Numerical simulation of the stability of three-dimensional elastic composite structures based on the finite element method

Yu I Dimitrienko and I O Bogdanov
Department of Computational Mathematics and Mathematical Physics, Bauman Moscow State Technical University, 5 Baumanskaya 2-ya, Moscow, 105005, Russia

E-mail: dimit.bmstu@gmail.com, ibogdanov@bmstu.ru

Abstract. A numerical method for solving the problem of the stability theory of linearly elastic bodies with small deformations in a general three-dimensional formulation is considered. The problems of this class are poorly studied in contrast to the two-dimensional problems of stability theory. At the same time, classical approaches do not allow one to take into account the effect on the structural stability of various three-dimensional effects: areas of compounds, zones of defects, etc. The study formulates a variation formulation of the problem of the three-dimensional stability theory. Based on the finite element method, a numerical statement is obtained in the form of a generalized eigenvalue problem with symmetric global stiffness matrix. The application of the proposed method is demonstrated by the example of calculating the stability of a composite plate under longitudinal compression. The simulation was carried out using the SMCM software package developed at the Department of Calculus Mathematics and Mathematical Physics of Bauman Moscow State Technical University.

1. Introduction
When operating products made of composite materials, of particular importance is the assessment of the influence of various three-dimensional effects (defects, joint areas, fastenings) on the stability of structures [1-5]. Using the classical approaches of one-dimensional and two-dimensional stability theories [6-17] does not allow to provide a sufficient level of accuracy in solving problems of this class. In this regard, the problem of studying the structures stability within the framework of the general three-dimensional theory has recently become relevant [18].

This paper describes a method for solving the three-dimensional problem of the stability theory of linearly elastic bodies with small deformations in a general three-dimensional formulation [19-27].

2. The mathematical model of the three-dimensional stability theory
We introduce three body configurations: reference \( \hat{K} \), actual \( K \), and varied \( \hat{K} \), which differs from the true actual configuration by small displacement. The indicated configurations are shown in Figure 1.
Figure 1. Reference, actual and varied body configurations

The stability problem consists in finding a varied configuration $\tilde{K}$ (i.e., a vector $\mathbf{w}$), and using it to search for a possible non-unique solution that determines the unstable state of the body. As shown in [18], in this case, the mathematical model of the stability theory of a linearly elastic body with small deformations consists in the formulation of two main problems solved together. The first of them is the equilibrium problem for the ground (stable) state has the form:

$$\nabla \cdot \mathbf{\sigma}^0 = 0,$$

$$\mathbf{\sigma}^0 = ^4 C \cdot \varepsilon^0, \; \varepsilon^0 = \frac{1}{2} \left( \nabla \otimes \mathbf{u}^0 + \nabla \otimes \mathbf{u}^{0T} \right),$$

$$\mathbf{n} \cdot \mathbf{\sigma}^0|_{\mathbf{\ell}_{\mathbf{w}}} = \lambda \mathbf{S}_e, \; \mathbf{u}^0|_{\mathbf{\ell}_{\mathbf{w}}} = \lambda \mathbf{u}_e,$$

where $\mathbf{\sigma}^0$ is stress tensor; $\varepsilon^0$ is small strain tensor; $\mathbf{u}^0$ is displacement vector; $^4 C$ is 4th rank elastic modules tensor; $\mathbf{S}_e$ and $\mathbf{u}_e$ are vectors of external surface forces and displacements, respectively; $\lambda$ is coefficient for vectors of external surface forces and displacements. The second task is actually the problem of stability theory:

$$\nabla \cdot \mathbf{\sigma} - \mathbf{\sigma}^0 \cdot (\mathbf{B} \cdot \mathbf{e}) = 0,$$

$$\mathbf{\sigma} = ^4 C \cdot \varepsilon (\mathbf{w}), \; \varepsilon (\mathbf{w}) = \frac{1}{2} \left( \nabla \otimes \mathbf{w} + \nabla \otimes \mathbf{w}^T \right),$$

