On Improving the Backjump Level in PB Solvers

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Abstract
Current PB solvers implement many techniques inspired by the CDCL architecture of modern SAT solvers, so as to benefit from its practical efficiency. However, they also need to deal with the fact that many of the properties leveraged by this architecture are no longer true when considering PB constraints. In this paper, we focus on one of these properties, namely the optimality of the so-called first unique implication point (1-UIP). While it is well known that learning the first assertive clause produced during conflict analysis ensures to perform the highest possible backjump in a SAT solver, we show that there is no such guarantee in the presence of PB constraints. We also introduce and evaluate different approaches designed to improve the backjump level identified during conflict analysis by allowing to continue the analysis after reaching the 1-UIP. Our experiments show that sub-optimal backjumps are fairly common in PB solvers, even though their impact on the solver is not clear.

1 Introduction

The CDCL architecture [14] and the use of efficient data structures and heuristics [4, 16] are at the core of the practical efficiency of modern SAT solvers. Even though these solvers can deal with very large benchmarks, containing millions of variables and clauses, some relatively small problems (with only few variables and clauses) remain completely out of their reach. This is particularly true for problems that require the ability to “count”, such as those known as pigeonhole-principle formulae, stating that \( n \) pigeons cannot fit into \( n - 1 \) holes. For such problems, the resolution proof system used internally by SAT solvers is too weak: it can only prove unsatisfiability with an exponential number of derivation steps [7].

This has motivated the generalization of the CDCL architecture to handle pseudo-Boolean (PB) problems [18]. Doing so, one can take advantage of the strength of the cutting planes proof system [6], which \( p \)-simulates resolution: any resolution proof can be translated into a cutting planes proof of polynomial size w.r.t. the size of the original proof. A common subset of cutting planes is the one known as generalized resolution [8]. It is implemented in many PB solvers [2, 3, 11, 19] to replace the use of the resolution proof system during conflict analysis to learn new constraints.

However, implementing the cutting planes proof system inside the CDCL architecture also requires to deal with many broken invariants. For instance, during conflict analysis, additional operations may be required to ensure that the conflict is preserved after each derivation step, and many different schemes have been proposed to this end [5, 10]. The need for such operations makes cutting planes-based PB solvers often slower in practice than resolution-based SAT solvers, especially on CNF instances: generalized resolution degenerates to resolution on such inputs.

In this paper, we consider another important difference between SAT solvers and PB solvers, regarding the optimality of the first unique implication point (1-UIP). In SAT solvers, it is well known that learning the first constraint that is assertive is optimal (i.e., gives the highest possible backjump level given the conflict) [1]. It also provides a convenient and efficient (syntactic)
criterion for determining when the conflict analysis must be stopped, that is, when only one literal in the derived clause is assigned at the decision level at which the conflict occurred. In PB solvers, this is not the case anymore. First, determining whether a PB constraint is assertive and at which level it is requires to perform arithmetic operations on the coefficients, which may be costly because of the use of arbitrary precision encodings. Second, learning the first assertive constraint encountered during the analysis does not guarantee to perform the highest possible backjump: we show that continuing the analysis may allow to find a better (i.e., higher) backjump level.

In this paper, we investigate different approaches towards improving the backjump levels computed by PB solvers. After having introduced the PB solving framework in Section 2, we illustrate the sub-optimality of the backjump levels computed by PB solvers in Section 3. Based on these examples, we introduce in Section 4 different strategies for improving the backjump levels in PB solvers by performing additional cancellation steps after having derived an assertive constraint. In particular, we present a criterion ensuring that the backjump level never gets worse when the analysis continues. We also introduce strategies allowing to detect when the analysis should stop. Finally, in Section 5, we empirically evaluate these approaches on the benchmarks of the PB competitions since their first edition [13]. We show that these new strategies allow to improve the computed backjump levels, even though the impact of this improvement on the performance of the solver is not clear.

2 Preliminaries

We consider a propositional setting defined on a finite set of propositional variables $V$. A literal $\ell$ is a variable $v \in V$ or its negation $\bar{v}$. Boolean values are represented by the integers 1 (true) and 0 (false), so that $\bar{v} = 1 - v$.

