ASYMPTOTICS OF INSTABILITY ZONES OF THE
HILL OPERATOR WITH A TWO TERM POTENTIAL

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Abstract. Let $\gamma_n$ denote the length of the $n$-th zone of instability of the Hill operator $Ly = -y'' - [4\alpha \cos 2x + 2\alpha^2 \cos 4x]y$, where $\alpha \neq 0$, and either both $\alpha, t$ are real, or both are pure imaginary numbers. For even $n$ we prove: if $t, n$ are fixed, then for $\alpha \to 0$

$$\gamma_n = \frac{8\alpha^n}{2^{n[(n-1)!]^2} n^{n/2} \prod_{k=1}^{n/2} (t^2 - (2k-1)^2)} \left(1 + O(\alpha)\right),$$

and if $\alpha, t$ are fixed, then for $n \to \infty$

$$\gamma_n = \frac{8|\alpha/2|^n}{2 \cdot 4 \cdot \cdots (n-2)!^2} \left|\cos\left(\frac{\pi}{2}\right)\right| \left[1 + O\left(\log \frac{n}{n}\right)\right].$$

Similar formulae (see Theorems 7 and 9) hold for odd $n$. The asymptotics for $\alpha \to 0$ imply interesting identities for squares of integers (see Sect. 4, Thm. 8).

1. Introduction. Main Results.

The Schrödinger operator, considered on $\mathbb{R}$,

$$Ly = -y'' + v(x)y,$$

with a real-valued periodic $L^2([0, \pi])$-potential $v(x)$, $v(x + \pi) = v(x)$, has spectral gaps, or instability zones, $(\lambda_n^-, \lambda_n^+), n \geq 1$, close to $n^2$ if $n$ is large enough. The points $\lambda_n^-, \lambda_n^+$ could be determined as eigenvalues of the Hill equation

$$-y'' + v(x)y = \lambda y,$$

considered on $[0, \pi]$ with boundary conditions

$$Per^+ : \quad y(0) = y(\pi), \quad y'(0) = y'(\pi),$$

for even $n$, and

$$Per^+ : \quad y(0) = -y(\pi), \quad y'(0) = -y'(\pi),$$

for odd $n$. See basics and details in [8, 16, 18, 19, 24].

The rate of decay of the sequence of spectral gaps $\gamma_n = \lambda_n^+ - \lambda_n^-$ is closely related to the smoothness of the potential $v$. We’ll mention now only Hochshtadt’s result [12] that an $L^2([0, \pi])$-potential $v$ is in $C^\infty$ if
and only if \((\gamma_n)\) decays faster than any power of \(1/n\), and Trubowitz’s result \[26\] that an \(L^2([0,\pi])\)-potential \(v\) is analytic if and only if \((\gamma_n)\) decays exponentially. See further references and later results in \[2, 3\].

In the case of specific potentials, like the Mathieu potential
\[
v(x) = 2a \cos 2x, \quad a \neq 0, \text{ real},
\]
or more general trigonometric polynomials
\[
v(x) = \sum_{-N}^{N} c_k \exp(ikx), \quad c_k = c_{-k}, \quad 0 \leq k \leq N < \infty,
\]
one comes to two classes of questions:
(i) Is the \(n\)-th zone of instability closed, i.e.,
\[
\gamma_n = \lambda_n^+ - \lambda_n^- = 0,
\]
or, equivalently, is the multiplicity of \(\lambda_n^+\) equal to 2?
(ii) If \(\gamma_n \neq 0\), could we tell more about the size of this gap, or, for large enough \(n\), what is the asymptotic behavior of \(\gamma_n = \gamma_n(v)\)?

E. L. Ince \[13\] answered in a negative way Question (i) in the case of the potential \[1.5\]: the Mathieu-Hill operator has only simple eigenvalues both for \(Per^+\) and \(Per^-\) boundary conditions, i.e., all zones of instability of the Mathieu–Schrödinger operator are open. His proof is presented in \[8\]; see other proofs of this fact in \[11, 20, 21\], and further references in \[8, 27\].

For fixed \(n\) and \(a \to 0\), D. Levy and J. Keller \[17\] gave the asymptotics of \(\gamma_n = \gamma_n(a)\), \(a \in \mathbb{R}\); namely
\[
\gamma_n = \lambda_n^+ - \lambda_n^- = \frac{8(|a|/4)^n}{[(n-1)!]^2} \left(1 + O(1)\right),
\]

Almost 20 years later, E. Harrell \[10\] found, up to a constant factor, the asymptotics of the spectral gaps of the Mathieu potential \[1.5\] as \(n \to \infty\). J. Avron and B. Simon \[1\] gave an alternative proof of E. Harrell’s asymptotics and found the exact value of the constant factor, which led to the following formula
\[
\gamma_n = \lambda_n^+ - \lambda_n^- = \frac{8(|a|/4)^n}{[(n-1)!]^2} \left(1 + O\left(\frac{1}{n^2}\right)\right).
\]

Let us mention that in \[4, 6\] we found the asymptotics of the spectral gaps of 1D Dirac operator with cosine potential and multiplicities of all periodic and antiperiodic eigenvalues.

J. Avron and B. Simon \[1\] raised the question about the asymptotics of spectral gaps in the case of a two term potential
\[
v(x) = a \cos 2x + b \cos 4x.
\]
A. Grigis [9] asked essentially the same question for the isospectral potential \( v(x) = a \sin 2x - b \cos 4x. \)

We found such asymptotics. Our results (see below) are announced in [5], and the present paper gives their detailed proofs.

Put for real \( a, b \neq 0 \)

\[
(1.11) \quad a = -4\alpha t, \quad b = -2\alpha^2,
\]

where either

(1.11a) both \( \alpha \) and \( t \) are real (if \( b < 0 \))

or

(1.11b) both \( \alpha \) and \( t \) are pure imaginary (if \( b > 0 \)).

This special parametrization comes from the Magnus-Winkler analysis [18, 27] of this Hill operator [Whittaker operator in their terminology]. Our paper [7] is essentially an algebraic introduction to the present article. In [7] we sharpen the Magnus-Winkler result on existence of finitely many zones of instability in the case of the integers \( t \) in (1.11). The special role of integer \( t \)'s is incorporated into coefficients in the asymptotics of \( \gamma_n(\alpha), \alpha \to 0 \), and \( \gamma_n(v), n \to \infty \), with \( v \in (1.10) + (1.11) \). Namely (see Thm. 7), if \( t \) and \( n \) are fixed, then for even \( n \)

\[
(1.12) \quad \gamma_n = \frac{8\alpha^n}{2^n[(n-1)!!]^2} \prod_{k=1}^{n/2} \left| t^2 - (2k - 1)^2 \right| \left( 1 + O(\alpha) \right),
\]

and for odd \( n \)

\[
(1.13) \quad \gamma_n = \frac{8\alpha^nt}{2^n[(n-1)!!]^2} \prod_{k=1}^{(n-1)/2} \left| t^2 - (2k)^2 \right| \left( 1 + O(\alpha) \right).
\]

This could be compared with the Levy-Keller statement (1.8) above.

If \( \alpha \neq 0, t \neq 0 \) are fixed, then (see Thm. 9) the following asymptotic formulae hold as \( n \to \infty \) : for even \( n \)

\[
(1.14) \quad \gamma_n = \frac{8|\alpha/2|^n}{[2 \cdot 4 \cdot \ldots (n-2)]^2} \left| \cos \left( \frac{\pi t}{2} \right) \right| \left[ 1 + O \left( \frac{\log n}{n} \right) \right],
\]

and for odd \( n \)

\[
(1.15) \quad \gamma_n = \frac{8|\alpha/2|^n}{[1 \cdot 3 \cdot \ldots (n-2)]^2 \pi} \left| \frac{2}{\sin \left( \frac{\pi t}{2} \right) \left[ 1 + O \left( \frac{\log n}{n} \right) \right]}.\]

This result could be compared with the Harrell–Avron–Simon formula (1.9) above.

Asymptotics (1.12)-(1.13) imply interesting identities for squares of integers (see Sect. 4, Thm 8).
Our proofs are based on an almost explicit formula (see Thm. 6) for \( \gamma_n \) in terms of Fourier coefficients of the potential

\[
v(x) = \sum_{k \text{ even}} V_k e^{ikx}, \quad x \in [0, 2\pi].
\]

We proved it in [2], Thm. 8. For convenience, we give in Sect. 2 all details to adjust this formula to both cases \( \alpha \to 0 \) (\( n, t \) fixed), and \( n \to \infty \) (\( \alpha, t \) fixed).

2. Preliminaries

1. Let \( v \) be a periodic function of a period \( \pi \). The differential operator

\[
L_0 y = -y'' + v(x)y, \quad x \in [0, \pi],
\]

considered with periodic boundary conditions

\[
\text{Per}^+: \quad y(0) = y(\pi), \quad y'(0) = y'(\pi),
\]

or antiperiodic boundary conditions

\[
\text{Per}^-: \quad y(0) = -y(\pi), \quad y'(0) = -y'(\pi),
\]

is known as the Hill operator with potential \( v \). It is self-adjoint for real-valued potentials.

Consider the operator

\[
L_0^0 y = -y''.
\]

The periodic and antiperiodic spectra of \( L_0^0 \) are discrete, and we have

\[
\sigma_{\text{Per}^+} = \{n^2, \ n \text{ even}\}, \quad \sigma_{\text{Per}^-} = \{n^2, \ n \text{ odd}\}.
\]

Moreover, each eigenvalue \( n^2 \neq 0 \) is of multiplicity 2, and

\[
e_{-n} = e^{-inx}, \quad e_n = e^{inx}
\]

are eigenfunctions corresponding to \( n^2 \). So, if we consider periodic boundary conditions, then \( \lambda = 0 \) is the only eigenvalue of \( L_0^0 \) of multiplicity 1, and the constant function \( e_0 = 1 \) is the corresponding normalized eigenfunction.

If \( L^2([0, \pi]) \) is considered with the scalar product

\[
(f, g) = \frac{1}{\pi} \int_0^\pi f(x)g(x)dx,
\]

then each of the families of functions \( \{e_{2k}, k \in \mathbb{Z}\} \) and \( \{e_{2k-1}, k \in \mathbb{Z}\} \) is an orthonormal basis in \( L^2([0, \pi]) \). The basis \( \{e_{2k}, k \in \mathbb{Z}\} \) (respectively, \( \{e_{2k-1}, k \in \mathbb{Z}\} \)) is used when we study the periodic (respectively, antiperiodic) spectra of \( L \).
We always assume that \( v \in L^2([0, \pi]) \) and denote by \( \|v\| \) its \( L^2 \)-norm. Since \( v \) has a period \( \pi \) its Fourier series can be written in the form

\[
v(x) = \sum_{m \in \mathbb{Z}} V(m) \exp(imx), \quad V(m) = 0 \text{ for odd } m;
\]

then \( \|v\|^2 = \sum |V(m)|^2 \).

It is well known (see [18], Thm 2.1, or [8], Thm 2.3.1) that the periodic and antiperiodic spectra of \( L \) are discrete, and moreover, there is a sequence of real numbers

\[
\lambda_0 < \lambda_1^- \leq \lambda_1^+ < \lambda_2^- \leq \lambda_2^+ < \lambda_3^- \leq \lambda_3^+ < \lambda_4^- < \cdots
\]

such that the terms with even (respectively, odd) indices give the periodic (respectively, antiperiodic) spectra of \( L \). We have

\[
\lambda_0 = 0, \quad \lambda_n^- = \lambda_n^+ = n^2, \quad \forall n \in \mathbb{N} \text{ if } v(x) \equiv 0.
\]

The Hill operator \( L = L^0 + v \) may be regarded as a perturbation of \( L^0 \). A perturbation type argument shows that if \( \|v\| \) is small then \( \lambda_0 \) is close to 0, and \( \lambda_n^- , \lambda_n^+ \) are close to \( n^2 \).

The following proposition states and proves this fact in a more precise form. Of course, different versions of this statement are well known but we give the version which is convenient for our purposes.

**Proposition 1.** (a) If \( \|v\| \leq 1/4 \), then

\[
(2.4) \quad |\lambda_0| \leq 4\|v\| \quad \text{and} \quad |\lambda_n^+ - n^2| \leq 4\|v\| \quad \text{for } n \in \mathbb{N}.
\]

(b) If \( V(0) = \frac{1}{\pi} \int_0^\pi v(x)dx = 0 \), then there is a constant \( N_0 = N_0(v) \) such that

\[
(2.5) \quad |\lambda_n^+ - n^2| < 1 \quad \text{for } n \geq N_0.
\]

**Proof.** A proof of Part (b) can be found in [19], Thm. 1.5.2, p. 76. We need only the case when Thm. 1.5.2 claims, as it is observed there on p. 241, (3.4.5), that if \( V(0) = 0 \), i.e., \( a_1 = 0 \) in (3.4.5), then

\[
(2.6) \quad \lambda_n^\pm = n^2 + \epsilon_n^\pm, \quad E^2 = \sum_{n=0}^{\infty} |\epsilon_n^\pm|^2 < \infty.
\]

If \( N_0 \) is chosen so that

\[
(2.7) \quad \sum_{n=N_0}^{\infty} |\epsilon_n^\pm|^2 < 1,
\]

then of course (2.6) and (2.7) imply (2.5). It is not said in [19] explicitly but, by following the proofs of Thm. 1.5.2 and its preliminaries, one can get estimates of \( E \) in terms of the norm \( \|v\| \) and explain (2.4), maybe with other (absolute) constants, instead of 1/4 and 4. To avoid any
doubts (or careful reading of tens pages in [19]), we give an alternative proof of Proposition 1 as an exercise in Perturbation Theory.

