On a question related to a basic convergence theorem of Harish-Chandra

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Abstract

In his first 1958 paper on zonal spherical functions Harish-Chandra proved an extremely delicate convergence theorem which was basic to his subsequent definition of his Schwartz space and his theory of cusp forms. This paper gives elementary proofs that a related integral converges for groups of real rank one, several groups of real rank 2 (including $SO(n,2)$, $Sp_4(\mathbb{R})$ and $Sp_4(\mathbb{C})$), $GL(n,\mathbb{R})$ and $GL(n,\mathbb{C})$. In fact, a stronger result has been proved in [B]. Applications of the question are also studied.

1 Introduction

Let $G$ be a real reductive group (for example $GL(n,\mathbb{R})$) with maximal compact subgroup $K$ ($O(n)$) and corresponding Cartan involution $\theta$ ($g \mapsto (g^T)^{-1}$) and let $G = \theta(N)AK$ be an Iwasawa decomposition of $G$ (the Gram-Schmidt process) with $N$ a maximal unipotent subgroup (upper triangular with ones on the main diagonal) and $A$ is the identity component of a maximal $\mathbb{R}$–split torus of $G$ normalizing $N$ (the diagonal matrices with positive entries). Let

$$\alpha^\rho = (\det Ad(a)_{\text{Lie}(N)})^{\frac{1}{2}}$$

for $g \in G$, $g = vak, v \in \theta(N), a \in A, k \in K$ set $a(g) = a$. One of Harish-Chandra’s most delicate results ([H], c.f. [RRG], Theorem 4.5.4 ) which is critical to his method of cusp forms in his proof of the Plancherel Theorem
for a real reductive group is that there exists \( d \) such that

\[
\int_{N} a(n)^{-\rho}(1 + \log a(n)^{\rho})^{-d} dn < \infty.
\]

To appreciate the delicacy of this result you should try to prove it by hand in the case of \( SL(4, \mathbb{R}) \). Here if

\[
X = \begin{bmatrix}
1 & x_1 & x_2 & x_3 \\
0 & 1 & x_4 & x_5 \\
0 & 0 & 1 & x_6 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

then the Harish-Chandra theorem, combined with his result that there exist \( c_1, c_2 > 0 \) such that

\[
c_1(1 + \log(1 + \|x\|^2)) \leq 1 + \log(a(n)^{\rho}) \leq c_2(1 + \log(1 + \|x\|^2)),
\]

is the same as the assertion that there exists \( r \) such that

\[
\int_{\mathbb{R}^6} \frac{dx}{\varphi(x)^{\frac{1}{2}}(1 + \log(1 + \|x\|^2)^r} < \infty
\]

with

\[
\varphi(x) = a(X)^{2\rho} = (1 + x_1^2 + x_2^2 + x_3^2) \times \]
\[
(1 + x_4^2 + x_5^2 + (x_2 - x_1x_4)^2 + (x_3 - x_1x_5)^2 + (x_3x_4 - x_2x_5)^2) \times \]
\[
(1 + x_6^2 + (x_5 - x_4x_6)^2 + (x_3 - x_1x_5 - x_2x_6 + x_1x_4x_6)^2)
\]

which is not obvious. In fact, except for the power of the log term in the Harish-Chandra result it is best possible. Indeed, on can easily show that if \( \varphi \) is a real polynomial on \( \mathbb{R}^n \) of degree 2\( n \) and \( \varphi(x) \geq 1 \) then

\[
\int_{\|x\| \leq r} \varphi(x)^{-\frac{1}{2}} dx \geq C(1 + \log(1 + r^2)).
\]

In this paper I study a related question. Is

\[
\int_{[N,N]} a(n)^{-\rho}(1 + \log a(n)^{\rho})^d dn < \infty
\]

for all \( d > 0 \)?
For $GL(4, \mathbb{R})$, noting that $n \in [N, N]$ if and only if $x_1 = x_4 = x_6 = 0$ one has

$$\varphi(0, x_2, x_3, 0, x_5, 0) = (1 + x_2^2 + x_3^2)((1 + x_2^2)(1 + x_5^2) + x_3^2)(1 + x_3^2 + x_5^2).$$

So one can show that

$$\varphi(0, x_2, x_3, 0, x_5, 0) \geq (1 + x_2^2)\tilde{\varphi}(1 + x_3^2)\tilde{\varphi}(1 + x_5^2)\tilde{\varphi}$$

and, thus, the answer to our question for $GL(4, \mathbb{R})$ is yes. However, In the case of $GL(5, \mathbb{R})$ the integral in the question is just as intractable as the Harish-Chandra integral for $GL(4, \mathbb{R})$.

An affirmative answer would imply that if $f$ satisfies Harish-Chandra’s weak inequality and if $P$ is a parabolic subgroup of $G$ with unipotent radical $N_P$ (see section 2 for the meanings of these terms) then

$$\int_{[N_P, N_P]} |f(n)| \, dn < \infty.$$ 

and if $g$ is in the Schwartz space of $[N_P, N_P] \backslash N_P$ (which is isomorphic with $\mathbb{R}^{\dim[N_P, N_P] \backslash N_P}$) then

$$\int_{[N_P, N_P] \backslash N_P} \int_{[N_P, N_P]} f(nx) dng(x) dx$$

defines a tempered distribution on $[N_P, N_P] \backslash N_P$. This can be used to give a more direct proof of Harish-Chandra’s formula for the Harish-Chandra transform of wave packets and the relationship between the Fourier transform of a Harish-Chandra wave packet and a Whittaker wave packet.

The main result of this paper is an elementary proof of an affirmative answer for $GL(n, F)$, $F = \mathbb{R}$ or $\mathbb{C}$ (in fact for any group whose commutator group is locally isomorphic with $SL(n, F)$). In addition it is observed that the answer is yes for real rank one groups and for certain groups of real rank 2 (including $Sp_4(F), F = \mathbb{R}$ or $\mathbb{C}, SO(n, 2)$ and $SU(2, 2)$).

The author has been informed that the answer is affirmative in all cases by Raphaël Beuzart-Plessis who points out that Proposition B.3.1 in [B] implies the stronger assertion

$$\int_{[N, N]} a(n)^{-p+\varepsilon} dn < \infty.$$ 

for some $\varepsilon > 0$. The methods in this paper are much more elementary than those in [B] so even though they are superseded they still may be useful. Especially to those who are interested in a simple proof of the Harish-Chandra Plancherel and Whittaker Plancherel Theorem for $GL(n)$.

We will now describe the organization of this paper. After some preliminaries (in which implications in this introduction are explained), we first prove that the answer to the question is yes for groups of real rank 1 and for some real rank 2 groups. The rest of the paper is devoted to the case when $G = SL(n,F), F = \mathbb{R}$ or $\mathbb{C}$. For this we first give a (perhaps) new proof of Harish-Chandra’s result which in particular shows that the $d$ in the statement can be taken to be $\dim N + \varepsilon$ for any $\varepsilon > 0$. We then follow the steps in that proof to prove that the integrals in question are finite by induction on $n$. In order to carry out the induction a stronger result is proved.

