FOURTH-ORDER SCHRÖDINGER TYPE OPERATOR WITH UNBOUNDED COEFFICIENTS IN $L^2(\mathbb{R}^N)$

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Abstract. In this paper we study generation results in $L^2(\mathbb{R}^N)$ for the fourth order Schrödinger type operator with unbounded coefficients of the form

$$A = a^2 \Delta^2 + V^2$$

where $a(x) = 1 + |x|^\alpha$ and $V = |x|^\beta$ with $\alpha > 0$ and $\beta > (\alpha - 2)^+$. We obtain that $(-A, D(A))$ generates an analytic strongly continuous semigroup in $L^2(\mathbb{R}^N)$ for $N \geq 5$. Moreover, the maximal domain $D(A)$ can be characterized for $N > 8$ by the weighted Sobolev space $D_2(A) = \{u \in H^4(\mathbb{R}^N) : V^2 u \in L^2(\mathbb{R}^N), |x|^{2\alpha-h} D^{4-h} u \in L^2(\mathbb{R}^N) \text{ for } h = 0, 1, 2, 3, 4\}$.

Mathematics Subject Classification. 47D08, 35J10, 47D06, 47F05, 35K35.

Keywords. Higher order elliptic equations, Maximal regularity, Schrödinger operators.

1. Introduction

Second order elliptic operators with unbounded coefficients and singular or unbounded potentials have been widely investigated, there exists nowadays a huge literature, see for example [18, 23, 17, 6, 8, 7, 4, 10, 11, 9, 12, 26, 28, 29, 30, 20, 16] and the references therein.

Recently, there is an increasing interest towards elliptic operators of higher order. They are involved in models of elasticity [25], condensation in graphene [32], free boundary problems [1] and non-linear elasticity [5].

The semigroup generated by a class of higher order elliptic operators with measurable coefficients has been systematically studied by Davies in [13]. Unbounded coefficients have also been considered in recent time. In [2], the authors study generation results for the square of the Kolmogorov operator $L = \Delta + \sum_\mu \mu \cdot \nabla$ in the weighted space $L^2(\mathbb{R}^N, d\mu)$ giving sufficient conditions for the characterization of both the domains of $L$ and $-L^2$. Perturbation of elliptic operators by singular or unbounded potentials is a classical problem studied by many authors. The most important features concern the characterization of the domain of the sum operator and kernel estimates for the solution. It is known that there is a strong relation between Schrödinger type operators and Hardy’s type inequalities. In this sense, relying on Rellich’s inequality, in [21], perturbation of the bi-Laplacian operator by...
a singular potential of the form \( c|x|^{-4}, c < \left( \frac{N(N-4)}{4} \right)^{2} \) has been considered. Non-existence result for the more general parabolic equation \( u_t = -(-\Delta)^m u + c|x|^{-2m} u \) in \( \Omega \times (0, T), m \geq 1 \) and \( c \) greater than a suitable Hardy constant has been stated in [19]. Special classes of potentials for higher order operators have been investigated such as Kato class in [15] and reverse Hölder class in [33, 22], see also the references therein.

In this paper we propose to start the investigation in \( L^2(\mathbb{R}^N) \) for the following Schrödinger type operator

\[
A = a^2 \Delta^2 + V^2
\]

with unbounded diffusion \( a(x) = 1 + |x|^\alpha \) and potential \( V = |x|^{\beta} \) where the constants \( \alpha, \beta \) are positive and the potential is supercritical, in the sense that \( \beta > \alpha - 2 \). In section 2, studying the properties of the form associated to \( A + \lambda \) where \( \lambda \geq \lambda_0 > 0 \) we prove that \((-A, D(A))\) is the generator of an analytic \( C_0\)-semigroup in \( L^2(\mathbb{R}^N) \) for \( N \geq 5 \). Moreover, in Section 3 we aim at characterizing the domain \( D(A) \) for suitable values of the dimension \( N \). Up to a transformation, we can consider the operator \((-\Delta)^2 + \tilde{V}^2 \) where \( \tilde{V} \) belongs to the reverse Hölder class. Making use of a priori estimates proved in [33] and some weighted higher order interpolation and Calderón-Zygmund estimates, for \( N > 8 \), we characterize the domain \( D(A) \) by the weighted Sobolev space

\[
D_2(A) = \{ u \in H^4(\mathbb{R}^N) : V^2 u \in L^2(\mathbb{R}^N), |x|^{2\alpha - h} D^{4-h} u \in L^2(\mathbb{R}^N) \text{ for } h = 0, 1, 2, 3, 4 \}.
\]

The constraints on the dimension \( N \geq 5 \), for the generation result, and \( N > 8 \), for the domain characterisation, correspond to the application of the Rellich inequality and the higher order Rellich inequality, respectively.

Finally, we prove that the spectrum of \( A \) is real and consists of a sequence of negative eigenvalues accumulating at \(-\infty\).

2. Generation in \( L^2(\mathbb{R}^N) \)

Consider the bi-Laplacian operator with unbounded diffusion and potential term

\[
A = a^2 \Delta^2 + V^2,
\]

where \( a(x) = 1 + |x|^\alpha \) and \( V = |x|^{\beta} \) with \( \alpha > 0 \) and \( \beta > (\alpha - 2)^+ \). Let \( u, v \in C_c^\infty(\mathbb{R}^N \setminus \{0\}) \), since

\[
\int_{\mathbb{R}^N} A u \nabla \varphi \, dx = \int_{\mathbb{R}^N} \left[ a^2 (\Delta^2 u) + V^2 u \right] \nabla \varphi \, dx = \int_{\mathbb{R}^N} \left[ \Delta u \Delta (a^2 \nabla \varphi) + V^2 u \nabla \varphi \right] \, dx
\]

we define the following sesquilinear form

\[
a_\lambda(u, v) = \int_{\mathbb{R}^N} \left[ a^2 \Delta^2 u \Delta \nabla v + 2 \nabla a^2 \cdot \nabla \nabla \Delta u \alpha - \alpha^2 \Delta u \varphi + (V^2 + \lambda) u \varphi \right] \, dx
\]

(2.1)

\[
= \int_{\mathbb{R}^N} \left\{ (1 + |x|^\alpha)^2 \nabla \varphi \Delta^2 u + 4 \alpha (1 + |x|^\alpha) |x|^2 x \cdot \nabla \nabla \Delta u + 2 \alpha \left[ (2\alpha - 2 + N)|x|^2 \alpha - (\alpha - 2 + N)|x|^{\alpha - 2} \right] \nabla \varphi \Delta u + (V^2 + \lambda) u \varphi \right\} \, dx
\]

(2.2)
on \( C_0^\infty(\mathbb{R}^N \setminus \{0\}) \) for some \( \lambda \geq \lambda_0 \), where \( \lambda_0 > 0 \) will be chosen later. Note that we propose to define \( \alpha_\lambda \) on \( C_0^\infty(\mathbb{R}^N \setminus \{0\}) \) for further purposes that will be clear in the sequel.

In the following we prove that the form \( \alpha_\lambda \) is densely defined, accretive, continuous and closable, and that the domain of the closure coincides with the weighted Sobolev space

\[
D = \{ u \in H^2(\mathbb{R}^N) : (1 + |x|^a)\Delta u, |x|^a-1\nabla u, |x|^a-2 u, Vu \in L^2(\mathbb{R}^N) \}
\]

endowed with the norm

\[
(2.3) \quad \| u \|_D = \| (1 + |x|^a)\Delta u \|_2 + \| |x|^a-1\nabla u \|_2 + \| |x|^a-2 u \|_2 + \| Vu \|_2 + \| u \|_2.
\]

We start with some estimate that will be useful in the sequel.

