INFINITE FAMILIES OF PAIRS OF CURVES OVER $\mathbb{Q}$ WITH ISOMORPHIC JACOBIANS

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Abstract. We present three families of pairs of geometrically non-isomorphic curves whose Jacobians are isomorphic to one another as unpolarized abelian varieties. Each family is parametrized by an open subset of $\mathbb{P}^1$. The first family consists of pairs of genus-2 curves whose equations are given by simple expressions in the parameter; the curves in this family have reducible Jacobians. The second family also consists of pairs of genus-2 curves, but generically the curves in this family have absolutely simple Jacobians. The third family consists of pairs of genus-3 curves, one member of each pair being a hyperelliptic curve and the other a plane quartic. Examples from these families show that in general it is impossible to tell from the Jacobian of a genus-2 curve over $\mathbb{Q}$ whether or not the curve has rational points — or indeed whether or not it has real points. Our constructions depend on earlier joint work with Franck Leprévost and Bjorn Poonen, and on Peter Bending’s explicit description of the curves of genus 2 whose Jacobians have real multiplication by $\mathbb{Z}[\sqrt{2}]$.

1. Introduction

Torelli’s theorem shows that a curve is completely determined by its polarized Jacobian variety, but it has been known since the late 1800s that distinct curves can have isomorphic unpolarized Jacobians. In particular, the unpolarized Jacobian of a curve may not reflect all of the curve’s geometric properties. Proving that a particular property of curves cannot always be determined from the Jacobian is equivalent to showing that there exist two curves, one with the given property and one without, whose Jacobians are isomorphic to one another. Thus, for example, the pairs of curves written down in [4] show that one cannot tell whether or not a curve of genus 3 over the complex numbers is hyperelliptic simply by looking at its Jacobian.

One would also like to find arithmetic properties of curves that are not determined by the Jacobian, but from an arithmetic perspective the heretofore-known explicit examples of distinct curves with isomorphic Jacobians (catalogued in the introduction to [4]) are not entirely satisfying. The primary complaint is that none of the examples involves curves that can be defined over $\mathbb{Q}$; in addition, for any given number field only finitely many of the examples can be defined over that field. Furthermore, all of the explicit examples in characteristic 0 known before now involve curves with geometrically reducible Jacobians, and the arithmetic of such curves differs qualitatively from that of curves whose Jacobians are irreducible.
In this paper we address these concerns by providing three new explicit families of pairs of non-isomorphic curves with isomorphic Jacobians. Each family is parametrized by an open subset of \( \mathbb{P}^1 \), so each family gives an infinite number of examples over \( \mathbb{Q} \). Also, the Jacobians of the curves in one of the families are typically absolutely simple. Using examples from these families, we show that the Jacobian of a genus-2 curve over \( \mathbb{Q} \) does not determine whether or not the curve has rational points, or indeed whether or not the curve has real points. Liu, Lorenzini, and Raynaud have used our results to show that the Jacobian of a genus-2 curve over \( \mathbb{Q} \) does not determine the number of components on the reduction of a minimal model of the curve modulo a prime.

Our first family of pairs of curves can be defined over an arbitrary field \( K \) whose characteristic is not 2. If \( t \) is an element of \( K \) with \( t(t + 1)(t^2 + 1) \neq 0 \) then the equation
\[
(t + 1)y^2 = (2x^2 - t)(4t^2x^4 + 4(t^2 + t + 1)x^2 + 1)
\]
defines a curve of genus 2 that we will denote \( C(t) \). Clearly the quotient of \( C(t) \) by the involution \( (x, y) \mapsto (-x, y) \) is an elliptic curve, so the Jacobian of \( C(t) \) splits over \( K \).

**Theorem 1.** Let \( K \) be a field of characteristic not 2 and suppose \( t \) is an element of \( K \) such that \( t(t^2 - 1)(t^2 + 1) \) is nonzero. Then \( C(t) \) and \( C(-t) \) are curves of genus 2 over \( K \) whose Jacobians are isomorphic over \( K \). Furthermore, \( C(t) \) and \( C(-t) \) are geometrically non-isomorphic unless \( K \) has characteristic 11 and \( t^2 \in \{-3, -4\} \).

Our next family takes a little more effort to describe. In order to do so we must define the Richelot duals of a genus-2 curve over a field \( K \) of characteristic not 2 (see [3 Ch. 9], [11 §3]). Suppose \( C \) is a genus-2 curve over \( K \) defined by an equation \( \delta y^2 = f \), where \( \delta \in K^\times \) and where \( f \) is a monic separable polynomial in \( K[x] \) of degree 6. Let \( \mathbb{K} \) be a separable closure of \( K \), and suppose \( f \) can be factored as a product \( g_1g_2g_3 \) of three monic quadratic polynomials in \( \mathbb{K}[x] \) that are permuted by \( \text{Gal}(\mathbb{K}/K) \). For each \( i \) write \( g_i = x^2 - t_ix + n_i \) and suppose the determinant
\[
d = \begin{vmatrix} 1 & t_1 & n_1 \\ 1 & t_2 & n_2 \\ 1 & t_3 & n_3 \end{vmatrix},
\]
which is an element of \( K \), is nonzero. Define three new polynomials by setting
\[
\begin{align*}
h_1 &= g_3 \frac{d g_2}{d x} - g_2 \frac{d g_3}{d x} \\
h_2 &= g_1 \frac{d g_3}{d x} - g_3 \frac{d g_1}{d x} \\
h_3 &= g_2 \frac{d g_1}{d x} - g_1 \frac{d g_2}{d x}.
\end{align*}
\]
Then the product \( h_1h_2h_3 \) is a separable polynomial in \( K[x] \) of degree 5 or 6.

**Definition.** The Richelot dual of \( C \) associated to the factorization \( f = g_1g_2g_3 \) is the genus-2 curve \( D \) defined by \( \delta \hat{y}^2 = h_1h_2h_3 \).

**Theorem 2.** Let \( K \) be a field of characteristic not 2, let \( v \) be an element of \( K \setminus \{0, 1, 4\} \) such that
\[
(v^2 - v + 4)(v^2 + v + 2)(v^2 + 3v + 4)(v^3 - 6v^2 - 7v - 4)(v^3 - 4v^2 + 7v + 4) \neq 0,
\]
let $w$ be a square root of $v$ in $\mathbb{K}$, and define numbers $\rho_1, \ldots, \rho_6$ by setting

$$
\rho_1 = \frac{(-2 + w)(1 + w)}{2w^2} \quad \rho_2 = \frac{(-2 - w)(1 - w)}{2w^2}
$$

$$
\rho_3 = \frac{-2(2 + w)}{(-2 + w)(1 + w)} \quad \rho_4 = \frac{(-2 - w)(1 - w)}{(-w)(-1 - w)}
$$

$$
\rho_5 = \frac{(-2 + w)(1 + w)}{w(-1 + w)} \quad \rho_6 = \frac{(-2 - w)(1 - w)}{(-w)(-1 - w)}
$$

The $\rho_i$ are distinct from one another, so that if we let $f = \prod (x - \rho_i)$ then the curve $D$ over $K$ defined by $y^2 = f$ has genus 2. Set

$$
g_1 = (x - \rho_1)(x - \rho_3) \quad g'_1 = (x - \rho_1)(x - \rho_3)
$$

$$
g_2 = (x - \rho_2)(x - \rho_4) \quad g'_2 = (x - \rho_2)(x - \rho_6)
$$

$$
g_3 = (x - \rho_3)(x - \rho_5) \quad g'_3 = (x - \rho_4)(x - \rho_5).
$$

The Richelot duals $C$ and $C'$ of $D$ with respect to the factorizations $f = g_1g_2g_3$ and $f = g'_1g'_2g'_3$ exist, and their Jacobians become isomorphic to one another over $K(\sqrt{v(v - 4)})$. The curves $C$ and $C'$ are geometrically non-isomorphic unless one of the following conditions holds:

(a) char $K = 3$ and $v^{10} - v^8 + v^7 - v^6 - v^5 + v + 1 = 0$;

(b) char $K = 19$ and $v + 1 = 0$;

(c) char $K = 89$ and $v + 36 = 0$;

(d) char $K = 1033$ and $v + 508 = 0$.

Furthermore, if $K$ has characteristic 0 and if $v$ is not an algebraic number, then the Jacobians of $C$ and $C'$ are absolutely simple.

In fact, when $K$ has characteristic 0 it is very easy to find algebraic numbers $v$ in $K$ for which the Jacobians in Theorem 2 are absolutely simple. For example, suppose $R$ is a subring of $K$ for which there is a homomorphism $\varphi$ to an extension of $\mathbb{F}_{13}$. We show in the proof of Theorem 2 that in this case the Jacobians of $D$ and $D'$ are geometrically irreducible whenever $v$ lies in $\varphi^{-1}(2)$ or $\varphi^{-1}(6)$.

