Blow-Up Solutions of Liouville’s Equation and Quasi-Normality

Jürgen Grahl\textsuperscript{1} · Daniela Kraus\textsuperscript{1} · Oliver Roth\textsuperscript{1}

Received: 29 March 2020 / Revised: 8 June 2020 / Accepted: 9 July 2020 / Published online: 13 August 2020
© Springer-Verlag GmbH Germany, part of Springer Nature 2020

Abstract
We prove that the family $\mathcal{F}_C(D)$ of all meromorphic functions $f$ on a domain $D \subseteq \mathbb{C}$ with the property that the spherical area of the image domain $f(D)$ is uniformly bounded by $C\pi$ is quasi-normal of order $\leq C$. We also discuss the close relations between this result and the well-known work of Brézis and Merle on blow-up solutions of Liouville’s equation. These results are completely in the spirit of Gromov’s compactness theorem, as pointed out at the end of the paper.

Keywords Quasi-normal families · Bubbling · Semilinear elliptic problem · Exponential non-linearities · Schwarz lemma

Mathematics Subject Classification Primary 30D45, 35J65; Secondary 30C80

1 Introduction
In their celebrated paper [4], Brézis and Merle have pioneered the study of bubbling phenomena for solutions of semilinear elliptic PDEs with exponential non-linearity in two dimensions. At the core of their work is the “compactness-concentration” principle, which roughly says that a loss of compactness of solutions of the twodimensional
non-linear equation

\[-\Delta u = V(z)e^{2u}.
\]

implies the existence of finitely many blow-up points (bubbles):

**Theorem A** (Brézis–Merle [4]) Let \( D \subseteq \mathbb{C} \) be a bounded domain and \( p \geq 1 \). Suppose that \((u_n)\) is a sequence of (weak) solutions of

\[-\Delta u_n = V_n(z)e^{2u_n} \text{ in } D,
\]

with \( 0 \leq V_n \leq C_1 \) and \( \| e^{2u_n} \|_{L^p(D)} \leq C_2 \) for some constants \( C_1 > 0 \) and \( C_2 > 0 \). Then—after taking a subsequence—one of the following alternatives holds:

1. either \((u_n)\) is locally bounded in \( D \) or \( u_n \to -\infty \) locally uniformly in \( D \);
2. there exists a finite non-empty set \( S \subseteq D \) with the following properties:

   - (Bubbling)
     \[ u_n \to -\infty \text{ locally uniformly in } D \setminus S \text{ and for each } p \in S \text{ there is a sequence } (z_n) \in D \text{ such that } z_n \to p \text{ and } u_n(z_n) \to +\infty. \]

   - (Mass Concentration)
     For each \( p \in S \) there is \( \alpha_p \geq 1 \) such that in the measure theoretic sense

     \[
     \frac{1}{\pi} e^{2u_n} \to \sum_{p \in S} \alpha_p \delta_p,
     \]

     that is,

     \[
     \frac{1}{\pi} \int_D e^{2u_n(z)} \psi(z) \, dx \, dy \to \sum_{p \in S} \alpha_p \psi(p) \tag{1.2}
     \]

     for any continuous function \( \psi : D \to \mathbb{R} \) with compact support in \( D \).

We focus on the constant case \( V_n = 4 \) which is related to numerous geometric and physical problems, see Sect. 2.8. In this case, Theorem A is in reality a result about locally univalent meromorphic functions. This follows from a classical result of Liouville [24] which asserts that every solution to \(-\Delta u = 4e^{2u}\) has (locally) the form

\[ u(z) = \log f^z(z) \]

for some locally univalent meromorphic function \( f \) and vice versa. Here, as is standard, \( f^z \) denotes the spherical derivative of \( f \) defined by

\[ f^z(z) = \frac{|f'(z)|}{1 + |f(z)|^2}. \]

In this note we prove:
**Theorem 1.1** Let $D \subseteq \mathbb{C}$ be a domain and $C > 0$. Denote by $\mathcal{F}_C$ the set of all functions $f$ meromorphic on $D$ such that

\[ \frac{1}{\pi} \int_{D} \left( f^\#(z) \right)^2 dx dy \leq C. \quad (1.3) \]

Then for every sequence $(f_n)$ in $\mathcal{F}_C$—after taking a subsequence—there is $f \in \mathcal{F}_C$ such that one of the following alternatives hold:

1. $(f_n)$ converges locally uniformly in $D$ to $f$ (w.r.t. the spherical metric);
2. There exists a finite non-empty $S \subseteq D$ with at most $C$ points with the following properties:
   2a. (Bubbling) $(f_n)$ converges locally uniformly in $D \setminus S$ to $f$ and for each $p \in S$ there is a sequence $(z_n) \in D$ such that $z_n \to p$ and $f_n^\#(z_n) \to +\infty$. If each $f_n$ is locally univalent, then $f$ is constant.
   2b. (Mass Concentration) For each $p \in S$ there is a real number $\alpha_p \geq 1$ such that in the measure theoretic sense
      \[ \frac{1}{\pi} \left( f_n^\# \right)^2 \to \sum_{p \in S} \alpha_p \delta_p + \frac{1}{\pi} (f^\#)^2. \]

Condition (1.3) means that the spherical area of the image domain $f(D)$ on the Riemann sphere $\hat{\mathbb{C}}$ is $\leq C\pi$ (counting multiplicities and with the normalization that the area of $\hat{\mathbb{C}}$ is $=\pi$).

2 Remarks and Questions

2.1 Theorem 1.1 vs. Theorem A

Theorem 1.1 extends the result of Brézis–Merle [4] (for $V_n = 4$, $p = 1$) from the case of locally univalent meromorphic functions to all meromorphic functions in $\mathcal{F}_C$. This follows from Liouville’s theorem mentioned above and Hurwitz’ theorem, which says that the limit function of a locally uniformly convergent sequence of locally univalent meromorphic functions is either (i) locally univalent or (ii) constant. In case (i), $(\log f_n^\#)$ is locally uniformly bounded in $D$; in case (ii), $\log f_n^\# \to -\infty$ locally uniformly in $D$.

