Analytical Solutions of the Riccati Equation with Coefficients Satisfying Integral or Differential Conditions with Arbitrary Functions

Tiberiu Harko\textsuperscript{1,*}, Francisco S. N. Lobo\textsuperscript{2}, M. K. Mak\textsuperscript{3}

\textsuperscript{1}Department of Mathematics, University College London, Gower Street, London WC1E 6BT, United Kingdom
\textsuperscript{2}Centro de Astronomia e Astrofísica da Universidade de Lisboa, Campo Grande, Ed. C8 1749-016 Lisboa, Portugal
\textsuperscript{3}Department of Computing and Information Management, Hong Kong Institute of Vocational Education, Chai Wan, Hong Kong, P. R. China

*Corresponding Author: t.harko@ucl.ac.uk

Abstract Ten new exact solutions of the Riccati equation $\frac{dy}{dx} = a(x) + b(x)y + c(x)y^2$ are presented. The solutions are obtained by assuming certain relations among the coefficients $a(x)$, $b(x)$ and $c(x)$ of the Riccati equation, in the form of some integral or differential expressions, also involving some arbitrary functions. By appropriately choosing the form of the coefficients of the Riccati equation, with the help of the conditions imposed on the coefficients, we obtain ten new integrability cases for the Riccati equation. For each case the general solution of the Riccati equation is also presented. The possibility of the application of the obtained mathematical results for the study of anisotropic general relativistic stellar models is also briefly considered.

Keywords Riccati equation; integrability condition; exact solutions: anisotropic relativistic stellar models; stars

1 Introduction

The Riccati equation is one of the most studied first order non-linear differential equations [1, 2, 3], and is given by

$$\frac{dy}{dx} = a(x) + b(x)y + c(x)y^2, \quad (1)$$

where $a$, $b$, $c$ are arbitrary real functions of $x$, with $a,b,c \in \mathbb{C}^{\infty}(I)$, defined on a real interval $I \subseteq \mathbb{R}$.

It is well-known that once a particular solution $y_p(x)$ of the Riccati equation is known, the general solution of Eq. (1) is given by

$$y[b(x),c(x),y_p(x)] = y_p(x) + \frac{e^{\int^x [b(\phi) + 2c(\phi)y_p(\phi)]d\phi}}{C - \int^x c(\psi)e^{\int^\psi [b(\phi) + 2c(\phi)y_p(\phi)]d\phi}d\psi}, \quad (2)$$

where $C$ is an arbitrary constant of integration and the particular solution $y_p(x)$ satisfies the Riccati equation

$$\frac{dy_p}{dx} = a(x) + b(x)y_p(x) + c(x)y_p^2(x). \quad (3)$$

If we know three particular solutions $y_i(x)$, $i = 1, 2, 3$, then the Riccati equation can be solved without quadratures [1, 2, 3].

Even when a particular solution is not known, the Riccati equation can be integrated exactly if the coefficients of the equation satisfy some conditions. For instance, if the coefficients of the Riccati equation satisfy the following specific condition

$$a(x) + b(x) + c(x) \equiv 0, \quad (4)$$

then the solution of the Riccati equation is given by

$$y = \frac{K + \int [c(x) + a(x)]E(x)dx - E(x)}{K + \int [c(x) + a(x)]E(x)dx + E(x)}, \quad (5)$$
where $K$ is an arbitrary constant of integration, and $E(x) = \exp(\int [c(x) - a(x)]dx$ [1].

If the coefficients of the Riccati equation satisfy the more general condition

$$\lambda^2 c(x) + \lambda \mu b(x) + \mu^2 a(x) \equiv 0,$$

where $\lambda$ and $\mu$ are arbitrary constants satisfying the condition $|\lambda| + |\mu| > 0$, then by means of the transformation $y(x) = \lambda/\mu + u(x)$, the Riccati equation is transformed into a Bernoulli type equation [1],

$$u'(x) = c(x)u^2(x) + \left[\frac{2\lambda}{\mu}c(x) + b(x)\right]u(x).$$

Another interesting case is if the coefficients $a(x)$, $b(x)$, $c(x)$ satisfy the relation

$$a(x) + b(x) + c(x) = \frac{d}{dx} \ln \left[\frac{\alpha(x)}{\beta(x)}\right] - \frac{\alpha(x) - \beta(x)}{\alpha(x)\beta(x)} [\alpha(x)c(x) - \beta(x)a(x)],$$

with $\alpha(x)$ and $\beta(x)$ properly chosen differentiable functions, such that $\alpha\beta > 0$, then the Riccati equation is integrable by quadratures [4]. If $c(x) \equiv 1$, and the functions $a(x)$ and $b(x)$ are polynomials satisfying the condition

$$\Delta = b^2(x) - 2\frac{db(x)}{dx} - 4a(x) \equiv \text{constant},$$

then

$$y\pm(x) = -\frac{b(x) \pm \sqrt{\Delta}}{2},$$

are both solutions of the Riccati Eq. (1) [1, 3, 5]. Very recently, the integrability condition given by Eq. (9) of the Riccati Eq. (1), and of the reduced Riccati equation of the form $dy/dx = a(x) + c(x)y^2$, have been generalized in [6, 7].

