Quantified separably injective spaces

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Abstract

Let $X$, $Y$ be two Banach spaces. Let $\varepsilon \geq 0$. A mapping $f : X \to Y$ is said to be a standard $\varepsilon$-isometry if $f(0) = 0$ and $|||f(x) - f(y)|| - ||x - y||| \leq \varepsilon$. In this paper we first show that if $Y^*$ has the point of $w^*$-norm continuity property (in short, $w^*$-PCP) or $Y$ is separable, then for every standard $\varepsilon$-isometry $f : X \to Y$ there exists a $w^*$-dense $G_\delta$ subset $\Omega$ of $ExtB_{X^*}$ such that there is a bounded linear operator $T : Y \to C(\Omega, \tau_{w^*})$ with $\|T\| = 1$ such that $Tf - Id$ is uniformly bounded by $4\varepsilon$ on $X$. As a corollary we obtain quantitative characterizations of separably injectivity of a Banach space and its dual that turn out to give a positive answer to Qian’s problem of 1995 in the setting of universality. We also discuss Qian’s problem for $L_{\infty, \lambda}$-spaces and $C(K)$-spaces. Finally, we prove a sharpen quantitative and generalized Sobczyk theorem.

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1 Introduction

That every surjective isometry between two Banach spaces $X$ and $Y$ is necessarily affine was proved by Mazur and Ulam [26] in 1932. Since then, properties of isometries and generalizations there of between Banach spaces has continued for 82 years. On this period, many significant problems about perturbation properties of surjective $\varepsilon$-isometries were proposed and solved by numerous mathematicians. In particular, we mention the Hyers-Ulam problem [21] (see, for instance, [17], [19], and [27]). In 1968, Figiel [16] showed the following remarkable result: For every standard isometry $f : X \to Y$ there is a linear operator $T : L(f) \to X$ with $\|T\| = 1$ so that $Tf = Id$ on $X$, where $L(f)$ is the closure of span $f(X)$ in $Y$ (see also [7] and [14]). In 2003, Godefroy and Kalton [18] studied the relationship between isometries and linear isometries and solved a long-standing problem: Does the existence of an isometry $f : X \to Y$ imply the existence of a linear isometry $U : X \to Y$?

Definition 1.1. Let $X,Y$ be two Banach spaces, $\varepsilon \geq 0$, and let $f : X \to Y$ be a mapping.

(1) $f$ is said to be an $\varepsilon$-isometry if

\[
\|f(x) - f(y)\| - \|x - y\| \leq \varepsilon \text{ for all } x, y \in X.
\]

In particular, a 0-isometry $f$ is simply called an isometry.

(2) We say an $\varepsilon$-isometry $f$ is standard if $f(0) = 0$.

(3) A standard $\varepsilon$-isometry $f$ is $(\alpha, \gamma)$-stable if there exist $\alpha, \gamma > 0$ and a bounded linear operator $T : L(f) \to X$ with $\|T\| \leq \alpha$ such that

\[
\|Tf(x) - x\| \leq \gamma \varepsilon, \text{ for all } x \in X.
\]

In this case, we also simply say $f$ is stable, if no confusion arises.

(4) A pair $(X,Y)$ of Banach spaces $X$ and $Y$ is said to be stable if every standard $\varepsilon$-isometry $f : X \to Y$ is $(\alpha, \gamma)$-stable for some $\alpha, \gamma > 0$.

(5) A pair $(X,Y)$ of Banach spaces $X$ and $Y$ is called $(\alpha, \gamma)$-stable for some $\alpha, \gamma > 0$ if every standard $\varepsilon$-isometry $f : X \to Y$ is $(\alpha, \gamma)$-stable.

The study of non-surjective $\varepsilon$-isometries has also been considered (see, for instance, [5], [10], [11], [12], [13], [27], [30], [32] and [34]). Qian [30] proposed the following problem in 1995.

Problem 1.2. Is it true that for every pair $(X,Y)$ of Banach spaces $X$ and $Y$ there exists $\gamma > 0$ such that every standard $\varepsilon$-isometry $f : X \to Y$ is $(\alpha, \gamma)$-stable for some $\alpha > 0$?
However, Qian [30] presented a counterexample showing that if a separable Banach space $Y$ contains an uncomplemented closed subspace $X$ then for every $\varepsilon > 0$ there is a standard $\varepsilon$-isometry $f : X \to Y$ which is not stable. Cheng, Dong and Zhang [10] showed the following weak stability version.

**Theorem 1.3** (Cheng-Dong-Zhang). Let $X$ and $Y$ be Banach spaces, and let $f : X \to Y$ be a standard $\varepsilon$-isometry for some $\varepsilon \geq 0$. Then for every $x^* \in X^*$, there exists $\phi \in Y^*$ with $\|\phi\| = \|x^*\| \equiv r$ such that

$$|\langle \phi, f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon r,$$ for all $x \in X$.

For study of the stability of $\varepsilon$-isometries of Banach spaces, the following question was proposed in [11].

**Problem 1.4.** Is there a characterization for the class of Banach spaces $X$ satisfying given any $X \in X$ and Banach space $Y$, the pair $(X,Y)$ is $((\alpha,\gamma), \text{resp.})$ stable?

Every space $X$ of this class is said to be a universally $((\alpha,\gamma), \text{resp.})$ left-stable space.

On one hand, Cheng, Dai, Dong et.al. [11] proved that every injective Banach space is a universally left-stable space. On the other hand, the first two authors Cheng and Dai, together with others [5] showed that every universally left-stable space is just a cardinality injective Banach space (i.e., a Banach space which is complemented in every superspace with the same cardinality) and they also showed that a dual space is injective if and only if it is a universally left-stable space, and further asked if every universally left-stable space is an injective Banach space. In Section 3, we will show that the second dual of a universally left-stable space is injective and that for a dual space, cardinality injectivity, separably injectivity and injectivity are equivalent to universal left-stability.

The following Problem 1.5 is also very natural.

**Problem 1.5.** Is there a characterization for the class of Banach spaces $S$ satisfying given any $X \in S$ and separable Banach space $Y$, the pair $(X,Y)$ is $((\alpha,\gamma), \text{resp.})$ stable?

Every space $X$ of this class is said to be a separably universally (resp. $(\alpha,\gamma)$) left-stable space. In Section 4, we will show that all of these spaces of the class $S$ coincide with separably injective Banach spaces. We here refer
the reader to a very excellent paper [4] by Avilés-Sánchez-Castillo-González-Moreno for further information about injective Banach spaces and separably injective Banach spaces where they resolve a long standing problem proposed by Lindenstrauss in the middle sixties.

In this paper, we first consider a weaker version of Problem 1.2 in Section 2. That is Theorem 2.4, by which we discuss Qian’s problem for $L_{\infty,\lambda}$-spaces and $C(K)$-spaces, and then conclude all of the results in Section 3 and Section 4 stated as follows.

In section 2, we use Theorem 1.3 and Lemma 2.2 to prove Theorem 2.4 that if $X$, $Y$ are Banach spaces, and $Y^*$ has the point of $w^*$-norm continuity property (in short, $w^*$-PCP) or $Y$ is separable, then there exists a $w^*$-dense $G_\delta$ subset $\Omega$ of Ext $(B_X)$ such that there is a bounded linear operator $T : Y \to C(\Omega, \tau_{w^*})$ such that

$$\|Tf(x) - x\| \leq 4\varepsilon, \text{ for all } x \in X.$$ 

In particular, we obtain a weak positive answer to Qian’s problem for $C(K)$-spaces (see Corollary 2.5).

