Density effects on the pion dispersion relation at finite temperature

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Abstract. We study the behavior of the pion dispersion relation in a pion medium at finite density and temperature, introducing a chemical potential to describe the finite pion number density. Such description is particularly important during the hadronic phase of a relativistic heavy-ion collision, between chemical and thermal freeze-out, where the pion number changing processes, driven by the strong interaction, can be considered to be frozen. We make use of an effective Lagrangian that explicitly respects chiral symmetry through the enforcement of the chiral Ward identities. The pion dispersion relation is computed through the computation of the pion self-energy in a non-perturbative fashion by giving an approximate solution to the Schwinger-Dyson equation for this self-energy. The dispersion relation is described in terms of a density and temperature dependent mass and an index of refraction which is also temperature, density as well as momentum dependent. The index of refraction is larger than unity for all values of the momentum for finite \( \mu \) and \( T \). Given the strong coupling between \( \rho \) vectors and pions, we argue that the modification of the pion mass due to finite pion density effects has to be taken into account self-consistently for the description of the in-medium modifications of \( \rho \)'s.

INTRODUCTION

In recent years, the possibility to produce a locally thermalized, deconfined state of matter, whose degrees of freedom are the quarks and gluons of quantum chromodynamics (QCD), the so called quark-gluon plasma (QGP), in high-energy heavy-ion collisions, has attracted a great deal of attention, both on the experimental and the theoretical aspects of the subject [1].

If the QGP is produced in these kind of reactions, the prevailing view portrays the evolution of such a system traversing a series of stages, the last of which consists of a large amount of hadrons strongly interacting in a finite volume, until a final freeze-out. For not too high temperatures, hadronic matter consists mainly of pions, therefore, the study of the propagation properties of pions within the above described environment represents an important ingredient for the understanding of the properties of the hadronic system at and just before freeze-out [2, 3].

The hadronic degrees of freedom are customarily accounted for by means of effective chiral theories that incorporate the Goldstone boson nature of pions. One of such theories is chiral perturbation theory (ChPT) which has been employed to show the well known result that at leading perturbative order and at low momentum, the modification of the pion dispersion curve in a pion medium at finite temperature is just a constant, temperature dependent, increase of the pion mass [4]. ChPT has also been used in a
two-loop computation of the pion self-energy [5] and decay constant [3]. A striking result obtained from such computations is that at second order, the shift in the temperature dependence of the pion mass is opposite in sign and about three times larger in magnitude than the first order shift, already at temperatures close to 150 MeV. This result might signal either the breakdown of the perturbative expansion at these temperatures or the need to perform such calculations using schemes other than perturbation theory.

However, a missing ingredient in the calculations of the pion dispersion curve is the treatment of the medium’s finite density. The conceptual difficulty is related to the fact that, though it is possible to assume that the system is in (at least local) thermal equilibrium, strictly speaking the only conserved charge that can be associated to the pion system is the electric charge and thus, for an electrically neutral pion system the corresponding chemical potential vanishes. The behavior of the pion mass in the presence of an isospin chemical potential has been recently studied in Ref. [7]. But in order to describe a situation in which the number of pions in thermal equilibrium is finite, we need to consider a chemical potential associated with the pion number, instead of its charge.

Recall that the pion number is not a conserved quantity due to either strong, weak or electromagnetic processes. Nevertheless, the characteristic time for electromagnetic and weak pion number-changing processes, is very large compared to the lifetime of the system created in relativistic heavy-ion collisions and therefore, these processes are of no relevance for the propagation properties of pions within the hadronic phase of the collision. As for the case of strong processes, it is by now accepted that they drive pion number toward chemical freeze-out at a temperature considerably higher than the thermal freeze-out temperature and therefore, that from chemical to thermal freeze-out, the pion system evolves with the pion abundance held fixed [8, 9]. Under these circumstances, it is possible to ascribe to the pion density a chemical potential and consider the pion number as conserved [10, 11]. In this context, the role of a finite pion chemical potential into a hadronic equation of state has been recently investigated in Ref. [12].

Furthermore, another important ingredient in the analysis is the well-known fact that in finite temperature field theories with either massless degrees of freedom or that exhibit spontaneous symmetry breaking [13], the perturbative expansion breaks down and thus the necessity to implement resummation techniques.

