Opinion Dynamics on Graphon: the piecewise constant case

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Abstract

The study of network supported opinion dynamics in large groups of autonomous agents is attracting an increasing interest during the last years. In this paper, we proposed the use of the recent graphon theory to model and simulate an interacting system describing the evolution of individual opinions over an arbitrary size networks. Specifically, we prove the existence and uniqueness of the limit problem that approximates a very large networks made by homogeneous groups of agents. The significant new example is the mean field analysis deduced from the graphon limit systems in the case of piecewise constant graphon.

Keywords: Opinion Dynamics, Dynamics on Networks, Graphon, mean field

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1. Introduction

Opinion dynamics is an active research area that deals with the evolution of opinions through the social interaction between a group of individuals or agents. In order to understand this dynamics we have to define the individual’s opinion, we should explain how the agents interact, and we should include any external influence factor \cite{1,2}.

Opinions are described by both discrete or continuous quantities. Some examples about the first models are Sznajd model \cite{3}, voter model \cite{4}, majority rule model \cite{5}, and the Latané model of the social impact. In continuous models, the distribution of opinions are used to be represented by real numbers. Examples from this class of models include the DeGroot model \cite{6}, the FJ model \cite{7}, the Hegselmann-Krause model \cite{8}, and the Deffuant-Weisbuch model \cite{9}.

Opinion formation is a complex process affected by the interplay of different elements, including the individual predisposition, the influence of positive and negative interactions with other individual, the information each individual is exposed to, and many others. A social network can be considered as a set of people (or even groups of people) that participate and interact sharing different kinds of information with the purpose of friendship, decision making, business exchange or marketing. Then, a crucial aspect is the analysis of the different structures in the network to understand what may either facilitate or not the formation of collective belief. A natural way to represent these interactions is a network which can be described by a weighted graph $G^M = (V_M, E_M, B_M)$, $M > 1$, where the nodes’ set $V_M = \{1, 2, ..., M\}$ represents the set of individuals (agents), and an edge set $E_M \subseteq V_M \times V_M$, representing pairwise interactions between each pair of agents \cite{11,12,13}. The edges in the set $E_M$ are unoriented, $e = (i, j) = (j, i)$, $i, j \in V$. In this paper we assign a weight $B_M(e)$ to each edge $e \in E$, where $B_M : E \to [-K, K]$, with $K > 0$. $B_M((i, j))$ models the persuasiveness of the agent $i$ with respect to the agent $j$. We point out that we allow $B_M$ to be negative, that means that the one agent might oppose to another one ending in a farther opinions, together with

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agents that mediate their opinion (\(B_M\) positive). This model may be seen as reacher than pure spreading or concentration models (see, e.g., [14]). External factors are made by media or by the so-called opinion leaders, who are more active agents that transmit information without being affected by other agents.

An equivalent representation of the graph \(G^M\) is obtained by the weighted adjacency matrix \(B^M\) which is defined to be the square matrix \(M \times M\) such that an element \(B^M_{ij}\) is \(B_M(\epsilon)\) when there is an edge \(\epsilon\) from node \(i\) to node \(j\), and zero otherwise. To each node \(i\) we associate an opinion value \(u^M_i(t) \in \mathbb{R}\) at continuous time \(t \geq 0\). Then, we model the opinion dynamics by the following first order system of differential equations (see e.g. [3])

\[
\dot{u}^M_i(t) = \frac{1}{M} \sum_{j=1}^{M} B^M_{ij} (u^M_j(t) - u^M_i(t)) , \quad i = 1, 2, ..., M, \tag{1}
\]

with an initial opinion value \(u^M_i(0) = u^M_{i,0}\), and where we normalized the weights with the factor \(1/M\).

In order to study the dynamics of (1) for a very large network, as in the case of the recent social media with virtual interactions between people and services, we will use the so-called graphon theory which is a new paradigm for understanding the continuum limit of a graph sequence when the number of the nodes \(M\) goes to infinity [15, 16]. To consider a graphon associated to the graph \(G\) we partition the interval \([0, 1]\) into \(M\) isometric intervals \(I^M_j := ((j - 1)/M, j/M], j = 1, 2, ..., M\). The graphon \(W_{GM}(x, y) : [0, 1] \times [0, 1] \to \mathbb{R}\) corresponding to the graph \(G^M\) is defined by setting \(W_{GM}(x, y) = B^M_{ij}\) if \((x, y) \in I^M_i \times I^M_j\) (this also called pixel diagram, see [17]). In this work a graphon is a bounded symmetric Lebesgue measurable functions \(W : [0, 1] \times [0, 1] \to [-K, K], K > 0\), which can be interpreted as weighted graphs on the node set \([0, 1]\). In the space of graphons we define the so-called cut-norm

