Topological domain wall states in a non-symmorphic chiral chain

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The Su-Schrieffer-Heeger (SSH) model, containing dimerized hopping and a constant onsite energy, has become a paradigmatic model for one-dimensional topological phases, soliton excitations and fractionalized charge in the presence of chiral symmetry. Motivated by the recent developments in engineering artificial lattices, we study an alternative model where hopping is constant but the onsite energy is dimerized. We find that it has a non-symmorphic chiral symmetry and supports topologically distinct phases described by a \(\mathbb{Z}_2\) invariant \(\nu\). In the case of multimode ribbon we also find topological phases protected by hidden symmetries and we uncover the corresponding \(\mathbb{Z}_2\) invariants \(\nu_n\). We show that, in contrast to the SSH case, zero-energy states do not necessarily appear at the boundary between topologically distinct phases, but instead these systems support a new kind of bulk-boundary correspondence: The energy of the topological domain wall states scales to zero as \(1/\xi\) or \(e^{-w/\xi}\), where \(\xi\) is an intrinsic length scale and \(w\) is the width of the domain wall separating phases with different \(\nu\) or different \(\nu_n\), respectively. We show that the spectral flow of these states and the charge of the domain walls are different than in the case of the SSH model.

The Su-Schrieffer-Heeger (SSH) model was originally introduced to describe the properties of conducting polymers, where the spontaneous symmetry breaking leads to dimerization of the sites along the chain \[1,3\]. Due to two-fold degeneracy of the ground state a new type of excitation, a domain wall (DW) between different bonding structures, can exist. For the conducting polymers the width of the DW excitations is large and they can propagate along the chain. Thus, they can be considered as solitons in analogy to the shape-preserving propagating solutions of the nonlinear differential equations \[3\]. Moreover, the solitons in the SSH model have a remarkable effect on the electronic spectrum leading to an appearance of a bound state in the middle of the energy gap. This midgap state is understood as a topologically protected boundary mode and the SSH model serves as a paradigmatic example of chiral symmetric topological insulator \[4,5\]. Namely, the chiral symmetry allows to block-off-diagonalize the Hamiltonian and the winding of the determinant \(z_k\) of the off-diagonal block around the origin as a function \(k\) determines a topological invariant \([6]\), where \(z_k\) is symmetric around zero-energy and each DW carries a fractionalized charge in the presence of chiral symmetry \[6\]. The interest for these models has revived because they can exhibit a very interesting feature of such a model is that its solitons can carry irrational charge \(q = \frac{1}{2}(1 \pm f)\) \[7\], where \(f\) describes the breaking of the chiral symmetry \[11\]. The interest for these models has revived because they can exhibit a very interesting feature of such a model is that its solitons can carry irrational charge \(q = \frac{1}{2}(1 \pm f)\) \[7\], where \(f\) describes the breaking of the chiral symmetry \[11\].
engineered in photonic quantum walks [15], optical lattices [16–21] and nanostructures [22–26] in a controlled way, and in these systems also their emergence from spontaneous symmetry breaking [27] and the properties of the solitons can be tuned using external parameters [28–33]. Motivated by the new possibilities opened by these recent developments we focus on a special case of the Rice-Mele model where all the hopping amplitudes are equal and only the mass term alternates. We show that in this case the model has an interesting non-symmorphic (NS) chiral symmetry and it supports a topologically nontrivial phase described by a non-symmorphic chiral $Z_2$ invariant $\nu$. This invariant was found by Shiozaki, Sato and Gomi in their pioneering work on non-symmorphic topological insulators [34] and therefore we name the special case of the Rice-Mele model as the Shiozaki-Sato-Gomi (SSG) model. The peculiar property of the SSG model is that the bulk topological invariant does not guarantee the existence of the end states in an open system, because the boundary always breaks the NS chiral symmetry [34].

In this paper we analytically derive exact phase diagrams of SSG nanoribbons of arbitrary width and uncover hidden symmetries relying on interchange of transverse and longitudinal modes. In addition to the NS chiral $Z_2$ invariant $\nu$ the multimode ribbons support $Z_2$ invariants $\nu_n$ protected by the hidden symmetries. These invariants lead to a new kind of bulk-boundary correspondence. The energy of the topological domain wall states scales to zero as $1/w$ or $e^{-w/\xi}$ (Fig. 2), where $\xi$ is an intrinsic length scale and $w$ is the width of the domain wall separating phases with different $\nu$ or different $\nu_n$, respectively. The NS chiral symmetry in SSG model leads to several important differences in comparison to the SSH model (Fig. 1). (i) In SSH model the topological zero-energy end or DW states come in pairs and have zero energy for any DW width, whereas the SSG model supports unpaired DW states approaching zero energy with increasing $w$. (ii) In SSH model the charge of the DWs is $q = \pm 1/2$, whereas for the SSG model we get irrational charges $q = \delta_{1,2} \pm \frac{1}{2}$ for solitons and antisolitons depending on whether the zero-energy state is occupied or empty. The DWs in SSG model separate regions without onsite energies $\pm m_1$ and $\pm m_2$ (mass terms) in the two sublattices (see Figs. 1 and 2) and $\delta = \frac{1}{2}(\zeta_1 - \zeta_2)$, where $\zeta_i$ is the difference of the bulk filling factors of the two sublattices in the region with mass $m_i$.

