q-Analogue Modified Laguerre Matrix Polynomials of Three Variables
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Abstract

In this paper, the q-analogue modified Laguerre matrix polynomials of three variables are introduced as finite series and Some properties of these matrix polynomials are obtained.

Keywords: q-analogue modified Laguerre matrix polynomials; generating functions; recurrence relations.

1. Introduction

Matrix generalization of special functions has become important in the last two decades. The reason of importance have many motivations. For instance, using special matrix functions provides solutions for some physical problems. Also, special matrix functions are in connection with different matrix functions.

J’odar et al introduced Laguerre matrix polynomials in [11]. Some important and different properties of Laguerre matrix polynomials were investigated (see [1,2,6,7,9,11,18,20] ). Throughout this paper, for a matrix $A$ in $C^{N \times N}$, its spectrum $\sigma(A)$ denotes the set of all eigenvalues of $A$.

The matrix analogues of Pochhammer symbol or shifted factorial is defined by [14]

$$(A)_n = A(A + 1)(A + 2) \ldots (A + (n - 1))I, \quad n \geq 1, (A)_0 = I,$$  

(1.1)

where, $A \in C^{N \times N}$. The hypergeometric matrix function $F(A; B; C; z)$ is defined by [14]

$$F(A; B; C; z) = \sum_{n\geq0} \frac{(A)_n(B)_n(C)_n^{-1}}{n!} z^n,$$  

(1.2)

for matrices $A, B, C$ in $C^{N \times N}$ such that $C + nl$ is invertible for all integers $n \geq 0$ and for $z < 1$ (see [10]).

Furthermore, for a matrix $A$ in $C^{N \times N}$, the authors exploited the following relation due to [10]:

$$(1 - y)^{-A} = \sum_{n=0}^{\infty} \frac{(A)_n}{n!} y^n, \quad |y| < 1.$$  

(1.3)

Also, for a matrix $A(k, n)$ in $C^{N \times N}$ for $n \geq 0$ and $k \geq 0$, the following relation is given by Defez and J’odar in [4]

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n - k).$$  

(1.4)

We conclude this section by recalling the Laguerre matrix polynomials. Let $A$ be a matrix in $C^{N \times N}$ such that $-k \notin \sigma(A)$ for every integer $k > 0$ and $\lambda$ be a complex number whose real part is positive. Then the Laguerre matrix polynomials $L^{(A, \lambda)}_n(x)$ are defined by [11]:

$$L^{(A, \lambda)}_n(x) = \sum_{k=0}^{n} \frac{(-1)^k(A + l)_k(A + l)_k^{-1}x^{k}}{k!(n-k)!},$$  

(1.6)

The generating function of Laguerre matrix polynomials is given in [11] by

$$(1 - t)^{-(A+1)} \exp \left( \frac{-\lambda t}{1-t} \right) = \sum_{n=0}^{\infty} L^{(A, \lambda)}_n(x) t^n, \quad t \in \mathbb{C}, \quad |t| < 1, \quad x \in \mathbb{C},$$

and Rodrigues formula is

$$L^{(A, \lambda)}_n(x) = \frac{x^{-(A+\lambda)} D^n \left[ x^{A+n\lambda} \exp(-\lambda x) \right]}{n!}, \quad n \geq 0.$$  

(1.7)

Also, Laguerre matrix polynomials satisfy the three-term recurrence relation

$$(n + 1)L^{(A, \lambda)}_{n+1}(x) + [\lambda x I - (A + (2n + 1)I)]L^{(A, \lambda)}_n(x) + (A + nl)L^{(A, \lambda)}_{n-1}(x) = 0,$$  

(1.8)

and second order matrix differential equation

\[ \frac{d}{dx}L^{(A, \lambda)}_n(x) + \frac{\lambda x - A}{n + 1}L^{(A, \lambda)}_n(x) = 0. \]
\[
xD^2 + ((A + I) - \lambda xI)D + \lambda nI L_n^{(A,\lambda)}(x) = 0. \tag{1.9}
\]

In [3], it is shown that an appropriate combination of methods, relevant to operational calculus and to matrix polynomials, can be a very useful tool to establish and treat a new class of two variable Laguerre matrix polynomials in the following form:

\[
L_{n,m}^{(A,\lambda)}(x, y) = \sum_{l=0}^{\ell} \sum_{m=0}^{m} \frac{(-1)^{l+k}(A + I)_{n,m}[(A + I)_{n,m}]^{-1}(\lambda x)^{l} \lambda y^{k}}{s(n-l)(m-k)!}, \ (n, m) \geq 0. \tag{1.10}
\]

The generating relation for the matrix function \(L_{n,m}^{(A,\lambda)}(x, y)\) is given by the formula:

\[
(1 - u - v) - (A + I) \exp \left(\frac{-\lambda xu + yv}{1 - u - v}\right) = \sum_{n,m=0}^{\infty} L_{n,m}^{(A,\lambda)}(x, y)u^n v^m,
\]
where \(\{x, y, u, v\} \in C\) and \(|u + v| < 1\).