2.2 Bubbling and Quasi-Normality

Recall that a family $\mathcal{F}$ of meromorphic functions on a domain $D$ is called quasi-normal if every sequence in $\mathcal{F}$ has a subsequence which converges locally uniformly with respect to the spherical metric on $D \setminus S$ where the set $S$ (which may depend on the
extracted subsequence) has no accumulation point in $D$. If $S$ always has at most $\nu \geq 0$ points, then $\mathcal{F}$ is said to be quasinormal of order $\leq \nu$. In particular, Theorem 1.1 says that $\mathcal{F}_C$ is quasi-normal of order at most $C$. The notion of quasi-normality generalizes the fundamental concept of normal families (= quasi-normal families of order zero), and had been introduced by Montel [26] as early as 1922. We refer to [33, App.] for background on quasi-normal families.

### 2.3 No Bubbling: Small Area and Bounded Weighted Area

(i) If $C < 1$, then Theorem 1.1 says that $\mathcal{F}_C$ is a compact family. For the case that $D$ is the unit disc $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$, this was already observed by Montel (1934, [29]); if $D$ is an arbitrary (hyperbolic) domain this result is stated explicitly in [40, Thm. 2].

(ii) If $V_n = 4$ and $p > 1$ in Theorem A, then there are no bubbles. This follows from the work of Aulaskari and Lappan (1988, [1]), who have shown that for each fixed $C > 0$ and $s > 2$ the family of all meromorphic functions $f$ in $\mathbb{D}$ satisfying

$$\frac{1}{\pi} \int_{\mathbb{D}} (f^s(z))^s \, dx \, dy \leq C$$

is a normal family. Hence Theorem 1.1 handles the borderline case $s = 2$ where bubbles actually occur (take e.g. $f_n(z) = nz$), but at most finitely many.

(iii) Reducing the exponent $s$ in (2.1) below the crucial threshold $s = 2$ does not even guarantee quasi-normality. A simple example is given by the family $\mathcal{F} = \{ f_n(z) := e^{inz} : n = 1, 2, \ldots \}$. It is easy to show that (2.1) holds for $s = 1$ with $C = 1$ for all $f \in \mathcal{F}$, but $f_n^s(z) \to +\infty$ as $n \to \infty$ for every $z \in \mathbb{D}$ with $\text{Im} z = 0$. Hence every point in the interval $S := (-1, 1)$ is a “bubble”. In particular, $\mathcal{F}$ is not quasi-normal on $\mathbb{D}$. Note that $(f_n)$ is normal in $\mathbb{D} \setminus S$ and in fact converges to a constant limit function on each of the components of $\mathbb{D} \setminus S$.

**Problem 2.1** Suppose $1 < s < 2$ and $\mathcal{F}$ is a family of meromorphic functions in $\mathbb{D}$ satisfying (2.1). It seems likely that $\mathcal{F}$ is not necessarily quasi-normal in $\mathbb{D}$. Is it possible to quantify the maximal size of possible exceptional sets $S$ of $\mathcal{F}$ in terms of the exponent $s$?

### 2.4 No Bubbling: Local Univalence

The intersection of Theorem A and Theorem 1.1 deals with the case of all locally univalent functions in $\mathcal{F}_C$. In fact, in this “locally univalent” case, there is the following general “no bubbles” criterion: For a domain $D \subseteq \mathbb{C}$ and a family $\mathcal{L}$ of locally univalent meromorphic functions on $D$, the following are equivalent:

(i) $\mathcal{L}$ is compact;

(ii) For each compact set $K \subseteq D$ there is a positive constant $c > 0$ such that

$$c \leq f^s(z) \quad \text{for all } f \in \mathcal{L} \text{ and all } z \in K.$$
The non-trivial implication is (ii) \(\implies (i)\) and follows from the general fact that each of the families

\[ L_c(D) := \{ f \text{ meromorphic in } D : f^2(z) \geq c \text{ for all } z \in D \}, \quad c > 0, \]

is compact (see [4, Cor. 8] and [18]). Hence compactness of \( L \) just means that the family \( \{ f^2 : f \in L \} \) is locally uniformly bounded away from zero. This fact plays a crucial role in our proof of Theorem 1.1. The set \( L_c(D) \) is also interesting in its own right and has been studied e.g. in [4,17,18,20,34,36]; see also Sect. 2.6 below.

**Remark 1** It is perhaps a bit surprising that \( L_c(D) \) is not empty only if \( c \leq 1/2 \), see [18]. The following beautiful proof was shown to us some years ago by Stephan (see also [16, Proof of Thm. 1.2]): Let \( f \in L_c(D) \). Postcomposing \( f \) with a rigid motion of the sphere does not change the spherical derivative, so we may assume w.l.o.g. that \( f(0) = 0 \). Then \( f(z)/zf'(z) \) is holomorphic in \( \mathbb{D} \) with value 1 at \( z = 0 \). Hence the maximum principle implies

\[
1 \leq \max_{|z|=\rho} \frac{|f(z)|}{z|f'(z)|} = \frac{1}{\rho} \max_{|z|=\rho} \left( \frac{|f(z)|}{1 + |f(z)|^2} \right) \frac{1}{\frac{f^2(z)}{f^2(z)}}
\]

so \( c \leq 1/2 \). It is now obvious that if \( D \) contains a disk of radius \( R > 0 \), then \( L_c(D) \) is possibly not empty only if \( c \leq 1/(2R) \).

**2.5 Quantification and Schwarz Lemmas: Small Area**

Let \( \mathcal{F} \) be a normal family of meromorphic functions on a domain \( D \subseteq \mathbb{C} \). Then, by Marty’s criterion, the quantity

\[ M_{\mathcal{F}}(z) := \sup_{f \in \mathcal{F}} f^2(z) \]

is finite for each \( z \in D \). Geometrically, \( M_{\mathcal{F}}(z) \) provides the maximal spherical-euclidean distortion of a function \( f \in \mathcal{F} \) at the point \( z \). Finding \( M_{\mathcal{F}} \) for a given family \( \mathcal{F} \) is a ubiquitous task in Geometric Function Theory, and quite often a challenging endeavour.

