NORMAL FORM FOR EDGE METRICS

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1. Introduction

An edge metric is a metric on the interior of a manifold-with-boundary which is singular at the boundary in a manner described by a given fibration of the boundary. The related edge differential operators arise in many settings and have been the subject of much research; see, for example, [Ma]. Our interest in edge metrics arises from the observation that they are a robust class of metrics naturally generalizing the product of a conformally compact metric with a metric on a compact manifold. The AdS/CFT correspondence in physics deals with such product metrics, and the arena of edge metrics appears to be a natural setting for the geometric and analytic questions which arise. The thesis [Ka] considers a problem concerning eleven-dimensional supergravity from this point of view. In this paper we derive a normal form for edge metrics which we expect will be useful in further studies. The normal form is the analogue of geodesic normal coordinates relative to the boundary at infinity.

Edge metrics reduce to conformally compact metrics in the special case that the fibers of the boundary are points. Asymptotically hyperbolic (AH) metrics are conformally compact metrics satisfying a particular scalar normalization at infinity. The normal form for AH metrics was derived in [GL] and a different proof was given in [JS2]. This normal form has been useful in a number of problems concerning AH metrics. The existence statement is that if \( g \) is AH on \( X \), then there is a diffeomorphism \( \psi \) from a neighborhood of \( \{0\} \times \partial X \) in \( [0, \infty) \times \partial X \) to a neighborhood of \( \partial X \) in \( X \) such that \( \psi|_{\partial X} = \text{Id} \) and

\[
\psi^* g = \frac{dx^2 + h_x}{x^2},
\]

where \( x \) is the coordinate in \( [0, \infty) \) and \( h_x \) is a 1-parameter family of metrics on \( \partial X \). The normal form for \( g \) is not unique and this is a crucial point. There is a conformal class of metrics on \( \partial X \), called the conformal infinity of \( g \), and the normal forms for \( g \) are parametrized precisely by the representative metrics in the conformal infinity: \( h_0 \) is the given conformal representative. Conformal rescalings on the boundary thus correspond via the normal form to diffeomorphism changes on the interior. One application of the AH normal form is to the renormalized volume of an AH Einstein metric (see [G]). The renormalized volume is defined in terms of the exhaustion determined by a defining function \( x \) in a normal form, and its invariance or noninvariance (measured by

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a conformal anomaly) is determined via properties of the diffeomorphism determined by a conformal change on the boundary. In [JS2] the normal form (1.1) arose in an inverse scattering context, where it was used to normalize the action of diffeomorphisms on AH metrics.

For edge metrics there is an analogue of the AH scalar normalization, which defines what we call a normalized edge metric. There is another necessary condition for an edge metric to have a normal form which is vacuous in the conformally compact case. It is that a 1-form on the boundary fibers constructed out of the metric and a defining function be globally smoothly exact; this condition is independent of the choice of defining function. We call an edge metric satisfying this condition an exact edge metric. An exact edge metric has a reduced conformal infinity analogous to the conformal infinity in the conformally compact case. In the edge case this is a conformal class of metrics on $TX|_{\partial X}/V$, where $V$ is the vertical bundle of the boundary fibration (i.e. the tangent bundle to the fibers), and any two metrics in the conformal class differ by a positive function which is locally constant on the boundary fibers. Our main result, Theorem 2.9, asserts that an exact, normalized edge metric $g$ can be put into normal form near the boundary by a diffeomorphism which restricts to the identity on the boundary, and the different normal forms for $g$ are parametrized precisely by the representatives for the reduced conformal infinity. Thus the result takes the same form as the result for AH metrics.

The proof of the normal form for AH metrics in [GL] is based on the observation that if $\psi$ puts $g$ into normal form as above, then $\hat{x} := x \circ \psi^{-1}$ satisfies the eikonal equation

$$\left| \frac{d\hat{x}}{\hat{x}} \right|_g^2 = 1. \tag{1.2}$$

By first solving this equation on $X$, one can therefore obtain directly the $x$-component of $\psi^{-1}$. And once one has $\hat{x}$, it is clear from the normal form (1.1) that the full map $\psi$ can be constructed by following the integral curves of the noncharacteristic vector field $\hat{x}^{-1}X_{\hat{x}}$, where $X_{\hat{x}}$ is the vector field dual to $\frac{d\hat{x}}{\hat{x}}$ with respect to $g$. Now (1.2) appears to be a singular equation. But by writing $\hat{x} = e^{x_0}x_0$, where $x_0$ is a fixed defining function and $\omega$ a new unknown, then expanding the left-hand side of (1.2) and moving a term to the right-hand side, and finally dividing by $x_0$, it becomes a noncharacteristic first order nonlinear pde for $\omega$, which is solvable by the method of characteristics. This method of reducing a singular eikonal equation to a noncharacteristic initial value problem at infinity and then constructing the diffeomorphism by following integral curves of an associated noncharacteristic vector field can be used to derive normal forms in a number of other settings. It gives a simple alternate derivation of the normal form for scattering metrics proved in [JS1], and is used in [GS] to derive a normal form for $\Theta$-metrics. (In the case of scattering metrics, the corresponding eikonal equation is $\left| \frac{d\hat{x}}{\hat{x}} \right|^2_g = 1$. One writes $\hat{x} = x_0 + \omega x_0^2$ for a new unknown $\omega$ and divides the equation by $x_0^2$ rather than $x_0$ to obtain a noncharacteristic initial value problem for $\omega$. The
diffeomorphism is constructed by following the integral curves of \( \hat{x}^{-2}X_{\hat{x}} \), where \( X_{\hat{x}} \) is the vector field dual to \( \frac{\partial}{\partial \hat{x}} \) with respect to \( g \).

In the edge case we were unable to reduce (1.2) to a noncharacteristic problem. The issue is the following. Equation (1.2) involves the components of the inverse metric \( g^{-1} \). In the cases of conformally compact or scattering or \( \Theta \)-metrics, all components of \( g^{-1} \) vanish at the boundary. Because of this, one can divide (1.2) by the correct power of \( x_0 \) as described above to obtain a noncharacteristic problem, and still have an equation with smooth coefficients up to the boundary. But for edge metrics, the components of \( g^{-1} \) along the fibers do not vanish at \( \partial X \). The coefficient of the derivative transverse to the boundary still vanishes there, and one cannot carry out the division by \( x_0 \) to make the problem noncharacteristic. So it seems that one is forced to deal with a singular equation.