A pseudo-Boolean (PB) constraint is an integral linear equation or inequation over Boolean variables of the form $\sum_{i=1}^{n} \alpha_i \ell_i \triangleq \delta$, in which the coefficients (or weights) $\alpha_i$ and the degree $\delta$ are integers, $\ell_i$ are literals and $\triangle \in \{<, \leq, =, \geq, >\}$. Such a constraint can be normalized in linear time into a conjunction of constraints of the form $\sum_{i=1}^{n} \alpha_i \ell_i \geq \delta$ in which the coefficients and the degree are all non-negative integers. In the following, we thus assume, without loss of generality, that all PB constraints are normalized. A cardinality constraint is a PB constraint in which all coefficients are equal to 1 and a clause is a cardinality constraint of degree 1. This definition illustrates that PB constraints are a generalization of clauses, and that clausal reasoning is a special case of PB reasoning.

PB solvers have thus been designed to extend the CDCL algorithm of classical SAT solvers. In particular, when looking for a solution, PB solvers need to assign variables, either by making a decision or by propagating a truth value. In the following, we use the notation $\ell(V \oplus D)$ to represent that the literal $\ell$ has been assigned value $V$ at decision level $D$, and $\ell(?@?)$ to represent that $\ell$ is unassigned. Let us remark that, contrary to clauses, PB constraints may trigger propagations multiple times, and even when some other literals are not assigned yet, as illustrated below. Additionally, a PB constraint may also become conflicting after having triggered a propagation at an earlier decision level.

**Example 1.** The PB constraint $8a(?@?) + 2b(?@?) + c(?@?) + d(?@?) \geq 10$ propagates the literal $a$ at decision level 0. Later on, if $d$ becomes falsified, say at decision level 3, then the constraint $8a(1@0) + 2b(?@?) + c(?@?) + d(0@3) \geq 10$ propagates $b$ under the current partial assignment.
To detect whether a PB constraint is assertive (i.e., propagates some literals), it is often convenient to compute the slack of this constraint [2,3].

**Definition 1 (Slack).** Let \( \chi \) be the PB constraint given by \( \sum_{i=1}^{n} \alpha_i \ell_i \geq \delta \). The slack of \( \chi \) under the current partial assignment is the value \( \sum_{i=1, \ell_i \neq 0}^{n} \alpha_i - \delta \).

**Example 2.** The slack of the constraint \( 8a(1@0) + 2b(?@?) + c(?@?) + d(0@3) \geq 10 \) is \( 8 + 2 + 1 - 10 = 1 \) at decision level 3.

Thanks to the slack of a constraint, it is possible to detect whether a constraint is assertive or conflicting. More precisely, a constraint propagates a literal \( \ell \) if the coefficient of \( \ell \) in the constraint is strictly greater than the slack, and a constraint is conflicting when its slack is negative.

**Example 3 (Example 2 cont’d).** At decision level 3, the slack of the constraint \( 8a(1@0) + 2b(?@?) + c(?@?) + d(0@3) \geq 10 \) is 1. This constraint propagates thus \( b \) at decision level 3, since \( 2 > 1 \).

When several assignments have been made by the solver, a conflict may occur (i.e., a constraint may be falsified). When this is the case, a conflict analysis is performed by applying the cancellation rule between the conflicting constraint and the reason for the propagation of some of its literals, so as to derive a new constraint ("LCM" denotes the least common multiple):

\[
\alpha \ell + \sum_{i=1}^{n} \alpha_i \ell_i \geq \delta \quad \beta \ell + \sum_{i=1}^{n} \beta_i \ell_i \geq \delta' \implies \mu \alpha = \nu \beta = \text{LCM}(\alpha, \beta) \quad \text{(canc.)}
\]

Contrary to the resolution rule used in SAT solvers, this operation does not guarantee that the derived constraint will be conflicting. To check that it will be the case, it is once again possible to use the slack, which is particularly convenient as it is subadditive: the slack of the constraint obtained by applying the cancellation rule between two constraints is at most equal to the sum of the slacks of these constraints. This allows to estimate the slack of the constraint that will be derived by providing an upper bound of its value: whenever this upper bound is not negative, the constraint may be non-conflictual. To preserve the conflict when this is the case, a possible approach is to apply the weakening and saturation rules on the reason until its slack becomes low enough to ensure the conflict to be preserved (only literals that are not falsified may be weakened away) [3].

