The Running Coupling from SU(3) Gauge Theory

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We present high precision results on the static $q\bar{q}$ potential on $32^4$ and smaller lattices, using the standard Wilson action at $\beta = 6.0, 6.2, 6.4,$ and 6.8 on the Connection Machine CM-2. Within our statistical errors ($\approx 1\%$) we did not observe any finite size effects affecting the potential values, on varying the spatial lattice extent from 0.9 fm up to 3.3 fm. We find violations of asymptotic scaling in the bare coupling up to $\beta = 6.8$. We demonstrate that scaling violations on the string tension can be considerably reduced by introducing effective coupling schemes, which allow for a safer extrapolation of $\Lambda_L$ to its continuum value. We are also able to see and to quantify the running of the coupling from the interquark force. From this we extract the ratio

$\sqrt{\sigma}/\Lambda_L$. Both methods yield consistent values for the $\Lambda$-parameter: $\Lambda_{\text{phys}} = 0.558^{+0.017}_{-0.007} \times \sqrt{\sigma} = 246^{+14}_{-15}$MeV.

1. Introduction

It is of great interest to extend lattice techniques into a regime where perturbative methods can be applied. We achieve a lattice spacing as small as $a \approx (6 \text{ GeV})^{-1}$ on a $32^4$ lattice at $\beta = 6.8$. From the static SU(3) quark-antiquark potential we can extract the running coupling “constant” within an accuracy that competes with current (real QCD) experiments.

2. Simulation

In our simulations we have realized lattice volumes of $L_S^3 \times L_T = 32^4$ for the $\beta$-values 6.0, 6.4, and 6.8, and a $24^3 \times 32$ lattice for $\beta = 6.2$. The corresponding spatial lattice extents in order of increasing $\beta$ are: 3.25, 2.74, 1.74, and 1.05 fm with the physical scale set by the value $\sqrt{\sigma} = 440$ MeV for the string tension. In order to investigate finite size effects (FSE) we simulated, moreover, a $16^4$ lattice at $\beta = 6.0$, $16^3 \times 32$, $24^3 \times 32$, and $32^3 \times 16$ lattices at $\beta = 6.4$, and a $16^3 \times 64$ lattice at $\beta = 6.8$.

We used the hybrid overrelaxation algorithm for updating the gauge fields, as advocated in Kennedy’s review talk. Great care was taken in optimizing the overrelaxation step.

To improve the projection of our operators onto the $q\bar{q}$ ground state we applied an iterative local gauge covariant smoothing procedure on the spatial links before constructing Wilson loops. Overlaps of 95(80)% for small (large) spatial $q\bar{q}$ separations $R$ were reached. Various smoothed on- and off-axis Wilson loops were measured every 100 sweeps. This separation was found to be sufficient to guarantee an (almost) independent sampling. Remaining autocorrelation effects were healed by blocking the data into bins of reasonable length before statistical analysis. On all volumes $O(100)$ measurements were performed.

The spatial $q\bar{q}$ separations $\mathbf{R} = ne_i$ with $e_i = (1, 0, 0), (1, 1, 0), (2, 1, 0), (1, 1, 1), (2, 1, 1), (2, 2, 1)$ were realized. $n$ was increased up to $L_S/2$ for $i = 1, 2, 4$, and up to $L_S/4$ for the remaining directions. Altogether this yields 72 different values of $\mathbf{R}$ on the $32^3 \times L_T$ lattices. The time separations $T = 1, 2, \ldots, 10$ were used to test exponential behaviour and for extraction of the potential values.

3. Results

$q\bar{q}$ potential: We follow Chris Michael and start from the ansatz

$$V(\mathbf{R}) = V_0 + KR - e \left( \frac{1}{R} + l G_L(\mathbf{R}) \right) + \frac{f}{R^2}. \quad (1)$$

The lattice propagator for the one gluon exchange...
Figure 1. Potential $V_C(R)$ at $\beta = 6.4$, $V = 32^4$.

Table 1
Fit parameters (see Eq. (1)).

| $\beta$ | $K$  | $e$  | $V_0$ | $l$  | $f$  |
|---------|------|------|-------|------|------|
| 6.0     | 0.0513(25) | 0.275(28) | 0.636(10) | 0.64(12) | 0.041(58) |
| 6.4     | 0.0148(3)  | 0.315(15)  | 0.601(4)   | 0.56(6)   | 0.075(18)  |
| $\beta = 6.5$ | 0.0114(2) | 0.311(14) | $-$     | 0.64(6) | 0.067(13) |
| $\beta = 6.8$ | 0.0053(2) | 0.311(10) | $-$     | 0.56(4) | 0.094(13) |

$G_L(R)$ has been calculated in the large volume limit. The parameter $l$ controls violations of rotational symmetry within this ansatz. The term $f/R^2$ simulates deviations from a pure Coulomb behaviour and is expected to be positive from asymptotic freedom, and to increase with the energy resolution.

