ELECTROWEAK DEFECTS

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A brief, non-technical review is given of certain defect-like configurations in the electroweak standard model, which may have played an important role in the physics of the early universe.

The history of the early universe is, for the most part, rather uneventful: the universe expands and cools down. There are, however, brief moments when something interesting does happen. These are, in reverse order, the epochs of

- recombination at temperature \( T = O(\text{eV}) \),
- nucleosynthesis at \( T = O(\text{MeV}) \),
- quark confinement at \( T = O(\text{GeV}) \),
- electroweak phase transition at \( T = O(\text{TeV}) \),

where the temperatures indicate the physics involved, namely the physics of the atomic, nuclear, chromodynamic and weak interactions (units are such that \( \hbar = c = k = 1 \)). What happened at even higher temperatures is a complete mystery, because we do not know how the elementary particles behave at collision energies significantly above \( 10^{12} \text{eV} \equiv 1 \text{TeV} \). Still, it is conceivable that already the electroweak phase transition epoch may have some bearing on the dark matter problem (the nature and origin of what constitutes the bulk of

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our present universe). We therefore present at this workshop some results on an unusual, but interesting, aspect of electroweak physics.

Elementary particle physics, over the years, has established a so-called standard model. It consists of two parts: the electroweak interactions and the strong (chromodynamic) interactions. A priori, these two interactions seem to be unrelated. The most intriguing part of the standard model is the electroweak sector, with the violation of parity (P) and charge conjugation (C) discovered in the 1950’s. The non-conservation of these two discrete symmetries is nowadays incorporated into the electroweak standard model, right from the start. In fact, the non-conservation already shows up in the fields present and their basic interaction structure (e.g. the absence of an interacting right-handed neutrino field).

This brings us to the main point of our talk, namely that the electroweak standard model has been established, so far, in perturbation theory only. The Feynman diagram of Fig. 1, for example, contributes to the scattering of two electrons ($e^-$) by the exchange of a virtual photon ($\gamma$) or neutral vector boson ($Z$). Essentially, the electrons are plane waves (wave packets, rather), which now and then emit or absorb a photon, with a probability amplitude proportional to the electric charge $e$ of the electron. Standard perturbation theory is the collection of all relevant Feynman diagrams, which can be ordered as an infinite series in powers of $e^2$ for the probabilities (Fig. 1, for example, contributes in order $e^4$ to the scattering cross-section). But there may be more to the electroweak theory than just Feynman diagrams. Indeed, we now believe that there are certain non-perturbative processes, proportional to $\exp[-1/e^2]$, which may lead to the violation of time reversal symmetry (T) and fermion number conservation (B + L, with B the baryon and L the lepton number operator). Remark that the probability amplitude $\exp[-1/e^2]$ has vanishing Taylor expansion around $e = 0$, so that there are no corresponding Feynman diagrams. For these processes there is no longer a meaningful distinction between a free and an interacting part of the Hamiltonian, all terms being equally
important. The field configurations involved are thus very different from plane waves (free particles) and generically go under the name of “defects”. Let us first discuss two such defects in simple models and then see whether or not defect-like configurations also appear in the electroweak standard model.

Consider the Abelian Higgs model in 3+1 dimensions. The cartesian coordinates are written as \((x^1, x^2, x^3, x^0)\) and, for later use, we define the cylindrical coordinates \(\rho, \varphi, z\) and \(t\) by setting \((\rho \cos \varphi, \rho \sin \varphi, z, t) = (x^1, x^2, x^3, x^0)\).

Static (time-independent) fields have an energy density

\[
\epsilon = \frac{1}{4} \left( \partial_k A_l - \partial_l A_k \right)^2 + \left| \partial_k \Phi + i \frac{e}{2} A_k \Phi \right|^2 + \lambda \left( \Phi^* \Phi - \frac{v^2}{2} \right)^2 ,
\]

with indices \(k, l\) running over 1, 2, 3, and \(\partial_k\) standing for the partial derivative \(\partial/\partial x^k\). Here, \(A_k\) is the real gauge field and \(\Phi = \phi_1 + i \phi_2\) the complex scalar Higgs field, with vacuum expectation value \(v/\sqrt{2}\), quartic coupling constant \(\lambda\) and electric charge \(\frac{1}{2} e\). The theory has an \(U(1)\) gauge invariance

\[
\Phi(x) \to e^{i \omega(x)} \Phi(x) ,
\]

\[
A_k(x) \to A_k(x) - \frac{2}{e} \partial_k \omega(x) ,
\]

with \(\omega(x)\) an arbitrary real function, so that \(e^{i \omega(x)} \in U(1)\). The \(U(1)\) gauge symmetry group is called Abelian, because a successive transformation with first \(\omega_1\) and then \(\omega_2\) gives the same result as with the order reversed (i.e. the group addition is commutative). The so-called Higgs symmetry breaking
mechanism transforms the gauge field $A$ into a massive vector field, with a vector boson mass given by $M_A = \frac{1}{2} e v$.

