Funnel-based Reachability Control of Unknown Nonlinear Systems using Gaussian Processes

Sandeep Gorantla1,†, Jeel Chatrola1,†, Jay Bhagiya1,†, Adnane Saoud2, and Pushpak Jagtap1

Abstract—This paper aims to synthesize a reachability controller for an unknown dynamical system. We first learn the unknown system using Gaussian processes and the (probabilistic) guarantee on the learned model. Then we use the funnel-based controller synthesis approach using this approximated dynamical system to design the controller for a reachability specification. Finally, the merits of the proposed method are shown using a numerical example.

I. INTRODUCTION

Existing controller synthesis approaches generally rely on a mathematical model of the system, such as physics-based first principal models. The formal guarantees provided by the synthesized controller are valid as long as the considered dynamical model is accurate. When dealing with complex dynamical systems, describing the system in a closed-form model is often complicated. In this case, a common practice is to resort to data-driven techniques.

The Gaussian process is a non-parametric learning-based approach that provides probabilistic approach to approximate and to synthesize controllers for unknown systems [1]. There are several works that utilize GPs for providing MPC scheme [2], adaptive control [3], tracking control [4], backstepping control [5], feedback linearization [6], safe optimization of controller [7], reinforcement learning [8], and control barrier functions for safety specification [9].

This work will consider the controller synthesis problem for reachability specification for unknown dynamical systems. In the past few decades, there have been several works in the literature addressing reachability problem (see [10], [11], [12], [13], [14]) for known dynamical systems. To solve this problem, we leverage the funnel-based control approaches [15] that have been extensively used for controlling systems with prescribed performance constraints (see [16] and references therein for examples). We first employ the Gaussian process learning to approximate the system dynamics using the noisy measurements along a probabilistic bound on approximation. Then, the synthesis of a closed-form funnel-based control law that ensures the satisfaction of the reachability specification, with a given confidence, using the learned dynamics from the GP model, is presented. Finally, we show the validity of our approach using a numerical example.

The organization of the paper is as follows. Section II introduces some notations and present the main problem addressed in the paper. Section III explains how Gaussian processes make it possible to learn unknown dynamical systems along with statistical guarantees. Section IV presents a solution to the reachability problem by combining learned Gaussian processes with funnel-based control techniques. Finally, Section V presents numerical results validating the merits of the proposed approach.

II. PROBLEM FORMULATION

A. Notations

The set of real, positive real, nonnegative real, and positive integer numbers are represented using \( \mathbb{R} \), \( \mathbb{R}^+ \), \( \mathbb{R}_0^+ \), and \( \mathbb{N} \), respectively. \( \mathbb{R}^p \) denotes \( p \)-dimensional Euclidean space and \( \mathbb{R}^{p \times q} \) denotes a space of real matrices with \( p \) rows and \( q \) columns. A diagonal matrix in \( \mathbb{R}^{p \times p} \) with diagonal entries \( d_1, \ldots, d_p \) is denoted by \( \text{diag}(d_1, \ldots, d_p) \). Given a matrix \( M \in \mathbb{R}^{p \times q} \), \( M^T \) represents transpose of matrix \( M \). For a vector \( x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \), we denote \( \text{sign}(x) = [\text{sign}(x_1), \ldots, \text{sign}(x_n)]^T \), where \( \text{sign}(x_i) = \begin{cases} -1 & \text{if } x_i < 0 \\ 1 & \text{if } x_i \geq 0 \end{cases} \). We use \( I_p \) to represent the identity matrix in \( \mathbb{R}^{p \times p} \).