Recently, q-calculus has served as a bridge between mathematics and physics. Therefore, there is a significant increase of activity in the area of the q-calculus due to its applications in mathematics, statistics and physics.

Let the q-analogues of Pochhammer symbol or q-shifted factorial be defined by [8]

\[
(a; q)_n = \begin{cases} 1, & n = 0 \\ \prod_{k=1}^{n-1}(1 - aq^k), & n \in N, \end{cases} \tag{1.11}
\]
also,

\[
(a; q)_{n+k} = (a; q)_n (aq^n; q)_k.
\]

Now, the q-shifted factorials, where \(k\) and \(n\) are nonnegative integers [21):

\[
(a; q)_{n-k} = \frac{(aq^n)/\bar{(a; q)_{n-k}}}{(q-a^{-1}; q)_{k}}, \ a \neq 0, k = 0, 1, 2, \ldots, n. \tag{1.12}
\]

The q-binomial coefficient (or Gaussian polynomial analogue to \(\binom{n}{k}\)) is defined by [8] and [21]:

\[
\begin{align*}
\left[\begin{array}{c} n \\ k \end{array}\right]_q &= \frac{(aq)_n}{(q; q)_n (aq^n; q)_k} = \left[\begin{array}{c} n \\ k \end{array}\right], \\
\left[\begin{array}{c} a \\ k \end{array}\right]_q &= \frac{(q^{-a}; q)_k}{(q; q)_k}, \ a \in C, k \in N_0.
\end{align*}
\tag{1.13}
\]

The q-analogue of the power (binomial) function \((x \pm y)^n\) ([17]) is given by

\[
(x \pm y)^n \equiv (x \pm y)^n \equiv x^n \left(\pm \frac{y}{x}\right)^n = x^n \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q q^{(k)} \left(\pm \frac{y}{x}\right)^k. \tag{1.15}
\]

The formulas for the q-difference \(D_q\) of a addition, a product and a quotient of functions are

\[
D_q(\lambda f(x) + \mu g(x)) = \lambda D_q f(x) + \mu D_q g(x), \tag{1.16}
\]

\[
D_q f(x, g(x)) = f(qx)D_q g(x) + g(x)D_q f(x), \tag{1.17}
\]

\[
D_q \left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(q(x))}, \quad g(x)g(q(x)) \neq 0. \tag{1.18}
\]

The q-exponential function is defined by [21]:

\[
e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n}. \tag{1.19}
\]

Moak (1981) introduced and studied the q-Laguerre polynomials [15]

\[
L_n^{(\alpha)}(x; \alpha) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \sum_{k=0}^{n} \frac{(q^{-\alpha}; q)_k (q^{\alpha+1}; q)_{k+k+1/2} (q^{n+\alpha+1}; q)_k}{[k]_q (q^{\alpha+1}; q)_k}, \quad \alpha > -1, n \in N_0. \tag{1.20}
\]

Fixed \(0 < q < 1\) and \(\alpha > 1\), the explicit form of the nth degree monic q-Laguerre polynomial reads ([15, 23])

\[
L_n^{(\alpha)}(x; \alpha) = (-1)^n (q^{\alpha+1}; q)_n \sum_{n=0}^{\infty} \frac{(q^{-\alpha}; q)_k (1-q)^k (q^{\alpha+1}; q)_{k+k+1/2}}{(1-q) k (q^{\alpha+1}; q)_k (q^{\alpha+1}; q)_{k+k+1/2}}, \quad n \in N_0. \tag{1.21}
\]

Mohsen and Alsarahi [16] introduced q-analogue modified Laguerre polynomial of two variables by the following:
\[ L^{(\alpha, \beta)}_{n,m}(x, y; q) = \frac{(q^{m}; q)_{n}\beta y)^{n}}{q^{mn}(q;q)_{n}} \Phi_{1} \left( q^{-n}; q^{m}; q, -q^{m+1}\frac{\alpha x}{\beta y} \right). \]  \hspace{1cm} (1.22)

where \( \Phi_{1} \) is the basic hypergeometric or \( q \)-hypergeometric function.