**Lemma 1.** Let \( N \geq 5 \) and \( \gamma > 0 \). For every real function \( u \in C_0^\infty(\mathbb{R}^N \setminus \{0\}) \), there exists a constant \( k \in \mathbb{R} \), which need not be positive, depending on \( \gamma \) and \( N \) such that

\[
\int_{\mathbb{R}^N} |x|^\gamma (\Delta^2 u) u \, dx \geq k \int_{\mathbb{R}^N} |x|^\gamma -4 u^2 \, dx.
\]

**Proof.** Let us first compute the derivatives of \( |x|^\gamma \)

\[
\nabla |x|^\gamma = \gamma |x|^{\gamma -2} x
\]

\[
D_{ij} |x|^\gamma = \gamma (\gamma -2) |x|^{\gamma -4} x_i x_j + \gamma |x|^{\gamma -2} \delta_{ij}
\]

\[
\Delta |x|^\gamma = \gamma (\gamma -2 + N) |x|^{\gamma -2} =: c_1 |x|^{\gamma -2}
\]

\[
\Delta^2 |x|^\gamma = \gamma (\gamma -2)(\gamma -2 + N)(\gamma -4 + N) |x|^{\gamma -4} =: c_2 |x|^{\gamma -4}
\]

and observe that \( c_1 \) is positive and the sign of \( c_2 \) depends on whether \( \gamma \) is bigger than 2 or not.

We now want to estimate \( \int_{\mathbb{R}^N} |x|^\gamma (\Delta^2 u) u \, dx \). Integrating by parts one has that

\[
\int_{\mathbb{R}^N} |x|^\gamma (\Delta^2 u) u \, dx = \int_{\mathbb{R}^N} |x|^\gamma (\Delta u)^2 \, dx + 2 \int_{\mathbb{R}^N} \nabla |x|^\gamma \cdot \nabla u \Delta u \, dx + \int_{\mathbb{R}^N} \Delta |x|^\gamma u \Delta u \, dx.
\]

The second term of the right-hand side can be written as

\[
2 \int_{\mathbb{R}^N} \Delta u \nabla u \cdot \nabla |x|^\gamma \, dx = -2 \int_{\mathbb{R}^N} \nabla u \cdot \nabla (\nabla u \cdot \nabla |x|^\gamma) \, dx = -2 \sum_{i,j=1}^N \int_{\mathbb{R}^N} D_i u D_i (D_j u D_j |x|^\gamma) \, dx
\]

\[
= -2 \sum_{i,j=1}^N \int_{\mathbb{R}^N} D_i u D_i D_j u D_j |x|^\gamma \, dx - 2 \sum_{i,j=1}^N \int_{\mathbb{R}^N} D_i u D_j u D_j |x|^\gamma \, dx
\]

\[
= - \sum_{i,j=1}^N \int_{\mathbb{R}^N} D_j (D_i u)^2 D_j |x|^\gamma \, dx - 2 \int_{\mathbb{R}^N} (D^2 |x|^\gamma) \nabla u \cdot \nabla u \, dx
\]

\[
= \int_{\mathbb{R}^N} |\nabla u|^2 \Delta |x|^\gamma \, dx - 2 \int_{\mathbb{R}^N} (D^2 |x|^\gamma) \nabla u \cdot \nabla u \, dx
\]

and since \( \Delta u^2 = 2u \Delta u + 2|\nabla u|^2 \), the third term can be written as

\[
\int_{\mathbb{R}^N} u \Delta u \Delta |x|^\gamma \, dx = \frac{1}{2} \int_{\mathbb{R}^N} \Delta u^2 \Delta |x|^\gamma \, dx - \int_{\mathbb{R}^N} |\nabla u|^2 \Delta |x|^\gamma \, dx
\]
Hence, one obtains
\[
\int_{\mathbb{R}^N} |x|^\gamma (\Delta^2 u) \, dx = \int_{\mathbb{R}^N} |x|^\gamma (\Delta u)^2 \, dx - 2 \int_{\mathbb{R}^N} (D^2 |x|^\gamma) \nabla \cdot \nabla u \, dx
\]
(2.4)
\[+ \frac{1}{2} \int_{\mathbb{R}^N} u^2 \Delta^2 |x|^\gamma \, dx.\]

Taking into account the derivatives of \(|x|^\gamma\) the equality (2.4) reads as
\[
\int_{\mathbb{R}^N} |x|^\gamma (\Delta^2 u) \, dx = \int_{\mathbb{R}^N} |x|^\gamma (\Delta u)^2 \, dx - 2\gamma (\gamma - 2) \int_{\mathbb{R}^N} |x|^{\gamma-4} \sum_{i,j=1}^{N} x_i x_j D_i u D_j u \, dx
\]
\[- 2\gamma \int_{\mathbb{R}^N} |x|^{\gamma-2} |\nabla u|^2 \, dx + \frac{1}{2} c_2 \int_{\mathbb{R}^N} |x|^{\gamma-4} u^2 \, dx.\]

We can estimate
\[
\sum_{i,j=1}^{N} x_i x_j D_i u D_j u = (\langle x, \nabla u \rangle)^2 \leq |x|^2 |\nabla u|^2,
\]
then \(-2\gamma (\gamma - 2) (\langle x, \nabla u \rangle)^2 \geq -2\gamma |\gamma - 2| |x|^2 |\nabla u|^2\). Therefore, one obtains
\[
\int_{\mathbb{R}^N} |x|^\gamma (\Delta^2 u) \, dx \geq \int_{\mathbb{R}^N} |x|^\gamma (\Delta u)^2 \, dx - 2\gamma (|\gamma - 2| + 1) \int_{\mathbb{R}^N} |x|^{\gamma-2} |\nabla u|^2 \, dx
\]
\[+ \frac{1}{2} c_2 \int_{\mathbb{R}^N} |x|^{\gamma-4} u^2 \, dx.\]

Since
\[
\int_{\mathbb{R}^N} |x|^{\gamma-2} |\nabla u|^2 \, dx = - \int_{\mathbb{R}^N} u \div (|x|^{\gamma-2} \nabla u) \, dx
\]
\[= - \int_{\mathbb{R}^N} |x|^{\gamma-2} u \Delta u \, dx - \frac{1}{2} (\gamma - 2) \int_{\mathbb{R}^N} |x|^{\gamma-4} x \cdot \nabla u^2 \, dx
\]
\[= - \int_{\mathbb{R}^N} |x|^{\gamma-2} u \Delta u \, dx + \frac{\gamma - 2}{2} (\gamma - 4 + N) \int_{\mathbb{R}^N} |x|^{\gamma-4} u^2 \, dx,
\]
setting \(c_3 = 2\gamma (|\gamma - 2| + 1) + 1\), \(c_4 = c_3 \frac{\gamma - 2}{2} (\gamma - 4 + N)\), \(c_5 = 2^{-\frac{1}{2}}\) and considering two suitable constants \(c_6, k \in \mathbb{R}\) one has
\[
\int_{\mathbb{R}^N} |x|^\gamma (\Delta^2 u) \, dx \geq \frac{1}{2} \int_{\mathbb{R}^N} |x|^\gamma (\Delta u)^2 \, dx + \int |x|^{\gamma-2} |\nabla u|^2 \, dx
\]
\[+ \frac{1}{2} \int_{\mathbb{R}^N} |x|^\gamma (\Delta u)^2 \, dx + c_3 \int_{\mathbb{R}^N} |x|^{\gamma-2} u \Delta u \, dx + \left( \frac{c_2}{2} - c_4 \right) \int_{\mathbb{R}^N} |x|^{\gamma-4} u^2 \, dx
\]
(2.5)
\[= \frac{1}{2} \int_{\mathbb{R}^N} |x|^\gamma (\Delta u)^2 \, dx + \int |x|^{\gamma-2} |\nabla u|^2 \, dx
\]
\[+ \int_{\mathbb{R}^N} \left( c_5 |x|^\frac{\gamma}{2} \Delta u + c_6 |x|^\frac{\gamma}{2} u^2 \right) \, dx + k \int_{\mathbb{R}^N} |x|^{\gamma-4} u^2 \, dx
\]
where we have rearranged the terms for future convenience.
Hence, \( \int_{\mathbb{R}^N} |x|^\gamma (\Delta^2 u) u \, dx \geq k \int_{\mathbb{R}^N} |x|^{-4} u^2 \, dx \).

