UNIVERSE EVOLUTION IN A 5D RICCI-FLAT COSMOLOGY

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We use Wetterich’s parameterization equation of state (EOS) of dark energy to a 5D Ricci-flat cosmological solution and we suppose the universe contains three major components: matter, radiation and dark energy. By using the relation between the scale factor and the redshift \( z \), we show that the two arbitrary functions contained in the 5D solution could be solved out analytically in terms of the variable \( z \). Thus the whole 5D solution could be constructed uniquely if the current values of the three density parameters \( \Omega_{m0}, \Omega_{r0}, \Omega_{x0} \), the EOS \( w_0 \), and the bending parameter \( b \) contained in the EOS are all known. Furthermore, we find that all the evolutions of the mass density \( \Omega_m \), the radiation density \( \Omega_r \), the dark energy density \( \Omega_x \), and the deceleration parameter \( q \) depend on the bending parameter \( b \) sensitively. Therefore it is deserved to study observational constraints on the bending parameter \( b \).

Keywords: cosmology; dark energy; bending parameter.

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1. Introduction

Observations of Cosmic Microwave Background (CMB) anisotropies indicate that the universe is flat and the total energy density is very close to the critical one with \( \Omega_{total} \approx 1 \). Meanwhile, observations of high redshift type Ia supernovae\(^2\) reveal the speeding up expansion of our universe and the surveys of clusters of galaxies show that the density of matter is very much less than the critical density\(^3\). These three tests nicely complement each other and indicate that an exotic component with negative pressure dubbed dark energy dominates the present universe. Various dark energy models have been proposed among them the most promising ones are probably those with a scalar field such as quintessence\(^4\), phantom\(^5\), K-essence\(^6\) and so on. For this kind of models, one can design many kinds of potentials\(^7\) and then study EOS for the dark energy. Another way is to use a parameterization of the

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EOS to fit the observational data, and then to reconstruct the potential and/or the evolution of the universe\cite{8}. The latter has the advantage that it does not depend on a specified model of the dark energy and, therefore, is also called a model-independent method\cite{16}.

Both the classical Kaluza-Klein theories and the modern string/brane theories require the existence of extra dimensions. If the universe has more than four dimensions, general relativity should be extended from 4D to higher dimensions. One of such extensions is the 5D Space-Time-Matter (STM) theory\cite{9,10} in which our universe is a 4D hypersurface floating in a 5D Ricci-flat manifold. This theory is supported by Campbell’s theorem which states that any analytical solution of the $N D$ Einstein equations can be embedded in a $(N + 1)D$ Ricci-flat manifold\cite{11}. A class of cosmological solutions of the STM theory is given by Liu, Mashhoon and Wesson\cite{12,13}. It was shown that dark energy models, similar as the 4D quintessence and phantom ones, can also be constructed in this 5D cosmological solution in which the scalar field is induced from the 5D vacuum\cite{14,15}. The purpose of this paper is to use a model-independent method to study the EOS of the dark energy and the evolution of the 5D universe.

Various parameterization of the EOS of dark energy have been presented and investigated\cite{16,17,18,19,20}. In this paper we will study one of them presented firstly by Wetterich\cite{21} where there is a bending parameter $b$ which describes the deviation of the EOS from a constant $w_0$. The paper is organized as follows. In Section 2, we introduce the 5D Ricci-flat cosmological solution and derive the densities for the three major components of the universe. In Section 3, we will reconstruct the evolution of the model with different values of the bending parameter $b$. Section 4 is a short discussion.

2. Density Parameters in the 5D Model

The 5D cosmological solutions read\cite{12}

$$dS^2 = B^2 dt^2 - A^2 \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) - dy^2,$$

with $d\Omega^2 \equiv (d\theta^2 + \sin^2 \theta d\phi^2)$ and

$$A^2 = (\mu^2 + k) y^2 + 2\nu y + \frac{\nu^2 + K}{\mu^2 + k},$$

$$B = \frac{1}{\mu} \frac{\partial A}{\partial t} = \frac{\dot{A}}{\mu},$$

(2)

where $\mu = \mu(t)$ and $\nu = \nu(t)$ are two arbitrary functions of $t$, $k$ is the 3D curvature index ($k = \pm 1, 0$), and $K$ is a constant. This solution satisfies the 5D vacuum equations $R_{AB} = 0$. So we have

$$I_1 \equiv R = 0, I_2 \equiv R^{AB} R_{AB} = 0,$$

$$I_3 = R_{ABCD} R^{ABCD} = \frac{72 K^2}{A^8},$$

(3)
which shows that $K$ determines the curvature of the 5D manifold. The Hubble and deceleration parameters are $
abla$

\[ H \equiv \frac{\dot{A}}{AB} = \frac{\mu}{A} \quad (4) \]

\[ q(t, y) \equiv -A \frac{d^2 A}{dt^2} \left( \frac{dA}{d\tau} \right)^2 = -\frac{A\dot{\mu}}{\mu A}. \quad (5) \]