$$\mathbf{B} = \nabla \otimes \mathbf{w}, \; \mathbf{\omega} = \frac{1}{2} \mathbf{e} \cdot \Omega (\mathbf{w}), \; \Omega (\mathbf{w}) = \frac{1}{2} \left( \nabla \otimes \mathbf{w} - \nabla \otimes \mathbf{w}^T \right),$$

$$\mathbf{n} \cdot (\mathbf{\sigma} - \mathbf{\sigma}^0 \cdot \mathbf{e} \cdot \mathbf{\omega})|_{\mathbf{\ell}_{\mathbf{w}}} = 0, \; \mathbf{w}^0|_{\mathbf{\ell}_{\mathbf{w}}} = 0,$$

where $\mathbf{\sigma}$ is stress tensor; $\varepsilon$ is small strain tensor; $\mathbf{w}$ is displacement vector in a varied configuration; $\mathbf{e}$ is Levi-Civita tensor.

The solution of problems (1) – (2) is carried out in accordance with the following algorithm:

1. Solve the problem (1) for the ground state with the parameter value $\lambda = 1$.
2. Calculate the $\mathbf{\sigma}^0(1)$ stress tensor field. Due to the linearity of the problem, the field of the stress tensor $\mathbf{\sigma}^0(\lambda) = \lambda \mathbf{\sigma}^0(1)$ corresponds to any other value of the parameter $\lambda$. 
3. Substituting the field $\sigma^0(\lambda)$ in (2), we obtain the stability theory problem (eigenvalue problem).

4. Find the system of eigenvalues $\lambda$ and eigenfunctions $w$.

3. Variation formulation of the problem

Consider the equilibrium problem for the ground (stable) state [18, 19]. We introduce the kinematically admissible field $\Psi^0 = \delta u^0$, where $\delta u^0$ is the variation of the displacement vector $u^0$, understood as the difference of two kinematically admissible fields. This field must satisfy the zero boundary condition on part of the surface $\Sigma_u$ of the region $V$. Multiplying the equilibrium equation from the system (1) by $\Psi^0$ and integrating the expression over the region $V$, taking into account the Gauss-Ostrogradsky theorem, we obtain the variation equation for the equilibrium problem in the ground state:

$$
\int_V \varepsilon^T \cdot \varepsilon^0 \left( u^0 \right) \cdot \delta \varepsilon^0 \left( u^0 \right) dV - \int_{\Sigma_u} \varepsilon_s \cdot \delta u^0 d\Sigma = 0.
$$

Applying a similar approach to the stability problem (2), we arrive at the variation equation:

$$
\int_V \left( \varepsilon^T C \cdot \varepsilon(w) + \sigma^0 \cdot \Omega(w) \right) \cdot \delta \varepsilon \left( u^0 \right) dV = 0.
$$

The relation (4) is an eigenvalue problem in which it is required to find the eigenvalues $\lambda$ and the corresponding eigenfunctions $w$. The least eigenvalue $\lambda_{min}$ is of most practical interest since it corresponds to the critical load $\sigma_{cr}$, which leads to the first form of stability loss. All other $\lambda$ values will correspond to other forms of structural stability loss.

4. The finite element method for solving the problem of stability theory

Consider the numerical formulation of the problem of stability theory. To formulate it in this paper, we used the finite element method [20, 21]. We assume that for the triangulation of the computational domain, a tetrahedral simplex element with three degrees of freedom at each node is used.

