The current phase relation in Josephson tunnel junctions.

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(Dated: 7 February 2005)

The $J(\varphi)$ relation in SFIFS, SNINS and SIS tunnel junctions is studied. The method for analytical solution of linearized Usadel equations has been developed and applied to these structures. It is shown that the Josephson current across the structure has the sum of $\sin \varphi$ and $\sin 2\varphi$ components. Two different physical mechanisms are responsible for the sign of $\sin 2\varphi$. The first one is the depairing by current which contributes positively to the $\sin 2\varphi$ term, while the second one is the finite transparency of SF or SN interfaces which provides the negative contribution. In SFIFS junctions, where the first harmonic vanishes at "0" - "\pi" transition, the calculated second harmonic fully determines the $J(\varphi)$ curve.

PACS numbers: 74.50.+r, 74.80.Dm, 75.30.Et

It is well known that tunnel SIS Josephson junctions have sinusoidal current-phase relation, while with the decrease of the barrier transparency deviations from $\sin \varphi$ take place (see\textsuperscript{1} for the review). The sign of second harmonic is important for many applications, in particular in junctions with a more complex structure like SNINS or SFIFS, where N is a normal metal and F is a weak metallic ferromagnet\textsuperscript{2,3,4}. To analyze this problem self-consistently, one should go beyond the approximation which is usually used and is called "Rigid boundary conditions" (RBC).

The RBC method is an effective tool extensively used earlier for theoretical study of the proximity and Josephson effects\textsuperscript{5,6}. This method is based on the assumption that all nonlinear and nonequilibrium effects in a Josephson structure are located in a "weak link" connecting two superconducting electrodes. The back influence of these effects on superconductivity in the electrodes is neglected. The RBC are valid if a junction has the constriction geometry. The quantitative criteria for the validity of RBC for planar SIS tunnel junctions, SS/S sandwiches and variable thickness bridges were studied only numerically for some parameter ranges\textsuperscript{7}. The main technical difficulty in formulating the analytical criteria of RBC validity is to find the solution of equations describing the perturbation of superconducting state in S electrodes. In this paper we will attack this problem by finding the solution of linearized Usadel equations\textsuperscript{8}. We will also use this solution to formulate the corrections to previous results obtained in RBC approximation.

The junction model. Let us consider the structure of SFIFS type, where for simplicity the parameters of the SF bilayers are equal to each other. We assume that the S layers are bulk and that the dirty limit conditions are fulfilled in the S and F metals. We assume further that F metals are weak monodomain ferromagnets with zero electron–phonon interaction constant and the FS interfaces are not magnetically active. We will restrict ourselves to the case of parallel orientation of the exchange fields $H$ in the ferromagnets. The results obtained for SFIFS junctions cross over to SNINS and SIS in corresponding limits.

Under the above assumptions the problem is reduced to the solution of the one-dimensional Usadel equations\textsuperscript{9} in S- and F-layers and matching these solutions by the appropriate boundary conditions\textsuperscript{10}. We choose the $x$ axis perpendicular to the plane of the interfaces with the origin at the central barrier I and introduce indexes $L$ (left), $R$ (right) and $I$ for description the materials and interfaces parameters of the SFIFS structure located on the left and right sides from the central barrier and at this central barrier, respectively.

The Usadel functions $G$ and $F$ obey the normalization condition $G^2_{(2)} + F_{(2)}^2 = 1$, which allows the following parametrization in terms of the new function $\Phi$:

$$G_\omega = \frac{\tilde{\omega}}{\sqrt{\omega^2 + F_\omega \Phi_{(2)}}}, \quad F_\omega = \frac{\Phi_\omega}{\sqrt{\omega^2 + F_\omega \Phi_{(2)}}}.$$ (1)

The quantity $\tilde{\omega} = \omega + iH$ corresponds to the general case when the exchange field $H$ is present. However, in the S layers $H = 0$ and we have simply $\tilde{\omega} = \omega$.

