Searching for Effects of Spatial Noncommutativity via Chern-Simons’ Processes *

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Abstract

The possibility of testing spatial noncommutativity in the case of both position-position and momentum-momentum noncommuting via a Chern-Simons’ process is explored. A Chern-Simons process can be realized by an interaction of a charged particle in special crossed electric and magnetic fields, in which the Chern-Simons term leads to non-trivial dynamics in the limit of vanishing kinetic energy. Spatial noncommutativity leads to the spectrum of the orbital angular momentum possessing fractional values. Furthermore, in both limits of vanishing kinetic energy and subsequent vanishing magnetic field, the Chern-Simons term leads to this system having non-trivial dynamics again, and the dominant value of the lowest orbital angular momentum being $\hbar/4$, which is a clear signal of spatial noncommutativity. An experimental verification of this prediction by a Stern-Gerlach-type experiment is suggested.

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1. Introduction

Studies of low energy effective theory of superstrings show that space is noncommutative \cite{1-8}. Spatial non-commutativity is apparent near the Planck scale. Its modifications to ordinary quantum theory are extremely small. We ask whether one can find some low energy detectable relics of physics at the Planck scale by current experiments. Such a possibility is inferred from the incomplete decoupling between effects at high and low energy scales. For the purpose of clarifying phenomenological low energy effects, quantum mechanics in noncommutative space (NCQM) is available. If NCQM is a realistic physics, all the low energy quantum phenomena should be reformulated in it. In literature, NCQM have been studied in detail \cite{9-15}; many interesting topics, from the Aharonov-Bohm effect to the quantum Hall effect have been considered \cite{16-25}. Recent investigations of the deformed Heisenberg-Weyl algebra (the NCQM algebra) in noncommutative space explore some new features of effects of spatial noncommutativity \cite{15}. The possibility of testing spatial noncommutativity via Rydberg atoms is explored. But there are two problems in the suggested experiment of Rydberg atoms: (1) The special arrangement of the electric field required in the experiment is difficult to realized in laboratories; (2) The measurement depends on a extremely high characteristic frequency which may be difficult to reach by current experiments.

In this paper we show a possibility of testing spatial noncommutativity via a Chern-Simons process. Chern-Simons’ processes \cite{26-28} exhibit interesting properties in physics. In laboratories a Chern-Simons process can be realized by an interaction of a charged particle in special crossed electric and magnetic fields, in which the experimental situation is different from one in the experiment of Rydberg atoms. Properties of the Chern-Simons process at the level of NCQM are investigated. Spatial noncommutativity leads to the spectrum of the orbital angular momentum possessing a fractional zero-point angular momentum. In the limit of vanishing kinetic energy the Chern-Simons term leads to this system having non-trivial dynamics. For the case of both position-position and momentum-momentum noncommuting in a further limit of the subsequent diminishing magnetic field this system possesses non-trivial dynamics again, and the dominant value
of the lowest orbital angular momentum in the process is $\hbar/4$. This result is a clear signal
of spatial noncommutativity, and can be verified by a Stern-Gerlach-type experiment, in
which two difficulties in the experiment of Rydberg atoms are resolved.

In Ref. [18] other electromagnetic effects of spatial noncommutativity was explored.

2. The Deformed Heisenberg-Weyl algebra

In the following we review the background of the deformed Heisenberg-Weyl Algebra.
In order to develop the NCQM formulation we need to specify the phase space and the
Hilbert space on which operators act. The Hilbert space can consistently be taken to be
exactly the same as the Hilbert space of the corresponding commutative system [9].

As for the phase space we consider both position-position noncommutativity (position-
time noncommutativity is not considered) and momentum-momentum noncommutativity.
There are different types of noncommutative theories, for example, see a review paper [8].

In the case of both position-position and momentum-momentum noncommuting the
consistent deformed Heisenberg-Weyl algebra [15] is:

$$
\begin{align*}
\{\hat{x}_I, \hat{x}_J\} &= i\xi^2 \theta_{IJ}, \\
\{\hat{x}_I, \hat{p}_J\} &= i\hbar \delta_{IJ}, \\
\{\hat{p}_I, \hat{p}_J\} &= i\xi^2 \eta_{IJ}, \\
(I, J &= 1, 2, 3)
\end{align*}
$$

where $\theta_{IJ}$ and $\eta_{IJ}$ are the antisymmetric constant parameters, independent of the position
and momentum. We define $\theta_{IJ} = \epsilon_{IJK} \theta_K$ (Henceforth the summation convention is used),
where $\epsilon_{IJK}$ is a three-dimensional antisymmetric unit tensor. We put $\theta_3 = \theta$ and the rest
of the $\theta$-components to zero (which can be done by a rotation of coordinates), then we
have $\theta_{ij} = \epsilon_{ij}\theta$ ($i, j = 1, 2$), where $\epsilon_{ij3}$ is rewritten as a two-dimensional antisymmetric unit
tensor $\epsilon_{ij}, \epsilon_{12} = -\epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0$. Similarly, we have $\eta_{ij} = \epsilon_{ij}\eta$. In Eqs. (1) the
scaling factor $\xi$ is

$$
\xi = \left(1 + \frac{1}{4\hbar^2} \theta \eta\right)^{-1/2}.
$$

It plays a role for guaranteeing consistent representations of $(\hat{x}_i, \hat{p}_j)$ in terms of the unde-
formed canonical variables $(x_i, p_j)$ (See Eqs. [17]).

