Exact Superconducting Instability in a Doped Mott Insulator

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Because the cuprate superconductors are doped Mott insulators, it would be advantageous to solve even a toy model that exhibits both Mottness and superconductivity. We consider the Hatsugai-Kohmoto \(^1\) model, an exactly solvable system that is a prototypical Mott insulator above a critical interaction strength at half filling. Upon doping or reducing the interaction strength, our exact calculations show that the system becomes a non-Fermi liquid metal with a superconducting instability. In the presence of a weak pairing interaction, the instability produces a thermal transition to a superconducting phase, which is distinct from the BCS state, as evidenced by a gap-to-transition temperature ratio exceeding the universal BCS limit. The elementary excitations of this superconductor are not Bogoliubov quasiparticles but rather superpositions of doublons and holons, composite excitations signaling that the superconducting ground state of the doped Mott insulator inherits the non-Fermi liquid character of the normal state.

Cooper’s \(^3\) demonstration that the normal state of a metal is unstable to a pairing interaction between two electrons above the Fermi surface paved the way to the eventual solution to the problem of superconductivity in elemental metals such as mercury. In modern language \(^4\)-\(^8\), the Cooper instability is understood as the only relevant perturbation along a Fermi surface given that all renormalizations due to short-ranged repulsive interactions are benign. The problem of high-temperature superconductivity in the copper-oxide ceramics persists because the normal state is a doped Mott insulator in which no organizing principle such as quasiparticles on a Fermi surface can be invoked. The question arises: Is there an analogue of Cooper’s argument for a doped Mott insulator? Such a demonstration would be non-trivial as the simplest model relevant to the cuprates, namely the Hubbard model, is intractable in \(d > 1\). Given this intractability, we seek a simplification. Namely, is there a simplified model which captures key features of Mottness but still permits a definitive answer to the Cooper problem?

We demonstrate such an instability for the Hatsugai-Kohmoto (HK) \(^1\) model of a doped Mott insulator. The minimum feature of Mottness \(^7\)-\(^9\), thereby setting it apart from a Fermi liquid, is a bifurcation of the spectral weight per momentum state into low and high-energy components. Such a bifurcation creates a surface of zeros of the single-particle Green function, connoted a Luttinger surface \(^9\), and is known to be essential to describing high-low energy mixing in doped Mott systems \(^7\)-\(^9\). As the HK model is the simplest example which captures how quasiparticles (poles of the single-particle Green function) on a Fermi surface are converted to zeros, any superconducting instability found in such a setup could ultimately illuminate the solution to the full problem. Of course, a full model would contain not just static mixing (HK model) between the high and low energy states but also dynamical spectral weight transfer \(^7\)-\(^9\), \(^10\)-\(^13\), the feature which makes the Hubbard model tractable. Nonetheless, the fact that the HK model enables controlled calculations in the presence of a Luttinger surface makes it inherently invaluable in the direction of analytical progress on the doped Mott problem. Recent progress has been made along these lines in the zero chemical potential limit \(^17\) in a phenomenological model for the Luttinger surface.

An analogy with Sachdev-Ye-Kitaev (SYK) \(^14\), \(^15\) models applied to the strange metal phase of the cuprates is appropriate as while they do not mirror the physics accurately, they do offer controlled analytics on non-Fermi liquid states. The HK model is probably more powerful in this regard as it actually models a Mott insulator with a Luttinger surface that gives rise to a non-Fermi liquid upon doping. With this in mind, we perform calculations in the non-Fermi liquid state and study the superconducting instability through an exact calculation of the pair-field susceptibility. We then include a weak pairing interaction and explore the nature of the superconducting ground state and its elementary excitations, finding fundamental differences with the BCS ground state that ultimately arise from the non-Fermi liquid nature of the doped Mott insulator.

As in the SYK model \(^14\), a key ingredient that makes the HK model tractable is the presence of all-to-all interactions. In the HK model,

\[
H_{\text{HK}} = -t \sum_{(j,l),\sigma} \left( c_{j\sigma}^\dagger c_{l\sigma} + \text{h.c.} \right) - \mu \sum_{j\sigma} c_{j\sigma}^\dagger c_{j\sigma} \tag{1}
\]

\[
+ U \sum_{j_1\cdots j_d} \delta_{j_1\cdots j_d} c_{j_1\uparrow}^\dagger c_{j_2\uparrow}^\dagger c_{j_3\downarrow} c_{j_4\downarrow}. \tag{2}
\]

the interaction term is not random but a constant, \(U\), and unlike SYK, a hopping term \((t)\) is present between nearest neighbors \((j,l)\) that gives the model dimensionality. An additional feature is the presence of a constraint \(j_1 + j_3 = j_2 + j_4\) that the electrons must satisfy for the interaction term \(U\) to be felt. Here \(\mu\) is the chemical potential and \(L^d\) is the number of lattice sites. While the SYK model is tractable only in the limit of a large number of flavors, the HK model is exactly solvable as can be seen from Fourier transforming to momentum space

\[
H_{\text{HK}} = \sum_k H_k = \sum_k \left( \xi_k (n_{k\uparrow} + n_{k\downarrow}) + Un_{k\uparrow} n_{k\downarrow} \right). \tag{3}
\]
Here \( n_{k\sigma} = c_{k\sigma}^\dagger c_{k\sigma} \) is the fermion number operator for the mode with momentum \( k \) and spin \( \sigma = \uparrow, \downarrow \). It is clear that the kinetic and potential energy terms commute. Consequently, momentum is a good quantum number, unlike the Hubbard model, and all eigenstates have a fixed unfluctuating occupancy in \( k \)-space. Here, the momenta are summed over a square Brillouin zone \( [−\pi, \pi]^d \), within which the quasiparticle spectrum \( \xi_k = \epsilon_k - \mu \) is set by the dispersion \( \epsilon_k = -(W/2d) \sum_{\mu=1}^d \cos k_{\mu} \) with non-interacting bandwidth \( W = 4dt \) and offset by a chemical potential \( \mu \).

