A Note on the Concrete Hardness of the Shortest Independent Vectors Problem in Lattices

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Abstract

Blömer and Seifert [BS99] showed that SIVP
2
is NP-hard to approximate by giving a reduc-
tion from CVP
2
for constant approximation factors as long as the CVP instance has a certain property. In order to formally define this requirement on the CVP instance, we introduce a new computational problem called the Gap Closest Vector Problem with Bounded Minima. We adapt the proof of [BS99] to show a reduction from the Gap Closest Vector Problem with Bounded Minima to SIVP for any ℓ
p
norm for some constant approximation factor greater than 1.

In a recent result, Bennett, Golovnev and Stephens-Davidowitz [BGS17] showed that under Gap-ETH, there is no 2
o( n)
-time algorithm for approximating CVP
p
up to some constant factor γ ≥ 1 for any 1 ≤ p ≤ ∞. We observe that the reduction in [BGS17] can be viewed as a reduction from Gap-3-SAT to the Gap Closest Vector Problem with Bounded Minima. This, together with the above mentioned reduction, implies that, under Gap-ETH, there is no 2
o( n)
-time algorithm for approximating SIVP
p
up to some constant factor γ ≥ 1 for any 1 ≤ p ≤ ∞.

1 Introduction

A lattice L is the set of all integer combinations of linearly independent basis vectors b1, . . . , bn ∈ R
d,

L = L(b1,...,bn) := \{ ∑ni=1zi bi : zi ∈ Z \}.

We call n the rank of the lattice L and d the dimension or the ambient dimension of the lattice L.

For i = 1, . . . , n, the ith successive minimum, denoted by λi(L), is the smallest ℓ such that there are i linearly independent lattice vectors that have length at most ℓ.

The Shortest Independent Vector Problem (SIVP) takes as input a basis for a lattice L ⊂ R
d and r > 0 and asks us to decide whether the largest successive minima is at most r, i.e., λn(L) ≤ r. Typically, we define length in terms of the ℓ
p
norm for some 1 ≤ p ≤ ∞, defined as

||x||p := (|x1|p + |x2|p + · · · + |xd|p)1/p

for finite p and

||x||∞ := max |xi|.

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In particular, the $\ell_2$ norm is the familiar Euclidean norm, and it is the most interesting case from our perspective. We write $\text{SIVP}_p$ for $\text{SIVP}$ in the $\ell_p$ norm (and just $\text{SIVP}$ when we do not wish to specify a norm).

Starting with the breakthrough work of Lenstra, Lenstra, and Lovász in 1982 [LLL82], algorithms for solving lattice problems in both its exact and approximate forms have found innumerable applications, including factoring polynomials over the rationals [LLL82], integer programming [Len83, Kan87, DPV11], cryptanalysis [Sha84, Odl90, JS98, NS01], etc. More recently, many cryptographic primitives have been constructed whose security is based on the (worst-case) hardness of $\text{SIVP}$ or closely related lattice problems [Ajt04, Reg09, GPV08, Pei10, Pei16]. In particular, the (worst-case) hardness of $\text{SIVP}$ for $\text{poly}(n)$ approximation factors implies the existence of several fundamental cryptographic primitives like one-way functions, collision-resistant hash functions, etc (see, for example, [GGH96], [Ajt98]). Such lattice-based cryptographic constructions are likely to be used on massive scales (e.g., as part of the TLS protocol) in the not-too-distant future [ADPS16, BCD+16, NIS].

Blömer and Seifert [BS99] showed that $\text{SIVP}$ is NP-hard to approximate for any constant approximation factor. While their result is shown only for the Euclidean norm, there proofs can easily be extended to arbitrary norms. As is true for many other lattice problems, $\text{SIVP}$ is believed to be hard to approximate up to factors polynomial in $n$, the rank of the lattice. In particular, the best known algorithms for $\text{SIVP}$, even for $\text{poly}(n)$ approximation factors run in time exponential in $n$ [ADRS15, ADS15].