The Usadel equations\textsuperscript{11} in the S and F layers have the form

$$\xi_S^2 \frac{\pi T_c}{\omega G_S} \frac{\partial}{\partial x} \left[ G^2_S \frac{\partial}{\partial x} \Phi_S \right] - \Phi_S = -\Delta,$$ (2)

$$\xi_F^2 \frac{\pi T_c}{\omega G_F} \frac{\partial}{\partial x} \left[ G^2_F \frac{\partial}{\partial x} \Phi_F \right] - \Phi_F = 0,$$ (3)

where $G_\omega = \tilde{\omega}/\sqrt{\omega^2 + F_\omega \Phi_{(2)}}$, $\tilde{\omega} = \omega + iH$ in a ferromagnet ($H$ is the exchange field), $\omega = \omega$ in S and N metals, $T_c$ and $\Delta$ are the critical temperature and the pair potential in a superconductor, $\omega = \pi T_c (2n + 1)$ are the Matsubara frequencies and $\xi_{S(F)}$ are the coherence lengths related to the diffusion constants $D_{S(F)}$ as $\xi_{S(F)} = \sqrt{D_{S(F)}/2\pi T_c}$. The pair potential satisfies the
self-consistency equations

\[ \Delta \ln \frac{T}{T_c} + \pi T \sum_{\omega = -\infty}^{\infty} \frac{\Delta - G_S \Phi_S \omega}{|\omega|} = 0. \]  

(4)

In the case of SFIFS tunnel junction in quasi-one-dimensional geometry the boundary conditions at the junction plane \((x = 0)\) read

\[ \xi_F \frac{G^{\prime}_{F,L}}{\omega_L} \frac{\partial}{\partial x} \Phi_{F,L} = \xi_F \frac{G^{\prime}_{F,R}}{\omega_R} \frac{\partial}{\partial x} \Phi_{F,R}, \]  

(5)

\[ \gamma_{BI} \frac{\xi_F G_{FL,R}}{\omega_L} \frac{\partial}{\partial x} \Phi_{FL,R} = \pm G_F R \left( \frac{\Phi_{F,L}}{\omega_R} - \frac{\Phi_{F,R}}{\omega_L} \right), \]  

(6)

with \( \gamma_{BI} = R_N A_I/\rho_F \xi_F, \)

where the indices \( L \) and \( R \) refer to the left- and right-hand side of the junction, respectively, \( R_N \) and \( A_I \) are the normal resistance and the area of FIF interface.

The boundary conditions at the SF interfaces \((x = \pm d_F)\) have the form

\[ \xi_F \frac{G^2_{S,k}}{\omega} \frac{\partial}{\partial x} \Phi_{S,k} = \gamma \frac{G^2_{F,k}}{\omega} \frac{\partial}{\partial x} \Phi_{F,k}, \]  

(7)

\[ \pm \gamma_B \frac{\xi_F G_{F,k}}{\omega} \frac{\partial}{\partial x} \Phi_{F,k} = G_S \left( \frac{\Phi_{F,L}}{\omega_R} - \frac{\Phi_{F,R}}{\omega_L} \right), \]  

(8)

with \( \gamma_B = R_B A_B/\rho_F \xi_F, \) \( \gamma = \rho_S \xi_S/\rho_F \xi_F, \)

where \( R_B \) and \( A_B \) are the resistance and the area of the SF interfaces; \( \rho_{SF}(F) \) is the resistivity of the S (F) layer; \( k = L, R. \) Both of these conditions ensure continuity of the supercurrent.

We will also suppose that due to low transparency of the FIF interface the Josephson current is much smaller that the depairing current of superconducting electrodes so that the suppression of superconductivity in the interior of the electrodes can be neglected and at \( x \to \pm \infty \)

\[ |\Phi_{S,k}| = \Delta_0, \]  

(9)

where \( \Delta_0 \) is the magnitude of bulk order parameter.

