Weather Forecast for Vacuum Fluctuations in QED

Maximilian Koegler\textsuperscript{1} and Marc Schneider\textsuperscript{2}

\textsuperscript{1}Arnold Sommerfeld Center for Theoretical Physics, Theresienstraße 37, 80333 München
\textsuperscript{2}EHU/UPV University of the Basque Country, Barrio Sarriena s/n, 48940 Leioa, Spain
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We derive closed analytic expressions for the Uehling and Serber contributions of the vacuum fluctuations in QED given by Meijer G-functions. The form of these potentials is analyzed, and their relevance for precision measurements in experiments is investigated. For the Uehling potential, we connect the solution with the propagation of photons through atmospheric turbulence.

Introduction. The hydrogen atom is one of the most intensively and accurately studied systems in theoretical and experimental physics \cite{1, 2}. In quantum mechanics, the hydrogen atom’s kinematics is ruled by the Coulomb potential $V_C(r) = -\frac{\alpha}{4\pi r}$ which determines the bound-state structure and the related energy levels of the electron. The vacuum state in quantum electrodynamics (QED) differs from the purely quantum mechanical description of the vacuum. This discrepancy was discovered because the predicted values from quantum mechanics did not match the measured energy levels. This feature of the spectrum, known as the Lamb shift \cite{3} could only be understood by the QED effects: electron self-energy, anomalous magnetic moment, and vacuum polarization \cite{4, 6}. The last contribution describes that the bound-state electron experiences a slightly different charge than the nominal charge of the proton. The physics of QED predict that the vacuum features virtual particles which interact with the photons that form the Coulomb potential leading to a slight weakening.

Through a description as a potential, Uehling \cite{7} was able to derive the shift in energy levels from the vacuum polarization. At the same time, Serber \cite{8} generalized this approach to time-dependent fluctuations. Albeit being major discoveries, one downside of their developments was the implicit integral representation of these potentials which complicates explicit analyses and cloaks physical intuition.

To explore the foundations of the QED vacuum from the perspective of atomic physics, an explicit functional form seems inevitable. In this article, we solve the long-standing integral representations of the Uehling and the Serber potential and provide an explicit form by Meijer G-functions. Furthermore, we confirm that these solutions match experimental data and agree with common approximations. The explicit form provides a novel perspective, because the Meijer G-function that describes how QED vacuum fluctuations affect the electron, occurs in the description of atmospheric fluctuations as well. The propagation of photons within the QED vacuum and their probability to scatter at atmospheric density fluctuations are both ruled by the same Meijer G-function, thus, painting a tentative picture of the QED vacuum as a turbulent dielectric medium.

The time-dependent Serber potential can be shown to vanish rapidly when moving away from the source \cite{9}. In fact, its contribution can only affect very short ranges in space and time which is in agreement with any measurements. In this article, our conventions are chosen to be consistent with the unit system of Serber, cf. \cite{8}, or the beginning of the End Matter section.

Vacuum effects. At first principles, vacuum fluctuations are expressed as a renormalized current density $\langle j^a(x)\rangle_{\text{ren}}$, obtained using standard renormalization techniques \cite{10, 11}. An elegant route to find $\langle j^a(x)\rangle_{\text{ren}}$ has been developed by Schwinger in terms of the (slowly varying) external current $J^a(x)$ through \cite{12, 15}

$$\langle j^a(x)\rangle_{\text{ren}} = \sum_{n=1}^{\infty} \frac{a_n}{m^2 n} \Box^n J^a(x), \quad (1)$$

where $\Box = \partial_b \partial^b$ and $a_n$ are coefficients that depend on the specifics of the system and are essentially constructed from the Green’s function (cf. \cite{13, 16} for the detailed treatment). The physical idea behind (1) is that the external current $J^a(x)$ (and variations thereof) induce a charge and current density $\langle j^a(x)\rangle_{\text{ren}}$ in vacuum. In QED, we would have $J^a(x) = \Box A^a - \partial_b \partial^b A^b$ \cite{15}, in which the specifics of $A_a$ contain the system’s details. How vacuum currents impact the energy levels of the atom requires to transform the contributions \cite{1}, in particular the $a_n$, into a potential form.

