ON EQUIVALENCE RELATIONS INDUCED BY LOCALLY COMPACT ABELIAN POLISH GROUPS

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Abstract. Given a Polish group $G$, let $E(G)$ be the right coset equivalence relation $G^\omega/c(G)$, where $c(G)$ is the group of all convergent sequences in $G$. The connected component of the identity of a Polish group $G$ is denoted by $G_0$.

Let $G, H$ be locally compact abelian Polish groups. If $E(G) \leq_B E(H)$, then there is a continuous homomorphism $S : G_0 \to H_0$ such that $\ker(S)$ is non-archimedean. The converse is also true when $G$ is connected and compact.

For $n \in \mathbb{N}^+$, the partially ordered set $P(\omega)/\text{Fin}$ can be embedded into Borel equivalence relations between $E(\mathbb{R}^n)$ and $E(\mathbb{T}^n)$.

1. Introduction

A topological space is Polish if it is separable and completely metrizable. For more details in descriptive set theory, we refer to [13]. It is an important application of descriptive set theory to study equivalence relations by using Borel reducibility. Given two Borel equivalence relations $E$ and $F$ on Polish spaces $X$ and $Y$ respectively, recall that $E$ is Borel reducible to $F$, denoted $E \leq_B F$, if there exists a Borel map $\theta : X \to Y$ such that for all $x, y \in X$,

$$x Ey \iff \theta(x) F \theta(y).$$

We denote $E \sim_B F$ if both $E \leq_B F$ and $F \leq_B E$, and denote $E \prec_B F$ if $E \leq_B F$ and $F \not\leq_B E$. We refer to [7] for background on Borel reducibility.

Polish groups are important tools in the research on Borel reducibility. A topological group is Polish if its topology is Polish. For a Polish group $G$, the authors [5] defined an equivalence relation $E(G)$ on $G^\omega$ by

$$xE(G)y \iff \lim_n x(n)y(n)^{-1} \text{ converges in } G$$

for $x, y \in G^\omega$. We say that $E(G)$ is the equivalence relation induced by $G$. Indeed, $E(G)$ is the right coset equivalence relation $G^\omega/c(G)$, where $c(G)$ is the group of all convergent sequences in $G$.

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In this article, we focus on equivalence relations induced by locally compact abelian Polish groups. Some interesting results have been found in some special cases. For instance, for \( c_0, e_0, c_1, e_1 \in \mathbb{N} \), \( E(\mathbb{R}^{c_0} \times \mathbb{T}^{e_0}) \leq_B E(\mathbb{R}^{c_1} \times \mathbb{T}^{e_1}) \) iff \( e_0 \leq e_1 \) and \( c_0 + e_0 \leq c_1 + e_1 \) (cf. [5, Theorem 5.19]).

Given a group \( G \), the identity element of \( G \) is denoted by \( 1_G \). If \( G \) is a topological group, the connected component of \( 1_G \) in \( G \) is denoted by \( G_0 \).

Recall that a Polish group \( G \) is non-archimedean if it has a neighborhood base of \( 1_G \) consisting of open subgroups.

**Theorem 1.1.** Let \( G \) and \( H \) be two locally compact abelian Polish groups. If \( E(G) \leq_B E(H) \), then there is a continuous homomorphism \( S : G_0 \to H_0 \) such that \( \ker(S) \) is non-archimedean.

By restricting attention to compact connected abelian Polish groups, we prove the following theorem.

**Theorem 1.2 (Rigid Theorem).** Let \( G \) be a compact connected abelian Polish group and \( H \) a locally compact abelian Polish group. Then \( E(G) \leq_B E(H) \) iff there is a continuous homomorphism \( S : G \to H \) such that \( \ker(S) \) is non-archimedean.

For every normal space \( X \), denoted by \( \text{dim}(X) \) the covering dimension of \( X \), where \( \text{dim}(X) \) is an integer \( \geq -1 \) or the “infinite number \( \infty \)”. Let \( G \) be an abelian topological group. The topological group \( \text{Hom}(G, \mathbb{T}) \) is called the dual group of \( G \), denoted by \( \hat{G} \) (see Section 4). For any finite dimensional compact abelian Polish group \( G \), \( \text{dim}(G) = \text{rank}(\hat{G}) \), the torsion-free rank of \( \hat{G} \) (cf. Lemma 8.13 and Corollary 8.26 of [11]). We say \( G \) is \( n \)-dimensional if \( \text{dim}(G) = n \) for some integer \( n \), or infinite dimensional if \( \text{dim}(G) \) is infinite.

Recall that \( \mathbb{T} \) is the multiplicative group of all complex numbers with modulus 1. For finite dimensional compact abelian Polish groups, we obtain the following results.

**Theorem 1.3.** Let \( G, H \) be locally compact abelian Polish groups.

1. If \( G \) is non-archimedean, then \( E(G) \leq_B E_0^{\omega} \).
2. If \( G \) is not non-archimedean, then \( E(\mathbb{R}) \leq_B E(G) \).
3. If \( G \) is not non-archimedean and \( G_0 \) is open, then \( E(G) \sim_B E(G_0) \).
4. If \( n \) is a positive integer, then \( E(\mathbb{T}^n) \leq_B E(G) \) iff \( \mathbb{T}^n \) embeds in \( G \).
5. If \( n \) is a positive integer and \( G \) is compact, then \( G \) is \( n \)-dimensional iff \( E(\mathbb{R}^n) \leq_B E(G) \leq_B E(\mathbb{T}^n) \).

Let \( \mathcal{P} \) denote the set of all primes. For \( P, Q \in \mathcal{P}^{\omega} \), \( Q \preceq P \) means that there is a co-finite subset \( A \) of \( \omega \) and an injection \( f : A \to \omega \) such that \( Q(n) = P(f(n)) \) for each \( n \in A \).

For \( P \in \mathcal{P}^{\omega} \), we consider the closed subgroup of \( \mathbb{T}^{\omega} \), named \( P \)-adic solenoid, \( \Sigma_P = \{ g \in \mathbb{T}^\omega : \forall l \ g(l) = g(l + 1)P(l) \} \) (cf. [8]).

**Theorem 1.4.** Let \( P \) and \( Q \) be in \( \mathcal{P}^{\omega} \). Then \( E(\Sigma_P) \leq_B E(\Sigma_Q) \) iff \( Q \preceq P \).
The partially ordered set $P(\omega)/\text{Fin}$ is so complicated that every Boolean algebra of size $\leq \omega_1$ embeds into it (see [2]). We usually express that some classes of Borel equivalence relations are extremely complicated under the order of Borel reducibility by showing that $P(\omega)/\text{Fin}$ embeds into them. For instance, Louveau-Velickovic [14] and Yin [18] showed that $P(\omega)/\text{Fin}$ embeds into both LV-equalities and Borel equivalence relations between $\ell_p$ and $\ell_q$ respectively. As an application, we prove that the partially ordered set $P(\omega)/\text{Fin}$ embeds into the partially ordered set of all $E(G)$’s under the ordering of Borel reducibility.

**Theorem 1.5.** Let $n \in \mathbb{N}^+$. Then for $A \subseteq \omega$, there is a $n$-dimensional compact connected abelian Polish group $G_A$ such that $E(\mathbb{R}^n) <_B E(G_A) <_B E(\mathbb{T}^n)$ and for $A, B \subseteq \omega$, we have
\[ A \subseteq^* B \iff E(G_A) \leq_B E(G_B). \]

We also get a sufficient and necessary condition concerning dual groups.

**Theorem 1.6 (Dual Rigid Theorem).** Let $G$ be a compact connected abelian Polish group and $H$ a locally compact abelian Polish group. Then $E(G) \leq_B E(H)$ iff there is a continuous homomorphism $S^*: \hat{H} \to \hat{G}$ such that $\hat{G}/\text{im}(S^*)$ is a torsion group.

**Notation convention:** In this article, the addition operation of any subgroup of $\mathbb{R}^n$ is denoted by $+$ and its identity element is denoted by $0$. Unless otherwise specified, for abstract abelian topological groups $G$, we still use multiplicative notation to express the group operation, and use $1_G$ to express the identity element of $G$, since we often consider subgroups of $\mathbb{T}^\omega$.

This article is organized as follows. In section 2, we prove theorems 1.1–1.3. In section 3, we consider $P$-adic solenoids and prove theorems 1.4 and 1.5. Finally, in section 4, we consider dual groups and prove Theorem 1.6.

