On Pólya–Szegö and Čebyšev type inequalities via generalized $k$-fractional integrals

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Abstract

In this paper, we introduce the generalized $k$-fractional integral in terms of a new parameter $k > 0$, present some new important inequalities of Pólya–Szegö and Čebyšev types by use of the generalized $k$-fractional integral. Our consequences with this new integral operator have the abilities to implement the evaluation of many mathematical problems related to real world applications.

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1 Introduction

There are numerous problems wherein fractional derivatives (non-integer order derivatives and integrals) attain a valuable position [1–25]. It must be emphasized that fractional derivatives exist in many technologies, especially they can be described in three different approaches, and any of these approaches can be used to solve many important problems in the real world. Every classical fractional operator is typically described in terms of a particular significance. There are many well-recognized definitions of fractional operators, we can also point out the Riemann–Liouville, Caputo, Grunwald–Letnikov, and Hadamard operators [26], whose formulations include integrals with singular kernels and which may be used to check, for example, issues involving the reminiscence effect [27]. However, within the year 2010, specific formulations of fractional operators appeared in the literature [28]. The new formulations diverge from the classical ones in numerous components. As an example, classical fractional derivatives are described in such a manner that in the limit wherein the order of the derivative is an integer, one recovers the classical derivatives in the sense of Newton and Leibniz. In addition, new fractional operators [29–31] with a corresponding integral whose kernel may be a non-singular mapping have been currently proposed; for instance, a Mittag-Leffler function [32]. In such instances, integer-order derivatives are rediscovered by supposing suitable limits for the values of their parameters.

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On the other hand, there are numerous approaches to acquire a generalization of classical fractional integrals. Many authors introduce parameters in classical definitions or in some unique specific function [4], as we shall do in what follows. Moreover, in the present paper, we introduce a parameter and enunciate a generalization for fractional integrals on a selected space, which we call generalized \( k \)-fractional integral, and further advocate a Pólya–Szegö and Čebyšev type inequalities modification of this generalization.

Inequalities and their potential applications are of great significance in pure mathematics and applied mathematics, many remarkable inequalities and their applications can be found in the literature [33–46]. In view of the broader applications, integral inequalities have received large interest [47–60]. Presently, many authors have provided the unique versions of such inequalities which may be beneficial within the study of diverse classes of differential and integral equations. Those inequalities act as far-reaching tools to look at the classes of differential and integral equations [61–70].

Čebyšev [71] introduced the well-known celebrated functional as follows:

\[
\mathcal{I}(\mathcal{U}, \mathcal{V}) = \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \mathcal{U}(\lambda) \mathcal{V}(\lambda) \, d\lambda - \left( \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \mathcal{U}(\lambda) \, d\lambda \right) \left( \frac{1}{\sigma_2 - \sigma_1} \int_{\sigma_1}^{\sigma_2} \mathcal{V}(\lambda) \, d\lambda \right),
\]

(1.1)

where \( \mathcal{U} \) and \( \mathcal{V} \) are two integrable functions on \([\sigma_1, \sigma_2]\). If \( \mathcal{U} \) and \( \mathcal{V} \) are synchronous, that is,

\[
(\mathcal{U}(\lambda) - \mathcal{U}(\omega))(\mathcal{V}(\lambda) - \mathcal{V}(\omega)) \geq 0
\]

for any \( \lambda, \omega \in [\sigma_1, \sigma_2] \), then \( \mathcal{I}(\mathcal{U}, \mathcal{V}) \geq 0 \).

Functional (1.1) has attracted the attention of many researchers due to its demonstrated applications in probability, numerical analysis, quantum theory, statistical and transform theory. Alongside facet with numerous applications, functional (1.1) has gained plenty of interest to yield a variety of fundamental inequalities [72–76].

Another interesting and fascinating aspect of the theory of inequalities is the Grüss type inequality [66] which states

\[
|\mathcal{I}(\mathcal{U}, \mathcal{V})| \leq \frac{(Q - q)(R - r)}{4},
\]

where the integrable functions \( \mathcal{U} \) and \( \mathcal{V} \) satisfy

\[
q \leq \mathcal{U}(\lambda) \leq Q
\]

and

\[
r \leq \mathcal{V}(\lambda) \leq R
\]

for all \( \lambda \in [\sigma_1, \sigma_2] \) and for some \( q, Q, r, R \in \mathbb{R} \).

Many famous versions mentioned in the literature are direct effects of the numerous applications in optimizations and transform theory. In this regard Pólya–Szegö integral
inequality is one of the most intensively studied inequalities. This inequality was introduced by Pólya and Szegö [76]:

\[
\frac{\int_{\sigma_1}^{\sigma_2} u^2(\lambda) d\lambda \cdot \int_{\sigma_1}^{\sigma_2} v^2(\lambda) d\lambda}{(\int_{\sigma_1}^{\sigma_2} u(\lambda) v(\lambda) d\lambda)^2} \leq \frac{1}{4} \left( \sqrt{\frac{QR}{qr}} + \sqrt{\frac{qr}{QR}} \right)^2. 
\] (1.2)

The constant \(\frac{1}{4}\) is a best possible constant such that inequality (1.2) holds, that is, it can’t be replaced by a smaller constant.

By using the Pólya–Szegö inequality, Dragomir and Diamond [75] proved that the inequality

\[ |\Sigma(\mathcal{U}, \mathcal{V})| \leq \frac{(Q - q)(R - r)}{4(\sigma_2 - \sigma_1)\sqrt{qrQR}} \int_{\sigma_1}^{\sigma_2} u(\lambda) d\lambda \int_{\sigma_1}^{\sigma_2} v(\lambda) d\lambda \]

holds for all \(\lambda \in [\sigma_1, \sigma_2]\) if the functions \(\mathcal{U}\) and \(\mathcal{V}\) defined on \([\sigma_1, \sigma_2]\) satisfy

\[ 0 < q \leq \mathcal{U}(\lambda) \leq Q < \infty, \quad 0 < r \leq \mathcal{V}(\lambda) \leq R < \infty. \]

It has been extensively discussed that Pólya–Szegö and Čebyšev type inequalities in continuous and discrete cases play a considerable role in examining the qualitative conduct of differential and difference equations. As a result of these studies, many new branches of mathematics have been opened up. Inspired by Pólya, Szegö, and Čebyšev [71, 76], we intend to show more general versions of Pólya–Szegö and Čebyšev type inequalities.

Our present paper has been inspired by the resource of the above-defined work. The principal aim of the present paper is to set up new Pólya–Szegö and Čebyšev types integral inequalities associated with generalized \(k\)-fractional integrals. We introduce parameter \(k > 0\) and generalize the results in such a way that the existing results can be explored too. Thus, the results provided in this research paper are more generalized as compared to the existing results.

2 Preliminaries

In this section, we demonstrate some important concepts from fractional calculus that will play a major role in proving the results of the present paper. The essential points of interest are exhibited in the monograph by Kilbas et al. [27].

