Parameterized Complexity of Locally Minimal Defensive Alliances *

Ajinkya Gaikwad
*Indian Institute of Science Education and Research, Pune, India

Soumen Maity*
*Indian Institute of Science Education and Research, Pune, India

Shuvam Kant Tripathi
Indian Institute of Science Education and Research, Pune, India

Abstract

A set $S$ of vertices of a graph is a defensive alliance if, for each element of $S$, the majority of its neighbours is in $S$. We consider the notion of local minimality in this paper. We are interested in locally minimal defensive alliance of maximum size. This problem is known to be NP-hard but its parameterized complexity remains open until now. We enhance our understanding of the problem from the viewpoint of parameterized complexity. The main results of the paper are the following: (1) Locally Minimal Defensive Alliance is NP-complete, even when restricted to planar graphs, (2) a randomized FPT algorithm for Exact Connected Locally Minimal Defensive Alliance parameterized by solution size, (3) Locally Minimal Defensive Alliance is fixed-parameter tractable (FPT) when parametrized by neighbourhood diversity, (4) Locally Minimal Defensive Alliance parameterized by treewidth is $W[1]$-hard and thus not FPT (unless $FPT=W[1]$), (5) Locally Minimal Defensive Al-

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*Corresponding author

Email addresses: ajinkya.gaikwad@students.iiserpune.ac.in (Ajinkya Gaikwad), soumen@iiserpune.ac.in (Soumen Maity), tripathi.shuvamkant@students.iiserpune.ac.in (Shuvam Kant Tripathi)
LIANCE can be solved in polynomial time for graphs of bounded treewidth.

**Keywords:** Parameterized Complexity, FPT, W[1]-hard, treewidth, neighbourhood diversity

1. Introduction

Throughout history, humans have formed communities, guilds, faiths etc in the hope of coming together with a group of people having similar requirements, visions and goals. Their reasons to do so, usually rest on the fact that any group with common interests often provides added mutual benefits to the union in fields of trade, culture, defense, etc as compared to the individual. Such activities are commonly seen in the present day, in areas of geo-politics, cultures, trades, economics, unions etc and are popularly termed as *alliances*. Based on the structure, formation and goals of an alliance, many variations of the problem exist in graph theory. A defensive alliance is usually formed with the aim of defending its members against non-members, and hence it is natural to ask that each member of the alliance should have more friends within the alliance (including oneself) than outside. Similarly, an offensive alliance is formed with the inverse goal of offending or attacking non-members of the alliance. It is known that the problems of finding small defensive and offensive alliances are NP-complete. We enhance our understanding of the problems from the viewpoint of parameterized complexity.

In 2004, Kristiansen, Hedetniemi, and Hedetniemi [25] introduced defensive, offensive, and powerful alliances, and these concepts were further studied by Shafique [16] and other authors [4, 1, 3, 33, 31, 32, 34, 8, 2, 10, 12, 20]. The theory of alliances in graphs has been studied intensively [13, 11] both from a combinatorial and from a computational perspective. As mentioned in [1], the focus has been mostly on finding small alliances, although studying large alliances does not only make a lot of sense from the original motivation of these notions, but was actually also delineated in the very first papers on alliances [25].
Note that being a defensive alliance is not a hereditary property, that is, a superset or subset of a defensive alliance is not necessarily a defensive alliance. Shafique [16] called an alliance a *locally minimal alliance* if the set obtained by removing any vertex of the alliance is not an alliance. Bazgan et al. [1] considered another notion of alliance that they called a *globally minimal alliance* which has the property that no proper subset is an alliance. In this paper we are interested in finding locally minimal alliances of size at least $k$. Bazgan et al. [1] proved that deciding if a graph contains a locally minimal defensive alliance of size at least $k$ is NP-complete, even when restricted to bipartite graphs with average degree less than 5.6. Clearly, the motivation is that big communities where every member still matters somehow are of more interest than really small communities. Also, there is a general mathematical interest in such type of problems, see [5, 28].

Motivation for maximum-minimal / minimum-maximal problems can be given as follows. Many local search heuristics for NP-hard optimisation problems can be modelled by defining a partial order on the feasible solutions that are iteratively improved upon. Once we reach a solution that is minimal or maximal with respect to the given partial order, the local search process terminates. For example, for INDEPENDENT SET, find a maximal independent set by selecting an unmarked vertex $v$, marking $v$ and all of its neighbours, iterating this approach until no unmarked vertex remains. The partial order is the strict subset relation. For CHROMATIC NUMBER, starting from an arbitrary graph colouring, try to “improve” it by using the following recolouring strategy. Try to find some colour $c$ such that each vertex of colour $c$ can be recoloured by one of the remaining colours, and iterate this process. The partial order here is one of partition refinement. See Chapter 2 of PhD thesis [28] for further details. We assume a relationship between the partial order used and the measure function of the source optimisation problem called “partial order measure monotonicity” – that is, every time we obtain a “local” improvement relative to the partial order, we obtain a “global” improvement (e.g., size of independent set goes up by 1, number of colours used goes down by 1) relative to the measure function.
The minimum-maximal or maximum-minimal problem then corresponds to the worst-case behaviour of such a local search heuristic. These problems can be interesting in their own right. Note that Minimum Maximal Independent Set is the same as the extensively studied Minimum Independent Dominating Set \[15\]. Similarly Maximum Minimal Chromatic Number is the same as a completely new problem called B-Chromatic Number, which ended up getting a lot of attention in the literature \[19, 28\].

Also note that enumerating all minimal (resp. maximal) solutions can be a useful strategy in relation to solving the original minimization (resp. maximization) problem. The enumeration problem asks to enumerate all minimal (resp. maximal) solutions for a given input instance. The existence of an enumeration algorithm which runs in time \(O^*(c^n)\) implies the existence of an \(O^*(c^n)\) for the source optimization problem. This follows from the fact that the solution to the minimization (resp. maximization) problem can be obtained by enumerating all minimal (resp. maximal) solutions and then look for the smallest (resp. largest) of the enumerated solutions. See \[22, 5\] for details.

2. Basic Notations

Throughout this article, \(G = (V, E)\) denotes a finite, simple and undirected graph of order \(|V| = n\). The (open) neighbourhood \(N_G(v)\) of a vertex \(v \in V(G)\) is the set \(\{u \mid (u, v) \in E(G)\}\). The closed neighbourhood \(N_G[v]\) of a vertex \(v \in V(G)\) is the set \(\{v\} \cup N_G(v)\). The degree of \(v \in V(G)\) is \(|N_G(v)|\) and denoted by \(d_G(v)\). The subgraph induced by \(S \subseteq V(G)\) is denoted by \(G[S]\). For a non-empty subset \(S \subseteq V\) and a vertex \(v \in V\), \(d_S(v)\) denotes the number of neighbours that \(v\) has in the vertex set \(S\). The complement of the vertex set \(S\) in \(V\) is denoted by \(S^c\).

**Definition 1.** A non-empty set \(S \subseteq V\) is a defensive alliance in \(G\) if for each \(v \in S\), \(d_S(v) + 1 \geq d_{S^c}(v)\).

We often use the terms defenders and attackers of an element \(v\) of a defensive alliance \(S\). By these we mean the sets \(N[v] \cap S\) and \(N[v] \setminus S\) respectively. Thus,
including itself, \( v \) has \( |N[v] \cap S| = d_S(v) + 1 \) defenders and \( v \) has \( |N[v] \setminus S| = d_{S'}(v) \) attackers in \( G \). A vertex \( v \in S \) is said to be protected if \( d_S(v) + 1 \geq d_{S'}(v) \). A vertex \( v \in S \) is said to be unprotected if \( d_S(v) + 1 < d_{S'}(v) \). A set \( S \subseteq V \) is a defensive alliance if every vertex in \( S \) is protected.

**Definition 2.** A vertex \( v \in S \) is said to be marginally protected if it becomes unprotected when any of its neighbours in \( S \) is moved from \( S \) to \( V \setminus S \). A vertex \( v \in S \) is said to be overprotected if it remains protected even when any of its neighbours is moved from \( S \) to \( V \setminus S \).

**Definition 3.** A defensive alliance \( S \) is called a locally minimal defensive alliance if for any \( v \in S \), \( S \setminus \{v\} \) is not a defensive alliance.

*It is important to note that \( S \) is a locally minimal defensive alliance in \( G \) if and only if for every vertex \( v \in S \), at least one of its neighbours in \( S \) is marginally protected.*

**Definition 4.** A defensive alliance \( S \) is called a globally minimal defensive alliance if no proper subset is a defensive alliance.

**Example 1.** Consider the tree in Figure 1. It has a locally minimal defensive alliance \( S_1 = \{2, 3, 4, 5, 6, 7, 9, 11, 13, 15\} \) of size 10 and a globally minimal defensive alliance \( S_2 = \{1, 2, 3\} \) of size 3. Note that, including itself, vertex 2 has two defenders and it has two attackers; so vertex 2 is marginally protected. Similarly, including itself, vertex 7 has two defenders and one attacker; so vertex 7 is marginally protected. It is easy to see that for every vertex \( v \in S_1 \), at least one of its neighbours in \( S_1 \) is marginally protected. Hence, \( S_1 \) is a locally minimal defensive alliance. Note that \( S_1 \) is not a globally minimal defensive alliance as \( \{2, 7\} \), a proper subset of \( S_1 \), is also a defensive alliance. On the other hand, no proper subset of \( S_2 \) is a defensive alliance, hence \( S_2 \) is a globally minimal defensive alliance. It may be noted that every globally minimal defensive alliance is also a locally minimal defensive alliance but the converse is not true.
A graph is said to be connected if there is a path between every pair of vertex. A locally minimal defensive alliance \( S \) is called a connected locally minimal defensive alliance if the subgraph induced by \( S \) is connected. Notice that any globally minimal defensive alliance is always connected. In this paper, we study Locally Minimal Defensive Alliance and Exact Connected Locally Minimal Defensive Alliance. We define the problems as follows:

**Locally Minimal Defensive Alliance**

**Input:** An undirected graph \( G = (V, E) \) and an integer \( k \leq |V(G)| \).

**Question:** Is there a locally minimal defensive alliance \( S \subseteq V(G) \) such that \( |S| \geq k \)?

**Exact Connected Locally Minimal Defensive Alliance**

**Input:** An undirected graph \( G = (V, E) \) and an integer \( k \leq |V(G)| \).

**Question:** Is there a connected locally minimal defensive alliance \( S \subseteq V(G) \) such that \( |S| = k \)?

The graph parameter that we explicitly use in this paper is treewidth. We review the concept of a tree decomposition, introduced by Robertson and Seymour in [30]. Treewidth is a measure of how “tree-like” the graph is.

