A Practical Approach to Interval Refinement for \texttt{math.h}/\texttt{cmath} Functions

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Abstract. Verification of C/C++ programs has seen considerable progress in several areas, but not for programs that use these languages' mathematical libraries. The reason is that all libraries in widespread use come without no guarantees about the computed results. This would seem to prevent any attempt at formal verification of programs that use them: without a specification for the functions, no conclusion can be statically drawn about the behavior of the program. We propose an alternative to surrender: even if we do not have a proper specification for the functions, we might have a partial specification; at the very least we have their implementation, for which a partial specification can sometimes be extracted. When even a partial specification is unavailable, we can still detect flaws in the program and facilitate its partial verification with testing. We propose a pragmatic, practical approach that leverages the fact that most \texttt{math.h}/\texttt{cmath} functions are almost piecewise monotonic: they may have glitches, often of very small size and in small quantities. We develop interval refinement techniques for such functions that enable verification via abstract interpretation, symbolic model checking and constraint-based test data generation.

1 Introduction

The use of floating-point computations for the implementation of critical systems is perceived as increasingly acceptable. Even in modern avionics, one of the most critical domains for software, floating-point numbers are now used, more often than not, instead of fixed-point arithmetic \cite{BDL12}.

Acceptance of floating-point computations in the design of critical systems was facilitated by the wide adoption of significant portions of the IEEE 754 standard for binary floating-point arithmetic \cite{IEEE754}. Most implementations of the C and C++ programming languages provide floating-point data types that conform to IEEE 754 as far as the basic arithmetic operations and the conversions are concerned. Some implementations push compliance further by providing other
operations required by IEEE 754, such as correctly rounded fused multiply-add\(^5\) and square root \cite{IEEE08} Section 5).

The C and C++ floating-point mathematical functions are part of the standard libraries of the respective languages. Different versions of the language standards define different, though backwards-compatible, sets of basic mathematical functions (see, e.g., \cite{Int11,Int14}). Access to such functions requires inclusion of the `math.h` header file in C, or the `cmath` header file in C++. While the IEEE 754 provides recommendations for these functions \cite{IEEE08} Section 9.2, no C/C++ implementation we know of complies to the recommendation that such functions be correctly rounded. An exception is CR-LIBM\(^6\), a truly remarkable research prototype that, however, has still not found industrial adoption, possibly because of the worst-case performance of some functions (the average case being usually very good). Another freely available library of correctly-rounded functions is libmcr\(^7\) by Sun Microsystems, Inc. It provides some double-precision transcendental elementary functions (\texttt{exp}, \texttt{log}, \texttt{pow}, \texttt{sin}, \texttt{cos}, \texttt{tan}, \texttt{atan}), but its development stopped in 2004. Even though we cannot exclude the existence of proprietary implementations of \texttt{libm} providing formalized precision guarantees, we were not able to find one.

For comparison, Numeric Annex (G) of the Ada Reference Manual \cite{TDB+06} prescribes that, when the result of the evaluation of an elementary function does not result into an exception, the numerical result belongs to an interval defined as the smallest floating-point interval in the format that contains all the values of the form \(f \cdot (1 + d)\), where

- \(f\) is the exact value of the corresponding mathematical function at the given parameter values;
- \(d\) is a real number such that \(|d|\) is less than or equal to the function’s maximum relative error.

Each elementary function is then given with a maximum relative error. This is expressed in terms of the “relative error of the format”, \(E_{\text{rel}}\), defined as the smallest positive number in the format satisfying \(1.0 \oplus E_{\text{rel}} \neq 1.0\), where \(\oplus\) denotes floating-point addition. Then:

- \texttt{sqrt}, \texttt{sin} and \texttt{cos} have a maximum relative error of \(2 \cdot E_{\text{rel}}\) (but see below for the restrictions concerning periodic functions);
- \texttt{log}, \texttt{exp}, \texttt{tan} and \texttt{cot} have a maximum relative error of \(4 \cdot E_{\text{rel}}\);
- the forward and inverse hyperbolic functions have a maximum relative error of \(8 \cdot E_{\text{rel}}\);
- the maximum relative error for exponentiation \texttt{x**y} depends on the parameter values \(x\) and \(y\): \(4 + |y \log(x)|/32 \cdot E_{\text{rel}}\).

\(^{5}\) Fused multiply-add applied to \(x, y\) and \(z\) yields \(x \cdot y + z\) rounded as per the current rounding mode.

\(^{6}\) See http://lipforge.ens-lyon.fr/www/crlibm/ last accessed on October 15th, 2016.

\(^{7}\) See https://github.com/simonbyrne/libmcr/blob/master/README last accessed on October 15th, 2016.
Note, though, that the maximum relative error given above for the forward
trigonometric functions applies only when the absolute value of the angle pa-
rameter is less than or equal to some implementation-defined angle threshold.
Finally, the following specifications also take precedence over the maximum rel-
ative error bounds:

– the absolute value of the result of the \( \sin \), \( \cos \), and \( \tanh \) functions never
  exceeds one;
– the absolute value of the result of the \( \coth \) functions is never less than one;
– the result of the \( \cosh \) functions is never less than one.

Summarizing, differently from the C and the C++ standard, the Ada standard
gives some guarantees about the result of mathematical functions, but such a
partial specification is still problematic from the point of view of formal verifica-
tion. In the C/C++ world, the only implementation we know that says something
about errors is GNU libc [LSM+16]:

"Therefore many of the functions in the math library have errors. The
table lists the maximum error for each function which is exposed by one
of the existing tests in the test suite. The table tries to cover as much as
possible and list the actual maximum error (or at least a ballpark figure)
but this is often not achieved due to the large search space."

So, nothing that can really be trusted in a safety-critical context. In the em-
bedded world, we checked the documentation of all major toolchain providers
and for two of them we found that the lack of guarantees is explicit mentioned,
whereas for all the others nothing is said in this regard. It is to mitigate this
state of affairs that test-suites like C11/C99 FPCE Test Suite by Tydeman Con-
sulting have been created. Solid Sands’ SuperTest, one of the leading test and
validation suites for C and C++ compilers, has been recently extended with many
more tests for the mathematical libraries.

To illustrate the concrete problem raised by the use of transcendental func-
tions in program verification settings, consider the code reproduc-
ed in Listing 1.1. It is a reduced version of a real-world example extracted from a critical
embedded system.\(^8\) Some of the questions to be answered for this code are:

1. Can infinities and NaNs be generated?
2. Can \( \sin \), \( \cos \), and \( \tan \) be invoked on ill-conditioned arguments?
3. If so, which inputs to the given functions may cause such anomalies?

Concerning the second question, we call the argument of a floating-point trigono-
metric function ill-conditioned if its absolute value exceeds some application-
dependent threshold. Ideally, this threshold should be just above \( \pi \). To under-
stand this often-overlooked programing error, consider that, if \( x \) is an IEEE 754

\(^8\) http://www.tybor.com/ last accessed on October 23rd, 2016.
\(^9\) http://www.solidsands.nl/ last accessed on October 23rd, 2016.
\(^10\) The original source code is available at http://paparazzi.enac.fr, Paparazzi UAV
(Unmanned Aerial Vehicle), v5.10.0, stable release, file sw/misc/satcom/tcp2ivy.c
last accessed on October 15th, 2016.
Listing 1.1. Code excerpted from a real-world avionic library

```c
#include <math.h>
#include <stdint.h>

#define RadOfDeg(x) ((x) * (M_PI/180.))
#define E 0.08181919106 /* Computation for the WGS84 geoid only */
#define LambdaOfUtmZone(utm_zone) RadOfDeg((utm_zone-1)*6-180+3)
#define CScal(k, z) { z.re *= k; z.im *= k; }
#define CAdd(z1, z2) { z2.re += z1.re; z2.im += z1.im; }
#define CSub(z1, z2) { z2.re -= z1.re; z2.im -= z1.im; }
#define CI(z) {
    float tmp = z.re; z.re = - z.im; z.im = tmp; 
}
#define CExp(z) {
    float e = exp(z.re); z.re = e*cos(z.im); \
    z.im = e*sin(z.im); 
}
#define CSin(z) { CI(z); struct complex _z = {-z.re, -z.im}; \
    float e = exp(z.re); float cos_z_im = cos(z.im); z.re = e*cos_z_im; \
    float sin_z_im = sin(z.im); z.im = e*sin_z_im; _z.re = cos_z_im/e; \
    _z.im = -sin_z_im/e; CSub(_z, z); CScal(-0.5, z); CI(z); }

static inline float isometric_latitude(float phi, float e) {
    return log(tan(M_PI_4 + phi / 2.0)) - e / 2.0 * log((1.0 + e * sin(phi)) / (1.0 - e * sin(phi)));
}

static inline float isometric_latitude0(float phi) {
    return log(tan(M_PI_4 + phi / 2.0));
}

void latlong_utm_of(float phi, float lambda, uint8_t utm_zone) {
    float lambda_c = LambdaOfUtmZone(utm_zone);
    float ll = isometric_latitude(phi, E);
    float dl = lambda - lambda_c;
    float phi_ = asin(sin(dl) / cosh(ll));
    float ll_ = isometric_latitude0(phi_);
    float lambda_ = atan(sinh(ll) / cos(dl));
    struct complex z_ = { lambda_, ll_ }; CScal(serie_coeff_proj_mercator[0], z_);
    uint8_t k;
    for(k = 1; k < 3; k++) {
        struct complex z = { lambda_, ll_ }; CScal(2*k, z); CSin(z); CScal(serie_coeff_proj_mercator[k], z); CAdd(z, z_);
    }
    CScal(N, z_);
    latlong_utm_x = XS + z_.im;
    latlong_utm_y = z_.re;
}
```
single-precision number and \( x \geq 2^{23} \), then the smallest single-precision range containing \([x, x + 2\pi]\) contains no more than three floating-point numbers. Current implementations of floating-point trigonometric functions contain precise range reduction algorithms that have no problem computing a very accurate result even for numbers much higher than \( 2^{23} \); the point is that the input is so inaccurate that the very accurate result is, from the point of view of the application, indistinguishable from a pseudo-random number.

We do not have a precise specification for the library functions that are assumed in the code of Listing 1.1. While the project it comes from is said to have been designed from the beginning with portability in mind, the sources and build machinery refer to GNU libc and to Newlib, but no specific versions are mentioned. We can probably assume a POSIX-compliant behavior with respect to special values, e.g.: if \( \log() \) is called with a negative number, a NaN is returned; if \( \text{atan}() \) is called with \( \pm 1 \) then an infinity is returned [IEE13]. This information is not sufficient in order to provide a general answer to the verification questions asked above. Things change if, in addition to assuming a compilation mode that strictly adheres to IEEE 754, we fix a specific implementation of the mathematical library.

In this paper, we propose a pragmatic, practical approach that enables verification of C/C++ programs using \texttt{math.h/cmath} functions, even with minimal or no specification in addition to the special cases mandated by the standards the implementation conforms to, e.g., POSIX [IEE13]. We observed that, for all the implementations of \texttt{libm} we tested, the piecewise monotonicity property of the corresponding real functions is almost preserved. Consider, for instance the \texttt{tanhf()} function, which is meant to approximate the hyperbolic tangent function over IEEE 754 single-precision floats. While \( y = \tanh(x) \) is monotonically (strictly) increasing over \(( -\infty, +\infty )\), \( y = \tanhf(x) \) can be monotonically non-decreasing over the full range of IEEE 754 single-precision floats, for some rounding mode, or it can be “almost monotonically non-decreasing.” By this we mean that, going from \(-\infty\) to \(+\infty\), there may be an occasional drop in the graph of \texttt{tanhf()}, but this is quickly recovered from, that is, the function starts increasing again. We use the term glitches to name such occasional drops: we observed that glitches are often shallow (most often just one ULP), narrow (most often just 2 ULPs), and, on average, not very frequent. We leverage this fact to provide general interval refinement algorithms that enable software verification with various techniques in all cases:

- If we have approximate but correct information about the maximal depth and width of glitches and, possibly, their number and their localization, then we can guarantee containment for the refined intervals, and this allows us to perform formal verification via abstract interpretation, symbolic model checking or automatic theorem proving:

11 Substitute \( 2^{23} \) with \( 2^{52} \) and the same holds for IEEE 754 double-precision numbers.
12 See [http://wiki.paparazziuav.org/](http://wiki.paparazziuav.org/) last accessed on October 20th, 2016.
13 See [https://sourceware.org/newlib/](https://sourceware.org/newlib/) last accessed on October 20th, 2016.
with precise information and small/few glitches (which includes the case
where there are no glitches at all, which, in turn, includes the case of
correctly rounded implementations), the refinement will result in tight in-
tervals, verification will be computationally cheaper and with less “don’t
knows”;
• with less precise information, verification is still possible, at a higher cost
and possibly more “don’t knows”.

– With possibly incorrect information about the presence and nature of glitches,
we can still automatically generate test input data. Things can go wrong in
two ways:
• the refined intervals are too tight, in which case we might fail to generate
test inputs for paths and conditions that are indeed feasible;
• the refined intervals are too lax, in which case we might generate test
inputs that fail to exercise the paths or conditions they are supposed to
exercise.

In both cases, this is not a big deal if we are still able to generate lots of
useful and accurate test inputs.

The plan of the paper is as follows: Section 2 recalls basic definitions and
introduces the required notation; Section 3 introduces the notions of monotonicity
glitch and quasi-monotonicity; Section 4 describes direct and indirect propaga-
tion algorithms that are able to deal with (at least) 75 of the math.h/cmath func-
tions; Section 5 explains how trigonometric functions, which are periodic, can
be treated by partitioning a subset of their graph into a set of quasi-monotonic
branches; Section 6 briefly describes the implementation in the context of the
ECLAIR software verification platforms and illustrates the result of an exper-
iment; Section 7 discusses the problems that remain to be solved and sketches
several ideas for future work; Section 8 concludes the main part of the paper. Ap-
pendix A presents additional data about the glitches in the single-precision func-
tions for several implementations of the math.h/cmath mathematical functions;
Appendices B and C contain more details on the algorithms of Section 4 along
with formal proofs of correctness and complexity results. Finally Appendix D
contains some theoretical and practical results concerning the implementation
of the interval-refinement algorithms for trigonometric functions.

2 Preliminaries

2.1 Floating-Point Numbers and Intervals

We will denote by \( \mathbb{R}_+ \) and \( \mathbb{R}_- \) the sets of strictly positive and strictly negative
real numbers, respectively.

Definition 1. (IEEE 754 binary floating-point numbers.) A set of IEEE 754
binary floating-point numbers \( \text{IEEE} \) is uniquely identified by: \( p \in \mathbb{N}, \) the num-
ber of significant digits (precision); \( \epsilon_{\text{max}} \in \mathbb{N}, \) the maximum exponent; \( \epsilon_{\text{min}} \in \mathbb{N}, \)
the minimum exponent (this can be either \( 1 - \epsilon_{\text{max}} \) or \( -\epsilon_{\text{max}} \), the former being preferred). The set of binary floating-point numbers \( \mathbb{F}(p, \epsilon_{\text{max}}, \epsilon_{\text{min}}) \) includes:
– all signed zero and non-zero numbers of the form \((-1)^s \cdot 2^e \cdot m\), where
  • \(s\) is the sign bit;
  • the exponent \(e\) is any integer such that \(e_{\text{min}} \leq e \leq e_{\text{max}}\);
  • the mantissa \(m\), with \(0 \leq m < 2^p\), is a number represented by a string of \(p\) binary digits with a “binary point” after the first digit:

\[ m = (d_0 \cdot d_1 d_2 \ldots d_{p-1})_2 = \sum_{i=0}^{p-1} d_i 2^{-i}; \]

– the infinities \(+\infty\) and \(-\infty\).

The smallest positive normal floating-point number is \(f_{\text{nor}} := 2^{e_{\text{min}}}\) and the largest is \(f_{\text{max}} := 2^{e_{\text{max}}} (2 - 2^{1-p})\). The non-zero floating-point numbers whose absolute value is less than \(2^{e_{\text{min}}}\) are called subnormals: they always have fewer than \(p\) significant digits. Every finite floating-point number is an integral multiple of the smallest subnormal magnitude \(f_{\text{min}} := 2^{e_{\text{min}}+1-p}\). Note that the signed zeroes \(+0\) and \(-0\) are distinct floating-point numbers.

Each IEEE 754 binary floating-point format also includes the representation of symbolic data called NaNs, from “Not a Number.” There are quiet NaNs, which are propagated by most operations without signaling exceptions, and signaling NaNs, which cause signaling invalid operation exceptions. The unintended and unanticipated generation of NaNs in a program (e.g., by calling the \(\log\) function on a negative number) is a serious programming error that could lead to catastrophic consequences.

In the sequel we will only be concerned with IEEE 754 binary floating-point numbers and we will write simply \(\mathbb{F}\) for \(\mathbb{F}(p, e_{\text{max}}, e_{\text{min}})\) when there is no risk of confusion.

**Definition 2. (Floating-point symbolic order.)** Let \(\mathbb{F}\) be any IEEE 754 floating-point format. The relation \(<\subseteq \mathbb{F} \times \mathbb{F}\) is such that, for each \(x, y \in \mathbb{F}\), \(x < y\) if and only if either: \(x = -\infty\) and \(y \neq -\infty\), or \(x \neq +\infty\) and \(y = +\infty\), or \(x = -0\) and \(y \in \{0\} \cup \mathbb{R}_+\), or \(x \in \mathbb{R}_- \cup \{-0\}\) and \(y = +0\), or \(x, y \in \mathbb{R}\) and \(x < y\). The function \((\cdot)^+ : (\mathbb{F} \setminus \{+\infty\}) \to \mathbb{F}\) is defined, for each \(x \in \mathbb{F} \setminus \{+\infty\}\), by \(x^+ := \min\{ y \in \mathbb{F} \mid x < y \}\). Similarly, function \((\cdot)^- : (\mathbb{F} \setminus \{-\infty\}) \to \mathbb{F}\) is defined, for each \(y \in \mathbb{F} \setminus \{-\infty\}\), by \(y^- := \max\{ x \in \mathbb{F} \mid x < y \}\). We will iteratively apply these functions, so that, e.g., for each \(n \in \mathbb{N}\), we will refer to the partial function \(((\cdot)^+)^n : \mathbb{F} \to \mathbb{F}\) given, for each \(x \in \mathbb{F}\), by

\[ \begin{cases} 
  x^{+0} := x; \\
  x^{+n+1} := (x^{+n})^+, \text{ if } x^{+n} \neq +\infty;
\end{cases} \]

The partial order \(<\subseteq \mathbb{F} \times \mathbb{F}\) is such that, for each \(x, y \in \mathbb{F}\), \(x \leq y\) if and only if either \(x < y\) or \(x = y\). We use the notation \(x >^n y\) to signify that \(x > y^{+n}\).

Note that \(\mathbb{F}\) is linearly ordered with respect to ‘<’.
For $x \in \mathbb{F}$, we will sometimes confuse the floating-point number with the extended real number it represents, the floats $-0$ and $+0$ both corresponding to the real number 0. Thus, when we write, e.g., $x < y$ we mean that $x$ is numerically less than $y$ (for example, we have $-0 \prec +0$ although $-0 \nleq +0$, but note that $x \leq y$ implies $x \leq y$ if $x$ and $y$ are finite).

**Definition 3. (Floating-point intervals.)** Let $\mathbb{F}$ be any IEEE 754 floating-point format. The set $\mathcal{I}_\mathbb{F}$ of floating-point intervals with boundaries in $\mathbb{F}$ is

$$\mathcal{I}_\mathbb{F} := \{ \emptyset \} \cup \{ [l, u] \mid l, u \in \mathbb{F}, l \preceq u \}.$$ 

$[l, u]$ denotes the set $\{ x \in \mathbb{F} \mid l \preceq x \preceq u \}$. $\mathcal{I}_\mathbb{F}$ is a bounded meet-semilattice with least element $\emptyset$, greatest element $[-\infty, +\infty]$, and the meet operation, which is induced by set-intersection, will be simply denoted by $\cap$.\footnote{A **bounded meet-semilattice** is a partially ordered set that has a meet (or greatest lower bound) for any nonempty finite subset and a greatest element.}

Floating-point intervals with boundaries in $\mathbb{F}$ allow to capture the extended numbers in $\mathbb{F}$: NaNs should be tracked separately.

Given a floating-point interval $[l, u] \in \mathcal{I}_\mathbb{F}$, we denote by $\#[l, u]$ the cardinality of the set $\{ x \in \mathbb{F} \mid l \preceq x \preceq u \}$.

### 2.2 Constraint Solving over Floating-Point Variables

We will now briefly recall the basic principles of interval-based consistency techniques over floating-point variables and constraints. We will do so by means of an example taken from line 31 of the listing in Listing 1.1, that is

```plaintext
31 float phi_ = asin(sin(dl) / cosh(ll));
```

Program analysis usually starts with the generation of an intermediate code representation in a form called *three-address code* (TAC). In this form, complex expressions and assignments are decomposed into sequences of assignment instructions where at most one operator is applied. Taking into account the implicit conversions, the above expression is transformed into

```plaintext
1 float phi_; double z1, z2, z3, z4, z5, z6;
2 z1 = (double) dl;
3 z2 = sin(z1);
4 z3 = (double) ll;
5 z4 = cosh(z3);
6 z5 = z2 / z4;
7 z6 = asin(z5);
8 phi_ = (float) z6;
```

A further transformation consists in ensuring that each variable is assigned to only once by labeling each assigned variable with a fresh name. In the resulting form, called *static single assignment form* (SSA), assignments can be considered as if they were equality constraints. This step can be avoided in our present
example and we can directly regard the lines with assignments as a system of constraints over floating-point numbers.

One approach to the solution of such systems is called \textit{interval-based consistency} and it amounts to iteratively narrow the floating-point intervals associated with each variable in a process called \textit{filtering}. In the literature on constraint propagation, the unary constraints associated to variables, e.g., intervals, are also called \textit{labels} [Dav87]. A \textit{projection} is a function that, given a constraint with \( n \) variables and the intervals associated to them, computes a possibly refined interval for one of the variables, i.e., an interval that is tighter than or equal to the original interval associated to that variable. For example, considering \( z_5 = z_2 / z_4 \), the projection over \( z_5 \) is called \textit{direct projection} (it goes in the same sense of the TAC assignment it comes from), while the projections over \( z_2 \) and \( z_4 \) are called \textit{indirect projections}. In this variant of constraint propagation, both direct and indirect projections are repeatedly applied in order to refine the intervals associated to the variables. We will call \textit{propagator} the entity that computes a projection and possibly refines an interval. The application of projections by means of propagators is governed by heuristic algorithms that go beyond the scope of this paper: for our purposes, suffices it to say that, whenever the interval associated to a variable is refined, all the propagators are inserted into a data structure of propagators that are ready to run; heuristics are used to select and remove from the data structure one of the ready propagators, which is then run; if that results into the refinement of the interval of one variable, all the propagators that depend on that variable are inserted into the same data structure, unless they are already present. This process continues until one of the intervals becomes empty, in which case propagation can be stopped as unsatisfiability of the given system of constraints has been proved, or the data structure becomes empty, i.e., propagation has reached \textit{quiescence} as no projection is able to infer more information, or propagation is artificially stopped, e.g., because a timeout has expired.

Let us see how this can be used for program verification. As a first example, let us consider the question whether the division \( z_5 = z_2 / z_4 \) can give rise to a division by zero. Assume all the intervals are initially full, i.e., they contain all possible numerical floating-point values and all propagators are ready to run. We modify the interval associated to \( z_4 \) to \([-0, +0]\) and start propagation. At some stage the inverse propagator for \texttt{cosh} will be called to possibly refine the interval for \( z_3 \) starting from the interval of \( z_4 \): a correct propagator correctly capturing a decent implementation of \texttt{cosh} will refine the label of \( z_3 \) to the empty interval, thus proving that division by zero is indeed not possible.\footnote{As we will see, all the implementations of \texttt{cosh()} we have examined are far from perfect, but none of them has a zero in its range.}

As another example, let us consider \( z_4 = \texttt{cosh}(z_3) \), and suppose the intervals associated to \( z_3 \) and \( z_4 \) are \([1, +\infty]\) and \([-\infty, +\infty]\), respectively. The direct projection for \texttt{cosh} would compute, on a machine we will later call \texttt{zoltan},\footnote{On \texttt{zoltan}, \texttt{float} and \texttt{double} are 32-bit and 64-bit IEEE 754 floating point numbers, respectively.}
the refining interval \([18B07551D9F550_{16} \cdot 2^{-52}, +\infty]\) for \(z4\), where we have \(18B07551D9F550_{16} \cdot 2^{-52} \approx 1.543\). Now suppose we want to determine for which values of \(z3\) the computation of \(z4 = \cosh(z3)\) results in an overflow, thereby binding \(z4\) to \(+\infty\). To answer this question we artificially refine the interval of \(z4\) to the the singleton \([+\infty, +\infty]\) and let the inverse propagator do its job: this will result into the refining interval \([1633CE8FB9F87E_{16} \cdot 2^{-43}, +\infty]\) for \(z3\), where \(1633CE8FB9F87E_{16} \approx 710.5\).

Suppose now we want to know whether a NaN can be generated by the invocation to \(\text{asin}\) in line 7, i.e., whether we can have \(z5 < -1\) or \(z5 > 1\). Let us concentrate on the latter constraint, which we impose together with the constraints saying that \(d1\) and \(ll\) are neither NaNs nor infinities; all the other variables can take any value. If we indicate with \(d_\ell\) and \(i_\ell\) the direct and the indirect projections for the constraint at line \(\ell\), respectively, here is what happens on a selected constraint propagation process on machine \(zoltan\), where the numbers have been rounded for increased readability:

\[
\begin{align*}
\xrightarrow{z5 > 1} & z5 \in [1.0000000000000002, 1.798 \cdot 10^{308}] \\
\xrightarrow{i_8} & z6 \in [+0, 1.571] \\
\xrightarrow{i_7} & z5 \in [-2^{-1074}, 1] \\
\xrightarrow{i_6} & z2 \in [-1.332 \cdot 10^{-15}, 1] \\
\xrightarrow{i_3} & z1 \in [-16, 15.71].
\end{align*}
\]

As the last constraint is unsatisfiable, the original constraint system is unsatisfiable. The same happens if \(z5 < -1\) is imposed, thereby proving that NaNs cannot be generated on line 7.

As a final example, in order to show the indirect projections for \(\text{asin}\) and \(\text{sin}\) at work, we consider a partial constraint propagation starting from state

\[
\begin{align*}
z1 & \in [-16, 16], & z5 & \in [-1, 1], \\
z2 & \in [-1, 1], & z6 & \in [-1.571, 1.571], \\
z4 & \in [1, 1.798 \cdot 10^{308}], & \phi_\ell & \in [+0, 1.571].
\end{align*}
\]

A possible sequence of propagation steps is the following:

\[
\begin{align*}
\xrightarrow{i_8} & z6 \in [+0, 1.571] \\
\xrightarrow{i_7} & z5 \in [-2^{-1074}, 1] \\
\xrightarrow{i_6} & z2 \in [-1.332 \cdot 10^{-15}, 1] \\
\xrightarrow{i_3} & z1 \in [-16, 15.71].
\end{align*}
\]

At this stage, the constraint system reaches quiescence and the labeling procedure comes into play: a variable is chosen and the corresponding interval, its
label, is divided into two or more parts and each part is searched independently. As soon as the interval for the chosen variable is instantiated to one of such parts, propagation restarts until a new quiescent state. Then labeling is performed again and so on. In this last example, after 7 labeling steps, a testcase is generated that falls off line 8 without generating NaN or infinities. This testcase is very simple indeed:

\[
\begin{align*}
\text{dl} &= +0, \\
\text{ll} &= +0, \\
\text{z1} &= +0, \\
\text{z2} &= +0, \\
\text{z3} &= +0, \\
\text{z4} &= 1, \\
\text{z5} &= +0, \\
\phi_i &= +0.
\end{align*}
\]

3 (Quasi-) Monotonicity and Glitches

A monotonic real-valued partial function \( \hat{f}: \mathbb{R} \to \mathbb{R} \) is called monotonic if it is order preserving —i.e., \( \hat{f}(x) \leq \hat{f}(y) \) whenever \( x \leq y \) and both \( \hat{f}(x) \) and \( \hat{f}(y) \) are defined—, in which case we call it isotonic, or if it is order reversing —i.e., \( \hat{f}(x) \geq \hat{f}(y) \) whenever \( x \leq y \) and both \( \hat{f}(x) \) and \( \hat{f}(y) \) are defined—, in which case we say \( \hat{f} \) is antitonic.

**Definition 4.** (Quasi-monotonicity.) Let \( I \subseteq \mathbb{F} \) be a floating-point interval and \( f: \mathbb{F} \to \mathbb{F} \) be a floating-point function meant to approximate a real-valued partial function \( \hat{f}: \mathbb{R} \to \mathbb{R} \). We say that \( f \) is quasi-monotonic/isotonic/antitonic on \( I \) if \( \hat{f} \) is always defined and monotonic/isotonic/antitonic on \( I \).

Let \( f: \mathbb{F} \to \mathbb{F} \) be a quasi-monotonic function. The best we can hope for is that \( f \) be monotonic over \((\mathbb{F}, \preceq)\) for all rounding modes. While this is often the case, it is not always the case: monotonicity is occasionally violated at spots we call monotonicity glitches.

**Definition 5.** (Monotonicity glitches.) Let \( f: \mathbb{F} \to \mathbb{F} \) be a quasi-isotonic function on \( I \subseteq \mathbb{F} \). An isotonicity glitch of \( f \) in \( I \) is an interval \([l, u] \subseteq I\) such that:

\[
\begin{align*}
\text{u} &\succ l^+ \quad \land \quad \forall x \in (l, u) : f(l) \succ f(x) \quad \land \quad f(l) \preceq f(u).
\end{align*}
\]

If \( f \) is quasi-antitonic, an antitonicity glitch of \( f \) in \( I \) is an isotonicity glitch of \(-f\) in \( I \). Isotonicity and antitonicity glitches are collectively called monotonicity glitches or, simply, glitches.

Let \( G = [l, u] \) be a monotonicity glitch of \( f \) in \( I \). The width and the depth of \( G \) are given, respectively, by

\[
\begin{align*}
\text{width}(G) &:= \# [l, u] - 1, \\
\text{depth}(G) &:= \# [m, f(u)] - 1, \quad \text{where } m = \min_{x \in (l, u)} f(x).
\end{align*}
\]

Note that, for each glitch \( G \), we have \( \text{width}(G) \geq 2 \) and \( \text{depth}(G) \geq 1 \). A glitch of \( f \) in \( I \) is called maximal if none of its proper supersets is a glitch of \( f \) in \( I \). Non-maximal glitches are also called sub-glitches.
See Figure 1 for an exemplification of these concepts: $G_1$ is a maximal glitch; $G_2$, being contained into $G_1$ is non-maximal; width($G_1$) = 5, depth($G_1$) = 4, width($G_2$) = depth($G_2$) = 2.

