Mesoscopic and macroscopic quantum correlations in photonic, atomic and optomechanical systems

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This paper reviews the progress that has been made in our knowledge of quantum correlations at the mesoscopic and macroscopic level. We begin by summarizing the Einstein-Podolsky-Rosen (EPR) argument and the Bell correlations that cannot be explained by local hidden variable theories. It was originally an open question as to whether (and how) such quantum correlations could occur on a macroscopic scale, since this would seem to counter the correspondence principle. The purpose of this review is to examine how this question has been answered over the decades since the original papers of EPR and Bell. We first review work relating to higher spin measurements which revealed that macroscopic quantum states could exhibit Bell correlations. This covers higher dimensional, multi-particle and continuous-variable EPR and Bell states where measurements on a single system give a spectrum of outcomes, and also multipartite states where measurements are made at multiple separated sites. It appeared that the macroscopic quantum observations were for an increasingly limited span of measurement settings and required a fine resolution of outcomes. Motivated by this, we next review correlations for macroscopic superposition states, and examine predictions for the violation of Leggett-Garg inequalities for dynamical quantum systems. These results reveal Bell correlations for coarse-grained measurements which need only distinguish between macroscopically distinct states, thus bringing into question the validity of certain forms of macroscopic realism. Finally, we review progress for massive systems, including Bose-Einstein condensates and optomechanical oscillators, where EPR-type correlations have been observed between massive systems. Experiments are summarized, which support the predictions of quantum mechanics in mesoscopic regimes.

I. INTRODUCTION

In recent decades, there has been huge progress made in the manipulation of quantum systems for the purpose of applications in the field of quantum information and quantum technologies. A large part of that progress is a direct result of our knowledge of quantum correlated systems. In this review, we summarize the status of what has been learned about macroscopic quantum correlations. Quantum correlations are defined as those correlations described by quantum mechanics that cannot also be described by classical theory or classical-like theories. Here, we restrict the meaning further, to imply those correlations whose source is a quantum entangled state, with emphasis given to the type of quantum correlations considered by Einstein, Podolsky and Rosen \cite{1} and Bell \cite{2}. Our approach is to present the progress as a historical timeline of discoveries relating to mesoscopic and macroscopic quantum correlations.

Arguably, the study of quantum correlations began with the Einstein-Podolsky-Rosen (EPR) argument \cite{1}. We begin therefore with a summary of the EPR paradox, and the theorem given by Bell \cite{2–5}, who proved how such correlations for two separated spin 1/2 systems cannot be explained by any local hidden variable (LHV) theory. This apparently resolved the paradox by implying that the premise of local realism on which the EPR argument was based was fundamentally invalid. The property of the correlations that violate Bell’s LHV theories is termed Bell nonlocality \cite{6}. A brief summary is given in Section II.

Originally, it was an open question as to whether such quantum correlations could manifest on a mesoscopic, or macroscopic, scale. It is the answer to this question that we analyze in the review. The anticipated answer may well have been no, in order to ensure compatibility with the correspondence principle. First, “macroscopic scale” could refer to systems of large size, i.e. to systems possessing a large number of particles. It was shown by Mermin that quantum mechanics predicts the failure of local hidden variable theories for two separated systems of higher spin \cite{7}. This translates to the prediction of failure of local hidden variable theories for separated systems possessing multiple particles \cite{8}, or to systems of higher dimensionality.

Another way to increase the system size is to consider three or more separated particles, or systems. Measurements can then be made locally on each of many systems. Svetlichny analyzed three systems to show how to confirm a genuine mesoscopic form of multipartite Bell nonlocality, that cannot be explained as correlations arising from a bipartite Bell nonlocality shared among just two systems \cite{9}. Greenberger, Horne and Zeilinger (GHZ) showed the extreme paradox associated with some types of tripartite quantum states \cite{10}, this being later extended to \textit{N} particles by Mermin \cite{11}, who demonstrated the possibility of an increasing amount of violation of a Bell inequality for systems of increasing size. The exact nature of the increasing size needed to be considered carefully however. For a genuine
multipartite nonlocality, it was revealed that the violations would be constant with increasing size. The macroscopic quantum observations appeared to be restricted to an increasingly limited span of measurement settings and also appeared to require a fine resolution of measurement outcomes. Later work countered some of these conclusions, for different systems and measurements. In Sections III, IV and VII, we review results for Bell nonlocality for higher dimensional systems, multi-particle systems, and multipartite systems.

One way to achieve macroscopic correlations is to amplify the size or dimensionality of each system involved in the correlations. A natural limit is provided where measurements give continuous variable (CV) outcomes. For the CV systems involving the measurement of quadrature field amplitudes, the number of photons at each site is amplified by introducing local oscillator fields that allow assignment of the correlations in terms of macroscopic Schwinger spin observables. We thus review quantum correlations in CV systems. It is feasible to generate EPR correlations and entanglement for fields, but the generation of Bell-nonlocal states that falsify local hidden variable theories for continuous-variable measurements is a more difficult task. This can be done, however, and we give a summary of continuous-variable quantum correlations in Section V.

Leggett argued that macroscopic quantum mechanics can only be rigorously tested using states that are superpositions of macroscopically distinct states, in the spirit of the Schrödinger cat paradox [12, 13]. In Sections V and VI, we thus review the quantum correlations associated with entangled cat states, which includes superpositions of two-mode number states (NOON states), the GHZ states, and entangled coherent states. The hybrid state involving a microscopic qubit entangled with a macroscopic qubit is also analyzed, as the prototype of the original Schrödinger cat state modeling a measurement apparatus [14]. We summarize interpretations, which gives insight into the measurement problem, in Section VI. The impact of a coarse graining of measurement outcomes is reviewed in Sections VII and VIII, and for some states and measurements is found not to be fundamentally limiting.

Leggett and Garg proposed to test macroscopic realism in a setting where measurements are made at different times, and where measurements need only distinguish between two macroscopically distinct states [15]. Motivated by this, we review in Sections VII and VIII the dynamics creating macroscopic superposition states (cat states), and the macroscopic correlations in time predicted to violate the Leggett-Garg inequalities. For suitably adapted systems, Bell correlations can be predicted using macroscopic measurements that distinguish only between two macroscopically distinct coherent states, which act as macroscopic qubits. This allows macroscopic quantum correlations under coarse-graining of measurement outcomes at a macroscopic level, although there is sensitivity to decoherence (losses) and to the imprecision of measurement settings. In Section VIII, we summarize the progress toward tests of Leggett-Garg’s macro-realism. Experiments realizing these correlations are more limited, but developments are summarized.

An alternative perspective is that for a system size to be truly macroscopic, the systems must have large mass. In Sections IX, we review the status of quantum correlations in atomic and Bose-Einstein condensate (BEC) systems, where experiments have demonstrated mesoscopic EPR correlations, both for atoms within a condensate and for separated condensates. In Section X, we give a summary of the significant progress made in achieving quantum correlations in optomechanical systems, including for EPR-type and Bell experiments. Overall, however, there has not yet been reported a rigorous violation of a Bell inequality where the hidden variables involved are for spatially separated objects of significant mass. The final section of this review gives a conclusion.

II. BELL CORRELATIONS FOR SMALL SYSTEMS

Bell considered the singlet state of two spin 1/2 particles,

$$|\psi_{Bell}\rangle = \frac{1}{\sqrt{2}} \{|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B\}$$

(1)

where $|\uparrow\rangle$ and $|\downarrow\rangle$ are the spin 1/2 eigenstates for the spin Z component $\hat{S}_z$ of the spin vector $\hat{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$ [2]. The particles are labelled A and B and become spatially separated. We will use superscripts to denote that the observables apply to the system A or B. For the singlet state, the outcomes of the two spin components $\hat{S}_z(A)$ and $\hat{S}_z(B)$ are anti-correlated. We also introduce the Pauli spin observables $\hat{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ where $\frac{1}{2}\hat{\sigma} = \hat{S}$, for which the outcomes of the measurements of the spin components are ±1. The rotated spin components are

$$\hat{\sigma}_\theta^{(A)} = \cos \theta \hat{\sigma}_x^{(A)} + \sin \theta \hat{\sigma}_y^{(A)}, \quad \hat{\sigma}_\phi^{(B)} = \cos \phi \hat{\sigma}_x^{(B)} + \sin \phi \hat{\sigma}_y^{(B)},$$

(2)

which can be measured on A and B respectively, by a Stern-Gerlach apparatus at each site. A transformation into a different spin basis reveals the anti-correlation between the outcomes at each site, for measurements of the spins $\hat{\sigma}_\theta^{(A)}$ and $\hat{\sigma}_\phi^{(B)}$ where $\theta = \phi$.

Most experiments to date use the photonic version of the Bell states, the rotations [2] being realized at each site by beam splitters with a variable transmission (or beam splitters with phase shifts), polarizing beam splitters (PBS),
or polarizers (refer [6, 10]). Each particle is a photon, and the Bell state [1] is mapped onto a four-mode state where $|↑⟩_A = |1⟩_a + |0⟩_a −$ and $|↓⟩_A = |0⟩_a + |1⟩_a −$. The restriction (6) leads to a bound on the correlations given by the Bell-Clauser-Horne-Shimony-Holt

$$\phi$$

that describe the state to be measured at $σ$. Here $P_{++}(θ, φ)$ may be realized as orthogonally polarized modes of the photon incident on a polarizer at $A (B)$. The horizontally and vertically polarized states are written as $|↑⟩_A = |1⟩_a + |0⟩_a − ≡ |H⟩_A$ and $|↓⟩_A = |0⟩_a + |1⟩_a − ≡ |V⟩_A$, and similarly for system B. As well as $|ψ_{Bell}⟩$, we also consider the positively correlated Bell state $|ψ_{Bell,+}⟩ = \sqrt{2}(|↑⟩_A |↑⟩_B + |↓⟩_A |↓⟩_B)$. The Bell state is then written conveniently as

$$|ψ_{Bell,+}⟩ = \frac{1}{\sqrt{2}}(|H⟩_A + |V⟩_A)$$

(3)

where $|H, H⟩ = |H⟩_A |H⟩_B$ and $|V, V⟩ = |V⟩_A |V⟩_B$. The photon emerges from the polarizer as for a beam splitter. The photon is detected to be either in a $+ \text{ position}$ or a $− \text{ position}$, which we identify as the spin result “up” and the spin result “down”, and which indicates the direction of polarization of the emerging photon. The polarizer (or beam splitter) provides the mode transformations

$$\hat{A}_± = \frac{\cos θ ± √2 sin θ}{2} \hat{a}_± + \frac{\sin θ ± √2 cos θ}{2} \hat{b}_±,$$

$$\hat{D}_± = \frac{± sin θ √2 cos θ}{2} \hat{b}_± + \frac{± cos θ √2 sin θ}{2} \hat{b}_±.$$  

(4)

Here $\hat{a}_±$ and $\hat{b}_±$ are the boson destruction operators for the modes $a_±$ and $b_±$, respectively. With only one photon incident on the polarizer $A (B)$, the spin observable $\hat{σ}_θ (A) (\hat{σ}_φ (B))$ is given by the mode number difference $\hat{c}_+ \hat{c}_− − \hat{c}_− \hat{c}_+$ ($\hat{d}_+ \hat{d}_− − \hat{d}_− \hat{d}_+$).

Calculation reveals that for systems $A$ and $B$ prepared in the Bell state [1], the expectation value for the Pauli spin product is

$$E(θ, φ) = \langle σ_θ^A σ_φ^B \rangle = − cos(θ − φ).$$  

(5)

A similar result is obtained for the photonic example, where for $|ψ_{Bell,+}⟩$ one obtains $E(θ, φ) = \langle (\hat{c}_+ \hat{c}_− − \hat{c}_− \hat{c}_+)(\hat{d}_+ \hat{d}_− − \hat{d}_− \hat{d}_+) \rangle = cos(θ − φ)$.

Bell’s first proof assumed a deterministic local realistic theory, where there are definite values $λ_θ^A$ and $λ_φ^B$ for the spin components $σ_θ (A)$ and $σ_φ (B)$ of both the particles [2, 3]. These values predetermine the result of the spin component if measured. Here, $λ_θ^A = ±1$, $λ_φ^B = ±1$, though the anti-correlation of the Bell state [1] would imply $λ_θ^A = −λ_φ^B$. Bell’s proof was later generalized to cover more general local realistic theories where there may be local stochastic interactions due to, for example, the measurement apparatus [4, 5, 16, 18, 20]. Bell’s more general work accounted for all theories that are local realistic and locally causal. Bell [3] and Clauser, Horne, Shimony and Holt (CHSH) [21] considered the predictions of all local hidden variable theories (LHV) for which the joint spin product satisfies

$$E(θ, φ) = \int ρ(λ) E(θ, λ) E(φ, λ) dλ.$$  

(6)

This LHV constraint may also be expressed in terms of the measurable joint probabilities $P_{++}(θ, φ)$ as

$$P_{++}(θ, φ) = \int ρ(λ) P_{++}^A(θ, λ) P_{++}^B(φ, λ) dλ.$$  

(7)

Here $P_{++}(θ, φ)$ is the joint probability of the outcome $+$ at both sites, with measurement settings $θ$ and $φ$ respectively. The $ρ(λ)$ is a distribution over a set of hidden variables $λ$ and $E(λ)$ is the expectation value of the measurement $σ_θ (A)$ at site $A$, given the measurement setting $θ$ at that site, for the hidden variable state $λ$. Similarly, $P_{++}^A(θ, λ)$ is the probability for outcome $+$ at site $A$, given $λ$. The $E(λ)$ is the expectation value of the measurement $σ_θ (A)$ at site $A$, given $λ$. The $E(λ)$ and $P_{++}^A(θ, λ)$ are dependent on the parameters $λ$ that describe the state to be measured at $A$, and are dependent on the measurement settings $θ$, but are assumed independent of the setting $φ$ chosen at the space-like separated site $B$. This is justified based on causality, and assuming independence of choices of the settings at each site (locality). The $E(λ)$ and $P_{++}^A(θ, λ)$ are defined similarly. The restriction (6) leads to a bound on the correlations given by the Bell-Clauser-Horne-Shimony-Holt (CHSH) inequality [21] $|B| ≤ 2$ where

$$B = E(θ, φ) − E(θ, φ') + E(θ', φ) + E(θ', φ').$$  

(8)
This is readily seen, by noting the algebraic constraint $|B| \leq 2$ on $B$ for any deterministic local hidden variable theory, where one assumes predetermined spin variables $A_1$, $A_2$, $B_1$ and $B_2$ for measurements $\sigma^A_0$, $\sigma^A_\phi$, $\sigma^B_\phi$, and $\sigma^B_\theta$, the values of the variables being either $+1$ or $-1$. Since a probability distribution compatible with the general local hidden variable theory (9) is convex, the same bound will apply to the more general local hidden variable theory. A convex set is fully determined by the extremal points, as any other points in the convex set can be expressed as convex combinations of these extremal points. Proofs are given in Scarani and Brunner et al. (8).

For the choices of angle $\theta = 0$, $\phi = \pi/4$, $\theta' = \pi/2$, $\phi' = 3\pi/4$, the quantum prediction of (5) for $|\psi_{Bell}\rangle$ violates the bound giving $B = -2\sqrt{2}$. Similarly, for state $|\psi_{Bell,+}\rangle$ the value is $B = 2\sqrt{2}$. In this way, rather dramatically, it is shown that the predictions of quantum mechanics cannot be compatible with any local realistic (or local causal) theory that embodies the very simple and reasonable premises, (6) and (7).

Bell’s original paper addressed the EPR paradox, of 1935 (11). The assumption (6) reduces to that of EPR’s local realism, when $\theta = \phi$. For any $\theta = \phi$, we note the anti-correlation between the spin outcomes at each site is maximum, as for the Bell state (11). EPR argued in their paper that if one can predict with certainty the result of a measurement on a system without disturbing that system, then the result of the measurement was predetermined and describable by a hidden variable. In Bohm’s version of the argument (22), the fact that one can infer the outcome for spin $\hat{\sigma}^{(A)}_\theta$ of system $A$ by measuring the spin $\hat{\sigma}^{(B)}_\theta$ of the space-like separated system $B$ implies the condition of EPR, since this measurement is justified to be noninvasive to system $A$. The EPR argument then implies that for any $\theta$, there exists a hidden variable $\lambda^A_\theta$ to predetermine the value of the measurement of $\hat{\sigma}^{(A)}_\theta$. Since this description is not consistent with any local quantum state for system $A$, EPR would conclude quantum mechanics to be incomplete. Bell’s paper thus showed that any hidden variable theory consistent with the assumption of local realism could not be compatible with the predictions of quantum mechanics.

Early evidence of a Bell state was given by Bleuler and Brandt (24) and Hanna (25) using Geiger counters, and by Wu and Shaknov (26) who used scintillation counters that have higher detection efficiency. These authors measured the correlation between two gamma photons generated by positron-electron annihilation. The annihilation produces two photons in a state $|\psi\rangle = \frac{1}{\sqrt{2}}(|HV\rangle - |VH\rangle)$. The coincidence measurements with polarizer angle settings $\theta$ and $\phi$ at 0 and $\pi/2$ gave an enhanced counting rate, in agreement with the theory proposed by Wheeler (27) and Pryce and Ward (28). This suggests correlations along the lines of an EPR paradox. An EPR paradox occurs when there is a maximum correlation between the elements of two pairs of non-commuting observables, such as $\{\hat{\sigma}_z^{(A)}, \hat{\sigma}_z^{(B)}\}$ and $\{\hat{\sigma}_x^{(A)}, \hat{\sigma}_x^{(B)}\}$. While the original EPR argument considered position and momentum (1), Bohm’s version considered non-commuting spin observables (23). In the ideal case, this correlation leads to a contradiction between the assumption of local realism, and the completeness of quantum mechanics. Bohm and Aharonov (29) later explained how the results of the Wu-Shaknov experiment were consistent with the Bell state $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle - \downarrow\rangle)$, and not for a non-separable classical mixture of the states $|\uparrow\rangle |\downarrow\rangle$ and $|\downarrow\rangle |\uparrow\rangle$. The Wu-Shaknov experiment thus gave early evidence of quantum correlation, in the form of Bohm’s EPR paradox and entanglement.

An early test of Bell’s theorem was performed by Freedman and Clauser using the correlation in polarization between photons emitted as a pair in an atomic cascade (30), following the proposal of Clauser, Horne, Shimony and Holt (21). This was followed by increasingly rigorous experimental tests including those by Aspect, Dalibard and Roger (31) and Aspect, Grangier and Roger (32,33), which supported the quantum predictions. Later, it was suggested by Reid and Walls (17) and Shih and Alley (34) to use the correlated photon pairs generated in parametric down conversion, modeled by the two-mode Hamiltonian $H = i\kappa E (\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b})$. The first such test was performed by Ou and Mandel, using the two-mode version and the equivalent of beam splitters to generate four modes (35). The four mode variants based on the interaction Hamiltonian

$$H = i\kappa E (\hat{a}^\dagger_+ \hat{b}^\dagger_+ + \hat{a}^\dagger_- \hat{b}^\dagger_- - \hat{a}_+ \hat{b}_+ - \hat{a}_- \hat{b}_-)$$

(9)

were outlined in Reid and Walls (17) and Horne, Shimony and Zeilinger (36), with proposals and an experimental violation given by Rarity and Tapster (37,38). A set of experiments based on parametric down conversion (38,42) further confirmed quantum predictions. Loopholes due to poor detection efficiencies and lack of spatial separation (needed to justify locality) have since been overcome (13,50), with more recent work focusing on multiple sources of loopholes (51,52). The experiments support quantum mechanics, giving a violation of Bell’s inequality. Bell correlations have found significant applications in quantum information, including for device-independent entanglement detection (53,54) and the certification of random numbers (50).
III. BELL CORRELATIONS FOR HIGHER SPIN, MULTI-PARTICLE AND HIGHER DIMENSIONAL SYSTEMS

Originally, it may have been speculated that the violation of Bell inequalities could only be possible for microscopic systems. The correspondence principle suggests Bell inequalities to be valid, for large systems. The extent to which this is true is an ongoing investigation, and is the topic of this review.

A. Mermin’s result for higher spin

The work of Mermin gave an advance in this direction, by showing that Bell’s theorem applies to higher spin systems \[57\]. Mermin considered the higher spin state that is the generalization of the singlet Bell state \[1\],

\[ \psi = \frac{1}{(2j+1)^{1/2}} \sum_{m=-j}^{j} (-1)^{j-m} \langle m, -m \rangle_{\tilde{n}, \tilde{a}} \]

where the state has zero total spin. Here, \( |m_1, m_2 \rangle_{\tilde{n}_1, \tilde{n}_2} \) denotes the state for two spin-\( j \) particles labelled \( i = 1, 2 \), which have a spin \( m_i \) along the axis \( \tilde{n}_i \). Defining \( m_i (a) \) to be the spin observable of particle \( i \) measured along the direction \( a \), Mermin started out with the inequality \( \langle m_1 (a) + m_1 (b) \rangle \geq -\langle m_1 (a) + m_1 (b) \rangle \) and the relation \( m_1 (c) = -m_2 (c) \), to derive an inequality of the form

\[ j\langle |m_1 (a) - m_2 (b)| \rangle \geq \langle m_1 (a) m_2 (c) \rangle + \langle m_1 (b) m_2 (c) \rangle . \]

Inequality \[11\] is satisfied if deterministic local realism is valid, where definite values for any component of either of the two correlated spins always exist. This assumption justifies the relation \( m_1 (c) = -m_2 (c) \) for the correlations of \[10\]. Thus, a violation of the inequality implies the contradiction between local realism and quantum theory.

Mermin showed that violations of \[11\] are possible for the state Eq. \[10\]. The averages in the inequality Eq. \[11\] were calculated using Eq. \[10\], which leads to the inequality

\[ \frac{1}{2j+1} \sum_{m,m'} |m - m'| \left| \langle m | e^{-2i\theta S_y} | m' \rangle \right| \geq \frac{2}{3} (j+1) \sin \theta . \]

Here, \( \pi - 2\theta \) is the angle between the spin directions \( a \) and \( b \), while the angle between \( a \) and \( c \) is identical to the angle between \( b \) and \( c \), and is given by \( \pi/2 + \theta \). Mermin found that there exists a range for \( \theta \), given by the condition \( 0 < \sin \theta < 1/2j \), such that inequality \[12\] will always be violated for a spin \( j \). This gives the surprising result that there is a violation of local realism for large spin \( j \). However, it was noted that for spin \( 1/2 \), the inequality \[12\] is violated for any \( \theta \). By contrast, in the limit of \( j \to \infty \), the range for \( \theta \) to show violation is of order \( 1/j \). This gives the possibility of local realism as a description of the quantum mechanical state in the classical limit. Since the control of the angle \( \theta \) is limited by experimental precision, consistency is given with the correspondence principle.

The higher spin system admits the \( 2j+1 \) outcomes \( -j, \ldots, j \) at each site, for systems \( A \) and \( B \), and thus is an example of a higher dimensional system of dimension \( d = 2j+1 \). The higher dimensionality can be achieved in different ways, with different mappings onto physical systems.

B. Multi-particle violations

The significance of the higher dimensions for macroscopic quantum mechanics is given by the work of Drummond, who considered the four-mode bosonic state \[58\]

\[ \Psi_N = \frac{1}{N!\sqrt{N+1}} \left( \hat{a}_+^{\dagger} \hat{b}_+^{\dagger} + \hat{a}_-^{\dagger} \hat{b}_-^{\dagger} \right)^N |0000\rangle \]

with \( N \) bosons at each site \( A \) and \( B \), modeled after \( |\psi_{Bell, +}\rangle \) for \( N = 1 \). This state maps onto the higher spin Bell state \[10\] considered by Mermin, thereby highlighting how the higher spin results lead to mesoscopic Bell violations involving multiple particles at each site. Here, \( |0000\rangle \) symbolizes the vacuum state of all four modes. A polarizer (or polarizing beam splitter) is placed at each site, aligned at an angle \( \theta \) or \( \phi \), the mode transformations being given as \[4\]. An incoming particle can be detected at either the + or - mode at the output of the polarizer, similar to the up or down state of a spin 1/2 particle. At each site, \( N \) indistinguishable bosons are incident on the polarizer, and the
possible outcomes are $N$, $N-1$, ..., 1, 0 bosons detected at the + output mode. The configuration therefore maps to a two systems of higher spin, with dimension $d = N + 1$. This work confirmed violations of Bell inequalities for the $N$-particle state \[13\] for arbitrarily large $N$.

