On $\mathcal{A}$-Transvections and Symplectic $\mathcal{A}$-Modules

Patrice P. Ntumba

Abstract

In this paper, building on prior joint work by Mallios and Ntumba, we show that $\mathcal{A}$-transvections and singular symplectic $\mathcal{A}$-automorphisms of symplectic $\mathcal{A}$-modules of finite rank have properties similar to the ones enjoyed by their classical counterparts. The characterization of singular symplectic $\mathcal{A}$-automorphisms of symplectic $\mathcal{A}$-modules of finite rank is grounded on a newly introduced class of pairings of $\mathcal{A}$-modules: the orthogonally convenient pairings. We also show that, given a symplectic $\mathcal{A}$-module $\mathcal{E}$ of finite rank, with $\mathcal{A}$ a PID-algebra sheaf, any injective $\mathcal{A}$-morphism of a Lagrangian sub-$\mathcal{A}$-module $\mathcal{F}$ of $\mathcal{E}$ into $\mathcal{E}$ may be extended to an $\mathcal{A}$-symplectomorphism of $\mathcal{E}$ such that its restriction on $\mathcal{F}$ equals the identity of $\mathcal{F}$. This result also holds in the more general case whereby the underlying free $\mathcal{A}$-module $\mathcal{E}$ is equipped with two symplectic $\mathcal{A}$-structures $\omega_0$ and $\omega_1$, but with $\mathcal{F}$ being Lagrangian with respect to both $\omega_0$ and $\omega_1$. The latter is the analog of the classical Witt’s theorem for symplectic $\mathcal{A}$-modules of finite rank.

Key Words: $\mathcal{A}$-homothecy, $\mathcal{A}$-transvections, adjoint $\mathcal{A}$-morphism, symplectic $\mathcal{A}$-modules, symplectic group sheaf, PID-algebra sheaf, orthogonally convenient pairings, locally free $\mathcal{A}$-module of varying finite rank.
Introduction

Here is a further attempt of investigating classical notions/results such as transvections, Witt’s theorem for symplectic vector spaces, the characterization of singular symplectic automorphisms of symplectic vector spaces of finite (even) dimension within the context of Abstract Differential Geometry (à la Mallios), [10, 11]. This endeavor, as already signalled in [14], is for the purpose of rewriting and/or recapturing a great deal of classical symplectic (differential) geometry without any use (at all!) of any notion of “differentiability” (differentiability is here understood in the sense of the standard differential geometry of $C^\infty$-manifolds).

Now, we take the opportunity to review succinctly the basic notions of Abstract Geometric Algebra which we are concerned with in this paper. Most of the concepts in this paper are defined on the basis of the classical ones; see to this effect, Artin [2], Berndt [4], Crumeyrolle [6], Deheuvels [7], Lang [9]. Our main reference is Mallios [10].

A $\mathbb{C}$-algebraized space on a topological space $X$ is a pair $(X, \mathcal{A})$, where $\mathcal{A} \equiv (\mathcal{A}, \tau, X)$ is a (preferably unital and commutative) sheaf of $\mathbb{C}$-algebras (or in other words, a $\mathbb{C}$-algebra sheaf). A sheaf of sets $\mathcal{E} \equiv (\mathcal{E}, \rho, X)$ on $X$ is a sheaf of groups (or a group sheaf) on $X$, provided the following conditions are satisfied: (i) Each stalk of $\mathcal{E}$ is a group; (ii) Given the set $\mathcal{E} \circ \mathcal{E} := \{(z, z') \in \mathcal{E} \times \mathcal{E} : \rho(z) = \rho(z')\}$, the map $\mathcal{E} \circ \mathcal{E} \longrightarrow \mathcal{E} : (z, z') \longmapsto z + z' \in \mathcal{E}_x \subseteq \mathcal{E}$ is continuous ($\mathcal{E} \circ \mathcal{E}$ is equipped with the topology induced from $\mathcal{E} \times \mathcal{E}$); A sheaf of $\mathcal{A}$-modules (or an $\mathcal{A}$-module) on $X$ is a sheaf $\mathcal{E} \equiv (\mathcal{E}, \rho, X)$ such that the following conditions hold: (i) $\mathcal{E}$ is a sheaf of abelian groups; (ii) For every point $x \in X$, the corresponding stalk $\mathcal{E}_x$ of $\mathcal{E}$ is a (left) $\mathcal{A}_x$-module; (iii) The exterior module multiplication in $\mathcal{E}$, viz. the map $\mathcal{A} \circ \mathcal{E} \longrightarrow \mathcal{E} : (a, z) \longmapsto a \cdot z \in \mathcal{E}_x \subseteq \mathcal{E}$ with $\tau(a) = \rho(z) = x \in X$, is continuous. An $\mathcal{A}$-module $\mathcal{E}$ is called a free $\mathcal{A}$-module of rank $n \ (n \in \mathbb{N})$, provided $\mathcal{E} \cong \mathcal{A}^n$ within an $\mathcal{A}$-isomorphism. The $\mathcal{A}$-module $\mathcal{A}^n$ is called the standard free $\mathcal{A}$-module of rank $n$. For an open subset $U \subseteq X$, the canonical basis of the $\mathcal{A}(U)$-module $\mathcal{A}^n(U)$ is the set $\{\epsilon_i^U\}_{1 \leq i \leq n}$, where $\epsilon_i^U := \delta_{ij} U \in \mathcal{A}^n(U) \cong \mathcal{A}(U)^n$ such that $\delta_{ii} U = 1$ for $i = j$ and $\delta_{ij} U = 0$ for $i \neq j$. So one gets, for any $x \in X$, $\epsilon_i^U(x) = (\delta_{ij} U(x))_{1 \leq j \leq n} \in \mathcal{A}_x^n$ ($1 \leq i \leq n$), where $\delta_{ii} U(x) = 1_x \in \mathcal{A}_x$, if $i = j$, and $\delta_{ij} U(x) = 0_x \in \mathcal{A}_x$, if $i \neq j$. 
Now suppose there is given a presheaf of unital and commutative \( \mathbb{C} \)-algebras \( \mathcal{A} \equiv (\mathcal{A}(U), \tau_V^U) \) and a presheaf of abelian groups \( \mathcal{E} \equiv (\mathcal{E}(U), \rho_V^U, \sigma_V^U) \), both on a topological space \( X \) and such that (i) \( \mathcal{E}(U) \) is a (left) \( \mathcal{A}(U) \)-module, for every open set \( U \subseteq X \), (ii) For any open sets \( U, V \) in \( X \), with \( V \subseteq U \), \( \rho_V^U(a \cdot s) = \tau_V^U(a) \cdot \rho_V^U(s) \) for any \( a \in \mathcal{A}(U) \) and \( s \in \mathcal{E}(U) \). We call such a presheaf \( \mathcal{E} \) a \emph{presheaf of \( \mathcal{A}(U) \)-modules} on \( X \), or simply an \( \mathcal{A} \)-presheaf on \( X \).