As in Theorem 1.1, we focus on the families \( \mathcal{F}_C \), for which we now write \( \mathcal{F}_C(D) \) in order to emphasize the dependence on the domain \( D \). If \( C < 1 \), then

\[ M_{\mathcal{F}_C(D)}(z) \leq \sqrt{\frac{C}{1-C}} \frac{1}{1-|z|^2}, \quad z \in \mathbb{D}. \]

This is essentially a classical result of Dufresnoy (1941, [15]), see also [33, p. 83]. The result of Dufresnoy has been extended from the disk \( \mathbb{D} \) to arbitrary hyperbolic
domains $D$ by Yamashita (2000, [40]), who showed that for any hyperbolic domain $D \subseteq \mathbb{C}$,

$$M_{\mathcal{F}_C(D)}(z) \leq \sqrt{\frac{C}{1 - C}} \lambda_D(z), \quad z \in D,$$

with equality for one—and then for any—point $z \in D$ if and only if $D$ is simply connected. Here, $\lambda_D(z)$ is the density of the Poincaré metric on $D$ (with curvature $-4$).

In the spirit of Schwarz’ lemma, these results say: for any $C < 1$ each map $f \in \mathcal{F}_C(D)$ is a Lipschitz-map from $D$ into the Riemann sphere $\hat{\mathbb{C}}$, provided both are equipped with their “natural” geometries (hyperbolic resp. spherical).

### 2.6 Quantification and Schwarz Lemmas: The Problem of Brézis–Merle

Let $\mathcal{L}_c(D)$ be the family of all meromorphic functions on $D$ with spherical derivative bounded from below by $c > 0$. Recall from Sect. 2.4 that $\mathcal{L}_c(D)$ is compact. Suppose that $K$ is a compact subset of $D$. Brézis and Merle [4, p. 1239, Open problem 3] have posed, amongst others, the problem to find

$$\max_{f \in \mathcal{L}_c(D), z \in K} f^\sharp(z).$$

It seems that little is known about this problem. Shafrir [34] has proved that

$$\sup_{z \in K} f^\sharp(z) \leq \frac{e^{C_2}}{c}, \quad f \in \mathcal{L}_c(D),$$

where $C_2 > 0$ is a constant depending only on $K$. In the case $D = \mathbb{D}$ Steinmetz [36] has quantified Shafrir’s estimate by proving that

$$\sup_{f \in \mathcal{L}_c(\mathbb{D})} f^\sharp(z) \leq \frac{1}{c \left(1 - |z|^2\right)^2}, \quad z \in \mathbb{D}. \quad (2.2)$$

In [17] this has been improved to

$$\sup_{f \in \mathcal{L}_c(\mathbb{D})} f^\sharp(z) \leq \frac{1 + \sqrt{1 - 4c^2 (1 - |z|^2)^2}}{2c (1 - |z|^2)^2}, \quad z \in \mathbb{D}, \quad (2.3)$$

showing that the dependence of the upper bound on the parameter $c$ found by Shafrir resp. Steinmetz is not best possible. Asymptotically however, both estimates (2.2) and (2.3) yield

$$\limsup_{|z| \to 1} \left(1 - |z|^2\right)^2 f^\sharp(z) \leq \frac{1}{c}, \quad f \in \mathcal{L}_c(\mathbb{D}). \quad (2.4)$$
Recent work of Gröhn [20, Thm. 3], which is based in parts on Carleson’s celebrated solution of the $H^\infty$-interpolation problem, shows that there is a function $f \in \bigcup_{c>0} \mathcal{L}_c(\mathbb{D})$ such that

$$\inf_{n \in \mathbb{N}} \left(1 - |z_n|^2\right)^2 f^\sharp(z_n) > 0$$

for some sequence $(z_n)$ in $\mathbb{D}$ with $|z_n| \to 1$. Hence, for sufficiently small values of $c > 0$ inequality (2.4) is sharp up to a multiplicative constant. It is shown in [17] that for all possible values of $c$ one can replace the number 1 on the right-hand side of (2.4) by $(3 - \sqrt{5})/2 \approx 0.38$.

We finally note that a complete solution of the Brézis–Merle problem in the very special case $\mathcal{D} = \mathbb{D}$ and $K = \{0\}$ has been given in [17], where it is shown that

$$\max_{f \in \mathcal{L}_c(\mathbb{D})} f^\sharp(0) = \frac{1 + \sqrt{1 - 4c^2}}{2c}.$$  

The problem to determine $\max_{f \in \mathcal{L}_c(\mathbb{D})} f^\sharp(z)$ for $z \neq 0$ remains open.

### 2.7 Mass Quantisation

It has been conjectured by Brézis and Merle [4, Open Problem 4] that under some mild regularity assumptions on the functions $V_n$ in Theorem A all the numbers $\alpha_p$ are in fact positive integers (Mass Quantisation). This conjecture has been confirmed in [22] under the condition that $V_n \in C(\overline{\mathcal{D}})$ and $V_n \to V$ in $C(\overline{\mathcal{D}})$. It therefore seems natural to expect that also in Theorem 1.1 each $\alpha_p$ is an integer. If in Theorem 1.1 each $f_n$ is locally univalent, then this follows at once from [22].

### 2.8 Remarks on Liouville’s Equation

As mentioned above, Liouville’s equation $-\Delta u = 4e^{2u}$ can be seen as the “governing” PDE of the set of locally univalent meromorphic functions. Despite, or maybe because of this, it also arises in a variety of diverse problems in analysis, geometry and physics. For instance, if $\mathcal{D}$ is a bounded domain in $\mathbb{R}^2$, then the Euler–Lagrange equation for the functional

$$J(v) = \frac{1}{2} \int_{\mathcal{D}} |\nabla v|^2 - 8\pi \log \int_{\mathcal{D}} e^v, \quad v \in W^{1,2}_0(\mathcal{D}),$$

which is closely tied to the well-known Moser–Trudinger inequality, is the Liouville-type equation

$$-\Delta v = \lambda \frac{e^v}{\int_{\mathcal{D}} e^v},$$

where $\lambda$ is a positive constant.
for some constant $\lambda > 0$. The functional (2.5) and Eq. (2.6) have been intensively studied, partly in view of many applications such as the problem of prescribing Gauss curvature [5,6,9], the theory of the mean field equation [7,8,12,13] and Chern–Simons theory [14,31,35,37,38].

