The EPRL intertwiners and corrected partition function

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Abstract
Do the $SU(2)$ intertwiners parametrize the space of the Engle, Pereira, Rovelli, Livine (EPRL) solutions to the simplicity constraint? What is the complete form of the partition function written in terms of this parametrization? We prove that the EPRL map is injective in the general $n$-valent vertex case for the Barbero–Immirzi parameter less than 1. We find, however, that the EPRL map is not isometric. In the consequence, a partition function can be defined either using the EPRL intertwiners Hilbert product or the $SU(2)$ intertwiners Hilbert product. We use the EPRL one and derive a new, complete formula for the partition function. Next, we view it in terms of the $SU(2)$ intertwiners. The result, however, goes beyond the $SU(2)$ spin-foam models’ framework and the original EPRL proposal.

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1. Introduction

The main technical ingredient of the spin-foam models of four-dimensional gravity is the so-called quantum simplicity constraint. Imposing a suitably defined constraint on the domain of the (discrete) path integral turns the $SU(2) \times SU(2)$ (or $SL(2,\mathbb{C})$) BF theory into the spin-foam model of Euclidean (respectively, Lorentzian) gravity [1]. The formulation of the simplicity constraint believed to be correct, or at least fitting gravity the best among the known approaches [5], is the one derived by Engle, Pereira, Rovelli, Livine (EPRL) [1] (and independently derived by Freidel and Krasnov [3]). The solutions to the EPRL simplicity constraint are EPRL $SU(2) \times SU(2)$ intertwiners. They are defined by the EPRL transformation, which maps each $SU(2)$ intertwiner into an EPRL solution of the simplicity constraint. An attempt is made in the literature [1] to parametrize the space of the EPRL solutions by the $SU(2)$ intertwiners. The vertex amplitude and the partition function of the EPRL model seem to be written in terms of that parametrization. The questions we raise and answer in this paper are as follows.
Is the EPRL map injective, doesn’t it kill any $SU(2)$ intertwiner?

Is the EPRL map isometric, does it preserve the scalar product between the $SU(2)$ intertwiners?

If not, what is a form of a partition function derived from the $SO(4)$ intertwiner Hilbert product written directly in terms of the $SU(2)$ intertwiners, the preimages of the EPRL map?

We prove that the EPRL map is injective in the general $n$-valent vertex case and for the Barbero–Immirzi parameter $|\gamma| < 1$. The proof in $|\gamma| > 1$ has already been provided in [4]. Hence, there are as many $SU(2) \times SU(2)$ EPRL intertwiners as there are the $SU(2)$ intertwiners. Owing to this result, the $SU(2)$ intertwiners indeed can be used to parametrize the space of the EPRL $SU(2) \times SU(2)$ intertwiners. However, we find that the EPRL map is not isometric. In consequence, there are two inequivalent definitions of the partition function.

One possibility is to use a basis in the EPRL intertwiners space orthonormal with respect to the $SO(4)$ representations. And this is what we do in this paper. A second possibility is to use the basis obtained as the image of an orthonormal basis of the $SU(2)$ intertwiners under the EPRL map. The partition function derived in [1] corresponds to the second choice, whereas the first one is ignored therein. The goal of this part of our paper is pointing out the first possibility and deriving the corresponding partition function. After the derivation, we compare our partition function with that of EPRL on a possibly simple example. We conjecture that the difference converges to zero for large spins.

To make the paper intelligible, we present the new results with the derivation of the partition function in section 2.4. The final formula for our proposal for the partition function for the EPRL model is presented in section 2.5. The lack of the isometricity of the EPRL map is illustrated in specific examples in section 2.6. Finally, the proof of the injectivity of the EPRL map takes the whole section 3.2.

This work is written in terms of the EPRL framework [1] combined with our previous paper [4] on the EPRL model.

2. Our proposal for a partition function of the EPRL model

2.1. Partition functions for the spin-foam models of 4-gravity: definition

Consider an oriented 2-complex $\kappa$ whose faces (2-cells) are labeled by $\rho$ with the irreducible representations of $G = SU(2) \times SU(2),

$\kappa^{(2)} \ni f \mapsto \rho(f),$

and denote by $\mathcal{H}(f)$ the corresponding Hilbert space. For every edge (1-cell) $e$, we have the set/set of incoming/outgoing faces, that is the faces which contain $e$ and whose orientation agrees/disagrees with the orientation of $e$. We use them to define the Hilbert space

$$\mathcal{H}(e) = \bigotimes_{f \text{ incoming}} \mathcal{H}(f) \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}(f')^\ast.$$  \hspace{1cm} (2.1)

The extra data we use are a subspace

$$\mathcal{H}(e)^\text{SIMPLE} \subset \mathcal{H}(e)$$ \hspace{1cm} (2.2)

defined by some constraints called the quantum simplicity constraints. In this paper, starting from the following section, we will be considering the subspace proposed by EPRL. For
According to this rule, given an edge \( e (e') \) contained in incoming (outgoing) face \( f \), the indices of \( P_e \) (\( P_{e'} \)) corresponding to \( H(f) \) are assigned to the beginning and, respectively, to the end of the edge. The oriented arc only marks the orientation of the polygonal face. The time being \( H(e)_{\text{SIMPLE}} \) is any subspace of the space of invariants of the representation
\[
\otimes_{\text{f incoming}} \rho(f) \otimes \otimes_{\text{f outgoing}} \rho(f')^*.
\]

\( (2.3) \)

(The subspace \( H_e^{\text{SIMPLE}} \) may be trivial for generic representations \( \rho(f) \) and \( \rho(f') \). Typically the simplicity constraints also constrain the representations themselves.) To every edge, we assign the operator of the orthogonal projection onto \( H(e)_{\text{SIMPLE}} \),
\[
P_e^{\text{SIMPLE}} : \left( \otimes_{\text{f incoming}} H(f) \otimes \otimes_{\text{f outgoing}} H(f')^* \right) \rightarrow \left( \otimes_{\text{f incoming}} H(f) \otimes \otimes_{\text{f outgoing}} H(f')^* \right).
\]