All our \( \mathcal{A} \)-modules and \( \mathcal{A} \)-presheaves in this paper are defined on a fixed topological space \( X \). \( \mathcal{A} \)-modules and \( \mathcal{A} \)-presheaves with their respective morphisms form categories which we denote \( \mathcal{A} \text{-Mod}_X \) and \( \mathcal{A} \text{-PSh}_X \) respectively. By virtue of the equivalence \( \mathcal{S}h_X \cong \mathcal{C}o\mathcal{P}sh_X \), an \( \mathcal{A} \)-morphism \( \phi : \mathcal{E} \longrightarrow \mathcal{F} \) of \( \mathcal{A} \)-modules \( \mathcal{E} \) and \( \mathcal{F} \) may be identified with the \( \mathcal{A} \)-morphism \( \overline{\phi} := (\phi_V)_{U \supseteq V, \text{open}} : \mathcal{E} \longrightarrow \mathcal{F} \) of the associated \( \mathcal{A} \)-presheaves. We shall most often denote by just \( \phi \) the corresponding \( \mathcal{A} \)-morphism associated with the \( \mathcal{A} \)-morphism \( \overline{\phi} \). The meaning of \( \phi \) will always be determined by the situation at hand.

Recall that given an \( \mathcal{A} \)-module \( \mathcal{E} \) and a sub-\( \mathcal{A} \)-module \( \mathcal{F} \) of \( \mathcal{E} \), the quotient \( \mathcal{A} \)-module of \( \mathcal{E} \) by \( \mathcal{F} \) is the \( \mathcal{A} \)-module generated by the presheaf sending an open \( U \subseteq X \) to an \( \mathcal{A}(U) \)-module \( S(U) := \Gamma(U, \mathcal{E})/\Gamma(U, \mathcal{F}) \equiv \mathcal{E}(U)/\mathcal{F}(U) \) such that for every restriction map \( \sigma_V^U : \mathcal{E}(U)/\mathcal{F}(U) \longrightarrow \mathcal{E}(V)/\mathcal{F}(V) \) one has \( \sigma_V^U(r + \mathcal{F}(U)) := \rho_V^U(r) + \mathcal{F}(V) \) (the \( \rho_V^U \) are the restriction maps for the \( \mathcal{A} \)-presheaf \( \mathcal{E} \)).

For the sake of easy referencing, we also recall some notions, which may be found in our recent papers such as [12], [13], [14], and [17]. Let \( \mathcal{F} \) and \( \mathcal{E} \) be \( \mathcal{A} \)-modules and \( \phi : \mathcal{F} \oplus \mathcal{E} \longrightarrow \mathcal{A} \) an \( \mathcal{A} \)-bilinear morphism. Then, we say that the triplet \( [\mathcal{F}, \mathcal{E}; \mathcal{A}] \equiv ([\mathcal{F}, \mathcal{E}; \phi]; \mathcal{A}] \) forms a \emph{pairing of \( \mathcal{A} \)-modules} or that \( \mathcal{F} \) and \( \mathcal{E} \) are \emph{paired through} \( \phi \) into \( \mathcal{A} \). The sub-\( \mathcal{A} \)-module \( \mathcal{F}^\perp \) of \( \mathcal{E} \) such that, for every open subset \( U \) of \( X \), \( \mathcal{F}^\perp (U) \) consists of all \( r \in \mathcal{E}(U) \) with \( \phi_V(\mathcal{F}(V), r|_V) = 0 \) for any open \( V \subseteq U \), is called the \emph{right kernel} of the pairing \( [\mathcal{F}, \mathcal{E}; \mathcal{A}] \). In a similar way, one defines the \emph{left kernel} of \([\mathcal{F}, \mathcal{E}; \phi]; \mathcal{A}] \) to be the sub-\( \mathcal{A} \)-module \( \mathcal{E}^\perp \) of \( \mathcal{F} \) such that, for any open subset \( U \) of \( X \), \( \mathcal{E}^\perp (U) \) is the set of all (local) sections \( r \in \mathcal{F}(U) \) such that \( \phi_V(r|_V, \mathcal{E}(V)) = 0 \) for every open \( V \subseteq U \). If \([\mathcal{F}, \mathcal{E}; \phi]; \mathcal{A}] \) is a pairing of free \( \mathcal{A} \)-modules, then, for every open subset \( U \) of \( X \), \( \mathcal{F}^\perp (U) = \mathcal{F}(U)^\perp := \{ r \in \mathcal{E}(U) : \phi_U(\mathcal{F}(U), r) = 0 \} \), and similarly \( \mathcal{E}^\perp (U) = \mathcal{E}(U)^\perp := \{ r \in \mathcal{F}(U) : \phi_U(r, \mathcal{E}(U)) = 0 \} \).
Now, let $[(\mathcal{E}, \mathcal{E}; \phi); \mathcal{A}]$ be a pairing such that if $r, s \in \mathcal{E}(U)$, where $U$ is an open subset of $X$, then $\phi_U(r, s) = 0$ if and only if $\phi_U(s, r) = 0$. The left kernel, $\mathcal{E}_l := \mathcal{E}^\perp$, is the same as the right kernel $\mathcal{E}_r := \mathcal{E}^\top$. In that case, we say that the $\mathcal{A}$-bilinear form $\phi$ is orthosymmetric and call $\mathcal{E}^\perp(= \mathcal{E}^\top)$ the radical sheaf (or sheaf of $\mathcal{A}$-radicals, or simply $\mathcal{A}$-radical) of $\mathcal{E}$, and denote it by $\text{rad}_\mathcal{A}\mathcal{E} \equiv \text{rad}\mathcal{E}$. An $\mathcal{A}$-module $\mathcal{E}$ such that $\text{rad}\mathcal{E} = 0$ (resp. $\text{rad}\mathcal{E} = 0$) is called isotropic (resp. non-isotropic); $\mathcal{E}$ is totally isotropic if $\phi$ is identically zero, i.e. $\phi_U(r, s) = 0$ for all sections $r, s \in \mathcal{E}(U)$, with $U$ open in $X$. For any open $U \subseteq X$, a non-zero section $r \in \mathcal{E}(U)$ is called isotropic if $\phi_U(r, r) = 0$.

