A HERETICAL VIEW
ON LINEAR REGGE TRAJECTORIES

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Abstract

We discuss a possibility that linear Regge trajectories originate not from gluonic strings connecting quarks, as it is usually assumed, but from pion excitations of light hadrons. From this point of view, at large angular momenta both baryons and mesons lying on linear Regge trajectories are slowly rotating thick strings of pion field, giving rise to a universal slope computable from the pion decay constant. The finite resonance widths are mainly due to the semiclassical radiation of pion fields by the rotating elongated chiral solitons. Quantum fluctuations about the soliton determine a string theory which, being quantized, gives the quantum numbers for Regge trajectories.

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1 Introduction

A long-standing problem of strong interactions is to understand linear Regge trajectories. Experimentally, most hadrons containing light quarks with spin $J$ have partners with the same quantum numbers but spin $J + 2, J + 4, \ldots$. Lines connecting these partners in the Chew–Frautchi plot have, with a few exceptions, parallel slopes $\alpha' \simeq 0.8 - 0.9 \text{GeV}^{-2}$. Certain trajectories seem to be parity and/or signature degenerate. In some cases parallel “daughter” trajectories are clearly seen. The present experimental status is shown in Figs. 1–4 where we plot all known non-strange hadrons from the 1986 Review of Particle Properties.

The usual qualitative explanation of resonances lying on linear trajectories is that they are rotating confining gluonic strings or flux tubes attached to quarks at the end points moving with the speed of light [1, 2]. The idea is rather old and is usually taken for granted. Good or bad, it ignores the spontaneous chiral symmetry breaking in QCD. Owing to it, nearly massless “current” quarks get a dynamical mass around 350 MeV, and
the lightest hadrons are (pseudo) Goldstone pions. It costs very little energy to produce a pion and hence the would-be strings of the pure glue world have to break [3].

Even if one takes for granted the flux tube model for Regge trajectories, there is still a long way to the realistic spectrum. Much work in that direction has been done [2, 4, 5, 6, 7] but to our knowledge many important questions remain unanswered. Why would 350 MeV quarks bound by a flux tube with a $\sqrt{\sigma} = 1/\sqrt{2\pi \alpha'} \simeq 425$ MeV string tension move with the speed of light is not very clear, but if they do not, the trajectories are not linear. Where, geometrically, is the baryons’ third quark? Is it close to one of the ends or smeared over the whole string? Why some trajectories are degenerate in signature and/or parity and some are not?

To point out one difficulty of the flux tube model, we note that the Regge spectrum apparently “knows” about the spontaneous chiral symmetry breaking. For instance, the pion trajectory starts from zero intercept exhibiting the Goldstone nature of pions. Another example: the baryon $1/2^+$ and $1/2^-$ trajectories seem to be almost degenerate, except for their first members, $N(940, 1/2^+)$ and $N(1535, 1/2^-)$. These particles cannot be degenerate if the chiral symmetry is spontaneously broken. Therefore, the $1/2^\pm$ trajectories have to split at the end. However the gluonic string would “feel” the chiral symmetry breaking only through quark loops, which is a $O(N_f/N_c)$ effect [8]. This quantity is considered to be small since it determines the ratio of the widths to the masses of resonances. But the ugly departure of the $1/2^-$ trajectory from a straight line does not look like a small effect.

In this paper we present an alternative view on linear Regge trajectories as due to the rotating pion fields. Let us give a few qualitative arguments in favor of such a heretical idea.

First, let us consider a “soft” (as opposed to rigid) rotating body (a hadron) and try to minimize its energy for given angular momentum $J$. Evidently, the minimum corresponds to a situation when the lightest piece of a body rotates at large distances around the heavy remnant. In other words, if a high $J$ hadron is considered as an excitation of a low $J$ one, the easiest way to get it is to excite the lightest degree of freedom, that is the pion field.

Second, large spin hadrons have presumably large sizes (at least in one direction) owing to the centrifugal forces. It means that fields inside such a hadron are slowly varying, in other words they have low momenta. The only degree of freedom of the strong interactions that survives at low momenta is the pseudo-Goldstone pion field. Therefore, it seems natural to describe the high $J$ hadrons in terms of pion excitations.

Third, it is known that even the lowest member of the nucleon Regge trajectory, i.e. the nucleon itself, can be understood as a soliton of the pion field. If the nucleon is a static chiral soliton, one is led to consider high $J$ excitations of nucleons as rotating chiral fields. The Skyrme model is the simplest realization of this idea [9]. A more fine model of a nucleon soliton has been proposed recently which seems to be more satisfactory, both philosophically and quantitatively [10].

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solitons.

Fourth, the pion itself belongs to a Regge trajectory which has approximately the same universal slope. It should not be accidental.

To complete the qualitative arguments, let us make a very rough estimate of the mass of a rotating chiral soliton. Since the characteristic size $r_0$ of the soliton will be shown to be parametrically large in $J$, it is sufficient to use only the kinetic energy term of the effective chiral action,

$$S_{\text{kin}} = \frac{F^2_\pi}{4} \int d^4x \sqrt{-g} \, g^{\mu\nu} \, \text{Tr} \, L_\mu L_\nu ,$$

where $L_\mu = i U^\dagger \partial_\mu U$, $U = \exp(i \pi^A \tau^A)$, $F_\pi \simeq 93 \text{ MeV}$. (1)

The energy of a static pion field configuration, as seen from Eq.(1), grows linearly with its size $r_0$, $E_{\text{rest}} \sim F^2_\pi r_0$ (we ignore numerical coefficients). If a soliton rotates, its rotation energy is $E_{\text{rot}} \simeq J^2/2I$, where $I$ is the moment of inertia. According to Eq.(1) $I$ grows as the third power of the size, $I \sim F^2_\pi r_0^3$. The total energy, $E_{\text{rest}} + E_{\text{rot}}$, is

$$E \sim F^2_\pi r_0 + \frac{J^2}{F^2_\pi r_0^3} .$$

This function has a minimum at $r_0 \sim \sqrt{J/F_\pi}$ (justifying that $r_0$ is large at large $J$) with the value at the minimum $E \sim F_\pi \sqrt{J}$. Hence, the mass $M_J$ of a rotating soliton with a large angular momentum $J$ satisfies the equation

$$J \simeq \alpha M_J^2 , \quad \alpha' \sim F^{-2}_\pi .$$

Thus we obtain linear Regge trajectories from a simple dimensional analysis. The Goldstone nature of pions is essential in this derivation.

However simple, this derivation has loopholes. First, we have introduced only one overall size $r_0$. It implies that the rotating soliton is spherically-symmetric (as in the case of a nucleon). But rotating hedgehogs have the same spin as isospin [9]. If one is interested in Regge trajectories with fixed isospin one has to consider non-spherically-symmetric solitons. Moreover, it is clear that centrifugal forces must stretch the rotating soliton into something cigar-like. Therefore, the above derivation has to be modified. This is performed in Sec.2. In fact we arrive to an effective string theory for large $J$ chiral solitons with a calculable slope $\alpha'$.