**Definition 2.1** (see [27, 77]) Let \(p \geq 1\) and \(r \in \mathbb{R}\). Then the function \(\mathcal{U}(\xi)\) is said to be in \(L_{pr}^{[\nu_1, \nu_2]}\) space if

\[
\|\mathcal{U}\|_{L_{pr}^{[\nu_1, \nu_2]}} = \left( \int_{\nu_1}^{\nu_2} |\mathcal{U}(\xi)|^p \xi^r d\xi \right)^{\frac{1}{p}} < \infty.
\]

In particular,

\[
L_{p,0}^{[\nu_1, \nu_2]} = L_p^{[\nu_1, \nu_2]} = \left\{ \mathcal{U} : \|\mathcal{U}\|_{L_{p}^{[\nu_1, \nu_2]}} = \left( \int_{\nu_1}^{\nu_2} |\mathcal{U}(\xi)|^p d\xi \right)^{\frac{1}{p}} < \infty \right\}.
\]
Definition 2.2 (see [78]) Let $p \geq 1$, $U \in L^1(0, \infty)$ and $\Psi$ be an increasing and positive monotone function defined on $[0, \infty)$ such that $\Psi'$ is continuous on $[0, \infty)$ and $\Psi(0) = 0$. Then $U$ is said to be in $\chi^p_\Psi[0, \infty)$ space if $\|U\|_{\chi^p_\Psi} < \infty$, where $\|U\|_{\chi^p_\Psi}$ is defined by

$$\|U\|_{\chi^p_\Psi} = \left( \int_0^\infty |U(\xi)|^p \Psi'(\xi) d\xi \right)^{\frac{1}{p}}$$

for $1 \leq p < \infty$ and

$$\|U\|_{\chi^\infty_\Psi} = \text{ess sup}_{0 \leq \xi < \infty} [\Psi'(\xi)|U(\xi)|].$$

In particular, if $\Psi(\lambda) = \lambda$, then $\chi^p_\Psi[0, \infty)$ coincides with $L^p[0, \infty)$; if $\Psi(\lambda) = \log \lambda$, then $\chi^p_\Psi[0, \infty)$ becomes $L^p_{-1}[0, \infty)$.

Definition 2.3 (see [27, 77]) Let $\sigma_1 < \sigma_2$ and $U \in L^1([\sigma_1, \sigma_2])$. Then the left and right Riemann–Liouville fractional integrals of order $\varrho > 0$ are defined by

$$J^{\varrho}_{\sigma_1+1}U(\lambda) = \frac{1}{\Gamma(\varrho)} \int_{\sigma_1}^{\lambda} (\lambda - \xi)^{\varrho-1}U(\xi) d\xi \quad (\lambda > \sigma_1)$$

and

$$J^{\varrho}_{\sigma_2-2}U(\lambda) = \frac{1}{\Gamma(\varrho)} \int_{\lambda}^{\sigma_2} (\xi - \lambda)^{\varrho-1}U(\xi) d\xi \quad (\lambda < \sigma_2),$$

respectively, where $\Gamma(\varrho) = \int_0^\infty t^{\varrho-1} e^{-t} dt$ is the gamma function [79–87].

Now, we recall the definition of $k$-fractional integral [88].

Definition 2.4 (see [88]) Let $\sigma_1 < \sigma_2$, $k > 0$, and $U \in L^1([\sigma_1, \sigma_2])$. Then the left and right $k$-fractional integrals of order $\varrho > 0$ are defined by

$$J^{\varrho k}_{\sigma_1+1}U(\lambda) = \frac{1}{k \Gamma_k(\varrho)} \int_{\sigma_1}^{\lambda} (\lambda - \xi)^{\varrho-1}U(\xi) d\xi \quad (\lambda > \sigma_1)$$

and

$$J^{\varrho k}_{\sigma_2-2}U(\lambda) = \frac{1}{k \Gamma_k(\varrho)} \int_{\lambda}^{\sigma_2} (\xi - \lambda)^{\varrho-1}U(\xi) d\xi \quad (\lambda < \sigma_2),$$

respectively, where $\Gamma_k(\varrho) = \int_0^\infty t^{\varrho-1} e^{-t^k} dt$ is the $k$-gamma function [89].

A generalization of the Riemann–Liouville fractional integrals with respect to another function is given in [27] as follows.

Definition 2.5 (see [27]) Let $\sigma_1 < \sigma_2$, $\varrho > 0$, and $\Psi(\xi)$ be an increasing and positive monotone function defined on $(\sigma_1, \sigma_2]$. Then the left and right generalized Riemann–Liouville...
fractional integrals of the function \( U \) with respect the function \( \Psi \) of order \( \varrho \) are defined by

\[
J_{\Psi, \sigma_1}^{\varrho} U(\lambda) = \frac{1}{\Gamma(\varrho)} \int_{\sigma_1}^{\lambda} \psi'(\xi) \left( \psi(\lambda) - \psi(\xi) \right)^{\varrho - 1} U(\xi) \, d\xi
\]  

(2.1)

and

\[
J_{\Psi, \sigma_2}^{\varrho} U(\lambda) = \frac{1}{\Gamma(\varrho)} \int_{\sigma_2}^{\lambda} \psi'(\xi) \left( \psi(\lambda) - \psi(\xi) \right)^{\varrho - 1} U(\xi) \, d\xi,
\]  

(2.2)

respectively.

Next, we present a new fractional integral operator which is known as the generalized \( k \)-fractional integral operator of a function with respect to another function.

**Definition 2.6** Let \( \sigma_1 < \sigma_2 \), \( \varrho, k > 0 \), and \( \Psi(\xi) \) be an increasing and positive monotone function defined on \( (\sigma_1, \sigma_2) \). Then the left and right generalized \( k \)-fractional integrals of the function \( U \) with respect to the function \( \Psi \) of order \( \varrho \) are defined by

\[
J_{\Psi, \sigma_1}^{\varrho, k} U(\lambda) = \frac{1}{k \Gamma_k(\varrho)} \int_{\sigma_1}^{\lambda} \psi'(\xi) \left( \psi(\lambda) - \psi(\xi) \right)^{\varrho - 1} U(\xi) \, d\xi
\]  

(2.3)

and

\[
J_{\Psi, \sigma_2}^{\varrho, k} U(\lambda) = \frac{1}{k \Gamma_k(\varrho)} \int_{\sigma_2}^{\lambda} \psi'(\xi) \left( \psi(\lambda) - \psi(\xi) \right)^{\varrho - 1} U(\xi) \, d\xi,
\]  

(2.4)

respectively.