Fig. 1. An example of monotonicity glitches

| id    | CPU      | OS            | compiler | libm version |
|-------|----------|---------------|----------|--------------|
| alpha | x86_64   | Ubuntu 14.04  | GCC 4.8.4| EGLIBC 2.19  |
| gcc10  | POWER7   | Fedora 20     | GCC 4.8.1| GNU libc 2.18|
| gcc11  | POWER7   | AIX 7         | GCC 4.8.1|              |
| gcc12  | POWER7   | Fedora 21     | GCC 4.9.2| GNU libc 2.20|
| gcc13  | AArch64  | Ubuntu 14.04  | GCC 4.8.4| EGLIBC 2.19  |
| igor   | x86_64   | Fedora 12     | GCC 4.4.4| GNU libc 2.11.2|
| macbook| x86_64   | Mac OS X 10.10.5| LLVM 6.1.0| Libm-3086.1  |
| raspi  | ARMv6 + VFPv2 | Raspbian Jessie | GCC 4.9.2| GNU libc 2.19|
| zoltan | x86_64   | Ubuntu 16.04  | GCC 5.4.0| GLIBC 2.23   |

Tables 1–3 show the relevant statistics about glitches for 25 functions provided by libm on several implementations. The tested implementations are introduced in Table 1 for each implementation the table lists its identification code, which is used in the other tables, the CPU architecture, the operating system, and, where known, the libm version.
Table 2. Glitch data for the gcc110/2/3 machines

| function | \(D_{\min}\) | \(D_M\) | near | up | down | zero |
|----------|----------------|-----------|------|----|-----|------|
| acosf    | -1             | 1         |      |    |     |      |
| acoshf   | 1 \(\infty\)   |           |      |    |     |      |
| asinf    | -1             | 1         |      |    |     |      |
| asinhf   | -\(\infty\) \(\infty\) |      |      |    |     |      |
| atanf    | -\(\infty\) \(\infty\) | 1 10^8 |      |    |     |      |
| atanhf   | -1             | 1         |      |    |     |      |
| cbrtf    | -\(\infty\) \(\infty\) 10^6 |      | 2 1 | 2 1 |     |      |
| coshf    | -\(\infty\) 456 | 1 2 462 | 2 442 | 1 | 2 10^6 |      |
| erff     | -\(\infty\) \(\infty\) |      |      |    |     |      |
| expf     | -\(\infty\) \(\infty\) |      |      |    |     |      |
| exp10f   | -\(\infty\) \(\infty\) |      |      |    |     |      |
| exp2f    | -\(\infty\) \(\infty\) | 2 1 | 2 |      |      |      |
| expmf    | -\(\infty\) \(\infty\) |      |      |    |     |      |
| lgammaf  | 2 \(\infty\) |      |      |    |     |      |
| logf     | 0 \(\infty\) |      |      |    |     |      |
| log10f   | 0 \(\infty\) |      |      |    |     |      |
| log1pf   | -1 \(\infty\) |      | 1 1 | 2 1 | 2 1 |      |
| log2f    | 0 \(\infty\) |      |      |    |     |      |
| sinhf    | -\(\infty\) \(\infty\) |      |      |    |     |      |
| sqrtf    | 0 \(\infty\) |      |      |    |     |      |
| tanhf    | -\(\infty\) \(\infty\) |      | 1 1 | 2 1 | 3 |      |
| tgammaf  | 2 \(\infty\) 10^4 | 2 3 | 10^4 | 4 4 | 10^4 | 2 3 10^4 | 3 4 |      |
| cosf     | -2^{23} 2^{23} | 10^4 | 1 3 | 10^4 | 1 3 10^4 | 1 3 |      |
| sinf     | -2^{23} 2^{23} |      |      |    |     |      |
| tanf     | -2^{23} 2^{23} |      |      |    |     |      |
Table 3. Glitch data for the zoltan machine

| function | $D_{min}$ | $D_{M}$ | near | up | down | zero |
|----------|-----------|---------|------|----|------|------|
|          | $n_g$ | $d_M$ | $w_M$ | $n_g$ | $d_M$ | $w_M$ | $n_g$ | $d_M$ | $w_M$ |
| acosf    | -1    | 1      |       |     |     |      |     |     |      |
| acoshf   | 1     | $\infty$ |        |     |     |      |     |     |      |
| asinf    | -1    | 1      |       |     |     |      |     |     |      |
| asinhf   | $-\infty$ | $\infty$ |        |     |     |      |     |     |      |
| atanf    | $-\infty$ | $\infty$ |        | 1   | $10^8$ |      |     |     |      |
| atanhf   | -1    | 1      |       |     |     |      |     |     |      |
| cbtf     | $-\infty$ | $\infty$ |        | $10^6$ | 1 | $2 \times 10^6$ |     |     |      |
| coshf    | $-\infty$ | $\infty$ |        | 454 | 1 | $4 \times 10^7$ |     |     |      |
| erf      | $-\infty$ | $\infty$ |        |     |     |      |     |     |      |
| expf     | $-\infty$ | $\infty$ |        |     |     |      |     |     |      |
| exp10f   | $-\infty$ | $\infty$ |        |     |     |      |     |     |      |
| exp2f    | $-\infty$ | $\infty$ |        | 1 | 2 | $2 \times 10^5$ |     |     |      |
| expm1f   | $-\infty$ | $\infty$ |        |     |     |      |     |     |      |
| lgammaf  | 2 | $163$ | 1 | $2 \times 166$ | 1 | $2 \times 161$ | 1 | $2 \times 161$ | 1 |
| logf     | 0 | $\infty$ |        |     |     |      |     |     |      |
| log10f   | 0 | $\infty$ |        |     |     |      |     |     |      |
| log1pf   | -1 | $\infty$ |        | 1 | 2 | 1 | 1 | 1 | 2 |
| log2f    | 0 | $\infty$ |        |     |     |      |     |     |      |
| sinhf    | $-\infty$ | $\infty$ |        |     |     |      |     |     |      |
| sqrtf    | 0 | $\infty$ |        |     |     |      |     |     |      |
| tanhf    | $-\infty$ | $\infty$ |        | 1 | 2 | 2 | 1 | 3 |      |
| tgammaf  | 2 | $10^5$ | 4 | $3 \times 10^5$ | 4 | $3 \times 10^5$ | 4 | $3 \times 10^5$ | 4 |
| cosf     | $-2^{23}$ | $2^{23}$ |        |     |     |      |     |     |      |
| sinf     | $-2^{23}$ | $2^{23}$ |        |     |     |      |     |     |      |
| tanf     | $-2^{23}$ | $2^{23}$ |        |     |     |      |     |     |      |

The data collected for two implementations are presented in Tables 2–3; data for the other implementations are reported in Appendix A. These tables present, for each function, its name and the minimum and maximum of the considered domain interval. For most of the functions, such interval is the natural one. The exceptions are the following: for lgammaf we start at 2, which is where monotonicity theoretically begins; for tgammaf we also start at 2 because the considered implementations behave in a bizarre way for arguments less than 2; for the trigonometric functions, at the bottom of the tables, we restrict the domain to a region where there are at least 12 floats per period, for reasons that will be discussed in Section 5.1.

For each function and each rounding mode (near, up, down, zero), Tables 2–3 give the number of glitches, $n_g$, their maximum depth, $d_M$, and their maximum width, $w_M$. For the trigonometric functions we report the cumulative results concerning all the quasi-isotonic and quasi-antitonic branches in the given range. In the columns labeled $n_g$, we report for these functions the maximum number of glitches in any such quasi-monotonic branch (we will refer to this quantity in Section 5.2 as $n_{gM}$).
The following observations can be made:

1. there are few glitches: many functions have no glitch at all, several functions have just a few glitches, a few functions have many glitches;
2. most glitches are very shallow;
3. with a notable exception, glitches are also very narrow.

It is important to observe that glitches are not simply bugs that will surely be fixed at the next release. For instance, the implementation of \texttt{tgammaf} of Ubuntu 16.04/x86\_64 has more numerous and deeper glitches than the one in Ubuntu 14.04/x86\_64. The point is that monotonicity is not one of the objective of most implementations of \texttt{math.h/cmath} functions. For instance, both the manual \cite{LSM16} and the FAQ of GNU libc explicitly exclude monotonicity from the accuracy goals of the library, so that bug reports about violated monotonicity are closed as invalid.\footnote{See, e.g., bug reports \url{https://sourceware.org/bugzilla/show_bug.cgi?id=15898} and \url{https://sourceware.org/bugzilla/show_bug.cgi?id=15899}}

We will now see how quasi-monotonicity can be exploited for the purposes of interval refinement and, in turn, of software verification. Afterwards, we will deal with the special case of the trigonometric functions, as they pose the additional problem of periodic slope inversions.

\section{Propagation Algorithms}

Let $\mathbb{F}$ be any IEEE 754 floating-point format and let $\mathcal{S} \subseteq 2^\mathbb{F}$ be a bounded meet-semilattice. A floating-point unary constraint over $\mathcal{A}$ is a formula of the form $x \in S$ for $S \in \mathcal{S}$.

Let $f : \mathbb{F} \to \mathbb{F}$ be a function and consider a constraint of the form $y = f(x)$ along with the unary constraints $x \in S_x$ and $y \in S_y$ with $S_x, S_y \in \mathcal{S}$.

Direct propagation amounts to compute a possibly refined set $S'_y \in \mathcal{S}$, such that

\begin{equation}
S'_y \subseteq S_y \land \forall x \in S_x : f(x) \in S_y \implies f(x) \in S'_y.
\end{equation}

Of course this is always possible by taking $S'_y = S_y$, but the objective of the game is to compute a “small”, possibly the smallest $S'_y$ enjoying \textcircled{1}, compatibly with the availability of information on $f$ and of computing resources. The smallest $S'_y \in \mathcal{S}$ that satisfies \textcircled{1} is such that

\begin{equation}
\forall S''_y \in \mathcal{S} : S''_y \subseteq S'_y \implies \exists x \in S_x . f(x) \in S_y \setminus S''_y.
\end{equation}

Inverse propagation for the same constraints, $y = f(x)$, $x \in S_x$ and $y \in S_y$, is the computation of a possibly refined set for $x$, $S'_x$, such that

\begin{equation}
S'_x \subseteq S_x \land \forall x \in S_x : f(x) \in S_y \implies x \in S'_x.
\end{equation}

Again, taking $S'_x = S_x$ is always possible and sometimes unavoidable. The best we can hope for is to be able to determine the smallest such set, i.e., satisfying

\begin{equation}
\forall S''_x \in \mathcal{S} : S''_x \subseteq S'_x \implies \exists x \in S_x \setminus S''_x . f(x) \in S_y.
\end{equation}
respectively, the total number of glitches given the implementation of a quasi-monotonic library function antitonic because of the presence of glitches. Yet, we devised algorithms that, above to “multi-interval consistency.”

Satisfying predicates \(2\) and \(4\) corresponds to enforcing and obtaining domain consistency \(^{[VSD98]}\) on our constraint set. Unfortunately, this goal is often difficult to reach, especially when the underlying variable domains are large. A less demanding approach to the problem is to seek interval consistency: we associate an interval \([x_l, x_u]\) to variable \(x\) and an interval \([y_l, y_u]\) to \(y\), and we try to obtain new intervals whose bounds satisfy the equation \(y = f(x)\).

If function \(f\) is isotonic, direct propagation can now be reduced to finding a new interval \([y'_l, y'_u]\) such that

\[
y'_l \geq y_l \land \forall x \in [x_l, x_u]: f(x) \geq y_l \implies f(x) \geq y'_l, \tag{5}
\]

\[
y'_u \leq y_u \land \forall x \in [x_l, x_u]: f(x) \leq y_u \implies f(x) \leq y'_u. \tag{6}
\]

Taking \(y'_l = y_l\) and \(y'_u = y_u\) would trivially satisfy these predicates. However, we aim to find an interval such that

\[
\forall y''_l \in F: y''_l > y'_l \implies \exists x \in [x_l, x_u]. y_l \leq f(x) < y''_l, \tag{7}
\]

\[
\forall y''_u \in F: y''_u < y'_u \implies \exists x \in [x_l, x_u]. y_u < f(x) \leq y''_u. \tag{8}
\]

Inverse propagation consists now in finding an interval \([x'_l, x'_u]\) such that

\[
x'_l \geq x_l \land \forall x \in [x_l, x_u]: f(x) \geq y_l \implies x \geq x'_l, \tag{9}
\]

\[
x'_u \leq x_u \land \forall x \in [x_l, x_u]: f(x) \leq y_u \implies x \leq x'_u. \tag{10}
\]

An optimal result would satisfy

\[
\forall x''_l \in F: x''_l > x'_l \implies \exists x \in [x_l, x_u'). f(x) \geq y_l, \tag{11}
\]

\[
\forall x''_u \in F: x''_u < x'_u \implies \exists x \in (x''_u, x_u]. f(x) \leq y_u. \tag{12}
\]

Similar predicates can be conceived for antitonic functions.

A possible compromise between domain and interval consistency is the use of multi-intervals. It consists in splitting domains in multiple intervals, thus allowing further granularity. It is not difficult to extend the predicates given above to “multi-interval consistency.”

Unfortunately, the functions we are concerned with are neither isotonic nor antitonic because of the presence of glitches. Yet, we devised algorithms that, given the implementation of a quasi-monotonic library function \(f: F \rightarrow \mathbb{R}\), an interval \([x_l, x_u]\) for \(x\) and an interval \([y_l, y_u]\) for \(y\), can compute refined bounds for both intervals, satisfying the correctness predicates defined above and, in particular cases, even the optimality predicates. These algorithms, that will be presented in the following sections, exploit some simple data describing the glitches of a specific function in order to overcome the issues generated by its quasi-monotonicity. Such data consist in safe approximations \(n_g, d_M\) and \(w_M\) of, respectively, the total number of glitches \(n_g\), their maximal depth \(d_M\) and width \(w_M\). Moreover, a safe approximation \(\alpha\) of where the first glitch starts, \(\alpha^l\), and a safe approximation \(\omega\) of where the last glitch ends inside the function’s domain, \(\omega^l\), are needed.
4.1 Direct Propagation

Given an interval \([x_l, x_u]\) for \(x\) and a function \(f\), finding a refined interval \([y'_l, y'_u]\) for \(y\) that satisfies constraint \(y = f(x)\) would be trivial if \(f\) was monotonic: computing \([y'_l, y'_u] = [f(x_l), f(x_u)]\) would suffice. However, the possible presence of glitches in quasi-monotonic functions requires some more care. In particular, it raises two main issues:

- there might be glitches in \([x_l, x_u]\) in which the value of the function is lower than \(f(x_l)\);
- \(x_u\) might be inside a glitch, and there might be values of \(x\) outside it such that \(f(x) > f(x_u)\).

If \([x_l, x_u]\) and \([\alpha, \omega]\) do not intersect, the simple approach for monotonic functions can be used. Otherwise, the problems mentioned can be tackled in the following ways, exploiting some minimal information about the glitches of the function.

**Lower bound** \(y'_l\): if \(x_l \in [\alpha, \omega]\), the worst-case scenario is that there is a glitch starting right after \(x_l\), where the graph of \(f\) goes lower than \(f(x_l)\). However, such glitch surely cannot be deeper than \(f(x_l)^{-d_M}\): we can take this value as \(y'_l\), the lower bound of the refined interval. If \(x_l\) is not in the "glitch-area," then we can consider the value of \(f\) in \(\alpha\): surely no glitch in \([\alpha, x_u]\) can go lower than \(f(\alpha)^{-d_M}\). In this case, we can set \(y'_l = \min\{f(x_l), f(\alpha)^{-d_M}\}\).

**Upper bound** \(y'_u\): if \(x_u \notin [\alpha, \omega]\), then it cannot be in a glitch, and \(f(x_u)\) is the highest value of \(f\) in \([x_l, x_u]\). Otherwise, it may be in a glitch, but surely not deeper than \(d_M\): the actual maximum value of the function, right outside the glitch, cannot be higher than \(f(x_u)^{+d_M}\). Consequently, \(y'_u\) can be set to the latter value.

If the maximum and minimum values taken by the function in its whole domain are also known, they can be compared to the obtained values of \(y'_l\) and \(y'_u\) in order not to return values outside the actual range of \(f\).

4.2 Indirect Propagation

Assume, again, that function \(f\) is quasi-isotonic. For inverse propagation, i.e., the process of inferring a new interval \([x'_l, x'_u]\) \(\subseteq [x_l, x_u]\) for \(x\) starting from the interval \([y_l, y_u]\) for \(y\), we handle the problem by looking for a lower bound for the values of \(x\) that satisfy equation \(y_l = f(x)\), and an upper bound for the values of \(x\) that satisfy equation \(y_u = f(x)\). We will then use the results of these two different searches to correctly refine interval \([x_l, x_u]\) into \([x'_l, x'_u]\), when it is possible. We therefore designed two different algorithms, **lower_bound** and **upper_bound**, capable of carrying out the tasks suggested by their names, for the equation \(y = f(x)\), where \(y\) is a given single value of \(y\).

For a sake of brevity, we describe in detail algorithm **lower_bound** in the next section, while Section 4.2 describes only the main features of the algorithm.
for computing the upper bound. However, the interested reader can find the pseudocode and theoretical results for algorithm upper_bound in Appendix C.

Algorithms lower_bound and upper_bound, which extend the well known dichotomic search method to quasi-isotonic functions, guarantee properties on whether the newly found bounds for \( x \) satisfy the equation \( y = f(x) \) that are expressed by the predicates presented below.

In more details, function lower_bound, given as Algorithm 11, returns a value \( l \) that satisfies one of the following predicates:

\[
\begin{align*}
p_0(y, x_l, x_u, l) &\equiv \forall x \in [x_l, x_u] : y \succ f(x), \\
p_1(y, x_l, x_u, l) &\equiv \forall x \in [x_l, l] : y \prec f(x), \\
p_2(y, x_l, x_u, l) &\equiv \forall x \in [l, x_u] : y \succ f(x), \\
p_3(y, x_l, x_u, l) &\equiv f(l) < y < f(l^+) \land \forall x \in [x_l, l] : y \succ f(x), \\
p_4(y, x_l, x_u, l) &\equiv y = f(l) \land \forall x \in [x_l, l] : y \succ f(x).
\end{align*}
\]

When condition \( p_0(y, x_l, x_u, l) \) holds, the equation \( y = f(x) \) has no solution over \([x_l, x_u]\). Moreover, even when \( p_1(y, x_l, x_u, l) \) holds with \( l = x_u \), the equation \( y = f(x) \) has no solution there. When \( p_1(y, x_l, x_u, l) \) holds with \( l < x_u \) or \( p_2(y, x_l, x_u, l) \) holds, the equation \( y = f(x) \) may have a solution and choosing \( x' = l^+ \) gives a correct refinement for \( x \). When \( p_3(y, x_l, x_u, l) \) holds, we identified the leftmost point in \([x_l, x_u]\) where \( f \) crosses \( y \) without touching it and we can set \( x' = l^+ \). Finally, when \( p_4(y, x_l, x_u, l) \) holds we identified the leftmost solution \( x = l \) of \( y = f(x) \) and we can set \( x' = l \).

The upper bound is computed by function upper_bound, which terminates returning a value \( u \) that satisfies one of the predicates listed below:

\[
\begin{align*}
p_5(y, x_l, x_u, u) &\equiv \forall x \in [x_l, x_u] : y \prec f(x), \\
p_6(y, x_l, x_u, u) &\equiv \forall x \in [u, x_u] : y \succ f(x), \\
p_7(y, x_l, x_u, u) &\equiv \forall x \in [u, x_u] : y \prec f(x), \\
p_8(y, x_l, x_u, u) &\equiv f(u^-) \prec y \prec f(u) \land \forall x \in (u, x_u] : y \prec f(x), \\
p_9(y, x_l, x_u, u) &\equiv y = f(u) \land \forall x \in (u, x_u] : y \prec f(x).
\end{align*}
\]

These predicates are the counterparts of those for lower_bound, previously presented. If one of predicates \( p_5(y, x_l, x_u, u) \) and \( p_6(y, x_l, x_u, u) \) holds, the latter only with \( u = x_l \), then equation \( y = f(x) \) has no solution in \([x_l, x_u]\). If \( p_6(y, x_l, x_u, u) \) holds with \( u < x_u \), or \( p_7(y, x_l, x_u, u) \) holds with any \( u \in [x_l, x_u] \), then equation \( y = f(x) \) has surely no solution in interval \([u, x_u]\), but it might have a solution somewhere in \([x_l, u]\). Setting \( x'_u = u^- \) is therefore correct. If \( p_8(y, x_l, x_u, u) \) holds, then the algorithm identified the rightmost point in \([x_l, x_u]\) where the graph of \( f \) crosses \( y \) without touching it. Setting \( x'_u = u^- \) is correct. Finally, when \( p_9(y, x_l, x_u, u) \) holds, the algorithm found the rightmost solution \( x = u \) of the equation \( y = f(x) \) and setting \( x'_u = u \) is correct.

Therefore, when function \( f \) is quasi-isotonic, the results of the invocation of lower_bound on \( y_l \) and upper_bound on \( y_u \) are combined to refine the interval \([x_l, x_u]\) into \([x'_l, x'_u]\), on the basis of which pair of predicates \( p_i \), with \( i \in \{0, \ldots, 9\} \), they satisfy.
If one of predicates \( p_0 \) or \( p_5 \) holds, then we are sure that there is no solution, because the entire graph of the function is either below or above the interval for \( y \). Note that \( p_0 \) implies \( p_6 \) and \( p_5 \) implies \( p_1 \).

If \( p_1 \) holds, then we must set \( x'_i = x_l \) even if \( l > x_l \). In fact, although we are sure the graph of the function is entirely above \( y_l \), there might be a value \( y'_i \) such that \( y_l < y'_i < y_u \) that satisfies \( y'_i = f(x) \) for some \( x \in [x_l, l] \). Excluding such interval would therefore undermine the correctness of the propagation process. This, of course, unless either \( y \) is a singleton, or also \( p_5 \) holds: in these cases we can correctly conclude that there are no solutions. The same reasoning can be done for the upper bound: unless \( y \) is a singleton or \( p_6 \) holds, \( x'_u \) must be set to \( x_u \) whenever \( p_6 \) holds.

If \( p_3 \) and \( p_8 \) hold together and \( l > u \), then we are sure that there is no solution for equation \( y = f(x) \).

For all other combinations of predicates, the new values \( x'_i \) and \( x'_u \) can be set as indicated below the definitions of the predicates.

The same algorithms can be used for the quasi-antitonic functions, by exploiting the fact that, if \( f \) is quasi-antitonic, then \( -f \) is quasi-isotonic. Both algorithms can therefore be called with \( f' = -f \), \( f'' = f \circ (-\text{id}) \), and \( -y_u \) instead of \( y_l \) for lower bound, and \( -y_l \) in place of \( y_u \) for upper bound. When they terminate, \( p_i(-y_u, x_l, x_u, l) \) and \( p_j(-y_l, x_l, x_u, u) \) with \( i \in \{0, \ldots, 4\} \) and \( j \in \{5, \ldots, 9\} \) hold with \( -f \). Note that, since \( -y \prec -f(x) \iff y \succ f(x) \) and \( -y \succ -f(x) \iff y \prec f(x) \), the same predicates do not hold on the original function \( f \). However, because \( y_l \) and \( y_u \) are switched, the same discussion regarding the treatment of the new bounds \( x'_i \) and \( x'_u \) depending on the post-condition predicates applies as for the isotonic case, obtaining the same case analysis on the values of \( i \) and \( j \).

**Computation of the Lower Bound for** \( y = f(x) \) **Given** A value for \( y \) and equation \( y = f(x) \), Algorithm 1 computes a correct lower bound that (possibly) allows us to refine the interval of \( x \). The preconditions are listed in the **Require** statement and demand a quasi-isotonic function \( f : \mathbb{F} \to \mathbb{F} \), a value \( y \in \mathbb{F} \), and an interval \([x_l, x_u]\) for \( x \) to be refined. In order to be as precise as possible, the algorithm needs safe approximations of the glitch data listed previously in this section. It also uses an inverse function \( f' : \mathbb{F} \to \mathbb{F} \), if available. Finally, to prevent complexity issues, parameter \( t \in \mathbb{N} \) fixes the maximum length of the linear searches that the algorithm performs in some cases, and \( s \in \mathbb{N} \) is the maximum number of times the function \( \logsearch_{lb} \) in Algorithm 8 can return an interval too wide to ensure the logarithmic complexity of the bisection algorithm.

The algorithm ends guaranteeing the post-conditions expressed in the **Ensure** statement, where the predicates \( p_r(y, x_l, x_u, l) \) for \( r \in \{0, \ldots, 4\} \) are those described previously. The post-conditions are divided into two parts: the **correctness** part is preceded by \( \Box \) whereas the **precision** part is identified by \( \Delta \).

The objective of the algorithm is to determine \( r \) and \( l \) by performing a number
of calls to the concerned library function $f$ bounded by a small constant $k$ (e.g.,
$k = 3$), times the logarithm of the cardinality of interval $[x_l, x_u]$.

**Algorithm 1** Inverse propagation: lower bound($f, y, [x_l, x_u], n_g, d_M, w_M, \alpha, \omega, f^\dagger, s, t$)

**Require:** $f : \mathbb{F} \to \mathbb{F}$, $y \in \mathbb{F}$, $[x_l, x_u] \in \mathbb{I}_\mathbb{F}$, $n_g \geq n_g^f$, $d_M \geq d_M^f$, $w_M \geq w_M^f$, $\alpha \leq \alpha^f$, $\omega \geq \omega^f$, $n_g > 0 \implies (x_l \leq \alpha \leq \omega \leq x_u)$, $f^\dagger : \mathbb{F} \to \mathbb{F}$, $s, t \in \mathbb{N}$.

**Ensure:** $\bigcirc l \in \mathbb{F}, r \in \{0, 1, 2, 3, 4\} \implies p_v(y, x_l, x_u, l)$

1: $i := \text{init}(y, [x_l, x_u], f^\dagger)$;
2: $\triangleright x_l \leq i \leq x_u$
3: $(lo, hi) := \text{gallop lb}(f, y, [x_l, x_u], d_M, i)$;
4: $\triangleright (x_l \leq lo \leq hi \leq x_u) \land (x_l \prec lo \implies y \succ d_M f(lo)) \land (x_u \succ hi \implies f(hi) \succ y)$
5: if $f(lo) \succ y$ then
6: if $n_g = 0 \lor f(lo) \succ d_M y$ then
7: $l := x_u; r := 1; \text{return}$
8: else
9: $l := \alpha; r := 1; \text{return}$
10: end if
11: else if $f(lo) = y$ then
12: $l := lo; r := 4; \text{return}$
13: end if;
14: if $f(hi) \prec y$ then
15: $(r, l, hi) := \text{findhi lb}(f, y, [x_l, x_u], n_g, d_M, w_M, \alpha, \omega, s, t)$;
16: if $r \in \{0, 2\}$ then return
17: end if
18: end if;
19: $lo := \text{bisect lb}(f, y, [x_l, x_u], n_g, d_M, w_M, \alpha, \omega, s, t, lo, hi)$;
20: while $f(lo^+) \prec y \land t > 0$ do
21: $lo := lo^+$;
22: $t := t - 1$
23: end while;
24: if $f(lo^+) \succ y$ then
25: $l := lo; r := 3$
26: else if $f(lo^+) = y$ then
27: $l := lo^+; r := 4$
28: else
29: $l := lo; r := 2$
30: end if

First, Algorithm 1 calls function init, that takes a value $y$, interval $[x_l, x_u]$ for $x$, and an inverse function $f^\dagger : \mathbb{F} \to \mathbb{F}$, if available. Its purpose is to simply return a point inside interval $[x_l, x_u]$. If the inverse function $f^\dagger$ is available and $x_l \leq f^\dagger(y) \leq x_u$, then init returns $f^\dagger(y)$; it returns the middle point between $x_l$ and $x_u$ otherwise.
Next, function $\text{gallop} \_\text{lb}$ is called. Its goal is to find values $lo$ and $hi$ that satisfy the precondition of algorithm $\text{bisect} \_\text{lb}$ and are suitable for the bisection process, i.e., such that $x_i \preceq lo \preceq hi \preceq x_u$. $x_i \preceq lo$ implies that $y \succ ^{dM} f(lo)$ and $x_u \succ hi$ implies $f(hi) \succ y$. Therefore, $\text{gallop} \_\text{lb}$ starts with $hi = i$ and increases it (e.g., by multiplying it by 2, if positive) until a new $hi$ such that $hi \preceq x_u$ and $f(hi) \succ y$ is found. If such $hi$ cannot be found in this way, it sets $hi = x_u$. Similarly, function $\text{gallop} \_\text{lb}$ starts with $lo = i$ and then decreases $lo$ (e.g., by dividing it by 2, if positive) until a new $lo$ such that $x_i \preceq lo$ and $y \succ ^{dM} f(lo)$ is found. If such $lo$ cannot be found in this way, it sets $lo = x_i$.

The latter case, in which $f(lo) \succ y$, is handled at lines 6-10. We need to determine if really $[x_i, x_u]$ does not contain any solution for $y$, that is if, for each $x \in [x_i, x_u]$, $y \succ f(x)$ holds. If we are not sure because glitches are too deep, a suboptimal value for $l$ is returned, and the algorithm terminates. If $f(lo) = y$, at line 11 the algorithm returns with the exact solution for the lower bound.

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**Algorithm 2** Inverse propagation: $\text{findhi} \_\text{lb}(f, y, [x_i, x_u], n_g, d_M, w_M, \alpha, \omega, t)$

**Require:** $f : F \to F$, $y \in F$, $[x_i, x_u] \in \mathcal{I} \text{F}$, $n_g \geq n_g^l$, $d_M \geq d_M^l$, $w_M \geq w_M^l$, $\alpha \approx \alpha^f$, $\omega \geq \omega^f$, $n_g > 0 \implies (x_i \preceq \alpha \preceq \omega \preceq x_u), t \in \mathbb{N}, f(x_u) \prec y$.

**Ensure:** $l \in F$, $r \in \{0, 2\} \implies p_v(y, x_i, x_u, l), r = 1 \implies (hi \in [x_i, x_u] \wedge f(hi) \succ y)$.

