Isolation probabilities in dynamic soft random geometric graphs

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We consider soft random geometric graphs, constructed by distributing points (nodes) randomly according to a Poisson Point Process, and forming links between pairs of nodes with a probability that depends on their mutual distance, the “connection function.” Each node has a probability of being isolated depending on the locations of the other nodes; we give analytic expressions for the distribution of isolation probabilities. Keeping the node locations fixed, the links break and reform over time, making a dynamic network; this is a good model of a wireless ad-hoc network with communication channels undergoing rapid fading. We use the isolation probabilities to investigate the distribution of the time to transmit information to all the nodes, finding good agreement with numerics.

Many kinds of complex networks such as transport, power, social and neuronal networks are spatial in character, that is, the nodes and perhaps also links have a physical location. Geometry structures the network in that the probability of a link between two nodes is related to their mutual distance.

Consider a wireless ad-hoc network where nodes (devices) communicate directly with each other rather than a central router and where their locations may be considered random; examples include sensor and vehicular networks and the Internet of Things. In wireless networks the probability of a link decreases with the distance between nodes. As time evolves, the links form a dynamic network. The communication channel exhibits rapid fading, so that some time later, the state of the system may be chosen independently with the same distance-dependent probabilities. Here we assume that the nodes remain in fixed locations, at least on the rapid fading timescale. See supplemental material for an animation in a square domain of length \( L = 8 \) with \( N = 100 \) nodes, showing connected components in different colours and pausing when the whole network is connected. The link probability between nodes of mutual distance \( r \) is Eq. (1) below with \( \eta = 2 \) and \( r_0 = 1 \).

If the link probability is either zero or one everywhere, there is randomness only due to the node locations. This is the case for the original random geometric graph (RGG) model, in which nodes connect if and only if their mutual distance is less than a threshold \( r_0 \). If the link probability somewhere lies strictly between 0 and 1, there are two sources of randomness, in the node locations and the links. Here, we fix the node locations (“quenched disorder”), and study the randomness due to the links, as in the above dynamic wireless network application. This system has also been studied using an approach based on graph entropy.

We distribute nodes and links according to the following spatial inhomogeneous random graph model: Place nodes in space according to a Poisson Point Process (PPP) with intensity measure \( \Lambda \) in \( d \)-dimensional space \( \mathbb{R}^d \); we usually consider \( d \in \{1, 2, 3\} \). This means that the number of nodes in a bounded set \( A \subset \mathbb{R}^d \) is Poisson distributed with mean \( \Lambda(A) \) and independent of the number of points in any set \( B \) disjoint with \( A \). Thus the average number of nodes in the whole system is \( \bar{N} = \Lambda(\mathbb{R}^d) \), possibly infinite. The simplest case is where \( \Lambda \) is proportional to Lebesgue measure, that is, \( \Lambda(A) = \rho \text{Vol}(A) \) where \( \rho \) is the (constant) density and \( \text{Vol}(A) \) is the volume of \( A \). In this case, we often replace \( \mathbb{R}^d \) by a cube \( [0, L]^d \) with opposite faces identified (a flat torus). Then \( \bar{N} = \rho L^d \).

Now we form links between each pair of nodes with locations \( \xi, \eta \), independently with probability \( \phi(\xi, \eta) \). Here we consider soft RGGs, for which \( \phi(\xi, \eta) = H(|\xi - \eta|) \) where \(|.|\) denotes the Euclidean (or in general some other) length and \( H : [0, \infty) \to [0, 1] \) is the connection function. It is possible with information about node locations and links to construct a connection function for any spatial network, and thus model it as a soft RGG. In practice, though, the link independence assumption may not be accurate. In the case of wireless communication networks, there are detailed theories of the physics of the communication channel leading to a variety of connection functions; see Refs. [10–12].