In this contribution we summarize the effects that the introduction of a finite pion chemical potential, associated to the finite pion density, has on the dispersion curve of pions at finite temperature. Starting from the linear sigma model, we use an effective Lagrangian [14] obtained by integrating out the heavy sigma modes and compute, in a non-perturbative fashion, the pion self-energy. We find that the dispersion curve is modified with respect to the vacuum in a way described by the introduction of an index of refraction larger than one, in addition to the thermal and density increase of the pion mass. Further details can be found in Ref. [15].
EFFECTIVE LAGRANGIAN

The Lagrangian for the linear sigma model, including only meson degrees of freedom and with an explicit chiral symmetry breaking term, can be written as \[16\]

\[
\mathcal{L} = \frac{1}{2} \left[ (\partial \pi)^2 + (\partial \sigma)^2 - m^2_\pi \pi^2 - m^2_\sigma \sigma^2 \right] - \lambda^2 f_\pi \sigma (\sigma^2 + \pi^2) - \frac{\lambda^2}{4} (\sigma^2 + \pi^2)^2, \quad (1)
\]

where \(\pi\) and \(\sigma\) are the pion and sigma fields, respectively, and the coupling \(\lambda^2\) is

\[
\lambda^2 = \frac{m^2_\sigma - m^2_\pi}{2f^2_\pi}. \quad (2)
\]

When interested in a given approximation scheme to build the Green’s functions of the theory at a given perturbative order, it is possible to exploit the relations that chiral symmetry imposes among them. These relations, better known as chiral Ward identities (ChWI), are a direct consequence of the fact that the divergence of the axial current may be used as an interpolating field for the pion \([16]\). In order to make sure that the approximation respects chiral symmetry, one needs to check that the modification of other Green’s functions respect the corresponding ChWI. For example, two of the ChWI satisfied –order by order in perturbation theory– by the functions \(\Delta_\pi(P)\), \(\Delta_\sigma(Q)\), \(\Gamma_{ij}^{kl}\) and \(\Gamma_{04}^{ijkl}\) are

\[
 f_\pi \Gamma_{04}^{ijkl}(;0,P_1,P_2,P_3) = \Gamma_{12}^{kl}(P_1;P_2,P_3) \delta^{ij} + \Gamma_{12}^{ij}(P_2;P_3,P_1) \delta^{ik} + \Gamma_{12}^{jk}(P_3;P_1,P_2) \delta^{il} \\
 f_\pi \Gamma_{12}^{ij}(Q;0,P) = [\Delta^{-1}_\sigma(Q) - \Delta^{-1}_\pi(P)] \delta^{ij}, \quad (3)
\]

where momentum conservation at the vertices is implied, that is \(P_1 + P_2 + P_3 = 0\) and \(Q + P = 0\). The notation for the functional dependence of the vertices in Eqs. (3) is such that the variables before and after the semicolon refer to the four-momenta of the sigma and pion fields, respectively \([14]\).

In Refs. \([14]\) it has been shown that in the kinematical regime where the pion momentum, the pion mass and the temperature are small compared to the sigma mass, the effective one-loop sigma propagator and one-sigma two-pion and four-pion vertices are given by

\[
i \Delta^*_\sigma(Q) = \frac{i}{Q^2 - m^2_\sigma + 6\lambda^2 f^2_\pi \mathcal{I}(Q)}, \quad (4)
\]

\[
i \Gamma_{12}^{ij}(Q;P_1,P_2) = -2i\lambda^2 f_\pi \delta^{ij} \left[ 1 - 3\lambda^2 \mathcal{I}(Q) \right], \quad (5)
\]

\[
i \Gamma_{04}^{ijkl}(;P_1,P_2,P_3,P_4) = 2i\lambda^2 \left\{ \times \left[ 1 - 3\lambda^2 \mathcal{I}(P_1 + P_2) \right] \delta^{ij}\delta^{kl} + \left[ 1 - 3\lambda^2 \mathcal{I}(P_1 + P_3) \right] \delta^{ik}\delta^{jl} + \left[ 1 - 3\lambda^2 \mathcal{I}(P_1 + P_4) \right] \delta^{il}\delta^{jk} \right\}, \quad (6)
\]
where in the imaginary-time formalism of thermal field theory (TFT), the function $\mathcal{I}^t$ is obtained as the time-ordered analytical continuation to real energies of the function $\mathcal{I}$ defined by

$$\mathcal{I}(Q) \equiv T \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{K^2 + m^2 \pi^2} \frac{1}{(K - Q)^2 + m^2 \pi^2}. \quad (7)$$

Here $Q = (\omega, \mathbf{q})$, $K = (\omega_n, \mathbf{k})$ are Euclidean space four-vectors, namely $Q^2 = \omega^2 + q^2$, $K^2 = \omega_n^2 + k^2$ with $\omega = 2m\pi T$ and $\omega_n = 2n\pi T$ ($m, n$ integers) being discrete boson frequencies, $T$ is the temperature and $q = |\mathbf{q}|$, $k = |\mathbf{k}|$.

It is easy to check that Eqs. (4), (5) and (6) satisfy the Ward identities in Eq. (3), this ensures that the approximation scheme adopted respects chiral symmetry.