The $k$-space SSG Hamiltonian for a multimode wire is

$$H_k = -m\sigma_z \tau_z + 2t_x \cos \frac{1}{2}\cos \frac{1}{2}\sigma_x - \sin \frac{1}{2}\sigma_y + t_y \tau_z + 2t_d \cos \frac{1}{2}\sin \frac{1}{2}\sigma_x + \cos \frac{1}{2}\sigma_y \tau_y,$$

where $m$ is the mass, $t_x$ and $t_y$ are hopping amplitudes in $x$ and $y$ directions and $t_d$ is the diagonal hopping amplitude with opposite signs in the two sublattices [Fig. 2(a)]. Here $\sigma_i$ are Pauli matrices describing the unit cell in the $x$ direction, $\tau_i$ are $L_x \times L_y$ matrices describing transverse hopping ($\tau_x = \delta_{1,2} \pm \frac{1}{2}$, $\tau_y = i\delta_{1,2} \pm \frac{1}{2}$), $\tau_z = \pm \frac{1}{2}$), $\delta_{1,2} = \frac{1}{2}(\zeta_1 - \zeta_2)$, where $\zeta_i$ is the difference of the bulk filling factors of the two sublattices in the region with mass $m_i$.

Figure 2. (a) Schematic view of the multimode SSG model. (b) Domain wall separating different mass $m$ regions with different NS chiral $Z_2$ invariants $\nu$ or hidden $Z_2$ invariants $\nu_n$. (c) Spectral flow of the DW states for multimode SSG chain as a function of the DW width $w$ for $L_y = 7$, $L_x = 10000$ and $t_y = t_x$. DW separates regions with masses $m_1 = 0.2t_x$, $m_2 = -20t_x$ and $t_y = 0.6t_x$ differing only by $\nu$. There is a single DW state whose energy approaches zero $n 1/w$. (d) The same for DW separating regions differing both by $\nu$ and $\nu_n$ with $m_1 = 0.25t_x$, $m_2 = -10t_x$ and $t_y = 0.5t_x$. In addition to the unpaired DW state with energy approaching zero $n 1/w$, there are hidden-symmetry protected DW states with energies approaching zero $\propto e^{-w/\xi}$, where $\xi$ is an intrinsic length scale.
times the trajectory $z_k$ crosses the positive real semiaxis for $k \in [0,2\pi]$ is a $\mathbb{Z}_2$ topological invariant because it cannot be changed without closing the gap or breaking the NS chiral symmetry (see Fig. 1). In our case the mirror symmetry $M_x = \cos \frac{k}{2} - i \sin \frac{k}{2} \sigma_z$ becomes identity in the eigenbasis of $S_y$ (see Appendix A) so that $z_k = z_{-k}$. For this reason $\Im z_\nu = 0$ and the formula for the $\nu$ gets simplified to

$$\nu = \text{sign } \Re z_x$$

(2)

in analogy to the simplification of the invariant for topological insulators in the presence of inversion symmetry \cite{34}. The band-inversion corresponding to a change of $\nu$ happens at $k = \pi$ and $m = 0$. We find that

$$\nu = \begin{cases} 
\frac{1}{2} \left[ 1 + \text{sign}(m) \right] & \text{if } L_y = 2n - 1 \text{ or } L_y = 2n (n \in \mathbb{N}_+) \\
0 & \text{if } L_y = 2n \text{ or } L_y = 2n + 1 (n \in \mathbb{N}_+) 
\end{cases}$$

(3)

In Fig. 3 we show the topological phase diagrams of the SSG model at $L_y = 6.7$ as functions of $m/t_x$ and $t_d/t_x$, setting $t_y = t_x$. Surprisingly we find more phases than predicted by the $\nu$ invariant. The gap closes not only for $m = 0$ at $k = \pi$ when $L_y$ is odd but also for any $L_y$ along lines $m = m_n$, $k = k_n$, where

$$m_n = \frac{t_d t_y}{t_x} 2^2 n, \quad k_n = \pm \arccos \left( \frac{t_y}{2t_x} \varepsilon_n \right), \quad \varepsilon_n = 2 \cos \frac{n \pi}{L_y + 1}$$

(4)

and $n = 1, 2, \ldots, \left\lfloor L_y/2 \right\rfloor$ provided that

$$\left| \frac{t_y}{2t_x} \varepsilon_n \right| \leq 1.$$ 

(5)

This means that in the limit of very wide ribbon ($L_y \to \infty$) the phase diagram consists of a quasi-continuous set of lines $t_d = \gamma_n m$ with slopes $\gamma_n$ ranging between $\frac{t_d}{t_x}$ and $\infty$. The natural question to ask now is what is the origin of these gap-closing lines? The answer are the hidden symmetries, that can be found at the magical $k_n$ points, yielding to new $\mathbb{Z}_2$ invariants $\nu_n$.