## 2. Locally Compact Abelian Polish groups

**Definition 2.1** ([5 Definition 6.1]). Let $G$ be a Polish group. We define equivalence relation $E_*(G)$ on $G^\omega$ as: for $x, y \in G^\omega$,
\[ xE_*(G)y \iff \lim_n x(0)x(1)\ldots x(n)y(n)^{-1}\ldots y(1)^{-1}y(0)^{-1} \text{ converges}. \]

The following is an easy but important observation.

**Proposition 2.2.** Let $G$ be a Polish group. Then $E(G) \sim_B E_*(G)$.

**Proof.** To see that $E(G) \leq_B E_*(G)$, for $x \in G^\omega$, we define $\theta(x) \in G^\omega$ as
\[ \theta(x)(n) = \begin{cases} x(0), & n = 0, \\ x(n-1)x(n), & n > 0. \end{cases} \]
Then $\theta$ witnesses that $E(G) \leq_B E_*(G)$.

To show the converse, for $x \in G^\omega$, we define $\vartheta(x) \in G^\omega$ as
\[ \vartheta(x)(n) = x(0)x(1)\ldots x(n). \]
Then \( \vartheta \) witnesses that \( E_*(G) \leq_B E(G) \). \( \square \)

In this article, we focus on abelian Polish groups. For abelian Polish groups \( G \), it is more convenient to take \( E_*(G) \) as research object than \( E(G) \).

Some reducibility results are obtained in [5]. Since we will use them again and again in this article, for the convenience of readers, we list them as follows.

**Proposition 2.3** ([5 Proposition 3.4]). Let \( G, H \) be two Polish groups. If \( G \) is topologically isomorphic to a closed subgroup of \( H \), then \( E(G) \leq_B E(H) \).

A metric \( d \) on a group \( G \) is called two sided invariant if \( d(ghl, gkl) = d(h, k) \) for all \( g, h, k, l \in G \). We say that a Polish group \( G \) is TSI if it admits a compatible two sided invariant metric. Any abelian Polish group is TSI (cf. [7 Exercise 2.1.4]).

**Lemma 2.4** ([5 Theorem 6.5]). Let \( G, H, K \) be three TSI Polish groups. Suppose \( \psi : G \to H \) and \( \varphi : H \to K \) are continuous homomorphisms with \( \varphi(\psi(G)) = K \) such that \( \ker(\varphi \circ \psi) \) is non-archimedean. If the interval \([0, 1] \) embeds into \( H \), then \( E(G) \leq_B E(H) \).

**Lemma 2.5** ([5 Theorem 6.13]). Let \( G, H \) be TSI Polish groups such that \( H \) is locally compact. If \( E(G) \leq_B E(H) \), then there exist an open normal subgroup \( G_c \) of \( G \) and a continuous map \( S : G_c \to H \) with \( S(1_G) = 1_H \) such that, for \( x, y \in G_c \), if \( \lim_n x(n)y(n)^{-1} = 1_G \), then

\[
xE_*(G_c)y \iff S(x)E_*(H)S(y),
\]

where \( S(x)(n) = S(x(n)), S(y)(n) = S(y(n)) \) for each \( n \in \omega \).

In particular, if \( G = G_c \) and the interval \([0, 1] \) embeds in \( H \), then the converse is also true.

**Remark 2.6.** Since \( G_c \) in the preceding lemma is an open subgroup, it is also closed. So \( G_0 \subseteq G_c \) as it is connected. Since \( S \) is continuous, we have \( S(G_0) \subseteq H_0 \). Moreover, for all \( x, y \in G_0' \), if \( \lim_n x(n)y(n)^{-1} = 1_G \), we have

\[
xE_*(G_0)y \iff S(x)E_*(H_0)S(y).
\]

The next lemma plays the key role in the proof of Theorem 2.8.

**Lemma 2.7.** Let \( G \) and \( H \) be two abelian Polish groups such that:

1. \( H \) is locally compact,
2. \( H_0 \subseteq \mathbb{R}^\omega \times \mathbb{T}^\omega \),
3. there is a nonzero continuous homomorphism \( f : \mathbb{R}^m \to G \) for some \( m \in \mathbb{N}^+ \).

If \( E_*(G) \leq_B E_*(H) \), then there is a continuous map \( S : G_0 \to H_0 \) such that the map \( S \) restricted on \( f(\mathbb{R}^m) \) is a homomorphism to \( H_0 \).

**Proof.** First, from Remark 2.6 we can obtain a continuous map \( S : G_0 \to H_0 \) with \( S(1_{G_0}) = 1_{H_0} \) such that, for \( x, y \in G_0' \), if \( \lim_n x(n)y(n)^{-1} = 1_{G_0} \), then

\[
xE_*(G_0)y \iff S(x)E_*(H_0)S(y),
\]
where \( S(x)(n) = S(x(n)), S(y)(n) = S(y(n)) \) for each \( n \in \omega \).

Since \( H_0 \subseteq \mathbb{R}^\omega \times \mathbb{T}^\omega \), without loss of generality we may assume that 
\( h(2k) \in \mathbb{R} \) and \( h(2k + 1) \in \mathbb{T} \) for all \( h \in H_0 \). For \( k \in \omega \), we define continuous homomorphisms \( \phi^{2k} : H_0 \to \mathbb{R} \) and \( \phi^{2k+1} : H_0 \to \mathbb{T} \) by \( \phi^j(h) = h(j) \).

Now fix \( g_0, g_1 \in f(\mathbb{R}^m) \) and find \( a_0, a_1 \in \mathbb{R}^m \) such that \( f(a_0) = g_0 \) and \( f(a_1) = g_1 \). For \( t \in [0, 1] \) and \( l \in \{1, 2\} \), define \( a^l(t) \in \mathbb{R}^m \) as

\[
a^l(t) = \begin{cases} 
a_0 + t(a_1 - a_0), & l = 1, \\
t(a_0 + a_1), & l = 2.
\end{cases}
\]

By the following claim, we can easily construct a continuous function \( F_j^l : [0, 1] \to \mathbb{R} \) for each \( l \in \{1, 2\} \) and \( j \in \omega \) such that

\[
(*) \quad F_j^{2k}(t) = \phi^{2k}(S(f(a^l(t)))) \quad \text{exp}(iF_j^{2k+1}(t)) = \phi^{2k+1}(S(f(a^l(t))))\.
\]

The nontrivial part of the construction, i.e., \( j = 2k+1 \), follows from a more general claim.

**Claim 1.** Given a continuous function \( \gamma : [0, 1] \to \mathbb{T} \) and \( t_0 \in [0, 1] \) with \( \exp(is_0) = \gamma(t_0) \) for some \( s_0 \in \mathbb{R} \), there exists a continuous function \( \tilde{\gamma} : [0, 1] \to \mathbb{T} \) such that \( \exp(i\tilde{\gamma}(t)) = \gamma(t) \) and \( \tilde{\gamma}(t_0) = s_0 \).

**Proof.** Note that the map \( t \mapsto \exp(it) \) is a covering map from \( \mathbb{R} \) to \( \mathbb{T} \), and the interval \( [0, 1] \) is simply connected (see Definitions A2.1 and Proposition A2.8 of [11]). So such a \( \tilde{\gamma} \) exists (cf. [11, Definition A2.6]).

For the convenience of readers, we briefly explain the construction of \( \tilde{\gamma} \). Since the map \( t \mapsto \exp(it) \) is a local homeomorphism, by the continuity of \( \gamma \), for each \( u \in [0, 1] \), there is an open interval \( J_u \) containing \( u \) and a continuous function \( \tilde{\gamma}_u : J_u \cap [0, 1] \to \mathbb{T} \) such that \( \sup_{t, t' \in J_u \cap [0, 1]} |\gamma(t) - \gamma(t')| < \frac{1}{2} \) and \( \exp(i\tilde{\gamma}_u(t)) = \gamma(t) \) for \( t \in J_u \cap [0, 1] \). Note that \( \exp(i(\tilde{\gamma}_u(t) + 2\pi n)) = \exp(i\tilde{\gamma}_u(t)) \) for each \( n \in \mathbb{Z} \). By the compactness of \([0, 1]\), there are \( u_0, u_1, \ldots, u_q \in [0, 1] \) such that \([0, 1] \subseteq \bigcup_{0 \leq i \leq q} J_{u_i} \). We can find \( p_0, p_1, \ldots, p_q \in \mathbb{Z} \) such that for each \( t \in J_{u_i} \cap J_{u_j} \cap [0, 1] \), we have \( \tilde{\gamma}_{u_i}(t) + 2p_i \pi = \tilde{\gamma}_{u_j}(t) + 2p_j \pi \). Then for \( t \in [0, 1] \cap J_{u_i} \), let \( \tilde{\gamma}'(t) = \tilde{\gamma}_{u_i}(t) + 2p_i \pi \). In the end, we put \( \tilde{\gamma}(t) = \tilde{\gamma}'(t) - \gamma(t_0) + s_0 \). It is obvious that \( \exp(i\tilde{\gamma}(t)) = \gamma(t) \) and \( \tilde{\gamma}(t_0) = s_0 \). \( \square \)

Note that \( S(f(a^2(0))) = 1_H \). We can assume that \( F_j^2(0) = 0 \) for each \( j \).

