Optimal cooperative control for formation flying spacecraft with collision avoidance

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Abstract
This article investigates optimal cooperative control algorithms for spacecraft formation flying system that can guarantee collision avoidance between spacecrafts. By selecting potential functions to avoid collisions and constructing index cost functions to describe optimal control, the optimal control algorithms based on the state-dependent Riccati equations, which only require local information vectors, are presented that can guarantee the formation spacecraft to track the reference trajectory without collisions and achieve optimal performance states. Finally, the corresponding proof analysis by Lyapunov stability theory shows that the closed-loop system for spacecraft formation flying is asymptotically stable. The simulation results demonstrate the effectiveness of the proposed optimal cooperative control algorithms with desired control objectives, including trajectory tracking, formation optimality, and collision avoidance.

Keywords
Spacecraft formation flying, cooperative control, optimal control, collision avoidance, optimal cooperative control

Introduction
Cooperative control of nonlinear multi-agent system has received increasing research in recent years especially for the spacecraft formation flying (SFF) system.¹⁻³ The optimal cooperative control has been proved to be an optimal condition for cooperative control, which can make the whole system reach the optimal performance index state. With the rapid practical applications for spacecraft
formation missions, the importance of optimal cooperative control is worth investigating in the theoretical research and practical applications.4–6

Optimal cooperative control has been identified as one of the most important cases to consider for cooperative control in SFF system,7,8 where the goal is to select optimal cost functions to obtain performance index and to predict the system state in real time. Due to this fact, many control algorithms have been presented to solve this problem by combining the optimal methods with cooperative control. Krstic and Tsiotras9 presented an approach to construct optimal feedback control laws for regulation of a rotating rigid spacecraft and gave a characterization of stability margins achieved with the inverse optimal control law. Another work associated with trajectory optimization was presented by Foderaro et al.,10 who presented a novel distributed optimal control for multi-agent and extended the capabilities with respect to classical optimal control. Considering the case of optimal control and collision avoidance simultaneously, a dual iteration algorithm was proposed to obtain optimal control with the effective penalty functions in the work by Li et al.11

In more practical applications for SFF system, the formation missions are accomplished by the following reference path to complete trajectory tracking. Moreover, the goal for SFF system is similar to the multi-agent system; a suitable trajectory should be designed that can drive spacecraft to track the desired position rather than random movement with collisions. A trajectory planning method for spacecraft was proposed by Hou et al.,12 which is based on σ-parameters that can avoid the impassable singular state. Liu et al.13 designed another novel trajectory tracking control law to deal with input saturation effect in which a stable dynamic formation system can be achieved. A similar type of control laws was formulated for multiple agents in the work by Mondal et al.;14 based on Barbalat’s lemma, the authors proposed control schemes for the problem of trajectory tracking, in which the system can achieve connectivity assurance and collision avoidance simultaneously. Motivated by the need for fast trajectory tracking, based on the modified electromagnetic satellite model, Huang et al.15 proposed an adaptive robust terminal sliding mode control which can achieve fast trajectory tracking and also can guarantee finite-time convergence in the presence of saturation.

In addition, the problem of collision avoidance is also one of the most crucial performance indexes in the SFF missions. The collision avoidance regions are constructed to guarantee that there are no collisions with each spacecraft in the formation flying, and the spacecraft can keep the reference trajectory out of the collision regions. Many methods, such as collision probability analysis and game theory, have been well used for collision avoidance control problem. Due to the advantages of global minimum in the desired state, potential functions have been widely studied by the researchers. The decentralized collision avoidance control law was presented by Lee et al.16 for SFF tracking control; a virtual leader trajectory was used with an artificial potential function to avoid possible collisions between each spacecraft. The authors dealt with formation configuration convergence control in which collision avoidance and network connectivity for multi-agents can be ensured. In more practical studies, cooperative collision avoidance control has
been originally formulated. In the work by Dušan et al., cooperative avoidance control laws with value functions were designed for individual agents which can guarantee collision-free conflict resolution. Moreover, the satellite formation containment flying control without collisions was researched by Chen et al.; it was worth noting that the convex hull and close-range omnidirectional are both considered for the system. However, it can be seen that the references mentioned above only consider the local control objective for the SFF system and need all spacecraft state information.

Motivated by the work by Liu et al., this article aims to solve the optimal cooperative control for SFF system with collision avoidance via constructing index cost functions and potential functions, respectively. The contributions of this article are twofold. First, the optimal cooperative control algorithms for SFF are presented, and the control objectives including reference trajectory tracking, collision avoidance, and state optimization can be achieved simultaneously. Moreover, by introducing potential functions and constructing index cost functions, based on the state-dependent Riccati equations, the optimal cooperative control algorithms require only local spacecraft information in the formation flying system.