Here, we use one of the simplest models: We assume Rayleigh fading, corresponding to diffuse scattering of the signal, which leads to an exponentially distributed channel gain \( |h|^2 \). The signal power decays as \( r^{-\eta} \) where \( \eta \in [2, 6] \) is called the path loss exponent. Free propagation gives the inverse square law \( \eta = 2 \), whilst more cluttered environments have a faster decay of the signal, leading to larger measured values of \( \eta \). A link may be broken if the signal to noise ratio, proportional to \( |h|^2 r^{-\eta} \), reaches a given threshold, leading to the connection probability

\[
H(r) = \exp (- (r/r_0)^\eta)
\] (1)

for some constant \( r_0 \) that determines the length scale; we measure length in these units and so take \( r_0 = 1 \) hereafter. Observe that the \( \eta \to \infty \) limit gives the original RGG model.
also find further information about the distribution of $P_{iso}$, namely for $\nu \in \mathbb{R}_+ \geq 0$ the $\nu$th moment is

$$\mathbb{E}(P_{iso}^{(\nu)}) = \exp \left\{ - \int [1 - (1 - H(|\xi' - \xi|))] \Lambda(\xi') \right\}$$

(2)

If the PPP has constant density we find

$$\mathbb{E}(P_{iso}^{(\nu)}) = \exp \left\{ -S_d \rho \int_0^\infty [1 - (1 - H(r))] r^{d-1} dr \right\}$$

independent of the location of the node. Here, $S_d$ is the total (solid) angle in $d$ dimensions, namely $S_d = \{2, 2\pi, 4\pi\}$ for $d = \{1, 2, 3\}$ respectively. Now, for Rayleigh fading, we have from Eq. [1],

$$\mathbb{E}(P_{iso}^{(\nu)}) = \exp \left\{ -S_d \rho \Gamma\left(\frac{d}{\eta} \right) H^{(d/\eta)}_\nu \right\}$$

(3)

where

$$H^{(s)}_\nu = \frac{1}{\Gamma(s)} \int_0^\infty (1 - (1 - e^{-x})^s) x^{s-1} dx$$

(4)

For integer $\nu = n$ we can expand the parentheses to yield a finite sum

$$H^{(s)}_n = \sum_{j=1}^n (-1)^{j-1} \binom{n}{j} j^{-s}$$

(5)

which is called the Roman harmonic number [13].

For $\nu$ not necessarily an integer, we can use Eq. [4]: Numerical integration provides an efficient and stable means of calculation, whilst asymptotically expanding the integral for large $\nu$ gives

$$H^{(s)}_\nu = (\ln \nu)^s + \frac{\gamma (\ln \nu)^{s-1}}{\Gamma(s)} + \frac{\eta \gamma (\ln \nu)^{s-1} - 6\gamma^2 + \pi^2}{12\pi (\ln \nu)^{s-2} + \ldots}$$

where $\gamma \approx 0.5772$ is the Euler constant. Thus we have

$$\mathbb{E}(P_{iso}^{(\nu)}) = \exp \left\{ -V_d \rho \left[ (\ln \nu)^s + \frac{d}{\eta} \frac{\gamma (\ln \nu)^{s-1}}{\Gamma(s)} + \frac{d}{\eta} (\ln \nu)^{s-2} \right] \right\}$$

(6)

where $V_d = S_d/d$ is the volume of the unit ball in $d$ dimensions.

When $\eta = d$, that is, $s = 1$, we have a further simplification

$$\mathbb{E}(P_{iso}^{(\nu)}) = \exp \left\{ -V_d \rho (\gamma + \psi(\nu + 1)) \right\}$$

(7)

$$= \exp \left\{ -V_d \rho \left[ \ln \nu + \gamma + \frac{1}{2\nu} - \frac{1}{12\nu^2} + \ldots \right] \right\}$$

where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ is the digamma function, and we have used its standard expansion for large argument. For integer $\nu = n$,

$$\gamma + \psi(n + 1) = H_n = \sum_{j=1}^n \frac{1}{j}$$

In order to understand transmission of information through a dynamic network, we must first analyse the instantaneous isolation probability of a node, that is, the probability that it has no links. This will be determined by the locations of nearby nodes; see Fig. 1. Considering all nodes together, there is a distribution of isolation probabilities.