\( (2.4) \)

Our index notation is as follows (we drop ‘SIMPLE’ for simplicity):
\[
(P_e v)^{A_{-}^\ldots B_{-}^\ldots} = P_e^{A_{+}^\ldots B_{+}^\ldots} v^{A_{+}^\ldots B_{+}^\ldots},
\]

\( (2.5) \)

where the upper/lower indices of any vector \( v \in \otimes_{\text{f incoming}} H(f) \otimes \otimes_{\text{f outgoing}} H(f')^* \) correspond to incoming/outgoing faces. In the operator \( P_e \) for each face containing \( e \), there are two indices, an upper and a lower one corresponding to the Hilbert space \( H(f) \). If \( f \) is incoming (outgoing), then we assign the corresponding lower/upper index of \( P_e \) to the beginning/end (end/beginning) of the edge. That rule is illustrated in figure 1. Now, for every pair of edges \( e \) and \( e' \), which belong to a same face \( f \) and share a vertex \( v \), the natural contraction at \( v \) of the corresponding vertex of \( P_e \) with the corresponding vertex of \( P_{e'} \) is defined. The contraction defines the following trace:
\[
\otimes_{e \in \kappa(1)} P_e^{\text{SIMPLE}} \rightarrow \text{Tr} \left( \otimes_{e \in \kappa(1)} P_e^{\text{SIMPLE}} \right).
\]

\( (2.6) \)

Define partition function \( Z(\kappa, \rho) \) to be the following number:
\[
Z(\kappa, \rho) := \prod_{f \in \kappa(2)} d(f) \text{Tr} \left( \otimes_{e \in \kappa(1)} P_e^{\text{SIMPLE}} \right) A(\text{boundary}), \quad d(f) := \dim H(f).
\]

\( (2.7) \)

where \( A(\text{boundary}) \) is a factor that depends only on the boundary of \( (\kappa, \rho) \), and we derive it elsewhere.
2.2. Partition functions for the spin-foam models of 4-gravity: the amplitude form

The partition function is usually rewritten in the spin-foam amplitude form [6–8]. For that purpose one needs an orthonormal basis in each Hilbert space \( \mathcal{H}(e) \), denote its elements by \( \iota_{e, \alpha} \in \mathcal{H}(e), \alpha = 1, 2, \ldots, n(e) \). Then

\[
P^\text{SIMPLE}_e = \sum_{\alpha=1}^{n(e)} \iota_{e, \alpha} \otimes \iota_{e, \alpha}^\dagger,
\]

(2.8)

where by ‘\( \dagger \)’, for every Hilbert space \( \mathcal{H} \), we denote the duality map

\( \mathcal{H} \ni v \mapsto v^\dagger \in \mathcal{H}^* \),

defined by the Hilbert scalar product. In the Dirac notation

\( \iota_{e, \alpha} = |e, \alpha \rangle \) and \( \iota_{e, \alpha}^\dagger = \langle e, \alpha | \).

Substituting the right-hand side of (2.8) for each \( P^\text{SIMPLE}_e \) in (2.7), one writes the partition function in terms of the vertex amplitudes in the following way:

• For each edge of \( \kappa \) choose an element of the corresponding orthonormal basis; denote this assignment by \( \iota : e \to \iota_{e, \alpha} \).

\( \iota : e \to \iota_{e, \alpha} \)  \( \text{(2.9)} \)

• At each vertex \( v \in \kappa(0) \):
  - take \( \iota_{e_1, \alpha_1}, \ldots, \iota_{e_m, \alpha_m} \), where \( e_1, \ldots, e_m \) are the incoming edges
  - take \( \iota_{e'_1, \alpha'_1}, \ldots, \iota_{e'_m, \alpha'_m} \), where \( e'_1, \ldots, e'_m \) are the outgoing edges
  - define the vertex amplitude

\[
A_v(\iota) := \text{Tr}\left( \iota_{e_1, \alpha_1} \otimes \cdots \otimes \iota_{e_m, \alpha_m} \otimes \iota_{e'_1, \alpha'_1} \otimes \cdots \otimes \iota_{e'_m, \alpha'_m} \right)
\]

(2.10)

where ‘\( \text{Tr} \)’ stands for the contraction (2.6) and can be defined by the evaluation of the spin-networks corresponding to the vertices (see [4]).

• To each face \( f \), assign the face amplitude \( d(f) \).

With these data, with the vertex amplitudes and face amplitudes, the partition function takes the famous form

\[
Z(\kappa, \rho) = \prod_{f \in \kappa(2)} d(f) \sum_{\iota} \prod_{v \in \kappa(0)} A_v(\iota) A(\text{boundary}).
\]

(2.11)

The result is independent of the choice of the orthonormal basis of each \( \mathcal{H}_e^\text{SIMPLE} \).

2.3. The EPRL map

Now we turn to the EPRL intertwiners. For every edge \( e \in \kappa(1) \)

\[
\mathcal{H}(e)^\text{SIMPLE} = \mathcal{H}(e)^\text{EPRL}, \quad P(e)^\text{SIMPLE} = P(e)^\text{EPRL}.
\]

(2.12)

The definition of \( \mathcal{H}(e)^\text{EPRL} \) uses a fixed number \( \gamma \in \mathbb{R} \) called the Barbero–Immirzi parameter.