The remaining parts of the article are organized as follows: the basic definitions and corresponding analysis for cooperative collision avoidance are briefly summarized in section “Background and preliminaries.” In section “Optimal cooperative control algorithm design,” the optimal cooperative algorithms and stability proofs are presented. Simulation results are given to verify the effectiveness of the proposed control algorithms in section “Simulations.” Finally, some conclusions are given in section “Conclusion.”

**Background and preliminaries**

**Basic graph theory**

For SFF system, the graph theory which is used to illustrate the information exchange can represent the spacecraft cooperative control problem briefly. In this article, the undirected graph \( G \) is defined as follows, which can present the information exchange in SFF system

\[
G = (v, e)
\]

where \( v = (v_1, v_2, \ldots, v_n) \) is the set of nodes which denotes the positions for formation spacecraft, and \( e \in v \times v \) is the set of edges which denotes the path following. The edge \((v_i, v_j)\) denotes that there has information exchange between node \( v_i \) and node \( v_j \). Furthermore, it depends on the limited information constraint for judging whether the \( i \)-th spacecraft has communication with the \( j \)-th spacecraft or not. The nonnegative adjacency matrix \( \lambda = [a_{ij}] \) is symmetric for the undirected graph and can be expressed as \( a_{ij} = a_{ji}, \ \forall i \neq j \). Then, the adjacency matrix \( \lambda \) can be defined as \((v_j, v_i) \in e \) if \( a_{ij} = 1 \), otherwise \( a_{ij} = 0 \). The Laplacian matrix \( L_A \) of graph \( G \) is given as follows:

\[
L_A = \begin{bmatrix}
    0 & a_{12} & \cdots & a_{1n} \\
a_{21} & 0 & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & 0
\end{bmatrix}
\]
\[
L_A = D - \lambda \in \mathbb{R}^{n \times n}
\]

where \(D = \text{diag}(d_0, d_1, \ldots, d_n)\), \(d_i = \sum_{j=0}^{n} a_{ij}\).

**Lemma 1.** For the undirected information exchange graph \(G\), \(L_A\) has zero eigenvalue with an associated eigenvector \(1\) and all the other eigenvalues are positive if and only if the graph is connected\(^{22}\). Therefore, \(L_A\) is positive semi-definite and has the following property

\[
L_A \bullet 1 = 0
\]

**Relative dynamics motion model**

The SFF system is assumed as a set of rigid bodies, which is orbited in the Earth-centered inertial (ECI) coordinate system \((OXYZ)\). The relative motion dynamics model is shown in Figure 1.

The reference spacecraft and the formation spacecraft are given by the following dynamic models

![Figure 1. Spacecraft formation flying system relative motion model.](image-url)
\[ \ddot{r} = -\frac{\mu}{r^3} r + \frac{F_{\text{sat}}}{m_{\text{sat}}} + \ddot{f}_{dr} \]  
\[ \ddot{r}_f = -\frac{\mu}{r_f^3} r_f + \ddot{f}_{df} \]  

where \( \mu \) is geocentric gravitational constant. In ECI coordinate frame, \( \ddot{f}_{dr} \) and \( \ddot{f}_{df} \) are perturbation accelerations for reference spacecraft and formation spacecraft, respectively; \( r \) and \( r_f \) are position vectors of the reference and formation spacecrafts; and \( F_{\text{sat}} \) is the control force which is acted on the reference spacecraft. The relative position vector \( \rho \) is described as \( \rho = r - r_f \)

\[ \dot{\rho} = \dot{r}_f - \dot{r} = -\frac{\mu}{r_f^3} r_f + \frac{\mu}{r^3} r + \ddot{f}_{df} - \ddot{f}_{dr} + \frac{F_{\text{sat}}}{m_{\text{sat}}} \]  

Furthermore, in order to describe the relative motion relationship in the LVLH coordinate system, equation (6) can be rewritten as follows

\[ \ddot{\rho} + 2\omega_r \times \dot{\rho} + \omega_r \times (\omega_r \times \rho) + \dot{\omega}_r \times \rho = -\mu \left( \frac{r_r}{r^3} - \frac{r_f}{r_f^3} \right) + \ddot{f}_{dr} - \ddot{f}_{df} + \frac{F_{\text{sat}}}{m_{\text{sat}}} \]  