To see the hidden symmetries we rotate $\sigma_\alpha$ matrices by angle $\frac{k}{2}$ around the $z$-axis and use the eigenbasis of $\tau_x$ to transform the operators $\tau_\alpha$ in a block-diagonal form, where the blocks are given by (see Appendix B)

$$\tau_{x,n} = \varepsilon_n \sigma_x', \quad \tau_{y,n} = \varepsilon_n \sigma_y', \quad \tau_{z,n} = \sigma_z'$$

and $\sigma'_\alpha$ is a new set of Pauli matrices. For odd $L_y$ the blocks $n = 0$ is given by $\tau_{x,0} = \tau_{y,0} = 0$ and $\tau_{z,0} = 1$. After this transformation the Hamiltonian (1) also has a block-diagonal form

$$\mathcal{H}_{k,n} = -m \sigma_z \sigma_x' + 2t_x \cos \frac{k}{2} \sigma_x$$

$$+ 2t_d \varepsilon_n \cos \frac{k}{2} \sigma_y \sigma_y' + t_y \varepsilon_n \sigma_z'$$

(6)

Now we notice that $\mathcal{H}_{k,n}'$ is invariant under interchange of $\sigma$ and $\sigma'$ operators if $2t_x \cos \frac{k}{2} = t_y \varepsilon_n$ which provides the condition for gap closing points in the $k$-space [Eq. 1].

The spin-interchange $X_{12} \sigma X_{12} = \sigma'$ and vice-versa is realized by operator $X_{12} = \frac{1}{2} (1 + \vec{\sigma} \cdot \vec{\sigma}')$ \cite{36}. The spectrum of $X_{12}$ consists of single $-1$ (singlet state) and three $+1$ (triplet states) eigenvalues. Thus $\mathcal{H}_{k,n}$ in the eigenbasis of $X_{12}$ becomes block-diagonal with one block being $1 \times 1$ and the other being $3 \times 3$. Therefore we can define a topological $\mathbb{Z}_2$ invariant based on the sign of the determinant of the $1 \times 1$ block. It changes at the gap closing lines defined by Eq. (4) so that it takes the form

$$\nu_n = \frac{1}{2} \left( 1 + \text{sign } [m - m_n] \right).$$

(7)

We conclude that the full topological description the SSG model is given by a vector $\{\nu, \nu_1, \nu_2, \ldots, \nu_{\left\lfloor L_y/2 \right\rfloor}\}$ because changes of these invariants coincide with all the gap closing lines in the phase diagram. Each hidden symmetry $n = 1, 2, \ldots, \left\lfloor L_y/2 \right\rfloor$ exists only if condition (4) is satisfied. Therefore, it is possible to connect phases with different $\nu_n$ without closing the bulk gap by choosing a path in the parameter space which goes outside the region where the hidden symmetry exists.
nal $\mathbb{Z}_2$ invariant does not generically support end states in one dimension because the boundary necessarily breaks the $S_k$ symmetry. This however does not exclude a special type of smooth DWs from having zero energy bound states. To obtain analytical insights we can develop a continuum model for odd $L_y$ by expanding $\mathcal{H}_k$ around gap closing point at $k = \pi$ and get
\begin{equation}
\mathcal{H}_{\text{eff}} = v k_0 \sigma_x - m \sigma_y \ (v > 0).
\end{equation}
Now we create a DW of width $w$ in the real space
\begin{equation}
m(x) = \frac{m_2 + m_1}{2} + \frac{m_2 - m_1}{2} \tanh \frac{x}{w}
\end{equation}
between regions with positive mass $m_1 = m_0$ and negative mass $m_2 = -m_0$ ($m_0 > 0$), separating phases with $\nu = 1$ and $\nu = 0$ [Figs. 2(b) and 4(a)], and we find that a zero-energy eigenstate of $\mathcal{H}_{\text{eff}}$ exists in a form
\begin{equation}
\psi(x) = (0 \ (\cosh \frac{x}{w})^{-m_0 w/v} )^T/N ,
\end{equation}
where $N$ is a normalization factor. This however does not take into account the fact that the chiral symmetry of $\mathcal{H}_{\text{eff}}$ becomes NS if one goes beyond linear order in $\delta k$. Therefore, we implemented numerically such a domain wall and calculated the energy of the DW state as a function of $w$ [Figs. 2(c)(d) and 4(a)]. This way we find that the energy of the topological DW state approaches zero as $1/w$ whereas the energies of the other bound states scale as $1/\sqrt{w}$. This means that the topological DW state can be distinguished from other bound states based on the scaling behavior because its energy approaches zero faster than the energies of the other states. The $1/w$ scaling can be intuitively understood by noticing that the NS chiral symmetry $S_k$ involves a lattice shift by one interatomic distance so that the breaking of $S_k$ scales as $1/w$. The $1/\sqrt{w}$ scaling of the bulk gap can be understood as well. By inserting an expansion $m(x) \simeq m_0 \frac{x}{w}$ around $x = 0$ to Hamiltonian $\mathcal{H}_{\text{eff}}$ and eliminating $\psi_1$ we obtain a Harmonic oscillator equation for $\psi_2$
\begin{equation}
-v^2 \psi''_2(x) + \frac{m_0^2}{w^2} x^2 \psi_2(x) = (E^2 + \frac{m_0 v}{w}) \psi_2(x).
\end{equation}
The energies of this problem are given by
\begin{equation}
E_n = \sqrt{2nm_0 v \frac{w}{w}}, \quad n = 0, 1, 2, ...
\end{equation}
The $n = 0$ solution gives the topological DW state and the energies of the other states scale as $1/\sqrt{w}$.

