Finite-Temperature and -Density QED: Schwinger-Dyson Equation in the Real-Time Formalism

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Abstract

Based on the real-time formalism, especially, on Thermo Field Dynamics, we derive the Schwinger-Dyson gap equation for the fermion propagator in QED and Four-Fermion model at finite-temperature and -density. We discuss some advantage of the real-time formalism in solving the self-consistent gap equation, in comparison with the ordinary imaginary-time formalism. Once we specify the vertex function, we can write down the SD equation with only continuous variables without performing the discrete sum over Matsubara frequencies which cannot be performed in advance without further approximation in the imaginary-time formalism. By solving the SD equation obtained in this way, we find the chiral-symmetry restoring transition at finite-temperature and present the associated phase diagram of strong coupling QED. In solving the SD equation, we consider two approximations: instantaneous-exchange and $p_0$-independent ones. The former has a direct correspondence in the imaginary time formalism, while the latter is a new approximation beyond the former, since the latter is able to incorporate new thermal effects which

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has been overlooked in the ordinary imaginary-time solution. However both approximations are shown to give qualitatively the same results on the finite-temperature phase transition.
1 Introduction

Quantum chromodynamics (QCD) exhibits very characteristic dynamics of confinement of quarks and gluons in long distance as well as the asymptotic freedom at short distance. According to the standard cosmology, however, our universe ought to have undergone the confinement-deconfinement transition of quarks and gluons in the early stage of evolution. This implies that the hadron phase at present should transfer into the quark-gluon-plasma phase, if the universe goes back to the past at high-temperature and -density [1, 2, 3, 4]. Furthermore, the quark-gluon-plasma is not merely a byproduct of the theory in the sense that its existence will be verified by experiments of heavy ion collision in the following several years [1].

At high temperature, the effective coupling constant of QCD gets small owing to the asymptotic freedom [5]. Hence it is natural to consider that the perturbation theory can be applied to QCD at high temperature successfully [6, 7, 8, 9]. However it becomes evident in the early stage of the investigation of high-temperature QCD that infrared (IR) singularity prevents one from calculating higher order terms in perturbation theory [10, 11, 12, 13, 14].

In view of this, our study aims at performing the non-perturbative analysis on the phase transition at finite-temperature and -density (FTD). For this we study the self-consistent (SC) equation represented by the Schwinger-Dyson (SD) equation and the Bethe-Salpeter (BS) equation. Such SC equations are simultaneous non-linear integral equations and are able to incorporate various non-perturbative effects self-consistently [15]. The concept of the SC equation is not new and indeed the SC equation is a well-known traditional method to solve the many body problem in condensed matter physics and nuclear physics [16, 17]. Nevertheless SC equations turn out to provide the powerful method also in elementary particle physics: Recently various types of SC or gap equations have been extensively studied to investigate the dynamical symmetry breaking in gauge field theories and have succeeded to reveal rich phase structure of strong coupling gauge theories [15]. In this paper we extend this type of research to the case of finite-temperature and density.

We desire to develop the method to study the phase transition at finite-temperature and -density from the quantum field theoretical point of view beyond the thermodynamic or phenomenological treatment. For this, we need at first the formulation of FTD field theory. Traditional Imaginary-Time Formalism (ITF) due to Matsubara [18] is very efficient to develop the perturbation theory of FTD field theory. In ITF, the field theory at zero-temperature can be transferred into the FTD field theory according to the following procedures: in a given Feynman diagram, each integral over the temporal component of a "fermion" loop momentum is replaced by an infinite sum over "odd" Matsubara frequencies according to the prescription:

$$
\int \frac{dp_0}{2\pi} f(p_0) \Rightarrow T \sum_{n=-\infty}^{+\infty} f(p_0 \rightarrow (2n + 1)\pi T - i\mu),
$$

(1)

where $n$ denotes the integer, $T$ the temperature and $\mu$ the chemical potential. In contrast, each integral over the temporal component of "bosonic" loop momentum is evaluated by
summing over "even" Matsubara frequencies,

$$\int \frac{dk_0}{2\pi} g(k_0) \Rightarrow T \sum_{n=-\infty}^{+\infty} g(k_0 \to 2n\pi T).$$  \quad (2)

However applying this method to the non-perturbative study is sometimes either convenient nor easy. Indeed the infinite sum is very cumbersome particularly in solving the SC equation, because the convergence of the infinite series is problematical and the discrete sum cannot be performed in advance before we know the solution, in sharp contrast with the perturbation theory where all the propagators are bare.

To write down explicitly the SC equation in the closed form, we must adopt the approximation or ansatz for the truncation of infinite hierarchy of SC equations in general field theories. Once such an ansatz (e.g., quenched ladder approximation in the SD equation) is adopted, we should not make further approximation to solve SC equations. However this forces us to solve the infinite set of integral equations distinguished by an integer \(n\) in ITF. In most studies based on the SC equation in IFT, therefore, infinite sum over discrete Matsubara frequencies was evaded by searching for frequency-independent approximate solution from the first \[14, 20, 21, 22, 23, 25, 26\]. To simplify further the equation, the constant (i.e., momentum-independent) solution has been investigated as in the NJL model in the leading 1/N approximation. By using such solutions, some qualitative features of FTD phase transition have been investigated in QED \(_4\) \[23, 26\], QED \(_3\) \[21, 22, 23\] and QCD \(_4\) \[19, 20\]. However it is evident that in gauge theories the zero-temperature limit of the momentum-independent approximate solution obtained in such a way does not necessarily coincide with the zero-temperature solution \[15\]. To avoid this type of discrepancy, we must obtain the momentum-dependent solution at finite-temperature as tried in \[24\] for QED \(_3\).

On the other hand, Thermo Field Dynamics (TFD), a realization of Real-Time Formalism (RTF) of FTD field theory \[27\], proposed by Takahashi and Umezawa \[28, 29\] needs no discrete sum and treats only continuous variables. Hence TFD is well suited to extend the usual field theoretical methods to the finite-temperature and is expected to be quite efficient to write down and solve the SC equation. Thanks to TFD, various methods developed so far in solving the integral equation can be extended to SC equations in the FTD case without much difficulty. Actually the infinite set of SC equations indexed by an integer in ITF is reduced to the corresponding single SC equation with an additional argument of continuous variable in TFD, as exemplified for the SD equation in this paper. Such a viewpoint has been overlooked to the best of our knowledge.

This paper is the first of a series of papers which treat the transition of finite-temperature and density in field theories by use of self-consistent equations based on RTF, particularly on TFD. In this paper, restricting to the Nambu-Jona-Lasinio (NJL) \[31\]-type four-fermion model and QED, we write down the SD equation for the fermion propagator based on TFD. Based on the solution of the SD equation for the fermion propagator, we show the existence of chiral-symmetry-restoring transition in QED \(_4\) by obtaining the critical line which separates the low-temperature phase where the chiral symmetry is spontaneously broken from the high-temperature phase where the chiral symmetry restores.
This paper is organized as follows. In section 2, the formalism of TFD is briefly reviewed to prepare the necessary materials and to fix the notation. In section 3, we treat the simplest case of NJL-type four-fermion model in the leading $1/N$ approximation, i.e., chain approximation. Within this approximation both formalism give the same SD equation for the fermion mass function. In section 4, we write down the SD equation in FTD QED based on TFD. In section 5, we solve the SD equation under the instantaneous-exchange approximation. In section 6, we consider another approximation and compare the result with the previous one. The final section is devoted to the conclusion and perspective.
2 Thermo-Field-Dynamics

In this section we recall basic materials of TFD which are necessary to derive the SD equation.

2.1 scalar

Corresponding to the lagrangian of free scalar field (at zero temperature),

\[ \mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi)^2 - \frac{1}{2} m_0^2 \phi^2, \]  

(3)

the free scalar propagator at zero-temperature reads

\[ \Delta(k) = \frac{i}{k^2 - m_0^2 + i\epsilon}. \]  

(4)

In the TFD, the free scalar propagator at finite-temperature is given by the matrix form [32, 33]:

\[ iD_{ab}(k) = U(\beta, k) \begin{pmatrix} \Delta(k) & 0 \\ 0 & \Delta^*(k) \end{pmatrix} U(\beta, k), \quad \beta \equiv 1/T, \]  

(5)

where

\[ U(\beta, k) = \begin{pmatrix} \cosh \theta_k & \sinh \theta_k \\ \sinh \theta_k & \cosh \theta_k \end{pmatrix}, \]  

(6)

with \[ \cosh \theta_k := \frac{1}{\sqrt{1 - e^{-\beta|k_0|}}}, \quad \sinh \theta_k := \frac{e^{-\beta|k_0|/2}}{\sqrt{1 - e^{-\beta|k_0|}}} \]  

(7)

In the zero-temperature limit \( \beta \to \infty \), \( U_{ab}(\beta, k) \to \delta_{ab} \) and hence the scalar particle \( \Delta(k) \) and the thermal ghost \( \Delta^*(k) \) decouple.