**Remark 2.7** Several existing fractional operators are the special cases of Definition 2.6. For example:

1. Let \( k = 1 \). Then Definition 2.6 reduces to Definition 2.5.
2. Let \( \Psi(\lambda) = \lambda \). Then Definition 2.6 reduces to Definition 2.4.
3. Let \( \Psi(\lambda) = \lambda \) and \( k = 1 \). Then Definition 2.6 reduces to 2.3.
4. Let \( \Psi(\lambda) = \log \lambda \) and \( k = 1 \). Then Definition 2.6 leads to the Hadamard fractional integral operators given in [27, 77].
5. Let \( \beta > 0 \), \( \Psi(\lambda) = \frac{\lambda^\beta}{\beta} \), and \( k = 1 \). Then Definition 2.6 leads to the Katugampola fractional integral operators in the literature [90].
6. Let \( \beta > 0 \), \( \Psi(\lambda) = \left( \frac{\lambda - a}{b} \right)^\beta \), and \( k = 1 \). Then Definition 2.6 becomes the conformable fractional integral operators defined by Jarad et al. in [91].
7. Let \( \Psi(\lambda) = \frac{\lambda^{\alpha \varphi}}{\alpha \varphi} \) and \( k = 1 \). Then Definition 2.6 becomes the generalized conformable fractional integrals defined by Khan et al. in [92].

Throughout this paper, we suppose that \( \Psi(\xi) \) is a strictly increasing function on \((0, \infty)\) and \( \psi'(\xi) \) is continuous, \( 0 \leq \sigma_1 < \sigma_2 \) with the condition that at any point \( \sigma_3 \in [\sigma_1, \sigma_2] \), we have \( \Psi(\sigma_3) = 0 \).
3 Pólya–Szegö type inequalities involving the generalized $\mathcal{K}$-fractional integrals

In this section, we derive certain Pólya–Szegö type integral inequalities for real-valued integrable functions via generalized Riemann–Liouville $k$-fractional integral defined in (2.3) and (2.4). Throughout this paper, we assume that $\Psi(\zeta)$ is an increasing, positive, and monotone function defined on $[0, \infty)$ such that $\Psi(0) = 0$, and $\Psi'(\zeta)$ is continuous on $[0, \infty)$.

**Lemma 3.1** Let $k, \lambda, \varrho > 0$, $U$ and $V$, $\rho_1$, $\rho_2$, $\chi_1$, and $\chi_2$ be six positive integrable functions defined on $[0, \infty)$ such that

\begin{align*}
0 < \rho_1(\zeta) &\leq U(\zeta) \leq \rho_2(\zeta), & 0 < \chi_1(\zeta) &\leq V(\zeta) \leq \chi_2(\zeta) \tag{3.1}
\end{align*}

for all $\zeta \in [0, \lambda]$. Then one has

\begin{align*}
\frac{1}{4} \left( \mathcal{I}_\Psi^{\alpha, k} \left[ (\rho_1 \chi_1 + \rho_2 \chi_2) U V \right](\lambda) \right)^2 &\geq \mathcal{I}_\Psi^{\alpha, k} \left[ \chi_1 \chi_2 U^2 \right](\lambda) \mathcal{I}_\Psi^{\alpha, k} \left[ \rho_1 \rho_2 V^2 \right](\lambda). \tag{3.2}
\end{align*}

**Proof** It follows from (3.1) that

\begin{align*}
\rho_2(\zeta) &\geq \frac{U(\zeta)}{\chi_1(\zeta)} \rho_1(\zeta), & \chi_1(\zeta) &\geq \frac{V(\zeta)}{\chi_2(\zeta)} \rho_2(\zeta) \tag{3.3}
\end{align*}

and

\begin{align*}
\rho_1(\zeta) &\geq \frac{U(\zeta)}{\chi_2(\zeta)} \rho_2(\zeta), & \chi_2(\zeta) &\geq \frac{V(\zeta)}{\chi_1(\zeta)} \rho_1(\zeta) \tag{3.4}
\end{align*}

for all $\zeta \in [0, \lambda]$.

Multiplying (3.3) and (3.4), we obtain

\begin{align*}
\left[ \rho_1(\zeta) \chi_1(\zeta) + \rho_2(\zeta) \chi_2(\zeta) \right] U(\zeta) V(\zeta) &\geq \chi_1(\zeta) \chi_2(\zeta) U^2(\zeta) + \rho_1(\zeta) \rho_2(\zeta) V^2(\zeta). \tag{3.5}
\end{align*}

Multiplying both sides of inequality (3.5) by

\begin{align*}
\frac{1}{k \Gamma_k(\varrho)} \Psi'(\zeta) (\Psi(\lambda) - \Psi(\zeta))^{{\frac{1}{2}}} - 1
\end{align*}

and integrating the obtained result with respect to $\zeta$ to $(0, \lambda)$, we get

\begin{align*}
\mathcal{I}_\Psi^{\alpha, k} \left[ (\rho_1 \chi_1 + \rho_2 \chi_2) U V \right](\lambda) &\geq \mathcal{I}_\Psi^{\alpha, k} \left[ \chi_1 \chi_2 U^2 \right](\lambda) + \mathcal{I}_\Psi^{\alpha, k} \left[ \rho_1 \rho_2 V^2 \right](\lambda).
\end{align*}

Applying the arithmetic-geometric inequality, we have

\begin{align*}
\mathcal{I}_\Psi^{\alpha, k} \left[ (\rho_1 \chi_1 + \rho_2 \chi_2) U V \right](\lambda) &\geq 2 \sqrt{\mathcal{I}_\Psi^{\alpha, k} \left[ \chi_1 \chi_2 U^2 \right](\lambda) \mathcal{I}_\Psi^{\alpha, k} \left[ \rho_1 \rho_2 V^2 \right](\lambda)},
\end{align*}

which leads to

\begin{align*}
\frac{1}{4} \left( \mathcal{I}_\Psi^{\alpha, k} \left[ (\rho_1 \chi_1 + \rho_2 \chi_2) U V \right](\lambda) \right)^2 &\geq \mathcal{I}_\Psi^{\alpha, k} \left[ \chi_1 \chi_2 U^2 \right](\lambda) \mathcal{I}_\Psi^{\alpha, k} \left[ \rho_1 \rho_2 V^2 \right](\lambda).
\end{align*}

Therefore, we obtain the desired inequality (3.1). □
Corollary 3.2 Let \( k, \lambda, q, r, \rho, Q, R > 0 \) with \( q \leq Q \) and \( r \leq R \), and \( U \) and \( V \) be two positive integrable functions defined on \([0, \infty)\) such that

\[
0 < q \leq U(\xi) \leq Q < \infty, \quad 0 < r \leq V(\xi) \leq R < \infty
\] (3.6)

for all \( \xi \in [0, \lambda] \). Then one has

\[
\frac{J_\psi^{0,k}U^2(\lambda)J_\psi^{0,k}V^2(\lambda)}{(J_\psi^{0,k}U(\lambda))^2} \leq \frac{1}{4} \left( \sqrt{qR} + \sqrt{QR} \right)^2.
\]

Corollary 3.3 Let \( k = 1 \). Then Lemma 3.1 reduces to the inequality for generalized Riemann–Liouville fractional integrals as follows:

\[
\frac{1}{4} \left( J_\psi^0 [(\rho_1 \chi_1 + \rho_2 \chi_2)U(\lambda)](\lambda) \right)^2 \geq J_\psi^0 [\chi_1 \chi_2 U^2](\lambda)J_\psi^0 [\rho_1 \rho_2 V^2](\lambda).
\] (3.7)

Corollary 3.4 Let \( \Psi(\lambda) = \lambda \). Then Lemma 3.1 leads to the inequality for \( k \)-fractional integral as follows:

\[
\frac{1}{4} \left( J^{0,k}_\psi [(\rho_1 \chi_1 + \rho_2 \chi_2)U(\lambda)](\lambda) \right)^2 \geq J^{0,k}_\psi [\chi_1 \chi_2 U^2](\lambda)J^{0,k}_\psi [\rho_1 \rho_2 V^2](\lambda).
\]

Remark 3.5 Let \( \Psi(\lambda) = \lambda \) and \( k = 1 \). Then Lemma 3.1 becomes Lemma 3.1 of [67].