2 Preliminaries

For lack of a term we will call a real reductive group tame if the answer to our question above is yes. Note that $G$ is tame if and only if every normal, simple subgroup of $G$ is tame. So in this paper we will assume that $G$ is simple. Let $K$ be a maximal compact subgroup of $G$ and let $G = NAK$ be an Iwasawa decomposition of $G$. Let $\Phi(N)$ be the set of weights of $A$ on $Lie(N)$. Set

$$A^+ = \{ a \in A | a^\alpha \geq 1, \alpha \in \Phi(N) \}.$$  

Let $a^\rho = \det(Ad(a))$, as in the introduction. Noting that $G = KA^+K$, set

$$\| k_1ak_2 \| = a^\rho, k_1, k_2 \in K, a \in A^+.$$  

We note that this expression depends only on $g = k_1ak_2$. Indeed, if we put an $Ad(G)$ invariant symmetric bilinear form on $Lie(G)$, $B$, such that if $\theta$ is the Cartan involution corresponding to $K$ then

$$\langle X, Y \rangle = -B(X, \theta Y)$$

defines an inner product on $Lie(G)$. Then if $n = \dim N$ then $\| g \|^2$ is equal to the operator norm of $\wedge^n Ad(g)$ relative to the norm corresponding to $\langle ..., ..., \rangle$ on $\wedge^n Lie(G)$. This implies that $\| ... \|$ satisfies the following properties  

1. $\| xy \| \leq \| x \| \| y \|$, $x, y \in G$.  
2. The sets $\{ g \in G | \| g \| \leq r \}$ are compact.
3. If $X \in \text{Lie}(G)$ satisfies $\theta X = -X$ and $t > 0$ then

$$\|\exp t X\| = \|\exp X\|^t.$$ 

Since the operator norm and the Hilbert-Schmidt norm are equivalent in a finite dimensional Hilbert space and if $X \in \text{Lie}(N)$, $\wedge^n \text{Ad}(\exp X)$ has coefficients that are polynomials in $X$ of degree at most $un$ if $(adX)^{u+1} = 0$. Thus there exists a constant, $C > 0$, such that

$$\|\exp X\| \leq C(1 + \|X\|)^{\frac{un}{2}}.$$ 

Also recall the following result of Harish-Chandra [H], c.f. [RRGI], Theorem 4.5.3.

**Theorem 1** There exist $C_1$ and $r$ such that

$$\|g\|^{-1} \leq \Xi(g) \leq C_1 \|g\|^{-1} (1 + \log \|g\|)^r.$$ 

We will now prove an estimate that is a consequence of tameness and show how it relates to the applications asserted in the introduction.

**Lemma 2** Let $Z$ be a subspace of $\text{Lie}(N)$ such that $\text{Lie}(N) = Z \oplus \text{Lie}([N,N])$. There exists $s \in \mathbb{R}$ and for each $\omega \subset G$ a compact subset a constant $C_\omega$ such that if $X \in Z$ and if $g \in \omega$ such that if $d$ is given there exists $C_d$ such that

$$\int_{[N,N]} \Xi(n \exp X g) (1 + \log \|n\|)^d dn \leq C_\omega C_d (1 + \|X\|)^s.$$ 

**Proof.** In light of the above theorem it is enough to prove that for all $d$

$$\int_{[N,N]} \|n \exp X g\|^{-1} (1 + \log \|n\|)^d dn \leq C_\omega C_d (1 + \|X\|)^s.$$ 

If $n \in [N,N], g \in \omega, X \in Z$ then

$$\|n\| = \|n \exp X g g^{-1} \exp(-X)\| \leq \|n \exp X g\| \|g^{-1} \exp(-X)\| \leq \|n \exp X g\| \|g^{-1}\| \|\exp(-X)\| \leq \|n \exp X g\| \|B_\omega\| C(1 + \|X\|)^{\frac{un}{2}}.$$ 

Since $\|g^{-1}\| \leq B_\omega$ for $g \in \omega$ and $\|\exp(-X)\| \leq C(1 + \|X\|)^{\frac{un}{2}}$ So

$$\|n \exp X g\|^{-1} \leq CB_\omega \|n\|^{-1} (1 + \|X\|)^{\frac{un}{2}}.$$ 

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Let $v$ be a non-zero element of $\wedge^n \theta \text{Lie}(N)$. Then if $n = ua(n)k$ with $u \in \theta N, a(n) \in A, k \in K$ then

$$\|n^{-1}\|^2 \|v\| \geq \|n^{-1}v\| = a(n)^2 \|v\|$$

so

$$\|n^{-1}\|^{-1} \leq a(n)^{-\rho}.$$ 

Finally, we have (here we are using $\|g\| = \|g^{-1}\|$)

$$\int_{[N,N]} \|n \exp Xg\|^{-1} (1 + \log \|n\|)^d d n \leq CB_\omega (1 + \|X\|)^{\frac{2}{d}} \int_{[N,N]} \|n^{-1}\|^{-1} (1 + \log \|n^{-1}\|)^d d n$$

$$= CB_\omega (1 + \|X\|)^{\frac{2}{d}} \int_{[N,N]} \|n^{-1}\|^{-1} (1 + \log \|n^{-1}\|)^d d n$$

$$\leq CB_\omega (1 + \|X\|)^{\frac{2}{d}} \int_{[N,N]} a(n)^{-\rho} (1 + \log \|n\|)^d d n.$$ 

Completing the proof of the lemma. \(\blacksquare\)

A subgroup $P$ of $G$ is said to be a parabolic subgroup if it contains a conjugate of the normalizer of $N$ in $G$ and it is its own normalizer. The unipotent radical of $P$, $N_P$, is the maximal normal nilpotent subgroup of $G$. Thus up to conjugacy we may assume that $P$ contains $N$. If so, we will call $P$ standard. Assume that $P$ is standard. Let $M_P = \theta(P) \cap P$ and $N_P^* = M_P \cap N$. The map $M_P \times N_P \to P$ given by multiplication is a diffeomorphism. This implies that $N = N_P N^*$ with unique expression. As in the proof of the preceding lemma the assertions that are consequences of tameness follow from the next Lemma and that a function, $f$, on $G$ is said to satisfy the weak inequality if there exist constants $C$ and $d$ such that

$$|f(g)| \leq C \Xi(g)(1 + \log \|g\|)^d.$$ 

Lemma 3 \(\int_{[N_P,N_P]} a(n)^{-\rho} (1 + \log \|n\|)^d d n < \infty.\)

Proof. We note that $[N, N] = [N_P, N_P][N_P^*, N_P^*]$ and the Haar measure on $[N, N]$ is after appropriate normalizations equal to the product measure. Thus Fubini’s Theorem implies that

$$\int_{[N,N]} a(n)^{-\rho} (1 + \log \|n\|)^d d n = \int_{[N_P, N_P] \times [N_P^*, N_P^*]} a(nn^*)^{-\rho} (1 + \log \|nn^*\|)^d d n d n^*$$

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and for almost all \( n^* \in [N_p, N_p] \)
\[
(\star) \int_{[N_p, N_p]} a(nn^*)^{-\rho} (1 + \log \|nn^*\|)^d \, dn < \infty.
\]

Now if \( v \) is as in the proof of the previous lemma then
\[
a(nn^*)^{2\rho} = \|n^*\|^{-1} a(n)^{2\rho} \leq \|n^*\|^{-1} a(n)^{2\rho}
\]
so
\[
a(n)^{2\rho} \geq \|n^*\|^{-1} a(nn^*)^{2\rho}
\]
also \( \|nn^*\| \leq \|n\| \|n^*\| \) and \( \|n\| = \|nn^*(n^*)^{-1}\| \leq \|n^*\| \|nn^*\| \)
so for fixed \( n^*, 1 + \log \|nn^*\| \geq C(1 + \log \|n\|) \) with \( C > 0 \). This if \( n^* \) satisfies (\( \star \)) then there exists a constant \( C_1 \) such that
\[
\int_{[N_p, N_p]} a(n)^{-\rho} (1 + \log \|n\|)^d \, dn \leq C_1 \int_{[N_p, N_p]} a(nn^*)^{-\rho} (1 + \log \|nn^*\|)^d \, dn < \infty.
\]

We will be using the following simple estimates often in this paper.