The generating relation for \( L^{(\alpha, \beta)}_{n,m}(x, y) \) is given by the formula:

\[ [1 - \beta ty]^m \exp_q \left[ \frac{-\alpha x}{1 - \beta ty} \right] = \sum_{n=0}^{\infty} L^{(\alpha, \beta)}_{n,m}(x, y; q) x^n. \]  \hspace{1cm} (1.23)

2. \textbf{q-Analogue Modified Laguerre Matrix Polynomials of Three Variables}

In this section, we introduce the \( q \)-analogue modified Laguerre matrix polynomial of three variables by the following generating function:

\[ [1 - \beta z(u + v)]^{(A+I)} \exp_q \left[ \frac{-\alpha (xu + yv)}{1 - \beta z(u+v)} \right] = \sum_{n,m=0}^{\infty} L^{(\alpha, \beta)}_{n,m}(x, y, z; q) u^n v^m, \]  \hspace{1cm} (2.1)

where \( u, v, x, y, z \in \mathbb{C}, \ |z(u + v)| < 1. \)

Now, we get the series representation of the \( q \)-analogue modified Laguerre matrix polynomials in the form of the following theorem:

**Theorem 2.1**

Let us assume that \( A \) is a matrix in \( C^{N \times N} \) and \( \alpha, \beta \) be a complex number whose real part is positive, then the series representation of the \( q \)-analogue modified Laguerre matrix polynomials \( L^{(\alpha, \beta)}_{n,m}(x, y, z; q) \) is given by:

\[ L^{(\alpha, \beta)}_{n,m}(x, y, z; q) = \sum_{s=0}^{n} \sum_{k=0}^{m} q^{k^2}(xu)^{s-k}(yv)^{k} q^{A+(s+1)I}(xu)^{s-k}(yv)^{k} (q; q)_{n}^{s-k} (q; q)_{n}^{k} \sum_{k=0}^{s} \frac{(q^{m}; q)_{n^{(m-n)}}}{(q^{m}; q)_{n}} \left( -q^{-A+(s+1)I} \right)^{n} \times (q; q)_{n}^{m} (q; q)_{n}^{m} \right)^{I} (\beta zq^{-(A+(s+1)I)})^{n} q^{(s-k)k} q^{(s-k)k} (q; q)_{n}^{k} q^{(s-k)k}. \] \hspace{1cm} (2.2)

**Proof.** Let us denote the left hand sides of (2.1) by \( W \), then, by using the \( q \)-exponential series (1.19), we get

\[ W = \sum_{n=0}^{\infty} \frac{(-1)^{n}u^{n}(xu+yv)^{n}}{(q;q)_{n}} \left[ 1 - \beta z(u + v) \right]^{(A+I)} \]  \hspace{1cm} (2.3)

by using the relation (1.15) in (2.3), we get

\[ W = \sum_{s=0}^{\infty} \frac{(-1)^{s}}{(q; q)_{s}} \sum_{k=0}^{s} \frac{q^{k^2}(xu)^{s-k} (yv)^{k} q^{A+(s+1)I} (xu)^{s-k} (yv)^{k}}{(q; q)_{s}^{k}} \sum_{k=0}^{s} \frac{(q^{m}; q)_{n^{(m-n)}}}{(q^{m}; q)_{n}} \left( -q^{-A+(s+1)I} \right)^{n} \times (q; q)_{n}^{m} (q; q)_{n}^{m}. \] \hspace{1cm} (2.4)

applying relations (1.13), (1.14) and (1.15) on (2.4), we obtain

\[ W = \sum_{s=0}^{\infty} \sum_{k=0}^{s} \frac{(-1)^{s} \alpha^{s} q^{k^2}(xu)^{s-k} (yv)^{k}}{(q; q)_{s}^{k}} \sum_{n=0}^{\infty} \frac{(q^{A+(s+1)I}) q^{n} (q^{-A+(s+1)I})^{n}}{(q; q)_{n}^{m}} \times (q; q)_{n}^{m} (q; q)_{n}^{m}. \] \hspace{1cm} (2.5)

which on using relation (1.4), gives

\[ W = \sum_{n,m=0}^{\infty} \sum_{k=0}^{m} \frac{(q^{A+(s+1)I})^{n} q^{m} x^{s} y^{k} (u^{n+m} v^{m+k})}{(q; q)_{n}^{m} (q; q)_{n}^{m}} \times (\beta zq^{-(A+(s+1)I)})^{n+m} x^{s} y^{k} (u^{n+m} v^{m+k}). \]  \hspace{1cm} (2.6)