We find even more striking bulk-boundary correspondence for phases described by $\nu_n$ invariants. In Figs. 2(d) and 4(c) we show the scaling of energies of the topological DW states and the non-topological states as a function of $w$ when the masses $m_1$ and $m_2$ are chosen so that the DW separates two different $\nu_n$ phases. The energies of the non-topological states behave in the same way as before, scaling as $1/\sqrt{w}$, but the energies of the hidden-symmetry protected topological DW states scale as $e^{-w/\xi}$. By expanding Hamiltonian around the gap closing points $\pm k_n$ we get two similar zero-energy solutions as $k = \pi$. Since the hidden symmetries are not non-symmporphic these solutions should give zero-energy states for any $w$, but because the gap closes at two different momenta these bound states hybridize leading to non-zero energy. In Appendix E we show using the properties of the Schwartz functions that their overlap vanishes exponentially fast with $w$ which explains the scaling of the DW states. By setting $m_1$ and $m_2$ is such a way that all the gap closing lines are crossed on the way from $m_1$ to $m_2$ we can always obtain extensive number of DW states $L_y$ both for even and odd $L_y$.

An interesting proterty of the SSG model is that when the width of the DW $w$ increases a single state separates from the bulk spectrum and tends to zero from above or below [Fig. 2(c)]. This asymmetric spectral flow needs to be taken into account when calculating the charges for solitons and antisolitons (see Appendix F). For $L_y = 1$ we obtain $\zeta_0 = \pm 2\frac{1}{\sqrt{5}}$, $\zeta_1 = \pm 1\frac{1}{\sqrt{5}}$ for solitons and antisolitons depending on whether the zero-energy state is occupied or empty. Here $\delta = \frac{2K - \zeta_1}{\sqrt{5}}$.

\begin{equation}
\zeta_i = \frac{2}{\pi} \frac{m_i}{\sqrt{m_i^2 + 4t_x^2}} K \left( \frac{4t_x^2}{m_i^2 + 4t_x^2} \right)
\end{equation}
are the differences of the bulk filling factors of the two
sublattices in the region with mass $m_1 > 0$ and $m_2 < 0$ and $K(x)$ is the complete elliptic integral of the first kind. The DWs between different hidden-symmetric topological phases carry charges $q_n = 0, \pm 1$, so that for a general DW the charge is the sum of $q_0$ and the charges $q_n$ contributed by the transverse modes supporting transitions between different hidden-symmetric topological phases.

To summarize, we have analytically described the topological properties of the SSG model and propose it as a paradigmatic model for NS chiral-symmetric topological phases. We show that a smooth DW supports zero energy state(s) if the DW separates regions with different NS chiral invariant $\nu$ or different hidden-symmetry invariants $\nu_n$. In addition to engineered artificial lattices [13, 20, 21, 22, 23, 24, 25, 26, 27] our findings are also relevant in the context of low-dimensional binary compounds supporting surface atomic steps. In these systems the surface steps lead to distinct phases can support DW states [37], providing a possible explanation for the zero-bias conductance peak observed in the recent experiment [38].

