Dynamics Analysis of the Arrow Distributed Directory Protocol in General Networks

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Abstract
The Arrow protocol is a simple and elegant protocol to coordinate exclusive access to a shared object in a network. The protocol solves the underlying distributed queueing problem by using path reversal on a pre-computed spanning tree (or any other tree topology simulated on top of the given network).

It is known that the Arrow protocol solves the problem with a competitive ratio of $O(\log D)$ on trees of diameter $D$. This implies a distributed queueing algorithm with competitive ratio $O(s \cdot \log D)$ for general networks with a spanning tree of diameter $D$ and stretch $s$. In this work we show that when running the Arrow protocol on top of the well-known probabilistic tree embedding of Fakcharoenphol, Rao, and Talwar [STOC 03], we obtain a randomized distributed queueing algorithm with a competitive ratio of $O(\log n)$ even on general network topologies. The result holds even if the queueing requests occur in an arbitrarily dynamic and concurrent fashion and even if communication is asynchronous. From a technical point of view, the main of the paper shows that the competitive ratio of the Arrow protocol is constant on a special family of tree topologies, known as hierarchically well separated trees.

Keywords: competitive analysis, distributed queueing, shared objects, tree embeddings

1 Introduction

Coordinating the access to shared data is a fundamental task that is at the heart of almost any distributed system. For example, when implementing a distributed shared memory system on top of a message passing system, each shared register has to be kept in a coherent state despite possibly a large number of concurrent requests to read or write the shared register. In a distributed transactional memory system, each transaction might need to operate on several shared objects, which need to be kept in a consistent state [15, 24, 27]. When implementing a shared object on top of large-scale network, a distributed directory protocol can be used to improve scalability of the system [1, 2, 4, 6, 7, 15, 24]. When a network node requires access to a shared object, the directory moves a copy of the object to the node requesting the object. If the node changes the state of the shared object, the directory protocol has to make sure that all existing copies of the object are kept in a consistent state.

Distributed Queueing: At the core of many distributed directory implementations is the following basic distributed queueing problem that allows to order potential concurrent access
requests to a shared object [16]. The nodes of a network issue queueing requests (e.g., requests to access a shared object) in a completely dynamic and possibly arbitrarily concurrent manner. A queueing protocol needs to globally order all the requests so that they can be acted on consecutively. Formally, each request has to find its predecessor request in the order. That is, when enqueueing a request \( r \) issued by some node \( v \), a queueing protocol needs to find the request \( r' \) that currently forms the tail of the queue and inform the node \( v' \) of request \( r' \) about the new request \( r \).

**The Arrow Protocol:** A particularly simple and elegant solution for this distributed queueing problem is given by the Arrow protocol, which was introduced by Raymond in the context of distributed mutual exclusion [22]. The Arrow protocol operates on a directed tree topology \( T = (V, E) \). In a quiescent state, the tree is rooted at the node \( u \) of the current tail of the queue, i.e., all edges of \( T \) are directed towards \( u \). When a new queueing request is issued at a node \( v \), the direction of the edges on the path between \( v \) and the previous tail \( u \) is reversed so that the tree is now rooted at \( v \). For a precise description of the protocol, we refer to Section 2. It has been shown in [8] that the Arrow protocol correctly solves the queueing problem even in an asynchronous system even if the requests are issued in a completely dynamic and possibly concurrent way. Moreover, the Arrow protocol guarantees that every request finds the node of its predecessor on a direct path (i.e., within \( D \) time units if \( D \) is the diameter of \( T \)). In [14], it was further shown that on a tree \( T \), the overall cost of the Arrow protocol for ordering a dynamic set of queueing requests is within a factor \( O(\log D) \) of the cost of an optimal offline queueing algorithm, which knows the request sequence in advance.\(^1\)

**Contribution:** In the present paper, we strengthen the result of [14] and we show that when run on the right underlying tree, the Arrow protocol is \( O(\log n) \)-competitive even on general network topologies. The best previously known competitive ratio for the distributed queueing problem with arbitrarily dynamically injected requests on general graphs is \( O(\log^2 n \cdot \log D) \) as shown in [25] for the hierarchical schemes defined of [4, 24]. This shows that (under some assumptions), the simple and elegant Arrow protocol outperforms all existing significantly more complicated distributed queueing protocols.\(^2\) For a more detailed comparison of our results with existing protocols, we refer to the discussion in Section 1.1.

More specifically, as our main technical result, we show that the Arrow protocol is \( O(1) \)-competitive when it is run on a special class of trees known as hierarchically well separated trees [5]. A hierarchically well separated tree (in the following referred to as an HST) is a weighted, rooted tree where on each level, all the nodes are at the same distance to the root and all the leaves are on the same level (and thus also at the same distance to the root). Further, the edge lengths decrease exponentially (by a constant factor per level) when going from the root towards the leaves. When running Arrow on an HST \( T \), we assume that all requests are issued at the leaves of \( T \). We show that the total cost of an Arrow execution on an HST \( T \) is within a constant factor of the total cost of an optimal offline algorithm for the given set of requests. Our result even holds if the communication on \( T \) is asynchronous.

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\(^1\)Note that this implies a competitive ratio of \( O(s \cdot \log D) \) for general graphs if a spanning tree \( T \) of diameter \( D \) and stretch \( s \) is given.

\(^2\)Our protocol is based on a randomized tree construction and its competitive ratio is w.r.t. an oblivious adversary. Other protocols with polylogarithmic competitive ratio are deterministic and they therefore also work in the presence of an adaptive adversary.
Theorem 1.1. Assume that we are given an HST $T$ with parameter 2 and queueing requests $R$ that arrive in an arbitrarily dynamic manner at the leaves of $T$. When using the Arrow protocol on tree $T$, the total cost for ordering the requests in $R$ is within a constant factor of the cost of an optimal offline algorithm for ordering the requests $R$ on $T$. This even holds if communication is asynchronous.

Remark 1.1. Because the statement of the theorem applies to the general asynchronous case, it also captures a synchronous scenario, where the delay on each edge is fixed, but might be smaller than the actual weight of the edge in the HST. Note that such executions are relevant because an HST is often built as an overlay graph on top of an underlying network graph $G$ and the delay of simulating a single HST edge might be smaller than the weight of the edge.

For a precise description of the Arrow protocol and the definition of queueing cost, we refer to Section 2. When combining Theorem 1.1 with the celebrated probabilistic tree embedding of Fakcharoenphol, Rao, and Talwar [9], we get our main result for general graphs. In [9], it is shown that there is a randomized algorithm that given an arbitrary $n$-point metric $(X,d)$ constructs an HST $T$ such all points $X$ are mapped to leaves of $T$, all distances in $(X,d)$ are upper bounded by the respective distances in $T$, and the expected distance between any two leaves in $T$ is within an $O(\log n)$ factor of the distance between the corresponding two points in $X$. When constructing such an HST $T$ for a given graph $G$ and when assuming an oblivious adversary\footnote{That is, when assuming that the sequence of requests is statistically independent of the randomness used to construct the HST $T$.}, this implies that the expected total cost of Arrow on $T$ is within an $O(\log n)$ factor of the optimal offline queueing cost on $G$. We also note that an efficient distributed construction of the HST embedding of [9] has been given in [10].

Theorem 1.2. Assume that we are given an arbitrary graph $G = (V,E)$ and queueing requests $R$ that arrive in an arbitrarily dynamic manner at the nodes of $G$. There is a randomized construction of an HST $T$ that can be simulated on $G$ such that when running Arrow on $T$, we get a distributed queueing algorithm for $G$ with competitive ratio at most $O(\log n)$ against an oblivious adversary providing the sequence of requests. This even holds if communication is asynchronous.

Organization of the Paper: The remainder of the paper is organized as follows. Section 2 formally defines the queueing problem, the Arrow protocol, as well as the cost model used in our paper. The section also contains some lemmas that establish some basic properties that are needed for the rest of the paper. Section 3 analyzes the cost of an optimal offline algorithm on an HST $T$ by relating it to the total weight of an MST defined on the set of requests. In Section 4, we introduce a general framework to analyze the queueing cost of distributed queueing algorithms on an HST $T$ and the framework is applied to synchronous executions of the Arrow protocol. The analysis of asynchronous executions appears in Section 5.

1.1 Related Work

The Arrow protocol has been introduced by Raymond [22] as a way to solve the mutual exclusion problem in a network. The protocol was later reinvented by Demmer and Herlihy [8], who used Arrow to implement a distributed directory [6]. Over the years, Arrow has
been used and analyzed in different contexts \cite{13, 14, 17, 18, 20, 26}. The protocol has been implemented as a part of Aleph Toolkit \cite{13} and shown to outperform centralized schemes significantly in practice \cite{18}. Several other tree-based distributed queueing protocols that are similar to the Arrow protocol have also been proposed in the literature. A protocol that combines the ideas of Arrow with path compression has been implemented in the Ivy system \cite{19}. The amortized cost to serve a single request is only $O(\log n)$ \cite{11}, however the protocol needs a complete graph as the underlying network topology. There are also other similar protocols that operate on fixed trees. The Relay protocol \cite{27} has been introduced as a distributed transactional memory protocol. It is run on top of a fixed spanning tree similar to Arrow, however to more efficiently deal with aborted transactions, it does not always move the shared object to the node requesting it. Further, in \cite{2}, a distributed directory protocol called Combine has been proposed. Combine runs on a fixed overlay tree and it is in particular shown in \cite{2} that Combine is starvation-free.

The first paper to study the competitive ratio of concurrent executions of a distributed queueing protocol is \cite{16}. The paper shows that in synchronous executions of Arrow on a tree $T$, if all requests are issued at time 0 (known as one-shot executions), the total cost of Arrow is within a factor $O(\log |R|)$ compared with the optimal queueing cost on tree $T$. This analysis has later been extended (and slightly strengthened) to the general concurrent setting where requests are issued in an arbitrarily dynamic fashion. In \cite{14}, it is shown that in this case, the total cost of Arrow is within a factor $O(\log D)$ of the optimal cost on the tree $T$. Later, the same bounds have also been proven for the Relay protocol \cite{27} and the Combine protocol \cite{2}. Typically, these protocols are run on a spanning tree or an overlay tree on top of an underlying general network topology. While the cost of all these protocols is small when compared with the optimal queueing cost on the tree, the cost of the protocols might be much larger when compared with the optimal cost on the underlying topology. In this case, the competitive ratio becomes $O(s \cdot \log D)$, where $s$ is the stretch of the tree. There are underlying graphs (e.g., cycles) for which every spanning tree and even every overlay tree has stretch $\Omega(n)$ \cite{12, 21}. The fact that even the best spanning tree might have large stretch initiated the work on distributed queueing protocols that run on more general hierarchical structures. In \cite{15}, a protocol called Ballistic is introduced and analyzed for the sequential and the one-shot case. Ballistic has competitive ratio $O(\log D)$, however the protocol requires the underlying distance metric to have bounded doubling dimension and it thus cannot be applied in general networks. The best protocol known for general networks is Spiral, which was introduced in \cite{24}. Spiral is based on a hierarchy of overlapping clusters that cover the graph. It’s general structure is thus somewhat resembling the classic sparse partitions and mobile objects solutions by Awerbuch and Peleg \cite{3, 4}. The competitive ratio of Spiral is shown to be $O(\log^2 n \cdot \log D)$ for sequential and one-shot executions in \cite{24}. In \cite{25}, a general framework to analyze the cost of concurrent executions of hierarchical queueing and directory protocols has been presented. In particular, in \cite{25}, the competitive analysis of Spiral and also of the classic mobile object algorithm of Awerbuch and Peleg \cite{3, 4} has been extended to the dynamic setting. In \cite{14}, a sketch is given of how the competitive analysis for Arrow generalized to the asynchronous case.
2 Model, Problem Statement, and Preliminaries

Communication Model: We consider a standard message passing model on a network modeled by a graph \( G = (V, E) \). In some cases, the edges of \( G \) have weights \( w : E \to \mathbb{R}_{>0} \), which are assumed to be normalized such that \( w(e) \geq 1 \) for all \( e \in E \). We distinguish between synchronous and asynchronous executions. In a synchronous execution, the delay for sending a message from a node \( u \) to a node \( v \) over an edge \( e \) connecting \( u \) and \( v \) is exactly 1 if the edge is unweighted and exactly \( w(e) \) otherwise. In an asynchronous execution, message delays are arbitrary, however when analyzing an asynchronous execution, we assume that the message delay over an edge \( e \) is upper bounded by the edge weight \( w(e) \) (or by 1 in the unweighted case).

The Distributed Queueing Problem: In the distributed queueing problem on a graph \( G = (V, E) \), a set \( R \) of queueing requests \( r_i = (v_i, t_i) \) are issued at the nodes of \( V \) in an arbitrarily dynamic fashion. The goal of a queueing algorithm is to order all the requests. Specifically, if a request \( r_i = (v_i, t_i) \) is issued at node \( v_i \) at time \( t_i \geq 0 \), the algorithm needs to enqueue the request \( r_i \) by informing the node \( v_j \) of the predecessor request \( r_j = (v_j, t_j) \) in the constructed global order. For this purpose, every queueing algorithm in particular has to send (possibly indirectly) a respective message from node \( v_i \) to \( v_j \). We assume that at time 0, when an execution starts, the tail of the queue is at a given node \( v_0 \in V \). Formally, this is modeled as a request \( r_0 = (v_0, 0) \) which has to be ordered first by any queueing protocol. We sometimes refer to \( r_0 \) as the dummy request. For a set \( R' \) of queueing request (and sometimes by overloading notation also for a set of request indexes), we define \( t_{\min}(R') \) and \( t_{\max}(R') \) to be the minimum and the maximum issue time \( t \) of any request \( r = (v, t) \in R' \), respectively.

The Arrow Protocol: The Arrow protocol [22] is a distributed queueing protocol that operates on a tree network \( T = (V, E) \). At each point in time, each node \( v \in V \) has exactly one outgoing link (arrow) pointing either to one of the neighbors of \( v \) or to the node \( v \) itself. In a quiescent state, the arrow of the node of the request at the tail of the queue points to itself and all other arrows point towards the neighbor on the path towards the tail of the queue (i.e., the tree is directed towards the current tail). When a new request at a node \( v \in V \) occurs, a “find predecessor” message is sent along the arrows until it finds the predecessor request. While following the path to the direction of the arrows are reversed. More formally, a request \( r \) at node \( v \) is handled as follows.