**Definition 5.** A tree decomposition of a graph \( G = (V, E) \) is a tree \( T \) together with a collection of subsets \( X_t \) (called bags) of \( V \) labeled by the vertices \( t \) of \( T \)
such that $\bigcup_{t \in T} X_t = V$ and (1) and (2) below hold:

1. For every edge $uv \in E(G)$, there is some $t$ such that $\{u, v\} \subseteq X_t$.
2. (Interpolation Property) If $t$ is a vertex on the unique path in $T$ from $t_1$ to $t_2$, then $X_{t_1} \cap X_{t_2} \subseteq X_t$.

**Definition 6.** [7] The *width* of a tree decomposition is the maximum value of $|X_t| - 1$ taken over all the vertices $t$ of the tree $T$ of the decomposition. The *treewidth* $tw(G)$ of a graph $G$ is the minimum width among all possible tree decomposition of $G$.

A special type of tree decomposition, known as a *nice tree decomposition* was introduced by Kloks [23]. The nodes in such a decomposition can be partitioned into four types:

**Definition 7.** [23] A tree decomposition is said to be *nice tree decomposition* if the following conditions are satisfied:

1. All bags correspond to leaves are empty. One of the leaves is considered as root node $r$. Thus $X_r = \emptyset$ and $X_l = \emptyset$ for each leaf $l$.
2. There are three types of non-leaf nodes:
   - **Introduce node**: a node $t$ with exactly one child $t'$ such that $X_t = X_{t'} \cup \{v\}$ for some $v \notin X_{t'}$; we say that $v$ is *introduced* at $t$.
   - **Forget node**: a node $t$ with exactly one child $t'$ such that $X_t = X_{t'} \setminus \{w\}$ for some $w \in X_{t'}$; we say that $w$ is *forgotten* at $t$.
   - **Join node**: a node with two children $t_1$ and $t_2$ such that $X_t = X_{t_1} = X_{t_2}$.

Note that, by the third property of tree decomposition, a vertex $v \in V(G)$ may be introduced several time, but each vertex is forgotten only once. It is known that if a graph $G$ admits a tree decomposition of width at most $tw$, then it also admits a nice tree decomposition of width at most $tw$, that has at most $O(n \cdot tw)$ nodes [6].
We recall the definitions of treedepth, vertex cover and feedback vertex set which are used in Section 6. A rooted forest is a disjoint union of rooted trees. Given a rooted forest $F$, its transitive closure is a graph $H$ in which $V(H)$ contains all the nodes of the rooted forest, and $E(H)$ contain an edge between two vertices only if those two vertices form an ancestor-descendant pair in the forest $F$.

**Definition 8.** The treedepth of a graph $G$ is the minimum height of a rooted forest $F$ whose transitive closure contains the graph $G$. It is denoted by $td(G)$.

**Definition 9.** A set $S \subseteq V(G)$ is a vertex cover of $G = (V, E)$ if each edge in $E$ has at least one endpoint in $S$. The size of a smallest vertex cover of $G$ is the vertex cover number of $G$.

**Definition 10.** A feedback vertex set of a graph $G$ is a set of vertices whose removal leaves $G$ without cycles. The minimum size of a feedback vertex set in $G$ is the feedback vertex set number of $G$, denoted by $fvs(G)$.

We now recall the definition of Iverson bracket that will be used in the proof of Theorem 5. Theorem 5 proves that Locally Minimal Defensive Alliance is fixed-parameter tractable when parameterized by the neighbourhood diversity of the input graph.

**Definition 11.** Let $S$ be a mathematical statement, then the Iverson bracket is defined by

$$[S] = \begin{cases} 
1 & \text{if } S \text{ is true} \\
0 & \text{if } S \text{ is false} 
\end{cases}$$

and corresponds to the so-called characteristic function or indicator function.

### 2.1. Parameterized Complexity

A parameterized problem is a language $L \subseteq \Sigma^* \times \mathbb{N}$, where $\Sigma$ is a fixed, finite alphabet. For an instance $(x, k) \in \Sigma^* \times \mathbb{N}$, $k$ is called the parameter. A parameterized problem $P$ is fixed-parameter tractable (FPT in short) if a given instance $(x, k)$ can be solved in time $f(k) \cdot |(x, k)|^c$ where $f$ is some (usually
computable) function, and \( c \) is a constant. Parameterized complexity classes are defined with respect to \( fpt\)-reducibility. A parameterized problem \( \mathcal{P} \) is \( fpt\)-reducible to \( \mathcal{Q} \) if in time \( f(k) \cdot |(x, k)|^c \), one can transform an instance \( (x, k) \) of \( \mathcal{P} \) into an instance \( (x', k') \) of \( \mathcal{Q} \) such that \( (x, k) \in \mathcal{P} \) if and only if \( (x', k') \in \mathcal{Q} \), and \( k' \leq g(k) \), where \( f \) and \( g \) are computable functions depending only on \( k \). Owing to the definition, if \( \mathcal{P} \) \( fpt\)-reduces to \( \mathcal{Q} \) and \( \mathcal{Q} \) is fixed-parameter tractable then \( \mathcal{P} \) is fixed-parameter tractable as well.

What makes the theory more interesting is a hierarchy of intractable parameterized problem classes above \( \text{FPT} \) which helps in distinguishing those problems that are not fixed parameter tractable. Central to parameterized complexity is the following hierarchy of complexity classes, defined by the closure of canonical problems under \( fpt\)-reductions: \( \text{FPT} \subseteq \text{W}[1] \subseteq \text{W}[2] \subseteq \ldots \subseteq \text{XP} \). All inclusions are believed to be strict. In particular, \( \text{FPT} \neq \text{W}[1] \) under the Exponential Time Hypothesis \([18]\). The class \( \text{W}[1] \) is the analog of \( \text{NP} \) in parameterized complexity. A major goal in parameterized complexity is to distinguish between parameterized problems which are in \( \text{FPT} \) and those which are \( \text{W}[1]\text{-hard} \), i.e., those to which every problem in \( \text{W}[1] \) is \( fpt\)-reducible. There are many problems shown to be complete for \( \text{W}[1] \), or equivalently \( \text{W}[1]\text{-complete} \), including the \text{MultiColored Clique (MCC)} problem \([7]\). We refer to \([6, 7]\) for further details on parameterized complexity.

### 2.2. Our results

In this paper, we study \textsc{Locally Minimal Defensive Alliance} and \textsc{Exact Connected Locally Minimal Defensive Alliance} mainly from the parameterized complexity point of view. We show both tractability and intractability results. Our results are the following:

- \textsc{Locally Minimal Defensive Alliance} is \text{NP}-complete, even when restricted to planar graphs.
• **Locally Minimal Defensive Alliance** is fixed-parameter tractable (FPT) when parameterized by neighbourhood diversity.

• **Locally Minimal Defensive Alliance** parameterized by treewidth is \(W[1]\)-hard and thus not FPT (unless FPT=\(W[1]\)).

• **Locally Minimal Defensive Alliance** problem is polynomial time solvable for graphs with bounded treewidth. That is, the problem can be solved in XP-time when parameterized by treewidth.

• Finally, we give a randomized FPT algorithm for **Exact Connected Locally Minimal Defensive Alliance** when parameterized by the solution size \(k\).

\[
\begin{array}{c}
    \text{vc}^* \\
    \text{tw}^* \\
    \text{pw}^* & \text{fvs}^* \\
    \text{tdl}^* & \text{nd}^* \\
    \text{vi} & \text{tc} \\
    \text{cvd} & \text{mw} \\
    \text{cw}^* \\
\end{array}
\]

Figure 2: Relationship between vertex cover (vc), neighbourhood diversity (nd), twin cover (tc), modular width (mw), cluster vertex deletion number (cvd), feedback vertex set (fvs), pathwidth (pw), treewidth (tw), vertex integrity (vi) and clique width (cw). Note that \(A \rightarrow B\) means that there exists a function \(f\) such that for all graphs, \(f(A(G)) \geq B(G)\). It also gives an overview of the parameterized complexity landscape for **Locally Minimal Defensive Alliance**. The problem is FPT parameterized by blue colored parameters and \(W[1]\)-hard when parameterized by red colored parameters. The stars highlight parameters that are covered in this paper. The problem remains unsettled when parameterized by mw, vi, cvd and tc.

**Known Results:** The decision version for several types of alliances have been
shown to be NP-complete. Carvajal et al. [4] proved that deciding if a graph contains a strong defensive alliance of size at most $k$ is NP-hard. The problem of deciding if a graph contains defensive alliance of size at most $k$ is NP-complete even when restricted to split, chordal and bipartite graphs [20]. Bazgan et al. [1] proved that deciding if a graph contains a locally minimal strong defensive alliance of size at least $k$ is NP-complete, even when restricted to bipartite graphs with average degree less than 3.6. Bazgan et al. [1] also proved that deciding if a graph contains a connected locally minimal strong defensive alliance or a connected locally minimal defensive alliance of size at least $k$ is NP-complete, even when restricted to bipartite graphs with average degree less than $2 + \epsilon$, for any $\epsilon > 0$.

3. **Locally Minimal Defensive Alliance in planar graphs is NP-complete**

Bazgan, Fernau and Tuza showed in [1] that the problem of deciding if a graph contains a locally minimal defensive alliance of size at least $k$ for bipartite graphs with average degree less than 5.6 and the problem of deciding if a graph contains a connected locally minimal defensive alliance of size at least $k$, even for bipartite graphs with average degree less than $2 + \epsilon$, for any $\epsilon > 0$, are NP-complete. Here we prove that Locally Minimal Defensive Alliance is NP-complete in planar graphs, via a reduction from Minimum Maximal Matching in cubic planar graph. Yannakakis and Gavril showed in [36] that the problem of finding a maximal matching of minimum size, is NP-hard in planar graphs of maximum degree 3. In [17], Horton and Kilakos obtained the NP-hardness of Minimum Maximal Matching in planar cubic graphs.