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Appendix A: Hamiltonian and its symmetries

The SSG Hamiltonian on a square lattice has a form

$$
\mathcal{H} = t_x \sum_j (c_j^\dagger c_{j+\hat{x}}^{} + h.c.) + t_y \sum_j (c_j^\dagger c_{j+\hat{y}}^{} + h.c.)
+ t_d \sum_{s=\pm 1} \sum_j (-1)^s \hat{c}_j^s \hat{t}_j^{s+1} \hat{c}_j^s \hat{c}_{j+\hat{s}}^s + h.c.)
+ m \sum_j (-1)^s \hat{t}_j^{s+1} \hat{c}_j^{\dagger s} \hat{c}_j^s \quad (A1)
$$

where $m$ is the mass term, $t_x$ and $t_y$ are hopping amplitudes along $x$ and $y$ directions and $t_d$ is the diagonal hopping amplitude. The $k$-space form is given by

$$
\mathcal{H}_k = -m \sigma_z \tau_z + 2t_x \cos \frac{k}{2} (\cos \frac{k}{2} \sigma_x - \sin \frac{k}{2} \sigma_y)
+ t_y \tau_x + 2t_d \cos \frac{k}{2} (\sin \frac{k}{2} \sigma_x + \cos \frac{k}{2} \sigma_y) \tau_y, \quad (A2)
$$

where $\tau_a$ operators are given by matrices

$$
\tau_x = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \ 0 & -1 & 0 & 0 & \cdots \ 0 & 0 & 1 & 0 & \cdots \ 0 & 0 & 0 & -1 \ \vdots & \vdots & \vdots & \vdots & \ddots \ \vdots & \vdots & \vdots & \vdots & \ddots \ \end{pmatrix}, \quad \tau_y = \begin{pmatrix} 0 & -i & 0 & 0 & \cdots \ i & 0 & i & 0 & \cdots \ 0 & -i & 0 & -i & \cdots \ 0 & 0 & i & 0 & \cdots \ \vdots & \vdots & \vdots & \vdots & \ddots \ \vdots & \vdots & \vdots & \vdots & \ddots \ \end{pmatrix}
\quad (A3)
$$

and

$$
\tau_z = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \ 0 & 0 & 0 & 1 & \cdots \ 0 & 0 & 1 & 0 & \cdots \ 0 & 0 & 0 & -1 \ \vdots & \vdots & \vdots & \vdots & \ddots \ \vdots & \vdots & \vdots & \vdots & \ddots \ \vdots & \vdots & \vdots & \vdots & \ddots \ \end{pmatrix}, \quad \tau_a = \begin{pmatrix} \cdots & 0 & 0 & 1 \ 0 & 0 & 1 & 0 && \cdots \ 1 & 0 & 0 & 0 \ \vdots & \vdots & \vdots & \ddots \ \vdots & \vdots & \vdots & \ddots \ 0 & 0 & 0 & 0 \ \end{pmatrix}.
\quad (A4)
$$

Here we defined additional matrix $\tau_a$ which is needed to construct some of the symmetry operators.

Depending on system width $L_y$ being even or odd the system have different symmetry properties. However, some symmetries are common for both cases. The one which is most relevant here is the non-symmetric (NS) chiral symmetry defined as $S_k = \sin \frac{k}{2} \sigma_x \tau_z + \cos \frac{k}{2} \sigma_y \tau_z$ which satisfies $S_k \mathcal{H}_k S_k^{-1} = -\mathcal{H}_k$. The $k$-dependence in $S_k$ is intrinsic and follows from the half lattice translation that is needed to go from one sublattice to the other. We also have a time-reversal symmetry for spinless particles $T \mathcal{H}_k T^{-1} = -\mathcal{H}_k$, where $T$ is complex conjugation. Finally, for any $L_y$ we have a symmetry with respect to a mirror line perpendicular to $x$ direction, passing through a lattice site, taking a form of $M_x = \cos \frac{k}{2} - i \sin \frac{k}{2} \sigma_z$ and acting as $M_x \mathcal{H}_k M_x^{-1} = -\mathcal{H}_k$.

Despite the $k$-dependence this is a symmorphic symmetry. By shifting a mirror line to cut a middle of a bond we can also get a chiral mirror symmetry $M_x = \sigma_y \tau_z$ yielding relation $M_x \mathcal{H}_k M_x^{-1} = -\mathcal{H}_k$.

For odd $L_y$ we have another mirror symmetry with respect to line perpendicular to $y$ direction $M_y \mathcal{H}_k M_y^{-1} = -\mathcal{H}_k$, where $M_y = \tau_y$. For even $L_y$ mirror $M_y$ does not exist but we have a particle-hole symmetry $C = i \mathcal{K} \sigma_z \tau_z$ and inversion symmetry $I = \sigma_x \tau_a$, yielding relations $C \mathcal{H}_k C^{-1} = -\mathcal{H}_k$ and $I \mathcal{H}_k I^{-1} = \mathcal{H}_k$.

