NONSMOOTH CRITICAL POINT THEOREMS
WITHOUT COMPACTNESS

YOUSSEF JABRI

Abstract. We establish an abstract critical point theorem for locally Lipschitz functionals that does not require any compactness condition of Palais-Smale type. It generalizes and unifies three other critical point theorems established in [10] for $C^1$-functionals under slightly stronger assumptions. Our approach uses continuous selections of multivalued mappings.

Mathematical Subject Classification: 58E05, 54C60, 49J35

Keywords. Critical point theorem, lack of compactness, quasi-concavity, multivalued mapping, continuous selections, Dolph’s method, locally Lipschitz functionals, Clarke subdifferential.

Abstract critical point theorems from the linking family like the mountain pass theorem of Ambrosetti-Rabinowitz and saddle point theorem of Rabinowitz, (cf. for example [16, 12, 17]) are important tools in nonlinear analysis. They require generally a compactness condition known as Palais-Smale condition, (PS) for short. In applications to nonlinear boundary value problems, verifying if (PS) holds or not is crucial and turns out often to be technical and/or too long.

In [10], the author and Moussaoui proved a critical point theorem (and some variants) with similar conditions to Rabinowitz saddle point theorem without requiring the Palais-Smale condition. They supposed instead a convexity assumption on Φ on a part of the underlying space.

In this paper, we are concerned with a twofold extension of the results of [10]. We will show that they follow easily from a more abstract critical point theorem valid for locally Lipschitz functionals with “less convexity” on Φ. The proof also is easier. A novelty in our approach is that we exploit the notion of continuous selections of multivalued mappings.

The following result is a reliable prototype of our critical point theorem without compactness.

Theorem 1. Let $E$ be a reflexive Banach space such that $E = V \oplus W$ where $\dim V < +\infty$ and $\Phi: E \to \mathbb{R}$ a $C^1$-functional that satisfies:
The origins of the minimax procedure that appears in Theorem 1 go back to 1949.

1. THE ORIGINS OF THE METHOD

The origins of the minimax procedure that appears in Theorem 1 go back to 1949.
Let $\Omega$ be a smooth bounded domain of $\mathbb{R}$, $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ a Carathéodory function. Consider the nonlinear Dirichlet problem

$$\begin{cases}
-\Delta u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}$$

and denote the potential associated to $f$ by

$$F(x, t) = \int_0^t f(x, s) \, ds.$$

Dolph [8] solved the problem $(P)$ when

$$\lambda_k < \mu_k \leq \liminf_{s \to \pm \infty} \frac{f(x, s)}{s} \leq \limsup_{s \to \pm \infty} \frac{f(x, s)}{s} \leq \mu_{k+1} < \lambda_{k+1}$$

where $\lambda_k$ and $\lambda_{k+1}$ are two consecutive eigenvalues of $-\Delta$ in $H_0^1(\Omega)$ (nonresonance between two consecutive eigenvalues of the Laplacian).

A similar condition in terms of the potential $F$ may be expressed as

$$\lambda_k < \mu_k \leq \liminf_{s \to \pm \infty} \frac{2F(x, s)}{s^2} \leq \limsup_{s \to \pm \infty} \frac{2F(x, s)}{s^2} \leq \mu_{k+1} < \lambda_{k+1}$$

In general, variational methods fail to handle the problem $(P)$ when only conditions on the potential, like (1.2), are required (cf. the discussion of [7] for example). This makes it impossible to verify the Palais-Smale condition required in linking theorems.

The first variational attempt to solve $(P)$ under condition (1.2), without requiring any assumption on $f$, is also due to Dolph. He required the following additional condition

$$\lambda_k < \mu_k \leq \liminf_{s \to \pm \infty} \frac{\Phi(u)}{s^2} \leq \limsup_{s \to \pm \infty} \frac{\Phi(u)}{s^2} \leq \mu_{k+1} < \lambda_{k+1}$$

The spaces $V = \bigoplus_{i \leq k} E(\lambda_i)$ and $W = V^\perp = \bigoplus_{i \geq k+1} E(\lambda_i)$ where $E(\lambda_i)$ is the eigenspace associated to the eigenvalue $\lambda_i$ of the Laplace operator with Dirichlet boundary data.

Later, Thews [18] treated $(P)$ under condition (1.2) allowing $\mu_k = \lambda_k$ and $\mu_{k+1} = \lambda_{k+1}$ but supposed more restrictive conditions than (DO).