The violations given in \[58\] were formulated in terms of the Clauser-Horne Bell inequalities \[52\]. Here, the outcome at $A$ ($B$) is assigned the value $+1$ if all $N$ bosons are detected in the + mode i.e. in the mode $c_+^N$ (or $d_+^N$), and otherwise are assigned the value 0. The Clauser-Horne (CH) inequality can be expressed as $-1 \leq CH \leq 0$, where

$$CH \equiv P_{++}(\theta, \phi) + P_{++}(\theta', \phi') + P_{++}(\theta, \phi') - P_{++}(\theta, \phi') - P_{++}(\theta', \phi) + P_{++}(\theta', \phi) .$$  

(14)

$P_{++}(\theta, \phi)$ is the joint probability for detecting $+1$ at both sites with polarizer angle settings $\theta$ and $\phi$, and $P_{++}(\theta)$ ($P_{++}(\phi)$) are the marginal probabilities for detecting $+1$ at the site $A$ ($B$) only. Where all bosons can be detected, the marginal probabilities become equivalent to the one-sided joint probabilities $P_{++}(\theta)$, $P_{++}(\phi)$ for detecting $+1$ at site $A$ ($B$) and a total of $N$ bosons at site $B$ ($A$). The fair sampling assumption (also called the no-enhancement axiom) justifies that the marginal probabilities can be measured as the one-sided joint probabilities, in situations of limited detection efficiency.

In the original paper \[58\], the detection probabilities were expressed as proportional to the higher-order normally-ordered moments

$$P_{++}(\theta, \phi) \propto \langle \psi | \left( \hat{c}_+^N \right)^N \left( \hat{d}_+^N \right)^N \hat{c}_+^N \hat{d}_+^N | \psi \rangle .$$  

(15)

For the state \[13\], quantum mechanics predicts that $P_{++}(\theta)$ and $P_{++}(\phi)$ are both independent of angle choice $\theta$ (and similarly that $P_{++}(\phi)$ and $P_{++}(\theta)$ are independent of $\phi$). The fair sampling assumption was used (along with the symmetry of the marginal probabilities) to justify an expression for the left-side of (14) in terms of the measurable ratio of joint to one-sided probabilities: $g(\theta, \phi) = P_{++}(\theta, \phi) / (P_{++}(\theta') + P_{++}(\phi)) \equiv P_{++}(\theta, \phi) / P_{++}(\theta, \phi)$. The quantum prediction

$$g(\theta, \phi) = \cos^{2N}(\theta - \phi)$$  

(16)

for the $2N$-boson state \[13\] gives a violation of the CH inequality \[13\] for suitable angle choices, for any $N$. The selected angles are expressed as $\theta = 0$, $\phi = \varphi$, $\theta' = 2\varphi$, $\phi' = 3\varphi$, and the value of $\varphi$ optimized. As for the higher spin case, the range of angle $\theta$ for which violation is possible reduces with increasing $N$. The use of the Clauser-Horne approach ensured that the result was not restricted to the assumption of local deterministic theories. Rather, all local hidden variable theories were ruled out, for all $N$. Drummond also considered the predictions where $J < N$ bosons are detected at the + mode, at each site \[58\]. The violations decreased with decreasing $J$. The reduced values of $J$ corresponded to the loss of information about the particle outcomes and the effect is similar to that expected for decoherence and detection efficiency losses. Overall, the work of \[58\] was consistent with Mermin’s prediction that the range of parameter space for which violations can be observed is reduced with increasing $N$.

The generation of the multi-particle bosonic Bell states \[13\] can be achieved from four-mode parametric down conversion \[9\], as was explained in \[60\]. The solution for the parametric Hamiltonian is $|\psi\rangle = \sum_{N=0}^{\infty} c_N |\Psi_N\rangle$, where $|\Psi_N\rangle$ is the quantum state in Eq. \[13\], $c_N = \sqrt{N+1} \tanh N r / \cosh^2 r$, and $r = \kappa t$. This implies the state $|\Psi_N\rangle$ can be generated, conditioned on the detection of a total of $N$ photons, at each site.

An experimental realization of a higher spin Bell experiment was given by Howell et al \[61\]. They used the method of Lamas-Linares et al \[62\] to generate a four-mode entangled state from parametric down conversion. Here the two photon polarization entangled modes are considered as spin–1 particles. In order to generate these spin–1 particles, polarization-entangled four photon states are created from a pulsed type-II parametric down-conversion process. The detection is performed using a post-selection measurement, in order to consider the second order term $|\Psi_2\rangle$ of the down-converted field which is given by:

$$\frac{1}{\sqrt{3}} \left( |2H, 2V\rangle - |HV, VH\rangle + |2V, 2H\rangle \right) .$$  

(17)

Here $|2V, 2H\rangle$ means that if Alice measures two vertical photons, Bob measures two horizontal photons. The possible outcomes for Alice (Bob) are three: $|2H\rangle$, $|HV\rangle$ ($|VH\rangle$) and $|2V\rangle$, which are denoted as $|1\rangle$, $|0\rangle$ and $|-1\rangle$ respectively, and all of them have the same probability. The assignment of values is the following: $+1$ for both measurements results $|1\rangle$ and $|-1\rangle$, while $-1$ for a $|0\rangle$ result. Using this assignment, it is possible to measure the probabilities for the CHSH Bell-type inequality given in \[8\]. A violation of the CHSH Bell-type inequality of $2.27 \pm 0.02$ is obtained, for polarizer analyzer settings corresponding to $\theta = -16^\circ, \phi' = 14^\circ, \theta' = 4^\circ$ and $\phi = 6^\circ$. 

C. Higher dimensional Bell inequalities

The higher spin Bell inequalities were the first examples of Bell inequalities for higher dimensions. The violation of local realism for two higher dimensional (higher spin) systems $A$ and $B$ has been studied extensively since Mermin’s original paper [63, 68]. It was pointed out that the violation of local realism could be obtained for a broader range of settings $\theta$ if different Bell inequalities were used [63, 64]. Garg and Mermin further considered higher spin systems in 1984 and used a geometrical method to derive Bell inequalities for $j = 1, 3/2$ and $5/2$ [79]. Gisin and Peres discovered perhaps surprisingly that for a pair of spin-$j$ particles in a singlet state, the Bell violation can be as large as for a pair of spin-1/2 particles [71]. Kaszlikowski et al. [73] numerically investigated violations of local realism for qudits using bipartite $d$-dimensional mixed states, suggesting that in fact the maximal violation increases monotonically with $d$. They argued that the limitation for violations was the restricted use of measurements.

In 2002, Collins, Gisin, Linden, Massar and Popescu (CGLMP) [74] constructed a family of Bell inequalities for bipartite systems of arbitrary dimension $d$, which for certain quantum states gave a violation as $d \to \infty$. They first considered two parties with two possible measurements $A_1, A_2, B_1, B_2$ each. Each measurement in turn has $d$ possible outcomes: $0, ..., d - 1$. Starting from a combination of these measurements, and noting that $(B_1 - A_1) + (A_2 - B_2) + (B_2 - A_2) + (A_1 - B_1) = 0$, they write down a Bell expression

$$ I = P (A_1 = B_1) + P (B_1 = A_2 + 1) + P (A_2 = B_2) + P (B_2 = A_1) $$

where $P (A_1 = B_1 + k) = \sum_{m=0}^{d-1} P (A_1 = m, B_1 = m + k \text{mod} d)$. Since only 3 of the probability distributions are allowed due to the constraint, $I$ that satisfies local realism obeys the inequality $I \leq 3$. The violation of this inequality implies nonlocal correlations. Collins et al. then considered generalized expressions for $I$, deriving a family of inequalities $I_d \leq 2$ which are satisfied by all local realistic theories, yet for nonlocal theories can obtain higher values. The restriction is not to deterministic local hidden variable theories. In fact, the task of deriving Bell inequalities has been turned into a geometric one [72, 73, 74]. In this approach, deterministic probability distributions form the extremal points in the convex set of probability distributions that are compatible with local variable model. Bell inequalities are the (hyper-)planes that bound the convex set and probability distributions that violate these inequalities are points that lie outside of these convex sets. To derive CGLMP inequalities (following Acín et al. [83]), one may start with the deterministic model, for which if $[(B_1 - A_1) + (A_2 - B_1) + (B_2 - A_2) + (A_1 - B_2) - 1] = d - 1$ where $|x|_d = x \text{mod} d$. From the inequality $|x|_d + |y|_d \geq |x + y|_d$, one finds $|B_1 - A_1|_d + |A_2 - B_1|_d + |B_2 - A_2|_d + |A_1 - B_2|_d \geq (B_1 - A_1) + (A_2 - B_1) + (B_2 - A_2) + (A_1 - B_2) - 1|_d$. Equality holds in the case of deterministic distributions and nonlocal correlation is demonstrated if $|B_1 - A_1|_d + |A_2 - B_1|_d + |B_2 - A_2|_d + |A_1 - B_2|_d > (|B_1 - A_1|_d + |A_2 - B_1|_d + |B_2 - A_2|_d + |A_1 - B_2|_d) > d - 1$. This leads to

$$ (|B_1 - A_1|_d + |A_2 - B_1|_d + |B_2 - A_2|_d + |A_1 - B_2|_d) > d - 1 $$

where $\langle |x|_d \rangle = \sum_{k=0}^{d-1} kP (|x|_d = k)$. It was shown that the maximally entangled states $|\psi\rangle = (1/\sqrt{d}) \sum_{k=0}^{d-1} |k, k\rangle$ violate Bell CGLMP inequalities [74, 71]. Here, $k = 0, ..., d - 1$ are the possible outcomes for the measurement made on the subsystems labelled $i = 1, 2$; $|k_1, k_2\rangle$ denotes a state with outcome $k_i$ at the site $i$. The maximally entangled state has outcomes that are fully correlated. The CGLMP inequalities have been violated experimentally up to $d = 12$ for photons entangled in orbital angular momentum [84] and up to $d = 16$ for polarization-entangled photon pairs [85].

Further work by Fu derived a CHSH inequality for bipartite systems of arbitrary dimension $d$ [72]. The natural and reasonable guess that a maximally entangled state should maximally violate CGLMP inequalities was shown not to be true by Acín et al. [83] for two qutrit and also two $d$-dimensional systems of up to $d = 8$. The result was generalized by Chen et al. [87], using the inequalities of Fu. Lee and Jaksch later derived optimal Bell inequalities [76], defined as Bell inequalities that are tight and maximally violated by maximally entangled states. The bipartite settings are identical to those of Fu (2004), where each party has two possible measurements and each measurement has $d$ possible outcomes.

IV. QUANTUM CORRELATIONS OF MULTIPARTITE SYSTEMS

The original set-up of Bell was bipartite, meaning two separated systems where local measurements are made on each system by independent observers, or parties. A different question was addressed by Svetlichny, who analyzed whether Bell nonlocality can genuinely exist between three or more separated systems [9]. Svetlichny showed that for certain quantum states, the tripartite correlations cannot be explained as arising from classical mixtures of two-party nonlocal states, i.e. from bipartite states displaying failure of bipartite LHV models [4] or [7]. Such states are called genuinely tripartite Bell nonlocal. In fact, quantum mechanics predicts that Bell nonlocality can exist genuinely
shared over an arbitrarily large number \( N \) of separated sites. Such states are called \( N \)-partite nonlocal, and are said to exhibit genuine \( N \)-partite Bell nonlocality.

### A. Multipartite Bell nonlocality with qubits

It might have been expected that the level of violation of local realism would be less for the three-party than for the two-party case. It was shown that, in some sense, the reverse is true \[10, 88, 89\]. The Greenberger-Horne-Zeilinger (GHZ) state generalized to \( N \) parties is a Schrödinger-cat extreme superposition state of the form \[11\]

\[
|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle^\otimes N - |\downarrow\rangle^\otimes N) .
\]

Here there are \( N \) spatially separated spin-1/2 systems, labelled by \( k = 1, 2, \ldots, N \). We use the notation \(|\uparrow\rangle^\otimes N = \prod_{k=1}^N |\uparrow\rangle_k = |\uparrow \uparrow \ldots \uparrow\rangle \) where \(|\uparrow\rangle_k\) is the eigenstate for the Pauli spin \( \sigma_z^{(k)} \) of the \( k \)-th system. For \( N = 3 \), this is the GHZ state examined in \[7\], based on the work of \[10, 88, 89\]. According to quantum mechanics, the measurement of \( \sigma_z^{(1)} \sigma_y^{(2)} \sigma_y^{(3)} \) (and the permutations with respect to \( k \)) always gives 1, whereas \( \sigma_z^{(1)} \sigma_x^{(2)} \sigma_y^{(3)} \) always gives \(-1\). By contrast, an LHV theory would always predict 1 for \( \sigma_z^{(1)} \sigma_x^{(2)} \sigma_y^{(3)} \), if it had been measured. The difference between the quantum and LHV predictions for the state \( (2') \) always gives a violation for the state \( (18) \), the left side being 4.

It was subsequently shown by Mermin that the GHZ state exhibits a violation of a Bell inequality by an amount that exponentially increases with \( N \). Mermin considered the state \(|\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow \uparrow \ldots \uparrow\rangle + i |\downarrow \downarrow \ldots \downarrow\rangle)\) and measurements of the spin of each particle either along the \( x \) or \( y \) axis. The product of all permutations of these measurements constitutes the operator \( \hat{A} = \prod_{k=1}^N A_k \) where \( s_k \in \{+, -\} \) and

\[
\hat{A}_k^\pm = \left( \sigma_x^{(k)} \pm i \sigma_y^{(k)} \right) e^{-i \theta_k} .
\]

Here, \( \theta_k \) is a phase shift chosen independently at each site that allows for a rotation of the spin axes of the measurement on system \( k \). This phase shift was selected zero in Mermin's paper. One may also consider a complex function \( F_k = X_k \pm i Y_k \) where \( X_k, Y_k \) are real, representing in a local hidden variable theory (LHV) the outcomes of the spin measurements. The average of \( \hat{A} \) is a complex number with real and imaginary parts, given as \( A = \langle \hat{A} \rangle = \text{Re} A + i \text{Im} A \) where the LHV prediction is of the form \( \langle \hat{A} \rangle = (\prod_{j=1}^N F_j^{(j)}) \). In this case, LHV theories are valid, then the expectation value for the imaginary part \( \text{Im} A \) satisfies the Bell-Mermin inequality

\[
\langle \text{Im} A \rangle_{\text{local}} \leq 2^{N/2} , \text{N even}
\]

\[
\langle \text{Im} A \rangle_{\text{local}} \leq 2^{(N-1)/2} , \text{N odd}
\]

while the quantum prediction is \( \langle \text{Im} A \rangle = 2^{N-1} \). This gives the exponential increase of violation with \( N \) noted by Mermin. For \( N = 3 \), \( \theta_k = 0 \) the Bell-Mermin inequality becomes:

\[
\langle \sigma_y^{(1)} \sigma_y^{(2)} \sigma_y^{(3)} \rangle + \langle \sigma_x^{(1)} \sigma_y^{(2)} \sigma_y^{(3)} \rangle + \langle \sigma_x^{(1)} \sigma_y^{(2)} \sigma_y^{(3)} \rangle - \langle \sigma_y^{(1)} \sigma_y^{(2)} \sigma_y^{(3)} \rangle \leq 2 .
\]

The similar Bell-Mermin inequality

\[
\langle \sigma_y^{(1)} \sigma_y^{(2)} \sigma_x^{(3)} \rangle + \langle \sigma_x^{(1)} \sigma_y^{(2)} \sigma_x^{(3)} \rangle + \langle \sigma_x^{(1)} \sigma_y^{(2)} \sigma_x^{(3)} \rangle - \langle \sigma_y^{(1)} \sigma_y^{(2)} \sigma_x^{(3)} \rangle \leq 2
\]

also applies, if one takes instead \( X_k = \sigma_y^{(k)} \) and \( Y_k = \sigma_x^{(k)} \). This gives a violation for the state \( (18) \), the left side being 4.

Ardehali \[90\] considered a different state \(|\phi\rangle = \frac{1}{\sqrt{2}} (|\uparrow \uparrow \ldots \uparrow\rangle - |\downarrow \downarrow \ldots \downarrow\rangle)\) and more general measurements, allowing \( \theta_k \) to be nonzero for the system \( k = N \). In this case, LHV theories imply

\[
\text{Re} \hat{A} + \text{Im} \hat{A} \leq 2^{N/2} , \text{N even}
\]

\[
\text{Re} \hat{A} + \text{Im} \hat{A} \leq 2^{(N+1)/2} , \text{N odd}
\]

whereas the quantum prediction is \( \text{Re} \hat{A} + \text{Im} \hat{A} = 2^{N-1/2} \). The inequalities obtained by Ardehali also work for the state

\[
|\phi\rangle = \frac{1}{\sqrt{2}} (|\uparrow \uparrow \uparrow \ldots \uparrow\rangle \pm |\uparrow \downarrow \ldots \downarrow\rangle)
\]

or

\[
|\phi\rangle = \frac{1}{\sqrt{2}} (|\downarrow \uparrow \uparrow \ldots \uparrow\rangle \pm |\uparrow \ldots \uparrow \ldots \uparrow\rangle)
\]

or

\[
|\phi\rangle = \frac{1}{\sqrt{2}} (|\uparrow \downarrow \ldots \uparrow \rangle \pm |\uparrow \uparrow \ldots \downarrow\rangle)
\]

or

\[
|\phi\rangle = \frac{1}{\sqrt{2}} (|\downarrow \ldots \uparrow \uparrow \rangle \pm |\downarrow \ldots \uparrow \downarrow\rangle)
\]
or any other state with distinct permutations of ↑ and ↓; and also the state \( |\phi\rangle = \frac{1}{\sqrt{2}} (|\downarrow\uparrow\uparrow\dots\uparrow\rangle \pm i |\downarrow\downarrow\dots\downarrow\rangle) \) or any other state with distinct permutations of ↑ and ↓. Results were further generalized by Belinski and Klyshko [11, 94]. In summary, introducing the operator

\[
\hat{V} = \begin{cases} 
\text{Re}\hat{A} + \text{Im}\hat{A}, & N \text{ is even} \\
\sqrt{2} \text{Im}\hat{A}, & N \text{ is odd}
\end{cases}
\]

leads to the formulation of the Mermin-Ardehali-Belinskii-Klyshko (MABK) Bell inequalities which gives the following inequality for any LHV theory

\[
V \equiv |\langle \hat{V} \rangle| \leq 2^{N/2}.
\]

The inequality is violated by quantum mechanics for the GHZ states and the measurement choices \( \theta_k \) given in [57, 60, 91]. The quantum prediction \( V_{QM} \) is:

\[
V = 2^{N-1/2}.
\]

The MABK inequality for \( N = 2 \) reduces to the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality [21], and for \( N = 3 \) is given by [21], and the similar inequality obtained by exchanging \( x \) with \( y \). As Mermin pointed out, as the number of particles \( N \) increases, the violation of the inequalities increases exponentially. Other papers investigating \( N \)-partite correlations include [82, 95–97].

Experimental evidence of the nonlocality of the GHZ correlations was given by Pan et al [98]. They produced GHZ states using the experimental set-up of Bouwmeester et al. [99], based on the proposal by Zeilinger et al. [100]. The proposal allows the generation of 3 entangled photons (GHZ state) from 2 pairs of polarization entangled photons. In order to demonstrate the creation of GHZ state, they followed [7, 10, 88, 89] and measured the values \( \langle \sigma_x^{(1)} \sigma_y^{(2)} \sigma_y^{(3)} \rangle, \langle \sigma_y^{(1)} \sigma_x^{(2)} \sigma_x^{(3)} \rangle, \langle \sigma_x^{(1)} \sigma_x^{(2)} \sigma_y^{(3)} \rangle \) to check the predictions of a GHZ state. The results are consistent with the violation of the MABK inequality [21].

A different type of multipartite entangled state is the W state, written for \( N = 3 \) as [101]

\[
|W\rangle = \frac{1}{\sqrt{3}} (|\uparrow\rangle|\downarrow\rangle|\downarrow\rangle + |\downarrow\rangle|\downarrow\rangle|\uparrow\rangle + |\downarrow\rangle|\uparrow\rangle|\downarrow\rangle).
\]

The \( W \) state with \( N = 4 \) was shown to be genuinely multipartite entangled in experiments performed by Papp et al [102]. The \( W \) states were generated using a photon source and a sequence of beam splitters and can also violate the Mermin inequalities [103, 104]. For higher \( N \), graph states are multi-qubit states including GHZ and cluster states with applications in quantum computing. Bell inequalities giving tests of local realism for graph states were developed by Gühne et al [103] and Toth et al [104], using the method of stabilizing operators. These authors showed that for certain families of graph states, Bell inequalities can be constructed such that the violation of local realism increases exponentially with \( N \). More generally, quantum networks may have several independent sources of entangled states distributed in the network. Rosset et al [107] and Tavakoli [108] derived nonlinear Bell inequalities which can detect the nonlocality of the correlations distributed to distant observers on such networks.

Experimentally, four-photon entangled states where generated by Pan et al. (2001) [109] and Zhao et al. [110]. Using higher pump powers and with improved photo-detection efficiency, six [111], eight [112], ten [113] and 18 [114, 115] qubit entangled photonic states have also been created. A multipartite entangled state of 14 atoms was created in an ion trap [116]. The violation of the MABK inequality was reported for \( N = 4 \) by Zhao et al [110], who generated a GHZ entanglement, the experimental result being \( V = 4.43 \). Other reports of violations of the Mermin inequality for \( N = 3 \) include Chen et al. [117], Hamel et al [118], Patel et al [119] and Li et al [120].

The signatures of multipartite entanglement do not always involve the MABK inequalities, however. Lu et al confirmed six-photon entanglement for graph states using an entanglement witness. Monz et al generated atomic GHZ states [110]. Here, the system was initialized into a state \( |1...1\rangle \), where each ion is in the electronic ground state \( |1\rangle \equiv S_{1/2} (m = -1/2) \). An entanglement interaction [121, 122] then produced GHZ states of the form \( (|000\rangle + |111\rangle) /\sqrt{2} \). The density matrix elements inferred in the measurement process gave evidence of the coherence and fidelity of the GHZ state, and the multi-particle entanglement was inferred by violation of inequalities deduced for separable states [123]. More recently, the experiments of Wei et al [124] and Song et al generated 18-qubit GHZ states using superconducting qubits [112], with Song et al creating multicomponent superpositions of atomic coherent states, involving up to 20 transmon qubits. Omran et al [125] created an atomic Schrödinger cat state in the form of an \( N \)-particle GHZ state.
generated in an array of Rydberg atoms, where \( N \approx 20 \). In the experiment, two-level Rydberg atoms are prepared in an \( N \)-partite GHZ superposition state

\[
|GHZ_N\rangle = \frac{1}{\sqrt{2}} (|0101...\rangle + |1010...\rangle)
\]  

(27)

where \(|0\rangle\) is the atomic ground state and \(|1\rangle\) is the excited Rydberg state. Lasers manipulate an ensemble of \( N \) atoms prepared in the ground state into the GHZ state. The signature of the GHZ state is the GHZ-state fidelity \( F > 0.5 \), this being a witness of the multipartite entanglement of the \( N \)-partite GHZ state.

