A PARALLEL ITERATIVE PROCEDURE FOR WEAK GALERKIN
METHODS FOR SECOND ORDER ELLIPTIC PROBLEMS

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Abstract. A parallelizable iterative procedure based on domain decomposition is presented
and analyzed for weak Galerkin finite element methods for second order elliptic equations. The
convergence analysis is established for the decomposition of the domain into individual elements
associated to the weak Galerkin methods or into larger subdomains. A series of numerical tests are
illustrated to verify the theory developed in this paper.

Key words. weak Galerkin, finite element methods, elliptic equation, parallelizable iterative,
domain decomposition.

AMS subject classifications. Primary, 65N30, 65N15, 65N12, 74N20; Secondary, 35B45,
35J50, 35J35

1. Introduction. This paper is concerned with an iterative procedure related
to domain decomposition techniques based on the use of subdomains as small as individual
elements for weak Galerkin (WG) methods for second order elliptic equations
in $\mathbb{R}^d (d = 2, 3)$. For simplicity, we consider the second order elliptic problem with
Dirichlet boundary condition

$$
-\nabla \cdot (a \nabla u) + cu = f, \quad \text{in } \Omega \subset \mathbb{R}^d,
$$

$$
u = g, \quad \text{on } \partial \Omega,
$$

where $d = 2, 3$. Assume the coefficients $a(x)$ and $c(x)$ satisfy

$$0 < a_0 \leq a(x) \leq a_1 < \infty, \quad 0 \leq c(x) \leq c_1 < \infty,$$

and are sufficiently regular so that the existence and uniqueness of a solution of (1.1)
in $H^s(\Omega)$ hold true for some $s > 1$ for reasonable $f$ and $g$. A weak formulation for
the model problem (1.1) reads as follows: Find $u \in H^1(\Omega)$ such that $u = g$ on $\partial \Omega$
satisfying

$$(a \nabla u, \nabla v) + (cu, v) = (f, v) \quad \forall v \in H^1_0(\Omega).$$

The WG finite element method is emerging as a new and efficient numerical
method for solving PDEs. The idea of WG method was first proposed by Wang and Ye
for solving second order elliptic equations in 2011 [25]. This method was subsequently
developed for various PDEs, see [11, 12, 14, 16, 23, 19, 22, 20, 18, 23, 25, 26, 13, 27, 28, 19, 21, 24]. To our best knowledge, there have not been any iterative algorithms designed for the WG methods along the line of domain decompositions. Due to the large size of the computational problem, it is necessary and crucial to design efficient and parallelizable iterative algorithms for the WG scheme. Our iterative procedure is motivated by Despres [5] for a Helmholtz problem and another Helmholtz-like problem related to Maxwell’s equations by Despres [6, 7]. It should be noted that the convergence in [5, 6, 7] were established for the differential problems in strong form where numerical results were presented to validate the iterative procedures for the discrete case. Douglas et al. [8] introduced a parallel iterative procedure for the second order partial differential equations by using the mixed finite element methods. The goal of this paper is to extend the result of Douglas into weak Galerkin finite element methods. In particular, based on the features of weak Galerkin methods, the iterative procedure developed in this paper can be very naturally and easily implemented on a massively parallel computer by assigning each subdomain to its own processor. In addition, the work is based on the hybridized weak Galerkin framework of [17] extended to non-overlapping subdomains.

The paper is organized as follows. In Section 2, we briefly review the weak differential operators and their discrete analogies. In Section 3 we describe the WG method for the model problem (1.1). In Section 4 we introduce domain decompositions and derive a hybridized formulation for the WG method. In Section 5 we present a parallel iterative procedure for the WG finite element method. In Section 6 we establish a convergence analysis for the parallel iterative scheme. Finally in Section 7 we report several numerical results to verify our convergence theory.

2. Weak Differential Operators. The primary differential operator in the weak formulation [1, 2] for the second order elliptic problem [1.1] is the gradient operator $\nabla$, for which a discrete weak version has been introduced in [26]. For completeness, let us briefly review the definition as follows.

Let $T$ be a polygonal or polyhedral domain with boundary $\partial T$. A weak function on $T$ refers to $v = \{v_0, v_b\}$ where $v_0 \in L^2(T)$ and $v_b \in L^2(\partial T)$ represent the values of $v$ in the interior and on the boundary of $T$ respectively. Note that $v_b$ may not necessarily be the trace of $v_0$ on $\partial T$. Denote by $W(T)$ the local space of weak functions on $T$; i.e.,

$$W(T) = \{v = \{v_0, v_b\} : v_0 \in L^2(T), v_b \in L^2(\partial T)\}.$$ 

The weak gradient of $v \in W(T)$, denoted by $\nabla_w v$, is defined as a linear functional on $[H^1(T)]^d$ such that

$$(\nabla_w v, w)_T = -(v_0, \nabla \cdot w)_T + (v_b, w \cdot n)_{\partial T}, \quad \forall w \in [H^1(T)]^d.$$ 

Denote by $P_r(T)$ the space of all polynomials on $T$ with total degree $r$ and/or less. A discrete version of $\nabla_w v$ for $v \in W(T)$, denoted by $\nabla_{w,r,T} v$, is defined as a unique polynomial vector in $[P_r(T)]^d$ satisfying

$$(\nabla_{w,r,T} v, w)_T = -(v_0, \nabla \cdot w)_T + (v_b, w \cdot n)_{\partial T}, \quad \forall w \in [P_r(T)]^d.$$ 

3. Weak Galerkin Algorithm. Let $T_h$ be a finite element partition of the domain $\Omega$ consisting of polygons or polyhedra that are shape-regular [26]. Denote by
the set of all edges or flat faces in $\mathcal{T}_h$ and $E^0_h = E_h \setminus \partial \Omega$ the set of all interior edges or flat faces. Denote by $h_T$ the meshsize of $T \in \mathcal{T}_h$ and $h = \max_{T \in \mathcal{T}_h} h_T$ the meshsize for the partition $\mathcal{T}_h$.

For any given integer $k \geq 1$, denote by $W_k(T)$ the local discrete space of the weak functions given by

$$(3.1) \quad W_k(T) = \{ \{ v_0, v_b \} : v_0 \in P_k(T), v_b \in P_{k-1}(e), e \subset \partial T \}.$$ 

Patching the local discrete space $W_k(T)$ with a single value on the element interface yields the global finite element space; i.e.,

$$W_h = \{ v = \{ v_0, v_b \} : v|_T \in W_k(T), v_b|_{\partial T_i \cap \partial T_j} = v_b|_{\partial T_j \cap \partial T_i}, T, T_i, T_j \in \mathcal{T}_h \},$$

where $v_b|_{\partial T_i \cap \partial T_j}$ is the value of $v_b$ on $\partial T_i \cap \partial T_j$ as seen from the element $T_i$. Denote by $W^g_h$ and $W^0_h$ the subspaces of $W_h$ with non-homogeneous and homogeneous boundary values; i.e.,

$$W^g_h = \{ \{ v_0, v_b \} \in W_h : v_b|_e = Q_b g, e \subset \partial \Omega \},$$

$$W^0_h = \{ \{ v_0, v_b \} \in W_h : v_b|_e = 0, e \subset \partial \Omega \},$$

where $Q_b$ is the $L^2$ projection onto the space $P_{k-1}(e)$.