**N.B.** We assume throughout the paper, unless otherwise mentioned, that the pair $(X, \mathcal{A})$ is an algebraized space, where $\mathcal{A}$ is a unital $\mathbb{C}$-algebra sheaf such that every nowhere-zero section of $\mathcal{A}$ is invertible. Furthermore, all free $\mathcal{A}$-modules are considered to be torsion-free, that is, for any open subset $U \subseteq X$ and nowhere-zero section $s \in \mathcal{E}(U)$, if $as = 0$, where $a \in \mathcal{A}(U)$, then $a = 0$.

## 1 $\mathcal{A}$-transvections and useful background

For the purpose of this section, we recall the following useful facts/results, which are found in our papers [12], [18] and [15]. For these results, $\mathcal{A}$ is just an arbitrary unital sheaf of $\mathbb{C}$-algebras with no other condition on.

**Theorem 1.1** Let $\mathcal{E}$ be an $\mathcal{A}$-module and $\mathcal{F}$ and $\mathcal{G}$ sub-$\mathcal{A}$-modules of $\mathcal{E}$. Then,

$$\mathcal{G}/(\mathcal{F} \cap \mathcal{G}) \cong S(\Gamma(\mathcal{F}) + \Gamma(\mathcal{G}))/\mathcal{F}.$$
Theorem 1.1 yields the following result.

**Corollary 1.1** Let \( \mathcal{E} \) be an \( A \)-module, \( \mathcal{F} \) and \( \mathcal{G} \) sub-\( A \)-modules of \( \mathcal{E} \) such that \( \mathcal{E} \cong \mathcal{F} \oplus \mathcal{G} \). Then,

\[ \mathcal{E}/\mathcal{F} \cong \mathcal{G}. \]

**Corollary 1.2** If \( \mathcal{F} \) is a free sub-\( A \)-module of a free \( A \)-module \( \mathcal{E} \), then the quotient \( \mathcal{E}/\mathcal{F} \) is free and for every open \( U \subseteq X \),

\[ (\mathcal{E}/\mathcal{F})(U) = \mathcal{E}(U)/\mathcal{F}(U) \]

within an \( A(U) \)-isomorphism. Moreover,

\[ \text{rank } \mathcal{F} + \text{corank } \mathcal{F} = \text{rank } \mathcal{E}, \]

where \( \text{corank } \mathcal{F} := \text{rank } (\mathcal{E}/\mathcal{F}) \).

Let us recall the following definition.

**Definition 1.1** Let \( \mathcal{E} \) be a free \( A \)-module and \( \mathcal{F} \) a free sub-\( A \)-module of \( \mathcal{E} \), complemented in \( \mathcal{E} \) be a free sub-\( A \)-module \( \mathcal{G} \). The rank of \( \mathcal{G} \cong \mathcal{E}/\mathcal{F} \) is called the **corank** of \( \mathcal{F} \), that is,

\[ \text{corank } \mathcal{F} = \text{rank}(\mathcal{E}/\mathcal{F}). \]

Furthermore, let us also state the following results.

**Theorem 1.2** Let \( \mathcal{E} \) be a free \( A \)-module, and \( \mathcal{F} \) and \( \mathcal{G} \) free sub-\( A \)-modules of \( \mathcal{E} \) such that \( \mathcal{F} \cap \mathcal{G} \) and \( \mathcal{F} + \mathcal{G} \) are free. Then,

\[
\begin{align*}
\text{rank } \mathcal{F} + \text{corank } \mathcal{F} & = \text{rank } \mathcal{E} \\
\text{rank } (\mathcal{F} + \mathcal{G}) + \text{rank } (\mathcal{F} \cap \mathcal{G}) & = \text{rank } \mathcal{F} + \text{rank } \mathcal{G} \\
\text{corank } (\mathcal{F} + \mathcal{G}) + \text{corank } (\mathcal{F} \cap \mathcal{G}) & = \text{corank } \mathcal{F} + \text{corank } \mathcal{G}.
\end{align*}
\]
**Theorem 1.3** Let $E$ be a free $A$-module of arbitrary rank. Then, for any open subset $U \subseteq X$, $\text{rank } E^*(U) = \text{rank } E(U)$. Fix an open set $U$ in $X$. If $\psi \equiv (\psi_V)_{V \supseteq U, \text{open}} \in E^*(U)$ and $\psi_U(s) = 0$ (which implies that $\psi_V(s|_V) = 0$ for any open $V \subseteq U$) for all $s \in E(U)$, then $\psi = 0$; on the other hand if $\psi_U(s) = 0$ for all $\psi \in E^*(U)$, then $s = 0$. Finally, let $\text{rank } E(U) = n$. To a given basis $\{s_i\}$ of $E(U)$, we can find a dual basis $\{\psi_i\}$ of $E^*(U) \cong E(U)$, where
\[ \psi_i, V(s_j|_V) = \delta_{ij, V} \in A(V) \]
for any open $V \subseteq U$.

**Theorem 1.4** Let $F$ and $E$ be $A$-modules paired into a $\mathbb{C}$-algebra sheaf $A$, and assume that $E^\perp = 0$. Moreover, let $F_0$ be a sub-$A$-module of $F$ and $E_0$ a sub-$A$-module of $E$. There exist natural $A$-isomorphisms into:
\[ \frac{E}{F^\perp_0} \longrightarrow F_0^*, \]
(1)
and
\[ E_0^\perp \longrightarrow (E/E_0)^*. \]
(2)

In keeping with the notations of Corollary 1.2, the free sub-$A$-module $F$ is called an $A$-hyperplane if $\text{corank } F = 1$, i.e. the quotient $A$-module $E/F$ is a line $A$-module. On the other hand, we notice that if $\phi \equiv (\phi_U)_{U \supseteq X, \text{open}}$ is an $A$-endomorphism of $E$, then $\phi$ induces on the line $A$-module $E/F$ an $A$-homothecy, which we denote by $\tilde{\phi}$. More explicitly, if $U$ is open in $X$ and $s$ a section of $E/F$ over $U$, then
\[ \tilde{\phi}(s) \equiv \tilde{\phi}_U(s) = a_U s \equiv a s \]
for some $a_U \equiv a \in A(U)$. The coefficient sections $a_U$ are such that $a_V = a_U|_V$ whenever $V$ is contained in $U$. The global section $a_X \equiv a$ is called the ratio of the $A$-homothecy $\tilde{\phi}$.