The limit of small \( F \) layer thickness. In this limit

\[ d_F \ll \min \left( \xi_F, \sqrt{\frac{D_F}{2H}} \right) \]  

(10)

the gradients in \( 4 \) are small and in the second approximation on \( d_F/\xi_F \) the solution of \( 4 \) has the form

\[ \Phi_{F,k} = A_k + B_k \frac{x}{\xi_F} + \frac{\varphi^*}{2 \pi T \xi^2_F} G_{F,k}, \]  

(11)

\[ G^2_{F,k} = \frac{\varphi^*}{\omega^2} + A_k^2(\omega), \]  

(12)

Integration constants \( \hat{A} \) and \( \hat{B} \) in \( 12 \) can be found from boundary conditions at \( x = 0 \)

\[ \frac{G^2_{F,L}}{\omega_L} B_L = \frac{G^2_{F,R}}{\omega_R} B_R = \frac{G_{FL,R} G_{F,R}}{\gamma_{BI}} \left( \frac{A_R}{\omega_R} - \frac{A_L}{\omega_L} \right), \]  

(13)

and at \( x = \pm d_F \)

\[ A_k = A_{0,k} \mp \gamma_B \frac{G_{F,k}}{\omega} \frac{\Delta_0}{\pi T_c} B_k, \]  

(14)

Expression \( 13 \) valid if \( \gamma_B \ll \gamma_{BI}. \) Substitution of \( 11 \) and \( 13 \) into the boundary condition at \( x = \pm d_F \) leads to

\[ \xi_S \frac{\partial}{\partial x} \Phi_{S,k} = \pm \gamma_M \frac{G_{F,k}}{G^2_{S,k}} \frac{\omega}{\pi T_c} A_k + \gamma \frac{\omega^2}{\omega_c} B_k, \]  

(15)

where \( \gamma_M = \gamma_H/\xi_F \) and reduce boundary problem \((2) – (4))\) to the solution of equations \((2), (4))\) in the S-layers with the boundary conditions \((9), (13))\). At \( H = 0 \) and \( (\gamma_{BI} d_F/\xi_F) \gg 1 \) expression \( 15 \) reduces to the known result for SN bilayer

The linearized Usadel equations. Following RBC approximation we will start with the assumption that the suppression of superconductivity in S layer is weak and the solution of Usadel equations in the superconductor has the form

\[ \Phi_{S,k}(\omega) = \Delta_0 + \Phi_{1,k}, \]  

\[ \Delta = \Delta_0 + \Delta_{1,k}, \]  

\[ G_{S,k} = G_0 + G_{1,k}, \]  

\[ G_0 = \frac{\omega}{\sqrt{\omega^2 + \Delta_0^2}} \]  

\[ G_{1,k} = -G_0 \frac{\Delta_{0,k}^* \Phi_{1,k} + \Delta_{0,k} \Phi^*_{1,k}}{2 \omega^2 + \Delta_0^2}, \]  

(16)

(17)

(18)

where \( \Delta_{0,k} = \Delta_0 \exp \{ \pm i \varphi/2 + iUx/\xi_S \} \), \( \varphi \) is order parameter phase difference across the barrier and the coefficient \( U \) describes the linear growth of phase difference due to the supercurrent in the electrodes. Corrections to \( \Delta_0 \) and \( \Phi_{S,k} \) are supposed to be small

\[ |\Delta_{1,k}| \ll \Delta_0, \]  

\[ |\Phi_{1,k}| \ll \Delta_0. \]  

The approximation is valid if the right hand side of Eq.15 is also small, so that

\[ \xi_S \frac{\partial}{\partial x} \Phi_{1,k} = \Xi_k(\omega), \]  

(19)

\[ \Xi_k(\omega) = \pm \gamma_M \frac{\omega G_{F0,k} A_{0,k}}{\pi T_c G^2_0} + \gamma \frac{\omega^2 G^2_{F0,k} B_k}{\omega_c G^2_0}, \]  

