Short Communication

A note on soliton solutions of fractional hybrid lattice equations

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ABSTRACT

This study is designed to propose a solitary-solution formulation method by applying transformation and ancient Chinese algorithm. This method is projected to obtain closed-form solution of fractional differential-difference equations (FDDEs). A closed-form solution of hybrid lattice is used to demonstrate the rationality and the large potential of the proposed method in finding the solutions of FDDEs. This method is very effective in obtaining the exact solutions of nonlinear FDDEs and it can be extended to obtain the solution of nonlinear FDDEs in mathematical physics. Comparison has been made with previous work.

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1. Introduction

The concept of differential-difference equations (DDEs) [1] has been motivated because of their numerous applications in different areas of physics [2,3] and engineering [4]. An important contribution of DDEs is perceived in modeling complex physical phenomenon such as current flow in electrical networks and vibrations of particle in lattices. In many nonlinear studies, DDEs have gained much attention ever since the work of Fermi et al. [5] in 1960s. Various conventional methods to acquire the solutions of nonlinear differential-difference equations (NDDEs) were presented in the theory of lattice-soliton [6,7], for instance, the bilinear form, symmetries [8], and some closed-form analytical solutions of several lattices in polynomial function of tanh [9]. A little attention was also devoted to obtain the closed-form solutions of NDDEs by using the symbolic computation. Baldwin et al. [10] have proposed the tanh-approach to solve NDDEs. Discrete nonlinear lattices have gained a significant attention in numerous branches of science. A considerable literature has grown up around the theme of nonlinear lattice differential equations (NLDEs) because of their use in physical and mathematical models of enormous real-world phenomena. Existing research recognizes the critical role played by discrete solitons in physical systems on very large scale, for example, biophysical systems, atomic chains with on-site cubic nonlinearities, electrical lattices, molecular crystals.
and recently in arrays of coupled nonlinear optical wave guides. Fractional order derivatives have also been the subject to describe several physical phenomena [11,12] such as diffusion processes, damping laws, rheology, and many other.

In this study, the fractional order transformation is used to transform the fractional differential-difference equation into ordinary differential-difference equation and then analyzed by frequency amplitude formulation [13–15]. It should be noted that the generalized hybrid lattice equation may be converted into discrete mKdV lattice and modified Volterra lattice equation for some particular cases. The proposed methodology allows us to precisely solve FDDEs with the support of symbolic computation.

2. Preliminaries

The Riemann–Liouville derivative of order \( \alpha \) is defined, for a function \( H(x) \), by

\[
D^\alpha H(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - \eta)^{\alpha-1}[H(\eta) - H(0)]d\eta \quad for \quad \alpha < 0, \tag{1}
\]

\[
D^\alpha H(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x - \eta)^{-\alpha}[H(\eta) - H(0)]d\eta \quad for \quad 0 < \alpha < 1, \tag{2}
\]

\[
D^\alpha H(x) = (H^{(n)}(x))^{(\alpha-n)} \quad for \quad n \leq \alpha \leq n+1, \quad n \geq 1, \tag{3}
\]

where \( H : R \rightarrow R \) is a continuous function.

Using the properties of the Riemann Liouville derivative from [12], we have

\[
D^\beta x^\alpha = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta - \alpha)} x^{\alpha-\beta}, \quad \beta > 0, \tag{4}
\]

\[
d^\alpha x(t) = \Gamma(\alpha + 1)dx(t) \tag{5}
\]

3. Method for solving fractional differential-difference equations

In this section, a method is described to solve NDDEs directly. Let’s suppose the FDDE in the form:

\[
d^\alpha u(t) = F(u_{t-1}, u_{t}, u_{t+1}), \quad t > 0, \quad 0 < \alpha \leq 1 \tag{6}
\]

where \( u_{t} \) is a dependent variable, and \( t \) is a continuous variable, and \( n \in \mathbb{Z} \).

