Black hole entropy: classical and quantum aspects

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An elementary introduction is given to the problem of black hole entropy as formulated by Bekenstein and Hawking, based on the so-called Laws of Black Hole Mechanics. Wheeler’s ‘It from Bit’ picture is presented as an explanation of plausibility of the Bekenstein-Hawking Area Law. A variant of this picture that takes better account of the symmetries of general relativity is shown to yield corrections to the Area Law that are logarithmic in the horizon area, with a finite, fixed coefficient. The Holographic hypothesis, tacitly assumed in the above considerations, is briefly described and the beginnings of a general proof of the hypothesis is sketched, within an approach to quantum gravitation which is non-perturbative in nature, namely Non-perturbative Quantum General Relativity (also known as Quantum Geometry). The holographic entropy bound is shown to be somewhat tightened due to the corrections obtained earlier. A brief summary of Quantum Geometry approach is included, with a sketch of a demonstration that precisely the log area corrections obtained from the variant of the It from Bit picture adopted earlier emerges for the entropy of generic black holes within this formalism.

I. INTRODUCTION

We begin with the startling fact, following basically from Newton’s law of gravitation, that if all matter (and radiation) on earth were to be squeezed into an ordinary marble of a couple of centimetres in diameter, the gravitational field of this marble would be so strong as to render it invisible (or ‘black’). While this is an interesting speculation, it actually portrays realistic situations. Consider the death of stars after the nuclear fuel that powers them is exhausted; the star dies, often with a spectacular explosion, and the corpse begins to contract due to its own gravitational attraction, in the absence of outward pressures generated by nuclear explosions that existed during its lifetime. If such a star-remnant weighs more than a couple of times the mass of the sun, the internal gravitational attraction supercedes all other repulsive forces (like Pauli degeneracy pressure) to cause it to collapse indefinitely under its own weight. During this gravitational collapse, the star contracts, and its density rises. A stage is reached when the resultant gravitational force is so strong as to trap everything, including light to the stellar surface. The particular size of the star for which this happens is given in terms of its remnant mass and is known as the Schwarzschild radius

\[ r_S = \frac{2GM}{c^2} \]

It is as though the surface of the star at this stage is a one-way membrane - matter and energy can go in through it but nothing can ever come out.

The collapse of the star under its own weight continues beyond this stage, until all the matter (and radiation) has collapsed to a single point! Of course this is a very special point of space, since the density at this point blows up. It is called a ‘singularity’ or a ‘hole’. It is as though all the matter of the dead star fell through the ‘hole’. Thus, a black hole has no structure! It is just pure gravitation devouring anything around, with a special point where all the stuff ends up, and a geometrical (not real) surface acting like a one-way membrane hiding this special singular point from view of external observers.

There is however an error in all of this: from the Newtonian perspective, photons are quite insensitive to gravitational forces. Our above considerations had endowed them with a small mass, but that is not quite right. To mediate long range electromagnetic forces as they do, photons must have strictly zero mass. So what went wrong? The point is that we already know that this Newtonian perspective is itself limited; it was replaced about 90 years ago by Einstein’s General Relativity (GR). According to GR, gravitation is no longer to be thought as a force, but only as a manifestation of curvature of spacetime; free particles, both massive and massless (like photons), in such a curved geometry are bound to follow trajectories that are not straight as in flat (Minkowski) spacetime, but bend around the source of the curvature. Also, given the special relativistic equivalence of mass and energy, it stands to reason that anything that has energy (and momentum, but no rest mass) can equally generate a gravitational field as spacetime curvature. This idea is enshrined in the celebrated Einstein equation of GR: \[ \text{Curvature} \propto \text{energy} - \text{momentum density} \] with Newton’s gravitational constant \( G \) appearing in the proportionality constant.

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How do black holes fit into GR? They fit in very nicely, as exact solutions to Einstein’s beautiful but very complicated equation. They describe spacetimes with curvature, but also possess the property that these spacetimes are singular - the curvature is infinite at a particular point, consistent with the density blowing up as discussed above. Secondly, the ‘trapped’ surfaces beyond which nothing escapes appear as null surfaces in these exact solutions - they are known as event horizons. Finally, there is this amazing morphology of black holes in GR: they are completely classified by three parameters - mass, electric charge and angular momentum. The astrophysicist S. Chandrasekhar summarised these properties rather eloquently when he described black holes as ‘the most perfect macroscopic objects there are in the universe. ... the simplest as well.’

