Restoring Lorentz Invariance in Classical N=2 String

Stefano Bellucci\textsuperscript{1} and Anton Galajinsky\textsuperscript{2}

INFN–Laboratori Nazionali di Frascati, C.P. 13, 00044 Frascati, Italia

Abstract

We study classical N=2 string within the framework of the N=4 topological formalism by Berkovits and Vafa. Special emphasis is put on the demonstration of a classical equivalence of the theories and the construction of an action for the N=4 topological string. The SO(2,2) Lorentz invariance missing in the conventional Brink–Schwarz action for the $N = 2$ string is restored in the $N = 4$ topological action.

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\textsuperscript{1} E-mail: Stefano.Bellucci@lnf.infn.it
\textsuperscript{2} On leave from Department of Mathematical Physics, Tomsk Polytechnical University, Tomsk, Russia
E-mail: Anton.Galajinsky@lnf.infn.it
1. Introduction

Nowadays, it seems to be a conventional wisdom to view string theories revealing an $N$–extended local supersymmetry on the world-sheets as models describing a coupling of an $N$–extended (conformal) $d = 2$ supergravity to matter multiplets. Technically, it suffices to start with an appropriate supergravity multiplet, the component fields being connected by a set of gauge symmetries, and require the coupling not to destroy the transformations. There has been considerable interest in the field [1]–[9] which culminated in the classification of the underlying $N$–extended superconformal algebras [10]–[13]. The $N \leq 4$ bound was encountered as the only one compatible with the presence of a central charge in the superconformal algebra (SCA). It should be stressed that with $N$ growing the construction of an action becomes technically involved. For example the $N = 4$ action of Ref. [3] appeals first to a dimensional reduction of the rigid $N = 2$, $d = 4$ nonlinear sigma model to $d = 2$ which is then coupled to a supergravity multiplet. Notice also that for $N > 1$ the existing actions [1, 3] lack manifest Lorentz invariance in a target.

The $N = 2$ case, which is also the subject of this paper, is of particular interest. In contrast to $N = 0$ and $N = 1$ strings, quantization of the model reveals only a finite number of physical states in the spectrum. What is more, one encounters a continuous family of sectors interpolating between $R$ and $NS$ which are connected by spectral flow [14]. The latter fact, in particular, allows one to stick with a preferred sector. All tree-level amplitudes with more than three external legs prove to vanish [15, 16] (for details of loop calculations see a recent work [17] and references therein), exhibiting a remarkable connection with self-dual gauge or gravitational theory in two spatial and two temporal dimensions [14].

Turning to problematic points, in spite of being a theory of an $N = 2$ $d = 2$ supergravity coupled to matter, there are no space-time fermions in the quantum spectrum. Furthermore, as has been mentioned above manifest Lorentz covariance is missing in the Brink–Schwarz action [1]. By this reason the notion of “spin” carried by a state is obscured and can be clarified only after explicit evaluation of string scattering amplitudes [16].

Recently, Berkovits and Vafa introduced a new topological theory based on a small $N = 4$ SCA [18] (a similar extension of $N = 2$ SCA has been considered earlier by Siegel [19]). After topological twisting, which incidentally does not treat all the fermionic currents symmetrically and breaks $SO(2,2)$ down to $U(1,1)$ [18], the $N = 4$ formalism turns out to be equivalent to the conventional $N = 2$ formalism. This has been demonstrated by explicit evaluation of scattering amplitudes [18]. It was revealed also that the topological prescription for calculating superstring amplitudes is free of total-derivative ambiguities (for further developments see Ref. [20]). It is noteworthy, that a global automorphism group of the small $N = 4$ SCA contains the full Lorentz group $SO(2,2) \simeq SU(1,1) \times SU(1,1)'$ which is larger than $U(1,1) \simeq U(1) \times SU(1,1)$ intrinsic to $N = 2$ SCA.

In the present paper we study classical N=2 string within the framework of the N=4 topological prescription. An action functional adequate for the topological formalism is
constructed, this revealing the full Lorentz invariance in the target space. The motivation for this work is two–fold. Firstly, it seems reasonable to use the $N = 4$ topological action for restoring the Lorenz invariance missing in the previous analysis on the $N = 2$ string scattering amplitudes. Secondly, being reduced to a chiral sector, the action is appropriate for describing a chiral half of a recently proposed $N = 2$ heterotic string with manifest space–time supersymmetry [21].

The organization of the work is as follows. In the next section we outline the main features concerning a conventional Lagrangian formulation for the $N = 2$ string. An extension by topological currents and the issue of the classical equivalence are addressed in Sect. 3. The global automorphism group of the $N = 4$ SCA is discussed in Sect. 4 which will give us a key to the construction of the action. In Sect. 5 we apply Noether procedure to install an extra $U(1, 1)$ global invariance in the conventional $N = 2$ string action, thus raising the global symmetry to the full Lorentz group. Beautifully enough, the world–sheet gauge fields fall in a multiplet of an $N = 4 d = 2$ supergravity. Sect. 6 contains Hamiltonian analysis for the model at hand which proves to be a necessary complement to that in Sect. 5. We summarize our results and discuss possible further developments in Sect. 7. Our notations are gathered in Appendix.

2. Classical N=2 string in the Lagrangian framework

As originally formulated in Ref. [1], the action of the classical $N = 2$ string describes a coupling of an $N = 2$ world–sheet supergravity (containing a graviton $e^a_n$ ($n$ stands for a flat index), a complex gravitino $\chi_{A\alpha}$, $A = 1, 2$, and a real vector field $A_A$) to a complex (on-shell) matter multiplet. The latter involves a complex scalar $z^n$ and a complex Dirac spinor $\psi_A^n$, with the target index $n$ taking values $n = 0, 1$ in the critical dimension. The Brink–Schwarz action reads

$$S_0 = -\frac{1}{2\pi} \int d\tau d\sigma \sqrt{-g} \left\{ g^{\alpha\beta} \partial_\alpha z \partial_\beta \bar{z} - i \bar{\psi} \gamma^n \partial_\alpha e^a_n + i \partial_\alpha \bar{\psi} \gamma^n \psi e_a^n + \bar{\psi} \gamma^n \psi A_\alpha e^\alpha_n + \left( \partial_\alpha z - \frac{1}{2} \chi_\alpha \psi \right) \bar{\psi} \gamma^m \chi_\beta e^\beta_n e^m - \left( \partial_\alpha \bar{z} - \frac{1}{2} \bar{\psi} \chi_\alpha \right) \chi_\beta \gamma^m \psi e^\alpha_ne^m \right\},$$

(1)

which explicitly lacks manifest Lorentz covariance. Due to the complex structure intrinsic to the formalism, the (spin cover of the) full target–space Lorentz group $Spin(2, 2) = SU(1, 1) \times SU(1, 1)'$ turns out to be broken down to $U(1) \times SU(1, 1) \simeq U(1, 1)$ which is the global symmetry group of the formalism.

Since in two space–time dimensions (on the world–sheet) all irreducible representations of the Lorentz group are one-dimensional, it seems convenient to get rid of the $\gamma$–matrices. This, in particular, allows one to avoid the extensive use of cumbersome Fierz identities in checking local symmetries of the action. Besides, the study of various heterotic theories taking a chiral half from the $N = 2$ string [22, 21] inevitably appeals to an action of such a type. For the case at hand this yields (our notations are gathered in Appendix)

$$S_0 = -\frac{1}{2\pi} \int d\tau d\sigma \sqrt{-g} \left\{ g^{\alpha\beta} \partial_\alpha z \partial_\beta \bar{z} + i \sqrt{2}(\psi^+(\partial_\alpha \bar{\psi}(+) + \bar{\psi}(+)\partial_\alpha \psi(+) )e^{-\alpha} +$$

$$+$$
and parameters Minkowskian form, the action (2) reveals two more bosonic transformations with real and an

\begin{align*}
\delta\varepsilon &= \bar{\varepsilon} - \varepsilon, \\
\delta\chi &= \bar{\chi} - \chi, \\
\delta A &= e^{-1}e_{\alpha\beta}g^{\beta\gamma}\partial_{\alpha}b, \\
\delta\psi_0 &= -\frac{i}{2}b\psi, \\
\delta\psi_+ &= \frac{i}{2}b\psi, \\
\delta\chi_+ &= -\frac{i}{2}b\chi, \\
\delta\chi_- &= \frac{i}{2}b\chi, \\
\delta A_+ &= \frac{1}{\sqrt{2}}g_{\alpha\beta}e^\beta e_\alpha \mu_\alpha, \\
\delta A_- &= \frac{1}{\sqrt{2}}g_{\alpha\beta}e^{-\beta}e_\alpha \mu_\alpha.
\end{align*}

Here the (±) indices indicate the chirality of an irreducible spinor representation with respect to the world–sheet (local) Lorentz group (see Appendix). These can also be used to mark the right and left moving fermionic modes on the world–sheet of the string.