Lemma 3.6 Let \( k, \lambda, \rho, \delta > 0 \) and \( U, V, \rho_1, \rho_2, \chi_1, \chi_2 \) be six positive integrable functions defined on \([0, \infty)\) such that \( (3.1) \) holds for all \( \lambda \in [0, \lambda] \). Then we have

\[
\frac{J_\psi^{0,k} \rho_1 \rho_2 \chi_1 \chi_2 U^2 (\lambda) J_\psi^{0,k} V^2 (\lambda) \left( J_\psi^{0,k} \rho_1 U(\lambda) J_\psi^{0,k} \chi_1 V(\lambda) + J_\psi^{0,k} \rho_2 U(\lambda) J_\psi^{0,k} \chi_2 V(\lambda) \right)^2}{(J_\psi^{0,k} \rho_1 U(\lambda) J_\psi^{0,k} \chi_1 V(\lambda))^2 (J_\psi^{0,k} \rho_2 U(\lambda) J_\psi^{0,k} \chi_2 V(\lambda))^2} \leq \frac{1}{4}.
\] (3.8)

Proof From (3.1) we clearly see that

\[
\frac{\rho_2 (\xi)}{\chi_1 (\eta)} - \frac{U(\xi)}{V(\eta)} \geq 0
\]

and

\[
\frac{U(\xi)}{V(\eta)} - \frac{\rho_1 (\xi)}{\chi_2 (\eta)} \geq 0,
\]

which imply that

\[
\left( \frac{\rho_1 (\xi)}{\chi_2 (\eta)} + \frac{\rho_2 (\xi)}{\chi_1 (\eta)} \right) \frac{U(\xi)}{V(\eta)} \geq \frac{U^2 (\xi)}{V^2 (\eta)} + \frac{\rho_1 (\xi) \rho_2 (\xi)}{\chi_1 (\eta) \chi_2 (\eta)}.
\] (3.9)

Multiplying both sides of inequality (3.9) by \( \chi_1 (\eta) \chi_2 (\eta) V^2 (\eta) \), we have

\[
\rho_1 (\xi) U(\xi) \chi_1 (\eta) V(\eta) + \rho_2 (\xi) U(\xi) \chi_2 (\eta) V(\eta) \geq \chi_1 (\eta) \chi_2 (\eta) U^2 (\xi) + \rho_1 (\xi) \rho_2 (\xi) V^2 (\eta).
\] (3.10)

Multiplying both sides of inequality (3.10) by
\[
\frac{1}{k I^k(\rho_1) k I^k(\delta)} \psi'(\zeta)(\psi(\lambda) - \psi(\zeta))^{\frac{d}{2} - 1} \psi'(\eta)(\psi(\lambda) - \psi(\eta))^{\frac{d}{2} - 1}
\]
and then integrating the obtained inequality with respect to \(\zeta\) and \(\eta\) from 0 to \(\lambda\), one has
\[
(J^k_\psi \rho_1 U)(\lambda)(J^{\delta k}_\psi \chi_1 V)(\lambda) + (J^k_\psi \rho_2 U)(\lambda)(J^{\delta k}_\psi \chi_2 V)(\lambda)
\geq (J^{\delta k}_\psi U^2)(\lambda)(J^{\delta k}_\psi \chi_1 \chi_2)(\lambda) + (J^{\delta k}_\psi V^2)(\lambda)(J^{\delta k}_\psi \rho_1 \rho_2)(\lambda).
\]
Making use of the arithmetic-geometric mean inequality, we obtain
\[
(J^k_\psi \rho_1 U)(\lambda)(J^{\delta k}_\psi \chi_1 V)(\lambda) + (J^k_\psi \rho_2 U)(\lambda)(J^{\delta k}_\psi \chi_2 V)(\lambda)
\geq 2\sqrt{(J^{\delta k}_\psi U^2)(\lambda)(J^{\delta k}_\psi \chi_1 \chi_2)(\lambda)(J^{\delta k}_\psi \rho_1 \rho_2)(\lambda)},
\]
which leads to the desired inequality (3.8).

**Corollary 3.7** For \(k, \lambda, \rho, \delta > 0\), and \(U\) and \(V\) being two positive integrable functions defined on \([0, \infty)\) such that inequality (3.6) holds for \(\xi \in [0, \lambda]\), we have
\[
\frac{J^{\delta k}_\psi U^2(\lambda) J^{\delta k}_\psi V^2(\lambda)}{(J^{\delta k}_\psi U(\lambda) J^{\delta k}_\psi V(\lambda))^2} \leq \frac{\Gamma(k+1) \Gamma(k+\delta)}{4(\psi(\lambda))^{\frac{2k}{2}} + \frac{\sqrt{\frac{q^r}{QR} + \sqrt{\frac{QR}{q^r}}}^2}{}.\]

**Corollary 3.8** Let \(k = 1\). Then Lemma 3.6 leads to a new inequality for generalized Riemann–Liouville fractional integral as follows:
\[
\frac{J^k_\psi \rho_1 \rho_2(\lambda)J^{\delta k}_\psi \chi_1 \chi_2(\lambda)J^k_\psi U^2(\lambda) J^{\delta k}_\psi V^2(\lambda)}{(J^k_\psi \rho_1 \rho_2(\lambda) J^{\delta k}_\psi \chi_1 \chi_2(\lambda) + J^k_\psi \rho_1 U(\lambda) J^{\delta k}_\psi \rho_2 V(\lambda))^2} \leq \frac{1}{4}. \tag{3.11}
\]

**Corollary 3.9** Let \(\psi(\lambda) = \lambda\). Then Lemma 3.6 leads to a new inequality for \(k\)-fractional integral as follows:
\[
\frac{J^k_\psi \rho_1 \rho_2(\lambda)J^{\delta k}_\psi \chi_1 \chi_2(\lambda)J^k_\psi U^2(\lambda) J^{\delta k}_\psi V^2(\lambda)}{(J^k_\psi \rho_1 \rho_2(\lambda) J^{\delta k}_\psi \chi_1 \chi_2(\lambda) + J^k_\psi \rho_1 U(\lambda) J^{\delta k}_\psi \rho_2 V(\lambda))^2} \leq \frac{1}{4}. \tag{3.12}
\]