\[
\|W\|_\square = \sup_{S, T \subseteq [0, 1]} \left| \int_{S \times T} W(x, y) dx dy \right|,
\]

with the supremum taken over all measurable subsets \(S\) and \(T\) of \([0, 1]\). The following inequalities hold between norms on a graphon \(W\):

\[
\|W\|_\square \leq \|W\|_1 \leq \|W\|_2 \leq \|W\|_\infty \leq \text{const}.
\]

Now the vector function of the opinion can be identify with the piecewise constant function \(u_M(x, t) = \sum_{i=1}^{M} u_j(t) I^M_j(x)\), where \(I^M_j\) is the characteristic function of the interval \(I^M_j\). We assume that the entries of the \(M\)-th adjacency matrix satisfy

\[
\sup_{M \in \mathbb{N}} \sup_{j=1,2,\ldots,M} \frac{1}{M} \sum_{k=1}^{M} B^M_{kj} \leq C,
\]

for a suitable constant \(C > 0\), and the graphons \(W_{GM}\) converge to a graphon \(W\) in the cut-norm (see [17] Section III and the references therein). Then it is possible to show (see, e.g., [17, Theorem 4]) that the sequence of opinion vector functions \(u_M(x, t)\) converges in \(L^2([0, T], L^2([0, 1]))\) to the solution of the following graphon Chauchy problem

\[
\begin{cases}
\frac{\partial u}{\partial t}(x, t) = \int_{[0, 1]} W(x, y)(u(y, t) - u(x, t)) dy \\
u(x, 0) = u_0(x)
\end{cases}
\tag{2}
\]

In this case we can understand the opinion dynamics on a very large graph \(G_M\) by using the analytical properties of the solution of the mean-field equation (2). Therefore with the graphon approach we can simplify the study of the behavior of the solution of (1) when a simpler analytical solution may be achieved by (2). In the following we will consider as a significant example the case of the step function graphons. This approach might be used in the clustering problems for image analysis described in [18, 19] or in dynamical systems on networks as in [20, 21].
2. Theoretical analysis of the step function graphons

We consider the case of \( N \) subgroups (parties, homogeneous subset of people), and we suppose different communication between people of each subgroup with respect to people of different groups. A good approximation of this realistic situation can be obtained by considering step function graphons. The dynamics of the system leads to the study of the possible reinforcement of groups or to the fragmentation of opinion. Let \( N \) be fixed; we define a partition \( \mathcal{i} \) of length \( N \) as \( \mathcal{i} = (i_0, i_1, i_2, \ldots, i_{N-1}, i_N) \), where \( 0 = i_0 < i_1 < i_2 < \cdots < i_{N-1} \leq i_N = 1 \). For any interval \( I \subseteq [0, 1] \), we call \( \mathcal{P}_I \) the set of real functions on \([0, 1]\) constant in \( I \) and 0 outside: \( \mathcal{P}_I = \{ u = a \mathbb{1}_I, a \in \mathbb{R} \} \). Given a partition \( \mathcal{i} \), let \( \mathcal{P}_i \) be the set of real functions on \([0, 1]\) constant on \( i \): \( \mathcal{P}_i = \{ u = \sum_{j=1}^N a_j \mathbb{1}_{(i_j-1, i_j]} : a_j \in \mathbb{R} \} \). Moreover, we call \( \mathcal{P}_{\times \mathcal{i}} \) the set of real functions on \([0, 1] \times [0, 1]\) constant on the rectangular subdivisions, and we assume that \( W \in \mathcal{P}_{\times \mathcal{i}} \) for \((x, y) \in [0, 1]^2\)

\[
W(x, y) = \sum_{j_1, j_2 = 1}^N b_{j_1j_2} \mathbb{1}_{(i_{j_1-1}, i_{j_1}]}(x) \mathbb{1}_{(i_{j_2-1}, i_{j_2}]}(y).
\]

