Quantum Gravity with Minimal Assumptions

Miyuki Nishikawa

Department of Physics, University of Tokyo,
Bunkyo-ku, Tokyo, 113–0033, Japan

Abstract

Several basic results are reviewed on purpose to construct the quantum field theory including gravity, based on physical assumptions as few as possible. Up to now, the work by Steven Weinberg probably suits this purpose the most. Motivated by these results we focus on the fact that the dimension of an operator is not unique unless the operand is identified. This leads to the classification of possible singularities for the relativistic Schrödinger equation.

1 Introduction

I first thank you for giving me the chance to overview in section 2 a thesis titled ‘Quantum Gravity with Minimal Assumptions’[1]. This is mainly the review of quantum gravity from particle point of view. The purpose is to construct the quantum field theory including gravity, based on physical assumptions as few as possible. This consists of 5 subjects, the last of which is an original consideration on the relation between essential singularity and renormalization. This subject is summarized in section 3-5, but please read a preprint[10] for more details.

2 Overview

The first subject, and probably suits this purpose the most is the work by Steven Weinberg, in which he derived the Einstein equation from the Lorentz invariance of the S-matrix. According to his old paper[1], gravity is derived without assuming a curved space-time. Therefore, the general covariance and the geometric property of gravity are possibly subsidiary or mere approximations.

The second subject is that, according to an effective field theory, we can make a prediction without knowing the underlying fundamental theory. For example, John F. Donoghue calculated one loop quantum corrections to the Newtonian potential explicitly, by assuming the Einstein-Hilbert action and fluctuations around the flat metric, and by making use of the result of ’t Hooft and Veltman. The potential naturally contains the classical corrections by general relativity[2].

As the third subject, we review what will happen if we loosen the assumption on coordinates in the standard model that all physical coordinates are transformed to the Minkowski space-time by a Poincaré transformation. And we review the troubles and the measures in treating gravitational field under classical approximations assuming a curved space-time[3]. It is known that for the standard model of elementary particles, the anomaly cancellation condition in a curved space-time with torsion is the same as in a flat space-time[4].

As the fourth subject, we clarify the inevitable ambiguities of a theory. The following works are reviewed. For example, the vacuum state in a curved space-time is not unique and there exist several theories those can not be distinguished by finite times of measurements[5]. This is a theorem on the ambiguity related to the problem of divergence. For another example, a higher-derivative theory includes non-physical solutions those can not be Taylor expanded. This can be the origin of the gauge ambiguity. If we exclude superfluous solutions by imposing the perturbative constraint conditions, it means a gauge fixing and the theory is reduced to local and lower-derivative[6]. This treatment is known to be equivalent to the treatment of a constraint system by Dirac brackets[7].

As the last subject, we consider the following problem. In usual dimensional counting, momentum has dimension one. But a function \( f(x) \), when differentiated \( n \) times, does not always behave like one

\[ f(x) \rightarrow \frac{d^n f}{dx^n} \]

\[ \text{dimension of } f(x) = \text{dimension of } x^n \]

\[ n \]

\[ \text{and } \text{dimension of } \frac{d^n f}{dx^n} = \text{dimension of } x^n \]

\[ n \]

\[ \text{but the result is not always true.} \]

\[ \text{This is a theorem on the dimension of operator.} \]

\[ \text{It is known that for the standard model of elementary particles, the anomaly cancellation condition in a curved space-time with torsion is the same as in a flat space-time[4].} \]

\[ \text{As the last subject, we consider the following problem. In usual dimensional counting, momentum has dimension one. But a function } f(x), \text{ when differentiated } n \text{ times, does not always behave like one} \]

\[ \text{This is a theorem on the dimension of operator.} \]

\[ \text{It is known that for the standard model of elementary particles, the anomaly cancellation condition in a curved space-time with torsion is the same as in a flat space-time[4].} \]
with its power smaller by \( n \). This inevitable uncertainty may be essential in general theory of renormalization, including quantum gravity. As an example, we classify possible singularities of a potential for the Schrödinger equation, assuming that a potential \( V \) has at least one \( C^2 \) class eigen function. The result crucially depends on the analytic property of the eigen function near its 0 points. Notice that neither super-symmetric, higher dimensional, nor grand unification theory is referred to.

3 Renormalization and Essential Singularity

For the rest of this article we are going to focus on the preprint titled Renormalization and Essential Singularity\[10\]. We consider the relativistic Schrödinger equation assuming a time independent and spherical symmetric \( U(1) \) potential \( \mathcal{A}^\mu := (\phi(r), 0, 0, 0) \). Then, the spherical part of an eigen function satisfies

\[
\left[ -\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) + \frac{l(l+1)}{r^2} \right] y = \frac{(E - e\phi)^2 - m^2c^4}{(\hbar/2\pi)^2c^2} y
\]

\[
=: -V(r)y \quad (1)
\]

From now on, \( V(r) \) defined in the R. H. S. is called a potential if and only if there exists a \( C^2 \) class eigen function \( y(r) \) satisfying (1). The problem is, how singular \( V(r) \) can be.