In fact, the violations of local realism for \( N \)-partite systems is possible for a broad range of states. The initial question of whether all bipartite entangled states can show Bell nonlocality was addressed by Gisin [126]. For any multipartite entangled state, this result was generalized in 2012 by Yu et al. [127]. These authors proved that all entangled pure states violate a CHSH-type Bell inequality involving two dichotomic measurements at each site. The proof is based on Hardy’s paradox, which is an all-or-nothing illustration of the violation of local realism, similar to the GHZ paradox.

**B. Genuine multipartite Bell nonlocality**

The \( N \)-partite GHZ states [12] are seen to illustrate a contradiction with the correspondence principle, that classical statistics follows for large systems. This is because violation of LHV theories is possible for an arbitrarily large \( N \), as shown by the Bell-MABK inequalities [23]. Furthermore, the amount of violation exponentially increases with \( N \). On the other hand, the Bell-MABK inequalities are derived assuming locality between every one of the \( N \) systems i.e. between all pairs of the \( N \) particles. The MABK inequalities if violated therefore imply a Bell nonlocality to exist at least between a pair of particles. It might be argued that it is then not surprising that the violation increases with an increasing number of such pairs.

This criticism can be overcome. The GHZ states and the multipartite generalizations illustrate a genuine \( N \)-partite Bell nonlocality of the type initially considered by Svetlichny [9]. In fact, a stricter set of inequalities can be derived which if violated will imply a genuine \( N \)-partite Bell nonlocality, meaning that the nonlocality cannot be described as arising from a mixing of states which allow genuine \( k \)-partite Bell nonlocality, where \( k < N \). The nonlocality is mutually shared between all \( N \) particles. The approach is to relax the assumptions made in the derivation of the \( N \)-partite Bell inequality. There is not the requirement that there is locality assumed between all of the \( N \) subsystems. Rather, one considers all possible bipartitions \( A_j - B_j \) of the \( N \) spatially separated subsystems, where \( A_j \) is a set of specific subsystems, and \( B_j \) denotes the complementary set. Hidden variables states are considered where locality is assumed between the \( A_j \) and \( B_j \), but not between the subsystems of \( A_j \) and \( B_j \). If one supposes the system to be modeled by a convex mixture of all such bilocal descriptions, then the correlations are constrained by Bell inequalities [128, 129]. The Bell inequality described by Collins, Gisin, Popescu, Roberts and Scarani (CGPRS) is [128]

\[
V_S \equiv ReA + ImA \leq 2^{N-1} .
\]  

(28)

Svetlichny’s inequality is a version for \( N = 3 \) [9]. The quantum prediction [24] maximizes to predict a violation, for even \( N \), by a constant amount: \( V_\text{QM} = \sqrt{2} \) [82]. It should be noted that since Svetlichny’s work, further improved definitions of genuine multipartite nonlocality have been constructed by Gallego et al and Bancal et al [130, 132]. These take into account no-signalling and the time ordering of measurements between the observers, and enable genuine \( N \)-partite nonlocality to be detected for a broader set of entangled states.

Experiments for \( N = 3 \) verifying the violation of Svetlichny’s inequality were performed by Lavioe et al [132] and Lu et al [134] using three-photon GHZ states with correlated polarization. Lavioe et al followed the approach of Bouwmeester et al. [99] to generate the GHZ state. The experiment setup by Lu et al. was based on the proposal by Rarity and Tapster [135], involving entangled photon pairs and a weak coherent state. They shone an infrared pulse with wavelength 780nm into a crystal that up converts into an ultraviolet pulse with wavelength 390nm. This pulse was split into two beams, where one beam was subsequently sent into another nonlinear crystal to produce a polarization entangled state \( (|H_3H_4\rangle + |V_2V_5\rangle)/\sqrt{2} \), while the other beam was prepared in a state \( (|H_1\rangle + |V_3\rangle)/\sqrt{2} \). Beam 1 and 2 are sent into a polarizing beam splitter such that the total state is a GHZ state \( (|HHH\rangle + |VVV\rangle)/\sqrt{2} \). The ion trap experiments of Barreiro et al [136] reported multipartite device-independent entanglement for up to \( N = 6 \) ions. When there are two measurement settings with two possible outcomes for each setting, the device-independent entanglement witnesses used by Barreiro et al. are equivalent to the \( n \)-partite Svetlichny inequalities. In this experiment, up to 6 ions were conclusively shown to have genuine multipartite quantum nonlocal correlations. Higher numbers of ions have the problem of cross-talks in the measurement process (local measurements on individual ions affect neighboring ions). Subsequent reports of violations of Svetlichny-CGPRS inequalities for three qubit GHZ-photon states include those
of Erven et al. (2014)\textsuperscript{137}, Hamel et al.\textsuperscript{118} and Patel et al.\textsuperscript{119}. Recently, violations of Svetlichny inequalities were verified using the IBM quantum computer for W and GHZ states\textsuperscript{104}. 

\section*{C. Multipartite Bell nonlocality with qudits}

After the original GHZ papers, further studies demonstrated that the “all or nothing” violation of local realism can apply even where there is a higher dimensionality at each of the sites. The paper by Reid and Munro\textsuperscript{138} generalized the three-party GHZ contradiction with local realism, for \(N\) particles at each of three sites, \(j = 1, 2, 3\). They considered the multipartite extension of the state \textsuperscript{(13)}, given by

\[ |\psi\rangle = \frac{\left(\hat{a}_{1+}^{\dagger} \hat{a}_{2+}^{\dagger} \hat{a}_{3+}^{\dagger} + \hat{a}_{1-}^{\dagger} \hat{a}_{2-}^{\dagger} \hat{a}_{3-}^{\dagger}\right)^N |0\rangle}{\sqrt{N! \left[ \sum_{r=0}^{N} r!(N-r)! \right]^{1/2}}} \]  

(29)

Here, the state \( |\psi\rangle = \frac{1}{\sqrt{2}} \left( \hat{a}_{1+}^{\dagger} \hat{a}_{2+}^{\dagger} \hat{a}_{3+}^{\dagger} + \hat{a}_{1-}^{\dagger} \hat{a}_{2-}^{\dagger} \hat{a}_{3-}^{\dagger} \right) |0\rangle \) with \(N = 1\) is a GHZ state similar to the spin version \textsuperscript{(13)}, since we may map the two-state system \(|1\rangle_0 |0\rangle_1 |0\rangle_2 \) onto spin qubits \(|\uparrow\rangle, |\downarrow\rangle\) for each mode given by \(a_j\), \(j = 1, 2, 3\). The \(\hat{a}_{1+}, \hat{a}_{1-}\) are boson operators for six orthogonal field modes, and the \(\pm\) denotes the orthogonal modes/polarizations at the same energy. The detected outputs for the analyzers (polarizers or beam splitters) at each site correspond to the following transformed modes: \(d_{j\pm} (\phi_j) = \frac{1}{\sqrt{2}} (\pm \hat{a}_{j\pm} + e^{i\phi} \hat{a}_{j\mp})\), similar to \textsuperscript{(11)} The choices for the settings are \(\phi = 0\) and \(\phi = \pi/2\), which gives measurements \(\hat{d}_x^{(j)}\) or \(\hat{d}_y^{(j)}\) for site \(j\). Incident on each of the three analyzers \((j = 1, 2, 3)\) are \(N\) bosons, some of which are detected in polarization mode + and the rest as −. Assuming the number of bosons with the final polarization + (or −) is measurable, one may treat the bosons as though distinguishable particles and determine the product \(S^N_{x\pm} (S^N_{y\pm})\) of the individual spin outcomes +1 or −1 associated with the \(N\) incident bosons at site \(j\). The spin products under consideration are \(S^N_{1y} S^N_{2y} S^N_{3y}, S^N_{1y} S^N_{2y} S^N_{3y}, S^N_{1x} S^N_{2x} S^N_{3x}\) and \(S^N_{1x} S^N_{2x} S^N_{3x}\). The expectations values of these products is calculated by rewriting the state \(|\psi\rangle\) given in Eq. \textsuperscript{(29)} in the transformed modes \(\hat{d}_{x\pm}\) and \(\hat{d}_{y\pm}\). The expectation values for \(N\) odd for the products \(S^N_{1x} S^N_{2y} S^N_{3y}, S^N_{1y} S^N_{2x} S^N_{3x}\) and \(S^N_{1y} S^N_{2y} S^N_{3y}\) is always −1, while for \(S^N_{1x} S^N_{2x} S^N_{3x}\) the state is transformed in the \(\hat{d}_{x\pm}\) modes, obtaining that the expectation value for this product is +1 for all \(N\). However, on calculating the classical predictions for this expectation value, the result is always −1, in disagreement with the previous result of +1. This is the “all or nothing” distinction between quantum and classical predictions, applied to arbitrary large values of odd \(N\), where one has a macroscopic state. The authors further showed how one may extend the approach to consider violation of the Mermin inequality for \(N\) particles at each site. GHZ correlations for multi-dimensional system without inequalities were also considered by Cabello\textsuperscript{139}.

Higher-dimensional multipartite Bell inequalities have since been studied extensively\textsuperscript{72,140,144}. Cabello extended the MABK inequality to \(n\) spin \(s\) particles, and showed that higher dimensional GHZ states maximally violate these inequalities\textsuperscript{140}. Cabello demonstrated that the violation for an arbitrary but fixed \(s\) increases exponentially with \(N\), thus extending the observation of Mermin\textsuperscript{11} to higher spin. Cabello also demonstrated that for arbitrary but fixed \(N\), the violation does not decrease with \(s\), thus generalizing the result of Gisin and Peres\textsuperscript{71} for \(N = 2\). Son, Lee and Kim derived generalized MABK inequalities for multipartite systems of arbitrary dimension i.e. for arbitrary \(d\) and \(N\)\textsuperscript{144}. They showed that the higher dimensional extensions of the GHZ states given as

\[ \frac{1}{\sqrt{d}} \{ \sum_{k=1}^{d-1} |k, k, k, \ldots \rangle \} \]

(30)

may violate these inequalities. Here \(|k, k, k, \ldots \rangle \equiv |k\rangle_1 |k\rangle_2 |k\rangle_3 |k\rangle_N\), the \(|k\rangle_i\) being an orthogonal basis set for states at the site \(j\). However, it was known that the MABK inequalities were not tight, in the sense that there exist entangled states that would not violate the MABK inequalities. For \(N = 3\), Chen et al presented two-setting inequalities that were shown numerically to be violated for all entangled states\textsuperscript{145}. In 2009, Chen and Deng extended this result, to derive a Bell inequality based on the CHSH correlation functions for \(N\) \(d\)-dimensional systems, which was shown tight for a range of systems, including up to \(d = 10\) for \(N = 3\).

Multipartite and higher dimensional Bell tests provide a way to overcome losses and noise, although this can also be achieved with multiple settings. It could be argued that the violations associated with the \(N\)-partite GHZ qubit states which have dichotomic outcomes \((d = 2)\) do not satisfy the requirement of a macroscopic Bell violation. For the Mermin inequality of type \textsuperscript{(21)}, for example, the products of the Pauli spins at each site are either +1 or −1.
Therefore, each spin system must be measured exactly, or else this product changes sign. A microscopic resolution of measurement outcomes is required. The Bell violations are thus lost when there is enough decoherence of the system so that one of the spins changes sign. This is made apparent by the fact that the violations are readily destroyed by reduced detection efficiencies. On the other hand, multi-setting Bell tests allow a choice of more than two measurement settings at each site and are known to enhance the tests of local realism, allowing violations to be obtained for greater levels of decoherence, as realized by losses associated with lower detection efficiencies. One prediction is for a failure of local realism with a detection efficiency as low as 43% at one detector, for a non-maximally entangled state, using three measurement settings \[148, 149\]. For highly entangled states however, violations of local realism are possible at 50% efficiency using combinations of multi-settings and/or multiple sites \[149, 153\]. The work of Durt, Kaszlikowski and Zukowski \[70\] showed the possibility that higher dimensional Bell tests are more robust with noise (refer Section VII).

V. CONTINUOUS-VARIABLE QUANTUM CORRELATIONS

A. EPR correlations

The original EPR argument was based on the correlations of position and momenta, the outcomes of which are continuous variables \[1\]. The EPR state is a two-party state, where each subsystem \(i = 1, 2\) has a position \(q_i\) and momentum \(p_i\). The EPR state is simultaneously an eigenvector of position difference \(q_1 - q_2\) and momenta sum \(p_1 + p_2\). This means that \((\Delta(q_1 - q_2))^2 \to \infty\) and \((\Delta(p_1 + p_2))^2 \to \infty\), where here we use the notation \((\Delta x)^2\) as the variance of \(x\).

The correlations can be realized for fields using the conjugate quadrature phase amplitude observables \(X_i\) and \(P_i\) of two modes \(i = 1, 2\), where (for a rotating frame) \(X_1 = \hat{a}_1 + \hat{a}_1^\dagger\) and \(P_1 = (\hat{a}_1 - \hat{a}_1^\dagger)/i\) \[154, 155\]. This was shown by Reid \[155\] for the output of the non-degenerate parametric amplifier, which is equivalent to the two-mode squeezed vacuum state. Here, \(\hat{a}_1\) and \(\hat{a}_1^\dagger\) are boson creation and destruction operators for the single mode field labelled by \(i\). The non-degenerate parametric amplifier is described by the interaction Hamiltonian

\[ H_I = i\kappa E(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2) \]  

(31)

where \(\kappa\) is the coupling strength between the modes and \(E\) the pump intensity. The unitary interaction generates a two-mode squeezed vacuum state of type \(|\text{TMSS}\rangle = \text{sech} r \sum_n \tanh^n r |n\rangle_1 |n\rangle_2\), where \(|n\rangle_1\) is the number state for mode \(i\). Following \[155\], one may solve \[31\] directly to confirm EPR correlations. We find \(\hat{a}_1 = \kappa E\hat{a}_2^\dagger\) and \(\hat{a}_2 = \kappa E\hat{a}_1\), implying \(\hat{X}_1 = \kappa EX_2, \hat{X}_2 = \kappa EX_1\) from which it is clear that the solution for \(\hat{X}_\pm = \hat{X}_1 \pm \hat{X}_2\) is \(\hat{X}_\pm(t) = X_\pm(0) e^{\pm \kappa Et}\). Similarly, defining \(\hat{P}_\pm = \hat{P}_1 \pm \hat{P}_2\), the solutions are \(\hat{P}_\pm(t) = \hat{P}_\pm(0) e^{\mp \kappa Et}\). Using that \((\Delta \hat{X}_\pm(0))^2 = 2\), this gives

\[ (\Delta(X_1 - X_2))^2 = 2e^{-2r}, \quad (\Delta(P_1 + P_2))^2 = 2e^{-2r} \]  

(32)

where \(r = \kappa Et\) is the two-mode squeezing parameter, leading to EPR correlations for \(r > 0\), as \(r \to \infty\).

The EPR correlations can also be realized using a beam splitter, with either one or two vacuum squeezed inputs \[155, 157, 158\]. In this case, the two incoming modes \(\hat{a}_1^{(in)}, \hat{a}_2^{(in)}\) are transformed into the two output modes \(\hat{a}_1, \hat{a}_2\) according to \(\hat{a}_1 = \cos \theta \hat{a}_1^{(in)} + \sin \theta \hat{a}_2^{(in)}\), \(\hat{a}_2 = \sin \theta \hat{a}_1^{(in)} - \cos \theta \hat{a}_2^{(in)}\). The \(\hat{a}_1^{(in)}\) and \(\hat{a}_2^{(in)}\) are the destruction boson operators for the modes. Let us assume the input \(\hat{a}_2^{(in)}\) to be squeezed along the \(X\) quadrature so that \(\Delta \hat{X}_2^{(in)} = e^{-r_2}\), and the input \(\hat{a}_1^{(in)}\) to be squeezed along \(P\) so that \(\Delta \hat{P}_1^{(in)} = e^{-r_1}\). Here, \(r_i > 0\) are the squeezing parameters. The solutions for the outputs of a 50/50 beam splitter are

\[ (\Delta(X_1 - X_2))^2 = 2e^{-2r_2}, \quad (\Delta(P_1 + P_2))^2 = 2e^{-2r_1}, \]  

(33)

which leads to ideal EPR correlations for large \(r_1\) and \(r_2\). More complete details are given elsewhere \[159\]. If \(r_2 = 0\) so that \(\hat{a}_2^{(in)}\) is not squeezed \((\Delta \hat{X}_2^{(in)} = \Delta \hat{P}_2^{(in)} = 1)\), but if \(\hat{a}_1^{(in)}\) remains squeezed with \(r_1 > 0\), then the correlations are no longer ideally EPR correlated. Nonetheless, as explained below, we will see that the system may be regarded as EPR correlated, since the correlations result in an EPR paradox. Later studies revealed that number states incident on beam splitters can also generate entanglement between the output modes \[160\].

More generally, ideal EPR correlations manifest for two conjugate (non-commuting) observables of a single system \(A\) \((i = 1)\), when both of those observables can be estimated with perfect accuracy by a measurement on a remote space-like separated system \(B\). The measurement at \(B\) will be different in each case. Let us denote the observables as
and $O_x$ and $O_y$, and take the special case where the commutator is of the form $[O_x, O_y] = C$, where $C$ is a constant. For position and momentum, $C = i\hbar$ and for quadratures, $C = [\hat{X}_i, \hat{P}_i] = 2i$. This implies an uncertainty $\Delta \hat{X}_i \Delta \hat{P}_i \geq 1$. Following [155], EPR correlations allow an inferred estimate for outcomes $X_i$ and $P_i$ of $\hat{X}_i$ and $\hat{P}_i$ that approaches zero in each case. Taking the estimate $X_{1,est}$ of $X_1$ at $A$ to be the optimal linear combination of quadrature amplitudes measurable at $B$, this estimate is quantified by the inference variance defined as $\Delta^2_{inf} \hat{X}_1 = (\Delta (\hat{X}_1 - X_{1,est}))^2$, where we use the notation $(\Delta_{inf} \hat{X}_1)^2 = \Delta^2_{inf} \hat{X}_1$ to simplify use of brackets. The EPR correlations are observed for $O_x$ and $O_y$ when the product reduces below that of the uncertainty bound, $\Delta_{inf} O_x \Delta_{inf} O_y < \frac{(C)}{2}$, which for $X$ and $P$ reduces to the EPR condition [155]

$$E_{1|2} \equiv \epsilon = \Delta_{inf} \hat{X}_1 \Delta_{inf} \hat{P}_1 < 1.$$  

Specifically, for parametric down conversion, the best linear estimate of $X_1$ is $X_{1,est} = gX_2$. Similarly, the best linear estimate of $P_1$ is $P_{1,est} = g'P_2$ where $g$ and $g'$ are real numbers. Therefore

$$E_{1|2} \equiv \Delta (\hat{X}_1 - g'X_2)\Delta (\hat{P}_1 + gP_2) < 1$$

is a condition sufficient to demonstrate correlations of the EPR paradox. The result for the two-mode squeezed state generated by down conversion is known to be [155, 156]

$$\Delta^2_{inf} \hat{X}_1 = \Delta^2_{inf} \hat{P}_1 = \frac{1}{\cosh 2r}$$

using $g = g' = \tanh 2r$ where $r = \kappa Et$. More details are given in [155]. For the beam-splitter configuration described above with two squeezed inputs, the result is given by the expression (39), but where $r$ denotes the squeeze parameter of the input fields. With only one squeezed input, we find $\Delta_{inf} \hat{X}_1 \Delta_{inf} \hat{P}_1 = \frac{1}{\cosh 2r}$. 

A further generalization of the EPR paradox is to consider the local hidden variable models, as in [6] and [7]. Here, one considers two separated systems labelled $i = 1, 2$ which in [6] and [7] were labelled $A$ and $B$, respectively. The structure [7] can be taken as a generalized definition of local realism, or of local causality. One may then also consider whether the expressions $P^{(A)}_+ (\phi, \lambda)$ for $A$ are consistent with a local quantum state description, which would be given by a density operator $\rho_{A,\lambda}$. We write

$$P_{++}(\theta, \phi) = \int \rho(\lambda) \, P^{(A)}_+(\theta, \lambda) \, P^{(B)}_+ (\phi, \lambda) \, d\lambda$$

where the subscript $Q$ denotes this extra condition on the local hidden state (LHS) at $A$. One is asking whether, within the framework of local hidden variables (based on a generalized premise of local realism, or local causality), the elements of reality (interpretable as the $\lambda$) are consistent with a local quantum state description at $A$. If no, one may interpret the result as a generalized EPR paradox, since the assumption of local realism / causality is shown to be incompatible with local hidden variable states for $A$ that are consistent with a local quantum state. The failure of the condition given as [37] is referred to as “steering”, or “EPR steering”. Steering is the term used by Schrödinger in his response to EPR’s original 1935 paper [161, 162]. The connection between the expression [7], “steering” and quantum tasks was given by Wiseman, Jones and Doherty [163, 164]. The failure of [37] gives the condition for a steering of system $A$.

The EPR condition [34] involving $E_{A|B}$ is sufficient for demonstration of EPR steering of system $A$ [165], and gives a one-sided device-independent condition for entanglement [166, 167]. The condition has been shown necessary and sufficient for two-mode steering where the systems $A$ and $B$ are Gaussian single-mode systems [168]. This implies restricting to Gaussian states and Gaussian measurements [169, 170]. Links between EPR-variance criteria for entanglement and steering and the criteria of Simon and Duan et al [171, 172] derived in the context of Gaussian states have been formalised by Marian and Marian [173, 174].

In order for the EPR correlations to be observed, the inferred uncertainties are compared relative to the value which is given as the commutator $C$. The correlation can then hardly be called “macroscopic”. On the other hand, the method of detection in optical physics is to amplify the quantum noise level, using local oscillator fields which in quantum mechanics are modeled (approximately) as coherent states $|\alpha\rangle$ with large amplitude $\alpha$. This implies that the detection involves large numbers of photons incident on detectors, in contrast with the detection of the Bell correlations described in Section III. A careful analysis shows that in some experiments, the amplification occurs prior to the choice of a phase angle which determines the measurement setting i.e. whether $X$ or $P$ is to be measured. This is the case for polarization entanglement experiments [173], where the local oscillator is combined with the signal field ahead of impinging on the polarizer beam splitter, the setting of which determines whether $X$ or $P$ is measured.
at the final detector. The continuous variable experiment can then be mapped onto an equivalent macroscopic spin EPR experiment, where the observables that are measured are (once the local oscillator mode is accounted for) the Schwinger spins \( \mathbf{S}_X \) and \( \mathbf{S}_Y \):

\[
\hat{S}_X^{(i)} = \frac{1}{2} \left( \hat{c}_i^\dagger \hat{a}_i + \hat{c}_i \hat{a}_i^\dagger \right), \quad \hat{S}_Y^{(i)} = \frac{1}{2i} \left( \hat{c}_i^\dagger \hat{a}_i - \hat{c}_i \hat{a}_i^\dagger \right), \quad \hat{S}_Z^{(i)} = \frac{1}{2} \left( \hat{c}_i^\dagger \hat{c}_i - \hat{a}_i^\dagger \hat{a}_i \right). \tag{38}
\]

Here, \( \hat{a}_i, \hat{a}_i^\dagger \) are the boson operators for the field mode labelled \( i \) that is being measured. The operators \( \hat{c}_i, \hat{c}_i^\dagger \) are the boson operators for the local oscillator fields associated with each mode \( i \), which are model-led as an intense coherent state of amplitude \( E = \alpha \) (taken to be real). In this limit, \( \hat{S}_X \to \frac{\alpha}{2} X \) and \( \hat{S}_Y \to \frac{\alpha}{2} P \) where \( X \) and \( P \) are the quadrature amplitudes of the field \( i \). Then we see that because \( E = \alpha \) is large, \( \langle \hat{c}_i^\dagger \hat{c}_i \rangle \gg \langle \hat{a}_i^\dagger \hat{a}_i \rangle \) and the values of the spins can also be large i.e. macroscopic. The combined system at each site comprises a single mode \( \hat{a}_i \) and a second very intense field \( \hat{c}_i \) and hence this system prior to the choice of measurement setting \( X \) or \( P \) is macroscopic.