We use Fourier analysis, by considering the basis \((e_k)_{k \in \mathbb{Z}}\) in the periodic case, and the basis \((e_k)_{k \in \mathbb{Z} - 1}\) in the antiperiodic case. Of course, each operator in \(L^2([0, \pi])\) is identified with the corresponding operator in \(\ell^2(2\mathbb{Z})\) or \(\ell^2(2\mathbb{Z} - 1)\) (and with the corresponding matrix representation).

Let \(V\) denote the operator \(y \mapsto v(x)y\), and let \(R_0^\lambda = (\lambda - L^0)^{-1}\). The matrix representations of \(R_0^\lambda\) and \(V\) are

\[
(R_0^\lambda)_{km} = \frac{1}{\lambda - m^2} \delta_{km}, \quad V_{km} = V(k - m).
\]

Since \(L = L^0 + V\) we have

\[
\lambda - L = \lambda - L^0 - V = (\lambda - L^0)(1 - R_0^\lambda V);
\]

thus, whenever \(\|R_0^\lambda V\| < 1\), the operator

\[
(2.8) \quad R_\lambda = (\lambda - L)^{-1} = (1 - R_0^\lambda V)^{-1} R_0^\lambda = \sum_{k=0}^{\infty} (R_0^\lambda V)^k R_0^\lambda.
\]

is well defined.

The matrix representations of \(R_0^\lambda V\) is

\[
(2.9) \quad (R_0^\lambda V)_{km} = \frac{V(k - m)}{\lambda - k^2}.
\]

Since the operator norm in \(\ell^2\)-norm does not exceed the Hilbert-Schmidt norm, we have

\[
(2.10) \quad \|R_0^\lambda V\|^2 \leq \sum_k \frac{1}{|\lambda - k^2|^2} \cdot \sum_m |V(k - m)|^2 = A(\lambda) \cdot \|v\|^2.
\]

with \(A(\lambda) = A^+(\lambda)\) if \(bc = Per^+\), and \(A(\lambda) = A^- (\lambda)\) if \(bc = Per^-,\)

where

\[
A^+(\lambda) = \sum_{k \in \mathbb{Z}} \frac{1}{|\lambda - k^2|^2}, \quad A^- (\lambda) = \sum_{k \in \mathbb{Z} - 1} \frac{1}{|\lambda - k^2|^2}.
\]

To estimate \(A(\lambda)\) we need the following lemma.

**Lemma 2.** For each \(n \in \mathbb{N}\)

\[
(2.11) \quad \sum_{k \neq \pm n \atop k \in n + 2\mathbb{Z}} 1 < \frac{9}{n^2} \quad \text{if} \quad (n - 1)^2 \leq \text{Re } \lambda \leq (n + 1)^2, \quad \lambda \in \mathbb{C}.
\]
Proof. Let \( k \in n + 2\mathbb{Z} \) in all sums that appear in the proof. The sum in (2.11) does not exceed \( 2S_1(n) + 2S_2(n) \), where

\[
S_1(n) = \sum_{0 \leq k < n-1} \frac{1}{[(n-1)^2 - k^2]^2}, \quad S_2(n) = \sum_{k > n+1} \frac{1}{[k^2 - (n+1)^2]^2}.
\]

Obviously,

\[
S_1(1) = 0, \quad S_1(2) = 1, \quad S_1(3) = \frac{1}{9}.
\]

If \( n \geq 4 \), then, by the inequality

\[
(n-1)^2 - k^2 = (n-1-k)(n-1+k) \geq (n-1-k)(n-1), \quad 0 \leq k < n-1,
\]

we have

\[
S_1(n) \leq \frac{1}{(n-1)^2} \sum_{0 \leq k < n-1} \frac{1}{(n-1-k)^2} < \frac{1}{(n-1)^2} \frac{\pi^2}{8} < \frac{2}{n^2} \frac{\pi^2}{8}, \quad n \geq 4.
\]

Also,

\[
k^2 - (n+1)^2 = (k+n+1)(k-n-1) > 2n(k-n-1) \quad \text{if} \quad k > n+1,
\]

and therefore,

\[
S_2(n) \leq \frac{1}{4n^2} \sum_{k > n+1} \frac{1}{(k-n-1)^2} = \frac{1}{4n^2} \frac{\pi^2}{8}.
\]

Now, (2.14) and (2.15) yield that the sum (2.11) does not exceed \( 9\pi^2/(16n^2) < 6/n^2 \) for \( n \geq 4 \). Since \( \pi^2 \leq 10 \), we obtain, by (2.15), that

\[
S_2(1) \leq \frac{\pi^2}{32} < \frac{1}{3}, \quad S_2(2) \leq \frac{\pi^2}{128} < \frac{1}{12}, \quad S_2(3) \leq \frac{\pi^2}{288} < \frac{1}{28};
\]

thus, in view of (2.13), the inequalities (2.11) hold for each \( n \in \mathbb{N} \). \( \square \)

Let

\[
H_0 = \{ z \in \mathbb{C} : \text{Re } z \leq 1 \}, \quad H_1 = \{ z \in \mathbb{C} : \text{Re } z \leq 4 \},
\]

(2.16) \( H_n = \{ z \in \mathbb{C} : (n-1)^2 \leq \text{Re } z \leq (n+1)^2 \}, \quad n \geq 2, \) and

(2.17) \( D_n(r) = \{ z \in \mathbb{C} : |z - n^2| < r \}, \quad r > 0, \ n \in \mathbb{Z}_+. \)

Next, we estimate the norm of \( R^\lambda V \), (or, \( A(\lambda) \), see (2.10)) for \( \lambda \in \mathbb{C} \setminus \mathcal{D}^+ \) if \( bc = \text{Per}^+ \), and for \( \lambda \in \mathbb{C} \setminus \mathcal{D}^- \) if \( bc = \text{Per}^- \), where

\[
D^+ = \bigcup_{k \in 2\mathbb{N}} D_k(r), \quad D^- = \bigcup_{k \in 2\mathbb{N}-1} D_k(r), \quad r = 4\|v\|.
\]

(2.18)
By (2.16) and (2.17),
\[ \mathbb{C} = \bigcup_{k \in 2\mathbb{N}} H_k = \bigcup_{k \in 2\mathbb{N} - 1} H_k, \]
and therefore, in view of (2.19),
\[ \mathbb{C} \setminus D^+ = \bigcup_{k \in 2\mathbb{N}} (H_k \setminus D_k(r)), \quad \mathbb{C} \setminus D^- = \bigcup_{k \in 2\mathbb{N} - 1} (H_k \setminus D_k(r)). \]
If \( \lambda \in H_n \setminus D_n(r), n \geq 2, \) then (2.11) from Lemma 2 yields
\[ A(\lambda) \leq \frac{2}{r^2} + 9, \quad \lambda \in H_n \setminus D_n(r), \ n \geq 2. \]
If \( \lambda \in H_1, \) then \( \sup\{A(\lambda) : \Re \lambda \leq 0\} \leq A(0); \) thus, by (2.11) with \( n = 1, \) we have
\[ A(\lambda) \leq \frac{2}{r^2} + 9, \quad \lambda \in H_1 \setminus D_1(r). \]
If \( \lambda \in H_0, \) i.e., \( \Re \lambda \leq 1, \) then
\[ \sum_{k \in 2\mathbb{N}} \frac{1}{|\lambda - k^2|^2} \leq \sum_{k \in 2\mathbb{N}} \frac{1}{|k^2 - 1|^2} \leq \frac{1}{9} \sum_{k \in 2\mathbb{N}} \frac{1}{(k-1)^2} = \frac{\pi^2}{72} \leq \frac{1}{7} \]
(because \( k^2 - 1 = (k+1)(k-1) \geq 3(k-1) \)). Thus,
\[ A(\lambda) \leq \frac{1}{r^2} + 2, \quad \lambda \in H_0 \setminus D_0(r). \]
Now, in view of (2.10) and (2.20)–(2.22), we have, with \( r = 4\|v\| \leq 1, \) that
\[ \|R^2_{\lambda}V\|^2 \leq A(\lambda)\|v\|^2 \leq 2/16 + 9/16 \leq \frac{3}{4}, \]
for \( \lambda \in \mathbb{C} \setminus D^+ \) if \( bc = Per^+, \) and for \( \lambda \in \mathbb{C} \setminus D^- \) if \( bc = Per^-, \) respectively. Therefore, \( R_{\lambda}(L_{Per^+}) \) is well defined for \( \lambda \in \mathbb{C} \setminus D^+, \) and \( R_{\lambda}(L_{Per^-}) \) is well defined for \( \lambda \in \mathbb{C} \setminus D^-, \) i.e.,
\[ \sigma(L_{Per^+}) \subset D^+ \quad \text{and} \quad \sigma(L_{Per^-}) \subset D^- \]
for all potentials \( v \) such that \( \|v\| \leq 1/4, \) and in particular for any \( v_\tau(x) = \tau v(x), \ \tau \in [0, 1]. \) For each even \( k \) (if \( bc = Per^+, \)) and for each odd \( k \) (if \( bc = Per^-), \) the resolvents \( R_{\lambda}(L_{v_\tau}) \) are analytic in \( \lambda \) for \( \lambda \in H_k \setminus D_k(4\|v\|), \) and continuous in \( \tau \) for \( \tau \in [0, 1], \) Thus, \( \dim P_k(\tau), \) where
\[ P_k(\tau) = \frac{1}{2\pi i} \int_{|z - \tau| = r} (z - L_\tau)^{-1}dz, \quad r = 4\|v\|, \]
being an integer, is a constant, i.e.,
\[ \dim P_k(1) = \dim P_k(0) = 2 \quad \text{for} \ k > 0, \quad \dim P_0(1) = \dim P_0(0) = 1. \]
Hence, if \( k > 0 \) then the disc \( D_k(4\|v\|) \) contains exactly two (periodic, if \( k \) is even, and antiperiodic, if \( k > 0 \) is odd) eigenvalues, and the disc \( D_0(4\|v\|) \) contains exactly one periodic eigenvalue. In view of (2.24) the latter proves Part (a), i.e., (2.4) holds.

(b) Next we prove Part (b), (2.5), by using the same notations, but with \( r = 1 \), and almost the same argument that has been used to prove Part (a). Thus, (2.5) will be proven if we explain that for large enough \( n \) the resolvent \( R_\lambda \) is defined for \( \lambda \in H_n \setminus D_n(1) \).

With \( T = R_\lambda^0 V \) we have, in view of (2.8), that
\[
R_\lambda = (1 - T)^{-1} R_\lambda^0 = (1 - T^2)^{-1} (1 + T) R_\lambda^0
\]
is well defined if \( \|T^2\| < 1 \). Thus, our goal is to show that there exists \( N_0 \) such that if \( n \geq N_0 \) then \( \|T^2\| < 1 \) for \( \lambda \in H_n \setminus D_n(1) \).

In view of (2.9), the matrix representation of \( T^2 \) is
\[
(T^2)_{km} = \sum_{s \in n + 2\mathbb{Z}} \frac{V(k - s)V(s - m)}{(\lambda - k^2)(\lambda - s^2)}, \quad k, m \in n + 2\mathbb{Z}.
\]

For \( \lambda \in H_n \setminus D_n(1) \) we have \( |\lambda - n^2| \geq 1 \), and therefore,
\[
\sum_{k, m} |(T^2)_{km}|^2 \leq 3 (\Sigma_+ + \Sigma_- + \Sigma_2),
\]
where
\[
\Sigma_\pm = \sum_{k, m} \frac{|V(k \pm n)^2|V(\mp n - m)^2}{|\lambda - k^2|^2}, \quad \Sigma_2 = \sum_{k, m} \frac{|V(k - s)V(s - m)}{(\lambda - k^2)(\lambda - s^2)}|^2.
\]

Taking into account that \( V(0) = 0 \) we have, by Lemma 2
\[
\Sigma_\pm = \left( \frac{|V(\pm 2n)|^2}{|\lambda - n^2|^2} + \sum_{k \neq \pm n} \frac{|V(k \pm n)|^2}{|\lambda - k^2|^2} \right) \sum_m |V(\mp n - m)|^2
\]
\[
\leq \left( \frac{|V(\pm 2n)|^2}{|\lambda - n^2|^2} + \frac{9}{n^2}\|v\|^2 \right) \|v\|^2.
\]
On the other hand, by the Cauchy inequality and Lemma 2
\[
\Sigma_2 \leq \sum_k \left( \sum_{s \neq \pm n} \frac{|V(k - s)|^2}{|\lambda - k^2|^2} \right) \left( \sum_m \sum_{s \neq \pm n} \frac{|V(s - m)|^2}{|\lambda - s^2|^2} \right)
\]
\[
\leq \left( \sum_k \frac{1}{|\lambda - k^2|^2} \right) \left( \sum_{s \neq \pm n} \frac{1}{|\lambda - s^2|^2} \right) \cdot \|v\|^4 \leq \left( 2 + \frac{9}{n^2} \right) \cdot \frac{9}{n^2} \cdot \|v\|^4.
\]
In view of (2.24), the estimates obtained in (2.25) and (2.26) show that the Hilbert-Schmidt norm of $T^2$, (and therefore, the operator norm in $\ell^2$ of $T^2$) goes to 0 as $n \to \infty$. Thus, we may choose $N_0$ so that
\[ \|T^2\| \leq 1/2 \quad \text{for } n \geq N_0, \lambda \in H_n \setminus D_n(1). \]
Then, as in the proof of (a), a homotopy type argument completes the proof. $\square$

2. In [2], Theorem 8, we obtained an asymptotic formula for the spectral gaps $\gamma_n = \lambda_n^+ - \lambda_n^-$ as $n \to \infty$. In this section we explain that, in fact, the same formula gives the asymptotics of $\gamma_n$ for each fixed $n$, if we consider $\gamma_n$ as a function of $v$ and look for its asymptotic as $\|v\| \to 0$.