For simplicity we first treat 1 dimensional case with the angular momentum \( l = 0 \). Then

\[
(1) \iff \frac{y''}{y} = V(r) \quad , \quad (2)
\]

so not \( y(r) \) itself but the ratio is important. For example, if \( y(r) \) is Taylor expanded, the second derivative of the constant and the linear term vanish. That is,

\[
\frac{y''}{y} = \frac{0 + 0 + 2cr + \cdots}{a + br + cr^2 + \cdots} .
\]

Therefore, the singularity of a potential depends on whether or not \( a, b = 0 \). In fact there are various kinds of singularities\[7\][8][9]. For example, we can replace the power of any term of a Taylor series with an arbitrary real number \( n \), or \( \log r \). An infinite power is called an essential singularity, and we can make more and more complex singularities by finite times of operations including summations, subtractions, multiplications, divisions, and compositions.

The most general shape of a singularity that is closed in these operations is like

\[
f(z) := (1)_k + (2)_j + \cdots + (m)_k ,
\]

\[
(1)_i := \left( \sum_{n \in \{n\}, m_1, \ldots, m_k \in \mathbb{C}} a_{nm_1, \ldots, m_k} z^n (-\log z)^{m_1} (-\log(-z/\log z))^{m_2} \cdots (-\log(-z/(-\log(-z/\log z))))^{m_k} \right) i ,
\]

\[
(2)_{\pm j} := \sum_{i \in \{i\}} (\pm) e^{\pm (1)_i} ,
\]

\[
(3)_{\pm k} := \sum_{j \in \{j\}_k} (\pm) e^{\pm (2)_j} ,
\]

\[
\vdots \quad (3)
\]

More precise construction and the meaning of this expansion are in \[10\]. Notice that this has several number of infinite series in one expansion and all the terms are partially ordered in the ascending powers of \( r \). In this case, the domain of the power of \( V(r) \) in the limit \( r \to +0 \) is

\[
V(r) \to r^{\nu}, -2 + \epsilon \lesssim \nu ; -1 \leq \nu ,
\]

\[
(4)
\]
where $\epsilon$ means an infinitesimal positive power like $-(\log r)^{-1}$.

Thus we can restrict the shape (i.e. power and sign) of a potential $V(r)$. This is the short distance limit case, but we can also treat the long distance limit case by the change of variables and in dimension $N$ there are 10 possible cases. Although precise version is in [10], we can see a shortcut version of the derivation of this main result in the next section.

4 Main Results

Here is the summary of the calculation. If we assume that the eigen function $y(r)$ is a $N$-dimensional spherical symmetric function $R(r)$ (i.e. orbital angular momentum $l = 0$), and that $R(r)$ is $C^2$ class, then (6) can be expanded as

$$R = a + br + \sum_{n=2}^{\infty} a_n r^n \sim + \cdots + \sum_{i < 0} (\pm) e^{-b_i r^i} \sim \cdots .$$

$$+ \sum_{j < 0} (\pm) e^{-c_j r^j} \sim \cdots + \sum_{k < 0} (\pm) e^{-d_k r^k} \sim \cdots . \quad (5)$$

For $a = 0$ and $N \neq 1$, the behavior of $V(r)$ in the limit $r \to +0$ is

$$\frac{\Delta R(r)}{R(r)} = \frac{R''}{R} + \frac{N - 1}{r} \frac{R'}{R} \to \begin{cases}
+ (N - 1) r^{-2} (b \neq 0) \\
+ n (n + N - 2) r^{-2} (b = 0 \text{ and } \exists a_n \neq 0) \\
+ (-i b) \epsilon \frac{z^{2 + \epsilon}}{r} (b = \nu a_n = 0 \text{ and } \exists b_1 > 0) \\
+ \infty (b = \nu a_n = \nu b_i = 0 \text{ and } \exists c_j \text{ or } d_k \text{ or } \cdots > 0) \\
\end{cases} \quad (6)$$

We can extend the results to $r \to +\infty$ case as follows. If we change the variable $r$ to $z := \frac{1}{r}$ and assume that $R(z)$ is $C^2$ class (expanded like above) (8) is clearly replaced by

$$\frac{\Delta R(r)}{R(r)} = \frac{1}{R(z)} \left\{ \frac{dz}{dr} \frac{d}{dz} \left( \frac{dz}{dr} \frac{dR(z)}{dz} \right) + (N - 1) \frac{dz}{dr} \frac{dR(z)}{dz} \right\}$$

