be a ruled surface equipped with a ruling $\pi : X \rightarrow Y$, and let $J$ be an almost-complex structure on $X$. We shall say that $J$ is compatible with the ruling $\pi : X \rightarrow Y$ if each fiber $\pi^{-1}(y)$ is $J$-holomorphic.

We wish to express our thanks to D. Alekseeva for sharing her proof of the following statement.
Proposition 1.6. Let \( \mathcal{J}(X, \pi) \) be the space of almost-complex structures on \( X \) compatible with \( \pi \).

i) \( \mathcal{J}(X, \pi) \) is contractible.

ii) Any structure \( J \in \mathcal{J}(X, \pi) \), as well as any compact family \( J_1 \in \mathcal{J}(X, \pi) \), is tamed by some symplectic form.

Proof. i) Let be \( \mathbb{J}(\mathbb{R}^4, \mathbb{R}^2) \) be the space of linear maps \( J: \mathbb{R}^4 \to \mathbb{R}^2 \) such that \( J^2 = -id \) and \( J(\mathbb{R}^2) = \mathbb{R}^2 \), i.e. it is the space of linear complex structures preserving \( \mathbb{R}^2 \). In addition, we assume \( \mathbb{R}^4 \) and \( \mathbb{R}^2 \) are both orientied and each \( J \in \mathbb{J}(\mathbb{R}^4, \mathbb{R}^2) \) induces the given orientations for both \( \mathbb{R}^4 \) and \( \mathbb{R}^2 \). We now prove the space \( \mathbb{J}(\mathbb{R}^4, \mathbb{R}^2) \) is contractible.

Indeed, let us take \( J \in \mathbb{J}(\mathbb{R}^4, \mathbb{R}^2) \). Fix two vectors \( e_1 \in \mathbb{R}^2 \) and \( e_2 \in \mathbb{R}^4 \setminus \mathbb{R}^2 \). The vectors \( e_1 \) and \( Je_1 \) form a positively oriented basis for \( \mathbb{R}^2 \). Therefore \( Je_1 \) is in the upper half-plane for \( e_1 \). Further, the vectors \( e_1, Je_1, e_2, Je_2 \) form a positively oriented basis for \( \mathbb{R}^4 \). Therefore \( Je_2 \) is in the upper half-space for the hyperplane spanned on \( e_1, Je_1, e_2 \).

We see that the space \( \mathbb{J}(\mathbb{R}^4, \mathbb{R}^2) \) is homeomorphic to the direct product of two half-spaces, and hence it is for sure contractible.

To finish the proof of i) we consider the subbundle \( V_x := \text{Ker} \pi(x) \subset T_xX, x \in X \), of the tangent bundle \( TX \) of \( X \). Every \( J \in \mathcal{J}(X, \pi) \) is a section of the bundle \( \mathbb{J}(TX, V) \to X \) whose fiber over \( x \in X \) is the space \( \mathbb{J}(T_xX, V_x) \). Since the fibers of \( \mathbb{J}(TX, V) \) are contractible; it follows that the space of section for \( \mathcal{J}(TX, V) \) is contractible as well.

ii) Again, we start with some linear algebra. Let \( V \) be a 2-subspace of \( W \cong \mathbb{R}^4 \), and let \( J \in \mathbb{J}(W, V) \). Choose a 2-form \( \tau \in \Lambda^2(W) \) such that the restriction \( \tau|_V \in \Lambda^2(V) \) of \( \tau \) to \( V \) is positive with respect to the \( J \)-orientation of \( V \), i.e. \( \tau(\xi,J\xi) > 0 \). Clearly, the subspace \( H := \text{Ker} \tau \subset W \) is a complement to \( V \). Further, let \( \sigma \in \Lambda^2(V) \) be any 2-form such that \( \sigma|_V \) vanishes, but \( \sigma|_H \) does not. If \( H \) is given the orientation induced by \( \sigma \), then the \( J \)-orientation of \( W \) agrees with that defined by the direct sum decomposition \( W \cong V \oplus H \). We now prove that \( J \) is tamed by \( \tau + K \sigma \) for \( K > 0 \) sufficiently large.

It is easy to show that there exists a basis \( e_1, e_2 \in V, e_3, e_4 \in H \) for \( W \) such that \( J \) takes the form

\[
J = \begin{pmatrix}
0 & -1 & 1 & 0 \\
1 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}.
\]

The matrix \( \Omega \) of \( \tau + K \sigma \) with respect to this basis is block-diagonal, say

\[
\Omega = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & K \sigma + \ldots \\
0 & 0 & -K \sigma + \ldots & 0 \\
\end{pmatrix}
\] for \( \sigma > 0 \).

It remains to check that the matrix \( \Omega J \) is positive definite, i.e. \( (\xi, \Omega J_\xi) > 0 \). A matrix is positive definite iff its symmetrization is positive definite. It is straightforward to check that \( \Omega J + (\Omega J)^t \) is of that kind for \( K \) large enough.

Let us go back to the ruled surface \( X \). The theorem of Thurston \([\text{Th}]\) ensures the existence of a closed 2-form \( \tau \) on \( X \) such that the restrictions of \( \tau \) to each fiber \( \pi^{-1}(y) \) is non-degenerate. Choose an area form \( \sigma \) on \( Y \). By the same reasoning as before, any \( J \in \mathcal{J}(X, \pi) \) is tamed by \( \tau + K \pi^* \sigma \) for \( K \) large enough. \( \Box \)
The following theorem by McDuff motivates the study of compatible almost-complex structures, see Lemma 4.1 in [McD-B].

**Theorem 1.7.** Let $X$ be an irrational ruled surface, and let $J \in \mathcal{J}(X)$. Then there exists a unique ruling $\pi : X \to Y$ such that $J \in \mathcal{J}(X, \pi)$.

1.6. **Diffeomorphisms.** Let $X$ be diffeomorphic to either $Y \times S^2$ or $Y \tilde{\times} S^2$, and let $\pi : X \to Y$ be a smooth ruling. Further, let $\text{Fol}(X)$ be the space of all smooth foliations of $X$ by spheres in the fiber class $F$.