(20)

where \( \partial_k = (G_0 + \varphi \gamma_{BM}/\pi T_c) \), and \( |\Xi(\omega)| \ll \Delta_0 \), From the structure of the linearized Usadel equations and the
boundary conditions (19) it follows that there are first order corrections only to the magnitudes Θ and Δ₁ of functions Φ₁ and Δ₁,k respectively, while the phases of all of these functions coincide with those of Δ₀,k. In this case
\[ Φ₁,k = Θ \exp \left\{ ±i \frac{φ}{2} \right\}, \quad Δ₁,k = Δ₁ \exp \left\{ ±i \frac{φ}{2} \right\} \tag{21} \]
and due to the symmetry of the structure we have
\[ \bar{ω}_R = \bar{ω}_L = \bar{ω}, \quad G_{F₀,k} = G_{F₀}, \quad \varθ_k = \varθ \]
\[ A_{0,k} \frac{G_{F₀}}{G₀} = C₀ \exp \left\{ ±i \frac{φ}{2} \right\}, \quad C₀ = \frac{ωG₀}{ω₀}, \]
\[ Ξ_κ(ω) = \frac{G_{F₀}}{G₀} \left[ ±γ_M \frac{ω}{πT_c} \cos \frac{φ}{2} + i \left( γ_M \frac{ω}{πT_c} + 2γ_G \frac{F₀}{G₀} \right) \sin \frac{φ}{2} \right]. \tag{22} \]
To write (22), we also used the fact that in the first order with respect to $|Ξ(ω)|$ the magnitudes of functions $Φ_{S,k}$ in (14) are equal to $Δ₀$ and that $G_S = G₀$.

Substituting (16), (21) into (2), (3), we arrive at the following boundary condition for Θ and Δ₁
\[ -\xi_S^2 \frac{πT_c}{√ω² + Δ₀²} \frac{∂²}{∂x²} Θ + Θ = Δ₁, \tag{24} \]
\[ Δ₁ \left[ \ln \frac{T}{T_c} + πT \sum_{ω=-∞}^{∞} \frac{1}{|ω|} \right] - πT \sum_{ω=-∞}^{∞} \frac{ωΘG₀}{(ω² + Δ₀²)} = 0, \tag{25} \]
\[ ξ_S \frac{∂}{∂x} Θ(±d_F) = \left[ \text{Re}Ξ_κ(ω) \cos \frac{φ}{2} ± \text{Im}Ξ_κ(ω) \sin \frac{φ}{2} \right], \tag{26} \]
\[ Θ(±∞) = 0. \tag{27} \]
Due to the symmetry of the problem it is enough to solve the equations (24), (27) only in one of the electrodes, namely, for $x ≥ d_F$. Using the equation for Δ₀(T)
\[ \ln \frac{T}{T_c} + πT \sum_{ω=-∞}^{∞} \frac{1}{|ω|} = πT \sum_{ω=-∞}^{∞} \frac{1}{√ω² + Δ₀²} \tag{28} \]
and the symmetry relation Θ(ω) = Θ(-ω) we can rewrite the selfconsistency equation in the form
\[ Δ₁Σ₂ = πT \sum_{ω>0} \frac{πT_cω²}{(ω² + Δ₀²)³/₂} Ξ_κ² \frac{∂²}{∂x²} Θ \tag{29} \]
\[ Σ₂ = πT \sum_{ω>0} \frac{Δ₀²}{(ω² + Δ₀²)³/₂}. \tag{30} \]
The solution of (24), (29) is
\[ Δ₁ = \sum_{Ω>0} δ_Ω \exp(-q_Ω \frac{x - d_F}{ξ_S}), \tag{31} \]
\[ Θ = \sum_{Ω>0} \frac{δ_Ω \sqrt{ω² + Δ₀²}}{√ω² + Δ₀² - πT_cq_Ω²} \exp(-q_Ω \frac{x - d_F}{ξ_S}), \tag{32} \]
where the coefficients $δ_Ω$ and $q_Ω$ satisfy the equation
\[ Σ₂ = πT \sum_{ω>0} \frac{ω²}{(ω² + Δ₀²)³/₂} q_Ω²\frac{πT_c}{q_Ω²} = -\frac{Δ₀P(ϕ, ω)}{√ω² + Δ₀²} \tag{33} \]
and $P(ϕ, ω) = \text{Re}Ξ_R(ω) \cos(φ/2) + \text{Im}Ξ_R(ω) \sin(φ/2)$. Multiplying Eq.(30) on $ω²/(ω² + Δ₀²)³/₂$, summing both sides of this equation on $ω$ and making use of (32), one can transform (33) into the system of equations for the coefficients $δ_Ω$ which yield
\[ δ_Ω = -πT \frac{πT_cΔ₀Ω²q_Ω}{Σ₂(Ω² + Δ₀²)²} Λ(Ω, ϕ), \tag{34} \]
where
\[ Λ(Ω, ϕ) = \left[ γ_M K₁(Ω) + \frac{γ_G}{γ_{BI}} K₂(Ω)(1 - \cos φ) \right] \]
\[ K₁(Ω) = \frac{Ω}{πT_cG₀} \sqrt{\frac{√p² + q² + p}{2}}, \tag{35} \]
\[ K₂(Ω) = \frac{pG₀ + (Hq + pΩ)γ_{BM}/πT_c}{G₀(π² + q²)}, \tag{36} \]
\[ q = 2γ_{BM} \frac{H}{πT_c} \frac{Ω}{γ_{BM}} + G₀, \tag{37} \]
Here $Ω = πT(2m + 1)$ are the Matsubara frequencies.
As a result, the solution of the boundary problem (24), (27) has the form
\[ Δ₁ = -πT \sum_{Ω>0} \frac{πT_cΔ₀Ω²q_Ω}{Σ₂(Ω² + Δ₀²)²} \exp(-q_Ω \frac{x - d_F}{ξ_S}) Λ(Ω, ϕ), \tag{38} \]
\[ \Theta = -\pi T \sum_{\Omega > 0} \frac{\pi T \Delta_0 \Omega^2 q_0 \Lambda(\Omega, \varphi) \exp(-q_0 \frac{\pi \omega}{\Omega})}{\Sigma_2(\Omega^2 + \Delta_0^2)^2 (1 - \pi T \Delta_0^2 G_0 / \omega)}. \] (39)