For \( a, b \in \mathbb{R} \) and \( a < b \), we use \((a, b)\) and \([a, b]\) to represent open and close intervals in \( \mathbb{R} \), respectively. For a function \( f : \mathbb{R}^p \to \mathbb{R}^q \), \( f_i : \mathbb{R}^p \to \mathbb{R} \), \( i \in \{1, 2, \ldots, p\} \) denotes the \( i \)-th component of \( f \). Consider a set \( X_a \subset \mathbb{R}^p \), its projection on \( i \)-th dimension, where \( i \in \{1, \ldots, p\} \), is given by an interval \( [X_{ai}^-, X_{ai}^+] \subset \mathbb{R} \), where \( X_{ai}^- := \min\{x_i \in \mathbb{R} \mid [x_1, x_2, \ldots, x_p] \in X_a\} \), \( X_{ai}^+ := \max\{x_i \in \mathbb{R} \mid [x_1, x_2, \ldots, x_p] \in X_a\} \), and \( \text{Int}(X_a) \) denotes interior of set \( X_a \). \( N(m, C) \) denotes multivariate Gaussian distribution, where \( m \in \mathbb{R}^p \) and \( C \in \mathbb{R}^{p \times p} \) are mean and covariance matrices of appropriate sizes, respectively. For events \( B_1, \ldots, B_p \), \( \bigcap_{i=1}^p B_i \) represents inner product of events.
B. Problem Formulation
Consider nonlinear control-affine system $S$:
\[ \dot{x} = f(x) + g(x)u, \]  
(1)
where $x \in X \subset \mathbb{R}^n$ is a state vector and $u$ is a control input. In this work, we assume that the map $f : X \to \mathbb{R}^n$ is unknown, the map $g : X \to \mathbb{R}^{n \times m}$ is known, and $g(x)g^T(x)$ is positive definite for all $x \in X$.

Assumption imposing a restriction on the complexity of the map $f$ through reproducing kernel Hilbert space (RKHS) norm is described below.

Assumption 2.1: For map $f : X \to \mathbb{R}^n$ in $S$, $\|f_i\|_k \leq \infty$ for all $i \in \{1, \ldots, n\}$ (i.e., the RKHS norm w.r.t. kernel $k$ is bounded).

Note that all continuous functions defined over compact state-space satisfy the above assumption for most of the commonly used kernels [17]. For more details on RKHS norm, we refer interested reader to [18].

\[ \text{III. GAUSSIAN PROCESS APPROXIMATION} \]

Gaussian processes (GPs) [20] is a non-parametric learning approach to approximate an unknown nonlinear function $f : X \to \mathbb{R}^n$ using samples. The data we obtain for each component of $f$ can be viewed as a collection of random variables having a joint multivariate Gaussian $f_i(x) \sim \mathcal{G}(0, \mu_i, \sigma^2_i I_n)$ where $\mu_i$ is the mean function which is set to 0 in practice, $K_i$ gives the covariance between $f_i(x)$ and $f_i(x')$ and is a function of corresponding $x, x' : K_i = k(x, x')$ known as kernel function. The kernel function can be any kind, provided that it generates a positive definite covariance matrix $K_i$ and is chosen according to problem. Some frequently used kernels include linear, squared exponential and Matèrn kernels [20]. Complete approximation of $f$ with $n$ independent GPs is therefore given by,

\[ \hat{f}(x) = \begin{cases} 
\hat{f}_1(x) \sim \mathcal{G}(0, k_1(x, x')) , \\
\vdots \\
\hat{f}_n(x) \sim \mathcal{G}(0, k_n(x, x')). 
\end{cases} \]

Now the posterior distribution of $f_i(x)$, conditioned on a given set of $N$ measurements $\{x^{(1)}, \ldots, x^{(N)}\}$ and $\{y^{(1)}, \ldots, y^{(N)}\}$, with $y^{(j)} = f(x^{(j)}) + w^{(j)}$, $j \in \{1, \ldots, N\}$, is Gaussian with mean and covariance
\[ \mu_i(x) = \hat{k}^T_i (K_i + \sigma^2_i I_n)^{-1} y_i, \]  
(2)
\[ \sigma^2_i(x) = k_i(x, x) - \hat{k}^T_i (K_i + \sigma^2_i I_n)^{-1} k_i, \]  
(3)
where $\hat{k} = [k_1(x^{(1)}), \ldots, k_N(x^{(N)})]^T \in \mathbb{R}^N$, $y_i = [y_i^{(1)}, \ldots, y_i^{(N)}]^T \in \mathbb{R}^N$, and

\[ K_i = \begin{bmatrix} k_1(x^{(1)}, x^{(1)}) & \cdots & k_1(x^{(1)}, x^{(N)}) \\
\vdots & \ddots & \vdots \\
k_N(x^{(N)}, x^{(1)}) & \cdots & k_N(x^{(N)}, x^{(N)}) \end{bmatrix} \in \mathbb{R}^{N \times N}. \]