Theorem 2 gives a 1-parameter family of pairs of non-isomorphic curves with isomorphic Jacobians. In fact, we shall see that there is a family of such pairs of curves parametrized by an elliptic surface; over $\mathbb{Q}$, this surface has positive rank.

Our third family of pairs of curves with isomorphic Jacobians is again easy to write down. Suppose $K$ is a field of characteristic not 2 and suppose $t$ is an element of $K$ with $t(t + 1)(t^2 + 1)(t^2 + t + 1) \neq 0$. Let $H(t)$ be the genus-3 hyperelliptic curve defined by the homogeneous equations

$$
W^2Z^2 = -\frac{(t^2 + 1)}{t(t + 1)(t^2 + t + 1)}X^4 - \frac{4(t^2 + 1)}{t(t + 1)(t^2 + t + 1)}Y^4 + \frac{1}{t}Z^4
$$

$$
0 = -X^2 + 2tY^2 + (t + 1)Z^2
$$

and let $Q(t)$ be the plane quartic

$$
X^4 + 4t^2Y^4 + (t + 1)^2Z^4 + (8t^2 + 4t + 8)X^2Y^2 - (4t^2 + 2t + 2)X^2Z^2 + (4t^2 + 4t + 8)Y^2Z^2 = 0.
$$

Theorem 3. Let $K$ be a field of characteristic not 2 and let $t$ be an element of $K$ such that $t(t + 1)(t^2 + 1)(t^2 + t + 1) \neq 0$. Then the Jacobians of the two genus-3 curves $H(t)$ and $Q(t)$ are isomorphic to one another over $K$. 
In Section 2 we mention some simple facts about abelian surfaces with two non-isomorphic principal polarizations and we show how Richelot isogenies can in principle be used to produce such surfaces from an abelian surface that has nontrivial automorphisms. In Section 3 we prove Theorem 1. In Section 4 we review a result of Bending that shows how to obtain genus-2 curves over a given field $K$ whose Jacobians have real multiplication by $\sqrt{2}$ over $K$, and we show how to adapt Bending’s result to obtain curves over $K$ with real multiplication by $\sqrt{2}$ over a quadratic extension of $K$. In Section 5 we give some Galois restrictions on our generalization of Bending’s construction that ensure that the curves we construct have two rational Richelot isogenies to curves with isomorphic Jacobians. In Section 6 we show that there is a positive-rank elliptic surface whose points give rise to pairs of genus-2 curves with isomorphic Jacobians, and we prove Theorem 2. In Section 7 we prove Theorem 3. Finally, in Section 8 we provide some explicit examples of curves over $\mathbb{Q}$ produced by our theorems, and we show that the Jacobian of a curve over $\mathbb{Q}$ does not determine whether or not the curve has rational points, or even whether or not it has real points.

We relied heavily on the computer algebra system Magma \cite{Magma} while working on this paper. Some of our Magma routines are available on the web: to find them, start at

\url{http://alumni.caltech.edu/~however/biblio.html}

and follow the links related to this paper.

2. Abelian surfaces with non-isomorphic polarizations

Weil \cite{Weil} showed that an abelian surface with an indecomposable principal polarization is a Jacobian, so one of our goals in this paper is to write down abelian surfaces with two non-isomorphic principal polarizations. In this section we will make a few observations about such surfaces.

Suppose $B$ is an abelian surface with two principal polarizations $\mu$ and $\mu'$, which we view as isogenies from $B$ to its dual variety $\hat{B}$. The polarized varieties $(B, \mu)$ and $(B, \mu')$ are isomorphic to one another if and only if there is an automorphism $\beta$ of $B$ such that $\mu' = \hat{\beta}\mu\beta$, where $\hat{\beta}$ is the dual of $\beta$. We would like to write down an abelian surface $B$ with two non-isomorphic principal polarizations $\mu$ and $\mu'$, so we would like to avoid the existence of such an automorphism $\beta$. We will accomplish this by obtaining $\mu'$ from $\mu$ through the use of an automorphism of a surface isogenous to $B$. Our main tool is the following well-known construction:

Suppose $(A, \lambda)$ is a principally-polarized abelian surface over a field $K$, suppose $n$ is a positive integer, and suppose $G$ is a rank-$n^2$ subschemes of the $n$-torsion $A[n]$ of $A$ that is isotropic with respect to the $\lambda$-Weil pairing on $A[n]$. Let $B$ be the quotient abelian surface $A/G$ and let $\varphi : A \to B$ be the natural map. Then there is a unique principal polarization $\mu$ of $B$ that makes the following diagram commute:

\[
\begin{array}{ccc}
A & \xrightarrow{n\lambda} & \hat{A} \\
\downarrow\varphi & & \uparrow\hat{\varphi} \\
B & \xrightarrow{\mu} & \hat{B}
\end{array}
\]
Now suppose that $A$ has an automorphism $\alpha$ such that $G' := \alpha(G)$ is also an isotropic subgroup of $A[n]$, and let $(B', \mu')$ be the principally polarized abelian surface obtained from $G'$ as above.

**Proposition 4.** The automorphism $\alpha$ of $A$ provides an isomorphism $B \to B'$. If we identify $B'$ with $B$ via this automorphism, then $\mu' = \hat{\beta}\mu\beta$, where $\beta$ is the image of $\alpha^{-1}$ in $(\text{End } B) \otimes \mathbb{Q}$.

**Proof.** We have a commutative diagram with exact rows:

$$
\begin{array}{cccccc}
0 & \to & G & \to & A & \phi \to & B & \to & 0 \\
\downarrow{\alpha} & & \downarrow{\alpha} & & \downarrow{\alpha} & & \downarrow{\alpha} & & \downarrow{\alpha} \\
0 & \to & G' & \to & A & \phi' \to & B' & \to & 0, \\
\end{array}
$$

where $\varphi$ and $\varphi'$ are the natural maps from $A$ to $B$ and $B'$, respectively. Completing the diagram, we find an isomorphism $B \to B'$. This proves the first statement of the theorem. The second statement follows by an easy diagram chase. \qed

This proposition leaves us some hope, because the $\beta$ in the proposition will not be an element of $\text{End } B$ if $G \neq G'$. Also, if we consider the case $n = 2$ and if the principally-polarized surface $(A, \lambda)$ is given to us explicitly as either a Jacobian or a product of polarized elliptic curves, then the theory of the Richelot isogeny will allow us to write down $(B, \mu)$ and $(B, \mu')$ explicitly as Jacobians, as we explain below. Thus, we would like to explicitly write down abelian surfaces with non-trivial automorphisms. In later sections we will consider two families of such explicitly-given surfaces: products of isogenous elliptic curves, and Jacobians with real multiplication by $\sqrt{2}$.

We close this section with a comment about Richelot duals and maximal isotropic subgroups. Suppose $C$ is a genus-2 curve defined by an equation $\delta y^2 = f$ and suppose $D$ is the Richelot dual of $C$ corresponding to a factorization $f = g_1g_2g_3$. Then there is an isogeny from the Jacobian of $C$ to the Jacobian of $D$ whose kernel is the order-4 subgroup $G$ of $\text{Jac } C$ containing the classes of the divisors $(a_i, 0) - (b_i, 0)$, where $a_i$ and $b_i$ are the roots of $g_i$ in $K$ (see [3, Ch. 9]). The subgroup $G$ is a maximal isotropic subgroup of the 2-torsion of $\text{Jac } C$ under the Weil pairing. Conversely, every $K$-defined maximal isotropic subgroup $G$ of $(\text{Jac } C)[2]$ arises in this way. Thus, given a $K$-defined maximal isotropic subgroup $G$, we can define the $G$-Richelot dual of $C$ to be the Richelot dual of $C$ with respect to the factorization $f = g_1g_2g_3$ that gives rise to $G$.

### 3. Proof of Theorem 1

In this section we will prove Theorem 1 by following the outline given in Section 2 in the case where $A$ is a split abelian surface.

Let $E$ and $E'$ be elliptic curves over a field $K$ of characteristic not 2 and suppose there is a 2-isogeny $\psi$ from $E$ to $E'$. Let $Q$ be the nonzero element of $E[2](K)$ in the kernel of $\psi$, and let $P$ and $R$ be the other two geometric 2-torsion points of $E$. Let $Q' = \psi(P) = \psi(R)$, so that $Q'$ is a nonzero element of $E'[2](K)$, and let $P'$ and $R'$ be the other geometric 2-torsion points of $E'$. Suppose the discriminants of $E$ and $E'$ are equal up to squares, so that the fields $K(P)$ and $K(P')$ are the same.
Let $A$ be the surface $E \times E'$ and let $\lambda$ be the product principal polarization on $A$. Let $\alpha$ be the automorphism of $A$ that sends a point $(U, V)$ to $(U, V + \psi(U))$. Let $G$ be the $K$-defined subgroup

$$G = \{(O, O), (P, P'), (Q, Q'), (R, R')\}$$

of $A[2]$ and let $G' = \alpha(G)$, so that $G'$ is the $K$-defined subgroup

$$G' = \{(O, O), (P, P'), (Q, Q'), (R, R')\}.$$ 

Let $(B, \mu)$ and $(B, \mu')$ be the principally-polarized surfaces obtained from $G$ and $G'$ as in Section 2. The polarizations $\mu$ and $\mu'$ will be indecomposable except in unusual circumstances, so there will usually be curves $C$ and $C'$ whose polarized Jacobians are isomorphic to $(B, \mu)$ and $(B, \mu')$, respectively. If $E$ and $E'$ are given to us by explicit equations, then $C$ and $C'$ can also be given by explicit equations — see [5, §3.2], where the unusual circumstances are also explained.