The Hilbert space \( \mathcal{H}(e)^\text{EPRL} \) can be non-empty only if the 2-complex \( \kappa \) is labeled by EPRL representations. A representation \( \rho = (\rho_-, \rho_+) \) of \( SU(2) \times SU(2) \), where \( j^\pm = \frac{1}{2} \in \mathbb{N} \) define the \( SU(2) \) representations in the usual way, is an EPRL representation, provided that there is \( k \in \frac{1}{2} \mathbb{N} \) such that

\[
|j^\pm| = \frac{|1 \pm \gamma|}{2} k.
\]

(2.13)
Therefore, we will be considering here labelings of the faces of the 2-complex \( \kappa \) with EPRL representations

\[
f \mapsto \rho(f) = (\rho_{j^{-}(f)}, \rho_{j^{+}(f)}),
\]

(2.14)
each of which is defined in the Hilbert space

\[
\mathcal{H}(f) = \mathcal{H}_{j^{-}(f)} \otimes \mathcal{H}_{j^{+}(f)}.
\]

Each labeling also defines a labeling with \( SU(2) \) representations given by (2.13),

\[
f \mapsto \rho_{k}(f),
\]

(2.15)
defined in the Hilbert space \( \mathcal{H}_{k}(f) \). Given an edge \( e \) and the corresponding Hilbert space

\[
\mathcal{H}(e) = \bigotimes_{f \text{ incoming}} \mathcal{H}_{j^{-}(f)} \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}_{j^{+}(f')},
\]

(2.16)
the natural isometric embeddings

\[
C : \mathcal{H}_{k} \to \mathcal{H}_{j^{-}} \otimes \mathcal{H}_{j^{+}},
\]

(2.17)
and the orthogonal projection operator

\[
P : \mathcal{H}(e) \to \mathcal{H}(e)
\]
onto the subspace \( \text{Inv}_{SU(2) \times SU(2)}(\mathcal{H}(e)) \) define the natural map, the EPRL map:

\[
\iota_{\text{EPRL}} : \text{Inv}_{SU(2)} \left( \bigotimes_{f \text{ incoming}} \mathcal{H}_{k}(f) \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}_{k}(f') \right)
\]

\[
\to \bigotimes_{f \text{ incoming}} \mathcal{H}_{j^{-}(f)} \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}_{j^{+}(f')}.
\]

Its image is promoted to the Hilbert space (2.2),

\[
\mathcal{H}_{e}^{\text{EPRL}} := \iota_{\text{EPRL}} \left( \text{Inv}_{SU(2)} \left( \bigotimes_{f \text{ incoming}} \mathcal{H}_{k}(f) \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}_{k}(f') \right) \right).
\]

(2.19)

2.4. The problem with the EPRL intertwiners

All the EPRL intertwiners can be constructed from the \( SU(2) \) intertwiners by using the EPRL map. The point is that one has to be more careful while doing that. First, one has to make sure that the map \( \iota_{\text{EPRL}}^{\text{EPRL}} \) is injective. If not, then the Hilbert space of the \( SU(2) \times SU(2) \) EPRL intertwiners is smaller than the corresponding space of the \( SU(2) \) intertwiners and we should know how big it is. For \( \gamma \geq 1 \), the injectivity was proved in [4]. In the next section we present a proof of the injectivity for \( |\gamma| < 1 \). Secondly, one should check whether or not the map \( \iota_{\text{EPRL}}^{\text{EPRL}} \) is isometric. Given an orthonormal basis \( \mathcal{I}_{\gamma,1}, \ldots, \mathcal{I}_{\gamma,n} \) of the Hilbert space \( \text{Inv}_{SU(2)} \left( \bigotimes_{f \text{ incoming}} \mathcal{H}_{k}(f) \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}_{k}(f') \right) \), we have a corresponding basis \( \iota_{\text{EPRL}}^{\text{EPRL}}(\mathcal{I}_{\gamma,1}), \ldots, \iota_{\text{EPRL}}^{\text{EPRL}}(\mathcal{I}_{\gamma,n}) \) of the corresponding Hilbert space \( \mathcal{H}_{e}^{\text{EPRL}} \). The question is whether or not the latter basis is also orthonormal. We show in section 2.6 that this is not the case. The direct procedure would be to orthonormalize the basis. We propose, however, a simpler solution.
2.5. A solution

An intelligent way is to go back to formula (2.7) for the partition function and repeat the step leading to (2.11) with each projection $P_e^\text{SIMPLE} = P_e^\text{EPRL}$ written in terms of the corresponding basis $i_e^\text{EPRL}(\mathcal{I}_{e,1}), \ldots, i_e^\text{EPRL}(\mathcal{I}_{e,a_e})$. The suitable formula reads

$$P_e^\text{EPRL} = \sum_{a, b=1}^{n_e} h_e^{ab} i_e^\text{EPRL}(\mathcal{I}_{e,a}) \otimes i_e^\text{EPRL}(\mathcal{I}_{e,b})^\dagger, \quad (2.20)$$

where $h_e^{ab}$, $a, b = 1, \ldots, n_e$, define the inverse matrix to the matrix

$$h_{e,ba} := (i_e^\text{EPRL}(\mathcal{I}_{e,b}) | i_e^\text{EPRL}(\mathcal{I}_{e,a})) \quad (2.21)$$
given by the Hilbert product $(\cdot | \cdot)$ in the Hilbert space (2.16). (In the Dirac notation, $P_e^\text{EPRL} = h_e^{ab} |e, a\rangle \langle e, b|$.)

Now, we are in a position to write the resulting spin-foam amplitude formula for the partition function. It is assigned to a fixed 2-complex $\kappa$ and a fixed labeling of the faces by the EPRL representations

$$f \mapsto \rho(f) = (\rho_{j_1}, \rho_{j_2}). \quad (2.22)$$

The labeling is accompanied by the corresponding labeling with the $SU(2)$ i

$$f \mapsto \rho_{k_f}, \quad (2.23)$$

according to (2.13). For every edge $e \in \kappa^{(1)}$, in addition to the Hilbert space $\mathcal{H}_e^\text{EPRL} \subset \text{Inv}_{SU(2) \times SU(2)}(\mathcal{H}_e)$, we also have its preimage, the Hilbert space

$$\text{Inv} \left( \bigotimes_{f \text{ incoming}} \mathcal{H}_{(k_f)} \otimes \bigotimes_{f' \text{ outgoing}} \mathcal{H}_{k_f}^* \right). \quad (2.24)$$

