Microstates of Four-Dimensional Rotating Black Holes from
Near-Horizon Geometry

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Abstract

We show that a class of four-dimensional rotating black holes allow five-dimensional embeddings as black rotating strings. Their near-horizon geometry factorizes locally as a product of the three-dimensional anti-deSitter space-time and a two-dimensional sphere ($AdS_3 \times S^2$), with angular momentum encoded in the global space-time structure. Following the observation that the isometries on the $AdS_3$ space induce a two-dimensional (super)conformal field theory on the boundary, we reproduce the microscopic entropy with the correct dependence on the black hole angular momentum.
Recent developments in nonperturbative string theory have provided a fruitful framework to consider quantum properties of black holes. In particular, extreme black holes with Ramond-Ramond (R-R) charges can be interpreted in higher dimensions as intersecting D-branes (the nonperturbative objects in string theory that carry such charges [1]), and this has lead to a counting of black hole quantum states that agrees precisely with the Bekenstein-Hawking (BH) entropy [2]. This counting is carried out in the weakly coupled regime where the D-brane constituents of the black hole experience flat space-time geometry; however, due to supersymmetry, it remains valid in the regime where the D-branes are strongly coupled, and the geometric space-time description of black holes emerges. Thus the microscopic derivation of the BH-entropy is justified, but it is difficult to explore the quantum black hole geometry in detail using D-branes.

The success of the D-brane counting overshadowed prior attempts to shed light on the microscopics of black holes in string theory. In pioneering work, Sen attempted to identify the microstates of extreme electrically charged black holes with perturbative excitations of string theory [3]. However, it was not until the discovery of extreme dyonic black holes in string theory — with regular horizons and thus finite BH-entropy — that a quantitative agreement between the microscopic and macroscopic entropy became feasible [4]. The microscopic features of these black holes are captured by string theory in the curved space-time geometry specified by their near-horizon region [5,6]. In particular, a $SL(2,\mathbb{Z}) \times SU(2)$ Wess-Zumino-Witten (WZW) model [6,8] reproduces, at least qualitatively, the extreme black hole entropy directly from the near-horizon geometry.

However, it is only very recently that a precise derivation of the BH-entropy from the near-horizon geometry was achieved by Strominger [9] (and also by Sachs et al. [10]). The

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1Such black holes were originally specified by four Neveu-Schwarz Neveu-Schwarz (NS-NS) charges [4], two electric and two magnetic ones. Note, however, that these can be mapped onto solutions with R-R charges, by exploiting duality symmetry; their space-time is thus the same.
central observation is that, when embedded in a higher dimensional space, the near-horizon geometry locally contains the three-dimensional anti-deSitter space-time \((AdS_3)\), whose quantum states are specified by a two-dimensional conformal field theory (CFT) on the boundary. This sets the stage for a remarkably robust microscopic counting, which precisely reproduces the BH-entropy. The result has stirred a renewed interest in addressing the details of black hole microscopics, as they emerge from features of the black hole near-horizon geometry \([11,12]\).

The most recent approach reproduces the BH-entropy of static near-extreme black holes in five \([9]\) and four \([13]\) space-time dimensions. However, the black hole solutions of the effective low-energy string theory include as special cases the familiar black holes of Maxwell-Einstein gravity; in particular, the four-dimensional (neutral rotating) Kerr black hole that is believed to be of astrophysical significance, and the (charged rotating) Kerr-Newman black hole. It is thus important to generalize the method to more general backgrounds, including rotating black holes.

In this paper we address the microscopics of near-extreme rotating black holes in four dimensions by exploring their near-horizon region; and we elucidate the role of angular momentum in the microscopic description. We interpret these black holes as charged rotating strings in five dimensions, with the string wrapped around the additional dimension. In M-theory this configuration corresponds to the intersection of a rotating configuration of three intersecting M5-branes with momentum along their one common direction, identified with the string in five dimensions. In the decoupling limit the geometry is locally a direct product of the three-dimensional anti-deSitter space-time and a two-dimensional sphere \((AdS_3 \times S^2)\); this factorized structure is obtained after a transformation of the angular coordinates into a “co-moving coordinate system”. By identifying the microstates with those of the CFT induced on the boundary of the \(AdS_3\) we reproduce the BH-entropy \(^2\).

