Local Coefficients Revisited

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January 8, 2018

Abstract
Two simple "simplicial approximation" tricks are invoked to prove basic results involving (co)-homology with local coefficients.

1 Introduction

Homology and cohomology with local coefficients have applications in a variety of topics including Obstruction Theory, Spectral Sequences, Generalized Poincare Duality and more. These homology and cohomology theories possess many properties analogous to those of homology and cohomology with constant coefficients. Yet some of the corresponding properties are missing. In particular, there is no Universal Coefficient Theorem linking homology with local coefficients with cohomology (there is a version in [7], p. 283, though its application is limited). This (among others) poses extra difficulties in proving many familiar (and useful) properties of these theories. In fact more than often the proof of a theorem, carried verbatim from the theory with constant coefficients, is fundamentally distinct in the case of local coefficients.

The purpose of this paper is to establish three basic properties of (co)-homology with local coefficients.

The First Property: A weak homotopy equivalence induces an isomorphism on cohomology with local coefficients.

The Second Property: For a CW-pair the long exact sequences for singular and cellular cohomology group with local coefficients are equivalent.

Despite being very basic, no written account of these properties could be located. Quite likely this was due to the suspicion that the technical difficulties in rigorous proofs presented a price too high to pay for the final product. Our proofs of these properties are based on a simple “simplicial approximation” trick which avoids most of the technicalities.

*The first author is supported by the Simons Foundation grant 281810. 2010 Mathematics Subject Classification 55P65(primary) 57P10(Secondary).
The Third Property: The Poincare Duality with local coefficients for closed, topological manifolds.

In this case the written (and complete) accounts of this property do exist. In fact, we are aware of two such accounts (cf. [8], [9]). Both of these accounts are surprisingly lengthy and technically quite demanding to say the least.

Our argument once again relies on a simple “simplicial approximation” trick of some what different nature than the one mentioned earlier. This leads to a conceptually satisfying and direct proof.

More comments of historical, motivational and mathematical nature are contained in corresponding sections dealing with the proofs.

2 Local Coefficient Systems

For more detail on this topic, the reader is referred to [11], Chapter VI, Section 1-4.

Let $G$ be a bundle of Abelian groups (local coefficient system) on a topological space $X$. Recall that the singular chain complex of $X$ with coefficient in $G$ is defined as

$$S_k(X; G) = \bigoplus_{\sigma: \Delta^k \to X} G(\sigma(e_0))$$

$$\partial(g\sigma) = G(\tilde{\sigma})(g)g \sigma_0 + \sum_{i=1}^{k} (-1)^i g\sigma_i$$

where $\Delta^k = \langle e_0, e_1, \cdots, e_k \rangle$ is the standard $k$-simplex, $g\sigma \in S_k(X; G)$ is the element that is $g$ on the $G(\sigma(e_0))$ factor and 0 otherwise, $\tilde{\sigma}$ is $\sigma$ composed with the straight path from $e_1$ to $e_0$ in $\Delta^k$, $\sigma_i$ is the restriction of $\sigma$ to the $i$-th face.

Note that the notations we are using are slightly different from the one used in Whitehead’s book.

It can be shown that $\{S_\ast(X; G), \partial\}$ is a chain complex, and its homology is called the singular homology of $X$ with coefficient in $G$, denoted as $H_\ast(X; G)$.

In a similar way, one could define the singular cochain complex $S^\ast(X; G)$

$$S^k(X; G) = \prod_{\sigma: \Delta^k \to X} G(\sigma(e_0))$$

$$(-1)^k(\delta c)(\sigma) = G(\tilde{\sigma})^{-1}c(\sigma_0) + \sum_{i=1}^{n+1} (-1)^i c(\sigma_i)$$

where $c \in S^k(X; G)$.

$\{S^\ast(X; G), \delta\}$ forms a cochain complex and its homology is called the singular cohomology of $X$ with coefficient in $G$, denoted as $H^\ast(X; G)$.