Apart from the usual reparametrization invariance, local Lorentz transformations and Weyl symmetry, which allow one to bring locally the world–sheet metric to the Minkowskian form, the action (B) reveals two more bosonic transformations with real parameters \(a\) and \(b\)

\begin{align*}
\delta A_\alpha &= \partial_\alpha a, \\
\delta\psi_\pm &= -\frac{i}{2}a\psi_\pm, \\
\delta\chi_\pm &= -\frac{i}{2}a\chi_\pm;
\end{align*}

\begin{align*}
\delta A_\alpha &= e^{-1}e_{\alpha\beta}g^{\beta\gamma}\partial_{\alpha}b, \\
\delta\psi_+ &= -\frac{i}{2}b\psi, \\
\delta\psi_- &= \frac{i}{2}b\psi, \\
\delta\chi_+ &= \frac{i}{2}b\chi, \\
\delta\chi_- &= -\frac{i}{2}b\chi,
\end{align*}

where \(e^{-1} = (\text{det}(e_{\alpha\beta}))^{-1} = \sqrt{-\gamma}\), these being sufficient to remove the corresponding gauge field \(A_\alpha\), as well as the super Weyl transformation with two complex fermionic parameters \(\mu_{\pm}\)

\begin{align*}
\delta\chi_{\alpha}(+) &= g_{\alpha\beta}\varepsilon^\beta \mu_{\alpha}, \\
\delta\chi_{\alpha}(-) &= g_{\alpha\beta}\varepsilon_\beta \mu_{\alpha}, \\
\delta A_\alpha &= \frac{1}{\sqrt{2}}g_{\alpha\beta}\varepsilon^\beta e_\alpha \gamma^\gamma (\bar{\mu}(+)\chi_\gamma(+) + \bar{\mu}(\)\chi_\gamma(-)\mu(+) + \\
&\quad + \frac{1}{\sqrt{2}}g_{\alpha\beta}\varepsilon^{-\beta}e_\alpha \gamma^\gamma (\bar{\mu}(-)\chi_\gamma(+) + \bar{\mu}(\)\chi_\gamma(-)\mu(-)),
\end{align*}

and an \(N = 2\) local world–sheet supersymmetry with fermionic complex parameter \(\varepsilon_{(-)}\)

\begin{align*}
\delta z &= i\bar{\varepsilon}_{(-)}\psi_-, \\
\delta\psi_- &= 0, \\
\delta\chi_{\alpha}(+) &= 0, \\
\delta\chi_{\alpha}(-) &= 0, \\
\delta e^\alpha &= 0, \\
\delta\varepsilon_{(-)} &= \partial_\alpha e_{(-)} + \frac{i}{2}e_{(-)} A_\alpha + \frac{i}{2\sqrt{2}}(\varepsilon_{(-)}\chi_{\alpha}(-) + \varepsilon_{(-)}\bar{\chi}_{\alpha}(-))\chi_{\gamma}(e_{\alpha}^- + \gamma) + \frac{i}{\sqrt{2}}(\varepsilon_{(-)}\chi_{\alpha}(-)^\gamma \bar{\chi}_{\gamma}(e_{\alpha}^- + \gamma), \\
\delta\psi_+ &= \frac{1}{\sqrt{2}}(\varepsilon_{(-)}\partial_\alpha e_{\alpha}^- - \frac{i}{\sqrt{2}}\varepsilon_{(-)}\bar{\psi}_+(\chi_{\gamma}(e_{\alpha}^- + \gamma) + \frac{i}{\sqrt{2}}\varepsilon_{(-)}\bar{\psi}_+(\varepsilon_{(-)}(\chi_{\gamma}(-) - e_{(-)}\bar{\chi}_{\gamma}(-))e_\gamma^+, \\
\delta e^- &= -\frac{i}{\sqrt{2}}e_{\alpha}^- \varepsilon_{(-)}\chi_{\gamma}(-) + (\varepsilon_{(-)}\bar{\chi}_{\gamma}(-))e_\gamma^-, \\
\delta A_\alpha &= \frac{1}{\sqrt{2}}(\varepsilon_{(-)}(\gamma_\alpha \chi_{\alpha}(\gamma) - \gamma_\alpha \bar{\chi}_{\alpha}(\gamma)) - \varepsilon_{(-)}(\gamma_\alpha \chi_{\alpha}(\gamma) - \gamma_\alpha \bar{\chi}_{\alpha}(\gamma))e_\gamma^+, \frac{i}{\sqrt{2}}(\varepsilon_{(-)}\bar{\chi}_{\alpha}(\gamma) - \varepsilon_{(-)}\chi_{\alpha}(\gamma))e_\gamma^+ - \frac{i}{4}(\varepsilon_{(-)}\bar{\chi}_{\alpha}(\gamma) - \varepsilon_{(-)}\chi_{\alpha}(\gamma))\chi_{\gamma}(e_{\alpha}^+)e^\delta,
\end{align*}

and \(\varepsilon_{(+)}\)

\begin{align*}
\delta z &= i\varepsilon_{(+)}\psi_-, \\
\delta\psi_- &= 0, \\
\delta\chi_{\alpha}(+) &= 0, \\
\delta\chi_{\alpha}(-) &= 0, \\
\delta e^\alpha &= 0,
\end{align*}
\[ \delta \chi_{\alpha(+)} = - \partial_\alpha \epsilon(+) - \frac{i}{2} \epsilon(+) A_\alpha - \frac{i}{2\sqrt{2}} (\epsilon(+) \chi_{\alpha(+)} + \epsilon(+) \bar{\chi}_{\alpha(+)} \chi_{\gamma(+)} e^{-\gamma} - \frac{i}{\sqrt{2}} \epsilon(+) \chi_{\alpha(+)} \bar{\chi}_{\gamma(+)} e^{-\gamma}, \]

\[ \delta \psi(-) = \frac{1}{\sqrt{2}} e(+) \partial_\alpha \psi(-) + \frac{i}{\sqrt{2}} e(+) \tilde{\psi}(+) \bar{\chi}_{\gamma(-)} e^{-\gamma} - \frac{i}{2\sqrt{2}} \psi(-) (\epsilon(+) \chi_{\gamma(+)} - \epsilon(+) \bar{\chi}_{\gamma(+)} \chi_{\gamma(+)} e^{-\gamma},\]

\[ \delta e_+^\alpha = \frac{1}{\sqrt{2}} e(-) (\epsilon(+) \chi_{\gamma(+)}) e_+^\gamma, \]

\[ \delta A_\gamma = - \frac{1}{\sqrt{2}} \left\{ (\epsilon(+) (\nabla_\alpha \chi_{\gamma(+)}) - \nabla_\alpha \chi_{\gamma(+)}) - \epsilon(+) (\nabla_\alpha \bar{\chi}_{\gamma(+)}) - \nabla_\gamma \chi_{\gamma(+)}) \right\} e_+^\alpha - \frac{1}{2\sqrt{2}} \left\{ A_\alpha (\epsilon(+) \bar{\chi}_{\gamma(+)}) + \tilde{\epsilon}(+) \chi_{\gamma(+)}) - A_\gamma (\epsilon(+) \chi_{\gamma(+)}) + \tilde{\epsilon}(+) \bar{\chi}_{\gamma(+)}) \right\} e_+^\alpha + \frac{i}{4} (\epsilon(+) \bar{\chi}_{\gamma(+)}) \tilde{\chi}_{\beta(+)}) \chi_{\delta(+) \chi_{\delta(+) e_+^\beta e_+^\delta}. \] (7)

Altogether these allow one to gauge away the gravitino fields \( \chi_{\alpha(\pm)} \). In the equations above \( \nabla_\alpha \) stands for the conventional world sheet covariant derivative \( \nabla_\alpha B_\beta = \partial_\alpha B - \Gamma^\gamma_{\alpha \beta} B_\gamma \).

