The emergence of big data has led to so-called convergence complexity analysis, which is the study of how Markov chain Monte Carlo (MCMC) algorithms behave as the sample size, $n$, and/or the number of parameters, $p$, in the underlying data set increase. This type of analysis is often quite challenging, in part because existing results for fixed $n$ and $p$ are simply not sharp enough to yield good asymptotic results. One of the first convergence complexity results for an MCMC algorithm on a continuous state space is due to Yang and Rosenthal (2019), who established a mixing time result for a Gibbs sampler (for a simple Bayesian random effects model) that was introduced and studied by Rosenthal (Stat Comput 6:269–275, 1996). The asymptotic behavior of the spectral gap of this Gibbs sampler is, however, still unknown. We use a recently developed simulation technique (Qin et al. Electron J Stat 13:1790–1812, 2019) to provide substantial numerical evidence that the gap is bounded away from 0 as $n \to \infty$. We also establish a pair of rigorous convergence complexity results for two different Gibbs samplers associated with a generalization of the random effects model considered by Rosenthal (Stat Comput 6:269–275, 1996). Our results show that, under a strong growth condition, the spectral gaps of these Gibbs samplers converge to 1 as the sample size increases.

**Keywords** Convergence rate · Geometric ergodicity · High-dimensional inference · Monte Carlo · Quantitative bound · Spectral gap · Total variation distance · Trace-class operator · Wasserstein distance

**1 Introduction**

Markov chain Monte Carlo (MCMC) is one of the most commonly used tools in modern Bayesian statistics. It is well known that the practical performance of an MCMC algorithm is
directly related to the speed at which the underlying Markov chain converges to its stationary distribution. Over the last three decades, a great deal of work has been done to establish the convergence properties of many different practical Monte Carlo Markov chains. Recently, with the emergence of big data, interest has shifted away from the analysis of individual Markov chains (for fixed data sets), and towards the study of how algorithms behave as the sample size, \( n \), and/or the number of parameters, \( p \), in the data set increase. This type of study, which is called convergence complexity analysis, is often quite challenging, in part because the dimension of the Markov chain typically increases as \( n \) and/or \( p \) grow, and existing results for fixed \( n \) and \( p \) are simply not sharp enough to yield good asymptotic results (see, e.g., Rajaratnam and Sparks 2015). Despite these difficulties, there has been a flurry of recent work on convergence complexity, which includes Rajaratnam and Sparks (2015), Yang et al. (2016), Qin and Hobert (2019a), Yang and Rosenthal (2019), Qin and Hobert (2019b), and Ekvall and Jones (2019). Some of these papers analyze mixing times, while others focus on convergence rates. In this paper, we study the asymptotic behavior of the spectral gaps of Gibbs samplers for a family of simple Bayesian random effects models.

One of the first convergence complexity results for MCMC (on a continuous state space) was developed by Yang and Rosenthal (2019), who studied a Gibbs sampler that was introduced and analyzed by Rosenthal (1996). Consider the following simple random effects model:

\[
y_i = \theta_i + e_i, \quad i = 1, \ldots, n, (1)
\]

where the components of \( \mathbf{\theta} = (\theta_1, \ldots, \theta_n)^T \) are iid \( N(\mu, A) \), the components of \( \mathbf{e} = (e_1, \ldots, e_n)^T \) are iid \( N(0, V) \), and \( \mathbf{\theta} \) is independent of \( \mathbf{e} \). The error variance, \( V \), is assumed known. We take \( A \) and \( \mu \) to be a priori independent with

\[
\pi(\mu) \propto 1 \text{ and } A \sim IG(a, b), (2)
\]

where \( a, b > 0 \), and we say \( X \sim IG(a, b) \) if its density is proportional to \( x^{-a-1} e^{-b/x} I_{(0, \infty)}(x) \). Denote the resulting posterior density as \( \pi(\mathbf{\theta}, \mu, A \mid y) \), where \( y = (y_1, \ldots, y_n)^T \). Consider a two-block Gibbs sampler with Markov transition density (Mtd) given by

\[
k(\mu', A', \mathbf{\theta} \mid \mu, A, \mathbf{\theta}) = \pi(\mu' \mid \mu', A', y)\pi(\mu', A' \mid \theta, y), (3)
\]

and let \( \{\mathbf{\theta}_m, (\mu_m, A_m)\}_{m=0}^\infty \) denote the corresponding \((n + 2)\)-dimensional Markov chain. (See Section 3 for the specific forms of the conditional densities.) Rosenthal (1996) used drift & minorization (d&m) conditions to study the convergence properties of this chain in the case where \( n \) is fixed. Unfortunately, as is the case for many d&m-based results, Rosenthal’s (1996) bounds are not sharp enough to provide useful information about the behavior of the chain as \( n \to \infty \). Rosenthal’s (2019) developed a modified version of Rosenthal (1995) general bound (on the total variation distance to stationarity), and used it to establish a convergence complexity result concerning the mixing time of the Gibbs sampler. In particular, they proved that the number of iterations required to get the total variation distance to stationarity below a prespecified threshold is constant as \( n \to \infty \). A precise statement of their result is given in Section 3.

While Yang and Rosenthal’s (2019) mixing time result is certainly a step in the right direction, it doesn’t provide any concrete information about the behavior of the spectral gap of the Gibbs sampler as \( n \to \infty \). One of our main contributions in this paper is to provide substantial numerical evidence that the spectral gap remains strictly positive as \( n \to \infty \). Our analysis centers on the marginal Markov chain, \( \{\mathbf{\theta}_m\}_{m=0}^\infty \), which is known to converge at the same rate as the full Gibbs chain, \( \{\mathbf{\theta}_m, (\mu_m, A_m)\}_{m=0}^\infty \) (Diaconis et al. (2008), Roberts and Rosenthal (2001), and Román et al. (2014)). Because \( \{\mathbf{\theta}_m\}_{m=0}^\infty \) is the marginal of a
two-block Gibbs chain, the corresponding Markov operator is self-adjoint and positive (Liu et al. 1994). We prove that this operator is also trace-class, which allows us to apply the simulation method of Qin et al. (2019) to estimate its spectral gap. We perform a large scale numerical study on seven different simulated data sets, each of size $n = 10^7$, to gain an understanding of how the spectral gap behaves as $n \to \infty$. Our results suggest that the gap is bounded away from zero as $n \to \infty$.

Our second contribution is a pair of rigorous convergence complexity results for Gibbs samplers associated with a generalization of Eq. 1. Indeed, consider a version of Eq. 1 with replicates:

$$y_{ij} = \theta_i + e_{ij}, \ i = 1, \ldots, n, \ j = 1, \ldots, r,$$  \hspace{1cm} (4)

where the components of $\theta$ are iid $N(\mu, A)$, the components of $e = (e_{11}, \ldots, e_{nr})^T$ are iid $N(0, V)$, $\theta$ is independent of $e$, and $V$ is known. We consider two different priors. The first is Eq. 2, and in the second, the flat prior on $\mu$ is replaced with a normal shrinkage prior whose variance decreases as $n, r \to \infty$. A recently developed method for studying convergence rates via Wasserstein distance (Qin and Hobert 2019b) is employed to analyze the corresponding two-block Gibbs samplers. We prove that, in each case, under a strong assumption about the rate at which $r = r(n)$ grows with $n$, not only is the spectral gap bounded away from 0 as $n \to \infty$, it actually converges to 1.

We made several serious attempts to use the Wasserstein-based techniques mentioned above to study (Rosenthal 1996) chain, but we were unable to make any headway. Thus, one surprise that comes out of our study is that it’s apparently easier to analyze the Gibbs samplers associated with the more complex likelihood (4) than it is to analyze Rosenthal’s (1996) chain. So the Gibbs sampler that Yang & Rosenthal chose to study (presumably because of its simple form) turns out to be a relatively tough nut to crack.

The remainder of the paper has the following organization. Section 2 provides the requisite background on Markov chain convergence in both total variation and Wasserstein distances. The Gibbs sampler studied by Rosenthal (1996) and Yang and Rosenthal (2019) is the topic of Section 3. In Section 4, we analyze the two Gibbs samplers associated with the likelihood (4). Section 5 contains some discussion. All of the proofs are relegated to the Appendix.

2 Markov Chain Background

Suppose that $X \subset \mathbb{R}^q$ and let $\mathcal{B}$ denote its Borel $\sigma$-algebra. Let $K : X \times \mathcal{B} \to [0, 1]$ be a Markov transition kernel (Mtk). For any $m \in \mathbb{N} := \{1, 2, 3, \ldots\}$, let $K^m$ be the $m$-step transition kernel, so that $K^1 = K$. For any probability measure $\mu : \mathcal{B} \to [0, 1]$ and measurable function $f : X \to \mathbb{R}$, denote $\int_X f(x) \mu(dx)$ by $\mu f$, $\int_X \mu(dx) K^m(x, \cdot)$ by $\mu K^m(\cdot)$, and $\int_X K^m(\cdot, dx) f(x)$ by $K^m f(\cdot)$. We often write $K^m(\cdot)$ instead of $K^m(x, \cdot)$. Also, let $L^2(\mu)$ denote the set of measurable, real-valued functions on $X$ that are square integrable with respect to $\mu(dx)$.

Assume that the Markov chain corresponding to $K$ is Harris ergodic (irreducible, aperiodic, and positive Harris recurrent), so it converges to a unique stationary distribution, which we denote by $\Pi$. The goal of convergence analysis is to understand how fast $\mu K^m$ converges to $\Pi$ as $m \to \infty$ for a large class of $\mu$s. The difference between $\mu K^m$ and $\Pi$ is usually measured using the total variation distance, which is defined as follows. For two
probability measures on \((X, \mathcal{B}), \mu \) and \(v\), their total variation distance is

\[ d_{TV}(\mu, v) = \sup_{A \in \mathcal{B}} |\mu(A) - v(A)|. \]

The Markov chain defined by \(K\) is \textit{geometrically ergodic} if there exist \(\rho < 1\) and \(M : X \to [0, \infty)\) such that, for each \(x \in X\) and \(m \in \mathbb{N}\),

\[ d_{TV}(K^m_x, \Pi) \leq M(x)\rho^m. \tag{5} \]

Define the \textit{geometric convergence rate} of the chain as

\[ \rho_* = \inf \{\rho \in [0, 1] : \text{Eq. 5 holds for some } M : X \to [0, \infty)\}. \]

Clearly, the chain is geometrically ergodic if and only if \(\rho_* < 1\).

The space of functions \(L^2(\Pi)\) is a Hilbert space with inner product \(\langle f, g \rangle = \int_X f(x)g(x)\Pi(dx)\) and norm of \(f\) given by \(\sqrt{\langle f, f \rangle}\). The Mk \(K\) defines an operator \(K : L^2(\Pi) \to L^2(\Pi)\) that maps \(f \in L^2(\Pi)\) to \(Kf\). If \(K\) is reversible with respect to \(\Pi\), then the Markov operator \(K\) is self-adjoint, and the Markov chain defined by \(K\) is geometrically ergodic if and only if the operator possesses a spectral gap (Roberts and Tweedie (2001) and Roberts and Rosenthal (1997)). (For a nice overview of this theory, see Jerison 2016.) If, in addition to being self-adjoint, the Markov operator \(K\) is also positive and compact, then for every probability measure \(v : \mathcal{B} \to [0, 1]\) that is absolutely continuous with respect to \(\Pi\) and satisfies \(\int_X (dv/d\Pi)^2 d\Pi < \infty\), there exists a constant \(M_v < \infty\) such that

\[ d_{TV}(vK^m, \Pi) \leq M_v\lambda_*^m, \tag{6} \]

where \(\lambda_*\) is the second largest eigenvalue of \(K\). (In this context, the spectral gap is \(1 - \lambda_*\).) It’s also known that \(\lambda_* \leq \rho_*\) (Roberts and Rosenthal, 1997).

The standard method of developing upper bounds on \(\rho_*\) requires the construction of drift and minorization (d&m) conditions for the chain under study (Rosenthal 1995; Roberts and Rosenthal 2004; Baxendale 2005). It is well known the d&m-based methods are often overly conservative, especially in high-dimensional situations (see, e.g., Rajaratnam and Sparks 2015, Qin and Hobert 2020). There is mounting evidence suggesting that convergence complexity analysis becomes more tractable when total variation distance is replaced with an appropriate Wasserstein distance (see, e.g., Hairer et al. 2011, Durmus and Moulines 2015, Qin and Hobert 2019b). In the remainder of this section, we describe a method of bounding \(\rho_*\) via Wasserstein distance.

