The Lusztig automorphism of the q-Onsager algebra

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Abstract

Pascal Baseilhac and Stefan Kolb recently introduced the Lusztig automorphism $L$ of the $q$-Onsager algebra $O_q$. In this paper, we express each of $L, L^{-1}$ as a formal sum involving some quantum adjoints. In addition, (i) we give a computer-free proof that $L$ exists; (ii) we establish the higher order $q$-Dolan/Grady relations previously conjectured by Baseilhac and Thao Vu; (iii) we obtain a Lusztig automorphism for the current algebra $A_q$ associated with $O_q$; (iv) we describe what happens when a finite-dimensional irreducible $O_q$-module is twisted via $L$.

Keywords. $q$-Onsager algebra, tridiagonal pair.

2010 Mathematics Subject Classification. Primary: 33D80. Secondary 17B40.

1 Introduction

Throughout this paper $\mathbb{F}$ denotes a field. Fix $0 \neq q \in \mathbb{F}$ that is not a root of unity. Recall the notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n = 0, 1, 2, \ldots$$

We will be discussing algebras. An algebra is meant to be associative and have a 1. A subalgebra has the same 1 as the parent algebra.

Definition 1.1. (See [2, Section 2], [23, Definition 3.9].) Let $O_q$ denote the $\mathbb{F}$-algebra with generators $A, B$ and relations

$$A^3B - [3]_qA^2BA + [3]_qABA^2 - BA^3 = (q^2 - q^{-2})^2(BA - AB), \quad (1)$$

$$B^3A - [3]_qB^2AB + [3]_qBAB^2 - AB^3 = (q^2 - q^{-2})^2(AB - BA). \quad (2)$$

We call $O_q$ the $q$-Onsager algebra. The relations (1), (2) are called the $q$-Dolan/Grady relations.

We now give some background on $O_q$; for more information see [24]. There is a family of algebras called tridiagonal algebras [23, Definition 3.9] that arise in the study of $(P$ and $Q)$-polynomial association schemes [21, Lemma 5.4] and tridiagonal pairs [16, Theorem 10.1], [23, Theorem 3.10]. The algebra $O_q$ is the “most general” example of a tridiagonal algebra [17].
p. 70]. Applications of \( \mathcal{O}_q \) to tridiagonal pairs can be found in [11, 15, 18, 22, 23, 27]. The algebra \( \mathcal{O}_q \) has applications to quantum integrable models [1, 10], reflection equation algebras [12], and coideal subalgebras [14, 19, 20]. There is an algebra homomorphism from \( \mathcal{O}_q \) into the algebra \( \square_q \) [20 Proposition 5.6], and the universal Askey-Wilson algebra [25, Sections 9, 10].

In [11] Pascal Baseilhac and Stefan Kolb found an automorphism \( L \) of \( \mathcal{O}_q \) that acts as follows:

\[
L(A) = A, \quad L(B) = B + \frac{qA^2B - (q + q^{-1})ABA + q^{-1}BA^2}{(q - q^{-1})(q^2 - q^{-2})}, \tag{3}
\]

\[
L^{-1}(A) = A, \quad L^{-1}(B) = B + \frac{q^{-1}A^2B - (q + q^{-1})ABA + qBA^2}{(q - q^{-1})(q^2 - q^{-2})}. \tag{4}
\]

They called \( L \) the Lusztig automorphism of \( \mathcal{O}_q \). In our view \( L \) is a profound discovery, and worthy of much further study. In this paper, we express each of \( L, L^{-1} \) as a formal sum that involves some quantum adjoints of \( A \). In addition, (i) we obtain a computer-free proof that \( L \) exists; (ii) we establish the higher order \( q \)-Dolan/Grady relations previously conjectured by Baseilhac and Thao Vu [13]; (iii) we obtain a Lusztig automorphism for the current algebra \( \mathcal{A}_q \) [12, Definition 3.1] associated with \( \mathcal{O}_q \); (iv) we describe what happens when a finite-dimensional irreducible \( \mathcal{O}_q \)-module is twisted via \( L \).

## 2 Statement of the main result

We will state our main result after a few comments. Recall the natural numbers \( \mathbb{N} = \{0, 1, 2, \ldots\} \) and integers \( \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\} \). Let \( \mathcal{A} \) denote an \( F \)-algebra. For \( A \in \mathcal{A} \), the corresponding adjoint map is \( \text{ad} A : \mathcal{A} \to \mathcal{A}, X \mapsto AX -XA \). For \( r \in \mathbb{Z} \) define the quantum adjoint map \( \text{ad}_r A : \mathcal{A} \to \mathcal{A}, X \mapsto q^r AX - q^{-r}XA \). We have \( \text{ad} A = \text{ad}_0 A \). Note that \( \text{ad}_r A, \text{ad}_s A \) commute for \( r, s \in \mathbb{Z} \).

We now state our main result.

**Theorem 2.1.** The Lusztig automorphism \( L \) of \( \mathcal{O}_q \) satisfies

\[
L = I + \sum_{n=1}^{\infty} \left( \frac{\text{ad} A}{q - q^{-1}} \prod_{r=1}^{n-1} \frac{(q^{2r} - q^{-2r})^2 I + (\text{ad}_r A)(\text{ad}_{-r} A)}{(q^{2r} - q^{-2r})(q^{2r+1} - q^{-2r-1})} \right) \frac{\text{ad}_n A}{q^{2n} - q^{-2n}}, \tag{5}
\]

\[
L^{-1} = I + \sum_{n=1}^{\infty} \left( \frac{\text{ad} A}{q - q^{-1}} \prod_{r=1}^{n-1} \frac{(q^{2r} - q^{-2r})^2 I + (\text{ad}_r A)(\text{ad}_{-r} A)}{(q^{2r} - q^{-2r})(q^{2r+1} - q^{-2r-1})} \right) \frac{\text{ad}_{-n} A}{q^{2n} - q^{-2n}}. \tag{6}
\]

Moreover for all \( X \in \mathcal{O}_q \), in the above sums the large parenthetical expression vanishes at \( X \) for all but finitely \( n \).