After the gauge fixing (for a more rigorous analysis see Sect. 6)

\[ e_+^0 = e_+^0 = e_+^1 = -e_+^1 = \frac{1}{\sqrt{2}}, \quad A_\alpha = 0, \quad \chi_{\alpha(\pm)} = 0, \] (8)

only the first three terms survive in the action (9) and one has to keep track of the \( N = 2 \) superconformal currents which arise from a variation of the action with respect to the world-sheet supergravity fields one had before fixing the gauge. We write these explicitly in the Hamiltonian form (see also Sect. 6)

\[ T = (p_z + \frac{1}{2\pi} \partial_1 z)(p_\bar{z} + \frac{1}{2\pi} \partial_1 \bar{z}) - \frac{1}{2\pi} (\psi(+)) \partial_1 \psi(+)) + \tilde{\psi}(+) \partial_1 \psi(+) = 0, \]

\[ G = (p_\bar{z} + \frac{1}{2\pi} \partial_1 \bar{z}) \psi(+) = 0, \quad \bar{G} = (p_\bar{z} + \frac{1}{2\pi} \partial_1 \bar{z}) \tilde{\psi}(+) = 0, \]

\[ J = \tilde{\psi}(+) \psi(+) = 0, \] (9)

where \( (z^n, p_z^n), n = 0, 1 \), form a canonical pair, \( (\bar{z}^n, p_{\bar{z}}^n) \), are complex conjugates and \( (\psi^n, \tilde{\psi}^n) \) are a couple of complex conjugate fermions

\[ \{ z^n(\sigma), p_z^m(\sigma') \} = \eta^{nm} \delta(\sigma - \sigma'), \quad \{ \bar{z}^n(\sigma), p_{\bar{z}}^m(\sigma') \} = \eta^{nm} \delta(\sigma - \sigma'), \]

\[ \{ \psi^n(+), p_z^m(+)(\sigma') \} = i\pi \eta^{nm} \delta(\sigma - \sigma'), \]

(10)

with \( \eta^{nm} = \text{diag}(-, +) \) the Minkowski metric. There is also an analogous set where \( (p_z + \frac{1}{2\pi} \partial_1 z) \) and \( \psi(+) \) are to be exchanged with \( (p_\bar{z} + \frac{1}{2\pi} \partial_1 \bar{z}), \psi(-), \) these for the right movers.

3. Adding topological currents

It is worth mentioning further that the \( N=2 \) superconformal currents given in Eq. (9) above are not the maximal closed set one can realize on the space of the matter fields. As was pointed out in Refs. [14, 15], two more bosonic currents

\[ e^{nm} \psi_{n(+)} \psi_{m(+)} = 0, \quad e^{nm} \tilde{\psi}_{n(+) \psi_{m(+) = 0}, \]

(11)

and two more fermionic ones

\[ e^{nm} (p_{\bar{z}n} + \frac{1}{2\pi} \partial_1 z_n) \psi_{m(+)} = 0, \quad e^{nm} (p_{\bar{z}n} + \frac{1}{2\pi} \partial_1 z_n) \tilde{\psi}_{m(+)} = 0, \]

(12)
extend the algebra up to a small $N=4$ SCA. After a topological twist, which does not treat all the fermionic currents symmetrically and breaks $SO(2, 2)$ down to $U(1, 1)$ \[18\], the $N=4$ extension turns out to be equivalent to the $N=2$ formalism. This has been demonstrated by explicit computation of scattering amplitudes \[18\]. Guided by the quantum equivalence, it seems quite natural to expect that a similar correspondence should hold also at the classical level. We now proceed to show that the extra currents \[12\] do not contain an additional information as compared to that implied by the $N=2$ currents.

Since equations of motion in both the cases are free and have the same form, this provides the classical equivalence. It seems convenient first to break a target space vector index into the light–cone components (see Eq. (A.3) of Appendix). Denoting $\Pi^+_z = (p^+_z + \frac{1}{2\pi} \partial_1 z^+)$ and $\Pi^-_z = (p^-_z + \frac{1}{2\pi} \partial_1 z^-)$, one finds for the currents \[9\] (for brevity we omit spinor indices carried by the fermions)

$$T = -\Pi^+_z \Pi^-_z - \Pi^-_z \Pi^+_z + \frac{1}{2\pi} (\psi^+ \partial_1 \bar{\psi}^- + \psi^- \partial_1 \bar{\psi}^+ + \bar{\psi}^+ \partial_1 \psi^- + \bar{\psi}^- \partial_1 \psi^+) = 0,$$

$$G = -\Pi^+_z \psi^- \Pi^-_z \psi^+ = 0, \quad \bar{G} = -\Pi^+_z \bar{\psi}^- \Pi^-_z \bar{\psi}^+ = 0,$$

$$J = -\bar{\psi}^+ \psi^- - \bar{\psi}^- \psi^+ = 0, \quad (13)$$

while the newly introduces ones acquire the form

$$\psi^+ \psi^- = 0, \quad \bar{\psi}^+ \bar{\psi}^- = 0,$$

$$\Pi^+_z \psi^- - \Pi^-_z \psi^+ = 0, \quad \Pi^+_z \bar{\psi}^- - \Pi^-_z \bar{\psi}^+ = 0. \quad (14)$$

Assuming now that $\Pi^+_z \neq 0$ \[4\] one can readily solve for $\psi^-, \bar{\psi}^-$ by making use of the second line of Eq. \[13\]

$$\psi^- = -\frac{\Pi^-_z}{\Pi^+_z} \psi^+, \quad \bar{\psi}^- = -\frac{\Pi^-_z}{\Pi^+_z} \bar{\psi}^+. \quad (15)$$

Being substituted in the expression for $J$ these yield

$$(\Pi^+_z \Pi^-_z + \Pi^-_z \Pi^+_z) \bar{\psi}^+ \psi^+ = 0. \quad (16)$$

It is trivial to observe now that the bosonic currents from Eq. \[14\] follow from Eq. \[15\]. The same turns out to be true for the fermionic ones, provided Eq. \[16\], $T \psi^+ = 0$ and $T \bar{\psi}^+ = 0$ have been used.

Thus, one can conclude that at the classical level the extra currents of the $N = 4$ topological description do not contain any new information. Our analysis here is in agreement with that of Siegel \[19\]. At the quantum level, as an alternative to the proof by Berkovits and Vafa which appeals to the explicit evaluation of string amplitudes \[18\], one could proceed directly from the small $N=4$ SCA to verify that the positive (half–integer) modes of the fermionic currents \[12\] kill all physical states, provided so do the zero modes of bosonic ones \[11\]. Since even for the smaller $N=2$ SCA a proper analysis

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\[^3\]This can be achieved, for example, by imposing a light–cone gauge $z^+ + \bar{z}^+ = z^+_0 + \bar{z}^+_0 + \tau (p^+_0 + p^+_0)$. The real part of $\Pi^+_z$ (and hence the modulus) then does not vanish and one can safely divide by $\Pi^+_z$. It should be stressed that, due to the presence of two temporal dimensions in the target, the light cone analysis for the $N = 2$ string seems to be less rigorous than in the Minkowski space.
shows the absence of excited states [23] and because the zero modes of the new bosonic currents annihilate the ground state, one arrives at the same spectrum as for the ordinary $N=2$ string.

