ON THE STEINNESS INDEX

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Abstract. We introduce the concept of Steinness index related to the Stein neighborhood basis. We then show several results: (1) The existence of Steinness index is equivalent to that of strong Stein neighborhood basis. (2) On the Diederich-Fornæss worm domains in particular, we present an explicit formula relating the Steinness index to the well-known Diederich-Fornæss index. (3) The Steinness index is 1 if a smoothly bounded pseudoconvex domain admits finitely many boundary points of infinite type.

1. Introduction

Let $\Omega \subset \mathbb{C}^n (n \geq 2)$ be a bounded domain with smooth boundary. A smooth function $\rho$ defined on a neighborhood $V$ of $\overline{\Omega}$ is called a (global) defining function of $\Omega$ if $\Omega = \{z \in V : \rho(z) < 0\}$ and $d\rho(z) \neq 0$ for all $z \in \partial \Omega$.

1.1. Diederich-Fornæss Index. The Diederich-Fornæss exponent of $\rho$ is defined by

$$\eta_\rho := \sup \{ \eta \in (0, 1) : -(-\rho)^\eta \text{ is strictly plurisubharmonic on } \Omega \}.$$

If there is no such $\eta$, then we define $\eta_\rho = 0$. The Diederich-Fornæss index of $\Omega$ is defined by

$$DF(\Omega) := \sup \eta_\rho,$$

where the supremum is taken over all defining functions $\rho$. We say that the Diederich-Fornæss index of $\Omega$ exists if $DF(\Omega) \in (0, 1]$. If $DF(\Omega)$ exists, then there exists a bounded strictly plurisubharmonic exhaustion function on $\Omega$. In other words, $\Omega$ becomes a hyperconvex domain. In 1977, Diederich and Fornæss ([3]) proved that $C^2$-smoothness of $\partial \Omega$ implies the existence of $DF(\Omega)$.

1.2. Steinness Index. We introduce the following definition. The Steinness exponent of $\rho$ is defined by

$$\tilde{\eta}_\rho := \inf \{ \tilde{\eta} > 0 : \rho^{\tilde{\eta}} \text{ is strictly plurisubharmonic on } \overline{\Omega} \setminus U \text{ for some neighborhood } U \text{ of } \partial \Omega \},$$

where $\overline{\Omega} ^\circ := \mathbb{C}^n \setminus \overline{\Omega}$. If there is no such $\tilde{\eta}$, then we define $\tilde{\eta}_\rho = \infty$. The Steinness index of $\Omega$ is defined by

$$S(\Omega) := \inf \tilde{\eta}_\rho.$$
where the infimum is taken over all defining functions $\rho$. We say that the Steinness index of $\Omega$ exists if $S(\Omega) \in [1, \infty)$. $\Omega$ is said to have a Stein neighborhood basis if for any neighborhood $V_1$ of $\Omega$, there exists a pseudoconvex domain $V_2$ such that $\Omega \subset V_2 \subset V_1$. If $S(\Omega)$ exists, then there exist a defining function $\rho$ and $\eta_2 \in (1, \infty)$ such that $\rho^{\eta_2}$ is strictly plurisubharmonic on $\overline{\Omega} \cap U$. Thus $\overline{\Omega}$ has a Stein neighborhood basis. In contrast with Diederich-Fornæss index, the smoothness of boundary does not implies the existence of $S(\Omega)$; Diederich-Fornæss worm domains provide an example ([4]). In section 3, we characterize the Steinness index by means of a differential inequality on the set of all weakly pseudoconvex boundary points (Theorem 3.1). This theorem plays a crucial role in this paper.

1.3. Strong Stein neighborhood basis. $\overline{\Omega} \subset\subset \mathbb{C}^n$ is said to have a strong Stein neighborhood basis if there exist a defining function $\rho$ of $\Omega$ and $\epsilon_0 > 0$ such that $\Omega_\epsilon := \{ z \in \mathbb{C}^n : \rho(z) < \epsilon \}$ is pseudoconvex for all $0 \leq \epsilon < \epsilon_0$. This implies the existence of a Stein neighborhood basis. In section 4, we show that the existence of $S(\Omega)$ is actually equivalent to the existence of a strong Stein neighborhood basis (Theorem 4.1).

1.4. Worm domains. In 1977, Diederich and Fornæss ([4]) constructed bounded smooth domains in $\mathbb{C}^2$ whose Diederich-Fornæss indices are strictly less than one. These examples are called worm domains and the only known domains in $\mathbb{C}^n$ which have non-trivial Diederich-Fornæss indices. Therefore, worm domains are worth to be studied.

Recently, Liu calculated the exact value of the Diederich-Fornæss index of worm domains (Definition 5.1) in 2017. In section 5, exploiting the idea of [10], we obtain a calculation of the exact values of the Steinness index of worm domains (Theorem 5.3). More precisely, the result is as follows.

**Theorem 1.1.** If $\Omega_\beta$ ($\beta > \frac{\pi}{2}$) is a worm domain, then the following 4 conditions are equivalent:

1. $\frac{1}{p} < DF(\Omega_\beta) < 1$.
2. $\Omega_\beta$ admits the Steinness index.
3. $\Omega_\beta$ admits the Stein neighborhood basis.
4. $\Omega_\beta$ admits a strong Stein neighborhood basis.

Moreover, if one of the above conditions holds, then

$$\frac{1}{DF(\Omega_\beta)} + \frac{1}{S(\Omega_\beta)} = 2.$$ 

1.5. Sufficient conditions for $DF(\Omega) = 1$ and $S(\Omega) = 1$. If a bounded domain $\Omega \subset \mathbb{C}^n$ is strongly pseudoconvex, then there exists a strictly plurisubharmonic defining function of $\Omega$, which implies $DF(\Omega) = 1$ and $S(\Omega) = 1$.

In fact, more is known: If a smoothly bounded pseudoconvex domain $\Omega$ is B-regular (i.e., for every $p \in \partial \Omega$, there exists a continuous peak function at $p$), then $DF(\Omega) = 1$ and $S(\Omega) = 1$ ([12], [13]). Since finite type domains in the sense of D’Angelo are B-regular by Catlin ([2]), both indices are equal to 1. In section 6, we give an alternative
proof of this fact using modified Theorem 3.1 (Corollary 6.4). Moreover, if the set of all infinite type boundary points is finite, then the Steinness index is 1 (Corollary 6.5).

In section 7, we also demonstrate that if \( \Omega \subset \subset \mathbb{C}^n \) is a \( C^1 \)-smooth convex domain, then \( DF(\Omega) = 1 \) and \( S(\Omega) = 1 \) (Corollary 7.2). For further results, we refer the reader to [5], [6], [7], [8] and [9].

2. Preliminary

We first fix the notation of this paper, unless otherwise mentioned.

• \( \Omega \) : a bounded pseudoconvex domain with smooth boundary in \( \mathbb{C}^n \).
• \( \rho \) : a defining function of \( \Omega \).
• \( \Sigma \) : the set of all weakly pseudoconvex points in \( \partial \Omega \).
• \( \Sigma_{\infty} \) : the set of all infinite type points in \( \partial \Omega \).
• \( g \) : the standard Euclidean complex Hermitian metric in \( \mathbb{C}^n \).
• \( \nabla \) : the Levi-Civita connection of \( g \).
• \( \nabla \rho \) : the real gradient of \( \rho \).
• \( U \) : a tubular neighborhood of \( \partial \Omega \).
• \( L_{\rho} \) : the Levi-form of \( \rho \).

Define

\[
N_{\rho} := \frac{1}{\sqrt{\sum_{j=1}^{n} \left| \frac{\partial \rho}{\partial z_j} \right|^2}} \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} \frac{\partial}{\partial z_j}.
\]

Let \( J \) be the complex structure of \( \mathbb{C}^n \) and \( T_p(\partial \Omega) \) be the real tangent space of \( \partial \Omega \) at \( p \in \partial \Omega \). Let \( T_p^c(\partial \Omega) := J(T_p(\partial \Omega)) \cap T_p(\partial \Omega) \). Then the complexified tangent space of \( T_p^c(\partial \Omega) \), \( \mathbb{C}T_p^c(\partial \Omega) := \mathbb{C} \otimes T_p^c(\partial \Omega) \), can be decomposed into the holomorphic tangent space \( T_p^{1,0}(\partial \Omega) \) and the anti-holomorphic tangent space \( T_p^{0,1}(\partial \Omega) \). We call \( X \) a \((1,0)\) tangent vector if \( X \in T_p^{1,0}(\partial \Omega) \).

Let \( X, Y, Z \) be complex vector fields in \( \mathbb{C}^n \). A direct calculation implies the following properties.

\[
g(N_{\rho}, N_{\rho}) = \frac{1}{2}, \quad N_{\rho} \rho = \frac{1}{\sqrt{\sum_{j=1}^{n} \left| \frac{\partial \rho}{\partial z_j} \right|^2}} \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j} \frac{\partial}{\partial z_j} = \frac{\|\nabla \rho\|}{2},
\]

\[
N_{\rho} + \nabla_{\rho} = 2 \text{Re}(N_{\rho}) = \frac{\nabla \rho}{\|\nabla \rho\|},
\]

\[
\mathcal{L}_{\rho}(X, Y) = g(\nabla_X \nabla \rho, Y) = X(\nabla \rho) - (\nabla_X \nabla Y) \rho,
\]

\[
Zg(X, Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y), \quad Z \rho = g(\nabla \rho, Z),
\]

\[
\nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X,Y]} = 0.
\]

For \( p \in \partial \Omega \) whenever we mention \( \lim_{z \to p} \), it means \( z \) approaches \( p \) along the real normal direction, and “smooth” means \( C^\infty \)-smooth, although \( C^3 \)-smoothness suffices for our purpose.

Now, we introduce two lemmas which we will use in the next section.
Lemma 2.1 (\cite{10}). Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary, and $\rho$ be a defining function of $\Omega$. Suppose that $L$ is a $(1,0)$ tangent vector field so that $L^j(L, L) = 0$ at $p \in \partial \Omega$. Assume that $T_j$ $(1 \leq j \leq n-2)$ are $(1,0)$ tangent vector fields and $T_1, T_2, \cdots, T_{n-2}, L$ are orthogonal at $p$. Then $L^j(L, T_j) = 0$ for $1 \leq j \leq n-2$ at $p$.