We mention two consequences of Theorem 2.1

**Corollary 2.2.** The automorphism \( L \) fixes every element of \( \mathcal{O}_q \) that commutes with \( A \).
Corollary 2.3. Pick $X \in \mathcal{O}_q$ such that

\[ A^3X - [3]_qA^2XA + [3]_qAXA^2 - XA^3 = (q^2 - q^{-2})^2(XA - AX). \]

Then $L$ sends

\[ X \mapsto X + \frac{qA^2X - (q + q^{-1})AXA + q^{-1}XA^2}{(q - q^{-1})(q^2 - q^{-2})}, \]

and $L^{-1}$ sends

\[ X \mapsto X + \frac{q^{-1}A^2X - (q + q^{-1})AXA + qXA^2}{(q - q^{-1})(q^2 - q^{-2})}. \]

We will obtain Theorem 2.1 as a consequence of a more general result, which we now summarize. Let $\mathcal{A}$ denote an $F$-algebra and let $A \in \mathcal{A}$. Consider the formal sums

\[ S = I + \sum_{n=1}^{\infty} \left( \frac{\text{ad } A}{q - q^{-1}} \prod_{r=1}^{n-1} \left( \frac{(q^{2r} - q^{-2r})^2I + (\text{ad}_r A)(\text{ad}_{-r} A)}{(q^{2r} - q^{-2r})(q^{2r+1} - q^{-2r-1})} \right) \frac{\text{ad}_n A}{q^{2n} - q^{-2n}} \right), \quad (7) \]

\[ S' = I + \sum_{n=1}^{\infty} \left( \frac{\text{ad } A}{q - q^{-1}} \prod_{r=1}^{n-1} \left( \frac{(q^{2r} - q^{-2r})^2I + (\text{ad}_r A)(\text{ad}_{-r} A)}{(q^{2r} - q^{-2r})(q^{2r+1} - q^{-2r-1})} \right) \frac{\text{ad}_{-n} A}{q^{2n} - q^{-2n}} \right). \quad (8) \]

An element $X \in \mathcal{A}$ is called $A$-standard whenever the large parenthetical expression in (7), (8) vanishes at $X$ for all but finitely many $n$. The algebra $\mathcal{A}$ is called $A$-standard whenever each element of $A$ is $A$-standard. Assume that $\mathcal{A}$ is $A$-standard. We will show that $S$ and $S'$ act on $\mathcal{A}$ as an automorphism, and these automorphisms are inverses. Also, we will show that the algebra $\mathcal{O}_q$ is $A$-standard and $S = L$, $S' = L^{-1}$.

3 Some identities for the quantum adjoint

As we work towards Theorem 2.1, our first goal is to establish some identities for the quantum adjoint, that apply to any $F$-algebra. Let $\mathcal{A}$ denote an $F$-algebra, and fix $A \in \mathcal{A}$. Recall the sums $S$, $S'$ from (7), (8). We will be discussing the terms in these sums. To simplify this discussion we introduce a “balanced” version of ad, called bad. Let $\mathbb{Z}^+$ denote the set of positive integers.

**Definition 3.1.** Define

\[ \text{bad}_q A = \frac{\text{ad } A}{q - q^{-1}} \]

and

\[ \text{bad}_n A = \frac{(q^{2n} - q^{-2n})^2I + (\text{ad}_n A)(\text{ad}_{-n} A)}{(q^{2n} - q^{-2n})(q^{2n+1} - q^{-2n-1})} \quad n \in \mathbb{Z}^+. \]

Further define

\[ (\text{bad } A)_n = \prod_{i=0}^{n-1} \text{bad}_i A \quad n \in \mathbb{N}. \quad (9) \]

We interpret $(\text{bad } A)_0 = I$. 

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Definition 3.2. Define $S_0 = I$ and
\[ S_n = \frac{(\text{bad } A)_n \text{ad}_n A}{q^{2n} - q^{-2n}} \quad n \in \mathbb{Z}^+. \]

Further define $S'_0 = I$ and
\[ S'_n = \frac{(\text{bad } A)_n \text{ad}_n' A}{q^{2n} - q^{-2n}} \quad n \in \mathbb{Z}^+. \]

Lemma 3.3. In the above notation the sums (7), (8) become
\[ S = \sum_{n \in \mathbb{N}} S_n, \quad S' = \sum_{n \in \mathbb{N}} S'_n. \]

Our next goal is to prove Proposition 3.10 below. To this end we give some identities that hold in $A$.

Lemma 3.4. For $i \in \mathbb{Z}$,
\[ \text{ad}_i A + \text{ad}_{-i} A = (q^i + q^{-i}) \text{ad } A. \]

Proof. Routine. \hfill \Box

Lemma 3.5. For $i \in \mathbb{Z}^+$,
\[ S_i + S'_i = \frac{(\text{bad } A)_i \text{ad } A}{q^i - q^{-i}}. \]

Proof. Use Definition 3.2 and Lemma 3.4. \hfill \Box

Lemma 3.6. For $i \in \mathbb{Z}^+$,
\[ \frac{(\text{ad}_i A)(\text{ad}_{-i} A)}{(q^{2i} - q^{-2i})^2} + I = \frac{q^{2i+1} - q^{-2i-1}}{q^{2i} - q^{-2i}} \text{bad}_i A. \]

Proof. Use Definition 3.1. \hfill \Box

Lemma 3.7. For $i \in \mathbb{Z}^+$,
\[ S_i S'_i + (\text{bad } A)_i^2 = \frac{q^{2i+1} - q^{-2i-1}}{q^{2i} - q^{-2i}} (\text{bad } A)_i (\text{bad } A)_{i+1}. \]

Proof. Use Definition 3.2 and Lemma 3.6. \hfill \Box

Lemma 3.8. For $i, j \in \mathbb{Z}^+$,
\[
\frac{(\text{ad}_i A)(\text{ad}_{-j} A) + (\text{ad}_{-i} A)(\text{ad}_j A)}{(q^{2i} - q^{-2i})(q^{2j} - q^{-2j})} + (q^{i-j} + q^{j-i}) I \\
= \frac{q^{2i+1} - q^{-2i-1}}{q^{i+j} - q^{-i-j}} \text{bad}_i A + \frac{q^{2j+1} - q^{-2j-1}}{q^{i+j} - q^{-i-j}} \text{bad}_j A.
\]
Proof. Routine using Definition 3.1.

Lemma 3.9. For $i, j \in \mathbb{Z}^+$,
\[
S_i S'_i + S'_j S_j + (q^{i-j} + q^{j-i})(\text{bad } A)_i (\text{bad } A)_j = \frac{q^{2i+1} - q^{-2i-1}}{q^{i+j} - q^{-i-j}} (\text{bad } A)_{i+1} (\text{bad } A)_j + \frac{q^{2j+1} - q^{-2j-1}}{q^{i+j} - q^{-i-j}} (\text{bad } A)_i (\text{bad } A)_{j+1}.
\]

Proof. Use Definition 3.2 and Lemma 3.8.

Proposition 3.10. For $n \in \mathbb{N}$,
\[
\left( \sum_{i=0}^{n} S_i \right) \left( \sum_{j=0}^{n} S'_j \right) = I + (\text{bad } A)_{n+1} \left( \sum_{r=0}^{n-1} \frac{q^{2n+1} - q^{-2n-1}}{q^{n+r+1} - q^{-n-r-1}} (\text{bad } A)_{r+1} \right).
\]

Proof. The proof is by induction on $n$. Let $D_n$ denote the left-hand side minus the right-hand side. We show that $D_n = 0$. One routinely obtains $D_0 = 0$, so assume $n \geq 1$. To show that $D_n = 0$, it suffices to show that $D_n - D_{n-1} = 0$. In the expression $D_n - D_{n-1}$, eliminate the terms $\{S_i S'_i \}_{i=0}^{n-1}$, $\{S'_j S_j \}_{j=0}^{n-1}$, $S_i S'_i$ using Lemmas 3.5, 3.7, 3.9. After a routine simplification we obtain $D_n - D_{n-1} = 0$, so $D_n = 0$.