It is important to notice that in the eigenbasis of $S_k$ the mirror symmetry operator $M_a$ operator transforms to identity. This is possible because if we put eigenvectors of $S_k$ in the columns of unitary matrix $U_k$ then $M_x$ is transformed as $M_x = U_k^\dagger M_a U_k$. Note that this is not similarity transformation so the spectrum of $M_x$ is not left invariant. The form of transformation is dictated by the mirror-symmetry relation with the Hamiltonian in the new basis. Denoting $\mathcal{H}_k = U_k^\dagger \mathcal{M}_x U_k$ we get that $M_x \mathcal{H}_k M_x^{-1} = \mathcal{H}_k$.

Appendix B: Hidden symmetries

To see the hidden symmetries we first transform $\mathcal{H}_k$ to $\mathcal{H}_k' = R_{k/2}^\dagger \mathcal{H}_k R_{k/2}$ using $R_{k/2} = \exp (i \frac{k}{2} \sigma_z)$ to get

$$
\mathcal{H}_k' = -m \sigma_z \tau_z + 2t_x \cos \frac{k}{2} \sigma_x + 2t_d \cos \frac{k}{2} \sigma_y \tau_y + t_y \tau_x. \quad (B1)
$$

The eigenfunctions of $\tau_a$ corresponding to eigenvalues $\varepsilon_n = \psi_n(j) = \sqrt{\frac{2}{L_y+1}} \sin \frac{n \pi}{L_y+1}$, where $j$ labels sites in
the $y$ direction and $n$ labels modes $(j, n = 1, 2, \ldots, L_y)$. We can use these transverse modes to construct a new basis $|\phi_{2n-1}\rangle = (|\psi_n\rangle + |\psi_{L_y-n+1}\rangle) / \sqrt{2}$ and $|\phi_{2n}\rangle = (|\psi_n\rangle - |\psi_{L_y-n+1}\rangle) / \sqrt{2}$ for $n = 1, 2, \ldots, [L_y/2]$ and if $L_y$ is odd $|\phi_0\rangle = |\psi_{L_y/2}\rangle$. In this basis $\tau_x$, $\tau_y$ and $\tau_z$ have block-diagonal forms with $[L_y/2]$ diagonal blocks given by

$$\tau_{x,n} = \varepsilon_n \sigma'_{x,n}, \quad \tau_{y,n} = \varepsilon_n \sigma'_{y,n}, \quad \tau_{z,n} = \sigma'_z,$$

where $\sigma'_\alpha$ is a new set of Pauli matrices, and for odd $L_y$ the block $n = 0$ is given by $\tau_{x,0} = \tau_{y,0} = 0$ and $\tau_{z,0} = 1$. Thus the Hamiltonian has a block-diagonal form

$$\mathcal{H}'_{k,n} = -m \sigma_z \sigma'_z + 2 t_x \cos \frac{k}{2} \sigma_x + 2 t_d \varepsilon_n \sigma'_y + t_y \varepsilon_n \sigma'_x \quad (B2)$$

supporting the hidden symmetries discussed in the main text.

**Appendix C: Fermi velocity at Dirac points $k = \pm k_n$**

We notice that the Hamiltonian $\mathcal{H}'_{k,n}$ given by Eq. (B2) commutes with $\mathcal{P} = \sigma_x \sigma'_x$. The bands that cross at $k = \pm k_n$ can be found in the block $\mathcal{P} = -1$ of the Hamiltonian that takes the form,

$$\mathcal{H}'_{k,n,-} = -4 \sin \frac{k + k_n}{4} \sin \frac{k - k_n}{4} (t_x \varepsilon_n \sigma_z) - (m - m_n) \sigma_z. \quad (C1)$$

The remaining $\mathcal{P} = +1$ block is related by a shift of $2\pi$ in the $k$-space and unitary transformation, namely $\mathcal{H}'_{k,n,+} = \sigma_z \mathcal{H}'_{k+2\pi,n,-} \sigma_z$. Consequently, the Fermi velocity at the Dirac points is given by

$$v_n = \pm t_x \sqrt{\left(1 - \frac{t_y^2 + \varepsilon_n^2}{4 t_x^2}ight) \left(1 + \frac{t_d^2 + \varepsilon_n^2}{4 t_x^2}ight)} \quad (C2)$$

**Appendix D: Overlap of two domain-wall functions**

We assume that the functional form of $m(x)$ is such that it only depends on parameter $x/w$. In analogy to the domain wall solution for a gap closing point at $k = \pi$ we get two solutions for $k = \pm k_n$ gap closings in a form (we ignore the spinor structure which would change as a function momentum and the normalization factor which is not important for the statements below)

$$\psi^\pm(x) = \exp \left[ \pm i k_n x - v_n^{-1} \int_0^x m \left(\frac{x'}{w}\right) dx' \right]. \quad (D1)$$