**Remark 3.10** If \(\psi(\lambda) = \lambda\) and \(k = 1\), then Lemma 3.6 reduces to Lemma 3.3 of [67].

**Theorem 3.11** Let \(k, \lambda, \rho, \delta > 0\), and \(U\), \(V\), \(\rho_1\), \(\rho_2\), \(\chi_1\), and \(\chi_2\) be six positive integrable functions defined on \([0, \infty)\) such that (3.1) holds for all \(\xi \in [0, \lambda]\). Then we have
\[
(J^k_\psi \rho_2 U)(\lambda)(J^{\delta k}_\psi \chi_1 V)(\lambda) \geq (J^k_\psi \rho_1 U^2(\lambda) J^{\delta k}_\psi V^2(\lambda). \tag{3.13}
\]

**Proof** It follows from (3.1) that
\[
\frac{1}{k I^k(\zeta)} \int_0^\lambda \psi'(\zeta)(\psi(\lambda) - \psi(\zeta))^{\frac{d}{2} - 1} \rho_2(\zeta) \chi_1(\zeta) U(\xi) V(\xi) d\zeta
\]
\[
\geq \frac{1}{k \Gamma(k)} \int_0^\lambda \Psi'(|\Psi(\lambda) - \Psi(\zeta)|)^{\frac{k}{\rho}} U^2(\zeta) d\zeta,
\]
which implies
\[
\mathcal{J}_\Psi^{\rho \lambda}(\frac{\rho U\mathcal{V}}{\chi_1})(\lambda) \geq \mathcal{J}_\Psi^{\rho \lambda} V^2(\lambda). \tag{3.14}
\]
Analogously, we obtain
\[
\geq \frac{1}{k \Gamma(k)} \int_0^\lambda \Psi'(\eta)(\Psi(\eta) - \Psi(\eta))^\frac{k}{\rho_1} \frac{\chi_2(\eta) U V d\eta}{\rho_1(\eta)}
\]
from which one has
\[
\mathcal{J}_\Psi^{\rho \lambda}(\frac{\rho_2 U\mathcal{V}}{\rho_1})(\lambda) \geq \mathcal{J}_\Psi^{\rho \lambda} V^2(\lambda). \tag{3.15}
\]
Multiplying (3.14) and (3.15), we get the desired inequality (3.13).

\[\square\]

**Corollary 3.12** For \(k, \lambda, \varrho, \delta > 0\), and \(U\) and \(V\) being two positive integrable functions defined on \([0, \infty)\) such that (3.6) holds for all \(\zeta \in [0, \lambda]\), we have
\[
\frac{\mathcal{J}_\Psi^{\rho \lambda} U^2(\lambda) \mathcal{J}_\Psi^{\rho \lambda} V^2(\lambda)}{\mathcal{J}_\Psi^{\rho \lambda} U V(\lambda) \mathcal{J}_\Psi^{\rho \lambda} V^2(\lambda)} \leq \frac{QR}{qr}.
\]

**Corollary 3.13** If \(k = 1\), then Theorem 3.11 gives the following new result for generalized Riemann–Liouville fractional integral:
\[
\mathcal{J}_\Psi^\rho(\frac{\rho U\mathcal{V}}{\chi_1})(\lambda) \mathcal{J}_\Psi^{\rho \lambda}(\frac{\rho_2 U\mathcal{V}}{\rho_1})(\lambda) \geq \mathcal{J}_\Psi^\rho U^2(\lambda) \mathcal{J}_\Psi^{\rho \lambda} V^2(\lambda).
\]

**Corollary 3.14** Let \(\Psi(\lambda) = \lambda\). Then Theorem 3.11 leads to the following new result for Riemann–Liouville \(k\)-fractional integral:
\[
\mathcal{J}_\Psi^{\rho \lambda}(\frac{\rho U\mathcal{V}}{\chi_1})(\lambda) \mathcal{J}_\Psi^{\rho \lambda}(\frac{\rho_2 U\mathcal{V}}{\rho_1})(\lambda) \geq \mathcal{J}_\Psi^{\rho \lambda} U^2(\lambda) \mathcal{J}_\Psi^{\rho \lambda} V^2(\lambda).
\]

**Remark 3.15** If \(\Psi(\lambda) = \lambda\) and \(K = 1\), then Theorem 3.11 reduces to Lemma 3.4 of [67].

4 **Pólya–Szegö type inequalities involving the generalized \(k\)-fractional integrals**

In this section, we present several Čebyšev type inequalities for generalized \(k\)-fractional integrals defined in (2.3) and (2.4).

**Theorem 4.1** Let \(k, \lambda, \varrho > 0\), and \(U\) and \(V\) be two integrable and synchronous functions on \([0, \infty)\). Then one has
\[
(\mathcal{J}_\Psi^{\rho \lambda} U V(\lambda)) \geq \frac{\Gamma_k(\varrho + K)}{(\Psi(\lambda))^{\frac{k}{K}}} (\mathcal{J}_\Psi^{\rho \lambda} U)(\mathcal{J}_\Psi^{\rho \lambda} V)(\lambda).
\]
Proof. It follows from the synchronism of the functions $\mathcal{U}$ and $\mathcal{V}$ on the interval $[0, \infty)$ that

$$\mathcal{U}(r)\mathcal{V}(r) + \mathcal{U}(s)\mathcal{V}(s) \geq \mathcal{U}(r)\mathcal{V}(s) + \mathcal{U}(s)\mathcal{V}(r). \quad (4.1)$$

Multiplying both sides of inequality (4.1) by

$$\frac{1}{k \Gamma_k(q)} \Psi'(r)(\Psi(\lambda) - \Psi(r))^\frac{q}{k-1}$$

for $\lambda \in \mathbb{R}$ gives

$$\frac{1}{k \Gamma_k(q)} \Psi'(r)(\Psi(\lambda) - \Psi(r))^\frac{q}{k-1} \mathcal{U}(r)\mathcal{V}(r) + \mathcal{U}(s)\mathcal{V}(s) \frac{1}{k \Gamma_k(q)} \Psi'(r)(\Psi(\lambda) - \Psi(r))^\frac{q}{k-1}$$

$$\geq \mathcal{V}(s) \frac{1}{k \Gamma_k(q)} \Psi'(r)(\Psi(\lambda) - \Psi(r))^\frac{q}{k-1} \mathcal{U}(r)\mathcal{V}(r) + \mathcal{U}(s) \frac{1}{k \Gamma_k(q)} \Psi'(r)(\Psi(\lambda) - \Psi(r))^\frac{q}{k-1} \mathcal{V}(r).$$

