EXISTENCE AND APPROXIMABILITY RESULTS FOR VARIATIONAL PROBLEMS UNDER UNIFORM CONSTRAINTS ON THE GRADIENT BY POWER PENALTY.

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Abstract. Variational problems under uniform quasiconvex constraints on the gradient are studied. In particular, existence of solutions to such problems is proved as well as existence of lagrange multipliers associated to the uniform constraint. They are shown to satisfy an Euler-Lagrange equation and a complementarity property. Our technique consists in approximating the original problem by a one-parameter family of smooth unconstrained optimization problems. Numerical experiments confirm the ability of our method to accurately compute solutions and Lagrange multipliers.

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1. Introduction. We study the following class of problems from the calculus of variations

\[
\inf \{ J(v) : |T(x, \nabla v(x))| \leq 1 \text{ a.e in } \Omega, \ v = g \text{ on } \partial \Omega \}. \tag{1.1}
\]

In particular, we prove existence and approximability of solutions and Lagrange multipliers associated to the uniform constraint on the gradient. We approximate the problem by a sequence of unconstrained problems penalizing the uniform constraint by a power term.

The model case of (1.1) is the problem of the elastoplastic torsion of a cylindrical bar of section \( \Omega \):

\[
\min_{v \in K_0} \frac{1}{2} \int_{\Omega} (|\nabla v(x)|^2 - h(x)v(x)) \, dx \tag{1.2}
\]

for \( K_0 = \{ v \in H^1_0(\Omega) \mid |\nabla v(x)| \leq 1 \text{ a.e } x \in \Omega \} \). Problem (1.2) has been extensively studied by Ting (1969); Brézis (1972); Caffarelli and Friedman (1979) and in the numerical aspects by Glowinski et al. (1981). Brézis (1972) proves the existence and uniqueness of a multiplier \( \lambda \in L^\infty \) satisfying the system

\[
\begin{align*}
\lambda & \geq 0 \quad \text{a.e on } \Omega \tag{1.3a} \\
\lambda(1 - |\nabla u|) & = 0 \quad \text{a.e on } \Omega \tag{1.3b} \\
-\Delta u - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} (\lambda \frac{\partial u}{\partial x_i}) & = h \quad \text{in } D' \tag{1.3c}
\end{align*}
\]

when the right hand side \( h \) is constant. Chiadò Piat and Percivale (1994) reconsider the problem for a general elliptic operator \( A \) and nonconstant right hand side \( h \), obtaining a measure multiplier satisfying a system analogous to (1.3b)-(1.3c). Brézis (1972) uses the characteristics method to
solve (1.3c) for \( \lambda \), obtaining a semi-explicit formula for the multiplier. Chiadò Piat and Percivale (1994) approximate the problem by a sequence of nonsmooth problems penalizing the violation of the constraint \( |\nabla u| \leq 1 \) a.e. Whether similar results could be obtained in the framework of a general duality theory stands as an open question for a long time. Ekeland and Temam (1976) show the insufficiency of the traditional duality theory for tackling this problem. The question was solved positively by Daniele et al. (2007) using a new infinite dimensional duality theory (see also Donato, 2011; Maugeri and Puglisi, 2014). Daniele et al. (2007) show, for a large class of problems including Problems (2.1) and (1.2), that if the problem is solvable and the solution satisfies a constraint qualification condition, then there exists a Lagrange multiplier \( \lambda \in L^\infty_+ \) satisfying (1.3b), which is indeed the solution of a dual problem. Concerning existence of solutions for the general Problem (1.1), we can cite the results of Ball (1977), showing existence for variational problems under constraints of the type \( T(\cdot, \nabla v(x)) \in C(x) \) for almost every \( x \in \Omega \). From this perspective, the existence of solutions as well as of Lagrange multipliers is well established. Nonetheless, at least two issues remain unsolved. The first is to have a practical way to approximate Problem (1.1) by simpler problems that can be solved using existing mature numerical methods. The second issue is closely related to the first, and has to do with choosing a particular solution in problems with lack of uniqueness. In this paper we address those open issues by providing an approximation scheme for Problem (1.1). The original problem is approximated by a sequence of unconstrained problems whose solution converges to a solution of the constrained problem. Moreover, by analyzing the optimality conditions we identify a term that is then showed to converge to a Lagrange multiplier associated to the uniform constraint on the gradient. In this way, we recover and in some cases improve the existence results and provide a practical approximation scheme. The effectiveness of our approach is illustrated through numerical simulations.

