PURSUIT-EVASION GAME WITH HYBRID SYSTEM OF DYNAMICS

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1. Formulation of the problem and result

In the space $l_2$ consisting of elements $a = (a_1, a_2, \ldots, a_m, \ldots)$, with $\sum_{m=1}^{\infty} a_m^2 < \infty$, and inner product $(a, b) = \sum_{m=1}^{\infty} a_m b_m$, the motions of the countably many pursuers $P_i$ and the evader $E$ are defined by the hybrid system of differential equations

$$
P : \dot{p} = \mu, \quad p(0) = p_0, \\
E : \ddot{e} = \nu, \quad \dot{e}(0) = e^1, \quad e(0) = e^0,$$

where $p, p_0, e, e^0, e^1, \nu \in l_2$, $\mu = (\mu_1, \mu_2, \ldots, \mu_m, \ldots)$ is the control parameter of the pursuer $P_i$, and $\nu = (\nu_1, \nu_2, \ldots, \nu_m, \ldots)$ is that of the evader $E$. Let $\varphi$ be a given positive number.

A ball of radius $\delta$ and center at the point $x_0$ is denoted by $B(x_0, \delta) = \{x \in l_2 : \|x - x_0\| \leq \delta\}$.

**Definition 1.1.** A function $\mu(\cdot), \mu : [0, \varphi] \to l_2$, such that $\mu_m : [0, \varphi] \to \mathbb{R}^1$, $m = 1, 2, \ldots$, are Borel measurable functions and

$$
\|\mu(\cdot)\|_2 = \int_{0}^{\varphi} \|\mu(s)\|^2 ds \leq \Gamma^2, \quad \|\mu\|^2 = \sum_{m=1}^{\infty} \mu_m^2,
$$

where $\Gamma$ is a given positive number, is called an *admissible control of the pursuer*.

**Definition 1.2.** A function $\nu(\cdot), \nu : [0, \varphi] \to l_2$, such that $\nu_m : [0, \varphi] \to \mathbb{R}^1$, $m = 1, 2, \ldots$, are Borel measurable functions and

$$
\|\nu(\cdot)\|_2 = \int_{0}^{\varphi} \|\nu(s)\|^2 ds \leq \Upsilon^2, \quad \|\nu\|^2 = \sum_{m=1}^{\infty} \nu_m^2,
$$

where $\Upsilon$ is a given positive number, is called an *admissible control of the evader*.

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Once the players’ admissible controls $\mu(\cdot)$ and $\nu(\cdot)$ are chosen, the corresponding motions $p(\cdot)$ and $e(\cdot)$ of the players are defined as

\[
p(t) = (p_1(t), p_2(t), \ldots, p_m(t), \ldots), \quad e(t) = (e_1(t), e_2(t), \ldots, e_k(t), \ldots),
\]

\[
p_m(t) = p_m^0 + \int_0^t \mu_m(s) \, ds, \quad e_m(t) = e_m^0 + t \epsilon_m + \int_0^s \nu_m(r) \, dr \, ds.
\]

One could observe that $p(\cdot), e(\cdot) \in C(0, \varphi; l_2)$, where $C(0, \varphi; l_2)$ is the space of functions $f(t) = (f_1(t), f_2(t), \ldots, f_m(t), \ldots) \in l_2, \quad t \geq 0,$ such that the following properties are valid.

1. $f_m(t), \quad 0 \leq t \leq \varphi, \quad m = 1, 2, \ldots$, are absolutely continuous functions;
2. $f(t), \quad 0 \leq t \leq \varphi,$ is a continuous function in the norm of $l_2$.

**Definition 1.3.** A function $\Xi(t, p, e, \nu), \Xi : [0, \infty) \times l_2 \times l_2 \times l_2 \rightarrow l_2$, such that the system

\[
\begin{align*}
\dot{p} &= \Xi(t, p, e, \nu), \quad p(0) = p^0, \\
\dot{e} &= \nu, \quad e(0) = e^0, \quad \dot{e}(0) = e^1,
\end{align*}
\]

has a unique solution $(p(\cdot), e(\cdot))$, with $p(\cdot), e(\cdot) \in C(0, \varphi; l_2)$, for an arbitrary admissible control $\nu = \nu(t), \quad 0 \leq t \leq \varphi$, of the evader $E$, is called a *strategy of the pursuer* $P$. A strategy $\Xi$ is said to be *admissible* if each control formed by this strategy is admissible.