**Proposition 1.1** Let $E$ be a free $A$-module, and $F$ a proper free sub-$A$-module of $E$. Then, the following assertions are equivalent.

(1) $F$ is an $A$-hyperplane of $E$. 
(2) For every (local) section $s \in \mathcal{E}(U)$ such that $s|_V \notin \mathcal{F}(V)$ for every open $V \subseteq U$,
\[ \mathcal{E}(U) \cong \mathcal{A}(U)s \oplus \mathcal{F}(U). \]

(3) For every open $U \subseteq X$, there exists a section $s \in \mathcal{E}(U)$ with $s|_V \notin \mathcal{F}(V)$, where $V$ is any open subset contained in $U$, such that
\[ \mathcal{E}(U) \cong \mathcal{A}(U)s \oplus \mathcal{F}(U). \]

(4) The free sub-$\mathcal{A}$-module $\mathcal{F}$ is a maximal sub-$\mathcal{A}$-module in the inclusion-ordered set of proper free sub-$\mathcal{A}$-modules of $\mathcal{E}$.

**Proof.** (1) $\Rightarrow$ (2): For every open $U \subseteq X$ and section $s \in \mathcal{E}(U)$ such that $s|_V \notin \mathcal{F}(V)$ for any open $V \subseteq U$, it is clear that $\mathcal{A}(U)s + \mathcal{F}(U)$ is a direct sum. On the other hand, the equivalence class containing $s$ is a nowhere-zero section of $\mathcal{E}/\mathcal{F}$; it spans $\mathcal{E}(U)/\mathcal{F}(U)$ since $\mathcal{E}(U)/\mathcal{F}(U)$ has rank 1. It thus follows that $\mathcal{E}(U) \cong \mathcal{A}(U)s + \mathcal{F}(U)$.

(2) $\Rightarrow$ (3): Evident.

(3) $\Rightarrow$ (1): Since
\[ \text{rank}(\mathcal{E}/\mathcal{F})(U) = \text{rank}\mathcal{E}(U)/\mathcal{F}(U) = \text{rank}\mathcal{A}(U)s = 1 \]
for every open $U \subseteq X$ and $s \in \mathcal{E}(U)$ with $s|_V \notin \mathcal{F}(V)$, where $V$ is any open subset contained in $U$.

(2) $\Rightarrow$ (4): Let $\mathcal{F}'$ be a free sub-$\mathcal{A}$-module of $\mathcal{E}$ containing $\mathcal{F}$ and such that $\text{rank}\mathcal{F}' > \text{rank}\mathcal{F}$. For every open $U$ there exists a section $s\mathcal{F}'(U)$ such that $s|_V \notin \mathcal{F}(V)$ for every open $V \subseteq U$. By (2), for every open $U \subseteq X$, $\mathcal{E}(U) \cong \mathcal{A}(U)s \oplus \mathcal{F}(U)$; but $\mathcal{A}(U)s \oplus \mathcal{F}(U)$ is contained in $\mathcal{F}(U)$, therefore $\mathcal{F}' = \mathcal{E}$ within an $\mathcal{A}$-isomorphism.

(4) $\Rightarrow$ (2): Let $U$ be an open set in $X$. There exists a section $s \in \mathcal{E}(U)$ with $s|_V \notin \mathcal{F}(V)$ for any open $V \subseteq U$; then $\mathcal{A}(U)s \oplus \mathcal{F}(U)$ contains strictly $\mathcal{F}(U)$, thus $\mathcal{A}(U)s \oplus \mathcal{F}(U) \cong \mathcal{E}(U)$, since $\mathcal{F}$ is maximal. \[\blacksquare\]

So we come to Theorem [1.5], which characterizes the notion of $\mathcal{A}$-transvection. For the classical notion, see [2, 5, p. 152, Proposition 12.9], [6, 7, p. 419 ff], [8, 9, p. 542-544].
Theorem 1.5 Let \( E \) be a free \( A \)-module, \( \mathcal{H} \) an \( A \)-hyperplane of \( E \), \( \phi \in \text{End}_A E \) such that \( \phi(s) \equiv \phi_U(s) = s \) for any \( s \in \mathcal{H}(U) \), where \( U \) is an arbitrary open subset of \( X \), and \( \sim \) the \( A \)-homothecy induced by \( \phi \) on the line \( A \)-module \( E/\mathcal{H} \). Moreover, let \( a \in \mathcal{A}(X) \) be the ratio of \( \sim \). Then, \( a \) is either zero or nowhere zero, and the following hold.

1. If \( a|_U \equiv a_U \neq 1 \) for every open \( U \subseteq X \), there exists a unique line \( A \)-module \( L \subseteq E \) such that \( E \sim H \oplus L \) and \( L \) is stable by \( \phi \), i.e. \( \phi(L) \cong L \).

2. If \( a = 1 \), then for every \( A \)-morphism \( \theta \in E^* \equiv \text{Hom}_A(E, A) \) with \( \ker \theta \equiv \mathcal{H} \), there exists, for every open subset \( U \subseteq X \), a unique section \( r \in \mathcal{H}(U) \) such that
   \[
   \phi(s) = s + \theta(s)r
   \]
   for every \( s \in E(U) \).

Proof. Clearly, as \( \sim \in \text{End}_A(E/\mathcal{H}) \) and \( E/\mathcal{H} \) is a line \( A \)-module, and

\[
\text{rank} \mathcal{H}(U) + \text{rank}(E/\mathcal{H})(U) = \text{rank} E(U),
\]
for every open \( U \subseteq X \), it follows that \( a \) is either zero or nowhere zero.