What is surprising about the HK model is that although the potential and kinetic energy terms commute, a correlated metal-insulator transition still exists\(^1\). The spectrum is easiest to deduce from the structure of the single-particle Green function

\[
G_{\sigma}(i\omega_n) \equiv -\int_0^\beta d\tau \langle c_{k\sigma}(\tau)c_{k\sigma}^\dagger(0) \rangle e^{i\omega_n \tau}
\]

\[
G_{\sigma}(i\omega_n \to z) = \frac{1 - \langle n_{k\sigma} \rangle}{z - \xi_k} + \frac{\langle n_{k\sigma} \rangle}{z - (\xi_k + U)}
\]

which is plotted in Fig. 1A-C. The two-pole structure is reminiscent of the atomic limit of the Hubbard model except now \( i\omega_n \) is replaced with \( i\omega_n - \xi_k \). The corresponding density of states shares a mutual energy region only for \( U < W \). Consequently, a gap \( (\Delta E = U - W) \) appears in the single-particle spectrum for \( U > W \) resulting in a Mott insulating state at half-filling (Fig. 1A), with \( \langle n_{k\sigma} \rangle = 1/2 \) for all \( k \). Doping away from half-filling or reducing interaction strength \( U < W \), the gap shifts away from the chemical potential or vanishes entirely, leading to a compressible metallic state (Fig. 1B and C).

In this metallic state, the momentum occupancy changes discontinuously from \( \langle n_k \rangle = \langle n_{k\uparrow} + n_{k\downarrow} \rangle = 1 \) for singly occupied momenta in \( \Omega_1 \) to 2 for the doubly occupied part \( \Omega_2 \) and 0 for the empty region \( \Omega_0 \).

Spin-rotation invariance of the HK model dictates that although singly-occupied momenta exist in the metal, they cannot appear as pure states. The metal is the mixed state consisting of a uniform ensemble over all spin states of the form

\[
|\Psi_G; \{\sigma_k\} \rangle = \prod_{k\in\Omega_1} \xi^c_{k\uparrow} \prod_{k\in\Omega_2} \xi^c_{k\sigma} \prod_{k\in\Omega_0} |0\rangle
\]

which results in a large ground-state degeneracy. Consequently, the excitations in the metallic state have no well-defined spin. In fact, they do not even have well-defined charge. Because of the lack of mixing between the two Hubbard bands, excitations in the lower band are created by \( \xi_k = c_{k\sigma}^\dagger (1 - n_{k\sigma}) \) at energy \( \xi_k \) while those in the upper by \( \eta_k = c_{k\sigma}^\dagger n_{k\sigma} \) at energy \( \xi_k + U \). As a result, the excitations in the metal are more akin to doublon and holon composite excitations and hence the metal of the HK model lacks any interpretation in terms of Fermi-liquid quasiparticles. This can be seen directly from the Green function

\[
G_\sigma(k, \omega) = \frac{1}{\omega - (\xi_k + U/2) - (U/2)^2}
\]

when the occupancies are equal. The self-energy now diverges at \( \omega = 0 \) along the surface defined by \( \xi_k = -U/2 \). Such a divergence obviates any attempt to define the quasiparticle weight as no pole is present in the Green function at \( \omega = \xi_k + U/2 \); the only discernible pole is in the self energy.

The two-pole structure determines the sign-changes of the Green function and is crucial in determining the Luttinger count \( 2 \sum_k \theta(\text{Re} G(k, 0)) \). Except for fine-tuned cases, such as at \( \mu = U/2 \) where there is exact particle-hole symmetry, or when \( U = 0 \) where the model is non-interacting, Luttinger’s theorem is violated in the HK model. The Luttinger count is decoupled from the true occupancy and can exceed it by up to a factor of 2 (Fig. 1D). This is evident deep in the doped Mott insulating regime (Fig. 1B), where \( \Omega_1 \) is singly occupied but contributes fully to the Luttinger count when the zeros of the Green function are above the chemical potential. The violation of Luttinger’s theorem throughout the phase diagram (Fig. 1D) indicates that the metallic

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**FIG. 1. Single-particle Green functions and phase diagram of the HK model.** (A - C) Poles and zeros of the single-particle Green function (Eq. 4). A 1d tight-binding dispersion is used for simplicity. Zeros are indicated by dashed blue lines. Poles with weight 0, 0.5, and 1 are indicated by dotted olive, thin orange lines, and bold red lines respectively. Upper Hubbard band (UHB) and lower Hubbard band (LHB) are labelled in A. Regions of occupancy \( n_k = i \) are labelled as \( \Omega_i \) for \( i = 0, 1, 2 \) in C. (D) Ground state phase diagram of the HK model. The non-Fermi liquid covers the entire diagram except for the half-filled Mott insulator for \( U > W \). Color represents the ratio of Luttinger count to filling; deviation from 1 (white) indicates violation of Luttinger’s theorem.
state of the HK model is incompatible with Fermi-liquid theory. Having established that we have a completely controlled non-Fermi liquid metallic state, we can address the question: is such a state unstable to pairing? To this end, we append the HK Hamiltonian with an attractive $(g > 0)$ pairing interaction,

\[
H = H_{\text{HK}} - g H_p, \quad H_p = \frac{1}{L^d} \Delta^\dagger \Delta
\]

where \( \Delta = \sum_k b_k = \sum_k c^{-k\downarrow} c^{k\uparrow} \) is the s-wave pair creation operator at zero total momentum. Seeking an analogue of Cooper’s argument, we first focus on the pair amplitude

\[
i \hbar \frac{\partial}{\partial t} \alpha_k(t = 0) = i \hbar \frac{\partial}{\partial t} \langle GS(t) | b_k | \psi(t) \rangle
= \langle GS(t) | [b_k, H] | \psi(t) \rangle,
\]

where \(|GS\rangle\) is a metallic state in the zero-temperature ensemble described by Eq. (6), and \(|\psi\rangle\) is the state with a single pair given by

\[
|\psi\rangle = \sum_{k \in \Omega_0} \alpha_k b_k^\dagger |GS\rangle.
\]

For clarity, we take a maximally polarised state for \(|GS\rangle = \left(\prod_{k \in \Omega_0} c^{k\uparrow}_k c^{k\downarrow}_k\right) |0\rangle\). (In Appendix A we show that the zero temperature Gibbs state recovers the same result.) From

\[
[b_k, H] = (2\xi_k + U(n_{k\downarrow} + n_{-k\uparrow})) b_k - \frac{g}{L^d} (1 - n_{k\downarrow} - n_{-k\uparrow}) \sum_{k'} b_{k'},
\]

the equations of motion take on the form

\[
(i \hbar \partial_t - 2\xi_k - U \langle n_{k\downarrow} + n_{-k\uparrow} \rangle) \alpha_k
= -\frac{g}{L^d} (1 - n_{k\downarrow} - n_{-k\uparrow}) \sum_{k'} \alpha_{k'}
\]

where \(\langle \cdots \rangle\) denotes an expectation value in the state \(|GS\rangle\).