We consider $\sigma^2_i = \max_{x \in X} \sigma^2_i(x)$. One can readily see that such a bound exists as the set X is compact and the kernels are continuous. The approximation of overall $f$ is as follows:

\[ \mu(x) := [\mu_1(x), \ldots, \mu_n(x)]^T, \]  
(4)
\[ \sigma^2(x) := [\sigma^2_1(x), \ldots, \sigma^2_n(x)]^T. \]  
(5)

In following proposition, we provide a probabilistic bound on the difference between the inferred mean $\mu_i(x)$ and the true value of $f_i(x)$.

Proposition 3.1: Consider a system $S$ with Assumptions 2.1 and 2.2, and GP approximation with mean $\mu$ in (4) and variance $\sigma^2$ in (5) obtained using $N$ measurements. Then, the approximation error is bounded by

\[ \mathbb{P}\{\|\mu(x) - \sigma(x)\| \leq \|f(x) - \mu(x) + \sigma(x)\|, \forall x \in X\} \geq (1 - \varepsilon)^n, \]  
(6)
with $\varepsilon \in (0, 1)$ and $\beta = \text{diag}\{\beta_1, \ldots, \beta_n\}$, where $\beta_i := \sqrt{2\|f_i\|_{k_i}^2 + 300\gamma_i \log^3(N+1)\|x\|}$, and $\gamma_i$ denotes information gain (c.f. Remark 3.2).

Proof: The proof can be found in [9].

Remark 3.2: The information gains $\gamma_i$ represents the maximum mutual information between data samples and unknown map $f_i$. Obtaining $\gamma_i$ is hard. However, there are techniques to over-approximate the value, see [21] for example.

Moreover, computing bound on RKHS norm $\|f_i\|_{k_i} \leq B_i$ is also in general hard. However, considering Lipschitz-like assumption, one can compute $B_i$ as discussed below.

Lemma 3.3: [22, Lemma 1] Consider a kernel function $k_i$ and we assume that $f_i(x)$ satisfies $|f_i(x) - f_i(y)| \leq L_i \sqrt{\|x - y\|}$ for all $x, y \in X$, then $B_i = \frac{L_i}{\sqrt{2}\|\mu_i\|_{\infty}}$. 

Now by utilizing the upper bound on RKHS norm $B_i$ in Lemma 3.3, one can provide a deterministic bound (i.e. with probability 1) for unknown dynamics $f_i(\cdot)$ as discussed in the following result.

Lemma 3.4: Consider a system $S$ with Assumptions 2.1 and 2.2 and GP approximation with mean $\mu$ and standard deviation $\sigma$ as given in (4) and (5), respectively. Then $\forall x \in X$, it follows that

\[ \mu_i(x) + \tilde{\beta}_i \sigma_i(x) \leq f_i(x) \leq \mu_i(x) + \tilde{\beta}_i \sigma_i(x), \]  
(7)
with $\tilde{\beta}_i = \sqrt{B_i^2 - y_i^T (K_i + \sigma^2_i I_n)^{-1} y_i + N}$, where $B_i$ is an upper bound on RKHS norm $\|f_i\|_{k_i}$ as defined.
in Lemma 3.3; $y_i$ and $K_i$ are defined in (2) and (3), respectively and $N$ is a number of data samples.

Proof: The proof is similar to that of [22, Lemma 2] and is omitted here.

Note that the bound obtained in (7) are very conservative. For more discussion, please refer to the case study in Section V.

