A Diagrammatic Equation for Oriented Planar Graphs

Razvan Gurau*

March 17, 2010

Abstract

In this paper we introduce a diagrammatic equation for the planar sector of square non hermitian random matrix models strongly reminiscent of Polchinski’s equation in quantum field theory. Our fundamental equation is first obtained by a graph counting argument and subsequently derived independently by a precise saddle point analysis of the corresponding random matrix integral. We solve the equation perturbatively for a generic model and conclude by exhibiting two duality properties of the perturbative solution.

1 Introduction

Random matrix models have wide applications in subjects ranging from statistical physics [1] to two dimensional quantum gravity [2]. They have long been related to triangulations of two dimensional surfaces [3] and various combinatorial counting problems [4,5,6] have been addressed with their help. Techniques inspired by quantum field theory have been adapted to such models [7], and non identically distributed random matrix models proved crucial in the study of non commutative quantum field theories [8,9,10,11].

For a large class of random matrix models, in the limit of large matrices (the so called “large $N$ limit”), the behavior of the partition function (and all correlation functions) is dominated by a saddle point. The planar sector

*Perimeter Institute for Theoretical Physics Waterloo, ON, N2L 2Y5, Canada
[3] is the first contribution to the evaluation of the random matrix integral by the saddle point method. The study of this saddle point is best addressed by means of the resolvent function which obeys a certain quadratic equation. In the large $N$ limit the resolvent is the generating function of planar graphs with one vertex, thus having a very natural graphical interpretation.

Rectangular non hermitian random matrices been extensively studied (see [5] and references therein). However the treatment presented [5] is not immediately applicable to square matrices: it is singular already for a free model (as it will be explained in detail in section 4). The solution implemented in [5] is to study rectangular non hermitian matrices and take the limit of square matrices after having taken the large $N$ limit.

In this paper we propose a new approach to square non hermitian random matrices. Our approach does not suffer from the singularities of [5] (and sheds light on there origin). A detailed analysis allows us to derive a tower of equations obeyed by the *coefficients* of resolvent of such models both by a graph counting argument (inspired by the Polchinski equation) and by a saddle point analysis. We will solve these equations in perturbations and prove certain duality properties of the solution. Our results encode the census of planar diagrams with one vertex generated by an arbitrary non hermitian random matrix model (which can be translated into the census of planar connected amputated graphs with one external face). Non hermitian matrix models have been studied in [12, 13] by the method of the “loop insertion equation” (very similar to our equation for the resolvent) which can be derived either by a combinatorial argument (for the resolvent itself) or by a functional manipulation. In [12, 13] the equation is solved by contour integrations (allowing rapid access to large order behavior of the solution) rather than trough the “brute force” approach we take here. Their results provide a check of our method and suggest possible roads to generalize our tower of equations for non planar graphs.

In higher dimensions, random matrix models generalize to group field theories [14, 15], associated to cellular complexes [16] with boundary [17]. A generalized notion of planarity adapted to group field theory has been proposed [18], and various power counting results [18, 19, 20] have been established. Our long term goal would be to generalize the results of this paper for such theories.

This paper is organized as follows. In section 2 we present the non hermitian random matrix models and the relation between the resolvent and the connected planar functions with one external face. In section 3 we derive
our fundamental equations by a graph counting argument, and recover them in section 4 by a saddle point analysis. Section 5 provides the perturbative solution of these equations. In section 6 we exhibit two duality properties of the perturbative solution. Section 7 draws the conclusions of our work.

2 Non Hermitian Random Matrix Models

In this section we review in some detail the non hermitian random matrix models. We use this opportunity to introduce our notations, starting with the most important: throughout this paper we will denote

\[ C^m_p = \frac{n!}{p!(n-p)!}, \tag{1} \]

the binomial coefficient.

Let \( M \) be an square \((N \times N)\) non hermitian \((M \neq M^\dagger)\) matrix. A non hermitian random matrix model is a probability measure

\[
d\mu_{V(\alpha M^\dagger M)} = \left( \frac{N}{2\pi i} \right)^N \left[ \prod_{ab} dM^\dagger_{ab} dM_{ab} \right] e^{-N \left( \text{Tr}(M^\dagger M) + \text{Tr}[V(\alpha M^\dagger M)] \right)}
\]

\[ V(\alpha M^\dagger M) = -\sum_{p=1}^{\infty} \frac{1}{p} g_p \alpha^p (M^\dagger M)^p. \tag{2} \]

with \( \alpha \) some small coupling constant. The normalization is chosen such that at \( \alpha = 0 \), \( d\mu_0 \) is a normalized gaussian measure. The partition function and the correlations of this measure are denoted

\[
Z = \langle 1 \rangle = \int d\mu_{V(\alpha M^\dagger M)}
\]

\[
\left\langle M^\dagger_{ab} \ldots M_{cd} \right\rangle = \int d\mu_{V(\alpha M^\dagger M)} M^\dagger_{ab} \ldots M_{cd}.
\tag{3}
\]

The arguments of a correlation, \( M^\dagger_{ab} \ldots M_{cd} \), are called “external points”. The partition function and the correlations are evaluated as sums of oriented ribbon graphs with external points build as follows

\[ \text{The exponent Tr}(M^\dagger M) + \text{Tr}[V(\alpha M^\dagger M)] \text{ in eq. (2) is called the action of the model. A rescaling of both } M \text{ and } M^\dagger \text{ by } \sqrt{\alpha} \text{ brings it in the more familiar form } \frac{1}{\alpha} \text{Tr}(M^\dagger M) + \text{Tr}[V(M^\dagger M)]. \]
• Vertices (drawn as crossroads of ribbons) are generated by $V(\alpha M^\dagger M)$. Each vertex has $2p$ alternating halflines (out of which $p$ are $M^\dagger$’s and $p$ are $M$’s) and a weight $N \frac{1}{p} g_p \alpha^p$.

• Lines (drawn as ribbons) fall in two categories: internal and external. The internal lines connect a $M^\dagger$ halfline and a $M$ halfline. The external lines connect an internal $M^\dagger$ (or $M$) halfline with an external $M$ (respectively $M^\dagger$) point. A line has a natural orientation (say from $M^\dagger$ to $M$) and a weight $\frac{1}{N}$.

• Faces are closed circuits formed by the sides of the ribbons. They also fall in two categories, internal and external. The internal faces are closed circuits which do not pass through any external point. They have weight $N$. The external faces (containing external points) have weight $1$.

The identification of the external faces of a graph is slightly non trivial. To count the external faces of a graph one “pinches” the external points, that is one connects the two sides of the ribbon arriving at any external point.

Figure depicts a typical ribbon graph. The “pinching” of the external points is represented by the dotted lines. This graph has two external points, two vertices, three internal lines, two external lines two internal faces and one external face.

The partition function $Z = \langle 1 \rangle$ is the sum of graphs with no external points (vacuum graphs). The non trivial correlations have the same number of $M$ and $M^\dagger$ external points. They write

$$\frac{1}{\langle 1 \rangle} \langle M_{\mu_1 \nu_1}^\dagger M_{\nu_1' \mu_1'} \cdots M_{\mu_N \nu_N}^\dagger M_{\nu_N' \mu_N'} \rangle = \sum_{\mathcal{G}} \mathcal{A}(\mathcal{G}) \ ,$$

(4)

We consider the lines connecting two external points as external.
where $\mathcal{G}$ are all ribbon graphs with no vacuum connected components (due to the division by $\langle 1 \rangle$) and $\mathcal{A}(\mathcal{G})$ is the amplitude of $\mathcal{G}$.

To compute $\mathcal{A}(\mathcal{G})$ we denote $V$ the number of vertices (of coordinations $2p_1, \ldots, 2p_V$), $L$ the number of internal lines, $N_e$ the number of external points, $F$ the total number of faces and $B$ the number of external faces of $\mathcal{G}$. We have the two topological relations

$$2 \sum_v p_v - N_e = 2L \quad V - L + F = 2 - 2g,$$

with $g$ the genus of the graph $\mathcal{G}$ (thus for the graph of figure 1 $g = 0$). Furthermore, for an external face $b$ of $\mathcal{G}$, we index the $N_b$ external points $M^\dagger$ by $i^{(b)}_1$ and the $N_b$ external points $M$ by $j^{(b)}_1$. We associate to the face $b$ the ordered set

$$b = \{ i^{(b)}_1, j^{(b)}_1, \ldots, i^{(b)}_{N_b}, j^{(b)}_{N_b} \},$$

and we have

$$b_1 \cap b_2 = \emptyset \quad \forall b_1 \neq b_2$$

$$\bigcup_b \{ i^{(b)}_1, \ldots, i^{(b)}_{N_b} \} = \{ 1 \ldots N_e \} \quad \bigcup_b \{ j^{(b)}_1, \ldots, j^{(b)}_{N_b} \} = \{ 1 \ldots N_e \}. \quad (7)$$

With these notations the amplitude of $\mathcal{G}$ writes

$$\mathcal{A}(\mathcal{G}) = K(\mathcal{G}) \alpha \sum_v p_v N^{V-(L+N_e)+(F-B)} \prod_{v=1}^V g_{p_v}$$

$$\prod_b \delta_{\nu^{(b)}_{i^{(b)}_1}} \delta_{\nu'^{(b)}_{j^{(b)}_1}} \cdots \delta_{\nu^{(b)}_{i^{(b)}_{N_b}}} \delta_{\nu'^{(b)}_{j^{(b)}_{N_b}}} \cdots \delta_{\nu^{(b)}_{i^{(b)}_1}} \delta_{\nu'^{(b)}_{j^{(b)}_1}}, \quad (8)$$

where $K(\mathcal{G})$ is some combinatorial coefficient.