1: $l = x_i$;
2: if $n_g = 0 \lor x_u \succ \omega \lor y \succ ^{dM} f(x_u)$ then
3:   $r := 0$
4: else if $n_g = 1 \land (w_M > t \lor f(\alpha^+) \prec f(\alpha))$ then
5:   if $y \succ f(\alpha)$ then
6:     if $f(\alpha^+) \prec f(\alpha)$ then
7:       $r := 0$
8: else
9:   $l := \alpha; r := 2$
10: end if
11: else
12:   $hi := \alpha; r := 1$
13: end if
14: else
15:   $(b, hi) := \text{linsearch} \_\text{geq}(f, y, [x_i, x_u], w_M, t)$;
16:   $\triangleright (b = 1 \land hi \in [x_i, x_u] \land f(hi) \succ y) \lor (b = 0 \land \forall x \in [\hat{x}, x_u] : f(x) \prec y)$
17:   $\triangleright$ where $v = \min\{t, w_M\}$ and $\hat{x} = \max\{x_i, x_u\}$
18: if $b = 1$ then
19:   $r := 1$
20: else if $t \geq w_M$ then
21:   $r := 0$
22: else
23:   $l := x_i; r := 2$
24: end if
25: end if

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On line 15 function \texttt{findhi lb} given by Algorithm 2 handles the special case where \texttt{gallop lb} ended up with \( hi = x_u \) and \( f(x_u) \prec y \). This might happen because for no \( x \in [x_l, x_u] \) we have \( f(x) \succ y \), or because such an \( x \) does exist, but \( x_u \) is in a glitch. The purpose of \texttt{findhi lb} is to quickly discriminate between these two cases and find a value for \( hi \) suitable for bisection, when possible. In this case function \texttt{findhi lb} returns \( r = 1 \).

- If \( x_u \) might be in a (wider than \( t \)) glitch, for the sake of efficiency we cannot exhaustively look for good candidates to apply bisection. However, we might still be able to set \( r = 0 \), to signify that \( \forall x \in [x_l, x_u] : f(x) \prec y \), or to set \( r = 1 \) and \( hi \) to an appropriate value for bisection. If we do not have enough information to chose one of these options, we may just set \( r = 2 \) and \( l \) to a valid (possibly suboptimal) lower bound.

- Otherwise, \texttt{linsearch geq}(\( f, y, [x_l, x_u], w_M, t \)) performs a backwards, float-by-float, search for at most the minimum between \( t \) and \( w_M \) steps, starting from \( hi = x_u \), and looking for the first value for \( hi \) such that \( f(hi) \succ y \).

The search stops in two cases, which are discriminated by the value given to variable \( b \).

**Case** \( b = 0 \) : within the allowed \( t \) steps of the search, no suitable value for \( hi \) was found. Still, if \( t \) was high enough to cover the glitch, we set \( r = 0 \) since we are sure that \( \forall x \in [x_l, x_u] : f(x) \prec y \). Otherwise we set \( r = 2 \) and we content ourselves with the trivial lower bound \( x_l \).

**Case** \( b = 1 \) : an appropriate value \( hi \) for bisection has been found.

At the end of line 17 we know that function \texttt{findhi lb} returned \( r = 1 \) and, therefore, a new value for \( hi \) was returned such that \( x_l \preceq hi \preceq x_u \) and \( f(hi) \succ y \).

Summarizing, before the invocation of function \texttt{bisect lb} at line 19 we are sure that \( f(lo) \prec y \preceq f(hi) \) and, therefore, \( lo \neq hi \). Function \texttt{bisect lb}, defined by Algorithm 3 employs the dichotomic method to refine the interval \([lo, hi]\) when \( f \) is a quasi-isotonic function. Each iteration of the \texttt{while} loop on line 1 uses function \texttt{split point} to pick the middle point \( mid \) of interval \([lo, hi]\), and compares the value of \( f(mid) \) with \( y \). When \( f(mid) \succ y \), the value of \( hi \) can be correctly updated with the value of \( mid \). The critical case is when \( f(mid) \prec y \). In this case, function \texttt{bisect lb} has to further discriminate whether \( lo \) can be correctly updated with the value of \( mid \) or other refinements of the interval \([lo, hi]\) are possible. In particular:

- if \( mid \) is surely not inside a glitch, the value of \( lo \) can correctly be updated with the value of \( mid \);
- if \( mid \) may be inside the unique glitch that either is bigger than \( t \) or starts exactly in \( \alpha \) (that is, \( f(\alpha^+) \prec f(\alpha) \)), the algorithm needs to test the value of \( f \) in \( \alpha \) and \( \omega \) in order to determine if and how the interval \([lo, hi]\) can be correctly refined;
- if \( mid \) may be inside a glitch narrower than \( t \), function \texttt{findfmax} finds the value \( b \) inside interval \([\max\{lo, \min^{-\alpha}, \omega\}, mid]\) for which value \( f(b) \) is the highest. Such value is then used to refine the interval \([lo, hi]\);
Algorithm 3 Inverse propagation: bisect\_lb\(f, y, [x_l, x_u], n_g, d_M, w_M, \alpha, \omega, s, t, lo, hi\)

Require: \(x_l \preceq lo \prec hi \preceq x_u, f(lo) < y \preceq f(hi), \forall x \in [x_l, lo]: f(x) < y f: \mathbb{F} \to \mathbb{F}, y \in \mathbb{F}, [x_l, x_u] \subseteq \mathbb{I}, n_g \geq n_g^l, d_M \geq d_M^l, w_M \geq w_M^l, \alpha \preceq \alpha^l, \omega \succeq \omega^l, n_g > 0 \implies (x_l \preceq \alpha \preceq \omega \preceq x_u), s, t \in \mathbb{N}.

Ensure: \(\bigcirc x_l \preceq lo \prec hi \preceq x_u, f(lo) \prec y \prec f(hi), \forall x \in [x_l, lo]: f(x) < y, f(lo^+) \succ y\)

1: while hi \succ^t lo do
2: \hspace{1em} mid := split\_point(lo, hi);
3: \hspace{1em} \triangleright \exists m, m' > 0 . |m - m'| \leq 1 \land mid = hi^{-m} = lo^{+m'}
4: \hspace{1em} if y \preceq f(mid) then
5: \hspace{2em} hi := mid
6: \hspace{1em} else if n_g = 0 \lor mid \preceq \alpha \lor mid \succeq \omega \lor y \succeq d_M \cdot f(mid) then
7: \hspace{2em} lo := mid
8: \hspace{1em} else if n_g = 1 \land (w_M \succ t \lor f(\alpha^+) \prec f(\alpha)) then
9: \hspace{2em} if f(\omega) \succ y then
10: \hspace{3em} hi := \alpha
11: \hspace{2em} else if f(\alpha^+) \prec f(\alpha) then
12: \hspace{3em} lo := mid
13: \hspace{2em} else if lo \prec \alpha then
14: \hspace{3em} lo := \alpha
15: \hspace{2em} else
16: \hspace{3em} break
17: \hspace{2em} end if
18: \hspace{2em} else lo := \omega
19: \hspace{2em} end if
20: \hspace{1em} end if
21: \hspace{1em} else if w_M \leq t then
22: \hspace{2em} b := find\_max(f, w_M, lo, mid);
23: \hspace{2em} \triangleright b \in \{\max\{lo, mid^{-w_M}\}, mid\} \land \forall x \in \{\max\{lo, mid^{-w_M}\}, mid\} : f(x) \preceq f(b)
24: \hspace{2em} if f(b) \succ y then
25: \hspace{3em} hi := b
26: \hspace{2em} else
27: \hspace{3em} lo := mid
28: \hspace{3em} end if
29: \hspace{2em} else
30: \hspace{3em} z := logsearch\_lb(f, d_M, lo, mid, y, s);
31: \hspace{3em} \triangleright z \in [lo, mid] \land (lo \prec z \implies f(z) \prec^{d_M} y)
32: \hspace{3em} if lo \prec z then
33: \hspace{4em} lo := z
34: \hspace{3em} else
35: \hspace{4em} break
36: \hspace{3em} end if
37: \hspace{2em} end if
38: \hspace{1em} end while
- if mid may be inside a glitch wider than $t$, for complexity reasons Algorithm 3 refrains from iterating the float-by-float search of function $\text{findfmax}$ and calls $\text{logsearch}_{lb}$ that finds, if it exists, a value $z \in [lo, mid]$ such that $f(z) \prec^{d_M} y$. If such $z$ is found, it is used to refine the interval $[lo, hi]$.

In more detail, $\text{logsearch}_{lb}$ performs a logarithmic search for finding, if it exists, a value $z \in [lo, mid]$ such that $f(z) \prec^{d_M} y$. Its argument $s \in \mathbb{N}$ constitutes (for efficiency reasons) an upper bound to the number of times the code of $\text{logsearch}_{lb}$ can return an excessively wide interval as refinement of $[x_l, x_u]$. If such $s$ is not reached, $\text{logsearch}_{lb}$ starts with $z = mid$ and decreases it (e.g., by dividing it by 2, if positive) until a new $z$ such that $f(z) \prec^{d_M} y$ is found. Otherwise, $z$ is set to $lo$.

After $\text{bisect}_{lb}$ has terminated, its post-condition ensures that $x_l \preceq lo \prec hi \preceq x_u$, $f(lo) \prec y \preceq f(hi)$ and $\forall x \in [x_l, lo] : f(x) \prec y$ hold. Then, a while loop is entered. This loop performs a float-by-float search (for no more than $t$ iterations) to approach the exact solution of $y = f(x)$. Afterwards, the invariant of the loop $\forall x \in [x_l, lo] : f(x) \prec y$ holds. At line 24 and line 28 we test if an optimal solution of $y = f(x)$ was found. If the body of the else statement of line 28 is reached, it means that the while loop of line 20 terminated because $t$ reached 0. In this case, $l$ is set to $lo$, a suboptimal solution of $y = f(x)$.

The pseudocode for functions $\text{init}$, $\text{gallop}_{lb}$, $\text{linsearch}_{geq}$, $\text{findfmax}$ and $\text{logsearch}_{lb}$ is not shown because it is mostly straightforward.

The next results assure us that Algorithms 1, 2 and 3 are correct.

**Lemma 41** Function $\text{findhi}_{lb}$ of Algorithm 2 satisfies its contract.

**Lemma 42** Function $\text{bisect}_{lb}$ of Algorithm 3 satisfies its contract.

As a consequence of the precision post-condition, the following result also shows that when function $f$ is isotonic or it has glitches narrower than $t$, Algorithm 1 finds a precise solution, that is, it returns either $r = 3$ or $r = 4$.

**Theorem 43.** Function $\text{lower bound}$ of Algorithm 1 satisfies its contract.

Therefore, algorithm $\text{lower bound}$ of Algorithm 1 ensures optimality of the bound in each one of the following cases:

- $n_g = 0$: the function is monotonic, or
- $w_M < t$: the glitches are not too large to perform linear searches, or
- $n_g = 1$ and $\alpha = \alpha^f$: the function has one glitch only and the position in which it begins is known exactly.

From the point of view of computational complexity, it can be proved that, for most functions, the worst-case computational complexity of $\text{lower bound}$ in Algorithm 1 measured as the number of calls to function $f$, has the form $k \log_2 (\#[x_l, x_u]) + k + c$, for small constants $k$ and $c$ that are related to $w_M$. 

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Theorem 44. If \( f : F \rightarrow F \) is an isotonic function, that is, \( n_g = 0 \), then, for each \([x_l, x_u] \in I_\emptyset, d_M, w_M, \alpha, \omega, f^i : F \rightarrow F, s, t \in \mathbb{N}\), computing lower_bound as per Algorithm 1 evaluates \( f \) at most \( 2 \log_2(\#[x_l, x_u]) + 4 \) times.

Moreover, if \( f \) has at least one glitch and \( w_M \leq t \), \( k \) is strictly related to \( w_M \).

Theorem 45. If \( f : F \rightarrow F \) has short glitches, that is, \( n_g > 0 \) but \( w_M < t \), then, for each \([x_l, x_u] \in I_\emptyset, d_M, \alpha, \omega, f^i : F \rightarrow F, s \in \mathbb{N}\), computing lower_bound as per Algorithm 1 evaluates \( f \) at most \((w_M + 1) \log_2(\#[x_l, x_u]) + w_M + 6 \) times.

The formal proofs of all results can be found in Appendix B.

Computation of the Upper Bound for \( y = f(x) \). The algorithm we conceived for the computation of the upper bounds is substantially similar to the one for the lower bounds in its structure and functioning. It employs the same arguments to obtain glitch data, and it ends ensuring the post-condition predicates listed earlier in this section.

The algorithm consists of a first phase in which it tries to find a sub-interval inside the initial one, \([x_l, x_u]\), that is suitable for the bisection process. If such interval cannot be found, it tries to quickly determine whether the equation \( y = f(x) \) has a solution or not, compatibly with the available glitch information.

Otherwise, the obtained interval is searched for an admissible upper bound with a dichotomic algorithm that takes into account the possible presence of glitches. This algorithm is similar to bisect_lb, except for the fact that it needs to ensure that the function is strictly greater than \( y \) in the whole interval between the found upper bound and \( x_u \). Another significant difference between the two algorithms is the behavior in case the function evaluated in the middle point \( \text{mid} \) is higher than \( y \). The computation should continue in the first half of the original interval, discarding the second one and making sure that the graph of the function is entirely above \( y \) in the latter. This means asserting that there are no glitches after \( \text{mid} \) which are deep enough to let the function reach \( y \). Data such as the maximum glitch depth \( d_M \) are almost always helpful in excluding this circumstance, along with \( \alpha \) and \( \omega \). Theoretically speaking, this is not always the case, e.g., if \( d_M \) is very high, or \( y \) is very close to \( f(\text{mid}) \). The former case seldom occurs in practice, as noted in Section B. The latter can occur in the last stages of the bisection process if the function increases very slowly. The experimental evaluation we performed, however, showed that this is not a substantial problem in practice. Anyways, should this circumstance manifest itself, if the function has only one glitch and it is sufficiently narrow, it can be searched float-by-float. Otherwise, the whole right interval should be searched for glitches, which is clearly unfeasible, if not for very small intervals.

The analogous issue with bisect_lb was making sure that \( \text{mid} \) was not inside a glitch, in order to exclude the left half of the interval. This situation could always be clarified if \( w_M < t \), by means of a linear search that could analyze the entire glitch. The same approach cannot clearly solve the analogous issue for the upper-bound algorithm, since all the glitches after \( \text{mid} \) should be analyzed.
This is the reason why the correctness post-condition is more demanding for the upper bound than for the lower bound. In particular, it ensures optimality of the bound if the function is monotonic, i.e., \( n_g = 0 \). Otherwise, it finds an optimal upper bound if

- \( n_g = 1 \): the function has one glitch only, and
- \( w_M < t \): it is not too large to perform a linear search, and
- the position of the glitch is known exactly, i.e., one of conditions \( \alpha = \alpha^f \), \( \omega = \omega^f \), or \( \alpha \prec k \omega \land k \leq t \) is true.

Nevertheless, the algorithm for the upper bound has the same order of complexity as \texttt{biselect.lb}.

An in-depth analysis of these aspects is available in Appendix C, where the pseudocode of the algorithm is given along with the proof of its correctness and more precise claims about complexity.

5 Trigonometric Functions

The development of interval refinement algorithms for periodic trigonometric functions (i.e., the floating-point approximations of \( \sin \), \( \cos \) and \( \tan \)) requires a more complex approach to quasi-monotonicity. Since the underlying theoretical (partial) functions in \( \mathbb{R} \to \mathbb{R} \) change their monotonicity periodically, Definition 5 would fail to distinguish between actual glitches caused by the implementation and the said “legitimate” monotonicity changes.

This issue can be overcome by exploiting the fact that monotonicity glitches are generally narrow and shallow. If we separately consider a quasi-monotonic branch of the periodic function that is significantly wider than the widest glitch in terms of ULPs, we can locally apply Definition 5. If we limit our reasoning to each quasi-monotonic branch separately, all the statements we made for quasi-monotonic functions locally hold, and we can therefore use the same methods we developed for those functions.

Therefore, in order to properly identify and characterize monotonicity glitches in the trigonometric functions, we need to find

- an interval in which the amount of floating point numbers in every quasi-monotonic branch of the graphs of the functions is sufficiently high;
- a way to identify the points where the monotonicity of the underlying function changes, thus being able to split its graph into quasi-monotonic branches.

5.1 An Appropriate Domain for Trigonometric Functions Analysis

When choosing a suitable domain for the search of monotonicity glitches, we must take into account the fact that, while the period of a trigonometric function is constant, the distance between two consecutive floating-point numbers increases with the exponent. Such distance is expressed by the value of the ulp: \( \mathbb{R} \to \mathbb{R} \) function: we will use the definition of ulp given in [Mul05, Definition 5]. A floating-point interval in which the idea of monotonicity glitch is well
defined should have a sufficient cardinality to allow for glitches that do not cover
the entire interval.

Let \( x \in \mathbb{F} \) be a positive normal floating-point number: then \( \text{ulp}(x) = x - x^- = 2^{\varepsilon_x - p+1} \), where \( p \) is the precision of the format, and \( \varepsilon_x \) the exponent of \( x^- \).

For the considered trigonometric functions, the size of those intervals in which the theoretical function is constantly monotonic or antitonic is \( \pi \). If we consider an interval \( [-\ell_{\text{max}}, \ell_{\text{max}}] \) in which for each \( x \in [-\ell_{\text{max}}, \ell_{\text{max}}] \) we have \( \text{ulp}(x) \leq 0.5 \), then each monotonic branch contains at least 6 or 7 floats, which is acceptable to be used with the propagation algorithms described in Section 4 if glitches have a width of 1 or 2 floats. In intervals with a higher ulp value, the definition of monotonicity glitches would be hardly meaningful.

Therefore, a maximum exponent \( e_{\ell_{\text{max}}} = p - 1 \) will be used, leading to a domain \( [-\ell_{\text{max}}, \ell_{\text{max}}] \) with \( \ell_{\text{max}} = 2^{p-1} \). In conclusion, for the IEEE 754 single-precision format, the value of \( \ell_{\text{max}} \) will be \( 2^{23} \), and for the double-precision format \( \ell_{\text{max}} = 2^{52} \).

As we noted in Section 1, however, domains like these are still excessively large for most real-world applications. For the experiments reported in Section 6.2 we used a bound \( \ell_{\text{max}} = 16 \), for example.

### 5.2 Recognizing Quasi-Monotonic Branches

As we just mentioned, the propagation algorithms for trigonometric functions exploit the fact that these functions are piece-wise quasi-isotonic or quasi-antitonic: the algorithms devised for regular quasi-monotonic functions are applied separately to each branch, in order to obtain refined intervals or multi-intervals. In order to split the domains of the functions into subintervals that contain exactly one monotonic branch, our algorithms need to be able to enumerate and recognize floating-point approximations of multiples of \( \pi/2 \), that are the “critical” points in which the trigonometric functions change their monotonicity.

Let us say \( x = [x_l, x_u] \) is the initial interval to be refined, with respect to a constraint involving a trigonometric function. It must be split in sub-intervals of the form \( \left[\left(2k - 2\right)\frac{\pi}{2}, k\frac{\pi}{2}\right] \). The possible values of \( k \in \mathbb{Z} \) depend on the function: for example, the sine functions changes its monotonicity in odd multiples of \( \pi/2 \), while the cosine functions does so in the even multiples (of the form \( 2k\frac{\pi}{2} \)). The tangent function has asymptotes in the odd multiples of \( \pi/2 \), and in the intervals between them it is isotonic.

Given \( x \in \mathbb{F} \), in order to identify the branch it belongs to, the number \( k = x\frac{\pi}{2} \) is computed. Then, \( k \) is incremented if we are dealing with the cosine function and the initial value of \( k \) was odd, or in the presence of either the sine or the tangent functions, with an even value of \( k \). The obtained value of \( k \) is such that the function changes its monotonicity or has a discontinuity in \( k\frac{\pi}{2} \) and \( (k - 2)\frac{\pi}{2} \), and \( x \in \left[\left((k - 2)\frac{\pi}{2}\right), (k\frac{\pi}{2})\right] \).  

18 Let \( c \in \mathbb{R} \). With \( \lceil c \rceil \), we denote the upper floating-point approximation of \( c \), i.e., \( \lceil c \rceil = \min\{ x \in \mathbb{F} \mid x \geq c \} \). Similarly, with \( \lfloor c \rfloor \), we denote the lower floating-point approximation of \( c \), so that \( \lfloor c \rfloor = \max\{ x \in \mathbb{F} \mid x \leq c \} \).
On the other hand, the algorithms need to compute the bounds \([x_i^{(l)}, x_i^{(u)}]\) of one of such monotonic intervals \(i\), given the value \(k \in \mathbb{Z}\) that identifies it. This means to compute \(x_i^{(l)} = \left(\frac{(k - 2)\pi}{2}\right)^{\uparrow}\), the upper floating-point approximation of the lower bound, and \(x_i^{(u)} = \left(\frac{k\pi}{2}\right)^{\downarrow}\), the lower floating-point approximation of the upper bound.

Both of the tasks described above can be fulfilled with the use of range reduction algorithms. Such techniques are based on the computation of multiplications of the mentioned coefficients \(k \in \mathbb{R}\) with approximations of either \(\frac{\pi}{2}\) or \(\frac{\pi}{2}\), performed with the appropriate (greater) precision. The literature on this subject is fairly extensive, especially with respect to the implementation of trigonometric library function. Our implementation of range-reduction algorithms is reported in Appendix D.2 along with some considerations that are useful to implement them in other environments.

5.3 Outline of the Propagation Algorithms

In this section we describe the functioning of the constraint propagation algorithms we devised for the trigonometric functions.

**Direct Propagation** The periodicity of trigonometric functions poses a fundamental issue: in each monotonic branch the function can cover its whole range. Therefore, if the interval \([x_l, x_u]\) to be used to refine \(y\) spans multiple branches, almost no refinement can be done. However, as floating-point numbers become sparser as the exponent grows, there is the possibility that some branches do not reach the ends of the range, because there are no points where the function takes those values in such branches.

Our algorithm for direct propagation in trigonometric functions tries to take advantage of these facts. First, it identifies the branches of the graph of the functions to which \(x_l\) and \(x_u\) belong, as described above. If they are both in the same monotonic branch, the refinement algorithm for regular functions described in Section 4.1 can be called. Otherwise, the refinement function should be called separately for each branch, and then the minimum of the found values for \(y_l\) and the maximum of the found values for \(y_u\) should be returned. Since the number of branches to be separately inspected can be high, a threshold \(g\) is imposed on the maximum number of branches to be analyzed. If \([x_l, x_u]\) spans more than \(g\) branches, the bounds of the function’s range are returned.

**Inverse Propagation** The algorithm for inverse propagation refines the interval for \(x\) by splitting its initial domain \([x_l, x_u]\) into intervals in which the function graph is monotonic, and then applies algorithms lower_bound and upper_bound locally in order to obtain a refined multi-interval. The multiplications by \(\frac{\pi}{2}\) and \(\frac{\pi}{2}\) needed by this algorithm are computed with sufficient precision, as explained in Appendix D.2.
Since interval \([x_l, x_u]\) can be wide, the number of sub-intervals it has to be split into can be excessively large. To overcome this issue, our algorithm accepts a parameter \(g\) that defines a maximum number of sub-intervals to be returned. The intervals nearest to 0 are split with maximum granularity (each monotonic branch is considered separately), because floating-point numbers in this area are denser, and the probability of finding an optimal solution is higher. If necessary, the remaining branches are gathered in two larger intervals, one to the left of the domain and starting with \(x_l\), and one to the right, ending with \(x_u\). These intervals are only refined in branches at the boundaries. Obviously, if the domain does not cross 0, then only one of these two big sub-intervals is made, and the \(g - 1\) branches nearest to 0 are treated individually in separate sub-intervals.

Functions lower\_bound and upper\_bound can be only called if \([x_l, x_u]\) is a subset of \([-\ell_{\text{max}}, \ell_{\text{max}}]\), because they cannot operate on sub-intervals whose length is too small (see section 5.1). If this algorithm is called repeatedly on the same interval, however, a value \(g > 1\) could still cause complexity issues: if the single-branch intervals are discarded because no solutions are found in them, a repeated call of this refinement algorithm would split the multi-branch sub-interval again and again. To avoid this problem, the domain in which the algorithm is called should be further reduced: good results can be obtained without falling into performance issues if a domain of a few periods is used. Actually it must be noted that, while this algorithm often succeeds in finding a refined interval if the function has the desired values \(y\) in one of the analyzed sub-intervals, it can say nothing if such value is not found. Indeed, if it does not succeed in finding the appropriate values of \(x\) in which the function takes the values in \(y\), it cannot be excluded that the function reaches them somewhere outside \([-\ell_{\text{max}}, \ell_{\text{max}}]\) (or a smaller domain, if chosen). This prevents us from being able to tell when the equation \(y = x\) has no solution, which could be done for the regular functions. However, the fact that floating-point numbers are denser near the value 0 implies that the functions take most of the values of their image in its vicinity. This allows our algorithm to find a solution very often (when present), making it particularly useful for automatic test-data generation.

This algorithm takes both \(y_l\) and \(y_u\) as arguments: \(y_l\) is used to refine the lower bounds, and \(y_u\) is used to refine the upper bounds. On quasi-antitonic intervals, function lower\_bound is called with \(-f\), which is quasi-isotonic in such intervals, and \(f^1 \circ (-\text{id})\) as the rough inverse of \(-f\). In this cases, \(-y_u\) is used instead of \(y_l\). We proceed similarly with upper\_bound.

The other required arguments, which provide information about glitches, are essentially the same as those for lower\_bound, except for \(n_{gM}\), which is a safe approximation of the maximum number of glitches in each quasi-monotonic branch of the function. For each branch, the value of \(n_g\) for the refinement function is set to 0 if the branch does not intersect \([\alpha, \omega]\), and to \(n_{gM}\) otherwise.
6 Implementation and Experiments

In this section we first describe the implementation of the algorithms introduced in this paper and its context. Then, by means of an experiment, we illustrate the kind of verification activities they make possible.

6.1 Implementation and Experiments

All the algorithms presented in this paper have been implemented and included in the ECLAIR software verification platform for C/C++ source code, Java source code and bytecode. For the analysis of integers and floating-point values, ECLAIR uses different domains: mainly multi-intervals with a judicious use of polyhedral approximations made available by (the commercial version of) the Parma Polyhedra Library (PPL) \cite{HZ08}. For reasoning on the floating-point arithmetic operations, ECLAIR uses:

- algorithms realizing optimal direct projections as well as correct and precise indirect projections: the result is similar to the projections defined in \cite{Mic02}, but the ECLAIR algorithms never require working with precision greater than the operation data type;
- algorithms that exploit properties of the binary floating-point representations in order to obtain enhanced precision \cite{CGG16};
- dynamic linear relaxation techniques \cite{MR12,DGD07} using the PPL to enhance constraint propagation with the relational information provided by convex polyhedra.

The algorithms defined in the present paper have been implemented in C++ and extensively tested on a variety of implementations with different characteristics in terms of the presence and nature of monotonicity glitches. These algorithms are now used in three components of the platform instantiation for C/C++: the semantic analysis engine based on abstract interpretation \cite{CC77}, the automatic generator of test inputs, and the symbolic model checker, the latter being both based on constraint solving \cite{GERS08,GERS00}. All components use multi-interval refinement, though in different ways: the test generator and symbolic model checker are driven by labeling and backtracking search. As these have a negative interaction with the searches controlled by the \( s \) and \( t \) parameters of the algorithms, their setting needs to be controlled more carefully (and they are better set to 0 when glitch data is precise) by these components, whereas they are used with values in the range 5–20 in the semantic analysis engine. Here, we only report about experimentation with the symbolic model checker and automatic generator of test input. One interesting feature of the latter is that it optionally produces a transformed source program that contains the original program, suitably instrumented, and a driver that runs one of the generated tests at a time. The instrumented code checks that each one of the generated tests achieves its target, e.g., it reaches a certain program point, or it causes an

\footnote{http://bugseng.com/products/eclair}
integer overflow or the generation of a floating-point NaN or infinite value. The validation of test inputs is thus completely automatic.

