The quantum strong subadditivity condition for systems without subsystems

Margarita A Man’ko and Vladimir I Man’ko

1. Introduction

Quantum correlations between the subsystems of composite systems provide specific entropic inequalities relating von Neumann entropies of the system to its subsystems. For example, quantum correlations are responsible for the violation of the classical entropic inequality for bipartite classical systems $H(1) \leq H(1,2)$, where $H(1,2)$ is the Shannon entropy of a bipartite classical system and $H(1)$ is the Shannon entropy [1] of its subsystem. This inequality, having an intuitively clear interpretation that the disorder in the total system is either the same as or larger than the disorder of its subsystems, is not true for a quantum bipartite system.

It is known that, for the two-qubit pure maximum entangled state with a density matrix $\rho(1,2)$, the von Neumann entropy $S(1,2) = -\text{Tr} \rho(1,2) \ln \rho(1,2) = 0$, but the von Neumann entropy for the one-qubit state $S(1) = -\text{Tr} \rho(1) \ln \rho(1)$ with $\rho(1) = -\text{Tr} \rho(1,2)$ has the maximum possible qubit value, i.e. $S(1) = \ln 2$. Thus, in this state, $S(1) > S(1,2)$, i.e. quantum correlations between two qubits in the composite system (consisting of two qubits) provide not only the violation of the Bell inequalities [2, 3] but also yield the violation of the classical entropic inequality.

For bipartite systems, both classical and quantum, there exist entropic inequalities, called the subadditivity conditions, which are the same for Shannon entropies and von Neumann entropies. The quantum subadditivity condition can be proved, e.g., by using the tomographic probability description of spin states [4]. Recent reviews of the tomographic representation of classical and quantum mechanics can be found in [5, 6]. For tripartite systems, both the classical and the quantum, there also exist entropic inequalities, called the strong subadditivity conditions, which have the same form for classical Shannon entropies in the classical case and for von Neumann entropies in the quantum case. Lieb and Ruskai [7] were the first to prove the quantum strong subadditivity condition. The tomographic probability approach to the strong subadditivity condition was discussed in [4]. Various aspects of entropic inequalities and the quantum strong subadditivity condition for tripartite systems can be found in [8–14].

Recently, it was shown that the subadditivity condition exists not only in bipartite quantum systems but also in systems which do not contain subsystems, e.g. for one qutrit [15]. An approach for deriving the subadditivity condition for the qutrit state is based on the method known as the qubit portrait of qudit states [16], later on used in [17] to study entanglement in two qubit systems.

The aim of this paper is to show that the strong subadditivity condition can be obtained for quantum systems which do not have subsystems. For this, we apply the qubit-portrait method (which is a generalization of the qubit-portrait method) that, in fact, is acting by a specific positive map on the density matrix. The map is described by the action on a vector by the matrix with matrix elements equal either to zero or unity. Such matrices are used to obtain density matrices of subsystems by partial tracing of the density matrices of the composite-system states. In this case, the system density matrix is first mapped onto the vector, and then the map matrix acts onto this vector. The obtained vector is mapped again onto the new density matrix. For composite systems, the portrait method is identical to taking the partial trace of the system density matrix. Since the $N$-dimensional density matrix of the composite system state and the state of a qubit in the $N$-dimensional Hilbert space have identical
properties, the possibility exists of obtaining and applying the strong subadditivity condition available for the composite system to a system without subsystems.

Our aim is to present the strong subadditivity condition for an arbitrary probability \( N \)-vector describing the classical system without subsystems. In the quantum case, we present the strong subadditivity condition for an arbitrary density \( N \times N \)-matrix describing the state of a system without subsystems.

This paper is organized as follows. In section 2, we consider the classical system described by the probability \( N \)-vector and show the example of \( N = 7 \) in detail. In section 3, we consider the quantum system state associated with the density \( N \times N \)-matrix and present the example of \( N = 7 \). We give our conclusions and perspectives in section 4, where we also discuss the possible consequences for the systems of qudits and quantum correlations in these systems in the context of the strong subadditivity conditions obtained. In the appendix, the entropic inequalities for tomograms of some qudit states are presented.

