General Properties of Two-dimensional Conformal Transformation in Electrostatics

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Abstract

Electrostatic properties of two-dimensional nanosystems can be described by their geometry resonances. In this paper we prove that these modes as well as the corresponding eigenvalues are invariant under any conformal transformation. This invariance further leads to a new way to studying the transformed structures. A special kind of three-dimensional transformations are further investigated.
One specular property of Maxwell’s equations is that their form will be invariant under arbitrary coordinate transformation if the field quantities as well as the material properties are transformed accordingly [1, 2], which leads to a powerful designing tool called transformation optics or transformation electromagnetism in 2006 [1, 2]. Since then, a wealth of novel and unique devices have been theoretically suggested and/or experimentally fabricated, see recent reviews [4, 5] and the references cited. An incomplete list includes electromagnetic cloaks [1, 2, 6–8], event cloaks [9], optical black holes [10] and field splitters [11].

Quasi-static approximations can be applied when electromagnetic wavelength is far longer than the characteristic size of the material. The interactions between light and medium hence can be approximately described by Laplace’s equation. Similar to the full-wave Maxwell equations, two-dimensional Laplacian equation is invariant under conformal transformations [12]. Furthermore, the electrostatic transformation does not alter the properties of the constituent material, in sharp contrast to the full-wave coordinate transformations which generally transform normal medium to exotic and strange one. Lately, this unique property of Laplacian equation is used to study two-dimensional plasmonic particles [13–18]. More specifically, two-dimensional complicated structures, by transforming to simple one-dimensional geometries such as finite-thickness metallic slab, can be studied analytically [13–18]. It is further found that these plasmonic nanoparticles, include crescent and touching or non-touching cylinder dimers, can be broadband and significantly enhance the local electric fields.

In this paper, we will study electrostatic conformal transformation in a new way, by taking advantage of the invariance of geometry resonances and their eigenvalues. We will show that transforming geometry can be equivalently explained as modifying the amplitudes of these geometry resonances by transforming the excitation source. Furthermore, a few energy quantities such as electrostatic energy possessed by the particle are found to be conserved under conformal transformation. Decreasing the area of the particle will then definitely increasing the degree of local field enhancement.

We briefly recall the spectral Bergman-Milton theory [19, 20]. It is stated that the electrostatic behavior of a two-constituent composite can be described by a set of eigenmodes $\varphi_n$ of the following generalized eigenproblem

$$\nabla \cdot [\theta(r)\nabla \varphi_n(r)] = s_n \nabla^2 \varphi_n(r),$$  \hspace{1cm} (1)
where \( s_n \) representing the corresponding eigenvalues, and the function \( \theta(\mathbf{r}) \) characterizing the geometry of the composite, equal to 1 inside one constituent, with a permittivity of \( \epsilon_1(\omega) \), and 0 inside the other medium with a permittivity of \( \epsilon_2(\omega) \) (we assume \( \epsilon_2 = 1 \) in the following to simplify the discussion). Since this equation depends exclusively on the geometry, but not on the material composition, the resultant eigenmodes are therefore named as geometry resonances \[19\]. Moreover, by defining a scalar product as

\[
(\phi|\psi) = \int \nabla \phi^* \cdot \nabla \psi \, dv,
\]

we will find that \( s_n \) must be real and limited as \( 0 \leq s_n \leq 1 \), and the normalized eigenmodes \( \phi_n \) form a complete orthonormal set \[19\]. Note that this theory has been successfully employed to predict spaser \[21\] and study ultrafast optical excitation in nanosystems \[22, 23\].

It is important to notice, for a two-dimensional system, the eigenequation (1) remains invariance under any conformal coordinate transformation. To prove this fact, we rewrite the left-hand side of equation (1) as

\[
\partial_i \mathbf{e}_i \cdot (\theta \partial_j \varphi_n \mathbf{e}_j) = g_{ij} \partial_i (\theta \partial_j \varphi_n) = g \partial_i (\theta \partial_i \varphi_n),
\]

where the Einstein summation convention is employed, and the metric tensor \( g_{ij} \equiv \mathbf{e}_i \cdot \mathbf{e}_j = g \delta_{ij} \) because of the conformal transformation \[2\]. Similarly the right-hand side can be reformulated as

\[
s_n \partial_i \mathbf{e}_i \cdot (\partial_j \varphi_n \mathbf{e}_j) = s_n g_{ij} \partial_i (\partial_j \varphi_n) = s_n g \partial_{ii} \varphi_n.
\]

Evidently the transformed equation is identical to the previous one if we interpret the new equation as being in a right-handed Cartesian system and keep \( \theta \) and \( \varphi_n \) unchanged.

