SINGULAR VALUE STATISTICS FOR THE SPIKED ELLIPTIC GINIBRE ENSEMBLE

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Abstract. The complex elliptic Ginibre ensemble is a complex Gaussian matrix interpolating between the Gaussian unitary ensemble and the Ginibre ensemble. Its eigenvalues form a determinantal point process in the complex plane, however, until recently singular values have been proved to build a Pfaffian point process by Kanazawa and Kieburg (arXiv:1804.03985). In this paper we turn to consider an extended elliptic Ginibre ensemble with correlated rows and columns, which connects GUE and the spiked Wishart matrix. We prove that the singular values still build a Pfaffian point process with correlation kernel expressed by a contour integral representation, and further observe a crossover transition of local eigenvalue statistics at the origin.

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1. Introduction and Main Results

1.1. Introduction. The study of non-Hermitian random matrices was first initiated in 1965 by Ginibre [23], who introduced Gaussian random matrices (complex, real and quaternion real) as a mathematical extension of Hermitian random matrix theory. At that time, unlike their Hermitian counterparts, it transpires that such random matrices did not have physical relevance. Although Ginibre was motivated by the works of Wigner, Dyson and Mehta on random Hamiltonians, he remarked that ‘apart from the intrinsic interest of the problem, one may hope that the methods and results will provide further insight in the cases of physical interest or suggest as yet lacking applications’. But now, non-Hermitian random matrices have many interesting and important applications in modelling of fractional quantum-Hall effect, Coulomb plasma [21], and stability of the complex ecological system [34, 8, 9] and others; see [31] for an overview.

Date: July 10, 2018.

Key words and phrases. Spiked Wishart matrix, Elliptic ensemble, Singular values, Pfaffian point processes.
In the case of the complex Ginibre ensemble, which is an $N \times N$ complex random matrix $X$ with the probability density being proportional to $\exp\{-\text{Tr}(X^*X)\}$, its eigenvalues form a determinantal point process in the complex plane [23] and corresponding eigenvalue statistics has been extensively studied, see e.g. [21]. Another important and well-studied model is the (complex) elliptic Ginibre ensemble which was introduced as an interpolation between Hermitian and non-Hermitian random matrices; see [24] and [18, 19]. Being precisely, let $H_1, H_2$ be two independent GUE matrices sampled from distribution proportional to $\exp\{-\text{Tr}(H_2)\}$, then $X = \sqrt{1 + \tau}H_1 + i\sqrt{1 - \tau}H_2$ with $\tau \in [0, 1]$ is an interpolating ensemble between the Ginibre ensemble ($\tau = 0$) and and the GUE ($\tau = 1$). The matrix $X = [X_{i,j}]$ has a probability density function (PDF for short)

$$
\tilde{P}_N(X) = \frac{1}{(\pi \sqrt{1 - \tau^2})^{N^2}} \exp \left[ -\frac{1}{1 - \tau^2} \text{Tr} \left( X^*X - \frac{\tau}{2}(X^2 + (X^*)^2) \right) \right].
$$

(1.1)

Here it is worth stressing that (i) $\{X_{j,j} : 1 \leq j \leq N\} \cup \{X_{i,j}, X_{j,i} : 1 \leq i < j \leq N\}$ are independent and (ii) $\tau$ is the correlation coefficient of $\text{Re}\{X_{i,j}\}$ and $\text{Re}\{X_{j,i}\}$ while $-\tau$ is the correlation coefficient of $\text{Im}\{X_{i,j}\}$ and $\text{Im}\{X_{j,i}\}$ for any $i < j$.

The model (1.1) has an interesting variation in Quantum chromodynamics (QCD) connecting with a certain of Dirac operator

$$
\mathcal{D} = \begin{bmatrix} 0 & iX \\ iX^* & 0 \end{bmatrix}.
$$

This chiral form has at least three major applications in QCD: 4D QCD at high temperature, 3D QCD at finite isospin chemical potential and 3D lattice QCD for staggered fermions; see [29, 30] and references therein for detailed discussion. An important characteristic of $\mathcal{D}$ is that its eigenvalues come in pairs like $\{\lambda_j, -\lambda_j\}$ because of the chiral symmetry, and this makes a world of difference with the Hermitian Wilson Dirac operator

$$
\mathcal{D}_5 = \begin{bmatrix} 0 & X \\ X^* & 0 \end{bmatrix} + \mu H
$$

where $H$ is an $2N \times 2N$ GUE matrix. The latter was considered by Akemann and Nagao [4] as an interpolating ensemble between GUE and chiral GUE.

The eigenvalue density of $X$ associated with (1.1) is given by

$$
\tilde{P}_N(z) = \frac{1}{(\pi \sqrt{1 - \tau^2})^{N^2}} \prod_{j=1}^N j! \exp \left[ -\frac{1}{1 - \tau^2} \sum_{j=1}^N \left( |z_j|^2 - \frac{\tau}{2}(z_j^2 + \bar{z}_j^2) \right) \right] \prod_{j<k} |z_k - z_j|^2,
$$

and thus forms a determinantal point process in the complex plane; see [18]. It is known that for fixed $\tau$ the limiting spectral measure is uniform on an ellipse

$$
\{z = x + iy \in \mathbb{C} : \frac{x^2}{(1 + \tau)^2} + \frac{y^2}{(1 - \tau)^2} \leq 1 \}.
$$

This is called the elliptic law in the literature and in fact holds true for iid type random matrices, see [24, 38]. When the parameter $\tau$ approaches 1, its support rapidly degenerates from an ellipse to an interval $[-1, 1]$ of the real line. Then certain crossover transitions may occur for local statistics at the critical rate of $1 - \tau$. Actually, with the help of Hermite polynomials, in the so-called weak non-Hermiticity situation that $1 - \tau = \kappa/N$, Fyodorov, Khoruzhenko and Sommers observed a cross-over transition of local bulk statistics from
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Wigner-Dyson to Ginibre statistics (see [18] or [31] and references therein in great detail). In another critical regime of $1 - \tau = \kappa N^{-1/3}$, Bender proved a edge cross-over transition from Poisson to Airy point processes. For recent results on local properties of the elliptic ensemble, see [3].

The squared singular values of the complex Ginibre ensemble $X$, equivalently, eigenvalues of $X^*X$ which is the well-known complex Wishart ensemble (also called Laguerre Unitary Ensemble), have a determinantal structure and have been studied quite well; see e.g. [20, 27, 28, 48, 49]. However, until recently a truly great breakthrough in singular values of the elliptic Ginibre ensemble has been established by Kanazawa and Kieburg [30]. Instead of a determinantal point process, it is proved that the singular values build a Pfaffian point process. Further, explicit formulas for skew orthogonal polynomials are constructed via the supersymmetry method and a finite sum of skew-orthogonal polynomials for correlation kernel is thus derived. This has opened up the possibility for asymptotic analysis of the local statistics, e.g., limiting distributions of both smallest and largest eigenvalues.

In the present paper we turn to consider an extended interpolating ensemble which connects GUE and the spiked Wishart ensemble. Explicitly, for $M \geq N$, let $A = [I_N \ 0_{N \times (M-N)}]$ and $\Sigma$ be an $N \times N$ positive definite matrix with eigenvalues $\sigma_1, \ldots, \sigma_N$, we define an $M \times N$ random matrix $X$ with PDF with respect to the Lebesgue measure on $\mathbb{C}^{M \times N}$

$$P_N(X) = C_N \exp \left[ -\eta_+ \operatorname{Tr} (X^*X\Sigma) + \frac{1}{2} \eta_- \operatorname{Tr} ((XA)^2 + (A^*X^*)^2) \right],$$

where the normalisation constant

$$C_N = \left( \frac{\eta_+}{\pi} \right)^M \left( \det \Sigma \right)^{M-N} \sqrt{ \det (\Sigma \otimes \Sigma) - (\eta_-/\eta_+)^2 I_{N^2} }. \quad (1.3)$$

We assume that $\eta_+ > 0$ and $\Sigma \otimes \Sigma > (\eta_-/\eta_+)^2 I_{N^2}$ for integrability and usually choose

$$\eta_+ = \frac{1}{1 - \tau^2}, \quad \eta_- = \frac{\tau}{1 - \tau^2}, \quad \tau \in (0, 1). \quad (1.4)$$

This may be treated as a spiked rectangular version of the complex elliptic ensemble, since when $M = N$ and $\Sigma = I_N$ it is the elliptic ensemble. In another special case of $\tau = 0$, $X^*X$ reduces to the complex Wishart matrix with a spike. Complex/Real spiked Wishart matrices are of great interest in statistics, signal processing and random growth models, and have been extensively studied in the literature; see e.g. [10, 40].