II. LAWS OF BLACK HOLE MECHANICS AND ENTROPY

This simple picture of black holes changed dramatically in the early and mid-seventies. It all started with Hawking’s discovery within GR that the area of the event horizon of a black hole cannot decrease in any physical process. In other words, if two black holes of horizon areas $A_1$ and $A_2$ coalesced into a single black hole, albeit with release of a lot of gravitational radiation, the resultant black hole cannot have an area less than $A_1 + A_2$. This was followed by the discovery, also within classical GR, of two other laws, the so-called Zeroth and First Laws of Black Hole Mechanics. The Zeroth law states that the geometrical quantity known as surface gravity (something like the Newtonian ‘acceleration due to the gravity’) remains constant on the horizon. The First law, for the Schwarzschild black hole (which is completely characterised by mass $M$ with charge $Q = 0$ and angular momentum $J = 0$), states that

$$dM = \frac{\kappa}{2\pi} dA,$$

where the surface gravity $\kappa \equiv 1/4M$ on the horizon. For black holes with all three parameters non-zero (the Kerr-Newman family), the first law takes the form

$$dM = \frac{\kappa}{2\pi} dA + \Phi dQ + \vec{\Omega} \cdot d\vec{L},$$

where, $\kappa \equiv (r_+ - r_-)/4A$, $\Phi \equiv 4\pi Q r_+ / A$, $\vec{\Omega} \equiv 4\pi \vec{L} / M A_{\text{hor}}$. Hawking’s earlier discovery of horizon area non-decrease is called the Second Law of Black Hole Mechanics.

The analogy of these three laws with the Zeroth, First and Second laws of ordinary thermodynamics is hard to miss, with the surface gravity being analogous to temperature, horizon area with entropy and mass with internal energy. Coupled with the usual description of the electrostatic potential $\Phi$ and the angular velocity $\Omega$ on the horizon, the $\kappa$-the second and third terms in (2) are together like the usual ‘$P dV$’ term of the First law of ordinary thermodynamics. But is this all there is to these laws of black hole mechanics?

Bekenstein was the first to go beyond mere analogizing and propose that black holes actually possess entropy - a truly bold hypothesis. He asserted that the horizon area was a measure of how much entropy a black hole could have, in sharp contrast to standard thermodynamic notions where entropy is supposed to be a function of volume. Drawing upon work of Christodolou and others, Bekenstein concluded that black hole entropy must be proportional to the horizon area: $S_{\text{bh}} \propto A$. In order that $S_{\text{bh}}$ be dimensionless, the proportionality constant must have dimensions of inverse squared length. However, if this relation is to be universally valid for all black holes, this constant should be independent of black hole parameters. Also, it should not depend on interaction constants of non-gravitational interactions. Thus, the only available fundamental length in this case is the Planck length, $\ell_P \equiv (G\hbar/c^3)^{1/2} \sim 10^{-33}$ cm. Thus,

$$S_{\text{bh}} = \eta \frac{A}{\ell_P^2},$$

where $\eta$ is a dimensionless constant of $O(1)$. Now, the Planck length is the distance from a point particle, for instance, at which its gravitational effect characterised by its Schwarzschild radius, becomes as significant as its quantum effect characterised by its Compton wavelength:

$$\ell_P \sim r_S \sim \lambda_C,$$

where, $r_S \equiv 2GM/c^2$, $\lambda_C \equiv h/Mc$; elimination of the mass $M$ from the second (approximate) inequality in (4) and substituting back in either $r_S$ or $\lambda_C$ yields the standard expression for $\ell_P$ given above. In this sense, this length typifies the length scale at which quantum gravitational effects can no longer be ignored. It follows that, black hole
entropy must therefore have origins that are actually quantum gravitational in nature. Indeed, an object of pure gravity with no structure classically should not possess any entropy at all. The only way it could possess entropy is through microstates that appear in a truly quantum description.

Bekenstein’s proposal of black hole entropy thus gives us a deep reason to worry about the programme of quantization of gravitation - a programme that has historically been steeped with difficulties and abandoned as hopeless by most practitioners of GR, until relatively recently. But it also raises fundamental issues:

• Why should black holes at all have entropy?
• In what sense is $T_{bh} \equiv \kappa/2\pi$ the temperature of the black hole?
• What becomes of the standard Second law of thermodynamics in presence of black holes?
• What are the microstates whose counting would yield the area law for black hole entropy?