1. If the arrow of \( v \) points to \( v \) itself, \( r \) is queued directly behind the previous request issued at \( v \). Otherwise if the arrow points to neighbor \( u \), atomically, a “find predecessor” message (including the information about request \( r \)) is sent to \( u \) and the arrow of \( v \) is redirected to \( v \) itself.

2. If a node \( u \) receives a “find predecessor” message for request \( r \) from a neighbor \( w \), if the arrow of \( u \) points to itself, atomically, the request \( r \) is queued directly behind the last request issued by node \( u \) and the arrow of \( u \) is redirected to node \( w \). Otherwise, if the arrow of \( u \) points to neighbor \( x \), atomically, the “find predecessor” message is forwarded to node \( x \) and the arrow of node \( u \) is redirected to node \( w \).

For a more detailed description of the Arrow protocol and of how Arrow handles concurrent requests, we refer the reader to [8, 14]. It was shown in [8] that the Arrow protocol correctly
orders a given sequence of requests even in an asynchronous network. Moreover as shown in [8, 14], when operating on tree \(T\), the protocol always finds the predecessor of a request on the direct path on \(T\). As a result, if two requests \(r'\) and \(r\) are at distance \(d\) on \(T\) and if \(r'\) is the predecessor of \(r\) in the queueing order, the “find predecessor” message initiated by request \(r\) finds the node of request \(r'\) in time exactly \(d\) in the synchronous setting and in time at most \(d\) in the asynchronous model. Further, it is shown in [14] that the successor request of a request \(r\) at node \(v\) in the queue is always the remaining request \(r''\) that first reaches \(v\) on a direct path. This “greedy” nature of the Arrow ordering was used in [16], where it was shown that in the one-shot case when all requests occur at time 0, the Arrow order corresponds to a greedy (nearest neighbor) TSP path through requests, whereas an optimal offline algorithm corresponds to an optimal TSP path on the request set. The competitive ratio on trees then follows from the fact that the nearest neighbor heuristic provides a logarithmic approximation of the TSP problem [23]. In [14], this analysis was extended and it was shown that even in the fully dynamic case, it is possible to reduce the problem to a (generalized) TSP nearest neighbor analysis. Formally, the greedy nature of the Arrow protocol in the synchronous setting is captured by Lemma 3.1 in Section 3, whereas the corresponding property in the asynchronous setting is formally discussed in Section 5.

Hierarchically Well Separated Trees: The notion of a hierarchically well separated tree (HST) was defined by Bartal in [5]. Given a parameter \(\alpha > 1\), an HST of depth \(h\) is a rooted tree with the following properties. All children of the root are at distance \(\alpha^{h-1}\) from the root. Further, every subtree of the root is an HST of depth \(h - 1\) that is characterized by the same parameter \(\alpha\) (i.e., the children 2 hops away from the root are at distance \(\alpha^{h-2}\) from their parents). The probabilistic tree embedding result of [9] shows that for every metric space \((X, d)\) with minimum distance normalized to 1 and for every constant \(\alpha > 1\), there is a randomized construction of an HST \(T\) with a bijection \(f\) of the points in \(X\) to the leaves of \(T\) such that for every \(x, y \in X\), \(d(x, y) \leq d_T(f(x), f(y))\) and thus such that the expected tree distance \(\mathbb{E}[d_T(f(x), f(y))] = O(\log |X|) \cdot d(x, y)\). Further, an efficient distributed implementation of the construction of [9] for the distances of a given network graph was given in [10].

The main technical result of this paper is an analysis of Arrow on an HST \(T\) if all requests are issued at leaves of \(T\). Throughout the paper, the HST parameter \(\alpha\) is set to \(\alpha = 2\). For convenience, we number the levels of an HST \(T\) of depth \(h\) from 0 to \(h\), where the level 0 nodes are the leaves and the single level \(h\) node is the root. For \(\ell \in \{0, \ldots, h\}\), \(\delta(\ell) := 2^{\ell+1} - 2\) denotes the distance between two leaves for which the least common ancestor is on level \(\ell\).

Cost Model: Assume when applying some queueing algorithm \(ALG\) to the dynamic set of request \(R\), the requests are ordered according to the permutation \(\pi_{ALG}\) such that the request ordered at position \(i\) in the order is \(r_{\pi_{ALG}(i)}\). For every \(i \in \{1, \ldots, |R| - 1\}\), we define the cost of ordering \(r_{\pi_{ALG}(i)}\) after \(r_{\pi_{ALG}(i-1)}\) as the time it takes a queueing algorithm to enqueue the request \(r_{\pi_{ALG}(i)}\) as the successor of \(r_{\pi_{ALG}(i-1)}\). More specifically, we assume that request \(r_{\pi_{ALG}(i)}\) can be enqueued as soon as the predecessor request \(r_{\pi_{ALG}(i-1)}\) is in the system and as soon as node \(v_{\pi_{ALG}(i-1)}\) knows about request \(r_{\pi_{ALG}(i)}\). Assume that algorithm \(ALG\) informs node \(v_{\pi_{ALG}(i-1)}\) (through a message) about \(r_{\pi_{ALG}(i)}\) at time \(t_{ALG}(i)\). The cost (latency) \(L_{ALG}(r_{\pi_{ALG}(i-1)}, r_{\pi_{ALG}(i)})\) incurred for enqueuing request \(r_{\pi_{ALG}(i)}\) and the overall cost (latency)
cost\textsubscript{ALG} of ALG are then defined as follows.

\[
L_{\text{ALG}}(r_{\pi_{\text{ALG}}(i-1)}, r_{\pi_{\text{ALG}}(i)}) := \max_{|R|-1} \{t_{\text{ALG}}(i), t_{\pi_{\text{ALG}}(i-1)}\} - t_{\pi_{\text{ALG}}(i)},
\]

\[
cost_{\text{ALG}}(\pi_{\text{ALG}}) := \sum_{i=1}^{\text{L}} L_{\text{ALG}}(r_{\pi_{\text{ALG}}(i-1)}, r_{\pi_{\text{ALG}}(i)}).
\]

We next specify the above cost more concretely for Arrow and for an optimal offline algorithm. Assume that we have an execution \(\mathcal{A}\) of the Arrow protocol that operates on a tree \(T\). Let \(\pi_{\mathcal{A}}\) be the ordering induced by the Arrow execution \(\mathcal{A}\). When the “find predecessor” message of a request \(r_{\pi_{\mathcal{A}}(i)}\) arrives at the node of the predecessor request \(r_{\pi_{\mathcal{A}}(i-1)}\), clearly the request \(r_{\pi_{\mathcal{A}}(i-1)}\) has already occurred and thus we always have \(L_{\mathcal{A}}(r_{\pi_{\mathcal{A}}(i-1)}, r_{\pi_{\mathcal{A}}(i)}) = t_{\mathcal{A}}(i) - t_{\pi_{\mathcal{A}}(i)}\) for any Arrow execution. Further note, that in a synchronous execution of arrow on tree \(T\), because Arrow always finds the predecessor on the direct path, this latency cost is always equal to the distance between the respective nodes in \(T\).

When studying the cost of an optimal offline queueing algorithm \(\mathcal{O}\), we assume that \(\mathcal{O}\) knows the whole sequence of requests in advance. However, \(\mathcal{O}\) still needs to send messages from each request to its predecessor request. The message delays are not under the control of the optimal offline algorithm. When lower bounding the cost of \(\mathcal{O}\), we can therefore assume that all communication is synchronous even in the asynchronous case. Note that a synchronous execution is a possible strategy of the asynchronous scheduler. When operating on a graph \(G\), the latency cost of \(\mathcal{O}\) for ordering a request \(r_j\) as the successor of a request \(r_i\) is then exactly \(L_{\mathcal{O}}^{G}(r_i, r_j) = \max \{t_i - t_j, d_G(v_i, v_j)\}\). As we analyze Arrow on an HST \(T\) that is simulated on top of an underlying network \(G\), we directly define the optimal offline w.r.t. synchronous executions on the tree \(T\) as follows.

\[
L_{\mathcal{O}}^{T}(r_{\pi_{\mathcal{O}}(i-1)}, r_{\pi_{\mathcal{O}}(i)}) := \max_{|R|-1} \{d_T(v_{\pi_{\mathcal{O}}(i-1)}, v_{\pi_{\mathcal{O}}(i)}), t_{\pi_{\mathcal{O}}(i-1)} - t_{\pi_{\mathcal{O}}(i)}\},
\]

\[
cost_{\mathcal{O}}^{T}(\pi_{\mathcal{O}}) := \sum_{i=1}^{\text{L}} L_{\mathcal{O}}^{T}(r_{\pi_{\mathcal{O}}(i-1)}, r_{\pi_{\mathcal{O}}(i)}).
\]

The ordering \(\pi_{\mathcal{O}}\) is chosen such that the total cost \(\text{cost}_{\mathcal{O}}^{T}(\pi_{\mathcal{O}})\) in (4) is minimized. The next lemma shows that when using the randomized HST construction of [9], the cost (4) is within a logarithmic factor of the optimal offline cost on the underlying network graph \(G\).

**Lemma 2.1.** Assume \(T\) is an HST that is constructed on top of an \(n\)-node network graph \(G\) by using the randomized algorithm of [9] and assume that there is a dynamic set of queueing requests issued at the nodes of \(G\). If the sequence of requests is independent of the randomness of the randomized HST construction, the expected optimal total cost on \(T\) (as defined in (4)) is within a factor \(O(\log n)\) of the optimal offline queueing cost on \(G\).

**Proof.** Let \(\pi_{\mathcal{O}}^{G}\) and \(\pi_{\mathcal{O}}^{T}\) be the optimal orderings w.r.t. the optimal offline costs \(L_{\mathcal{O}}^{G}(r_i, r_j)\) and
\[ L^T_O(r_i, r_j) \text{ on } G \text{ and } T, \text{ respectively, as defined above. We have} \]

\[
\mathbb{E} \left[ \text{cost}^T_O(\pi^T_O) \right] = \mathbb{E} \left[ \sum_{i=1}^{\left| R \right|-1} L^T_O(r_{\pi^T_G(i-1)}, r_{\pi^T_G(i)}) \right] \\
\leq \sum_{i=1}^{\left| R \right|-1} \mathbb{E} \left[ L^T_O(r_{\pi^T_G(i-1)}, r_{\pi^T_G(i)}) \right] \\
= \sum_{i=1}^{\left| R \right|-1} \mathbb{E} \left[ \max \left\{ d_T(v_{\pi^T_G(i-1)}, v_{\pi^T_G(i)}), t_{\pi^T_G(i-1)} - t_{\pi^T_G(i)} \right\} \right] \\
\leq 2 \cdot \sum_{i=1}^{\left| R \right|-1} \max \left\{ \mathbb{E} \left[ d_T(v_{\pi^T_G(i-1)}, v_{\pi^T_G(i)}) \right], t_{\pi^T_G(i-1)} - t_{\pi^T_G(i)} \right\} \\
\leq 2 \cdot \max \left\{ O(\log n) \cdot d_G(v_{\pi^T_G(i-1)}, v_{\pi^T_G(i)}), t_{\pi^T_G(i-1)} - t_{\pi^T_G(i)} \right\} \\
\leq O(\log n) \cdot \sum_{i=1}^{\left| R \right|-1} \max \left\{ d_G(v_{\pi^T_G(i-1)}, v_{\pi^T_G(i)}), t_{\pi^T_G(i-1)} - t_{\pi^T_G(i)} \right\} \\
\leq O(\log n) \cdot \text{cost}^T_G(\pi^T_G). 
\]

The first inequality follows from the fact that \( \pi^T_O \) is an optimal ordering w.r.t. the cost \( L^T_O(r_i, r_j) \) and by linearity of expectation. The second inequality follows because for every non-negative random variable \( X \) and every fixed (possibly negative) constant \( c \), it holds that \( \mathbb{E}[\max \{X, c\}] \leq 2 \cdot \max \{\mathbb{E}[X], c\} \). The third inequality follows from the expected stretch bound of the HST construction of [9], and the fourth inequality follows because for all values \( \lambda \geq 1, a \geq 0 \) and \( b \in \mathbb{R} \), it holds that \( \max \{\lambda a, b\} \leq \lambda \cdot \max \{a, b\} \).

Given Theorem 1.1 (which will be proven as the main technical result of the paper) and Lemma 2.1, we immediately get Theorem 1.2. We note in light of the remark following the statement of Theorem 1.1 in Section 1, the statement of Theorem 1.2 is also true for synchronous executions on the underlying graph \( G \).

**Manhattan Cost:** In the dynamic competitive analysis of Arrow on general trees in [14], it has been shown that it is useful to study the optimal ordering w.r.t. to the following Manhattan cost on a tree \( T \) between two queueing requests \( r_i = (v_i, t_i) \) and \( r_j = (v_j, t_j) \).

\[
c^T_m(r_i, r_j) := d_T(v_i, v_j) + |t_i - t_j|.
\]

As the cost function \( c^T_m(r_i, r_j) \) defines a metric space on the request set, the problem of finding an optimal ordering w.r.t. the cost \( c^T_m(r_i, r_j) \) is a metric TSP problem.\(^4\) As a result, we will for example use that the total weight of an MST on the set of request w.r.t. the weight function \( c^T_m(r_i, r_j) \) is within a factor 2 of the cost of an optimal TSP path. The following definition is inspired by Lemma 3.12 in [14].

\(^4\)The relation of Arrow and the TSP problem was already exploited in [14] when analyzing Arrow on general trees.
Definition 2.1 (Condensed Request Set). A set $R$ of queueing requests $r_i = (v_i, t_i)$ on a tree $T$ is called condensed if for any two requests $r_i = (v_i, t_i)$ and $r_j = (v_j, t_j)$ that are consecutive w.r.t. time of occurrence, there exits requests $r_a = (v_a, t_a)$ and $r_b = (v_b, t_b)$ such that $t_a \leq t_i$, $t_b \geq t_j$, and $d_T(v_a, v_b) \geq t_b - t_a$.