The data matrix $X$ defined in (1.2) is also of high interest in statistics. Typically when $\tau = 0$ in eq.(1.2), $X$ consists of $M$ independent column variables (samples) each with covariance matrix $\Sigma^{-1}$. However, for many types of microarray datasets, e.g., genetic and financial data, there are complicated correlations among row variables; see e.g. [8, 14] and references therein. It raises a serious problem about the seemingly natural independence assumption of samples and suggests the need for conducting independence test among samples. This challenge arising from correlation among samples has recently been taken on by Allen and Tibshirani [8], Chen and Liu [12], Pan et al. [41]. Besides, correlated sample matrices also appear MIMO channels in wireless communications [26, 35]. When $\tau \in (0, 1)$, the data matrix $X$ defined in (1.2) is in fact transposable, meaning that both rows and columns are potentially correlated [8]. At this point, mixing effect from
correlation between columns caused by $\Sigma$ and correlation between entries $\text{Re}\{X_{i,j}\}$ and $\text{Re}\{X_{j,i}\}$ ($i \neq j$) by $\tau$ shows that both rows and columns are correlated. To the best of our knowledge, this is the first example of transposable data matrices for which the PDF for singular values is exactly derived. Moreover, local eigenvalue statistics can be analysed in details; cf. Theorems 1.1-1.3 below.

One fundamental problem in random matrix theory (RMT) is to prove universality of local eigenvalue statistics, typically according to the spectral position of limiting eigenvalue density and symmetry class of matrix entries. In the Hermitian case the universal phenomenon is usually described by Airy kernel at the soft edge, sine kernel in the bulk and Bessel kernel at the hard edge. In the literature there are a lot of relevant articles on universality phenomenon, say, [15, 16, 45, 46, 47] or recent monographs [1, 21, 17] and references therein. Except for above three typical universal patterns, two new families of hard edge kernels which are so-called Meijer G- and Pearcey Meijer G- kernels have been found in a rapid developing topic of products of random matrices respectively in [32] and [22]. Also, there exist crossover transitions between different limiting kernels in products of two coupled random matrices; see [2] [3] [4] [33]. Particularly in a non-central complex Wishart matrix, a phase transition for smallest singular values is established at the hard edge from Bessel kernel to Pearcey kernel and to finite LUE kernel; see [22]. This is a hard-edge analog of BBP transition for largest eigenvalues first proved in [10].

Furthermore, RMT affords a versatile description of crossover transition phenomena between different symmetry classes. Two classical interpolating random matrix ensembles were introduced and solved by Mehta and Pandey [43, 37]. These described the transition between the Gaussian unitary ensemble (GUE), relevant for systems without time-reversal invariance, and the Gaussian orthogonal ensemble as well as the Gaussian symplectic ensemble for systems with time-reversal symmetry and even or odd spin, respectively. As mentioned before, the elliptic complex Ginibre ensemble interpolates between GUE to the Ginibre ensemble, and so its eigenvalues describe a transition from complex to real eigenvalues in the weak non-Hermiticity regime; see [11, 18, 19] or [3] for recent development. As to its singular values it’s an entirely reasonable prediction that there should be a crossover transition as $\tau$ changes from 0 to 1. Actually, in one previous paper [29], Kanazawa and Kieburg conjectured that there would be a crossover transition of smallest singular values at the the origin. Moreover, they employed the supersymmetry method and Monte Carlo simulations to claim an expression of the microscopic level density at the origin; see [29, eq.(14)].

1.2. Main Results. Our goal is to prove the singular value PDF as a Pfaffian point process, derive a double contour integral for the corresponding correlation kernel and further investigate scaling limits for correlation functions. The explicit expression form for singular value PDF and correlation kernel will be different for finite even and odd $M$, however, we just collect the main results in the even case for convenience.

Our first result is the derivation of the probability density function of eigenvalues of $W = X^*X$ as following.
Theorem 1.1. With $X$ defined as in (1.2) and notations as in (1.3), for even $M$ the joint probability density function of eigenvalues $\lambda_1, \ldots, \lambda_N$ of $W = X^*X$ is given by

$$f_N(\lambda) = \frac{1}{Z_{M,N}} \det[e^{-(\eta_1 + \sigma_i + \eta_j)}\lambda_{ij}]_{i,j=1}^N \text{Pf} \left[ \frac{E(\eta_j - \eta_j - \lambda_k)}{-g_\lambda(\lambda_k)} \right]_{j,k=1,\ldots,N}^{a,b=1,\ldots,M-N},$$

where $Z_{M,N}$ is the normalisation constant and with the modified Bessel function $I_0$

$$E(u, v) = \int_{\mathbb{R}^2} dx dy \frac{x - y}{x + y} e^{-\frac{1}{2}(x^2 + y^2)} \left( I_0(2x\sqrt{uv}) I_0(2y\sqrt{uv}) - I_0(2y\sqrt{u}) I_0(2x\sqrt{v}) \right),$$

$$g_\lambda(u) = \eta^{a-1} \int_{\mathbb{R}^2} dx dy \frac{x - y}{x + y} e^{-\frac{1}{2}(x^2 + y^2)} \left( x^{2(a-1)} I_0(2y\sqrt{\eta u}) - y^{2(a-1)} I_0(2x\sqrt{\eta u}) \right),$$

$$\alpha_{a,b} = \eta^{a+b-2} \int_{\mathbb{R}^2} dx dy \frac{x - y}{x + y} e^{-\frac{1}{2}(x^2 + y^2)} \left( x^{2(a-1)} y^{2(b-1)} - y^{2(a-1)} x^{2(b-1)} \right).$$

We remark that $Z_{M,N}$ can be expressed in terms of the determinant of the Gram type moment matrix in Sect. 3 cf. eq(3.25) below. When some of parameters $\sigma_1, \ldots, \sigma_N$ coincide, (1.5) reduces to a limiting density by applying L’Hospital’s rule. Particularly when $M = N$ and $\Sigma = I_N$, it proves to be Kanazawa-Kieburg’s result [30, Sect. 2]. When $\tau$ tends to 0, the PDF indeed converges to that eigenvalues of complex spiked Wishart matrix.

Next, we will prove that the eigenvalue PDF of (1.5) forms a Pfaffian point process and the associated correlation kernel $K_N(u, v)$ can be given by double contour integrals. If so, recalling the definition of $k$-point correlation functions (see e.g. [21, 36]) by

$$R_N^{(k)}(\lambda_1, \ldots, \lambda_k) = \frac{N!}{(N-k)!} \int \cdots \int f_N(\lambda_1, \ldots, \lambda_N) d\lambda_{k+1} \cdots d\lambda_N,$$

then we have

$$R_N^{(k)}(\lambda_1, \ldots, \lambda_k) = \text{Pf} [K_N(\lambda_i, \lambda_j)]_{i,j=1}^k.$$

For this, let

$$\varphi_k = \frac{\tau}{\sigma_k + \tau}, \quad k = 1, 2, \ldots, N.$$

Note that $\varphi_k < 1/2$ for all $k$. Also let $\sqrt{z}$ denote the principal value such that it is positive for positive $z$. For convenience in later using, let $C_A$ be an anti-clockwise contour encircling all the points in the set $A$ but not any other poles of the integrand.

Theorem 1.2. With the same notations as in Theorem 1.1 the correlation kernel associated with (1.3) is given by

$$K_N(u, v) = \begin{bmatrix} D_S_N(u, v) & S_N(u, v) \\ -S_N(v, u) & I S_N(u, v) \end{bmatrix},$$

where

$$D_S_N(u, v) = \eta^2 \int_{C(\varphi_1, \ldots, \varphi_N)} \frac{dw}{2\pi i} \int_{C(1-\varphi_1, \ldots, 1-\varphi_N)} \frac{dz}{2\pi i} e^{\frac{w - \eta - u}{v - \eta - v}} \frac{1}{\sqrt{1 - 2w\sqrt{2(z - w) - 1}}} \times \frac{1 - z - w}{w - z} \prod_{k=1}^N \frac{z - \varphi_k}{w - \varphi_k} \frac{1 - z - \varphi_k}{w - \varphi_k} \frac{1 - z - \varphi_k}{w - \varphi_k}$$

for even $M$.
\[ S_N(u, v) = \int_0^\infty \mathcal{E}(\eta_v, \eta_t) DS_N(u, t) dt, \quad (1.14) \]
\[ IS_N(u, v) = -\mathcal{E}(\eta_u, \eta_v) + \int_0^\infty \int_0^\infty \mathcal{E}(\eta_v, \eta_t) DS_N(s, t) \mathcal{E}(\eta_u, \eta_s) ds dt; \quad (1.15) \]

cf. Figure 2 for an illustration of the two contours which do not intersect.