Uehling potential. To study the modifications of the hydrogen atom inflicted by QED, we choose the gauge $A_0(r) = V_C(r)$. By working with static fields, that is, a static external current, \cite{1} reduces (in linear order) to the form found by Uehling \cite{7}

$$\langle j^a(\vec{x})\rangle_{\text{ren}} = \int d^3x' V_U(r) \Delta J^a(\vec{x}'). \quad (2)$$

where the Uehling potential $V_U(r)$, with $r = |\vec{x} - \vec{x}'|$, has been defined through the integral \cite{1} as

$$V_U(r) = -\frac{\alpha}{16\pi^4} \int_0^{\frac{\pi}{2}} d\varpi \cos^3(\varpi) \int \frac{d^3k}{k^2} e^{i\vec{k}\vec{x}'} \times \ln \left(1 + \frac{k^2}{4} \cos^2(\varpi)\right). \quad (3)$$

This expression provides a consistent way to understand the shift in energy levels, which is consistent with the unit system of Serber, cf. \cite{8}, or the beginning of the End Matter section.
Performing the $k$-integration in polar coordinates, the Uehling potential becomes

$$V_U(r) = \frac{\alpha}{4\pi^2 r} \int_0^\infty d\varpi \cos^3(\varpi) \text{Ei}(2r \sec(\varpi)),$$

with exponential integral $\text{Ei}(x) = \int_x^\infty \frac{e^{-s}}{s} \, ds$. We were able to find the integral in [4] for the first time [17] and find the explicit form for the Uehling potential to be

$$V_U(r) = -\frac{\alpha}{16\pi^2 r} G^{4,0}_{2,4} \left(0, 0, \frac{1}{2}, \frac{1}{2} \mid r^2 \right),$$

where $G^{4,0}_{2,4}$ denotes a Meijer G-function (cf. appendix B for details) [18]. Note, there exists another, more commonly used representation which we also solve for details) [18]. Note, there exists another, more commonly used representation which we also solve for details) [18].

Generically, the Uehling potential contributes to the Lamb shift of the hydrogen atom as follows [7],

$$\Delta E_{nl} = \frac{\alpha}{\pi} \langle n| V_U | n \rangle = \frac{\alpha}{\pi} \int d^3 x V_U(r) |\psi_{nl}(\vec{x})|^2,$$

where the electron states $\psi_{nl}(x)$ and a brief summary of the hydrogen atom are discussed in appendix A. As such the values for the energy shifts yield: $\Delta E_{10} = -8.8959 \cdot 10^{-7}$ eV, $\Delta E_{20} = -1.1120 \cdot 10^{-7}$ eV, and $\Delta E_{30} = -3.2947 \cdot 10^{-8}$ eV. These values concur with the commonly employed formula in the low momentum transfer expansion, $\Delta E_{nl}^{\text{approx}} = \frac{-4m_e k^2 \delta_{0l}}{15\pi^2}$, to an accuracy of 99.69%. This approximation only shifts states with $l = 0$, as detailed in [21], while we report $\Delta E_{21} = -3.1658 \cdot 10^{-13}$ eV. In the study of the Lamb shift, as discussed by [22], an analytic and closed form of the Uehling potential would enhance precision and accelerate numerical computations, such as in the context of superheavy elements and molecules where the Lamb shift is significantly amplified, reaching magnitudes of up to 100 eV [20, 23, 24]. The commonly used fitted parametric representations of the Uehling potential can now be complemented by Eq. (5).
small- and large-scale scattering cells within the atmosphere.