3 Quasi-Normality and Wandering Exceptional Functions

An essential step in the proof of Theorem 1.1 is to show that the family $F_C$ is quasi-normal. In order to prove this one can apply a quasi-normality criterion which has already been established by Montel [27] and Valiron [39]. The Montel–Valiron criterion might be viewed as a “quasi-normal” version of Carathéodory’s [33, p. 104] extended Fundamental Normality Test (FNT) for a family $F$ of meromorphic functions. In fact, we prove here an extension of the result of Montel and Valiron, which shows that one can replace the exceptional values in the Montel–Valiron criterion by exceptional meromorphic functions “wandering” on the sphere. Here “wandering” means that the exceptional functions are allowed to depend on the individual members of the family $F$.

We first need to recall some standard terminology. $M(D)$ denotes the set of all meromorphic functions on $D$ and $\sigma$ denotes the spherical distance on $\hat{C}$. A function $f$ is understood to assume the function $a$ if there is a point $z_0 \in D$ such that $f(z_0) = a(z_0)$. We call such a point $z_0$ an $a$-point of $f$. In the case $f(z_0) = a(z_0) = \infty$ it isn’t assumed that $z_0$ is a zero of $f - a$. (Actually, for $a \equiv \infty$ the latter wouldn’t make sense.) Furthermore, we do not take multiplicities into account, i.e. if (in the finite case) $z_0$ is a multiple zero of $f - a$, it is counted only once.

Theorem 3.1 Let $F$ be a family of functions meromorphic on a domain $D \subseteq \mathbb{C}$, $\varepsilon > 0$ and $p, q, r$ non-negative integers with $p \leq q \leq r$. Assume that for each $f \in F$ there exist functions $a_f, b_f, c_f \in M(D) \cup \{\infty\}$ such that $f$ assumes the function $a_f$ at most $p$ times, the function $b_f$ at most $q$ times and the function $c_f$ at most $r$ times in $D$ (ignoring multiplicities), and such that

$$\min\{\sigma(a_f(z), b_f(z)), \sigma(a_f(z), c_f(z)), \sigma(b_f(z), c_f(z))\} \geq \varepsilon$$

for all $z \in D$. Then $F$ is a quasi-normal family of order at most $q$.

Remark 2 (Exceptional Values vs. Exceptional Functions and Wandering vs. non-wandering) Normality criteria in the spirit of Montel’s or Carathéodory’s FNT, but with exceptional functions instead of exceptional values, have been established by Bargman, Bonk, Hinkkanen and Martin [2] as well as by the first named author and Nevo [19]. In [2] a version of Montel’s FNT for exceptional continuous, but not wandering functions with disjoint graphs is proved, while in [19] a version of Carathéodory’s extended FNT for exceptional meromorphic functions wandering on the sphere is established. Theorem 3.1 extends the main result of [19] to the case of quasi-normal families. It extends as well the result of Montel and Valiron mentioned earlier, which is just the special case of Theorem 3.1 that the exceptional functions are constants, but might be
wandering. Even though it is tempting, it is not possible to replace the meromorphic exceptional functions by continuous exceptional functions in Theorem 3.1. In fact, for merely continuous wandering exceptional functions we neither have quasi-normality nor $Q_\alpha$-normality for any ordinal number $\alpha$. (For the exact definition of $Q_\alpha$-normality we refer to [30].) This is shown by the same counterexample as in [19] or Sect. 2.3 (iii): The functions $f_n(z) := e^{nz}$ omit the three continuous functions $a_n := 0$, $b_n := \infty$, $c_n(z) := -e^{i\pi \text{Im}(z)}$ which clearly satisfy (3.1), but all points on the imaginary axis are points of non-normality, so in the unit disk $(f_n)_n$ is neither quasi-normal nor $Q_\alpha$-normal for any ordinal number $\alpha$.

4 Proofs

4.1 Proof of Theorem 3.1

An essential tool in the proof of Theorem 3.1 is the following normality criterion from [19, Thm. 2]. It is concerned with pairs of meromorphic functions “wandering” on the sphere which “uniformly stay away from each other”.

Lemma 4.1 Let $G$ be a family of pairs of meromorphic functions on a domain $D$ and $\varepsilon > 0$. Assume that

$$\sigma(a(z), b(z)) \geq \varepsilon \quad \text{for all } (a, b) \in G \text{ and all } z \in D.$$  

(4.1)

Then the families $\{a \mid (a, b) \in G\}$ and $\{b \mid (a, b) \in G\}$ are normal on $D$.

Using this lemma, Theorem 3.1 can be deduced from [19, Thm. 1] basically in the same way as the well-known special case of Theorem 3.1 that $a_f$, $b_f$, $c_f$ are constant (see [33, Thm. A.5 and Thm. A.9]) is deduced from the FNT. In order to make the paper self-contained and since the proof in [33, Thm. A.5] is given only for the special case of analytic functions and fixed exceptional values not depending on $f$, we provide full details.

In what follows we denote by $K_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$ the euclidean open disk with center $z_0 \in \mathbb{C}$ and radius $r > 0$. For a quasi-normal sequence $(f_n)$ in $\mathcal{M}(D)$ a point $p \in D$ is called an irregular point for $(f_n)$ if the sequence $(f_n)$ fails to be normal at $p$.

Proof of Theorem 3.1 Let $(f_n)_n$ be a sequence in $\mathcal{F}$. We write $a_n := a_{f_n}$, $b_n := b_{f_n}$, $c_n := c_{f_n}$.

