ON THE GROWTH RATES FOR A THREE-LAYER FLOW IN POROUS MEDIA

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We study the displacement of three immiscible Stokes fluids with constant viscosities in a porous medium. The middle-layer fluid is contained in a bounded region. We give an analysis of the linear stability of this process. This stability problem has no solution (in general).

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1. INTRODUCTION

The flow porous media are often studied by using the Hele-Shaw model. This is obtained by an average procedure of a Stokes flow in the narrow gap between two parallel plates. The fluid velocity is related to pressure gradients via a Darcy-type law - see [1], [11].

The Saffman-Taylor instability [14] appears when a less viscous Stokes fluid is displacing a more viscous one in a two-layer Hele-Shaw cell. It is important to find flow-models which could minimize this instability. A possible strategy is to put some intermediate liquids between the initial fluids. In [12], [13] we proved that this strategy is not giving us an almost stable flow, even if the number of intermediate layers is very large.

In this paper we study the linear stability of the flow with a single intermediate liquid layer with a constant viscosity, first studied in [2]. The growth constants of perturbations are the eigenvalues of the stability system and are contained in the boundary conditions, then we have also a compatibility condition. The eigenfunctions are the amplitudes of the linear perturbations, and must be bounded (only small perturbations are allowed). From the compatibility condition it results some (unexpected) restrictions on the three viscosities, which can lead to physical contradictions. Thus the linear stability problem has no solution (in general).

2. THE 3-LAYER FLOW WITH CONSTANT VISCOSITIES

We consider a horizontal Hele-Shaw cell in the fix plane $x_1 Oy$, filled by three immiscible Stokes fluids with constant viscosities. The averaged velocity $(u, v)$ is related with the pressure $p$ by the Darcy-type law [2].
The displacing fluid is pushing the middle-layer fluid with the far-upstream velocity \((U, 0)\). The middle-layer fluid contained in a bounded region is pushing the displaced liquid (say, oil). This model is described in paper \([2]\). We use the moving reference \(x = x_1 - Ut\). The flow is governed by the equations (1a), (1b) of \([2]\):

\[
\nabla \cdot \mathbf{u} = 0, \quad \mathbf{u} = (u, v),
\]

\[
\nabla p = -\nu \mathbf{u}.
\]

Here the viscosity \(\nu\) verifies the conditions

\[
\nu = \mu, \quad x \in (a, b); \quad 0 < \mu_L < \mu < \mu_R; \quad b \leq 0;
\]

\[
\nu = \mu_L, \quad x < a; \quad \nu = \mu_R, \quad x > b.
\]

The system (1)-(3) is stationary along with two planar interfaces \(x = a, x = b\) separating these three fluid layers - see \([10]\). On \(x = a, x = b\) we assume the existence of the positive surface tensions \(T(a), T(b)\). The Laplace-Young law will be used on both interfaces.

3. THE LINEAR STABILITY ANALYSIS

The perturbations of velocity and pressure are denoted by \(u', v', p'\). We consider a small positive number \(\varepsilon\), the wavenumbers \(k\), the growth rates \(\sigma\), the amplitude \(f\) and the Fourier decomposition

\[
u' = \varepsilon f(x) \exp(iky + \sigma t),
\]

\[
f(x) = f(a)e^{k(x-a)}, \quad \forall x \leq a; \quad f(x) = f(b)e^{-k(x-b)}, \quad \forall x \geq b.
\]

The last condition is imposed in \([14]\): \(f\) must decay to zero in the far field and is continuous - see also \([10]\). The amplitude \(f\) must be bounded also in the intermediate region. Only small perturbations are allowed in the frame of the linear perturbations. Thus we assume

\[
-\infty < f(k, x) < \infty, \quad \forall x \in [a, b], \quad \forall k \geq 0.
\]

The restriction (5) is not considered in \([2]\).