IV. REACHABILITY USING FUNNEL-BASED CONTROL

In this section, we propose the use of funnel-based control approach [15] to solve Problem 2.3. Consider a funnel representing time-varying bounds for the trajectory $x_i$, $i \in \{1, \ldots, n\}$ given as follows

$$-c_i\rho_i(t) < x_i(t) - \eta_i < d_i\rho_i(t)$$

(8)

for all $t \in \mathbb{R}_0^+$, where $\rho_i : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, $i \in \{1, \ldots, n\}$ are positive, smooth, and strictly decreasing funnel functions, $c_i, d_i \in \mathbb{R}_0^+$ and $\eta_i \in \mathbb{R}$ are some constants.

In this work, we consider the following form of funnel function

$$\rho_i(t) = \rho_{i0}e^{-\epsilon_i t} + \rho_{i\infty},$$

where $\rho_{i0}, \rho_{i\infty}, \epsilon_i \in \mathbb{R}_0^+$ are positive constants and $\rho_{i\infty} = \lim_{t \to \infty} \rho_i(t)$. Now, by normalizing $x_i(t) - \eta_i$ with respect to the performance function $\rho_i(t)$, the modulating error is defined as $\hat{x}_i(t) := \frac{x_i(t) - \eta_i}{\rho_i(t)}$ and the corresponding performance region $\hat{D}_i := \{\hat{x}_i \mid \hat{x}_i \in (-c_i, d_i)\}$. Then, we transform the modulated error through a strictly increasing transformation function $T_i : \hat{D}_i \to \mathbb{R}$ such that $T_i(0) = 0$ and is chosen as

$$T_i(\hat{x}_i) = \ln \left(\frac{d_i(c_i + \hat{x}_i)}{c_i(d_i - \hat{x}_i)}\right).$$

(10)

The transformed error is then defined as $\xi_i(\hat{x}_i(t), \rho_i(t)) := T_i(\hat{x}_i)$. It can be verified that if the transformed error is bounded, then the modulated error $\hat{x}_i$ is constrained within the region $\hat{D}_i$. This also implies that $x_i(t) - \eta_i$ evolves within the bounds given in (8). Differentiating $\xi_i$ with respect to time, we obtain transformed error dynamics for ith dimension as

$$\dot{\xi}_i = \phi_i(\hat{x}_i, t)[\dot{x}_i + \alpha_i(x_i - \eta_i)],$$

(11)

where $\phi_i(\hat{x}_i, t) := \frac{1}{\rho_i(t)} \frac{\partial T_i(\hat{x}_i)}{\partial \hat{x}_i} > 0$ for all $\hat{x}_i \in (-c_i, d_i)$ and $\alpha_i(t) := -\frac{\phi_i(t)}{\rho_i(t)} > 0$ for all $t \in \mathbb{R}_0^+$ are the normalized Jacobian of the transformation function $T_i$ and the normalized derivative of the performance function $\rho_i$, respectively. Now, by stacking all the transformed error dynamics, one gets

$$\dot{\xi} = \Phi_i(\dot{x} + \alpha_i(x - \eta)),$$

where $\xi = [\xi_1, \ldots, \xi_n]^T$, $\Phi_i = \text{diag}\{\phi_1(\hat{x}_i, t), \ldots, \phi_n(\hat{x}_i, t)\}$, $\alpha_i = \text{diag}\{\alpha_1(t), \ldots, \alpha_n(t)\}$, and $\eta = [\eta_1, \ldots, \eta_n]^T$. The following theorem provides the result for enforcing reachability specification by utilizing the funnel approach.

Theorem 4.1: Consider the system $S$, the learned GP approximation with mean $\mu$ (4) and standard deviation $\sigma$ (5), sets $X_a, X_b \subset X$, $\mathcal{E}_i := [X_{ai}, X_{bi}]$, $\mathcal{E}_i := \min\{X_{ai}, X_{bi}\}$, $X_i := \max\{X_{ai}, X_{bi}\}$, an arbitrarily chosen state $\eta = [\eta_1, \eta_2, \ldots, \eta_n]^T \in \text{Int}(X_b)$ satisfying

$$\eta_i \in \bigg\{\begin{array}{ll}
\Xi_i & \text{if } \Xi_i \neq \emptyset \\
\{\Xi_i \cup \mathcal{E}_i \} & \text{if } \Xi_i = \emptyset,
\end{array}$$