We analyze these issues one by one. A gas in a container has a very large number of molecules in random motion. A microscopic description of this system in terms of canonical coordinates and momenta of each particle is actually impossible. Thus, a statistical description, in terms of average macroscopic variables like temperature, pressure, etc. is the best one can have of such a system. This means that one has less than complete information about the system, and this lack of information manifests as entropy. In case of black holes, this lack of information is contained in our lack of information about the very nature of gravitational collapse. Thus, the parameters characterising black holes in GR actually do not specify individual black holes; rather they, like the temperature and pressure of a gas, are average parameters describing equivalence classes of black holes, each of which may have collapsed from a very different star via a very different process.

The temperature $T_{bh}$ is not the temperature of the horizon of the black hole in the sense that a freely falling observer will sense it as she crosses the horizon. At the horizon, the redshift factor vanishes, so the observer detects no temperature at all. The notion of black hole temperature is made unambiguous if we imagine placing the black hole in a background of black body radiation in equilibrium at a temperature $T < T_{bh}$. In his celebrated work, Hawking showed that, for an observer located at infinity with a vanishing ambient temperature, a black hole actually radiates in a Planckian spectrum like a black body at a temperature $T_{bh}$! The mean number of particles emitted at a frequency $\omega$ is given by the Planckian formula

$$n_{\omega} = \frac{|t_{\omega}|^2}{\exp(h\omega/T_{bh}) - 1},$$

where, $t_{\omega}$ is the absorption coefficient and we have set the Boltzmann constant $k_B = 1$. The larger the mass of the black hole, the smaller is this equilibrium (or Hawking) temperature $T_{bh}$, so that for most stellar black holes, this temperature $T_{bh} \ll 2.7 \text{deg.K}$, so that Hawking radiation from such black holes is swamped by the cosmic microwave background. In fact, these black holes absorb rather than emit, radiation. One other outcome of Hawking’s work is fixing the constant $\eta = \frac{1}{4}$ in (3); this law, $S_{BH} = A/4\ell_P^2$, is called the Bekenstein-Hawking Area Law (BHAL).

As for the status of the second law of thermodynamics in presence of black holes, it is true that when matter falls across the event horizon into a black hole, the entropy of the matter is lost. One might think of this as a decrease in the entropy of the universe. However, as Bekenstein pointed out, when matter falls into a black hole, the mass of the black hole, and hence its area, increases. The hypothesis of black hole entropy then says that the entropy of the black hole must increase as a consequence, by at least the same amount as the entropy lost by the the part of the universe outside the event horizon. Thus, in presence of black holes, the Second law of thermodynamics is modified into the statement that the total entropy, defined as the sum of the entropy outside the event horizon and the black hole entropy, can never decrease in any physical process,

$$\delta(S_{\text{ext}} + S_{bh}) \geq 0.$$

This relation is called the Generalized Second Law of thermodynamics and its enunciation marks the beginning of the subject known as Black Hole Thermodynamics.

III. MICROSTATES

A. Generalities

We now come to the issue of microstates. As mentioned earlier, black hole entropy originates from microstates that must be essentially quantum gravitational in nature. This means that one must consider at least the kinematical
aspects of a quantum theory of gravitation. Since entropy entails counting the degrees of freedom of the theory rather than studying the dynamics of these degrees of freedom, a priori one may hope that a kinematical approach may work, especially since there is no complete theory of quantum gravitation yet at our disposal.

Before subscribing to a particular proposal for a quantum GR, one may make some progress in terms of very general pictures of what a black hole may look like in quantum gravity. A crucial role is played by the Planck length $\ell_P$, which is like $\hbar$ in the quantum theory of spacetime. Thus, the spacetime continuum may degenerate, at such a length scale, to some sort of a lattice structure with adjacent lattice sites separated by this length scale. However, recall that classical GR has a huge symmetry - those under coordinate diffeomorphisms in spacetime. Quantum GR should least retain at least some aspects of this symmetry, and from purely kinematical considerations, this might be taken to imply that there is no restriction on the kind of lattice structure one should confine to.

Furthermore, we are interested in a canonical picture, i.e., with what happens at a given instant of time, and not the whole of spacetime. To this effect, it is enough to deal with a foliation of a spacetime containing a black hole, by a three dimensional spacelike hypersurface. Such a foliation of the event horizon, which recall is a null hypersurface which behaves as a one-way membrane, can be taken to be some (preferred) spacelike two dimensional spheres. For what follows, this 2-sphere will play the role of the horizon on a spatial slice.