Lemma 2.2. Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary, and $\rho$ be a defining function of $\Omega$. Let $\psi$ be a smooth function defined on $\Omega$ and $\rho = \rho e^{\psi}$. We denote $\tilde{N} = N_{\tilde{r}}$, $N = N_\rho$. Let $\tilde{L}$, $L$ be $(1,0)$ tangent vector fields on $\Omega$ such that $\tilde{L} = L$ on $\partial \Omega$, and $\tilde{L}\rho = 0$, $L\rho = 0$ on $\Omega$. Define

$$\Sigma_L := \{p \in \partial \Omega : L^j(L, L)(p) = 0\}.$$

Then

$$\frac{L^j(\tilde{L}, \tilde{N})}{\|\nabla \tilde{\rho}\|} = \frac{L^j(L, N)}{\|\nabla \rho\|} + \frac{1}{2} (L\psi)$$

and

$$\tilde{N}L^j(\tilde{L}, \tilde{L}) = \frac{N L^j(L, L)}{\|\nabla \rho\|} + \frac{1}{2} L^j(\psi, L) - \frac{1}{2} |L\psi|^2 - 2Re \left[ \frac{L^j(L, N)}{\|\nabla \rho\|} (\tilde{\psi}) \right]$$

on $\Sigma_L$. In particular,

$$\frac{|L^j(\tilde{L}, \tilde{N})|^2}{\|\nabla \tilde{\rho}\|^2} + \frac{1}{2} \frac{|N L^j(L, L)|}{\|\nabla \rho\|^2} = \frac{|L^j(L, N)|^2}{\|\nabla \rho\|^2} + \frac{1}{2} \frac{N L^j(L, L)}{\|\nabla \rho\|} + \frac{1}{4} L^j(\psi, L, L)$$

on $\Sigma_L$.

Proof. Since $\tilde{L} = L$, $\tilde{N} = N$, and $\|\nabla \tilde{\rho}\| = e^{\psi}\|\nabla \rho\|$ on $\partial \Omega$,

$$L^j(\tilde{L}, \tilde{N}) = L^j(L, N) = e^{\psi} L^j(L, N) + \rho L^j(\psi, L) + (L\rho)(N e^{\psi}) + (Le^{\psi})(N \rho) = e^{\psi} L^j(L, N) + e^{\psi} L^j(\psi).$$

Therefore, we have

$$\frac{L^j(\tilde{L}, \tilde{N})}{\|\nabla \tilde{\rho}\|} = \frac{L^j(L, N)}{\|\nabla \rho\|} + \frac{1}{2} (L\psi)$$

on $\Sigma_L$. Now, we prove (2.1) holds at $p \in \Sigma_L$.

$$\tilde{N}L^j(\tilde{L}, \tilde{L}) = N L^j(L, L) = g(\nabla \tilde{L}, \nabla \tilde{L}) + g(\nabla N \nabla \tilde{L}, \tilde{L}) + g(\nabla L \nabla \tilde{L}, \nabla \tilde{L})$$

If $L$ vanishes at $p$, then the last three terms of (2.2) are all zero and so $\tilde{N}L^j(\tilde{L}, \tilde{L}) = N L^j(L, L) = 0$. Thus (2.1) holds at $p$. Therefore, we assume that $L \neq 0$ at $p$ and normalize $L$ and $\tilde{L}$ as $g(L, L) = g(\tilde{L}, \tilde{L}) = \frac{1}{2}$. This normalization is for convenience and
does not affect the result. Now, we will compute the last three terms of \((2.2)\) one by one. Let \(\{\sqrt{2T_1}, \ldots, \sqrt{2T_{n-2}}, \sqrt{2\tilde{L}}\}\) be an orthonormal basis of \(T_{\rho}^{1,0}(\partial\Omega)\). Then, first,
\[
g(\nabla_L \nabla \tilde{\rho}, \nabla_N \tilde{L})
= g\left(\nabla_L \nabla \tilde{\rho}, \sum_{j=1}^{n-2} g(\nabla_N \tilde{L}, \sqrt{2T_j}) \sqrt{2T_j} + g(\nabla_N \tilde{L}, \sqrt{2\tilde{L}}) \sqrt{2\tilde{L}} + g(\nabla_N \tilde{L}, \sqrt{2N}) \sqrt{2N}\right)
= \sum_{j=1}^{n-2} 2g(\nabla_N \tilde{L}, T_j) \mathcal{L}_{\tilde{\rho}}(\tilde{L}, \tilde{T_j}) + 2g(\nabla_N \tilde{L}, L) \mathcal{L}_{\tilde{\rho}}(\tilde{L}, L) + 2g(\nabla_N \tilde{L}, N) \mathcal{L}_{\tilde{\rho}}(\tilde{L}, N)
= 2g(\nabla_N \tilde{L}, N) \mathcal{L}_{\tilde{\rho}}(\tilde{L}, N)
= 2g(\nabla_N \tilde{L}, \mathcal{L}_{\tilde{\rho}}(\tilde{L}, N)) = 2 \mathcal{L}_{\tilde{\rho}}(\tilde{L}, N).
\]
Here, we used Lemma [2.1] in the third equality above. Second, by the same argument as above,
\[
g(\nabla_{[N,L]} \nabla \tilde{\rho}, \tilde{L}) = g(\nabla_L \nabla \tilde{\rho}, [N, \tilde{L}]) = 2g([N, \tilde{L}], \mathcal{L}_{\tilde{\rho}}(\tilde{L}, N)).
\]
Third,
\[
g(\nabla_L \nabla_N \nabla \tilde{\rho}, \tilde{L})
= g(\nabla_L \nabla_N \nabla \tilde{\rho}, L)
= g(\nabla_N \nabla_L \nabla \tilde{\rho}, L) - g(\nabla_{[N,L]} \nabla \tilde{\rho}, L)
= N g(\nabla_L \nabla \tilde{\rho}, L) - g(\nabla_L \nabla \tilde{\rho}, \nabla_N L) - g(\nabla_{[N,L]} \nabla \tilde{\rho}, L)
= N \mathcal{L}_{\tilde{\rho}}(L, L) - 2g(\nabla_N \tilde{L}, N) \mathcal{L}_{\tilde{\rho}}(\tilde{L}, N) - 2g([N, L], N) \mathcal{L}_{\tilde{\rho}}(\tilde{L}, N).
\]
Therefore, we have
\[
\tilde{N} \mathcal{L}_{\tilde{\rho}}(\tilde{L}, \tilde{L}) = N \mathcal{L}_{\tilde{\rho}}(L, L)
+ 2 \left(g(\nabla_N \tilde{L}, N) - g(\nabla_N \tilde{L}, \tilde{\rho})\right) \mathcal{L}_{\tilde{\rho}}(L, N)
+ 2 \left(g([N, \tilde{L}], N) - g([N, L], N)\right) \mathcal{L}_{\tilde{\rho}}(L, N).
\]
Now,
\[
\mathcal{L}_{\tilde{\rho}}(L, L) = e^{\psi} \mathcal{L}_{\rho}(L, L) + \rho \mathcal{L}_{e^{\psi}}(L, L) + (L \rho)(\mathcal{T} e^{\psi}) + (Le^{\psi})(\mathcal{T} \rho),
\]
and
\[
N \mathcal{L}_{\tilde{\rho}}(L, L) = e^{\psi} \left(N \mathcal{L}_{\rho}(L, L) + (N \rho) \mathcal{L}_{\psi}(L, L) + (N \rho)|L \psi|^2\right),
\]
and
\[
\mathcal{L}_{\rho}(N, L) = N (\mathcal{T} \rho) - (\nabla_N \mathcal{T}) \rho = - (\nabla_N \mathcal{T}) \rho
\Rightarrow - \mathcal{L}_{\rho}(L, N) = (\nabla_N \mathcal{T}) \rho = 2g(\nabla_N \tilde{L}, N) \rho = g(\nabla_N \tilde{L}, \nabla \rho)
\Rightarrow g(\nabla_N \tilde{L}, N) = - \frac{\mathcal{L}_{\rho}(L, N)}{\|\nabla \rho\|.}
By the same argument,
\[
g(\nabla_N\bar{L}, N) = \frac{\mathcal{L}_\rho(L, N)}{\|\nabla \rho\|} - \frac{\mathcal{L}_\rho(L, N)}{\|\nabla \rho\|} - \frac{1}{2} (L\psi)
\]
and
\[
g([N, L], N) = g(\nabla_N L, N) - g(\nabla_L N, N) = g(\nabla_N N, N) + g(\nabla_N \bar{L}, N) - g(\nabla_L N, N),
g([\bar{N}, L], N) = g(\nabla_N \bar{L}, N) - g(\nabla_L N, N) = g(\nabla_N \bar{N}, N) + g(\nabla_N \bar{L}, N) - g(\nabla_L N, N).
\]
Here, \(g(\nabla_L N, N) = g(\nabla_L N, N)\) on \(\Sigma_L\) and since \(N - \bar{N}\) is a real tangent vector field on \(\partial \Omega\), \(g(\nabla_N \bar{N} L, N) = g(\nabla_N \bar{N} L, N)\) on \(\Sigma_L\). Therefore,
\[
g([N, L], N) - g([N, L], N) = g(\nabla_N \bar{L}, N) - g(\nabla_N \bar{N}, N) = -\frac{1}{2} (L\psi).
\]
Combining (2.3), (2.4), and (2.5), we have
\[
\bar{N}\mathcal{L}_\psi(L, \bar{L}) = e^\psi \left( N\mathcal{L}_\rho(L, L) + (N\rho)\mathcal{L}_\psi(L, L) + (N\rho)|L\psi|^2 \right)
- (L\psi) \left( e^\psi \mathcal{L}_\rho(L, N) + e^\psi (L\psi)(N\rho) \right)
- (\rho) \left( e^\psi \mathcal{L}_\rho(L, N) + e^\psi (L\psi)(N\rho) \right)
= e^\psi \left[ N\mathcal{L}_\rho(L, L) + (N\rho)\mathcal{L}_\psi(L, L) - (N\rho)|L\psi|^2 - 2 \text{Re} (\mathcal{L}_\rho(L, N)(\overline{L}\psi)) \right].
\]
Since \(\|\nabla \rho\| = e^\psi \|\nabla \rho\|\) on \(\partial \Omega\), we conclude
\[
\frac{\bar{N}\mathcal{L}_\psi(L, \bar{L})}{\|\nabla \rho\|} = \frac{N\mathcal{L}_\rho(L, L)}{\|\nabla \rho\|} + \frac{1}{2} \mathcal{L}_\psi(L, L) - \frac{1}{2} |L\psi|^2 - 2 \text{Re} \left[ \frac{\mathcal{L}_\rho(L, N)}{\|\nabla \rho\|}(\overline{L}\psi) \right]
\]
at \(p \in \Sigma_L\). 

3. Equivalent Definition for Steinness index

In order to check whether \(S(\Omega)\) exists, by its definition, it is necessary to find a defining function \(\rho\) and \(\eta_2 > 1\) such that \(\rho^{\eta_2}\) is strictly plurisubharmonic on \(\overline{\Omega^C} \cap U\). In this section, we replace this condition on \(\overline{\Omega^C} \cap U\) to another condition on \(\Sigma\), so that we only need to check it at weakly pseudoconvex boundary points. Now, we introduce the main theorem of this section.