Integrating the above inequality with respect to $r$ over $(0, \lambda)$ leads to

$$\frac{1}{k \Gamma_k(q)} \int_0^\lambda \Psi'(r)(\Psi(\lambda) - \Psi(r))^\frac{q}{k-1} \mathcal{U}(r)\mathcal{V}(r) \, dr$$

$$+ \mathcal{U}(s)\mathcal{V}(s) \frac{1}{k \Gamma_k(q)} \int_0^\lambda \Psi'(r)(\Psi(\lambda) - \Psi(r))^\frac{q}{k-1} \, dr$$

$$\geq \mathcal{V}(s) \frac{1}{k \Gamma_k(q)} \int_0^\lambda \Psi'(r)(\Psi(\lambda) - \Psi(r))^\frac{q}{k-1} \mathcal{U}(r) \, dr$$

$$+ \mathcal{U}(s) \frac{1}{k \Gamma_k(q)} \int_0^\lambda \Psi'(r)(\Psi(\lambda) - \Psi(r))^\frac{q}{k-1} \mathcal{V}(r) \, dr.$$

Therefore, we get

$$(\mathcal{J}_\psi^\alpha \mathcal{U}\mathcal{V})(\lambda) + \mathcal{U}(s)\mathcal{V}(s) \frac{1}{k \Gamma_k(q)} \int_0^\lambda \Psi'(r)(\Psi(\lambda) - \Psi(r))^\frac{q}{k-1} \, dr$$

$$\geq \mathcal{V}(s)(\mathcal{J}_\psi^\alpha \mathcal{U})(\lambda) + \mathcal{U}(s)(\mathcal{J}_\psi^\alpha \mathcal{V})(\lambda)$$

and

$$(\mathcal{J}_\psi^\alpha \mathcal{U}\mathcal{V})(\lambda) + \mathcal{U}(s)\mathcal{V}(s) \frac{(\Psi(\lambda))^\frac{q}{k}}{\Gamma_k(q + k)}$$

$$\geq \mathcal{V}(s)(\mathcal{J}_\psi^\alpha \mathcal{U})(\lambda) + \mathcal{U}(s)(\mathcal{J}_\psi^\alpha \mathcal{V})(\lambda), \quad (4.2)$$

where

$$\frac{1}{k \Gamma_k(q)} \int_0^\lambda \Psi'(r)(\Psi(\lambda) - \Psi(r))^\frac{q}{k-1} \, dr = \frac{(\Psi(\lambda))^\frac{q}{k}}{\Gamma_k(q + k)}.$$

Multiplying both sides of inequality (4.2) by

$$\frac{1}{k \Gamma_k(q)} \Psi'(s)(\Psi(\lambda) - \Psi(s))^\frac{q}{k-1} \quad (\lambda \in \mathbb{R})$$
leads to the conclusion that
\[
(J^0 \mathcal{U}) (\lambda) \frac{1}{k \Gamma_k (q)} \Psi' (s) (\Psi (\lambda) - \Psi (s))^{\frac{q}{q-1}} \\
+ \frac{1}{k \Gamma_k (q)} \Psi' (s) (\Psi (\lambda) - \Psi (s))^{\frac{q}{q-1}} \mathcal{U} (s) \frac{(\Psi (\lambda))^{\frac{q}{q-1}}}{\Gamma_k (q + k)} \\
\geq (J^{q,k} \mathcal{U}) (\lambda) \frac{1}{k \Gamma_k (q)} \Psi' (s) (\Psi (\lambda) - \Psi (s))^{\frac{q}{q-1}} \mathcal{V} (s) \\
+ (J^{q,k} \mathcal{V}) (\lambda) \frac{1}{k \Gamma_k (q)} \Psi' (s) (\Psi (\lambda) - \Psi (s))^{\frac{q}{q-1}} \mathcal{U} (s).
\]

Integrating the above inequality over \((0, \lambda)\) reveals
\[
(J^{q,k} \mathcal{U}) (\lambda) \frac{1}{k \Gamma_k (q)} \int_0^\lambda \Psi' (s) (\Psi (\lambda) - \Psi (s))^{\frac{q}{q-1}} ds \\
+ \frac{1}{k \Gamma_k (q)} \int_0^\lambda \Psi' (s) (\Psi (\lambda) - \Psi (s))^{\frac{q}{q-1}} \mathcal{U} (s) ds \frac{(\Psi (\lambda))^{\frac{q}{q-1}}}{\Gamma_k (q + k)} \\
\geq (J^{q,k} \mathcal{U}) (\lambda) \frac{1}{k \Gamma_k (q)} \int_0^\lambda \Psi' (s) (\Psi (\lambda) - \Psi (s))^{\frac{q}{q-1}} \mathcal{V} (s) ds \\
+ (J^{q,k} \mathcal{V}) (\lambda) \frac{1}{k \Gamma_k (q)} \int_0^\lambda \Psi' (s) (\Psi (\lambda) - \Psi (s))^{\frac{q}{q-1}} \mathcal{U} (s) ds.
\]

Therefore,
\[
\frac{(\Psi (\lambda))^{\frac{q}{q-1}}}{\Gamma_k (q + k)} (J^{q,k} \mathcal{U}) (\lambda) + (J^{q,k} \mathcal{U}) (\lambda) \frac{(\Psi (\lambda))^{\frac{q}{q-1}}}{\Gamma_k (q + k)} \\
\geq (J^{q,k} \mathcal{U}) (\lambda) (J^{q,k} \mathcal{V}) (\lambda) + (J^{q,k} \mathcal{V}) (\lambda) (J^{q,k} \mathcal{U}) (\lambda).
\]

This completes the proof of Theorem 4.1.

\[\square\]

**Corollary 4.2** Let \( k = 1 \). Then Theorem 4.1 leads to a new result for generalized Riemann–Liouville fractional integrals as follows:
\[
(J^0 \mathcal{U}) (\lambda) \geq \frac{\Gamma (q + 1)}{(\Psi (\lambda))^q} (J^q \mathcal{U}) (\lambda) (J^0 \mathcal{V}) (\lambda).
\]

**Corollary 4.3** If \( \Psi (\lambda) = \lambda \), then Theorem 4.1 provides a new inequality for \( k \)-fractional integral as follows:
\[
(J^{q,k} \mathcal{U}) (\lambda) \geq \frac{\Gamma_k (q + k)}{\lambda^{\frac{q}{q-1}}} (J^{q,k} \mathcal{U}) (\lambda) (J^{q,k} \mathcal{V}) (\lambda).
\]

**Corollary 4.4** Let \( \Psi (\lambda) = \lambda \) and \( k = 1 \). Then Theorem 4.1 leads to a new result for Riemann–Liouville fractional integral as follows:
\[
(J^0 \mathcal{U}) (\lambda) \geq \frac{\Gamma (q + 1)}{\lambda^{q}} (J^q \mathcal{U}) (\lambda) (J^0 \mathcal{V}) (\lambda).
\]
**Theorem 4.5** Let $k, \lambda, \varrho, \delta > 0$, and $U$ and $V$ be two integrable and synchronous functions on $[0, \infty)$. Then

\[
\frac{(J^\alpha_U U V)(\lambda)}{\Gamma_k(\delta + k)} + \frac{(\Psi(\lambda))^{\frac{1}{\alpha}}(J^\alpha_U U V)(\lambda)}{\Gamma_k(\varrho + k)} \\
\geq (J^\alpha_U U)(\lambda)(J^\alpha_U V)(\lambda) + (J^\alpha_U V)(\lambda)(J^\alpha_U U)(\lambda).
\]