Next we claim that \( F_j^l \) are linear functions.

**Claim 2.** \( F_j^l(t) = F_j^l(0) + t(F_j^{l+1}(1) - F_j^{l+1}(0)) \) for \( t \in [0, 1] \).

**Proof.** We only verify the claim for \( l = 1 \). It is similar for \( l = 2 \).

Fix \( j_0 \in \omega \). Define \( \gamma : [0, 1] \to \mathbb{R} \) as \( \gamma(t) = F_j^1(t) - F_j^1(0) - t(F_j^1(1) - F_j^1(0)) \). Note that \( \gamma \) is continuous and \( \gamma(0) = \gamma(1) = 0 \). We only need to prove that \( \gamma(t) = 0 \) for all \( t \in (0, 1) \).

If not, without loss of generality we may assume that \( \gamma(t_0) > 0 \) for some \( t_0 \in (0, 1) \). Similar to the proof of [5] Lemma 6.17, we can find \( 0 < \xi < \frac{1}{2} \) with
\[ \xi_1 < \xi_2 < \cdots < \xi < 1 \] such that \( \gamma(\xi_k) = \frac{k+1}{k+2} \gamma(t_0) \) for each \( k \in \omega \), and

\[ 1 > \zeta_0 > \zeta_2 > \cdots > \zeta > 0, K \in \omega \] such that, for \( k \geq K \), we have

\[ \xi - \xi_k > \zeta_k - \zeta > \xi - \xi_{k+1}. \]

\[ \lim_k \xi_k = \xi, \lim_k \zeta_k = \zeta, \gamma(\xi) = \gamma(t_0), \text{ and } \gamma(\zeta) > \gamma(\zeta_k) \text{ for each } k. \]

Note that \( f : \mathbb{R}^m \rightarrow G \) is a nonzero continuous homomorphism. For \( p \in \omega \), we set

\[ x(p) = \begin{cases} f(a^1(\xi_k)), & p = 2k; \\ f(a^1(\zeta_k)), & p = 2k + 1, \end{cases} \]

\[ y(p) = \begin{cases} f(a^1(\xi_k)), & p = 2k; \\ f(a^1(\zeta_k)), & p = 2k + 1. \end{cases} \]

From the alternating series test, the following series

\[ (\xi - \xi_0) + (\zeta - \zeta_0) + \cdots + (\xi - \xi_k) + (\zeta - \zeta_k) + \cdots \]

is convergent. Then

\[
x(0)x(1)\cdots x(2k)y(2k)^{-1}\cdots y(1)^{-1}y(0)^{-1}
= x(0)y(0)^{-1}x(1)y(1)^{-1}\cdots x(2k)y(2k)^{-1}
= f(((\xi - \xi_0) + (\zeta - \zeta_0) + \cdots + (\xi - \xi_k))(a_1 - a_0)).
\]

Since \( f \) is continuous and \( \lim_{p}x(p)y(p)^{-1} = 1_G \), we have \( xE_s(G)y \). And hence, by Remark 2.4, we have \( S(x)E_s(H)S(y) \).

On the other hand, we have

\[
\sum_k(\gamma(\xi) - \gamma(\xi_k) + \gamma(\zeta) - \gamma(\zeta_k)) \geq \sum_k(\gamma(\xi) - \gamma(\xi_k)) = \sum_k \frac{\gamma(t_0)}{k+2} = \infty.
\]

Note that

\[
F^1_{j_0}(\xi) - F^1_{j_0}(\xi_k) + F^1_{j_0}(\zeta) - F^1_{j_0}(\zeta_k)
= \gamma(\xi) - \gamma(\xi_k) + \gamma(\zeta) - \gamma(\zeta_k) + (\xi - \xi_k + \zeta - \zeta_k)(F^1_{j_0}(1) - F^1_{j_0}(0)).
\]

If \( j_0 = 2i \), then

\[
\phi^{2i}(S(x(0))S(x(1))\cdots S(x(2k))S(y(2k))^{-1}\cdots S(y(1))^{-1}S(y(0))^{-1})
= F^1_{j_0}(\xi) - F^1_{j_0}(\xi_0) + F^1_{j_0}(\zeta) - F^1_{j_0}(\zeta_0) + \cdots + F^1_{j_0}(\xi) - F^1_{j_0}(\xi_k).
\]

Thus \( S(x)E_s(H)S(y) \) fails. We get a contradiction. If \( j_0 = 2i + 1 \), following similar arguments, we can also get a contradiction. This complete the proof of the claim.

Now by Claim 2 and \( F^2_j(0) = 0 \), we know that

\[
F^2_j(1/2) = F^1_j(0) + (F^1_j(1) - F^1_j(0))/2 = (F^1_j(0) + F^1_j(1))/2,
\]

\[
F^2_j(1/2) = F^2_j(1)/2.
\]

By comparing equation (*) before Claim 1, it follows that

\[
S(f(a^1(1/2)))^2 = S(f(a_0))S(f(a_1)) = S(g_0)S(g_1),
\]

\[
S(f(a^2(1/2)))^2 = S(f(a_0 + a_1)) = S(g_0g_1).
\]

Since \( a^1(1/2) = a^2(1/2) \), we have \( S(g_0)S(g_1) = S(g_0g_1) \).

So, the map \( S : f(\mathbb{R}^m) \rightarrow H_0 \) is a continuous homomorphism. \qed
Let us recall the structure of Hausdorff locally compact abelian groups. Let $G$ be a Hausdorff locally compact abelian group, then $G$ is topologically isomorphic to the group $\mathbb{R}^n \times H$, where $H$ is a locally compact abelian group containing a compact open subgroup (cf. [10 Theorem 24.30]). Moreover, if $G$ is also connected, then it is a direct product of a compact connected abelian group $K$ and the group $\mathbb{R}^n$ (cf. [10 Theorem 9.14]). Any locally compact connected metrizable abelian group can be embedded as a closed subgroup of $\mathbb{R}^n \times \mathbb{T}^\omega$. In particular, all compact metrizable abelian groups can be embedded in $\mathbb{T}^\omega$ (see Page 119 of [1]). $G$ is said to be solenoidal if there is a continuous homomorphism $f : \mathbb{R} \to G$ such that $f(\mathbb{R})$ is dense in $G$ (see [10 (9.2)]). It is well known that a compact metrizable abelian group is solenoidal iff it is connected (see Page 13 and Proposition 5.16 of [1]). Thus for each locally compact connected metrizable abelian group $G$, there is a continuous homomorphism $f : \mathbb{R}^m \to G$ which satisfies $f(\mathbb{R}^m) = G$. For more details on locally compact abelian groups, we refer to [10].

By applying Lemma 2.7 for locally compact abelian Polish groups, we get the following results.

**Theorem 2.8.** Let $G$ and $H$ be two locally compact abelian Polish groups. If $E(G) \leq_B E(H)$, then there is a continuous homomorphism $S : G_0 \to H_0$ such that $\ker(S)$ is non-archimedean.

*Proof.* If $E(G) \leq_B E(H)$, then $E_\ast(G) \leq_B E_\ast(H)$. Without loss of generality we may assume that $G_0$ is nontrivial. First note that $H_0$ can be embedded into $\mathbb{R}^n \times \mathbb{T}^\omega$. Thus we may assume without loss of generality that $H_0 \subseteq \mathbb{R}^\omega \times \mathbb{T}^\omega$. Let $f$ be a continuous homomorphism from $\mathbb{R}^m$ to $G_0$ with $f(\mathbb{R}^m) = G_0$. Then by Lemma 2.7 there exists a continuous map $S : G_0 \to H_0$ such that the map $S$ restricted on $f(\mathbb{R}^m)$ is a homomorphism to $H_0$. Since $f(\mathbb{R}^m)$ is dense in $G_0$, we see that $S$ is a homomorphism from $G_0$ to $H_0$.