Therein, we fix an orthonormal basis

$$\mathcal{I}_{e,a}, \quad a = 1, 2, \ldots, n_e. \quad (2.25)$$

To define the partition function we proceed as follows:

- assign to every edge of $\kappa$ a pair of elements of the basis,

$$\mathcal{I} \mathcal{I} : e \mapsto (\mathcal{I}_{e,a_e}, \mathcal{I}_{e,b_e}^\dagger), \quad (2.26)$$

more specifically, $\mathcal{I}_{e,a_e}$ is assigned to the end point and $\mathcal{I}_{e,b_e}^\dagger$ to the beginning point of $e$, and we denote the assignment by the double symbol $\mathcal{I} \mathcal{I}$;

- define for every edge, an edge amplitude to be

$$h_e(\mathcal{I} \mathcal{I}) := h^{b_e a_e},$$

- to every vertex $v$ of $\kappa$, assign the vertex amplitude with the trace defined by figure 1, (2.6) and (2.20):

$$A_v(\mathcal{I} \mathcal{I}) := \text{Tr} \left( \bigotimes_{e \text{ incoming}} i_e^\text{EPRL}(\mathcal{I}_{e,a_e}) \otimes \bigotimes_{e' \text{ outgoing}} i_{e'}^\text{EPRL}(\mathcal{I}_{e',b_{e'}})^\dagger \right),$$

- to every face $f$, assign the amplitude $d_f$. 

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Finally, the spin-foam amplitude formula for the partition function reads

\[
Z(\kappa, \rho) = \prod_f d_f \sum_{\mathcal{I}I} \sum_c h_c(\mathcal{I}I) \prod_v A_v(\mathcal{I}I) A(\text{boundary}).
\]  

(2.27)

The matrix (2.21) can be written in terms of the EPRL fusion coefficients,

\[
\mathcal{I}^{\text{EPRL}}(f_{\text{a}},b) =: f^e_{\text{a}} c \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \circle
As an example, we give the result of the calculation of the \( h_{ab} \) matrix for \( \gamma = \frac{1}{2}, j_1 = 2, j_2 = 4, j_3 = 4 \): \( a, b \in \{2, \ldots, 6\} \):

\[
\begin{pmatrix}
53.723 & -2265.\sqrt{7} & 5093.\sqrt{7} & -3\sqrt{5} & 0 \\
175.166 & 30.176 & 1053.056 & -25.088 & 0 \\
2265.\sqrt{7} & 117.853 & -12.805 & 45.\sqrt{7} & -1\sqrt{11} \\
5093.\sqrt{7} & 301.056 & 3912.320 & 7168 & 53.0 \\
3\sqrt{5} & 32.088 & -761.117 & 752.640 & 3120. & 0 \\
0 & 3\sqrt{7} & 5376 & 0 & 10 \\
\end{pmatrix}
\]

We used the analytic expression for the fusion coefficient presented in [9]. Clearly this matrix is nondiagonal. It shows that the EPRL map is not isometric. The edge amplitude \( \text{EPRL} : I_{n} \rightarrow I_{n} \) is given by the inverse matrix:

\[
\begin{pmatrix}
46.976.713 & 31.729.718 & -76.194.882 & -3865.813 & 13.006.606 \\
10.329.715 & 76.823.836 & 67.172.795 & -31.127.222 & 721.244.772 \\
75.194.882 & 257.849.955 & 580.322.625 & 112.636.131 & 1305.090.458 & -70.908.737.236 \\
53.722.715 & 721.204.474 & 579.006.785 & 1305.090.458 & 70.908.737.236 & -192.524.374 \\
13.066.606 & 721.204.474 & 155.006.785 & 12462.943 & 85.031.744.497 & -27.540.436 \\
773.549.865 & 155.006.785 & 12462.943 & 85.031.744.497 & 19.338.746.625 & 27.540.436 \\
\end{pmatrix}
\]

3. Injectivity of the map \( \mathcal{I} \mapsto \iota_{\text{EPRL}}(\mathcal{I}) \)

This part of the paper is devoted to the injectivity of EPRL intertwiner. More explicitly, we will prove the result stated in theorem 1.

3.1. Statement of the result

We assume that \( \gamma \in \mathbb{R} \) and \( |\gamma| < 1 \). Suppose that

\[
(k_1, \ldots, k_n) \in \left( \frac{1}{2} \mathbb{N} \right)^n
\]

\[
\forall j \in \left( \frac{1}{2} \mathbb{N} \right) \quad j^+ = \frac{1 \pm \gamma}{2} k_i \in \left( \frac{1}{2} \mathbb{N} \right)
\]

We consider the EPRL map

\[
\iota_{\text{EPRL}} : \text{Inv}(\mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n}) \rightarrow \text{Inv}(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_n}) \otimes \text{Inv}(\mathcal{H}_{j_{1'}} \otimes \cdots \otimes \mathcal{H}_{j_{n'}})
\]

\[
\iota_{\text{EPRL}}(\mathcal{I})_{j_1', \ldots, j_n', j_{1'}, \ldots, j_{n'}} = I_{k_1, k_2, \ldots, k_n} C^{k_1, k_2, \ldots, k_n}_{j_1', j_2', \ldots, j_n'} E_{k_1, k_2, \ldots, k_n} F^{k_1, k_2, \ldots, k_n}_{j_1', j_2', \ldots, j_n'} E_{j_1', j_2', \ldots, j_n'}
\]

with \( P \) standing for the orthogonal projections onto the subspaces of the SU(2) invariants of the Hilbert spaces \( \mathcal{H}_{j_1'} \otimes \cdots \otimes \mathcal{H}_{j_{n'}} \), and respectively, \( \mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_n} \).

Now we can state our result.