Second, strictly speaking, there exists no classical solution of the equations of motion, corresponding to a stationary rotation. Examples of this “no go” theorem are presented in Refs. [11, 12]. Its general cause is the same as in electrodynamics: accelerated charges must radiate e.m. fields. In our case a rotating chiral soliton, being an accelerated source of isospin, must radiate pion fields. Therefore, one can speak only of an approximately stationary rotation – as far as the energy loss per period is much less than the energy itself. However, high $J$ Regge excitations are not expected to be stable. Moreover, their
lifetime can be calculated from the classical pion radiation theory. This is performed in Sec. 3. A systematic way to study quantum corrections to rotating solitons is outlined in Sec. 4. Finally, Sec. 5 contains conclusions and an outlook.

2 String-like chiral solitons

Let us consider a chiral soliton rotating around the $z$ axis with angular velocity $\omega$. We mark by primes coordinates in the body-fixed frame. One has

$$
\begin{align*}
    x' &= x \cos \omega t + y \sin \omega t, \\
    y' &= -x \sin \omega t + y \cos \omega t, \\
    z' &= z, \\
    t' &= t.
\end{align*}
$$

The metric tensor in the body-fixed frame is

$$

g^{\alpha\beta}(x') = \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} g^{(0)\mu\nu} = \begin{pmatrix}
    1 & \omega y' & -\omega x' & 0 \\
    \omega y' & \omega^2 y'^2 - 1 & -\omega^2 x'y' & 0 \\
    -\omega x' & -\omega^2 x'y' & \omega^2 x'^2 - 1 & 0 \\
    0 & 0 & 0 & -1
\end{pmatrix}.
$$

The kinetic energy term of the effective chiral action (1) in this frame is

$$
S = \frac{F^2}{4} \int d^4x' g^{\mu\nu}(x') \Tr L_\mu L_\nu, \quad L_\mu = iU^\dagger \frac{\partial}{\partial x'^\mu} U.
$$

The Euler–Lagrange equation of motion reads

$$
\frac{\partial}{\partial x'^\mu} \left[ g^{\mu\nu}(x') U^\dagger(x') \frac{\partial}{\partial x'^\nu} U(x') \right] = 0.
$$

We look for a solution of this equation, which is (i) time-independent in the body-fixed frame, (ii) independent of one coordinate (we choose it to be $x'$ orthogonal to the rotation axis $z = z'$). With such restrictions Eq.(5) takes the form

$$
(1 - \omega^2 x'^2) \frac{\partial}{\partial y'} \left( U^\dagger \frac{\partial}{\partial y'} U \right) + \frac{\partial}{\partial z'} \left( U^\dagger \frac{\partial}{\partial z'} U \right) = 0.
$$

Introducing a Lorentz-contracted variable

$$
\tilde{y} = \frac{y'}{\sqrt{1 - \omega^2 x'^2}}, \quad \tilde{z} = z',
$$

we rewrite Eq.(6) as

$$
\frac{\partial}{\partial \tilde{y}} \left( U^\dagger \frac{\partial}{\partial \tilde{y}} U \right) + \frac{\partial}{\partial \tilde{z}} \left( U^\dagger \frac{\partial}{\partial \tilde{z}} U \right) = 0.
$$
Actually, it is the extremum condition for the transverse energy or energy per unit length in the \(x'\) direction,

\[
E_\perp = \frac{F^2}{4} \int d\tilde{y} d\tilde{z} \text{Tr} \left( \partial_{\tilde{y}} U \partial_{\tilde{y}} U^\dagger + \partial_{\tilde{z}} U \partial_{\tilde{z}} U^\dagger \right). \tag{9}
\]

It has been shown in Ref. [13] that Eq.(8) for the \(SU_2\) chiral field has no non-trivial solutions other than embeddings of two-dimensional grassmannian instantons. The simplest embedding is a two-dimensional hedgehog:

\[
U = \exp(i \pi A^A \tau^A), \quad \pi^1 = 0, \quad \pi^2 = \frac{\tilde{y}}{\sigma} P(\sigma),
\]

\[
\pi^3 = \frac{\tilde{z}}{\sigma} P(\sigma), \quad \sigma = \sqrt{\tilde{y}^2 + \tilde{z}^2}. \tag{10}
\]

The transverse energy (9) becomes

\[
E_\perp = \pi F^2 \pi \int_0^\infty d\sigma \sigma \left[ \left( \frac{\partial P}{\partial \sigma} \right)^2 + \frac{\sin^2 P}{\sigma^2} \right]. \tag{11}
\]

This functional has a non-trivial extremum corresponding to a solution of a first-order “self-duality” equation:

\[
\frac{\partial P}{\partial \sigma} = -\frac{\sin P}{\sigma}. \tag{12}
\]

Its solution (with a unity two-dimensional topological charge) is

\[
P(\sigma) = 2 \arctan \left( \frac{\sigma_0}{\sigma} \right), \quad P(0) = \pi, \tag{13}
\]

where the transverse scale \(\sigma_0\) is arbitrary. Eqs. (10,13) give a solution of Eq.(8) as well. The integral in Eq.(9) is dimensionless, therefore \(E_\perp\) is independent of the string thickness \(\sigma_0\). We find at the extremum

\[
E_\perp^{(0)} = 4\pi F^2. \tag{14}
\]

Substituting the ansatz (10) into Eq.(4) we get for the action

\[
S = -E_\perp^{(0)} \int dt \int dx' \sqrt{1 - \omega^2 x'^2}. \tag{15}
\]

This expression coincides with the Nambu string action [1]

\[
S_{\text{Nambu}} = -\frac{1}{2\pi \alpha'} \int d^2 \xi \sqrt{(\dot{x}_\mu x'_\mu)^2 - \dot{x}_\mu x'^2}. \tag{16}
\]

written for a rigid rotating string. Indeed, using the parametrization

\[
x_0 = t = \xi_0, \quad \dot{x}_i x'_i = 0, \tag{17}
\]
(hence, $x_0 = 1$, $x_0' = 0$) and taking into account that $x_i^2 = \omega^2 s^2$ where $s$ is the length along the string $(ds/d\xi_1 = \sqrt{x_i^2})$ and $\omega$ is the rotation angular velocity, one gets from Eq.(16)

$$S_{\text{Nambu}} = -\frac{1}{2\pi\alpha'} \int dt \int ds \sqrt{1 - \omega^2 s^2},$$

which coincides in form with Eq.(15), the Regge slope being

$$\alpha' = \frac{1}{2\pi E_\perp} = \frac{1}{8\pi^2 F_\pi^2}.$$  

Let us check directly that the chiral soliton of the type given by Eq.(10) leads to a linear Regge trajectory, without referring to the Nambu lagrangian. To this end let us calculate the angular momentum of the rotating pion field. The general expression is

$$J_i = \frac{F_\pi^2}{2} \int d^2x \varepsilon_{ijk} \text{Tr} \partial_0 U \partial_j U^\dagger x_k .$$

Taking into account that the time dependence of the pion field $U$ comes only through global rotation (see Eq.(2)) one can rewrite $J_3$ in terms of body-fixed coordinates (denoted by a prime):

$$J_3 = \frac{F_\pi^2 \omega}{2} \int dx' dy' dz' \text{Tr} (x' L_{yy'} - y' L_{xx'})^2,$$

$$L_{x',y'} = iU^+ \frac{\partial}{\partial x', y'} U .$$

Using the ansatz (10) we find:

$$J_3 = \frac{F_\pi^2 \omega}{2} \int dx' \frac{x'^2}{\sqrt{1 - \omega^2 x'^2}} \int dy' dz' \text{Tr} L_y^2 + O(\omega^0)$$

$$= \frac{\omega E_\perp}{\sqrt{1 - \omega^2 x'^2}} + O(\omega^2) = \frac{\pi E_\perp}{2\omega^2} + O(\omega^0) .$$

We, thus, get the following relation between the angular momentum and the angular velocity, which is typical for an expandable string:

$$J = \frac{2\pi^2 F_\pi^2}{\omega^2} .$$

The larger $J$, the smaller is the rotation velocity $\omega$. This is because the string length (along the $x'$ axis) is $L = 2/\omega$, its ends rotating with the speed of light.