Joshi studied the normal form for \( b \)-metrics in [J]. A \( b \)-metric is an edge metric in the opposite extreme case from a conformally compact metric: the case in which there is only one boundary fiber, the boundary itself. Although geometrically simpler than general edge metrics, \( b \)-metrics already exhibit the essential analytic difficulty of the general case as far as derivation of the normal form. By formal calculations, Joshi derived the normal form for exact, normalized \( b \)-metrics modulo error terms vanishing to infinite order at the boundary using the method of [JS1], [JS2], but was unable to obtain the result in an open set. In this regard he commented, “it is not clear how to proceed”. The normal form modulo infinite-order vanishing error terms was derived for general exact, normalized edge metrics in [Ka] by formal analysis of (1.2). (We remark that the literature concerning \( b \)-metrics, for instance [Mc] and [J], defines an exact \( b \)-metric to be what we refer to as an exact, normalized \( b \)-metric. That is, we separate the two separately invariant conditions: exactness and normalization.)

To solve (1.2) in the edge case, we derive a result concerning existence and uniqueness of certain characteristic nonlinear first-order initial value problems for a real scalar unknown. We consider problems of the form

\[
(1.3) \quad x \partial_x \omega = F(x, y, \omega, \partial_y \omega), \quad \omega(0, y) = \omega_0(y).
\]

Here \((x, y)\) are coordinates on \( \mathbb{R}^n, n \geq 1 \), with \( x \geq 0, y \in \mathbb{R}^{n-1} \), and the unknown function \( \omega(x, y) \) is real-valued. \( F(x, y, \omega, q) \) is a smooth real function of \((x, y, \omega, q) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R}^{n-1} \) and the initial value \( \omega_0 \) is a smooth function of \( y \).

**Theorem 1.1.** Suppose that for all \( y \) one has

\[
(1.4) \quad F(0, y, \omega_0(y), \partial_y \omega_0(y)) = 0, \quad F_q(0, y, \omega_0(y), \partial_y \omega_0(y)) = 0
\]

and

\[
(1.5) \quad F_\omega(0, y, \omega_0(y), \partial_y \omega_0(y)) < 1.
\]

Then there exists a unique smooth solution of (1.3) for sufficiently small \( x \geq 0 \).
We prove Theorem 1.1 by an adaptation of the method of characteristics. In the noncharacteristic case this proceeds by solving the ordinary differential equations corresponding to a Hamiltonian flow-out in the first jet bundle of the solution. Because our initial value problem is characteristic, the Hamiltonian vector field vanishes identically on the initial submanifold, so there is no flow-out in the usual sense. Nonetheless, we are able to construct a flow-out by considering what we call characteristic integral curves emanating from a zero of a vector field. We show these exist under an appropriate hypothesis on the eigenvalues of the linearization of the vector field at the zero (Theorem 3.3) and the union of the characteristic integral curves of the Hamiltonian vector field starting from the initial submanifold gives the 1-jet of the solution. It seems likely that Theorem 1.1 and the method of using these characteristic integral curves will be useful in other problems. Once we have solved (1.2), the second part of the proof of the normal form, constructing the diffeomorphism by flowing along integral curves, works just as before: the relevant vector field $\hat{x}^{-1}X_\hat{x}$ is smooth and noncharacteristic.

In §2 we define edge metrics and study the geometric structure they induce at infinity. We work with edge metrics of arbitrary signature, under an additional nondegeneracy hypothesis which we call horizontal nondegeneracy. This is automatic in the case of definite signature. We define the notions of normalization, exactness, and reduced conformal infinity referred to above, and show that an arbitrary horizontally nondegenerate edge metric invariantly induces a metric on each fiber of the boundary. We formulate the normal form condition and state the main result, Theorem 2.9 which asserts the existence and uniqueness of the normal form. We show that Theorem 2.9 follows from the solvability of the eikonal equation, and reduce the eikonal equation to a problem of the form (1.3). In §3 we prove Theorem 1.1.

Throughout, by smooth we mean infinitely differentiable, and all objects are assumed smooth unless explicitly stated otherwise.

2. Edge Metrics

Let $(X, \partial X)$ be a manifold-with-boundary. Suppose that $\partial X$ is the total space of a fibration

$$
\begin{align*}
F & \longrightarrow \partial X \\
\downarrow \pi & \\
Y
\end{align*}
$$

with fiber $F$ and base $Y$. One says that $(X, \partial X)$ is an edge, or boundary-fibered, manifold. A motivating special case is that of a product $X = M \times F$, where $(M, \partial M)$ is a manifold-with-boundary. Then $\partial X = \partial M \times F$ with projection $\pi : \partial X \to \partial M = Y$.

Let $V = \ker \pi_* \subset T\partial X$ denote the vertical vector bundle over $\partial X$. A vector field $\xi$ on $X$ is said to be an edge vector field if its restriction to $\partial X$ is tangent to the fibers. Equivalently, one requires that $\xi|_{\partial X} \in \Gamma(V)$.

Near a point of $\partial X$ one can choose local coordinates $(x, y^\alpha, z^A)$ so that $x$ is a defining function for $\partial X$ with $x > 0$ in $X^o$, $y^\alpha$ are coordinates on $X$ whose restrictions to $\partial X$
are lifts of local coordinates on $Y$ (so the $g^\alpha$ are constant on the fibers), and $z^A$ are coordinates on $X$ whose restrictions to each fiber of $\partial X$ are local coordinates on the fiber. The edge vector fields are then $\text{span}_{C^\infty(X)} \{x\partial_x, x\partial_y^\alpha, x\partial_z^A\}$.

There is a vector bundle $\mathfrak{e}TX$ on $X$, the *edge tangent bundle*, which can be characterized by the requirement that its smooth sections are the edge vector fields. $x\partial_z$ and the $x\partial_y^\alpha$ define sections of $\mathfrak{e}TX$ which are nonvanishing on $\partial X$, and in local coordinates near $\partial X$ one has $\mathfrak{e}T_pX = \text{span}\{x\partial_x, x\partial_y^\alpha, x\partial_z^A\}$ for $p \in X$. The dual edge cotangent bundle $\mathfrak{e}T^*X$ has fibers spanned by the dual basis $dx_x, dy_x^\alpha, dz_x^A$. There is a well-defined evaluation map $\text{Eval} : \mathfrak{e}TX \to TX$ with the property that $\text{Eval}_p : \mathfrak{e}T_pX \to T_pX$ is an isomorphism for $p \in X \setminus \partial X$, but $\text{Eval}_p(\mathfrak{e}T_pX) = V_p$ for $p \in \partial X$. For $p \in \partial X$, we define the horizontal bundle $H_p = \text{ker} \text{Eval}_p \subset \mathfrak{e}T_pX$, so that $H_p = \text{span}\{x\partial_x, x\partial_y^\alpha\}$.