2. The classical strong subadditivity condition

We consider a classical system for which one has a random variable. The probabilities of obtaining the values of this random variable are described by a probability vector \( \vec{p} = (p_1, p_2, \ldots, p_N) \), where \( p_k \geq 0 \) and \( \sum_{k=1}^{N} p_k = 1 \). The system has no subsystems, and the order in this system is described by the Shannon entropy

\[
H = - \sum_{k=1}^{N} p_k \ln p_k
\]

which satisfies the inequality \( H \geq 0 \), takes the maximum value for \( \vec{p} \) with components \( p_k = N^{-1} \) and equals \( H_{\text{max}} = \ln N \).

If the classical system has two subsystems 1 and 2 and two random variables, the probability of obtaining the two values of these random variables is described by the nonnegative numbers \( P_{kj} \), \( k = 1, 2, \ldots, N_1 \) and \( j = 1, 2, \ldots, N_2 \). The probabilities satisfy the normalization condition \( \sum_{k=1}^{N_1} \sum_{j=1}^{N_2} P_{kj} = 1 \), and the Shannon entropy of the system state reads

\[
H(1, 2) = - \sum_{k=1}^{N_1} \sum_{j=1}^{N_2} P_{kj} \ln P_{kj}.
\]

The joint probability distribution \( P_{kj} \) provides the marginal distributions for systems 1 and 2 as follows:

\[
P_{1k} = \sum_{j=1}^{N_2} P_{kj}, \quad P_{2j} = \sum_{k=1}^{N_1} P_{kj}.
\]

Thus, we have two Shannon entropies associated with marginal distributions (3), and they read

\[
H(1) = - \sum_{k=1}^{N_1} P_{1k} \ln P_{1k}, \quad H(2) = - \sum_{j=1}^{N_2} P_{2j} \ln P_{2j}.
\]

It is known that these entropies satisfy the subadditivity condition written in the form of the inequality

\[
H(1) + H(2) \geq H(1, 2)
\]

and the Shannon information is defined as the difference

\[
I = H(1) + H(2) - H(1, 2).
\]

If the classical system has three subsystems (1, 2 and 3) with three random variables, the joint probability distribution describing the results of the measurement of the random variables is related to the nonnegative numbers \( \Pi_{kjl} \) (\( k = 1, 2, \ldots, N_1 \), \( j = 1, 2, \ldots, N_2 \) and \( l = 1, 2, \ldots, N_3 \)). The nonnegative numbers determine the marginal probability distributions

\[
P_{1k}^{(12)} = \sum_{j=1}^{N_2} \Pi_{kjl}, \quad P_{2j}^{(23)} = \sum_{k=1}^{N_1} \Pi_{kjl}, \quad P_{3l}^{(3)} = \sum_{k=1}^{N_1} \sum_{j=1}^{N_2} \Pi_{kjl}.
\]

The Shannon entropies associated with these probability distributions satisfy the strong subadditivity condition

\[
H(1, 2) + H(2, 3) \geq H(1, 2, 3) + H(2),
\]

where

\[
H(1, 2, 3) = - \sum_{k=1}^{N_1} \sum_{j=1}^{N_2} \sum_{l=1}^{N_3} \Pi_{kjl} \ln \Pi_{kjl}
\]

and entropies \( H(1, 2), H(2, 3) \) and \( H(2) \) associated with distributions \( P_{1k}^{(12)}, P_{2j}^{(23)} \) and \( P_{3l}^{(3)} \) are given by (2) and (4) with obvious substitutions.