Using the 4D part of the 5D metric to calculate the 4D Einstein tensor, one obtains

\[ (4) G_{00} = 3 \left( \frac{\mu^2 + k}{A^2} \right), \quad (4) G_{11} = (4) G_{22} = (4) G_{33} = 2 \frac{\mu \dot{\mu}}{A^2} \left( \frac{\mu^2 + k}{A^2} \right). \quad (6) \]

So the 4D induced energy-momentum tensor can be defined as $T_{\alpha\beta} = (4) G_{\alpha\beta}$. In this paper we consider the case where the 4D induced matter $T_{\alpha\beta}$ is composed of three components: matter $\rho_m$, radiation $\rho_r$ and dark energy $\rho_x$, which are minimally coupled to each other. So we have

\[ 3 \left( \frac{\mu^2 + k}{A^2} \right) = \rho_m + \rho_r + \rho_x, \]

\[ \frac{2\mu \dot{\mu}}{A \dot{A}} + \frac{\mu^2 + k}{A^2} = -p_m - p_r - p_x, \quad (7) \]

with

\[ \rho_m = \rho_m A_0^3 A^{-3}, \quad p_m = 0, \quad \rho_r = \rho_r A_0^4 A^{-4} = 3p_r, \quad (8) \]

\[ p_x = w_x \rho_x. \quad (9) \]

From Eqs. (7) - (9) and for $k = 0$, we obtain the EOS of the dark energy

\[ w_x = \frac{p_x}{\rho_x} = -2 \frac{\mu \dot{\mu}}{3 \mu^2 A^2 - \rho_m A_0^3 A^{-3} - \rho_r A_0^4 / 3}, \quad (10) \]

and the dimensionless density parameters

\[ \Omega_m = \frac{\rho_m}{\rho_m + \rho_r + \rho_x} = \frac{\rho_m A_0^3}{3 \mu^2 A}, \quad (11) \]

\[ \Omega_r = \frac{\rho_r}{\rho_m + \rho_r + \rho_x} = \frac{\rho_r A_0^4}{3 \mu^2 A^2}, \quad (12) \]

\[ \Omega_x = 1 - \Omega_m - \Omega_r. \quad (13) \]

where $\rho_{m0}$ and $\rho_{r0}$ are the current values of matter and radiation densities, respectively.

Consider equation (10), where $A$ is a function of $t$ and $y$. However, on a given $y = constant$ hypersurface, $A$ becomes $A = A(t)$. As noticed before, the term $\mu \dot{\mu}/A$ in (10) can now be rewritten as $d\mu/dA$. Furthermore, we use the relation

\[ A_0/A = 1 + z, \quad (14) \]
Fig. 1. EOS $w_x$ of the dark energy as a function of the redshift $z$ with its current value $w_0 = -1.1$ and the bending parameter $b = 0, 1/2, 1, 2, 4$, respectively.

and define $\mu^2/\mu^2 = f(z)$ (with $f(0) \equiv 1$), and then we find that equations (10) - (13) and (6) can be expressed in term of the redshift $z$ as

$$w_x = -\frac{1 + \Omega_r + (1 + z)dln f(z)/dz}{3(1 - \Omega_m - \Omega_r)}, \quad (15)$$

$$\Omega_m = \Omega_{m0}(1 + z)f(z), \quad (16)$$

$$\Omega_r = \Omega_{r0}(1 + z)^2f(z), \quad (17)$$

$$\Omega_x = 1 - \Omega_m - \Omega_r, \quad (18)$$

$$q = -\frac{1 + z}{2}dln f(z)/dz. \quad (19)$$

Now we conclude that if the function $f(z)$ is given, the evolutions of all the cosmic observable parameters in (15) - (19) could be determined uniquely.

3. The Function $f(z)$ and the Evolutions of Cosmic Parameters

The parameterization of EOS of the dark energy given by Wetterich has been extensively studied. \cite{24, 25} It is of the form \cite{21}

$$w_x(z, b) = \frac{w_0}{1 + b \ln(1 + z)}, \quad (20)$$

where $w_x(z, b)$ is the EOS parameter with its current value as $w_0$, and $b$ is a bending parameter describing the deviation of $w_x$ from $w_0$ as $z$ increases. Let $w_0 = -1.1$, we plot the function (20) in Fig. 1 where $b$ takes the value $0, 1/2, 1, 2, 4$, respectively. From this figure we see that $w_x$ varies with $z$ sensitively at low redshift. At high redshift, it is near to a constant. By properly choosing the two parameters $w_0$ and $b$, the transition from $w_x < -1$ to $w_x > -1$ and the last scattering point at $z \approx 1100$ can be adjusted easily to fit cosmic observations.
Consider equation (15) now. With use of (16), (17) and (20), we find that (15) is actually a nonlinear first-order differential equation. This equation can be integrated out analytically, giving the solution

$$f(z, b) = (1 + z)^{-1}[(1 + b \ln(1 + z))^{3w_0/b} + \Omega_{m0} - (1 + b \ln(1 + z))^{3w_0/b}\Omega_{m0} + \Omega_{r0} + z\Omega_{r0} - (1 + b \ln(1 + z))^{3w_0/b}\Omega_{r0}]^{-1}.$$  (21)

For $b = 0$ we have $w(z, 0) = w_0$. For $b \to 0$, we find $f(z, b) \to f(z, 0)$ with

$$f(z, 0) = (1 + z)^{-1}[(1 + z)^{3w_0} + \Omega_{m0} - (1 + z)^{3w_0}\Omega_{m0} + \Omega_{r0} + z\Omega_{r0} - (1 + z)^{3w_0}\Omega_{r0}]^{-1}.$$  (22)

Furthermore, for $z = 0$ we have $f(0, 0) = 1$ as it should be by it’s definition.