Introducing the matrix analogues of the quantities included in the variation formulation (3), and then applying the classical procedure of the finite element method, we obtain a numerical formulation of the equilibrium problem in the ground state, which can be written in the form:

$$
\begin{bmatrix}
K_{12} & F_{12}
\end{bmatrix}_{12} = \begin{bmatrix}
F
\end{bmatrix}_{12},
$$

where the notations are introduced:

$$
\begin{bmatrix}
K_{12} & F_{12}
\end{bmatrix}_{12} = \int_V \begin{bmatrix}
B^T & C
\end{bmatrix}_{12} \begin{bmatrix}
S \end{bmatrix}_{12} dV,
\begin{bmatrix}
F
\end{bmatrix}_{12} = \int_{\Sigma_u} \begin{bmatrix}
S \end{bmatrix}_{12} d\Sigma,
\begin{bmatrix}
B_i
\end{bmatrix} = \begin{bmatrix}
L_i
\end{bmatrix}_{3 \times 12},
\end{bmatrix}_{3 \times 12}.
$$

Here $[C]$ is the matrix of the elastic modulus tensor $4C$ components, $[N]$ is the matrix of the shape functions, $[L_i]$ is the differentiation operator, $[S \Sigma]$ is the vector of external surface forces.

Let us consider the numerical formulation of the problem for a varied state. To do this, we introduce the following notation:

$$
\begin{bmatrix}
\sigma^R
\end{bmatrix}_6 = \begin{bmatrix}
\sigma_{11} & \sigma_{22} & \sigma_{33} & \sigma_{23} & \sigma_{13} & \sigma_{12}
\end{bmatrix}_6
$$

– vector of stress tensor components;

$$
\begin{bmatrix}
\varepsilon^R
\end{bmatrix}_6 = \begin{bmatrix}
\varepsilon_{11} & \varepsilon_{22} & \varepsilon_{33} & 2\varepsilon_{23} & 2\varepsilon_{13} & 2\varepsilon_{12}
\end{bmatrix}_6
$$

– vector of the small strain tensor components;

$$
\begin{bmatrix}
W^R
\end{bmatrix}_3 = \begin{bmatrix}
W_1 & W_2 & W_3
\end{bmatrix}_3
$$

– finite element (FE) displacement vector in a varied configuration;
the displacement vector in the nodes of the finite element in a varied configuration;

\[
\left\{ w \right\}_{12}^T = (w_{11} w_{12} w_{13} w_{21} w_{22} w_{23} w_{31} w_{32} w_{33} w_{41} w_{42} w_{43})
\]

– matrix of stress tensor components in the ground state;

\[
\{R\} = (R_{11} R_{12} R_{13} R_{21} R_{22} R_{23} R_{31} R_{32} R_{33})
\]

– a vector whose components are derivatives of the form \( \partial w_j / \partial x_i \);

\[
[T] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

– transpose matrix.

Given the notation introduced, the generalized Hooke’s law (the second of the equations in (2)) can be written in the following equivalent form:

\[
\{ \sigma \} = [C] \{ \varepsilon \},
\]

and the Cauchy relations (the third of the equations in (2)) take the form:

\[
\{ \varepsilon \} = [L] \{ W \},
\]

where the displacement vector of a finite element (FE) in a varied configuration is related to the displacement vector of its nodes as:

\[
\{ W \} = [N] \{ w \}.
\]

Variations of the vector of the small strain tensor \( \{ \varepsilon \} \) components and vector \( \{ R \} \) can be expressed as follows, respectively:

\[
\{ \delta \varepsilon \} = [L] \{ \delta W \}, \quad \{ \delta R \} = [L] \{ \delta W \}, \quad \{ \delta W \} = [N] \{ \delta w \}.
\]
where the following notations are introduced:

\[
\begin{align*}
[K] &= \int \left( [B^T] [C] [B] \right) dV, \\
[S] &= \frac{1}{2} \int \left( [B^T_2] [\Sigma^0] [B_2] \right) dV, \\
[S^0] &= \frac{1}{2} \left( [\Sigma^0] + [\Sigma^0]^T \right), \\
[\Sigma^0] &= \left( \begin{bmatrix} \Sigma^0_{yx} \\ \Sigma^0_{zy} \end{bmatrix} \right), \\
\left( [T] - [E] \right) &= \left[ \begin{bmatrix} \sigma_{yx} \\ \sigma_{zy} \end{bmatrix} \right], \\
[B_2] &= \left[ L_2 \right] [N].
\end{align*}
\]

Here \([L_2]\) is the differentiation operator, \([B^T_2]\) and \([B_2]\) are the matrices of derivative functions of the form.