For $v \in W_h$, denote by $\nabla_w v$ the discrete weak action $\nabla_{w,k-1,T} v$ computed by using (2.1) on each element $T$; i.e.,

$$(3.2) \quad (\nabla_w v)|_T = \nabla_{w,k-1,T}(v|_T).$$

Next, we introduce the following bilinear forms in $W_h \times W_h$:

$$(3.3) \quad s(u, v) = \sum_{T \in \mathcal{T}_h} s_T(u, v),$$

$$(3.4) \quad a(u, v) = \sum_{T \in \mathcal{T}_h} a_T(u, v),$$

where

$$s_T(u, v) = h_T^{-1}(Q_b u_0 - u_b, Q_b v_0 - v_b)_{\partial T},$$

$$a_T(u, v) = (a \nabla_w u, \nabla_w v)_T + (cu_0, v_0)_T.$$

The weak Galerkin finite element scheme for the second order problem (1.1) based on the variational formulation (1.2) can be stated as follows:

**Weak Galerkin Algorithm 3.1.** Find $\bar{u}_h \in W^g_h$ such that

$$(3.5) \quad s(\bar{u}_h, v) + a(\bar{u}_h, v) = (f, v_0) \quad \forall v \in W^0_h.$$ 

**4. Weak Galerkin based on Domain Decompositions.** Let $\{ \Omega_j : j = 1, \cdots, M \}$ be a partition of $\Omega$ such that

$$(4.1) \quad \Omega = \bigcup_{j=1}^M \Omega_j, \quad \Omega_j \cap \Omega_k = \emptyset, j \neq k.$$
In practice, with the exception of a few $\Omega_j$'s along $\partial \Omega$, each $\Omega_j$ is convex with a piecewise-smooth boundary. We introduce

$$\Gamma = \partial \Omega, \quad \Gamma_j = \Gamma \cap \partial \Omega_j, \quad \Gamma_{jk} = \Gamma_{kj} = \partial \Omega_j \cap \partial \Omega_k.$$  

Assume that the edges/faces of the elements in $T_h$ align with the interface $I_f = \bigcup_{j,k=1}^{M} I_{jk}$. The partition $T_h$ can be grouped into $M$ sets of elements denoted by $T_h^j = T_h \cap \Omega_j$, so that each $T_h^j$ provides a finite element partition for the subdomain $\Omega_j$ for $j = 1, \ldots, M$. The intersection of the partitions $T_h^j$ and $T_h^k$ also introduces a finite element partition for the interface $\Gamma_{jk}$, denoted by $\Gamma_{jk}^h$.

Let us introduce the Lagrange multipliers on the edge $\Gamma_{jk}$ as seen from $\Omega_j$ as follows

$$\Lambda_{jk} = \{ \lambda_{jk} : \lambda_{jk} \in P_{k-1}(\Gamma_{jk} \cap \Omega_j), \Gamma_{jk} \neq \emptyset \} \text{ is of dimension } d-1, \forall j, k = 1, \ldots, M \}.$$  

Note that there are two copies of $P_{k-1}(e)$ assigned to the interface $\Gamma_{jk}$ as seen from $\Omega_j$ and $\Omega_k$ respectively; i.e., $\Lambda_{jk}$ and $\Lambda_{kj}$. Note that the Lagrangian space $\Lambda_{jk}$ is defined only on $d - 1$ dimensional interfaces $\Gamma_{jk} \subset \mathbb{R}^{d-1}$.

Define, for $j = 1, \ldots, M$, the finite element space on each subdomain $\Omega_j$:

$$W_h(\Omega_j) = \{ v|_{\Omega_j} : v \in W_h \},$$
$$W_h^0(\Omega_j) = \{ v \in W_h(\Omega_j) : v|_{\Gamma_j} = 0 \}.$$

The weak Galerkin finite element method (3.5) restricted in the subdomain $\Omega_j$ ($j = 1, \ldots, M$) is as follows: Find $u_j = \{ u_{j,0}, u_{j,b} \} \in W_h(\Omega_j)$, such that $u_{j,b} = Q_h g$ on $\Gamma_j$, $\lambda_{jk} \in \Lambda_{jk}$, $j, k = 1, \ldots, M$,

$$\begin{align*}
(a \nabla u_j, \nabla v_j)_{\Omega_j} + s_j(u_j, v_j) &= \sum_{k=1}^{M} (\lambda_{jk}, v_{j,b})_{\Gamma_{jk}} \\
&+ (cu_{j,0}, v_{j,0})_{\Omega_j}, \quad \forall v_j \in W_h^0(\Omega_j),
\end{align*} \tag{4.2}$$

where $(\cdot, \cdot)_{\Omega_j} = \sum_{T \in T_h^j} (\cdot, T)$, $s_j(u_j, v_j) = \sum_{T \in T_h^j} s_T(u, v)$, and $[u_b]_{\Gamma_{jk}}$ is the jump of $u_b$ on $\Gamma_{jk}$ defined as follows:

$$[u_b]_{\Gamma_{jk}} = u_{b,j}|_{\Gamma_{jk} \cap \Omega_j} - u_{b,k}|_{\Gamma_{kj} \cap \Omega_k} \tag{4.3}$$

where $u_{b,j}|_{\Gamma_{jk} \cap \Omega_j}$ and $u_{b,k}|_{\Gamma_{kj} \cap \Omega_k}$ represent the values of $u_b$ on $\Gamma_{jk}$ as seen from $\Omega_j$ and $\Omega_k$ respectively.

**Lemma 4.1.** Let $u_j = \{ u_{j,0}, u_{j,b} \} \in W_h(\Omega_j)$ and $\lambda_{jk} \in \Lambda_{jk}$ be the solution of the algorithm (4.2). Then, we have $[u_b]_{\Gamma_{jk}} = 0$ for $j, k = 1, \ldots, M$, so that $u_h|_{\Omega_j} = u_j$ is a function in the finite element space $W_h$. Furthermore, this function $u_h$ satisfies the WG scheme (3.3). In other words, we have $u_h \equiv \bar{u}_h$.  

Proof. By letting \( \mu = [u_0]_{\Gamma_{jk}} \in \Lambda_{jk} \) and then using the second equation in (4.2) we arrive at

\[
0 = \sum_{j,k=1}^{M} \int_{\Gamma_{jk}} [u_0]^2 \, ds,
\]

which implies that \([u_0]_{\Gamma_{jk}} = 0 \) for \( j, k = 1, \ldots, M \).