**Theorem 1.** **Locally Minimal Defensive Alliance is NP-complete, even when restricted to planar graphs.**

*Proof.* Clearly, the decision version of the problem belongs to NP. In order to obtain the NP-hardness result for Locally Minimal Defensive Alliance, we obtain a polynomial reduction from Minimum Maximal Matching on cubic planar graphs proved NP-hard in [17]. Given an instance $I = (G, k)$ of
Minimum Maximal Matching where $G$ is a cubic planar graph, we construct an instance $I' = (G', k')$ of Locally Minimal Defensive Alliance where $G'$ is planar. See Figure 3, which provides an illustration of the construction.

The graph $G'$ that we construct has vertex set $A \cup B$, where $A = V(G) = \{v_1, v_2, \ldots, v_n\}$, the vertex set of $G$ and $B = E(G) = \{e_1, e_2, \ldots, e_m\}$, the edge set of $G$. We make $v_i$ adjacent to $e_j$ if and only if $v_i$ is an endpoint of $e_j$. Further we add two sets of vertices $V^\Box = \{v_1^\Box, \ldots, v_n^\Box\}$ and $E^\Box = \{e_1^\Box, e_2^\Box, \ldots, e_m^\Box\}$; and make $v_i$ adjacent to $v_i^\Box$ and $e_i$ adjacent to $e_i^\Box$ and $e_i^\Box$. For each $x \in V^\Box \cup E^\Box$, we add a cycle $C_x$ of length $6(n + m)$ with vertices $x_1, x_2, \ldots, x_{6(n + m)}$ and make

![Figure 3: Reducing Minimum Maximal Matching on planar cubic graphs to Locally Minimal Defensive Alliance on planar graphs. (a) An undirected graph $G = (V,E)$ with minimum maximal matching $M = \{(a,b),(c,d)\}$. (b) The planar graph $G'$ produced by the reduction algorithm that has locally minimal defensive alliance $D = A \cup B \setminus \{ab,cd\} \cup \bigcup_{x \in V^\Box \cup E^\Box} \{x_i \mid 1 \leq i \leq 60 \& i \text{ is not divisible by } 3\}$. A red circle represents a cycle of length 60 and a red line between $x$ and $C_x$ indicates that $x$ is adjacent to all the vertices of $C_x$. Note that $G'$ can be redrawn in a way in which no edges cross.](image-url)
all the vertices of $C_x$ adjacent to $x$. This completes the construction of $G'$. It is easy to note that $G'$ is a planar graph. Set $k' = 4(n+m)(n+2m) + (n+m-k)$. To complete the proof, we show that $G$ has a maximal matching of size at most $k$ if and only if $G'$ has a locally minimal defensive alliance of size at least $k'$.

Suppose $G$ has a maximal matching $M$ of size at most $k$. Let

$$C^* = \bigcup_{x \in V \cup E} \{x_i \mid 1 \leq i \leq 6(m+n) \& i \text{ is not divisible by 3}\}.$$ 

We claim that $D = A \cup (B \setminus M) \cup C^*$ is a locally minimal defensive alliance of size at least $k'$ in $G'$. Let $x$ be an arbitrary element of $D$.

**Case 1.** Suppose $x$ is an element of $A$. As $G$ is cubic, $x$ has three neighbours in $B$. As $M$ is a matching, out of three neighbours of $x$ in $B$, at most one is in $M$. Therefore, $x$ has at least two neighbours in $B \setminus M$. Thus, including itself, $x$ has at least three defenders and it has at most two attacker. Thus every vertex of $A$ is protected.

**Case 2.** If $x$ is an element of $(B \setminus M)$, we will prove that $x$ is marginally protected. The attackers of $x$ consist of its two neighbours in $E^\square$ and the defenders of $x$ consist of its two neighbours in $A$. Thus, including itself, $x$ has three defenders and two attackers. Hence $x$ is marginally protected.

**Case 3.** If $x$ is an element of $C^*$, we will prove that $x$ is marginally protected. Without loss of generality suppose $x = x_{3i+1}$ for some $i \geq 1$. Then the defender of $x$ is $x_{3i+2}$ and the attackers of $x$ consist of $x_{3i}$ and its only neighbour in $V^\square \cup E^\square$. Therefore, including itself, $x$ has two defenders and it has two attackers. Hence $x$ is marginally protected.

This show that $D$ is a defensive alliance. In order to prove that $D$ is a locally minimal defensive alliance, we prove that for any $v \in D$, $D \setminus \{v\}$ is not a defensive alliance. Since $M$ is maximal, it is not possible to move a vertex from $B \setminus M$ to $D^c$. If we move a vertex from $B \setminus M$ to $D^c$, then some vertex from $A \subseteq D$ will become unprotected. Similarly it is not possible to move a vertex from $A$ to $D^c$. If we move a vertex from $A$ to $D^c$ then some vertices from $B \setminus M \subseteq D$ will become unprotected. As every vertex in $C^*$ has a marginally protected neighbour, we cannot move a vertex from $C^*$ to $D^c$. This shows that
is indeed a locally minimal defensive alliance.

For the reverse direction, suppose that \( G' \) has a locally minimal defensive alliance \( D \) of size at least \( k' \). Before we continue with the proof for the reverse direction, we will prove two crucial properties:

**Claim 1.** Let \( D \) be a locally minimal defensive alliance in \( G' \). Then for each \( x \in V \square \cup E \square \), \( C_x \) can contribute at most \( 4(m + n) \) vertices in \( D \).

**Proof.** Let \( C_x = \{x_1, \ldots, x_{6(m+n)}\} \). We define the following three sets of vertices

\[
C_x^0 = \{x_{3i} \mid 1 \leq i \leq 2(n + m)\}, C_x^1 = \{x_{3i+1} \mid 0 \leq i \leq 2(n + m) - 1\} \quad \text{and} \quad C_x^2 = \{x_{3i+2} \mid 0 \leq i \leq 2(n + m) - 1\}.
\]

Suppose \( x \notin D \). Then the union of any two of the above sets, say, \( C_x^1 \cup C_x^2 \) forms a locally minimal defensive alliance of size \( 4(m + n) \) and can be part of \( D \). The defender of \( x_{3i+1} \) is \( x_{3i+2} \) and the attackers of \( x_{3i+1} \) is \( x \) and \( x_{3i} \). Thus, including itself, \( x_{3i+1} \) has two defenders and two attackers. So \( x_{3i+1} \) is marginally protected. Similarly \( x_{3i+2} \) is also marginally protected. As every vertex of \( C_x^1 \cup C_x^2 \) is protected and has a marginally protected neighbour, it is a locally minimal defensive alliance and can be part of \( D \). Now we prove that \( C_x \) cannot contribute more than \( 4(m + n) \) vertices in \( D \). For the sake of contradiction, assume that \( C_x \) contributes \( 4(m + n) + 1 \) vertices in \( D \). Then \( D \) contains at least three consecutive vertices, say, \( x_{3i}, x_{3i+1}, x_{3i+2} \) for some \( i \). Again suppose \( x \notin D \). Then the defenders of \( x_{3i+1} \) consist of \( x_{3i}, x_{3i+2} \) and the attacker of \( x_{3i+1} \) is \( x \) only. Thus, including itself, \( x_{3i+1} \) has three defenders and one attacker. Thus \( x_{3i+1} \) is overprotected. It may be verified that \( x_{3i} \) and \( x_{3i+2} \) are marginally protected but they do not have a marginally protected neighbour in \( D \). Thus \( D \) is not a locally minimal defensive alliance, a contradiction.

**Claim 2.** Let \( D \) be a locally minimal defensive alliance of size at least \( k' \) in \( G' \) and let \( x \in V \square \cup E \square \). Then \( x \) does not lie in \( D \).

**Proof.** For the sake of contradiction assume that \( x \in D \). This means \( x \) is either marginally protected or overprotected in \( D \). We consider two cases:

**Case 1.** Suppose \( x \) is marginally protected in \( D \). Note that \( x \) has \( 6(m + n) + 1 \)
neighbours in \( G' \) and the neighbours of \( x \) in \( G' \) consists of \( 6(m+n) \) vertices of \( C_x \) and one neighbour in \( V \Box \cup E \Box \). Since \( x \) is marginally protected, including itself, it can have \( 3(m+n)+1 \) neighbours in \( D \). Thus the cycle \( C_x \) can contribute at most \( 3(n+m) \) vertices in \( D \), as otherwise \( x \) is not marginally protected. Recall that \( k' = 4(n+m)(n+2m) + (n+m-k) \) and \( |V \Box \cup E \Box| = (n+2m) \). By the above claim we know, for every \( y \in V \Box \cup E \Box \), \( C_y \) can contribute at most \( 4(m+n) \) vertices in \( D \). Assume that for each \( y \in V \Box \cup E \Box \) (\( y \neq x \)), \( y \) does not belong to \( D \) and \( C_y \) contributes exactly \( 4(m+n) \) vertices in \( D \). On the other hand \( x \) is marginally protected in \( D \) and \( C_x \) contributes at most \( 3(m+n) \) vertices in \( D \). Even if we include all the vertices of \( (A \cup B) \setminus \{x\} \) in \( D \), its size reaches the value \( 4(m+n)(n+2m-1) + 3(m+n) + (m+n-1) < k' \) and this is a contradiction. Thus \( x \) does not lie in \( D \).