6.2 Experiments

The development of the algorithms of Sections 4 and 5 has been paired by the development of a test suite to exercise them in the most disparate ways. However, to better illustrate the potentiality such algorithms have, let us consider again our introductory example, which is reproduced in Listing 1.2 suitably annotated. Let us pretend we know nothing about such code (which is realistic, as there are no comments besides the one reported). So, we initially assume that the entry point is `latlong_utm_of()`; as there are no assertions, we also assume all inputs are possible. For an exploratory analysis, we use ECLAIR’s symbolic model checker in order to detect the possible presence of run-time anomalies: overflow, division by zero and other source of undefined and implementation-defined behavior over the integers, inexact integer-to-floating conversions, finite-to-infinite and numeric-to-NaN transitions over floating point numbers. A finite-to-infinite (resp., numeric-to-NaN) transition is a computation whereby the inputs to a floating-point operation or `math.h`/`cmath` function is finite (resp., numeric) and the output is infinite (resp, NaN). We also set an analysis parameter asking ECLAIR to flag all the invocations of trigonometric functions whose argument has an absolute value greater than, say, 16. Not surprisingly, we obtain three test inputs showing that this is indeed possible; they concern the following program points:

\[ p_2 : (-0x86487f.p - 18F,+0.0F,1), \] where \(-0x86487f.p - 18F \approx -33.570797,\]

\[ p_3 : (-0x8a3ae7.p - 19F,+0.0F,1), \] where \(-0x8a3ae7.p - 19F \approx -17.278761,\]

\[ p_5 : (+0.0F, -0x98b6c1.p - 19F,1), \] where \(-0x98b6c1.p - 19F \approx -19.089235.\]

Of course, the latter input causes the same phenomenon at program point \(p_7\) as well. Fair enough: perhaps `latlong_utm_of()` callers only pass smaller values for \(\phi\) and \(\lambda\); even if that is not the case, then perhaps the only problem is a slight precision issue. But ECLAIR produces two other test inputs, with the specification that they trigger number-to-NaN transitions:

\[ p_1 : (+0xc90fdb.p - 23F,+0.0F,1), \] where \(+0xc90fdb.p - 23F \approx 1.570796,\]

\[ p_4 : (-0xa8a9483.p - 53F,+0xcfc398.p - 23F,1), \] where we can give the approximations \(-0xa8a9483.p - 53F \approx -1.0083 \cdot 10^{-9},\) and \(+0xcfc398.p - 23F \approx -17.278761.\]

As \(+0xc90fdb.p - 23F \approx 1.570796\) converted to double precision is slightly greater than \(\text{M}_\pi,2\), the round-to-nearest, double-precision approximation of \(\frac{\pi}{2}\) defined in `math.h`, we make the hypothesis that \(\phi\) has to be less than or equal to \(\text{M}_\pi,2\). Indeed, looking at the function callers (there is only one in the program), we come to the realization that \(\phi\) and \(\lambda\) are a latitude and longitude in radians, respectively. We attempt validation of this hypothesis by adding the assertions
Listing 1.2. Code excerpted from a real-world avionic library, annotated

1 #include <math.h>
2 #include <stdint.h>
3 #define RadOfDeg(x) ((x) * (M_PI/180.))
4 #define E 0.08181919106 /* Computation for the WGS84 geoid only */
5 #define LambdaOfUtmZone(utm_zone) RadOfDeg((utm_zone-1)*6-180+3)
6 #define CScal(k, z) { z.re *= k; z.im *= k; }
7 #define CAdd(z1, z2) { z2.re += z1.re; z2.im += z1.im; }
8 #define CSub(z1, z2) { z2.re -= z1.re; z2.im -= z1.im; }
9 #define CI(z) {
10 float tmp = z.re; z.re = - z.im; z.im = tmp; }
11 #define CExp(z) {
12 float e = exp(z.re); z.re = e*cos(z.im); 
13 z.im = e*sin(z.im); }
14 #define CSin(z) { CI(z); struct complex _z = {-z.re, -z.im}; 
15 float e = exp(z.re); float cos_z_im = cos(z.im); z.re = e*cos_z_im; 
16 float sin_z_im = sin(z.im); z.im = e*sin_z_im; _z.re = cos_z_im/e; 
17 _z.im = -sin_z_im/e; CSub(_z, z); CScal(-0.5, z); CI(z); }
18 static inline float isometric_latitude(float phi, float e) {
19 return log(p1(tan(p2(phi) / 2.0)) - e / 2.0 * log((1.0 + e * sin(p3(phi)) / (1.0 - e * sin(phi))));
20 }
21 static inline float isometric_latitude0(float phi) {
22 return log(p1(tan(M_PI_4 + phi / 2.0));
23 }
24 void latlong_utm_of(float phi, float lambda, uint8_t utm_zone) {
25 float lambda_c = LambdaOfUtmZone(utm_zone);
26 float ll = isometric_latitude(phi, E);
27 float dl = lambda - lambda_c;
28 float phi_ = asin(sin(phi) / cosh(ll));
29 float ll_ = isometric_latitude0(phi_);
30 float lambda_ = atan(sin(ll) / cos(ll));
31 struct complex z_ = { lambda_, ll_ }; 
32 CScal(serie_coeff_proj_mercator[0], z_);
33 uint8_t k;
34 for(k = 1; k < 3; k++) {
35 struct complex z = { lambda_, ll_ }; 
36 CScal(2*k, z);
37 CSin(z);
38 CScal(serie_coeff_proj_mercator[k], z);
39 CAdd(z, z_); 
40 }
41 CScal(N, z_);
42 latlong_utm_x = XS + z_.im;
43 latlong_utm_y = z_.re; 
44 }
45 }
Listing 1.3. Calling context of `latlong_utm_of()`

```c
int gps_lat, gps_lon; /* 1e7 deg */
unsigned char nav_utm_zone0;
/* [...] */
static gboolean
read_data(GIOChannel *chan, GIOCondition cond, gpointer data) {
    int count;
    char buf[BUFSIZE];
    /* receive data packet containing formatted data */
    count = recv(sock, buf, sizeof(buf), 0);
    if (count > 0) {
        if (count == 23) {
            // FillBufWith32bit(com_trans.buf, 1, gps_lat);
            gps_lat = buf2uint(&buf[0]);
            // FillBufWith32bit(com_trans.buf, 5, gps_lon);
            gps_lon = buf2uint(&buf[4]);
            /* [...] */
        }
        nav_utm_zone0 = (gps_lon/10000000+180) / 6 + 1;
        latlong_utm_of(RadOfDeg(gps_lat/1e7), RadOfDeg(gps_lon/1e7),
                       nav_utm_zone0);
        assert(-M_PI_2 <= phi && phi <= M_PI_2);
        assert(-M_PI <= lambda && lambda <= M_PI);
    }
    return FALSE;
}
```

at the beginning of `latlong_utm_of()` and repeat the analysis. We obtain another ill-conditioned trigonometric function argument test input for program point $p_5$:

$p_5: (\pm 0.0F, +0xc90fda.p - 22F, 255)$, where $-0x98b6c1.p - 19F \approx 3.141593$

(surely `utm_zone = 255` is not among the expected inputs) and another numeric-to-NaN transition input:

$p_4: (-0x8a1e4e.p - 53F, +0xc5b5fe.p - 22F, 15)$, where we have the approximations $-0x8a1e4e.p - 53F \approx -1.00494 \cdot 10^{-9}$, and $+0xc5b5fe.p - 22F \approx 3.089233$.

In order to understand what are the intended inputs for `latlong_utm_of()` we now take into account its calling context, summarized in Listing 1.3. Basically, the inputs to `latlong_utm_of()` depend on two 32-bit signed integers, `gps_lat` and `gps_lon`, that are received from a communication channel: no check is made upon them after reading the values out of the input buffer. Taking into account the caller context, ECLAIR generates three reports: if `gps_lat = 0` and `gps_lon = -1920000000` at line 20 in Listing 1.3 then the
conversion in the assignment marked with ‘∗’ on the same line causes an unsigned wraparound (−1 mod 256 = 255, so that, yes, `latlong_utm_of()` can be called with `utm_zone = 255`); the same input also generates an ill-conditioned trigonometric function argument for program point `p5` in Listing 1.2 most importantly, if `gps_lon = 900000059` and `gps_lat = −1920000000`, then we have a numeric-to-NaN transition at program point `p1`. This probably means that if the equipment at the other hand of the communication channel is defective or if there is a communication error, things can go horribly wrong. However, let us now suppose that there are no problems of this kind and that we have |gps_lat · 10−7| ≤ 90 and |gps_lon · 10−7| ≤ 180 as the code seems to assume. The analysis with ECLAIR shows this is not enough: the numeric-to-NaN transition at program point `p1` is still possible with `gps_lat = −899999991` and `gps_lon = −1800000000`. In a couple more iterations we add the assertions

18′ assert(−899999990 <= gps_lat && gps_lat <= 899999990);  
19′ assert(−1800000000 <= gps_lon && gps_lon <= 1800000000);

and the final ECLAIR run shows no report. This, per se, does not mean much. However, this experiment was done on the `zoltan` machine, for which we have precise glitch data for the single-precision functions (which are not used in the code considered) and we have the maximum known errors provided by the GNU libc manual for the double-precision functions [LSM+16]. As explained in Section 7.2, this data provides imprecise and possibly incorrect information about glitches that our algorithms can exploit. In turn, all this means that:

- if the numbers in [LSM+16] do really provide upper bounds to the maximum errors of the used functions, and
- if the caller guarantees that the values of `gps_lat` and `gps_lon` do satisfy the “stay away from the poles” assertions at lines 18′–19′ in modified Listing 1.3
- then, in the context of such call, all the 154 potential run-time anomalies in the 90 potentially problematic program points of Listing 1.2 cannot occur on `zoltan`.

Just to mention one potential problem, division by zero and consequent finite-to-infinite transition at program point `p6`, cannot happen on that implementation.

7 Discussion and Further Work

In this section we discuss some aspects related to the applicability of our proposal: every such aspect immediately suggests directions for further work.

7.1 Access to the Target Library

For the purposes of true verification, our approach requires execution access to the mathematical library used by the target. Of course, when the host and the

20 This point is roughly 10 cm from the Geographic South Pole.
21 More precisely: 4 integer overflows, 4 inexact conversions, 10 ill-conditioned trigonometric function arguments, 70 finite-to-infinity and 66-numeric-to-NaN transitions.
target computer can coincide, i.e., when the target is powerful enough to run
the verifier code, this is no problem. Alternatively, the host computer might
provide an implementation that is fully equivalent to the one used on the target
computer: this is the case, e.g., of targets where all floating-point support is
implemented in software. In other cases, using an emulator is required. Luckily,
emulation technology is improving very rapidly due to the increasing adoption
of virtualization.

This can be seen as the major drawback of our approach. However, in our
opinion the question should be put in the following terms: in order to prop-
erly verify a piece of code using library functions against, say, the absence of
run-time anomalies, the library functions have to be fully specified. If a speci-
fication of the form “all functions are POSIX-compliant [IEEE13] and compute
correctly-rounded results” is available, then fine. Otherwise there really is no
other way than supplementing the partial specification available with the miss-
ing bits: providing execution access to the library during the analysis along with
correct bounds on the size of the glitches might well be the less expensive option.

7.2 Obtaining Glitch Data

The other requirement of the approach described in this paper concerns the
availability of (possibly imprecise) information about the monotonicity glitches.
Some ways in which such information can (at least in principle) be obtained are
the topics of the next sections.

**Brute Force** For single-precision IEEE 754 (unary) functions collection of pre-
cise glitch data by brute force is perfectly feasible. Glitch data presented in this
paper have been obtained in this way. For the 25 functions studied in this pa-
per, it takes less than two hours on ordinary PC hardware. With current less
powerful CPUs used on embedded systems, it might take ten or twenty times
as much. This is not really a problem as glitch data must be collected only
once for each implementation of the `math.h`/`cmath` functions. And, especially in
safety-critical sectors, the mathematical (and other) libraries will rarely if ever
be changed once they have been selected. Of course, this method cannot be used
for double-precision or extended-precision implementations of the functions.

**Using Precision Guarantees When Available** When the target mathemat-
ic library comes equipped with information on the maximum errors for each
function, such information can be used to determine safe approximations of the
glitch parameters required by our algorithms. Given a function and an archi-
tecture, the maximum error is measured in ULP and can be used as an upper
bound for the maximal depth of the glitches \( w_M \). Given an interval \([x_l, x_u]\), in
absence of more precise information, the cardinality of the floating-point inter-
val \([x_l, x_u]\) is an upper bound to the maximum width of the glitches and, of
course \( x_l \) and \( x_u \) are safe approximations of where the glitches begin and end,
respectively. Finally, setting \( n_\delta > 1 \) allows us to call the indirect propagation
algorithms to refine the interval \([x_l, x_u]\) of the function domain with respect to a given interval \([y_l, y_u]\) of the function range. Even with such a rough information on the glitches, the algorithms would allow us to refine the interval \([x_l, x_u]\) using the logarithmic searches (\texttt{logsearch\_lb} and \texttt{logsearch\_ub}) and returning \([x'_l, x'_u] \subseteq [x_l, x_u]\) such that \(f(x'_l)\) is smaller than \(y_l\) by more than \(w_M\) ULPs and \(f(x'_u)\) is bigger than \(y_u\) by more than \(w_M\) ULPs. Therefore, the cardinality of the resulting refined interval is strictly related to the speed of growth of the considered function. For future work, we intend to investigate how information on the maximum error, coupled with the knowledge of the function and of the interval to be refined, allow computing sound and tight bounds to the width of glitches on that interval.

**Analysis of the Implementation** The transcendental functions are usually implemented computing polynomial approximations. When speed is more important than precision, such computations are carried out in the same floating-point format as the the function being approximated; otherwise extended precision can be used to reduce the error. Whereas the total error accumulation can be bounded, the ordinary techniques used to do so do not allow relating the rounding errors for different input values to one another. So, if the error bound is small enough to imply monotonicity, fine. Otherwise, as things stand today, we are left with the approach of the previous section. However, we conjecture that (some of) the implementation algorithms can be analyzed with other techniques in order to obtain mode precise glitch data: this is another direction for future work.

### 7.3 Remaining Functions in math.h/cmath

In this paper we described how to deal with 75 of the standard C/C++ mathematical functions: but there are many others. Several of them are not problematic, as they are fully specified and their treatment poses no problem (e.g., the families of operations \texttt{round}, \texttt{trunc}, \texttt{floor}, \texttt{ceil}, \texttt{fma}, \texttt{fabs}, \texttt{floor}, \texttt{ceil}, \texttt{next} . . . ). In future work we will focus on the remaining functions, particularly the functions with two inputs, such as \texttt{atan2} and \texttt{pow}, and then the complex functions.

### 7.4 The Case of Correctly Rounded Libraries

When a function is correctly rounded for the rounding modes of interest, then the approach presented in this paper is clearly applicable: all the quasi-monotonic branches are also monotonic, zero glitches. However, if a library provides correctly rounded versions of both \(f: \mathbb{R} \rightarrow \mathbb{R}\) and \(f^{-1}: \mathbb{R} \rightarrow \mathbb{R}\) for all rounding modes, monotonicity allows a much more efficient approach: \cite{Mic02} presents optimal projections for isotonic unary functions over one argument that can be generalized to all piecewise monotonic functions.
7.5 Better Labeling Strategies for Constraint-Based Reasoning

Constraint-based symbolic model checkers and test input generators are based on search algorithms that operate by interleaving two rather different processes:

1. *constraint propagation*, in which constraints are used to refine the domains of variables (intervals or multi-intervals in our case), and
2. *labeling*, whereby a variable is chosen and its domain is partitioned into two or more subsets, each of which is explored separately.

It is the second process that drives the first one: when constraint propagation goes to *quiescence*, i.e., when no further refinement of the domains can be achieved, labeling splits the domain of the chosen variable, and this triggers a new phase of constraint propagation. This goes on until a solution has been found or one of the domains becomes empty, in which case unsatisfiability of the original problem is proved.

In this paper we only dealt with constraint propagation, but different choices, as far as the labeling strategies are concerned, have an enormous influence on performance. Unfortunately, there is no such a thing as *the* good labeling strategy (though it is possible to recognize some definitely bad labeling strategies): it is a matter of heuristics, and strategies that work well for one problem may still work badly for another problem. It is worth observing that test input generation and symbolic model checking give rise to constraint problems that may be of very different nature: while the latter are very often over-constrained (that is, there are few or no solutions at all, as the program in question exhibits very few or no run-time anomalies), this is not the case for the former (e.g., in the extreme case, a function body that is a basic block can be covered by a single test-input chosen more or less at random). This implies that input generation and symbolic model checking can both profit from the choice of different labeling strategies.

During the experimental part of our work on mathematical functions, we got a strong feeling that the current labeling strategy employed by ECLAIR can be significantly enhanced by defining heuristics that take into account how variables are constrained by invocations to such functions. Work on these new heuristics is ongoing.

8 Conclusion

There is a popular quotation in the software verification and validation community, whereby “Without a specification, a system cannot be right or wrong, it can only be surprising!” This captures quite well the current state of affairs for C/C++ software that uses the functions declared in the standard *math.h/cmath* header files. Despite the progress made on the development of correctly rounded functions all implementations in widespread use, especially in the world of

\[^22\] Paraphrased from YBK85.

\[^23\] See, e.g., the very interesting MetaLibm project at [http://www.metalibm.org/](http://www.metalibm.org/) (last accessed on October 23rd, 2016).
embedded systems, offer little or no guarantees about the computed results. As a consequence, the verification of programs using such functions is always painful and expensive and, for these reasons, more often than not it is only partially performed through testing. As the search space can be huge, testing can only cover a tiny fraction of all the possible value combinations: this cannot exclude the manifestation of unexpected results, certainly not with the level of confidence that is required for mission- and safety-critical applications.

The aim of this work is to improve upon the current situation now, i.e., without waiting for the wider adoption of correctly-rounded implementations. While such adoption is generally desirable and will certainly take place, at some stage and in some application domains, it is not clear whether correctly-rounded implementations can meet the efficiency criteria of all application domains, particularly in the field of embedded systems. Studying different implementations of the standard C/C++ mathematical functions, we realized that what they have in common is a piecewise quasi-monotonicity property: monotonicity is either preserved or only perturbed by small and, on average, not too frequent “glitches.” Based on this observation, we developed direct and indirect propagation algorithms for the refinement of intervals and multi-intervals. These algorithms can be integrated into abstract interpreters as well as symbolic model checkers and automatic test input generators based on constraint propagation.

The techniques proposed here are now used in the C/C++ semantic analysis components of the ECLAIR software verification platform and the initial experiences are quite positive. We are now able to properly verify absence of run-time anomalies for code using the C/C++ standard functions that, before, was completely out of reach. Verification in the strong sense is only feasible modulo the possibility of bounding the size of glitches (this can always be done for the single-precision functions) and the possibility to query the underlying implementation of the functions during the analysis. For the cases where the first condition cannot be granted, we can still detect many definite program issues, even though we cannot draw conclusions from the fact issues have not been found. When the second condition cannot be met, still it may be possible to use a reference implementation with significant commonalities with the target implementation (the case where libraries for different architectures are derived from the same code base is quite common), and we can nonetheless detect high-severity, possible program issues.

Even though much remains to be done before we can claim that the problem of the verification of C/C++ programs using the standard mathematical functions has been solved from the practical point of view, we believe the present work is a definite step forward in the right direction, and one that has the potential of improving, starting from today, the current state of the art.

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A Glitch Data for Other Implementations of \texttt{libm}

In this section, we provide additional data about the glitches in the single-precision functions for other implementations of the \texttt{math.h/cmath} mathematical functions. The reader is referred to Table 1 for information on the specific implementations.

Note that what appears as a large glitch in Table 7 for function \texttt{tanf} rounded down on the \texttt{macbook} machine, is actually due to a clear bug in the range reduction algorithm used there.

Table 4. Glitch data for the \texttt{alpha} machine

| function | $D_{\min}$ | $D_M$ | near | up | down | zero |
|----------|------------|-------|------|----|------|------|
| acosf    | $-1$       | $1$   |  $1$ |  $2$ |  $1$ |  $1$ |  $2$ |
| acoshf   | $1$        | $\infty$ |      |    |      |      |
| asinf    | $-1$       | $1$   |  $2$ |  $1$ |  $2$ |  $1$ |  $2$ |
| asinhf   | $-\infty$  | $\infty$ |      |    |      |      |
| atanf    | $-\infty$  | $\infty$ |  $1$ | $10^8$ |      |      |
| atanhf   | $-1$       | $1$   |  $2$ |  $1$ |  $2$ |  $1$ |  $2$ |
| cbrtf    | $-\infty$  | $10^6$ |  $1$ |  $2$ | $10^6$ | $1$ |  $2$ | $10^6$ |  $1$ |  $2$ |
| coshf    | $-\infty$  | $454$ |  $2$ |  $466$ |  $1$ |  $2$ | $442$ |  $1$ |  $2$ | $448$ |  $1$ |  $2$ |
| erff     | $-\infty$  | $\infty$ |      |    |      |      |
| expf     | $-\infty$  | $\infty$ |      |    |      |      |
| exp10f   | $-\infty$  | $\infty$ |      |    |      |      |
| exp2f    | $-\infty$  | $\infty$ |  $1$ |  $2$ |  $3$ |  $1$ |  $2$ |  $1$ |  $2$ |  $1$ |  $1$ |  $2$ |
| expm1f   | $-\infty$  | $\infty$ |      |    |      |      |
| lgammaf  | $2\infty$  | $163$ |  $1$ |  $2$ | $10^4$ |  $2$ |  $2$ | $10^4$ |  $1$ |  $2$ | $10^4$ |  $1$ |  $2$ | $10^4$ |  $1$ |  $2$ | $10^4$ |  $1$ |  $2$ | $10^4$ |  $1$ |  $2$ |
| logf     | $0\infty$  | $164$ |  $1$ |  $2$ | $166$ |  $1$ |  $2$ | $161$ |  $1$ |  $2$ |
| log10f   | $0\infty$  | $\infty$ |      |    |      |      |
| log1pf   | $-1\infty$ | $\infty$ |  $1$ |  $2$ |  $1$ |  $2$ |  $1$ |  $2$ |
| log2f    | $0\infty$  | $\infty$ |      |    |      |      |
| sinhf    | $-\infty$  | $\infty$ |      |    |      |      |
| sqrtf    | $0\infty$  | $\infty$ |      |    |      |      |
| tanhf    | $-\infty$  | $\infty$ |  $1$ |  $2$ |  $2$ |  $1$ |  $3$ |      |      |
| tgammf   | $2\infty$  | $10^4$ |  $2$ |  $3$ | $10^4$ |  $2$ |  $4$ | $10^4$ |  $3$ |  $3$ | $10^4$ |  $3$ |  $4$ |      |      |
| cosf     | $-2^{23}$  | $2^{23}$ |      |    |      |      |
| sinh     | $-2^{23}$  | $2^{23}$ |      |    |      |      |
| tanf     | $-2^{23}$  | $2^{23}$ |      |    |      |      |

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Table 5. Glitch data for the gcc111 machine

| function | $D_{min}$ | $D_{M}$ | near | up | down | zero |
|-----------|-----------|---------|------|----|------|------|
| acoshf    | 1         | $\infty$|      |    |      |      |
| asinhf    | $-\infty$| $\infty$|      |    |      |      |
| atanf     | $-\infty$| $\infty$|      |    |      |      |
| atanhf    | -1        | 1       |      |    |      |      |
| coshf     | $-\infty$| $\infty$|      |    |      |      |
| erff      | $-\infty$| $\infty$|      |    |      |      |
| lgammaf   | 2         |         |      |    |      |      |
| sinhf     | $-\infty$| $\infty$|      |    |      |      |
| tanhf     | $-\infty$| $\infty$|      |    |      |      |
| tgammaf   | 2         |         |      |    |      |      |
| cosf      | $-2^{23}$ | $2^{23}$|      |    |      |      |
| sinf      | $-2^{23}$ | $2^{23}$|      |    |      |      |
| tanf      | $-2^{23}$ | $2^{23}$|      |    |      |      |

Table 6. Glitch data for the igor machine

| function | $D_{min}$ | $D_{M}$ | near | up | down | zero |
|-----------|-----------|---------|------|----|------|------|
| acoshf    | -1        | 1       |      |    |      |      |
| acoshf    | 1         | $\infty$|      |    |      |      |
| asinhf    | -1        | 1       |      |    |      |      |
| asinhf    | $-\infty$| $\infty$|      |    |      |      |
| atanf     | $-\infty$| $\infty$|      |    |      |      |
| atanhf    | -1        | 1       |      |    |      |      |
| cbrtf     | $-\infty$| $10^6$  | 1    | 2 | $10^6$| 1    |
| coshf     | $-\infty$| 454     | 1    | 2 | 466  | 1    |
| erff      | $-\infty$| $\infty$|      |    |      |      |
| expf      | $-\infty$| $\infty$|      |    |      |      |
| exp10f    | $-\infty$| $\infty$|      |    |      |      |
| exp2f     | $-\infty$| $\infty$| 1    | 2 | 3    | 1    |
| expm1f    | $-\infty$| $\infty$|      |    |      |      |
| lgammaf   | 2         | $163$   | 1    | 2 | 164  | 1    |
| logf      | 0         | $\infty$|      |    |      |      |
| log10f    | 0         | $\infty$|      |    |      |      |
| log1pf    | -1        | $\infty$|      |    |      |      |
| log2f     | 0         | $\infty$|      |    |      |      |
| sinhf     | $-\infty$| $\infty$|      |    |      |      |
| sqrtf     | 0         | $\infty$|      |    |      |      |
| tanhf     | $-\infty$| $\infty$|      |    |      |      |
| tgammaf   | 2         | $155$   | 2    | 1 | 122  | 2    |
| cosf      | $-2^{23}$ | $2^{23}$| $10^4$| 1 | 3    | $10^4$|
| sinf      | $-2^{23}$ | $2^{23}$| $10^4$| 1 | 3    | $10^4$|
| tanf      | $-2^{23}$ | $2^{23}$| $10^4$| 1 | 3    | $10^4$|
Table 7. Glitch data for the macbook machine

| function | $D_{\text{min}}$ | $D_M$ | near | up | down | zero |
|----------|-----------------|-------|------|----|------|------|
| acosf    | $-1$            | 1     |      |    |      |      |
| acoshf   | $1$             | $\infty$ |      |    |      |      |
| asinf    | $-1$            | 1     |      |    |      |      |
| asinhf   | $-\infty$       | $\infty$ |      |    |      |      |
| atanf    | $-\infty$       | $\infty$ |      |    |      |      |
| atanhf   | $-1$            | 1     |      |    |      |      |
| cbrtf    | $-\infty$       | $\infty$ |      |    |      |      |
| coshf    | $-\infty$       | $\infty$ |      |    |      |      |
| erff     | $-\infty$       | $\infty$ |      |    |      |      |
| expf     | $-\infty$       | $\infty$ |      |    |      |      |
| exp10f   | $-\infty$       | $\infty$ |      |    |      |      |
| exp2f    | $-\infty$       | $\infty$ |      |    |      |      |
| expm1f   | $-\infty$       | $\infty$ |      |    |      |      |
| lgammaf  | 2               | $\infty$ |      |    |      |      |
| logf     | 0               | $\infty$ |      |    |      |      |
| log10f   | 0               | $\infty$ |      |    |      |      |
| log1pf   | $-1$            | $\infty$ |      |    |      |      |
| log2f    | 0               | $\infty$ |      |    |      |      |
| sinhf    | $-\infty$       | $\infty$ |      |    |      |      |
| sqrtf    | 0               | $\infty$ |      |    |      |      |
| tanhf    | $-\infty$       | $\infty$ |      |    |      |      |
| tgammaf  | 2               | $\infty$ |      |    |      |      |
| cosf     | $-2^{23}$       | $2^{23}$ |      |    |      |      |
| sinf     | $-2^{23}$       | $2^{23}$ |      |    |      |      |
| tanf     | $-2^{23}$       | $2^{23}$ |      |    |      |      |

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Table 8. Glitch data for the raspi machine

| function | $D_{min}$ | $D_M$ | near | up | down | zero |
|-----------|-----------|-------|------|----|------|------|
| acosf     | $-1$     | $1$   |      |    |      |      |
| acoshf    | $1$      | $\infty$ | 1 | 1 | 2 | 1 | 2 |
| asinf     | $-1$     | $1$   |      |    |      |      |
| asinhf    | $-\infty$ | $\infty$ | 2 | 1 | 2 | 2 | 1 | 2 |
| atanf     | $-\infty$ | $\infty$ | 1 | 1 | $10^8$ |      |
| atanhf    | $-1$     | $1$   |      |    |      |      |
| cbrtf     | $-\infty$ | $\infty$ | $10^6$ | 2 | 1 | $10^6$ | 1 | 2 | $10^6$ | 1 | 2 | 2 | 1 | 2 |
| coshf     | $-\infty$ | $\infty$ | 454 | 1 | 2 | 466 | 1 | 2 | 442 | 1 | 2 | 448 | 1 | 2 |
| erf       | $-\infty$ | $\infty$ |      |    |      |      |
| exp       | $-\infty$ | $\infty$ |      |    |      |      |
| exp10f    | $-\infty$ | $\infty$ |      |    |      |      |
| exp2f     | $-\infty$ | $\infty$ | 1 | 1 | 2 |      |      |
| expm1f    | $-\infty$ | $\infty$ |      |    |      |      |
| lgammaf   | $2$      | $\infty$ | 163 | 2 | 164 | 2 | 166 | 1 | 2 | 161 | 1 | 2 |      |      |
| log       | $0$      | $\infty$ |      |    |      |      |
| log10f    | $0$      | $\infty$ |      |    |      |      |
| log1pf    | $-1$     | $\infty$ |      |    |      |      |
| log2f     | $0$      | $\infty$ |      |    |      |      |
| sinh      | $-\infty$ | $\infty$ |      |    |      |      |
| sinhf     | $-\infty$ | $\infty$ |      |    |      |      |
| sqrt      | $0$      | $\infty$ |      |    |      |      |
| sqrtf     | $0$      | $\infty$ |      |    |      |      |
| tanhf     | $-\infty$ | $\infty$ |      |    |      |      |
| tgammaf   | $2$      | $\infty$ | $10^4$ | 2 | 3 | $10^4$ | 2 | 4 | $10^4$ | 3 | 3 | $10^4$ | 3 | 4 |      |
| cosf      | $-2^{23}$ | $2^{23}$ | $10^3$ | 1 | 3 | $10^4$ | 1 | 3 | $10^4$ | 1 | 3 |      |      |
| sinf      | $-2^{23}$ | $2^{23}$ | $10^4$ | 1 | 3 | $10^4$ | 1 | 3 | $10^4$ | 1 | 3 |      |      |
| tanf      | $-2^{23}$ | $2^{23}$ |      |    |      |      |
B Computation of Lower Bounds: Proofs and Complexity

Lemma 41 Function $\text{findhi}_\text{lb}$ of Algorithm 2 satisfies its contract.

Proof. The proof begins by assuming the precondition for $\text{findhi}_\text{lb}(f, y, [x_l, x_u], n_\text{g}, d, w, \omega, \alpha, t)$ is satisfied: in particular, $f(x_u) \prec y$. The condition on line 2 holds in the following cases:

- $n_\text{g} = 0$: as $n_\text{g} \geq n_\text{f}$ we have $n_\text{f} = 0$, that is, $f$ is isotonic on $[x_l, x_u]$; thus $f(x_u) \prec y$ implies $f(x) \prec y$ for each $x \in [x_l, x_u]$.
- $x_u \succ \omega$: as $\omega \geq \omega^f$, we know $x_u \succ \omega^f$. This implies that $x_u$ cannot be inside a glitch, since, by definition of $\omega^f$, $f$ is isotonic on $[\omega^f, x_u]$. Therefore $f(x) \lneq f(x_u) \prec y$ for each $x \in [x_l, x_u]$.
- $y \succ^t f(x_u)$: this means that $f(x_u)$ is so low that, even under the worst-case assumption $x_u$ is the minimum of a maximal-depth glitch, the value of $f$ just before the glitch would still be below $y$. Therefore, $f(x) \prec y$ for each $x \in [x_l, x_u]$.

In all the circumstances listed above, the post-condition for the case $r = 0$ is proved.

Line 3 contains an else statement: we know that $n_\text{g} > 0$, $x_u \ll \omega$ and $y \ll^{\text{last}} f(x_u)$. Further, let us assume that the condition on line 4 is true, hence $n_\text{g} = 1$. The reasoning must be split as follows:

- $y \succ f(\alpha)$: we must check whether a glitch starts in $\alpha$ or not.
  - $f(\alpha^+) \prec f(\alpha)$: in this case, $f$ has exactly one glitch in $[x_l, x_u]$ beginning in $\alpha^f = \alpha$. Hence, $f(x) \lneq f(\alpha^f)$ for each $x \in [x_l, \alpha^f]$ and, since $y \succ f(\alpha)$, we also have $f(x) \prec y$ for each $x \in [x_l, \alpha^f]$. Moreover, the precondition of the algorithm entails that $f(x_u) \prec y$, which, together with $n_\text{f} = 1$, allows us to conclude that $f(x) \prec y$ also in interval $(\alpha^f, x_u]$. Summing up, $[x_l, x_u]$ does not contain any solutions for $y = f(x)$ and setting $r = 0$ in line 7 satisfies the post-condition.
  - $f(\alpha^+) \succ f(\alpha)$: in this case, $f$ either has no glitch or it has exactly one glitch strictly past $\alpha$ (that is, $\alpha^f > \alpha$). As glitches can be too wide (i.e., $w > t$), we refrain from searching a suitable value for $\text{hi}$ to start bisection with. However, as $y \succ f(\alpha)$, we have $y \succ f(x)$ for each $x \in [x_l, \alpha^f]$: predicate $p_2(y, x_l, x_u, \alpha)$ is satisfied, and setting $l = \alpha$ and $r = 2$ in line 9 is correct.
- $y \lneq f(\alpha)$: setting $\text{hi} = \alpha$ and $r = 1$, as done in line 12 is guaranteed to satisfy the post-condition. In fact, $n_\text{g} = 1$ implies $x_l \ll \alpha \ll \omega \ll x_u$, therefore $\text{hi} \in [x_l, x_u]$ and $f(\text{hi}) \succ y$.

