Bosonization of interacting fermions in arbitrary dimension beyond the Gaussian approximation

Peter Kopietz, Joachim Hermisson and Kurt Schönhammer

Institut für Theoretische Physik der Universität Göttingen,
Bunsenstr.9, D-37073 Göttingen, Germany
(February 20, 1995)

Abstract

We use our recently developed functional bosonization approach to bosonize interacting fermions in arbitrary dimension $d$ beyond the Gaussian approximation. Even in $d = 1$ the finite curvature of the energy dispersion at the Fermi surface gives rise to interactions between the bosons. In higher dimensions scattering processes describing momentum transfer between different patches on the Fermi surface (around-the-corner processes) are an additional source for corrections to the Gaussian approximation. We derive an explicit expression for the leading correction to the bosonized Hamiltonian and the irreducible self-energy of the bosonic propagator that takes the finite curvature as well as around-the-corner processes into account. In the special case that around-the-corner scattering is negligible, we show that the self-energy correction to the Gaussian propagator is negligible if the dimensionless quantities $(q_c/k_F)^d F_0 [1 + F_0]^{-1} \frac{\mu}{\nu^\alpha} |\partial \nu^\alpha / \partial \mu|$ are small compared with unity for all patches $\alpha$. Here $q_c$ is the cutoff of the interaction in wave-vector space, $k_F$ is the Fermi wave-vector, $\mu$ is the chemical potential, $F_0$ is the usual dimensionless Landau interaction-parameter, and $\nu^\alpha$ is the local density of states associated with patch $\alpha$. We also show that the well known cancellation between vertex- and self-energy corrections in one-dimensional systems, which is responsible for the fact that the random-phase approximation for the density-density correlation function is exact in $d = 1$, exists also in $d > 1$, provided (1) the interaction cutoff $q_c$ is small compared with $k_F$, and (2) the energy dispersion is locally linearized at the Fermi surface. Finally, we suggest a new systematic method to calculate corrections to the RPA, which is based on the perturbative calculation of the irreducible bosonic self-energy arising from the non-Gaussian terms of the bosonized Hamiltonian.

PACS numbers: 05.30Fk, 05.30.Jp, 11.10.Ef, 71.27.+a
I. INTRODUCTION

One of the most powerful methods to analyze systems of interacting fermions in \( d = 1 \) dimension is the bosonization approach [1]-[4]. Over the past 30 years numerous interesting results have been obtained with this essentially non-perturbative method, and the study of one-dimensional fermions via bosonization techniques continues to be a field of active research. Because interacting fermions in \( d = 1 \) are not Fermi-liquids [5], conventional many-body perturbation theory is not applicable to these models. The success of bosonization in \( d = 1 \) hinges on the linearization of the energy dispersion at the two Fermi points. As soon as the curvature of the energy dispersion is taken into account, bosonization maps an interacting Fermi system onto an effective interacting Bose system, which in general cannot be solved exactly. Of course, one can circumvent this problem by choosing the energy dispersion of the Fermi system to be linear by definition [6] (Tomonaga-Luttinger model), but then the physical relevance of this model needs further justification, because in realistic materials the energy dispersion is never exactly linear.

In higher dimensions conventional many-body perturbation theory is at least consistent, provided the interaction is not too singular [4]. Because within the framework of perturbation theory there is no sign for the breakdown of Fermi-liquid theory, there was no need to develop non-perturbative methods for interacting fermions in dimensions higher than one. Luther’s pioneering attempt to describe higher dimensional Fermi-surface degrees of freedom by bosonization was not further pursued [5]. The discovery of the high-temperature superconductors and Anderson’s suggestion [9,10] that the normal state properties of these materials are a manifestation of non-Fermi-liquid behavior in \( d = 2 \) has revived the interest to develop non-perturbative methods for analyzing interacting Fermi systems in \( d > 1 \). Because of the success of bosonization in \( d = 1 \) it was natural to further develop this approach in higher dimensions. Recent progress in this field is based on ideas due to Haldane [11,12], who realized that Luther’s construction [8] must be generalized such that the direction of the net momentum of particle-hole pairs is not restricted. (In Luther’s approach particle-hole pairs carry a net momentum that is strictly normal to the local Fermi vector).

For later reference it is useful to summarize at this point the basic features of Haldane’s ”tomographic” construction. Consider the many-body Hamiltonian

\[
\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \quad ,
\]

\[
\hat{H}_0 = \sum_k \epsilon_k \hat{c}_k^\dagger \hat{c}_k \quad ,
\]

\[
\hat{H}_{\text{int}} = \frac{1}{2V} \sum_{qkk'} f_q^{kk'} \hat{c}_k^\dagger \hat{c}_{k+q}^\dagger \hat{c}_{k'-q} \hat{c}_{k'} \quad ,
\]

where \( \hat{c}_k \) annihilates an electron with wave-vector \( k \), \( V \) is the volume of the system, and \( f_q^{kk'} \) are generalized Landau-parameters that depend not only on the momentum transfer, but also on the momenta of the incoming particles. For simplicity we have suppressed the spin label. The spin is easily taken into account by defining \( k \) and \( k' \) to be collective labels for wave-vector and spin. We are implicitly assuming that only a single band with energy dispersion \( \epsilon_k \) of the underlying lattice model lies in the vicinity of the Fermi energy, and that...
the effect of the other bands is either completely negligible or can be taken into account via the definition of effective parameters $\epsilon_k$ and $f_{qk'}$. The Fermi surface is the $d-1$-dimensional hypersurface in $k$-space which satisfies

$$\xi_k \equiv \epsilon_k - \mu = 0 \quad ,$$

(1.4)

where $\mu$ is the chemical potential, measured with respect to the bottom of the conduction band. In $d = 1$ the Fermi surface consists of two distinct points $\pm k_F$, where $k_F$ is the Fermi wave-vector, while in $d > 1$ it is a $d-1$-dimensional manifold, the topology of which depends on the form of $\xi_k$. The crucial step in Haldane’s construction is the subdivision of the Fermi surface into disjoint patches with volume $\Lambda^{d-1}$. The precise shape of the patches and the size of $\Lambda$ should be chosen such that within a given patch the curvature of the Fermi surface can be locally neglected, so that the variations of the local normal vector can be ignored within a patch. Note that in this way sufficiently flat Fermi surfaces can be covered with only a small number of patches. The extreme case is the Fermi surface of an array of chains in $d = 3$ without inter-chain hopping, which consists of two parallel planes that can be identified with the patches. Obviously, in this case two patches are sufficient to cover the entire Fermi surface [13,14].

We introduce a label $\alpha$ to enumerate the patches in some convenient ordering and denote the patch labelled $\alpha$ by $K_{\alpha}^\Lambda$. Note that by definition $K_{\alpha}^\Lambda$ is a subset of the Fermi surface, i.e. it is also a $d-1$-dimensional hypersurface. To describe all degrees of freedom, we extend each patch $K_{\alpha}^\Lambda$ into a $d$-dimensional "squat box" $K_{\alpha,\lambda}^\Lambda$ of radial height $\lambda$, such that the union $\bigcup_\alpha K_{\alpha,\lambda}^\Lambda$ agrees with the wave-vector space necessary to label all degrees of freedom in the system, see Fig.1. By introducing a radial cutoff $\lambda$, we are implicitly assuming that all states with wave-vectors outside a shell with thickness $\lambda$ around the Fermi surface have been integrated out, and that the associated finite renormalizations have been taken into account via the proper definition of the bare parameters $\epsilon_k$ and $f_{qk'}$ in Eqs.1.2 and 1.3. Of course, this construction is only useful if the physical quantities of interest do not depend on the choice of the cutoffs $\Lambda$ and $\lambda$. For sufficiently smooth Fermi surfaces this is indeed the case as long as we are interested in correlation functions at separations $|r-r'| \gg \max\{\Lambda^{-1}, \lambda^{-1}\}$, and the interaction is dominated by momentum transfers $|q| \ll \min\{\Lambda, \lambda\}$ [16]. However, the slightly vague picture of integrating out the high-energy degrees of freedom without really doing this calculation is not very useful if we are interested in the precise numerical value of some physical quantity. To describe all degrees of freedom, we should therefore remove the radial cutoff $\lambda$ and work with boxes $K_{\alpha}^\Lambda$ that are constructed such that $\bigcup_\alpha K_{\alpha}^\Lambda$ covers not only a thin shell around the Fermi surface, but also the high-energy degrees of freedom. Of course, if we would like to linearize the energy dispersion around the Fermi surface, we should keep the radial cutoff finite, because the linearization is only good close to the Fermi surface. However, our bosonization approach does not rely on the linearization, so that in some cases we can remove the radial cutoff and can dispose of the picture of having integrated out the high-energy degrees of freedom. For example, to discuss the homogeneous electron gas in $d = 2$ we take $\epsilon_k = \frac{k^2}{2m}$, $f_{qk'} = \frac{2\pi e^2}{|q|}$, and choose the boxes $K_{\alpha}^\Lambda$ shown in Fig.2. We shall come back to the cutoff problem in the radial direction in Sec. IV C.

Because the union of all $K_{\alpha}$ agrees by construction with the total relevant $k$-space, we have
\[
\sum_{\alpha} \Theta^\alpha(k) = 1, \tag{1.5}
\]

where
\[
\Theta^\alpha(k) = \begin{cases} 1 & \text{if } k \in K^\alpha \\ 0 & \text{else} \end{cases}. \tag{1.6}
\]

For simplicity we have omitted the cutoff label of \(K^\alpha\). In the relevant Hilbert space Eq.1.3 is correct for both type of boxes \(K^\alpha\) and \(K^\alpha_{\lambda\Lambda}\) discussed above. Let us denote by \(k^\alpha\) a vector on the Fermi surface \((\xi_{k^\alpha} = 0)\) that points to the (suitably defined) center of patch \(\alpha\). The set of vectors \(\{k^\alpha\}\) define the origins of local coordinate systems on the Fermi surface. Such a collection of coordinate systems is also called an \textit{atlas}. Let us now measure the wave-vector \(k\) locally with respect to the origin of the appropriate coordinate system. Using Eq.1.5, we may write

\[
\xi_k = \sum_{\alpha} \Theta^\alpha(k) \xi_k = \sum_{\alpha} \Theta^\alpha(k) \xi^\alpha_{k-k^\alpha}, \tag{1.7}
\]

where the quantity \(\xi^\alpha_q\) is defined by
\[
\xi^\alpha_q \equiv \xi_{k^\alpha+q}. \tag{1.8}
\]

The crucial step in the conventional bosonization approach is the local linearization of the energy dispersion within a given patch,

\[
\xi^\alpha_q = v^\alpha \cdot q + O(q^2), \tag{1.9}
\]

where
\[
v^\alpha = \nabla_k \xi_k |_{k=k^\alpha}, \tag{1.10}
\]

is the local Fermi velocity at patch \(\alpha\), and we measure frequencies in units of energy, which amounts to formally setting \(\hbar = 1\). Note that such a linearization would not be possible in a rigid coordinate system. It is important to stress that this linearization should be made around the true chemical potential of the interacting many-body system \([15]\), and that \(v^\alpha\) should be identified with the renormalized local Fermi velocity, taking effective mass renormalizations into account \([16]\).

The operator-approach to bosonization follows then closely the usual construction for one-dimensional systems \([1]\). This approach has been further developed by Houghton, Kwon, Marston and Shankar \([17]-[20]\), and independently by Castro Neto and Fradkin \([21]-[23]\). The fundamental objects are the local density operators associated with the patches,

\[
\hat{\rho}^\alpha_q = \sum_k \Theta^\alpha(k) \hat{c}^\dagger_{k+q} \hat{c}_{k+q}. \tag{1.11}
\]

In terms of these operators the interaction-part of \(\hat{H}\) can be written as

\[
\hat{H}_{int} = \frac{1}{2V} \sum_q \sum_{\alpha\alpha'} f^\alpha_{\alpha'} \hat{\rho}^\alpha_q \hat{\rho}^{\alpha'}_q, \tag{1.12}
\]
where \( \ldots : \) denotes normal ordering, and it is assumed that the variations of \( f_{q}^{kk'} \) are negligible if \( k \) and \( k' \) are restricted to given patches, so that it is allowed to introduce the coarse-grained interaction function

\[
f_{q}^{\alpha \alpha'} = \frac{\sum_{kk'} \Theta^{\alpha}(k) \Theta^{\alpha'}(k') f_{q}^{kk'}}{\sum_{kk'} \Theta^{\alpha}(k) \Theta^{\alpha'}(k')}. \tag{1.13}
\]

By considering the commutation relations of the operators \( \hat{\rho}_{q}^{\alpha} \) with each other and with \( \hat{H} \) in the restricted Hilbert space of states with wave-vectors close to the Fermi surface, it is possible to show that the \( \hat{\rho}_{q}^{\alpha} \) approximately obey bosonic commutation relations, and that \( \hat{H} \) is approximately quadratic in the local density operators, so that it is exactly solvable \([12],[17]-[23]\).

It should be mentioned that an alternative non-perturbative approach to study interacting fermions in dimensions higher than one has been advanced by Castellani, Di Castro and Metzner \([24]\). This method is based on Ward identities and leads to results that are very similar to those of Refs. \([17]-[23]\).

Recently two of us have developed a functional integral method to bosonize interacting Fermi systems in arbitrary dimensions \([25]\), which avoids the algebraic manipulations of commutators that is necessary in the operator approach. A similar method has been developed independently by Fröhlich and collaborators \([26]\), although in their work the physical content is somehow hidden in elegant mathematics. In the context of the one-dimensional Tomonaga-Luttinger model the functional bosonization technique has first been discussed by Fogedby \([27]\), and later by Lee and Chen \([28]\). Besides reproducing all of the results of Refs. \([17]-[23]\) with a minimum of algebra \([16,25]\), our functional bosonization approach has several other advantages. First of all, it can cope with retardation and non-locality in a very simple way, because path integrals are the natural language to describe these important many-body effects. Hence, the inclusion of electron-phonon interactions or the treatment of the retarded interaction mediated by transverse gauge-fields is straightforward in our approach \([16]\). A second important advantage of our approach is that it opens the way for a controlled calculation of corrections to the non-interacting boson approximation. As already mentioned, these non-Gaussian corrections exist in any dimension if the energy dispersion is not linearized at the Fermi surface. Some time ago Haldane pointed out in his seminal article on one-dimensional interacting Fermi systems \([4]\) that the perturbation theory for the effective interacting boson-problem, which arises from the bosonization mapping, is most likely well behaved. In this paper we shall verify this hypothesis not only in \( d = 1 \), but calculate explicitly the leading correction to the non-interacting boson approximation in arbitrary dimension. We shall show that this correction is closely related to the local field correction to the random-phase approximation (RPA) for the density-density correlation function.