Of particular interest in the sequel is the resolvent defined as

$$\omega_N(z) = \frac{1}{N} \frac{\langle \text{Tr} \left( \frac{1}{z-M^\dagger M} \right) \rangle}{\langle 1 \rangle} = \sum_{p=0}^\infty \frac{1}{z^{p+1} N} \frac{\langle \text{Tr} \left[ (M^\dagger M)^p \right] \rangle}{\langle 1 \rangle}.$$

We will denote its limit when $N \to \infty$ by $\omega(z)$. The coefficient

$$d_{p;N} = \frac{1}{N} \frac{1}{\langle 1 \rangle} \langle \text{Tr} \left[ (M^\dagger M)^p \right] \rangle,$$

where $|p;N| = p+1$. For $p = 0$ it is $d_{0;N} = 1$. For $p = 1$ it is $d_{1;N} = \frac{1}{N} \frac{1}{\langle 1 \rangle} \langle \text{Tr} \left( M^\dagger M \right) \rangle$.
is the sum over all connected vacuum graphs having a special vertex of co-
modation $2p$ and weight 1 (the trace and matrix product over the external
points allows us to reinterpret them as the halflines of this special vertex).
The global power in $N$ of a graph contributing to $d_{p;(N)}$ is, using eq. (8),
\[ \frac{1}{N} N^{(V-1) - L + F} = N^{-2g} , \] (11)
and we recognize the classical result that $d_p = \lim_{N \to \infty} d_{p;(N)}$ is the sum
over connected planar vacuum graphs with a special vertex of weight 1 and
coordination $2p$.

Before concluding this section we present the relation between the con-
nected planar one vertex functions $d_p$ and the connected amputated planar
functions with one external face. Our motivation is twofold: first, from a
quantum field theory perspective, the latter dominate the floating Wilsonian
action\footnote{To be distinguished from the effective action which is generated by one particle irre-
ducible graphs.} and second, the reasoning we present below is similar to the one
we will use in section 3 to derive our fundamental equation.

A planar connected amputated function with one external face and $2t$
external points writes
\[ N^{2t} \left< M_{a_1 b_1}^\dagger M_{b_1' a_2} \cdots M_{b_t' a_t} \right>^{g=0, B=1} = G_t \delta_{b_1 b_1'} \delta_{a_2 a_2} \cdots \delta_{b_t b_t'} \delta_{a_t a_t} . \] (12)

Consider a graph $G$ contributing to $d_p$ (depicted in figure 2). $G$ is a
connected planar vacuum graph with a special vertex of coordination $2p$
and weight 1. The special vertex is represented by dashed lines in figure 2.
Consider a halfline of the special vertex, denoted $A_1$ in figure 2. As $G$
is planar, if we erase the special vertex $A_1$ becomes an external point of some
planar connected component $C$ with one external face, represented as shaded
in figure 2. Say the $C$ has 2$t$ external points. All of them originate from
halflines on the special vertex, $A_1$, $A_2$, $A_3$, etc. We denote $2k_1$ the number of
halflines on the special vertex between $A_1$ and $A_2$, $2k_2$ the number of halflines
between $A_2$ and $A_3$ etc. up to $2k_{2t}$ the number of halflines between $A_{2t}$ and
$A_1$. As $G$ is planar, the halflines $2k_1$ must connect among themselves and
therefore form a graph corresponding to $d_{k_1}$. The same is true for $2k_2$, $2k_3$
etc. Therefore $d_p$ writes

$$d_p = \sum_{t=1}^{p} G_t \sum_{k_1 \ldots k_{2t}=0 \atop k_1 + \ldots + k_{2t}=p-t} d_{k_1} \ldots d_{k_{2t}} .$$  \hspace{1cm} (13)$$

Figure 2: A graph $\mathcal{G}$ contributing to $d_p$.

Equations (13) can be solved iteratively for $G_t$. The first equations are

$$d_1 = G_1$$
$$d_2 = G_2 + 2G_1 d_1$$
$$d_3 = G_3 + 4G_2 d_1 + G_1 (d_1^2 + 2d_2) ,$$ \hspace{1cm} (14)

which are solved by

$$G_1 = d_1$$
$$G_2 = d_2 - 2G_1 d_1 = d_2 - 2d_1^2$$
$$G_3 = d_3 - 4G_2 d_1 - G_1 (d_1^2 + 2d_2) = d_3 - 6d_1 d_2 + 7d_1^3 .$$ \hspace{1cm} (15)

Before concluding this section we introduce a notation. As $V(x) = - \sum_p \frac{1}{p} g_p x^p$ is a formal power series, we will denote its derivative (in the sense of formal power series) by

$$V'(x) = - \sum g_p x^{p-1} .$$ \hspace{1cm} (16)
3 The Diagrammatic Equation

In quantum field theory the Polchinski equation \cite{22} is an equation for the floating Wilsonian action $S$ at scale $\Lambda$ and writes

$$\partial_\Lambda S = \frac{1}{2} \int_{x,y} \frac{\delta S}{\delta \phi(x)} \partial_\Lambda K(x,y) \frac{\delta S}{\delta \phi(y)} - \frac{1}{2} \int_{x,y} \frac{\delta^2 S}{\delta \phi(x) \delta \phi(y)} \partial_\Lambda K(x,y) \, , \quad (17)$$

where $K(x,y)$ is the propagator with ultraviolet cutoff $\Lambda$. The action $S$ is the sum of connected amputated graphs, and the propagator $K$ is associated to lines. This equation naturally translates in a graph classification: a line in a graph contributing to the action $S$ can either be a “tree line” separating two distinct effective vertices (the first term in eq. (17)) or it can be a “loop line” for some effective vertex (the second term in eq. (17)). Adapting this idea to ribbon graphs will lead to our fundamental equation. There are two important aspects one needs to take into account. First, in stead of dealing with the floating action, we will deal with the resolvent of the matrix model. Second, we will look for an equation valid in the $N \to \infty$ limit, thus involving only planar graphs.

![Diagram](image)

**Figure 3:** A loop line on the special vertex.

First consider a free model, obtained by setting $\alpha = 0$ in eq. (2). We will denote (the $N \to \infty$ limit of) the resolvent of the free model $\omega^{(0)}(z)$ and its coefficients $d_p^{(0)}$.

With this notations, $d_{q+1}^{(0)}$ is the sum of all planar vacuum graphs with one special vertex of coordination $2q + 2$ and weight 1. Consider a line starting on the special vertex. As the graph contains no other vertices, this line must also end on the special vertex. It is therefore always a loop line for the special vertex. The line encloses a certain number of halflines, say $2k$, and
leaves on the exterior $2q - 2k$ halflines as in figure 3. As the initial graph is planar, the interior (exterior) halflines necessarily connect into planar graphs with one special vertex of coordination $2k$ (respectively $2q - 2k$). Therefore

$$d_{q+1}^{(0)} = \sum_{k=0}^{q} d_k^{(0)} d_{q-k}^{(0)}, \quad d_0^{(0)} = 1.$$  \quad (18)

The attentive reader will note that up to now we did nothing new: eq. (18) is nothing but equation eq. (13) supplemented by the condition that the connected amputated functions of a free model are trivial, $G_t = \delta_{t1}$.

![Figure 4: A tree line in the interacting model.](image)

For an interacting model, a line originating on the special vertex of $d_{q+1}$ can either (as for the free model) end on the special vertex, or it can end on a $g_n$ vertex as in figure 4. Up to a weight factor, the graphs contributing to the latter are in one to one correspondence with the connected planar vacuum graphs with a special vertex of coordination $2q + 2n$. To compute the mismatch in weight factors, recall that the weight of the vertex $g_n$ is $\frac{1}{n} g_n \alpha^n$ and note that the line has a choice of one among the $n$ halflines of appropriate orientation on $g_n$, therefore

$$d_{q+1} = \sum_{k=0}^{q} d_k d_{q-k} + \sum_{n=1}^{\infty} g_n \alpha^n d_{q+n}.$$  \quad (19)

This are our fundamental diagrammatic equations. The rest of this paper is devoted to their analysis.

4 The Saddle Point Analysis

In this section we derive the equations (19) by an independent method. First, by a carefull saddle point analysis adapted to a square non hermitian random
matrix model, we will derive an equation obeyed by the resolvent \( \omega(z) \) and then, by developing in powers of \( z \), we will rederive the equations \([19]\) for the coefficients \( d_p \).

To evaluate an integral with measure \([2]\) by a saddle point method one needs to change variables from \( M_{ab}^*, M_{ab} \) to the eigenvalues of \( M^\dagger M \) \([5], [23]\).

At this stage one has a choice. One can either choose to change variables to \( \lambda_i \) the eigenvalues of \( M^\dagger M \), or to change variables to \( y_i \), the eigenvalues of \( \sqrt{M^\dagger M} \). For example in \([5]\) the author chooses the former. However, for square matrices, there is a heavy price to be paid in doing this choice.