Next, by restricting \( v \in W^0_{h} \) in the first equation in (4.2) and using the fact that \( \lambda_{jk} + \lambda_{kj} = 0 \) we have

\[
\sum_{j,k=1}^{M} (\lambda_{jk}, v_{jk})_{\Gamma_{jk}} = 0,
\]

which leads to

\[
a(u_h, v) + s(u_h, v) = (f, v_0) \quad \forall v \in W^0_{h}.
\]

It follows from \( u_{j,b} = Q_{b}g \) on \( \Gamma_j \) that \( u_h \in W^0_{h} \), which together with the last equation shows that \( u_h \) is a solution of the WG scheme (3.5). Finally, from the solution uniqueness for (3.5) we have \( u_h = \bar{u}_h \). This completes the proof. \( \square \)

5. A Parallel Iterative Scheme. A parallel iterative scheme for the weak Galerkin finite element method can be designed by using the equivalent numerical scheme (4.2). The motivation of the iterative procedure comes from the observation that, in (4.2), the following consistency conditions

\begin{align}
(5.1) \quad & u_{j,b} = u_{k,b}, \quad \text{on } \Gamma_{jk}, \\
& \lambda_{jk} = -\lambda_{kj}, \quad \text{on } \Gamma_{jk},
\end{align}

are equivalent to

\begin{align}
(5.2) \quad & \beta u_{j,b} + \lambda_{jk} = \beta u_{k,b} - \lambda_{kj}, \quad \text{on } \Gamma_{jk}, \\
& \beta u_{k,b} + \lambda_{kj} = \beta u_{j,b} - \lambda_{jk}, \quad \text{on } \Gamma_{kj},
\end{align}

for any non-zero function \( \beta \) on \( \bigcup_{j,k=1}^{M} \Gamma_{jk} \). This gives rise to

\[
(5.3) \quad \langle \lambda_{jk}, v_{j,b} \rangle_{\Gamma_{jk}} = \langle \beta(u_{k,b} - u_{j,b}) - \lambda_{kj}, v_{j,b} \rangle_{\Gamma_{jk}},
\]

where \( v_{j} = \{v_{j,0}, v_{j,b}\} \in W_h(\Omega_j) \). The use of the equation (5.3) in the numerical scheme (4.2) is a critical step in the design of the following parallel iterative procedure.

Starting from any initial guess \( u_{j}^{(0)} = \{u_{j,0}^{(0)}, u_{j,b}^{(0)}\} \in W_h(\Omega_j) \) with \( u_{j,b}^{(0)} = Q_{b}g \) on \( \Gamma_{j} \), \( \lambda_{jk}^{(0)} \in \Lambda_{jk} \) and \( \lambda_{kj}^{(0)} \in \Lambda_{kj} \), we solve for \( u_{j}^{(n)} = \{u_{j,0}^{(n)}, u_{j,b}^{(n)}\} \in W_h(\Omega_j) \), \( \lambda_{jk}^{(n)} \in \Lambda_{jk} \), and \( \lambda_{kj}^{(n)} \in \Lambda_{kj} \) such that \( u_{j,b}^{(n)} = Q_{b}g \) on \( \Gamma_{j} \) and satisfying the following system of linear equations

\[
(5.4) \quad \begin{cases}
(a \nabla u_{j}^{(n)}, \nabla v_{j})_{\Omega_j} + s_{j}(u_{j}^{(n)}, v_{j}) + \sum_{k=1}^{M} \langle \beta u_{j,b}^{(n)}, v_{j,b} \rangle_{\Gamma_{jk}} + \{cu_{j,0}^{(n)}, v_{j,0}\}_{\Omega_j} \\
= \sum_{k=1}^{M} \langle \beta u_{j,b}^{(n-1)} - \lambda_{kj}^{(n-1)}, v_{j,b} \rangle_{\Gamma_{jk}} + (f, v_{j,0})_{\Omega_j} \quad \forall v_{j} \in W^0_{h}(\Omega_j), \\
\lambda_{jk}^{(n)} = \beta(u_{k,b}^{(n-1)} - u_{j,b}^{(n)}) - \lambda_{kj}^{(n-1)}.
\end{cases}
\]
6. Convergence. Let us establish the convergence for the iteration procedure defined in (5.4). To this end, for $e = \{e_j : e_j \in W_h(\Omega_j)\}$ and $\mu = \{\mu_{jk} : \mu_{jk} \in \Lambda_{jk}\}$, denote by

$$(a\nabla_w e, \nabla_w e) = \sum_{j=1}^{M} (a\nabla_w e_j, \nabla_w e_j)_{\Omega_j},$$

$$(ce_0, e_0) = \sum_{j=1}^{M} (ce_{j,0}, e_{j,0})_{\Omega_j},$$

$$s(e, e) = \sum_{j=1}^{M} s_j(e_j, e_j).$$

Moreover, denote by

$$E(\{e, \mu\}) = \sum_{j=1}^{M} \beta^2 |e_{j,b}|_{0, B_j}^2 + \sum_{j,k=1}^{M} |\mu_{jk}|_{0, \Gamma_{jk}}^2$$

$$+ 2\beta \{(a\nabla_w e, \nabla_w e) + s(e, e) + (ce_0, e_0)\},$$

where $B_j = \partial \Omega_j \setminus \Gamma_j$.

Let $\{u_j, \lambda_{jk}\}$ be the solution of the domain-decomposition-based numerical scheme [4.2]. We define the error functions:

$$(6.1) \quad e^{(n)}_j = u_j - u^{(n)}_j, \quad \mu_{jk} = \lambda_{jk} - \lambda^{(n)}_{jk}, \quad \mu_{kj} = \lambda_{kj} - \lambda^{(n)}_{kj}.$$  

The error equations for the iterative procedure (5.4) can be written in the form:

$$\left\{ \begin{align*}
(a\nabla_w e^{(n)}_j, \nabla_w v_j)_{\Omega_j} &+ s_j(e^{(n)}_j, v_j) - \sum_{k=1}^{M} (\mu^{(n)}_{jk}, v_{j,b})_{\Gamma_{jk}} \\
+ (ce^{(n)}_{j,0}, v_{j,0})_{\Omega_j} &= 0, \quad \forall v_j \in W_h(\Omega_j), \\
\mu^{(n)}_{jk} &= \beta (e^{(n-1)}_{j,b} - e^{(n)}_{j,b}) - \mu^{(n-1)}_{kj},
\end{align*} \right.$$  

(6.2)

**Lemma 6.1.** Let $E^{(n)} = E(\{e^{(n)}, \mu^{(n)}\})$. The following identities hold true:

$$E^{(n-1)} - E^{(n)} = 4\beta \{(a\nabla_w e^{(n-1)}, \nabla_w e^{(n-1)}) + s(e^{(n-1)}, e^{(n-1)}) + (ce_0^{(n-1)}, ce_0^{(n-1)})\},$$

(6.3)

$$E^{(n)} = \sum_{k=1}^{M} |\beta e^{(n)}_{j,0}|_{0, B_j}^2 + \mu^{(n)}_{jk}|_{0, \Gamma_{jk}}^2$$

$$= \beta^2 |e^{(n)}_{j,b}|_{0, B_j}^2 + \sum_{k=1}^{M} \mu^{(n)}_{jk}|_{0, \Gamma_{jk}}^2$$

$$+ 2\beta \{(a\nabla_w e^{(n)}_j, \nabla_w e^{(n)}_j)_{\Omega_j} + s_j(e^{(n)}_j, e^{(n)}_j) + (ce^{(n)}_{j,0}, ce^{(n)}_{j,0})_{\Omega_j}\},$$

(6.4)
Proof. Note that \( e_{j,b}^{(n)} = 0 \) on \( \Gamma_j \). By letting \( v_j = e_j^{(n)} \) in the first equation of (6.2) we obtain

\[
(a \nabla w e_j^{(n)}, \nabla w e_j^{(n)})_{\Omega_j} + s_j (e_j^{(n)}, e_j^{(n)}) - \sum_{k=1}^{M} (\mu_{jk}^{(n)}, e_j^{(n)})_{\Gamma_{jk}} + (ce_{j,0}^{(n)}, e_j^{(n)})_{\Omega_j} = 0.
\]

From (6.5) we have

\[
E^{(n)} = \sum_{k=1}^{M} |\beta e_{j,b}^{(n)} + \mu_{jk}^{(n)}|^2 \|_{0, \Gamma_{jk}}
\]

\[
= \beta^2 |e_{j,b}^{(n)}|^2 + \sum_{k=1}^{M} |\mu_{jk}^{(n)}|^2 \|_{0, \Gamma_{jk}} + 2\beta \sum_{k=1}^{M} (e_{j,b}^{(n)}, \mu_{jk}^{(n)})_{\Gamma_{jk}}
\]

\[
= \beta^2 |e_{j,b}^{(n)}|^2 + \sum_{k=1}^{M} |\mu_{jk}^{(n)}|^2 \|_{0, \Gamma_{jk}}
\]

\[
+ 2\beta \{(a \nabla w e_j^{(n)}, \nabla w e_j^{(n)})_{\Omega_j} + s_j (e_j^{(n)}, e_j^{(n)}) + (ce_{j,0}^{(n)}, e_j^{(n)})_{\Omega_j}\},
\]

which verifies the identity (6.4).