Let \(\phi(x, \cdot)\) denote the usual Euclidean distance on \(\mathbb{R}^d\), i.e., \(\phi(x, y) = \|x - y\|\), and assume that \((X, \phi)\) is a Polish metric space. For two probability measures on \((X, \mathcal{B}), \mu \) and \(v\), their Wasserstein distance is defined as

\[ d_W(\mu, v) = \inf_{\xi \in \tau(\mu, v)} \int_{X \times X} \|x - y\| \xi(dx, dy), \]

where \(\tau(\mu, v)\) is the set of all couplings of \(\mu\) and \(v\), that is, the set of all probability measures \(\xi(\cdot, \cdot)\) on \((X \times X, \mathcal{B} \times \mathcal{B})\) having marginals \(\mu\) and \(v\). One way to bound the Wasserstein distance between \(K^m_x\) and \(K^m_y\) is via coupling, and coupling is often achieved through random mappings, which we now describe. On a probability space \((\Omega, \mathcal{F}, P)\), let \(\theta : \Omega \to \Omega\) be a random element, and let \(\tilde{f} : X \times \Theta \to X\) be a Borel measurable function. Define \(f(x) = \tilde{f}(x, \theta)\) for all \(x \in X\). Then \(f\) is called a random mapping on \(X\). The evolution of a Markov chain can often be viewed as being governed by a random mapping. If \(f(x) \sim K_x(\cdot)\) for all \(x \in X\), then we say that \(f\) induces \(K\). For example, suppose that \(X = \mathbb{R}\) and \(K(x, dy) = (2\pi)^{-1/2} \exp\{-(y - x/2)^2/2\}dy\). Let \(Z\) be standard normal, and define \(\tilde{f}(x, Z) = x/2 + Z\). Then the random mapping \(f(x) = x/2 + Z\) induces \(K\).
Assuming that $f$ induces $K$, let $\{f_i\}_{i=1}^{\infty}$ be iid copies of $f$, and let $F_m = f_m \circ f_{m-1} \circ \cdots \circ f_1$ for $m \geq 1$. Then, for all $x, y \in X$, $\{(F_m(x), F_m(y))\}_{m=0}^{\infty}$ defines a Markov chain such that $(F_m(x), F_m(y)) \in \tau(K^m_x, K^m_y)$ for all $m \geq 1$. The following result is well known (see, e.g., Ollivier 2009).

**Proposition 1** Assume that $c(x) = f_x \|x - y\|K_x(dy) < \infty$ for all $x \in X$. Suppose that the random mapping $f$ induces $K$, and that there exists a $\gamma < 1$ such that, for every $x, y \in X$,

$$E\|f(x) - f(y)\| \leq \gamma \|x - y\|,$$

Then for each $x \in X$ and each $m \in \mathbb{N}$, we have

$$d_W(K^m_x, \Pi) \leq \frac{c(x)}{1-\gamma} \gamma^m.$$

The next result provides a connection between Wasserstein distance and total variation distance.

**Theorem 2** (Madras and Sezer 2010) Assume that $K_x(\cdot)$ has a density $k(\cdot | x)$ with respect to some dominating measure $\mu$ for all $x \in X$. If there exists a constant $C < \infty$ such that, for all $x, y \in X$,

$$\int_X \left| k(z | x) - k(z | y) \right| \mu(dz) \leq C\|x - y\|,$$

then, for all $m \in \{2, 3, 4, \ldots\}$, we have

$$d_{TV}(K^m_x, \Pi) \leq \frac{C}{2} d_W(K^{m-1}_x, \Pi).$$

Suppose that we are able to show that $d_W(K^m_x, \Pi) \leq M(x) \gamma^m$ where $\gamma \in [0, 1)$ and $M : X \rightarrow [0, \infty)$. Then, if the conditions in Theorem 2 are satisfied, we have $\rho_* \leq \gamma$. Finally, the following result provides a tractable upper bound for $E\|f(x) - f(y)\|$.

**Lemma 3** (Qin and Hobert 2019b) Assume that $X \subset \mathbb{R}^q$ is convex and that $f$ is a random mapping on $X$. Let $x, y \in X$ be fixed. Suppose that, with probability 1, $\frac{d}{dt} f(x + t(y - x))$, as a function of $t \in [0, 1]$, exists and is integrable. Then

$$E\|f(x) - f(y)\| \leq \sup_{t \in [0, 1]} E \left\| \frac{d}{dt} f(x + t(y - x)) \right\|.$$

The reader may wonder why we are concerned with converting Wasserstein bounds into total variation bounds, instead of simply being satisfied with convergence in Wasserstein distance. One reason is the existence of central limit theorems (CLTs), which are extremely important for the application of MCMC in practice (see, e.g., Flegal and Jones 2011). Let $\{X_m\}_{m=0}^{\infty}$ denote the Markov chain corresponding to $K$, and suppose that $f : X \rightarrow \mathbb{R}$ is such that $\Pi|f| < \infty$. Then because the chain is Harris ergodic, $\hat{f}_m := m^{-1} \sum_{i=0}^{m-1} f(X_i)$ is a strongly consistent estimator of $\Pi f$. If, in addition, $K$ satisfies (5) (so the chain is geometrically ergodic with respect to total variation distance), and $\Pi|f|^{2+\delta} < \infty$ for some $\delta > 0$, then $\sqrt{n}(\hat{f}_m - \Pi f)$ has a Gaussian limit distribution. On the other hand, if we replace total variation convergence with Wasserstein convergence, then the $2 + \delta$ moment is no longer sufficient for a CLT, and stronger conditions on $f$ (such as $f$ being a Lipschitz function) are required (Komorowski and Walczuk 2012).

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3 Rosenthal’s Gibbs Sampler

3.1 What is Known?

As in the Introduction, let \( \pi(\theta, \mu, A | y) \) denote the posterior density that results when the likelihood associated with Eq. 1 is combined with the prior (2). We now describe the two conditionals that define (3). Of course, \( \pi(A | \theta, y) \propto \pi(\theta, A | y) = \int_{\mathbb{R}} \pi(\theta, \mu, A | y) d\mu \), and it follows that

\[
A | \theta, y \sim IG\left(a + \frac{n - 1}{2}, b + \frac{1}{2} \sum_{i=1}^{n} (\theta_i - \bar{\theta})^2\right),
\]

where \( \bar{\theta} \) is the mean of the \( \theta_i \)s. Also,

\[
\mu | \theta, A, y \sim N(\bar{\theta}, A/n).
\]

Clearly, the product of these two conditional densities equals \( \pi(\mu, A | \theta, y) \). Thus, given \( \theta \), we can sample from \( \pi(\mu, A | \theta, y) \) by first drawing from \( \pi(A | \theta, y) \), and then drawing from \( \pi(\mu | \theta, A, y) \). It’s also easy to show that, conditional on \( \theta \), \( \mu \), and \( y \), the elements of \( \theta \) are independent with

\[
\theta_i | \theta_{-i}, A, \mu, y \sim N\left(\frac{V \mu + Ay_i}{A + V}, \frac{AV}{A + V}\right).
\]

The fact that the Mtd (3) is strictly positive on \((\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n) \times (\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n)\) implies that the Markov chain it defines is Harris ergodic (see, e.g., Asmussen and Glynn 2011), where \( \mathbb{R}_+ := (0, \infty) \).

Note that we are actually considering an entire family of chains here. Indeed, on the left-hand side of Eq. 3 we are suppressing dependence on the sample size \( n \in \mathbb{N} \), the data \( y \in \mathbb{R}^n \), the known error variance \( V \), and the hyperparameter \((a, b) \in \mathbb{R}_+ \times \mathbb{R}_+ \). Since the convergence behavior of the Gibbs chain may depend on \( (n, y, V, a, b) \), we will denote the geometric convergence rate by \( \rho(n, y, V, a, b) \). Rosenthal (1996) used drift and minorization (d&m) conditions in conjunction with results in Rosenthal (1995) to establish that every member of the family is geometrically ergodic, and his results lead to an explicit function, \( \hat{\rho}(n, y, V, a, b) \), such that, for each fixed \( (n, y, V, a, b) \), \( \rho(n, y, V, a, b) \leq \hat{\rho}(n, y, V, a, b) < 1 \).

Our interest centers on the convergence behavior of the Gibbs sampler as \( n \to \infty \). Rosenthal (1996) upper bound, \( \hat{\rho}(n, y, V, a, b) \), converges (rapidly) to 1 as \( n \to \infty \) (Yang and Rosenthal 2019), which suggests that the chain may behave poorly when \( n \) is large. However, as we shall see below, there is actually a great deal of evidence pointing in the opposite direction. One might be tempted to attribute this disconnect to the fact that d&m-based methods often break down in high dimensional situations, but, as we now explain, increasing dimension is not the culprit. Recall that the marginal chains \( \{\mu_m, A_m\}_{m=0}^{\infty} \) and \( \{\theta_m\}_{m=0}^{\infty} \) have the same convergence rate as the full Gibbs chain, \( \{\theta_m, (\mu_m, A_m)\}_{m=0}^{\infty} \), and note that \( \{\mu_m, A_m\}_{m=0}^{\infty} \) always has dimension 2, regardless of \( n \). The Mtd of this chain is given by

\[
k_1(\mu', A' | \mu, A) = \int_{\mathbb{R}^n} \pi(\mu', A' | \theta, y) \pi(\theta | \mu, A, y) d\theta.
\]

Because the integrand of \( k_1 \) contains the \( n \)-dimensional density \( \pi(\theta | \mu, A, y) \), it’s possible that the chain \( \{\mu_m, A_m\}_{m=0}^{\infty} \) is not completely immune to increasing dimension. Note, however, that \( \pi(\mu, A | \theta, y) \) depends on \( \theta \) only through two univariate functions of \( \theta: \bar{\theta} \) and \( \sum_{i=1}^{n} (\theta_i - \bar{\theta})^2 \). Thus, we can perform a change of variables on the right-hand side of Eq. 7.
that reduces the dimension of the integral from \( n \) to 2. Of course, the new integrand would involve the variable \( n \), but \( n \) would no longer represent a dimension. Thus, there is no sense in which the chain \( \{\mu_{mm}, A_m\}_{m=0}^{\infty} \) depends on dimension \( n \) in any way other than as a parameter. We have made multiple attempts to prove that \( \rho_*(n, y, V, a, b) \) is bounded away from 1 as \( n \to \infty \) by analyzing each of the two marginal chains using both d&m methods and Wasserstein methods, and all have been unsuccessful. It seems quite difficult to get a handle on the asymptotic behavior of \( \rho_*(n, y, V, a, b) \), and the difficulty goes beyond increasing dimension.

Yang and Rosenthal (2019) attacked this convergence complexity problem in a different way. Instead of focusing on the geometric convergence rate, they studied the mixing time. In particular, these authors showed that, under a weak assumption on the asymptotic behavior of \( (n-1)^{-1} \sum_{i=1}^{n} (y_i - \bar{y})^2 \), and for a particular starting value \((\theta_0, A_0, \mu_0)\), there exist an \( N \in \mathbb{N} \), positive constants \( C_1, C_2, C_3 \), and \( \gamma \in (0, 1) \) such that, for all \( n \geq N \) and all \( m \),

\[
d_{TV}(K^m_{(\theta_0, A_0, \mu_0)}, \Pi_n) \leq C_1 \gamma^m + C_2 \frac{m(m+1)}{n} + C_3 \frac{m}{\sqrt{n}},
\]

(8)

where \( K^m_{(\theta_0, A_0, \mu_0)} \) denotes the \( m \)-step Mtk of the Gibbs sampler started at \((\theta_0, A_0, \mu_0)\) based on sample size \( n \), and \( \Pi_n \) denotes the corresponding posterior distribution. Note that the right-hand side of Eq. 8 is a decreasing function of \( n \). Thus, if for some fixed \( m' \) and \( n' \geq N \), the total variation distance is less than some threshold, then this remains so for all \( n > n' \) with the same \( m' \). So, in this sense, the mixing time is constant in \( n \). While this result certainly suggests that the chain is reasonably well-behaved when \( n \) is large, it does not provide us with any information about the asymptotic behavior of the geometric convergence rate as \( n \to \infty \). Indeed, Eq. 8 does not even imply that the chain is geometrically ergodic.

In the next subsection, we apply a simulation technique developed in Qin et al. (2019) to produce evidence suggesting that the spectral gap is bounded away from 0 as \( n \to \infty \).

