Unquenching the Topological Susceptibility with an Overlap Action
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We estimate the quark-mass dependence of the topological susceptibility with dynamical overlap and clover fermions. Unquenching effects on the susceptibility turn out to be well approximated by a reweighting of a quenched ensemble with a low-eigenmode truncation of the fermionic determinant. We find that it is most likely due to the explicit chiral symmetry breaking of the fermion action that present day dynamical simulations do not show the expected suppression of the topological susceptibility.

One of the most profound effects the inclusion of dynamical fermions is expected to have on the QCD vacuum is the suppression of fluctuations of the topological charge. Due to the index theorem in the continuum, the Dirac operator has $|Q|$ zero modes in the charge $Q$ sector. In the presence of $N_f$ degenerate flavours of light dynamical quarks, in the charge $Q$ sector the fermionic determinant is proportional to $m^{(Q|N_f)}$ (to lowest order in $m$). Light dynamical fermions are therefore expected to suppress higher topological sectors and thus also the topological susceptibility, $\chi = \langle Q^2 \rangle / V$.

On a more quantitative level, for small enough quark masses chiral perturbation theory predicts that

$$\chi_{LS} = \frac{\Sigma m}{N_f} = \frac{f_\pi^2 m_\pi}{2N_f},$$

where $\Sigma$ is the chiral condensate and $m_\pi$ and $f_\pi$ are the pion mass and decay constant [1]. This holds provided the volume is large enough that chiral symmetry is effectively broken, i.e. $x = \Sigma m V \gg 1$.

At the other extreme, i.e. for heavy quarks, one expects the susceptibility to approach a constant, namely the quenched value $\chi_{\text{quenched}} = (203 \pm 5\text{MeV})^4$ [2]. The interpolation of the susceptibility from light to heavy quarks has recently been discussed by S. Dürr [3]. He noticed that besides the light dynamical quarks the finite volume of the box also suppresses fluctuations of the topological charge, since only a finite number of instantons can be accommodated in a finite box. Assuming that the two suppression mechanisms work independently, Dürr derived and essentially unique formula for the susceptibility. In terms of the pion mass it reads

$$\chi(m_\pi) = \left( \frac{2N_f}{m_\pi^2 f_\pi^2} + \frac{1}{\chi_{\text{quenched}}} \right)^{-1},$$

having two free parameters, $f_\pi$ and $\chi_{\text{quenched}}$. Dürr also pointed out that — most probably due to scaling/chirality violations — all the presently available dynamical susceptibility data lie significantly above this curve.

We expect that the suppression of the susceptibility depends crucially on the index theorem which in general does not hold on the lattice [4]. The reason for this is that Dirac operators that explicitly break chiral symmetry (e.g. the Wilson operator) do not have exact zero modes and besides, the very definition of the topological charge is ambiguous on the lattice. It is thus not clear how exactly light dynamical fermions suppress higher $|Q|$ sectors on the lattice. The aim of the present work is to check whether improving the chirality of the fermion action can have a significant effect on the topological susceptibility. In what follows, the topological charge is always defined with my overlap Dirac operator by counting zero modes. Note that this implies an exact “in-
Theorem” which in fact is a tautology with this charge definition.

Unfortunately chiral fermion actions are still tremendously expensive and a full-fledged dynamical simulation is out of question. It turns out however that in the present case that might not be needed. A reasonable estimate of the quark-mass dependence of the susceptibility can be obtained by reweighting a quenched ensemble with a low-mode truncation of the fermionic determinant. Computing the lowest $N$ eigenvalues of the Dirac operator $D$, the determinant can be written as

$$\det(D + m) \approx \prod_{k=1}^{N} (\lambda_k + m) \times \ldots$$

which in turn provides the full quark-mass and $N_f$ dependence of the topological susceptibility. Notice that while the truncation (3) with $N$ small, does not provide a good approximation to the full determinant, the topological charge is correlated only with the low-end of the Dirac spectrum. Therefore, the difference in the effective action between the various charge sectors can be reliably estimated even with $N$ reasonably small.

Reweighting is generally a bad procedure since control over the errors is lost exponentially fast with increasing volume. In certain cases, however it can happen that there is still a large enough window in the volume where interesting results can be obtained (see e.g. [9] for a recent example). In the following I give some reasons why reweighting could work for the topological susceptibility.

First of all, there are two sources of the fluctuations in the fermionic determinant. Eigenvalues $\lambda_k$ with $k > N$, which is by far the largest contribution since with our choice of $N$, most of the eigenvalues belong to this sector. This part does not appear in our approximation. For light quarks an important contribution to the fluctuations comes from the zero modes but this is not harmful since this is exactly the quantity we want to measure: the difference in the effective action between the various charge sectors. Secondly, as the sea quark-mass changes, in principle, all the other parameters of the theory (lattice spacing, etc.) change. But this is not the case if only the low eigenvalues are used for the reweighting. Clearly e.g. the heavy quark potential is expected to be the same in all topological sectors. In spite of all this, the results that I present here have to be considered as preliminary. The error bars are estimated with a jack-knife analysis and a more careful error analysis will appear in a future publication. This is certainly needed to be able to judge how far one can trust the results in the small quark mass limit.

The simulations were done with the Wilson plaquette action at $\beta = 5.85$ corresponding to a lattice spacing of about $0.12 - 0.13$fm. The volume of the box was set to $2.5 fm^4$ which turned out to be a good compromise; this is already a physically reasonable volume (if somewhat small) and at the same time the reweighting still works reasonably well. The overlap action that I use is constructed from 10 times APE smeared fat links [6] and the $c_{sw} = 1.0$ clover action. The results are based on 300 configurations and the full run took about 10-20 PC-months.

![Figure 1. The spectral density of the overlap Dirac operator in different $Q$ sectors.](image)
around $\lambda = 0.2 - 0.4$ which at the given volume translates into taking the lowest $N = 10 - 30$ eigenvalues.

In Fig. 2 I show the pion mass dependence of the topological susceptibility computed with truncations of $N = 16, 24, 32$. The steep line starting at the origin is the Leutwyler-Smilga prediction \cite{leutwyler1992} and Dürr’s formula is also included using $f_\pi = 93\text{MeV}$ and $\chi_{\text{quenched}} = (200\text{MeV})^4$. The horizontal shaded region is the quenched estimate for the susceptibility from the same data set. One expects the predictions based on broken chiral symmetry to be valid only if $x \gg 1$. At the given volume the value of the parameter $x$ happens to roughly coincide with $(m_\pi r_0)^2$, so chiral symmetry is effectively broken if $(m_\pi r_0)^2 \gg 1$. There are some indications that even in that regime, there are considerable finite volume corrections to the unquenched susceptibility, much more so than in the quenched case. A more detailed study of this question will appear in a separate publication.

The same method can be used to compute the susceptibility with the clover action. For this I take the non-perturbatively determined clover coefficient $c_{\text{sw}} = 1.91 \,^{[7]}$. The results are shown in Fig. 3 in a fashion similar to Fig. 2. Here I also plotted the UKQCD results \cite{ukqcd2001} obtained with $N_f = 2 \, O(a)$ improved Wilson fermions.

On the same set of configurations the clover action produces a significantly higher susceptibility than the overlap. This strongly indicates that chirality of the fermion action can be important for the topology. The error bars in the clover case are bigger than for the overlap, mainly due to the weaker correlation between the topological sectors and the fermionic determinant. This is even more pronounced in the case of unimproved Wilson fermions (not shown in the Fig.), where this makes it impossible to obtain sensible results using reweighting and the available statistics.

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