The rest of this paper is organized as follows: In Sec. II we shall give a detailed description of our functional bosonization technique, and develop the general formalism that is necessary to calculate the corrections to the non-interacting boson approximation. Sec. III contains the calculation of the bosonized Hamiltonian and the density-density correlation function within the Gaussian approximation. The calculation of the leading correction to the Gaussian approximation will be presented in Sec. IV. We show that the non-Gaussian correction...
terms to the bosonized Hamiltonian correspond to the local field corrections to the RPA, and calculate the hidden small parameter which determines the range of validity of the Gaussian approximation. Finally, in Sec. V we present our conclusions and discuss some open problems.

II. GENERAL FORMALISM OF FUNCTIONAL BOSONIZATION

In this section we shall describe our functional bosonization approach and prove that under certain conditions there exists for all \( d \) a remarkable cancellation between self-energy and vertex corrections, which implies that the Gaussian approximation becomes exact.

A. The effective action for collective density fluctuations

It is convenient to derive the bosonized Hamiltonian by first considering the imaginary-frequency density-density correlation function of our interacting many-body system, which is defined by

\[
\Pi(q, i\omega_m) \equiv \Pi(q) = \frac{1}{\beta V} \int_0^\beta d\tau \int_0^\beta d\tau' e^{-i\omega_m(\tau-\tau')} < T [\hat{\rho}_q(\tau)\hat{\rho}_{-q}(\tau')] > ,
\]

(2.1)

where \( \beta = 1/T \) is the inverse temperature, \( \hat{\rho}_q = \sum_k \hat{c}_k^\dagger \hat{c}_{k+q} \) is the operator representing the Fourier transform of the total density, and \( T \) denotes time-ordering in imaginary time. Throughout this work \( q = [q, i\omega_m] \) is a collective label for wave-vector \( q \) and bosonic Matsubara frequency \( \omega_m = 2\pi m T \). The average in Eq. 2.1 denotes thermal average with respect to all degrees of freedom in the system. An essential first step is the decomposition of Eq. 2.1 into contributions from the various patches. Using Eq. 1.5, we may write

\[
\Pi(q) = \sum_{\alpha\alpha'} \Pi^{\alpha\alpha'}(q) ,
\]

(2.2)

where

\[
\Pi^{\alpha\alpha'}(q) = \frac{1}{\beta V} \int_0^\beta d\tau \int_0^\beta d\tau' e^{-i\omega_m(\tau-\tau')} < T [\hat{\rho}_q^\alpha(\tau)\hat{\rho}_{-q}^{\alpha'}(\tau')] > .
\]

(2.3)

We shall refer to \( \Pi(q) \) as the global density-density correlation function, and to \( \Pi^{\alpha\alpha'}(q) \) as the patch density-density correlation function. \( \Pi^{\alpha\alpha'}(q) \) can be represented as a functional integral over a Grassmann-field \( \psi \) and an auxiliary field \( \phi_q^\alpha \) that mediates the interaction between the patch densities [16,25,29,30],

\[
\Pi^{\alpha\alpha'}(q) = \frac{\beta}{V} \int \mathcal{D} \{ \psi \} \mathcal{D} \{ \phi^\alpha \} \rho_q^\alpha \rho_{-q}^{\alpha'} e^{-S_1(\psi)-S_2(\psi,\phi^\alpha)-S_3(\phi^\alpha)} ,
\]

(2.4)

where \( \rho_q^\alpha \) is now a quadratic functional of the Grassmann-fields,

\[
\rho_q^\alpha = \sum_k \Theta^\alpha(k) \psi_k^\dagger \psi_{k+q} ,
\]

(2.5)
Here $\psi_k$ is the Fourier component of the Grassmann-field, where $k$ is again a collective label for wave-vector $k$ and fermionic Matsubara frequency $\tilde{\omega}_n = 2\pi (n + \frac{1}{2}) T$. The three contributions to the action in Eq. (2.4) are

$$S_1 \{ \psi \} = \beta \sum_k \psi_k^\dagger \left[ -i \tilde{\omega}_n + \xi_k \right] \psi_k ,$$

$$S_2 \{ \psi, \phi^\alpha \} = i \sum_q \sum_\alpha \phi^\alpha_{-q} \rho^\alpha_q ,$$

$$S_3 \{ \phi^\alpha \} = \frac{1}{2} \sum_q \sum_{\alpha \alpha'} \left[ \tilde{f}_q^{-1} \right]^{\alpha \alpha'} \phi^\alpha_{-q} \phi^{\alpha'}_q ,$$

where $\tilde{f}_q$ is a dimensionless matrix in the patch labels, which is defined in terms of the coarse-grained Landau parameters $f^\alpha_{\alpha'}_q$ of Eq. (1.13) via

$$\frac{V}{\beta} [\tilde{f}_q]^{\alpha \alpha'} = f^\alpha_{\alpha'}_q .$$

The field $\phi^\alpha_q$ can be considered as the dual field of the composite fermionic field $\rho^\alpha_q$. We have intentionally chosen our notation for the actions $S_1$, $S_2$, and $S_3$ such that it matches the notation used by Feynman and Hibbs \cite{31}: $S_1$ is the action for the matter degrees of freedom, $S_2$ describes the coupling between matter and a bosonic field $\phi^\alpha_q$ that mediates the interaction between the matter, and $S_3$ is the action of the bosonic field. In the case of the three dimensional Coulomb interaction the field $\phi^\alpha_q$ can be identified with the scalar potential of electromagnetism in the Coulomb gauge, and it is straightforward to include also the transverse radiation field in our formalism \cite{16}.

In order to introduce a collective bosonic field that represents the bosonized local density, we decouple $S_3 \{ \phi^\alpha \}$ by means of another Hubbard-Stratonowich transformation \cite{16,25,28}. The integration over the Grassmann-field $\psi$ is then quadratic and can be formally carried out, so that we can eliminate the composite fermionic field $\rho^\alpha_q$ in favour of a collective bosonic field $\tilde{\rho}^\alpha_q$. We arrive at the following exact expression for the patch density-density correlation function,

$$\Pi^{\alpha \alpha'}(q) = \frac{\beta V}{\beta} \int \mathcal{D} \{ \tilde{\rho}^\alpha \} e^{-S_{\text{int}}(\tilde{\rho}^\alpha)} \int \mathcal{D} \{ \phi^\alpha \} e^{i \sum_{q \alpha} \tilde{\rho}^\alpha_{-q} \phi^\alpha_q} e^{-S_{\text{kin}}(\phi^\alpha)} ,$$

where

$$S_{\text{int}} \{ \tilde{\rho}^\alpha \} = \frac{1}{2} \sum_q \sum_{\alpha \alpha'} \left[ \tilde{f}_q \right]^{\alpha \alpha'} \tilde{\rho}^\alpha_{-q} \tilde{\rho}^{\alpha'}_q ,$$

$$S_{\text{kin}} \{ \phi^\alpha \} = - Tr \ln \left[ \hat{1} - \hat{G}_0 \tilde{V} \right] .$$

Here the trace is over all wave-vectors and frequencies, and $\hat{G}_0$ and $\tilde{V}$ are infinite matrices in wave-vector and frequency space, with matrix elements given by

$$[\hat{G}_0]_{kk'} = \delta_{kk'} G_0(k) , \quad G_0(k) = \frac{1}{i \tilde{\omega}_n - \xi_k} .$$
\[ [\hat{V}]_{kk'} = \frac{i}{\beta} \sum_{\alpha} \Theta^{\alpha}(k) \phi^\alpha_{k-k'} . \tag{2.14} \]

The subscript on \( S_{\text{kin}} \{ \phi^\alpha \} \) indicates that this quantity is closely related to the bosonized kinetic energy. To be precise, below we shall show that the bosonized kinetic energy is given by the negative logarithm of the functional-Fourier transform of \( e^{-S_{\text{kin}} \{ \phi^\alpha \}} \), i.e.

\[ S_{\text{kin}} \{ \tilde{\rho}^\alpha \} = -\ln \left[ \int \mathcal{D} \{ \phi^\alpha \} e^{i \sum_{q\alpha} \phi^\alpha_q \tilde{\rho}^\alpha_q e^{-S_{\text{kin}} \{ \phi^\alpha \}}} \right] . \tag{2.15} \]

With this definition Eq.\( \ref{2.10} \) can also be written as

\[ \Pi^{\alpha\alpha'}(q) = \beta \frac{\mathcal{D} \{ \tilde{\rho}^\alpha \} e^{-S_{\text{eff}} \{ \tilde{\rho}^\alpha \}} \tilde{\rho}^\alpha_{-q} \tilde{\rho}^\alpha_{q'}} \mathcal{D} \{ \tilde{\rho}^\alpha \} e^{-S_{\text{eff}} \{ \tilde{\rho}^\alpha \}}}{\mathcal{V}} , \tag{2.16} \]

with

\[ S_{\text{eff}} \{ \tilde{\rho}^\alpha \} = S_{\text{int}} \{ \tilde{\rho}^\alpha \} + S_{\text{kin}} \{ \tilde{\rho}^\alpha \} . \tag{2.17} \]

Note that the electron-electron interaction has been treated exactly, while the bosonized kinetic \( S_{\text{kin}} \{ \tilde{\rho}^\alpha \} \) can in general only be calculated approximately. The calculation consists of two steps. First of all, one should calculate the functional \( S_{\text{kin}} \{ \phi^\alpha \} \). In praxis, this has to be done by expanding the logarithm in Eq.\( \ref{2.12} \) and performing the traces in each term separately,

\[ S_{\text{kin}} \{ \phi^\alpha \} = \sum_{n=1}^{\infty} S_{\text{kin},n} \{ \phi^\alpha \} . \tag{2.18} \]

\[ S_{\text{kin},n} \{ \phi^\alpha \} = \frac{1}{n} \text{Tr} \left[ \hat{G}_0 \hat{V} \right]^n . \tag{2.19} \]

To obtain the bosonized kinetic energy \( S_{\text{kin}} \{ \tilde{\rho}^\alpha \} \) it is necessary to perform the functional Fourier transformation in Eq.\( \ref{2.13} \). Of course, in general also this step has to be carried out perturbatively. If we ignore all terms with \( n \geq 3 \) in the expansion \( \ref{2.18} \), we obtain the Gaussian approximation. In Ref. [25] we have pointed out that in the one-dimensional Tomonaga-Luttinger model all higher order terms vanish identically, so that we have exactly

\[ -\text{Tr} \ln \left[ 1 - \hat{G}_0 \hat{V} \right] = \text{Tr} \left[ \hat{G}_0 \hat{V} \right] + \frac{1}{2} \text{Tr} \left[ \hat{G}_0 \hat{V} \right]^2 . \tag{2.20} \]

The functional Fourier transformation of \( S_{\text{kin}} \{ \phi^\alpha \} \) involves then a simple Gaussian integral, so that the bosonized kinetic energy can be calculated exactly. In the next subsection we shall show that in a certain physically relevant limit the remarkable cancellation between self-energy and vertex corrections that is responsible for the validity of Eq.\( \ref{2.20} \) in \( d = 1 \) happens also in higher dimensions.

**B. Generalized closed loop theorem**

Graphically, the trace in Eq.\( \ref{2.19} \) can be represented as a closed fermion loop with \( n \) external \( \phi^\alpha \)-fields, see Fig. 3. Performing the trace in Eq.\( \ref{2.19} \) we obtain
where the dimensionless vertices $U_n$ are given by

\[
U_n(q_1\alpha_1 \ldots q_n\alpha_n) = \delta_{q_1+\ldots+q_n,0} \left( \frac{i}{\beta} \right)^n \frac{1}{n!} \sum_{P(1\ldots n)} \sum_k \Theta^{\alpha_1}_{\alpha_i}(k) \times \Theta^{\alpha_2}_{\alpha_i}(k+q_{P_1}) \cdots \Theta^{\alpha_n}_{\alpha_i}(k+q_{P_{n-1}}) \\
\times G_0(k)G_0(k+q_{P_1})\cdots G_0(k+q_{P_{n-1}}).
\]

Here $\delta_{q_1+\ldots+q_n,0}$ denotes a Kronecker-$\delta$ in wave-vector and frequency space. We have used the invariance of $S_{\text{kin},n}\{\phi^\alpha\}$ under relabeling of the fields to symmetrize the vertices $U_n$ with respect to the interchange of any two labels. The sum $\sum_{P(1\ldots n)}$ is over the $n!$ permutations of $n$ integers, and $P_i$ denotes the image of $i$ under the permutation. Note that the vertices $U_n$ are uniquely determined by the energy dispersion $\epsilon_k - \mu$ that defines the free fermionic action. The amazing point is now that there exists a physically interesting limit where all vertices $U_n$ with $n \geq 3$ vanish. This limit is characterized by the requirement that the following two approximations (A1) and (A2) become accurate:

(A1): High density-limit or small momentum-transfer limit: Suppose that there exists a cutoff $q_c \ll k_F$ such that the contribution from fields $\phi^\alpha_q$ with $|q| \geq q_c$ to physical observables becomes negligibly small. Because the fields $\phi^\alpha_q$ mediate the interaction between the fermions, this condition is equivalent with the requirement that the nature of the bare interaction $\tilde{\mathcal{L}}$ should be such that the resulting effective screened interaction (which takes into account the modification of the bare interaction between two particles due to the presence of all other particles) is negligibly small for $|q| \geq q_c$. If this condition is satisfied, we may approximate in Eq.\ref{eq:2.22}

\[
\Theta^{\alpha_1}_{\alpha_i}(k)\Theta^{\alpha_2}_{\alpha_i}(k+q_{P_1}) \cdots \Theta^{\alpha_n}_{\alpha_i}(k+q_{P_{n-1}}) \\
\approx \delta^{\alpha_1\alpha_2}\delta^{\alpha_2\alpha_3}\cdots\delta^{\alpha_n\alpha_1}\Theta^{\alpha_1}_{\alpha_i}(k),
\]

because the $k$-sum in Eq.\ref{eq:2.22} is dominated by wave-vectors of the order of $k_F$. This approximation is correct to leading order in $q_c/k_F$, and becomes exact in the limit $q_c/k_F \rightarrow 0$. Note that this limit is approached either at high densities, where $k_F \rightarrow \infty$ at constant $q_c$, or in the limit that the range $q_c$ of the effective interaction in momentum space approaches zero while $k_F$ is held constant. It follows that up to higher order corrections in $q_c/k_F$ the vertex $U_n(q_1\alpha_1 \ldots q_n\alpha_n)$ is diagonal in all patch labels,

\[
U_n(q_1\alpha_1 \ldots q_n\alpha_n) = \delta^{\alpha_1\alpha_2}\cdots\delta^{\alpha_1\alpha_n}U_n^{\alpha_1}(q_1 \ldots q_n),
\]

with

\[
U_n^{\alpha}(q_1 \ldots q_n) = \delta_{q_1+\ldots+q_n,0} \left( \frac{i}{\beta} \right)^n \frac{1}{n!} \sum_{P(1\ldots n)} \sum_k \Theta^\alpha(k) \\
\times G_0(k)G_0(k+q_{P_1})\cdots G_0(k+q_{P_{n-1}}).
\]
Below we shall refer to the approximation \(2.23\) as the diagonal-patch approximation. It is important to note that at finite \(q_c/k_F\) this approximation can only become exact in \(d = 1\), because in this case the Fermi surface consists of two widely separated points. Except for special cases \([13,14,16]\), the covering of the Fermi surface in \(d > 1\) involves at least some adjacent patches, so that in higher dimensions there exist always patches which can be connected by an arbitrarily small momentum transfer \(q\). These “around-the-corner” processes are ignored within the diagonal-patch approximation \((A1)\).