Our next goal is to prove Proposition 3.19 below. To this end we give some more identities that hold in $\mathcal{A}$.

Lemma 3.11. For distinct $i, j \in \mathbb{Z}$ and $X, Y \in \mathcal{A}$,
\[
XA = \frac{q^i \text{ad}_i A - q^i \text{ad}_j A}{q^{i-j} - q^{j-i}} (X), \quad AY = \frac{q^{-j} \text{ad}_j A - q^{-i} \text{ad}_j A}{q^{i-j} - q^{j-i}} (Y).
\]

Proof. Routine.

Lemma 3.12. For $i \in \mathbb{Z}$ and $j \in \mathbb{Z}^+$ and $X, Y \in \mathcal{A}$,
\[
\text{ad}_i A (\text{bad } A)_j (X) = q^{i-j}(q^{2j} - q^{-2j})S_j (X) + q^{-j}(q^{i-j} - q^{j-i})(\text{bad } A)_j (X)A,
\]
\[
\text{ad}_i A (\text{bad } A)_j (Y) = q^{i-j}(q^{2j} - q^{-2j})S_j (Y) + q^{i}(q^{i-j} - q^{-j})(\text{bad } A)_j (Y).
\]

Proof. Use Definition 3.2 and Lemma 3.11.

Lemma 3.13. For $i \in \mathbb{Z}$ and $j \in \mathbb{N}$ and $X, Y \in \mathcal{A}$,
\[
\text{ad}_i A S_j (X) = q^{i+j}(q^{2j+1} - q^{-2j-1})(\text{bad } A)_{j+1} (X) - q^{i+j}(q^{2j} - q^{-2j}) (\text{bad } A)_{j+1} (X) + q^{j}(q^{i+j} - q^{-i-j}) (S_j (X))A,
\]
\[
\text{ad}_i A S_j (Y) = q^{-i-j}(q^{2j+1} - q^{-2j-1})(\text{bad } A)_{j+1} (Y) - q^{-i-j}(q^{2j} - q^{-2j}) (\text{bad } A)_{j+1} (Y) + q^{i-j}(q^{i+j} - q^{-i-j}) (S_j (Y))A.
\]

Proof. Use Definitions 3.11, 3.12 and Lemma 3.11.
Lemma 3.14. For $h, i, j \in \mathbb{Z}$ and $X, Y \in \mathcal{A}$,
\[\operatorname{ad}_h A (XY) = q^{h-i} (\operatorname{ad}_i A (X)) Y + q^{j-h} X (\operatorname{ad}_j A (Y)) + q^{j-i} (q^{h-i-j} - q^{i+j-h}) XAY.\]

Proof. Routine. \hfill \Box

The next four lemmas are routinely obtained using Lemmas 3.11–3.14.

Lemma 3.15. For $h \in \mathbb{Z}$ and $i, j \in \mathbb{Z}^+$ and $X, Y \in \mathcal{A}$,
\[\operatorname{ad}_h A \left( (\operatorname{bad} A)_i (X) \right) \left( (\operatorname{bad} A)_j (Y) \right) = q^{h-i} \left( q^{2i} - q^{-2i} \right) (S_i(X)) \left( (\operatorname{bad} A)_j (Y) \right) + q^{j-h} \left( q^{2j} - q^{-2j} \right) (S_i(X)) \left( (\operatorname{bad} A)_j (Y) \right) + q^{j-i} \left( q^{h-i-j} - q^{i+j-h} \right) (S_i(X)) \left( (\operatorname{bad} A)_j (Y) \right).\]

Lemma 3.16. For $h \in \mathbb{Z}$ and $i, j \in \mathbb{N}$ and $X, Y \in \mathcal{A}$,
\[\operatorname{ad}_h A \left( (S_i(X)) (S_j (Y)) \right) = q^{h-i} \left( q^{2i+1} - q^{-2i-1} \right) (S_j (Y)) + q^{j-h} \left( q^{2j+1} - q^{-2j-1} \right) (S_j (Y)) + q^{j-i} \left( q^{h+i+j} - q^{-h-i-j} \right) (S_j (Y)) A (S_j (Y)).\]

Lemma 3.17. For $h \in \mathbb{Z}$ and $i \in \mathbb{N}$ and $j \in \mathbb{Z}^+$ and $X, Y \in \mathcal{A}$,
\[\operatorname{ad}_h A \left( (S_i(X)) (\operatorname{bad} A)_j (Y) \right) = q^{h-i} \left( q^{2i+1} - q^{-2i-1} \right) (\operatorname{bad} A)_{i+1} (X) \left( \operatorname{bad} A \right)_{j} (Y) + q^{j-h} \left( q^{2j} - q^{-2j} \right) (S_j (Y)) + q^{j-i} \left( q^{h+i-j} - q^{-h-j} \right) (S_j (Y)) A \left( \operatorname{bad} A \right)_{j} (Y).\]

Lemma 3.18. For $h \in \mathbb{Z}$ and $i \in \mathbb{Z}^+$ and $j \in \mathbb{N}$ and $X, Y \in \mathcal{A}$,
\[\operatorname{ad}_h A \left( (\operatorname{bad} A)_{i} (X) \right) \left( S_j (Y) \right) = q^{h-j} \left( q^{2j+1} - q^{-2j-1} \right) (\operatorname{bad} A)_{i} (X) \left( \operatorname{bad} A \right)_{j+1} (Y) + q^{j-h} \left( q^{2j} - q^{-2j} \right) (S_j (Y)) + q^{j-i} \left( q^{h+i-j} - q^{-h-j} \right) (S_j (Y)) A \left( \operatorname{bad} A \right)_{i} (X) \left( \operatorname{bad} A \right)_{j} (Y).\]
Proposition 3.19. For \( n \in \mathbb{N} \) and \( X, Y \in \mathcal{A} \),

\[
\sum_{i=0}^{n} S_i(XY) = \sum_{r+s \leq n} S_r(X)S_s(Y) + \sum_{r+s = n-1} \left((\text{bad } A)_{r+1}(X)\right) \left((\text{bad } A)_{s+1}(Y)\right) q^{s-r}.
\]

\[
(\text{bad } A)_{n+1}(XY) = \sum_{r+s \leq n} S_r(X) \left((\text{bad } A)_{s+1}(Y)\right) q^{-r} + \sum_{r+s = n-1} \left((\text{bad } A)_{r+1}(X)\right) S_s(Y) q^{s-r} - \sum_{r+s = n-1} \left((\text{bad } A)_{r+1}(X)\right) A \left((\text{bad } A)_{s+1}(Y)\right).
\]

Proof. The proof is by induction on \( n \). Let \( S(n) \) (resp. \( B(n) \)) denote the first (resp. second) displayed equation in the proposition statement. The equation \( S(0) \) holds since \( S_0 \) the identity map. The equation \( B(0) \) holds by Lemma [3.14] at \( h = 0, i = 0, j = 0 \). To get \( S(n+1) \) from \( S(n) \) and \( B(n) \), apply \( \text{ad}_{n+1} A \) to each side of \( B(n) \), and evaluate the result using Lemmas [3.15] [3.18]. One obtains \( S(n+1) - S(n) \) after a brief calculation. For the rest of this proof, assume that \( n \geq 1 \). To get \( B(n) \) from \( S(n) \) and \( S(n-1) \), apply \( \text{ad}_{n-1} A \) to each side of \( S(n) - S(n-1) \), and evaluate the result using Lemmas [3.15] [3.18]. One obtains \( B(n) \) after a brief calculation. The result follows from these comments. \( \square \)

The following result is a variation on Proposition 3.19.

