NORMAL SALLY MODULES OF RANK ONE

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Dedicated to Professor Shiro Goto on the occasion of his 70th birthday

Abstract. In this paper, we explore the structure of the normal Sally modules of rank one with respect to an m-primary ideal in a Nagata reduced local ring which is not necessary Cohen-Macaulay. As an application of this result, when the base ring is Cohen-Macaulay analytically unramified, the extremal bound on the first normal Hilbert coefficient leads to the Cohen-Macaulayness of the associated graded rings with respect to a normal filtration.

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1. Introduction

Throughout this paper, let R be an analytically unramified Noetherian local ring with the maximal ideal m and d = dim R > 0. Let I be an m-primary ideal of R and suppose that I contains a parameter ideal Q = (a1, a2, ..., ad) of R as a reduction. Let ℓR(M) denote the length of an R-module M and \overline{I^{n+1}} denote the integral closure of I^{n+1} for each n ≥ 0. Since R is an analytically unramified, there are integers \{e_i(I)\}_{0 \leq i \leq d} such that the equality

\[ \ell_R(R/\overline{I^{n+1}}) = e_0(I) \left( \frac{n + d}{d} \right) - e_1(I) \left( \frac{n + d - 1}{d - 1} \right) + \ldots + (-1)^d e_d(I) \]

holds true for all integers n ≫ 0, which we call the normal Hilbert coefficients of R with respect to I. We will denote by \{e_i(I)\}_{0 \leq i \leq d} the ordinary Hilbert coefficients of R.

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with respect to $I$. Let 
\[ R = R(I) := R[It] \] and 
\[ T = R(Q) := R[Qt] \subseteq R[t] \]
denote, respectively, the Rees algebra of $I$ and $Q$, where $t$ stands for an indeterminate over $R$. Let 
\[ R' = R'(I) := R[It, t^{-1}] \] and 
\[ G = G(I) := R'/t^{-1}R' \cong \oplus_{n \geq 0} I^n/T^{n+1} \]
denote, respectively, the extended Rees algebra of $I$ and $Q$ with respect to $I$. Let 
\[ R'' = R''(I) := R'[It, t^{-1}] \] and 
\[ G' = G(I) := R''/t^{-1}R'' \cong \oplus_{n \geq 0} I^n/T^{n+1} \]
denote, respectively, the extended Rees algebra of $I$ and $Q$ with respect to $I$. Let $\overline{R}$ denote the integral closure of $R$ in $R[t]$ and $G = G(I) := R'/t^{-1}R'$ denote the associated graded ring of the normal filtration \{\(I^n\)\}_{n \in \mathbb{Z}}. Then $\overline{R} = \oplus_{n \geq 0} I^n/T^n$ and $\overline{R}$ is a module-finite extension of $R$ since $R$ is analytically unramified (see [23, Corollary 9.2.1]). For the reduction $Q$ of $I$, the reduction number of \{\(I^n\)\}_{n \in \mathbb{Z}} with respect to $Q$ is defined by 
\[ r_Q(\{I^n\}_{n \in \mathbb{Z}}) = \min\{r \in \mathbb{Z} \mid I^{n+1} = QI^n, \forall n \geq r\}. \]

The notion of Sally modules of normal filtrations was introduced by [3] in order to find the relationship between a bound on the first normal Hilbert coefficients $e_1(I)$ and the depth of $G$ when $R$ is an analytically unramified Cohen-Macaulay rings $R$. Following [3], we generalize the definition of normal Sally modules to the non-Cohen-Macaulay cases, and we define the normal Sally modules $\overline{S} = \overline{S}_Q(I)$ of $I$ with respect to a minimal reduction $Q$ to be the cokernel of the following exact sequence 
\[ 0 \rightarrow \overline{T}T \rightarrow \overline{R}_+(1) \rightarrow \overline{S} \rightarrow 0 \]
of graded $T$-modules. Since $\overline{R}$ is a finitely generated $T$-module, so is $\overline{S}$ and we get 
\[ \overline{S} = \oplus_{n \geq 1} \overline{I}^{n+1}/Q^nT \]
by the following isomorphism 
\[ \overline{R}_+(1) \xrightarrow{t^{-1}} \sum_{n \geq 0} \overline{I}^{n+1}t^n (\supseteq \sum_{n \geq 0} (Q^nT)t^n = \overline{T}T) \]
of graded $T$-modules.

To state the results of this paper, let us consider the following four conditions:
(C0) The sequence $a_1, a_2, \ldots, a_d$ is a $d$-sequence in $R$ in the sense of [9]
(C1) The sequence $a_1, a_2, \ldots, a_d$ is a $d^+$-sequence in $R$, that is for all integers $n_1, n_2, \ldots n_d \geq 1$ the sequence $a_1^{n_1}, a_2^{n_2}, \ldots, a_d^{n_d}$ forms a $d$-sequence in any order.
(C2) $(a_1, a_2, \ldots, \hat{a}_i, \ldots, a_d) :_R a_i \subseteq I$ for all $1 \leq i \leq d$
(C3) depth$R > 0$ and depth$R > 1$ if $d \geq 2$. 
These conditions \((C_0), (C_1),\) and \((C_2)\) are exactly the same as in [6]. The conditions \((C_1), (C_2),\) and \((C_3)\) are automatically satisfied if \(R\) is Cohen-Macaulay. Conditions \((C_1)\) and \((C_3)\) imply the ring \(R\) has the property \((S_2)\).

The main result of this research is as follows.

**Theorem 1.1.** Let \(R\) be a Nagata and reduced local ring with the maximal ideal \(m\) and \(d = \dim R > 0\). Let \(I\) be an \(m\)-primary ideal of \(R\) and suppose that \(I\) contains a parameter ideal \(Q\) of \(R\) as a reduction. Assume that conditions \((C_1), (C_2),\) and \((C_3)\) are satisfied. Then the followings are equivalent to each other.