using the relation (1.5), we find
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\[ W = \sum_{n,m=0}^{\infty} \sum_{s=0}^{m} \sum_{k=0}^{s} \frac{(-1)^{s+k} q^{s+k} q^{k} \sum_{q=0}^{m-k} \left( q^{A+(s+k+1)l}; q \right)_{n+m-(s+k)} \left( q; q \right)_{q} \left( q; q \right)_{s} \left( q; q \right)_{m-k}}{\left( q; q \right)_{n-s} \left( q; q \right)_{m-k}} \times (\beta q^{-A+(s+k+1)l})^{n+m-(s+k)}, \]  

(2.7)

Applying relation (1.12) on (2.7), we obtain

\[ W = \sum_{n,m=0}^{\infty} \sum_{s=0}^{m} \sum_{k=0}^{s} \frac{(-1)^{s+k} q^{s+k} q^{k} \sum_{q=0}^{m} \left( q^{A+(s+k+1)l}; q \right)_{n+m} \left( q; q \right)_{q} \left( q; q \right)_{s} \left( q; q \right)_{m-k}}{\left( q; q \right)_{n-s} \left( q; q \right)_{m-k}} \times \left[ \left( q^{-A+(n+s+k+3)l}; q \right)_{s+k} \right]^{-1} (\beta q^{-A+(s+k+3)l})^{n+s+k} (\frac{qax}{\beta z})^{s} (\frac{qay}{\beta z})^{k}, \]  

(2.8)

by equating the coefficients of \( u^n v^m \), we obtain the relation (2.2).

Next, we derive some recurrence relations for the polynomials \( l_{n,m}^{(A,a,b)}(x,y,z;q) \) in the form of the following theorems:

**Theorem 2.2**

The q-analogue modified Laguerre matrix polynomials of three variables \( l_{n,m}^{(A,a,b)}(x,y,z;q) \) satisfy the following relations:

\[ \frac{\partial^r}{\partial x^r} l_{n,m}^{(A,a,b)}(x,y,z;q) = (-a)^r \sum_{s=0}^{n-r} \sum_{k=0}^{m} q^{k} \left( q^{A+(s+k+1)l}; q \right)_{n+r} \left( q; q \right)_{q} \left( q; q \right)_{s} \left( q; q \right)_{m-k} \times \left[ \left( q^{-A+(n+r+s+k+3)l}; q \right)_{s+k} \right]^{-1} (\beta q^{-A+(s+k+3)l})^{n+s+k} (\frac{qax}{\beta z})^{s} (\frac{qay}{\beta z})^{k}, \]  

(2.9)

**Proof.** Differentiating both sides of (2.1) with respect to \( x \), we get

\[ \sum_{n,m=0}^{\infty} \frac{\partial}{\partial x} l_{n,m}^{(A,a,b)}(x,y,z;q) u^n v^m \]

\[ = -\alpha u \sum_{s=0}^{\infty} \sum_{k=0}^{s} \frac{(-1)^{s} \alpha^s \left[ x u + y v \right]^s}{\left( q; q \right)_{s} \left( q; q \right)_{k}} \left[ 1 - \beta z (u + v) \right]_{q}^{-(A+(s+2)l)}, \]  

(2.10)

Applying relation (1.15) in (2.10), we get

\[ \sum_{n,m=0}^{\infty} \frac{\partial}{\partial x} l_{n,m}^{(A,a,b)}(x,y,z;q) u^n v^m = -\alpha u \sum_{s=0}^{\infty} \sum_{k=0}^{s} \left[ x u + y v \right]^s \frac{(-1)^{s} \alpha^s}{\left( q; q \right)_{s} \left( q; q \right)_{k}} q^{(k)}(x u)^s (y v)^k \]

\[ \times \sum_{n=0}^{\infty} \left[ -(A+(s+2)l) \right]_{q}^{n} \left[ -\beta z (u + v) \right]_{q}^{n}, \]  