Theorem 1. For any sequence \((k_1, \ldots, k_n) \in \frac{1}{2} \mathbb{N} \) such that \( \text{Inv}(\mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n}) \) is nontrivial, the map

\[
\iota_{\text{EPRL}} : \text{Inv}(\mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n}) \rightarrow \text{Inv}(\mathcal{H}_{j_1} \otimes \cdots \otimes \mathcal{H}_{j_n}) \otimes \text{Inv}(\mathcal{H}_{j_{1'}} \otimes \cdots \otimes \mathcal{H}_{j_{n'}})
\]

is injective.
3.2. Proof of the theorem

In order to make the proof transparent, we divide it into subsections. In subsection 3.2.1, some auxiliary definitions are introduced. We also state an inductive hypothesis, which will be proved in subsection 3.2.5. The injectivity of the EPRL map follows from that result. The main technical tool of the proof is placed in subsection 3.2.3, where map 7 is defined.

3.2.1. Auxiliary definitions

Let us introduce some notations.

**Definition 2.** For \( x \in \mathbb{R} \) we define

- \( \lfloor x \rfloor_+ \) as the only half-integer number in the interval \( \left( x - \frac{1}{4}, x + \frac{1}{4} \right] \),
- \( \lfloor x \rfloor_- \) as the only half-integer number in the interval \( \left[ x - \frac{1}{4}, x + \frac{1}{4} \right) \),

and

**Definition 3.** A sequence of half natural numbers \((k_1, \ldots, k_n)\) satisfies triangle inequality if

\[
\forall i \quad k_i \leq \sum_{j \neq i} k_j.
\]

One can define the map \( \iota \) under the following condition

**Con n:** Sequences of half natural numbers \((k_1, \ldots, k_n)\) and \((j^\pm_1, \ldots, j^\pm_n)\) are such that

- \((k_1, \ldots, k_n)\) satisfies triangle inequality,
- \(j^+_i + j^-_i = k_i\) for \(i = 1, \ldots, n\),
- \(j^+_i = \frac{1 + \gamma}{2} k_i\) for \(i \neq 1\) and

\[
j^+_1 = \left[ \frac{1 + \gamma}{2} k_1 \right], \quad j^-_1 = \left[ \frac{1 - \gamma}{2} k_1 \right],
\]

or

\[
j^+_i = \left[ \frac{1 + \gamma}{2} k_1 \right], \quad j^-_i = \left[ \frac{1 - \gamma}{2} k_1 \right].
\]

Let us define

\[
\iota_{k_1, \ldots, k_n} : \text{Inv}\left( \mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n} \right) \rightarrow \text{Inv}\left( \mathcal{H}_{j^+_1} \otimes \cdots \otimes \mathcal{H}_{j^+_n} \right) \otimes \text{Inv}\left( \mathcal{H}_{j^-_1} \otimes \cdots \otimes \mathcal{H}_{j^-_n} \right)
\]

\[
\iota(\mathcal{I})_{j^+_1, \ldots, j^+_n, k_1, \ldots, k_n} = C_{j^+_1, \ldots, j^+_n, k_1, \ldots, k_n} \mathcal{I}_{j^+_1, \ldots, j^+_n, E_1, \ldots, E_n}
\]

with \( P \) standing for projections onto invariant subspaces. We will use the letter \( \iota_{k_1, \ldots, k_n} \) for all sequences \((k_1, \ldots, k_n)\), \((j^+_1, \ldots, j^+_n)\) if it does not cause any misunderstanding.

We will base our prove on the following inductive hypothesis:

**Hyp n:** Suppose that \((k_1, \ldots, k_n)\) and \((j^+_1, \ldots, j^+_n)\) satisfy condition **Con n** and that \( \mathcal{I} \in \text{Inv}\left( \mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n} \right) \). Then, there exists

\[
\phi \in \text{Inv}\left( \mathcal{H}_{j^+_1} \otimes \cdots \otimes \mathcal{H}_{j^+_n} \right) \otimes \text{Inv}\left( \mathcal{H}_{j^-_1} \otimes \cdots \otimes \mathcal{H}_{j^-_n} \right)
\]

such that \( \langle \iota_{k_1, \ldots, k_n} (\mathcal{I}), \phi \rangle \neq 0 \).

This in fact proves the injectivity.
3.2.2. Useful inequalities. Both \([x]_\pm\) are increasing functions and satisfy \((x, y \in \mathbb{R}, j \in \frac{1}{2} \mathbb{N})\)

(a) \([x + j] = [x] + j\) and \([j]_\pm = j\).
(b) if \(x > y\) then \([x]_- > [y]_-\) and if \(x \geq y\) then \([x]_+ = [y]_+\).
(c) if \(x + y \in \mathbb{Z}\) then \([x]_+ + [y]_- = x + y\),
(d) if \(x + y \geq j\) then \([x]_+ + [y]_- \geq j\).

In order to prove the last point, we note that \([x]_+ > x - \frac{1}{2}\) and \([y]_- \geq y - \frac{1}{2}\) so \([x]_+ + [y]_- > x + y - \frac{1}{2} \geq j - \frac{1}{2}\) but as \(j\) is an half-integer number \([x]_+ + [y]_- \geq j\).

Lemma 4. Suppose that \((k, l, j)\) satisfies the triangle inequality and that \(\frac{1 + j}{2} k \in \frac{1}{2} \mathbb{N}\); then

\[
\begin{align*}
&\text{both triples } (\frac{1 + y}{2} k, [\frac{1 + y}{2} l]_\pm, [\frac{1 + y}{2} j]_\pm) \text{ satisfy triangle inequalities if } k + l = j \text{ or } k + j = l, \\
&\text{both triples } (\frac{1 + y}{2} k, [\frac{1 + y}{2} l]_\pm, [\frac{1 + y}{2} j]_\pm) \text{ satisfy triangle inequalities if } k + l > j \text{ and } k + j > l.
\end{align*}
\]

Proof. In the first case suppose that \(k + l = j\) holds; then \(\frac{1 + j}{2} k + [\frac{1 + y}{2} l]_\pm = [\frac{1 + y}{2} j]_\pm\) that proves triangle inequality.