However, NP-Hardness itself does not exclude the possibility of sub-exponential time algorithms since it merely shows that there does not exist a polynomial time algorithm unless P = NP. To rule out such algorithms, we typically rely on a fine-grained complexity-theoretic hypothesis—such as the Strong Exponential Time Hypothesis (SETH), the Exponential Time Hypothesis (ETH), or the Gap-Exponential Time Hypothesis (Gap-ETH). To that end, a few recent results showed quantitative hardness results for the Closest Vector Problem ($\text{CVP}_p$) [BGS17], and the Shortest Vector Problem ($\text{SVP}_p$) [AS18] which are closely related problems. In particular, assuming SETH, [BGS17] showed that there is no $2^{(1-\varepsilon)n}$-time algorithm for $\text{CVP}_p$ or $\text{SVP}_\infty$ for any $\varepsilon > 0$ and “almost all” $1 \leq p \leq \infty$ (not including $p = 2$). Under ETH, [BGS17] showed that there is no $2^{o(n)}$-time algorithm for $\text{CVP}_p$ for any $1 \leq p \leq \infty$. Under Gap-ETH, [BGS17] showed that there is no $2^{o(n)}$-time algorithm for approximating $\text{CVP}_p$ up to some constant factor $\gamma \geq 1$ for any $1 \leq p \leq \infty$. Similar, but slightly weaker, results were obtained for $\text{SVP}_p$ in [AS18].

### 1.1 Our results and techniques.

Blömer and Seifert [BS99] showed that SIVP$_2$ is NP-hard by giving a reduction from CVP$_2$ to SIVP$_2$. This reduction can easily be extended to all $\ell_p$ norms, and increases the rank of the lattice by 1. Thus, combined with the SETH hardness result from [BGS17], it implies the following.

**Theorem 1.** Under the SETH, there is no $2^{(1-\varepsilon)n}$-time algorithm for $\text{SIVP}_p$ for any $\varepsilon > 0$ and for all but finitely many values of $p$ in $[1, \infty)$. Furthermore, under randomized ETH, There is no $2^{o(n)}$-time algorithm for $\text{SIVP}_p$ for any $1 \leq p \leq \infty$.

Note that the latter result is due to [BGS17].

A closer look at their reduction reveals that it cannot be extended to showing NP-hardness of approximate SIVP directly (even though CVP is known to be NP-hard for almost polynomial approximation factors) in that for the lattice $\mathcal{L}$ when given as a part of a CVP instance, $\lambda_n(\mathcal{L})$ might be much larger than the distance of the target from the lattice, in which case, an oracle for
approximating SIVP up to a constant factor, does not tell anything about the distance of the target from the lattice.

To overcome this difficulty, [BS99], the CVP instance obtained from a reduction from the minimum label cover problem has a guarantee that for the CVP instance \((\mathcal{L}, t)\), \(\lambda_n(\mathcal{L})\) is “not much larger” than the distance of \(t\) from \(\mathcal{L}\).

We introduce a new computational problem called the Gap Closest Vector Problem with Bounded Minima (GapCVP\(^\tau\)), which captures the above mentioned requirement on the CVP instance that \(\lambda_n(\mathcal{L})\) has an upper bound depending on the parameter \(\tau\). We observe that the reduction from Gap-3-SAT to GapCVP in [BGS17] (which implies approximate hardness of approximate-CVP is actually a reduction from Gap-3-SAT to GapCVP\(^\tau\) for an appropriate choice of \(\tau\). We then show a reduction similar to [BS99] from GapCVP\(^\tau\) to SIVP, which implies the following result.

**Theorem 2.** Under the (randomised) Gap Exponential Time Hypothesis, for any \(p \geq 1\), there exists \(\gamma' > 1, \epsilon > 0\) such that \(\gamma' - \text{SIVP}_p\) with rank \(n\) is not solvable in \(2^{\epsilon n}\) time.

2 Basic Definitions

**Lattices**

Let \(\mathbb{R}^n\) be a real vector space, with an \(\ell_p\)-norm on the vectors such that \(v \in \mathbb{R}^n, \|v\|_p := \sum_{i=1}^{n} |v_i|^p\). Then a lattice \(L\) is defined as the set of all integer linear combinations of a finite set \(B = \{b_1, b_2, \ldots, b_n\}\) of linearly independent vectors in \(\mathbb{R}^n\):

\[
L = \left\{ \sum_{i=1}^{m} c_i \cdot b_i \mid c_i \in \mathbb{Z} \right\}
\]

We will then call such a set \(B\) the basis of the lattice. Note that the dimension of the subspace spanned by \(B\) (called the rank of the lattice) is a subspace of the space in which the basis vectors are obtained. Thus the rank of the lattice may be less than the dimension of the lattice. Cases where the rank of the lattice is equals to the dimension of the lattice are referred to as full-rank lattices.