In [15], the suggestion was made to obtain an analogue of the subadditivity condition (5) for a system without subsystems. The general scheme for obtaining such an inequality is to write the probability vector \( \vec{p} \) with components \( p_k \) (\( k = 1, 2, \ldots, N \)) in a matrix form with matrix elements \( P_{kj} \). Then, inequality (5) can be obtained in view of the above procedure. Here, we apply this method to map the probability vector \( \vec{p} \) onto a table of numbers with three indices \( \Pi_{kjl} \). As a result, we can obtain the strong subadditivity condition for a system without subsystems. We demonstrate this procedure on the example of \( \vec{p} \) with eight components.

Let us define a map given by the equalities

\[
p_1 = \Pi_{111}, \quad p_2 = \Pi_{112}, \quad p_3 = \Pi_{121}, \quad p_4 = \Pi_{122},
\]

\[
p_5 = \Pi_{211}, \quad p_6 = \Pi_{212}, \quad p_7 = \Pi_{221}, \quad p_8 = \Pi_{222}.
\]

The map introduced provides an inequality which is the strong subadditivity condition associated with the table \( \Pi_{kjl} \). To point out a peculiarity of the strong subadditivity condition, we consider the case of \( N = 7 \). It is the prime number, and the system with the probability vector has no subsystems. Thus, we have seven nonnegative numbers \( p_1, p_2, \ldots, p_7 \) and the normalization condition \( p_1 + p_2 + \ldots + p_7 = 1 \). Also we add an extra component \( p_8 = 0 \) to the probability vector. We added zero components to the probability vector since there is a mismatch of numbers 2\( n \) and 2\( k+1 \) (in the case under consideration, numbers 8 and 7). This means that, in the
previous picture of the eight-dimensional probability vector, we consider the probability distribution with the constraint \( p_8 = 0 \) that provides the constraint \( \Pi_{22} = 0 \) in map (10).

Applying inequality (8) and formula (9), we obtain the strong subadditivity condition in the case of the probability 7 vector, which we express in an explicit form in terms of the vector components

\[
-\sum_{k=1}^{7} p_k \ln p_k \leq -(p_1 + p_2 + p_5 + p_6) \ln(p_1 + p_2 + p_5 + p_6) \\
-(p_3 + p_4 + p_7) \ln(p_3 + p_4 + p_7) \leq -(p_1 + p_2) \ln(p_1 + p_2) \\
-(p_3 + p_4) \ln(p_3 + p_4) - (p_5 + p_6) \ln(p_5 + p_6) - p_7 \ln p_7 \\
-(p_1 + p_5) \ln(p_1 + p_5) - (p_2 + p_6) \ln(p_2 + p_6) \\
-(p_3 + p_7) \ln(p_3 + p_7) - p_4 \ln p_4.
\]

(11)

This inequality is valid if one makes an arbitrary permutation of 7! permutations of the vector components of the probability vector \( \vec{p} \). Inequality (11) can be presented in a form where the terms \(-p_2 \ln p_4\) and \(-p_7 \ln p_7\) are removed from both sides of the inequality.

Analogously, we can write the subadditivity condition following the approach of [15]. For example, we have

\[
-\sum_{k=1}^{7} p_k \ln p_k \leq -(p_1 + p_2 + p_5 + p_6) \ln(p_1 + p_2 + p_5 + p_6) \\
-(p_3 + p_4 + p_7) \ln(p_3 + p_4 + p_7) -(p_1 + p_3) \\
\times \ln(p_1 + p_3) - (p_2 + p_4) \ln(p_2 + p_4) \\
-(p_3 + p_7) \ln(p_3 + p_7) - p_4 \ln p_4.
\]

(12)