The explicit form of the propagator reads

\[ iD_{ab}(k) = \begin{pmatrix} \cosh^2 \theta_k \Delta(k) + \sinh^2 \theta_k \Delta^*(k) & \cosh \theta_k \sinh \theta_k [\Delta(k) + \Delta^*(k)] \\ \cosh \theta_k \sinh \theta_k [\Delta(k) + \Delta^*(k)] & \sinh^2 \theta_k \Delta(k) + \cosh^2 \theta_k \Delta^*(k) \end{pmatrix}. \]  

(8)

Using the identities, \( \cosh^2 \theta_k - \sinh^2 \theta_k = 1 \) and \( \cosh^2 \theta_k + \sinh^2 \theta_k = 2 \cosh^2 \theta_k - 1 = \coth \frac{\beta|k_0|}{2} \), we obtain

\[ iD^{11}(k) = \cosh^2 \theta_k \Delta(k) + \sinh^2 \theta_k \Delta^*(k) \]  

\[ = \Delta(k) + \sinh^2 \theta_k [\Delta(k) + \Delta^*(k)] \]  

\[ = \Delta(k) + 2\pi \delta(k^2 - m_0^2) N_B(k) \]  

\[ = \frac{i}{k^2 - m_0^2 + i\epsilon} + 2\pi \delta(k^2 - m_0^2) N_B(k) \]  

\[ = \frac{i}{k^2 - m_0^2 + \pi\delta(k^2 - m_0^2) \coth \frac{\beta|k_0|}{2}}. \]  

(9)

\[ ^1 \text{The notation } A := B \text{ implies that } A \text{ is defined by } B. \]
where $N_B(k)$ is the Bose-Einstein distribution function:

$$N_B(k) := \frac{1}{\exp(\beta |k_0|) - 1}. \quad (10)$$

The full propagator will be assumed to be of the form

$$i\mathcal{D}^{ab}(k) = U(\beta, k) \begin{pmatrix} \mathcal{D}(k) & 0 \\ 0 & \mathcal{D}^*(k) \end{pmatrix} U(\beta, k), \quad (11)$$

with $\mathcal{D}(k)$ being a complex function.

As a consequence of the SD equation:

$$i\mathcal{D}^{ab}(k) = iD^{ab}(k) + iD^{ac}(k)(-i\Pi^{cd})iD^{db}(k), \quad (12)$$

the self-energy matrix is written as

$$-i\Pi^{ab}(k) = U^{-1}(\beta, k) \begin{pmatrix} -i\Pi(k) & 0 \\ 0 & i\Pi^*(k) \end{pmatrix} U^{-1}(\beta, k), \quad (13)$$

where each component is related as follows:

$$\Pi^{12}(k) = \Pi^{21}(k) = -2i \cosh \theta_k \sinh \theta_k \Im \Pi(k) = -i \tanh 2\theta_k \Im \Pi^{11}(k),$$
$$\Pi^{22}(k) = -\Pi^{11*}(k). \quad (14)$$

This allows us to write the function $\mathcal{D}(k)$ as

$$\mathcal{D}(k) = \frac{i}{k^2 - m_0^2 - \Pi + i\epsilon}. \quad (15)$$

Hence the SD equation for the scalar propagator reads

$$i\mathcal{D}^{-1}(k) = i\Delta^{-1}(k) - \Pi(k), \quad (16)$$

where the real and imaginary part is related to $\Pi^{11}(p)$ as

$$\Re \Pi(k) = \Re \Pi^{11}(k),$$
$$\Im \Pi(k) = \epsilon(k_0) \tanh\left[\frac{\beta}{2} k_0\right] \Im \Pi^{11}(k). \quad (17)$$

2.2 fermion

For the lagrangian of free fermion field,

$$\mathcal{L} = \bar{\psi}(i\not\partial - m_0)\psi, \quad (18)$$

\[\text{This is proved if there is the spectral representation which implies the Kubo-Martin-Schwinger (KMS) equilibrium condition }\]
the free fermion propagator is obtained as

\[ S(p) = \frac{i}{p - m_0 + i\epsilon} = i - \frac{p}{p^2 - m_0^2 + i\epsilon}. \]  (19)

Then the free fermion propagator at finite-temperature (and -density) is given by

\[ iS^{ab}(k) = V(\beta, p, \mu) \left( \begin{array}{cc} S(p) & 0 \\ 0 & S^*(p) \end{array} \right) V(\beta, p, \mu), \]  (20)

where \( S^*(p) \) denotes

\[ S^*(p) = \frac{-i}{p - m_0 - i\epsilon} = -i - \frac{p}{p^2 - m_0^2 - i\epsilon}, \]  (21)

and

\[ V(\beta, p, \mu) = \begin{pmatrix} \cos \varphi_{p+\mu} & \epsilon(p_0)e^{-\mu/2} \sin \varphi_{p+\mu} \\ \epsilon(p_0)e^{\beta/2} \sin \varphi_{p+\mu} & \cos \varphi_{p+\mu} \end{pmatrix}, \]  (22)

with \( \epsilon(p_0) := \theta(p_0) - \theta(-p_0) \) and

\[ \cos \varphi_{p+\mu} = \frac{\theta(p_0)e^{\beta(p_0+\mu)/4} + \theta(-p_0)e^{-\beta(p_0+\mu)/4}}{\sqrt{e^{\beta(p_0+\mu)/2} + e^{-\beta(p_0+\mu)/2}}}, \]  (23)

\[ \sin \varphi_{p+\mu} = \frac{\theta(p_0)e^{-\beta(p_0+\mu)/4} + \theta(-p_0)e^{\beta(p_0+\mu)/4}}{\sqrt{e^{\beta(p_0+\mu)/2} + e^{-\beta(p_0+\mu)/2}}}. \]  (24)

In the zero-temperature limit \( \beta \to \infty \), \( V_{ab}(\beta, p, \mu) \to \delta_{ab} \) and hence the thermal part \( S^*(p) \) is separated.

Explicitly writing down the matrix element:

\[ iS^{ab}(p) = \begin{pmatrix} \cos^2 \varphi_{p+\mu}S(p) - \sin^2 \varphi_{p+\mu}S^*(p) & -\epsilon(p_0)e^{-\beta/2} \cos \varphi_{p+\mu} \sin \varphi_{p+\mu}[S + S^*](p) \\ \epsilon(p_0)e^{\beta/2} \sin \varphi_{p+\mu} \sin \varphi_{p+\mu}[S + S^*](p) & -\sin^2 \varphi_{p+\mu}S(p) + \cos^2 \varphi_{p+\mu}S^*(p) \end{pmatrix}. \]  (25)

Note that, since \( \cos^2 \varphi_{p+\mu} + \sin^2 \varphi_{p+\mu} = 1 \), and \( \cos^2 \varphi_{p+\mu} - \sin^2 \varphi_{p+\mu} = 2 \cos^2 \varphi_{p+\mu} - 1 = \epsilon(p_0) \tanh \frac{\beta(p_0+\mu)}{2} \), the 1-1 component of the fermion propagator is rewritten as

\[ iS^{11}(p) = \cos^2 \varphi_{p+\mu}S(p) - \sin^2 \varphi_{p+\mu}S^*(p) \\
= S(p) - \sin^2 \varphi_{p+\mu}[S(p) + S^*(p)] \\
= S(p) - 2\pi \epsilon(p_0) \delta(p^2 - m_0^2)(\not{p} + m_0)N_F(p) \\
= \frac{i}{\not{p} - m_0 + i\epsilon} - 2\pi \epsilon(p_0) \delta(p^2 - m_0^2)(\not{p} + m_0)N_F(p) \\
= \frac{i}{\not{p} - m_0 + i\epsilon} + \pi \epsilon(p_0) \delta(p^2 - m_0^2)(\not{p} + m_0) \tanh \frac{\beta(p_0 + \mu)}{2}, \]  (26)

where \( N_F(p) \) is the Fermi-Dirac distribution function:

\[ N_F(p) := \sin^2 \varphi_{p+\mu} = \frac{1}{e^{\beta(p_0+\mu)}} + \frac{1}{e^{-\beta(p_0+\mu)}} \theta(p_0) + \frac{1}{e^{-\beta(p_0+\mu)}} - 1 - \theta(-p_0). \]  (27)
The full propagator will have the form

\[ iS^{ab}(p) = V(\beta, p) \begin{pmatrix} S(p) & 0 \\ 0 & S^*(p) \end{pmatrix} V(\beta, p), \]  

(28)

with \( S(p) \) being a complex function.

