Null Physical States in String Models

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Abstract

This note is a brief addendum to my article Nucl. Phys. B 864 (2012) 285, [arXiv: 1110.5510], which discusses the noghost theorem in Ramond sectors of string models. In this addendum we derive additional information about the structure of null physical states in the Ramond-Neveu-Schwarz model.
1 Introduction

The structure of null states lies at the heart of the proofs of the no ghost theorems for string models. This is true in the original versions [1–3], their improved versions [4, 5], and also in the modern BRST-based versions [4–6]. In all cases, one shows that in the critical dimension $(D = 26$ for the bosonic models and $D = 10$ for strings based on Ramond-Neveu-Schwarz models [7]) all physical states, i.e. all states that couple to physical on-shell processes, can be expressed as

$$|\text{phys}\rangle = |\text{null}\rangle + |T\rangle$$

where the transverse states $|T\rangle$ span a positive definite subspace of the physical states, and the null states have zero overlap with themselves and with all physical states. The state spaces of the superstring and other derivative string models lie within the state spaces of these parent critical string models, and so are also covered by these theorems.

While some properties of the null states are derived in the course of proving the no ghost theorem, there are more detailed facts about them that require further argumentation to establish. For example, in the appendix of my paper [4], which streamlined the original Goddard-Thorn proof [1], I proved that all on-shell ($[L_0 - 1]|\text{null}\rangle = 0$) null states in the bosonic model can be expressed in the form

$$|\text{null}\rangle = L_{-1}|\text{phys}\rangle_1 + \left( L_{-2} + \frac{3}{2} L_{-1}^2 \right)|\text{phys}\rangle_2.$$  \hspace{1cm} (2)

where $L_n$ are the generators of the Virasoro algebra. The states $|\text{phys}\rangle_1,2$ are annihilated by all $L_n$ with $n > 0$. In the language of conformal field theory this means they are primary states, and the above equation states that all null states are either of two particular descendants of primary states. Such a classification of null states has proved useful in some investigations, for example Witten’s recent treatise on superstring perturbation theory [8].

Analogous facts are true of the non-bosonic string models. We will use the notation of the original papers: the super-Virasoro generators will be denoted $F_n, L_n$ in Ramond sectors and $G_r, L_n$ in Neveu-Schwarz sectors. Indices $m, n$ will run over all integers, and indices $r, s$ will run over all half-odd integers. In Neveu-Schwarz sectors of such models the analog of (2) reads

$$|\text{null}\rangle = G_{-1/2}|\text{phys}\rangle_1 + \left( G_{-3/2} + \frac{1}{2} G_{-1/2} L_{-1} \right)|\text{phys}\rangle_2.$$  \hspace{1cm} (3)

the proof of which is a straightforward generalization of the one in the appendix of [4]. However the corresponding generalization to Ramond sectors

$$|\text{null}\rangle = F_0 L_{-1}|\text{phys}\rangle_1 + F_0 F_{-1}|\text{phys}\rangle_2$$  \hspace{1cm} (4)

is less straightforward because of zero mode complications, so we devote the remainder of this short note to explaining it. The corresponding argument for Neveu-Schwarz sectors is completely parallel but with the absence of zero-mode complications. In the following section 2 I recall some results from [5] that are necessary to complete the proof of the validity of (4), after which I complete the proof of (4) in section 3.
2 An ordered basis

Here we gather results from [5] that we will need later. The super-Virasoro algebra in $D$ spacetime dimensions reads:

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{D}{8} n^3 \delta_{n,-m} \quad (5)$$
$$[L_n, F_m] = \left(\frac{n}{2} - m\right) F_{n+m} \quad (6)$$
$$\{F_n, F_m\} = 2L_{n+m} + \frac{D}{2} n^2 \delta_{n,-m} \quad (7)$$

In the following we will always assume the critical dimension $D = 10$. A lightlike vector $k^\mu$ is chosen to define $D_n = k \cdot d_n$ and $K_n = k \cdot a_n$, where $d_n^\mu$ and $a_n^\mu$ are the fermionic and bosonic modes respectively of the Ramond sector. We will always work in the eigenspace of only the values 0 or 1. A conjugate (or "dual") to each element (12) is defined by

$$D \quad (12)$$

are bosonic partitions of two nonnegative integers. Similarly $|$ and $\langle$ are fermionic partitions of two nonnegative integers. Fermionic simply means that each $f_i$ and $d_i$ assumes only the values 0 or 1. A conjugate (or "dual") to each element (12) is defined by

$$\langle f, \lambda \rangle \neq 0 \quad (11)$$

Defining the norm with this inner product, the transverse states have nonnegative norm, relative to an overall constant factor.