Then we only need to check that $\ker(S)$ is non-archimedean. Assume toward a contradiction that $\ker(S)$ is not non-archimedean. Note that $\ker(S)$ is an abelian Polish group. Fix a compatible two sided invariant metric on $\ker(S)$. Let $V_k \subseteq \ker(S)$, $k \in \omega$ be an open symmetric neighborhood base of $1_{\ker(S)} = 1_G$ with $\lim_k \text{diam}(V_k) = 0$. Then there exists a $k_0 \in \omega$ such that $V_{k_0}$ does not contain any open subgroup of $\ker(S)$. Since $V_k$ is symmetric, $\bigcup_m V^m_k$ is an open subgroup of $\ker(S)$, so $\bigcup_m V^m_k \supseteq V_{k_0}$ for each $k$. Thus we can find an $m_k \in \omega$ and $g_{k,0}, \ldots, g_{k,m_k-1} \in V_k$ such that $g_{k,0}g_{k,1} \cdots g_{k,m_k-1} \notin V_{k_0}$.

Denote $M_{-1} = 0$ and $M_k = m_0 + m_1 + \cdots + m_k$ for $k \in \omega$. Now for $n \in \omega$, define

$$x(n) = \begin{cases} g_{k,j}, & n = M_{k-1} + j, 0 \leq j < m_k, \\ 1_G, & \text{otherwise.} \end{cases}$$

Therefore $x E_\ast(G) 1_{G^\omega}$ fails. Note that we have $\lim_n x(n) = 1_G$ and $S(x(n)) = 1_H$ for each $n$. So it is trivial that $S(x)E_\ast(H_0)S(1_{G^\omega})$, where $S(x(n)) = S(x(n))$, contradicting Lemma 2.5. \qed
In particular, if $G$ is compact connected, then the converse of Theorem 2.8 is also true.

**Theorem 2.9** (Rigid Theorem). Let $G$ be a compact connected abelian Polish group and $H$ a locally compact abelian Polish group. Then $E(G) \leq_B E(H)$ iff there is a continuous homomorphism $S : G \to H$ such that ker($S$) is non-archimedean.

**Proof.** Let $S$ be a continuous homomorphism from $G$ to $H$ such that ker($S$) is non-archimedean. Since $G$ is compact, $S(G)$ is a compact, thus closed subgroup of $H$. So we have $E(S(G)) \leq_B E(H)$.

Note that $S(G)$ is also a compact connected abelian Polish group. Let $f$ be a continuous homomorphism $f : \mathbb{R} \to S(G)$ such that $f(\mathbb{R}) = S(G)$. Then ker($f$) is a proper closed subgroup of $\mathbb{R}$. Hence ker($f$) is a discrete group. This gives that the interval $[0, 1]$ embeds in $S(G)$. Then by Lemma 2.4 we get that $E(G) \leq_B E(S(G)) \leq_B E(H)$.

On the other hand, if $E(G) \leq_B E(H)$, by Theorem 2.8 there is a continuous homomorphism $S : G_0 \to H_0$ such that ker($S$) is non-archimedean. Since $G$ is connected, we have $G = G_0$. □

**Corollary 2.10.** Let $G$ be a compact connected abelian Polish group and $H$ a locally compact abelian Polish group. Suppose $H_0 \cong \mathbb{R}^n \times K$, where $K$ is a compact connected abelian Polish group. Then $E(G) \leq_B E(H)$ iff $E(G) \leq_B E(K)$.

**Proof.** ($\Leftarrow$) part is trivial, since $E(K) \leq_B E(H_0) \leq_B E(H)$.

($\Rightarrow$). If $E(G) \leq_B E(H)$, then there exists a continuous homomorphism $S : G \to H$ such that ker($S$) is non-archimedean, so $S(G)$ is a connected compact subgroup of $H$, thus $S(G) \subseteq H_0$. Without loss of generality, we assume that $H_0 = \mathbb{R}^n \times K$. Let $\pi : H_0 \to \mathbb{R}^n$ and $\pi_K : H_0 \to K$ be canonical projections. Then $\pi(S(G))$ is a compact subgroup of $\mathbb{R}^n$, so $\pi(S(G)) = \{0\}$. It follows that ker($\pi_K \circ S$) = ker($S$). Applying Theorem 2.9 on $\pi_K \circ S : G \to K$, we have $E(G) \leq_B E(K)$. □

Recall that a topological group $G$ is totally disconnected if $G_0 = \{1_G\}$. For any locally compact abelian Polish group, it is totally disconnected iff it is non-archimedean (cf. [11] Theorem 1.34).

For every normal space $X$, denoted by dim($X$) the covering dimension of $X$, where dim($X$) is an integer $\geq -1$ or the “infinite number $\infty$”. We omit the definition of covering dimension since it is very complicated (see page 54 of [6]). We recall the following useful facts concerning compact abelian group $G$: dim($G$) = $n < \infty$ iff $G$ has a totally disconnected closed subgroup $\Delta$ such that $G/\Delta \cong \mathbb{T}^n$ iff there is a compact totally disconnected subgroup $N$ of $G$ and a continuous surjective homomorphism $\varphi : N \times \mathbb{R}^n \to G$ which has a discrete kernel (see Theorem 8.22 and Corollary 8.26 of [11]). In this case, we say that $G$ is finite dimensional (cf. [11] Definitions 8.23). Clearly, dim($G$) = 0 iff $G$ is totally disconnected. For more details on compact abelian groups, see [11].
Now we recall two equivalence relations $E^\omega_0$ and $E(M; 0)$ (see [4, Definition 3.2]). The equivalence relation $E^\omega_0$ on $2^{\omega \times \omega}$ defined by

$$xE_0^\omega y \iff \forall k \exists m \forall n \geq m \, (x(n, k) = y(n, k)).$$

Fix a metric space $M$. The equivalence relation $E(M; 0)$ on $M^\omega$ defined by

$$xE(M; 0)y \iff \lim_n d(x(n), y(n)) = 0.$$

From the above discussions, we can establish the following theorem.

**Theorem 2.11.** Let $G, H$ be locally compact abelian Polish groups.

1. If $G$ is non-archimedean, then $E(G) \leq_B E_0^\omega$.
2. If $G$ is not non-archimedean, then $E(\mathbb{R}) \leq_B E(G)$.
3. If $G$ is not non-archimedean and $G_0$ is open, then $E(G) \sim_B E(G_0)$.
4. If $n$ is a positive integer, then $E(\mathbb{T}^n) \leq_B E(G)$ iff $\mathbb{T}^n$ embeds in $G$.
5. If $n$ is a positive integer and $G$ is compact, then $G$ is $n$-dimensional iff $E(\mathbb{R}^n) <_B E(G) \leq_B E(\mathbb{T}^n)$.

**Proof.** (1) It follows from [5, Theorem 3.5.(3)].

(2) Note that $G$ is not totally disconnected (cf. [11, Theorem 1.34]), so $G_0$ contains at least two points. We have $G_0 \cong \mathbb{R}^n \times K$, where $K$ is a compact connected abelian group. If $n > 0$, it is trivial that $E(\mathbb{R}) \leq_B E(G)$. By Proposition 2.3, $E(K) \leq_B E(G_0) \leq_B E(G)$. Thus we may assume that $G$ is compact connected and $G \subseteq \mathbb{T}^\omega$. Note that there is a continuous homomorphism $f : \mathbb{R} \to G$ such that $f(\mathbb{R}) = G$. For $g \in G \subseteq \mathbb{T}^\omega$ and $p \in \omega$, let $\phi_p(g) = g(p)$. Since $G$ contains at least two points, we can find $p_0 \in \omega$ such that $\phi_{p_0}(f(\mathbb{R})) \neq \{1_\omega\}$, so $\phi_{p_0}(f(\mathbb{R})) = T$. By [11, Corollary 8.24], the interval $[0, 1]$ embeds in $G$. Then by Theorem 2.4, we have $E(\mathbb{R}) \leq_B E(G)$.

(3) By [10, §24.45], we have $G \cong G_0 \times G/G_0$. Since $G_0$ is open, $G/G_0$ is countable and discrete. By [5, Corollary 3.6], this implies that $E(G_0 \times G/G_0) \sim_B E(G_0)$ and thus $E(G) \sim_B E(G_0)$.

(4) The “if” part follows Proposition 2.3. Assume that $E(\mathbb{T}^n) \leq_B E(G)$. By Theorem 2.3 and [7, Corollary 2.3.4], there is a closed subgroup $\Delta$ of $\mathbb{T}^n$ such that the group $\mathbb{T}^n/\Delta$ can be embedded in $G$, where $\Delta$ is non-archimedean. It is obvious that $\mathbb{T}^n/\Delta$ is a locally connected, connected and compact abelian Polish group. By [1, Proposition 8.17], $\mathbb{T}^n/\Delta \cong \mathbb{T}^n$.