In noncommutative space questions about whether the concept of identical particles
being still meaningful and whether Bose-Einstein statistics being still maintained should be
answered. Bose - Einstein statistics can be investigated at two levels: the level of quantum
field theory and the level of quantum mechanics. On the fundamental level of quantum field theory the annihilation and creation operators appear in the expansion of the (free) field operator \( \Psi(\hat{x}) = \int d^3ka_k(t)\Phi_k(\hat{x}) + h.c. \). The consistent multi-particle interpretation requires the usual (anti)commutation relations among \( a_k \) and \( a_k^\dagger \). Introduction of the Moyal type deformation of coordinates may yield a deformation of the algebra between the creation and annihilation operators \[29\]. Whether the deformed Heisenberg - Weyl algebra is consistent with Bose - Einstein statistics is an open issue at the level of quantum field theory.

In this paper our study is restricted in the context of non-relativistic quantum mechanics. Following the standard procedure in the ordinary quantum mechanics in commutative space we construct the deformed annihilation-creation operators \( \hat{a}_i, \hat{a}_i^\dagger \) which are related to the deformed canonical variables \( (\hat{x}_i, \hat{p}_i) \). In order to maintain the physical meaning of \( \hat{a}_i \) and \( \hat{a}_i^\dagger \) the relations among \( (\hat{a}_i, \hat{a}_i^\dagger) \) and \( (\hat{x}_i, \hat{p}_i) \) should keep the same formulation as the ones in commutative space. For a system with mass \( \mu \) and frequency \( \omega = \omega_p/2 \) (Here the reason of introducing \( \omega_p/2 \) is that in the Hamiltonian \([13]\) the potential energy takes the same form as one of harmonic oscillator) the \( \hat{a}_i \) reads

\[
\hat{a}_i = \sqrt{\frac{\mu \omega_p}{4\hbar}} \left( \hat{x}_i + i\frac{2}{\mu \omega_p} \hat{p}_i \right).
\]

From Eq. (3) and the deformed Heisenberg-Weyl algebra (1) we obtain the commutation relation between the operators \( \hat{a}_i \) and \( \hat{a}_j \):

\[
[\hat{a}_i, \hat{a}_j] = i\xi^2 \mu \omega_p \epsilon_{ij} \left( \theta - 4\eta/\mu^2 \omega_p^2 \right)/4\hbar.
\]

When the state vector space of identical bosons is constructed by generalizing one-particle quantum mechanics, in order to maintain Bose-Einstein statistics at the deformed level described by \( \hat{a}_i \) the basic assumption is that operators \( \hat{a}_i \) and \( \hat{a}_j \) should be commuting. This requirement leads to a consistency condition

\[
\eta = \frac{1}{4} \mu^2 \omega_p^2 \theta.
\]

which puts constraint between the parameters \( \eta \) to \( \theta \). The commutation relations of \( \hat{a}_i \) and \( \hat{a}_j^\dagger \) are

\[
[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} + \frac{1}{2\hbar} \xi^2 \mu \omega_p \theta \epsilon_{ij}, \quad [\hat{a}_i, \hat{a}_j] = 0.
\]

Here, the three equations \([\hat{a}_1, \hat{a}_2^\dagger] = [\hat{a}_2, \hat{a}_1^\dagger] = 1, \quad [\hat{a}_1, \hat{a}_2] = 0\) are the same boson algebra as
the one in commutative space. The equation

$$[\hat{a}_1, \hat{a}_2^\dagger] = i \frac{1}{2\hbar} \xi^2 \mu \omega_p \theta$$

(6)
is a new type. Different from the case in commutative space, it correlates different degrees of freedom to each other, so it is called the correlated boson commutation relation. It encodes effects of spatial noncommutativity at the deformed level described by \((\hat{a}_i, \hat{a}_j^\dagger)\), and plays essential roles in dynamics \[15\].

It is worth noting that Eq. (6) is consistent with all principles of quantum mechanics and Bose-Einstein statistics.

If momentum-momentum were commuting, \(\eta = 0\), we could not obtain \([\hat{a}_i, \hat{a}_j] = 0\). It is clear that in order to maintain Bose-Einstein statistics for identical bosons at the deformed level we should consider both position-position noncommutativity and momentum-momentum noncommutativity. In this paper momentum-momentum noncommutativity means the \textit{intrinsic} noncommutativity. It differs from the momentum-momentum noncommutativity in an external magnetic field; In that case the corresponding noncommutative parameter is determined by the external magnetic field. Here both parameters \(\eta\) and \(\theta\) should be extremely small, which is guaranteed by the consistency condition (4).