Now consider the same object within the very loose lattice structure considered above. We can think of a two dimensional ‘floating’ (as opposed to a rigid) lattice basically covering the sphere. Since the area of a tiny ‘plaquette’ of this lattice may be taken to be Planck area $\ell_P^2$, it stands to reason that the ratio of the macroscopic area of the event horizon to the area of an elementary plaquette $A/\ell_P^2 \gg 1$. This latter inequality defines our notion of a macroscopic black hole; our treatment in this review will focus on such black holes, and not to those very interesting but difficult cases where this number is $O(1)$.

### B. It from Bit

Let us now place, following \cite{6}, binary variables (‘bits’) on the lattice sites (or equivalently, at the centre of the plaquettes). Then the number of such variables $p = \xi A/\ell_P^2 \gg 1$, where $\xi = O(1)$. Without any loss of generality, $p$ can be taken to be an even integer. Now, we can think of the two values of each binary variable as characterising two quantum states, so that the size of the ‘Hilbert space’ of states on the (latticized) horizon is $\mathcal{N}_{bh} = 2^p$. The entropy corresponding to this system is defined as $S_{bh} = \log \mathcal{N}_{bh}$; choosing the constant $\xi = (4 \log 2)^{-1}$, we obtain

$$S_{bh} = \frac{A}{\ell_P^2} = S_{BH}.$$

The generality of the above scenario makes it appealing vis-a-vis a quantum theory of black holes in particular and of quantum gravity in general. There is however one crucial aspect of any quantum approach to black hole physics which seems to have been missed in the above, – the aspect of symmetry. Indeed, the mere random distribution of spin 1/2 (binary) variables on the lattice which approximates the black hole horizon, without regard to possible symmetries, possibly leads to a far bigger space of states than the physical Hilbert space, and hence to an overcounting of the number of the degrees of freedom, i.e., a larger entropy.

### C. The physical Hilbert (sub)space

But what is the most plausible symmetry that one can impose on states so as to identify the physical subspace? The elementary variables are binary or spin 1/2 variables which can be considered to be standard spin 1/2 variables under spatial rotations (more precisely $SU(2)$ doublets). Recall now that at every point on a curved spacetime one can erect a local Lorentz frame where the basic variables can be subjected to a local Lorentz transformation. Here, of course, we are interested in a spatial slice of the curved spacetime, and so the transformation of interest is a ‘local spatial rotation’. Thus, the symmetry that one would like to impose on the degrees of freedom obtained so far would be invariance under these local spatial rotations in three dimensions. However, since one is dealing with black holes of very large area, this amounts to considering ‘global’ or ‘rigid’ spatial rotations or $SU(2)$ transformations. On very general grounds then, the most natural symmetry of the physical subspace must be this $SU(2)$.

One is thus led to a symmetry criterion which defines the physical Hilbert space $\mathcal{H}_S$ of horizon states contributing to black hole entropy: $\mathcal{H}_S$ consists of states that are compositions of elementary $SU(2)$ doublet states with vanishing total spin ($SU(2)$ singlets). Observe that this criterion has no allusions whatsoever to any specific proposal for a quantum theory of gravitation. Nor does it involve any gauge redundancies (or any other infinite dimensional symmetry like conformal invariance) at this point. It is the most natural choice for the symmetry of physical horizon states simply
because in the ‘It from bit’ picture, the basic variables are spin 1/2 variables. Later on we shall show however that this symmetry arises very naturally in the Non-perturbative Quantum GR approach known also as Quantum Geometry. It will emerge from that approach that horizon states of large macroscopic black holes are best described in terms of spin 1/2 variables at the punctures of a punctured two-sphere which represents (a spatial slice of) the event horizon.

The criterion of $SU(2)$ invariance leads to a simple way of counting the dimensionality of the physical Hilbert space $\mathcal{H}_S$. For $p$ variables, this number is given by

$$\dim \mathcal{H}_S \equiv N(p) = \binom{p}{p/2} - \binom{p}{p/2 - 1}. \quad (8)$$

There is a simple intuitive way to understand the result embodied in (8). This formula counts the number of ways of making $SU(2)$ singlets from $p$ spin 1/2 representations. The first term corresponds to the number of states with net $J_z$ quantum number $m = 0$ constructed by placing $m = \pm 1/2$ on the punctures. However, this term by itself overcounts the number of $SU(2)$ singlet states, because even non-singlet states (with net integral spin, for $p$ is an even integer) have a net $m = 0$ sector. Beside having a sector with total $m = 0$, states with net integer spin have, of course, a sector with overall $m = \pm 1$ as well. The second term basically eliminates these non-singlet states with $m = 0$, by counting the number of states with net $m = \pm 1$ constructed from $m = \pm 1/2$ on the $p$ sites. The difference then is the net number of $SU(2)$ singlet states that represents the dimensionality of $\mathcal{H}_S$.