The group $\text{Diff}(X)$ acts transitively on $\text{Fol}(X)$ as well as the group $\text{Diff}_0(X)$ acts transitively on a connected component $\text{Fol}_0(X)$ of $\text{Fol}(X)$. This gives rise to a fibration sequence

$$\mathcal{D} \cap \text{Diff}_0(X) \to \text{Diff}_0(X) \to \text{Fol}_0(X),$$

where $\mathcal{D}$ is the group of fiberwise diffeomorphisms of $X$. By the definition of $\mathcal{D}$ there exists a projection homomorphism $\tau : \mathcal{D} \to \text{Diff}(Y)$ such that for every $F \in \mathcal{D}$ we have a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{F} & X \\
\downarrow & \downarrow & \downarrow \\
Y & \xrightarrow{\tau(F)} & Y,
\end{array}
$$

which induces the corresponding commutative diagram for homology

$$
\begin{array}{ccc}
H_1(X;\mathbb{Z}) & \xrightarrow{F_*} & H_1(X;\mathbb{Z}) \\
\downarrow & & \downarrow \\
H_1(Y;\mathbb{Z}) & \xrightarrow{\tau(F)_*} & H_1(Y;\mathbb{Z}).
\end{array}
$$

Notice that $\tau(F)$ is isotopic to the identity only if $\tau(F)_* = id$. Since $\pi_*$ is an isomorphism, it follows that the subgroup $\mathcal{D} \cap \text{Diff}_0(X)$ of $\mathcal{D}$ is mapped by $\tau$ to $\text{Diff}_0(Y)$, so we end up with the restricted projection homomorphism

$$\tau : \mathcal{D} \cap \text{Diff}_0(X) \to \text{Diff}_0(Y).$$

Since we shall exclusively be considering this restricted homomorphism, we use the same notation $\tau$ for this.

Given an isotopy $f_t \in \text{Diff}_0(Y)$, $f_0 = id$, one can lift it to an isotopy $F_t \in \mathcal{D} \cap \text{Diff}_0(X)$, $F_0 = id$ such that $\tau(F_t) = f_t$. This immediately implies that the inclusion $\ker \tau \in \mathcal{D} \cap \text{Diff}_0(X)$ induces an epimorphism

$$\pi_0(\ker \tau) \to \pi_0(\mathcal{D} \cap \text{Diff}_0(X)). \quad (1.3)$$

Because of this property we would like to look at the group $\ker \tau$ in more detail, but first introduce some useful notion.

Let $X$ be a smooth manifold, and let $f$ be a self-diffeomorphism $X$. Define the **mapping torus** $T(X, f)$ as the quotient of $X \times [0, 1]$ by the identification $(x, 1) \sim (f(x), 0)$. For the diffeomorphism $f$ to be isotopic to identity it is necessary to have the mapping torus be diffeomorphic to $T(X, id) \cong X \times S^1$.

Let us go back to the group $\ker \tau$ that consists of bundle automorphisms of $\pi : X \to Y$. Let $F \in \ker \tau$ be a bundle automorphism of $\pi$, and let $\gamma$ be a simple closed curve on $Y$. By $F_\gamma$ denote the restriction of $F$ to $\pi^{-1}(\gamma) \cong S^1 \times S^2$. The mapping torus
$T(\pi^{-1}(\gamma), F_\gamma)$ is either diffeomorphic to $S^2 \times T$ or $S^2 \tilde{\times} T$. In the later case we shall say that the automorphism $F$ is **twisted** along $\gamma$.

**Lemma 1.8.** Let $X$ be diffeomorphic to either $Y \times S^2$ or $Y \tilde{\times} S^2$, and let $F \in \text{Ker} \, \tau$. Then $F$ is isotopic to the identity through $\text{Ker} \, \tau$ iff $Y$ contains no curve for $F$ to be twisted along.

**Proof.** The closed orientable genus $g$ surface $Y$ has a cell structure with one 0-cell, $2g$ 1-cells, and one 2-cell. Clearly, $F$ can be isotopically deformed to $\text{id}$ over the 0-skeleton of $Y$. The obstruction for extending this isotopy to the 1-skeleton of $Y$ is a well-defined cohomology class $c(F) \in H^1(X; \mathbb{Z}_2)$; the obstruction cochain $c(F)$ is the cochain whose value on a 1-cell $e$ equals 1 if $F$ is twisted along $e$ and 0 otherwise. It is evident that $c(F)$ is a cocycle.

By assumption $c(F) = 0$. Consequently there is an extension of our isotopy to a neighbourhood of the 1-skeleton of $Y$, but such an isotopy always can be extended to the rest of $Y$. $\square$

A short way of represent the issue algebraically is by means of the **obstruction homomorphism**

$$c : \text{Ker} \, \tau \to H^1(X; \mathbb{Z}_2)$$

defined in the lemma; any two elements $F, G \in \text{Ker} \, \tau$ are isotopic to each other through $\text{Ker} \, \tau$ iff $c(F) = c(G)$.

**Lemma 1.9.** Let $X$ be diffeomorphic to $S^2 \times Y^2$, and let $F \in \text{Ker} \, \tau$, then $Y$ contains no curve for $F$ to be twisted along. This means that the obstruction homomorphism is the null homomorphism.

**Proof.** The converse would imply that the mapping torus $T(X, F)$ is not spin, but $T(X, \text{id}) \cong S^2 \times Y \times S^1$ is a spin 5-manifold. $\square$

The following result is due to McDuff [McD-B], but the proof follows by combining **Lemma 1.9** with **Lemma 1.8**.

**Proposition 1.10.** Let $X$ be diffeomorphic to $S^2 \times Y^2$, then the group $\mathcal{D} \cap \text{Diff}_0(X)$ is connected.

In what follows we need a non-spin analogue of this Proposition for the case of elliptic ruled surfaces.

**Proposition 1.11.** Let $X$ be diffeomorphic to $S^2 \tilde{\times} T^2$, then the group $\mathcal{D} \cap \text{Diff}_0(X)$ is connected.

**Proof.** Fix any cocycle $c \in H^1(X; \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, then we claim there exists $F \in \text{Ker} \, \tau$ such that $c(F) = c$ and, moreover, $F$ is isotopic $\text{id}$ through $\mathcal{D} \cap \text{Diff}_0(X)$. It follows from Suwa’s model, see subsection [1.2] that the automorphism group for the complex ruled surface $X_A$ contains the complex torus $T$ as a subgroup. By construction, it is clear that $T$ is a subgroup of $\mathcal{D} \cap \text{Diff}_0(X)$. Besides that, the 2-torsion subgroup $T_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ of $T$ is a subgroup of $\text{Ker} \, \tau$. We trust the reader to check $T_2$ is mapped isomorphically by the obstruction homomorphism to $H^1(X; \mathbb{Z}_2)$. 