To make this outline explicit and to prove Theorem 1 we must start with an explicit 2-isogeny $\psi : E \to E'$, where the discriminants of $E$ and $E'$ are equal up to squares. Let $t$ be an element of $K$ such that $t(t^2 + 1)$ is nonzero, and let $E$ and $E'$ be the elliptic curves

$$E : \quad y^2 = x(x^2 - 4t^2 + 1)x + 4(t^2 + 1)$$

$$E' : \quad y^2 = x(x^2 + 8t^2 + 1)x + 16t^2(t^2 + 1).$$

It is easy to check that the discriminants of $E$ and $E'$ are both equal to $t^2 + 1$, up to squares.

Let $s$ be a square root of $t^2 + 1$ in an algebraic closure of $K$, so that the 2-torsion points of $E$ are

$$P = (2t^2 + 2 + 2st, 0)$$
$$Q = (0, 0)$$
$$R = (2t^2 + 2 - 2st, 0)$$

and the 2-torsion points of $E'$ are

$$P' = (-4t^2 - 4 + 4s, 0)$$
$$Q' = (0, 0)$$
$$R' = (-4t^2 - 4 - 4s, 0).$$

It is easy to check that the map

$$(x, y) \mapsto \left(\frac{y^2}{x^2}, \frac{(x^2 - 4(t^2 + 1))y}{x^2}\right)$$

defines a 2-isogeny $\psi : E \to E'$ that kills $Q$ and that sends $P$ to $Q'$ (see [10, Example III.4.5]). Let $G$ and $G'$ be the subgroups of $A[2]$ defined above and let $(B, \mu)$ and $(B, \mu')$ be the principally-polarized surfaces obtained from $G$ and $G'$ as above. If we apply [5, Prop. 4] we find that $(B, \mu)$ is isomorphic over $K$ to the polarized Jacobian of the curve $y^2 = h_t$, where

$$h_t = 2^{38}t^6(t + 1)^3(t^2 + 1)^{12}(2x^2 - t)(4t^2x^4 + 4(t^2 + t + 1)x^2 + 1).$$

Furthermore, $(B, \mu')$ is isomorphic to the polarized Jacobians of the curve $y^2 = h_{-t}$. Scaling $h_t$ and $h_{-t}$ by squares in $K$, we find that $y^2 = h_t$ is isomorphic to the curve $C(t)$ of Theorem 1 and that $y^2 = h_{-t}$ is isomorphic to the curve $C(-t)$.
To complete the proof of Theorem 1 we must show that $C(t)$ and $C(-t)$ are geometrically non-isomorphic, except for the special cases listed in the theorem. The simplest way to do this is to use Igusa invariants (see [7], [9]). Facilities for computing Igusa invariants are included in the computer algebra package Magma [2].

Let us begin by working over the ring $\mathbb{Z}[t]$, where $t$ is an indeterminate. Let $J_2(t), J_4(t), J_6(t), J_8(t),$ and $J_{10}(t)$ be the Igusa invariants of the twist $y^2 = (2x^2 - t)(4x^4 + 4(t^2 + t + 1)x^2 + 1)$ of $C(t)$. (The invariants $J_{2i}(t)$ of this curve, scaled by $4^i$, can be computed in Magma using the function $\text{ScaledIgusaInvariants}.$)

Let

$$R_2 = \frac{J_4(t)J_2(-t)^2 - J_4(-t)J_2(t)^2}{t(t^2 + 1)^3},$$

$$R_3 = \frac{J_6(t)J_2(-t)^3 - J_6(-t)J_2(t)^3}{t^3(t^2 + 1)^3},$$

$$R_5 = \frac{J_{10}(t)J_2(-t)^5 - J_{10}(-t)J_2(t)^5}{t^3(t^2 + 1)^7},$$

all of which we view as elements of $\mathbb{Z}[t]$. If $C(t)$ and $C(-t)$ are isomorphic for a given value of $t$ in a given field $K$, then the polynomials $R_2, R_3,$ and $R_5$ must all evaluate to 0 at this value. But we compute that

$$\text{gcd}(\text{resultant}(R_2, R_3), \text{resultant}(R_2, R_5)) = 2^{980}3^{48}11^8,$$

so if the characteristic of $K$ is neither 3 nor 11 then the two curves $C(t)$ and $C(-t)$ are geometrically non-isomorphic for every value of $t$ in $K$ with $t(t^2 + 1)(t^2 - 1) \neq 0$.

We repeat the above calculation in the ring $\mathbb{F}_3[t]$, only now we define

$$R_2 = \frac{J_4(t)J_2(-t)^2 - J_4(-t)J_2(t)^2}{t(t^2 - 1)^2(t^2 + 1)^7},$$

$$R_3 = \frac{J_6(t)J_2(-t)^3 - J_6(-t)J_2(t)^3}{t^3(t^2 + 1)^9}.$$

We find that $\text{gcd}(R_2, R_3) = 1$, so the two curves $C(t)$ and $C(-t)$ are geometrically non-isomorphic for every value of $t$ in characteristic 3, as long as $t(t^2 + 1)(t^2 - 1) \neq 0$.

Next we repeat the above calculation in the ring $\mathbb{F}_{11}[t]$, with

$$R_2 = \frac{J_4(t)J_2(-t)^2 - J_4(-t)J_2(t)^2}{t(t^2 + 1)^3},$$

$$R_3 = \frac{J_6(t)J_2(-t)^3 - J_6(-t)J_2(t)^3}{t^3(t^2 + 1)^3}.$$

We find that $\text{gcd}(R_2, R_3) = (t^2 + 3)(t^2 + 4)$, so that the two curves are geometrically non-isomorphic in characteristic 11 except possibly when $t^2$ is $-3$ or $-4$.

Finally we note that in characteristic 11, when $t^2$ is $-3$ or $-4$ the curve $C(t)$ is geometrically isomorphic to the supersingular curve $y^2 = x^6 + x^4 + 4x^2 + 7$. Thus, $C(t)$ and $C(-t)$ are geometrically isomorphic for these values of $t$. □

Remark. It was not critical in our construction that the isogeny $\psi : E \to E'$ have degree 2. Similar constructions can be made with other kinds of isogenies.
4. Jacobians with real multiplication by $\sqrt{2}$

In this section we review a construction of Bending [1] that produces every genus-
2 curve over a given field $K$ whose Jacobian has a $K$-rational endomorphism that
is fixed by the Rosati involution and whose square is 2. We will give a variant
of Bending’s construction that produces curves over $K$ with a not-necessarily $K$-
rationale endomorphism that is fixed by Rosati and whose square is 2. We do not
claim that our construction will produce all such curves.

First we recall Bending’s construction. Let $K$ be a field of characteristic not 2
and let $A$, $P$, and $Q$ be elements of $K$ with $P$ nonzero. Define

$$B = (APQ - Q^2 + 4P^2 + 1)/P^2$$
$$C = 4(AP - Q)/P$$
$$R = 4P$$

and let $\alpha_1$, $\alpha_2$, $\alpha_3$ be the roots of $T^3 + AT^2 + BT + C$ in a separable closure $\overline{K}$
of $K$. For $i = 1, 2, 3$ let

$$G_i = X^2 - \alpha_iX + P\alpha_i^2 + Q\alpha_i + R,$$

and suppose that the product $G_1G_2G_3 \in K[X]$ has nonzero discriminant. Let $D$
be a nonzero element of $K$.

**Theorem 5.** The Jacobian of the genus-2 curve $DY^2 = G_1G_2G_3$ has a $K$-rational
endomorphism that is fixed by the Rosati involution and whose square is 2. Further-
more, if $\#K > 5$ then every curve over $K$ whose Jacobian has such an endo-
morphism is isomorphic to a curve that arises in this way from some choice of $A$,
$P$, $Q$, and $D$ in $K$.

**Proof.** See [1, Theorem 4.1]. Bending assumes that the base field $K$ has charac-
teristic 0, but his proof works over an arbitrary field $K$ of characteristic not 2 so long
as every genus-2 curve over $K$ can be written in the form $y^2 = (sextic)$. This is the
case for every field with more than 5 elements. $\square$

The endomorphism of $DY^2 = G_1G_2G_3$ whose existence is claimed by Theorem 5
is obtained by noting that the obvious Richelot dual of the curve is isomorphic over
$K$ to the curve itself. Thus the degree-4 isogeny from the Jacobian of the curve to
the Jacobian of its dual can be viewed as an endomorphism of the curve’s Jacobian,
and this endomorphism has the properties claimed in the theorem.