1. \(e_1(I) = e_0(I) - \ell_R(R/I) + 1.\)
2. \(mS = (0)\) and \(\text{rank}_B S = 1,\) where \(B = T/mT.\)
3. \(S \cong B(-q)\) as graded \(T\)-modules for some integer \(q \geq 1.\)

When this is the case

(a) \(S\) is a Cohen-Macaulay \(T\)-module.
(b) Put \(t = \text{depth} R.\) Then

\[
\text{depth} G \geq \begin{cases} 
  d - 1 & \text{if } t \geq d - 1, \\
  t & \text{if } t < d - 1.
\end{cases}
\]

(c) for all \(n \geq 0,\)

\[
\ell_R(R/T^{n+1}) = e_0(I) \left(\frac{n+d}{d}\right) - \{e_0(I) + e_1(Q) - \ell_R(R/T) + 1\} \binom{n+d-1}{d-1} 
+ \sum_{i=2}^{d} (-1)^i \{e_{i-1}(Q) + e_i(Q)\} \binom{n+d-i}{d-i}
\]

if \(n < q,\) and

\[
\ell_R(R/T^{n+1}) = e_0(I) \left(\frac{n+d}{d}\right) - \{e_0(I) + e_1(Q) - \ell_R(R/T) + 1\} \binom{n+d-1}{d-1} 
+ \sum_{i=2}^{d} (-1)^i \{e_{i-1}(Q) + e_i(Q) + \binom{q}{i-1}\} \binom{n+d-i}{d-i}
\]

if \(n \geq q.\) Hence \(e_i(I) = e_{i-1}(Q) + e_i(Q) + \binom{q}{i-1}\) for all \(2 \leq i \leq d.\)

The relationship between the equality \(e_1(I) = e_0(I) - \ell_R(R/T) + 1\) and the depth of \(G\) in an analytically unramified Cohen-Macaulay local ring was examined in [3]. In their paper, they proved that if \(R\) is an analytically unramified Cohen-Macaulay ring possessing a canonical module \(\omega_R,\) then \(e_1(I) = e_0(I) - \ell_R(R/T) + 1\) makes \(\text{depth} G \geq d - 1\) ([3 Theorem 2.6]). Moreover if \(d \geq 3\) and \(e_3(I) = 0,\) then this equality \(e_1(I) = e_0(I) - \ell_R(R/T) + 1\) leads to the Cohen-Macaulayness of \(G\) ([3 Proposition 3.4]).
The assumption $R$ has a canonical module assures that $\overline{R}$ satisfies the Serre condition $(S_2)$ as a ring, which is essential for the proofs of their results. In this paper, as an application of Theorem 1.1 we will prove that the above results [3, Theorem 2.6 and Proposition 3.4] still hold true even when we delete the assumptions that the base ring possessing a canonical module $\omega_R$ and the vanishing of $e_2(I)$, as stated in the following.

**Theorem 1.2.** Let $R$ be an analytically unramified Cohen-Macaulay local ring with the maximal ideal $m$, dimension $d = \dim R \geq 2$, and $I$ an $m$-primary ideal of $R$ containing a parameter ideal $Q$ of $R$ as a reduction. Then $e_1(I) = e_0(I) - \ell_R(R/\overline{R}) + 1$ if and only if $e_3(I) = 1$. When this is the case $G$ is Cohen-Macaulay, $\ell_R(\overline{T}/Q\overline{I}) = 1$, and $r_Q(\{T^n\}_{n \in \mathbb{Z}}) = 2$.

Here we notice that Theorem 1.2 was proved in [18, Proposition 4.9]. However there is a gap in their proof because $\overline{T}^2 = Q\overline{T}$ does not imply $r_Q(\{T^n\}_{n \in \mathbb{Z}}) \leq 1$ as mentioned in [3] Remark 2.7. Theorem 1.1 helps us correct the proof of [18, Proposition 4.9].

Now it is a position to explain how this paper is organized. This paper contains 4 sections. The introduction part is this present section. In Section 2 we will collect some auxiliary results on Sally module and normal Hilbert functions. We will prove Theorem 1.1 in Section 3. In the last Section 4, we prove Theorem 1.2 and explore some consequences of Theorem 1.1 in the Cohen-Macaulay case.

## 2. Auxiliaries

In this section we will collect properties of the normal Sally module and the normal Hilbert coefficients which are essential for the proof of our main results. Throughout this section, let $R$ be an analytically unramified Noetherian local ring with the maximal ideal $m$ and $\dim R = d \geq 1$. Let $I$ be an $m$-primary ideal and assume that $I$ contains 1 minimal reduction $Q = (a_1, ..., a_d)$.

Let us begin with the following lemma which play an important role on computing the normal Hilbert functions and on examining the structure of $S$.

**Lemma 2.1.** [6, Lemma 2.1] Suppose that conditions $(C_0)$ and $(C_2)$ are satisfied. Then

$$T/\overline{T} \cong (R/\overline{T})[X_1, X_2, ..., X_d]$$

as graded $R$-algebras, where $(R/\overline{T})[X_1, X_2, ..., X_d]$ denotes the polynomial ring with $d$ indeterminates over Artinian local ring $R/\overline{T}$. Hence $T/\overline{T}$ is a Cohen-Macaulay ring with $\dim T/\overline{T} = d$. 
Under our generalized assumption, the results [3] Proposition 2.2] on the associated prime ideals set and the dimension of \( \overline{S} \) do not change, and moreover we obtain a formula on \( \text{depth} \overline{G} \) as follows.

**Lemma 2.2.** The following assertions hold true.

1. [3] Lemma 2.1] \( m^\ell \overline{S} = 0 \) for integer \( \ell \gg 0 \). Hence \( \text{dim}_T \overline{S} \leq d \).

2. [3] Lemma 2.3] Suppose that conditions \((C_0), (C_2) \) and \((C_3) \) are satisfied and \( T \) is a \((S_2) \) ring. Then \( \text{Ass}_T(\overline{S}) \subseteq \{mT\} \). Hence \( \text{dim}_T \overline{S} = d \) provided \( S \neq (0) \).

3. Suppose that conditions \((C_0), (C_2) \) and \((C_3) \) are satisfied. Then \( \text{depth} \overline{G} \geq \text{depth} R \) if \( \overline{S} = (0) \) and

\[
\text{depth} \overline{G} \geq \begin{cases} 
\text{depth} R & (\text{depth}_{T, \overline{S}} \geq \text{depth} R + 1), \\
\text{depth}_{T, \overline{S}} - 1 & (\text{depth}_{T, \overline{S}} \leq \text{depth} R \text{ or } \text{depth} R = d - 1)
\end{cases}
\]

if \( \overline{S} \neq (0) \).