It may be pointed out that the first term in (8) also has another interpretation. It counts the number of ways binary variables corresponding to spin-up and spin-down can be placed on the sites to yield a vanishing total spin. Alternatively, one can think of the binary variables as unit positive and negative $U$ then corresponds to the dimensionality of the Hilbert space of $U(1)$ invariant states. As already shown in (8), this corresponds to a binomial rather than a random distribution of binary variables.

In the limit of very large $p$, one can evaluate the factorials in (8) using the Stirling approximation. One obtains

$$N(p) \approx \frac{2^p}{p^{\frac{3}{2}}}. \quad (9)$$

Clearly, the dimensionality of the physical Hilbert space is smaller than what one had earlier, as would be an obvious consequence of imposing $SU(2)$ symmetry. Using the relation between $p$ and the classical horizon area $A_S$ discussed in the last section, with the constant $\xi$ chosen to take the same value as in that section, (8) can be shown to lead to the following formula for black hole entropy,

$$S_{bh} \equiv \log N(p) \approx \frac{A}{4\ell_p^2} - \frac{3}{2} \log \left( \frac{A}{4\ell_p^2} \right) + \text{const.} + O(A^{-1}). \quad (10)$$

The logarithmic correction to the BHAL is not unexpected if we think of $S_{bh}(A)$ as a power series for large $A$ with $\frac{1}{4}A$ as the leading term; indeed, various approaches to computation of black hole entropy (like the Euclidean path integral [4], Non-perturbative Quantum GR [11], [12], boundary conformal field theory [12], and so on) have been used, and a general result like

$$S_{bh}(A) = \frac{A}{4\ell_p^2} + C \log \left( \frac{A}{4\ell_p^2} \right), \quad (11)$$

has been found, with various values of $C$, both positive and negative. In some of the perturbative approaches, there is an added constant $C' \sim \log(A)$ where $A$ is a length cut-off needed to yield a finite result for $S_{bh}$ [13]. This is quite in contrast to our result (10) where $S_{bh}$ is intrinsically finite. Note also that according to the Second Law of Black Hole Mechanics, if two black holes coalesce, the minimum area of the resultant black hole is the sum of the two horizon areas. For such a coalescence, it is easy to see that, for $C > 0$,

$$S_{bh}(A_1 + A_2) < S_{bh}(A_1) + S_{bh}(A_2). \quad (12)$$

Assuming isolated eternal black holes which coalesce adiabatically with no emission of gravitational waves, this property is perhaps not too desirable from the point of view of the Second Law of Thermodynamics. From this point of view also our result $C = -\frac{3}{2}$ appears more preferable. This is precisely the result that was obtained earlier from Non-perturbative Quantum GR (also called Quantum Geometry) [10] on the basis of incipient contributions in ref. [11].

Even when the resultant black hole has a horizon area larger than the sum of the areas, this preference for $C < 0$ seems to hold, although in a slightly weaker form. Also, from the theory of isolated horizons which incorporates radiation present in the vicinity of the horizon without crossing it, this result seems to have a greater appeal than those others with $C > 0$. 

5
Having identified the kinematical quantum states characterising a black hole horizon, the question that immediately comes to mind is whether there are other states that describe black hole physics. Although the ‘It from bit’ picture tends to imply that the entire information lies with the horizon states, this has been more sharply articulated in the so-called Holographic hypothesis [14]. According to this hypothesis, the horizon states exhaust the Hilbert space of a black hole, encoding the entire information of gravitationally collapsed matter in terms of macroscopic observables like the horizon area.

While a rigorous proof of this hypothesis is not available yet for asymptotically flat spacetimes\(^1\), the beginnings of a general proof should first demonstrate the primacy of quantum gravitational states associated with some spacetime boundary rather than the quantum states describing bulk spacetime. Let us see heuristically how this may come about.