Note that the Riccati equation plays a significant role in many fields of applied and fundamental science [8]. Some applications of the integrability conditions for the case of the damped harmonic oscillator with time dependent frequency and for solitonic wave have been briefly discussed in [6]. The applications of the integrability conditions of the reduced Riccati equation for the integration of the Schrödinger and Navier-Stokes equations have also been studied in [7]. The integrability conditions derived in [6, 7] have been applied in [9] to obtain a general solution of the Einstein’s gravitational field equations for the static spherically symmetric gravitational interior space-time of an isotropic fluid sphere. The astrophysical analysis indicates that this solution can be used as a realistic model for static general relativistic high density objects, such as neutron stars. Riccati equations also play an important role in cosmology. The mechanism of the initial inflationary scenario of the Universe and of its late-time acceleration can be described by assuming the existence of some gravitationally coupled scalar fields $\phi$, with the inflaton field generating inflation and the quintessence field being responsible for the late accelerated expansion of the Universe. In the case of a scalar field dominated Universe the dynamics of the expansion can be described by the solutions of a Riccati type equation [10]. In fact, the integrability conditions of the Riccati equation, given by Eq. (1), and its applications to stellar and cosmological models have been extensively discussed in the literature, and we refer the reader to [11, 12, 13, 14].

Due to the nonlinear structure of the Riccati equation, the general solution of the Riccati equation (1), cannot be easily found. Therefore, one has to use numerical techniques, or approximate methods for obtaining its solutions. Recently, a numerical method, using hybrid of block-pulse functions, and Chebyshev polynomials for solving the Riccati equation has been presented in [15]. Various numerical methods such as Adomian decomposition method, He’s variational iteration method, homotopy perturbation method, Taylor matrix method and Legendre wavelet method for solving the Riccati equation have been proposed in [16, 17, 18, 19, 20, 21, 22].

It is the purpose of the present paper to present some further integrability conditions of the Riccati equation, given by Eq. (1). The relations among the coefficients of the Riccati equation involve some integrals or differential representations, as well as the presence of some arbitrary functions. By appropriately choosing the form of the coefficients of the Riccati equation we obtain ten new integrability conditions. For each case the general solution of the Riccati equation is also obtained. The possibility of the application of the obtained mathematical results for the study of the anisotropic general relativistic stellar models is also briefly considered.

The present paper is organized as follows. The ten new exact solutions of the Riccati Eq. (1) are presented in Section 2. Some astrophysical applications of the solutions of the Riccati equation are presented in Section 3. We conclude our results in Section 4.

## 2 Integrability conditions for the Riccati equation

From the algebraic point of view the Riccati Eq. (1) is a quadratic equation in $y(x)$. We consider that its particular solutions $y^p_\pm(x)$ take the form

$$y^p_\pm(x) = \frac{-b(x) \pm \sqrt{b^2(x) - 4a(x)c(x) + 4c(x)\frac{du}{dx}}}{2c(x)}.$$  

(11)
In order to obtain the general solution of the Riccati Eq. (1) with the help of Eq. (11), we introduce the new generating function \( f_1 (x) \), satisfying the differential condition given by

\[
b^2 (x) + 4c(x) \frac{dy^p}{dx} = f_1 (x),
\]

representing a first order differential equation in \( y^p(x) \), and which can be immediately integrated to give the particular solution of the Riccati Eq. (1). Therefore the general solution of the Riccati Eq. (1) can be obtained through quadratures, since its particular solution is known.

Based on the above algorithm, we shall show the detailed calculations in obtaining Theorem 1 in Section 2.1, case 1. In order to make the paper readable and not to have similar calculations, we shall not present the detailed calculations for the rest of the theorems here. However, one may follow the same procedures to obtain the Theorems 2-9 presented in this paper. In this Section, we shall present ten new integrability cases of the Riccati Eq. (1). For each case the general solution of the Riccati equation is explicitly obtained.