At large $J$, $\omega$ is small, the chiral string is long, and one can neglect departures from a simple infinite string-like solution (10). It should be stressed that, strictly speaking, Eq.(10) is not a solution of the full equation of motion (5). It ceases to be a solution i) at
the end-points $x' \approx \pm 1/\omega$, ii) far away from the string axis, *viz.* at $y', z' \geq 1/\omega$. However the corresponding corrections are small in $\omega$ and are irrelevant to the calculation of the Regge slope. Therefore, we neglect $L_x$ in Eq.(21) and arrive to Eq.(23).

Let us now calculate directly the energy of the rotating chiral string. We have

$$E = \frac{F^2}{4} \int d^3x \ Tr \left( \partial_0 U \partial_0 U^\dagger + \partial_i U \partial_i U^\dagger \right)$$

$$= \frac{F^2}{4} \int dx' dy' dz' Tr \left[ (\omega^2 x'^2 + 1) \partial_y U \partial_y U^\dagger + \partial_z U \partial_z U^\dagger \right]$$

$$= \int dx' \sqrt{1 - \omega^2 x'^2} \frac{F^2}{4} \int d\tilde{y} d\tilde{z} \ Tr \left( 1 + \frac{\omega^2 x'^2}{1 - \omega^2 x'^2} \frac{L^2_y}{L^2} + \frac{L^2_z}{L^2} \right)$$

$$= E_\perp \int dx' \frac{\pi E_\perp}{\omega} = \frac{4\pi^2 F^2}{\omega}. \quad (24)$$

Comparing this result with Eq.(23) we see that the mass squared of a rotating chiral string grows linearly with the angular momentum:

$$M^2_J = \frac{8\pi^2 F^2}{\omega} = \frac{1}{\alpha'} J, \quad \alpha' = \frac{1}{8\pi^2 F^2}. \quad (25)$$

This result coincides with that obtained previously from comparison with the Nambu action and reveals what is called a linear Regge trajectory.

Numerically we get from Eq.(25) $\alpha' \simeq 1.45 \text{ GeV}^{-2}$ which is a factor of 1.5 larger than the phenomenological Regge slope. Possibly the discrepancy is eliminated when quantum corrections to the transverse energy are taken into account (see Sec.4).

It is interesting that our leading-order considerations do not fix the transverse size of the chiral string $\sigma_0$ (see Eq.(13)). In order to fix $\sigma_0$ we have to consider corrections to the transverse energy (14).

First, there are corrections to the effective chiral action itself. A popular way of modelling the higher derivatives terms is to add to the kinetic energy term (4) the so-called Skyrme term [9],

$$S_{\text{Skyrme}} = N_c e^2 \int d^4x \sqrt{-g} g^{\alpha\beta} g^{\mu\nu} \ Tr \left[ L_\alpha L_\mu \right] \left[ L_\beta L_\nu \right], \quad (26)$$

where $e^2$ is a numerical constant of the order of unity. Substituting the ansatz (10),(13) into (26) we get a correction to $E_\perp$ of the form:

$$\delta E^{(1)}_\perp \sim \frac{e^2 N_c}{\sigma_0^2}. \quad (27)$$

Perhaps a more realistic four-derivative term of the effective chiral action has been suggested in Ref. [10]. It also leads to a positive correction to the transverse energy.
of the same form (27). Such a term prevents the chiral string from shrinking to zero thickness.

What prevents it from infinite swelling in the transverse direction? The answer to this question is less obvious. Probably corrections in $\omega$ mentioned above are important here. They come from taking into account the string end points and large transverse distances $\sigma \geq 1/\omega$. Since the correction to the energy cannot depend on the direction of rotation, at small $\omega$ it must be quadratic in $\omega$. On dimension grounds one finds then that the correction should be of the form:

$$\delta E^{(2)}_\perp \sim F_\pi^2 \sigma_0^2 \omega^2.$$  \hspace{1cm} (28)

Adding up (27) and (28) and minimizing their sum in $\sigma_0$ we obtain the transverse size

$$\sigma_0 \sim \left( \frac{e^2 N_c}{F_\pi^2 \omega^2} \right)^{1/4} \sim \frac{1}{F_\pi} (N_c J)^{1/4}.$$  \hspace{1cm} (29)

This expression should be compared with that for the string length

$$L = \frac{2}{\omega} = \frac{\sqrt{2J}}{F_\pi} = 4\sqrt{J \alpha'}. \hspace{1cm} (30)$$

We see that the transverse size $\sigma_0$ of the rotating chiral string grows with the angular momentum $J$ (thus justifying the use of the long wave-length limit of the chiral action) although more slowly than the longitudinal size $L$.

3 Classical radiation by rotating chiral solitons

Let us first of all show that a strictly stationary rotating soliton cannot exist: it has to lose its energy through radiation of classical pion fields.

Indeed, let us consider the far-distance tail of the rotating soliton field assuming that the pion field is already small there, so that one can linearize the equation of motion. Then it is just the d’Alembert equation. Assuming that the soliton rotates with the angular velocity $\omega$ around the $z$ axis which means that the field depends on $r$, $\vartheta$ and $\varphi' = \varphi - \omega t$, the d’Alembert equation can be rewritten as

$$\Delta \pi^A - \omega^2 \frac{\partial^2}{\partial \varphi'^2} \pi^A = 0.$$  \hspace{1cm} (31)

We look for its solution in the form

$$\pi^A(r, \vartheta, \varphi') = \sum_{\ell,m} R_{\ell m}^A(r) Y_{\ell m}(\vartheta, \varphi'),$$  \hspace{1cm} (32)
where radial functions satisfy the equation
\[
\left[ \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} - \frac{\ell(\ell + 1)}{r^2} + m^2 \omega^2 \right] R_{\ell m}^A (r) = 0 .
\]

Its solution are spherical Bessel functions \( j_\ell (m \omega r) \) and \( y_\ell (m \omega r) \) [14]. Their asymptotics are \( j_\ell (z) \sim z^{-1} \sin(z - \ell \pi/2) \) and \( y_\ell (z) \sim -z^{-1} \cos(z - \ell \pi/2) \). Hence, the pion field (32) decreases as \( 1/r \) at large distances. Note that such behavior is typical for the radiation field. The energy and the angular momentum of such field diverge at \( r \to \infty \):
\[
E = \frac{F_\pi^2}{2} \int d^3r \left[ (\partial_0 \pi^A)^2 + (\partial_i \pi^A)^2 \right] = \infty , \quad J_3 = -F_\pi^2 \int d^3r \partial_0 \pi^A \partial_\varphi \pi^A = \infty .
\]