**Lemma 4** Let \( A_1, \ldots, A_n \geq 0, a_1, \ldots, a_n \geq 0 \) then
\[
(1 + A_1 + \ldots + A_n)^{a_1 + \ldots + a_n} \geq (1 + A_1)^{a_1} (1 + A_2)^{a_2} \ldots (1 + A_n)^{a_n}
\]
and also
\[
\sum_{i=1}^{n} \log(1 + A_i) \geq \log(1 + \sum_{i=1}^{n} A_i) \geq \frac{1}{n} \sum_{i=1}^{n} \log(1 + A_i).
\]

**Proof.**
\[
(1 + A_1 + \ldots + A_n)^{a_1 + \ldots + a_n} = \prod_{i=1}^{n} (1 + A_1 + \ldots + A_n)^{a_i} \geq \prod_{i=1}^{n} (1 + A_i)^{a_i}
\]
proving the first assertion. Using it we have
\[
(1 + A_1 + \ldots + A_n)^n \geq \prod_{i=1}^{n} (1 + A_i)
\]
implying the right side of the second inequality. The inequality on the follows from
\[
\prod_{i=1}^{n} (1 + A_i) \geq 1 + \sum_{i=1}^{n} A_i.
\]
3 Groups of real rank 1

Let $G$ be a connected, real reductive group of split rank 1 with finite center
let $K$ be a maximal compact subgroup and $\theta$ the corresponding Cartan involution, let $P$ be a non-trivial proper parabolic subgroup of $G$ with unipotent radical $N$. Let $G = \tilde{N}AK$ be an Iwasawa decomposition of $G$. If $n = \text{Lie}(N)$ and $a = \text{Lie}(A)$ then $\Phi(P, A) = \{\lambda\}$ if and only if $n$ is commutative and $\Phi(P, A) = \{\lambda, 2\lambda\}$ otherwise. If the $\alpha$ root space in $n$ is denoted $n_\alpha$ then $[n, n] = n_{2\lambda}$. If $X \in n_\lambda$ and $Y \in n_{2\lambda}$ and if $m_{i\lambda} = \dim(n_{i\lambda})$ then (c.f. [W] Lemma 8.10.13)

$$a_p(\exp(X + Y)^\rho = ((1 + \frac{\lambda(H_\lambda)}{2} \|X\|^2)^2 + 2\lambda(H_\lambda) \|Y\|^2)^{\frac{m_\lambda}{4} + \frac{m_{2\lambda}}{2}}$$

with $H_\lambda$ the element of $a$ such that $(H_\lambda, h) = \lambda(h)$, $h \in a$ and $(..., ...)$ is the inner product on $\text{Lie}(G)$ given by $(U, V) = -B(U, \theta V)$ with $B$ the killing form and $\|U\|^2 = (U, U)$, as usual. Thus

$$\int_{[N,N]} a_p(n)^{-\rho} (1 + \log \|n\|)^r dn =$$

$$\int_{n_{2\lambda}} (1 + 2\lambda(H_\lambda) \|Y\|^2)^{-\frac{m_\lambda}{4} - \frac{m_{2\lambda}}{2}}(1 + \log(1 + (1 + \|Y\|^2))^r dY < \infty$$

for all $r$ since $m_\lambda > 1$ if $n_{2\lambda} \neq 0$.

4 $Sp_4(\mathbb{R})$ and $Sp_4(\mathbb{C})$

Let $F = \mathbb{R}$ or $\mathbb{C}$. Set

$$L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$J = \begin{bmatrix} 0 & L \\ -L & 0 \end{bmatrix}.$$ 

We realize $G = Sp_4(F)$ as

$$G = \{g \in GL(2, F) | gJg^{-1}\}.$$
We choose $P$ to be the upper triangular elements of $G$. Thus the lie algebra of $N$ is
\[
\mathfrak{n} = \left\{ \begin{bmatrix} 0 & x & y & z \\ 0 & 0 & w & y \\ 0 & 0 & 0 & -x \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| x, y, z, w \in F \right\}.
\]
So
\[
[\mathfrak{n}, \mathfrak{n}] = \left\{ \begin{bmatrix} 0 & 0 & y & z \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| y, z \in F \right\}.
\]
So
\[
[N_0, N_0] = \left\{ n(y, z) = \begin{bmatrix} 1 & 0 & y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \middle| y, z \in F \right\}.
\]
If $g \in G$ and if $g_1, g_2$ are the last two columns of $g^{-1}$ then
\[
a(g)^{\rho_{a_0}} = \|g_2\|^{\dim F} \|g_1 \wedge g_2\|^{\dim F}
\]
so if $n(y, z) \in [N, N]$ then
\[
a(n)^{\rho_a} = (1 + |y|^2 + |z|^2)^{\frac{\dim F}{2}} \frac{\dim F}{2} ((1 + |y|^2)^{\frac{1}{2}} + |z|^2)^{\frac{1}{2}}
\]
Observing that
\[
(1 + |y|^2 + |z|^2) \geq (1 + |y|^2)^{\frac{1}{4}} (1 + |z|^2)^{\frac{3}{4}}
\]
and
\[
((1 + |y|^2)^{\frac{1}{2}} + |z|^2) \geq (1 + |y|^2)^{\frac{1}{2}} (1 + |z|^2)^{\frac{1}{2}}.
\]
So
\[
a(n)^{\rho_a} \geq ((1 + |y|^2)^{\frac{1}{4}} (1 + |z|^2)^{\frac{3}{4}}) \frac{\dim F}{2}
\]
From this it is easy to see that
\[
\int_{[N_0, N_0]} a(n)^{\rho_a(1 + \rho(\log a(n)))^r} dn < \infty
\]
for all $r \in \mathbb{R}$.
5 \ SO(n, 2)

In this section \( G = SO(n + 2, 2) \) which we realize as follows: Let

\[
L = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

and

\[
H = \begin{bmatrix} 0 & 0 & L \\ 0 & I & 0 \\ L & 0 & 0 \end{bmatrix}
\]

where \( I \) denotes the \( n \times n \) identity matrix and \( G \) is the group of all elements of \( SL(n + 4, \mathbb{R}) \) such that

\[
gHg^T = H.
\]

Let \( K = G \cap SO(n + 4) \) then \( K \) is isomorphic with \( S(SO(2) \times SO(n + 2)) \).