### 2.1 Case 1: \( a (x) \)

We assume that the arbitrary functions \( b (x), c(x) \) and \( f_1(x) \) satisfy the differential condition

\[
b^2 (x) + 4c(x) \frac{dy^p}{dx} = f_1 (x),
\]

where we have introduced a generating function \( f_1 (x) \). The particular solutions \( y^p_\pm (x) \) of the Riccati Eq. (1) take the form

\[
y^p_\pm (x) = \frac{-b (x) \pm \sqrt{f_1 (x) - 4a(x)c(x)}}{2c(x)} = \frac{1}{2} \left[ f_1 (\phi) - b^2 (\phi)/2c(\phi) \right] d\phi - C_1,
\]

where we have used and integrated Eq. (13), and \( C_1 \) is an arbitrary constant of integration. By differentiating Eq. (14) with respect to \( x \) yields the result

\[
\frac{d}{dx} \left[ \frac{-b (x) \pm \sqrt{f_1 (x) - 4a(x)c(x)}}{c(x)} \right] = \frac{f_1 (x) - b^2 (x)}{2c(x)}.
\]

Eq. (15) can be integrated to give the coefficient \( a (x) \) of the Riccati Eq. (1) as

\[
a (x) = \frac{f_1 (x) - \left\{ b (x) + c(x) \left[ \int f_1 (\phi) - b^2 (\phi)/2c(\phi) d\phi - C_1 \right] \right\}^2}{4c(x)},
\]

where \( C_1 \) is an arbitrary constant of integration. By substituting Eq. (16) into the Riccati Eq. (1), the latter can be expressed as

\[
\frac{dy}{dx} = \frac{f_1 (x) - \left\{ b (x) + c(x) \left[ \int f_1 (\phi) - b^2 (\phi)/2c(\phi) d\phi - C_1 \right] \right\}^2}{4c(x)} + b(x)y + c(x)y^2.
\]

Therefore we obtain the following:

**Theorem 1.** If the coefficient \( a(x) \) of the Riccati Eq. (1) satisfies the integral condition (16), then the general solution of the Riccati Eq. (1) is given by

\[
y(x) = \frac{e^{\int \left\{ b(\phi) + c(\phi) \left[ \int f_1 (\phi) - b^2 (\phi)/2c(\phi) d\phi - C_1 \right] \right\} d\phi}}{C_0 - \int c(\psi) e^{\int \left\{ b(\psi) + c(\psi) \left[ \int f_1 (\psi) - b^2 (\psi)/2c(\psi) d\psi - C_1 \right] \right\} d\psi} d\psi} + \frac{1}{2} \left[ \int f_1 (\phi) - b^2 (\phi)/2c(\phi) d\phi - C_1 \right],
\]

where \( C_0 \) is an arbitrary constant of integration.

### 2.2 Case 2: \( a (x) \)

Next, we assume that the arbitrary functions \( a(x) \) satisfies the differential condition

\[
a (x) = \frac{d}{dx} \left[ \frac{-b (x) \pm \sqrt{f_2 (x) + b^2 (x)}}{2c(x)} \right] - \frac{f_2 (x)}{4c(x)}.
\]
where we have introduced a new arbitrary function \( f_2(x) \in C^\infty(I) \) defined on a real interval \( I \subseteq \mathbb{R} \). By substituting Eq. (19) into the Riccati Eq. (1), the latter can be expressed as

\[
\frac{dy_\pm}{dx} = \left\{ \frac{d}{dx} \left[ -\frac{b(x) \pm \sqrt{f_2(x) + b^2(x)}}{2c(x)} \right] - \frac{f_2(x)}{4c(x)} \right\} + b(x)y_\pm + c(x)y_\pm^2. \tag{20}
\]

Therefore we obtain the following:

**Theorem 2.** If the coefficient \( a(x) \) of the Riccati Eq. (1) satisfies the differential condition (19), then the general solutions of the Riccati Eq. (20) are given by

\[
y_\pm(x) = \frac{e^{\pm \int \sqrt{f_2(x) + b^2(x)} \, d\phi}}{C_{\pm 2} - \int^x_c \left( \int \sqrt{f_2(x) + b^2(x)} \, d\phi \right) \, d\psi} + \left[ -\frac{b(x) \pm \sqrt{f_2(x) + b^2(x)}}{2c(x)} \right], \tag{21}
\]

where \( C_{\pm 2} \) are arbitrary constants of integration.

### 2.3 Case 3: \( b(x) \)

We now assume that the arbitrary function \( b(x) \) satisfies the integral condition

\[
b(x) = \frac{f_2(x) - 4c^2(x)}{4c(x)} \left\{ \int^x_c a(\phi) + \frac{f_2(\phi)}{4c(\phi)} \, d\phi - C_3 \right\}^2, \tag{22}
\]

where \( C_3 \) is an arbitrary constant. By substituting Eq. (22) into the Riccati Eq. (1), the latter can be expressed as

\[
\frac{dy}{dx} = a(x) + \frac{f_2(x) - 4c^2(x)}{4c(x)} \left\{ \int^x_c a(\phi) + \frac{f_2(\phi)}{4c(\phi)} \, d\phi - C_3 \right\} y + c(x)y^2. \tag{23}
\]