The deformed Heisenberg-Weyl algebra (1) has different realizations by undeformed variables \((x_i, p_i)\) \[14\]. We consider the following consistent ansatz of a linear representation of the deformed variables \((\hat{x}_i, \hat{p}_j)\) by the undeformed variables \((x_i, p_j)\):

$$\hat{x}_i = \xi (x_i - \frac{1}{2\hbar} \theta \epsilon_{ij} p_j), \quad \hat{p}_i = \xi (p_i + \frac{1}{2\hbar} \eta \epsilon_{ij} x_j).$$

(7)

where \(x_i\) and \(p_i\) satisfy the undeformed Heisenberg-Weyl algebra \([x_i, x_j] = [p_i, p_j] = 0, [x_i, p_j] = i \hbar \delta_{ij}\). It is worth noting that the scaling factor \(\xi\) is necessary for guaranteeing that the Heisenberg commutation relation \([\hat{x}_i, \hat{p}_j] = i \hbar \delta_{ij}\) is maintained by Eq. (7).

The last paper in Ref. \[15\] clarified that though the deformed \(\hat{x}_i\) and \(\hat{p}_j\) are related to the undeformed \(x_i\) and \(p_j\) by the linear transformation (7), the deformed Heisenberg-Weyl algebra is related to the undeformed one by a similarity transformation with a non-orthogonal real matrix and a unitary similarity transformation which transforms two algebras to each other does not exist, thus two algebras are not unitarily equivalent.
3. Chern-Simons’ Interactions

Physical systems confined to a space-time of less than four dimensions show a variety of interesting properties. There are well-known examples, such as the quantum Hall effect, high $T_c$ superconductivity, cosmic string in planar gravity, etc. In many of these cases the Chern-Simons interaction $^{[26][28]}$, which exists in 2+1 dimensions and is associated with the topologically massive gauge fields, plays a crucial role. In laboratories a Chern-Simons’ process can be realized by an interaction of a charged particle in special crossed electric and magnetic fields, an example is a Penning trap $^{[30][32]}$, in which an analog of the Chern-Simons term reads

$$\epsilon_{ij}\hat{x}_i\hat{p}_j.$$ 

This term leads to non-trivial dynamics in the limit of vanishing kinetic energy, and in turn a testable effect of spatial noncommutativity.

The objects trapped in a Penning trap are charged particles or ions. The trapping mechanism combines an electrostatic quadrupole potential

$$\hat{\phi} = \frac{V_0}{2d^2}(-\frac{1}{2}\hat{x}_i^2 + \hat{x}_3^2), (i = 1, 2)$$

and a uniform magnetic field $\mathbf{B}$ aligned along the $z$ axis. The vector potential $\hat{A}_i$ corresponding to the uniform magnetic field $B$ reads

$$\hat{A}_i = \frac{1}{2}\epsilon_{ij}B\hat{x}_j.$$ 

The parameters $V_0 (> 0)$ and $d$ are the characteristic voltage and length. The particle oscillates harmonically with an axial frequency $\omega_z = (qV_0/\mu d^2)^{1/2}$ (charge $q > 0$) along the axial direction (the $z$-axis), and in the $(1, 2)$ - plane, executes a superposition of a fast circular cyclotron motion of a cyclotron frequency $\omega_c = qB/\mu c$ with a small radius, and a slow circular magnetron drift motion of a magnetron frequency $\omega_m \equiv \omega_z^2/2\omega_c$ in a large orbit. Typically the quadrupole potential superimposed upon the magnetic field is a relatively weak addition in the sense that the hierarchy of frequencies is

$$\omega_m << \omega_z << \omega_c.$$ 

(10)
The Hamiltonian $\hat{H}$ of this system can be decomposed into a two-dimensional Hamiltonian $\hat{H}_2$ and a one-dimensional harmonic Hamiltonian $\hat{H}_z$:

$$\hat{H} = \frac{1}{2\mu} (\hat{p}_i - \frac{q}{c} \hat{A}_i)^2 + q \hat{\phi} = \hat{H}_2 + \hat{H}_z,$$

$$\hat{H}_z = \frac{1}{2\mu} \hat{p}_3^2 + \frac{1}{2} \mu \omega^2 \hat{x}_3^2,$$

and $\hat{H}_2$ is [30, 31]

$$\hat{H}_2 = \frac{1}{2\mu} \hat{p}_i^2 + \frac{1}{8} \mu \omega^2 \hat{x}_i^2 - \frac{1}{2} \omega_c \epsilon_{ij} \hat{x}_i \hat{p}_j,$$

where $\mu$ is the particle mass, $\omega_p \equiv \omega_c (1 - 4 \omega_m / \omega_c)^{1/2}$. If NCQM is a realistic physics, low energy quantum phenomena should be reformulated in this framework. In the above the noncommutative Hamiltonian (13) is obtained by reformulating the corresponding commutative one $H_2 = p_i^2 / 2\mu + \mu \omega^2 \hat{x}_i^2 / 8 - \omega_c \epsilon_{ij} x_i p_j / 2$ in commutative space in terms of the deformed canonical variables $\hat{x}_i$ and $\hat{p}_i$.