Long exact sequence of pairs, homotopy invariance, excision and additivity (with respect to disjoint union) are still valid as in the case of constant coefficients (cf. [11]. VI. 2). As is the equivalence between singular and cellular homology/cohomology (cf. [11]. VI. 4).
3 Weak Homotopy Equivalence and Cohomology

In this section, we will provide a proof of the following result:

**Theorem 1** Let \( f : X \to Y \) be a weak homotopy equivalence, \( G \) be a coefficient system on \( Y \), then \( f_* : H_*(X; f^*G) \to H_*(Y; G), f^* : H^*(Y; G) \to H^*(X; f^*G) \) are isomorphisms, where \( f^*G \) is the pull-back of \( G \) via \( f \).

3.1 Background

It is a standard result in homotopy theory that weak homotopy equivalences (continuous maps which induce isomorphisms of all homotopy groups with all choices of basepoints) induce isomorphism on singular homology and cohomology. One may ask whether this is true for homology and cohomology with local coefficients. This result is needed for the following definition in Obstruction Theory.

Suppose \((K, L)\) is a relative CW-complex, \( p : X \to B \) is a fibration with \((n - 1)\)-connected fiber \( F \) and we are given a commutative diagram:

\[
\begin{array}{ccc}
L & \xrightarrow{f} & X \\
\downarrow & & \downarrow p \\
K & \xrightarrow{\phi} & B
\end{array}
\]

The diagram induces an element \( \bar{\gamma}^{n+1}(f) \in H^{n+1}(K, L; \phi^*\pi_n(F)) \), called the primary obstruction to extending \( f \) ([11], p.298). The name comes from the fact that \( f \) can be extended to a partial lifting \( f_{n+1} : K^{n+1} \to X \) of \( \phi \) if and only if \( \bar{\gamma}^{n+1}(f) = 0 \).

Sometimes it is useful to have primary obstruction defined when \((K, L)\) is replaced by an arbitrary pair \((P, Q)\). In particular, one needs such definition when defining the Whitney class of a vector bundle over an arbitrary base (not necessarily homotopic to a CW-complex), or constructing the Leray-Serre spectral sequences of a fibration over an arbitrary base.

To this end, we can take a CW-approximation \( \varphi : (K, L) \to (P, Q) \), i.e., a map of pairs such that \((K, L)\) is a CW-pair and \( \varphi, \varphi|_L \) are both weak homotopy equivalences. Thus we have a diagram:

\[
\begin{array}{ccc}
L & \xrightarrow{\varphi} & Q & \xrightarrow{f} & X \\
\downarrow & & \downarrow & & \downarrow p \\
K & \xrightarrow{\varphi} & P & \xrightarrow{\phi} & B
\end{array}
\]
The element $\tilde{\gamma}^{n+1}(f \circ \varphi) \in H^{n+1}(K, L; \varphi^* \phi^* \pi_n(\mathcal{F}))$ is well-defined. If we know that
\[ \varphi^* : H^{n+1}(P, Q; \phi^* \pi_n(\mathcal{F})) \longrightarrow H^{n+1}(K, L; \varphi^* \phi^* \pi_n(\mathcal{F})) \]
is an isomorphism (in [11] p.300 this is assumed without any explanation), then one could define $\tilde{\gamma}^{n+1}(f) := \varphi^{-1} \tilde{\gamma}^{n+1}(f \circ \varphi) \in H^{n+1}(P, Q; \phi^* \pi_n(\mathcal{F}))$. An easy argument of naturality shows that this is independent of the CW-approximation $\varphi$. Hence $\tilde{\gamma}^{n+1}(f)$ is well-defined.