To draw direct comparisons with our atomic case, we calculate the intensity of the electric field originating from the potential $\Psi(x) = -\partial_t V(x)$. Utilizing identities for the Meijer G-function and its derivative from appendix A, the intensity results in

$$I(r) \propto |E(r)|^2 \approx \frac{\alpha^2}{16\pi^4 r^4} \left[ 1 + \frac{\alpha}{\pi} G_{1,4}^{3,1} \left( 0, \frac{3}{2}, \frac{5}{2}, 2, 0 \right) r^2 \right].$$

(11)

Since this particular Meijer G-function is negative for all $r$ the intensity $I$ is damped due to vacuum fluctuations. Analogous to interference phenomena, e.g. the double slit experiment, where the intensity of electromagnetic fields correlates with the photon detection probability ($I \sim P$) [29], the similarity between the atmospheric photon propagation and the nucleus’s electric field plus QED perturbations points towards a profound connection mediated by the same special function in (10) and (11).

This analogy suggests that an atomic nucleus’s electric field is influenced by the QED vacuum in a manner akin to atmospheric disturbances affecting photons. Comparing the two expressions, (10) and (11), reveals that, in the atomic context, only the small-cell turbulence parameter $\alpha$ contributes. This aligns with the known behavior of vacuum fluctuations in QED, i.e. virtual electron-positron loops contribute on scales much smaller than the photon’s wavelength due to the uncertainty principle. Such insights enable us to propose a new perspective on the QED vacuum and how its impact on physical processes can be described.

**Serber potential.** Serber generalized Uhling’s analysis to time-dependent external currents [8, 9] for which the vacuum current density [1] has to be considered in the full, four-dimensional case. Following [8], we transform into hyperbolic coordinates in $k$-space such that, $k_0 = K \sinh(\theta)$ and $k = K \cosh(\theta)$ with $\theta \in [0, \infty)$, thus,

$$\langle j^a(x) \rangle_{ren} = \int d^4x' \int_0^\infty d\varpi \cos^3(\varpi)$$

$$\times \Lambda(x - x'; \varpi) \Box J^a(x').$$

(12)

where $\Lambda(x - x'; \varpi)$ denotes the integration kernel. Under appropriate boundary conditions, cf. [8], the integral kernel consists of a static and a dynamic contribution, i.e. $\Lambda(x - x'; \varpi) = \Lambda_{stat}(r; \varpi) + \Lambda_{dyn}(r, \varpi)$ where

$$\Lambda_{stat}(r; \varpi) = -\frac{\alpha}{16\pi^2 r} \cos^2(\varpi) \int_1^\infty \frac{dK}{K^3} e^{-2K r \sec(\varpi)},$$

(13)

$$\Lambda_{dyn}(r; \varpi) = \frac{\alpha}{8\pi^2} \cos(\varpi) \int_1^\infty \frac{dK}{K^3} J_1(2K r \sec(\varpi)).$$

(14)

Here, $J_1(x)$ is the Bessel function of the first kind and we introduced the shorthand notation for the geodesic distance $\tau = \sqrt{(r - r')^2 - (t - t')^2}$. The boundary conditions further infer that $\Lambda_{dyn}(\tau; \varpi)$ has solely support in the future directed light-cone and vanishes outside $[8, 9]$.

Since we are particularly interested in the dynamical modification, we extract the Serber potential $V_S(t, r)$ by the usual method

$$\langle j^a(x) \rangle_{ren} = \int d^4x' V_S(t, r) \Box J^a(x').$$

(15)

where, similar to the time-independent derivation, the potential is found through the $\varpi$-integral in (12)

$$V_S(t, r) = \int_0^\infty d\varpi \cos^3(\varpi) \Lambda_{dyn}(t, r; \varpi).$$

(16)

The $K$-integral in (14) can be evaluated directly to:

$$\Lambda_{dyn}(y) = \frac{\alpha}{8\pi^2} \left[ \frac{1}{2} - (\ln(y) + \gamma) + \frac{y^2}{4} F_3 \left( \frac{1}{2}, 2, 3 ; y^2 \right) \right].$$