\[ \square \]

**Proposition 1.** Let \( N \geq 5, \alpha > 0, \beta > (\alpha - 2)^+ \). The form \( a_\lambda \) is accretive. Moreover, for every \( u \in C_c^\infty(\mathbb{R}^N \setminus \{0\}) \) the following inequality holds

\[
\text{Re } a_\lambda(u, u) \geq \frac{1}{4} \left( (1 + |x|^\alpha) \Delta u \right)^2 + \|x|^{\alpha-1} \|\nabla u\|^2 + \|x|^{\alpha-2} \|\nabla^2 u\|^2 + \frac{1}{2} \| Vu \|^2 + \| u \|^2.
\]

**Proof.** If \( u \) is a real function, by (2.1) we can write

\[
a_\lambda(u, u) = \int_{\mathbb{R}^N} (1 + |x|^\alpha)^2 (\Delta^2 u) u \, dx + \int_{\mathbb{R}^N} (V^2 + \lambda) u^2 \, dx
\]

\[
= \int_{\mathbb{R}^N} (\Delta u)^2 \, dx + 2 \int_{\mathbb{R}^N} |x|^\alpha (\Delta^2 u) u \, dx + \int_{\mathbb{R}^N} |x|^{2\alpha} (\Delta^2 u) u \, dx + \int_{\mathbb{R}^N} (V^2 + \lambda) u^2 \, dx.
\]

By Lemma 1 there exists a constant such that the term \( 2 \int_{\mathbb{R}^N} |x|^\alpha (\Delta^2 u) u \, dx \geq k_1 \int_{\mathbb{R}^N} |x|^{-4} u^2 \, dx \). Moreover, one can estimate the term \( \int_{\mathbb{R}^N} |x|^{2\alpha} (\Delta^2 u) u \, dx \) as in (2.5) to obtain that

\[
a_\lambda(u, u) \geq \frac{1}{2} \int_{\mathbb{R}^N} (\Delta u)^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} (\Delta u)^2 \, dx + k_1 \int_{\mathbb{R}^N} |x|^{-4} u^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |x|^{2\alpha} (\Delta u)^2 \, dx
\]

\[
\quad + \int_{\mathbb{R}^N} |x|^{2\alpha-2} |\nabla u|^2 \, dx + k_2 \int_{\mathbb{R}^N} |x|^{2\alpha-4} u^2 \, dx + \int_{\mathbb{R}^N} \left( |x|^{2\beta} + \lambda \right) \, dx
\]

\[
\geq \int_{\mathbb{R}^N} \left[ \frac{1}{4} (1 + |x|^\alpha)^2 (\Delta u)^2 + |x|^{2\alpha-2} |\nabla u|^2 + \frac{1}{2} (\Delta u)^2 \right] \, dx
\]

\[
\quad + \int_{\mathbb{R}^N} \left[ \frac{c_0}{|x|^4} + k_1 |x|^{\alpha-4} + k_2 |x|^{2\alpha-4} + |x|^{2\beta} + \lambda \right] u^2 \, dx
\]

\[
= \int_{\mathbb{R}^N} \left[ \frac{1}{4} (1 + |x|^\alpha)^2 (\Delta u)^2 + |x|^{2\alpha-2} |\nabla u|^2 + |x|^{2\alpha-4} u^2 + \frac{1}{2} |x|^{2\beta} u^2 + u^2 \right] \, dx
\]

\[
\quad + \int_{\mathbb{R}^N} \left[ \frac{c_0}{|x|^4} + k_1 |x|^{\alpha-4} + (k_2 - 1) |x|^{2\alpha-4} + \frac{1}{2} |x|^{2\beta} u^2 + (\lambda - 1) \right] u^2 \, dx
\]

(2.6)

\[
\geq \int_{\mathbb{R}^N} \left[ \frac{1}{4} (1 + |x|^\alpha)^2 (\Delta u)^2 + |x|^{2\alpha-2} |\nabla u|^2 + |x|^{2\alpha-4} u^2 + \frac{1}{2} |x|^{2\beta} u^2 + u^2 \right] \, dx \geq 0
\]

for some \( \lambda \geq \lambda_0 \) since the growth at infinity and in 0 is led by the terms \( \frac{1}{2} |x|^{2\beta} \) and \( c_0 |x|^{-4} \), respectively, by the relation \( 2\beta > 2\alpha - 4 > \alpha - 4 > -4 \).

If \( u \) is complex valued then one can write \( u = \text{Re } u + i \text{ Im } u \) and in view of \( \text{Re } a_\lambda(u, u) = a_\lambda(\text{Re } u, \text{Re } u) + a_\lambda(\text{Im } u, \text{Im } u) \) the thesis follows.

Since the form \( a_\lambda \) is accretive on \( C_c^\infty(\mathbb{R}^N \setminus \{0\}) \) we can associate to \( a_\lambda \) the norm

\[
\|u\|_{a_\lambda}^2 = \text{Re } a_\lambda(u, u) + \| u \|_2^2.
\]
By Proposition 1 it follows that
\[ \|u\|_D \leq C\|u\|_{a_\lambda} \text{ for all } u \in C_c^\infty(\mathbb{R}^N \setminus \{0\}). \]

We are now ready to study the properties of the form \( a_\lambda \).

**Proposition 2.** Let \( N \geq 5, \alpha > 0, \beta > (\alpha - 2)^+ \). The form \( a_\lambda \) is densely defined, accretive and continuous on \( C_c^\infty(\mathbb{R}^N \setminus \{0\}) \).

**Proof.** \( C_c^\infty(\mathbb{R}^N \setminus \{0\}) \subset D \) then \( a_\lambda \) is densely defined. The accretivity is stated in (2.6). As regards the continuity we observe that from (2.2) we have
\[
|a_\lambda(u, v)| \leq \|[(1 + |x|^\alpha)\Delta u]_2[(1 + |x|^\alpha)\Delta v]_2 + 4\alpha\|[(1 + |x|^\alpha)\Delta u]_2\|x^\alpha_2v\|_2 \\
+ 2\alpha(2\alpha - 2 + N)\|x^\alpha u\|_2\|x^{\alpha - 2} v\|_2 \\
+ 2\alpha(2\alpha - 2 + N)\|\Delta u\|_2\|x^\alpha_2 v\|_2 + \|Vu\|_{L^2}\|Vv\|_2 + \lambda\|u\|_2\|v\|_2
\]
(2.8)
Then, by (2.7)
\[
|a_\lambda(u, v)| \leq C\|u\|_{a_\lambda}\|v\|_{a_\lambda}.
\]

Combining (2.7) and (2.8) we can deduce that the norms \( \| \cdot \|_{a_\lambda} \) and \( \| \cdot \|_D \) are equivalent on \( C_c^\infty(\mathbb{R}^N \setminus \{0\}) \).