It is shown in [14] that for condensed request sets, the total optimal Manhattan cost is within a constant factor of the optimal offline queueing cost.

Lemma 2.2 (Lemma 3.17 in [14] rephrased). If the request set $R$ is condensed, then on any tree $T$ and for every ordering $\pi$ on the requests, it holds that

$$
\sum_{i=1}^{\lfloor R \rfloor - 1} c_M^T(r_{\pi(i-1)}, r_{\pi(i)}) \leq 12 \cdot \sum_{i=1}^{\lfloor R \rfloor - 1} L_D^T(r_{\pi(i-1)}, r_{\pi(i)}).
$$

For synchronous executions on trees, it is also shown in [14] that every request set $R$ can be transformed into a condensed request set without changing the ordering (and the cost) of Arrow and without increasing the optimal offline cost.

Lemma 2.3 (Lemma 3.11 in [14] rephrased). Let $R$ be a set of queueing requests issued on a tree $T$ and let $r_i = (v_i, t_i)$ and $r_j = (v_j, t_j)$ be two requests of $R$ that are consecutive w.r.t. time of occurrence. Further, choose two requests $r_a = (v_a, t_a)$ with $t_a \leq t_i$ and $r_b = (v_b, t_b)$ with $t_b \geq t_j$ minimizing $\delta := t_b - t_a - d_T(v_a, v_b)$. If $\delta > 0$, every request $r = (v, t)$ with $t \geq t_j$ can be replaced by a request $r' = (v, t - \delta)$ without changing the synchronous Arrow order and without increasing the optimal offline cost.

Lemma 2.3 implies that every request set $R$ can be transformed into a condensed set $R'$ without changing the synchronous order of Arrow and without increasing the optimal offline cost. For the analysis of Arrow in synchronous systems, we can thus w.l.o.g. assume that the request set is condensed. In Section 5, we show that this also holds in asynchronous systems.

3 Analysis of the Optimal Offline Cost

This and the next section discuss the main technical contribution of the paper and analyzes the total cost of a synchronous Arrow execution when run on an HST $T$. Throughout this section, we assume that a fixed HST $T$, a set of dynamic requests $R$ placed at the leaves of $T$, and a synchronous execution of Arrow with request set $R$ on $T$ are given. For convenience, we relabel the requests in $R$ so that they are ordered according to the queueing order resulting from the given Arrow execution on $T$. That is, we assume that for all $i \in \{0, \ldots, |R| - 1\}$, request $r_i = (v_i, t_i)$ is the $i^{th}$ request in Arrow’s order. Note that $r_0 = (v_0, 0)$ is still the dummy request defining the initial tail of the queue. As discussed in Section 2, the Arrow order can be seen as a greedy ordering in the following sense. Given the first $i - 1$ requests in the order, the $i^{th}$ request $r_i$ is a request $r = (v, t)$ from the subset of the remaining requests that can reach the node $v_{i-1}$ of request $r_{i-1}$ first immediately sending a message at time $t$ from node $v$ to node $v_{i-1}$. This greedy behavior is captured by the following basic lemma. The generalization of this basic greedy property to the asynchronous setting is discussed in Section 5. For a more thorough discussion, we also refer to [14].
**Lemma 3.1.** Consider a synchronous execution of \textit{Arrow} on tree \(T\) and consider two arbitrary requests \(r_i\) and \(r_j\) for which \(1 \leq i < j\) (i.e., \(r_j\) is ordered after \(r_i\) by \textit{Arrow}). Then it holds that

1. \(t_i + d_T(v_{i-1}, v_i) \leq t_j + d_T(v_{i-1}, v_j)\) and
2. \(t_i \leq t_j + d_T(v_i, v_j)\).

**Proof.** The first claim of the lemma follows immediately from Definition 3.5 and from Lemma 3.8 and Lemma 3.9 in [14]. The second claim follows the first claim of the lemma and the triangle inequality.

Before delving into the details of the analysis, we give a short outline. In the first step in Section 3.1, we study the ordering generated by \textit{Arrow} in more detail and show that it implies a hierarchical partition of the requests \(R\) in a natural way. To simplify the next Section 3.2 transforms the given HST \(T\) into a new tree such that inside each subtree, if ordering the request by time of occurrence, the gap between the times of consecutive requests cannot be too large (whenever such a gap is too large, we split the corresponding subtree into two trees). Section 3.3 then shows that the optimal offline cost can be characterized by the total Manhattan cost of a spanning tree that respects the hierarchical structure of the HST \(T\) in a best given way. Finally, in Section 4, we give a general framework to compare the queueing cost of an online distributed algorithm on an HST \(T\) to the optimal offline cost on \(T\) and we apply this method to synchronous \textit{Arrow} executions. In Section 5, we show that the same framework can also be applied to general asynchronous \textit{Arrow} executions.

### 3.1 Characterizing \textit{Arrow} By A Hierarchical Partition of \(R\)

We hierarchically partition the requests \(R\) according to the \textit{Arrow} queueing order and the hierarchical structure of the HST \(T\). On each level \(\ell\) of \(T\), we partition the requests into blocks, where a block of requests is a maximal set of requests that are ordered consecutively by \textit{Arrow} inside some level-\(\ell\) subtree of \(T\). In the following, for non-negative integers \(s\) and \(t\), we use the abbreviations \([s] := \{0, \ldots, s - 1\}\) and \([s, t] := \{s, \ldots, t\}\). Formally, instead of partitioning the set of requests \(R\) directly, we partition the set of indexes \([|R|]\). Recall that the requests in \(R\) are indexed consecutively according to the queueing order of \textit{Arrow}.

**Definition 3.1 (Hierarchical Block Partition).** For each level \(\ell \in [0, h]\), we partition \([|R|]\) into \(n(\ell)\) blocks \(\{b^\ell_0, b^\ell_1, \ldots, b^\ell_{n(\ell)-1}\}\) such that

1. each block is a consecutive set of integers (i.e., a consecutively ordered set of requests),
2. for every block \(b^\ell_i\), all requests \(r_p\) for \(p \in b^\ell_i\) are in the same level-\(\ell\) subtree of \(T\), and
3. for all \(i, j \in [n(\ell)]\) and all \(p \in b^\ell_i\) and \(q \in b^\ell_j\), \(i < j \implies p < q\).

For each block \(b\), we further define the first request of \(b\) to be the one that has minimum index in \(b\).

Note that for each level \(\ell\) and for the first block of this level, the first request of the block has index 0. The block partition defined in **Definition 3.1** is illustrated in Figure 1. Figure 1a shows the blocks within the HST structure, whereas Figure 1b shows the hierarchical partition...
Figure 1: The partition of $R$. (a) An HST with height 2 and 5 leaves. The leaves issue requests at different times. The issued requests by nodes $v_1$, $v_2$, and $v_3$ are partitioned into the blocks $b_0^1$ and $b_2^1$ on level 1. These two blocks are called neighbor blocks at a subtree rooted at height 1. (b) The corresponding 4 level-wise partition based on Arrow’s order that forms a parent-child relation between the blocks on different levels. Blue boxes include the requests that are ordered first by Arrow among all requests in blocks $b_i^q$ for all $i \in [0,9]$. 

induced by the blocks. To simplify the presentation of our analysis, we also define a level $-1$ block $b_i^{-1}$ for each individual request $r_i$. Note that we have $n(-1) = |R|$. The following definition allows to navigate through the block hierarchy.

**Definition 3.2 (Children Blocks).** The set of children blocks of a block $b_i^\ell$ on a level $\ell \in [0,h]$ is defined as $\text{child}(b_i^\ell) := \{b_j^{\ell-1} : b_j^{\ell-1} \subseteq b_i^\ell\}$. Block $b_i^\ell$ is called the parent block of each of the blocks in $\text{child}(b_i^\ell)$.

In Figure 1b, block $b_2^1$ is the parent block of its children blocks $b_5^0$ and $b_6^0$. Block $b_1^1$ has only one child block $b_4^1$ and thus $b_1^1 = b_4^1$.

The blocks $\{b_0^\ell, b_1^\ell, \ldots, b_{n(\ell)-1}^\ell\}$ of level $\ell$ belong to the subtrees rooted at height $\ell$ of the HST $T$. Note that by the definition of the block partition, no two consecutive blocks at the same level $\ell$ belong to the same level-$\ell$ subtree of $T$. The next definition specifies notation to argue about blocks of the same subtree of $T$.

**Definition 3.3 (Blocks of Same Subtree).** If two blocks $b_i^\ell$ and $b_j^\ell$ belong to the same level-$\ell$ subtree of $T$, this is denoted by $\hat{b_i^\ell b_j^\ell}$. Moreover, $|\hat{b_i^\ell b_j^\ell}| := \left|\{w : i < w < j \land \hat{b_i^\ell b_w} \text{ holds}\}\right|$.

Two blocks $b_i^\ell$ and $b_j^\ell$ are called neighbor blocks if $\hat{b_i^\ell b_j^\ell}$ and $|\hat{b_i^\ell b_j^\ell}| = 0$.

In Figure 1a, blocks $b_0^0$, $b_2^0$, and $b_8^0$ are within the same subtree rooted at node $v_1$. Blocks $b_0^0$ and $b_2^0$ are not neighbor blocks, however blocks $b_0^0$ and $b_3^0$, as well as blocks $b_2^0$ and $b_3^0$ are neighbor blocks. The next lemma lists a number of simple properties of the block partition.

**Lemma 3.2.** The block partition of Definition 3.1 satisfies the following properties:

1. For every block $b_i^\ell$ and for all $p, q \in b_i^\ell$, we have $d_T(v_p, v_q) \leq \delta(\ell)$.

2. For each level $\ell$ and all level-$\ell$ blocks $b_i^\ell$ and $b_j^\ell$, if $\hat{b_i^\ell b_j^\ell}$ holds, for any $p \in b_i^\ell$ and $q \in b_j^\ell$, we have $d_T(v_p, v_q) \leq \delta(\ell)$.

3. For each level $\ell$ and all level-$\ell$ blocks $b_i^\ell$ and $b_j^\ell$, if $\hat{b_i^\ell b_j^\ell}$ does not hold, for all $p \in b_i^\ell$ and $q \in b_j^\ell$, we have $d_T(v_p, v_q) \geq \delta(\ell + 1)$.
4. Assume \( \ell < h \) and consider two blocks \( b_i^\ell \) and \( b_j^\ell \) that have a common parent block \( b_w^{\ell+1} \), but for which \( \widetilde{b_i^\ell b_j^\ell} \) does not hold. Then, for all \( p \in b_i^\ell \) and \( q \in b_j^\ell \), we have \( d_T(v_p,v_q) = \delta(\ell + 1) \).

Proof. Recall that the distance between two leaves \( u,v \) of the HST \( T \) is equal to \( \delta(\ell) \) if the least common ancestor of \( u \) and \( v \) is on level \( \ell \). The first claim then holds because all requests in a block \( b_i^\ell \) at level \( \ell \) are issued at nodes in the same level-\( \ell \) subtree of \( T \) and therefore the least common ancestor of any two of them is on level at most \( \ell \). The second claim holds for a similar reason. If \( \widetilde{b_i^\ell b_j^\ell} \) holds for two blocks \( b_i^\ell \) and \( b_j^\ell \), both blocks consist of requests in the same level-\( \ell \) subtree of \( T \). For the third claim, note that when \( \widetilde{b_i^\ell b_j^\ell} \) does not hold for two blocks \( b_i^\ell \) and \( b_j^\ell \), the two blocks do not belong to the same subtree at level \( \ell \). Therefore for any two requests \( p \in b_i^\ell \) and \( q \in b_j^\ell \), the least common ancestor has to be on level at least \( \ell + 1 \) and thus the distance \( d_T(v_p,v_q) \geq \delta(\ell + 1) \). Finally, the fourth claim holds by combining the second claim (applied to block \( b_w^{\ell+1} \) on level \( \ell + 1 \)) and the third claim. \( \square \)

We have seen that in a synchronous Arrow execution, the latency cost for ordering request \( r_{i+1} \) as the successor of \( r_i \) is exactly the distance \( d_T(v_i,v_{i+1}) \) between the nodes of the two requests. The total cost of Arrow therefore directly follows from the structure of the block partition.

**Lemma 3.3.** The total cost of a synchronous Arrow execution on the HST \( T \) with corresponding hierarchical block partition is given by

\[
\text{cost}_A(\pi_A) = \sum_{\ell=0}^{h-1} \left( n(\ell) - n(\ell + 1) \right) \cdot \delta(\ell + 1).
\]

Proof. It follows from claim 4 of Lemma 3.2 that for any two requests \( r \) and \( r' \), \( d_T(r,r') = \delta(\ell + 1) \) for the smallest \( \ell \) for which \( r \) and \( r' \) are in the same level-\( \ell \) block. The block partition implies that for every level \( \ell \), there are \( n(\ell) - 1 \) consecutive requests \( r_i \) and \( r_{i+1} \) which are in different level-\( \ell \) blocks. For every \( \ell \in \{0, \ldots, h - 1\} \), the number of consecutive request pairs at distance at least \( \delta(\ell + 1) \) is therefore equal to \( n(\ell) - 1 \). The claim of the lemma now follows because \( \text{cost}_A(\pi_A) = \sum_{i=1}^{\lceil |R| \rceil - 1} d_T(v_{i-1},v_i) \).

\( \square \)

### 3.2 HST Conversion

In this section, a recursive (top-down) splitting procedure is provided so that the original HST is converted into a new HST with better properties. The conversion does not change the total cost of ordering the requests by Arrow (in fact, it does not change the block partition). Further, the total Manhattan cost of optimal offline algorithm’s order asymptotically remains unchanged as well. We describe how the splitting procedure works and we then argue its properties.