An *edge metric* $g$ on an edge manifold $X$ is a smooth nondegenerate section of $S^2(\mathfrak{e}T^*X)$. (We allow metrics of arbitrary signature.) In local coordinates near $\partial X$, if we write

\begin{equation}
\tag{2.1}
g = \begin{pmatrix}
\frac{dx_x}{x} & \frac{dy_x^\alpha}{x} & dz_x^A
\end{pmatrix}
\begin{pmatrix}
\tilde{g}_{00} & \tilde{g}_{0\beta} & \tilde{g}_{0B} \\
\tilde{g}_{\alpha0} & \tilde{g}_{\alpha\beta} & \tilde{g}_{\alpha B} \\
\tilde{g}_{A0} & \tilde{g}_{A\beta} & \tilde{g}_{AB}
\end{pmatrix}
\begin{pmatrix}
\frac{dx_x}{x} \\
\frac{dy_x^\alpha}{x} \\
\frac{dz_x^A}{x}
\end{pmatrix},
\end{equation}

then this is the requirement that the $\tilde{g}$-matrix be smooth and nondegenerate up to $\partial X$.

An edge metric restricts to a usual metric on $X^\circ$ which is singular at $\partial X$ with the form of the singularity determined by the boundary fibration. In the special case that the fiber $F$ is a point, there are no $z$ variables and an edge metric is just a conformally compact metric on the manifold-with-boundary $(X, \partial X)$. In the special case that $Y$ is a point, there are no $y$ variables, and in this case edge metrics are called $b$-metrics on the manifold-with-boundary $(X, \partial X)$. For a product edge manifold $X = M \times F$ as above, a product edge metric is a metric of the form $g = g_M + g_F$, where $g_M$ is a conformally compact metric on $(M, \partial M)$ and $g_F$ is a metric on $F$.

We will say that an edge metric is *horizontally nondegenerate* if $g|_{H_p}$ is nondegenerate for all $p \in \partial X$. Clearly any positive definite edge metric is horizontally nondegenerate. A horizontally nondegenerate metric induces a metric on the bundle $H \subset \mathfrak{e}TX|_{\partial X}$. This induced metric may be reinterpreted as a conformal class of metrics on $TX|_{\partial X}/V$ as follows. Multiplication by a defining function $x$ induces a map $T_pX \to \mathfrak{e}T_pX$ for $p \in \partial X$ with kernel $V_p$ and range $H_p$. This induces an isomorphism $H \cong TX|_{\partial X}/V$ dependent on the choice of $x$ only up to scale. Via this isomorphism we can transfer the metric $g|_H$ to a conformal class of metrics on $TX|_{\partial X}/V$, called the conformal infinity of $g$. (This choice of terminology is slightly at odds with usual usage in the conformally compact case, where the conformal infinity is typically regarded as a metric on $T\partial X$ rather than on $TX|_{\partial X}$. One reason for our choice is that $T\partial X/V$ has rank 0 for the special case of $b$-metrics.) Of course one can realize the representatives in the conformal infinity directly without recourse to $H$: the metric $(x^2g)|_{X^\circ}$ extends smoothly to $X$ as a section of $S^2T^*X$ and $(x^2g)|_{\partial X}$ annihilates $V$, so it induces a quadratic form on the bundle $TX|_{\partial X}/V$ over $\partial X$ which is nondegenerate iff $g$ is horizontally nondegenerate.
There is a generalization to edge metrics of the normalization condition that a conformally compact metric be asymptotically hyperbolic. The section $\frac{dx}{x}$ of $eT^*X$ restricts to a section of $H^*$ which is independent of $x$. This is because if $\tilde{x} = ax$ with $0 < a \in C^\infty(X)$, then $\frac{dx}{\tilde{x}} = \frac{dx}{x} + \frac{da}{a}$, and the restriction of $\frac{da}{a}$ to $H$ vanishes. If $g$ is horizontally nondegenerate, then we can consider the length squared of $\frac{dx}{x}|_H$ with respect to $g|_H$, and this will be an $\mathcal{H}$-invariant of $g$.

**Definition 2.1.** An edge metric $g$ is said to be normalized if $g$ is horizontally nondegenerate and if

$$\left|\frac{dx}{x}|_H\right|^2 = 1 \quad \text{on } \partial X.$$

**Remark 2.2.** In the case of indefinite signature, one could equally well consider the condition $\left|\frac{dx}{x}|_H\right|^2 = -1$. Our treatment applies to this case upon replacing $g$ by $-g$.

We can also consider the length squared of $\frac{dx}{x}$ with respect to $g$ on all of $eT^*X$. In general this will depend on $x$ since $\frac{da}{a}|_{\partial X}$ is a nontrivial section of $eT^*X|_{\partial X}$ if $a$ varies along the fibers.

**Definition 2.3.** Let $g$ be an edge metric and $x$ a defining function for $\partial X$. $x$ is said to be $g$-normalized if $\left|\frac{dx}{x}\right|^2 = 1$ on $\partial X$.

We make these invariant conditions explicit in local coordinates. If $g$ is written as (2.1), then $g|_H$ is represented by $\begin{pmatrix} \bar{g}_{00} & \bar{g}_{0\beta} \\ \bar{g}_{\alpha0} & \bar{g}_{\alpha\beta} \end{pmatrix}$. Horizontal nondegeneracy of $g$ is the requirement that this quadratic form be nondegenerate at $\partial X$. In this case, the dual metric is given by the inverse matrix, so we write

$$\begin{pmatrix} \bar{g}_{00} & \bar{g}_{0\beta} \\ \bar{g}_{\alpha0} & \bar{g}_{\alpha\beta} \end{pmatrix}^{-1} = \begin{pmatrix} C & * \\ * & * \end{pmatrix},$$

and $C = \left|\frac{dx}{x}\right|^2|_H$. Thus $g$ is normalized means exactly that $C = 1$, and this condition is independent of the choice of all the coordinates.

On the other hand, $\frac{dx}{x}$ is a dual basis vector in the full frame. We write

$$\begin{pmatrix} \bar{g}_{00} & \bar{g}_{0\beta} & \bar{g}_{0B} \\ \bar{g}_{\alpha0} & \bar{g}_{\alpha\beta} & \bar{g}_{\alpha B} \\ \bar{g}_{A0} & \bar{g}_{A\beta} & \bar{g}_{AB} \end{pmatrix}^{-1} = \begin{pmatrix} \bar{g}^{00} & \bar{g}^{0\beta} & \bar{g}^{0B} \\ \bar{g}^{\alpha0} & \bar{g}^{\alpha\beta} & \bar{g}^{\alpha B} \\ \bar{g}^{A0} & \bar{g}^{A\beta} & \bar{g}^{AB} \end{pmatrix},$$

and then $\bar{g}^{00} = \left|\frac{dx}{x}\right|^2$. So $x$ is $g$-normalized means $\bar{g}^{00} = 1$. This condition is independent of the choice of $y^a$, $z^A$, but in general does depend on the choice of $x$.