(5) If $n = \dim(G)$, then we have $(N \times \mathbb{R}^n)/\Delta_1 \cong G$ and $G/\Delta_2 \cong \mathbb{T}^n$, where $N, \Delta_1$, and $\Delta_2$ are totally disconnected, and hence are non-archimedean. Then Proposition 2.3 and Lemma 2.4 imply that

$$E(\mathbb{R}^n) \leq_B E(N \times \mathbb{R}^n) \leq_B E(G) \leq_B E(\mathbb{T}^n).$$

So we only need to show that $E(G) \not\leq_B E(\mathbb{R}^n)$. To see this, assume toward a contradiction that $E(G) \leq_B E(\mathbb{R}^n)$. By Theorem 2.3, there exists a continuous homomorphism $S : G_0 \to \mathbb{R}^n$ such that $\ker(S)$ is non-archimedean. Note that $\mathbb{R}^n$ has no nontrivial compact connected subgroup. So this implies that $S(G_0) = \{0\}$, contradicting that $\ker(S)$ is non-archimedean.
On the other hand, suppose $E(\mathbb{R}^n) <_B E(G) \leq B E(\mathbb{T}^n)$. Let $m = \dim(G)$. By (1) we have $m > 0$. Assume for contradiction that $m = \infty$, then there exists a continuous homomorphism $S : G_0 \to \mathbb{T}^n$ such that $\ker(S)$ is non-archimedean. Then we have $\dim(G_0/\ker(S)) = \infty$, and hence $[0,1]^\omega$ embeds into $G_0/\ker(S)$ (cf. [11] Corollary 8.24). By [7] Corollary 2.3.4, $S$ induce an embedding from $G_0/\ker(S)$ to $\mathbb{T}^n$. So $[0,1]^\omega$ embeds into $\mathbb{T}^n$, contradicting that $n$ is finite. Therefore, we have $0 < m < \infty$, and hence $E(\mathbb{R}^m) <_B E(G) \leq B E(\mathbb{T}^m)$. Then [5] Theorem 6.19 gives $m = n$. □

**Remark 2.12.** Let $G$ and $H$ be two locally compact abelian Polish groups. Suppose that $G_0$ is an open subgroup of $G$, and that $G_0$ is compact or $G_0 \cong \mathbb{R}$. Then theorems 2.8, 2.9 and 2.11(2)–(3) imply that $E(G) \leq_B E(H)$ iff there is a continuous homomorphism $S : G_0 \to H_0$ such that $\ker(S)$ is non-archimedean. This generalizes Rigid Theorem, i.e., Theorem 2.9. We don’t know whether this can be generalized to all locally compact abelian Polish groups.

**Question 2.13.** Does the converse of Theorem 2.8 hold for all locally compact abelian Polish groups?

**Question 2.14.** Let $G$ be a locally compact abelian Polish group. If $G$ is not non-archimedean, does $E(G) \sim_B E(G_0)$?

### 3. P-adic Solenoids

Let $P = (P(0), P(1), \ldots)$ be a sequence of integers greater than 1. Recall that the $P$-adic solenoid $S_P$ is defined by

$$S_P = \{g \in \mathbb{T}^\omega : \forall l \, (g(l) = g(l + 1)^{P(l)})\}.$$  

In particular, if for each $i$, $P(i)$ is a prime number, then the $P$-adic solenoid is denoted by $\Sigma_P$ (cf. [8]). Let $\mathcal{P}$ denote the set of all primes. The group $S_P$ is topologically isomorphic to $\Sigma_{P'}$ for some $P' \in \mathcal{P}^\omega$ satisfying that $P(l) = P'(i_l) \cdots P'(i_{l+1} - 1)$, where $0 = i_0 < i_1 < \cdots < i_l < \cdots$. For example, we have $S_{(4,6,8,9,\ldots)} \cong \Sigma_{(2,2,2,2,2,2,2,3,3,3,3,3,\ldots)}$.

It is well known that, the group $\Sigma_P$ is a compact connected abelian group which is neither locally connected (cf. [8]), nor arcwise connected (see [11] Theorem 8.27]). Every nontrivial proper closed subgroup $H$ of a $P$-adic solenoid is totally disconnected (cf. [12] Proposition 2.7]), and thus $H$ is non-archimedean. Clearly, $\Sigma_P$ is an 1-dimensional and metrizable group.

Denote $\Omega = \{\mathbb{R} \times T, \Sigma_P : P \in \mathcal{P}\}$.

**Lemma 3.1.** Let $m,n \in \mathbb{N}^+$ and let $G_1,G_2,\ldots,G_m,H_1,H_2,\ldots,H_n \in \Omega$. Then the following are equivalent:

1. $E(G_1 \times G_2 \times \cdots \times G_m) \leq_B E(H_1 \times H_2 \times \cdots \times H_n)$.
2. There is an injective map $\theta^* : \{1,2,\ldots,m\} \to \{1,2,\ldots,n\}$ such that $E(G_i) \leq_B E(H_{\theta^*(i)})$ for $1 \leq i \leq m$.

In particular, $E(G_i^n) \leq_B E(H_i^n)$ iff $m \leq n$ and $E(G_1) \leq_B E(H_1)$. 

Proof. (2) ⇒ (1) is obvious. We only prove (1) ⇒ (2).

Denote \( G = G_1 \times G_2 \times \cdots \times G_m \) and \( H = H_1 \times H_2 \times \cdots \times H_n \). For \( 1 \leq i \leq m \), let \( e^i \) be the canonical injection of \( G_i \) into \( G_1 \times \cdots \times G_m \), i.e., \( e^i(g) = (1_{G_1}, \ldots, 1_{G_{i-1}}, g, 1_{G_{i+1}}, \ldots, 1_{G_m}) \).

Suppose \( E(G) \leq_B E(H) \). Since \( G \) and \( H \) are both connected, by Theorem 2.8, there is a continuous homomorphism \( S : G \rightarrow H \) such that \( \ker(S) \) is non-archimedean. For each \( 1 \leq j \leq n \), let \( \pi_j \) be the canonical projection from \( H \) onto \( H_j \).

Note that, except for \( \mathbb{R} \), all groups in \( \Omega \) are compact. By rearranging, we may assume that there is an \( i_0 \leq m \) such that, \( G_i \) is compact for \( 1 \leq i \leq i_0 \), and \( G_i = \mathbb{R} \) for \( i_0 < i \leq m \).

For any \( 1 \leq i \leq i_0 \), since \( \ker(S) \) is non-archimedean, there exists \( j \) satisfying that \( \pi_j(S(e^i(G_i))) \neq \{1_{H_j}\} \). Note that \( H_j \) has no nontrivial proper connected compact subgroup. It follows that \( \pi_j(S(e^i(G_i))) = H_j \). Now we construct a bipartite graph \( G[X, Y] \) as follows. Let \( X = \{G_1, G_2, \ldots, G_{i_0}\} \), \( Y = \{H_j : \exists i \ (1 \leq i \leq i_0) \text{ and } \pi_j(S(e^i(G_i))) = H_j \} \).

For \( G_i \in X \) and \( H_j \in Y \), we put an edge between \( G_i \) and \( H_j \) if \( \pi_j(S(e^i(G_i))) = H_j \). Given \( K \subseteq X \), we denote the set of all neighbors of the vertices in \( K \) by \( N(K) \).

Next we show that \( |N(K)| \geq |K| \) for all \( K \subseteq X \). Given \( K \subseteq X \), denote
\[
G^K = \{x \in G : x(i) = 1_{G_i} \text{ for all } G_i \not\in K\},
\]
\[
H^{N(K)} = \{z \in H : z(j) = 1_{H_j} \text{ for all } H_j \not\in N(K)\}.
\]
Then the restriction of \( S \) on \( G^K \) is a continuous homomorphism to \( H^{N(K)} \). By Theorem 2.9 \( E(G^K) \leq_B E(H^{N(K)}) \). Again by Theorem 2.11(5), this implies \( E(\mathbb{R}^{|K|}) \leq_B E(\mathbb{T}^{|N(K)|}) \). Then [5] Theorem 6.19] gives \( |N(K)| \geq |K| \).

By Hall’s theorem (cf. 3 Theorem 16.4)], there is an injective map \( \theta^* : \{1, 2, \ldots, i_0\} \rightarrow \{1, 2, \ldots, n\} \) such that \( \pi_{\theta^*(i)}(S(e^i(G_i))) = H_{\theta^*(i)} \). Since every proper closed subgroup of \( G_i \) is non-archimedean, from Theorem 2.9 we have \( E(G_i) \leq_B E(H_{\theta^*(i)}) \).