Assertion (1). Uniqueness. Let \( L \) be a line \( A \)-module that complements \( \mathcal{H} \) in \( E \) and stable by the \( A \)-morphism \( \phi \), and \( s \) a nowhere-zero section of \( L \) on \( X \) (such a section \( s \) does exist because \( L \cong A \) and \( A \) is unital). Therefore, there exists \( b \in \mathcal{A}(X) \) such that \( \phi(s) = bs \). Next, assume that \( q \equiv (q_U)_{X \supseteq U, \text{open}} \) is the canonical \( A \)-morphism of \( E \) onto \( E/\mathcal{H} \). It is clear that \( \sim_X(q_X(s)) = bq_X(s) \equiv bq(s) \); thus \( \sim_X \) is a homothecy of ratio \( a = b \), hence, by hypothesis, \( b \) is nowhere 1. Now, let \( u \) be an element of \( E(X) \) such that \( u \notin \mathcal{H}(X) \); then there exists a non-zero \( \lambda \in \mathcal{A}(X) \) and an element \( t \in \mathcal{H}(X) \) such that

\[
u = \lambda s + t.
\]

It follows that

\[
\phi(u) = \lambda bs + t.
\]

Of course, \( \phi(u) \) and \( u \) are colinear if and only if \( t = 0 \). Thus, we have proved that every section \( u \in E(X) \) which is colinear with its image \( \phi(u) \) belongs
to $L(X)$. A similar argument holds should we consider the decomposition $E(U) \cong H(U) \oplus L(U)$, where $U$ is any other open subset $U$ of $X$. Hence, $L$ is the unique complement of $H$ in $E$, up to $A$-isomorphism, and stable by $\phi$.

**Existence.** Since $a$ is nowhere 1 on $X$, there exists a nowhere-zero section $s \in E(X)$ such that

$$\tilde{\phi}_U(q_U(s|_U)) := \tilde{\phi}_U(q_U(s_U)) \neq q_U(s_U) = q_U(s|_U)$$

for any open $U \subseteq X$. As $\tilde{\phi}q = q\phi$, it follows that $r_U := \phi_U(s_U) - s_U$ does not belong to $H(U)$, for any open $U \subseteq X$. The line $A$-module $L := [r_U]_{U \supseteq V}$, open clearly complements $H$. It remains to show that $L$ is stable by $\phi$: To this end, we first observe that every $s_U$ does not belong to the corresponding $H(U)$, and, by Proposition 1.1, $E(U) \cong A(U)s_U \oplus H(U)$. So, since $r_U \notin H(U)$ for every open $U \subseteq X$, there exists for every $r_U$ sections $\alpha_U \in A(U)$ and $t_U \in H(U)$ such that

$$r_U = \alpha_U s_U + t_U. \quad (4)$$

We deduce from (4) that

$$\phi_U(r_U) = (\alpha_U + 1)r_U,$$

and the proof is complete.

**Assertion 2. Uniqueness.** Let us fix an open set $U$ in $X$. The uniqueness of $r$ such that (3) holds is immediate, as $\theta_U(s) \equiv \theta(s) \neq 0$ for some $s \in E(U)$. Relation (4) also shows that if $s \in E(U)$ and $\theta(s)$ is nowhere zero, then necessarily

$$r = (\theta(s))^{-1}(\phi(s) - s).$$

**Existence.** Suppose given a section $s_0 \in E(U)$ such that $s_0|_V \notin H(V)$ for any open $V \subseteq U$. Let us consider the section $r = (\theta(s_0))^{-1}(\phi(s_0) - s_0)$. Clearly, $r \in H(U)$; indeed

$$(q \circ \phi)(s_0) - q(s_0) = (\tilde{\phi} \circ q)(s_0) - q(s_0) = 0.$$ 

The two $A(U)$-morphisms $s \mapsto \phi(s)$ and $s \mapsto s + \theta(s)r$ are equal, since they take on, on one hand, the same value at $s_0$, and, on the other hand, the same value at every $s \in H(U)$.
**Definition 1.2** Let $\mathcal{E}$ be a free $\mathcal{A}$-module, $\mathcal{H}$ an $\mathcal{A}$-hyperplane of $\mathcal{E}$, $\phi \in \text{End}_\mathcal{A}\mathcal{E}$ such that $\phi(s) \equiv \phi_U(s) = s$ for every $s \in \mathcal{H}(U)$, where $U$ is any open subset of $X$, and that the $\mathcal{A}$-homothecy $\widetilde{\phi}$, induced by $\phi$, has ratio $a \in \mathcal{A}(X): a = 1$. Then, $\phi$ is called an $\mathcal{A}$-transvection of $\mathcal{E}$, with respect to the $\mathcal{A}$-hyperplane $\mathcal{H}$.

We shall now be led to adjoints of $\mathcal{A}$-morphisms. More precisely,

**Definition 1.3** Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{A}$-modules and $\theta$ an $\mathcal{A}$-morphism of $\mathcal{E}$ into $\mathcal{F}$. The $\mathcal{A}$-morphism $^t\theta \in \text{Hom}_\mathcal{A}(\mathcal{F}^*, \mathcal{E}^*)$ such that, for every open set $U \subseteq X$ and sections $\psi \in \mathcal{F}^*(U)$, $s \in \mathcal{E}(V)$, where $V \subseteq U$ is open,

$$[(^t\theta)_U(\psi)](s) := \psi_V(\theta_V(s))$$

is called the transpose of the $\mathcal{A}$-morphism $\theta$.

It is clear from Definition 1.3 that every $\mathcal{A}$-morphism $\theta \in \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{F})$ admits a unique transpose $^t\theta \in \text{Hom}_\mathcal{A}(\mathcal{F}^*, \mathcal{E}^*)$. Note also that, in general, for every open subset $U \subseteq X$,

$$(^t\theta)_U \neq ^t(\theta_U).$$

Indeed, $(^t\theta)_U$ is the $\mathcal{A}|_U$-map $\mathcal{F}^*(U) \to \mathcal{E}^*(U)$, given by the formula above, whereas $^t(\theta_U)$ is the $\mathcal{A}(U)$-map $(\mathcal{F}(U))^* \to (\mathcal{E}(U))^*$ such that if $\psi \in (\mathcal{F}(U))^*$, i.e. an $\mathcal{A}(U)$-linear map on $\mathcal{F}(U)$, and $s \in \mathcal{E}(U)$, then

$$^t(\theta_U)(\psi)(s) := \psi(\theta_U(s)).$$

The inequality above means that transposes do not commute with restrictions.