A horizontally nondegenerate edge metric invariantly induces a pseudo-Riemannian metric on the fibers of $\partial X$. Let $g$ be an edge metric. The induced dual metric on $eT^*X$ is a section $g^{-1}$ of $S^2(eTX)$. Thus $\text{Eval}(g^{-1})$ is a smooth section of $S^2TX$. On $\partial X$, this section of $S^2TX$ degenerates: its restriction to $\partial X$ is a smooth section of $S^2V \subset S^2TX$. Elementary linear algebra (most easily carried out in terms of the explicit formulation
of these conditions below) shows that the condition that $g$ is horizontally nondegenerate is equivalent to the condition that $\text{Eval}(g^{-1})$ is a nondegenerate section of $S^2V$. So if $g$ is horizontally nondegenerate, $\text{Eval}(g^{-1})$ defines a metric on $V^\ast$. Its dual is a metric on $V$, or equivalently a pseudo-Riemannian metric on each fiber of $\partial X$. We denote this induced metric on the fibers by $g_F$.

Concretely:

$$g^{-1} = \begin{pmatrix} x\partial_x & x\partial_y & \partial_z \end{pmatrix} \begin{pmatrix} \tilde{g}^{00} & \tilde{g}^{0\beta} & \tilde{g}^{0B} \\ \tilde{g}^{\alpha0} & \tilde{g}^{\alpha\beta} & \tilde{g}^{\alpha B} \\ \tilde{g}^{A0} & \tilde{g}^{A\beta} & \tilde{g}^{AB} \end{pmatrix} \begin{pmatrix} x\partial_x \\ x\partial_y \\ \partial_z \end{pmatrix},$$

so $\text{Eval}(g^{-1})|_{\partial X} = \tilde{g}^{AB}\partial_A\partial_B$. Nondegeneracy of $\begin{pmatrix} \tilde{g}^{00} & \tilde{g}^{0\beta} \\ \tilde{g}^{\alpha0} & \tilde{g}^{\alpha\beta} \\ \tilde{g}^{A0} & \tilde{g}^{A\beta} \end{pmatrix}$ is equivalent to nondegeneracy of $\tilde{g}^{AB}$. The induced metric on the fibers is $(g_F)_{AB}dz^Adz^B$, where $(g_F)_{AB} = (\tilde{g}^{AB})^{-1}$. This metric $g_F$ is independent of the choice of all coordinates.

Next we introduce the notion of an exact edge metric. Let $g$ be an edge metric and $x$ a defining function. Now $\frac{dx}{x}$ is a smooth section of $e^T^*X$. Let $X_x$ be the edge vector field dual to $\frac{dx}{x}$ with respect to $g$. $\text{Eval}(X_x)|_{\partial X}$ is then a section of $V$. If $g$ is horizontally nondegenerate, we can define a section $\alpha_x$ of $V^\ast$ to be its dual with respect to $g_F$. Thus we have associated to each defining function $x$ a 1-form $\alpha_x$ on the fibers of $\partial X$. If $\hat{x} = ax$ is another defining function, then $\frac{d\hat{x}}{\hat{x}} = \frac{dx}{x} + \frac{da}{a} = \frac{dx}{x} + d\log a$. Following through the definition shows that the corresponding 1-forms are related by $\alpha_{ax} = \alpha_x + d_V \log a$, where $d_V \log a = d\log a|_V$. Thus $\alpha_x$ changes by an exact form under change of defining function.

**Definition 2.4.** An edge metric $g$ is said to be exact if $g$ is horizontally nondegenerate and if for each defining function $x$, there is $f \in C^\infty(\partial X)$ so that $\alpha_x = d_V f$.

The above reasoning shows that if this holds for one $x$, it holds for all $x$. If $g$ is exact, then by correct choice of $a$ we can find $x$ so that $\alpha_x = 0$.

**Definition 2.5.** If $g$ is an exact edge metric, a defining function $x$ is said to be $g$-related if $\alpha_x = 0$.

If $x$ is $g$-related, then another defining function $\hat{x} = ax$ is also $g$-related if and only if $d_V a = 0$, i.e. $a|_{\partial X}$ is locally constant on the fibers. So $g$-related defining functions are determined precisely up to multiplication by a positive function whose restriction to $\partial X$ is locally constant on the fibers.

Clearly $x$ is $g$-related if and only if $\text{Eval}(X_x)|_{\partial X} = 0$. Since $g^{-1}$ is given by (2.2), we deduce that $x$ is $g$-related if and only if $\tilde{g}^{0B}|_{\partial X} = 0$, and this condition is independent of the choice of $g^\alpha$ and $z^A$.

Recall that a horizontally nondegenerate edge metric $g$ induces a conformal class of metrics on $TX|_{\partial X}/V$ with representatives $(x^2g)|_{\partial X}$. If $g$ is exact, we can restrict to representatives of the conformal class which arise from $g$-related defining functions $x$. We will call this the reduced conformal infinity of $g$. Metrics in the reduced conformal
infinity are determined up to rescaling by functions which are locally constant on the fibers of \( \partial X \). Choosing a representative metric in the reduced conformal infinity is entirely equivalent to choosing a \( g \)-related defining function \( x \) to first order at \( \partial X \).

The following lemma will be useful in the sequel.

**Lemma 2.6.** Let \( g \) be an exact edge metric and \( x \) a \( g \)-related defining function. Then 
\[
\left| \frac{dx}{x} \right|_{H|_{\mathcal{U}}}^2 = \left| \frac{dx}{x} \right|_g^2 \text{ on } \partial X.
\]
In particular, if \( g \) is exact and normalized, then every \( g \)-related defining function is \( g \)-normalized.

**Proof.** Choose local coordinates \((x, y^\alpha, z^A)\), taking \( x \) to be the given \( g \)-related defining function. Recall that if 
\[
M = \begin{pmatrix} T & U \\ V & W \end{pmatrix}
\]
is an invertible matrix in block form with \( T \) and \( W \) square and \( T \) invertible, then its inverse can be written
\[
M^{-1} = \begin{pmatrix} T^{-1} + T^{-1}US^{-1}VT^{-1} & -T^{-1}US^{-1} \\ -S^{-1}VT^{-1} & S^{-1} \end{pmatrix},
\]
where \( S = W - VT^{-1}U \) necessarily is invertible. Apply this with
\[
M = \begin{pmatrix} \bar{g}_{00} & \bar{g}_{0\beta} & \bar{g}_{0B} \\ \bar{g}_{a0} & \bar{g}_{a\beta} & \bar{g}_{aB} \\ \bar{g}_{A0} & \bar{g}_{A\beta} & \bar{g}_{AB} \end{pmatrix},
\]
\[
T = \begin{pmatrix} \bar{g}_{00} & \bar{g}_{0\beta} \\ \bar{g}_{a0} & \bar{g}_{a\beta} \end{pmatrix}, \quad W = (\bar{g}_{AB}), \quad U = V^t = \begin{pmatrix} \bar{g}_{0B} \\ \bar{g}_{0A} \\ \bar{g}_{aB} \end{pmatrix},
\]
all evaluated at \( \partial X \). The hypothesis that \( x \) is \( g \)-related says exactly that the first row of \( T^{-1}US^{-1}VT^{-1} \) vanishes. Hence the first row of \( T^{-1}US^{-1}VT^{-1} \) also vanishes. Hence the first row of \( T^{-1} + T^{-1}US^{-1}VT^{-1} \) agrees with the first row of \( T^{-1} \). In particular, their \( 00 \) components agree, which is the desired conclusion. \( \square \)

**Remark 2.7.** A simpler proof can be given if one assumes that \( \left| \frac{dx}{x} \right|_g^2 \neq 0 \) on \( \partial X \). This is of course automatic for \( g \) positive definite. Under this hypothesis one can make a change of the \( y \)-variables \( y^\alpha \to y^\alpha + \lambda^\alpha x \), with \( \lambda^\alpha|_{\partial X} \) chosen to make \( \bar{g}^{0\alpha}|_{\partial X} = 0 \). In the new coordinates one has \( \bar{g}^{0\alpha} = 0 \) and \( \bar{g}^{0A} = 0 \) on \( \partial X \), and the conclusion is clear.