**Theorem 3.1.** Let \(\Omega \subset \subset \mathbb{C}^n\) be a pseudoconvex domain with smooth boundary, and \(\rho\) be a defining function of \(\Omega\). Let \(L\) be an arbitrary \((1, 0)\) tangent vector field on \(\partial \Omega\). Define
\[
\Sigma_L : = \{ p \in \partial \Omega : \mathcal{L}_\rho(L, L)(p) = 0 \}.
\]
Let $\eta_\rho$ be the infimum of $\eta_2 \in (1, \infty)$ satisfying

$$\frac{1}{\eta_2 - 1} \frac{|\mathcal{L}_\rho(L, N)|^2}{\|\nabla \rho\|^2} - \frac{1}{2} \frac{N \mathcal{L}_\rho(L, L)}{\|\nabla \rho\|} \leq 0$$

on $\Sigma_L$ for all $L$. Here, $N = N_\rho$ and we extend $L$ so that $L \rho = 0$ on $U$. Then

$$S(\Omega) = \inf \eta_\rho$$

where the infimum is taken over all smooth defining functions $\rho$.

The following sequence of lemmas are essential towards the proof of the theorem above.

**Lemma 3.2.** Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary, and $\rho$ be a defining function of $\Omega$. For $p \in \partial \Omega$, let $U_p$ be a neighborhood of $p$ in $\mathbb{C}^n$. Let $L$ be a smooth $(1, 0)$ tangent vector field in $U_p$ such that $L \rho = 0$ and $\mathcal{L}_\rho(L, L) = 0$. Fix $\eta_2 \in (1, \infty)$. We denote $N_\rho$ by $N$. Then

$$\mathcal{L}_{\rho^2}(aL + bN, aL + bN) > 0$$

for all $(a, b) \in \mathbb{C}^2 \setminus (0, 0)$ on $\overline{\Omega} \cap U_p$ implies

$$\frac{1}{\eta_2 - 1} \frac{|\mathcal{L}_\rho(L, N)|^2}{\|\nabla \rho\|^2} - \frac{1}{2} \frac{1}{\|\nabla \rho\|} \leq 0$$

at $p$.

**Proof.** By using $L \rho = 0$ and the assumption, we have

$$\mathcal{L}_{\rho^2}(aL + bN, aL + bN)$$

$$= \eta_2 \rho^{\eta_2 - 1} \left[ |a|^2 \mathcal{L}_\rho(L, L) + 2 \text{Re} \left( \overline{ab} \mathcal{L}_\rho(L, N) \right) ight] + |b|^2 \left( \mathcal{L}_\rho(N, N) + \frac{\eta_2 - 1}{\rho} |N \rho|^2 \right) > 0$$

for all $(a, b) \in \mathbb{C}^2 \setminus (0, 0)$ on $\overline{\Omega} \cap U_p$, and it is equivalent to

(3.1) \quad $|a|^2 \mathcal{L}_\rho(L, L) - 2 |a||b| |\mathcal{L}_\rho(L, N)| + |b|^2 \left( \mathcal{L}_\rho(N, N) + \frac{\eta_2 - 1}{\rho} |N \rho|^2 \right) > 0$

for all $(a, b) \in \mathbb{C}^2 \setminus (0, 0)$ on $\overline{\Omega} \cap U_p$. This is because $\eta_2 \rho^{\eta_2 - 1} > 0$ and by rotating $a$ or $b$, one can make $2 \text{Re} \left( \overline{ab} \mathcal{L}_\rho(L, N) \right) = -2 |a||b| |\mathcal{L}_\rho(L, N)|$. We may assume that $\mathcal{L}_\rho(N, N) + \frac{\eta_2 - 1}{\rho} |N \rho|^2 > 0$ on $\overline{\Omega} \cap U_p$, because it blows up as point goes to the boundary. By dividing (3.1) by $|a|^2$ and letting $x = \frac{|b|}{|a|}$, (3.1) is equivalent to

(3.2) \quad $\mathcal{L}_\rho(L, L) - 2x |\mathcal{L}_\rho(L, N)| + x^2 \left( \mathcal{L}_\rho(N, N) + \frac{\eta_2 - 1}{\rho} |N \rho|^2 \right) > 0$

for all $x \geq 0$ on $\overline{\Omega} \cap U_p$. The axis of symmetry of the quadratic equation above is

$$\frac{|\mathcal{L}_\rho(L, N)|}{\mathcal{L}_\rho(N, N) + \frac{\eta_2 - 1}{\rho} |N \rho|^2},$$
which is always positive on $\overline{\Omega} \cap U_p$. Thus, (3.2) if and only if the determinant
\begin{equation}
|\mathcal{L}_\rho(L, N)|^2 - (\mathcal{L}_\rho(L, L)) \left( \mathcal{L}_\rho(N, N) + \frac{\eta_2 - 1}{\rho} |N\rho|^2 \right) < 0
\end{equation}
onumber
on $\overline{\Omega} \cap U_p$. Now, taking a limit to $p$ on (3.3) along the real normal direction yields
\begin{equation}
|\mathcal{L}_\rho(L, N)|^2 - (\eta_2 - 1) |N\rho|^2 \lim_{z \to p} \frac{\mathcal{L}_\rho(L, L)}{\rho} \leq 0
\end{equation}
at $p \in \partial \Omega$. Here,
\begin{equation}
\lim_{z \to p} \frac{\mathcal{L}_\rho(L, L)(z)}{\rho(z)} = \frac{-(N + \overline{N})\mathcal{L}_\rho(L, L)(p)}{-(N + \overline{N})\rho(p)} = \frac{N\mathcal{L}_\rho(L, L)(p)}{N\rho(p)}.
\end{equation}
The explanation of second equality of (3.5) is following. Since $\mathcal{L}_\rho(L, L) = 0$ at $p \in \partial \Omega$, $\mathcal{L}_\rho(L, L)$ attains the local minimum at $p$ on $\partial \Omega$. Thus, the directional derivative of $\mathcal{L}_\rho(L, L)$ along the real tangent vector $N - \overline{N}$ at $p$ is zero, and $(N - \overline{N})\mathcal{L}_\rho(L, L) = 0$ implies $N\mathcal{L}_\rho(L, L) = \overline{N}\mathcal{L}_\rho(L, L)$ at $p$. Finally, since $N\rho = \frac{\|\nabla\rho\|}{2}$, (3.4) is equivalent to
\[
\frac{1}{\eta_2 - 1} \frac{|\mathcal{L}_\rho(L, N)|^2}{\|\nabla\rho\|^2} - \frac{1}{2} \frac{N\mathcal{L}_\rho(L, L)}{\|\nabla\rho\|} \leq 0
\]
at $p \in \partial \Omega$. \hfill \Box

Lemma 3.3. Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary, and $\rho$ be a defining function of $\Omega$. Let $\{U_\alpha\}$ be a chart of $U$ and $L$ be a non-vanishing smooth $(1, 0)$ tangent vector field in $U_\alpha$ such that $L\rho = 0$. We normalize $L$ as $g(L, L) = \frac{1}{2}$. Fix $\eta_2 \in (1, \infty)$. We denote $N\rho$ by $N$. Define
\[
\Sigma_L^\alpha := \{ p \in \partial \Omega \cap U_\alpha : \mathcal{L}_\rho(L, L)(p) = 0 \}.
\]
If
\begin{equation}
\frac{1}{\eta_2 - 1} \frac{|\mathcal{L}_\rho(L, N)|^2}{\|\nabla\rho\|^2} - \frac{1}{2} \frac{N\mathcal{L}_\rho(L, L)}{\|\nabla\rho\|} < 0
\end{equation}
onumber
on $\Sigma_L^\alpha$, then there exists a neighborhood $V_L^\alpha$ of $\Sigma_L^\alpha$ in $U_\alpha$ such that
\[
\mathcal{L}_{\rho^a}(aL + bN, aL + bN) > 0
\]
for all $(a, b) \in \mathbb{C}^2 \setminus (0, 0)$ on $\overline{\Omega} \cap V_L^\alpha$.

Proof. First, note that the assumption (3.6) is equivalent to
\[
|\mathcal{L}_\rho(L, N)|^2 - (\eta_2 - 1)(N\rho)(N\mathcal{L}_\rho(L, L)) < 0
\]
on $\Sigma_L^\alpha$. Now define $F : \overline{\Omega} \cap U_\alpha \to \mathbb{R}$ by
\[
F(z) := |\mathcal{L}_\rho(L, N)(z)|^2 - (\mathcal{L}_\rho(L, L)(z) - \mathcal{L}_\rho(L, L)(p)) \mathcal{L}_\rho(N, N)(z)
- (\eta_2 - 1)\frac{\mathcal{L}_\rho(L, L)(z) - \mathcal{L}_\rho(L, L)(p)}{\rho(z)} |N\rho(z)|^2
\]
for all \( z \in \overline{\Omega} \cap U_\alpha \), where \( p \in \partial \Omega \cap U_\alpha \) is the closest point to \( z \), and
\[
F(z) := |\mathcal{L}_\rho(L, N)(z)|^2 - (\eta_2 - 1)(N \rho(z))(N \mathcal{L}_\rho(L, L)(z))
\]
for all \( z \in \partial \Omega \cap U_\alpha \). Since \( \lim_{z \to p} F(z) = F(p) \) for all \( p \in \partial \Omega \cap U_\alpha \), \( F \) is continuous on \( \Omega^c \cap U_\alpha \). By the assumption \( \mathcal{L}_\rho \), there exists a neighborhood \( V^\alpha_L \) of \( \Sigma^\alpha_L \) in \( U_\alpha \) such that
\[
|\mathcal{L}_\rho(L, N)(z)|^2 - (\mathcal{L}_\rho(L, L)(z) - \mathcal{L}_\rho(L, L)(p)) \mathcal{L}_\rho(N, N)(z)
- (\eta_2 - 1) \frac{\mathcal{L}_\rho(L, L)(z) - \mathcal{L}_\rho(L, L)(p)}{\rho(z)}|N \rho(z)|^2
\]
for all \( z \in \Omega^c \cap V^\alpha_L \). We may assume that \( \mathcal{L}_\rho(N, N)(z) + \frac{\eta_2 - 1}{\rho(z)}|N \rho(z)|^2 > 0 \) for all \( z \in \Omega^c \cap V^\alpha_L \), because it blows up as \( z \) goes to the boundary. Therefore, the following quadratic function
\[
(\mathcal{L}_\rho(L, L)(z) - \mathcal{L}_\rho(L, L)(p)) - 2x|\mathcal{L}_\rho(L, N)(z)|
+ x^2 \left( \mathcal{L}_\rho(N, N)(z) + \frac{\eta_2 - 1}{\rho(z)}|N \rho(z)|^2 \right) > 0
\]
for all \( z \in \Omega^c \cap V^\alpha_L \), \( x \geq 0 \). By letting \( x = \frac{|b|}{|a|} \) and multiplying \( |a|^2 \), we have
\[
|a|^2 (\mathcal{L}_\rho(L, L)(z) - \mathcal{L}_\rho(L, L)(p)) - 2|a||b| |\mathcal{L}_\rho(L, N)(z)|
+ |b|^2 \left( \mathcal{L}_\rho(N, N)(z) + \frac{\eta_2 - 1}{\rho(z)}|N \rho(z)|^2 \right) > 0
\]
for all \( z \in \Omega^c \cap V^\alpha_L \), \( (a, b) \in \mathbb{C}^2 \setminus (0, 0) \). If \( |a| = 0 \), then \( \mathcal{L}_\rho(N, N)(z) + \frac{\eta_2 - 1}{\rho(z)}|N \rho(z)|^2 > 0 \), which is automatically satisfied. Now, \( \mathcal{L}_\rho \) implies
\[
|a|^2 (\mathcal{L}_\rho(L, L)(z) - \mathcal{L}_\rho(L, L)(p)) + 2 \text{Re} (a \overline{b} (\mathcal{L}_\rho(L, N)(z)))
+ |b|^2 \left( \mathcal{L}_\rho(N, N)(z) + \frac{\eta_2 - 1}{\rho(z)}|N \rho(z)|^2 \right) > 0
\]
for all \( z \in \Omega^c \cap V^\alpha_L \), \( (a, b) \in \mathbb{C}^2 \setminus (0, 0) \) because \( \text{Re} (a \overline{b} (\mathcal{L}_\rho(L, N))) > -|a||b| |\mathcal{L}_\rho(L, N)| \). Since \( \Omega \) is pseudoconvex, \( \mathcal{L}_\rho(L, L)(p) \geq 0 \) for all \( p \in \partial \Omega \), and thus
\[
|a|^2 (\mathcal{L}_\rho(L, L)(z)) + 2 \text{Re} (a \overline{b} (\mathcal{L}_\rho(L, N)(z)))
+ |b|^2 \left( \mathcal{L}_\rho(N, N)(z) + \frac{\eta_2 - 1}{\rho(z)}|N \rho(z)|^2 \right) > 0
\]
for all \( z \in \Omega^c \cap V^\alpha_L \), \( (a, b) \in \mathbb{C}^2 \setminus (0, 0) \). Finally, by using \( L \rho = 0 \), \( \mathcal{L}_\rho \) is equivalent to
\[
\mathcal{L}_\rho(aL + bN, aL + bN)(z) > 0
\]
for all \( z \in \Omega^c \cap V_\alpha \), \((a, b) \in \mathbb{C}^2 \setminus (0, 0)\).