Therefore we obtain the following:

**Theorem 3.** If the coefficient \( b(x) \) of the Riccati Eq. (1) satisfies the integral condition (22), then the general solution of the Riccati Eq. (23) is given by

\[
y(x) = -\frac{f_2(\omega) + \left\{ \int^x_c a(\phi) + \frac{f_2(\phi)}{4c(\phi)} \, d\phi - C_3 \right\}^2 \, d\omega}{C_4 - \int^x_c \left\{ \int \sqrt{f_2(x) + b^2(x)} \, d\phi \right\} \, d\psi} - \int^x_c \left\{ \int \sqrt{f_2(x) + b^2(x)} \, d\phi \right\} \, d\psi - \frac{1}{2c(x)} \left\{ \int^x_c a(\phi) + \frac{f_2(\phi)}{4c(\phi)} \, d\phi - C_3 \right\}^2 \right\}, \tag{24}
\]

where \( C_4 \) is an arbitrary constant of integration.

### 2.4 Case 4: \( c(x) \)

We assume that the arbitrary function \( c(x) \) satisfies the integral condition

\[
c(x) = \frac{-\frac{b(x) \pm \sqrt{f_2(x) + b^2(x)}}{2c(x)} e^{\int^x_c \sqrt{f_2(x) + b^2(x)} \, d\phi}}{2 \left\{ \int^x_c a(\psi) e^{\int^\psi_c \sqrt{f_2(x) + b^2(x)} \, d\phi} \, d\psi \right\}} \left[ \frac{f_2(x)}{-b(x) \pm \sqrt{f_2(x) + b^2(x)}} \right] \tag{25}
\]

\[
-\frac{1}{2} \int^x_c a(\phi) + \frac{f_2(\phi)}{4c(\phi)} \, d\phi - C_3 \right\} \right\}^2 \right\}, \tag{24}
\]

where \( C_4 \) is an arbitrary constant of integration.
where $C_{\pm 5}$ are arbitrary constants. By substituting Eq. (25) into the Riccati Eq. (1), the latter can be expressed as

$$\frac{dy}{dx} = \frac{a(x) + b(x)y}{2C_{\pm 5} + \int z a(\psi) e^{-\frac{1}{4} \int x f_{2}(\phi) d\phi} - \frac{1}{2} \int x f_{2}(\phi) d\phi - b(x)^{2}}.$$  (26)

Therefore we obtain the following:

**Theorem 4.** If the coefficient $c(x)$ of the Riccati Eq. (1) satisfies the integral condition (25), then the general solutions of the Riccati Eq. (26) are given by

$$y(x) = \frac{e^{\frac{1}{4} \int x f_{2}(\phi) d\phi} - \frac{1}{2} \int x f_{2}(\phi) d\phi - b(x)^{2}}{C_{\pm 6} - \int x a(\psi) e^{-\frac{1}{4} \int x f_{2}(\phi) d\phi} - \frac{1}{2} \int x f_{2}(\phi) d\phi - b(x)^{2}}.$$  (27)

where $C_{\pm 6}$ are arbitrary constants of integration.

### 2.5 Case 5: $a(x)$

$$a(x) = \frac{1}{4} \left\{ f_{3}(x) c(x) - 2b(x) \left[ \int x f_{3}(\phi) d\phi - C_{7} \right] - c(x) \left[ \int x f_{3}(\phi) d\phi - C_{7} \right]^{2} \right\}.$$  (28)

Assume now that the arbitrary function $a(x)$ satisfies the integral condition

$$a(x) = \frac{1}{4} \left\{ f_{3}(x) c(x) - 2b(x) \left[ \int x f_{3}(\phi) d\phi - C_{7} \right] - c(x) \left[ \int x f_{3}(\phi) d\phi - C_{7} \right]^{2} \right\} + b(x) y + c(x) y^{2}.$$  (29)

Therefore we obtain the following:

**Theorem 5.** If the coefficient $a(x)$ of the Riccati Eq. (1) satisfies the integral condition (28), then the general solution of the Riccati Eq. (29) is given by

$$y(x) = \frac{e^{\frac{1}{4} \int x f_{2}(\phi) d\phi} - \frac{1}{2} \int x f_{2}(\phi) d\phi - b(x)^{2}}{C_{8} - \int x c(\omega) e^{-\frac{1}{4} \int x f_{2}(\phi) d\phi} - \frac{1}{2} \int x f_{2}(\phi) d\phi - b(x)^{2}}.$$  (30)

where $C_{8}$ is an arbitrary constant of integration.