The main steps of this method are precised as follows:

**Step 1.** Consider the FDDE as in Eq. (6)

**Step 2.** To find the exact solution of Eq. (6), introduce the variable transformation

\[
u_0(n, t) = u_0(n, \eta), \quad \eta = t^\alpha
\]

Hence, Eq. (6) by using Eq. (5), can be written as:

\[
\frac{d\nu_0}{d\eta} = \frac{1}{\Gamma(\alpha + 1)} F(u_{t-1}, u_{t}, u_{t+1}) \tag{8}
\]

**Step 3.** Using the ancient Chinese algorithm (frequency amplitude formulation) initially applied to nonlinear oscillators in 2006 [16]; two test functions are selected in the forms:

\[
u_{1.1}(n, \eta) = f(\xi_n + \omega_1 \eta), \quad f : R \rightarrow R \tag{9}
\]

\[
u_{1.2}(n, \eta) = g(\xi_n + \omega_2 \eta), \quad g : R \rightarrow R \tag{10}
\]

where \( \xi_n = an + \xi_0 \). \( \xi_0 \) is arbitrary, \( f, g \) are known functions, \( \omega_1, \omega_2 \) are the freely chosen test frequencies, and \( \omega \) is the frequency of the nonlinear oscillator. A bell solitary solution of a DDE is deliberated.

The test functions are selected in a fashion:

\[
u_{1.1}(n, \eta) = \frac{A}{e^{i\omega_1 n + \xi_0} e^{-i\omega_1 n + \xi_0}} \omega_1 = 1, \tag{11}
\]

\[
u_{1.2}(n, \eta) = \frac{A}{e^{i\omega_2 n + \xi_0} e^{-i\omega_2 n + \xi_0}} \omega_2 = \omega. \tag{12}
\]

Which should be compatible for \( u_0 \), \( u_{t-1} \) and \( u_{t+1} \); Eqs. (11) and (12) are now given as:

\[
u_{1.1.1}(n, \eta) = \frac{A}{e^{ih} e^{-ih} \pm e^{-i(h/h - h \omega)}} \tag{13}
\]

\[
u_{1.1.2}(n, \eta) = \frac{A}{e^{i(h/h + \omega_1) + e^{i(h/h + \omega_2)}} \tag{14}
\]

\[
u_{1.2.1}(n, \eta) = \frac{A}{e^{ih} e^{-ih} \pm e^{-i(h/h - h \omega)}} \tag{15}
\]

\[
u_{1.2.2}(n, \eta) = \frac{A}{e^{i(h/h + \omega_1) + e^{i(h/h + \omega_2)}} \tag{16}
\]

where \( h \) is small increment in \( \xi_0 \).

**Step 4.** Define residual function

\[
\mathcal{R}(\eta) = \frac{d\nu_0}{d\eta} = \frac{1}{\Gamma(\alpha + 1)} F(u_{t-1}, u_{t}, u_{t+1}). \tag{17}
\]

On utilizing Eq. (11)–(16) into Eq. (8), the residual functions \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) can be achieved as:

\[
\mathcal{R}_1(\eta) = \frac{d\nu_{1.1}}{d\eta} = \frac{1}{\Gamma(\alpha + 1)} F(u_{t-1,1}, u_{t,1}, u_{t+1,1}) \tag{18}
\]

\[
\mathcal{R}_2(\eta) = \frac{d\nu_{1.2}}{d\eta} = \frac{1}{\Gamma(\alpha + 1)} G(u_{t-1,2}, u_{t,2}, u_{t+1,2}). \tag{19}
\]

**Step 5.** Using frequency amplitude formulation the frequency of the nonlinear oscillator \( \omega \) is assumed to be as follows:

\[
\omega^2 = \frac{\omega_1^2 \mathcal{R}_1(0) - \omega_2^2 \mathcal{R}_2(0)}{\mathcal{R}_1(0) - \mathcal{R}_2(0)} \tag{20}
\]

where \( \omega_1 = 1 \) and \( \omega_2 = \omega \).
Step 6. Equating each coefficient $e^{\xi_n}, e^{2\xi_n}, e^{3\xi_n}, \ldots$ of the exponential equation to zero gives a set of algebraic equations for $\omega, A$.