Proposition 3.20. For \( n \in \mathbb{N} \) and \( X, Y \in \mathcal{A} \),

\[
\sum_{i=0}^{n} S'_i(XY) = \sum_{r+s \leq n} S'_r(X)S'_s(Y) + \sum_{r+s = n-1} \left((\text{bad } A)_{r+1}(X)\right) \left((\text{bad } A)_{s+1}(Y)\right) q^{s-r},
\]

\[
(\text{bad } A)_{n+1}(XY) = \sum_{r+s \leq n} S'_r(X) \left((\text{bad } A)_{s+1}(Y)\right) q^{-r} + \sum_{r+s = n-1} \left((\text{bad } A)_{r+1}(X)\right) S'_s(Y) q^{s-r} + \sum_{r+s = n-1} \left((\text{bad } A)_{r+1}(X)\right) A \left((\text{bad } A)_{s+1}(Y)\right).
\]

Proof. In Proposition 3.19, replace \( q \) by \( q^{-1} \) and evaluate the result using Definitions [3.1] [3.2]. \( \square \)

Strictly speaking we do not need the following result; we mention it for the sake of completeness.

Proposition 3.21. For \( n \in \mathbb{N} \) and \( X, Y \in \mathcal{A} \),

\[
(\text{bad } A)_{n+1}(XY) = \sum_{r+s \leq n} S_r(X) \left((\text{bad } A)_{s+1}(Y)\right) q^{-r} + \sum_{r+s = n-1} \left((\text{bad } A)_{r+1}(X)\right) S'_s(Y) q^{s-r} - \sum_{r+s = n-1} \left((\text{bad } A)_{r+1}(X)\right) A \left((\text{bad } A)_{s+1}(Y)\right).\]

\[
= \sum_{r+s \leq n} S_r'(X) \left((\text{bad } A)_{s+1}(Y)\right) q^{-r} + \sum_{r+s = n-1} \left((\text{bad } A)_{r+1}(X)\right) S_s(Y) q^{s-r}
\]

\[
= \sum_{r+s \leq n} S_r'(X) \left((\text{bad } A)_{s+1}(Y)\right) q^{-r} + \sum_{r+s = n-1} \left((\text{bad } A)_{r+1}(X)\right) S_s(Y) q^{s-r}
\]

\[
= \sum_{r+s \leq n} S_r'(X) \left((\text{bad } A)_{s+1}(Y)\right) q^{-r} + \sum_{r+s = n-1} \left((\text{bad } A)_{r+1}(X)\right) S_s(Y) q^{s-r}
\]

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Proof. Using Definition 3.2 and Lemma 3.11 we find that for \( i \in \mathbb{Z}^+ \),

\[
q^{-i}S_i(X) - q^iS'_i(X) = \left((\text{bad } A)_i(X)\right)A,
\]

(10)

\[
q^iS_i(Y) - q^{-i}S'_i(Y) = A\left((\text{bad } A)_i(Y)\right).
\]

(11)

In the relations from the proposition statement, eliminate the terms \( S'_r(X), S'_s(Y) \) using (10), (11) and compare the results with the second equation in Proposition 3.19.

Proposition 3.22. Given \( r, s \in \mathbb{N} \) and \( X, Y \in A \) such that

\[
(\text{bad } A)_{r+1}(X) = 0, \quad (\text{bad } A)_{s+1}(Y) = 0.
\]

Then

\[
(\text{bad } A)_{r+s+1}(XY) = 0.
\]

Proof. Use the second equation in Proposition 3.19 or 3.20. Alternatively use either equation in Proposition 3.21.

4 The subalgebra \( A^\vee \)

We continue to work with the element \( A \) of the \( \mathbb{F} \)-algebra \( A \).

Definition 4.1. For \( n \in \mathbb{N} \) let \( A^{(n)} \) denote the set of elements in \( A \) at which \( (\text{bad } A)_{n+1} \) vanishes. Note that \( A^{(n)} \) is a subspace of the \( \mathbb{F} \)-vector space \( A \).

Example 4.2. The subspace \( A^{(0)} \) consists of the elements in \( A \) that commute with \( A \).

Example 4.3. The subspace \( A^{(1)} \) consists of the elements \( X \) in \( A \) such that

\[
A^3X - [3]qA^2XA + [3]qAXA^2 -XA^3 = (q^2-q^{-2})^2(XA-AX).
\]

Lemma 4.4. We have \( A^{(n)} \subseteq A^{(n+1)} \) for \( n \in \mathbb{N} \).

Proof. By (9) and Definition 4.1.

Lemma 4.5. Pick \( n \in \mathbb{N} \). Then for \( r > n \) the maps \( S_r, S'_r \) vanish on \( A^{(n)} \). Moreover on \( A^{(n)} \),

\[
S = \sum_{r=0}^{n} S_r, \quad S' = \sum_{r=0}^{n} S'_r.
\]

(12)

Proof. By (9) the map \( (\text{bad } A)_{n+1} \) is a factor of \( (\text{bad } A)_r \). By Definition 3.2 the map \( (\text{bad } A)_r \) is a factor of \( S_r \) and \( S'_r \). Consequently \( S_r \) and \( S'_r \) vanish on \( A^{(n)} \). The equations (12) are from Lemma 3.3.

By Lemma 4.5, \( S \) and \( S' \) are well defined \( \mathbb{F} \)-linear maps on \( A^{(n)} \) for all \( n \in \mathbb{N} \).

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Lemma 4.6. For $n \in \mathbb{N}$ the subspace $\mathcal{A}^{(n)}$ is invariant under $S$ and $S'$.

Proof. The map $(\text{bad } A)_{n+1}$ commutes with $S_r$ and $S'_r$ for $r \in \mathbb{N}$. □

Lemma 4.7. For $n \in \mathbb{N}$ the maps $S : \mathcal{A}^{(n)} \to \mathcal{A}^{(n)}$ and $S' : \mathcal{A}^{(n)} \to \mathcal{A}^{(n)}$ are inverses.

Proof. By Proposition 3.10 □

Example 4.8. The maps $S$ and $S'$ fix everything in $\mathcal{A}^{(0)}$.