We solve this equation in the standard way by letting \(\alpha_k(t) = e^{-iE_k t/\hbar} \alpha_k(t = 0)\). Dividing by the coefficient on the left-hand side and performing the sum over momentum, we obtain

\[
1 = -\frac{g}{L^d} \sum_{k \in \Omega_0} \frac{\langle 1 - n_{k\downarrow} - n_{-k\uparrow} \rangle}{E - 2\xi_k - U \langle n_{k\downarrow} + n_{-k\uparrow} \rangle}
\]

as the familiar criterion for a superconducting instability. In the range of integration, \(\langle n_{k\sigma} \rangle = 0\), resulting in the simplified expression,

\[
1 = -g \int_{\mu}^{W/2} \frac{d\epsilon}{E - 2\epsilon + 2\mu} \rho(\epsilon),
\]

having converted the sum to an integral weighted by the density of states \(\rho(\epsilon)\) for the band \(\epsilon_k\). This is, up to the limits of integration and density of states, exactly of the BCS form. It therefore results in a bound-state energy \(E < 0\) for any \(g > 0\), which we plot in Fig. 2 for \(d = 1, 2, 3\). In the case of half-filled metal (\(\mu = U/2\)) in one dimension, for example, the binding energy

\[
E_b = -E \sim W (1 - (U/W)^2) e^{-\pi W \sqrt{1 - (U/W)^2}/g}
\]

is exponentially small in \(1/g\). The full \(d\)-dimensional dependence is shown in Fig. (2). Hence, the HK model exhibits an analogue of the instability Cooper found for a Fermi liquid.

The advantage of the HK model is that we need not settle on the pair-binding calculation to determine whether a superconducting instability exists. We can compute the pair susceptibility

\[
\chi(i\nu_n) = \frac{1}{L^d} \int_0^\beta d\tau e^{i\nu_n \tau} \langle T \Delta(\tau) \Delta^\dagger \rangle_g
\]

exactly at all temperatures, though below we will emphasize the low temperature regime \(T \ll U, W\). As we show in Appendix B, \(\chi(i\nu_n)\) is related to the ‘bare’ susceptibility, \(\chi_0(\nu_n)\) at \(g = 0\), through the Dyson equation

\[
\chi = \chi_0 + g \chi_0 \chi,
\]

where \(\langle \cdots \rangle_g\) (resp. \(\langle \cdots \rangle_0\)) is an expectation value in the Gibbs state \(e^{-\beta H}/Z\) (resp. \(e^{-\beta H_{\text{HK}}}/Z_0\)), and \(\nu_n = 2\pi n/\beta\) is a bosonic Matsubara frequency. Then the fluctuation propagator \(L \equiv -g\chi/\chi_0\) satisfies the usual equation \(16\)

\[
L = -g + g \chi_0 L = \frac{1}{\chi_0 - 1/g}
\]

such that \(L(\omega = 0)\) diverges when \(\chi_0(0)\big|_{T=T_c}= 1/g\), thereby fixing the critical superconducting temperature \(T_c\). In order to compute \(\chi_0\), we first simplify

\[
\langle T \Delta(\tau) \Delta^\dagger \rangle_0 = \sum_{k,p} \langle T_c - k\downarrow(\tau)c^{k\uparrow}_p c^{k\downarrow}_{-p}\rangle_0
= \sum_k \langle T_c - k\downarrow(\tau)c^{k\uparrow}_k c^{k\downarrow}_{-k}\rangle_0
= \sum_k G_{-k\downarrow}(\tau) G_{k\uparrow}(\tau)
\]

up to an unimportant subextensive contribution from coincident terms with \(k = -k\), having used that the ensemble consists only of Fock states (guaranteed by the preservation of \(n_{k\sigma}\) as a good quantum number in the HK model) in the second line, and in the third line that the Gibbs state \(e^{-\beta H_{\text{HK}}}/Z_0\) factorizes in \(k\)-space. We note that despite the appearance of Eq. (21), we have not utilized Wick’s theorem, which does not apply in general to the HK model. Writing out the single-particle Green function,

\[
-G_{k\sigma}(\tau) = \langle c_{k\sigma}(\tau) c_{k\sigma}^\dagger \rangle_0 = n_{k\sigma} f(-\xi_k) e^{-\tau\xi_k^2} + (\tau \to u)
\]

(22)
FIG. 2. Cooper pair binding energy $E_b$ at half-filling in one (left), two (middle), and three (right) dimensions. The linear regime $\ln(E_b/W) \sim -(g/W)^{-1}$ is achieved at large values of the inverse pair coupling $(g/W)^{-1}$.

FIG. 3. Superconducting temperature $T_c$ (solid) and mean density of states $\bar{\rho} = \frac{1}{2}(\rho(\mu) + \rho(\mu - U))$ (dotted), for selected values of the Hatsugai-Kohmoto coupling $U$ and fixed pair coupling $g/W = 0.1$, over a range of hole dopings $x = 1 - \sum_{k,\sigma} \langle n_{k\sigma} \rangle / L^d$ away from half-filling. Hole dopings here exceed the regime $|\delta \mu| < W - U$ described in the text.

for $\xi_k^l = \xi_k$ and $\xi_k^u = \xi_k + U$, $n_k^u = \langle n_{k\uparrow} \rangle_0$ and $n_k^l = 1 - n_k^u$, and $f(\omega)$ the Fermi function at temperature $T$, we have

$$\chi_0(i\nu_n) = \chi_0^u + \chi_0^l + \chi_0^{ul}$$

(23)

$$\chi_0^{ab} = \frac{1}{L^d} \sum_{k_1, k_2} n_{k_1}^a n_{-k_2}^b f(\omega_{k_1}^a) + f(\omega_{-k_2}^b) - 1$$