\( n = \text{Lie}(N) \) is the group of elements of \( M_{n+4}(\mathbb{R}) \) of the form

\[
\begin{bmatrix}
0 & x & Y & z \\
0 & 0 & -Y^T L \\
0 & 0 & -x \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

with \( Y \) of size \( 2 \times n \). One checks that \([n, n]\) is the space of matrices

\[
X = \begin{bmatrix}
0 & 0 & y & z & 0 \\
0 & 0 & 0 & 0 & -z \\
0 & 0 & 0 & 0 & -y^T \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

with \( y \) of size \( 1 \times n \). With this notation we have

\[
\exp(X) = \begin{bmatrix}
1 & 0 & y & z & -\frac{\|y\|^2}{2} \\
0 & 1 & 0 & 0 & -z \\
0 & 0 & I & 0 & -y^T \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

and one calculates
Lemma 5 With $X$ as above

$$a(\exp(X))^{2\rho} = \left(1 + \frac{\|y\|^2}{2}\right)^2 + z^2)(1 + z^2 + \frac{\|y\|^2}{2})^n.$$ 

Thus,

$$a(\exp(X))^{-\rho} \leq (1 + \frac{\|y\|^2}{2})^{-\frac{n}{12}}(1 + z^2)^{-\frac{1}{4} - \frac{1}{12}}.$$ 

Proof. We note that if

$$m = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & U & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

with $U \in SO(n)$ then $m \in G$ and

$$a(m \exp(X)m^{-1}) = \begin{bmatrix}
1 & 0 & yU & z & -\frac{\|y\|^2}{2} \\
0 & 1 & 0 & 0 & -z \\
0 & 0 & I & 0 & -U^Ty^T \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

thus we may assume that

$$\exp X = \begin{bmatrix}
1 & 0 & re_1 & z & -\frac{r^2}{2} \\
0 & 1 & 0 & 0 & -z \\
0 & 0 & I & 0 & -re_1^T \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

with $e_i$ the $1 \times n$ matrix with a 1 in the $i$th position and zeros elsewhere. If $a$ is a diagonal matrix with diagonal entries $a_1, ..., a_m$ then we define $\mu_j(a) = a_1 \cdots a_j$ for $j \leq m$. On $\wedge^j \mathbb{R}^{n+4}$ we put the inner product that makes $e_{i_1} \wedge \cdots \wedge e_{i_j}$ for $1 \leq i_1 < \ldots < i_j \leq n + 4$ an orthonormal basis then if $g \in G$

$$a(g)^{\mu_j} = \|e_1g \wedge \cdots \wedge e_jg\|.$$
Now
\[ e_i \exp X = \begin{cases} 
 e_1 + re_3 + 2e_{n+3} - \frac{1}{2}e_{n+4}, & \text{if } i = 1 \\
 e_2 - ue_{n+4}, & \text{if } i = 2 \\
 e_3 - re_{n+4}, & \text{if } i = 3 \\
 e_i, & \text{if } i > 3 
\end{cases} \]

we also note that if \( V_i \) is the span of \( \{ e_j | j \neq i \} \) then the map \( \wedge^j V_i \to \wedge^{i+1} \mathbb{R}^{n+4} \) given by \( u \mapsto u \wedge e_i \) is isometric if \( j < n+4 \). This implies that if \( 3 < i < n+3 \) then
\[ a(\exp X)^{\mu_i} = a(\exp X)^{\mu_3}. \]

We leave it to the reader to show that
\[ a(\exp X)^{2\mu_1} = \left( 1 + \frac{r^2}{2} \right)^2 + z^2 \]
and
\[ a(\exp X)^{2\mu_2} = a(\exp X)^{2\mu_3} = (1 + z^2 + \frac{r^2}{2})^2 \]
thus
\[ a(\exp X)^{2\mu_j} = (1 + z^2 + \frac{r^2}{2})^2 \text{ for } 3 < j < n + 3. \]

Now if \( n = 2k \) then \( \rho = \mu_1 + \ldots + \mu_{k+1} \) and if \( n = 2k + 1 \) then \( \rho = \mu_1 + \ldots + \mu_{k+1} + \frac{1}{2} \mu_{k+2} \). Thus using the observations above we have in both cases
\[ a(\exp X)^{2\rho} = (\left( 1 + \frac{r^2}{2} \right)^2 + z^2)(1 + z^2 + \frac{r^2}{2})^n. \]

The inequality is proved by observing that if \( 0 < \varepsilon < 1 \) and \( 0 < \delta < 1 \) then Lemma \[ \]
\[ (1 + \frac{r^2}{2})^2 + z^2)(1 + z^2 + \frac{r^2}{2})^n \geq \left( 1 + \frac{r^2}{2} \right)^{2\varepsilon + n(1-\delta)} (1 + z^2)^{1-\varepsilon + n\delta}. \]

Taking \( \varepsilon = \frac{1}{3} \) and \( \delta = \frac{1}{2n} \) yields the inequality \[ \]
So \( SO(n+2, 2) \) is tame. Note that the identity component of \( SO(4, 2) \) is locally isomorphic with \( SU(2, 2) \) thus \( SU(2, 2) \) is tame. We also note that the identity component of \( SO(3, 2) \) is locally isomorphic with \( Sp_4(\mathbb{R}) \) which reproves that it is tame.
6 Harish-Chandra’s convergence theorem for $SL(n, F), F = \mathbb{R}$ or $\mathbb{C}$

In this section we will give a proof of Harish-Chandra’s convergence theorem for $SL(n, F)$, which will serve as a template for our proof of the tameness of $SL(n, F)$.

Here $G_n = SL(n, F), F = \mathbb{R}$ or $\mathbb{C}, K_n$ is $SO(n)$ if $F = \mathbb{R}$ or $SU(n)$ if $F = \mathbb{C}, N_n$ is the group of upper triangular matrices in $G_n$ with ones on the main diagonal and $A_n$ is the group of diagonal matrices in $G_n$ with real positive entries. $\bar{N}_n = N_n^T$ and $G_n = \bar{N}_n A_n K_n$ the corresponding Iwasawa decomposition. Write $g = \bar{n}(g) a_n(g) k_n(g)$ for $g \in G_n$ as usual. If $v \in N_n$ then writing $v$ out in terms of its columns

$$v = [v_1 v_2 \ldots v_n], v_1 = [1, 0, \ldots, 0]^T, v_i = [x^{i-1}_1, \ldots, x^{i-1}_{i-1}, x^{i}_1, 1, 0, \ldots],$$

Let $\Lambda_i$ be the basic highest weights (that is the highest weights of the representations $\wedge i \mathbb{R}^n$ respectively) then if $\xi_i = e_{n-i+1} \wedge \cdots \wedge e_n$ and if $g \in SL(n, F)$ then

$$a_n(g)^{\Lambda_i} = \| g^{-1} \xi_i \|$$

so

$$a_n(v^{-1})^{\Lambda_{i-1}} = \| v_i \wedge \cdots \wedge v_n \|$$

for $i = 2, \ldots, n$. Thus

$$a_n(v^{-1})^{2\rho} = \prod_{i=2}^{n} \| v_i \wedge \cdots \wedge v_n \|^{2 \dim F}.$$