In particular, at \( x = d_F \) from (38) and (39) we have

\[ \frac{\Theta(d_F)}{A_0} = -\gamma_M \Sigma F_1 - \frac{\gamma}{\gamma_{BI}} \Sigma F_2 (1 - \cos \varphi), \] (40)

\[ \Sigma F_1 = \pi T \sum_{\Omega > 0} \frac{\pi T \Omega^2 q_0 K_1(\Omega)}{\Sigma_2(\Omega^2 + \Delta_0^2)^2 (1 - \pi T \Delta_0^2 G_0 / \omega)}. \] (41)

\[ \Sigma F_2 = \pi T \sum_{\Omega > 0} \frac{\pi T \Omega^2 q_0 K_2(\Omega)}{\Sigma_2(\Omega^2 + \Delta_0^2)^2 (1 - \pi T \Delta_0^2 G_0 / \omega)}. \] (42)

To calculate the sums (41), (42) one needs to know the expression for the coefficients \( q_0 \) which can be in general obtained from numerical solution of Eq. (32). Since the main contribution to the sums (41), (42) comes from large \( \Omega \), the asymptotic behavior of \( q_0 \) at large \( \Omega \) can be used

\[ q_0^2 = \alpha \frac{\Omega^2 + \Delta_0^2}{\pi T_0}, \quad \alpha = 1 - \frac{\pi T^2}{\Omega T_0} \ln \frac{\sqrt{\Omega^2 + \Delta_0^2}}{\pi T}. \] (43)

The developed method is valid if the following condition is fulfilled

\[ (\gamma_M + \frac{\gamma}{\gamma_{BI}}) \max \left\{ 1, \ln \left[ \frac{H^2 + (\pi T_0)^2}{\min \{\gamma_{BM}^2, \gamma_{M}^2\} (\pi T)^2} \right] \right\} < 1, \] (44)

\[ \gamma_B \ll \gamma_{BI}. \]

Therefore for the function \( \Phi_{S,k} \) in Eq. (14) we get

\[ \Phi_{S,k} = \delta_0 + \Theta(d_F) \exp \left\{ \frac{\mp i \varphi}{2} \right\}, \] (45)

and substituting (45) into (13) we finally obtain

\[ A_k = \left[ \delta_0 + \frac{\omega \mu C_0}{\omega} \Theta(d_F) \right] C_0 \exp \left\{ \pm i \varphi / 2 \right\} \mp \frac{2}{\gamma_{BI}} \frac{\bar{\omega} G_0 G_F \Delta_0}{\omega T} \sin \frac{\varphi}{2}, \] (46)

\[ \mu = 1 + G_0 \bar{\omega} \gamma_{BM} / \pi T_0 \]

From the structure of coefficients \( A_{R,L} \) we see that the corrections to the supercurrent across the SFIFS tunnel junction leads not only to the reduction of the critical current of the structure, but also to changes in the \( J_s(\varphi) \) relation.