The isolation probability of a node at $\xi$ in some configuration $X$ of the PPP is

$$P_{iso}(\xi) = \prod_{\xi' \in X \backslash \{\xi\}} (1 - H(|\xi' - \xi|))$$

We note that in a PPP, the distribution of points found by conditioning on a node at a fixed position is unaffected (ie Palm distribution of a PPP is the same PPP); see Ref. [13] for the theory of PPPs. To find the distribution of $P_{iso}$, we use the probability generating functional (PGF)

$$G_X(u) = \mathbb{E} \prod_{\xi \in X} u(\xi) = \exp \left\{ - \int (1 - u(\xi)) \Lambda(d\xi) \right\}$$

for arbitrary function $u(\xi)$ where the first equality is the definition and second follows for a PPP. The function needs to satisfy some mild conditions, for example (a) $\check{N} < \infty$, or (b) $u \in [0, 1]$ and $\int |\log u(\xi)| \Lambda(d\xi) < \infty$ as the case here. In particular,

$$\mathbb{E}(P_{iso}(\xi)) = \exp \left\{ - \int H(|\xi' - \xi|) \Lambda(\xi') \right\}$$

disappear in some configuration $X$ of the PPP for arbitrary function $u(\xi)$ where the first equality is the definition and second follows for a PPP. The function needs to satisfy some mild conditions, for example (a) $\check{N} < \infty$, or (b) $u \in [0, 1]$ and $\int |\log u(\xi)| \Lambda(d\xi) < \infty$ as the case here. In particular,

$$\mathbb{E}(P_{iso}(\xi)) = \exp \left\{ - \int H(|\xi' - \xi|) \Lambda(\xi') \right\}$$

which is the connectivity mass, important for understanding the overall (multihop) connection probability of an ad-hoc network when $d \geq 2$ [11]. However, we can

FIG. 1. A Poisson Point Process with density $\rho = 2$. Nodes are coloured by isolation probability, using the Rayleigh connection function, Eq. [1] with $\eta = 2$ and periodic boundary conditions.
is the usual harmonic number.

We can now attempt to extract the probability density function (pdf) of \( \rho_{\text{iso}} \), which we will denote as \( f(x) \) with \( x \in (0, 1) \), from these moments. For a general distribution on a finite interval this is called the Hausdorff moment problem, and the solution is unique if it exists. For the most general numerical approach we follow Mnatshanov [15], who gives for a general pdf determined from integer moments \( \mu_n = \mathbb{E}(\rho_{\text{iso}}^n) \), an approximation depending on a positive integer parameter \( \alpha \):

\[
f_{\alpha}(x) = \frac{\Gamma(\alpha + 2)}{\Gamma(\alpha + 1)} \sum_{m=0}^{\alpha-a} \frac{(-1)^m \mu_{m+a}}{m!(\alpha - a - m)!}, \quad a = \lfloor \alpha x \rfloor
\]

The function depends on \( x \) only through \( a \), and hence piecewise constant for any fixed \( \alpha \). It converges to the correct function as \( \alpha \to \infty \). It is possible to use this to get a numerical approximation to \( f(x) \), using high precision arithmetic to overcome problems from cancellations; see Fig. 2. We see that \( f(x) \) is singular at \( x = 0 \) or \( x = 1 \) or both; when \( \eta = d = 2 \) it is almost symmetrical at \( \rho = 0.22 \). It is never quite symmetrical: For \( \mathbb{E}(\rho_{\text{iso}}) = 1/2 \) we must have \( \rho = \frac{\ln 2}{\eta} \approx 0.220636 \) and then the third central moment is \( 2^{-11/6} - 3 \times 2^{-5/2} + 2^{-2} \approx 0.000285 \neq 0 \).

In the case of \( \eta = d \), we can write

\[
\mathbb{E}(\rho_{\text{iso}}^\nu) = \int_0^\infty f(x)x^\nu dx = \exp\{ -V_d \rho(\nu + 1) + \gamma \}
\]

giving a representation as an inverse Mellin transform

\[
f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \exp\{ -V_d \rho(\psi(s) + \gamma) \} ds
\]

This is however intractable either analytically or numerically.