By Lemma 4.1 the families $\{a_f \mid f \in \mathcal{F}\}$, $\{b_f \mid f \in \mathcal{F}\}$ and $\{c_f \mid f \in \mathcal{F}\}$ are normal. Therefore there exists a subsequence of $((f_n, a_n, b_n, c_n))_n$ which we continue to denote by $((f_n, a_n, b_n, c_n))_n$ such that $(a_n)_n$, $(b_n)_n$ and $(c_n)_n$ converge locally uniformly in $D$ (with respect to the spherical metric) to limit functions $a, b, c \in \mathcal{M}(D) \cup \{\infty\}$.

After heavily extracting further subsequences we may assume that there are at most $p$ accumulation points of the $a_n$-points of $f_n$. More precisely: We can assume that there is a set $A \subset D$ consisting of at most $p$ points such that the following holds:
1. For each $z_0 \in A$ there exists a sequence $(z_k)_k$ in $D$ converging to $z_0$ and a subsequence $(f_n)_k$ such that $f_n(z_k) = a_n(z_k)$ for all $k$.

2. For each $z_0 \in D \setminus A$ there exists an $r > 0$ such that the disk $K_r(z_0)$ contains only finitely many points $z$ which are $a_n$-points of $f_n$ for some $n$.

In the same way we find a set $B \subset D$ consisting of at most $q$ points and a set $C \subset D$ consisting of at most $r$ points such that, in the sense described above, the points in $B$ and $C$, respectively, are the only accumulation points of the $b_n$-points and of the $c_n$-points, respectively, of $f_n$.

Set $E := A \cup B \cup C$. In each compact subset of $D \setminus E$ there are only finitely many $a_n$-points, $b_n$-points and $c_n$-points of the functions $f_n$. Therefore, by [19, Thm. 1], $(f_n)_n$ is normal in $D \setminus E$, so by moving to a further subsequence we can assume that $(f_n)_n$ converges locally uniformly in $D \setminus E$ (with respect to the spherical metric) to a limit function $F \in \mathcal{M}(D \setminus E) \cup \{\infty\}$.

This shows that there are at most $p + q + r$ irregular points of $(f_n)_n$. In order to improve this bound we use the well-known fact that, by the maximum principle, the locally uniform convergence of a sequence of analytic functions on a punctured disk implies the locally uniform convergence on the whole disk.

So let some point $z_0 \in E$ be given. Then there exists a rigid motion $T$ of the Riemann sphere (i.e. an isomorphism with respect to the spherical metric) such that $T(a(z_0)) \neq \infty$, $T(b(z_0)) \neq \infty$ and $T(c(z_0)) \neq \infty$. Since the normality of $(f_n)_n$ on a subdomain of $D$ or at a point in $D$ is equivalent to the normality of $(T \circ f_n)_n$ (due to the fact that the spherical derivative is invariant under post-composition with rigid motions of the sphere and due to Marty’s theorem) and since (3.1) as well as the other assumptions of our theorem are invariant under rigid motions, we can replace $((f_n, a_n, b_n, c_n))_n$ by $((T \circ f_n, T \circ a_n, T \circ b_n, T \circ c_n))_n$ if required. Therefore, without loss of generality we may assume that $a(z_0) \neq \infty$, $b(z_0) \neq \infty$ and $c(z_0) \neq \infty$. Now we consider two cases.

**Case 1:** $F \not\equiv a$

Assume that $z_0 \notin A$. We choose $R > 0$ such that the punctured disk $K_{3R}(z_0) \setminus \{z_0\}$ is contained in $D \setminus E$ and such that $a$ is analytic in $K_{3R}(z_0)$. Then $K_{3R}(z_0) \cap A = \emptyset$, hence all but finitely many $f_n$ omit $a_n$ in $K_{2R}(z_0)$ since otherwise the points where $f_n$ assumes $a_n$ would have an accumulation point in $K_{3R}(z_0)$, contradicting the definition of $A$. This means that $g_n := 1/(f_n - a_n)$ is analytic in $K_{2R}(z_0)$ for all but finitely many $n$. In the compact annulus $K = K_{2R}(z_0) \setminus K_R(z_0)$ the sequence $(g_n)_n$ converges uniformly to $G := 1/(F - a) \neq \infty$. By the maximum principle, the uniform convergence carries over to the whole disk $K_{2R}(z_0)$. So $(f_n)_n$ itself converges uniformly (with respect to the spherical metric) in this disk to $1/G + a$.

We conclude that the only possible irregular points of $(f_n)_n$ are the points in $A$. In particular, there are at most $p \leq q$ such points.

**Case 2:** $F \equiv a$

Assume that $z_0 \notin B$. Then as in Case 1 we find an $R > 0$ such that $g_n := 1/(f_n - b_n)$ is analytic in $K_{2R}(z_0)$ for all but finitely many $n$ and such that $(g_n)_n$ converges uniformly in the compact annulus $K = K_{2R}(z_0) \setminus K_R(z_0)$ to the analytic limit function $G := 1/(F - b) = 1/(a - b)$. Again, by the maximum principle we deduce the uni-
form convergence of this sequence in the whole disk $K_{2R}(z_0)$, hence the uniform convergence of $(f_n)_n$ in this disk.

So in this case irregular points of $(f_n)_n$ can occur only at the points of $B$. Therefore their number is bounded by $q$. 

\[\square\]

### 4.2 Proof of Theorem 1.1

We start with the following elementary lemma.