**Splitting Procedure:** We describe the splitting procedure as it is applied to a subtree \( T' \) that is rooted at a given level \( \ell \in \{0, \ldots, h\} \) of \( T \). If \( \ell = 0 \), the tree \( T' \) is returned unchanged. Otherwise (\( \ell \geq 1 \)), we go through all level-(\( \ell - 1 \)) subtrees \( T'' \) of \( T' \). As long as the tree \( T'' \)
has two neighbor blocks $b_{i-1}^x$ and $b_{j-1}^y$ (for $i < j$) for which the following condition (6) is true, the subtree $T''$ is split into two separate subtrees $T_1''$ and $T_2''$ of $T'$.

$$t_{\min}(b_{j-1}^x) - t_{\max}(b_{i-1}^y) \geq \delta(\ell).$$

The splitting of $T''$ into $T_1''$ and $T_2''$ works as follows. The topology of $T_1''$ and $T_2''$ is identical to the topology of $T''$. Each request $r = (v,t)$ that is issued at some node $v$ of $T''$ is either placed on the isomorphic copy of $v$ in $T_1''$ or in $T_2''$. All requests $r$ in blocks $b_{x-1}^x$ of $T''$ for $x \leq i$ are placed in tree $T_1''$ and all request in blocks $b_{y-1}^y$ of $T''$ for $y \geq j$ are placed in tree $T_2''$. We perform such splittings for trees $T'$ of level $\ell$ as long as there are subtrees of $T'$ on level $\ell - 1$ with neighbor blocks that satisfy Condition (6). As soon as no such neighbor blocks exist, the procedure is applied recursively to all trees $T''$ at level $\ell - 1$ (including the new subtrees). The whole conversion is started by applying the procedure to the complete HST $T$.

**Lemma 3.4.** The above splitting procedure does not change the hierarchical block partition and it thus also preserves Arrow’s queueing order $\pi_A$ and its total cost $\text{cost}_A(\pi_A)$.

**Proof.** We prove that a single splitting step does not change the block partition or the Arrow cost. The lemma then follows by induction on the number of splits in the above procedure. Assume that we are working on tree $T'$ on level $\ell$ and that we are splitting subtree $T''$ of $T'$ into $T_1''$ and $T_2''$ as a result of two neighbor blocks $b_{i-1}^x$ and $b_{j-1}^y$ satisfying Condition (6).

We first show that w.r.t. Arrow’s ordering $\pi_A$ before the splitting step, the block partition remains the same. W.r.t. the ordering $\pi_A$, the block partition can only change if some block of level $\ell' \leq \ell - 1$ at a subtree of $T''$ is split into two blocks. Note that any subtree $\tau$ of $T$ that is rooted at some node $v$ outside $T''$ either does not contain any node of $T''$ or it contains the whole subtree $T''$. In both cases, the request set of $\tau$ does not change and w.r.t. ordering $\pi_A$ therefore also their blocks on the level of node $v$ remain the same. Because the blocks at some level $\ell' < \ell - 1$ of tree $T''$ are a refinement of the blocks on level $\ell - 1$, if some block of some level $\ell' \leq \ell - 1$ at a subtree of $T''$ is split, there is also a level-$(\ell - 1)$ block of tree $T''$ is split into two blocks. However this cannot happen because the splitting procedure moves each level-$(\ell - 1)$ block of $T''$ either completely to $T_1''$ or to $T_2''$. Hence, w.r.t. the ordering $\pi_A$ before the splitting, the block partition remains the same.

We next show that this implies that for all pairs of requests $(r_i, r_{i+1})$ ordered consecutively by Arrow, the tree distance $d_T(v_i, v_{i+1})$ remains the same. If it does not remain the same, it means that $v_i$ and $v_{i+1}$ are both within $T''$ and thus before the split $d_T(v_i, v_{i+1}) \leq \delta(\ell - 1)$ (their least common ancestor is some node in $T''$). Hence, $r_i$ and $r_{i+1}$ are in the same block on level $\ell - 1$. To see this, recall that the blocks of level $\ell - 1$ of $T''$ are the maximal set of requests inside tree $T''$ that are ordered consecutively by Arrow. Because $r_i$ and $r_{i+1}$ are ordered consecutively, they therefore have to be in the same level $\ell - 1$ block of $T''$. After the split, we then have $d_T(v_i, v_{i+1}) = \delta(\ell)$ and thus $r_i$ and $r_{i+1}$ cannot be in the same block at level $\ell$ any more. As the splitting does not change the block partition (w.r.t. the original ordering $\pi_A$), this cannot happen. Hence, we have that for every $i \in \{0, \ldots, |R| - 2\}$, $d_T(v_i, v_{i+1})$ remains unchanged. All other distances can only increase. Hence, even after the split, for every $i \in \{0, \ldots, |R| - 2\}$, request $r_{i+1}$ still minimizes $t + d_T(v_i, v)$ among all non-ordered requests $r = (v, t)$. Lemma 3.1 therefore implies that $\pi_A$ is still a valid Arrow ordering. Because the block partition remains the same, Lemma 3.3 also immediately implies that $\text{cost}_A(\pi_A)$ remains unchanged. Because when splitting tree $T''$, every level-$(\ell - 1)$ block
of \( T'' \) either completely goes to tree \( T''_1 \) or to tree \( T''_2 \), the splitting does not divide any block. Hence, if we assume that the queueing order \( \pi_A \) is preserved, also the block partition is preserved.

The next lemma shows that if a tree \( T'' \) is split into two trees \( T''_1 \) and \( T''_2 \) such that all requests in \( T''_1 \) are ordered before all requests in \( T''_2 \), there is a significant time of occurrence gap between the requests ending up in subtrees \( T''_1 \) and \( T''_2 \).

**Lemma 3.5.** Assume that we are performing a single splitting. Further, assume that we are working on a tree \( T' \) on level \( \ell \) and that we are splitting a subtree \( T'' \) of \( T' \) into \( T''_1 \) and \( T''_2 \) such that \( T''_1 \) obtains the blocks that are scheduled first by \( \text{Arrow} \). If \( R_1 \) and \( R_2 \) are the request sets of \( T''_1 \) and \( T''_2 \), respectively, we have \( t_{\min}(R_2) - t_{\max}(R_1) \geq \delta(\ell) - \delta(\ell - 1) \).

**Proof.** Assume that the split of the tree \( T'' \) is caused by two neighbor blocks \( b_{i-1}^x \) and \( b_j^{x-1} \) satisfying Condition (6). We first show that \( t_{\min}(R_2) = t_{\min}(b_{j-1}^x) \). To see this, we generally show that for any subset of blocks \( b_{i_1}^x, b_{i_2}^x, \ldots \) of some tree \( T \) rooted at level \( x \), if \( b_{i_1}^x \) is the first of these blocks ordered by \( \text{Arrow} \), then the first request ordered in \( b_{i_1}^x \) has the smallest time of occurrence among all requests in blocks \( b_{i_1}^x, b_{i_2}^x, \ldots \). To see this, note that whenever \( \text{Arrow} \) enters a level-\( x \) block \( b_{i_1}^x \) of tree \( T \), the predecessor request \( r \) is at a node \( v \) outside tree \( T \). As a consequence, all leaf nodes in \( u \in T \) and thus all requests in \( T \) are at the same distance from \( v \) in the HST \( T \). Therefore Lemma 3.1 implies that the successor of \( r \) is a request with minimum time of occurrence.

It remains to show that

\[
    t_{\max}(R_1) \leq t_{\max}(b_{i-1}^x) + \delta(\ell - 1).
\]

Assume that \( r_p = (v_p, t_p) \) is a request from \( R_1 \) with \( t_p = t_{\max}(R_1) \). Further, assume that \( r_q = (v_q, t_q) \) is the last request ordered by \( \text{Arrow} \) among the requests in \( R_1 \). Note that request \( r_q \) needs to be inside block \( b_{i-1}^x \) because that is the last level-\( (\ell - 1) \) block that is assigned to tree \( T''_1 \). Hence, we clearly have \( t_q \leq t_{\max}(b_{i-1}^x) \). Therefore, if \( r_p = r_q \) (7) clearly holds. We can therefore assume that \( r_p \) is ordered before \( r_q \) by \( \text{Arrow} \). Consider the predecessor \( r_{p-1} \) of request \( r_p \). From the second part of Lemma 3.1, we have

\[
    t_p - t_q \leq d_T(v_p, v_q).
\]

Since both \( r_p \) and \( r_q \) are in \( T'' \) then \( d_T(v_p, v_q) \leq \delta(\ell - 1) \) thus (7) holds. 

It remains to show that the splitting also does not affect the optimal offline cost in a significant way. The following lemma shows that the Manhattan cost \( c_M(r, r') \) for any two requests \( r \) and \( r' \) can increase by at most a factor 3. Hence, also the total Manhattan cost of an optimal ordering cannot increase by more than a factor 3.

**Lemma 3.6.** For any two requests \( r \) and \( r' \), the splitting procedure does not increase the Manhattan cost \( c_M(r, r') \) by more than a factor 3.

**Proof.** We prove that a) by every single splitting, the Manhattan cost \( c_M(r, r') \) can at most increase by a factor of 3 and b) the Manhattan cost \( c_M(r, r') \) is affected by at most one splitting. Assume that \( r = (v, t) \) and \( r' = (v', t') \). Clearly, the issue times \( t \) and \( t' \) are not affected by the splitting. The Manhattan cost can therefore only change because \( d_T(v, v') \) changes. We
first show that this can happen at most once. When working on tree $T'$ at level $\ell$, a splitting divides a subtree $T''$ at level $\ell - 1$ into two subtrees $T'_1''$ and $T'_2''$. Hence, when working on level $\ell$, if two nodes are affected by the splitting their distance in $T'$ increases from at most $\delta(\ell - 1)$ to exactly $\delta(\ell)$. Therefore, after separating two nodes $v$ and $v'$ because of a splitting for a tree $T'$ on level $\ell$, the two nodes cannot be affected by another splitting on a level $\ell' \geq \ell$. Claim b) now follows because we do the splitting in a top-down way, i.e., throughout the splitting procedure the levels on which we split are monotonically non-increasing.

To prove claim a), let us assume that $r = (v,t)$ and $r' = (v',t')$ are affected by a splitting when a tree $T''$ at level $\ell - 1$ is split into two trees $T'_1''$ and $T'_2''$. We have already seen that this implies that after the splitting, we have $d_T(v,v') = \delta(\ell)$. It further follows from Lemma 3.5 that $|t - t'| \geq \delta(\ell) - \delta(\ell - 1) > \delta(\ell)/2$. Hence, before the splitting, we have $c_u(r,r') \geq |t - t'|$ and after the splitting, we have $c_u(r,r') \leq |t - t'| + d_T(v,v') < 3 \cdot |t - t'|$. 

For the remainder of the analysis in this section (and also in Section 5), we assume that the HST $T$ is an HST that is obtained after applying the splitting procedure recursively. We therefore assume that for every level $\ell$ and every subtree $T'$ at level $\ell$, there is no level-$(\ell - 1)$ subtree $T''$ of $T'$ that contains two neighbor blocks that satisfy Condition (6).

### 3.3 Lower Bounding The Optimal Manhattan Cost

In this section, we construct a tree $S^*$ that spans all requests in $R$. The tree $S^*$ has a nice hierarchical structure: For each subtree $T'$ of $T$, the set edges of $S^*$ induced by the request set of the subtree $T'$ forms a spanning tree of the request set of $T'$. Apart from this useful structural property, we will show that the total Manhattan cost of the spanning tree $S^*$ is within a constant factor of minimum spanning tree (MST) of the request set $R$ w.r.t. the Manhattan cost. We have seen that on condensed request sets, the optimal TSP path of the request set w.r.t. the Manhattan cost is within a constant factor of the optimal offline queueing cost. Note that because any TSP path is also a spanning tree, this implies that the total Manhattan cost of the MST and thus also the total Manhattan cost of the tree $S^*$ are lower bounding the optimal offline queueing cost within a constant multiplicative factor.

Throughout this section, for convenience, we add one more level to the HST $T$. Instead of placing the requests at the leaves on level 0, we assume that each level 0 node $v$ has a child node on level $-1$ for each of the requests issued at node $v$. Hence, the new leaf nodes are on level $-1$ and each leaf node receives exactly one request.\(^5\) The distance between a level $-1$ node and its parent on level 0 is set to be 0.

**Spanning Tree Construction:** The spanning tree $S^*$ is constructed greedily in a bottom-up fashion. For each subtree $T'$ of $T$, we recursively define a tree $S^*(T')$ as follows. For the leaf nodes on level $-1$, the tree consists of the single request placed at the node. For a tree $T'$ rooted at a node $v$ on level $\ell \geq 0$, the tree $S^*(T')$ consists of the recursively constructed trees $S^*(T'_1), S^*(T'_2), \ldots$ of the subtrees $T'_1, T'_2, \ldots$ of $T''$ and of edges connecting the trees $S^*(T'_1), S^*(T'_2), \ldots$ to a spanning tree of the set of request issued at leaves of tree $T'$. The edges for connecting the trees $S^*(T'_1), S^*(T'_2), \ldots$ are chosen so that they have minimum total Manhattan cost. That is, to connect the trees $S^*(T'_1), S^*(T'_2), \ldots$, we compute an MST of the graph we get if each of the trees $S^*(T''_i)$ is contracted to a single node. We can therefore

\(^5\)Note that subtrees of $T$ that do not have any queueing requests can be ignored and therefore, we can w.l.o.g. assume that every leaf node issues some queueing request.
for example choose the edges to connect the trees $S^*(T_1'''), S^*(T_2'')$, . . . in a greedy way: Always add the lightest (w.r.t. Manhattan cost) edge that does not close a cycle with the already existing edges, including the edges of the trees $S^*(T_1'), S^*(T_2')$, . . . .

**MST Approximation:** In the following, it is shown that the total Manhattan cost of the tree $S^* = S^*(T)$ is within a constant factor of the cost of an MST w.r.t. the Manhattan cost. Where convenient, we identify a tree $\tau$ with its set of edges, i.e., we also use $S^*$ to denote the set of edges of the tree $S^*$. Further, the cost of an edge $e = \{r, r'\}$ is the Manhattan cost $c_M(r, r')$. We also slightly abuse notation and use $c_M(e)$ to denote this cost. The proof applies a general MST approximation result that appears in Theorem A.1 in Appendix A. Together with the following technical lemma, Theorem A.1 directly implies that the total Manhattan cost of $S^*$ is within a factor 4 of the MST Manhattan cost. For a subtree $T'$ of $T$, we use $R(T')$ to denote the subset of the requests $R$ that are issued at nodes of $T'$.