This easily implies

**Lemma 6** If

$$v = \begin{bmatrix} I & y \\ 0 & 1 \end{bmatrix}$$

with

$$y = [y_1, \ldots, y_{n-1}]^T$$

then

$$a_n(v)^{2\rho} = \prod_{i=1}^{n-1} (1 + \sum_{j=1}^{i} y_i^2)^{\dim F}.$$
Note that $N_n = N^*V$ with

$$N^* = \begin{bmatrix} N_{n-1} & 0 \\ 0 & 1 \end{bmatrix}, V = \begin{bmatrix} I_{n-1} & F^{n-1} \\ 0 & 1 \end{bmatrix}.$$ 

Also if $v^* \in N^*$ then

$$v^* = \bar{v}^* a^* k^*$$

with the constituents giving the Iwasawa decomposition in $G_{n-1} 0 1$. Define for $v^* \in N^*$ $w(v^*) \in N_{n-1}$ by

$$v^* = \begin{bmatrix} w(v^*) & 0 \\ 0 & 1 \end{bmatrix}$$

Note that

$$a_{P_n}(v^*)^{\rho_n} = a_{P_{n-1}}(w(v^*))^{\rho_{n-1}}, a_{P_n}(v^*)^{\Lambda_{n-1}} = 1.$$ 

If $v^* \in N^*, v_1 \in V$ then

$$v^* v_1 = \bar{n}(v^*) a_n(v^*) k_n(v^*) v_1 = \bar{n}(v^*) a_n(v^*) k_n(v^*) v_1 k_n(v^*)^{-1} k(v^*).$$

If

$$k_n(v^*) v_1 k_n(v^*)^{-1} = \bar{v} a_n(k_n(v^*) v_1 k_n(v^*)^{-1}) k_n$$

is its Iwasawa decomposition then

$$v^* v_1 = \bar{n}(v^*) (a_n(v^*) \bar{v} a_n(v^*)^{-1}) a_n(v^*) a_n(k_n(v^*) v_1 k_n(v^*)^{-1}) k_n(v^*)$$

So

$$a_n(v^*) v_1 = a_n(v^*) a_n(k_n(v^*) v_1 k_n(v^*)^{-1}).$$

We now show how this material gives a simple inductive proof of a slight strengthening of Harish-Chandra’s result in the case of $SL(n, F)$.

**Theorem 7** If $n \geq 2$ and $\varepsilon > 0$ then

$$\int_{N_n} a_n(v)^{-\rho_n} (1 + \rho(\log a_n(v))^{-\frac{n}{2}})^{-\varepsilon} dv < \infty.$$ 

**Proof.** We prove the result by induction on $n$. If $F = C$ then we identify $F$ with $R^2$ and $dx$ will denote Lebesgue measure. If $n = 2$ then we are looking at

$$\int_F \frac{dx}{(1 + |x|^2)^{\frac{\dim F}{2}} (1 + \log(1 + |x|^2))^{1+\varepsilon}}$$

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which finite if \( \varepsilon > 0 \) (We will give simple argument after the proof is completed). So assume the result for \( n - 1 \). We have

\[
\int_{N_n} a_n(v)^{-\rho_n} (1 + \rho_n(\log a(v)))^{-(\frac{n}{2}) - \varepsilon} dv = \int_{N^*} a_{n-1}(w(v^*))^{-\rho_{n-1}} \times \\
\int_V a_n(k_n(v^*)v_1k_m(v^*)^{-1})^{-\rho_n}(1+\rho_{n-1}(\log a_{n-1}(w(v^*))+\rho_n(\log a_n(k_n(v^*)v_1k_n(v^*)^{-1})))^{-(\frac{n}{2}) - \varepsilon} dv^* dv
\]

We note that the map \( v_1 \rightarrow gv_1g^{-1} \) preserves \( V \) if \( g \in \begin{bmatrix} G_{n-1} & 0 \\ 0 & 1 \end{bmatrix} \) and preserves the invariant measure on \( V \). Thus the integral becomes

\[
\int_{N^*} a_{n-1}(w(v^*))^{-\rho_{n-1}} \int_V a_n(v_1)^{-\rho_n}(1+\rho_{n-1}(\log a_{n-1}(w(v^*))+\rho_n(\log a_n(v_1)))^{-(\frac{n}{2}) - \varepsilon} dv^* dv \leq \\
\int_{N^*} a_{n-1}(w(v^*))^{-\rho_{n-1}}(1+\rho_{n-1}(\log a_{n-1}(w(v^*))))^{-(\frac{n}{2}) - \varepsilon} dv^* \times \\
\int_V a_n(v_1)^{-\rho_n}(1+\rho_n(\log a_n(v_1)))^{-n+1-\frac{\varepsilon}{2}} dv_1.
\]

The first integral is covered by the inductive hypothesis. To analyze the second integral, the above lemma implies that it is equal to

\[
\int_{F^{n-1}} (1 + \sum_{i=1}^{n-1} \log(1 + \sum_{j=1}^{i} |y_i|^2))^{-n+1-\frac{\varepsilon}{2}} \prod_{i=1}^{n-1} (1 + \sum_{j=1}^{i} |y_i|^2)^{-\frac{\text{dim}_F}{2}} dy.
\]

since we can write

\[
v = \begin{bmatrix} I_{n-1} & y^T \\ 0 & 1 \end{bmatrix}, y = [y_1y_2...y_{n-1}].
\]

This integral converges since it is less than or equal to

\[
\left( \int_F \frac{dx}{(1 + |x|^2)^{\frac{\text{dim}_F}{2}}(1 + \log(1 + |x|^2))^{1+\frac{\varepsilon}{2(n-1)}}} \right)^{n-1} < \infty.
\]

\[\blacksquare\]

**Lemma 8** If \( \varepsilon > 0 \) then

\[
\int_F \frac{dx}{(1 + |x|^2)^{\frac{\text{dim}_F}{2}}(1 + \log(1 + |x|^2))^{1+\varepsilon}} < \infty.
\]
Proof. Integrating in polar coordinates the integral is
\[2\pi^{\dim_R F - 1}\int_0^\infty\frac{r^{\dim_R F - 1}dr}{(1 + r^2)^{\dim_F F \cdot \frac{1}{2}}(1 + \log(1 + r^2))^{1+\varepsilon}}.\]
Which we can write as
\[2\pi^{\dim_R F - 1}\int_0^2\frac{r^{\dim_R F - 1}dr}{(1 + r^2)^{\dim_F F \cdot \frac{1}{2}}(1 + \log(1 + r^2))^{1+\varepsilon}} + 2\pi^{\dim_R F - 1}\int_2^\infty\frac{r^{\dim_R F - 1}dr}{(1 + r^2)^{\dim_F F \cdot \frac{1}{2}}(1 + \log(1 + r^2))^{1+\varepsilon}}.\]
The first integral is clearly finite. As for the second it is less than or equal to a constant times
\[\int_2^\infty\frac{dr}{r^{1+\varepsilon}} = \int_\log 2^{1+\varepsilon} < \infty.\]

7 \(SL(n, F), F = \mathbb{R} \text{ or } \mathbb{C}\)

We retain the notation of the previous section. That the group \(SL(n, F)\) is tame is an immediate consequence of the following result the proof of which will be the topic of the rest of this section. The extra factor is necessary in its inductive proof.