4. $U(1, 1)$ and $U(1, 1)_{outer}$

As has been mentioned above, after the topological twist the $N=4$ prescription is equivalent to the $N=2$ formalism. There is an intimate connection between this point and automorphism groups corresponding to the algebras. Since this will give us a key to construction of the action, we turn to discuss this issue in more detail.

As we have seen earlier, the global symmetry group of the conventional $N=2$ string, which is also a trivial automorphism of an $N = 2$ SCA, is given by $U(1, 1)$. It seems instructive to give here the explicit realization. Combining $z^n$ and $\bar{z}^n$ in a single column $Z^A = \begin{pmatrix} z^n \\ \bar{z}^n \end{pmatrix}$ with $A=1, 2, 3, 4$, one has for the infinitesimal $U(1, 1)$ transformation

$$\delta Z^A = i\alpha_i L^A_{iB} Z^B, \quad L^T_i \eta = \eta L_i, \quad \eta_{AB} = \text{diag}(-, +, -, +),$$

where $\alpha^i$ are four real parameters and $L_i$ form a basis in the $u(1, 1)$ algebra,

$$L_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 0 & +1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix}.$$ (18)

Because the matrices are block–diagonal the corresponding representation of the $u(1, 1)$ algebra is obviously decomposed into a direct sum and the fields $z^n$ and $\bar{z}^n$ do not get mixed under these transformations.

Beautifully enough, one can find another realization of the group on the space of the fields at hand that do mix $z^n$ and $\bar{z}^n$ (generators are off-diagonal) and, what is more important, it automatically generates the currents of the small $N = 4$ SCA when applied to those of the $N = 2$ SCA [21]. We stick with the terminology of Ref. [20] and call this $U(1, 1)_{outer}$. The explicit form of the generators is
\[ L_1 = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \]

\[ L_3 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix}. \] (19)

Notice that a combination of the form \( \bar{z}^n \eta_{nm} y^m \), which is trivially invariant under the action of \( U(1,1) \), does not hold invariant under the action of \( U(1,1)_{outer} \), and a more general object like \( \bar{Z}^A \eta_{AB} Y^B \) is to be handled with in a formalism respecting the latter. This, in particular, causes certain difficulties, when trying to construct vertex operators for a recent \( N = 2 \) heterotic string with manifest space–time supersymmetry \([21]\).

In the next section we shall install \( U(1,1)_{outer} \) in the \( N = 2 \) string action by making use of the Noether procedure. Before concluding this section we collect some technical points regarding the action of \( U(1,1) \) and \( U(1,1)_{outer} \) on the elementary objects like \( \psi^n \eta_{nm} \bar{\varphi}^m \) and \( \psi^n \epsilon_{nm} \varphi^m \).

Independently of the statistics of the fields \( \psi \) and \( \varphi \), one finds for the outer group

| \( U(1,1)_{outer} \) | \( \delta_{\alpha_1} \) | \( \delta_{\alpha_2} \) | \( \delta_{\alpha_3} \) | \( \delta_{\alpha_4} \) |
|---------------------|----------------|----------------|----------------|----------------|
| \( \psi \bar{\varphi} \) | 0 | \( \alpha_2(\psi \varphi - \bar{\psi} \bar{\varphi}) \) | 0 | \( i \alpha_4(\psi \varphi + \bar{\psi} \bar{\varphi}) \) |
| \( \psi \epsilon \varphi \) | 0 | \( \alpha_2(\bar{\psi} \bar{\varphi} - \bar{\psi} \bar{\varphi}) \) | \( -2i \alpha_3(\psi \epsilon \varphi) \) | \( -i \alpha_4(\psi \varphi - \bar{\psi} \bar{\varphi}) \) |
| \( \bar{\psi} \varphi \) | 0 | \( -\alpha_2(\bar{\psi} \bar{\varphi} - \bar{\psi} \bar{\varphi}) \) | \( 2i \alpha_3(\psi \varphi) \) | \( -i \alpha_4(\bar{\psi} \bar{\varphi} - \psi \varphi) \) |

(20)

while the variations with respect to the ordinary group prove to be much simpler

| \( U(1,1) \) | \( \delta_{\alpha_1} \) | \( \delta_{\alpha_2} \) | \( \delta_{\alpha_3} \) | \( \delta_{\alpha_4} \) |
|--------------|----------------|----------------|----------------|
| \( \psi \bar{\varphi} \) | 0 | 0 | 0 | 0 |
| \( \psi \epsilon \varphi \) | 0 | \( -2i \alpha_2(\psi \epsilon \varphi) \) | 0 | 0 |
| \( \bar{\psi} \epsilon \varphi \) | 0 | \( 2i \alpha_2(\bar{\psi} \epsilon \varphi) \) | 0 | 0 |

(21)

When realizing \( U(1,1)_{outer} \) in the \( N = 2 \) string action, a successive inspection of the terms entering Eq. (2) simplifies considerably with the use of the tables (20), (21).
5. Restoring Lorentz invariance

No work has to be done with the kinetic terms, these prove to be invariant with respect to both $U(1, 1)$ and $U(1, 1)_{outer}$. The next two contributions involving the vector field $A_\alpha$ are to be accompanied by

$$S_1 = -\frac{1}{2\pi} \int d\tau d\sigma \sqrt{-g} \{ \sqrt{2}(\psi_-(\bar{\psi}_-)^{\alpha}(B_\alpha + iC_\alpha) - \sqrt{2}(\bar{\psi}_-(\bar{\psi}_-)^{\alpha}(B_\alpha - iC_\alpha) \},$$

where $B_\alpha$ and $C_\alpha$ are two new real vector (gauge) fields. The whole piece is inert under $U(1, 1)_{outer}$ provided the following transformation rules for the gauge fields

$$\delta \alpha_1 A_\alpha = 0, \quad \delta \alpha_2 B_\alpha = -4\alpha_2 C_\alpha, \quad \delta \alpha_3 C_\alpha = 2\alpha_3 B_\alpha, \quad \delta \alpha_4 C_\alpha = -4\alpha_4 C_\alpha.$$  \hspace{1cm} (23)

For the ordinary $U(1, 1)$ the triplet of vector fields proves to be invariant under $\delta \alpha_1$, $\delta \alpha_3$ and $\delta \alpha_4$, while for $\delta \alpha_2$ one has to set

$$\delta \alpha_2 A_\alpha = 0, \quad \delta \alpha_2 B_\alpha = -2\alpha_2 C_\alpha, \quad \delta \alpha_2 C_\alpha = 2\alpha_2 B_\alpha.$$ \hspace{1cm} (24)

An important point to notice is that the transformations on the gauge fields do satisfy the $u(1, 1)$ algebra. Furthermore, one infers from Eq. (23) that they transform as a triplet of $SU(1, 1)$ subgroup of the full $U(1, 1)_{outer}$, so that one can set a single field $A'_\alpha = (A_\alpha, B_\alpha, C_\alpha), I = 1, 2, 3.$

Let us now proceed to the terms linear in $\partial z, \partial \bar{z}$. In order to make them $U(1, 1)_{outer}$ invariant it suffices to introduce a couple of complex fermions $\mu_{\alpha(+)}$ and $\mu_{\alpha(-)}$ which bring the following contribution to a full action

$$S_2 = -\frac{1}{2\pi} \int d\tau d\sigma \sqrt{-g} \{ 2i\partial_\alpha z\bar{\epsilon}\psi_-(\bar{\psi}_-)^{\alpha} + 2i\partial_\alpha \bar{z}\bar{\epsilon}\bar{\psi}_-(\bar{\psi}_-)^{\alpha} + 2i\partial_\alpha \bar{z}\bar{\epsilon}\bar{\psi}_+(\bar{\psi}_+)^{\alpha} - 2i\partial_\alpha \epsilon\psi_+(\bar{\psi}_+)^{\alpha} \bar{\mu}_{\bar{\beta}(-)} e_{- \bar{\beta}} + 2i\partial_\alpha \bar{z}\bar{\epsilon}\bar{\psi}_+(\bar{\psi}_+)^{\alpha} e_{- \beta} - 2i\partial_\alpha \bar{z}\bar{\epsilon}\bar{\psi}_+(\bar{\psi}_+)^{\alpha} e_{- \beta} \}.$$ \hspace{1cm} (25)