\(\text{(A2): Local linearization of the energy dispersion at the Fermi-surface:}\) Suppose we linearize the energy dispersion locally by approximating \(\xi_\alpha \approx v_\alpha \cdot q\), see Eqs.\([1.9,1.10]\). Shifting the summation wave-vector in Eq.\([2.25]\) according to \(k = k_\alpha + q\) and defining the linearized Greens-function

\[
G_{0}^\alpha(q) \equiv G_{0}^\alpha(q, i\tilde{\omega}_n) = \frac{1}{i\tilde{\omega}_n - v_\alpha \cdot q},
\]

Eq.\([2.25]\) reduces to

\[
U_n^\alpha(q_1 \ldots q_n) = \delta_{q_1+\ldots+q_n,0} \left( \frac{i}{\beta} \right)^n \frac{1}{n!} \sum_{P(1 \ldots n)} \sum_q \Theta^\alpha(k_\alpha + q) \\
\times G_0^\alpha(q)G_0^\alpha(q + q_p_1) \ldots G_0^\alpha(q + q_p_1 + \ldots + q_{P-1}) .
\]

Having made the approximations \((A1)\) and \((A2)\), we are now ready to show that in arbitrary dimension the vertices \(U_n^\alpha(q_1 \ldots q_n)\) with \(n \geq 3\) vanish in the limit \(q_c/\lambda \to 0\), so that in this limit the Gaussian approximation becomes exact! In the context of the Tomonaga-Luttinger model the vanishing of the \(U_n\), \(n \geq 3\) has been called “weak closed loop theorem”, and is discussed and proved in unpublished lecture notes by T. Bohr \([32]\). Under assumptions \((A1)\) and \((A2)\) the proof goes through in any dimension without changes. Note that the validity of \((A1)\) and \((A2)\) is implicitly built into the Tomonaga-Luttinger model by definition. The vanishing of \(U_n\) for \(n \geq 3\) is equivalent with the statement that the RPA for the density-density correlation function is exact in this model. This is due to a complete cancellation between vertex- and self-energy corrections. In \(d = 1\) this cancellation has been discovered by Dzyaloshinskii and Larkin \([33]\). Because the proof given in Ref. \([33]\) is rather cryptic and the lecture notes by T. Bohr \([32]\) are not published, we briefly outline the basic features of the proof. It is perhaps simpler to formulate the proof in the space-time domain \([32,34]\), but for our purpose it is more convenient to work in momentum-space, because here the Fermi-surface and the patching construction are defined. The following two properties of our locally linearized Greens-function defined in Eq.\([2.26]\) are essential,

\[
G_{0}^\alpha(-q) = -G_{0}^\alpha(q),
\]

\[
G_0^\alpha(q)G_0^\alpha(q + q') = G_0^\alpha(q') [G_0^\alpha(q) - G_0^\alpha(q + q')] .
\]

Note that Eq.\([2.28]\) follows trivially from the definition, while Eq.\([2.29]\) is nothing but the partial-fraction representation of the product of two rational functions. To show that the
odd vertices $U_3, U_5, \ldots$ vanish, we only need Eq.\[2.28\] and the fact that the patch $\alpha$ in Eq.\[2.27\] has inversion symmetry with respect to $k^\alpha$, so that the domain for the $q$-sum is invariant under $q \rightarrow -q$. Then it is easy to see that the contribution from a given permutation $(P_1 P_2 \ldots P_{n-1} P_n)$ is exactly cancelled by the contribution from the permutation $(P_n P_{n-1} \ldots P_2 P_1)$ in which the loop is traversed in the opposite direction. As already pointed out by T. Bohr [32], the vanishing of the odd vertices is a direct consequence of Furry’s theorem [35]. To show that the even vertices $U_n, n = 4, 6, \ldots$ vanish, we use Eq.\[2.29\] $n$-times for the pairs

$$G_0^\alpha(q)G_0^\alpha(q + qP_1),$$

$$G_0^\alpha(q + qP_1)G_0^\alpha(q + qP_1 + qP_2),$$

$$\ldots,$

$$G_0^\alpha(q + qP_1 + \ldots + qP_{n-2})G_0^\alpha(q + qP_1 + \ldots + qP_{n-1}),$$

$$G_0^\alpha(q + qP_1 + \ldots + qP_{n-1})G_0^\alpha(q),$$

and take into account that we may replace $qP_1 + \ldots + qP_{n-1} = -qP_n$ because of overall energy-momentum conservation. Then it is easy to show that

$$G_0^\alpha(q)G_0^\alpha(q + qP_1) \cdots G_0^\alpha(q + qP_1 + \ldots + qP_{n-1})$$

$$= \frac{1}{n} \left[ G_0^\alpha(qP_n) - G_0^\alpha(qP_1) \right] G_0^\alpha(q + qP_1) \cdots G_0^\alpha(q + qP_1 + \ldots + qP_{n-1}).$$

Substituting Eq.\[2.31\] in Eq.\[2.27\], shifting the summation label $q \rightarrow q - qP_1 + qP_n$, and using the fact that we may rename $qP_1 \leftrightarrow qP_n$, because we sum over all permutations, it is evident that the resulting expression vanishes. This argument is not valid for $n = 2$, because in this case $G_0^\alpha(qP_2) - G_0^\alpha(qP_1) = 2G_0^\alpha(qP_1)$ due to energy-momentum conservation. We shall discuss the vertex $U_2$ in detail in Sec.III A. Note that the shift $q \rightarrow q - qP_1 + qP_n$ affects also the patch-cutoff, $\Theta^\alpha(k^\alpha + q) \rightarrow \Theta^\alpha(k^\alpha + q - qP_1 + qP_n)$, but this leads to corrections of higher order in $q_c$. Because we have already ignored higher order terms in $q_c$ by making the diagonal-patch approximation (A1), it is consistent to ignore this shift.

The above proof of the generalized closed loop theorem uses the language of Greens-functions in the momentum-frequency domain. Such a formulation naturally emerges from our functional bosonization approach. After completion of this work we have learnt that independently from us W. Metzner has also realized that the closed loop theorem can be generalized to higher dimensions [38]. His proof is based on the analysis of the commutation relations and equations of motion of the patch density operators $\rho_q^\alpha$ defined in Eq.[1.11].

In fermionic language, the vanishing of the higher order vertices is due to a complete cancellation between self-energy and vertex corrections. This cancellation is automatically incorporated in our bosonic formulation via the symmetrization of the vertices $U_n$. We would like to emphasize again that this remarkable cancellation happens not only in $d = 1$ [33, 32] but in arbitrary dimensions. The existence of these cancellations in the perturbative calculation of the dielectric function of the homogeneous electron gas in $d = 3$ has already been noticed by Geldart and Taylor more than 20 years ago [36], although the origin for this cancellation has not been identified. The generalized closed loop theorem discussed in
the present work gives a clear mathematical explanation for this cancellation to all orders in perturbation theory.

It is important to stress that the cancellation does not depend on the nature of the external fields that enter the closed loop; in particular, it occurs also in models where the fermionic current density is coupled to transverse gauge-fields \[16\]. The one-loop corrections to the RPA for the gauge-invariant two-particle Greens-functions of electrons interacting with transverse gauge-fields have recently calculated by Kim et al. \[37\]. They found that at long wavelengths and low frequencies the leading self-energy and vertex corrections cancel. In the light of the generalized closed loop theorem this cancellation is not surprising. However, the generalized closed loop theorem is a much stronger statement, because it implies a cancellation between the leading self-energy and vertex corrections to all orders in perturbation theory.

A system where the approximations (A1) and (A2) are exact is the natural generalization of the Tomonaga model to higher dimensions \([1]\). If we follow Luttinger’s construction \([6]\) and remove the radial cutoff \(\lambda\) by extending the locally linearized energy dispersion beyond the squat boxes (so that it is allowed to shift \(q\)), we obtain an interacting higher dimensional Fermi system that is exactly solvable in precisely the same sense as the Tomonaga-Luttinger model in \(d = 1\). As already mentioned, however, for finite \(q_c/k_F\) and realistic Fermi surfaces the condition (A1) can never be exactly satisfied in \(d > 1\).

III. GAUSSIAN APPROXIMATION

In this section we shall calculate the density-density correlation function and the bosonized Hamiltonian within the Gaussian approximation. We also show that at long wavelengths the resulting bosonized Hamiltonian is identical with the Hamiltonian obtained from the operator approach \([18,22]\).

A. The first two vertices

Within Gaussian approximation only the vertices \(U_1\) and \(U_2\) are retained, and all higher order vertices are set equal to zero. From the previous subsection we know that the Gaussian approximation is justified in a parameter regime where (A1) and (A2) are accurate. Let us start with the vertex \(U_1\), which is given by

\[
U_1(q\alpha) = \delta_{q,0} \frac{i}{\beta} \sum_k \Theta^\alpha(k) \frac{1}{i\tilde{\omega}_n - \xi_k} = i\delta_{q,0} N_0^\alpha,
\]

(3.32)

where

\[
N_0^\alpha = \sum_k \Theta^\alpha(k) f(\xi_k),
\]

(3.33)

is the number of occupied states in patch \(\alpha\) in the non-interacting limit. Here \(f(\epsilon) = [e^{\beta \epsilon} + 1]^{-1}\) is the Fermi-function. Thus,
\[ S_{\text{kin},1} \{ \phi^\alpha \} = i \sum_\alpha \phi^\alpha_0 N^\alpha_0. \]  

(3.34)

Combining this term with the \( \phi - \rho \)-coupling in the functional Fourier transform in Eq.2.13, we can write
\[
i \sum_q \sum_\alpha \phi^-_q \tilde{\rho}^\alpha_q - S_{\text{kin},1} \{ \phi^\alpha \} = i \sum_q \sum_\alpha \phi^-_q \left[ \tilde{\rho}^\alpha_q - \delta_{q_0} N^\alpha_0 \right].
\]

(3.35)

Thus, \( S_{\text{kin},1} \{ \phi^\alpha \} \) simply shifts the collective density field according to
\[
\tilde{\rho}^\alpha_q \to \tilde{\rho}^\alpha_q - \delta_{q_0} N^\alpha_0.
\]

(3.36)

i.e. the uniform component is shifted. We conclude that the action \( S_{\text{kin}} \{ \tilde{\rho}^\alpha \} \) is actually a functional of the shifted field. For simplicity we shall from now on redefine the collective field according to Eq.3.36. Note also that the term \( q = 0 \) in \( S_{\text{int}} \{ \tilde{\rho}^\alpha \} \) is usually excluded due to charge neutrality, so that the effective action \( S_{\text{eff}} \{ \tilde{\rho}^\alpha \} \) can be considered as functional of the shifted field.

The second-order vertex is given by
\[
U_2(q_1 \alpha_1, q_2 \alpha_2) = -\delta_{q_1+q_2} \frac{1}{2 \beta^2} \sum_k \left[ \Theta^\alpha_1(k) \Theta^\alpha_2(k + q_1) G_0(k) G_0(k + q_1) \right. \\
+ \Theta^\alpha_2(k) \Theta^\alpha_1(k + q_2) G_0(k) G_0(k + q_2) \right].
\]

(3.37)

Performing the frequency sum we obtain
\[
U_2(-q \alpha, q \alpha') = \frac{V}{\beta} \Pi^\alpha_0(q) \left( \phi^-_q \phi^\alpha_0 \right).
\]

(3.38)

where
\[
\Pi^\alpha_0(q) = \frac{1}{2V} \sum_k \left[ \Theta^\alpha(k) \Theta^\alpha'(k - q) \frac{f(\xi_{k-q}) - f(\xi_k)}{\xi_k - \xi_{k-q} - i\omega_m} + (\alpha \leftrightarrow \alpha', q \to -q) \right].
\]

(3.39)

Hence \( S_{\text{kin},2} \{ \phi^\alpha \} \) is given by
\[
S_{\text{kin},2} \{ \phi^\alpha \} = \frac{V}{2 \beta} \sum_q \sum_{\alpha \alpha'} \Pi^\alpha_0(q) \phi^-_q \phi^\alpha_0 \phi^\alpha_0. \]

(3.40)

Note that \( \Pi^\alpha_0(q) \) is the patch polarization in the absence of interactions, i.e. the exact result of Eq.2.3 if we set \( \tilde{f}_q = 0 \). In this case our complicated transformations are of course not necessary and it is much simpler to calculate the density-density correlation function directly from its definition in terms of fermionic operators. For \( |q| \ll k_F \) the diagonal-patch approximation (A1) is justified, so that in Eq.3.39 we may replace \( \Theta^\alpha(k) \Theta^\alpha'(k - q) \approx \delta^{\alpha \alpha'} \Theta^\alpha(k) \). To leading order in \( |q|/k_F \) we have therefore in any dimension
\[
\Pi^\alpha_0(q) \approx \delta^{\alpha \alpha'} \Pi^\alpha_0(q) \quad \Pi^\alpha_0(q) = \nu^\alpha \frac{v^\alpha \cdot q}{v^\alpha \cdot q - i\omega_m},
\]

(3.41)
where
\[
\nu^\alpha = \frac{1}{V} \frac{\partial N_0^\alpha}{\partial \mu} = \frac{1}{V} \sum_k \Theta^\alpha(k) \left[ -\frac{\partial f(\xi_k)}{\partial \xi_k} \right]
\]  
(3.42)
is the "local" density of states associated with patch \( \alpha \), and \( \nu^\alpha \) is the local Fermi velocity, see Eq.1.10. Note that Eq.3.41 is valid for small \( |q|/k_F \) but for arbitrary frequencies.