**Proof.** The proof of (2) is almost the same as that of [3] Lemma 2.3, let us include a proof for the sake of completeness. We may assume that \( \overline{S} \neq (0) \). Let \( P \in \text{Ass}_T \overline{S} \). Then \( mT \subseteq P \). Assume that \( P \neq mT \). Then \( \text{ht}_T P \geq 2 \) since \( \text{ht}_T mT = 1 \). Therefore \( \text{depth} T \geq 2 \) by condition \((S_2) \). Now we consider the following exact sequences

\[ 0 \rightarrow (\overline{T})_P \rightarrow (\overline{R}_+(1))_P \rightarrow \overline{S}_P \rightarrow 0 \]

and

\[ 0 \rightarrow (\overline{T})_P \rightarrow T_P \rightarrow T_P/(\overline{T})_P \rightarrow 0 \]

of graded \( T_P \)-modules. Since \( \text{depth}_{T_P}(\overline{R}_+(1))_P \geq 1 \) and \( \text{depth}_{T_P} \overline{S}_P = 0 \), \( \text{depth}_{T_P}(\overline{T})_P = 1 \). Therefore \( \text{depth}_{T_P} T_P/(\overline{T})_P = 0 \) by the second exact sequence. Moreover since conditions \((C_0) \) and \((C_2) \) are satisfied, \( T/\overline{T} \) is a Cohen-Macaulay ring by [3] Proposition 2.2, so is \( T_P/(\overline{T})_P \). Therefore \( P \in \text{Min}_T T_P/(\overline{T})_P = \{mT\} \), which is a contradiction. Thus \( P = mT \) as desired.

The statement (3) follows by comparing depths of \( T \)-modules in the following exact sequences

1. \( 0 \rightarrow \overline{T} \rightarrow T \rightarrow T/\overline{T} \rightarrow 0 \).
2. \( 0 \rightarrow \overline{T} \rightarrow \overline{R}_+(1) \rightarrow \overline{S} \rightarrow 0 \)
3. \( 0 \rightarrow \overline{R}_+(1) \rightarrow \overline{R} \rightarrow \overline{G} \rightarrow 0 \),
4. \( 0 \rightarrow \overline{R}_+ \rightarrow \overline{R} \rightarrow R \rightarrow 0 \)

of graded \( T \)-modules. \( \square \)
Applying Lemma 2.2 to the case where the base ring \( R \) is analytically unramified Cohen-Macaulay, we get \( \text{depth} \S \geq \text{depth}_T \S - 1 \) (\cite{3} Proposition 2.4(b)) and \( \S \) is Cohen-Macaulay if \( S = (0) \) (\cite{3} Proposition 2.4(a)).

**Remark 2.3.** Suppose that conditions \((C_1)\) and \((C_3)\) are satisfied. Then \( T \) is a \((S_3)\)-ring by \cite{25} Theorem 6.2. Therefore if we assume that conditions \((C_1), (C_2)\) and \((C_3)\) are satisfied, then by Lemma 2.2 \( \text{Ass}_T(\S) \subseteq \{mT\} \). Hence \( \dim_T \S = d \) provided \( \S \neq (0) \).

The following lemma play a crucial role on computing the normal Hilbert polynomial in Theorem 1.11.

**Lemma 2.4.** Suppose that conditions \((C_0)\) and \((C_2)\) are satisfied. Then the following assertions hold true.

1. \cite{6} Proposition 2.4] For every \( n \geq 0 \)
   \[
   \ell_R(R/T^{n+1}) = e_0(I)\binom{n+d}{d} - \{e_0(I) + e_1(Q) - \ell_R(R/T)\}\binom{n+d-1}{d-1} \\
   + \sum_{i=2}^{d} (-1)^i\{e_{i-1}(Q) + e_i(Q)\binom{n+d-i}{d-i}\} - \ell_R(\S_n).
   \]

2. \cite{6} Proposition 2.5] \( \overline{e}_1(I) = e_0(I) + e_1(Q) - \ell_R(R/T) + \ell_T(mT(\S_{mT})) \), whence \( \overline{e}_1(I) \geq e_0(I) + e_1(Q) - \ell_R(R/T) \).

We omit the proof of the above lemma because they are the same as in \cite{6}. Here we notice that under the condition \((C_0)\), \( \ell_R(R/Q^{n+1}) = \sum_{i=0}^{d}(-1)^i\{e_i(Q)\binom{n+d-i}{d-i}\} \) for every \( n \geq 0 \) by \cite{24} Theorem 4.1.

The following lemma shows that the equality \( \overline{e}_1(I) = e_0(I) + e_1(Q) - \ell_R(R/T) \) corresponds to the case where either \( \S \) vanishes or the reduction number of the normal Hilbert filtration is at most one. And the equality \( \overline{e}_1(I) = e_0(I) + e_1(Q) - \ell_R(R/T) + 1 \) corresponds to the normal Sally module of rank one.

**Lemma 2.5.** Assume that conditions \((C_1), (C_2), \) and \((C_3)\) are satisfied. Then the following assertions hold true.

1. The followings are equivalent to each other
   (a) \( \overline{e}_1(I) = e_0(I) + e_1(Q) - \ell_R(R/T) \)
   (b) \( \S = 0 \).
   (c) \( r_Q(\{T^n\}_{n \in \mathbb{Z}}) \leq 1 \).

   When this is the case we get the followings.
   (i) \( \text{depth} \S \geq \text{depth} R \).
(ii) for all $n \geq 0$
\[
\ell_R(R/I^n_{n+1}) = e_0(I) \left( \frac{n + d}{d} \right) - \{ e_0(I) + e_1(Q) - \ell_R(R/T) \} \left( \frac{n + d - 1}{d - 1} \right)
+ \sum_{i=2}^{d} (-1)^i \{ e_{i-1}(Q) + e_i(Q) \} \left( \frac{n + d - i}{d - i} \right).
\]

(2) [3] Theorem 2.9 \[ \overline{e_1}(I) = e_0(I) + e_1(Q) - \ell_R(R/T) + 1 \text{ if and only if } m \overline{S} = 0 \text{ and } \text{rank}_B \overline{S} = 1. \]

Proof. The statement (2) is by [3, Theorem 2.9]. Now we will prove (1). Since conditions $(C_1), (C_2), \text{ and } (C_3)$ are satisfied, $\text{Ass}_T S \subseteq \{ mT \}$ by Lemma 2.1 and we get $S_{mT} = (0)$ if and only if $S = (0)$. Moreover by Lemma 2.1(2) \[ \overline{e_1}(I) = e_0(I) + e_1(Q) - \ell_R(R/T) + \ell_{T_{mT}}(S_{mT}), \] therefore \[ \overline{e_1}(I) = e_0(I) + e_1(Q) - \ell_R(R/T) \text{ if and only if } S = 0. \] The equivalence of $(b)$ and $(c)$ is clear. Now assume that $S = (0)$. The statement $(i)$ follows by Lemma 2.2(3). The last assertion $(ii)$ follows by Lemma 2.2(1). \hfill \box

3. Proof of Theorem 1.1

This section is devoted for presenting the proofs of Theorem 1.1. In order to do that we need the result that the integral closure of $R(I)$ in $R[t]$ is a $(S_2)$-ring. Here we notice that a Noetherian ring $R$ is called Nagata if for every $P \in \text{Spec}R$, for any finite extension $L$ of $Q(R/P)$, the integral closure of $R/P$ in $L$ is a finite $R/P$-module, where $Q(R/P)$ denotes the quotient field of $R/P$ (see [17, 31.A DEFINITIONS]). Let $\mathcal{I} = \{ I_n \}_{n \in \mathbb{Z}}$ be a filtrations of ideals in $R$, that is $I_n$ is an ideal of $R$ for every $n \in \mathbb{Z}$, $I_0 = R$, $I_n \supseteq I_{n+1}$ for every $n \in \mathbb{Z}$, and $I_m I_n \subseteq I_{mn}$ for all $m, n \in \mathbb{Z}$. Then we get the following result which is belong to Shiro Goto.