We insert the perturbations in equations (1)-(2). The linearized disturbance equations and also the linearized dynamic and kinematic interfacial conditions given in \([2], [9]\) are used. Consider \(a = -L < b \leq 0\). We get the problem (2a), (2b), (3) of \([2]\) (see also the relation (20) in \([3]\):
\( f_{xx} - k^2 f = 0, \quad x \in (a, b), \quad \forall k \geq 0; \)
\( f_x(a) = [-E_a(k)/\sigma + s]f(a); \)
\( f_x(b) = [E_b(k)/\sigma + q]f(b); \)
\( E_b(k) := \frac{(\mu_R - \mu)U k^2 - T(b)k^4}{\mu}, \quad q = -\frac{\mu_R k}{\mu}; \)
\( E_a(k) := \frac{(\mu - \mu_L)U k^2 - T(a)k^4}{\mu}, \quad s = \frac{\mu_L k}{\mu}. \)

The same growth constants appear in the boundary conditions, thus:
\( \sigma(k) = \frac{kE_b(k)f(b)}{\mu f_x(b) + \mu_R k f(b)} = \frac{kE_a(k)f(a)}{\mu_L k f(a) - \mu f_x^+(a)}, \quad \forall k \geq 0. \)

Here \( f_x^-(b), f_x^+(a) \) are the limit values of \( f_x \) in the points \( b, a \). This condition is not mentioned in \([2]\).

The possible solutions of (6) (considered also in \([2]\)) are
\( f(x) = A(k)e^{kx} + B(k)e^{-kx}. \)

In fact, in the cited paper is only specified that the \( A, B \) are constants. We will often use the notation \( A(k), B(k) \). This is a new element of our paper. The condition (10) is equivalent with the relations (12), (13) below
\( E_b/E_a = [\mu F(k, b) + \mu_R]/[\mu_L - \mu F(k, a)], \)
\( F(k, x) = \frac{Ae^{kx} - Be^{-kx}}{Ae^{kx} + Be^{-kx}}. \)

From the boundary conditions (7)-(8) we obtain the relations
\[ [\mu f_x(a) - \mu_L f(a)] = -f(a)E_a/\sigma, \quad [\mu f_x(b) + \mu_R f(b)] = f(b)E_b/\sigma. \]

Therefore \( A, B \), are verifying the equations
\[ \mu(Ae^{ka} - Be^{-ka}) = (-E_a/\sigma + \mu_L)(Ae^{ka} + Be^{-ka}), \]
\[ \mu(Ae^{kb} - Be^{-kb}) = (E_b/\sigma - \mu_R)(Ae^{kb} + Be^{-kb}), \]
\[ Ae^{ka}(\mu - \mu_L + E_a/\sigma) + Be^{-ka}(-\mu - \mu_L + E_a/\sigma) = 0, \]
\[ Ae^{kb}(\mu + \mu_R - E_b/\sigma) + Be^{-kb}(-\mu + \mu_R - E_b/\sigma) = 0. \]
and we get the homogeneous system

\[ Ae^{k_\alpha} c + Be^{-k_\alpha} d = 0, \quad Ae^{k_\beta} g + Be^{-k_\beta} h = 0; \]

\[ c = (\mu_L - \mu - E_a/\sigma), \quad d = (\mu_L + \mu - E_a/\sigma), \]

\[ g = (\mu_R + \mu - E_b/\sigma), \quad h = (\mu_R - \mu - E_b/\sigma). \]  

(14)

A solution \((A, B) \neq (0, 0)\) exist if the following conditions is verified

\[ e^{2k(a-b)}ch - gd = 0, \quad \forall k \geq 0. \]  

(15)

It is difficult to verify whether the compatibility condition (12) is fulfilled. To this end, we derive some properties of the growth rates for large \(k\). The numerical values of \(\sigma(k)\) are obtained in [2], but only for \(k \leq 3.5\).

There are an infinity of solutions for the system (14). For an arbitrary \(B(k)\) we get \(A(k)\) from (14), due to the relationship (15).

**Proposition 1.** From (15) we get

\[ \lim_{k \to \infty} \frac{E_b(k)}{\sigma(k)} = \mu_R + \mu, \quad \lim_{k \to \infty} \frac{E_b(k)}{\sigma(k)} = \mu_L + \mu. \]

**Proof.** We introduce the notations \(Q, \Delta, i, j, m, n\) below and (15) gives us

\[ \sigma^2(k)(Qij - mn) + \sigma(k)[(mE_b + nE_a) - Q(iE_b + jE_a)] \]

\[ + (QE_a E_b - E_a E_b) = 0; \quad Q = e^{2k(a-b)}. \]  

(16)

\[ i = \mu_L - \mu; \quad j = \mu_R - \mu; \quad m = \mu_L + \mu; \quad n = \mu_R + \mu; \]

\[ \Delta := (mE_b + nE_a)^2 - 2Q(mE_b + nE_a)(iE_b + jE_a) \]

\[ + Q^2(iE_b + jE_a)^2 - 4(Qij - mn)(QE_a E_b - E_a E_b). \]  