Suppose that $\lambda = n^2 + z$, $n \in \mathbb{N}$ is a periodic (or antiperiodic) eigenvalue of $L$, with $|z| < 1$, and $y \neq 0$ is a corresponding eigenfunction. Let $E^0_n = [e_{-n}, e_n]$ be the eigenspace of $L^0$ corresponding to $n^2$, and let $P^0_n$ be the orthogonal projector on $E^0_n$. We set
\[ Q^0_n = 1 - P^0_n. \]
Then the equation $(n^2 + z - L)y = 0$ is equivalent to the following system of two equations:
\begin{align}
(2.27) \quad & Q^0_n(n^2 + z - L^0 - V)Q^0_n y + Q^0_n(n^2 + z - L^0 - V)P^0_n y = 0, \\
(2.28) \quad & P^0_n(n^2 + z - L^0 - V)Q^0_n y + P^0_n(n^2 + z - L^0 - V)P^0_n y = 0.
\end{align}
Taking into account that
\[ P^0_n Q^0_n = Q^0_n P^0_n = 0, \quad P^0_n L^0 Q^0_n = Q^0_n L^0 P^0_n = 0, \quad L^0 P^0_n y = n^2 P^0_n y, \]
we obtain that (2.27) and (2.28) can be rewritten as
\begin{align}
(2.29) \quad & Q^0_n(n^2 + z - L^0 - V)Q^0_n y - Q^0_n V P^0_n y = 0, \\
(2.30) \quad & -P^0_n V Q^0_n y - P^0_n V P^0_n y + z P^0_n y = 0
\end{align}

Let $\mathbb{H}(n)$ denote the range of the operator $Q^0_n$. The operator
\begin{align}
(2.31) \quad & A = A(n, z) := Q^0_n(n^2 + z - L^0 - V)Q^0_n : \mathbb{H}(n) \to \mathbb{H}(n)
\end{align}
is invertible if $3\|v\|/n < 1$ (see below Lemma 3). Thus, solving (2.29) for $Q^0_n y$, we obtain
\[ Q^0_n y = A^{-1} Q^0_n V P^0_n y, \]
where \( P_n^0 y \neq 0 \) (otherwise \( Q_n^0 y = 0 \), which implies \( y = P_n^0 y + Q_n^0 y = 0 \)). Therefore (2.30) implies
\[
(2.32) \quad [P_n^0 VA^{-1} Q_n^0 VP_n^0 + P_n^0 V P_n^0 - z]P_n^0 y = 0
\]
with \( P_n^0 y \neq 0 \). Let
\[
\begin{pmatrix}
S_{11} & S_{12} \\
S_{21} & S_{22}
\end{pmatrix}
\]
be the matrix representation of the two-dimensional operator
\[
(2.33) \quad S := P_n^0 VA^{-1} Q_n^0 VP_n^0 + P_n^0 V P_n^0 : \ E_n^0 \rightarrow E_n^0
\]
with respect to the basis \( e_{-n}, e_n \). Then we have
\[
(2.34) \quad S_{11} = \langle Se_{-n}, e_{-n} \rangle, \quad S_{22} = \langle Se_n, e_n \rangle, \quad S_{21} = \langle Se_{-n}, e_n \rangle, \quad S_{12} = \langle Se_n, e_{-n} \rangle.
\]
Since \( P_n^0 y \neq 0 \), (2.32) implies
\[
(2.35) \quad \begin{vmatrix}
S_{11} - z & S_{12} \\
S_{21} & S_{22} - z
\end{vmatrix} = 0.
\]

In the selfadjoint case (where \( v \) is real-valuead), if \( \lambda \) is a double eigenvalue, then there exists another eigenvector \( \tilde{y} \) (corresponding to \( \lambda \)), such that \( y \) and \( \tilde{y} \) are linearly independent. Then \( P_n^0 y \) and \( P_n^0 \tilde{y} \) are linearly independent also. Indeed, if \( P_n^0 y = cP_n^0 \tilde{y} \) then
\[
Q_n^0 y = A^{-1} Q_n^0 VP_n^0 y = cA^{-1} Q_n^0 VP_n^0 \tilde{y} = cQ_n^0 \tilde{y},
\]
which leads to a contradiction:
\[
y = P_n^0 y + Q_n^0 y = c (P_n^0 \tilde{y} + Q_n^0 \tilde{y}) = c \tilde{y}.
\]
Thus \( S \equiv 0 \), i.e., if \( \lambda = \pi n + z \) is a double eigenvalue of a self-adjoint Schrödinger operator \( L \), then (for large enough \( n \))
\[
(2.36) \quad S_{11} - z = 0, \quad S_{12} = 0, \quad S_{21} = 0, \quad S_{22} - z = 0.
\]

Next, in order to obtain explicit formulas for \( S_{11}, S_{22}, S_{12} \) and \( S_{21} \), we compute the matrix representations of the operators \( A \) and \( S \) with respect to the basis \( \{ e_{2k}, k \in \mathbb{Z} \} \) for even \( n \), and with respect to the basis \( \{ e_{2k-1}, k \in \mathbb{Z} \} \) for odd \( n \). The operator
\[
Q_n^0 (n^2 + z - L^0)Q_n^0 : \ H(n) \rightarrow H(n)
\]
is invertible for any \( z \) with \( |z| < 1 \). Let \( D_n \) denote its inverse operator. Obviously, the matrix representing \( D_n \) is
\[
(2.37) \quad (D_n)_{km} = \frac{1}{n^2 - k^2 + z} \delta_{km}, \quad k, m \in (n + 2\mathbb{Z}) \setminus \{ \pm n \},
\]
where \( \delta_{km} = 0 \) for \( k \neq m \) and \( \delta_{km} = 1 \) for \( k = m \).
The operator $A$ from (2.31) can be written as

$$A = \left[ Q^0_n(n^2 + z - L^0_n)Q^0_n \right](1 - T_n),$$

where

$$(2.38) \quad T_n = D_nQ^0_nVQ^0_n.$$  

Thus $A = A(n, z)$ is invertible if and only if $1 - T_n$ is invertible, and in this case

$$(2.39) \quad A^{-1} = (1 - T_n)^{-1}D_n.$$

The matrix representation of the operator of multiplication by $v(x) = \sum V(k) \exp(ikx)$ is

$$(2.40) \quad V_{km} = V(k - m).$$

Now, by (2.37), (2.38) and (2.40) we obtain that the matrix representation of the operator $T_n$ is given by

$$(2.41) \quad (T_n)_{km} = \frac{V(k - m)}{n^2 - k^2 + z}, \quad k, m \in (n + 2\mathbb{Z}) \setminus \{\pm n\}.$$

**Lemma 3.** If $|z| \leq 1$, then, for each $n \in \mathbb{N}$, the norm of the operator $T_n : H_n \to H_n$ satisfies

$$(2.42) \quad \|T_n\| \leq \frac{3\|v\|}{n}.$$  

**Proof.** Since the $\ell^2$-norm does not exceed the Hilbert-Schmidt norm, we obtain, in view of (2.41) and Lemma 2

$$\|T_n\|^2 \leq \sum_k \frac{1}{|n^2 - k^2 + z|^2} \sum_m |V(k - m)|^2 \leq \frac{9\|v\|^2}{n^2},$$

where $k, m \in (n + 2\mathbb{Z}) \setminus \{\pm n\}$. \qed

Let us consider $n \geq 9\|v\|$ until the end of this section. Then, by (2.42), $\|T_n\| \leq 3\|v\|/n \leq 1/3$, so (2.39) yields

$$(2.43) \quad A^{-1} = \sum_{m=0}^{\infty} T^m_n D_n,$$

and therefore, by (2.33),

$$(2.44) \quad S = P^0_nV P^0_n + \sum_{m=0}^{\infty} P^0_nVT^m_n D_nQ^0_nV P^0_n.$$  

Now, by (2.37), (2.40) and (2.41) we obtain, in view of (2.34), that

$$(2.45) \quad S^{ij}(n, z) = \sum_{k=0}^{\infty} S^{ij}_k(n, z), \quad i, j = 1, 2,$$
where

\[(2.46) \quad S_0^{11} = S_0^{22} = 0, \quad S_0^{12} = V(-2n), \quad S_0^{21} = V(2n),\]

and for each \(k = 1, 2, \ldots,\)

\[(2.47) \quad S_{k}^{11}(n, z) = \sum_{j_1, \ldots, j_k \neq \pm n} \frac{V(-n - j_1)V(j_1 - j_2) \cdots V(j_{k-1} - j_k)V(j_k + n)}{(n^2 - j_1^2 + z) \cdots (n^2 - j_k^2 + z)}\]

\[(2.48) \quad S_{k}^{22}(n, z) = \sum_{j_1, \ldots, j_k \neq \pm n} \frac{V(n - j_1)V(j_1 - j_2) \cdots V(j_{k-1} - j_k)V(j_k - n)}{(n^2 - j_1^2 + z) \cdots (n^2 - j_k^2 + z)}\]

\[(2.49) \quad S_{k}^{12}(n, z) = \sum_{j_1, \ldots, j_k \neq \pm n} \frac{V(-n - j_1)V(j_1 - j_2) \cdots V(j_{k-1} - j_k)V(j_k + n)}{(n^2 - j_1^2 + z) \cdots (n^2 - j_k^2 + z)}\]

\[(2.50) \quad S_{k}^{21}(n, z) = \sum_{j_1, \ldots, j_k \neq \pm n} \frac{V(n - j_1)V(j_1 - j_2) \cdots V(j_{k-1} - j_k)V(j_k - n)}{(n^2 - j_1^2 + z) \cdots (n^2 - j_k^2 + z)}\]

The above series converge absolutely and uniformly for \(|z| \leq 1.\)

**Lemma 4.** (a) For any (even complex-valued) potential \(v\)

\[(2.51) \quad S^{11}(n, z) = S^{22}(n, z).\]

(b) If \(v\) is a real-valued potential, then

\[(2.52) \quad S^{12}(n, z) = S^{21}(n, \overline{z}).\]

**Proof.** (a) By (2.47) and (2.48), for each \(k = 1, 2, \ldots,\) the change of sumation indices

\[i_s = -j_{k+1-s}, \quad s = 1, \ldots, k,\]

proves that \(S_{k}^{11}(n, z) = S_{k}^{22}(n, z).\) Thus, in view of (2.45) and (2.46), (2.51) holds.

(b) If \(v\) is real-valued, we have for its Fourier coefficients the identity \(V(-m) = \overline{V(m)}.\) By (2.45),

\[S_0^{12}(n, z) = V(-2n) = \overline{V(2n)} = S_0^{21}(n, \overline{z}).\]

Also, for each \(k = 1, 2, \ldots,\) the change of sumation indices

\[i_s = j_{k+1-s}, \quad s = 1, \ldots, k,\]

explains that \(S_{k}^{12}(n, z) = S_{k}^{21}(n, \overline{z}),\) thus (2.52) holds. \(\square\)
In this paper we consider only real-valued potentials \( v \). For convenience we set

\[
\alpha_n(z) = S^{11}(n, z) = S^{22}(n, z), \quad \beta_n(z) = S^{21}(n, z) = S^{12}(n, z).
\]

Under these notations the basic equation (2.35) becomes

\[
(z - \alpha_n(z))^2 - |\beta_n(z)|^2 = 0,
\]

which splits into two equations

\[
z - \alpha_n(z) - |\beta_n(z)| = 0,
\]

\[
z - \alpha_n(z) + |\beta_n(z)| = 0.
\]

**Lemma 5.** If \(|z| \leq 1\) and \(\|v\|/n \leq 1/9\), then

\[
|\frac{d}{dz} \alpha_n(z)| \leq \frac{\|v\|^2}{n^2}, \quad |\frac{d}{dz} \beta_n(z)| \leq \frac{\|v\|^2}{n^2}.
\]

**Proof.** By (2.31) and (2.33)

\[
\frac{d}{dz} S(n; z) = -P_n^0 V Q_n^0 (A^{-1})^2 Q_n^0 V P_n^0,
\]

and therefore, in view of (2.34) and (2.53), we have

\[
\alpha'_n(z) = -\langle P_n^0 V Q_n^0 (A^{-1})^2 Q_n^0 V P_n^0 e_n, e_n \rangle,
\]

\[
\beta'_n(z) = -\langle P_n^0 V Q_n^0 (A^{-1})^2 Q_n^0 V P_n^0 e_{-n}, e_n \rangle.
\]

Set

\[
f_{\pm n} = Q_n^0 V P_n^0 e_{\pm n}, \quad h_n(x) = \langle P_n^0 V Q_n^0 x, e_n \rangle.
\]

With these notations we have, by (2.38) and (2.39),

\[
\alpha'_n(z) = -h_n [(A^{-1}(z))^2 f_n], \quad \beta'_n(z) = -h_n [(A^{-1}(z))^2 f_{-n}],
\]

and therefore,

\[
|\alpha'_n(z)| \leq \|h_n\| \cdot \|A^{-1}\|^2 \cdot \|f_n\|, \quad |\beta'_n(z)| \leq \|h_n\| \cdot \|A^{-1}\|^2 \cdot \|f_{-n}\|.
\]