In section 3, combined Theorem 2.4 with some results from Johnson [22] and Avilés-Sánchez-Castillo-González-Moreno [4] we show that (a) $X^{**}$ is an injective Banach space if $X$ is universally left-stable. (b) If $X^{**}$ is $\lambda$-injective, then for every standard $\varepsilon$-isometry $f : X \to Y$, there is a bounded linear operator $S : Y \to X^{**}$ with $\|S\| \leq \lambda$ such that $Sf - Id$ is uniformly bounded by $4\varepsilon$ on $X$. (c) If $X$ is a $L_{\infty,\lambda}$-space, then for every standard $\varepsilon$-isometry $f : X \to Y$, there is a bounded linear operator $T : Y \to X^{**}$ such that $Tf - Id$ is uniformly bounded by $4\lambda\varepsilon$ on $X$. If, in addition, $X$ is isomorphic to a dual space $M^*$, then $X$ is universally $(\lambda\alpha, 4\lambda\alpha)$ left-stable for each $\alpha > d(X, M^*)$, which further yields that $X$ is $\lambda\alpha$-injective. Therefore, a dual space is separably injective if and only if it is universally left-stable.

In section 4, combined Theorem 2.4 together with some results from [24] by Johnson-Oikhberg (see also Rosenthal [29], Sánchez [31] and Castillo-Moreno [9]) and from [4] by Avilés-Sánchez-Castillo-González-Moreno, a quantitative characterization of separably injective Banach spaces is given. That is, we show that (i) if $X$ is a $\lambda$-separably injective Banach space, then the pair $(X, Y)$ is $(3\lambda, 12\lambda)$ stable for every separable Banach space $Y$. (ii) If the pair $(X, Y)$ is $(\lambda, 4\lambda)$ stable for every separable Banach space $Y$, then $X$ is a $\lambda$-separably injective Banach space. For example, (a) for every compact $F$-space $K$ (resp. compact $K$ of height $n$), the pair $(C(K), Y)$ is $(3, 12)$
(resp. $(6n - 3, 24n - 12)$) stable for every separable Banach space $Y$. In particular, $\ell_\infty/c_0$ is separably universally $(3, 12)$ left-stable. (b) If $\{E_i\}_{i \in \Lambda}$ is a family of $\lambda$-separably injective space, then the pair $((\sum_{i \in \Lambda} E_i)_\ell_\infty, Y)$ (resp. $((\sum_{i \in \Lambda} E_i)c_0, Y)$) is $(3\lambda, 12\lambda)$ (resp. $(6\lambda^2, 24\lambda^2)$) stable for every separable Banach space $Y$. (iii) In particular, by the Cheng-Dong-Zhang theorem (Theorem 1.3) we prove a sharpen quantitative and generalized Sobczyk theorem [33], that is, Theorem 4.6 if either $E_i = c_0(\Gamma_i)$ or $E_i$ is $\lambda$-injective for each $i \in \Lambda$.

All symbols and notations in this paper are standard. We use $X$ to denote a real Banach space and $X^*$ its dual. $B_X$, $\text{Ext}(B_X^*)$ and $S_X$ denote the closed unit ball of $X$, the set of all extremal points of $B_X^*$, and the unit sphere of $X$, respectively. Given a bounded linear operator $T : X \to Y$, $T^* : Y^* \to X^*$ stands for its conjugate operator. For a subset $A \subset X$, $2^A$, $\overline{A}$, $\text{card}(A)$ and $\text{dens}(A)$ stand respectively for the power set of $A$, the closure of $A$, the cardinality of $A$, the density character of $A$. We denote by $d(X,Y) = \inf \{\|T\| \cdot \|T^{-1}\| : T$ is an isomorphism between $X$ and $Y\}$ the Banach-Mazur distance between $X$ and $Y$.

2 $\varepsilon$-Isometric embedding into Banach spaces whose dual has the $w^*$-PCP

Recall that $\mathcal{S}$ is the class of Banach spaces satisfying given any $X \in \mathcal{S}$ and separable Banach space $Y$, the pair $(X,Y)$ is $((\alpha,\gamma),-$ resp.) stable. Every space $X$ of this class is said to be a separably universally $((\alpha,\gamma),$ resp.) left-stable space. In this section, we will consider a weaker version of Problem 1.2. That is, Theorem 2.4 by which we will discuss Qian’s problem for $C(K)$-spaces (Corollary 2.5) and $L_{\infty,\lambda}$-spaces (Theorem 3.13), and then show that for a dual space, cardinality injectivity, separably injectivity and injectivity are equivalent to universal left-stability. Moreover, we completely solve Problem 1.5 in Section 4. That is, we prove that all of these spaces of the class $\mathcal{S}$ coincide with separably injective Banach spaces.

Recall that a dual Banach space $Y^*$ is said to have the point of weak star to norm continuity property (in short, $w^*$-PCP) if every nonempty bounded subset of $Y^*$ admits relative weak star neighborhoods of arbitrarily small norm diameter. For example, if $Y$ is an Asplund space, then $Y^*$ has the $w^*$-PCP (see, for instance, [28]).

Recall that a set valued mapping $F : X \to 2^Y$ is said to be usco provided it is nonempty compact valued and upper semicontinuous, i.e., $F(x)$
is nonempty compact for each \( x \in X \) and \( \{ x \in X : F(x) \subset U \} \) is open in \( X \) whenever \( U \) is open in \( Y \). We say that \( F \) is usco at \( x \in X \) if \( F \) is nonempty compact valued and upper semicontinuous at \( x \), i.e., for every open set \( V \) of \( Y \) containing \( F(x) \) there exists a open neighborhood \( U \) of \( X \) such that \( F(U) \subset V \). Therefore, \( F \) is usco if and only if \( F \) is usco at each \( x \in X \).

Recall that a mapping \( \varphi : X \to Y \) is called a selection of \( F \) if \( \varphi(x) \in F(x) \) for each \( x \in X \), moreover, we say \( \varphi \) is a continuous (linear) selection of \( F \) if \( \varphi \) is a continuous (linear) mapping. We denote the graph of \( F \) by \( G(F) \equiv \{ (x,y) \in X \times Y : y \in F(x) \} \), we write \( F_1 \subset F_2 \) if \( G(F_1) \subset G(F_2) \).

A usco mapping \( F \) is said to be minimal if \( E = F \) whenever \( E \) is a usco mapping and \( E \subset F \) (see, for instance, [12, page 19, 102-109]).

The following Problem 2.1 is equivalent to Problem 1.2.

**Problem 2.1.** Does there exist a constant \( \gamma > 0 \) depending only on \( X \) and \( Y \) with the following property: For each \( \varepsilon \)-isometry \( f : X \to Y \) with \( f(0) = 0 \) there is a \( w^* - w^* \) continuous linear selection \( Q \) of the set-valued mapping \( \Phi \) from \( X^* \) into \( 2^{L(f)^*} \) defined by

\[
\Phi(x^*) = \{ \phi \in L(f)^* : |\langle \phi, f(x) \rangle - \langle x^*, x \rangle| \leq \gamma \|x^*\| \varepsilon, \text{ for all } x \in X \},
\]

where \( L(f) = \overline{\text{span}} f(X) \)?

The following Lemma 2.2 was motivated by Dai et.al. in [12, Lemma 4.2]. By an analogous argument we conclude the result on \( w^* - w^* \) usco mappings, which will be used to prove the main results.