Next, from the second equation in (6.2) we have

\[
E^{(n)} = \sum_{j=1}^{M} \sum_{k=1}^{M} |\beta e_{j,b}^{(n)} + \mu_{jk}^{(n)}|^2 \|_{0, \Gamma_{jk}}
\]

\[
= \sum_{j=1}^{M} \sum_{k=1}^{M} |\beta e_{k,b}^{(n-1)} - \mu_{kj}^{(n-1)}|^2 \|_{0, \Gamma_{kj}}
\]

\[
= E^{(n-1)} - 4\beta \sum_{k=1}^{M} \{(a \nabla w e_k^{(n-1)}, \nabla w e_k^{(n-1)})_{\Omega_k} + s_k (e_k^{(n-1)}, e_k^{(n-1)})
\]

\[
+ (ce_{k,0}^{(n-1)}, e_{k,0}^{(n-1)})_{\Omega_k}\},
\]

which leads to the identity (6.3). □

Theorem 6.2. Let \( \{u_j^{(n)}, \lambda_{jk}^{(n)}\} \) be the solution of the iterative scheme (5.4) and \( \{e_j^{(n)}, \mu_{jk}^{(n)}\} \) be the error functions defined in (6.1) for the numerical scheme (4.2). For any \( \beta > 0 \), the following convergence holds true:

\[
u_j^{(n)} \to u_j,
\]

\[
\lambda_{jk}^{(n)} \to \lambda_{jk}
\]

as \( n \to \infty \).

Proof. Note that, from Lemma 6.1, \( \{E^{(n)}\} \) is a decreasing sequence of nonnegative numbers. Thus, we have

\[
\sum_{n=0}^{\infty} \{(a \nabla w e^{(n)}, \nabla w e^{(n)}) + s(e^{(n)}, e^{(n)}) + (ce_{0}^{(n)}, e_{0}^{(n)})\} < \infty.
\]
Note that \( a > 0 \). If \( c > 0 \), it is easy to obtain \( e_0^{(n)} \to 0 \) on each subdomain \( \Omega_j \), which, using \( s(e^{(n)}, c^{(n)}) \to 0 \), gives \( e_b^{(n)} \to 0 \) and further \( e^{(n)} \to 0 \) as \( n \to \infty \). For the case of \( c \geq 0 \), the above argument would not go through so that new approaches are necessary. The rest of the proof assumes the general case of \( c \geq 0 \).

For \( \mu^{(n)} = \{\mu_j^{(n)}\} \), we construct \( u^* = \{v_j^*\} \), where \( v_j^* \in W_h(\Omega_j) \) assumes the value of \( \mu_j^{(n)} \) on \( \Gamma_{jk} \) and zero otherwise. It is easy to show that

\[
\sum_{j,k} \mu_{j,k}^{(n)} \leq \sum_{j,k} \mu_{j,k}^{(n)} \leq C h^{-1} \sum_{j,k} \mu_{j,k}^{(n)} \|
\]

Substituting \( v_j \) by \( v_j^* \) in (6.2) yields

\[
(a \nabla_w e_j^{(n)}, \nabla_w v_j^*)_{\Omega_j} + s_j(e_j^{(n)}, v_j^*) - \sum_{k=1}^M \mu_{j,k}^{(n)} \|
\]

Hence,

\[
\sum_{j,k} \|
\sum_{k=1}^M \|
\sum_{j,k} \|
\sum_{j,k} \|
\]

Using the estimate (6.10) in the above inequality gives

\[
\sum_{j,k} \|
\sum_{j,k} \|
\sum_{j,k} \|
\sum_{j,k} \|
\]

It follows that \( \sum_{j,k=1}^M \|
\to 0 \) as \( n \to \infty \), which asserts the convergence for the Lagrangian multiplier \( \lambda_{jk}^{(n)} \).

To prove the convergence of \( u_j^{(n)} \), we note from (6.4) that \( \{e_j^{(n)}\} \) is a bounded sequence in \( L^2(B_j) \), and hence from Lemma 6.3 \( \{e_j^{(n)}\} \) is a bounded sequence with respect to the norm \( ||\cdot||_{1, \Omega_j} \). It follows that \( \{e_j^{(n)}\} \) must have a convergent subsequence. Without loss of generality, we may assume that \( \{e_j^{(n)}\} \) is convergent so that

\[
e_j^{(n)} \to e_j^* \quad \text{as} \quad n \to \infty.
\]

By passing to the limit of \( n \to \infty \) and using the face that \( \mu_j^{(n)} \to 0 \), from (6.2) we obtain the following equations:

\[
(a \nabla_w e_j^*, \nabla_w v_j)_{\Omega_j} + s_j(e_j^*, v_j) + (ce_j^*, v_j)_0 = 0, \quad \forall v_j \in W_h(\Omega_j),
\]

which implies that \( e_j^* \equiv 0 \) for \( j = 1, \cdots, M \). This completes the proof of the theorem. \( \Box \)
The rest of this sections is concerned with two technical results that support the convergence analysis for the parallel iterative procedure.

**Lemma 6.3.** There exists $C_1 > 0$ and $C_2 > 0$ such that
\[
C_1((\nabla v_0, \nabla v_0)_{\Omega_j} + s_j(v,v)) \leq (\nabla_w v, \nabla_w v)_{\Omega_j} + s_j(v,v)
\]
\[
\leq C_2((\nabla v_0, \nabla v_0) + s_j(v,v))
\]
for any $v \in W_h(\Omega_j)$.

**Proof.** For each $T \in \mathcal{T}^h_j$, using (2.1) and the usual integration by parts yields
\[
(\nabla_w v, w)_T = -(v_0, \nabla \cdot w)_T + \langle v_b, w \cdot n \rangle_{\partial T}
\]
\[
= (\nabla v_0, w)_T + \langle v_b - Q_nv_0, w \cdot n \rangle_{\partial T}.
\]
Now from the Cauchy-Schwarz inequality and the trace inequality we obtain
\[
\|\nabla v\|_T \leq \|\nabla v_0\|_T \|w\|_T + \|Q_nv_0 - v_b\|_{\partial T} \|w \cdot n\|_{\partial T}
\]
\[
\leq C_2 \|\nabla v_0\|_T \|w\|_T + \|Q_nv_0 - v_b\|_{\partial T} h^{-\frac{1}{2}} \|w\|_T
\]
\[
\leq C_2 (\|\nabla v_0\|_T + h^{-\frac{1}{2}} \|Q_nv_0 - v_b\|_{\partial T}).
\]
Summing over all $T \in \mathcal{T}^h_j$ gives rise to the upper-bound estimate of $(\nabla_w v, \nabla_w v)_{\Omega_j} + s_j(v,v)$. The lower-bound estimate of $(\nabla_w v, \nabla_w v)_{\Omega_j} + s_j(v,v)$ can be established analogously. This completes the proof of the lemma. \( \square \)