However, in this article, we assume that the SFF system runs on the Kepler elliptical orbit, and the relative orbital dynamics equation of the SFF can be converted into the form of components and linearized as follows (Appendix 1)

\[ \begin{cases} 
\ddot{x} - 2\dot{\theta} \dot{y} - \left( \dot{\theta}^2 - \frac{\mu \dot{\theta}^{3/2}}{h^{3/2}} \right) x - \ddot{\theta} y = dx \\
\ddot{y} + 2\dot{\theta} \ddot{x} - \left( \dot{\theta}^2 + 2 \left( \frac{\mu \dot{\theta}^{3/2}}{h^{3/2}} \right) \right) y + \ddot{\theta} x = dy \\
\ddot{z} + \left( \frac{\mu \dot{\theta}^{3/2}}{h^{3/2}} \right) z = dz 
\end{cases} \]  

where \( x, y, z \) denote the formation spacecraft position coordinates with respect to the reference spacecraft; \( D = [d_x \ d_y \ d_z]^T \) denotes the relative perturbation acceleration; and \( U = [u_x \ u_y \ u_z]^T \) denotes the control acceleration of the formation spacecraft.

In this article, we assume that the relative perturbation acceleration of SFF is \( D = 0 \). According to each spacecraft, we associate the following form: \( X = [x \ y \ z \ \dot{x} \ \dot{y} \ \dot{z}]^T \) denotes the formation system state, \( Y = [x \ y \ z]^T \) denotes the formation system output, then equation (8) can be expressed by the following state equations

\[ \dot{X} = AX + BU \]
\[ Y = CX \]  

where

\[ A = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & \dot{\theta}^2 - \mu \dot{\theta}^3/\hbar^{3/2} & 0 & 0 & 0 & 2\dot{\theta} \\
-\dot{\theta} & \dot{\theta}^2 + 2\mu \dot{\theta}^3/\hbar^{3/2} & 0 & -2\dot{\theta} & 0 & 0 \\
0 & 0 & -\mu \dot{\theta}^3/\hbar^{3/2} & 0 & 0 & 0 \\
\end{bmatrix} \]  

\[ B^T = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix} \]  

\[ C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\end{bmatrix} \]

The path following for SFF is shown in Figure 2 which describes the trajectory tracking and formation keeping. It can be seen that the spacecraft formation \( \Lambda \) is a constant vector and enables to track the reference trajectory \( \mathbf{r}_{ref} \) with a desired velocity \( \dot{\mathbf{r}}_{ref} \) while maintaining a specific formation pattern without collisions.

The desired state vector of spacecraft formation is defined as follows

\[ \mathbf{X}_{des} = [x_{des} \ y_{des} \ z_{des} \ \dot{x}_{des} \ \dot{y}_{des} \ \dot{z}_{des}]^T \]

Then, the error state vector of the spacecraft formation is defined as follows

\[ \mathbf{X} = \mathbf{X} - \mathbf{X}_{des} \]
The error state equation for SFF system can be expressed as follows

$$\dot{X} = X - \dot{X}_{\text{des}} = AX + BU - AX_{\text{des}} = A\dot{X} + BU$$  \hspace{1cm} (15)

where \( \dot{X}_{\text{des}} = AX_{\text{des}} + BU_{\text{des}} = AX_{\text{des}} \), since \( U_{\text{des}} = 0 \) when the spacecraft reaches the desired position.

**Cooperative collision avoidance control**

The SFF system is assumed as a group of \( N \)-independent rigid bodies and the relative motions are expressed as equations (9) and (10). The cooperative collision avoidance control is aimed to guarantee the spacecraft to obtain the desired formation pattern without collisions. In order to achieve this control objective, according to each spacecraft in the formation flying system, the potential functions can be defined as follows

$$V_{ij}(r_i, r_j) = \left\{ \begin{array}{ll} \min \left(0, \frac{\| r_i - r_j \|^2 - r_{\text{out}}^2}{\| r_i - r_j \|^2 - r_{\text{in}}^2} \right) \right. & , \; i, j \in N, i \neq j \end{array} \right. \hspace{1cm} (16)$$

where \( r_{\text{out}} \) denotes the radius of the region in which the spacecraft can communicate with other spacecraft. \( r_{\text{in}} \) denotes the avoidance region in which the smallest safe distance between the spacecrafts can be seen. It can be noticed that equation (16) is related to penalty and barrier functions which can be used for designing optimal control algorithms. The gradient of \( V_{ij}(r_i, r_j) \) with respect to the vector \( r_i \) is given as follows