As a further restriction we impose $z$-independence, so that we have an essentially 2-dimensional theory with fields $\Phi(x^1, x^2)$ and $A_k(x^1, x^2)$ for $k = 1, 2$. Field configurations with finite string tension (total energy per unit length in the $z$-direction) obey the following condition:

$$\lim_{\rho \to \infty} |\Phi| \equiv |\Phi^\infty| = \frac{v}{\sqrt{2}},$$

otherwise the integrated scalar potential gives an infinite contribution to the string tension. The field values $|\Phi| = v/\sqrt{2}$ define the vacuum manifold $M_{\text{vac}}$, which has the topology of a circle $S_1$. Continuous, finite string tension field configurations are thus characterized by their behaviour at spatial infinity $S_1^\infty$ (a limiting circle in the plane $z = 0$, say). In short, there is the map

$$\Phi^\infty : S_1^\infty \to M_{\text{vac}} = S_1.$$  

This map $\Phi^\infty(\varphi) \in M_{\text{vac}}$ classifies the different field configurations according to the number ($n$) of times the complex phase of the field $\Phi$ winds around, for an azimuthal angle $\varphi$ running from 0 to $2\pi$; see Fig. 2.

Intuitively, it is clear that any configuration in the $n = 0$ sector can relax to the classical vacuum solution ($A_k = 0, \Phi = v/\sqrt{2}$, modulo gauge transformations), whereas a configuration in the $n = \pm 1$ sector cannot. One therefore expects the existence of another classical solution to which $n = 1$ configurations may relax. This is the well-known vortex solution:

$$\Phi = \bar{f}(\rho) e^{i\varphi} \frac{v}{\sqrt{2}},$$

$$\vec{A} = -\frac{2}{e \rho} \bar{g}(\rho) \hat{e}_\varphi,$$

with $\hat{e}_\varphi$ the azimuthal unit vector and $\bar{f}(\rho), \bar{g}(\rho)$ radial functions solving the reduced field equations under the boundary conditions

$$\bar{f}(0) = \bar{g}(0) = 0, \quad \bar{f}(\infty) = \bar{g}(\infty) = 1.$$
The anti-vortex solution, with winding number \( n = -1 \), has the signs of \( \varphi \) and \( \hat{e}_\varphi \) in (3) reversed. Only the behaviour of the scalar field was sketched in Fig. 2, but the gauge field plays an equally important role. The asymptotic gauge field (5) cancels the gradient of \( \Phi \) and makes the first term on the right-hand side of (1) vanish as \( \rho \to \infty \), thereby keeping the string tension finite.

Moreover, the gauge field carries a quantized magnetic flux in the \( z \)-direction. The solution (5, 6) is therefore a flux tube in the \( z \)-direction, with the Higgs symmetry breaking absent at the core (\( |\Phi| = 0 \) for \( \rho = 0 \)), so that locally the vector field is massless again.

The existence of the vortex solution is governed by the topological classification \( n = \pm 1 \), as sketched in Fig. 2. For this reason, the vortex solution is called a topological defect. Also, if the highly symmetric vortex configuration (5, 6) is the lowest energy classical solution in the \( n = 1 \) sector, then it must necessarily be stable, since there is no other solution to decay to (admittedly, the argument is somewhat circular, but local stability of the vortex solution can be proven rigorously). To return to a point made at the beginning, all terms of the energy density (1) are of equal importance in the vortex solution.
and there is no separation possible into a free and an $O(e)$ interacting part. Indeed, the interaction term $|e A_k \Phi|^2$ is $O(1)$, due to the large, $O(1/e)$, amplitude of the vortex gauge field (5). All in all, the vortex solution is very different from the classical vacuum, or small fluctuations around it.

Another interesting example of a topological defect occurs in $SO(3)$ Yang-Mills-Higgs theory. We now consider fully 3-dimensional static fields and introduce the spherical coordinates $r, \theta$ and $\varphi$ by setting $(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta) = (x^1, x^2, x^3)$. This theory has a real Higgs triplet

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix},$$  

(7)
on which a non-Abelian (non-commutative) gauge symmetry operates, $\Phi(x) \rightarrow \Omega(x)\Phi(x)$ with $\Omega(x) \in SO(3)$. Finite total energy requires $|\Phi| \rightarrow v / \sqrt{2}$ for $r \rightarrow \infty$ and the vacuum manifold has the topology of a sphere, $M_{\text{vac}} = S^2$. The relevant map is from the sphere at spatial infinity into the vacuum manifold

$$\Phi^\infty : S^2 \rightarrow M_{\text{vac}} = S^2.$$  

(8)

There is now a 3-dimensional generalization of Fig. 2 with winding number $n = 1$ (imagine the field vectors pointing outwards, just as for a frightened hedgehog). Such a configuration with $n = 1$ cannot relax to the vacuum solution, but may evolve towards another classical solution, the magnetic-monopole solution. This concludes our brief review of so-called topological defects. Generally speaking, these stable defects are relevant to the static properties of the theory, for example the spectrum of the Hamiltonian.