Similarly in the case of the scalar field, the SD equation:

\[ iS^{ab}(p) = iS^{ac}(p) + iS^{cd}(p)(-i\Sigma^{cd})iS^{db}(p), \]  

(29)

is compatible with the self-energy matrix written as

\[ -i\Sigma^{ab}(p) = V^{-1}(\beta, p) \begin{pmatrix} -i\Sigma(p) & 0 \\ 0 & i\Sigma^*(p) \end{pmatrix} V^{-1}(\beta, p), \]  

(30)

where

\[ \Sigma^{12}(p) = -e^{-\beta \mu} \Sigma^{21}(p) = i\epsilon(p_0) e^{-\beta \mu/2} \tan 2\varphi_{\mu+p} \Sigma^{11}(p), \]

\[ \Sigma^{22}(p) = -\Sigma^{11*}(p). \]  

(31)

This allows us to write the function \( S(p) \) as

\[ S(p) = \frac{i}{\gamma_\mu p_\mu - m_0 - \Sigma(p) + i\epsilon}. \]  

(32)

Hence the SD equation for the fermion propagator takes the form:

\[ iS^{-1}(p) = iS^{-1}(p) - \Sigma(p), \]  

(33)

where

\[ \Re \Sigma(p) = \Re \Sigma^{11}(p), \]

\[ \Im \Sigma(p) = \epsilon(p_0) \coth \left[ \frac{\beta}{2} (p_0 + \mu) \right] \Im \Sigma^{11}(p). \]  

(34)

### 2.3 photon

Given the lagrangian for the photon field

\[ \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2\alpha} (\partial^\mu A_\mu)^2, \]  

(35)

with \( \alpha \) being the gauge-fixing parameter, the photon propagator is given by

\[ iD^{ab}_{\mu\nu}(k) = \left[ -g_{\mu\nu} - (1 - \alpha) k_\mu k_\nu \frac{\partial}{\partial k^2} \right] iD^{ab}(k)_{m=0}, \]  

(36)

where \( D^{ab}(k)_{m=0} \) is the massless scalar propagator.
For the full photon propagator $D_{ab}^{\mu\nu}(k)$ obeying the SD equation

$$iD_{ab}^{\mu\nu}(k) = iD_{ab}^{\mu\nu}(k) + iD_{ac}^{\mu\rho}(k)(-i\Pi_{\rho\sigma}^{cd})iD_{db}^{\sigma\nu}(k),$$

(37)

the vacuum polarization function is introduced

$$i\Pi_{\mu\nu}^{ab}(k) = U^{-1}(\beta, k) \left( \begin{array}{cc} -i\Pi_{\mu\nu}^{(k)} & 0 \\ 0 & -i\Pi_{\mu\nu}^{(k)} \end{array} \right) U^{-1}(\beta, k),$$

(38)

where

$$\Pi_{\mu\nu}^{12}(k) = \Pi_{\mu\nu}^{21}(k) = -2i\cosh k\theta \sinh k\Im \Pi_{\mu\nu}(k) = -i\tanh 2\beta \Im \Pi_{\mu\nu}^{11}(k),$$

$$\Pi_{\mu\nu}^{22}(k) = -\Pi_{\mu\nu}^{11}(k).$$

(39)

It is expected to satisfy the relation

$$iD_{\mu\nu}^{-1}(k) = iD_{\mu\nu}^{-1}(k) - \Pi_{\mu\nu}(k),$$

(40)

where

$$\Re \Pi_{\mu\nu}(k) = \Re \Pi_{\mu\nu}^{11}(k),$$

$$\Im \Pi_{\mu\nu}(k) = \epsilon(k_0) \tanh[2\beta k_0] \Im \Pi_{\mu\nu}^{11}(k).$$

(41)

### 2.4 remark on the full propagator

In this paper we discuss only the real part of the self-energy function as explained in section 4.

In the general case of $\Im \Sigma(p) \neq 0$, the full boson propagator should take the form

$$iD_{1}^{11}(k) = \frac{i}{k^2 - \Pi(k) + i\epsilon} + \left[ \frac{i}{k^2 - \Pi(k) + i\epsilon} - \frac{i}{k^2 - \Pi^*(k) - i\epsilon} \right] N_B(k),$$

(42)

and the full fermion propagator

$$iS_{1}^{11}(p) = \frac{i}{p^2 - \Sigma(p) + i\epsilon} - \left[ \frac{i}{p^2 - \Sigma(p) + i\epsilon} - \frac{i}{p^2 - \Sigma^*(p) - i\epsilon} \right] N_F(p).$$

(43)
We consider the model with four-fermion interaction of Nambu-Jona-Lasinio (NJL) type \[31\] whose lagrangian is given by
\[
L = \bar{\psi}^a i \not \partial \psi^a - m_0 \bar{\psi}^a \psi^a + \frac{1}{2N} \left[ (\bar{\psi}^a \psi^a)^2 + (\bar{\psi}^a i \gamma_5 \psi^a)^2 \right], \quad (a = 1, ..., N) \tag{44}
\]
which is equivalent to
\[
L = \bar{\psi}^a i \not \partial \psi^a - \bar{\psi}^a (\sigma + i \gamma_5 \pi) \psi^a - \frac{N}{G} \left[ \frac{1}{2} (\sigma^2 + \pi^2) - m_0 \sigma \right], \tag{45}
\]
where we have introduced the scalar and the pseudoscalar auxiliary fields \(\sigma\) and \(\pi\) respectively.

Due to the Yukawa interaction of the fermion with the auxiliary scalar field, two tadpole diagrams contribute to the fermion self-energy in the NJL model in the leading order of \(1/N\) expansion (Fig.1), which is the chain approximation:

\[
\Sigma^{11} = - \int \frac{d^Dp}{(2\pi)^D} \left\{ D^{11}(0) \text{tr}[i \mathcal{S}^{11}(p)] + D^{12}(0) \text{tr}[i \mathcal{S}^{22}(p)] \right\}, \tag{46}
\]
where \(D^{ab}(k)\) is the auxiliary field propagator. Since \(D^{11}(0) = G/N\), \(D^{12}(0) = 0\), it turns out that the second diagram has vanishing contribution.

When \(m_0 = 0\), we notice that
\[
\text{tr}[i \mathcal{S}^{11}(p)] = i \text{tr}(1) \left[ \frac{M}{p^2 - M^2} + i \epsilon(p_0) \pi \delta(p^2 - M^2) M \tanh \frac{\beta(p_0 + \mu)}{2} \right]. \tag{47}
\]
Hence we obtain
\[
D^{11}(0) \text{Re}[\text{tr}[i \mathcal{S}^{11}(p)]] = -\pi G \text{tr}(1) \epsilon(p_0) \delta(p_0^2 - E_P^2) M \tanh \frac{\beta(p_0 + \mu)}{2} - \pi G \frac{\delta(p_0 - E_P) - \delta(p_0 + E_P)}{2E_P} M \tanh \frac{\beta(p_0 + \mu)}{2}, \tag{48}
\]
where
\[
E_P := \sqrt{P^2 + M^2}, \quad P = |\vec{P}|. \tag{49}
\]
Then the integration with respect to the time-component is straightforward:
\[
\int \frac{dp_0}{2\pi} D^{11}(0) \text{Re}[\text{tr}[i \mathcal{S}^{11}(p)]] = -G \frac{\text{tr}(1)}{4} \frac{M}{E_P} \left[ \tanh \frac{\beta(E_P + \mu)}{2} + \tanh \frac{\beta(E_P - \mu)}{2} \right]. \tag{50}
\]
Identifying the fermion mass \(M\) with the real part of \(\Sigma\):
\[
M = \text{tr}(\text{Re}\Sigma)/\text{tr}(1) = \text{tr}(\text{Re}\Sigma^{11})/\text{tr}(1), \tag{51}
\]
\[3\] See the next section for the reason.
therefore, the gap equation in the NJL$_D$ ($D \geq 2$) model at FTD is obtained:

\[
M = \frac{\text{tr}(1)}{4} G \int \frac{d^{D-1}P}{(2\pi)^{D-1}} \frac{M}{E_P} \left[ \tanh \frac{\beta(E_P + \mu)}{2} + \tanh \frac{\beta(E_P - \mu)}{2} \right]. \tag{52}
\]

The solution obtained from this gap equation is momentum-independent.