(17)

For large values of \(k\) we have

\[ e^{-2k}k^3 \approx 0, \quad e^{-2k}k^3 \approx 0; \quad e^{-2k}k^6 \approx 0, \quad e^{-2k}k^6 < k^6 \Rightarrow \]

\[ (mE_b + nE_a) - Q(iE_b + jE_a) \approx (mE_b + nE_a), \]

\[ (Qij - mn) \approx -mn, \quad QE_a E_b - E_a E_b \approx -E_a E_b, \]

\[ 2Q(mE_b + nE_a)(iE_b + jE_a) \approx 0, \quad Q^2(iE_b + jE_a)^2 \approx 0, \]

\[ \Delta \approx (mE_b + nE_a)^2 - 4mnE_a E_b = (mE_b - nE_a)^2, \]
\[
\sigma(k) \approx \frac{-(mE_b + nE_a)}{-2mn} + |mE_b - nE_a|.
\]

We see that \(E_a(k), E_b(k) \neq 0\) if \(k\) is large enough. So we have:

\[
\lim_{k \to \infty} \frac{E_b(k)}{\sigma(k)} = \mu_R + \mu, \quad \lim_{k \to \infty} g(k) = 0, \quad \lim_{k \to \infty} h(k) = -2\mu;
\]

\[
\lim_{k \to \infty} \frac{E_a(k)}{\sigma(k)} = \mu_L + \mu, \quad \lim_{k \to \infty} d(k) = 0, \quad \lim_{k \to \infty} c(k) = -2\mu.
\]

### Proposition 2.
The eigenfunctions (11), with \(A\) and \(B\) given by the system (14), do not check the compatibility condition (12).

**Proof.** The relations (10), (12), (13) give us

\[
\sigma(k) = \frac{E_b(k)}{\mu F(k, b) + \mu_R} = \frac{E_a(k)}{\mu_L - \mu F(k, a)}, \quad \forall k \geq 0.
\]

From (19), (20), (21) we obtain the existence of the limits \(\lim_{k \to \infty} F(k, b), \lim_{k \to \infty} F(k, a)\). We use the notations below and get

\[
\lim_{k \to \infty} F(k, b) = F(b), \quad \lim_{k \to \infty} F(k, a) = F(a),
\]

\[
F(b) \neq 1 \Rightarrow \lim_{k \to \infty} \frac{E_b(k)}{\sigma(k)} \neq \mu + \mu_R;
\]

\[
F(a) \neq -1 \Rightarrow \lim_{k \to \infty} \frac{E_a(k)}{\sigma(k)} \neq \mu_L + \mu.
\]

The last two relations are in contradiction with both possible relations (14), (13). Thus we obtain \(F(b) = 1, \quad F(b) = -1\). The relation (27) for large \(k\) gives us

\[
\frac{T(b)}{T(a)} = \frac{\mu + \mu_R}{\mu_L + \mu}
\]

Therefore we obtain the following unexpected restrictions, which were not initially imposed:

\[
\frac{T(b)}{T(a)} = \frac{\mu_R + \mu}{\mu_L + \mu} \iff \mu = \frac{\mu_R T(a) - \mu_L T(b)}{T(b) - T(a)},
\]

\[
0 < \mu_L < \left[\mu_R T(a) - \mu_L T(b)\right]/\left[T(b) - T(a)\right] < \mu_R.
\]
The physical significance (related to fluid mechanics) of these restrictions is not clear. Moreover, if we assume $T(b) < T(a)$, we need $\mu_R T(a) < \mu_L T(b)$, so $\mu_R < \mu_L$. Thus, in general, the condition (12) is not fulfilled.

**Remark 1.** The eigenfunctions (11) do not check the compatibility condition (12). So the growth rates of the problem (5)-(10) do not exist. In fact, in general, the problem (3)-(10) doesn’t make sense. However, in [2] are given some estimations for the growth rates $\sigma$, by using relation (15). Moreover, the results obtained in [2] are used in the papers [3] - [8]. We proved in [12] that the multi-layer model with constant viscosities is not useful for minimizing the Saffman-Taylor instability, when the coefficients $A, B$ in (11) are absolute constants. The present paper can be considere an improvement of [12] for the case $A = A(k), B = B(k)$.

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