Now we prove the closability of the form \( a_\lambda \).

**Proposition 3.** Let \( N \geq 5, \alpha > 0, \beta > (\alpha - 2)^+ \). The form \( a_\lambda \) is closable on \( C_c^\infty(\mathbb{R}^N \setminus \{0\}) \).

**Proof.** Let \( (u_n) \in C_c^\infty(\mathbb{R}^N \setminus \{0\}) \) be such that \( u_n \to 0 \) in \( L^2(\mathbb{R}^N) \) and \( \|u_n - u_m\|_{a_\lambda} \to 0 \) as \( n, m \to \infty \). By estimate (2.7) it follows that \( (u_n) \) and \( (\Delta u_n) \) are Cauchy sequences in \( L^2(\mathbb{R}^N) \). Using the interpolation inequality
\[
\|\nabla u\|_2 \leq C\|\Delta u\|_2\|u\|_2
\]
it follows that \( (u_n) \) is a Cauchy sequence in \( H^2(\mathbb{R}^N) \) and then it converges to 0 in \( H^2(\mathbb{R}^N) \).

Taking into account the expression (2.3) for the norm \( \| \cdot \|_D \), we have that \( w_n = (1 + |x|^\alpha)\Delta u_n \) and \( v_n = |x|^\alpha_2\nabla u_n \) are Cauchy sequences in \( L^2(\mathbb{R}^N) \). Since they converge to 0 a.e. they also converge to 0 in \( L^2(\mathbb{R}^N) \) and one has
\[
\int_{\mathbb{R}^N} (1 + |x|^\alpha)\Delta u_n^2 \, dx \to 0 \text{ and } \int_{\mathbb{R}^N} |x|^{\alpha_2} v_n^2 \, dx \to 0
\]
as \( n \to \infty \). Moreover (2.3) gives also that
\[
\int_{\mathbb{R}^N} |x|^{2\alpha - 4} u_n^2 \, dx \to 0
\]
and
\[
\int_{\mathbb{R}^N} |x|^{2\beta} u_n^2 \, dx \to 0
\]
as \( n \to \infty \).

Then we have proved that \( \|u_n\|_D \to 0 \) as \( n \to \infty \). Since by (2.8) it follows that
\[
|a_\lambda(u_n, u_n)| \leq C\|u_n\|^2_D,
\]
we have that \( a_\lambda(u_n, u_n) \to 0 \) as \( n \to \infty \) and then \( a_\lambda \) is closable. \( \square \)

Next proposition gives a characterization of the domain of the closure of the form.

**Proposition 4.** Let \( N \geq 5, \alpha > 0, \beta > (\alpha - 2)^+ \). The domain of the closure of \( a_\lambda \) coincides with the weighted Sobolev space

\[
D = \{ u \in H^2(\mathbb{R}^N) : (1 + |x|^\alpha)\Delta u, |x|^\alpha - 2u, V u \in L^2(\mathbb{R}^N) \}.
\]

**Proof.** We recall that \( D \) is a Banach space with respect to the norm \( \| \cdot \|_D \). Since \( C^\infty(\mathbb{R}^N \setminus \{0\}) \subset D \) and since the norm \( \| \cdot \|_D \) is equivalent to the form norm \( \| \cdot \|_D \) on \( C^\infty(\mathbb{R}^N \setminus \{0\}) \), we have just to prove that \( C^\infty_c(\mathbb{R}^N \setminus \{0\}) \) is dense in \( D \) with respect the norm \( \| \cdot \|_D \).

It is enough to prove that the set of functions in \( H^2(\mathbb{R}^N) \) with compact support contained in \( \mathbb{R}^N \setminus \{0\} \) is dense in \( D \). Take \( u \in D \) and consider \( u_n = u \varphi_n \), where \( \varphi_n \in C^\infty_c(\mathbb{R}^N \setminus \{0\}) \) is such that

\[
\begin{aligned}
\varphi_n &= 0 \text{ in } B(\frac{1}{n}) \cup B^c(2n), \\
\varphi_n &= 1 \text{ in } B(n) \setminus B(\frac{2}{n}), \\
0 \leq \varphi_n \leq 1, \\
|\nabla \varphi_n(x)| &\leq C \frac{1}{|x|^2}, \\
|D^2 \varphi_n(x)| &\leq C \frac{1}{|x|^4}.
\end{aligned}
\]

Observe that such a function exists. Indeed, let \( \varphi \in C^\infty([0, +\infty)) \) such that \( 0 \leq \varphi(t) \leq 1 \), \( \varphi(t) = 1 \) for \( 0 \leq t \leq 1 \), \( \varphi(t) = 0 \) if \( t \geq 2 \). Then the function

\[
\varphi_n(x) = \begin{cases} 
1 - \varphi(n|x|) & \text{in } B(\frac{2}{n}), \\
1 & \text{in } B(n) \setminus B(\frac{2}{n}), \\
\varphi \left( \frac{|x|}{n} \right) & \text{in } B^c(n)
\end{cases}
\]

satisfies the desired properties.

We show now that \( u_n \to u \) with respect to the norm \( \| \cdot \|_D \). Indeed, \( u_n|x|^{\alpha} \to u|x|^{\alpha} \) and \( u_n|x|^{\alpha - 2} \to u|x|^{\alpha - 2} \) in \( L^2(\mathbb{R}^N) \) by dominated convergence.

As regards the first order term we observe that

\[
|x|^{\alpha - 1} |\nabla u_n - \nabla u| = |x|^{\alpha - 1} |(\varphi_n - 1)\nabla u + u \nabla \varphi_n| \\
\leq |x|^{\alpha - 1} (1 - \varphi_n) |\nabla u| + |x|^{\alpha - 1} |u| |\nabla \varphi_n|.
\]

Therefore, it suffices to prove that the second term of the right-hand side of the above estimate converges to 0 in \( L^2(\mathbb{R}^N) \). Indeed it converges a.e. to 0, and since

\[
|x|^{\alpha - 1} |u| |\nabla \varphi_n| \leq C |x|^{\alpha - 2} |u| |\chi_{K_n} \in L^2(\mathbb{R}^N)
\]

where \( K_n = B(\frac{2}{n}) \setminus B(\frac{1}{n}) \cup B(2n) \setminus B(n) \), the convergence is in \( L^2(\mathbb{R}^N) \).

As regards the diffusion term we can argue in a similar way, since

\[
(1 + |x|^{\alpha}) |\Delta u_n - \Delta u| = (1 + |x|^{\alpha}) |\varphi_n \Delta u + 2 \nabla \varphi_n \cdot \nabla u + u \Delta \varphi_n - \Delta u| \\
\leq (1 + |x|^{\alpha}) (1 - \varphi_n) |\Delta u| + 2(1 + |x|^{\alpha}) |\nabla u||\nabla \varphi_n| + (1 + |x|^{\alpha}) |u||\Delta \varphi_n|,
\]
and
\[ 2(1 + |x|^\alpha)|\nabla u|\nabla \varphi_n| \leq C \left( \frac{1}{|x|} |\nabla u| + |x|^{\alpha - 1}|\nabla u| \right) \chi_{K_n} \in L^2(\mathbb{R}^N) \]
by Hardy’s inequality, and
\[ (1 + |x|^\alpha)|u|\Delta \varphi_n| \leq C \left( \frac{1}{|x|^2} |u| + |x|^{\alpha - 2}|u| \right) \chi_{K_n} \in L^2(\mathbb{R}^N) \]
by Rellich’s inequality. \(\square\)

We observe that \(C_c^\infty(\mathbb{R}^N) \subset D\) and therefore we have that also \(C_c^\infty(\mathbb{R}^N)\) is a core for \(a_\lambda\).