**Case 2.** Suppose \( x \) lies in \( D \) and it is overprotected. Let \( x_i \) be a neighbour of \( x \) in \( C_x \cap D \). As \( D \) is a locally minimal defensive alliance, \( x_i \) must have a marginally protected neighbour, say \( x_{i+1} \), in \( C_x \). The defenders of \( x_i \) consist of \( x_{i+1} \) and \( x \). Thus, including itself, the number of defenders of \( x_i \) is at least 3, and the number of attackers is at most 1. Hence \( x_i \) is not marginally protected; similarly \( x_{i+1} \) is also not marginally protected. This implies that \( x_i \) and \( x_{i+1} \) do not have a marginally protected neighbour, a contradiction to the assumption that \( D \) is a locally minimal defensive alliance. Thus \( x \) does not lie in \( D \). This proves the claim that \( (V \Box \cup E \Box) \cap D = \emptyset \). \( \square \)

We now define \( M = B \cap D^c \) and claim that there exists a set \( M' \subseteq M \) such that \( M' \) is a maximal matching. Assume, for the sake of contradiction, that no subsets of \( M \) form a maximal matching. If no subsets of \( M \) form a maximal matching then clearly there exists an edge \( (u,v) \in E \) such that it can still be added to \( M \). This means no edges incident with \( u \) and \( v \) are in \( M \). It implies that all the edges incident with \( u \) and \( v \) are in \( D \) and hence \( u \) and \( v \) are overprotected. Note that vertex \( e = uv \in B \) has four neighbours \( u, v, uv \Box \) and \( uv \Box \). By the above claim \( uv \Box \) and \( uv \Box \) are not in \( D \). The other two neighbours \( u \) and \( v \) are overprotected in \( D \). Therefore, vertex \( e = uv \in B \) does not have
a marginally protected neighbour in $D$ as both of its neighbours $u$ and $v$ are overprotected, a contradiction to the fact that $D$ is a locally minimal defensive alliance. This shows that there must exist a set $M' \subseteq M$ such that $M'$ is a maximal matching and $|M'| \leq |M| \leq k$. This completes the proof of Theorem 1.

4. A randomized FPT algorithm for Exact Connected Locally Minimal Defensive Alliance parameterized by solution size

In this section, we give a randomized FPT algorithm for Exact Connected Locally Minimal Defensive Alliance parameterized by the solution size $k$. Let $G = (V, E)$ be a graph and let $S \subseteq V$ be a subset of size $k$. Every vertex in $G$ is colored independently with one colour from the set \{red, green\} with uniform probability. Denote the obtained coloring by $\chi : V(G) \rightarrow \{\text{red, green}\}$. A connected locally minimal defensive alliance $S$ in $G$ is called a green connected locally minimal defensive alliance in $G$ with coloring $\chi$ if all the vertices in $S$ are colored with green color and all the vertices in $N(S)$ are colored red.

Lemma 2. Let $G$ be a graph and let $\chi : V(G) \rightarrow \{\text{red, green}\}$ be a colouring of its vertices with two colours, chosen uniformly at random. Let $S \subseteq V$ be a connected locally minimal defensive alliance of size $k$ in $G$. Then the probability that the elements of $S$ are coloured with green colour and elements of $N(S)$ are coloured with red colour is at least $\frac{1}{2^{k^2+k}}$.

Proof. As $S$ is a (connected locally minimal) defensive alliance of size $k$ in $G$, each element $v$ in $S$ is protected and therefore $v$ can have at most $k$ neighbours outside $S$. It follows that $|N(S)| \leq |S|k = k^2$. There are $2^n$ possible colorings $\chi$; and there are $2^{n-k^2-k}$ possible colorings where the $k$ vertices of $S$ are coloured green and at most $k^2$ neighbours of $S$ are colored red. Hence the lemma follows.

Lemma 3. Let $G$ be a graph and let $\chi : V(G) \rightarrow \{\text{red, green}\}$ be a colouring of its vertices with two colours. Then there exists an algorithm that checks in
time $O(n + m)$ whether $G$ contains a green connected locally minimal defensive alliance of size $k$ and, if this is the case, returns one such an alliance.

Proof. Let $V_g$ and $V_r$ be a partitioning of $V(G)$ such that all vertices in $V_g$ are coloured green and all vertices in $V_r$ are coloured red. A connected component $C$ is said to be a green connected component if the vertices of $C$ are colored green.

Run DFS to identify all green connected components $C_1, C_2, \ldots, C_\ell$ of $G[V_g]$ in $O(m + n)$ time. Then verify in linear time if there exists a green connected component $C_i$ of size $k$ that forms a locally minimal defensive alliance in $G$. □

We now combine Lemma 2 and Lemma 3 to obtain the main result of this section.

**Theorem 4.** There exists a randomized algorithm that, given an Exact Connected Locally Minimal Defensive Alliance instance $(G, k)$, in time $2^{O(k^2 + k)}(n + m)$ either reports a failure or finds a connected locally minimal defensive alliance of size exactly $k$ in $G$. Moreover, if the algorithm is given a yes-instance, it returns a solution with a constant probability.

Proof. Given an input instance $(G, k)$, we uniformly at random color the vertices of $V(G)$ with two colors green and red. That is, every vertex is colored independently with either green or red color with uniform probability. Denote the obtained coloring by $\chi : V(G) \rightarrow \{\text{red, green}\}$. We run the algorithm of Lemma 3 on the graph $G$ with coloring $\chi$. If it returns a green connected locally minimal defensive alliance $S$ of size $k$, then we return this $S$ as connected locally minimal defensive alliance of size $k$ in $G$. Otherwise, we report failure.

It remains to bound the probability of finding a connected locally minimal defensive alliance of size $k$ in the case $(G, k)$ is a yes-instance. To this end, suppose $G$ has a connected locally minimal defensive alliance $S$ of size $k$. By Lemma 2, $S$ becomes a green connected locally minimal defensive alliance of size $k$ in the colouring $\chi$ with probability at least $\frac{1}{2^{k^2 + k}}$. If this is the case, the algorithm of Lemma 3 finds a green connected locally minimal defensive alliance
of size \( k \) (not necessarily \( S \) itself), and the algorithm returns a connected locally minimal defensive alliance of size \( k \) in \( G \).

Thus we have an algorithm that runs in time \( O(m + n) \) and given a yes-instance, returns a solution with probability at least \( \frac{1}{2^{k^2+k}} \). Clearly, by repeating the algorithm independently \( 2^{k^2+k} \) times, we obtain the running time bound and the success probability at least \( 1 - \frac{1}{e} \).

5. FPT algorithm parameterized by neighbourhood diversity

In this section, we present an FPT algorithm for \textsc{Locally Minimal Defensive Alliance} parameterized by neighbourhood diversity. That is, we prove the following theorem.

**Theorem 5.** \textsc{Locally Minimal Defensive Alliance} is fixed-parameter tractable when parameterized by the neighbourhood diversity.

In a graph \( G = (V, E) \), we say two vertices \( u \) and \( v \) have the same type if and only if \( N(u) \setminus \{v\} = N(v) \setminus \{u\} \). The relation of having the same type is an equivalence relation. The idea of neighbourhood diversity is based on this type structure.

**Definition 12.** [20] The neighbourhood diversity of a graph \( G = (V, E) \), denoted by \( \text{nd}(G) \), is the least integer \( k \) for which we can partition the set \( V \) of vertices into \( k \) classes, such that all vertices in each class have the same type.

If neighbourhood diversity of a graph is bounded by an integer \( k \), then there exists a partition \( \{T_1, T_2, \ldots, T_k\} \) of \( V(G) \) into \( k \) type classes. It is known that such a minimum partition can be found in linear time using fast modular decomposition algorithms [35]. Notice that each type class could either be a clique or an independent set by definition. For algorithmic purpose it is often useful to consider a type graph \( H \) of graph \( G \), where each vertex of \( H \) is a type class in \( G \), and two vertices \( T_i \) and \( T_j \) are adjacent if and only if there is complete bipartite clique between these type classes in \( G \). It is not difficult to see that there will be either a complete bipartite clique or no edges between any two type classes.
The key property of graphs of bounded neighbourhood diversity is that their type graphs have bounded size. For example, a graph $G$ with neighbourhood diversity four and its corresponding type graph $H$ is illustrated in Figure 4.

Outline of the algorithm. Given an $n$-vertex graph $G$ with $\text{nd}(G) \leq k$, we find a partition $\{T_1, T_2, \ldots, T_k\}$ of $V(G)$ into $k$ type classes. It is known that such a minimum partition can be found in linear time using fast modular decomposition algorithms [35]. Suppose $S$ is a hypothetical locally minimal defensive alliance in $G$. Next we guess whether a type class $T_i$ contributes no vertices, one vertex or at least two vertices to $S$. There are at most $3^k$ guesses as each $T_i$ has three options: either it contributes no vertex, one vertex, or at least two vertices to $S$. Finally we reduce the problem of finding $S$ to an integer linear programming (ILP) optimization with at most $k$ variables. Since ILP optimization is fixed-parameter tractable when parameterized by the number of variables [9], we can conclude that our problem is fixed-parameter tractable when parameterized by the neighbourhood diversity.

5.1. Characterization of a locally minimal defensive alliance $S$ with type classes

Let $G$ be a connected graph such that $\text{nd}(G) = k$. In this section we assume that we have the partition of $V(G)$ into sets of type classes $T_1, \ldots, T_k$. We as-
sume \( k \geq 2 \) since otherwise the problem becomes trivial. We prove the following lemma.

**Lemma 6.** Suppose \( S_1, S_2 \subseteq V(G) \) are such that \(|S_1 \cap T_i| = |S_2 \cap T_i| \) for all \( i \in [k] \). Then \( S_1 \) is a locally minimal defensive alliance in \( G \) if and only if \( S_2 \) is also a locally minimal defensive alliance in \( G \).