In this way, one can argue that the continuous variable (CV) correlations are “macroscopic”. The analysis of the CV experiment gives an example of amplification due to measurement, and the analogy with the Schrödinger cat gedanken experiment has been given in [179].

Schrödinger noted in his response to the EPR argument of 1935 that the paradoxical correlations occur when the states are “entangled” [12]. This led to the concept of entanglement defined within quantum theory; two systems \( A \) and \( B \) are entangled if the density operator \( \rho \) for the combined systems cannot be expressed in the separable form \( \rho = \sum_i P_i \rho_{i(A)} \rho_{i(B)} \) where here the system \( A \) \((B) \) is identified with system \( i = 1 \) \((2) \). The verification of entanglement between two single modes generally requires a less strict bound than for EPR steering. As an example, a criterion for entanglement between systems 1 and 2 is given by

\[
\Delta_{\text{prod}} = \Delta(\hat{X}_1 - g \hat{X}_2) \Delta(\hat{P}_1 + g \hat{P}_2) < 1 + gg' \tag{39}
\]

as derived by Giovannetti et al [180] in their Eq (5), with \( a_1 = b_1 = 1 \). \( a_2 = g \), \( b_2 = g' \). From this, we see that the output modes in the beam splitter configurations above are entangled for all \( r \). The product entanglement criterion \( \Delta_{\text{prod}} = \Delta(\hat{X}_1 - \hat{X}_2) \Delta(\hat{P}_1 + \hat{P}_2) < 2 \) derived earlier by Tan [181] is a special case of \( (39) \) for \( g = g' = 1 \). On noting that for any real numbers \( x \) and \( y \), \( x^2 + y^2 \geq 2 \) \( xy \), we see that the sum criterion

\[
\Delta_{\text{sum}} = \left[ \Delta^2(\hat{X}_1 - g' \hat{X}_2) + \Delta^2(\hat{P}_1 + g \hat{P}_2) \right] < 2(1 + gg') \tag{40}
\]

derived by Duan et al and Simon [171, 172] for \( g = g' \), is also a special case of \( (39) \). Schrödinger’s historical responses to the EPR argument motivated the classifications of entanglement, steering, Bell nonlocality. More details are given in the reviews [6, 182, 183].

There have been many examples of realizations of the continuous variable (CV) EPR quantum correlations, beginning with the demonstration of the CV EPR paradox by Ou et al [184]. Since then, there have been reports of both entanglement and EPR steering in optical CV systems, including those referred to in the review [182] (see also [185, 188]). Experiments have also demonstrated EPR correlations between pairs of photons, using the EPR criterion \( (33) \) [189, 190]. We see from the expressions \( (32), (34) \) and \( (39) \) that the EPR entanglement and steering conditions are also conditions for squeezing, since a reduction in the noise below that governed by the uncertainty principle is required. The most significant results for two-mode squeezing, EPR entanglement and EPR steering have been reported in a set of optical experiments, which achieve as low as \( \epsilon \sim 0.1 \) [158, 191, 192].

It is also possible to construct EPR conditions for spin operators, defined as \( (38) \), which have discrete outcomes. For spin 1/2, this corresponds to Bohm’s version of the EPR paradox [22]. Conditions to realize Bohm’s EPR paradox for spin systems have been put forward in [182, 196]. These apply the uncertainty relations involving spin commutation relations. In fact, the macroscopic realizations of EPR correlations obtained using the Stokes spin observables (defined similarly to \( (33) \)) may be interpreted as a higher spin version of Bohm’s paradox [175]. The relation \( \Delta \hat{S}_X \Delta \hat{S}_Y \geq \frac{|\langle \hat{S}_Z \rangle|^2}{2} \) for the Schwinger relations \( (38) \) is used, in which case \( c_i \) are intense local oscillator fields and the value of the \( |\langle \hat{S}_Z \rangle| \) becomes large, giving a macroscopic level of quantum noise. In the experiment of Bowen et al [175], the correlated source is generated by non-degenerate down conversion modeled as \( (31) \). The modes are combined with strong fields \( c_i \) using beam splitters and then passed through polarizers at each site. This achieves the scenario where an EPR correlation is expressed in terms of spin operators, with a macroscopic level of quantum noise.

### B. Bell tests

An important question is whether one may demonstrate Bell nonlocality for continuous variable measurements. This would rule out all local hidden variable theories, which is a stronger result than confirming either entanglement
or EPR steering. As we have seen, the CV measurements using optical homodyne methods can be expressed in terms of the Schwinger operators, for certain arrangements at least, which would allow more macroscopic tests of LHV theories.

Bell violations for continuous variable measurements were proposed by Bell. Bell considered a quantum mechanical state with a Wigner function

\[
W(q_1, q_2, p_1, p_2) = Ke^{-q^2/2}e^{-p^2/2} \delta(p_1 + p_2) \left[ (q_1^2 + q_2^2)^2 - 5q_1^2 - q_2^2 + \frac{11}{4} \right]
\]

(41)

where \( K \) is a constant, \( q = q_1 - q_2 \), and \( p = (p_1 - p_2)/2 \). This Wigner function is negative in certain regions, for example at \( p = 0, q = 1 \), and was used by Bell to show a violation of a Bell inequality. This is carried out by calculating the sign correlation function between particle 1 at time \( t_1 \) after free evolution, and particle 2 at time \( t_2 \).

It was understood by Bell that a positive Wigner function would provide a local hidden variable theory for the measurements of \( q_1 \) and \( p_1 \). Leonhardt and Vaccaro [197] obtained the wave-function

\[
\tilde{\psi}(p_1, p_2) \propto [(p_1 - p_2)^2 - 4] e^{-\frac{1}{2}[(p_1 - p_2)^2]} \delta(p_1 + p_2)
\]

(42)

in the momentum representation from the Wigner function Eq. (41). They noted that the two features in Eq. (42) giving rise to a nonlocal correlation are the Dirac delta function \( \delta(p_1 + p_2) \) and the admission of negative values in the Wigner function. Leonhardt and Vaccaro [197] proposed mixing a squeezed vacuum state with a superposition of Fock state to produce a state with these two features. A squeezed vacuum has a momentum wave-function of the form \( W_{sq}(p) \propto \delta(p) \) in the strong squeezing limit. In order to introduce negativity into the Wigner function, a superposition of \( |0 \rangle \) and \( |2 \rangle \) is mixed with the squeezed vacuum using a 50 : 50 beam splitter. The momentum wave-function of the output state from the beam splitter resembles that of the wave-function Eq. (42) and violates a Bell inequality. The optical equivalence of free particle evolution in the Bell work is the homodyne detection of the rotated quadratures \( q_0 \equiv q \cos \theta - p \sin \theta \) with the angle \( \theta = \arctan t \), where \( t \) is the time as in Bell’s work. The approach was to create dichotomic outcomes from the continuous variable outcomes by binning according to the sign of \( q_0 \), so that a traditional Bell inequality based on dichotomic observables could be implemented.

Gilchrist, Deuar and Reid showed how local hidden variable theories can be excluded for correlations based on pair-coherent states and superpositions of squeezed two-mode cat states [198, 199]. Cat states are the superpositions \( |\psi\rangle \sim |\alpha\rangle + e^{i\theta}|\alpha\rangle - \alpha \) of two macroscopically distinct coherent states, \( |\alpha\rangle \) and \( |\alpha\rangle \) [200]. These states have been generated experimentally in optical [201] and microwave systems [202, 203]. Collapse and revivals reported for matter waves suggest similar states to be generated in a Bose-Einstein condensate (BEC) [204]. These authors studied entangled cat states based on two spatially separated modes \( A \) and \( B \). In particular, pair-coherent states are the continuous superpositions of coherent states with a fixed amplitude but arbitrary phase \( \zeta \) [205]

\[
|\Psi\rangle_m = \mathcal{N} \int_0^{2\pi} e^{-im\zeta} |\alpha_0 e^{i\zeta} \rangle_A |\alpha_0 e^{-i\zeta} \rangle_B d\zeta.
\]

(43)

Here \( \mathcal{N} \) is a normalization constant, \( m \) is the particle/photon number difference between modes \( A \) and \( B \), and \( \alpha_0 \) is the amplitude of the coherent state. Gilchrist et al. [199] also considered a squeezed entangled cat state. The entangled cat state is defined as

\[
|\text{cat}\rangle = N \left( |\alpha_0\rangle_A |\beta_0\rangle_B + e^{i\theta} |\alpha_0\rangle_A |\alpha_0\rangle_B - |\beta_0\rangle_B \right)
\]

(44)

where \( N \) is the normalization constant and \( \alpha_0, \beta_0 \) are the amplitudes of the coherent states for each mode. The squeezed two-mode cat state is generated under the evolution of the squeezing interaction Hamiltonian \( H_1 \) such that the density operator of the system is \( \rho_{sc} = e^{\left(-\frac{t \hat{H}_1}{\hbar}\right)} |\text{cat}\rangle |\text{cat}\rangle |\text{cat}\rangle |\text{cat}\rangle e^{\left(\frac{t \hat{H}_1}{\hbar}\right)} \). Rotated quadrature phase amplitudes are defined, as \( \hat{X}_\theta = \hat{X} \cos \theta + \hat{P} \sin \theta \), and binning was used, to classify the outcomes as either +1 or −1 depending on the sign of \( \hat{X}_\theta \). Similar to the approach of [197], violations are then evident as violations of the Clauser-Horne-Shimony-Holt (CHSH) or Clauser-Horne (CH) Bell inequalities. It is interesting that the bin can distinguish the dead or alive aspect of the cat-like state, determined by the sign of the amplitude. In this case, however, in the limit where \( \alpha \) is large, the violations vanish or become increasingly small.

Recent work by Kumar, Saxena and Arvind shows how the CH inequalities based on polarization can be used to confirm the nonlocality of both the entangled cat state \( |\text{cat}\rangle \) and the pair coherent state [200]. These authors considered modes of definite polarization, where intensity correlations are detected after passing through polarizers, after some mode transformations. The outcomes are binned according to whether photons are detected at the output location, or not. Violations of the CH inequality are found possible, for coherent amplitudes of order \( \sqrt{2} \), where the two coherent states \( |\alpha_0\rangle \) and \( |\alpha_0\rangle \) for each mode are distinctly separated.
Banaszek and Wodkiewicz considered the two-mode entangled cat states \( |\text{cat} \rangle \) given by (1) and showed how to achieve Bell violations for these states, in a way that does not decay for larger \( \alpha \) and \( \beta \), using phase space distributions (207). The work was extended by Jeong et al. (208) and Milman et al. (209). Banaszek and Wodkiewicz demonstrated how the phase-space \( Q \) and Wigner functions can be used to infer nonlocal correlations. These quasi-probability distributions are functions of continuous variables, defined by a complex amplitude for each mode. The approach considered a two-mode cat-state that is displaced before measurements. If detectors detect no photons the event is assigned 0, and 1 otherwise. The joint probability of no-count events in both detectors is \( Q_{AB}(\alpha, \beta) \) where, \( Q_{AB}(\alpha, \beta) = \langle Q_A(\alpha) Q_B(\beta) \rangle \) and \( Q_A(\alpha) = \hat{D}(\alpha)|0\rangle\langle0|\hat{D}^\dagger(\alpha) = |\alpha\rangle \langle \alpha | \). Here, \( \hat{D}(\alpha) \) and \( \hat{D}(\beta) \) are the standard displacement operators with amplitude \( \alpha \) and \( \beta \) for modes labelled \( A \) and \( B \) respectively. The Clauser-Horne (CH) Bell inequality can be written as \(-1 \leq CH \leq 0\), where

\[
CH \equiv Q_{AB}(0, 0) + Q_{AB}(\alpha, 0) + Q_{AB}(0, \beta) - Q_{AB}(\alpha, \beta) - Q_A(0) - Q_B(0) .
\]  

(45)

The realization that \( Q \) is the Husimi \( Q \)-function, defined as \( Q(\alpha) = \langle \alpha | \rho | \alpha \rangle / \pi \) (210), thus implies that \( Q \)-functions can display nonlocal quantum correlations. The authors then considered another measurement where the number of photons detected can be resolved. A parity value \( P \) of \(+1\) \((-1)\) is assigned to even (odd) number of detected photons. The operators corresponding to these measurements are \( \hat{\Pi}_A(\alpha) = \hat{D}(\alpha) \sum_{k=0}^{\infty} |2k\rangle\langle2k|\hat{D}^\dagger(\alpha) \) and \( \hat{\Pi}_B(\beta) \). The correlation function is shown to be

\[
\hat{\Pi}_{AB}(\alpha, \beta) = \langle \hat{\Pi}^+_A(\alpha) - \hat{\Pi}^-_A(\alpha) \rangle \otimes \langle \hat{\Pi}^+_B(\beta) - \hat{\Pi}^-_B(\beta) \rangle
\]

\[
= \hat{D}_A(\alpha) \hat{D}_B(\beta) (-1)^{\hat{n}^+_A + \hat{n}^+_B} \hat{D}^\dagger_A(\alpha) \hat{D}^\dagger_B(\beta)
\]  

(46)

where \( \langle \hat{\Pi}_{AB}(\alpha, \beta) \rangle = \pi^2 W(\alpha, \beta)/4 \) corresponds to a scaled two-mode Wigner function \( W(\alpha, \beta) \) (211). The \( \langle \hat{\Pi}_{AB}(\alpha, \beta) \rangle \) corresponds to the average parity product \( P_A P_B \) after the displacements. It follows that the outcomes \( \Pi_{AB}(\alpha, \beta) \) are bounded by \( \pm1 \), implying the Bell inequality \( |B| \leq 2 \) where

\[
B = \langle \hat{\Pi}_{AB}(\alpha', \beta') \rangle + \langle \hat{\Pi}_{AB}(\alpha', \beta') \rangle + \langle \hat{\Pi}_{AB}(\alpha', \beta') \rangle - \langle \hat{\Pi}_{AB}(\alpha, \beta) \rangle .
\]  

(47)

Thus, Wigner functions can also be used to demonstrate nonlocal correlations. Violations are predicted for the entangled cat state, which for optimally selected values of the state and of the displacements \( \alpha \) and \( \beta \) allow the maximum violation \( B = 2\sqrt{2} \) for \( \alpha_0, \beta_0 \rightarrow \infty \) (209).

Milman et al. (209) proposed the generation of the two-mode cat states (14) in a cavity QED setting. The setting consists of a Rydberg atom and two superconducting cavities. Firstly, the atom is prepared in a superposition state \( \langle g | + | e \rangle / \sqrt{2} \), which is then passed through two cavities that are both in a coherent state \( |\lambda \rangle \). The atom interacts with the cavity field in such a way that a phase shift \( \phi \) is induced, depending on the state of the atom: \( |e\rangle |\lambda \rangle \rightarrow e^{i\phi}|e\rangle |\lambda e^{i\phi} \rangle \) and \( |g\rangle |\lambda \rangle \rightarrow e^{-i\phi}|g\rangle |\lambda e^{-i\phi} \rangle \). By choosing the interaction time and detuning, the phase \( \phi \) can be fixed and they focused on the case \( \phi = \pi/2 \), so that the atom-cavity state after the interaction is given by \( (-|e\rangle |\alpha_0 \rangle + |g\rangle - |\alpha_0 \rangle - |\alpha_0 \rangle ) / \sqrt{2} \) with \( \alpha_0 = i\lambda \). Finally, a \( \pi/2 \) pulse is applied to the atom, transforming \( |e\rangle \rightarrow |e\rangle + |g\rangle \) \( \sqrt{2} \) and \( |g\rangle \rightarrow |e\rangle + |g\rangle / \sqrt{2} \). This brings the atom-cavity state to be:

\[
|\hat{\phi} \rangle = \frac{1}{2} \left( |e\rangle (|\alpha_0 \rangle |\alpha_0 \rangle + | -\alpha_0 \rangle |\alpha_0 \rangle - |\alpha_0 \rangle | -\alpha_0 \rangle - | -\alpha_0 \rangle \right) \rangle .
\]  

(48)

A detection of the state \( |e\rangle \) prepares the two-mode cat state \( |\phi^+ \rangle \propto |\alpha_0 \rangle |\alpha_0 \rangle + | -\alpha_0 \rangle | -\alpha_0 \rangle \) with an even photon number (even parity), while a detection of the state \( |g\rangle \) prepares the two-mode cat state \( |\phi^- \rangle \propto |\alpha_0 \rangle |\alpha_0 \rangle - | -\alpha_0 \rangle | -\alpha_0 \rangle \) with an odd photon number (odd parity). In order to measure the correlation function \( \Pi_{AB}(\alpha, \beta) = Tr\left[\rho D_A(\alpha) D_B(\beta) (-1)^{\hat{n}^+_A + \hat{n}^+_B} D^\dagger_A(\alpha) D^\dagger_B(\beta) \right] \), the cavity states \( A \) and \( B \) are displaced by amplitudes \(-\alpha\) and \(-\beta\) respectively. A second atom is then passed through both cavities, where the subsequent measurement on the atom gives the measurement outcome of \( \Pi_{AB}(\alpha, \beta) \).

Wang et al. (213) have experimentally created a two-mode cat state for microwave fields. The two-mode cat state generation process is based on the modified proposal by Leghtas et al. (213), who showed how to map an arbitrary two-mode qubit state onto a two-mode cat-state superposition in a cavity. As above, Wang et al. prepared the atom in a superposition state, with two cavity states being in the vacuum state. Instead of passing the atom through two cavities that are both in a coherent state \( |\lambda \rangle \), the cavity states are only transformed to coherent states if the atom is in the \( |g\rangle \) state, while both cavities remain in the vacuum state if the atom is in the \( |e\rangle \) state. The state of the whole system up to this point is given by \( (|g\rangle |2\lambda \rangle + |e\rangle |0\rangle |0\rangle ) / \sqrt{2} \). Another conditional interaction puts the atom in the \( |e\rangle \) state back to the \( |g\rangle \) state, while leaving the \( |g\rangle \) state unchanged, thus disentangling the atomic state from the
cavity states. Finally, a displacement operator with a displacement amplitude $-\lambda$ is applied to both cavities, which prepares the whole system in the cat state, as

$$|g\rangle (|\lambda\rangle + | -\lambda\rangle) / \sqrt{2}.$$  \hfill (49)

Following these proposals, Wang et al.\textsuperscript{[211]} experimentally measured the joint parity and observed a violation of a Bell inequality, based on the observation of the four points of the Wigner function, as in \textsuperscript{[144]}. We note that the measurements involved in the above schemes require to distinguish between a 0 or 1 photon count, or else to determine the parity of the cat state, which refers to whether the cat state has an odd or even photon number. These distinctions are not macroscopic. If one infers the correlations using the phase-space analysis by measuring $W(\alpha, \beta)$, the measurements do not distinguish between amplitudes (say, $\alpha_0$ and $-\alpha_0$ for the cat state \textsuperscript{[44]}) that are macroscopically separated in phase space. While one can infer failure of LHV theories for a macroscopic state with $\alpha_0$ large, it could be argued that this is not a macroscopic correlation in the sense of a fully macroscopic measurement.

Further different approaches have been taken. Ketterer et al.\textsuperscript{[214]} considered testing the CHSH inequality without a prior choice of binning procedure and with no prior knowledge of the Hilbert space dimension of the system, following from work by Horodecki\textsuperscript{[215]}. They proposed modular variables to turn unbounded observables into bounded ones and then to check for violation of the CHSH inequality. Arora and Asadian also propose a macroscopic Bell scheme by transforming an unbounded momentum operator $\hat{p}$ into a bounded one \textsuperscript{[216]}. They consider the observable

$$\hat{X} \equiv \cos \left( \frac{\hat{p}L}{\hbar} \right)$$ \hfill (50)

so that the value is bounded by $\pm 1$, and define two states $|\psi_0\rangle$ and $|\psi_1\rangle$, which are superposition of position states. The position superposition state can be implemented with a grating with $N = 2M$ slits. A linear combination of these two states can be formed where $|\psi_{\pm}\rangle = \frac{1}{\sqrt{2}} (|\psi_0\rangle \pm |\psi_1\rangle)$. These states are such that the bounded operator $\hat{X}$ have the following expectation values: $\langle \psi_{\pm} | \hat{X} | \psi_{\pm}\rangle = \pm \frac{N-1}{N}$. In other words, depending on the state $|\psi_{+}\rangle$ or $|\psi_{-}\rangle$, the expectation value of $\hat{X}$ either returns a positive or negative value, for large $N$, which are justified as macroscopically distinguishable. In order to implement different measurement settings as required in CHSH inequality, they considered a unitary operator $U$ such that $U (\phi) |\psi_0\rangle = e^{i\phi/2} |\psi_0\rangle$ and $U (\phi) |\psi_1\rangle = e^{-i\phi/2} |\psi_1\rangle$. For a state $|\Psi\rangle = (|\psi_{+}\rangle |\psi_{-}\rangle_2 - |\psi_{-}\rangle |\psi_{+}\rangle_2)/2$, the correlation function is found to be

$$\langle \hat{X} (\phi) \otimes \hat{X} (\theta) \rangle = - \left( \frac{N - 1}{N} \right) \cos (\phi - \theta),$$ \hfill (51)

giving the Bell parameter $|\langle \hat{B} \rangle | = \left( \frac{N-1}{N} \right)^2 2 \sqrt{2}$. The authors suggest a physical implementation based on entangled beams in polarization. A similar scheme has been proposed by Huang et al.\textsuperscript{[217]}

Recent work by Thearle et al.\textsuperscript{[218]} infers an experimental violation of a Bell inequality using intensities and CV measurements. They justify that the violations are continuous variable, since in their scheme the photon correlation intensity functions are measured via continuous-variable quadrature-phase amplitude measurements. This work is based on the earlier proposals, which evaluate Bell correlations in terms of normalized intensity correlations \textsuperscript{[219, 220]}. Reid and Walls originally considered four-mode states generated from parametric down conversion \textsuperscript{[9]}, with two polarization modes $a_{\pm}$ and $b_{\pm}$ at the respective sites $A$ and $B$. They considered bounded observables at each site according to a normalized intensity difference \textsuperscript{[13]}. At $A$ the normalized intensity is defined as $(I_{+}^{(A)}(\theta) - I_{-}^{(A)}(\theta))/(I_{+}^{(A)}(\theta) + I_{-}^{(A)}(\theta))$, where $I_{\pm}^{(A)}(\theta)$ are the outcomes for $I_{\pm}^{(A)}(\theta) = \hat{c}_{\pm}^{\dagger} \hat{c}_{\pm}$ given in \textsuperscript{[44]}. Similar intensity differences are defined at $B$ in terms of the angle $\phi$. The normalization motivates the application of CHSH-type Bell inequalities in terms of intensity correlations, where more than a single photon might be detected. Thearle et al. generated a 4-mode entangled state from optical parametric oscillation (OPO), and the output is sent to two parties. Each party is free to apply the mixer that allows the polarization along any axis to be measured. Explicitly, the correlation function of interest has the form

$$R^{IJ}(\theta, \phi) = \langle R_{A}^{i} (\theta) R_{B}^{j} (\phi) \rangle = \langle \hat{a}_{i}^{\dagger} \hat{a}_{j} R_{A}^{i} R_{B}^{j} \rangle,$$ \hfill (52)

where $\theta, \phi$ are the measurement settings for party $A$ and $B$ respectively, while $i, j \in \{+, -\}$ characterized the detector that detects the mode $\pm$. In the scheme, the operators $\hat{a}_{i}^{\dagger} \hat{a}_{i}$ are measured in terms of quadrature phase amplitude observables, which required additional assumptions about vacuum corrections. It is important to measure the vacuum intensity and this is carried out in the experiment by randomly swapping between measuring the quadratures and vacuum intensity. They inferred a Bell violation with $|\langle \hat{B} \rangle | = 2.31 > 2$. 
The work of Zukowski, Wiesniak and Laskowski [221] demonstrates how to obtain violation of Bell inequalities for the four-mode intensity outputs of the parametric down conversion process given by [9], directly. In a series of papers [221, 222], Zukowski et al rigorously formalised the renormalisation approach, by giving a careful treatment of the vacuum state, for which the normalized intensities become undefined. This led to new predictions of violation of Bell inequalities at higher intensities. A similar renormalisation procedure also using the Moore-Penrose operator was developed by He et al [222] to justify the use of EPR entanglement criteria involving normalized atomic detection counts, which were applied to give predictions for EPR correlations in a Bose-Einstein condensate [224, 225]. 