**Lemma 2.2.** Suppose that \( X, Y \) are Banach spaces. Let \( \varepsilon \geq 0 \). Assume that \( f \) is a \( \varepsilon \)-isometry from \( X \) into \( Y \) with \( f(0) = 0 \), \( H \) is a Baire subspace contained in \( S_{X^*} \). If we define a set-valued mapping \( \Phi_1 : S_{X^*} \to 2^{S_{L(f)^*}} \) by

\[
\Phi_1(x^*) = \{ \phi \in S_{L(f)^*} : |\langle \phi, f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon, \text{ for all } x \in X \},
\]

where \( L(f) = \overline{\text{span}} f(X) \). Then

(i) \( \Phi_1 \) is convex \( w^* \)-usco at each point of \( S_{X^*} \).

(ii) There exists a minimal convex \( w^* - w^* \) usco mapping contained in \( \Phi_1 \).

(iii) If, in addition, \( Y^* \) has the \( w^* \)-PCP (especially, if \( Y \) is an Asplund space) or \( Y \) is separable, then there exists a selection \( Q \) of \( \Phi_1 \) such that \( Q \) is \( w^* - w^* \) continuous on a \( w^* \)-dense \( G_\delta \) subset of \( H \).

**Proof.** (i) It follows easily from [12, Lemma 4.2 (i)].
(ii) By Zorn Lemma (see [12, Lemma 4.2 (ii)] or [28, Prop. 7.3, p. 103]) there exists a minimal convex $w^*-w^*$ usco mapping contained in $\Phi_1$.

(iii) By (ii) there is a minimal convex $w^*-w^*$ usco mapping $F \subset \Phi_1$, and $H$ itself is a Baire space with respect to $w^*$ topology, and $Y^*$ has the $w^*$-PCP (especially, if $Y$ is an Asplund space) or $Y$ is separable, which follows easily from [28, Lemma 7.14, p.106-107] and [12, Lemma 4.2 (iii)].

Remark 2.3. The above Lemma 2.2 also holds if we substitute $Y^*$ and $S_Y$ for $L(f)^*$ and $S_{L(f)^*}$, respectively.

Theorem 2.4. Suppose that $X, Y$ are Banach spaces. Let $\varepsilon \geq 0$. Assume that $f$ is a $\varepsilon-$ isometry from $X$ into $Y$ with $f(0) = 0$. Then

(1) for every $w^*$-dense subset $\Omega \subset \text{Ext} (B_X^*)$ there is a bounded linear operator $T : Y \to \ell_\infty(\Omega)$ such that

$$\|Tf(x) - x\| \leq 4\varepsilon, \text{ for all } x \in X.$$ 

(2) If $Y^*$ has the $w^*$-PCP or $Y$ is separable, then there exists a $w^*$-dense $G_\delta$ subset $\Omega \subset \text{Ext} B_X^*$ such that there is a bounded linear operator $T : Y \to C(\Omega, \tau_{w^*})$ such that

$$\|Tf(x) - x\| \leq 4\varepsilon, \text{ for all } x \in X.$$ 

Proof. (1) By Theorem 1.3, for every $x^* \in \Omega$, there exists a functional $Q(x^*) \in S_{Y^*}$ such that

$$|\langle Q(x^*), f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon, \text{ for all } x \in X.$$ 

We now define a mapping $T : Y \to \ell_\infty(\Omega)$ by

$$T(y) = \{Q(x^*)(y)\}_{x^* \in \Omega}.$$ 

It is clear that $T$ is a bounded linear operator with norm one and

$$\|Tf(x) - x\| = \sup_{x^* \in \Omega} |Q(x^*)f(x) - x^*(x)| \leq 4\varepsilon, \text{ for all } x \in X.$$ 

(2) Since $\text{Ext}(B_X^*)$ itself is a Baire space in its relative $w^*$-topology (see [20, p. 217, line 17-19]), it follows from Lemma 2.2 that there is a $w^*$-dense $G_\delta$ subset $\Omega$ in $\text{Ext}(B_X^*)$ such that there is a $w^*-w^*$ continuous selection $Q$ of $\Phi_1$ on $\Omega$ satisfying that for every $x \in X$ and $x^* \in \Omega$, the following inequality holds:

$$|\langle Q(x^*), f(x) \rangle - \langle x^*, x \rangle| \leq 4\varepsilon.$$
Let \( T : Y \to \ell_\infty(\Omega) \) be defined as in (i). Therefore, \( T(y) \in C(\Omega, \tau_{w^*}) \) and
\[
\|Tf(x) - x\| \leq 4\varepsilon, \quad \text{for all} \ x \in X.
\]
\(\square\)

**Corollary 2.5.** Suppose that \( X = C(K) \) for a compact Hausdorff space \( K \) and \( Y^* \) has the \( w^* \)-PCP (especially, if \( Y \) is an Asplund space) or \( Y \) is separable. Let \( \varepsilon \geq 0 \). Assume that \( f \) is a standard \( \varepsilon \)-isometry from \( X \) into \( Y \). Then there exists a dense \( G_\delta \) subset \( \Omega \) of \( K \) such that there is a bounded linear operator \( T : Y \to C(\Omega) \) such that \( Tf - Id \) is uniformly bounded by \( 4\varepsilon \) on \( X \).

**Proof.** It suffices to note that \( \text{Ext } (B_{X^*}) = \{\pm \delta_t : t \in K\} \) and \( \{\delta_t : t \in K\} \) is a compact Baire space norming for \( X \), and then apply Lemma 2.2 and Theorem 2.4 to conclude the results we desired by substituting \( \{\delta_t : t \in K\} \) respectively for \( H \) and \( \text{Ext } (B_{X^*}) \) everywhere.
\(\square\)

### 3 A quantitative characterization of separably injective dual spaces

This section is based on a communication with W.B. Johnson, and the author would like to thank him for discussion. In this section, combined Theorem 2.4 with some results from Avilés-Sánchez-Castillo-González-Moreno [4] and Johnson [22] we show that

(a) \( X^{**} \) is an injective Banach space if \( X \) is universally left-stable.

(b) If \( X^{**} \) is \( \lambda \)-injective, then for every standard \( \varepsilon \)-isometry \( f : X \to Y \), there is a bounded linear operator \( S : Y \to X^{**} \) with \( \|S\| \leq \lambda \) such that \( Sf - Id \) is uniformly bounded by \( 4\lambda\varepsilon \) on \( X \).

(c) If \( X \) is a \( \mathcal{L}_{\infty,\lambda} \)-space, then for every standard \( \varepsilon \)-isometry \( f : X \to Y \), there is a bounded linear operator \( S : Y \to X^{**} \) with \( \|S\| \leq \lambda \) such that \( Sf - Id \) is uniformly bounded by \( 4\lambda\varepsilon \) on \( X \). If, in addition, \( X \) is isomorphic to a dual space \( M^* \), then \( X \) is universally \((\lambda\alpha, 4\lambda\alpha)\) left-stable for each \( \alpha > d(X, M^*) \), which further yields that it is \( \lambda\alpha \)-injective. Therefore, a dual space is separably injective if and only if it is universally left-stable.

Recall that a Banach space \( X \) is said to be \( \lambda \)-(resp. separably injective) injective if it has the following extension property: Every bounded linear operator \( T \) from a closed subspace of a (resp. separable) Banach space into \( X \) can be extended to be a bounded operator on the whole space with its
norm at most $\lambda \|T\|$ (see, for instance, [1], [4], [15], [35], [36]). In this case, $X$ is said to be injective (resp. separably injective) if it is $\lambda$-(resp. separably injective) injective for some $\lambda \geq 1$.

The following Proposition 3.1 follows easily from Remark 3.3.