There are at least two ways to prove that a weak homotopy equivalence induces isomorphism on homology (with constant coefficients) in the literature. One approach uses Hurewicz Theorem ([7] 7.5.9, 7.6.25), the other proof ([2], Proposition 4.21) is by a construction that relies heavily on the finiteness of singular chains. The analogous result for cohomology with constant coefficients follows from this via the Universal Coefficient Theorem.

| Homology | Constant coefficients | Local coefficients |
|----------|----------------------|-------------------|
| Hurewicz/Construction | Construction | ? |
| Cohomology | Universal Coefficient Theorem | ? |

Table 1:

When it comes to local coefficients, Hurewicz Theorem is no longer available. The constructive proof still works for homology with local coefficients. Yet due to the absence of Universal Coefficient Theorem, the result for cohomology does not follow automatically. Our proof turns out to be quite different from those above. As far as we know, no alternative exists in the literature for cohomology.

### 3.2 Singular Complex

We begin with the notion of singular complex, which is central in our proof.

Let $X$ be a topological space. Take the disjoint union of $k$-simplexes, one for each continuous map $\sigma : \Delta^k \rightarrow X$. Do this for all integer $k \geq 0$. Glue the simplexes according to restriction of maps to faces. The resulted CW-complex is called the singular complex of $X$, denoted by $SX$. In fact, $SX$ is a $\Delta$-complex ([2], p.164).

A continuous map $f : X \rightarrow Y$ induces (with obvious definition) $Sf : SX \rightarrow SY$.

Since the simplexes of $SX$ corresponds to continuous maps $\Delta^k \rightarrow X$, there is a canonical map $I_X : SX \rightarrow X$ mapping each simplex via the map defining it. $I_X$ is natural with respect to continuous map $f : X \rightarrow Y$, i.e., the following diagram commutes:
Theorem 2 For any topological space $X$, $I_X$ is a weak homotopy equivalence.

Proof See [5], Chapter III, Theorem 6.7, or [1] Theorem 16.43 on p.149. 

The following result will be useful for our purpose:

For a coefficient system $G$ on a $\Delta$-complex $K$, there is a version of simplicial homology/cohomology. The definition of chains/cochains and boundary/coboundary maps are the same with that of singular ones, except that the direct sum/product is now over all simplices of $K$. We denote the simplicial chain/cochain of $K$ with coefficient in $G$ by $\Delta^\ast(X;G)$ and $\Sigma^\ast(X;G)$. We still have:

Theorem 3 The canonical injection $\Delta^\ast(K;G) \xrightarrow{j^\#} S^\ast(K;G)$ and projection $S^\ast(K;G) \xrightarrow{j^\#} \Delta^\ast(K;G)$ are chain $I_X^\ast : H^\ast(X;G) \rightarrow H^\ast(SX;I_X^G)$ maps that induce isomorphism on homology/cohomology groups.

Proof The proof for homology is the same as in the case of constant coefficients (cf. [2], 2.27). Although Universal Coefficient Theorem is not available, one could easily adapt the above mentioned proof to the case of cohomology, with little change.

Now suppose $G$ is a coefficient system on $X$. Let $I_X^G$ be the pullback of $G$ via $I_X$. We have

$$S^\ast(X;G) \xrightarrow{I_X^\#} S^\ast(SX;I_X^G) \xrightarrow{j^\#} \Delta^\ast(SX;I_X^G)$$

It is easy to see that the composition $j^\# \circ I_X^\#$ identifies $S^k(X;G) = \prod_{\sigma : \Delta^k \rightarrow X} G(\sigma(e_0)) = \Delta^\ast(SX;I_X^G)$. In particular, $j^\# \circ I_X^\#$ is a chain isomorphism and $j^\ast \circ I_X^\ast : H^\ast(X;G) \rightarrow H^\ast(SX;I_X^G)$ is also an isomorphism. By a similar argument one can show that $I_X^\ast \circ j^\ast : H^\ast(SX;I_X^G) \rightarrow H^\ast(X;G)$ is an isomorphism. Combined with Theorem 3 we have shown:

Theorem 4 The map $I_X$ induces isomorphisms on homology and cohomology. To be more precise, for any coefficient system $G$ on $X$, $I_X^\ast : H_\ast(SX;I_X^G) \rightarrow H_\ast(X;G)$ and $I_X^\ast : H^\ast(X;G) \rightarrow H^\ast(SX;I_X^G)$ are isomorphisms.
3.3 Proof of Theorem 1

**Proof** Let \( f : X \to Y \) be a weak homotopy equivalence. As noted above there’s a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\uparrow{I_X} & & \uparrow{I_Y} \\
SX & \xrightarrow{Sf} & SY
\end{array}
\]

in which \( I_X, I_Y \) are weak homotopy equivalences by Theorem 2.