**Lemma 4.2** Let $\alpha > 0$ be a real number. Then there exists a $\delta > 0$ such that any measurable subset $E$ of the Riemann sphere with spherical area at least $\alpha$ contains three points $a, b, c \in E$ such that

$$\min\{\sigma(a, b), \sigma(a, c), \sigma(b, c)\} \geq \delta.$$

**Proof** Without loss of generality, and to simplify our considerations, we replace the sphere by the euclidean plane $\mathbb{C}$ and the spherical metric by the euclidean metric. Let $E \subseteq \mathbb{C}$ be some fixed measurable set with euclidean area $\geq \alpha$. First, we can find two points $a, b \in E$ such that

$$|a - b| \geq \sqrt{\frac{\alpha}{\pi}} =: r.$$

(Otherwise, $E$ would be contained in an euclidean disc of radius less than $r$, and hence of area less than $\pi r^2 = \alpha$.) Then, we can find a third point $c \in E$ such that

$$\min\{|a - c|, |b - c|\} \geq \frac{r}{\sqrt{2}}$$

since otherwise $E$ would be contained in the union of two discs centered at $a$ and $b$, respectively, and with radii less than $r/\sqrt{2}$, and the area of the union of these discs would be less than

$$2\pi \cdot \left(\frac{r}{\sqrt{2}}\right)^2 = \pi r^2 = \alpha.$$

So the assertion of our lemma holds with $\delta := r/\sqrt{2}$. 

\[\square\]

We are now in a position to prove

**Theorem 4.3** Let $D \subseteq \mathbb{C}$ be a domain, $C > 0$, and $\mathcal{F}$ be a family of functions meromorphic on $D$ such that

$$\frac{1}{\pi} \int_D (f^#(z))^2 \, dx \, dy \leq C \quad \text{for all } f \in \mathcal{F}. \tag{4.2}$$

Then $\mathcal{F}$ is quasi-normal of order at most $C$. 

\[\bowtie\] Springer
Example 4.4 The estimate for the order of quasi-normality in Theorem 4.3 is best possible. This is illustrated by the family of the functions $f_n := n \cdot P$ on the unit disk $\mathbb{D}$ where $P$ is an arbitrary polynomial of degree $m$ with exactly $m$ distinct zeros in the unit disk (for example, $P(z) = 2z^m - 1$). Clearly, $(f_n)$ is quasi-normal of order $m$, the irregular points being the zeros of $P$. Since $f_n$ assumes each value in $\mathbb{C}$ at most $m$ times, (4.2) holds with $C = m$.

Proof of Theorem 4.3 Recall that the integral

$$\int_D \left( f^\#(z) \right)^2 \, dx \, dy$$

is the area of the image domain $f(D)$ of $D$ under the mapping $f$ on the Riemann sphere $\hat{\mathbb{C}}$, determined with respect to multiplicities, while $\pi$ is the area of the whole Riemann sphere. Let $m$ denote the largest integer less than or equal to $C$ and let

$$\varepsilon := \pi \cdot \left( 1 - \frac{C}{m + 1} \right) > 0.$$

Then for each $f \in \mathcal{F}$ there is a set $E_f \subseteq \hat{\mathbb{C}}$ of spherical measure at least $\varepsilon$ such that all values in $E_f$ are assumed by $f$ at most $m$ times. Otherwise, all values in a set of spherical measure larger than $\pi - \varepsilon$ would be assumed at least $m + 1$ times, which would imply

$$\int_D \left( f^\#(z) \right)^2 \, dx \, dy > (m + 1)(\pi - \varepsilon) = \pi \cdot C,$$

contradicting (4.2). By Lemma 4.2 for each $f \in \mathcal{F}$ we can find for each $f \in \mathcal{F}$ three points $a_f, b_f, c_f \in E_f$ such that

$$\min\{\sigma(a_f, b_f), \sigma(a_f, c_f), \sigma(b_f, c_f)\} \geq \delta$$

with an universal constant $\delta > 0$ depending only on $\varepsilon$, but not on the function $f \in \mathcal{F}$. Since each $f \in \mathcal{F}$ assumes the values $a_f, b_f, c_f$ at most $m$ times, the family $\mathcal{F}$ is quasi-normal of order at most $m$ by Theorem 3.1

By the converse of Bloch’s principle [33, Ch. 4], Theorem 4.3 should encapsulate a corresponding removable singularity criterion. Such a criterion is in fact a simple consequence of Picard’s Great Theorem:

Lemma 4.5 Let $f$ be meromorphic on the punctured disk $K_R(z_0) \setminus \{z_0\}$ such that

$$\int_{K_R(z_0) \setminus \{z_0\}} \left( f^\#(z) \right)^2 \, dx \, dy < \infty. \quad (4.3)$$

Then $f$ has a meromorphic extension to $K_R(z_0)$. 

Springer
In order to see how this follows from Picard’s Great Theorem note that (4.3) is equivalent to the convergence of the series
\[ \sum_{n=1}^{\infty} \int_{\frac{\rho}{2^k} < |z-z_0| < \frac{\rho}{2^{k-1}}} (f^\#(z))^2 \, dx \, dy. \]

Hence, for sufficiently small \( \rho > 0 \), the function \( f \) restricted to \( K_\rho(z_0) \setminus \{z_0\} \) would omit a set of spherical area less than \( \pi/2 \), say, so \( f \) would omit there (at least) three values, so cannot have an essential singularity at \( z_0 \) by Picard’s Great Theorem.

**Proof of Theorem 1.1** We know from Theorem 4.3 that \( F_C \) is quasi-normal of order at most \( C \). Let \( (f_n) \) be a sequence in \( F_C \). Then the quasi-normality of \( F_C \) implies that there exists a subsequence which we still denote by \( (f_n) \) and a finite set \( S \subset D \) such that \( (f_n) \) converges locally uniformly on \( D \setminus S \) to some limit function \( f \) which is meromorphic in \( D \setminus S \) and such that \( (f_n) \) fails to be normal in any neighborhood of any point of \( S \).

If \( S = \emptyset \), then \( f \) is meromorphic on \( D \) and clearly also belongs to \( F_C \), so alternative (1) in Theorem 1.1 holds. We assume from now on that \( S \) is not empty. In order to prove that (2a) holds, let \( p \in S \) and choose \( \epsilon > 0 \) such that the closed disk \( K \) of radius \( \epsilon \) centered at \( p \) does not contain any other point of \( S \). Then \( (f_n) \) converges spherically uniformly on each annulus \( \delta \leq |z-p| \leq \epsilon \) to \( f \), so
\[ \frac{1}{\pi} \int_{K \setminus \{p\}} (f^\#(z))^2 \, dx \, dy \leq C. \]

Therefore, \( f \) has a meromorphic extension to \( K \) by Lemma 4.5, and hence to all of \( D \). We denote this extension also by \( f \). It follows at once that \( f \in F_C \).