By using the above effective vertices and propagator, it is possible to construct the two-loop modification to the pion self-energy in the same kinematical regime with the result [14]

$$\Pi_2(P) = \left(\frac{m^2 \pi}{2f^2 \pi}\right) T \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{K^2 + m^2 \pi^2} \left\{ 3 - \left(\frac{m^2 \pi}{2f^2 \pi}\right) \left[ 9\mathcal{I}^t(0) + 6\mathcal{I}^t(P + K) \right] \right\}. \quad (8)$$

Equation (8) reproduces the leading order result obtained from ChPT [4]. Furthermore, we observe that Eq. (8) can be formally obtained by means of the effective Lagrangian

$$\mathcal{L} = \frac{1}{2} \left(\partial_\mu \phi\right)^2 - \frac{1}{2} m^2 \pi \phi^2 - \frac{\alpha}{4!} \left(\phi^2\right)^2, \quad (9)$$

where $\alpha = 6(m^2 \pi/2f^2 \pi)$ and the factor 6 comes from considering the interaction of like-isospin pions in the vertex

$$i\Gamma_{ijkl}^4 = -2i \left(\frac{m^2 \pi}{2f^2 \pi}\right) \left( \delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right). \quad (10)$$

Eqs. (8) and (9) mean that in the kinematical regime where the sigma mass is large compared to the pion mass, the momentum and the temperature, the linear sigma model Lagrangian reduces to a $\phi^4$ Lagrangian for effective like-isospin pions with an effective coupling given by $\alpha$. In essence, the theory thus constructed and summarized by the effective Lagrangian in Eq. (9) can be thought of a theory for the effective coupling $\alpha$. By restricting ourselves to the above kinematical regime, we will proceed on working with the Lagrangian given by Eq. (9).

**NON-PERTURBATIVE PION SELF-ENERGY**

It is well known that quantum field theories at finite temperature present certain subtleties such as the breakdown of the perturbative expansion [17]. This breakdown becomes manifest in two important cases: the appearance of infrared divergences in theories with massless degrees of freedom, and the compensation of powers of the coupling...
constant with powers of $T$ for large temperatures. In both situations, the resummation of certain classes of diagrams represents an important improvement for the study of the physical properties of such theories. Even for cases where neither there is a massless degree of freedom, nor the temperature is extremely large, it is important to consider a resummation scheme, particularly for the case of theories with spontaneous symmetry breaking near critical behavior.

In order to consider the above mentioned general situation and with the purpose of studying the pion dispersion curve at finite density and temperature, let us formally consider the Schwinger-Dyson equation for the pion self-energy

\[
\Pi(P) = \frac{\alpha}{2} T \sum_n \int \frac{d^3 k}{(2\pi)^3} \frac{1}{K^2 + m^2 + \Pi}
\]

\[
- \frac{\alpha^2}{6} T^2 \sum_{n_1, n_2} \int \frac{d^3 k_1}{(2\pi)^3} \frac{d^3 k_2}{(2\pi)^3} \frac{1}{K_1^2 + m^2 + \Pi} \times \frac{1}{K_2^2 + m^2 + \Pi_0 (K_1 + K_2 - P)^2 + m^2 + \Pi_0},
\]

where for internal lines, we make the substitution $i\omega \rightarrow i\omega + \mu$. Notice that since the interaction Lagrangian contains only a quartic term, there is no need to dress the vertices in the above equation.

Equation (11) represents an integral equation for the function $\Pi(P)$, which, needless to say, is very difficult to be solved exactly. In order to find an approximate solution let us write

\[
\Pi(P) \approx \Pi_0 + \tilde{\Pi}(P)
\]

and consider $\tilde{\Pi}(P) \ll \Pi_0$. As we will see, such assumption is justified given that in our approximation, $\tilde{\Pi} \sim \mathcal{O}(\alpha^2)$. Keeping only the lowest order contribution in $\Pi$, we find

\[
\Pi_0 \equiv \frac{\alpha}{2} T \sum_n \int \frac{d^3 k}{(2\pi)^3} \frac{1}{K^2 + m^2 + \Pi_0},
\]

\[
\tilde{\Pi}(P) \equiv - \frac{\alpha^2}{6} T^2 \sum_{n_1, n_2} \int \frac{d^3 k_1}{(2\pi)^3} \frac{d^3 k_2}{(2\pi)^3} \frac{1}{K_1^2 + m^2 + \Pi_0} \times \frac{1}{K_2^2 + m^2 + \Pi_0 (K_1 + K_2 - P)^2 + m^2 + \Pi_0}.
\]

Equation (13) represents a self consistent equation for the (momentum independent) constant $\Pi_0$. This is the well known resummation for the superdaisy diagrams which constitute the dominant contribution in the large-$N$ expansion of the Lagrangian in Eq. (9). On the other hand, Eq. (14) represents a first approximation to the momentum dependent piece of $\Pi(P)$. 