$$\nabla_{r_i} V_{ij}(r_i, r_j) = \left\{ \begin{array}{ll} 0, & \| r_i - r_j \| \geq r_{\text{out}} \\ 4 \frac{(r_{\text{out}}^2 - r_{\text{in}}^2)(\| r_i - r_j \|^2 - r_{\text{out}}^2)}{(\| r_i - r_j \|^2 - r_{\text{in}}^2)^3} (r_i - r_j)^T, & r_{\text{in}} < \| r_i - r_j \| < r_{\text{out}} \\ \text{not defined}, & \| r_i - r_j \| = r_{\text{in}} \\ 0, & \| r_i - r_j \| < r_{\text{in}} \end{array} \right. \hspace{1cm} (17)$$

The SFF system with collision avoidance, shown in Figure 3, can express the communication graph and the connected graph directly. In this article, the information exchange topology for SFF is assumed to be undirected and connected. Then, the problem can be stated as follows: the spacecraft communicates with its neighbor spacecraft through communication range \( R \), in which the topology is based on \( r_{\text{out}} \), where \( 0 < r_{\text{out}} < R \).

The relevant constraints for SFF system can be defined by the following regions. For collision avoidance region

$$\Xi_{ij} \triangleq \{ r \in \mathbb{R}^3, 0 < r_{\text{in}} < \| r_i - r_j \| \} \hspace{1cm} (18)$$

For connectivity preservation region
For communication region

\[ T_{ij} \triangleq \{ r \mid r \in \mathbb{R}^3, \| r_i - r_j \| \leq r_{out} \} \]  

The total potential function is defined as follows

\[ \Psi_{ij} = \sum_{i=1}^{N} V_{ij}(r_i, r_j) \]  

Then, \( \Psi_{ij} \) is taken as a smooth nonnegative function in the region \( \| r_i - r_j \| > r_{in} \), satisfying \( \lim_{\| r_i - r_j \| \rightarrow r_{in}^-} V_{ij}(r_i, r_j) = +\infty, \quad \forall i = 1, \ldots, n. \)

### Optimal cooperative control algorithm design

In this section, the optimal cooperative control algorithms for SFF are proposed that guarantee the entire formation system asymptotically stable. In addition, an important corresponding lemma is first given to illustrate the optimality of the proposed cooperative control algorithms.

**Lemma 2.** The nonlinear controlled dynamical system is considered as

\[ \dot{\bar{X}}(t) = f(\bar{X}(t), U(t)), \bar{X}(0) = X_0, \quad t \geq 0 \]  

with \( f(0, 0) = 0 \) and a cost function is given by

\[ J(\bar{X}_0(t), U(\cdot)) \triangleq \int_0^{\infty} \Gamma(\bar{X}(t), U(t)) dt \]
where $U(\bullet)$ is an available control. Assume that there exists a continuously differentiable function $V : K \rightarrow \mathbb{R}^n$ in which $K \subset \mathbb{R}^n$ is an open set and a control law $\Phi : K \rightarrow \Omega$, then

$$V(0) = 0$$ (24)

$$V(\bar{X}(t)) > 0, \bar{X}(t) \in \mathbb{R}^n, \bar{X}(t) \neq 0$$ (25)

$$\Phi(0) = 0$$ (26)

$$\dot{V}(\bar{X}(t)) + f(\bar{X}(t), \Phi(\bar{X}(t))) < 0, \bar{X}(t) \in \mathbb{R}^n, \bar{X}(t) \neq 0$$ (27)

$$H(\bar{X}(t), \Phi(\bar{X}(t))) = 0, \bar{X}(t) \in \mathbb{R}^n$$ (28)

$$H(\bar{X}(t), U(t)) \geq 0, \bar{X}(t) \in \mathbb{R}^n, U(t) \in \Omega$$ (29)

where $H(\bar{X}(t), U(t)) \triangleq \Gamma(\bar{X}(t), U(t)) + \dot{V}(\bar{X}(t))f(\bar{X}(t), U(t))$

is the Hamiltonian function.