\[
\alpha \ell + \sum_{i=1}^{n} \alpha_i \ell_i \geq \delta \quad \sum_{i=1}^{n} \alpha_i \ell_i \geq \delta - \alpha \\ \text{(weakening)}
\]

\[
\sum_{i=1}^{n} \alpha_i \ell_i \geq \delta \quad \sum_{i=1}^{n} \min(\delta, \alpha_i) \ell_i \geq \delta \\ \text{(saturation)}
\]

The conflict analysis procedure of PB solvers consists of successive applications of the cancellation rule (and of the weakening and saturation rules when needed), following in reversed order the assignments on the solver’s trail (i.e., the assignment stack). We note that current PB solvers do not implement chronological backtracking techniques as those presented in, e.g., [15,17], so that the assignments in the trail are ordered by their decision level. The conflict analysis ends when the first assertive constraint is derived, as in SAT solvers.

Since PB constraints may propagate different literals at different decision levels (see Example 1), checking whether such a constraint is assertive and at which level it is is harder than when considering clauses or cardinality constraints. Indeed, in the case of a PB constraint, one needs to compute, for each literal \( \ell \) of the constraint, whether the constraint propagates a
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literal at the decision level at which \( \ell \) had been assigned (based on the slack at this decision level, for instance). Moreover, the PB constraint produced by the conflict analysis procedure may be assertive at different decision levels. In this case, the backjump must be performed at the first decision level at which the constraint is assertive, as current PB solvers always perform the propagations as soon as they appear (once again, no chronological backtracking techniques are implemented in such solvers).

3 Sub-Optimal Backjumps in PB Solvers

The conflict analysis procedure described in the previous section takes for granted the fact that learning a clause that is a first unique implication point (1-UIP) provides the optimal backjump level. While it is well known that, in SAT solvers, learning the first assertive clause guarantees to perform the highest possible backjump [1], this is not the case in PB solvers. In particular, we exhibit in this section two examples of sub-optimal backjump levels that may be computed when learning the first assertive constraint.

3.1 Pigeonhole-Principle Formulae

It is well known that, contrary to classical SAT solvers, PB solvers are very efficient at solving pigeonhole-principle formulae. In particular, while SAT solvers may need an exponential number of conflicts to prove the unsatisfiability of such formulae, PB solvers based on cutting planes can do so with a linear number of conflicts.

Let us solve such a problem with 4 pigeons and only 3 holes following the approach of a PB solver. Consider the following encoding of this problem, in which \( p_{i,j} \) (1 \( \leq i \leq 4, 1 \leq j \leq 3 \)) represents that pigeon \( i \) is put in hole \( j \):

\[
H_1 \equiv p_{1,1} + p_{2,1} + p_{3,1} + p_{4,1} \leq 1 \quad \quad P_1 \equiv p_{1,1} + p_{1,2} + p_{1,3} \geq 1
\]

\[
H_2 \equiv p_{1,2} + p_{2,2} + p_{3,2} + p_{4,2} \leq 1 \quad \quad P_2 \equiv p_{2,1} + p_{2,2} + p_{2,3} \geq 1
\]

\[
H_3 \equiv p_{1,3} + p_{2,3} + p_{3,3} + p_{4,3} \leq 1 \quad \quad P_3 \equiv p_{3,1} + p_{3,2} + p_{3,3} \geq 1
\]

The constraints on the left specify that “each hole cannot contain more than one pigeon”, while the constraints on the right specify that “each pigeon must be put in a hole”. The normalized form of the constraints above is given by:

\[
H_1 \equiv \bar{p}_{1,1} + \bar{p}_{2,1} + \bar{p}_{3,1} + \bar{p}_{4,1} \geq 3 \quad \quad P_1 \equiv p_{1,1} + p_{1,2} + p_{1,3} \geq 1
\]

\[
H_2 \equiv \bar{p}_{1,2} + \bar{p}_{2,2} + \bar{p}_{3,2} + \bar{p}_{4,2} \geq 3 \quad \quad P_2 \equiv p_{2,1} + p_{2,2} + p_{2,3} \geq 1
\]

\[
H_3 \equiv \bar{p}_{1,3} + \bar{p}_{2,3} + \bar{p}_{3,3} + \bar{p}_{4,3} \geq 3 \quad \quad P_3 \equiv p_{3,1} + p_{3,2} + p_{3,3} \geq 1
\]