Commutativity implies that \( Sf \) is also a weak homotopy equivalence. Since \( SX, SY \) are CW-complexes, Whitehead Theorem (cf. [2]) implies that \( Sf \) is a homotopy equivalence, hence induce isomorphism on homology/cohomology with local coefficients.

Now apply Theorem 4 to the induced (commutative) diagram on homology/cohomology we have the desired result. \( \square \)

4 Identifying Singular and Cellular Long Exact Sequences

4.1 Background

Let \((K, L)\) be a CW-pair, \( G \) be a coefficient system on \( K \). There is long exact sequence

\[
\cdots \to H_n(L; G) \to H_n(K; G) \to H_n(K, L; G) \to \cdots \tag{3}
\]

and

\[
\cdots \to H^n(K, L; G) \to H^n(K; G) \to H^n(L; G) \to \cdots \tag{4}
\]

There are also naturally defined short exact sequences of cellular chain/cochain complexes

\[
0 \to \Gamma_*(L; G) \to \Gamma_*(K; G) \to \Gamma_*(K, L; G) \to 0
\]

and

\[
0 \to \Gamma^*(K, L; G) \to \Gamma^*(K; G) \to \Gamma^*(L; G) \to 0
\]

which induce long exact sequences

\[
\cdots \to H_n(\Gamma_*(L); G) \to H_n(\Gamma_*(K); G) \to H_n(\Gamma_*(K, L); G) \to \cdots \tag{5}
\]

and

\[
\cdots \to H^n(\Gamma^*(K, L); G) \to H^n(\Gamma^*(K); G) \to H^n(\Gamma^*(L); G) \to \cdots \tag{6}
\]

The groups in (3) and (5) (resp. (4) and (6)) are term-wise isomorphic. It is natural to ask whether the the long exact sequences (viewed as chain complexes)
are chain isomorphic. This is used in the proof of [11] Theorem VI.6.9 (again this is used without any justification). It is natural to expect that this problem can be solved by a diagram chasing, since cellular homology/cohomology are themselves defined by certain diagrams. Yet as far as we know, no such proof has been given. In fact the only relevant result in the literature is given in Schubert’ book (cf. [6], p. 303). Schubert constructed an intermediate between the singular and cellular chain complex, called normal chain complex and used it to show (3) and (5) are chain isomorphic. The construction (again!) depends heavily on the finiteness of singular chains, thus fails to prove the result for cohomology with local coefficients (though for constant coefficients one could still use Universal Coefficient Theorem to dualize everything).

Our goal is to prove:

**Theorem 5** The long exact sequences (3) and (5) (resp. (4) and (6)) are chain isomorphic.

We shall prove the result for cohomology, the proof for homology is analogous.

### 4.2 Proof of Theorem 5

**Proof** Since the identification of singular and cellular homology/cohomology groups are natural with respect to cellular maps (whether the coefficient is constant or local), it suffice to check the commutativity of the diagram (coefficients omitted):

\[
\begin{array}{c}
H^n(\Gamma^*(L)) \xrightarrow{\delta} H^{n+1}(\Gamma^*(K,L)) \\
\downarrow \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
H^n(L) \xrightarrow{\delta} H^{n+1}(K,L)
\end{array}
\]

Note that \(SL\) can be identified canonically with a subspace of \(SK\) and \(I_K : SK \to K\) restricts to \(I_L\) on \(SL\). Thus we have a weak homotopy equivalence \(I_K : (SK, SL) \to (K, L)\). Since both the domain and codomain are CW-complexes, \(I_K\) is actually a homotopy equivalence. Homotope \(I_K\) to a cellular map \(J_K\) and consider:
in which vertical arrows are isomorphism between singular and cellular cohomology groups, horizontal arrows are boundary maps in the corresponding long exact sequence and arrows going down left are induced by $J_K$. Coefficients are obvious and omitted.