### 3.2 A Numerical Investigation of the Asymptotic Properties of \( \lambda_* \)

Recall the Markov operator, \( K \), from Section 2. If \( K \) is self-adjoint, positive and compact, then it has a pure eigenvalue spectrum, and the eigenvalues are all in the set \([0, 1]\). If, in addition, the eigenvalues are summable, then \( K \) is called trace-class. (See Qin et al. (2019) for more details.) Qin et al. (2019) provide a method of estimating the second largest eigenvalue of such a \( K \), which, as we know from Eq. 6, dictates the rate of convergence.

Here’s the basic idea. Let \( \{\lambda_i\}_{i=0}^{\kappa} \) denote the non-zero eigenvalues of \( K \) in decreasing order, so \( \lambda_0 = 1, \lambda_i \in (0, 1) \) for all \( i \in \{1, 2, \ldots, \kappa\} \), and \( \kappa \) could be \( \infty \). (In the sequel, we use \( \lambda_1 \) and \( \lambda_* \), interchangeably.) Now, fix a positive integer \( l \), and define

\[
s_l = \sum_{i=0}^{\kappa} \lambda_i^l.
\]

The fact that the chain is trace-class implies that this sum is finite for any \( l \in \mathbb{N} \). Qin et al. (2019) show that \( u_l = (s_l - 1)^{1/l} \) is an upper bound on \( \lambda_* \), which decreases to \( \lambda_* \) as \( l \to \infty \). These authors also develop a classical Monte Carlo estimator for \( s_l \) that is asymptotically normal, and this leads to an asymptotically normal estimator for \( u_l \). We will apply this method to the marginal chain \( \{\theta_m\}_{m=0}^{\infty} \) whose Mtd is given by

\[
k_2(\theta' \mid \theta) = \int_{\mathbb{R}_+} \int_{\mathbb{R}} \pi(\theta' \mid \mu, A, y) \pi(\mu, A \mid \theta, y) d\mu dA.
\]
Again, the Markov operators associated with the marginal chains of any two-block Gibbs sampler are reversible and positive, so all we have left to do is to show that the Markov operator associated with $k_2$ is trace-class (which implies compactness). By Qin et al. (2019) Theorem 2, it suffices to show that
\[ \int_{\mathbb{R}^n} k_2(\theta \mid \theta) d\theta < \infty. \]
A proof of the following result is provided in Appendix A.

**Proposition 4** The Markov operator defined by $k_2$ is trace-class whenever $n \geq 3$.

In order to apply the Monte Carlo algorithm, we must specify an auxiliary density $\omega(\mu, A)$ that is positive (almost) everywhere on $\mathbb{R} \times \mathbb{R}_+$. We will use $\omega(\mu, A) = \omega(\mu \mid A) \omega(A)$, where
\[ \omega(A) = IG(a, b) \quad \text{and} \quad \omega(\mu \mid A) = N\left( \tilde{y}, \frac{(A + V)(A + 4V)}{nA} \right). \]
The (strongly consistent) Monte Carlo estimator of $s_t$ is given by
\[ \hat{s}_t = \frac{1}{N} \sum_{i=1}^{N} \frac{\pi(\mu_i^*, A_i^* \mid \theta_i^*)}{\omega(\mu_i^*, A_i^*)}, \] (9)
where the random vectors $\{(\theta_i^*, \mu_i^*, A_i^*)\}_{i=1}^{N}$ are iid and each is generated according to Algorithm 1.

**Algorithm 1** Drawing $(\theta^*, \mu^*, A^*)$ in order to estimate $s_t$.

1. Draw $(\mu^*, A^*) \sim \omega(\cdot, \cdot)$.
2. Given $(\mu^*, A^*) = (\mu, A)$, for $i = 1, \ldots, n$, draw
\[ \theta_i^* \overset{ind}{\sim} N\left( \frac{V \mu + Ay_i}{A + V}, \frac{AV}{A + V} \right), \]
and set $\theta' = (\theta_1^*, \ldots, \theta_n^*)^T$.
3. If $l = 1$, set $\theta^* = \theta'$. If $l \geq 2$, draw $\theta^* \sim k_2^{(l-1)}(\cdot \mid \theta')$ by running $l - 1$ iterations of the two-block Gibbs sampler.

By Qin et al. (2019) Theorem 4, the Monte Carlo estimator Eq. 9 has finite variance if the following condition is satisfied:
\[ \int_{\mathbb{R}_+} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{\pi^3(A, \mu \mid \theta) \pi(\theta \mid A, \mu)}{\omega^2(A, \mu)} d\theta d\mu dA < \infty. \]
A proof of the following result is provided in Appendix B.

**Proposition 5** The Monte Carlo estimator (9) has finite variance whenever $n \geq 3$.

In order to estimate $u_t$, the upper bound on the second largest eigenvalue of the Markov operator defined by $k_2$, we have to apply Algorithm 1 $N$ times, where $N$ is the Monte
Carlo sample size, and within each iteration of Algorithm 1, we must generate \( n \) univariate normals \( l \) times. This becomes quite burdensome when \( n \) is large, which, unfortunately, is precisely the case on which we are focused. However, there is a simple way of circumventing this problem. Recall that \( \pi(\mu, A \mid \theta, y) \) depends on \( \theta \) only through two univariate functions of \( \theta \): \( \tilde{\theta} \) and \( \sum_{i=1}^{n} (\theta_i - \tilde{\theta})^2 \). Moreover, if the components of \( \theta \) are independent with \( \theta_i \sim N\left( \frac{V\mu + Ay_i}{A + V}, \frac{AV}{A + V} \right) \),

then it follows that \( \tilde{\theta} \) and \( \sum_{i=1}^{n} (\theta_i - \tilde{\theta})^2 \) are independent,

\[
\tilde{\theta} \sim N\left( \frac{V\mu + A\bar{y}}{A + V}, \frac{AV}{n(A + V)} \right) \quad \text{and} \quad \frac{(A + V)}{AV} \sum_{i=1}^{n} (\theta_i - \tilde{\theta})^2 \sim \chi^2_{n-1}(\phi),
\]

where \( \phi = (A\Delta)/(2V(A + V)) \) is the non-centrality parameter, and \( \Delta = \sum_{i=1}^{n} (y_i - \bar{y})^2 \). Therefore, when running Algorithm 1, each time we are required to make a draw from \( \pi(\theta \mid \mu, A, y) \), which nominally requires making \( n \) independent univariate normal draws, we can instead simply draw one univariate normal and one non-central \( \chi^2 \). This maneuver is a massive time saver when \( n \) and/or \( l \) are large.

We now employ Qin et al.’s (2019) method to gain some insight into the behavior of the convergence rate of the \( \theta \)-chain as \( n \) becomes large. Again, we have a family of chains indexed by \( (n, y, V, a, b) \), so \( \lambda_\ast = \lambda_\ast(n, y, V, a, b) \). Our idea is to consider a sequence of Gibbs samplers based on a growing data set (with \( a, b \) and \( V \) fixed) to study whether there is a noticeable relationship between the value of \( \lambda_\ast \) and increasing dimension. We simulated seven different data sets, that is, seven different versions of \( y \), each of length \( 10^7 \).

The simulations were based on different values of \( A \) and \( V \). In three of the the cases, we set \( A = V \) with values \( \{1, 10, 100\} \), in two cases we took \( A \) larger than \( V \) (\( A = 10, V = 1 \) and \( A = 100, V = 10 \)), and in the final two cases, we took \( V \) larger than \( A \) (\( A = 1, V = 10 \) and \( A = 10, V = 100 \)). For each of the seven configurations, we simulated \( \theta_i \overset{\text{iid}}{\sim} N(0, A), \quad i = 1, \ldots, 10^7 \), and then we simulated \( y_i \overset{\text{iid}}{\sim} N(\theta_i, V), \quad i = 1, \ldots, 10^7 \). Then, for each of the seven configurations, we considered six different \( \theta \)-chains corresponding to six different samples sizes: \( n = 10^2, 10^3, \ldots, 10^7 \). We then applied Qin et al.’s (2019) method to each of the six chains. So, overall, we estimated an upper bound on \( \lambda_\ast \) for \( 42 \) different Markov chains. Of course, in order to use Qin et al.’s (2019) algorithm, we need to specify values for \( a \) and \( b \). We simply chose \( a \) and \( b \) such that \( b/(a - 1) \) equals the value of \( A \) that was used to simulate the data. (See Fig. 1 for the exact values.)

Of course, there is still the issue of choosing the tuning parameter, \( l \). Qin et al. (2019) recommend increasing \( l \) until \( u_l \) is strictly less than 1. In practice, one can observe \( u_l \) steadily fall as \( l \) increases before hitting a point of volatility, where it begins to produce unreliable estimates. (The variance of the estimator of \( s_l \) is stable as \( l \to \infty \), but that of \( u_l \) is not.) We utilized a “Goldilocks” strategy, choosing values of \( l \) that were large enough to have \( u_l \) appear to be a good estimator of the upper bound but small enough to ensure the variance of \( u_l \) remains as low as possible. In each case, we used a Monte Carlo sample size of \( N = 5,000,000 \), that is, for each of the 42 different Markov chains that we studied, once we identified a reasonable value of \( l \), we used Algorithm 1 to produce \( N = 5,000,000 \) draws of \( (\theta^*, \mu^*, A^*) \), and those were then used to estimate \( s_l \) (and \( u_l \)).

The results are presented in Fig. 1. There is one plot for each of the seven configurations, and in each case, it appears that, as the sample size, \( n \), becomes large, the estimated upper bounds on \( \lambda_\ast \) approach an asymptote that is strictly below 1. Note that the values of \( n \) in
Fig. 1 Plots of the Monte Carlo estimator of the upper bound $M$ of $\lambda_*$ for each of 42 different Gibbs samplers. There is one plot for each of the seven simulated data sets. The values of $\alpha$ and $\beta$ that were used to simulate the data are provided, as are the values of $\alpha$ and $\beta$ that were used to run the Gibbs samplers. Each plot increase by a factor of 10 each time. It is clear that different underlying values of $\alpha$, $\beta$, and $\nu$ can result in different convergence rates, and that $\lambda_*$ can grow as $n$ increases, but, in each case, $\lambda_*$ appears to be bounded away from 1 as $n$ grows. Because each of the 42 estimates is based on a very large Monte Carlo sample size ($5 \times 10^6$), the standard errors are all relatively small, and certainly not large enough to change the takeaway that $\lambda_*$ seems to be bounded away from 1.

Our numerical work suggests that $\lambda_*$ is bounded away from 1 as $n \to \infty$, and if this is true, one would think that $\rho_*$ probably behaves similarly. On the other hand, it is true that $\lambda_* \leq \rho_*$, so, while it seems unlikely, it is possible that $\rho_*$ behaves poorly even when $\lambda_*$ does not. In the next section, we show that a more complex random effects model (containing replicates) leads to Gibbs samplers that are actually easier to analyze than those studied in this section.