This gap equation indeed coincides with that obtained by performing the sum over Matsubara frequencies in ITF.

\[
\Sigma = -T \sum_{n=\pm \infty} \int \frac{d^{D-1}P}{(2\pi)^{D-1}} \text{tr}[S(p)] D(0), \tag{53}
\]

where

\[
S(p) = \frac{1}{p - \Sigma} = \frac{p + \Sigma}{p^2 - \Sigma^2}, \quad p_0 = (2n + 1)i\pi T + \mu,
\]

\[
D(0) = \frac{G}{N}. \tag{54}
\]

The self-consistent solution $\Sigma$ is $p$-independent and hence we can perform the discrete sum:

\[
\Sigma = \text{tr}(1) G T \sum_{n=-\infty}^{n=+\infty} \int \frac{d^{D-1}P}{(2\pi)^{D-1}} \frac{\Sigma}{p_0^2 - E_P^2}
\]

\[
= \text{tr}(1) G \int \frac{d^{D-1}P}{(2\pi)^{D-1}} \frac{\Sigma}{4E_P} \left[ \tanh \frac{\beta(E_P + \mu)}{2} + \tanh \frac{\beta(E_P - \mu)}{2} \right], \tag{55}
\]

with $E_P := \sqrt{P^2 + \Sigma^2}$.
4 QED at finite-temperature and -density

It should be mentioned on treatment of imaginary part of self-energy function in this paper. In TFD, two types of field indexed by 1 and 2 appear in the theory where the type-1 field is the usual field and the type-2 newly introduced tilde field corresponding to the ghost field in the heat bath [32].

The self-energy function $\Sigma(p)$ differs from $\Sigma^{11}(p)$, although the real parts of $\Sigma(p)$ and $\Sigma^{11}(p)$ coincide [33]:

\[
\Re \Sigma(p) = \Re \Sigma^{11}(p), \quad \Im \Sigma(p) = \epsilon(p_0) \coth \frac{\beta(p_0 + \mu)}{2} \Im \Sigma^{11}(p).
\] (56)

In this paper we consider only the real part of the self-energy $\Re \Sigma(p)$. In other words, we search for a self-consistent solution $\Sigma(p)$ satisfying $\Im \Sigma(p) \equiv 0$:

\[
\Re \Sigma(p) \equiv \Re \Sigma^{11}(p), \quad \Im \Sigma(p) \equiv 0.
\] (57)

This is one possible solution in the scheme of SC equations. This standpoint is different from the finite-temperature perturbation theory [33, 34]. An imaginary part of the self-energy at finite temperature describes the approach to equilibrium and can be related to the dissipative transport coefficients of viscosity and heat conductivity [34]. In this paper we consider only the equilibrium case and neglect the imaginary part which will be discussed in the subsequent paper. $^4$

4.1 derivation of the SD equation from TFD

We consider the fermion self-energy function $\Sigma^{11}(p)$ in QED in $D$-dimensions (QED$_D$). In the bare vertex approximation, the first non-trivial contribution to $\Sigma^{11}(p)$ is written by using only $S^{11}$ and $D^{11}_{\mu\nu}$ as diagrammatically shown in Fig.2:

\[
\Sigma^{11}(p) = e^2 \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} (2\pi)^D \delta(p - q - k) \gamma_\mu iS^{11}(q)\gamma_\nu iD^{11}_{\mu\nu}(k),
\] (58)

since the 1-1 component $S^{11}$ of the full fermion propagator corresponds to two physical type-1 external legs.

In this paper, moreover, we search for the self-consistent (real) solution of the full fermion propagator in the following form:

\[
iS^{11}(p) = Z(p) \left[ \frac{i}{\not{p} - M(p)} + \pi\epsilon(p_0)\delta(p^2 - M^2(p)) \tanh \frac{\beta(p_0 + \mu)}{2}(\not{p} + M(p)) \right],
\] (59)

$^4$The limit $\Im \Sigma(p) \to 0$ should be taken from the negative side $\Im \Sigma(p) < 0$. 

and we choose the photon propagator in the Feynman gauge \(^5\) at finite-temperature:

\[
i D_{\mu\nu}^{11}(k) = -g_{\mu\nu} \left[ \frac{i}{k^2 - \Pi(k)} + \pi \delta(k^2 - \Pi(k)) \coth \left( \frac{\beta |k_0|}{2} \right) \right] = -g_{\mu\nu} i D_{\mu\nu}^{11}(k), \tag{60}\]

where \(\Pi(k)\) is the vacuum polarization function of the photon. Then the real part of \(\Sigma^{11}(p)\) is given by

\[
\Re \Sigma^{11}(p) = -e_0^2 \pi \int \frac{d^D q}{(2\pi)^D} \int \frac{d^D k}{(2\pi)^D} (2\pi)^D \delta(p - q - k) \delta(p - q) Z(q) \gamma_\mu(q + M(q)) \gamma_\mu(k) \\
\left[ \frac{\epsilon(q_0) \delta(q^2 - M^2(q)) \tanh \left( \frac{\beta(q_0 + \mu)}{2} \right)}{k^2 - \Pi(k)} + \frac{\delta(k^2 - \Pi(k)) \coth \left( \frac{\beta |k_0|}{2} \right)}{q^2 - M^2(q)} \right]. \tag{61}\]

Here we define the wave function renormalization function \(Z(p)\) and the fermion mass function \(M(p)\) through

\[
S(p) = \frac{i}{p - \Sigma(p)} = \frac{iZ(p)}{p - M(p)}. \tag{62}\]

Hence we obtain

\[
M(p) = \frac{\text{tr}[\Re \Sigma(p)]}{\text{tr}(1)} = \frac{\text{tr}[\Re \Sigma^{11}(p)]}{\text{tr}(1)}, \\
Z^{-1}(p) = 1 - \frac{\text{tr}[\Re \Sigma(p)]}{p^2 \text{tr}(1)} = 1 - \frac{\text{tr}[\Re \Sigma^{11}(p)]}{p^2 \text{tr}(1)}. \tag{63}\]

Thus we can write down the coupled SD equation in FTD QED\(_D\) as

\[
Z^{-1}(p) = 1 - (2 - D)e_0^2 \pi \int \frac{d^D q}{(2\pi)^D} \frac{p \cdot q}{p^2} Z(q) \left[ \frac{\epsilon(q_0) \delta(q^2 - M^2(q)) \tanh \left( \frac{\beta(q_0 + \mu)}{2} \right)}{(p - q)^2 - \Pi(p - q)} \right. \\
\left. + \frac{\delta((p - q)^2 - \Pi(p - q)) \coth \left( \frac{\beta |p_0 - q_0|}{2} \right)}{q^2 - M^2(q)} \right], \tag{64}\]

\[
M(p) = -De_0^2 \pi \int \frac{d^D q}{(2\pi)^D} Z(q) M(q) \left[ \frac{\epsilon(q_0) \delta(q^2 - M^2(q)) \tanh \left( \frac{\beta(q_0 + \mu)}{2} \right)}{(p - q)^2 - \Pi(p - q)} \right. \\
\left. + \frac{\delta((p - q)^2 - \Pi(p - q)) \coth \left( \frac{\beta |p_0 - q_0|}{2} \right)}{q^2 - M^2(q)} \right]. \tag{65}\]

It is easy to see that if \(M(p)\) is a solution of the SD equation, then \(-M(p)\) is also a solution.

In ITF we must solve infinite number of integral equations (indexed by an integer) for the function \(M_n(p)\) with one argument, see Section 5.1. In the TFD, on the other hand, we have only to solve the single integral equation for the function \(M(p_0, P)\) with two arguments.

\(^5\) At zero temperature, the bare vertex approximation and the free photon propagator in the Landau gauge leads to no wave function renormalization of the fermion propagator, \(Z(p) \equiv 1\), \(^{13}\) which is consistent with the bare vertex approximation in light of the Ward identity, \(Z_1 = Z_2\). This property is not preserved at finite temperature in the exact sense. However the deviation of \(Z(p)\) from 1 is at most logarithmic in the momentum and does not substantially change the result in this paper. This will be discussed in detail in the subsequent paper.
4.2 finite-temperature and -density QED\textsubscript{4}

In what follows, we put $Z(p) \equiv 1$ as explained in the footnote. For convenience, we decompose the self-energy function $\Sigma(p_0, P; T, \mu)$, right-hand-side of the above equation, into two parts, $\Sigma_f(p_0, P; T, \mu)$ and $\Sigma_{ph}(p_0, P; T)$. Thus the SD equation for the fermion mass function $M(p_0, P)$ is written as

$$M(p_0, P) = \Sigma_f(p_0, P; T, \mu) + \Sigma_{ph}(p_0, P; T).$$

(66)

In what follows, we replace for simplicity the vacuum polarization function of the photon with its infrared value, i.e., mass of the photon generated by finite-temperature and -density effects [36].

$$\Pi(k) \Rightarrow \Pi(0) \equiv m^2 = m^2(T, \mu).$$

(67)

Improvement of this approximation will be tackled in the subsequent paper.