Also we can rewrite this inequality removing the term \(-p_4 \ln p_4\) from both sides of the inequality. We see that this inequality is valid for a system without subsystems. For example, in the case of a quantum particle with spin \( j = 3 \), the state of this particle is determined by the spin tomogram \( w(m, \vec{n}) \) [18, 19], where the spin projection \( m = -3, -2, -1, 0, 1, 2, 3 \), and the unit vector \( \vec{n} \) determines the quantization axes. The tomographic probability distribution (spin tomogram) of any qudit state with a density matrix \( \rho \) is determined by diagonal matrix elements of the rotated density matrix as \( w(m, \vec{n}) = |m\ U_{\rho}^u |m\rangle \), where the unitary matrix \( u \) is the matrix of an irreducible representation of the rotation group, and depends on the Euler angles determining the unit vector \( \vec{n} \). Thus, the tomogram is the probability distribution of the spin projection \( m \) in the direction \( \vec{n} \). We can identify the components of the probability vector \( \vec{p} \) with the tomographic probabilities. Then we have the inequality—the subadditivity condition for the spin tomographic probabilities:

\[
-\sum_{m=-3}^{3} w(m, \vec{n}) \ln w(m, \vec{n}) \leq -[w(-3, \vec{n}) + w(-2, \vec{n}) + w(1, \vec{n})] \\
+ w(2, \vec{n}) \ln[w(-3, \vec{n}) + w(-2, \vec{n}) + w(1, \vec{n}) + w(2, \vec{n})] \\
- [w(-1, \vec{n}) + w(0, \vec{n}) + w(3, \vec{n})] \ln[w(-1, \vec{n}) + w(0, \vec{n})] \\
+ w(3, \vec{n})] - [w(-3, \vec{n}) + w(1, \vec{n})] \ln[w(-3, \vec{n}) + w(1, \vec{n})] \\
- [w(-2, \vec{n}) + w(2, \vec{n})] \ln[w(-2, \vec{n}) + w(2, \vec{n})] - [w(-1, \vec{n}) + w(3, \vec{n})] \ln[w(-1, \vec{n}) + w(3, \vec{n})] - w(0, \vec{n}) \ln w(0, \vec{n}).
\]

(13)

This inequality describes some properties of quantum correlations in the spin system with \( j = 3 \). In spite of the fact that this system does not have subsystems, inequality (13), which corresponds to the subadditivity condition, is valid for any direction of the vector \( \vec{n} \). Other examples of tomographic inequalities are given in the appendix.

3. The strong subadditivity condition for the one qudit state

In this section, we obtain the strong subadditivity condition for a system without subsystems written in the form of an inequality for von Neumann entropies associated with the initial density matrix of the spin-\( j \) state and its qubit (or qudit) portraits. The qubit (or qudit) portrait [16, 20] of the initial density matrix is a specific positive map of this matrix obtained using the following procedure. Any \( N \times N \)-matrix \( \rho_{jk} \) is considered as the column vector \( \vec{p} \) with components \( \rho_{11}, \rho_{12}, \ldots, \rho_{1N}, \rho_{21}, \rho_{22}, \ldots, \rho_{2N}, \ldots, \rho_{N1}, \rho_{N2}, \ldots, \rho_{NN} \). We multiply this vector by the matrix \( M \) which contains only units and zeros, and the units and zeros are matrix elements to provide the new vector \( \rho_{jk} M \) obtained is considered as a new matrix \( \rho_{jk} M \) being the density matrix. It is easy to prove that, for any density matrix of a multi-qudit system \( \rho(1, 2, \ldots, M) \), one can calculate the density matrix of an arbitrary subsystem of qudits \( \rho(1, 2, \ldots, M') \) by means of a portrait of the initial density matrix. In view of this observation, we extend the entropic inequalities available for composite systems of qudits to arbitrary density matrices including the density matrices of a single qudit. We show the result of such an approach on the example of the strong subadditivity condition known for three partite quantum systems [7].