$$= z^4 \frac{R''(z)}{R(z)} - z^{3(N - 3)} \frac{R(z)}{R(z)}$$

$$\to \begin{cases}
(3 - N) \frac{z^3}{a} (a \neq 0 \text{ and } b \neq 0 \text{ and } N \neq 3) \\
(n - N + 2) n \frac{z^{2 + \epsilon}}{a} (a \neq 0 \text{ and } b = 0 \text{ and } \exists a_n \neq 0 \text{ and } N \neq 3) \\
(n - 1) n \frac{z^{2 + \epsilon}}{a} (a \neq 0 \text{ and } \exists a_n \neq 0 \text{ and } N = 3) \\
(\pm) (a \neq 0 \text{ and } b = \nu a_n = 0 \text{ and } \exists b_1 \text{ or } c_j \text{ or } d_k \text{ or } \cdots > 0) \\
(3 - N) z^2 (a = 0 \text{ and } b \neq 0 \text{ and } N \neq 3) \\
(n - 1) n \frac{z^{2 + \epsilon}}{a} (a = 0 \text{ and } b \neq 0 \text{ and } \exists a_n \neq 0 \text{ and } N = 3) \\
(\pm) (a = 0 \text{ and } b \neq 0 \text{ and } \exists a_n = 0 \text{ and } \exists b_1 \text{ or } c_j \text{ or } d_k \text{ or } \cdots > 0 \text{ and } N = 3) \\
(n - N + 2) n \frac{z^{2 + \epsilon}}{a} (a = b = \nu a_n = 0 \text{ and } \exists b_1 > 0) \\
+ \infty (a = b = \nu a_n = \nu b_i = 0 \text{ and } \exists c_j \text{ or } d_k \text{ or } \cdots > 0) \end{cases} \quad (7)$$

Noting that $2 \leq n$ and $i < 0$, we conclude that the power of potential $V(r) \to r^\nu$ as $r \to \infty$ is $\nu \leq -3$; $-2 - \epsilon \leq \nu$. There is no reason to assume that $R(z)$ is $C^2$ class, but more natural normalizability condition that $R(r)$ is a $L^2$ function leads to small modification $a = b = 0$ and $N < 2n$ (instead of $2 \leq n$) in (3) and (7). Notice that (8) for more general case of $N, a$ can be obtained from (3) by the trivial replacement $N \to 4 - N$ and $z \to r$ with its power smaller by 4. Furthermore, above results show that for a physical dimension $N = 1, 2, 3$, the sign of a potential $V$ must be positive for $\nu \leq -2 + \epsilon$ ($r \to 0$) and $-2 - \epsilon \leq \nu$ ($r \to \infty$), but can be negative for other cases.

---

2 The coefficients are all real and $b_i, c_j, d_k, \cdots$ are positive if exist.
We considered here an example of the real scalar field, and the fermion field equation is of course another story. As a future work, I’d like to apply the result to the general theory of renormalization, and renormalons appearing in the perturbative QCD.

5 Discussions

A potential with $C^2$ class eigen function is only an assumption. The definition of a potential here is local, and valid only in this paper. But my conjecture is that the analyticity of an eigen function is such a property that cannot be distinguished by finite times of measurements. Therefore we assumed that a physical eigen function is $C^2$ class. This is also only a local constraint and very weak self consistent condition. Therefore it is not a sufficient condition and normalizability is another problem, particularly, in the long distance limit case.

The motivations to introduce such a condition are as follows. The first is that we want to clarify the inevitable ambiguity of a theory. The second is that there are subtle physical problems: which is the more fundamental, a matter field or a potential? Can a potential be measured in the absence of a matter field? It is a future work to clarify the relation between all the different facts that a quantity is realistic, physically measurable, reduced inevitably from other properties, and calculable.

Acknowledgements

I am grateful to Izumi Tsutsui and Toyohiro Tsurumaru for useful discussions. This article is partially based on some implications given by Tsutomu Kambe and Kazuo Fujikawa. I also appreciate Tsutomu Yanagida and Ken-ichi Izawa, and all related people.

References

[1] S. Weinberg, THE QUANTUM THEORY OF FIELDS. VOL. 1,2 (Cambridge, UK, 1995); Physical Review 135 (1964) B1049; Physical Review 138 (1965) B988.
[2] J.F. Donoghue, Physical Review D50 (1994) 3874.
[3] R.M. Wald, Quantum Field Theory in Curved space-time and Black Hole Thermodynamics (The University of Chicago Press, Chicago, 1994).
[4] A. Dobado, A. Gómez−Nicola, A.L. Maroto, and J.R. Peláez, Effective Lagrangians for the Standard Model (Springer, New York, 1997).
[5] J.Z. Simon, Physical Review D41 (1990) 3720.
[6] P.A.M. Dirac, Lectures on Quantum Mechanics (Yeshiva University, New York, 1964).
[7] M.B. Pour-El and J.I. Richards, Computability in analysis and physics (Springer, New York, 1989).
[8] R. Penrose, The emperor’s new mind: concerning computers, minds, and the laws of physics (Oxford University Press, Oxford, 1989).
[9] L.V. Ahlfors, Complex analysis: an introduction to the theory of analytic functions of one complex variable (Mcgraw-Hill, New York, 1953).
[10] M. Nishikawa, preprint hep-th/0110093 (submitted to Progress of Theoretical Physics).
[11] M. Nishikawa, master thesis, 2001.