In Eq. (13), the term $\omega_c \epsilon_{ij} \hat{x}_i \hat{p}_j / 2$ plays an interesting role of realizing analogs of the Chern-Simons theory [26–28].

In order to explore the new features of such a system our attention is focused on the investigation of $\hat{H}_2$ and the $z$ component of the orbital angular momentum. There are different ways to define the deformed angular momentum in noncommutative space.

(i) As a generator of rotations at the deformed level the deformed angular momentum $\hat{J}_z'$ should transform $\hat{x}_i$ and $\hat{p}_j$ as two dimensional vectors [14]:

$$[\hat{J}_z', \hat{x}_i] = i \epsilon_{ij} \hat{x}_j, \quad [\hat{J}_z', \hat{p}_i] = i \epsilon_{ij} \hat{p}_j.$$

Comparing to the case in commutative space, the deformed angular momentum $\hat{J}_z'$ acquires $\theta-$ and $\eta-$ dependent scalar terms $\hat{x}_i \hat{x}_j$ and $\hat{p}_i \hat{p}_j$,

$$\hat{J}_z' = \frac{\hbar^2}{\hbar^2 - \xi^4 \theta \eta} \left( \epsilon_{ij} \hat{x}_i \hat{p}_j + \frac{\xi^2 \eta}{2\hbar} \hat{x}_i \hat{x}_i + \frac{\xi^2 \theta}{2\hbar} \hat{p}_i \hat{p}_i \right).$$

(ii) The quantum mechanical system described by the deformed Hamiltonian Eq. (13), or equivalently Eq. (18), possesses a full rotational symmetry in (1, 2) - plane. The generator of those rotations is given as

$$J_z = \epsilon_{ij} x_i p_j,$$
i.e., all quantities $\hat{x}_i, \hat{p}_i, x_i, p_i$ transforms as two dimensional vectors.

(iii) The third point of view is as follows: If NCQM is a realistic physics, all deformed observables (the deformed Hamiltonian, the deformed angular momentum, etc.) in non-commutative space can be obtained by reformulating the corresponding undeformed ones in commutative space in terms of deformed canonical variables. Thus the deformed angular momentum $\hat{J}_z$, like the deformed Hamiltonian (13), keeps the same representation as the undeformed one $J_z$, but is reformulated in terms of $\hat{x}_i$ and $\hat{p}_i$, i.e., the Chern-Simons term

$$\hat{J}_z = \epsilon_{ij} \hat{x}_i \hat{p}_j.$$  \hfill (16)

Our starting point is at the deformed level. Because of the scalar terms in $\hat{J}'_z$ have nothing to do with the angular momentum, so in this paper we prefer to take Eq. (16) as the definition of $\hat{J}_z$. Eq. (6) modifies the commutation relations between $\hat{J}_z$ and $\hat{x}_i, \hat{p}_i$. From the NCQM algebra (1) we obtain

$$[\hat{J}_z, \hat{x}_i] = i\epsilon_{ij} \hat{x}_j + i\xi^2 \theta \hat{p}_i, \quad [\hat{J}_z, \hat{p}_i] = i\epsilon_{ij} \hat{p}_j - i\xi^2 \eta \hat{x}_i.$$

Comparing with the commutative case, the above commutation relations acquires $\theta-$ and $\eta-$ dependent terms which represent effects in noncommutative space. From the above commutation relations we conclude that $\hat{J}_z$ plays approximately the role of the generator of rotations at the deformed level.

All quantities $\hat{J}_z, \hat{J}'_z, J_z$ and $\hat{H}_2$ commute each other, thus have common eigenstates. For example, from Eqs. (1), (4), (13) and (16) it follows that

$$[\hat{J}_z, \hat{H}_2] = 0.$$  \hfill (17)

Here cancellations between $\theta-$ and $\eta-$ dependent terms, provided by the consistency condition (4), is crucial for obtaining Eq. (17).

The $\theta-$ and $\eta-$ dependent terms of the Chern-Simons term $\hat{J}_z$ have no direct relation to the angular momentum, see Eq. (20). The essential point is whether the interaction with magnetic field (the Stern-Gerlach part of the apparatus) is mediated by $J_z$ or $\hat{J}_z$. The definition of a generator of rotations elucidates that the interaction with magnetic field is mediated by $J_z$. It is worth noting that in both limits of vanishing kinetic energy and
subsequent vanishing magnetic field $\hat{J}_z$ is proportional to $J_z$, see Eqs. (39)-(41). This means that in the particular limit the eigenvalue of the Chern-Simons term $\hat{J}_z$ should appear in the spectrum of angular momentum.