Since we wish to have inputs of bounded size, we can assume that an \(n\)-dimensional lattice \(L\) is generated by basis vectors from \(\mathbb{Q}^n\). Additionally, this can be scaled to integral values. Thus we may assume that lattices are generated by vectors from \(\mathbb{Z}^n\).

**Successive Minima**

Denoted by \(\lambda_i(L)\), the \(i^{th}\) successive minimum denotes the minimum length such that there are exactly \(i\) linearly independent lattice vectors that are at most this length.

Minkowski’s second theorem states the following with regards to the successive minima:

**Theorem 3.** For any full-rank lattice \(L\) we have that:

\[
\left( \prod_{i=1}^{n} \lambda_i(L) \right)^{\frac{1}{n}} \leq n^\frac{1}{2} (\text{det}(L))^{\frac{1}{n}}
\]
2.1 Computational problems

**Gap-Closest Vector Problem (γ-GapCVP \(_p\)):** Given a lattice \( L \), a target vector \( t \in \mathbb{Z}^n \) (which may or may not be in the lattice) and a value \( d \) output YES if there exists a vector \( v \) in the lattice such that \( \|v - t\|_p \leq d \) (i.e. the closest vector in the lattice to the vector \( t \) has a distance to the target of less than \( d \)), and output NO if all the vectors in the lattice are of distance greater than \( \gamma \cdot d \) to the target.

**Gap-Closest Vector Problem with Bounded Minima (γ-GapCVP \(_p^\tau\)):** Given a lattice \( L \), a target vector \( t \in \mathbb{Z}^n \) (which may or may not be in the lattice), and a value \( d \) output YES if there exists a vector \( v \) in the lattice such that \( \|v - t\|_p \leq d \) (i.e. the closest vector in the lattice to the vector \( t \) has a distance to the target of less than \( d \)), and output NO if all the vectors in the lattice are of distance greater than \( \gamma \cdot d \) to the target with the added guarantee that there exists a \( \tau > 0 \) such that \( \lambda_n(L)^p \leq \tau d^p \). Note that the bound on the minima hold for both the YES and NO instances.

**Gap-Shortest Independent Vector Problem (γ-GapSIVP \(_p\)):** Given a lattice \( L \), and value \( d \), output YES if there exists a set of linearly independent vectors \( \{b_1, b_2, ..., b_n\} \) that are in \( L \) such that the longest vector in the set has length less than \( d \), and output NO if all such sets have a vector of length greater than \( \gamma \cdot d \).

For the above gap problems, the non-gap variants are the exact cases where \( \gamma = 1 \), and thus the \( \gamma \)-prefix will be omitted.

**k-SAT:** Given a boolean formula in conjunctive normal form over \( n \) variables, i.e. as a conjunction of \( m \) clauses where each clause is a disjunction of \( k \) literals, decide if there is a assignment (of either true or false) to the variables such that the boolean formula evaluates to true.

**\((\delta, \epsilon)\)-Gap-k-SAT:** Given a boolean formula in conjunctive normal form and a two constants \( 0 \leq \delta < \epsilon \leq 1 \), output YES if there exists an assignment such that it satisfies at least \( \epsilon \) fraction of the clauses, and output NO if for all assignments they only satisfy at most \( \delta \) fraction of the clauses. For convenience at times the \((\delta, \epsilon)\)-prefix may be omitted when unnecessary.

2.2 ETH, SETH and Gap-ETH-hardness

[IP01] introduced conjectures that will be used as main assumptions to derive the hardness results that we have.

**Definition 1 (Exponential Time Hypothesis).** The Exponential Time Hypothesis (ETH) states that for every \( k \geq 3 \) there is exists a constant \( \epsilon > 0 \) such that no algorithm solves \( k\)-SAT formulas with \( n \) variables in \( 2^{\epsilon n} \) deterministic time.

**Definition 2 (Strong Exponential Time Hypothesis).** The Strong Exponential Time Hypothesis (SETH) states that for all \( \epsilon > 0 \), there exists a \( k \geq 3 \) such that no algorithm solves \( k\)-SAT formulas with \( n \) variables in \( 2^{(1-\epsilon)n} \) deterministic time.