\(^2\)Related work on five-dimensional rotating black holes is presented in detail elsewhere \([14]\).
The starting point is a large class of four-dimensional black holes (of toroidally compactified string theory), whose explicit space-time metric is given in [15]. They are specified by their mass $M$, four $U(1)$ charges $Q_i$ and the angular momentum $J$ or, more conveniently, in terms of the non-extremality parameter $m$, four boosts $\delta_i$ and the angular parameter $l$:

$$G_4M = \frac{1}{4}m \sum_{i=0}^{3} \cosh 2\delta_i,$$
$$G_4Q_i = \frac{1}{4}m \sinh 2\delta_i ; \quad i = 0, 1, 2, 3,$$
$$G_4J = ml(\prod_{i=0}^{3} \cosh \delta_i - \prod_{i=0}^{3} \sinh \delta_i),$$

where $G_4$ is the four-dimensional Newton’s constant. The Kerr-Newman black hole corresponds to the case where the four charges are identical. The extreme limit is obtained by taking, $m \to 0$ and $l \to 0$ while keeping $Q_{0,1,2,3}$ finite; in this case $J = 0$. From the explicit solution one finds the BH-entropy [15]:

$$S \equiv \frac{A_4}{4G_4} = \frac{\pi}{4G_4} \left[8m^2(\prod_{i=0}^{3} \cosh \delta_i + \prod_{i=0}^{3} \sinh \delta_i) + 8m\sqrt{m^2 - l^2}(\prod_{i=0}^{3} \cosh \delta_i - \prod_{i=0}^{3} \sinh \delta_i)\right], \quad (1)$$

where $A_4$ is the area of the outer horizon.

A specific representation of the metric and its accompanying matter fields is given in [15] in terms of the NS-NS fields, e.g., its higher-dimensional interpretation is that of a rotating fundamental string with winding and momentum modes, superimposed with the Kaluza-Klein monopole and the H-monopole [18]. A particular duality transformation leaves the four-dimensional space-time invariant, while mapping this configuration to three intersecting $M5$-branes of $M$-theory (specified by $Q_{1,2,3}$), with momentum (specified by $Q_0$) along the common string. This $M$-theory configuration can be interpreted as a rotating string in five dimensions after toroidal compactification. The space-time metric of the rotating string is

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3The notation follows [15]. The $r_0$ of [14] is $r_0 = 2m$, the $\mu$ of [17] is $m = 4\mu$, the $l$ of [17] is $l_{\text{here}} = 4l_{\text{there}}$, the $Q_i$ of [13] is $Q_{\text{here}} = 2Q_{\text{there}}$. 

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rather complicated, and we were unable to write it in a relatively compact form. However, the metric simplifies significantly in the near-horizon region $r \ll Q_{1,2,3}$, when the dilute gas condition $\delta_{1,2,3} \gg 1$ is satisfied. Then the metric of the five-dimensional rotating string in the Einstein frame becomes:

$$
\begin{align*}
\frac{ds_5^2}{\lambda} &= \frac{2}{\lambda} \left[ (r - \frac{l^2}{2m}) \cos^2 \theta (-d\tilde{t}^2 + d\tilde{y}^2) + 2m \left(1 - \frac{l^2}{2m^2}\right) \cos^2 \theta d\tilde{t}^2 - \frac{l^2}{m} \cos^2 \theta d\tilde{t} d\tilde{y} \right] \\
&\quad + \frac{\lambda^2}{4} \left[ \frac{1}{r^2 - 2mr + l^2} dr^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right] - \sqrt{\frac{\lambda l^2}{m}} (d\tilde{y} + d\tilde{t}) \sin^2 \theta d\phi ,
\end{align*}
$$

where the boosted variables (specifying the momentum along the string) are:

$$
\begin{align*}
\tilde{t} &= \cosh \delta_0 dt - \sinh \delta_0 dy , \quad \tilde{y} = \cosh \delta_0 dy - \sinh \delta_0 dt , \\
\end{align*}
$$

and the characteristic length scale $\lambda$ is defined as $\lambda \equiv (Q_1 Q_2 Q_3)^{\frac{1}{3}}$. Note that the metric (2) retains nontrivial dependence on the angular momentum; however, the Kerr-Newman black holes are not compatible with the limit considered here.

Introducing the shifted coordinate:

$$
\begin{align*}
\tilde{\phi} &= \phi - \frac{2l}{\sqrt{\lambda^3 m}} (d\tilde{y} + d\tilde{t}) , \\
\end{align*}
$$

yields the factorized metric:

$$
\begin{align*}
\frac{ds_5^2}{\lambda} &= \frac{2}{\lambda} \left[ -(r - 2m + \frac{l^2}{2m}) dt^2 - \frac{l^2}{m} d\tilde{t} d\tilde{y} + (r - \frac{l^2}{2m}) dy^2 \right] + \\
&\quad + \frac{\lambda^2}{4} \left[ \frac{dr^2}{r^2 - 2mr + l^2} + d\theta^2 + \sin^2 \theta d\phi^2 \right] .
\end{align*}
$$