$\hat{H}_2$ and $\hat{J}_z$ constitute a complete set of observables of the two-dimensional sub-system. Using Eqs. (7) the Hamiltonian $\hat{H}_2$ is represented by undeformed variables $x_i$ and $p_i$ as

$$\hat{H}_2 = \frac{1}{2M}(p_i + \frac{1}{2}G\epsilon_{ij}x_j)^2 - \frac{1}{2}Kx_i^2 = \frac{1}{2M}p_i^2 + \frac{1}{2M}G\epsilon_{ij}p_ix_j + \frac{1}{8}M\Omega_p^2 x_i^2,$$

(18)

where the effective parameters $M, G, \Omega_p$ and $K$ are defined as

$$\frac{1}{M} \equiv \xi^2 \left( b_1^2/\mu - qV_0\theta^2/8d^2\hbar^2 \right), \quad \frac{G}{M} \equiv \xi^2 \left( 2b_1b_2/\mu - qV_0\theta/2d^2\hbar \right),$$

$$M\Omega_p^2 \equiv \xi^2 \left( 4b_2^2/\mu - 2qV_0/d^2 \right), \quad K \equiv (G^2/M - M\Omega_p^2)/4,$$

(19)

and $b_1 = 1 + qB\theta/4c\hbar$, $b_2 = qB/2c + \eta/2\hbar$. The parameter $K$ consists of the difference of two terms. It is worth noting that the dominant value of $K$ is $qV_0/2d^2 = \mu\omega_z^2/2$, which is positive.

Similarly, from Eq. (16) and Eqs. (7) the Chern-Simons term $\hat{J}_z$ is rewritten as

$$\dot{J}_z = \epsilon_{ij}x_ip_j - \frac{1}{2\hbar}\xi^2 (\theta p_ip_i + \eta x_ix_i) = J_z - \frac{1}{2\hbar}\xi^2 (\theta p_ip_i + \eta x_ix_i).$$

(20)

The deformed Heisenberg-Weyl algebra and the undeformed one are, respectively, the foundations of noncommutative and commutative quantum theories. Because of the unitary un-equivalency between two algebras it is expected that the spectrum of deformed observables (the Hamiltonian $\hat{H}_2$, the angular momentum $\dot{J}_z$, etc.) may be different from the spectrum of the corresponding undeformed ones ($H_2, J_z$, etc.).

4. Dynamics in the limiting case of vanishing kinetic energy

In the following we are interested in the system (18) for the limiting case of vanishing kinetic energy. In this limit the Hamiltonian (18) has non-trivial dynamics, and there are constraints which should be carefully considered [15, 33, 34]. For this purpose it is more convenient to work in the Lagrangian formalism. The limit of vanishing kinetic energy in the Hamiltonian formalism identifies with the limit of the mass $M \to 0$ in the Lagrangian
formalism. In Eq. (18) in the limit of vanishing kinetic energy, 
\[ \frac{1}{2M} \left( p_i + \frac{1}{2} G \epsilon_{ij} x_j \right)^2 = \frac{1}{2} M \dot{x}_i \dot{x}_i \rightarrow 0, \]
the Hamiltonian \( \hat{H}_2 \) reduces to

\[ H_0 = -\frac{1}{2} K x_i x_i. \]  

(21)

The Lagrangian corresponding to the Hamiltonian (18) is

\[ L = \frac{1}{2} M \dot{x}_i \dot{x}_i - \frac{1}{2} G \epsilon_{ij} \dot{x}_i x_j + \frac{1}{2} K x_i x_i. \]  

(22)

In the limit of \( M \rightarrow 0 \) this Lagrangian reduces to

\[ L_0 = -\frac{1}{2} G \epsilon_{ij} \dot{x}_i x_j + \frac{1}{2} K x_i x_i. \]  

(23)

From \( L_0 \) the corresponding canonical momentum is \( p_{0i} = \partial L_0 / \partial \dot{x}_i = -\frac{1}{2} G \epsilon_{ji} x_j \), and the corresponding Hamiltonian is \( H'_0 = p_{0i} \dot{x}_i - L_0 = -\frac{1}{2} K x_i x_i = H_0 \). Thus we identify the two limiting processes. Here the point is that when the potential is velocity dependent the limit of vanishing kinetic energy in the Hamiltonian does not correspond to the limit of vanishing velocity in the Lagrangian. If the velocity approached zero in the Lagrangian, there would be no way to define the corresponding canonical momentum, thus there would be no dynamics.

The massless limit have been studied by Dunne, Jackiw and Trugenberger [28].

The first equation of (18) shows that in the limit \( M \rightarrow 0 \) there are constraints

\[ C_i = p_i + \frac{1}{2} G \epsilon_{ij} x_j = 0, \]  

(24)

which should be carefully treated. In this example the Faddeev- Jackiw’s symplectic method [35] leads to the same results as the Dirac method for constrained quantization, and the representation of the symplectic method is much streamlined. In the following we adopt the Dirac method [36]. The Poisson brackets of the constraints are \( \{ C_i, C_j \} = G \epsilon_{ij} \neq 0 \), so that the corresponding Dirac brackets of the canonical variables \( x_i, p_j \) can be determined,

\[ \{ x_i, p_j \}_D = \frac{1}{2} \delta_{ij}, \quad \{ x_1, x_2 \}_D = -\frac{1}{G}, \quad \{ p_1, p_2 \}_D = -\frac{1}{4} G. \]  

(25)