**Proof** Multiplying both sides of inequality (4.2) by

\[
\frac{1}{k \Gamma_k(\delta)} \Psi'(s)(\Psi(\lambda) - \Psi(s))^{\frac{1}{\alpha} - 1} (\lambda \in \mathbb{R})
\]

gives

\[
(J^\alpha_U U V)(\lambda) \frac{1}{k \Gamma_k(\delta)} \Psi'(s)(\Psi(\lambda) - \Psi(s))^{\frac{1}{\alpha} - 1} \\
+ \frac{1}{k \Gamma_k(\delta)} \Psi'(s)(\Psi(\lambda) - \Psi(s))^{\frac{1}{\alpha} - 1} U(s)V(s) \frac{(\Psi(\lambda))^{\frac{1}{\alpha}}}{\Gamma_k(\varrho + k)} \\
\geq (J^\alpha_U U)(\lambda) \frac{1}{k \Gamma_k(\delta)} \Psi'(s)(\Psi(\lambda) - \Psi(s))^{\frac{1}{\alpha} - 1} V(s) \\
+ (J^\alpha_U V)(\lambda) \frac{1}{k \Gamma_k(\delta)} \Psi'(s)(\Psi(\lambda) - \Psi(s))^{\frac{1}{\alpha} - 1} U(s).
\]

Integrating both sides of the above inequality with respect to $s$ over $(0, \lambda)$ leads to

\[
\frac{(J^\alpha_U U V)(\lambda)}{\Gamma_k(\delta + k)} \int_0^\lambda \Psi'(s)(\Psi(\lambda) - \Psi(s))^{\frac{1}{\alpha} - 1} ds \\
+ \frac{(\Psi(\lambda))^{\frac{1}{\alpha}}}{\Gamma_k(\varrho + k)} \int_0^\lambda \Psi'(s)(\Psi(\lambda) - \Psi(s))^{\frac{1}{\alpha} - 1} U(s)V(s) ds \\
\geq \frac{(J^\alpha_U U)(\lambda)}{k \Gamma_k(\delta)} \int_0^\lambda \Psi'(s)(\Psi(\lambda) - \Psi(s))^{\frac{1}{\alpha} - 1} V(s) ds \\
+ \frac{(J^\alpha_U V)(\lambda)}{k \Gamma_k(\delta)} \int_0^\lambda \Psi'(s)(\Psi(\lambda) - \Psi(s))^{\frac{1}{\alpha} - 1} U(s) ds.
\]

Therefore,

\[
\frac{(J^\alpha_U U V)(\lambda)(\Psi(\lambda))^{\frac{1}{\alpha}}}{\Gamma_k(\delta + k)} + \frac{(\Psi(\lambda))^{\frac{1}{\alpha}}(J^\alpha_U U V)(\lambda)}{\Gamma_k(\varrho + k)} \\
\geq (J^\alpha_U U)(\lambda)(J^\alpha_U V)(\lambda) + (J^\alpha_U V)(\lambda)(J^\alpha_U U)(\lambda),
\]

which is the proof of Theorem 4.5. □

**Remark 4.6** Let $\varrho = \delta$. Then Theorem 4.5 becomes Theorem 4.1.
Corollary 4.7 Let $k = 1$. Then Theorem 4.5 provides a new result for generalized Riemann–Liouville fractional integrals as follows:

\[
\frac{(J_\psi^0 U)(\lambda)(\Psi(\lambda))^\delta}{\Gamma(\delta + 1)} + \frac{(\Psi(\lambda))^\theta (J_\psi^2 U)(\lambda)}{\Gamma(\theta + 1)} \\
\geq (J_\psi^0 U)(\lambda)(J_\psi^2 V)(\lambda) + (J_\psi^0 V)(\lambda)(J_\psi^2 U)(\lambda).
\]

Corollary 4.8 If $\Psi(\lambda) = \lambda$ and $k = 1$, then Theorem 4.5 gives a new result for Riemann–Liouville fractional integral as follows:

\[
\frac{\lambda^\delta (J_\psi^0 U)(\lambda)}{\Gamma(\delta + 1)} + \frac{\lambda^\theta (J_\psi^2 U)(\lambda)}{\Gamma(\theta + 1)} \\
\geq (J_\psi^0 U)(\lambda)(J_\psi^2 V)(\lambda) + (J_\psi^0 V)(\lambda)(J_\psi^2 U)(\lambda).
\]

Theorem 4.9 Let $k, \lambda, \sigma > 0$, $\sigma_1, \sigma_2 \in \mathbb{R}$ with $\sigma_1 < \sigma_2$, and $U_j (1 \leq j \leq \sigma)$ be a real-valued increasing function on $[\sigma_1, \sigma_2]$. Then

\[
\left( J_\psi^0 \prod_{j=1}^\sigma U_j \right)(\lambda) \geq \left[ \frac{\Gamma_{k}(\rho + k)}{\Gamma(\lambda)^{\frac{k}{\rho}}} \right] \prod_{j=1}^{\gamma} (J_\psi^0 U_j)(\lambda). \tag{4.3}
\]

**Proof** We use mathematical induction on $\gamma \in \mathbb{N}$ to prove Theorem 4.9. We clearly see that inequality (4.3) holds for $\gamma = 1$.

For $\gamma = 2$, since $U_1, U_2$ are increasing, we have

\[
|U_1(\lambda) - U_1(\omega), U_2(\lambda) - U_2(\omega)| \geq 0.
\]

Note that the left-hand side of inequality (4.3) for $\gamma = 2$ is the same as that of Theorem 4.1. Therefore, inequality (4.3) also holds for $\gamma = 2$.