2. Statement of the problem and main results. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) with \( N \geq 1 \) and \( T: \Omega \times \mathbb{R}^{m \times N} \to [0, \infty) \) a Carathéodory function. Let \( s \geq 1 \) and consider a functional \( J: W^{1,s}(\Omega; \mathbb{R}^m) \to \mathbb{R} \cup \{+\infty\} \), which is supposed to be bounded from below and sequentially lower semicontinuous in the weak topology of \( W^{1,s}(\Omega; \mathbb{R}^m) \). We are interested in the minimization problem

\[
\inf \{ J(v) \mid \|T(\cdot, \nabla v)\|_{\infty, \Omega} \leq 1, \ v \in g + W^{1,s}_0(\Omega; \mathbb{R}^m) \}, \tag{2.1}
\]

where

\[
\|T(\cdot, \nabla v)\|_{\infty, \Omega} = \text{ess-sup} \{ T(x, \nabla v(x)) \mid x \in \Omega \},
\]

and \( g \in W^{1,\infty}(\Omega; \mathbb{R}^m) \cap C(\overline{\Omega}; \mathbb{R}^m) \) is a given function satisfying

\[
J(g) < +\infty \text{ and } T(x, \nabla g(x)) \leq 1 \text{ for a.e. } x \in \overline{\Omega}. \tag{2.2}
\]

Define \( J_\infty: W^{1,s}(\Omega; \mathbb{R}^m) \to \mathbb{R} \cup \{+\infty\} \) by

\[
J_\infty(v) = \begin{cases} 
J(v) & \text{if } \|T(\cdot, \nabla v)\|_{\infty, \Omega} \leq 1, \\
+\infty & \text{otherwise.}
\end{cases}
\]

Then (2.1) may be rewritten as

\[
\inf \{ J_\infty(v) \mid v \in g + W^{1,s}_0(\Omega; \mathbb{R}^m) \}. \tag{2.3}
\]
By (2.2), we have that $J_\infty(g) < +\infty$.

From now on, we assume that $T$ is quasiconvex in the sense of Morrey, i.e. for almost for every $x_0 \in \Omega$ and any $\xi_0 \in \mathbb{R}^{m \times N}$

$$T(x_0, \xi_0) \leq \frac{1}{\mathcal{L}_N(D)} \int_D T(x_0, \xi_0 + \nabla \phi(x)) dx,$$

where $D$ is an arbitrary bounded domain in $\mathbb{R}^N$ and $\phi$ is any function in $W^{1,\infty}_0(D; \mathbb{R}^m)$. Here, $\mathcal{L}_N$ stands for the Lebesgue measure in $\mathbb{R}^N$. Suppose also that

$$\alpha_1 (1 + |\xi|^r) \leq T(x, \xi) \leq \beta_1 (1 + |\xi|^r)$$

where $0 < \alpha_1 \leq \beta_1$ and $1 \leq r < \infty$. Concerning the functional $J$, in most interesting applications it will take the integral form

$$J(u) = \int_{\Omega} f(x, u(x), \nabla u(x)) dx$$

where $f : \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^{m \times N}$ is a Carathéodory integrand satisfying, for almost every $x \in \Omega$, for every $(u, \xi) \in \mathbb{R}^N \times \mathbb{R}^{m \times N}$,

$$\xi \mapsto f(x, u, \xi) \text{ is quasiconvex}$$

$$\gamma_1(x) \leq f(x, u, \xi) \leq \beta_2 (|\xi|^s + |u|^t) + \gamma_2(x)$$

where $\beta_2 \geq 0$, $\gamma_1, \gamma_2 \in L^1(\Omega)$ and $1 \leq t < \infty$.

For each $p \in \max(r, s, \infty)$ define the $p$-power penalty functional $J_p : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$J_p(v) = J(v) + \frac{r}{p} \int_{\Omega} T_v(x)^{p/r} dx,$$

where

$$T_v(x) = T(x, \nabla v(x))$$

and consider the penalized problems

$$\inf \{ J_p(v) \mid v \in g + W^{1,p}_0(\Omega; \mathbb{R}^m) \}.$$  

Under the above conditions, the existence of solutions $u_p$ to (2.8) follows from a standard application of the direct method of the calculus of variations (cf. Dacorogna [2007], Theorem 8.29). In this direction, notice that the quasiconvexity of $T$ yields the quasiconvexity of $T^p$ for every $1 < p < \infty$.

Any selection of solutions to Problems (2.8) uniformly converges to a solution of Problem (1.1). We do not assume a priori existence of solutions to Problem (1.1), therefore the following is an existence and approximability result.

**Theorem 2.1.** Under the previous assumptions, we have that:
(i) For every \( q \geq \max\{N + 1, r, s\} \), the net \( \{u_p \mid p \geq q, p \to \infty\} \) is bounded in \( W^1,q(\Omega; \mathbb{R}^m) \) and relatively compact in \( C^\alpha(\Omega; \mathbb{R}^m) \), where \( \alpha = 1 - N/q \).

(ii) If \( u_\infty \) is a cluster point of \( \{u_p \mid p \to \infty\} \) in \( C(\Omega; \mathbb{R}^m) \), then \( u_\infty \) is an optimal solution to (2.1) and, moreover,

\[
\lim_{p \to \infty} \min_{u_p} J_p = \lim_{p \to \infty} J_p(u_p) = \lim_{p \to \infty} J(u_p) = J(u_\infty) = \min J.
\]

Next we address the existence and approximability of Lagrange multipliers for the uniform constraint on the gradient. The underlying rationale bears some resemblances to some methods for showing existence of Lagrange multipliers without recourse to separation theorems, such as the Fritz John optimality conditions in nonlinear programming. Let us consider the Lagrange functional

\[
L(u, \lambda) = J(u) + \int \lambda(x)(Tu(x) - 1)dx \tag{2.9}
\]