By Lemma 3 and (2.43), if \(\|v\|/n < 1/9\) then

\[
\|A^{-1}\| \leq \left( \sum_{k=0}^{\infty} \|T_n\|^k \right) \cdot \|D_n\| < \frac{3}{2} \|D_n\|.
\]

The operator \(D_n\) is diagonal, and therefore, by (2.37),

\[
\|D_n\| = \max_{k \neq \pm n} \frac{1}{|n^2 - k^2 + z|}.
\]
If \( n = 1 \), then (2.64) implies
\[
\|D_1\| = \max_{k \neq \pm 1} \frac{1}{|k^2 - 1 - z|} \leq \frac{1}{7}.
\]

For \( n \geq 2 \) we obtain, by (2.64),
\[
\|D_n\| \leq \frac{1}{4n - 5} \leq \frac{2}{3n}.
\]

Now (2.63), (2.65) and (2.66) yield
\[
\|A^{-1}\| \leq \frac{1}{n} \text{ if } \|v\|/n < 1/9 \text{ and } |z| \leq 1.
\]

On the other hand, by (2.40),
\[
\|f_{\pm n}\| = \left\| \sum_{k \neq \pm n} V(k \mp n) e_k \right\| \leq \|v\|.
\]
and, with \( x = \sum_x k e_k \),
\[
\|h_n(x)\| = \left| \sum_{k \neq \pm n} V(n - k)x_k \right| \leq \|v\| \cdot \|x\|,
\]
thus
\[
\|h_n\| \leq \|v\|.
\]

Now (2.67)-(2.69) yield, in view of (2.62), that (2.57) holds. \( \square \)

**Theorem 6.** (a) If \( \|v\| \leq 1/9 \), then for each \( n = 1, 2, \ldots \) there exists \( z = z_n \) such that
\[
|z| \leq 4\|v\|,
\]
and
\[
2|\beta_n(z)| \left( 1 - 3\|v\|^2/n^2 \right) \leq \gamma_n \leq 2|\beta_n(z)| \left( 1 + 3\|v\|^2/n^2 \right).
\]

(b) If \( V(0) = \frac{1}{\pi} \int_0^\pi v(x)dx = 0 \), then there is \( N_0 = N_0(v) \) such that (2.71) holds for \( n \geq N_0 \) with \( z = z_n \),
\[
|z_n| < 1.
\]

**Proof.** We will prove (2.71) simultaneously in both cases (a) and (b).

In Case (a) we know, by part (a) of Proposition \( \square \) that, for each \( n \in \mathbb{N} \), there are exactly two eigenvalues \( \lambda_n^\pm = n^2 + z_n^\pm \) of \( L \) (periodic for even \( n \) and antiperiodic for odd \( n \)) such that \( |z_n^\pm| < 4\|v\| < 1 \).

By part (b) of Proposition \( \square \) the same is true in Case (b) if \( n \) is large enough. Let \( N_0 = N_0(v) \) be chosen so that part (b) of Proposition \( \square \) holds for \( n \geq N_0 \), and
\[
\|v\|/n \leq 1/9 \quad \text{for } n \geq N_0.
\]
Fix an $n \geq N_0$ in Case (b), and let $n \in \mathbb{N}$ in Case (a).

We know, in both cases (a) and (b), that the numbers $z_n^\pm$ are roots of (2.71), and therefore, of (2.55). If $z_n^+ = z_n^- = z^*$, then $\lambda = \pi n + z^*$ is a double eigenvalue and (2.36) yields $\beta_n(z^*) = 0$, thus (2.70) holds.

If $z_n^+ \neq z_n^-$, set

$$
(2.74) \quad \zeta_n^+ = z_n^+ - \alpha_n(z_n^+), \quad \zeta_n^- = z_n^- - \alpha_n(z_n^-).
$$

Then, by (2.55) and (2.56),

$$
(2.75) \quad |\zeta_n^+| = |\beta_n(z_n^+)|, \quad |\zeta_n^-| = |\beta_n(z_n^-)|.
$$

By (2.74),

$$
\zeta_n^+ - \zeta_n^- = \int_{z_n^-}^{z_n^+} (1 - \alpha'_n(z)) \, dz.
$$

Thus, in view of Lemma 5 and (2.73) in Case (b), or the inequality $\|v\| \leq 1/9$ in Case (a), we obtain

$$
(2.76) \quad (z_n^+ - z_n^-)(1 - \|v\|^2/n^2) \leq |\zeta_n^+ - \zeta_n^-| \leq (z_n^+ - z_n^-)(1 + \|v\|^2/n^2),
$$

which yields

$$
(2.77) \quad |\zeta_n^+ - \zeta_n^-| \left(1 - \frac{\|v\|^2}{n^2}\right) \leq z_n^+ - z_n^- \leq |\zeta_n^+ - \zeta_n^-| \left(1 + \frac{9\|v\|^2}{8n^2}\right) \leq \frac{9}{8}|\zeta_n^+ - \zeta_n^-|.
$$

Since $z_n^+$ and $z_n^-$ are roots of (2.54), each of these numbers is a root of either (2.55), or (2.56). There are two cases: (i) $z_n^+$ and $z_n^-$ are roots of different equations; (ii) $z_n^+$ and $z_n^-$ are roots of one and the same equation.

In Case (i) we have, by (2.55), (2.56) and (2.75), that

$$
(2.78) \quad |\zeta_n^+ - \zeta_n^-| = |\beta_n(z_n^+)| + |\beta_n(z_n^-)| = |\zeta_n^+| + |\zeta_n^-|.
$$

On the other hand, since $\beta_n(z_n^+) - \beta_n(z_n^-) = \int_{z_n^-}^{z_n^+} \beta'_n(t) \, dt$, Lemma 3 and (2.73) in Case (b), or the inequality $\|v\| \leq 1/9$ in Case (a), imply that

$$
(2.79) \quad |\beta_n(z_n^+) - \beta_n(z_n^-)| \leq (z_n^+ - z_n^-) \frac{\|v\|^2}{n^2} \leq |\zeta_n^+ - \zeta_n^-| \frac{9\|v\|^2}{8n^2}.
$$

Thus (2.76) and (2.78) yield

$$
|\zeta_n^+| - |\zeta_n^-| = |\beta_n(z_n^+)| - |\beta_n(z_n^-)| \leq (|\zeta_n^+| + |\zeta_n^-|) \frac{9\|v\|^2}{8n^2}.
$$

So, since $2|\zeta_n^+| = (|\zeta_n^+| + |\zeta_n^-|) + (|\zeta_n^+| - |\zeta_n^-|)$,

$$
(|\zeta_n^+| + |\zeta_n^-|) \left(1 - \frac{9\|v\|^2}{8n^2}\right) \leq 2|\zeta_n^+| \leq (|\zeta_n^+| + |\zeta_n^-|) \left(1 + \frac{9\|v\|^2}{8n^2}\right),
$$
and therefore, since $\|v\|^2/n^2 < 1/9$,

\[(2.80)\quad 2|\zeta_n^+| \left(1 - \frac{9\|v\|^2}{8n^2}\right) \leq |\zeta_n^+| + |\zeta_n^-| \leq 2|\zeta_n^+| \left(1 + \frac{8\|v\|^2}{7n^2}\right)\]

Since $\gamma_n = z_n^+ - z_n^-$, (2.77), (2.78) and (2.80) yield (2.71) with $z = z_n^+$.

Case (ii), where $z_n^+$ and $z_n^-$ are simultaneously roots of one of the equations (2.55) and (2.56), is impossible. Indeed, by (2.79) we would have (since $\|v\|^2/n^2 < 1/9$)

\[|\zeta_n^+ - \zeta_n^-| = |\beta_n(z_n^+) - \beta_n(z_n^-)| \leq |\zeta_n^+ - \zeta_n^-| \cdot \frac{9\|v\|^2}{8n^2} \leq \frac{1}{72} |\zeta_n^+ - \zeta_n^-|,\]

which implies $\zeta_n^+ = \zeta_n^-$. But then (2.77) yield $z_n^+ = z_n^-$, which is a contradiction to our assumption that $z_n^+ \neq z_n^-$. □

3. Asymptotic formula for the spectral gaps of a Schrödinger operator with a two term potential

In this section we apply the general asymptotic formula (2.71) from Theorem 6 to get a corresponding formula for a Hill operator with a potential of the form

\[v(x) = a \cos 2x + b \cos 4x, \quad a = -4\alpha t, \quad b = -2\alpha^2.\]

The next theorem gives the asymptotics of $\gamma_n = \gamma_n(\alpha, t)$, for fixed $n$ and $t$, as $\alpha \to 0$.

**Theorem 7.** Let $\gamma_n$, $n \in \mathbb{N}$ be the lengths of instability zones of the Hill operator

\[(3.1)\quad Ly = -y'' - \left[4\alpha t \cos 2x + 2\alpha^2 \cos 4x\right]y,\]

where either both $\alpha$ and $t$ are real, or both are pure imaginary numbers.

If $t$ is fixed and $\alpha \to 0$, then for even $n$

\[(3.2)\quad \gamma_n = \left|\frac{8\alpha^n}{2^n [(n - 1)!!]^2} \prod_{k=1}^{n/2} \left(t^2 - (2k - 1)^2\right)\right| (1 + O(\alpha)),\]

and for odd $n$

\[(3.3)\quad \gamma_n = \left|\frac{8\alpha^n t}{2^n [(n - 1)!!]^2} \prod_{k=1}^{(n-1)/2} \left(t^2 - (2k)^2\right)\right| (1 + O(\alpha)).\]

**Proof.** For convenience the proof is divided into several steps.

**Step 1.** First we apply (for small enough $\alpha$) Theorem 6 to the Hill operator with the potential $v(x) = -4\alpha t \cos 2x - \alpha^2 \cos 4x$, i.e., with

\[(3.4)\quad V(\pm 2) = -2t\alpha, \quad V(\pm 4) = -\alpha^2, \quad V(k) = 0 \text{ if } k \neq \pm 2, \pm 4.\]
Since
\[(3.5) \quad \|v\|^2 = 8|t|^2|\alpha|^2 + 2|\alpha|^4\]
we obtain, by (2.70) and (2.71), that
\[(3.6) \quad \gamma_n = \pm 2 \left( V(2n) + \sum_{k=1}^{\infty} \beta_k(n, z) \right) \left( 1 + O(|\alpha|^2) \right), \]
where \(z = z(n)\),
\[(3.7) \quad z = O(\alpha), \]
\[(3.8) \quad \beta_k(n, z) = \sum_{j_1, \ldots, j_k \neq \pm n} \frac{V(n+j_1)V(j_2-j_1) \cdots V(n-j_k)}{(n^2-j_1^2+z) \cdots (n^2-j_k^2+z)}, \]
and all series converge absolutely and uniformly for small enough \(\alpha\).

Observe that each non-zero term in (3.8) corresponds to a \(k\)-tuple of indices \((j_1, \ldots, j_k)\) such that
\[(3.9) \quad (n+j_1) + (j_2-j_1) + \cdots + (j_k-j_{k-1}) + (n-j_k) = 2n \]
and, by (3.4),
\[(3.10) \quad (n+j_1), (j_2-j_1), \ldots, (j_k-j_{k-1}), (n-j_k) \in \{\pm 2, \pm 4\}. \]

Therefore, in view of (3.9) and (3.10), there is one-to-one correspondence between the non-zero terms of (3.8) and the walks from \(-n\) to \(n\) with steps \(\pm 2\) and \(\pm 4\).

By (3.4), each non-zero expression of the form
\[(3.11) \quad V(n+j_1)V(j_2-j_1) \cdots V(n-j_k) \]
is a monomial in \(\alpha\) of degree
\[\frac{1}{2} (|n+j_1| + |j_2-j_1| + \cdots + |n-j_k|). \]
Therefore, \(n\) is the minimal possible degree, and each such monomial of degree \(n\) corresponds to a walk from \(-n\) to \(n\) with positive steps. In addition, each expression of the form (3.11) is also a monomial in \(t\).

Taking into account the above remark, we obtain from (3.6) - (3.8) that
\[(3.12) \quad \gamma_n = \pm P_n(t)\alpha^n + O(\alpha^{n+1}). \]
Our next goal is to find the polynomial $P_n(t)$ for each $n \in \mathbb{N}$. In view of (3.6) – (3.8), the above discussion shows that

$$P_n(t)\alpha^n = 2V(2n) + 2 \sum_{-n < j_1 < \cdots < j_s < n} \frac{V(n + j_1)V(j_2 - j_1) \cdots V(n - j_s)}{(n^2 - j_1^2) \cdots (n^2 - j_s^2)},$$

where each non-zero term corresponds to a walk with positive steps. Moreover, the coefficient in front of $t^n$ is coming from the term corresponding to a walk with steps of length 2 only. Thus we have

$$P_n(t) = C_n t^n + \cdots,$$

where

$$C_n = 2(-2)^n \left( \prod_{j=1}^{n-1} (n^2 - (-n + 2j)^2) \right)^{-1} = \frac{8(-1)^n}{2^n ((n-1)!)^2}.$$

For even $n$, each walk from $-n$ to $n$ has even number of steps with length 2. Thus, by (3.13), we obtain that $P_n(t)$ is a sum of monomials of even degrees, so for $n = 2m$ we have

$$P_{2m}(t) = C_{2m} t^m \prod_{k=1}^{m} (t^2 - x_k),$$

where $x_k$, $k = 1, \ldots, m$, depend on $m$.