These should transform according to the rules
The action of $U(1,1)_{\text{outer}}$ proves to be nontrivial only for $\delta_{a_2}$ and looks like

$$\delta_{a_2}\mu_{a(+)} = 2i\alpha_2\mu_{a(+)}; \quad \delta_{a_2}\mu_{a(-)} = 2i\alpha_2\mu_{a(-)}.$$  

It is noteworthy, that the fermionic fields $\chi_{\alpha(\pm)}$, $\mu_{a(\pm)}$ furnish a two–dimensional complex representation of $SU(1,1)_{\text{outer}}$ (to be more precise, the representation is realized on a four component “Majorana” spinor $(\mu, \chi, \bar{\mu}, \bar{\chi})$; the indices $\alpha$ and $(\pm)$ hold inert). One can again set a single complex fermionic field $\varphi_{\alpha(\pm)}$, $i=1,2$, this composed of $\chi_{\alpha(\pm)}$, $\mu_{a(\pm)}$ and forming a doublet representation of $SU(1,1)_{\text{outer}}$. As we shall see below no extra fields are necessary to make the full action $U(1,1)$ and $U(1,1)_{\text{outer}}$ invariant. One eventually comes to the conclusion that on the world–sheet the theory is described by $(e_0^a, A^I_0, \varphi_{\alpha A})$. Beautifully enough, this set coincides with the $N=4$, $d=2$ supergravity multiplet (see e.g. [3]). Thus, within the framework of the $N=4$ topological formalism the action describes a coupling of the $N=4$, $d=2$ supergravity to matter multiplets.

We now turn to discuss the last two terms in Eq. (3). Introducing a further amendment to the action

$$S_3 = -\frac{1}{2\pi} \int d\tau d\sigma \sqrt{-g} \{ (\psi_{\alpha(+)} e^{\psi_{\alpha(+)}} - \bar{\psi}_{\alpha(+)} e^{\bar{\psi}_{\alpha(+)}}) (\mu - \bar{\mu})_{\alpha(-)} (\chi + \bar{\chi})_{\beta(-)} + (\psi_{\alpha(+)} e^{\psi_{\alpha(+)}} + \bar{\psi}_{\alpha(+)} e^{\bar{\psi}_{\alpha(+)}}) (\mu + \bar{\mu})_{\alpha(-)} (\chi + \bar{\chi})_{\beta(-)} + (\psi_{\alpha(-)} e^{\psi_{\alpha(-)}} - \bar{\psi}_{\alpha(-)} e^{\bar{\psi}_{\alpha(-)}}) (\mu - \bar{\mu})_{\alpha(+)} (\chi + \bar{\chi})_{\beta(+)} + (\psi_{\alpha(-)} e^{\psi_{\alpha(-)}} + \bar{\psi}_{\alpha(-)} e^{\bar{\psi}_{\alpha(-)}}) (\mu + \bar{\mu})_{\alpha(+)} (\chi + \bar{\chi})_{\beta(+)} \} \times \frac{1}{4} (e_+^a e_-^\beta + e_+^\beta e_-^a),$$

one can compensate their variations. It remains to discuss two terms involving both $\psi_{\alpha(+)}$ and $\psi_{\alpha(-)}$ and entering the fourth line in Eq. (3). Here the analysis turns out to be more intricate. The most general form of the compensators is

$$S_4 = -\frac{1}{2\pi} \int d\tau d\sigma \sqrt{-g} \{ (\psi_{\alpha(-)} e^{\psi_{\alpha(-)}} - \bar{\psi}_{\alpha(-)} e^{\bar{\psi}_{\alpha(-)}}) \Sigma_{\alpha\beta} + (\psi_{\alpha(+)} e^{\psi_{\alpha(+)}} + \bar{\psi}_{\alpha(+)} e^{\bar{\psi}_{\alpha(+)}}) \Omega_{\alpha\beta} + 2\psi_{\alpha(-)} \bar{\psi}_{\alpha(+)} \Lambda_{\alpha\beta} + 2\psi_{\alpha(+)} \bar{\psi}_{\alpha(-)} \Pi_{\alpha\beta} \} e_+^a e_-^\beta;$$

where $\Sigma_{\alpha\beta}$, $\Omega_{\alpha\beta}$, $\Lambda_{\alpha\beta}$ and $\Pi_{\alpha\beta}$ are some as yet unspecified tensors. In the following, we shall assume that $\Sigma, \Omega, \Lambda, \Pi \rightarrow 0$ as $\mu_{a(\pm)} \rightarrow 0$. This seems natural because in the limit the action we are searching for should reduce to the $N=2$ string action. In order that the action be real the compensators must obey

$$(\Sigma_{\alpha\beta})^* = \Sigma_{\alpha\beta}, \quad (\Omega_{\alpha\beta})^* = -\Omega_{\alpha\beta}, \quad (\Lambda_{\alpha\beta})^* = \Pi_{\alpha\beta}. \quad (30)$$
A variation of the whole piece with respect to $U(1)$ and $U(1)_{\text{outer}}$ yields a set of constraints on the newly introduced fields. For example, the $\delta_2$ transformation implies

$$
\delta_2 \Sigma_{\alpha\beta} + 2 \alpha_2 [\Lambda_{\alpha\beta} + \Pi_{\alpha\beta} + \bar{\chi}_{\alpha (+)} \chi_{\beta (-)} + \bar{\chi}_{\beta (-)} \chi_{\alpha (+)}] = 0,
$$

$$
\delta_2 \Lambda_{\alpha\beta} + \alpha_2 [\Sigma_{\alpha\beta} - \bar{\chi}_{\alpha (+)} (\mu - \bar{\mu})_{\beta (-)} - \chi_{\beta (-)} (\mu - \bar{\mu})_{\alpha (+)}] = 0,
$$

$$
\delta_2 \Pi_{\alpha\beta} + \alpha_2 [\Sigma_{\alpha\beta} - \bar{\chi}_{\beta (-)} (\mu - \bar{\mu})_{\alpha (+)} - \chi_{\alpha (+)} (\mu - \bar{\mu})_{\beta (-)}] = 0,
$$

(31)

with $\Omega_{\alpha\beta}$ being inert. With the help of the tables (26) and (27) one can readily solve the variational equations. Surprisingly enough, the global invariance does not fix the compensators uniquely. One encounters the following solution

$$
\Sigma_{\alpha\beta} = c_1 (\chi - \bar{\chi})_{\beta (-)} (\mu + \bar{\mu})_{\alpha (+)} + c_1 (\chi - \bar{\chi})_{\alpha (+)} (\mu + \bar{\mu})_{\beta (-)} + c_2 (\chi - \bar{\chi})_{\alpha (-)} (\mu + \bar{\mu})_{\beta (+)} + c_2 (\chi - \bar{\chi})_{\beta (+)} (\mu + \bar{\mu})_{\alpha (-)} + \chi_{\alpha (+)} \mu_{\beta (-)} - \bar{\chi}_{\alpha (+)} \bar{\mu}_{\beta (-)} + \chi_{\beta (-)} \mu_{\alpha (+)} - \bar{\chi}_{\beta (-)} \bar{\mu}_{\alpha (+)},
$$

$$
\Omega_{\alpha\beta} = c_1 (\chi - \bar{\chi})_{\beta (-)} (\mu - \bar{\mu})_{\alpha (+)} + c_1 (\chi - \bar{\chi})_{\alpha (+)} (\mu - \bar{\mu})_{\beta (-)} + c_2 (\chi - \bar{\chi})_{\alpha (-)} (\mu - \bar{\mu})_{\beta (+)} + c_2 (\chi - \bar{\chi})_{\beta (+)} (\mu - \bar{\mu})_{\alpha (-)} + \chi_{\alpha (+)} \mu_{\beta (-)} + \bar{\chi}_{\alpha (+)} \bar{\mu}_{\beta (-)} + \chi_{\beta (-)} \mu_{\alpha (+)} + \bar{\chi}_{\beta (-)} \bar{\mu}_{\alpha (+)},
$$

$$
\Lambda_{\alpha\beta} = c_1 (\mu_{\alpha (+)} \bar{\mu}_{\beta (-)} + \mu_{\beta (-)} \bar{\mu}_{\alpha (+)}) + c_2 (\mu_{\alpha (-)} \bar{\mu}_{\beta (+)} + \mu_{\beta (+)} \bar{\mu}_{\alpha (-)}) - \bar{\mu}_{\alpha (+)} \mu_{\beta (-)},
$$

(32)

with $c_1$ and $c_2$ being arbitrary real numbers. Obviously, the missing information needed to fix them is encoded in local symmetries an ultimate action must possess. As is clear from our discussion above, one expects the action to exhibit $N = 4$ local supersymmetry.