\[ \square \]

**Lemma 3.4.** Let \( \Omega \subset \subset \mathbb{C}^n \) be a pseudoconvex domain with smooth boundary, and \( \rho \) be a defining function of \( \Omega \). Let \( \{U_\alpha\} \) be a chart of \( U \) and \( L \) be a non-vanishing smooth \((1, 0)\) tangent vector field in \( U_\alpha \) such that \( L \rho = 0 \). We normalize \( L \) as \( g(L, L) = \frac{1}{2} \). Fix \( \eta_2 \in (1, \infty) \). We denote \( N_\rho \) by \( N \). Define
\[
\Sigma^\alpha_L := \{ p \in \partial \Omega \cap U_\alpha : \mathcal{L}_\rho(L, L)(p) = 0 \}.
\]

If
\[
\frac{1}{\eta_2 - 1} \left| \frac{\mathcal{L}_\rho(L, N)}{\|
abla \rho\|^2} - \frac{1}{2} \frac{N \mathcal{L}_\rho(L, L)}{\|
abla \rho\|} \right| \leq 0
\]
on \( \Sigma^\alpha_L \), then there exists a defining function \( \tilde{\rho} \) such that for any small \( \nu > 0 \) so that \( \frac{1}{\eta_2 - 1} - \nu > 0 \),
\[
\frac{1}{\tilde{\eta}_2 - 1} \left| \frac{\mathcal{L}_{\tilde{\rho}}(\tilde{L}, \tilde{N})}{\|
abla \tilde{\rho}\|^2} - \frac{1}{2} \frac{N \mathcal{L}_{\tilde{\rho}}(\tilde{L}, \tilde{L})}{\|
abla \tilde{\rho}\|} \right| < 0
\]
on \( \Sigma^\alpha_L \), where \( \tilde{\eta}_2 = 1 + \frac{1}{\eta_2 - 1} - \nu \) and \( \tilde{N} = N_{\tilde{\rho}} \). \( \tilde{L} \) is a non-vanishing smooth \((1, 0)\) tangent vector field in \( U_\alpha \) such that \( \tilde{L} = L \) on \( \partial \Omega \cap U_\alpha \), and \( \tilde{L} \rho = 0 \) on \( U_\alpha \).

**Proof.** First, notice that \( \frac{1}{\eta_2 - 1} = \frac{1}{\eta_2 - 1} - \nu \). Define \( \tilde{\rho} = \rho e^{\epsilon \psi} \), where \( \psi = \|z\|^2 \), \( \epsilon \) is a small positive number and we will decide \( \epsilon \) later. Then by Lemma 2.2
\[
\frac{1}{\tilde{\eta}_2 - 1} \left| \frac{\mathcal{L}_{\tilde{\rho}}(\tilde{L}, \tilde{N})}{\|
abla \tilde{\rho}\|^2} - \frac{1}{2} \frac{N \mathcal{L}_{\tilde{\rho}}(\tilde{L}, \tilde{L})}{\|
abla \tilde{\rho}\|} \right| = \left( \frac{1}{\eta_2 - 1} - \nu \right) \frac{|\mathcal{L}_\rho(L, N)|^2}{\|
abla \rho\|^2} - \frac{1}{2} \frac{N \mathcal{L}_\rho(L, L)}{\|
abla \rho\|}
\]
\[
+ \left( \frac{1}{\eta_2 - 1} + 1 - \nu \right) \left( \frac{\epsilon^2}{4} |L \psi|^2 + \epsilon \Re \left[ \frac{\mathcal{L}_\rho(L, N)}{\|
abla \rho\|} (\overline{L} \psi) \right] \right) - \frac{\epsilon}{8}
\]
\[
\leq \left( \frac{1}{\eta_2 - 1} - \nu \right) \frac{|\mathcal{L}_\rho(L, N)|^2}{\|
abla \rho\|^2} - \frac{1}{2} \frac{N \mathcal{L}_\rho(L, L)}{\|
abla \rho\|}
\]
\[
+ \left( \frac{1}{\eta_2 - 1} + 1 - \nu \right) \frac{\epsilon^2}{4} |L \psi|^2 + \frac{\nu}{2} \frac{|\mathcal{L}_\rho(L, N)|^2}{\|
abla \rho\|^2} + \frac{\epsilon}{8} \leq \frac{1}{2} \frac{N \mathcal{L}_\rho(L, L)}{\|
abla \rho\|}
\]
\[
- \frac{\nu}{2} \frac{|\mathcal{L}_\rho(L, N)|^2}{\|
abla \rho\|^2} + \left( \frac{1}{\eta_2 - 1} + 1 - \nu \right) \left( \frac{1}{4} + \frac{1}{2\nu} \right) |L \psi|^2 \epsilon^2 - \frac{\epsilon}{8}
\]
on \( \Sigma^\alpha_L \). Now \( \frac{1}{\eta_2 - 1} \frac{|\mathcal{L}_\rho(L, N)|^2}{\|
abla \rho\|^2} - \frac{1}{2} \frac{N \mathcal{L}_\rho(L, L)}{\|
abla \rho\|} \leq 0 \) by the assumption, and \( \frac{\nu}{2} \frac{|\mathcal{L}_\rho(L, N)|^2}{\|
abla \rho\|^2} \leq 0 \). We choose sufficiently small \( \epsilon > 0 \) so that \( \left( \frac{1}{\eta_2 - 1} + 1 - \nu \right) \left( \frac{1}{4} + \frac{1}{2\nu} \right) |L \psi|^2 \epsilon^2 - \frac{\epsilon}{8} < 0 \). All together, we conclude that
ON THE STEINNESS INDEX

\[ \frac{1}{\eta_2 - 1} \frac{|\mathcal{L}_\rho(\tilde{L}, \tilde{N})|^2}{\|\nabla \rho\|^2} - \frac{1}{\eta_2} \frac{\tilde{N} \cdot \mathcal{L}_\rho(\tilde{L}, \tilde{L})}{\|\nabla \rho\|^2} < 0 \]
on $\Sigma_L$. \hfill \Box

Lemma 3.5. Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary, and $\Pi$ be the set of all strongly pseudoconvex points in $\partial \Omega$. Then for any defining function $\rho$ of $\Omega$ and $\eta_2 \in (1, \infty)$, there exists a neighborhood $V$ of $\Pi$ in $\mathbb{C}^n$ such that $\rho^{\eta_2}$ is strictly plurisubharmonic on $\Omega^c \cap V$.

Proof. Let $V$ be a neighborhood of $\Pi$ in $\mathbb{C}^n$, and $\{V_\alpha\}$ be a chart of $V$. We denote $N = N_\rho$. It is enough to show that

\[ (3.9) \quad \mathcal{L}_{\rho^{\eta_2}}(aL + bN, aL + bN) > 0 \]

for all $(a, b) \in \mathbb{C}^2 \setminus (0, 0)$ on $\Omega^c \cap V_\alpha$, where $L$ is an arbitrary non-vanishing smooth $(1, 0)$ tangent vector field in $V_\alpha$. As in the proof of Lemma 3.2, (3.9) is equivalent to

\[ (3.10) \quad |\mathcal{L}_\rho(L, N)|^2 - (\mathcal{L}_\rho(L, L)) \left( \mathcal{L}_\rho(N, N) + \frac{\eta_2 - 1}{\rho} |N\rho|^2 \right) < 0 \]
on $\Omega^c \cap V_\alpha$. After possibly shrinking $V$, (3.10) holds because $\mathcal{L}_\rho(L, L) > 0$ on $\Pi$ and $\mathcal{L}_\rho(N, N) + \frac{\eta_2 - 1}{\rho} |N\rho|^2$ blows up as point approaches to $\partial \Omega$. \hfill \Box

Proof of Theorem 3.1. First, we prove $S(\Omega) \geq \inf \eta_\rho$. For a defining function $\rho$ and $\eta_2 \in (1, \infty)$, if $\rho^{\eta_2}$ is strictly plurisubharmonic on $\Omega^c \cap U$, then

\[ \mathcal{L}_{\rho^{\eta_2}}(aL + bN, aL + bN) > 0 \]

for all $(a, b) \in \mathbb{C}^2 \setminus (0, 0)$ on $\Omega^c \cap U$. By Lemma 3.2 this implies that

\[ \frac{1}{\eta_2 - 1} \frac{|\mathcal{L}_\rho(L, N)|^2}{\|\nabla \rho\|^2} - \frac{1}{\eta_2} \frac{\tilde{N} \cdot \mathcal{L}_\rho(L, L)}{\|\nabla \rho\|^2} \leq 0 \]
on $\Sigma_L$ for all $L$. Therefore, $S(\Omega) \geq \inf \eta_\rho$.