(24)

where the superscripts $ab$ may represent $ll$, $uu$, $lu$, or $ul$. Because of the factors $f(\omega_{k_1}^a) + f(\omega_{-k_2}^b) - 1$, the cross terms $\chi_0^{lu}$ and $\chi_0^{ul}$ (between the lower and upper Hubbard bands) contribute no low-energy spectral weight when $T \ll U$ and are dropped hereafter. Using $\xi_k = \xi_{-k}$ and $\langle n_{k\uparrow} \rangle = \langle n_{k\downarrow} \rangle$ and changing variables, we finally arrive

$$\chi_0(0) = \int d\omega N'(\omega) \tanh \frac{\delta \omega}{2\omega}$$

(25)

$$L^d N'(\omega) = \sum_{k \in \Omega_0} \delta(\omega - \xi_k^l) + \sum_{k \in \Omega_2} \delta(\omega - \xi_k^u)$$

(26)

$$+ \frac{1}{4} \sum_{k \in \Omega_1} \delta(\omega - \xi_k^l) + \delta(\omega - \xi_k^u).$$

(27)

Here, $N'(\omega)$ is an effective density of states, similar to but not equal to the HK model’s single-particle density of states $N(\omega) = \frac{1}{\pi} \sum_k \frac{1}{2} \text{Im} G(k, \omega + i0^+)$, which has a factor of $\frac{1}{2}$ rather than $\frac{1}{4}$ before the sum over the singly occupied region. By the same manipulations as in the free fermion case, it is clear that $\chi_0(0)$ grows as $\ln \frac{1}{\delta}$.
at low temperature. Hence with any nonzero pairing strength $g$, $\chi(0)$ diverges at the transition temperature $T_c \propto e^{-1/(N''(0))g}$ (see Appendix C). In Fig. 3, the transition temperatures are calculated explicitly for a variety of parameters.

From both the Cooper argument and a direct calculation of the pair susceptibility, we have established that the HK model has a superconducting instability. We now seek to characterize the ground state of the model in the presence of a nonzero pairing interaction. We work with the variational wavefunction

$$|\psi\rangle = \prod_{k>0} \left( x_k + y_k b_k^\dagger b_{-k}^\dagger + \frac{z_k}{\sqrt{2}} (b_k^\dagger + b_{-k}^\dagger) \right) |0\rangle \quad (28)$$

normalized by $|x_k|^2 + |y_k|^2 + |z_k|^2 = 1$. This is a generalization of the BCS wavefunction, which corresponds to $x_k = u_k^2$, $y_k = v_k^2$, $z_k = \sqrt{2} u_k v_k$. The utility of this generalized wavefunction is that the state with $x_k = 1$ for $k \in \Omega_0$, $y_k = 1$ for $k \in \Omega_2$, and $z_k = 1$ for $k \in \Omega_1$, is a ground state of the HK model. Minimizing the energy variationally leads to two equations

$$\xi_k^l x_k z_k = (x_k^2 - z_k^2) \frac{g}{\sqrt{2}} \sum_{p>0} (x_p z_p + z_p y_p) \quad (29)$$

$$\xi_k^l y_k z_k = (z_k^2 - y_k^2) \frac{g}{\sqrt{2}} \sum_{p>0} (x_p z_p + z_p y_p), \quad (30)$$

after taking the limit $g \ll U, W$. Details are provided in Appendix D. After a few changes of variables, we obtain a gap equation

$$1 = \frac{g}{2} \int d\omega \frac{N''(\omega)}{\sqrt{\omega^2 + \Delta^2}} \quad (31)$$

$$N''(\omega) = \sum_{k \in \Omega_0} \delta(\omega - \xi_k^l) + \sum_{k \in \Omega_2} \delta(\omega - \xi_k^l) \quad (32)$$

$$+ \sum_{k \in \Omega_1} \delta(\omega - \xi_k^l) + \delta(\omega - \xi_k^l), \quad (33)$$

This is the BCS gap equation except for the effective density of states $N''(\omega)$. The solution has $\Delta \propto e^{-1/N''(0)g}$, which is verified in Appendix C for a specific example. Note that $N''(\omega)$ is different from $N'(\omega)$ which controls $T_c$, as there is no factor of $\frac{1}{\omega}$ before the sum $\sum_{k \in \Omega_0}$. Because this sum over the singly occupied region affects the low energy spectra, $N''(0) > N'(0)$ and the superconducting gap-to-transition temperature ratio diverges as $e^{3/(10\rho(0)g)}$ for $g \to 0$. This is in contrast to the universal BCS result $\frac{\rho}{\Delta} = 3.53\ldots$ for $s$-wave pairing in the weak coupling limit. Therefore, despite apparent mathematical similarities between pairing in the HK model and BCS pairing of free fermions, the presence of the singly occupied region $\Omega_1$ leads to qualitatively different phenomena.

The elementary excitations of a BCS superconductor are Bogoliubov quasiparticles $\gamma_{k\sigma} = u_k c_k^\dagger - \sigma v_k e_{-k}^\dagger$. The excitations of the superconducting state of the HK model with pairing cannot be the same. This is for the same reason that $c_{-k}^\dagger$ is not an elementary excitation of the HK model, namely that in the singly occupied region, both the upper and lower Hubbard bands have nonzero spectral weight. On the other hand, the Green functions for the holon and doublon excitations, $\xi_{k\sigma}$ and $\eta_{k\sigma}$, have weight only in the lower and upper Hubbard bands, respectively; these composite operators describe the elementary excitations of the HK model. Upon turning on pairing and entering the superconducting state, the new excitations are given by mixing of the composite operators and their conjugates

$$\gamma_{k\sigma}^l \propto \sqrt{2} x_k \xi_{k\sigma} - \sigma z_k \eta_{-k\sigma} \quad (34)$$

$$\gamma_{k\sigma}^u \propto z_k \eta_{k\sigma} - \sigma \sqrt{2} y_k \eta_{-k\sigma}. \quad (35)$$

It is straightforward to check that

$$\gamma_{k\sigma}^l |\psi\rangle = 0 \quad (36)$$

$$\langle \psi| H^\dagger \gamma_{k\sigma}^l |\psi\rangle = \langle \psi| H|\psi\rangle + E_{k\sigma}^u, \quad (37)$$

where $E_{k\sigma}^u = \sqrt{\xi_{k\sigma}^u/\Delta + \Delta^2}$. Therefore, $\gamma_{k\sigma}^u$ are analogous to the Bogoliubov quasiparticle excitations of a BCS superconductor but composed of doublons or holons, indicating that the non-Fermi liquid nature of the metallic state carries over into the superconducting state.