### 2.6 Case 6: $b(x)$

$$b(x) = \frac{f_{3}(x) - 4a(x) c(x) - c^{2}(x) \left[ \int x f_{2}(\phi) d\phi - C_{7} \right]^{2}}{2c(x) \left[ \int x f_{2}(\phi) d\phi - C_{7} \right]}.$$  (31)

In this case, we assume that the coefficient $b(x)$ of the Riccati Eq. (1) satisfies the integral condition

$$b(x) = \frac{f_{3}(x) - 4a(x) c(x) - c^{2}(x) \left[ \int x f_{2}(\phi) d\phi - C_{7} \right]^{2}}{2c(x) \left[ \int x f_{2}(\phi) d\phi - C_{7} \right]}.$$  (31)

By substituting Eq. (31) into the Riccati Eq. (1), the latter can be expressed as

$$\frac{dy}{dx} = a(x) + \frac{f_{3}(x) - 4a(x) c(x) - c^{2}(x) \left[ \int x f_{2}(\phi) d\phi - C_{7} \right]^{2}}{2c(x) \left[ \int x f_{2}(\phi) d\phi - C_{7} \right]} y + c(x) y^{2}.$$  (32)
Therefore we obtain the following:

**Theorem 6.** If the coefficient \( b(x) \) of the Riccati Eq. (1) satisfies the integral condition (31), then the general solution of the Riccati Eq. (32) is given by

\[
y(x) = \frac{-f^2}{c} \left\{ \frac{f_3(x) - 4a(x)c(x) - c^2(x) \left[ f_4(x) \right]^{2} - 2a(x)c(x) - 2b(x)f_4(x) + f_1(x) \right\} - 4a(x)c(x) + f_3(x) + \right\} + \frac{1}{2c(x)} \left\{ \right.\]

where \( C_9 \) is an arbitrary constant of integration.

**2.7 Case 7:** \( a(x) = \frac{1}{4c(x)} \left[ 2c(x) \frac{d}{dx} \left( \frac{f_4}{c} \right) - f_4^2(x) - 2b(x)f_4(x) \right] \)

We assume that the coefficient \( a(x) \) of the Riccati Eq. (1) satisfies the differential condition

\[
a(x) = \frac{1}{4c(x)} \left[ 2c(x) \frac{d}{dx} \left( \frac{f_4}{c} \right) - f_4^2(x) - 2b(x)f_4(x) \right],
\]

where \( f_4(x) \in C^\infty(I) \) is an arbitrary function defined on a real interval \( I \subseteq \mathbb{R} \).

By substituting Eq. (34) into the Riccati Eq. (1), the latter can be expressed as

\[
\frac{dy}{dx} = \frac{1}{4c(x)} \left[ 2c(x) \frac{d}{dx} \left( \frac{f_4}{c} \right) - f_4^2(x) - 2b(x)f_4(x) \right] + b(x)y + c(x)y^2.
\]

Therefore we obtain the following:

**Theorem 7.** If the coefficient \( a(x) \) of the Riccati Eq. (1) satisfies the differential condition (34), then the general solution of the Riccati Eq. (35) is given by

\[
y(x) = \frac{e^{\int^x b(\phi) + f_4(\phi) d\phi}}{C_{10} - \int^x c(\psi) e^{\int^\infty b(\phi) + f_4(\phi) d\phi} d\psi} + \frac{f_1(x)}{2c(x)}.
\]

where \( C_{10} \) is an arbitrary constant of integration.

**2.8 Case 8:** \( b(x) = \frac{1}{f_4(x)} \left[ c(x) \frac{d}{dx} \left( \frac{f_4}{c} \right) - \frac{f_4^2(x)}{2} - 2a(x)c(x) \right] \)

We assume that the coefficient \( b(x) \) of the Riccati Eq. (1) satisfies the differential condition

\[
b(x) = \frac{1}{f_4(x)} \left[ c(x) \frac{d}{dx} \left( \frac{f_4}{c} \right) - \frac{f_4^2(x)}{2} - 2a(x)c(x) \right].
\]

By substituting Eq. (37) into the Riccati Eq. (1), the latter can be expressed as

\[
\frac{dy}{dx} = a(x) + \frac{1}{f_4(x)} \left[ c(x) \frac{d}{dx} \left( \frac{f_4}{c} \right) - \frac{f_4^2(x)}{2} - 2a(x)c(x) \right] + b(x)y + c(x)y^2.
\]

Therefore we obtain the following:

**Theorem 8.** If the coefficient \( b(x) \) of the Riccati Eq. (1) satisfies the differential condition (37), then the general solution of the Riccati Eq. (38) is given by

\[
y(x) = \frac{e^{\int^x \frac{1}{\phi^2} \left[ c(\phi) \frac{d}{d\phi} \left( \frac{f_4}{c} \right) - \frac{f_4^2(\phi)}{2} - 2a(\phi)c(\phi) \right] + f_4(\phi) \right] d\phi}}{C_{11} - \int^x c(\psi) e^{\int^\infty \frac{1}{\phi^2} \left[ c(\phi) \frac{d}{d\phi} \left( \frac{f_4}{c} \right) - \frac{f_4^2(\phi)}{2} - 2a(\phi)c(\phi) \right] + f_4(\phi) \right] d\phi} + \frac{f_1(x)}{2c(x)}.
\]

where \( C_{11} \) is an arbitrary constant of integration.
2.9 Case 9: \( c(x) = \frac{f_4(x)e^{-\int^\phi f_4(\phi)\,d\phi}}{C_{12} + 2 \int^\phi a(\psi)e^{-\int^\phi f_4(\phi)\,d\phi}\,d\psi} \)

We assume that the coefficient \( c(x) \) of the Riccati Eq. (1) satisfies the integral condition

\[
c(x) = \frac{f_4(x)e^{-\int^\phi f_4(\phi)\,d\phi}}{C_{12} + 2 \int^\phi a(\psi)e^{-\int^\phi f_4(\phi)\,d\phi}\,d\psi},
\]

where \( C_{12} \) is an arbitrary constant. By substituting Eq. (40) into the Riccati Eq. (1), the latter can be expressed as

\[
dy{dx} = a(x) + b(x)y + \frac{f_4(x)e^{-\int^\phi f_4(\phi)\,d\phi}}{C_{12} + 2 \int^\phi a(\psi)e^{-\int^\phi f_4(\phi)\,d\phi}\,d\psi} y^2.
\]

Therefore we obtain the following:

**Theorem 9.** If the coefficient \( c(x) \) of the Riccati Eq. (1) satisfies the integral condition (40), then the general solution of the Riccati Eq. (41) is given by

\[
y(x) = \frac{e^{\int^x f_4(\phi)\,d\phi} f_4(\psi)e^{-\int^\phi f_4(\phi)\,d\phi}}{C_{13} - \int^\phi f_4(\psi)e^{-\int^\phi f_4(\phi)\,d\phi} C_{12} + 2 \int^\phi a(\omega)e^{-\int^\phi f_4(\phi)\,d\phi}\,d\omega} e^{\int^\phi f_4(\phi)\,d\phi} d\psi} + \frac{1}{2} \left( C_{12} + 2 \int^\phi a(\omega)e^{-\int^\phi f_4(\phi)\,d\phi}\,d\omega \right) e^{\int^\phi f_4(\phi)\,d\phi} d\psi
\]

where \( C_{13} \) is an arbitrary constant of integration.

2.10 Case 10: \( a(x) = \frac{b^2(x)-4c^2(x)f_2^2(x)}{4c(x)} + \frac{d}{dx} \left( -\frac{b}{2c} \pm f_5 \right) \)

We assume that the coefficient \( a(x) \) of the Riccati Eq. (1) satisfies the differential condition

\[
a(x) = \frac{b^2(x)-4c^2(x)f_2^2(x)}{4c(x)} + \frac{d}{dx} \left( -\frac{b}{2c} \pm f_5 \right),
\]

where we have introduced an arbitrary function \( f_5(x) \in C^\infty(I) \) defined on a real interval \( I \subseteq \mathbb{R} \). By substituting Eq. (43) into the Riccati Eq. (1), the latter can be expressed as

\[
dy{dx} = \left[ \frac{b^2(x)-4c^2(x)f_2^2(x)}{4c(x)} + \frac{d}{dx} \left( -\frac{b}{2c} \pm f_5 \right) \right] + b(x)y_\pm + c(x)y_\pm^2.
\]

Therefore we obtain the following:

**Theorem 10.** If the coefficient \( a(x) \) of the Riccati Eq. (1) satisfies the differential condition (43), then the general solutions of the Riccati Eq. (44) are given by

\[
y_\pm(x) = \frac{e^{\pm\int^x c(\phi)f_5(\phi)d\phi}}{C_{14} + \int^x c(\psi)e^{\pm\int^\phi c(\phi)f_5(\phi)d\phi}d\psi} b(x) \pm f_5(x),
\]

where \( C_{14} \) are arbitrary constants of integration.