(17)

where we introduced $y := \tau \sec(\varpi)$ for simplicity. Taking (17), we perform the $\varpi$-integration in (16) and find the closed form of the Serber potential to be

$$V_S(t, r) = \frac{\alpha}{8\pi^2} \int_1^\infty \frac{dK}{K^3} e^{2K r \sec(\varpi)}.$$

(18)

with constant $c_S = -\frac{\sqrt{\pi}}{4} C_{3,5}^{2,2} \left( 1, 2, -1, 0, 0 \right) r^2 - \frac{2 \ln(\tau)}{3} + c_S$

Clearly, the Serber potential shows a divergence at the origin and a damped oscillation around zero towards large $\tau$ values. This can be understood from asymptotically expanding $V_S(t, r)$: for $\tau \ll 1$, we can approximate

$$V^0_S(t, r) \sim -\alpha \frac{3(\ln(\tau) + \gamma) + 1 - \ln(8)}{36\pi^2} + O(\tau),$$

(19)
The asymptotic expansion for the far interior, we see a, oscillation of the order $2 \cdot \text{orange hues}$. At the boundary, the potential diverges while in values are represented by magenta hues and negative ones by orange hues. At the boundary, the potential diverges while in the far interior, we see a, oscillation of the order $2 \cdot \text{orange hues}$.

Finding that the divergence is logarithmic at $\tau = 0$, while for $\tau \gg 1$, $V_S(t, r)$ goes into a damped oscillation

$$V_S^\infty(t, r) \sim \alpha \frac{4\pi \cos(2\tau) + 11 \sin(2\tau)}{128 \pi^2 r^4} + O \left( \frac{1}{\tau^5} \right). \quad (20)$$

The asymptotic expansion for $V_S^\infty(\tau)$ agrees with the dynamical kernel for $\tau < 0.5$ while $V_S^\infty(\tau)$ matches $V_S(\tau)$ for $\tau > 8$. Note, appendix D shows that $\frac{13}{3}$ vanishes asymptotically as well.

Nevertheless, this analysis captured only the behavior in the geodesic distance while a better understanding of $V_S(t, r)$ would be gained in terms of $t$ and $r$. Fig. 3 shows the support of the Serber potential within a $t$-$r$-diagram. The ceasing of the oscillations occurs very rapidly; at $t \approx 20$, they reach amplitudes that challenge the numerical resolution. From Fig. 2 and 4 and (20), we discover that for time-spans of about two Compton times the modulation from $V_S(t, r)$ turns into damped oscillations around zero. Hence, we would not expect a significant net modification of $V_C(r)$ from $V_S(t, r)$, but rather a modulation that averages out (temporally and spatially) away from the source. Hence, $V_S(t, r)$ contributes only on short time spans. Since QED fermions are massive, they propagate inside the lightcone which confines any potential influence of $V_S(t, r)$ to the vicinity of the source where the Serber potential remains monotonous.

**Discussion.** Deriving the explicit forms of the Uehling and Serber potential allowed us to explore the features of the QED vacuum from a new perspective. Because hydrogen atoms are extremely accurately measured, we substantiated our solution since all predictions from the Meijer G-functions matched the results in the literature accurately. We furthermore saw that the Serber potential may only affect time-resolved measurements while averaging out effectively, whereas the Uehling contributions persists.

The occurrence of the Meijer G-function revealed a remarkable resemblance with the propagation of photons through the atmosphere. Albeit being a tentative connection so far, the similarity appears logical. Scattering on atmospheric density fluctuations seem to be translated into a scattering on virtual fermion loops. If we adopt this picture for a moment, this means that some photons that form the Coulomb potential will not reach the bound state electron due to interaction with the vacuum. Therefore, the effective potential is a smidgen lifted through the Uehling potential which contributes negatively to the observed Lamb-shifted energy levels.