Step 7. The exact solutions for Eq. (6) can be obtained by substituting the obtained results into Eq. (11) and (12) with $\eta = t^\nu$, by solving the algebraic equations found in step 6.

4. Application

The well-known hybrid lattice equation with fractional order derivative can be written as

$$\frac{d^n u_n}{dt^n} = (\lambda + \beta u_n + \gamma u_n^2)(u_{n-1} - u_{n+1}),$$

where $\lambda, \beta, \gamma \neq 0$ are constants.

On utilizing the transformation $\nu = t^n$ in Eq. (21) yields

$$\frac{du_n}{d\nu} = \frac{1}{n!}(\lambda + \beta u_n + \gamma u_n^2)(u_{n-1} - u_{n+1}).$$

On following the steps 2–5 as mentioned in the previous section, with the help of a symbolic computer software Mathematica 8, Eq. (22) can be transformed as:

$$-\frac{\omega^2}{(e^{-i} + e^{i})^2} \lambda \left( \frac{A(-e^{-i} + e^{i})}{(e^{-i} + e^{i})^2} \right) \left( \frac{A(-e^{-i} + e^{i})}{(e^{-i} + e^{i})^2} \right) \lambda + \frac{\beta}{2} \left( \frac{A(-e^{-i} + e^{i})}{(e^{-i} + e^{i})^2} \right)^2 = 0$$

(23)

Considering $s = e^\xi$ yields

$$-\frac{(A\xi(-1 + c) - (-\lambda + e^{i\beta} - \lambda + e^{i\beta})\omega I(\alpha + 1))s}{I(\alpha + 1)}$$

$$+ \frac{A^2e^{-i\beta}(-1 + e^{i\beta})(-1 + \omega^2)\beta s^2}{I(\alpha + 1)}$$

$$+ \frac{(Ae^{-i\lambda}(1 + e^{i\beta})(1 + e^{i\beta})(1 + \omega^2\beta)s^3)}{I(\alpha + 1)} + \frac{A^3e^{-3i\lambda}(1 + e^{i\beta})(1 + e^{i\beta})(1 + \omega^2\beta)s^4}{I(\alpha + 1)} + \ldots = 0$$

(24)

Now, setting the coefficients of $s^i$ to be zero give

$$s^1: -\frac{Ae^{-i\lambda}(1 + \omega^2)}{\omega I(\alpha + 1)} = 0$$

$$s^2: -\frac{A^2e^{-i\lambda}(1 + e^{i\beta})(-1 + \omega^2)}{\omega I(\alpha + 1)} = 0$$

$$s^3: \frac{A^3e^{-i\lambda}(1 + \omega^2)}{\omega I(\alpha + 1)} = 0$$

$$s^4: \frac{A^4e^{-i\lambda}(1 + e^{i\beta})(1 + e^{i\beta})(1 + \omega^2\beta)}{\omega I(\alpha + 1)} = 0$$

(25)

Now solving set of Eq. (25) simultaneously gives

$$\omega = -\frac{2i \sinh(h)}{I(\alpha + 1)}$$

(26)

$$A = 2\frac{\sqrt{\gamma}}{\sqrt{\gamma}}$$

(27)

Therefore, the required soliton-solution is

$$u_n = \sqrt{\frac{\gamma}{\gamma}} \sinh(h) \sec h \left( \frac{2i \sinh(h)}{I(\alpha + 1)} \right)$$

(28)

5. Conclusion

This study has proposed a method to find the solitary wave solution for nonlinear fractional order DDEs. The soliton-solution is proposed for fractional order DDE by a transformation and the Chinese algorithm. The closed-form solutions of has been obtained. It should be noted that the soliton solution of discrete mKdV lattice and modified Volterra lattice can be found by using particular values of $a, \lambda, \beta, \gamma$. This study also reveals that the symbolic computation software plays a vital role in finding the exact solution of DDEs. Moreover, this method may also be applicable for a system of DDEs and partial DDEs with fractional order derivatives. This method is the generalization of [17] and the solution can obtained for $0 < \alpha < 1$.

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