Let us now try to assign some variables, until we get a conflict. We first assign \( p_{1,1} \) to 0 at decision level 1 and \( p_{1,2} \) to 0 at decision level 2. At this point, we have that:

- \( P_1 \) propagates \( p_{1,3} \) to 1, and
- \( H_3 \) propagates then \( p_{2,3}, p_{3,3} \) and \( p_{4,3} \) to 0.
If we now assign \( p_{2,1} \) to 0, we have that:

- \( P_2 \) propagates \( p_{2,2} \) to 1,
- \( H_2 \) propagates then \( p_{3,2} \) and \( p_{4,2} \) to 0,
- \( P_3 \) and \( P_4 \) now propagate \( p_{3,1} \) and \( p_{4,1} \) to 1, respectively, and
- \( H_1 = \bar{p}_{1,1}(1@1) + \bar{p}_{2,1}(1@3) + \bar{p}_{3,1}(0@3) + \bar{p}_{4,1}(0@3) \geq 3 \) is conflicting.

The conflict is thus analyzed by applying successively the cancellation rule between the conflicting constraint \( H_1 \) and the reasons for \( p_{4,1} \) and \( p_{3,1} \) being propagated to 1, i.e., \( P_4 \) and \( P_3 \), respectively.

\[
\begin{array}{c|c}
H_1 & P_4 \\
\hline
\bar{p}_{1,1}(1@1) + \bar{p}_{2,1}(1@3) + \bar{p}_{3,1}(0@3) + \bar{p}_{4,2}(0@3) + p_{4,3}(0@2) \geq 3 & P_3 \\
\bar{p}_{1,1}(1@1) + \bar{p}_{2,1}(1@3) + p_{3,2}(0@3) + p_{3,3}(0@2) + p_{4,2}(0@3) + p_{4,3}(0@2) \geq 3 \\
\end{array}
\]

The constraint is still not assertive, so we apply the cancellation rule between the inferred constraint and the reason for both \( p_{3,2} \) and \( p_{4,2} \) being propagated to 0, i.e., \( H_2 \).

\[
\begin{array}{c|c}
\bar{p}_{1,1}(1@1) + \bar{p}_{2,1}(1@3) + p_{3,2}(0@3) + p_{3,3}(0@2) + p_{4,2}(0@3) + p_{4,3}(0@2) \geq 3 & H_2 \\
\bar{p}_{1,1}(1@1) + \bar{p}_{1,2}(1@2) + \bar{p}_{2,1}(1@3) + p_{2,2}(0@3) + p_{3,3}(0@2) + p_{4,3}(0@2) \geq 4 \\
\end{array}
\]

Observe that this constraint propagates \( p_{2,1} \) and \( p_{2,2} \) to 1 at decision level 2. This constraint is thus assertive, so the solver learns it, and backjumps to decision level 2, before continuing to explore the search space. But can we do better than this?

Suppose we continue the analysis, and apply the cancellation rule between the assertive constraint above and the reason for \( p_{2,2} \) being assigned to 1, i.e., \( P_2 \).

\[
\begin{array}{c|c}
\bar{p}_{1,1}(1@1) + \bar{p}_{1,2}(1@2) + \bar{p}_{2,1}(1@3) + p_{2,2}(0@3) + p_{3,3}(0@2) + p_{4,3}(0@2) \geq 4 & P_2 \\
\bar{p}_{1,1}(1@1) + \bar{p}_{1,2}(1@2) + p_{2,3}(0@2) + p_{3,3}(0@2) + p_{4,3}(0@2) \geq 3 \\
\end{array}
\]

We note that there is no more literal assigned at decision level 3 in the derived constraint, but it is *conflicting* at decision level 2. This is not so surprising, as finding a conflict at a higher decision level may also happen when performing a “regular” conflict analysis in a PB solver. Let us thus continue to analyze this conflict. The next cancellation to perform is with the reason for \( p_{2,3} \), \( p_{3,3} \) and \( p_{4,3} \) being assigned to 0, i.e., \( H_3 \).

\[
\begin{array}{c|c}
\bar{p}_{1,1}(1@1) + \bar{p}_{1,2}(1@2) + p_{2,3}(0@2) + p_{3,3}(0@2) + p_{4,3}(0@2) \geq 3 & H_3 \\
\bar{p}_{1,1}(1@1) + \bar{p}_{1,2}(1@2) + \bar{p}_{1,3}(0@2) \geq 3 \\
\end{array}
\]