In the second case, we focus our attention to \((\frac{1 + y}{2} k, [\frac{1 + y}{2} l], [\frac{1 + y}{2} j])\). We have

\[
\begin{align*}
&\frac{1 + y}{2} k + [\frac{1 + y}{2} j]_- \geq [\frac{1 + y}{2} l], \text{ because } \frac{1 + y}{2} k + [\frac{1 + y}{2} l] > [\frac{1 + y}{2} j], \\
&\frac{1 + y}{2} k + [\frac{1 + y}{2} j]_+ \geq [\frac{1 + y}{2} l], \text{ because } \frac{1 + y}{2} k + [\frac{1 + y}{2} j] \geq [\frac{1 + y}{2} l], \\
&\frac{1 + y}{2} k \leq [\frac{1 + y}{2} l]_- + [\frac{1 + y}{2} j]_- \text{ from property (d) listed above.}
\end{align*}
\]

The case of \((\frac{1 + y}{2} k, [\frac{1 + y}{2} l]_+, [\frac{1 + y}{2} j]_+)_\) is analogous. \(\square\)

Lemma 5. Suppose that \((k_1, \ldots, k_n)\) satisfies the triangle inequality and that \(\frac{1 + y}{2} k_i \in \frac{1}{2} \mathbb{N}\) for \(i = 2, \ldots, n; \) then \(([\frac{1 + y}{2} k_1]_+, \frac{1 + y}{2} k_2, \ldots, \frac{1 + y}{2} k_n)\) also satisfy triangle inequalities.

Proof. It follows from the monotonicity of functions \([x]_\pm\) and the fact that in the inequality

\[
\frac{1 + y}{2} k_i \leq \sum_{j \neq i} \frac{1 + y}{2} k_j
\]

all terms but one are half-integer. \(\square\)

Lemma 6. Suppose that \((k_1, \ldots, k_n)\) satisfies the triangle inequality; then

\[
\text{Inv}(\mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n})
\]

is nontrivial.

Proof. We will find \(k_a\) such that both \((k_a, k_1, k_2)\) and \((k_a, k_3, \ldots, k_n)\) satisfy triangle inequalities. By induction there would be

\[
0 \neq \phi \in \text{Inv}(\mathcal{H}_{k_a} \otimes \mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n}),
\]

and then

\[
0 \neq c_{k_A}^A \phi_{k_A} \phi_{k_B} \phi_{k_C} \cdots \phi_{k_n}
\]

proves nontriviality. Now we extract conditions on \(k_a\) from triangle inequalities (we assume for simplicity that \(k_1 \geq k_2\))

\[
\sum_{i \geq 3} k_i \geq k_a \geq k_1 - k_2
\]

\[
k_i \geq k_a \geq k_i - \sum_{j \neq i, j \geq 3} k_j,
\]

\(i \geq 3.\)
For the existence of such $k_a$ we need only to show that
\[
  k_1 + k_2 \geq k_i - \sum_{j \neq i, j \geq 3} k_j, \quad i \geq 3
\]
\[
  \sum_{i \geq 3} k_i \geq k_1 - k_2,
\]
but these are exact conditions for $(k_1, \ldots, k_n)$ to satisfy the triangle inequality. \hfill \square

3.2.3. Important maps. Every $\mathcal{T} \in \text{Inv} (\mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n})$ may be uniquely written as
\[
  \mathcal{T}_{k_1,k_2,k_3,\ldots,k_n} = \sum_{k_a} C_{k_1,k_2,k_3,\ldots,k_n}^{k_a} \mathcal{T}_{k_a},
\]
where $\mathcal{T}_{k_a} \in \text{Inv} (\mathcal{H}_{k_a} \otimes \cdots \otimes \mathcal{H}_{k_a})$. Summation is taken over such $k_a$ that $(k_a, k_1, k_2)$ and $(k_a, k_1, \ldots, k_n)$ satisfy the triangle inequality.

This gives us decomposition of $\text{Inv} (\mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n})$ into orthogonal subspaces
\[
  \oplus \mathcal{H}_a,
\]
where each $\mathcal{H}_a$ is isomorphic to $\text{Inv} (\mathcal{H}_{k_a} \otimes \mathcal{H}_{k_a} \otimes \cdots \otimes \mathcal{H}_{k_a})$. Let us define maps which assign these partial isometries
\[
  Q_{k_a} : \text{Inv} (\mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n}) \rightarrow \text{Inv} (\mathcal{H}_{k_a} \otimes \mathcal{H}_{k_a} \otimes \cdots \otimes \mathcal{H}_{k_a}), \quad Q_{k_a} \mathcal{T} = \mathcal{T}_{k_a}.
\]
Adjoint to them are embeddings $Q_{k_a}^*$:
\[
  Q_{k_a}^* (\mathcal{T}_{k_1,k_2,k_3,\ldots,k_n}) = C_{k_1,k_2,k_3,\ldots,k_n}^{k_a} \mathcal{T}_{k_a}.
\]
These maps are also well defined in a case where $\alpha$ does not occur in the decomposition $\oplus \mathcal{H}_a$ but $(k_a, k_1, k_2)$ satisfies triangle inequalities. Then the space $\text{Inv} (\mathcal{H}_{k_a} \otimes \mathcal{H}_{k_a} \otimes \cdots \otimes \mathcal{H}_{k_a})$ is trivial and the maps $Q_{k_a}$ and $Q_{k_a}^*$ too.

Let us fix $(k_1, \ldots, k_n)$ and $(j_1^+, \ldots, j_n^+)$ satisfying triangle inequalities and such that $j_1^+ + j_1^- = k_1$.