Assume that the Hilbert space \( \mathcal{H} \) of quantum GR in the presence of an inner boundary (and the usual outer boundaries at asymptotic null infinity) can be decomposed into states describing the geometry of ‘bulk’ spacetime and the boundary,

\[
\mathcal{H} \sim \mathcal{H}_{\text{bulk}} \otimes \mathcal{H}_{\text{bdy}} .
\]  

(13)

The canonical partition function \( Z(\beta) \) can be written formally as

\[
Z(\beta) = Tr \exp\{ -\beta \hat{H} \} ,
\]

(14)

where, \( \hat{H} \) is the full Hamiltonian operator of quantum gravitation. Now, we know that, in the canonical description of classical GR [16]

\[
\mathcal{H} = \mathcal{H}_{\text{bulk}} + \mathcal{H}_{\text{bdy}} ,
\]

(15)

where,

\[
\mathcal{H}_{\text{bulk}} \approx 0 ,
\]

(16)

being a sum of first class constraints generating general coordinate transformations and internal (local Lorentz) gauge transformations on the basic canonical variables (metric or tetrad, connections) of GR. The boundary Hamiltonian \( \mathcal{H}_{\text{bdy}} \) would, in the case of asymptotically flat spacetime, give the ADM mass etc., and so does not vanish.

Let us assume that the decomposition of the Hamiltonian (15) and the constraint relation (16) hold for the full quantum gravitational operator Hamiltonian, i.e.,

\[
\hat{H}_{\text{bulk}} |\Psi_{\text{bulk}}\rangle = 0 ,
\]

(17)

where, \( |\Psi_{\text{bulk}}\rangle \in \mathcal{H}_{\text{bulk}} \). The partition function (14) can now be written

\[
Z(\beta) = Tr_{\text{bulk}} \exp\{ -\beta \hat{H}_{\text{bulk}} \} \cdot Tr_{\text{bdy}} \exp\{ -\beta \hat{H}_{\text{bdy}} \} .
\equiv Z_{\text{bulk}}(\beta) \cdot Z_{\text{bdy}}(\beta) .
\]

(18)

Observe that \( Z_{\text{bulk}} \) can be rewritten

\[
Z_{\text{bulk}}(\beta) = \sum_{\Psi_{\text{bulk}}} \langle \Psi_{\text{bulk}} | \exp\{ -\beta \hat{H}_{\text{bulk}} \} | \Psi_{\text{bulk}} \rangle = \dim \mathcal{H}_{\text{bulk}} ,
\]

(19)

using the operator constraint eq. (17). While the dimensionality of \( \mathcal{H}_{\text{bulk}} \) can indeed be infinity, the important point is that it is a constant independent of the inverse temperature \( \beta \). One thus has

\[
Z(\beta) = \dim \mathcal{H}_{\text{bulk}} \cdot Z_{\text{bdy}}(\beta) .
\]

(20)

\(^1\)For anti-de Sitter spacetimes the so-called adS-CFT Correspondence conjecture [13] makes this hypothesis plausible, although the conjecture itself warrants a proof.
It follows that the entire non-trivial dependence on $\beta$ (and thermodynamic implications thereof) of the partition function lies in the ‘boundary’ part of the partition function; the bulk part merely contributes an overall constant scale factor (an additive constant in the free energy) which is not of much relevance thermodynamically.

Notice that the behaviour depicted in eq. (20) extends to the situation with matter as well, since even in that case, without making any approximations, one expects the quantum constraint (Wheeler-de Witt equation) (17) to hold. So, within the tenets of non-perturbative quantum GR (or quantum geometry) in the presence of an inner boundary like a black hole horizon, non-trivial information appears to remain confined to the boundary rather than the bulk. The situation changes dramatically when $H_{\text{bulk}}$ is approximated by assuming a fixed classical background and the quantum degrees of freedom are assumed to be small perturbative fluctuations about this background, like in a theory of gravitons. The full Wheeler-de Witt equation (17) is then replaced by an inhomogeneous equation which exhibits only the linearized symmetries at the quantum level to leading order. Under such conditions the primacy of the boundary states vis-a-vis physical information is no longer obvious. This is the situation, for instance, in (perturbative) string theory which is basically a theory of gravitons. The bulk partition function in this case is quite non-trivial. The plausibility argument given above must therefore be relegated to the status of an unproven hypothesis, in order that it can have physical applications, like the calculation of black hole entropy. In contrast, a relation like (21) appears to demystify to a large extent why black hole entropy must be a function of horizon area rather than some bulk property.