Additionally, [Din16] and [MR17] introduced an equivalent version for Gap-\( k \)-SAT.

**Definition 3 (Gap Exponential Time Hypothesis).** There exists constants \( \delta < 1 \) and \( \epsilon > 0 \) such that no algorithm solves \((\delta, 1) - \text{Gap-3-SAT}\) instances with \( n \) variables in \( 2^{\epsilon n} \) deterministic time.

The above formulation is from [BGS17].
3 Related Results

The main result that has led to subsequent hardness proofs in other lattice problems was derived by [BGS17] through the construction of isolating parallelepipeds that encode assignments from instances of Gap-k-SAT to choices of vectors such that each clause contributes the same distance regardless of how many literals are as long at least one literal is satisfied, however unsatisfied clauses would contribute a much greater distance.

3.1 SETH-hardness of CVP\(_2\) under also any \(p\)-norm

Theorem 4. Solving exact CVP\(_p\) under all \(p\)-norms where \(p\) is not even and \(\leq k - 1\) is not possible in time \(2^{(1-\epsilon)n}\) where \(\epsilon > 0\).

The same proof works for \(p\) in general instead of 2.

3.2 Gap-ETH-hardness of approximating CVP\(_p\) within a constant factor

Theorem 5 ([BGS17]). There exists a reduction from \((\delta, \epsilon)\)-Gap-2-SAT with \(n\) variables and \(m\) clauses to \(\gamma\)-GapCVP\(_p\) for any \(p\)-norm, so that the rank of the lattice in the resulting instance is the same as the number of variables in the original instance. Furthermore, \(\gamma\) is given as:

\[
\left( \frac{\delta + (1 - \delta)3^p}{\epsilon + (1 - \epsilon)3^p} \right)^\frac{1}{p}
\]

We will provide their construction of the \(\gamma\)-CVP\(_p\) instance here. Let \(t\) be a target vector defined by the following:

\[
t_i = 3 - \eta_i
\]

where \(\eta_i\) denotes the number of negated literals in the \(i^{th}\) clause, the distance \(d\) be \((\epsilon + (1 - \epsilon)3^p)^\frac{1}{p}\), and \(B\) a set of basis (column) vectors \(\{b_1, b_2, \ldots, b_k\}\) defined by the following:

\[
b_{i,j} = \begin{cases} 
2 & x_j \in C_j \\
-2 & -x_j \in C_j \\
0 & \text{else}
\end{cases}
\]

We will make the following claim about the reduction that was proposed in their paper as they will be useful to us in our reduction: In the resulting lattice, both \(\lambda_n^p\) and the length of the target vector is upper bounded by \(\frac{3^p}{\epsilon + (1 - \epsilon)3^p} \cdot d^p\), where \(d^p\) is proportional to the number of clauses in the \((\delta, \epsilon)\)-Gap-2-SAT instance. Thus we can say that the resulting instance is also an instance of \(\gamma\)-CVP\(_p\), where \(\tau = \frac{3^p}{\epsilon + (1 - \epsilon)3^p} \cdot d^p\).

Proof. Consider the construction provided in [BGS17], the basis vectors that are then provided have values of either \(-2, 2, 0\), thus in the worst case, we obtain a set of linearly independent vectors with the longest vector having all \(2\) or \(-2\)'s. \(\Box\)
3.3 Gap-ETH-hardness of \((\delta, \epsilon)\)-Gap-2-SAT

**Theorem 6 ([GJS76]).** \(\forall \delta, \epsilon\) such that 0 \(\leq\) \(\delta\) \(<\epsilon\) \(\leq 1\), there exists a polynomial time reduction from \((\delta, \epsilon)\)-Gap-3-SAT with \(n\) variables and \(m\) clauses to an instance of \((\frac{6+\delta}{10}, \frac{6+\epsilon}{10})\)-Gap-2-SAT, with \(n+m\) variables and 10\(m\) clauses.