Our exact calculation shows for the first time that a non-Fermi liquid metal derived from a Mott insulator has a superconducting transition. As remarked previously, recent work on the Luttinger surface applies strictly in the zero-chemical potential limit, that is the insulator where the susceptibility appears to diverge, exhibiting SYK dynamics. Given the recent spate of papers on superconductivity in the absence of quasiparticles, our approach offers a systematic Hamiltonian-based approach to the breakdown of the quasiparticle picture without invoking randomness. In particular, we have shown that even in the weak coupling limit of pairing, the superconducting state of a doped Mott insulator is different from a BCS superconductor arising from a Fermi liquid normal state. The lessons we have learned here for the HK model have broad implications for understanding the behavior of superconductivity in doped Mott insulators. Most importantly, when both upper and lower Hubbard bands carry spectral weight, the essence of Mottness, the fundamental excitations of either the metallic or superconducting states of a doped Mott insulator cannot be described by conventional quasiparticles.

This is a conclusion that applies also to the cuprates, in which the spectral weight of the upper Hubbard band in hole-doped compounds has been observed and compared to calculations of the Hubbard model. Although we do not know precisely the excitations of the strange metal normal state of cuprates, there is overwhelming evidence that they are not Fermi liquid quasiparticles. These arguments extend this notion to the superconducting state.
Our findings for the HK model thus challenge the assumption that the appearance of coherent peaks in the spectral function of superconducting cuprates is a signature of regular Bogoliubov quasiparticles. Conversely, they suggest that detailed studies of the superconducting state and its excitations can help unravel the mysteries of the normal state.

While the HK model is complex enough to capture zeros of the Green function and their associated consequences on the metallic state and on the superconducting instability, it does not support dynamical spectral weight transfer. In fact, it is because of the absence of the latter that the model is tractable. A promising line of inquiry is how stable the present results are to such dynamical mixing. Whether a renormalization principle can be established to show that the excitations on a zero surface are impervious to such mixing remains an open question.

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Appendix A: Cooper instability with zero temperature Hatsugai-Kohmoto Gibbs state

Given the Gibbs state \( \rho = e^{-\beta H_{\text{HK}}}/Z = \sum_n e^{-\beta E_n} |n\rangle \langle n| \), we fix the purification on the doubled Hilbert space with \( \mathcal{H}_A \approx \mathcal{H}_B \)

\[
|\beta\rangle = \sum_n e^{-\beta E_n/2} |n\rangle \otimes |n\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \tag{A1}
\]

satisfying \( \text{tr}_B |\beta\rangle \langle \beta| = \rho \), and restrict

\[
A^\dagger = \sum_{k \notin \Omega_2} \alpha_k b_k^\dagger \tag{A2}
\]

to the singly-occupied and unoccupied regions \( \Omega_1 \) and \( \Omega_0 \) of the Brillouin zone. Now \( |\psi\rangle = A^\dagger \otimes 1 |\infty\rangle \) has overlap

\[
\langle \infty | b_k \otimes 1 |\psi\rangle = \text{tr}_{A,B} |\infty\rangle \langle \infty| (b_k A^\dagger \otimes 1) \tag{A3}
\]

\[
= \text{tr}_A p_b A^\dagger = (b_k A^\dagger) \tag{A4}
\]

\[
\begin{cases}
0 & \text{if } k \in \Omega_2, \\
\frac{1}{4}\alpha_k & \text{if } k \in \Omega_1, \\
\alpha_k & \text{if } k \in \Omega_0.
\end{cases}
\tag{A5}
\]

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With \([b_k, H]\) as before, for each \(k \in \Omega_1, \Omega_0\) we have
\[
C_k \hbar \partial_0 \alpha_k(t = 0) = \langle \infty | e^{i(H \otimes 1 + \frac{1}{\hbar} H')} (b_k \otimes 1) e^{-i(H \otimes 1 + \frac{1}{\hbar} H')} | \psi \rangle |_{t = 0}
\]
(A6)
\[
= \langle \infty | [b_k, H] \otimes 1 | \psi \rangle
= (2\xi_k + U\langle n_{k} + n_{-k} \rangle) C_k \alpha_k
- \frac{g}{L^d} (1 - n_{k} + n_{-k}) \left[ \sum_{q \in \Omega_1} \frac{1}{4} \alpha_q + \sum_{q \in \Omega_0} \alpha_q \right]
\]
(A8)
where \(C_k = 1/4\) if \(k \in \Omega_1\) and \(C_k = 1\) if \(k \in \Omega_0\). Then taking \(\alpha_k(t) = e^{-iEt/\hbar} \alpha_k(0)\) recovers the same consistency equation
\[
1 = -\frac{g}{L^d} \sum_{k \in \Omega_1, \Omega_0} \frac{(1 - n_{k} + n_{-k})}{E - 2\xi_k - U\langle n_{k} + n_{-k} \rangle}
= -g \int_{\mu}^{W/2} d\epsilon \frac{\rho(\epsilon)}{E - 2\epsilon + 2\mu}
\]
(A9)
since the numerator vanishes in the singly-occupied region \(\Omega_1\).