The invariance of the eigenmodes \( \varphi_n \) and their corresponding eigenvalue \( s_n \) immediately suggest a way to studying the electrostatic response of the transformed structure. Note that we can expand its total potential \( \varphi_t \) in terms of these eigenmodes,

\[
\varphi_t(\omega) = \sum_n \frac{s(\omega)}{s(\omega) - s_n (1 - i k^2/8)} (\varphi_n|\varphi_0) \varphi_n \equiv \sum_n \beta_n (\varphi_n|\varphi_0) \varphi_n,
\]

where \( k = \omega/c \), \( s(\omega) = 1/(1 - \epsilon_1) \) being a spectral parameter and \( \varphi_0 \) being the external potential. The radiation damping has been included by adding the factor \( i k^2/8 \) \[24, 25\]. Consequently, once these eigenmodes of the original or seed geometry are known, we can obtain the potential \( \varphi_t \) of any transformed geometry by just calculating the expansion coefficients \( (\varphi_n|\varphi_0) \). In other words, transforming geometry is equivalent to transforming the
external source \( \varphi_0 \), or more precisely, changing the expansion coefficients of each eigenmodes. Furthermore, the time-averaged power absorbed by the structure can be written as

\[
P_a = \frac{\omega \epsilon_0}{2} \text{Im}(\epsilon_1) \sum_n s_n |\beta_n(\varphi_n|\varphi_0)|^2 = \frac{\omega \epsilon_0}{2} \text{Im}(\epsilon_1) I_e,
\]

(6)

where \( I_e \) represents the integration of the electric field intensity \( |E|^2 \) inside the particle. In a similar way, the total power, the extinction, taken out by the particle has an expression as

\[
P_{ex} = \frac{\omega \epsilon_0}{2} \sum_n s_n (|\varphi_n|\varphi_0)|^2 \text{Im}[(\epsilon_1 - 1)|\beta_n|].
\]

(7)

When we transform the external field \( \varphi_0 \) accordingly, i.e., \( \varphi_0(x, y) = \varphi'_0(u, v) \) with \( u \) and \( v \) being the new coordinates, all these energy quantities, \( P_a, I_e \) and \( P_{ex} \), are evidently invariant. On the other hand, if the external potential is fixed in the \( w \) coordinate with \( w = u + iv \), we will find that these energy quantities are proportional to \( a^2 \) when the conformal mapping has a form as \( a \times w(z) \) with \( z = x + iy \) and \( a \) is real. In other words, bigger the particle area, stronger the absorption and extinction.

It should be mentioned that the geometry resonances actually are the surface modes of the particle \[26, 27\], and the resonance condition for the \( n \)th mode is strictly \( s(\omega) = s_n \), or equivalently \( \epsilon_1 = 1 - 1/s_n \), when we do not include the radiation loss. Notice that the corresponding permittivity of the particle \( \epsilon_1 \) should be real and negative since \( 0 < s_n \leq 1 \) \[26\]. The most striking property of the surface mode is that the resultant total electrostatic energy of the whole system, including the free space and the particle, is exactly zero \[28\]. Furthermore, the complex frequency of the \( n \)th surface plasmonic resonance (SPR) is given by \[19, 21\]

\[
s(\omega_n - i\gamma_n) = s_n,
\]

(8)

with \( \omega_n \) being real resonant frequency and \( \gamma_n \) being the relaxation rate \[29\]. Since the eigenvalues \( s_n \) are conserved under any conformal transformation, the transformed structure hence have same SPRs as the original structure at the identical resonant frequencies. For instance, since a metallic cylinder can be transformed from a metal-dielectric interface by \( w = e^z \), its SPR hence can be determined by nonretarded surface-plasmon condition, \( \epsilon_1 = -1 \), of the metal-dielectric interface \[30\].

As a proof of principle, we consider an one-dimensional dielectric slab with finite thickness (Figure (1a)) and its derivative systems obtained through different conformal transforma-
FIG. 1: An one-dimensional finite-thickness slab in the $xy$ coordinates (a) is transformed to an annulus (b) with a conformal transformation of $w = e^z$, a crescent (c) and two kissing cylinders (d) with a conformal transformation of $w = 1/z$, in the new $uv$ coordinates. Here $w = u + iv$ and $z = x + iy$.