The rectangles on top of the above diagram commute since a map (in this case $J_K$) induces a chain map between singular long exact sequences. Similarly, the cellular map $J_K$ induce chain map between cellular long exact sequences, hence the commutativity of the bottom rectangle. The rectangles on the left and right commute thanks to naturality of the identification of singular cohomology with cellular cohomology under cellular maps.

All down left arrows are isomorphisms since $J_K$ is a (cellular) homotopy equivalence. In particular the map $J_K^* : H^{n+1}(\Gamma^*(K, L)) \to H^{n+1}(\Gamma^*(SK, SL))$ is an isomorphism.

We intend to prove the commutativity of the rectangle in the back. As indicated by the above argument, it suffice to show that for the front rectangle. In other words, we have reduced the problem to the case where $(K, L)$ is a pair of $\Delta$-complexes. We shall assume this from now.

For a $\Delta$-complex pair $(K, L)$ and a coefficient system $G$ on $K$, there is an isomorphism $\Phi : \Delta^*(K, L; G) \to \Gamma^*(K, L; G)$ defined by the identification

$$
\Delta^*(K, L; G) \longleftrightarrow \Pi G(\sigma(e_0)) \longleftrightarrow \Pi H^n(\Delta^a, \partial \Delta^a; \sigma^* G) \longleftrightarrow \Gamma^*(K, L; G)
$$

where the direct products are over all $n$-simplexes of $K - L$. It is easy to check (by a diagram chasing) that $\Phi$ commute with boundary maps of the two chain complexes and hence is a chain isomorphism.

$j^\#$ and $\Phi$ induces the following commutative diagram joining the singular, simplicial, and cellular short exact sequences of $(K, L)$ (coefficients omitted)

$$
\begin{array}{c}
0 \longrightarrow S^*(K, L) \longrightarrow S^*(K) \longrightarrow S^*(L) \longrightarrow 0 \\
0 \longrightarrow \Delta^*(K, L) \longrightarrow \Delta^*(K) \longrightarrow \Delta^*(L) \longrightarrow 0 \\
0 \longrightarrow \Gamma^*(K, L) \longrightarrow \Gamma^*(K) \longrightarrow \Gamma^*(L) \longrightarrow 0
\end{array}
$$
which induces a commutative diagram for the boundary homomorphisms in the corresponding long exact sequences

\[
\begin{align*}
H^n(L) & \xrightarrow{\delta} H^{n+1}(K, L) \\
\downarrow j^* & \downarrow j^* \\
H^\Delta_n(L) & \xrightarrow{\delta} H^\Delta_{n+1}(K, L) \\
\downarrow \Phi^* & \downarrow \Phi^* \\
H^n(\Gamma^*(L)) & \xrightarrow{\delta} H^{n+1}(\Gamma^*(K, L))
\end{align*}
\]

The theorem will follow from the lemma below.  

\textbf{Lemma 6} \ The isomorphism \( \Phi^* \circ j^* \) is exactly the canonical identification between cellular and singular cohomology.

\textbf{Proof} \ We shall prove this for the absolute case (i.e. \( L = \emptyset \)). The proof of the relative case is similar. The coefficient system \( G \) will be omitted unless necessary.

Consider the following diagram:

\[
\begin{array}{ccc}
H^n(K) & \xrightarrow{j^*} & H^n(K^n) \\
\downarrow & & \downarrow \kappa \\
H^n(\Gamma^*(K)) & \xrightarrow{\Phi^*} & H^n(\Gamma^*(L))
\end{array}
\]

where the \( \iota \) is induced by the inclusion \( K^n \hookrightarrow K \) and \( \kappa \) is induced by the homomorphism (induced by identity) \( \Gamma^n(K) = H^n(K^n, K^{n-1}) \to H^n(K^n) \).