In the end, since \( \dim(G) = m \) and \( \dim(H) = n \), by Theorem 2.11(5), we have \( E(\mathbb{R}^m) \leq_B E(\mathbb{T}^n) \). So \( m \leq n \). Since \( E(\mathbb{R}) \leq_B E(H_j) \) for each \( j \), we can trivially extend \( \theta^* \) to an injection from \( \{1, 2, \ldots, m\} \) to \( \{1, 2, \ldots, n\} \) such that \( E(G_i) \leq_B E(H_{\theta^*(i)}) \) for each \( i \).

Let \( P \) and \( Q \) be in \( \mathcal{P}^\omega \). We write \( Q \leq_P \) provided there is a co-finite subset \( A \) of \( \omega \) and an injection \( f : A \rightarrow \omega \) such that \( Q(n) = P(f(n)) \) for each \( n \in A \) (for more details, see [8, 9, 16]).

**Lemma 3.2** (folklore). Let \( P \) and \( Q \) be in \( \mathcal{P}^\omega \). Then the following are equivalent:

1. There is a nonzero continuous homomorphism \( f : \Sigma_P \rightarrow \Sigma_Q \).
2. There is a surjective continuous homomorphism \( g : \Sigma_P \rightarrow \Sigma_Q \).
There is a surjective continuous map \( h : \Sigma_P \to \Sigma_Q \).

\( Q \preceq P \).

**Proof.** (2) \( \Rightarrow \) (1) and (2) \( \Rightarrow \) (3) are obvious. (1) \( \Rightarrow \) (2) follows immediately from the fact that each nontrivial proper closed subgroup of a \( P \)-adic solenoid is totally disconnected. The equivalence of (3) and (4) follows from \([16, \text{Theorem 4.4}]\).

It remains to show (3) \( \Rightarrow \) (1). Let \( h \) be a surjective continuous map from \( \Sigma_P \) to \( \Sigma_Q \). Without loss of generality assume that \( h(1_{\Sigma_P}) = 1_{\Sigma_Q} \). Then there exists a continuous homomorphism \( f : \Sigma_P \to \Sigma_Q \) such that \( h \) is homotopic to \( f \) (cf. \([17, \text{Corollary 2}]\)). Since \( \Sigma_Q \) is not arcwise connected, \( \ker(f) \neq \Sigma_P \). \( \square \)

**Theorem 3.3.** Let \( P \) and \( Q \) be in \( \mathcal{P}^\omega \). Then \( E(\Sigma_P) \leq_B E(\Sigma_Q) \iff Q \preceq P \iff \) there is a nonzero continuous homomorphism \( f : \Sigma_P \to \Sigma_Q \).

**Proof.** Note that every nontrivial proper closed subgroup of \( \Sigma_P \) is non-archimedean. Then this follows from Theorem 2.9 and Lemma 3.2. \( \square \)

Let \( \text{Fin} \) denote the set of all finite subsets of \( \omega \). For \( A, B \subseteq \omega \), we use \( A \preceq^* B \) to denote \( A \setminus B \in \text{Fin} \).

We prove that, for \( n \in \mathbb{N}^+ \), the partially ordered set \( \mathcal{P}(\omega)/\text{Fin} \) can be embedded into Borel equivalence relations between \( E(\mathbb{R}^n) \) and \( E(\mathbb{T}^n) \).

**Lemma 3.4.** Let \( P \) be in \( \mathcal{P}^\omega \). Then \( E(\mathbb{R}) \prec_B E(\Sigma_P) \prec_B E(\mathbb{T}) \).

**Proof.** By Theorem 2.11 (5), we have that \( E(\mathbb{R}) \prec_B E(\Sigma_P) \leq_B E(\mathbb{T}) \).

Assume toward a contradiction that \( E(\mathbb{T}) \leq_B E(\Sigma_P) \). From Theorem 2.11 (4), \( \mathbb{T} \) embeds in \( \Sigma_P \). This is impossible, since \( \Sigma_P \) is not arcwise connected and every proper closed subgroup of \( \Sigma_P \) is non-archimedean. \( \square \)

For \( P \in \mathcal{P}^\omega \) and \( \gamma \in \mathcal{P} \), we define \( t^P(\gamma) \in \omega \cup \{\omega\} \) as

\[
t^P(\gamma) = \begin{cases} \omega, & \exists \infty j \in \omega (P(j) = \gamma), \\ \{j : P(j) = \gamma\}, & \text{otherwise.} \end{cases}
\]

Given \( P, Q \in \mathcal{P}^\omega \), denote

\[
D(P, Q) = \{ \gamma \in \mathcal{P} : t^P(\gamma) < t^Q(\gamma) \}.
\]

From the definition of \( Q \preceq P \), we can easily see that

\[
E(\Sigma_P) \leq_B E(\Sigma_Q) \iff Q \preceq P \iff \sum_{\gamma \in D(P, Q)} (t^Q(\gamma) - t^P(\gamma)) \text{ is finite.}
\]

**Lemma 3.5.** Let \( P, Q \in \mathcal{P}^\omega \) with \( E(\Sigma_Q) \leq_B E(\Sigma_P) \). Suppose that \( D(P, Q) \) is infinite. Then for \( A \subseteq \omega \), there is a group \( \Sigma_{P_A} \) such that \( E(\Sigma_Q) \prec_B E(\Sigma_{P_A}) \prec_B E(\Sigma_P) \) and for \( A, B \subseteq \omega \), we have

\[
A \preceq^* B \iff E(\Sigma_{P_A}) \leq_B E(\Sigma_{P_B}).
\]
Proof. Enumerate $D(P, Q)$ as $d_0 < d_1 < d_2 < \cdots$. Let $P^*_0 \in \mathcal{P}^\omega$ such that $P^*_0(i) = d_{3i}$ for all $i \in \omega$.

For $L, M \in \mathcal{P}^\omega$, we define an element $L \oplus M \in \mathcal{P}^\omega$ as

$$(L \oplus M)(n) = \begin{cases} L(k), & n = 2k, \\ M(k), & n = 2k + 1. \end{cases}$$

It is clear that

$$t^{L \oplus M}(\gamma) = \begin{cases} \omega, & t^L(\gamma) = \omega \text{ or } t^M(\gamma) = \omega, \\ t^L(\gamma) + t^M(\gamma), & \text{otherwise}. \end{cases}$$

Given a set $A \subseteq \omega$, define $P_A \in \mathcal{P}^\omega$ as follows. If $\omega \setminus A$ is finite, put $P_A = P^*_0 \oplus P$. Then

$$t^{P_A}(\gamma) = \begin{cases} t^P(\gamma) + 1, & \gamma = d_{3i}, i \in \omega, \\ t^P(\gamma), & \text{otherwise}. \end{cases}$$

If $\omega \setminus A$ is infinite, enumerate it as $a_0 < a_1 < a_2 < \cdots$. Define $P^*_A \in \mathcal{P}^\omega$ as $P^*_A(j) = d_{1+3a_j}$ for $j \in \omega$, and put $P_A = P^*_A \oplus (P^*_0 \oplus P)$. Then

$$t^{P_A}(\gamma) = \begin{cases} t^P(\gamma) + 1, & \gamma = d_{3i}, i \in \omega \text{ or } \gamma = d_{1+3a}, a \in (\omega \setminus A), \\ t^P(\gamma), & \text{otherwise}. \end{cases}$$

Next we show that $E(\Sigma_Q) <_B E(\Sigma_{P_A}) <_B E(\Sigma_P)$ for all $A \subseteq \omega$.

First, since $t^P(\gamma) \leq t^{P_A}(\gamma)$ for all $\gamma \in \mathcal{P}$, we have $D(P_A, P) = \emptyset$. So $P \preceq P_A$, and hence $E(\Sigma_{P_A}) \leq_B E(\Sigma_P)$.

Since $E(\Sigma_Q) \leq_B E(\Sigma_P)$, by Theorem 3.3, we have $P \preceq Q$, and hence

$$\sum_{\gamma \in D(P, P)} (t^P(\gamma) - t^Q(\gamma)) \text{ is finite.}$$

Note that $t^{P_A}(\gamma) = t^P(\gamma) + 1$ only occurs when $t^Q(\gamma) > t^P(\gamma)$ holds, in which case we always have $\gamma \notin D(Q, P_A)$. So we have $D(Q, P_A) = D(Q, P)$ and $t^{P_A}(\gamma) = t^P(\gamma)$ for all $\gamma \in D(Q, P_A)$. This gives $E(\Sigma_Q) \leq_B E(\Sigma_{P_A})$.