Since \( (f_n) \) is not normal at \( p \), Marty’s theorem [25] yields a sequence \( (z_n) \) in \( D \) converging to \( p \) such that \( f_n^\#(z_n) \rightarrow +\infty \).

If each \( f_n \) is locally univalent on \( D \), then \( f \) is either a constant function or also locally univalent in \( D \setminus S \) by the locally uniform convergence of \( (f_n) \) to \( f \) on \( D \setminus S \) and Hurwitz’ theorem. If \( f \) were locally univalent on \( D \setminus S \), then with \( p \in S \) and \( \epsilon > 0 \) as above, there would be a constant \( c > 0 \) such that \( f_n^\#(z) \geq 2c \) for all \( |z-p| = \epsilon \). Hence \( f_n^\#(z) \geq c \) for all \( |z-p| = \epsilon \) and all \( n \) sufficiently large. Since each \( f_n \) is locally univalent on \( D \), so \( f_n^\# \) is never zero in \( D \), the equation \( -\Delta \log f_n^\# = 4(f_n^\#)^2 \) shows that \( \log f_n^\# \) is superharmonic on \( |z-p| < \epsilon \) and thus \( f_n^\#(z) \geq c \) for all \( |z-p| \leq \epsilon \) by the minimum principle. But then as we have already pointed out in Sect. 2.4, the main result in [18] implies that \( (f_n) \) is normal in \( |z-p| < \epsilon \), contradicting \( p \in S \). Hence, \( f \) is in fact constant on \( D \), and we have proved (2a).

We finally prove (2b). Since
\[ \left( \frac{(f_n^\#)^2}{f_n^\#} \right) \]

 Springer
is bounded in $L^1(D)$ we can apply the weak-$\ast$ compactness principle for bounded Borel measures which yields a (unique) non-negative bounded Borel measure $\mu$ on $D$ such that—after taking a subsequence—we have

$$(f_n^\#)^2 \to \mu$$

in the measure theoretic sense, that is,

$$\int_D (f_n^\#(z))^2 \psi(z) \, dx \, dy \to \int_D \psi \, d\mu \quad (4.4)$$

for any $\psi \in C_c(D)$. Here, $C_c(D)$ denotes the set of continuous functions on $D$ with compact support. It follows that for any such $\psi$,

$$\int_D \psi \, d\mu = \sum_{p \in S} \left( \lim_{n \to \infty} \int_{K_\varepsilon(p)} (f_n^\#(z))^2 (\psi(z) - \psi(p)) \, dx \, dy ight) + \psi(p) \lim_{n \to \infty} \int_{K_\varepsilon(p)} (f_n^\#(z))^2 \, dx \, dy$$

$$+ \lim_{n \to \infty} \int_{D \setminus \bigcup_{p \in S} K_\varepsilon(p)} (f_n^\#(z))^2 \psi(z) \, dx \, dy$$

for any $\varepsilon > 0$ sufficiently small. Since $f_n \to f$ locally uniformly in $D \setminus S$, $f_n \in \mathcal{F}_C$ and $\psi$ is continuous, this yields

$$\int_D \psi \, d\mu = \sum_{p \in S} \mu(\{p\}) \psi(p) + \int_D (f^\#(z))^2 \psi(z) \, dx \, dy.$$ 

Hence,

$$\mu = \sum_{p \in S} \mu(\{p\}) \delta_p + (f^\#)^2$$

in the sense of measures. Now, $\mu(\{p\}) \geq \pi$ for each $p \in S$, since otherwise there would exist a function $\psi \in C_c(D)$ with $0 \leq \psi \leq 1$ such that $\psi \equiv 1$ on some disk centered at $p$ and

$$\int_D \psi \, d\mu < \pi.$$
In view of (4.4), this would imply that eventually

$$\int_{K_\varepsilon(p)} \left( f_n^\sharp(z) \right)^2 \, dxdy < \pi,$$

so \( (f_n) \) would be normal on \( K_\varepsilon(p) \) by Theorem 4.3, contradicting \( p \in S \). Hence \( \alpha_p := \mu(\{ p \}) / \pi \geq 1. \)

\[\Box\]

5 Final Remark: Gromov’s Compactness Theorem for Pseudoholomorphic Curves

Theorem A and Theorem 1.1 are mere examples of a general phenomenon. Thinking of

$$\int_D \left( f^\sharp(z) \right)^2 \, dxdy$$

as “Energy”, the “Mass Concentration Property” (2b) tells us that a discrete loss of energy when passing from \( (f_n) \) to the limit \( f \) has to be compensated by “bubbles” appearing as \( n \) tends to infinity. This is a key idea in Geometric Analysis, which is impressively demonstrated in the proof of Gromov’s compactness theorem [21] or the work of Sacks and Uhlenbeck [32] on minimal immersions of 2-spheres.