Then, with the feedback control law $U(\bullet) = \Phi(\bar{X}(\bullet))$, the solution $\bar{X}(t) = 0, \ t \geq 0$ of the closed-loop system is asymptotically stable, and it exists as a neighborhood of the origin $K_0 \subseteq K$ such that

$$J(\bar{X}_0(t), \Phi(\bar{X}(\bullet))) = V(\bar{X}_0(t))$$ (30)

Furthermore, the feedback control laws $U(\bullet) = \Phi(\bar{X}(\bullet))$ minimized $J(\bar{X}_0(t), U(\bullet))$ in the case that

$$J(\bar{X}_0(t), \Phi(\bar{X}(\bullet))) = \min_{U(\bullet) \in U(\bar{X}_0)} J(\bar{X}_0(t), U(\bullet))$$ (31)

where $U(\bar{X}_0)$ denotes the set of asymptotically stabilizing control laws for each initial condition $\bar{X}_0(t) \in K$. Finally, if $K = \mathbb{R}^n$, $\Omega = \mathbb{R}^m$, then $V(\bar{X}(t)) \rightarrow \infty$ as $||\bar{X}(t)|| \rightarrow \infty$. The solution $\bar{X}(t) = 0$ for the closed-loop system is globally asymptotically stable.

Moreover, in this article, the cost functions are presented theoretically for measuring optimal control performance for SFF system, including formation error control cost, control constraint cost, and collision avoidance cost.

The optimal index cost functions are given as follows

$$\min : J(\bar{X}(t), U(t)) = J_1(\bar{X}(t), U(t)) + J_2(\bar{X}(t), U(t)) + J_3(\bar{X}(t), U(t))$$

$$s.t. \quad \dot{\bar{X}}(t) = A\bar{X}(t) + BU(t)$$ (32)

where $J_1(\bar{X}(t), U(t))$ denotes the comprehensive error cost between the actual error and the given state error, $J_2(\bar{X}(t), U(t))$ denotes constraint cost for control total energy, and $J_3(\bar{X}(t), U(t))$ denotes collision avoidance cost.

The index cost functions can be divided into three parts:
Part 1. Formation error index cost $J_1(\tilde{X}(t), U(t))$

$$J_1(\tilde{X}(t), U(t)) = \frac{1}{2} \int_0^\infty \dot{X}^T(t)Q(t)\dot{X}(t)dt$$ \hspace{1cm} (33)

where $Q(t)$ is the weighted matrix of state variables. The optimal control algorithms for SFF system can track reference trajectory and reach desired position by minimizing $J_1(\tilde{X}(t), U(t))$.

Part 2. Control energy index cost $J_2(\tilde{X}(t), U(t))$

$$J_2(\tilde{X}(t), U(t)) = \frac{1}{2} \int_0^\infty U^T(t)R(t)U(t)dt$$ \hspace{1cm} (34)

where $R(t)$ is the weighted matrix of control effort and positive definite.

Part 3. Collision avoidance index cost $J_3(\tilde{X}(t), U(t))$

$$J_3(\tilde{X}(t), U(t)) = \frac{1}{2} \int_0^\infty \Psi_{ij}[\tilde{X}(t)]dt$$ \hspace{1cm} (35)

However, $\Psi_{ij}[\tilde{X}(t)]$ can be denoted as the following form

$$\Psi_{ij}[\tilde{X}(t)] = \frac{1}{4} \dot{\dot{V}}^T_{ij}BR^{-1}B^T(t)\dot{V}_{ij} - \dot{\dot{V}}^T_{ij}(A - BR^{-1}(t)B^TP(t))\tilde{X}(t)$$ \hspace{1cm} (36)

where $V_{ij}$ is given by equation (16). And $P(t)$ is the solution for the state-dependent Riccati equation.

**Theorem 1.** If the communication graph is undirected and connected, the proposed optimal control algorithms (equation (37)) can guarantee the optimality and global asymptotic stability for the SFF system (equations (9)–(10))

$$U(t) = -\frac{1}{2}R^{-1}(t)B^T\dot{V}_{ij} - R^{-1}(t)B^TP(t)\tilde{X}(t)$$ \hspace{1cm} (37)

The optimal index cost functions $J(\tilde{x}(t), u(t))$ with respect to $t \to \infty$ has the following form

$$P(t)A + A^TP(t) - P(t)BR^{-1}(t)B^TP(t) + Q(t) = 0$$ \hspace{1cm} (38)

Moreover, the feedback gain matrix $K(t)$ can be expressed as follows

$$K(t) = R^{-1}(t)BP(t)$$ \hspace{1cm} (39)
Proof 1. The optimal cooperative control problem mentioned above can be converted to the following form

$$\Theta(\dot{X}(t), U(t)) = \dot{X}^T(t)Q(t)\dot{X}(t) + U^T(t)R(t)U(t) + \Psi_{ij}[\dot{X}(t)]$$  \hspace{1cm} (40)$$

$$f(\dot{X}(t), U(t)) = A\dot{X}(t) + BU(t)$$  \hspace{1cm} (41)$$

The following Lyapunov function for formation system is constructed as follows

$$V(\dot{X}(t)) = V_{ij} + \dot{X}^T(t)P(t)\dot{X}(t)$$  \hspace{1cm} (42)$$

For obtaining the optimal cooperative control algorithms, the case of $$(\partial/\partial U(t))H(\dot{X}(t), U(t), \dot{V}^T(\dot{X}(t)))$$ should be equal to zero.