The Dirac brackets of \( C_i \) with any variables \( x_i \) and \( p_j \) are zero so that the constraints are strong conditions, and can be used to eliminate the dependent variables. If we select \( x_1 \)
and $p_1$ as the independent variables, from the constraints we obtain $x_2 = -2p_1/G$, $p_2 = Gx_1/2$. We introduce new canonical variables $q = \sqrt{2}x_1$ and $p = \sqrt{2}p_1$ which satisfy the Heisenberg quantization condition $[q, p] = i\hbar$, and define the effective mass $\mu^*$ and the effective frequency $\omega^*$ as

$$\mu^* \equiv \frac{G^2}{2K}, \quad \omega^* \equiv \frac{K}{G}, \quad (26)$$

then the Hamiltonian $\hat{H}_2$ reduces to

$$H_0 = -\left(\frac{1}{2\mu^*}p^2 + \frac{1}{2}\mu^*\omega^* q^2\right). \quad (27)$$

We define an annihilation operator

$$A = \sqrt{\frac{\mu^*\omega^*}{2\hbar}} q + i\sqrt{\frac{1}{2\hbar\mu^*\omega^*}} p, \quad (28)$$

The annihilation and creation operators $A$ and $A^\dagger$ satisfies $[A, A^\dagger] = 1$, and the eigenvalues of the number operator $N = A^\dagger A$ is $n' = 0, 1, 2, \cdots$.

The Hamiltonian $H_0$ is rewritten as

$$H_0 = -\hbar\omega^* \left(A^\dagger A + \frac{1}{2}\right). \quad (29)$$

Similarly, the angular momentum $J_z$ and the Chern-Simons term $\hat{J}_z$ reduce, respectively, to the following $J'_z$ and $\hat{J}'_z$

$$J'_z = \hbar \left(A^\dagger A + \frac{1}{2}\right), \quad \hat{J}'_z = \hbar \mathcal{J}^* \left(A^\dagger A + \frac{1}{2}\right) = \mathcal{J}^* J'_z, \quad (30)$$

where

$$\mathcal{J}^* = 1 - \xi^2 \left(\frac{G\theta}{4\hbar} + \frac{\eta}{G\hbar}\right). \quad (31)$$

The eigenvalues of $H_0$ and $\hat{J}_z$ are, respectively,

$$E^*_n = -\hbar\omega^* \left(n' + \frac{1}{2}\right), \quad (32)$$

$$\mathcal{J}^*_n = \hbar \mathcal{J}^* \left(n' + \frac{1}{2}\right), \quad (33)$$

The eigenvalue of $H_0$ is negative, thus unbound. This motion is unstable. It is worth noting that the dominant value of $\omega^*$ is the magnetron frequency $\omega_m$, i.e. in the limit
of vanishing kinetic energy the surviving motion is magnetron-like, which is more than adequately metastable \[30, 31\].

The \(\theta\)– and \(\eta\)– dependent terms of \(\mathcal{J}^*\) take fractional values. Thus the Chern - Simons term \(\hat{J}_z\) possesses fractional eigenvalues and fractional intervals.

Using the consistency condition (4) we rewrite the \(\mathcal{J}^*\) in Eq. (31) as \(\mathcal{J}^* = 1 + O(\theta)\).

From Eq. (33) it follows that the zero-point value \(\mathcal{J}_0^*\) reads

\[
\mathcal{J}_0^* = \frac{1}{2} \hbar + O(\theta). \tag{34}
\]

For the case of both position-position and momentum-momentum noncommuting we can consider a further limiting process. After the sign of \(V_0\) is changed, the definition of \(\Omega_p\) shows that the limit of magnetic field \(B \to 0\) is meaningful, and the survived system also has non-trivial dynamics. In this limit the frequency \(\omega_p\) reduces to \(\bar{\omega}_p = \sqrt{2} \omega_z\), the consistency condition (4) becomes a reduced consistency condition

\[
\eta = \frac{1}{2} \mu^2 \omega_z^2 \theta, \tag{35}
\]

and the scaling parameter \(\xi\) in Eq. (2) reduces to

\[
\bar{\xi} = \left(1 + \frac{1}{8 \hbar^2 \mu^2 \omega_z^2 \theta^2}\right)^{-1/2} = 1 + O(\theta^2). \tag{36}
\]

The effective parameters \(M, G, \Omega_p\) and \(K\) reduce, respectively, to the following effective parameters \(\tilde{M}, \tilde{G}, \tilde{\Omega}_p\) and \(\tilde{K}\), which are defined by

\[
\begin{align*}
\tilde{M} &\equiv \left[\xi^2 \left(\frac{1}{\mu} + \frac{1}{8 \hbar^2 \mu^2 \omega_z^2 \theta^2}\right)\right]^{-1} = \mu, \\
\frac{\tilde{G}}{\tilde{M}} &\equiv \bar{\xi}^2 \left(\frac{2 \hbar}{\mu \hbar} + \frac{1}{2 \hbar^2 \mu^2 \omega_z^2 \theta^2}\right) = \frac{1}{\hbar} \mu \omega_z^2 \theta + O(\theta^3), \\
\tilde{\Omega}_p^2 &\equiv \bar{\xi}^2 \left(\frac{\eta^2}{\mu^2 \hbar^2} + 2 \omega_z^2\right) = 2 \omega_z^2 + O(\theta^2), \\
\tilde{K} &\equiv \frac{1}{4} \left(\frac{\tilde{G}^2}{\tilde{M}} - \tilde{M} \tilde{\Omega}_p^2\right) = -\frac{1}{2} \mu \omega_z^2 + O(\theta^2). \tag{37}
\end{align*}
\]