This is because we calculate here, in fact, not the energy and the angular momentum of a soliton but of a soliton together with its radiation field. The “proper” energy and angular momentum of the soliton itself can be found by subtracting the contribution of the radiation to these quantities. Let us show how it is done in the case of a spherically-symmetric soliton (a hedgehog). We shall consider slow rotations, \( \omega r_0 \ll 1 \), where \( r_0 \) is the typical size of the hedgehog. If \( \omega r_0 \geq 1 \) the problem loses any sense: in this case the radiation is so strong that one cannot speak of a stationary rotation. At \( r \gg r_0 \) the pion field of the soliton is small, and one can use the asymptotic form for the profile function of the hedgehog: \( U = \exp i (n \tau) P(r) \), \( P(r) \to A/r^2 \), \( A = \text{const} \cdot r_0^2 \). In the range \( r_0 \ll r \ll 1/\omega \) one has for the rotating soliton:
\[
\begin{align*}
\pi^1 &= \frac{A}{r^2} \sin \theta \cos(\varphi - \omega t), \\
\pi^2 &= \frac{A}{r^2} \sin \theta \sin(\varphi - \omega t), \\
\pi^3 &= \frac{A}{r^2} \cos \theta .
\end{align*}
\]

At the same time, the pion fields at \( r \gg r_0 \) must satisfy the d’Alembert equation \( \Box \pi^A = 0 \), to which the general non-linear equation of motion is reduced when the fields are linearized. In addition, at \( r \to \infty \) the pion field must satisfy the radiation condition,
\[
\frac{\partial (\pi^A r)}{\partial r} + \frac{\partial (\pi^A r)}{\partial t} = 0 .
\]

In principle, one could as well look for a solution which has the form of incoming waves at infinity. That would correspond to a situation when one keeps the rotation by pumping energy from infinity. But now we are interested in a free soliton which looses energy and not vice versa.
To find the field at large \( r \geq 1/\omega \), one has to solve the d’Alembert Eq.(31) with the boundary conditions (35),(36) we get to which the solution must reduce at \( r \ll 1/\omega \).

Such solution is readily found:

\[
\begin{align*}
\pi^1 &= \frac{A}{r^2} \sin \vartheta (\cos \alpha + \omega r \sin \alpha), \\
\pi^2 &= \frac{A}{r^2} \sin \vartheta (\sin \alpha - \omega r \cos \alpha), \\
\pi^3 &= \frac{A}{r^2} \cos \vartheta, \quad \alpha = \omega r + \varphi - \omega t. 
\end{align*}
\]

(37)

At \( \omega r \ll 1 \) it becomes the rotating soliton field (35) and at \( \omega r \gg 1 \) the \( \pi^1, \pi^2 \) components fall off as \( 1/r \) corresponding to the outgoing radiation waves.

Let us find the intensity of the pion radiation by the soliton. Using the general expression for the stress-energy tensor,

\[
\vartheta_{\mu\nu} = \frac{F^2}{2} \left( \text{Tr} \partial_{\mu} U \partial_{\nu} U^\dagger - \frac{1}{2} g_{\mu\nu} \text{Tr} \partial_{\alpha} U \partial_{\alpha} U^\dagger \right),
\]

(38)

we get for the momentum flow at infinity:

\[
\vartheta_{\vartheta i} \xrightarrow{r \to \infty} F^2 \vartheta_0 \pi^A \partial_\vartheta \pi^A \to \frac{F^2 A^2 \omega^4 \sin^2 \vartheta n_i}{r^2}.
\]

(39)

The momentum flow goes in the radial direction as it is proportional to the unit vector \( \mathbf{n} \) but its intensity depends on the angle \( \vartheta \) from the rotation axis.

In a slightly different context, this formula has been known to Lorentz, coinciding in its angular and frequency dependence with the intensity of the electromagnetic dipole radiation [15]. It is not accidental: owing to its Goldstone nature the pion field couples to the isospin source through a gradient; owing to gauge invariance only the field strength counts in the e.m. radiation, which also has a gradient coupling to charges. In the non-relativistic limit we are now considering only the dipole component of the radiation survives in both cases.

The energy loss owing to the radiation is, according to the energy-momentum conservation law \( \partial_{\mu} \vartheta_{\mu\nu} = 0 \),

\[
-\frac{dE}{dt} = W = \lim_{r \to \infty} \int d\Sigma_i \vartheta_{\vartheta i} r^2 = F^2 \pi A^2 \omega^4 2\pi \int_0^\pi d\vartheta \sin^3 \vartheta \frac{8\pi F^2 A^2 \omega^4}{3}.
\]

(40)

We notice, that the coefficients \( A \) in the asymptotics of hedgehog profile function is directly related [9] to the nucleon axial constant \( g_A = 8\pi AF^2/3 \) and the latter quantity — through the Goldberger–Treiman relation — to the pion-nucleon coupling constant \( g_{\pi NN} = g_A M/F_\pi \). Using these formulae, the radiation intensity can be rewritten as

\[
W = \frac{3g_{\pi NN}^2 \omega^4}{8\pi M^2},
\]

(41)
where $M$ is the mass of the soliton.

The radiation carries away not only the energy but also the angular momentum which decreases as

$$\frac{dJ_3}{dt} = \lim_{r \to \infty} \int d\Sigma_i \varepsilon_{3jk} x_j \vartheta_{ik} = F^2 \int d\Sigma_i \frac{\partial \pi_A}{\partial x_i} \frac{\partial \pi^A}{\partial \varphi} = -\frac{W}{\omega} . \quad (42)$$

The “proper” energy and angular momentum of a rotating soliton can be found by subtracting volume integrals over, respectively, the energy density and the angular momentum density of the radiation [15]:

- Energy:
  $$E_{\text{prop}} = \int d^3 r \vartheta^\text{prop}_{00} = \int d^3 r (\vartheta_{00} - n_i \vartheta_{i0}) , \quad (43)$$
  $$J^\text{prop}_3 = \int d^3 r \varepsilon_{3jk} \vartheta^\text{prop}_{0j} x_k = \int d^3 r \varepsilon_{3jk} (\vartheta_{0j} - n_i \vartheta_{ij}) x_k . \quad (44)$$

It can be seen from Eqs. (37) that at $r \to \infty$ the integrands in Eqs. (43,44) behave as

$$\vartheta_{00} - n_i \vartheta_{i0} \xrightarrow{r \to \infty} \frac{F^2 A^2}{r^4} \left( \frac{6}{r^2} + \omega^2 \sin^2 \vartheta \right) ,$$

$$\varepsilon_{3jk} (\vartheta_{0i} - n_i \vartheta_{ij}) x_k \xrightarrow{r \to \infty} \frac{F^2 A^2}{r^4} 2\omega \sin^2 \vartheta . \quad (45)$$

As a result both quantities, $E_{\text{prop}}$ and $J^\text{prop}_3$, are now convergent. Moreover, it can be shown on general grounds that these quantities satisfy familiar relations for slowly rotating bodies:

$$E_{\text{prop}} = E_{\text{rest}} + \frac{I \omega^2}{2} + O(\omega^4) ,$$

$$J^\text{prop}_3 = I \omega + O(\omega^3) , \quad (46)$$

where $E_{\text{rest}}$ is the soliton rest mass and $I$ is its moment of inertia. Eqs. (45) evidently correspond to the more general relations (46) giving, in fact, the large-distance contributions to $E_{\text{rest}}$ and $I$. It is important that the energy loss (40) is $O(\omega^4)$. To this accuracy rotation is approximately stationary and $O(\omega^2)$ corrections to the rest mass make sense.