We will want to consider curves over $K$ whose Jacobians have real multiplication
by $\sqrt{2}$ that is not necessarily defined over $K$. For this reason, we will require the
following variant of Bending’s construction:

Suppose $r$, $s$, and $t$ are elements of a field $K$ of characteristic not 2, with $s \neq 0$,
$s \neq 1$, and $t \neq 1$. Let

$$c_2 = r + 4t$$
$$c_1 = 4t(r + s^3 - s^2t - 2s^2 + 5s + t)$$
$$c_0 = 4t(s - 1)(rs^2 - rst - rs - rt - 8st)$$

and suppose that the polynomial $T^3 - c_2T^2 + c_1T - c_0$ has three distinct roots $\beta_1$,
$\beta_2$, $\beta_3$ in $\overline{K}$. For $i = 1, 2, 3$ let

$$g_i = x^2 - 2\beta_i x + (1 - s)\beta_i^2 - 4s(s - 1)^2 t(s - t - 1),$$
and suppose that the discriminant of the product \( f = g_1g_2g_3 \) is nonzero. Let \( \mathcal{C}(r,s,t) \) be the curve over \( K \) defined by \( y^2 = f \).

**Theorem 6.** The Richelot dual of \( \mathcal{C}(r,s,t) \) associated to the factorization \( f = g_1g_2g_3 \) is isomorphic over \( K(\sqrt{st}) \) to \( \mathcal{C}(r,s,t) \). The endomorphism of \( \text{Jac}\mathcal{C}(r,s,t) \) (over \( K(\sqrt{st}) \)) obtained by composing the Richelot isogeny with the natural isomorphism from the dual curve to \( \mathcal{C}(r,s,t) \) is fixed by the Rosati involution, and its square is the multiplication-by-2 endomorphism.

**Proof.** This theorem can be proven by direct calculation, but here we will prove it by relating the curve \( \mathcal{C}(r,s,t) \) back to Bending’s construction. Let \( Q \) be a square root of \( st \) in \( K \), let \( P = (1 - s)/4 \), let \( A = (r + 6st - 2t)/(4PQ) \), let \( D = 1 \), and let \( C \) be the curve over \( K(Q) \) defined by using this \( A \), \( P \), \( Q \), and \( D \) in Bending’s construction. Then the \( \alpha_i \) are related to the \( \beta_i \) by

\[
Q\alpha_i = \beta_i/(s - 1) + 2t,
\]

and the curve \( Y^2 = G_1G_2G_3 \) is isomorphic \( y^2 = g_1g_2g_3 \) via the relation \( x = 2(s - 1)(QX + t) \). This shows that \( \mathcal{C}(r,s,t) \) is isomorphic to its Richelot dual over \( K(Q) \). The rest of the theorem follows from Bending’s theorem.

**Remark.** Bending’s family of curves has three “geometric” parameters \( A, P, \) and \( Q \) and one “arithmetic” parameter \( D \) (which parametrizes quadratic twists of the curve determined by \( A, P, \) and \( Q \)). Since the moduli space \( M \) of genus-2 curves with real multiplication by \( \sqrt{2} \) is a two-dimensional rational variety, one might hope to replace Bending’s three-geometric-parameter family with a two-parameter family. But there is an obstruction, which stems from the fact that \( M \) is a coarse moduli space and not a fine one: A \( K \)-rational point on \( M \) does not necessarily give rise to a curve over \( K \). Indeed, Mestre [9] has shown that to every \( K \)-rational point \( P \) on the moduli space of genus-2 curves there is naturally associated a genus-0 curve over \( K \), and \( P \) corresponds to a curve over \( K \) if and only if the genus-0 curve has a \( K \)-rational point.

5. **Galois restrictions**

In order to prove Theorem 6 we will apply the construction outlined in Section 2 to a Jacobian with real multiplication by \( \sqrt{2} \) that we will obtain from Theorem 4. We will take the automorphism \( \alpha \) to be \( 1 + \sqrt{2} \). The construction requires that we find a Galois-stable maximal isotropic subgroup \( G \) of the 2-torsion of the Jacobian such that \( G' = (1 + \sqrt{2})(G) \) is a maximal isotropic subgroup different from \( G \). This requirement imposes some restrictions on the values of \( r, s, \) and \( t \) that we will be able to use in Theorem 4. In this section we will make these restrictions explicit, and in Section 4 we will find an elliptic surface that parametrizes a subset of the allowable values of \( r, s, \) and \( t \).

Recall the basic outline of Theorem 4. Given three elements \( r, s, t \) of our base field \( K \), we define a polynomial \( h = T^3 - c_2T^2 + c_1T - c_0 \) in the polynomial ring \( K[T] \), and we assume that \( h \) is separable. We use the roots \( \beta_1, \beta_2, \beta_3 \) of \( h \) to define three polynomials \( g_1, g_2, g_3 \) in the polynomial ring \( K[x] \), we assume that the product \( f = g_1g_2g_3 \) is separable, and we define a curve \( C \) by \( y^2 = f \). Then we show that the Richelot dual of \( C \) corresponding to the factorization \( f = g_1g_2g_3 \) is geometrically isomorphic to \( C \) itself.
Let $L$ be the quotient of the polynomial ring $K[T]$ by the ideal generated by the polynomial $h$ and let $\beta$ be the image of $T$ in $L$. Since $h$ is separable, the algebra $L$ is a product of fields. Let $g \in L[x]$ be the polynomial

$$g = x^2 - 2\beta x + (1 - s)\beta^2 - 4s(s - 1)^2(t - 1).$$

Let $\Delta \in K^*$ be the discriminant of $h$ and let $\Delta' \in L^*$ be the discriminant of $g$.

**Theorem 7.** There are distinct Galois-stable maximal isotropic subgroups $G$ and $G'$ of $(\text{Jac} C)[2]$ with $G' = (1 + \sqrt{2})(G)$ if and only if $\Delta \Delta'$ is a square in the algebra $L$.

**Proof.** Let the roots of $g_1$ (respectively, $g_2$, $g_3$) be $r_1$ and $r_2$ (respectively, $r_3$ and $r_4$, $r_5$ and $r_6$). For each $i$ let $W_i$ be the Weierstraß point of $C$ corresponding to the root $r_i$ of $f = g_1g_2g_3$. The kernel $H$ of the Richelot isogeny multiplication-by-$\sqrt{2}$ on the Jacobian $J$ of $C$ is the order-4 subgroup containing the divisor classes $[W_1 - W_2]$, $[W_3 - W_4]$, and $[W_5 - W_6]$.

Suppose there are distinct Galois-stable maximal isotropic subgroups $G$ and $G'$ of $(\text{Jac} C)[2]$ with $G' = (1 + \sqrt{2})(G)$. Then clearly $G \neq H$, so $\#(G \cap H)$ is either 2 or 1. Suppose $\#(G \cap H) = 2$. By renumbering the polynomials $g_i$ and by renumbering their roots, we may assume that $G$ is the order-4 subgroup

$$G = \{0, [W_1 - W_2], [W_3 - W_5], [W_4 - W_6]\}.$$

Then $\sqrt{2}$ kills the first two elements of $G$ and sends the second two elements to $[W_1 - W_2]$, and it follows that $(1 + \sqrt{2})(G) = G$, contradicting our assumption that $G$ and $G'$ are distinct.

So now we know that $G \cap H = \{0\}$. By renumbering the polynomials $g_i$ and by renumbering their roots, we may assume that $G$ is the order-4 subgroup

$$G = \{0, [W_1 - W_5], [W_2 - W_4], [W_3 - W_6]\}.$$

It is not hard to show that the automorphism $1 + \sqrt{2}$ of $J$ sends $[W_1 - W_5]$ to $[W_2 - W_6]$, $[W_2 - W_4]$ to $[W_1 - W_3]$, and $[W_3 - W_6]$ to $[W_4 - W_5]$, so we have

$$G' = \{0, [W_1 - W_5], [W_2 - W_6], [W_4 - W_5]\}.$$

Suppose $\sigma$ is an element of the Galois group such that $r_1^\sigma = r_2$. Since $G$ is Galois stable, it follows that $\sigma$ sends $[W_1 - W_5]$ to $[W_2 - W_4]$, and therefore $r_5^\sigma = r_4$. But since $G'$ is Galois stable, we see that $\sigma$ must send $[W_1 - W_5]$ to itself, and it follows that $r_5^\sigma = r_5$. Continuing in this manner, we find that $r_2^\sigma = r_1$ and $r_6^\sigma = r_3$ and $r_3^\sigma = r_6$. Thus, $\sigma$ acts on the roots of $f$ according to the permutation $(12)(36)(45)$ of the subscripts.