**Proposition 3.1.** A (resp. separable) Banach space $X$ is $\lambda$-(resp. separably injective) injective if and only if it is $\lambda$-complemented in every (resp. separable) superspace (i.e., a normed linear space which contains $X$).

The following Proposition 3.2 was proved by Avilés, Sánchez, Castillo, González and Moreno (see [4, Prop. 3.2]).

**Proposition 3.2.** (1) If a Banach space $X$ is $\lambda$-separably injective, then it is $3\lambda$-complemented in every superspace $Y$ such that $Y/X$ is separable.

(2) If a Banach space $X$ is $\lambda$-complemented in every superspace $Y$ such that $Y/X$ is separable, then $X$ is $\lambda$-separably injective.

**Remark 3.3.** For any set $\Gamma$, that $\ell_\infty(\Gamma)$ is 1-injective follows from the Hahn-Banach theorem.

**Definition 3.4.** A Banach space $X$ is said to be cardinality injective if it is complemented in every superspace (a normed linear space containing $X$) with the same cardinality.

**Proposition 3.5.** A Banach space $X$ is cardinality injective if and only if every bounded linear operator $T$ from a subspace $Z$ of a normed linear space $Y$ with $\text{card}(Y) \leq \text{card}(X)$ into $X$ can be extended to be a bounded operator on the whole space with its norm at most $\lambda \|T\|$, where $\lambda$ depends only on $X$. In this case, we say $X$ is $\lambda$-cardinality injective.

**Proof.** Sufficiency. It is trivial.

Necessity. It is clear that $J(X)$ is also cardinality injective where $J$ is the canonical embedding from $X$ into $\ell_\infty(B_{X^*})$. Let $\bar{S} : Y \to \ell_\infty(B_{X^*})$ be a norm-preserving extension of operator $J \cdot T : Z \to \ell_\infty(B_{X^*})$. Let $Y'' = \text{span} \{ J(X) \cup \bar{S}(Y) \}$. So there is a projection from $Y''$ onto $J(X)$. Hence $\bar{T} = J^{-1} \cdot P \cdot \bar{S}$ is an extension of $T$ such that $\|\bar{T}\| \leq \|P\| \|T\|$.

We now show that there is a constant $\lambda$ depending only on $X$ such that for every $Y$ with $\text{card} Y \leq \text{card} X$, every subspace $Z$ and every operator $T : Z \to X$, there is an extension $\bar{T}$ of $T$ satisfying $\|\bar{T}\| \leq \lambda \|T\|$. To the contrary, for each $n \in \mathbb{N}$ there exist a normed linear space $Y_n$ with card $Y_n \leq \text{card} X$, a subspace $Z_n$ of $Y_n$ and an operator $T_n : Z \to X$ such that for every extension $\bar{T}_n$ of $T_n$, $\|\bar{T}_n\| \geq n \|T_n\|$. Let $Y = (\Sigma Y_n)_{c_0}$ endowed
the norm \(\|\cdot\|_{\ell_1}\) and \(Z = (\Sigma Z_n)c_0 \subset Y\). Obviously, \(\text{card}\ (Y) \leq \text{card}\ (X)\) and let \(T : Z \rightarrow X\) be defined for all \(z = \{z_n\} \in Z\) by \(T(z) = \sum \frac{T_n z_n}{\|T_n z_n\|}\) and \(\|T\| = 1\). If \(\tilde{T}\) is an extension of \(T\), then \(\|\tilde{T}\| \geq n\) for every \(n \in N\), which is a contradiction.

\(\square\)

The following Lemma 3.6 follows from Qian’s counterexample in [30] (see also [11]).

**Lemma 3.6.** Let \(X\) be a closed subspace of Banach space \(Y\). If \(\text{card}\ (X) = \text{card}\ (Y)\), then for every \(\varepsilon > 0\) there is a standard \(\varepsilon\)-isometry \(f : X \rightarrow Y\) such that

1. \(L(f) \equiv \overline{\text{span}}\ f(X) = Y\);
2. \(X\) is complemented whenever \(f\) is stable.

The following Lemma 3.7 and Lemma 3.8 are due to W.B.Johnson based on a communication (see [22]).

**Lemma 3.7.** \(\text{card}\ (X) = \text{dens}\ (X)^{\aleph_0}\).

**Proof.** It is clear that \(\text{card}\ (X) \leq \text{dens}\ (X)^{\aleph_0}\). It suffices to show that \(\text{card}\ (X) \geq \text{dens}\ (X)^{\aleph_0}\). By the Riesz’s lemma and axiom of choice, there exists a set \(\{x_i : 0 \leq i < \text{dens}\ (X)\}\) such that for each \(1 \leq j < \text{dens}\ (X)\),

\[d(x_j, \overline{\text{span}}\{x_i : i < j\}) > \frac{1}{2}\]

It follows for each \(i \neq j\) that \(\|x_i - x_j\| > \frac{1}{2}\).

We now define a mapping \(g\) for each \(i \in N\) and \(0 \leq n_i < \text{dens}\ (X)\) by \(g(\{x_{n_i}\}_{i=0}^{\infty}) = \sum_{i=0}^{\infty} \frac{1}{2^i} x_{n_i}\). For each \(\{x_{n_i}\}_{i=0}^{\infty} \neq \{x_{m_i}\}_{i=0}^{\infty}\), let \(k \in N\) be the least cardinal number such that \(x_{n_k} \neq x_{m_k}\). It follows from the triangle inequality that \(\|2^k \sum_{i=k}^{\infty} \frac{1}{2^i} x_{n_i} - 2^k \sum_{i=k}^{\infty} \frac{1}{2^i} x_{m_i}\| > 0\). Hence \(\|g(\{x_{n_i}\}_{i=0}^{\infty}) - g(\{x_{m_i}\}_{i=0}^{\infty})\| > 0\) and we complete the proof.

\(\square\)

**Lemma 3.8.** Every Banach space is linearly isometric to a subspace of some \(C(K)\)-space with the same cardinality, where \(K\) is a compact Hausdorff space.

**Proof.** Let \(X\) be identified with a subspace of \(C(B_X^*, \tau_{w^*})\) denoted by \(J(X)\): \(J(x)(x^*) = x^*(x)\) for all \(x^* \in B_X^*\). Let \(X_0\) be a dense set of \(X\) such that \(\text{card}\ (X_0) = \text{dens}\ (X)\) by the well-ordering principle of cardinals. Let \(P(X_0)\) be defined to be a subspace consisting of all polynomials with rational coefficients by

\[P(X_0) = \{q_m x_1^{p_{11}} x_2^{p_{22}} \cdots x_m^{p_{mm}} : m, p_m \in \mathbb{N}, q_m \in Q\ \text{and}\ x_i \in J(X_0)\}.\]
By the Stone-Weierstrass theorem, the closure of $P(X_0)$ forms a subalgebra which contains all constants and separates all points of $B_{X^*}$, hence

\[ \overline{P(X_0)} = C(B_{X^*}, \tau_{w^*}). \]

It is easy to see that $\text{card } (P(X_0)) = \text{card } (X_0)$, thus $\text{dens } (C(B_{X^*}, \tau_{w^*})) \leq \text{dens } (X)$. Therefore, by Lemma 3.7, $\text{card } (X) = \text{card } (C(K))$, where $K = B_{X^*}$ endowed the usual weak star topology $\tau_{w^*}$.

\[ \square \]

**Proposition 3.9.** $X$ is complemented in every complete superspace $Y$ with the same cardinality if and only if it is complemented in every superspace which is isomorphic to a $C(K)$ space with the same cardinality, where $K$ is a compact Hausdorff space.