Next we formulate the normal form condition. If \( X \) is an edge manifold, then \( \partial X \) is the total space of a fibration. Consider \([0, \infty) \times \partial X \) as a manifold-with-boundary, with boundary \( \{0\} \times \partial X \cong \partial X \). The given fibration of \( \partial X \) induces a natural edge manifold structure on \([0, \infty) \times \partial X \). The coordinate \( x \) of the first factor is a canonical defining function on \([0, \infty) \times \partial X \).

**Definition 2.8.** An edge metric \( g \) on a neighborhood \( \mathcal{U} \) of \( \{0\} \times \partial X \) in \([0, \infty) \times \partial X \) is in **normal form** if \( g = \frac{dx^2}{x^2} + k \), where \( k \) is a smooth section of \( \mathcal{T}^*\mathcal{U} \) satisfying \( x\partial_x \cdot k = 0 \) everywhere.
This is equivalent to requiring that $g$ have the form
\[
g = \left( \begin{array}{ccc}
\frac{dx}{x} & \frac{dy}{x} & dz \\
0 & \bar{g}_{\alpha\beta} & 0 \\
0 & 0 & \bar{g}_{AB}
\end{array} \right)
\left( \begin{array}{c}
\frac{dx}{x} \\
\frac{dy}{x} \\
\frac{dz}{B}
\end{array} \right),
\]
and $k$ is given by
\[
k = \left( \begin{array}{c}
\frac{dy}{x} \\
\frac{dz}{A}
\end{array} \right)
\left( \begin{array}{ccc}
\bar{g}_{\alpha\beta} & 0 & \bar{g}_{\alpha B} \\
0 & \bar{g}_{AB} & 0 \\
0 & 0 & \bar{g}_{AB}
\end{array} \right)
\left( \begin{array}{c}
\frac{dy}{x} \\
\frac{dz}{B}
\end{array} \right).
\]
Observe that $g$ is exact and normalized. Also $x$ is $g$-normalized and $g$-related.

The main theorem asserts that any exact, normalized edge metric $g$ can be put into normal form, and the normal forms for $g$ are parametrized by the $g$-related defining functions to first order, or equivalently by the representatives $f$ or the reduced conformal infinity.

**Theorem 2.9.** Let $X$ be an edge manifold and $g$ an exact, normalized edge metric. If $x_0$ is a $g$-related defining function, then there is a unique diffeomorphism $\psi$ from a neighborhood of $\{0\} \times \partial X$ in $[0, \infty) \times \partial X$ to a neighborhood of $\partial X$ in $X$, such that $\psi|_{\partial X} = \text{Id}$, $\psi^*g$ is in normal form, and $\psi^*x_0 = x + O(x^2)$.

The main step in the proof is to solve the eikonal equation:

**Proposition 2.10.** Let $X$ be an edge manifold and $g$ an exact, normalized edge metric. If $x_0$ is a $g$-related defining function, then in a neighborhood of $\partial X$ there is a $g$-related defining function $\hat{x}$, uniquely determined by the conditions
\[
\left| \frac{d\hat{x}}{\hat{x}} \right|_g = 1, \quad \hat{x} = x_0 + O(x_0^2).
\]

Theorem 2.9 follows from Proposition 2.10 by the usual argument of flowing along integral curves:

**Proof of Theorem 2.9.** Let $\hat{x}$ be as in Proposition 2.10. Recall that $X_{\hat{x}}$ is the edge vector field dual to $d\hat{x}/\hat{x}$ with respect to $g$, and $\text{Eval}(X_{\hat{x}}) = 0$ on $\partial X$ since $\hat{x}$ is $g$-related. Consequently $N := \hat{x}^{-1}\text{Eval}(X_{\hat{x}})$ is a smooth vector field up to $\partial X$, and $N\hat{x} = \left| \frac{d\hat{x}}{\hat{x}} \right|_g^2 = 1$. In particular, $N$ is transverse to $\partial X$. For $x \geq 0$ and $p \in \partial X$, define $\psi(x, p)$ to be the point obtained by following the integral curve of $N$ emanating from $p$ for $x$ units of time. Since $N\hat{x} = 1$, we have $\psi^*\hat{x} = x$, and $N$ is orthogonal to the level sets of $\hat{x}$ since $X_{\hat{x}}$ is dual to $d\hat{x}/\hat{x}$. Thus $\psi^*g$ has the desired form. \hfill \Box

We conclude this section by reducing Proposition 2.10 to the solution of a singular initial value problem of the form considered in Theorem 1.1. It suffices to prove Proposition 2.10 locally in a neighborhood of a boundary point, since the uniqueness implies that the local solutions will piece together to form a global solution. Relabel $x_0$ as $x$.
and write $\tilde{x} = e^{\omega} x$. Our new unknown is $\omega$, with boundary condition $\omega = 0$ at $x = 0$. Now $d\tilde{x} = dx + d\omega$, so the equation $|d\tilde{x}|^2_g = 1$ becomes

$$
(2.3) \quad 2X_x\omega + |d\omega|^2_g = 1 - \left|\frac{dx}{x}\right|^2_g,
$$

where we now neglect the distinction between $X_x$ and $\text{Eval}(X_x)$. Lemma 2.6 shows that $x$ is $g$-normalized, so the right-hand side vanishes at $\partial X$. Work in local coordinates $(x, y, z, x^A)$ as above. The left-hand side is a quadratic polynomial in $x\partial_x\omega$, $x\partial_y\omega$, and $\omega$ with no constant term and with coefficients smooth up to the boundary. It follows that (2.3) can be written as

$$
(2.4) \quad Q(x, y, z, x\partial_x\omega, \partial_y\omega, \partial_z\omega) = f(x, y, z),
$$

where $Q$ is a quadratic polynomial in $(x\partial_x\omega, \partial_y\omega, \partial_z\omega)$ with no constant term and with coefficients depending on $(x, y, z)$ which are smooth up to $x = 0$, and $f$ is smooth with $f(0, y, z) = 0$. (We have absorbed the $x$ multiplying $\partial_y\omega$ into the coefficients.) We make the following observations about $Q$. First, the coefficient of the linear term $x\partial_x\omega$ is nonzero at $x = 0$, since $X_x = x\left|\frac{dx}{x}\right|^2_g$. Second, the coefficients of the linear terms $\partial_y\omega$ and $\partial_z\omega$ vanish at $x = 0$, since $x$ is $g$-related so that $X_x = 0$ at $x = 0$. Third, all of the arguments $(x\partial_x\omega, \partial_y\omega, \partial_z\omega)$ themselves vanish at $x = 0$ when evaluated on any function $\omega$ satisfying the initial condition $\omega = 0$ at $x = 0$. In particular, the partial derivative of the quadratic terms of $Q$ with respect to any of $x\partial_x\omega, \partial_y\omega, \partial_z\omega$ vanishes at $x = 0$ when evaluated on the initial data.