Let us now calculate the lifetime of a highly excited ($J \gg 1$) rotational state of a hedgehog. This can be done in two different ways: “quantum” and “classical”. In the quantum approach one has to calculate the transition amplitude between the states $J$ and $J - 1$ with the emission of a pion. We remind the reader that the rotational excitations of a spherically-symmetric soliton which we are now considering, have isospin $T = J$ [9]. The rotational wave functions are Wigner $D$-functions which depend on a unitary matrix $R$ characterizing the soliton orientation in spin-isospin space [9, 10]:

$$\Psi^I_{J_3, T_3}(R) = \sqrt{2J + 1} (-1)^{J + J_3} D^I_{-T_3, T_3}(R) . \quad (47)$$
The pion-soliton coupling is [9, 16]:
\[
-\frac{3g_{\pi NN}}{2M} \frac{1}{2} \text{Tr} \left( R^+ \tau^A R \sigma_i \right) ik_i,
\]
where \( k_i \) is the 3-momentum of the pion, \( A \) is its isotopic component. Sandwiching (48) between the initial and final wave functions (47) we get for the \( J \rightarrow J-1 \) transition amplitude squared (averaged over the initial and summed over the final spin and isospin states)
\[
\left( \frac{3g_{\pi NN}}{2M} \right)^2 \frac{2J-1}{2J+1} \frac{|k|^2}{3}.
\]
(49)
To get the decay width one has to multiply (49) by the phase space factor \( |k|/2\pi \). We obtain
\[
\Gamma_{J \rightarrow J-1} = \frac{3g_{\pi NN}^2}{8\pi M^2} \frac{2J-1}{2J+1} \frac{M_{J-1}}{M_J} |k|^3,
\]
(50)
In the chiral limit we put the pion mass \( \mu_\pi = 0 \) and obtain at \( J \gg 1 \):
\[
\Gamma = \frac{3g_{\pi NN}^2}{8\pi M^2} \left( \frac{J}{I} \right)^3.
\]
(51)

Just as in the case of a highly excited state of an atom, the lifetime can be also calculated from the classical radiation theory. We have already found the energy loss per unit time as due to the classical radiation of the pion field, see Eqs.(40,41). The lifetime of a rotating soliton with given \( J \gg 1 \) can be determined as the time during which the energy of the soliton decreases from \( M_J \) to \( M_{J-1} \). This prescription is known as the Bohr correspondence principle: the lifetime of an excited state of an atom is not the time during which the electron looses all its energy through radiation but only a small portion corresponding to the transition to the nearest lower state. In our case we have therefore:
\[
\Gamma = \frac{1}{t} = \frac{W}{M_J - M_{J-1}} = \frac{3g_{\pi NN}^2}{8\pi M^2} \left( \frac{J}{I} \right)^3,
\]
which coincides exactly with the quantum-mechanical result (51). (In the last equation we have used Eq.(41) and the relation (46) for the angular velocity, \( \omega = J/I \). The same result follows immediately from Eq.(42): one can determine the lifetime as the time during which the soliton looses one unit of its angular momentum:
\[
\Gamma = \frac{1}{dJ/dt} \left| \frac{dJ}{dt} \right| \frac{1}{J - (J-1)} = \frac{W}{\omega} = \frac{3g_{\pi NN}^2}{8\pi M^2} \left( \frac{J}{I} \right)^3.
\]
We leave it for the reader to check that the time during which a soliton looses one unit of isospin owing to the classical pion radiation, is also given by the same formula.

We notice in passing that Eq.(50) gives the right numerical value for the width of Δ resonance, \( \Gamma_\Delta \approx 110 \text{ MeV} \). In this estimate we use experimental values of the \( N \) and Δ masses and of the \( \pi N \) coupling, \( g_{\pi NN} \approx 13.6 \). For the exotic \( J = T = 5/2 \) state with \( M_{5/2} \approx 1700 \text{ MeV} \) Eq.(50) predicts \( \Gamma(5/2 \rightarrow 3/2) \approx 760 \text{ MeV} \). Such a big width explains perhaps why the \((5/2, 5/2)\) resonance has not been definitely observed.

Let us summarize. We start with a spherically-symmetric chiral soliton (a skyrmion) which presumably reproduces the nucleon. We try to rotate it. If \( \omega \) is small, the form of the soliton does not change. It remains spherically-symmetric, hence, only \( T = J \) states are allowed. However a rotating soliton inevitably radiates pion fields. We have demonstrated that the classical radiation is in direct correspondence with the quantum-mechanical calculation of the widths.

At \( \omega r_0 \approx 1 \) (where \( r_0 \) is the characteristic size of the soliton) the radiation becomes very strong, the widths blow up. Fast rotating hedgehogs, actually with \( T = J = O(N_c) \), do not exist. Deformation due to centrifugal forces cannot be further ignored. If we insist on getting a soliton with still larger \( J \), it must be stretched in the direction perpendicular to the rotation axis. But when we pass to a cigar-like (eventually to a string-like) soliton, the angular velocity \( \omega \) is no more proportional to \( J \). On the contrary, it decreases with \( J \), see Eq.(23). Since the radiation grows with \( \omega \), it means that a rotating string-like soliton may be relatively stable with respect to classical radiation (we shall estimate the corresponding width in a moment). A non-spherically symmetric soliton does not necessarily have isospin \( T = J \). In fact \( T \) may remain fixed. We come to linear Regge trajectories described in Sec. 2.

If we plot angular velocity \( \omega \) as a function of \( J \) we see that there exists a maximal \( \omega \) at which a crossover from a spherically-symmetric to a string-like soliton occurs, see Fig. 5.

Finally, let us estimate the intensity of the classical radiation of a rotating string-like soliton. Similar to the case of a spherically-symmetric hedgehog discussed above, one has to find the solution of the d’Alembert equation together with the radiation condition (36) and with the boundary conditions being the soliton field itself. Since the soliton field decreases far away from the string axis as \( \sigma_0/\sigma \) (see Eq.(13)) where \( \sigma \) is the distance from the axis, the radiation intensity \( W \) is proportional to \( \sigma_0^2 \) where \( \sigma_0 \) is the string thickness. On dimensional grounds we find then:

\[
W \sim F_\pi^2 \sigma_0^2 \omega^2 \sim F_\pi^4 \sigma_0^2 / J .
\]

The width due to classical radiation is \( \Gamma = W / (M_J - M_{J-1}) \). Taking into account that \( M_J^2 = J/\alpha' \) and that \( 1/\alpha' \sim F_\pi^2 \) we get:

\[
\Gamma_{\text{class}} \sim \frac{F_\pi^3 \sigma_0^2}{\sqrt{J}} .
\]
Finally, we use Eq.(29) for the transverse size $\sigma_0$ and find for the width of a state $J$:

$$\Gamma_{\text{class}} \sim F_\pi \sqrt{N_c},$$

(54)

which is linear in $N_c$ but independent of $J$ (cf. with the width of a spherically-symmetric rotator (Eq.(51)), which grows as $J^3$). The width-to-mass ratio is

$$\frac{\Gamma_{\text{class}}}{M} \sim \sqrt{\frac{N_c}{J}}.$$  

(55)

Let us mention that there is also a purely quantum contribution to the width of a Regge resonance which we calculate in the next section.