**Proof.** It suffices to note that $X \subset Y \subset C(B_{Y^*}, \tau_{w^*})$.

\[ \square \]

**Lemma 3.10.** Suppose that $X$ is $\lambda$-cardinality injective. Then $X^{**}$ is $\lambda$-injective. If, in addition, $X$ is isomorphic to a dual space, then $X$ is even an $\alpha$-injective Banach space for every $\alpha > d(X, M^*)\lambda$.

**Proof.** By Lemma 3.8, $X$ is $\lambda$-complemented in some $C(K)$-space for a compact Hausdorff space $K$. Hence $X^{**}$ is $\lambda$-complemented in the 1-injective Banach space $C(K)^{**}$. Thus $X^{**}$ is $\lambda$-injective Banach space. If, in addition, $X$ is isomorphic to a dual space $M^*$, then $X$ is even an $\alpha$-injective Banach space for every $\alpha > d(X, M^*)\lambda$ since a dual space is complemented in its second dual.

\[ \square \]

**Theorem 3.11.** Suppose that $X$ is a Banach space such that for every Banach space $Y$ and every standard $\varepsilon$-isometry $f : X \to Y$, there exist $\gamma > 0$ and a bounded linear operator $T : L(f) \to X$ satisfying that

\[ \|Tf(x) - x\| \leq \gamma \varepsilon, \text{ for all } x \in X. \]

Then $X^{**}$ is an injective Banach space. If, in addition, $X$ is isomorphic to a dual space, then $X$ is injective.

**Proof.** Let $Y = C(B_{X^*}, \tau_{w^*})$. By Lemma 3.8, $\text{card } (X) = \text{card } (Y)$. Let $f : X \to Y$ be defined as in Lemma 3.6. Thus, $Y = L(f) \equiv \text{span } f(X)$ and $X$ is complemented in $Y$, hence that follows from Lemma 3.10.

\[ \square \]

By an analogous argument of Theorem 2.4 we have the following Corollary.
Corollary 3.12. Suppose that $X^{**}$ is $\lambda$-injective, $Y$ is a Banach space, Let $\varepsilon \geq 0$. Assume that $f$ is a standard $\varepsilon$-isometry from $X$ into $Y$. Then there is a bounded linear operator $S : Y \to X^{**}$ such that $\|S\| \leq \lambda$ such that

$$\|Sf(x) - x\| \leq 4\lambda\varepsilon, \text{ for all } x \in X.$$

Proof. It suffices to note that $X^{**}$ is $\lambda$-complemented in $\ell_\infty(\Omega)$ for every norm-dense set of Ext $(B_{X^*})$. By an analogous argument of Theorem 2.4 there is a bounded linear operator $T : Y \to \ell_\infty(\Omega)$ such that $Tf - Id$ is uniformly bounded by $4\varepsilon$ on $X$. Let $S = PT : Y \to X^{**}$ for a projection $P : \ell_\infty(\Omega) \to X^{**}$ with $\|P\| \leq \lambda$. Therefore, $Sf - Id$ is uniformly bounded by $4\lambda\varepsilon$.

Recall that a Banach space $X$ is said to be a $\mathcal{L}_{\infty,\lambda}$-space if every finite dimensional subspace $F$ of $X$ is contained in another finite dimensional subspace $E$ of $X$ such that $d(E, \ell_{\dim E}) \leq \lambda$ (see, for instance, [3], [4], [8]).

Theorem 3.13. Suppose that $X$ is a $\mathcal{L}_{\infty,\lambda}$-space and $Y$ is a Banach space. Then

(i) for every standard $\varepsilon$-isometry $f : X \to Y$, there is a bounded linear operator $T : Y \to X^{**}$ such that $Tf - Id$ is uniformly bounded by $4\lambda\varepsilon$ on $X$.

(ii) If, in addition, $X$ is isomorphic to a dual space $M^*$, then $X$ is universally $(\lambda\alpha, 4\lambda\alpha)$ left-stable for each $\alpha > d(X, M^*)$. Hence, $X$ is $\lambda\alpha$-injective.

Proof. (i) By Theorem 2.4 (i), for every $w^*$-dense subset $\Omega \subset \text{Ext} (B_{X^*})$ there is a bounded linear operator $T : Y \to \ell_\infty(\Omega)$ such that $Tf - Id$ uniformly bounded by $4\varepsilon$ on $X$. Let $X = \bigcup_{i \in I} E_i$ be such that for every $i, j \in (I, \geq)$, $i \geq j$ if and only if $E_i \supseteq E_j$ satisfying that for each $i \in I$, $\dim E_i < \infty$ and $d(E_i, \ell_{\dim E_i}) \leq \lambda$. Hence for each $i \in I$, there exists a projection $P_i : \ell_\infty(\Omega) \to E_i$ such that $\|P_i\| < \lambda + \frac{1}{1 + \dim E_i}$. Since $\{P_i\}_{i \in I}$ is uniformly bounded on $\ell_\infty(\Omega)^*$, it follows from the Arzelà–Ascoli theorem that there is a subnet $\{\delta_{i}^{\prime}\}_{i \in \Lambda}$ of $I$ for an partial order set $\Lambda$ such that $P : \ell_\infty(\Omega) \to X^{**}$ is well defined by

$$P(y) = w^* - \lim_{i \in \Lambda} P_{\delta_{i}^\prime}(y), \text{ for all } y \in \ell_\infty(\Omega),$$

which yields that $\|P\| \leq \lambda$ and $P|_X = Id$. 

Hence
\[ \| PTf(x) - x \| \leq 4\varepsilon\lambda, \quad \text{for all } x \in X, \]
where \( PT : Y \to X \) with \( \| PT \| \leq \lambda \).

(ii) By the assumption, there exists an isomorphism \( S : X \to M^* \) such that \( \| S \| \cdot \| S^{-1} \| < \alpha \). Clearly, \( SP : \ell_\infty(\Omega) \to M^* \) is uniformly bounded on \( B_{\ell_\infty(\Omega)^*} \). It follows from (i) that there is a subnet \( \{ \delta_i \}_{i \in \Lambda} \) such that \( Q : \ell_\infty(\Omega) \to M^* \) is well defined by
\[ Q(y) = w^* - \lim_{i \in \Lambda} SP_{\delta_i}(y), \quad \text{for all } y \in \ell_\infty(\Omega). \]

Hence \( S^{-1}QT : Y \to X \) is a bounded linear operator with \( \| S^{-1}QT \| \leq \alpha\lambda \) such that \( S^{-1}Q|_X = Id \) and
\[ \| S^{-1}QTf(x) - x \| \leq 4\varepsilon\alpha\lambda, \quad \text{for all } x \in X. \]
Thus, it follows from Lemma 3.10 that \( X \) is \( \lambda\alpha \)-injective and we complete the proof. \qed

Combined Theorem 3.13 with Theorem 3.11, we have the following Corollary 3.14.

**Corollary 3.14.** A dual Banach space is separably injective if and only if it is universally left-stable.

**Proof.** It suffices to note that a dual space is complemented in its second dual, hence sufficiency follows from Theorem 3.11. Note that a \( \lambda \)-separably injective Banach space is \( \mathcal{L}_{\infty,9\lambda} \)-space (see [1, p.199, Prop.3.5 (a)]). Hence, necessity follows from Theorem 3.13 (ii). \qed

**Remark 3.15.** For a dual space, cardinality injectivity, separably injectivity and injectivity are equivalent to universal left-stability.