Figure 1. Contours of double integrals for \( DS_N \)

Finally, we investigate the local statistical behaviour of eigenvalues as the matrix size \( N \) goes to infinity. What is most interesting about the singular values of the spiked elliptic ensemble is a possible crossover as the parameter \( \tau \) changes. This does indeed seem to be the case. At the hard edge of the spectrum (at the origin), we observe a transition of singular values from Ginibre to GUE exactly at a critical value of \( \tau = \frac{1}{\kappa/N} \) with \( \kappa \in (0, \infty) \). So in the subsequent sections \( \tau = \tau_N \) may depend on \( N \), but we omit the notation \( N \) for simplicity.

For \( \kappa \in (0, \infty) \), \( \rho_1, \ldots, \rho_n \in (0, 1/2) \) and a non-negative integer \( \alpha \), define a function of two variables

\[
DS(u, v) = \text{PV} \frac{1}{\kappa^2} \int_{C_L} \frac{dw}{2\pi i} \int_{C_R} \frac{dz}{2\pi i} e^{-\frac{\kappa}{w} - \frac{1}{w} + \frac{1}{\kappa w} - \frac{1}{z} - \frac{1}{\kappa z}} \frac{1}{\sqrt{1 - 2w\sqrt{2z - 1}}} \\
\times \frac{1 - z - w}{w - z} \left( \frac{1 - w}{w} \right)^\alpha \prod_{k=1}^n \frac{z - \rho_k}{w - \rho_k} \frac{1}{1 - \rho_k} \frac{1}{1 - z - \rho_k} \\
+ \frac{1}{\pi \kappa^2} \int_0^1 ds \left( \frac{2}{\kappa^2 + s^2} \right) \left( \frac{2}{\kappa^2 + s^2} \right) \left( u - v \right), \quad (1.16) \]

where \( \text{PV} \) denotes the Cauchy principal value integral and correspondingly

\[ S(u, v) = \int_0^\infty \mathcal{E}(v/\kappa, t/\kappa) DS(u, t) dt; \quad (1.17) \]
\[ IS(u, v) = -\mathcal{E}(u/\kappa, v/\kappa) + \int_0^\infty \int_0^\infty \mathcal{E}(v/\kappa, t/\kappa) DS(s, t) \mathcal{E}(u/\kappa, s/\kappa) ds dt; \quad (1.18) \]

Here the two contours \( C_L \) and \( C_R \) are illustrated in Figure 2 with two intersections \( e_3 = (1 + i)/2 \) and \( \bar{e}_3 = (1 - i)/2 \).

It’s ready to state the third main result of the present paper.

**Theorem 1.3.** With the same notations as in Theorem 1.2 and with fixed nonnegative integers \( n \) and \( \alpha := M - N \), assume that

\[
\sigma_{n+1} = \cdots = \sigma_N = 1 \quad \text{and} \quad \rho_i := 1/(\sigma_i + 1) \in (0, 1/2), \quad i = 1, \ldots, n. \quad (1.19)\]
If $N(1-\tau) \to \kappa \in (0, \infty)$ as $N \to \infty$, then the scaled correlation functions hold true
\[
\lim_{N \to \infty} \left( \frac{2}{N} \right)^k R_N^{(k)} \left( \frac{2u_1}{N}, \ldots, \frac{2u_k}{N} \right) = \text{Pf} \left[ DS(u_i, u_j) \begin{array}{c} S(u_i, u_j) \\ -S(u_j, u_i) \end{array} IS(u_i, u_j) \right]_{i,j=1}^k.
\] (1.20)

The rest of this article is organised as follows. In the next Section 2 we are devoted to the derivation of the joint density function of squared singular values as a Pfaffian point process. In Section 3 we show that all three sub kernels of the $2 \times 2$ block correlation kernel permit contour integral representations. The last Section 4 focuses on asymptotic behaviour analysis of correlation functions at the origin.

2. Singular value PDF

In this section we are devoted to the derivation of the probability density function of squared singular values of $X$, by following the similar steps as in [30].

Proof of Theorem 1.1. Write $n = M - N$ for simplicity. Use the singular value decomposition $X = U\Lambda V^*$, where two unitary matrices $U \in U(M)$, $V \in U(N)$, and
\[
\Lambda = \begin{bmatrix} \Lambda_0 \\ 0_{n \times N} \end{bmatrix}, \quad \Lambda_0 = \text{diag} \left( \sqrt{\lambda_1}, \ldots, \sqrt{\lambda_N} \right).
\]

Under this change of variables, the standard approach of calculating the Jacobian as in [21] shows
\[
[dX] = C_{N,M} \Delta_N^2(\lambda_1, \ldots, \lambda_N) \prod_{k=1}^N \lambda_k^n d\lambda_k d\mu_M(U) d\mu_N(V),
\] (2.1)

where $[dX]$ denotes the Lebesgue measure on $C^{M\times N}$, $\Delta_N(\lambda_1, \ldots, \lambda_N)$ the Vandermande determinant, and $\mu_N$ is the Haar measure on the space of unitary $N \times N$ matrices $U(N)$. If we denote the normalization of the probability density function of $W$ induced from (1.2) by $Z_N$, then the eigenvalue PDF for $W$ reads
\[
f_N(\lambda) = \frac{C_{N,M} \Delta_N^2(\lambda) \prod_{k=1}^N \lambda_k^n \int_{U(N)} d\mu_N(V) \exp \left\{ -\eta_+ \text{Tr} \left( V\Lambda_0^2 V^*\Sigma \right) \right\}}{Z_N} \int_{U(M)} d\mu_M(U) \exp \left\{ \frac{\eta-}{2} \text{Tr} \left( (AA \text{diag}(V^*, I_n)U)^2 + (U^* \text{diag}(V, I_n)A^*A)^2 \right) \right\},
\] (2.2)

where we have used the simple fact that $V^*A = A \text{diag}(V^*, I_n)$. 

![Figure 2. Contours of double integrals for $DS$](image-url)
Absorb the factor \( \text{diag}(V^*, I_n) \) into the group \( \mathcal{U}(M) \) and we rewrite
\[
f_N(\lambda) = \frac{C_{N,M}}{Z_N} \Delta_N^2(\lambda) \prod_{k=1}^{N} \lambda_k \nu \ I_1 I_2
\]  
where
\[
I_1 = \int_{\mathcal{U}(N)} d\mu_N(V) \exp \left\{ -\eta_+ \ \text{Tr} \left( V I_0^2 V^* \Sigma \right) \right\},
\]
and
\[
I_2 = \int_{\mathcal{U}(M)} d\mu_M(U) \exp \left\{ \frac{n}{2} \ \text{Tr} \left( (\Lambda A U)^2 + (U^* A^* \Lambda^*)^2 \right) \right\}.
\]

Apply the Harish-Chandra-Itzykson-Zuber (HCIZ) integral formula (cf. [21, 36]) and we arrive at
\[
I_1 = \prod_{k=0}^{N-1} \frac{k! \det[e^{-\eta_+ \sigma_i \lambda_j}]_{i,j=1}^{N}}{\Delta_N(\sigma) \Delta_N(\lambda)}.
\]  
To the integral \( I_2 \), noting \( \Lambda A = \text{diag}(\Lambda_0, 0_n) \), introduce an extended \( M \times M \) matrix
\[
\tilde{\Lambda} = \text{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_N}, \sqrt{\lambda_{N+1}}, \ldots, \sqrt{\lambda_M}).
\]
By a new integral formula over the unitary group due to Kanazawa and Kieburg [30, Appendix A], we obtain
\[
I_2 = \lim_{\lambda_{N+1, \ldots, \lambda_M} \to 0} \int_{\mathcal{U}(M)} \exp \left\{ \frac{1}{2} \eta_+ \ \text{Tr} \left( (\tilde{\Lambda} U)^2 + (U^* \tilde{\Lambda}^*)^2 \right) \right\} d\mu_M(U)
\]
\[
= \prod_{k=0}^{M-1} \frac{k! \eta^{-(M-1)/2} \int_{\mathbb{R}^M} dx \prod_{k=1}^{M} \frac{e^{-\frac{1}{2} x_k^2} \Delta_M(x_1, \ldots, x_M)}{\Delta_M(x_1^2, \ldots, x_M^2)}}{\Delta_M(\lambda_1, \ldots, \lambda_M) \det \left[ I_0(2x_i \sqrt{\eta_+ - \lambda_j}) \right]_{1 \leq i, j \leq M}}.
\]
Thus, application of the L'Hôpital's rule gives us to
\[
I_2 = \lim_{\lambda_{N+1, \ldots, \lambda_M} \to 0} \prod_{k=0}^{M-1} \frac{k! \prod_{k=1}^{N} \lambda_k^{-n} e^{-\eta_+ - \lambda_k}}{\Delta_N(\lambda) \prod_{k=1}^{N} \lambda_k^{-\eta} \cdot \Delta_M(x_1, \lambda_1, \ldots, x_M) \det \left[ I_0(2x_i \sqrt{\eta_+ - \lambda_j}) \right]_{1 \leq i, j \leq M}}.
\]
By the Schur Pfaffian identity
\[
\frac{\Delta_M^2(x_1, \ldots, x_M)}{\Delta_M(x_1^2, \ldots, x_M^2)} = \begin{cases} 
\text{Pf} \left[ \frac{x_i - x_j}{x_i + x_j} \right]_{i,j=1}^{M}, & \text{M even;} \\
\text{Pf} \left[ \frac{x_i - x_j}{-1 + x_i + x_j} \right]_{i,j=1}^{1 \times M} & \text{M odd,}
\end{cases}
\]
Let Lemma 2.1. Here Pf

For \( m \) with the same notations as in Theorem 1.1.

with the same notations as in Theorem 1.1.