Our particular Meijer G-function, hence, supports the idea of the QED vacuum as a fluctuating environment in which tiny (by Heisenberg’s uncertainty principle) inhomogeneities are created. Any propagating photon could in principle interact with these fermionic vacuum bubbles. While the cumulative net effect is captured by the Uehling potential, the Serber potential describes the influence of such a fluctuation dynamically and contributes only within a small (with respect to Heisenberg’s uncertainty principle) space-time region around the event. It is understood that the QED fluctuations are in charge rather than molecular density. However, the similarity of the solutions supports the intuition of viewing the vacuum as some sort of medium. A promising fact is that non-linear effects in QED can be traced back to such loop couplings as well [31, 32]. How far the analogy can be spun, however, has to be investigated in further research.
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* m.koegler@physik.uni-muenchen.de
† marc.schneider@ehu.eus

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[17] Note1. We managed to solve this integral on an Apple MacBook Pro with an M3 Pro core with MacOS 15.1.1 using mathematica 14.1. On any other system, e.g. a local cluster, the integral remained unsolved.

[18] Note2. Unluckily, the Meijer G-function was discovered exactly in the year after Uehling’s and Serber’s article [33].

[19] Note3. There exists an analytic formula in terms of recursive integrals [34, 35].

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The associated Laguerre polynomials are defined as:

\[ L_n(x) = \frac{e^x}{n!} \frac{d^n}{dx^n} (xe^{-x}) \]

where \( n \) is a non-negative integer.

The Hamilton operator is defined as:

\[ H = -\frac{\hbar^2}{2m_e} \frac{d^2}{dx^2} + \frac{e^2}{4\pi \varepsilon_0 \varepsilon_r} \frac{1}{x} \]

where \( m_e \) is the electronic mass, \( e \) is the electronic charge, and \( \varepsilon_r \) is the relative permittivity.

The energy levels are determined by the eigenstates of the Hamilton operator, \( \psi_n(x) = E_n^e \psi_n(x) \), where \( E_n^e \) are the energy eigenvalues.

For convenience, we will present the most important states here, that will be used in the article:

- The s-states
  \[ \psi_{100}(x) = \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{x}{a_0}} \]

- The p-states
  \[ \psi_{210}(x) = \frac{\cos(\theta)}{4\sqrt{2\pi a_0^3}} a_0 e^{-\frac{x}{2a_0}} \]

where \( a_0 = \frac{\hbar}{m_e e} \) is the Bohr radius.

The Meijer G-function satisfies a plethora of remarkable properties, for example, it can be expressed as Meijer G-function features several nice properties, for example, it can be expressed as:

\[ f(x) = G_{m,n}^{p,q} \left( \begin{array}{c} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \end{array} \mid x \right) \]

along one path \( C \) within the complex plane. The Meijer G-function satisfies several nice properties, for example, it is closed under various operations like the reflections \( x \to -x \) and \( x \to x^2 \), scalar multiplication, Laplace and Euler transformation, and the convolution. One remarkable property is that many elementary functions can be expressed as Meijer G-function, e.g.

\[ e^x = G_{1,0}^{0,1}(x, 0, 1) \]

Moreover, the Meijer G-function satisfies a plethora of interesting identities; the following have been used in this article. The first is a very elementary one for \( \ell \in \mathbb{Z} \),

\[ G_{m,n}^{p,q} \left( \begin{array}{c} a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \end{array} \mid x \right) = (-1)^q G_{m+1,n-1}^{p+1,q+1} \left( \begin{array}{c} a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_p, \ell \\ a+\ell, b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \end{array} \mid x \right) \]
which allows to exchange the indices while changing the parameters. Another identity involves the derivative
\[
\frac{d}{dx} \left[ x^{-b_i} G_{p,q}^{m,n} \left( a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \mid b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \right) x \right] = -x^{-b_i+1} G_{p,q}^{m,n} \left( a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \mid b_1+1, \ldots, b_m, b_{m+1}, \ldots, b_q \right).
\]

Others can be found in standard collections of tables, integrals, and special functions, e.g. [38].