## 4 A quantitative characterization of separably injective Banach spaces

In this section, combined Theorem 2.4 (ii) with some results from [24] by Johnson-Oikhberg (Lindenstrass [25], Rosenthal [29], Sánchez [31] and Castillo-Moreno [9]) and from [4] by Avilés-Sánchez-Castillo-González-Moreno, we conclude a quantitative characterization of separably injective Banach space which completely solves Problem 1.5. That is, we show that:
(i) If $X$ is a $\lambda$-separably injective Banach space, then the pair $(X, Y)$ is $(3\lambda, 12\lambda)$ stable for every separable Banach space $Y$;

(ii) If the pair $(X, Y)$ is $(\lambda, 4\lambda)$ stable for every separable Banach space $Y$, then $X$ is a $\lambda$-separably injective Banach space;

As a corollary, (a) for every compact $F$-space $K$ (for example, $K = \beta\mathbb{N}\setminus\mathbb{N}$), the pair $(C(K), Y)$ (resp. $(\ell_\infty/c_0, Y)$) is $(3, 12)$ stable for every separable Banach space $Y$;

(b) For every compact space $K$ of height $n$, the pair $(C(K), Y)$ is $(6n - 3, 24n - 12)$ stable for every separable Banach space $Y$;

(c) If $\{E_i\}_{i \in \Lambda}$ is a family of $\lambda$-separably injective space, then the pair $((\sum_{i \in \Lambda} E_i)\ell_\infty, Y)$ (resp. $((\sum_{i \in \Lambda} E_i)c_0, Y)$) is $(3\lambda, 12\lambda)$ (resp. $(6\lambda^2, 24\lambda^2)$) stable for every separable Banach space $Y$;

(iii) If either $E_i = c_0(\Gamma_i)$ or $E_i$ is a $\lambda$-injective Banach space for each $i \in \Lambda$, then by Theorem 1.3 we have a sharpen estimate for the constant pair $(\alpha, \gamma)$ in Theorem 4.6, which could be seen as a quantitative and generalized Sobczyk theorem.

**Theorem 4.1.** (i) If $X$ is a $\lambda$-separably injective Banach space, then the pair $(X, Y)$ is $(3\lambda, 12\lambda)$ stable for every separable Banach space $Y$.

(ii) If the pair $(X, Y)$ is $(\lambda, 4\lambda)$ stable for every separable Banach space $Y$, then $X$ is a $\lambda$-separably injective Banach space.

**Proof.** (i) Since $Y$ is separable, it follows from Theorem 2.4 (ii) that for every $w^*$-dense subset $\Omega \subset \text{Ext} (B_{X^*})$, there is a bounded linear operator $T : Y \to C(\Omega, \tau_{w^*})$ such that

$$\|Tf(x) - x\| \leq 4\varepsilon, \text{ for all } x \in X.$$ 

Hence, it could be reduced to ask if $X$ is complemented in $Z = \text{Span} \{Tf(X) \cup X\}$. It follows from the continuity of $T$ that $Z/X$ is separable quotient space since $Y$ is separable. Since $X$ is $\lambda$- separably injective, it follows from Proposition 3.2 that $X$ is $3\lambda$-complemented in $Z$. Therefore, there is a bounded linear operator $P : Z \to X$ with $\|P\| \leq 3\lambda$ such that

$$\|PTf(x) - x\| = \|PTf(x) - Px\| \leq 12\varepsilon, \text{ for all } x \in X,$$

where $PT : L(f) \to X$ satisfies that $\|PT\| \leq 3\lambda$.

(ii) By Proposition 3.2, it suffices to show that $X$ is $\lambda$-complemented in every superspace $Y$ such that $Y/X$ is separable. Let $Y = X + Y/X$ be the algebraic direct sum. Since $Y/X$ is separable, $\text{card}(X) = \text{card}(Y)$. It follows from Qian’s counterexample (i.e., Lemma 3.6) that there is an
isometry \( f: X \to Y \) such that \( Y = L(f) \) and \( f(0) = 0 \). Hence by the assumption, there is a projection \( P: Y \to X \) with \( \|P\| \leq \lambda \) such that

\[
\|Pf(x) - x\| \leq 4\epsilon, \text{ for all } x \in X,
\]

and we complete the proof.

Recall that a compact Hausdorff space \( K \) is said to be an \( F \)-space if disjoint open \( F_\sigma \) sets have disjoint closures. For example, \( \beta \mathbb{N} \), the Čech-Stone compactification of \( \mathbb{N} \) and \( \beta \mathbb{N}\setminus \mathbb{N} \) are \( F \)-spaces. Since \( C(K) \) is 1-separably injective for every \( F \)-space \( K \) (see, for instance, [4, p.202-203], [25]), we have

**Corollary 4.2.** For every compact \( F \)-space \( K \) (for example, \( K = \beta \mathbb{N}\setminus \mathbb{N} \)), the pair \( (C(K), Y) \) (resp. \( (\ell_\infty/c_0, Y) \) ) is (3, 12) stable for every separable Banach space \( Y \).

**Proof.** It is sufficient to note that \( \ell_\infty/c_0 \) is linearly isometric to \( C(\beta \mathbb{N}\setminus \mathbb{N}) \).

Recall that a compact space \( K \) has height \( n \) if \( K^{(n)} = \emptyset \), where we write \( K' \) for the derived set of \( K \) and \( K^{(n+1)} = (K^{(n)})' \). Since \( C(K) \) is \((2n - 1)\)-separably injective for every \( K \) of height \( n \) (see, for instance, [4, p.203]), we have

**Corollary 4.3.** For every compact space \( K \) of height \( n \), the pair \( (C(K), Y) \) is \((6n - 3, 24n - 12)\) stable for every separable Banach space \( Y \).

Combined Theorem 4.1 with the results of Johnson-Oikhberg [24] that for every family of \( \lambda \)-separably injective spaces \( \{E_i\}_{i \in A} \), \( (\sum_{i \in A} E_i)_{\ell_\infty} \) and \( (\sum_{i \in A} E_i)c_0 \) are respectively \( \lambda \)-separably injective and \( 2\lambda^2 \)-separably injective, which was also proved by Rosenthal [29], Sánchez [31] and Castillo-Moreno [9] with the estimates for the constant, respectively \( \lambda(1+\lambda)^+, (3\lambda^2)^+ \) and \( 6(1+\lambda) \), we have the following corollaries.

**Corollary 4.4.** The pair \( ((\sum_{i \in A} E_i)_{\ell_\infty}, Y) \) is \((3\lambda, 12\lambda)\) stable for every separable Banach space \( Y \), where \( \{E_i\}_{i \in A} \) is a family of \( \lambda \)-separably injective spaces.

**Corollary 4.5.** The pair \( ((\sum_{i \in A} E_i)c_0, Y) \) is \((6\lambda^2, 24\lambda^2)\) (resp. \((3\lambda(1+\lambda)^+, 12\lambda(1+\lambda)^+)\) \((9\lambda^2)^+, (36\lambda^2)^+) \) and \((18(1+\lambda), 72(1+\lambda))\) stable for every separable Banach space \( Y \), where \( \{E_i\}_{i \in A} \) is a family of \( \lambda \)-separably injective spaces.
If either \( E_i = c_0(\Gamma_i) \) or \( E_i \) is a \( \lambda \)-injective Banach spaces for each \( i \in \Lambda \), then by Theorem [1.3] we have the following Theorem [4.6] which gives a sharpen estimate for the constant pair \((\alpha, \gamma)\) by contrast with Corollary [4.4] and Corollary [4.5] respectively. In some sense, it could be seen as a quantitative and generalized Sobczyk theorem [33].