Theorem 1 may be seen as a continuation of the attempt of the author and Moussaoui in [10] to provide an abstract framework to Dolph’s method. In [10], this approach was applied to get a variant of a result proved earlier by Mawhin and Willem in [11] by combining the least dual action of Clarke-Ekeland [6] and an approximation method of Brézis [4].
2. SOME ELEMENTS OF NONSMOOTH CRITICAL POINT THEORY AND MULTIVALUED ANALYSIS

Let $\Phi: X \to \mathbb{R}$ be a locally Lipschitz functional. For each $x, v \in X$, the generalized directional derivative of $\Phi$ at $x$ in the direction $v$ is

$$\Phi^o(x; v) = \limsup_{y \to x, t \downarrow 0} \frac{\Phi(y + tv) - \Phi(y)}{t}.$$ 

It follows by the definition of locally Lipschitz functionals that $\Phi^o(x; v)$ is finite and $|\Phi^o(x; v)| \leq C||v||$.

Moreover, $v \mapsto \Phi^o(x; v)$ is positively homogenous and subadditive and $(x, v) \mapsto \Phi^o(x; v)$ is u.s.c.

The generalized gradient (Clarke subdifferential) of $\Phi$ at $x$ is the subset $\partial \Phi(x)$ of $X^*$ defined by

$$\partial \Phi(x) = \left\{ x^* \in X^* ; \Phi^o(x; v) \geq \langle x^*, v \rangle, \text{ for all } v \in X \right\}.$$ 

It enjoys the following properties:

a) For each $x \in X$, $\partial \Phi(x)$ is non-empty, convex weak-* compact subset of $X^*$.

b) For each $x, v \in X$, we have

$$\Phi^o(x; v) = \max\{\langle x^*, v \rangle ; x^* \in \partial \Phi(x)\}$$

c) $\partial(\Phi + \Psi)(x) \subset \partial \Phi + \partial \Psi$, where $\Phi$ and $\Psi$ are locally Lipschitz at $x$.

Theorem 2 (Lebourg mean value theorem). If $x$ and $y$ are two distinct points in $X$, then there exists $z = x + \tau(y - x), 0 < \tau < 1$ such that

$$\Phi(y) - \Phi(x) \in \langle \partial \Phi(z), y - x \rangle.$$ 

The notion of critical point of a locally Lipschitz functional is the following.

Definition 1. Let $\Phi: X \to \mathbb{R}$ be locally Lipschitz. A point $x \in X$ is a critical point of $\Phi$ if $0 \in \partial \Phi(x)$. A real number $c$ is called a critical value of $\Phi$ if $\Phi^{-1}(c)$ contains a critical point $x$.

We recall now some results on multivalued mappings. Let $M$ and $N$ be two topological spaces.

Definition 2. A multivalued mapping $T: M \to 2^N$ is a map which assigns to each point $m \in M$ a subset $T(x)$ in $N$.

A multivalued mapping $T: M \to 2^N$ is upper semi-continuous if and only if $T^{-1}(A)$ is closed for all closed subsets $A$ of $N$, where the preimage $T^{-1}(A)$ is defined by

$$T^{-1}(A) = \left\{ m \in M ; T(m) \cap A \neq \emptyset \right\}.$$
The multivalued mapping \( T \) is lower semi-continuous if and only if \( T^{-1}(A) \) is open for all open subsets \( A \) of \( N \).

And \( T \) is continuous if and only if it is both lower and upper semi-continuous.

**Remark 1.** Notice that when \( T \) is single-valued, the lower semi-continuity (resp. upper semi-continuity) of \( T \) as multivalued coincides with continuity.

Let \( T: M \to 2^N \) be a multivalued mapping. By a selection of \( T \), we mean a single-valued mapping \( s: M \to N \) with

\[ s(m) \in T(m), \quad \text{for all } m \in M. \]

**Theorem 3** (Michael’s selection theorem). A lower semi-continuous multivalued mapping \( T: M \to 2^N \) has a continuous selection \( s: M \to N \) if the following three conditions are satisfied:

(i) \( M \) is paracompact,
(ii) \( N \) is a Banach space,
(iii) The set \( T(m) \) is nonempty, closed and convex for all \( m \in M \).

**Theorem 4** (Minimal selection theorem). Let \( T: M \to 2^N \) be a continuous multivalued mapping, where \( M \) is a metric space and \( N \) is a Hilbert space. Suppose that \( T(m) \) is nonempty, closed and convex for all \( m \in M \). Denote by \( m(T(m)) \) the unique element of the set \( T(m) \) with smallest norm.