If a solution \( u \) to Problem (2.1) satisfies a constraint qualification condition, then there exists \( \lambda \in L^\infty \) such that \( (u, \lambda) \) is a saddle point of \( L \). Let \( (u, \lambda) \) be a saddle point of \( L \), and suppose that \( T \) is differentiable with respect to its second argument. The minimality condition for \( u \) reads

\[
J'(u)[v] + \int \lambda DT(x, \nabla u(x))\nabla v(x) = 0 \quad \forall v \in C_0^\infty
\]

On the other hand, the optimality conditions for the penalized problem (2.8) yields

\[
J'(u_p)[v] + \int (Tu_p)^{p-1}DT(x, \nabla u_p(x))\nabla v(x) = 0 \quad \forall v \in C_0^\infty.
\]

Suppose that \( J'(u_p) \to J'(u) \) as \( p \to \infty \), then

\[
\int (Tu_p)^{p-1}DT(x, \nabla u_p(x))\nabla v(x) \to \int \lambda DT(x, \nabla u(x))\nabla v(x) \quad \forall v \in C_0^\infty \tag{2.10}
\]

Equation (2.10) strongly suggests that the sequence \( \{(Tu_p)^{p-1}\}_{p \geq p_1} \) must play the role of a Lagrange multiplier as \( p \) goes to infinity. The main difficulty of this part is to prove the convergence of that sequence in \( L^\infty(\Omega) \), which is required in order to obtain results supporting the numerical approximation of the multipliers. We use differential equations methods in this part, therefore the class of considered problems is more restrictive than in Theorem (2.1). For those problems we prove the following

**Theorem 2.2.** Let \( u \) be a cluster point of \( \{u_p\}_{p \geq p_1} \) in \( C(\Omega) \). There exists \( \lambda \in L^\infty(\Omega) \) such that

(i) The sequence \( \{\nabla u_p\}_{p \geq p_1} \) weakly-\( * \) converges to \( \lambda \nabla u \), up to subsequence.

(ii) The primal-dual pair \( (u, \lambda) \) satisfy the system

\[
\text{div}(W'(\nabla u^2)\nabla u) + \text{div}(\lambda \nabla u) = -\phi'(u) \quad \text{in } D'. \tag{2.11}
\]

\[
\lambda(x) \geq 0 \quad \text{a.e in } \Omega. \tag{2.12}
\]

\[
\lambda(x)(|\nabla u(x)| - 1) = 0 \quad \text{a.e in } \Omega \tag{2.13}
\]

For the elastoplastic torsion problem (1.2), \( \text{Brézis} \, 1972 \) proved the uniqueness of \( \lambda \in L^\infty(\Omega) \) verifying (2.11)–(2.13). Moreover, using the known explicit solution for the primal problem on the disk, we obtain an explicit expression for \( \lambda \), to which the whole sequence \( \{\nabla u_p\}_{p \geq p_1} \) must converge. These explicit solutions make possible to validate numerically our method.
3. Primal convergence results. In this section we provide the proof of Theorem 2.1. The proof is divided into a series of lemmas. For clarity of the exposition we put \( r = 1 \), the general case being completely analogous.

**Lemma 3.1 (Compactness).** we have that:

(i) \( \sup_{p \geq s} \frac{1}{p} \| T u_p \|^p_{p, \Omega} < +\infty \), where

\[
\| T u_p \|^p_{p, \Omega} = \int \limits_{\Omega} T(x, \nabla u_p(x))^p dx
\]

(ii) Let \( p_1 = \max \{ N + 1, s \} \). For every \( q > 1 \), \( \{ u_p \}_{p \geq p_1} \) is bounded in \( W^{1,q}(\Omega; \mathbb{R}^m) \)

(iii) \( \{ u_p \}_{p \geq p_1} \) is relatively compact in \( C(\overline{\Omega}; \mathbb{R}^m) \).

(iv) For every uniform cluster point \( u_\infty \) of \( \{ u_p \}_{p \geq p_1} \), we have that

\[ u_\infty \in g + W^{1,\infty}_0(\Omega; \mathbb{R}^m). \]

(v) If \( u_{p_j} \to u_\infty \) in \( C(\overline{\Omega}; \mathbb{R}^m) \) then \( u_{p_j} \rightharpoonup u_\infty \) weakly in \( W^{1,q}(\Omega; \mathbb{R}^m) \) for every \( q \in [p_1, \infty[. \)

**Proof.** From the optimality of \( u_p \) it follows that

\[ \alpha + \frac{1}{p} \| T u_p \|^p_{p, \Omega} \leq J(g) + \frac{1}{p} \| T g \|^p_{\infty, \Omega} \mathcal{L}(\Omega), \]

where \( \alpha = \inf \{ J(v) \mid v \in W^{1,s}(\Omega; \mathbb{R}^m) \} \in \mathbb{R} \) (recall that \( J \) is supposed to be bounded from below).