**Theorem 9** If \(n \geq 3\) if \(0 \leq \alpha < 1\) then
\[\int_{[N_n, N_n]} a_n(v)^{-\rho_n} a_n(v)^{\alpha \dim_F F \Lambda_{n-1}} (1 + \rho_n(\log(a_n(v))))^r dv < \infty\]
for all \(r\).

We first note that
\[[N, N] \cap N^* = [N^*, N^*] = \begin{bmatrix} [N_{n-1}, N_{n-1}] & 0 \\ 0 & 1 \end{bmatrix}\]
Also, if
\[v_1 \in [N, N] \cap V\]
then

\[ v_1 = \begin{bmatrix} I_{n-1} & y^T \\ 0 & 1 \end{bmatrix}, \; y = [y_1, \ldots, y_{n-2}, 0, 1]. \]

Note that the map

\( ([N, N] \cap N^*) \times ([N, N] \cap V) \to [N, N] \)

given by multiplication defines a measure preserving diffeomorphism if the Haar measures on the three groups are properly normalized.

If \( v^* \in N^* \) with

\[ v^* = \begin{bmatrix} w(v^*) & 0 \\ 0 & 1 \end{bmatrix}, \; w(v^*) \in N_{n-1} \]

then

\[ k_n(v^*) = \begin{bmatrix} k_{n-1}(w(v^*)) & 0 \\ 0 & 1 \end{bmatrix}. \]

So we have

\[ k_n(v^*)v_1k_n(v^*)^{-1} = \begin{bmatrix} I_{n-1} & k_{n-1}(w(v^*))y^T \\ 0 & 1 \end{bmatrix} \]

Set \( V_1 = V \cap [N, N] \), as in the proof of Harish-Chandra’s convergence theorem we have

\[
\int_{[N_n, N_n]} a_n(v)^{-\rho_n} a_n(v)_{(\alpha \dim F A_{n-1}}(1 + \rho(\log(a_n(v))))^r dv = \\
\int_{N^* \cap [N_n, N_n]} a_{n-1}(w(v^*))^{-\rho_{n-1}} \int_{V_1} a_n(k(v^*)v k(v^*)^{-1})^{-\rho_n(\alpha \dim F A_{n-1})} x (1 + \rho_{n-1}(\log a_{n-1}(w(v^*)))) + \rho_n(\log a_n(k(v^*)v k(v^*)^{-1})^r dv^* dv \leq \\
\int_{V_1} a_n(k(v^*)v k(v^*)^{-1})^{-\rho_n(\alpha \dim F A_{n-1})} x (1 + \rho_n(\log a_n(k(v^*)v k(v^*)^{-1})^r dv_1 dv^*. \]

To continue the argument we must analyze the following integral with \( v^* \in \) \([N^*, N^*] \) fixed

\[
\int_{V_1} a_n(k(v^*)v_1k(v^*)^{-1})^{-\rho_n(\alpha \dim F A_{n-1})} x (1 + \rho_n(\log a_n(k(v^*)v_1k(v^*)^{-1})^r dv_1
\]
with \( v^* \in [N^*, N^*] \) fixed. We note that if
\[
u = \begin{bmatrix} I & u' \\ 0 & 1 \end{bmatrix}
\]
with
\[
u/ = [u_1, u_2, ..., u_{n-1}]
\]
then
\[
\rho_n(\log(a_n(u))) = \dim \mathcal{R} F^{n-1} \sum_{i=1}^{n-1} \log(1 + \sum_{j=1}^{i} |u_j|^2) \leq \frac{\dim \mathcal{R} F}{2} (n-1) \log(1 + \sum_{i=1}^{n-1} |u_i|^2).
\]

We write
\[
k_n(w(v^*)) = \begin{bmatrix} A & b \\ c & d \end{bmatrix}
\]
as above. Noting that if \( v^* \in N^* \) then Lemma 10 implies that, since the \( "L" \) in the lemma is the identity,
\[
|\det(A)| = a_{n-1}(w(v^*))^{-\Lambda n-2}.
\]
Writing \( Ay = [(Ay)_1 (Ay)_2 ...(Ay)_{n-2}]^T \) for \( y = [y_1...y_{n-2}]^T \) and
\[
v_1 = \begin{bmatrix} I & y' \\ 0 & 1 \end{bmatrix}
\]
\[
\int_{V_1} a_n(k(v^*)v_1k(v^*)^{-1}) \rho_n(\log a_n(k(v^*)v_1k(v^*)^{-1}))^r dv_1 \leq
\]
\[
(n-1)^r \int_{F^{n-2}} (1 + \log(1 + \sum_{j=1}^{n-2} |(Ay')_j|^2 + |cy'|^2)^r (1 + \sum_{j=1}^{n-2} |(Ay')_j|^2 + |cy'|^2)^{\dim F \frac{2}{2}} (1 + \sum_{j=1}^{n-2} |(Ay')_j|^2 + |cy'|^2)^{\dim F \frac{2}{2}} \prod_{i=1}^{n-2} (1 + \sum_{j=1}^{n-2} |(Ay')_j|^2 + |cy'|^2)^{\dim F \frac{2}{2}} dy'.
\]
After the change of variables \( Ay' \to y' \) the integral becomes
\[
(n - 1)^r |\det A|^{-\dim F} \times
\]
\[
\int_{F^{n-2}} (1 + \log(1 + \sum_{j=1}^{n-2} |y_j|^2 + |cA^{-1}y'|^2)^r (1 + \sum_{j=1}^{n-2} |y_j|^2 + |cA^{-1}y'|^2)^{\dim F \frac{2}{2}} \prod_{i=1}^{n-2} (1 + \sum_{j=1}^{n-2} |y_j|^2 + |cA^{-1}y'|^2)^{\dim F \frac{2}{2}} dy'.
\]
Let $0 < \delta < 1 - \alpha$. There exists $B_{\delta,r}$ such that
\[
(1 + \log(1 + \sum_{j=1}^{n-2} y_j^2 + (cA^{-1}y')^2))^r \leq B_{\delta,r}(1 + \sum_{j=1}^{n-2} y_j^2 + (cA^{-1}y')^2)^{\frac{r}{2}}.
\]

So if we set $\bar{u} = cA^{-1} \in \mathbb{C}^{n-2}$ then since $0 < a + \delta < 1$ we see that Lemma 12 implies that
\[
\int_{V_1} a_P(w(n^*)v_{k(n^*)}^{-1})^{-\rho_n - \alpha \dim \mathbb{R}^n} (1 + \rho_n (\log a_P(w(n^*)v_{k(n^*)}))^r \leq (n-1)^r \frac{|\det A|^{-\dim \mathbb{F}_2} B_{\delta,r} \int_{\mathbb{R}^{n-2}} \varphi_{1-\alpha-\delta,n}(x)^{-\frac{\dim \mathbb{R}^F}{2}} dx \leq (n-1)^r \frac{|\det A|^{-\dim \mathbb{F}_2} B_{\delta,r} C_{\varepsilon,\alpha,m} (1 + \|u\|^2)^{-\frac{(1-\varepsilon)(1-\alpha-\delta)}{2} \dim \mathbb{R}^F}}
\]
for all $0 < \varepsilon < 1$. Lemma 11 implies that
\[
cA^{-1} = -\frac{b^*}{d}.
\]