The eigenvalues of the slab are given by

$$s_{k, \pm} = \frac{1 \pm e^{-|k|(d_2 - d_1)}}{2},$$

where $k$ being real and nonzero, and the corresponding eigenmodes are

$$\alpha_{k, \pm, \varphi_{k, \pm}}e^{-iky} = \begin{cases} [e^{-2|k|d_1} \mp e^{-|k|(d_1 + d_2)}]e^{|k|x}, & x \leq d_1 \\ \mp e^{-|k|(d_1 + d_2)}e^{|k|x} + e^{-|k|x}, & d_1 < x < d_2 \\ (1 \mp e^{|k|(d_2 - d_1)})e^{-|k|x}, & x \geq d_2 \end{cases}$$

where the normalize coefficient $\alpha_{k, \pm}^2 = 8\pi|k|e^{-|k|d_1} (e^{-|k|d_1} \mp e^{-|k|d_2})$. Evidently, in terms of the symmetry of electric field, these eigenmodes can be cataloged into two groups. The subscript $+(-)$ corresponds to even (odd) mode respectively, while their eigenvalues are bigger (smaller) than 0.5. Since the electric fields of the odd modes penetrate the metal weakly, they can propagate longer along the metallic surface than the even modes. Furthermore, the resonant condition, approximately given by $\text{Re}[s(\omega_n)] = s_n$ when the
relaxation rate is weak, suggests that only the odd modes will be excited resonantly when the real part of the dielectric permittivity, Re(ε₁), is smaller than −1. Moreover, these eigenmodes and eigenvalues above can be directly applied to the complementary structure of the finite slab, i.e. a free-space gap sandwiched by two semi-infinite metallic spaces. Since these two structures have identical geometry, they therefore share same eigenvalues and eigenmodes. To obtain the induced potential of the complementary structure, we only need to interchange ε₁ with ε₂ in the definition of s(ω). As a direct consequence, when Re(ε₁) is smaller than −1, only the even modes will be resonantly excited in the semi-infinite structure [18].

We now calculate the resultant expansion coefficients of a structure transformed from the finite slab through a conformal mapping of w = w(z). It is further assumed that the external potential ϕ₀(u, v) = pₑu + pᵥv, which corresponding a uniform electric field −pₑₑₑₑ − pᵥₑᵥ.

The corresponding expansion coefficients can be written as

\[
(ϕ_k, \pm |ϕ₀) = \left(\frac{2k}{s_k, \pm α_k, \pm} [g₂pF_k - g₁p^* F_{-k}]\right),
\]

where

\[
F_k = \int_{d₁}^{d₂} dx \int_{-∞}^{∞} dy \frac{dw}{dz} w^{kz^*}, \quad p = \frac{1}{2}(p_u - ip_v).
\]

In addition, g₁ = 1 and g₂ = ±e^{-|k|(d₁+d₂)} for positive k, and they should exchange when k < 0.

To validate our approach, we first transforms the slab to two coaxial cylinders (Figure 1b)) with w = e^z, and the radius of the two cylinders are r₁ = e^{d₁} and radius r₂ = e^{d₂} respectively. The expansion coefficients are found to be

\[
(ϕ_k, \pm |ϕ₀) = \pm \frac{α_k, \pm}{4} p_u e^{d₁+d₂} δ(\pm |k| - 1),
\]

here pᵥ is assumed to be zero because of the structural symmetry. Evidently, only eigenmodes with k = ±1 are excited. We further use these coefficients to calculate the induced potential ϕᵢ

\[
\frac{σϕ_i}{ϕ₀} = \begin{cases} (r₁² - r₂²)(1 - ε₂)², & r ≤ r₁ \\ r₁²(1 - ε₂)² + r₂²(1 - ε₂)² - 2(1 - ε₂)r₁²r₂²/r², & r₁ < r < r₂ \\ r₂²(r₂² - r₁²)(1 - ε₂)/r², & r ≥ r₂ \end{cases}
\]

with σ = r₂²(1 + ε₂)² − r₁²(1 − ε₂)². Our results are exactly identical to the one obtained by expanding the potential with Bessel functions [26]. Furthermore, by letting r₁ → 0 or
$r_2 \to \infty$, the results above can be used to describe a dielectric cylinder of a cylinder void.