The implicit function theorem (or the quadratic formula) implies that in a neighborhood of $(x, y, z, x\partial_x\omega, \partial_y\omega, \partial_z\omega) = (0, y, z, 0, 0, 0)$, (2.4) may be solved for $x\partial_x\omega$. So it may be written in the form

$$
(2.5) \quad x\partial_x\omega = F(x, y, z, \partial_y\omega, \partial_z\omega),
$$

where $F$ is a smooth function of its arguments satisfying $F(0, y, z, 0, 0) = 0$. Moreover, the observations above show that $F_{\partial_y\omega}(0, y, z, 0, 0) = 0$ and $F_{\partial_z\omega}(0, y, z, 0, 0) = 0$. Equation (2.5) with initial condition $\omega = 0$ at $x = 0$ is of the form (1.3), where $y$ in (1.3) plays the role of $(y, z)$ in (2.5). Condition (1.5) holds since $F$ in (2.5) is independent of $\omega$, so that $F_{\omega} \equiv 0$. Thus Proposition 2.10 follows from Theorem 1.1.

3. Singular Initial Value Problems

In this section we prove Theorem 1.1. First observe that the conclusion in Theorem 1.1 fails without the hypothesis $F_{\omega}(0, y, \omega_0(y), \partial_y\omega_0(y)) < 1$. For instance, the equation $x\partial_x\omega = \omega$ has infinitely many smooth solutions $\omega = cx$ satisfying $\omega(0) = 0$, and the equation $x\partial_x\omega = \omega + x$ has no smooth solutions (the general solution is $\omega = cx + x \log x$). Also note that if $0 < F_{\omega}(0, y, \omega_0(y), \partial_y\omega_0(y)) < 1$, then the smooth solution need not be the only continuous solution. For example, if $0 < \alpha < 1$, then $\omega = cx^\alpha$ solves $x\partial_x\omega = \alpha \omega$ with $\omega(0) = 0$ for any $c \in \mathbb{R}$. In this case the unique smooth solution is $\omega = 0$. 


We first use a standard reduction technique via Taylor expansion to reduce the equation to a simpler form. In the following we denote \( F_\omega(0)(y) = F_\omega(y, \omega(y), \partial_y \omega(y)) \) and similarly for other derivatives of \( F \) evaluated on the initial data.

Observe first that differentiating (1.3) with respect to \( x \) at \( x = 0 \) and solving for \( \omega_x \) shows that if \( \omega \) is a smooth solution, then
\[
\omega_x(0, y) = \frac{F_x(0)(y)}{1 - F_\omega(0)(y)} := \omega_1(y).
\]

We can write
\[
\omega(x, y) = \omega_0(y) + x(\omega_1(y) + u(x, y))
\]
for a smooth function \( u(x, y) \), and regard \( u \) as the new unknown.

**Proposition 3.1.** In terms of \( u \), (1.3) becomes
\[
(3.1) \quad x \partial_x u = (F_\omega(0)(y) - 1)u + xG(x, y, u, \partial_y u), \quad u(0, y) = 0,
\]
where \( G \) is a smooth function of \((x, y, u, q)\).

**Proof.** It is clear from the discussion above that the initial condition on \( u \) is \( u(0, y) = 0 \). Set \( q_0(y) = \partial_y \omega_0(y) \). The second order Taylor expansion of \( F(x, y, \omega, q) \) about \((0, y, \omega_0(y), q_0(y))\) takes the form
\[
F(x, y, \omega, q) = F_x(0)(y)x + F_\omega(0)(y)(\omega - \omega_0(y)) + Q(x, \omega - \omega_0(y), q - q_0(y)),
\]
where \( Q \) is a homogeneous quadratic polynomial of its arguments with coefficients which are smooth functions of \((x, y, \omega, q)\). We have
\[
\partial_x \omega = \omega_1 + (x \partial_x + 1)u, \quad \partial_y \omega = q_0(y) = x(\partial_y \omega_1 + \partial_y u).
\]

Substituting and then dividing by \( x \) shows that (1.3) becomes
\[
\omega_1 + (x \partial_x + 1)u = F_x(0) + F_\omega(0)(\omega_1 + u) + xG(x, y, u, \partial_y u)
\]
for a smooth function \( G \). The definition of \( \omega_1 \) shows that \( \omega_1 = F_x(0) + F_\omega(0) \omega_1 \), so this reduces to (3.1). \( \Box \)

**Proposition 3.2.** Let \( b(y) \) and \( G(x, y, u, q) \) be smooth and suppose \( b(y) < 0 \). Then the IVP
\[
(3.2) \quad x \partial_x u = b(y)u + xG(x, y, u, \partial_y u), \quad u(0, y) = 0
\]
has a unique smooth solution for sufficiently small \( x \geq 0 \).

We prove Proposition 3.2 by an adaptation of the method of characteristics. The main tool is a result asserting the existence and uniqueness of smooth “characteristic integral curves” of time-dependent vector fields vanishing at an initial point.

Let \( M \) be a smooth manifold and \( p_0 \in M \). Suppose that \( V(t, p) \) is a smooth time-dependent vector field defined for \( t \) near 0 and \( p \) in a neighborhood of \( p_0 \), such that
V(0,p₀) = 0. By a characteristic integral curve for V at p₀ we mean a curve γ : [0, ε) → M for some ε > 0 such that

\[ t \frac{d}{dt} \gamma(t) = V(t, \gamma(t)), \quad \gamma(0) = p₀. \tag{3.3} \]

Recall that the linearization of a vector field at a zero is the endomorphism DV of Tp₀M such that V(p) = DV(p − p₀) to first order at p₀. For a time-dependent vector field this refers to the linearization of the vector field in the space variables with t fixed.

**Theorem 3.3.** Let V(t, p) be a smooth time-dependent vector field such that V(0, p₀) = 0. Suppose that all eigenvalues λ of DV(0, p₀) satisfy Re λ < 1. Then on a sufficiently small time interval there exists a unique smooth characteristic integral curve for V at p₀. This characteristic integral curve depends smoothly on variations of the initial point p₀ for which the conditions V(0, p₀) = 0 and Re λ < 1 continue to hold.