Proposition 3.1. Assume that $R$ be a Noetherian local ring with the maximal ideal $m$, $d = \text{dim} R > 0$ such that $R$ is a reduced, Nagata, and $(S_2)$-ring. Let $\mathcal{I} = \{ I_n \}_{n \in \mathbb{Z}}$ be a filtrations of ideals in $R$ such that $I_1 \neq R$. Suppose that $\text{ht}_R I_1 \geq 2$, and $R(I) \subseteq R[t]$ is Noetherian. Then the integral closure of $R(I)$ in $R[t]$ is a $(S_2)$-ring.

Proof. Put $\mathcal{R} := R(I)$, $F := Q(R[t])$ and let $S$ and $T$ denote the integral closure of $\mathcal{R}$ in $R[t]$ and $F$, respectively. Since $\text{ht}_R I_1 \geq 2$, there exists an $R$-regular element $a \in I_1$. Put $f = at$. Then $Q(R[f]) = Q(R[t]) \supseteq Q(R)$. Let $\overline{R}$ be the integral closure of $R$ in $Q(R)$. Then $\overline{R}$ is a finitely generated graded $R$-module. Since $\overline{R}[t]$ is integral closed in $F$, $T \subseteq \overline{R}[t]$. Therefore $S \subseteq T$ and $S = T \cap R[t]$. Since $R$ is Nagata and $\mathcal{R}$ is Noetherian, $T$ is a finitely generated graded $\mathcal{R}$-module and hence $S$ is Noetherian.
Assume on the contrary that $S$ is not a $(S_2)$-ring. Then there is a prime ideal $P$ of $S$ such that

$$\text{depth}_P S < \inf \{2, \dim S_P\}.$$  

If $\dim S_P \leq 1$, then $\text{depth}_P S = 0, \dim S_P = 1$, which contradicts to the fact that $S$ is reduced. Therefore $\dim S_P \geq 2$, $\text{depth}_P S = 1$ and $P$ is graded. Put $p = P \cap R$. Then $p \supseteq I_1$.

**Claim 3.2.** For all prime ideal $P$ of $S$ such that $\text{ht} P \geq 2$, we have $\text{ht}_T Q \geq 2$ for all prime ideal $Q$ of $T$ such that $Q \cap S = P$.

**Proof.** Take $Q_0 \in \text{Min} T$ such that $Q_0 \subseteq Q$ and $\dim T_Q = \dim T_Q/Q_0T_Q$. Since $Q_0 \in \text{Ass}_T \overline{R}[t]$, there is $U \in \text{Ass}_\overline{R}[t]$ such that $U \cap T = Q_0$. Put $W = Q_0 \cap \overline{R}$. Then $W \in \text{Ass}_\overline{R} R$ and $U = W \overline{R}[t]$. Put $p_0 = W \cap R$. Then $p_0 \in \text{Ass} R$ and therefore $\text{ht}_{Rp_0} = 0$ as $R$ is $(S_1)$. Since

$$(Q_0 \cap \mathcal{R}) \cap R = p_0 	ext{ and } \mathcal{R}/(Q_0 \cap \mathcal{R}) \cong \mathcal{R}/\{(I_n + p_0)/p_0\}.$$  

Therefore $\dim \mathcal{R}/(Q_0 \cap \mathcal{R}) = d + 1$, where $d = \dim R \geq 2$. Similarly we also get $S/(Q_0 \cap S) \cong \mathcal{R}/\{(\overline{T}_n + p_0)/p_0\}$ and hence $\dim S/(Q_0 \cap S) = d + 1$. Now let $M$ be the graded maximal of $S$. Then $P \subseteq M$. Since the extension $S \subseteq T$ is finite, by Going Up theorem, there exists a graded maximal ideal $N$ of $T$ such that $\dim T_N/QT_N = \dim S_M/PS_M =: \alpha$.

\[
\begin{array}{c}
Q_0 \quad Q \quad \exists N \subseteq T \\
\downarrow \quad \downarrow \quad \downarrow \\
Q_0 \cap S \quad P \quad \alpha \\
\downarrow \quad \downarrow \\
Q_0 \cap \mathcal{R} \quad Q \cap \mathcal{R} \\
\downarrow \quad \downarrow \\
\mathcal{R}
\end{array}
\]

Since the extension $S/(Q_0 \cap S)_M \subseteq T/Q_0 N$ is finite and $R$ is universal catenary, $S/(Q_0 \cap S)_M$ is universal catenary local domain. Therefore $\text{ht} N/Q_0 = \text{ht} M/(Q_0 \cap S)$ and hence $\dim T/Q_0 = d + 1$. Now we assume on the contrary that $\dim T_Q \leq 1$. Then $\alpha \geq d$. On the other hand, since $\text{ht} P \geq 2$, $d + 1 = \alpha + \text{ht} P \geq \alpha + 2$. Hence $\alpha \leq d - 1$ which yields a contradiction. Thus $\dim T_Q \geq 2$.

Therefore $\text{depth}_{S_p} T \geq 2$ because of the following fact which we omit the proof.

**Claim 3.3.** Let $S$ be a Noetherian local ring with the maximal ideal $m$ and $T$ is a finite extension of $S$ with $(S_2)$ property. If for every maximal ideal $n$ of $T$, $\dim T_n \geq 2$, then $\text{depth}_S T \geq 2$. 