Proof. On $\mathcal{A}^{(0)}$ we have $S = S_0 = I$ and $S' = S'_0 = I$. □

Example 4.9. Pick $X \in \mathcal{A}^{(1)}$. Then $S$ sends

$$X \mapsto X + \frac{qA^2X - (q + q^{-1})AXA + q^{-1}XA^2}{(q - q^{-1})(q^2 - q^{-2})},$$

and $S'$ sends

$$X \mapsto X + \frac{q^{-1}A^2X - (q + q^{-1})AXA + qXA^2}{(q - q^{-1})(q^2 - q^{-2})}.$$

Proof. On $\mathcal{A}^{(1)}$ we have $S = S_0 + S_1$ and $S' = S'_0 + S'_1$. □

Lemma 4.10. We have $\mathcal{A}^{(r)}\mathcal{A}^{(s)} \subseteq \mathcal{A}^{(r+s)}$ for $r, s \in \mathbb{N}$.

Proof. By Proposition 3.22 □

Definition 4.11. Define $\mathcal{A}^\vee = \bigcup_{n \in \mathbb{N}} \mathcal{A}^{(n)}$.

Lemma 4.12. The set $\mathcal{A}^\vee$ is a subalgebra of $\mathcal{A}$ that contains $A$.

Proof. By Definition 4.1 and Lemma 4.4 $\mathcal{A}^\vee$ is a subspace of the $\mathbb{F}$-vector space $\mathcal{A}$. By Example 4.2 the subspace $\mathcal{A}^\vee$ contains $1$ and $A$. By Lemma 4.10 the subspace $\mathcal{A}^\vee$ is closed under multiplication. The result follows. □

By Definition 4.11 along with Lemma 4.6 and the comment above it, we obtain $\mathbb{F}$-linear maps $S : \mathcal{A}^\vee \to \mathcal{A}^\vee$ and $S' : \mathcal{A}^\vee \to \mathcal{A}^\vee$.

Proposition 4.13. The maps $S$ and $S'$ act on the algebra $\mathcal{A}^\vee$ as an automorphism, and these automorphisms are inverses.

Proof. To get the first assertion use Propositions 3.19, 3.20. The last assertion follows from Lemma 4.7. □
5  A-Standard algebras and their Lusztig automorphism

We continue to work with the element $A$ of the $\mathbb{F}$-algebra $\mathcal{A}$.

**Definition 5.1.** An element $X \in \mathcal{A}$ is called $A$-standard whenever there exists a positive integer $n$ such that $(\text{bad } A)_n(X) = 0$. Note that $\mathcal{A}^\vee$ consists of the $A$-standard elements of $\mathcal{A}$.

**Definition 5.2.** The algebra $\mathcal{A}$ is called $A$-standard whenever each element of $\mathcal{A}$ is $A$-standard.

**Lemma 5.3.** The following (i)--(iii) are equivalent:

(i) $\mathcal{A}$ is $A$-standard;

(ii) $\mathcal{A}^\vee = \mathcal{A}$;

(iii) $\mathcal{A}$ has a generating set whose elements are $A$-standard.

**Proof.** (i) $\iff$ (ii) By Definitions 5.1, 5.2

(i) $\Rightarrow$ (iii) Clear.

(iii) $\Rightarrow$ (ii) By Lemma 4.12 $\mathcal{A}^\vee$ is a subalgebra of $\mathcal{A}$. By Definition 5.1 $\mathcal{A}^\vee$ contains each $A$-standard element of $\mathcal{A}$. The result follows. 

**Theorem 5.4.** Assume that $\mathcal{A}$ is $A$-standard. Then $S$ and $S'$ act on $\mathcal{A}$ as an automorphism, and these automorphisms are inverses.

**Proof.** Apply Proposition 4.13 to the algebra $\mathcal{A}^\vee = \mathcal{A}$. 

Recall the $q$-Onsager algebra $\mathcal{O}_q$ and its generators $A, B$.

**Proposition 5.5.** For $\mathcal{O}_q$ the following (i)--(iv) hold:

(i) $A \in \mathcal{O}_q^{(0)}$ and $B \in \mathcal{O}_q^{(1)}$;

(ii) the algebra $\mathcal{O}_q$ is $A$-standard;

(iii) $S$ sends $A \mapsto A$, $B \mapsto B + \frac{qA^2B - (q + q^{-1})ABA + q^{-1}BA^2}{(q - q^{-1})(q^2 - q^{-2})}$

and $S'$ sends $A \mapsto A$, $B \mapsto B + \frac{q^{-1}A^2B - (q + q^{-1})ABA + qBA^2}{(q - q^{-1})(q^2 - q^{-2})}$;

(iv) $S = L$ and $S' = L^{-1}$.
Proof. (i) We have $A \in \mathcal{O}_q^{(0)}$ by Example 4.2, and $B \in \mathcal{O}_q^{(1)}$ by Example 4.3.
(ii) The generators $A, B$ are $A$-standard by (i) above. Now $\mathcal{O}_q$ is $A$-standard by Lemma 5.3.
(iii) By (i) above and Examples 4.8, 4.9.
(iv) Compare (3), (4) with (iii) above.

Theorem 2.1 follows from Theorem 5.4 and Proposition 5.5. Combining Theorem 5.4 and Proposition 5.5(i)–(iii), we get a computer-free proof that there exists an automorphism $L$ of $\mathcal{O}_q$ that satisfies (3), (4).

We return our attention to the algebra $\mathcal{A}$, and the element $A \in \mathcal{A}$. The following definition is motivated by Proposition 5.5.

Definition 5.6. Assume that $A$ is $A$-standard, and consider its automorphism $S$ from Theorem 5.4. We call $S$ the Lusztig automorphism of $\mathcal{A}$ that corresponds to $A$.

6 The higher order $q$-Dolan/Grady relations

In this section we establish the higher order $q$-Dolan/Grady relations conjectured by Baseilhac and Vu [13]. Let $\mathcal{A}$ denote an $F$-algebra and fix $A \in \mathcal{A}$.

Theorem 6.1. Given $X \in \mathcal{A}$ such that

$$A^3X - [3]_q A^2XA + [3]_q AXA^2 - XA^3 = (q^2 - q^{-2})^2(XA - AX).$$

Then

$$(\text{bad } A)_{r+1}(X^r) = 0 \quad r \geq 1.$$ 

We are using the notation (9).

Proof. By Example 4.3 we have $X \in \mathcal{A}^{(1)}$. By Lemma 4.10 we have $X^r \in \mathcal{A}^{(r)}$. The result follows by Definition 4.1.