Using (2.2) we deduce that

\[
\sup_{p \geq s} \frac{1}{p} \| T u_p \|^p_{p, \Omega} < +\infty,
\]

hence

\[ C_1 := \sup_{p \geq s} \| T u_p \|^p_{p, \Omega} < +\infty. \]

In particular,

\[ \| \nabla u_p \|^p_{p, \Omega} \leq \alpha_1 C_1. \]

On the other hand, the Poincaré inequality yields

\[ \| u \|^p_{p, \Omega} \leq C(\Omega, p) (\| \nabla u \|^p_{p, \Omega} + \| \nabla g \|^p_{p, \Omega} + \| g \|^p_{p, \Omega}), \]

for every \( u \in g + W^{1,p}_0(\Omega; \mathbb{R}^m) \) and a suitable constant \( C(\Omega, p) > 0 \). Combining these estimates, and recalling [Adams 1975] that the constant \( C(\Omega, p) \) may be chosen such that

\[ \sup_{p \in [N+1, \infty[} C(\Omega, p) < +\infty, \]

we deduce that there exists a constant \( C_2 > 0 \) such that

\[ \forall p \in [p_1, +\infty[, \quad \| u_p \|_{1,p, \Omega} = \| u_p \|^p_{p, \Omega} + \| \nabla u_p \|^p_{p, \Omega} \leq C_2, \]
where $p_1 = \max\{N + 1, s\}$. In particular, $\{u_p := u_p - g\}_{p \geq p_1}$ is bounded in $W_{0}^{1,q}(\Omega; \mathbb{R}^m)$ for each $q \geq p_1$, hence for every $q > 1$ by Hölder inequality. Since $p_1 > N$, we deduce that $\{u_p\}_{p \geq p_1}$ is relatively compact in $C(\Omega; \mathbb{R}^m)$ by the Rellich-Kondrachov theorem (since we deal with $W_{0}^{1,p_1}$ we do not require any regularity condition on $\partial\Omega$). Thus, we deduce that $\{u_p\}_{p \geq p_1}$ is relatively compact in $C(\Omega; \mathbb{R}^m)$.

Let $u_\infty$ be a cluster point of $\{u_p\}_{p \geq p_1}$ in $C(\Omega; \mathbb{R}^m)$. First, we prove that $u_\infty \in W^{1,\infty}(\Omega; \mathbb{R}^m)$. By Morrey’s theorem there exists a constant $\varepsilon > 0$ such that

$$|u_p(x) - u_p(y)| \leq C'(\Omega, p)\|u_p\|_{1,p,\Omega}|x - y|^{1 - N/p}$$

for every $x, y \in \Omega$. In fact, the constant can be chosen in such a way that

$$\sup_{p \in [q, \infty[} C'(\Omega, p) < +\infty$$

for every $q > N$ (see Adams (1975)). Therefore, we conclude that for a suitable constant $C_3 > 0$,

$$|u_p(x) - u_p(y)| \leq C_3|x - y|^{1 - N/p},$$

for every $x, y \in \Omega$ and $p \in [p_1, \infty]$. We deduce that

$$|u_\infty(x) - u_\infty(y)| \leq C_3|x - y|,$$

then $u_\infty \in W^{1,\infty}(\Omega; \mathbb{R}^m)$. Of course, $u_\infty = g$ on $\partial\Omega$.

Next, fix $q \in [1, \infty[$. From our previous analysis it follows that $\{u_p\}_{p \in [p_1, \infty[}$ is bounded in $W^{1,q}(\Omega; \mathbb{R}^m)$ and therefore relatively compact for the weak topology of $W^{1,q}(\Omega; \mathbb{R}^m)$. Consequently, if $p_j \to \infty$ is a sequence such that $u_{p_j} \to u_\infty$ uniformly on $\Omega$, then $u_{p_j} \to u_\infty$ weakly in $W^{1,q}(\Omega; \mathbb{R}^m)$.

**Lemma 3.2.** If $u_\infty$ is a cluster point of $\{u_p \mid p \to \infty\}$ in $C(\Omega; \mathbb{R}^m)$ then $\|Tu_\infty\|_{q,\Omega} \leq 1$. Moreover, $u_\infty$ is an optimal solution to (2.3), and we have that

$$\lim_{p \to \infty} J_p(u_p) = \lim_{p \to \infty} J(u_p) = J(u_\infty) = \min J_\infty.$$

**Proof.** Let $u_{p_j} \to u_\infty$ in $C(\Omega; \mathbb{R}^m)$ and fix $q \in [1, \infty[$. By Lemma 3.1, $u_{p_j} \to u_\infty$ weakly in $W^{1,q}(\Omega; \mathbb{R}^m)$. It follows from the weak lower semicontinuity in $W^{1,q}(\Omega; \mathbb{R}^m)$ of $v \to \|Tv\|_{q,\Omega}$, that

$$\|Tu_\infty\|_{q,\Omega} \leq \lim_{j \to \infty} \inf_{j \to \infty} \|Tu_{p_j}\|_{q,\Omega}.$$  

For every $p \in [q, \infty[$, the Hölder inequality yields

$$\|Tu_p\|_{q,\Omega} \leq \|Tu_p\|_{p,\Omega} \mathcal{L}_N(\Omega)^{\frac{1}{q} - \frac{1}{p}}.$$  