where

$$H(\dot{X}(t), U(t), \dot{V}^T(\dot{X}(t))) = \Theta(\dot{X}(t), U(t)) + \dot{V}^T(\dot{X}(t))f(\dot{X}(t), U(t))$$

$$= \dot{X}^T(t)Q(t)\dot{X}(t) + U^T(t)R(t)U(t) + \Psi_{ij}[\dot{X}(t)]$$

Then, equation (37) can be rewritten in the following form

$$U(t) = \Phi(\dot{X}(t)) = \frac{-1}{2}R^{-1}B^T\dot{V}(\dot{X}(t))$$

$$= -R^{-1}B^T P(t)\dot{X}(t) - \frac{1}{2}R^{-1}B^T \dot{V}_{ij}$$ \hspace{1cm} (44)$$

By equation (42), the derivative of Lyapunov can be obtained as follows

$$\dot{V}^T(\dot{X}(t))f(\dot{X}(t), \Phi(\dot{X}(t))) = \dot{X}^T(t)(A^T P(t) + P(t)A - 2P(t)BR^{-1}(t)B^T)$$

$$- \frac{1}{2} \dot{V}_{ij}BR^{-1}(t)B^T V_{ij} - \dot{X}^T(t)P(t)BR^{-1}(t)B^T \dot{V}_{ij}$$

$$+ \dot{V}^T_{ij}(A - BR^{-1}(t)B^T P(t))\dot{X}(t)$$ \hspace{1cm} (45)$$

Substituting the feedback gain matrix (equation (39)), equation (45) can be rewritten as follows

$$\dot{V}^T(\dot{X}(t))f(\dot{X}(t), \Phi(\dot{X}(t))) = \dot{X}^T(t)(A^T P(t) + P(t)A - 2K(t)B^T)\dot{X}(t)$$

$$- \frac{1}{2} \dot{V}_{ij}BR^{-1}(t)B^T V_{ij} - \dot{X}^T(t)K(t)B^T \dot{V}_{ij}$$

$$+ \dot{V}^T_{ij}(A - BR^{-1}(t)B^T P(t))\dot{X}(t)$$ \hspace{1cm} (46)$$

The part of Hamiltonian function can be rewritten as follows
where the parts of \( P(t) \) and \( Q(t) \) satisfy with equation (38).

In order to let equation (47) to be zero, the following equation should be equal to zero

\[
\dot{V}_{ij}^T (A - BR^{-1}(t)B^T P(t)) \ddot{X}(t) + \dot{\Psi}_{ij} - \frac{1}{4} \dot{V}_{ij}^T BR^{-1}(t)B^T \dot{V}_{ij} = 0 \quad (48)
\]

Then, with equations (38), (43), and (48), equation (47) can be rewritten as follows

\[
H(\dddot{X}(t), U(t), \dot{V}^T(\dddot{X}(t))) = U^T(t)R(t)U(t) + \frac{1}{4} \dot{V}_{ij}^T BR^{-1}(t)B^T V_{ij}
\]

\[
+ \dot{V}_{ij}^T BR^{-1}(t)B^T P(t) \ddot{X}(t) + \dot{X}^T(t)P(t)BR^{-1}(t)B^T P(t) \dddot{X}(t)
\]

\[
+ (2 \dddot{X}(t)P(t) + \dot{V}_{ij}^T)BU(t)
\]

\[
= U^T(t)R(t)U(t) + (2 \dddot{X}(t)P(t) + \dot{V}_{ij}^T)BU(t)
\]

\[
+ \frac{1}{4} (2 \dddot{X}(t)P(t) + \dot{V}_{ij}^T)BR^{-1}(t)B^T (2 \dddot{X}(t)P(t) + \dot{V}_{ij}^T)^T
\]

\[
= U^T(t)R(t)U(t) + \frac{1}{4} \dot{V}^T(\dddot{X}(t)) BR^{-1}(t)B^T \dot{V}(\dddot{X}(t))
\]

\[
+ U^T(t)B^T \dot{V}(\dddot{X}(t))
\]

\[
= U^T(t)R(t)U(t) + \Phi^T(\dddot{X}(t))R(t)\Phi(\dddot{X}(t))
\]

\[
- 2U^T(t)R(t)\Phi(\dddot{X}(t))
\]

\[
= (U(t) - \Phi(\dddot{X}(t)))^T R(t)(U(t) - \Phi(\dddot{X}(t)))
\]

\[
\geq 0
\]

\[
(49)
\]

From equation (49), we can conclude that condition (29) in Lemma 2 is proved.