For $v_j = \{v_{j,0}, v_{j,b}\} \in W_h(\Omega_j)$, we define a semi-norm by setting
\[
(6.13) \quad \|v_j\|_{1,\Omega_j}^2 = \sum_{T \in \mathcal{T}^h_j} (a\nabla v_{j,0}, \nabla v_{j,0})_T + s_j(v_j, v_j) + \|v_{j,b}\|_{\partial \Omega_j}^2.
\]

**Lemma 6.4.** The semi-norm $\|\cdot\|_1,\Omega_j$ defined in (6.13) is a norm in $W_h(\Omega_j)$.  

**Proof.** It suffices to verify the positivity property for $\|\cdot\|_1,\Omega_j$. To this end, assume $\|v_j\|_1,\Omega_j = 0$ for a weak function $v = \{v_{j,0}, v_{j,b}\} \in W_h(\Omega_j)$. It follows that $\nabla v_{j,0} = 0$ on each $T \in \mathcal{T}_h$, $Q_b v_{j,0} = v_{j,b}$ on each $\partial T$, and $v_{j,b} = 0$ on $\partial \Omega_j$. Therefore, we have $v_{j,0} = const$ on each $T \in \mathcal{T}^h_j$. Using $Q_b v_{j,0} = v_{j,b}$ on $\partial T$, we have $v_{j,0} = const$ in the subdomain $\Omega_j$. Using $v_{j,b} = 0$ on $\partial \Omega_j$ and $Q_b v_{j,0} = v_{j,b}$ on $\partial T$, we obtain $v_{j,0} = 0$ in $\Omega_j$. Again from $Q_b v_{j,0} = v_{j,b}$ on $\partial T$ we have $v_{j,b} = 0$ in $\Omega_j$. This completes the proof of the lemma. \( \square \)

**7. Numerical Experiments.** In this section, we shall report some numerical results to demonstrate the performance of the iterative procedure for the weak Galerkin finite element method (3.5) for the second order elliptic model problem (1.1).

**7.1. Test Example 1.** The configuration of the test is set up as follows: the coefficients are $a(x,y) = 2 - x(1-x)$ and $c = 1$; the exact solution is
\[
(7.1) \quad u(x,y) = 2^6 x^2 (1-x)^2 y^2 (1-y)^2;
\]
and the domain is $\Omega = (0, 1)^2$. Note that this corresponds to a homogeneous Dirichlet boundary value problem. The triangular grids shown as in Figures 7.1 and 7.2 are employed in the numerical tests. The 4-subdomain iterations and 16-subdomain iterations are computed respectively. In Tables 7.1-7.6, we list the computational results for the $\{P_k, P_{k-1}\}$ weak Galerkin finite element defined in (3.1) when the weak gradient is discretized by $[P_{k-1}(T)]^2$ defined in (3.2). Note that $Q_hu = \{Q_0u, Q_hu\}$ where $Q_0$ is the element-wise $L^2$ projection to space $P_k(T)$ and $Q_h$ is the edge-wise $L^2$ projection to $P_{k-1}(e)$. In the computation, the iteration stops when the iterative error reaches the truncation error.

![Fig. 7.1. The level 2, 3 and 4 grids for the 4-subdomain computation.](image)

![Fig. 7.2. The level 2, 3 and 4 grids for the 16-subdomain computation.](image)

In Table 7.1, we choose $\beta = 8$ and the $\{P_1, P_0\}$ WG finite element (3.1). We can see from Table 7.1 that we have an order two convergence in the $L^2$-norm and an order one convergence in the energy norm. We list the number of domain-decomposition iterations when the domain is subdivided into 4 subdomains and 16 subdomains respectively. In theory, the number of iterations may increase on higher level grids. However, the number of iterations appears to be steady when using 4 subdomains and 16 subdomains.

In Table 7.2, we employ the $\{P_2, P_1\}$ WG finite element (3.1) and $\beta = 8$. We can see from Table 7.2 that we have an order three convergence in the $L^2$-norm and an order two convergence in the energy norm. In addition, the number of iterations needed for the 16-subdomain iteration is slightly higher than that of the 4-subdomain iteration.

In Table 7.3, $\beta = 8$ and $\{P_3, P_2\}$ weak Galerkin finite element are taken for the 4-subdomain and 16-subdomain iterations respectively. It can be seen from Table 7.3 that an order four convergence in the $L^2$-norm and an order three convergence in
Table 7.1
Error profile of \( \{P_1, P_0\} \) solutions for (7.1) on Figures 7.1–7.2 grids.

| Grid | \( \| Q_0 u - u_0 \|_0 \) rate | \( \| \nabla w(Q_h u - u_h) \|_0 \) rate | # iteration |
|------|----------------------------------|----------------------------------|------------|
| The \( \{P_1, P_0\} \) WG element (3.1), by 4-subdomain iteration. |
| 1    | 0.211E+00 0.0                    | 0.614E-01 0.0                    | 6          |
| 2    | 0.684E-01 1.6                    | 0.181E+00 0.0                    | 7          |
| 3    | 0.178E-01 1.9                    | 0.110E+00 0.7                    | 9          |
| 4    | 0.448E-02 2.0                    | 0.579E-01 0.9                    | 11         |
| 5    | 0.112E-02 2.0                    | 0.293E-01 1.0                    | 11         |
| 6    | 0.280E-03 2.0                    | 0.147E-01 1.0                    | 13         |
| 7    | 0.692E-04 2.0                    | 0.734E-02 1.0                    | 13         |
| The \( \{P_1, P_0\} \) WG element (3.1), by 16-subdomain iteration. |
| 1    | 0.211E+00 0.0                    | 0.614E-01 0.0                    | 6          |
| 2    | 0.684E-01 1.6                    | 0.181E+00 0.0                    | 7          |
| 3    | 0.178E-01 1.9                    | 0.110E+00 0.7                    | 9          |
| 4    | 0.448E-02 2.0                    | 0.579E-01 0.9                    | 11         |
| 5    | 0.112E-02 2.0                    | 0.293E-01 1.0                    | 11         |
| 6    | 0.280E-03 2.0                    | 0.147E-01 1.0                    | 13         |
| 7    | 0.692E-04 2.0                    | 0.734E-02 1.0                    | 13         |

Table 7.2
Error profile of \( \{P_2, P_1\} \) solutions for (7.1) on Figures 7.1–7.2 grids.

| Grid | \( \| Q_0 u - u_0 \|_0 \) rate | \( \| \nabla w(Q_h u - u_h) \|_0 \) rate | # iteration |
|------|----------------------------------|----------------------------------|------------|
| The \( \{P_2, P_1\} \) WG element (3.1), by 4-subdomain iteration. |
| 1    | 0.106E+00 0.0                    | 0.110E+00 0.0                    | 6          |
| 2    | 0.261E-01 2.7                    | 0.351E-01 1.7                    | 9          |
| 3    | 0.327E-04 3.0                    | 0.708E-03 2.0                    | 17         |
| 4    | 0.411E-05 3.0                    | 0.183E-03 2.0                    | 20         |
| The \( \{P_2, P_1\} \) WG element (3.1), by 16-subdomain iteration. |
| 1    | 0.106E+00 0.0                    | 0.110E+00 0.0                    | 6          |
| 2    | 0.161E-01 2.7                    | 0.352E-01 1.6                    | 8          |
| 3    | 0.207E-02 3.0                    | 0.103E-01 1.8                    | 10         |
| 4    | 0.262E-03 3.0                    | 0.275E-02 1.9                    | 14         |
| 5    | 0.327E-04 3.0                    | 0.709E-03 2.0                    | 22         |
| 6    | 0.410E-05 3.0                    | 0.182E-03 2.0                    | 25         |