Since the eigenvalues of the finite slab cover a wide region $[0, 1]$ (note that the eigenvalues of the eigenproblem (1) are limited between 0 and 1), any negative Re($\epsilon_1$) will excite a surface mode as long as its expansion coefficient is not zero. Consequently the transformed system will present broadband response in principle. One example is the crescent studied in Ref. [13], which can be obtained by using $w = 1/z$ when $d_1 > 0$ (Figure (1c)). The expansion coefficients are found to be

$$(\varphi_{k,\pm}|\varphi_0) = \frac{\alpha_{k,\pm}}{4} [p_u + isgn(k)p_v]. \quad (15)$$

Evidently, each eigenmode is excited by the external potential. Furthermore, $|(\varphi_{k,\pm}|\varphi_0)|$ depends on the amplitude of the external electric field exclusively. The energy quantities such as the absorption hence do not depend on the direction of the incident field [13], a property which is not so obviously. Another example is the kissing cylinders suggested in Ref. [18], which is obtain by transforming two semi-infinite slabs, with $d_1 < 0$ and $d_2 > 0$, with $w = 1/z$ (Figure (1d)). The corresponding expansion coefficients are calculated as

$$(\varphi_{k,\pm}|\varphi_0) = \frac{\alpha_{k,\pm}}{4} \left[ p_u \frac{e^{ik|d_2+d_1|}}{e^{ik|d_2-d_1|}} \pm 1 + isgn(k)p_v \frac{e^{ik|d_2+d_1|}}{e^{ik|d_2-d_1|}} \mp 1 \right]. \quad (16)$$

Again all the eigenmodes can be excited by the external source. Note that when $d_1 = -d_2$ the cylinders have same radius. The kissing cylinders then possess both $u$ and $v$ mirror symmetry. Consequently, the even mode $\varphi_{k,+}$ or the odd mode $\varphi_{k,-}$ can be excited by $u$ or $v$- polarized electric field alone [15, 17].

As suggested by the equation (6), the integration of the electric field intensity $I_e$ inside the particle is invariant under conformal transformation. On the other hand, the area $A$ of the particle is generally changed significantly. As a direct result, the degree of the local field localization, approximately measured by $I_e/A$, will be modified by the transformation. Taking the kissing cylinders or the crescent above as an example. Before transformation, the area of the one-dimensional slab is infinite. The transformed structure, on the other hand, has a finite area. Consequently, the kissing cylinders or the crescent strongly enhances the local field [13–18].

A few words regarding three-dimensional transformation optics techniques in electrostatic, which has been developed in Ref. [31] lately. Under a general transformation, the
three-dimensional eigen-equation, equation (1), will be no longer invariant and the resultant expression usually is very complicated. One exception is that the Jacobian matrix \( J \) of the transformation satisfies \( JJ^T = f^{-1}(r')I \), with \( I \) being the unit \( 3 \times 3 \) matrix and \( f(r') \) being an arbitrary function of \( r' \) (the transformation used in Ref. [31] possesses this property). Under this condition, the new eigenmodes \( \phi_n \) connect to the previous eigenmodes \( \varphi_n \) of the original system as

\[
\nabla \phi_n = \sqrt{f} \nabla \varphi_n.
\]  (17)

The corresponding eigenvalues \( t_n \) are also different with the previous \( s_n \)

\[
t_n = \frac{\int \theta \nabla \varphi_n dv}{\int f \nabla \varphi_n dv}, \quad s_n = \frac{\int \theta \nabla \varphi_n dv}{\int \nabla \varphi_n dv},
\]  (18)

when the integrations are performed over the previous system. Once the new eigenmodes and eigenvalues are obtained, equation (5-7) can be directly employed without any modification. Note that for a two-dimensional conformal transformation equation (17,18) still work, by simply replacing the function \( f \) with a constant number. Again, we find that the eigenmodes and eigenvalues are invariant under any two-dimensional conformal transformation.

In conclusion, we proved that the electrostatic eigenmodes and eigenvalues of two-dimensional nanosystems are invariant under any conformal transformation. Based on this property, we suggested an new approach to studying the electrostatic responses of the transformed structures. Namely, transforming a geometry is equivalent to transforming the external potential or the expansion coefficients of the invariant eigenmodes. Furthermore, except the potential, energy quantities such as absorption, extinction and electrostatic energy are also found to be conserved. Moreover, the significant conditions to designing broadband nanosystems are the eigenvalues of the system should cover a wide portion of \([0, 1]\) as well as the external source should excite nearly all the eigenmodes.

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