**Proof.** Suppose \( S_1 \) is a locally minimal defensive alliance in \( G \). For each \( i \in [k] \), the vertices in \( S_1 \cap T_i \) and \( S_2 \cap T_i \) have the same neighbourhood in \( G \) as the vertices in \( T_i \) have the same neighbourhood in \( G \). Therefore \( S_2 \) is also a locally minimal defensive alliance in \( G \). The converse part of the lemma also holds. This completes the proof of Lemma 6. \( \square \)

Suppose \( S \) is a hypothetical locally minimal defensive alliance. Let \( \mathcal{T}_0 \) (resp. \( \mathcal{T}_1 \)) be the set of all type classes that contribute zero vertices (one vertex) to \( S \). Similarly, let \( \mathcal{T}_{\geq 2} \) be the set of all type classes that contribute at least two vertices to \( S \). More formally, we define the following sets:

\[
\mathcal{T}_0 = \{ T_i : |S \cap T_i| = 0 \}; \hspace{1em} \mathcal{T}_1 = \{ T_i : |S \cap T_i| = 1 \}; \hspace{1em} \mathcal{T}_{\geq 2} = \{ T_i : |S \cap T_i| \geq 2 \}.
\]

Given \( \mathcal{T}_0, \mathcal{T}_1 \) and \( \mathcal{T}_{\geq 2} \), our goal here is to find a largest locally minimal defensive alliance \( S \) of \( G \), with \(|S \cap T_i| = 0 \) when \( T_i \in \mathcal{T}_0 \), \(|S \cap T_i| = 1 \) when \( T_i \in \mathcal{T}_1 \) and \(|T_i \cap S| \geq 2 \) when \( T_i \in \mathcal{T}_{\geq 2} \). Let \( x_i = |S \cap T_i| \) for \( i \in [k] \). We partition \([k]\) into \( I_0, I_1 \) and \( I_{\geq 2} \) as follows:

\[
\begin{align*}
  i \in & \begin{cases}
    I_0 & x_i = 0 \\
    I_1 & x_i = 1 \\
    I_{\geq 2} & x_i \geq 2
  \end{cases}
\end{align*}
\]

By Lemma 6, the variables \( x_i \) determine \( S \) uniquely. Therefore, given \( I_0, I_1, I_{\geq 2} \), the goal is to maximize the sum

\[
\sum_{i \in [k]} 1 \cdot [T_i \in \mathcal{T}_0] + \sum_{i \in [k]} x_i \cdot [T_i \in \mathcal{T}_{\geq 2}] = |I_1| + \sum_{i \in I_{\geq 2}} x_i
\]

under the conditions: \( x_i = 0 \) for \( i \in I_0 \), \( x_i = 1 \) for \( i \in I_1 \), \( 2 \leq x_i \leq |T_i| = n_i \) for \( i \in I_{\geq 2} \) and two additional conditions (Type 1 and Type 2) described below.
Here \([T_i \in T_1]\) and \([T_i \in T_{\geq 2}]\) are the Iverson brackets; see Definition \[11\]

**Type 1 Condition:** For each \(i \in I_1 \cup I_{\geq 2}\), we add the condition given in Equation 1. This is called type 1 condition. Type 1 conditions ensure that \(S\) is a defensive alliance, that is, the vertices in \(T_i \cap S\) are protected for all \(i \in I_1 \cup I_{\geq 2}\). Define

\[
K = \text{the collection of all clique type classes.}
\]

A vertex \(u \in T_i \cap S\) is protected if and only if \(d_S(u) \geq \frac{d_G(u) - 1}{2}\), that is,

\[
(x_i - 1)[T_i \in K] + \sum_{j \in [k]} 1 \cdot [T_j \in N_H(T_i) \cap T_i] + \sum_{j \in [k]} x_j \cdot [T_j \in N_H(T_i) \cap T_{\geq 2}] \\
\geq \frac{d_G(u) - 1}{2} \quad (1)
\]

The left-hand side expression of Eq. \([1]\) is equal to \(d_S(u)\). The first term of the expression stands for the number of neighbours of \(u\) in \(S \cap T_i\) if \(T_i\) is a clique type class; the second term stands for the number of neighbours of \(u\) in \(S \cap T_j\) if \(T_j \in T_1\) and \(T_j\) is a neighbour of \(T_i\) in the type graph \(H\); the third term stands for the number of neighbours of \(u\) in \(S \cap T_j\) if \(T_j \in T_{\geq 2}\) and \(T_j\) is a neighbour of \(T_i\) in the type graph \(H\).

**Type 2 Condition:** Type 2 condition ensures that \(S\) is a locally minimal defensive alliance, that is, for any \(v \in S\), \(S \setminus \{v\}\) is not a defensive alliance.

**Lemma 7.** Suppose \(S\) is a defensive alliance in \(G\). Then, given \(I_0, I_1, I_{\geq 2}\), \(S\) is a locally minimal defensive alliance in \(G\) if and only if there is a function \(f : I_1 \cup I_{\geq 2} \to I_1 \cup I_{\geq 2}\) with \(f(i) \neq i\) when \(i \in I_1\) such that for each \(i\) the vertices of \((S \setminus \{v\}) \cap T_{f(i)}\) are unprotected in \(S \setminus \{v\}\) for any \(v \in S \cap T_i\).

**Proof.** The proof follows directly from the definition of locally minimal defensive alliance. Suppose \(S\) is a locally minimal defensive alliance in \(G\). We know for any \(v \in S\), \(S \setminus \{v\}\) is not a defensive alliance, that is, some vertices of \(S \setminus \{v\}\) are unprotected. In terms of type classes, we can say there exists a function \(f : I_1 \cup I_{\geq 2} \to I_1 \cup I_{\geq 2}\) with \(f(i) \neq i\) when \(i \in I_1\) such that for each \(i\) the vertices of \((S \setminus \{v\}) \cap T_{f(i)}\) are unprotected in \(S \setminus \{v\}\) for any \(v \in S \cap T_i\).
To prove the reverse direction, suppose $S$ is a defensive alliance and there is a function $f : I_1 \cup I_{\geq 2} \rightarrow I_1 \cup I_{\geq 2}$ with $f(i) \neq i$ when $i \in I_1$ such that for each $i$ the vertices of $(S \setminus \{v\}) \cap T_{f(i)}$ are unprotected in $S \setminus \{v\}$ for any $v \in S \cap T_i$. This implies that for each $i$, every vertex of $S \cap T_i$ has a marginally protected neighbour in $S \cap T_{f(i)}$. Thus $S = \bigcup_{i \in I_1 \cup I_{\geq 2}} S \cap T_i$ is a locally minimal defensive alliance. This completes the proof of Lemma 7.

Given $(I_0, I_1, I_{\geq 2})$ and $f$, we want $S$ to satisfy the conditions in Equation 2. This is called type 2 condition. Type 2 conditions ensure that $S$ is a locally minimal defensive alliance. By Lemma 7 the vertices in $(S \setminus \{v\}) \cap T_{f(i)}$ must be unprotected in $S \setminus \{v\}$ for any $v \in T_i$. A vertex $u$ from $(S \setminus \{v\}) \cap T_{f(i)}$ is unprotected in $S \setminus \{v\}$ if and only if the number of neighbours of $u$ in $S \setminus \{v\}$ is strictly less than $\frac{d_{G(u)} - 1}{2}$, that is,

$$(x_{f(i)} - 1) [T_{f(i)} \in \mathcal{K}] + (x_i - 1) [T_i \in N_H(T_{f(i)})] + \sum_{j \in [k] \setminus j \neq i} 1 \cdot [T_j \in N_H(T_{f(i)}) \cap \mathcal{T}_1] +$$

$$\sum_{j \in [k] \setminus j \neq i} x_j \cdot [T_j \in N_H(T_{f(i)}) \cap \mathcal{T}_{\geq 2}] < \frac{d_{G(u)} - 1}{2}$$

(2)

The left-hand side expression of Eq. (2) is equal to $d_{S \setminus \{v\}}(u)$ where $u \in (S \setminus \{v\}) \cap T_{f(i)}$. The first term of the expression stands for the number of neighbours of $u$ in $S \cap T_{f(i)}$ if $T_{f(i)}$ is a clique type class; the second term stands for the number of neighbours of $u$ in $(S \setminus \{v\}) \cap T_i$ if $T_i$ is a neighbour of $T_{f(i)}$ in the type graph $H$; the third term stands for the number of neighbours of $u$ in $S \cap T_j$ if $T_j \in T_1$ and $T_j$ is a neighbour of $T_{f(i)}$ in the type graph $H$; the fourth term stands for the number of neighbours of $u$ in $S \cap T_j$ if $T_j \in T_{\geq 2}$ and $T_j$ is a neighbour of $T_{f(i)}$ in the type graph $H$.

5.2. ILP formation for Annotated LMDA

Our algorithm for Locally Minimal Defensive Alliance (LMDA) will use the following annotated problem as subroutine. In the Annotated LMDA problem, we are given a graph $G$ with $\text{nd}(G) = k$, type classes $T_1, T_2, \ldots, T_k$ of $G$, a partition of $[k]$ into three parts $I_0, I_1, I_{\geq 2}$, a function $f : I_1 \cup I_{\geq 2} \rightarrow I_1 \cup I_{\geq 2}$.
with $f(i) \neq i$ if $i \in I_1$ and the goal is to find a largest locally minimal defensive alliance $S \subseteq V(G)$ such that $|S \cap T_i| = 0$ if $i \in I_0$, $|S \cap T_i| = 1$ if $i \in I_1$ and $|S \cap T_i| \geq 2$ if $i \in I_{\geq 2}$.

Let $(G, (T_1, \ldots, T_k), I_0, I_1, I_{\geq 2}, f)$ be an instance of Annotated LMDA. We reduce the problem of solving Annotated LMDA to an integer linear programming optimization with at most $k$ variables as follows:

Maximize $|I_1| + \sum_{i \in I_{\geq 2}} x_i$

Subject to

$x_i = 0$ for $i \in I_0$

$x_i = 1$ for $i \in I_1$

$2 \leq x_i \leq |C_i| = n_i$ for $i \in I_{\geq 2}$

Equation (1) for $i \in I_1 \cup I_{\geq 2}$

Equation (2) for $i \in I_1 \cup I_{\geq 2}$

5.3. Running time for Annotated LMDA

Lenstra [27] showed that the feasibility version of $k$-ILP is FPT with running time doubly exponential in $k$, where $k$ is the number of variables. Later, Kannan [21] proved an algorithm for $k$-ILP running in time $k^O(k)$. In our algorithm, we need the optimization version of $k$-ILP rather than the feasibility version. We state the minimization version of $k$-ILP as presented by Fellows et al. [9].

$k$-Variable Integer Linear Programming Optimization ($k$-Opt-ILP):
Let matrices $A \in \mathbb{Z}^{m \times k}$, $b \in \mathbb{Z}^{k \times 1}$ and $c \in \mathbb{Z}^{1 \times k}$ be given. We want to find a vector $x \in \mathbb{Z}^{k \times 1}$ that minimizes the objective function $c \cdot x$ and satisfies the $m$ inequalities, that is, $A \cdot x \geq b$. The number of variables $k$ is the parameter. Then they showed the following:
Lemma 8. \[ k\text{-Opt-ILP} \text{ can be solved using } O(k^{2.5+o(k)} \cdot L \cdot \log(MN)) \] arithmetic operations and space polynomial in \( L \). Here \( L \) is the number of bits in the input, \( N \) is the maximum absolute value any variable can take, and \( M \) is an upper bound on the absolute value of the minimum taken by the objective function.