Next we consider the exact sequence

$$0 \longrightarrow S_P \longrightarrow T_P \longrightarrow (T/S)_P \longrightarrow 0$$

of graded $S_P$-modules. Applying the Depth lemma to the above exact sequence we get $\text{depth}_{S_P}(T/S)_P = 0$, since $\text{depth}_{S_P}, T_P \geq 2$, by Claim 3.2 and $\text{depth}_{S_P} = 1$. Therefore $P \in \text{Ass}_S(T/S)$ and $P \in \text{Ass}_{S/S}(\overline{T}/[T])$ since $T \cap [T] = S$. Hence $P = Q \cap S$ for some $Q \in \text{Ass}_{S/S}(\overline{T}/[T])$. Moreover since $\overline{T}/[T] \cong (\overline{R}/R) \otimes_R R[t]$ and $\text{Ass}_{S/S}(\overline{T}/[T]) = \bigcup_{p \in \text{Ass}_{S/S}(\overline{T}/[T])} \text{Ass}_{S/S}(\overline{T}/[T] / Q)$, there is $p_1 \in \text{Ass}_{S/S}(\overline{T}/[T])$ such that $Q = p_1 \overline{T}/[T] / p_1 R[t]$. Since $Q = p_1 R[t]$, $p_1 = Q \cap R = P \cap R = p$. Therefore $p \in \text{Ass}_{S/S}(\overline{T}/[T])$. Furthermore since $p \geq I_1$, $\dim R_p \geq 2$ and hence $\text{depth}_R(p) \geq 2$ as $R$ is (S2). On the other hand $\text{depth}_{R_p}(\overline{R}_p) > 0$ as there is an $R$-regular element in $R$ which is also $\overline{R}$-regular. Now applying Depth lemma to the following exact sequence

$$0 \longrightarrow R_p \longrightarrow (\overline{R}_p) \longrightarrow (\overline{R}/R)_p \longrightarrow 0$$

of $R_p$-modules we get a contradiction. Thus $S$ is an (S2)-ring. \qed

Now it is a position to prove Theorem 1.1.

Proof of Theorem 1.1. Since conditions (C1), (C2), and (C3) are satisfied, we get $e_1(I) = e_0(I) + e_1(Q) - \ell_R(R/T) + \ell_{mT}(\overline{S}/mT)$, by Lemma 2.3(2) and $\text{Ass}_S(S) \subseteq \{mT\}$, by Lemma 2.2(2).

(3) $\Rightarrow$ (2) This is obvious.

(2) $\Leftarrow$ (1) This is by Lemma 2.5(2).

(1) $\Rightarrow$ (3) Assume that $e_1(I) = e_0(I) + e_1(Q) - \ell_R(R/T) + 1$. Then $S \neq (0)$ by Lemma 2.3 and hence $\text{Ass}_S(S) = \{mT\}$. Therefore $S$ is a torsion free $B$-module. If $d = 1$, then $B$ is a PID. Hence $\overline{S}$ is $B$-free because every torsion free module over a PID are free. Now we consider the case where $d \geq 2$. We will show that $S$ is a (S2) module over $B$. When this is the case, since $\text{rank}_B \overline{S} = 1$ and $B$ is an UFD, $\overline{S}$ is a reflexive $B$-module and hence a free $B$-module. Therefore $\overline{S}/B_+^Q = (R/m)\overline{\varphi}$ for some homogeneous element $\varphi \in \overline{S}/m$ of degree $q \geq 1$. Hence $\overline{S} = B\varphi + B_+^Q$ and $\overline{S}/B_+ = \overline{S}/B_+^Q$ by the graded Nakayama lemma. So $(\overline{S}/B\varphi)_{B_+} = 0$ and $\overline{S}/B\varphi = 0$. Thus $\overline{S} \cong B(-q)$ as desired.

Now we assume on the contrary that $\overline{S}$ is not a (S2) module over $B$. Then

$$\text{depth}_{B_p}(\overline{S}_P) < \inf\{2, \dim_{B_p}(\overline{S}_P)\}$$

for some prime ideal $P \in \text{Supp}_{B_p} \overline{S}$. Therefore $\dim_{B_p}(\overline{S}_P) \geq 2$, $\text{depth}_{B_p}(\overline{S}_P) = 1$, and $P$ is a graded ideal of $B$. Here we notice that $\dim \overline{S}_P = \dim B_P$. Let $p \in \text{Spec}T$ such that $P = p + mT$. Then $p$ is also graded as $mT$ is graded. Moreover $\text{ht}_{TP} \geq 3$ because
\[ \text{ht}_B P \geq 2, \ mT \subseteq p \text{ and } \text{ht}_T mT = 1. \] We will prove that \( \text{depth}_{T_p}(\mathcal{R})_p \geq 2. \) In order to prove this, it is enough to show the following.

**Claim 3.4.** For all graded prime ideal \( p \) of \( T \) such that \( \text{ht}_P \geq 3 \), we have \( \text{ht}_\mathcal{R} Q \geq 2 \) for all prime ideal \( Q \) in \( \mathcal{R} \) with \( Q \cap T = p \).

When this Claim 3.4 holds true, since \( \mathcal{R} \) is a \((S_2)\)-ring by Proposition 3.1, we get \( \text{depth}_{T_p}(\mathcal{R})_p \geq 2 \) by applying Claim 3.3.

**proof of Claim 3.4.** Assume on the contrary that there exists a prime ideal \( Q \) of \( \mathcal{R} \) such that \( Q \cap T = p \) but \( \text{ht}_\mathcal{R} Q \leq 1 \). Take \( Q_0 \in \text{Min} \mathcal{R} \) such that \( Q_0 \subseteq Q \) and \( \dim \mathcal{R}_Q = \dim \mathcal{R}_{Q_0}/\mathcal{R}_Q \). Then \( Q_0 = qR[t] \cap \mathcal{R} \), for some \( q \in \text{Min} R \), and \( Q_0 \cap T = qR[t] \cap T \). Therefore \( (Q_0 \cap T) \cap R = q \) and hence \( T/(Q_0 \cap T) \cong \mathcal{R}((q + I)/q) \). Since \( I \not\subseteq q \), \( \dim T/(Q_0 \cap T) = \dim \mathcal{R}((q + I)/q) = \dim R/q + 1 = d + 1 \) and we get \( Q_0 \cap T \in \text{Min} T \).

Let \( \mathcal{M} \) be the unique graded maximal ideals of \( \mathcal{R} \) such that \( \dim \mathcal{R}_{\mathcal{M}}/\mathcal{R}_{\mathcal{M}} = \dim T_{\mathcal{M}}/pT_{\mathcal{M}} = \alpha \).