The form \(a_\lambda\) defined in (2.2) is densely defined, accretive, continuous and closable. Therefore, its closure \(\overline{a_\lambda}\) is associated to a closed operator \((A_\lambda, D(A_\lambda))\) on \(L^2(\mathbb{R}^N)\) defined by
\[
D(A_\lambda) := \{ u \in D : \exists v \in L^2(\mathbb{R}^N) s.t. \overline{a_\lambda}(u, h) = \langle v, h \rangle, \forall h \in D \} \\
A_\lambda u := v.
\]

Taking into consideration the properties of the form \(a_\lambda\) and [31, Proposition 1.51 and Theorem 1.52], we obtain the following generation theorem on \(L^2(\mathbb{R}^N)\).

**Theorem 1.** Let \(N \geq 5, \alpha > 0, \beta > (\alpha - 2)^+\). The operator \((-A_\lambda, D(A_\lambda))\) generates a strongly continuous analytic contraction semigroup on \(L^2(\mathbb{R}^N)\).

Since for \(u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})\) by (2.1) one has that
\[
a_\lambda(u, v) = \int_{\mathbb{R}^N} (Au + \lambda u)v \, dx,
\]
and \(C_c^\infty(\mathbb{R}^N \setminus \{0\})\) is a core for \(\overline{a_\lambda}\), the equality holds for every \(v \in D\). Then, the operator \((-A_\lambda, D(A_\lambda))\) is an extension of \((-A, C_c^\infty(\mathbb{R}^N \setminus \{0\}))\), more precisely, \(Au + \lambda u = A_\lambda u\). Therefore, there is \((-A, D(A))\), an extension of \((-A, C_c^\infty(\mathbb{R}^N \setminus \{0\}))\), that generates the analytic \(C_0\)-semigroup \(e^{-tA} = e^{\lambda t}e^{-tA_\lambda}\) in \(L^2(\mathbb{R}^N)\). In the next section we aim at characterizing the domain \(D(A)\).

### 3. Domain Characterization

In this section we aim at characterizing \(D(A)\) for suitable values of the dimension \(N\). By definition
\[
D(A) = \{ u \in D : \exists v \in L^2(\mathbb{R}^N) s.t. \overline{a_\lambda}(u, h) = \langle v, h \rangle, \forall h \in D \} \\
Au + \lambda u = v.
\]
where
\[
D = \{ u \in H^2(\mathbb{R}^N) : (1 + |x|^\alpha)\Delta u, |x|^{\alpha - 1}\nabla u, |x|^{\alpha - 2}u, Vu \in L^2(\mathbb{R}^N) \}.
\]

Let \(u \in D(A)\). Since \(u \in H^2(\mathbb{R}^N)\) one can integrate by parts and obtain
\[
\int_{\mathbb{R}^N} Auv \, dx = \int_{\mathbb{R}^N} uA^*v \, dx
\]
for every \(v \in C_c^\infty(\mathbb{R}^N \setminus \{0\})\) where
\[
A^*v = a^2\Delta^2 v + 4\nabla a^2 \cdot \nabla \Delta v + 2\Delta a^2 \Delta v + 4TrD^2a^2D^2v
\]
Thus, consider the Schrödinger operator $\tilde{\Delta}^2 u + \nabla^2 v + V^2v$.

By local elliptic regularity, see [3], since the coefficients of $A$ are bounded in every compact set contained in $\mathbb{R}^N \setminus \{0\}$ we have that $u \in H^4_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$.

Now let $f = Au$, then $f \in L^2(\mathbb{R}^N)$ and dividing by $a^2$ one obtains

$$\Delta^2 u + \left(\frac{V}{a}\right)^2 u = \frac{f}{a^2} \in L^2(\mathbb{R}^N).$$

Thus, consider the Schrödinger operator $\tilde{A} = \Delta^2 u + \tilde{V}^2 u$ where $\tilde{V} = \frac{V}{a}$. In [33] the author established maximal estimates for $\tilde{H}$, assuming that the potential $\tilde{V}$ belongs to the reverse Hölder class $B_q$ for some $q \geq \frac{N}{2}$. We recall that a nonnegative locally $L^q$-integrable function $\tilde{V}$ on $\mathbb{R}^N$ is said to be in $B_q$, $1 < q < \infty$, if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B|} \int_B \tilde{V}^q(x) dx\right)^{1/q} \leq C \left(\frac{1}{|B|} \int_B \tilde{V}(x) dx\right)$$

holds for every $x \in \mathbb{R}^N$ and for every ball $B$ in $\mathbb{R}^N$.

In the same way as in [6, Section 3] one verifies that

$$C_1(1 + |x|^\beta - \alpha) \leq \tilde{V} \leq C_2(1 + |x|^\beta - \alpha) \quad \text{if } \beta \geq \alpha,$$

$$C_3 \frac{1}{1 + |x|^{\alpha - \beta}} \leq \tilde{V} \leq C_4 \frac{1}{1 + |x|^{\alpha - \beta}} \quad \text{if } \alpha - 2 < \beta < \alpha,$$

for some positive constants $C_1, C_2, C_3, C_4$. Then it can be proved that $\tilde{V} \in B_q$ if $\beta - \alpha > -\frac{N}{q}$. In our case, since $\beta > \alpha - 2$, the potential $\tilde{V} \in B_{\frac{N}{2}}$.

The following theorem was proved by Sugano [33, Theorem 1].

**Theorem 2.** Let $H = (-\Delta)^2 + V^2, f \in L^p(\mathbb{R}^N)$ and $j = 0, 1, 2, 3$. Suppose $V \in B_{\frac{N}{2}}$ and there exists a constant $C$ such that $V(x) \leq C m(x, V)^2$, then there exist constants $C_j$ such that

$$||V^{2 - \frac{1}{2}j} H^{-1} f||_p \leq C_j ||f||_p$$

for $1 < p \leq \infty$.

where $\frac{1}{m(x, V)} = \sup \left\{ r > 0 : \frac{x^2}{|B_r(x)|} \int_{B_r(x)} V(y) \, dy \leq 1 \right\}$. Moreover, there exists $C'$ such that

$$||\nabla^4 H^{-1} f||_p \leq C' ||f||_p$$

for $1 < p < \infty$.

In our case $\tilde{V} \in B_{\frac{N}{2}}$ and since $\tilde{V}$ behaves as a non-negative polynomial, the condition $\tilde{V}(x) \leq C m(x, \tilde{V})^2$ is fulfilled, cf. [33, Remark 5]. Therefore it holds $\tilde{V}^2 u \in L^2(\mathbb{R}^N)$ and $\Delta^2 u \in L^2(\mathbb{R}^N)$ with

$$||\tilde{V}^2 u||_2 \leq C \left\| \frac{f}{a} \right\|_2, \quad ||\Delta^2 u||_2 \leq C' \left\| \frac{f}{a} \right\|_2$$

for some constant $C > 0$. Then $u \in H^4(\mathbb{R}^N)$ and

$$D(A) = \{ u \in H^4(\mathbb{R}^N) \cap D : Au \in L^2(\mathbb{R}^N) \}.$$
Proposition 5. Let $N \geq 5, \alpha > 0, \beta > (\alpha - 2)^+$. There exists $\lambda_0 > 0$ such that for every $\lambda \geq \lambda_0$ and for every $u \in C^\infty_c(\mathbb{R}^N \setminus \{0\})$ the following estimate holds

$$\|V^2 u\|_2 \leq C\|Au + \lambda u\|_2.$$  

Proof. Let $u \in C^\infty_c(\mathbb{R}^N \setminus \{0\})$ be a real function. Recall that by Lemma 1, for every $\gamma > 0$ we have