This constraint is assertive at decision level 0, and propagates to 0 the variables \( p_{1,1} \), \( p_{1,2} \) and \( p_{1,3} \). We can still perform a last cancellation step, between the constraint above and the reason for \( p_{1,3} \) being propagated to 1, i.e., \( P_1 \).

\[
\begin{array}{c|c}
\bar{p}_{1,1}(1@1) + \bar{p}_{1,2}(1@2) + \bar{p}_{1,3}(0@2) \geq 3 & P_1 \\
0 \geq 1 \\
\end{array}
\]
We have thus been able to derive $0 \geq 1$ (i.e., $\bot$) in a single conflict analysis for this pigeonhole instance. Actually, it is easy to see that, for any pigeonhole instance with $n$ pigeons to put in $n-1$ holes, it is always possible to prove unsatisfiability with a single conflict if we do not stop the analysis when the first assertive constraint is derived. This also shows that learning the first assertive constraint is not optimal in general. However, continuing conflict analysis does not guarantee to always improve the backjump level, as shown in the next example.

3.2 Choosing the Right Constraints

As already mentioned, it is well known that it is not possible to find a higher backjump level than that provided by the 1-UIP in SAT solvers. In the following example, we show that the backjump level may also increase when performing additional cancellation steps with PB constraints.

Suppose that the solver is run on a given instance, and that, during the search, the constraint $\chi \equiv 4a(0@10) + 4b(1@30) + 3c(0@20) + 3d(0@30) + 2e(1@30) + f(0@40) + g(0@40) + z(0@40) \geq 8$ becomes conflicting. Suppose that the reason for $f$ and $g$ being propagated to 0 is $\rho_1 \equiv 3i(0@20) + 3j(0@40) + 2f(1@40) + 2g(1@40) + h(1@40) \geq 5$. The cancellation rule is applied between these two constraints.

\[
\chi \equiv 4a(0@10) + 4b(1@30) + 3c(0@20) + 3d(0@30) + 2e(1@30) + f(0@40) + g(0@40) + z(0@40) \geq 8
\]

\[
\rho_1 \equiv 3i(0@20) + 3j(0@40) + 2f(1@40) + 2g(1@40) + h(1@40) \geq 5
\]

Let $\chi'$ be the constraint derived above. $\chi'$ is assertive, and propagates $b$ at decision level 20. This constraint is thus learned, and a backjump is performed. Once again, let us try to find a higher backjump level. Suppose that the reason for $j$ is $\rho_2 \equiv 6c(1@20) + 6d(1@30) + 3j(1@40) + 3k(0@40) + 3l(0@30) \geq 15$, and let us apply the cancellation rule between this constraint and $\chi'$.

\[
\chi' \equiv 4a(0@10) + 4b(1@30) + 3c(0@20) + 3d(0@30) + 2e(1@30) + f(0@40) + g(0@40) + h(1@40) \geq 17
\]

\[
\rho_2 \equiv 6c(1@20) + 6d(1@30) + 3j(1@40) + 3k(0@40) + 3l(0@30) \geq 15
\]

The constraint obtained here is also assertive and propagates now $b$ at decision level 10. We thus managed to improve the backjump level. However, suppose that, instead of applying the cancellation rule with the reason for $j$ above, we had applied it with the reason for $z$ given by $\rho_3 \equiv 10w(1@25) + 10x(0@25) + y(0@40) + z(1@40) \geq 11$.

\[
\chi'' \equiv 20w(1@25) + 20x(0@25) + 8a(0@10) + 8b(1@30) + 6c(0@20) + 6d(0@30) + 4e(1@30) + 3i(0@20) + 3j(0@40) + 2g(0@40) + h(1@40) \geq 37
\]

Even though the constraint is still assertive, it propagates now $b$ at decision level 25. So, continuing conflict analysis may also worsen the backjump level, as it is the case for clauses. As shown in this example, it may even depend on the order of the propagations on the trail. To make sure that continuing the analysis never worsens the backjump level, we thus need to design appropriate criteria for deciding when to continue.