On the contrary, if the condition on line 2 is false, control flow reaches the else statement on line 14 and we know that either $n_\text{g} > 1$ or $w \leq t$. In this case function $\text{linesearch_geq}(f, y, [x_l, x_u], w, t)$ searches backwards the first value for $\text{hi}$ such that $f(\text{hi}) \succ y$; it dows so float-by-float, starting from $x_u$, and it stops after the minimum between $t$ and $w$ steps, but without going beyond $x_l$. 

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Lemma 42 Function bisect \_ lb of Algorithm 3 satisfies its contract.

Proof. Let us assume that the precondition for bisect \_ lb (f, y, [x \_ l, x \_ u], n \_ e, d \_ M, w \_ M, \alpha, \omega, n \_ g, s, t) is satisfied initially. We will consider the following while loop invariant:

$$\text{Inv} \equiv (x_\ell \leq y \wedge (f(x_\ell) < y \wedge y \leq f(h_\ell)) \wedge (\forall x \in [x_\ell, l_0] : f(x) < y).$$

We will show that when the while loop of bisect \_ lb finishes, the post-condition in the Ensure statement holds. To this aim, we will prove that the while loop of bisect \_ lb satisfies the following properties.

Initialization: the invariant Inv holds prior to the first loop iteration because it is entailed by the precondition, that is, the Require statement.

Maintenance: assume that Inv holds at the beginning of an arbitrary loop iteration: we will prove that it holds at the end of that iteration, as well. If the guard on line 11 hi >\^1 lo, is false, then Inv trivially holds at the end of the loop. Therefore, assuming that hi >\^1 lo holds, a stronger property can be proved: Inv holds and either the loop is exited with a break statement, or the new value hi’ or lo’, that is updated during the iteration, is contained into the open interval (lo, hi).

After the invocation of function split \_ point, at line 4 mid = hi^{-m} = lo^{+m’}, with m, m’ > 0, holds. Hence, lo < mid < hi holds and this property is exploited in the following two main cases, tested beginning at line 4

$$y \leq f(mid): \text{in this case hi’ = mid. By assuming that Inv holds before the}$$

iteration, we need to prove that $$x_\ell \leq y \wedge hi’ < hi \leq x_\ell$$ and $$f(lo) < y \leq f(hi’).$$ Note that this is a direct consequence of lo < mid < hi and $$y \leq f(mid).$$ Since the value of lo was not updated, the remaining part of
invariant Inv holds as it did before this iteration. Moreover, in the rest of the proof we will show that \( \forall x \in [x_l, l_0'] : f(x) \prec y \) only when the value of \( l_0 \) is being updated.

\( y \succ f(\text{mid}) \): here, further cases need to be distinguished.

\[ \begin{align*}
\text{If } l_0 < \text{mid} < h_i \text{ and } y \succ f(\text{mid}), \text{ we can derive that } x_l < l_0 < l_0' < h_i \leq x_u \text{ and } f(l_0') \prec y \preceq f(h_i). \text{ We are left to prove that } \forall x \in [x_l, l_0'] : f(x) \prec y. \text{ If the function } f \text{ is isotonic or, at least, we are sure that point mid is not inside a glitch, that is, } \text{mid} \preceq \alpha \preceq \alpha_l \text{ or } \text{mid} \preceq \omega \preceq \omega_l, \text{ } y \succ f(\text{mid}) \text{ implies that } \forall x \in [x_l, l_0' = \text{mid}] : f(x) \prec y \text{ and the invariant Inv is proved. For the remaining case } y \succ^d \text{ mid } f(\text{mid}), \text{ note that } f(\text{mid}) \text{ is so low that the value of } f \text{ just before the glitch would be smaller than } y \text{ even under the worst-case assumption mid is the minimum of a maximal-depth glitch. Therefore, we can conclude that } \forall x \in [x_l, l_0' = \text{mid}] : f(x) \prec y.
\end{align*} \]

\[ n_g = 1 \text{ and either } \mu_0 \succ t \text{ or } f(\alpha^+) \prec y(\alpha) : \text{ since line } 8 \text{ is the else-if guard of the if statement at line } 4 \text{ we are sure that } \alpha \prec \text{mid} \prec \omega. \text{ This implies that mid may be inside a glitch. For this reason, at line } 9 \text{ the condition } f(\omega) \succ y \text{ is tested and if it holds, at line 10 condition } f(\alpha) \succ y \text{ is tested. If it is satisfied, we set } h_i = \alpha. \text{ Since mid} \prec h_i \leq x_u \text{ and } \alpha \prec \text{mid}, \text{ we can conclude that } \alpha \prec h_i. \text{ Moreover, since } f(l_0) \prec y, f(\alpha) \succ y \text{ and } \alpha \preceq \alpha_l, \text{ we have } l_0 \prec \alpha. \text{ Hence, we can conclude } x_l \leq l_0 \prec h_i' = \alpha \prec h_i \leq x_u \wedge f(l_0) \prec y \preceq f(h_i') \text{ holds.} \]

\[ \text{At line } 12 \text{ we know that } f(\alpha) \prec y \preceq f(\omega). \text{ Then we test if } f(\alpha^+) \prec f(\alpha) : \text{ if this is true, we can be sure that the unique glitch of function } f \text{ starts exactly at point } \alpha, \text{ that is, } \alpha = \alpha_l. \text{ Therefore, we have } f(\alpha_l') \prec y \preceq f(\omega). \text{ In this case } l_0' = \text{mid}. \text{ From } l_0 < \text{mid} < h_i \text{ and } y \succ f(\text{mid}), \text{ we can derive that } x_l \leq l_0 < l_0' < h_i \leq x_u \wedge f(l_0') \prec y \preceq f(h_i) \text{ and } \forall x \in [x_l, l_0' = \alpha'] : f(x) \prec y, \text{ since } f(\alpha) \prec y \text{ and } \alpha \preceq \alpha_l. \text{ This proves the invariant Inv.} \]

\[ \text{At line } 14 \text{ we know that } l_0 \prec \alpha \text{ and we exit from the loop. Note that, in this case, the invariant Inv still holds, trivially.} \]

\[ \text{When line } 19 \text{ is reached we are sure that } f(\omega) \prec y \text{ and we set } l_0' = \omega. \text{ Since } f(\omega) \prec y, \omega \preceq \omega_l \text{ and } y \preceq f(h_i), \text{ we can conclude that } \omega \prec h_i \leq x_u. \text{ Moreover, observe that } \omega \preceq l_0 \text{ cannot hold: since the control flow reached this point, the else-if-guard of line } 6 \text{ is false, and mid} \preceq \omega. \text{ This would imply } \text{mid} \preceq l_0, \text{ which contradicts the post-condition of function split point}. \text{ Therefore, } x_l \leq l_0 \prec \omega \prec h_i \leq x_u. \text{ Moreover, by } f(\omega) \prec y \text{ and } \omega \preceq \omega_l \text{ we have } \forall x \in [x_l, l_0' = \omega] : f(x) \prec y. \text{ Setting } l_0 := \omega \text{ satisfies the invariant.} \]
\(w_M \leq t\) and the previous conditions are false: at line 22 function \(\text{findmax}(f, w_M, \text{lo}, \text{mid})\) is called, returning a value \(b\) that satisfies the post-condition of line 22. The if statement of line 24 distinguishes between two cases:

- \(f(b) \succ y\): \(hi' = b\) is set. Note that by the post-condition of line 22 we have \(b \in [\max\{\text{lo}, \text{mid}^{-w_M}\}, \text{mid}]\). Since \(\text{lo} \prec \text{mid} \prec \text{hi}\) and \(f(b) \succ y\) while \(f(\text{lo}) \prec y\), we have \(\text{lo} \prec b \prec \text{hi}\). Therefore, \(x_t \not\in \text{lo} \prec hi' = b \prec hi \not\in x_u\).

- \(f(b) \prec y\): \(lo' = \text{mid} \) is set. Since \(\text{lo} \prec \text{mid} \prec \text{hi}\) we have \(x_l \not\in \text{lo} \prec lo' = \text{mid} \prec \text{hi} \not\in x_y\). Moreover, \(f(lo') \prec y \not\in f(\text{hi})\), since line 27 is in the else body of the if construct of line 4. Finally, we have to prove that \(\forall x \in [x_l, lo'] : f(x) \prec y\). From post-condition at line 22 we know that \(\forall x \in [\max\{\text{lo}, \text{mid}^{-w_M}\}, \text{mid}] : f(x) \not\prec f(b)\).

We have now two cases:

1. \(\max\{\text{lo}, \text{mid}^{-w_M}\} = \text{lo}\): in this case, the post-condition at line 22 implies that \(\forall x \in [\text{lo}, \text{mid}] : f(x) \not\prec f(b) \prec y\).

2. \(\max\{\text{lo}, \text{mid}^{-w_M}\} = \text{mid}^{-w_M}\): since \(w_M \geq w_M'\), even in the worst-case scenario, that is, mid is inside a glitch at the maximal distance from the its beginning, interval \([\text{mid}^{-w_M}, \text{mid}]\) contains the last point \(x_m\) before the glitch started. Therefore, we have \(\forall x \in [\text{lo}, x_m] : f(x) \not\prec f(x_m)\). Together with \(\forall x \in [\text{mid}^{-w_M}, \text{mid}] : f(x) \not\prec f(b) \prec y\), this implies \(\forall x \in [\text{lo}, \text{mid}] : f(x) \not\prec f(b) \prec y\), which proves that \(\forall x \in [x_l, \text{lo}'] : f(x) \prec y\).

In both cases, invariant \(\text{Inv}\) holds.

Otherwise: Function \(\text{logsearch lb}(f, d_M, \text{lo}, \text{mid}, \text{y}, \text{s})\), which returns a value \(z\) satisfying the post-condition stated at line 31 is called at line 23. The if statement of line 24 distinguishes between two cases:

- \(\text{lo} \prec z\): we set \(\text{lo}' = z\). From the post-condition stated at line 31 we have \(z \in [\text{lo}, \text{mid}]\). Together with \(\text{lo} \prec z\), this gives \(x_l \not\in \text{lo} \prec \text{lo}' = z \prec \text{hi} \not\in x_u\). Moreover, the post-condition stated at line 31 implies that \(f(z) \prec^{d_M} y\): therefore also \(f(lo') \prec y \not\prec f(\text{hi})\) holds. Finally, we have to prove that \(\forall x \in [x_l, \text{lo}' = z] : f(x) \prec y\).

Since \(f(z) \prec^{d_M} y\), the value of \(f(z)\) is so low that, even under the worst-case assumption \(z\) is the minimum of a maximal-depth glitch, the value of \(f\) just before the glitch would still be below \(y\). This implies that \(\forall x \in [x_l, \text{lo}' = z] : f(x) \prec y\).

- \(\text{lo} = z\): in this case we exit the loop without any change and, therefore, invariant \(\text{Inv}\) trivially holds.

**Termination:** We have just proved that invariant \(\text{Inv}\) holds and either we exit the loop with a break statement, or the new value \(hi'\) or \(lo'\) is contained into the interval \((\text{lo}, \text{hi})\). Therefore, \#(\text{lo}, \text{hi}) decreases at each iteration. The guard of the while loop at line 11 tests the condition \(hi > 1 \text{lo}\), that is equivalent to \#(\text{lo}, \text{hi}) > 2. This assures that the while loop always terminates.

**Correctness:** We will prove that, whenever the loop invariant and the loop exit-condition both hold, then the post-condition stated in the Ensure statement
holds. Since the correctness post-condition coincides with the invariant Inv, we only need to prove that the precision post-condition holds. Note that, under the hypothesis that \( n_g = 0 \), \( w_M < t \) or \( n_g = 1 \land \alpha = \alpha^f \), the control flow of the program can never reach the break statements at lines 17 or 25. If we did not exit with one of the above mentioned break statements, \(#[lo, hi] = 2\) finally holds, that is, \( lo^+ = hi\). Therefore we can conclude that, when we exit the loop, \( lo^+ = hi\) holds. The fact that, according to the invariant Inv, we have \( y \preceq f(hi) \), implies that \( f(lo^+) \succ y\). This concludes the proof. □

**Theorem 43.** Function `lower_bound` of Algorithm 1 satisfies its contract.

*Proof.* The precondition for `lower_bound`\((y, [x_l, x_u], n_k, d_M, w_M, \alpha, \omega, f^1, t)\) will be assumed to be satisfied, initially. First, we will prove the correctness post-condition. As described in Section 4.2 function `init`\((y, [x_l, x_u], f^1)\) has been designed to return a point inside the interval \([x_l, x_u]\), which, therefore, meets the condition at line 2. Moreover, recall that function `gallop_lb`\((f, y, [x_l, x_u], d_M, i)\) starts with \( lo = hi = i \), where \( x_l \leq i \leq x_u \) and it is made in such a way that it returns new values for \( lo \) and \( hi \) satisfying the condition at line 4. The goal now is to verify if such \( lo \) and \( hi \) can be used for bisection.

We first focus on the value of \( lo \): the value \( f(lo) \) is compared to \( y \) on line 5 and on line 11.

\( f(lo) \succ y \): In this case, according to the post-condition at line 4 we can be sure that \( lo = x_l \). Since \( f(x_l) \succ y \), we further need to distinguish the case in which we are guaranteed that \( \forall x \in \[x_l, x_u\] : f(x) \succ y \) from the case we are not. The first case surely occurs in the following conditions, which are part of the guard of the if statement on line 6:

\( n_g = 0 \): in this case, since \( n_g \geq n_g^f \), we can conclude that \( n_g^f = 0 \) and, therefore, that function \( f \) is isotonic in interval \([x_l, x_u]\). Since \( f(lo) \succ y \) we can be sure that \( \forall x \in \[x_l, x_u\] : f(x) \succ y \).

\( f(\alpha) \succ d_M y \): since \( d_M \geq d_M^f \) (from the precondition), we can conclude that \( f(\alpha) \succ d_M y \). Moreover, \( \alpha \preceq \alpha^f \) and the fact that \( f \) is quasi-isotonic allow us to conclude that \( f(\alpha^f) \succ f(\alpha) \succ d_M y \). By definition of \( \alpha^f \) we know that function \( f \) is isotonic on interval \([x_l, \alpha^f]\), and since \( f(x_l) \succ y \) we can conclude that \( \forall x \in \[x_l, \alpha^f\] , f(x) \succ y \). Moreover, \( f(\alpha^f) \succ d_M y \) assures us that the value \( f(\alpha^f) \) is so high that, even under the worst case assumption that the glitch starting at \( \alpha^f \) has maximal depth, the value of \( f \) could not decrease enough to reach \( y \). Therefore, we can be sure that \( \forall x \in \[x_l, x_u\] : f(x) \succ y \).

Hence, in all circumstances in which the condition on line 6 is true, the post-condition for values \( r = 1 \) and \( l = x_u \) is proved.

At line 8 we are on the else statement, and therefore we cannot conclude that \( \forall x \in \[x_l, x_u\] : f(x) \succ y \), but we cannot apply bisection either, since we are under the assumption \( f(lo) \succ y \). From \( \alpha \preceq \alpha^f \) and \( f(x_l) \succ y \), we have \( \forall x \in \[x_l, \alpha]\) , \( f(x) \succ y \). Therefore, setting \( l = \alpha \prec x_u \) satisfies the post-condition for the case \( r = 1 \).
\( f(lo) = y \): In this case, according to the condition at line 4, we can be sure that \( x_l = lo \). Therefore, setting \( l = lo \) satisfies the post-condition for the case \( r = 4 \).

Since each branch of the \texttt{if-else} instructions considered until now contains a return statement, if the control flow reaches the end of line 13 we are sure that \( f(lo) \prec y \).

At line 14, we focus on the value of \( hi \). If \( f(hi) \prec y \), using the condition at line 4 we are sure that \( hi = x_u \) and, hence, \( f(x_u) \prec y \). Under this condition, the value of \( hi = x_u \) cannot be used for bisection: function \texttt{findhi}\_\texttt{lb} \((f, y, [x_l, x_u], n_g, d_M, w_M, \alpha, \omega, t)\) specified in Algorithm 2 is called. The post-condition of function \texttt{findhi}\_\texttt{lb} holds. At line 16, we test if function \texttt{findhi}\_\texttt{lb} returned \( r = 0 \) or \( r = 2 \): in these cases the post-condition of function \texttt{findhi}\_\texttt{lb} is satisfied. At the end of line 17, we know that function \texttt{findhi}\_\texttt{lb} returned \( r = 1 \) and, therefore, a new value such that \( x_l \prec hi \prec x_u \) and \( f(hi) \succeq y \) was assigned to \( hi \).

Summarizing, before the invocation of function \texttt{bisect}\_\texttt{lb} at line 19 we are sure that \( f(lo) \prec y \prec f(hi) \) and, therefore, \( lo \neq hi \).

In order to prove that the preconditions of function \texttt{bisect}\_\texttt{lb} are met, we need to prove that the values of \( lo \) and \( hi \) at line 19 satisfy the following predicates: \( x_l \prec lo \prec hi \prec x_u \) and \( \forall x \in [x_l, lo] : f(x) \prec y \). Note, indeed, that we have already proved that at line 19 \( f(lo) \prec y \prec f(hi) \) holds and that the remaining requirements of function \texttt{bisect}\_\texttt{lb} are implied by the precondition of function \texttt{lower bound}. Moreover, we will first prove that the value of \( lo \) at line 19 satisfies \( x_l \prec lo \) and that \( \forall x \in [x_l, lo] : f(x) \prec y \). Observe that the value of \( lo \) is the one returned by function \texttt{gallop}\_\texttt{lb}, since the code at lines 5-19 tests the value of \( lo \) without changing it. Therefore, by the post-condition of line 4 we have \( x_l \prec lo \). As for the remaining clause, we have two cases:

\( x_l \prec lo \): by the post-condition of line 4 we have \( y \succ^{d_M} f(lo) \). This means that \( f(lo) \) is so low that, even under the worst-case assumption \( lo \) is the minimum of a maximal-depth glitch, the value of \( f \) just before the glitch would still be below \( y \). This implies that \( \forall x \in [x_l, lo] : f(x) \prec y \).

\( x_l = lo \): since at line 14 we are sure that \( f(x_l) \prec y \), in this case \( \forall x \in [x_l, lo] : f(x) \prec y \) holds, trivially.

In order to prove that the value of \( hi \) at line 19 satisfies \( lo \prec hi \prec x_u \), we need to distinguish between the following cases:

\( y \prec f(hi) \): the value of \( hi \) is the one computed by \texttt{gallop}\_\texttt{lb}, since the guard of the \texttt{if} instruction at line 14 is false. Therefore, by the condition of line 4 we have \( lo \prec hi \prec x_u \). Since at line 19 we are sure that \( lo \neq hi \), we can conclude that \( lo \prec hi \prec x_u \).

\( y \succ f(hi) \): in this case, the current value of \( hi \) is the one chosen by function \texttt{findhi}\_\texttt{lb}, which must have returned \( r = 1 \), since we are at line 19. Therefore, by the post-condition of function \texttt{findhi}\_\texttt{lb}, we know that the new
value of hi satisfies hi \in [x_l, x_u] and f(hi) \succ y. Since we have proved that 
\forall x \in [x_l, lo] : f(x) \prec y, we can conclude that lo \prec hi \preceq x_u.

Hence, since the preconditions of function bisect_lb are met, by Lemma 42
we are sure that, after it returns, x_l \preceq lo \prec hi \preceq x_u, f(lo) \prec y \preceq f(hi) and
\forall x \in [x_l, lo] : f(x) \prec y hold, for the new values of lo and hi. At line 20 a while
loop is entered. This loop performs a float-by-float search (for a maximum of t
iterations) to approach the exact solution of y = f(x). We want to prove that
the predicate \forall x \in [x_l, lo] : f(x) \prec y, which is also the invariant of the loop,
holds at line 23. To this aim, we will prove the following loop properties:

Initialization: the invariant \forall x \in [x_l, lo] : f(x) \prec y holds prior to the first loop
iteration because it is entailed by the post-condition of function bisect_lb.

Maintenance: assume \forall x \in [x_l, lo] : f(x) \prec y holds at the beginning of an
arbitrary loop iteration. Then such assumption, together with the guard of
the loop f(lo^+) \prec y and the assignment lo' := lo^+ in the body of the loop,
allows us to conclude that \forall x \in [x_l, lo'] : f(x) \prec y holds also at the end of
that iteration.

Termination: the while loop terminates because we are assured by the post-
condition of bisect lb that there exists a value hi such that lo \prec hi and
y \preceq f(hi). Moreover, the loop can end before reaching such value, because
the parameter t \in \mathbb{N} is decremented inside the loop, until it reaches 0.

Correctness: as a consequence, at the end of the while loop, the property
\forall x \in [x_l, lo] : f(x) \prec y holds and either f(lo^+) \succeq y or t = 0.

Then, at line 24 we test if f(lo^+) \succ y. Since \forall x \in [x_l, lo] : f(x) \prec y, setting l
to the value of lo satisfies the post-condition for the case r = 3. With the else-if
instruction at line 26 we test if f(lo^+) = y. In this case, since \forall x \in [x_l, lo] :
f(x) \prec y, setting l to the value of lo^+ satisfies the post-condition for the case
r = 4. Furthermore, if we are on the else instruction of line 28 it means that the
while-loop of line 20 terminated because t became equal to 0. In this case, since
\forall x \in [x_l, lo] : f(x) \prec y, setting l to the value of lo satisfies the post-condition
for the case r = 2.

We will now prove the precision post-condition under the hypothesis that
f(x_l) \preceq y \preceq f(x_u) holds and either n_x = 0, w_M < t or n_x = 1 \land \alpha = \alpha'
holds. Reasoning as before, at line 3 we are sure that the function gallop_lb(f, y, [x_l, x_u],
d_M, i) returns values for lo and hi such that post-condition of line 4 is satisfied.
The latter, together with the assumption f(x_l) \preceq y \preceq f(x_u), allows us to con-
clude that, in the cases we are interested in, f(lo) \preceq y \preceq f(hi). The condition
tested by the if statement at line 5 is surely false. Therefore, in this case, the
condition at line 11 needs to be tested. We have the following cases:

f(lo) = y: function lower_bound returns with r = 4. Therefore, the precision
post-condition is satisfied.
f(lo) \prec y: since y \preceq f(hi), the test of line 14 is false and before line 19 we are
sure that x_l \preceq lo \prec hi \preceq x_u, since f(x_l) \prec y and y \preceq f(x_u). Therefore, since
the preconditions of function bisect lb are met, by Lemma 42 we know
that the post-condition holds. In more details, since we have assumed that
\( n_g = 0 \lor w_M < t \lor (n_g = 1 \land \alpha = \alpha^f) \), from the precision post-condition of
function \( \text{bisect}_\ell \) we can derive that \( f(lo^+) \succ y \). In this case, the while
loop of line \( 20 \) is never executed. The value of \( f(lo^+) \succ y \) is then tested at lines
24 and 26. If \( f(lo^+) \succ y \) then \( r = 3 \) is returned, if \( f(lo^+) = y \), then
\( r = 4 \) is returned. In both cases, the precision post-condition is satisfied.

This completes the proof. \( \square \)

**Theorem 44.** If \( f : \mathbb{F} \rightarrow \mathbb{F} \) is an isotonic function, that is, \( n_g = 0 \), then, for
each \( [x_l, x_u] \in \mathcal{I}_g, d_M, w_M, \alpha, \omega, f^1 : \mathbb{F} \rightarrow \mathbb{F}, s, t \in \mathbb{N} \), computing lower_bound
as per Algorithm 1 evaluates \( f \) at most \( 2 \log_2(|[x_l, x_u]|) + 4 \) times.

**Proof.** At the beginning of Algorithm 1 function \( \text{init}(y, [x_l, x_u], f^1) \) is called and it returns a point inside interval \( [x_l, x_u] \), meeting the condition at line 2. It never calls function \( f \). Then, function \( \text{gallop}_\ell(f, y, [x_l, x_u], d_M, t) \) is called with \( lo = hi = i \). First, we will assume that \( d_M = 0 \). Since either \( y \succ 0 f(i) \) (that is, \( y \succ f(i) \)) or \( f(i) \succeq y \) holds, the worst cases are the following:

- \( i = x_l \) and \( \forall x \in [x_l, x_u] : y \succ f(x) \): in this case \( hi = x_u \) is found after
  \( \log_2(|[x_l, x_u]|) + 1 \) calls to function \( f \);
- \( i = x_u \) and \( \forall x \in [x_l, x_u] : f(x) \succeq y \): in this case \( lo = x_l \) is found after
  \( \log_2(|[x_l, x_u]|) + 1 \) calls to \( f \).

If, on the contrary, \( d_M > 0 \), then the worst-case scenario for function \( \text{gallop}_\ell \)
is when it starts at some point \( i \) inside interval \( [x_l, x_u] \), but it terminates only
when \( lo = x_l \) and \( hi = x_u \). In this case, \( \text{gallop}_\ell \) invokes \( \log_2(|[x_l, x_u]|) + 1 \)
times function \( f \), as well.

After calling \( \text{gallop}_\ell \), the algorithm discerns between the following cases.

- \( f(lo) \succ y \): condition \( f(lo) \succ y \) is tested at line 5 of Algorithm 1. Then, since
  \( n_g = 0 \) the function returns at line 7. Condition \( f(lo) = y \) is tested at line 11
  and the function returns at line 12.
- \( f(hi) \prec y \): function \( \text{findhi}_\ell(f, y, [x_l, x_u], n_g, d_M, w_M, \alpha, \omega, t) \) is called but,
  since \( n_g = 0 \), \( r = 0 \) is returned to the calling function lower_bound, that
  terminates at line 16 without invoking function \( f \).
- \( f(lo) \prec y \prec f(hi) \): at line 19 we call function \( \text{bisect}_\ell(f, y, n_g, d_M, w_M, \alpha, \omega, 
  s, t, lo, hi) \). At each iteration, it computes the middle point mid between \( lo \) and \( hi \), and \( f(mid) \). Since \( n_g = 0 \), it updates either the value of \( hi \) or the
  value of \( lo \) with mid until \( lo^+ = hi \). Since mid satisfies the post-condition

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that \( d_M \geq d_M^f \) and \( w_M \geq w_M^f \). In particular, although if \( n_g = n_g^f = 0 \) it would
surely make sense to have \( d_M = 0 \) and \( w_M = 0 \), the execution of the algorithms
with \( d_M > 0 \) and \( w_M > 0 \) cannot be excluded. Nevertheless, the algorithms are
correct even in such occurrence, and the overall computational complexity is not
affected. However, in a practical implementation, it would be advisable to enforce
\( n_g = 0 \implies d_M = 0 \land w_M = 0 \), in order to prevent function \( \text{gallop}_\ell \) from doing
useless work.

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of function \texttt{split\_point} at line 8 of Algorithm 8, we can conclude that \(\#[\text{mid}, \text{hi}] \leq \lfloor \#[\text{lo}, \text{hi}] / 2 \rfloor\) and \(\#[\text{lo}, \text{mid}] \leq \lfloor \#[\text{lo}, \text{hi}] / 2 \rfloor\). Hence, in the worst-case scenario, function \texttt{bisect\_lb} calls \(f \log_2(\#[x_l, x_u])\) times. In the case \(n_g = 0\), the \textit{precision} post-condition of \texttt{bisect\_lb} holds and we are sure that \(f(\text{lo}^+) \gg y\). At line 20 of function \texttt{lower\_bound}, there is a \texttt{while} loop. Since \(\text{lo}^+ = \text{hi}\) and \(f(\text{lo}^+) \gg y\), such loop is never executed. Then, either the function \texttt{lower\_bound} returns at line 26 because \(f(\text{lo}^+) \gg y\), or it returns at line 27 because \(f(\text{lo}^+) = y\). In any case, function \texttt{lower\_bound} called function \(f\) at most \(2 \log_2(\#[x_l, x_u]) + 4\) times.

This concludes the proof. \qed

**Theorem 45.** If \(f : \mathbb{F} \to \mathbb{F}\) has short glitches, that is, \(n_g > 0\) but \(w_M < t\), then, for each \([x_l, x_u] \subseteq \mathbb{F}, d_M, \alpha, \omega, f^1 : \mathbb{F} \to \mathbb{F}, s \in \mathbb{N}\), computing \texttt{lower\_bound} as per Algorithm 1 evaluates \(f\) at most \((w_M + 1) \log_2(\#[x_l, x_u]) + w_M + 6\) times.

**Proof.** Algorithm 11 describing function \texttt{lower\_bound}, starts by calling function \texttt{init}(\(y, [x_l, x_u], f\)), which returns a point inside interval \([x_l, x_u]\) without calling function \(f\). Then, at line 8 function \texttt{gallop\_lb}(\(f, y, [x_l, x_u], d_M, i\)) is invoked. As we discussed in the proof of Theorem 44 in the worst-case scenario, \texttt{gallop\_lb} invokes \(\log_2(\#[x_l, x_u]) + 1\) times function \(f\).

Then, the two following cases must be distinguished.

- \(f(\text{lo}) \gg y\): condition \(f(\text{lo}) \gg y\) is tested in line 5. Then, since \(n_g > 0\), function \texttt{lower\_bound} of Algorithm 1 returns either on line 11 or on line 14 depending on whether \(f(\alpha) \gg d_M y\) holds or not. If \(f(\text{lo}) \ll y\), condition \(f(\text{lo}) = y\) is tested in line 11 and if it holds, function \texttt{lower\_bound} returns on line 12. Therefore, in this case function \texttt{lower\_bound} terminates after calling function \(f\) at most 2 times.

- \(f(\text{lo}) \ll y\) and \(f(\text{hi}) \ll y\): function \texttt{findhi\_lb}(\(f, y, [x_l, x_u], n_g, d_M, w_M, \alpha, \omega, t\)) is called after 2 calls to \(f\) (to test the values of \(f(\text{lo})\) and \(f(\text{hi})\)). We will now prove that the call to \texttt{findhi\_lb} will invoke function \(f\) at most \(w_M + 2\) times, in the worst case.

Since \(n_g > 0\) and \(w_M < t\), the behavior of function \texttt{findhi\_lb} changes according to the following cases.

- \(x_u \gg \omega\) or \(y \gg d_M f(x_u)\): the call to \texttt{findhi\_lb} terminates at line 13 returning the value \(r = 0\). This causes function \texttt{lower\_bound} to terminate with 2 invocations to function \(f\).