Combine (2.3), (2.6) and (2.10), we thus complete the proof of Theorem 1.1.

\[ \text{Proof.} \ \text{We proceed by induction.} \]

Obviously, the desired result is true for \( m = 2 \). Suppose it also holds true when \( m - 2 \). For \( m \), starting from the recursive definition of the Pfaffian, we have

\[ \text{Pf} \left[ \epsilon(x_1, x_j) \right]_{1 \leq i, j \leq m} = \sum_{k=2}^{m} (-1)^{k} \epsilon(x_1, x_k) \text{Pf} \left[ \epsilon(x_1, x_j) \right]_{i, j \notin \{1, k\}}. \]  

(2.13)

Here \( \text{Pf} \left[ \epsilon(x_1, x_j) \right]_{i, j \notin \{1, k\}} \) is the Pfaffian of the matrix \( [\epsilon(x_r, x_s)]_{1 \leq r, s \leq m} \) with both \( i \)-th and \( j \)-th rows and columns removed. Correspondingly, take the Laplace expansion of the determinant \( \det[\phi_i(x_j)] \) as

\[ \det[\phi_i(x_j)] = \sum_{1 \leq i < j \leq m} (-1)^{i+j+k+1} \det \begin{bmatrix} \phi_i(x_1) & \phi_i(x_k) \\ \phi_j(x_1) & \phi_j(x_k) \end{bmatrix} \det[\phi_r(x_s)]_{r \notin \{i, j\}, s \notin \{1, k\}}. \]

(2.14)

Taking (2.13) and (2.14) into consideration, we obtain

\[
\text{LHS of (2.12)} = \sum_{k=2}^{m} \sum_{1 \leq i < j \leq m} (-1)^{i+j+k+1} (-1)^{k} \int_{\Omega^m} \epsilon(x_1, x_k) \text{Pf} \left[ \epsilon(x_1, x_j) \right]_{i, j \notin \{1, k\}}
\]

\[
\times \det \begin{bmatrix} \phi_i(x_1) & \phi_i(x_k) \\ \phi_j(x_1) & \phi_j(x_k) \end{bmatrix} \det[\phi_r(x_s)]_{r \notin \{i, j\}, s \notin \{1, k\}} \prod_{i=1}^{m} d\nu(x_i)
\]

\[
= \sum_{k=2}^{m} \sum_{1 \leq i < j \leq m} (-1)^{i+j+1} \int_{\Omega^2} \epsilon(x, y) \det \begin{bmatrix} \phi_i(x) & \phi_i(y) \\ \phi_j(x) & \phi_j(y) \end{bmatrix} d\nu(x)d\nu(y)
\]

\[
\times \int_{\Omega^{m-2}} \text{Pf} \left[ \epsilon(x_1, x_j) \right]_{1 \leq i, j \leq m-2} \det[\phi_r(x_s)]_{r \notin \{i, j\}, 1 \leq s \leq m-2} \prod_{i=1}^{m-2} d\nu(x_i).
\]
By the induction hypothesis, we further get
\[
\text{LHS of (2.12)} = \frac{m - 1}{2^{m-1}} \sum_{1 \leq i,j \leq m} (-1)^{i+j} \epsilon_{i,j} \text{Pf} [\epsilon_{r,s}]_{r,s \notin \{i,j\}}
\]
\[
= \frac{m - 1}{2^{m-1}} \sum_{1 \leq i,j \leq m} (-1)^{i+j+1} \Theta(i-j) \epsilon_{i,j} \text{Pf} [\epsilon_{r,s}]_{r,s \notin \{i,j\}}
\]
\[
= \frac{1}{2^{m-1}} \text{Pf} [\epsilon_{i,j}]_{1 \leq i,j \leq m},
\]
where $\Theta$ denotes the Heaviside step function. We thus complete the proof. $\Box$

At last we conclude this section with a remark about the limit of $\tau \to 0$ by following the method of [30, Sect.6.2]. A different limit of $\tau \to 1$, relevant to singular values of GUE, is much more technical; see [13, 30]). Since each entry from the Pfaffian in (2.10) tends to 0 as $\tau \to 0$, we need to expand it into a series in $\eta_-$. Recall the series definition of the modified Bessel function $I_0$, we have
\[
E(\eta_- \lambda_j, \eta_- \lambda_k) = \sum_{r,s=0}^{\infty} \beta_{r,s}(\eta_- \lambda_j)^r (\eta_- \lambda_k)^s,
\]
\[
g_a(\lambda_j) = \eta_-^{a-1} \sum_{r=0}^{\infty} \beta_{r,s}(\eta_- \lambda_j)^r,
\]
where
\[
\beta_{r,s} = \int_{\mathbb{R}^2} dx dy \frac{x-y}{x+y} e^{-\frac{1}{4}(x^2+y^2)} \frac{1}{(r!s!)^2} (x^{2r}y^{2s} - x^{2s}y^{2r}).
\]
Introduce an infinite matrix
\[
B = [\beta_{r,s}]_{r,s=0}^{\infty} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}
\]
with $B_{11}$ denoting its left-upper sub-matrix of order $N + n$, and a Vandermonde matrix
\[
V = [(\eta_- \lambda_r)^{s-1}]_{r=1,\ldots,N,s=1,\ldots,\infty} =: [V_1, V_2]
\]
where $V_1$ is the subblock consisting of first $N + (n - 1)! \eta_-^{n-1}$ columns. Let
\[
D = \text{diag}(1, \eta_-, \ldots, (n - 1)! \eta_-^{n-1}),
\]
we then obtain
\[
\begin{bmatrix} E(\eta_- \lambda_j, \eta_- \lambda_k) & g_b(\lambda_j) \\ -g_a(\lambda_k) & \alpha_{a,b} \end{bmatrix} = \begin{bmatrix} V_1 & V_2 \\ D & 0 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} V_1^t & D \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} V_1 D & [V_1^t D] \\ V_2 D & [V_2 D] \end{bmatrix} + \begin{bmatrix} V_1^t & 0 \\ 0 & 0 \end{bmatrix} B_{21} \begin{bmatrix} V_1 D & [V_1^t D] \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} V_2^t & 0 \\ 0 & 0 \end{bmatrix} B_{22} \begin{bmatrix} V_2^t & 0 \\ 0 & 0 \end{bmatrix}.
\]
Recalling the definition of $\eta_-$ in (1.4), we arrive at
\[
\lim_{\tau \to 0} \frac{1}{\eta_-(M-1)^{1/2}} \text{Pf} \left[ \begin{array}{c|c} \mathcal{E}(\eta_-, \lambda_j, \eta_-, \lambda_k) & g_{\alpha, \beta} \\ \hline -g_{\alpha, \beta} & \end{array} \right]_{j, k = 1, \ldots, N; a, b = 1, \ldots, n}
\]
\[
= \text{Pf}(\widetilde{V} B_{11} \tilde{V}^t) = \det(\widetilde{V}) \text{Pf}(B_{11}) \propto \prod_{k=1}^{N} \lambda_k^n \Delta_N(\lambda),
\]
where
\[
\widetilde{V} = \left[ \begin{array}{c|c} (\lambda_j^{-1})_{i=1,\ldots,N; j=1,\ldots,N+n} & \\
\hline \text{diag}(1, 1!, \ldots, (n-1)!) & 0_{n \times N} \end{array} \right].
\]

Together with (2.3), (2.6) and (2.10), we have shown that when $\tau \to 0$ (1.5) in Theorem 1.1 reduces to the eigenvalue PDF for the complex spiked Wishart matrix.

3. Double integrals for correlation kernels

In order to derive a compact expression of correlation kernel associated with the Pfaffian point process (1.5), we need to generalize the result of [42, Theorem 1.1] which is inspired by the results of [50, Sect. 7]. For a determinantal analog of the following proposition, see [2, Proposition 3.1].