B. Density-density correlation function

Within Gaussian approximation the action for the dual field \( \phi^\alpha_q \) is quadratic, so that the functional Fourier transform in Eq.2.15 can be calculated trivially, with the result
\[
S_{\text{kin}} \{ \tilde{\rho}^\alpha \} = S_{\text{kin},0}^{(0)} + \frac{1}{2} \sum_q \sum_{\alpha\alpha'} \Gamma_{q}^{\alpha\alpha'} \tilde{\rho}_{-q}^\alpha \tilde{\rho}_{q}^{\alpha'} ,
\]  
(3.43)
where \( S_{\text{kin},0}^{(0)} \) is a constant independent of the \( \tilde{\rho}^\alpha \)-field, and \( \Gamma_{q}^{\alpha\alpha'} \) is the propagator of the \( \phi^\alpha \)-field with respect to the Gaussian action \( S_{\text{kin},2} \{ \phi^\alpha \} \),
\[
\langle \phi^\alpha_q \phi^{\alpha'}_q \rangle_0 = \delta_{q+q',0} \Gamma_{q}^{\alpha\alpha'} .
\]  
(3.44)
Here \( < ... >_0 \) denotes Gaussian average with probability measure \( e^{-S_{\text{kin},2}(\phi^\alpha)} \), i.e. for any functional \( F \{ \phi^\alpha \} \)
\[
\langle F \{ \phi^\alpha \} \rangle_0 = \frac{\int \mathcal{D} \{ \phi^\alpha \} F \{ \phi^\alpha \} e^{-S_{\text{kin},2}(\phi^\alpha)}}{\int \mathcal{D} \{ \phi^\alpha \} e^{-S_{\text{kin},2}(\phi^\alpha)}} .
\]  
(3.45)
Note that \( \Gamma_{q}^{\alpha\alpha'} \) is the inverse of the matrix \( U_2(-q\alpha, q\alpha') \) in the space spanned by the patch-indices, i.e.
\[
\sum_{\alpha'} U_2(-q\alpha, q\alpha') \Gamma_{q}^{\alpha\alpha'} = \delta^{\alpha\alpha''} .
\]  
(3.46)
Thus, \( \Gamma_{q}^{\alpha\alpha'} \) is proportional to the inverse of the non-interacting patch polarization. Absorbing for simplicity the constant \( S_{\text{kin},0}^{(0)} \) in the re-definition of the integration measure \( \mathcal{D} \{ \tilde{\rho}^\alpha \} \) and combining the contribution from the kinetic energy with the interaction contribution, we obtain for the bosonized action \( S_{\text{eff}} \{ \tilde{\rho}^\alpha \} \) defined in Eq.2.17 in Gaussian approximation
\[
S_{\text{eff}} \{ \tilde{\rho}^\alpha \} = \frac{1}{2} \sum_q \sum_{\alpha\alpha'} \left[ \tilde{f}_q^{\alpha\alpha'} + \Gamma_{q}^{\alpha\alpha'} \right] \tilde{\rho}_{-q}^\alpha \tilde{\rho}_{q}^{\alpha'} .
\]  
(3.47)
Note that the inclusion of interactions is trivial, while the major difficulty lies in the representation of the kinetic energy as a functional of the collective density field. The Gaussian propagator of the \( \tilde{\rho}^\alpha \)-field is then
\[
\frac{\int \mathcal{D} \{ \tilde{\rho}^\alpha \} e^{-S_{\text{eff}}(\tilde{\rho}^\alpha)} e^{\tilde{f}_q^\alpha \tilde{\rho}_{-q}^{\alpha'}}}{\int \mathcal{D} \{ \tilde{\rho}^\alpha \} e^{-S_{\text{eff}}(\tilde{\rho}^\alpha)}} = \delta_{q+q',0} \left[ \tilde{L}_q + \tilde{L}_q \right]^{-1} .
\]  
(3.48)
where $\Gamma_q$ is a matrix in the patch-labels, with $[\Gamma_q]^{\alpha\alpha'} = \Gamma^{\alpha\alpha'}_q$. From Eq.2.16 we obtain for the density-density correlation function

$$\Pi^{\alpha\alpha'}(q) = \frac{\beta}{V} \left[ [\tilde{f}_q + \Gamma_q]^{-1} \right]^{\alpha\alpha'} = \left[ [\Pi_0^{-1}(q) + f_q]^{-1} \right]^{\alpha\alpha'},$$

(3.49)

where $\Pi_0(q)$ is again a matrix in the patch-labels, with

$$[\Pi_0(q)]^{\alpha\alpha'} = \frac{\beta}{V} \left[ [\Gamma_q^{-1}]^{-1} \right]^{\alpha\alpha'} \equiv \Pi_0^{\alpha\alpha'}(q),$$

(3.50)

see Eqs.3.38, 3.39 and 3.46. The matrix elements of $f_q$ are the usual Landau-parameters defined in Eq.1.13, i.e. $[f_q]^{\alpha\alpha'} = f^{\alpha\alpha'}_q$. Eq.3.49 is nothing but the RPA for the patch density-density correlation function. To obtain the standard RPA-result for the total density-density correlation function, we should according to Eq.2.2 sum Eq.3.49 over both patch-labels,

$$\Pi(q) = \sum_{\alpha\alpha'} \left[ [\Pi_0^{-1}(q) + f_q]^{-1} \right]^{\alpha\alpha'}.$$  

(3.51)

For simplicity let us assume that $[f_q]^{\alpha\alpha'} = f_q$ is independent of the patch-indices. Expanding

$$[\Pi_0^{-1}(q) + f_q]^{-1} = \Pi_0(q) - \Pi_0(q) f_q \Pi_0(q) + \Pi_0(q) f_q \Pi_0(q) f_q \Pi_0(q) - \ldots,$$

(3.52)

and taking matrix elements, we see that Eqs.3.51 and 3.52 reduce to

$$\Pi(q) = \frac{\Pi_0(q)}{1 + f_q \Pi_0(q)},$$

(3.53)

where

$$\Pi_0(q) = \sum_{\alpha\alpha'} \Pi_0^{\alpha\alpha'}(q)$$

(3.54)

is the total non-interacting polarization. We would like to emphasize that up to this point we have not linearized the energy dispersion, so that Eq.3.53 is the exact RPA result for all wave-vectors, including the short wavelength regime.

C. Bosonization of the Hamiltonian

To make contact with the operator-formalism to bosonization [17]- [23] let us now derive a bosonic Hamiltonian that is in the limit of long wavelengths equivalent to our Gaussian action $S_{eff} \{ \tilde{\rho}^a \}$ in Eq.3.47. The key observation is that the patch polarization is in the limit of high densities and long wavelengths (i.e. in the limit where the diagonal-patch approximation (A1) is correct) diagonal in the patch indices, and of the form given in Eq.3.41. From Eqs.3.38, 3.39 and 3.46 it is then easy to see that the matrix elements of $\Gamma_q$ are given by
\[ \Gamma^{\alpha\alpha'}_q \approx \delta^{\alpha\alpha'} \frac{\beta}{V \nu^\alpha} \frac{\mathbf{v}^\alpha \cdot \mathbf{q} - i \omega_m}{\mathbf{v}^\alpha \cdot \mathbf{q}} , \quad |\mathbf{q}| \ll k_F . \] (3.55)

Hence the Gaussian action in Eq. 3.47 can be written as

\[ S_{\text{eff}} \{ \tilde{\rho}^\alpha \} = \frac{\beta}{2V} \sum_{q} \sum_{\alpha} \left[ f^\alpha_q \delta^{\alpha\alpha'} + \delta^{\alpha\alpha'} \frac{\mathbf{v}^\alpha \cdot \mathbf{q} - i \omega_m}{\nu^\alpha \mathbf{v}^\alpha \cdot \mathbf{q}} \right] \tilde{\rho}^\alpha_{-q} \tilde{\rho}^{\alpha'}_q . \] (3.56)

Obviously the term proportional to \( i \omega_m \) defines the dynamics of the \( \tilde{\rho}^\alpha \)-field. We now recall that in the functional integral for canonically quantized bosons the coefficient of the term proportional to \( -i \omega_m \) should be precisely \( \beta \). Thus, to write our effective action in terms of a canonical boson field \( b_q^\alpha \), we should rescale the \( \tilde{\rho}^\alpha \)-field accordingly. This is achieved by substituting

\[ \tilde{\rho}^\alpha_q = (V \nu^\alpha |\mathbf{v}^\alpha \cdot \mathbf{q}|)^{\frac{1}{2}} \left[ \Theta(\mathbf{v}^\alpha \cdot \mathbf{q})b_q^\alpha + \Theta(-\mathbf{v}^\alpha \cdot \mathbf{q})b^\alpha_q \right] \] (3.57)

in Eq. 3.56. The \( \Theta \)-functions are necessary to make the coefficient of \( -i \omega_m \) equal to \( \beta \) for all patches, because the sign of \( i \omega_m \) in Eq. 3.56 depends on the sign of \( \mathbf{v}^\alpha \cdot \mathbf{q} \). Our final result for the bosonized action \( S \{ b^\alpha \} \equiv S_{\text{eff}} \{ \tilde{\rho}^\alpha(b^\alpha) \} \) is

\[ S \{ b^\alpha \} = \beta \sum_q \sum_\alpha \Theta(\mathbf{v}^\alpha \cdot \mathbf{q})(-i \omega_m)b_q^\alpha b^\alpha_q \]
\[ + \beta [H_{\text{kin}} \{ b^\alpha \} + H_{\text{int}} \{ b^\alpha \}] , \] (3.58)

\[ H_{\text{kin}} \{ b^\alpha \} = \sum_q \sum_\alpha \Theta(\mathbf{v}^\alpha \cdot \mathbf{q})\mathbf{v}^\alpha \cdot \mathbf{q}b_q^{\alpha\dagger}b^\alpha_q , \] (3.59)

\[ H_{\text{int}} \{ b^\alpha \} = \frac{1}{2} \sum_q \sum_{\alpha\alpha'} \Theta(\mathbf{v}^\alpha \cdot \mathbf{q}) \frac{1}{|\mathbf{v}^\alpha \cdot \mathbf{q}|} \left[ \Theta(\mathbf{v}^{\alpha'} \cdot \mathbf{q}) \left( f^{\alpha\alpha'}_q b_q^{\alpha\dagger}b^\alpha_q + f^{\alpha'\alpha}_q b_q^{\alpha'}b^{\alpha'}_q \right) + \Theta(-\mathbf{v}^{\alpha'} \cdot \mathbf{q}) \left( f^{\alpha\alpha'}_q b_q^{\alpha\dagger}b^\alpha_q + f^{\alpha'\alpha}_q b_q^{\alpha'}b^{\alpha'}_q \right) \right] , \] (3.60)

where \( f^{\alpha\alpha'}_q = \sqrt{\nu^\alpha \nu^{\alpha'}} f^{\alpha\alpha'}_q \) are dimensionless couplings. The functional integral for the \( b^\alpha \)-field is now formally identical with a standard bosonic functional integral. The corresponding second quantized bosonic Hamiltonian is therefore \( \hat{H}^b = \hat{H}^b_{\text{kin}} + \hat{H}^b_{\text{int}} \), where \( \hat{H}^b_{\text{kin}} \) and \( \hat{H}^b_{\text{int}} \) are simply obtained by replacing the bosonic fields \( b_q^\alpha \) in Eqs. 3.59, 3.60 by operators \( \hat{b}_q^\alpha \) satisfying \( [\hat{b}_q^\alpha, \hat{b}_{q'}^{\alpha'}] = \delta_{\alpha\alpha'} \delta_{q,q'} \). The resulting \( \hat{H}^b \) agrees with the bosonized Hamiltonian derived in Refs. 18,22 by means of an operator approach.

Note, however, that the above identification with a canonical bosonic Hamiltonian is only possible in the limit of long wavelengths and high densities, so that our parametrization of the effective Gaussian action in Eq. 3.47 is more general. Moreover, for practical calculations the substitution in Eq. 3.57 is not very useful, because it maps the very simple form of \( S_{\text{eff}} \{ \tilde{\rho}^\alpha \} \) given in Eq. 3.47 onto the complicated effective action \( S \{ b^\alpha \} \) in Eqs. 3.58–3.60 without containing new information.
IV. BEYOND THE GAUSSIAN APPROXIMATION

In this section we shall calculate the one-loop correction to the Gaussian approximation for the effective action of the collective density field. In this way we determine the hidden parameter which determines the range of validity of the Gaussian approximation. We also show that bosonization leads to a new systematic method to calculate corrections to the RPA for the dielectric function.

A. General perturbative expansion of \( S_{\text{kin}} \{ \tilde{\rho}^\alpha \} \)

In this subsection we develop a systematic perturbative method to calculate the bosonized kinetic energy \( S_{\text{kin}} \{ \tilde{\rho}^\alpha \} \) defined in Eq.2.15. This method is based on a loop-wise expansion of the functional Fourier transformation with the help of the linked cluster theorem. Defining \( S'_{\text{kin}} \{ \phi^\alpha \} \) to be the sum of all non-Gaussian terms in the expansion of the effective action of the dual field (see Eq.2.18),

\[
S'_{\text{kin}} \{ \phi^\alpha \} = \sum_{n=3}^\infty S_{\text{kin},n} \{ \phi^\alpha \},
\]

we need to calculate

\[
e^{-S_{\text{kin}} \{ \tilde{\rho}^\alpha \}} = e^{-S_{\text{kin,0}}^{(0)} - S_{\text{kin,2}}^{(0)} \{ \tilde{\rho}^\alpha \}} \left< e^{-S'_{\text{kin}} \{ \phi^\alpha \}} \right>_{\tilde{\rho}},
\]

where we have defined

\[
S_{\text{kin,2}}^{(0)} \{ \tilde{\rho}^\alpha \} = \frac{1}{2} \sum_q \sum_{\alpha\alpha'} \Gamma^\alpha_{q q} \tilde{\rho}_q^\alpha \tilde{\rho}_q^{\alpha'},
\]

and

\[
\left< F \{ \phi^\alpha \} \right>_{\tilde{\rho}} = \frac{\int D \{ \phi^\alpha \} F \{ \phi^\alpha \} e^{i \sum_q \phi^\alpha_q \tilde{\rho}_q^\alpha - S_{\text{kin,2}} \{ \phi^\alpha \}}}{\int D \{ \phi^\alpha \} e^{i \sum_q \phi^\alpha_q \tilde{\rho}_q^\alpha - S_{\text{kin,2}} \{ \phi^\alpha \}}}
\]

for any functional \( F \{ \phi^\alpha \} \). Performing the shift-transformation

\[
\phi^\alpha_q \to \phi^\alpha_q + i \sum_{\alpha'} \Gamma^\alpha_{q q} \tilde{\rho}_q^{\alpha'},
\]

it is easy to see that

\[
\left< F \{ \phi^\alpha \} \right>_{\tilde{\rho}} = \left< F\{ \phi^\alpha + i \sum_{\alpha'} \Gamma^\alpha_{q q} \tilde{\rho}_q^{\alpha'} \} \right>_{0}.
\]