(2.11)

by using the relations (1.13), (1.14) and (1.15) in (2.11), we obtain.
\[\sum_{n,m=0}^{\infty} \frac{\partial}{\partial x} L_n^{(a, \beta)}(x, y, z; q) u^n v^m \]

\[= -\alpha u \sum_{n,m=0}^{\infty} \sum_{s=0}^{\infty} \sum_{k=0}^{s} \frac{(-1)^s \alpha^2 q^{(k)}(xu)^{s-k}(yv)^k}{(q; q)_s(q; q)_k} \frac{(qA^{+(s+2)}l; q)_n}{(q; q)_n} \]

\[\times \left(-q^{-(A+(s+2))l}\right)^n \sum_{m=0}^{\infty} \frac{(qz_m^nq_{1/2}^m)}{(q; q)_{n-m}(q; q)_m} (-\beta z)^n(u)^{n-m}(v)^m, \quad (2.12)\]

using the relation (1.4), we find

\[\sum_{n,m=0}^{\infty} \frac{\partial}{\partial x} L_n^{(a, \beta)}(x, y, z; q) u^n v^m \]

\[= -\alpha \sum_{n,m=0}^{\infty} \sum_{s=0}^{\infty} \sum_{k=0}^{s} \frac{(-1)^s \alpha^2 q^{(k)}(xu)^{s-k}(yv)^k}{(q; q)_s(q; q)_k} \frac{(qA^{+(s+2)}l; q)_{n+m}}{(q; q)_{n+m}} \]

\[\times \left(-q^{-(A+(s+k+2))l}\right)^{n+m} x^s y^k (u)^{n+s+1}(v)^{m+k}, \quad (2.13)\]

which on using relation (1.5), gives

\[\sum_{n,m=0}^{\infty} \frac{\partial}{\partial x} L_n^{(a, \beta)}(x, y, z; q) u^n v^m \]

\[= -\alpha \sum_{n,m=0}^{\infty} \sum_{s=0}^{n-1} \sum_{k=0}^{s} \frac{(-1)^s \alpha^2 q^{(k)}(xu)^{s-k}(yv)^k}{(q; q)_s(q; q)_k} \frac{(qA^{+(s+k+2)}l; q)_{(n-1)+m-(s+k)}}{(q; q)_{(n-1)+m-(s+k)+1}} \]

\[\times \left(-q^{-(A+(s+k+2))l}\right)^{(n-1)+m-(s+k)} x^s y^k u^n v^m, \quad (2.14)\]

applying relation (1.12) in (2.14), we obtain

\[\sum_{n,m=0}^{\infty} \frac{\partial}{\partial x} L_n^{(a, \beta)}(x, y, z; q) u^n v^m \]

\[= -\alpha \sum_{n,m=0}^{\infty} \sum_{s=0}^{n-1} \sum_{k=0}^{s} \frac{(-1)^s \alpha^2 q^{(k)}(xu)^{s-k}(yv)^k}{(q; q)_s(q; q)_k} \frac{(qA^{+(s+k+2)}l; q)_{(n-1)+m}}{(q; q)_{(n-1)+m-(s+k)+1}} \]

\[\times \left(-q^{-(A+(n+m+s+k+2))l}\right)^{(n-1)+m-(s+k)} (ax)^s(ay)^k u^n v^m, \quad (2.15)\]

by equating the coefficients of \(u^n v^m\), we obtain

\[\frac{\partial}{\partial x} L_n^{(a, \beta)}(x, y, z; q) \]

\[= -\alpha \sum_{s=0}^{n-1} \sum_{k=0}^{m} \frac{q^{(k)}(xu)^{s-k}(yv)^k}{(q; q)_s(q; q)_k} \frac{(qA^{+(s+k+1)}l; q)_{(n-1)+m}}{(q; q)_{(n-1)+m-(s+k)+1}} \]

\[\times \left(-q^{-(A+(n+m+s+k+2))l}\right)^{(n-1)+m} (ax)^s(ay)^k u^n v^m. \quad (2.16)\]

Thus, by same manner as above, we can obtain

\[\frac{\partial^2}{\partial x^2} L_n^{(a, \beta)}(x, y, q) \]

\[= \alpha^2 \sum_{s=0}^{n-2} \sum_{k=0}^{m} \frac{q^{(k)}(xu)^{s-k}(yv)^k}{(q; q)_s(q; q)_k} \frac{(qA^{+(s+k+2)}l; q)_{(n-2)+m}}{(q; q)_{(n-2)+m-(s+k)+1}} \]

\[\times \left(-q^{-(A+(n+m+s+k+2))l}\right)^{(n-2)+m} (ax)^s(ay)^k u^n v^m. \quad (2.17)\]

Hence, by continuing the above steps, we get the required relation (2.8).
Similarly, differentiating (2.1), with respect to \( y \), we get relation (2.9).