Then, with equations (38) and (47), equation (45) can be rewritten as follows

\[
\dot{V}^T(\dddot{X}(t))f(\dddot{X}(t), \Phi(\dddot{X}(t))) = -[\dddot{X}^T(t)Q(t)\dddot{X}(t) + \Psi_{ij}] + (\dddot{X}(t)P(t) + \frac{1}{2} \dot{V}_{ij}^T BR^{-1}(t)B^T (P(t) \dddot{X}(t) + \frac{1}{2} \dot{V}_{ij})] < 0
\]

\[
(50)
\]

Thereby, condition (27) in Lemma 2 is obtained that guarantees

\[
\dot{V}^T(\dddot{X}(t))f(\dddot{X}(t), \Phi(\dddot{X}(t))) < 0.
\]
Therefore, according to the evidence mentioned above, the proposed optimal cooperative control algorithms can guarantee the stability for the closed-loop system and has no collisions with each spacecraft which can also obtain system optimality.

**Simulations**

To verify the effectiveness of the proposed control algorithms in this article, the formation flying system of three spacecrafts is simulated in this section. The formation spacecraft orbital parameters and other parameters set at the initial time are shown in Table 1.

The Laplacian matrix $L_A$ for the undirected and connected communication topology is given as

$$L_A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 1 \end{bmatrix}$$

The initial formation configuration for spacecraft is centered on the reference spacecraft in which the radius is $r = 500$ m. The phases of three spacecraft configurations in the initial time with respect to the original configuration are $0, 2\pi/3, -2\pi/3$, respectively. Maximum control acceleration constraint vector in trajectory tracking condition is given as $u_{\text{max}} = [0.01 \ 0.01 \ 0.01]^T$ m/s$^2$. Eigenvalues of the given matrix are $Q(t) = \text{diag}([1 \ 1 \ 1 \ 1 \ 1 \ 1]), \ R(t) = \text{diag}([3 \ 3 \ 3])$ The minimum allowable distance for collision avoidance is set as $r_{\text{min}} = 20$ m; the potential function is given as follows

$$V_{ij}(r_i; r_j) = \sum_{i=1}^{3} \frac{225,000}{\|r_i - r_j\|^2 - 20^2}$$

The relative state initial values for three spacecrafts are shown in Table 2. The simulation results for the formation system with the proposed control algorithms are shown in Figures 4–7.

Figure 4 shows the three-dimensional tracking trajectory for SFF system. It can be seen that all the spacecrafts converge into the desired formation pattern and the distances between any two spacecrafts in the formation has no changes and without collisions among spacecrafts after around $t = 150$ s, achieving the requirements of tracking trajectories and formulating desired formation pattern.
The velocity tracking error and relative position of formation spacecraft are shown in Figures 5 and 6, respectively. The velocity component on the three-axis keeps and converges to constant because the desired velocity has the component. From Figure 5, it can be seen that the velocities of three spacecrafts in the formation converge to the constant velocity within $t = 120$ s. Figure 6 shows the relative position between the spacecrafts in the formation. It can be seen that the distances between any two spacecrafts are never less than 20 m, and no collision occurs in the regions of communication $T_{ij}$ and connectivity preservation $Z_{ij}$. In addition, the optimal control input of formation spacecraft is shown in Figure 7, which indicates that the system is optimal by the proposed control algorithms.

| Formation spacecraft | $x$ (m) | $y$ (m) | $z$ (m) | $\dot{x}$ (m/s) | $\dot{y}$ (m/s) | $\dot{z}$ (m/s) |
|----------------------|---------|---------|---------|----------------|----------------|----------------|
| $\Lambda_1$          | -250    | 0       | -435    | 0              | 0.5400         | 0              |
| $\Lambda_2$          | 125     | 435     | 217     | 0.2440         | -0.2750        | 0.4100         |
| $\Lambda_3$          | 124     | -435    | 217     | -0.2440        | -0.2750        | -0.4100        |

Table 2. Relative state initial values.

Figure 4. The relative position trajectory of formation spacecraft.