References

1. Aulaskari, R., Lappan, P.: Some integral conditions involving the spherical derivative, Complex analysis, Proc. 13th Rolf Nevanlinna-Colloq., Joensuu/Finn. 1987, Lect. Notes Math. 1351 (1988), 28–37
2. Bargmann, D., Bonk, M., Hinkkanen, A., Martin, G.J.: Families of meromorphic functions avoiding continuous functions. J. Anal. Math. 79, 379–387 (1999)
3. Bartolucci, D., Tarantello, G.: Liouville type equations with singular data and their applications to periodic multivortices for the electroweak theory. Comm. Math. Phys. 229, 3–47 (2002)
4. Brézis, H., Merle, F.: Uniform estimates and blow-up behavior for solutions of \( \Delta u = V(x)e^u \) in two dimensions. Comm. Partial Diff. Eq. 16, 1223–1253 (1991)
5. Chang, S.Y.A., Gursky, M.J., Yang, P.C.: The scalar curvature equation on 2- and 3-spheres. Calc. Var. Partial Diff. Eq. 1, 205–229 (1993)
6. Chang, S.Y.A., Yang, P.C.: Prescribing Gaussian curvature on \( S^2 \). Acta Math. 159, 215–259 (1987)
7. Chen, C.C., Lin, C.S.: Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces. Comm. Pure Appl. Math. 55, 728–771 (2002)
8. Chen, C.C., Lin, C.S.: Topological degree for a mean field equation on Riemann surfaces. Comm. Pure Appl. Math. 56, 1667–1727 (2003)
9. Chen, W., Ding, W.: Scalar curvatures on \( S^2 \). Trans. Am. Math. Soc. 303, 365–382 (1987)
10. Chen, X.: Remarks on the existence of branch bubbles on the blowup analysis of equation \( -\Delta u = e^{2u} \) in dimension two. Comm. Anal. Geom. 7(2), 295–302 (1999)
11. Chuang, C.-T.: Normal Families of Meromorphic Functions. World Scientific, Singapore (1993)
12. Ding, W., Jost, J., Li, J., Wang, G.: The differential equation \( \Delta u = 8\pi 8\pi e^u \) on a compact Riemann surface. Asian J. Math. 1, 230–248 (1997)
13. Ding, W., Jost, J., Li, J., Wang, G.: Existence results for mean field equations. Ann. Inst. H. Poincaré Anal. Non Linéaire, 16(5), 653–666 (1999)
14. Ding, W., Jost, J., Li, J., Wang, G.: An analysis of the two-vortex case in the Chern–Simons–Higgs model. Calc. Var. Partial Diff. Eq. 7, 87–97 (1998)
15. Dufresnoy, J.: Sur les domaines couverts par les valeurs d’une fonction méromorphe ou algebroide. Ann. Sci. Éc. Norm. Sup. 58(3), 179–259 (1941)
16. Fournier, R., Ruscheweyh, St.: Free boundary value problems for analytic functions in the closed unit disk, Proc. Amer. Math. Soc. 127, 3287–3294 (1999)
17. Fournier, R., Kraus, D., Roth, O.: A Schwarz lemma for locally univalent meromorphic functions, Proc. Am. Math. Soc. to appear, arXiv:1902.07242
18. Grahl, J., Nevo, S.: Spherical derivatives and normal families. J. Anal. Math. 117, 119–128 (2012)
19. Grahl, J., Nevo, S.: Exceptional functions wandering on the sphere and normal families. Isr. J. Math. 202, 21–34 (2014)
20. Gröhn, J.: Converse growth estimates for ODEs with slowly growing solutions. arXiv:1811.08736
21. Gromov, M.: Pseudoholomorphic curves in symplectic manifolds. Invent. Math. 82, 307–347 (1985)
22. Li, Y.Y., Shafrir, I.: Blow-up Analysis for Solutions of $\Delta u = Ve^u$ in Dimension Two. Indiana. Math. J. 43, 1255–1270 (1994)
23. Lina, C.-S., Tarantello, G.: When “blow-up” does not imply “concentration”: A detour from Brézis–Merle’s result, C. R. Acad. Sci. Paris Sér. I Math 354, 493–498 (2016)
24. Liouville, J.: Sur l’équation aux différences partielles $d^2 \log \lambda \over du dv \pm \lambda^2 a^2 = 0$. J. de Math. 16, 71–72 (1853)
25. Marty, F.: Recherches sur la répartition des valeurs d’une fonction méromorphe. Ann. Fac. Sci. Univ. Toulouse 23(3), 183–261 (1931)
26. Montel, P.: Sur les familles quasi-normales de fonctions holomorphes. Mem. Acad. Roy. Belgique 6(2), 1–41 (1922)
27. Montel, P.: Sur les familles quasi-normales de fonctions analytiques. S. M. F. Bull. 52, 85–114 (1924)
28. Montel, P.: Leçons sur les familles normales de fonctions analytiques et leurs applications, Gauthier-Villars, Paris (1927)
29. Montel, P.: Le rôle des familles normales. Enseign. Math. 33, 5–21 (1934)
30. Nevo, S.: Transfinite extension to $Q_m$-normality theory. Results Math. 44, 141–156 (2003)
31. Nolasco, M., Tarantello, G.: Double vortex condensates in the Chern–Simons–Higgs theory. Calc. Var. Partial Diff. Eq. 9, 31–94 (1999)
32. Sacks, J., Uhlenbeck, K.: The existence of minimal immersions of 2-spheres. Ann. Math. 113, 1–24 (1981)
33. Schiff, J.: Normal Families. Springer, New-York (1993)
34. Shafrir, I.: A sup+inf inequality for the equation $\Delta u = Ve^u$, C. R. Acad. Sci. Paris Sér. I Math. 315, 159–164 (1992)
35. Spruck, J., Yang, Y.: On multivortices in the electroweak theory I: Existence of periodic solutions. Comm. Math. Phys. 144, 1–16 (1992)
36. Steinmetz, N.: Normal families and linear differential equations. J. Anal. Math. 117, 129–132 (2012)
37. Struwe, M., Tarantello, G.: On the multivortex solutions in the Chern–Simons gauge theory, Boll. U.M.I. Sez. B Artic. Ric. Mat. 1, 109–121 (1998)
38. Tarantello, G.: Multiple condensate solutions for the Chern–Simons–Higgs theory. J. Math. Phys. 37, 3769–3796 (1996)
39. Valiron, G.: Familles normales et quasi-normales de fonctions méromorphes, Gauthier-Villars (Mémo- rial Sc. Math. Fasc. 38), Paris (1929)
40. Yamashita, S.: Spherical derivative of meromorphic function with image of finite spherical area. J. Inequal. Appl. 5, 191–199 (2000)
41. Zalcman, L.: Normal families: new perspectives. Bull. Am. Math. Soc. 35, 215–230 (1998)