By considering the other choices for $r_1^\sigma$ and using the same reasoning as above, we find that the image of the absolute Galois group of $K$ in the symmetric group on the roots of $f$ is contained in the subgroup

$$S = \{\text{Id}, (12)(36)(45), (13)(24)(56), (146)(235), (15)(26)(34), (164)(253)\};$$

here of course we identify the root $r_i$ with the integer $i$. In particular, note that the action of $\sigma$ on the $r_i$ is determined by the action of $\sigma$ on the $\beta_i$.

To show that $\Delta \Delta'$ is square in the algebra $L$, we will consider three cases, depending on the splitting of the polynomial $h$.

**Case 1.** Suppose $h$ is irreducible. Then $L$ is a field, and the condition that $\Delta \Delta'$ be a square in $L$ is equivalent to saying that $g$ defines the Galois closure $M$ of $L$ over $K$. So suppose, to obtain a contradiction, that $g$ does not define $M$ over $L$. 

There are two ways that this can happen: either the roots of $g$ do not lie in $M$, or $M$ is a quadratic extension of $L$ and the roots of $g$ lie in $L$.

Suppose that the roots of $g$ do not lie in $M$. Then there is an element $\sigma$ of the absolute Galois group of $K$ that fixes $M$ but that moves the roots of $g$. But this contradicts the fact that the action of $\sigma$ on the roots of $g$ is determined by the action of $\sigma$ on the $\beta_i$.

Suppose $M$ is a quadratic extension of $L$, so that the image of the absolute Galois group in the symmetric group on the $\beta_i$ is the full symmetric group. Then the image of the absolute Galois group in the symmetric group on the $r_i$ must be the entire group $S$ given above, which acts transitively on the $r_i$. But if the roots of $g$ lie in $L$, then the $r_i$ will form two orbits under the action of the absolute Galois group, giving a contradiction.

**Case 2.** Suppose $h$ factors as a linear polynomial times an irreducible quadratic. Then one of the $\beta_i$, say $\beta_1$, lies in $K$, while $\beta_2$ and $\beta_3$ are conjugate elements in a quadratic extension $M$ of $K$. With the labelings we have chosen, this means that the image of the absolute Galois group in the symmetric group on the $r_i$ must be equal to the two-element group $S' = \{\text{Id}, (12)(36)(45)\}$; in particular, we see that $r_3$ and $r_6$ (and $r_4$ and $r_5$) are quadratic conjugates of one another, so they all must be elements of $M$. This means that the image of $g$ in $K[x]$ (obtained by sending $\beta$ to $\beta_1$) is an irreducible polynomial that defines $M$, while the image of $g$ in $M[x]$ (obtained by sending $\beta$ to $\beta_2$) splits into two linear factors. Thus, the discriminant $\Delta'$ of $g$ in $L = K \times M$ is equal to the discriminant of $M$ (up to squares) in the first component, and is a square in the second component. But $\Delta$ has this same property, so the product $\Delta \Delta'$ is a square in $L$.

**Case 3.** Suppose $h$ splits over $K$ into three linear factors, so that $\Delta$ is a square in $K$. Then the absolute Galois group of $K$ acts trivially on the $\beta_i$, so it must act trivially on the $r_i$ as well. This means that the discriminant $\Delta'$ of $g$ must be a square in each factor of $L = K \times K \times K$, so $\Delta \Delta'$ is a square in $L$ as well.

We see that if there are subgroups $G$ and $G'$ as in the statement of the theorem then $\Delta \Delta'$ must be a square in $L$. We leave the details of the proof of the converse statement to the reader; the point is that in each of the three cases above, the reasoning is reversible.

---

6. Application of our construction to curves with real multiplication

In this section we will follow the outline given in Section 2 in the case where $A$ is a Jacobian with real multiplication by $\sqrt{2}$ that has appropriate Galois-stable subgroups. Theorem 2 will follow quickly from the result we obtain.

We will continue to use the notation from previous sections:

- $r$, $s$, and $t$ will be elements of a field $K$;
- $h$ will be a polynomial in $K[T]$ defined in terms of $r$, $s$, and $t$;
- $L$ will be the algebra $K[T]/(h)$;
- $\beta$ will be the image of $T$ in $L$;
- $g$ will be a polynomial in $L[x]$ defined in terms of $r$, $s$, $t$, and $\beta$;
- $\Delta \in K^*$ will be the discriminant of $h$; and
- $\Delta' \in L^*$ will be the discriminant of $g$. 


Theorem 8. Let $K$ be a field of characteristic not 2, suppose $r, s, t$ are elements of $K$ that satisfy the hypotheses appearing before the statement of Theorem 6 and let $C$ be the curve $C(r, s, t)$ from Theorem 5. Suppose further that the product $\Delta'\Delta''$ is a square in $L^*$, so that there are Galois-stable subgroups $G$ and $G'$ of $(\text{Jac } C)[2]$ as in the statement of Theorem 7. Then the Jacobian of the $G$-Richelot dual of $C$ is isomorphic over $K(\sqrt{s})$ to the Jacobian of the $G'$-Richelot dual of $C$.

Proof. Let $D$ be the $G$-Richelot dual of $C$ and let $D'$ be the $G'$-Richelot dual of $C$. Theorem 6 and Theorem 7, combined with the argument in Section 2, show that the Jacobian of $D$ becomes isomorphic to the Jacobian of $D'$ when the base field is extended to $K(\sqrt{s})$. \hfill $\Box$

Remark. Since $D$ and $D'$ are defined over $K$, and since their Jacobians become isomorphic over $K(\sqrt{s})$, it is tempting to think that $\text{Jac } D$ must be isomorphic over $K$ to either the Jacobian of $D'$ or the Jacobian of the standard quadratic twist of $D'$ over $K(\sqrt{s})$. But in fact this is not the case. It is true that $\text{Jac } D'$ is a $K(\sqrt{s})/K$-twist of $\text{Jac } D$, but the twist is by an automorphism of $\text{Jac } D$ that does not come from an automorphism of $D$. Indeed, generically the automorphism group of $D$ contains 2 elements, while the automorphism group of $\text{Jac } D$ is isomorphic to the unit group of $\mathbb{Z}[2\sqrt{2}]$.

Proposition 9. Suppose $t = s - 1$ and let $u = r + 2$. Then $\Delta'\Delta''$ is a square in $L$ if and only if $(u^2 + a)^2 + 8bu + 4c$ is a square in $K$, where

\[
\begin{align*}
a &= -4s(s^2 + 11s - 11) \\
b &= -8s^2(s - 1)(4s - 1) \\
c &= -16s^2(s - 1)(28s^2 - 19s + 1).
\end{align*}
\]

Proof. When $t = s - 1$ and $r = u - 2$, we find that the coefficients of the polynomial $h$ used to define the algebra $L$ are

\[
\begin{align*}
c_2 &= 4s + u - 6 \\
c_1 &= -4(s - 1)(s^2 - 6s - u + 3) \\
c_0 &= 4(s - 1)^3(-8s - u + 2),
\end{align*}
\]

and we compute that

\[
\Delta = 16s(s - 1)^2((u^2 + a)^2 + 8bu + 4c),
\]

where $a$, $b$, and $c$ are as in the statement of the proposition. Furthermore, the polynomial $g \in L[\delta]$ defined in Section 5 is $x^2 - 2\beta x + (1 - s)\beta^2$, so that $\Delta' = 4s\beta^2$. We see that $\Delta'\Delta''$ is a square in $L$ if and only if the element $\delta = (u^2 + a)^2 + 8bu + 4c$ of $K$ is a square in $L$. If $L$ is a field then it is a cubic extension of $K$, and $\delta$ is a square in $L$ if and only if it is a square in $K$. If $L$ is not a field then it has $K$ as a factor, and again $\delta$ is a square in $L$ if and only if it is a square in $K$. \hfill $\Box$

Proposition 10. Let $K = \mathbb{Q}(s)$ be the function field in the variable $s$ over $\mathbb{Q}$, let

\[
\begin{align*}
a &= -4s(s^2 + 11s - 11) \\
b &= -8s^2(s - 1)(4s - 1) \\
c &= -16s^2(s - 1)(28s^2 - 19s + 1),
\end{align*}
\]
let $F$ be the curve over $K$ defined by
$$z^2 = (u^2 + a)^2 + 8bu + 4c,$$
and let $E$ be the elliptic curve over $K$ defined by
$$y^2 = x^3 - ax^2 - cx + b^2.$$

Then
(a) the map $u = (y - b)/x, z = 2x - u^2 - a$ gives an isomorphism from $E$ to $F$, whose inverse is $x = (z + u^2 + a)/2, y = ux + b$;
(b) the point $P = (0, b)$ on $E$ has infinite order;
(c) the point $T = (4s^2(1 - s), 0)$ on $E$ has order 2;
(d) the isomorphism in statement (a) takes the involution $(u, z) \mapsto (u, -z)$ on $F$ to the involution $Q \mapsto -Q - P$ on $E$;
(e) the isomorphism in statement (a) takes $-P$ and the origin of $E$ to the two infinite points on $F$.