$$\int_{\mathbb{R}^N} |x|^\gamma (\Delta^2 u) u \, dx \geq k \int_{\mathbb{R}^N} |x|^{\gamma - 4} u^2 \, dx,$$

where $k$ is a real constant which depends on $\gamma$ and $N$. Now we evaluate $a_\lambda(u, (1 + V^2)u) = \int_{\mathbb{R}^N} (A + \lambda)u(1 + V^2)u \, dx$. By (3.1) and Rellich’s inequality there exist $k_1, k_2, k_3, k_4, k_5 \in \mathbb{R}$ and a positive $c_0$ such that

$$\int_{\mathbb{R}^N} \left((1 + |x|^{\alpha})^2 \Delta^2 u + |x|^{2\beta} u + \lambda u\right)(1 + |x|^{2\beta})u \, dx$$

$$= \int_{\mathbb{R}^N} (\Delta u)^2 + (2|x|^{\alpha} + |x|^{2\alpha} + |x|^{2\beta} + 2|x|^{\alpha + 2\beta} + |x|^{2(\alpha + \beta)})(\Delta^2 u)u$$

$$+ (|x|^{4\beta} + (1 + \lambda)|x|^{2\beta} + \lambda)u^2 \, dx$$

$$\geq \int_{\mathbb{R}^N} \left(c_0|x|^{-4} + k_1|x|^{\alpha - 4} + k_2|x|^{2\alpha - 4} + k_3|x|^{2\beta - 4} + k_4|x|^{\alpha + 2\beta - 4} + k_5|x|^{2(\alpha + \beta - 2)}

+ |x|^{4\beta} + (1 + \lambda)|x|^{2\beta} + \lambda\right)u^2 \, dx$$

$$= \int_{\mathbb{R}^N} \left(c_0|x|^{-4} + k_1|x|^{\alpha - 4} + k_2|x|^{2\alpha - 4} + k_3|x|^{2\beta - 4} + k_4|x|^{\alpha + 2\beta - 4} + k_5|x|^{2(\alpha + \beta - 2)}

+ \frac{3}{4}|x|^{4\beta} + \left(\frac{1}{2} + \lambda\right)|x|^{2\beta} + \lambda - \frac{3}{4}\right)u^2 \, dx + \int_{\mathbb{R}^N} \left(\frac{1}{4}|x|^{4\beta} + \frac{1}{2}|x|^{2\beta} + \frac{1}{4}\right)u^2 \, dx.$$  

We can choose $\lambda_0$ such that

$$c_0|x|^{-4} + k_1|x|^{\alpha - 4} + k_2|x|^{2\alpha - 4} + k_3|x|^{2\beta - 4} + k_4|x|^{\alpha + 2\beta - 4} + k_5|x|^{2(\alpha + \beta - 2)}$$

$$+ \frac{3}{4}|x|^{4\beta} + \left(\frac{1}{2} + \lambda_0\right)|x|^{2\beta} + \lambda_0 - \frac{3}{4} \geq 0$$

and then

$$\int_{\mathbb{R}^N} (Au + \lambda u)(1 + V^2)u \, dx \geq C \int_{\mathbb{R}^N} (1 + V^2)^2 u^2 \, dx$$

for every $\lambda \geq \lambda_0$. By Hölder’s inequality

$$\|(1 + V^2)u\|_2^2 \leq C\|Au + \lambda u\|_2\|(1 + V^2)u\|_2$$

so that

$$\|V^2 u\|_2 \leq \|(1 + V^2)u\|_2 \leq C\|Au + \lambda u\|_2.$$  

$\square$
For every function $u$ with derivative up to order 4 we set

$$|D^4 u| = \left( \sum_{i,j,k,l=1}^{N} |D_{ijkl}u|^2 \right)^{\frac{1}{2}}, \quad |D^3 u| = \left( \sum_{i,j,k=1}^{N} |D_{ijk}u|^2 \right)^{\frac{1}{2}},$$

$$|D^2 u| = \left( \sum_{i,j=1}^{N} |D_{ij}u|^2 \right)^{\frac{1}{2}}, \quad |Du| = \left( \sum_{i=1}^{N} |D_i u|^2 \right)^{\frac{1}{2}}.$$

By applying \([27, \text{Lemma 4.4}]\) to the derivatives of $u$, for every $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\}), h = 1, 2, 3$ and $\gamma \in \mathbb{R}$ the following weighted interpolation inequalities holds

$$\| |x|^{\gamma} D^h u \|_2 \leq \varepsilon \| |x|^{\gamma+1} \|_2 + C \| |x|^{-1} D^{h-1} u \|_2.$$

**Proposition 6.** Let $N > 8, \alpha > 0, \beta > (\alpha - 2)^+$. For every $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ we have

$$\| |x|^{2\alpha-h} D^h u \|_2 \leq C \| Au \|_2 + \| u \|_2$$

for $h = 0, 1, 2, 3, 4$.

**Proof.** Let $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$. We propose now to use repeatedly (3.2) to prove that

$$\| |x|^{2\alpha-3} D^3 u \|_2 \leq \varepsilon \| |x|^{2\alpha-2} D^2 u \|_2 + C \| |x|^{2\alpha-4} u \|_2,$$

$$\| |x|^{2\alpha-2} D^2 u \|_2 \leq \varepsilon \| |x|^{2\alpha-3} u \|_2 + C \| |x|^{2\alpha-4} u \|_2,$$

$$\| |x|^{2\alpha-3} D u \|_2 \leq \varepsilon \| |x|^{2\alpha-4} u \|_2 + C \| |x|^{2\alpha-3} D u \|_2.$$

In the sequel we will denote the constants as $C$ and $\varepsilon$ even if they may change from line to line. We have

$$\| |x|^{2\alpha-3} D u \|_2 \leq \varepsilon \| |x|^{2\alpha-2} D^2 u \|_2 + C \| |x|^{2\alpha-4} u \|_2,$$

$$\leq \varepsilon (\varepsilon \| |x|^{2\alpha-1} D^3 u \|_2 + C \| |x|^{2\alpha-3} D u \|_2) + C \| |x|^{2\alpha-4} u \|_2.$$ 

then

$$\| |x|^{2\alpha-3} D u \|_2 \leq \varepsilon \| |x|^{2\alpha-1} D^3 u \|_2 + C \| |x|^{2\alpha-4} u \|_2,$$

Now

$$\| |x|^{2\alpha-1} D^3 u \|_2 \leq \varepsilon \| |x|^{2\alpha-1} D^4 u \|_2 + C \| |x|^{2\alpha-2} D^2 u \|_2,$$

$$\leq \varepsilon \| |x|^{2\alpha-1} D^4 u \|_2 + C (\varepsilon \| |x|^{2\alpha-1} D^3 u \|_2 + C \| |x|^{2\alpha-3} D u \|_2)$$

$$\leq \varepsilon \| |x|^{2\alpha-1} D^4 u \|_2 + \varepsilon \| |x|^{2\alpha-3} D^3 u \|_2 + C (\varepsilon \| |x|^{2\alpha-1} D^3 u \|_2 + C \| |x|^{2\alpha-4} u \|_2)$$

therefore, one obtains (3.3).

A combination of (3.6) and (3.3) gives (3.5).