Next, we prove $S(\Omega) \leq \inf \eta_\rho$. By Lemma 3.5 we only need to consider it in a neighborhood of $\Sigma$ in $\mathbb{C}^n$. Let $\eta_0 = \inf \eta_\rho$. Fix $\tilde{\eta}_2 > \eta_0$ and choose small $\nu > 0$ so that

\[ \eta_2 := 1 + \frac{1}{\eta_2 - 1 + \nu} \quad (\Leftrightarrow \tilde{\eta}_2 = 1 + \frac{1}{\eta_2 - 1 + \nu}) \]

satisfies $\tilde{\eta}_2 > \eta_2 > \eta_0$. Let $\{U_\alpha\}$ be a chart of $U$ and $L$ be a non-vanishing smooth $(1, 0)$ tangent vector field in $\partial \Omega \cap U_\alpha$ such that $g(L, L) = \frac{1}{2}$. Since $\eta_2 > \eta_0$, there exists $\rho$ such that

\[ \frac{1}{\eta_2 - 1} \frac{|\mathcal{L}_\rho(L, N)|^2}{\|\nabla \rho\|^2} - \frac{1}{\eta_2} \frac{\tilde{N} \cdot \mathcal{L}_\rho(L, L)}{\|\nabla \rho\|^2} \leq 0 \]
on $\Sigma_L^\alpha$ for all $L$. Here, we extend $L$ so that $L\rho = 0$ on $U_\alpha$. By Lemma 3.3, there exists a defining function $\tilde{\rho}$ such that
\[
\frac{1}{\eta_2 - 1} \frac{\left| L\rho(\tilde{L}, \tilde{N}) \right|^2}{\|\nabla\tilde{\rho}\|^2} - \frac{1}{2} \frac{\tilde{N}. L\rho(\tilde{L}, \tilde{L})}{\|\nabla\tilde{\rho}\|^2} < 0
\]
on $\Sigma_L^\alpha$ for all $L$. Since $\eta_2$ is arbitrary, by Lemma 3.3 we have $S(\Omega) \leq \inf \eta_\rho$. □

Together with Lemma 2.2, we have

**Corollary 3.6.** Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary, and $\rho$ be a defining function of $\Omega$. Let $L$ be an arbitrary $(1, 0)$ tangent vector field on $\partial \Omega$. Define
\[
\Sigma_L := \{ p \in \partial \Omega : L\rho(L, L)(p) = 0 \}.
\]
Let $\psi$ be a smooth function defined on $U$, and $\eta_\psi$ be the infimum of $\eta_2 \in (1, \infty)$ satisfying
\[
\left( \frac{1}{\eta_2 - 1} + 1 \right) \frac{\left| L\rho(L, N) \right|^2}{\|\nabla\rho\|^2} + \frac{1}{2} (L\psi)^2 - \left( \frac{\left| L\rho(L, N) \right|^2}{\|\nabla\rho\|^2} + \frac{1}{2} \frac{N.L\rho(L, L)}{\|\nabla\rho\|^2} \right) - \frac{1}{4} L\rho(L, L) \leq 0
\]
on $\Sigma_L$ for all $L$. Here, $N = N_\rho$ and we extend $L$ so that $L\rho = 0$ on $U$. Then
\[
S(\Omega) = \inf \eta_\psi,
\]
where the infimum is taken over all smooth functions $\psi$.

### 4. Strong Stein neighborhood basis

In 2012, Sahutoğlu ([11]) gave several characterizations for $\Omega$ to have a strong Stein neighborhood basis. In this section, using one of the characterizations, we prove that the existence of the Steinness index and the existence of a strong Stein neighborhood basis are equivalent.

**Theorem 4.1.** Let $\Omega \subset \subset \mathbb{C}^n$ be a pseudoconvex domain with smooth boundary. Then $S(\Omega)$ exists if and only if $\Omega$ has a strong Stein neighborhood basis.

**Proof.** First, assume $S(\Omega)$ exists. Then there exists a defining function $\rho$ of $\Omega$ and $\eta_2 \in (1, \infty)$ such that $\rho^{\eta_2}$ is strictly plurisubharmonic on $\mathbb{C}^n \cap U$. Since level sets of $\rho^{\eta_2}$ and $\rho$ are same, $\rho$ is the desired defining function.

Now, suppose that $\Omega$ has a strong Stein neighborhood basis. Then by Sahutoğlu ([11]), there exist a defining function $\rho$ of $\Omega$ and $c > 0$ such that
\[
L\rho(L, L) \geq c \rho \|L\|^2
\]
on $\mathbb{C}^n \cap U$, where $L$ is a smooth $(1, 0)$ tangent vector field in $U$ with $L\rho = 0$. By letting $\|L\|^2 = \frac{1}{2}$ and dividing it by $\rho$, we have
\[
\frac{L\rho(L, L)}{\rho} \geq \frac{c}{2}
\]
Define $\Sigma_L := \{ p \in \Sigma : \mathcal{L}_\rho(L, L)(p) = 0 \}$ and denote $N = N_\rho$. Taking a limit to $p \in \Sigma_L$ along the real normal direction gives
\[
\frac{1}{2} \frac{N \mathcal{L}_\rho(L, L)}{\| \nabla \rho \|^2} \geq \frac{c}{8}
\]
on $\Sigma_L$. On the other hand, since $\Sigma$ and $\{L \in T^1_\rho(\partial \Omega) : \| L \|^2 = \frac{1}{2} \}$ are compact, $\frac{| \mathcal{L}_\rho(L, N) |^2}{\| \nabla \rho \|^2}$ has the maximum value $M \geq 0$ on $\Sigma$. Therefore, for $\eta_2 \in (1, \infty)$
\[
\frac{1}{\eta_2 - 1} \frac{| \mathcal{L}_\rho(L, N) |^2}{\| \nabla \rho \|^2} - \frac{1}{2} \frac{N \mathcal{L}_\rho(L, L)}{\| \nabla \rho \|^2} \leq \frac{M}{\eta_2 - 1} - \frac{c}{8}
\]
on $\Sigma_L$. By choosing $\eta_2 > \frac{8M}{c} + 1$, we have
\[
\frac{1}{\eta_2 - 1} \frac{| \mathcal{L}_\rho(L, N) |^2}{\| \nabla \rho \|^2} - \frac{1}{2} \frac{N \mathcal{L}_\rho(L, L)}{\| \nabla \rho \|^2} \leq 0
\]
on $\Sigma_L$. By Theorem 3.1 we proved $S(\Omega)$ exists. \hfill \Box

5. Steinness index of worm domains

In this section, we calculate the exact value of the Steinness index of worm domains, and prove Theorem 1.1. Recall first the definition of worm domains.

**Definition 5.1.** The worm domain $\Omega_\beta (\beta > \frac{\pi}{2})$ is defined by
\[
\Omega_\beta := \left\{ (z, w) \in C^2 : \rho(z, w) = \left| z - e^{i \log |w|^2} \right|^2 - (1 - \phi(\log |w|^2)) < 0 \right\}
\]
where $\phi : \mathbb{R} \to \mathbb{R}$ is a fixed smooth function with the following properties:
1. $\phi(x) \geq 0$, $\phi$ is even and convex.
2. $\phi^{-1}(0) = I_{\beta - \frac{\pi}{2}} = [-(\beta - \frac{\pi}{2}), \beta - \frac{\pi}{2}]$.
3. $\exists a > 0$ such that $\phi(x) > 1$ if $x < -a$ or $x > a$.
4. $\phi'(x) \neq 0$ if $\phi(x) = 1$.

Let $\rho(z, w) = \left| z - e^{i \log |w|^2} \right|^2 - (1 - \phi(\log |w|^2))$, and $U$ be a neighborhood of $\partial \Omega_\beta$. Define
\[
L = \frac{1}{\sqrt{\frac{\partial \rho}{\partial z}^2 + \frac{\partial \rho}{\partial w}^2}} \left( \frac{\partial \rho}{\partial w} \frac{\partial}{\partial z} - \frac{\partial \rho}{\partial z} \frac{\partial}{\partial w} \right),
\]
\[
N = \frac{1}{\sqrt{\frac{\partial \rho}{\partial z}^2 + \frac{\partial \rho}{\partial w}^2}} \left( \frac{\partial \rho}{\partial \bar{z}} \frac{\partial}{\partial z} + \frac{\partial \rho}{\partial \bar{w}} \frac{\partial}{\partial w} \right).
\]
Let $\Sigma$ be the set of all weakly pseudoconvex points in $\partial \Omega_\beta$. Then

$$\Sigma = \left\{ (0, w) \in \mathbb{C}^2 : |\log |w|^2| \leq \beta - \frac{\pi}{2} \right\}.$$

By direct calculation, we have

$$\|\nabla \rho\| = 2, \quad L = -e^{i \log |w|^2} \frac{\partial}{\partial w}, \quad N = e^{i \log |w|^2} \frac{\partial}{\partial z},$$

$$\mathcal{L}_\rho(L, N) = \frac{i}{w} e^{-i \log |w|^2}, \quad N \mathcal{L}_\rho(L, L) = -\frac{1}{|w|^2}$$
on $\Sigma$.

**Lemma 5.2** (Riccati equations [10]). For $a, b > 0$ and $t > 0$, the following Riccati equation

$$\frac{d}{dt} s(t) = a(s(t))^2 - \frac{s(t)}{t} + \frac{b}{t^2}$$

has the solution

$$s(t) = -\sqrt{\frac{\cot \left( \sqrt{ab} \log t + \phi \right)}{a}}$$

for arbitrary $\phi$.