Lee and Jaksh (2009) extended their approach of obtaining an optimal Bell inequality to continuous variable systems. They considered the two-mode squeezed state $|TMSS\rangle = \text{sech} \sum_{n=0}^{\infty} \tanh^n r |n\rangle |n\rangle$. The measurement basis is obtained by carrying out quantum Fourier transform on the basis state $|n\rangle$. This gives the phase state: $|\theta, k\rangle = (1/\sqrt{s+1}) \sum_{n=0}^{\infty} \exp(\text{in} \theta) |n\rangle |n\rangle$. The correlation operator $\hat{E}(\theta, \phi) = \Pi(\theta) \otimes \Pi(\phi)$, where $\Pi(\theta) = \sum_{k=0}^{\infty} (-1)^k |\theta, k\rangle \langle \theta, k|$. They find the Bell correlation function $B_{QM}$ arbitrarily close to $2\sqrt{2}$ when the squeezing strength $r \to \infty$.

Many of the above methods rely on a binning, or renormalisation, of outcomes in order to obtain Bell violations for continuous variables measurements. This provides an observable with outcomes bounded by $\pm 1$, so that one may take advantage of the CHSH or CH Bell inequalities. Cavalcanti, Foster, Reid and Drummond (CFRD) showed that the falsification of local hidden variable theories could be obtained without this binning, directly from the continuous variable spectrum [226]. These authors adapted the approach of Mermin [11] for continuous variable outcomes. They considered the function

$$F_j^\pm = X_j \pm i Y_j$$

(53)

of measurement outcomes $X_j, Y_j$ at each site $j$, where the quantum observable for these measurements is $\hat{X}_j, \hat{Y}_j$ respectively. For a local hidden variable (LHV) theory, it is always true that $|\langle \prod_{j=1}^{N} F_j^{\pm} \rangle|^2 \leq \int d\lambda \prod_{j=1}^{N} |\langle F_j^{\pm} \rangle\lambda|^2$. Since $|\langle F_j^{\pm} \rangle\lambda|^2 = \langle X_j^2 \rangle_\lambda + \langle Y_j^2 \rangle_\lambda$, it follows from the non-negativity of variances that for any LHV state, $|\langle F_j^{\pm} \rangle\lambda|^2 \leq \langle X_j^2 \rangle_\lambda + \langle Y_j^2 \rangle_\lambda$. This leads to the bound

$$|\langle \prod_{j=1}^{N} F_j^{\pm} \rangle|^2 \leq \left( \prod_{j=1}^{N} (X_j^2 + Y_j^2) \right)^{1/2}$$

(54)

giving the CFRD Bell inequality.

Cavalcanti et al [227] pointed out that if one constrains the local hidden variables for some of the sites, say labelled $r = 1, \ldots, T$, to be consistent with quantum predictions, then there is a further restriction given by the uncertainty relation. They consider quantum uncertainty relations of the form $(\Delta \hat{X}_j)^2 + (\Delta \hat{Y}_j)^2 \geq C_j$, where $C_j$ may depend on the operators associated with $x_j$ and $y_j$. This will imply for quantum states that $|\langle F_j^{\pm} \rangle\lambda|^2 \leq \langle X_j^2 \rangle_\lambda + \langle Y_j^2 \rangle_\lambda - C_j$, leading to the unified nonlocality inequalities

$$|\langle \prod_{j=1}^{N} F_j^{\pm} \rangle|^2 \leq \left( \prod_{j=1}^{T} (X_j^2 + Y_j^2 - C_j) \prod_{j=T+1}^{N} (X_j^2 + Y_j^2) \right)^{1/2}$$

(55)

If one takes observables $X$ and $Y$ to be spin, as in Mermin’s original approach given by [19], then the inequalities (54) becomes the MABK inequalities, if $T = 0$. For $T = N$, a set of entanglement criteria is derived, corresponding to the criteria of Roy [228].

The original work of CFRD considered $X$ and $Y$ to be continuous-variable outcomes of quadrature phase amplitudes, $X$ and $P$ [226]. Symbolizing $\hat{a}^+ = \hat{a}^\dagger$ and $\hat{a}^- = \hat{a}$, the nonlocality inequalities (55) will be violated when

$$|\langle \hat{a}_{1}^{+} \cdots \hat{a}_{N}^{+} \rangle| > \left( \prod_{j=1}^{T} \bar{n}_j \prod_{j=T+1}^{N} (\bar{n}_j + 1/2) \right)^{1/2}$$

(56)

For $T = 0$, these are the CV Bell inequalities derived by CFRD [226]. These authors showed that certain GHZ states with $N > 9$ will violate the inequalities. If $T = N$, one arrives at inequalities for entanglement given by Hillery and Zubairy [229], whereas for intermediate $T$, the inequalities are steering inequalities [227].

Further work by Salles et al [230] and He et al [231] studied the CFRD Bell inequalities. Salles et al showed violations of generalizations of the inequalities for three settings using spin qubits and GHZ states, for $N \geq 3$. The approach was adapted by Shchukin and Vogel [232] who derived multi-setting inequalities closely related to the algebra of quaternions and octonions. Violations were predicted for GHZ states with $N \geq 3$ [153]. The unified approach to
deriving Bell and steering inequalities was applied to spin qubits by Cavalcanti et al \[227\], who derived MABK steering inequalities. Criteria were also developed by Jebaratnam et al \[233\]. Li et al have experimentally investigated the violation of Mermin steering inequalities for qubits where \( N = 3 \) \[120\].

VI. QUANTUM CORRELATIONS OF CAT STATES

The Schrödinger cat gedanken experiment analyses the correlated state of type \[12, 13\]

\[|\psi\rangle_{cat} = \frac{1}{\sqrt{2}}(|\alpha\rangle_A|\uparrow\rangle_B + e^{i\theta}|\alpha\rangle_A|\downarrow\rangle_B) \]  

(57)

where, for large \( \alpha \), a microscopic system (the spin) is coupled to a macroscopic system (modeled as a field mode). Here, \(|\alpha\rangle\) is a coherent state. Where \( A \) and \( B \) refer to different systems, this state is entangled. In Schrödinger’s original argument, a microscopic system \( B \) is prepared in superposition state, and is then coupled to a macroscopic system \( A \) which represents the measurement device. In the above analogy, system \( B \) is originally in the superposition \((|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}\) and system \( A \) is prepared in a coherent state \(|\alpha\rangle\), where \( \alpha \) is large. A coupling \( H \) between the systems creates the state after an interaction time. This models the measurement procedure, since the result of a measurement of the coherent-state amplitude (whether \( \alpha \) or \(-\alpha\)) is correlated with the result of the spin measurement \( \hat{\sigma}_x \) on \( B \).

Correlated cat states such as \((57)\) have been generated experimentally. The experiment of Monroe et al \[234\] created a cat state similar to \((57)\) where the correlation is between the internal degree of freedom of an atom and its spatial degree of freedom. The atom is trapped in a harmonic potential and is prepared in the state \(|\downarrow,0\rangle\). Here, \(|\downarrow\rangle\) is the internal state and \(|0\rangle\) is the motional ground state of the atom. A \(\pi/2\) pulse puts the atom in a superposition and a displacement beam excites the motional state to a coherent state \(|\alpha e^{-i\phi/2}\rangle\), conditioned on the internal state being \(|\uparrow\rangle\). This brings the state to \(((|\uparrow,0\rangle + |\uparrow,\alpha e^{-i\phi/2}\rangle)/\sqrt{2}\). After a \(\pi\) pulse and further displacement, the cat state \(((|\uparrow,\alpha e^{i\phi/2}\rangle + |\downarrow,\alpha e^{-i\phi/2}\rangle)/\sqrt{2}\) is created.

On the other hand, the experiment of Brune et al \[14\] creates a cat state of type \((57)\) by entangling Rydberg atoms one at a time with a microwave field. The Rydberg rubidium atom is prepared in a superposition of two atomic states \(e\) and \(g\), and then passed through a high \(Q\) microwave cavity prepared in a coherent state \(|\alpha\rangle\) of a few photons. A dispersive interaction between the atom and cavity generates the entangled cat state. The superposition was inferred by observation of coherence using two-atom correlations, based on a proposal by Davidovich et al \[235\]. The experiment modeled the coupling of a system to a meter, in which the decay of the coherence was found to increase with the separation \(\phi\) in phase space of the two coherent states, in agreement with predictions \[200, 236, 240\]. The experiment thus models the rapid decay of a macroscopic pointer from the macroscopic superposition \((57)\) of two distinct coherent states into a classical mixture of the two states.

The entangled cat state \((57)\) is paradoxical, since the concept of realism is challenged by the correlations. The correlations are readily shown to be of the EPR-type. We find it useful to specify the entangled cat state

\[|\psi\rangle_{cat} = \frac{1}{\sqrt{2}}(|\alpha\rangle_A|\uparrow\rangle_B + i|\alpha\rangle_A|\downarrow\rangle_B) \]  

(58)

The measurement of the spin \(\hat{S}_z^{(B)}\) of the system \(B\) indicates the sign of the amplitude \(\hat{X}_A\) of the field mode \(A\) (taking \(\alpha\) to be real). If we consider the inference of \(X_A\) based on the measurement of \(\hat{S}_z^{(B)}\), then as \(\alpha \to \infty\), the system \(A\) is projected into either \(|\alpha\rangle\) or \(|-\alpha\rangle\). Hence \(\Delta_{inf}\hat{X} = 1\). This can be verified by a complete calculation of the conditional distributions for \(X\) conditioned on the outcome \(\pm 1\) for spin at \(B\). On the other hand, if one measures \(\hat{S}_z^{(B)}\) at \(B\), then we consider

\[|\psi\rangle_{cat} = (|\alpha\rangle + i|\alpha\rangle)|\uparrow\rangle_x + (|\alpha\rangle - i|\alpha\rangle)|\downarrow\rangle_x \]  

(59)

which implies the conditional variance for \(\hat{P}\) given an outcome \(\pm 1\) for \(\hat{S}_z^{(B)}\). The result for the spin \(\hat{S}_z\) projects the system into one of the cat-states \((|\alpha\rangle \pm i|\alpha\rangle)/\sqrt{2}\) considered by Yurke and Stoler \[200\], these states having the same variance \((\Delta\hat{P}_A)^2\) in \(\hat{P}_A\). This gives \(156, 241, 242\)

\[\Delta_{inf}^2\hat{P} = (\Delta\hat{P}_A)^2 = 1 - 2\alpha^2 e^{-4\alpha^2} \]  

(60)

Hence, the EPR-paradox and EPR steering condition \([34]\) (for steering of \(A\)) is satisfied, with \(\epsilon < 1\). If one assumes an EPR-type local realism, which is the assumption that the system is described by a local realistic theory, then the localized state of the system cannot be directly modeled as a quantum state.
An EPR-steering paradox has been explained for other two-mode correlated cat states, such as the NOON states given by \[ |N\rangle_J \text{ is the number state for the mode labelled } I. \] These states have been generated experimentally. \[ EPR \text{ correlations can be detected using an EPR criterion of the type } \] but based on number and phase measurements, as shown in \[ \text{Similar NOON-type states are generated in macroscopic or multimode versions of the Hong-Ou-Mandel effect. } \] \[ \text{EPR correlations are also evident in the correlated cat states given by. } \] The important question of whether one can illustrate failure of local realism directly for an entangled cat state was answered by the analyses that gave violations of Bell LHV theories for the state. \[ \text{These analyses are summarized in the previous section. Relevant to the two-mode cat state is the analysis of Banaszek and Wódkiewicz, which allows violations of Bell inequalities that do not decay with } \alpha. \text{ This shows that the cat states are not compatible with the predictions of any LHV model, indicating that the premise of local realism/local causality underlying the EPR argument is itself invalid. A similar result applies to the Schrödinger cat state. } \] \[ \text{Wódkiewicz investigated the nonlocal Bell correlations for the hybrid cat state (57), based on the Banaszek and Wódkiewicz approach. The correlation is expressed in terms of the Wigner function and shows the entanglement and interference of the spin and the cat system. Violations of the Banaszek-Wódkiewicz Bell inequality for the hybrid cat state are obtained for certain spin orientations, showing the nonlocality of this state. } \] Bell correlations for the NOON states were demonstrated by Wildfeuer, Lund and Dowling. \[ \text{They first applied the Bell scheme of Banaszek and Wódkiewicz to NOON states, i.e. to coherently displace the state before detection. They found a maximal violation of the CH-Bell inequality for } N = 1, \text{ with the degree of violation reducing with } N, \text{ making experimental observation difficult for } N \geq 3. \text{ Similar conclusions were reached in the parity measurements based on the Wigner function. In this case, the CHSH inequality is violated for } N = 1 \text{ but not for } N > 1. \text{ However, for different Bell inequalities, the authors found Bell violations for a NOON state that are independent of } N. \text{ The argument is that the CH and CHSH inequalities involve only four joint probability distributions and are not sufficiently sensitive to detect the nonlocal correlations in a NOON state. Bell inequalities with more joint probability distributions are needed. They considered four Bell-type inequalities derived by Janssens et al. that have six joint probability distributions. Written in terms of the Q-functions, these inequalities are given by } \] \[ \text{The authors numerically determined the parameters } \alpha, \beta, \gamma, \delta \text{ that optimally violate each of these inequalities. } J_1, J_2, \text{ and } J_4 \text{ are found to be violated by a constant amount for all } N, \text{ while } J_3 \text{ shows a decrease in violation as a function of } N. \] \[ \text{VII. COARSE-GRAINING AND DECOHERENCE} \] The argument could be put forward that the quantum correlations considered so far are not genuinely macroscopic because, although describing macroscopic systems, they require measurements that are microscopically or mesoscopically resolving i.e. they require measurements that distinguish at some point between states that are only microscopically or mesoscopically distinct. In fact, it was originally an open question whether, in a Bell test with } N \text{ particles at each site, violations would be possible with the loss of information about the result of just } one \text{ of the particles. For some Bell inequalities and states, the answer was negative, as for the MABK inequalities with GHZ states (refer Section IV). More generally, this has been shown to not be the case. } \] Peres pointed out that additional noise in } x \text{ and } p \text{ of order the quantum noise level } (\sim \hbar) \text{ would damp out any Bell violations arising from the } x \text{ and } p \text{ measurements. } \text{This is apparent because the addition of such noise transforms any negative Wigner function } W(x_i, p_i) \text{ (for two modes } i = 1, 2) \text{ into a positive distribution, given by the } Q \text{ function. There then exists a local hidden variable theory which models the measured } x_i \text{ and } p_i \text{ moments as arising from a joint probability distribution, } W. \text{ From this point of view, the measurements must be highly resolved if one is to observe Bell correlations using continuous variable (CV) } \hat{x} \text{ and } \hat{p} \text{ measurements. There will be a threshold value of noise, as measured by a standard deviation } \sigma_i \text{ (of order the quantum limit), beyond which Bell violations will not be possible. } \]
CV Bell violations can be observed in the presence of significant coarse-graining, however, if measured in an absolute sense. As an example, examination of the two-mode Schwinger operators given by Eq. (38) indicates that Bell violations are possible for large absolute noise values in the spins $S_X$ and $S_Y$. These observables are measured when continuous variable (CV) measurements are carried out by optical or atomic homodyne. Consider a CV measurement, where $\sigma_t$ is the threshold value of noise in $X$ and $P$ beyond which there can be no violation of a CV Bell inequality. In terms of the Schwinger operators, this threshold corresponds to an amplified noise value of $\sim E\sigma_t$. The $E\sigma_t$ reflects the allowed noise in the two-mode Schwinger photon number difference given by Eq. (38), but is nonetheless small relative to the total mode occupation numbers, of order $\sim E^2$. The relative threshold noise level scales as $1/\sqrt{N}$ where $N$ is the total field number.

The work reviewed in Sections III and IV examines Bell violations using measurements with discrete outcomes. Violation of local realism is possible for arbitrarily large systems (of size $N$) and for arbitrarily large dimension ($d$). The effect of coarse-grained measurements on Bell nonclassicality for discrete measurements has been well studied. After the work of Mermin [57], Busch developed a mathematical framework that characterizes the sharpness of observables [258]. With this formalism, Busch showed that an EPR experiment with two spin-1/2 particles in the singlet state no longer violates the Bell inequality when the unsharpness of the observables exceeds certain threshold. The incompatibility with local realism is uncovered by very precise measurement selections, and was shown sensitive to noise. In this way, the transition from quantum to classical can be viewed as arising from measurement, as it becomes increasingly coarse-grained [245].

Durt, Kaszlikowski and Zukowski showed explicitly that the higher dimensional Bell inequalities may allow a greater robustness to noise [70]. They extended the work of Mermin and Garg [63] to values of $d = 16$. The state of interest is the mixed state

$$\rho_d \left( F_d \right) = F_d \rho_{\text{noise}} + \left( 1 - F_d \right) |\Psi_{\text{max}}^d \rangle \langle \Psi_{\text{max}}^d |$$

where $F_d \leq 1$ is a parameter that characterizes the transition of Bell violations. The entangled state is $|\Psi_{\text{max}}^d \rangle = \frac{1}{\sqrt{d}} \sum_{m=1}^{d} |m, m\rangle$, where $m$ is a mode that contains a photon, which can be prepared using parametric down conversion. The measurement for each party involves a $2d$-multiport and $d$ phase shifters. A $2d$-multiport is an optical device that lets a photon enter one of its $d$ input ports, and that photon has equal probability of exiting from one of the $d$ output ports. By imposing that the quantum prediction can be expressed in terms of joint probability distributions that satisfy local realism, the authors obtained $4d^2$ linear equations that must be true for $d^2$ local hidden probabilities $P^{HV}$. This is a linear optimization problem where the minimal noise threshold $F_d$ is allowed for larger $d$. The behavior for detection efficiencies $\eta$ was also studied. Values of $\eta < 1$ reflect loss (or decoherence) from the system, so that information can be accessed by another party. The critical detection efficiency $\eta_d$ below which local realism holds, was found to decrease very slowly but continuously with $d$. This extended the result of Mermin and Garg for $d = 2$ [63]. However, the conclusions were limited to maximally entangled states with certain observables only.

The degree of sensitivity to noise and decoherence will depend on the state and measurements being considered. For a cat state, different to a maximally entangled state [59], the superposition is for two macroscopically distinct states only. In this work, the impact of decoherence with increasing size $N$ is more extreme [200, 238, 240], as shown by Brune et al [14]. In the work of Yurke and Stoler, the impact of detection efficiency $\eta$ was considered for the cat state $\frac{1}{\sqrt{2}} (|\alpha\rangle \pm i |\alpha\rangle)$, with $\alpha$ real. The quantum coherence was measured by the fringes in the distribution $P(P)$ of the quadrature phase measurement $\hat{P}$. The term contributing to the fringes was found to decay as $\sim e^{-2(1-\eta)|\alpha|^2}$, showing an exponentially increased sensitivity with $\alpha$. Similarly, Kennedy and Walls showed that thermal noise $n_{th}$ destroys the fringes and there is an increased sensitivity of the decay with $|\alpha|$ [60]. We expect that the steering signature for the entangled cat state [59] as measured by the conditional variance [60] will show a similar sensitivity to the efficiency $\eta$ and thermal noise $n_{th}$ of the steered system. This is because $\frac{1}{\sqrt{2}} (|\alpha\rangle + i |\alpha\rangle)$ and $\frac{1}{\sqrt{2}} (|\alpha\rangle - i |\alpha\rangle)$ are the states of the steered system, conditioned on the measurement of the spin $S_z^{(B)}$ of the micro-system, as given by Eq. (39). This was confirmed for the decoherence of the steering of a cat state by Rosales-Zarate et al [156]. However, in these analyses, the measurements being considered were limited to $X$ and $P$.

Jeong, Paternostro and Ralph [261] gave an explicit example of how Bell violations could be obtained for coarse grained measurements based on macroscopic superposition states, using more general measurements involving the displacement operator and homodyne detection. They considered an entangled thermal cat state [262, 263], with a density operator

$$\rho = \mathcal{N} \left[ \rho_j^{th} (V, d) \otimes \rho_B^{th} (V, d) + \sigma_A (V, d) \otimes \sigma_B (V, d) + \sigma_A (V, -d) \otimes \sigma_B (V, -d) + \rho_j^{th} (V, -d) \otimes \rho_B^{th} (V, -d) \right].$$

Here, $\rho_j^{th} (V, d) = \int d^2 \alpha P^{th} (V, d) |\alpha\rangle_j \langle \alpha|$ is a displaced thermal state for system $j = A, B$ where $P^{th} (V, d) =$
\[
\frac{2}{\pi |V|^2} \exp \left( -\frac{2|\alpha|^2}{V^2} \right) \]

with \( d \) being the displacement in phase space and \( V \) the variance associated with the distribution; \( \sigma_i^2 (V, d) = \int d^2 \alpha P^0 (V, d) |\alpha_i\rangle \langle \alpha_i| \) and \( N = 2 \left[ 1 + \exp \left( -\frac{4|\alpha|^2}{V^2} \right) /V^2 \right] \). We note that \( d \), the effective separation between the two states of the superposition, can be large. The local measurements comprise a displacement operator \( D_j (i \theta_j / 2d) = e^{i \theta_j a_d} a_i d a^\dagger_i a_d^i \) where \( \theta_j \) determines the measurement setting for party \( j \), and also involve a unitary operator that represents the Kerr nonlinear interaction. Finally, a parameter \( \eta \) denotes the homodyne detector efficiency that characterizes the final measurement outcome. A Bell-CHSH inequality based on the binned positive or minus outcomes of the quadrature phase amplitude measurements was evaluated. The authors gave numerical results for \( d = 150 \) and found strong Bell violations (approaching \( B = 2\sqrt{2} \)) with a detection efficiency as low as \( \eta = 0.05 \). It is argued that this observation goes against the viewpoint that a coarsening of measurement outcomes (i.e. measurement-outcomes imprecision) contributes to quantum-classical transition.

In a later paper, Jeong, Lim and Kim [264] further studied how Bell violations are destroyed by coarse-grained measurements, in a discrete measurement setting. The authors evaluated the effect of imprecision in measurement settings and measurement outcomes on the violation of a CHSH Bell inequality, for an entangled state with a varying distribution with a standard deviation \( \sigma \). The coherence is signified by the revival of a coherent state \( |\psi\rangle \sim |\alpha\rangle + i|\alpha\rangle \), where \( \alpha \) is real [265]. The coherence is signified by the revival of a coherent state \( |\alpha\rangle \), after application of the nonlinear unitary dynamics \( U_{NL} = e^{i \hat{\alpha} \hat{a}} / \sqrt{\hat{\alpha}^2 + \hat{\alpha}^2} \). Here, the measurements need only distinguish between the macroscopically distinct outcomes \( a_0 \) and \( -a_0 \), and hence allow macroscopic coarse-graining. Similar to the results of Jeong, Lim and Kim, however, the authors show that an imprecise transformation \( U_{NL} \) will inevitably affect the outcome. More explicitly, instead of \( \pi/2 \) phase, the authors allow \( \pi/2 + \phi \) where \( \phi \) satisfies a Gaussian distribution with a standard deviation \( \sigma \). Wang et al calculated that instead of obtaining a state \( |\alpha\rangle \langle \alpha| \) after applying the unitary transformation on \( |\psi\rangle \langle \psi| \), the measurement-setting imprecision of the transformation leads to a state

\[
C_{\sigma} \left( |\alpha\rangle \langle \alpha| \right) = e^{-\alpha^2} \sum_{n, n'} \frac{e^{-\frac{1}{2} \sigma^2 \left( \frac{n^2 - n'^2}{\sqrt{n! n'^!}} \right)^2 \frac{\alpha_n + \alpha_n'}{\sqrt{n! n'^!}}}} \right| n \rangle \langle n' |. \tag{65}
\]

For large \( |\alpha| \), \( \sigma \neq 0 \) will make it hard to distinguish the two states in the final measurement (even if the measurement is ideal), and there will be a threshold (depending on \( |\alpha| \)) where it is impossible to distinguish the states.