4 Gibbs Samplers for Models with Replicates

4.1 An Alternative Blocking Strategy

Here we consider the posterior that results when we combine the likelihood defined by Eq. 4 with the prior (2). It turns out to be more convenient to work with a simple transformation...
of the resulting posterior. Let \( \eta_i = \sqrt{n} \mu_i, \eta_i = \theta_i - \mu, i = 1, \ldots, n, \) and \( B = 1/A. \) Then the new posterior density is given by
\[
\pi(\eta, B \mid y) \propto B^{a + \frac{n}{2} - 1} \exp \left\{ -\frac{U}{2} \sum_{i=1}^{n} \sum_{j=1}^{r} \left( y_{ij} - \frac{\eta_i}{\sqrt{n}} - \eta_j \right)^2 - B \left( b + \frac{1}{2} \sum_{i=1}^{n} \eta_i^2 \right) \right\} I_{\mathbb{R}_+}(B),
\]
where \( \eta = (\eta_0, \ldots, \eta_n)^T \) and \( U = 1/V. \) Routine calculations show that
\[
\eta_0 \mid B, y \sim N\left( \sqrt{n} \bar{y}, \frac{B + rU}{rBU} \right),
\]
where \( \bar{y} = (nr)^{-1} \sum_{i,j} y_{ij}. \) Moreover, conditional on \((\eta_0, B, y), \eta_1, \ldots, \eta_n\) are independent with
\[
\eta_i \mid \eta_{-i}, B, y \sim N\left( \frac{rU}{B + rU} \left( \bar{y}_i - \frac{\eta_0}{\sqrt{n}} \right), \frac{1}{B + rU} \right).
\]
Thus, we can draw from \( \pi(\eta \mid B, y) \) in a sequential manner. Finally, it’s easy to show that
\[
B \mid \eta, y \sim \text{Gamma}\left( a + \frac{n}{2}, b + \frac{1}{2} \sum_{i=1}^{n} \eta_i^2 \right).
\]
Consider the two-block Gibbs sampler with blocks \( \eta \) and \( B. \) Note that here, unlike in Section 3, all of the location parameters are in one block, and the second block has just a single, univariate parameter. We will study the \( \eta \)-marginal of this two-block sampler, which we denote by \( \{\eta_m\}_{m=0}^{\infty}. \) Let \( K \) denote its Mtk and \( \Pi \) its stationary distribution. The corresponding Mtst is given by
\[
k(\eta' \mid \eta) = \int_0^\infty \pi(\eta' \mid B, y) \pi(B \mid \eta, y) dB.
\]
We now describe a random mapping that induces \( \{\eta_m\}_{m=0}^{\infty}. \) Let \( \bar{y}_i = \frac{1}{r} \sum_{j=1}^{r} y_{ij}. \) Also, let \( J \) and \( (N_i)_{i=0}^{n} \) be independent and such that \( J \sim \text{Gamma}\left( a + \frac{n}{2}, 1 \right) \) and \( (N_i)_{i=0}^{n} \) are iid \( \text{N}(0, 1). \) Fix \( \eta \) and define
\[
\tilde{B}(\eta) = \frac{J}{b + \frac{1}{2} \sum_{i=1}^{n} \eta_i^2},
\]
\[
\tilde{\eta}_0(\eta) = \sqrt{n} \bar{y} + \sqrt{\frac{\tilde{B}(\eta) + rU}{r \tilde{B}(\eta) U}} N_0
\]
\[
\tilde{\eta}_i(\eta) = \frac{rU}{\tilde{B}(\eta) + rU} \left( \bar{y}_i - \frac{\tilde{\eta}_0(\eta)}{\sqrt{n}} \right) + \sqrt{\frac{1}{\tilde{B}(\eta) + rU}} N_i, i = 1, 2, \ldots, n.
\]
Now let \( f(\eta) = (\tilde{\eta}_0(\eta), \tilde{\eta}_1(\eta), \ldots, \tilde{\eta}_n(\eta))^T. \) It’s clear that \( f(\eta) \sim K_\eta(J). \) Our main result involves the following conditions:
\[
(A1) \quad \frac{r(n)^2}{n^2} \rightarrow \infty \text{ as } n \rightarrow \infty,
\]
\[
(A2) \quad \text{For all large } n, n^{-1} \sum_{i=1}^{n} (\bar{y}_i - \bar{y})^2 < C, \text{ for some } C < \infty.
\]
A proof of the following result is given in Appendix C.

**Proposition 6** Let \( \rho_* (n, r(n), y, U, a, b) \) denote the geometric convergence rate of \( \{\eta_m\}_{m=0}^{\infty} \) and assume that (A1) and (A2) hold. Then \( \rho_* (n, r(n), y, U, a, b) \rightarrow 0 \) as \( n \rightarrow \infty. \)
Remark 7 Qin and Hobert (2019b) proved a similar result for a more complex model in which \( V = 1/U \) is considered unknown, and has an IG prior. Our proof is quite similar to theirs.

Proposition 6 constitutes a strong convergence complexity result. Indeed, not only is the geometric convergence rate bounded below 1 as \( n, r(n) \to \infty \), but it actually converges to 0. Of course, \( r(n)_n^2 \to \infty \) is a strong assumption.

4.2 A shrinkage prior

Here we consider the same model as in Section 4.1, except that we change the prior on \( \mu \). Instead of the flat prior on \( \mu \), we employ a shrinkage prior: \( \mu \sim N(w, z^{-1}) \), where \( w \) is fixed, but \( z = z(n) \). It turns out to be more convenient to work with a simple transformation.

Let \( \beta_i = \theta_i - \mu \), \( i = 1, \ldots, n \), and \( B = 1/A \). Then the resulting posterior density is given by

\[
\pi(\beta, \mu, B | y) \propto B^{a+\frac{n}{2}-1} \exp \left\{ -\frac{U}{2} \sum_{i=1}^{n} \sum_{j=1}^{r} (y_{ij} - (\beta_i + \mu))^2 - B \left( b + \frac{1}{2} \sum_{i=1}^{n} \beta_i^2 \right) - \frac{z}{2} (\mu - w)^2 \right\} I_{B, \infty}(B),
\]

where \( \beta = (\beta_1, \ldots, \beta_n)^T \) and \( U = 1/V \). It’s easy to see that, conditional on \((\beta, y)\), \( \mu \) and \( B \) are independent, and that

\[
B \mid \mu, \beta, y \sim \text{Gamma} \left( a + \frac{n}{2}, b + \frac{1}{2} \sum_{i=1}^{n} \beta_i^2 \right),
\]

and

\[
\mu \mid B, \beta, y \sim N \left( \frac{nrU(\bar{y} - \tilde{\beta}) + zw}{nrU + z}, \frac{1}{nrU + z} \right).
\]

Conditional on \((\mu, B, y)\), the components of \( \beta \) are independent with

\[
\beta_i \mid \beta_{-i}, \mu, B, y \sim N \left( \frac{rU}{B + rU} (\bar{y}_i - \mu), \frac{1}{B + rU} \right).
\]

Consider the two-block Gibbs sampler with blocks \((\mu, B)\) and \( \beta \). We will study the \( \beta \)-marginal of this two-block sampler, which we denote by \( \{ \beta_m \}_{m=0}^\infty \). Let \( K \) denote its \( \text{Mtk} \) and \( \Pi \) its stationary distribution. The corresponding \( \text{Mtd} \) is given by

\[
k(\beta' | \beta) = \int_0^\infty \int_k \pi(\beta' | \mu, B, y) \pi(\mu, B | \beta, y) d\mu dB.
\]

We now describe a random mapping that induces \( \{ \beta_m \}_{m=0}^\infty \). Let \( J \) and \( \{ N_i \}_{i=0}^n \) be independent and such that \( J \sim \text{Gamma} \left( a + \frac{n}{2}, 1 \right) \) and \( \{ N_i \}_{i=0}^n \) are iid \( N(0, 1) \). Fix \( \beta \) and define

\[
\bar{\beta}(\beta) = \frac{J}{b + \frac{1}{2} \sum_{i=1}^{n} \beta_i^2},
\]

\[
\bar{\mu}(\beta) = \frac{nrU(\bar{y} - \tilde{\beta}) + zw}{nrU + z} + \frac{N_0}{\sqrt{nrU + z}},
\]

\[
\bar{\beta}_i(\beta) = \frac{rU}{\bar{B}(\beta) + rU} (\bar{y}_i - \bar{\mu}(\beta)) + \frac{N_i}{\sqrt{\bar{B}(\beta) + rU}}, i = 1, 2, \ldots, n.
\]
Now let $f(\beta) = (\hat{\beta}_1^{(\beta)}, \hat{\beta}_2^{(\beta)}, \ldots, \hat{\beta}_n^{(\beta)})^T$. It’s clear that $f(\beta) \sim K(\cdot)$. Our main result involves the following conditions:

(A3) $r(n) \rightarrow \infty$ as $n \rightarrow \infty$.
(A4) $|\gamma|$ is bounded above for large $n$.

A proof of the following result is given in Appendix D.

**Proposition 8** Let $\rho_n(n, r(n), y, U, a, b, w, z(n))$ denote the geometric convergence rate of $(\beta_m)_{m=0}^\infty$, and assume that (A1)-(A4) hold. Then $\rho_n(n, r(n), y, U, a, b, w, z(n)) \rightarrow 0$ as $n \rightarrow \infty$.

## 5 Discussion

It should be noted that all of the Gibbs samplers analyzed in this paper can be considered “toys” in the following sense. In each case, it is possible to make an exact draw from the posterior by drawing one $(n + 1)$-dimensional multivariate normal random vector, and one random variable from an intractable univariate density. For example, consider the posterior density, $\pi(\theta, \mu, A | y)$, from Section 3. Routine calculations reveal that $\pi(\theta, \mu, A | y)$ is multivariate normal, and, moreover, it is straightforward to construct a simple rejection sampler to draw from the univariate density $\pi(A | y)$ (see, e.g., Jones 2001, pp. 123-126). Given the choice between a correlated sample from the Gibbs sampler and an iid sample, one would probably choose that latter. On the other hand, the fact that these Gibbs samplers would probably not be used in practice doesn’t render them easy to analyze. Indeed, it is still unknown whether the convergence rate of Rosenthal’s Gibbs sampler remains bounded away from 1 as dimension grows, or if not, how quickly the rate approaches 1 as dimension grows.

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**Appendix A: Proof of Proposition 4**

We begin with two simple lemmas whose proofs are straightforward.

**Lemma 9** Suppose that $X$ is a continuous random variable with positive support, and let $s$ and $t$ be real numbers such that $1 \leq s \leq t < \infty$. If $1 \leq EX' < \infty$, then $EX' \leq 1 + EX' \leq 2EX'$.

**Lemma 10** Suppose that $X \sim \chi^2_k(\phi)$ (non-central $\chi^2$ with $k$ degrees of freedom, and non-centrality parameter $\phi$) where $k \geq 1$. If $r \in \mathbb{N}$, then

$$E[X^r] \leq C(k + r\phi)^r,$$

where $C$ is a positive constant that does not depend on $\phi$ (but may depend on $k$ and $r$).
Proof of Proposition 4  We begin with an overview of the argument, and then fill in the details. The goal is to show that

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^n} I(A, \mu, \theta) d\theta d\mu dA < \infty,$$

where $I(A, \mu, \theta) = \pi(\theta | A, \mu) \pi(A, \mu | \theta)$. We first show that

$$\int_{\mathbb{R}^n} I(A, \mu, \theta) d\theta$$

can be bounded above by $h_1(A)h_2(\mu, A)h_3(A)$, where $h_1(A)$ is the expectation of a function of a non-central $\chi^2$ random variable, $h_2(\mu, A)$ is a univariate normal density in the variable $\mu$, and $h_3(A)$ is a simple function of $A$. Hence,

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^n} I(A, \mu, \theta) d\theta d\mu \leq h_1(A)h_3(A).$$

We then use Lemmas 9 and 10 to show that $h_1(A)$ is bounded above by a constant. Finally, a routine argument shows that

$$\int_{\mathbb{R}^+} h_3(A) dA < \infty,$$

which completes the argument.

We now provide the details. Observe that

$$I(A, \mu, \theta) = C_1 \left[ b + \frac{1}{2} \sum_{i=1}^{n} (\theta_i - \bar{\theta})^2 \right]^{a + \frac{n-1}{2}} A^{-a - \frac{n-1}{2} - 1} \exp \left\{ \frac{-1}{A} \left[ b + \frac{1}{2} \sum_{i=1}^{n} (\theta_i - \bar{\theta})^2 \right] \right\} \times \exp \left\{ -\frac{n}{2A} (\mu - \bar{\theta})^2 \right\} \left( \frac{AV}{A + V} \right)^{-\frac{n}{2}} \exp \left\{ -\frac{A+V}{2AV} \sum_{i=1}^{n} (\theta_i - \left( \frac{V\mu + Ay_i}{A + V} \right))^2 \right\},$$

where, throughout the proof, the $C_i$ are positive constants that do not depend on $(\theta, \mu, A)$. We begin by showing that $\int_{\mathbb{R}^n} I(A, \mu, \theta) d\theta$ can be expressed as an expectation with respect to a multivariate normal distribution. We have

$$\frac{1}{A} \sum_{i=1}^{n} (\theta_i - \bar{\theta})^2 + \frac{n}{A} (\mu - \bar{\theta})^2 + \frac{A + V}{AV} \sum_{i=1}^{n} \left( \theta_i - \left( \frac{V\mu + Ay_i}{A + V} \right) \right)^2$$

$$= \sum_{i=1}^{n} \theta_i^2 \left[ \frac{1}{A} \left( \frac{A + V}{AV} \right) + \frac{n}{A} \left( \bar{\theta}^2 - \bar{\theta}^2 + \mu^2 - 2\mu\bar{\theta} \right) \right]$$

$$+ \frac{A + V}{AV} \sum_{i=1}^{n} \left( \left( \frac{V\mu + Ay_i}{A + V} \right)^2 - 2\theta_i \left( \frac{V\mu + Ay_i}{A + V} \right) \right)$$

$$= \frac{A + 2V}{AV} \sum_{i=1}^{n} \theta_i^2 + \frac{n}{A} \mu^2 - \frac{2}{AV} \sum_{i=1}^{n} \theta_i (2V\mu + Ay_i) + \frac{1}{AV (A + V)} \sum_{i=1}^{n} (V\mu + Ay_i)^2$$

$$= \frac{A + 2V}{AV} \sum_{i=1}^{n} \left( \theta_i - \left( \frac{2V\mu + Ay_i}{A + 2V} \right) \right)^2 + \frac{n}{A} \mu^2$$

$$+ \frac{\sum_{i=1}^{n} (V\mu + Ay_i)^2}{AV (A + V)} - \frac{\sum_{i=1}^{n} (2V\mu + Ay_i)^2}{AV (A + 2V)}.$$
Letting \( G = \frac{A + 2V}{AV} \sum_{i=1}^{n} \left( \theta_i - \left( \frac{2V \mu + Ay_i}{A + 2V} \right) \right)^2 \), we have

\[
\frac{1}{A} \sum_{i=1}^{n} (\theta_i - \bar{\theta})^2 + \frac{n}{A} (\mu - \bar{\mu})^2 + \frac{A + V}{AV} \sum_{i=1}^{n} \left( \theta_i - \left( \frac{V \mu + Ay_i}{A + V} \right) \right)^2 \\
= G + \frac{1}{A} \left[ n \mu^2 + \frac{1}{V(A + V)} \left( nV^2 \mu^2 + 2V An \bar{y} \mu + A^2 \sum_{i=1}^{n} y_i^2 \right) \right] \\
- \frac{1}{V(A + 2V)} \left( 4nV^2 \mu^2 + 4V An \bar{y} \mu + A^2 \sum_{i=1}^{n} y_i^2 \right) \\
= G + \frac{nA}{(A + V)(A + 2V)} (\mu^2 - 2 \mu \bar{y}) + \frac{A}{(A + V)(A + 2V)} \sum_{i=1}^{n} y_i^2 \\
= G + \frac{nA}{(A + V)(A + 2V)} (\mu - \bar{y})^2 + \frac{A}{(A + V)(A + 2V)} \Delta,
\]

where \( \Delta = \sum_{i=1}^{n} (y_i - \bar{y})^2 \).