For any \([b] \in H^n(K)\), where \( b \in S^n(K) \). Let \([\alpha] \in H^n(\Gamma(K))\) be the element corresponding to \([b]\) via the canonical identification between singular and cellular cohomology. Then \( \kappa([\alpha]) = \iota([b]) \), hence \( \alpha = [a] \) for some \( a \in S^n(K^n, K^{n-1}) \subset S^n(K^n) \) such that \( a - b_{|K^n} = \delta c \) for some \( c \in S^{n-1}(K^n) \).

It suffice then to show that \( \Phi^{*-1}([\alpha]) = j^*([b]) \).

We know that

\[
\begin{align*}
j^#(a) - j^#(b) \\
= j^#(a - b_{|K^n}) \\
= j^#(\delta c) \\
= \delta j^#(c) \in \Delta^n(K^n) = \Delta^n(K)
\end{align*}
\]

In other words, \( j^#(a), j^#(b) \) are cohomologous.
Also, one can check that
\[
j^!(a) = \Phi^!(\alpha) = \Delta^n(K) = \Pi G(e_0)
\]
by looking at their value on each \(n\)-simplex \(\sigma\).
Thus \(\Phi^!(\alpha) = j^!(a) = j^!(b) = j^!(b)\).
\[\blacksquare\]

5 Simplicial Approximation and Poincaré Duality

We now turn to another type of simplicial approximation. Let \(\mathcal{M}\) be a \(n\)-dimensional topological manifold, \(R\) be a principal ideal domain and \(G\) be a bundle of right \(R\)-modules.

The Poincaré Duality Theorem (with local coefficients) states that
\[
H^i_{\text{c}}(\mathcal{M}; G) \xrightarrow{\Delta_{\mu}} H_{n-i}(G \otimes_R \mathcal{M}_R), \quad 0 \leq i \leq n
\]
where \(H^i_{\text{c}}\) stands for singular cohomology with compact support, \(\mu_{M}\) is the (generalized) fundamental class and \(\mathcal{M}_R\) is the orientation bundle of \(\mathcal{M}\) with coefficient in \(R\).

For relevant definitions and proof of the theorem, see [8] or [9].

There is a version of this duality for compact triangulated manifolds with or without boundary (see [4], or [10] Theorem 2.1 p.23), which dates back much earlier i.e., the original proof given by S. Lefschetz. This proof is short and purely geometric (it uses the dual decomposition of the corresponding simplicial complex). Thus it would be nice if one could reduce the general case to the case of triangulated manifolds. This is when simplicial approximation comes into the picture.

Assume, for simplicity, that \(\mathcal{M}\) is closed and orientable.

**Theorem 7** Let \(G\) be a coefficient system on \(\mathcal{M}\), then \(H^i(\mathcal{M}; G) \xrightarrow{\Delta_{\mu}} H_{n-i}(G)\) is an isomorphism for all \(0 \leq i \leq n\).

**Proof** By [3], there is a \(k\)-disk bundle \(p : E \to \mathcal{M}\) such that \(E\) admits a triangulation. Also, \(E\) is embedded in \(\mathbb{R}^{n+k}\). In particular, \(E\) is orientable as a manifold with boundary. There is a Poincaré Duality
\[
H^{i+k}(E, \partial E; p^*G) \xrightarrow{\Delta_{\mu_E}} H_{n-i}(E; p^*G)
\]
where \(\mu_E\) is the fundamental class of \(E\) and \(p^*G\) is the pull-back bundle. Since \(E\) is triangulable, this is an isomorphism.
Next we prove that $E$ is orientable as a disk bundle, i.e., there exist $U \in H^k(E, \partial E) = H^k(E, \partial E; \mathbb{Z})$ that restrict to a generator $U|_x \in H^k(E_x, \partial E_x)$ for every $x \in \mathcal{M}$. Here $(E_x, \partial E_x)$ stands for the fiber over $x$.