Appendix C: Other representation. Several other integral representations of \(-\frac{2}{\pi} \overline{V}_U(r)\) exist, e.g. the one by Uehling [7]
\[
\overline{V}_U(r) = -\frac{2\alpha^2}{3\pi r} \int_1^\infty dx \frac{e^{-2\pi^2 x^2}}{2x^4} \left( 2x^2 + 1 \right) \sqrt{x^2 - 1}. \tag{34}
\]

Again, this integral is solvable in terms of Meijer G-functions
\[
\int_1^\infty dx \frac{e^{-2\pi^2 x^2}}{2x^4} \left( 2x^2 + 1 \right) \sqrt{x^2 - 1} = \frac{3}{2\pi\lambda} r + \frac{2}{3\pi\lambda^3} r^3 + \frac{1}{2} G_{2,4}^{2,0} \left( \frac{1}{2}, \frac{3}{2}, 1 \mid \frac{r^2}{\lambda^2} \right) \frac{1}{4} G_{2,4}^{2,0} \left( \frac{1}{2}, \frac{5}{2}, 1 \mid \frac{r^2}{\lambda^2} \right). \tag{35}
\]

Unfortunately, there is no identity or representation of the Meijer G-function known to us that allows to represent polynomials. To verify our result, we numerically confirmed that the difference \(\overline{V}_U(r) - \left(-\frac{2}{\pi} \overline{V}_U(r)\right)\), using [35] and [38], yielded zero. From [7], we know that [1] can be represented by (34) which in turn leads to a novel identity for the Meijer G-function and a representation of polynomials in terms of Meijer G-functions.

Appendix D: Kernel asymptotics. To understand the fading impact of the Serber contribution at long distances, let us analyze the two kernels [13], and [14] directly. We consider first the dynamical kernel
\[
\Lambda_{\text{dyn}}(y) = \frac{\alpha}{8\pi^2} \left[ \frac{1}{2} - \left( \ln(y) + \gamma \right) + \frac{y^2}{4} F_3 \left( \frac{1}{2}, 1, 1 \mid -y^2 \right) \right],
\]
where the hypergeometric function for large arguments \(y \to \infty\),
\[
F_3 \left( \frac{1}{2}, 1, 2, 2 \mid -y^2 \right) \sim \frac{4}{y^2} \left( \ln(y) + \gamma - \frac{1}{2} + \mathcal{O} \left( \frac{1}{y^2} \right) \right).
\]

Comparison with \(\Lambda_{\text{dyn}}(y)\) shows that these terms exactly cancel and the contribution goes asymptotically to zero.

The same conclusion holds true for the static part for which the \(K\)-integral yields
\[
\Lambda_{\text{stat}}(r; \omega) = \frac{\alpha}{16\pi^2 r} \cos^2(\omega) \left[ e^{-2r\sec(\omega)} \left( 1 - 2r \sec(\omega) \right) \right. \]
\[
\left. + r^2 \sec^2(\omega) E_1 \left( r \sec(\omega) \right) \right], \tag{38}
\]

involving the exponential integral \(E_1(x) = \int_1^\infty ds \frac{e^{-sx}}{s}\). For large radii, it is clear that the first two terms are suppressed by the damping exponential and we are left with a contribution \(x E_1(x)\). By using the representation through Tricomi’s confluent hypergeometric function \(U(a, b, x)\) one can derive the asymptotics
\[
E_1(x) = e^{-x} U(1, 1; x) \xrightarrow{x \to \infty} \frac{e^{-x}}{x} \left( 1 + \mathcal{O} \left( \frac{1}{x} \right) \right). \tag{39}
\]

Therefore, the static contribution’s asymptotics is completely ruled by a damping exponential and thus complies with the vanishing of the Serber potential far from the source.