**Proposition 3.1.** Let $N > 0$ and $n$ be nonnegative integers such that $M = N + n$ is even. Let $(\Omega, \nu)$ be a measure space, let $\phi_1, \ldots, \phi_N, h_1, \ldots, h_n$ be functions from $\Omega$ to $\mathbb{R}$, and let $\epsilon$ be an antisymmetric function from $\Omega \times \Omega$ to $\mathbb{R}$. For any probability distribution on $\Omega^N$ with density of the form
\[
f(x_1, \ldots, x_N) \propto \det \left[ \epsilon(x_j, x_k) \right]_{1 \leq j, k \leq N} \text{Pf} \left[ \begin{array}{c|c} \epsilon(x_j, x_k) & h_{\alpha, \beta} \\ \hline -h_{\alpha, \beta} & \end{array} \right]_{1 \leq j, k \leq N; 1 \leq a, b \leq n},
\]
where $[\alpha_{a, b}]_{1 \leq a, b \leq n}$ is an antisymmetric matrix, introduce
\[
\epsilon \phi_k(x) = \begin{cases} 
\int_{\Omega} \epsilon(x, z) \phi_k(z) d\nu(z), & k = 1, \ldots, N, \\
n_{k-N}(x), & k > N,
\end{cases}
\]
and the Gram matrix $G = [g_{jk}]_{1 \leq j, k \leq M}$ with
\[
g_{jk} = \begin{cases} 
\int_{\Omega} \phi_j(x) \epsilon \phi_k(x) d\nu(x), & 1 \leq j \leq N, 1 \leq k \leq M, \\
-\int_{\Omega} \epsilon \phi_j(x) \phi_k(x) d\nu(x), & N < j \leq M, 1 \leq k \leq N, \\
\alpha_{j, k}, & N < j, k \leq M.
\end{cases}
\]
Assume that $G$ is invertible and let $C = G^{-1}$, then (3.1) forms a Pfaffian point process with correlation kernel
\[
K_N(x, y) = \begin{bmatrix} DS_N(x, y) & S_N(x, y) \\
-S_N(y, x) & IS_N(x, y) \end{bmatrix},
\]
(3.4)
where

\[ DS_N(x, y) = \sum_{j,k=1}^{N} \phi_j(x) c_{kj} \phi_k(y), \quad (3.5) \]

\[ S_N(x, y) = \sum_{j=1}^{N} \phi_j(x) \sum_{k=1}^{N+n} c_{kj} \phi_k(y), \quad (3.6) \]

\[ IS_N(x, y) = -\epsilon(x, y) + \sum_{j,k=1}^{N+n} \epsilon \phi_j(x) c_{kj} \phi_k(y). \quad (3.7) \]

**Proof.** To matrix \([K_N(x, x_k)]_{1 \leq j, k \leq N}\), interchange the \((2k - 1)\)-th and \((k - 1)\)-th columns for any \(k\), and also interchange the corresponding rows, then its determinant doesn’t change. That is, with a \(2 \times 2\) block form \(C = [C_{i,j}]_{i,j=1,2}\),

\[
\det [K_N(x, x_k)]_{1 \leq j, k \leq N} = \det \left[ \frac{\sum_{r,s=1}^{N} \phi_r(x) c_{sr} \phi_s(x_k)}{\sum_{r=1}^{N+n} \phi_r(x) c_{sr} \phi_s(x_k)} \right] \]

\[
= \det \left[ \begin{array}{c|c} \Phi C_{1,1}^t \Phi^t & \Phi C_{2,1}^t H^t \\ \hline \Phi C_{1,2}^t \Phi^t + HC_{1,2}^t \Phi^t & -E + \epsilon \Phi C_{1,2}^t \Phi^t + \epsilon \Phi C_{1,2}^t H^t + HC_{1,2} \end{array} \right]
\]

where \(\Phi = [\phi_k(x_k)]_{1 \leq j, k \leq N}, \phi \Phi = [\epsilon \phi_k(x_k)], H = [h_k(x_k)]_{1 \leq j, k \leq N_{1 \leq k \leq n}}\) and \(E = [\epsilon(x, x_k)]\). Notice that, for any given \(x_1, \ldots, x_N\) such that \(\Phi\) is invertible, \(\epsilon \Phi\) is a linear transformation of \(\Phi\). Thus, we obtain

\[
\det [K_N(x, x_k)] = \det \left[ \begin{array}{c|c} \Phi C_{1,1}^t \Phi^t & \Phi C_{2,1}^t H^t \\ \hline \Phi C_{1,2}^t \Phi^t + HC_{1,2}^t \Phi^t & -E + \epsilon \Phi C_{1,2}^t \Phi^t + \epsilon \Phi C_{1,2}^t H^t + HC_{1,2} \end{array} \right]
\]

\[
= \det \left\{ \begin{array}{cc} \Phi & \Phi C_{2,1}^t H^t \\ \hline \Phi C_{1,2}^t \Phi^t + HC_{1,2}^t \Phi^t & -E + \epsilon \Phi C_{1,2}^t \Phi^t + \epsilon \Phi C_{1,2}^t H^t + HC_{1,2} \end{array} \right\}
\]

\[
= (\det \Phi)^2 \lim_{\delta \rightarrow 0} \det \left[ \delta I_N + C_{1,1} \right] \left[ \begin{array}{c|c} C_{1,2}^t H^t & \Phi C_{2,1}^t H^t \\ \hline HC_{2,1} & E + H C_{2,2} \end{array} \right] \left[ \begin{array}{c} \Phi^t \Phi \Phi^t \Phi \\ \hline \Phi C_{2,1}^t \Phi^t + HC_{2,2}^t \Phi^t \end{array} \right]
\]

Here we can always let \(\delta\) change in such a way that all involved inverses below exist.

To continue these calculations, we will make use of the well-known fact about the Schur complement of a \(2 \times 2\) block matrix. That is, let \(A_1 \in \mathbb{R}^{N \times N}, A_2, A_3 \in \mathbb{R}^{N \times n}\) and \(A_4 \in \mathbb{R}^{n \times n}\), if all involved inverses exist, then

\[
\det \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \det(A_1) \det(A_4 - A_3 A_1^{-1} A_2) = \det(A_4) \det(A_1 - A_2 A_4^{-1} A_3), \quad (3.8)
\]

and

\[
\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}^{-1} = \begin{bmatrix} A_1^{-1} + A_1^{-1} A_2 (A_4 - A_3 A_1^{-1} A_2)^{-1} A_3 A_1^{-1} & -A_1^{-1} A_2 (A_4 - A_3 A_1^{-1} A_2)^{-1} \\ -A_1^{-1} (A_4 - A_3 A_1^{-1} A_2)^{-1} A_3 A_1^{-1} & (A_4 - A_3 A_1^{-1} A_2)^{-1} \end{bmatrix}, \quad (3.9)
\]
By use of the formula (3.8) we get
\[
\det [K_N(x_j, x_k)] = (\det \Phi)^2 \lim_{\delta \to 0} \det[\delta I_N + C_{1,1}] \det [E + H((\delta I_n + C_{2,2}) - C_{2,1}(\delta I_N + C_{1,1})^{-1}C_{1,2})H^t]
\]
\[
= (\det \Phi)^2 \lim_{\delta \to 0} \frac{\det(\delta I_N + C_{1,1})}{\det((\delta I_n + C_{2,2}) - C_{2,1}(\delta I_N + C_{1,1})^{-1}C_{1,2})}
\times \det \left[ E \begin{bmatrix} -H^t & \alpha \end{bmatrix} \right].
\] (3.10)

On the other hand, we easily see from the formula (3.9) that
\[
G = \lim_{\delta \to 0} \left[ \begin{array}{c|c} \delta I_N + C_{1,1} & -C_{1,2} \\ \hline C_{2,1} & \delta I_n + C_{2,2} \end{array} \right]^{-1}
\]
\[
= \left[ \begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & \lim_{\delta \to 0} \left( (\delta I_n + C_{2,2}) - C_{2,1}(\delta I_N + C_{1,1})^{-1}C_{1,2} \right)^{-1} \end{array} \right].
\]

Compare the right-lower block of both sides and we arrive at
\[
\lim_{\delta \to 0} \left( (\delta I_n + C_{2,2}) - C_{2,1}(\delta I_N + C_{1,1})^{-1}C_{1,2} \right)^{-1} = \alpha. \quad (3.11)
\]

Combination of (3.10) and (3.11) gives us
\[
\text{Pf} [K_N(x_j, x_k)] = \sqrt{\det [K_N(x_j, x_k)]} \propto \det \Phi \text{Pf} \left[ E \begin{bmatrix} -H^t & \alpha \end{bmatrix} \right]. \quad (3.12)
\]

This means that the Pfaffian Pf\([K_N(x_j, x_k)]_{1 \leq j,k \leq N}\) exactly defines the same probability distribution as in (3.11).