From now on, we assume that the graphon kernel \( W \in \mathcal{P}_{\times \mathcal{i}} \). For any interval \( I \subseteq [0, 1] \), we call \( L_0^2(I) \) the set of real functions \( u \in L^2([0, 1]) \) which are zero outside \( I \) and with zero mean: \( L_0^2(I) = \{ u \in L^2([0, 1]) : u = u \mathbb{1}_I, \int_{[0, 1]} u = 0 \} \). It is obvious that \( L^2([0, 1]) = \mathcal{P}_{\times \mathcal{i}} \oplus \bigoplus_{j=1}^N L_0^2((i_{j-1}, i_j]) \), where the sum involves orthogonal closed subspaces of \( L^2([0, 1]) \). In fact, for any \( u \in L^2([0, 1]) \), we have

\[
u(x) = \sum_{i=1}^N \mathbb{1}_{(i_{j-1}, i_j]}(x) \left( u(x) - \frac{\int_{(i_{j-1}, i_j]} u(y)dy}{i_j - i_{j-1}} + \frac{\int_{(i_{j-1}, i_j]} u(y)dy}{i_j - i_{j-1}} \right)
\]

\[
= \left( \sum_{i=1}^N \mathbb{1}_{(i_{j-1}, i_j]}(x) \left( \frac{\int_{(i_{j-1}, i_j]} u(y)dy}{i_j - i_{j-1}} \right) + \sum_{i=1}^N \mathbb{1}_{(i_{j-1}, i_j]}(x) \right) \left( u(x) - \frac{\int_{(i_{j-1}, i_j]} u(y)dy}{i_j - i_{j-1}} \right) \tag{3}
\]

We prove the following theorem.

**Theorem 2.1.** Let \( W \in \mathcal{P}_{\times \mathcal{i}} \). There exists a unique solution \( u \in C([0, T], L^2([0, 1])) \) of (\textit{2}) for any initial condition \( u_0 \in L^2([0, 1]) \) and \( T > 0 \).

**Proof. Uniqueness.** Multiplying both terms of (\textit{2}) by \( u(x, t) \) and integrating on \( x \) we obtain

\[
\frac{d}{dt} \int_0^1 u^2(x, t)dx = \int_0^1 \left( \int_{[0, 1]} W(x, y)u(y, t) - u(x, t)dy \right) u(x, t)dx \leq 2C \int_0^1 u^2(x, t)dx
\]

which implies the uniqueness by Gronwall Lemma, since \( ||u_t||_2 \leq ||u_0||_2 \exp(2Ct) \).

**Existence.** The superposition principle allows us to find the solutions of the Graphon Cauchy problem (\textit{2}) on each component of the orthogonal decomposition \( L^2([0, 1]) = \mathcal{P}_{\times \mathcal{i}} \oplus \bigoplus_{j=1}^N L_0^2((i_{j-1}, i_j]) \) given above. In fact, it is sufficient to use (\textit{3}) to decompose the initial condition, if \( u_0 \in L^2([0, 1]) \).

**Solution starting in \( \mathcal{P}_i \).** When \( u_0 \in \mathcal{P}_i \), it may be written as \( u_0(x) = \sum_{j=1}^N u_{0,j} \mathbb{1}_{(i_{j-1}, i_j]}(x) \). Denote by \( u_0 = (u_0_1, \ldots, u_0_N)^T \) and by \( \mathbb{1}_x = (\mathbb{1}_{(i_{0}, i_1]}, \mathbb{1}_{(i_1, i_2]}, \ldots, \mathbb{1}_{(i_{N-1}, i_N]})^T \), so that \( u_0(x) = \mathbb{1}_x^T u_0 \). We claim that the (unique) solution of (\textit{2}) belongs to \( \mathcal{P}_i \) for any \( t \geq 0 \) and it has the form

\[
u(x, t) = \mathbb{1}_x^T u(t) = \mathbb{1}_x^T \exp(-\Delta_{W_i}t)u_0,
\]

where \( \Delta_{W_i} \) is the Laplacian of the \( N \times N \) matrix \( W_i \) given by

\[
[W_i]_{j_1j_2} = b_{j_1j_2}(i_{j_2} - i_{j_2-1}) = \int_{[0, 1]} \mathbb{1}_{(i_{j_2-1}, i_{j_2}]}(y)W(x, y)dy \quad \text{for any } x \in \mathbb{1}_{(i_{j_1-1}, i_{j_1}]}. 
\]
In fact:

\[ \mathbb{I}_4(x)^\top \text{diag}(W_4 \mathbf{1}) = \left( \int_{[0,1]} W(x, y) dy \right) \mathbb{I}_4(x)^\top, \quad \mathbb{I}_4(x)^\top W_4 = \int_{[0,1]} W(x, y) \mathbb{I}_4(y)^\top dy. \]  