Observe that the case M = ℝ of Theorem 3.3 coincides precisely with the special case n = 1 of Theorem 1.1, upon relabeling t as x, γ as ω, and V as F. In particular, the examples above show that existence and uniqueness of smooth solutions can fail if λ = 1, and there may be continuous solutions which are not smooth if 0 < λ < 1.

The first step in the proof of Theorem 3.3 is to perform a Taylor expansion analogous to the one made above for the pde. Work in local coordinates on M and let x₀ denote the coordinates of p₀. Differentiating (3.3) with respect to t at t = 0 and solving for γ′(0) gives

\[ γ′(0) = [I - DV(0, x₀)]^{-1}V_t(0, x₀) := γ_1. \]

Write

\[ γ(t) = x₀ + t(γ_1 + σ(t)). \tag{3.4} \]

Upon Taylor expanding V(t, x) about (0, x₀), substituting (3.4), and simplifying as in the proof of Proposition 3.1, one finds that when written in terms of σ, (3.3) takes the form

\[ tσ′ + Aσ = tG(t, σ), \quad σ(0) = 0. \tag{3.5} \]

Here A = I − DV(0, x₀) has the property that all of its eigenvalues have positive real part. G is smooth, and A and G depend smoothly on the initial point x₀. Initial value problems of the form (3.5) are studied in Chapter 5 of [K]. The results formulated there assume that A is independent of the parameters, but the same arguments apply to our situation. We briefly outline a proof that (3.5) has a unique smooth solution varying smoothly with the parameters x₀ if the eigenvalues of A have positive real part. Theorem 3.3 is then a consequence by the reduction above.

The problem (3.5) can be reformulated as the integral equation

\[ σ(t) = (Tσ)(t) := t \int_0^1 s^A G(st, σ(st))ds. \tag{3.6} \]

The hypothesis that the eigenvalues of A have positive real part implies that the operators s^A are uniformly bounded for 0 < s ≤ 1. A standard contraction mapping/fixed
point argument proves the existence and uniqueness of a continuous solution. To establish smoothness in \( t \), rewrite (3.6) as

\[
\sigma(t) = t^{-A} \int_0^t s^A G(s, \sigma(s)) ds.
\]

This shows that \( \sigma \) is \( C^1 \) for \( t > 0 \). Differentiate in \( t \) and change variables back to obtain

(3.7) \quad \sigma'(t) = G(t, \sigma(t)) - A \int_0^1 s^A G(st, \sigma(st)) ds.

Thus \( \sigma \) is \( C^1 \) up to \( t = 0 \). Now successively differentiating (3.7) shows that \( \sigma \) is \( C^\infty \). Smoothness of \( \sigma \) with respect to the parameters is a consequence of the implicit function theorem applied to the equation \( \sigma - T \sigma = 0 \).

**Proof of Proposition 3.2.** We construct a singular version of a Hamiltonian flow-out in the first jet bundle of the solution \( u \). The argument follows the usual reasoning for the non-characteristic case, substituting Theorem 3.3 in an appropriate parameterization for the existence and uniqueness of integral curves of the Hamiltonian vector field.

Let \( J \) denote the first jet bundle of a scalar function \( u \) on \( \mathbb{R}^n \), with coordinates \((x, y, u, p, q)\) where \( p \) is the variable dual to \( x \), and projection \( \pi : J \to \mathbb{R}^n \) given by \( \pi(x, y, u, p, q) = (x, y) \). Set \( x = (x, y) \) and \( p = (p, q) \). The 1-jet of a function \( u \) on \( \mathbb{R}^n \) is the section of \( J \) given by \( x \to (x, u(x), du(x)) \). We denote its image \( \{(x, u(x), du(x))\} \) by \( S_u \); this is a submanifold of \( J \) of dimension \( n \). The tautological contact form is \( \theta = du - p_i dx^i \). \( S_u \) is a Legendrian submanifold relative to \( \theta \); i.e. the pullback of \( \theta \) to \( S_u \) vanishes.

Recall that if \( H(x, u, p) \) is a smooth real function on \( J \), the associated Hamiltonian vector field is

\[
\xi_H = H_p \partial_x + p_i H_{pi} \partial_u - (H_x + p_i H_{pi}) \partial_{p_i}.
\]

It is uniquely determined by the conditions

(3.8) \quad \xi_H \mathcal{J} \partial \theta = dH \mod \theta \quad \theta(\xi_H) = 0

and satisfies

(3.9) \quad \xi_H \mathcal{J} d \theta = dH - H_u \theta, \quad \xi_H H = 0.

If \( u \) is a solution of \( H(x, u, du) = 0 \), then \( \xi_H \) is tangent to \( S_u \) at all points of \( S_u \).

Take \( H \) to be the Hamiltonian corresponding to (3.2), i.e.

\[
H(x, y, u, p, q) = xp - b(y)u - xG(x, y, u, q).
\]

Differentiating (3.2) at \( x = 0 \) shows that a solution \( u \) must satisfy

(3.10) \quad \partial_x u(0, y) = \frac{G(0, y, 0, 0)}{1 - b(y)} := p_0(y).

Define a smooth submanifold \( \mathcal{I} \subset J \) of dimension \( n - 1 \) by

\[
\mathcal{I} = \{(0, y, 0, p_0(y), 0)\}.
\]
Hamilton’s equations for the integral curves of $\xi_H$ take the form

$$
\begin{align*}
\frac{dx}{ds} &= x \\
\frac{dy}{ds} &= -xG_q \\
\frac{du}{ds} &= x(p - q_iG_{q_i}) \\
\frac{dp}{ds} &= p(b(y) - 1) + G + x(G_x + pG_u) \\
\frac{dq}{ds} &= b_y(y)u + b(y)q + x(G_y + qG_u).
\end{align*}
$$

Observe that $\xi_H$ vanishes identically on $\mathcal{I}$. So all integral curves of $\xi_H$ beginning on $\mathcal{I}$ are constant; there is no Hamiltonian flow-out in the usual sense. Instead we consider characteristic integral curves of $\xi_H$ beginning on $\mathcal{I}$. $\xi_H$ is time-independent and constants are also characteristic integral curves. But the characteristic integral curves are not unique: $D\xi_H$ on $\mathcal{I}$ has $\lambda = 1$ as an eigenvalue arising from the first equation in the system above. By using $x$ as the parameter, we will obtain unique nonconstant characteristic integral curves of $\xi_H$ emanating from $\mathcal{I}$ whose union will form the submanifold $S_u$ giving the solution $u$.