In our case we have to calculate

\[
\left< e^{-S_{\text{kin}} \{ \phi^\alpha \}} \right>_{\tilde{\rho}} = \left< e^{-S'_{\text{kin}} \{ \phi^\alpha + i \sum_{\alpha'} \Gamma^\alpha_{q q} \tilde{\rho}_q^{\alpha'} \} \right>_{0}.
\]
Consider now the term of order \((\phi^\alpha)^n\) in the expansion of \(S'_{\text{kin}}\{\phi^\alpha\}\) given in Eq.\(4.11\). Clearly the substitution \(\phi^\alpha_q \to \phi^\alpha_q + i\sum_{\alpha'} \Gamma^\alpha_{q\alpha'} \tilde{\phi}^\alpha_q\) generates also a term of order \((\tilde{\rho}^\alpha)^n\) which does not depend on the \(\phi^\alpha\)-field, and can be pulled out of the average in Eq.\(4.7\). Let us denote this term by \(S^{(0)}_{\text{kin},n}\{\tilde{\rho}^\alpha\}\). From Eq.\(2.22\) it is easy to see that \(S^{(0)}_{\text{kin},n}\{\tilde{\rho}^\alpha\}\) is obtained by replacing \(\phi^\alpha_q \to i\sum_{\alpha'} \Gamma^\alpha_{q\alpha'} \tilde{\rho}^\alpha_{q'}\) in \(S_{\text{kin},n}\{\phi^\alpha\}\), so that it is given by

\[
S^{(0)}_{\text{kin},n}\{\tilde{\rho}^\alpha\} = S_{\text{kin},n}\{i\sum_{\alpha'} \Gamma^\alpha_{q\alpha'} \tilde{\rho}^\alpha_q\} = \frac{1}{n} \sum_{q_1 \ldots q_n} \sum_{\alpha_1 \ldots \alpha_n} \Gamma^{(0)}_{n}(q_1\alpha_1 \ldots q_n\alpha_n) \tilde{\rho}^{\alpha_1}_{q_1} \cdots \tilde{\rho}^{\alpha_n}_{q_n},
\]

(4.8)

where for \(n \geq 3\) the vertices \(\Gamma^{(0)}_{n}\) are

\[
\Gamma^{(0)}_{n}(q_1\alpha_1 \ldots q_n\alpha_n) = i^n \sum_{\alpha'_1 \ldots \alpha'_n} U_n(q_1\alpha'_1 \ldots q_n\alpha'_n) \Gamma^{\alpha'_1}_{q_1} \cdots \Gamma^{\alpha'_n}_{q_n}, \quad n \geq 3 .
\]

(4.9)

Obviously the Gaussian action \(S^{(0)}_{\text{kin},2}\{\tilde{\rho}^\alpha\}\) in Eq.\(4.3\) is also of the form \(4.8\), with

\[
\Gamma^{(0)}_2(q_1\alpha_1 q_2\alpha_2) = \delta_{q_1+q_2,0} \Gamma^{\alpha_1\alpha_2}.
\]

(4.10)

The vertex \(U_1\) has been absorbed into the re-definition of \(\tilde{\rho}^\alpha_q\) (see Eq.\(3.36\)), so that \(S^{(0)}_{\text{kin},1}\{\tilde{\rho}^\alpha_q\} = 0\). Defining

\[
S^{(0)}_{\text{kin}}\{\hat{\rho}^\alpha\} = S^{(0)}_{\text{kin},0} + \sum_{n=2}^{\infty} S^{(0)}_{\text{kin},n}\{\tilde{\rho}^\alpha\},
\]

(4.11)

\[
S''_{\text{kin}}\{\phi^\alpha, \hat{\rho}^\alpha\} = S'_{\text{kin}}\{\phi^\alpha + i \sum_{\alpha'} \Gamma^\alpha_{\alpha'} \hat{\rho}^\alpha_{\alpha'}\} = S'_{\text{kin}}\{i \sum_{\alpha'} \Gamma^\alpha_{\alpha'} \hat{\rho}^\alpha_{\alpha'}\},
\]

(4.12)

the general perturbative expansion for \(S_{\text{kin}}\{\hat{\rho}^\alpha\}\) is

\[
S_{\text{kin}}\{\hat{\rho}^\alpha\} = S^{(0)}_{\text{kin}}\{\tilde{\rho}^\alpha\} - \ln \left[ 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left\langle \left[S''_{\text{kin}}\{\phi^\alpha, \hat{\rho}^\alpha\}\right]^k\right\rangle_0 \right].
\]

(4.13)

According to the linked cluster theorem \(39\) the logarithm eliminates all disconnected diagrams, so that Eq.\(4.13\) can also be written as

\[
S_{\text{kin}}\{\hat{\rho}^\alpha\} = S^{(0)}_{\text{kin}}\{\tilde{\rho}^\alpha\} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left\langle \left[S''_{\text{kin}}\{\phi^\alpha, \hat{\rho}^\alpha\}\right]^k\right\rangle_0,\]

(4.14)

where the subscript \(c\) means that all different connected diagrams should be retained \(39\). From this expression it is easy to see that \(S_{\text{kin}}\{\hat{\rho}^\alpha\}\) is in general of the following form

\[
S_{\text{kin}}\{\hat{\rho}^\alpha\} = S_{\text{kin},0} + \sum_{n=1}^{\infty} S_{\text{kin},n}\{\tilde{\rho}^\alpha\},
\]

(4.15)

where \(S_{\text{kin},0}\) is a constant independent of the fields that cancels in the calculation of correlation functions, and for \(n \geq 1\)
\[ S_{\text{kin},n} \{ \tilde{\rho}^\alpha \} = \frac{1}{n} \sum_{q_1...q_n} \sum_{\alpha_1...\alpha_n} \Gamma_n(q_1\alpha_1...q_n\alpha_n) \tilde{\rho}_{q_1}^{\alpha_1} \cdots \tilde{\rho}_{q_n}^{\alpha_n} , \] 

(4.16)

where the vertices \( \Gamma_n \) have an expansion of the form

\[ \Gamma_n(q_1\alpha_1...q_n\alpha_n) = \sum_{k=0}^{\infty} \Gamma_n^{(k)}(q_1\alpha_1...q_n\alpha_n) . \] 

(4.17)

Here \( \Gamma_n^{(k)} \) describes the interaction between \( n \) collective density fields \( \tilde{\rho}_q^\alpha \), that are generated from all diagrams in the linked cluster expansion \( 4.14 \) containing \( k \) internal loops of the \( \phi^\alpha \)-field. Note that the vertices \( \Gamma_n^{(0)} \) in Eq.\( 4.9 \) do not contain any internal \( \phi^\alpha \)-loops, and are therefore the tree-approximation for the exact vertices \( \Gamma_n \). Because each internal \( \phi^\alpha \)-loop attached to a vertex \( U_m \) reduces the number of external \( \phi \)-fields by 2, it is clear that for \( k \geq 1 \) the vertices \( \Gamma_n^{(k)} \) can only be determined by vertices \( U_m \) with \( m > n \). Within the Gaussian approximation the vertices \( U_m \) with \( m \geq 3 \) are set equal to zero, while the contribution from \( U_1 \) can be absorbed into the re-definition of the \( q = 0 \)-part of \( \tilde{\rho}_q^\alpha \), see Eq.\( 3.36 \). Hence in Gaussian approximation

\[ \Gamma_2(-q\alpha,q\alpha') \approx \Gamma_2^{(0)}(-q\alpha,q\alpha') = \Gamma_q^{\alpha\alpha'} , \] 

(4.18)

where \( \Gamma_q^{\alpha\alpha'} \) is defined in Eq.\( 3.46 \), and

\[ \Gamma_n^{(k)} = 0 \quad \text{for} \quad n \geq 3 \quad \text{or} \quad k \geq 1 . \] 

(4.19)

Although \( \Gamma_1 = 0 \) at the level of the Gaussian approximation, the higher order terms will in general lead to a finite value of \( \Gamma_1 \), which describes the fluctuations of the total number of occupied states in the patches. These terms do not contribute to correlation functions at finite \( q \), but are certainly important for the calculation of the free energy. The formalism developed above is the starting point for a systematic calculation of corrections to the Gaussian approximation.

**B. Explicit calculation of the leading correction to the Gaussian approximation**

The leading correction to the Gaussian approximation is obtained from the one-loop correction in our effective bosonic theory, which amounts to a two-loop calculation at the fermionic level. Note that we have mapped the problem of calculating a two-particle Greens-function of the original fermionic model onto the problem of calculating a one-particle Greens-function of an effective bosonic model. The latter is conceptually simpler, because the symmetrized vertices \( U_n \) and \( \Gamma_n^{(k)} \) automatically contain the relevant self-energy and vertex corrections of the underlying fermionic problem. This will become evident below.

At one-loop order, it is sufficient to truncate the expansion of the interaction part \( S''_{\text{kin}} \{ \phi^\alpha \} \) of the effective action of the \( \phi^\alpha \)-field in Eq.\( 4.1 \) at the fourth order, i.e. we may approximate
\[ S^{\prime}_{\text{kin}}\{\phi^\alpha\} \approx S^{\prime}_{\text{kin},3}\{\phi^\alpha\} + S^{\prime}_{\text{kin},4}\{\phi^\alpha\} \]
\[ = \frac{1}{3} \sum_{q_1q_2q_3} \sum_{\alpha_1\alpha_2\alpha_3} U_3(q_1\alpha_1q_2\alpha_2q_3\alpha_3)\phi^{\alpha_1}_{q_1}\phi^{\alpha_2}_{q_2}\phi^{\alpha_3}_{q_3} \]
\[ + \frac{1}{4} \sum_{q_1q_2q_3q_4} \sum_{\alpha_1\alpha_2\alpha_3\alpha_4} U_4(q_1\alpha_1q_2\alpha_2q_3\alpha_3q_4\alpha_4)\phi^{\alpha_1}_{q_1}\phi^{\alpha_2}_{q_2}\phi^{\alpha_3}_{q_3}\phi^{\alpha_4}_{q_4}, \quad (4.20) \]

where the vertices \( U_3 \) and \( U_4 \) are defined in Eq. 2.22. According to the general formalism outlined above, the bosonized kinetic energy \( S^{\prime}_{\text{kin}}\{\bar{\rho}^\alpha\} \) is obtained by calculating the functional Fourier transform of \( S^{\prime}_{\text{kin}}\{\phi^\alpha\} \). Within one-loop approximation it is sufficient to retain only the term \( k = 1 \) in the linked cluster expansion (4.14), so that

\[ S^{\prime}_{\text{kin}}\{\bar{\rho}^\alpha\} = S^{(0)}_{\text{kin}}\{\bar{\rho}^\alpha\} + \langle S^{\prime\prime}_{\text{kin}}\{\phi^\alpha, \bar{\rho}^\alpha\}\rangle_0, \quad (4.21) \]

where

\[ S^{(0)}_{\text{kin}}\{\bar{\rho}^\alpha\} = S^{(0)}_{\text{kin},0} + \frac{1}{2} \sum_q \sum_{\alpha\alpha'} \Gamma^{\alpha\alpha'}_q \bar{\rho}_{-q}\bar{\rho}_q' \]
\[ + \frac{1}{3} \sum_{q_1q_2q_3} \sum_{\alpha_1\alpha_2\alpha_3} \Gamma^{(0)}_3(q_1\alpha_1q_2\alpha_2q_3\alpha_3)\bar{\rho}^{\alpha_1}_{q_1}\bar{\rho}^{\alpha_2}_{q_2}\bar{\rho}^{\alpha_3}_{q_3} \]
\[ + \frac{1}{4} \sum_{q_1q_2q_3q_4} \sum_{\alpha_1\alpha_2\alpha_3\alpha_4} \Gamma^{(0)}_4(q_1\alpha_1q_2\alpha_2q_3\alpha_3q_4\alpha_4)\bar{\rho}^{\alpha_1}_{q_1}\bar{\rho}^{\alpha_2}_{q_2}\bar{\rho}^{\alpha_3}_{q_3}\bar{\rho}^{\alpha_4}_{q_4}, \quad (4.22) \]

with

\[ \Gamma^{(0)}_3(q_1\alpha_1q_2\alpha_2q_3\alpha_3) = -i \sum_{\alpha_1'\alpha_2'\alpha_3'} U_3(q_1\alpha_1'q_2\alpha_2'q_3\alpha_3')\Gamma^{\alpha_1'\alpha_1}_{q_1}\Gamma^{\alpha_2'\alpha_2}_{q_2}\Gamma^{\alpha_3'\alpha_3}_{q_3}, \quad (4.23) \]
\[ \Gamma^{(0)}_4(q_1\alpha_1q_2\alpha_2q_3\alpha_3q_4\alpha_4) = \sum_{\alpha_1'\alpha_2'\alpha_3'\alpha_4'} U_4(q_1\alpha_1'q_2\alpha_2'q_3\alpha_3'q_4\alpha_4')\Gamma^{\alpha_1'\alpha_1}_{q_1}\Gamma^{\alpha_2'\alpha_2}_{q_2}\Gamma^{\alpha_3'\alpha_3}_{q_3}\Gamma^{\alpha_4'\alpha_4}_{q_4}. \quad (4.24) \]