**Theorem 5.3.** Let $\Omega_\beta$ ($\beta > \frac{\pi}{2}$) be a worm domain. Then

$$S(\Omega_\beta) = \begin{cases} \frac{\pi}{2(\pi - \beta)} & \text{for } \frac{\pi}{2} < \beta < \pi \\ \infty & \text{for } \pi \leq \beta \end{cases}$$

**Proof.** We will use Corollary 3.6. Let $\alpha = \frac{1}{\eta_2 - 1} + 1$ for $\eta_2 \in (1, \infty)$. Suppose that there exists a smooth function $\psi$ defined on $U$ such that

$$\alpha \left| \frac{\mathcal{L}_\rho(L, N)}{\|\nabla \rho\|} + \frac{1}{2} (L \psi) \right|^2 - \left( \frac{\|\mathcal{L}_\rho(L, N)\|^2}{\|\nabla \rho\|^2} + \frac{1}{2} \frac{N \mathcal{L}_\rho(L, L)}{\|\nabla \rho\|^2} \right) - \frac{1}{4} \mathcal{L}_\psi(L, L) \leq 0$$
on $\Sigma$. By the calculation above,

$$\frac{\|\mathcal{L}_\rho(L, N)\|^2}{\|\nabla \rho\|^2} + \frac{1}{2} \frac{N \mathcal{L}_\rho(L, L)}{\|\nabla \rho\|^2} = \frac{1}{4 |w|^2} - \frac{1}{4 |w|^2} = 0$$
on $\Sigma$. Consequently, (5.1) is equivalent to

$$(5.2) \quad \alpha \left| \frac{i}{w} + \frac{\partial \psi}{\partial w} \right|^2 - \frac{\partial^2 \psi}{\partial w \partial \bar{w}} \leq 0$$
on $\Sigma$. Let $w = re^{i\theta}$. Then (5.2) implies

$$\alpha \left( \frac{\partial \psi}{\partial w} : \frac{\partial \psi}{\partial w} + 1 \right|w|^2 + 2 \text{Re} \left( \frac{\partial \psi}{\partial w} : \frac{i}{w} \right) \right) - \frac{\partial^2 \psi}{\partial w \partial \bar{w}}$$

$$= \frac{\alpha}{4} \psi_r^2 + \frac{\alpha}{4r^2} \psi_\theta^2 + \frac{\alpha}{r^2} \psi_\theta - \frac{1}{4} \psi_{rr} - \frac{1}{4r} \psi_r - \frac{1}{4r^2} \psi_{\theta \theta} \leq 0$$

(5.3)
on $\Sigma$. Notice that $\int_{0}^{2\pi} \psi_{\theta} d\theta = 0$, $\int_{0}^{2\pi} \psi_{\theta\theta} d\theta = 0$ and $\int_{0}^{2\pi} \psi_{\theta}^{2} d\theta \geq 0$. Also by Schwarz's lemma $\left(\int_{0}^{2\pi} \psi_{\gamma} d\theta\right) \leq \left(\int_{0}^{2\pi} d\theta\right) \left(\int_{0}^{2\pi} \psi_{\gamma}^{2} d\theta\right)$. Thus integrating on the both sides of (5.3) with respect to $\theta$ gives

\begin{equation}
\frac{\alpha}{8\pi} \left(\int_{0}^{2\pi} \psi_{\gamma} d\theta\right)^{2} + \frac{2\pi \alpha}{r^{2}} - \frac{1}{4} \int_{0}^{2\pi} \psi_{\gamma\gamma} d\theta - \frac{1}{4r} \int_{0}^{2\pi} \psi_{\gamma} d\theta \leq 0
\end{equation}

for all $r \in \left[e^{-\left(\frac{\pi}{2} - \frac{\pi}{2}\right)}, e^{\frac{\pi}{2} - \frac{\pi}{2}}\right]$. Define $\Psi(r) := \frac{1}{2\pi} \int_{0}^{2\pi} \psi(r, \theta) d\theta$. Then (5.4) is equivalent to

$$\Psi\left(\frac{\pi \alpha}{2} \psi_{r}^{2} + \frac{2\pi \alpha}{r^{2}} - \frac{\pi}{2} \psi_{rr} - \frac{\pi}{2r} \Psi_{r} \leq 0$$

for all $r \in \left[e^{-\left(\frac{\pi}{2} - \frac{\pi}{2}\right)}, e^{\frac{\pi}{2} - \frac{\pi}{2}}\right]$, where $\Psi_{r}(r) = \frac{d}{dr} \Psi(r)$. Letting $s(r) = \Psi_{r}(r)$, we have

$$-s' + \alpha s^{2} - \frac{s}{r} + \frac{4\alpha}{r^{2}} \leq 0$$

for all $r \in \left[e^{-\left(\frac{\pi}{2} - \frac{\pi}{2}\right)}, e^{\frac{\pi}{2} - \frac{\pi}{2}}\right]$. Suppose $s(1) = s_{0}$. By the comparison principle of ordinary differential equation, and Lemma 5.2

$$s(r) \geq -2 \cot\left(\frac{2\alpha \log r + \phi_{0}}{r}\right)$$

for all $r \in \left[e^{-\left(\frac{\pi}{2} - \frac{\pi}{2}\right)}, e^{\frac{\pi}{2} - \frac{\pi}{2}}\right]$, where $\phi_{0}$ is a constant such that

$$s_{0} = -2 \cot(\phi_{0}).$$

Since $s(r)$ is a smooth function on $\left[e^{-\left(\frac{\pi}{2} - \frac{\pi}{2}\right)}, e^{\frac{\pi}{2} - \frac{\pi}{2}}\right]$, the period of the cotangent function is $\pi$, and $-\alpha \left(\frac{\pi}{2} - \frac{\pi}{2}\right) \leq 2\alpha \log r \leq \alpha \left(\frac{\pi}{2} - \frac{\pi}{2}\right)$, the following must hold.

$$\alpha \left(\frac{\pi}{2} - \frac{\pi}{2}\right) < \frac{\pi}{2}$$

which is equivalent to

\begin{equation}
\frac{1}{\eta_{2} - 1} < \frac{2(\pi - \beta)}{2\beta - \pi}.
\end{equation}

If $\beta \geq \pi$, then (5.5) never holds. This proves that $S(\Omega_{\beta}) = \infty$ for all $\beta \geq \pi$.

Now we assume $\frac{\pi}{2} < \beta < \pi$. Then (5.5) is equivalent to

\begin{equation}
\eta_{2} > \frac{\pi}{2(\pi - \beta)}.
\end{equation}

This shows that there does not exist any smooth function $\psi$ defined on $U$ satisfying (5.1) if $\eta_{2} \leq \frac{\pi}{2(\pi - \beta)}$. Therefore, $S(\Omega_{\beta}) \geq \frac{\pi}{2(\pi - \beta)}$. Next, we prove that there exists a smooth function $\psi$ defined on $U$ satisfying (5.1) if $\eta_{2} > \frac{\pi}{2(\pi - \beta)}$. It is sufficient to find a smooth function $\psi$ on $\Sigma$ because we can extend $\psi$ to $U$. By the argument above, (5.1) is equivalent to

\begin{equation}
\frac{\alpha}{4} \psi_{r}^{2} + \frac{\alpha}{4r^{2}} \psi_{\theta}^{2} + \frac{\alpha}{r^{2}} - \frac{\alpha}{r^{2}} \psi_{r} - \frac{1}{4} \psi_{rr} - \frac{1}{4r} \psi_{r} - \frac{1}{4r^{2}} \psi_{\theta\theta} \leq 0.
\end{equation}
We assume $\psi(r, \theta)$ is independent of $\theta$ and let $s(r) = \psi_r$. Then (5.7) becomes
\[
-s' + \alpha s^2 - \frac{s}{r} + \frac{4\alpha}{r^2} \leq 0.
\]
By Lemma 5.2, $-s' + \alpha s^2 - \frac{s}{r} + \frac{4\alpha}{r^2} = 0$ has a solution
\[
s(r) = -2\frac{\cot(2\alpha \log r + \frac{\pi}{2})}{r}.
\]
Since $\eta_2 > \frac{\pi}{2(\pi - \beta)}$, $s(r)$ is a well-defined smooth function on $[e^{-\left(\frac{\pi}{\eta_2}-\pi\right)}, e^{\frac{\beta}{\eta_2}-\frac{\pi}{2}}]$ and hence $\psi(r, \theta) = \int s(r) dr$ satisfies (5.1) on $\Sigma$. Therefore, by Corollary 3.6, $S(\Omega_\beta) = \frac{\pi}{2(\pi - \beta)}$. \qed

**Proof of Theorem 1.1.** Notice that the second and fourth conditions are equivalent by Theorem 1.1. We will show that the ranges of $\beta$ for each condition are same. First, Liu ([11]) showed $DF(\Omega_\beta) = \frac{\pi}{2\beta}$, which implies that $\frac{1}{2} < DF(\Omega_\beta) < 1$ if and only if $\frac{\pi}{2} < \beta < \pi$. Next, by Theorem 5.3, the existence of $S(\Omega_\beta)$ is equivalent to $\frac{\pi}{2} < \beta < \pi$. Finally, one can prove that the third condition is equivalent to $\frac{\pi}{2} < \beta < \pi$ using Theorem 5.1 and 5.2 in [1] (see also section 5 in [1]). If one of the conditions holds, then $DF(\Omega_\beta) = \frac{\pi}{2\beta}$ and $S(\Omega_\beta) = \frac{\pi}{2(\pi - \beta)}$ implies the last equality. \qed

6. Steinness index of finite type domains

Theorem 3.1 says that the Steinness index is characterized by some differential inequality on the set of all weakly pseudoconvex boundary points $\Sigma$. Here, we show that considering the set $\Sigma_\infty$ of infinite type boundary points suffices to characterize the Steinness index (Theorem 6.3).

**Lemma 6.1 ([11]).** Let $\Omega \subset\subset \mathbb{C}^n$ be a domain with smooth boundary, and $K$ be a compact subset of $\partial \Omega$. Assume that $z$ is of finite type for every $z \in K$ and $h \in C^\infty(\overline{\Omega})$ is given. Then for every $j > 0$ there exists $h_j \in C^\infty(\overline{\Omega})$ such that $|h_j - h| \leq \frac{1}{j}$ uniformly on $\overline{\Omega}$ and $\mathscr{L}_{h_j}(X, X) \geq j\|X\|^2$ for all $X \in T_p^{1,0}(\mathbb{C}^n)$ on $K$.

**Lemma 6.2.** Let $\Omega \subset\subset \mathbb{C}^n$ be a domain with smooth boundary, and $\Sigma_\infty \subset \partial \Omega$ be the set of all infinite type boundary points. Assume that there exist a neighborhood $V$ of $\Sigma_\infty$ in $\mathbb{C}^n$, a defining function $\rho$ of $\Omega$, and $\eta_2 > 1$ such that $\rho^{12}$ is strictly plurisubharmonic on $\overline{\Omega} \cap V$. Then there exists a defining function $\tilde{\rho}$ such that $\tilde{\rho}^{12}$ is strictly plurisubharmonic on $\overline{\Omega} \cap \cup$.

**Proof (Based on [11]).** Let $\Sigma_f$ be the set of all finite type boundary points, and $\Pi$ be the set of all strongly pseudoconvex boundary points. Let $\Sigma_0 := \Sigma_f \setminus \Pi$. If $\Sigma_0 = \emptyset$, then by Lemma 3.3, $\rho^{12}$ is strictly plurisubharmonic on $\overline{\Omega} \cap U$. Assume that $\Sigma_0$ is non-empty.