Now let \( \theta_{A, \mu} \) denote the \( n \times 1 \) vector whose \( i \)th entry is \( \theta_{A, \mu, i} = \frac{2V \mu + Ay_i}{A + 2V} \). Let \( \mathbb{E}_* \{ f(\theta) \} \) denote the expected value of a function \( f(\theta) \) when \( \theta \sim N(\theta_{A, \mu}, \frac{AV}{A + 2V} I_n) \). Since \( \frac{A}{(A + V)(A + 2V)} \Delta > 0 \), we have

\[
\int_{\mathbb{R}^n} I(A, \mu, \theta) d\theta \leq C_2 A^{-\alpha - \frac{n}{2} - 1} e^{-\frac{A}{2} \left( \frac{A + V}{A + 2V} \right)^2} \mathbb{E}_* \left[ \left( b + \frac{1}{n} \theta^T \left( I - \frac{1}{n} J \right) \theta \right)^{a + \frac{n+1}{2}} \right] \\
\times \exp \left\{ \frac{nA}{2(A + 2V)(A + V)} (\mu - \bar{y})^2 \right\}. \tag{10}
\]

Now, it follows from basic distribution theory that, if \( \theta \sim N(\theta_{A, \mu}, \frac{AV}{A + 2V} I_n) \), then

\[
\frac{A + 2V}{AV} \theta^T \left( I - \frac{1}{n} J \right) \theta \sim \chi_{n-1}^2(\phi),
\]

where the non-centrality parameter is given by

\[
\phi = \frac{A}{2V(A + 2V)} \Delta.
\]

We deduce from this that the expectation on the right-hand side of Eq. 10 does not depend on \( \mu \) (but does depend on \( A \)). Hence, we have

\[
\int_{\mathbb{R}} \int_{\mathbb{R}^n} I(A, \mu, \theta) d\theta d\mu \leq C_3 A^{-\left( a + \frac{n+1}{2} \right)} e^{-\frac{A}{2} \left( \frac{A + V}{A + 2V} \right)^2} \mathbb{E}_* \left[ \left( b + \frac{1}{n} \theta^T \left( I - \frac{1}{n} J \right) \theta \right)^{a + \frac{n+1}{2}} \right] \\
\leq C_3 A^{-\left( a + \frac{n+1}{2} \right)} e^{-\frac{A}{2} \left( A + V \right)} \mathbb{E}_* \left[ \left( b + \frac{1}{n} \theta^T \left( I - \frac{1}{n} J \right) \theta \right)^{a + \frac{n+1}{2}} \right]. \tag{11}
\]

Let \( N = \lceil a + \frac{n-1}{2} \rceil \), where \( \lceil \cdot \rceil \) returns the smallest integer that exceeds the argument. Since \( n \geq 3, a + \frac{n-1}{2} > 1 \), and \( N \geq 2 \). Now,

\[
\mathbb{E}_* \left[ \frac{A + 2V}{AV} \theta^T \left( I - \frac{1}{n} J \right) \theta \right] = n - 1 + 2\phi > 1.
\]
Therefore, Jensen’s inequality implies that
\[
E_x \left[ \left( \frac{A + 2V}{AV} \theta^T \left( I - \frac{1}{n} J \right) \theta \right)^N \right] > 1.
\]
Applying Lemma 9 yields
\[
E_x \left[ \left( \frac{A + 2V}{AV} \theta^T \left( I - \frac{1}{n} J \right) \theta \right)^{a + \frac{n-1}{2}} \right] \leq 2E_x \left[ \left( \frac{A + 2V}{AV} \theta^T \left( I - \frac{1}{n} J \right) \theta \right)^N \right].
\]
Now, using the fact that \(A/(A + 2V) < 1\) and applying Lemma 10, we have
\[
E_x \left[ \left( \frac{A + 2V}{AV} \theta^T \left( I - \frac{1}{n} J \right) \theta \right)^{a + \frac{n-1}{2}} \right] = E_x \left[ \left( \frac{A}{A + 2V} \theta^T \left( I - \frac{1}{n} J \right) \theta \right)^{a + \frac{n-1}{2}} \right] \leq 2 \left( \frac{A}{A + 2V} \right)^{a + \frac{n-1}{2}} E_x \left[ \left( \frac{A + 2V}{AV} \theta^T \left( I - \frac{1}{n} J \right) \theta \right)^N \right] \leq C_4 \left( \frac{A}{A + 2V} \right)^{a + \frac{n-1}{2}} \left( n - 1 + N \frac{A}{2V(A + 2V)D} \right)^N \leq C_5.
\]
Since \((u + v)^p \leq (2u)^p + (2v)^p\) whenever all three variables are positive, we have
\[
E_x \left[ \left( \frac{A + 2V}{AV} \theta^T \left( I - \frac{1}{n} J \right) \theta \right)^{a + \frac{n-1}{2}} \right] \leq (2b)^{a + \frac{n-1}{2}} + E_x \left[ \left( \theta^T \left( I - \frac{1}{n} J \right) \theta \right)^{a + \frac{n-1}{2}} \right] \leq C_6.
\]
Combining this with Eq. \(11\), we have
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} I(A, \mu, \theta) \, d\theta \, d\mu \leq C_7 A^{-\left(a + \frac{p}{2} + \frac{1}{2}\right)} e^{-\frac{b}{\lambda}} (A + V).
\]
Finally, it’s clear that
\[
\int_{\mathbb{R}^n} A^{-\left(a + \frac{p}{2} + \frac{1}{2}\right)} (A + V) e^{-\frac{b}{\lambda}} dA < \infty,
\]
and the proof is complete. \(\square\)

**Appendix B: Proof of Proposition 5**

**Proof** We begin with an overview of the argument, and then fill in the details. The goal is to show that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{I'(A, \mu, \theta)}{\omega^2(A, \mu)} \, d\theta \, d\mu \, dA < \infty,
\]
where \(I'(A, \mu, \theta) = \pi^3(A, \mu | \theta) \pi(\theta | A, \mu)\). Arguments similar to those used in the proof of Proposition 4 show that
\[
\int_{\mathbb{R}^n} I'(A, \mu, \theta) \, d\theta \leq h'(\mu, A) h'_3(A),
\]
\(\square\) Springer
where $h_2'(\mu, A)$ is a univariate normal density in the variable $\mu$ and $h_3'(A)$ is a simple function of $A$. It is then shown that

$$\frac{h_2'(\mu, A)h_3'(A)}{\omega^2(\mu, \mu)} \leq h_2''(\mu, A)h_3''(A),$$

where $h_2''(\mu, A)$ is another univariate normal density in the variable $\mu$ and $h_3''(A)$ is another simple function of $A$. It follows that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} \frac{I'(A, \mu, \theta)}{\omega^2(A, \mu)} d\theta d\mu \leq h_3''(A),$$

and the result follows by establishing that

$$\int_{\mathbb{R}^+} h_3''(A) dA < \infty.$$

Here are the details. Observe that $\pi^3(A, \mu \mid \theta_{\pi}(\theta \mid A, \mu)$ is given by

$$C_1 \left[ b + \frac{1}{2} \sum_{i=1}^{n} (\theta_i - \bar{\theta})^2 \right] 3^{\left( a + \frac{n-1}{2} \right)} A^{-3a - \frac{3n}{2} - 3} \exp \left\{ -\frac{3}{A} \left[ b + \frac{1}{2} \sum_{i=1}^{n} (\theta_i - \bar{\theta})^2 \right] \right\} \times \exp \left\{ -\frac{3n}{2A} (\mu - \bar{\mu}) \right\} \left( \frac{AV}{A + V} \right)^{-\frac{n}{2}} \exp \left\{ -\frac{A + V}{2AV} \sum_{i=1}^{n} \left( \theta_i - \left( \frac{V\mu + Ay_i}{A + V} \right) \right)^2 \right\},$$

where, throughout the proof, the $C_i$ are positive constants that do not depend on $(\theta, \mu, A)$. Calculations similar to those in the proof of Proposition 4 show that

$$\frac{3}{A} \sum_{i=1}^{n} (\theta_i - \bar{\theta})^2 + \frac{3n}{A} (\mu - \bar{\mu})^2 + \frac{A + V}{AV} \sum_{i=1}^{n} \left( \theta_i - \left( \frac{V\mu + Ay_i}{A + V} \right) \right)^2 = \frac{A + 4V}{AV} \sum_{i=1}^{n} \left( \theta_i - \left( \frac{4V\mu + Ay_i}{A + 4V} \right) \right)^2 + \frac{3nA}{(A + V)(A + 4V)} (\mu - \bar{\mu})^2 + \frac{3A}{(A + V)(A + 4V)} \Delta.$$

Now let $\theta_{A,\mu}$ denote the $n \times 1$ vector whose $i$th entry is $\theta_{A,\mu,i} = \frac{4V\mu + Ay_i}{A + 4V}$. Let $E_a[f(\theta)]$ denote the expected value of a function $f(\theta)$ when $\theta \sim N(\theta_{A,\mu}, \frac{AV}{A + 4V} I_n)$. Since $\frac{3A}{(A + V)(A + 4V)} \Delta > 0$, we have

$$\int_{\mathbb{R}^n} \pi^3(A, \mu \mid \theta) \pi(\theta \mid A, \mu) d\theta \leq C_2 A^{-3a - \frac{3n}{2} - 3} e^{-\frac{3n}{2}} \left( \frac{A + V}{A + 4V} \right)^\Delta \times E_a \left[ \left( b + \frac{1}{2} \theta^T \left( I - \frac{1}{n} J \right) \theta \right) \right]^{3^{\left( a + \frac{n-1}{2} \right)}} \exp \left\{ -\frac{3nA}{2(A + V)(A + 4V)} (\mu - \bar{\mu})^2 \right\}.$$

Now, it follows from basic distribution theory that, if $\theta \sim N(\theta_{A,\mu}, \frac{AV}{A + 4V} I_n)$, then

$$\frac{A + 4V}{AV} \theta^T \left( I - \frac{1}{n} J \right) \theta \sim \chi^2_{n-1}(\phi),$$

where the non-centrality parameter is given by

$$\phi = \frac{A}{2V(A + 4V)} \Delta.$$
An argument similar to one used in the proof of Proposition 4 shows that

$$\text{E}_* \left[ \left( b + \frac{1}{2} \theta^T \left( I - \frac{1}{n} J \right) \theta \right)^3 \left( a + \frac{n-1}{2} \right) \right] \leq C_3.$$ 

Thus,

$$\int_{\mathbb{R}^n} \pi^3(A, \mu \mid \theta)^\circ(\theta \mid A, \mu) d\theta \leq C_4 A^{-\frac{3n}{2} - 3} e^{-\frac{3n}{2}} \left( \frac{A + V}{A + 4V} \right)^{\frac{n}{2}} \exp \left\{ -\frac{3nA(\mu - \bar{y})^2}{2(A + V)(A + 4V)} \right\}.$$ 

It follows that

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}^n} \frac{\pi^3(A, \mu \mid \theta)^\circ(\theta \mid A, \mu)}{\omega^2(A, \mu)} d\theta d\mu dA \leq C_5 \int_{\mathbb{R}^+} \int_{\mathbb{R}^n} A^{-\frac{3n}{2} - 3} e^{-\frac{3n}{2}} \left( \frac{A + V}{A + 4V} \right)^{\frac{n}{2}} (A + V)^{\frac{1}{2}} (A + 4V)^{\frac{3}{2}} d\mu dA \leq C_6 \int_{\mathbb{R}^+} A^{-\frac{3n}{2} - 3} e^{-\frac{3n}{2}} \left( \frac{A + V}{A + 4V} \right)^{\frac{n}{2}} (A + V)^3 dA \leq C_6 \int_{\mathbb{R}^+} A^{-\frac{3n}{2} - 3} e^{-\frac{3n}{2}} (A + V)^3 dA < \infty.$$ 

\[\square\]

**Appendix C: Proof of Proposition 6**

We prove Proposition 6 by utilizing Theorem 2 and Proposition 1. In particular, we first show that the hypothesis of Theorem 2 holds for our chain. It then follows from Theorem 2 that, if the Wasserstein geometric rate of convergence of our chain goes to zero as $n \to \infty$, then the TV geometric rate of convergence goes to zero as well. We then use Proposition 1 to show that the Wasserstein geometric rate of convergence does indeed go to zero as $n \to \infty$.