In the formulation for Annotated LMDA, we have at most \( k \) variables. The value of objective function is bounded by \( n \) and the value of any variable in the integer linear programming is also bounded by \( n \). The constraints can be represented using \( O(k^2 \log n) \) bits. Lemma 8 implies that we can solve the problem in FPT time.

5.4. An algorithm for Locally Minimal Defensive Alliance (LMDA)

Lemma 9. If there exists an FPT algorithm for Annotated LMDA then there exists an FPT algorithm for LMDA.

Proof. Suppose there exists an FPT algorithm for Annotated LMDA parameterized by neighbourhood diversity \( k \). Note that there are \( 3^k k^k \) candidates for Annotated LMDA instances. The reason is this. There are at most \( 3^k \) candidates for \((I_0, I_1, I_{\geq 2})\) as each \( T_i \) has three options: either in \( I_0 \), \( I_1 \) or \( I_{\geq 2} \); there are at most \( k^k \) candidates for \( f \). In order to obtain a locally minimal defensive alliance of maximum size, we first solve all Annotated LMDA instances, then consider a largest solution over all Annotated LMDA instances. Therefore, LMDA can be solved in FPT time parameterized by neighbourhood diversity \( k \).

We proved that the ILP formula for an Annotated GMDA can be solved in FPT time. By Lemma 9, thus Theorem 3 holds.

6. Hardness of Locally Minimal Defensive Alliance parameterized by treewidth

In this section we show that Locally Minimal Defensive Alliance is W[1]-hard parameterized by treewidth, via a reduction from a variant of
Defensive Alliance. The input of Defensive Alliance consists of a graph $G$, and an integer $k$, the task is to decide if $G$ has a defensive alliance of size at most $k$. Gaikwad and Maity [14] proved that Defensive Alliance is $W[1]$-hard when parameterized by the size of a vertex deletion set into collection of stars, i.e., the size of a subset $D$ of the vertices of the graph such that every component in the graph, after removing $D$, is a star. They proved the following result.

Theorem 10. [14] Defensive Alliance is $W[1]$-hard when parameterized by the size of a vertex deletion set into collection of stars, even when restricted to bipartite graphs.

While Defensive Alliance asks for a defensive alliances of size at most $k$, we consider Exact Defensive Alliance that concerns defensive alliances of size exactly $k$. As a consequence of Theorem 10 we have the following result:

Corollary 1. Exact Defensive Alliance is $W[1]$-hard when parameterized by the size of a vertex deletion set into collection of stars, even when restricted to bipartite graphs.

In this section, we prove the following theorem:

Theorem 11. Locally Minimal Defensive Alliance is $W[1]$-hard when parameterized by the size of a vertex deletion set into collection of stars.

Proof. Let $I = (G, k)$ be an instance of Exact Defensive Alliance. We construct an instance $I' = (G', k')$ of Locally Minimal Defensive Alliance as follows. See Figure 5 which provides an illustration of the construction. The construction of $G'$ starts with $G' := G$ and then add some new vertices and edges. Without loss of generality, we can assume that $k < \frac{n}{2} - 1$. First, we introduce a set $A = \{a_0, a_1, \ldots, a_{n-2k}\}$ of $n - 2k + 1$ new vertices and a new vertex $b_0$ into $G'$. For every vertex $a \in A \setminus \{a_0\}$, we introduce a set $V_a = \{v^1_a, v^2_a, \ldots, v^{2n^3}_a\}$ of $2n^3$ vertices into $G'$ and make them adjacent to $a$. Make $a_0$ adjacent to $a_i$ for every $1 \leq i \leq n - 2k$. Finally, we make $a_0$ and $b_0$
adjacent to every vertex of $G$. This completes the construction of graph $G'$. We set $k' = (n - k + 1) + (n^3 - 1)(n - 2k)$. Let $D$ be a set of vertices such that $V(G) \setminus D$ is a collection of stars. One can easily see that deletion of $D \cup \{a_0, b_0\}$ from $G'$ results in a collection of stars.

To prove the correctness of the reduction, we claim that $G$ has a defensive alliance of size exactly $k$ if and only if $G'$ admits a locally minimal defensive alliance of size at least $k'$. Assume first that $G$ has a defensive alliance of size exactly $k$. We claim that $S' = S \cup A \bigcup_{a \in A \setminus \{a_0\}} \bigcup_{i=1}^{n^3-1} \{v_i^a\}$ is a locally minimal defensive alliance of size $k'$ in $G'$. Clearly $|S'| \geq k'$. Let $x$ be an arbitrary element of $S'$.

Case 1. If $x$ is an element of $S$, then we show that it is protected in $S'$. The only new neighbour of $x$ in $G'$ that is part of $S'$ is $a_0$; the only new neighbour of $x$ in $G'$ that is outside $S'$ is $b_0$. Thus the number of defenders and attackers of $x$ in $G'$ increase by 1. Therefore, as $x$ was protected in $S$, it remains protected in $S'$.

Case 2. If $x \in \bigcup_{a \in A \setminus \{a_0\}} V_a$, then the only neighbour of $x$ in $G'$ is in $S'$, so $x$ can trivially defend itself.

Case 3. If $x = a_0$, the defenders of $x$ consist of all elements of $A \setminus \{a_0\}$ and all elements of $S$. The attackers of $x$ consist of all elements of $V(G) \setminus S$. Hence $x$ has $n - k$ defenders in $G'$; and $x$ has $n - k$ attackers in $G'$. This shows that $a_0$ is marginally protected.

Case 4. If $x \in A \setminus \{a_0\}$, then the defenders of $x$ consist of $n^3 - 1$ elements of $V_x$ and $a_0$; and the attacker of $x$ consist of $n^3 + 1$ elements of $V_x$. Hence $x$,
including itself, has $n^3 + 1$ defenders in $G'$; and $x$ has $n^3 + 1$ attackers in $G'$.
This shows that $x$ is marginally protected.

As $|S'| > 1$ and every vertex of $S'$ has a marginally protected neighbour, $S'$
is a locally minimal defensive alliance in $G'$.

To prove the reverse direction of the equivalence, suppose $G'$ has a locally
minimal defensive alliance $S'$ of size at least $k'$.

First, we claim $A \subseteq S'$. To show this, we first observe that if a degree 1
vertex is in some locally minimal defensive alliance of size at least 2 then its
only neighbour must be marginally protected. Note that $v^i_a$ is a degree 1 vertex.
Therefore, if $v^i_a \in V_a$ is present in $S'$ for some $i$ then $a$ has to be in $S'$ and $a$
has to be marginally protected in $S'$. Furthermore $a$ must also have a marginally
protected neighbour in $S'$. Since all the vertices in $V_a \cap S'$ are strongly protected,
the only other neighbour $a_0$ of $a$ must be inside $S'$ and $a_0$ must be marginally
protected. As $a$ is marginally protected, we have $|S' \cap V_x| = n^3 - 1$. Otherwise,
if $a \not\in S'$ then $V_a \cap S' = \emptyset$. This would imply that $|S'| < k'$ as for all $x \in A \setminus \{a_0\}$,
we have $|S' \cap V_x| \leq n^3 - 1$. Therefore we get that $A \subseteq S'$ and all the vertices
in $A$ are marginally protected in $S'$. This proves the claim. Note that $a_0$ is
marginally protected in $S'$ and $A \subseteq S'$ imply that $|S' \cap V(G)| = k$.

We now claim that $b_0 \not\in S'$. Assume for the sake of contradiction that
$b_0 \in S'$. Then $b_0$, including itself, has $k + 1$ defenders and $n - k$ attackers in $G'$.
As $k < \frac{n}{2} - 1$, $b_0$ is not protected in $S'$, a contradiction.

Finally we claim that $S = S' \cap V(G)$ is a defensive alliance of size exactly $k$. It
is clear that $|S| = k$. For $S \subseteq V(G)$ and $u \in V(G)$, $d_S(G, u)$ denotes the number
of neighbours that $u$ has in $S$ and $d_{S^c}(G, u)$ denotes the number of neighbours
that $u$ has outside $S$ of $G$. Note that for each $u \in S$, $d_S(G, u) = d_{S'}(G', u) - 1$ and
$d_{S^c}(G, u) = d_{S^c}(G', u) - 1$ As $S'$ is a defensive alliance in $G'$, we have
d_{S'}(G', u) + 1 \geq d_{S^c}(G', u)$ for all $u \in S$. This implies $d_S(G, u) + 1 \geq d_{S^c}(G, u)$
for all $u \in S$. This shows that $S$ is a defensive alliance. This completes the proof. 27
Clearly stars are trivially acyclic. Moreover, it is easy to verify that stars have pathwidth $24$ and treedepth $29$ at most two, which implies:

**Theorem 12.** Locally Minimal Defensive Alliance is $W[1]$-hard when parameterized by any of the following parameters:

- the feedback vertex set number,
- the treewidth and clique width of the input graph,
- the pathwidth and treedepth of the input graph.

7. **XP algorithm parameterized by treewidth**

This section presents an XP-algorithm for Locally Minimal Defensive Alliance problem parameterized by treewidth. We prove the following theorem:

**Theorem 13.** Given an $n$-vertex graph $G$ and its nice tree decomposition $T$ of width at most $k$, the size of a maximum locally minimal defensive alliance of $G$ can be computed in $O(18^k n^{4k+10})$ time.

Let $(T, \{X_t\}_{t \in V(T)})$ be a nice tree decomposition rooted at node $r$ of the input graph $G$. For a node $t$ of $T$, let $V_t$ be the union of all bags present in the subtree of $T$ rooted at $t$, including $X_t$. We denote by $G_t$ the subgraph of $G$ induced by $V_t$. Here we distinguish not only if a vertex is in the solution or not, but if it is in the solution we also distinguish if it is marginally protected or not.

A coloring of bag $X_t$ is a mapping $f : X_t \to \{b, w, r\}$ assigning three different colours to vertices of the bag. We give intuition behind the three colours.

- **White**, represented by $w$. The meaning is that all white vertices have to be contained in the partial solution in $G_t$.

- **Black**, represented by $b$. The meaning is that all black vertices have to be contained in the partial solution in $G_t$; additionally, all black vertices must be marginally protected in the final solution.
- **Red**, represented by \( r \). The meaning is that all red vertices are not contained in the partial solution in \( G_t \).