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4 Towards an Improvement of the Backjump Level

In this section, we introduce an approach for improving the backjump level computed by PB solvers by allowing to continue the analysis after an assertive constraint has been derived. To this end, we extend the classical conflict analysis procedure of PB solvers by considering two new criteria that are used after the first assertive constraint has been produced to decide whether additional cancellations should be performed, and on which constraints. These criteria are designed to make sure that an assertive constraint will indeed be derived at the end of the analysis, while optimizing the height of the computed backjump level.

4.1 Non-Worsening Backjump Level

As observed in the previous section, continuing the analysis does not guarantee to improve the computed backjump level. In the worst case, the assertion may even be lost, which would break the CDCL algorithm. We thus need to introduce a new invariant for the conflict analysis procedure: if the conflicting constraint is assertive at decision level \(b\), the constraint that will be obtained after applying the cancellation rule between this constraint and a reason must be either assertive at decision level \(b\) or conflicting at decision level \(b\). It is easy to see that such an invariant guarantees that an assertive constraint at decision level \(b\) or higher will be eventually derived. Let us now describe how to maintain it.

Suppose that the conflicting constraint is assertive at decision level \(b\), and that we want to perform a cancellation with a reason. The resulting constraint will be assertive at decision level \(b\) if it propagates (at least) one of its literals. In particular, (at least) one of its literals must have a coefficient that is strictly greater than the slack of this constraint computed at decision level \(b\). We note that, in this particular case, it is preferable to compute the exact slack (rather than using the subadditive property), to limit the number of false negative. Similarly, the slack may be used to check whether the constraint will be conflicting at decision level \(b\).

When the value of the slack guarantees that the constraint will remain assertive or conflicting, the cancellation rule can be applied safely. Otherwise, one can simply ignore the reason and go on to the next propagation on the trail (as a "regular" conflict analysis would have ignored it anyway). Another solution is to use an approach similar to that used to preserve conflicts during a "regular" conflict analysis. More precisely, one can try to weaken away some of the literals of the reason until the constraint obtained after applying the cancellation is assertive or conflicting at decision level \(b\).

Example 4. Suppose that, during conflict analysis, the constraint
\[
4a(0@5) + 4b(0@3) + 4c(0@4) + 4d(0@3) + e(0@1) + f(0@2) + g(0@4) + h(0@5) \geq 4
\]
is derived. This constraint is assertive at decision level 4, and propagates \(a\) to 0 at this decision level. Suppose now that the reason for \(a\) being propagated to 1 is the constraint
\[
2a(1@5) + 2b(1@3) + 2c(1@4) + 2g(0@4) + 2h(0@5) \geq 5.
\]
If we resolve this reason with the assertive constraint above, the derived constraint is a tautology, and is thus not assertive at any decision level. However, if the literals \(b\) and \(c\) are weakened away from the reason to get the clause
\[
a(1@5) + g(0@4) + h(0@5) \geq 1\]
(which still propagates \(a\) at decision level 5), the constraint obtained after applying the cancellation with the assertive constraint above is
\[
2b(0@3) + 2c(0@4) + 2d(0@3) + 2g(0@4) + 2h(0@5) + e(0@1) + f(0@2) \geq 2
\]
which is still assertive at decision level 4, and propagates now \(h\) to 1 at this decision level.

Note that, contrary to the case in which the weakening and saturation rules are applied to preserve the conflict, there is no guarantee that applying these rules will eventually preserve the assertivity or the conflict at decision level \(b\). In particular, the successive weakening operations may produce a tautology from the reason, and the cancellation step must be ignored. We also
note that this is compatible with the 1-UIP scheme in SAT solvers, as weakening a literal from a clause always produces a tautology.

Remark that, when a new assertive constraint is derived, the approach above guarantees that it propagates a literal at decision level \( b \). However, this literal may be propagated at a decision level \( b' \) that is higher than \( b \). This is precisely what allows to improve the backjump level. For the subsequent application of the cancellation rule, the decision level \( b' \) must thus be used instead of \( b \) when ensuring to maintain the invariant.

4.2 Ending Conflict Analysis

Since deriving an assertive constraint is no longer a sufficient stop condition for the conflict analysis, we need a new criterion for determining when to stop it.