4 Quantum corrections, zero modes and all that

In Sec. 2 we have found an extremum of the chiral action (4), corresponding to a thick rotating string. The solution is given by Eqs. (19,13). Let us expand the chiral field $U$ near the classical solution which we denote by $U_0$. Introducing hermitean quantum fluctuations $r$:

$$U = U_0 \left( 1 + i r - \frac{r^2}{2} + \cdots \right),$$

(56)

we obtain for the action

$$S = -\frac{2\pi^2 F_\pi^2}{\omega} \int dt + \frac{F_\pi^2}{4} \int d^4 x' g^{\mu\nu}(x') \text{Tr} \left\{ \partial_\mu r \partial_\nu r + i \partial_\mu r [r, L_\nu] + O(r^3) \right\}.$$  

(57)

Neglecting terms of the order of $\omega y'$ and $\omega z'$ in the metric tensor given by Eq.(3) (to that accuracy the classical equation of motion has been solved in Sec. 2) we get the following quadratic form for quantum fluctuations:

$$S^{(2)} = \frac{F_\pi^2}{4} \int d^4 x' \text{Tr} \left\{ (\partial_i r)^2 - \frac{2\omega x'}{\sqrt{1 - \omega^2 x'^2}} \partial_i r (\partial_\phi r + i[r, L_\phi]) \right. 
- (\partial_x r)^2 - \partial_y r (\partial_\phi + i[r, L_\phi]) - \partial_z r (\partial_\phi + i[r, L_\phi]) \left. \right\}.  \quad (58)$$

We remind the reader that the primes refer to the body-fixed frame and the tildes to the Lorentz-contracted coordinates (see Sec. 2).

Let us consider the transverse Laplace operator

$$\Delta_\perp \equiv \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - i \left[ L_\phi, \frac{\partial}{\partial y} \ldots \right] - i \left[ L_\phi, \frac{\partial}{\partial z} \ldots \right].$$

(59)

Let its eigenfunctions be $v_n^{A, T}$ with eigenvalues $\lambda_n$:

$$-\Delta_\perp v_n^{A, T} = \lambda_n v_n^{A, T}. \quad (60)$$
The functions $v^A_n(\tilde{y}, \tilde{z})$ form a complete set of orthonormalized functions. Therefore, we can decompose the general fluctuation field $r^A$ as

$$r^A(t, x', y', z') = \sum_n c_n(t, x') v^A_n(\tilde{y}, \tilde{z}).$$  \hspace{1cm} (61)

Substituting (61) into Eq.(58) we get:

$$S^{(2)} = \frac{F_2^2}{2} \int dt dx' \sqrt{1 - \omega^2 x'^2} \sum_n \left[ \left( \frac{\partial c_n}{\partial t} \right)^2 - \left( \frac{\partial c_n}{\partial x'} \right)^2 - \lambda_n c_n^2 \right].$$  \hspace{1cm} (62)

Note that the second term in Eq.(58) cancels out. The relativistic square root arises here when one passes from $y'$ to $\tilde{y}$.

Eq.(62) is the action for excitations which "live" on the string and have masses $\sqrt{\lambda_n}$. Their zero-point oscillations give the quantum correction to the transverse energy (or energy per unit length) of the string:

$$E_{\perp}^{\text{quant}} = \sum_n \int \frac{dk_x}{2\pi} \frac{1}{2} \left( \sqrt{\lambda_n + k_x^2} - \sqrt{\lambda_n^{(0)} + k_x^2} \right),$$  \hspace{1cm} (63)

where $\lambda_n^{(0)}$ are eigenvalues of the free Laplace operator.

If there is a negative eigenvalue $\lambda_-$, it leads to a non-zero imaginary part of the energy. It means that the soliton is unstable with respect to quantum fluctuations in a particular direction in Hilbert space. The corresponding width is

$$\Gamma = 2 \text{Im} E = 2 \int dx' \sqrt{1 - \omega^2 x'^2} \left[ \int_{-\sqrt{\lambda_-}}^{\sqrt{\lambda_-}} \frac{dk_x}{2\pi} \frac{1}{2} \sqrt{\lambda_-} - k_x^2 \right] = \frac{\pi \sqrt{|\lambda_-|}}{8\omega},$$  \hspace{1cm} (64)

where $\lambda_-$ is the negative eigenvalue of the transverse Laplace operator (59).

Let us give a qualitative argument that there must be at least one negative eigenvalue. The point is that, while in 3 dimensions a hedgehog belongs to a non-trivial homotopy class thanks to $\pi_3(SU_2) = Z$, in 2 dimensions it is reducible by a continuous deformation to a zero field, since $\pi_2(SU_2) = 0$. Therefore, it is natural to expect that the extremum we have found in Sec. 2 is in fact a saddle point. The exact calculation below confirms this expectation.

In order to perform exact calculations one has to diagonalize the $\Delta_\perp$ operator (59). If the external field $U_0$ is a 2-dimensional hedgehog given by Eqs.(10,13) $\Delta_\perp$ commutes with $K = (T + L)_{x'}$ — the projection of isospin plus orbital moment on the string axis. Let us parametrize the transverse coordinates: $\tilde{y} = \sigma \cos \Psi$, $\tilde{z} = \sigma \sin \Psi$. Using the concrete transverse profile function $P(\sigma) = 2 \arctan(\sigma_0/\sigma)$ and putting temporarily $\sigma_0 = 1$ we obtain:

$$\Delta_\perp = \left( \frac{\partial^2}{\partial \sigma^2} + \frac{1}{\sigma} \frac{\partial}{\partial \sigma} + \frac{1}{\sigma^2} \frac{\partial^2}{\partial \Psi^2} \right) - \frac{i}{\sigma^2 + 1} \left[ e^{i\Psi} \tau^- + e^{-i\Psi} \tau^+, \frac{\partial}{\partial \sigma} \ldots \right].$$
where $\tau^\pm = \tau^2 \pm i\tau^3$. Apparently this operator does not mix states with different $K$. Therefore, we look for the eigenfunctions in the form:

$$v^A_K(\tilde{y}, \tilde{z}) = if_K(\sigma)e^{iK\Psi}\tau^1 + iq_K(\sigma)e^{i(K+1)\Psi}\tau^- + h_K(\sigma)e^{i(K-1)\Psi}\tau^+.$$  (66)