5. Stability of the composite plate

Let us demonstrate the application of the considered model by the example of a multilayer composite plate with a hole. The test plate consists of two layers of ST-12026 twill weaving carbon fabric with reinforcement structure \((0^\circ/90^\circ)\) and one layer of biaxial diagonal fiberglass SM-42020 with reinforcement structure \((+45^\circ/-45^\circ)\) (see Figure 2). The total thickness is 0.01 m. The effective elastic constants of this material are shown in Table 1 and were calculated separately [22].

![Figure 2. Reinforcement scheme of a multilayer composite plate](image)

| Effective elastic constants | Designation | Value |
|-----------------------------|-------------|-------|
| Young’s modulus along the \(x\) axis, MPa | \(E_1\) | 47752.5 |
| Young’s modulus along the \(y\) axis, MPa | \(E_2\) | 9039.68 |
| Young’s modulus along the \(z\) axis, MPa | \(E_3\) | 47752.5 |
| The shear modulus in the \(xy\) plane, MPa | \(G_{12}\) | 3831.02 |
| The shear modulus in the \(xz\) plane, MPa | \(G_{13}\) | 10430.6 |
| The shear modulus in the \(yz\) plane, MPa | \(G_{23}\) | 3831.02 |
| Poisson's ratio in the \(xy\) plane | \(v_{12}\) | 0.315505 |
| Poisson's ratio in the \(xz\) plane | \(v_{13}\) | 0.158262 |
| Poisson's ratio in the \(yz\) plane | \(v_{23}\) | 0.0597261 |

The paper considers two calculations for stability under longitudinal compression. In each of the calculations, the plate had a size of \(0.5 \text{ m} \times 1 \text{ m}\). The initial load, set at the short end of the plate, was 0.2 GPa, the opposite end was rigidly fixed (zero components of the displacement vector were specified). In the first case, a hole with a diameter of 0.05 m was located in the centre in the upper part of the plate, in the second, in the lower right.
Table 2 shows the calculated critical loads. Figure 3 shows examples of the results of solving the problem for the ground state with symmetrical location of the hole, Figure 4 shows examples of the results of solving the problem for the varied state with symmetrical location of the hole, Figures 5-6 show examples of results, respectively, for the ground and varied states with an asymmetric location of the hole.

Figure 3. Examples of fields obtained by solving the problem for the ground state with the symmetrical location of the hole: a) component $u_i^0$ of the displacement vector; b) component $u_i^0$ of the displacement vector; c) component $\sigma_{33}^0$ of the stress tensor

Figure 4. Examples of fields obtained by solving the problem for a varied state with the symmetrical location of the hole: a) component $w_i$ of the displacement vector; b) component $w_i$ of the displacement vector; c) component $w_i$ of the displacement vector

Figure 5. Examples of fields obtained by solving the problem for the ground state with an asymmetric location of the hole: a) component $u_i^0$ of the displacement vector; b) component $u_i^0$ of the displacement vector; c) component $\sigma_{33}^0$ of the stress tensor

Figure 6. Examples of fields obtained by solving the problem for a varied state with an asymmetric location of the hole: a) component $w_i$ of the displacement vector; b) component $w_i$ of the displacement vector; c) component $w_i$ of the displacement vector

From the results shown in Table 2, it is seen that the critical loads at different positions of the holes are quite close.

Table 2. Values of calculated critical loads

| Hole position   | The value of the critical load, MPa |
|-----------------|-------------------------------------|
| Top center      | 17.449                              |
6. Conclusions
The work considers a method for solving the stability theory of linearly elastic bodies with small deformations in a general three-dimensional formulation. The main advantage of this approach is the ability to take into account the influence of three-dimensional effects on the loss of structural stability. In the paper, a variation formulation of the problem is formulated and reduced to a generalized eigenvalue problem based on the use of the finite element method. The results of applying the considered model are presented on the example of studying the stability of a multilayer composite plate with a hole.

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