7 The current algebra $\mathcal{A}_q$ and its Lusztig automorphism

In [12] Baseilhac and K. Shigechi introduce the current algebra $\mathcal{A}_q$ for $\mathcal{O}_q$, and they discuss how $\mathcal{A}_q$ is related to $\mathcal{O}_q$. This relationship is discussed further in [7], where it is conjectured that $\mathcal{O}_q$ is a homomorphic image of $\mathcal{A}_q$ [7, Conjecture 2]. The algebra $\mathcal{A}_q$ is defined by generators and relations [12, Definition 3.1]. The generators are denoted $W_{-k}, W_{k+1}, G_{k+1}, G_{k+1}$, where $k \in \mathbb{N}$. In [7, Lemma 2.1], Baseilhac and S. Belliard display some central elements $\{\Delta_{k+1}\}_{k \in \mathbb{N}}$ for $\mathcal{A}_q$. In [7, Corollary 3.1], it is shown that $\mathcal{A}_q$ is generated by these central elements together with $A = W_0$ and $B = W_1$. The elements $A, B$ are known to satisfy the $q$-Dolan/Grady relations (11), (12) [7, eqn. (3.7)]. In this section we show that $\mathcal{A}_q$ is $A$-standard, and describe how the corresponding Lusztig automorphism acts on the elements mentioned above. We now recall the definition of $\mathcal{A}_q$. 

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Definition 7.1. (See \cite{12} Definition 3.1.) Let $\mathcal{A}_q$ denote the $\mathbb{F}$-algebra with generators $\mathcal{W}_{-k}, \mathcal{W}_{k+1}, \mathcal{G}_{k+1}, \tilde{\mathcal{G}}_{k+1} (k \in \mathbb{N})$ and the following relations:

$$
\begin{align*}
[W_0, W_{k+1}] &= [W_{-k}, W_1] = (\tilde{G}_{k+1} - G_{k+1})/(q + q^{-1}), \\
[W_0, G_{k+1}] &= [G_{k+1}, W_0] = \rho W_{-k-1} - \rho W_{k+1}, \\
[G_{k+1}, W_1] &= [W_1, G_{k+1}] = \rho W_{k+2} - \rho W_{-k}, \\
[W_{-k}, W_{-\ell}] &= 0, \quad [W_{k+1}, W_{\ell+1}] = 0, \\
[W_{-k}, W_{\ell+1}] + [W_{k+1}, W_{-\ell}] &= 0, \\
[W_{-k}, G_{\ell+1}] + [G_{k+1}, W_{-\ell}] &= 0, \\
[W_{k+1}, G_{\ell+1}] + [G_{k+1}, W_{\ell+1}] &= 0, \\
\tilde{G}_{k+1} G_{\ell+1} + [G_{k+1}, \tilde{G}_{\ell+1}] &= 0, \\
\tilde{G}_{k+1} G_{\ell+1} + [G_{k+1}, \tilde{G}_{\ell+1}] &= 0.
\end{align*}
$$

In the above equations $\ell \in \mathbb{N}$ and $\rho = -(q^2 - q^{-2})^2$. We are using the notation $[X, Y] = XY - YX$ and $[X, Y]_q = qXY - q^{-1}YX$.

For the algebra $\mathcal{A}_q$, consider the element $A = W_0$ and the corresponding subspaces $\mathcal{A}_q^{(0)}$, $\mathcal{A}_q^{(1)}$ from Examples \cite{4, 2, 13}.

Lemma 7.2. For the algebra $\mathcal{A}_q$ the following (i)–(v) hold for $k \in \mathbb{N}$:

(i) $\mathcal{W}_{-k} \in \mathcal{A}_q^{(0)}$;  
(ii) $\mathcal{W}_{k+1} \in \mathcal{A}_q^{(1)}$;  
(iii) $\mathcal{G}_{k+1} \in \mathcal{A}_q^{(1)}$;  
(iv) $\tilde{\mathcal{G}}_{k+1} \in \mathcal{A}_q^{(1)}$;  
(v) $\Delta_{k+1} \in \mathcal{A}_q^{(0)}$.

Proof. (i) The elements $\mathcal{W}_{-k}, \mathcal{W}_0$ commute by (16). The result follows in view of Example \cite{4}.

(ii) We show that

$$
[W_0, [W_0, [W_0, W_{k+1}]]_q]_q^{-1} = \rho [W_0, W_{k+1}].
$$

By (13),

$$
[W_0, [W_0, [W_0, W_{k+1}]]_q]_q^{-1} = \frac{[W_0, [W_0, \tilde{G}_{k+1} - G_{k+1}]]_q}{q + q^{-1}}.
$$

Using linear algebra and (14),

$$
[W_0, [W_0, \tilde{G}_{k+1}]]_q^{-1} = [W_0, [W_0, \tilde{G}_{k+1}]]_q = -[W_0, [\tilde{G}_{k+1}, W_0]]_q
$$

$$
= -[W_0, [\tilde{G}_{k+1}, W_0]]_q.
$$

(26)
Using in order (25), (26), (14), (13) we obtain
\[
[W_0, [W_0, [W_0, W_{k+1}]_{q^{-1}}]_{q^{-1}} = \frac{[W_0, [W_0, \tilde{G}_{k+1} - G_{k+1}]_{q^{-1}}]}{q + q^{-1}} - \frac{[W_0, [W_0, G_{k+1}]_q + [W_0, [W_0, G_{k+1}]_{q^{-1}}]}{q + q^{-1}}
\]
\[
= -[W_0, [W_0, G_{k+1}]_q]
\]
\[
= \rho[W_0, W_{k+1} - W_{-k-1}]
\]
\[
= \rho[W_0, W_{k+1}].
\]

We have shown (24). The result follows in view of Example 4.3

(iii) We show that
\[
[W_0, [W_0, [W_0, G_{k+1}]_{q^{-1}} = \rho[W_0, G_{k+1}].
\]
(27)

Using in order (14), (16), (13) we obtain
\[
[W_0, [W_0, [W_0, G_{k+1}]_{q^{-1}}]_{q^{-1}} = \frac{[W_0, [W_0, W_{-k-1} - W_{k+1}]_{q^{-1}}]}{q + q^{-1}} = -\rho[W_0, [W_0, W_{k+1}]_{q^{-1}}
\]
\[
= \rho[\tilde{G}_{k+1} - G_{k+1}]_{q^{-1}}
\]
\[
= \rho[\tilde{G}_{k+1} - G_{k+1}, W_0]_q
\]
\[
= \rho \frac{[G_{k+1} - \tilde{G}_{k+1}]_q}{q + q^{-1}}.
\]

Now in (27), the left-hand side minus the right-hand side is equal to
\[
\rho \frac{[\tilde{G}_{k+1}, W_0]_q - [G_{k+1}, W_0]_q - \rho[W_0, G_{k+1}]}{q + q^{-1}}
\]
\[
= \rho \frac{[\tilde{G}_{k+1}, W_0]_q - [W_0, G_{k+1}]_q}{q + q^{-1}},
\]
and this is zero by (14). We have shown (27). The result follows in view of Example 4.3

(iv) We show that
\[
[W_0, [W_0, [W_0, \tilde{G}_{k+1}]_{q^{-1}} = \rho[W_0, \tilde{G}_{k+1}].
\]
(28)