**Lemma 7.** Suppose $k_a, j_a^\pm$ are such that $j_a^+ + j_a^- = k_a$ and $(k_a, k_1, k_2)$ and $(j_a^+, j_1^+, j_2^+)$ satisfy triangle inequalities. Then there exists an operator
\[
  G_{j_a^+ j_a^-} : \text{Inv} (\mathcal{H}_{j_a^+} \otimes \mathcal{H}_{j_a^-} \otimes \cdots \otimes \mathcal{H}_{j_a^-}) \otimes \text{Inv} (\mathcal{H}_{j_a^+} \otimes \mathcal{H}_{j_a^-} \otimes \cdots \otimes \mathcal{H}_{j_a^-})
\]
\[
  \rightarrow \text{Inv} (\mathcal{H}_{j_a^+} \otimes \cdots \otimes \mathcal{H}_{j_a^-}) \otimes \text{Inv} (\mathcal{H}_{j_a^+} \otimes \cdots \otimes \mathcal{H}_{j_a^-})
\]
such that for all $\mathcal{T} \in \text{Inv} (\mathcal{H}_{j_a^+} \otimes \mathcal{H}_{j_a^-} \otimes \cdots \otimes \mathcal{H}_{j_a^-})$ and $\phi \in \text{Inv} (\mathcal{H}_{j_a^+} \otimes \mathcal{H}_{j_a^-} \otimes \cdots \otimes \mathcal{H}_{j_a^-})$ and $\mathcal{N}(\mathcal{H}_{j_a^+} \otimes \cdots \otimes \mathcal{H}_{j_a^-})$
\[
  \langle \psi_{k_1,\ldots,k_n} Q_{j_a^+ j_a^-}^* \mathcal{T}, G_{j_a^+ j_a^-} \phi \rangle = \begin{cases} 1, & k_{j_a} = k_a \\ 0, & k_{j_a} < k_a \\ 0, & k_{j_a} > k_a \end{cases}.
\]

**Proof.** We define $G_{j_a^+ j_a^-}$ as
\[
  G_{j_a^+ j_a^-} (\phi) j_a^+ A_{j_a^+} A_{j_a^-} B_{j_a^+} \cdots B_{j_a^-} = \beta C_{j_a^+ A_{j_a^+} A_{j_a^-}}^B C_{j_a^- B_{j_a^-} A_{j_a^-}} B_{j_a^+} A_{j_a^+} \phi j_a^+ A_{j_a^+} A_{j_a^-} B_{j_a^+} \cdots B_{j_a^-},
\]
with $\beta$ a nonzero constant to be defined later. Let us compute
\[
  \langle \psi_{k_1,\ldots,k_n} Q_{j_a^+ j_a^-}^* \mathcal{T}, G_{j_a^+ j_a^-} \phi \rangle.
\]
In the definition of $t$, one can skip projection because both $\phi$ and $G_{k,j\ell}^\alpha \phi$ are invariants. Let us write explicitly $|k_1, \ldots, k_n \rangle_{I_1} \otimes I_2 \otimes G_{k,j \ell}^\alpha \phi$. We have

$$\mathcal{I}_{k_1 A} \cdots k_n A \beta C_{k_1 A_1}^{k_2 A_2} C_{j_1 B_1 j_1 C_1}^{k_2 A_2} \cdots C_{k_n A_n}^{k_2 A_2}$$

$$= \beta C_{k_1 A_1}^{k_2 A_2} C_{j_1 B_1 j_1 C_1}^{k_2 A_2} C_{j_2 B_2 j_2 C_2}^{k_2 A_2} \cdots C_{j_n B_n j_n C_n}^{k_2 A_2}$$

$$\times \mathcal{I}_{k_1 A} \cdots k_n A \beta C_{k_1 A_1}^{k_2 A_2} C_{j_1 B_1 j_1 C_1}^{k_2 A_2} \cdots C_{k_n A_n}^{k_2 A_2}$$

We only need to show that

$$\beta C_{k_1 A_1}^{k_2 A_2} C_{j_1 B_1 j_1 C_1}^{k_2 A_2} C_{j_2 B_2 j_2 C_2}^{k_2 A_2} \cdots C_{j_n B_n j_n C_n}^{k_2 A_2}$$

is equal to zero or equivalently the same for $C_{k_2 A_2}^{k_1 A_1}$. In the definition of $\text{Class. Quantum Grav.} (2010) 165020$, we include it for the sake of completeness.

3.2.4. Relation among intertwiners. We know that $C_{k_1 A_1}^{k_2 A_2} C_{j_1 B_1 j_1 C_1}^{k_2 A_2} C_{j_2 B_2 j_2 C_2}^{k_2 A_2} C_{j_3 B_3 j_3 C_3}^{k_2 A_2} C_{j_4 B_4 j_4 C_4}^{k_2 A_2} C_{j_5 B_5 j_5 C_5}^{k_2 A_2} C_{j_6 B_6 j_6 C_6}^{k_2 A_2}$ is proportional to $C_{j_6 B_6 j_6 C_6}^{k_2 A_2}$. In order to prove that the factor of proportionality is nonzero, we will show that

$$C_{k_1 A_1}^{k_2 A_2} C_{j_1 B_1 j_1 C_1}^{k_2 A_2} C_{j_2 B_2 j_2 C_2}^{k_2 A_2} C_{j_3 B_3 j_3 C_3}^{k_2 A_2} C_{j_4 B_4 j_4 C_4}^{k_2 A_2} C_{j_5 B_5 j_5 C_5}^{k_2 A_2} C_{j_6 B_6 j_6 C_6}^{k_2 A_2} \neq 0$$

and that would be $\beta^{-1}$. In fact it is enough to show that the intertwiner

$$C_{k_1 A_1}^{k_2 A_2} C_{j_1 B_1 j_1 C_1}^{k_2 A_2} C_{j_2 B_2 j_2 C_2}^{k_2 A_2} C_{j_3 B_3 j_3 C_3}^{k_2 A_2} C_{j_4 B_4 j_4 C_4}^{k_2 A_2} C_{j_5 B_5 j_5 C_5}^{k_2 A_2} C_{j_6 B_6 j_6 C_6}^{k_2 A_2}$$

is not equal to zero or equivalently the same for

$$C_{k_1 A_1}^{k_2 A_2} C_{j_1 B_1 j_1 C_1}^{k_2 A_2} C_{j_2 B_2 j_2 C_2}^{k_2 A_2} C_{j_3 B_3 j_3 C_3}^{k_2 A_2} C_{j_4 B_4 j_4 C_4}^{k_2 A_2} G_{k,j \ell}^\alpha \phi$$

We only sketch the proof. First of all, we recall some facts about intertwiners and diagrammatic notation.