For the qudit state with \( j = 3 \) and the density matrix \( \rho \) with matrix elements \( \rho_{jk}, k, j = 1, 2, \ldots, 7 \), the strong subadditivity condition found turns out to be

\[
-\text{Tr} (\rho \ln \rho) - \text{Tr} (R_2 \ln R_2) \leq -\text{Tr} (R_{12} \ln R_{12}) - \text{Tr} (R_{23} \ln R_{23}),
\]

(14)

where the density matrix \( R_{12} \) has matrix elements expressed in terms of the density matrix \( \rho_{jk} \) as follows:

\[
R_{12} = \begin{pmatrix}
\rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} & \rho_{15} + \rho_{26} & \rho_{17} \\
\rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} & \rho_{35} + \rho_{46} & \rho_{37} \\
\rho_{51} + \rho_{62} & \rho_{53} + \rho_{64} & \rho_{55} + \rho_{66} & \rho_{57} \\
\rho_{71} & \rho_{73} & \rho_{75} & \rho_{77}
\end{pmatrix}.
\]

(15)

The density matrix \( R_{23} \) reads

\[
R_{23} = \begin{pmatrix}
\rho_{11} + \rho_{55} & \rho_{12} + \rho_{56} & \rho_{13} + \rho_{57} & \rho_{14} \\
\rho_{21} + \rho_{65} & \rho_{22} + \rho_{66} & \rho_{23} + \rho_{67} & \rho_{24} \\
\rho_{31} + \rho_{75} & \rho_{32} + \rho_{76} & \rho_{33} + \rho_{77} & \rho_{34} \\
\rho_{41} & \rho_{42} & \rho_{43} & \rho_{44}
\end{pmatrix},
\]

(16)

while the matrix \( R_2 \) is

\[
R_2 = \begin{pmatrix}
\rho_{11} + \rho_{22} + \rho_{55} + \rho_{66} & \rho_{13} + \rho_{24} + \rho_{57} \\
\rho_{31} + \rho_{42} + \rho_{75} & \rho_{33} + \rho_{44} + \rho_{77}
\end{pmatrix}.
\]

(17)
The inequality for von Neumann entropies associated with the matrices \( \rho, R_{12}, R_{23} \), and \( R_2 \) has the form of the strong subadditivity condition for a three partite system with the density matrix \( \rho(1, 2, 3) \) obtained in [7].

The other entropic inequality for the spin-2 state with the density matrix \( \rho_{jk}, j, k = 1, 2, 3 \) was presented in [14] with the matrices \( R_{12}, R_{23} \) and \( R_2 \) as follows:

\[
R_{12} = \begin{pmatrix}
\rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} & \rho_{15} \\
\rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} & \rho_{35} \\
\rho_{51} & \rho_{53} & \rho_{55}
\end{pmatrix},
\]

\[
R_{23} = \begin{pmatrix}
\rho_{11} + \rho_{55} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\
\rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\
\rho_{41} & \rho_{42} & \rho_{43} & \rho_{44}
\end{pmatrix},
\]

\[
R_2 = \begin{pmatrix}
\rho_{11} + \rho_{22} + \rho_{55} & \rho_{13} + \rho_{24} \\
\rho_{31} + \rho_{42} & \rho_{33} + \rho_{44}
\end{pmatrix}.
\]

### 4. Conclusions

To conclude, we list our main results.

We proved matrix inequalities for arbitrary nonnegative Hermitian \( N \times N \)-matrices with trace equal to unity. If the matrix is identified with the density matrix of the qudit state, the matrix inequalities obtained are entropic inequalities characterizing quantum correlations in the system.

Employing the positive map of an arbitrary density matrix corresponding to the qubit (or qudit) portrait of the density matrix of a multiqudit state identified with the calculation of the subsystem-state density matrices, we obtained an analogue of the strong subadditivity condition for the state of a system which does not contain any subsystems. This result is an extension of the approach in [15], where the subadditivity condition was obtained for quantum systems without subsystems. We derived the entropic inequalities for the qudit-state tomograms and showed examples of the subadditivity condition and the strong subadditivity condition for the spin states with \( j = 2 \) and 3, respectively. We presented the entropic inequalities for density matrices—anallogues of the strong subadditivity condition for \( j = 3 \)—in the form of an explicit matrix inequality. We formulated the approach to find new entropic inequalities for both cases: (i) the probability distributions and related Shannon entropies and (ii) the density matrices and related von Neumann entropies.