**Definition 1.4** Let $\mathcal{E}$ be an $\mathcal{A}$-module and $\phi$ an $\mathcal{A}$-bilinear form on $\mathcal{E}$. The $\mathcal{A}$-bilinear form $\phi^*$ on $\mathcal{E}$ such that, for any open set $U$ in $X$ and sections $s, t \in \mathcal{E}(U)$,

$$\phi^*(s, t) \equiv \phi^*_U(s, t) = \phi_U(t, s) \equiv \phi(t, s)$$

is called the adjoint of $\phi$. 
Clearly, $\phi^* = \phi$ if and only if $\phi$ is symmetric; for $\phi^* = -\phi$ it is necessary and sufficient that $\phi$ be antisymmetric. An $\mathcal{A}$-bilinear form $\phi$ is called self-adjoint (resp. skew-adjoint) if $\phi^* = \phi$ (resp. $\phi^* = -\phi$).

In classical multilinear algebra (see e.g. [5, p. 339, Definition 20.1], [9, pp. 144, 145]), one may associate with a given bilinear form two linear maps: the right insertion map and the left insertion map. The corresponding situation for $\mathcal{A}$-bilinear forms is as follows.

**Definition 1.5** Let $\mathcal{E}$ and $\mathcal{F}$ be $\mathcal{A}$-modules, and $\phi : \mathcal{E} \oplus \mathcal{F} \rightarrow \mathcal{A}$ an $\mathcal{A}$-bilinear form. The $\mathcal{A}$-morphism

$$\phi^R \in \text{Hom}_\mathcal{A}(\mathcal{F}, \mathcal{E}^*) \equiv \text{Hom}_\mathcal{A}(\mathcal{F}, \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{A}))$$

(7)

such that, for any open subset $U \subseteq X$ and sections $t \in \mathcal{F}(U)$ and $s \in \mathcal{E}(V)$, where $V \subseteq U$ is open,

$$\phi^R_U(t)(s) \equiv (\phi^R)_U(t)(s) := \phi_V(s, t|_V)$$

is called the right insertion $\mathcal{A}$-morphism associated with the $\mathcal{A}$-bilinear form $\phi$. Similarly, for every open subset $U \subseteq X$ and sections $s \in \mathcal{E}(U)$ and $t \in \mathcal{F}(V)$, where $V \subseteq U$ is open,

$$\phi^L_U(s)(t) \equiv (\phi^L)_U(s)(t) := \phi_V(s|_V, t)$$

defines an $\mathcal{A}$-morphism $\phi^L$ of $\mathcal{E}$ into $\mathcal{F}^*$, i.e.

$$\phi^L \in \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{F}^*) \equiv \text{Hom}_\mathcal{A}(\mathcal{E}, \text{Hom}_\mathcal{A}(\mathcal{F}, \mathcal{A})).$$

The $\mathcal{A}$-morphism $\phi^L$ is called the left insertion $\mathcal{A}$-morphism associated with $\phi$.

It is clear in the light of Definition 1.5 that if the $\mathcal{A}$-bilinear form $\phi : \mathcal{E} \oplus \mathcal{F} \rightarrow \mathcal{A}$ is non-degenerate, then both insertion $\mathcal{A}$-morphisms $\phi^R$ and $\phi^L$ are injective.

**Definition 1.6** Let $\mathcal{E}$ and $\mathcal{E}'$ be $\mathcal{A}$-modules, $\phi$ and $\phi'$ non-degenerate $\mathcal{A}$-bilinear forms on $\mathcal{E}$ and $\mathcal{E}'$, respectively. Moreover, let $\psi$ be an $\mathcal{A}$-morphism
of $\mathcal{E}$ into $\mathcal{E}'$. An $\mathcal{A}$-morphism $\theta \in \text{Hom}_\mathcal{A}(\mathcal{E}', \mathcal{E})$ such that, for every open subset $U \subseteq X$ and sections $s \in \mathcal{E}(U)$, $t \in \mathcal{E}'(U)$, 
\[ \phi'(\psi(s), t) \equiv \phi'_U(\psi_U(s), t) = \phi_U(s, \theta_U(t)) \equiv \phi(s, \theta(t)) \]
is called an adjoint of $\psi$, and is denoted $\psi^*$. 

Keeping with the notations of Definition 1.6 above, we have

**Proposition 1.2** $\theta$ is unique whenever it exists.

**Proof.** Let $\theta_1$ and $\theta_2$ be $\mathcal{A}$-morphisms of $\mathcal{E}'$ into $\mathcal{E}$ such that, given any open subset $U \subseteq X$ and sections $s \in \mathcal{E}(U)$, $t \in \mathcal{E}'(U)$,
\[ \phi'_U(\psi_U(s), t) = \phi_U(s, \theta_1_U(t)) = \phi_U(s, \theta_2_U(t)). \] (8)
Using the right insertion $\mathcal{A}$-morphism $\phi^R$, it follows from (8) that
\[ \phi^R_U(\theta_1_U(t))(s) = \phi^R_U(\theta_2_U(t))(s). \]
Since $s$ is arbitrary in $\mathcal{E}(U)$,
\[ \phi^R_U(\theta_1_U(t)) = \phi^R_U(\theta_2_U(t)). \]
But $\phi^R$ is injective, therefore
\[ \theta_1_U = \theta_2_U. \]
Finally, since $U$ is arbitrary, $\theta_1 = \theta_2$. ■

Let us now enquire the existence of the adjoint of an $\mathcal{A}$-morphism $\psi \in \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{E}')$, where $\mathcal{E}$ and $\mathcal{E}'$ are $\mathcal{A}$-modules equipped with $\mathcal{A}$-bilinear forms $\phi$ and $\phi'$, respectively.

**Proposition 1.3** Let $\mathcal{E}$ and $\mathcal{E}'$ be $\mathcal{A}$-modules, equipped with non-degenerate $\mathcal{A}$-bilinear forms $\phi$ and $\phi'$, respectively. If $\mathcal{E}$ is free and of finite rank, then for every $\mathcal{A}$-morphism $\psi \in \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{E}')$ there exists an adjoint, denoted $\psi^*$, which is given by
\[ \psi^* = (\phi^R)^{-1} \circ t\psi \circ \phi^R, \]
where $t\psi : (\mathcal{E}')^* \rightarrow \mathcal{E}^*$ is the transpose of $\psi$. 