Proof. An easy calculation shows that statement (a) is true; the particular values of $a, b$, and $c$ are irrelevant to the calculation.

To show that the point $P$ has infinite order, it suffices to show that when we specialize $s$ to a particular value the specialized $P$ has infinite order. For example, if we set $s = 2$, then $E$ becomes the curve
$$y^2 = x^3 + 120x^2 + 4800x + 50176$$
and $P$ becomes the point $(0, -224)$. Translating $x$ by 40, we find a new equation for $E$: $y^2 = x^3 - 13824$, where now $P = (40, -224)$. But 7 divides 224 and 7 does not divide 13824, so by the Lutz-Nagell theorem [10, Cor. VIII.7.2] the point $P$ has infinite order. This proves statement (b).

Statement (c) is clear.

Let $R$ be a point $(u, z)$ on $F$ and let $\tilde{R} = (u, -z)$ be its involute. Let $Q$ and $\tilde{Q}$ be the images of $R$ and $\tilde{R}$ on $E$. Clearly $Q$ and $\tilde{Q}$ both lie on the line $y = ux + b$, and the third intersection point of this line with $E$ is easily seen to be $P$. Thus, the involution on $E$ satisfies $\tilde{Q} + Q = -P$, and this is statement (d).

The equations for the isomorphism show that $-P$ is mapped to an infinite point on $F$, and statement (d) shows that $O_E$ gets mapped to an infinite point as well. □

Remark. If we view the curve $F \cong E$ as an elliptic surface $S$ over $\mathbb{Q}$, then the points $P$ and $T$ of Proposition [10] can be viewed as rational curves on $S$. By adding multiples of $P$ and $T$ together, we get a countable family of rational curves on $S$. But $S$ contains more rational curves than just the ones in this family. For example, we have the curves

$$s = 5/4$$
$$u = (4w^2 + 5w + 40)/(4w)$$
$$z = (2w^4 + 5w^3 - 50w - 200)/(2w^2)$$

and

$$s = -1$$
$$u = (2w^2 - 10w - 4)/w$$
$$z = (4w^4 - 40w^3 - 80w - 16)/w^2,$$
where \( w \) is a parameter; the curve
\[
\begin{align*}
s &= (5 - w^2)/4 \\
u &= (-w^3 + w + 7w^2 - 5w - 10)/(4w + 8) \\
z &= (w^8 + 9w^7 + 22w^6 - 18w^5 - 135w^4 - 135w^3)/(8w^2 + 32w + 32),
\end{align*}
\]
which corresponds to a 3-torsion point on \( E \) defined over a genus-0 extension of the function field \( \mathbb{Q}(s) \); five curves in which \( u \) is a linear expression in \( s \), for example
\[
\begin{align*}
s &= 4(w^2 + 9w + 19)/w \\
u &= -s \\
z &= 16(16w^6 + 283w^5 + 1555w^4 - 29545w^2 - 102163w - 109744)/w^3;
\end{align*}
\]
and three curves in which \( u \) is a quadratic expression in \( s \), for example
\[
\begin{align*}
s &= (w^2 + 3w + 1)/w \\
u &= 4s^2 - 6s \\
z &= 4(4w^8 + 35w^7 + 105w^6 + 119w^5 - 119w^3 - 105w^2 - 35w - 4)/w^4.
\end{align*}
\]

Remark. One can check that the image of the elliptic surface \( S \) in the moduli space of genus-2 curves is 2-dimensional. To check this, one need only write explicitly the Igusa invariants of the genus-2 curve obtained from a pair \((s, u)\) and verify that the rank of the Jacobian matrix of the mapping from \((s, u)\)-pairs to Igusa invariants at some arbitrary point is 2.

We now have enough machinery available to prove Theorem 2.

Proof of Theorem 2. Consider the point \( P = (0, b) \) from statement (b) of Proposition 10. The \( u \)-coordinate of its image on the curve \( F \) is
\[
u = (28s^2 - 19s + 1)/(1 - 4s).
\]
So given any \( s \in K \), we will obtain a curve satisfying the conclusion of Theorem 5 if we set
\[
\begin{align*}
t &= s - 1 \\
r &= -2 + (28s^2 - 19s + 1)/(1 - 4s)
\end{align*}
\]
and set \( C = C(r, s, t) \). Given a \( v \in K \setminus \{0, 1, 4\} \), let us apply the preceding observation with \( s = v/4 \), so that
\[
\begin{align*}
s &= v/4 \\
t &= (v - 4)/4 \\
r &= (7v^2 - 11v - 4)/(4 - 4v).
\end{align*}
\]
The coefficients of the polynomial \( h \) used in the construction of Section 4 are
\[
\begin{align*}
c_2 &= \frac{3v^2 + 9v - 20}{4(1 - v)} \\
c_1 &= \frac{(v - 4)(v^3 + 3v^2 - 4v - 32)}{16(1 - v)} \\
c_0 &= \frac{(v - 4)^3(v^2 + 3v + 4)}{64(1 - v)},
\end{align*}
\]
and over $K(w)$ the roots of $h$ are
\[
\begin{align*}
\beta_1 &= \frac{(2+w)(2-w)}{4} \\
\beta_2 &= \frac{-(2+w)^2(2-w+w^2)}{4(1+w)} \\
\beta_3 &= \frac{-(2-w)^2(2+w+w^2)}{4(1-w)}.
\end{align*}
\]
Each polynomial $g_i$ is $x^2 - 2\beta_i x + (1-s)\beta_i^2$ and has roots $\beta_i(1 \pm w/2)$, so we calculate that the roots of $g_1$ are
\[
\begin{align*}
r_1 &= -(1/8)(2-w)^2(2+w) \\
r_2 &= -(1/8)(2-w)(2+w)^2,
\end{align*}
\]
the roots of $g_2$ are
\[
\begin{align*}
r_3 &= -(1/8)(2+w)^3(2-w+w^2)/(1+w) \\
r_4 &= -(1/8)(2-w)(2+w)^2(2-w+w^2)/(1+w),
\end{align*}
\]
and the roots of $g_3$ are
\[
\begin{align*}
r_5 &= -(1/8)(2-w)^2(2+w)(2+w+w^2)/(1-w) \\
r_6 &= -(1/8)(2-w)^2(2+w+w^2)/(1-w).
\end{align*}
\]
These roots are indexed in a manner consistent with the indexing of the roots in the proof of Theorem 4. Note that the $r_i$ are related to the $\rho_i$ of Theorem 2 by the relation
\[
r_i = 4s(s-1)\rho_i - 2(s-1)^2,
\]
so the $r_i$ are distinct exactly when the $\rho_i$ are distinct. It is easy to check that when $(v^2 + 3v + 4)(v^2 - v + 4)(v^3 - 6v^2 - 7v - 4) \neq 0$ the $\rho_i$ are distinct, so in this case the curve $D$ of Theorem 2 has genus 2. Furthermore, we see that $D$ is isomorphic to the curve $\mathcal{C}(r, s, t)$.

For each $i$ let $W_i$ be the point $(\rho_i, 0)$ of $D$. Let $G$ be the Galois-stable subgroup of the Jacobian of $D$ that consists of the divisor classes
\[
\{[0], [W_1 - W_5], [W_2 - W_4], [W_3 - W_6]\}
\]
and let $G'$ be the Galois-stable subgroup
\[
\{[0], [W_1 - W_3], [W_2 - W_6], [W_4 - W_5]\}.
\]
An easy computation shows that when $v^3 - 4v^2 + 7v + 4$ and $v^2 + v + 2$ are nonzero the $G$-Richelot dual of $D$ and the $G'$-Richelot dual of $D$ are defined (that is, the determinants mentioned in the definition of the two Richelot duals are nonzero). Then the results of Section 4 show that the $G$-Richelot dual of $D$ and the $G'$-Richelot dual of $D$ become isomorphic over $K(\sqrt{st}) = K(\sqrt{v(v-4)})$.

The proof that these two Richelot duals of $D$ are geometrically non-isomorphic to one another (except in the special cases listed in the theorem) is a computation along the same lines as the proof of the corresponding statement of Theorem 1. We leave the details to the reader.

Finally, suppose that $K$ has characteristic 0 and suppose that there is a ring homomorphism from $\mathbb{Z}[v]$ to $\mathbb{F}_{13}$ that takes $v$ to either 2 or 6. Then the curve $D$
and for each \(i\) and for each \(i\)

We define 2-torsion points

Note that the \(\Delta\)

E curve

with

A/G

a plane quartic whose Jacobian is

of the 2-torsion of

A

to

G

will continue to call our base field

K

the case that

v

results of \([5, \S 4]\). Our notation will be chosen to match that of \([5]\), except that we

have Jacobians isogenous to that of \(D\), we see that their Jacobians are absolutely simple too. The final statement of the theorem then follows from the observation

that if \(v\) is not algebraic, then there is a homomorphism \(\mathbb{Z}[v] \to \mathbb{F}_{13}\) that sends \(v\) to any given element.

\[\square\]

Remark. We used the field \(\mathbb{F}_{13}\) at the end of the proof simply because it is the smallest prime field that contains values of \(v\) that give rise to absolutely simple Jacobians. Other prime fields have a larger proportion of good values of \(v\). For example, there are 341 values of \(v\) in \(\mathbb{F}_{769}\) that give rise to absolutely simple Jacobians. For three-digit primes \(p\) the number of good \(v\)-values is typically greater than 0.3\(p\).