In this limit \(H_0\) and \(\hat{J}_z^*\) reduce, respectively, to the following \(\tilde{H}_0\) and \(\tilde{J}_z\):

\[
\tilde{H}_0 = -\frac{1}{2} \tilde{K} x_i^2 = \hbar \bar{\omega} \left(\tilde{A}^\dagger \tilde{A} + \frac{1}{2}\right), \tag{38}
\]

12
\[ \tilde{J}_z = \hbar \tilde{\mathcal{J}} \left( \tilde{A}^\dagger \tilde{A} + \frac{1}{2} \right) = \tilde{\mathcal{J}} J'_z. \]  

(39)

In this limit the angular momentum \( J'_z \) is not changed, but can be rewritten as

\[ J'_z = \hbar \left( \tilde{A}^\dagger \tilde{A} + \frac{1}{2} \right). \]  

(40)

In Eq. (39)

\[ \tilde{\mathcal{J}} = 1 - \tilde{\xi}^2 \left( \frac{1}{4\hbar} G \theta + \frac{\eta}{G \hbar} \right). \]  

(41)

In the above the annihilation operator is defined as

\[ \tilde{A} = \sqrt{\tilde{\mu} \tilde{\omega}^2} q + i \sqrt{\frac{1}{2\hbar \tilde{\mu} \tilde{\omega}}} p, \]  

(42)

and the effective mass \( \tilde{\mu} \) and the effective frequency \( \tilde{\omega} \) are

\[ \tilde{\mu} \equiv - \frac{\tilde{G}^2}{2K} (> 0), \quad \tilde{\omega} \equiv \frac{\tilde{K}}{G}. \]  

(43)

The annihilation and creation operators \( \tilde{A} \) and \( \tilde{A}^\dagger \) satisfy \([\tilde{A}, \tilde{A}^\dagger] = 1\), and the eigenvalues of the number operator \( \tilde{N} = \tilde{A}^\dagger \tilde{A} \) is \( n = 0, 1, 2, \ldots \).

Eqs. (38)-(41) show that \( \tilde{H}_0, \tilde{J}_z \) and \( J'_z \) commute each other, thus have common eigenstates. The eigenvalues of \( \tilde{H}_0, \tilde{J}_z \) and \( J'_z \) are, respectively,

\[ \tilde{E}_n = \hbar \tilde{\omega} \left( n + \frac{1}{2} \right), \]  

(44)

\[ \tilde{\mathcal{J}}_n = \hbar \tilde{\mathcal{J}} \left( n + \frac{1}{2} \right), \quad J'_n = \hbar \left( n + \frac{1}{2} \right). \]  

(45)

It is worth noting that in both limits of vanishing kinetic energy and subsequent vanishing magnetic field we have \( \tilde{J}_n = \tilde{\mathcal{J}} J'_n \).

Now we estimate the dominant value of the constant \( \tilde{\mathcal{J}} \). A dominant value of an observable means its \( \theta \)– and \( \eta \)– independent term. Generally, the dominant value is just the value in commutative space. In some special case the consistency condition (4) or the reduced consistency condition (35) may provides a cancellation between \( \theta \) and \( \eta \) in some term of an observable. This leads to that the dominant value is different from one in commutative space.
In the third term of $\tilde{J}$ in Eq. (41), unlike the term $\eta/G\bar{h} = O(\eta)$ of $J^*$ in Eq. (31), the reduced consistency condition (35) provides a fine cancellation between $\theta$ and $\eta$. Using Eqs. (35) - (37), this term reads $\eta/G\bar{h} = 1/2$, which leads to

$$\tilde{J} = \frac{1}{2} + O(\theta^2).$$

(46)

From Eqs. (45) and (46) it follows that the zero-point value $\tilde{J}_0$ is

$$\tilde{J}_0 = \frac{1}{4}\bar{h} + O(\theta^2),$$

(47)

and the interval $\Delta \tilde{J}_n$ of the Chern-Simons term reads

$$\Delta \tilde{J}_n = \frac{1}{2}\bar{h} + O(\theta^2).$$

(48)

The dominant values of the zero-point value and the interval of the Chern-Simons term are, respectively, $\bar{h}/4$ and $\bar{h}/2$, which are different from the values in commutative space. These unusual results explore the essential new feature of spatial noncommutativity.

5. Testing Spatial Noncommutativity via a Penning Trap

The dominant value $\bar{h}/4$ of the lowest Chern-Simons term in a Penning trap can be measured by a Stern-Gerlach-type experiment. The experiment consists of two parts: the trapping region and the Stern-Gerlach experimental region. The trapping region serves as a source of the particles for the Stern-Gerlach experimental region. After establishing the trap, the experiment includes three steps.