Their overlap is

$$\langle \psi^- | \psi^+ \rangle = \int_{-\infty}^{+\infty} \exp \left[ 2 i k_n x - 2 v_n^{-1} \int_0^x m \left(\frac{x'}{w}\right) dx' \right] dx \quad (D2)$$

and substituting $x = yw$ we get

$$\langle \psi^- | \psi^+ \rangle = w \int_{-\infty}^{+\infty} \exp \left[ 2 i k_n y w - 2 w v_n^{-1} M(y) \right] dy, \quad (D3)$$

where $M(y) = \int_0^y m(y) dy$. Assuming that $M(y) > 0$, $m(\infty) > 0$ and $m(-\infty) < 0$ as we expect from a domain wall, we notice that $f(y) = \exp \left[ -2 w v_n^{-1} M(y) \right]$ is a Schwartz function. This property does not depend on the details of the model (even if one goes beyond the linear order expansion in momentum) or the shape of the DW as long as the solutions $\psi^\pm$ are smooth functions decaying exponentially (or faster) far away from the DW. The Fourier transform of a Schwartz function is also a Schwartz function and therefore the overlap $\langle \psi^- | \psi^+ \rangle$ must vanish quicker than any power of $1/(k_n w)$.

To see the exponential dependence explicitly we can assume for example $m(x/w) = m_0 x/w$. Then we obtain

$$\langle \psi^- | \psi^+ \rangle = w \int_{-\infty}^{+\infty} \exp \left[ 2 i k_n y w - \frac{w m_0 v_n^{-1} y^2}{2} \right] dy = \frac{\pi v_n}{w m_0} \exp \left[ -w k_n^2 v_n^2 / m_0 \right]. \quad (D4)$$

**Appendix E: Charge of a domain wall**

To understand the charge of the DW we start by calculating the charge appearing at the end of single-mode SSG chain with open boundary conditions

$$\mathcal{H} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & t & m & t & \cdot \\ \cdot & m & t & m & \cdot \\ \cdot & \cdot & m & t & \cdot \\ \cdot & \cdot & \cdot & t & m \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \quad (E1)$$

Assuming even number of sites $L = 2N$ the Hamiltonian can be Block diagonalized into 2x2 blocks similarly as in Appendix B and the eigenstates can be found by diagonalizing these blocks. This way we find that the eigenenergies are $(n = 1, 2, \ldots, N)$

$$E_{n\pm} = \pm E_n, \quad E_n = \sqrt{m^2 + 4t^2 \cos^2 \left( \frac{n \pi}{2N+1} \right)}. \quad (E2)$$

The eigenstate corresponding to $E_{n-}$ at lattice site $j$ is

$$\psi_{n-,j} = \frac{\sin \left( \frac{n \pi j}{2N+1} \right)}{\sqrt{2(2N+1)}} \left[ 1 - (-1)^j \right] \sqrt{1 + \frac{m}{E_n}} \left[ 1 + \frac{m}{E_n} \right]. \quad (E3)$$
The total end charge $q_{\text{end}}$ at lattice sites $j = 1, \ldots, 2\xi$ (relative to the corresponding bulk charge $\xi$) is

$$
q_{\text{end}} = \frac{2}{2N + 1} \sum_{n=1}^{N} \sum_{j=1}^{2\xi} \sin^2\left(\frac{n j 2\pi}{2N + 1}\right) \left[1 - (-1)^j \frac{m}{E_n}\right] - \xi
$$

$$
= \frac{1}{2N + 1} \sum_{n=1}^{N} \sum_{j=1}^{2\xi} (-1)^j \cos\left(\frac{n j 2\pi}{2N + 1}\right) \frac{m}{E_n}
$$

$$
= -\frac{\zeta}{4} + \frac{1}{2N + 1} \sum_{n=1}^{N} \cos\left(\frac{2(2\xi + 1 + \frac{N j 2\pi}{2N + 1})}{2N + 1}\right) \frac{m}{E_n}
$$

$$
= \frac{\text{sign}(m) - \zeta}{4}, \tag{E4}
$$

where

$$
\zeta(m) = \frac{2}{2N + 1} \sum_{n=1}^{N} \frac{m}{E_n} = \frac{2}{2\pi} \int_0^\pi dk \frac{m}{\sqrt{m^2 + 4t^2}\cos^2(k/2)}
$$

$$
= \frac{2}{\pi} \frac{m}{\sqrt{m^2 + 4t^2}} K\left(\frac{4t^2}{m^2 + 4t^2}\right), \tag{E5}
$$

is the difference of the bulk filling factors of the two sublattices and

$$
K(x) = \int_0^{\pi/2} d\theta \frac{1}{\sqrt{1 - x \sin^2(\theta)}}
$$

is the complete elliptic integral of the first kind. Here we have used

$$
\frac{1}{4\pi} \int_0^\pi dk \frac{m \cos\left[\frac{(2\xi + 1)k}{2}\right]}{\sqrt{m^2 + 4t^2}\cos^2\left(\frac{k}{2}\right)\cos\left(\frac{k}{2}\right)} = \frac{\text{sign}(m)}{4},
$$

(E7)

which is valid up to corrections which decay exponentially with increasing $\xi$. At the other end of the chain there is a charge $-q_{\text{end}}$.