**Lemma 3.7.** Consider the constructed spanning tree $S^*$ and consider an arbitrary edge $e$ of $S^*$. Let $S_1^*$ and $S_2^*$ the two subtrees that result when removing edge $e$ from $S^*$. Further, assume $e^*$ be an edge that connects the two subtrees $S_1^*$ and $S_2^*$ and that has minimum Manhattan cost among all such edges. We then have $c_M(e) \leq 4 \cdot c_M(e^*)$.

**Proof.** Assume that the edge $e = \{r_p, r_q\} \in S^*(\tau)$ is an edge that connects two subtrees of a subtree $\tau$ of $T$ that is rooted at some level $\ell \in [0, h]$. Further, let $V_{S_1^*}$ and $V_{S_2^*}$ be the node sets of the two subtrees of $S_1^*$ and $S_2^*$.

Let us first assume that $|t_p - t_q| \leq 3 \cdot \delta(\ell)$. All edges including $e^*$ from the metric $(R, c_M)$ that cross the cut $(V_{S_1^*}, V_{S_2^*})$ have length at least $\delta(\ell)$ since $d_T(v_w, v_z) \geq \delta(\ell)$ for all $v_w \in V_{S_1^*}$ and $v_z \in V_{S_2^*}$. Since $d_T(v_p, v_q) = \delta(\ell)$, we then have $c_M(e) \leq 4 \cdot \delta(\ell)$. Hence, the claim of the lemma holds.

Let us therefore assume that $|t_p - t_q| > 3 \cdot \delta(\ell)$ and let $\ell' \in [\ell, h]$ be the largest level for which $|t_p - t_q| > 3 \cdot \delta(\ell')$ and let $T'$ be the subtree of $T$ that is rooted on level $\ell'$ and that contains both requests $r_p$ and $r_q$ (see Figure 2). Note that this implies that

\[ |t_p - t_q| \leq 3 \cdot \delta(\ell' + 1) \quad \text{and thus} \quad c_M(e) \leq 3 \cdot \delta(\ell' + 1) + \delta(\ell). \quad (9) \]

We can partition each of the sets $V_{S_1^*}$ and $V_{S_2^*}$ into two sets where one of the sets in each case includes the requests in the subtree $T'$ and the other set includes the requests outside subtree $T'$ (see Figure 3). The edge $e$ obviously connects the two components $V_{S_1^*} \cap R(T')$ and $V_{S_2^*} \cap R(T')$.
and $V_{S_2} \cap R(T')$ since $r_p$ and $r_q$ are both in $R(T')$. If the edge $e$ is removed then edge $e^*$ is an edge connecting one of the two components $V_{S_1} \cap R(T')$ and $V_{S_2} \setminus R(T')$ in $S_1$ to one of the two components $V_{S_2} \cap R(T')$ and $V_{S_2} \setminus R(T')$ in $S_2$. The four different types of such edges are shown by the dashed edges in Figure 3.

Any edge that connects the two components $V_{S_1} \setminus R(T')$ and $V_{S_2} \cap R(T')$ has length at least $\delta(\ell' + 1)$ since $d_T(v_w, v_z) \geq \delta(\ell' + 1)$ for all $v_w \in V_{S_1} \setminus R(T')$ and $v_z \in V_{S_2} \cap R(T')$. By symmetry, the same also holds for the edges that connect the two components $V_{S_1} \cap R(T')$ and $V_{S_2} \setminus R(T')$. Hence, if $e^*$ is an edge of one of these two types, we have $c_{\ast}(e^*) \geq \delta(\ell' + 1)$. It then follows directly from (9) that $c_{\ast}(e^*) \leq 4 \cdot c_{\ast}(e)$ and thus the claim of the lemma holds.

Let us therefore move to the case where $e^*$ connects the two components $V_{S_1} \setminus R(T')$ and $V_{S_2} \setminus R(T')$, i.e., $e^* = \{r_x, r_y\}$ connects to nodes $v_x$ and $v_y$ outside tree $T'$. Recall that the tree $S^*$ is constructed in a bottom-up way such that the subtree $S^*(T'')$ of $S^*$ is connected for every subtree $T''$ of $T$. Hence, removing edge $e$ inside subtree $T'$ does not affect subtrees $S^*(T'')$ for trees $T''$ that do not contain $T'$. Therefore if two nodes $u$ and $v$ outside tree $T'$ end up on different sides of the cut $(V_{S_1}, V_{S_2})$, the least common ancestor of $v_x$ and $v_y$ has to be an ancestor of $T'$ and it is thus at level at least $\ell + 1$. Hence, if $e^*$ connects the two components $V_{S_1} \setminus R(T')$ and $V_{S_2} \setminus R(T')$, we also have $c_{\ast}(e^*) \geq \delta(\ell' + 1)$ and therefore again (9) implies the claim of the lemma.

It remains to show that all edges that connect the two components $V_{S_1} \cap R(T')$ and $V_{S_2} \cap R(T')$ are also large enough. W.l.o.g., we assume that $p < q$, i.e., the request $r_p$ is ordered before the request $r_q$ by Arrow. Further w.l.o.g., we assume that the dummy request is in $V_{S_1}$.

We next show that $t_q > t_p$. If $p = 0$ then the $t_q \geq t_p$ because $t_p = 0$ and because $|t_p - t_q| > 3 \cdot \delta(\ell) \geq 0$. Otherwise, for the sake of contradiction, let us assume that $t_q \leq t_p$. By the second part of Lemma 3.1 we have

$$t_p - t_q \leq d_T(v_p, v_q) \leq \delta(\ell).$$

This together with our assumption $t_q \leq t_p$ contradicts the fact that $|t_p - t_q| > 3 \cdot \delta(\ell)$. Therefore, $t_q > t_p$.

Recall that $e$ connects the two requests $r_p$ and $r_q$ inside level-$\ell$ tree $\tau$. Consider the subtree $S^*(\tau)$ of $S^*$ and let $S_1^*(\tau)$ and $S_2^*(\tau)$ be the two subtrees of $S^*(\tau)$ that are obtained when removing edge $e$ from $S^*(\tau)$. By the construction of the tree $S^*$, the edge $e$ is one with
minimum Manhattan cost among all edges connecting the requests in \( S^1_x(\tau) \) and \( S^2_z(\tau) \). We know that for all \( r_w \in V^1_{S^1_x(\tau)} \) and \( r_z \in V^2_{S^2_z(\tau)} \) we have \( d(v_w, v_z) = \delta(\ell) \). These facts imply that \( t_p = t_{\text{max}}(V^1_{S^1_x(\tau)}) \) and \( t_q = t_{\text{min}}(V^2_{S^2_z(\tau)}) \).

Now we show that there is an Arrow edge \((r_x, r_{x+1})\) where \( r_x \in V^1_{S^1_x(\tau)} \) and \( r_{x+1} \in V^2_{S^2_z(\tau)} \). For any two neighbor blocks \( b^x_i \) and \( b^z_j \) at subtree \( \tau \) and with \( i < j \), we know that

\[
t_{\text{min}}(b^x_i) - t_{\text{max}}(b^z_j) < \delta(\ell + 1)
\]

as otherwise because of the split condition (6), the subtree \( \tau \) would have been split. Thus, we have

\[
t_{\text{min}}(b^x_i) - t_{\text{max}}(b^z_j) < 3 \cdot \delta(\ell)
\]

since \( \delta(\ell + 1) \leq 3 \cdot \delta(\ell) \) for \( \alpha = 2 \). Let \( b^x_{i_1}, b^x_{i_2}, \ldots, b^x_{i_s} \) be the level-\( \ell \) blocks of the subtree \( \tau \) and assume that \( i_1 < i_2 < \cdots < i_s \). As \( t_q - t_p > 3 \cdot \delta(\ell) \) and because \( t_p = t_{\text{max}}(V^1_{S^1_x(\tau)}) \) and \( t_q = t_{\text{min}}(V^2_{S^2_z(\tau)}) \), for any two neighbor blocks \( b^x_{i_j} \) and \( b^z_{i_{j+1}} \), the requests \( r = (v, t) \) from \( b^x_{i_j} \) with \( t = t_{\text{max}}(b^x_{i_j}) \) and the requests \( r' = (v', t') \) from \( b^z_{i_{j+1}} \) with \( t' = t_{\text{min}}(b^z_{i_{j+1}}) \) either all have to be in in \( V^1_{S^1_x(\tau)} \) or they all have to be in \( V^2_{S^2_z(\tau)} \). We show that this implies that there has to be a block \( b^z_{i_j} \) at tree \( \tau \) for which the first request is in \( V^1_{S^1_x(\tau)} \) and which contains some request from \( V^2_{S^2_z(\tau)} \). First note that because of Lemma 3.1 and because we assumed that the dummy request is in \( V^1_{S^1_x(\tau)} \), the first request of \( b^z_{i_j} \) is in \( V^1_{S^1_x(\tau)} \). If all the first requests of blocks \( b^z_{i_j} \) are in \( V^2_{S^2_z(\tau)} \), it follows from the fact that \( V^2_{S^2_z(\tau)} \) needs to be non-empty that there has to be a block \( b^x_{i_j} \) for which the first request is in \( V^2_{S^2_z(\tau)} \) and which contains some request from \( V^1_{S^1_x(\tau)} \).

Otherwise, assume that \( b^x_{i_j} \) (for \( j \geq 2 \)) is the first block for which the first request is in \( V^1_{S^1_x(\tau)} \). Because by Lemma 3.1, the first request of a block is always one with smallest issue time, the above observation implies that the request with the largest issue time in \( b^x_{i_{j-1}} \) is in \( V^2_{S^2_z(\tau)} \) and then \( b^x_{i_{j-1}} \), which has the first request is in \( V^1_{S^1_x(\tau)} \) and which contains some request from \( V^2_{S^2_z(\tau)} \). In a block, where the first request is from \( V^1_{S^1_x(\tau)} \) and there is some request from \( V^2_{S^2_z(\tau)} \), there also have be two consecutive requests \( r_x \) and \( r_{x+1} \) (and thus an Arrow edge), such that \( r_x \in V^1_{S^1_x(\tau)} \) and \( r_{x+1} \in V^2_{S^2_z(\tau)} \).

We next show that the Arrow edge \((r_x, r_{x+1})\) is the only such Arrow edge even with respect to the tree \( T' \) containing tree \( \tau \). Specifically, we show that for all \( r_w \in V^1_{S^1_x(\tau)} \cap R(T') \) and all \( r_z \in V^2_{S^2_z(\tau)} \cap R(T') \) we have \( w \leq x \) and \( z \geq x + 1 \). In other words, \( r_x \) is the last request ordered in \( V^1_{S^1_x(\tau)} \cap R(T') \) and \( r_{x+1} \) is the first request ordered in \( V^2_{S^2_z(\tau)} \cap R(T') \). For the sake of contradiction, let us assume that there is a request \( r_w \in V^1_{S^1_x(\tau)} \cap R(T') \) for which \( w > x \) or that there is a request \( r_z \in V^2_{S^2_z(\tau)} \cap R(T') \) for which \( z < x + 1 \). We first assume the existence of request \( r_w \). Since \((x, x+1)\) is an Arrow edge, we have \( w > x + 1 \) and using the second part of Lemma 3.1 we get

\[
t_{x+1} - t_w \leq d(v_w, v_{x+1}) \leq \delta(\ell').
\]

However, we know that \( t_q - t_p \leq t_{x+1} - t_w \) and therefore

\[
t_q - t_p \leq \delta(\ell').
\]

This contradicts the fact that \( t_q - t_p > 3 \cdot \delta(\ell') \). Consequently, there does not exist any requests \( r_w \in V^1_{S^1_x(\tau)} \cap R(T') \) for which \( w > x \). Now, let us assume that there is a request \( r_z \in V^2_{S^2_z(\tau)} \cap R(T') \)

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for which \( z < x + 1 \). Again since \( (x, x + 1) \) is an Arrow edge, we have \( z < x \) and using the second part of Lemma 3.1 we get
\[
t_z - t_x \leq d(v_z, v_x) \leq \delta(\ell').
\]
However, we know that \( t_q - t_p \leq t_z - t_x \) and therefore
\[
t_q - t_p \leq \delta(\ell').
\]
Again, this is a contradiction to the fact that \( t_q - t_p > 3 \cdot \delta(\ell') \). Consequently, there does not exist any requests \( r_z \in V_{S_2} \cap R(T') \) with \( z < x + 1 \).

Finally we show that for all \( r_w \in V_{S_1} \cap R(T') \) and all \( r_z \in V_{S_2} \cap R(T') \) the Manhattan cost \( c_M(r_w, r_g) \) is at most \( 3 \cdot \delta(\ell') \). Using the second part of Lemma 3.1 we have
\[
t_{x+1} - t_z \leq d(v_{x+1}, v_z) \leq \delta(\ell').
\]
We can similarly bound \( t_w - t_x \). If \( w = 0 \) we have \( t_w \leq t_x \) and otherwise, using the second part of Lemma 3.1 we have
\[
t_w - t_x \leq d(v_w, v_x) \leq \delta(\ell').
\]
Using (10) and (11) we then get
\[
t_{x+1} - t_x \leq t_z - t_w + 2 \cdot \delta(\ell').
\]
We know that the Manhattan cost of \( (r_x, r_{x+1}) \) is at least the Manhattan cost of \( (r_p, r_q) \) because \( t_q - t_p \leq t_{x+1} - t_x \) and because for all \( r_f \in V_{S_1} \cap R(T') \) and \( r_g \in V_{S_2} \cap R(T') \), we have \( d(v_f, v_g) = \delta(\ell') \). That is, we have
\[
c_M(r_p, r_q) \leq c_M(r_x, r_{x+1}).
\]
Further, because for all \( r_f \in V_{S_1} \cap R(T') \) and \( r_g \in V_{S_2} \cap R(T') \), we have \( d(v_f, v_g) \geq \delta(\ell') \), by using (12), we obtain
\[
c_M(r_p, r_q) \leq c_M(r_x, r_{x+1}) \leq c_M(r_w, r_z) + 2 \cdot \delta(\ell').
\]
Therefore, by using the facts that \( t_q - t_p > 3 \cdot \delta(\ell') \) and \( t_q - t_p \leq t_{x+1} - t_x \), and by using (12), we get that \( t_z - t_w \geq t_{x+1} - t_x - 2 \delta(\ell') \geq \delta(\ell') \) and we thus have \( c_M(r_w, r_z) \geq \delta(\ell') \). By applying (13), we thus get that
\[
c_M(r_p, r_q) < 3 \cdot c_M(r_w, r_z).
\]
Consequently, also if \( e^* \) connects the two components \( V_{S_1} \cap R(T') \) and \( V_{S_2} \cap R(T') \), its Manhattan cost is within a factor 3 of the Manhattan cost of \( e \). Hence, the claim of the lemma holds.