Since $d_{3i} \in D(P, P_A)$ for $i \in \omega$, $D(P, P_A)$ is infinite, so $E(\Sigma_P) \nleq_B E(\Sigma_{P_A})$. Similarly, since $t^{P_A}(d_{2+3i}) = t^P(d_{2+3i}) < t^Q(d_{2+3i})$, we have $d_{2+3i} \in D(P_A, Q)$ for $i \in \omega$, so $E(\Sigma_{P_A}) \nleq_B E(\Sigma_Q)$.

Given $A, B \subseteq \omega$, note that $A \subseteq^* B$ iff $(\omega \setminus B) \setminus (\omega \setminus A) = (A \setminus B)$ is finite. We will check that $A \subseteq^* B$ iff $P_B \preceq P_A$. We consider four cases as follows. (1) If both $\omega \setminus A$ and $\omega \setminus B$ are finite, then we have $A \subseteq^* B$ and $P_A = P_B = P^*_0 \oplus P$. (2) If $\omega \setminus A$ is infinite and $\omega \setminus B$ is finite, then we have $A \subseteq^* B$ and $P_B = P^*_0 \oplus P \preceq P^*_0 \oplus (P^*_0 \oplus P) = P_A$, since $t^{P_B}(\gamma) \leq t^{P_A}(\gamma)$ for all $\gamma \in \mathcal{P}$. (3) If $\omega \setminus A$ is finite and $\omega \setminus B$ is infinite, then $A \not\subseteq^* B$ and $P_B = P^*_0 \oplus (P^*_0 \oplus P) \notin P^*_0 \oplus P = P_A$, since $t^{P_A}(d_{1+3b}) < t^{P_B}(d_{1+3b})$ for $b \in (\omega \setminus B)$. (4) If both $\omega \setminus A$ and $\omega \setminus B$ are infinite, then $t^{P_A}(\gamma) < t^{P_B}(\gamma)$ iff $\gamma = d_{1+3b}$ for some $b \in (\omega \setminus B) \setminus (\omega \setminus A) = (A \setminus B)$. Moreover, $t^{P_B}(d_{1+3b}) = t^P(d_{1+3b}) + 1 = t^{P_A}(d_{1+3b}) + 1$ for all $b \in (A \setminus B)$. So

$$\sum_{\gamma \in D(P_A, P_B)} (t^{P_B}(\gamma) - t^{P_A}(\gamma)) = |A \setminus B|,$$
and hence \( A \subseteq^* B \) iff \( P_B \preceq P_A \).

Again by Theorem 3.3 we have \( A \subseteq^* B \) iff \( E(P_A) \leq_B E(P_B) \). \( \square \)

**Theorem 3.6.** Let \( n \in \mathbb{N}^+ \). Then for \( A \subseteq \omega \), there is an \( n \)-dimensional compact connected abelian Polish group \( G_A \) such that \( E(\mathbb{R}^n) \leq_B E(G_A) \leq_B E(\mathbb{T}^n) \) and for \( A, B \subseteq \omega \), we have

\[
A \subseteq^* B \iff E(G_A) \leq_B E(G_B).
\]

**Proof.** It follows from Theorem 2.11.(5), lemmas 3.1, 3.4, and 3.5. \( \square \)

4. **Dual groups**

Let \( G \) and \( H \) be two abelian topological groups. Denote the class of all continuous homomorphisms of \( G \) to \( H \) by \( \text{Hom}(G, H) \), which is an abelian group under pointwise addition. We always equip \( \text{Hom}(G, H) \) with compact-open topology. The abelian topological group \( \text{Hom}(G, \mathbb{T}) \) is called the dual group of \( G \), denoted by \( \hat{G} \) (cf. [11, Definition 7.4]).

Let \((A, +)\) be an abelian group whose identity element denoted by \( 0_A \).

We say that \((A, +)\) is a torsion group if each element of \( A \) is finite order. We say that \((A, +)\) is torsion-free if \( n \cdot g \neq 0_A \) for all \( g \in A \) with \( g \neq 0_A \) and \( n \in \mathbb{N}^+ \). A subset \( X \) of \( A \) is free if any equation \( \sum_{x \in X} n_x \cdot x = 0_A \) implies \( n_x = 0 \) for all \( x \in X \). The torsion-free rank of \( A \), written \( \text{rank}(A) \), is the cardinal number (uniquely determined) of any maximal free subset of \( A \).

Each Hausdorff locally compact abelian group \( G \) is reflexive, thus it is topologically isomorphic to the double dual group \( \hat{\hat{G}} \) (cf. [11, Theorem 7.63]). A Hausdorff locally compact abelian group is compact and metrizable iff its dual group is a countable discrete group (cf. Proposition 7.5.(i) and Theorem 8.45 of [11]). Let \( G \) be a Hausdorff compact abelian group, then \( G \) is connected iff \( \hat{G} \) is torsion-free; and \( G \) is totally disconnected iff \( \hat{G} \) is torsion (cf. [11, Corollary 8.5]). For any finite dimensional compact abelian Polish group \( G \), the covering dimension of \( A \) is equal to \( \text{rank}(\hat{G}) \) (cf. Lemma 8.13 and Corollary 8.26 of [11]).

If \( H \) is a subset of an abelian topological group \( G \), then the subgroup

\[
H^\perp = \{ \gamma \in \hat{G} : \forall x \in H (\gamma(x) = 1_T) \}
\]

is called the annihilator of \( H \) in \( \hat{G} \) (cf. [11, Definition 7.12]).

Now we focus on compact connected abelian Polish groups.

**Theorem 4.1 (Dual Rigid Theorem).** Let \( G \) be a compact connected abelian Polish group and \( H \) a locally compact abelian Polish group. Then \( E(G) \leq_B E(H) \) iff there is a continuous homomorphism \( S^* : \hat{H} \to \hat{G} \) such that \( \hat{G}/\text{im}(S^*) \) is a torsion group.

**Proof.** (\( \Rightarrow \)). We assume that \( E(G) \leq_B E(H) \). By Theorem 2.8 there is a continuous homomorphism \( S : G \to H \) such that \( \ker(S) \) is non-archimedean. This implies that there is a homomorphism \( S^* \) from \( \hat{H} \) to \( \hat{G} \) such that
ker(S) \cong \text{im}(S^*)^\perp \quad (\text{cf. [11] P.22 and P.23(a)}). \text{ By [11] Lemma 7.13(ii)}, we have that ker(S) \cong (\hat{G}/\text{im}(S^*))^\perp, and hence ker(S) \cong \hat{G}/\text{im}(S^*)$. Since ker(S) is non-archimedean, thus is totally disconnected, so \( \hat{G}/\text{im}(S^*) \) is a torsion group.

\((\Leftarrow)\). Since \( G \cong \hat{G} \) and \( H \cong \hat{H} \), we can define \( S : G \to H \) via \((S^*)^* : \hat{G} \to \hat{H} \) (cf. [10] (24.41)). Then the similar arguments as the preceding paragraph give the desired result. \( \square \)

**Corollary 4.2.** Let \( G \) be a compact connected abelian Polish group and \( H \) a locally compact abelian Polish group. If \( E(G) \leq_B E(H) \), then there is a nonzero continuous homomorphism \( S^* : \hat{H} \to \hat{G} \).

**Proof.** It follows from Theorem 4.1 and that \( \hat{G} \) is non-torsion. \( \square \)

**Example 4.3.** \( \hat{T} \cong \mathbb{Z} \) (cf. [10] Examples 23.27(a)). Fix a \( P \in \mathcal{P}^\omega \), then \( \hat{\Sigma}_P \cong \left\{ \frac{m}{P(0)P(1)\cdots P(n)} : m \in \mathbb{Z}, n \in \mathbb{N} \right\} \) (see [10] 25.3). In view of Corollary 4.2, we get \( E(\mathbb{T}) \not\leq_B E(\Sigma_P) \) again.

Recall that \( \hat{\mathbb{Q}} \cong S_{[2,3,4,5,6,\ldots]} \) (see [10] 25.4). We have the following.

**Corollary 4.4.** Let \( G \) be a \( n \)-dimensional compact abelian Polish group with \( n \in \mathbb{N}^+ \). Then \( E((\hat{\mathbb{Q}})^n) \leq_B E(G) \).