Use $x$ as a parameter for the characteristic integral curves. The first equation above gives $d/ds = xd/dx$. Substituting in the remaining equations gives the system

$$
\begin{align*}
\frac{dy}{dx} &= -xG_q \\
\frac{du}{dx} &= x(p - q_iG_{q_i}) \\
\frac{dp}{dx} &= p(b(y) - 1) + G + x(G_x + pG_u) \\
\frac{dq}{dx} &= b_y(y)u + b(y)q + x(G_y + qG_u).
\end{align*}
$$

This has the form (3.3), where $x$ plays the role of $t$. Choose $y_0 \in \mathbb{R}^{n-1}$ and impose initial conditions

$$
(3.12) \quad y(0) = y_0, \quad u(0) = 0, \quad p(0) = p_0(y_0), \quad q(0) = 0.
$$

The linearization of the right-hand side of (3.11) evaluated at $x = 0$ and at the given initial conditions for $(y, u, p, q)$ is

$$
D = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & b(y_0) - 1 & * & * \\
0 & * & b(y_0) & 0
\end{pmatrix},
$$
where the blocks have sizes \( n-1,1,1,1 \). Here * denotes a quantity whose value will be irrelevant and \( I \) denotes the \((n-1) \times (n-1)\) identity matrix. The eigenvalues of \( D \) are 0 with multiplicity \( n \), \( b(y_0) - 1 \) with multiplicity 1, and \( b(y_0) \) with multiplicity \( n-1 \). These are all real and less than 1, so Theorem 3.3 implies that there is a unique smooth solution \((y(x,y_0), u(x,y_0), p(x,y_0), q(x,y_0))\) of (3.11), (3.12) for sufficiently small \( x \) varying smoothly with \( y_0 \).

Define a map \( \Phi \) into \( \mathcal{J} \) by

\[
\Phi(x,y_0) = (x, y(x,y_0), u(x,y_0), p(x,y_0), q(x,y_0)).
\]

Since \( y(0,y_0) = y_0 \), it follows that \( \Phi \) is a diffeomorphism from a neighborhood of \( \{x = 0\} \) to a submanifold \( \mathcal{F} \subset \mathcal{J} \) of dimension \( n \). We claim that \( H = 0 \) on \( \mathcal{F} \). Since \( \xi_H x = x \) and \( y_0 \) is constant on the solution curves, it follows that \( \Phi^* \xi_H = x \partial_x \). Since \( \xi_H H = 0 \), we have \( x \partial_x (\Phi^* H) = 0 \). Since \( H = 0 \) on \( \mathcal{I} \), one concludes that \( H = 0 \) on \( \mathcal{F} \) as claimed.

We now prove existence in Proposition 3.2. The projection \( \pi : \mathcal{J} \to \mathbb{R}^n \) restricts to a diffeomorphism from \( \mathcal{F} \) to a neighborhood of \( \{x = 0\} \) (possibly after shrinking \( \mathcal{F} \)). Therefore on \( \mathcal{F} \) we can regard \( u, p, q \) as functions of \( (x,y) \). In particular this defines a smooth function \( u(x,y) \). We claim that

\[
(3.13) \quad p(x,y) = \partial_x u(x,y), \quad q_i(x,y) = \partial_{y^i} u(x,y).
\]

This is equivalent to saying that \( \mathcal{F} = \mathcal{S}_u \). Existence in Proposition 3.2 follows immediately, as then the equation \( H = 0 \) on \( \mathcal{F} \) together with the initial condition become the statement that \( u \) satisfies (3.2).

Since \( \theta = du - pdx - q_i dy^i \), in order to prove (3.13) it suffices to show that the pullback of \( \theta \) to \( \mathcal{F} \) vanishes. Recalling (3.8), (3.9), observe that

\[
\mathcal{L}_{\xi_H} \theta = \xi_H \mathcal{J} \theta + d(\theta(\xi_H)) = dH - H_0 \theta,
\]

where \( \mathcal{L} \) denotes the Lie derivative. For the pullback to \( \mathcal{F} \) we therefore obtain \( \mathcal{L}_{\xi_H} \theta = -H_0 \theta \). Pulling back by \( \Phi \) gives \( \mathcal{L}_{x \partial_x} \Phi^* \theta = -\Phi^* H_u \Phi^* \theta \). If we write \( \Phi^* \theta = \theta_0 dx + \theta_i dy^i \), then this becomes

\[
(3.14) \quad x \partial_x \theta_0 = (-\Phi^* H_u - 1) \theta_0, \quad x \partial_x \theta_i = (-\Phi^* H_u) \theta_i.
\]

For each \( y_0 \), these are scalar ode’s of the form \( x \partial_x v = \beta(x)v \), where \( v = \theta_0 \) or \( \theta_i \) and \( \beta = -\Phi^* H_u - 1 \) or \( -\Phi^* H_u \). Since \( H_u = -b(y) \) at \( x = 0 \) and \( b < 0 \), we have \( \beta < 0 \) near \( x = 0 \). The general solution is

\[
v = c \exp \int^x \frac{\beta(s)}{s} ds = cx^{\beta(0)} h(x),
\]

where \( c \in \mathbb{R} \) and \( h \) is a nonvanishing smooth function. Since \( \beta(0) < 0 \) and \( \theta \) is smooth, we must have \( c = 0 \), so we obtain \( \theta_0 = \theta_i = 0 \) as desired. (Alternately, the vanishing of \( \theta_0 \) and \( \theta_i \) follows from uniqueness in Theorem 3.3 applied to (3.14).)

Finally we prove uniqueness in Proposition 3.2. We show that if \( u \) is any smooth solution of (3.2), then \( \mathcal{S}_u = \mathcal{F} \) near \( \mathcal{I} \). We have already observed that \( \partial_x u(0,y) \) is given
by (3.10), so that $S_u \cap \{x = 0\} = \mathcal{I} = \mathcal{F} \cap \{x = 0\}$. The system
\[
    x \frac{dy}{dx} = -x G_q(x, y, u(x, y), \partial_y u(x, y)), \quad y(0) = y_0
\]
for unknown $y(x)$ has a unique smooth solution (either by Theorem 3.3 or by cancelling $x$ and quoting usual ode theory). As $y_0$ varies, the corresponding curves $(x, y(x))$ fill out a neighborhood of $\{x = 0\}$ in $\mathbb{R}^n \cap \{x \geq 0\}$. Therefore near $\mathcal{I}$, $S_u$ is the union over $y_0$ of the lifts
\[
    (3.15) \quad x \mapsto (y(x), u(x, y(x)), \partial_x u(x, y(x)), \partial_y u(x, y(x))).
\]
The curves
\[
    x \mapsto (y(x), u(x, y(x)), \partial_x u(x, y(x)), \partial_y u(x, y(x)))
\]
solve (3.11) since $\xi_H$ is everywhere tangent to $S_u$. Since $\mathcal{F}$ was defined to be the union of all curves (3.15) corresponding to solutions of (3.11), it follows that $S_u = \mathcal{F}$ near $\mathcal{I}$.

\[
\Box
\]

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