For odd $n$, say $n = 2m - 1$, each walk from $-n$ to $n$ has odd number of steps with length 2, and therefore, in this case (3.13) implies that $P_{2m-1}$ is a sum of monomials of odd degrees. Thus we have

$$P_{2m-1}(t) = C_{2m-1} t \prod_{k=1}^{m-1} (t^2 - y_k),$$

where $y_k$, $k = 1, \ldots, m - 1$, depend on $m$.

Taking into account the combinatorial meaning of the non-zero terms in (3.13), it is easy to compute $P_1(t), P_2(t), P_3(t)$ and $P_4(t)$. We have

$$P_1(t) = -4t, \quad P_3(t) = -\frac{1}{4} t(t^2 - 2^2),$$

$$P_2(t) = 2(t^2 - 1), \quad P_4(t) = \frac{1}{72} (t^2 - 1)(t^2 - 3^2).$$

Indeed, if $n = 1$, then there is only one walk with step 2 from -1 to 1, so we obtain $P_1(t)\alpha = 2V(2) = -4t\alpha$.

If $n = 3$, then there are exactly three walks from -3 to 3 with positive steps: (2,2,2), (2,4), (4,2). Now, from (3.13), it follows that

$$P_3(t)\alpha^3 = \frac{-2(2t\alpha)^3}{(3^2 - (-1)^2)(3^2 - 1^2)} + \frac{2(2t\alpha)\alpha^2}{3^2 - (-1)^2} + \frac{2(2t\alpha)\alpha^2}{3^2 - 1^2},$$
and therefore, the second formula in (3.15) holds. The formulae (3.16) can be obtained in the same way.

Obviously, the theorem will be proved, if we show that similar formulae hold for each \( n \), i.e., for \( n = 2m \)

\[
P_{2m} = C_{2m}(t^2 - 1)(t^2 - 3^2) \cdots (t^2 - (2m - 1)^2),
\]

and for \( n = 2m - 1 \)

\[
P_{2m-1}(t) = C_{2m-1}t(t^2 - 2^2) \cdots (t^2 - (2m - 2)^2).
\]

**Step 2.** In the following, till the end of the proof of the theorem, we assume that both \( \alpha \) and \( t \) are real numbers. Next we recall some facts from [13], Chapter VII, that will be used in the proof. (We discuss these constructions in detail in [7] where we sharpen the Magnus-Winkler results and analyze their connections to the theory of quasi-exactly solvable differential equations [24].)

The eigenvalue equation for the operator (3.1) is

\[
y'' + [\lambda + 4t\alpha \cos 2x + \alpha^2 \cos 4x]y = 0.
\]

The substitution

\[
y = u e^{\alpha \cos 2x}
\]

carries (3.19) into the equation

\[
u'' - 4\alpha (\sin 2x)u' + [\mu + 4(t - 1)\alpha \cos 2x]u = 0,
\]

where \( \mu = \lambda + 2\alpha^2 \). If \( y(x) \) is a periodic (respectively antiperiodic) solution of (3.19), then \( u(x) = y(x) e^{-\alpha \cos 2x} \) is a periodic (respectively antiperiodic) solution of (3.21), and v.v., if \( u(x) \) is a periodic (respectively antiperiodic) solution of (3.21), then \( \tilde{u}(x) \) is a periodic (respectively antiperiodic) solution of (3.19).

Since \( \sin 2x \) is an odd function and \( \cos 2x \) is an even function, it is easy to see that if \( u(x) \) is a periodic (or antiperiodic) solution of (3.21), then the function \( \tilde{u}(x) = u(-x) \) is also a periodic (respectively antiperiodic) solution of (3.21). On the other hand \( u(x) + \tilde{u}(x) \) is an even function, and \( u(x) - \tilde{u}(x) \) is odd function. Therefore, if for some \( \mu \) the equation (3.21) has a non-zero solution, then it has also either an even non-zero solution, or an odd non-zero solution, or both.

In other words, when solving (3.21), we may look for periodic solutions of the form

\[
u(x) = A_0 + \sum_{k \in 2\mathbb{N}} A_k \cos kx, \quad w(x) = \sum_{k \in 2\mathbb{N}} B_k \sin kx,
\]
or antiperiodic solutions of the form

\[ u(x) = \sum_{k \in 2\mathbb{N}-1} A_k \cos kx, \quad w(x) = \sum_{k \in 2\mathbb{N}-1} B_k \sin kx. \]  

Observe that only even indices \( k \) are used in (3.22), while only odd \( k \) appear in (3.23).

By substituting (3.22) into (3.21) we obtain that \( u(x) \) is a periodic even solution (respectively \( v(x) \) is a periodic odd solution) if and only if \( A_k, k = 0, 2, 4, \ldots \), satisfy the recurrence relations

\[ -\mu A_0 + 2\alpha(t-1)A_2 = 0, \]

\[ 4\alpha(t+1)A_0 + (2^2 - \mu)A_2 + 2\alpha(t-3)A_4 = 0, \]

\[ 2\alpha(t-1+k)A_{k-2} + (k^2 - \mu)A_k + 2\alpha(t-1-k)A_{k+2} = 0, \quad k \geq 4 \]

and respectively \( B_k, k = 2, 4, \ldots \), satisfy

\[ (2^2 - \mu)B_2 + 2\alpha(t-3)B_4 = 0, \]

\[ 2\alpha(t-1+k)B_{k-2} + (k^2 - \mu)B_k + 2\alpha(t-1-k)B_{k+2} = 0, \quad k \geq 4. \]

If we substitute (3.23) into (3.21), then it follows that \( u(x) \) is an antiperiodic even solution (respectively \( w(x) \) is an antiperiodic odd solution) if and only if \( A_k, k = 1, 3, 5, \ldots \), satisfy the relations

\[ (1 - \mu + 2\alpha t)A_1 + 2\alpha(t-2)A_3 = 0, \]

\[ 2\alpha(t-1+k)A_{k-2} + (k^2 - \mu)A_k + 2\alpha(t-1-k)A_{k+2} = 0, \quad k \geq 3, \]

and respectively \( B_k, k = 1, 3, 5, \ldots \), satisfy

\[ (1 - \mu - 2\alpha t)B_1 + 2\alpha(t-2)B_3 = 0, \]

\[ 2\alpha(t-1+k)B_{k-2} + (k^2 - \mu)B_k + 2\alpha(t-1-k)B_{k+2} = 0, \quad k \geq 3. \]

**Step 3.** Now we prove (3.17). Observe, that to prove (3.17) (since \( \deg P_{2m} = 2m \), and \( P_{2m} \) is even) it is enough to show that 1 is a root of the polynomials \( P_2, P_4, \ldots \), 3 is a root of the polynomials \( P_4, P_6, \ldots \), and so on. So, we are going to prove the following statement.

\[ \text{In [15], p. 95, formula (7.17) for } n = 1 \text{ (which is equivalent to (3.25)) gives a coefficient 2 in front of } \alpha(t+1)A_0 \text{ although 4 is the correct coefficient.} \]
Claim. For each \( m = 1, 2, \ldots \) the number \( 2m - 1 \) is a joint root of the polynomials \( P_{2m}, P_{2m+2}, \ldots \).

First we prove the claim for \( m = 1 \). By (3.33), it is easy to see that

\[
\|v\| < 1/4 \quad \text{if} \quad t \geq 1 \quad \text{and} \quad |\alpha| < 1/(12t).
\]

Choose \( t = 1 \), fix an even \( n \geq 2 \), and let \( |\alpha| < 1/(12t) \). By (3.33), if \( \alpha < 1/(12t) \), then \( \|v\| < 1/4 \), and therefore, by part (a) of Proposition \( \Pi \) there exists a periodic eigenvalue \( \lambda = \lambda(\alpha) \) of the operator \( L \) such that \( |\lambda - n^2| < 1 \), and with this \( \lambda \) the equation (3.19) has a non-zero solution. Then, with \( \mu = \lambda + 2\alpha^2 \geq \lambda > 0 \), the equation (3.21) has a non-zero periodic solution. Moreover, by Step 2, we know that the equation (3.21) has an even, or an odd solution of the form (3.22), and the corresponding coefficients \( A_k \) (respectively \( B_k \)) satisfy (3.24)-(3.26) (respectively (3.27) and (3.28)).

If there is a non-zero periodic solution of (3.21) of the form \( u(x) = A_0 + \sum_{k \in 2N} A_k \cos kx \), then with \( t = 1 \) and \( A_0 = 0 \), we obtain, by (3.24), that \( u(x) = \sum_{k \in 2N} A_k \sin kx \) is a non-zero periodic solution. Moreover, by Step 2, we know that the equation (3.21) has an even, or an odd solution of the form (3.22), and the corresponding coefficients \( A_k \) (respectively \( B_k \)) satisfy (3.24)-(3.26) (respectively (3.27) and (3.28)).

The same argument shows that if there is a non-zero odd solution of (3.21) of the form \( w(x) = \sum_{k \in 2N} B_k \sin kx \), then \( u(x) = \sum_{k \in 2N} B_k \cos kx \) is also a non-zero periodic solution.

So, in both cases the equation (3.21) has two linearly independent periodic solutions \( u(x) \) and \( w(x) \), and therefore, (3.19) has also two linearly independent periodic solutions \( u(x) \exp(\alpha \cos 2x) \) and \( w(x) \exp(\alpha \cos 2x) \). Thus \( \lambda \) is an eigenvalue of multiplicity 2, and therefore, the \( n \)-th instability zone is closed. Hence we have for each small enough \( \alpha \) that

\[
\gamma_n = \pm P_n(1)\alpha^n + O(\alpha^{n+1}) \equiv 0,
\]

which implies that \( P_n(1) = 0 \).

Next we consider the general case. Fix \( m \geq 1 \), choose \( t = 2m + 1 \), and let \( |\alpha| < 1/(12t) \). By (3.33), if \( \alpha < 1/(12t) \), then \( \|v\| < 1/4 \). Fix an even \( n > 2m \) and choose, by part (a) of Proposition \( \Pi \) a periodic eigenvalue \( \lambda = \lambda(\alpha) \) such that \( |\lambda - n^2| < 1 \), and with this \( \lambda \) the equation (3.19) has a non-zero periodic solution (an eigenfunction corresponding to \( \lambda \)), and therefore, (3.21) has a non-zero periodic solution. By Step 2 we know that
(3.21) has either a non-zero even periodic solution, or a non-zero odd periodic solution.

Suppose the first case occurs. Let \( u(x) = A_0 + \sum_{k \in 2\mathbb{N}} A_k \cos kx \) be a non-zero solution of (3.21). Then the coefficients \( A_k, k = 0, 2, 4, \ldots \), satisfy the system of equations (3.24)–(3.26).

Consider the first \( m+1 \) equations. Since \( t = 2m+1 \), the coefficient \( 2\alpha(t-1-2m) \) of \( A_{2m+2} \) vanishes, so we have \( m+1 \) homogeneous equations in unknowns \( A_0, \ldots, A_m \). The corresponding coefficient determinant \( D \) has on its main diagonal \(-\mu, 2^2 - \mu, \ldots, (2^m)^2 - \mu\). Since all other non-zero terms are multiples of \( \alpha \) we obtain that

\[
D = -\mu(2^2 - \mu) \cdots ((2^m)^2 - \mu) + O(\alpha^2).
\]

By (3.34) we have \( \mu > 4m^2 + 1 \). Therefore, we may choose a positive number \( \varepsilon < 1/(12t) \) so that \( D \neq 0 \) for each \( \alpha \) such that \( |\alpha| < \varepsilon \). Hence \( P_n(2m+1) = 0 \), which completes the proof of (3.17).

Since the proof of (3.18) is an analogue of the proof of (3.17), with (3.20)–(3.32) playing the role of (3.24)–(3.28), we omit it.

### 4. A collateral result

By analyzing the proof of Theorem 7 we obtain some interesting identities.

**Theorem 8.** (a) If \( k \) and \( m, 1 \leq k \leq m \), are fixed, then

\[
\sum_{s=1}^{k} \prod_{i_s} (m^2 - i_s^2) = \sum_{1 \leq j_1 < \cdots < j_k \leq m} \prod_{s=1}^{k} (2j_s - 1)^2,
\]

where the left sum is over all indices \( i_1, \ldots, i_k \) such that

\[-m < i_1 < \cdots < i_k < m, \quad |i_s - i_r| \geq 2 \text{ if } s \neq r.\]

(b) If \( k \) and \( m, 1 \leq k \leq m-1 \), are fixed, then

\[
\sum_{s=1}^{k} \prod_{i_s} [(2m-1)^2 - (2i_s - 1)^2] = \sum_{1 \leq j_1 < \cdots < j_k \leq m-1} \prod_{s=1}^{k} (4j_s)^2,
\]
where the left sum is over all indices \( i_1, \ldots, i_k \) such that
\[-m + 1 < i_1 < \cdots < i_k < m, \quad |i_s - i_r| \geq 2 \quad \text{if} \quad s \neq r.\]

**Remark.** The terms in (4.1) and (4.2) look to be similar to the terms in the identity conjectured by V. Kac and M. Wakimoto [14] and proved by S. Milne [22], and later by D. Zagier [28]; see details and further bibliography in [23], in particular, Sect. 7 and Cor. 7.6, pp. 120-121. Our asymptotic analysis involves eigenvalues of Schrödinger operators. This occurrence of eigenvalues suggests a possible link with advanced determinant calculus developed by G. Andrews (see C. Krattenthaler [15] and references there) and Hankel determinants in S. Milne [23].