With this choice of coordinates it is apparent that the geometry is a direct product of a two-sphere $S^2$, with radius $\frac{1}{2}$, and a Banados, Teitelboim and Zanelli (BTZ) black hole in three space-time dimensions with a negative cosmological constant $\Lambda = -\lambda^2$. Indeed, introducing the coordinates: $\tau \equiv \frac{t}{R_{11}}$, $\sigma \equiv \frac{y}{R_{11}}$, and $\rho^2 \equiv \frac{2R_{11}^2}{\lambda} \left[ r + 2m \sinh^2 \delta_0 - \frac{l^2}{2m} (\cosh \delta_0 - \sinh \delta_0)^2 \right]$, we obtain:

4The complications associated with the angular momentum are similar to those of adding an additional charge (the “fifth parameter”) to the configuration [9].
where $R_{11}$ is the radius of the dimension wrapped by the string, we find the standard BTZ metric [20]:

$$
 ds_5^2 = -N^2 d\tau^2 + N^{-2} d\rho^2 + \rho^2 (d\sigma - N_\sigma d\tau)^2 + \frac{1}{4} \lambda^2 d\tilde{\Omega}_3^2 ,
$$

$$
 N^2 = \frac{\rho^2}{\lambda^2} - M_3 + \frac{16 G_3 J_3^2}{\rho^2} , \quad N_\sigma = \frac{4 G_3 J_3}{\rho^2} ,
$$

where the effective BTZ mass $M_3$ and angular momentum $J_3$ are:

$$
 M_3 = \frac{R_{11}^2}{\lambda^3} \left[ (4m - \frac{2l^2}{m}) \cosh 2\delta_0 + \frac{2l^2}{m} \sinh 2\delta_0 \right] ,
$$

$$
 8G_3 J_3 = \frac{R_{11}^2}{\lambda^2} \left[ \frac{2l^2}{m} \cosh 2\delta_0 + (4m - \frac{2l^2}{m}) \sinh 2\delta_0 \right] ,
$$

and the effective three-dimensional gravitational coupling $G_3$ is related to the four-dimensional one $G_4$ as [13]:

$$
 \frac{1}{G_3} = \frac{1}{G_4} \frac{A_2}{2\pi R_{11}} = \frac{1}{G_4} \frac{\lambda^2}{2R_{11}} , \quad (4)
$$

where $A_2$ is the area of the two-sphere $S^2$. The BTZ geometry is locally $AdS_3$ but global identifications ensure causal structures that are similar to those familiar from four-dimensional black holes. For our purposes it is crucial that the geometry is asymptotically $AdS_3$, because then the isometries induce a CFT on the boundary at infinity [21][9]. Its central charge $c$ is determined by the effective cosmological constant $-\lambda^2$ as [21]:

$$
 c = \frac{3\lambda}{2G_3} = 6 \frac{Q_1 Q_2 Q_3}{8G_4 R_{11}} , \quad (5)
$$

and the conformal weights $h_{L,R}$ (eigenvalues of the Virasoro operators $L_0$, $\bar{L}_0$, respectively) are related to the BTZ parameters as:

$$
 h_{L,R} = \frac{\lambda M_3 \pm 8G_3 J_3}{16G_3} . \quad (6)
$$

Collecting the formulae (3) and (6) we find, in the semi-classical regime where the conformal weights are large, the statistical entropy:

$$
 S = 2\pi \left( \sqrt{\frac{c}{6} h_L} + \sqrt{\frac{c}{6} h_R} \right) = \frac{\pi}{4G_4} \sqrt{Q_1 Q_2 Q_3} \left[ \sqrt{m} e^{\delta_0} + \sqrt{m - \frac{l^2}{m}} e^{-\delta_0} \right] . \quad (7)
$$
On the other hand, in the dilute gas limit, i.e. $\delta_{1,2,3} \gg 1$, the BH-entropy (1) becomes:

$$S \simeq \frac{\pi}{4G_4} \sqrt{Q_1 Q_2 Q_3} \left[ \sqrt{m} e^{\delta_0} + \sqrt{m - \frac{l^2}{m}} e^{-\delta_0} \right].$$

(8)

This is in precise agreement with the microscopic calculation (7). It also agrees with the D-brane motivated counting given in [16].

The derivation of statistical black hole entropy does not rely on the details of the underlying quantum theory, but the relation to M-theory is interesting. In M-theory units $R_{11} = g\sqrt{\alpha'}$, the Planck length is $l_p = (2\pi g)^{\frac{3}{4}} \sqrt{\alpha'}$, and $G_4 = \frac{1}{8} \frac{(\alpha')^4 g^2}{R_1 R_2 R_3 R_4 R_5 R_6}$ where the $R_i$ are the radii of the compact dimensions and $g$ is the string coupling constant.