**Corollary 3.8.** The total Manhattan cost of the spanning tree \( S^* \) is at most 4 times the total Manhattan cost of an MST spanning all the requests.

**Proof.** Follows directly from Lemma 3.7 and Theorem A.1.
4 Analysis of the Online Queueing Cost

In this section, we give a general framework to compare the queueing cost of an online queueing algorithm on HST $T$ with the bound of the offline queueing cost as established in Section 3. At the end of the section, we apply the method to analyze synchronous Arrow executions on $T$. As in Section 3.3, for convenience, we add one more level to the HST $T$ so that each level 0 node $v$ has a child node on level $-1$ for each of the requests issued at node $v$. The new leaf nodes are on level $-1$ and each leaf node receives exactly one request.

We first state two basic locality properties of Arrow and possibly other online queueing protocols. We will then show that those properties are sufficient to prove a constant competitive ratio compared to the optimal offline queueing cost on $T$. We define the notion of a distance-respecting queueing order and the notion of distance-respecting latency cost of a queueing algorithm.

**Definition 4.1 (Distance-Respecting Order).** Let $R$ be a set of requests $r_i = (v_i, t_i)$ issued at the nodes of a tree $T$ and let $\pi$ be permutation on $[0, |R| - 1]$. The ordering $r_{\pi(0)}, r_{\pi(1)}, \ldots, r_{\pi(|R| - 1)}$ induced by $\pi$ is called distance-respecting if whenever $\pi(i) < \pi(j)$, we have $t_i - t_j \leq d_T(v_i, v_j)$.

**Definition 4.2 (Distance-Respecting Latency Cost).** An online distributed queueing algorithm $ALG$ is said to have distance-respecting latency cost if for any request set $R$ and any possible queueing order $\pi_{ALG}$ of $ALG$, for all $1 \leq i < j < |R|$, it holds that

$$t_{\pi_{ALG}(i)} + L_{ALG}(r_{\pi_{ALG}(i), \pi_{ALG}(i-1)}) \leq t_{\pi_{ALG}(j)} + d_T(v_{\pi_{ALG}(j)}, v_{\pi_{ALG}(i-1)})$$

4.1 Constructing a Spanning Tree

As the first part of the online queueing cost analysis, we construct a new tree $S$ that spans all requests in $R$. It will be shown that the total Manhattan cost of $S$ asymptotically equals the total Manhattan cost of the tree $S^*$ constructed in the previous section.

We construct a new tree $S$ on $R$ based on an ordering $\pi$ of the set of requests. We assume that the ordering of the requests given by $\pi$ is $r_{\pi(0)}, r_{\pi(1)}, \ldots, r_{\pi(|R| - 1)}$. For each index $i$ with $i \in [0, |R| - 2]$, we define the local successor as

$$next(i) := \min \left\{ j \in [i + 1, |R| - 1] : d_T(v_{\pi(i)}, v_{\pi(j)}) = \min_{k \in [i + 1, |R| - 1]} d_T(v_{\pi(i)}, v_{\pi(k)}) \right\}.$$  \hspace{1cm} (14)

Hence, among the requests ordered after $r_{\pi(i)}$ by order $\pi$, $next(i)$ is the position of a request in the order $\pi$ with minimum tree distance to $v_{\pi(i)}$ and among those, of the first one ordered by $\pi$. Note that this means that for all requests $r_{\pi(k)}$ for which $i < k < next(i)$, we have $d_T(v_{\pi(i)}, v_{\pi(k)}) > d_T(v_{\pi(i)}, v_{\pi(next(i))})$ and for all requests $r_{\pi(k)}$ for which $k \geq next(i)$, we have $d_T(v_{\pi(i)}, v_{\pi(k)}) \geq d_T(v_{\pi(i)}, v_{\pi(next(i))})$.

The spanning tree $S$ is constructed as follows. For every request $r_{\pi(i)}$ for all $i \in [0, |R| - 2]$, we add the edge $\{r_{\pi(i)}, r_{\pi(next(i))}\}$ to the tree $S$. Note that $S$ is indeed a spanning tree: If directing each edge from $r_{\pi(i)}$ to $r_{\pi(next(i))}$, each node has out-degree 1 and we cannot have cycles because $next(i) > i$. The following observation shows that in addition, $S$ has the same useful hierarchical structure as the tree $S^*$ constructed in Section 3.3.
Observation 4.1. As the tree $S^*$, also the tree $S$ has the property that for any subtree $T'$ of $T$, the subgraph of $S$ induced by only the requests at nodes in $T'$ is a connected subtree of $S$. This follows directly from the definition of the local successor $r_{\pi(x)}$. Except for the last ordered request inside $T'$, the local successor of any other request of $T'$ is inside $T'$ (because the local successor is a request with minimum tree distance).

In light of Observation 4.1, for any subtree $T'$ of $T$, we use $S(T')$ to denote the subtree of $S$ induced by the requests issued at nodes in $T'$.

4.2 Bounding the Manhattan Cost of the Spanning Tree

The following lemma shows that if the spanning tree $S$ is constructed by using a distance-respecting ordering $\pi$, the total Manhattan cost of the spanning tree $S$ is asymptotically equal to the total Manhattan cost of $S^*$.

Lemma 4.2. Let $C_m(S)$ and $C_m(S^*)$ be the total Manhattan costs of $S$ and of $S^*$. If the tree $S$ is constructed using a distance-respecting ordering $\pi$, we have $C_m(S) \leq 3 \cdot C_m(S^*)$.

Proof. Consider some subtree $\tau$ of $T$ that is rooted at a node on level $\ell \in [0, h]$. Assume that $v$ has $m$ children an that the subtrees of $T$ rooted at the $m$ children are $\tau_1, \tau_2, \ldots, \tau_m$. Using Observation 4.1, we know that $S(\tau_1), S(\tau_2), \ldots, S(\tau_m)$ are subtrees of $S(\tau)$ trees that are connected to each other with $m - 1$ edges to form the spanning tree $S(\tau)$. Let us call this set of edges $I(\tau)$. Note that for $\ell = 0$ the subtrees of $\tau$ are single requests at level $-1$. Similarly, the construction of $S^*$ implies that the spanning tree $S^*(\tau)$ results from connecting the spanning trees $S^*(\tau_1), S^*(\tau_2), \ldots, S^*(\tau_m)$ with $m - 1$ edges. Let $I^*(\tau)$ denote this set of these $m - 1$ edges. Recall that the edges in $I^*(\tau)$ are chosen such that they have minimum total Manhattan cost among all sets of $m$ edges connecting the trees $S^*(\tau_1), S^*(\tau_2), \ldots, S^*(\tau_m)$. We also emphasize that for all $i \in [1, m]$, the trees $S(\tau_i)$ and $S^*(\tau_i)$ consist of the same set of nodes (the requests inside tree $\tau_i$). Let $C_m(I(\tau))$ and $C_m(I^*(\tau))$ be the total Manhattan costs of the edges in $I(\tau)$ and $I^*(\tau)$, respectively. To prove the lemma, it suffices to show that

$$\forall \text{ subtree } \tau \text{ of } T: C_m(I(\tau)) \leq 3 \cdot C_m(I^*(\tau)).$$

Let $e = (r_{\pi(w)}, r_{\pi(z)}) \in I(\tau)$ be an arbitrary edge of $I(\tau)$ and let $S_1(\tau)$ and $S_2(\tau)$ be the two subtrees of $S(\tau)$ resulting from removing $e$ from $I(\tau)$. Let $V_{S_1(\tau)}$ and $V_{S_2(\tau)}$ be the set of nodes (requests) of the trees $S_1(\tau)$ and $S_2(\tau)$ and assume, w.l.o.g., that $w < z$ and that $r_{\pi(w)} \in V_{S_1(\tau)}$ and $r_{\pi(z)} \in V_{S_2(\tau)}$. Also, consider an edge $e^*$ that crosses the cut $(V_{S_1(\tau)}, V_{S_2(\tau)})$ and has minimum Manhattan cost among all edges in $S^*(\tau)$ that cross this cut. Note that because for all $i$ the trees $S(\tau_i)$ and $S^*(\tau_i)$ consist of the same set of node, node $e^*$ must be from the set $I^*(\tau)$. In order to prove (15), it suffices to show that

$$c_m(e) \leq 3 \cdot c_m(e^*).$$

Inequality (15) then directly follows from Theorem A.1.

From the definition of local successor, we know that $z = next(w)$. This implies that for all requests $r_{\pi(x)}$ where $w < x < z$, we have $d_T(v_{\pi(w)}, v_{\pi(x)}) > \delta(\ell)$ since $d_T(v_{\pi(w)}, v_{\pi(z)}) = \delta(\ell)$. Therefore, all requests that are ordered between $r_{\pi(w)}$ and $r_{\pi(z)}$ by Arrow are not in $R(\tau)$ (i.e., in the set of requests of tree $\tau$). This means that all requests in $R(\tau)$ are ordered
either before \( r_{\pi(w)} \) or after \( r_{\pi(z)} \) by Arrow. More precisely, the claim is that for all requests \( r_{\pi(x)} \in V_{S_1(\tau)} \) we have \( x \leq w \) and for all requests \( r_{\pi(z)} \in V_{S_2(\tau)} \) we have \( x \geq z \). To show this, we first observe that by the definition of \( e \), \( S_1(\tau) \) and \( S_2(\tau) \), among all edges of \( S(\tau) \), the edge \( e = \{ r_{\pi(w)}, r_{\pi(z)} \} \) is the only edge that crosses the cut \((V_{S_1(\tau)}, V_{S_2(\tau)})\).

We now first show that for all requests \( r_{\pi(x)} \in V_{S_2(\tau)} \) we have \( x \geq z \). For contradiction, let us assume that there is a request \( r_{\pi(x)} \in V_{S_2(\tau)} \) for which \( x < z \) and therefore \( x < w \). This implies that there must be a largest \( y < w \) such that \( r_{\pi(y)} \in V_{S_2(\tau)} \). Note that because \( r_{\pi(y)} \) is not the last request ordered in \( \tau \), \( r_{\pi(next(y))} \) must be in \( \tau \) and it therefore must be in \( V_{S_1(\tau)} \). This implies that the edge \( \{ r_{\pi(y)}, r_{\pi(next(y))} \} \) of \( S(\tau) \) crosses the cut \((V_{S_1(\tau)}, V_{S_2(\tau)})\), which is not possible because the edge \( \{ r_{\pi(w)}, r_{\pi(z)} \} \) is the only edge of \( S(\tau) \) crossing this cut.

We next show that for all requests \( r_{\pi(x)} \in V_{S_1(\tau)} \), we have \( x \leq w \). Again assume that there is a request \( r_{\pi(x)} \in V_{S_1(\tau)} \) such that \( x > w \) and thus \( x > z \). Therefore, there must be smallest \( y > w \) for which \( r_{\pi(y)} \in V_{S_1(\tau)} \). This implies that \( r_{\pi(y)} \) is the local successor of some request in \( V_{S_2(\tau)} \). This again contradicts the fact that the edge \( e = \{ r_{\pi(w)}, r_{\pi(z)} \} \) is the only edge of \( S(\tau) \) crossing the cut \((V_{S_1(\tau)}, V_{S_2(\tau)})\).

Finally we show that for all \( r_{\pi(p)} \in V_{S_1(\tau)} \) and \( r_{\pi(q)} \in V_{S_2(\tau)} \) the Manhattan cost of \( e \) is at most \( 3 \cdot c_M(r_{\pi(p)}, r_{\pi(q)}) \). Because \( \pi \) is distance-respecting, we have

\[
t_{\pi(z)} - t_{\pi(q)} \leq d_T(v_{\pi(q)}, v_{\pi(z)}) \leq \delta(\ell). \tag{17}
\]

Further, if \( p = 0 \), we have \( t_{\pi(p)} = 0 \) and thus \( t_{\pi(p)} \leq t_{\pi(w)} \). Otherwise, because \( \pi \) is distance-respecting, we get

\[
t_{\pi(p)} - t_{\pi(w)} \leq d_T(v_{\pi(p)}, v_{\pi(w)}) \leq \delta(\ell). \tag{18}
\]

Using (17) and (18) we have

\[
t_{\pi(z)} - t_{\pi(w)} \leq t_{\pi(q)} - t_{\pi(p)} + 2 \cdot \delta(\ell). \tag{19}
\]

We continue by distinguishing the two cases \( t_{\pi(z)} \geq t_{\pi(w)} \) and \( t_{\pi(w)} > t_{\pi(z)} \). First assume that \( t_{\pi(z)} \geq t_{\pi(w)} \). Then, using \( d_T(v_{\pi(z)}, v_{\pi(w)}) = d_T(v_{\pi(p)}, v_{\pi(q)}) = \delta(\ell) \) and (19) we obtain

\[
c_M(r_{\pi(z)}, r_{\pi(w)}) \leq c_M(r_{\pi(p)}, r_{\pi(q)}) + 2 \cdot \delta(\ell).
\]

Moreover, because \( d_T(v_{\pi(p)}, v_{\pi(q)}) = \delta(\ell) \), we know that \( \delta(\ell) \leq c_M(r_{\pi(p)}, r_{\pi(q)}) \). Thus,

\[
c_M(e) \leq 3 \cdot c_M(r_{\pi(p)}, r_{\pi(q)}).
\]

Let us therefore consider the second case where \( t_{\pi(w)} > t_{\pi(z)} \). It is clear that \( w \neq 0 \) as otherwise \( t_{\pi(w)} = 0 \) and thus \( t_{\pi(z)} \geq t_{\pi(w)} \). Because \( \pi \) is distance-respecting, we have

\[
t_{\pi(w)} - t_{\pi(z)} \leq d_T(v_{\pi(w)}, v_{\pi(z)}) = \delta(\ell).
\]

Using the assumption that \( t_{\pi(w)} > t_{\pi(z)} \), we then have

\[
c_M(r_{\pi(z)}, r_{\pi(w)}) = |t_{\pi(w)} - t_{\pi(z)}| + d_T(v_{\pi(w)}, v_{\pi(z)}) = t_{\pi(w)} - t_{\pi(z)} + \delta(\ell) \leq 2 \cdot \delta(\ell).
\]

Finally, we can again use that \( c_M(r_{\pi(p)}, r_{\pi(q)}) \geq d_T(v_{\pi(p)}, v_{\pi(q)}) = \delta(\ell) \) and thus get that

\[
c_M(e) \leq 2 \cdot c_M(r_{\pi(p)}, r_{\pi(q)}).
\]

This concludes the proof of the lemma.
4.3 Bounding the Total Latency Cost

It remains to prove the main claim and show that the total online queueing cost on the HST $T$ is within a constant factor of the optimal offline queueing cost on $T$. The following theorem states that this is generally true for algorithms with distance-respecting latency cost (Definition 4.2) and which produce distance-respecting queueing orders (Definition 4.1), as long as the request set $R$ is condensed (Definition 2.1).