Then, Lemma 3.1 ensures that

$$\|Tu_p\|_{q,\Omega} \leq (pC)^{\frac{1}{q}} \mathcal{L}_N(\Omega)^{\frac{1}{q} - \frac{1}{p}}$$

for some constant $C > 0$. Hence

$$\|Tu_\infty\|_{q,\Omega} \leq \mathcal{L}_N(\Omega)^{\frac{1}{q}}.$$  

Letting $q \to \infty$, we get the desired inequality.
Let $v \in g + W^{1,\infty}(\Omega; \mathbb{R}^m)$ with $\|Tv\|_{\infty,\Omega} \leq 1$. By optimality of $u_p$ we have that

$$J(u_p) \leq J_p(u_p) \leq J_p(v) = J(v) + \frac{1}{p}\|Tv\|_{p,\Omega}^p.$$ 

Since $\|Tv\|_{\infty,\Omega} \leq 1$, we have that

$$\limsup_{p \to \infty} J(u_p) \leq \limsup_{p \to \infty} J_p(u_p) \leq \limsup_{p \to \infty} J_p(v) = J(v).$$

As $v$ is arbitrary, we obtain that

$$\limsup_{p \to \infty} J(u_p) \leq \limsup_{p \to \infty} J_p(u_p) \leq \inf J_\infty.$$ 

Now, let $u_{p_j} \to u_\infty$ in $C(\overline{\Omega}; \mathbb{R}^m)$. By the weak lower semicontinuity of $J$, we have that

$$J(u_\infty) \leq \liminf_{j \to \infty} J(u_{p_j}),$$

and due to the previous lemmas, we know that $J(u_\infty) = J_\infty(u_\infty)$. This proves the optimality of $u_\infty$ and moreover

$$\lim_{j \to \infty} J_{p_j}(u_{p_j}) = \lim_{j \to \infty} J(u_{p_j}) = \min J_\infty.$$ 

Finally, note that, up to a subsequence, the same is valid for an arbitrary sequence $\{p_k\}_{k \in \mathbb{N}}$ with $p_k \to \infty$. This fact together with a compactness argument proves indeed the result. □

4. Dual convergence results. In this section we are concerned with the existence and approximation of Lagrange multipliers for the constrained problem (2.1). The techniques used to this end does not allow the great degree of generality as the primal results of Section 3. We shall restrict ourselves to particular cases where the regularity of solutions is known. More precisely, we consider the following instances of (2.1)

$$\min \left\{ J(v) := \int_{\Omega} \left[ W(|\nabla v|^2) + \phi(v) \right] : |\nabla v| \leq 1, \ v \in g + H^1_0(\Omega) \right\}.$$ 

We suppose that $g$ is a real constant, and additionally

$$t \mapsto W(t^2) \text{ and } \phi \text{ are convex and of class } C^2(\mathbb{R})$$

$$G(s) := W'(s) + 2sW''(s) > 0, \text{ for } s > 0.$$ (4.1) (4.2)

Let us consider the penalized problem

$$\min \left\{ J(v) + \frac{1}{p} \int |\nabla u|^p : v \in g + H^1_0(\Omega) \right\}.$$ 

By the convexity assumptions on the functions $W$ and $\phi$, that problem has a unique solution $u_p$ which is a weak solution of the Euler-Lagrange equation:

$$\text{div}((W'(|\nabla u|^2) + |\nabla u|^p - 2)\nabla u) = -\phi'(u_p).$$ (4.3)
Let us define:
\[
\Psi(x; \alpha) = \int_0^{\|\nabla u_p\|^2} G(s)\,ds + \frac{2p-1}{p}\|\nabla u_p\|^p + \alpha \phi(u_p)
\]

Note that by \((4.2)\), \(|\nabla u_p|^p + \alpha \phi(u_p) \leq \Psi(x; \alpha)\). Therefore if we succeed in obtaining uniform bounds for \(\Psi\) we can deduce thereof bounds for the sequence \(|\nabla u_p|^p\). Maximum principles of Payne and Philippin \cite{Payne1977, Philippin1979} state that under mild conditions the maximum of \(\Psi(\cdot, \alpha)\) is attained at a critical point of \(u_p\). In such a point \(\Psi(x, \alpha) = \alpha \phi(u_p(x))\) and we can conclude using the uniform bounds on \(u_p\) obtained in Section 3. The application of maximum principle techniques require to work with classical \((C^2(\Omega))\) solutions. Results of Uhlenbeck \cite{Uhlenbeck1977}, Tolksdorf \cite{Tolksdorf1984} and Lieberman \cite{Lieberman1988} show that bounded solutions to equations of the type \((4.3)\) are \(C^{1,\alpha}(\Omega)\)-regular, provided that hypothesis \((4.2)\) holds. Higher regularity can be obtained by a bootstrap argument at points where \(|\nabla u_p| \neq 0\). However, if the function \(G\) defined in \((4.2)\) is degenerate, \(i.e.\) \(G(0) = 0\), a further regularization is necessary \cite{Kawohl1990}. Following a classic procedure \(\text{see eg. Evans and Gangbo} \ 1999\), Bhattacharya et al. \(1989\), Sakaguchi \(1987\), DiBenedetto \(1983\) the term \(|\nabla u_p|^p\) is regularized by \((\varepsilon^2 + |\nabla u_p|^2)^{p/2}\) to obtain a sequence of regular functions \(u_p^\varepsilon\) converging to \(u_p\) pointwise and in \(W^{1,p}\) norm as \(\varepsilon \to 0\). In this way estimations on \(u_p\) can be obtained by approximation.

