ON THE CLUSTER STRUCTURES IN COLLATZ LEVEL SETS

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Abstract. The cluster structures that can be observed in the first few level sets of the Collatz tree are maintained through all its levels, provided that the orbit steadiness
\[
\prod_{k \in R(n) \mod 6 \equiv 4} \frac{k - 1}{k}
\]
of the elements \( n \) of the Collatz tree is suitably bounded from below, where \( R(n) \) denotes the Collatz orbit of \( n \).

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1. The Question

Let \( \mathbb{N} \) be the set of positive integers. By \( c : \mathbb{N} \to \mathbb{N} \) we denote the Collatz function (see [1]), defined by
\[
c(n) := \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even}, \\
3n + 1 & \text{if } n \text{ is odd}.
\end{cases}
\]
We are interested in the level sets \( L_\nu := \{ n \in \mathbb{N} : c^\nu(n) = 1 \land c^i(n) \neq 1 \text{ for } i < \nu \} \) for \( \nu \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), whose elements are listed in [2]. It is easy to see that
\[
L_{\nu+1} = \{2n : n \in L_\nu\} \uplus \left\{ \frac{n - 1}{3} : 4 < n \in L_\nu, n \equiv 4 \pmod{6} \right\}.
\]
(1)
The level sets for the first some dozens values of \( \nu \) exhibit a cluster structure. For example, the 72 elements of \( L_{20} \) are given by the seven clusters
\[
\{18, 19\},
\{112, 116, 117, 120, 122\},
\{704, 720, 724, 725, 736, 738, 739, 744, 746, 753, 802, 803, 804, 805, 806\},
\{4352, \langle 20 \rangle, 4849\},
\{24576, \langle 17 \rangle, 29126\},
\{163840, 172032, 174080, 174592, 174720, 174752, 174760, 174762\},
\{1048576\},
\]
where \( \langle x \rangle \) is short for a list of \( x \) intermediate elements. Of course the largest number 1048576 equals \( 2^{20} \), and [1] makes plausible that the elements of a cluster have about six times the size of the elements of the previous cluster: Each cluster \( C \) of \( L_\nu \) gives a cluster \( 2C \) of \( L_{\nu+1} \), and the second set in [1] contributes to the cluster below that one. An additional cluster appears at the bottom end in \( L_{\nu+1} \) whenever there exists an \( n \equiv 4 \pmod{6} \) in the lowest cluster of \( L_\nu \). The question suggests itself whether the pattern extends to all level sets or whether, for large \( \nu \), the clusters eventually dissolve so much that they overlap.

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2. The answer (under a provision)

A closer investigation of the issue shows that the product of quotients $\frac{k-1}{k}$ for orbit elements $k \equiv 4 \pmod{6}$ controls the evolution of the cluster shapes. We introduce the orbit steadiness function $\sigma : \mathbb{N} \to [0, 1]$ by

$$\sigma(n) := \prod_{k \in R(n), k \equiv 4 \pmod{6}} \frac{k-1}{k} \quad \text{for } n \in \mathbb{N},$$

where $R(n) := \{c^i(n) : i \in \mathbb{N}_0\}$ is the orbit set of $n$, and with $\mathcal{C} := \bigcup_{\nu \in \mathbb{N}_0} L_\nu$

$$\sigma_0 := \inf_{n \in \mathcal{C}} \sigma(n).$$

For $\nu \in \mathbb{N}_0$, by

$$S_{\nu, \kappa} := \left[\sigma_0, \frac{2^\nu}{6^\kappa}, \frac{2^\nu}{6^\kappa}\right] \quad \text{for } \kappa \in \mathbb{N}_0$$

we define “slots” for the clusters of the level set $L_\nu$. Then, given $\nu \in \mathbb{N}_0$ and $n \in L_\nu$, setting $I_\alpha := \{i \in \{1, \ldots, \nu\} : c^{i-1}(n) \equiv \alpha \pmod{2}\}$ for $\alpha \in \{0, 1\}$ and $\kappa := |I_1|$ yields

$$n = \prod_{i=1}^\nu \frac{c^{i-1}(n)}{c^i(n)} = \prod_{i \in I_0} \frac{c^{i-1}(n)}{c^i(n)} \cdot \prod_{i \in I_1} \frac{c^{i-1}(n)}{c^i(n)} = 2^{|I_0|} \prod_{i \in I_1} \frac{c^i(n)-1}{3^\kappa} = \frac{2^\nu-\kappa}{3^\kappa} \sigma(n) = \frac{2^\nu}{6^\kappa} \sigma(n),$$

which gives $n \leq \frac{2^\nu}{6^\kappa}$ and $n \geq \sigma_0 \frac{2^\nu}{6^\kappa}$, hence $n \in S_{\nu, \kappa}$. This proves

$$L_\nu \subset \bigcup_{\kappa \in \mathbb{N}_0} S_{\nu, \kappa} \quad \text{for all } \nu \in \mathbb{N}_0.$$

For $\nu \in \mathbb{N}_0$, the slots $S_{\nu, \kappa}$ for $\kappa \in \mathbb{N}_0$ are pairwise disjoint if $\sigma_0 > \frac{1}{6}$. Numerical evidence indicates $\sigma_0 \approx 0.5152$. The slightly weaker assumption $\sigma_0 > \frac{1}{2}$ gives $\max_{\kappa \in \mathbb{N}} S_{\nu, \kappa} < \frac{1}{3} \min_{\kappa \in \mathbb{N}} S_{\nu, \kappa-1}$ for all $\kappa \in \mathbb{N}$, i.e. a clear separation of the slots and the persistence of the cluster pattern in all level sets of the Collatz tree.

To get a trustworthy statement, a proven lower bound for $\sigma_0$ would of course be preferable.

3. A remark

We admitted only elements of the Collatz tree in the definition of $\sigma_0$. This does not make a difference if the Collatz conjecture is true, because then $\mathcal{C} = \mathbb{N}$. However, at this point it cannot be excluded that, in case of the falsehood of the Collatz conjecture, the orbit steadiness of some $n \in \mathbb{N} \setminus \mathcal{C}$, i.e. some $n$ with a non-trivial cyclic or a diverging orbit, might be smaller than $\sigma_0$. This is why, in an abundance of caution, and because it is sufficing for the application, we decided for $\mathcal{C}$ instead of $\mathbb{N}$.

References

[1] J. C. Lagarias, The $3x + 1$ problem: An Overview, pp. 3–29 in The Ultimate Challenge: The $3x + 1$ Problem (J. C. Lagarias, Ed.), Amer. Math. Society, Providence, RI2010.

[2] T. D. Noe, Sequence [A127824] in The Online Encyclopedia of Integer Sequences (2010), published electronically at https://oeis.org.

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