Suppose that inequality (4.3) holds for some $\gamma \geq 2$. We observe that $U = \prod_{j=1}^{\gamma} U_j$ is increasing due to $U_j$ is increasing. Let $V = U_{\gamma+1}$. Then applying the case $\gamma = 2$ to the functions $U$ and $V$ produces

\[
\left( J_\psi^0 \prod_{j=1}^\gamma U_j U_{\gamma+1} \right)(\lambda) \geq \left[ \frac{\Gamma_{k}(\rho + k)}{\Gamma(\lambda)^{\frac{k}{\rho}}} \right] \left( J_\psi^0 \prod_{j=1}^\gamma U_j \right) (J_\psi^0 U_{\gamma+1})(\lambda)
\geq \left[ \frac{\Gamma_{k}(\rho + k)}{\Gamma(\lambda)^{\frac{k}{\rho}}} \right] \prod_{j=1}^{\gamma+1} (J_\psi^0 U_j)(\lambda),
\]

in which the induction hypothesis for $\gamma$ is used inside the deduction of second inequality. The proof of Theorem 4.9 is completed. \hfill \Box

Corollary 4.10 Let $k = 1$. Then Theorem 4.9 leads to the following new result for generalized Riemann–Liouville fractional integral:

\[
\left( J_\psi^0 \prod_{j=1}^\gamma U_j \right)(\lambda) \geq \left[ \frac{\Gamma(\rho + 1)}{\Gamma(\lambda)^{\frac{1}{\rho}}} \right] \prod_{j=1}^{\gamma} (J_\psi^0 U_j)(\lambda).
\]
Corollary 4.11 If $\Psi(\lambda) = \lambda$, then Theorem 4.9 leads to a new result for $k$-fractional integral as follows:

$$
\left(\mathcal{J}^\alpha \prod_{j=1}^{\gamma} \mathcal{U}_j\right)(\lambda) \geq \left[\frac{\Gamma(\alpha + k)}{\lambda^\alpha}\right]^{\gamma-1} \prod_{j=1}^{\gamma} \left(\mathcal{J}^\alpha \mathcal{U}_j\right)(\lambda). 
$$

(4.4)

Corollary 4.12 Let $\Psi(\lambda) = \lambda$ and $k = 1$. Then Theorem 4.9 provides a new result for Riemann–Liouville fractional integral as follows:

$$
\left(\mathcal{J}^\alpha \prod_{j=1}^{\gamma} \mathcal{U}_j\right)(\lambda) \geq \left[\frac{\Gamma(\alpha + 1)}{\lambda^\alpha}\right]^{\gamma-1} \prod_{j=1}^{\gamma} \left(\mathcal{J}^\alpha \mathcal{U}_j\right)(\lambda).
$$

(4.5)

Theorem 4.13 Let $k, \lambda, \varrho > 0$, $\mathcal{U}$ and $\mathcal{V}$ be two positive functions defined on $[0, \infty)$ such that $\mathcal{U}$ is increasing and $\mathcal{V}$ is differentiable, and $\vartheta = \inf_{\mu \in [0, \infty)} \mathcal{V}'(\mu)$. Then one has

$$
\left(\mathcal{J}^\alpha \mathcal{U}\mathcal{V}\right)(\lambda) \geq \frac{\Gamma(\alpha + k)}{(\Psi(\lambda))^{\varphi}} \left(\mathcal{J}^\alpha \mathcal{U}\right)(\lambda)\left(\mathcal{J}^\alpha \mathcal{V}\right)(\lambda)
$$

$$
- \frac{\vartheta \lambda(\Psi(\lambda))^{\varphi}}{\Gamma(\alpha + k)} \left(\mathcal{J}^\alpha \mathcal{U}\right)(\lambda) + \vartheta \left(\mathcal{J}^\alpha \mathcal{U}\right)(\lambda),
$$

where $I(\lambda)$ is the identity mapping.

Proof Let $\mathcal{U}\mathcal{V} = \mathcal{V} - \vartheta \lambda$ and $\mathcal{I}(\lambda) = \vartheta \lambda$. Then we clearly see that $\mathcal{U}\mathcal{V}$ is differentiable and increasing on $[0, \infty)$, and from the proof of Theorem 4.9 we know that

$$
\left(\mathcal{J}^\alpha \mathcal{U}\mathcal{V}\right)(\lambda) = \frac{\Gamma(\alpha + k)}{(\Psi(\lambda))^{\varphi}} \left(\mathcal{J}^\alpha \mathcal{U}\right)(\lambda)\left(\mathcal{J}^\alpha \mathcal{V}\right)(\lambda)
$$

$$
- \frac{\vartheta \lambda(\Psi(\lambda))^{\varphi}}{\Gamma(\alpha + k)} \left(\mathcal{J}^\alpha \mathcal{U}\right)(\lambda) + \vartheta \left(\mathcal{J}^\alpha \mathcal{U}\right)(\lambda),
$$

(4.6)

where

$$
\left(\mathcal{J}^\alpha \mathcal{U}\mathcal{V}\right)(\lambda) = \left(\mathcal{J}^\alpha \mathcal{U}\mathcal{V}\right)(\lambda) - \vartheta \left(\mathcal{J}^\alpha \mathcal{U}\right)(\lambda)
$$

(4.7)

and

$$
\left(\mathcal{J}^\alpha \mathcal{V}\right)(\lambda) = \frac{\vartheta \lambda(\Psi(\lambda))^{\varphi}}{\Gamma(\alpha + k)} .
$$

(4.8)

Substituting (4.7) and (4.8) into (4.6) leads to the desired result. \hfill \Box

Corollary 4.14 Let $k = 1$. Then Theorem 4.13 leads to a new result for generalized Riemann–Liouville fractional integral as follows:

$$
\left(\mathcal{J}^\alpha \mathcal{U}\mathcal{V}\right)(\lambda) \geq \frac{\Gamma(\alpha + 1)}{(\Psi(\lambda))^{\varphi}} \left(\mathcal{J}^\alpha \mathcal{U}\right)(\lambda)\left(\mathcal{J}^\alpha \mathcal{V}\right)(\lambda)
$$

(4.9)
Corollary 4.15 If $\Psi(\lambda) = \lambda$, Theorem 4.13 provides the following new result for $k$-fractional integral:

$$
\left( J_k^\varphi U V \right)(\lambda) \geq \frac{\Gamma'\left(\varphi + k\right)}{\lambda^k}\cdot \left( J_k^\varphi U \right)(\lambda)\cdot \left( J_k^\varphi V \right)(\lambda) - \frac{\varphi(\lambda)^{k+1}}{\Gamma'\left(\varphi + k\right)}\cdot \left( J_k^\varphi U \right)(\lambda) + \varphi \left( J_k^\varphi U \right)(\lambda).
$$

Corollary 4.16 Let $\Psi(\lambda) = \lambda$ and $k = 1$. Then Theorem 4.13 leads to a new inequality for Riemann–Liouville fractional integral as follows:

$$
\left( J^\varphi U V \right)(\lambda) \geq \frac{\Gamma'(\varphi + 1)}{\lambda^\varphi}\cdot \left( J^\varphi U \right)(\lambda)\cdot \left( J^\varphi V \right)(\lambda) - \frac{\varphi(\lambda)^{\varphi+1}}{\Gamma'(\varphi + 1)}\cdot \left( J^\varphi U \right)(\lambda) + \varphi \left( J^\varphi U \right)(\lambda).
$$

5 Conclusion

In the article, we have established some new Pólya–Szegö and Čebyšev-type inequalities for two synchronous functions via generalized $k$-fractional integrals. Our obtained results are very general and can be specialized to discover numerous interesting fractional integral inequalities, and our approach may lead to a lot of follow-up research. Furthermore, they are expected to find some applications for establishing the uniqueness of solutions in fractional boundary value problems of the fractional partial differential equations.

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