$i \in \{1, \ldots, n\}$, and funnel function (9) with $\epsilon_i \in \mathbb{R}_0^+$,

$$\rho_{i0} = \frac{\max\{|\eta_i - X_{ai}|, |\eta_i - X_{bi}|\}}{\rho_{i0}}, \quad \rho_{i0} \neq 0 \quad \text{if } \Xi_i \neq \emptyset;$$

$$\rho_{i0} = \frac{\max\{|\eta_i - X_{ai}|, |\eta_i - X_{bi}|\}}{\rho_{i0}} \quad \text{if } \Xi_i = \emptyset;$$

and $\rho_{i\infty}$ is such that $\prod_{i \in \{1, \ldots, n\}} \eta_i^{-c_i \rho_{i\infty}, d_i \rho_{i\infty}} \subset X_1.1$

Then under time-varying control law:

$$u(x, \rho) = -g(x)^T(g(x)g(x)^T)^{-1}(\mu(x) + (\text{sign}(x - \eta))^T \beta \sigma(x) + \xi(x, \rho) + \tau(x - \eta))^T,$$

(13)

where

$$\xi(x, \rho) = [\xi_1(x_1, \rho_1), \ldots, \xi_n(x_n, \rho_n)]^T := \ln\left(\frac{d_i(c_i + x_i - \eta_i)}{c_i(d_i - x_i - \eta_i)}\right), \ldots, \ln\left(\frac{d_n(c_n + x_n - \eta_n)}{c_n(d_n - x_n - \eta_n)}\right)$$

is a transformation error as discussed above, $\tau := \max_{i \in \{1, \ldots, n\}} \epsilon_i \text{sign}(x - \eta) = [\text{sign}(x_1 - \eta_1), \ldots, \text{sign}(x_n - \eta_n)]^T$, $\beta = \text{diag}\{\beta_1, \ldots, \beta_n\}$, one can ensure that $\exists t \in \mathbb{R}_0^+$ such that $x_{ao}(t) \cap X_b = \emptyset$ for all $x_0 \in X_a$ with probability $(1 - \epsilon)^n$. In other words, the trajectory starting from any initial point in $X_a$, will reach $X_b$ in a finite time under the control law (13) with a minimum probability of $(1 - \epsilon)^n$.

Proof: To improve readability, we will drop the arguments $x$ and $\rho$ of the map $\xi$. Consider Lyapunov like function $V = \frac{1}{2} \xi^T \xi$ and

$$\dot{V} = \xi^T \Phi_i(f(x) + g(x)u + \alpha_i(x - \eta))$$

$$= \xi^T \Phi_i(f(x) - g(x)g(x)^T(g(x)g(x)^T)^{-1}(\mu(x) + (\text{sign}(x - \eta))^T \beta \sigma(x) + \tau(x - \eta) + \alpha_i(x - \eta))$$

$$- \xi^T \Phi_i(\mu(x) + (\text{sign}(x - \eta))^T \beta \sigma(x) - f(x)) - \xi^T \Phi_i \tau(x - \eta) + \xi^T \Phi_i \alpha_i(x - \eta).$$

(14)

Considering the construction of transformed error $\xi$ and (6), one can obtain that the first term of last equality is always non-positive with a probability greater than $(1 - \epsilon)^n$. To elaborate more, for an $i \in \{1, \ldots, n\}$, we consider the following two cases:

$^1$One can choose $\rho_{i\infty}$ arbitrary small in order to satisfy this condition.

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
Case I: $\xi_i < 0$ implies that $(x_i - \eta_i) < 0$ (this is due to $\xi_i(\hat{x}_i)$ is strictly increasing and $\xi_i(0) = 0$). It follows that
\[-\xi_i\phi_i(\hat{x}_i, t)(\mu_i(x) + \text{sign}(x_i - \eta_i)\beta_i\sigma_i(x) - f_i(x)) = -\xi_i\phi_i(\hat{x}_i, t)(\mu_i(x) - \beta_i\sigma_i(x) - f_i(x)) \leq 0.\]
The last inequality is due to $\xi_i < 0$, $\phi_i(\hat{x}_i, t) > 0$, and $\mu_i(x) - \beta_i\sigma_i(x) - f_i(x) \leq 0$.