The algebra behind this argument is expressed by a commutative diagram

\[
\begin{array}{ccc}
T_2 & \xrightarrow{i} & \text{Ker} \tau \\
\downarrow & & \downarrow \\
\pi_0(T_2) & \xrightarrow{i_*} & \pi_0(\text{Ker} \tau) \\
\end{array}
\xrightarrow{j} \begin{array}{c} \text{Diff}_0(X) \\
\downarrow \\
\pi_0(\text{Diff}_0(X)),
\end{array}
\]

where \( i_* \) is an isomorphism, \( j_* \circ i_* \) is the null homomorphism, and therefore \( j_* \) is the null homomorphism as well. But we already know that \( j_* \) is an isomorphism, and hence \( \pi_0(\text{Diff}_0(X)) \) is trivial. \( \square \)

1.7. Vanishing of elliptic twists. Here is the part where a proof of Theorem 0.1 comes. We split it into a few pieces. Let \( X \) be the symplectic ruled 4-manifold \( (S^2 \times T^2, \omega_\mu) \), \( [\omega_\mu] = T_+ - \mu T_- \). Recall that the space \( J(X) \) was defined, see subsection 1.4, to be the space of almost-complex structures on \( X \) tamed by some symplectic from. Here this definition is somewhat modified; here we take the connected component of \( J(X) \) which contains the space \( J(X, \omega_\mu) \); the same applies to \( J_A \) and \( J_{1-2k} \).

Lemma 1.12. \( J_A \subset J(X, \omega_\mu) \) for every \( \mu \in (-1, 1) \).

**Proof.** For every \( J \in J_A \) we take any symplectic form \( \omega \) such that \( J \) is \( \omega \)-tamed. Then inflate \( \omega \) along the classes \( Y_+ - Y_- \) and \( Y_+ + Y_- \), and then rescale it. \( \square \)

Lemma 1.13. \( J_A = J(X, \omega_\mu) \) for every \( \mu \in (-1, 0] \).

**Proof.** It is clear that \( J(X, \omega_\mu) \) does no contain the structures \( J_{1-2k} \) for \( \mu \in (-1, 0] \), and hence by (1.2) and Lemma 1.12 the proof follows. \( \square \)

This means that there is no topology change for the space \( J(X, \omega_\mu) \) when \( \mu \) is being varied in \((-1, 0]\). In particular,

\[
\pi_1(J(X, \omega_\mu)) = \pi_1(J_A(X)) \quad \text{for } \mu \in (-1, 0] \]

Lemma 1.14. \( J_{-k} \subset J(X, \omega_\mu) \) iff \( \mu \in \left(1 - \frac{1}{k}, 1\right) \).

**Proof.** The “only if” part is obvious, while the “if” can be proved by deflating along \( Y_+ - k(Y_+ - Y_-) \) and inflating along \( Y_+ - Y_- \). \( \square \)

Combining Lemma 1.12 with Lemma 1.14 as well as the fact that the higher codimension submanifolds \( J_{-k}, k \geq 2 \) do not affect on the fundamental group of \( J(X, \omega_\mu) \), we see that there is no topology change for \( \pi_1(J(X, \omega_\mu)) \) when \( \mu \) is being varied in \((0, 1)\), i.e. we have

\[
\pi_1(J(X, \omega_\mu)) = \pi_1(J(X)) \quad \text{for } \mu \in (0, 1). \quad (1.4)
\]

To derive the symplectic mapping class group the expected way is to use diagram 0.1 which requires to know the image of the homomorphism

\[
\nu_* : \pi_1(\text{Diff}_0(X)) \to \pi_1(J(X)).
\]

**Lemma 1.15.** \( \nu_* \) is an epimorphism.

**Proof.** Though the map \( \nu : \text{Diff}_0(X) \to J(X) \) is not fibration, it can be extended to a homotopy fibration

\[
\text{Diff}_0(X) \to J(X) \to \text{Fol}_0(X),
\]
where the last arrows is a homotopy equivalence, see Theorem \ref{6} and Proposition \ref{1}. Thus we end up with the homotopy exact sequence
\[
\ldots \to \pi_1(\text{Diff}_0(X)) \to \pi_1(\mathcal{J}_0(X)) \to \pi_0(\mathcal{D} \cap \text{Diff}_0(X)).
\]
If $X$ is of genus 1, the group $\pi_0(\mathcal{D} \cap \text{Diff}_0(X))$ is trivial by Propositions \ref{11} and \ref{11}. This finishes the proof.

By virtue of (0.1) and (1.4) this lemma immediately implies

**Proposition 1.16.** $\pi_0(\mathcal{J}(X,\omega_\mu)) = 0$ for every $\mu \in (0,1)$.

In order to compute the group $\pi_0(\mathcal{J}(X,\omega_\mu))$ for $\mu \in (-1,0)$ it is necessary to know better the fundamental group of $\mathcal{J}_A$. The space $\mathcal{J}_A$ is the complement to (the closure of) the elliptic divisorial locus $\mathcal{D}_{T_-}$ in the ambient space $\mathcal{J}(X)$. We denote by $i$ the inclusion
\[
i : \mathcal{J}_A(X) \to \mathcal{J}(X).
\]
By Lemma \ref{15} every loop $J(t) \in \pi_1(\mathcal{J}_A)$ can be decomposed into a product $J(t) = J_0(t) \cdot J_1(t)$, where $J_0(t) \in \text{Im} \nu_\ast$, and $J_1(t) \in \text{Ker} i_\ast$.

It follows from Lemma \ref{12} that every points of $\mathcal{D}_{T_-}$ satisfies the wall-crossing property. Thus the loops which lie in $\text{Ker} i_\ast$ could contribute drastically to the symplectic mapping class group via the corresponding elliptic twists. But this is what will not happen, because the following holds.

**Lemma 1.17.** $\text{Ker} i_\ast \subset \text{Im} \nu_\ast$.

**Proof.** Choose some $J_\ast \in \mathcal{D}_{T_-}$, and let $\Delta$ be a 2-disc which intersects $\mathcal{D}_{T_-}$ transversally at the single point $J_\ast$. Denote by $J(t)$ the boundary of $\Delta$. By Lemma \ref{18} one simply needs to show that the homotopy class of $J(t)$ comes from the natural action of $\text{Diff}_0(X)$ on $\mathcal{J}_A$, and the lemma will follow.