Next, we need to verify that the kernel \(K_N(x, y)\) indeed defines a Pfaffian point process. It is sufficient to calculate the integral identity
\[
\int_{\Omega} \text{Pf}[K_N(x_j, x_k)]_{j,k=1}^l d\nu(x_l) = (N - l + 1) \text{Pf}[K_N(x_j, x_k)]_{j,k=1}^{-1}.
\]

Actually, this can be completed by expanding \(\text{Pf}[K_N(x_j, x_k)]_{1 \leq j,k \leq l}\) along the bottom two rows, with the help of the following integrals:
\[
\int_{\Omega} -S_N(x_l, x_l) d\nu(x_l) = -N, \quad (3.13)
\]
\[
\int_{\Omega} -S_N(x_k, x_l)DS_N(x_l, x_j) d\nu(x_l) = -DS_N(x_k, x_j), \quad (3.14)
\]
\[
\int_{\Omega} -S_N(x_k, x_l)S_N(x_l, x_j) d\nu(x_l) = -S_N(x_k, x_j), \quad (3.15)
\]
\[
\int_{\Omega} IS_N(x_l, x_k)DS_N(x_l, x_j) d\nu(x_l) = 0, \quad (3.16)
\]
\[
\int_{\Omega} IS_N(x_l, x_k)S_N(x_l, x_j) d\nu(x_l) = 0. \quad (3.17)
\]

These integrals can be easily verified according to the definitions (3.5)- (3.7). \(\square\)
Proposition 3.1 affords us an explicit approach of expressing correlation kernel for a Pfaffian point process. However, it is usually hard to give a nice formula suitably for asymptotic analysis. Our subsequent goal is to derive a double contour integral for correlation kernel associated with the squared singular values PDF (1.3). For this, we find that it is much convenient by introducing an extended ensemble.

**Theorem 3.2.** With the notations in Proposition 3.1, let $\Omega = (0, \infty)$ and $\nu$ be the Lebesgue measure. Let $\epsilon(u, v) = \mathcal{E}(\eta_u, \eta_v)$, $\phi_j(u) = e^{-(\eta_u + \sigma_j + \eta_v)u}$ for $j = 1, \ldots, N$ and for $1 \leq a, b \leq n$

\[
h_a(u) = \eta_a^{-1} \int_{\mathbb{R}^2} dx dy e^{-\frac{1}{2}(x^2 + y^2)} \frac{x - y}{x + y} \left( I_0(2y\sqrt{\eta_u})e^{\theta_{a+N}x^2} - I_0(2x\sqrt{\eta_u})e^{\theta_{a+N}y^2} \right),
\]

\[
o_{a,b} = \eta_a^{a+b-2} \int_{\mathbb{R}^2} dx dy e^{-\frac{1}{2}(x^2 + y^2)} \frac{x - y}{x + y} \left( e^{\theta_{a+N}x^2}e^{\theta_{a+N}y^2} - e^{\theta_{b+N}x^2}e^{\theta_{b+N}y^2} \right).
\]

If $\varrho_1, \ldots, \varrho_{N+n} \in (0,1/2)$ are pairwise distinct, then

\[
S_N(u, v) = \int_0^\infty \mathcal{E}(\eta_u, \eta_v) \widetilde{DS}_N(u, t) dt,
\]

\[
IS_N(u, v) = -\mathcal{E}(\eta_u, \eta_v) + \int_0^\infty \int_0^\infty \mathcal{E}(\eta_u, \eta_v) \widetilde{DS}_N(s,t) \mathcal{E}(\eta_u, \eta_s) ds dt,
\]

and

\[
DS_N(u, v) = \eta_u^{-2} \int_{C(\epsilon_1, \ldots, \epsilon_N)} \frac{dw}{2\pi i} \int_{C(1-\epsilon_1, \ldots, 1-\epsilon_N)} \frac{dz}{2\pi i} e^{-\frac{w_u - \eta_u}{w - \eta}} \frac{1}{\sqrt{(2z - 1)(1 - 2w)}} \times \frac{1 - z - w}{w - z} \prod_{k=1}^M \frac{z - \varrho_k}{w - \varrho_k} \frac{1 - w - \varrho_k}{1 - z - \varrho_k},
\]

while $\widetilde{DS}_N = DS_N$ but with the contour $C(1-\epsilon_1, \ldots, 1-\epsilon_N) \mapsto C(1-\varrho_1, \ldots, 1-\varrho_{N+n})$ and $\widetilde{\widetilde{DS}}_N = \widetilde{DS}_N$ but with the contour $C(\epsilon_1, \ldots, \epsilon_N) \mapsto C(\varrho_1, \ldots, \varrho_{N+n})$.

**Proof.** First, we need to calculate the Gram matrix $G = [g_{jk}]_{j,k \leq M}$ in (3.3).

When $1 \leq j, k \leq N$, using the simple fact (cf. [39, 10.43.23])

\[
\int_0^\infty I_0(\beta \sqrt{t}) e^{-\omega t} dt = \frac{1}{\alpha \sqrt{\pi}},
\]

we see from Fubini’s theorem that

\[
g_{jk} = \int_{\mathbb{R}^2} dx dy \phi_j(u) \phi_k(v) \mathcal{E}(\eta_u, \eta_v)
\]

\[
= \int_{\mathbb{R}^2} dx dy e^{-\frac{1}{2}x^2 - \frac{1}{2}y^2} \frac{x - y}{x + y} \int_{\mathbb{R}^2} dx dy e^{-(\eta_1 + \sigma_j + \eta_k)u} e^{-(\eta_1 + \sigma_k + \eta_v)v}
\]

\[
\times \left( I_0(2y\sqrt{\eta_u})I_0(2y\sqrt{\eta_v}) - I_0(2\eta_1\sqrt{\eta_u})I_0(2\eta_1\sqrt{\eta_v}) \right)
\]

\[
= \frac{1}{(\eta_1 + \sigma_j + \eta_k)(\eta_1 + \sigma_k + \eta_v)} \int_{\mathbb{R}^2} dx dy e^{-\frac{1}{2}x^2 - \frac{1}{2}y^2} \frac{x - y}{x + y} \left( e^{\eta_1 x^2 + \eta_k y^2} - e^{\eta_1 y^2 + \eta_k x^2} \right),
\]
where \( g_j \) is given by (1.11). Apply Lemma 3.4 to the above integral, we obtain
\[
g_{jk} = \frac{4\pi}{\eta^2} \frac{\theta_j \theta_k (\theta_j - \theta_k)}{(1 - \theta_j - \theta_k) \sqrt{1 - 2\theta_j} \sqrt{1 - 2\theta_k}}, \quad 1 \leq j, k \leq N. \tag{3.22}
\]

Similarly, when \( 1 \leq j \leq N < k \leq N + n \), we obtain
\[
g_{jk} := \int_{\mathbb{R}_+} du \phi_j(u) h_{k-N}(u) = \eta_{k-N-1}^{k-N-1} \int_{\mathbb{R}^2} dx dy e^{\frac{1}{2} (x^2 + y^2)} \frac{x - y}{x + y}
\times \int_{\mathbb{R}_+} du e^{-\left(\eta_j \sigma_j + \eta_k \sigma_k\right) u} \left( I_0(2x \sqrt{\eta_j u}) e^{\eta_j y^2} - I_0(2y \sqrt{\eta_k u}) e^{\eta_k x^2} \right)
= 4\pi \eta_{k-N}^{k-N-2} \frac{\theta_j (\theta_j - \theta_k)}{(1 - \theta_j - \theta_k) \sqrt{1 - 2\theta_j} \sqrt{1 - 2\theta_k}}. \tag{3.23}
\]

Also, when \( N < j, k \leq N + n \), it’s easy to see from Lemma 3.4 that
\[
g_{jk} := \alpha_{j-N,k-N} = 4\pi \eta_j^{j+k-2N-2} \frac{\theta_j - \theta_k}{(1 - \theta_j - \theta_k) \sqrt{1 - 2\theta_j} \sqrt{1 - 2\theta_k}}. \tag{3.24}
\]

The we can rewrite the Gram matrix as
\[
G = 4\pi D_1 D_2 \tilde{G} D_2 D_1, \tag{3.25}
\]
where
\[
D_1 = \text{diag}(\theta_1/\eta, \ldots, \theta_N/\eta, 1, \eta, \ldots, \eta^{n-1}), \tag{3.26}
\]
\[
D_2 = \text{diag}((1 - 2\theta_1)^{-1/2}, \ldots, (1 - 2\theta_{N+n})^{-1/2}), \tag{3.27}
\]
\[
\tilde{G} = [\tilde{g}_{jk}] := \left[ \begin{array}{c} \theta_j - \theta_k \\ 1 - \theta_j - \theta_k \end{array} \right]_{1 \leq j, k \leq N + n}. \tag{3.28}
\]