Then \( m: M \to N \) is a continuous selection.

### 3. Continuous Selections of Multivalued Mappings

We give now our main abstract critical point theorem that contains all the subsequent results as special cases.

**Theorem 5.** Let \( E \) be a Banach space such that \( E = V \oplus W \) and \( \Phi: E \to \mathbb{R} \) a locally Lipschitz functional. Suppose the following assumptions.

(a) \( \forall w \in W, \) the set

\[ V(w) = \{ v \in V; \varphi(w) = \Phi(v + w) = \max_{g \in V} \Phi(g + w) \} \neq \emptyset. \]

(b) The functional \( \varphi: W \to \mathbb{R} \) is bounded below and achieves its minimum at some point \( \overline{w} \).

(c) There exists a continuous selection \( s: W \to V \) such that \( s(w) \in V(w) \) for all \( w \in W \).
Then, $\overline{u} = s(\overline{w}) + \overline{w}$ is a critical point such that

$$\Phi(\overline{u}) = \min_{w \in W} \max_{v \in V} \Phi(v + w).$$

**Proof.** Take $g \in V$, then

$$\Phi(\overline{u} + t(-g)) - \Phi(\overline{u}) \leq 0, \quad \forall t > 0.$$  
Divide by $t$ and let $t$ go to infinity to obtain

$$\Phi^o(\overline{u}; -g) = -\Phi^o(\overline{u}; -g) \leq 0.$$  
This is true for all $g$ in the linear space $V$, so

$$\Phi^o(\overline{u}; g) = 0, \quad \text{for all } g \in V.$$  

On the other side, we know by Lebourg mean value theorem, that

$$\Phi(\overline{u} + sk) - \Phi(\overline{u}) = \langle \xi, sk \rangle = s \langle \xi, k \rangle \leq s \Phi^o(z; k)$$

where $\xi \in \partial \Phi(z)$ for some $z = \overline{u} + \tau w$, $0 < \tau < 1$ and $k \in X$.

So, for $h \in W$, if we write $w_t = \overline{w} + th$, $0 < t \leq 1$ and $v_t = s(w_t)$.

Consider a sequence $t_n \downarrow 0$ and denote by $\overline{v} = s(\overline{w}) = \lim_{n \to \infty} s(w_{t_n})$.

Since $\Phi(\overline{w} + \overline{v}) \geq \Phi(\overline{w} + s(w_{t_n}))$ because $\overline{v} \in V(\overline{w})$, we have

$$\frac{\Phi(\overline{w} + t_n h + v_{t_n}) - \Phi(\overline{w} + v_{t_n})}{t_n} \geq \frac{\Phi(\overline{w} + t_n h + v_{t_n}) - \Phi(\overline{w} + \overline{v})}{t_n} \geq 0.$$  
So,

$$\Phi^o(z_n, h) \geq 0, \quad \text{where } z_n \in \overline{w} + v_{t_n}, \overline{w} + v_{t_n} + t_n h[.$$  
At the limit we get by the u.s.c. of $\Phi^o(., .)$,

$$\Phi^o(\overline{u}, h) \geq \limsup_{n \to \infty} \Phi^o(z_n, h) \geq 0.$$  
And there too, we get

$$\Phi^o(\overline{u}; h) = 0, \quad \text{for all } h \in W.$$  
So, we have finally that

$$0 \in \partial \Phi(\overline{u}).$$

Using this result and Michael’s selection theorem, we obtain the following immediate consequence.

**Corollary 6.** Suppose that $E$ is as in the former theorem and that $\Phi$ is locally Lipschitz on $X$ and satisfies (a) and (b).

Suppose also that the multivalued mapping, $T: X \to 2^V$, $w \mapsto V(w)$ is lower semi-continuous and $T(w)$ is convex for all $w \in W$. Then, the conclusion of Theorem 5 holds true.
Corollary 7. Suppose that $E$ and $\Phi$ are as above and satisfy (i), (ii$''$), (iii) and (iv).

Then, the conclusion of Theorem 5 holds true.

This result has been proved in a direct way in [11, page 372].