**Proof.** These formulae come from the combinatorial meaning of the coefficients of the polynomials \( P_n(t) \) in the proof of Theorem 4. We give details of the proof for (4.1) only; the proof of (4.2) is the same.

Let \( D_{2(m-k)} \) denote the coefficient of \( P_{2m}(t) \) in front of \( t^{2(m-k)} \). By (3.17) we obtain that
\[
D_{2(m-k)} = (-1)^k C_{2m} \sum_{1 \leq j_1 < \cdots < j_k \leq m} (2j_1 - 1)^2 \cdots (2j_k - 1)^2 \tag{4.3}
\]

On the other hand \( P_{2m}(t) \) is defined by (3.13) as a sum of \( t \)-monomials such that each monomial corresponds to a walk \((j_1, \ldots, j_k)\) with positive steps of length 2 or 4. Moreover, by (3.14), the degree of each such monomial equals the number of steps of length 2. Thus we obtain, by (3.13), that
\[
D_{2(m-k)} = 2 \prod_{i=1}^{2m-1} \frac{(-2\alpha t)^{2i}(-\alpha^2)^i}{((2m)^2 - (\nu_1)^2) \cdots ((2m)^2 - (\nu_s)^2)}, \tag{4.4}
\]

where \( s = 2m - k - 1 \) and \( J_s \) is the set of all \( s \)-tuples of indices \((\nu_1, \ldots, \nu_s)\) that correspond to walks from \(-2m\) to \(2m\) with \(2m - 2k\) steps of length 2 and \(k\) steps of length 4.

Each term of the sum in (4.4) may be rewritten in the form
\[
\frac{(-1)^k 2^{2(m-k)} \alpha^{2m} \alpha^{2m} \alpha^{2m} \alpha^{2m}}{\prod_{i=1}^{2m-1} [(2m)^2 - (\nu_1)(2i)]}(2m)^2 \cdots (2m)^2 \cdots (2m)^2, \tag{4.5}
\]

where the \( k \)-tuple \((2i_1, \ldots, 2i_k)\) complements the \((2m - k - 1)\)-tuple \((\nu_1, \ldots, \nu_s)\) to the \((2m - 1)\)-tuple \((-2m + 2i)^{2m-1}\). Thus (4.4) implies, in view of (3.14),
\[
D_{2(m-k)} = (-1)^k C_{2m} \sum_{-m < i_1 < \cdots < i_k < m} (m^2 - i_1^2) \cdots (m^2 - i_k^2). \tag{4.6}
\]

Obviously, (4.3) and (4.5) imply (4.1). \( \square \)
The asymptotics of spectral gaps for large $n$

In this section we prove the following theorem.

**Theorem 9.** Let $\gamma_n$ be the $n$-th spectral gap of the Hill operator

$$Ly = -y'' - (4\alpha t \cos 2x + 2\alpha^2 \cos 4x)y,$$

where either both $\alpha$ and $t \neq 0$, are real, or both are pure imaginary numbers. Then the following asymptotic formulae hold for fixed $\alpha, t$ and $n \to \infty$:

for even $n$

$$\gamma_n = \frac{8|\alpha|^n}{2^n[(n-2)!!]^2} \left| \cos \left( \frac{\pi t}{2} \right) \right| \left[ 1 + O \left( \frac{\log n}{n} \right) \right],$$

and for odd $n$

$$\gamma_n = \frac{8|\alpha|^n}{2^n[(n-2)!!]^2} \left| \sin \left( \frac{\pi t}{2} \right) \right| \left[ 1 + O \left( \frac{\log n}{n} \right) \right],$$

where

$$(2m-1)!! = 1 \cdot 3 \cdots (2m-1), \quad (2m)!! = 2 \cdot 4 \cdots (2m).$$

**Proof.** The case where $\alpha = 0$ is trivial (then $\gamma_n \equiv 0$), so we assume that $\alpha \neq 0$. For convenience the proof is divided into several steps.

**Step 1.** Consider all possible walks from $-n$ to $n$. Each such walk is determined by the sequence of its steps

$$x = (x_1, \ldots, x_{\nu+1}),$$

or by its vertices

$$j_s = -n + \sum_{k=1}^{s} x_k, \quad s = 1, \ldots, \nu.$$
where \( j_s \) are given by (5.5). With these notations part (b) of Theorem 6 gives

\[
\gamma_n = 2 \left| \sum_{x \in X} B_n(x, z) \right| \left( 1 + O \left( \frac{1}{n^2} \right) \right),
\]

where \( z = z_n \) depends on \( n \), but \( |z| < 1 \).

In particular, the same formula holds for the operator (5.1). Moreover, since in that case

\[
V(m) = 0 \text{ if } m \neq \pm 2, \pm 4,
\]

it is enough to take into account only the walks with steps \( \pm 2 \) and \( \pm 4 \), so further we may think that \( X \) denotes the set of all walks from \(-n\) to \( n \) with steps \( \pm 2 \) and \( \pm 4 \).

**Step 2.** Let \( X^+ \) denote the set of all walks from \(-n\) to \( n \) with positive steps equal to 2 or 4. By the proof of Theorem 7 (see the text from (3.11) to (3.14)) we know that for even \( n \)

\[
2 \sum_{\xi \in X^+} B_n(\xi, 0) = \frac{8 \alpha^n}{2^n [(n-1)!]^2} \prod_{k=1}^{n/2} (t^2 - (2k - 1)^2),
\]

and for odd \( n \)

\[
2 \sum_{\xi \in X^+} B_n(\xi, 0) = \frac{-8 \alpha^n t}{2^n [(n-1)!]^2} \prod_{k=1}^{(n-1)/2} (t^2 - (2k)^2).
\]

Theorem 7 says that the sums (5.8) and (5.9) give the main part of the asymptotics of \( \gamma_n \) as \( \alpha \to 0 \). We are going to prove, for fixed \( \alpha \) and \( t \), that the same expressions give the asymptotics of \( \gamma_n \) for large \( n \).

Since for even \( n \) we have

\[
\cos \left( \frac{\pi}{2} t \right) = \prod_{k=1}^{\infty} \left( 1 - \frac{t^2}{(2k - 1)^2} \right) = \prod_{k=1}^{n/2} \left( 1 - \frac{t^2}{(2k - 1)^2} \right) \left[ 1 + O \left( \frac{1}{n} \right) \right],
\]

and for odd \( n \)

\[
\sin \left( \frac{\pi}{2} t \right) = \frac{\pi t}{2} \prod_{k=1}^{\infty} \left( 1 - \frac{t^2}{(2k)^2} \right) = \frac{\pi t}{2} \prod_{k=1}^{(n-1)/2} \left( 1 - \frac{t^2}{(2k)^2} \right) \left[ 1 + O \left( \frac{1}{n} \right) \right],
\]

(5.8) can be rewritten for even \( n \) as

\[
2 \sum_{\xi \in X^+} B_n(\xi, 0) = \frac{\pm 8 \alpha^n}{2^n [(n-2)!]^2} \cos \left( \frac{\pi}{2} t \right) \left[ 1 + O \left( \frac{1}{n} \right) \right],
\]
while (5.9) gives for odd \( n \):

\[
2 \sum_{\xi \in X^+} B_n(\xi, 0) = \pm 8\alpha^n \cdot \frac{\pi}{2^n \sqrt{(n - 2)!!}} \cdot \frac{2}{\pi} \sin \left( \frac{\pi t}{2} \right) \left[ 1 + O \left( \frac{1}{n} \right) \right].
\]

In view of (5.7), (5.10) and (5.11), to accomplish the proof of Theorem 9 we need to show that

\[
\sum_{x \in X} B_n(x, z) \approx \sum_{\xi \in X^+} B_n(\xi, 0) \quad \text{as} \quad n \to \infty,
\]

where \( z = z_n \) with \( |z| < 1 \). In Lemma 11 below it is proven that

\[
\sum_{\xi \in X^+} B_n(\xi, z) \approx \sum_{\xi \in X^+} B_n(\xi, 0).
\]

The remaining part of the proof shows that \( \sum_{x \in X \setminus X^+} B_n(x, z) \) is relatively small in comparison with \( \sum_{\xi \in X^+} B_n(\xi, z) \).

**Step 3.** Two technical lemmas.

**Lemma 10.**

\[
\frac{\log n}{2n} \leq \sum_{0<i<n} \frac{1}{n^2 - (n - 2i)^2} \leq \frac{1 + \log n}{2n}.
\]

\[
\sum_{i \neq 0, n} \left| \frac{1}{n^2 - (n - 2i)^2} \right| \leq \frac{1 + \log n}{n}.
\]

The proof is elementary, and it is omitted.

**Lemma 11.** If \( \xi \in X^+ \) and \( n \geq 3 \) then for \( z \in [0, 1) \)

\[
1 - z \frac{\log n}{n} \leq \frac{B_n(\xi, z)}{B_n(\xi, 0)} \leq 1 - z \frac{\log n}{4n},
\]

and for \( z \in (-1, 0] \)

\[
1 + |z| \frac{\log n}{2n} \leq \frac{B_n(\xi, z)}{B_n(\xi, 0)} \leq 1 + |z| \frac{2 \log n}{n}.
\]

**Proof.** By (5.6),

\[
\frac{B_n(\xi, z)}{B_n(\xi, 0)} = \prod_{0<i<n} \left( 1 + \frac{z}{n^2 - (n^2 - 2i)^2} \right)^{-1}.
\]

One can easily see (since \( 1 + x \leq e^x \), \( \forall x \in \mathbb{R} \)), that if either \( y_i \geq 0 \), \( i = 1, \ldots, m \), or \( y_i \in [-1, 0] \), \( i = 1, \ldots, m \), then

\[
1 + \sum_{i=1}^m y_i \leq (1 + y_1) \cdots (1 + y_m) \leq \exp \left( \sum_{i=1}^m y_i \right).
\]
Now from (5.16) it follows that if \( y_i \in (-1, 0) \) \( \forall i \), or \( y_i \geq 0 \) \( \forall i \), then
\[
1 - \sum_{i=1}^{m} y_i \leq \exp \left( -\sum_{i=1}^{m} y_i \right) \leq \left( \prod_{i=1}^{m} (1 + y_i) \right)^{-1} \leq \left( 1 + \sum_{i=1}^{m} y_i \right)^{-1}.
\]

We use the inequalities (5.17) with \( y_i = z / \left( n^2 - (n - 2i)^2 \right) \), \( i = 1, \ldots, n - 1 \). If \( z \in [0, 1) \), then by (5.17) and (5.12), we have
\[
1 - z \frac{1 + \log n}{2n} \leq \frac{B_n(\xi, z)}{B_n(\xi, 0)} \leq \left( 1 + z \frac{\log n}{2n} \right)^{-1}.
\]
Taking into account that \((1 + \varepsilon)^{-1} < 1 - \varepsilon / 2\) with \( \varepsilon = z \log n / (2n) < 1 \), we obtain (5.14).

If \( z \in (-1, 0] \) then, again by (5.17) and (5.12), we obtain
\[
1 - |z| \frac{1 + \log n}{2n} \leq \frac{B_n(\xi, z)}{B_n(\xi, 0)} \leq \left( 1 - |z| \frac{1 + \log n}{2n} \right)^{-1}.
\]
If \( n \geq 3 \) then, with \( \varepsilon = |z|(1 + \log n) / (2n) \leq 1 / 2 \), we have
\[
(1 - \varepsilon)^{-1} \leq 1 + 2\varepsilon \leq 1 + 2|z| \frac{1 + \log n}{2n} \leq 1 + |z| \frac{2 \log n}{n},
\]
which proves (5.15).

**Step 4.** This step contains some constructions and inequalities that are crucial for the estimate of the sum \( \sum_{x \in X \setminus X^+} B_n(x, z) \). For each \( \xi \in X^+ \) let \( X_\xi \) denote the set of all walks \( x \in X \setminus X^+ \) such that each vertex of \( \xi \) is a vertex of \( x \) also. It is easy to see that
\[
X \setminus X^+ = \bigcup_{\xi \in X^+} X_\xi,
\]
i.e., for each \( x \in X \setminus X^+ \) there exists a \( \xi \in X^+ \) such that each vertex of \( \xi \) is a vertex of \( x \). Indeed, fix a walk \( x \in X \setminus X^+ \). If \( (v_k)_{k=1}^{r} \) is the sequence of its vertices, then we define a strictly increasing subsequence \( (v_{k_\nu})_{\nu=1}^{r} \) of it as follows:
\[
k_1 = \min \{ k : -n < v_k < n \}, \quad k_s = \min \{ k : v_{k_{s-1}} < v_k < n \}.
\]
For some \( \nu \leq r \) one would get that such a choice cannot continue anymore and stop: this gives the last term \( v_{k_\nu} \) of the subsequence. Since each step of \( x \) is equal to \( \pm 4 \) or \( \pm 2 \), the distance between every two consecutive terms of the subsequence \( (v_{k_{\nu}})_{\nu=1}^{r} \) is equal to 2 or 4, and the same is true for the differences \( v_{k_1} - (-n) \) and \( n - v_{\nu} \). Thus \( (v_{k_{\nu}})_{\nu=1}^{r} \) is the sequence of the vertices of a walk \( \xi \in X^+ \) such that each vertex of \( \xi \) is a vertex of \( x \).
For each $\xi \in X^+$ and $\mu \in \mathbb{N}$ let $X_{\xi,\mu}$ be the set of all $x \in X_\xi$ such that $x$ has $\mu$ more vertices than $\xi$. Then we have
\begin{equation}
X_\xi = \bigcup_{\mu=1}^{\infty} X_{\xi,\mu}.
\end{equation}

Moreover, for each $\mu$-tuple $(i_1, \ldots, i_\mu)$ of integers in $I_n = (n + 2\mathbb{Z}) \setminus \{\pm n\}$ we define $X_\xi(i_1, \ldots, i_\mu)$ as the set of all walks $x$ with $\nu + 1 + \mu$ steps such that $(i_1, \ldots, i_\mu)$ and the sequence of the vertices of $\xi$ are complementary subsequences of the sequence of the vertices of $x$. Then
\begin{equation}
X_{\xi,\mu} = \bigcup_{(i_1, \ldots, i_\mu) \in (I_n)\mu} X_\xi(i_1, \ldots, i_\mu).
\end{equation}

**Lemma 12.** Under the above notations, for each walk $\xi \in X^+$ and each $\mu$-tuple $(i_1, \ldots, i_\mu) \in (I_n)\mu$,
\begin{equation}
\# X_\xi(i_1, \ldots, i_\mu) \leq 5^\mu.
\end{equation}

**Proof.** Fix $\xi \in X^+$ and let $(j_n)_{n=1}^{\infty}$ be the sequence of its vertices. If $x \in X_\xi(i_1, \ldots, i_\mu)$ then the sequence of the vertices of $x$ may be obtained by adding $i_1, \ldots, i_\mu$, one by one, at appropriate places, as new terms to the sequence $(j_n)_{1}^{\nu+1}$.