Let us now return to the issue of black hole entropy. The above discussion of holography seems to justify focusing on the surface degrees of freedom alone in the It from Bit picture and the subsequent Physical State Criterion. Thus, black hole entropy can be taken to represent the maximal possible entropy of a spacetime whose spatial slice has a boundary that coincides with the intersection of this spatial slice with the horizon. Now, it can be shown [8] that eq. (21) actually translates into a bound on black hole entropy, given by

$$S_{\text{max}} = \log \left( \exp \frac{S_{\text{BH}}}{S_{\text{BH}}^{3/2}} \right).$$  \hspace{1cm} (21)

Thus, it follows that all 3-spaces with boundary have an entropy bounded from above by (21). That this is extremely plausible follows from the following argument, based on *reductio ad absurdum* [17]: we assume, for simplicity that the spatial slice of the boundary of an asymptotically flat spacetime has the topology of a 2-sphere on which is induced a spherically symmetric 2-metric. Let this spacetime contain an object whose entropy exceeds the entropy bound given in eq. (21). Certainly, such a spacetime cannot have a black hole horizon as a boundary, since then, its entropy would have been subject to (21). But, in that case, its energy should be less than that of a black hole which has the 2-sphere as its horizon. Let us now add energy to the system, so that it does transform adiabatically into a black hole with the said horizon, but without affecting the entropy of the exterior. But we have already seen above that a black hole with such a horizon must obey the bound (21); it follows that the starting assumption of the system having an entropy exceeding the bound (21) must be incorrect.

Thus, we have indeed obtained an upper bound on the entropy of a large class of spacetimes. Notice that this bound *tightens* the semiclassical Bekenstein bound [4], which is of course expected because of its quantum kinematical underpinning. We now turn to a brief, mostly qualitative, exposé of aspects of Non-perturbative Quantum GR (also called Quantum Geometry) which is a candidate theory that provides such underpinning.

**V. ENTROPY FROM QUANTUM GEOMETRY**

Quantum Geometry (QGeo) is a framework for non-perturbative canonical quantization of four dimensional general relativity, consistent with all symmetries of the latter. These symmetries include general coordinate transformations and local Lorentz transformations; within GR these symmetries are generated by first class constraints just as in Maxwell electrodynamics the Gauss law constraint generates (time-independent) gauge transformations which are symmetries of the theory. In the quantum theory, one expects that the Hilbert space will be spanned by states that are annihilated by the constraints, now appearing as quantum mechanical operators. Thus, one could attempt to find the Hilbert space by solving these operator constraints. In practice, this turns out to be a formidable task and all parts of this programme have not been completed yet for QGeo. One finds a *kinematical* Hilbert space whose elements solve some of the constraints, but not all. As we have already mentioned, this kinematical Hilbert space turns out to be sufficient for the calculation of black hole entropy.

What sort of wave-functions span the kinematical Hilbert space? We know for example that the configuration space of classical scalar field theory in Minkowski spacetime consists of square integrable function(al)s of smooth scalar fields. However, the situation changes dramatically for the quantum configuration space where *non-smooth* Dirac delta
function-like distributions must also be included in addition to smooth scalar fields. Similarly, the configuration space of classical GR consists of smooth metrics, but the wave functions of QGeo are function(al)s of metrics that are not necessarily smooth. Thus, one has to consider metrics (and hence curvatures) which are actually generalized functions of spacetime coordinates (i.e., like Dirac delta functions), being non-vanishing only over certain subspaces of the classical continuum.

It turns out that a good basis for the kinematical Hilbert space of QGeo is the so-called Spin Network basis. Typically, a spin network is a three dimensional ‘floating’ lattice embedded in the three dimensional spacelike hypersurface that slices spacetime. It consists of links each carrying a spin $j$ which can take integral as well as half-odd integral values. These links meet at vertices where one inserts certain tensors which are invariant under local spatial rotations. Specifying the spins on all links and the invariant tensors at all vertices gives a complete specification of the spin network state. No links are left sticking out of the network, so that the net spin of the state vanishes. Also, the lengths of links are quite immaterial for all physical considerations. These properties ensure that the state is invariant under local spatial rotations and time independent coordinate transformations respectively. However, whether it is invariant under time dependent coordinate transformations is not yet clear. States constructed out of such networks form a complete orthonormal basis for the kinematical Hilbert space. Now, from a classical geometrical standpoint, the curvature is non-vanishing only on the network - it is non-smooth in that sense.

What are the observables in this framework? These are geometrical in nature, like length of a curve, area of a two-surface, volume of a three-surface etc. The spin network basis turns out to be an eigenbasis for these observables. For instance, a two-surface $S$ with classical area $A$ (w.r.t. some classical metric) has a well-defined area operator $\hat{A}$ associated with it, such that the spin network basis is an eigenbasis of this operator, with $A = 1\text{ Planck area}$ of the classical area are given by $A = \frac{1}{\sqrt{\pi}} \gamma \ell_\text{P}^2 \sum_{i=1}^{P} \sqrt{j_i(j_i + 1)}$, where, $\gamma$ is the so-called Barbero-Immirzi parameter characterising inequivalent quantizations of the same classical theory, something akin to the $\theta$ parameter in Yang-Mills theory.