**Theorem 4.1.** Under hypothesis \((4.1) - (4.2)\), if \(\Omega\) is convex and \(\partial \Omega \in C^2\), then the sequence \(|\nabla u_p|^p\) is uniformly bounded in \(L^\infty(\Omega)\).

**Proof.** By Payne and Philippin \cite{Payne1977} Corollary 1), the function \(\Psi(x; 2)\) attains its maximum at a critical point of \(u_p\), therefore
\[
|\nabla u_p|^p + 2\phi(u_p) \leq \Psi(x; \alpha) \leq \max \Psi(x; 2) \leq 2 \max \phi(u_p(x)),
\]
whence
\[
|\nabla u_p|^p \leq 4 \max \phi(u_p) < +\infty
\]
and conclude by Theorem \((2.1)\) and the continuity of \(\phi\). \(\square\)

**Corollary 4.2.** Let \(u_\infty \in C^{0,1}(\Omega)\) be a cluster point of \(\{u_p\}_{p \geq p_1}\). Then, passing if necessary to a further subsequence,
\begin{enumerate}
\item \(\nabla u_p(x) \to \nabla u_\infty(x)\) for \(a.e\. x \in \Omega\).
\item \(\nabla u_p \rightharpoonup \nabla u_\infty\) in the weak--* topology of \(L^\infty(\Omega)\).
\item There exists \(A \in L^\infty(\Omega)^n\) such that the sequence \(|\nabla u_p|^{p-2}\nabla u_p\) converges to \(A\) in the \(\text{weak}--*\) topology.
\end{enumerate}

**Proof.** Assertion \((i)\) is obtained from Boccardo and Murat \cite{Boccardo1992} using hypothesis \((4.1)\). Points \((ii)\) and \((iii)\) are consequences of the Banach–Alaoglu Theorem. \(\square\)

We are now in position to state our existence and approximability result for both primal and dual solutions of \((2.1)\).

**Theorem 4.3.** Let \(u\) be a cluster point of \(\{u_p\}_{p \geq p_1}\) in \(C(\Omega)\) achieving the convergences of Corollary \((4.2)\). There exists \(\lambda \in L^\infty(\Omega)\) such that
\begin{enumerate}
\item The sequence \(|\nabla u_p|^{p-2}\nabla u_p\) weakly--* converges to \(\lambda \nabla u\).
\end{enumerate}
(ii) The primal-dual pair \((u, \lambda)\) satisfy the system

\[
\text{div}(W(|\nabla u|^2)\nabla u) + \text{div}(\lambda \nabla u) = -\phi'(u) \text{ in } \mathcal{D}'.
\]

\[
\lambda(x) \geq 0 \text{ a.e in } \Omega. \tag{4.5}
\]

\[
\lambda(x)(|\nabla u(x)| - 1) = 0 \text{ a.e in } \Omega. \tag{4.6}
\]

**Proof.** The first step of the proof consists in showing that the limit field \(A\) in Corollary 4.2 (iii) verifies

\[
|A| = A \cdot \nabla u \text{ a.e in } \Omega. \tag{4.7}
\]

Using \(u - g\) as test function in (4.3) we have

\[
\int \Omega |\nabla u_p|^{p-2} \nabla u_p \nabla u = - \int \Omega W'(|\nabla u_p|^2) |\nabla u_p| \nabla u_p + \phi'(u_p) (u - g). \tag{4.8}
\]

Then by Corollary 4.2

\[
\int \Omega A \nabla u = - \int \Omega W'(|\nabla u|^2) |\nabla u|^2 + \phi'(u)(u - g). \tag{4.9}
\]

The same procedure using \(u_p - g\) as test shows that

\[
\int \Omega |\nabla u_p|^p \rightarrow - \int \Omega W'(|\nabla u|^2) |\nabla u|^2 + \phi'(u)(u - g). \tag{4.10}
\]

and therefore

\[
\int \Omega |\nabla u_p|^p \rightarrow \int \Omega A \nabla u \tag{4.11}
\]

whence

\[
\int \Omega |A| \leq \int \Omega A \cdot \nabla u \tag{4.12}
\]

and (4.7) follows using \(|\nabla u| \leq 1\) a.e (Theorem 2.1). The existence of \(\lambda \in L^\infty(\Omega)\) satisfying (4.5) & (4.6) follows from (4.7). Taking the limit in (4.3) using Theorem 2.1 Corollary 4.2 and the representation (4.7) gives (4.6). \(\square\)