This implies that for a “randomly chosen” rational number \(v\), it is almost certainly the case that \(v\) will give rise to an absolutely simple Jacobian.

7. Proof of Theorem \([5]\)

The proof of Theorem \([5]\) is very much like the proof of Theorem \([4]\). We will produce three elliptic curves \(E_1, E_2, E_3\), two maximal isotropic subgroups \(G, G'\) of the 2-torsion of \(A = E_1 \times E_2 \times E_3\), and an automorphism \(\alpha\) of \(A\) that takes \(G\) to \(G'\). Then we will produce a hyperelliptic curve whose Jacobian is \(A/G\) and a plane quartic whose Jacobian is \(A/G'\). To produce these curves we will use the results of \([5, \S 4]\). Our notation will be chosen to match that of \([5]\), except that we will continue to call our base field \(K\), instead of \(k\).

Let \(K\) be an arbitrary field of characteristic not 2 and let \(t\) be an element of \(K\) with \(t(t + 1)(t^2 + 1)(t^2 + t + 1) \neq 0\). Let \(s = -(t^2 + t + 1)\) and let \(r\) be a square root of \(t^2 + 1\) in an algebraic closure of \(K\). Let

\[
A_1 = -2(t^2 + 1)s \quad B_1 = (t^2 + 1)s^2
\]

\[
A_2 = 4(t^2 + 1)s \quad B_2 = 4t^2(t^2 + 1)s^2
\]

\[
A_3 = -2(t^2 + t + 1)s \quad B_3 = (t + 1)^2(t^2 + 1)s^2
\]

and for each \(i\) let

\[
\Delta_1 = A_1^2 - 4B_1 = 4t^2(t^2 + 1)s^2
\]

\[
\Delta_2 = A_2^2 - 4B_2 = 16(t^2 + 1)s^2
\]

\[
\Delta_3 = A_3^2 - 4B_3 = 4t^2s^2.
\]

Note that the \(\Delta_i\) and the \(B_i\) are nonzero, so we may define for each \(i\) an elliptic curve \(E_i\) by

\[
y^2 = x(x^2 + A_i x + B_i).
\]

We define 2-torsion points \(P_i\) on the \(E_i\) by setting

\[
P_1 = ((t^2 + 1)s - rts, 0)
\]

\[
P_2 = (-2(t^2 + 1)s - 2rs, 0)
\]

\[
P_3 = ((t^2 + t + 1)s - ts, 0)
\]

and for each \(i\) we let \(Q_i\) be the 2-torsion point \((0, 0)\) on \(E_i\) and we let \(R_i = P_i + Q_i\).
Let $A = E_1 \times E_2 \times E_3$ and let $G$ be the subgroup of $A[2]$ generated by $(P_1, P_2, P_3)$, $(Q_1, Q_2, 0)$, and $(Q_1, 0, Q_3)$. Associated to these choices of $A$ and $G$ there is a quantity called the *twisting factor* $T$ (see §4). Using the formula in §4 we find that for our $A$ and $G$ the twisting factor is 0, so we may apply Prop. 14 to find a hyperelliptic genus-3 curve whose Jacobian is isomorphic over $K$ to $A/G$. The curve given by Prop. 14 is defined by two equations in $\mathbb{P}^3$, namely

\begin{align}
W^2Z^2 &= aX^4 + bY^4 + cZ^4 \\
0 &= dX^2 + eY^2 + fZ^2
\end{align}

where

\begin{align}
a &= 4t(t + 1)(t^2 + 1)^3s^5 \\
b &= 16t^2(t + 1)(t^2 + 1)^3s^5 \\
c &= 4t(t + 1)^2(t^2 + 1)^2s^6 \\
d &= -2t(t + 1)(t^2 + 1)s^2 \\
e &= (t + 1)(t^2 + 1)s^2 \\
f &= 2t(t^2 + 1)s^2.
\end{align}

If we replace $W$ by $2t(t + 1)(t^2 + 1)s^3W$ in Equation 4 and divide out common factors, we get Equation 11 and if we multiply Equation 5 by $2(t(t + 1)(t^2 + 1)s^2$ we get Equation 12. This shows that the Jacobian of $H(t)$ is isomorphic to $A/G$.

Now let $G'$ be the subgroup of $A[2]$ generated by $(P_1, P_2, R_3)$, $(Q_1, Q_2, 0)$, and $(Q_1, 0, Q_3)$, and let $T'$ be the twisting factor associated to $A$ and $G'$. The formula in §4 shows that

$$T' = -64(t^2 + 1)^2(t^2 + t + 1)s^3 = 64(t^2 + 1)^2s^4,$$

so the twisting factor is a nonzero square. Then Prop. 15 shows that there is a plane quartic whose Jacobian is isomorphic (over $K$) to $A/G'$. The plane quartic is given by

\begin{equation}
B_1X^4 + B_2Y^4 + B_3Z^4 + d'X^2Y^2 + e'X^2Z^2 + f'Y^2Z^2 = 0
\end{equation}

where

\begin{align}
d' &= 4(t^2 + 1)(2t^2 + t + 2)s^2 \\
e' &= -2(t^2 + 1)(2t^2 + t + 1)s^2 \\
f' &= 4(t^2 + 1)(t^2 + t + 2)s^2.
\end{align}

Dividing Equation 13 by $(t^2 + 1)s^2$ gives Equation 14 so the Jacobian of $Q(t)$ is isomorphic to $A/G'$.

To complete the proof we must show that $A/G \cong A/G'$. Note that there is a 2-isogeny $\psi$ from $E_1$ to $E_2$ that kills $Q_1$ and that takes $P_1$ and $R_1$ to $Q_2$ (see Example III.4.5]). Consider the automorphism $\alpha$ of $A$ that sends a point $(S_1, S_2, S_3)$ to $(S_1, S_2 + \psi(S_1), S_3)$. It is easy to check that $\alpha(G) = G'$, and it follows that $A/G \cong A/G'$, as desired, so the Jacobians of $H(t)$ and $Q(t)$ are isomorphic over $K$.  \[\square\]
8. Examples

Example 11. The curves

\[ 3y^2 = (x^2 - 4)(x^4 + 7x^2 + 1) \]

and

\[ -y^2 = (x^2 + 4)(x^4 + 3x^2 + 1) \]

over \( \mathbb{Q} \) are geometrically non-isomorphic, and yet their Jacobians are isomorphic to one another over \( \mathbb{Q} \).

Proof. If we take the two curves obtained by taking \( t = 2 \) in Theorem 11, replace \( x \) by \( x/2 \) in each equation, and twist both curves by 2, we get the two curves given above.

Example 12. The curves

\[ 5y^2 = -6x^6 - 64x^5 - 113x^4 + 262x^3 - 331x^2 + 584x + 232 \]

and

\[ 2y^2 = -21x^6 - 236x^5 + 45x^4 - 440x^3 - 615x^2 - 76x - 553 \]

are geometrically non-isomorphic, but their Jacobians become isomorphic to one another over \( \mathbb{Q}(\sqrt{-1}) \). Furthermore, their Jacobians are absolutely simple.

Proof. Take \( v = 2 \) in Theorem 12. We find that \( \rho_1 = -w/4, \rho_2 = w/4, \rho_3 = 2w + 2, \rho_4 = w - 1, \rho_5 = -w - 1, \) and \( \rho_6 = -2w + 2, \) where \( w = \sqrt{2} \). The curves \( C \) and \( C' \) in the theorem are \( y^2 = f_1 \) and \( y^2 = f_2 \), where

\[ f_1 = -(30625/32)x^6 - (67375/16)x^5 - (305025/64)x^4 - (23765/16)x^3 + (28665/16)x^2 + (1715/2)x - (735/2) \]

and

\[ f_2 = -(553/2)x^6 + 38x^5 - (615/2)x^4 + 220x^3 + (45/2)x^2 + 118x - (21/2). \]

Replacing \( x \) with \(-2/(x+1)\) in \( f_1 \) and multiplying the result by \((1/5)(2/7)^2(x+1)^6\) gives rise to the first curve given in the example. Replacing \( x \) with \(-1/x\) in \( f_2 \) and multiplying the result by \( 2x^6 \) gives rise to the second curve. Thus the Jacobians of the two curves become isomorphic to one another over \( \mathbb{Q}(\sqrt{\nu(v - 4)}) = \mathbb{Q}(\sqrt{-1}) \). The Jacobians are simple because we chose our \( v \) to be 2 modulo 13.