In [5] we established that the basis set of the whole Ramond sector state space,

$$|\{f\} \{\lambda\}, \{d\} \{\kappa\} \rangle = F_{0}^{f_{0}} L_{\lambda_1}^{\lambda_{1}} \cdots F_{l_{1}}^{f_{j_{1}}} L_{\lambda_{l_{1}}}^{\lambda_{l_{1}}} D_{-k}^{d_{k}} K_{-l}^{\kappa_{l}} \cdots K_{-k}^{\kappa_{k}} |T\rangle \quad (12)$$

where $|T\rangle$ are arbitrary transverse states, is linearly independent. The labels $\{\lambda\}$ and $\{\kappa\}$ are bosonic partitions of two nonnegative integers. Similarly $\{f\}$ and $\{d\}$ are fermionic partitions of two nonnegative integers. Fermionic simply means that each $f_i$ and $d_i$ assumes only the values 0 or 1. A conjugate (or "dual") to each element $f_1, \cdots, f_n$ is defined by

$$\langle \{f\} \{\lambda\}, \{d\} \{\kappa\}, C \rangle = F_{0}^{-f_{0}} F_{l_{1}}^{d_{l_{1}}} L_{\lambda_{1}}^{\lambda_{l_{1}}} \cdots F_{-k}^{d_{k}} L_{-l}^{\kappa_{l}} D_{-k}^{f_{l_{1}}} \cdots D_{-l}^{f_{1}} K_{l}^{\lambda_{l}} \cdots K_{l}^{\lambda_{l}} |T\rangle \quad (13)$$

The inner product of each basis element with its conjugate is not zero.

We also defined an ordering of this basis by ordering the partitions $\{f\}, \{\lambda\}$, according to $\langle \{f, \lambda\} \rangle < \langle \{f', \lambda'\} \rangle$ if the first nonzero entry of the sequence

$$\sum_i i(f_i - f_i' + \lambda_i - \lambda_i') \quad f_0 - f_0', \quad f_1 - f_1', \quad \lambda_1 - \lambda_1', \quad f_2 - f_2', \quad \cdots \quad (14)$$

2
is positive. A similar ordering is defined for the other pair of partition labels \( \{d\kappa\} \). Then we order the entire basis according to \((\{f\lambda\},\{d\kappa\}) < (\{f'\lambda'\},\{d'\kappa'\})\) if \(f\lambda < f'\lambda'\) or if \(f\lambda = f'\lambda'\) and \(\{d\kappa\} > \{d'\kappa'\} \). With this ordering we then quote a crucial result of [5], which will also be needed in the following section:

\[
\langle\{f\}\{\lambda\},\{d\}\{\kappa\},C\{f'\}\{\lambda'\},\{d'\}\{\kappa'\}\rangle = 0, \quad \text{if} \quad (\{f\lambda\},\{d\kappa\}) < (\{f'\lambda'\},\{d'\kappa'\}) , \quad (15)
\]

which is to say that the corresponding matrix of inner products is lower triangular.

### 3 Null States

We first enumerate all physical states, those annihilated by \( L_n \) and \( F_n \) for all \( n > 0 \), on and off shell. They are spanned by the basis

\[
|\{f\}\{\lambda\}, \text{phys}\rangle = F_0 F_{-1} F_{-1}^\lambda L_{-1}^\lambda \cdots F_{-1}^n L_{-1}^\lambda |T\rangle + \text{Terms with } \{d, \kappa\} \neq 0 . \quad (16)
\]

For each fixed \( \{f, \lambda\} \) the unlisted terms are uniquely determined.\(^2\) The first term, which completely determines each such physical state will be called the leading term. In the following we will frequently be working with that term alone with all the others implied.