In the next section, we examine results by Thenabadu et al [253, 270, 271] showing how violations of local realism can be obtained for macroscopic coarse-grained measurements that need only distinguish between the two distinct states \( |\alpha\rangle \) and \( |-\alpha\rangle \), where \( \alpha \to \infty \). Watts, Halpern and Harrow have also shown how nonlinear Bell inequalities, which have additional assumptions, may be violated for macroscopic measurements [272].

Furthermore, calculations indicate that the decoherence causing the fragility of the macroscopic quantum coherence and correlations of the cat state can be controlled by modifying the environment to which the cat system is coupled [13]. For the simple cat state \( \frac{1}{\sqrt{N}} \left( |\alpha\rangle \pm |\alpha\rangle \right) \), Meccozzi and Tombesi showed that coupling to a squeezed reservoir slows down the otherwise rapid decay of quantum coherence of the cat state as \( \alpha \) increases [273, 274]. Kennedy and Walls and Serafini et al [275, 276] showed a similar effect for the decoherence caused by a thermal reservoir [260]. In both cases, it was important to orientate the squeezed quadrature in suitable direction. Similar results have been predicted [277, 278] for the even and odd cat states \( \frac{1}{\sqrt{N}} \left( |\alpha\rangle \pm |-\alpha\rangle \right) \) generated in parametric oscillation above threshold [274, 283], and for cat states generated in optomechanical systems [284]. The progress made towards creating and manipulating squeezed states of light in a variety of systems (for example, [285, 293]) suggests the realisation of more macroscopic cat states to be possible in the future, which also motivates tests of quantum mechanics [294].

We have seen how quantum mechanics violates local realism in macroscopic regimes. Another approach was taken by Navascues and Wunderlich [292]. They considered the bounds placed on bipartite correlations if consistency with classical mechanics in the macroscopic regime is to be upheld in the form of a mechanism that they called macroscopic locality. They first considered a source that produces a pair of particles which are sent to two spatially separated parties, Alice and Bob. Each of them can change their measurement settings by applying one out of \( s \)
possible interactions on their particle. The possible measurement settings are indexed from 1 to \( s \) for Alice, and \( s + 1 \) to \( 2s \) for Bob. There will be a set of detectors that determine the measurement outcomes, labelled by \( D(a) \) or \( D(b) \), where \( D(a) \) is the detector that returns outcome \( a \) for Alice, and similarly \( D(b) \) returns outcome \( b \) for Bob. The experiment is characterized by the joint probability \( P(z_j) \). To define the notion of macroscopic locality, they described a macroscopic experiment. Instead of a pair of particles, \( N \) independent pairs of particles are produced and Alice and Bob each receive a beam of particles. In the case of one particle, a chosen measurement setting (one out of \( s \) interactions) will lead to one detector being triggered. In the macroscopic case, the choice of one measurement setting will lead to the trigger of all detectors, one for each particle. The measurement outcomes are a distribution of intensities from all the detectors of either Alice or Bob. Let the intensities detected by Alice (Bob) be \( \vec{I}_{1s} \) (\( \vec{I}_{2s} \)), where \( X \) (\( Y \)) is one of the \( s \) interactions by Alice (Bob). In a local hidden variable model, a joint probability density \( P(\vec{I}_1, ..., \vec{I}_{2s}) \) characterizes the experiment, where \( \vec{I}_j \) is the hidden variable state giving the result of the measurement with setting \( j \). The marginal probability density \( P(\vec{I}_X, \vec{I}_Y) \) can be estimated. When \( N \gg 1 \) is large, the marginal probability density is demanded to be consistent with classical physics, and hence admits local hidden variable models. The marginal probability density then satisfies

\[
P(\vec{I}_X, \vec{I}_Y) = \int \left( \prod_{Z \neq X,Y} d\vec{I}_Z \right) P(\vec{I}_1, ..., \vec{I}_{2s}).
\]

If this is the case, then the system is said to satisfy \textit{macroscopic locality}. The set of correlations that satisfies macroscopic locality is completely characterized by a set \( Q_1 \). Navascues and Wunderlich inferred that all quantum correlations (given by the set of correlations \( Q \)) are macroscopically local in the bipartite case (\( Q \subset Q^1 \)).

\section{VIII. Leggett-Garg Correlations and Macroscopic Realism}

In 1985, Leggett and Garg gave an explicit proposal to test the predictions of macroscopic realism against those of quantum mechanics. In the proposal, it is only necessary to make measurements distinguishing between two macroscopically distinct states of a system \cite{13}. The measurements are thus macroscopic and allow a macroscopic coarse graining, at least in principle. Leggett and Garg considered systems which evolve dynamically with time. In their analysis, it was however necessary to consider a specific definition of macroscopic realism, which is referred to as macrorealism.

\subsection{A. Leggett-Garg inequalities and macro-realism}

The question of how to define a truly macroscopic quantum regime was raised by Leggett \cite{297, 298}. Leggett suggested that in order to confirm the presence of a quantum superposition or quantum coherence, some observables/quantitative measures that allow the distinction between a superposition and a classical mixture have to be designed. In the paper \cite{297}, Leggett defined disconnectivity as one possible measure. As an example, an \( N \)-particle state

\[
\psi = (a\phi_1 + b\phi_r)^N
\]

is not regarded as a macroscopic quantum state based on the disconnectivity measure, but an \( N \)-particle state

\[
\psi_1 = a\phi_1^N + b\phi_r^N
\]

would be. Leggett also suggested to observe quantum tunneling as evidence of macroscopic quantum state. In the later paper \cite{298}, Leggett proposed another feature to be included in the notion of ‘macroscopic distinctness’. It is named the extensive difference of the measurement outcomes. For a superposition of two states, the difference between the measurement outcome of both states has to be large, relative to a reference value of an observable which is typical at the atomic scale. Leggett gave the Bohr magneton as a reference value when the observable in consideration is the magnetic moment of the state.

In 1985, Leggett and Garg proposed a test of macroscopic realism, where one makes measurements distinguishing between two macroscopically distinct states of a system \cite{13}. A review is given in \cite{299}. The assumption of macroscopic realism (MR) is that the system prior to the measurement is actually in one or other of the two macroscopically distinct
states. One may then assign a hidden variable $\lambda_M$ to the system to denote the outcome of the measurement, should it be performed. For example, for the cat state ($\alpha_0$ is real)

$$|\psi\rangle \sim |\alpha_0\rangle + e^{i\theta} | - \alpha_0\rangle$$

(69)
of a system $A$, the measurement $\hat{S}$ defined as the sign of the quadrature amplitude $\hat{X}_A$ distinguishes between the two states $|\alpha_0\rangle$ and $| - \alpha_0\rangle$. One assigns $\lambda_M = +1$ if the system is in state giving outcome $+1$, and $\lambda_M = -1$ if the system is in a state giving outcome $-1$. We note the system cannot be regarded as being in either state $|\alpha_0\rangle$ or $| - \alpha_0\rangle$. The superposition $|\psi\rangle$ can be distinguished from the mixture $\rho_{\text{mix}}$ of states $|\alpha_0\rangle$ and $| - \alpha_0\rangle$, for example, by fringes in the distribution of $\hat{P}_A$.

Macroscopic realism asserts that the system is in a “state” with a definite value for the result for $S$, and does not propose details about the microscopic nature of that state.

Leggett and Garg’s work considered a system dynamically evolving in such a way that the system at times $t_i$ gives outcomes $+1$ or $-1$ for a measurement, these outcomes corresponding to the macroscopically distinct states. Here, we consider the measurement $\hat{S}_i$ of $\hat{S}$ at the time $t_i$, and denote the outcomes by $S_i \equiv \hat{S}(t_i)$. They made two assumptions, the first being macroscopic realism (MR). The second assumption, noninvasive measurability (NIM), is that the value of $\lambda_M$ can be measured without a disturbance to the future (macroscopic) dynamics of the system. The two assumptions (referred to as macro-realism) imply that certain inequalities will be satisfied. The Leggett-Garg inequalities involve the two-time correlation functions $\langle S(t_i)S(t_j) \rangle$. One of the inequalities is

$$LG \equiv \langle S(t_i)S(t_2) \rangle + \langle S(t_2)S(t_3) \rangle - \langle S(t_1)S(t_3) \rangle \leq 1$$

(70)
The inequalities can be violated by macroscopic two-state systems whose correlation function is given as $\langle S(t_i)S(t_j) \rangle = \cos \Omega(t_1 - t_2) \Omega$ where $\Omega$ is a constant. This is clear, if we choose, for example, $t_1 = 0$, $t_2 = \pi/(4\Omega)$ and $t_3 = \pi/(2\Omega)$. The no-signaling-in-time inequality gives a test necessary and sufficient for macrorealism. Higher dimensional studies have also been given by Halliwell and Mawby. Systems that violate Leggett-Garg inequalities can be considered to exhibit macroscopic correlations with respect to time. In these tests, one requires only to make macroscopic coarse-grained measurements, which need only distinguish between the macroscopically distinct states, e.g. $|\alpha_0\rangle$ and $| - \alpha_0\rangle$, at the given time $t_i$.

There have been many predictions and some realizations of violation of Leggett-Garg inequalities, including and those referenced in Emary et al. In many of these (e.g. 311, 315), the two states $+1$ and $-1$ are realized as a photonic qubit similar to the qubit Bell experiments. In other analyses, the outcomes are binned to create a dichotomic observable. Kohler and Brukner showed how the Leggett-Garg inequality could be used to demonstrate non-classical behavior in high spin systems, and to illustrate the quantum to classical transition under coarse graining.

They tested a Leggett-Garg inequality for a large spin $j$ system with the Hamiltonian $\hat{H} = j^2/2I + \omega J_z$ and an initial maximally mixed state $\rho(0) = \frac{1}{2j+1} \sum_{m=-j}^{j} |m\rangle\langle m|$. Here, the spin vector is $j \equiv (J_x, J_y, J_z)$ and $|m\rangle$ are the eigenstates of the spin $z$ component $J_z$. The parity measurement $\hat{Q} \equiv \sum_{m=-j}^{j} (-1)^{j-m} |m\rangle\langle m|$ is carried out at different times. In terms of these parity operators, the Leggett-Garg inequality they considered has the form

$$K \equiv C_{12} + C_{23} + C_{34} - C_{14} \leq 2,$$

where $C_{ij} \equiv \langle Q(t_i)Q(t_j) \rangle$. They found a violation of the Leggett-Garg inequality for arbitrarily high spin $j$, even for a maximally mixed initial state, provided that the values $m$ can be resolved perfectly. They then considered the system initially in a spin-$j$ coherent state

$$|\theta_0, \phi_0\rangle = \sum_{m} \left( \frac{2j}{j+m} \right)^{1/2} \cos^{j+m} \frac{\theta_0}{2} \sin^{j-m} \frac{\theta_0}{2} e^{-im\theta_0} |m\rangle. \quad (71)$$

The state at time $t$, under the time evolution unitary operator $U_t = e^{-i\omega t J_z}$, is given by $|\theta, \phi\rangle = U_t |\theta_0, \phi_0\rangle$. The probability that a $J_z$ measurement at time $t$ for large spin $j$ gives outcome $m$ is

$$p(m,t) = |\langle m|\theta, \phi\rangle|^2 \approx \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{(m-\mu)^2}{2\sigma^2}}, \quad (72)$$

which is a Gaussian. Here, $\mu = j \cos \theta$ and $\sigma = \sqrt{j/2} \sin \theta$. Sharp measurements will allow this Gaussian distribution to be resolved and a violation of the Leggett-Garg inequality observed. In order to introduce a finite resolution, the authors subdivide the $2j + 1$ possible outcomes into $(2j+1) / \Delta m$ coarse-grained values, where $\Delta m$ determines the resolution of the measurement. If $\Delta m$ is larger than the standard deviation $\sigma$ in the distribution of $m$, then the Gaussian function cannot be distinguished.

The noninvasive measurability assumption (NIM) is the additional assumption in the definition of macrorealism, and must be justified in an experiment. This has been done in different ways, most notably using stationarity, weak measurements, or ideal negative-result measurements. The ideal negative-result
measurement was proposed originally by Leggett and Garg \[15\], and is based on the assumption that MR is correct. In some scenarios, this then implies the experimentalist may measure a result $-1$ without disturbing the system by registering, for example, an absence of a photon or current, at a given location. The weak measurement can be shown to have negligible impact on the quantum system in some limit, yet with results yielding agreement with the quantum prediction for the ensemble average. The weak measurement involves postselection and is an ambiguous one, returning results that can be outside the normal eigenvalue range, which in this case corresponds to the set $\{-1, 1\}$. More sophisticated approaches are given in Uola, Vitagliano and Budroni \[321\].

Macroscopic tests giving evidence of violations of a Leggett-Garg inequality have been realized for superconducting experiments (see e.g. \[309, 312, 322\]). The recent experiment of Knee et al reports violations for macroscopic superconducting qubits, where the noninvasive measurability assumption is validated by a control experiment in which the macroscopic states are prepared and the impact of the measurement calibrated \[309\]. This however involves the assumption that the macroscopic ‘state’ of the system is actually the state prepared in the laboratory. Proposals to test violation of Leggett-garg inequalities in other systems have been put forward e.g. for optomechanics \[316\], atomic states \[323\], and for NOON states and two-well Bose-Einstein condensates \[308, 324\]. A proposal to demonstrate violations of the inequality \[70\] using cat states and a suitable dynamics is given in \[270, 325\].

B. Two-party Leggett-Garg tests

In this review, we are mainly concerned with macroscopic quantum correlations connected with spatial separation i.e. with entangled states. However, it is possible to link the Bell and Leggett-Garg approaches, to obtain a situation where a two-party Leggett-Garg test is given, corresponding to Bell violations using macroscopic measurements that only distinguish between two macroscopically distinguishable states e.g. between $|\alpha\rangle$ and $|-\alpha\rangle$.

Dressel et al \[315\] extended the Leggett-Garg inequality to multipartite systems, as well as including ambiguous detections results (i.e. weak measurements). They derived a two-party generalized Leggett-Garg inequality. First, a pair of particles are created at time $t_0$. After some time at $t_1$, particle 1 interacts with an imperfect detector and returns a generalized value $\alpha_1 \in S$, where $S$ is a set with $\min S \leq -1$ and $\max S \geq 1$ (a weak measurement). At yet a later time $t_2$, both particles 1 and 2 are measured with unambiguous detectors (a strong measurement) with detection results $b_1, b_2 \in \{-1, 1\}$. The correlation function considered is $C = \alpha_1 + \alpha_1 b_1 b_2 - b_1 b_2$, which has the inequality

$$- |1 - 2\min S| \leq \langle C \rangle \leq |2\max S - 1|$$

that must be satisfied for the macroscopic realism model. This inequality is just one of the many possible correlation functions that can be formed involving $\alpha_1$, $b_1$ and $b_2$. The setup was realized in an optical experiment. The degenerate type-II down conversion process generates entangled photon pairs where the polarization of these photon pairs are orthogonal to each other. The measurements that correspond to the correlation function in theory are given by: $\alpha_1 \leftrightarrow -\sigma_z^{(1)}$, $b_1 \leftrightarrow \sigma_0^{(1)}$ and $b_2 \leftrightarrow \sigma_z^{(2)}$. The weak measurement $\alpha_1$ is made by passing the photon beam through a coverslip before measuring the polarization state $\sigma_z^{(1)}$. They found violation of the generalized Leggett-Garg inequality.

A proposal to avoid loopholes using a hybrid Leggett-Garg-Bell inequality was put forward by Dressel and Korotkov \[314\]. They combined the generalized Leggett-Garg inequality with Bell locality. The assumption of non-invasiveness of measurements is replaced by Bell locality, where a measurement on a system cannot disturb/ influence the measurements made on the other system that is space-like separated from it. The hybrid Bell-Leggett-Garg inequality is aimed to circumvent the non-invasiveness problem, as well as the disjoint sampling loophole in Bell inequalities, where different experimental settings are required to check for the Bell inequalities. They considered a similar setup as in Dressel et al \[315\], the difference being that the weak measurement is also carried out on particle 2. The correlation function considered is

$$C = \alpha_1 \alpha_2 + \alpha_1 b_2 + \alpha_2 b_1 - b_1 b_2,$$

with the average value of $C$ satisfying the inequality $|\langle C \rangle| \leq 2$ in the hybrid Leggett-Garg Bell locality model. Related experiments investigating violation of hybrid inequalities in the context of weak measurements were performed by White et al \[322\] for transmon qubits and Higgens et al \[326\] using entangled photons.

It is possible to test macroscopic realism where the noninvasiveness assumption is replaced by that of a macroscopic Bell locality, if one considers Bell inequalities derived for macroscopically distinct qubit states. This allows a situation where all the necessary measurements are macroscopic, in the sense that one only requires to distinguish between two macroscopically distinguishable states, for all choices of measurement setting. Thenabado et al \[270\] considered macroscopic Bell-CHSH inequalities where the two outcomes of $\hat{S}$ correspond to detecting one or other of the states $|N\rangle|0\rangle$ or $|0\rangle|N\rangle$, where $|N\rangle$ is a number state. This replaces the microscopic qubits $\{|1\rangle|0\rangle, |0\rangle|1\rangle\}$
with mesoscopic qubits \( \{|N\rangle|0\rangle, |0\rangle|N\rangle \} \), distinct by \( N \) quanta for each mode. The work of \[270\] considered two space-like separated systems \( A \) and \( B \). The overall system is prepared in a four mode NOON-type Bell state, e.g. \( \frac{1}{\sqrt{2}}(|N\rangle_{a_+}|0\rangle_{a_-}|0\rangle_{b_+}|N\rangle_{b_-} + e^{i\theta}|0\rangle_{a_+}|N\rangle_{a_-}|N\rangle_{b_+}|0\rangle_{b_-}) \). The rotations at each site corresponding to the standard Bell experiments to polarizer or Stern-Gerlach rotations \[41\] are provided by a nonlinear Josephson interaction. This is given for site \( A \) by the Hamiltonian \( H^{(A)}_{NL} = \kappa (\hat{a}^\dagger \hat{a} - \hat{a} \hat{a}^\dagger) + g a^{12} \hat{a}^2 + g a^{12} \hat{a}^2 \) \[327, 328\]. Here, \( \hat{a}, \hat{a}^\dagger \) are the boson operators for the corresponding fields modeled as single modes \( a_+ \) and \( a_- \), and \( \kappa \) and \( g \) are the interaction constants. A similar interaction \( H^{(B)}_{NL} \) is defined for site \( B \). The solutions for \( H^{(A)}_{NL} \) confirm that to an excellent approximation the state created after a time \( t_a \) from an initial state \( |N\rangle_{a_+}|0\rangle_{a_-} \) is

\[
|\psi(t)\rangle \sim \cos \theta |N\rangle_{a_+}|0\rangle_{a_-} + i \sin \theta |0\rangle_{a_+}|N\rangle_{a_-} \tag{75}
\]

where \( \theta \) is proportional to the interaction time \( t_a \). The solution for \( H^{(B)}_{NL} \) is similar, with interaction time \( t_b \). This implies one can map the microscopic qubit Bell experiment involving qubits \( \{|1\rangle|0\rangle, |0\rangle|1\rangle \} \) onto a mesoscopic one involving the qubits \( \{|N\rangle|0\rangle, |0\rangle|N\rangle \} \), distinct by \( N \) quanta at each site. The settings \( \theta \) and \( \phi \) correspond to the interaction times \( t_a \) and \( t_b \). For all relevant choices of settings \( \theta \) and \( \phi \), the Bell violation can be obtained where the measurement makes only the distinction between \( 0 \) or \( N \) photons at each site. Leggett and Garg’s macroscopic realism is applied at the level of \( N \) quanta, to assert that the system \( A \) or \( B \) is predetermined to be in a state giving the outcome of \( \sim N \) or \( \sim 0 \). The noninvasive measurability assumption of Leggett and Garg is justified in this case as an \( N \)-scopic Bell locality, that the measurement on system \( B \) cannot induce a change of \( \sim N \) to the outcomes at \( A \) (and vice versa). The predictions for the violations were confirmed numerically for up to \( N = 100 \). This gives a rigorous prediction for violation of Leggett-Garg’s macrorealism at the level of \( \sim 100 \) quanta.

Thenabadu et al continued to confirm violation of a macroscopic Bell-CHSH inequality, using the macroscopically distinct outcomes provided by multi-component entangled superpositions of coherent states \( |\alpha_0\rangle, |\alpha_0 e^{i\theta}\rangle \), for distinct and fixed \( \theta \) where \( \alpha_0 \) is real \[270\]. At each of two sites, the two outcomes for the measurement of the sign of the quadrature amplitude \( X \) correspond to macroscopically distinct states, as \( \alpha_0 \to \infty \). In this case, the unitary rotation corresponding to the choice of measurement at each site is achieved using the interaction \[201\]

\[
H_{NL} = \Omega \hat{n}^k \tag{76}
\]

This is applied independently at \( A \) and \( B \), where \( k = 2 \). Here, \( \Omega \) is a constant and \( \hat{n} \) is the field mode number operator. For certain times \( t \), the system in the coherent state \( |\alpha_0\rangle \) evolves to a superposition of the two states, \( |\alpha_0\rangle \) and \( |-\alpha_0\rangle \), and the application of the Leggett-Garg premise of macroscopic realism at those times. The measurement settings \( \theta \) and \( \phi \) correspond to those certain times of interaction, \( t_a \) and \( t_b \). For all relevant choices of settings \( \theta \) and \( \phi \), the Bell violation can be obtained where the measurements make only the distinction between a negative or positive value of amplitude \( X_A \) (or \( X_B \)), where these values are increasingly macroscopically separated in phase space, as \( \alpha_0 \to \infty \). The noninvasive measurability assumption of Leggett and Garg is justified in this case as a macroscopic Bell locality, that the measurement on system \( B \) cannot make a macroscopic change to the outcomes at \( A \) (and vice versa). Violations of the Leggett-Garg-Bell inequalities were predicted in the macroscopic regime, for arbitrarily large \( \alpha_0 \) \[270\].