For a given arbitrary integer \( N \), one can construct many integers \( N = N' + K \), such that \( N' = n_1 n_2 \), where \( n_1 \) and \( n_2 \) are integers. If there exists the probability vector with \( N \) components, a new probability vector with \( N' \) components can be constructed, and the \( K \) components of the constructed vector can be assumed to be zero components. Then, the numbers 1, 2, \ldots, \( N' \) can be mapped onto pairs of integers \( (1, 1), (1, 2), \ldots, (n_1, 1), (n_2, 1), (2, 2), \ldots, (2, n_2), \ldots, (n_1, 1), (n_2, 1), \ldots, (n_1 n_2) \). This means that the probability vector constructed is mapped onto a matrix with matrix elements analogous to the joint probability distribution of two random variables. In view of the known subadditivity condition for this joint probability distribution, one has the entropic inequality, which can be expressed in terms of the components of the initial probability \( N \)-vector. We used such a procedure to obtain both the classical and the quantum strong subadditivity conditions.

The physical interpretation of the obtained strong subadditivity condition requires extra clarification. It is possible to connect the new entropic inequalities with such state characteristics as purity or such parameters as \( \text{Tr} \rho^o \), as well as with correlations between different groups of measurable quantities. The entropic interpretation can be given to the correlations between groups of tomographic probability values.

The tomographic distributions and their relations to different quasi-distributions obtained in [21] can be used to derive entropic inequalities associated with analytical signals.

New relations for \( q \)-entropies obtained for multipartite systems in [20] and associated with entropic inequalities discussed in [22] can also be considered for systems without subsystems, in view of the approach developed. We apply this procedure to find new equalities and inequalities for probability distributions and density matrices of quantum states in a future publication.

### Acknowledgments

This study was motivated by our discussions with Professor Rui Vilela Mendes and Professor Mary Beth Ruskai during the Madeira Math Encounters XXVI (3–11 October 2003) Quantum Information, Control and Computing. We thank Rui Vilela Mendes for reading the manuscript before publication and giving useful comments. This work was supported by the Russian Foundation for Basic Research under project number 11-02-00456 a. We are grateful to the Organizers of the XX Central European Workshop on Quantum Optics (Stockholm, Sweden, 16–20 June 2013) and especially to Professor Gunnar Bjork for his kind hospitality.

### Appendix

We present the entropic inequalities—the subadditivity conditions for the spin tomographic probability distributions \( w(m, n) \) for one qudit with spin \( j = 2 \) and 3 as follows:

\[
\begin{align*}
\text{for } j = 2, m = -2, -1, 0, 1, 2 \quad \text{and } n = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \\
&= -\ln \left[ w(-2, n) + w(-1, n) + w(0, n) \right] \ln \left[ w(-2, n) + w(-1, n) \right] \\
&+ w(0, n) - \ln \left[ w(2, n) + w(2, n) \right] \ln \left[ w(1, n) + w(2, n) \right] \\
&= -\ln \left[ w(-2, n) + w(1, n) \right] \ln \left[ w(-2, n) + w(1, n) \right] \\
&- \ln \left[ w(0, n) + w(2, n) \right] \ln \left[ w(0, n) + w(2, n) \right] \\
&\geq -\ln \left[ w(-2, n) + w(0, n) \right] \ln \left[ w(0, n) + w(2, n) \right] \\
&+ w(1, n) \ln \left[ w(1, n) + w(2, n) \right] \ln \left[ w(2, n) \right].
\end{align*}
\]
\( j = 3, \ m = -3, -2, -1, 0, 1, 2, 3 \) and \( \vec{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \),