Figure 1 shows the behavior of the quantity $\sqrt{m_\pi^2 + \pi_0}$ as a function of (a) the temperature $T$ for different values of the chemical potential ranging from $\mu = 0$ to $\mu = 130$ MeV, from bottom to top, and as a function of (b) the chemical potential $\mu$ for different values of the temperature ranging from $T = 50$ to $T = 150$ MeV, from bottom to top.

In both cases, $\sqrt{m_\pi^2 + \pi_0}$ grows monotonically with both $T$ and $\mu$.

The pion dispersion relation is given as the solution to

$$p_0^2 - [p^2 + m_\pi^2 + \pi_0 + \Re \Pi'(p_0, p)] = 0$$

for positive $p_0$. In Eq. (15), $\Pi'$ denotes the retarded version of $\Pi$ upon analytical continuation to Minkowski space. The left panel of Fig. 2 shows plots of $p_0$ as a function of $p$ for different values of $\mu$ and a temperature $T = 120$ MeV. As can be seen from this figure, $\Pi(p_0, p = 0)$ contributes to the increase of the pion mass. Also, for large $p$, the dispersion curves approach the light cone, always from within the causal region $p_0^2 > p^2$.

We also notice that the dispersion curves can be parametrized by a function of the form

$$p_0 = \sqrt{n^{-1}(T, \mu)p^2 + M^2(T, \mu)}.$$  

This is shown in the right panel of Fig. 2 where the dots represent the computed values of the dispersion relation and the continuous curves the fits. We thus observe that the presence of a finite chemical potential has two dramatic effects on the behavior of the pion dispersion curve compared to the case where only the temperature is considered: first, there is a significant increase in the pion mass and second, the index of refraction parameter $n$ becomes larger than unity.

**SUMMARY AND CONCLUSIONS**

In this paper we have considered the effects of a finite pion density on the pion dispersion curve at finite temperature. The finite density is described in terms of a finite
pion chemical potential. We have argued that such description is important during the hadronic phase of a collision of heavy nuclei at high energies between chemical and thermal freeze-out when the strong pion-number changing processes have driven the pion number to a fixed value.

In order to consider a general scenario that takes into account resummation effects, we have presented an approximate solution to the Schwinger-Dyson equation for the momentum-dependent pion self-energy, writing this as \( \Pi(P) \simeq \Pi_0 + \tilde{\Pi}(P) \) and considering \( \tilde{\Pi}(P) \ll \Pi_0 \), which is justified given that in our approximation, \( \tilde{\Pi} \sim O(\alpha^2) \), whereas the perturbative expansion of \( \Pi_0 \) starts at \( O(\alpha) \).

The pion dispersion relation thus obtained at finite density and temperature deviates from the vacuum dispersion relation and can be described in terms of a density and temperature dependent mass and an index of refraction, also temperature, density as well as momentum dependent. This index of refraction is larger than unity for all values of the momentum. Similar results have been obtained in Ref. [19] using also a linear sigma model without the introduction of a finite chemical potential but rather as a consequence of the loss of Lorentz invariance when a particle travels in a medium at finite temperature. We note however that our result is more general than that of the former reference where the computation of the pion dispersion relation was carried from the onset in the weak coupling regime, that is to say, \( \lambda^2 \ll 1 \), whereas our approach is valid for arbitrary values of \( \lambda^2 \) [14].

Recall that one of the most salient features resulting from the analyses of collisions of heavy nuclei at SPS energies is the low mass dilepton spectra which shows an increase around the free \( \rho \) peak. Such behavior has been explained in terms of possible in-medium modifications of the \( \rho \) with either a shift toward lower values of its mass or an increase in its width [20]. Given that the \( \rho \) meson couples strongly to the two pion channel, any possible in-medium modifications of the pion propagation properties should translate into induced modifications of the \( \rho \) meson properties and should therefore be properly accounted for in a self-consistent manner in these kind of analyses. In particular, the increase of the pion mass with density modifies the phase space for the

\[ p_0 \text{ (MeV)} \]

\[ \mu = 0, 120, 130 \text{ MeV} \]

\[ T = 120 \text{ MeV} \]

The dots are the computed values of the dispersion relation.

\[ \Pi_0 \simeq \Pi_0 + \tilde{\Pi}(P) \]
production and decay of \( \rho \)'s in equilibrium and consequently it affects the production of \( e^+ e^- \) pairs. This matters will be the subject of a future work [21].

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