We now come to the crucial question: can these stable, topological defects be carried over to the electroweak standard model (EWSM)? The short answer is: no. The reason is that the electroweak theory has a complex Higgs doublet

$$\Phi = \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix},$$  

(9)
so that $M_{\text{vac}} = S_3$. This implies that the mappings from either the circle $S_1^\infty$ or the sphere $S_2^\infty$ at infinity into $M_{\text{vac}} = S_3$ are topologically trivial (in Coleman’s words: you cannot lasso a basketball). Hence, there are no static, stable topological defects in the $3 + 1$ dimensional EWSM. Still (and this is the long answer to the question above), there are other types of defects in the EWSM, but they turn out to be unstable. It is believed that these unstable defects may be relevant to the dynamic properties of the theory. Here, we will discuss two such electroweak defects, the sphaleron and the Z-string.

The existence of the sphaleron solution in the EWSM again follows from a topological argument. However, rather than considering the behaviour of the field configurations directly, such as was done in Fig. 2, one considers instead the topology of configuration space. Configuration space here is the abstract, mathematical space of 3-dimensional, finite energy configurations (with the gauge freedom eliminated). Each point of configuration space corresponds to a snapshot of the fields. There is, for example, one point in configuration space which identifies the vacuum configuration (Higgs field constant and gauge fields vanishing). The energy functional $E_B$ defines a surface over configuration space, with the stationary points corresponding to the solutions of the classical field equations. This configuration space is infinite dimensional and non-compact. Moreover, the topology of configuration space turns out to be highly non-trivial: configuration space is not simply an infinite dimensional euclidian space $\mathbb{R}^\infty$. In fact, there are holes and the point at the “top” of one such hole (to be discussed further in the next paragraph) corresponds to a new classical solution, the so-called sphaleron $\mathcal{S}$. The sphaleron $S$, whose explicit field configurations are somewhat complicated, has typical field energy

$$E_S \equiv E_B[\mathcal{S}] = O(M_W/\alpha_w) \sim 10 \text{ TeV},$$

with $M_W \equiv \frac{1}{2} g v = 80.4 \text{ GeV}$ the mass of the $W^\pm$ vector bosons, $v = 247 \text{ GeV}$ the Higgs vacuum expectation value and $\alpha_w \equiv g^2/4\pi = 1/29.6$ the fine-structure constant of the $SU(2)$ gauge interactions. The shape of the sphaleron is a

\[\text{From the Greek adjective ‘sphaleros’, meaning ready to fall.}\]
sphaleron $S$ is slightly elongated (cigar-like) and its electromagnetic field has a large magnetic dipole moment, which may be interpreted as coming from a tight monopole - antimonopole pair. In this way, we find magnetic monopoles already in the EWSM!

This particular hole in configuration space is captured by the following non-trivial map:

$$S_1 \times S_2^\infty \rightarrow M_{\text{vac}} = S_3,$$

where $S_1$ parametrizes a loop of configurations, each of which is characterized by its behaviour at spatial infinity $S_2^\infty$. The sphaleron solution is just one “point” on this non-contractible loop in configuration space (Fig. 3, left-hand side) and by itself has trivial topology ($S_2^\infty \rightarrow S_3$), in contrast to, for example, the single magnetic monopole solution of the $SO(3)$ theory based on the non-trivial map (8).

Without saying explicitly, we have considered up till now only bosonic fields (scalar and gauge fields) with total energy $E_B$. We now look at the response of quantized fermionic fields to these classical bosonic fields. Specifically, we take a loop of bosonic field configurations around the configuration space hole mentioned above, starting and ending at the vacuum $V$, while passing through the sphaleron $S$. This is a non-contractible loop (NCL) of configurations, precisely because it encircles a hole. If we now consider the Dirac Hamiltonian eigenvalues ($E_F$) for the different bosonic configurations of the NCL, we observe the remarkable phenomenon of spectral flow. The NCL starts and ends at the same configuration ($V$) and the Dirac eigenvalue spectrum there is the same, but along the NCL there is a non-trivial flow of eigenvalues (Fig. 3, right-hand side). The non-trivial spectral flow, whose origin traces back to the Atiyah-Singer index theorem, has also been observed in numerical calculations. In the electroweak context this spectral flow leads to the violation of fermion number conservation.