The first self-energy part reads

$$\Sigma_f(p_0, P; T, \mu) = -e^2_0 \pi D \int \frac{dD q}{(2\pi)^D} M(q_0, Q) \frac{\epsilon(q_0)\delta(q^2 - M^2(q)) \tanh \frac{\beta(q_0 + \mu)}{2}}{(p_0 - q_0)^2 - (P - Q)^2 - m^2}. \tag{68}$$

Decomposing the integration measure into the angular part and the radial one, we obtain for $D > 3$

$$\Sigma_f(p_0, P; T, \mu) = -e^2_0 \pi DC_D \int_{-\infty}^{+\infty} \frac{dq_0}{2\pi} \int_0^\Lambda Q^{D-2}dQ \int_0^\pi d\vartheta \sin^{D-3}\vartheta \frac{M(q_0, Q) \epsilon(q_0)\delta(q^2 - M^2(q)) \tanh \frac{\beta(q_0 + \mu)}{2}}{(p_0 - q_0)^2 - (P^2 + Q^2 + m^2) + 2PQ \cos \vartheta}, \tag{69}$$

where $P = |\vec{P}|$, $Q = |\vec{Q}|$, $\cos \vartheta = \vec{P} \cdot \vec{Q}/PQ$ and

$$C_D = \frac{2}{(2\pi)^{D-1}} \frac{\pi^{D-2}}{\Gamma(D-2)^2} = \frac{1}{2^{D-2}-\pi^{D-2}\Gamma(D-2)^2}, \quad (D > 3). \tag{70}$$

Here we have introduced the ultraviolet (UV) momentum cutoff $\Lambda$.

For $D = 4$, after performing the angular integration, we obtain

$$\Sigma_f(p_0, P; T, \mu) = \frac{\alpha}{\pi} \int_{-\infty}^{+\infty} dq_0 \int_0^\Lambda dQ \epsilon(q_0)\delta(q^2 - M^2(q)) \frac{M(q_0, Q) \tanh[\frac{\beta}{2}(q_0 + \mu)]Q}{P} \ln \frac{(p_0 - q_0)^2 - (P + Q)^2 - m^2}{(p_0 - q_0)^2 - (P - Q)^2 - m^2}, \tag{71}$$

where we have introduced a new coupling constant ($C_4 = 1/(4\pi^2)$)

$$\alpha := \frac{e^2_0}{4\pi} \tag{72}$$

\footnote{\textit{D = 3 case (QED\textsubscript{3}) will be discussed in a separate paper.}}
Next we carry out the integration with respect to $q_0$ in the formal way. Assuming that the equation

$$p_0^2 - P^2 - M^2(p_0, P) = 0 \tag{73}$$

has two solutions: $p_0 = \pm E_P^\pm$ ($E_P^\pm > 0$),

$$\delta(p^2 - M^2(p_0, P)) = \delta(p_0^2 - P^2 - M^2(p_0, P))$$

$$= \frac{\delta(p_0 - E_P^+)}{|2E_P^+ - \frac{d}{dp_0}M^2(p_0, P)|_{p_0=+E_P^+}} + \frac{\delta(p_0 + E_P^-)}{|-2E_P^- - \frac{d}{dp_0}M^2(p_0, P)|_{p_0=-E_P^-}}. \tag{74}$$

Using this formula, the first self-energy part $\Sigma_f(p_0, P; T, \mu)$ reads

$$\Sigma_f(p_0, P; T, \mu) = \frac{\alpha}{\pi} \int_0^\Lambda dQ \frac{Q}{P} \left[ \frac{M(E_Q^+, Q) \tanh \frac{\beta(E_Q^+ + \mu)}{2}}{|2E_Q^+ - \frac{d}{dp_0}M^2(q_0, Q)|_{q_0=+E_Q^+}} \ln \frac{(p_0 - E_Q^+)^2 - (E_m^+)^2}{(p_0 - E_Q^-)^2 - (E_m^-)^2} \right.$$ 

$$+ \frac{M(-E_Q^-, Q) \tanh \frac{\beta(E_Q^- - \mu)}{2}}{|-2E_Q^- - \frac{d}{dp_0}M^2(q_0, Q)|_{q_0=-E_Q^-}} \ln \frac{(p_0 + E_Q^-)^2 - (E_m^+)^2}{(p_0 + E_Q^+)^2 - (E_m^-)^2} \right], \tag{75}$$

where

$$E_m^\pm := \sqrt{|P \pm Q|^2 + m^2}. \tag{76}$$

On the other hand, the second self-energy part $\Sigma_{ph}(p_0, P; T)$ reads ($D > 3$)

$$\Sigma_{ph}(p_0, P; T) = -\pi D e_0^2 \int \frac{d^D q}{(2\pi)^D} M(q_0, Q) \frac{\delta((p-q)^2 - m^2) \coth \frac{\beta(p-q)_0}{2}}{q_0^2 - Q^2 - M^2(q_0, Q)}$$

$$= -\pi D e_0^2 \int_{-\infty}^{+\infty} \frac{dq_0}{2\pi} \int_0^\Lambda Q^{D-3} dQ \int_0^\pi d\vartheta \sin^{D-3} \vartheta M(q_0, Q) \coth \frac{\beta(p-q)_0}{2}$$

$$\delta((p_0 - q_0)^2 - (P^2 + Q^2 + m^2) + 2PQ \cos \vartheta). \tag{77}$$

Note that $\Sigma_{ph}(p_0, P; T)$ has no $\mu$-dependence.

For $D = 4$, after performing the angular integration, we obtain

$$\Sigma_{ph}(p_0, P; T) = -\frac{\alpha}{\pi} \int_0^\Lambda dQ \int_D dQ_0 \frac{Q}{P} M(q_0, Q) \coth \frac{\beta(p-q)_0}{2}. \tag{78}$$

Note that the range of integration $-1 < \cos \vartheta \leq 1$ is equivalent to $D = \{\sqrt{|P - Q|^2 + m^2} \leq |p_0 - q_0| < \sqrt{(P + Q)^2 + m^2}\}$ from the fact that the delta function has the support at $|p_0 - q_0| > \sqrt{(P + Q)^2 + m^2}$.

---

\(^7\)We assume that such a pair of solutions exists. Note that invariance of the mass function $M(p_0, P) = M(-p_0, P)$ under the reflection $p_0 \rightarrow -p_0$ does not hold in the presence of the chemical potential $\mu \neq 0$. If the solution has no $p_0$-dependence, $M(p_0, P) = \tilde{M}(P)$, then $p_0 = \pm E_P$, $E_P = \sqrt{P^2 + \tilde{M}(P)}$. 

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16
Integration with respect to $q_0$ cannot be performed for $\Sigma_{ph}(p_0, P; T)$, unless we know the dependence of the solution $M(q_0, Q)$ on $q_0$.

Thus the SD equation of FTD QED$_4$ for the fermion mass function is written as

$$M(p_0, P) = \frac{\alpha}{\pi} \int_0^\Lambda dQ \frac{Q}{P} \left[ \int_{p_0-E_m^+}^{p_0-E_m^-} dq_0 M(q_0, Q) \frac{\coth \frac{\beta(q-p_0)}{2}}{q_0^2 - Q^2 - M^2(q_0, Q)} - \int_{p_0+E_m^-}^{p_0+E_m^+} dq_0 M(q_0, Q) \frac{\coth \frac{\beta(q-p_0)}{2}}{q_0^2 - Q^2 - M^2(q_0, Q)} \right]. \quad (79)$$

### 4.3 limiting cases of QED$_4$

The SD equation in the various limiting cases is obtained in the following replacements.