The numerical results for the \( \{P_4, P_3\}, \{P_5, P_4\} \) and \( \{P_6, P_5\} \) WG finite element solutions by the 4-subdomain and 16-subdomain iterations are respectively listed in Tables 7.3, 7.4 and 7.5 with corresponding \( \beta = 32, \beta = 19 \) and \( \beta = 32 \). In all these computations, we have observed the optimal order of convergence in the \( L^2 \)-norm and the

the energy norm are observed. Table 7.3 shows that the number of iterations for the 16-subdomain iteration is somewhat higher than that of the 4-subdomain iteration. Note that the two iterations are the same as there are only 4 squares on the first level grid.
Table 7.3
Error profile of \( \{P_3, P_2\} \) solutions for (7.1) on Figures 7.1–7.2 grids.

| Grid | \( \|Q_0 u - u_0\|_0 \) rate | \( \|\nabla_w(Q_h u - u_h)\|_0 \) rate | \# iteration |
|------|-------------------|-------------------|---------|
| The \( \{P_3, P_2\} \) WG element (3.1), by 4-subdomain iteration. | | | |
| 1 | 0.366E-01 0.0 | 0.279E-01 0.0 | 10 |
| 2 | 0.236E-02 4.0 | 0.632E-02 2.1 | 12 |
| 3 | 0.151E-03 4.0 | 0.947E-03 2.7 | 16 |
| 4 | 0.947E-05 4.0 | 0.126E-03 2.9 | 25 |
| 5 | 0.592E-06 4.0 | 0.162E-04 3.0 | 44 |

| Grid | \( \|Q_0 u - u_0\|_0 \) rate | \( \|\nabla_w(Q_h u - u_h)\|_0 \) rate | \# iteration |
|------|-------------------|-------------------|---------|
| The \( \{P_3, P_2\} \) WG element (3.1), by 16-subdomain iteration. | | | |
| 1 | 0.366E-01 0.0 | 0.279E-01 0.0 | 10 |
| 2 | 0.236E-02 4.0 | 0.631E-02 2.1 | 14 |
| 3 | 0.151E-03 4.0 | 0.939E-03 2.7 | 19 |
| 4 | 0.946E-05 4.0 | 0.125E-03 2.9 | 35 |
| 5 | 0.592E-06 4.0 | 0.161E-04 3.0 | 59 |

Table 7.4
Error profile of \( \{P_4, P_3\} \) solutions for (7.1) on Figures 7.1–7.2 grids.

| Grid | \( \|Q_0 u - u_0\|_0 \) rate | \( \|\nabla_w(Q_h u - u_h)\|_0 \) rate | \# iteration |
|------|-------------------|-------------------|---------|
| By \( \{P_4, P_3\} \) element (3.1), 4-subdomain iteration, \( \beta = 32 \). | | | |
| 1 | 0.579E-02 0.0 | 0.123E-01 0.0 | 29 |
| 2 | 0.243E-03 4.6 | 0.114E-02 3.4 | 55 |
| 3 | 0.861E-05 4.8 | 0.802E-04 3.8 | 96 |
| 4 | 0.278E-06 5.0 | 0.522E-05 3.9 | 121 |
| 5 | 0.889E-08 5.0 | 0.331E-06 4.0 | 78 |

| Grid | \( \|Q_0 u - u_0\|_0 \) rate | \( \|\nabla_w(Q_h u - u_h)\|_0 \) rate | \# iteration |
|------|-------------------|-------------------|---------|
| By \( \{P_4, P_3\} \) element (3.1), 16-subdomain iteration, \( \beta = 32 \). | | | |
| 1 | 0.579E-02 0.0 | 0.123E-01 0.0 | 29 |
| 2 | 0.244E-03 4.6 | 0.115E-02 3.4 | 59 |
| 3 | 0.861E-05 4.8 | 0.802E-04 3.8 | 97 |
| 4 | 0.278E-06 5.0 | 0.522E-05 3.9 | 121 |
| 5 | 0.877E-08 5.0 | 0.331E-06 4.0 | 139 |

energy norm. It seems that larger \( \beta \) may reduce the number of iterations for higher order finite elements. It is surprising that 78-iteration shows up for the \( \{P_3, P_3\} \)-element with 4-subdomain iterations on the fifth-level grid. Another noticeable surprise is that the number of iterations for the 4-subdomain iteration increases much less than that of the 16-subdomain iteration, when the \( \{P_5, P_4\} \) and \( \{P_6, P_5\} \) finite elements move to higher level grids. We conjecture that it might be due to non-hanging subdomains in the 4-subdomain iterations.

7.2. Test Example 2. We solve the elliptic boundary value model problem (1.1) where the configuration is set up as follows: \( a(x, y) = 1; \ c = 0; \) the exact solution (7.2)
\[
    u(x, y) = 4(x - x^3)(y - y^3);
\]
and the domain \( \Omega = (0, 1)^2 \). The polygonal grids of quadrilaterals and pentagons, shown as in Figures 7.3 and 7.4 are employed in this test. The 4-subdomain iterations and 16-subdomain iterations are computed respectively.
Table 7.5
Error profile of \( \{ P_5, P_4 \} \) solutions for (7.1) on Figures 7.1–7.2 grids.

| Grid | \( \| Q_0 u - u_0 \|_0 \) rate | \( \| \nabla w(Q_h u - u_h) \|_0 \) rate | \# iteration |
|------|-----------------|-----------------|-------------|
| By \( \{ P_5, P_4 \} \) element (3.1), 4-subdomain iteration, \( \beta = 19 \). | | | |
| 1 | 0.146E-02 0.0 | 0.394E-02 0.0 | 31 |
| 2 | 0.357E-04 5.4 | 0.154E-03 4.7 | 55 |
| 3 | 0.621E-06 5.8 | 0.513E-05 4.9 | 71 |
| 4 | 0.996E-08 6.0 | 0.165E-06 5.0 | 78 |
| By \( \{ P_5, P_4 \} \) element (3.1), 16-subdomain iteration, \( \beta = 19 \). | | | |
| 1 | 0.342E-03 0.0 | 0.711E-03 0.0 | 43 |
| 2 | 0.348E-05 6.6 | 0.129E-04 5.8 | 67 |
| 3 | 0.289E-07 6.9 | 0.211E-06 5.9 | 99 |
| 4 | 0.232E-09 7.0 | 0.337E-08 6.0 | 94 |

Table 7.6
Error profile of \( \{ P_6, P_5 \} \) solutions for (7.1) on Figures 7.1–7.2 grids.