- \(n_g = 1\), \(f(\alpha^+) < f(\alpha)\) and \(y \gg f(\alpha)\): the call to \texttt{findhi\_lb} terminates at line 14 returning the value \(r = 0\). Again, \texttt{lower\_bound} terminates with 2 calls to function \(f\).

- \(n_g = 1\), \(f(\alpha^+) < f(\alpha)\) and \(y \ll f(\alpha)\): the call to \texttt{findhi\_lb} terminates at line 12 returning the value \(r = 1\) and invoking the function \(f\) just 2 times.

- \(n_g > 1\): after 2 invocations of function \(f\), \texttt{findhi\_lb} calls \texttt{linsearch\_geq}(\(f, y, [x_l, x_u], w_M, t\)). The latter performs a float-by-float backwards search of the first value of \(hi\) such that \(f(\text{hi}) \gg y\). It stops after at most \(w_M\) steps.
(recall \(w_M < t\), in this case). Hence, function \texttt{linsearch\_eq} invokes function \(f\) at most \(w_M\) times. Therefore, we can conclude that \texttt{findhi\_lb} calls function \(f\) at most \(w_M + 2\) times. From the point of view of complexity, the worst-case scenario is when \texttt{findhi\_lb} returns a value of \(hi\) which meets the preconditions of Algorithm \texttt{bisect\_lb}, allowing to start bisection (in this case \texttt{findhi\_lb} returns \(r = 1\), see lines \([12]\) and \([19]\) of Algorithm \([2]\)). Hence, at line \([19]\) of Algorithm \([4]\) function \texttt{bisect\_lb}(\(f, y, n_k, d_M, w_M, \alpha, \omega, s, t, lo, hi\)) is called. In each iteration, it computes the middle point \(mid\) between \(lo\) and \(hi\) and it evaluates \(f(mid)\). If \(y \leq f(mid)\), function \texttt{bisect\_lb} updates the value of \(hi\) with \(mid\) on line \([6]\) As we have already discussed in the proof of Theorem \([44]\), \(mid\) satisfies the post-condition of function \texttt{split\_point} at line \([8]\) of Algorithm \([6]\) Therefore, \#\([lo, mid]\) \(\leq \lceil\#\([lo, hi]\)/2\rceil\). Alternatively, when \(y > f(mid)\), we have the following cases:

mid \(\preceq\) \(\alpha\) or mid \(\succ\) \(\omega\) or \(y >\) \(\) \(d_M\) \(f(mid)\); function \texttt{bisect\_lb} updates the value of \(lo\) with \(mid\) at line \([1]\) Since mid satisfies the post-condition of function \texttt{split\_point} at line \([3]\) of Algorithm \([6]\) we can conclude that \#\([mid, hi]\) \(\leq \lceil\#\([lo, hi]\)/2\rceil\).

\[\begin{align*}
\alpha < mid < \omega, n_k = 1, f(\alpha^+) < f(\alpha), f(\omega) \geq y \text{ and } f(\alpha) \geq y: \text{ at line } [11] \\
\text{the value of } hi \text{ is updated with } \alpha. \text{ Since } \alpha < mid < \omega, \text{ we have that } \#\([lo, \alpha]\) < \lceil\#\([lo, hi]\)/2\rceil. \\
\alpha < mid < \omega, n_k = 1, f(\alpha^+) < f(\alpha), f(\omega) \geq y \text{ and } f(\alpha) < y: \text{ at line } [13] \\
\text{the value of } lo \text{ is updated with } mid. \text{ Is is sufficient to remember that } \#\([mid, hi]\) \leq \lceil\#\([lo, hi]\)/2\rceil.
\end{align*}\]

\(n_k > 1:\) at line \([22]\) function \texttt{findfmax}(\(f, w_M, lo, mid\)) is called. It finds a value \(b\) inside interval \([\max\{lo, mid\} \wedge w_M\}, \text{mid}\) such that the value \(f(b)\) is the highest possible. Hence, \texttt{findfmax} calls \(w_M\) times function \(f\). Then, two cases must be distinguished based on value \(b\), returned by \texttt{findfmax}:

\(f(b) \geq y: \) at line \([25]\) the value of \(hi\) is updated with \(b\). Since \(x_l \leq b \leq \) mid, we have \#\([lo, b]\) \(\leq \lceil\#\([lo, hi]\)/2\rceil\).

\(f(b) > y: \) at line \([27]\) the value of \(lo\) is updated with \(mid\) and, as discussed above, \#\([mid, hi]\) \(\leq \lceil\#\([lo, hi]\)/2\rceil\) holds.

Hence, in the worst-case scenario, function \texttt{bisect\_lb} calls function \texttt{findfmax} at every step. In that case Algorithm \texttt{bisect\_lb} calls \(w_M \log_2(\#\{x_l, x_u\}) + 1\) times function \(f\), in particular.

Since the \texttt{precision} post-condition of \texttt{bisect\_lb} holds \((w_M < t)\), we are sure that \(f(lo^+) \geq y\). At line \([20]\) of function \texttt{lower\_bound}\), there is a \texttt{while} loop.
Since \(lo^+ = hi\) and \(f(lo^+) \geq y\), the said loop is never executed. Then, either function \texttt{lower\_bound} returns at line \([26]\) because \(f(lo^+) > y\), or it returns at line \([27]\) because \(f(lo^+) = y\). In any case, function \texttt{lower\_bound} called function \(f\) at most \((w_M + 1) \log_2(\#\{x_l, x_u\}) + w_M + 6\) times.

This concludes the proof. \(\square\)
Algorithm 4 tries to refine the interval for \( x \) by finding a correct upper bound. The preconditions for this algorithm are listed in its \texttt{Require} statement, and they are the same as those of Algorithm 1.

The algorithm terminates satisfying the post-condition presented in its \texttt{Ensure} statement. The predicates \( p_i \) that appear in the statement are those defined at the beginning of Section 4.2.

The way algorithm \texttt{upper bound} operates is substantially similar to the one of \texttt{lower bound}. First, function \texttt{init} tries to guess initial values for \( lo \) and \( hi \) by using the rough inverse \( f' \). Then, \texttt{gallop ub} tries to quickly find values for \( lo \) and \( hi \) suitable for the call to \texttt{bisection ub}. In particular, \( f(lo) \preceq y \prec f(hi) \) and \( \forall x \in [hi, xu] : f(x) \succ y \) must hold. If this is not the case, the \texttt{if} statement of the subsequent lines handle the situation.

If \( f(hi) \prec y \), we must discern the case when there is no solution from the case when \( xu \) is in a glitch, and there is actually a value of \( x \) such that \( f(x) \) reaches \( y \) somewhere outside of the glitch. Function \texttt{findhi ub} is called for this purpose. It operates mostly like \texttt{findhi lb}: if we can be sure that \( xu \) is not in a glitch, either because there are no glitches, or because it is outside of \([\alpha, \omega]\), or if anyways no glitch could be deep enough for the function to reach \( y \) outside of it, we can claim that the function is lower than \( y \) for the whole interval. Otherwise, if there is only one glitch and we know its position, a case analysis can be done to conclude whether there is a solution or not. If none of these conditions apply, a float-by-float search is performed in order to try to reach the beginning of the glitch where \( xu \) is, if any. When \texttt{findhi ub} terminates, predicate \( p_6 \) always holds: therefore this function cannot find a value of \( hi \) suitable for bisection, and it is only useful when the interval for \( y \) is a singleton.

When \( f(lo) \succ y \), if the function has no glitches then we are sure its graph is completely below \( y \), and equation \( y = f(x) \) has no solution. Otherwise, there might be glitches in \([xl, xu]\) in which the function decreases until it reaches \( y \). If there is only one glitch and it is not too wide, function \texttt{checkglitch} searches it float-by-float to find out whether the function actually reaches \( y \). If not, we can state that our constraint is unsatisfiable; the appropriate predicate index is returned otherwise.

If \( lo \) and \( hi \) satisfy its precondition, Algorithm \texttt{bisection ub} is eventually called on line 28. It has been designed to implement the dichotomic method on interval \([lo, hi]\), taking into account the fact that \( f \) may not be isotone due to glitches. In this function, \texttt{checkglitch} is used again in those cases where we do not know whether the function’s graph is completely above \( y \) in the upper half of interval \([lo, hi]\) due to the possible presence of glitches. If performing the linear search is unfeasible because of the excessive number or width of glitches, function \texttt{logsearch ub} tries to quickly find a value for \( hi \) such that the distance between \( f(x) \) and \( y \) is too large for a glitch to be deep enough to let the function reach \( y \). Since this could make the partitioning uneven and undermine the logarithmic


Algorithm 4 Inverse propagation: \texttt{upper\_bound}(f, y, [x_l, x_u], n_g, d_M, w_M, \alpha, \omega, f^i, s, t)

\textbf{Require:} \( f: \mathcal{F} \to \mathcal{F}, y \in \mathcal{F}, [x_l, x_u] \in \mathcal{I}_f, n_g \geq n_g^f, d_M \geq d_M^f, w_M \geq w_M^f, \alpha \approx \alpha^f, \omega \approx \omega^f, n_g > 0 \implies (x_l \leq \alpha \leq \omega \leq x_u), f^i: \mathcal{F} \to \mathcal{F}, s, t \in \mathbb{N} \).

\textbf{Ensure:} \( \exists u \in \mathcal{F}, r \in \{5, 6, 7, 8, 9\} \implies p_i(y, x_l, x_u, u) \)

\[
\begin{align*}
&\quad\quad (f(x_l) \not\approx y \not\approx f(x_u)) \\
&\quad\quad \land (n_g = 0 \lor (n_g = 1 \land w_M < t \land (\alpha = \alpha^f \lor \omega = \omega^f \lor (\alpha \not\approx \omega \land k \leq t)))) \\
&\implies r \in \{8, 9\} \\
&1. \quad i := \text{init}(y[x_l, x_u], f^i); \quad \triangleright x_l \approx i \approx x_u \\
&2. \quad (lo, hi) := \text{gallop\_ub}(f, y[x_l, x_u], d_M, i); \\
&3. \quad (x_l \not\approx lo \not\approx hi \not\approx x_u) \land (x_l \not\approx lo \implies y \not\approx f(lo)) \land (x_u \not\approx hi \implies f(hi) \not\approx d_M y) \\
&4. \quad \text{if } f(lo) > y \text{ then} \\
&5. \quad u := \text{findhi\_ub}(f, y[x_l, x_u], n_g, d_M, w_M, \alpha, \omega, t); \\
&6. \quad r := 6; \text{ return} \\
&7. \quad \text{else if } f(hi) = y \text{ then} \\
&8. \quad u := hi; r := 9; \text{ return} \\
&9. \quad \text{end if} \\
&10. \quad \text{if } f(lo) < y \text{ then} \\
&11. \quad \text{if } n_g = 0 \lor f(\alpha) \not\approx d_M y \text{ then} \\
&12. \quad r := 5; \text{ return} \\
&13. \quad \text{else} \\
&14. \quad (b, z) := \text{check\_glitch}(f, y[x_l, x_u], n_g, d_M, w_M, \alpha, \omega, t, lo, lo, hi); \\
&15. \quad \text{if } b = 0 \text{ then} \\
&16. \quad r := 5; \text{ return} \\
&17. \quad \text{else if } b = 1 \text{ then} \\
&18. \quad \text{if } f(z) = y \text{ then} \\
&19. \quad \quad u := z; r := 9; \text{ return} \\
&20. \quad \quad \text{else} \\
&21. \quad \quad \quad u := z^+; r := 8; \text{ return} \\
&22. \quad \quad \text{end if} \\
&23. \quad \quad \text{else} \\
&24. \quad \quad \quad u := \text{min}\{hi, \omega\}; r := 7; \text{ return} \\
&25. \quad \quad \text{end if} \\
&26. \quad \quad \text{end if} \\
&27. \quad \text{end if}; \\
&28. \quad hi := \text{biset\_ub}(f, y[x_l, x_u], n_g, d_M, w_M, \alpha, \omega, n_g, s, t, lo, hi); \\
&29. \quad \text{while } f(hi^-) \not\approx y \land t > 0 \text{ do} \\
&30. \quad \quad hi := hi^-; \\
&31. \quad \quad t := t - 1 \\
&32. \quad \text{end while}; \\
&33. \quad \text{if } f(hi^-) < y \text{ then} \\
&34. \quad \quad u := hi; r := 8 \\
&35. \quad \text{else if } f(hi^-) = y \text{ then} \\
&36. \quad \quad u := hi^-; r := 9 \\
&37. \quad \text{else} \\
&38. \quad \quad u := hi; r := 7 \\
&39. \quad \text{end if} \\
\]
Algorithm 5 Inverse propagation: \texttt{findhi\_ub}(f, y, [x_l, x_u], n_g, d_M, w_M, \alpha, \omega, t)

Require: \(f : F \to F\), \(y \in F\), \([x_l, x_u] \in I_F\), \(n_g \geq n_g^f\), \(d_M \geq d_M^f\), \(w_M \geq w_M^f\), \(\alpha \leq \alpha^f\), \(\omega \geq \omega^f\), \(n_g > 0 \implies (x_l \leq \omega \leq x_u), t \in \mathbb{N}, f(x_u) \prec y\).

Ensure: \(u \in F, p_6(y, x_l, x_u, u)\).

1: if \(n_g = 0 \lor x_u \succ \omega \lor y \succ f(x_u)\) then
2: \hspace{1em} \(u := x_l\)
3: else if \(n_g = 1 \land (w_M > t \lor (f(\alpha^+) < f(\alpha) \land y \succ f(\alpha)))\) then
4: \hspace{1em} if \(y < f(\alpha) \lor f(\alpha^+) \succ f(\alpha)\) then
5: \hspace{2em} \(u := x_u\)
6: \hspace{1em} else if \(y \preceq f(\alpha)\) then
7: \hspace{2em} \(u := \alpha^+\)
8: \hspace{1em} else
9: \hspace{2em} \(u := x_l\)
10: \hspace{1em} end if
11: else
12: \hspace{1em} \((b, \text{hi}, \hat{x}) := \text{linsearch\_geq}(f, y, [x_l, x_u], w_M, t);\)
13: \hspace{2em} \(\triangleright (b = 1 \land \text{hi} \in [x_l, x_u] \land f(\text{hi}) \succeq y \land \forall x \in [\text{hi}^+, x_u] : f(x) \prec y)\)
14: \hspace{2em} \(\triangleright (b = 0 \land \forall x \in [\hat{x}, x_u] : f(x) \prec y)\)
15: \hspace{2em} \(\triangleright \text{where } v = \min\{t, w_M\} \text{ and } \hat{x} = \max\{x_l, x_u - v\}\)
16: \hspace{1em} if \(b = 1\) then
17: \hspace{2em} \(u := \text{hi}^+\)
18: \hspace{1em} else if \(t \geq w_M\) then
19: \hspace{2em} \(u := x_l\)
20: \hspace{1em} else
21: \hspace{2em} \(u := \hat{x}\)
22: \hspace{1em} end if
23: end if
Algorithm 6: Inverse propagation: check_glitch\(f, y, [x_l, x_u], n_k, d_M, w_M, \alpha, \omega, f, t, lo, m, hi\)

**Require:** \(x_l \leq lo \leq m \leq hi \leq x_u, f(m) \succ y, f(hi) \succ y, f: \mathcal{F} \rightarrow \mathcal{F}, y \in \mathcal{F}, [x_l, x_u] \in \mathcal{I}_x, n_k \geq n^l_k, d_M \geq d_M^l, w_M \geq w_M^l, \alpha \neq \alpha', \omega \neq \omega', n_k > 0, x_l \leq \alpha \leq \omega \leq x_u,\)

**Ensure:** \(b \in \{0, 1, 2\},\)

\[n_k = 1 \land w_M < t \land (\alpha = \alpha' \lor \omega = \omega' \lor (\alpha \prec^k \omega \land \alpha \leq t)) \implies b \in \{0, 1\},\]

\[b = 0 \implies \exists x \in [m, hi] : f(x) \succ y,\]

\[b = 1 \implies z \in \mathcal{F} \land lo \leq z \leq hi \land \forall x \in (z, hi) : f(x) \succ y \land f(z) \leq y.\]

1: if \(n_k = 1 \land w_M \leq t\)
\[\land (f(\omega') \prec f(\omega) \lor f(\alpha^+) \prec f(\alpha) \lor (\alpha \prec^k \omega \land k \leq t))\]
then
2: \(s_l := \max(\alpha, lo);\)
3: if \(f(\omega') \prec f(\omega) \lor (\alpha \prec^k \omega \land k \leq t)\)
then
4: \(s_u := \min(\omega, hi);\)
5: else
6: \(s_u := \min(\alpha^+ w_M, hi);\)
7: end if:
8: \((b, z) := \text{linsearch} _{\text{leq}}(f, y, w_M, s_l, s_u)\)
9: \(\triangleright (b = 0 \land z = \hat{x} \land \forall x \in [z, s_u] : f(x) \succ y)\)
10: \(\triangleright (b = 1 \land z \in [\hat{x}, s_u] \land f(z) \leq y \land \forall x \in (z, s_u) : f(x) \succ y)\)
11: \(\triangleright \text{where } \hat{x} = \max(s_l, s_u - t^1)\)
12: else
13: \(b = 2\)
14: end if

The complexity of the process, the number of such calls of \texttt{logsearch} _\texttt{ub} is limited by threshold \(s\).

Finally, on line 29 a \texttt{while} loop goes backwards float-by-float for a maximum of \(t\) steps, until it reaches a value for \(hi\) such that \(f(\text{hi}) \ll y\), and returns it as the upper bound. If such value is not found, the algorithm returns the current value of \(hi\) with predicate \(p_7\), which means an optimal value for the upper bound was not found.

The correctness of Algorithms 4, 5 and 6 is formally proved in the rest of this section. The number of calls to function \(f\) performed at most by \texttt{upper} _\texttt{bound} has the form \(k \log_2(\#[x_l, x_u]) + c\), where \(k\) and \(c\) are small constants that depend on \(w_M, s\) and \(t\). The exact values of these constants in some special glitch data configurations are given in Theorems 2 and 3.

**Lemma 1.** Function \texttt{findhi} _\texttt{ub} specified in Algorithm 5 satisfies its contract.

**Proof.** We assume that the precondition for \texttt{findhi} _\texttt{ub}(\(f, y, [x_l, x_u], n_k, d_M, w_M, \alpha, \omega, t\)) is satisfied. In particular, \(f(x_u) \prec y\) holds.

The guard of the if statement on line 1 is the same as the one of Algorithm 2. When it is satisfied we know for sure that either \(x_u\) is not in a glitch, or that \(x_u\) might be inside a glitch but not deep enough for the function to reach the value \(y\) elsewhere in \([x_l, x_u]\). Together with \(f(x_u) \prec y\), this allows us to state that the equation \(y = f(x)\) has no solution, and setting \(u = x_l\) satisfies \(p_6\).
Algorithm 7 Inverse propagation: $\texttt{bisect\_ub}(f, y, [x_l, x_u], n_g, d_M, w_M, \alpha, \omega, s, t, \text{lo}, \text{hi})$

**Require:** $x_l \preceq \text{lo} \prec x_u$, $f(\text{lo}) \preceq y \prec f(\text{hi})$, $\forall x \in [\text{hi}, x_u] : f(x) \succ y$ $f : \mathbb{F} \rightarrow \mathbb{F}$, $y \in \mathbb{F}$, $[x_l, x_u] \subseteq \mathbb{F}$, $n_g \geq n_{g_f}$, $d_M \geq d_{M_f}$, $w_M \geq w_{M_f}$, $\alpha \preceq \omega \prec \omega_f$, $\forall x \in [\text{hi}, x_u]$ if $f(x)$.

**Ensure:** $\forall x \in [\text{hi}, x_u] : f(x) \succ y$ $f : \mathbb{F} \rightarrow \mathbb{F}$, $y \in \mathbb{F}$, $[x_l, x_u] \subseteq \mathbb{F}$, $n_g \geq n_{g_f}$, $d_M \geq d_{M_f}$, $w_M \geq w_{M_f}$, $\alpha \preceq \omega \prec \omega_f$, $n_g > 0$$

1: \textbf{while} \ \text{hi} \succ \text{lo} \ \textbf{do} \\
2: \quad \text{mid} := \texttt{split\_point}(\text{lo}, \text{hi}); \\
3: \quad \triangleright \ \exists m, m' > 0, |m - m'| \leq 1, \text{mid} = \text{hi}^{-m} = \text{lo}^{+m'} \\
4: \quad \textbf{if} \ f(\text{mid}) \preceq y \ \textbf{then} \\
5: \quad \quad \text{lo} := \text{mid} \\
6: \quad \textbf{else if} \ n_g = 0 \lor \text{hi} \preceq \alpha \lor \text{mid} \succeq \omega \lor y \preceq f(\text{mid}) \ \textbf{then} \\
7: \quad \quad \text{hi} := \text{mid} \\
8: \quad \textbf{else} \\
9: \quad \quad (b, z) := \texttt{check\_glitch}(f, y, n_g, d_M, w_M, \alpha, \omega, t, \text{lo}, \text{mid}, \text{hi}); \\
10: \quad \quad \textbf{if} \ b = 0 \ \textbf{then} \\
11: \quad \quad \quad \text{hi} := \text{mid} \\
12: \quad \quad \textbf{else if} \ b = 1 \ \textbf{then} \\
13: \quad \quad \quad \text{hi} := z^+; \\
14: \quad \quad \quad \textbf{break} \\
15: \quad \textbf{else} \\
16: \quad \quad z := \texttt{logsearch\_ub}(f, d_M, \text{mid}, \text{hi}, y); \\
17: \quad \quad \triangleright z \in [\text{mid}, \text{hi}] \land (z \prec \text{hi}) \implies f(z) \succ d_M y) \\
18: \quad \quad \textbf{if} \ z \preceq \text{hi} \ \textbf{then} \\
19: \quad \quad \quad \text{hi} := z \\
20: \quad \quad \textbf{else} \\
21: \quad \quad \quad \textbf{break} \\
22: \quad \quad \textbf{end if} \\
23: \quad \textbf{end if} \\
24: \quad \textbf{end if} \\
25: \textbf{end while}
The purpose of the if block which starts at line 3 is to distinguish between a few cases when we know from $n_\varpi = 1$ and $n_\varpi \geq n_\varpi^2$ that $f$ has at most one glitch. If such block is entered, $x_u \equiv \omega$ and $y \equiv d_M f(x_u)$ hold, which comes from the negation of the guard of the previous if statement. Therefore, $x_u$ may be inside a glitch. We compute the value of the function in $\alpha$ to discern between the cases listed below.

$y \prec f(\alpha)$: This, together with the precondition of this algorithm, implies that $f(x_u) \prec y \prec f(\alpha)$. Therefore, there must exist some $x \in [\alpha, x_u]$ such that $f(x) \leq y$. If $w_M > t$, then we refrain from searching such $x$ because the glitch may be too wide, and setting $u = x_u$ trivially satisfies $p_6$. This is captured by the if guard of line 3. Otherwise, the control flow can reach line 12.

$y \succ f(\alpha) \land f(\alpha^+) \succ f(\alpha)$: The second term of this condition implies that $\alpha \prec \alpha^+$. Therefore, we cannot exclude that the function further increases between $\alpha$ and $\alpha^+$, reaching $y$. If $w_M > t$ there is nothing more we can do to refine the upper bound, and setting $u = x_u$ trivially satisfies $p_6$. This case is also captured by the if statement of line 4. If $w_M \leq t$, line 12 can be executed.

$y = f(\alpha) \land f(\alpha^+) < f(\alpha)$: The second term of this condition implies that $\alpha = \alpha^+$. Since $y \succ f(x_u)$ and $n_\varpi = 1$, we know for sure that the whole interval $(\alpha, x_u]$ is a glitch, and the function is strictly lower than $y$ in that interval. Therefore, we just found the rightmost point $x = \alpha$ where $f(x) = y$, and predicate $p_6$ is satisfied if $u = \alpha^+$ is set. When this condition is true, control flows into the if statement of line 6.

$y > f(\alpha) \land f(\alpha^+) \prec f(\alpha)$: The same reasoning we did for the previous case can be made: $\forall x \in [\alpha, x_u]: y > f(x)$. Together with the fact that $y > f(\alpha)$ and that $f$ is isotonic to the right of $\alpha$, this allows us to state that the equation $y = f(x)$ has no solutions in $[x_l, x_u]$, and we can set $u = x_l$, satisfying $p_6$.

This last case is caught by the else statement of line 8.

The if guard of line 3 captures all the conditions listed above if $w_M > t$. Otherwise, if $w_M \leq t$, only the last two conditions are captured, because if the first two of them occur, then proceeding with the linear search described below can find a better value for $u$.

In line 12, function linsearch_geq is invoked. Starting from $x_u$, it searches backwards for a point where the functions reaches or exceeds $y$. If it does not find such value in at most $t$ steps and before reaching $x_l$, $b = 0$ is set. Otherwise, $b = 1$ is set and $hi$ is set to such value.

- If $b = 1$, the post-condition of linsearch_geq states that $\forall x \in [hi^+, x_u]: y > f(x)$, and $hi$ is the rightmost point where $y \prec f(x)$. Therefore, setting $u = hi^+$ satisfies $p_6$.
- If $b = 0$, then linsearch_geq could not find a point where the function is greater or equal than $y$, and the function is always lower than $y$ in the interval $[\hat{x}, x_u]$. This could happen because of either of the two reasons described below, which can be distinguished by comparing $t$ and $w_M$.

If $i \geq w_M$, linsearch_geq analyzed the entire glitch, and the value of $f$ in the point where the glitch starts is still lower than $y$. Since the function
is quasi-isotonic, it can only decrease further proceeding towards \( x_l \).
Therefore, we can state that the equation \( y = f(x) \) has no solution, and set \( u = x_l \), which satisfies \( p_6 \).

\( t < w_M \) : This last case occurs when the \textbf{else} branch of line 20 is entered.
The glitch was too wide to be analyzed completely, and we can only set \( u = \hat{x} \), satisfying \( p_6 \).

\( \square \)

**Lemma 2.** Function \texttt{check_glitch} specified in Algorithm 6 satisfies its contract.

**Proof.** The precondition of function \texttt{check_glitch} requires three values \( l_0, m \) and \( h_i \) such that \( x_l \leq l_0 \leq m \leq h_i \leq x_u \) and both \( f(m) > y \) and \( f(h_i) > y \). In this situation, we would like to claim that the graph \( f \) is always higher than \( y \), and that there is no solution to equation \( y = f(x) \) in interval \([m, h_i]\). However, there might be a glitch inside that interval that lets the graph of function \( f \) reach the value \( y \). The purpose of this algorithm is to search the glitch float-by-float when possible, in order to either find the rightmost point where \( f \) evaluates to \( y \), or to be able to claim that \( f \) never reaches \( y \) in said interval.

The algorithm proceeds with the linear search only if there is at most one glitch, and it is sufficiently tight. This condition is checked by the guard of the \textbf{if} statement on line 11. If it does not hold, we refrain from searching the glitch because it is computationally too expensive, and \( b \) is set to 2 in line 13, which means that nothing can be said about the solution of the equation \( y = f(x) \) in interval \([m, h_i]\).

Otherwise, function \texttt{linsearch_leq} searches the glitch backwards starting from \( s_u \) for a maximum of \( w_M \) floats, until it either finds a value \( z \) such that \( f(z) \leq y \), or it reaches \( s_l \), \( s_l \) and \( s_u \) are chosen so that the whole glitch is checked by \texttt{linsearch_leq}. Therefore, \( s_l \) is set to the maximum between \( \alpha \leq \alpha_f \) and \( l_0 \).

As for \( s_u \), it is chosen depending on the following cases:

- \( f(\omega^-) < f(\omega) \) : This implies that \( \omega = \omega_f \), and the glitch finishes in \( \omega \). Setting \( s_u \) to the minimum between \( \omega \) and \( h_i \) lets \texttt{linsearch_leq} search the entire glitch (or the part of it inside the interval of interest), because it is surely not too wide (\( w_M < t \)).

- \( \alpha \prec^k \omega \land k \leq t \) : We may not know exactly where the glitch ends, but the distance between \( \alpha \) and \( \omega \) is small enough to search the whole interval delimited by them. Again, setting \( s_u := \min\{\omega, h_i\} \) lets \texttt{linsearch_leq} analyze the entire glitch.

- \( f(\alpha^+) < f(\alpha) \) : This implies \( \alpha = \alpha_f \). Therefore, the glitch starts exactly in \( \alpha \) and, since its width is lower than \( t \), setting \( s_u := \min\{\alpha + w_M, h_i\} \) allows \texttt{linsearch_leq} to search the entire glitch.

On line 34 function \texttt{linsearch_leq} is finally called. It can only return \( b = 0 \) or \( b = 1 \), which proves part of the post-condition of this algorithm, i.e.,

\[ n_g = 1 \land w_M < t \land (\alpha = \alpha_f \lor \omega = \omega_f \lor (\alpha \prec^k \omega \land k \leq t)) \implies b \in \{0, 1\}. \]

It also satisfies the two other claims of the **Ensure** statement, depending on the value of \( b \).
Lemma 3. Function $\text{bisect}_{\omega \beta}$ specified in Algorithm 7 satisfies its contract.

Proof. We assume that the precondition of $\text{bisect}_{\omega \beta} (f, y, [x_l, x_u], n_{\beta}, d_{\omega}, w_{\omega}, \alpha, \omega, s, t, lo, hi)$ is satisfied before the first iteration of the while loop that starts at line 4. We will now prove that the loop invariant

$$\text{Inv} \equiv (x_l \leq lo < hi \leq x_u) \land (f(lo) \leq y < f(hi)) \land (\forall x \in [hi, x_u] : f(x) \succ y)$$

holds during and after the execution of the loop, satisfying the post-condition expressed in the Ensure statement.

Initialization: The invariant Inv is implied by the precondition of this algorithm. Therefore, it holds before the execution of the loop.

Maintenance: At the beginning of the loop body, we assume that both Inv and $hi \succ l_1$ lo, the guard of the loop, hold. We will now prove that either the loop terminates with a break statement, or it continues and one between $lo$ and $hi$ takes a new value $lo'$ or $hi'$. The said new value will be part of the interval $(lo, hi)$. Note that this last statement implies $x_l \leq lo < lo' \leq hi' \leq hi < x_u$, and therefore any new value for $lo$ and $hi$ that satisfies this condition, also satisfies the first condition of the invariant. We will prove that the invariant holds at the end of the loop body.