We notice that \( p \subseteq \mathcal{N} \), as \( p \) is graded, and \( \text{ht}_{\mathcal{N}}/Q_0 = \text{ht}_{\mathcal{M}}/(Q_0 \cap T) = d + 1 \), because \( R \) is universal catenary and the extension \( T/(Q_0 \cap T) \hookrightarrow \mathcal{R}/Q_0 \) is finite. Therefore \( d + 1 = \text{ht}_{\mathcal{N}}/Q_0 \leq 1 + \alpha \) and \( d + 1 = \text{ht}_{\mathcal{M}}/(Q_0 \cap T) \geq 3 + \alpha \), a contradiction. \( \square \)

**Claim 3.5.** \( \text{depth}_{T_p}(\mathcal{R}_+)_p \geq 2 \).

**proof of Claim 3.5.** We consider the following exact sequence of \( T_p \)-modules.

\[
0 \to (\mathcal{R}_+)_p \to \mathcal{R}_p \to R_p \to 0.
\]

If \( R_p = (0) \), then \( \text{depth}_{T_p}(\mathcal{R}_+)_p = \text{depth}_{T_p}(\mathcal{R})_p \geq 2 \) by Claim 3.4. If \( R_p \neq (0) \), then \( p = \mathcal{M} \), the unique graded maximal ideal of \( T \), because \( p \) is a graded ideal. Hence \( \text{depth}_{T_p} R_{\mathcal{M}} = \text{depth} R \geq 1 \) by the condition \((C_3)\). Now use Claim 3.4 and apply Depth lemma to the above exact sequence we get the desired result. \( \square \)

By Claim 3.4, Claim 3.5, and comparing depths of \( T_p \)-modules in the following exact sequences

\[
0 \to (\mathcal{I}T)_p \to (\mathcal{R}_+)(1)_p \to S_p \to 0
\]

and

\[
0 \to (\mathcal{T}_p) \to T_p \to T_p/(\mathcal{I}T)_p \to 0
\]
of \( T_p \)-modules, we get \( \text{depth} T_p = 2 \). On the other hand since \( T \) is an \( (S_3) \)-ring and \( \text{ht}_T \mathfrak{p} \geq 3 \), we get \( 2 = \text{depth} T_p \geq \inf \{3, \dim T_p\} = 3 \) which is a contradiction. Thus \( \overline{\mathfrak{m}} \) is a \( (S_2) \) module over \( B \) as desired.

The statement (b) is by Lemma\(^{2.4}(3)\). Lastly we will prove the statement (c). Since \( \overline{\mathfrak{m}} \cong B(-q) \) for some integer number \( q \geq 1 \), \( \ell_R(\overline{\mathfrak{m}}) = \ell_R(B_{n-q}) \) for all \( n \geq 0 \). If \( n < q \) then \( \ell_R(\overline{\mathfrak{m}}) = \ell_R(B_{n-q}) = (0) \) and hence

\[
\ell_R(R/T^{n+1}) = e_0(I) \begin{pmatrix} n + d \\ d \end{pmatrix} - \{e_0(I) + e_1(Q) - \ell_R(R/T)\} \begin{pmatrix} n + d - 1 \\ d - 1 \end{pmatrix} + \sum_{i=2}^{d} (-1)^i \{e_{i-1}(Q) + e_i(Q)\} \begin{pmatrix} n + d - i \\ d - i \end{pmatrix}
\]

by Lemma\(^{2.4}(1)\). If \( n \geq q \) then

\[
\ell_R(B_{n-q}) = \begin{pmatrix} n - q + d - 1 \\ d - 1 \end{pmatrix} = \sum_{i=0}^{q} (-1)^i \begin{pmatrix} q \\ i \end{pmatrix} \begin{pmatrix} n + d - 1 - i \\ d - 1 - i \end{pmatrix}.
\]

Therefore

\[
\ell_R(R/T^{n+1}) = e_0(I) \begin{pmatrix} n + d \\ d \end{pmatrix} - \{e_0(I) + e_1(Q) - \ell_R(R/T)\} \begin{pmatrix} n + d - 1 \\ d - 1 \end{pmatrix} + \sum_{i=2}^{d} (-1)^i \{e_{i-1}(Q) + e_i(Q)\} \begin{pmatrix} n + d - i \\ d - i \end{pmatrix} - \sum_{i=0}^{q} (-1)^i \begin{pmatrix} q \\ i \end{pmatrix} \begin{pmatrix} n + d - 1 - i \\ d - 1 - i \end{pmatrix} - \sum_{i=2}^{d} (-1)^i \{e_{i-1}(Q) + e_i(Q)\} \begin{pmatrix} n + d - i \\ d - i \end{pmatrix} + \{e_0(I) + e_1(Q) - \ell_R(R/T) + 1\} \begin{pmatrix} n + d - 1 \\ d - 1 \end{pmatrix} + \sum_{i=2}^{d} (-1)^i \{e_{i-1}(Q) + e_i(Q) + e_2(Q) + \begin{pmatrix} q \\ i - 1 \end{pmatrix} \} \begin{pmatrix} n + d - i \\ d - i \end{pmatrix}
\]

also by Lemma\(^{2.4}(1)\). Thus the last statement (c) follows. \( \square \)

4. The Cohen-Macaulay case.

In this section, we will prove Theorem\(^{1.2}\) and examine Theorem\(^{1.4}\) in the Cohen-Macaulay case. In order to prove Theorem\(^{1.2}\) we need the following result of Shiroh Itoh which is very important for the induction step.

Lemma 4.1. [14] THEOREM 1] Let \( R \) be a analytically unramified Cohen-Macaulay local ring with the maximal ideal \( \mathfrak{m} \) and \( d = \dim R > 0 \). Let \( I \) be a parameter ideal of \( R \). Then there exists a system of generator \( a_1, \ldots, a_d \) of \( I \) such that, if we put \( C = R(T)/(\sum_{i=1}^{d} a_i T_i) \) and \( J = IC \), where \( R(T) = R[T]_{\mathfrak{m}R[T]} \) and \( T = (T_1, \ldots, T_d) \) a set of indeterminates, then

(1) \( J^n \cap R = \mathfrak{m}^n \) for every \( n \geq 0 \).
(2) \( \overline{J} = \overline{T}C \).
(3) \( \overline{T}^n = \overline{T}C \cong \overline{T}^n R(T)/(\sum_{i=1}^d a_i T_i)\overline{T}^{n-1} R(T) \).
(4) \( \overline{e}_i(I) = \overline{e}_i(J) \) for all \( 0 \leq i \leq d - 1 \).

Then we get the following.

**Lemma 4.2.** Let \( R \) be a analytically unramified Cohen-Macaulay local ring with the maximal ideal \( \mathfrak{m} \), dimension \( d = \dim R \geq 2 \), and \( I \) an \( \mathfrak{m} \)-primary ideal of \( R \) containing a parameter ideal \( Q \) of \( R \) as a reduction. Then \( \overline{e}_1(I) = e_0(I) - \ell_R(R/\overline{I}) + 1 \) if and only if \( \overline{e}_2(I) = 1 \). When this is the case \( \ell_R(T^2/Q\overline{T}) = 1 \), and \( r_Q(\{T^n\}_{n\in\mathbb{Z}}) = 2 \).