5. **Numerical experiments.** We solved numerically the elastoplastic torsion problem in a variety of domains, that permitted to gain some insight on the method. The problem

\[
\min \left\{ \frac{1}{2} \int \Omega |\nabla u|^2 - \int \Omega hu \middle| \begin{array}{l}
|\nabla u| \leq 1 \text{ a.e in } \Omega \\
u = 0 \text{ on } \partial \Omega
\end{array} \right. \right\} \tag{5.1}
\]

is approximated by the sequence of unconstrained problems

\[
\min \left\{ \frac{1}{2} \int \Omega |\nabla u_p|^2 + \frac{1}{p} \int |\nabla u_p|^p - \int \Omega hu_p \middle| u_p = 0 \text{ on } \partial \Omega \right. \right\} \tag{5.2}
\]
Algorithm 1 The primal-dual algorithm

Given $p > 2$ and an initial point $u_0$, choose $c_1, \varepsilon$. Set $n := 0$ and iterate:

1. Compute the multiplier $\lambda_n = |\nabla u_n|^{p-2}$.
2. Find the primal descent direction $w_n$, by solving
   \[ \int_{\Omega} (1 + \lambda_n) \nabla w_n \nabla v dx = - \int_{\Omega} \lambda_n \nabla u_n \nabla v dx + \int_{\Omega} f v dx \quad \forall v \]
3. Perform a line-search with sufficient decrease condition, i.e., find $\alpha_n > 0$ satisfying $J(u_n + \alpha_n w_n) \leq J(u_n) + c_1 \alpha_n J'(u_n)[w_n]$
4. Set $u_{n+1} = u_n + \alpha_n w_n$.
5. If $\|J'(u_{n+1})\| \leq \varepsilon$, stop. Otherwise update $n = n + 1$ and go to step 1.

which possess an unique and regular solution. Besides, results of Brezis and Stampacchia (1968) ensure that solutions $u_\infty$ of (5.1) are of class $C^1(\Omega)$ for regular domains. For Problem (5.2) we solve the Euler equation
\[ \int_{\Omega} \langle \nabla u_p, \nabla v \rangle + \int_{\Omega} |\nabla u_p|^{p-2} \langle \nabla u_p, \nabla v \rangle - \int_{\Omega} hv = 0 \quad \forall v \in V, \]
where $V$ stands for the space of continuous functions whose restriction to any element of a regular mesh of $\Omega$ is polynomial of degree 2. Since we are dealing with a nonlinear problem, we cannot apply the finite elements method directly; the use of an iterative procedure is necessary. However, for large $p$ the convergence and stability of such an iterative procedure is a delicate issue. Huang et al. (2007) proposed to use the term $|\nabla u_p|^{p-2}$ as a preconditioner in a gradient-type method with good results (cf. Algorithm 1). Incidentally, the term used as a preconditioner by Huang et al. (2007) coincides with the approximating multiplier, and therefore, in the light of Theorem 4.3, their algorithm can be viewed as a primal-dual algorithm with a multiplier computed explicitly from the primal solution, instead of maximizing a saddle-point function. For the tests presented here, we implemented Algorithm 1 in C++ using the deal.II finite elements library (Bangerth et al., 2007).

Denote by $D$ the unit disk of $\mathbb{R}^2$, i.e $D = \{ x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1 \}$. When $\Omega = D$ and $h$ is constant, (5.1) has an explicit solution (Glowinski et al., 1981). For $h \equiv 4$ the solution is given by:
\[
  u(x) = \begin{cases} 
    1 - r & \text{if } 1/2 \leq r \leq 1 \\
    -r^2 + 3/4 & \text{if } 0 \leq r \leq 1/2
  \end{cases} 
\]
(5.3)

where $r = \sqrt{x^2 + y^2}$. Since $\Omega$ is convex, in this case the multiplier $\lambda$ is continuous and it is also given by an explicit formula (Brézis, 1972),
\[
  \lambda(x) = \begin{cases} 
    2r - 1 & \text{if } 1/2 \leq r \leq 1 \\
    0 & \text{if } 0 \leq r \leq 1/2
  \end{cases} 
\]
(5.4)

The norm of the gradient of the computed solution and the multiplier are plot in Figure 5.1. In Table 5.1 we show the error with respect to the explicit solutions for different values of $p$. It is
Table 5.1

| Mesh info | Primal error | Dual error |
|-----------|--------------|------------|
| p | # cells | # dofs | $L^2$-norm | $H^1$-norm | $W^{1,\infty}$-norm | $L^1$-norm | $L^\infty$-norm |
| 10 | 65708 | 280049 | 4.585e-02 | 1.355e-01 | 1.756e-01 | 2.645e-01 | 2.133e-01 |
| 50 | 65348 | 273345 | 9.876e-03 | 3.255e-02 | 5.680e-02 | 5.115e-02 | 6.058e-02 |
| 100 | 123917 | 517501 | 4.989e-03 | 1.705e-02 | 3.366e-02 | 2.555e-02 | 3.530e-02 |
| 300 | 442940 | 1883001 | 1.674e-03 | 5.956e-03 | 1.416e-02 | 8.716e-03 | 2.963e-02 |
| 500 | 857396 | 3698513 | 1.006e-03 | 3.624e-03 | 9.358e-03 | 5.267e-03 | 2.705e-02 |

Fig. 5.1. Plot of the norm of the gradient $|\nabla u_p|$ and the multiplier $\lambda_p = |\nabla u_p|^{p-2}$ for $p = 500$ on a circle.

shown that for a working precision, a parameter $p$ in the order of few hundreds is enough, preserving in this way the numerical stability of the algorithm.