Recall that the Mtd associated with $K$ is given by

$$k(\eta' \mid \eta) = \int_0^\infty \pi(\eta' \mid B, y) \pi(B \mid \eta, y) dB.$$ 

The following result shows that the hypothesis of Theorem 2 holds for our Markov chain.

**Lemma 11** Assume that $n \geq 2$ and $r \geq 1$. There exists a constant $c = c(a, b) < \infty$ such that, for all $\eta, \eta' \in \mathbb{R}^{n+1}$,

$$\int_{\mathbb{R}^{n+1}} |k(\eta'' \mid \eta) - k(\eta'' \mid \eta')| \eta'' d\eta'' \leq cn\|\eta - \eta'\|.$$
Proof We begin with an overview of the argument, and then fill in the details. First, it is shown that
\[ \int_{\mathbb{R}^{n+1}} |k(\eta'' | \eta) - k(\eta'' | \eta')| \, d\eta'' \leq \int_0^\infty |\pi(B | \eta, y) - \pi(B | \eta', y)| \, dB. \] (12)
Thus, we can work with an integral on \( \mathbb{R}_+ \) rather than an integral on \( \mathbb{R}^{n+1} \). Recall that
\[ B | \eta, y \sim \text{Gamma} \left( a + \frac{n}{2}, b + \frac{\sum_{i=1}^n \eta_i^2}{2} \right). \]

We provide a closed-form expression for the integral on the right-hand side of Eq. 12, and then it is shown that this expression is bounded above by an explicit function of \( \| \eta - \eta' \| \), call it \( g(\| \eta - \eta' \|) \). We then apply the mean value theorem to the function \( g(\cdot) \) to show that there exists a finite constant \( c \) such that
\[ \int_{\mathbb{R}_+} |k(\eta'' | \eta) - k(\eta'' | \eta')| \, d\eta'' \leq c \| \eta - \eta' \| \]
whenever \( \| \eta - \eta' \| < 1/n \). Finally, the triangle inequality is used to extend the result to pairs \( (\eta, \eta') \) for which \( \| \eta - \eta' \| \geq 1/n \). This completes the argument.

Here are the details. Note that
\[ 1 - d_{1_2} 1 \leq d_{0} 1 \leq d_{0}. \] (13)
Define \( \delta(\eta, \eta') = S(\eta') - S(\eta) \), where, for \( x = (x_0, x_1, \ldots, x_n)^T \in \mathbb{R}^{n+1} \), \( S(x) := b + \frac{1}{2} \sum_{i=1}^n x_i^2 \). (Note that \( S(\eta) \) is free of \( \eta_0 \)). Now
\[ \delta(\eta, \eta') = \frac{1}{2} \sum_{i=1}^n (\eta_i' - \eta_i)^2 = \frac{1}{2} \sum_{i=1}^n (\eta_i' - \eta_i)^2 - (\eta_i' - \eta_i)^2 + (\eta_i')^2 - (\eta_i)^2 \]
\[ = \sum_{i=1}^n \left[ \eta_i (\eta_i' - \eta_i) + \frac{(\eta_i' - \eta_i)^2}{2} \right] \leq \left\{ \left( \sum_{i=1}^n \eta_i (\eta_i' - \eta_i) \right)^2 \right\}^{1/2} + \frac{1}{2} \| \eta - \eta' \|^2 \]
\[ \leq \left\{ \left( \sum_{i=1}^n \eta_i^2 \right) \| \eta - \eta' \|^2 \right\}^{1/2} + \frac{1}{2} \| \eta - \eta' \|^2 \leq \| \eta - \eta' \|^2 \leq \sqrt{2S(\eta)} \| \eta - \eta' \|^2 + \frac{1}{2} \| \eta - \eta' \|^2. \]
WLOG, assume that \( S(\eta') \geq S(\eta) \). Define the point
\[ t_1 = \frac{\delta(\eta, \eta')}{\delta(\eta, \eta')} \log \left[ 1 + \frac{\delta(\eta, \eta')}{S(\eta)} \right]. \]
Observe that \( \pi(B | \eta', y) \geq \pi(B | \eta, y) \) if and only if
\[ \frac{S(\eta')^{a+n/2}}{\Gamma(a+n/2)} B^{a+n/2-1} e^{-BS(\eta')} \geq \frac{S(\eta)^{a+n/2}}{\Gamma(a+n/2)} B^{a+n/2-1} e^{-BS(\eta)}, \]
which happens if and only if
\[
\left( \frac{S(\eta')}{S(\eta)} \right)^{a+n/2} \geq e^{B(S(\eta')-S(\eta))},
\]
which happens if and only if \( B \leq t_1 \).

We now use these results to bound the right-hand side of Eq. 13. We have

\[
\int_0^\infty \left| \pi(B \mid \eta, \eta') - \pi(B \mid \eta', \eta) \right| dB \\
= 2 \int_0^{t_1} \left[ \frac{S(\eta')^{a+n/2}}{\Gamma(a+n/2)} u^{a+n/2-1} e^{-u S(\eta')} - \frac{S(\eta)^{a+n/2}}{\Gamma(a+n/2)} u^{a+n/2-1} e^{-u S(\eta)} \right] du \\
= 2 \int_0^{t_1} \frac{S(\eta')^{a+n/2}}{\Gamma(a+n/2)} u^{a+n/2-1} e^{-u S(\eta)} \left[ \left( \frac{S(\eta')}{S(\eta)} \right)^{a+n/2} e^{u(S(\eta')-S(\eta))} - 1 \right] du \\
\leq 2 \left[ \left( \frac{S(\eta')}{S(\eta)} \right)^{a+n/2} - 1 \right] \int_0^{t_1} \frac{S(\eta)^{a+n/2}}{\Gamma(a+n/2)} u^{a+n/2-1} e^{-u S(\eta)} du \\
\leq 2 \left[ \left( \frac{S(\eta')}{S(\eta)} \right)^{a+n/2} - 1 \right] \int_0^\infty \frac{S(\eta)^{a+n/2}}{\Gamma(a+n/2)} u^{a+n/2-1} e^{-u S(\eta)} du \\
\leq 2 \left[ \left( \frac{S(\eta) + \delta(\eta, \eta')}{S(\eta)} \right)^{a+n/2} - 1 \right] \\
= 2 \left[ \left( 1 + \frac{\delta(\eta, \eta')}{S(\eta)} \right)^{a+n/2} - 1 \right] \\
\leq 2 \left[ \left( 1 + \sqrt{\frac{2}{S(\eta)}} \|\eta - \eta'\| + \frac{\|\eta - \eta'\|^2}{2S(\eta)} \right)^{a+n/2} - 1 \right] \\
\leq 2 \left[ \left( 1 + \sqrt{\frac{2}{b}} \|\eta - \eta'\| + \frac{\|\eta - \eta'\|^2}{2b} \right)^{a+n/2} - 1 \right].
\]

Now consider the function \( f : [0, \infty) \rightarrow (0, \infty) \) given by

\[
f(v) = \left( 1 + \sqrt{\frac{2}{b}} v + \frac{v^2}{2b} \right)^{a+n/2}.
\]

Note that

\[
f'(v) = \left( a + \frac{n}{2} \right) \left( 1 + \sqrt{\frac{2}{b}} v + \frac{v^2}{2b} \right)^{a+n/2-1} \left( \sqrt{\frac{2}{b}} + \frac{v}{b} \right).
\]

If \( \|\eta - \eta'\| \leq \frac{1}{n} \), then for any \( c \in [0, \|\eta - \eta'\|] \), we have

\[
\frac{1}{n} f'(c) \leq w(n) := \frac{1}{n} \left( a + \frac{n}{2} \right) \left( 1 + \sqrt{\frac{2}{b}} \frac{1}{n} + \frac{1}{2bn^2} \right)^{a+n/2-1} \left( \sqrt{\frac{2}{b}} + \frac{1}{bn} \right).
\]
Some analysis reveals that \( \lim_{n \to \infty} w(n) < \infty \). Thus, as \( n \to \infty \), \( \frac{1}{n} f'(c) \) is bounded above. Let \( c = \max_{n \geq 1} w(n) \), which is finite. Then, by the mean value theorem, if \( \| \eta - \eta' \| \leq \frac{1}{n} \), we have

\[
\int_{\mathbb{R}^{n+1}} |k(\eta'' | \eta) - k(\eta'' | \eta')| \, d\eta'' \leq 2cn \| \eta - \eta' \|.
\]

We now extend the result to pairs \((\eta, \eta')\) such that \( \| \eta - \eta' \| > \frac{1}{n} \). Divide the vector \( \eta - \eta' \) into segments whose lengths are less than \( \frac{1}{n} \). In particular, let \( \{\eta(i)\}_{i=0}^{N} \) be points in \( \mathbb{R}^{n+1} \) such that

\[
\sum_{i=1}^{N} (\eta(i) - \eta(i-1)) = \eta - \eta'
\]

where \( \eta(0) = \eta' \), \( \eta(N) = \eta \), \( \eta(i) - \eta(i-1) \) has the same direction as \( \eta - \eta' \), and \( \| \eta(i) - \eta(i-1) \| < \frac{1}{n} \). Then, by what have already shown, we have

\[
\int_{\mathbb{R}^{n+1}} |k(\eta'' | \eta) - k(\eta'' | \eta')| \, d\eta'' \leq 2cn \sum_{i=1}^{N} \int_{\mathbb{R}^{n+1}} |k(\eta'' | \eta(i)) - k(\eta'' | \eta(i-1))| \, d\eta''
\]

\[
\leq 2cn \sum_{i=1}^{N} \| \eta(i) - \eta(i-1) \|
\]

\[
= 2cn \| \eta - \eta' \|.
\]

\( \blacksquare \)

**Proof of Proposition 6** With Lemma 11 in hand, it suffices to show that the Wasserstein geometric rate of convergence of our chain goes to zero as \( n \to \infty \). This is accomplished using Proposition 1, which requires that we bound \( \mathbb{E} \| f(\eta) - f(\eta') \| \), where \( f(\eta) \) is defined in Section 4.1. Now, Lemma 4 and Jensen’s inequality yield

\[
\mathbb{E} \| f(\eta) - f(\eta') \| \leq \sup_{t \in [0, 1]} \mathbb{E} \left[ \frac{d}{dt} \left. f(\eta + t(\eta' - \eta)) \right| \right]
\]

\[
= \sup_{t \in [0, 1]} \mathbb{E} \left[ \sum_{i=0}^{n} \left( \frac{d}{dt} \eta_i \eta_i(\eta + t(\eta' - \eta)) \right)^2 \right]
\]

\[
\leq \sup_{t \in [0, 1]} \mathbb{E} \left[ \sum_{i=0}^{n} \left( \frac{d}{dt} \eta_i \eta_i(\eta + t(\eta' - \eta)) \right)^2 \right].
\]