The correction term due to one internal \( \phi^\alpha \)-loop is

\[ \langle S^{\prime\prime}_{\text{kin}}\{\phi^\alpha, \bar{\rho}^\alpha\}\rangle_0 = S^{(1)}_{\text{kin},0} + S^{(1)}_{\text{kin},1}\{\bar{\rho}^\alpha\} + S^{(1)}_{\text{kin},2}\{\bar{\rho}^\alpha\}, \quad (4.25) \]

where

\[ S^{(1)}_{\text{kin},0} = \frac{3}{2} \sum_{q\alpha}\sum_{\alpha_2\alpha_3} U_4(-q\alpha_1, q'\alpha_2, -q'\alpha_3, q'\alpha_4)\Gamma^{\alpha_2\alpha_1}_{q}\Gamma^{\alpha_3\alpha_3}_{q'}. \quad (4.26) \]
\[ S^{(1)}_{\text{kin},1}\{\bar{\rho}^\alpha\} = \sum_{\alpha} \Gamma^{(1)}_1(\alpha)\bar{\rho}_\alpha^\alpha, \quad (4.27) \]
\[ S^{(1)}_{\text{kin},2}\{\bar{\rho}^\alpha\} = \frac{1}{2} \sum_{q\alpha}\sum_{\alpha'} \Gamma^{(1)}_2(-q\alpha, q\alpha')\bar{\rho}_{-q}\bar{\rho}_q', \quad (4.28) \]

with

\[ \Gamma^{(1)}_1(\alpha) = i \sum_{\alpha_1\alpha_2\alpha_3} U_3(-q\alpha_1, q\alpha_2, 0\alpha_3)\Gamma^{\alpha_2\alpha_1}_{q}\Gamma^{\alpha_3\alpha_3}_{0}. \quad (4.29) \]
\[ \Gamma^{(1)}_2(-q\alpha, q\alpha') = -3 \sum_{\alpha_1\alpha_2\alpha_3\alpha_4} U_4(-q\alpha_1, q\alpha_2, -q'\alpha_3, q'\alpha_4)\Gamma^{\alpha_2\alpha_1}_{q}\Gamma^{\alpha_3\alpha_3}_{q} \Gamma^{\alpha_4\alpha_4}_{q'}. \quad (4.30) \]
Recall that the superscript (1) indicates that these terms contain one internal bosonic loop. Thus, within the one-loop approximation the constant in Eq. (4.15) is $S_{\text{kin,0}} = S_{\text{kin,0}}^{(0)} + S_{\text{kin,0}}^{(1)}$ and the vertices $\Gamma_n$ in Eq. (4.17) are approximated by

$$
\Gamma_1(q\alpha) = \Gamma_1^{(1)}(q\alpha),
\Gamma_2(-q\alpha, q\alpha') = \Gamma_2^{(0)} + \Gamma_2^{(1)}(-q\alpha, q\alpha'),
\Gamma_3(q_1\alpha_1q_2\alpha_2q_3\alpha_3) = \Gamma_3^{(0)}(q_1\alpha_1q_2\alpha_2q_3\alpha_3),
\Gamma_4(q_1\alpha_1q_2\alpha_2q_3\alpha_3q_4\alpha_4) = \Gamma_4^{(0)}(q_1\alpha_1q_2\alpha_2q_3\alpha_3q_4\alpha_4),
$$

and all $\Gamma_n$ with $n \geq 5$ are set equal to zero. The term with $\Gamma_1$ can be ignored for a calculation of correlation functions at finite $q$, because it involves only the $q = 0$ component of the density fields. Furthermore, at the level of a one-loop calculation we may also ignore the term $\Gamma_3^{(0)}$, because the Gaussian expectation value of a product of three $\tilde{\rho}^\alpha$-fields vanishes. Combining the relevant contributions from the kinetic energy with the interaction contribution, we finally arrive at the following effective action

$$
S_{\text{eff}} \{\tilde{\rho}^\alpha\} = \frac{1}{2} \sum_q \sum_{\alpha\alpha'} \left[ \tilde{f}_q \right]^{\alpha\alpha'} + \Gamma_2^{(1)}(-q\alpha, q\alpha') \tilde{\rho}_q \tilde{\rho}_{q'} + \frac{1}{2} \sum_q \sum_{\alpha\alpha'} \Gamma_2^{(1)}(q\alpha, q\alpha') \tilde{\rho}_q \tilde{\rho}_{q'} + \frac{1}{4} \sum_{q_1q_2q_3q_4} \sum_{\alpha_1\alpha_2\alpha_3\alpha_4} \Gamma_4^{(0)}(q_1\alpha_1q_2\alpha_2q_3\alpha_3q_4\alpha_4) \tilde{\rho}_{q_1\alpha_1} \tilde{\rho}_{q_2\alpha_2} \tilde{\rho}_{q_3\alpha_3} \tilde{\rho}_{q_4\alpha_4}. \tag{4.35}
$$

We emphasize that this effective action is only good for the purpose of calculating the one-loop corrections to the Gaussian approximation. At two-loop order one should also retain the terms with $\Gamma_3$ and $\Gamma_6$. The last two terms in Eq. (4.35) contain the one-loop corrections to the non-interacting boson approximation for the bosonized collective density fluctuations. In the limit of long wavelengths we may again write down an equivalent effective Hamiltonian of canonically quantized bosons by using the substitution 3.57. However, we shall not even bother writing down this complicated expression, because this mapping is only valid at long wavelengths and high densities, and does not lead to any simplification. For all practical purposes the parametrization in terms of the collective density-field $\tilde{\rho}^\alpha$ is superior. We shall now use this parametrization to calculate the leading correction to the bosonic propagator and determine in this way the hidden small parameter which controls the range of validity of the Gaussian approximation.

C. First order self-energy and hidden small parameter

Let us define a dimensionless proper (or irreducible) self-energy matrix $\Sigma_i^s$ via

$$
\langle \tilde{\rho}_q^\alpha \tilde{\rho}_{q'}^{\alpha'} \rangle = \delta_{q+q',0} \left[ \tilde{f}_q + \tilde{L} + \Sigma_i^s \right]^{-1} \delta^{\alpha\alpha'}, \tag{4.36}
$$

where the probability distribution for the average is determined by the exact effective action $S_{\text{eff}}\{\tilde{\rho}^\alpha\}$, see Eqs. (3.16) and (3.17). From Eq. (3.48) it is clear that the self-energy $\Sigma_i^s$ contains by definition all corrections to the RPA. Introducing the exact proper polarization matrix $\Pi^s(q)$ via
\[
(\Pi^*(q))^{-1} = (\Pi_0(q))^{-1} - \frac{V}{\beta} \Sigma^*_q,
\]
the exact total density-density correlation function can be written as
\[
\Pi(q) = \sum_{\alpha \alpha'} \left[ (\Pi^*(q))^{-1} + \tilde{f}_q \right]^{-1}^{\alpha \alpha'}.
\]
If all matrix elements of \(\tilde{f}_q\) are identical and equal to \(f_q\), we may repeat the manipulations in Eqs. 3.51-3.53, so that Eq. 4.38 reduces to \[39\]
\[
\Pi(q) = \frac{\Pi^*(q)}{1 + f_q \Pi^*(q)},
\]
where
\[
\Pi^*(q) = \sum_{\alpha \alpha'} [\Pi^*(q)]^{\alpha \alpha'}.
\]
is the total proper polarization. With the help of the dielectric function
\[
\epsilon(q) = 1 + f_q \Pi^*(q),
\]
Eq. 4.39 can also be written as
\[
\Pi(q) = \frac{\Pi^*(q)}{\epsilon(q)}.
\]
To first order in an expansion in the number of bosonic loops, we simply have to add the two diagrams shown in Fig. 4. Because we have symmetrized the vertices, the loop diagram in Fig. 4 has a combinatorial factor of three, so that to first order \(\Sigma^* \approx \Sigma^{(1)}\), with
\[
(\Sigma^{(1)}_{\alpha \alpha'}) = -\Gamma^{(1)}_2(-q \alpha, q \alpha') - 3 \sum_{q'} \sum_{\alpha_3 \alpha_4} \Gamma^{(0)}_4(-q \alpha, q \alpha', -q' \alpha_3, q' \alpha_4) \left[ f_{\bar{q}'} + \Sigma_{\bar{q}'} \right]^{-1}^{\alpha_4 \alpha_3}.
\]
Using the definitions of \(\Gamma^{(1)}_2\) and \(\Gamma^{(0)}_4\) (see Eqs. 4.30 and 4.24), it is easy to show that Eq. 4.43 can also be written as
\[
(\Sigma^{(1)}_{\alpha \alpha'}) = 3 \sum_{q'} \sum_{\alpha_1 \alpha_2} \sum_{\alpha_3 \alpha_4} U_4(-q \alpha_1, q \alpha_2, -q' \alpha_3, q' \alpha_4) \Gamma^{\alpha_1 \alpha_2} \Gamma^{\alpha_3 \alpha_4} \times \left[ \Sigma_{\bar{q}'} - \Sigma_{\bar{q}'} f_{\bar{q}'} + \Sigma_{\bar{q}'} \right]^{-1}^{\alpha_4 \alpha_3}.
\]
A simple manipulation of the matrix in the last line gives
\[
\Sigma_{\bar{q}} - \Sigma_{\bar{q}} \left[ f_{\bar{q}} + \Sigma_{\bar{q}} \right]^{-1} \Sigma_{\bar{q}} = \tilde{f}_{\bar{q}} \left[ 1 + \Sigma_{\bar{q}}^{-1} \tilde{f}_{\bar{q}} \right]^{-1} = \beta \Sigma_{\bar{q}}^{RPA},
\]
where
where the matrix $f^{RPA}_q$ is defined by

$$f^{RPA}_q = f_q \left[ 1 + \Pi_q(q) f_q \right]^{-1} . \quad (4.46)$$

It follows that Eq.4.44 reduces to

$$[\Sigma^{(1)}_q]^{\alpha \alpha'} = 3 \frac{\beta}{V} \sum_{q'} \sum_{\alpha_1 \alpha_2 \alpha_3} \Gamma^{\alpha \alpha_1}_q \Gamma^{\alpha_2 \alpha'}_{q'} U_4(-q \alpha_1, q \alpha_2, -q' \alpha_3, q' \alpha_4) \left[ f^{RPA}_{q'} \right]^{\alpha_3 \alpha_4} . \quad (4.47)$$

Note that $\Sigma^{(1)}_q$ is proportional to the RPA-screened interaction and vanishes in the non-interacting limit, as it should. Eq.4.47 is the general result for the leading correction to the Gaussian approximation in an arbitrary patch-geometry, and for an arbitrary interaction matrix $f_q$.

To make further progress, we shall now restrict ourselves to the high-density limit and assume that the interaction is negligibly small for wave-vectors larger than a cutoff $q_c \ll \min\{\Lambda, \lambda\} \ll k_F$. Choosing also the magnitude of the external wave-vector $q$ in Eq.4.47 small compared with the characteristic size of the patches, the diagonal-patch approximation (A1) is justified, so that $\Pi_q$ and $U_4(-q \alpha_1, q \alpha_2, -q' \alpha_3, q' \alpha_4)$ are diagonal in the patch-indices, see Eqs.3.55 and 2.24. Then Eq.4.47 reduces to

$$[\Sigma^{(1)}_q]^{\alpha \alpha'} = \delta^{\alpha \alpha'} \frac{\beta}{V} \frac{v^\alpha \cdot q - i \omega_m}{v^\alpha \cdot q} A^\alpha_q , \quad (4.48)$$

where the dimensionless function $A^\alpha_q$ is given by

$$A^\alpha_q = \frac{3 \beta}{V} \sum_{q'} U^\alpha_{q'} (-q, q, -q', q') \left[ f^{RPA}_{q'} \right]^{\alpha \alpha_4} . \quad (4.49)$$

We conclude that in the high-density- and long wavelength limit we have to first order in the screened interaction

$$\Gamma^{\alpha \alpha'}_q - [\Sigma^{(1)}_q]^{\alpha \alpha'} = \frac{\beta}{V} \left[ [\Pi^{\star}_q(q)]^{-1} \right]^{\alpha \alpha'} = \frac{\beta}{V} \frac{\delta^{\alpha \alpha'}}{\Pi^{\star}_q(q)} , \quad (4.50)$$

with

$$\frac{1}{\Pi^{\star}_q(q)} = \frac{(1 - A^\alpha_q) v^\alpha \cdot q - (1 - 2 A^\alpha_q) i \omega_m - A^\alpha_q (i \omega_m)^2}{v^\alpha \cdot q} . \quad (4.51)$$

Comparing this expression with Eq.3.55, we conclude that the non-interacting boson approximation is quantitatively correct provided the condition

$$|A^\alpha_q| \ll 1 \quad (4.52)$$

is satisfied for all patches $\alpha$, because then the corrections to the propagator of the locally defined collective density field $\tilde{\rho}^\alpha_q$ are small.
For simplicity, let us now assume that all matrix elements of $f_q$ are identical. Using the same procedure as in Eq. 2.22, it is then easy to show that $[L_q^{RPA}]^{\alpha \alpha} = f_q^{RPA}$ is also independent of the patch label. From the general definition of the vertices $U_n$ in Eq. 2.22, we obtain in this case

$$A_q^\alpha = \frac{1}{\nu_\alpha \beta V} \sum_k \Theta^\alpha(k) \left\{ G_0(k) \Sigma_F^{(1)}(k) G_0(k) [G_0(k+q) + G_0(k-q)] + \frac{1}{2} G_0(k) \Gamma_F^{(1)}(k,q) G_0(k+q) + \Gamma_F^{(1)}(k,-q) G_0(k-q) \right\},$$

with

$$\Sigma_F^{(1)}(k) = \frac{1}{\beta V} \sum_{q'} f_q^{RPA} G_0(k+q'),$$

$$\Gamma_F^{(1)}(k,q) = \frac{1}{\beta V} \sum_{q'} f_q^{RPA} G_0(k+q') G_0(k+q+q).$$

Note that $A^\alpha_q = A^\alpha_0$ due to the symmetrization of the vertex $U_4$ with respect to the interchange of any two labels. It is now obvious that the vertices of our effective bosonic action automatically contain the relevant self-energy and vertex corrections of the underlying fermionic problem [30]. The first term in Eq. 4.53 corresponds to the two self-energy corrections to the non-interacting polarization bubble shown in Fig. 5 (a) and (b), while the last term is due to the vertex correction shown in Fig. 5 (c).