Case II: $\xi_i \geq 0$ implies that $(x_i - \eta_i) \geq 0$. It follows that
\[-\xi_i\phi_i(\hat{x}_i, t)(\mu_i(x) + \text{sign}(x_i - \eta_i)\beta_i\sigma_i(x) - f_i(x)) = -\xi_i\phi_i(\hat{x}_i, t)(\mu_i(x) + \beta_i\sigma_i(x) - f_i(x)) \leq 0.\]
The last inequality is due to $\xi_i > 0$, $\phi_i(\hat{x}_i, t) > 0$, and $\mu_i(x) + \beta_i\sigma_i(x) - f_i(x) \geq 0$. This implies that the first term of (14) is non-positive with probability of at least $(1 - \epsilon)^n$.

Next, following the facts that $\Phi_k$ and $\alpha_k$ are positive definite matrices, $\alpha_k < \tau := \max_{x \in \mathbb{R}_+} e_i$, $\xi^T(x - \eta) \geq 0$ (this is due to $\xi_i(\hat{x}_i)$ is strictly increasing and $\xi_i(0) = 0$), one obtains $V \leq -\xi^T\Phi_k\xi$. This implies that $\xi(t)$ is bounded for all $t \in \mathbb{R}_+$ and hence we guarantee (8) that $-c_i\rho_i(t) + \eta_i < x_i(t) < d_i\rho_i(t) + \eta_i$ with probability of at least $(1 - \epsilon)^n$. From the choice of $\eta$ and constants $\rho_0$, $\rho_{i\infty}$, $c_i$, $d_i$ for all $i \in \{1, \ldots, n\}$, one can readily ensure that $X_{\eta} \subseteq \prod_{i \in \{1, \ldots, n\}} [-c_i\rho_i(0) + \eta_i, d_i\rho_i(0) + \eta_i]$ and
\[
\lim_{t \to \infty} \prod_{i \in \{1, \ldots, n\}} [-c_i\rho_i(t) + \eta_i, d_i\rho_i(t) + \eta_i] = \prod_{i \in \{1, \ldots, n\}} \eta_i + [-c_i\rho_{i\infty}, d_i\rho_{i\infty}] \subset X_b.\]
This implies that exist $t \in \mathbb{R}_+$ such that $x_{\eta}(t) \cap X_b \neq \emptyset$ for all $x_0 \in X_{\eta}$ with probability of at least $(1 - \epsilon)^n$. This concludes the proof.

V. CASE STUDY

Here, we demonstrates the efficacy of the proposed result using a numerical example adapted from [19].

\[ f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} x_1 + (\cos(x_1) - 1)x_2 \\ -s(x_1) + x_2 \end{bmatrix}, g(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \]

where $s(x_1) = \frac{1}{1 + \exp(-2x_1)} - 0.5$ is the shifted sigmoid function. We consider a compact state-space $X = [-5, 5] \times [-5, 5]$, initial state-set $X_{\eta} = [-2, -3] \times [-2, -3]$, and the goal set $X_b = [1, 3] \times [1, 3]$. The functions $f_1(\cdot)$ and $f_2(\cdot)$ are continuous, has therefore a finite RKHS norm under the squared exponential kernel on a compact set and complies to Assumption 2.1.

For solving Problem 2.3, we first approximated the unknown dynamics using GPs with 50 collected measurements of $x$ and corresponding $y$’s, $y = f(x) + w$, where $w \sim \mathcal{N}(0, \sigma_f^2\mathbf{I})$, $\sigma_f = 0.01$, by running the simulated system with different start states. We used exponential quadratic kernel [20] defined as $k_i(x, x') = \sigma_i^2_0 \exp \left(-\sum_{j=1}^{\infty} \frac{(x_j - x'_j)^2}{2l_j^2}\right), i \in \{1, 2\}$, where $\sigma_{i1} = 316$ and $\sigma_{i2} = 25.3$ are signal variances and $l_{11} = 2.9$, $l_{12} = 177$, $l_{21} = 1.67$, and $l_{22} = 50.5$ are length scales. We use Limited Memory Broyden–Fletcher–Goldfarb–Shanno (L-BFGS-B) algorithm [23], [24] to obtain these parameters. The inferred mean and variance are as in (4) and (5) with $\mathbf{p}_{\max} = \max\{\mathbf{p}_{1}, \mathbf{p}_{2}\} = 0.0616, \mathbf{p}_{1} = 0.022, \mathbf{p}_{2} = 0.0616$. Figure 1 depicts the original and the approximated map $f(x)$.