**The \( J_s(\varphi) \) relation.** Using the the standard expression for the supercurrent, the boundary condition (10) we can write down the supercurrent \( I \) across the SFIFS junction in the form

\[ I = (J_0 + J_{11}) \sin \varphi + J_{12} \sin 2\varphi, \] (48)

where

\[ J_0 = \frac{\pi T}{e R_N} \sum_{\omega = -\infty}^{\infty} \frac{\Delta_0^2 C_0}{(\omega^2 + C_0^2 \Delta_0^2)}, \quad C_0 = \frac{\bar{\omega} G_0}{\omega T}. \] (49)

\[ J_{11} = \frac{-2 \pi T}{e R_N} \sum_{\omega = -\infty}^{\infty} \frac{\Delta_0^2 C_0}{(\omega^2 + C_0^2 \Delta_0^2)^2} \left[ \frac{\gamma_B}{\gamma_{BI}} \frac{\bar{\omega} G_0 \Delta_0}{\omega T} \right], \] (50)

\[ J_{12} = \frac{-\pi T}{e R_N} \sum_{\omega = -\infty}^{\infty} \frac{\Delta_0^2 C_0}{(\omega^2 + C_0^2 \Delta_0^2)^2} \left[ \frac{\gamma_B}{\gamma_{BI}} \frac{G_F \Delta_0}{\omega T} - \frac{\gamma_{BI}}{\gamma_{BM}} \frac{\bar{\omega} G_0}{\omega T} \right]. \] (51)

Expression (49) has been obtained previously in [11]. The \( \varphi \)-independent correction to it, \( J_{11} \), is negative and describes the suppression of \( \sin \varphi \) component of the supercurrent. The first term in Eq. (50) proportional to \( \gamma_M \) takes into account the suppression of superconductivity in S electrodes due to proximity with thin F layer. The last two terms proportional to \( \gamma_{BI}^{-1} \) describe the suppression of superconductivity by the current across the junction. The larger \( \gamma_B \) and \( \gamma \) the weaker is the superconductivity induced into F layer and the stronger is the influence of this effect.

The sign of the second harmonic \( J_{12} \) depends on the relation between \( \gamma_B \) and \( \gamma \). At \( \gamma_B = 0 \) it is positive and \( J(\varphi) \) relation (13) has a maximum at \( \varphi = \varphi_{max} < \pi/2 \). Such a shift was predicted earlier near \( T_c \) for SIS tunnel junctions and is due to the suppression of superconductivity near the barrier by a supercurrent. Increase of \( \gamma_B \) leads to additional phase shifts at both SF interfaces and provides the mechanism for the shift of the \( \varphi_{max} \) into the region \( \varphi > \pi/2 \). As a result, at sufficiently large \( \gamma_B \) the amplitude \( J_{12} \) changes its sign and \( \varphi_{max} \) shifts to \( \varphi > \pi/2 \). Such a competition between suppression by a supercurrent and by proximity effect was first analyzed in the SNS junctions at \( T \approx T_c \). This fact is in the full agreement with the results of numerical calculations summarized in [2].