In fact, it is possible to obtain a direct mapping between microscopic and macroscopic versions of the Bell-Leggett-Garg experiments, involving spin qubits \( \{|\uparrow\rangle, |\downarrow\rangle \} \) and macroscopic qubits \( \{|\alpha_0\rangle, |\alpha_0 e^{i\theta}\rangle \} \) respectively \[253, 271\]. For large \( \alpha_0 \), the two coherent states are orthogonal and one may define Schwinger spin measurements \[48\] as \( \hat{S}_z = \frac{1}{2} |\alpha_0\rangle \langle \alpha_0 | - |\alpha_0\rangle \langle -\alpha_0 | \) and \( \hat{S}_x = \frac{1}{2} (|\alpha_0\rangle \langle -\alpha_0 | + |\alpha_0\rangle \langle \alpha_0 |) \). Similar observables were considered by Wang et al \[328\]. The authors Thenabadu and Reid consider two sites prepared in the entangled cat state \[41\], and propose a macroscopic Bell violation using the macroscopic qubits \( |\alpha_0\rangle \) and \( |-\alpha_0\rangle \), and \( |\beta_0\rangle \) and \( |-\beta_0\rangle \), at each site. In this case, the local interaction that brings about the unitary rotations for certain crucial values of \( \theta \) and \( \phi \) is realized by the nonlinear Hamiltonian \[476\] with \( k = 4 \) \[201, 325\]. The systems evolve independently at each site according to the interactions \( H^{(A)}_{NL} \) and \( H^{(B)}_{NL} \), for times \( t_a \) and \( t_b \) respectively. At site \( A \), for certain interaction times \( t_a = t_\theta \), a system prepared in a coherent state \( |\alpha_0\rangle \) evolves to the superposition

\[
|\alpha_0\rangle \to e^{-iH_{NL}t/\hbar} |\alpha_0\rangle = e^{-i\theta}(\cos \theta |\alpha_0\rangle + i \sin \theta | -\alpha_0\rangle) \tag{77}
\]

where \( \theta = t_\theta /2 \). For \( k = 4 \), the result is valid for interaction times \( t_\theta = m\pi /8 \) where \( m \) is a non-negative integer. Similarly at \( B \), the system prepared in \( |\beta_0\rangle \) evolves to \( e^{-i\phi}(\cos \phi |\beta_0\rangle + i \sin \phi | -\beta_0\rangle) \), where \( \phi = t_\phi /2 \) after a time \( t_b = t_\phi \). The solutions imply that for the system prepared initially in the two-mode cat state \[41\] with \( \alpha_0 = \beta_0 \), violations of Bell-CHSH inequalities \[8\] will be obtained for the choice of measurement settings \( t_a \) and \( t_b \) corresponding to \( \theta \) and \( \phi \) as given for \[8\]. It is evident that all the measurements are macroscopic, because the outcomes of \( X \)
for each measurement setting distinguish the amplitudes \( +\alpha_0 \) and \(-\alpha_0 \), as associated with the macroscopic qubit. Violations of the Leggett-Garg-Bell inequalities were thus predicted in the macroscopic regime, for arbitrarily large separations of outcomes of order \( \sim \alpha_0 \rightarrow \infty \), for all measurement settings.

IX. QUANTUM CORRELATIONS FOR ATOMIC SYSTEMS

Early experimental investigations of quantum correlations focused on two separated photonic systems, which could not be called macroscopic correlations. Another interpretation of the meaning of “macroscopic” is that the systems involved possess mass. Quantum correlations have been realized for atomic systems. An early experiment of Lamel-Rachti and Mittig investigated the quantum correlations for pairs of protons [330], although additional assumptions were necessary to infer quantum correlations [16].

A. Quantum correlations between two massive particles

A two-level atom is an example of a spin 1/2 system, the two levels corresponding to the states \(| \uparrow \rangle \) and \(| \downarrow \rangle \). A spin for each atom can therefore be assigned in terms of two internal atomic levels, the spin components being constructed from pseudo-Schwinger spins, where the \( a^\dagger \) and \( b^\dagger \) operators refer to the ‘creation’ of each level. This technique has been used to detect quantum correlations in atomic systems. In 2001, Rowe et al reported violations of Bell inequalities for two spin 1/2 systems given by the internal levels of two ions, in an ion trap [331]. Significantly, this experiment overcame detection efficiency loopholes for violation of a Bell inequality, but the correlations were observed without the spatial separation required for a rigorous Bell test. Hensen et al have since demonstrated conclusive loophole-free violations of Bell inequalities for electron spins in diamond separated by 1.3 km, and Rosenfield et al for two Rb atoms separated by 398 m [47–49].

The two atomic internal states of a single atom could not be called macroscopically distinct however, and give a weak gravitational interaction. Other origins of entanglement have been investigated. In 2019, Shin et al realized Einstein-Podolsky-Rosen-type correlations between spatially separated propagating atoms [332]. In their experiment, spin-entangled pairs of ultra-cold He atoms are created from two colliding spin-polarized Bose Einstein condensates. The range of settings for each particle was insufficient to claim a complete test of Bell’s theorem however. Bergschneider et al [333] similarly demonstrated entanglement between ultra-cold fermions in coupled wells. Here, the quantum correlation is for the momenta and positions of the atoms, modeled after the photon experiments of Rarity and Tapster [38] and related theoretical work that gives mechanisms for achieving Bell violations [334, 335]. Other experiments have used macroscopic entangled matter-waves to create interferometers operating beyond the usual classical limits [330] and correlated matter-waves to suppress atomic fluctuations [337]. As of yet, there is no violation of a Bell inequality reported.

B. Multi-atom quantum correlations and depth of entanglement

Quantum correlations were originally defined along the lines of EPR and Bell, as existing between separated systems and detected by local measurements on each system. Such correlations may falsify local hidden variable theories. Another strategy is to identify quantum correlations within the framework of quantum mechanics, by measuring collective observables. This approach has proved useful for certifying the existence of quantum correlations within a system of \( N \) atoms, where \( N \) is large. As a related example, the genuine entanglement of \( N \) photonic systems was confirmed for \( N = 4 \) in the optical experiment of Papp et al [102], by measuring collective operators involving all \( N \) systems at one converging site.

Sorensen et al showed that the entanglement of many atoms in a Bose Einstein condensate can be inferred from the observation of spin squeezing, as defined for the collective atomic spin operators [338, 339]. The total collective spin of \( N \) atoms is given as \( \hat{J}_\theta = \sum_{i=1}^{N} \hat{j}_\theta^{(i)} \) where \( \hat{j}_\theta^{(i)} \) is the spin of the \( i \)-th atom in the group of \( N \) atoms. The spin squeezing relation is determined by the uncertainty relation \( \Delta \hat{J}_x \Delta \hat{J}_y \geq \frac{|\langle \hat{J}_x \rangle \langle \hat{J}_y \rangle |}{2 \langle \hat{J}_z \rangle} \). Spin squeezing occurs when \( \Delta \hat{J}_x < |\langle \hat{J}_x \rangle|/2 \) [340, 341]. In experiments where \( N \) atoms are prepared in the same initial state \(| \uparrow \rangle \), and evolve according to the same Hamiltonian, one may express the collective system in terms of a spin \( J = N/2 \). The collective spin \( \hat{J}_z \) then gives the population difference between the two levels. A rotation from one basis to another is realized by a Rabi rotation, and prepares the atoms in a superposition of the two levels. Spin squeezing and hence multi-particle entanglement can be created using interactions modeled as \( H = \hat{J}_x^2 \) or \( \hat{J}_x^2 - \hat{J}_y^2 \) [340, 341]. This has confirmed possible in sophisticated multimode models of Bose Einstein condensates [342, 343].
If there is no entanglement between the $N$ atoms, then separability of the entire spin 1/2 system implies that the density matrix factorizes as

$$\rho = \sum_R P_R \prod_i^N \rho^{(i)}_R.$$  \hspace{1cm} (78)

Here, the $P_R$ are probabilities, $\sum_R P_R = 1$, and $\rho^{(i)}_R$ is the individual density operator for the $i$-th atom. This assumption leads to the result that the spin squeezing parameter defined as $\xi_N = \frac{\sqrt{27}}{\langle \Delta J \rangle}$ is constrained \[^{338}\]. The constraint can be understood in the following way. It is clear that for a spin 1/2 system, the maximum variance is 1/4, so that $(\Delta J_y)^2 \leq 1/2$, which gives a limit on the amount of spin squeezing: $(\Delta J_x) \geq |\langle J_x \rangle|$. For a fully separable system, we then see that for any decomposition of the density operator, the variance in $J_x$ has a lower bound, $(\Delta J_x)^2 \geq \sum_R P_R \sum_i^N (\Delta J_{x(i)}^2) \geq \sum_R P_R (\sum_i^N |\langle J_{x(i)} \rangle|^2)$. This follows because for the mixture, $(\langle \hat{O} \rangle) = \sum_R P_R (\hat{O})_R$ and the overall variance cannot be less than the weighted average of the variances of its components \[^{172, 346}\]. Here, we denote the average of an operator $\hat{O}$ for a system in the state $\rho_R$ by the subscript $R$: $(\langle \hat{O} \rangle)_R = Tr(\rho_R \hat{O})$ and $(\langle \hat{O} \rangle)^2_R = (\langle \hat{O} \rangle)_R^2 - (\langle \hat{O} \rangle^2)_R$. The Cauchy Schwarz inequality implies $(\sum_i^N \frac{1}{N}|\langle J_{x(i)} \rangle|)^2 \geq |\sum_i^N \frac{1}{N}|\langle J_{x(i)} \rangle|^2|^2$. Noting that $\langle J_z \rangle_R = \sum_i^N |\langle J_z \rangle_R|^2$, and that the Cauchy-Schwarz inequality also implies $(\sum_R P_R |\langle J_z \rangle_R|^2)^2 (\sum_R P_R \geq |\sum_R P_R |\langle J_z \rangle_R|^2|^2$, one finally obtains that for a fully separable state, the $N$-atom system satisfies

$$(\Delta J_x)^2 \geq |\langle J_x \rangle|^2/N .$$  \hspace{1cm} (79)

Where $\langle J_z \rangle = N/2$, this reduces to $(\Delta J_x)^2 < |\langle J_x \rangle|^2/N = N/4$, which is the condition for spin squeezing: $\xi_N = \frac{\sqrt{27}}{\langle J_z \rangle} < 1$ \[^{340, 341}\]. Hence, the observation of spin squeezing implies non-separability i.e. entanglement between at least one pair of atoms \[^{338}\].

The result was used by Esteve et al \[^{347}\] and Riedel et al \[^{348}\] to deduce entanglement in a Bose Einstein condensate of hundreds of atoms. However, the question arises as to whether this is truly a macroscopic effect relating to all $N$ atoms, since logically, the violation of the spin squeezing inequality can come from the entanglement of just one pair of atoms. The number of atoms genuinely involved in the entanglement is referred to as the “depth of entanglement”.

The concept of depth of entanglement was developed by Sorensen and Mølmer \[^{349}\]. They demonstrated that the minimum depth of entanglement can be inferred, if the spin squeezing is sufficiently extreme i.e. if the variance $(\Delta J_x)^2$ is reduced below a certain level. For systems of a finite dimensionality (corresponding to a fixed nonzero spin $J$) and where there is a nonzero $|\langle J_z \rangle|$, the amount of squeezing possible is limited i.e. $(\Delta J_x)^2$ cannot be zero. This is because the variance in $J_y$ cannot be infinite for finite $J$. However, as $J$ increases, the lower bound for $(\Delta J_x)^2$ approaches zero. Sorensen and Mølmer considered a spin $J$ system, and determined the minimum value of $(\Delta J_x)^2$ that is possible, for a given measured value of $\langle J_z \rangle$. The result is a function $F_J(\langle J_z \rangle)/J$. For each $J$, one can show

$$(\Delta J_x)^2/J \geq F_J(\langle J_z \rangle)/J.$$  \hspace{1cm} (80)

The curves $F_J$ are convex and monotonically increasing with $\langle J_z \rangle/J$, and, for a given $\langle J_z \rangle/J$, monotonically decreasing with $J$. This allowed the authors to derive the inequality that holds for $N$ separable systems of spin $J$: $(\Delta J_x)^2/NJ \geq F_J(\langle J_z \rangle)/N$. The inequality provided a calibration: for a given measured variance $(\Delta J_x)^2$, it is possible to determine $J_0$ such that $(\Delta J_x)^2/NJ_0 < F_J(\langle J_z \rangle)/N$. The conclusion is that the factorization breaks down, and that there exists a subsystem $\rho^{(i)}$ with a total spin greater than $J_0$. This implies a block of at least $n_0 = 2J_0$ mutually entangled atoms. The experiments of Gross et al \[^{350}\] measured the spin squeezing in a multi-well Bose Einstein condensate to infer the multi-particle entanglement involving at least $\sim 100$ atoms. Here, two hyperfine states of Rb act as the two modes of a nonlinear interferometer. A similar multi-particle entanglement was inferred by Riedel et al \[^{348}\], using atom-chip based interferometry.

Tura et al extended the approach that uses collective operators to infer the existence of Bell correlations within the ensemble of atoms \[^{351, 352}\]. This was measured by Schmied et al \[^{353}\] for a Bose Einstein condensate, and by Engelsen et al \[^{354}\] for ultra-cold but not Bose-condensed atoms, at higher temperatures. There was however no assessment of the collective number of atoms mutually sharing the Bell nonlocality. These measurements also involved the collective atomic spin-squeezing parameter.

In fact, a debate had arisen around how to interpret entanglement criteria when applied to the atoms of a Bose Einstein condensate (BEC) \[^{355, 358}\]. In a BEC, the atoms are identical bosonic particles which are indistinguishable.
and which obey the symmetrization principle. Super-selection rules apply for massive particles that exclude the possibility of superpositions of states with different atom number in a single mode. A resolution was put forward by Killoran, Cramer and Plenio [358], who gave a connection between the so-called particle entanglement and mode entanglement approaches. Using criteria based on super-selection rules, Cramer et al. [359] were able to quantify the large-scale entanglement of ultra-cold bosons in $10^5$ sites of an optical lattice. Dalton et al. [360–363] adopted second quantization to derive conditions based on super-selection rules for both entanglement and steering between the modes associated with the two atomic levels. This allowed the conclusion that the experimental observation of interference in a two-mode BEC interferometer is sufficient to imply entanglement [360] and steering [363] between the modes, since the modes are distinguishable. Moreover, the number of atoms collectively involved in the mode-entanglement could be quantified, using the measurable fringe visibility or higher moments [363]. A particular atom interferometer experiment was examined [360, 361], to infer 40,000 atoms genuinely involved in the two-mode steering [364].

In 2015, Islam et al. used quantum interference to directly measure the amount of entanglement in a lattice of ultracold bosonic atoms [368]. The measure was based on entanglement entropy. Experiments have now demonstrated macroscopic superpositions at the time-scale of everyday life [369], an entanglement of 3000 atoms with a non-positive Wigner function [370], and 16 million genuinely entangled atoms entangled through their electronic states in a solid environment [371]. The work of Frowis et al. establishes the extraordinary level of genuine entanglement based on an entanglement depth witness [371]. In another approach, the quantum propagation of an attractive Bose gas soliton was analyzed. This showed theoretically that such a system would evolve dynamically to have nonlocal pair correlations, due to the creation of a superposition of different types of fragments, caused by quantum instabilities not present in the usual classical analysis [372].

C. EPR entanglement, steering and multipartite entanglement between atomic groups

In 2001, Julsgaard et al. experimentally demonstrated entanglement between two macroscopic spatially-separated ensembles of $\sim 10^{12}$ Cesium atoms at room temperature [373–376]. Their method identifies the spin 1/2 system as the two-level atom associated with an internal hyperfine atomic transition. In [372], macroscopic spin operators are then defined for each ensemble, and the correlations confirmed using variance criteria applied to collective macroscopic spin observables, $J_X$ and $J_Y$. The entanglement is created by first transmitting an off-resonant polarized laser pulse through two atomic ensembles with opposite mean macroscopic spins, in order to correlate the spins $J_X$ of each ensemble ($i = 1, 2$). The process is then repeated with a second pulse to correlate the atomic spins $J_Y$ [377, 378]. The final simultaneous correlation of both $J_X$ and $J_Y$ for the ensembles gives the correlations necessary for an EPR entanglement and (if strong enough) for an EPR paradox. The correlation is inferred by the measurements of $J_{X1} + J_{X2}$ and $J_{Y1} + J_{Y2}$, where $J_{X1}$ and $J_{Y1}$ refer to the spins of the ensemble labelled $i$. These measurements are made on the outputs of the polarized pulses, which according to the theory have values for the Stokes observables that are correlated with the spin sums. The experiment reported entanglement between the ensembles using a variance measure similar to type [39], but the correlation did not satisfy [34] as necessary for an EPR paradox (refer [379]).

To explore quantum correlations in the strictest sense, it is necessary to obtain evidence of correlations where the values of the observables at each spatially separated site are obtained by a local measurement, as in the Bell tests. This motivated theoretical investigations which analyzed how to achieve EPR-type quantum correlations at a mesoscopic level, between groups of atoms [380, 384]. In a step towards this goal, the experimental observation of the entanglement between two distinct groups of atoms in a BEC was reported by Gross et al. [382]. Here, the correlations were detected using the equivalent of an optical homodyne technique for each system, as described for continuous variable measurements in Section V. The atomic homodyne involved a second group of atoms that form the local oscillator [380]. In the atom-optics equivalent to the photonic scheme, the beam splitter interaction that combines the local oscillator with the signal field is carried out with a Rabi rotation. Using atomic homodyne detection, the two-mode squeezing criterion of type [39] allowed an inference of entanglement between the two systems of atoms. While the two groups were distinguishable, there was limited spatial separation. The stronger correlations required for an EPR paradox and for steering were generated experimentally by Peise et al. [387]. The correlations were verified using the atomic homodyne method and the EPR criterion [41], although spatial separation of the atomic groups was limited.

A significant advance came in 2018 from three experiments which confirmed quantum correlations between the spatially separated atomic clouds of a split Bose Einstein condensate (BEC). Entanglement, an EPR paradox and EPR steering were detected between spatially separated groups of several hundreds of atoms [388–391]. Kunkel et al. demonstrated entanglement and bipartite EPR steering between the clouds of hundreds of Rb atoms in an expanding BEC [388]. The steering was measured using spatially resolved spin read-outs, and properties such as monogamy of steering were also investigated. Kunkel et al. certified the genuine multipartite entanglement of five spatially separated
mesoscopic groups of atoms, using witnesses constructed by modifying techniques applied previously to continuous variable systems [391, 392]. Fadel et al [388] used high-resolution imaging to infer EPR steering based on the spin correlations between spatially separated parts (∼100 of atoms) of a spin-squeezed Bose-Einstein condensate generated on an atom chip. Variance criteria were also used to infer the correlations. Lange et al [390] similarly demonstrated entanglement between two spatially separated mesoscopic clouds of hundreds of Rb atoms, obtained by splitting an ensemble of ultra-cold identical particles prepared in a twin Fock state. The method of generation of entanglement is analogous to that described in Section V, where a squeezed field combines with a vacuum state on a beam splitter to create EPR entangled outputs. These experiments give evidence of entanglement distributed over several hundred atoms. It remains to rigorously quantify the number of atoms genuinely entangled from each group, but a step in this direction was provided in [179]. Arguments validating a large depth of entanglement can also be made based on the indistinguishability of the atoms of the BEC [388, 390]. As of yet, it remains to demonstrate Bell nonlocal correlations between spatially separated groups of atoms.

X. QUANTUM CORRELATIONS IN OPTOMECHANICS

The question of the existence of quantum correlations in systems that are macroscopic by mass is addressable in the field of optomechanics. Quantum correlations in optomechanics are expected to play a significant role in fundamental tests of quantum mechanics. The idea that separated quantum systems may decohere was proposed by Furry [393], as a possible resolution of the Einstein-Podolsky-Rosen paradox [1]. Spontaneous decoherence is not observed for low-mass systems like photons [40], electrons [47] or atoms [332], where entanglement has been verified for separated masses. The idea that gravitational effects may be involved in causing quantum decoherence [395, 397] or changes in commutation relations [398], has led to substantial interest in quantum superpositions of more massive objects than atoms [392]. Experimental success in cooling massive optomechanical systems to their quantum ground state [400] has resulted in the generation of entanglement in optomechanics. The simplest theoretical schemes to generate quantum correlations in optomechanics produce entanglement between the optical and mechanical subsystems in a single optomechanical system. Entanglement between two separated optomechanical systems requires a more complex approach, involving at least two mechanical oscillator subsystems. Since one of the motivating factors in this work is to test for the combined effects of quantum mechanics and gravity, experiments that combine both entangled massive oscillators and a controllable degree of spatial separation appear to be of greatest interest. Recently there has been an interest in entanglement resulting directly from gravitational interactions, as a possible direct test of quantum gravity, explained below.

Typical experimental masses in cryogenic optomechanical experiments are $m \sim 50 \text{pg}$, using aluminum cantilevers or capacitor plates for the mechanical sub-system, together with a superconducting microwave LC circuit for the ‘optical’ component. The number of atoms is therefore of order $n_a \sim 10^{12}$, which means that these devices are both macroscopic and massive. Even larger optomechanical systems exist in the form of the LIGO gravitational wave detectors, with mirror masses of over $10 kg$, so that $n_a \sim 10^{26}$. However, these are generally at room temperatures, making it difficult to observe quantum effects.

A. Entanglement between modes in an optomechanical system

We first review the generation of entanglement within an optomechanical system. These schemes typically rely upon the interaction Hamiltonian between the optical $a$ and mechanical $b$ modes due to radiation pressure. The fundamental Hamiltonian has the form [401–403]

$$H/\hbar = \omega_0 \hat{a} \hat{a}^\dagger + \omega_m \hat{b} \hat{b}^\dagger + \hbar \chi \hat{a} \hat{a}^\dagger (\hat{b} + \hat{b}^\dagger),$$

(81)

where $\omega_0, \omega_m$ are the optical and mechanical oscillator resonance frequencies respectively, while $\chi$ is the nonlinear coupling strength between modes $a$ and $b$. Here, $\hat{a}$ and $\hat{b}$ are the boson destruction operators for the optical and mechanical modes respectively. The Hamiltonian generates a unitary transformation operator that contains a nonlinear Kerr term $(\hat{a}^\dagger \hat{a})^2$ [404]. Mancini et al [404] and Bose et al [405] obtained different entangled states between the optical and mechanical modes, depending on specific times in the system evolution. Similar couplings have been engineered between two optical modes in a superconducting device [406], but these are less interesting from the viewpoint of gravitational effects. Typical experimental values in cryogenic nano-mechanical systems are $\omega_0/2\pi \sim 100 \text{GHz}$, $\omega_m/2\pi \sim 10 \text{MHz}$ and $\chi/2\pi \sim 100 \text{Hz}$ [407]. These are microwave electromagnetic frequencies, and the systems are cooled to temperatures of around $T \sim 10 \text{mK}$ to reduce thermal excitations of the oscillator. There is also some optical and mechanical damping due to reservoirs, causing optical and mechanical decays at typical rates.
of $\gamma_0/2\pi \sim 500\text{kHz}$, $\gamma_m/2\pi \sim 50\text{Hz}$, respectively. In the weak coupling regime, where $\chi$ is small compared to the optical damping rates, the coupling between the optical and mechanical modes is enhanced by an external driving field. Depending on the driving field frequency $\omega_d$, the optomechanical system behaves differently.

For a driving field that has a frequency such that $\omega_d = \omega_m + \omega_\phi$, where $\omega_m$ and $\omega_\phi$ are the mechanical and optical mode frequencies respectively, the driving field is said to be blue detuned, defined by the detuning parameter $\Delta \equiv \omega_o - \omega_d = -\omega_m$. In this case the blue-detuned optomechanical system has an effective interaction Hamiltonian

$$H_{int}^b = \hbar g \left( \hat{a} \hat{b} + \hat{a}^\dagger \hat{b}^\dagger \right)$$

(82)

where $g = \chi \sqrt{N}$ is the effective coupling strength due to the driving field, for an internal stored microwave photon number of $N$. This Hamiltonian is known to generate entanglement between $a$ and $b$ modes [415], and here is the physical source of the entanglement between the optical and mechanical modes. The dynamics from this effective optomechanical Hamiltonian in the presence of noise and losses was studied by Vitali et al [408, 409], Genes et al [411], and Hofer et al [411] using linearized Langevin equations. Hofer et al and Vanner et al analyzed pulsed schemes for the generation of non-classical states [411, 412].