**Appendix B: Dyson equation for pair susceptibility**

In order to relate the pair susceptibility
\[
\chi(i\nu_n) \equiv \frac{1}{L^d} \int_{0}^{\beta} d\tau \ e^{i\nu_n \tau} \langle T \Delta(\tau) \Delta^\dagger \rangle_g
\]
(B1)
\[
\frac{1}{L^d} \langle T \left( \int_{0}^{\beta} d\tau_1 H_p(\tau_1) \right)^m \Delta(\tau) \Delta^\dagger \rangle_0
= \frac{1}{(L^d)^{m+1}} \int_{0}^{\beta} \cdots d\tau_1 \cdots d\tau_m \langle T \Delta^\dagger(\tau_1) \Delta(\tau_1) \cdots \Delta^\dagger(\tau_m) \Delta(\tau_m) \Delta(\tau) \Delta^\dagger \rangle_0
\]
(B5)

factorizes, as \(\chi_0\) does in Eq. (21), because each pair annihilation \(b_k(\tau)\) in the sum
\[
\Delta(\tau) = \sum_k b_k(\tau) = \sum_k e^{-\tau(2\xi_k + U\langle n_{k} + n_{-k} \rangle - 1)} b_k
\]
(B6)
evolves (in the interaction picture under \(H_{HK}\)) as a multiple of an unevolved pair annihilation \(b_k\). The denominator then removes all disconnected factorizations with either \(\langle T \Delta^\dagger(\tau_1) \Delta(\tau_1) \rangle_0\) for any time \(\tau_1\) or \(\langle T \Delta(\tau) \Delta^\dagger \rangle_0\). This leaves only those factorizations with the form of an \(m\)-wise convolution, resulting in
\[
\chi(i\nu_n) = \sum_{m=1}^{\infty} g^{m-1} (\chi_0(i\nu_n))^m = \frac{\chi_0(i\nu_n)}{1 - g\chi_0(i\nu_n)}.
\]
(B7)
to the bare pair susceptibility, \(\chi_0(i\nu_n)\) at \(g = 0\), we work in the interaction picture where
\[
\langle T \Delta(\tau) \Delta^\dagger \rangle_g = \frac{\langle TS(\beta, 0) \Delta(\tau) \Delta^\dagger \rangle_0}{\langle TS(\beta, 0) \rangle_0}
\]
(B2)

for \(\Delta(\tau)\) evolved in the Heisenberg picture (under \(H\)) in \(\langle \cdots \rangle_g\) and evolved in the interaction picture (under \(H_{HK}\)) in \(\langle \cdots \rangle_0\). Here
\[
S(\beta, 0) = e^{\beta H_{HK}} e^{-\beta H} = T_\tau \exp g \int_{0}^{\beta} d\tau_1 H_p(\tau_1),
\]
(B3)

for \(H_p(\tau_1)\) evolved in the interaction picture. Then the numerator takes the form of a power series in \(g\),
\[
\langle TS(\beta, 0) \Delta(\tau) \Delta^\dagger \rangle_0
= \sum_{m=0}^{\infty} \frac{g^m}{m!} \langle T \left( \int_{0}^{\beta} d\tau_1 H_p(\tau_1) \right)^m \Delta(\tau) \Delta^\dagger \rangle_0
\]
(B4)

where each term
\[
\langle T \Delta^\dagger(\tau_1) \Delta(\tau_1) \cdots \Delta^\dagger(\tau_m) \Delta(\tau_m) \Delta(\tau) \Delta^\dagger \rangle_0
\]

**Appendix C: Example of \(T_\tau\) and \(\Delta\) calculation**

To ease calculations, consider \(\epsilon_k\) such that \(\rho(\epsilon) = \frac{1}{L^d} \sum_k \delta(\epsilon - \epsilon_k) = \frac{1}{W/2} \text{ for } -\frac{W}{2} < \epsilon < \frac{W}{2}\). We will focus on the half-filled metal i.e. \(U < W\) and \(\mu = U/2\).

The single-particle density of states (DOS) can be broken into contributions from different regions of momentum space
\[
N(\omega) = N_0(\omega) + N_2(\omega) + \frac{1}{2} N_1(\omega)
\]
(C1)
\[
N_0(\omega) = \theta(\omega) \rho(\omega + U/2)
\]
(C2)
\[
N_2(\omega) = \theta(-\omega) \rho(\omega - U/2)
\]
(C3)
\[
N_1(\omega) = \theta(-\omega) \rho(\omega + U/2)
+ \theta(\omega) \rho(-\omega + U/2)
\]
(C4)
The effective DOS for calculating $T_c$ and $\Delta$ are

\begin{align*}
N'(\omega) &= N_0(\omega) + N_2(\omega) + \frac{1}{4} N_1(\omega) \\
N''(\omega) &= N_0(\omega) + N_2(\omega) + N_1(\omega)
\end{align*}

(C6) \hspace{1cm} (C7) \hspace{1cm} (C8)

1. $T_c$

The susceptibility diverges when

\[ \frac{1}{g} = \chi_0(0) = \int d\omega N'(\omega) \frac{\tanh \beta \omega}{2 \omega}. \]  \hspace{1cm} (C9)

Set $x = \beta \omega/2$ and integrate by parts

\[ \frac{1}{g} = -\frac{1}{2} \int dx \ln x \left[ N'(\frac{2x}{\beta}) \text{sech}^2 x \right. \\
\left. + \left( \frac{d}{dx} N'(\frac{2x}{\beta}) \right) \tanh x \right]. \]  \hspace{1cm} (C10) \hspace{1cm} (C11)

For $T \ll U,W$ this becomes

\begin{align*}
\frac{1}{g} &= \frac{1}{W} \ln \frac{\beta(W-U)}{4} + \frac{1}{4} \frac{\ln \beta U}{W} - N'(0) \left( -\ln \left( \frac{4}{\pi} \right) - \gamma \right) \\
&= \frac{W}{g} \ln \frac{\beta(W-U)}{4} + \frac{1}{4} \frac{\ln \beta U}{4} - 5 \left( -\ln \left( \frac{4}{\pi} \right) - \gamma \right)
\end{align*}

(C12) \hspace{1cm} (C13) \hspace{1cm} (C14)

where $\gamma \approx 0.577$ is Euler’s constant. The solution gives the transition temperature

\[ T_c = (W-U)^{4/5} U^{1/5} e^{\frac{6\gamma}{\pi}} \frac{W}{e^{\frac{4\gamma}{\pi}}}. \]  \hspace{1cm} (C15)

2. $\Delta$

The gap equation is given by

\[ 1 = \frac{g}{2} \int d\omega \frac{N''(\omega)}{\sqrt{\omega^2 + |\Delta|^2}} \]  \hspace{1cm} (C16)

\[ = \frac{g}{W} \sinh^{-1} \left( \frac{W-U}{2\Delta} \right) + \frac{\alpha g}{W} \sinh^{-1} \left( \frac{U}{2\Delta} \right) \]  \hspace{1cm} (C17)