For convenience we put $j_0 = -n$ and $j_{\mu+1} = n$. The integer $i_1$ could appear as a vertex in a prospective walk $x \in X_\xi(i_1, \ldots, i_\mu)$ after some $\xi$-vertex $j_{s_1}$. But then $i_1 - j_{s_1} = \pm 2, \pm 4$, so there are only four possible choices for $s_1$, i.e., at most 4 choices where to place $i_1$. If $i_1, \ldots, i_m$, $1 \leq m < \mu$ have been properly placed, then the next vertex $i_{m+1}$ could appear immediately after $i_m$ (one option) or after some $\xi$-vertex $j_{s_{m+1}}$, but then $i_{m+1} - j_{s_{m+1}} = \pm 2, \pm 4$, so there are at most 5 options for the spot where $i_{m+1}$ could be placed. Thus, for each $k$, $1 \leq k \leq \mu$, there are at most 5 choices for a place for $i_k$, and therefore, the cardinality of $X_\xi(i_1, \ldots, i_\mu)$ does not exceed $5^\mu$.

**Remark.** We could use instead of $X_\xi(i_1, \ldots, i_\mu)$ its subset $X'_\xi(i_1, \ldots, i_\mu)$ of the walks $x$ which lead to $\xi$ after restructuring defined by (5.19). Then one can show that $\# X'_\xi(i_1, \ldots, i_\mu) \leq 3^\mu$.

**Lemma 13.** If $\xi \in X^+$ and $|z| \leq 1$ then there exists $n_1$ such that for $n \geq n_1$
\begin{equation}
\sum_{x \in X_\xi} |B_n(x, z)| \leq |B_n(\xi, z)| \cdot \frac{K \log n}{n}
\end{equation}
where \( K = 160(|t| + |\alpha|)^2 \).

Proof. By (5.20)

\[
\sum_{x \in X_\xi} |B_n(x, z)| = \sum_{\mu = 1}^{\infty} \sum_{x \in X_{\xi, \mu}} |B_n(x, z)|,
\]

We are going to show that

\[
\sum_{x \in X_{\xi, \mu}} |B_n(x, z)| \leq |B_n(\xi, z)| \left( \frac{20C \log n}{n} \right)^\mu
\]

where \( C = 4(|t| + |\alpha|)^2 \). If (5.25) is proven, then with \( n_1 \) chosen so that \( (20C \log n)/n \leq 1/2 \) for \( n \geq n_1 \) one would obtain, by (5.24),

\[
\sum_{x \in X_\xi} |B_n(x, z)| \leq \sum_{\mu = 1}^{\infty} \left( \frac{20C \log n}{n} \right)^\mu \leq 40C \log n n,
\]

i.e., (5.23) would hold with \( K = 40C \).

By (5.20),

\[
\sum_{x \in X_{\xi, \mu}} |B_n(x, z)| \leq \sum_{(i_1, \ldots, i_\mu) \in X_\xi(i_1, \ldots, i_\mu)} \sum_{x \in X_{\xi}} |B_n(x, z)|,
\]

where the first sum on the right is taken over all \( \mu \)-tuples \((i_1, \ldots, i_\mu)\) of integers \( i_s \in n + 2\mathbb{Z} \) such that \( i_s \neq \pm n \).

Fix \((i_1, \ldots, i_\mu)\). If \( x \in X_\xi(i_1, \ldots, i_\mu) \), then, in view of (5.6),

\[
\frac{B_n(x, z)}{B_n(\xi, z)} = \prod_{k \in J_s} V(x_k) \cdot \frac{1}{(n^2 - i_1^2 + z) \cdots (n^2 - i_\mu^2 + z)}.
\]

If each step of \( \xi \) is a step of \( x \), then (since \( V(x_k) = -2\alpha t \) if \( x_k = \pm 2 \), and \( V(x_k) = -\alpha^2 \) if \( x_k = \pm 4 \))

\[
\prod_{k \in J_s} |V(x_k)| \leq C^\mu
\]

(because \( x \) has \( \mu \) steps more). The same is true in the general case also.

Indeed, let \((j_s)_{s=1}^\nu = 1\) are the vertices of \( \xi \), and let us put for convenience \( j_0 = -n \) and \( j_{\nu + 1} = n \). Since each vertex of \( \xi \) is a vertex of \( x \), for each \( s, 1 \leq s \leq \nu + 1 \),

\[
\xi_s = j_s - j_{s-1} = \sum_{k \in J_s} x_k
\]

where \( x_k, k \in J_s \), are the steps of \( x \) between the vertices \( j_{s-1} \) and \( j_s \).

Fix an \( s, 1 \leq s \leq \nu + 1 \). If \( \xi_s = 2 \), then there is a step \( x_k^*, k^* \in J_s \),
such that $|x_k^*| = 2$ (otherwise $\xi_s$ would be a multiple of 4). Therefore, $|V(\xi_s)| = |V(x_k^*)|$, and

$$\tag{5.29} \prod_{J_s} |V(x_k)| \leq C^{b_s-1}, \quad \text{where } b_s := \# J_s. \leqno{\text{(5.29)}}$$

Suppose $\xi_s = 4$. If there is $k^* \in J_s$ with $|x_k^*| = 4$, then $|V(\xi_s)| = |V(x_k^*)|$, so (5.29) holds. Otherwise, there are $k', k'' \in J_s$ such that $|x_{k'}| = |x_{k''}| = 2$, and therefore,

$$\frac{|V(x_{k'}) V(x_{k''})|}{|V(\xi_s)|} = \frac{4|\alpha|^2 |t|^2}{|\alpha|^2} = 4|t|^2 \leq C,$$

which implies (5.29).

Since $\sum_s (b_s - 1) = \mu$, (5.29) yields (5.28). Therefore, using the elementary inequality

$$|n^2 - i^2 + z|^{-1} \leq 2|n^2 - i^2|^{-1}, \quad i \neq \pm n, \quad |x| \leq 1,$$

we obtain, by (5.27) and (5.28), that

$$\frac{|B_n(x, z)|}{|B_n(\xi, z)|} \leq \frac{(2C)^\mu}{|n^2 - i^2_1| \cdots |n^2 - i^2_\mu|}, \quad x \in X_\xi(i^1, \ldots, i_\mu).$$

Now, by Lemma 12

$$\sum_{x \in X_\xi(i^1, \ldots, i_\mu)} |B_n(x, z)| \leq \sum_{i \in (n+2\mathbb{Z}) \setminus \{\pm n\}} \frac{10C^\mu}{|n^2 - i^2|} \leq \left( \sum_{i \in (n+2\mathbb{Z}) \setminus \{\pm n\}} \frac{10C}{|n^2 - i^2|} \right)^\mu \leq \left( \frac{20C}{n} \log n \right)^\mu,$$

i.e., (5.25) holds. This completes the proof of Lemma 13.

\[ \square \]

\textit{Step 5.} This step completes the proof of Theorem 9 for even $n$. If $t = 2k - 1$, $k = 1, 2, \ldots$, then, by Theorem 11 in [7], $\gamma_n = 0$ for $n > 2k$, thus (5.2) holds.

Suppose that $\alpha$ and $t$ are nonzero real numbers, and $t \neq 2k - 1$, $k \in \mathbb{N}$. 
By (5.18), \( X \setminus X^+ = \bigcup_{\xi \in X^+} X_\xi \). Let us choose disjoint sets \( X'_\xi \subset X_\xi \) so that
\[
(5.30) \quad X' = \bigcup_{\xi \in X^+} X'_\xi.
\]
Then
\[
(5.31) \quad \sum_{x \in X \setminus X^+} B_n(x, z) = \sum_{\xi \in X^+} \left( \sum_{x \in X'_\xi} B_n(x, z) \right)
\]
and therefore, by (5.31), we have
\[
(5.32) \quad \sum_{x \in X} B_n(x, z) = \sum_{\xi \in X^+} \left( B_n(\xi, z) + \sum_{x \in X'_\xi} B_n(x, z) \right) = \Sigma^- + \Sigma^+
\]
where
\[
(5.33) \quad \Sigma^- = \sum_{\xi : B_n(\xi, 0) < 0} \ldots, \quad \Sigma^+ = \sum_{\xi : B_n(\xi, 0) > 0} \ldots.
\]
Set
\[
(5.34) \quad \sigma^-_n = \sum_{\xi : B_n(\xi, 0) < 0} B_n(\xi, 0), \quad \sigma^+_n = \sum_{\xi : B_n(\xi, 0) > 0} B_n(\xi, 0),
\]
and
\[
(5.35) \quad \sigma_n = \sum_{\xi \in X^+} B_n(\xi, 0) = \sigma^-_n + \sigma^+_n.
\]
By (5.14) and (5.15) in Lemma 11 and by (5.23), we obtain, for each \( \xi \) with \( B_n(\xi, 0) < 0 \),
\[
\left[ 1 + C \frac{\log n}{n} \right] B_n(\xi, 0) \leq B_n(\xi, z) + \sum_{x \in X'_\xi} B_n(x, z) \leq \left[ 1 - C \frac{\log n}{n} \right] B_n(\xi, 0),
\]
and therefore,
\[
(5.36) \quad \left[ 1 + C \frac{\log n}{n} \right] \sigma^-_n \leq \Sigma^- \leq \left[ 1 - C \frac{\log n}{n} \right] \sigma^-_n,
\]
where the constant \( C > 0 \) depends on \( \alpha \) and \( t \) only.

In an analogous way it follows
\[
(5.37) \quad \left[ 1 - C \frac{\log n}{n} \right] \sigma^+_n \leq \Sigma^+ \leq \left[ 1 + C \frac{\log n}{n} \right] \sigma^+_n
\]
(with the same constant \( C \), otherwise we may take a greater constant \( C \) in (5.36)).
Now (5.32) - (5.37) yield
\[-C(|\sigma_n^-| + \sigma_n^+) \log \frac{n}{n} \leq \sum_{x \in X} B_n(x, z) - \sigma_n \leq C(|\sigma_n^-| + \sigma_n^+) \log \frac{n}{n},\]
and therefore,
\[
\left|\frac{1}{|\sigma_n|} \sum_{x \in X} B_n(x, z) - 1\right| \leq C \frac{|\sigma_n^-| + \sigma_n^+}{|\sigma_n|} \log \frac{n}{n}.
\]

In view of (5.38) and (5.35),
\[
\sigma_n = \sum_{\xi \in X^+} B_n(\xi, 0) = \frac{4\alpha^n}{2^n[(n-1)!]^2} \prod_{k=1}^{n/2} (t^2 - (2k-1)^2),
\]
thus \(\sigma_n \neq 0\) (since \(t \neq 2k-1, k \in \mathbb{N}\)). By (5.34),
\[
|\sigma_n^-| + \sigma_n^+ = \frac{4\alpha^n}{2^n[(n-1)!]^2} \prod_{k=1}^{n/2} (t^2 + (2k-1)^2),
\]
so (for \(n > |t|\))
\[
\frac{|\sigma_n^-| + \sigma_n^+}{|\sigma_n|} = \frac{\prod_{k=1}^{n/2} (t^2 + (2k-1)^2)}{\prod_{k=1}^{n/2} (t^2 - (2k-1)^2)} = \frac{\prod_{k=1}^{n/2} \left(1 + \frac{t^2}{(2k-1)^2}\right)}{\prod_{k=1}^{n/2} \left|1 - \frac{t^2}{(2k-1)^2}\right|} \leq \frac{\prod_{k=1}^{\infty} \left(1 + \frac{t^2}{(2k-1)^2}\right)}{\prod_{k=1}^{\infty} \left|1 - \frac{t^2}{(2k-1)^2}\right|} = \left|\cosh \left(\frac{\pi t}{2}\right)\right|. \tag{5.41}
\]
Hence, by (5.38),
\[
\sum_{x \in X} B_n(x, z) = \sigma_n \left[1 + O\left(\frac{\log n}{n}\right)\right]. \tag{5.42}
\]

If \(\alpha\) and \(t\) are pure imaginary, then the situation is more simple because \(B(\xi, 0) > 0\) for each \(\xi \in X^+\). Thus we have \(\sigma_n = \sigma_n^+ > 0\) and \(\sum_{x \in X} = \Sigma^+\), so (5.37) yields immediately (5.42).