3 Applications in physics: anisotropic stars in general relativity

In standard coordinates \( x^i = (t, r, \chi, \phi) \), the line element for a static spherically symmetric space-time takes the form [23]

\[
ds^2 = A^2(r)\,dt^2 - V^{-1}(r)\,dr^2 - r^2(d\chi^2 + \sin^2\chi\,d\phi^2).
\]

For the metric (46), Einstein’s gravitational field equations (where natural units \( 8\pi G = c = 1 \) have been used throughout) describing the evolution of the star take the form [23]

\[
\rho(x) = \frac{1 - V(x)}{x} - 2\frac{dV}{dx},
\]

\[
p_r(x) = 4V(x)\frac{d\ln A}{dx} + \frac{V(x) - 1}{x},
\]

and

\[
[1 - 2\rho(x)]\frac{d^2A}{dx^2} - \left[ x\frac{d\eta}{dx} + \eta(x) \right] \frac{dA}{dx} - \left[ \frac{1}{2} \frac{d\eta}{dx} + \frac{\Delta(x)}{4x} \right] A = 0,
\]
where we have used the coordinate transformation $x = r^2$, $\rho(x)$ is the energy density, $p_r(x)$ is the pressure in the direction $\chi^i$ (normal pressure), and $p_{\perp}(x)$ is the pressure orthogonal to $\chi_i$ (transversal pressure), respectively. $\chi^i$ is the unit space-like vector defined as $\chi^i = \sqrt{\gamma^i_\perp}$. We assume that $p_r(x) \neq p_{\perp}(x)$.

The anisotropy parameter $\Delta(x)$ is defined as $\Delta(x) = p_{\perp}(x) - p_r(x)$. We have also defined the following functions,

$$V(x) = 1 - 2x\eta(x), \quad \eta(r) = \frac{m(r)}{r^3},$$

and

$$2m(r) = \int_0^r \xi^2 \rho(\xi) d\xi,$$

respectively. The function $m(r)$ represents the mass distribution within radius $r$. By introducing a new function $u(x)$, defined as

$$u(x) = \frac{1}{A} \frac{dA}{dx},$$

provides the metric function $A(x)$ in the form

$$A(x) = A_0 e^{\int u(\phi) d\phi},$$

with $A_0$ an arbitrary constant of integration. By substituting Eq. (52) into Eq. (49), the latter becomes a Riccati equation of the form

$$\frac{du}{dx} = \frac{1}{2} \frac{dn}{dx} + \frac{\Delta(x)}{1 - 2x\eta(x)} + \frac{x}{1 - 2x\eta(x)} u - u^2.$$  

The physical quantities $\eta(x)$ and $\Delta(x)$ are free arbitrary functions here. Now, by comparing the Riccati Eq. (54) with the Riccati Eq. (1), one gets the following settings

$$y \to u, \quad a(x) \to \frac{1}{2} \frac{dn}{dx} + \frac{\Delta(x)}{1 - 2x\eta}, \quad b(x) \to \frac{x}{1 - 2x\eta}, \quad c(x) \to -1,$$

respectively. By inserting these settings into the ten theorems presented in Section II, we can obtain ten solutions of the Riccati Eq. (54), and, consequently, ten solutions of the interior Einstein’s gravitational field equations. In order not to have repetitive calculations, we shall not present these results in the present paper.

## 4 Conclusions

In the present paper, we have obtained a number of integrability conditions of the Riccati Eq. (1). All the theorems presented in this paper have been obtained with the help of Eq. (2). By using Eq. (3), Eq. (2) can be expressed equivalently as

$$y[a(x), c(x), y_p(x)] = \frac{y_p(x) e^{\int e^{-[c(\phi)y_p(\phi) - a(\phi)/y_p(\phi)]} d\phi}}{C - \int e^{-[c(\phi)y_p(\phi) - a(\phi)/y_p(\phi)]} d\phi} + y_p(x),$$

and

$$y[a(x), b(x), y_p(x)] = \frac{y_p^2(x) e^{-\int e^{-[b(\phi) + 2a(\phi)/y_p(\phi)]} d\phi}}{C - \int e^{-[b(\phi) + 2a(\phi)/y_p(\phi)]} d\phi} + y_p(x).$$

Note that the complexity of the general solutions Eqs. (2), (56), and (57) depends on the coefficients $b(x)$ and $c(x)$, $a(x)$ and $c(x)$, $a(x)$ and $b(x)$ respectively, and on the particular solution $y_p(x)$, satisfying the Riccati Eq. (1). Therefore, one may choose some simple particular solutions to obtain the theorems in the present paper.

We have also discussed a physical application of the presented results. The study of general relativistic compact objects is of fundamental importance for astrophysics. After the discovery of the pulsars and the explanation of their properties by assuming that they are rapidly rotating neutron stars, the theoretical investigation of superdense stars has been done using both numerical and analytical methods and the parameters of neutron stars have been worked out by using a general relativistic approach.