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[^8]: Note that the Dirac sea (the set of filled negative energy states of the Dirac Hamiltonian) is truly bottomless, which allows for an overall shift of the occupied states as in Fig. 3.
Figure 3: Non-contractible loop and spectral flow.

\[ \Delta(B + L) = \pm 2N_{\text{fam}}, \quad (12) \]

where \(N_{\text{fam}}(= 3?)\) is the number of families of quarks and leptons. The selection rule (12) is simply the total contribution of all left-handed fermion doublets (quarks and leptons) responding to the \(SU(2)\) gauge fields of the NCL.

Originally, \(B + L\) violation was considered to proceed via quantum tunneling (instantons), but the picture of Fig. 3 suggests the possibility of \(B + L\) violation at large temperatures or collision energies, where one passes over the energy barrier \(E_S\), not through. As expected, the tunneling rate is extremely small, with a WKB factor \(\exp[-16\pi^2/g^2\hbar] \sim \exp[-372]\), whereas the thermal rate, with a Boltzmann factor \(\exp[-E_S/kT]\), may become significant for large enough temperatures. These thermal reactions turn out to have been important in the very early universe. However, the precise mechanism for the creation of the present baryon number of the universe is not known, even though the necessary ingredients are available, namely the violation of C, CP, B and thermal equilibrium. This remains one of the outstanding problems of cosmology and particle physics.

Another classical solution in the EWSM is the Z-string, which is simply
the embedded vortex solution $\mathcal{E}$,  

$$ \Phi = \bar{f}(\rho) \begin{pmatrix} 0 \\ e^{i\phi} \end{pmatrix} \frac{\nu}{\sqrt{2}}, $$

$$ \vec{Z} = -\frac{2\cos\theta_w}{g} \frac{\bar{g}(\rho)}{\rho} \hat{e}_\phi, $$

$$ \vec{W}^\pm = 0, $$

$$ \vec{A} = 0, $$

(13)

with $A$ the massless photon field and $Z, W^\pm$ the massive vector boson fields (mass ratio $M_W/M_Z = \cos\theta_w$, in terms of the weak mixing angle $\theta_w = 0.491$).

The extra degrees of freedom of the EWSM turn the Z-string into an unstable solution. In fact, it can be shown that there is a hole in the space of $z$-independent configurations, around which a non-contractible sphere (NCS) may be constructed, with the Z-string at the top (Fig. 4, left hand side, where $E_B$ indicates the 2-dimensional “energy”, i.e. the string tension in the 3-dimensional context). The instability of the Z-string is manifest, with at least two unstable directions. The topologically non-trivial map is now

$$ S_2 \times S^\infty_1 \rightarrow M_{\text{vac}} = S_3, $$

(14)
where $S_2$ parametrizes a sphere of configurations, each of which is characterized by its behaviour at spatial infinity $S_1^{\infty}$. Again, there is spectral flow of the Dirac eigenvalues over the NCS. The fermionic levels over the NCS display a cone-like structure and the behaviour along an arbitrary meridian is sketched in Fig. 4, right-hand side. In this case there is no net change of quantum numbers, since fermions and anti-fermions are created simultaneously. Still, the non-trivial spectral flow over the NCS may have implications for the consistency of the theory.

To conclude, we mention an interesting observation, which relates these two electroweak solutions. The observation is that linked Z-string loops and twisted Z-string segments (Fig. 5) have non-vanishing Chern-Simons number, just as the sphaleron $S$, and may therefore lead to violation of $B + L$ (whether or not this actually occurs needs to be verified, though). This ties in with our earlier remark that the sphaleron solution $S$ can be viewed as a bound state of a monopole and an antimonopole, which are then connected by a very short segment of twisted Z-string.

All these unstable defects of the electroweak standard model sketched in Fig. 5 are expected to play an important role in the electroweak phase transition of the early universe ($T \sim 100 - 1000$ GeV). Extended models of the

\[ \text{These particular configurations are, strictly speaking, not exact static solutions of the field equations and evolve with time.} \]
electroweak and strong interactions may even have (meta-)stable defects, which could have been created at an earlier phase transition \( T > 1 \text{ TeV} \). Remarkably, there may be the possibility of simulating the cosmological creation of defects in appropriate low-temperature condensed matter physics experiments. It is to be hoped that the coming decennia will provide us with clues, from both elementary particle and condensed matter physics experiments, to help explore the cosmological epoch of the electroweak phase transition.

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