| Grid | \( \| Q_0 u - u_0 \|_0 \) rate | \( \| \nabla w(Q_h u - u_h) \|_0 \) rate | \# iteration |
|------|-----------------|-----------------|-------------|
| By \( \{ P_6, P_5 \} \) element (3.1), 4-subdomain iteration, \( \beta = 32 \). | | | |
| 1 | 0.342E-03 0.0 | 0.711E-03 0.0 | 43 |
| 2 | 0.348E-05 6.6 | 0.129E-04 5.8 | 67 |
| 3 | 0.289E-07 6.9 | 0.211E-06 5.9 | 99 |
| 4 | 0.232E-09 7.0 | 0.337E-08 6.0 | 94 |
| By \( \{ P_6, P_5 \} \) element (3.1), 16-subdomain iteration, \( \beta = 32 \). | | | |
| 1 | 0.342E-03 0.0 | 0.711E-03 0.0 | 43 |
| 2 | 0.348E-05 6.6 | 0.129E-04 5.8 | 97 |
| 3 | 0.289E-07 6.9 | 0.211E-06 5.9 | 133 |
| 4 | 0.232E-09 7.0 | 0.334E-08 6.0 | 156 |

The computational results for the \( \{ P_2, P_1 \} \) weak Galerkin finite element are listed in Table 7.7. Note that \( Q_h = \{ Q_0u, Q_b u \} \) where \( Q_0 \) is the element-wise \( L^2 \) projection to the space \( P_k(T) \) and \( Q_b \) is the edge-wise \( L^2 \) projection to the space \( P_{k-1}(e) \). In the computation, the iterative process is stopped when the iterative error achieves the truncation error. The numerical solution converges at the optimal order in the \( L^2 \)-norm and in the energy norm respectively. The number of iterations for the 4-subdomain method is slightly less than that of the 16-subdomain method.

The errors for the \( \{ P_3, P_2 \} \) and \( \{ P_4, P_3 \} \) weak Galerkin finite elements [3.1] are listed in Table 7.8. The numerical solutions converge at the optimal order in the \( L^2 \)-norm and in the energy norm respectively. Due to the use of polygonal meshes, the round-off error is accumulated to be very large for the computation of \( \{ P_4, P_3 \} \) element on the 4th-level grid. We observe that for the high-order finite elements, the number of iteration may not increase when the number of the grid level increases.

7.3. Test Example 3. We shall test the subdomain iterations for a superconvergent WG finite element method [29]. We consider the Poisson equation on the domain \( \Omega = (0, 1)^2 \) with the exact solution \( u(x, y) = \sin(\pi x)\sin(\pi y) \). The superconvergent
{P_k, P_k} WG finite element was introduced in [29] as follows:

\[(7.3) \quad \{v_0, v_b\} : v_0 \in P_k(T), \ v_b \in P_k(e), \ e \subset T, \ T \in T_h.\]

For this superconvergent \(\{P_k, P_k\}\) WG finite element, the weak gradient is defined by

\[\nabla_w v_h \in RT_k = P_k(T)^2 + xP_k(T),\]

satisfying

\[\langle \nabla_w v_h, q \rangle_T = -(v_0, \text{div} \ q)_T + (v_b, q \cdot n_T)_{\partial T}, \quad \forall q \in RT_k.\]

(7.4)

For this WG finite element, the stabilizer \(\langle \cdot, \cdot \rangle\) is dropped from the WG scheme (3.5), in order to get one-order superconvergence as discussed in [29].

We take \(\beta = 4\) in all \(\{P_k, P_k\}\) weak Galerkin finite element computations. Note that \(Q_k u = \{Q_0 u, Q_b u\}\) where \(Q_0 u\) is the triangle-wise \(L^2\) projection of \(u\) onto the polynomial space \(P_k(T)\) and \(Q_b\) is the edge-wise \(L^2\) projection of \(u\) onto the polynomial space \(P_k(e)\). Again the iteration is stopped when the iterative error is about the truncation error. In Tables 7.9–7.12, we have observed one-order superconvergence for the \(\{P_k, P_k\}\) weak Galerkin finite element defined in (7.3), in both the \(L^2\)-norm and the energy norm.

As expected, the number of 16-subdomain iterations would be more than that of 4-subdomain iterations for the \(\{P_1, P_1\}\) weak Galerkin finite element as we can see from Table 7.9.

In Table 7.10 we list the computational results for the \(\{P_2, P_2\}\) weak Galerkin finite element. Worse than the \(\{P_1, P_1\}\) computation, the number of iterations for 16-subdomains is much more than that of 4-subdomain. But on the other side, a better parallelization is possible for the computation with 16-subdomains.
Table 7.7
Error profile of $\{P_2,P_1\}$ (3.1) solutions on Figures 7.3–7.4 grids, and the number of iterations with 4-subdomains and 16-subdomains.

| Grid | $\|Q_0 u - u_0\|_0$ | rate | $\|\nabla_w (Q_h u - u_h)\|_0$ | rate | # iteration |
|------|-----------------|------|-----------------|------|----------------|
|      | $\{P_2,P_1\}$ WG element | 4-sub | 16-sub |
| 1    | 0.180E-01       | 0.0  | 0.817E-01       | 0.0  | 20             |
| 2    | 0.228E-02       | 3.0  | 0.202E-01       | 2.0  | 24             |
| 3    | 0.287E-03       | 3.0  | 0.491E-02       | 2.0  | 32             |
| 4    | 0.349E-04       | 3.0  | 0.121E-02       | 2.0  | 43             |
| 5    | 0.432E-05       | 3.0  | 0.299E-03       | 2.0  | 54             |
| 6    | 0.543E-06       | 3.0  | 0.745E-04       | 2.0  | 60             |

Table 7.8
Error profile of $\{P_k,P_{k-1}\}$ (3.1) solutions on Figures 7.3–7.4 grids, and the number of iterations with 4-subdomains and 16-subdomains.

| Grid | $\|Q_0 u - u_0\|_0$ | rate | $\|\nabla_w (Q_h u - u_h)\|_0$ | rate | # iteration |
|------|-----------------|------|-----------------|------|----------------|
|      | $\{P_3,P_2\}$ WG element | 4-sub | 16-sub |
| 1    | 0.174E-02       | 0.0  | 0.933E-02       | 0.0  | 39             |
| 2    | 0.110E-03       | 4.0  | 0.126E-02       | 2.9  | 51             |
| 3    | 0.687E-05       | 4.0  | 0.162E-03       | 3.0  | 84             |
| 4    | 0.438E-06       | 4.0  | 0.207E-04       | 3.0  | 59             |
| 5    | 0.278E-07       | 4.0  | 0.260E-05       | 3.0  | 69             |
|      | $\{P_4,P_3\}$ WG element | 4-sub | 16-sub |
| 1    | 0.115E-03       | 0.0  | 0.417E-03       | 0.0  | 44             |
| 2    | 0.364E-05       | 5.0  | 0.274E-04       | 3.9  | 64             |
| 3    | 0.114E-06       | 5.0  | 0.176E-05       | 4.0  | 99             |
| 4    | 0.469E-08       | —    | 0.113E-06       | 4.0  | 95             |

Table 7.11 shows that the number of iterations for 16-subdomains is about the same as that of 4-subdomains for the $\{P_3,P_3\}$ weak Galerkin finite element.