(4)

In fact:

- by definition, \( u(\cdot, t) \in \mathcal{P}_4 \) for any \( t \geq 0 \) and \( u(x, 0) = u_0(x) \);
- by definition of exponential matrix and by (1), the equation (2) is satisfied, since

\[
\frac{\partial u}{\partial t}(x, t) = \mathbb{I}_4(x)^\top ( -\Delta W_4 ) \exp( -\Delta W_4 t ) u_0 = \mathbb{I}_4(x)^\top (W_4 - \text{diag}(W_4 \mathbf{1})) \exp( -\Delta W_4 t ) u_0 \\
= \int_{[0,1]} W(x, y) \left[ \mathbb{I}_4(y)^\top \exp( -\Delta W_4 t ) u_0 \right] dy - \int_{[0,1]} W(x, y) \left[ \mathbb{I}_4(x)^\top \exp( -\Delta W_4 t ) u_0 \right] dy \\
= \int_{[0,1]} W(x, y) (a(y, t) - u(x, t)) dy.
\]

Remark. Note that the Laplacian of a square matrix does not depend on the values on its diagonal. As an expected consequence, \( (W_4)_{jj} \) does not affect the action of (2) on \( \mathcal{P}_4 \), but it has an effect on \( \oplus_{j=1}^N L_0^2((i_{j-1}, i_j)) \).

Solution starting in \( L_0^2((i_{j-1}, i_j)) \). Fixed \( j \in \{1, \ldots, N \} \), for \( u_0 \in L_0^2((i_{j-1}, i_j)) \), we claim that the (unique) solution of (2) has the form \( u(x, t) = u_0(x) \exp( -\mu t ) \), where \( \mu = \sum_{j=2}^N b_{j,j}(i_{j-1} - i_{j-2}) = (W_4 \mathbf{1})_j \).

In fact:

- since \( u_0 \in L_0^2((i_{j-1}, i_j)) \) and \( u_0(\cdot) = u_0 \mathbb{I}_{(i_{j-1}, i_j)}(\cdot) = u(x, 0) \), then \( u(\cdot, t) \in L_0^2((i_{j-1}, i_j)) \) for any \( t \geq 0 \);
- the equation (2) is satisfied, again since \( u_0(\cdot) = u_0(\cdot) \mathbb{I}_{(i_{j-1}, i_j)}(\cdot) \), and hence

\[
\frac{\partial u}{\partial t}(x, t) = -\mu u_0(x) \exp( -\mu t ) \\
= \exp( -\mu t ) \left( 0 - u_0(x) \sum_{j_2=1}^N b_{j,j}(i_{j-1} - i_{j-2}) \right) \\
= \exp( -\mu t ) \left( \left[ \sum_{j_2=1}^N b_{j,j} \mathbb{I}_{(i_{j-1}, i_j)}(x) \right] \int_{i_{j-1}}^{i_j} u_0(y) dy \\
- u_0(x) \int_{[0,1]} \mathbb{I}_{(i_{j-1}, i_j)}(x) \sum_{j_2=1}^N b_{j,j} \mathbb{I}_{(i_{j-2}, i_{j-2})}(y) dy \right) \\
= \exp( -\mu t ) \left( \int_{0}^{1} W(x, y) u_0(y) \mathbb{I}_{(i_{j-1}, i_j)}(y) dy - u_0(x) \mathbb{I}_{(i_{j-1}, i_j)}(x) \int_{0}^{1} W(x, y) dy \right) \\
= \exp( -\mu t ) \left( \int_{0}^{1} W(x, y) (u(y, t) - u(x, t)) dy \right) = \int_{[0,1]} W(x, y) (u(y, t) - u(x, t)) dy.
\]