To understand the problem consider first Euler’s gamma function

\[
\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt .
\]

(20)

It is well known that this integral is governed by a saddle point \( t_0 = n - 1 \), and the saddle point evaluation of it proves Stirling’s formula. However a polynomial change of variables \( u = t^n \) yields

\[
\Gamma(n) = \frac{1}{n} \int_0^\infty e^{-u^{1/n}}du ,
\]

(21)

and in this new form the integral is not governed by a saddle point: the Jacobian of the polynomial change of variables absorbed it.

This same phenomenon occurs for square non hermitian matrices. Following \([5]\) one writes (up to some normalization) the partition function

\[
Z = \left[ \int_0^\infty \prod_k d\lambda_k \right] \prod_{i<j} (\lambda_j - \lambda_i)^2 e^{-N\left( \sum_i \lambda_i + \sum_i V(\alpha \lambda_i) \right)} .
\]

(22)

For simplicity let us set \( \alpha = 0 \). Raising the Van Der Monde determinant in the exponent yields

\[
\sum_i \lambda_i - \frac{2}{N} \sum_{j<i} \ln |\lambda_j - \lambda_i| .
\]

(23)

The integral \((22)\) is not governed by a saddle point: the would be saddle point equations

\[
1 - \frac{2}{N} \sum_{j\neq i} \frac{1}{\lambda_i - \lambda_j} = 0, \quad \forall i
\]

(24)
are incompatible: adding them all up yields $N = 0$. Considering, as in [5], rectangular matrices solves this problem. In that case the saddle point equations pick up an extra term and are compatible. As previously stated, their results can be applied for square matrices only after taking the large $N$ limit.

In the sequel we will take the second alternative and use the variables $y_i$, eigenvalues of $\sqrt{M^\dagger M}$. The Jacobian of this change of variables has been computed in [23] and the partition function writes (again up to some normalization)

$$Z = \left[ \int_0^\infty \prod_k dy_k \right] \prod_i (y_i^2 - y_i^2)^2 \prod_i y_i e^{-N\left(\sum y_i^2 + \sum V(\alpha y_i^2)\right)}$$

$$= \left[ \int_0^\infty \prod_k dy_k \right] e^{-NF\{y_i\}} ,$$

(25)

where we denoted

$$F\{\{y_i\}\} = \sum_i y_i^2 + \sum_i V(\alpha y_i^2) - \frac{1}{N} \sum_i \ln y_i - \frac{2}{N} \sum_{i<j} \ln |y_j^2 - y_i^2| .$$

(26)

We have the following lemma

**Lemma 1.** $\omega(z)$ obeys the quadratic equation

$$\omega(z)^2 - P(z)\omega(z) + Q(z) = 0 ,$$

(27)

with

$$P(z) = 1 + \alpha V'(\alpha z)$$

$$Q(z) = \frac{1}{z} + \frac{1}{z} \frac{\alpha}{N} \sum_i \frac{zV'(\alpha z) - m_i^2 V'(\alpha m_i^2)}{z - m_i^2} .$$

(28)

**Proof:** The proof is a variation on the standard derivation of the quadratic equation obeyed by the resolvent of a hermitian matrix model.

We start by writing the resolvent as

$$\omega_N(z) = \frac{1}{N} \frac{1}{\langle 1 \rangle} \left\langle \text{Tr} \left( \frac{1}{z - M^\dagger M} \right) \right\rangle$$

$$= \frac{1}{N} \left[ \int_0^\infty \prod_k dy_k \right] \sum_i \frac{1}{z - y_i^2} e^{-NF(y_i)}$$

$$= \frac{1}{N} \left[ \int_0^\infty \prod_k dy_k \right] e^{-NF(y_i)} .$$

(29)
By slight abuse of notations we denote the leading contribution (in powers of $\frac{1}{N}$) in eq. (29) also by $\omega_N(z)$. At large $N$ both the numerator and the denominator above are dominated by the same saddle point $y_i = m_i$, with $m_i$ the solution of the equations

$$\frac{\partial}{\partial y_i} F = 2y_i + \alpha 2y_i V'(\alpha y_i^2) - \frac{1}{N} \frac{1}{y_i} - \frac{2}{N} \sum_{j \neq i} \frac{2y_i}{y_i^2 - y_j^2} = 0,$$  \hspace{1cm} (30)

where $V'$ is the formal power series in eq. (16). The resolvent (at leading order in $\frac{1}{N}$) is then

$$\omega_N(z) = \frac{1}{N} \sum_i \frac{1}{z - m_i^2}.$$

(31)

Eq. (25) ensures that all $m_i$ are strictly positive. In order to find the equation for $\omega_N(z)$ we note that any linear combination of the saddle point equations is zero, hence

$$0 = \frac{1}{N} \sum_i \frac{1}{z - m_i^2} \frac{1}{2m_i} \left( \frac{\partial}{\partial y_i} F \right)_{y_i = m_i}$$

(32)

$$= \frac{1}{N} \sum_i \frac{1}{z - m_i^2} \left( 1 + \alpha V'(\alpha m_i^2) - \frac{1}{2N} \frac{1}{m_i^2} - \frac{2}{N} \sum_{j \neq i} \frac{1}{m_i^2 - m_j^2} \right).$$

In the case of a hermitian matrix model one employs a similar trick, but without dividing by $2m_i$. Each term in the last line of eq. (32) receives a different treatment. The first term (also present for hermitian matrices) is identified as

$$\frac{1}{N} \sum_i \frac{1}{z - m_i^2} = \omega_N(z).$$

(33)

The second term (again present for hermitian matrices) rewrites by adding and subtracting $\alpha V'(\alpha z)$ as

$$\frac{1}{N} \sum_i \frac{1}{z - m_i^2} \left( \alpha V'(\alpha m_i^2) - \alpha V'(\alpha z) + \alpha V'(\alpha z) \right)$$

$$= \alpha V'(\alpha z) \omega_N(z) + \frac{1}{N} \sum_i \frac{\alpha V'(\alpha m_i^2) - \alpha V'(\alpha z)}{z - m_i^2}.$$

(34)
The third term receives a similar treatment namely it is rewritten as

\[- \frac{1}{N^2} \sum_i \frac{1}{z - y_i^2} \left( \frac{1}{y_i^2} - \frac{1}{z} + \frac{1}{z} \right) = - \frac{1}{2N} \frac{1}{z} \omega_N(z) + \frac{1}{2N} \frac{1}{z} \omega_N(0). \tag{35} \]

This is a new term, not present in the hermitian matrices case. Finally the last term computes (as for of hermitian matrices) to

\[- \frac{2}{N^2} \sum_i \sum_{j \neq i} \frac{1}{m_i^2 - m_j^2} \frac{1}{z - m_i^2} = - \left( \omega_N(z)^2 + \frac{1}{N} \frac{d\omega_N(z)}{dz} \right). \tag{36} \]

Substituting everything into eq. (32) yields

\[0 = \omega_N(z) + \alpha V'(\alpha z) \omega_N(z) + \frac{\alpha}{N} \sum_i \frac{V'(\alpha m_i^2) - V'(\alpha z)}{z - m_i^2} \]
\[\quad - \frac{1}{2N} \frac{1}{z} \omega_N(z) + \frac{1}{2N} \frac{1}{z} \omega_N(0) - \left( \omega_N(z)^2 + \frac{1}{N} \frac{d\omega_N(z)}{dz} \right). \tag{37} \]

The new terms of equation (35) prevent us from taking \( N \to \infty \) at this stage: although \( \omega_N(0) \) is well defined for any finite \( N \), its limit \( \omega(0) \) is ill defined hence one can not conclude anything about the limit of \( \frac{1}{N} \omega_N(0) \). To get around this problem we use again the saddle point equations (30) to write

\[0 = \frac{1}{N} \sum_i \frac{1}{2m_i} \partial_i F(m_i) = 1 + \frac{\alpha}{N} \sum_i V'(\alpha m_i^2) + \frac{1}{2N} \omega_N(0). \tag{38} \]

Substituting eq. (38) into eq. (37) yields

\[0 = - \frac{1}{N} \frac{d\omega_N(z)}{dz} - \omega_N(z)^2 + \omega_N(z) - \frac{1}{2N} \frac{1}{z} \omega_N(z) + \alpha V'(\alpha z) \omega_N(z) \]
\[\quad + \frac{\alpha}{N} \sum_i \frac{V'(\alpha y_i^2) - V'(\alpha z)}{z - m_i^2} + \frac{1}{z} \left[ - \frac{\alpha}{N} \sum_i V'(\alpha m_i^2) - 1 \right], \tag{39} \]

which rewrites after rearranging the terms as

\[0 = - \frac{1}{N} \frac{d\omega_N(z)}{dz} - \frac{1}{2N} \frac{1}{z} \omega_N(z) - \omega_N(z)^2 + \left( 1 + \alpha V'(\alpha z) \right) \omega_N(z) - \frac{1}{z} \]
\[\quad + \frac{\alpha}{N} \sum_i \left( \frac{V'(\alpha m_i^2) - V'(\alpha z)}{z - m_i^2} - \frac{1}{z} V'(\alpha m_i^2) \right). \tag{40} \]
In the large $N$ limit the first two terms vanish and a straightforward com-
putation proves the lemma \[1\] \[\square\]

Recall that $\omega(z) = \sum_p \frac{1}{z^{p+1}} d_p$, therefore lemma \[1\] can be translated into
an equation for the coefficients $d_p$. In the reminder of this section we will
show that this equation is exactly our diagrammatic equation.