**Theorem 2.3**

The q-analogue modified Laguerre matrix polynomials of three variables \( L_{n,m}^{(A,\alpha,\beta)}(x,y,z; q) \) satisfy the following relations:

\[
[n + 1]_q L_{n+1,m}^{(A,\alpha,\beta)}(x,y,z; q) = -\alpha x \sum_{s=0}^{n} \sum_{k=0}^{m} \sum_{t=0}^{m-s} \sum_{p=0}^{k} \frac{(-1)^{s+k} q^{k}(z)^{s} + (m-k-p)_{t} + t (q^{A+2l}; q)t+p}{(q; q)_{s}(q; q)_{k}(q; q)_{t}(q; q)_{p}} \]

\[
\times \left( q^{A+s(k+1)}; q \right)_{n+m} \left[ q^{-A+(n+m+s+k)}; q \left( s+t+k+1 \right) \right]^{-1} \left( q^{-A+l} \right)_{t+p} \]

\[
\times (\beta z q^{-(A+s(k+1))})^{n+m} \left( q^{A+s(k+1)}; q \right)_{z}^{-n+m} \left( z^{-(s+t+k+1)} \right) \left( q^{A+s(k+2)}; q \right)_{n+m} \left( q^{A+s(k+1)}; q \right)_{z}^{-n+m} \left( q^{A+s(k+1)}; q \right)_{z}^{-n+m} \frac{\alpha x}{\beta z} \frac{q \alpha y}{\beta z} k, \]

\[
\tag{2.16}
\]

and

\[
[m + 1]_q L_{n+1,m}^{(A,\alpha,\beta)}(x,y,z; q) = -\alpha x \sum_{s=0}^{m} \sum_{k=0}^{n} \sum_{t=0}^{n-s} \sum_{p=0}^{s-k} \frac{(-1)^{s+k} q^{k}(z)^{s} + (m-k-p)_{t} + t (q^{A+2l}; q)t+p}{(q; q)_{s}(q; q)_{k}(q; q)_{t}(q; q)_{p}} \]

\[
\times \left( q^{A+s(k+1)}; q \right)_{n+m} \left[ q^{-A+(n+m+s+k)}; q \left( s+t+k+1 \right) \right]^{-1} \left( q^{-A+l} \right)_{t+p} \]

\[
\times (\beta z q^{-(A+s(k+1))})^{n+m} \left( q^{A+s(k+1)}; q \right)_{z}^{-n+m} \left( z^{-(s+t+k+1)} \right) \left( q^{A+s(k+2)}; q \right)_{n+m} \left( q^{A+s(k+1)}; q \right)_{z}^{-n+m} \frac{\alpha x}{\beta z} \frac{q \alpha y}{\beta z} k, \]

\[
\tag{2.16}
\]
\[ +\beta z(A + 1) \sum_{s=0}^{m} \sum_{k=0}^{m} \frac{\left(-1\right)^{s+k} q^{(k)_2} (s+k)_2}{(q; q)_s (q; q)_k} \left[q^{A+(s+k+1)l}; q\right]_{n+m} \frac{(s+k)_2}{I}\frac{z^{(n+m)(s+k)}}{(n+m)(s+k)} \]

\[\times \left[q^{A+(s+k+1)l}; q\right]_{(s+k)}^{-1} \left[\beta z q^{-A+(s+k+1)l}\right]^{n+m} \frac{\left(qz\right)_{s+k}}{\beta z} (qz)^k. \]  

(2.17)

**Proof.** Differentiating the both sides of (2.1), with respect to \(u\) and using relations (1.17),(1.18), we get

\[\sum_{n,m=0}^{\infty} [n]_{q}^{t(A,\alpha,\beta)} (x, y, z; q) u^{n-1} v^{m} = \left[1 - \beta z (q u + v)\right]^{-A+1} \left[q^{-\alpha x + \beta z v(x-y)} \left(1 - \frac{-\alpha x + \beta z v(x-y)}{1-\beta z (u+v)}\right) \exp_q \left(-\alpha x u + \beta z v(x-y)\right) \right]^{A+1} (u+v) q^{-A+2l} \exp_q \left(-\alpha x u + \beta z v(x-y)\right), \]  