\[
- \left[ w(-3, \vec{n}) \ln w(-3, \vec{n}) + w(-2, \vec{n}) \ln w(-2, \vec{n}) + w(-1, \vec{n}) \right] \\
\times \ln w(-1, \vec{n}) + w(1, \vec{n}) \ln w(1, \vec{n}) + w(2, \vec{n}) \ln w(2, \vec{n}) \\
- \left[ w(-1, \vec{n}) + w(0, \vec{n}) + w(3, \vec{n}) \right] \ln \left[ w(-1, \vec{n}) + w(0, \vec{n}) + w(3, \vec{n}) \right] \\
\times \ln \left[ w(-3, \vec{n}) + w(-2, \vec{n}) + w(1, \vec{n}) + w(2, \vec{n}) \right] \\
\leq - \left[ w(-3, \vec{n}) + w(-2, \vec{n}) \right] \ln \left[ w(-3, \vec{n}) + w(-2, \vec{n}) \right] \\
- \left[ w(-1, \vec{n}) + w(0, \vec{n}) \right] \ln \left[ w(-1, \vec{n}) + w(0, \vec{n}) \right] \\
- \left[ w(1, \vec{n}) + w(2, \vec{n}) \right] \ln \left[ w(1, \vec{n}) + w(2, \vec{n}) \right] \\
- \left[ w(-3, \vec{n}) + w(1, \vec{n}) \right] \ln \left[ w(-3, \vec{n}) + w(1, \vec{n}) \right] \\
- \left[ w(-2, \vec{n}) + w(2, \vec{n}) \right] \ln \left[ w(-2, \vec{n}) + w(2, \vec{n}) \right] \\
- \left[ w(-1, \vec{n}) + w(3, \vec{n}) \right] \ln \left[ w(-1, \vec{n}) + w(3, \vec{n}) \right].
\]

One can easily obtain the other inequalities by arbitrary permutations of the spin projections, i.e. the set of all \( m \) can be replaced by arbitrary permutations of the values of spin projections.

These inequalities can also be checked experimentally.

References

[1] Shannon C E 1948 Bell Syst. Tech. J. 27 379
[2] Bell J 1964 Physics I 195
[3] Clauser J F, Horne M A, Shimony A and Holt R 1969 Phys. Rev. Lett. 23 880
[4] Man’ko M A, Man’ko V I and Vilela Mendes R 2006 J. Russ. Laser Res. 27 507
[5] Man’ko M A, Man’ko V I, Marmo G, Simon A and Ventriglia F 2013 Nuovo Cimento C 36 163
[6] Man’ko O V and Chernega V N 2013 JETP Lett. 97 557
[7] Lieb E H and Ruskai M B 1973 J. Math. Phys. 14 1938
[8] Ruskai M B 2004 arXiv:quant-ph/0404126v4
[9] Carlen E A and Lieb E H 2008 Lett. Math. Phys. 83 107
[10] Kim I H 2011 arXiv:1210.5190
[11] Ohya M and Petz D 2004 Quantum Entropy and its Use 2nd edn (Heidelberg: Springer)
[12] Ruskai M B 2007 Rep. Math. Phys. 60 1
[13] Frank L R and Lieb E H 2012 arXiv:1204.0825v1
[14] Man’ko V I and Man’ko V I 2011 Found. Phys. 41 330
[15] Chernega V N and Man’ko O V 2013 J. Russ. Laser Res. 34 383
[16] Chernega V N and Man’ko V I 2007 J. Russ. Laser Res. 28 103
[17] Lupo C, Man’ko V I and Marmo G 2007 J. Phys. A: Math. Theor. 40 13091
[18] Dodonov V V and Man’ko V I 1997 Phys. Lett. A 229 335
[19] Man’ko V I and Man’ko O V 1997 J. Exp. Theor. Phys. 85 430
[20] Man’ko M A and Man’ko V I 2013 J. Russ. Laser Res. 34 203
[21] Man’ko M A, Man’ko V I and Vilela Mendes R 2001 J. Phys. A: Math. Gen. 34 8321
[22] Rastegin A E 2012 arXiv:1210.6742