If $J_\ast$ is integrable, then one can choose $\Delta$ such that $J(t)$ is indeed an orbit of the action of a certain loop in $\text{Diff}_0(X)$, see the description of the complex-analytic family constructed in subsection \ref{3}. Thus it remains to check that every structure $J_\ast \in \mathcal{D}_{T_-}$ can be deformed to be integrable through structures on $\mathcal{D}_{T_-}$. This will be proved by Lemma \ref{19} below.

**Lemma 1.18.** Let $x,y \in \mathcal{J}_A$, and let $H(t) \in \mathcal{J}_A, t \in [0,1]$ be a path joining them such that $H(0) = x, H(1) = y$. If a loop $J(t) \in \pi_1(\mathcal{J}_A,y), t \in [0,1]$ lies in the image of $\pi_1(\text{Diff}_0(X),\text{id}) \to \pi_1(\mathcal{J}_A,y)$, then $H^{-1} \cdot J \cdot H \in \pi_1(\mathcal{J}_A,x)$ lies in the image of $\pi_1(\text{Diff}_0(X),\text{id}) \to \pi_1(\mathcal{J}_A,x)$.

**Proof.** Without loss of generality we assume that there exists a loop $f(t) \in \pi_1(\text{Diff}_0,\text{id})$ such that $J(t) = f_\ast(t) \cdot H(0)$. Let $H_\ast$ be the piece of the path $H$ that joins the points $H(0) = x$ and $H(s)$. To prove the lemma it remains to consider the homotopy
\[
J(s,t) := H^{-1}_\ast \cdot f_\ast(t) \cdot H(s) \cdot H_\ast,
\]
where $J(1,t) = H^{-1} \cdot J \cdot H$ and $J(0,t) = f_\ast(t) \cdot H(0)$.

**Lemma 1.19.** Every connected component of $\mathcal{J}_{-1}$ contains at least one integrable structure.

**Proof.** Take a structure $J \in \mathcal{J}_{-1}$, and denote by $C$ the corresponding smooth elliptic curve in class $[C] = T_{-1}$. Let $\pi : X \to C$ be the ruling such that $J \in \mathcal{J}(X,\pi)$, see Theorem
Further, let $X'$ be a complex ruled surface over $C$ such that $C$ is embedded in $X'$ as a holomorphic $(-1)$-section, and let $\pi': X' \to C$ be the corresponding projection. Apart from the section given by $C$, we now choose two more sections $C_1$ and $C_2$ of $\pi : X \to C$ such that the both $C_1$ and $C_2$ are disjoint from $C$ and they intersect each other transversally at a single point. We then choose a similar pair of section $C'_1$ and $C'_2$ for $\pi' : X' \to C$.

We claim that there exists a unique map $f : X \to X'$ such that

(a) the diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\pi & \downarrow & \pi' \\
C & \xrightarrow{id} & C
\end{array}
$$

commute,

(b) the fibers of $\pi$ are mapped biholomorphically by $f$ those of $\pi'$, and

(c) the sections $C_1$ and $C_2$ are mapped by $f$ respectively to $C'_1$ and $C'_2$.

By virtue of this map there exists an integrable complex structure $J' \in J_{-1}$ on $X$ such that $J' \in J(X, \pi)$, the curve $C$ is $J'$-holomorphic, and, moreover, $J'$ coincides with $J$ when is restricted to the bundle $TX|_C$ over $C$.

By Proposition 1.16 there is a symplectic form $\omega$ taming the both structures $J$ and $J'$. Given a symplectic curve, say $C$, in $X$, and an almost-complex structure, say $J$, defined along $C$ (i.e. on $TX|_C$) and tamed by $\omega$. There exists an $\omega$-tamed almost-complex structure on $X$ which extends the given one. Moreover, such an extension is homotopically unique. In particular, one can always construct a family $J_t$ joining $J$ and $J'$ such that $C$ is keep being $J_t$-holomorphic, and the lemma is proved. □

Summarizing the results of Lemma 1.15 and Lemma 1.17 we obtain

**Lemma 1.20.** $\pi_1(Diff_0(X)) \to \pi_1(J_0(X))$ is epimorphic.

Again, it is implied by diagram 0.1 that the following holds.

**Proposition 1.21.** $\pi_0(Symp^*(X, \omega_\mu)) = 0$ for every $\mu \in (-1,0]$.

Together with Proposition 1.16 this statement covers what is claimed in Theorem 0.1

### 2. Blow up once

#### 2.1. Rational $(-1)$-curves.

Let $(Z, \omega)$ be a symplectic ruled 4-manifold diffeomorphic to $S^2 \times Y^2 \# \overline{CP}^2$. Here we study homology classes in $H_2(Z; \mathbb{Z})$ that can be represented by a symplectically embedded $(-1)$-sphere. Given a symplectically embedded $(-1)$-sphere $A$, it satisfies

$$
[A]^2 = -1, \quad c_1([A]) = 1.
$$

A simple computation shows that there are but two homology classes satisfying (2.1), namely, $[A] = E$ and $[A] = F - E$.

The following lemma will be used in the sequel often without any specific reference.

**Lemma 2.1.** Let $(Z, \omega)$ be a symplectic ruled 4-manifold diffeomorphic to $S^2 \times Y^2 \# \overline{CP}^2$. Then for every choice of $\omega$-tamed almost-complex structure $J$, both the classes $E$ and $F - E$ are represented by smooth rational $J$-holomorphic curves.
Therefore, we have

\[ 0 < \int_{A_i} \omega < \int_A \omega. \]  

(2.2)

Because \( c_1(F - E) = 1 \), there exists at least one irreducible component of the curve \( A \), say \( A_1 \), with \( c_1([A_1]) \geq 1 \).

Note that spherical homology classes in \( H_2(Z;\mathbb{Z}) \) are generated by \( F \) and \( E \). Hence, we have \( [A_1] = pF - qE \), which particularly implies \( [A_1]^2 = -q^2 \leq 0 \), with an equality iff \( [A_1] = pF \). But the latter is prohibited by (2.2) because

\[ \int_{F - E} \omega > 0. \]

Therefore, we have \( [A_1]^2 \leq -1 \). Further, one may use the adjunction formula to obtain that \( A_1 \) is a smooth rational curve with \( [A_1]^2 = -1 \) and \( c_1(A_1) = 1 \). Note that it is not possible for \( A_1 \) to be in the class \( F - E \) because of (2.2). Hence, we have \( [A_1] = E \).