Using in order (14), (16), (13) we obtain
\[
[W_0, [W_0, [W_0, \tilde{G}_{k+1}]_{q^{-1}}]_{q^{-1}} = [W_0, [W_0, [W_0, \tilde{G}_{k+1}]_{q^{-1}}]_q
\]
\[
= -[W_0, [W_0, [\tilde{G}_{k+1}, W_0]_q]_q
\]
\[
= \rho[W_0, [W_0, W_{k+1} - W_{-k-1}]_q
\]
\[
= \rho[W_0, [W_0, W_{k+1}]_q
\]
\[
= \rho \frac{[W_0, \tilde{G}_{k+1} - G_{k+1}]_q}{q + q^{-1}}.
\]
Now in (28), the left-hand side minus the right-hand side is equal to
\[
\rho \frac{[\mathcal{W}_0, \tilde{\mathcal{G}}_{k+1}]_q - [\mathcal{W}_0, \mathcal{G}_{k+1}]_q}{q + q^{-1}} - \rho [\mathcal{W}_0, \tilde{\mathcal{G}}_{k+1}] = \rho \frac{[\tilde{\mathcal{G}}_{k+1}, \mathcal{W}_0]_q - [\mathcal{W}_0, \mathcal{G}_{k+1}]_q}{q + q^{-1}},
\]
and this is zero by (14). We have shown (28). The result follows in view of Example 4.3.

(v) The element $\Delta_{k+1}$ is central, so it commutes with $\mathcal{W}_0$. The result follows in view of Example 4.2. 

\begin{proof}
Consider the generators of $\mathcal{A}_q$ from Definition 7.1. By Lemma 7.2 these generators are $A$-standard. Now $\mathcal{A}_q$ is $A$-standard by Lemma 7.3.
\end{proof}

Since the algebra $\mathcal{A}_q$ is $A$-standard, we may speak of the corresponding Lusztig automorphism $S$ of $\mathcal{A}_q$, from Theorem 5.4 and Definition 5.6.

\begin{proposition}
For $k \in \mathbb{N}$ the automorphism $S$ sends
\begin{align*}
\mathcal{W}_{-k} & \rightarrow \mathcal{W}_{-k}, \\
\mathcal{W}_{k+1} & \rightarrow \mathcal{W}_{k+1} + \frac{q\mathcal{W}_0^2\mathcal{W}_{k+1} - (q + q^{-1})\mathcal{W}_0\mathcal{W}_{k+1}\mathcal{W}_0 + q^{-1}\mathcal{W}_{k+1}\mathcal{W}_0^2}{(q - q^{-1})(q^2 - q^{-2})}, \\
\mathcal{G}_{k+1} & \rightarrow \mathcal{G}_{k+1} + \frac{q\mathcal{W}_0^2\mathcal{G}_{k+1} - (q + q^{-1})\mathcal{W}_0\mathcal{G}_{k+1}\mathcal{W}_0 + q^{-1}\mathcal{G}_{k+1}\mathcal{W}_0^2}{(q - q^{-1})(q^2 - q^{-2})}, \\
\tilde{\mathcal{G}}_{k+1} & \rightarrow \tilde{\mathcal{G}}_{k+1} + \frac{q\mathcal{W}_0^2\tilde{\mathcal{G}}_{k+1} - (q + q^{-1})\mathcal{W}_0\tilde{\mathcal{G}}_{k+1}\mathcal{W}_0 + q^{-1}\tilde{\mathcal{G}}_{k+1}\mathcal{W}_0^2}{(q - q^{-1})(q^2 - q^{-2})}, \\
\Delta_{k+1} & \rightarrow \Delta_{k+1}.
\end{align*}
Moreover $S^{-1}$ sends
\begin{align*}
\mathcal{W}_{-k} & \rightarrow \mathcal{W}_{-k}, \\
\mathcal{W}_{k+1} & \rightarrow \mathcal{W}_{k+1} + \frac{q^{-1}\mathcal{W}_0^2\mathcal{W}_{k+1} - (q + q^{-1})\mathcal{W}_0\mathcal{W}_{k+1}\mathcal{W}_0 + q\mathcal{W}_{k+1}\mathcal{W}_0^2}{(q - q^{-1})(q^2 - q^{-2})}, \\
\mathcal{G}_{k+1} & \rightarrow \mathcal{G}_{k+1} + \frac{q^{-1}\mathcal{W}_0^2\mathcal{G}_{k+1} - (q + q^{-1})\mathcal{W}_0\mathcal{G}_{k+1}\mathcal{W}_0 + q\mathcal{G}_{k+1}\mathcal{W}_0^2}{(q - q^{-1})(q^2 - q^{-2})}, \\
\tilde{\mathcal{G}}_{k+1} & \rightarrow \tilde{\mathcal{G}}_{k+1} + \frac{q^{-1}\mathcal{W}_0^2\tilde{\mathcal{G}}_{k+1} - (q + q^{-1})\mathcal{W}_0\tilde{\mathcal{G}}_{k+1}\mathcal{W}_0 + q\tilde{\mathcal{G}}_{k+1}\mathcal{W}_0^2}{(q - q^{-1})(q^2 - q^{-2})} = \mathcal{G}_{k+1}, \\
\Delta_{k+1} & \rightarrow \Delta_{k+1}.
\end{align*}
\end{proposition}

\begin{proof}
By Theorem 5.4, Lemma 7.2 and Examples 4.8 and 4.9.
\end{proof}
8 Finite-dimensional $\mathcal{O}_q$-modules

Recall the generators $A$, $B$ for the $q$-Onsager algebra $\mathcal{O}_q$. Throughout this section $V$ denotes a finite-dimensional irreducible $\mathcal{O}_q$-module on which $A$ and $B$ are diagonalizable. To avoid trivialities, we always assume that $V$ has dimension at least 2. We describe what happens when $V$ is twisted via the Lusztig automorphism $L$ of $\mathcal{O}_q$. By [23, Theorem 3.10] the elements $A$, $B$ act on $V$ as a tridiagonal pair. The tridiagonal pair concept is defined in [16, Definition 1.1], and described further in [15, 17, 18, 24]. In what follows, we freely invoke the notation and theory of tridiagonal pairs. Fix a standard ordering in [16, Definition 1.1], and described further in [15, 17, 18, 24]. In what follows, we freely invoke the notation and theory of tridiagonal pairs. Fix a standard ordering in [16, Definition 1.1], and described further in [15, 17, 18, 24]. In what follows, we freely invoke the notation and theory of tridiagonal pairs. Fix a standard ordering in [16, Definition 1.1], and described further in [15, 17, 18, 24].

By [16, Lemma 2.4] the following hold for $0 \leq i, j \leq d$:

$$E_i^* A E_j^* = \begin{cases} 0 & \text{if } |i - j| > 1; \\ \neq 0 & \text{if } |i - j| = 1 \end{cases}$$

$$E_i B E_j = \begin{cases} 0 & \text{if } |i - j| > 1; \\ \neq 0 & \text{if } |i - j| = 1. \end{cases}$$

(29)

The following result can be found in [16, Theorem 11.2]; we give a short proof for the sake of completeness.