In the above we assumed the near-horizon approximation $r \ll Q_{1,2,3}$ and the dilute gas limit $\delta_{1,2,3} \gg 1$. These become exact in the formal decoupling limit [22]:

$$(l_p, r, m, l) \to 0, \text{ with } (r \sim l_p^3, m \sim l_p^3, l \sim l_p^3, R_{1,\ldots,6} \sim l_p, R_{11} \sim 1),$$

(9)

where the field theory on the intersection of the M-branes decouples from gravity. Note, in particular, that angular momentum is compatible with decoupling. This appears only to be the case for configurations that correspond to regular black holes in four and five dimensions; the near-horizon geometry of, e.g., the D3-brane and the M5-brane do not have rotating versions. Thus only the induced CFTs in two dimensions seems to have worldvolume currents with charges that can be interpreted as angular momenta.

The quantization conditions on the D-brane charges are [1] $Q_i = \frac{1}{2\pi} \frac{(2\pi \sqrt{\alpha'})^3}{R_{2i-1} R_{2i}} n_i g$, where $n_{1,2,3}$ is the number of coincident M5-branes with a given orientation, so $Q_1 Q_2 Q_3 = n_1 n_2 n_3 8G_4 R_{11}$; and from (3) the quantized form of the central charge $c$ becomes $c = 6n_1 n_2 n_3$ as expected [23–25]. A heuristic microscopic interpretation of this formula is that each of the M-branes traverse the intersection string $n_i$ times, giving a total of $n_1 n_2 n_3$ distinct topological sectors, each associated with 6 degrees of freedom.

The quantum numbers $\epsilon$ and $p$ for the string energy and momentum, respectively, are introduced through:

$$E = 2m \cosh 2\delta_0 = 8G_4 \frac{\epsilon}{R_{11}}, \quad Q_0 = 2m \sinh 2\delta_0 = 8G_4 \frac{p}{R_{11}}.$$

(10)
and then the conformal weights $h_{L,R}$ can be written as:

$$h_L = \frac{R_{11}}{8G_4} mc^{2\delta_0} = \frac{1}{2}(\epsilon + p), \quad h_R = \frac{R_{11}}{8G_4} (m - \frac{l^2}{m}) e^{-2\delta_0} = \frac{1}{2}(\epsilon - p) - \frac{1}{n_1n_2n_3} J^2. \quad (11)$$

The space-time angular momentum is normalized so that $J$ is measured in units of $\hbar$. Thus, from semi-classical reasoning, we expect that $J$ is quantized as an integer. By introducing a single unit of angular momentum we see that the $h_R$ is quantized in units of $1/n_1n_2n_3$.

The angular momentum of the black hole breaks rotational invariance of the background, so it is not guaranteed by symmetries that the near-horizon geometry contains a two-sphere $S^2$. In the present model the linking of $AdS_3$ and $S^2$ is accomplished by the global features contained in the boundary conditions at infinity and encoded in the coordinate shift (3). It is therefore surprisingly simple to include angular momentum, and thus more realism, while preserving full analytical control. This makes the present model an attractive setting to study angular momentum. The precise value of the shift can be understood as follows: the potentials conjugate to the left- and right-moving string energies are:

$$\beta_L = \frac{\pi}{2} \lambda^\frac{3}{2} m^{-\frac{1}{2}} e^{-\delta_0}, \quad \beta_R = \frac{\pi}{2} \lambda^\frac{3}{2} m^\frac{1}{2} (m^2 - l^2)^{-\frac{1}{2}} e^{\delta_0}, \quad (12)$$

respectively; and the rotational velocity $\Omega$ is given through $\beta_H \Omega = \frac{2\pi l}{\sqrt{m^2 - l^2}}$, where $\beta_H = \frac{1}{2}(\beta_L + \beta_R)$ is the inverse of the Hawking temperature. Thus, in the “co-moving” frame where the $\tilde{\phi}$, given in (3), is fixed, we have:

$$\left. \frac{d\phi}{dt} \right|_{t=\tilde{y}, \tilde{\phi}} = \frac{4l}{\sqrt{\lambda^3 m}} e^{-\delta_0} = \frac{\beta_H \Omega}{\beta_R}, \quad (13)$$

so the azimuthal angle $\phi$ is essentially shifted by the angular velocity $\Omega$. The factors of inverse temperatures and their significance for the wave functions of black hole perturbations are similar to the ones discussed for five-dimensional black holes in [14].

The direct connection between the near-horizon geometry and the underlying CFT appears to be valid for black holes in the near-extreme limit only. Eventually, it will be important to test its validity and limitations away from the near-extreme limit. The structure indicated by angular momentum may play an important role in this endeavour [15,26,27].
Acknowledgments: MC would like to thank J. Polchinski and other participants in
the duality program at ITP, Santa Barbara, for discussions. This work is supported in part
by DOE grant DOE-FG02-95ER40893. Work at the ITP was further supported by the NSF
under grant PHY94-07194.
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