Take another irreducible component, say \( A_2 \). If \( A_2 \) does not intersect \( A_1 \), then \( [A_2] = pF \), which contradicts (2.2). Thus \( A_2 \) intersects \( A_1 \), positively. Hence, \( [A_2] = pF - qE \) for \( q \) positive. The same argument works for the other irreducible components \( A_2, A_3, \ldots \) of the curve \( A \). But note that \( [A_2] : [A_3] < 0 \), and hence there are no other components of \( A \), except \( A_1 \) and \( A_2 \). We thus have \( m_2[A_2] = F - (m_1 + 1)E \) for \( m_1, m_2 \geq 1 \). The class \( F - (m_1 + 1)E \) is prime, and hence \( m_2 = 1 \). Further, this class cannot be represented by a rational curve, which can be easily checked using the adjunction formula. We thus proved the lemma for the class \( F - E \); the case of \( E \) is analogous.

This lemma leads to the following generalization of Theorem 1.7 for ruled but not geometrically ruled symplectic 4-manifolds.

Lemma 2.2. Let \((Z, \omega)\) be a symplectic ruled 4-manifold diffeomorphic to \( S^2 \tilde{\times} Y^2 \# \mathbb{CP}^2 \), and let \( J \) be an \( \omega \)-tamed almost-complex structure. Then \( Z \) admits a singular ruling given by a proper projection \( \pi : Z \to Y \) onto \( Y \) such that

i) there is a singular value \( y^* \in Y \) such that \( \pi \) is a spherical fiber bundle over \( Y - y^* \), and each fiber \( \pi^{-1}(y) \), \( y \in Y - y^* \), is a \( J \)-holomorphic smooth rational curve in class \( F \);

ii) the fiber \( \pi^{-1}(y^*) \) consists of the two exceptional \( J \)-holomorphic smooth rational curves in classes \( F - E \) and \( E \).

Proof. By Lemma 2.1, our manifold \( Z \) contains a unique smooth \( J \)-holomorphic rational curve \( E \) representing the class \( E \), and a unique smooth \( J \)-holomorphic rational curve \( E' \) representing the class \( F - E \).

Denote by \( S \) the union of the curves \( E \) and \( E' \). To prove the lemma it suffices to check that the complement \( Z - S \) to the singular curve \( S \) in \( Z \) is fibered by smooth \( J \)-holomorphic rational curves of class \( F \).

Choose any point \( P \in Z - S \). Take a \( C^0 \)-small perturbation \( \tilde{J} \) of \( J \) which is integrable a small neighbourhood \( \mathcal{U}(S) \) of the curve \( S \) and such that \( S \) remains \( \tilde{J} \)-holomorphic; the structures \( J \) and \( \tilde{J} \) coincide away from \( \mathcal{U}(S) \), and the neighbourhood \( \mathcal{U}(S) \) can be chosen small enough to not contain the point \( P \).
Let $X$ be the blow-down of $E$ from $Z$. To finish the proof we check that $X$ contains a unique smooth pseudoholomorphic curve in class $F$ that pass through $P$. But that is what Theorem 1.7 states. \hfill $\Box$

### 2.2. Straight structures

Let $Z \cong S^2 \times T^2 \# \mathbb{CP}^2$ be a complex ruled surface, and let $E$ be a smooth rational $(-1)$-curve in $E \in H_2(Z;\mathbb{Z})$. The blow-down of $E$ from $Z$, which is a non-spin geometrically ruled genus one surface, will be denoted by $X$. The surface $Z$ is said to be an affine surface if $X$ is biholomorphic to the surface $X_A$, see subsection 1.2.

Let $p \in X$ be the image of $E$ under the contraction map. Recall that $X_A$ contains the triple of bisections, which are smooth elliptic curves in class $B \in H_2(X;\mathbb{Z})$. The surface $Z$ is called a straight affine surface if there is no bisection passing through $p$ in $X$.

In other words, a straight affine surface contains a triple of smooth curves in homology class $B$, while a non-straight affine surface contains a smooth elliptic $(-1)$-curve in class $B - E \in H_2(Z;\mathbb{Z})$. We remark that it follows from Theorem 1.3 that straight affine surfaces can be characterized as those for which there exists a smooth elliptic $(-1)$-curve in homology class $2B - E \in H_2(Z;\mathbb{Z})$.

Let $\pi$ be the ruling of $X$, and let $S$ be the fiber of $\pi$ that pass through $p$. When $Z$ is affine, there are three bisection $B_i \subset X$, each of which intersects $W$ at precisely two distinct points. The following result was established in subsection 1.2 when Suwa’s model for $X_A$ was described.

**Lemma 2.3.** There exists a complex coordinate $s$ on $S$ such that the intersection points $B_i \cap S$ are as follows:

$$B_1 \cap S = \{0, \infty\}, \quad B_2 \cap S = \{-1, 1\}, \quad B_3 \cap S = \{-i, i\}.$$  \hspace{1cm} (2.3)

We then claim

**Lemma 2.4.** There exists a complex-analytic family $Z_s$ of affine surfaces depending on a parameter $s$ ranging over $\mathbb{CP}^2$ such that for $s$ equals one of these exceptional values

$$\{0, \infty\}, \quad \{-1, 1\}, \quad \{-i, i\},$$

the surface $Z_s$ is not a straight affine surface, while for other parameter values, $Z_s$ is straight affine.

**Proof.** Let $X$ be a ruled surface of the $X_A$ complex type. Choose any fiber $F$ of the ruling of $X$. Now consider the complex submanifold $F \times \mathbb{CP}^1 \subset X \times \mathbb{CP}^1$, and denote by $S$ the diagonal in $F \times \mathbb{CP}^1$. We construct $Z$ as the blow-up of $X \times \mathbb{CP}^1$ along $S$. The 3-fold $Z$ forms the complex-analytic family $Z \rightarrow \mathbb{S}$ that was claimed to exist in the lemma. \hfill $\Box$

The notion of the straight affine complex structure can be generalized to almost-complex geometry as follows. Choose a tamed almost-complex structure $J \in \mathcal{J}(Z)$. We will call $J$ straight affine, or simply straight, if each $J$-holomorphic representaitve in class $B \in H_2(Z;\mathbb{Z})$ is smooth. Clearly, the space of straight structures $\mathcal{J}_{st}(Z)$ is an open dense submanifold in $\mathcal{J}(Z)$. Instead of $\mathcal{J}(Z)$ or $\mathcal{J}_{st}(Z)$ we write $\mathcal{J}$ and $\mathcal{J}_{st}$ for short. This definition of the straightforwardness is motivated by the following lemma the proof of which is left to the reader because it is similar to the proof of Proposition 1.5 (but the modified version of Theorem 1.7 given by Lemma 2.2 should be used).