The quadratic equation \[27\] for $\omega(z)$ can be written as
\[\omega(z) - \frac{1}{z} - \omega^2(z) = -\alpha V'(\alpha z) \omega(z) + \frac{1}{z} \frac{\alpha}{N} \sum_i \frac{z V'(\alpha z) - m_i^2 V'(\alpha m_i^2)}{z - m_i^2},\]
and substituting $\omega(z) = \sum_{p=0}^{\infty} z^{-p-1} d_p$, the left hand side becomes
\[\sum_{p=0}^{\infty} \frac{1}{z^{p+1}} d_p - \frac{1}{z} - \sum_{p=0}^{\infty} \frac{1}{z^{p+q+2}} d_p d_q\]
\[= \sum_{q=0}^{\infty} \frac{1}{z^{q+2}} (d_{q+1} - \sum_{k=0}^{q} d_k d_{q-k}).\] \[42\]

The right hands side in eq. \[41\] needs a bit more work. We first express
the coefficients $d_p$ in terms of the saddle point solution $m_i$ as
\[\omega(z) = \sum_{p=0}^{\infty} \frac{1}{z^{p+1}} d_p = \frac{1}{N} \sum_i \frac{1}{z - m_i^2} \Rightarrow d_p = \frac{1}{N} \sum_i (m_i^2)^p,\]
and substituting $V'$ from eq. \[16\], the second term on the right hand side of
\[41\] becomes
\[-\frac{1}{z} \frac{\alpha}{N} \sum_i \sum_{n=1}^{\infty} g_n \alpha^{n-1} z^n - (m_i^2)^n \frac{z - m_i^2}{z - m_i^2} = -\frac{1}{z} \sum_{n=1}^{\infty} g_n \alpha^n \sum_{t=0}^{n-1} z^{n-1-t} \frac{1}{N} \sum_i (m_i^2)^t\]
\[= -\frac{1}{z} \sum_{n=1}^{\infty} g_n \alpha^n \sum_{t=0}^{n-1} z^{n-1-t} d_t.\] \[44\]
Using again the definition of $V'$, the right hand side of (41) becomes

$$\alpha \sum_{n=1}^{\infty} g_n \alpha^{n-1} z^{n-1} \sum_{t=0}^{\infty} \frac{d_t}{z^{t+1}} = \sum_{n=1}^{\infty} g_n z^{n-1} \alpha^{n} \sum_{t=0}^{\infty} \frac{d_t}{z^{t+1}}$$

$$= \sum_{n=1}^{\infty} g_n z^{n-1} \alpha^{n} \sum_{t=n}^{\infty} \frac{d_t}{z^{t+1}} = \sum_{n=1}^{\infty} g_n z^{n-1} \alpha^{n} \sum_{q=0}^{\infty} \frac{d_{q+n}}{z^{q+n+1}}$$

$$= \sum_{q=0}^{\infty} \frac{1}{z^{q+2}} \sum_{n=1}^{\infty} g_n d_{q+n} \alpha^{n}.$$ (45)

Equating the coefficients of $z^{-q-2}$ in eq. (42) and (45) yields

$$d_{q+1} = \sum_{k=0}^{q} d_k d_{q-k} + \sum_{n=1}^{\infty} g_n \alpha^{n} d_{q+n}.$$ (46)

which is exactly our diagrammatic equations (19).

A selfconsistency check is provided by the free model ($\alpha = 0$). In this case eq. (27) writes

$$\omega(z)^2 - \omega(z) + \frac{1}{z} = 0 \Rightarrow \omega(z) = \frac{1}{2} \left(1 - \sqrt{1 - \frac{4}{z^2}}\right),$$ (47)

as $\omega(z) \sim z \rightarrow \infty z^{-1}$. The eigenvalues distribution of the free model reads

$$\tilde{\rho}(\lambda^2) = \frac{1}{2\pi i} [\omega(\lambda^2 + i0) - \omega(\lambda^2 - i0)] = \frac{1}{N} \sum \delta(\lambda^2 - m_i^2)$$

$$= \tilde{\rho}(\lambda^2) = \frac{1}{2\pi \lambda} \sqrt{4 - \lambda^2},$$ (48)

and we recover Wigner’s semicircle law

$$\rho(\lambda)d\lambda = \tilde{\rho}(\lambda^2)d(\lambda^2) = \frac{1}{\pi} \sqrt{4 - \lambda^2} d\lambda.$$ (49)

5 The Perturbative Solution

In this section we will solve our diagrammatic equation in perturbations. We start by the free model, and subsequently work our way to the solution of a generic model.
Lemma 2. The solution \( d_p^{(0)} \) of the free model (equations (18)) is

\[
d_k^{(0)} = \frac{1}{k+1} C_k^{2k}, \quad \forall k.
\]

Proof: A direct computation shows that \( d_0^{(0)} = 1 \). We proceed by induction. Suppose that

\[
d_k^{(0)} = \frac{1}{k+1} C_k^{2k}, \quad \forall k \leq q.
\]

Eq. (18) then reads

\[
d_{q+1}^{(0)} = \sum_{k=0}^{q} \frac{1}{k+1} \frac{1}{q-k+1} C_k^{2k} C_{q-k}^{2q-2k}
\]

\[
= \frac{1}{q+2} \sum_{k=0}^{q} \left( \frac{1}{k+1} + \frac{1}{q-k+1} \right) C_k^{2k} C_{q-k}^{2q-2k}
\]

\[
= \frac{1}{q+2} \sum_{k=0}^{q} \frac{2}{k+1} C_k^{2k} C_{q-k}^{2q-2k},
\]

where in the second line a change of the dummy variable identifies the two terms. Separating the term \( q = k \) in the sum gives

\[
d_{q+1}^{(0)} = \frac{1}{q+2} \left( \frac{2}{q+1} C_q^{2q} + \sum_{k=0}^{q-1} \frac{2}{k+1} C_k^{2k} C_{q-k}^{2q-2k} \right)
\]

\[
= \frac{1}{q+2} \left( \frac{2}{q+1} C_q^{2q} + \sum_{k=0}^{q-1} \frac{2}{k+1} C_k^{2k} \left( 4 - \frac{2}{q-k} \right) C_{q-1-k}^{2q-2k-2k} \right),
\]

which further computes to

\[
d_{q+1}^{(0)} = \frac{1}{q+2} \left( \frac{2}{q+1} C_q^{2q} + 4 \sum_{k=0}^{q-1} \frac{2}{k+1} C_k^{2k} C_{q-1-k}^{2q-2k-2k} \\
- 4 \sum_{k=0}^{q-1} \frac{1}{k+1} C_k^{2k} \frac{1}{(q-1)-k+1} C_{q-1-k}^{2q-2k-2k} \right).
\]
Taking into account the first and third lines in eq. (52) allows us to write

\[
d_{q+1}^{(0)} = \frac{1}{q+2} \left( \frac{2}{q+1} C_{2q}^2 + 4(q+1) d_q^{(0)} - 4d_q^{(0)} \right)
\]

\[
= \frac{1}{q+2} \left( \frac{2}{q+1} C_{2q}^2 + 4 \cdot \frac{q}{q+1} C_{2q}^2 \right) = \frac{1}{q+2} C_{q+1}^{2q+2} . \quad (54)
\]

We now turn our attention to the interacting model. We will solve the equation (19) in perturbations around the \( \alpha = 0 \) solution we just derived. We substitute

\[
d_q = \sum_i d_q^{(i)} \alpha^i , \quad d_q^{(0)} = \frac{1}{q+1} C_{2q}^2 , \quad d_q^{(i)} = 0, i \geq 1 , \quad (55)
\]

in eq. (19) to get

\[
\sum_{s=0}^{\infty} d_{q+1}^{(s)} \alpha^s = \sum_{k=0}^{q} \sum_{s=0}^{\infty} \left( \sum_{i=0}^{s} d_k^{(i)} d_{q-k}^{(s-i)} \right) \alpha^s + \sum_{s=1}^{\infty} \sum_{n=1}^{s} g_n d_{q+n}^{(s-n)} \alpha^s . \quad (56)
\]

Equating the powers of \( s \) on the left hand side and right hand side yields

\[
d_{q+1}^{(s)} = \sum_{k=0}^{q} \sum_{i=0}^{s} d_k^{(i)} d_{q-k}^{(s-i)} + \sum_{n=1}^{s} g_n d_{q+n}^{(s-n)} \quad \forall s \geq 1 . \quad (57)
\]

This equation mixes all orders of perturbation from (0) to (s). Solving it comes to write the terms at order (s) as functions of terms at lower orders. To this end we define

\[
T_{q+1}^{(1)} = g_1 d_{q+1}^{(0)}
\]

\[
T_{q+1}^{(s)} = \sum_{n=1}^{s} g_n d_{q+n}^{(s-n)} + \sum_{k=0}^{q} \sum_{i=1}^{s-1} d_k^{(i)} d_{q-k}^{(s-i)} , \quad s \geq 2 , \quad (58)
\]

such that at the order (s), \( T^{(s)} \) depends only on lower orders \( d^{(0)} \) to \( d^{(s-1)} \).