Finally, we have

$$\| |x|^{2\alpha-2} D^2 u \|_2 \leq \varepsilon \| |x|^{2\alpha-1} D^3 u \|_2 + C \| |x|^{2\alpha-3} D u \|_2,$$

$$\leq \varepsilon \| |x|^{2\alpha-1} D^4 u \|_2 + C \| |x|^{2\alpha-4} u \|_2$$

and then (3.4) is also proved.
Now we prove the following weighted Calderón-Zygmund type estimate
\begin{equation}
\|x|^{2α} D^4 u\|_2 \leq C \left( \|x|^{2α} \Delta^2 u\|_2 + \|x|^{2α-4} u\|_2 \right).
\end{equation}
We have
\[
\|x|^{2α} D^4 u\|_2 \leq \|D^4 (|x|^{2α} u)\|_2
+ C \left( \|x|^{2α-1} D^3 u\|_2 + \|x|^{2α-2} D^2 u\|_2 + \|x|^{2α-3} Du\|_2 + \|x|^{2α-4} u\|_2 \right)
\leq \|\Delta^2 (|x|^{2α} u)\|_2
+ C \left( \|x|^{2α-1} D^3 u\|_2 + \|x|^{2α-2} D^2 u\|_2 + \|x|^{2α-3} Du\|_2 + \|x|^{2α-4} u\|_2 \right)
\leq \|x|^{2α} \Delta^2 u\|_2 + \varepsilon \|x|^{2α} D^4 u\| + C |||x|^{2α-4} u||
\]
and then (3.7) follows.

So for \( h = 0, 1, 2, 3 \) we have proved that
\[
\|x|^{2α-h} D^{4-h} u\|_2 \leq C \left( \|x|^{2α} \Delta^2 u\|_2 + \|x|^{2α-4} u\|_2 \right).
\]
Now, from [14, Corollary 14] for \( N > 8 \) the higher order Rellich inequality holds
\[
\left\| \frac{u}{|x|^4} \right\|_2 \leq C \|\Delta^2 u\|_2
\]
and since by the assumptions on \( α \) and \( β \) one can estimate \( |x|^{2α-4} \leq C \left( 1 + \frac{1}{|x|^7} + |x|^{2β} \right) \), it follows that
\[
\|x|^{2α-4} u\|_2 \leq C \left( \|\Delta^2 u\|_2 + \|V^2 u\|_2 + \|u\|_2 \right).
\]
Thus, for \( h = 0, 1, 2, 3, 4 \) we have
\[
\|x|^{2α-h} D^{4-h} u\|_2 \leq C \left( \|\Delta^2 u\|_2 + \|x|^{2α} \Delta^2 u\|_2 + \|V^2 u\|_2 + \|u\|_2 \right)
\leq C \left( \|Au\|_2 + \|V^2 u\|_2 + \|u\|_2 \right)
\]
and the thesis follows by Proposition 5.

\[ \square \]

**Remark 1.** Note that for \( h = 0, 1, 2, 3 \) the weighted interpolation inequalities
\[
\|x|^{2α-h} D^{4-h} u\|_2 \leq C \left( \|x|^{2α} \Delta^2 u\|_2 + \|x|^{2α-4} u\|_2 \right)
\]
hold for any \( u \in C^\infty_c(\mathbb{R}^N \setminus \{0\}), N \geq 1 \), and any real \( α \). In particular, setting \( α = 0 \) the inequalities read as
\[
\|x|^{-h} D^{4-h} u\|_2 \leq C \left( \|\Delta^2 u\|_2 + \|x|^{-4} u\|_2 \right)
\]
then, if \( N > 8 \), by the higher order Rellich’s inequality one obtains
\[
\|x|^{-h} D^{4-h} u\|_2 \leq C \|\Delta^2 u\|_2
\]
for \( h = 0, 1, 2, 3, 4 \).
Moreover, for $u \in H^4(\mathbb{R}^N)$, reasoning as above with the standard interpolation inequality $\|Du\|_2 \leq C (\|D^2u\|_2 + \|u\|_2)$ one obtains that $\|u\|_{H^4(\mathbb{R}^N)} \leq C (\|D^4u\|_2 + \|u\|_2)$ and then for $u \in D(A)$

\begin{equation}
\|u\|_{H^4(\mathbb{R}^N)} \leq C\|u\|_A.
\end{equation}

Now arguing as in Proposition 4 we prove that $C^\infty_c(\mathbb{R}^N \setminus \{0\})$ is a core for $A$ in

$D_2(A) = \{ u \in H^4(\mathbb{R}^N) : V^2u \in L^2(\mathbb{R}^N), |x|^{2\alpha-h}D^{4-h}u \in L^2(\mathbb{R}^N) \mbox{ for } h = 0, 1, 2, 3, 4\}$.

**Proposition 7.** Let $N > 8, \alpha > 0, \beta > (\alpha - 2)^+$. Then $C^\infty_c(\mathbb{R}^N \setminus \{0\})$ is dense in $D_2(A)$ with respect to the operator norm. There exists $C \geq 0$ such that for every $u \in D_2(A)$

\begin{align*}
\| |x|^{2\alpha-h}D^{4-h}u\|_2 &\leq C (\|Au\|_2 + \|u\|_2) \mbox{ for } h = 0, 1, 2, 3, 4, \\
\|V^2u\|_2 &\leq C (\|Au\|_2 + \|u\|_2).
\end{align*}

**Proof.** It is enough to prove that the set of functions in $H^4(\mathbb{R}^N)$ with compact support contained in $\mathbb{R}^N \setminus \{0\}$ is dense in $D_2(A)$ with respect to the operator norm. Take $u \in D_2(A)$ and consider $u_n = u\varphi_n$, where $\varphi_n \in C^\infty_c(\mathbb{R}^N \setminus \{0\})$ is such that

\begin{align*}
\varphi_n &= 0 \mbox{ in } B\left(\frac{1}{n}\right) \cup B^c(2n), \\
\varphi_n &= 1 \mbox{ in } B(n) \setminus B\left(\frac{2}{n}\right), \\
0 &\leq \varphi_n \leq 1, \\
|\nabla \varphi_n(x)| &\leq C\frac{1}{|x|}, \\
|D^2\varphi_n(x)| &\leq C\frac{1}{|x|^2}, \\
|D^3\varphi_n(x)| &\leq C\frac{1}{|x|^3}, \\
|D^4\varphi_n(x)| &\leq C\frac{1}{|x|^4}.
\end{align*}

Observe that here we can consider the same sequence as in Proposition 4.

We have

\begin{align*}
\Delta^2(u\varphi_n) &= \Delta (\Delta u\varphi_n + 2\nabla u \cdot \nabla \varphi_n + u \Delta \varphi_n) \\
&= \varphi_n \Delta^2 u + u \Delta^2 \varphi_n + 4\nabla \Delta u \cdot \nabla \varphi_n + 4\nabla u \cdot \nabla \Delta \varphi_n \\
&+ 4 \sum_{i=1}^N \nabla D_i u \cdot \nabla D_i \varphi_n + 2\Delta u \Delta \varphi_n.
\end{align*}