The rest of the proof is simply brute force analysis of

\[
\mathbb{E} \left[ \left( \frac{d}{dt} \eta_i \eta_i(\eta + t(\eta' - \eta)) \right)^2 \right],
\]
for \( i = 0, 1, \ldots, n \). Henceforth, we shall abbreviate using \( \alpha = \eta' - \eta \), so that \( \eta + t (\eta' - \eta) = \eta + t \alpha \). We begin by calculating \( \frac{d}{dt} \tilde{B}^{(q + i \alpha)} \). We have

\[
\frac{d}{dt} \tilde{B}^{(q + i \alpha)} = \frac{d}{dt} \left( b + \frac{1}{2} \sum_{i=1}^{n} (\eta_i + t \alpha_i)^2 \right) = -\frac{J}{\left[ b + \frac{1}{2} \sum_{i=1}^{n} (\eta_i + t \alpha_i)^2 \right]^{\frac{3}{2}}} \sum_{i=1}^{n} (\eta_i + t \alpha_i) \alpha_i.
\]

Thus, by Cauchy-Schwarz, we can see that

\[
\left( \frac{d}{dt} \tilde{B}^{(q + i \alpha)} \right)^2 = \frac{\left( \tilde{B}^{(q + i \alpha)} \right)^4}{J^2} \left[ \sum_{i=1}^{n} (\eta_i + t \alpha_i) \alpha_i \right]^2 \leq \frac{\left( \tilde{B}^{(q + i \alpha)} \right)^4}{J^2} \left[ \sum_{i=1}^{n} (\eta_i + t \alpha_i)^2 \right] ||\alpha||^2
\]

\[
= \frac{2 \left( \tilde{B}^{(q + i \alpha)} \right)^4}{J^2} \left[ \frac{1}{2} \sum_{i=1}^{n} (\eta_i + t \alpha_i)^2 \right] ||\alpha||^2 \leq \frac{2 \left( \tilde{B}^{(q + i \alpha)} \right)^3}{J} ||\alpha||^2. \quad (15)
\]

Next, observe that

\[
\frac{d}{dt} \tilde{B}^{(q + i \alpha)} = \frac{d}{dt} \sqrt{\frac{\tilde{B}^{(q + i \alpha)} + r U}{\tilde{B}^{(q + i \alpha)} U}} N_0 = \frac{N_0}{2} \sqrt{r \tilde{B}^{(q + i \alpha)} U + r U} \left[ -\frac{d}{dt} \tilde{B}^{(q + i \alpha)} \right]. \quad (16)
\]

Hence, using Eq. 15, we have

\[
\left( \frac{d}{dt} \tilde{B}^{(q + i \alpha)} \right)^2 = \frac{N_0^2}{4} \left( \frac{r \tilde{B}^{(q + i \alpha)} U}{\tilde{B}^{(q + i \alpha)} + r U} \right) \left( \frac{1}{\tilde{B}^{(q + i \alpha)}} \right)^4 \left( \frac{d}{dt} \tilde{B}^{(q + i \alpha)} \right)^2 \leq \frac{N_0^2}{2} \frac{r U}{J \tilde{B}^{(q + i \alpha)} + r U} ||\alpha||^2 \leq \frac{N_0^2}{2J} ||\alpha||^2.
\]

It follows that

\[
E \left[ \left( \frac{d}{dt} \tilde{B}^{(q + i \alpha)} \right)^2 \right] \leq \frac{1}{2a + n - 2} ||\alpha||^2.
\]

Next, we calculate \( \frac{d}{dt} \tilde{B}^{(q + i \alpha)} \) for \( i = 1, \ldots, n \). We have
\[
\frac{d}{dt} \tilde{\eta}_i^{(q+i\alpha)} = \frac{d}{dt} \left[ \frac{rU}{\tilde{B}(q+i\alpha) + rU} \left( \tilde{y}_i - \frac{\tilde{\eta}_0^{(q+i\alpha)}}{\sqrt{n}} \right) + \frac{1}{\sqrt{n}(\tilde{B}(q+i\alpha) + rU)} N_i \right] \\
= \frac{rU}{(\tilde{B}(q+i\alpha) + rU)^2} \left( \frac{\tilde{\eta}_0^{(q+i\alpha)}}{\sqrt{n}} - \tilde{y}_i \right) \left( \frac{d}{dt} \tilde{B}(q+i\alpha) \right) - \frac{rU}{\sqrt{n}(\tilde{B}(q+i\alpha) + rU)} \left( \frac{d}{dt} \tilde{\eta}_0^{(q+i\alpha)} \right) \\
- \frac{N_i}{2(\tilde{B}(q+i\alpha) + rU)^{3}} \left( \frac{d}{dt} \tilde{B}(q+i\alpha) \right)
\]

= \text{T}_{1,i} + \text{T}_{2,i} + \text{T}_{3},

where

\[
\text{T}_{1,i} = \frac{rU}{(\tilde{B}(q+i\alpha) + rU)^2} (\tilde{y}_i - \tilde{y}_i) \left( \frac{d}{dt} \tilde{B}(q+i\alpha) \right),
\]

\[
\text{T}_{2,i} = -\frac{N_i}{2(\tilde{B}(q+i\alpha) + rU)^{3}} \left( \frac{d}{dt} \tilde{B}(q+i\alpha) \right),
\]

and

\[
\text{T}_{3} = \frac{\sqrt{rU} N_0}{\sqrt{n}(\tilde{B}(q+i\alpha) + rU)^{3}} \left( \frac{d}{dt} \tilde{B}(q+i\alpha) \right) - \frac{rU}{\sqrt{n}(\tilde{B}(q+i\alpha) + rU)} \left( \frac{d}{dt} \tilde{\eta}_0^{(q+i\alpha)} \right).
\]

Equation 16 shows that \(\frac{d}{dt} \tilde{\eta}_0^{(q+i\alpha)}\) has a factor of \(N_0\). Thus, \(\text{T}_{1,i}\) has no normal terms in it, \(\text{T}_{2,i} = cN_i\), and \(\text{T}_{3} = dN_0\), where \(c\) and \(d\) do not have any normal terms in them. Since all the normal random variables are independent of each other (and of \(J\)), we have

\[
E \left[ \left( \frac{d}{dt} \tilde{\eta}_i^{(q+i\alpha)} \right)^2 \right] = E \left[ (\text{T}_{1,i} + \text{T}_{2,i} + \text{T}_{3})^2 \right] = E \left[ \text{T}_{1,i}^2 \right] + E \left[ \text{T}_{2,i}^2 \right] + E \left[ \text{T}_{3}^2 \right].
\]

Let \(S(x) = b + \frac{1}{2} \sum_{i=1}^{n} x_i^2\) for any \(x \in \mathbb{R}^n\). Letting \(\Delta' = \sum_{i=1}^{n} (\tilde{y}_i - \tilde{y})^2\), we have

\[
\sum_{i=1}^{n} E \left[ \text{T}_{1,i}^2 \right] = \Delta' E \left[ \frac{(rU)^2}{(\tilde{B}(q+i\alpha) + rU)^4} \left( \frac{d}{dt} \tilde{B}(q+i\alpha) \right)^2 \right] \\
\leq \Delta' E \left[ \frac{1}{(rU)^2} \left( \frac{d}{dt} \tilde{B}(q+i\alpha) \right)^2 \right] \\
\leq 2\Delta' E \left[ \frac{n \left( \tilde{B}(q+i\alpha) \right)^3}{J(rU)^2} \|\alpha\|^2 \right] \\
= 2\Delta' E \left[ \frac{nJ^2}{(rU)^2 S^3(q+i\alpha)} \|\alpha\|^2 \right] \\
= \frac{\Delta' n(2a + n)(2a + n + 2)}{n \left( 2(rU)^2 S^3(q+i\alpha) \right)} \|\alpha\|^2 \\
\leq \frac{\Delta' n(2a + n)(2a + n + 2)}{2(rU)^2 b^3} \|\alpha\|^2.
\]
Now
\[
\sum_{i=1}^{n} \mathbb{E} \left[ T_{2,i}^2 \right] = \frac{n}{4} \mathbb{E} \left[ \frac{1}{(\hat{B}(\eta + \tau \alpha) + rU)^3} \left( \frac{d}{dt} \hat{B}(\eta + \tau \alpha) \right)^2 \right] \\
\leq \frac{n}{2} \mathbb{E} \left[ \frac{\left( \hat{B}(\eta + \tau \alpha) \right)^3}{J(\hat{B}(\eta + \tau \alpha) + rU)^3} \right] \| \alpha \|^2 \\
= \frac{n}{2} \mathbb{E} \left[ \frac{\left( \hat{B}(\eta + \tau \alpha) \right)^2}{S(\eta + t\alpha)(\hat{B}(\eta + \tau \alpha) + rU)^3} \right] \| \alpha \|^2 \\
\leq \frac{n}{2brU} \| \alpha \|^2.
\]

Finally, using the fact that \((u + v)^2 \leq 2u^2 + 2v^2\), we have
\[
\sum_{i=1}^{n} \mathbb{E} \left[ T_{3,i}^2 \right] \leq n \mathbb{E} \left[ \frac{2rU N_0^2}{n \hat{B}(\eta + \tau \alpha)(\hat{B}(\eta + \tau \alpha) + rU)^3} \left( \frac{d}{dt} \hat{B}(\eta + \tau \alpha) \right)^2 + \frac{2(rU)^2}{n(\hat{B}(\eta + \tau \alpha) + rU)^2} \left( \frac{d}{dt} \eta_i(\eta + \tau \alpha) \right)^2 \right] \\
\leq \mathbb{E} \left[ \frac{4rU N_0^2}{J(\hat{B}(\eta + \tau \alpha) + rU)^3} \left( \frac{d}{dt} \hat{B}(\eta + \tau \alpha) \right)^2 + \frac{(rU)^2}{J(\hat{B}(\eta + \tau \alpha) + rU)^2} \left( \frac{d}{dt} \eta_i(\eta + \tau \alpha) \right)^2 \right] \| \alpha \|^2 \\
= \mathbb{E} \left[ \frac{4rU}{J(\hat{B}(\eta + \tau \alpha) + rU)^3} \left( \frac{d}{dt} \hat{B}(\eta + \tau \alpha) \right)^2 + \frac{(rU)^2}{J(\hat{B}(\eta + \tau \alpha) + rU)^2} \right] \| \alpha \|^2 \\
\leq \| \alpha \|^2 \mathbb{E} \left[ 5f^{-1} \right] \\
= \frac{10}{2a + n - 2} \| \alpha \|^2.
\]

Therefore, combining Eq. 14 with the bounds developed above, we have
\[
\mathbb{E} \| f(\eta) - f(\eta') \| \leq \sup_{\tau \in [0,1]} \sqrt{\mathbb{E} \sum_{i=0}^{n} \left( \frac{d}{dt} \eta_i(\eta + r(t - \eta)) \right)^2} \leq \gamma_{n,r} \| \eta - \eta' \|,
\]
where
\[
\gamma_{n,r} = \gamma_{n,r}(y, U, a, b) \\
= \sqrt{\frac{\Delta' n(2a + n)(2a + n + 2)}{2(aU)^2b^3} + \frac{n}{2brU} + \frac{11}{2a + n - 2}}
\]
Under (A1) and (A2), \(\gamma_{n,r} \to 0\) as \(n \to \infty\). Thus, for all large \(n\), \(\gamma_{n,r} < 1\), and Proposition 1 implies that, for every \(\eta \in \mathbb{R}^{n+1}\), we have
\[
dW(K^m_{\eta}, \Pi) \leq \frac{c(\eta)}{1 - \gamma_{n,r}} \gamma_{n,r}^m,
\]
where \(c(\eta) = c(\eta; n, r, y, U, a, b)\). The proof is now complete. \(\square\)
Appendix D: Proof of Proposition 8

The proof of Proposition 8 is very similar to the proof of Proposition 6. Recall that the Mtd associated with $K$ is given by

$$k(\beta' | \beta) = \int_0^\infty \int_\mathbb{R} \pi(\beta' | \mu, B, y)\pi(B | \beta, y)\pi(\mu | \beta, y)d\mu dB.$$

The following result shows that the hypothesis of Theorem 2 holds for our chain.