After function $\text{split} \_ \text{point}$ is invoked at line 4, $lo < mid < hi$. Note that this implies that, every time $lo'$ or $hi'$ are set to mid, the new value is part of the interval $(lo, hi)$. Then, the value of $f(mid)$ is compared to $y$ in order to decide whether mid can be a new value for $lo$ or $hi$. The following cases may occur:

- $f(mid) < y$: $lo'$ is set to mid. Since $lo < mid < hi$ and because Inv holds, also $x_l \leq lo' \leq mid < hi \leq x_u$ holds. Again, Inv and $f(mid) < y$ imply that $f(lo') \leq y < f(hi)$. The third part of the invariant trivially holds, because hi remained the same.

- $f(mid) \succ y$: If $n_\omega = 0$, then $hi < \alpha \lor mid \succ \omega \lor y \prec d_{\omega}$ and $f(mid)$ we are sure that $\forall x \in [mid, hi] : f(x) \succ y$, because there is no glitch in $[mid, hi]$ where

$b = 0$: By the post-condition of $\text{linsearch}_{\omega \beta}$, $\forall x \in [z, s_u] : f(x) \succ y$ holds, with $z = \hat{x} = \max\{s_l, s_u - w_{\omega}\}$. As we previously explained, $s_l$ and $s_u$ are chosen so that the only glitch is entirely contained in $[s_l, s_u]$, and it is within $w_{\omega}$ floats from $s_u$: the glitch is contained in interval $[z, s_u]$, which was analyzed by $\text{linsearch}_{\omega \beta}$. Therefore, function $f$ is actually isotonic in $[m, z]$ and in $[s_u, hi]$. This, together with $f(m) \succ y$, $f(\alpha) \leq f(\omega)$, and the post-condition of $\text{linsearch}_{\omega \beta}$, allows us to claim that $\forall x \in [m, hi] : f(x) \succ y$.

$b = 1$: The post-condition of $\text{linsearch}_{\omega \beta}$ states that we have $z \in [\hat{x}, s_u]$, with $\hat{x} = \max\{s_l, s_u - w_{\omega}\} \succ lo$ and $s_u \leq hi$: this implies that $lo \prec z \leq hi$. Moreover, $s_u$ was chosen so that $f$ is isotonic in the interval $[s_u, hi]$. This, together with $\forall x \in (z, s_u] : f(x) \succ y$ (which is part of the post-condition of $\text{linsearch}_{\omega \beta}$), proves that $\forall x \in (z, hi] : f(x) \succ y$.

Finally, the post-condition of $\text{linsearch}_{\omega \beta}$ states that $f(z) \preceq y$, which concludes the proof of this part of the post-condition of $\text{check} \_ \text{glitch}$. □
the function could become lower than or equal to $y$. This can be stated because either we are sure there are no glitches at all in that interval (first three conditions of the if guard of line 6), or $f(mid)$ is so higher than $y$ that no glitch can be sufficiently deep for the function to touch $y$ (last condition). Therefore, on line 7 $hi'$ is set to mid, which satisfies the invariant. In fact, $hi' \in (lo, hi)$ holds because $lo < mid < hi$, and $f(mid) > y$ implies that $f(lo) \preceq y < f(hi')$ holds. The last condition of the invariant is satisfied by this choice of $hi'$, as proved at the beginning of this paragraph.

Otherwise, the control flow reaches the else body starting at line 9. At this point, we are not sure whether there is a glitch between mid and hi where the graph of the function reaches $y$ or not. If there is only one glitch, its maximum width is small enough, and we know exactly where it starts, function check_glitch searches it float-by-float for a point $z$ such that $f(z) \succeq y$. Whether it succeeded or not can be exerted from the value of $b$.

$b = 0$: In this case, check_glitch could search the glitch, but it did not find any point in which the function was lower than or equal to $y$: setting $hi' = mid$ is correct. The first condition of the invariant is satisfied because of the post-condition of split_point, and the second condition holds because $lo$ remained the same and, since we are into one of the else statements of the if at line 4, $f(mid) > y$.

The third condition is also satisfied because the post-condition of check_glitch ensures that $\forall x \in [mid, hi]: f(x) \succeq y$.

$b = 1$: Function check_glitch succeeded in finding a value $z$ such that $f(z) \preceq y$ inside the glitch. $hi'$ is then set to $z^+$. $xl \preceq lo < hi' \preceq xu$ holds because the post-condition of check_glitch states that $mid \preceq z \preceq hi$, and both mid and hi are enclosed in $xl$ and $xu$. The said post-condition also states that $\forall x \in (z, hi]: f(x) \succeq y$: this maintains the last condition of the invariant, and implies that $f(z^+) > y$, which satisfies $f(lo) \preceq y < f(hi')$. This part of the post-condition of check_glitch, together with $\forall x \in [hi, xu]: f(x) \succeq y$, also implies that $z$ is the rightmost point such that $f(z) \preceq y$. The while loop can therefore be broken, as the upper bound found until now cannot be refined further.

$b = 2$: In this case the conditions that would allow check_glitch to search the glitch did not hold. Hence, $lo$ and $hi$ were left untouched, and the algorithm proceeds with the else block at line 14. Here, function logsearch_ub is called to find a suitable value for $hi$. Its post-condition allows us to distinguish between the two following cases:

$z < hi$: logsearch_ub found a value $z$ such that $f(z) >_{d_{M}} y$: this assures us that there is no glitch in $[z, hi]$ deep enough for the graph of the function to reach $y$. Therefore, setting $hi' = z$ satisfies the last two terms of the invariant. Also, $hi' \in (lo, hi)$ holds because $z \in (mid, hi)$.
Termination: In the previous paragraphs we proved that, either:

- the loop terminates with a \texttt{break} statement;
- the loop continues and one of \texttt{lo} or \texttt{hi} is set to a new value contained into the interval \((\texttt{lo}, \texttt{hi})\). In this case, the distance between \texttt{lo} and \texttt{hi} decreases at each iteration: the guard of the \texttt{while} loop, \(\texttt{hi} \succ 1 \texttt{lo}\), will be eventually negated, terminating the loop.

Correctness: We have already proved that the invariant \texttt{Inv} holds after each iteration of the loop, and whenever it terminates with a \texttt{break} statement. \texttt{Inv} coincides with the correctness part of the \texttt{Ensure} statement, which is therefore also proved.

If \((\texttt{n_g} = 0 \lor (\texttt{n_g} = 1 \land \texttt{w_M} < \texttt{t} \land (\alpha = \alpha^f \lor \omega = \omega^f \lor (\alpha \prec^k \omega \land k \leq \texttt{t}))))\) holds, the post-condition of \texttt{check.glitch} assures us that either \(\texttt{b} = 0\) or \(\texttt{b} = 1\). Therefore, the only \texttt{break} statement that can be reached is the one at line 14. In this case, the same post-condition implies that \(f(\texttt{z}) \preceq y\): setting \(\texttt{hi'} = \texttt{z}^+\) satisfies the precision statement, because \(f(\texttt{hi'}^-) = f(\texttt{z}) \preceq y\). In all the other cases, the control flow can only reach lines 5, 7 or 11 which let the loop continue. If the loop is not terminated by the \texttt{break} statement at line 14, the same reasoning presented at the end of the proof of Algorithm 3 can be made: whenever the loop terminates because its guard is negated, we have \(#[\texttt{lo}, \texttt{hi}] = 2\). This, together with \texttt{Inv}, implies \(f(\texttt{hi}^- \lor \texttt{lo}) \preceq y\). \qed

\textbf{Theorem 1.} Function \texttt{upper\_bound}, specified in Algorithm 4, satisfies its contract.

\textit{Proof.} Calls to functions \texttt{init} and \texttt{gallop\_ub}, on lines 1 and 2 respectively, have the purpose of finding values for \texttt{lo} and \texttt{hi} suitable for the bisection phase. That is, as specified in the \texttt{Require} statement of function \texttt{bisect\_ub} (Algorithm 7), \(f(\texttt{lo}) \preceq y \prec f(\texttt{hi})\) and \(\forall x \in [\texttt{hi}, \texttt{lo}] : f(x) \succ y\). Function \texttt{gallop\_ub} is specular to its \texttt{lower\_bound} counterpart: for more details about those functions, see the proof of Theorem 43.

The \texttt{if} statements before the call to \texttt{bisect\_ub} on line 28 deal with the cases in which the said functions fail in their purpose. First, the value of \(f(\texttt{hi})\) is checked by the \texttt{if} statements on lines 4 and 7, leading to the following cases:

- \(f(\texttt{hi}) \prec y\): In this case \(\texttt{hi}\) clearly does not satisfy the precondition of \texttt{bisect\_ub} and, because of the post-condition of \texttt{gallop\_ub} stated on line 8 \(\texttt{hi} = \texttt{x_u}\). Therefore, function \texttt{findhi\_ub} is called on line 9 to further discern whether the equation \(y = f(x)\) has no solution, or \(\texttt{hi} = \texttt{x_u}\) is inside a glitch and there is, in fact, a solution. Since function \texttt{findhi\_ub} always sets \(u\) to a value satisfying \(p_8\) (see Lemma 1), \(r\) can be set to 6, and function \texttt{upper\_bound} can terminate.
- \(f(\texttt{hi}) = y\): Again, according to the post-condition of \texttt{gallop\_ub}, we have \(\texttt{hi} = \texttt{x_u}\). Since \(\texttt{hi}\) is the highest value of interval \([\texttt{x_l}, \texttt{x_u}]\), setting \(r = 9\) and \(u = \texttt{hi}\) is correct.
If control flow is not caught by the `if` statements described above, then $f(hi) \succ y$ and, by the post-condition of `gallop`, either of the following holds:

$x_u \succ hi$: This implies that $f(hi) \succ^{dist} y$. Therefore, no glitch in interval $[hi, x_u]$ can be deep enough for function $f$ to reach $y$: $\forall x \in [hi, x_u]: f(x) \succ y$ holds.

$x_u = hi$: In this case, $\forall x \in [hi, x_u] \equiv [x_u, x_u]: f(x) \succ y$ trivially holds.

Therefore, $hi$ satisfies the preconditions of `bisection`, either of the following holds:

Then, the value of $f$ in $lo$ is checked on line 11. If $f(lo) \succ y$, $lo$ is not suitable for bisection, and the body of the `if` statement on the said line tries to understand whether $y = f(x)$ has no solutions, or a solution is contained in a glitch. Note that the post-condition of `gallop` entails that, in this case, $lo = x_l$. The guard of the `if` statement on line 11 catches the two following cases:

$n_k = 0$: In this case, function $f$ is isotonic. Therefore, $\forall x \in [lo, x_u]: f(x) \succ f(lo)$. Together with $f(lo) \succ y$, this implies that $\forall x \in [lo, x_u] \equiv [x_l, x_u]: f(x) \succ y$. This lets us state that equation $y = f(x)$ has no solution, and setting $r = 5$ on line 12 is correct.

$f(\alpha) \succ^{dist} y$: This condition signifies that $f(\alpha)$ is too high for the graph of $f$ to reach $y$ inside a glitch after $\alpha$. This, together with the definition of $\alpha$ and $\omega$, brings the following conclusions:

- $\forall x \in [x_l, \alpha]: f(x) \succ y$ ($f(lo = x_l) \succ y$ and isotonicity of $f$);
- $\forall x \in [\alpha, \omega]: f(x) \succ y$;
- $\forall x \in [\omega, x_u]: f(x) \succ y$ ($f(\omega) \succ f(\alpha) \succ y$ and isotonicity of $f$).

Therefore, equation $y = f(x)$ has no solution in interval $[x_l, x_u]$, and we can set $r = 5$.

If none of the above conditions apply, there might be glitches deep enough to allow the function to take the value $y$ somewhere in interval $[x_l, x_u]$. Function `check_glitch` is called with $m = lo$ in line 14 to search such glitch, if it is only one and it is not too wide. The correctness of the said function is discussed in Lemma 2. The three values of $b$ that this function can return are distinguished in the following `if` statements.

$b = 0$: By the post-condition of `check_glitch`, $\forall x \in [lo, hi]: f(x) \succ y$. As we previously noted, $\forall x \in [hi, x_u]: f(x) \succ y$ holds at this point of the algorithm. This, together with the fact that $lo = x_l$, implies that $\forall x \in [x_l, x_u]: f(x) \succ y$. Therefore, equation $y = f(x)$ has no solution and we can set $r = 5$.

$b = 1$: The post-condition of `check_glitch` assures us that a value $z \in \mathbb{F}$ is returned, and $lo \preceq z \preceq hi \land \forall x \in (z, hi]: f(x) \succ y \land f(z) \prec y$ holds. We also know that $\forall x \in [hi, x_u]: f(x) \succ y$ as stated before. This means that $\forall x \in (z, x_u]: f(x) \succ y$ and there is no solution after $z$. The algorithm must now distinguish whether $z$ is a solution or not, and it does so starting from line 18.

$f(z) = y$: $z$ is a solution to equation $y = f(x)$, and it is also the rightmost, for the reasons stated above. Therefore, setting $u = z$ and $r = 9$ is correct, and no further action is required, so the algorithm can `return`.

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The correctness of this function is discussed in Lemma 6 after its call, the post-condition stated in the Ensure statement of Algorithm 7 is satisfied.

On line 29 a while loop approaches the solution of $y = f(x)$ by going backwards float-by-float. This loop is specular to the one starting on line 20, and is the proof of its correctness and termination. See Theorem 43 for more details, keeping in mind that the loop invariant is $\forall x \in [hi, x_u]: f(x) \succ y$.

If control does not flow into the body of the if statement on line 10 then we have $f(lo) \leq y$, which satisfies the precondition of $\text{bisect}_ub$. Both lo and hi are suitable for the bisection phase, which is started on line 28 by calling $\text{bisect}_ub$.

The loop invariant, $\forall x \in [\alpha, \omega]: f(x) \succ y$, is checked, leading to the following cases:

$f(\text{hi}^-) \prec y$: If $u = \text{hi}$ is set, we have $f(u^-) \prec y \prec f(u)$. This, together with the while-loop invariant, satisfies $p_8$, and the algorithm can terminate.

$f(\text{hi}^-) = y$: The rightmost solution of $y = f(x)$ was found. $u$ can be set to $\text{hi}^-$ and $r$ to 9, because $p_9$ is satisfied.

$f(\text{hi}^-) \succ y$: This condition holds in the body of the else statement on line 27. The loop invariant, $\forall x \in [\alpha, \omega]: f(x) \succ y$, assures that setting $u = \text{hi}$ satisfies $p_7$; and $u$ and $r$ are set accordingly on line 28.

The correctness statement of this algorithm is therefore proved.

In order to prove the precision statement, we assume that $f(x_l) \leq y \preceq f(x_u)$ holds, together with either $n_\delta = 0$ or $n_\delta = 1 \wedge w_M < t \wedge (\alpha = \alpha^f \vee \omega = \omega^f \vee (\alpha \preceq \omega \wedge k \leq t))$. This, together with the post-condition of $\text{gallop}_ub$, implies that after line 2 $f(lo) \preceq y$ and $f(hi) \succ y$ hold, even if $lo = x_l$ or $hi = x_u$. Therefore, $f(lo)$ is always suitable for $\text{bisect}_ub$, and the if statement of line 10 cannot be entered. As for $f(hi)$, the following two cases must be distinguished:

$f(hi) = y$: The body of the if statement on line 11 is entered, and $r = 9$ is returned, satisfying the precision post-condition.

$f(hi) \succ y$: $f(hi)$ satisfies the precondition of $\text{bisect}_ub$, that is called on line 28.

After this call, according to the precision post-condition of $\text{bisect}_ub$, $f(hi^-) \preceq y$. The while loop on line 29 will therefore assign $hi^-$ to hi, so that $f(hi) \preceq y$.

Only the bodies of the if statements on lines 8 and 11 can now be entered.
\( r \) can only be set to 8 or 9, respectively. This satisfies the precision post-condition of this algorithm. \( \Box \)

**Theorem 2.** In case \( f : \mathbb{F} \rightarrow \mathbb{F} \) is isotonic, that is, \( n_g = 0 \), then, for each \( [x_l, x_u] \in \mathcal{I}_x \), \( d_M, w_M, \alpha, \omega, f^i : \mathbb{F} \rightarrow \mathbb{F}, s, t \in \mathbb{N} \), the call to the function \( \text{upper_bound}(f, y, [x_l, x_u], 0, d_M, w_M, \alpha, \omega, f^i, s, t) \) in Algorithm 4 can be executed calling at most \( 2 \log_2(\#[x_l, x_u]) + 4 \times \text{the function } f \).

**Proof.** The proof of this theorem is analogous to the proof of Theorem 44. It is therefore omitted.

**Theorem 3.** In case the function \( f : \mathbb{F} \rightarrow \mathbb{F} \) has small glitches, that is, \( n_g > 0 \) but \( w_M < t \) then, for each \( [x_l, x_u] \in \mathcal{I}_x \), \( d_M, \alpha, \omega, f^i : \mathbb{F} \rightarrow \mathbb{F}, s \in \mathbb{N} \), the call to function \( \text{upper_bound}(f, y, [x_l, x_u], n_g, d_M, w_M, \alpha, \omega, f^i, s, t) \) in Algorithm 4 can be executed calling at most \((w_M + 1) \log_2(\#[x_l, x_u]) + 8 \times \text{function } f \) if the precision clause holds, or at most \((s + 2) \log_2(\#[x_l, x_u]) - s + t + 7 \) if it does not.

**Proof.** The proof of this theorem is extremely similar to the one of Theorem 45; only the most significant differences will be discussed here. Moreover, as noted in the said proof, the control flow paths that present the highest number of calls to the function are those that include the call to \text{bisect,ub}. We will therefore restrict the discussion of this theorem to those paths.

If the precision post-condition of Algorithm 4 holds then, on line 9 of function \text{bisect,ub}, function \text{check_glitch} is invoked. It performs the linear search of line 10 which behaves like the call to \text{findfmax} on line 22. Since in the worst case scenario this search is repeated in all iterations of the while loop of \text{bisect,ub}, the number of calls to \( f \) performed is \( w_M \log_2(\#[x_l, x_u]) \), plus 4 extra calls to evaluate the guard of the \textbf{if} statement of line 11 of function \text{check_glitch}. Therefore, the overall number of calls to \( f \) of \text{upper_bound} is \((w_M + 1) \log_2(\#[x_l, x_u]) + 8 \) if the precision condition holds.

Otherwise, function \text{logsearch,ub} is called for a maximum of \( s \) times on line 17 of algorithm \text{bisect,ub}. It calls \( f \) for at most \( \log_2(\#[\text{mid}, \text{hi}]) \) times. Since in every iteration of the main while loop of \text{bisect,ub} we have \#[lo, hi] = \#[mid, hi], the actual number of function calls performed by \text{logsearch,ub} is \( \log_2(\#[\text{lo}, \text{hi}]/2) \). Because \text{logsearch,ub} returns a value \( z \) which, in the worst case, is \text{hi}−, we can consider as if it was always called on the whole initial \([\text{lo}, \text{hi}]\) interval. Since \text{logsearch,ub} is called for a maximum of \( s \) times, it leads to an additional number of calls to \( f \) at most \( s(\log_2(\#[\text{lo}, \text{hi}]) - 1) \).

Then, after the call to \text{bisect,ub} has returned, the while loop of line 29 may be executed for up to \( t \) iterations, because the precision condition does not hold.

The final count of the calls to function \( f \) sums up to

\[(s + 2) \log_2(\#[x_l, x_u]) - s + t + 7. \Box \]
D Implementation of the Trigonometric Algorithms

This Section contains some theoretical and practical results that concern the implementation of the interval-refinement algorithms for trigonometric functions. These results were omitted from the main article for brevity and simplicity of exposition, but they are thoroughly described here as a reference to those willing to implement the described concepts.

D.1 Glitches in Trigonometric Functions

In this section, the concept of monotonicity glitches in trigonometric functions is analyzed in a more formal way, and new definitions of glitches that take into account the problematic behavior of these functions are given. In particular, we define the set of the quasi-monotonic intervals, which captures in a formal way the fact that trigonometric functions change their monotonicity periodically, and their graphs consist of a succession of monotonic branches. These concepts will be required in the proofs of the presented algorithms.

With the definitions of glitches given in Section [3], the normal changes in monotonicity of trigonometric functions would be detected as glitches. Suppose we analyzed, say, the $\sin f$ function: Definition 5 would recognize all the intervals $I_{\text{mono},k} \in I_f$ of the form

$$I_{\text{iso},k} = \left[\left[\frac{3}{2} \pi + 2k\pi\right], \left[\frac{\pi}{2} + 2k\pi\right]\right]$$

as isotonicity glitches, and intervals $I_{\text{anti},k}$ of the form

$$I_{\text{anti},k} = \left[\left[-\frac{\pi}{2} + 2k\pi\right], \left[\frac{3}{2} \pi + 2k\pi\right]\right]$$

as antitonicity glitches, with $k \in \mathbb{Z}$. Actual glitches would be recognized as sub-glitches of these intervals, thus being difficult to discern.

On the other hand, Definition 5 is still useful if we separately consider those intervals in which the functions are quasi-isotonic or quasi-antitonic. The following definitions let us denote appropriate intervals in which the functions have such properties. Definition 6 describes the maximal sets of intervals in which the function has a constant monotonicity, while Definition 7 applies the definition of glitches to those intervals.

**Definition 6. (Set of the quasi-isotonic (quasi-antitonic) intervals.)** Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function, and let $\mathcal{M}_f^\text{iso} (\mathcal{A}_f^\text{iso})$ be the set of the maximal isotonic (resp., antitonic) intervals of $f$ in $\mathbb{R}$:

$$\mathcal{M}_f^\text{iso} := \left\{ I \in \mathcal{I}_\mathbb{R} \mid \begin{array}{l} f \text{ is isotonic on } I \\ \forall I' \in \mathcal{I}_\mathbb{R} : I \subset I' \implies f \text{ is not isotonic on } I' \end{array} \right\}.$$
Let $D \subseteq F$ such that, for each $x \in D$, $f(x) \in \mathbb{R}$. The set of the quasi-isotonic intervals of $f$ in $D$ is a set $M^D_f \subseteq I_F$ (Definition 3) such that:

$$M^D_f := \{ F \in I_F \mid F \subseteq D \land \exists I \in M^R_f . I = I \cap D \}.$$  

When an interval $I \in M^R_f$ has 0 as the lower or upper bound, the corresponding interval in $M^D_f$ has $+0$ or $-0$ as the corresponding bound. The sets of the (maximal) quasi-antitonic intervals $A^R_f$ and $A^D_f$ are defined similarly.

**Definition 7.** (Isotonicity glitch in a periodic function.) Let $f : F \to F$ be the implementation of a periodic function, let $I \subseteq F$ such that, for each $x \in I$, $f(x) \in \mathbb{R}$, and let $M^I_f$ be the set of the isotonic intervals of $f$ in $I$. An isotonicity glitch of $f$ in $I$ is an interval $G \subseteq I$ such that:

1. there exists an interval $I_M \in M^I_f$ such that $G \subseteq I_M$;
2. $G$ is an isotonicity glitch of $f$ in $I_M$ according to Definition 5.

Antitonicity glitches in periodic functions are defined similarly, but substituting $M^I_f$ with $A^I_f$. Glitch width and depth in periodic functions are consequently defined according to Definition 5.

### D.2 Range Reduction for Trigonometric Functions

Now that we have theoretically characterized the behavior of trigonometric functions, we present a method for bringing these concepts into practice.

In order to split the domain of trigonometric functions into quasi-monotonic intervals, algorithms capable of discerning close floating-point approximations of multiples of $\frac{\pi}{2}$ are needed. Such algorithms will be very similar to the range reduction algorithms used to implement trigonometric library functions: see, e.g., [BKDK+05, DMMM96, Ng92, PH83]. Let $x \in F$ be the argument of a trigonometric function: argument reduction is aimed at finding values $k \in \mathbb{Z}$ and $r \in \mathbb{R}$ such that $x = k \frac{\pi}{2} + r$. In order to obtain these values, $k = \lfloor x \frac{\pi}{2} \rfloor$ and an approximation of $r = \frac{\pi}{2}(x \frac{\pi}{2} - k)$ are computed with an appropriate increased precision. Then, $r$ is used to compute the value of the function in a reduced interval, and the last digits of $k$ are used to identify the quadrant of $x$. A notable implementation of range reduction algorithms is the one in fdlibm\(^{25}\); it is also included in some other libm implementations, such as GNU libc\(^{26}\) which uses it for some architectures. Other important implementations are the ones of the mentioned CR-LIBM and libmcr libraries.

As we shall point out, the algorithms needed for our analysis do not have the same precision requirements as actual range reduction algorithms: since we will not actually compute the functions, $r$ is not needed in full precision. An

\(^{25}\) Developed in 1993 by Sun Microsystems, Inc. See [http://www.netlib.org/fdlibm/](http://www.netlib.org/fdlibm/), file e_rem_pio2.c. Last accessed on October 17th, 2016.

\(^{26}\) See GNU libc 2.22, file sysdeps/ieee754/dbl-64/e_rem_pio2.c.
algorithm capable of computing integer division by $\frac{2}{\pi}$ is needed, as well as a generator of floating-point approximations of $\frac{2}{\pi}$.

The computations of such operations poses the problem of deciding how many digits of the approximation of $\frac{2}{\pi}$ are needed in order to obtain correctly rounded results. This is also known as the Table Maker’s Dilemma [LMT98]. Lemma 4 allows us to answer this question, by computing the rounding error generated by the multiplication of a floating point number $k$, which shall be considered exact, and a floating-point approximation $[x]_r$ of a real number, $x$. The rounding mode used to generate the approximation and to perform the multiplication can be anyone of those defined in the IEEE 754 Standard. In order to prove Lemma 4, the following Definition 8 is needed.

Definition 8. Let $x \in \mathbb{R}$, such that $\exists n \in \mathbb{Z} \cdot |x| = 2^n$ and $\neg f_{\text{max}} < |x| < f_{\text{max}}$ (i.e., $x$ is not a power of 2, an infinity or subnormal). Let $[x]_r \in \mathbb{F}$ be a floating-point approximation of $x$ with $p$ significant digits and exponent $e_x$.

Then, the rounding error for $[x]_r$ with respect to $x$ when rounding towards zero, positive or negative infinity is $|\nabla(x)| \leq \text{ulp}(x) = 2^{e_x-p+1}$. Moreover, with any of the roundToNearest rounding modes, the rounding error is $|\nabla(x)| \leq \frac{\text{ulp}(x)}{2} = 2^{e_x-p}$.

Lemma 4. Let $k \in \mathbb{F}$ be a number in any of the IEEE 754 binary floating-point formats, with an exponent $e_k$. Let $x \in \mathbb{R}$, such that $\exists n \in \mathbb{Z} \cdot |x| = 2^n$ and $f_{\text{min}} < |x| < f_{\text{max}}$. Let $[x]_r \in \mathbb{F}$ be an approximation of $x$ with $p$ significant digits and exponent $e_x$. Then, the absolute error of the correctly rounded floating-point multiplication between $k$ and $[x]_r$ is $|\nabla(k \boxdot [x]_r)| \leq 2^{e_k+e_x-p+4}$.

Proof. The purpose of this lemma is to find a sufficiently tight upper bound for the rounding error of the said multiplication, keeping into account the initial rounding error of the second floating-point operand $[x]_r$. This can be written as the sum of the real number $x$ plus its rounding error, i.e., $[x]_r = x + \nabla(x)$. Since we are assuming the multiplication is correctly rounded, it can also be written as the sum of its exact value plus its absolute error:

$$ k \boxdot [x]_r = \lfloor k(x + \nabla(x)) \rfloor_r $$

$$ = \lfloor kx + k \nabla(x) \rfloor_r $$

$$ = kx + k \nabla(x) + \nabla(kx + k \nabla(x)) $$

$$ = kx + \nabla(k \boxdot [x]_r). $$

We already have an upper bound for the rounding error of $[x]_r$, which is $|\nabla(x)| \leq 2^{e_x-p+1}$. An upper bound for $|k|$ is given by $|k| < 2^{e_k+1}$. Their product is thus bounded by the product of these two bounds.
An upper bound for $|kx + k \nabla(x)|$ can be found in the following way:

$$|kx + k \nabla(x)| \leq kx + k|\nabla(x)|$$
$$\leq 2^{e_k+1}2^v + 2^{e_k+1}2^v - p + 1$$
$$= 2^{e_k+v+2} + 2^{e_k+v} - p + 2$$
$$= 2^{e_k+v+2}(1 + 2^{-p})$$
$$< 2^{e_k+v+3}$$

Thus, the exponent of $kx + k \nabla(x)$ is at most $2^{e_k+v+2}$, and

$$\left| \nabla(kx + k \nabla(x)) \right| \leq \text{ulp}(kx + k \nabla(x)) = 2^{e_k+v} - p + 3.$$

We can now further derive the upper bound for the magnitude of the absolute error of the multiplication:

$$\left| \nabla(k \div [x]) \right| = \left| k \nabla(x) + \nabla(kx + k \nabla(x)) \right|$$
$$\leq k |\nabla(x)| + |\nabla(kx + k \nabla(x))|$$
$$\leq 2^{e_k+v} - p + 2 + 2^{e_k+v} - p + 3$$
$$= 2^{e_k+v} - p + 2(1 + 2)$$
$$< 2^{e_k+v} - p + 4$$

This proves that $|\nabla(k \div [x])| \leq 2^{e_k+v} - p + 4$. \hfill \Box

The algorithms for the propagation of constraints with trigonometric functions, which have been introduced in Section 5.3, divide the domains of trigonometric functions into quasi-monotonic intervals. They need functions to identify approximations of multiples of $\pi$ (by computing $\lceil x \div \pi \rceil$, $x \in \mathbb{R}$), and to generate such upper and lower approximations $\lceil k\pi \rceil$ and $\lfloor k\pi \rfloor$, $k \in \mathbb{Z}$— in the floating-point format in which the studied trigonometric functions operate, which will be called the “target” format.

**Integer division by $\frac{\pi}{2}$, rounded upwards** This procedure has the purpose of computing a number $k$ such that $(k-1)\frac{\pi}{2} \leq x \leq k\frac{\pi}{2}$. It does so by multiplying $x$ by $\frac{2}{\pi}$, and then rounding the result to an integer towards positive infinity. The major caveat in this procedure is the precision at which the multiplication should be performed, and consequently the number of digits of $\frac{2}{\pi}$ that should be stored. For $x \times \frac{2}{\pi}$ to be rounded to the correct integer, we need a precision $p$ sufficient to preserve at least the sign and order of magnitude of the difference $\Delta x = x\frac{2}{\pi} - k$, where $k$ is the integer nearest to $x\frac{2}{\pi}$:

$$\left| \nabla \left( x \div \left\lceil \frac{2}{\pi} \right\rceil \right) \right| \leq 2^{e_{\Delta x}}.$$
where $e_{\Delta x}$ is the exponent of the difference $\Delta x$. According to Lemma 4, this means that, for each $x \in [-\ell_{\max}, \ell_{\max}]$ the following relation must hold:

$$2^{e_{\hat{x}}+e_x-p+4} < 2^{e_{\Delta x}} \iff p > e_x - e_{\Delta x} + e_{\hat{x}} + 4.$$  

The minimum value for $p$ can be obtained by finding the float $\hat{x} \in [-\ell_{\max}, \ell_{\max}]$ that maximizes the value of $e_{\hat{x}} - e_{\Delta \hat{x}}$. This task can be fulfilled by running the algorithm given in [KM84].