Let \( \tilde{C} = [\tilde{c}_{jk}] \) be the inverse of matrix \( \tilde{G} \), we claim that there exists a system of identical relations between entries of \( \tilde{C} \). For this, introduce some rational functions as follows
\[
f_j(z) = \sum_{l=1}^{N+n} \frac{z - \theta_l}{1 - z - \theta_l} \tilde{c}_{lj}, \quad j = 1, \ldots, N + n. \tag{3.29}
\]

For fixed \( j \), it immediately follows from the orthogonal relations
\[
f_j(\theta_k) = \sum_{l=1}^{N+n} \tilde{g}_{kl} \tilde{c}_{lj} = \delta_{kj} \tag{3.30}
\]
that \( f_j(z) \) has \( N + n - 1 \) zeros at \( \theta_1, \ldots, \theta_j-1, \theta_{j+1}, \ldots, \theta_{N+n} \). On the other side, since \( \tilde{C} \) is anti-symmetric which means \( \tilde{c}_{jj} = 0 \), it has \( N + n - 1 \) poles at \( 1 - \theta_1, \ldots, 1 - \theta_{j-1}, 1 - \theta_{j+1}, \ldots, 1 - \theta_{N+n} \). Since \( \theta_k \)'s are pairwise distinct, recalling that a rational function is uniquely determined by its zeros, poles, and the value of the function at one extra point,
we get

\[
f_j(z) = \prod_{k \neq j, k=1}^{N+n} \frac{z - \varrho_k}{1 - z - \varrho_k} \frac{1 - \varrho_j - \varrho_k}{\varrho_j - \varrho_k}. \tag{3.31}
\]

In order to obtain explicit form of correlation kernels as in Proposition 3.1, recalling the definition (3.2) and \( C = [c_{jk}] \) being the inverse of \( G \), we need to evaluate the following summation for each \( j \)

\[
\sum_{k=1}^{N+n} c_{kj} \phi_k(v) = \int_0^\infty dt \mathcal{E}(\eta_v, \eta_t) \frac{1}{4 \pi d_{1j} d_{2j}} \sum_{k=1}^{N+n} \tilde{c}_{kj} \sqrt{1 - 2 \varrho_k e^{-\frac{\eta_t}{\varrho_k}}} \int dt \mathcal{E}(\eta_v, \eta_t) e^{-\frac{\eta_t}{\varrho_a+N}}. \tag{3.32}
\]

By Cauchy’s residue theorem, it is easy to see that for \( 1 \leq k \leq N + n \)

\[
\sqrt{1 - 2 \varrho_k} e^{-\frac{\eta_t}{\varrho_k}} = -\frac{1}{2 \pi i} \int_{C_{\varrho_k}} \frac{dz}{z - \varrho_k} \frac{1}{1 - \varrho_k} e^{-\frac{\eta_t}{\varrho_k}}, \tag{3.33}
\]

where \( C_{\varrho_k} \) is an anticlockwise contour encircling \( 1 - \varrho_k \). Noting

\[
\frac{1}{d_{1j} d_{2j}} = \frac{\eta_j}{\varrho_j} \sqrt{1 - 2 \varrho_j}, \quad j = 1, \ldots, N,
\]

when \( 1 \leq j \leq N \) we obtain

\[
\sum_{k=1}^{N+n} c_{kj} \phi_k(v) = \int_0^\infty dt \mathcal{E}(\eta_v, \eta_t) \frac{-\eta_t^2}{2 \pi i} \frac{\sqrt{1 - 2 \varrho_j}}{\varrho_j} \times \int_{C_{\varrho_k}} \frac{dz}{z - \varrho_k} \frac{1}{\sqrt{2z - 1}} e^{-\frac{\eta_t}{\varrho_k}}, \tag{3.34}
\]

Substituting the evaluation determined by (3.29) and (3.31) into the above shows

\[
\sum_{k=1}^{N+n} c_{kj} \phi_k(v) = \int_0^\infty dt \mathcal{E}(\eta_v, \eta_t) \frac{-\eta_t^2}{2 \pi i} \frac{\sqrt{1 - 2 \varrho_j}}{\varrho_j} \int_{C_{\varrho_k}} \frac{dz}{z - \varrho_k} \frac{1}{\sqrt{2z - 1}} \prod_{k \neq j, k=1}^{N+n} \frac{z - \varrho_k}{1 - \varrho_j - \varrho_k} \frac{1 - \varrho_j - \varrho_k}{\varrho_j - \varrho_k}. \tag{3.35}
\]

Apply the residue theorem for the following function of variable \( w \) at all \( \varrho_k \) and we arrive at

\[
S_N(u, v) = \int_0^\infty \mathcal{E}(\eta_v, \eta_t) \tilde{S}_N(u, t) dt, \tag{3.36}
\]
with
\[
\tilde{D}_{SN}(u, v) = \eta^2 \sqrt{\pi} \int_{\mathcal{C}_{(e_1, \ldots, e_N)}} \frac{dw}{2\pi i} \int_{\mathcal{C}_{(e_1, \ldots, \bar{e}_{N+n})}} \frac{dz}{2\pi i} e^{\frac{\eta s - v}{\eta s}} \frac{1}{\sqrt{(2z - 1)(1 - 2w)}}
\times \frac{1 - z - w}{w - z} \frac{1}{1 - w - \varrho_k} \frac{1}{1 - z - \varrho_k},
\] (3.37)

To find an integral representation for \( D_{SN}(u, v) \), introduce some auxiliary functions
\[
0 = \frac{-1}{2\pi i} \int_{\mathcal{C}_{(e_1, \ldots, \bar{e}_{N+N})}} dz \frac{z - e^\varrho_k}{1 - z - e^\varrho_k} \frac{1}{\sqrt{2z - 1}} \frac{1}{1 - z} e^\frac{\eta s}{\eta s^2}, \quad k > N. \tag{3.38}
\]
Combining (3.33) and using similar argument as in the derivation of \( S_{N}(u, v) \), we will see that \( D_{SN}(u, v) \) is just equal to \( D_{SN}(u, v) \) except that the \( z \)-contour \( \mathcal{C}_{(e_1, \ldots, \bar{e}_{N+N})} \) is changed to \( \mathcal{C}_{(e_1, \ldots, \bar{e}_{N})} \). In fact,
\[
D_{SN}(u, v) = \eta^2 \sum_{j=1}^{\nu} \sqrt{1 - 2\varrho_j} e^{-\eta s} \sum_{k=1}^{\nu} \hat{c}_{kj} \sqrt{1 - 2\varrho_k} e^{-\eta s}
\times \int_{\mathcal{C}_{(e_1, \ldots, \bar{e}_{N})}} \frac{dz}{2\pi i} e^{-\frac{\eta s}{\eta s^2}} \frac{1}{\sqrt{2z - 1}} \frac{1}{1 - z} \sum_{k=1}^{\nu} \hat{c}_{kj} \frac{z - e^\varrho_k}{1 - z - e^\varrho_k}
\]
from which the integral representation of \( D_{SN}(u, v) \) immediately follows via Cauchy’s residue theorem.

Finally, we turn to the representation of \( I_{SN}(u, v) \). Similar to the computation of \( S_{N}(u, v) \), we obtain
\[
\sum_{j=1}^{\nu} \phi_j(u) \sum_{k=1}^{\nu} c_{kj} \tilde{c}_{kj}(v) = \eta^2 \int_0^\infty ds \int_0^\infty dt E(\eta - u, \eta - s) E(\eta - v, \eta - t)
\times \sum_{j=1}^{\nu} \frac{1}{\varrho_j} \sqrt{1 - 2\varrho_j} e^{-\frac{\eta s}{\eta s^2}} \sum_{j=1}^{\nu} \hat{c}_{kj} \frac{1}{\varrho_k} \sqrt{1 - \varrho_k} e^{-\frac{\eta t}{\eta t}}
\]
\[
= \int_0^\infty \int_0^\infty E(\eta - u, \eta - s) \tilde{D}_{SN}(s, t) E(\eta - v, \eta - t) ds dt \tag{3.39}
\]
with a defined contour \( \tilde{D}_{SN}(u, v) \).

Remark 3.3. When all \( \varrho_{N+1}, \ldots, \varrho_{N+n} \to 0 \) in the integral representation of \( D_{SN}(u, v) \) (3.38), we can choose contours of \( D_{SN}(u, v) \) such that \( \mathcal{C}_{(e_1, \ldots, e_N)} \) encircles \( \varrho_1, \ldots, \varrho_N \) but
not 0 while \( C_{(1-\eta_1,...,1-\eta_N)} \) encircles \( 1-\eta_1,\ldots,1-\eta_N \) but not 1. This is,

\[
DS_N(u, v) = n^2 \int \left( \frac{dw}{2\pi i} \int \frac{dz}{2\pi i} \right) \frac{1}{\sqrt{(2z-1)(1-2w)}} \times \frac{1}{w-z} \frac{1}{(1-z)w} \left( \frac{z}{w} \right)^n \prod_{k=1}^N \frac{1}{w-\eta_k} \equiv 0 \quad \text{by noting the factor on the RHS of (3.35)}.
\]

The reason is that the LHS of (3.33) goes to zero and we can choose the \( z \)-contour on the RHS not encircling 1. Similarly, we can choose the \( w \)-contour that does not contain 0 by noting the factor on the RHS of (3.35).