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth increasing convex function such that $\chi(t) = 0$ if $t \leq 0$ and $\chi(t) > 0$ if $t > 0$. Let $\beta = \frac{1}{\eta_2 - 1} + 1$ and $A := \max\{4 + \beta \chi'(t) : 0 \leq t \leq 2\}$. Then
by Lemma 6.1, there exist a sequence of \( \phi_j \in C^\infty(\overline{\Omega}) \) and neighborhoods \( V_1, V_2, V_3 \) in \( \mathbb{C}^n \) with \( \Sigma_\infty \subset \subset V_1 \subset \subset V_2 \subset \subset V_3 \subset \subset V \) such that:

1. \(-6 \ln A < \phi_j < -\ln A \) on \( \overline{\Omega} \).
2. \(-6 \ln A < \phi_j < -3 \ln A \) on \( V_2 \cap \partial \Omega \).
3. \(-2 \ln A < \phi_j < -\ln A \) on \( \partial \Omega \setminus V_3 \).
4. \( \mathcal{L}_{\phi_j}(X,X) > j A^6 \|X\|^2 \) for all \( X \in T_p^{1,0}(\mathbb{C}^n) \) on \( \partial \Omega \setminus V_1 \).

Let \( h_j = e^{\phi_j}, a = A^{-3}, \chi_a(t) = \chi(t-a), \psi_j = \chi_a \circ h_j \) and \( \tilde{\rho} = \rho e^{\psi_j} \). Then \( \psi_j \equiv 0 \) on a neighborhood \( U_1 \) of \( \overline{V_2} \cap \partial \Omega \) in \( \mathbb{C}^n \), hence \( \tilde{\rho}^n \) is strictly plurisubharmonic on \( \overline{\Omega} \cap U_1 \).

Also,

\[
(6.1) \quad \mathcal{L}_{h_j}(X,X) = e^{\phi_j} \mathcal{L}_{\phi_j}(X,X) + e^{\phi_j} |X\phi_j|^2 = e^{\phi_j} \mathcal{L}_{\phi_j}(X,X) + e^{-\phi_j} |Xh_j|^2 > j \|X\|^2 + A |Xh_j|^2
\]

for all \( X \in T_p^{1,0}(\mathbb{C}^n) \) on \( \partial \Omega \setminus V_1 \).

Let \( \{U_\alpha\} \) be a chart of \( U \). Let \( \tilde{L}, L \) be non-vanishing smooth \((1,0)\) tangent vector fields in \( U_\alpha \) such that \( \tilde{L} = L \) on \( \partial \Omega \cap U_\alpha \), and \( L\rho_j = 0, L\rho = 0 \) on \( U_\alpha \). We normalize \( \tilde{L}, L \) as \( \|\tilde{L}\|^2 = \|L\|^2 = \frac{1}{2} \). Denote \( \tilde{N} = N_{\rho_j}, N = N_\rho \). Define

\[ \Sigma^\alpha_L := \{ p \in (\Sigma_0 \setminus V_1) \cap U_\alpha : \mathcal{L}_{\rho}(L,L)(p) = 0 \}. \]

Let \( F(\rho, \eta_2) = \frac{1}{m-1} \frac{|\mathcal{L}_{\rho}(L,N)|^2}{\|\nabla \rho\|^2} - \frac{1}{2} \frac{N_{\mathcal{L}_{\rho}(L,L)}}{\|\nabla \rho\|^2} \). Then by using Lemma 2.2 on \( \Sigma^\alpha_L \),

\[ F(\rho, \eta_2) = \beta \frac{|\mathcal{L}_{\rho}(L,N)|}{\|\nabla \rho\|} |L_{\psi_j}| = \beta \chi_a'(h_j) \frac{|\mathcal{L}_{\rho}(L,N)|}{\|\nabla \rho\|} |Lh_j| \leq \chi_a'(h_j) \frac{1}{4} \left( \frac{\beta^2 |\mathcal{L}_{\rho}(L,N)|^2}{4 \|\nabla \rho\|^2} + |Lh_j|^2 \right), \]

\[ \frac{\beta}{4} |L_{\psi_j}|^2 = \frac{\beta}{4} (\chi_a'(h_j))^2 |Lh_j|^2, \]

\[ -\frac{1}{4} \mathcal{L}_{\psi_j}(L,L) = -\frac{1}{4} \chi_a'(h_j) \mathcal{L}_{h_j}(L,L) - \frac{1}{4} \chi_a''(h_j) |Lh_j|^2 < -\frac{\chi_a'(h_j)}{8} j - \frac{\chi_a'(h_j)}{4} A |Lh_j|^2 - \frac{\chi_a''(h_j)}{4} |Lh_j|^2. \]

All together, the right-hand side of (6.2) is negative if and only if

\[ (6.3) \quad \frac{F(\rho, \eta_2)}{\chi_a'(h_j)} + \frac{\beta^2 |\mathcal{L}_{\rho}(L,N)|^2}{4 \|\nabla \rho\|^2} + \left( 1 + \frac{\beta \chi_a'(h_j)}{4} - \frac{A}{4} - \frac{\chi_a''(h_j)}{4} \right) |Lh_j|^2 - \frac{j}{8} < 0. \]

Since \( h_j - a < A^{-1} - A^{-3} < 2 \) on \( \overline{\Omega} \), \( 4 + \beta \chi_a'(h_j) \leq A \) by the definition of \( A \). Since \( \partial \Omega \) is compact, \( \frac{\beta^2 |\mathcal{L}_{\rho}(L,N)|^2}{4 \|\nabla \rho\|^2} \) is bounded. Since \( \rho^{n^2} \) is strictly plurisubharmonic on \( V \), \( F(\rho, \eta_2) \leq 0 \) on \( \partial \Omega \cap V \) by Lemma 3.2. For the outside of \( V \), \( F(\rho, \eta_2) \) may be positive, but since
\[ \chi'_a(h_j) > c \text{ on } \partial \Omega \setminus V^3 \text{ for some positive constant } c > 0, \quad \frac{F(\rho, \eta_2)}{\chi'_a(h_j)} \text{ is bounded. Therefore, there exists a sufficiently large } j > 0 \text{ such that } (6.3) \text{ holds. Hence, } F(\tilde{\rho}_j, \eta_2) < 0 \text{ on } \Sigma_L^{\alpha}, \text{ for all } \alpha \text{ and } \tilde{L}. \text{ By Lemma 3.3, there exists a neighborhood } U_2 \text{ of } \Sigma_0 \setminus V_1 \text{ such that } \tilde{\rho}_j^{\eta_2} \text{ is strictly plurisubharmonic on } \overline{\Omega} \cap U_2. \text{ Finally, by Lemma 3.5, there exists a neighborhood } U_3 \text{ of } \Pi \text{ such that } \tilde{\rho}_j^{\eta_2} \text{ is strictly plurisubharmonic on } \overline{\Omega} \cap U_3. \text{ Since } U_1 \cup U_2 \cup U_3 \text{ is a neighborhood of } \partial \Omega, \text{ the proof is completed.} \]

**Theorem 6.3.** Let \( \Omega \subset \subset \mathbb{C}^n \) be a pseudoconvex domain with smooth boundary, and \( \rho \) be a defining function of \( \Omega \). Let \( L \) be an arbitrary \((1,0)\) tangent vector field on \( \partial \Omega \), and \( \Sigma_{\infty} \) be the set of all infinite type boundary points. Define

\[ \Sigma_{\infty,L} := \{ p \in \Sigma_{\infty} : \mathcal{L}_\rho(L,L)(p) = 0 \}. \]

Let \( \eta_\rho \) be the infimum of \( \eta_2 \in (1, \infty) \) satisfying

\[ \frac{1}{\eta_2 - 1} \frac{\lvert \mathcal{L}_\rho(L,N) \rvert^2}{\lVert \nabla \rho \rVert^2} - \frac{\lvert N \mathcal{L}_\rho(L,L) \rvert}{2 \lVert \nabla \rho \rVert} \leq 0 \]

on \( \Sigma_{\infty,L} \) for all \( L \). Here, \( N = N_\rho \) and we extend \( L \) so that \( L \rho = 0 \) on \( U \). Then

\[ S(\Omega) = \inf \eta_\rho \]

where the infimum is taken over all smooth defining functions \( \rho \).

**Proof.** The proof is same as that of Theorem 3.1 except that we use Lemma 6.2 instead of Lemma 3.3. \( \square \)

**Corollary 6.4.** Let \( \Omega \subset \subset \mathbb{C}^n \) be a pseudoconvex domain with smooth boundary. Assume that all boundary points of \( \Omega \) are of finite type. Then \( S(\Omega) = 1 \).

**Corollary 6.5.** Let \( \Omega \subset \subset \mathbb{C}^n \) be a pseudoconvex domain with smooth boundary. Assume that the set of all infinite type boundary points of \( \Omega \) is finite. Then \( S(\Omega) = 1 \).

**Proof.** Suppose that the set of all infinite type boundary points is one point, says \( \Sigma_{\infty} = \{ p_0 \} \). We denote \( \tilde{N} = N_\tilde{\rho}, N = N_\rho \). Let \( \tilde{L}, L \) be \((1,0)\) tangent vector fields on \( U \) such that \( \tilde{L} = L \text{ on } \partial \Omega, \tilde{L} \tilde{\rho} = 0, L \rho = 0 \) on \( U \), and \( \mathcal{L}_\rho(L,L) = 0 \) at \( p_0 \). First, by Lemma 2.2

\[ (6.4) \quad \frac{\lvert \mathcal{L}_{\tilde{\rho}}(\tilde{L}, \tilde{N}) \rvert^2}{\lVert \nabla \tilde{\rho} \rVert^2} + \frac{1}{2} \frac{\lvert \nabla \tilde{\rho} \rvert^2}{\lVert \nabla \rho \rVert^2} = \frac{\lvert \mathcal{L}_\rho(L,N) \rvert^2}{\lVert \nabla \rho \rVert^2} + \frac{1}{2} \frac{\lvert N \mathcal{L}_\rho(L,L) \rvert}{\lVert \nabla \rho \rVert} + \frac{1}{4} \mathcal{L}_{\phi}(L,L), \]

where \( \tilde{\rho} = \rho e^\phi \). By letting \( \phi = \alpha \lVert z \rVert^2 \) and choosing \( \alpha > 0 \) sufficiently large, we can make the left-hand side of (6.4) positive at \( p_0 \) for all \( \tilde{L} \). Hence, we may assume that there exists a defining function \( \rho \) of \( \Omega \) such that

\[ (6.5) \quad \frac{\lvert \mathcal{L}_\rho(L,N) \rvert^2}{\lVert \nabla \rho \rVert^2} + \frac{1}{2} \frac{\lvert N \mathcal{L}_\rho(L,L) \rvert}{\lVert \nabla \rho \rVert} > 0 \]

for all \( L \) at \( p_0 \). Now, if \( L(z) = \sum_{j=1}^n a_j(z) \frac{\partial}{\partial x_j}, \) then \( -2 \frac{\mathcal{L}_\rho(L,N)}{\lVert \nabla \rho \rVert} \) can be represented by

\[ -2 \frac{\mathcal{L}_\rho(L,N)}{\lVert \nabla \rho \rVert}(z) = \sum_{j=1}^n b_j(z) a_j(z) \]
for some complex-valued functions $b_j(z)$. Define $\psi(z) = \sum_{j=1}^{n} b_j(p_0)z_j$. Then

$$L\psi(p_0) = -2\frac{\mathcal{L}_{\rho}(L, N)}{\|\nabla \rho\|}(p_0), \quad \mathcal{L}_{\psi}(L, L)(p_0) = 0.$$ 