We remark that, if the assertive constraint we have derived is assertive at decision level \( b \), we should not apply the cancellation rule between this constraint and reasons for literals propagated at decision level \( b \) or higher. While doing so may still allow to find a better backjump level, there is no guarantee that it will be the case, which may lead to practical issues. Indeed, as assignments are undone at each cancellation step (even those that are ignored), it would be needed to redo all of them if the backjump level remains the same, and the backjump would become a forejump, which seems unreasonable. So, a naive criterion is to stop the analysis when the highest decision level on the trail is the latest computed backjump level.

Of course, this does not apply to the particular case in which the derived constraint has become conflicting at decision level \( b \). In this case, a “regular” conflict analysis must start again with the new conflicting constraint.

Let us note that the criterion above may be optimized in different ways. For instance, if at some point the derived assertive constraint propagates a literal at decision level 0, it will not be possible to find a higher backjump level, so the analysis can stop (even though one could still derive \( \perp \), as in the pigeonhole-principle example presented in the previous section). Similarly, if the constraint is conflicting at decision level 0, the analysis can stop early, as the problem is unsatisfiable in this case. We could also decide to stop the analysis when we consider the backjump level to be “high enough”, based on (parameterizable) numerical criteria.

We remark that the stopping criteria presented in this section are designed towards improving the backjump level computed by the conflict analysis procedure. However, different criteria may also be considered for determining when to stop the analysis, such as the quality of the derived constraint (e.g., with different quality measures as introduced in [12]), or its propensity to generate short proofs (e.g., by ensuring to derive constraints having a low slack).

5 Experimental Results

In this section, we present an empirical evaluation of the approach described in Section 4. All experiments presented in this section have been run on a cluster equipped with quadcore bi-processors Intel XEON E5-5637 v4 (3.5 GHz) and 128 GB of memory. The time limit was set to 1200 seconds and the memory limit to 32 GB. The whole set of decision benchmarks containing “small” integers used in the PB competitions since the first edition [13] was considered as input.
5.1 Implementation Details

To evaluate our approach, we implemented it in Sat4j-CuttingPlanes [11]. This implementation, which is available in a dedicated repository\(^1\), provides different variants of the two criteria presented in the previous section. They are described below.

In order to preserve the invariant that the derived constraint remains either assertive at a decision level \(b\) or conflicting at this decision level, we experimented the following variants:

- **never-weaken**: if the invariant is not preserved, no weakening is applied and the cancellation step is ignored.
- **weaken-any**: if the invariant is not preserved, a literal that is not falsified at decision level \(b\) is weakened iteratively until the invariant is restored or the reason becomes a tautology (in which case the cancellation step is ignored).
- **weaken-ordered**: similar to **weaken-any**, but weakened literals are preferably those unassigned or, assigned at the lowest decision level (in this order).

We also implemented different strategies for detecting when to stop the conflict analysis after having derived an assertive constraint. Suppose that the first literal propagated by this constraint is propagated at decision level \(b\). Our strategies are:

- **until-bjlevel**: the conflict analysis stops when the highest decision level on the trail is \(b\) (i.e., when all assignments made at a decision level higher than \(b\) have been undone).
- **until-toplevel**: the conflict analysis stops when the highest decision level on the trail is \(b\), or when \(b\) is 0.
- **until-highlevel**: the conflict analysis stops when the highest decision level on the trail is \(b\), or when \(b\) is 10\% of the highest decision level on the trail, so as to avoid making too many additional (and costly) cancellation steps.

We evaluated all these strategies, and the complete analysis of their results may be retrieved in a dedicated repository\(^2\). As there is no clear difference between them, this section only reports the results of the combination of **weaken-any** and **until-bjlevel**.

5.2 Effectiveness of our Approach

Let us first analyze to what extent continuing conflict analysis after having produced an assertive constraint allows to find higher backjump levels. To this end, we consider the percentage of conflict analyses for which our approach allowed to improve the backjump level. The results are given in Figure 1.

The boxplots on this figure show that our approach indeed allows to improve the backjump levels initially computed with the first assertive constraint, and that this is the case for many families of benchmarks. Moreover, we can see that for some families, this happens quite often in practice. For instance, for the **subsetcard** family, more than 20\% of the computed backjump levels are sub-optimal in average (and the median is also near 20\%).

Let us remark that, as constraints learned by the solver differ when using our approach, the boxplots here does not represent the percentage of sub-optimal backjump levels performed by the solver when it applies a “regular” conflict analysis, but those that we were able to improve with our approach.