Alternatively, one could proceed to the Hamiltonian formalism and fix the constants there from the requirement that only the set of currents generating a small $N = 4$ SCA remains after completion of the Dirac procedure. Extra constrains are not allowed. Because the Lagrangian and Hamiltonian methods are in one-to-one correspondence, this will yield a correct answer. It should be mentioned that, generally, it may happen to be an intricate task to reveal all the local symmetries just from the form of an action at hand. For example, the $b$–symmetry of the $N = 2$ string action displayed above in Eq. (14) was missing in the original paper [1] and has been discovered only four years later in Ref. [25]. As is well known, a straightforward way to discover how many local symmetries one should expect to find in a Lagrangian theory is prompted by the Hamiltonian framework. For models with irreducible constraints it suffices to count the number of Lagrange multipliers corresponding to primary first class constrains (see e.g., [25], [26]). In order to elucidate this point for the model under consideration and for the sake of coherence we then proceed to the Hamiltonian formalism to fix the missing constant.

### 6. Hamiltonian analysis

Introducing momenta associated to the configuration space variables one find the following primary constraints (we define a momentum conjugate to a fermionic variable to be the right derivative of a Lagrangian with respect to velocity)

$$
p_c = 0, \quad p_A = 0, \quad p_B = 0, \quad p_C = 0, \quad p_\chi = 0, \quad p_\bar{\chi} = 0, \quad p_\mu = 0, \quad p_{\bar{\mu}} = 0,
$$

(33)
\[ \Theta_{\psi(\pm)} \equiv \frac{P_\psi(\pm) + i\sqrt{2}p_\psi(\pm)e_-}{2\pi}e^0 = 0, \quad \Theta_{\psi(\mp)} \equiv \frac{P_\psi(\mp) + i\sqrt{2}p_\psi(\mp)e_+}{2\pi}e^0 = 0, \]

where \( e = \text{det}(e_{\a\b}) \) and \( p_q \) stands for a momentum canonically conjugate to a configuration space variable \( q \). It is worth mentioning now that, a Hamiltonian of a diffeomorphism invariant theory is given by a linear combination of constraints (Dirac’s theorem). Remarkably, the form of the coefficients \( c_1 \) and \( c_2 \) is uniquely fixed already from the form of the Hamiltonian. The key point to notice is that, since the currents under consideration do no involve \( \psi(+) \) and \( \psi(-) \) simultaneously, such terms should disappear from the Hamiltonian. This automatically holds in the Hamiltonian of the \( N = 2 \) string

\[ H_{N=2} = \int d\sigma \{ p_e \lambda_e + p_\lambda \lambda_e + p_\lambda \lambda_\chi + \Theta_{\psi(\pm)} \lambda_{\psi(\pm)} + \Theta_{\psi(\mp)} \lambda_{\psi(\mp)} + \Theta_{\psi(\mp)} \lambda_{\psi(\mp)} + \frac{i\sqrt{2}}{2\pi} \langle \psi(+) \rangle \partial_1 \psi(+) + \psi(+) \partial_1 \psi(+) \rangle e_-^0 + \frac{i\sqrt{2}}{2\pi} \langle \psi(-) \rangle \partial_1 \psi(-) + \psi(-) \partial_1 \psi(-) \rangle e_+^0 + \frac{1}{2\pi} \langle \psi(+) \rangle \partial_1 \psi(+) + \psi(+) \partial_1 \psi(+) \rangle e_+^0 - \frac{1}{2\pi} \langle \psi(-) \rangle \partial_1 \psi(-) + \psi(-) \partial_1 \psi(-) \rangle e_-^0 \}

(34)

For the full Hamiltonian this turns out to be true only if the constants take the following specific value

\[ c_1 = -1, \quad c_2 = 0. \]

(36)

In this case the full Hamiltonian acquires the form

\[ H_{N=4} = H_{N=2} + \int d\sigma \{ p_B \lambda_B + p_C \lambda_C + p_\chi \lambda_\chi + p_\bar{\chi} \lambda_{\bar{\chi}} + \frac{i\sqrt{2}}{2\pi} \langle \psi(-) \rangle \psi(+) \epsilon \psi(-) \epsilon e_+^\alpha + \psi(+) \psi(-) \epsilon e_-^\alpha \}

(37)

\[ \langle B_\alpha + iC_\alpha \rangle = -\frac{i\sqrt{2}}{2\pi} \langle \psi(-) \rangle \psi(+) \epsilon \psi(-) \epsilon e_+^\alpha + \psi(+) \psi(-) \epsilon e_-^\alpha \langle B_\alpha - iC_\alpha \rangle + i(p_\bar{\zeta} - \frac{1}{2\pi} \partial_1 \bar{\zeta}) \psi(-) \times \]

\[ \langle B_\alpha + iC_\alpha \rangle = -\frac{i\sqrt{2}}{2\pi} \langle \psi(-) \rangle \psi(+) \epsilon \psi(-) \epsilon e_+^\alpha + \psi(+) \psi(-) \epsilon e_-^\alpha \langle B_\alpha - iC_\alpha \rangle + i(p_\bar{\zeta} - \frac{1}{2\pi} \partial_1 \bar{\zeta}) \psi(-) \times \]

\[ \langle B_\alpha + iC_\alpha \rangle = -\frac{i\sqrt{2}}{2\pi} \langle \psi(-) \rangle \psi(+) \epsilon \psi(-) \epsilon e_+^\alpha + \psi(+) \psi(-) \epsilon e_-^\alpha \langle B_\alpha - iC_\alpha \rangle + i(p_\bar{\zeta} - \frac{1}{2\pi} \partial_1 \bar{\zeta}) \psi(-) \times \]

\[ \langle B_\alpha + iC_\alpha \rangle = -\frac{i\sqrt{2}}{2\pi} \langle \psi(-) \rangle \psi(+) \epsilon \psi(-) \epsilon e_+^\alpha + \psi(+) \psi(-) \epsilon e_-^\alpha \langle B_\alpha - iC_\alpha \rangle + i(p_\bar{\zeta} - \frac{1}{2\pi} \partial_1 \bar{\zeta}) \psi(-) \times \]

\[ \langle B_\alpha + iC_\alpha \rangle = -\frac{i\sqrt{2}}{2\pi} \langle \psi(-) \rangle \psi(+) \epsilon \psi(-) \epsilon e_+^\alpha + \psi(+) \psi(-) \epsilon e_-^\alpha \langle B_\alpha - iC_\alpha \rangle + i(p_\bar{\zeta} - \frac{1}{2\pi} \partial_1 \bar{\zeta}) \psi(-) \times \]
which, as we shall shortly see, is indeed a linear combination of the constraints. In the relations above \(\lambda\)'s stand for the Lagrange multipliers associated to the primary constraints. Given the form of the constants \(c_1\) and \(c_2\), the solutions (32) simplify considerably

\[
\Lambda_{\alpha\beta} = -\mu_{\alpha(+)}\bar{\mu}_{\beta(-)}, \quad \Sigma_{\alpha\beta} = \bar{\chi}_{\alpha(+)}\mu_{\beta(-)} + \tilde{\chi}_{\beta(-)}\mu_{\alpha(+)} - \chi_{\alpha(+)}\bar{\mu}_{\beta(-)} - \chi_{\beta(-)}\bar{\mu}_{\alpha(+)} ,
\]

\[
\Pi_{\alpha\beta} = \bar{\mu}_{\alpha(+)}\mu_{\beta(-)}, \quad \Omega_{\alpha\beta} = \chi_{\alpha(+)}\bar{\mu}_{\beta(-)} + \bar{\chi}_{\beta(-)}\mu_{\alpha(+)} + \tilde{\chi}_{\alpha(+)}\mu_{\beta(-)} + \tilde{\chi}_{\beta(-)}\mu_{\alpha(+)}. \quad (38)
\]

An ultimate form of the \(N = 4\) topological string has thus been fixed and is given by the sum

\[
S = S_0 + S_1 + S_2 + S_3 + S_4. \quad (39)
\]

It has to be stressed that \(SO(2, 2)\) global invariance missing in our starting point \(S_0\) is restored in the full action \(S\).