Additionally, Bennett et al. used Dinur's result in [Din16] to derive the following result:

**Theorem 7 ([BGST17]).** \(\forall \delta, \delta'\) such that 0 \(\leq\) \(\delta\) \(<\delta'\) \(< 1\), there is a polynomial time-randomised reduction from a \((\delta, 1)\)-Gap-k-SAT with \(n\) variables and \(m\) clauses, to instances of \((\delta', 1)\)-Gap-k-SAT with \(n\) variables and \(O(n)\) clauses.

This implies it is almost always possible to reduce the number of clauses in \((\delta, 1)\)-Gap-k-SAT instances so that reductions that run linear in \(m\) may also be considered linear in \(n\), so that Gap-ETH may still apply. However, since the reduction is randomised, existence of sub-exponential time algorithms that solve the resulting instances only imply existence of randomised sub-exponential time algorithms for \((\delta, 1)\)-Gap-k-SAT in the general case (i.e. when \(m = \omega(n)\)).

3.4 SETH-hardness of exact CVP\(_p\) under almost any \(p\)-norm

**Theorem 8 ([BGST17]).** There exists a polynomial time reduction from \(k\)-SAT to CVP\(_p\) such that the rank of the resulting lattice is the same as the number of variables in the original \(k\)-SAT instance, for all \(p\) that is not even and less or equals to \(k-1\).

**Corollary 1.** Solving exact CVP\(_p\) under all \(p\)-norms where \(p\) is not even is not possible in time \(2^{(1-\epsilon)n}\) where \(\epsilon > 0\).

[BSS99] had also previously constructed a reduction that was tight in the resulting instance size since it only increased the rank by 1 by intuitively treating the target vector as the \((n+1)^{\text{th}}\) vector in an \(\text{SIVP}\) instance. To do this, an extra value that was large enough was padded to the bottom of the target vector to ensure it would be long enough to be considered the \((n+1)^{\text{th}}\) successive minima.

4 Main Contribution

We now present our main contribution, that is showing hardness of approximating \(\gamma\)-SIVP\(_p\) within a constant factor \(\gamma\).

**Theorem 9.** For any \(\tau = \tau(n) > 0\), and \(\gamma \geq 1\) there exists an efficient reduction from \(\gamma\)-GapCVP\(_p^\tau\) to \(\gamma'-\text{SIVP}_p\) for any \(p\)-norm where \(p\in[1,\infty)\), with \(\gamma'^p \leq \frac{2\gamma^p}{2^{\tau} - 1 - \gamma^p}\). Moreover, the rank of the lattice in the \(\gamma'-\text{SIVP}_p\) instance is equals to \(n+1\) where \(n\) is the rank \(\gamma\)-CVP\(_p^\tau\) instance.

**Proof.** We will let \((\mathcal{L}, b, d)\) denote a \(\gamma\)-GapCVP\(_p^\tau\) instance and \((\mathcal{L}', d')\) denote a \(\gamma'-\text{SIVP}_p\) instance. Likewise, we will let \(\lambda_n = \lambda_n(\mathcal{L})\) denote the \(n^{\text{th}}\) minimum for the \(\gamma\)-GapCVP\(_p\) whereas \(\lambda_{n+1}' = \lambda_{n+1}'(\mathcal{L}')\) denotes the \((n+1)^{\text{th}}\) minimum for the \(\gamma'-\text{SIVP}_p\).

Given a basis for the \(\gamma\)-CVP\(_p\) instance as \(\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n\) and the target vector \(\mathbf{t}\), we construct the basis for \(\mathcal{L}'\):

\[
\begin{bmatrix}
\mathbf{b}_1 \\
\mathbf{b}_2 \\
\vdots \\
\mathbf{b}_n \\
0 \\
0 \\
\vdots \\
0 \\
r
\end{bmatrix}
\]
where \( r \) is some value that we are able to tweak — we will choose \( r \) such that \( r^p = \frac{2^p \gamma^p}{2^p - 1 + \gamma^p} \). We will firstly analyse how the YES and NO instances of \( \gamma \text{-CVP}_p^p \) translate into the corresponding YES and NO instances of \( \gamma' \text{-SIVP}_p \), and will then show that there exist possible values for \( r \) such that the reduction holds.