**Proof.** By [11] Theorem 8.22.(4)], \( G_0 \cong (\hat{\mathbb{Q}})^n/\Delta \), where \( \Delta \) is a compact totally disconnected subgroup of \( (\hat{\mathbb{Q}})^n \). Again by Theorem 2.9 this means that \( E((\hat{\mathbb{Q}})^n) \leq_B E(G_0) \), and thus \( E((\hat{\mathbb{Q}})^n) \leq_B E(G) \). \( \square \)

From the arguments above, if \( \Gamma \) is a countable discrete torsion-free abelian group, then \( \hat{\Gamma} \) is a compact connected abelian Polish group.

**Remark 4.5.** Let \( G \) be a compact connected Polish group with \( E(\mathbb{R}^n) \leq_B E(G) \leq_B E(\mathbb{T}^n) \) for some \( n > 0 \). By Theorem 2.11(5), \( \dim(G) = n \), so \( \text{rank}(\hat{G}) = n \). Thus \( \hat{G} \) is isomorphic to a subgroup of \( \mathbb{Q}^n \) (cf. [7] Exercise 13.4.3]). In particular, if \( n = 1 \), we have either \( \hat{G} \cong \mathbb{T} \) or there exists a \( P \in \mathcal{P}^\omega \) such that \( G \cong \Sigma_P \).

The following proposition shows that, if \( n > 1 \), the structure of \( G \) can be more complicated.

**Proposition 4.6.** There is a 2-dimensional compact connected Polish group \( G \) such that \( E(G) \not\leq_B E(\Sigma_{P_0} \times \Sigma_{P_1} \times \cdots \times \Sigma_{P_n}) \) for \( n \in \mathbb{N} \) and each \( P_i \in \mathcal{P}^\omega \). Moreover, if \( |\{i \in \omega : P(i) = 2\}| < \infty \), then \( E(\Sigma_P) \not\leq_B E(G) \).

**Proof.** Pontryagin has constructed a countable torsion-free abelian group \( \Gamma \subseteq \mathbb{Q}^2 \) whose rank is two (cf. [15] Example 2). Then \( \hat{\Gamma} \) is a 2-dimensional compact connected abelian Polish group. The group \( \Gamma \) defined by its generators \( \eta, \xi_i, (i = 0, 1, 2, \ldots) \) and relations,

\[
2^{k_i+1}\xi_{i+1} = \xi_i + \eta, \tag{**}
\]
where $i \in \omega$ and $k_i \in \mathbb{N}^+$ such that $\sup\{k_i : i \in \omega\} = \infty$.

Put $G = \hat{\Gamma}$. We claim that $E(G) \not\leq_B E(\Sigma_{P_n} \times \Sigma_{P_1} \times \cdots \times \Sigma_{P_m})$. Otherwise, by Corollary 4.2 and [10, Theorem 23.18], there exists $i \leq n$ such that there is a nonzero continuous homomorphism $f$ from $\Sigma_{P_i}$ to $\hat{G}$. Note that for any $a \in \Sigma_{P_i}$, there are infinitely many positive integers $n$ such that the equation $nx = a$ has a solution. But any element in $\Gamma$ does not admit such property. This implies that $f(\Sigma_{P_i}) = \{1\}$ contradicting that $f$ is a nonzero homomorphism.

Now assume that $E(\Sigma_{P_i}) \leq_B E(G)$ for some $P \in \mathcal{P}^{\omega}$. We show that $\{i \in \omega : P(i) = 2\}$ is infinite. By Corollary 4.2, there is a nonzero homomorphism $f$ from $\hat{G}$ to $\hat{\Sigma}_P$. Without loss of generality we may assume $\hat{G} = \Gamma$ and $\hat{\Sigma}_P = \{P_0^{m_0}P_1^{m_1} \cdots P_n^{m_n} : m \in \mathbb{Z}, n \in \mathbb{N}\} \subseteq \mathbb{Q}$. From (**), a straightforward calculation shows that

$$2^{k_1+k_2+\cdots+k_i} \xi_i = \xi_0 + \eta(1 + 2^{k_1} + 2^{k_1+k_2} + \cdots + 2^{k_1+k_2+\cdots+k_i-1}).$$

So we have

$$2^{k_1+k_2+\cdots+k_i} f(\xi_i) = f(\xi_0) + f(\eta)(1 + 2^{k_1} + 2^{k_1+k_2} + \cdots + 2^{k_1+k_2+\cdots+k_i-1}).$$

Note that $\lim_i 2^{-(k_1+k_2+\cdots+k_i)} f(\xi_0) = 0$ and

$$\frac{1 + 2^{k_1} + 2^{k_1+k_2} + \cdots + 2^{k_1+k_2+\cdots+k_i-1}}{2^{k_1+k_2+\cdots+k_i}} f(\eta) \leq \frac{f(\eta)}{2^{k_i-1}} \to 0 \ (i \to \infty).$$

This implies that $\lim_i f(\xi_i) = 0$.

Let $f(\xi_0) = a/b$ and $f(\eta) = c/d$ for some integers $a, b, c, d$ with $c, d > 0$. Note that $2^{k_i+1} f(\xi_i+1) = f(\xi_i) + f(\eta)$. Since $f$ is a nonzero homomorphism, there can be at most one $f(\xi_i) = 0$. For large enough $i$, we have $f(\xi_i) \neq 0$. So there exist integers $m_i, m'_i, c', d'$ with $m_i, m'_i \neq 0$ and $c', d' > 0$ such that

$$f(\xi_i) = \frac{m_i}{2^{k_1+k_2+\cdots+k_i} cd} = \frac{m'_i}{2^{k_i} c'd' \cdots k_i},$$

where $m'_i$ and $2^{k_i} c'd'$ are coprime and $c'|d', d'|d$. It follows that

$$|f(\xi_i)| \geq \frac{1}{2^{k_i} c'd'} \geq \frac{1}{2^i cd} \to 0 \ (i \to \infty).$$

So $l_i \to \infty$ as $i \to \infty$, and hence $\{i \in \omega : P(i) = 2\}$ is infinite.

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**References**

[1] D.L. Armacost, The Structure of Locally Compact Abelian Groups, Monographs and Textbooks in Pure and Applied Mathematics, vol. 68, Marcel Dekker, Inc., New York, 1981.

[2] A. Bella, A. Dow, K.P. Hart, M. Hrusak, J. van Mill, P. Ursino, Embeddings into $P(\mathbb{N})$/fin and extension of automorphism, Fund. Math. 174 (2002) 271–284.

[3] J.A. Bondy, U.S.R. Murty, Graph Theory, Graduate Texts in Mathematics, vol. 244, Springer-Verlag, 2008.
[4] L. Ding, Borel reducibility and Hölder(α) embeddability between Banach spaces, J. Symb. Logic 77 (2012) 224–244.
[5] L. Ding, Y. Zheng, On equivalence relations induced by Polish groups, arXiv:2204.04594, 2022.
[6] R. Engelking, Dimension Theory, North Holland, 1978.
[7] S. Gao, Invariant Descriptive Set Theory, Monographs and Textbooks in Pure and Applied Mathematics, vol. 293, CRC Press, 2009.
[8] R.N. Gumerov, On finite-sheeted covering mappings onto solenoids, Proc. Amer. Math. Soc. 133 (2005) 2771–2778.
[9] A. Gutek, Solenoids and homeomorphisms on the Cantor set, Annales Societatis Mathematicae Polonae, Series I: Commentationes Mathematicae XXI (1979) 299–302.
[10] E. Hewitt, K. A. Ross, Abstract harmonic analysis I, Springer-Verlag, 1963.
[11] K.H. Hofmann, S.A. Morris, The Structure of Compact Groups, De Gruyter Studies in Mathematics, vol. 25, De Gruyter, 2013.
[12] B. Kadri, Characterization of locally compact groups by closed totally disconnected subgroups, Monatsh. Math. (2022).
[13] A.S. Kechris, Classical Descriptive Set Theory, Graduate Texts in Mathematics, vol. 156, Springer-Verlag, 1995.
[14] A. Louveau, B. Velickovic, A note on Borel equivalence relations, P. Amer. Soc. 120 (1994) 255–259.
[15] L. Pontrjagin, The theory of topological commutative groups, Ann. of Math. 35 (1934) 361–388.
[16] J.R. Prajs, Mutual aposyndesis and products of solenoids, Topology Proc. 32 (2008) 339–349.
[17] W. Scheffer, Maps between topological groups that are homotopic to homomorphisms, Proc. Amer. Math. Soc. 33 (1972), 562–567.
[18] Z. Yin, Embeddings of $P(\omega)/\text{Fin}$ into Borel equivalence relations between $\ell_p$ and $\ell_q$, J. Symb. Logic 80 (2015), 917–939.

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