We now turn to outline the Dirac procedure. That only the currents of a small \(N = 4\) SCA remain as essential constraints on the system provides a consistency check for the formalism developed so far.

Following Dirac’s method, one requires primary constraints to be conserved in time. In other words, a point constrained to lie on a surface at some initial instant of time is not allowed to leave the latter in the process of time evolution. Being applied to \(p_A = 0\), \(p_B = 0\) and \(p_C = 0\) from Eq. (33) this gives

\[
\bar{\psi}_{(+)\bar{\psi}_{(+)}} = 0, \quad \psi_{(+)\epsilon\psi_{(+)}} = 0, \quad \bar{\psi}_{(+)}\epsilon\bar{\psi}_{(+)} = 0, \quad (40)
\]

plus analogous equations where a "\((+)\)" should be exchanged with a "\((-)\)". These are to be viewed as secondary constraints. In deriving the relations above Eq. (A.5) of Appendix proves to be helpful. Analogously, the fermionic primary constraints from Eq. (33) induce further relations

\[
(p_z + \frac{1}{2\pi}\partial_1 \bar{z})(p_{\bar{z}})\psi_{(+)} = 0, \quad (p_z - \frac{1}{2\pi}\partial_1 \bar{z})\psi_{(-)} = 0,
\]

\[
(p_{\bar{z}} + \frac{1}{2\pi}\partial_1 z)\epsilon\psi_{(+)} = 0, \quad (p_{\bar{z}} - \frac{1}{2\pi}\partial_1 z)\epsilon\psi_{(-)} = 0, \quad (41)
\]

as well as their complex conjugates. The remaining equation \(p_e = 0\) in (33) yields two more secondary constraints

\[
(p_z - \frac{1}{2\pi}\partial_1 \bar{z})(p_{\bar{z}} - \frac{1}{2\pi}\partial_1 z) + \frac{i\sqrt{2}}{2\pi e^0}(\psi_{(-)}\partial_1 \bar{\psi}_{(-)} - \bar{\psi}_{(-)}\partial_1 \psi_{(-)})\epsilon^0 + 0 = 0,
\]

\[
(p_{\bar{z}} + \frac{1}{2\pi}\partial_1 z)(p_z + \frac{1}{2\pi}\partial_1 \bar{z}) - \frac{i\sqrt{2}}{2\pi e^0}(\bar{\psi}_{(+)}\partial_1 \psi_{(+)} + \bar{\psi}_{(+)}\partial_1 \bar{\psi}_{(+)})\epsilon^0 = 0. \quad (42)
\]

In their turn the \(\Theta\)-equations from Eq. (34) do not lead to any new constraints but determine the value of the Lagrange multipliers \(\lambda_{\psi(\pm)}\) and \(\lambda_{\bar{\psi}(\pm)}\) (indicating the presence of second class constraints). These prove to be complicated expressions. Of direct relevance to the forthcoming discussion are some of their consequences which we list below

\[
\bar{\psi}_{(+)}\lambda_{\psi_{(+)}} = -\bar{\psi}_{(+)}\partial_1 \psi_{(+)}\epsilon^1_{0}, \quad \bar{\psi}_{(-)}\lambda_{\psi_{(-)}} = -\bar{\psi}_{(-)}\partial_1 \psi_{(-)}\epsilon^1_{0}, \quad \psi_{(+)}\epsilon\lambda_{\psi_{(+)}} \approx 0,
\]

\[
\psi_{(-)}\epsilon\lambda_{\psi_{(-)}} \approx 0, \quad (p_z - \frac{1}{2\pi}\partial_1 \bar{z})\lambda_{\psi_{(-)}} = -(p_{\bar{z}} - \frac{1}{2\pi}\partial_1 z)\lambda_{\bar{\psi}_{(-)}}\epsilon^1_{0} - \frac{e^0}{\sqrt{2}\epsilon^0} \times
\]

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\[(p_z - \frac{1}{2\pi} \partial_1 z)(p_z - \frac{1}{2\pi} \partial_1 z)\bar{\chi}_{\beta(+)} e_{-\beta}, \quad (p_z + \frac{1}{2\pi} \partial_1 z)\lambda_{\psi(+)} = -(p_z + \frac{1}{2\pi} \partial_1 z)\partial_1 \bar{\psi}_{(+)} e_{-\beta} + \frac{\pi e}{\sqrt{2e_{-\beta} e_{+\alpha}}} (p_z + \frac{1}{2\pi} \partial_1 z)(p_z + \frac{1}{2\pi} \partial_1 z)\bar{\chi}_{\beta(-)} e_{-\beta}, \quad (p_z - \frac{1}{2\pi} \partial_1 z)\epsilon \lambda_{\psi(-)} = -(p_z - \frac{1}{2\pi} \partial_1 z)\times \epsilon \partial_1 \psi(-) e_{-\beta} + \frac{\pi e}{\sqrt{2e_{+\alpha} e_{-\beta}}} (p_z - \frac{1}{2\pi} \partial_1 z)(p_z - \frac{1}{2\pi} \partial_1 z)\bar{\mu}_{\beta(+)} e_{+\beta}, \quad (p_z + \frac{1}{2\pi} \partial_1 z)\epsilon \lambda_{\psi(+)} = -(p_z + \frac{1}{2\pi} \partial_1 z)\epsilon \partial_1 \psi(+) e_{-\beta} - \frac{\pi e}{\sqrt{2e_{-\alpha} e_{+\beta}}} (p_z + \frac{1}{2\pi} \partial_1 z)(p_z + \frac{1}{2\pi} \partial_1 z)\bar{\mu}_{\beta(-)} e_{-\beta}, \quad (43)\]

plus their complex conjugates (notice that \((\lambda_{\psi(\pm)})^* = \lambda_{\bar{\psi}(\pm)}\)). The symbol \(\approx\) stands for an equality up to a linear combination of the constraints.

An important next point to check is that the secondary constraints (40), (41), (42) are preserved in time. It is straightforward, although a bit tedious, exercise to verify that this is indeed the case and no tertiary constraints arise or Lagrange multipliers get fixed. The Dirac algorithm thus ends here.