**Proof.** The implication \( \overline{e}_2(I) = 1 \) implies \( \overline{e}_0(I) - \ell_R(R/\overline{I}) + 1 \) was proved by [18, Proposition 4.9]. Let we sketch a proof. Assume that \( \overline{e}_2(I) = 1 \). Then

\[
1 = \overline{e}_0(I) - \ell_R(R/\overline{I}) \geq 0,
\]

where the second inequality follows by [14, THEOREM 2] and the last inequality is by Lemma 2.4(2). If \( \overline{e}_1(I) = e_0(I) - \ell_R(R/\overline{I}) = 0 \), then \( \overline{e}_2(I) = 0 \), by Lemma 2.5(1), which is absurd. Hence \( \overline{e}_1(I) = e_0(I) - \ell_R(R/\overline{I}) + 1 \) as desired.

For the converse, assume that \( \overline{e}_1(I) = e_0(I) - \ell_R(R/\overline{I}) + 1 \). We will prove that \( \overline{e}_2(I) = 1 \) by using induction on \( d \). Assume that \( d = 2 \). Since \( \overline{a}R = \overline{a}\widehat{R} \) for every \( \mathfrak{m} \)-primary ideal \( \mathfrak{a} \) in \( R \), by passing to the \( \mathfrak{m} \)-adic completion \( \widehat{R} \) of \( R \), without lost of generality we may assume that \( R \) is complete. Therefore \( R \) is Nagata and hence, by Theorem 1.1 for all \( n \geq 0 \)

\[
\ell_R(R/T^n+1) = e_0(I) \left( \frac{n+d}{d} \right) - \left\{ e_0(I) - \ell_R(R/\overline{I}) + 1 \right\} \left( \frac{n+d-1}{d-1} \right) + q \left( \frac{n+d-2}{d-2} \right),
\]

i.e. the Hilbert function coincides with the Hilbert polynomial for all \( n \geq 0 \). Then \( T^n = QT^{n-1} \) for all \( n \geq 3 \), by [11, Theorem 4.4], and \( \overline{G} \) is Cohen-Macaulay, by [11, Theorem 4.6(ii)], because \( T^2 Q = QT \) by [13, THEOREM 1]. Since \( \text{depth}(\overline{G}) \geq d - 1 \), \( \overline{e}_1(I) = \sum_{n \geq 1} \ell_R(T^n/Q\overline{T}^{n-1}) \) and \( \overline{e}_2(I) = \sum_{n \geq 1} (n-1) \ell_R(T^n/Q\overline{T}^{n-1}) \) by [12, Corollary 4.6]. Hence \( q = \overline{e}_2(I) = \ell_R(T^2/Q\overline{T}) = 1 \) and \( r_Q(\{T^n\}_{n \in \mathbb{Z}}) = 2 \).

Now assume that \( d \geq 3 \) and the statement holds true for \( d - 1 \). Let \( J \) and \( C \) as in Lemma 4.1. Since \( \overline{e}_1(J) = \overline{e}_1(I) = e_0(I) - \ell_R(R/\overline{I}) + 1 = e_0(J) - \ell(R/\overline{J}) + 1 \), by Lemma 4.1 we get \( \overline{e}_2(J) = 1 \), by the induction hypothesis, and hence \( \overline{e}_2(J) = 1 \). The last statement follows by Lemma 4.1(3).

Since \( \overline{e}_2(I) \geq \overline{e}_1(I) - e_0(I) + \ell_R(R/\overline{I}) \geq 0 \) by [14, THEOREM 2], small values of \( \overline{e}_2(I) \) will lead to special bounds on \( \overline{e}_1(I) \) as follows.
Corollary 4.3. Let $R$ be a analytically unramified Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$. Let $d = \dim R \geq 2$ and $I$ an $\mathfrak{m}$-primary ideal of $R$. Then the following assertions hold true.

(1) ([18, Theorem 4.8] and [11, Theorem 4.5]) The followings are equivalent to each other.

(a) $e_2(I) = 0$.
(b) $e_1(I) = e_0(I) - \ell_R(R/\overline{T})$.
(c) $r_\mathcal{Q}(\{\overline{\mathfrak{m}}\}_{n \in \mathbb{Z}}) \leq 1$.

When this is the case $\overline{G}$ is Cohen-Macaulay.

(2) If $e_2(I) = 2$ then $e_1(I) = e_0(I) - \ell_R(R/\overline{T}) + 2$.

Proof. By [14, THEOREM 2] and Lemma 2.4(2) we have

$$2 = e_2(I) \geq e_1(I) - e_0(I) + \ell_R(R/\overline{T}) \geq 0.$$ 

Therefore $e_2(I) = 0$ implies $e_1(I) - e_0(I) + \ell_R(R/\overline{T}) = 0$. The converse is by Lemma 2.5(1) and we get the equivalence of (a) and (b). The equivalence of (b) and (c) and the last assertion of (1) are by Lemma 2.4(1).

Next, we assume that $e_2(I) = 2$. Then $e_1(I) - e_0(I) + \ell_R(R/\overline{T}) > 1$ by (1). Moreover if $e_1(I) - e_0(I) + \ell_R(R/\overline{T}) = 1$, then $e_2(I) = 1$, by Theorem 1.2 which contradicts to our assumption. Thus $e_1(I) = e_0(I) - \ell_R(R/\overline{T}) + 2$. □

When $R$ is a Nagata reduced Cohen-Macaulay ring, the number $q$ in Theorem 1.4 turns into exactly 1 and the equality $e_1(I) = e_0(I) - \ell_R(R/\overline{T}) + 1$ makes $\overline{G}$ to be Cohen-Macaulay as follows.

Corollary 4.4. Assume that $R$ be a Nagata reduced Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim R \geq 2$. Let $I$ be an $\mathfrak{m}$-primary ideal of $R$ and suppose that $I$ contains a parameter ideal $\mathcal{Q}$ of $R$ as a reduction. Then the followings are equivalent to each other.

(1) $e_1(I) = e_0(I) - \ell_R(R/\overline{T}) + 1$.
(2) $\mathfrak{m}\overline{S} = (0)$ and $\operatorname{rank}_B\overline{S} = 1$, where $B = T/\mathfrak{m}T$.
(3) $\overline{S} \cong B(-1)$ as graded $T$-modules.
(4) $e_2(I) = 1$.