It turns out that, for the IEEE 754 single-precision floating point format, such $\hat{x}$ is $1.4AC55C_{16} \cdot 2^{21}$, with $|\Delta \hat{x}| = |\hat{x} \frac{2}{\pi} - \hat{k}| \approx 1.4112D_{16} \cdot 2^{-27}$. Running the algorithm in [KM84] with a “threshold” of $|\Delta x| \leq 2^{-25}$ suffices to show that for no other exponent $e_{\hat{x}'}$ there exists an $\hat{x}'$ such that $|\Delta \hat{x}'| < |\Delta \hat{x}|$. It follows that $p > 21 - 1 + 4 + 27 = 51$. Proceeding in the same way, the minimum precision needed for the double-precision format is $p > 109$.

Since we established that for the single-precision format a precision of at least 52 bits is needed, IEEE 754 double-precision numbers may be used to perform the multiplication $x \times [\frac{2}{\pi}]$. The obtained result can be rounded to the nearest integer towards positive infinity using functions provided by the C and C++ standard math libraries. The cases of $x = -0.0$ and $x = +0.0$ must be handled separately: the values returned by this function must be 0 and 1 respectively, because of Definition 6.

**Generation of lower $\frac{2}{\pi}$ multiple approximations** A similar reasoning can be done to compute $[k \frac{2}{\pi}]$ for a given integer $k$, that is, the maximum floating point number $x \in F$ such that $x < k \frac{2}{\pi}$. This can be achieved by multiplying $k$ by the constant $\frac{2}{\pi}$ with a sufficient precision, and then rounding it to the target precision using the roundTowardNegative rounding direction.

The precision used for this multiplication must satisfy

$$p > e_k - e_{x-k \frac{2}{\pi}} + e_{\frac{2}{\pi}} + 4$$

according to Lemma [1]. The value $\hat{x}$ that maximizes $e_x - e_{\Delta x}$ is the same as for the previous section. Since for each $x \in [-\ell_{\max}, \ell_{\max}]$ we have $|\Delta x| = |x \frac{2}{\pi} - k| \leq |x - k \frac{2}{\pi}|$, where $k$ is the integer nearest to $x \frac{2}{\pi}$, $e_x - e_k \frac{2}{\pi} \geq e_{\Delta x}$ holds. Also, because of the criteria we used to define $\ell_{\max}$ (i.e., the ulp in the interval should be sufficiently large, and greater than 1), $k$ is always representable in our target floating-point format. Moreover, $e_k \leq e_x$, because $|k| \leq |[k \frac{2}{\pi}]| = |x|$. Consequently, $\hat{x}$ and $\hat{k}$ maximize also $e_k - e_{x-k \frac{2}{\pi}}$.

For the single-precision format we can state

$$p > e_{\hat{x}} + e_{\frac{2}{\pi}} + 4 - e_{\Delta \hat{x}} = 21 + 0 + 4 + 27 = 52,$$

which imposes a required precision of at least 53 bits. This allows us to use the IEEE 754 double-precision format to perform this multiplication. In order to round the result to the single-precision format, the rounding direction can be set to roundTowardNegative. It is also possible to avoid switching rounding mode.
by adjusting the obtained result according to the rounding direction in use. If $k = 0$, the function should return $x = -0.0$. This case can be handled separately.

To obtain the result in the double-precision format, $p > 110$ is required. The computation can be performed using a technique for the implementation of multiple precision arithmetic, such as the one described in [Dek71] with an extended-precision format of at least 55 bits of precision, or the one given in [HLB01] with the double-precision format. For formats requiring higher precision, see [MPT16].

D.3 Algorithms for Trigonometric Functions

The constraint propagation algorithms we developed for trigonometric functions are described in their principles in Section D.2. They make use of the concepts and range-reduction techniques presented above.

Since the direct propagation algorithm is substantially simple in its functioning, we will not describe it in further detail. In this section, we will instead present the inverse propagation algorithm in pseudocode, and provide an argument for its correctness.

Algorithm compute_bounds_trig, presented as Algorithm 8, has the main purpose of splitting the interval to be refined, $[x_l, x_u]$, into the corresponding set of the monotonic intervals, making use of the range-reduction procedures presented in Section D.2. It then invokes algorithms lower_bound and upper_bound through the helper functions branch_lb and branch_ub, one of which is shown in as Algorithm 9.

The algorithms presented in this section require the same glitch data as the algorithms for regular functions: please refer to Section 4 for an explanation. Argument $g \in \mathbb{N}$ is a parameter that fixes the maximum number of monotonic sub-intervals that $[x_l, x_u]$ should be split into. Argument $p$ holds information about the behavior of function $f$, allowing the algorithm to correctly recognize whether the graph of the function is isotonic or antitonic in each sub-interval, and to correctly invoke the other propagation algorithms. They can take one of values even, odd, and oddc, which carry the following meanings:

- **even**: the function changes its monotonicity in even multiples of $\frac{\pi}{2}$, i.e., in numbers of the type $2k\frac{\pi}{2}$, with $k \in \mathbb{Z}$. If $f = \cos$, this value of $p$ is passed.
- **odd**: the function changes its tonicity in odd multiples of $\frac{\pi}{2}$, numbers of the form $(2k + 1)\frac{\pi}{2}$, $k \in \mathbb{Z}$. This is the behavior of the sine function.
- **oddc**: the function has a discontinuity in odd multiples of $\frac{\pi}{2}$, but it remains isotonic. The tangent has this behavior.

For a deeper understanding of how the algorithms work, the reader can refer to their correctness proofs, which follow.

**Lemma 5.** If function $f$ is quasi-monotonic over $[i.x_l, i.x_u]$, then the function branch_lb, specified in Algorithm 9 satisfies its contract.
Algorithm 8 Inverse propagation: compute_bounds_trig($f, f^i, p, y, \{x_i, x_u\}, n_{gM}, d_M, w_M, \alpha, \omega, g, s, t$)

Require: $f : F \to F, f^i : F \to F, p \in \{\text{even}, \text{odd}, \text{odd}_c\}, \{x_i, x_u\}, \{y, y_u\} \in \mathcal{I}_f, n_{gM} \geq n_{gM}' \geq d_M, d_M' \geq n_{gM}, w_M \geq w_M', \forall \alpha \leq \alpha', \omega \geq \omega', n_{gM} > 0 \implies (x_i \leq \alpha \leq \omega \leq x_u), g, s, t \in \mathbb{N}.$

Ensure: $|I| \leq g, \bigcup_{i \in I} [i.x_i, i.x_u] = [x_i, x_u], \forall i \in I : i.l, i.u \in F$

\[ k_1 := \text{geq\_tonicity\_change}(p, [x_i^\frac{g}{2}]^\frac{u}{2}); \]
\[ \tau k_1 = \min\{ k \in \mathbb{Z} | k \geq [x_i^\frac{g}{2}]^\frac{u}{2} \} \]
\[ k_u := \text{geq\_tonicity\_change}(p, [x_u^\frac{g}{2}]^\frac{u}{2}); \]
\[ \tau k_u = \min\{ k \in \mathbb{Z} | k \geq [x_u^\frac{g}{2}]^\frac{u}{2} \} \]

3: if even($p$) then $k_c := 0$
4: else $k_c := -1$
5: end if;
6: if $g = 1$ then $g_l := 1; g_r := 0; k_c := k_u$
7: else if $x_u \leq \frac{k_c}{2}$ then $g_l := g; g_r := 0$
8: else if $x_i \geq \frac{k_c}{2}$ then $g_l := 0; g_r := g$
9: else $g_l := \frac{k_c}{2}; g_r := g - g$
10: end if;
11: if $g_i > 0$ then
12: $i.x_i := x_i; i.x_u := \min\{x_u, [k_i]\}$
13: (i.l, i.r) := branch_lb($f, f^i, p, [y, y_u], \{x_i, x_u\}, k_l, n_{gM}, d_M, w_M, \alpha, \omega, s, t$);
14: $k_{u, i} := \min\{k_{u, i}, k_c\}$
15: $i.k := \max\{k_i, k_{u, i} - 2(g_i - 1)\}$
16: if $i.r = 0 \land k_i < i.k$ then
17: $i.l := i.x_u; i.r := 2$
18: end if;
19: $c_{e_i} := \max\{x_i, \{i.k - 2\frac{w}{2}\}\}$
20: $i.x_u := \min\{x_u, [i.k + \frac{w}{2}]\}$
21: $i.u, i.r_a := \text{branch\_ub}(f, f^i, p, [y, y_u], \{x_i, x_u\}, i.k, n_{gM}, d_M, w_M, \alpha, \omega, s, t)$
22: if $i.r_a = 5 \land k_i < i.k$ then
23: $i.u := i.x_i; i.r_a := 7$
24: end if;
25: $I.\text{add}(i)$
26: $i.k := i.k + 2$
27: while $i.k < k_{u, i}$ do
28: $i.x_i := i.x_u^+; i.x_u := \min\{i.k + \frac{w}{2}; i.x_u\}$
29: (i.l, i.r) := branch_lbd($f, f^i, p, [y, y_u], \{x_i, x_u\}, i.k, n_{gM}, d_M, w_M, \alpha, \omega, s, t$);
30: $i.u, i.r_a := \text{branch\_ub}(f, f^i, p, [y, y_u], \{x_i, x_u\}, i.k, n_{gM}, d_M, w_M, \alpha, \omega, s, t)$
31: $I.\text{add}(i)$
32: $i.k := i.k + 2$
33: end while
34: end if;
35: if \( g_i > 0 \) then
36: \( i.k := \max\{k_l, k_u + 2\} \);
37: \( i.x_u := \max\{x_i^-, \left(\frac{k_u - 2}{2}\right)^+\} \);
38: \( \text{while } i.k < k_u \land g_i > 1 \text{ do} \)
39: \( i.x_l := i.x_u^+ \); \( i.x_u := \min\{x_u, \left(\frac{i.k + 2}{2}\right)^+\} \);
40: \( (i.l, i.r_l) := \text{branch}_l(b(f, f^i, p, [y_l, y_u], [i.x_l, i.x_u], i.k, n_{\text{gM}}, d_M, w_M, \alpha, \omega, s, t); \)
41: \( (i.u, i.r_u) := \text{branch}_u(b(f, f^i, p, [y_l, y_u], [i.x_l, i.x_u], i.k, n_{\text{gM}}, d_M, w_M, \alpha, \omega, s, t); \)
42: \( I.\text{add}(i); \)
43: \( i.k := i.k + 2; \)
44: \( g_i := g_i - 1 \);
45: \( \text{end while}; \)
46: \( i.x_l := i.x_u^+; i.x_u := \min\{x_u, \left(\frac{i.k + 2}{2}\right)^+\} \);
47: \( (i.l, i.r_l) := \text{branch}_l(b(f, f^i, p, [y_l, y_u], [i.x_l, i.x_u], i.k, n_{\text{gM}}, d_M, w_M, \alpha, \omega, s, t); \)
48: \( \text{if } i.r_l = 0 \land i.k < k_u \text{ then} \)
49: \( i.l := i.x_l; i.r_l := 2 \);
50: \( \text{end if}; \)
51: \( c_{x_l} := \max\{x_l, \left(\frac{k_u - 2}{2}\right)^+\}; i.x_u := x_u; \)
52: \( (i.u, i.r_u) := \text{branch}_u(b(f, f^i, p, [y_l, y_u], [c_{x_l}, i.x_u], k_u, n_{\text{gM}}, d_M, w_M, \alpha, \omega, s, t); \)
53: \( \text{if } i.r_u = 5 \land i.k < k_u \text{ then} \)
54: \( i.u := c_{x_l}; i.r_u := 7 \);
55: \( \text{end if}; \)
56: \( I.\text{add}(i) \);
57: \( \text{end if} \)

\begin{algorithm}
\caption{Inverse propagation: \text{branch}_l(b(f, f^i, p, [y_l, y_u], [i.x_l, i.x_u], i.k, n_{\text{gM}}, d_M, w_M, \alpha, \omega, s, t))}
\begin{algorithmic}
\Require \( f: F \to F, f^i: F \to F, p \in \{\text{even}_\omega, \text{odd}_\omega, \text{odd}_t\}, y \in F, [i.x_l, i.x_u], [y_l, y_u] \in \mathcal{I}_F, i.k \in \mathbb{Z}, n_{\text{gM}} \geq n_{\text{gM}}^f, d_M \geq d_M^f, w_M \geq w_M^f, \alpha \leq \alpha^f, \omega \geq \omega^f, n_{\text{gM}} > 0 \implies (\alpha \leq \omega), s, t \in \mathbb{N}. \)
\Ensure \( (\bigcirc) i.l \in F, \)
\( i.r_l \in \{0, 1, 2, 3, 4\} \implies (p_i^r_f(y_l, i.x_l, i.x_u, i.l) \lor p_i^l_f(-y_u, i.x_l, i.x_u, i.l)) \).
\State if \( n_{\text{gM}} = 0 \lor i.x_l > \omega \lor i.x_u < \alpha \) then
\State \( i.z_{\text{rg}} := 0 \)
\State else
\State \( i.\omega_{\text{rg}} := n_{\text{gM}}; \)
\State \( i.\omega_l := \max\{\alpha, i.x_l\}; \)
\State \( i.\omega_u := \min\{\omega, i.x_u\} \)
\State end if;
\State if quasi_isotonic(i.k, p) then
\State \( (i.l, i.r_l) := \text{lower_bound}(f, y_l, [i.x_l, i.x_u], i.\omega_{\text{rg}}, d_M, w_M, i.\omega_l, i.\omega_u, f^i, s, t); \)
\State else
\State \( (i.l, i.r_l) := \text{lower_bound}(-f, -y_u, [i.x_l, i.x_u], i.\omega_{\text{rg}}, \) \( d_M, w_M, i.\omega_l, i.\omega_u, f^i \circ (\text{id}), s, t) \)
\State end if
\end{algorithmic}
\end{algorithm}
Proof. The if statement on line 4 prepares variables \(i.n_k\), \(i.\alpha\) and \(i.\omega\) to be used as arguments for lower_bound. If there are no glitches in \([i.x_l, i.x_u]\), \(i.n_k\) is set to 0. Otherwise, the else statement ensures that \(i.x_l \leq i.\alpha \leq i.\omega \leq x_u\). Along with the preconditions of this algorithm, this satisfies all preconditions of lower_bound.

The only thing that remains to do, is to distinguish whether \(f\) is quasi-isotonic or quasi-antitonic in \([i.x_l, i.x_u]\). This is done by predicate quasi_isotonic \((i.k,p)\), by checking the value of \(m = i.k \mod 4\): if \(p = \text{even}\), \(f\) is isotone when \(m \in \{0,3\}\); if \(p = \text{odd}\), \(f\) is isotone when \(m \in \{1,2\}\). If \(f\) is quasi-isotonic, lower_bound can be called normally. Otherwise, lower_bound is called on line 12 with \(-f\) instead of \(f\), \(-y_u\) instead of \(y_l\), and the inverse function \(f_i\) with the sign of its argument changed. Note that, if \(f\) is quasi-antitonic, then \(-f\) is quasi-isotonic, which allows us to invoke lower_bound normally. Also, \(-y_u\) must be passed as the lower bound for \(y\) because if \(y_l \leq y_u\), then \(-y_u \leq -y_l\).

Since \(i.l\) and \(i.r\) are set by lower_bound, they satisfy its post-condition. So \(p^{f}_{i,rl}(y_l, i.x_l, i.x_u, i.l)\) holds if \(f\) is isotonic in \([i.x_l, i.x_u]\), or \(p^{f}_{i,rl}(-y_u, i.x_l, i.x_u, i.l)\) if \(f\) is antitonic in that interval. Note that, by \(p^{f}_{i,rl}\), we intend predicate \(p^{f}_{i,rl}\) with all occurrences of \(f\) substituted with \(-f\). Consequentially, also the Ensure statement of this algorithm holds.

The algorithm listing and the proof of branch_ub are omitted, since they are very similar to those of branch_lb.

**Theorem 4.** Function compute_bounds_trig, specified in Algorithm 8 satisfies its contract.

Proof. The algorithm starts by identifying which branches of the function graph contain \(x_l\) and \(x_u\). This operation is performed at lines 11 and 2 where function geq_tonicity_change is called. The lowest integer that, multiplied by \(\frac{2}{n}\), gives a number greater than \(x_l\) where the monotonicity of the function changes is assigned to \(k_l\). The same is done for \(x_u\), and the value is assigned to \(k_u\).

On line 3 a value for \(k_c\) is chosen. The approximations of \(k_c\) will be used to split \([x_l, x_u]\) in two intervals \([x_l, \frac{k_c + 1}{2}]\) and \([\frac{k_c + 2}{2}, x_u]\), which will be referred to as “left” and “right” intervals respectively. Branches of the function nearest to \(k_c\) will not be included into the two large intervals near the bounds of the domains, and will be processed individually. If \(f\) changes monotonicity in even multiples of \(\frac{2}{n}\) (such as the cosine), \(k_c = 0\). Otherwise, \(k_c\) is set to \(-1\): in this way the branch between \(-1\) and \(\frac{2}{n}\) will be the leftmost one in the right interval.

Now, values for \(g_l\) and \(g_r\) are chosen. They are, respectively, the number of sub-intervals in which the left and right intervals will be divided. The if statement at line 6 distinguishes among the four cases listed below.

\(g = 1:\) Only one interval must be returned, corresponding to \([x_l, x_u]\), but refined in the outward branches of the function. In this case, \(g_l\) is set to 1, and \(g_r\) to 0. In this way, the body of the if statement on line 11 will take care of the said interval, as it would normally do with the leftmost interval. Also, \(k_c := k_u\) is set, since the above mentioned code uses \(k_c\) to chose the upper bound of the said interval.
\[ g > 1 \land x_u \ll \left[k_i \frac{x}{2}\right]_1 : [x_l, x_u] \text{ lies entirely into the left interval. Therefore, } g_l \text{ must be set to } g \text{ and } g_r \text{ to } 0. \]

\[ g > 1 \land x_l \gg \left[k_i \frac{x}{2}\right]_1 : [x_l, x_u] \text{ lies entirely into the right interval. Therefore, } g_l \text{ must be set to } 0 \text{ and } g_r \text{ to } g. \]

\[ g > 1 \land x_l < \left[k_i \frac{x}{2}\right]_1 \land x_u \gg \left[k_i \frac{x}{2}\right]_1 : [x_l, x_u] \text{ must be divided into left and right intervals. } g_l \text{ is set to } \left[\frac{g}{2}\right] \text{ and } g_r \text{ is set to } g - g_l. \text{ This way, half of the sub-interval will be in the left interval, and half in the right one. Note that, if the function has less than } g_l \text{ monotonic branches in the left interval or less then } g_r \text{ in the right one, the interval will be only split in as many intervals as the branches.} \]

In all four cases, \( g_l + g_r = g \).

The purpose of the body of the if statement on line 11 is to divide the left interval \([x_l, \min\{x_u, [k_i \frac{x}{2}]_1\}]\) into the appropriate sub-intervals, and refine them by calling \texttt{branch} \_l and \texttt{branch} \_ub. First, the leftmost sub-interval is generated. Each sub-interval is identified by its initial bounds \(i.x_l\) and \(i.x_u\), by the refined bounds \(i.l\) and \(i.u\), by \(i.r_l\) and \(i.r_u\) (the predicates that hold after calling \texttt{lower} \_bound and \texttt{upper} \_bound), and by \(i.k\), which identifies the branch (or one of the branches) of the function in which the sub-interval is contained. That is, it defines the interval \([[(i.k - 2) \frac{x}{2}]_1, [i.k \frac{x}{2}]_1]\).

If the number of monotonic branches in the left interval is lower than or equal to \(g_l\), then this sub-interval will correspond to a single branch; it will include multiple branches otherwise. On line 12 \(i.x_l\) and \(i.x_u\) are set to the bounds of the leftmost branch, \([x_l, \min\{x_u, [k_i \frac{x}{2}]_1\}]\). Because of how \(k_l\) was set, this interval is part of the leftmost monotonic function of the branch, and it covers the end of it if \(x_u \geq [k_l]_1\). Then since \(f\) is monotonic in \([i.x_l, i.x_u]\), \texttt{branch} \_l\_b can be called and \(i.l\) and \(i.r_l\) are set, satisfying \(p_{l_{r_1}}^f(y_l, i.x_l, i.x_u, i.l)\) if \(f\) is isotonic in that interval, or \(p_{l_{r_1}}^{-f}(y_l, i.x_l, i.x_u, i.l)\) if it is antitonic, according to the post-condition of \texttt{branch} \_l\_b.

The left interval will end with the branch defined by \(k_i\), or with \(x_u\) if \(x_u \ll [k_i \frac{x}{2}]_1\). On line 13, \(k_u\) is set accordingly to \(\min\{k_u, k_i\}\). Then, on line 15 the rightmost branch of the left interval is chosen: it will end in \([i.k \frac{x}{2}]_1\). Since the rightmost \(g_l - 1\) branches will be treated individually by the \texttt{while} loop on line 27, \(i.k\) is set to \(\max\{k_l, k_u - (g_l - 1)\}\). \(g_l - 1\) is multiplied by 2 because each branch has a length of \(\pi\) (two times \(\frac{x}{2}\)), and subtracted from \(k_u\), which identifies the rightmost possible branch in the left interval.

Before passing to the refinement of the upper bound of the sub-interval, the if statement on line 16 checks whether \(i.r_l\), the predicate number returned by \texttt{lower} \_bound, is 0. If \(k_l = i.k\) it means this sub-interval contains a single branch, and no more work is needed, because we know there are no solutions to \(y_l = f(x)\) in this sub-interval. If also \(k_l < i.k\) holds, there are more branches after \(i.x_u\) in this sub-interval. According to \(p_0\) we have \(\forall x \in [i.x_l, i.x_u] : y > f(x)\) but we know nothing about the other branches. Since \(i.x_u\) will be later set to the upper bound of this sub-interval, \(i.l\) must be set to the current value of \(i.x_u\), and \(i.r_l\) to
2. This way, we have $\forall x \in [i.x_l, i.l] : y \succ f(x)$, and either $p_2^f$ or $p_{2^-}^f$ is satisfied even if the value of $i.x_u$ is later increased.

Then, the rightmost branch of the leftmost sub-interval is refined. On line 19 $c_n$ is set to the maximum between $x_l$ (remain inside the domain, in case this sub-interval contains only one branch), and $\lceil\{i.k-2\frac{\pi}{2}\}\rceil$. This is the lower bound of the rightmost branch of this sub-interval (note that $i.k_u$ was previously set to the corresponding value). Similarly, $i.x_u$ is set to the minimum between $x_u$ and $\lceil\{i.k+2\frac{\pi}{2}\}\rceil$ on line 20. Now, $f$ is monotonic over $[c_x, i.x_u]$, and we are ready to refine the upper bound of this sub-interval, by calling $\text{branch}\_\text{ub}$.

The if statement of line 22 checks whether $i.r_u = 5$. Again, if $k_l = i.k$ nothing needs to be done. If $k_l < i.k$, we cannot return $i.r_u = 5$, because we do not know if there are solutions or not in the branches we did not analyze. Therefore, $i.u$ is set to $i.x_l$ and $i.r_u$ to 7. Since the post-condition of $\text{branch}\_\text{ub}$ ensures that $p_5$ holds, we have $\forall x \in [i.x_l, i.x_u] : y \prec f(x)$: then also $\forall x \in [i.u, i.x_u] : y \prec f(x)$ holds, which satisfies $p_2^f$ or $p_{2^-}^f$.

At this point, both bounds of the leftmost interval have been refined, and $i$, with the current values of its fields, can be added to $I$. Then we can pass to the next sub-interval, identified by the current $i.k+2$. $i.k$ is updated accordingly on line 26.

The while loop starting on line 27 refines all other sub-intervals in the left interval, which all correspond to a single monotonic branch of the function. We will now prove that loop invariant

$$\text{Inv} \equiv \bigcup_{h \in I} \{h.x_l, h.x_u\} = [x_l, i.x_u] \quad (13)$$

$$\land \forall h \in I : h.l, h.u \in \mathbb{R}$$

$$\land h.r_l \in \{0, 1, 2, 3, 4\} \implies \begin{cases} p_{h.r_l}^f(y_l, h.x_l, h.x_u, h.l) \\ \lor \not p_{h.r_l}^f(y_l, h.x_l, h.x_u, h.l) \end{cases} \quad (14)$$

$$\land h.r_u \in \{5, 6, 7, 8, 9\} \implies \begin{cases} p_{h.r_u}^f(y_u, h.x_l, h.x_u, h.u) \\ \lor \not p_{h.r_u}^f(y_u, h.x_l, h.x_u, h.u) \end{cases} \quad (15)$$

$$\land i.x_u = \min\left\{\lceil\{i.k-2\frac{\pi}{2}\}\rceil, x_u\right\} \quad (16)$$

holds before and after each iteration of the loop, unless the loop guard is false before the loop is entered the first time. In this latter case, clause (17) does not hold, but the other do.

Before starting with the proof of the loop, we need to prove the following fact: right after lines 20 and 28

$$x_u \leq \lceil\{i.k+2\frac{\pi}{2}\}\rceil \implies i.k + 2 \geq k_u. \quad (18)$$

Because of how $k_u$ was set on line 2 we also have $\lceil\{k_u-2\frac{\pi}{2}\}\rceil \leq x_u$. This implies $\lceil\{k_u-2\frac{\pi}{2}\}\rceil \leq \lceil\{i.k+2\frac{\pi}{2}\}\rceil$ and $k_u - 2 \leq i.k$. Line 14 ensures that $k_u \leq k_u$: we have $k_u - 2 \leq k_u - 2 \leq i.k$, or $i.k + 2 \geq k_u$. 

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Initialization: Before of the first iteration of the loop, \( I \) contains only one interval \( i \). Since \( i.x_l \) was set to \( x_l \) on line 12 and left untouched, and also \( i.x_u \) was not changed after the interval was added to \( I \), clause (13) of the invariant is satisfied. As for \( i.l \), it satisfies predicate \( p_{i.r} \) because, either:

\( i.r_l \neq 0 \): it was set by \texttt{branch_lb} and left untouched. Even if \( i.x_u \) has been possibly increased after \texttt{branch_lb} was called, all predicates \( p_{i,r_l} \) with \( i.r_l \neq 0 \) still hold, because they do not imply anything so specific about interval \((i.l,i.x_u]\) to be invalidated by increasing \( i.x_u \).

\( i.r_l = 0 \): the \texttt{if} statement on line 10 changed \( i.l \) and \( i.r_l \) so that \( p_{i,r_l} \) still holds, as was proved before.

A very similar reasoning can be done for \( i.u \) and \( i.r_u \), distinguishing whether \texttt{branch_ub} returned \( i.r_u = 5 \) or not. Therefore, clauses (15) and (16) of the invariant are also proved.

On line 20, \( i.x_u \) was set to \( x_u := \min\{x_u, \lfloor i.k \frac{x}{2} \rfloor \} \) and, since \( i.k \) was incremented by 2 on line 26, clause (17) is satisfied.

Maintenance: If the loop body was entered, the guard condition is true: so \( i.k < k_h \). This, together with implication (18), assures us that \( x_u \triangleright \lfloor i.k \frac{x}{2} \rfloor \). Because of clause (17) of the invariant, \( x_u = \lceil (i.k-2) \frac{x}{2} \rceil \) and \( x_u^+ = \lceil (i.k-2) \frac{x}{2} \rceil \). \( i.x_l \) is set to this value on line 28. On the same line, \( i.x_u \) is set to the minimum between \( x_u \) and \( \lfloor i.k \frac{x}{2} \rfloor \).

Therefore, \([i.x_l, i.x_u]\) is contained in a single monotonic branch, and functions \texttt{branch_lb} and \texttt{branch_ub} can be called to refine the bounds of this sub-interval. The post-conditions of the said functions satisfy clauses (15) and (16) of the invariant. Sub-interval \( i \) is then added to \( I \). Since on line 28 \( i.x_l \) was set to the successor of the previous value of \( i.x_u \), and because the invariant ensures that previously \( \bigcup_{h \in I} [h.x_l, h.x_u] = [x_l, i.x_u] \), adding the new interval to \( I \) leaves clause (13) satisfied. Eventually, after line 32 clause (17) also holds.

Termination: On line 13, \( i.k \) was set to a value such that \( k_u - i.k \leq 2(g_1 - 1) \).

Since \( i.k \) is incremented by 2 at the end of the loop body, it reaches \( k_u \) after at most \( g_1 - 1 \) iterations: the guard then evaluates to false, and the loop terminates.

Since the loop iterates at most \( g_1 - 1 \) times, the number of sub-intervals it adds to \( I \) is lower than or equal to \( g_1 - 1 \). Together with the fact that the leftmost sub-interval is always added to \( I \) before the loop, this implies \( |I| \leq g_1 \).

At this point, if \( k_u \leq k_c \), then \( k_u = k_u \) and the last sub-interval ended in \( x_u \): \( i.x_u = x_u \) and clause \( \bigcup_{h \in I} [i.x_l, i.x_u] = [x_l, x_u] \) of the post-condition is proved.

In this case, \( x_u \triangleright \lfloor k_c \frac{x}{2} \rfloor \) and \( g_t = 0 \) according to line 14 the \texttt{if} statement of line 35 is not entered, and the algorithm terminates. Since there are no more sub-intervals to be analyzed, \( |I| \leq g_1 = g \) and the loop invariant satisfy the rest of the post-condition.

Otherwise, if \( x_u \) has not been reached yet, the \texttt{if} body of line 35 processes the remaining sub-intervals. First, it splits the first \( g_t - 1 \) branchs of \( f \) and refines them with \texttt{branch_lb} and \texttt{branch_ub}. The loop on line 35 terminates
when either the rightmost branch is refined, or \( g_r = 1 \). Then, a sub-interval made of the remaining branches is processed in a way similar to the one of the leftmost interval. Since this part of the algorithm is similar to the if statement of line [11] the proof of its correctness is omitted.

After the body of this last if statement is executed, a number of sub-intervals less or equal than \( g_r \) has been added to \( I \). Together with the fact that \( |I| \leq g_l \) held before, this implies that at the end \( |I| \leq g \). The fact that it processes all branches between \( [k_c, \pi/2]_l \) and \( x_u \) by calling branch\_lb and branch\_ub, and it corrects their output for the rightmost interval if needed, proves the rest of the post-condition of this algorithm. \( \square \)