Computing $\|f_i\|_{\bar{p}_i}$ and $\gamma_j, i \in \{1, 2\}$, is intractable in general. Thus, we used Monte-Carlo method to get the probability bound for the confidence interval given in Proposition 3.1.

For a fixed value of $\beta_i\mathbf{p}_{i}$, $i = 0, 1, 2$, we get a probability interval for the probability in (6) as $P\{\{\mu(x) - \beta_i\sigma_i(x) \leq f(x) \leq \mu(x) + \beta_i\sigma_i(x)\}, \forall x \in X\} \in [0.9894, 0.9907]$ with confidence $1 - 10^{-10}$ using $10^6$ realizations. Thus, one can choose the lower bound $(1 - \epsilon)^2$ as 0.9894.

We also computed the value of $\tilde{\beta}_1 = 7.0878$ and $\tilde{\beta}_2 = 7.0710$ as shown in Lemma 3.4. To compare the conservativeness of the bounds, we compare the value of $\beta_i\mathbf{p}_i$ with Monte-Carlo approach to obtain probability of 1 with confidence of $1 - 10^{-10}$. Using Monte-Carlo approach, the obtained values are $\tilde{\beta}_1\mathbf{p}_1 = 0.016$ and $\tilde{\beta}_2\mathbf{p}_2 = 0.0442$ and the values obtained using results of Lemma 3.4 are $\tilde{\beta}_1\mathbf{p}_1 = 0.1599$ and $\tilde{\beta}_2\mathbf{p}_2 = 0.4366$, respectively. One can readily see the conservativeness in the bounds obtained using results of Lemma 3.4.

With the help of learned mean and variance, we simulate the results using the proposed control law (13). The parameters of the controllers and construction of corresponding funnel functions are as per the Theorem 4.1.

The trajectory of the system reaching $X_b$ from $X_{\eta}$ is shown in Figure 2. One can readily see from Figure 3, the trajectories $x_1$ and $x_2$ satisfy the constructed funnel bounds.

VI. CONCLUSION AND FUTURE WORK

The work proposed a scheme for designing closed-form controller for unknown nonlinear control systems enforcing reachability specifications. We provide a control
policy using a funnel-based approach by approximating unknown system dynamics using Gaussian processes. We verified the proposed method using a numerical example. Future research includes incorporating constraints on the input space and extending the results to more complex specifications like linear/signal temporal logic specifications.

References

[1] J. Kocijan, Modelling and control of dynamic systems using Gaussian process models. Springer, 2016.
[2] L. Hewing, J. Kaba, and M. N. Zeilinger, “Cautious model predictive control using Gaussian process regression,” IEEE Transactions on Control Systems Technology, 2019.
[3] G. Chowdhary, H. A. Kingravi, J. P. How, and P. A. Vela, “Bayesian nonparametric adaptive control using Gaussian processes,” transactions on neural networks and learning systems, vol. 26, no. 3, pp. 537–550, 2014.
[4] T. Beckers, D. Kulić, and S. Hirche, “Stable gaussian process based tracking control of euler–lagrange systems,” Automatica, vol. 103, pp. 390–397, 2019.
[5] A. Capone and S. Hirche, “Backstepping for partially unknown nonlinear systems using gaussian processes,” Control Systems Letters, vol. 3, no. 2, pp. 416–421, 2019.
[6] J. Umlauft, T. Beckers, M. Kimmel, and S. Hirche, “Feedback linearization using Gaussian processes,” in 56th Annual Conference on Decision and Control (CDC). IEEE, 2017, pp. 5249–5255.
[7] F. Berkenkamp, A. P. Schoellig, and A. Krause, “Safe controller optimization for quadrotors with Gaussian processes,” in International Conference on Robotics and Automation (ICRA). IEEE, 2016, pp. 491–496.
[8] A. K. Akametalu, J. F. Fisac, J. H. Gillula, S. Kaynama, M. N. Zeilinger, and C. J. Tomlin, “Reachability-based safe learning with Gaussian processes,” in 53rd Conference on Decision and Control. IEEE, 2014, pp. 1424–1431.
[9] P. Jagtap, G. J. Pappas, and M. Zamani, “Control barrier functions for unknown nonlinear systems using Gaussian processes,” in 59th Conference on Decision and Control (CDC).
IEEE, 2020, pp. 3699–3704.