Proof. Let $U$ be an open subset of $X$, $s \in \mathcal{E}(U)$ and $t \in \mathcal{E}'(U)$. Using the right insertion $\mathcal{A}$-morphism $\phi'^R$, one has

$$\phi'_U(\psi_U(s), t) = \phi'^R_U(t)(\psi_U(s)) = (t\psi)_U(\phi'^R_U(t))(s).$$

(9)

Since $\mathcal{E}$ has finite rank and $\phi$ is non-degenerate, $\phi^R$ is an $\mathcal{A}$-isomorphism of $\mathcal{E}$ onto $\mathcal{E}^*$; so $t\psi \circ \phi'^R$ may be written

$$t\psi \circ \phi'^R = \phi^R \circ ((\phi^R)^{-1} \circ t\psi \circ \phi'^R).$$

It follows from (9) that

$$\phi'_U(\psi_U(s), t) = [\phi'^R_U(((\phi^R)^{-1} \circ (t\psi)_U \circ \phi'^R_U(t)))](s) = \phi_U(s, ((\phi^R)^{-1} \circ (t\psi)_U \circ \phi'^R_U(t))),$$

which ends the proof. 

Corollary 1.3 Adjoints commute with restrictions.

Proof. Let $\mathcal{E}$ and $\mathcal{E}'$ be $\mathcal{A}$-modules, $\phi$ and $\phi'$ non-degenerate $\mathcal{A}$-bilinear forms on $\mathcal{E}$ and $\mathcal{E}'$, respectively. Assume that $\psi \in \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}')$. Let $U$ be an open subset of $X$, and $s$, $t$ be sections of $\mathcal{E}$ and $\mathcal{E}'$ on $U$, respectively. By Definition 1.6 we have

$$\phi'_U(\psi_U(s), t) = \phi_U(s, (\psi^*)_U(t)).$$

On the other hand, since $\phi_U$ and $\phi'_U$ are non-degenerate and

$$\psi_U \in \text{Hom}_{\mathcal{A}(U)}(\mathcal{E}(U), \mathcal{E}'(U)),$$

then by virtue of [5] pp. 385, 386], we have

$$\phi'_U(\psi_U(s), t) = \phi_U(s, (\psi_U)^*(t)).$$

On account of uniqueness of adjoints, we have

$$(\psi^*)_U = (\psi_U)^*,$$

as desired. 

2 Symplectic $\mathcal{A}$-modules. Witt’s theorem

In this section, for the sake of self-containedness of the paper, we first recall the notion of symplectic $\mathcal{A}$-modules, and then describe how to characterize symplectomorphic $\mathcal{A}$-modules. We refer the reader to [14] and [17] for useful details regarding symplectic $\mathcal{A}$-modules and symplectic bases (of sections). Sheaves of symplectic groups arise in a natural way when one considers $\mathcal{A}$-isomorphisms between symplectic $\mathcal{A}$-modules which respect the symplectic structures involved. Finally, the section ends with a version of Witt’s theorem for symplectic $\mathcal{A}$-modules. For some other versions of the Witt’s theorem, see [13] and [19].

Definition 2.1 Let $\mathcal{E}$ be a free $\mathcal{A}$-module of finite rank, endowed with a skew-symmetric non-degenerate $\mathcal{A}$-bilinear morphism $\omega : \mathcal{E} \oplus \mathcal{E} \to \mathcal{A}$. Then, the pair $(\mathcal{E}, \omega)$ is called a symplectic $\mathcal{A}$-module.

Definition 2.2 Let $(\mathcal{E}, \omega)$ and $(\mathcal{E}', \omega')$ be symplectic $\mathcal{A}$-modules. An $\mathcal{A}$-morphism $\varphi \in \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}')$ is called symplectic if

$$\varphi^* \omega' := \omega' \circ (\varphi \times \varphi) = \omega,$$

that is, for any sections $s, t \in \mathcal{E}(U)$,

$$(\varphi_U^* \omega')(s, t) := \omega'_U \circ (\varphi_U(s), \varphi_U(t)) = \omega_U(s, t).$$

A symplectic $\mathcal{A}$-isomorphism is called an $\mathcal{A}$-symplectomorphism. Symplectic $\mathcal{A}$-modules $\mathcal{E}, \omega)$ and $(\mathcal{E}', \omega')$ are called symplectomorphic if there is an $\mathcal{A}$-symplectomorphism $\varphi$ between them.

The following result is not hard to prove (see [14, pp. 187-189, Lemma 4] for a proof thereof), and introduces a particular case of the notion of symplectic group sheaf (or sheaf of symplectic groups).

Lemma 2.1 Let $\mathcal{E} \equiv (\mathcal{E}, \omega)$ be a symplectic $\mathcal{A}$-module and let

$$(Sp \mathcal{E})(U) \subseteq \text{Aut}_{\mathcal{A}|U}(\mathcal{E}|U),$$

where $Sp \mathcal{E}$ denotes the symplectic group associated with the symplectic $\mathcal{A}$-module $\mathcal{E}$. 


where $U$ varies over the topology of $X$, be the group (under composition) of all $\mathcal{A}|_U$-symplectomorphisms of $\mathcal{E}|_U$ into $\mathcal{E}|_U$. Then, mappings

$$U \mapsto (Sp\mathcal{E})(U)$$

together with the obvious restriction maps yield a complete presheaf of groups on $X$. If $Sp\mathcal{E}$ the sheaf on $X$, generated by the aforesaid presheaf, one has

$$(Sp\mathcal{E})(U) = (Sp\mathcal{E})(U)$$

up to a group isomorphism, for every open $U \subseteq X$. The sheaf $Sp\mathcal{E}$ is called the \textit{symplectic group sheaf} of $\mathcal{E}$ (in fact, of $(\mathcal{E},\omega)$).