**Theorem 4.6.** Let \( \Lambda \) and \( \Gamma_i \) for each \( i \in \Lambda \) are index sets. Suppose that one of the following three statements holds

i) \( X \) is isomorphic to \( Z = (\sum_{i \in \Lambda} c_0(\Gamma_i))_{\ell_\infty} \) and \( \lambda > d(X, Z) \);

ii) \( X \) is isomorphic to \( Z = (\sum_{i \in \Lambda} \ell_\infty(\Gamma_i))_{c_0} \) and \( \lambda > d(X, Z) \);

iii) \( X = (\sum_{i \in \Lambda} E_i)c_0 \) and \( \{E_i\}_{i \in \Lambda} \) is a family of \( \lambda \)-injective Banach spaces.

Then \((X, Y)\) is \((2\lambda, 8\lambda)\)-stable for every separable Banach space \( Y \).

**Proof.**

i) Let \( X \) be a Banach space isomorphic to \((\sum_{i \in \Lambda} c_0(\Gamma_i))_{\ell_\infty} \) and \( T : X \to (\sum_{i \in \Lambda} c_0(\Gamma_i))_{\ell_\infty} \) be an isomorphism such that \( \|T\| \cdot \|T^{-1}\| < \lambda \). For each \( n \in \Lambda \) and \( m \in \Gamma_n \), let \( e_{nm} \in (\sum_{i \in \Lambda} c_0(\Gamma_i))_{\ell_\infty} \) with the standard biorthogonal functionals \( e^*_nm \in (\sum_{i \in \Lambda} c_0(\Gamma_i))_{\ell_\infty}^* \) such that \( e^*_{ij}(e_{nm}) = \delta_{mn}\delta_{jm}. \)

For all \( n \in \Lambda \) and \( m \in \Gamma_n \), let \( x_{nm} \in X \) be such that \( T(x_{nm}) = e_{nm} \). Let \( T^*: Z^* \to X^* \) be the conjugate operator of \( T \). Then

\[
T(x) = \{ \sum_{m \in \Gamma_n} (T^*e^*_{nm})(x)e_{nm} \}_{n \in \Lambda}
\]

and

\[
x = T^{-1}\{ \sum_{m \in \Gamma_n} (T^*e^*_{nm})(x)e_{nm} \}_{n \in \Lambda} \quad \text{for all } x \in X.
\]

For all \( n \in \Lambda \) and \( m \in \Gamma_n \), let \( x^*_{nm} = T^*e^*_{nm} \in \|T\|B_{X^*}. \) It follows from Theorem [1.3] that for each \( n \in \Lambda \) and \( m \in \Gamma_n \), there exists a functional \( \phi_{nm} \in \|T\|B_{Y^*} \) with \( \|\phi_{nm}\| = \|x^*_{nm}\| \) such that

\[
(4.1) \quad |\langle \phi_{nm}, f(x) \rangle - \langle x^*_{nm}, x \rangle| \leq 4\varepsilon\|T\|, \quad \text{for all } x \in X.
\]

It follows from the \( w^* - w^* \) continuity of \( T^* \) that for each \( n \in \Lambda \), \( x^*_{nm} \to 0 \) in the \( w^* \)-topology of \( X^* \) Since \( e^*_{nm} \to 0 \) in the \( w^* \)-topology of \( Z^* \). Let

\[
K = \{ \psi \in \|T\|B(Y^*) : |\langle \psi, f(x) \rangle| \leq 4\varepsilon\|T\|, \quad \text{for all } x \in X \}.
\]

Then \( K \) is a nonempty \( w^* \)-compact subset of \( Y^* \). Since \( Y \) is separable, \((\|T\|B_{Y^*}, w^*)\) is metrizable. Let \( d \) be a metric such that \((\|T\|B_{Y^*}, d)\) is homeomorphic to \((\|T\|B_{Y^*}, w^*)\). Since for each \( n \in \Lambda \), \( x^*_{nm} \) is a \( w^* \)-null net in \( X^* \), inequality \((4.1)\) implies that for each \( n \in \Lambda \), every \( w^* \)-cluster point
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\( \phi \) of \((\phi_{nm})\) is in \( K \) such that \( \|\phi\| \leq \|T\| \), which yields that \( d(\phi_{nm}, K) \to 0 \) for each \( n \in \Lambda \). Hence, for each \( n \in \Lambda \), there is a net \((\psi_{nm}) \subset K\) such that \( d(\phi_{nm}, \psi_{nm}) \to 0 \), or equivalently, \( \phi_{nm} - \psi_{nm} \to 0 \) in the \( w^*\)-topology of \( Y^* \).

Let \( S : Y \to X \) be defined for every \( y \in Y \) by

\[
S(y) = T^{-1}\{ \sum_{m \in \Gamma_n} \langle \phi_{nm} - \psi_{nm}, y \rangle e_{nm} \}_{n \in \Lambda} \in X.
\]

Hence

\[
\|S\| \leq 2\|T\| \cdot \|T^{-1}\| < 2\lambda
\]

and

\[
\|Sf(x) - x\| = \|T^{-1}\{ \sum_{m \in \Gamma_n} \langle \phi_{nm} - \psi_{nm}, f(x) \rangle e_{nm} \}_{n \in \Lambda} - T^{-1}\{ \sum_{m \in \Gamma_n} \langle x_{nm}^*, x \rangle e_{nm} \}_{n \in \Lambda}\|
\]

\[
\leq \|T^{-1}\| \cdot \sup_{n \in \Lambda} \left( \| \sum_{m \in \Gamma_n} \langle \phi_{nm} - \psi_{nm}, f(x) \rangle e_{nm} - \sum_{m \in \Gamma_n} \langle x_{nm}^*, x \rangle e_{nm} \| \right)
\]

\[
\leq \|T^{-1}\| \cdot \sup_{n \in \Lambda} \| \langle \phi_{nm}, f(x) \rangle - \langle x_{nm}^*, x \rangle - \langle \psi_{nm}, f(x) \rangle \|
\]

\[
\leq \|T^{-1}\| \left( \sup_{n \in \Lambda} \| \langle \phi_{nm}, f(x) \rangle - \langle x_{nm}^*, x \rangle \| + \sup_{n \in \Lambda} \| \langle \psi_{nm}, f(x) \rangle \| \right)
\]

\[
\leq 8\varepsilon \|T\| \cdot \|T^{-1}\| < 8\varepsilon \lambda.
\]

ii-iii) For each \( i \in \Lambda \), \( \Gamma_i \) denotes by \( B_{E_i^*} \). It suffices to show this case that \( X = \left( \sum_{i \in \Lambda} E_i \right)c_0 \). Let \( J : X = \left( \sum_{i \in \Lambda} E_i \right)c_0 \to \left( \sum_{i \in \Lambda} \ell\infty(B_{E_i^*}) \right)c_0 = \left( \sum_{i \in \Lambda} \ell\infty(\Gamma_i) \right)c_0 \) be the canonical embedding. For each \( n \in \Lambda \), let \( Q_n : \left( \sum_{i \in \Lambda} \ell\infty(\Gamma_i) \right)c_0 \to \ell\infty(\Gamma_n) \) be the canonical projection. Let \( P_n : \ell\infty(\Gamma_n) \to E_n \) be a family of projections with \( \|P_n\| \leq \lambda \). For each \( n \in \Lambda \) and \( m \in \Gamma_n \), let \( e_{nm} \in \left( \sum_{i \in \Lambda} \ell\infty(\Gamma_i) \right)c_0 \) with the standard biorthogonal functionals \( e_{nm}^* \in \left( \left( \sum_{i \in \Lambda} \ell\infty(\Gamma_i) \right)c_0 \right)^* \) such that \( e_{ij}^*(e_{nm}) = \delta_{in}\delta_{jm} \). Then

\[
x = \sum_{n \in \Lambda} \{ (e_{nm}^*)(x) \}_{m \in \Gamma_n} \text{ for all } x \in X.
\]