When this is the case

(a) $\overline{S}$ is a Cohen-Macaulay $T$-module.
(b) $\overline{G}$ is Cohen-Macaulay.
(c) $\overline{R}$ is Cohen-Macaulay if $d \geq 3$ and $\overline{R}$ is not Cohen-Macaulay if $d = 2$.
(d) $\ell_R(\overline{T}^2/\overline{Q\overline{T}}) = 1$ and $r_\mathcal{Q}(\{\overline{\mathfrak{m}}\}_{n \in \mathbb{Z}}) = 2$. 
Proof. The equivalent statements (1) ⇔ (4) and the statement (d) follow by Lemma 4.2. The equivalent statement (1) ⇔ (2) follows by Lemma 2.5(2). The implication (1) ⇒ (3) follows by Theorem 1.1 and Theorem 1.2. Now we assume that 
\[ e_1(I) = e_0(I) - \ell(R/R/I) + 1. \]
The statements (a) is by Theorem 1.1. The statements (e) follows by Theorem 1.1 (c) and the fact that \( e_2(I) = 1. \) For a proof of (b), [11, Theorem 4.4] proves (b) for the case where \( d = 2. \) If \( d \geq 3 \) then
\[ e_i(I) = \sum_{n \geq i} \binom{n-1}{i-1} \ell(R/T^n/Q/T^n-I) \]
for \( 1 \leq i \leq d, \) by [12, Proposition 4.6], since depth \( G \geq d - 1 \) by Theorem 1.1(b). Therefore \( T^n = Q/T^n-I \) for all \( n \geq 3, \) as \( e_3(I) = 0, \) and \( \overline{e_2}(I) = \ell(R/T^2/Q/T). \) Hence, by [2, Theorem 3.12], \( G \) is Cohen-Macaulay and we get (b). It is left to prove (c). By [12, Proposition 4.10] \( \overline{R} \) is Cohen-Macaulay if and only if \( \overline{e_1}(I) = \ell(R/T/I) + \ell(R/T^2 + Q/Q). \) Therefore \( \overline{R} \) is Cohen-Macaulay when \( d \geq 3 \) because \( \ell(R/T^2 + Q/Q) = \ell(R/T^2/Q/T) = 1. \) When \( d = 2, \) \( \overline{R} \) is not Cohen-Macaulay because \( \overline{e_1}(I) > \ell(R/T/I) \) and again by using by [12, Corollary 4.10]. Thus we get the statement (c) which complete the proof of Corollary 4.4.

Example 4.5. Let \( S = k[x, y] \) be the polynomial ring over a field \( k \) and \( A = k[x^2, xy, xy^2] (\subseteq S). \) We set \( R = A_M \) where \( M = (x^2, xy, xy^2)A \) and let \( m \) denote the maximal ideal of \( R. \) We then have the following.

(1) \( R \) is a Gorenstein local integral domain such that \( \dim R = 2, e_0(m) = 3, \) and 
\[ m^3 = Qm^2 \]
where \( Q = (x^2 - xy^2, xy), \) but \( \ell_R(m^2/Qm) = 1. \)

(2) \( R \) is not a normal ring but \( m \) is a normal ideal in \( R. \)

(3) \( \ell_R(R/m^{n+1}) = 3\binom{n+2}{2} - 3\binom{n+1}{1} + 1 \) for all \( n \geq 0. \)

(4) \( S_Q(m) = S_Q(m) \cong B(-1) \) as a graded \( T-\)module.

(5) \( G(m) \) is a Cohen-Macaulay ring with \( a(G(m)) = 0, \) so that \( R(m) \) is not a Cohen-Macaulay ring.

Proof. Since \( A \cong k[x, y, z]/(z^4 - xy^2) \) where \( k[x, y, z] \) denotes the polynomial ring, we have \( \dim R = 2 \) and \( e_0(m) = 3. \) It is direct to check the rest of Assertion (1). The ring \( A \) is clearly not normal, whence so is \( R. \) To check that \( m \) is normal, one needs some computation which we leave to readers. Assertions (3) and (4) now follow from
Theorem 1.1. As \( G(\mathfrak{m}) \cong k[x, y, z]/(xy^2) \), \( G(\mathfrak{m}) \) is a Cohen-Macaulay ring with \( a(G(\mathfrak{m})) = 0 \). Therefore \( \mathcal{R}(\mathfrak{m}) \) is not a Cohen-Macaulay ring (see, e.g., [7]). □

By using Corollary 4.4, we can prove Theorem 1.2 as follows.

Proof of Theorem 1.2. By Lemma 4.2 it is enough to prove that \( e_1(I) = e_0(I) - \ell_R(R/I) + 1 \) implies \( G \) is Cohen-Macaulay. Moreover since \( aR = a\hat{R} \) for every \( \mathfrak{m} \)-primary ideal \( a \) in \( R \), by passing to the \( \mathfrak{m} \)-adic completion \( \hat{R} \) of \( R \), without lost of generality we may assume that \( R \) is complete. Therefore \( R \) is a Nagata reduced Cohen-Macaulay ring and the desired result is exactly Corollary 4.4(b). □

Remark 4.6. When \((R, \mathfrak{m})\) is a one-dimensional analytically unramified Cohen-Macaulay local ring, with the same notations for as above, \( G \) is a Cohen-Macaulay ring and \( \mathcal{R} \) is a Cohen-Macaulay ring if and only if \( R \) is a discrete valuation ring. In fact, the fact that \( G \) is a Cohen-Macaulay ring is by [16, Proposition 3.25]. For a proof of the second fact, if \( R \) is a discrete valuation ring, then \( \mathcal{T} \) is a parameter ideal and hence \( \mathcal{R} \) is Cohen-Macaulay. Conversely let \( a \in I \) such that \( (a) \subseteq I \) as a reduction. Notice that by the module version of [7, Theorem 1.1], \( \mathcal{R} \) is Cohen-Macaulay if and only if \( G \) is Cohen-Macaulay and \( a(G) < 0 \), where \( a(G) \) denotes the \( a \)-invariant of \( G \) (Definitions 3.1.4)). Since

\[
\mathcal{G}/atG \cong G(\{I^n + (a)/(a)\}_{n \in \mathbb{Z}}),
\]

the associated graded ring of the filtration \( \{I^n + (a)/(a)\}_{n \in \mathbb{Z}} \), \( a(G/atG) = a(G) + 1 \leq 0 \). Therefore for all \( n \geq 1 \), \( I^n \subseteq (a) + I^{n+1} \). Moreover since \( \mathcal{T} = aI^{\ell-1} \) for \( \ell \gg 0 \), \( I^n \subseteq (a) \) for all \( n \geq 1 \) and hence \( I^n = aI^{n-1} \) for all \( n \geq 1 \). In particular \( \mathcal{T} = (a) \), whence \( R \) is a discrete valuation ring by [3, Corollary 2.5].

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