**Lemma 12** Assume that $n \geq 2$ and $r \geq 1$. There exists a constant $c = c(a, b, U) < \infty$ such that, for all $\beta, \beta' \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |k(\beta'' | \beta) - k(\beta'' | \beta')| d\beta'' \leq c(n + \sqrt{r})\|\beta - \beta'\|.$$

**Proof** We begin by noting that

$$\int_{\mathbb{R}^n} |k(\beta'' | \beta) - k(\beta'' | \beta')| d\beta''$$

$$\leq \int_{\mathbb{R}^n} \int_0^\infty \pi(\beta'' | B, \mu, y)\pi(B | \beta, y)\pi(\mu | \beta, y) - \pi(B | \beta', y)\pi(\mu | \beta', y)\pi(\mu | \beta, y) d\mu dB d\beta''$$

$$= \int_{\mathbb{R}^n} \int_0^\infty \pi(B | \beta, y)\pi(\mu | \beta, y) - \pi(B | \beta', y)\pi(\mu | \beta', y) d\mu dB$$

$$\leq \int_{\mathbb{R}^n} \int_0^\infty \pi(B | \beta', y)\pi(\mu | \beta', y) - \pi(B | \beta', y)\pi(\mu | \beta', y) d\mu dB$$

$$+ \int_{\mathbb{R}^n} \int_0^\infty \pi(B | \beta', y)\pi(\mu | \beta', y) - \pi(B | \beta', y)\pi(\mu | \beta', y) d\mu dB$$

$$= \int_0^\infty \int_{\mathbb{R}^n} \pi(B | \beta, y) - \pi(B | \beta', y) dB + \int_{\mathbb{R}^n} \pi(\mu | \beta, y) - \pi(\mu | \beta', y) d\mu.$$  (17)

Arguments similar to those used in the proof of Lemma 11 can be used to show that

$$\int_0^\infty \pi(B | \beta, y) - \pi(B | \beta', y) dB \leq cn\|\beta - \beta'\|,$$  (18)

where $c = c(a, b) < \infty$ is a constant. We now go to work on $\int_{\mathbb{R}^n} \pi(\mu | \beta, y) - \pi(\mu | \beta', y) d\mu$. It’s easy to show that

$$\int_{\mathbb{R}} \pi(\mu | \beta, y) - \pi(\mu | \beta', y) d\mu = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} \left| e^{-u^2/2} - e^{-(u-m)^2/2} \right| du,$$

where

$$m = \frac{nrU}{\sqrt{nrU + z}}(\bar{\beta} - \bar{\beta'}).$$

Assume that $\bar{\beta} \geq \bar{\beta}'$, so $m \geq 0$. A straightforward calculation shows that the term inside the absolute value is non-negative if and only if $u \leq m/2$. Therefore,

$$\int_{\mathbb{R}} \pi(\mu | \beta, y) - \pi(\mu | \beta', y) d\mu = 2 \int_{-\infty}^{m/2} \frac{1}{\sqrt{2\pi}} \left[ e^{-\frac{1}{2}u^2} - e^{-(u-m)^2/2} \right] du$$

$$= 2 \left[ \Phi \left( \frac{m}{2} \right) - \Phi \left( -\frac{m}{2} \right) \right],$$
where $\Phi(\cdot)$ denotes the standard normal cdf. Similar consideration of the case $\tilde{\beta} < \tilde{\beta}'$ leads to the following:

$$
\int_\mathbb{R} |\pi(\mu | \beta, y) - \pi(\mu | \beta', y)| \, d\mu = 2 \left[ \Phi \left( \frac{|m|}{2} \right) - \Phi \left( -\frac{|m|}{2} \right) \right] = 4 \left[ \frac{1}{2} - \Phi \left( -\frac{|m|}{2} \right) \right].
$$

By the mean value theorem, there exists $d \in [-|m|/2, 0]$ such that

$$
4 \left[ \frac{1}{2} - \Phi \left( -\frac{|m|}{2} \right) \right] = 4 \Phi'(d) \frac{|m|}{2} \leq \frac{2|m|}{\sqrt{2\pi}}.
$$

By Cauchy-Schwarz, we have $|\tilde{\beta} - \tilde{\beta}'| \leq \|\beta - \beta'\|/\sqrt{n}$, and it follows that

$$
\int_\mathbb{R} |\pi(\mu | \beta, y) - \pi(\mu | \beta', y)| \, d\mu \leq \frac{2nrU}{\sqrt{2\pi(nrU + z)}} \|\tilde{\beta} - \tilde{\beta}'\| \leq \frac{2U}{\sqrt{r}} \|\beta - \beta'\|.
$$

Combining (17), (18), and (19) yields the result.

Proof of Proposition 8 With Lemma 12 in hand, it suffices to show that the Wasserstein geometric rate of convergence of our chain goes to zero as $n \to \infty$. We accomplish this via Proposition 1, which requires that we bound $E\|f(\beta) - f(\beta')\|$, where $f(\beta)$ is defined in Section 4.2. As in the proof of Proposition 6, Lemma 3 and Jensen’s inequality yield

$$
E\|f(\beta) - f(\beta')\| \leq \sup_{t \in [0,1]} \left\{ E \sum_{i=1}^n \left( \frac{d}{dt} \tilde{\beta}_i \big(\beta + t(\beta' - \beta)\big) \right)^2 \right\}.
$$

The rest of the proof is simply brute force analysis of

$$
E \left[ \left( \frac{d}{dt} \tilde{\beta}_i \big(\beta + t(\beta' - \beta)\big) \right)^2 \right],
$$

for $i = 1, \ldots, n$. Henceforth, we shall abbreviate using $\alpha = \beta' - \beta$, so that $\beta + t(\beta' - \beta) = \beta + t\alpha$. Calculations similar to those used in the proof of Proposition 6 show that

$$
\left( \frac{d}{dt} \tilde{\beta}_i \big(\beta + t(\beta' - \beta)\big) \right)^2 \leq \frac{2 \left( \tilde{\beta}_i \big(\beta + t\alpha\big) \right)^3}{J} \|\alpha\|^2.
$$

Now, plugging in the value of $\bar{\mu}^{(\beta + t\alpha)}$ and rearranging yields

$$
\tilde{\beta}_i \big(\beta + t\alpha\big) = \frac{rU}{\tilde{B}^{(\beta + t\alpha)} + rU} \left( \tilde{\gamma}_i - \bar{\mu}^{(\beta + t\alpha)} \right) + \frac{N_i}{\sqrt{\tilde{B}^{(\beta + t\alpha)} + rU}}
$$

$$
\quad = \frac{rU(\tilde{\gamma}_i - \tilde{\gamma})}{\tilde{B}^{(\beta + t\alpha)} + rU} + \frac{nr^2U^2(\tilde{\beta} + t\alpha)}{\tilde{B}^{(\beta + t\alpha)} + rU} - \frac{rUz(w - \tilde{\gamma})}{\tilde{B}^{(\beta + t\alpha)} + rU} + \frac{N_i}{\sqrt{nrU + z}}
$$

$$
\quad - \frac{rUN_0}{\tilde{B}^{(\beta + t\alpha)} + rU} \sqrt{nrU + z} + \frac{N_i}{\tilde{B}^{(\beta + t\alpha)} + rU}.
$$

We then have
Denote the right-hand side as \[ \sum_{j=1}^4 a_j + a_5 N_0 + a_6 N_i. \] Then, since \( \{N_i\}_{i=0}^n \) are iid standard normal, we have

\[
E \left[ \left( \sum_{j=1}^4 a_j + a_5 N_0 + a_6 N_i \right)^2 \right] = \left( \sum_{j=1}^4 a_j \right)^2 + a_5^2 + a_6^2 \leq 2a_1^2 + 4a_2^2 + 8a_3^2 + 8a_4^2 + a_5^2 + a_6^2.
\]

where we have used the fact that \((u + v)^2 \leq 2u^2 + 2v^2\) three times. Letting \( \Delta' = \sum_{i=1}^n (\bar{y}_i - \bar{y})^2 \), we have

\[
E \left[ \sum_{i=1}^n \left( \frac{d}{dt} \tilde{\beta}(\bar{y}_i + t\alpha_0) \right)^2 \right] \leq E \left[ \frac{4r^2 U^2 \Delta' \left( \tilde{B}(\bar{y}_i + t\alpha_0) \right)^3}{J \left( \tilde{B}(\bar{y}_i + t\alpha_0) + rU \right)^4} \|\alpha\|^2 + \frac{4n^3 r^4 U^4 \alpha^2}{J \left( \tilde{B}(\bar{y}_i + t\alpha_0) + rU \right)^4 (nrU + z)^2} \|\alpha\|^2 \right.
\]

\[
+ \frac{16n^3 r^4 U^4 (\bar{y} + t\alpha_0)^2 \left( \tilde{B}(\bar{y}_i + t\alpha_0) \right)^3}{J \left( \tilde{B}(\bar{y}_i + t\alpha_0) + rU \right)^4 (nrU + z)^2} \|\alpha\|^2 + \frac{16nr^2 U^2 (w - \bar{y})^2 \left( \tilde{B}(\bar{y}_i + t\alpha_0) \right)^3}{J \left( \tilde{B}(\bar{y}_i + t\alpha_0) + rU \right)^4 (nrU + z)^2} \|\alpha\|^2
\]

\[
+ \frac{2nr^2 U^2 \left( \tilde{B}(\bar{y}_i + t\alpha_0) \right)^3}{J \left( \tilde{B}(\bar{y}_i + t\alpha_0) + rU \right)^4 (nrU + z)} \|\alpha\|^2 + \frac{n \left( \tilde{B}(\bar{y}_i + t\alpha_0) \right)^3}{2J \left( \tilde{B}(\bar{y}_i + t\alpha_0) + rU \right)^3} \|\alpha\|^2 \right].
\]

By Cauchy-Schwarz, \( n\alpha^2 \leq \|\alpha\|^2 \), and

\[
n(\bar{y} + t\alpha_0)^2 \tilde{B}(\bar{y}_i + t\alpha_0) = \frac{n(\bar{y} + t\alpha_0)^2 J}{b + \frac{1}{2} \sum_{i=1}^n (\bar{y}_i + t\alpha_i)^2} \leq \frac{2Jn(\bar{y} + t\alpha_0)^2}{\sum_{i=1}^n (\bar{y}_i + t\alpha_i)^2} \leq 2J.
\]

Thus,
Note that $\tilde{B}^{(\beta + \alpha)} \leq J/b$. Hence,

$$E \left[ \sum_{i=1}^{n} \left( \frac{d}{dt} \tilde{\beta}_{i}^{(\beta + \alpha)} \right)^{2} \right] \leq E \left[ \frac{4r^{2}U^{2} \Delta (\tilde{B}^{(\beta + \alpha)})}{J \left( \tilde{B}^{(\beta + \alpha)} + ru \right)^{4}} + \frac{4n^{2}r^{4}U^{4}}{J \left( \tilde{B}^{(\beta + \alpha)} + ru \right)^{4}} \right] \lVert \alpha \rVert^{2} \leq \frac{32n^{2}r^{4}U^{4}}{J \left( \tilde{B}^{(\beta + \alpha)} + ru \right) \lVert \alpha \rVert^{2}} - \frac{16nr^{2}U^{2}(w - \bar{y})^{2}}{J \left( \tilde{B}^{(\beta + \alpha)} + ru \right) \lVert \alpha \rVert^{2}} + \frac{2n^{2}U^{2}}{J \left( \tilde{B}^{(\beta + \alpha)} + ru \right)^{4}} \lVert \alpha \rVert^{2}.$$

Therefore, using Eq. 20, we have

$$E \| f(\beta) - f(\beta') \| \leq \sup_{\beta \in [0,1]} \left\{ \frac{E \sum_{i=1}^{n} \left( \frac{d}{dt} \tilde{\beta}_{i}^{(\beta + (\beta - \beta'))} \right)^{2}}{\gamma_{n,r} \lVert \beta - \beta' \rVert} \right\} \leq \gamma_{n,r} \lVert \beta - \beta' \rVert,$$

where

$$\gamma_{n,r} = \gamma_{n,r}(y, U, a, b, w, z) = \sqrt{\frac{(2a + n + 2)^{2}}{4} \left( \frac{4 \Delta^{2}}{b^{2}r^{2}U^{2}} + \frac{32}{b^{2}r^{2}U^{2}} + \frac{16n(w - \bar{y})^{2}}{b^{2}r^{2}U^{2}} + \frac{2}{b^{2}r^{2}U^{2}} + \frac{4n^{2}r^{2}U^{2}}{z^{2}} + \frac{n}{2brU} \right)}.$$

Under (A1)-(A4), $\gamma_{n,r} \to 0$ and $n \to \infty$. Thus, for all large $n$, $\gamma_{n,r} < 1$, and Proposition 1 implies that, for every $\beta \in \mathbb{R}^{n}$, we have

$$d_{W}(K_{\beta}^{m}, \Pi) \leq c(\beta) \frac{\gamma_{n,r}^{m}}{1 - \gamma_{n,r}},$$

where $c(\beta) = c(\beta; n, r, y, U, a, b, w, z)$. The proof is now complete.

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