So
\[
\|u\|^2 = 1 - \frac{|d|^2}{|d|^2}.
\]

And we have seen that if $w(n^*)^{-1}$ has last column $[x_1 \ldots x_{n-3} 01]^T$ then
\[
|d|^2 = \frac{1}{1 + \sum_{i=1}^{n-3} |x_i|^2}
\]

Thus
\[
\|u\|^2 = \sum_{i=1}^{n-3} |x_i|^2.
\]

Hence
\[
1 + \|u\|^2 = a_{n-1}(w(n^*))^{2\Lambda_{n-2}}.
\]

Also $|\det A| = |d|^{-1}$ so
\[
\int_{V_1} a_n(k(v^*)v_{k(n^*)}^{-1})^{-\rho_n - \alpha \dim \mathbb{R}^n} (1 + \rho_n (\log a_P(w(n^*)v_{k(n^*)}))^r \leq (n-1)^r B_{\delta,r} C_{\varepsilon,\alpha,m} a_{n-1}(w(n^*))^{(1-\varepsilon)(1-\alpha-\delta)} \dim \mathbb{R}^F \mathbb{A}_{n-2}.
\]
Hence
\[
\int_{[N_n,N_n]} a_n(v)^{-\rho_n} a_n(v)^{\alpha \dim_F \Lambda_{n-1}} (1 + \rho_n (\log(a_n(v))))^r dv \leq \\
(n - 1)^r B_{\delta,r} C_{\varepsilon,\alpha,m} \int_{[N_{n-1},N_{n-1}]} a_{n-1}(z)^{-\rho_{n-1}} \times \\
a_{n-1}(z)^{(1-(1-\varepsilon)(1-\alpha-\delta))) \dim_F \Lambda_{n-2}} \log(1 + \rho_{n-1}(\log(a_{n-1}(z))))^r dz
\]
which is finite by the inductive hypothesis, since 0 < (1 - \varepsilon)(1 - \alpha - \delta) < 1.

8 Appendices

In the following appendices $F$ denotes either $\mathbb{R}$ or $\mathbb{C}$.

8.1 Some linear algebra

The purpose of this appendix is to point about two simple linear algebra lemmas.

Lemma 10 If $g \in SL(n, F)$ with
\[
g = \begin{bmatrix} L & u \\ v & w \end{bmatrix},
\]
if the last column of $g^{-1}$ is
\[
x = [x_1, \ldots, x_n]^T,
\]
and if
\[
k_n(g) = \begin{bmatrix} A & b \\ c & d \end{bmatrix}
\]
with $A, L$ matrices of size $(n - 1) \times (n - 1)$ then
\[
\det A = \frac{\det L}{\sqrt{\sum_{i=1}^{n} |x_i|^2}} = (\det L) a_{P_n}(g)^{-\Lambda_{n-1}}.
\]
Proof. We write the Iwasawa decomposition

\[ g = \begin{bmatrix} \tilde{n} & 0 \\ z & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & \det(a)^{-1} \end{bmatrix} \begin{bmatrix} A & b \\ c & d \end{bmatrix} = \begin{bmatrix} \tilde{n}aA & \tilde{n}ab \\ za & \det(a)^{-1} \end{bmatrix} = \begin{bmatrix} \tilde{n}A & \tilde{n}ab \\ uA & uaA + \det(a)^{-1}c + uab + \det(a)^{-1}d \end{bmatrix}. \]

Thus

\[ \det L = \det a \det A. \]

To compute \( \det a \), note that

\[ [x_1, \ldots, x_n]^T = g^{-1}e_n = k_n(g)^{-1} \det(a)e_n. \]

Thus

\[ \det(a) = \| [x_1, \ldots, x_n] \| = \sqrt{\sum |x_i|^2} = a_P(g)^{\Lambda_n-1}. \]

The lemma follows. ■

If \( a \in \mathbb{R} \) then we write \( \bar{a} = a \). If \( R \) is a \( p \times q \) matrix with coefficients in \( F \) then \( R^* \) will mean \( R^T \).

Lemma 11 If

\[ B = \begin{bmatrix} A & b \\ c & d \end{bmatrix} \in K_n \]

with \( A \) an \((n-1) \times (n-1)\) matrix then

\[ \det(A) = d \det(B) \]

so

\[ |\det A| = |d| = \sqrt{1 - \| b \|^2} \]

If \( d \neq 0 \) then

\[ cA^{-1} = -\frac{b^*}{d}. \]

Proof. By definition of \( K_n \), \( BB^* = I \). Multiplying out we have

\[ AA^* + cc^* = I, Aec^* + b\bar{d} = 0, cc^* + |d|^2 = 1 \]

so

\[ \begin{bmatrix} A & b \\ c & d \end{bmatrix} \begin{bmatrix} I & c^* \\ 0 & \bar{d} \end{bmatrix} = \begin{bmatrix} A & 0 \\ c & cc^* + |d|^2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ c & 1 \end{bmatrix} \]
thus
\[ \bar{d} \det B = \det A. \]

The first assertion of the lemma follows since \( \|b\|^2 + |d|^2 = 1 \).

To prove the second part \( B^*B = I \) implies that
\[ A^*b + dc^* = 0. \]

Taking adjoints
\[ b^*A = -\bar{d}c \]
so if \( d \neq 0 \)
\[ cA^{-1} = -\frac{b^*}{d}. \]

\[ \blacksquare \]

8.2 An elementary estimate

Let for \( u \in F^m \) and \( 0 < \alpha < 1 \)
\[ \psi_{\alpha,u,m}(x) = (1 + \sum_{i=1}^{m} |x_i|^2 + |\langle u, x \rangle|^2) \prod_{i=1}^{m} (1 + |x_i|^2) \]
then

Lemma 12 If \( 0 \leq \varepsilon < 1 \) then there exists \( 0 < C_{\alpha,m,\varepsilon} < \infty \) such that
\[ \int_{F^m} \psi_{\alpha,u,m}(x)^{-\frac{\dim F}{2}} dx \leq C_{\alpha,m,\varepsilon} (1 + \|u\|^2)^{-\alpha \varepsilon \frac{\dim F}{2}}. \]

If \( u = 0 \) then since
\[ \psi_{\alpha,0,m}(x) = (1 + \sum_{i=1}^{m} |x_i|^2) \prod_{i=1}^{m} (1 + |x_i|^2) \geq \prod_{i=1}^{m} (1 + |x_i|^2)^{1 + \frac{1}{m}} \]
we see that
\[ I_{\alpha,m} = \int_{F^m} \psi_{\alpha,u,m}(x)^{-\frac{\dim F}{2}} dx < \infty. \]