For a node \( t \), there are \( 3^{|X_t|} \) colourings \( X_t \). Now, for each node \( t \) in \( T \), we construct a table \( dp_t(f, \mathbf{p}, \mathbf{a}, \mathbf{v}, \alpha, \pi, \beta, \beta^*, \gamma) \in \{\text{true, false}\} \) where \( f \) is a colouring of \( X_t \), \( \mathbf{p} \) is a vector of length \( n \) such that

\[
\mathbf{p}(i) = \begin{cases} 
0 \text{ or } 1 & \text{if } v_i \in X_t \text{ and } f(v_i) \in \{b, w\} \\
* & \text{otherwise;}
\end{cases}
\]

\( \mathbf{a} \) and \( \mathbf{v} \) are vectors of length \( n \), and their \( i \)-th coordinates are positive only if \( v_i \) is in \( X_t \) and it is coloured \( b \) or \( w \); \( \alpha, \pi, \beta, \beta^* \) and \( \gamma \) are integers between 0 to \( n \). We set \( dp_t(f, \mathbf{p}, \mathbf{a}, \mathbf{v}, \alpha, \pi, \beta, \beta^*, \gamma) = \text{true} \) if and only if there exists a set \( A_t \subseteq V_t \) such that

1. \( \alpha = |A_t| = |\{v \in V_t : f(v) \in \{b, w\}\}| \)
2. \( f^{-1}\{b, w\} = A_t \cap X_t = A \), which is the set of vertices of \( X_t \) colored black or white.
3. the \( i \)-th coordinate of vector \( \mathbf{p} \) is

\[
\mathbf{p}(i) = \begin{cases} 
1 & \text{if } v_i \in X_t, f(v_i) \in \{b, w\} \text{ and } v_i \text{ has a black neighbour in } A_t \\
0 & \text{if } v_i \in X_t, f(v_i) \in \{b, w\} \text{ and } v_i \text{ has no black neighbours in } A_t \\
* & \text{otherwise}
\end{cases}
\]

4. the \( i \)-th coordinate of vector \( \mathbf{a} \) is

\[
\mathbf{a}(i) = \begin{cases} 
d_{A_t}(v_i) & \text{if } v_i \in X_t \text{ and } f(v_i) \in \{b, w\} \\
0 & \text{otherwise}
\end{cases}
\]

That is, \( \mathbf{a}(i) \) denotes the number of neighbours of vertex \( v_i \) in \( A_t \) if \( v_i \in X_t \) and \( f(v_i) \in \{b, w\} \).

5. the \( i \)-th coordinate of vector \( \mathbf{v} \) is

\[
\mathbf{v}(i) = \begin{cases} 
d_{V_t}(v_i) & \text{if } v_i \in X_t \text{ and } f(v_i) \in \{b, w\} \\
0 & \text{otherwise.}
\end{cases}
\]
That is, \( v(i) \) denotes the number of neighbours of vertex \( v_i \) in \( V_i \) if \( v_i \in X_t \) and \( f(v_i) \in \{ b, w \} \).

6. \( \pi \) is the number of vertices \( v \in A_t \) that are protected, that is, \( d_{A_t}(v) \geq \frac{d_G(v) - 1}{2} \).

7. \( \beta \) is the number of black vertices in \( A_t \).

8. \( \beta^* \) is the number of black vertices \( v \) in \( A_t \) such that \( N(v) \subseteq V_i \) and \( d_{A_t}(v) = \lceil \frac{d_G(v) - 1}{2} \rceil \). Thus \( \beta^* \) is the number of black vertices \( v \) in \( A_t \) that are marginally protected when all its neighbours are introduced in \( G_i \). The intuition here is that we want every back vertex to be marginally protected when all its neighbours are introduced.

9. \( \gamma \) is the number of vertices in \( A_t \) who has a black neighbour. In other words, \( \gamma \) is the number of good vertices in \( A_t \).

We compute all entries \( dp_t(f, p, a, v, \alpha, \pi, \beta, \beta^*, \gamma) \) in a bottom-up manner. Since \( tw(T) \leq k \), there are \( O(3^k \cdot 2^k \cdot n^k \cdot (n + 1)^5) = O(6^k n^{2k+5}) \) possible tuples \((f, p, a, v, \alpha, \pi, \beta, \beta^*, \gamma)\). Thus, to prove Theorem 13 it suffices to show that each entry \( dp_t(f, p, a, v, \alpha, \pi, \beta, \beta^*, \gamma) \) can be computed in \( O(3^k n^{2k+5}) \) time, assuming that the entries for the children of \( t \) are already computed.

**Lemma 14.** For a leaf node \( t \), \( dp_t(f, p, a, v, \alpha, \pi, \beta, \beta^*, \gamma) \) can be computed in \( O(1) \) time.

**Proof.** For leaf node \( t \) we have that \( X_t = \emptyset \). Thus \( dp_t(f, p, a, v, \alpha, \pi, \beta, \beta^*, \gamma) \) is true if and only if \( f = \emptyset, p = 0, a = 0, v = 0, \alpha = 0, \pi = 0, \beta = 0, \beta^* = 0 \) and \( \gamma = 0 \). These conditions can be checked in \( O(1) \) time.

**Lemma 15.** For an introduce node \( t \), \( dp_t(f, p, a, v, \alpha, \pi, \beta, \beta^*, \gamma) \) can be computed in \( O(1) \) time.

**Proof.** Suppose \( t \) is an introduce node with child \( t' \) such that \( X_t = X_{t'} \cup \{ v_i \} \) for some \( v_i \notin X_{t'} \). Let \( f \) be any coloring of \( X_t \). We consider three cases:

*Case (i):* Let \( f(v_i) = r \). In this case \( dp_t(f, p, a, v, \alpha, \pi, \beta, \beta^*, \gamma) \) is true if and only if \( dp_{t'}(f |_{X_{t'}}, p, a, v', \alpha, \pi, \beta, \beta^*, \gamma) \) is true where
\[ v(j) = \begin{cases} 
v'(j) + 1 & \text{if } j \neq i, v_j \in X_t, f(v_j) \in \{b, w\} \text{ and } v_j \in N_{X_t}(v_i) \\
v'(j) & \text{otherwise} \end{cases} \]

Case (ii): Let \( f(v_i) = b \). Here \( dp_t(f, p, a, v, \alpha, \pi, \beta, \beta^*, \gamma) \) is true if and only if there exist a tuple \((f', p', a', v', \alpha', \pi', \beta', \beta^*', \gamma')\) such that

1. \( f|_{X_t \setminus \{v_i\}} = f'|_{X_t \setminus \{v_i\}} \); 
2. \( p(j) = \begin{cases} 1 & \text{if } v_j \in X_t, f(v_j) \in \{b, w\} \text{ and } v_j \in N(v_i) \\
p'(j) & \text{otherwise} \end{cases} \)
3. \( a(j) = \begin{cases} a'(j) + 1 & \text{if } j \neq i, v_j \in X_t, f(v_j) \in \{b, w\} \text{ and } v_j \in N_{X_t}(v_i) \\
|N_A(v_i)| & \text{if } j = i \\
a'(j) & \text{otherwise} \end{cases} \)

where \( A = A_t \cap X_t \).
4. \( v(j) = \begin{cases} v'(j) + 1 & \text{if } j \neq i, v_j \in X_t, f(v_j) \in \{b, w\} \text{ and } v_j \in N_{X_t}(v_i) \\
|N_{X_t}(v_i)| & \text{if } j = i \\
v'(j) & \text{otherwise} \end{cases} \)
5. \( \alpha = \alpha' + 1; \)
6. \( \pi = \pi' + l; \) here \( l \) is the cardinality of the set \( \left\{ v_j \in X_t \mid f(v_j) \in \{b, w\}, a'(j) < \frac{d_G(v_j) - 1}{2}; a(j) \geq \frac{d_G(v_j) - 1}{2} \right\} \).

That is, to compute \( \pi \) from \( \pi' \) we need to add the number \( l \) of vertices \( v_j \in X_t \) which are not protected in \( X_t' \) but protected in \( X_t \).
7. \( \beta = \beta' + 1; \)
8. $\beta^* = \beta' + \delta$

where $\delta$ is the number of black vertices $v_j \in X_t$ such that $a'(j) \neq \lceil \frac{d_G(v_j) - 1}{2} \rceil$ or $v'(j) \neq d(v_j)$ but it satisfies the conditions $a(j) = \lceil \frac{d_G(v_j) - 1}{2} \rceil$ and $v(j) = d(v_j)$.

9. $\gamma = \gamma' + \left| \{ v_j \in A \mid p(j) = 1 \text{ but } p'(j) = 0 \} \right|$. 

Case (iii): Let $f(v_i) = w$. Here $dp_t(f, p, a, \alpha, \pi, \beta, \beta^*, \gamma)$ is true if and only if there exist a tuple $(f', p', a', \alpha', \pi', \beta', \beta^*, \gamma')$ such that $dp_t'(f', p', a', \alpha', \pi', \beta', \beta^*, \gamma') = $true, where

1. $f|_{X_t \setminus \{v_i\}} = f'$;

2. 

\[
p(j) = \begin{cases} 
p'(j) & \text{if } j \neq i \\
1 & \text{if } j = i \text{ and } v_i \text{ has a black neighbour in } X_t \\
0 & \text{if } j = i \text{ and } v_i \text{ has no black neighbours in } X_t
\end{cases}
\]

3. 

\[
a(j) = \begin{cases} 
a'(j) + 1 & \text{if } j \neq i, v_j \in X_t, f(v_j) \in \{b, w\} \text{ and } v_j \in N_{X_t}(v_i) \\
\left| N_A(v_i) \right| & \text{if } j = i \\
a'(j) & \text{otherwise}
\end{cases}
\]

where $A = A_t \cap X_t$.