**Lemma 2.5.** Let $(Z, \omega)$ be a symplectic ruled 4-manifold diffeomorphic to $S^2 \times T^2 \# \mathbb{CP}^2$, and let $J$ be an $\omega$-tamed almost-complex structure. Then every $J$-holomorphic representative in class $B$ is either irreducible smooth or contains a smooth component in one of the classes $T_+ - kF, B - E$.  

Similarly to Proposition 1.3, this lemma leads to a natural stratification of the space \( J \) of tamed almost-complex structures. Namely, this space can be presented as the disjoint union

\[
J = J_{st} + D_{T_-} + D_{B-E} + \ldots,
\]

where \( D_{T_-} \) and \( D_{B-E} \), which are submanifolds of real codimension 2 in \( J \), are the elliptic divisorial locus for respectively the classes \( T_- \) and \( B-E \). Here we omitted the terms of real codimension greater than 2, because they do not affect the fundamental group of \( J \).

Coming to the symplectic side of straightness, we claim that if a symplectic form \( \omega \) on \( Z \) satisfies both these two period conditions

\[
\int_{T_-} \omega < 0, \quad \int_{B-E} \omega < 0,
\]

then \( J(Z,\omega) \subset J_{st} \). Moreover, a somewhat inverse statement holds, at least for integrable structures.

**Lemma 2.6.** Every complex straight affine surface \( Z \) has a symplectic form which tames the given complex structure on \( Z \) and satisfies the period conditions. Moreover, the cohomology class of such a form can chosen to be the same for each straight affine structure.

**Proof.** If \( Z \) is affine then it is the surface \( X_A \cong S^2 \times T^2 \) blown-up once. Since \( X_A \) admits a symplectic structure which satisfies the first period condition, then so does \( Z \). It is not hard to check that the cohomology class can be chosen to be the same for each affine structure. Further, the second period condition can be achieved by means of deflation along a smooth elliptic curve in class \( 2B-E \); such a curve indeed exists thanks to the straightness of \( Z \). \[\Box\]

### 2.3. Refined Gromov invariants.

In this subsection, we work with an almost-complex manifold \((Z,J)\) equipped with a straight structure \( J \in J_{st} \), i.e. every \( J \)-holomorphic curve of class \( B \in H_2(Z;\mathbb{Z}) \) in \( Z \) is smooth. We also note that such a curve is not multiply-covered, because the homology class \( B \) is prime. The universal moduli space \( \mathcal{M}(B;J_{st}) \) of embedded non-parametrized pseudoholomorphic curves of class \( B \) is a smooth manifold, and the natural projection \( \text{pr} : \mathcal{M}(B;J_{st}) \to J_{st} \) is a Fredholm map, see \([\text{Lv-Sh-1, McD-Sa-3}]\). Given a generic \( J \in J_{st} \), the preimage \( \text{pr}^{-1}(J) \) is canonically oriented zero-dimensional manifold, see \([\text{Tb}]\) where it is explained how this orientation is chosen. Further, the oriented bordism class of \( \text{pr}^{-1}(J) \) does not depends on any particular choice of \( J \in J_{st} \). Therefore the degree of \( \text{pr} \) can be defined to be equal to \( \text{Gr}(B) \).

It is stated by Corollary 1.4 that \( \text{Gr}(B) = 3 \), and hence \( Z \) contains not one but several curves in class \( B \). Once we restricted almost-complex structures to those with the straightness property, the following modification of Gromov invariants can be proposed: given the image \( G \) of a certain homomorphism \( \mathbb{Z}^2 \to H_1(X;\mathbb{Z}) \), instead of counting pseudoholomorphic curves \( C \) such that \([C] = B\), we will count curves \( C \) such that \([C] = B\) and the embedding \( i : C \hookrightarrow X \) satisfies \( \text{Im}i = G \). The definitions of Gromov invariants \( \text{Gr}(B,G) \), moduli space \( \mathcal{M}(B,G;J_{st}) \), and so forth are completely analogous to those in “usual” Gromov’s theory.

Suppose \( J \) is an integrable straight affine structure, then the complex surface \((Z,J)\) contains precisely 3 smooth elliptic curves \( C_1, C_2, \) and \( C_3 \) in homology class \( B \). We denote by \( G_k \) the subgroup of \( H_1(X;\mathbb{Z}) \) generated by cycles on \( C_k \); these subgroups \( G_k \) are pairwise distinct, as it can be deduced, for example, from Suwa’s model of \( X_A \), see subsection 1.2.
It is clear now that the space $\mathcal{M}(B; \mathcal{J}_{st})$ is disconnected and can be presented as the union

$$\mathcal{M}(B; \mathcal{J}_{st}) = \bigcup_{k=1}^{3} \mathcal{M}(B, G_{k}; \mathcal{J}_{st}).$$

We define the moduli space of bisections to be the fiber product

$$\mathcal{M}_{3B} = \{(m_1, m_2, m_3) \mid m_k \in \mathcal{M}(B, G_{k}; \mathcal{J}_{st}), \text{ pr}(m_1) = \text{ pr}(m_2) = \text{ pr}(m_3)\}.$$  

Similarly to $\mathcal{M}(B; \mathcal{J}_{st})$, the moduli space $\mathcal{M}_{3B}$ is a smooth manifold equipped with the projection $\text{ pr} : \mathcal{M}_{3B} \rightarrow \mathcal{J}_{st}$, which is a smooth map of degree one. We close this section by stating an obvious property of the projection map that we shall use in the sequel.

**Lemma 2.7.** The projection map $\text{ pr} : \mathcal{M}_{3B} \rightarrow \mathcal{J}_{st}$ is a diffeomorphism, when is restricted to integrable straight affine complex structures.