Finally, by (5.38) and (5.42), (5.43) implies
\[
2 \sum_{x \in X} B_n(x, z) = \frac{\pm 8\alpha^n}{2^n[(n-2)!!]^2} \cos \left(\frac{\pi t}{2}\right) \left[1 + O\left(\frac{\log n}{n}\right)\right]. \tag{5.43}
\]
In view of (5.7) the estimate (5.43) proves Theorem 9 for even \(n\), i.e., (5.2) holds.

Step 6. This step completes the proof of Theorem 9 for odd \(n\).
If \( t = 2k, \ k = 0, 1, 2, \ldots \), then, by Theorem 11 in [7], \( \gamma_n = 0 \) for \( n > 2k + 1 \), thus (5.3) holds.

Suppose that \( \alpha \) and \( t \) are non-zero real numbers, and \( t \neq 2k, \ k \in \mathbb{N} \). Using the same argument and notations as in Step 5, we obtain (see (5.30)-(5.38)) that (5.38) holds for odd \( n \).

In view of (5.3), (5.35), (5.44)

\[
\sigma_n = -4\alpha^n t \frac{(n-1)/2}{2^n((n-1)!)^2} \prod_{k=1}^{(n-1)/2} (t^2 - (2k)^2),
\]

thus \( \sigma_n \neq 0 \) (since \( t \neq 2k, \ k \in \mathbb{N} \)). By (5.34),

\[
|\sigma_n^-| + \sigma_n^+ = \frac{4\alpha^n |t|}{2^n((n-1)!)^2} \prod_{k=1}^{(n-1)/2} (t^2 + (2k)^2),
\]

so (for \( n > |t| \))

\[
\frac{|\sigma_n^-| + \sigma_n^+}{|\sigma_n|} = \frac{\prod_{k=1}^{(n-1)/2} (t^2 + (2k)^2)}{\prod_{k=1}^{(n-1)/2} (t^2 - (2k)^2)} = \frac{\prod_{k=1}^{(n-1)/2} \left(1 + \frac{t^2}{(2k)^2}\right)}{\prod_{k=1}^{(n-1)/2} \left|1 - \frac{t^2}{(2k)^2}\right|} \\
\leq \frac{\prod_{k=1}^{\infty} \left(1 + \frac{t^2}{(2k)^2}\right)}{\prod_{k=1}^{\infty} \left|1 - \frac{t^2}{(2k)^2}\right|} = \left|\frac{\sinh \left(\frac{\pi}{2} t\right)}{\sin \left(\frac{\pi}{2} t\right)}\right|.
\]

Hence, by (5.38),

\[
\sum_{x \in X} B_n(x, z) = \sigma_n \left[1 + O \left(\frac{\log n}{n}\right)\right].
\]

If \( \alpha \) and \( t \) are pure imaginary, then \( B(\xi, 0) > 0 \) for each \( \xi \in X^+ \). Thus we have \( \sigma_n = \sigma_n^+ > 0 \) and \( \sum_{x \in X} = \Sigma^+ \), so (5.37) yields immediately (5.46).

Finally, by (5.3), (5.31), (5.46) implies

\[
2 \sum_{x \in X} B_n(x, z) = \frac{\pm 8\alpha^n t}{2^n((n-2)!)^2} \frac{2}{\pi} \sin \left(\frac{\pi}{2} t\right) \left[1 + O \left(\frac{\log n}{n}\right)\right].
\]

In view of (5.7) the estimate (5.47) proves Theorem 9 for odd \( n \), i.e., (5.3) holds. \( \square \)
6. Comments and generalizations

1. In Sections 3-5 we consider only two term potentials of the form $v(x) = a \cos 2x + b \cos 4x$. Now we would like to make some comments about the more general case where $v$ is a real-valued trigonometric polynomial of the form

\begin{equation}
6.1 \quad v(x) = \sum_{k=1}^{K} (a_k e^{2ikx} + \overline{a_k} e^{-2ikx}).
\end{equation}

For each $n \in \mathbb{N}$ let $X_n$ and $X_n^+$ denote, respectively, the set of all walks from $-n$ to $n$ with steps $\pm 2, \ldots, \pm 2K$, and the subset of $X_n$ of all walks with positive steps. With these notations the general asymptotic formula (2.71) from Theorem 6 becomes

\begin{equation}
6.2 \quad \gamma_n = 2 \left| \sum_{x \in X_n} B_n(x, z) \right| \left( 1 + O \left( \frac{\|v\|^2}{n^2} \right) \right), \quad z = z_n,
\end{equation}

where either $|z_n| \leq 4\|v\|$ for all $n$ (if $\|v\| < 1/9$), or $|z_n| < 1$ for large enough $n$.

Consider the parametrization

\begin{equation}
6.3 \quad a_1 = t_1 \alpha, \quad a_2 = t_2 \alpha^2, \ldots, a_{K-1} = t_{K-1} \alpha^{K-1}, \quad a_K = \alpha^K.
\end{equation}

Then, by part (a) of Proposition 11 $|z| \leq 4\|v\| = O(|\alpha|)$ for small $\alpha$, so (6.2) yields (as in the proof of Theorem 7) that

\begin{equation}
6.4 \quad \gamma_n = 2 \left| \sum_{\xi \in X_n^+} B_n(\xi, 0) \right| + O \left( |\alpha|^{n+1} \right).
\end{equation}

If $K = 1$ then there is only one walk with positive steps from $-n$ to $n$, namely $\xi^* = (2, \ldots, 2)$ (i.e., $X_n^+ = \{ \xi^* \}$). Since

\[ B(\xi^*, 0) = \frac{\alpha^n}{\prod_{j=1}^{n-1} [n^2 - (-n + 2j)^2]} = \frac{\alpha^n}{4^{n-1}[(n-1)!]^2}, \]

we obtain

\begin{equation}
6.5 \quad \gamma_n = 2 |B_n(\xi^*, 0)| + O \left( |\alpha|^{n+1} \right) = \frac{|\alpha|^n}{4^{n-1}[(n-1)!]^2} (1 + O(|\alpha|)),
\end{equation}

which gives the Levy-Keller’s formula (1.8) for the Mathieu potential.

2. If $K > 1$ then the computation of $\sum_{\xi \in X_n^+} B_n(\xi, 0)$ is not trivial (unless all $t$’s vanish and there is only one walk $\xi$ with positive steps that gives a non-zero term $B_n(\xi, 0)$).
As in the proof of Theorem 7 one can easily see, for each \( K \), that
\[
\sum_{x \in X_k^+} B_n(x, 0) = \alpha^n \cdot P_n(t_1, \ldots, t_{K-1}),
\]
where \( P \) is a polynomial in \( t_1, \ldots, t_{K-1} \).

Our main achievement in Theorem 7 (where \( K = 2 \)) is the explicit form of the corresponding polynomials \( P_n, n \in \mathbb{N} \). In Theorem 7 we consider potentials of the form
\[
v(x) = -4\alpha t \cos 2x - 2\alpha^2 \cos 4x,
\]
where \( \alpha \) and \( t \) are simultaneously real or pure imaginary, while the potentials that comes from (6.1) for \( K = 2 \) are more general, namely
\[
v(x) = a_1 e^{2ix} + a_1^* e^{-2ix} + a_2 e^{4ix} + a_2^* e^{-4ix}, \quad a_1, a_2 \in \mathbb{C}, \ a_2 \neq 0,
\]
or equivalently,
\[
v(x) = A \cos 2x + B \sin 2x + C \cos 4x + D \sin 4x,
\]
where
\[
A = 2 \text{Re} a_1, \quad B = -2 \text{Im} a_1, \quad C = 2 \text{Re} a_2, \quad D = -2 \text{Im} a_2.
\]

Using the same parametrization as in Theorem 7 (it is slightly different from (6.3)), we may write each potential \( v \in \{6.7\}, \{6.8\} \) as
\[
v(x) = -2\alpha t e^{2ix} - 2\alpha^* t e^{-2ix} - \alpha^2 e^{4ix} - \alpha^* \alpha e^{-4ix},
\]
where \( \alpha \) and \( t \) are complex numbers such that \( \alpha^2 = -a_2, \ 2\alpha t = -a_1 \).

Observe that formally the expression \( \sum_{x \in X_n^+} B_n(x, 0) \) is exactly the same that has been used in the proof of Theorem 7, because \( \alpha \) and \( t \) would appear in \( B_n(x, 0) \) only if \( x \) has negative steps. Therefore, the same argument that proves Theorem 7 shows that the following more general statement holds.

**Theorem 14.** Let \( \gamma_n, n \in \mathbb{N} \) be the lengths of instability zones of the Hill operator which potential \( v \) is given by (6.4), with \( \alpha, t \in \mathbb{C} \) and \( \alpha \neq 0 \). If \( t \) is fixed and \( \alpha \to 0 \), then for even \( n \)
\[
\gamma_n = \left| \frac{8\alpha^n}{2^n [(n-1)!]^2} \prod_{k=1}^{n/2} \left( t^2 - (2k-1)^2 \right) \right| (1 + O(\alpha)),
\]
and for odd \( n \)
\[
\gamma_n = \left| \frac{8\alpha^n t}{2^n [(n-1)!]^2} \prod_{k=1}^{(n-1)/2} \left( t^2 - (2k)^2 \right) \right| (1 + O(\alpha)).
\]
If \( K > 2 \) we don’t know any explicit formula for the asymptotics of \( \gamma_n \) as \( \alpha \to 0 \) besides the simple extensions of Theorem 14 and (6.5) that one can obtain by using the following elementary statement (see [6], Prop. 20 and 24).

**Proposition 15.** Suppose \( m \in \mathbb{N}, m > 1 \), is fixed. Let \( \gamma_n, n \in \mathbb{N} \) be the lengths of instability zones of the Hill operator with a potential \( v \), and let \( \tilde{\gamma}_n \) be the lengths of instability zones of the Hill operator which potential is \( \tilde{v}(x) = m^2 v(mx) \). Then

\[
(6.12) \quad \tilde{\gamma}_mn = m^2 \gamma_n, \quad \tilde{\gamma}_k = 0 \text{ if } k \notin m\mathbb{N}.
\]

3. If we fix \( \alpha \) and \( t \), then the proof of Theorem 9 (with only a slight change in its Steps 5 and 6) proves the following more general claim.

**Theorem 16.** Let \( \gamma_n, n \in \mathbb{N} \) be the lengths of instability zones of the Hill operator which potential \( v \) is given by (6.9), with \( \alpha, t \in \mathbb{C}, \alpha \neq 0 \). Then, for fixed \( \alpha \) and \( t \), the asymptotic formula (5.2) holds for even \( n \to \infty \), while the asymptotic formula (5.3) holds for odd \( n \to \infty \).

But here we would like to reformulate Theorem 16 so that to have the asymptotics of \( \gamma_n \) given explicitly in terms of the coefficients \( a_1 \) and \( a_2 \). In fact, we proved (see Step 2 in the proof of Theorem 9), that (5.8) and (5.9) give the asymptotics of \( \gamma_n \), respectively, for even and odd \( n \).

Replacing, respectively in (5.2) and (5.3), \( \cos(\frac{\pi}{2} t) \) and \( \frac{2}{\pi} \sin(\frac{\pi}{2} t) \) with the right-hand sides of (5.8) and (5.9), and taking into account that

\[
 a_1 = -2\alpha t \quad \text{and} \quad a_2 = -\alpha^2,
\]

we obtain, for \( a_2 \neq 0 \), the following theorem.

**Theorem 17.** Let \( \gamma_n, n \in \mathbb{N} \) be the lengths of instability zones of the Hill operator \( Ly = -y'' + v(x)y \), where

\[
(6.13) \quad v(x) = a_1e^{2ix} + \bar{a}_1e^{-2ix} + a_2e^{4ix} + \bar{a}_2e^{-4ix},
\]

where \( a_1, a_2 \in \mathbb{C} \). Then, for even \( n \),

\[
(6.14) \quad \gamma_n = \left\lfloor \frac{8}{2^n[(n-1)!]^2} \prod_{k=1}^{n/2} \left( \frac{a_1^2}{4} + (2k-1)^2 a_2 \right) \right\rfloor \left[ 1 + O \left( \frac{\log n}{n} \right) \right],
\]

and for odd \( n \),

\[
(6.15) \quad \gamma_n = \left\lfloor \frac{8}{2^n[(n-1)!]^2} \frac{a_1}{2} \prod_{k=1}^{(n-1)/2} \left( \frac{a_1^2}{4} + (2k)^2 a_2 \right) \right\rfloor \left[ 1 + O \left( \frac{\log n}{n} \right) \right].
\]
Observe, that Theorem 17 holds for $a_2 = 0$ as well, because then (6.14) and (6.15) come from (1.9). Of course, one can give an alternative direct proof of (6.14) and (6.15) in the case where $a_2 = 0$ by the same argument that has been used to prove Theorem 9.

If $K > 2$, then, besides the simple extensions of Theorem 17 that come from Proposition 15, we don’t know any explicit formula (in terms of the coefficients $a_k$ in (6.1)) for the exact asymptotics of $\gamma_n$ as $n \to \infty$ although some general formulas for the asymptotics in the case of trig-polynomial potentials could be found, for example, in [9].

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