**Theorem 4.3.** Assume that we are given an HST $T$ and a condensed set of requests issued at the leaves of $R$. Further, assume that we are given a distributed queueing algorithm ALG that has distance-respecting latency cost and that always produces a distance-respecting queueing order $\pi$. Then, the total latency cost of ALG is within a constant factor of the optimal offline cost on $T$.

**Proof.** Because the request set $R$ is condensed, Lemma 2.2 implies that the optimal offline cost is within a constant factor of the Manhattan cost of an optimal TSP path connecting all the requests. The optimal offline cost therefore also is within a constant factor of the total Manhattan cost of an MST of the request set. Hence, Corollary 3.8 implies that also the total Manhattan cost of $S^*$ is within a constant factor of the cost of an optimal offline solution on $T$. Because the ordering $\pi$ generated by ALG is distance-respecting, by Lemma 4.2, the same is true for the total Manhattan cost $C_M(S)$ of the tree $S$. It therefore remains to show that $cost^T_{ALG}(\pi) = O(C_M(S))$.

Because ALG has distance-respecting latency cost, for all $i \in [0, |R| - 2]$, we have

$$t_{\pi(i+1)} + L^T_{ALG}(r_{\pi(i)}, r_{\pi(i+1)}) \leq t_{\pi(next(i))} + d_T(v_{\pi(i)}, v_{\pi(next(i))}).$$

Note that we have $next(i) \geq i + 1$. Subtracting $t_{\pi(i)}$ on both sides yields

$$t_{\pi(i+1)} - t_{\pi(i)} + L^T_{ALG}(r_{\pi(i)}, r_{\pi(i+1)}) \leq t_{\pi(next(i))} - t_{\pi(i)} + d_T(v_{\pi(i)}, v_{\pi(next(i)))}$$

If we sum up the above inequality for all $i \in [0, |R| - 2]$, we get

$$\sum_{i=0}^{|R|-2} (t_{\pi(i+1)} - t_{\pi(i)} + d_T(v_{\pi(i)}, v_{\pi(i+1)})) \leq \sum_{i=0}^{|R|-2} (t_{\pi(next(i))} - t_{\pi(i)} + d_T(v_{\pi(i)}, v_{\pi(next(i)))})$$

The sum of the latencies on the left-hand side exactly equals the total queueing cost $cost^T_{ALG}(\pi)$ of ALG. To bound the right-hand side, note that $t_{\pi(next(i))} - t_{\pi(i)} + d_T(v_{\pi(i)}, v_{\pi(next(i)))} \leq c_M(r_{\pi(i)}, r_{\pi(next(i)))}$. Together, we get

$$t_{\pi(|R|-1)} - t_{\pi(0)} + cost^T_{ALG}(\pi) \leq C_M(S).$$

As specified in Section 2, we assume that $t \geq 0$ for every request $r = (v, t)$ and that every queueing algorithm first has to order the dummy request $r_0 = (v_0, 0)$. We therefore have $t_{\pi(|R|-1)} \geq 0$ and $t_{\pi(0)} = t_0 = 0$, which completes the proof of the theorem. $\blacksquare$

**Corollary 4.4.** The total latency cost of a synchronous execution of Arrow on an HST $T$ is within a constant factor of the optimal offline queueing cost on $T$. 

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Proof. First note that by Lemma 2.3, w.l.o.g., for synchronous Arrow executions, we can assume that the request set \( R \) is condensed. The corollary therefore follows from Theorem 4.3 if we show that synchronous Arrow’s ordering is distance-respecting and that synchronous Arrow has distance-respecting latency cost. The former follows from claim 2 of Lemma 3.1, the latter follows from claim 1 of Lemma 3.1 and the fact that the latency cost of synchronous Arrow for ordering a request \( r_i \) as the predecessor of request \( r_{i+1} \) is exactly \( d_T(v_i, v_{i+1}) \).

\[ \square \]

Remark 4.1. The above corollary proves Theorem 1.1 (cf. Section 1) for synchronous executions on the HST \( T \). The full statement of Theorem 1.1 for general asynchronous executions is proven in Section 5. There, it is shown that also for asynchronous executions, Arrow has distance-respecting latency cost and produces distance-respecting queueing orders. In addition, we also show that we can still restrict attention to condensed request sets. The claim of Theorem 1.1 for the asynchronous case then follows from Theorem 4.3 in the same way as in the above corollary.

5 Queueing Cost in the Asynchronous Model

In this section, we show that the generic analysis of Section 4 also applies to asynchronous executions of the Arrow protocol on \( T \). In order to use the framework of Section 4 in the asynchronous setting, we mostly importantly need to show that Arrow has distance-respecting latency cost (Definition 4.2) and that it generates distance-respecting queueing orders (Definition 4.1) also in the asynchronous case. To show this, we need asynchronous variants of the basic Lemma 2.3 and Lemma 3.1. In addition, we also need to generalize Lemma 2.3 to show that also in the asynchronous setting, w.l.o.g., we can assume that the given request set \( R \) is condensed (Definition 2.1).

As in Section 3, we relabel the requests for convenience. Throughout the section, we assume that an asynchronous execution \( \pi_A \) of Arrow is given and we label the requests according the order \( \pi_A \). That is, \( r_0 \) is the dummy request and for every \( i \geq 1 \), \( r_i \) is the \( i^{th} \) non-dummy request ordered by the asynchronous Arrow execution.

5.1 Basic Properties of Asynchronous Arrow Executions

We have seen that a synchronous Arrow execution can be seen as a greedy queueing order in the following sense. Assume that requests \( r_0, \ldots, r_{i-1} \) of the queueing order are known and let \( v_{i-1} \) be the node at which request \( r_{i-1} \) has been issued. Then, request \( r_i \) the first one among the remaining requests that reaches node \( v_{i-1} \) on a direct path. In the asynchronous setting, an analogous property is true. However, we need to be a bit more careful and argue the arrival time of the “find predecessor” message on the whole path from the node of a request to its predecessor.

Let us assume that we are given a tree \( T \), a dynamic set of requests \( R \) issued at the nodes of \( T \), as well as an asynchronous execution of Arrow that orders the requests \( r_0, r_1, \ldots, r_{|R|-1} \) in this order. It has been shown in [8] that even in a concurrent asynchronous Arrow execution, every request \( r_i \) finds the node \( v_{i-1} \) of its predecessor \( r_{i-1} \) on a direct path. To formally specify the greedy property of Arrow in the asynchronous setting, we need to study the progress of messages on the whole path from a request to its predecessor. For any two nodes \( u, v \) of \( T \), we
use $P_{u,v}$ to denote the direct path from $u$ to $v$ on tree $T$. The following Lemma 5.1 formally establishes the greedy behavior of asynchronous Arrow executions.

We first introduce some terminology defined in [14]. For all $i \in [0, |R| - 1]$, we define $F_i$ to be a configuration of the tree network, where all arrows are pointing towards the node $v_i$ of request $r_i$. Further, let $R_i$ be the set of requests $[r_i+1, |R| - 1]$ that are ordered after request $r_i$. Finally, let $E_i$ be an execution of the Arrow protocol starting from configuration $F_i$ and in which only the requests in $R_i$ are issued. It is shown in Lemma 3.7 in [14] that for all $i$, except for request $r_i$, no request in $R_{i-1}$ can distinguish locally between executions $E_{i-1}$ and $E_i$. More specifically, all these requests see exactly the same arrows in both executions. This implies that the “find predecessor” message of every request $r_i$ sees exactly the same arrows as if the network started in configuration $F_{i-1}$ and only request $r_i$ was issued. To study the behavior of the requests in $R_{i-1}$, it therefore suffices to study an execution that starts in configuration $F_{i-1}$ and where only the requests in $R_{i-1}$ are issued.

**Lemma 5.1.** Consider an asynchronous Arrow execution for a request set $R$ on a tree $T$. Let $i \in [1, |R| - 1]$ and consider the path $P_{u_i,v_i-1} = (u_0, u_1, \ldots, u_s)$ from node $u_0 = v_i$ of request $r_i$ to the node $u_s = v_{i-1}$ of the predecessor $r_{i-1}$. For every node $u_k$ on the path, the “find predecessor” message of request $r_i$ is the first “find predecessor” message that reaches node $u_k$ (or is generated at node $u_k$) among all the “find predecessor” of requests $r_j$ for $j \in [i, |R| - 1]$.

**Proof.** In order to prove the claim of the lemma, we can assume that requests $r_0, \ldots, r_{i-1}$ have already found their predecessors and therefore the tree is in configuration $F_{i-1}$. Lemma 3.7 in [14] implies that this does not affect the behavior of any of the remaining queuing requests in $R_{i-1}$.

Assume for contradiction that the claim of the lemma is not true. Let $x \in [0, \ldots, s]$ be the maximal value such that the “find predecessor” message of request $r_i$ is not the first one among the requests in $R_{i-1}$ reaching $u_k$. Note that we need to have $k < s$ because by the definition of the Arrow protocol, the first message reaching $u_s = v_{i-1}$ is the successor request of $r_{i-1}$. Let $r = (v, t)$ be the first request in $R_{i-1}$ that reaches node $u_s$. In configuration $F_{i-1}$, the arrow of node $u_k$ points to $u_{k+1}$. In order to change this, a “find predecessor” message first has to be sent from node $u_k$ to $u_{k+1}$. Because $r$ is the first request reaching $u_k$, when the “find predecessor” message of $r$ reaches $u_k$, this has not happened and therefore the arrow still points from $u_k$ to $u_{k+1}$. When reaching $u_s$, in an atomic step, the “find predecessor” message of $r$ is therefore forwarded to $u_{k+1}$. As long as the message is in transit between the two nodes, there is no arrow across the edge $\{u_k, u_{k+1}\}$ and therefore the “find predecessor” message of $r$ also reaches $u_{k+1}$ before the “find predecessor” message of $r_i$ reaches $u_{k+1}$. This is a contradiction to the assumption on the maximality of $k$ and therefore the claim of the lemma holds.

The above lemma shows that if the “find predecessor” messages of two requests reach the same node $v$, then the earlier ordered request reaches $v$ first. To have an analogous statement for Lemma 3.1, we would like to have a statement saying that a request $r$ reaches a node $v$ on the path to the predecessor request before any request $r'$ that is ordered after $r$ (not only for a request $r'$ that actually reaches $v$). To achieve this, we extend a given Arrow execution to simplify the analysis. Whenever a request $r = (v, t)$ is issued at node $v$ at time $t$, a “find predecessor” message leaves $v$ at time $t$ and it travels on the direct path to the predecessor request $r'$ of $r$. For the proof, we assume that instead of only going to the predecessor, the
“find predecessor” message is sent as a broadcast to the whole network. We think of the additional messages to complete this broadcast as virtual messages that are only used for the analysis and have no influence on the queueing protocol. Given an asynchronous execution of Arrow, we assume that the actual messages sent by the Arrow protocol keep their message delays (to ensure an equivalent execution). All the virtual messages are assumed to have the maximum possible message delay. That is, the delay of sending a virtual message from $u$ to $v$ is equal to the length $d_T(u, v)$ of the respective tree edge. Further, to make sure that virtual messages can never overtake real messages, if a real message and a virtual message reach a node at the same time, the node always first processes the real message. In this way, for every request $r = (v, t)$, the delay of the respective “find predecessor” message is defined for all nodes. For a request $r$ and a node $u \in V$, we introduce the following notation:

$$\Delta(r, u) := \text{time of “find predecessor” message of request } r \text{ to reach node } u. \quad (20)$$

We note that for $r = (v, t)$ and any node $u \in V$, we have $\Delta(r, u) \leq d_T(u, v)$ (recall that in the asynchronous setting, for the analysis, the delay of a message is assumed to be at most the length of the respective edge). The next lemma will be used as a replacement of the main statement of Lemma 3.1 in the asynchronous analysis.

**Lemma 5.2.** Consider an asynchronous execution of Arrow for a set of requests $R$ on tree $T$ and consider two arbitrary requests $r_i$ and $r_j$ for which $1 \leq i < j$ (i.e., $r_j$ is ordered after $r_i$ by Arrow). Then for any node $v$ on the path from $v_i$ to $v_{i-1}$, it holds that

$$t_i + \Delta(r_i, v) \leq t_j + \Delta(r_j, v).$$

**Proof.** Similarly to the proof of Lemma 5.1, we apply Lemma 3.7 from [14] and we assume that the network starts in configuration $F_i-1$. Consequently, initially, all arrows are pointing towards $v_{i-1}$ and only the requests in $R_{i-1}$ still need to be ordered.