In order to determine the range of validity of the non-interacting boson approximation, we have to calculate the dependence of $A_q^\alpha$ on the various parameters in the problem. In the limit of long wavelengths and low energies, it is to leading order in $|\nu_\alpha \cdot q|$ and $|\omega_m|$ consistent to replace in Eq. 4.52 $A_q^\alpha \to A_0^\alpha$. Actually, the $q \to 0$ limit of $A_q^\alpha$ should be taken in such a way that the ratio $\frac{\omega_m}{\nu_\alpha \cdot q}$ is held constant, because in this case we obtain the low-energy behavior of $A_q^\alpha$ close to the poles of the Gaussian propagator. However, since we are only interested in the order of magnitude of $A_q^\alpha$ for small $|\omega_m|$ and $|q|$, it is sufficient to consider the "$q$-limit" $A_q^\alpha = \lim_{q \to 0} \lim_{\omega_m \to 0} A_q^\alpha$. For simplicity, let us now ignore the frequency-dependence of the RPA-interaction. This amounts to the static approximation for the dielectric constant, which seems reasonable to obtain the correct order of magnitude of $A_q^\alpha$. The "$q$-limit" is obtained by setting $q = 0$ under the summation sign and performing the Matsubara sums before doing the wave-vector integrations. For $\beta \to \infty$ we obtain

$$A_0^\alpha = \frac{1}{\nu_\alpha V^2} \sum_{q'k} \Theta^\alpha(k) f_{q'}^{RPA} \left\{ f(\xi_{k+q'}) \left[ \frac{\partial^2}{\partial \mu^2} f(\xi_k) + \frac{\partial}{\partial \mu} f(\xi_{k+q'}) \right] \frac{\partial}{\partial \mu} f(\xi_k) \right\}.$$  \hspace{1cm} (4.56)

Because the $k$-sum extends over the entire box $K^\alpha$ and by assumption the $q'$-sum is cut off by the interaction at $q_c \ll \min\{\Lambda, \lambda\}$, we may set $\xi_{k+q'} \approx \xi_k$ in the Fermi-functions of Eq. 4.56. Then the summations factorize, and we obtain

$$A_0^\alpha = \left[ \frac{1}{V} \sum_q f_q^{RPA} \right] \frac{1}{\nu_\alpha} \int_{-\infty}^{\infty} d\xi \nu^\alpha(\xi) \left[ f(\xi) f'(\xi) + f'(\xi) f''(\xi) \right].$$ \hspace{1cm} (4.57)
where \( \nu^\alpha(\xi) \) is the energy dependent patch density of states, 

\[
\nu^\alpha(\xi) = \frac{1}{V} \sum_k \Theta^\alpha(k) \delta(\xi - \xi_k).
\] (4.58)

Note that from the definition of \( \nu^\alpha \) in Eq. 3.42

\[
\nu^\alpha = \int_{-\infty}^{\infty} d\xi \nu^\alpha(\xi) [-f'(\xi)] .
\] (4.59)

Integrating in Eq. 4.57 by parts, we obtain in the limit \( \beta \to \infty \)

\[
\int_{-\infty}^{\infty} d\xi \nu^\alpha(\xi) [f(\xi)f''(\xi) + f'(\xi)f'(\xi)] = \frac{1}{2} \frac{\partial \nu^\alpha}{\partial \mu} .
\] (4.60)

Integrating in Eq. 4.57 by parts, we obtain in the limit \( \beta \to \infty \)

\[
\int_{-\infty}^{\infty} d\xi \nu^\alpha(\xi) \frac{\partial}{\partial \xi} [f(\xi)f'(\xi)] = \frac{1}{2} \frac{\partial \nu^\alpha}{\partial \mu} .
\] (4.61)

Because by assumption \( f^\text{RPA}_q \) becomes negligibly small for \( |q| \geq q_c \), the first factor in Eq. 4.57

is for \( V \to \infty \)

\[
\frac{1}{V} \sum_q f^\text{RPA}_q \propto f^\text{RPA}_0 q^d .
\] (4.62)

Ignoring a numerical factor of the order of unity, the final result for \( A^\alpha_0 \) can be written as

\[
A^\alpha_0 \approx \frac{q^d_c f^\text{RPA}_{0}}{\mu} C^\alpha ,
\] (4.63)

where the dimensionless parameter

\[
C^\alpha = \frac{\mu}{V^\alpha} \frac{\partial V^\alpha}{\partial \mu} = \frac{\mu \frac{\partial^2 N^\alpha}{\partial \mu^2}}{V^\alpha \frac{\partial N^\alpha}{\partial \mu}}
\] (4.64)

is a measure for the local curvature of the Fermi surface in patch \( \tilde{K}_\alpha^\lambda \). For sufficiently smooth Fermi surfaces the patch density of states \( \nu^\alpha \) is proportional to \( \Lambda^{d-1} \). However, the cutoff dependence cancels in Eq. 4.64, because it appears in the numerator as well as in the denominator. Therefore \( C^\alpha \) is a cutoff-independent quantity. If we linearize the energy dispersion in patch \( \tilde{K}_\alpha^\lambda \), then the local density of states is replaced by a constant independent of \( \mu \), so that \( C^\alpha \) vanishes in this case. Then there is no correction to the Gaussian approximation. Of course, we already know from the closed loop theorem that the Gaussian approximation is exact if in addition to the diagonal-patch approximation (A1) the energy dispersion is linearized for all patches. As usual, we introduce the dimensionless interaction \( F^\text{RPA}_0 = \nu^\text{RPA}_0 f^\text{RPA}_0 \), which measures the strength of the potential energy relative to the kinetic energy. Note that for regular interactions \( F^\text{RPA}_0 \approx \frac{\nu f^\text{RPA}_0}{1 + \nu f^\text{RPA}_0} \). Using the fact to the global density of states is in \( d \) dimensions proportional to \( k^{d-2} F \), we conclude that in the high-density limit the Gaussian approximation is quantitatively correct provided the condition
is satisfied. This is the main result of this section.

The appearance of three parameters that control the accuracy of the Gaussian approximation has a very simple intuitive interpretation. First of all, if at all points on the Fermi surface the curvature is intrinsically small (i.e. \(|C^\alpha| \ll 1\) for all patches \(\alpha\)) then the corrections to the linearization of the energy dispersion are small, and hence the Gaussian approximation becomes accurate. Note that in the one-dimensional Tomonaga-Luttinger model \(C^\alpha = 0\), because the energy dispersion is linear by definition. However, for realistic energy dispersions of the form \(\epsilon_k = \frac{k^2}{2m}\) the dimensionless curvature parameter \(C^\alpha\) is of the order of unity. But even then the Gaussian approximation is accurate, provided the nature of the interaction is such that it involves only small momentum transfers. This is also intuitively obvious, because in this case the scattering processes probe only a thin shell around the Fermi surface and do not feel the deviations from linearity. Finally, it is clear that also the strength of the effective interaction should determine the range of validity of Gaussian approximation, because in the limit that the strength of the interaction approaches zero all corrections to the Gaussian approximation vanish.

Finally, we would like to discuss a subtlety associated with the choice of the radial patch cutoff \(\lambda\) and the picture of having integrated out all degrees of freedom outside a thin shell around the Fermi surface. Consider the three dimensional Coulomb interaction in the homogeneous electron gas, which in the static screening approximation corresponds in the regime \(|q| \ll k_F\) to

\[
F^{\text{RPA}}_q = \frac{\kappa^2}{q^2 + \kappa^2},
\]

where \(\kappa = [4\pi e^2 \nu]^{1/2}\) is the Thomas-Fermi screening wave-vector. For \(|q| \approx k_F\) the Coulomb potential reduces to a constant of the order of \(\frac{e^2}{k_F}\). If we introduce a radial patch cutoff \(\lambda \approx \kappa\) and define our bare model such that it contains only the degrees of freedom close to the Fermi surface (see Sec.\[\text{I}\]), then we see from Eq.\[4.66\] that we must choose \(q_c \approx \kappa\) and \(F^{\text{RPA}}_0 \approx 1\), so that Eq.\[4.65\] gives

\[
A_0^\alpha \approx \left(\frac{\kappa}{k_F}\right)^3, \quad \lambda \approx \kappa,
\]

where we have ignored numerical constants of the order of unity. Note that \(C^\alpha = O(1)\) for free electrons in \(d = 3\). On the other hand, if we do not introduce a radial patch cutoff but treat also the high-energy degrees of freedom explicitly, then our simple estimate in Eq.\[4.65\] is not valid, because \(F^{\text{RPA}}_q\) in Eq.\[4.66\] does not fall off sufficiently fast, so that the summations in Eq.\[4.56\] do not factorize. Substituting Eq.\[4.66\] into \[4.56\], it is not difficult to show that in this case the value of \(A_0^\alpha\) is essentially determined by the short-wavelength regime \(\kappa \leq |q| \leq k_F\), and that it is given by

\[
A_0^\alpha \approx \left(\frac{\kappa}{k_F}\right)^2, \quad \lambda = \infty.
\]
Although in both cases the corrections to the Gaussian approximation are small at high densities (where $\kappa \ll k_F$), the corrections are smaller by a factor of $\kappa/k_F$ in the model with a radial patch cutoff. This is evidently due to the fact that in the cutoff model the high-energy degrees of freedom have already been taken into account implicitly via the re-definition of the parameters in the Gaussian approximation, so that only the much smaller number of degrees of freedom in a thin shell around the Fermi surface can give rise to corrections. However, our approach does not depend on the linearization of the energy dispersion, so that the radial cutoff can be removed and also the high-energy degrees of freedom can be explicitly taken into account. This enables us to use our bosonization approach as a basis to develop a new systematic method of calculating corrections to the RPA. We shall discuss this method in the following subsection.

D. Calculating corrections to the RPA via bosonization

The corrections to the RPA are usually expressed in terms of the dynamic local field factor $g(q)$, which is defined by writing the exact irreducible polarization (see Eq.(4.41)) in the form [39] [41]

$$\Pi^*(q) = \frac{\Pi_0(q)}{1 - g(q)\Pi_0(q)} \tag{4.69}$$

Note that this equation can also be written as

$$[\Pi^*(q)]^{-1} = [\Pi_0(q)]^{-1} - g(q) \tag{4.70}$$

which has the structure $G^{-1} = G_0^{-1} - \Sigma$, i.e. it resembles the Dyson equation for the single-particle Greens-function of a bosonic problem, with the proper polarization and the local field factor playing the role of the exact Greens-function and the irreducible self-energy. This analogy is well known [39] [41], although it seems that it has not been thoroughly exploited as a guide to develop systematic methods to calculate corrections to the RPA. Such a method naturally emerges from our bosonization approach. Defining a local-field correction matrix

$$g(q) = \frac{V}{\beta} \Sigma^\times_{q} \tag{4.71}$$

our matrix Dyson equation [4.37] for the patch proper polarization takes the form

$$[\Pi^*(q)]^{-1} = [\Pi_0(q)]^{-1} - g(q) \tag{4.72}$$

Comparison with Eq.(4.70) shows that the matrix elements $[\Sigma^\times_{q}]^{\alpha\alpha'}$ of the irreducible self-energy of our effective bosonic model can be identified physically with generalized local field corrections $[g(q)]^{\alpha\alpha'}$, which differentiate between the contributions from the various patches. For simplicity let us assume that the diagonal-patch approximation (4.1) is justified, so that Eq.(4.72) reduces to an equation for the diagonal elements, which can be written as
\[ \Pi^\alpha(q) = \frac{\Pi_0^\alpha(q)}{1 - g^\alpha(q)\Pi_0^\alpha(q)} \approx \Pi_0^\alpha(q) + g^\alpha(q)\Pi_0^\alpha(q) + \ldots . \] (4.73)

Here \( \Pi^\alpha(q) = [\Pi^\alpha(q)]^\alpha \), \( g^\alpha(q) = [g(q)]^\alpha \), and \( \Pi_0^\alpha(q) \) is at long wavelengths given in Eq.3.41. The difference between our approach, which is based on the perturbative calculation of the inverse proper polarization, and the naive perturbative approach is now clear. In the latter method the corrections to the proper polarization bubble are determined by direct expansion of \( \Pi^\alpha(q) \) in powers of the interaction \([40]\). Such a procedure does not correspond to the perturbative calculation of the self-energy, but is equivalent to a direct expansion of the Greens-function. To first order, only the leading correction in the expansion of the Dyson equation (the second line in Eq.4.73) is kept in this method, so that the total proper polarization is approximated by

\[
\Pi^\alpha(q) \approx \sum_\alpha \left[ \frac{\nu^\alpha v^\alpha \cdot q}{\nu^\alpha v^\alpha \cdot q - i\omega_m} + \nu^\alpha A_q \right]
= \Pi_0^\alpha(q) + \frac{1}{\beta V} \sum_k \left\{ G_0(k)\Sigma_F^{(1)}(k)G_0(k)[G_0(k + q) + G_0(k - q)] 
+ \frac{1}{2}G_0(k)[\Gamma_F^{(1)}(k, q)G_0(k + q) + \Gamma_F^{(1)}(k, -q)G_0(k - q)] \right\}, \quad (4.74)
\]

see Eqs.4.53-4.57. In the three dimensional Coulomb-problem, the correction term in Eq.4.74 has been discussed in Refs. [36,40]. Note, however, that these authors evaluate the fermionic self-energy \( \Sigma_F^{(1)}(k) \) and vertex correction \( \Gamma_F^{(1)}(k, q) \) with the bare Coulomb interaction. Holas et al. [40] find that the expansion in Eq.4.74 leads to unphysical singularities in the dielectric function close to the plasmon-poles.

In contrast to the direct expansion of \( \Pi^\alpha(q) \) in powers of the bare interaction, in our method we first calculate the irreducible self-energy of the effective bosonized Hamiltonian in powers of the RPA-screened interaction, and then re-sum the perturbation series via the Dyson equation. The crucial point is that the problem of calculating corrections to the RPA can be completely mapped onto an effective bosonic problem: our functional bosonization method allows us to explicitly construct the interacting bosonic Hamiltonian. Once we accept the validity of this mapping, we can use standard many-body theory for bosonic systems, which leaves us no choice: The corrections to the propagator of this effective bosonic theory should be calculated by expanding its irreducible self-energy \( \Sigma_q^\alpha \) (i.e. the inverse Greens-function) in the number of internal bosonic loops, and then solving the Dyson equation. A similar re-summation has been suggested in Refs. [10,12,13], but it is not so easy to justify this procedure at the fermionic level. Our bosonization approach provides the natural justification for this re-summation. The unphysical singularities [10] that are encountered in the direct perturbative approach might be understandable from the point of view of bosonization: they are artificially generated because one attempts to calculate the bosonic single-particle Greens-function by direct expansion. This expansion is bound to fail close to the poles of the Greens-function, i.e. close to the plasmon poles.

Based on the insights gained from our bosonization approach, we would like to suggest that corrections to the RPA should be calculated by expanding the generalized local field
corrections $g(q)$ in powers of the RPA-interaction and then solving the bosonic Dyson equation \textsuperscript{4.72}. We suspect that in this way unphysical singularities in the dielectric function can be avoided. In the high-density limit we obtain from the first line in Eq.\textsuperscript{4.73} for the total proper polarization

$$\Pi^\star(q) = \sum_\alpha \frac{\nu^\alpha}{1 - A_q^\alpha} v^\alpha \cdot q - i\omega_m(1 - B_q^\alpha) - \frac{(i\omega_m)^2}{v^\alpha \cdot q} B_q^\alpha,$$

(4.75)

with

$$B_q^\alpha = \frac{A_q^\alpha}{1 - A_q^\alpha}.$$ (4.76)

A detailed analysis of Eq.\textsuperscript{4.73} and the resulting dielectric function $\epsilon(q) = 1 + f_q\Pi^\star(q)$ in various dimensions requires a careful analysis of the analytic properties of the function $A_q^\alpha$ and extensive numerical work, which will be presented elsewhere. Here, we would like to restrict ourselves to the discussion of the compressibility $K_\rho$, which can be obtained as the "$q$-limit" of the exact proper polarization (compressibility sum rule) \textsuperscript{39},

$$K_\rho = \lim_{q \to 0} \left[ \lim_{\omega_m \to 0} \Pi^\star(q, i\omega_m) \right].$$ (4.77)

As already mentioned, for sufficiently smooth Fermi surfaces the dependence of $A_q^\alpha$ on the patch-index $\alpha$ can be ignored. Setting $A_0 = \lim_{q \to 0} \lim_{\omega_m \to 0} A_q^\alpha$, we obtain from Eq.\textsuperscript{4.75}

$$K_\rho = \frac{\nu}{1 - A_0}.$$ (4.78)

If we combine this expression with the estimate for $A_0$ given in Eq.\textsuperscript{4.68}, we obtain

$$K_\rho = \frac{\nu}{1 - \gamma \left( \frac{\kappa}{k_F} \right)^2},$$ (4.79)

where $\gamma$ is a numerical constant of the order of unity. Eq.\textsuperscript{4.78} agrees with the expression given by Singwi et al. \textsuperscript{44}, who present also numerical results for $\gamma$ as function of the density of the electron gas. Note that according to Eq.\textsuperscript{4.78} the compressibility diverges and the system becomes unstable for $A_0 \to 1$. Eq.\textsuperscript{4.78} should be compared with the expression for the compressibility resulting from approximation \textsuperscript{4.74}, $K_\rho \approx \nu(1 + A_0)$. This is the leading term in the expansion of Eq.\textsuperscript{4.78} and does not predict any instabilities.