Let $P^k$ stand for projection onto symmetric subspace in $\mathcal{H}_{1/2}^{(2k)}$ equivalent to $\mathcal{H}_k$ ($k$ is a half natural number):

$$P^k : \mathcal{H}_{1/2}^{(2k)} \rightarrow \mathcal{H}_{1/2}^{(2k)}$$

In this subsection, we regard $\mathcal{H}_k$ as the subspace of $\mathcal{H}_{1/2}^{(2k)}$. Let us also denote the canonical map $\varepsilon : C \mapsto \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$. Let $H_{1/2} \leq \mathcal{H}$.

The intertwiner $H_{1/2}^{(2k)} : C \mapsto \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2} \otimes \mathcal{H}_{1/2}$ is proportional to $P^{k_1} \otimes P^{k_2} \otimes P^{k_3}$.

In the diagrammatic language, this can be depicted as in figure 2. We skip the index $k$ in $P^k$ in the diagrams for notations’ brevity. The line with symbol $k$ denotes $\mathcal{H}_{1/2}^{(2k)}$.

We have to note important properties which are shown in figures 3 and 4 in diagrammatic language.

Our intertwiner can be written as shown in figure 5.

1 Although it seems to be standard, we include it for the sake of completeness.
Figure 2. An intertwiner proportional to $C_{k_1k_2}^{k_3}$, $k_{12} = k_1 + k_2 - k_3$ etc.

\[
\begin{array}{c}
k_1 \\ P \\ k_2 \\ P \\ k_3
\end{array}
\]

Figure 3. An equality between $\mathcal{H}_{1/2}^{k_1} \otimes \mathcal{H}_{1/2}^{k_2}$ and $\mathcal{H}_{1/2}^{k_1+k_2}$.

\[
\begin{array}{ccc}
k_1 & k_2 & = \\
\end{array}
\]

Figure 4. An equality $p^{k_1+k_2+k_3} \circ p^{k_1+k_2} \otimes I = p^{k_1+k_2+k_3}$.

Now using the properties mentioned earlier, we see that one can merge $j^+_{ij}$ with $j^-_{ij}$ into $k_{ij}$ and that the intertwiner is equal to the intertwiner shown in figure 2 and is nonzero.

3.2.5. Inductive steps. Induction starts with $n = 1$. In this case $k_1 = 0$ and so also $j^+_{ij} = 0$.

The map $i_0 = C_{00}^0 : \mathbb{C} \rightarrow \mathbb{C} \otimes \mathbb{C}$ is the identity.

Suppose now that we have just proved Hyp $n - 1$. 

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In the decomposition of given $I \in \text{Inv} (\mathcal{H}_{k_1} \otimes \cdots \otimes \mathcal{H}_{k_n})$ into subspaces $\mathcal{H}_\alpha$, we choose minimal $k_\alpha$ such that $Q_{k_\alpha} I$ is nonzero. We know, by lemmas 4 and 5, that for either

$$(j_{\alpha}, j_{-\alpha}) = (\left\lceil \frac{1 + \gamma}{2} k_\alpha \right\rceil, \left\lfloor \frac{1 - \gamma}{2} k_\alpha \right\rfloor)$$

or

$$(j_{\alpha}^+, j_{-\alpha}^-) = (\left\lfloor \frac{1 + \gamma}{2} k_\alpha \right\rfloor, \left\lceil \frac{1 - \gamma}{2} k_\alpha \right\rceil)$$

all necessary assumptions of lemma 7 are satisfied. From Hyp $n - 1$ for the sequences $(k_\alpha, k_3, \ldots, k_n)$, $(j_{\alpha}^+, j_3^+, \ldots, j_n^+)$, we know that there exists

$$\phi \in \text{Inv} (\mathcal{H}_{j_1}^+ \otimes \cdots \otimes \mathcal{H}_{j_n}^+) \otimes \text{Inv} (\mathcal{H}_{j_1^-}^- \otimes \cdots \otimes \mathcal{H}_{j_n^-}^-)$$

such that

$$\langle i_{k_3 \cdots k_n} Q_{k_\alpha} I, \phi \rangle \neq 0.$$

We have

$$\langle i_{k_3 \cdots k_n} I, G_{k_3 \cdots k_n} \phi \rangle = \sum_{k_3 \geq k_\alpha} \langle i_{k_3 \cdots k_n} Q_{k_3}^+ Q_{k_\alpha} I, G_{k_3 \cdots k_n} \phi \rangle = \langle i_{k_3 \cdots k_n} Q_{k_\alpha} I, \phi \rangle \neq 0.$$

This finishes inductive step and the proof.

4. Short discussion

In this paper, we studied the properties of the solutions to the EPRL simplicity constraints which were derived in [1]. We also pointed out two different possibilities of defining the partition function out of them. Our definition is (2.7). It uses only the subspace of the $SO(4)$ intertwiners which solve the EPRL simplicity constraint. The comparison and contrast between our definition and that of [1] is provided by (2.27) and the comments which follow.
that equality. The difference follows from the fact proven in section 2.6 that the EPRL map is not isometric. The example considered in that section also gives a quantitative idea of the difference. The question of which definition of the partition function is correct cannot be answered at this stage. Finally, we studied the ‘size of the space of the EPRL solutions’. We have shown that the EPRL map does not kill any $SU(2)$ intertwiner. The proof is presented in detail in section 3.

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