**Lemma 3.** The solution of (57) at the order of perturbations (s) is

\[
d_{q+1}^{(s)} = \sum_{p=0}^{q} C_{q-p}^{2q-2p} T_{p+1}^{(s)} . \quad (59)
\]
**Proof:** The proof of this lemma is somewhat technical. We start by inserting the definition of $T^{(s)}$ in eq. (57) to get (for $s \geq 1$)

$$d_{q+1}^{(s)} = T_{q+1}^{(s)} + 2 \sum_{k=0}^{q} d_{q-k}^{(0)} d_k^{(s)}.$$  \hspace{1cm} (60)

We will solve this equation as a recursion over $q$. However note that the right hand side involves all $d_k^{(s)}$ with $k < q + 1$. We will first derive $d_{q+1}^{(s)}$ as a function of only $d_q^{(s)}$ and $T^{(s)}$. Then we will solve this recursion and find $d_{q+1}^{(s)}$ as a function of only $T^{(s)}$.

**Step 1:** Recalling that $d_0^{(s)} = 0$ for $s \geq 1$, we have

$$d_{q+1}^{(s)} = T_{q+1}^{(s)} + 2 \sum_{k=1}^{q} d_{q-k}^{(0)} d_k^{(s)} = T_{q+1}^{(s)} + 2 \sum_{p=0}^{q-1} d_{q-1-p}^{(0)} d_{p+1}^{(s)}.$$  \hspace{1cm} (61)

Reinserting eq. (60) yields

$$d_{q+1}^{(s)} = T_{q+1}^{(s)} + 2 \sum_{p=0}^{q-1} d_{q-1-p}^{(0)} T_{p+1}^{(s)} + 4 \sum_{p=0}^{q-1} \sum_{t=0}^{p} d_{q-1-p}^{(0)} d_t^{(0)} d_{p-t}^{(s)}$$

$$= T_{q+1}^{(s)} + 2 \sum_{k=1}^{q} d_{q-k}^{(0)} T_k^{(s)} + 4 \sum_{t=0}^{q-1} \sum_{p=t}^{q-1} d_{q-1-p}^{(0)} d_t^{(0)} d_{p-t}^{(s)}$$

$$= T_{q+1}^{(s)} + 2 \sum_{k=1}^{q} d_{q-k}^{(0)} T_k^{(s)} + 4 \sum_{t=0}^{q-1} \sum_{u=0}^{q-1-t} d_{q-1-t-u}^{(0)} d_u^{(0)} d_t^{(s)}.$$  \hspace{1cm} (62)

The sum over $u$ reproduces the right hand side of eq. (18), thus

$$d_{q+1}^{(s)} = T_{q+1}^{(s)} + 2 \sum_{k=1}^{q} d_{q-k}^{(0)} T_k^{(s)} + 4 \sum_{t=0}^{q-1} d_{q-t}^{(0)} d_t^{(s)}.$$  \hspace{1cm} (63)

Using $d_0^{(0)} = 1$ and again eq. (60), we have

$$d_{q+1}^{(s)} = T_{q+1}^{(s)} + 2 \sum_{k=1}^{q} d_{q-k}^{(0)} T_k^{(s)} + 4 \left( \sum_{t=0}^{q} d_{q-t}^{(0)} d_t^{(s)} - d_q^{(s)} \right)$$

$$= T_{q+1}^{(s)} + 2 \sum_{k=1}^{q} d_{q-k}^{(0)} T_k^{(s)} + 4 \left( \frac{d_{q+1}^{(s)} - T_{q+1}^{(s)}}{2} - d_q^{(s)} \right).$$  \hspace{1cm} (64)
and regrouping \( d_{q+1}^{(s)} \) on the left hand side yields

\[
d_{q+1}^{(s)} = 4d_q^{(s)} - 2 \sum_{k=1}^{q} d_{q-k}^{(0)} T_k^{(s)} + T_{q+1}^{(s)}. \tag{65}
\]

Our first objective is now achieved, as \( d_{q+1}^{(s)} \) is expressed solely in terms of \( d_q^{(s)} \) and \( T^{(s)} \).

**Step 2:** Reinserting eq. (65) into itself \( p \) times we find

\[
d_{q+1}^{(s)} = 4^{p+1} d_{q-p}^{(s)} - \sum_{k=1}^{q-p} \left( \sum_{s=0}^{p} 2 \cdot 4^s d_{q-k-s}^{(0)} \right) T_k^{(s)} + T_{q+1}^{(s)}. \tag{66}
\]

Using \( d_1^{(s)} = T_1^{(s)} \), setting \( p = q - 1 \) and grouping the first two terms yields

\[
d_{q+1}^{(s)} = \sum_{k=1}^{q} \left( 4^{q+1-k} - \sum_{s=0}^{q-k} 2 \cdot 4^s d_{q-k-s}^{(0)} \right) T_k^{(s)} + T_{q+1}^{(s)}. \tag{67}
\]

By now we have expressed \( d^{(s)} \) only in terms of \( T^{(s)} \). To simplify this expression, we use the result of appendix A

\[
S_q = \sum_{k=0}^{q} \frac{1}{4k+1} d_k^{(0)} = \frac{1}{2} \left( 1 - \frac{1}{4q+1} C_{q+1}^{2q+2} \right), \tag{68}
\]

and write

\[
d_{q+1}^{(s)} = \sum_{k=1}^{q} \left( 4^{q+1-k} - 2 \cdot 4^q S_{q-k} \right) T_k^{(s)} + T_{q+1}^{(s)} \tag{69}
\]

\[
= \sum_{k=1}^{q} \left[ 4^{q+1-k} - 2 \cdot 4^q \frac{1}{2} \left( 1 - C_{q-k}^{2q-2k+2} \frac{1}{4q+1} \right) \right] T_k^{(s)} + T_{q+1}^{(s)},
\]

and finally

\[
d_{q+1}^{(s)} = \sum_{k=1}^{q} C_{q-k+1}^{2q-2k+2} T_k^{(s)} + T_{q+1}^{(s)} = \sum_{p=0}^{q} C_{q-p}^{2q-2p} T_{p+1}^{(s)}. \tag{70}
\]
For practical computations it is useful to recast the perturbative solution of lemma 3 in terms of generating functions.

### 5.1 Generating Functions

Let us define the generating functions at order $(s)$

$$\omega^{(s)}(z) = \sum_{q=0}^{\infty} \frac{1}{z^{q+1}} d^{(s)}_q, \quad T^{(s)}(z) = \sum_{q=0}^{\infty} \frac{1}{z^{q+1}} T^{(s)}_q. \quad (71)$$

where we set $T^{(s)}_0 = 0$ for all $s \geq 1$. The solution of the non-interacting model of lemma 2 gives us the generating function at order zero

$$\omega^{(0)}(z) = \sum_{q=0}^{\infty} \frac{1}{z^{q+1}} \frac{1}{q+1} C^{2q}_q = \frac{1}{2} \left( 1 - \sqrt{1 - \frac{4}{z}} \right). \quad (72)$$

For all $s \geq 1$ $d^{(s)}_0 = 0$, thus lemma 3 translates for the generating functions at order of perturbations $(s)$ as

$$\omega^{(s)}(z) = \sum_{q=0}^{\infty} \frac{1}{z^{q+1}} \frac{1}{q+1} \sum_{p=0}^{q} C^{2q-2p}_q T^{(s)}_{p+1}$$

$$= \sum_{p=0}^{\infty} \frac{1}{z^{p+2}} T^{(s)}_{p+1} \sum_{q=p}^{\infty} C^{2q-2p}_q \frac{1}{z^{q-p}} = \frac{1}{\sqrt{1 - \frac{4}{z}}} T^{(s)}(z). \quad (73)$$

The last ingredient one needs is to translate the definition of $T^{(s)}_{q+1}$ (eq. (58)) for generating functions. We first treat the case $s = 1$

$$T^{(1)}(z) = \sum_{q=0}^{\infty} \frac{1}{z^{q+1}} T^{(1)}_q = \sum_{q=0}^{\infty} \frac{1}{z^{q+2}} T^{(1)}_{q+1}$$

$$= g_1 \sum_{q=0}^{\infty} \frac{1}{z^{q+2}} d^{(0)}_{q+1} = g_1 \left( \omega^{0}(z) - \frac{1}{z} \right). \quad (74)$$
Then, for all $s \geq 2$ we have

$$T^{(s)}(z) = \sum_{q=0}^{s} \frac{1}{z^{q+2}} T^{(s)}_{q+1} = \sum_{q=0}^{\infty} \frac{1}{z^{q+2}} \left( \sum_{n=1}^{s} g_n d^{(s-n)}_{q+n} + \sum_{k=0}^{q-1} \sum_{i=1}^{s} d^{(i)}_{k} d^{(s-i)}_{q-k} \right)$$

$$= \sum_{n=1}^{s} z^{n-1} g_n \sum_{q=0}^{\infty} d^{(s-n)}_{q+n} \frac{1}{z^{q+n+1}} + \sum_{i=1}^{s-1} \sum_{q=0}^{\infty} \frac{1}{z^{q+2}} \sum_{k=0}^{q} d^{(i)}_{k} d^{(s-i)}_{q-k} \tag{75}$$

$$= \sum_{n=1}^{s} z^{n-1} g_n \left( \omega^{(s-n)}(z) - \sum_{q=0}^{n-1} \frac{1}{z^{q+1}} d^{(s-n)}_{q} \right) + \sum_{i=1}^{s-1} \omega^{(i)}(z) \omega^{(s-i)}(z).$$

The equations (72), (73), (74) and (75) allow one to compute the generating functions $\omega^{(s)}$ order by order in $(s)$. In appendix B we use them to compute $\omega^{(1)}$, $\omega^{(2)}$ and $\omega^{(3)}$.