In Figure 1 the lattice corrected data points $V_C(R) = V(R) + e l (G_L(R) - 1/R)$ are plotted together with the interpolating fit curve. The figure demonstrates the success of ansatz Eq. (1) as all data points line up on a smooth curve. Our potential fits yield $\chi^2/N_{DF} < 1$ as long as the first two data points are excluded.

We have included the corresponding fit parameters from the recent UKQCD investigation on a $36^4$ lattice at $\beta = 6.5$ in Table 1. For $\beta \geq 6.4$ all Coulomb coefficients $e$ are definitely different (but remarkably stable) from the string vibration value $\pi/12 \approx 0.262$. $V_0$ decreases with the coupling. We emphasize that the parameter $f$ is established to increase with $\beta$ as expected. $l$ seems to vary only slightly.

Finite size effects: On varying the lattice extent at constant lattice spacing from 1.7 to 3.3 fm ($\beta = 6.0$), and from 0.9 up to 1.8 fm ($\beta = 6.4$) we found no evidence of FSE on our potential data within statistical accuracy ($\approx 1\%$). However, a comparison of results on the $\beta = 6.8$ lattices (0.53 and 1.05 fm) shows deviations (up to $3\sigma$), mainly for small $R$ separations where statistical errors are small. We conclude that a lattice extent $aL_s > 0.9$ fm $\approx 2/\sqrt{\sigma}$ suffices in our case. The same limit was also found in Ref. [3]. This condition coincides with the transition into the deconfined phase at $aL_T > 1/T_C \approx 2/\sqrt{\sigma}$.

Note, that a small volume can also cause problems of another kind: It is difficult to fit the potential to a small number of data points with some confidence. This effect becomes dominant for $\beta \leq 5.9$ where the physical limit is $L_s \geq 8$. In any case, it pays to work on a large lattice. The computational task (except memory requirement) remains comparable because self averaging suppresses fluctuations and reduces the number of required Monte Carlo sweeps.

Table 2
$\Lambda^{-1}_{L}$ in units of the string tension.

| $\beta$ | $\sqrt{\sigma}/\Lambda_L$ | $\sqrt{\sigma}/\Lambda_{L}^{(1)}$ | $\sqrt{\sigma}/\Lambda_{L}^{(2)}$ |
|---------|--------------------------|---------------------------------|---------------------------------|
| 5.7     | 124.7 (07)               | 63.3 (04)                       | 55.7 (03)                       |
| 5.8     | 112.4 (10)               | 63.0 (06)                       | 55.6 (05)                       |
| 5.9     | 102.9 (14)               | 61.2 (08)                       | 54.3 (07)                       |
| 6.0     | 96.5 (23)                | 60.0 (15)                       | 53.4 (13)                       |
| 6.2     | 86.4 (10)                | 56.9 (07)                       | 50.8 (06)                       |
| 6.4     | 81.3 (08)                | 55.7 (05)                       | 50.0 (05)                       |
| 6.5     | 80.0 (14)                | 55.7 (10)                       | 50.1 (09)                       |
| 6.8     | 76.9 (13)                | 55.7 (09)                       | 50.4 (08)                       |
| $\infty$| $54^{+18}_{-15}$          | $53.2^{+2.6}_{-7.3}$            | $49.1^{+2.3}_{-5.9}$            |
Scaling: Within our range of lattice spacings, scaling of physical quantities works nicely: In the present investigation all potential data, measured at different \( \beta \) values, fall onto a single curve when scaled to \( V_0 = 0 \), and \( K = 1 \). However, though scaling seems to be restored, it still does not follow the perturbatively expected two-loop behaviour \( a_\sigma = f(\beta)/\Lambda_L \) with \( a_\sigma = \sqrt{K}/\sqrt{\sigma} \), and \( f(\beta) = \) the integrated two-loop \( \beta \)-function. This is reflected in the first column of Table 2. It was argued that the lack of asymptotic scaling may be caused by a bad choice of expansion parameter. For remedy, a variety of schemes with redefined expansion parameters have been proposed in the past.