In the zero-temperature limit $\beta \rightarrow \infty$,

$$\tanh \frac{\beta(E_Q - \mu)}{2} \rightarrow \theta(E_Q - \mu) - \theta(\mu - E_Q),$$

$$\tanh \frac{\beta(E_Q + \mu)}{2} \rightarrow 1 \equiv \theta(E_Q - \mu) + \theta(\mu - E_Q),$$

$$\coth \frac{\beta|p_0 - q_0|}{2} \rightarrow 1. \quad (81)$$

In the zero-temperature $\beta \rightarrow \infty$ and the zero-density limit $\mu \rightarrow 0$,

$$\tanh \frac{\beta(E_Q - \mu)}{2}, \tanh \frac{\beta(E_Q + \mu)}{2} \rightarrow 1,$$

$$\coth \frac{\beta|p_0 - q_0|}{2} \rightarrow 1. \quad (82)$$
5 Instantaneous-exchange approximation

5.1 instantaneous-exchange approximation in ITF

In the imaginary-time formalism, the full fermion propagator as the solution of the SD equation has the form (when the wave function renormalization can be neglected):

$$S^{-1}(p) = p + \Sigma_m(P), \quad p_0 = (2m + 1)\pi T - i\mu,$$

which becomes frequency-dependent. Then we can not perform the discrete sum in advance, before we find the solution of the self-consistent SD equation:

$$\Sigma_m(P) = e_0^2 T \sum_{n=-\infty}^{+\infty} \int \frac{d^{D-1}Q}{(2\pi)^{D-1}} D_{\mu\nu}(k_0 = p_0 - q_0, \vec{K} = \vec{P} - \vec{Q}) \frac{\Sigma_n(Q)}{q_0^2 + Q^2 + \Sigma_n^2(Q)},$$

with \(q_0 = (2n + 1)\pi T - i\mu\). This fact renders the analytical treatment as well as the numerical one extremely cumbersome.

To avoid this problem, the instantaneous-exchange (IE) approximation\(^{[22]}\) is adopted:

$$D_{\mu\nu}(k_0, \vec{K}) \simeq D_{\mu\nu}(k_0 = 0, \vec{K}).$$

In this approximation the solution \(\Sigma_m(P)\) becomes frequency-independent \(\Sigma(P)\) and the summation over discrete frequencies can be performed explicitly as

$$\Sigma(P) = e_0^2 \int \frac{d^{D-1}Q}{(2\pi)^{D-1}} D_{\mu\nu}(0, \vec{P} - \vec{Q}) \frac{\Sigma(Q)}{4E_Q} \left[ \tanh \left( \frac{\beta(E_Q + \mu)}{2} \right) + \tanh \left( \frac{\beta(E_Q - \mu)}{2} \right) \right],$$

where

$$E_Q = \sqrt{Q^2 + \Sigma^2(Q)}.$$  

For our choice

$$D_{\mu\nu}(0, \vec{K} = \vec{P} - \vec{Q}) = \frac{g_{\mu\nu}}{(\vec{P} - \vec{Q})^2 + m^2},$$

the angular integration is performed for \(D > 3\) as

$$\Sigma(P) = DC_D e_0^2 \int_0^\Lambda Q^{D-2}dQ \frac{\Sigma(Q)}{4E_Q} \left[ \tanh \left( \frac{\beta(E_Q + \mu)}{2} \right) + \tanh \left( \frac{\beta(E_Q - \mu)}{2} \right) \right] \times \int_0^\pi d\vartheta \frac{\sin^{D-3}\vartheta}{P^2 + Q^2 + m^2 - 2PQ\cos\vartheta}.$$  

Then the SD equation for QED\(_4\) in IE approximation is obtained

$$\Sigma(P) = \frac{\alpha}{\pi} \int_0^\Lambda dQ \frac{Q\Sigma(Q)}{PE_Q} \left[ \tanh \left( \frac{\beta(E_Q + \mu)}{2} \right) + \tanh \left( \frac{\beta(E_Q - \mu)}{2} \right) \right] \ln \frac{(P + Q)^2 + m^2}{(P - Q)^2 + m^2}. $$
5.2 instantaneous-exchange approximation in TFD

We consider the similar approximation of the SD equation in TFD, which we call also the IE approximation. If we put \( k_0 = 0 \) in the photon propagator, \( iD_{\mu\nu}^{11}(k_0, \vec{K}) \), the temperature-dependence piece vanishes from the nature of the delta-function \( \delta(-\vec{K}^2 - m^2) = 0 \) as long as \( m^2 > 0 \), i.e.,

\[
iD_{\mu\nu}^{11}(k_0 = 0, \vec{K}) = -g_{\mu\nu} \frac{i}{-\vec{K}^2 - m^2}.
\]  

(91)

Therefore, in our IE approximation

\[
\Sigma_{\text{ph}}(p_0, P; T) \equiv 0.
\]  

(92)

Furthermore, \( \Sigma_f(p_0, P; T, \mu) \) becomes \( p_0 \)-independent, so that the fermion mass function \( M(p_0, P) \) obtained in the self-consistent way should be \( p_0 \)-independent, \( M(P) \). Thus the SD equation in the IE approximation reads

\[
M(P) = \frac{\alpha}{2\pi} \int_0^\Lambda dQ \frac{QM(Q)}{PE_Q} \left[ \frac{\tanh(\frac{\beta(E_Q + \mu)}{2}) + \tanh(\frac{\beta(E_Q - \mu)}{2})}{2} \right] \ln \frac{(P + Q)^2 + m^2}{(P - Q)^2 + m^2}.
\]  

(93)

In the IE approximation our choice of the photon propagator gives the same SD equation in the two formalism: ITF and RTF (or TFD).

In the zero-temperature limit (at finite-density \( \mu \neq 0 \)), the SD equation reads

\[
M(P) = \frac{\alpha}{\pi} \int_0^\Lambda dQ \frac{Q}{PE_Q} M(Q) \theta(E_Q - \mu) \ln \frac{(P + Q)^2 + m^2_{T=0}}{(P - Q)^2 + m^2_{T=0}}.
\]  

(94)

5.3 numerical results (\( \mu = 0 \))

We have solved the integral equation:

\[
M(P) = \frac{\alpha}{\pi} \int_0^\Lambda dQ \frac{Q}{PE_Q} M(Q) \tanh(\frac{\beta E_Q}{2}) \ln \frac{(P + Q)^2 + m^2_T}{(P - Q)^2 + m^2_T}, E_Q = \sqrt{Q^2 + M(Q)^2},
\]  

(95)

where the photon mass is borrowed from the 1-loop calculation [36]:

\[
m^2_T = \frac{1}{3} e^2 T^2 = \frac{4\pi}{3} \alpha T^2.
\]  

(96)

In the numerical calculation the dimensionful quantities e.g., \( T, P, Q, M(P), E_Q, \ldots \) is normalized by the cutoff \( \Lambda \), and hence the SD equation can be rewritten in terms of dimensionless variables where the range of integration is the finite interval \([0, 1]\).

First of all, we study the zero-temperature limit \( T = 0 \):

\[
M(P) = \frac{\alpha}{\pi} \int_0^\Lambda dQ \frac{Q}{PE_Q} M(Q) \ln \frac{(P + Q)^2}{(P - Q)^2}, E_Q = \sqrt{Q^2 + M(Q)^2}.
\]  

(97)
The fermion mass is dynamically generated due to the nontrivial solution $M(p) \neq 0$ and the chiral symmetry is spontaneously broken in the strong coupling region $\alpha > \alpha_c$ with the critical coupling $\alpha_c = 0.39$.

Fig. 3 shows the coupling-constant $\alpha$ dependence of the fermion mass $M(0)$ which is identified with a pole position of the fermion propagator: $p^2 = M^2(p^2) \simeq M^2(0)$. Fig. 4 exhibits the fermion mass function $M(P)$ which is monotonically decreasing in $P$. These features are qualitatively the same as the well-known results obtained from the zero-temperature SD equation in QED [15].

At finite temperature $T > 0$, Fig. 5 corresponds to Fig. 3 at zero-temperature. The critical coupling $\alpha_c$ depends on the temperature and $\alpha_c(T)$ is greater than $\alpha_c$ at zero-temperature. Temperature-dependence of $M(0)$ for a fixed $\alpha(\geq \alpha_c)$ is exhibited in Fig. 6(a). This implies that there exists a critical temperature $T_c$ above which the chiral symmetry restores in the sense that the dynamical fermion mass vanishes: $M(0) = 0$ for $T > T_c$.

In Fig. 7 we compare the fermion mass functions $M(P)$ for various temperature $T$ at a fixed $\alpha$. For relatively low-temperature, there exists a peak for $M(P)$. However the peak disappears as the temperature increases and the critical temperature is approached.

The phase diagram for two-coupling space ($\alpha, T$) is depicted in Fig. 8 where the low-temperature strong coupling phase where the chiral symmetry is spontaneously broken is separated by the critical line from the high-temperature chiral-symmetry-restoring phase. This shows that no matter how large the coupling constant $\alpha$ is taken there exists a finite critical temperature $T_c < \infty$, namely, the chiral symmetry always restores at a finite temperature $T_c$.

To see the scaling of the dynamical fermion mass, the dimensionless quantity $M(0)/T_c$ in the region $T/T_c \approx 1$ is plotted in Fig. 9 for various values of $\theta := (\alpha - \alpha_c)/\alpha_c$, $\alpha = \alpha(T \to 0) = 0.36$. Fig. 9 suggests the existence of the scaling function, i.e., $M(0)/T_c$ is written as a single function of $T/T_c$ near the critical temperature $T_c$. Our data are still insufficient to specify the critical exponent associated with the finite-temperature transition. More detailed data will be given in the subsequent paper.