Finally the computational results by using $\{P_4,P_1\}$, $\{P_5,P_5\}$, $\{P_6,P_6\}$ weak Galerkin finite elements are illustrated in Table 7.12. In all these cases, the number of 16-subdomain iterations is slightly bigger than that of the 4-subdomain iterations.
## Table 7.9

*Error profile for the \( \{P_1, P_1\} \) solution on Figures 7.1–7.2 grids.*

| Grid | \( \|Q_0u - u_0\|_0 \) rate | \( \|\nabla_u(Q_hu - u_h)\|_0 \) rate | # iteration |
|------|-----------------|-----------------|--------------|
| The \( \{P_1, P_1\} \) WG element (7.3), by 4-subdomain iteration. |
| 1    | 0.208E-01 0.0   | 0.500E+00 0.0   | 7            |
| 2    | 0.305E-02 2.8   | 0.140E+00 1.8   | 9            |
| 3    | 0.393E-03 3.0   | 0.362E-01 2.0   | 13           |
| 4    | 0.494E-04 3.0   | 0.912E-02 2.0   | 12           |
| 5    | 0.624E-05 3.0   | 0.229E-02 2.0   | 17           |
| 6    | 0.789E-06 3.0   | 0.572E-03 2.0   | 26           |
| 7    | 0.994E-07 3.0   | 0.143E-03 2.0   | 42           |
| The \( \{P_1, P_1\} \) WG element (7.3), by 16-subdomain iteration. |
| 1    | 0.208E-01 0.0   | 0.500E+00 0.0   | 9            |
| 2    | 0.304E-02 2.8   | 0.140E+00 1.8   | 11           |
| 3    | 0.392E-03 3.0   | 0.362E-01 2.0   | 13           |
| 4    | 0.496E-04 3.0   | 0.913E-02 2.0   | 13           |
| 5    | 0.631E-05 3.0   | 0.229E-02 2.0   | 18           |
| 6    | 0.795E-06 3.0   | 0.572E-03 2.0   | 30           |
| 7    | 0.101E-06 3.0   | 0.143E-03 2.0   | 48           |

## Table 7.10

*Error profile for the \( \{P_2, P_2\} \) solution on Figures 7.1–7.2 grids.*

| Grid | \( \|Q_0u - u_0\|_0 \) rate | \( \|\nabla_u(Q_hu - u_h)\|_0 \) rate | # iteration |
|------|-----------------|-----------------|--------------|
| The \( \{P_2, P_2\} \) WG element (7.3), by 4-subdomain iteration. |
| 1    | 0.320E-02 0.0   | 0.116E+00 0.0   | 12           |
| 2    | 0.203E-03 4.0   | 0.158E-01 2.9   | 12           |
| 3    | 0.127E-04 4.0   | 0.202E-02 3.0   | 14           |
| 4    | 0.802E-06 4.0   | 0.254E-03 3.0   | 25           |
| 5    | 0.507E-07 4.0   | 0.318E-04 3.0   | 48           |
| 6    | 0.321E-08 4.0   | 0.398E-05 3.0   | 92           |
| The \( \{P_2, P_2\} \) WG element (7.3), by 16-subdomain iteration. |
| 1    | 0.320E-02 0.0   | 0.116E+00 0.0   | 16           |
| 2    | 0.202E-03 4.0   | 0.158E-01 2.9   | 12           |
| 3    | 0.128E-04 4.0   | 0.202E-02 3.0   | 15           |
| 4    | 0.801E-06 4.0   | 0.254E-03 3.0   | 29           |
| 5    | 0.508E-07 4.0   | 0.318E-04 3.0   | 55           |
| 6    | 0.321E-08 4.0   | 0.398E-05 3.0   | 107          |

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Table 7.11
Error profile for the \( \{P_3, P_3\} \) solution on Figures 7.2 grids.

| Grid   | \( \|Q_0u - u_0\|_0 \) rate | \( \|\nabla_w(Q_hu - u_h)\|_0 \) rate | # iteration |
|--------|-----------------------------|-------------------------------------|-------------|
| The \( \{P_3, P_3\} \) WG element \([7.3]\), by 4-subdomain iteration. |
| 1      | 0.478E-03 0.0              | 0.222E-01 0.0                     | 12          |
| 2      | 0.179E-04 4.7             | 0.149E-02 3.9                     | 24          |
| 3      | 0.593E-06 4.9             | 0.944E-04 4.0                     | 44          |
| 4      | 0.189E-07 5.0             | 0.593E-05 4.0                     | 82          |
| 5      | 0.597E-09 5.0             | 0.372E-06 4.0                     | 142         |
| The \( \{P_3, P_3\} \) WG element \([7.3]\), by 16-subdomain iteration. |
| 1      | 0.478E-03 0.0              | 0.222E-01 0.0                     | 12          |
| 2      | 0.179E-04 4.7             | 0.149E-02 3.9                     | 22          |
| 3      | 0.593E-06 4.9             | 0.944E-04 4.0                     | 46          |
| 4      | 0.189E-07 5.0             | 0.593E-05 4.0                     | 80          |
| 5      | 0.596E-09 5.0             | 0.372E-06 4.0                     | 161         |

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Table 7.12
Error profiles for the $P_4$, $P_5$ and $P_6$ WG solution on Figures 7.1–7.2 grids.

| Grid | $\|Q_0 u - u_0\|_0$ | rate | $\|\nabla_w(Q_h u - u_h)\|_0$ | rate | # iteration |
|------|--------------------|------|-----------------------------|------|-------------|
| The $\{P_4, P_4\}$ WG element [7.3], by 4-subdomain iteration. | 2 | 0.120E-05 | 6.0 | 0.122E-03 | 4.9 | 34 |
| | 3 | 0.190E-07 | 6.0 | 0.388E-05 | 5.0 | 54 |
| | 4 | 0.301E-09 | 6.0 | 0.122E-06 | 5.0 | 123 |
| The $\{P_4, P_4\}$ WG element [7.3], by 16-subdomain iteration. | 2 | 0.119E-05 | 6.0 | 0.122E-03 | 4.9 | 24 |
| | 3 | 0.190E-07 | 6.0 | 0.387E-05 | 5.0 | 62 |
| | 4 | 0.300E-09 | 6.0 | 0.123E-06 | 5.0 | 133 |
| The $\{P_5, P_5\}$ WG element [7.3], by 4-subdomain iteration. | 1 | 0.938E-05 | 0.0 | 0.541E-03 | 0.0 | 18 |
| | 2 | 0.827E-07 | 6.8 | 0.896E-05 | 5.9 | 36 |
| | 3 | 0.655E-09 | 7.0 | 0.141E-06 | 6.0 | 95 |
| The $\{P_5, P_5\}$ WG element [7.3], by 16-subdomain iteration. | 1 | 0.938E-05 | 0.0 | 0.541E-03 | 0.0 | 18 |
| | 2 | 0.816E-07 | 6.8 | 0.890E-05 | 5.9 | 44 |
| | 3 | 0.647E-09 | 7.0 | 0.141E-06 | 6.0 | 111 |
| The $\{P_6, P_6\}$ WG element [7.3], by 4-subdomain iteration. | 1 | 0.115E-05 | 0.0 | 0.710E-04 | 0.0 | 24 |
| | 2 | 0.460E-08 | 8.0 | 0.579E-06 | 6.9 | 66 |
| | 3 | 0.185E-10 | 8.0 | 0.459E-08 | 7.0 | 162 |
| The $\{P_6, P_6\}$ WG element [7.3], by 16-subdomain iteration. | 1 | 0.115E-05 | 0.0 | 0.710E-04 | 0.0 | 24 |
| | 2 | 0.459E-08 | 8.0 | 0.578E-06 | 6.9 | 72 |
| | 3 | 0.185E-10 | 8.0 | 0.461E-08 | 7.0 | 166 |

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