### 2.4. A cocycle on $\mathcal{M}_{3B}$.

The map $\nu : \mathcal{D}iff_0(Z) \rightarrow \mathcal{J}_{st}$ defined by

$$\mathcal{D}iff_0(X) \xrightarrow{\nu} \mathcal{J}(X, \omega) : f \rightarrow f_*J,$$

can be naturally lifted to a map $\mathcal{D}iff_0(Z) \rightarrow \mathcal{M}_{3B}$. Indeed, take a point $m \in \mathcal{M}_{3B}$, which is a quadruple $[J, B_1, B_2, B_3](m)$ consisting of an almost-complex structure $J(m) \in \mathcal{J}_{st}$ on $Z$ and of the triple of smooth $J(m)$-holomorphic elliptic curves $B_1(m)$, $B_2(m)$, and $B_3(m)$ in $Z$. Then one can define

$$\mathcal{D}iff_0(X) \xrightarrow{\nu} \mathcal{M}_{3B} : f \rightarrow [f_*J, f(B_1), f(B_2), f(B_3)].$$

Here we construct a cocycle $\Lambda \in H^1(\mathcal{M}_{3B}; \mathbb{Q}^2)$ such that this homomorphism

$$\pi_1(\mathcal{D}iff_0(Z)) \xrightarrow{\nu} \pi_1(\mathcal{M}_{3B}) \xrightarrow{\Lambda} \mathbb{Q}^2$$

is the null-homomorphism.

To start we consider the tautological bundle $Z \cong \mathcal{M}_{3B} \times Z$ over $\mathcal{M}_{3B}$ whose fiber over a point $m \in \mathcal{M}_{3B}$ is the almost-complex manifold $(Z, J(m))$.

It was claimed in **Lemma 2.2** that every almost-complex manifold $(Z, J(m))$ contains a unique smooth rational $(-1)$-curve $S(m)$ in class $F - E$. Thus one can associate to $Z$ an auxiliary bundle $S$ whose fiber over $m \in \mathcal{M}_{3B}$ is the rational curve $S(m)$.

Note that each $B_i(m)$ intersects $S(m)$ at precisely 2 distinct points denoted by $P_{i,1}$ and $P_{i,2}$. Hence we can mark out 3 distinct pairs of points $(P_{i,1}, P_{i,2}), i = 1, 2, 3$ on each fiber $S(m)$ of $S$.

Besides that, every $(Z, J(m))$ contains a unique smooth rational curve $E(m)$ in class $E$. The curve $E(m)$ intersects $S(m)$ at precisely one point, say $Q(m)$. This point $Q$ does not coincide with any of the point $P_{i,1}, P_{i,2}$, because $J(m)$ is assumed to be a straight one. Therefore $S$ can be considered as a fiber bundle over $\mathcal{M}_{3B}$ whose fiber is the rational curve $S(m)$ with 7 distinct marked points, partially ordered in such a way that the first six points form the three ordered pairs, points inside every pair are not ordered, and the last point is of number seven.

One more fiber bundle, or better to say, a covering, we work with is the bundle $\mathcal{N}$ whose fiber over $m \in \mathcal{M}_{3B}$ consists of the six points $P_{i,1}(m), P_{i,2}(m), i = 1, 2, 3$, so $\mathcal{N}$ is a covering space of the covering group $G := \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This covering is not necessary trivial, i.e. it is possible for $\mathcal{N}$ to have a monodromy along certain loop in $\mathcal{M}_{3B}$.

Let $p : \widetilde{\mathcal{M}}_{3B} \rightarrow \mathcal{M}_{3B}$ be the Galois covering, so the pullback $p^*\mathcal{N}$ has no monodromy. Then the the bundle $\tilde{S} := p^*S$ can be considered as a fiber bundle over $\widetilde{\mathcal{M}}_{3B}$ whose fiber over $m \in \widetilde{\mathcal{M}}_{3B}$ is the rational curve $S(m)$ with 7 marked points, distinct and ordered.
Denote by $\tilde{S}$ the punctured projective line $\mathbb{CP}^1 \setminus \{0, 1, i\}$. We now construct a map $\lambda: \tilde{\mathcal{M}}_{3B} \to \tilde{S}$ as follows: choose $m \in \mathcal{M}_{3B}$, and consider the corresponding fiber $S(m)$ of the bundle $\tilde{S}$. Let $P_{1,1}, P_{1,2}$, and $Q$ be the corresponding marked points on $W(m)$; then there is a unique complex coordinate $s$ on $S(m)$ such that $s(P_{1,1}) = 0, s(P_{2,1}) = 1, s(P_{3,1}) = i$. We set $\lambda(m) := s(Q)$.

The following obvious property of $\lambda$ will be used soon.

**Lemma 2.8.** The map $\lambda$ is invariant w.r.t. to the natural action of $\mathcal{Diff}_0$ on $\tilde{\mathcal{M}}_{3B}$.

Now choose a basis for $H_1(\tilde{S}; \mathbb{Q}) \cong \mathbb{Q}^2$ consisting of two small loops going respectively around the points $s = 0$ and $s = 1$. One can think of the induced map $\lambda_*: H_1(\mathcal{M}_{3B}; \mathbb{Q}) \to H_1(\tilde{S}; \mathbb{Q})$ as a $\mathbb{Q}^2$-valued 1-cocycle on $\tilde{\mathcal{M}}_{3B}$.

In order to get an element in $H^1(\mathcal{M}_{3B}; \mathbb{Q}^2)$ we average the cocycle $\Lambda \in H^1(\tilde{\mathcal{M}}_{3B}; \mathbb{Q}^2)$ over the action of $G$ on $\tilde{\mathcal{M}}_{3B}$. We keep the notation $\Lambda$ for this new element in $H^1(\mathcal{M}_{3B}; \mathbb{Q}^2)$; the statement below follows from **Lemma 2.3**.

**Corollary 2.9.** $\text{Im} \nu_* \subset \text{Ker} \Lambda$.

2.5. Loops in $\mathcal{J}_{st}$. Here the group $\pi_1(\mathcal{J}_{st})$ will be proved to contain infinite order elements; the elements to be presented will not lie in the image of the homomorphism $\nu_*: \pi_1(\mathcal{Diff}_0(Z)) \to \mathcal{J}_{st}$.

The first ingredient we use is the complex-analytic family $Z_s$, $s \in S$, where $S \cong \mathbb{CP}^1$, given by **Lemma 2.4**. The surface $Z_s$ satisfies the straightness property for all but finitely many $s \in S$, see subsection 2.2; there are, however, these six exceptional values

$$\{0, \infty\}, \{-1, 1\}, \{-i, i\},$$

for which the corresponding surface $Z_s$ violates the mentioned property. These exceptional surfaces, which are affine but not straight affine, are of interest for us because they contain a smooth elliptic curve of class $B - E$, and hence they will correspond to the points of the elliptic divisorial locus $\mathcal{D}_{B - E}$.