### 6 Dualities

In this last section we will exhibit two duality properties of the perturbative solution. In fact $d_p$ is a power series not only in $\alpha$ but also in $g_k$, that is it writes

$$d_p = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_k=0}^{\infty} \ldots \alpha^{(n_1+2n_2+\ldots+kn_k+\ldots)} g_1^{n_1} g_2^{n_2} \ldots g_k^{n_k} \ldots$$

and the coefficient $d^{(n_1+\ldots+kn_k+\ldots)}_{p;g_1^{n_1} \ldots g_k^{n_k} \ldots}$ is the number of unlabeled, planar, orientable graphs having a special vertex of coordination $2p$ and $n_1$ vertices $g_1$, $n_2$ vertices $g_2$ and so on and so fort.

It is clear intuitively that these coefficients are not independent. First, the number of unlabeled graphs with a choice of vertices is a combinatorial quantity and does not depend on which of the vertices is the “special vertex”. Second, the vertices of weight $g_1$ have coordination two thus they can be viewed as decorations on the lines of a simpler graph. We present below the precise duality properties of the coefficients $d^{(n_1+\ldots+kn_k+\ldots)}_{p;g_1^{n_1} \ldots g_k^{n_k} \ldots}$ mirroring these two observations. To this end we start from the definition of $d_p$

$$d_p = \frac{1}{N} \left\langle \text{Tr}[(M^TM)^p] \right\rangle , \tag{77}$$
and denoting \( \langle \ldots \rangle_0 \) the gaussian (\( \alpha = 0 \)) correlation, \( d_p \) writes

\[
d_p = \frac{1}{N} \frac{1}{\langle 1 \rangle} \left\langle \text{Tr}[(M^\dagger M)^p] e^{N \sum_{q=1}^{\infty} \frac{g_q}{q} \text{Tr}[(M^\dagger M)^q]} \right\rangle_{g=0}^\phi,
\]

where the superscript indicates that only planar graphs contribute to the Gaussian average. In the sequel, we only count the planar contributions in all Gaussian averages, but, to simplify notations, we drop the superscript. A perturbative development in all the coupling constants \( g \) yields

\[
d_p = \frac{1}{N} \frac{1}{\langle 1 \rangle} \sum_{n_1, \ldots, n_k, \ldots = 0}^\infty N^{n_1 + \ldots + n_k + \ldots} \alpha^{n_1 + \ldots + kn_k + \ldots} \frac{1}{n_1!} \left( \frac{g_1}{1} \right)^{n_1} \cdots \frac{1}{n_k!} \left( \frac{g_k}{1} \right)^{n_k} \cdots \left\langle \text{Tr}[(M^\dagger M)^n] \text{Tr}[(M^\dagger M)^{n_1} \ldots \text{Tr}[(M^\dagger M)^{n_k} \ldots] \right\rangle_0,
\]

which allows us to identify

\[
d_p^{n_1 + 2n_2 + \ldots + kn_k + \ldots} = \frac{1}{N} \frac{1}{\langle 1 \rangle} \sum_{n_1, \ldots, n_k, \ldots = 0}^\infty N^{n_1 + \ldots + n_k + \ldots} \frac{1}{n_1!n_2! \ldots k!n_k \ldots} \times \left\langle \text{Tr}[(M^\dagger M)^p] \text{Tr}[(M^\dagger M)^{n_1} \ldots \text{Tr}[(M^\dagger M)^{n_k} \ldots] \right\rangle_0.
\]

In the gaussian average above the special vertex does not play any distinguished role,

\[
\left\langle \text{Tr}[(M^\dagger M)^p] \text{Tr}[(M^\dagger M)^{n_1} \ldots \text{Tr}[(M^\dagger M)^{n_k} \ldots] \right\rangle_0 = \left\langle \text{Tr}[(M^\dagger M)^{n_1} \ldots \text{Tr}[(M^\dagger M)^p]^{n_p+1} \ldots \text{Tr}[(M^\dagger M)^k]^{n_k} \ldots] \right\rangle_0.
\]

The only thing which distinguishes between the special vertex and any other vertex is the combinatorial weight. Balancing this combinatorial proves the first duality property

\[
\frac{1}{p} \frac{1}{n_p} d^{n_1 + \ldots + n_p + \ldots + q(n_q+1) + \ldots} = \frac{1}{q} \frac{1}{n_q} d^{n_1 + \ldots + p(n_p+1) + \ldots + q(n_q+1) + \ldots},
\]

for all \( q \) with \( n_q \geq 0 \).
To prove the second duality we introduce the modified Gaussian measure

\[ \int \left[ \prod_{ab} dM_{ab} \right] dM_{ab} e^{-xN \left( \text{Tr}(M^\dagger M) \right)} \] (83)

and we denote the correlations of this measure \( \langle \ldots \rangle_{0;x} \). With the help of the modified measure we can write

\[ d^{(n_1+1+\ldots+n_k+\ldots)}_{p;g_1^{n_1+1} \ldots g_k^{n_k} \ldots} = \frac{1}{N} \frac{1}{\langle 1 \rangle (n_1 + 1)! \ldots n_k!} \frac{1}{1^{n_1+1}2^{n_2} \ldots k^{n_k} \ldots} \left\langle \text{Tr}[(M^\dagger M)^p] \text{Tr}[(M^\dagger M)^{n_1+1} \ldots \text{Tr}[(M^\dagger M)^k]^{n_k} \ldots] \right\rangle_0 \]

\[ = \frac{1}{N} \frac{1}{\langle 1 \rangle (n_1 + 1)! \ldots n_k!} \frac{1}{1^{n_1+1}2^{n_2} \ldots k^{n_k} \ldots} \left( -\frac{1}{N} \frac{d}{dx} \right) \bigg|_{x=1} \left( x^{-(p+\sum_{j=1}^\infty jn_j)} \right) \left\langle \text{Tr}[(M^\dagger M)^p] \text{Tr}[(M^\dagger M)^{n_1} \ldots \text{Tr}[(M^\dagger M)^k]^{n_k} \ldots] \right\rangle_0 . \] (84)

as the derivative acting on the gaussian measure produces an extra \( \text{Tr}(M^\dagger M) \) insertion. But the correlations \( \langle \ldots \rangle_{0;x} \) of the modified gaussian measure are sums over the same graphs as the usual gaussian correlations \( \langle \ldots \rangle_0 \), with the only difference that the lines of the graphs have a weight \( \frac{1}{xN} \) instead of \( \frac{1}{N} \). All graphs contributing to the averages in (84) have the same number of lines, namely \( p + \sum_{j=1}^\infty jn_j \). Hence

\[ d^{(n_1+1+\ldots+n_k+\ldots)}_{p;g_1^{n_1+1} \ldots g_k^{n_k} \ldots} = \frac{1}{N} \frac{1}{\langle 1 \rangle (n_1 + 1)! \ldots n_k!} \frac{1}{1^{n_1+1}2^{n_2} \ldots k^{n_k} \ldots} \left( -\frac{1}{N} \frac{d}{dx} \right) \bigg|_{x=1} \left( x^{-(p+\sum_{j=1}^\infty jn_j)} \right) \left\langle \text{Tr}[(M^\dagger M)^p] \text{Tr}[(M^\dagger M)^{n_1} \ldots \text{Tr}[(M^\dagger M)^k]^{n_k} \ldots] \right\rangle_0 . \] (85)

Tracking again the weight factors one has

\[ d^{(n_1+1+2n_2+\ldots+n_k+\ldots)}_{p;g_1^{n_1+1} \ldots g_k^{n_k} \ldots} = \sum_{j=1}^\infty jn_j + p \frac{d^{(n_1+1+2n_2+\ldots+n_k+\ldots)}}{n_1 + 1} . \] (86)

These dualities are very useful in concrete computations. Using the results of appendix \( \text{B} \), \( d_p \) writes up to third order in \( \alpha \) as

\[ d_q = \frac{1}{q+1} C_q^2 + \alpha g_1 \frac{q}{q+1} C_q^2 + \alpha^2 g_1^2 \frac{q}{2} C_q^2 + \alpha^2 g_2 \frac{3q}{q+2} C_q^2 + \alpha^3 \frac{3q(q+2)}{6} C_q^2 + \alpha^3 g_1 g_2 3q C_q^2 + \alpha^3 g_3 \frac{10q}{q+3} C_q^2 . \] (87)
The first duality \((82)\) implies for instance

\[
\frac{1}{2} d_{2,g_3}^{(3)} = \frac{1}{3} d_{3,g_2}^{(2)} \quad d^{(3)}_{1,g_3} = \frac{1}{3} d_{3,g_1}^{(1)} \quad \frac{1}{2} d_{1,g_1,g_2}^{(3)} = \frac{1}{2} d_{2,g_1}^{(2)} \quad \frac{1}{2} d_{1,g_2}^{(2)} = \frac{1}{4} d_{2,g_1}^{(1)}, \tag{88}
\]

whereas the second duality \((86)\) implies

\[
\begin{align*}
d^{(3)}_{q,g_1,g_2} &= (q+2) d^{(2)}_{q,g_2} \quad d^{(3)}_{q,g_1} = \frac{q+2}{3} d^{(2)}_{q,g_1} \\
d^{(2)}_{q,g_1} &= \frac{q+1}{2} d^{(1)}_{q,g_1} \quad d^{(1)}_{q,g_1} = q d^{(0)}_{q}.
\end{align*} \tag{89}
\]

7 Conclusions

In this paper we introduced a tower of diagrammatic equations for the planar sector of a square non hermitian random matrix model. We solved the equations in perturbations and proved two duality properties of the solution.