4. 

\[
v(j) = \begin{cases} 
v'(j) + 1 & \text{if } j \neq i, v_j \in X_t, f(v_j) \in \{b, w\} \text{ and } v_j \in N_{X_t}(v_i) \\
\left| N_{X_t}(v_i) \right| & \text{if } j = i \\
v'(j) & \text{otherwise}
\end{cases}
\]

5. $\alpha = \alpha' + 1$;

6. $\pi = \pi' + l$; here $l$ is the cardinality of the set

\[
\left\{ v_j \in X_t \mid f(v_j) \in \{b, w\}, a'(j) < \frac{d_G(v_j) - 1}{2}; a(j) \geq \frac{d_G(v_j) - 1}{2} \right\}.
\]

That is, to compute $\pi$ from $\pi'$ we need to add the number $l$ of vertices $v_j \in X_t$ which are not protected in $X_{\nu'}$ but protected in $X_t$.
7. $\beta = \beta'$;
8. $\beta^* = \beta'^* + \delta$

where $\delta$ is the number of black vertices $v_j \in X_t$ such that $a'(j) \neq \lceil \frac{d_G(v_j) - 1}{2} \rceil$ or $v'(j) \neq d(v_j)$ but it satisfies the conditions $a(j) = \lceil \frac{d_G(v_j) - 1}{2} \rceil$ and $v(j) = d(v_j)$.

9. $\gamma = \gamma' + 1$ if $v_i$ is adjacent to a vertex in $X_t$ which is coloured black; otherwise $\gamma = \gamma'$.

For introduce node $t$, $dp_t(f, p, a, \alpha, \pi, \beta, \beta^*, \gamma)$ can be computed in $O(1)$ time as there is only one candidate of such tuple $(f', p', a', \alpha', \pi', \beta', \beta'^*, \gamma')$. 

**Lemma 16.** For a forget node $t$, $dp_t(f, p, a, v, \alpha, \pi, \beta, \beta^*, \gamma)$ can be computed in $O(n)$ time.

**Proof.** Suppose $t$ is a forget node with child $t'$ such that $X_t = X_{t'} \setminus \{v_i\}$ for some $v_i \in X_{t'}$. Here $dp_t(f, p, a, v, \alpha, \pi, \beta, \beta^*, \gamma)$ is true if and only if $dp_{t'}(f', p', a', \alpha', \pi', \beta', \beta'^*, \gamma')$ is true, where

1. $f' = f_{v_i \rightarrow b}, f_{v_i \rightarrow w}$ or $f_{v_i \rightarrow r}$.
2. $p(j) = \begin{cases} p'(j) & \text{if } j \neq i \\ * & \text{if } j = i \end{cases}$
3. $a(j) = a'(j)$ for all $j \neq i$ and $a(i) = 0$;
4. $v(j) = v'(j)$ for all $j \neq i$ and $v(i) = 0$;
5. $\alpha = \alpha'$;
6. $\pi = \pi'$;
7. $\beta = \beta'$;
8. $\beta^* = \beta'^*$;
9. $\gamma = \gamma'$. 

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There are \( n + 1 \) choices for \( a'(i) \) and \( v'(i) \) each. Thus the lemma follows as there are \( O(n) \) candidates of such tuples \((f', p', a', v', \alpha', \beta', \beta'^*, \gamma')\). This completes the proof of the lemma.

\[ \square \]

**Lemma 17.** For a join node \( t \), \( dp_t(f, p, a, v, \alpha, \pi, \beta, \beta^*, \gamma) \) can be computed in \( O(3^{k} n^{3k+5}) \) time.

**Proof.** Suppose \( t \) is a join node with children \( t_1 \) and \( t_2 \) such that \( X_t = X_{t_1} = X_{t_2} \). Then \( dp_t(f, p, a, v, \alpha, \pi, \beta, \beta^*, \gamma) \) is true if and only if there exist \((f_1, p_1, a_1, v_1, \alpha_1, \pi_1, \beta_1, \beta_1^*, \gamma_1)\) and \((f_2, p_2, a_2, v_2, \alpha_2, \pi_2, \beta_2, \beta_2^*, \gamma_2)\) such that \( dp_{t_1}(f_1, p_1, a_1, v_1, \alpha_1, \pi_1, \beta_1, \beta_1^*, \gamma_1) \) and \( dp_{t_2}(f_2, p_2, a_2, v_2, \alpha_2, \pi_2, \beta_2, \beta_2^*, \gamma_2) \) are true, where

1. \( f = f_1 = f_2 \);
2. \( p(i) = 1 \) if \( p_1(i) = 1 \) or \( p_2(i) = 1 \);
3. \( a(i) = a_1(i) + a_2(i) - d_A(v_i) \) for all \( v_i \in A \), and \( a(i) = 0 \) if \( v_i \notin A \) where \( A = \{ v \in X_t \mid f(v) \in \{b, w\} \} \);
4. \( v(i) = v_1(i) + v_2(i) - d_{X_t}(v_i) \) for all \( v_i \in A \), and \( v(i) = 0 \) if \( v_i \notin A \);
5. \( \alpha = \alpha_1 + \alpha_2 - |A| \);
6. \( \pi = \pi_1 + \pi_2 - l_1 + l_2 \);
   where \( l_1 \) is the cardinality of the set
   \[
   \{ v_j \in A \mid a_1(j) \geq \frac{d_G(v_j) - 1}{2}; \ a_2(j) \geq \frac{d_G(v_j) - 1}{2} \}
   \]
   and \( l_2 \) is the cardinality of the set
   \[
   \{ v_j \in A \mid a_1(j) < \frac{d_G(v_j) - 1}{2}; \ a_2(j) < \frac{d_G(v_j) - 1}{2}; \ a(j) \geq \frac{d_G(v_j) - 1}{2} \}
   \]
   To compute \( \pi \) from \( \pi_1 + \pi_2 \), we need to subtract the number of those \( v_j \) which are protected in both the branches and add the number of vertices \( v_j \) which are not protected in either of the branches \( t_1 \) and \( t_2 \) but protected in \( t \).
7. \( \beta = \beta_1 + \beta_2 - |\{ v \in A \mid f(v) = b \}| \);

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8. $\beta^* = \beta_1^* + \beta_2^* + \delta_1 + \delta_2 - \delta_{12}$.

Here $\delta_1$ is the number of black vertices $v_j$ in $X_i$ such that $a_1(j) \neq [\frac{d_2(v_j) - 1}{2}]$ or $v_1(j) \neq d(v_j)$ but it satisfies the conditions $a(j) = [\frac{d_2(v_j) - 1}{2}]$ and $v(j) = d(v_j)$. Similarly, $\delta_2$ is the number of black vertices $v_j$ in $X_i$ such that either $a_2(j) \neq [\frac{d_2(v_j) - 1}{2}]$ or $v_2(j) \neq d(v_j)$ but it satisfies the conditions $a(j) = [\frac{d_2(v_j) - 1}{2}]$ and $v(j) = d(v_j)$. Finally $\delta_{12}$ is the number of black vertices $v_j$ in $X_i$ such that $a_1(j) \neq [\frac{d_2(v_j) - 1}{2}]$ or $v_1(j) \neq d(v_j)$, and $a_2(j) \neq [\frac{d_2(v_j) - 1}{2}]$ or $v_2(j) \neq d(v_j)$ but it satisfies the conditions $a(j) = [\frac{d_2(v_j) - 1}{2}]$ and $v(j) = d(v_j)$.

9. $\gamma = \gamma_1 + \gamma_2 - |\{v \in A \mid p_1(v) = p_2(v) = 1\}|$.

For join node $t$, there are at most $3^k$ possible pairs for $(p_1, p_2)$ as $(p_1(i), p_2(i)) \in \{(1, 0), (0, 1), (1, 1)\}$ when $p(i) = 1$ and $(p_1(i), p_2(i)) = (0, 0)$ when $p(i) = 0$; there are $n^k$ possible pairs for $(a_1, a_2)$ as $a_2$ is uniquely determined by $a_1$; there are $n^k$ possible pairs for $(v_1, v_2)$ as $v_2$ is uniquely determined by $v_1$; $n + 1$ possible pairs for $(\alpha_1, \alpha_2)$; $n + 1$ possible pairs for $(\pi_1, \pi_2)$; $n + 1$ possible pairs for $(\beta_1, \beta_2)$; $n + 1$ possible pairs for $(\beta_1^*, \beta_2^*)$; and $n + 1$ possible pairs for $(\gamma_1, \gamma_2)$.

In total, there are $O(3^k n^{2k+5})$ candidates, and each of them can be checked in $O(1)$ time. Thus, for join node $t$, $dp_t(f, p, a, v, \alpha, \pi, \beta, \beta^*, \gamma)$ can be computed in $O(3^k n^{2k+5})$ time.

At the root node $r$, we look at all records such that $dp_r(\emptyset, \emptyset, \emptyset, \emptyset, \alpha, \pi, \beta, \beta^*, \gamma)$ = true, $\beta = \beta^*$ (that is, all black vertices in the solution are marginally protected) and $\alpha = \pi = \gamma$ (that is, every vertex in the solution is protected and has a black or marginally protected neighbour). The size of a maximum locally minimal defensive alliance is the maximum $\alpha$ satisfying $dp_r(\emptyset, \emptyset, \emptyset, \emptyset, \alpha, \pi, \beta, \beta^*, \gamma)$ = true, $\alpha = \pi = \gamma$ and $\beta = \beta^*$.

**Remark.** The above algorithm implies that **Locally Minimal Defensive Alliance** can be solved in polynomial time on trees. It is not difficult to modify the above algorithm to find a connected locally minimal defensive al-
liance of maximum size. This means that one can also get an XP algorithm for \textsc{Connected Locally Minimal Defensive Alliance} parameterized by treewidth.

8. Conclusion

The main contributions in this paper are that \textsc{Locally Minimal Defensive Alliance} is W[1]-hard when parameterized by any of the following parameters: feedback vertex set number, treewidth, clique width, pathwidth and treedepth of the input graph, and the problem is XP in treewidth. We also proved that \textsc{Locally Minimal Defensive Alliance} is NP-complete on planar graphs and FPT when parameterized by neighbourhood diversity. We gave a randomized FPT algorithm for \textsc{Exact Connected Locally Minimal Defensive Alliance}. We list some nice problems emerge from the results here. The question whether \textsc{Locally Minimal Defensive Alliance} is FPT when parameterized by solution size has still remained open. Noting that the result for neighbourhood diversity implies that the problem is FPT in vertex cover, it would be interesting to consider the parameterized complexity with respect to twin cover. The modular width parameter also appears to be a natural parameter to consider here, and since there are graphs with bounded modular-width and unbounded neighbourhood diversity; we believe this is also an interesting open problem. The parameterized complexity of \textsc{Locally Minimal Defensive Alliance} remains unsettled when parameterized by other important structural graph parameters like vertex integrity and cluster vertex deletion.

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