Then

\begin{align*}
|Au_n(x) - Au(x)| &\leq |1 - \varphi_n(x)||Au(x)| \\
&+ C \sum_{i,j,k=1}^N \left( |x|^{-1} + |x|^\alpha - 1 + |x|^{2\alpha - 1} \right) |D_{ijk} u(x)| \chi_{K_n} \\
&+ C \sum_{i,j=1}^N \left( |x|^{-2} + |x|^\alpha - 2 + |x|^{2\alpha - 2} \right) |D_{ij} u(x)| \chi_{K_n} \\
&+ C \sum_{i=1}^N \left( |x|^{-3} + |x|^\alpha - 3 + |x|^{2\alpha - 3} \right) |D_i u(x)| \chi_{K_n}
\end{align*}
\[ + C \left( |x|^{-4} + |x|^{2 \alpha - 4} + |x|^{2 \alpha - 4} \right) |u(x)| |\chi|_{K_n} \]
\[ \leq |1 - \varphi_n(x)||Au(x)| \]
\[ + C \sum_{i,j,k=1}^{N} (|x|^{-1} + |x|^{2 \alpha - 1}) |D_{ijk}u(x)| |\chi|_{K_n} \]
\[ + C \sum_{i,j=1}^{N} (|x|^{-1} + |x|^{2 \alpha - 1}) |D_{ij}u(x)| |\chi|_{K_n} \]
\[ + C \sum_{i=1}^{N} (|x|^{-2} + |x|^{2 \alpha - 2}) |D_iu(x)| |\chi|_{K_n} \]
\[ + C \left( |x|^{-4} + |x|^{2 \alpha - 4} \right) |u(x)| |\chi|_{K_n} \]

where \( K_n = B\left(\frac{2}{n}\right) \setminus B\left(\frac{1}{n}\right) \cup B(2n) \setminus B(n) \). All the terms in the right hand side converge to 0 pointwisely and hence in \( L^2(\mathbb{R}^N) \) since \( |x|^{2 \alpha - h} D^{4-h} u(x) \chi_{K_n} \leq |x|^{2 \alpha - h} D^{4-h} u(x) \in L^2(\mathbb{R}^N) \) for \( h = 0, 1, 2, 3, 4 \) and by Remark 1 also for \( \alpha = 0 \).

Finally we prove that \( D(A) \) coincides with \( D_2(A) \).

**Theorem 3.** Assume that \( N > 8, \alpha > 0, \beta > (\alpha - 2)^+ \). Then maximal domain \( D(A) \) coincides with \( D_2(A) \).

**Proof.** We have to prove only the inclusion \( D(A) \subseteq D_2(A) \).

Let \( \tilde{u} \in D(A) \) and set \( f = A\tilde{u} + \lambda\tilde{u} \). The operator \( A \) in \( C\left(\frac{1}{p}, \rho\right) \) := \( B(\rho) \setminus B\left(\frac{1}{p}\right) \), \( \rho > 0 \), is an elliptic operator with bounded coefficients, then for a suitable \( \lambda \) the problem
\[
\begin{align*}
Au + \lambda u &= f & \text{in } C\left(\frac{1}{p}, \rho\right), \\
u &= 0 & \text{on } \partial C\left(\frac{1}{p}, \rho\right), \\
Dv &= 0 & \text{on } \partial C\left(\frac{1}{p}, \rho\right),
\end{align*}
\]

admits a unique solution \( u_\rho \) in \( W^{4,2} \left( C\left(\frac{1}{p}, \rho\right) \right) \cap W^{1,2}_0 \left( C\left(\frac{1}{p}, \rho\right) \right) \) (cf. [24, Section 3.2]). Now \( u_\rho \in D_2(A) \) and by Proposition 7 and (3.8)
\[
|||x|^{2 \alpha} D^4 u_\rho||_{L^2(B(\rho))} + |||x|^{2 \alpha - 1} D^3 u_\rho||_{L^2(B(\rho))} + |||x|^{2 \alpha - 2} D^2 u_\rho||_{L^2(B(\rho))} + |||x|^{2 \alpha - 3} Du_\rho||_{L^2(B(\rho))} \\
+ |||x|^{2 \alpha - 4} u_\rho||_{L^2(B(\rho))} + \|V^2 u_\rho\|_{L^2(B(\rho))} + ||u||_{H^4(B(\rho))} \leq C \left( \|Au_\rho\|_2 + \|u_\rho\|_2 \right). 
\]

By a standard weak compactness argument it is possible to construct a sequence \( (u_{\rho_n}) \) which converges to a function \( u \) in \( W^{4,2}_0(\mathbb{R}^N) \) such that \( Au + \lambda u = f \). Since the estimates above are independent of \( \rho \), also \( u \in D_2(A) \). Then we have \( A\tilde{u} + \lambda\tilde{u} = Au + \lambda u \) and since \( D_2(A) \subseteq D(A) \) and \( A + \lambda \) is invertible on \( D(A) \) being the generator of a \( C_0 \)-semigroup, it is possible to conclude that \( \tilde{u} = u \).

\[ \square \]

In the last part of this section we investigate on the spectrum of the operator \( A \).
Proposition 8. For any $\alpha > 0$, $\beta > (\alpha - 2)^+$, and $N > 8$ the spectrum of $A$ consists of a sequence of negative real eigenvalues which accumulates at $-\infty$.

Proof. We prove that $D(A)$ is compactly embedded into $L^2(\mathbb{R}^N)$, this will prove that the spectrum of $A$ consists of eigenvalues. By Proposition 7 and Theorem 3 we have
\[
\int_{\mathbb{R}^N} |x|^{4\beta}|u|^2 \, dx \leq C (\|Au\|_2 + \|u\|_2).
\]

Then we consider the unitary ball of $D(A)$ that is $B = \{ u \in D(A) \mid \|Au\|_2 + \|u\|_2 \leq 1 \}$ and for any $u \in B$ we have
\[
\int_{\mathbb{R}^N} |x|^{4\beta}|u|^2 \, dx \leq C.
\]

Now fix $\varepsilon > 0$, there exists $M > 0$ such that $\frac{C}{M^{4\beta}} < \varepsilon^2$, so that
\[
\int_{|x| > M} |u|^2 \, dx \leq \frac{1}{M^{4\beta}} \int_{|x| > M} |x|^{4\beta}|u|^2 \, dx \leq \frac{C}{M^{4\beta}} < \varepsilon^2.
\]

Let us now consider the set of the restriction to $B(M)$, the ball of center 0 and radius $M$, of the functions in $B$ that we denote by $B_M$. We observe that for every $u \in D(A)$, we have $\|u\|_{H^4(\mathbb{R}^N)} \leq C (\|Au\|_2 + \|u\|_2)$ then $B_M$ is bounded in $H^4(B(M))$. Since $H^4(B(M))$ is compactly embedded into $L^2(B(M))$ we have that $B_M$ is totally bounded in $L^2(B(M))$. Then there exists $u_1, \ldots, u_n \in L^2(B(M))$ such that
\[
B_M \subset \bigcup_{i=1}^n \{ u \in L^2(B(M)), \|u - u_i\|_{L^2(B(M))} < \varepsilon \}.
\]

Now, for $i = 1, \ldots, n$, we set $\tilde{u}_i(x) = u_i(x)$ if $|x| \leq M$ and $\tilde{u}_i(x) = 0$ otherwise. For every $u \in B$ we have $\|u - \tilde{u}_i\|_2 \leq 2\varepsilon$ and
\[
B \subset \bigcup_{i=1}^n \{ u \in L^2(\mathbb{R}^N), \|u - \tilde{u}_i\|_2 < 2\varepsilon \}.
\]

So $B$ is totally bounded in $L^2(\mathbb{R}^N)$.

Now let $\lambda \in \mathbb{C}$ be an eigenvalue of $A$, there exists $u \in D(A)$ such that $Au + \lambda u = 0$ and then
\[
\Delta^2 u + \frac{V}{a^2} u + \lambda \frac{u}{a^2} = 0.
\]

Multiplying by $u$ and integrating by parts we obtain, recalling that $u \in H^4(\mathbb{R}^N)$,
\[
\int_{\mathbb{R}^N} \left( (\Delta u)^2 + \frac{V}{a^2} u^2 \right) \, dx + \lambda \int_{\mathbb{R}^N} \frac{u^2}{a^2} \, dx = 0
\]
from which follows that $\lambda$ is real and negative. \hfill \Box

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