V. CONCLUSIONS

The main results of this work can be summarized as follows:

1. We have developed a functional integral method which allows us to bosonize interacting fermions in arbitrary dimensions. In general, bosonization maps a system of interacting fermions onto an effective system of interacting bosons. The corrections to the non-interacting boson approximation can be calculated in a systematic way.
(2) There exists a physically interesting limit where the Gaussian approximation for the effective bosonic Hamiltonian becomes exact. This limit is characterized by the requirements that the diagonal-patch approximation \((A1)\) and the local linearization \((A2)\) are correct. If the approximations \((A1)\) and \((A2)\) are made and the radial cutoff \(\lambda\) is removed, the resulting interacting Fermi system is exactly solvable in any dimension due to an exact cancellation of self-energy and vertex corrections (generalized closed loop theorem). In the one-dimensional Tomonaga-Luttinger model the condition \((A2)\) is satisfied by definition, while \((A1)\) is satisfied as long as the two Fermi points are sufficiently separated, so that they cannot be connected by the maximal momentum transfer of the interaction. Because for realistic Fermi surfaces in \(d > 1\) there exist always neighboring patches that can be connected by arbitrarily small momentum transfers, \((A1)\) is only approximately correct in higher dimensions.

(3) If the diagonal-patch approximation \((A1)\) is made but the energy dispersion is not linearized, we have explicitly calculated the leading correction to the Gaussian approximation, and have shown that the effect of interactions between the bosons is negligible as long as the parameter \(\left(\frac{\xi_0}{k_F}\right)^d |F^{\text{RPA}}_0| |C^\alpha|\) is small compared with unity. The dimensionless quantity \(C^\alpha\) is proportional to the derivative of the density of states with respect to the chemical potential, and vanishes if the energy dispersion is linearized. The origin of an additional small parameter \(C^\alpha\) in the correction to the RPA lies in the cancellation between self-energy and vertex corrections, as described by the generalized closed loop theorem.

(4) Our bosonization approach maps the calculation of the density-density correlation function of the Fermi system on the problem of calculating the single-particle Greens-function of an effective bosonic model. This mapping explains the origin of unphysical singularities that arise in direct perturbative expansions of the proper polarization, and strongly suggests that corrections to the RPA should be calculated by expanding the generalized local field corrections in powers of the RPA-interaction.

The present work is the first step to go beyond the non-interacting boson approximation. We have calculated the bosonized Hamiltonian as well as the density-density correlation function beyond the Gaussian approximation. The problem of calculating the non-Gaussian corrections to the leading bosonization expression for the single-particle Greens-function remains open. This calculation is more difficult, because it involves the solution of a partial differential equation \([10, 23, 25, 27]\). If the energy dispersion is linearized, \(\xi^\alpha_q \approx v^\alpha \cdot q + O(q^2)\), this differential equation is linear and first order, and can be solved exactly. Inclusion of nonlinear terms of order \(q^2\) and higher in the expansion of \(\xi^\alpha_q\) lead to higher spatial derivatives, which are not so easy to handle. Following the method used by Khveshchenko and Stamp \([46]\), it might be possible to set up some kind of perturbation theory for the solution of this higher order partial differential equation, but so far we have not pursued this possibility.

It is tempting to take the form of the Greens-function within the Gaussian approximation as a basis to speculate on the effect of the non-Gaussian terms. Within the Gaussian approximation the general bosonization result for the real-space imaginary-time single-particle Greens-function associated with patch \(\alpha\) can be written as \([25]\)

\[
G^\alpha(r, \tau) = G_0^\alpha(r, \tau)e^{Q^\alpha(r, \tau)},
\]

(5.1)
where \( G_0^\alpha(r, \tau) \) is the non-interacting Greens-function, and

\[
Q^\alpha(r, \tau) = R^\alpha - S^\alpha(r, \tau), \quad R^\alpha = \lim_{r, \tau \to 0} S^\alpha(r, \tau), \tag{5.2}
\]

\[
S^\alpha(r, \tau) = \frac{1}{\beta V} \sum_q \left[ f_{q}^{\text{RPA}} \right]_{\alpha \alpha} \frac{\cos(q \cdot r - \omega_m \tau)}{(i\omega_m - v^\alpha \cdot q)^2}. \tag{5.3}
\]

Note that this expression depends exclusively on the RPA-screened interaction matrix \( f_{q}^{\text{RPA}} \), as defined in Eq.4.46. Due to the linearization of the energy dispersion the non-interacting real-space Greens-function \( G_0^\alpha(r, \tau) \) is proportional to \( \delta^{(d-1)}(r_\perp) \), where \( r_\perp \) is the \( d-1 \)-dimensional projection of \( r \) perpendicular to \( v^\alpha \). Therefore at the level of the Gaussian approximation we may replace

\[
Q^\alpha(r, \tau) \to Q^\alpha(r_\parallel \hat{v}^\alpha, \tau) \tag{5.4}
\]

in Eq.5.1, where \( \hat{v}^\alpha \) is a unit vector in the direction of \( v^\alpha \) and \( r_\parallel = r \cdot \hat{v}^\alpha \). If the non-linear terms in the energy dispersion are taken into account, we expect that the RPA-interaction is replaced by the exact effective interaction, which takes the local field corrections to the bare polarization bubbles into account. Furthermore, the term \( v^\alpha \cdot q \) in the denominator of Eq.5.3 should be replaced by the full dispersion \( \xi^\alpha_q \), and the substitution 5.4 cannot be made any more. However, non-Gaussian terms will certainly also give rise to further modifications, so that it is not clear if the Greens-function retains at least approximately the above simple structure.

Finally, it would be interesting to calculate the corrections to the RPA by means of the method outlined in Sec.[IV D]. This method is different from a straight-forward expansion of the proper polarization, because it is based on the loop-wise calculation of generalized local field corrections associated with the patches. The calculation of corrections to the RPA has been discussed intensely in the 70’s, but today the interest in this problem has waned. Our bosonization approach sheds new light on the various approximations that can be found in the literature, and perhaps will revive the interest in this problem.

ACKNOWLEDGEMENTS

We are grateful to Lorentz Bartosch for checking a large part of the algebra presented in this work and for suggesting some useful modifications. We would also like to thank Walter Metzner for discussions, for giving us insight into his unpublished notes, and for very useful comments on the manuscript. Finally, we would like to thank Volker Meden for his critical reading of the manuscript.
REFERENCES

[1] S. Tomonaga, Prog. Theor. Phys. 5, 544 (1950).
[2] D. C. Mattis and E. H. Lieb, J. Math. Phys. 65, 304 (1965).
[3] J. Sólyom, Adv. Phys. 28, 201 (1979).
[4] F. D. M. Haldane, J. Phys. C 14, 2585 (1981).
[5] This statement is only true if the Fourier transform \( f_\mathbf{q} \) of the interaction does not vanish for \( \mathbf{q} \to 0 \). An interaction of the form \( f_\mathbf{q} \propto |\mathbf{q}|^\eta \) with \( \eta > 0 \) gives rise to Fermi-liquid behavior even in \( d = 1 \), see Refs. [13,14].
[6] J. M. Luttinger, J. Math. Phys. 4, 1154 (1963).
[7] J. R. Engelbrecht and M. Randeria, Phys. Rev. Lett. 65, 1032 (1990).
[8] A. Luther, Phys. Rev. B 19, 320 (1979).
[9] P. W. Anderson, Phys. Rev. Lett. 64, 1839 (1990).
[10] P. W. Anderson and Y. Ren, in High Temperature Superconductivity, edited by K. S. Bedell, D. E. Meltzer, D. Pines, and J. R. Schrieffer (Addison-Wesley, Reading, MA, 1990).
[11] F. D. M. Haldane, Helv. Phys. Acta. 65, 152 (1992).
[12] F. D. M. Haldane, in Perspectives in Many-Particle Physics, Proceedings of the International School of Physics ”Enrico Fermi”, Course 121, (North Holland, Amsterdam, 1994).
[13] P. Kopietz, V. Meden and K. Schönhammer, accepted for publication in Phys. Rev. Lett..
[14] K. Schönhammer, V. Meden and P. Kopietz, accepted for publication in J. of Low Temp. Physics.
[15] P. W. Anderson, Phys. Rev. Lett. 71, 1220 (1993).
[16] P. Kopietz, Habilitationsschrift, Universität Göttingen, 1995, in preparation.
[17] A. Houghton and J. B. Marston, Phys. Rev. B 48, 7790 (1993).
[18] A. Houghton, H.-J. Kwon, and J. B. Marston, Phys. Rev. B 50, 1351 (1994).
[19] A. Houghton, H.-J. Kwon, J. B. Marston, and R. Shankar, J. Phys. C 6, 4909 (1994).
[20] H.-J. Kwon, A. Houghton, and J. B. Marston, Theory of Fermion liquids, preprint, January 1995.
[21] A. H. Castro Neto and E. Fradkin, Phys. Rev. Lett. 72, 1393 (1994).
[22] A. H. Castro Neto and E. Fradkin, Phys. Rev. B 49, 10877 (1994).
[23] A. H. Castro Neto and E. Fradkin, Bosonization of Fermi Liquids II: Landau fixed point, Green’s functions and screening, preprint, 1994.
[24] C. Castellani, C. Di Castro, and W. Metzner, Phys. Rev. Lett. 72, 316 (1994).
[25] P. Kopietz and K. Schönhammer, Functional Bosonization of Interacting Fermions in Arbitrary Dimensions, preprint, July 1994.
[26] J. Fröhlich, R. Götschmann, and P. A. Marchetti, Bosonization of Fermi Systems in Arbitrary Dimension in Terms of Gauge Forms, preprint, May 1994.
[27] H. C. Fogedby, J. Phys. C 9, 3757 (1976).
[28] D. K. K. Lee and Y. Chen, J. Phys. A 21, 4155 (1988).
[29] V. N. Popov, Functional Integrals and Collective Excitations, (Cambridge University Press, Cambridge, 1987).
[30] J. W. Negele and H. Orland, *Quantum Many-Particle Physics*, (Addison-Wesley, Redwood City, 1988).

[31] R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, (McGraw-Hill, New York, 1965), chapter 9.

[32] T. Bohr, Nordita preprint 81/4, *Lectures on the Luttinger Model*, 1981 (unpublished).

[33] I. E. Dzyaloshinskii and A. I. Larkin, Zh. Eksp. Teor. Fiz. 65, 411 (1973) [Sov. Phys. JETP 38, 202 (1974)].

[34] J. Hermisson, Diplomarbeit, Universität Göttingen, 1995, in preparation.

[35] See, for example, C. Itzykson and J.-B. Zuber, *Quantum Field Theory*, (McGraw-Hill, New-York, 1980), p.276.

[36] D. J. W. Geldart and R. Taylor, Canadian J. Phys. 48, 150 and 167 (1970).

[37] Y. B. Kim, A. Furusaki, X.-G. Wen, and P. A. Lee, *Gauge-invariant response functions of fermions coupled to gauge fields*, preprint (1994).

[38] W. Metzner, unpublished notes (1995).

[39] G. D. Mahan, *Many-Particle Physics*, (Plenum, New York, 1981).

[40] A. Holas, P. K. Aravind, and K. S. Singwi, Phys. Rev. B20, 4912 (1979).

[41] V. D. Gorabchenko, V. N. Kohn and E. G. Maksimov, in *Modern Problems in Condensed Matter Sciences, Vol. 24: The Dielectric Function of Condensed Matter Systems*, edited by L. V. Keldysh, D. A. Kirzhnitz and A. A. Maradudin, (North Holland, Amsterdam, 1989).

[42] A. K. Rajagopal and K. P. Jain, Phys. Rev. A 5, 1475 (1972).

[43] M. W. C. Dharma-wardana, J. Phys. C 9, 1919 (1976).

[44] K. S. Singwi, M. P. Tossi, R. H. Land, and A. Sjölander, Phys. Rev. 157, 589 (1968).

[45] J. Schwinger, Phys. Rev. 128, 2425 (1962).

[46] D. V. Khveshchenko and P. C. E. Stamp, Phys. Rev. Lett. 71, 2118 (1993).
FIGURES

FIG. 1. Graph of a squat box $K_{\Lambda,\Lambda}^\alpha$ with patch cutoff $\Lambda$ and radial cutoff $\lambda$ in three dimensions. The vector $k^\alpha$ points on the Fermi surface to the center of patch $\bar{K}_\Lambda^\alpha$.

FIG. 2. The boxes $K_{\Lambda}^\alpha$ for a spherical Fermi surface in $d = 2$. The dashed lines mark the corresponding squat boxes $K_{\Lambda,\Lambda}^\alpha$ that contain only the degrees of freedom in the vicinity of the Fermi surface.

FIG. 3. Feynman-diagram representing $Tr[\hat{G}_0\hat{V}]^\alpha$, see Eqs. 2.19 and 2.21. The lines with arrows denote fermionic Greens-functions, while the wavy lines denote external $\phi^\alpha$-fields.

FIG. 4. Leading contributions to the bosonic self-energy, see Eq. 4.43. The dark circle denotes the vertex $\Gamma_2^{(1)}$ defined in Eq. 4.30 and the shaded square represents the vertex $\Gamma_4^{(0)}$ defined in Eq. 4.24. The dashed arrows denote the collective density field $\tilde{\rho}^\alpha$, and the dashed loop is the Gaussian propagator of the $\tilde{\rho}^\alpha$-field, see Eq. 3.48.

FIG. 5. Leading local-field corrections to the polarization bubble. (a) and (b) are the leading self-energy corrections (see Eq. 4.54), while (c) is the leading vertex correction (see Eq. 4.55). The thick wavy line denotes the RPA-interaction, as defined in (d). The thin wavy line in (d) represents the bare interaction.