The function $f(z, b)$ shown in (21), including the bending parameter $b$ given in (20), could, in principle, be determined by observational data. Be aware that $f \equiv \mu_0^2/\mu^2$, so we arrive at a conclusion that the arbitrary function $\mu$ contained in the 5$D$ solution (1) - (2) could be determined, in terms of the redshift $z$, by observational data. As for another arbitrary function $\nu(z)$, it can be expressed via $\mu(z)$ and $A(z)$ by solving (2) itself. Therefore, $\nu(z)$ is now not arbitrary anymore but is determined by observational data too. In this way, the whole 5$D$ solution could be determined in principle.

Return back to (16) - (19). If $f(z, b)$ is known, all the three densities $\Omega_m$, $\Omega_r$, $\Omega_x$ and the deceleration parameter $q$ are also known. The evolutions of these three densities and the deceleration parameter are plotted in Figs. 2-6.

From Fig. 2 to Fig. 4 we see that as the redshift $z$ increases, both $\Omega_m$ and $\Omega_r$ increase too while $\Omega_x$ decreases, and $\Omega_r$ increases almost linearly at low redshift. The effect of the bending parameter $b$ on the three densities is sensitive. At high redshift it becomes much larger. Clearly, there is a transition at $z = z_e$ at which
Fig. 3. Evolution of the density $\Omega_r$ of radiation versus redshift $z$ with $w_0 = -1.1$, $\Omega_{m0} = 0.3$, $\Omega_{r0} = 0.00005$, $\Omega_{x0} = 0.7$.

Fig. 4. Evolution of the density $\Omega_x$ of dark energy versus redshift $z$ with $w_0 = -1.1$, $\Omega_{m0} = 0.3$, $\Omega_{r0} = 0.00005$, $\Omega_{x0} = 0.7$.

$\Omega_m = \Omega_x$. When $z < z_c$, dark energy governs the universe; when $z > z_c$, matter becomes the dominant part of the universe. For several different values of $b$ we plot the transitions $z_c$ in Fig. 5. If we traced back much earlier (around $z = 5000$), there must be another transition $z = z_r$ at which $\Omega_r = \Omega_m$ and before which (at $z > z_r$) the density $\Omega_r$ dominates the universe. From Fig. 6 we can find how the bending parameter $b$ affect the deceleration parameter $q$. The transition from deceleration to acceleration can easily be seen from this figure.

4. Conclusions

The 5D Ricci-flat cosmological solution (1) - (2) contains two arbitrary functions $\mu(t)$ and $\nu(t)$, and usually it is not easy to be determined for a real universe model. In this paper we have considered the case where the universe is composed of three major components: matter, radiation, and dark energy, and we have supposed the
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Fig. 5. This figure shows how the bending parameter $b$ affect the transition point $z_e$. The bigger the bending parameter is, the earlier the transition from matter dominated to dark energy dominated happen.

Fig. 6. Evolution of the deceleration parameter $q$ versus redshift $z$ with $w_0 = -1.1$, $\Omega_{m0} = 0.3$, $\Omega_{r0} = 0.00005$, $\Omega_{x0} = 0.7$.

equation of state of dark energy is of Wetterich’s parameterization form. Then we show that with use of the relation between the scale factor $A$ and the redshift $z$, $A = A_0/(1 + z)$, one can easily change the arbitrary function $\mu(t)$ to another arbitrary function $f(z)$. Furthermore, we show that this $f(z)$ could be integrated out analytically. Thus, if the current values of the three density parameters $\Omega_{m0}$, $\Omega_{r0}$, $\Omega_{x0}$, the EOS $w_0$, and the bending parameter $b$ contained in the EOS are all known, this $f(z)$ could be determined uniquely, and then both $\mu(z)$ and $\nu(z)$ could be determined too. In this way, the whole 5D solution could be constructed and this 5D solution could, in principle, provide us a global cosmological model to simulate our real universe. We have also studied the evolutions of the mass density $\Omega_m$, the radiation density $\Omega_r$, the dark energy density $\Omega_x$, and the deceleration parameter $q$, and we find that they all are sensitively dependent on the values of the bending parameter $b$. Thus we expect that more accurate observational constraints,
such as that on the last-scattering surface and those about the transition points from $\Omega_r$ dominated to $\Omega_m$ dominated, from $\Omega_m$ dominated to $\Omega_x$ dominated, and from decelerating expansion to accelerating expansion of $q$ could help greatly to determine the bending parameter $b$ and then to determine the global evolution of the universe.

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