Finally some remarks on the numerical calculation are in order. In our numerical calculation, we adopt the Double Exponential (DE) formula to choose sample points. DE formula is efficient when there is a weak singularity at the end of the interval of integration. Actually the integral kernel of the SD equation at $T = 0$ has a singular point at $P = Q$. At finite-temperature $T \neq 0$, on the other hand, the singularity at $P = Q$ disappear due to the existence of the photon mass $m_T$. Therefore we can calculate the integrand also at $P = Q$ without difficulty for high-temperature. In our numerical calculations we need at least 200 sample points according to DE formula to guarantees the precision of $10^{-5}$ for the solution $M(P)$. However we need more sample points to obtain the stable result in the relatively low-temperature region. As the number of sample points becomes larger, the low-temperature result becomes more stable, i.e., the rapid change of $M(0)$ at $T \approx 0$ disappear even in the low-temperature region ($T/\Lambda < 0.01$) as shown in Fig. 6(b). In order to check the influence of the singularity to a minumum, we divided the interval $[0,1]$ into subintervals so that each of the subinterval contains 50 sample points which are chosen according to DE formula. For more details, see [38].
In this section we consider another approximation to the SD equation. In the IE approximation the second self-energy part $\Sigma_{ph}$ including the factor $\coth\frac{\beta}{2}q_0$ has automatically dropped from the SD equation and the mass function $M(p_0, P)$ gets $p_0$-independent. To go beyond the IE approximation, we must include the effect of the second self-energy part $\Sigma_{ph}$. However the integral equation with two arguments:

$$M(p_0, P) = \Sigma_f(p_0, P; T, \mu) + \Sigma_{ph}(p_0, P; T),$$  

(98)

is rather difficult to solve even in the numerical way. Therefore we consider $p_0$-independent solution $M(P)$ of the SD equation by requiring

$$\frac{\partial M(q_0, Q)}{\partial q_0} \approx 0.$$  

(99)

We try to determine the $p_0$-independent solution $M(P)$ self-consistently so as to satisfy this condition. The previous IE approximation is nothing but a sufficient condition for the solution to be $p_0$-independent.

### 6.1 finite-temperature case ($\mu = 0$)

From the $q_0$-independence of the solution, the first self-energy part reads

$$\Sigma_f(p_0, P; T, 0) = \frac{\alpha}{2\pi} \int_0^\Lambda dQ \frac{Q}{PE_Q} M(Q) \tanh \frac{\beta E_Q}{2} \times$$

$$\times \left[ \ln \frac{(p_0 - E_Q)^2 - (E_m^+)^2}{(p_0 - E_Q)^2 - (E_m^-)^2} + \ln \frac{(p_0 + E_Q)^2 - (E_m^+_Q)^2}{(p_0 + E_Q)^2 - (E_m^-_Q)^2} \right].$$  

(100)

with

$$E_Q := \sqrt{Q^2 + M(Q)^2}.$$  

(101)

Here the value of $p_0$ is specified below. Note that the right-hand-side of the above equation can be rewritten in the form which is invariant under the reflection: $p_0 \rightarrow -p_0$:

$$\Sigma_f(p_0, P; T, 0) = \frac{\alpha}{2\pi} \int_0^\Lambda dQ \frac{Q}{PE_Q} M(Q) \tanh \frac{\beta E_Q}{2} \times$$

$$\times \ln \frac{[p_0^2 - (E_m^+ + E_Q)^2][p_0^2 - (E_m^- + E_Q)^2]}{[p_0^2 - (E_m^- + E_Q)^2][p_0^2 - (E_m^- - E_Q)^2]}.$$  

(102)

On the other hand, the second self-energy part reads

$$\Sigma_{ph}(p_0, P; T) = \frac{\alpha}{2\pi} \int_0^\Lambda dQ \frac{Q}{PE_Q} M(Q) I_\beta(P, Q, M),$$  

(103)
where
\[
I_\beta(P, Q, M) = \int_{E_m^-}^{E_m^+} dt \frac{1}{t - p_0 - E_Q} \coth \frac{\beta t}{2}
\]
\[
+ \int_{E_m^+}^{E_m^n} dt \frac{1}{t + p_0 + E_Q} \coth \frac{\beta t}{2}
\]
\[
= 2 \int_{E_m^n}^{E_m^+} dq_0 \left[ \frac{q_0 + E_Q}{(q_0 + E_Q)^2 - p_0^2} - \frac{q_0 - E_Q}{(q_0 - E_Q)^2 - p_0^2} \right] \coth \frac{\beta q_0}{2}. \tag{105}
\]

Hence \( \Sigma_{ph}(p_0, P; T) \) is also invariant under the replacement: \( p_0 \to -p_0 \). In the zero-temperature limit \( (T = 0) \),
\[
\Sigma_{ph}(p_0, P; 0) = \frac{\alpha}{2\pi} \int_{0}^{A} dQ \frac{Q}{PE_Q} M(Q) \ln \frac{[p_0^2 - (E_m^+ + E_Q)^2][p_0^2 - (E_m^- - E_Q)^2]}{[p_0^2 - (E_m^+ + E_Q)^2][p_0^2 - (E_m^- - E_Q)^2]}. \tag{106}
\]

In the zero-temperature and zero-density limit \( (T = 0 = \mu) \), therefore, the SD equation \( M(P) = \Sigma_f(p_0, P; 0) + \Sigma_{ph}(p_0, P; 0) \) reads
\[
M(P) = \frac{\alpha}{\pi} \int_{0}^{A} dQ \frac{Q M(Q)}{P} \ln \frac{[p_0^2 - (P + Q + E_Q)^2]}{[p_0^2 - (|P - Q| + E_Q)^2]}. \tag{107}
\]

Now we discuss how to choose \( p_0 \). To determine \( p_0 \), we require positivity of the nontrivial solution \( M(P) \) obtained self-consistently, at least in the neighborhood \( P \simeq 0 \), which is satisfied if the integrand is positive for \( P \simeq 0 \). In the zero-temperature limit \( T = 0 \), this follows from
\[
p_0 < E_Q + Q. \tag{108}
\]

In the finite-temperature case \( T > 0 \), we notice
\[
\Sigma_f(p_0, P; T, 0) = \frac{\alpha}{2\pi} \int_{0}^{A} dQ \frac{Q}{PE_Q} M(Q) \tanh \frac{\beta E_Q}{2}
\]
\[
\times \ln \left[ 1 + \frac{4(E_Q + \sqrt{Q^2 + m^2})Q P}{[E_Q + \sqrt{Q^2 + m^2}]^2 - p_0^2 \sqrt{Q^2 + m^2}} + \mathcal{O}(P^2) \right]
\]
\[
\times \left[ 1 - \frac{4(E_Q - \sqrt{Q^2 + m^2})Q P}{[E_Q - \sqrt{Q^2 + m^2}]^2 - p_0^2 \sqrt{Q^2 + m^2}} + \mathcal{O}(P^2) \right], \tag{109}
\]

\footnote{For the \( p_0 \)-independent \( M(P) \), the integration with respect to \( q_0 \) can be done in principle, but the result cannot be expressed by the elementary function.}
where we have used

\[ E_m^\pm = \sqrt{Q^2 + m^2} \pm \frac{QP}{\sqrt{Q^2 + m^2}} + \mathcal{O}(P^2). \] (110)

In the total (self-consistent) mass function \( M(P) = \Sigma_f + \Sigma_{ph} \), we assume the first self-energy part \( \Sigma_f(p_0, P; T, 0) \) is the dominant part. Hence \( \Sigma_f(p_0, P; T, 0) \) has the same signature as \( M(P) \), since it is not difficult to show \( p_0 \) cannot be chosen such that both \( \Sigma_f(p_0, P \simeq 0; T, 0) \) and \( \Sigma_{ph}(p_0, P \simeq 0; T) \) are simultaneously positive (or negative). Thus positivity requirement of both the total mass function \( M(P) \) and the first self-energy part \( \Sigma_f(p_0, P \simeq 0; T, 0) \) restrict the range of \( p_0 \) as

\[ 0 < E_Q - \sqrt{Q^2 + m^2} < p_0 < E_Q + \sqrt{Q^2 + m^2}. \] (111)

Here \( E_Q > \sqrt{Q^2 + m^2} \), i.e., \( M(Q) > m = \sqrt{\frac{4\pi}{3}\alpha N_f T} \) is satisfied for sufficiently strong coupling \( \alpha > \alpha_c \approx 1 \) and sufficiently low-temperature \( T/\Lambda \ll 1 \) in the case of finite fermion flavor \( N_f \), while it is always satisfied in the quenched limit \( N_f \to 0 \). This condition does not guarantee positivity of the second part \( \Sigma_{ph}(p_0, P; T) \) even at \( T = 0 \). The apparently simplest choice \( p_0 = 0 \) is excluded by this condition.