Proof. To see that it is a consequence of Theorem 5, it suffices to show that the single valued $w \mapsto s(w)$ where $s(w)$ is the unique element in $V$ (by strict quasi-concavity) that achieves that maximum of $v \mapsto \Phi(v+w)$ in $V$ is continuous. Consider a sequence $w_n \to \mathfrak{w} \in W$. There exists $A > 0$, such that $||v|| \geq A$ implies (by (iv)) that

$$\Phi(v + \mathfrak{w}) \leq \Phi(v + \mathfrak{w}) < \inf \Phi \leq \Phi(w_t).$$

So, $||s(w_t)|| \leq A$. Because otherwise we get the following contradiction with the definition $v_\varepsilon(w_t)$:

$$\Phi(\varphi_\varepsilon(w_t) + w_t) < \Phi(w_t).$$

Hence, there is a sequence $t_n \to 0$ such that $\varphi_\varepsilon(w_{t_n}) \to v_0 \in V$.

While by definition of $s(w_n)$, we have

$$\Phi(s(w_n) + w_n) \geq \Phi(v + w_n), \forall v \in V.$$

At the limit, we get

$$\Phi(v_0 + w_0) \geq \Phi(v + w_0), \forall v \in V,$$

that is, $v_0 = s(w_0)$. 

Corollary 8. Suppose that $E$ is a Hilbert space and $\Phi$ satisfies (i), (ii$''$), (iii) and (iv).

Then, the conclusion of Theorem 5 holds true.

Proof. Consider the single valued mapping $w \in W \mapsto s(w) = \max_{V(w)}(\Phi(v+w) - ||v||)$. It is well defined because, the sets $V(w)$ are convex (by (ii$'$) and closed (they are even compact by (iii) and dim $V < +\infty$). And $s(w)$ is indeed the minimal selection because in $V(w)$, $\Phi(v+w)$ is constant and the element with smallest norm of the closed convex set $V(w)$ in the Hilbert $E$, is unique. Moreover it realizes the maximum of $v \in V(w) \mapsto \Phi(v+w) - ||v||$. 


Let us show that $s$ is continuous in $W$. Consider $w_n \to \overline{w}$. Then, $s(w_n)$ is bounded. Otherwise, there would be $N > 0$ such that for all $n \geq N$,

$$\Phi(s(w_n) + w_n) - ||s(w_n)|| < \inf_{w} \Phi \leq \Phi(0 + w_n) - ||0|| = \Phi(w_n).$$

A contradiction since $s(w_n)$ achieves the maximum of $\Phi$ on $w_n + V$. So, $s(w_n) \to \overline{v}$.

First $\overline{v} \in V(\overline{w})$, indeed

$$\Phi(\overline{v} + \overline{w}) = \lim_n \Phi(w_n + s(w_n)) \\
\geq \lim_n \Phi(w_n + v), \quad \forall v \in V \\
\geq \Phi(\overline{w} + v), \quad \forall v \in V.$$  

On the other hand, we have by definition $s(w_n)$,

$$\Phi(s(w_n) + w_n) - ||s(w_n)|| \geq \Phi(v + w_n) - ||v||, \quad \forall v \in V.$$  

And at the limit, we get

$$\Phi(v_0 + v_0) - ||v_0|| \geq \Phi(v + v_0), \quad \forall v \in V,$$

and $v_0 = s(w_0)$.

Finally, we come to the proof of Theorem 1 which is in fact also a simple corollary of Theorem 5.

Proof of Theorem 1. The proof of the former corollary applies verbatim except for the justification of the definiteness of “the single valued” mapping $s$. In the proof of the former corollary (Hilbertian case), we used the fact that $V(w)$ is convex and closed. So, a unique element of minimal norm exists. In our new situation, this follows from the strict quasi-convexity of $v \mapsto \Phi(v + w) - ||v||$ (by (ii) and the strict convexity of the norm in a reflexive Banach space [2]) and from the compactness of $V(w)$ for each $w \in W$.

We can remove the assumption that $\dim V < +\infty$ by requiring that $\Phi$ is weakly upper semi-continuous. This improves Theorem 11 in [10] and shows that it is also a particular case of Theorem 5.

Theorem 9. In a Banach reflexive space $X = V \oplus W$ such that (i), (ii), (iii) are satisfied. Suppose moreover that $\Phi$ is weakly upper semi-continuous. Then, the conclusion of Theorem 5 holds true.

Proof of Theorem 9. The reflexive character of $V$ with the anti-coerciveness of $v \mapsto \Phi(v + w)$ suffice to guarantee that $V(w)$ is nonempty and bounded. Since it is also closed and convex (by (ii)) it is weakly compact. So, the strict quasi-concave weakly upper semi-continuous functional $v \mapsto \Phi(v + w) - ||v||$ achieves its maximum in a unique point $s(w)$. 
Claim 1. The selection \( w \in W \mapsto s(w) \in V(w) \) is continuous.