Still for \( \eta = d \), the large \( \nu \) asymptotics does however give information on the behaviour of \( f(x) \) near \( x = 1 \), the distribution of highly isolated nodes. Making an ansatz

\[
f(1-\epsilon) = \sum_{i=0}^\infty g_i \epsilon^{\delta + i}
\]

multiplying by \( (1-\epsilon)^\nu \approx \exp(-\nu \epsilon) \) and integrating gives

\[
\mathbb{E}(\rho_{\text{iso}}^\nu) = \sum_{i=0}^\infty g_i \frac{\Gamma(\delta + i + 1)}{\rho^{\delta + i + 1}}
\]

which by comparison with Eq. (7) yields

\[
f(1-\epsilon) = e^{V_d \rho -1} \frac{e^{-V_d \rho \epsilon}}{\Gamma(V_d \rho)} \left[ 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{24} + \frac{3V_d \rho \epsilon}{V_d \rho + \ldots} \right]
\]

For moderate \( \epsilon \) it is more accurate to keep just the first term, as shown in Fig. 2.

Alternatively, we can take the limit \( d \to \infty \). If \( \eta \) increases proportional to \( d \), we see from Eq. (8) that the only effect is to change the effective density \( \rho \). If \( \eta \) is constant, we find \( H_{\rho}^{(d/\eta)} \approx \nu \), so that \( \mathbb{E}(\rho_{\text{iso}}^\nu) \approx x_\nu^\nu \) with

\[
x_* = \exp \left[ -\frac{S_d \rho}{\eta} \frac{\Gamma(d)}{\eta} \right]
\]

and so corresponds to a distribution that is sharply peaked at \( x = x_* \).

Now, we return to the problem of a dynamic network, assuming \( d > 1 \), constant density and neglecting boundary effects. The network chooses links anew each time \( \tau \). In order to ensure information can reach every node in the network, we need to ensure that no node is isolated for the considered time interval \( T \tau \) where \( T \) is the number of time steps. At each time step, the isolation probabilities of the nodes are a PPP on \([0, 1]\) with intensity \( dA = N f(x) dx \). The probability that a node with \( \rho_{\text{iso}} = x \) is isolated for \( T \) consecutive timesteps is simply \( x^T \). Denoting the event that none of the nodes are isolated during these \( T \) timesteps by \( C_T \) we can again use the PGF (noting that the number of nodes is almost surely finite):

\[
\mathbb{P}(C_T | X) = \prod_i (1 - x_i^T)
\]
But the integral is just from which we find that the time to ensure all nodes are connected at least once is

\[ T \approx \exp \left( \frac{\ln(\rho L^d)}{V_d \rho} \right)^{\eta/d} \]

Eq. (10) has been confirmed by numerical simulation; see Fig. 3. Thus, for low density the required time grows as a stretched exponential, controlled by the path loss exponent \( \eta \). When \( \eta = d \), it reduces to simply \( T \approx (\rho L^d)^{1/(\nu d \rho)} \), with the probability distribution determined by the highly isolated nodes as in Eq. (9).

Strictly speaking all our results for isolation probabilities apply to one dimensional networks. However, in this case, transmission of information is limited by large gaps, rather than nodes that are likely to be isolated. In the original RGG, transmission can occur if and only if there are no gaps larger than the link range \( r_0 \); see Ref. [10]. For the soft RGG, it is quite likely that link may be made between nodes that are not directly adjacent to the gaps, and estimating the probability of connectivity, even at a single point in time, remains an open problem.

In conclusion, we have investigated the distribution of isolation probabilities in quenched soft random geometric graphs. This has allowed an analysis of the performance of a dynamic soft RGG model of wireless ad-hoc networks with fixed nodes and rapid channel fading. We obtained explicit formulas for the probability that no node will be isolated for \( T \) time steps, with good numerical agreement. In contrast to networks with mobile nodes, the transmission of information is greatly hindered by extremes of the quenched disorder, namely highly isolated nodes. In the future it would be interesting to consider boundary effects, and other nonuniform node distributions, which are characteristic of many spatial complex networks.

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