\(^1\)https://gitlab.ow2.org/romain_wallon/sat4j/-/tree/assertion-level
\(^2\)https://gitlab.com/pb-backjump-level/pos21-experiments
Figure 1: Boxplots of the percentage of improved backjumps per family. Each box displays the quartiles with the vertical bars and the estimated minimum and maximum with the horizontal bars, computed from the percentage of improved backjumps per instance in each family. Points represent outliers, which are instances for which the percentage of improved backjumps is above the estimated maximum.
5.3 Impact of the Improved Backjump Levels

In order to evaluate the impact of the improved backjump levels, we first consider the size of the proofs built by the solver, as the approach we presented is penalized by the additional operations it requires (especially, the computation of the exact slack of the constraints). The results are given in Figures 2 and 3.

Figure 2: Scatter plot of the number of conflicts. Only instances that are solved by both approaches are reported, as the proof must be complete for a fair comparison.

Figure 3: Scatter plot of the number of cancellations. Only instances that are solved by both approaches are reported, as the proof must be complete for a fair comparison.

Figure 2 shows that, as expected, pigeonhole instances (from the roussel family) are always solved with a single conflict. Quite interestingly, the number of cancellations remains the same on these instances: in fact the same proof is built by a “regular” conflict analysis, but several conflicts are required to find it.
In Figure 3, we can see that the proof built by the solver on instances of the \textit{vertexcover} family is exponentially shorter when the conflict analysis continues. We looked closer at the behavior of the solver on these instances, and what we observed is that the first conflict that is analyzed occurs on the constraint \(\sum_{i=1}^{n} 2\bar{x}_i \geq n\). During the analysis of this conflict, the cancellation rule is successively applied between this constraint and all clauses of the form \(x_1 + x_j \geq 1\) \((2 \leq j \leq n)\), producing \(x_1 \geq 1\). As noticed in [9], the constraint that is learned when the conflict analysis stops at the first assertive constraint contains a number of irrelevant literals that cause the proof to be exponentially larger. When the analysis continues, all these irrelevant literals get cancelled out, thus allowing to find the shorter proof found when removing explicitly irrelevant literals.

Regarding the other families, there is no clear difference between continuing the analysis or stopping it at the first assertive constraint. For some families, continuing the analysis makes almost no difference. For some other families, there exist instances for which the proof is shorter when continuing the analysis, and others for which the proof is shorter when stopping at the first assertive constraint. This suggests that we lack an additional heuristic for determining when the analysis should continue, or that our criterion for determining when to stop the analysis is not good enough.

Let us now compare the runtime of our approach with that of the solver when it runs a “regular” conflict analysis. The results are given in Figure 4. Clearly, our approach remains costly in many cases, especially because checking that our invariant is preserved requires to perform a number of arithmetic operations, whose cost is increased by the use of arbitrary precision encodings. On the scatter plot, we can observe that only few instances are solved faster with our approach. These instances are those for which the size of the proof is reduced enough to counter-balance the added cost of the additional operations.
6 Conclusion

In this paper, we showed that, contrary to classical SAT solvers, the backjump level computed by PB solvers when learning the first assertive constraint derived during conflict analysis is not optimal in general. More precisely, if the conflict analysis continues after reaching such a constraint, it is possible to decrease the backjump level, and thus to backjump higher in the search tree. To make it possible, we presented and evaluated different strategies designed to preserve the invariants of the solver and to make sure to never worsen the computed backjump level while performing additional cancellation steps. While doing so indeed allows to improve the backjump levels computed during conflict analysis, the impact of this approach on the performance of the solver is unclear.

A perspective for future work is to find heuristics to decide when the conflict analysis should be continued after having derived the first assertive constraint and when it should not, for instance based on the quality of the derived constraints or on the (expected) size of the proof. Improving our different strategies for choosing on which constraints the additional cancellation steps should be applied is also an important perspective, especially to make them more efficient. It would also be interesting to investigate how to combine these strategies with chronological backtracking techniques that have recently been investigated for SAT solvers, but not exploited yet in PB solvers. Another avenue to explore is to find ways of continuing the analysis asynchronously, so as to allow the solver to learn the first assertive constraint and backjump as usual, while the learned constraint is refined to trigger a better backjump, only if it is worth doing so.

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