### 6.2 numerical results \((\mu = 0)\)

In the numerical calculation we take the simplest choice:

\[ p_0 = E_Q, \] (112)

and study the effect of the second part of the self-energy contribution. For this choice, the SD equation for the mass function reads \( M(P) = \Sigma_f(E_Q, P; T, 0) + \Sigma_{ph}(E_Q, P; T) \) which can be rewritten in the form:

\[ M(P) = \frac{2\alpha}{\pi} \int_0^\Lambda dQ \frac{Q}{PE_Q} M(Q) \int_{E^-_m}^{E^+_m} dq_0 \left[ q_0^2 - 2E_Q^2 \right] \frac{\tanh[\frac{\beta}{2}E_Q] - E_Qq_0 \coth[\frac{\beta}{2}q_0]}{q_0[q_0^2 - (2E_Q)^2]}, \] (113)

where

\[ E_m^\pm := \sqrt{|P + Q|^2 + m_T^2}, \quad m_T^2 = \frac{4\pi}{3}\alpha T^2. \] (114)

At first, we study the zero-temperature limit.

\[ M(P) = \frac{\alpha}{\pi} \int_0^\Lambda dQ \frac{Q}{PE_Q} M(Q) \ln \frac{(P + Q)(P + Q + 2E_Q)}{|P - Q|(|P - Q| + 2E_Q)}. \] (115)

The chiral symmetry is spontaneously broken in the strong coupling region \( \alpha > \alpha_c = 0.62 \) due to dynamical generation of the fermion mass \( M(0) \), as shown in Fig.10. The dynamical
fermion mass function $M(P)$ is shown in Fig. 11. These results are qualitatively in good agreement with those in the IE approximation.

Next, we study the finite-temperature case. Fig. 12 shows the temperature-dependence of the fermion mass $M(0)$. The critical temperature $T_c$ exists at $T_c \sim 0.0651$, above which the chiral symmetry restores.

Temperature-dependence of the fermion mass function is shown in Fig. 13. This shows the same tendency as the case under IE approximation.

### 6.3 Comparison with results in two approximations

Finally, we compare two results obtained in IE approximation and in $p_0$-independent approximation.

Fig. 14 is the plot of the dynamical mass $M(0)$ obtained at $T=0$ under two approximations where the horizontal axis is the coupling constant normalized respectively by each critical coupling. This shows that $M(0)$ under $p_0$-indep. approximation is slightly smaller than IE approximation for all over the range of the coupling constant. In other words, additional effects from $\Sigma_{ph}$ work to decrease $M(P)$ at $T=0$.

Fig. 15 shows the plot of the $M(0)/T_c$ versus $T/T_c$ for the same ratio $\alpha_c(T = 0)/\alpha = 0.618$. Hence $M(0)/T_c$ in IE approximation exhibits larger value for $T/T_c \simeq 0$ and smaller value for $T/T_c \simeq 1$ than $p_0$-indep. approximation.

Concerning the transition temperature, IE approximation gives slightly smaller critical temperature ($T_c/\Lambda = 0.063$) than $p_0$-indep. approximation ($T_c/\Lambda = 0.065$). This implies that effect of the retarded propagation of the photon has a tendency of raising the critical temperature.
7 Conclusion and Discussion

As the formalism which can treat the field theory at finite-temperature and finite-density, we know two formalism: Imaginary-Time Formalism (ITF) and Real-Time Formalism (RTF). In this paper we have derived the Schwinger-Dyson (SD) equation at finite-temperature and non-zero density as a tool to analyze the non-perturbative effect in the Nambu-Jona-Lasinio (NJL) model of four-fermion interaction and QED. In section 3, we have shown that two formalism give the same SD equation for the NJL model in the leading order of 1/N expansion, i.e., chain approximation.

In D-dimensional QED (QED$_D$) at finite-temperature and nonzero-density we have derived the SD equation for the fermion propagator under the bare vertex ansatz (approximation). This is a coupled integral equation for the fermion mass function $M(p_0, P)$ and the wave function renormalization function $Z(p_0, P)$ with two-variables $p_0, P = |\vec{P}|$.

In order to obtain the explicit solution, we have put $Z(p) \equiv 1$ from the outset and searched for the real solution of the SD equation for the fermion mass function $M(p)$ in QED$_4$ at finite-temperature and nonzero-density. Even in this stage, solving the two-variable integral equation is rather difficult even in the numerical calculation. Therefore we consider the approximation so that the SD equation reduces to the one-variable integral equation whose solution is denoted by $M(P)$. Actually we have adopted two types of approximations: instantaneous-exchange (IE) approximation and the $p_0$-independent approximation. The IE approximation in TFD adapted in this paper leads to the same SD equation as that derived under the corresponding IE approximation in ITF [22]. In IE approximation only one temperature-dependent factor \( \tanh[\beta E_P] \) appears in the SD gap equation and hence all the effects of temperature comes from this factor. Moreover IE approximation greatly simplifies the gap equation, since IE approximation automatically render the solution $M(p)$ $p_0$-independent: $M(P)$.

The SD equation derived in this paper based on RTF immediately allows us to do another approximation, i.e., $p_0$-indep. approximation which goes beyond IE approximation. This approximation requires, from the first, that the solution $M(p)$ is $p_0$-independent. However this approximation is able to incorporate a new temperature-effect coming from the factor $\coth[\beta q_0]$, which is disregarded in IE approximation.

Numerical calculations under the two approximations have shown qualitatively the same result. For any coupling constant $\alpha(>\alpha_c)$ belonging to the strong coupling region where the chiral symmetry is spontaneously broken in the zero-temperature, there exists a finite critical temperature $T_c(\alpha) < \infty$ above which the chiral symmetry restores. No matter how the coupling $\alpha$ may be strong, the critical temperature $T_c$ is kept finite.

In this paper we have not analyzed the finite-density transition. In the presence of chemical potential $\mu \neq 0$, the symmetry $p_0 \rightarrow -p_0$ will be lost in the SD equation and we are forced to tackle with the original two-variable integral equation. Therefore approximations adopted in this paper must be improved to treat such a case. This will be a subject in the forthcoming paper.
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9 Figure Captions

Fig.1: Fermion self-energy contribution in the NJL model.

Fig.2: Fermion self-energy contribution in QED.

Fig.3: $M(0)$ vs $1/\alpha$ at $T = 0$ in IE approximation, ($\alpha_c = 0.39$).

Fig.4: Fermion mass function at $T = 0$ in IE approximation, ($\alpha = 0.40$).

Fig.5: $M(0)/\Lambda$ vs $1/\alpha$ at $T/\Lambda = 0.0794$ in IE approximation, ($\alpha_c = 0.70$).

Fig.6: (a) Temperature-dependence of the fermion mass in IE approximation ($\alpha = 0.63$, $T_c/\Lambda = 0.0628$), (b) Dependence of $M(0)$ on the number of sample points $N$ in the low-temperature region $T/\Lambda < 0.01$.

Fig.7: Temperature-dependence of the fermion mass function in IE approximation for $T/\Lambda = 10^{-9}, 10^{-7}, 10^{-5}, 10^{-3}, 0.04$ ($\alpha = 0.70$).

Fig.8: Phase diagram of QED$_4$ at finite-temperature in IE approximation.

Fig.9: $M(0)/T_c$ vs $T/T_c$ in IE approximation where $\theta = 0.08, 0.75, 0.94, 1.7$.

Fig.10: $M(0)/\Lambda$ vs $1/\alpha$ at $T = 0$ in $p_0$-indep. approximation, ($\alpha_c = 0.618$).

Fig.11: Fermion mass function at $T = 0$ in $p_0$-indep. approximation, ($\alpha = 1.0$).

Fig.12: Temperature-dependence of the fermion mass in $p_0$-indep. approximation ($\alpha = 1.0$, $T_c/\Lambda = 0.0651$).

Fig.13: Temperature-dependence of the fermion mass function in $p_0$-indep. approximation for $T/\Lambda = 10^{-5}, 10^{-3}, 0.024$ ($\alpha = 1.0$).

Fig.14: Comparison of IE and $p_0$-indep. approximations: $M(0)/\Lambda$ vs $\alpha_c/\alpha$.

Fig.15: Comparison of IE and $p_0$-indep. approximations: $M(0)/T_c$ vs $T/T_c$ for $\alpha_c/\alpha = 0.618$. 