Consider a sequence \((w_n) \subset W\) such that \( w_n \rightarrow \varpi \). Then, the sequence \( v_n = s(w_n) \) is bounded. Indeed, by (iv), there exists \( A > 0 \) such that
\[
\Phi(v + w_n) - \|v\| < \inf_W \Phi < \Phi(w_n) = \Phi(w_n + 0) - \|0\|, \quad \forall v \in V, \|v\| \geq A.
\]
So, \( \|v_n\| \leq A \), because otherwise we would get the contradiction
\[
\Phi(v_n + w_n) < \Phi(w_n).
\]
Therefore, \( v_n \rightharpoonup \varpi \in V \). This weak limit \( \varpi \) belongs to \( V(\varpi) \). Indeed,
\[
\Phi(\varpi + \varpi) \geq \limsup_n \Phi(v_n + w_n) \\
\geq \limsup_n \Phi(v + w_n), \quad \forall v \in V \\
\geq \Phi(v + \varpi), \quad \forall v \in V.
\]
Moreover, we know that
\[
\Phi(v_n + w_n) - \|v_n\| \geq \Phi(v + w_n) - \|v\|, \quad \forall v \in V.
\]
By the weak upper semi-continuity of \( \Phi \), the weak lower semi-continuity of the norm and since \( w_n + v_n \rightharpoonup \varpi + \varpi \),
\[
\Phi(\varpi + \varpi) - \|\varpi\| \geq \Phi(v + \varpi) - \|v\|, \quad \forall v \in V;
\]
i.e., \( \varpi = s(\varpi) \).

Some applications to nonlinear boundary value problems will be investigated in a forthcoming paper.

References

[1] A. Ambrosetti and P.H. Rabinowitz, Dual variational methods in critical point theory and applications. J. Funct. Anal., 14, 349–381, (1973).
[2] E. Asplund, Averaged norms. Israel J. Math., 5, 227–233, (1967).
[3] J.-P. Aubin and A. Cellina, Differential Inclusions. Springer-Verlag, (1983).
[4] H. Brézis, Periodic solutions of nonlinear vibrating strings and duality principles. Bull. Amer. Math. Soc. (N.S.), 8, no. 3, 409–426, (1983).
[5] F. Clarke, Optimization and nonsmooth analysis. Canadian Mathematical Series of Monographs and advanced Texts, Wiley-Interscience, 1983. Second edition, SIAM, Philadelphia, PA, 1990.
[6] F.H. Clarke and I. Ekeland, Hamiltonian trajectories having prescribed minimal period. Comm. Pure Appl. Math., 33, no. 2, 103–116, (1980).
[7] D.G. Costa and A.S. Oliveira, Existence of solution for a class of semilinear elliptic problems at double resonance. Bol. Soc. Bras. Mat., 19, no.1, 21–37 (1988).
[8] C.L. Dolph, Nonlinear integral equations of the Hammerstein type. Trans. Amer. Math. Soc., 66, 289–307, (1949).
[9] I. Ekeland and R. Témam, Convex analysis and variational problems. North Holland, (1976).
Y. Jabri and M. Moussaoui, A critical point theorem without compactness and applications. *Nonlinear Anal.*, 32, no. 3, 363–380, (1998).

J. Mawhin and M. Willem, Critical points of convex perturbations of some indefinite quadratic forms and semilinear boundary value problems at resonance. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 3, no. 6, 431–453, (1986).

J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems*. Applied Mathematical Sciences, 74, Springer-Verlag, New York, 1989.

E. Michael, Continuous selections. I. *Ann. of Math.*, 63, 361–382, (1956).

E. Michael, Continuous selections. II. *Ann. of Math.*, 64, 562–580, (1956).

E. Michael, Continuous selections. III. *Ann. of Math.*, 65, 375–390, (1957).

P.H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*. CBMS Regional Conference Series in Mathematics, 65, Amer. Math. Soc., Providence, RI, (1986).

M. Struwe, *Variational methods, Applications to nonlinear partial differential equations and Hamiltonian systems*. Springer-Verlag, Berlin, 1990.

K. Thews, A reduction method for some nonlinear Dirichlet problems. *Nonlinear Anal.*, 3, no. 6, 795–813, (1979).

University Mohamed I, Department of Mathematics, Faculty of Sciences, Box 524, 60000 Oujda, Morocco

E-mail address: jabri@sciences.univ-oujda.ac.ma