The velocity tracking error and relative position of formation spacecraft are shown in Figures 5 and 6, respectively. The velocity component on the three-axis keeps and converges to constant because the desired velocity has the component. From Figure 5, it can be seen that the velocities of three spacecrafts in the formation converge to the constant velocity within $t = 120$ s. Figure 6 shows the relative position between the spacecrafts in the formation. It can be seen that the distances between any two spacecrafts are never less than 20 m, and no collision occurs in the regions of communication $T_{ij}$ and connectivity preservation $Z_{ij}$. In addition, the optimal control input of formation spacecraft is shown in Figure 7, which indicates that the system is optimal by the proposed control algorithms.
Figure 5. The velocity tracking error of formation spacecraft.

Figure 6. The relative position of formation spacecraft.
Conclusion

In this article, the novel optimal cooperative control associated with collision avoidance for SFF system was investigated. Moreover, it can be seen that the state matrix of relative motion for spacecraft formation is a linear form of the information vectors that only use the local information from the spacecraft. Based on the index cost functions to evaluate system state performance and potential functions to avoid collisions, the optimal cooperative control algorithms were proposed that can guarantee the asymptotic stability and optimal performance of the system. And the simulation results demonstrated that the effectiveness and the robustness of the proposed control algorithms can track the reference trajectory without collision avoidance.

Declaration of conflicting interests

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding

The author(s) received no financial support for the research, authorship, and/or publication of this article.

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**Appendix 1**

The semi-major axis, eccentricity, mean angular velocity, and true anomaly for the reference spacecraft orbit are denoted by $a, e, h, \theta$, respectively

\[
\begin{align*}
    h &= \sqrt{\frac{\mu}{a^3}} \quad \text{(53)} \\
    r_t &= \frac{a(1 - e^2)}{1 + e \cos \theta} \quad \text{(54)} \\
    \dot{\theta} &= \frac{n(1 + e \cos \theta)^2}{(1 - e^2)^{3/2}} \quad \text{(55)} \\
    \ddot{\theta} &= -\frac{2n^2 e \sin \theta (1 + e \cos \theta)^3}{(1 - e^2)^{3}} \quad \text{(56)}
\end{align*}
\]

Follow equation (7), according to the reference spacecraft orbit coordinate system $(oxyz)$

\[
\begin{align*}
    \mathbf{\rho} &= \begin{bmatrix} x & y & z \end{bmatrix}^T \\
    \dot{\mathbf{\rho}} &= \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^T, \quad \mathbf{\omega}_r = \begin{bmatrix} 0 & 0 & \dot{\theta} \end{bmatrix}^T, \quad \mathbf{\dot{\omega}}_r = \begin{bmatrix} 0 & 0 & \ddot{\theta} \end{bmatrix}^T
\end{align*}
\]
Then, substitute into equation (7)

\[
\begin{align*}
\ddot{x} - 2\dot{\theta}\dot{y} - \ddot{\theta}x + \mu r_r^{-3}(1 - 2y/r_r + (x^2 + y^2 + z^2)/r_r^2)^{-3/2}x &= dx \\
\ddot{y} + 2\dot{\theta}\dot{x} - \ddot{\theta}y + \mu r_r^{-3}((1 - 2y/r_r + (x^2 + y^2 + z^2)/r_r^2)^{-3/2}(y - r_r) + r_r) &= dy \\
\ddot{z} + \mu r_r^{-3}(1 - 2y/r_r + (x^2 + y^2 + z^2)/r_r^2)^{-3/2}z &= dz
\end{align*}
\] (57)

Furthermore, the relative distance between the spacecrafts is much smaller than the distance from the spacecraft to the earth center; then, the nonlinear part \((1 - 2y/r_r + (x^2 + y^2 + z^2)/r_r^2)^{-3/2}\) in equation (57) can be simplified as follows

\[
(1 - 2y/r_r + (x^2 + y^2 + z^2)/r_r^2)^{-3/2} = 1 + 3y/r_r
\] (58)

In this article, we assume that the spacecraft formation flying system runs on the Kepler elliptical orbit, then the linearized motion equation of spacecraft formation flying can be expressed as follows

\[
\begin{align*}
\ddot{x} - 2\dot{\theta}\dot{y} - \ddot{\theta}x - \mu\dot{\theta}^2/h^3/2 x - \ddot{\theta}y &= dx \\
\ddot{y} + 2\dot{\theta}\dot{x} - \ddot{\theta}y + 2\mu\dot{\theta}^2/h^3/2 y + \ddot{\theta}x &= dy \\
\ddot{z} + \mu\dot{\theta}^2/h^3/2 z &= dz
\end{align*}
\] (59)