The following are on-shell null states \((L_0 = 0)\):

\[
F_0 F_{-1} |\text{phys}\rangle_1 , \quad F_0 L_{-1} |\text{phys}\rangle_2 \quad (17)
\]

as can be seen by a short direct calculation. The on-shell condition means that the \( L_0 \) eigenvalues of \(|\text{phys}\rangle_1,2\) are always -1. In the following we show that these are all of the on-shell null states. We can enumerate the states \(|\text{phys}\rangle_1,2\) via the basis \((16)\), but there are linear dependences among the states \((17)\) in that labeling. First of all, from \(F_0^2 = L_0 = 0\) on-shell and the superconformal algebra, we have the proportionalties

\[
F_0 F_{-1} F_0 F_{-1} \propto F_0 L_{-1} F_{-1} , \quad F_0 F_{-1} F_0 \propto F_L L_{-1} \quad (18)
\]

\[
F_0 L_{-1} F_0 F_{-1} \propto F_0 F_{-1} F_0 \propto F_0 F_{-1} \quad (19)
\]

Thus those basis states contributing to \(|\text{phys}\rangle_1,2\) with leading terms with \(f_0 = 1\) give the same contribution to the null state as those with \(f_0 = 0\). (Recall that the Null states are physical and that the contribution of each basis element is uniquely fixed by the leading term).

Each on-shell basis element has \(f_0 = 1\). Substituting in turn each of the basis elements with \(f_0 = f_1 = 0\) for \(|\text{phys}\rangle_1\), we see that we will generate all on-shell physical basis elements

\(^2\)To see this one applies in turn, in the order highest to lowest according to \((14)\), the monomials \(L_{-1}^\lambda F_{-1}^n \cdots L_{-1}^\lambda F_{-1}\) to a general linear combination of the basis states \((12)\). Then because of the triangularity \((15)\) the action of the monomial picks out one by one the terms with \(\{d, \kappa\} \neq 0\) which produce a term with \(\{d, \kappa\} = 0\). This unique term can only be cancelled by states produced by the action of the monomial on terms with \(\{d, \kappa\} = 0\). Thus all physical states must have at least one term with \(\{d, \kappa\} = 0\), and further the structure of the states \((16)\) is uniquely determined.
with $f_0 = f_1 = 1$. To obtain the states with $f_1 = 0$, we examine the second class of states. Substituting in turn each of the basis elements with $f_0 = f_1 = 0$ for $|\text{phys}\rangle_2$, we produce all on-shell physical basis elements with $f_0 = 1, f_1 = 0, \lambda_1 \geq 1$.

We proceed step by step. Next substitute each of the basis elements with $f_0 = 0, f_1 = 1, \lambda_1 = 0$ for $|\text{phys}\rangle_2$, we find a leading term that starts with $F_0 L_{-1} F_{-1} \cdots$, which is out of canonical order. We then use the algebra to rearrange

$$F_0 L_{-1} F_{-1} = F_0 F_{-1} L_{-1} + \frac{1}{2} F_0 F_{-2}$$  \hspace{1cm} (20)

which puts the factors in canonical order. We can subtract a null state of the first type to cancel away the physical state associated with the leading term from the first term on the right from the null state we just formed, so that what remains is all of the physical states associated with a leading term with $f_0 = f_1 = \lambda_1 = 0$ and $f_2 = 1$. To find the states with $f_0 = 1, f_1 = \lambda_1 = f_2 = 0$, we substitute basis elements with $f_1 = 1, \lambda_1 = f_2 = 0$ for $|\text{phys}\rangle_1$. Then, because $F^2_{-1} = L_{-2}$, the leading term of the resulting null state has $f_0 = 1, f_1 = \lambda_1 = f_2 = 0$, and $\lambda_2 \geq 1$.