3. Dynamics for symmetric cases with \( N = 3 \) groups

Let \( N = 3 \) and \( i = (i_0 = 0, i_1, i_2, i_3 = 1) = 1 \) and assume \( W_4 \) to be symmetric. Then

\[
W_i = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}, \quad \Delta W_i = \begin{pmatrix} a_{12} + a_{13} & -a_{12} & -a_{13} \\ -a_{12} & a_{23} + a_{12} & -a_{23} \\ -a_{13} & -a_{23} & a_{13} + a_{23} \end{pmatrix}
\]
where \( a_{j_1j_2} = b_{j_1j_2}(i_{j_2} - i_{j_2-1}) \) and \( 0 < j_1 \leq j_2 \leq 3 \). Denoting by

\[
\Delta = \sqrt{a_{12}^2 + a_{13}^2 + a_{23}^2 - a_{12}a_{13} - a_{12}a_{23} - a_{13}a_{23}} = \sqrt{\frac{(a_{12} - a_{13})^2 + (a_{12} - a_{23})^2 + (a_{13} - a_{23})^2}{2}}
\]

we get the eigenvalues of \( \Delta W_i \):

\[
\lambda_1 = 0, \quad \lambda_2 = a_{12} + a_{13} + a_{23} + \Delta, \quad \lambda_3 = a_{12} + a_{13} + a_{23} - \Delta
\]

and the three corresponding orthogonal eigenvectors

\[
v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} a_{23} - a_{12} - \Delta \\ a_{12} - a_{13} + \Delta \\ a_{13} - a_{23} \end{pmatrix}, \quad v_3 = \begin{pmatrix} a_{23} - a_{12} + \Delta \\ a_{12} - a_{13} - \Delta \\ a_{13} - a_{23} \end{pmatrix}, \quad V = (v_1 | v_2 | v_3)
\]

so that \( \Delta W_i V = V \text{diag}(\lambda_1, \lambda_2, \lambda_3) \). Moreover, any symmetric matrix has orthogonal eigenspaces, and hence \( V^\top V = \text{diag}(||v_1||^2, ||v_2||^2, ||v_3||^2) \). We note that \( \exp(-\Delta W_i t) = V \text{diag}(1, e^{-\lambda_2 t}, e^{-\lambda_3 t}) V^{-1} \), where \( V^{-1} = \text{diag}(1/||v_1||^2, 1/||v_2||^2, 1/||v_3||^2) V^\top \).

Now, let \( u_0(x) \in L^2([0, 1]) \) be the initial condition; we can immediately find the solution \( u(x, t) = \mathbb{1}_i(x)^\top u(t) \) of the component in \( \mathcal{P}_i \) with

\[
\begin{align*}
\mathbf{u}_0 &= \begin{pmatrix}
\int_{i_{-1}}^{i_{1}} u_0(y)dy \\
\int_{i_{1}}^{i_{2}} u_0(y)dy \\
\int_{i_{3}}^{1} u_0(y)dy
\end{pmatrix}
\mathbf{u}(t) &= \exp(-\Delta W_i t) \mathbf{u}_0 \\
&= V \text{diag}(1, e^{-\lambda_2 t}, e^{-\lambda_3 t}) \text{diag}(1/||v_1||^2, 1/||v_2||^2, 1/||v_3||^2) V^\top \mathbf{u}_0.
\end{align*}
\]

The vector \( \mathbf{u}_0 \) gives that barycenter of the three groups’ opinion. Note that the “barycenter” \( b(t) \) of the whole system does not change with \( t \), since \( v_1 = \mathbb{1}^\top \) is a (left) kernel vector of \( \Delta W_i \), and hence of \( \exp(-\Delta W_i t) \).

In fact, let \( b(t) = \mathbb{1}^\top \mathbf{u}(t) \). Since \( \mathbb{1}^\top V = (1, 0, 0)^\top \), we have \( b(t) = \mathbb{1}^\top \mathbf{u}_0 = \mathbb{1}^\top \mathbf{u} = b(0) \).

By superposing the solution with that of \( E \bigcup ((i_{j_1}, i_{j_2})] \), we find the final solution:

\[
\begin{align*}
group 1: \quad & \forall x \in [0, t_1], \quad u(x, t) = u_1(t) + (u_0(x) - u_01) \exp(-a_{11} + a_{12} + a_{13})t \\
group 2: \quad & \forall x \in (t_1, t_2], \quad u(x, t) = u_2(t) + (u_0(x) - u_02) \exp(-a_{12} + a_{22} + a_{23})t \\
group 3: \quad & \forall x \in (t_2, 1], \quad u(x, t) = u_3(t) + (u_0(x) - u_03) \exp(-a_{13} + a_{23} + a_{33})t.
\end{align*}
\]