Recall that in \( \gamma \text{-CVP}_p^p \), in the YES instances are when the shortest possible distance from the target vector \( \mathbf{t} \) to the given lattice is less than or equals to \( d \), and otherwise in the NO instances the shortest possible distance from the target vector \( \mathbf{t} \) is at least \( \gamma d \). Then in the resulting instance, we obtain the following inequalities:

\[
\begin{align*}
\text{YES} & : \text{dist}(L, \mathbf{t}) \leq d \\
\text{NO} & : \text{dist}(L, \mathbf{t}) > \gamma d
\end{align*}
\]

Let \( \mathbf{v} \) be the vector closest to the target \( \mathbf{t} \). Let \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) be a set of linearly independent vectors in \( L \) such that

\[
\max(||\mathbf{v}_1||, \ldots, ||\mathbf{v}_n||)
\]
is minimized.

Notice that \( \mathbf{v}_1, \ldots, \mathbf{v}_n, (\mathbf{v} - \mathbf{t}, r)^T \) is a set of linearly independent vectors in \( L' \). Thus, if the CVP instance is a YES instance, \( \lambda'_{n+1} \) is upper bounded by the maximum of \( (d^p + r^p)^{1/p} \) and \( \lambda_n \).

Also, any set of linearly independent vectors must have at least one vector with a non-zero coefficient for the last vector \( (\mathbf{t}, r)^T \). So, if the CVP instance is a NO instance, then if the coefficient is 1 or \(-1\), then the length of the vector is at least \( (\gamma^p \cdot d^p + r^p)^{1/p} \), and if the coefficient has absolute value at least \( 2 \), then the length is at least \( 2r \).

From this we obtain:

\[
\begin{align*}
\text{YES} & : \lambda'_{n+1} \leq \max(d^p \cdot r, d^p + r^p) \\
\text{NO} & : \lambda'_{n+1} > \min((\gamma d)^p + r^p, (2r)^p)
\end{align*}
\]

For all cases, we will pick \( r^p \) to be \( \frac{2^p \gamma^p}{2^p - 1 + \gamma^p} \), it will always be the case that

\[
\gamma^p \leq \min \left( \frac{(\gamma d)^p + r^p}{d^p + r^p}, \frac{2^p r^p}{d^p + r^p} \right)
\]

**CASE 1:** \( \tau \leq 1 \). Since \( r^p + d^p \geq d^p \tau \), then we get that \( \gamma^p \leq \frac{\gamma^p 2^p}{2^p - 1 + \gamma^p} \).

**CASE 2:** \( 1 + \frac{\gamma^p}{2^p - 1} \geq \tau > 1 \). The in the YES case, we have that \( \lambda'_{n+1} \leq \max(d^p + d^p \frac{\gamma^p}{2^p - 1}, d^p + r^p) \).

Ergo, by our choice of \( r^p \) again, we get \( \gamma^p \leq \frac{\gamma^p 2^p}{2^p - 1 + \gamma^p} \).

**CASE 3:** \( \tau > 1 + \frac{\gamma^p}{2^p - 1} \). In this case we have that \( r^p \geq d^p(\tau - 1) \). So \( d^p + r^p \geq \tau d^p \). Then we have that \( \gamma^p \) is upper bounded by:

\[
\min \left\{ \frac{\gamma^p + \tau - 1}{\tau}, \frac{2^p(\tau - 1)}{\tau} \right\}
\]

This reduction is clearly runs in polynomial time. \( \square \)

From this, we can conclude that if we were to set \( r^p \) to \( \frac{d^p + \gamma^p}{2^p - 1} \), we would get that \( \gamma^p < \frac{2^p \gamma^p}{2^p - 1 + \gamma^p} \).
Theorem 10. Under the randomised Gap Exponential Time Hypothesis, there exists $\gamma' > 1$, $\epsilon > 0$ such that $\gamma'$-SIVP with rank $n$ is not solvable in $2^{\epsilon n}$ time.

Proof. This can be achieved by considering of the instances throughout the chain of reductions from $(\delta, \epsilon)$-Gap-3-SAT to $(\delta', \epsilon')$-Gap-2-SAT to $\gamma$-GapCVP and finally $\gamma'$-SIVP.

In the original $(\delta, \epsilon)$-Gap-3-SAT instance with $n$ variables and $m$ clauses, we obtain a $\gamma'$-SIVP with rank $n + m + 1$ with high probability. Thus under the randomised Gap-ETH, there is no sub-exponential time algorithm for $\gamma'$-SIVP, for all $p \in [1, \infty)$.

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