We next look for states with $f_0 = 1, f_1 = \lambda_1 = f_2 = \lambda_2 = 0$. First substitute the physical states with leading term for which $f_0 = 0, f_1 = 1, \lambda_1 = f_2 = \lambda_2 = 0$ for $|\text{phys}\rangle_2$. Then rearrange

$$L_{-1} F_{-1} L_{-1} = \frac{1}{2} F_{-2} L_{-1} + F_{-1} L^2_{-1} = -\frac{1}{2} \cdot \frac{3}{2} F_{-3} + \frac{1}{2} L_{-1} F_{-2} + F_{-1} L^2_{-1}$$ \hspace{1cm} (21)

which puts all the operators in canonical order. The contributions from the second two terms will produce the leading terms of null states previously accounted for, so they can be cancelled away leaving all the null physical states with leading terms with $f_0 = 1, f_1 = \lambda_1 = f_2 = \lambda_2 = 0$ and $f_3 = 1$, i.e. of the form

$$F_0 F_{-3} L^{\lambda_3}_{-3} \cdots |T\rangle.$$  \hspace{1cm} (22)

To get states with $f_3 = 0$, we substitute basis elements with $f_1 = 1, \lambda_1 = f_2 = \lambda_2 = 0$ for $|\text{phys}\rangle_1$. Then, using $F^2_{-1} = L_{-2}$, we rearrange

$$L_{-2} L_{-1} = -L_{-3} + L_{-1} L_{-2}$$ \hspace{1cm} (23)

The second term produces null stated already accounted for. Subtracting them leaves all null physical states with $f_0 = 1, f_1 = \lambda_1 = f_2 = \lambda_2 = f_3 = 0$ and $\lambda_3 \geq 1$.

From here on we just continue this process recursively. At the $n$th step we first substitute the physical states with leading term for which $f_0 = 0, f_1 = 1, \lambda_1 = n, f_2 = \lambda_2 = f_3 = \lambda_3 = \cdots = f_{n+1} = \lambda_{n+1} = 0$ for $|\text{phys}\rangle_2$. Then rearrange

$$L_{-1} F_{-1} L^n_{-1} = F_{-1} L^{n+1}_{-1} + \frac{1}{2} F_{-2} L^n_{-1}$$

$$= F_{-1} L^{n+1}_{-1} + \frac{1}{2} L_{-1} \sum_{k=0}^{n-1} k! \binom{-3/2}{k} \binom{n}{k} L^{n-k-1}_{-1} F_{-(k+2)}$$

$$+ (-)^n \frac{1}{2} \cdot \frac{3}{2} \cdots \frac{2n+1}{2} F_{-n}$$ \hspace{1cm} (24)
all terms but the last term on the right produce the leading terms of null states previously accounted for, so they can be cancelled away leaving the null states with \( f_0 = 1, f_1 = \lambda_1 = f_2 = \cdots f_{n+1} = \lambda_{n+1} = 0 \) and \( f_{n+2} = 1 \).

The second part of the \( n \)th step is to substitute physical states with leading terms for which \( f_0 = 0, f_1 = 1, \lambda_1 = n, f_2 = \lambda_2 = f_3 = \lambda_3 = \cdots = f_{n+1} = \lambda_{n+1} = f_{n+2} = 0 \) for \(|\text{phys}\rangle_1\).

Then, using \( F^2_1 = L_{-2} \), we rearrange

\[
L_{-2}L_{-1}^n = \sum_{k=0}^{n-1} (-)^k k! \binom{n}{k} L_{-1}^{n-k} L_{-2-k} + (-)^n n! L_{-n-2}
\]

so all terms are in canonical order. All terms but the last term on the right produce the leading terms of null states previously accounted for, so they can be cancelled away leaving the null states with \( f_0 = 1, f_1 = \lambda_1 = f_2 = \cdots f_{n+1} = \lambda_{n+1} = f_{n+2} = 0 \) and \( \lambda_{n+2} \geq 1 \).

Induction on \( n \) then shows that every element in the basis of on-shell \((L_0 = 0)\) physical states \((16)\) with \( f_0 = 1 \) are contained in the list of null states \((17)\), which is thus complete.

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