The eigenvalue Eq. (60) becomes a system of 3 ordinary equations:

$$-f''_K - \frac{1}{\sigma}f'_K + \frac{K}{\sigma^2}f_K + \frac{4}{\sigma^2 + 1}(g_K - h_K)' = \frac{4(\sigma^2 - 1)}{\sigma(\sigma^2 + 1)^2}[(K - 1)h_K + (K + 1)g_K] = \lambda f_K;$$

$$-g''_K - \frac{1}{\sigma}g'_K + \frac{(K + 1)^2}{\sigma^2}g_K - \frac{2}{\sigma^2 + 1}f'_K + \frac{8(K + 1)}{(\sigma^2 + 1)^2}g_K - \frac{2K(\sigma^2 - 1)}{\sigma(\sigma^2 + 1)^2}f_K = \lambda g_K;$$

$$-h''_K - \frac{1}{\sigma}h'_K + \frac{(K - 1)^2}{\sigma^2}h_K + \frac{2}{\sigma^2 + 1}f'_K + \frac{8(K - 1)}{(\sigma^2 + 1)^2}h_K - \frac{2K(\sigma^2 - 1)}{\sigma(\sigma^2 + 1)^2}f_K = \lambda h_K.$$  (67)

It can be seen that at $K = 0$ this system splits into an equation for $(g + h)_0$ and a system of two equations for $f_0$ and $(g - h)_0$:

$$-f''_0 - \frac{1}{\sigma}f'_0 + \frac{4}{\sigma^2 + 1}(g - h)_0' - \frac{4(\sigma^2 - 1)}{\sigma(\sigma^2 + 1)^2}(g - h)_0 = \lambda f_0;$$

$$-(g - h)_0'' - \frac{1}{\sigma}(g - h)_0' + \frac{1}{\sigma^2}(g - h)_0 - \frac{4}{\sigma^2 + 1}f'_0 - \frac{8}{(\sigma^2 + 1)^2}(g - h)_0 = \lambda (g - h)_0;$$

and

$$-(g + h)_0'' - \frac{1}{\sigma}(g + h)_0' + \frac{1}{\sigma^2}(g + h)_0 - \frac{8}{(\sigma^2 + 1)^2}(g + h)_0 = \lambda (g + h)_0.$$  (68)

It is not clear if these equations can be solved analytically. However, the zero modes ($\lambda = 0$) can be found exactly from symmetry considerations.

**Zero modes**

We expect 6 zero modes of the operator $\Delta_\perp$: two translations, three global isospin rotations and one dilatation. The last one is related to the fact that the transverse energy does not depend on the string thickness $\sigma_0$.

The infinitesimal variation of the chiral field with translation of the string axis in the $(\tilde{y}, \tilde{z})$ plane is

$$\delta U = -\frac{\partial U_0}{\partial x_\alpha}\delta x_\alpha \equiv U_0i\alpha_0\delta x_\alpha, \quad x_\alpha = (\tilde{y}, \tilde{z}).$$
It corresponds to two translational zero modes ($\alpha = 1, 2$):

$$r_\alpha = iU_0^+ \partial_\alpha U_0 = L_\alpha.$$  \hfill (70)

One can choose their combinations $r_\pm = i(L_y \pm iL_z)/2$. Being written in the form (66) these functions have quantum numbers $K = \pm 1$, respectively. The corresponding $f, g, h$ functions are:

$$f_{+1} = \frac{-2\sigma}{(\sigma^2 + 1)^2}, \quad g_{+1} = \frac{\sigma^2}{(\sigma^2 + 1)^2}, \quad h_{+1} = \frac{1}{(\sigma^2 + 1)^2};$$

$$f_{-1} = \frac{2\sigma}{(\sigma^2 + 1)^2}, \quad g_{-1} = \frac{1}{(\sigma^2 + 1)^2}, \quad h_{-1} = \frac{\sigma^2}{(\sigma^2 + 1)^2}. \quad (71)$$

These functions satisfy Eqs. (67) with $\lambda = 0$, hence they are, indeed, zero modes.

Similarly, there are three rotational zero modes,

$$r^1 = U_0^+ \tau' U_0 - \tau', \quad r^\pm = U_0^+ \tau^\pm U_0 - \tau^\pm. \quad (72)$$

Being standartized to the form given by Eq.(66) these functions appear to have quantum numbers $K = 0$ and $K = \pm 1$, respectively. We have:

$$f_0 = \frac{4\sigma^2}{(\sigma^2 + 1)^2}, \quad (g - h)_0 = -\frac{\sigma(\sigma^2 - 1)}{(\sigma^2 + 1)^2}, \quad (g + h)_0 = 0; \quad (73)$$

$$f_{\pm 1} = \mp \frac{\sigma(\sigma^2 - 1)}{(\sigma^2 + 1)^2}, \quad g_{\pm 1} = \pm \frac{\sigma^2}{(\sigma^2 + 1)^2}, \quad h_{\pm 1} = \mp \frac{\sigma^2}{(\sigma^2 + 1)^2}. \quad (74)$$

The functions (73) satisfy Eqs. (68) with $\lambda = 0$ and the function (74) satisfy Eqs.(67) at $K = \pm 1$ also with a zero right-hand side.

Finally, the dilatational zero mode is

$$r = -iU_0^+ \frac{\partial U_0}{\partial \sigma_0} \big|_{\sigma_0 = 1} = \frac{2(n\tau)\sigma}{\sigma^2 + 1}, \quad (75)$$

which corresponds to a state with $K = 0$:

$$f_0 = 0, \quad (g - h)_0 = 0, \quad (g + h)_0 = \frac{\sigma}{\sigma^2 + 1}. \quad (76)$$

This function is a zero mode of Eq.(69).

We notice that the rotational (Eqs.(73,74)) and the dilatational (Eq.(76)) zero modes, in contrast to the translational ones, decrease as $1/\sigma$ at large distances from the string axis. Therefore, they are not normalizable and cannot, strictly speaking, be considered as zero modes. Actually, they belong to the continuous spectrum.

For this reason, connected with the particular form of our “instanton” solution (10), one is left only with the translation zero modes, and the low-energy effective string theory
is just that of Nambu (Eq.(16)). In practical terms it means, in particular, a degeneracy of trajectories in isospin — a property which seems to be realized in nature but looks totally unexpected for a string made of chiral fields!

It should be stressed that since the resulting string theory is an effective (not microscopic) one, taking into account only the long-wave excitations of the string, one should not be confused by ghosts, tachyons and other inconsistencies which arise in the Nambu string at \( d = 4 \).

**Negative mode**

As anticipated above from topological considerations, the operator \( -\Delta_\perp \) has a negative eigenvalue. It belongs to the \( K = 0 \) sector (Eq.(68)). There are no negative eigenvalues in other sectors. Solving Eq.(68) numerically we find \( \lambda_\perp \simeq -2.64 \) (in units of \( 1/\sigma_0^2 \)). According to Eq.(64) we get for the quantum width:

\[
\Gamma_{\text{quant}} = \frac{2.64\pi}{8\sigma_0^2\omega} = \frac{2.64\sqrt{J}}{8\sqrt{2}F_\pi\sigma_0^2},
\]

(77)

where the relation between \( \omega \) and \( J \) (Eq.(23)) has been used. Further on, if we take a slowly growing transverse size \( \sigma_0 \) as given by Eq.(29) we get:

\[
\Gamma_{\text{quant}} \sim \frac{F_\pi}{\sqrt{N_c}},
\]

(78)

which is independent of \( J \) and is \( N_c \) times smaller than the classical width \( \Gamma_{\text{class}} \), see Eq.(54). It means that the widths of the resonances lying on Regge trajectories are mainly determined by the classical radiation of pion fields.