**Lemma 8.1.** (See [16, Theorem 11.2].) There exist nonzero $a, b \in \mathbb{F}$ such that

$$\theta_i = a q^{d - 2i} + a^{-1} q^{2i - d}, \quad \theta_i^* = b q^{d - 2i} + b^{-1} q^{2i - d}$$

(30)

for $0 \leq i \leq d$.

**Proof.** We verify the equation on the left in (30). For $0 \leq i, j \leq d$ we multiply each side of (11) on the left by $E_i$ and on the right by $E_j$. Simplify the result to get

$$0 = E_i B E_j (\theta_i - \theta_j) \left( \theta_i^2 - (q^2 + q^{-2}) \theta_i \theta_j + \theta_j^2 + (q^2 - q^{-2})^2 \right).$$

(31)

Now assuming $|i - j| = 1$ and using $E_i B E_j \neq 0$,

$$0 = \theta_i^2 - (q^2 + q^{-2}) \theta_i \theta_j + \theta_j^2 + (q^2 - q^{-2})^2.$$  

(32)

Let $p(i, j)$ denote the right-hand side of (32). For $1 \leq j \leq d - 1$,

$$\theta_{j-1} - (q^2 + q^{-2}) \theta_j + \theta_{j+1} = \frac{p(j - 1, j) - p(j, j + 1)}{\theta_{j-1} - \theta_{j+1}}$$

$$= 0.$$
By the above recurrence there exist \( u, v \in \mathbb{F} \) such that
\[ \theta_i = uq^{d-2i} + vq^{2i-d} \quad (0 \leq i \leq d). \] (33)
Since \( d \geq 1 \) we have the equation \( 0 = p(0, 1) \). Evaluate this equation using (33) to obtain \( uv = 1 \). This yields the equation on the left in (30). The equation on the right in (30) is similarly obtained.

**Definition 8.2.** Define
\[ t_i = a^{2i}q^{2i(d-i)} \quad (0 \leq i \leq d). \]
The following calculation will be useful.

**Lemma 8.3.** For \( 0 \leq i, j \leq d \) such that \(|i - j| \leq 1\),
\[ \frac{t_j}{t_i} = 1 + \frac{q\theta_i^2 - (q + q^{-1})\theta_i\theta_j + q^{-1}\theta_j^2}{(q - q^{-1})(q^2 - q^{-2})} \] (34)
and
\[ \frac{t_i}{t_j} = 1 + \frac{q^{-1}\theta_i^2 - (q + q^{-1})\theta_i\theta_j + q\theta_j^2}{(q - q^{-1})(q^2 - q^{-2})}. \] (35)

**Proof.** Each side of (34) is equal to \( q^{4i-2d-2}a^{-2} \) (if \( i - j = 1 \)), 1 (if \( i = j \)), and \( q^{2d+2-4i}a^2 \) (if \( i - j = -1 \)). Each side of (35) is equal to \( q^{2d+2-4i}a^2 \) (if \( i - j = 1 \)), 1 (if \( i = j \)), and \( q^{4j-2d-2}a^{-2} \) (if \( i - j = -1 \)).

**Definition 8.4.** Define
\[ \Psi = \sum_{i=0}^{d} t_i E_i. \] (36)

**Lemma 8.5.** The map \( \Psi \) is invertible, and
\[ \Psi^{-1} = \sum_{i=0}^{d} t_i^{-1} E_i. \] (37)

**Proof.** Since \( I = \sum_{i=0}^{d} E_i \) and \( E_i E_j = \delta_{i,j} E_i \) for \( 0 \leq i, j \leq d \).

**Theorem 8.6.** For \( X \in \mathcal{O}_q \) the following holds on \( V \):
\[ L(X) = \Psi^{-1}X\Psi. \] (38)

**Proof.** It suffices to show \( L(A) = \Psi^{-1}A\Psi \) and \( L(B) = \Psi^{-1}B\Psi \). Certainly \( L(A) = \Psi^{-1}A\Psi \), since \( L(A) = A \) and \( A \) commutes with \( \Psi \). We now verify \( L(B) = \Psi^{-1}B\Psi \). Since \( I = \sum_{i=0}^{d} E_i \) it suffices to show \( E_i L(B) E_j = E_i \Psi^{-1}B\Psi E_j \) for \( 0 \leq i, j \leq d \). Let \( i, j \) be given. Using the definition (3) of \( L \), one finds that \( E_i L(B) E_j \) is equal to \( E_i B E_j \) times the scalar on the right in (34). Using Definition 8.4 one finds that \( E_i \Psi^{-1}B\Psi E_j \) is equal to \( E_i B E_j \) times the scalar on the left in (34). For the moment assume \(|i - j| \leq 1\). Then \( E_i L(B) E_j = E_i \Psi^{-1}B\Psi E_j \) by Lemma 8.3. Next assume \(|i - j| > 1\). Then \( E_i L(B) E_j = E_i \Psi^{-1}B\Psi E_j \) since \( E_i B E_j = 0 \). We have shown \( L(B) = \Psi^{-1}B\Psi \).
In Lemma 8.3 we related the parameters \( \{ t_i \}_{i=0}^{d} \) and \( \{ \theta_i \}_{i=0}^{d} \). We now give a more general result along this line.

**Theorem 8.7.** For \( 0 \leq i, j \leq d \) we have

\[
\frac{t_j}{t_i} = 1 + \sum_{n=1}^{\infty} \left( \frac{\theta_i - \theta_j}{q - q^{-1}} \prod_{r=1}^{n-1} \left( q^{2r} - q^{-2r} \right)^2 \right) \frac{q^n \theta_i - q^{-n} \theta_j}{q^{2n} - q^{-2n}}
\]

and

\[
\frac{t_i}{t_j} = 1 + \sum_{n=1}^{\infty} \left( \frac{\theta_i - \theta_j}{q - q^{-1}} \prod_{r=1}^{n-1} \left( q^{2r} - q^{-2r} \right)^2 \right) \frac{q^{-n} \theta_i - q^n \theta_j}{q^{2n} - q^{-2n}}.
\]

Moreover, in the above sums the large parenthetical expression is zero for \( n > |i - j| \).

**Proof.** We verify the first displayed equation. Since the \( \mathcal{O}_q \)-module \( V \) is irreducible, there exists \( X \in \mathcal{O}_q \) such that \( E_i X E_j \neq 0 \). For this \( X \), equation (38) holds on \( V \). In equation (38), multiply each side on the left by \( E_i \) and on the right by \( E_j \). Evaluate the results using (5), (36), (37) together with \( E_i X E_j \neq 0 \). This yields the first displayed equation after a brief calculation. The second displayed equation is similarly obtained using \( L^{-1}(X) = \Psi X \Psi^{-1} \). The last assertion of the theorem statement can be checked directly using (30).

**Note 8.8.** It is natural to ask how we discovered Theorem 2.1. The answer is that we first discovered Theorem 8.7, and then considered the implications for \( L \).

9 Acknowledgment

The author thanks Pascal Baseilhac and Stefan Kolb for sharing their preprint [11] prior to publication.

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