If \( \|u\| \leq 1 \) then
\[ (1 + \|u\|^2)^{-\alpha \varepsilon \frac{\dim F}{2}} \geq 2^{-\alpha \varepsilon \frac{\dim F}{2}}. \]
Also, 

\[ \psi_{\alpha,u,m}(x) \geq \psi_{\alpha,0,m}(x) \]

so

\[ \int F^m \psi_{\alpha,u,m}(x) \frac{\dim_{F} F}{2} dx \leq 2^{-\alpha \frac{\dim_{F} F}{2}} I_{\alpha,m}(1 + \|u\|^2)^{-\alpha \frac{\dim_{F} F}{2}}. \]

Thus we need only prove the result for \( \|u\| \geq 1 \). Set \( v = \frac{u}{\|u\|} \). Let \( W_{u,1} = \{ x \in \mathbb{R}^m | |(v, x)| \leq 1 \} \) and \( W_{u,2} = \{ x \in \mathbb{R}^m | |(v, x)| \geq 1 \} \). Then

\[ \int F^m \psi_{\alpha,u,m}(x) \frac{\dim_{F} F}{2} dx = \int_{W_{u,1}} \psi_{\alpha,u,m}(x) \frac{\dim_{F} F}{2} dx + \int_{W_{u,2}} \psi_{\alpha,u,m}(x) \frac{\dim_{F} F}{2} dx. \]

We have on \( W_{u,2} \)

\[ \psi_{\alpha,u,m}(x) \geq (1 + \sum_{i=1}^{m} |x_i|^2 + \|u\|^2)^\alpha \prod_{i=1}^{m} (1 + |x_i|^2) \geq (1 + \|u\|^2)^\alpha \psi_{(1-\epsilon)\alpha,m}(x). \]

So

\[ \int_{W_{u,2}} \psi_{\alpha,u,m}(x) \frac{\dim_{F} F}{2} dx \leq I_{(1-\epsilon)\alpha,m}(1 + \|u\|^2)^{-\alpha \frac{\dim_{F} F}{2}}. \]

We now consider the integral over \( W_{u,1} \). Note that

\[ \int F^m \psi_{\alpha,u,m}(x) \frac{\dim_{F} F}{2} dx = \int F^m \psi_{\alpha,T(u),m}(x) \frac{\dim_{F} F}{2} dx \]

with \( T(u) = (\xi_1 u_1, \ldots, \xi_m u_m) \) and \( |\xi_i| = 1 \). Thus we may assume that \( u_i \in \mathbb{R} \) and \( u_i \geq 0 \) in the proof of the lemma. Let \( q \) be such that \( u_q \) is maximum. Define new linear coordinates by

\[ y_i = x_i, \quad i < q, \quad y_q = \langle x, v \rangle, \quad y_i = x_i, \quad q < i \leq m. \]

Then the Jacobian of the transformation \( x \mapsto y \) is \( \psi_q \dim_{F} F \). We also note that since \( \|v\| = 1, 1 \geq v_q \geq \frac{1}{\sqrt{m}} \). Also,

\[ x_q = \frac{y_q - \sum_{i \neq q} v_i y_i}{v_q} \]

and

\[ \langle x, u \rangle = \|u\| y_q. \]
Writing out $\psi_{\alpha,u,m}(x)$ in terms of the $y_i$ we have ($0 < \varepsilon < 1$)

\[
\prod_{i < q}(1 + |y_i|^2) \left( 1 + \left| \frac{y_q - \sum_{i \neq q} v_i y_i}{v_q} \right|^2 \right) \times
\]

\[
\prod_{i > q}(1 + |y_i|^2)(1 + \sum_{j \neq q} |y_j|^2 + \left| \frac{y_q - \sum_{i \neq q} v_i y_i}{v_q} \right|^2 + \|u\|^2 |y_q|^2)^\alpha \geq
\]

\[
\prod_{i \neq q}(1 + \sum_{j \leq i} |y_j|^2) \left( 1 + \sum_{j \neq q} |y_j|^2 + \|u\|^2 |y_q|^2 \right)^{(1-\varepsilon)\alpha} \geq
\]

\[
\prod_{i \neq q}(1 + \sum_{j \leq i} |y_j|^2) \left( 1 + \sum_{j \neq q} |y_j|^2 \right)^{(1-\varepsilon)\alpha} (1 + \|u\|^2 |y_q|^2)^\varepsilon \alpha = \psi_{(1-\varepsilon),\alpha,0,m-1}(y) (1 + \|u\|^2 |y_q|^2)^\varepsilon \alpha.
\]

Thus (since the real Jacobian of the transformation $x \to y$ is $\frac{1}{y_q^{\dim_F \mathbb{R}}}$) and $y_q \geq \frac{1}{\sqrt{m}}$

\[
\int_{W_{u,1}} \psi_{\alpha,u,m}(x) \frac{\dim_F \mathbb{R}}{\dim_F \mathbb{R}} dx \leq \frac{1}{m^{\dim_F \mathbb{R}} \alpha} \int_{|z| \leq 1} \int \frac{dz}{(1 + \|u\|^2 |z|^2)^{\varepsilon \alpha \dim_F \mathbb{R}}}.
\]

Thus, to complete the argument in the case at hand we need to show that if $a \in \mathbb{R}$, $a \geq 1$, $0 < \beta < 1$ then

\[
\int_{|y| \leq 1} (1 + a^2 |y|^2)^{-\beta/2} dy \leq C_\beta (1 + a^2)^{-\beta}.
\]

$F = \mathbb{R}$: then

\[
\int_0^1 \frac{dx}{(1 + a^2 x^2)^{\beta/2}} \leq a^{-\beta} \int_0^1 \frac{dx}{x^{\beta}} = \frac{1}{1 - \beta} a^{-\beta}.
\]

Since $a \geq 1$ we have $2a^2 \geq 1 + a^2$ so $2^{-\beta} a^{-\beta} \leq (1 + a^2)^{-\beta/2}$. Take

\[
C_\beta = \frac{1}{1 - \beta^2 \beta}.
\]
\( F = \mathbb{C} : \)

\[
\int_{|x| \leq 1} \frac{dx}{(1 + a^2 |x|^2)^{\beta/2}} = 2\pi \int_0^1 \frac{sds}{(1 + a^2 s^2)^{\beta/2}} \\
= \pi \int_0^1 \frac{ds}{(1 + a^2 s)^{\beta/2}} \leq \frac{\pi}{a^\beta} \int_0^1 \frac{ds}{s^{3/2}} \leq \frac{\pi}{a^\beta(1 - \beta/2)}
\]

Take

\[
C_\beta = \frac{\pi}{1 - \beta/2} 2^\beta.
\]

References

[H] Harish-Chandra, Spherical Functions on a Semisimple Lie Group, 1, Amer. J. Math. 80 (1958), 241-310

[B] Raphaël Beaupré-Plessis, A local trace formula for the Gan-Gross-Pasad conjecture for unitary groups: the Archimedean case, Astérisque, Livre-Tom 418, 2020.

[RRGI] Nolan R. Wallach, Real Reductive Groups I, Academic Press, Boston, 1988

[W] Nolan R. Wallach, Harmonic Analysis on Homogeneous Spaces, Second Edition, Dover, Mineola, 2018