We first show that for every arrow pointing from a node $v_1$ to a node $v_2$ in configuration $F_{i-1}$, the first message sent from $v_1$ to $v_2$ has to be a real message. For contradiction, assume otherwise and assume that the first arrow along which a virtual message is sent before a real message is pointing from node $w_1$ to node $w_2$. Further, assume that message $M$ is the first such message that is sent by $w_1$ over the edge. Note that this also implies that $M$ is the first message sent from $w_1$ to $w_2$. Assume that this virtual message $M$ belongs to a request $r = (v, t)$. First note that $M$ is the first message arriving at node $w_1$. Otherwise, some other message would have been sent from $w_1$ to $w_2$. If message $M$ arrives at $w_1$ as a real message, it is forwarded as a real message to node $w_2$. We can therefore conclude that message $M$ reaches $w_1$ as a virtual message (say from neighbor $w_0$). Because $M$ is the first message that reaches $w_1$, it is also the first message sent from $w_0$ to $w_1$ (note that as a virtual message, it has the maximum possible message delay, so it cannot overtake any other message). Because in configuration $F_{i-1}$, there also is an arrow from $w_0$ to $w_1$, this is a contradiction to the assumption that the arrow from $w_1$ to $w_2$ is the first on which a virtual message is sent before a real one.

To conclude the proof, observe that in configuration $F_{i-1}$, all neighbors $u$ of the path $P_{v_i, v_{i-1}} = (u_0, \ldots, u_s)$ from $u_0 = v_i$ to $u_s = v_{i-1}$ have an arrow pointing from $u$ to the neighbor on the path. Hence, on each edge connecting to the path, the first message that reaches the path is a real message. The same is true for all edges of the path in the direction
from node \( u_0 = v_i \) to node \( u_x = v_{i-1} \). The only way a virtual message can therefore reach a node \( u_k \) of the path before a real message does is when a virtual message for a request \( r \) is sent from a node \( u_{k+1} \) to node \( u_k \). Assume that this is the case and assume that \( u_x \) for \( x \geq k + 1 \) is the first node on the path that is reached by the message of \( r \). There are two cases to consider, either the message of \( r \) reaches node \( u_x \) from a neighbor outside the path \( P_{v_i,v_{i-1}} \) or the request is issued at node \( u_x \). Because the first message reaching the path \( P_{v_i,v_{i-1}} \) from a neighbor of the path has to be a real message, Lemma 5.1 implies that the “find predecessor” message of request \( r_i \) reaches \( u_x \) before any message from outside the path reaches \( u_x \). However, in that case, the “find predecessor” message of \( r_i \) also reaches all earlier nodes on path \( P_{v_i,v_{i-1}} \) (and thus in particular node \( u_k \)) before the message of \( r \) does. If the request \( r \) is issued at node \( u_x \), Lemma 5.1 also implies that this has to happen after the “find predecessor” message of \( r_i \) reaches \( u_x \).

The following lemma is a simple consequence of Lemma 5.2.

**Lemma 5.3.** Consider an asynchronous execution of Arrow for a given set of requests \( R \) on a tree \( T \) and consider two arbitrary requests \( r_i \) and \( r_j \) for which \( i < j \) (i.e., \( r_i \) is ordered before \( r_j \)). Then, the following two statements hold:

1. \( t_i - t_j \leq d_T(v_i,v_j) \),
2. if \( i \geq 1 \), \( t_i + \Delta(r_i,v_{i-1}) \leq t_j + d_T(v_{i-1},v_j) \).

**Proof.** If \( i = 0 \), we only need to prove the first claim, which in this clearly holds because \( t_0 = 0 \) and \( t_j \geq 0 \) for all \( r_j \in R \). Let us therefore assume that \( i \geq 1 \). We consider the part of the tree \( T \) induced by the paths between the nodes \( v_i, v_j \), and the node \( v_{i-1} \) of the predecessor request \( r_{i-1} \) of \( r_i \). Let \( x \) be the (unique) node on the tree on which the three paths \( P_{v_i,v_j}, P_{v_i,v_{i-1}}, \) and \( P_{v_j,v_{i-1}} \) intersect. Because \( x \) in particular is a node on the path \( P_{v_i,v_{i-1}} \), from Lemma 5.2, we get that

\[
t_i + \Delta(r_i,x) \leq t_j + \Delta(r_j,x). \tag{21}
\]

The term \( \Delta(r_j,x) \) is the delay of the message of request \( r_j \) to reach node \( x \) from node \( v_j \). Because the message delay is upper bounded by the length of the path and because \( x \) is on the path \( P_{v_i,v_j} \), we have \( \Delta(r_j,x) \leq d_T(v_j,x) \leq d_T(v_j,v_i) \) and thus, the first claim of the lemma follows directly from (21) (note that \( \Delta(r_i,x) \geq 0 \)). The second claim can also be proved based on (21):

\[
t_i + \Delta(r_i,v_{i-1}) = t_i + \Delta(r_i,x) + (\Delta(r_i,v_{i-1}) - \Delta(r_i,x)) \leq (21) \leq t_j + \Delta(r_j,x) + (\Delta(r_i,v_{i-1}) - \Delta(r_i,x)) \leq t_j + d_T(v_j,x) + d_T(x,v_{i-1}) = t_j + d_T(v_{i-1},v_j).
\]

The second inequality follows because the message delay of an edge is at most the length of the edge.

It remains to adapt the basic Lemma 2.3 to the asynchronous setting.
**Lemma 5.4.** Let \( R \) be a set of queueing requests issued on a tree \( T \) and let \( r_i = (v_i, t_i) \) and \( r_j = (v_j, t_j) \) be two requests of \( R \) that are consecutive w.r.t. time of occurrence. Further, choose two requests \( r_a = (v_a, t_a) \) with \( t_a \leq t_i \) and \( r_b = (v_b, t_b) \) with \( t_b \geq t_j \) minimizing \( \delta := t_b - t_a - d_T(v_a, v_b) \). If \( \delta > 0 \), every request \( r = (v, t) \) with \( t \geq t_j \) can be replaced by a request \( r' = (v, t - \delta) \) without decreasing the worst-case cost of \( \text{Arrow} \) and without increasing the optimal offline cost.

**Proof.** Because the optimal offline cost is computed w.r.t. synchronous executions, the proof that the optimal offline cost is not increased follows directly from Lemma 2.3. To show that the worst-case \( \text{Arrow} \) cost does not decrease, we show that if all the message delays remain the same, the execution can still produce the same \( \text{Arrow} \) order with the same total cost.

Let \( R_\leq \) be the set of requests with issue time \( \leq t_i \) and let \( R_\geq \) be the set of requests with issue time \( \geq t_j \). Note that \( R = R_\leq \cup R_\geq \). We first show that when replacing every request \( r = (v, t) \) in \( R_\geq \) by a request \( r' = (v, t - \delta + \varepsilon) \) for an arbitrary \( \varepsilon > 0 \), if we do not change any of the message delays, we obtain exactly the same \( \text{Arrow} \) ordering and cost.\(^6\) To see this, first observe that in this case, claim 1 of Lemma 5.3 implies that all requests in \( R_\leq \) are ordered before any request in \( R_\geq \) is ordered. Let \( r_x = (v_x, t_x) \) be the last request ordered in \( R_\leq \) and let \( r_y = (v_y, t_y) \) be the first request ordered in \( R_\leq \) and \( R_\geq \) in the original execution. Because all requests in \( R_\geq \) are shifted by the same amount and they are still all ordered after the requests in \( R_\leq \), also after the shifting, the “find predecessor” request of \( r_y \) is the first one to arrive at node \( v_x \) and therefore \( r_y \) still is the successor of \( r_x \). Because the time differences inside \( R_\geq \) do not change, also the rest of the ordering does not change. The argument holds even if we let \( \varepsilon \) go arbitrarily close to 0. In the limit, the argument therefore still holds as long as whenever a node receives several messages at the same time, the asynchronous scheduler processes messages corresponding to requests in \( R_\leq \) before processing messages corresponding to \( R_\geq \).

We have therefore shown that for every initial \( \text{Arrow} \) execution, the asynchronous scheduler can enforce an equivalent execution with the same cost with the shifted request. This proves the claim of the lemma.

We now have everything needed to prove Theorem 1.1 stating that the total cost of an asynchronous execution of \( \text{Arrow} \) on an HST \( T \) is within a constant factor of the optimal offline queueing cost on \( T \).

**Proof of Theorem 1.1.** The above Lemma 5.4 shows that we can (iteratively) transform the initial request set \( R \) into a condensed set of requests without decreasing the cost of \( \text{Arrow} \) and without increasing the optimal offline cost. We can therefore assume that we are given a condensed set of requests. The claim of the theorem now follows if we can show that the latency cost of asynchronous \( \text{Arrow} \) is distance-respecting and that any asynchronous \( \text{Arrow} \) execution generates a distance-respecting queueing order. However, these statements follow directly from claims 2 and 1 of Lemma 5.3, respectively.\(^\dagger\)
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A Minimum Spanning Tree Approximation

In this section, we prove a general minimum spanning tree (MST) approximation result. Assume that we are given a spanning tree \( \tau = (V, E_\tau) \) of a graph \( G = (V, E) \). Together with \( \tau \), every edge \( e \in E_\tau \) induced a cut of \( G \) as follows. When removing \( e \) from \( \tau \), we obtain a spanning forest consisting of two connected subtrees of \( \tau \). Let \( S \) and \( V \setminus S \) be the node sets of these two connected components. We say that \( (S, V \setminus S) \) is the cut induced by removing \( e \) from \( \tau \). The next theorem shows that if for every edge \( e \in E_\tau \), the weight of \( e \) is within a factor \( \lambda \) of the weight of the heaviest edge crossing the cut induced by removing \( e \) from \( \tau \), then the total weight of \( \tau \) is within a factor \( \lambda \) of the weight of an MST. We expect that this results is already known, however, we have not found a proof of it in the literature. The next theorem proves a slightly more general statement.

**Theorem A.1.** Let \( \lambda \geq 1 \) be some number and let \( G = (V, E, w) \) be a weighted connected graph with non-negative edge weights \( w(e) \geq 0 \) and let \( \tau \subseteq E \) and \( \tau^* \subseteq E \) be two arbitrary spanning trees of \( G \). If for every edge \( e \) of \( \tau \), the lightest edge \( e' \) of \( \tau^* \) crossing the cut induced by removing \( e \) from \( \tau \) has weight \( w(e') \geq w(e)/\lambda \), then the total weight of all edges in \( \tau \) is at most a \( \lambda \)-factor larger than the total weight of the edges in \( \tau^* \).

**Proof.** In the following, we slightly abuse notation and we identify a spanning tree \( \tau \) with the set of edges contained in \( \tau \). For an edge set \( F \subseteq E \), we also use \( w(F) \) to denote the total weight of the edges in \( F \). We prove the stronger statement that

\[
w(\tau \setminus \tau^*) \leq \lambda \cdot w(\tau^* \setminus \tau). \tag{22}
\]

We show (22) by induction on \( |\tau \setminus \tau^*| = |\tau^* \setminus \tau| \). First note that if \( |\tau \setminus \tau^*| = 0 \), we have \( \tau = \tau^* \) and thus (22) is clearly true. Further, if \( |\tau \setminus \tau^*| = 1 \), there is exactly one edge \( e \in \tau \setminus \tau^* \) and exactly one edge \( f \in \tau^* \setminus \tau \). Because \( \tau \) and \( \tau^* \) are spanning trees, \( f \) connects the two sides of the cut \((V_{e,1}, V_{e,2})\) induced by removing \( e \) from \( \tau \) and we therefore have \( w(f) \leq \lambda \cdot w(e) \), implying (22).

Let us therefore assume that \( |\tau \setminus \tau^*| = k \geq 2 \) and let \( e \) be a maximum weight edge of \( \tau \setminus \tau^* \). Let \((V_{e,1}, V_{e,2})\) be the cut induced by removing \( e \) from \( \tau \). Further, let \( \tau' \) be a spanning tree of \( G \) that is obtained by removing \( e \) from \( \tau \) and by adding some edge \( f \in \tau^* \setminus \tau \) that connects \( V_{e,1} \) and \( V_{e,2} \). Note that by the assumptions of the theorem, we have \( w(e) \leq \lambda \cdot w(f) \). To prove (22), it thus suffices to show that \( w(\tau' \setminus \tau^*) \leq \lambda \cdot w(\tau^* \setminus \tau') \). We have \( |\tau' \setminus \tau^*| = k - 1 \) and thus, if the spanning tree \( \tau' \) satisfies the conditions of the theorem, \( w(\tau' \setminus \tau^*) \leq \lambda \cdot w(\tau^* \setminus \tau') \) and (22) follows from the induction hypothesis. We therefore need to show that \( \tau' \) satisfies the conditions of the theorem.

Consider an arbitrary edge \( e' \in \tau' \setminus \tau^* \) and let \((U_1, U_2)\) be the partition of \( V \) induced by removing \( e' \) from tree \( \tau' \). Since \( e' \) is an edge of one of the two subtrees of \( \tau \) resulting after removing \( e \), \( e' \) either connects two nodes in \( V_{e,1} \) or two nodes in \( V_{e,2} \). W.l.o.g., assume that \( e' \) connects two nodes in \( V_{e,2} \) and let \( V_{e,2,1} \) and \( V_{e,2,2} \) be the partition of \( V_{e,2} \) induced by removing \( e' \) from the subtree of \( \tau \) induced by \( V_{e,2} \). We need to show that for every edge \( f' \in \tau^* \) connecting \( U_1 \) and \( U_2 \), it holds that \( w(e') \leq \lambda \cdot w(f') \). Any edge \( f' \) crossing the cut has to either connect \( V_{e,1} \) with \( V_{e,2} \) or it has to connect \( V_{e,2,1} \) with \( V_{e,2,2} \). In the first case, we have \( w(e') \leq w(e) \leq \lambda \cdot w(f') \) (recall that we chose \( e \) to be the heaviest edge from \( \tau \setminus \tau' \)). In the second case, \( f' \) also crosses the cut induced by removing \( e' \) from the original tree \( \tau \) and therefore we also have \( w(e') \leq \lambda \cdot w(f') \). This concludes the proof. \( \square \)