Our results can be generalized automatically to rectangular non hermitian random matrix models. The generalizations to hermitian matrix models is a bit more subtle as the latter generate also vertices of odd coordination. We are however confident that one can find corresponding diagrammatic equations related to the quadratic equation obeyed by the resolvent.

Our equation encodes the census of oriented planar diagrams in a very transparent way. Although much more involved, a generalization to graphs of arbitrary genus should be possible.

Another direction of future research is to generalize these equations to group field theories. In order to do this several major points still require clarification. First, although we have a proposition for a generalization of the notion of planarity (type 1 graphs in \([18]\)), we still need a proof that only these graphs dominate the partition function. Second, in higher dimensions, one deals with random tensors, not with random matrices. In order to perform an in-depth saddle point approximation one would ideally need to find an appropriate generalization of the notion of eigenvalues for tensors.

Acknowledgements

The author would like to thank Jan Ambjorn for suggesting references \([12, 13]\) and for very stimulating discussions regarding the loop insertion equation.
Also, the author would like to thank the theoretical physics department at McGill University for hosting him in an impromptu visit at an early stage of this work.

Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation.

### A A simple sum

**Lemma 4.** Let $S_q = \sum_{k=0}^{q} \frac{1}{4^{k+1}} d_k^{(0)}$. Then

$$S_q = \frac{1}{2} \left( 1 - \frac{1}{4^{q+1}} C_{q+1}^{2q+2} \right)$$

(90)

**Proof:** Recall that $d_q^{(0)} = \frac{1}{q+1} C_{q}^{2q}$. We have the Taylor expansion

$$\frac{1}{2} \left( 1 - \sqrt{1 - x} \right) = \sum_{q=0}^{\infty} \frac{1}{4^{q+1}} d_q^{(0)} x^{q+1},$$

(91)

and therefore the generating function of $S_q$ is

$$S(x) = \sum_{q=0}^{\infty} S_q x^{q+1} = \sum_{q=0}^{\infty} \sum_{k=0}^{q} \frac{1}{4^{k+1}} d_k^{(0)} x^{q+1} = \sum_{k=0}^{\infty} \sum_{q=k}^{\infty} \frac{1}{4^{k+1}} d_k^{(0)} x^{q+1}$$

$$= \sum_{k=0}^{\infty} \frac{1}{4^{k+1}} d_k^{(0)} \frac{x^{k+1}}{1-x} = \frac{1}{1-x} \sum_{k=0}^{\infty} \frac{1}{4^{k+1}} d_k^{(0)} x^{k+1}$$

$$= \frac{1}{2} \left( \frac{1}{1-x} - \frac{1}{\sqrt{1-x}} \right).$$

(92)

But

$$\frac{1}{\sqrt{1-x}} = 1 + \sum_{q=0}^{\infty} \frac{1}{4^{q+1}} C_{q+1}^{2q+2} x^{q+1},$$

(93)

hence (90) holds.
B  Perturbative computations

B.1  First order

At first order, eq. (72), (73) and (74) yield

\[ \omega^{(1)}(z) = g_1 \left( \frac{1}{2} \sqrt{1 - \frac{4}{z}} - \frac{1}{2} - \frac{1}{z} \sqrt{1 - \frac{4}{z}} \right), \]  

(94)

and using

\[ \frac{1}{\sqrt{1 - \frac{4}{z}}} = \sum_{q=0}^{\infty} \frac{1}{z^q C_q^{2q}}, \]  

(95)

we have

\[ \omega^{(1)}(z) = g_1 \left( \sum_{q=0}^{\infty} \frac{1}{z^q} \frac{1}{2} C_q^{2q} - \frac{1}{2} - \sum_{q=0}^{\infty} \frac{1}{z^{q+1}} \frac{1}{2} C_q^{2q} \right) \]
\[ = g_1 \sum_{q=0}^{\infty} \frac{1}{z^{q+1}} \left( \frac{1}{2} C_q^{2q+2} - C_q^{2q} \right). \]  

(96)

Therefore

\[ d_q^{(1)} = g_1 \left( \frac{1}{2} \left( \frac{2q + 2}{(q+1)^2} - 1 \right) \right) C_q^{2q} = g_1 \frac{q}{q+1} C_q^{2q}. \]  

(97)

B.2  Second order

At second order we have

\[ T^{(2)}(z) = g_1 \left( \omega^{(1)}(z) - d_0^{(1)} \right) + g_2 z \left( \omega^{(0)}(z) - \frac{d_0^{(0)}}{z} - \frac{d_1^{(0)}}{z^2} \right) \]
\[ + \omega^{(1)}(z) \omega^{(1)}(z), \]  

(98)

which simplifies to

\[ T^{(2)}(z) = g_1 \omega^{(1)}(z) + g_2 z \left( \omega^{(0)}(z) - \frac{1}{z} - \frac{1}{z^2} \right) + \omega^{(1)}(z) \omega^{(1)}(z). \]  

(99)
We group together the terms containing \( g_1 \)

\[
g_1 \omega^{(1)}(z) + [\omega^{(1)}(z)]^2,
\]

which compute to

\[
g_1^2 \left( \frac{1}{2\sqrt{1 - \frac{4}{z}}} - \frac{1}{z\sqrt{1 - \frac{4}{z}}} - \frac{1}{2} \right) \left( \frac{1}{2\sqrt{1 - \frac{4}{z}}} - \frac{1}{z\sqrt{1 - \frac{4}{z}}} + \frac{1}{2} \right) = g_1^2 \left[ \frac{1}{1 - \frac{4}{z}} - \frac{1}{4} \right] = g_1^2 \frac{1}{z^2(1 - \frac{4}{z})},
\]

thus

\[
\omega^{(2)}(z) = g_1^2 \frac{1}{z^2(1 - \frac{4}{z})^{3/2}} + g_2 z \left( \frac{1}{2\sqrt{1 - \frac{4}{z}}} - \frac{1}{2} \right) - \frac{1}{z\sqrt{1 - \frac{4}{z}}} - \frac{1}{z^2\sqrt{1 - \frac{4}{z}}}.
\]

Using

\[
\frac{1}{(1 - \frac{4}{z})^{3/2}} = \sum_{q=0}^{\infty} \frac{q + 1}{2} C_{q+1}^{2q+2} \frac{1}{z^q} = \sum_{t=1}^{\infty} \frac{t}{2} C_{t-1}^{2t} \frac{1}{z^{t-1}},
\]

\( \omega^{(2)}(z) \) writes

\[
g_1^2 \sum_{q=1}^{\infty} \frac{q}{2} C_q^{2q} \frac{1}{z^{q+1}} + g_2 \left( \sum_{q=0}^{\infty} \frac{1}{2} C_q^{q+1} \frac{1}{z^{q+1}} - \frac{z}{2} - \sum_{q=0}^{\infty} C_q^{2q} \frac{1}{z^{q+1}} - \sum_{q=0}^{\infty} C_q^{2q} \frac{1}{z^{q+1}} \right).
\]

A direct computation shows that the terms \( z, z^0, z^{-1} \) cancel. Shifting each sum by the appropriate amount yields

\[
g_1^2 \sum_{q=0}^{\infty} \frac{q}{2} C_q^{2q} \frac{1}{z^{q+1}} + g_2 \left( \sum_{q=0}^{\infty} \frac{1}{2} C_q^{2q+4} \frac{1}{z^{q+1}} - \sum_{q=0}^{\infty} C_q^{2q+2} \frac{1}{z^{q+1}} - \sum_{q=0}^{\infty} C_q^{2q} \frac{1}{z^{q+1}} \right),
\]
and we conclude
\[
d_q^{(2)} = g_2 \left( \frac{1}{2} C_{q+2}^{2q+4} - C_{q+1}^{2q+2} - C_q^{2q} \right) + g_1^2 \frac{q}{2} C_q^{2q} \\
= g_1^2 \frac{q}{2} C_q^{2q} + g_2 \frac{3q}{q + 2} C_q^{2q}.
\] (106)

**B.3 Third order**

To the third order we have
\[
T^{(3)}(z) = g_1 \omega^{(2)}(z) + g_2 z \left( \omega^{(1)}(z) - \frac{g_1}{z^2} \right) + g_3 z^2 \left( \omega^{(0)}(z) - \frac{1}{z} - \frac{1}{z^2} - \frac{2}{z^3} \right) + 2 \omega^{(1)} \omega^{(2)}.
\] (107)

We start by computing
\[
g_1 + 2 \omega^{(1)} = g_1 \frac{1}{\left(1 - \frac{4}{z}\right)^{1/2}} \left(1 - \frac{2}{z}\right),
\] (108)

therefore, factoring $\omega^{(2)}$ in the first and last term of (107) yields
\[
T^{(3)}(z) = g_1 g_2 \left( \frac{z}{2 \sqrt{1 - \frac{4}{z}}} - \frac{z}{2} - \frac{1}{\sqrt{1 - \frac{4}{z}}} - \frac{1}{z} \right) + \\
+ g_3 \left( \frac{z^2}{2} - \frac{z^2}{2} \sqrt{1 - \frac{4}{z}} - z - 1 - \frac{2}{z} \right) + \\
+ g_1 \frac{1}{\left(1 - \frac{4}{z}\right)^{1/2}} \left(1 - \frac{2}{z}\right) \left[ g_1^2 \frac{1}{z^2} \left(1 - \frac{4}{z}\right)^{3/2} \right] \\
+ g_2 z \left( \frac{1}{2 \sqrt{1 - \frac{4}{z}}} - \frac{1}{2} - \frac{1}{z \sqrt{1 - \frac{4}{z}}} - \frac{1}{z^2 \sqrt{1 - \frac{4}{z}}} \right). 
\] (109)