By Theorem 1.3 for each \( n \in \Lambda \) and \( m \in \Gamma_n \), there exists \( \phi_{nm} \in B_{Y^*} \) with \( \|\phi_{nm}\| = \|e_{nm}^*\| \) such that

\[
|\langle \phi_{nm}, f(x) \rangle - \langle e_{nm}^*, x \rangle| \leq 4\varepsilon, \text{ for all } x \in X.
\]

Clearly, \( e_{nm}^* \to 0 \) uniformly for each \( m \in \Gamma_n \) in the \( w^*\)-topology of \( Z^* \). Let

\[
K = \{ \psi \in B(Y^*) : |\langle \psi, f(x) \rangle| \leq 4\varepsilon, \text{ for all } x \in X \}.
\]
Since $\Gamma_n$ can be well ordered for every $n \in \Lambda$, we write
\[ \Gamma_n = \{0, 1, 2, \ldots, w_0, w_0 + 1, \ldots, w_1, \ldots \prec \Gamma_n\}, \]
where $\Gamma_n$ also denotes by its ordinal number. It follows from i) that for each $n \in \Lambda$, there is a net $(\psi_{n0}) \subset K$ such that $d(\phi_{n0}, \psi_{n0}) \to 0$. We can choose $(\psi_{nm}) \subset K$ such that for every $n \in \Lambda$ and $m \in \Gamma_n$, $d(\phi_{nm}, \psi_{nm}) \leq d(\phi_{n0}, \psi_{n0})$ or equivalently, $(\phi_{nm} - \psi_{nm}) \to 0$ uniformly for each $m \in \Gamma_n$ in the $w^*$-topology of $Y^*$. Let $Q : Y \to (\sum_{i \in \Lambda} \ell_\infty(\Gamma_i))c_0$ be defined for all $y \in Y$ by
\[ Q(y) = \sum_{n \in \Lambda} \{(\phi_{nm} - \psi_{nm}, y)\} = \sum_{n \in \Lambda} \{\langle \phi_{nm} - \psi_{nm}, y \rangle\} \in (\sum_{i \in \Lambda} \ell_\infty(\Gamma_i))c_0, \]
which yields that
\[ \|Q(y)\| \leq (\sup_{n \in \Lambda, m \in \Gamma_n} \|\phi_{nm} - \psi_{nm}\|)\|y\| \leq 2\|y\|. \]
Thus
\[ \|Q\| \leq 2. \]
Let $S : Y \to X$ be defined for all $y \in Y$ by
\[ S(y) = \sum_{n \in \Lambda} P_n Q_n(y) = \sum_{n \in \Lambda} P_n \{\langle \phi_{nm} - \psi_{nm}, y \rangle\} \in (\sum_{i \in \Lambda} \ell_\infty(\Gamma_i))c_0. \]
Hence
\[ \|S\| = \sup_{n \in \Lambda} \|P_n Q_n\| \leq 2\lambda \]
and
\[ \|Sf(x) - x\| = \left\| \sum_{n \in \Lambda} P_n \{\langle \phi_{nm} - \psi_{nm}, f(x) \rangle\} - \sum_{n \in \Lambda} P_n \{\langle e_{nm}^*, x \rangle\} \right\| \]
\[ \leq \lambda \sup_{n \in \Lambda, m \in \Gamma_n} |\langle \phi_{nm}, f(x) \rangle - \langle e_{nm}^*, x \rangle| + \langle \psi_{nm}, f(x) \rangle| \]
\[ \leq \lambda (\sup_{n \in \Lambda} |\langle \phi_{nm}, f(x) \rangle - \langle e_{nm}, x \rangle| + \sup_{n \in \Lambda} |\langle \psi_{nm}, f(x) \rangle|) \]
\[ \leq 8\varepsilon \lambda. \]
Thus, our proof is completed.

\[ \square \]

**Remark 4.7.** There are many other examples for separably injective Banach spaces, such as the Johnson-Lindenstrauss spaces [23], Benyamini-space which is an M-space nonisomorphic to a $C(K)$-space [6] and the WCG nontrivial twisted sums of $c_0(\Gamma)$ constructed by Argyros, Castillo, Granero, Jimenez and Moreno [2] (see, for instance, [4]).
Qian [30] proved that the pair \((L_p, L_p)\) is stable for \(1 < p < \infty\). Šemrl and Väisälä [32] gave a sharp estimate for the constant pair \((\alpha, \gamma)\) with \(\gamma = 2\). Therefore, it is very natural to ask:

**Problem 4.8.** Is it true that the following pairs are stable for \(1 \leq p \leq \infty\) and \(p \neq q < \infty\)?

\[
\begin{align*}
(1) \quad & ((\sum_{n=1}^{\infty} l_p^n) c_0, (\sum_{n=1}^{\infty} l_p^n) c_0); \\
(2) \quad & ((\sum_{n=1}^{\infty} l_p^n) \ell_\infty, (\sum_{n=1}^{\infty} l_p^n) \ell_\infty); \\
(3) \quad & ((\sum_{n=1}^{\infty} \ell_\infty l_p, (\sum_{n=1}^{\infty} \ell_\infty l_p); \\
(4) \quad & ((\sum_{n=1}^{\infty} \ell_p \ell_\infty, (\sum_{n=1}^{\infty} \ell_p \ell_\infty); \\
(5) \quad & ((\sum_{n=1}^{\infty} L_p \ell_\infty, (\sum_{n=1}^{\infty} L_p \ell_\infty); \\
(6) \quad & ((\sum_{n=1}^{\infty} c_0 l_p, (\sum_{n=1}^{\infty} c_0 l_p); \\
(7) \quad & ((\sum_{n=1}^{\infty} L_p c_0, (\sum_{n=1}^{\infty} L_p c_0); \\
(8) \quad & ((\sum_{n=1}^{\infty} \ell_p c_0, (\sum_{n=1}^{\infty} \ell_p c_0); \\
(9) \quad & ((\sum_{n=1}^{\infty} l_p l_q, (\sum_{n=1}^{\infty} l_p l_q); \\
(10) \quad & ((\sum_{n=1}^{\infty} L_p l_q, (\sum_{n=1}^{\infty} L_p l_q).
\end{align*}
\]

It is true for (1), (2), (3), (4) and (5) if \(p = \infty\) as we have proved. In this case, it is not true for (6), (7) and (8) since \((\sum_{n=1}^{\infty} c_0 l_p, (\sum_{n=1}^{\infty} L_p c_0)\) and \((\sum_{n=1}^{\infty} \ell_\infty l_p, (\sum_{n=1}^{\infty} L_p \ell_\infty)\) are not complemented in \(\ell_\infty\). If \(1 \leq p < \infty\), then it is also not true for (3), (4) and (5) since \((\sum_{n=1}^{\infty} \ell_\infty l_p, (\sum_{n=1}^{\infty} \ell_p \ell_\infty)\) and \((\sum_{n=1}^{\infty} L_p \ell_\infty, (\sum_{n=1}^{\infty} L_p \ell_\infty)\) are not complemented in \(\ell_\infty\). However, we do not know if it is true or not for the above problem 4.8 in general case.

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