3.1. Right, center, and left parties

Assume \( 0 \leq \min(a_{12}, a_{23}) \) and \( a_{12} + a_{23} > 0 \). In other words, we think that the second group acts as a political “center”, or mediator, that tries to attract the other groups toward a common center. The other groups may have a conflict relationship \( (a_{13} < 0) \) and the strength of this conflict may lead to different scenarios. We start by noticing that the assumption on the second group implies that, when \( a_{13} < 0 \),\( \Delta^2 \geq \frac{a_{13}^2 + (a_{12} - a_{23})^2 + a_{12}^2}{2} \), \( \Delta^2 \geq |a_{13}| \), which implies that \( \lambda_2 > 0 \) \( ( \text{that is trivially true for } a_{13} \geq 0) \). Now, \( \lambda_2 > 0 \) forces the initial component in \( v_2 \) to collapse to 0 (see (5)). The possible different scenarios depend on the sign of \( \lambda_3 \), which is always lower than \( \lambda_2 \).

Dynamics of the barycenter of the three groups as function of \( \lambda_3 \). We have, for what regards the dynamics on \( \mathcal{P}_i \), with respect to \( a_{13} \)

\[
\lambda_3 > 0: \quad \text{this result is achieved when } a_{13} > -\frac{a_{12} + a_{23}}{a_{12} + a_{23}}. \quad \text{With } \lambda_3 > 0, \text{ also the component in } v_3 \text{ is forced to tend to 0 by (5). Then the limit invariant subspace of } \mathcal{P}_i \text{ is the constant one: } \mathbf{u}(t) \rightarrow b(0) \mathbf{1}, \text{ i.e. all the three barycenter collapse into a single point determined by the initial conditions;}
\]
\[ \lambda_3 = 0: \text{ when } a_{13} = -\frac{a_{12}a_{23}}{a_{12} + a_{23}}, \text{ by substituting we get } \lambda_3 = 2(a_{12} + a_{13} + a_{23}) = 2\frac{a_{12}^2 + a_{23}^2}{a_{12} + a_{23}}, \text{ and} \]

\[ \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -2a_{12} + \frac{a_{12}a_{23}}{2a_{12} + a_{23}} \\ 2a_{12} + a_{23} \\ -a_{23} \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2a_{23} + \frac{a_{12}a_{23}}{a_{12} + a_{23}} \\ -a_{23} - \frac{a_{12}a_{23}}{a_{12} + a_{23}} \\ -a_{23} \end{pmatrix}. \]

The fact that \( \lambda_3 = 0 \) leaves the component on \( \mathbf{v}_3 \) unchanged as that on \( \mathbf{v}_1 \), and then by (5) there will be the formation of three different barycenter of the groups in

\[ \mathbf{u}_\infty = \lim_{t \to \infty} \mathbf{u}(t) = \frac{1}{3} \mathbf{u}_0 + \mathbf{v}_3 \mathbf{u}_0 / \| \mathbf{v}_3 \|. \]

\( \lambda_3 < 0: \text{ for } a_{13} < -\frac{a_{12}a_{23}}{a_{12} + a_{23}}, \text{ the barycenter of the whole system remains constant (the component on } \mathbf{v}_1), \text{ while the barycenter of the three clusters diverge by } (5). \]

**Dynamics inside each group with respect to its own barycenter.**

The dynamic in each \( L_0^3(\{i_{j-1, i_j}\}) \) depends on the sign of \( \mu_j = (W_j^t)_{ij} = \sum_{j=1}^{3} a_{jjj} \). In fact, as a consequence of the proof above, \( (u(x, t) - u_j(t))\mathbb{I}_{\{i_{j-1, i_j}\}}(x) = \exp(-\mu_j t). \) As an example, for the second group (the other are similar), we have \( \mu_2 = a_{12} + a_{22} + a_{23}. \) Hence

\( \mu_2 > 0: \text{ the second group will collapse on its barycenter;} \)

\( \mu_2 = 0: \text{ in this equilibrium phase, the points of the second group are translated rigidly with their barycenter.} \) This happens when the repulsion inside the center party equalizes the action of this party on the opposite ones: \( a_{22} = -(a_{12} + a_{23}); \)

\( \mu_2 < 0: \text{ the repulsion inside the center party is so big } (a_{22} < -(a_{12} + a_{23})) \text{ that its points will explode exponentially fast around its barycenter.} \)

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