For the admissible control $\nu(t) = (\nu_1(t), \nu_2(t), \ldots), \quad 0 \leq t \leq \varphi, \quad$ of the evader $E$, according to (1.1) we have

\[
e(\varphi) = e^0 + e^1 \varphi + \int_0^\varphi \int_0^t \nu(s) \, ds \, dt = e^0 + e^1 \varphi + \int_0^\varphi (\varphi - t) \nu(t) \, dt,
\]

and using (1.2) one can see

\[
e(\varphi) = e_0 + \int_0^\varphi (\varphi - t) \nu(t) \, dt = e^0 + e^1 \varphi + \int_0^\varphi (\varphi - t) \nu(t) \, dt.
\]

Therefore, instead of differential game described by (1.1) we can use an equivalent differential game with the same payoff function as the following:

\[
\begin{align*}
P : \dot{p}(t) &= \mu(t), \quad p(0) = p_0, \\
E : \dot{e}(t) &= (\varphi - t) \nu(t), \quad e(0) = e_0 = e^1 \varphi + e^0.
\end{align*}
\]

**Proposition 1.4.** The attainability domain of the pursuer $P$ at time $\varphi$ from the initial state $p_0$ at time $t_0 = 0$ is the closed ball $B(p_0, \Gamma \sqrt{\varphi})$. 

Proof. By Cauchy-Schwartz inequality we obtain

\[ \|p(\varphi) - p_0\| = \left\| \int_0^{\varphi} \mu(s) \, ds \right\| \leq \int_0^{\varphi} \|\mu(s)\| \, ds \leq \left( \int_0^{\varphi} 1^2 \, ds \right)^{1/2} \cdot \left( \int_0^{\varphi} \|\mu(s)\|^2 \, ds \right)^{1/2} \leq \Gamma \sqrt{\varphi}. \]

Let \( \bar{p} \in B(p_0, \Gamma \sqrt{\varphi}) \). If the pursuer \( P \) uses the control

\[ \mu(t) = \frac{\bar{p} - p_0}{\varphi}, \quad 0 \leq t \leq \varphi, \]

then we obtain

\[ p(\varphi) = p_0 + \int_0^{\varphi} \mu(t) \, dt = p_0 + \int_0^{\varphi} \frac{\bar{p} - p_0}{\varphi} \, dt = p_0 + \bar{p} - p_0 = \bar{p}. \]

The above pursuer’s control is admissible. Indeed,

\[ \int_0^{\varphi} \|\mu(t)\|^2 \, dt = \int_0^{\varphi} \left( \frac{\bar{p} - p_0}{\varphi} \right)^2 \, dt = \frac{\|\bar{p} - p_0\|^2}{\varphi} \leq \frac{1}{\varphi} \Gamma^2 \varphi = \Gamma^2. \]

\[ \square \]

**Proposition 1.5.** The attainability domain of the evader \( E \) at time \( \varphi \) from the initial state \( e_0 \) at time \( t_0 = 0 \) is the closed ball \( B \left( e_0, \Upsilon \sqrt{\frac{\varphi^3}{3}} \right) \).

Proof. We have

\[ \|e(\varphi) - e_0\| = \left\| \int_0^{\varphi} (\varphi - t) \nu(t) \, dt \right\| \leq \int_0^{\varphi} \|(\varphi - t) \nu(t)\| \, dt \leq \left( \int_0^{\varphi} (\varphi - t)^2 \, dt \right)^{1/2} \cdot \left( \int_0^{\varphi} \|\nu(t)\|^2 \, dt \right)^{1/2} \leq \Upsilon \sqrt{\frac{\varphi^3}{3}}. \]

Let \( \bar{e} \in B \left( e_0, \Upsilon \sqrt{\frac{\varphi^3}{3}} \right) \). If the evader \( E \) uses the control
\[ \nu(t) = 3(\varphi - t) \frac{\bar{e} - e_0}{\varphi^3}, \quad 0 \leq t \leq \varphi, \]

then we obtain

\[ e(\varphi) = e_0 + \int_0^\varphi (\varphi - t) \nu(t) \, dt = e_0 + 3 \frac{\bar{e} - e_0}{\varphi^3} \int_0^\varphi (\varphi - t)^2 \, dt = e_0 + \bar{e} - e_0 = \bar{e}. \]

The above evader’s control is admissible. Indeed,

\[ \int_0^\varphi \|\nu(t)\|^2 \, dt = \int_0^\varphi \|3(\varphi - t) \frac{\bar{e} - e_0}{\varphi^3}\|^2 \, dt = 9 \frac{\|\bar{e} - e_0\|^2}{\varphi^6} \int_0^\varphi (\varphi - t)^2 \, dt \leq \frac{9}{\varphi^6} \frac{\gamma^2}{3} \frac{\varphi^3}{3} = \gamma^2. \]

\[ \square \]

The problem is to construct a winning strategy for the pursuer in the game (1.1) that guarantees the equality \( p(\varphi) = e(\varphi) \), for any admissible control of the evader.