Since the Einstein’s gravitational field equations describing the interior of compact stellar objects can be reduced to a Riccati equation, the mathematical results obtained in the present paper can be used to model massive stars, such as neutron stars and pulsars. The results may be useful whenever there is some uncertainty regarding the actual equation of state of the dense matter inside the general relativistic star. Note, however, that in order to have physical stellar models, the interior solution for static fluid spheres of Einstein’s gravitational field equations must satisfy specific general physical requirements. The following conditions have been generally recognized to be crucial for anisotropic fluid spheres: (i) the density $\rho$, the radial pressure $p_r$, and the tangential pressure $p_{\perp}$ should be positive inside the star; (ii) the gradients $dp/dr$, $dp_r/dr$ and $dp_{\perp}/dr$ should be negative; (iii) inside the static
configuration the speed of sound should be less than the speed of light, i.e. $0 \leq dp_r/d\rho \leq 1$, $0 \leq dp_r/d\rho \leq 1$; (iv) the interior metric should be joined continuously with the exterior Schwarzschild metric and; (v) the radial pressure $p_r$ must vanish at the boundary $r = R$ of the sphere, but the tangential pressure $p_\perp$ may not vanish. The physical interpretation of the structure and properties of the compact relativistic star models obtained by using the previous theorems as applied to the Riccati Eq. (54) will be presented in a future publication.

Acknowledgements

We would like to thank to the anonymous referee for comments and suggestions that helped us to improve our manuscript. FSNL acknowledges financial support of the Fundação para a Ciência e Tecnologia through the grants CERN/FP/123615/2011 and CERN/FP/123618/2011.

REFERENCES

[1] E. Kamke, Differentialgleichungen: Lösungsmethoden und Lösungen, Chelsea, New York (1959).
[2] A. D. Polyanin, V. F. Zaitsev, Handbook of exact solutions for ordinary differential equations, Boca Raton, Chapman & Hall/CRC (2003).
[3] M. V. Soare, P. P. Teodorescu and I. Toma, Ordinary differential equations with applications to mechanics, Dordrecht, Springer (2007).
[4] V. M. Strelchenya, A new case of integrability of the general Riccati equation and its application to relaxation problems, J. Phys. A: Math. Gen. 24, 4965 (1991).
[5] E. D. Rainville, Necessary conditions for polynomial solutions of certain Riccati equations, The American Mathematical Monthly 43, 473-476 (1936).
[6] M. K. Mak, T. Harko, New integrability case for the Riccati equation, Applied Mathematics and Computation 218, 10974 (2012).
[7] M. K. Mak, T. Harko, New further integrability case for the Riccati equation, Applied Mathematics and Computation 219, 7465 (2013).
[8] W. T. Reid, Riccati Differential Equations, New York, USA: Academic Press, (1972).
[9] M. K. Mak, T. Harko, Isotropic stars in general relativity, The European Physical Journal C 73, 2585 (2013).
[10] T. Harko, F. S. N. Lobo, M. K. Mak, Arbitrary scalar field and quintessence cosmological models, arXiv:1310.7167 (2013).
[11] T. Harko, M. K. Mak, Anisotropic charged fluid spheres in D space-time dimensions, J. Math. Phys. 41, 4752 (2000).
[12] M. K. Mak, J. A. Belinchon, T Harko, Causal bulk viscous dissipative isotropic cosmologies with variable gravitational and cosmological constants, International Journal of Modern Physics D, 11, 1265 (2002).
[13] Chiang-Mei Chen, T. Harko, M. K. Mak, Viscous dissipative effects in isotropic brane cosmology, Phys. Rev. D64, 124017 (2001).
[14] M. K. Mak, T. Harko, Exact dissipative cosmologies with stiff fluid, Europhysics Letter. 56, 762 (2001).
[15] Changqing Yang, Jianhua Hou, Beibo Qin, Numerical solution of Riccati differential equations by using hybrid functions and tau method, International Journal of Mathematical and Computational Sciences 6, 216 (2012).
[16] M. A. El-Tawil, A. A. Bahnasawi, A. Abdel-Naby, Solving Riccati differential equation using Adomians decomposition method, Applied Mathematics and Computation 157, 503 (2004).
[17] P. Y. Tsai, C. K. Chen, An approximate analytical solution of the nonlinear Riccati differential equation, J. Franklin Inst. 347, 1850 (2011).
[18] S. Abbasbandy, Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomian decomposition method, Applied Mathematics and Computation 172, 485 (2006).
[19] S. Abbasbandy, Iterated He’s homotopy perturbation method for quadratic Riccati equation, Applied Mathematics and Computation 175, 581 (2006).
[20] S. Abbasbandy, A new application of He’s variational iteration method for quadratic Riccati differential equation by using Adomian’s polynomials, J. Comput. Appl. Math. 207, 59 (2007).
[21] M. Gülsu, M. Sezer, On the solution of the Riccati equation by the Taylor matrix method, Applied Mathematics and Computation 176, 414 (2006).
[22] F. Mohammadi, M. M. Hosseini, *A comparative study of numerical methods for quadratic Riccati differential equations*, J. Franklin Inst. **348**, 156 (2011).

[23] M. K. Mak, T. Harko, *Anisotropic stars in general relativity*, Proc. Roy. Soc. Lond. **A56**, 393 (2003).