To fix the gauge freedom one encounters in solutions of equations of motion due to the presence of the unspecified Lagrange multipliers, it is customary to impose extra gauge conditions. For the case at hand one can choose them in the form

\[e_+^0 = e_-^0 = e_+^1 = e_-^1 = \frac{1}{\sqrt{2}}, \quad A_\alpha = 0, \quad B_\alpha = 0, \quad C_\alpha = 0, \quad \chi_{\alpha(\pm)} = 0, \quad \bar{\chi}_{\alpha(\pm)} = 0, \quad \mu_{\alpha(\pm)} = 0, \quad \bar{\mu}_{\alpha(\pm)} = 0. \quad (44)\]

The conservation in time of the gauges indeed specifies the Lagrange multipliers

\[\lambda_e = \lambda_A = \lambda_B = \lambda_C = \lambda_\chi = \lambda_\bar{\chi} = \lambda_\mu = \lambda_{\bar{\mu}} = 0, \quad (45)\]

and singles out a unique trajectory from the bunch of those connected by gauge transformations. Notice also that the complicated expressions for \(\lambda_{\psi}, \lambda_{\bar{\psi}}\) simplify considerably in the gauge chosen

\[\lambda_{\psi(+)} = \partial_1 \psi(+), \quad \lambda_{\psi(-)} = -\partial_1 \psi(-), \quad \lambda_{\bar{\psi}(+)} = \partial_1 \bar{\psi}(+), \quad \lambda_{\bar{\psi}(-)} = -\partial_1 \bar{\psi}(-). \quad (46)\]

It remains to notice that the second class constraints from Eq. (44) can be considered to be fulfilled strongly, provided a conventional Dirac bracket has been introduced. This removes the momenta \(P_{\psi(+)}^{(\pm)}, P_{\bar{\psi}}^{(\pm)}\), while leads \(\psi, \bar{\psi}\) to obey

\[\{\psi_{(+)}^n, \bar{\psi}_{(+)}^m\}_D = i\pi \eta^{nm}, \quad \{\psi_{(-)}^n, \bar{\psi}_{(-)}^m\}_D = i\pi \eta^{nm}. \quad (47)\]

Finally, the full Hamiltonian boils down to

\[H_{N=4}^{\text{gauge}} = \int d\sigma \{-\frac{1}{2\pi} (\psi_{(+)} \partial_1 \bar{\psi}_{(+)} + \bar{\psi}_{(+)} \partial_1 \psi_{(+)}) + \frac{1}{2\pi} (\psi_{(-)} \partial_1 \bar{\psi}_{(-)} + \bar{\psi}_{(-)} \partial_1 \psi_{(-)}) + 2\pi (p_z \bar{z} + \frac{1}{(2\pi)^2} \partial_1 \bar{z} \partial_1 \bar{z})\}, \quad (48)\]

which also guarantees the free dynamics for the physical fields \(z, \bar{z}, \psi, \bar{\psi}\) remaining in the question.
Now let us comment on the number of local symmetries one can expect to reveal within the Lagrangian framework. Restricting ourselves first to the case of the conventional \( N = 2 \) string Hamiltonian \( H_{N=2} \), one sees that, before the gauge fixing, the Lagrange multipliers \( \lambda_{\epsilon \pm \alpha} \), \( \lambda_{A \alpha} \), \( \lambda_{\chi \alpha(\pm)} \), \( \lambda_{\bar{\chi} \alpha(\pm)} \) are not determined by the Dirac procedure and bring degeneracy into solutions of equations of motion. This is known to be in a one-to-one correspondence with the presence of local symmetries in the Lagrangian framework. One indeed finds two local diffeomorphisms, the Weyl symmetry and the local Lorentz symmetry corresponding to \( \lambda_{\epsilon \pm \alpha} \). Associated to \( \lambda_{A \alpha} \) are the local \( a \) and \( b \) symmetries we considered in Sect. 2. The remaining four complex fermionic Lagrange multipliers \( \lambda_{\chi \alpha(\pm)} \) correspond to the super Weyl transformations (two complex parameters) and the \( N = 2 \) local world-sheet supersymmetry (two complex parameters). Turning to the full Hamiltonian \( H_{N=4} \), one concludes that the action (39) must exhibit four more bosonic symmetries associated to \( \lambda_{B \alpha} \) and \( \lambda_{C \alpha} \), the corresponding parameters presumably forming a triplet with \( a \) and \( b \) from Eqs. (3), (4) under the action of \( SU(1,1)_{outer} \). The latter point is also prompted by the fact that the corresponding gauge fields \( A_{\alpha}, B_{\alpha}, C_{\alpha} \) do fall in the triplet. Besides, one expects to reveal a set of fermionic symmetries with four complex parameters, these corresponding to \( \lambda_{\mu \alpha(\pm)} \). Because the gauge fields \( \chi_{\alpha(\pm)}, \mu_{\alpha(\pm)} \) proved to form a spinor representation of \( SU(1,1)_{outer} \), one can indeed expect a doubling of the super Weyl transformations and the \( N = 2 \) local supersymmetry intrinsic to the conventional \( (U(1,1) \text{ covariant}) \) formalism. Although we did not perform the Lagrangian analysis explicitly, on the basis of the Hamiltonian analysis outlined above one can conclude that the full action (39) must exhibit an \( N = 4 \) local supersymmetry.

7. Conclusion

Thus, in the present paper we extended the \( N = 2 \) string action to the one adequate for the \( N = 4 \) topological prescription by Berkovits and Vafa. The major advantage of the new formulation is that the Lorentz invariance holds manifest. The approach proposed in this work differs from the previous ones. We neither use gauging of global supersymmetry [1] nor appeal to dimensional reductions [3]. Based on a simple observation that an automorphism group of the small \( N = 4 \) SCA involves an extra \( U(1,1) \) we constructed the \( N = 4 \) topological string action just by installing the latter in the \( N = 2 \) string. To guarantee the new global invariance extra world-sheet fields are to be introduces. Remarkably, they proved to complement the \( N = 2 \) \( d = 2 \) supergravity multiplet to that of the \( N = 4 \) \( d = 2 \) world-sheet supergravity.

Turning to possible further developments, the obvious point missing in this paper is to explicitly reveal extra local symmetries indicated in Sect. 6. This seems to be a technical point rather than an ideological one. A more tempting question is whether it is possible to make use of the action for a covariant calculation of scattering amplitudes. It is worth mentioning also, that the action functional, when reduced to a chiral half, describes the right movers of a recent \( N = 2 \) heterotic string with manifest space time supersymmetry [21]. Adding a Lagrangian describing the left movers might give a heterotic action missing in Ref. [21] and suggests another interesting problem.


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Appendix

It seems customary to use purely imaginary $\gamma$–matrices to describe spinors on the world sheet of a string. A conventional basis is

$$
\gamma_0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma_3 = \gamma_0 \gamma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

(A.1)

It is trivial to check the algebraic properties

$$
\{\gamma_m, \gamma_n\} = -2 \eta_{mn}, \quad \gamma_m \gamma_n = -\eta_{mn} - \epsilon_{mn} \gamma_3, \quad (\gamma_0 \gamma_m)^T = \gamma_0 \gamma_m,
$$

(A.2)

where $\eta_{mn} = \text{diag}(-, +)$ and $\epsilon_{mn}$ is the 2d Levi-Civita totally antisymmetric tensor, $\epsilon_{01} = -1$.

Since in $d = 2$ irreducible representations of the Lorentz group are one–dimensional, it is convenient to use the light-cone notation for vectors and spinors

$$
A_\pm = \frac{1}{\sqrt{2}} (A_0 \pm A_1), \quad A^\alpha B_\beta = -A_+ B_- - A_- B_+, \quad \Psi_A = \begin{pmatrix} \psi(+) \\ \psi(-) \end{pmatrix},
$$

(A.3)

which makes the latter point more transparent. Actually, the Lorentz transformation acquires the form

$$
\delta A_\pm = \pm \omega A_\pm, \quad \delta \psi(\pm) = \pm \frac{1}{2} \psi(\pm),
$$

(A.4)

and the invariance is kept, for example, by contracting a “+” with a “−” (one could fairly well contract a “(+)” with a “(−)” or a “+” with two “(−)”). In the light cone notation one can get rid of $\gamma$–matrices and work explicitly in terms of irreducible components of tensors under consideration.

Introducing the zweibein $e^m_\alpha$ on the world sheet, $\eta_{mn} = e^n_\alpha g_{\alpha \beta} e^m_\beta$, $g^{\alpha \beta} = e^n_\alpha \eta^{mn} e^m_\beta$, where $\alpha$ stands for a curved index and $m$ for a flat one, one can easily verify the relations

$$
g^{\alpha \beta} = -e_+^\alpha e_-^\beta - e_-^\alpha e_+^\beta, \quad \epsilon^{mn} e^m_\alpha e^n_\beta = -e_+^\alpha e_-^\beta + e_-^\alpha e_+^\beta = e e^{\alpha \beta},
$$

(A.5)

$$
\eta_{++} = \eta_{--} = 0, \quad \eta_{+-} = \eta_{-+} = -1,
$$

where $e = \text{det}(e^m_\alpha)$.

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