For $\Delta \ll U,W$ this becomes

\[ \frac{1}{g} = \frac{1}{W} \ln \left( \frac{W-U}{\Delta} \right) + \frac{\alpha}{W} \ln \left( \frac{U}{\Delta} \right) \]  \hspace{1cm} (C18)

which can be solved to find

\[ \Delta = (W-U)^{1/2} U^{1/2} e^{-\frac{W}{4g}} \]  \hspace{1cm} (C19)

Appendix D: Variational ground state

Consider the variational wave function

\[ |\psi\rangle = \prod_{k>0} \left( x_k + y_k b_k^\dagger b_{-k}^\dagger + \frac{z_k}{\sqrt{2}} \left( b_k^\dagger + b_{-k}^\dagger \right) \right) |0\rangle. \]  \hspace{1cm} (D1)

\[ \langle \psi | \psi \rangle = 1 \] is satisfied if $|x_k|^2 + |y_k|^2 + |z_k|^2 = 1$. This generalizes the BCS wavefunction, which corresponds to $x_k = u_k^\star$, $y_k = v_k^\star$, $z_k = \sqrt{2} u_k v_k$. Furthermore, the state defined by $x_k = 1$ for $k \in \Omega_0$, $z_k = 1$ for $k \in \Omega_1$, and $y_k = 1$ for $k \in \Omega_2$ is a ground state of the HK model. Note that although one signal of pair condensation in the BCS wavefunction is the presence of nonzero $u_k v_k \propto z_k$, this state is not a pair condensate.

In the free fermion case, the ground state in the absence of pairing is the filled Fermi sea, with $u_k = 1$ for $k \in \Omega_0$ and $v_k = 1$ for $k \in \Omega_2$. For a small pairing interaction $g$, the variational ground state with pairing is very similar but with both $u_k$ and $v_k$ non-zero near the boundary of $\Omega_0$ and $\Omega_2$, namely the Fermi surface. In the HK model with weak pairing ($g \ll U,W$), we similarly expect that both $x_k$ and $z_k$ become nonzero near the boundary of $\Omega_0$ and $\Omega_1$ and both $y_k$ and $z_k$ become nonzero near the boundary of $\Omega_1$ and $\Omega_2$.

Again we try to minimize $\langle \psi | H | \psi \rangle$. For all $k > 0$, $p > 0$, and $k \neq p$,

\begin{align*}
\langle \psi | n_{k\sigma} | \psi \rangle &= |y_k|^2 + \frac{|z_k|^2}{2} \\
\langle \psi | n_{k\uparrow} n_{k\downarrow} | \psi \rangle &= |y_k|^2 \\
\langle \psi | b_k^\dagger b_k | \psi \rangle &= |y_k|^2 + \frac{|z_k|^2}{2} \\
\langle \psi | b_k^\dagger b_{-k}^\dagger b_{-k} b_k | \psi \rangle &= \frac{1}{2} (x_k^* z_k + z_k^* x_k) \\
\langle \psi | \langle \psi | b_{k\uparrow}^\dagger b_{p\uparrow} | \psi \rangle &= \frac{1}{2} (x_{kp} x_k + y_k z_k) (x_{kp}^* z_p + z_{kp}^* y_p). \\
\langle \psi | H | \psi \rangle &= \sum_{k>0} \xi_k \left( 4 |y_k|^2 + 2 |z_k|^2 \right) + U \left( 2 |y_k|^2 \right) \\
&\quad - g \sum_{k,p>0; k \neq p} 2 (z_{kp} x_k + y_k z_k) (x_{kp}^* z_p + z_{kp}^* y_p) \\
&\quad - 2 g' \sum_{k,p>0; k \neq p} (z_{kp} x_k + y_k z_k) (x_{kp}^* z_p + z_{kp}^* y_p)
\end{align*}

The same equations apply if we take $k \rightarrow -k$, $p \rightarrow -p$ on the left hand sides. Combining everything, and ignoring terms like $g' \sum_k \cdots$ that do not scale extensively in the thermodynamic limit,

\begin{align*}
\langle \psi | H | \psi \rangle &= \sum_{k>0} \xi_k \left( 4 |y_k|^2 + 2 |z_k|^2 \right) + U \left( 2 |y_k|^2 \right) \\
&\quad - g \sum_{k,p>0; k \neq p} 2 (z_{kp} x_k + y_k z_k) (x_{kp}^* z_p + z_{kp}^* y_p) \\
&\quad - 2 g' \sum_{k,p>0; k \neq p} (z_{kp} x_k + y_k z_k) (x_{kp}^* z_p + z_{kp}^* y_p)
\end{align*}

(D8) \hspace{1cm} (D9) \hspace{1cm} (D10) \hspace{1cm} (D11)
For each $k$, introduce a lagrange multiplier $\lambda_k$ to enforce normalization.

$$0 = \frac{\partial}{\partial x_k} \left[ (\psi | H | \psi) + \lambda_k \left( |x_k|^2 + |y_k|^2 + |z_k|^2 - 1 \right) \right]$$

$$= \lambda_k x_k \frac{\partial}{\partial x_k} (x_k^* z_p + z_p^* y_p)$$

$$= \frac{2 x_k^*}{x_k} O,$$  \hspace{1cm} (D12)

$$= \frac{2 x_k^*}{x_k} O,$$  \hspace{1cm} (D13)

$$\lambda_k = \frac{2 x_k^*}{x_k} O,$$  \hspace{1cm} (D14)

where $O = g' \sum_{p > 0, p \neq k} (x_p^* z_p + z_p^* y_p)$ (now including the contribution $p = k$, which is a $O(1/L^d)$ difference).