Choose a closed path $s(t) \in S$ avoiding the exceptional values. Since $S \cong \mathbb{CP}^1$ is simply-connected, there exists a disc $\Delta$ which bounds $s(t)$; note that, in general, such a disc cannot be mapped into $S$ in such way to avoid the exceptional points.

Since $\Delta$ is the disc, the family $Z_s$ is smoothly trivial when is restricted to $\Delta$. Once a trivialization for, say $Z_{s(0)}$, is chosen, one can map $\Delta$ into the space $\mathcal{J}_{st}$ by extending the trivialization for $Z_{s(0)}$ to the trivialization for the family $Z_s$ over $\Delta$; note that such an extension is not unique, though all possible extensions are homotopic to each other. We also note that if the mapping of $\Delta$ into $S$ was chosen to be transversal to the exceptional values of $s$, then the constructed mapping of $\Delta$ into $\mathcal{J}_{st}$ would be transversal to the elliptic divisorial locus $\mathcal{D}_{B - E}$. Denote by $J(t)$ the loop in $\mathcal{J}_{st}$ which bounds $\Delta$.

Recall that there is a smooth map $\text{pr}: \mathcal{M}_{3B} \to \mathcal{J}_{st}$ of degree 1, see subsection 2.3. Since $J(t)$ consists of integrable structures, the preimage $m(t) := \text{pr}^{-1}(J(t))$ is a loop in $\mathcal{M}_{3B}$, see **Lemma 2.7**. The key property we need is:

**Lemma 2.10.** If a loop $J(t)$ is homotopic to zero in $\mathcal{J}_{st}$, then $m(t)$ is homologous to zero in $\mathcal{M}_{3B}$. In particular, we have $\Lambda(m(t)) = 0$.

**Proof.** Let $\Delta$ be a disc which bounds $J(t)$. By Sard-Smale theorem we can arrange that $\Delta$ is transverse to $\text{pr}$, and the preimage $\text{pr}^{-1}(J(t))$ is a smooth orientable surface that bounds $m(t)$. \qed
This lemma together with Corollary 2.9 imply

**Lemma 2.11.** If \( J(t) \in \text{Im} \nu_\ast \), then \( \Lambda(m(t)) = 0 \).

Now set \( s(t) = \varepsilon e^{it} \), where \( \varepsilon > 0 \) small enough, and consider the corresponding loops \( J(t) \in J_{st} \) and \( m(t) \in M_{3B} \). To compute \( \Lambda(m(t)) \) we note that \( \tilde{m}(t) := p^{-1}(m(t)) \in \tilde{M}_{3B} \) consists of eight distinct closed curves; one of them is

\[
(P_{1,1}(t), P_{1,2}(t), P_{2,1}(t), P_{2,2}(t), P_{3,1}(t), P_{3,2}(t), Q(t)) = (0, \infty, 1, -1, i, -i, \varepsilon e^{it}).
\]

with respect to some trivialization of the bundle \( S \).

We divide these curves into two groups:

\[
\tilde{m}_{1,k}(t) = (0, \infty, \ldots, \varepsilon e^{it}), \quad \tilde{m}_{2,k}(t) = (\infty, 0, \ldots, \varepsilon e^{it}), \quad k = 1, \ldots, 4.
\]

For each \( k \) we have that \( \lambda \circ \tilde{m}_{1,k}(t) \in \tilde{S} \) is a small simple closed path going around the point \( z = 0 \), while \( \lambda \circ \tilde{m}_{2,k}(t) \in \tilde{S} \) is a small path around \( z = \infty \). For the homology class \( [\lambda \circ m_{1,k}] \in H_1(\tilde{S}; \mathbb{Q}) \) we have \( [\psi \circ x_{1,k}] = (1, 0) \) with respect to the chosen basis for \( H_1(\tilde{S}; \mathbb{Q}) \), while \( [\lambda \circ m_{2,k}] \in H_1(\tilde{S}; \mathbb{Q}) \) clearly vanishes. It follows that

\[
\Lambda(m(t)) \neq 0.
\]

If \( s(t) \) was given by

\[
s(t) = 1 + \varepsilon e^{it} \quad \text{or} \quad s(t) = i + \varepsilon e^{it},
\]

then a similar argument would work to prove that \( \Lambda(m(t)) \neq 0 \).

**2.6. Let’s twist again.** Here we outline the proof of Theorem 1.2 referring the reader to the previous subsections for details.

We start with a symplectic 4-manifold \( Z \cong S^2 \times Y^2 \# \mathbb{CP}^2 \) for which the symplectic mapping class group mapping class group will be proved to contain elliptic twists.

Let \( X \) be a complex surface biholomorphic to \( X_A \), see subsection 1.2. By Theorem 1.3 we know that \( X \) contains a triple of smooth elliptic curves \( C_1, C_2, \) and \( C_3 \) in homology class \( B \in H_2(X; \mathbb{Z}) \), which are bisections of the corresponding ruling. Therefore, the procedure given in subsection 0.4 can be applied to prove the existence of elliptic twist for \( X \# \mathbb{CP}^2 \).

In subsection 2.5 the corresponding three loops, say \( J_{C_1}, J_{C_2}, \) and \( J_{C_3} \), are constructed as loops contained in the space \( J_{st} \) of the straight almost-complex structures. We then prove these loops do not lie in the image of \( \nu_\ast : \pi_1(\text{Diff}_0(Z)) \to \pi_1(J_{st}) \), see subsection 2.5. Because these loops consist of integrable structure, it follows from Lemma 2.9 there exists a symplectic form, say \( \theta \), such that \( J_{C_i} \in J(Z, \theta) \) and the inclusion \( J(Z, \theta) \subset J_{st} \) holds.

Since the inclusion \( J(Z, \theta) \subset J_{st} \) is equivariant w.r.t. to the natural action of \( \text{Diff}_0(Z) \) on these spaces, it follows that \( J_{C_i} \) do not lie in the image of \( \nu_\ast : \pi_1(\text{Diff}_0(Z)) \to J(Z, \theta) \). Therefore the elements \( \psi_\ast(J_{C_i}) \) are not in the kernel of \( \partial : \pi_1(J(Z, \theta)) \to \pi_0(\text{Symp}^\ast(Z, \theta)) \) and the theorem follows.
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