[10] H. Ravanbakhsh, S. Sankaranarayanan, and S. A. Seshia, “Formal policy learning from demonstrations for reachability properties,” in International Conference on Robotics and Automation (ICRA). IEEE, 2019, pp. 6037–6043.

[11] M. Rungger and M. Zamani, “SCOTS: A tool for the synthesis of symbolic controllers,” in Proceedings of the 19th international conference on hybrid systems: Computation and control, 2016, pp. 99–104.

[12] J. Lygeros, C. Tomlin, and S. Sastry, “Controllers for reachability specifications for hybrid systems,” Automatica, vol. 35, no. 3, pp. 349–370, 1999.

[13] S. Junges, N. Jansen, and S. A. Seshia, “Enforcing almost-sure reachability in pomdps,” in International Conference on Computer Aided Verification. Springer, 2021, pp. 602–625.

[14] R. Vignali and M. Prandini, “A method for detecting relevant inputs while satisfying a reachability specification for piecewise affine systems,” in Conference on Control Applications (CCA). IEEE, 2016, pp. 538–543.

[15] C. P. Bechlioulis and G. A. Rovithakis, “A low-complexity global approximation-free control scheme with prescribed performance for unknown pure feedback systems,” Automatica, vol. 50, no. 4, pp. 1217–1226, 2014.

[16] X. Bu, “Prescribed performance control approaches, applications and challenges: A comprehensive survey,” Asian Journal of Control, 2021.

[17] M. W. Seeger, S. M. Kakade, and D. P. Foster, “Information consistency of nonparametric Gaussian process methods,” Transactions on Information Theory, vol. 54, no. 5, pp. 2376–2382, 2008.

[18] V. I. Paulsen and M. Raghupathi, An introduction to the theory of reproducing kernel Hilbert spaces. Cambridge University Press, 2016, vol. 152.

[19] J. Umlauf, L. Pöhler, and S. Hirche, “An uncertainty-based control Lyapunov approach for control-affine systems modeled by Gaussian process,” IEEE Control Systems Letters, vol. 2, no. 3, pp. 483–488, 2018.

[20] C. K. Williams and C. E. Rasmussen, Gaussian processes for machine learning. MIT press Cambridge, MA, 2006, vol. 2, no. 3.

[21] N. Srinivas, A. Krause, S. M. Kakade, and M. W. Seeger, “Information-theoretic regret bounds for Gaussian process optimization in the bandit setting,” Transactions on Information Theory, vol. 58, no. 5, pp. 3250–3265, 2012.

[22] K. Hashimoto, A. Saoud, M. Kishida, T. Ushio, and D. Dimarogonas, “Learning-based symbolic abstractions for nonlinear control systems,” in arXiv preprint arXiv:2004.01879, 2021.

[23] R. H. Byrd, P. Lu, J. Nocedal, and C. Zhu, “A limited memory algorithm for bound constrained optimization,” SIAM Journal on Scientific Computing, vol. 16, no. 5, pp. 1190–1208, 1995. [Online]. Available: https://doi.org/10.1137/0916069

[24] C. Zhu, R. H. Byrd, P. Lu, and J. Nocedal, “Algorithm 778: L-BFGS-B: fortran subroutines for large-scale bound-constrained optimization.” ACM Trans. Math. Softw., vol. 23, no. 4, p. 550–560, dec 1997.