$$0 = \frac{\partial}{\partial y_k^*} \left[ \ldots \right] = (4 \xi_k + 2 U) y_k - 2 z_k O + \lambda_k y_k$$

$$= (4 \xi_k + 2 U) y_k - 2 \left( z_k - \frac{z_k^* y_k}{x_k} \right) O$$

$$2 \xi_k + U = \left( \frac{z_k}{y_k} - \frac{x_k}{x_k} \right) O$$

$$= \left( \frac{z_k}{y_k} - \frac{x_k}{x_k} \right) O$$

$$= \left( \frac{z_k}{y_k} - \frac{x_k}{x_k} \right) O$$

$$= \left( \frac{z_k}{y_k} - \frac{x_k}{x_k} \right) O$$

$$\xi_k x_k^* z_k = \left( |x_k|^2 - |z_k|^2 \right) O + \lambda_k^* y_k O^*$$

$$\xi_k = \left( \frac{x_k}{z_k} - \frac{z_k}{x_k} \right) O + \frac{y_k}{z_k} O^*.$$  \hspace{1cm} (D19)

$$\xi_k^u = \left( \frac{x_k}{z_k} - \frac{z_k}{x_k} \right) O + \frac{y_k}{z_k} O^*.$$  \hspace{1cm} (D20)

$$\xi_k + U = \left( \frac{z_k}{y_k} - \frac{x_k}{x_k} \right) O - \frac{y_k}{z_k} O^*.$$  \hspace{1cm} (D21)

In the last lines, we take the limit $L^d \to \infty$, so we ignore the $g'$ on the LHS of and also replace the sum in $O$ with a sum over all momentum. Subtracting (D21) from (D17) gives

$$\xi_k^l = \frac{x_k}{z_k} - \frac{z_k}{x_k} \right) O + \frac{y_k}{z_k} O^*.$$  \hspace{1cm} (D22)

Using $\xi_k^l = \xi_k$ and $\xi_k^u = \xi_k + U$, and assuming everything is real,

$$\xi_k^l = \left( \frac{x_k}{z_k} + \frac{y_k}{z_k} - \frac{z_k}{x_k} \right) O$$

$$\xi_k^u = \left( \frac{x_k}{z_k} + \frac{y_k}{z_k} - \frac{z_k}{x_k} \right) O.$$  \hspace{1cm} (D23)

It is straightforward to check for $U = 0$, combining these equations produces exactly the BCS result, even though we started with a more general wavefunction. This system of two equations is possible to solve analytically, but requires finding the roots of a quartic equation.

1. **Weak coupling $g \ll U$ and $g \ll W$**

First, rewrite

$$\xi_k^l x_k z_k = (\xi_k^l)^2 - (\xi_k^l)^2 + x_k y_k)O$$  \hspace{1cm} (D25)

$$\xi_k^u y_k z_k = (\xi_k^u)^2 - (\xi_k^u)^2 - x_k y_k)O,$$  \hspace{1cm} (D26)

which now looks very similar to the BCS case, apart from the $x_k y_k$ terms.

If $g \ll U$, we expect that there are still well defined regions $\Omega_0, \Omega_1, \Omega_2$, such that mixing occurs only between $x_k$ and $z_k$ or between $y_k$ and $z_k$ and never $x_k$ and $y_k$. Is it safe to drop the $x_k y_k$ terms from (D25) and (D26)? Consider a $k$ point where $\xi_k^l < \xi_k^u < 0$. Here we expect $1 \approx y_k \gg z_k \gg x_k$. If $x_k, y_k, z_k$ are all positive, (D25) can only be satisfied if $z_k^2 \gg x_k y_k$. A similar argument can be made for (D26). Therefore we drop $x_k y_k$ from both equations and work with

$$\xi_k^l x_k z_k = (\xi_k^l)^2 - (\xi_k^l)^2)O$$

$$\xi_k^u y_k z_k = (\xi_k^u)^2 - (\xi_k^u)^2)O.$$  \hspace{1cm} (D27)

(D28)

Change variables to

$$x_k^2 - z_k^2 = \frac{\xi_k^l}{E_k} (1 - y_k^2), \quad 2 x_k z_k = \frac{\Delta_k^l}{E_k} (1 - y_k^2)$$

$$E_k^l = \sqrt{\xi_k^l^2 + \Delta_k^l}$$

$$z_k^2 - y_k^2 = \frac{\xi_k^u}{E_k} (1 - x_k^2), \quad 2 y_k z_k = \frac{\Delta_k^u}{E_k} (1 - x_k^2)$$

$$E_k^u = \sqrt{\xi_k^u^2 + \Delta_k^u}$$

to get

$$\Delta_k^l = g' \sum_{p > 0} \frac{\Delta_p^l}{E_p} (1 - y_k^2) + \frac{\Delta_p^u}{E_p} (1 - x_k^2)$$

$$\Delta_k^u = g' \sum_{p > 0} \frac{\Delta_p^l}{E_p} (1 - y_k^2) + \frac{\Delta_p^u}{E_p} (1 - x_k^2)$$

from which we see that there is only a single momentum-independent parameter $\Delta$ defined by

$$1 = g' \sum_{k > 0} \frac{1 - y_k^2}{\sqrt{\xi_k^l^2 + \Delta^2}} + \frac{1 - x_k^2}{\sqrt{\xi_k^u^2 + \Delta^2}}$$

$$1 = g' \sum_{d \in W} \frac{N''(\omega)}{\sqrt{\omega^2 + \Delta^2}}.$$  \hspace{1cm} (D35)

This is the same as the BCS gap equation, but with an effective density of states

$$N''(\omega) = \frac{1}{L^d} \sum_k \delta(\omega - \xi_k^l)(1 - y_k^2) + \delta(\omega - \xi_k^u)(1 - x_k^2)$$

Because we are considering $g \ll U$, to a very good approximation $1 - y_k^2 = \theta(\xi_k^l)$ and $1 - x_k^2 = \theta(-\xi_k^l)$.

$$N''(\omega) = \frac{1}{L^d} \sum_k \delta(\omega - \xi_k^l)\theta(\xi_k^u) + \delta(\omega - \xi_k^u)\theta(-\xi_k^l)$$

(D37)
Note that $N''(\omega)$ is not the single-particle density of states of the HK model. In fact it is larger than or equal to it for all $\omega$, and $\int d\omega N''(\omega) \geq 1$. (D38) may be rewritten as

$$N''(\omega) = \sum_{k \in \Omega_0} \delta(\omega - \xi^1_k) + \sum_{k \in \Omega_2} \delta(\omega - \xi^u_k)$$

$$+ \sum_{k \in \Omega_1} \delta(\omega - \xi^l_k) + \delta(\omega - \xi^v_k).$$