(2.18)

applying relation (1.19) in (2.18), we get

\[\sum_{n,m=0}^{\infty} [n+1]_{q}^{t(A,\alpha,\beta)} (x, y, z; q) u^{n} v^{m} = \left[1 - \beta z (q u + v)\right]^{-A+1} \left[q^{-\alpha x + \beta z v(x-y)} \left(1 - \frac{-\alpha x + \beta z v(x-y)}{1-\beta z (u+v)}\right) \exp_q \left(-\alpha x u + \beta z v(x-y)\right) \right]^{A+1} (u+v) q^{-A+2l} \exp_q \left(-\alpha x u + \beta z v(x-y)\right), \]  

(2.19)

by using the relation (1.15) in (2.19), we get

\[\sum_{n,m=0}^{\infty} [n+1]_{q}^{t(A,\alpha,\beta)} (x, y, z; q) u^{n} v^{m} = \left[1 - \beta z (q u + v)\right]^{-A+1} \left[q^{-\alpha x + \beta z v(x-y)} \left(1 - \frac{-\alpha x + \beta z v(x-y)}{1-\beta z (u+v)}\right) \exp_q \left(-\alpha x u + \beta z v(x-y)\right) \right]^{A+1} (u+v) q^{-A+2l} \exp_q \left(-\alpha x u + \beta z v(x-y)\right), \]  

(2.20)

which using relations (1.13) and (1.14) in (2.20), gives

\[\sum_{n,m=0}^{\infty} [n+1]_{q}^{t(A,\alpha,\beta)} (x, y, z; q) u^{n} v^{m} = \left[1 - \beta z (q u + v)\right]^{-A+1} \left[q^{-\alpha x + \beta z v(x-y)} \left(1 - \frac{-\alpha x + \beta z v(x-y)}{1-\beta z (u+v)}\right) \exp_q \left(-\alpha x u + \beta z v(x-y)\right) \right]^{A+1} (u+v) q^{-A+2l} \exp_q \left(-\alpha x u + \beta z v(x-y)\right), \]  

(2.20)
\[
\sum_{m=0}^{n} \frac{(q; q)_n}{(q; q)_{n-m}(q; q)_m} \frac{q^m z^m}{z^2} (-\beta z)^n (u)^{n-m} (v)^m \\
+ \beta z(A + 1) \sum_{s=0}^{\infty} \sum_{k=0}^{s} \frac{(-1)^s \alpha^s q^k}{(q; q)_s(q; q)_k} \sum_{m=0}^{\infty} \frac{(q^{A+s+2l}; q)_n}{(q; q)_n} \\
\times \left( -q^{-A+s+2l} \right)^n \sum_{m=0}^{n} \frac{(q; q)_n}{(q; q)_{n-m}} \frac{q^m z^m}{z^2} (-\beta z)^n (u)^{n-m} (v)^m,
\]

which, using relation (1.5) in (2.21), gives

\[
\sum_{n,m=0}^{\infty} [n + 1] q^{(A, \alpha, \beta)} (x, y, z; q) u^n v^m = (-\alpha x + \alpha \beta z v(x - y)) \\
\times \frac{q^{(p)} (q^{A+2l}; q)_t + p}{(q; q)_t(q; q)_p} \frac{(-1)^s \alpha^s q^k}{(q; q)_s(q; q)_k} \frac{(q^{A+s+k+1}); q)_{n+m}}{(q; q)_n(q; q)_m} \\
\times \left( \beta zq^{-A+s+k+1} \right)^{n+m} (\alpha x)^s (\alpha y)^k (u)^{n+s} (v)^{m+k}.
\]

By using relation (1.4) in (2.22), we get

\[
\sum_{n,m=0}^{\infty} [n + 1] q^{(A, \alpha, \beta)} (x, y, z; q) u^n v^m \\
= -\alpha x \sum_{n,m=0}^{\infty} \sum_{s=0}^{n} \sum_{k=0}^{m} \sum_{t=0}^{n-s-m-k} \frac{(-1)^s \alpha^s q^k}{(q^{A+s+k+1}); q)_{n+m+(s+t+k+1)}} \\
\times \left( \beta zq^{-(A+s+k+1)} \right)^{n+m-(s+t+k+1)} (\alpha x)^s (\alpha y)^k (u)^{n+s} (v)^{m+k}.
\]