Let’s now consider a sharp DW between regions with mass $m_i$ and $m_j$. If we first assume that we turn off the hopping connecting these two regions, we find that the region with mass $m_i$ gives rise to the charge $-q_{\text{end},i}$ and the region with mass $m_j$ gives rise to a charge $q_{\text{end},j}$ at the DW so that the total charge is

$$
q_{\text{DW}} = \frac{\text{sign}(m_j) - \text{sign}(m_i)}{4} - \frac{\delta_{ji}}{2}, \tag{E8}
$$

where we have defined

$$
\delta_{ji} = \frac{\zeta(m_j) - \zeta(m_i)}{2}. \tag{E9}
$$

This charge is exponentially localized at the DW and therefore when the hopping connecting the regions is turned on it causes only a local perturbation in the Hamiltonian (slightly redistributing the charge density locally) but does not influence the total charge localized at the DW. If both $m_i$ and $m_j$ have the same sign the regions are in the same topological phase and the $q_{\text{DW}} = -\delta_{ji}/2$. If the signs of the masses are different we have DW between two topologically distinct phases. We now consider two possible DWs: (i) Soliton where mass changes from $m_1 > 0$ to $m_2 < 0$ as a function of increasing $x$ and (ii) antisoliton where mass changes from $m_2 < 0$ to $m_1 > 0$ as a function of increasing $x$ (see Figs. 1 and 2 in the main text.) We denote $\delta = \delta_{ji}$.

(i) In the case of soliton the sharp DW carries a charge $q_{\text{DW}} = -(1 + \delta)/2$. By increasing the width of the DW we find that the spectral flow is such that a state will approach zero energy from positive energies (see Fig. 2 in the main text). This means that if this zero-energy state is unoccupied the charge of the DW is $q_{\text{DW}}$ and if it is occupied the charge is $q_{\text{DW}} + 1$. Thus we can summarize that the possible charges for soliton are

$$
q_{\text{DW}} = \pm \frac{1}{2}, \tag{E10}
$$

(ii) For antisoliton the charge of the sharp DW is $q_{\text{DW}} = (1 + \delta)/2$. By increasing the width of the DW we find that the spectral flow is such that a state will approach zero energy from negative energies. This means that if this zero-energy state is unoccupied the charge of the DW is $q_{\text{DW}} - 1$ and if it is occupied the charge is $q_{\text{DW}}$. Thus we can summarize that the possible charges for antisoliton are

$$
q_{\text{DW}} = \pm \frac{1}{2}. \tag{E11}
$$

In the case of multimode SSG system we can separate the transverse modes $n$ as discussed in Appendix B. We consider the cases (a) $L_y$ even and (b) $L_y$ odd separately.

(a) When $L_y$ is even the Hamiltonian can be decomposed into 4x4 blocks given by Eq. (B2). Each of these blocks $n = 1, \ldots, L_y/2$ supports symmorphic chiral symmetries $C_{yx} = \sigma_x \sigma_y$ and $C_{zy} = \sigma_z \sigma_y$. Therefore using the argument given in Ref. [9] we find that each transverse mode $n$ carries possible charges $q_n = -N_n/2, \ldots, N_n/2$, where $N_n$ is the number of zero-energy states at the DW supported by transverse mode $n$ and the value of charge $q_n$ is determined by the number of occupied zero-energy states. In the case of smooth DWs each transverse mode supports $N_n = 2$ zero-energy states if $\nu_n$ is different on the two sides of the DW so that $q_n = -1, 0, 1$ (with two possible states corresponding to $q_n = 0$). If $\nu_n$ is the same on both sides then $N_n = 0$ and $q_n = 0$. The total charge of the DW is

$$
q_{\text{DW}} = \sum_{n=1}^{L_y/2} q_n. \tag{E12}
$$

(b) When $L_y$ is odd the Hamiltonian can be decomposed into $|L_y/2|$ 4x4 blocks obeying the same symmorphic chiral symmetries. Each of these modes carries charges $q_n = -1, 0, 1$ ($n = 1, \ldots, |L_y/2|$). Additionally there exists one transverse mode $n = 0$ which
is similar as the one studied in the case of single-mode SSG chain. Thus, this transverse mode carries a charge $q_0 = (\pm 1 - \delta)/2$ in the case of solitons and $q_0 = (\pm 1 + \delta)/2$ in the case of antisolitons. The total charge of the DW is

$$q_{DW} = \sum_{n=0}^{[L_y/2]} q_n.$$  \hspace{1cm} (E13)