Example 13. The curves

\[ y^2 + (x^3 + x^2 + x)y = 31x^6 - 38x^5 - 217x^4 - 380x^3 + 304x^2 + 501x - 366 \]

and

\[ 11y^2 = -49x^6 - 378x^5 + 755x^4 + 110x^3 - 2285x^2 + 732x - 1368 \]

are geometrically non-isomorphic, but their Jacobians are isomorphic to one another over \( \mathbb{Q} \). Furthermore, their Jacobians are absolutely simple.

Proof. Take \( v = -4/3 \) in Theorem 13. The curves \( C \) and \( C' \) we obtain are \( y^2 = f_1 \) and \( y^2 = f_2 \), where

\[ f_1 = (28125/268912)x^6 - (11250/16807)x^5 + (3154875/1882384)x^4 - (812325/470596)x^3 - (57675/470596)x^2 + (26325/16807)x - (2025/2401) \]
and
\[
f_2 = -(131769/38416)x^6 + (11979/343)x^5 \\
- (5595645/38416)x^4 + (62535/196)x^3 \\
- (3735435/9604)x^2 + (86229/343)x - (23199/343).
\]
If we replace \( x \) with \((2x + 2)/(x + 2)\) in \(f_1\), multiply the result by \((343/5)^2(x + 2)^6\), and twist by 3, we get the curve
\[
y^2 = 125x^6 - 150x^5 - 865x^4 - 1518x^3 + 1217x^2 + 2004x - 1464;
\]
replacing \( y \) with \(2y + (x^3 + x^2 + x)\) gives the first curve in the example. If we replace \( x \) with \((x + 2)/(x + 1)\) in \(f_2\), multiply the result by \((1/11)^{196/2}(x + 1)^6\), and twist by 3, we get the second curve in the example. The Jacobians of the two curves become isomorphic to one another over \(\mathbb{Q}(\sqrt{v(v - 4)}) = \mathbb{Q}\). The Jacobians are absolutely simple because their reductions modulo 17 are absolutely simple. □

Remark. It is easy to see that the first curve in Example 13 has real-valued points, while the second curve does not. It follows that the real topology of a curve over \(\mathbb{Q}\) is not determined by its Jacobian. Furthermore, suppose we choose a positive integer \(d\) such that the quadratic twist of the second curve by \(d\) has rational points. The quadratic twist of the first curve by \(d\) will still not have any real points, let alone any rational points, so we see that the existence of rational points on a genus-2 curve over \(\mathbb{Q}\) is not determined by its Jacobian, even if the Jacobian is absolutely simple.

There are also triples \((r, s, t)\) that satisfy the hypotheses of Theorem 8 but that do not lie on the elliptic surface discussed in Section 6.

Example 14. The curves
\[
y^2 = x^6 - 24x^4 + 80x^3 - 63x^2 - 24x - 2
\]
and
\[
y^2 = -2x^6 + 6x^5 + 9x^4 - 48x^3 + 162x - 171
\]
are geometrically non-isomorphic, but their Jacobians become isomorphic to one another over \(\mathbb{Q}(\sqrt{2})\). Furthermore, their Jacobians are absolutely simple.

Proof. We take \( r = -7/4 \) and \( s = 1/2 \) and \( t = 1/4 \) in Theorem 8. Let \( \xi \) be a root of the irreducible polynomial
\[
x^6 + 6x^4 + 9x^2 + 16
\]
and let \( K \) be the number field generated by \( \xi \). The polynomial \( h \) of Section 5 is
\[
h = T^3 + 3/4T^2 + 9/16T + 3/64
\]
and its roots are the elements
\[
\beta_1 = (-\xi^4 - 7\xi^2 - 12)/16 \\
\beta_2 = (-\xi^5 + \xi^4 - 5\xi^3 + 7\xi^2 - 10\xi)/32 \\
\beta_3 = (\xi^5 + \xi^4 + 5\xi^3 + 7\xi^2 + 10\xi)/32
\]
of \( K \). The polynomials \( g_i \) are given by
\[
g_i = x^2 - 2\beta_i x + \beta_i^2/2 + 3/32,
\]
and their roots (indexed in accordance with the proof of Theorem 7) are

\[
\begin{align*}
    r_1 &= (-\xi^4 - \xi^3 - 7\xi^2 - 7\xi - 12)/16 \\
    r_2 &= (-\xi^4 + \xi^3 - 7\xi^2 + 7\xi - 12)/16 \\
    r_3 &= (-\xi^5 - \xi^4 - 3\xi^3 - 3\xi^2 - 8\xi - 8)/32 \\
    r_4 &= (-\xi^5 + 3\xi^4 - 7\xi^3 + 17\xi^2 - 12\xi + 8)/32 \\
    r_5 &= (\xi^5 + 3\xi^4 + 7\xi^3 + 17\xi^2 + 12\xi + 8)/32 \\
    r_6 &= (-\xi^5 - \xi^4 + 3\xi^3 - 3\xi^2 + 8\xi - 8)/32
\end{align*}
\]

We note that the two subgroups \( G \) and \( G' \) that appear in the proof of Theorem 7 are indeed Galois stable. We compute that the two Richelot duals are \( y^2 = f_1 \) and \( y^2 = f_2 \), where

\[
\begin{align*}
    f_1 &= -(81/512)x^6 - (1215/1024)x^5 - (21141/8192)x^4 - (8991/8192)x^3 \\
    &\quad - (19683/131072)x^2 - (2187/262144)x + (729/2097152) \\
    f_2 &= -(1863/256)x^6 - (3159/512)x^5 - (26973/4096)x^4 - (11421/4096)x^3 \\
    &\quad - (76545/65536)x^2 - (28431/131072)x - (13851/1048576).
\end{align*}
\]

Evaluating \( f_1 \) at \((2-x)/(4x)\) and multiplying the result by \((256/9)^2 x^6\) gives the first curve mentioned in the example; evaluating \( f_2 \) at \(-x/(4x-8)\) and multiplying the result by \((128/9)^2 (x-2)^6\) gives the second curve. These curves are geometrically non-isomorphic, and their Jacobians become isomorphic over \( \mathbb{Q}(\sqrt{3}) = \mathbb{Q}(\sqrt{2}) \). Furthermore, their Jacobians are absolutely simple because their reductions modulo 7 are absolutely simple.

\[\square\]

**Remark.** We obtained Example 16 from a triple \((r,s,t)\) that does not lie on the elliptic surface from Section 6, but the same example can be obtained from the triple \((r,s,t) = (-10,-1,-2)\), which does lie on the surface.

We computed all triples \((r,s,t)\) of naïve height at most 20 for which the curve \( C(r,s,t) \) has two \( \mathbb{Q} \)-rational Richelot duals whose Jacobians are absolutely simple and isomorphic over \( \mathbb{Q} \). Of all the examples we found, the triple \((r,s,t) = (-19/3,-6,-1/6)\) gave rise to the curves with the smallest coefficients:

**Example 15.** The curves

\[
y^2 = -9x^6 + 6x^5 - 47x^4 - 14x^3 - 5x^2 - 36x - 72
\]

and

\[
y^2 = 8x^6 - 60x^5 + 235x^4 - 186x^3 - 239x^2 - 30x - 1
\]

are geometrically non-isomorphic, but their Jacobians are isomorphic to one another over \( \mathbb{Q} \). Furthermore, their Jacobians are absolutely simple.

\[\square\]

**Example 16.** The Jacobian of the hyperelliptic curve

\[
3v^2 = -17u^8 + 56u^7 - 84u^6 + 56u^5 - 70u^4 - 56u^3 - 84u^2 - 56u - 17
\]

and the Jacobian of the plane quartic

\[
x^4 + 4y^4 + 4z^4 + 20x^2y^2 - 8x^2z^2 + 16y^2z^2 = 0
\]

are isomorphic to one another over \( \mathbb{Q} \).
Proof. We take $t = 1$ in Theorem 3. The plane quartic $Q(1)$ from the theorem is the plane quartic given in the example. The hyperelliptic curve from the theorem is given by the pair of homogeneous equations

\begin{align*}
W^2Z^2 &= -(1/3)X^4 - (4/3)Y^4 + Z^4 \\
0 &= -X^2 + 2Y^2 + 2Z^2.
\end{align*}

We dehomogenize the equations by setting $Z = 1$, and we parametrize the conic given by Equation 8 by setting

\begin{align*}
X &= 2(u^2 + 1)/(u^2 + 2u - 1) \\
Y &= (u^2 - 2u - 1)/(u^2 + 2u - 1).
\end{align*}

Taking $W = v/(u^2 + 2u - 1)^4$ then gives us the hyperelliptic curve in our example. \hfill \Box

Remark. We note that the discriminant of the degree-8 polynomial used to define the hyperelliptic curve in Example 16 is $2^{94}$!

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