**Theorem 1.6.** Let

\[ Z = \left\{ \zeta \in l^2 : 2(e_0 - p_0, \zeta) \leq \varphi \left( \Gamma^2 - \frac{\varphi^5}{5} \right) + \|e_0\|^2 - \|p_0\|^2, \ e_0 \neq p_0 \right\}. \]

If \( e(\varphi) \in Z \) (phase constraint), then the pursuer has a winning strategy.

**Proof.** Let’s define the following strategy as a winning strategy for the pursuer.

\[ \Xi(t) = \frac{e_0 - p_0}{\varphi} + (\varphi - t) \nu(t), \quad 0 \leq t \leq \varphi. \]

Let’s show that the above strategy is admissible. Since the evader is satisfied to the phase constraint, we have

\[ 2(e_0 - p_0, e(\varphi)) \leq \varphi \left( \Gamma^2 - \frac{\varphi^5}{5} \right) + \|e_0\|^2 - \|p_0\|^2. \]

Using above inequality we have the following:

\[ 2 \left( e_0 - p_0, \int_0^\varphi (\varphi - t) \nu(t) \, dt \right) \leq \varphi \left( \Gamma^2 - \frac{\varphi^5}{5} \right) - \|e_0 - p_0\|^2. \]
Indeed,

\[
2 \left( e_0 - p_0, \int_0^\varphi (\varphi - t)\nu(t) \, dt \right) = 2 (e_0 - p_0, e(\varphi) - e_0) = 2 (e_0 - p_0, e(\varphi)) - 2 (e_0 - p_0, e_0) = 2 (e_0 - p_0, e(\varphi)) - 2\|e_0\|^2 + 2 (p_0, e_0) \\
\leq \varphi \left( \Gamma^2 - \Upsilon^2 \sqrt{\frac{\varphi^5}{5}} \right) + \|e_0\|^2 - \|p_0\|^2 - 2\|e_0\|^2 + 2 (p_0, e_0) \\
\leq \varphi \left( \Gamma^2 - \Upsilon^2 \sqrt{\frac{\varphi^5}{5}} \right) - \|e_0\|^2 - \|p_0\|^2 + 2 (p_0, e_0) \\
\leq \varphi \left( \Gamma^2 - \Upsilon^2 \sqrt{\frac{\varphi^5}{5}} \right) - \|e_0\|^2 - \|p_0\|^2 + 2 (p_0, e_0) \\
\leq \varphi \left( \Gamma^2 - \Upsilon^2 \sqrt{\frac{\varphi^5}{5}} \right) - \|e_0 - p_0\|^2.
\]

Thus, taking contribution of above inequality

\[
\int_0^\varphi \|\Xi(t)\|^2 \, dt = \int_0^\varphi \frac{\|e_0 - p_0\|^2}{\varphi} + (\varphi - t)\nu(t)\|\|^2 \, dt \\
= \int_0^\varphi \left( \frac{\|e_0 - p_0\|^2}{\varphi} + 2 \left( \frac{e_0 - p_0}{\varphi}, (\varphi - t)\nu(t) \right) + \|\varphi - t\|\nu(t)\|\|^2 \right) \, dt \\
= \int_0^\varphi \frac{\|e_0 - p_0\|^2}{\varphi^2} \, dt + 2 \int_0^\varphi \left( \frac{e_0 - p_0}{\varphi}, (\varphi - t)\nu(t) \right) \, dt + \int_0^\varphi (\varphi - t)\|\nu(t)\|^2 \, dt \\
\leq \frac{\|e_0 - p_0\|^2}{\varphi} + 2 \varphi \left( e_0 - p_0, \int_0^\varphi (\varphi - t)\nu(t) \, dt \right) \\
+ \left( \int_0^\varphi (\varphi - t)^4 \, dt \right)^{\frac{1}{2}} \left( \int_0^\varphi \|\nu(t)\|^4 \, dt \right)^{\frac{1}{2}} \\
\leq \frac{\|e_0 - p_0\|^2}{\varphi} + \frac{1}{\varphi} \left( \varphi \left( \Gamma^2 - \Upsilon^2 \sqrt{\frac{\varphi^5}{5}} \right) - \|e_0 - p_0\|^2 \right) + \Upsilon^2 \sqrt{\frac{\varphi^5}{5}} \\
\leq \Gamma^2,
\]

and therefore the strategy \( \Xi \) is admissible.
Now we show that $\Xi$ is a winning strategy for the pursuer. Indeed,

$$p(\varphi) = p_0 + \int_0^\varphi \left( \frac{e_0 - p_0}{\varphi} + (\varphi - t)\nu(t) \right) ds$$

$$= p_0 + \int_0^\varphi \left( \frac{e_0 - p_0}{\varphi} \right) ds + \int_0^\varphi (\varphi - t)\nu(t) ds$$

$$= p_0 + e_0 - p_0 + \int_0^\varphi (\varphi - t)\nu(t) ds = e(\varphi).$$

□

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