In this framework, it is not yet possible to describe black holes as quantum states that solve all operator constraints. Instead, the idea is to treat the classical event horizon as an inner boundary of spacetime which is canonically quantized using the spin network basis. The boundary conditions imposed on this inner boundary of course correspond strictly to those on the black hole horizon. Very detailed analyses of these boundary conditions have been performed $[19]$. These boundary conditions require that the gravitational action be augmented by the action of a Chern-Simons theory living on the horizon. Boundary states of the Chern-Simons theory constitute precisely the microstates that contribute to the entropy. These states correspond to conformal blocks of the two-dimensional Wess-Zumino model that lives on the spatial slice of the horizon, which is a 2-sphere of area $A$. The dimensionality of the boundary Hilbert space has been calculated thus $[10]$ by counting the number of conformal blocks of two-dimensional $SU(2)_k$ Wess-Zumino model, for arbitrary level $k$ and number of punctures $p$ on the 2-sphere. The Chern Simons coupling constant $k \sim A$. We shall show, from the formula for the number of conformal blocks specialized to macroscopic black holes characterized by large $k$ and $p$ $[10]$, that eq. (10) ensues.

Let us start with the formula for the number of conformal blocks of two-dimensional $SU(2)_k$ Wess-Zumino model that lives on the punctured 2-sphere. For a set of punctures $P$ with spins $\{j_1, j_2, \ldots, j_p\}$ at punctures $\{1, 2, \ldots, p\}$, this number is given by $[10]$, for large $k \rightarrow \infty$

$$N^P = \sum_{j_1}^{j_1} \cdots \sum_{j_p}^{j_p} \sum_{m_1 = -j_1}^{j_1} \cdots \sum_{m_p = -j_p}^{j_p} \left[ \delta (\sum_{n=1}^{p} m_n), 0 \right] - \frac{1}{2} \delta (\sum_{n=1}^{p} m_n), 1 \right] - \frac{1}{2} \delta (\sum_{n=1}^{p} m_n), -1 \right].$$

We consider a large fixed classical area of the horizon, and ask what the largest value of number of punctures $p$ should be, so as to be consistent with $[23]$: this is clearly obtained when the spin at each puncture assumes its lowest nontrivial value of $1/2$, so that, the relevant number of punctures $p$ is given by

$$p = \frac{A}{4\ell_\text{P}^2 \gamma_0},$$

where, $\gamma_0 = 1/\pi\sqrt{3}$. We are of course interested in the case of very large $p$ appropriate to a macroscopic black hole.
Now, with the spins at all punctures set to $1/2$, the number of states for this set of punctures $P$ is given by

$$N^P = \sum_{m_1=-1/2}^{1/2} \cdots \sum_{m_p=-1/2}^{1/2} \left[ \delta(\sum_{n=1}^p m_n), 0 - \frac{1}{2} \delta(\sum_{n=1}^p m_n), 1 - \frac{1}{2} \delta(\sum_{n=1}^p m_n), -1 \right]$$

The summations can now be easily performed, with the result given precisely by the rhs of eq. (10).

This establishes on a microscopic basis the validity of the extension of the ‘It from bit’ picture proposed by us earlier. The central role played by variables in the doublet representation of a (global) $SU(2)$ group, which we identified with the binary variables on the lattice approximating the horizon, is now clarified. This completes the derivation of our physical space criterion and the ensuing entropy formula and holographic bound on the basis of a quantum kinematical formulation.

VI. CONCLUSION

Quantum Geometry as a non-perturbative approach to quantum GR has had a measure of success in providing an ab initio way to calculate the entropy of generic non-rotating four dimensional black holes. The results obtained within this formalism establish on firm foundations heuristic results that can be obtained using somewhat more intuitive arguments. That all the seemingly disparate elements of the formalism converge to these results points to their inherent robustness.

It would be most gratifying to have a complete rigorous proof of the so-called holographic hypothesis, so germane to all considerations of entropy of spacetimes with horizons. Hopefully, we have been able to provide a reasonable intuitive argument as to how such a proof might go. A test of the approach given here would be the derivation of logarithmic corrections to black hole entropy that we have found, from a path integral framework, using only the boundary action and fluctuations around its classical value (which is known to yield the BHAL in the Euclidean approach).

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