Solving the problem in different domains gives some intuition about the extensibility of Theorem 4.3 to more general situations. In Figures 5.2 and 5.3 we show the solutions of Problem 5.1 in a rectangle and a domain with an interior corner, respectively. We also plot the approximate multipliers. It is seen that in the rectangle, a convex domain with piecewise smooth border, we are still able to compute satisfactorily both the solution and the multiplier. On the contrary, in the piecewise smooth nonconvex domain, even if the are able to compute the solution with a good accuracy, it is not enough to have the multiplier bounded. The difficulty relies on the concentration effect occurring near the interior corners. However, the plot with a truncated scale shows that far from the concentrations we are computing the right multiplier, suggesting that our method combined with some truncation mechanism (see eg. Li [1995] Section 4) should be able to cope with a more general class of problems.

References.
Fig. 5.2. Plot of the norm of the gradient $|\nabla u_p|$ and the multiplier $\lambda_p = |\nabla u_p|^{p-2}$ for $p = 300$ on a rectangle.

Fig. 5.3. Plot of the norm of the gradient $|\nabla u_p|$ and the multiplier $\lambda_p = |\nabla u_p|^{p-2}$ for $p = 700$ on a domain with an interior corner. The scale in the plot of the multiplier is truncated.

Adams R (1975) Sobolev spaces. Academic Press, New York, first edn.
Ball JM (1977) Convexity conditions and existence theorems in nonlinear elasticity. Arch Rational Mech Anal 63(4):337–403
Bangerth W, Hartmann R, Kanschat G (2007) deal.II – a general purpose object oriented finite element library. ACM Trans Math Softw 33(4):24/1–24/27
Bhattacharya T, DiBenedetto E, Manfredi J (1989) Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems. Rend Sem Mat Univ Politec Torino (Special Issue):15–68 (1991), some topics
in nonlinear PDEs (Turin, 1989)
Boccardo L, Murat F (1992) Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. Nonlinear Anal 19(6):581–597
Brézis H (1972) Multiplicateur de Lagrange en torsion elasto-plastique. Arch Rational Mech Anal 49:32–40
Brézis H, Stampacchia G (1968) Sur la régularité de la solution d’inéquations elliptiques. Bull Soc Math France 96:153–180
Caffarelli LA, Friedman A (1979) The free boundary for elastic-plastic torsion problems. Trans Amer Math Soc 252:65–97
Chiadò Piat V, Percivale D (1994) Generalized Lagrange multipliers in elastoplastic torsion. J Differential Equations 114(2):570–579
Dacorogna B (2007) Direct methods in the calculus of variations, vol. 78 of Applied Mathematical Sciences. Springer
Daniele P, Giuffrè S, Idone G, Maugeri A (2007) Infinite dimensional duality and applications. Math Ann 339(1):221–239
DiBenedetto E (1983) $C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations. Nonlinear Anal 7(8):827–850
Donato MB (2011) The infinite dimensional lagrange multiplier rule for convex optimization problems. Journal of Functional Analysis 261(8):2083 – 2093
Ekeland I, Temam R (1976) Convex Analysis and Variational Problems, vol. 28 of Classics In Applied Mathematics. Second edn.
Evans L, Gangbo W (1999) Differential equations methods for the Monge-Kantorovich mass transfer problem, vol. 137 of Memoirs of the American Mathematical Society. American Mathematical Society, Providence, Rhode Island
Glowinski R, Lions JL, Trémolières R (1981) Numerical analysis of variational inequalities, vol. 8 of Studies in Mathematics and its Applications. North-Holland Publishing Co., Amsterdam, translated from the French
Huang Y, Li R, Liu W (2007) Preconditioned descent algorithms for p-laplacian. Journal of Scientific Computing 32(2):343–371
Kawohl B (1990) On a family of torsional creep problems. J reine angew Math 410:1–22
Li ZP (1995) A numerical method for computing singular minimizers. Numer Math 71(3):317–330
Lieberman GM (1988) Boundary regularity for solutions of degenerate elliptic equations. Nonlinear Anal 12(11):1203–1219
Maugeri A, Puglisi D (2014) A new necessary and sufficient condition for the strong duality and the infinite dimensional lagrange multiplier rule. Journal of Mathematical Analysis and Applications 415(2):661 – 676
Payne L, Philippin G (1977) Some applications of the maximum principle in the problem of torsional creep. SIAM J Appl Math 33(3):446–455
Payne L, Philippin G (1979) Some maximum principles for nonlinear elliptic equations in divergence form with applications to capillarity surfaces and to surfaces of constant mean curvature. Nonlin Anal 3(2):193–211
Sakaguchi S (1987) Concavity properties of solutions to some degenerate quasilinear elliptic dirichlet problems. Ann Sc Norm Sup di Pisa (IV) 14(3):403–421
Tolksdorf P (1984) Regularity for a more general class of quasilinear elliptic equations. J Differential Equations 51(1):126–150
Uhlenbeck K (1977) Regularity for a class of non-linear elliptic equations. Acta Math 48:217–238