5 Discussion and conclusions

In the last years there has been progress in understanding nucleons as chiral solitons. The \( \Delta \) resonance with \( T = J = 3/2 \) fits nicely the idea of being the first rotational excitation of the nucleon soliton [9, 10]. Theoretically speaking, rotations can be considered as quantum corrections as long as \( J \ll N_c \).

In Sec. 3 we have considered the range \( 1 \ll J \ll N_c \). Although hardly realized in nature, this range of angular momenta is very interesting from the theory point of view since the rotation can be considered simultaneously in quantum and classical framework. The main result of Sec. 3 is that rotating chiral solitons inevitably radiate pion fields quite similarly to the rotating electric charges. We have calculated explicitly the radiation intensity and hence the “classical” width of a state with given \( J \). In accordance with Bohr correspondence principle the widths calculated classically coincide at \( J \gg 1 \) with the exact quantum-mechanical result.
At $J \sim N_c$ the rotation of a spherical-symmetric soliton becomes so fast that the classical radiation of pions blows up and the widths become comparable to the masses. In order to “survive”, the soliton with $J \geq N_c$ has to expand in a direction perpendicular to the rotation axis. This is exactly what one expects from the action of centrifugal forces. The angular velocity of an expandable cigar-like soliton decreases with the growth of $J$. As a result the lifetime of a rotating soliton becomes stable with $J$ even as one goes to higher and higher values of angular momentum.

Unfortunately, the crossover region at $J \sim N_c$ is too complicated to be studied analytically. A significant simplification is achieved at $J \gg N_c$. In this case a string-like analytical solution of the equation of motion for a rotating chiral field can be found (Sec. 2). Actually the solution looks more like a double-blade kayak paddle, being squeezed at the end points by Lorentz contraction. Knowing the transverse pion field distribution inside the string (or the kayak paddle) one can easily calculate the energy per unit length or the “string tension” and hence the Regge slope. We find $\alpha' \simeq (8 \pi^2 F_\pi^2)^{-1} \simeq 1.45 \text{ GeV}^{-2}$ which is a factor of 1.5 larger than the phenomenological value $\alpha' = 0.8 - 0.9 \text{ GeV}^{-2}$. It means that our resonances are $\sqrt{1.5}$ times lighter, for given $J$, than in reality. The discrepancy is possibly eliminated when quantum corrections to the classical soliton energy are added. A systematic way to study quantum fluctuations is outlined in Sec. 4.

We do not also exclude the possibility that other, perhaps more adequate classical solutions may be found. For example, an obvious modification of our ansatz (10) is to take the winding number $n_w$ in the instant transverse plane to the string axis to be larger than 1 [17]. One gets then for the transverse profile function $P(\sigma) = \arctan(\sigma_0/\sigma)^{n_w}$ and for the Regge slope $\alpha' = (8 \pi^2 F_\pi^2 n_w)^{-1}$. [Curiously, the phenomenological value of $\alpha'$ corresponds to $n_w = 3/2$.] Ultimately, the true classical solution should be chosen from the requirement of minimal width due to classical radiation – the minimum-of-mass criterium is senseless for the whole Regge trajectory. For a given classical rotating solution, one has to investigate the quadratic form for quantum oscillations about it and extract the zero modes since they determine the low-energy string theory (see Sec. 4). Finally, one should check that the quantization of the effective string theory gives the right quantum numbers.

In this paper we did not specify what Regge trajectories we were dealing with – meson or baryon. At $1 \sim J \ll N_c$ only baryons can be treated as chiral solitons since there is no large parameter in the meson case allowing for a semiclassical approach. However at $J \geq N_c$ both baryon and meson resonances can be understood as rotating solitons discussed in this paper, with quantum corrections suppressed as $1/N_c$.

Where does the difference between meson and baryon trajectories come in? To answer this question we have to go one step back and consider the $\sigma$-model-type quark–pion lagrangian suggested in Ref. [10]. Integrating out quarks one gets the effective chiral action whose derivative expansion starts with the kinetic energy term (1).

In terms of the quark-pion lagrangian the problem of getting a classical extremum is to find the pion field providing an extremum for the quark determinant in that field. If
we take the pion field in the form of a rotating string, the problem is reduced to solving
the two-dimensional transverse Dirac equation in the background chiral field given by
Eq.(10). Thanks to two dimensions there is always a discrete bound state whose position
\( \varepsilon_0 \) depends on the string thickness \( \sigma_0 \). All levels are \( N_c \) times degenerate. The squared
eigenvalues of the transverse Dirac operator \( \varepsilon_n^2 \) play the rôle of the squared masses for
fermions which “live” on the string (cf. the boson excitations considered in Sec. 4). Their
energies are \( \pm \sqrt{\varepsilon_n^2 + k_x^2} \).

The zero baryon number states (mesons) are obtained by filling in all energy levels
which come from the negative-energy continuum when the trial pion field is switched in
(see Fig. 6). It can be easily shown that at \( \sigma_0 M \gg 1 \) where \( M \) is the dynamical quark
mass, \( M \approx 350 \, \text{MeV} \) [10], the aggregate energy of the occupied levels coincides with the
kinetic-energy term of the chiral action, given by Eq.(24). Therefore, nothing has to be
modified in the above.

To get the unity baryon number states (baryons) one has to fill in the lowest level
coming from the upper Dirac continuum (also shown in Fig. 6). It increases the energy
given by Eq.(24), \( \text{viz. } E = O(J) \), by \( |\varepsilon_0| \). However this quantity is independent of \( J \) and
hence does not alter the Regge slope.

We conclude that in our chiral approach meson and baryon Regge trajectories auto-
matically have the same slope.

To summarize: We have shown how a rotating relativistic string can be formed from
massless chiral fields, giving rise to linear Regge trajectories. Both the longitudinal and
transverse sizes of the string grow with angular momentum. Therefore, large-J resonances
are huge in all directions, and it seems reasonable to understand and describe them in
terms of the lightest degrees of freedom – the pions – and not as gluonic strings which
anyhow have to break because of the existence of light pions.

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Figure 1: Baryon states with isospin $T = 1/2$. The signs denote parity; their sizes correspond to the status of a resonance.

Figure 2: Baryon states with isospin $T = 3/2$. The signs denote parity; their sizes correspond to the status of a resonance.
Figure 3: Meson states with $T = 0$. $f$-trajectory has $T^G = 0^+$; $\omega$-trajectory has $T^G = 0^-$. Both have "natural" parity. Omitted are states with presumably large admixtures of strange quarks or having a low status.

Figure 4: Meson states with $T = 1$. $\rho$-trajectory has $T^G = 1^+$, $\pi$- and $a_0$-trajectories have "natural" whereas $\pi$- and $a_1$-trajectories have "unnatural" parity.
Figure 5: Dependence of the angular velocity of a rotating soliton on its momentum. $\omega_{1,2}$ are two regimes corresponding to low and large $J$, respectively. The solid line is a possible interpolation.

Figure 6: Quark spectrum in a background string-like chiral field. The upper and lower Dirac continua are shaded. The crosses denote occupied levels, $N_c$ times degenerate in color. The upper level is filled in baryons and empty in mesons. It results in a shift of baryons masses as compared to mesons, by a value independent of the angular momentum.