The composition

$$H^k(E, \partial E) \xrightarrow{\sim \mu_k} H_n(E) \xrightarrow{p_*} H_n(\mathcal{M})$$

is an isomorphism since $p$ is a homotopy equivalence. Define $U \in H^k(E, \partial E)$ by $U \sim \mu_E = p_1^{-1}(\mu_\mathcal{M})$.

Note that $\mathcal{M}$ embeds in $E$ by the zero-section. For $x \in \mathcal{M}$, consider the following diagram, where vertical maps are induced by inclusions

$$H^k(E, \partial E) \otimes H_{n+k}(E, \partial E) \xrightarrow{\sim} H^n(E) \xrightarrow{p_*} H_n(\mathcal{M})$$

Obviously the rightmost box is commutative, and $i_1$ is an isomorphism. Let $U' = i_1^{-1}(U) \in H^k(E|\mathcal{M})$. Define $\mu_{(x,0)} = i_2(\mu_E) \in H_{n+k}(E|(x,0))$. Then by naturality of cap products, $U' \sim \mu_{(x,0)} = i_3p_*^{-1}\mu_\mathcal{M}$. By commutativity and definition of the fundamental class $\mu_\mathcal{M}$, $p_*i_3p_*^{-1}\mu_\mathcal{M} = i_4\mu_\mathcal{M}$ is a generator of $H_n(\mathcal{M}|x)$. Since $p : (E|E_x) \to (\mathcal{M}|x)$ is a homotopy equivalence, $i_3p_*^{-1}\mu_\mathcal{M}$ is also a generator.

Now choose an open disk neighborhood $W$ of $x$ in $\mathcal{M}$ upon which $E$ admits a local trivialization $\Phi : p^{-1}(W) \to W \times D^k$. Let $\Phi_x : E_x \to D^k$ be the restriction of $\Phi$. Consider the following diagram

$$H^k(E|\mathcal{M}) \otimes H_{n+k}(E|(x,0)) \xrightarrow{\sim} H^n(E|E_x)$$

where vertical homomorphisms are induced by $\Phi$.

By excision, $\Phi_2, \Phi_3$ are isomorphisms. Hence by naturality $\Phi_1(U') \sim \Phi_2^{-1}(\mu_{(x,0)}) = \Phi_3i_3p_*^{-1}\mu_\mathcal{M}$, which is a generator of $H_n(W \times D^k|E_x) \cong \mathbb{Z}$. Note that cross product induces an isomorphism

$$H^k(W \times D^k|W \times 0) \leftrightarrow H^k(D^k|0) \otimes H^0(W)$$

and $\Phi_1(U')$ corresponds to $\Phi_{x*}(U|_x) \times 1$. This forces $\Phi_{x*}(U|_x)$ and thus $U|_x$ to be a generator. This proves that $U$ is an orientation for the disk bundle $E$.

Now we have a diagram

$$H^*(\mathcal{M}; G) \xrightarrow{\sim \mu_\mathcal{M}} H_{n-i}(\mathcal{M}; G)$$

$$H^{i+k}(E, \partial E; p^*G) \xrightarrow{\mu_E} H_{n-i}(E; p^*G)$$

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in which the left vertical map is the Thom isomorphism (cf. [7] p.283) sending
\( \alpha \) to \( p^*(\alpha) \sim U \).

This diagram commutes:

\[
\begin{align*}
p_*(((p^*\alpha) \sim U) \sim \mu_E) \\
= p_*((p^*\alpha \sim (U \sim \mu_E)) \\
= \alpha \sim p_* (U \sim \mu_E) \quad \text{(naturality)} \\
= \alpha \sim \mu_M \quad \text{(definition of } U) 
\end{align*}
\]

The top arrow is thus an isomorphism since all others are isomorphisms. \(\square\)

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