Here, we attempt to achieve an improvement by using the (inverse) coupling \( \beta_E \) that has originally been invented by Parisi, and successfully applied in Ref. [1] to gauge theories. This coupling is defined by truncating the weak order expansion of the plaquette (measured in the Monte Carlo simulation) after the first order term, and inverting the resulting relation. As a check, we investigate an alternative \( \beta_E^{(2)} \) scheme, defined by truncating the expansion after the second order term, instead. The improvement can be seen from Table 2 where \( \Lambda_L^{(1)} \) corresponds to the \( \beta_E \), and \( \Lambda_L^{(2)} \) to the \( \beta_E^{(2)} \) scheme.

Extrapolations to the continuum limit were done by fitting the data to the leading order expectation \( \Lambda_L^{-1}(\alpha) = \Lambda_L^{-1}(0) + C/(\sqrt{\sigma} \ln(Da\sqrt{\sigma})) \). There is a considerable bandwidth in the extrapolations, depending on the number of points included in the fits. The effective schemes help to decrease this uncertainty and teach us that linear extrapolations are misleading. Our best estimate for \( \Lambda_L(0) \) is \( \sqrt{\sigma} = 50.8^{+1.0}_{-0.9} \) GeV.

An interesting program would be to take other nonperturbative quantities (like small Wilson loops) that can be measured accurately on small lattices, and use them to define different effective couplings. Since one can translate, at a given \( \beta \), one scheme into another, self consistent extrapolations (and interpolations) become possible. By exploiting all available nonperturbative information, one can in this way reconstruct the underlying \( \beta \)-function beyond the two-loop approximation.

Running coupling: So far we have computed a mass \( (\sqrt{\sigma}) \) as function of a coupling. In the following, we reverse this procedure and determine the coupling \( \alpha_{q\bar{q}}(R) \). This is achieved by numerically differentiating the (lattice corrected) potential, and multiplying the resulting interquark force with \( \sqrt{R^2} \). The results (Figure 3) are in fairly good agreement with \( \alpha_{\text{MS}}(a^{-1}) \) \( \sqrt{\alpha_{\text{MS}}(q)} = \alpha_{q\bar{q}}(r) + 0.05\alpha_{q\bar{q}}^{05}(r) + \cdots \) with \( q = 1/r \), calculated from the scaling of the string tension in the manner proposed in Ref. [1].

We fit our data to the expectation \( \alpha_{q\bar{q}}^{-1}(r) = 4\pi (b_0 \ln(Ra\Lambda_R)^{-2} + b_1/b_0 \ln(\ln(Ra\Lambda_R)^{-2}) \). Exploiting the relation \( \Lambda_R = 30.19\Lambda_L \), we arrive at \( \sqrt{\sigma} = 53.7(2.1)\Lambda_L^2 \) which is in agreement with the value extrapolated from the scaling of the string tension. We find the data to be in good agreement with the perturbative formula for \( q \geq 5\sqrt{\sigma} \approx 2 \) GeV. At small energies (large \( R \)) our data tends to lie above the two-loop expression since \( \alpha_{q\bar{q}} \propto 1/q^2(q \to 0) \).

Averaging the two independently calculated \( \Lambda \) parameters gives \( \sqrt{\sigma} = 51.6^{+0.7}_{-0.8}\Lambda_L \) or \( \Lambda_{\text{MS}} = 0.558^{+1.1}_{-0.7}\sqrt{\sigma} = 246^{+7}_{-3}\) MeV.

The approach towards this asymptotic limit for different definitions of effective couplings is illustrated in Figure 4. We observe that the values for \( \Lambda_L^{-1} \), extracted from the running of the cou-
Figure 3. The running coupling $\alpha_{q\bar{q}}(R)$.

Figure 4. Violations of asymptotic scaling.

4. Conclusion

For our results, it has been important to study both infrared, and ultraviolet aspects in order to verify the reliability of the continuum extrapolation. We might say that we have been lucky to get hold of asymptotia within our means. This is due to the discovery that the running coupling “constant” is well described within this theory by the two-loop formula down to a scale of 1–2 GeV.

If nature continues to be nice to us it is possible to predict experimental numbers like $\alpha_S(M_Z)$ or $\alpha_S^{(4)}_{\overline{MS}}$, as explained in Ref. [3]. In order to remove additional systematic uncertainties caused by the fact that experimentalists still have not discovered how to switch off some quark flavours and deliver us a quenched value for the string tension, it is preferable to repeat this study in full QCD on the level of TeraFLOPS power. Meanwhile, further improvements of lattice techniques are of great interest.

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