The physical reason for different signs of \( J_{12} \) can be easily understood if we consider the two cases separately. Suppose first that \( \gamma_B \) is finite. In this case the SFIFS structure may be considered as a system of three Josephson junctions in series as shown schematically in Fig.1. For rough estimates one can assume that the phase \( \chi \) of \( \Phi_{F,k} \) does not depend on \( \omega \). Demanding the equality of...
the currents across FIF and FS interfaces and taking into
to account that $I_C \propto \gamma_B^{-1} \ll I_{C1} \propto \gamma_B^{-1}$ for $\chi$ we will have
$$\chi = \varphi/2 - \frac{I_C}{I_{C1}} \sin 2\chi.$$  

Substituting this $\chi$ into the expression for the supercurrent across FIF interface, we get
$$I = I_C \sin(\varphi - \frac{I_C}{I_{C1}} \sin \varphi) \approx I_C \sin \varphi - \frac{\gamma_B}{\gamma_B} \sin 2\varphi. \quad (53)$$  

Therefore with increasing $\gamma_B$ the phase partly jumps at the FS interfaces leading to a continuous crossover from the Josephson effect lumped at $x = 0$ to the phase drop distributed at $|x| \leq d_F$. In a full agreement with the theory of double barrier devices, this crossover results in appearance of second harmonic in $J_S(\varphi)$ with negative sign which provides maximum $J_S(\varphi)$ achieved at $\varphi \geq \pi/2$.

If $\gamma_B = 0$, the structure is always lumped at $x = 0$ and the main effect is the suppression of superconductivity by supercurrent in the vicinity of FIF interface as shown schematically in Fig. 2. The resulting contribution to the full current is
$$I_\omega \propto \gamma_B^{-1}(\Delta_0 - \xi S \frac{\partial \Theta}{\partial x}) \sin \varphi \propto \gamma_B^{-1} \Delta_0 \left(1 - \frac{\sin^2 \varphi}{2}\right) \sin \varphi. \quad (54)$$  

It follows directly from (54) that the amplitude of the second harmonic is positive.

The competition of the above two mechanisms of $I(\varphi)$ deformation is clearly seen from Eq. (54).

The general expressions (49) - (51) can be simplified in several limiting cases.

In the symmetric SNINS tunnel junctions $H = 0$ in both electrodes and in the first approximation from (49), the earlier result from (13) is reproduced
$$J_0 = \frac{2\pi T C_0}{eR_N} \sum_{\omega > 0} \frac{\Delta_0^2}{(\omega^2 + \Delta_0^2)} \Theta(\omega), \quad (55)$$  

$$\Theta(\omega) = \left(1 + 2G_0 \omega \gamma_{BM}/\pi T_e + (\omega \gamma_{BM}/\pi T_e)^2\right)$$  

while (50) and (51) reduce to
$$J_{11} = -\frac{4\pi T}{eR_N} \left[\gamma_M \Sigma_4 + \frac{\gamma_B}{\gamma_B} \Sigma_5 + \frac{2}{\gamma_B} \Sigma_6\right], \quad (56)$$  

$$J_{12} = -\frac{2\pi T}{eR_N} \left[\frac{\gamma_B}{\gamma_B} \Sigma_7 - \frac{\gamma_B}{\gamma_B} \Sigma_6\right], \quad (57)$$  

where
$$\Sigma_4 = \sum_{\omega > 0} \left(\omega^2 + \Delta_0^2\right) \Theta(\omega), \quad (58)$$  

$$\Sigma_5 = \sum_{\omega > 0} \frac{\Delta_0^2}{(\omega^2 + \Delta_0^2)} \Theta(\omega), \quad (59)$$  

$$\Sigma_6 = \sum_{\omega > 0} \frac{G_0 \Delta_0 \Theta(\omega)}{(\omega^2 + \Delta_0^2) \Theta(\omega)}, \quad (60)$$  

$$\Sigma_7 = \sum_{\omega > 0} \frac{\Delta_0^4}{(\omega^2 + \Delta_0^2) \Theta(\omega)}, \quad (61)$$  

and $G_0 = \omega/\sqrt{\omega^2 + \Delta_0^2}$.