Applying relation (1.12) in (2.14), we obtain

\[
\sum_{n,m=0}^{\infty} [n + 1] q^{(A, \alpha, \beta)} (x, y, z; q) u^n v^m \\
= -\alpha x \sum_{n,m=0}^{\infty} \sum_{s=0}^{n} \sum_{k=0}^{m} \sum_{t=0}^{n-s-m-k} \sum_{p=0}^{t} \frac{(-1)^s \alpha^s q^k}{(q^{A+s+k+1}); q)_{n+m-(s+t+k+1)}} \\
\times \left( \beta zq^{-(A+s+k+1)} \right)^{n+m-(s+t+k+1)} (\alpha x)^s (\alpha y)^k (u)^{n+s} (v)^{m+k}.
\]
\[\sum_{n,m=0}^{\infty} [n+1]_q q^{(A,a,b)}_{n,m}(x,y,z;q) u^m v^n = -a x \sum_{n,m=0}^{\infty} \sum_{s=0}^{n} \sum_{k=0}^{m} \sum_{m-s-k=0}^{m} \sum_{p=0}^{m-k} \frac{(-1)^s k q^{(k+1)}}{z^2} \frac{(q^2)^{s+k} q^{(m-k-p)} q^p}{(q; q)_s (q; q)_k (q; q)_t (q; q)_p} \times \frac{q^{(A+s+k+1)}; q}_{n+m} \left[ q^{(n+m+s+k+1)}; q \right]_{(s+t+k+p)} \]
\[
\sum_{t=0}^{\infty} \left( q^{A+2t}; q \right)_t \frac{(-q^{-(A+l)}; q)^t}{(q; q)_{m-k-p}} \left[ \left( q^{-(A+(n+m+s+k-2)l)}; q \right)_{(s+t+k+p+1)} \right]^{-1} \\
\times \left( q^{A+(s+k+1)l}; q \right)_{n+m-2} \left( \beta z q^{-(A+(s+k+1)l)} \right)^{(n-1)+m} \frac{q\alpha x^s q\alpha y^k}{\beta z} \\
- \alpha \beta y \sum_{s=0}^{n-1} \sum_{k=0}^{m-2} \sum_{t=0}^{m-k-2} \sum_{p=0}^{m-k-2} \frac{q^{(k+2n-2)}(q; q)_{s+t} (q; q)_p}{(q; q)_{n-s-t} (q; q)_{m-k-p-2}} \\
\times \left( q^{A+2l}; q \right)_t \frac{(-q^{-(A+l)}; q)^t}{(q; q)_{m-k-p-2}} \left[ \left( q^{-(A+(n+m-2)+s+k)l)}; q \right)_{(s+t+k+p+1)} \right]^{-1} \\
\times \left( q^{A+(s+k+1)l}; q \right)_{n+(m-2)} \left( \beta z q^{-(A+(s+k+1)l)} \right)^{(n-1)+m} \frac{q\alpha x^s q\alpha y^k}{\beta z} \\
+ \beta (A+1) \sum_{s=0}^{n-1} \sum_{k=0}^{m-1} \frac{q^{(k+2n-2)}(q; q)_{s+t} (q; q)_p}{(q; q)_{n-s-t} (q; q)_{m-k-1}} \\
\times \left( q^{A+(s+k+1)l}; q \right)_{(s+k)} \left[ \left( q^{-(A+(s+k+2)l)}; q \right)_{(s+k)} \right]^{-1} \beta z q^{-(A+(s+k+2)l)} \\
+ \beta (A+1) \sum_{s=0}^{n-1} \sum_{k=0}^{m-1} \frac{q^{(k+2n-2)}(q; q)_{s+t} (q; q)_p}{(q; q)_{n-s-t} (q; q)_{m-k-1}} \\
\times \left[ \left( q^{A+(s+k+1)l}; q \right)_{(s+k)} \right]^{-1} \beta z q^{-(A+(s+k+2)l)} \\
+ \beta (A+1) (u+v) \sum_{s=0}^{\infty} \frac{(-1)^s \alpha^s [xu+yv]^s}{(q; q)_s} [1 - \beta z (u+v)]_{q}^{(A+(s+1)l)} \\
\times \left[ 1 - \beta z (u+v) \right]_{q}^{(A+2l)} \sum_{s=0}^{\infty} \frac{(-1)^s \alpha^s [xu+yv]^s}{(q; q)_s} [1 - \beta z (u+v)]_{q}^{(A+(s+2)l)} \\
+ \beta (A+1) (u+v) \sum_{s=0}^{\infty} \frac{(-1)^s \alpha^s [xu+yv]^s}{(q; q)_s} [1 - \beta z (u+v)]_{q}^{(A+(s+2)l)}, \tag{2.26}
\]

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الملخص
في هذه البحث قدمنا مصفوفة كثيرات حدود لأجير المعدلة الأساسية ذات ثلاثة متغيرات كمسلسلات
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