If $\tilde{\rho} = \rho e^{\psi}$, then by Lemma 2.2 and (6.5) at $p_0$

$$\frac{1}{\eta_2 - 1} \left( \frac{|\mathcal{L}_\rho(\tilde{L}, \tilde{N})|^2}{\|\nabla \tilde{\rho}\|^2} + \frac{1}{2} \frac{\tilde{N}.\mathcal{L}_\rho(\tilde{L}, \tilde{L})}{\|\nabla \tilde{\rho}\|^2} \right) = \left( \frac{1}{\eta_2 - 1} + 1 \right) \left( \frac{|\mathcal{L}_\rho(L, N)|^2}{\|\nabla \rho\|^2} + \frac{1}{2} \frac{N.\mathcal{L}_\rho(L, L)}{\|\nabla \rho\|^2} \right) - \frac{1}{4} \frac{\mathcal{L}_\rho(L, L)}{\|\nabla \rho\|^2} \mathcal{L}_\rho(L, L)

$$

for all $L$ and $\eta_2 > 1$. Therefore, by Theorem 6.3, $S(\Omega) = 1$. If the number of points in $\Sigma_{\infty}$ is more than 1, then one can construct a smooth function $\psi$ on $U$ satisfying (6.6) at all infinite type points using similar argument as above. □

7. Steininess index of convex domains

Assume that $\Omega \subset \subset \mathbb{C}^n$ is a domain with $C^k(k \geq 1)$-smooth boundary. The primary goal is to prove that if $\Omega$ is convex then $DF(\Omega) = 1$ and $S(\Omega) = 1$. In fact, we prove more: there exists a defining function which is strictly convex on $\mathbb{C}^n \setminus \partial \Omega$. For this, we need the following notion of smooth maximum. For $\epsilon > 0$, let $\chi_\epsilon : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\chi_\epsilon$ is strictly convex for $|t| < \epsilon$ and $\chi_\epsilon = |t|$ for $|t| \geq 0$. Then we define a smooth maximum by

$$\tilde{\max}_\epsilon(x, y) := \frac{x + y + \chi_\epsilon(x - y)}{2}.$$ 

Note that $\tilde{\max}_\epsilon(x, y) = \max(x, y)$ if $|x - y| \geq \epsilon$. Moreover, the smooth maximum of two $C^k$-smooth (strictly) convex functions is $C^k$-smooth (strictly) convex. Let $\text{dist}(x, \partial \Omega) := \inf\{\|x - y\| : y \in \partial \Omega\}$. Let $\delta : \mathbb{R}^n \to \mathbb{R}$ be the distance function of $\Omega$ defined by

$$\delta(x) := \begin{cases} -\text{dist}(x, \partial \Omega) & \text{if } x \in \Omega, \\ \text{dist}(x, \partial \Omega) & \text{if } x \in \Omega^c. \end{cases}$$

Define $\Omega_\epsilon := \{x \in \mathbb{R}^n : \delta(x) < \epsilon\}$.

**Theorem 7.1.** Let $\Omega \subset \subset \mathbb{R}^n$ be a convex domain with $C^k(k \geq 1)$-smooth boundary. Then there exists a $C^k$-smooth function $\rho : \mathbb{R}^n \to \mathbb{R}$ satisfying

1. $\rho$ is a defining function of $\Omega$.
2. $\rho$ is strictly convex on $\mathbb{R}^n \setminus \partial \Omega$.

The author would like to acknowledge that he learned the formulation of this theorem as well as the proof from Professor N. Shcherbina.
Proof. We may assume that $0 \in \Omega$. Consider the Minkowski function of $\Omega$. Define a function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 
\lambda, & \text{where } \frac{1}{\lambda} x \in \partial \Omega \quad \text{if } x \neq 0 \\
0 & \text{if } x = 0.
\end{cases}$$

Then $f$ is well-defined, $C^k$-smooth on $\mathbb{R}^n \setminus \{0\}$, and $\nabla f \neq 0$ on $\partial \Omega$. Convexity of $\Omega$ implies that $f$ is a convex function. Consequently, $\sigma := f - 1$ is a convex defining function of $\Omega$.

We first construct a defining function of $\Omega$ which is strictly convex inside of $\Omega$. For all $\epsilon > 0$ such that $0 \in \Omega_{-\epsilon}$, we claim that there exist $\delta_1, \delta_2 > 0$ such that $\delta_1 \|x\|^2 - \delta_2 < 0$ on $\partial \Omega$, and $\delta_1 \|x\|^2 - \delta_2 > \sigma$ on $\Omega_{-\epsilon}$. Let $S_\epsilon := \max\{\sigma(x) : x \in \Omega_{-\epsilon}\} < 0$, $m_\epsilon := \min\{\|x\|^2 : x \in \Omega_{-\epsilon}\} > 0$, $M_\epsilon := \max\{\|x\|^2 : x \in \partial \Omega\} > 0$. Then one may choose sufficiently small $\delta_1, \delta_2 > 0$ such that

$$\delta_1 M_\epsilon < \delta_2 < \delta_1 m_\epsilon + (-S_\epsilon).$$

Therefore

$$\delta_1 \|x\|^2 - \delta_2 \geq \delta_1 m_\epsilon - \delta_2 > S_\epsilon > \sigma(x)$$
on $\Omega_{-\epsilon}$, and

$$\delta_1 \|x\|^2 - \delta_2 \leq \delta_1 M_\epsilon - \delta_2 < 0$$
on $\partial \Omega$. The claim is proved.

Let $V_\epsilon := \{x \in \mathbb{R}^n : \sigma - (\delta_1 \|x\|^2 - \delta_2) < 0\}$ and $m := \min\{\dist(\partial V_\epsilon, \partial \Omega_{-\epsilon}), \dist(\partial V_\epsilon, \partial \Omega)\}$. Define

$$\rho_\epsilon := \max\{\sigma, \delta_1 \|x\|^2 - \delta_2\}.$$Then $\rho_\epsilon$ is a $C^k$-smooth function on $\mathbb{R}^n$ and $\rho_\epsilon = \delta_1 \|x\|^2 - \delta_2$ on $\Omega_{-\epsilon}$ which is strictly convex. Assume that $0 \in \Omega_1$ (the number 1 is not important here and one may choose a number smaller than 1). Let $U_\epsilon := \{x \in \mathbb{R}^n : \|x\| < \epsilon\}$ and for $j \in \mathbb{N}$

$$c_j := \sup_{U_\epsilon} \sup_{0 \leq \alpha \leq k} \left| \frac{\partial^\alpha}{\partial x^\alpha} \rho_\epsilon \right|, \quad \eta_j := \frac{1}{2^j c_j}.$$Here, we used multi-index $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}_0^n$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$. Define

$$\sigma_1(x) := \sum_{j=1}^{\infty} \eta_j \rho_\epsilon(x).$$Then for all $j \in \mathbb{N}$, $0 \leq |\alpha| \leq k$, since $\left| \frac{\partial^\alpha}{\partial x^\alpha} \rho_\epsilon \right| \leq \frac{1}{2^j}$ on $U_j$ and $\sum_{j=1}^{\infty} \frac{1}{2^j}$ converges,

$$\sum_{j=1}^{\infty} \eta_j \frac{\partial^\alpha}{\partial x^\alpha} \rho_\epsilon(x)$$is uniformly convergent and hence continuous on $\mathbb{R}^n$. Note that $\nabla \sigma_1 = \sum_{j=1}^{\infty} \eta_j \nabla \rho_\epsilon \neq 0$ on $\partial \Omega$. Therefore $\sigma_1$ is a well-defined $C^k$-smooth function on $\mathbb{R}^n$ which is strictly convex.
ON THE STEINNESS INDEX

Now, we consider the outside of $\Omega$. The argument is similar to that above. There exist $\tilde{\delta}_1, \tilde{\delta}_2 > 0$ such that $\sigma(x) + \tilde{\delta}_1\|x\|^2 - \tilde{\delta}_2 < 0$ on $\partial\Omega$, and $\sigma(x) + \tilde{\delta}_1\|x\|^2 - \tilde{\delta}_2 > 0$ on $\partial\Omega^c$. Let $\tilde{V}_\epsilon := \{x \in \mathbb{R}^n : \sigma + \tilde{\delta}_1\|x\|^2 - \tilde{\delta}_2 < 0\}$ and $\tilde{m} := \min\{\text{dist}(\partial\tilde{V}_\epsilon, \partial\Omega), \text{dist}(\partial\tilde{V}_\epsilon, \partial\Omega^c)\}$.

Then $\tilde{\rho}_\epsilon := \max_{\bar{\Omega}}(0, \sigma(x) + \tilde{\delta}_1\|x\|^2 - \tilde{\delta}_2)$ is $C^k$-smooth function on $\mathbb{R}^n$ which is strictly convex on $\Omega^c_\epsilon$. Finally, for $\tilde{\eta}_j > 0$ such that $\tilde{\eta}_j \searrow 0$ sufficiently fast,

$$\rho(x) := \sigma_1(x) + \sum_{j=1}^{\infty} \tilde{\eta}_j \tilde{\rho}_j(x)$$

is the desired defining function of $\Omega$.

**Corollary 7.2.** For a convex domain $\Omega \subset \subset \mathbb{C}^n$ with $C^k(k \geq 1)$-smooth boundary, $DF(\Omega) = 1$ and $S(\Omega) = 1$.

**Proof.** By Theorem 7.1 there exists a defining function $\rho$ of $\Omega$ which is strictly convex on $\mathbb{C}^n \setminus \partial\Omega$. Hence $\rho$ is strictly plurisubharmonic on $\mathbb{C}^n \setminus \partial\Omega$. This implies that $DF(\Omega) = 1$ and $S(\Omega) = 1$. □

**Remark 7.3.** If the boundary regularity of a convex domain is $C^\infty$-smooth, then there is another way to prove Corollary 7.2. $\sigma$ in the proof of Theorem 7.1 is $C^\infty$-smooth plurisubharmonic defining function of $\Omega$. Hence the theorem by Fornæss and Herbig [7] implies $DF(\Omega) = 1$ and $S(\Omega) = 1$.

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