The scheme of Hofer et al is based on the description given above, where a blue-detuned pulse first entangles the optical and mechanical modes [411]. The entanglement verification process requires the readout of the mechanical mode, which is is achieved by applying a red-detuned pulse that transfers the mechanical state to an optical state where measurements are made using optical homodyne. A red-detuned driving field ($\Delta = \omega_o - \omega_d = \omega_m$) leads to an effective interaction Hamiltonian

$$H_{int}^r = \hbar g \left( \hat{a} \hat{b}^\dagger + \hat{a}^\dagger \hat{b} \right)$$

(83)

which was shown to enable the transfer of a quantum state between the optical and mechanical modes by Zhang, Peng and Braunstein [413]. In the theory of Hofer et al, dissipation and noise are included and the optomechanical system evolves according to the Langevin equations (in the rotating wave approximation)

$$\dot{\hat{a}}_c = -\kappa \hat{a}_c - ig \hat{a}^\dagger_m - \sqrt{2} \kappa \hat{a}_{in}$$

$$\dot{\hat{a}}_m = -ig \hat{a}_c^\dagger$$

(84)

where $\hat{a}_c, \hat{a}_m$ are the boson operators for cavity optical and mechanical modes respectively, $g$ is the effective optomechanical coupling strength, $\kappa$ is the cavity decay rate, and $\hat{a}_{in}$ contains the quantum noise entering the cavity. In the limit of large cavity decay rate $\kappa \gg g$, the adiabatic approximation allows the cavity optical and mechanical modes to have the following expressions:

$$\hat{a}_c (t) \approx -\frac{g}{\kappa} \hat{a}^\dagger_m (t) - \sqrt{\frac{2}{\kappa}} \hat{a}_{in} (t)$$

$$\hat{a}_m (t) \approx e^{Gt} \hat{a}_m (0) + i \sqrt{2G} e^{Gt} \int_0^t e^{-G\tau} \hat{a}^\dagger_{in} (s) \, ds$$

(85)

where $G = g^2/\kappa$. In particular, the cavity optical mode satisfies the input-output relation, given by $\hat{a}_{out} = \sqrt{2\kappa} \hat{a}_c + \hat{a}_m$, where $\hat{a}_{out}$ is the output of the cavity [414]. This relation is useful as it is the output field from the cavity that is usually being measured. Hofer et al also defined normalized temporal light modes: $A_{in} = \sqrt{\frac{2G}{1-e^{-2\kappa}}} \int_0^t e^{-G\tau} \hat{a}_{in} (t) \, dt$ and $A_{out} = \sqrt{\frac{2G}{1-e^{-2\kappa}}} \int_0^t e^{G\tau} \hat{a}_{out} (t) \, dt$ and the mechanical modes $B_{in} = \hat{a}_m (0)$, $B_{out} = \hat{a}_m (\tau)$, showing that it is the quadrature amplitudes of these modes that become entangled. They showed that for $G\tau \to \infty$, the mechanical state is perfectly transferred to the optical mode, apart from a phase shift.

Entanglement between the optical and mechanical modes in an optomechanical system was demonstrated experimentally by Palomaki et al [407], following the scheme of Hofer et al. They realized the optomechanical system using an electromechanical circuit where an LC oscillator corresponds to the optical mode and one of the capacitor plates is moveable, behaving like a mechanical mode. The entanglement between modes is then generated using the interaction Hamiltonian [82], with a blue-detuned microwave field. The experiment measured the quadrature amplitudes of the entangled optical ($\hat{X}_1, \hat{P}_1$) and mechanical modes ($\hat{X}_2, \hat{P}_2$), where $\hat{X}_i = \hat{a}_i + \hat{a}_i^\dagger$ and $\hat{P}_i = (\hat{a}_i - \hat{a}_i^\dagger)/i$. The measured statistical moments of these quadrature amplitudes allow the inseparability parameter $\Delta_{sum}$ to be determined, as defined in [41] with $g = g' = 1$. The stronger criterion [59] allowing $g, g' \neq 1$, although not measured in the Palomaki experiment, was predicted from simulations to a give a more sensitive measure [415].
Since linearization can fail when there are strong laser driving fields, quantum phase-space simulations without any linearization approximations were carried out by Kiesewetter et al. \[415\]. The theory uses the exact positive-P representation \[410\] to transform the nonlinear quantum master equation into stochastic equations, which can be numerically simulated. This allows the full multi-mode output fields to be calculated, giving excellent quantitative agreement with the pulsed optomechanical entanglement experiment of Palomaki et al and justifying the linearization regime. Nonlinear effects were studied in Teh et al. \[417\], and can be significant due to the strong pump fields that are often used in experiments.

### B. Entanglement between optomechanical systems

Different methods can be used to entangle two massive systems, but the common feature is to entangle by interacting the masses with an optical field. For example, in the scheme of Hofer et al, the outgoing red pulse can be propagated through a second oscillator, and the state of the pulse transferred onto the second oscillator, thus entangling both oscillators.

In 2018, two experiments reported entanglement between oscillators \[418\]. \[419\]. Discrete variable entanglement between two mechanical modes separated by \(\sim 20\) cm was demonstrated in the experiment of Riedinger et al \[418\]. Here, a pump field is sent into one of the two optomechanical systems, which creates a phonon-photon pair in one of the systems. The photon leaks out of the optomechanical system and is subsequently sent into a beam splitter and detected. As the whole process does not provide information on which optomechanical system the phonon-photon pair is created, the detection of the photon at the beam splitter output heralds the mechanical mode into a superposition state of one phonon in mechanical oscillator \(A\) or \(B\), given by \(|\Psi\rangle = (|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B) / \sqrt{2}\). Here \(|1\rangle_A|0\rangle_B\) is a state with a single excitation in the mechanical oscillator \(A\), with the mechanical oscillator \(B\) is in its ground state; and \(|0\rangle_A|1\rangle_B\) is similarly defined. An entanglement witness involving second-order coherences was used to certify the entanglement.

Ockeloen-Korppi et al demonstrated entanglement between two electromagnetic systems \[419\]. The method was based on the idea of reservoir engineering to prepare the two cavity-coupled mechanical oscillators into a steady state that is entangled, as proposed by Woolley and Clerk \[420\], Tan, Li and Meystre \[421\], and Wang and Clerk \[422\]. Here, two mechanical modes with frequencies \(\omega_{m,1}\) and \(\omega_{m,2}\) are coupled to a single microwave cavity mode with frequency \(\omega_c\). Two driving fields are applied such that the effective interaction Hamiltonian has the form

\[
H_{eff} = g_+ \left[ (\hat{a} + \hat{b}) \hat{c} + H.c. \right] + g_- \left[ (\hat{a} + \hat{b}) \hat{c}^\dagger + H.c. \right],
\]

(86)

which consists of a sum of the quantum state transfer term and entanglement generation term. H.c. refers to hermitian conjugate. By tuning the amplitude of the driving fields, a two-mode squeezed state was generated between the two mechanical modes. Barzanjeh et al also carried out an experiment in a similar electromechanical setting using two microwave fields \[423\]. The entanglement is in the continuous variable quadratures of the modes, and the inseparability parameter \[10\] with \(g = g'\) is used to verify the entanglement.

Entanglement between two optomechanical systems has also been recently reported in the electromechanical experiment by Kotler et al \[424\]. Based on the theory similar to previous works \[403\], \[321\], \[422\], \[425\], a blue-detuned pulse is first applied to entangle the cavity with a drum, producing a quantum correlated photon-phonon pair. A red-detuned pulse is subsequently applied to transfer the photon state to a phonon state in a second drum, and hence realizing the generation of entanglement between the phonon-phonon pair.

More recent work by Mercier de Lepinay et al \[426\] generated a two-mode squeezed state using four driving fields. This method has the feature that it involves a ‘quantum-mechanics free subsystem’, a manifold where commutation relations are nearly zero. This is closely related to planar spin squeezing \[224\], \[383\], which has been achieved in macroscopic atomic spin systems \[427\].

The entanglement of an optomechanical system with an atomic spin ensemble was realized by Thomas et al \[428\], based on the proposal by Hammerer et al \[429\]. The light-spin interaction is first established by sending a polarized light beam through a sample of \(10^9\) atoms with a collective macroscopic spin along a certain direction, similar to the experiment of Julsgaard et al \[373\]. The propagated light is then sent into a cavity that contains a mechanical dielectric membrane, where the mechanical mode interacts with the light so that the mechanical mode becomes entangled with the atomic ensemble.

Alternatively, one can use a direct state transfer from an optical to mechanical system, using the red-detuned Hamiltonian \[430\]. \[413\]. The red-detuned Hamiltonian (for vacuum inputs) does not generate entanglement, and an external source of optical entanglement is therefore required. The two entangled light fields generated by parametric oscillation can be sent to two spatially separated optomechanical systems, where the entangled optical modes are transferred to the mechanical modes via the red-detuned effective Hamiltonian. This has the advantage in principle
that because the optical entanglement has been generated for large spatial distances, one may be able to obtain an arbitrary separation, for tests of massive entanglement at different distances, as in proposals to test Furry’s hypothesis \[430\]. This type of scheme was first studied using linearization and a steady state approach by Zhang et al \[413\]. However, to study the possible dynamics of gravitational decoherence, a pulsed entanglement approach is important \[412\], and a full dynamical study without a linearization approximation was numerically carried out by Kiesewetter et al \[430\]. A similar treatment gave a proposal to transfer a cat state from an optical to mechanical mode \[431\], which could in principle be extended to generate entangled cat states in optomechanics. Vanner has proposed another mechanism to generate cat states, using a conditional pulsed measurement scheme \[432\]. A further analysis has been given by Hoff et al \[433\].

C. EPR steering

A proposal to realize the correlations of an EPR paradox through radiation pressure in optomechanics was first put forward by Giovannetti et al \[434\]. For a pulsed system, the correlations of the EPR paradox and of EPR steering were investigated by He and Reid \[435\]. These authors first considered the generation of entanglement between the optical and mechanical modes of a single optomechanical system as considered by Hofer et al. The entangled state is characterized by a squeezing parameter \(r\) that is proportional to the coupling strength between the optical and mechanical modes. In order to quantify the steering of the entangled state, they use the steering criterion \[34\], which becomes

\[
E_{m|c} = \Delta \left( \hat{X}_m - g_x \hat{P}_c \right) \Delta \left( \hat{P}_m + g_p \hat{X}_c \right) < 1
\]

as given by \[35\]. Here, \(\hat{X}_m, \hat{P}_m\) are the quadratures of the mechanical mode, and \(\hat{X}_c, \hat{P}_c\) are the quadratures of the optical mode. The \(g_x, g_p\) are real numbers that can be chosen to minimize \(E_{m|c}\). Steering of the mechanical mode by optical mode is confirmed when \(E_{m|c} < 1\). The presence of thermal noise degrades the quantum correlation. Using this steering criterion, which is necessary and sufficient for a two-mode Gaussian system, the authors evaluated the minimal squeezing strength required to show steering for a given thermal occupation number \(n_0\). They found that the required squeezing strength for steering does not grow indefinitely with \(n_0\) but asymptotically approaches \(r = 0.5 \ln 2\) as \(n_0 \to \infty\). On the other hand, using the steering criterion \(E_{c|m} < 1\) for the steering of optical mode by the mechanical mode, no such minimum squeezing strength is required to demonstrate steering, and there is always steering of the optical mode by the mechanical mode as long as \(r \neq 0\). The authors argued that the steering of the mechanical mode was of interest, because then the “elements of reality” considered by Einstein-Podolsky-Rosen related to the massive object, rather than to the field. The model used by the authors is essentially that of a two-mode squeezed state with asymmetric reservoirs for the two modes, and illustrated the sudden death of EPR steering that occurs with a certain threshold amount of thermal noise on the steered system \[156\]. A multimode model appropriate for a pulsed treatment was put forward by Kiesewetter et al and supported these predictions \[414\].

Next, the authors propose to entangle two oscillators by first entangling the mechanical mode of mechanical oscillator \(M1\) and cavity optical mode as before. The cavity optical mode is then transferred to the mechanical mode of a second mechanical oscillator \(M2\), and hence entangling \(M1\) and \(M2\). They study the steering in the entanglement of two optomechanical systems as a function of thermal noise using the steering criterion of the form Eq. \[87\] and provide the squeezing strength threshold required to observe steering. The same steering criterion is used in the work of Kiesewetter et al \[430\] where steering is studied for different mechanical modes storage times, as well as for thermal noise.

Other proposals using the steering criterion are given by Sun et al \[436\] in a different setting. In that work, a dielectric membrane is placed in a cavity that divides the cavity into two independent cavity modes. Two pump fields enter these cavity modes and create entanglement among the two cavity and mechanical modes. By varying the phase difference between the two pump fields, the degree of bipartite entanglement between the mechanical mode and one of the optical mode varies. This is then extended to the case of steering where the condition on the phase difference required to show steering is established.

For Gaussian states, the criterion \[34\], and hence \[57\] with the optimally selected values of \(g\) and \(g'\), has been shown to be necessary and sufficient for steering in two-mode Gaussian systems \[168\]. This means it is possible to construct a measure of such steering. A Gaussian steering quantifier \(G^{A\rightarrow B}_{\sigma_{AB}}\) that quantifies the steerability of mode \(B\) by mode \(A\) was put forward by Kogias et al \[437\], and has the form

\[
G^{A\rightarrow B}_{\sigma_{AB}}(\sigma_{AB}) = \max \left\{ 0, \frac{1}{2} \ln \frac{\det A}{\det \sigma_{AB}} \right\}
\]

\[88\]
where $\sigma_{AB}$ is the covariance matrix of the bipartite system $AB$, and $\det A$ is the determinant of the covariance matrix of the subsystem $A$. This steering quantifier is shown to be related to the steering parameter $E_{B\mid A}$ in Eq. (87) via the expression $(E_{B\mid A})_{\text{opt}} = e^{-2G^{A\rightarrow B}}$, where here $(E_{B\mid A})_{\text{opt}}$ is the value of $E_{B\mid A}$ for optimally chosen $g$ and $g'$ as given by (35). However, the Gaussian quantifier only applies under the assumption of Gaussian states, whereas the condition (34) (and hence (87)) holds for all states, as a witness to EPR steering and as a one-sided device-independent witness to entanglement (167). While the steering parameter $E_{B\mid A}$ has the advantage of clear operational interpretations, the steering quantifier $G^{A\rightarrow B}$ allows some mathematical properties such as convexity, additivity and monotonicity under quantum operations to be readily proven (437). This quantifier is studied in the work of Tan and Zhan (438), and El Qars et al (439). Zhong et al use the EPR parameter $E_{B\mid A}$ and the Gaussian steering quantifier to study one-way EPR steering between two macroscopic magnons located in optically driven cavities (440).

D. Bell nonlocality

Schemes that generate Bell nonlocality in the optomechanical systems may follow closely the schemes that generate entanglement. The difference lies in the quantum correlation verification process. The variance entanglement criteria used for entanglement verification cannot be applied to demonstrate Bell nonlocality. Rather, a CHSH inequality $|B| = \left| E(a, b) - E(a, b') + E(a', b) + E(a', b') \right| \leq 2$ is to be checked, where $|B| > 2$ is required to demonstrate Bell nonlocality. Here, $E$ is a correlation function and it is a function of different settings $a, a', b, b'$.

In the theoretical work by Vivoli et al (441), Hofer, Lehnert and Hammerer (442), and Manninen et al (443), entanglement is generated between the mechanical and optical modes using the blue-detuned driving field as described in the previous section. The difference in those works lies in the verification of Bell nonlocality of the entangled state generated. Vivoli et al and Hofer et al propose to coherently displace the optical fields before measuring the photons, similar to earlier continuous variable Bell approaches. The amplitudes of coherent displacement $\alpha, \alpha', \beta, \beta'$ constitute the different settings in the correlation function $E(\alpha, \beta)$, which are then used to check against the CHSH inequality.

Yet another possible choice of measurement settings is considered in the work of Manninen et al (444). In that work, quadrature phases of the modes are measured using homodyne detection and the different settings are the phases of the local oscillator in the homodyne detection scheme. This measurement scheme has been experimentally shown to violate a Bell inequality in an optical system (218). Finally, although not a Bell test per se, we mention that other quantum paradoxes, including a delayed choice wave-particle duality experiment and a Leggett-Garg test of macro-realism, have been theoretically proposed for mechanical resonators (316, 444).

E. Experimental optomechanical Bell test

The first experimental Bell test involving an optomechanical system has been carried out by Marinkovic et al (445). Here, the origin of the entanglement involves two-particle interference between four photonic modes as in the earlier photonic interferometric proposals (17, 38). This experiment measures coincidences in two detectors using photon counting and is thus a discrete variable Bell test, unlike the theoretical schemes in the previous section, where continuous variables are measured. The physical system of Marinkovic et al involves two nano-mechanical resonators with $10^{10}$ atoms, whose entanglement is mediated by photons. As discussed previously, a blue-detuned pulse is used to generate entanglement between the optical and mechanical modes.

However, different from other schemes, this pulse is not sent directly into the optomechanical system. Rather, the pulse is first sent into an interferometer with a beam splitter where the output from it is then sent into either one of the identical optomechanical systems that is located in each arm of the interferometer. An electro-optical modulator is present in one of the interferometer arms to induce a phase difference $\phi_b$ between the two arms. The optomechanical system that receives the pulse will have entanglement between its optical and mechanical modes where a cavity photon and phonon are created. Up to this point, an entangled state between the optical and mechanical modes is generated in one of the optomechanical systems in this interferometer.

In the optomechanical system that contains the entangled state, the cavity photon leaks out of the cavity and goes through a beam splitter before being detected by photon detectors. The photon detection implies the existence of a single phonon, while the beam splitter before the photon detection erases the information on where the detected photon is originated from. This puts the state of the whole system into a superposition of single phonon state in one optomechanical system or the other. In other words, the photon detection heralds an entangled state between the mechanical modes of two optomechanical systems.

A red-detuned pulse is also sent into the interferometer some time after the blue-detuned pulse. Similarly, the phase shift $\phi_r$ in one of the interferometer arms is controlled by an electro-optical modulator. This red-detuned pulse
transfers the mechanical state into the cavity optical state that leaks out of the optomechanical system, which is then measured just as in the case for blue-detuned pulse.

The observables measured are the number of coincidences \( n_{ij} \) in the two detectors when both the blue and red detuned pulses are sent into the interferometer. Here, \( i, j (i, j = 1, 2) \) correspond to the detection when the blue and red detuned pulses are sent, respectively. For instance, \( n_{12} \) is the number of times the blue drive triggers a photon detection at detector 1 and a subsequent detection at detector 2 by the red drive. The correlation function \( E(\phi_b, \phi_r) \) is related to these coincidences by

\[
E(\phi_b, \phi_r) = \frac{n_{11} + n_{22} - n_{12} - n_{21}}{n_{11} + n_{22} + n_{12} + n_{21}}
\]  

where \( \phi_b \) and \( \phi_r \) are the phase difference acquired in the arm of the interferometer when the blue and red detuned pulses are sent, respectively. Marinkovic et al obtain \( B = 2.174^{+0.041}_{-0.042} \) and show a Bell violation by more than 4 standard deviations. While a significant step forward, this test requires a fair-sampling assumption, and also does not use spatially separated detectors. Moreover, the measurement settings are for the phases of the photon part of the interferometer, rather than the mechanical oscillator amplitudes. The settings are selected prior to the photon entering the mechanical oscillator, which suggests that the hidden variables being tested relate to the photon rather than oscillator fields.

\section{F. Gravitational quantum entanglement}

In the proposals discussed and experimentally demonstrated above, the entanglement is generated through optomechanical interactions. The entanglement of two oscillators is achieved by interacting with an optical field, the quantum nature of which is essential to the mechanism. These methods may potentially test for gravitationally induced decoherence, owing to the presence of a massive test particle.

A more direct test of the quantum features of gravity would come from the direct gravitational coupling of two massive test particles, as proposed by Marletto and Vedral \cite{marletto2015quantum}, and Bose et al \cite{bose2015quantum}. There have been a number of related proposals, including \cite{ali2015quantum, ali2016entanglement}. The basic idea is to create a superposition state in an optical field, and transfer this to a massive mirror, which can interact gravitationally with a second massive mirror. The result is a quantum entanglement of two mirrors obtained through gravitational interactions, and hence implicitly involving a superposition of two distinct metric tensors, thus providing evidence for quantum gravity. This is a quantum version of the Cavendish experiment \cite{cavendish1798experiments}, which measured the gravitational constant \( G \), and hence the density of the earth.

Due to the extreme weakness of gravitational forces, the original experiment was by no means trivial. Generating quantum entanglement purely through gravitational interactions is considerably more difficult, and has not been achieved as yet. However, with improvements in technology since 1798, one may hope that a quantum Cavendish experiment is not impossible in future. This might require test masses as large as the kilogram-mass LIGO mirrors, or even a space-based experiment. There are a number of technical challenges, including the elimination of unwanted nongravitational interactions, as well as the problem of decoherence caused by external microgravity sources. Recently, the first step was achieved, of measuring entanglement of large, LIGO-scale masses with an optical field \cite{ali2015quantum}.

\section{XI. Conclusions}

This review summarizes the developments in our understanding of macroscopic quantum correlations since the original papers of Einstein, Podolsky and Rosen (EPR) and Bell. The original papers pertained to just two systems of one particle each, and the correlations between the two systems ruled out all local hidden variable theories. Contrary to what might have been expected at the time of EPR and Bell, quantum mechanics has since been shown to predict such correlations for higher spin systems comprising many particles — either with many particles at just two locations, or with single particles at a large number of locations, or with many particles at multiple sites. Furthermore, it was found that the difference between the predictions of quantum mechanics and local hidden variables theories can (in a certain context) be more extreme as systems become larger. While the larger systems become more sensitive to decoherence which will erase the difference, this is in principle controllable. In fact, all pure entangled states can exhibit Bell nonlocality. Counterintuitively, strong Bell violations can be predicted in the presence of a macroscopic coarse-graining of measurement outcomes.

Experiments have so far supported the predictions of quantum mechanics in the mesoscopic regime. Higher dimensional Bell inequalities have been violated in photonic systems, and Einstein-Podolsky-Rosen paradoxes have been confirmed for high optical fluxes incident on detectors. In a step towards demonstrating the nonlocality of a cat state, the quantum coherence of superpositions of coherent states well separated in phase space has been measured, for
microwave fields in a cavity. The genuine multipartite nonlocality associated with multipartite photonic states has also been verified.

Moreover, there is evidence of quantum correlations in massive systems. Experiments confirm the existence of multi-particle Bell correlations inferred within a Bose-Einstein condensate, and the genuine entanglement of tens of ions in a trap and of millions of atoms in a solid, certified by rigorous theoretical methods. EPR-type entanglement has been detected for spatially-separated propagating atoms, and for ensembles of atoms at room temperature. EPR correlations in the form of a rigorous paradox (and multipartite entanglement) has been demonstrated for the atomic clouds of a split Bose-Einstein condensate, where each cloud contains hundreds of atoms. These are the first steps towards a rigorous demonstration of Bell correlations between massive systems, and between freely propagating massive particles. As we approach more rigorous testing of the quantum correlations in a macroscopic regime, there is a potential for fundamental tests of quantum mechanics. This is especially true for tests involving more massive objects. To date, there has been no Bell test involving the position and momentum of well-separated massive particles or objects, and no Bell test where the hidden variables are directly associated with spatially separated macroscopic objects.

Leggett and Garg in 1985 explained the possibility of testing for the incompatibility between macroscopic realism and quantum mechanics. This is not necessarily achieved by confirming Bell correlations in macroscopic systems. Leggett and Garg argued such tests might be carried out for dynamically evolving macroscopic superposition states. Recently, Leggett and Garg’s definition of macroscopic realism (macro-realism) has been tested for macroscopic superconducting qubits, with results supporting the quantum predictions that counter macrorealism. While loopholes remain, there is now an expanding number of theoretical proposals, including for optomechanical Leggett-Garg tests and for Bell-Leggett-Garg tests involving dynamically evolving cat states at separated sites.

ACKNOWLEDGMENTS

This work was funded through the Australian Research Council Discovery Project scheme under Grants DP180102470 and DP190101480. The authors also wish to thank NTT Research for their financial and technical support.

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