(i) Taking the limit of vanishing kinetic energy. In an appropriate laser trapping field the speed of atoms can be slowed to the extent that the kinetic energy term may be removed \[37\]. In the limit of vanishing kinetic energy the situations of the cyclotron motion, the harmonic axial oscillation and the magnetron-like motion in a Penning trap are different \[30, 31\]. The energy in the cyclotron motion is almost exclusively kinetic energy. The energy in the harmonic axial oscillation alternates between kinetic and potential energy. Reducing the kinetic energy in either of these motions reduces their amplitude. In contrast to these two motions, the energy in the magnetron-like motion is almost exclusively potential energy. Thus in the limit of vanishing kinetic energy the harmonic axial oscillation and the
cyclotron motion disappear, only the magnetron-like motion survives. Any process that removes energy from the magnetron-like motion increases the magnetron radius until the particle strikes the ring electrode and is lost from the trap. The magnetron-like motion is unstable. Fortunately, its damping time is on the order of years \[30\], so that it is more than adequately metastable. In this limit, the survived magnetron-like motion slowly drifts in a large orbit in the (1, 2) - plane. At the quantum level, in the limit of vanishing kinetic energy the mode with the frequency \(\omega^*\) survives. As we noted before, the dominant value of \(\omega^*\) is the magnetron frequency \(\omega_m\), i.e. the surviving mode is magnetron-like.

(ii) Changing the sign of the voltage \(V_0\) and subsequently diminishing the magnetic field \(B\) to zero. The voltage \(V_0\) is weak enough so that when the magnetic field \(B\) approaches zero the trapped particles can escape along the tangent direction of the circle from the trapping region and are injected into the Stern-Gerlach experimental region.

(iii) Measuring the \(z\)-component of the lowest Chern-Simons term in the Stern-Gerlach experimental region. As noticing before, the commutation relations between \(\hat{J}_z\) and \(\hat{x}_i, \hat{p}_i\) show that \(\hat{J}_z\) plays approximately the role of the generator of rotations at the deformed level. Eqs. (38) - (41) and (45) elucidate that the lowest dominant value \(\hbar/4\) of the Chern-Simons term \(\tilde{J}_z\) can be read out from spectrum of the angular momentum which are measured from the deflection of the beam in the Stern-Gerlach experimental region.

6. Discussions

As is well known, a direct measurement of the magnetism and the gyromagnetic ratio for free electrons are impossible. Thus in the above suggested experiment, the trapped object are chosen as ions.

When ions are injected into the Stern-Gerlach experimental region, in order to avoid a disturbance of the Lorentz motion in the inhomogeneous magnetic field, they should first go through a region of revival and are restored to neutral atoms. We should choose ions of the first class atoms in periodic table of the elements. An advantage of choosing such ions is that in the ordinary case of commutative space the revived atoms are in the \(S\)-state.

Now we estimate the possibility of changing the state of revived atoms by effects of spatial noncommutativity. The effective frequency \(\bar{\omega}\) in Eq. (43) depends on noncommuta-
tive parameters. There are different bounds on the parameter $\theta$ set by experiments. The existing experiments on the Lorentz symmetry violation placed strong bounds on $\theta$ \[38\]:

$$\frac{\theta}{(hc)^2} \leq (10 \text{ TeV})^{-2};$$

Measurements of the Lamb shift \[9\] give a weaker bound; Clock-comparison experiments \[39\] claim a stronger bound. The magnitudes of $\theta$ and $\eta$ are surely extremely small. From Eq. (43) it follows that the dominate value of the frequency $\tilde{\omega}$ reads

$$\tilde{\omega} = |\tilde{K}|/\tilde{G} \approx \hbar/(2\mu\theta).$$

If we take $\mu c^2 = 2 \text{ GeV}$ and $\theta/(hc)^2 \leq (10^4 \text{ GeV})^{-2}$ we obtain

$$\tilde{\omega} \geq 10^{32} \text{ Hz}.$$  Eq. (44) shows that the corresponding energy interval $\Delta \tilde{E}_n$ is extremely large. Thus revived atoms can not transit to higher exciting states, they are definitely preserved in the ground state.

The result obtained in this paper is different from results obtained in literature. All effects of spatial noncommutativity explored in literature depend on extremely small noncommutative parameters $\theta$ and/or $\eta$, thus can not be tested in the foreseeable future. Because of a direct proportionality between $\theta$ and $\eta$ provided by the reduced consistency condition \[35\], in Eq. (41) there is a fine cancellation between $\theta$ and $\eta$. This leads to a $\theta$– and $\eta$– independent effect of spatial noncommutativity which can be tested by current experiments.

In both limits of vanishing kinetic energy and subsequent diminishing magnetic field for the case of only position-position noncommuting dynamics of the Penning trap is trivial; But for the case of both position-position and momentum-momentum noncommuting its dynamics is non-trivial, and the dominant value $\hbar/4$ of the lowest Chern-Simons term in the Penning trap is different from the value in commutative space. The above suggested experiment can distinguish the case of both position-position and momentum-momentum noncommuting from the case of only position-position noncommuting.

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