In the limit $\gamma \to 1$, $H, \gamma_M, \gamma_B, \gamma_{BM} \to 0$ the SFIFS structure transforms into SIS tunnel junction. In this case
$$C_0 = 1, \quad A_{pR,L} = [\Delta_0 + \Theta(d_F)] \exp \left\{\pm i\varphi/2\right\}, \quad (62)$$  

$$\Theta(d_F) = -\frac{2}{\gamma_B} \pi T \sum_{\Omega > 0} \frac{\pi T e \Delta_0 \Omega q_{01} \sin^2 \frac{\varphi}{2}}{\Delta_0^2 \Omega (\Omega^2 + \Delta_0^2)(1 - \pi T_e q_{01} G_0/\omega)}, \quad (63)$$  

and for the supercurrent $I$ in the first approximation we have the well known result of Ambegavaokar-Baratoff theory
$$I = \frac{2\pi T}{eR_N} \sum_{\omega > 0} \frac{\Delta_0^2}{\omega^2 + \Delta_0^2} \sin \varphi. \quad (64)$$
Using (52) for $J_{11}$ and $J_{12}$ it is easy to get

$$J_{11} = -\frac{\Delta_0}{eR_N} \Sigma_3, \quad J_{12} = \frac{\Delta_0}{eR_N} \Sigma_3,$$

(65)

$$\Sigma_3 = \frac{4}{\gamma_B} \pi T \sum_{\Omega > 0} \frac{\Delta_0 \Omega^2}{(\Omega^2 + \Delta_0^2)^2} q_{11},$$

(66)

and the full current across the tunnel junctions is

$$I = \frac{\Delta_0}{eR_N} \left[ \frac{\pi}{2} \tanh \frac{\Delta_0}{2T} - 2 \Sigma_3 \right] \sin \phi + \frac{\Delta_0 \Sigma_3}{eR_N} \sin 2\phi.$$

(67)

The critical current achieves at phase difference $\varphi_c$

$$\varphi_c = \frac{\pi}{2} - 4 \Sigma_3 \frac{\pi}{\pi} \tanh^{-1} \frac{\Delta_0}{2T},$$

(68)

and equals to

$$I_c \approx \frac{\Delta_0}{eR_N} \left[ \frac{\pi}{2} \tanh \frac{\Delta_0}{2T} - 2 \Sigma_3 \right],$$

and at $T \to 0$ the $I(\phi)$ simplifies to

$$I_c \approx \left[ \frac{\Delta_0}{eR_N} \frac{\pi}{2} - 1.92 \frac{\pi T_c}{\gamma_B \Delta_0^3/2} \right].$$

(69)

At $T \approx T_c$ Eqs. (65) transform to the result obtained in 13.

Conclusions. In summary, we have studied the current-phase relations $J_S(\phi)$ in SFIFS, SNINS and SIS junctions in the regime when the second harmonic of $J_S(\phi)$ is not small. To solve this problem selfconsistently, we have developed the analytical method for solving the linearized Usadel equations. This solution describes a weak suppression of superconducting state in a superconductor caused by proximity with normal or ferromagnetic material or by a current in composite SN or SF proximity systems. The method is rather general and can be applied to a wide spectrum of proximity problems.

We have demonstrated that the full current across the structure 13 consists of the sum $\sin \phi$ and $\sin 2\phi$ components and have calculated the amplitudes $(J_0 + J_{11})$ and $J_{12}$ of these components. In SIS and SNINS structures the corrections $J_{11}$ and $J_{12}$ to the previously calculated critical current $J_0$ are small. The $J(\phi)$ curve is slightly deformed so that the maximum value of the supercurrent achieved at phase difference $\varphi_c$ which can be smaller or larger $\pi/2$ for positive and negative sign of $J_{12}$ respectively. In SFIFS junctions $J_0 = 0$ at the point of the transition from "0" to "n" state. It means that in this case the calculated values $J_{11}$ and $J_{12}$ determine the $J(\phi)$ curve. Since the amplitudes $J_{11}$ and $J_{12}$ may have comparable magnitude, the $J(\phi)$ measured experimentally can be essentially different from $\sin \phi$. The validity of the developed approach is determined by inequalities 13 and $\gamma_B \ll \gamma_B$. These conditions also determine the validity of rigid boundary conditions in the models describing the properties of SFIFS, SNINS and SIS tunnel junctions.

This work has been supported in part by Russian Ministry of Education and Science, RFBR Grant N 04 0217397-a, INTAS Grant 01-0809, NWO-RFBR cooperation programme 047.016.005 and ESF PiShift programme.

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