**Lemma 3.4.** For real numbers \( \alpha, \beta < 1/2 \), we have

\[
\int_{\mathbb{R}^2} \int dy \ e^{-\frac{1}{2}x^2-\frac{1}{2}y^2} \frac{x-y}{x+y} e^{\alpha y^2+\beta y^2} = \frac{4\pi(\alpha-\beta)}{\sqrt{1-2\alpha-2\beta}}.
\]

*Proof.* After change of variables \( x = u + v, y = u - v \), the LHS of (3.41) becomes

\[
2 \int_{-\infty}^{\infty} du \frac{1}{u} e^{-(1-\alpha-\beta)u^2} \left( \int_{-\infty}^{\infty} dv \ e^{-(1-\alpha-\beta)v^2} (e^{2(\alpha-\beta)uv} - e^{-2(\alpha-\beta)uv}) \right).
\]

Integrate out the variable \( v \) with the help of the expectation of normal variables first, then the LHS of (3.41) further reduces to

\[
\frac{4\pi(\alpha-\beta)}{\sqrt{1-2\alpha-2\beta}} \int_{-\infty}^{\infty} du \ e^{-(1-\alpha-\beta)u^2},
\]

from which the requested integral formula immediately follows. \[\square\]

At last, with previous preparation we can easily complete the proof of Theorem 1.2 as follows.

*Proof of Theorem 1.2.* With Remark 3.3 in mind, let \( \eta_{N+1}, \ldots, \eta_{N+n} \to 0 \) in Theorem 3.2 and we thus complete the proof. \[\square\]

4. Scaling limits

We use the double integral representations for correlation kernels to investigate local statistical properties of singular values, first at the hard edge limit and leaving bulk and soft edge limits in the forthcoming article.

*Proof of Theorem 1.3.* According to (1.10) it is sufficient to prove scaling limits of the three sub-kernels for \( K_N(u, v) \).

For this, let’s deform the integral contours \( C_{(\eta_1,...,\eta_N)} \) and \( C_{(1-\eta_1,...,1-\eta_N)} \) as illustrated in Figure 4 into new contours \( \tilde{C}_L \) and \( \tilde{C}_R \) respectively illustrated in Figure 3. For simplicity, let \( f_{u,v}(w, z) \) denote the integrand in (1.13). We rewrite \( DS_N(u, v) \) as

\[
DS_N(u, v) = \left( \text{PV} \int_{\tilde{C}_L} dw \int_{\tilde{C}_R} dz - \int_{\tilde{C}_L^+} dw \int_{\tilde{C}_R} dz - \int_{\tilde{C}_L^-} dw \int_{\tilde{C}_R} dz \right) f_{u,v}(w, z)
\]

\[
:= I_1(u, v) - I_2(u, v) - I_3(u, v),
\]

where \( \tilde{C}_L \) is a contour starting at \( e_0 \), moving across \( e_4, e_3, e_1, e_2, \tilde{e}_3, \tilde{e}_4 \) in turn and finally returning to \( e_0 \), while \( \tilde{C}_R \) starting at \( e_0' \), moving across \( e_4, e_3, e_1', e_2, e_3' \) and returning
to \( e'_0 \); \( \tilde{C}_{L+} \) is a contour connecting \( e_0, e_4, e_3, e_1 \) and \( e_0 \), while \( \tilde{C}_{L-} \) is the complex conjugate curve of \( \tilde{C}_{L+} \). Here as will seen below we can choose \( e_3 = (1 + i)/2 \).

\[
\begin{array}{c}
\includegraphics{figure3.png}
\end{array}
\]

**Figure 3.** Deformed contours of double integrals

First, we prove the scaling limit for \( DS_N \). According to the assumption we see that both \( \tau/(1 + \tau) \) and \( 1/(1 + \tau) \) approach 1 as \( N \to \infty \). Throughout this process the points \( e_0 \) and \( e'_0 \) come close to \((1/2, 0)\) respectively from left and right. Therefore, both contours \( \tilde{C}_L \) and \( \tilde{C}_R \) are forced to change to the curves as Figure 2 by letting \( e_4 \) and \( e'_4 \) approach \((1/2, 0)\) and noting that they encircle \( \tau/(1 + \tau) \) and \( 1/(1 + \tau) \) respectively. On the other hand, we get from the assumptions of \( \sigma_{n+1} = \cdots = \sigma_N = 1 \) and \( N(1 - \tau) \to \kappa \in (0, \infty) \) that

\[
\left( \frac{z - \frac{\tau}{1 + \tau}}{w - \frac{\tau}{1 + \tau}} \right)^{N-n} = \left( 1 + \frac{1 - \tau}{(1 + \tau)z - 1} \right)^{N-n} \left( 1 + \frac{1 - \tau}{(1 + \tau)w - 1} \right)^{-N+n}
\]

as \( N \to \infty \). Hence, the rescaled integral

\[
\frac{4}{N^2} I_1\left( \frac{2u}{N}, \frac{2v}{N} \right)
\]

converges to the Principal value integral on the right-hand side of (1.16).

To \( I_2 \) and \( I_3 \), we first integrate out variable \( w \). It is easy to see that only the curve segments from \( \tilde{C}_R \) lying inside \( \tilde{C}_{L+} \) lead to nonzero contribution, so applying Cauchy’s residue theorem gives us

\[
\frac{4}{N^2} \left( I_2\left( \frac{2u}{N}, \frac{2v}{N} \right) + I_3\left( \frac{2u}{N}, \frac{2v}{N} \right) \right) = \frac{4\eta_2}{N^2} \int_{e_4}^{e_4'} \frac{dz}{2\pi i} e^{-\frac{2\pi i}{N} - \frac{\pi i}{N - \pi} - \frac{\pi i}{1 - \pi}} \frac{1}{i(1 - z)}
\]

\[
+ \frac{4\eta_2}{N^2} \int_{e_4}^{e_4'} \frac{dz}{2\pi i} e^{-\frac{2\pi i}{N} - \frac{\pi i}{N - \pi} - \frac{\pi i}{1 - \pi}} \frac{1}{-i(1 - z)}.
\]

Note that as \( N \to \infty \) the point \( e_4 \) converges to \((1/2, 0)\), \( \tau \sim 1 - \kappa/N \) and \( \eta_\tau = \tau/(1 - \tau^2) \), we have

\[
\lim_{N \to \infty} \frac{4}{N^2} \left( I_2\left( \frac{2u}{N}, \frac{2v}{N} \right) + I_3\left( \frac{2u}{N}, \frac{2v}{N} \right) \right) = \frac{1}{\pi \kappa^2} \int_0^1 \frac{1}{1 + z^2} e^{-\frac{2\pi i}{N} - \frac{\pi i}{N - \pi} - \frac{\pi i}{1 - \pi}} \sin \left( \frac{2(v - u)z}{\kappa(1 + z^2)} \right) dz.
\]

Thus the proof of \( DS_N \) is complete.
Recalling (1.14) and (1.15), and noting
\[ E\left(\frac{2\eta_N}{u}, \frac{2\eta_N}{v}\right) \rightarrow E(u/\kappa, v/\kappa), \]
we easily arrive at
\[
\lim_{N \to \infty} \frac{2}{N} S_N\left(\frac{2u}{N}, \frac{2v}{N}\right) = \lim_{N \to \infty} \frac{4}{N^2} \int_0^\infty E\left(\frac{2\eta_N}{v}, \frac{2\eta_N}{t}\right) DS_N\left(\frac{2u}{N}, \frac{2t}{N}\right) dt
\]
\[
= \int_0^\infty E\left(\frac{v}{\kappa}, \frac{t}{\kappa}\right) DS(u, t) dt,
\]
and
\[
\lim_{N \to \infty} IS_N\left(\frac{2u}{N}, \frac{2v}{N}\right) = -E\left(\frac{u}{\kappa}, \frac{v}{\kappa}\right) + \int_0^\infty \int_0^\infty E\left(\frac{v}{\kappa}, \frac{t}{\kappa}\right) DS(s, t) E\left(\frac{u}{\kappa}, \frac{s}{\kappa}\right) ds dt.
\]

Acknowledgements: D.-Z. Liu was supported by the Natural Science Foundation of China # 11771417, the Youth Innovation Promotion Association CAS #2017491, Anhui Provincial Natural Science Foundation #1708085QA03 and the Fundamental Research Funds for the Central Universities #WK0010450002.

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