The coefficient of $g_1 g_2$ is
\[
\frac{z}{2 \sqrt{1 - \frac{4}{z}}} - \frac{z}{2} - \frac{1}{\sqrt{1 - \frac{4}{z}}} - \frac{1}{z} + \frac{1}{\left(1 - \frac{4}{z}\right)^{1/2}} \left(1 - \frac{2}{z}\right) \\
\times \left( \frac{z}{2 \sqrt{1 - \frac{4}{z}}} - \frac{z}{2} - \frac{1}{\sqrt{1 - \frac{4}{z}}} - \frac{1}{z \sqrt{1 - \frac{4}{z}}} \right),
\] (110)
which, after a tedious but straightforward computation, simplifies to
\[
\frac{6}{z^2} \frac{1}{1 - \frac{4}{z}}.
\] (111)

We can now write \(\omega^{(3)}\) as
\[
\omega^{(3)} = g_1^3 \frac{1}{z^2 \left(1 - \frac{4}{z}\right)^{5/2}} \left(1 - \frac{2}{z}\right)
\]
\[+\]
\[
6 g_1 g_2 \frac{1}{z^2 \left(1 - \frac{4}{z}\right)^{3/2}}
\]
\[+\]
\[
g_3 \frac{z^2}{\sqrt{1 - \frac{4}{z}}} \left(\frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4}{z} - \frac{1}{z} - \frac{1}{z^2} - \frac{2}{z^3}}\right).
\] (112)

Using
\[
\frac{1}{\left(1 - \frac{4}{z}\right)^{5/2}} = \sum_{t=0}^{\infty} \frac{(t+2)(t+1)}{12} C_{t+2}^4 \frac{1}{z^t},
\] (113)

the coefficient of \(g_3^2\) in (112) is
\[
\sum_{t=0}^{\infty} \frac{(t+2)(t+1)}{12} C_{t+2}^4 \frac{1}{z^t+2} - \sum_{t=0}^{\infty} \frac{(t+2)(t+1)}{6} C_{t+2}^4 \frac{1}{z^{t+3}}
\] (114)
\[
= \frac{1}{6} C_2^4 \frac{1}{z^2} + \sum_{q=2}^{\infty} \left(\frac{q+1}{12} C_{q+2}^{2q+2} - \frac{q(q-1)}{6} C_{q}^{2q}\right) \frac{1}{z^{q+1}}
\]
\[
= \sum_{q=0}^{\infty} \frac{q(q+2)}{6} C_{q}^{2q} \frac{1}{z^{q+1}},
\]

while the coefficient of \(g_1 g_2\) in (112) computes
\[
\frac{6}{z^2} \frac{1}{\left(1 - \frac{4}{z}\right)^{3/2}} = \sum_{q=0}^{\infty} \frac{6 q C_{q}^{2q+1}}{2} \frac{1}{z^{q+1}}.
\] (115)

Finally the coefficient of \(g_3\) in (112),
\[
\frac{z^2}{2 \sqrt{1 - \frac{4}{z}}} - \frac{z^2}{2} - \frac{1}{z} \sqrt{1 - \frac{4}{z}} - \frac{1}{z} \sqrt{1 - \frac{4}{z}} - \frac{2}{z} \sqrt{1 - \frac{4}{z}},
\] (116)
has the Taylor development
\[ -\frac{z^2}{2} + \sum_{q=0}^{\infty} \left( \frac{1}{2} C_{q}^{2q} \frac{1}{z^{q-2}} - C_{q}^{2q} \frac{1}{z^{q-1}} - C_{q}^{2q} \frac{1}{z^{q}} - 2C_{q}^{2q} \frac{1}{z^{q+1}} \right). \] (117)

Again the terms \(z^2, z, z^0\) and \(z^{-1}\) cancel and shifting each sum by the appropriate amount, eq. (117) writes
\[
\sum_{q=0}^{\infty} \left( \frac{1}{2} C_{q+3}^{2q+6} - C_{q+2}^{2q+4} - C_{q+1}^{2q+2} - 2C_{q}^{2q} \right) \frac{1}{z^{q+1}} = \sum_{q=0}^{\infty} \frac{10q}{q+3} C_{q}^{2q} \frac{1}{z^{q+1}} . \] (118)

Thus, the third order in perturbations is
\[
d_{q}^{(3)} = g_1 q(q + 2) \frac{6}{C_{q}^{2q}} + g_1 g_2 3q C_{q}^{2q} + g_3 \frac{10q}{q+3} C_{q}^{2q} . \] (119)

References

[1] T. Guhr, A. Muller-Groeling and H. A. Weidenmuller, “Random matrix theories in quantum physics: Common concepts,” Phys. Rept. 299, 189 (1998) [arXiv:cond-mat/9707301].

[2] P. Di Francesco, P. H. Ginsparg and J. Zinn-Justin, “2-D Gravity and random matrices,” Phys. Rept. 254, 1 (1995) [arXiv:hep-th/9306153].

[3] E. Brezin, C. Itzykson, G. Parisi and J. B. Zuber, “Planar Diagrams,” Commun. Math. Phys. 59, 35 (1978).

[4] J. Bouttier, P. Di Francesco and E. Guitter, “Census of Planar Maps: From the One-Matrix Model Solution to a Combinatorial Proof,” Nucl. Phys. B 645, 477 (2002) [arXiv:cond-mat/0207682].

[5] P. Di Francesco, “Rectangular Matrix Models and Combinatorics of Colored Graphs,” Nucl. Phys. B 648, 461 (2003) [arXiv:cond-mat/0208037].

[6] P. Zinn-Justin and J. B. Zuber, “Matrix Integrals and the Generation and Counting of Virtual Tangles and Links,” J. Knot Theor. Ramifications 13, 325 (2004) [arXiv:math-ph/0303049].

[7] E. Brezin and J. Zinn-Justin, “Renormalization group approach to matrix models,” Phys. Lett. B 288, 54 (1992) [arXiv:hep-th/9206035].
[8] H. Grosse and R. Wulkenhaar, “Renormalisation of phi**4 theory on noncommutative R**4 in the matrix base,” Commun. Math. Phys. 256, 305 (2005) [arXiv:hep-th/0401128].

[9] H. Grosse and R. Wulkenhaar, “Power-counting theorem for non-local matrix models and renormalisation,” Commun. Math. Phys. 254, 91 (2005) [arXiv:hep-th/0305066].

[10] M. Disertori, R. Gurau, J. Magnen and V. Rivasseau, “Vanishing of beta function of non commutative phi(4)**4 theory to all orders,” Phys. Lett. B 649, (2007) 95 [arXiv:hep-th/0612251].

[11] J. B. Geloun, R. Gurau and V. Rivasseau, “Vanishing beta function for Grosse-Wulkenhaar model in a magnetic field,” Phys. Lett. B 671, 284 (2009) [arXiv:0805.4362 [hep-th]].

[12] J. Ambjorn, C. F. Kristjansen and Yu. M. Makeenko, “Higher Genus Correlators For The Complex Matrix Model,” Mod. Phys. Lett. A 7 (1992) 3187 [arXiv:hep-th/9207020].

[13] J. Ambjorn, L. Chekhov, C. F. Kristjansen and Yu. Makeenko, “Matrix model calculations beyond the spherical limit,” Nucl. Phys. B 404 (1993) 127 [Erratum-ibid. B 449 (1995) 681] [arXiv:hep-th/9302014].

[14] L. Freidel, “Group field theory: An overview,” Int. J. Theor. Phys. 44, 1769 (2005) [arXiv:hep-th/0505016].

[15] D. Oriti, in Quantum Gravity, B. Fauser, J. Tolksdorf and E. Zeidler, eds., Birkhaeuser, Basel, (2007), [arXiv: gr-qc/0512103]

[16] R. Gurau, “Colored Group Field Theory”, [arXiv:0907.2582 [hep-th]].

[17] R. Gurau, “Topological Graph Polynomials in Colored Group Field Theory,” arXiv:0911.1945 [hep-th].

[18] L. Freidel, R. Gurau and D. Oriti, “Group field theory renormalization - the 3d case: power counting of divergences,” Phys. Rev. D 80, 044007 (2009) [arXiv:0905.3772 [hep-th]].

[19] J. Magnen, K. Noui, V. Rivasseau and M. Smerlak, “Scaling behaviour of three-dimensional group field theory,” Class. Quant. Grav. 26, 185012 (2009) [arXiv:0906.5477 [hep-th]].
[20] J. B. Geloun, T. Krajewski, J. Magnen and V. Rivasseau, “Linearized Group Field Theory and Power Counting Theorems,” arXiv:1002.3592 [hep-th].

[21] M. Salmhofer, Renormalization: An Introduction, Springer; ISBN-10: 3540646663, ISBN-13: 978-3540646662

[22] J. Polchinski, “Renormalization And Effective Lagrangians,” Nucl. Phys. B 231, 269 (1984).

[23] T. R. Morris, “Checkered surfaces and complex matrices,” Nucl. Phys. B 356, 703 (1991).