For an example of an $\mathcal{A}$-symplectic form, consider the $\mathcal{A}$-bilinear form, denoted $( | )$, on the standard free $\mathcal{A}$-module $\mathcal{A}^2n$ such that, given any open set $U$ in $X$ and sections $a \equiv (a_1, \ldots, a_{2n})$, $b \equiv (b_1, \ldots, b_{2n}) \in \mathcal{A}^{2n}(U) = \mathcal{A}(U)^{2n}$, one has

$$(a | b) := \sum_{i=1}^{n} a_i b_{i+n} - a_{i+n} b_i.$$  

The $\mathcal{A}$-bilinear form $( | )$ is called the \textit{standard $\mathcal{A}$-symplectic scalar product} or \textit{standard $\mathcal{A}$-symplectic form} on $\mathcal{A}^{2n}$, and the pair $(\mathcal{A}^{2n}, ( | ))$ the \textit{standard symplectic $\mathcal{A}$-module of rank 2n}. When there is no confusion about the symplectic $\mathcal{A}$-structure, the standard symplectic $\mathcal{A}$-module $(\mathcal{A}^{2n}, ( | ))$ will simply be denoted by $\mathcal{A}^{2n}$. The symplectic group sheaf of the standard symplectic $\mathcal{A}$-module $\mathcal{A}^{2n}$ is denoted by $Sp(2n;\mathcal{A})$ (or just $Sp(n;\mathcal{A})$).

An element of $Sp(2n;\mathcal{A})(U)$, where $U$ is an open subset of $X$, is called a \textit{symplectic section-matrix}.

Recall (see [14], [17]) that if a pair $(\mathcal{E},\omega)$ is a symplectic $\mathcal{A}$-module, then given any open subset $U$ of $X$, $\mathcal{E}(U)$ may be equipped with a basis $s_1, \ldots, s_n, t_1, \ldots, t_n$ with respect to which the symplectic form $\omega_U$ is represented by the matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$
in other words
\[ \omega_U(s_i, s_j) = \omega_U(t_i, t_j) = 0, \quad \omega_U(s_i, t_j) = \delta_{ij} \quad \text{for } 1 \leq i, j \leq n \]

A basis \((s_1, \ldots, s_n, t_1, \ldots, t_n)\) is called a *symplectic basis* of \(\mathcal{E}(U)\).

If \((\mathcal{E}, \omega)\) is a symplectic \(\mathcal{A}\)-module, and \(A\) and \(B\) two column (coordinate) matrices representing sections \(s\) and \(t\), respectively, with respect to a symplectic basis of \(\mathcal{E}(U)\), then
\[ \omega_U(s, t) = {}^t A JB, \]
where \({}^t A\) denotes the transpose of \(A\).

Applying the classical symplectic algebra machinery (see for instance \([5\text{, pp. 407, 408]}\) and \([7\text{, pp. 410-413}]\)) and in view of Lemma 3.1 a \(2n \times 2n\) section-matrix \(M \equiv \left( \begin{array}{cc} M_{11} & M_{12} \\ M_{21} & M_{22} \end{array} \right) \in GL(2n, \mathcal{A})(U) = GL_{2n}(\mathcal{A})(U) = GL_{2n}(\mathcal{A}(U))\) (where \(GL(2n, \mathcal{A})\) is the general linear group sheaf generated by the (complete) sub-presheaf \(U \mapsto \text{GL}(2n, \mathcal{A}(U))\), cf. \([10\text{, pp. 280-285}]\) is symplectic if and only if for any elements (sections) \(A\) and \(B\) of \((\mathcal{A}^{2n}(U), (\ | \ )) = (\mathcal{A}(U)^{2n}, (\ | \ ))\)
\[ (MA | MB) = {}^t (MA)JMB = {}^t A^t MJMB = {}^t AJB, \]
that is
\[ {}^t MJM = J. \quad (10) \]
Equation (10) splits into three:

1. \( {}^t M_{11}M_{21} = {}^t M_{21}M_{11} \) (i.e. \( {}^t M_{11}M_{21} \) is symmetric),
2. \( {}^t M_{12}M_{22} = {}^t M_{22}M_{12} \) (i.e. \( {}^t M_{12}M_{22} \) is symmetric),
3. \(-{}^t M_{21}M_{12} + {}^t M_{11}M_{22} = I_n\).
We will now prove an analog of the Witt’s theorem within the context of Abstract Differential Geometry. For the classical Witt’s theorem, see [1, pp. 368-387], [2, pp. 121, 122], [4, p. 21], [21], [6, pp. 11, 12], [7, pp. 148-152], [9, pp. 591, 592], [20, p. 9]. But, first we need the following definition (cf. [16]).

**Definition 2.3** A sheaf of algebras $\mathcal{A}$ is called a PID-algebra sheaf if for every open $U \subseteq X$, the algebra $\mathcal{A}(U)$ is a PID-algebra. In other words, given a free $\mathcal{A}$-module $E$ and a sub-$\mathcal{A}$-module $F \subseteq E$, one has that $F$ is “section-wise free.” That is, $F(U)$ is a free $\mathcal{A}(U)$-module, for any open $U \subseteq X$.

**Proposition 2.1** Let $\mathcal{A}$ be a PID algebra sheaf, $(\mathcal{E}, \omega)$ a symplectic free $\mathcal{A}$-module of rank $2n$, $\mathcal{F}$ a Lagrangian (free) sub-$\mathcal{A}$-module of $\mathcal{E}$ and $\mathcal{G}$ any sub-$\mathcal{A}$-module of $\mathcal{E}$ such that $\mathcal{F}$ and $\mathcal{G}$ are supplementary. Then, using $\mathcal{G}$ we can construct a Lagrangian sub-$\mathcal{A}$-module $\mathcal{H}$ of $\mathcal{E}$ such that $\mathcal{E} \cong \mathcal{F} \oplus \mathcal{H}$.

**Proof.** The restriction $\omega'$ of $\omega$ to $\mathcal{F} \oplus \mathcal{G} \subseteq \mathcal{E} \oplus \mathcal{E}$ is also non-degenerate. In fact, let $\mathcal{F}^\perp_{\omega'}$ and $\mathcal{G}^\perp_{\omega'}$ denote the kernels of $\mathcal{F}$ and $\mathcal{G}$ respectively. More precisely, for every open $U \subseteq X$,

$$\mathcal{F}^\perp_{\omega'}(U) = \{ r \in \mathcal{G}(U) | \omega'(\mathcal{F}(V), r|_V) = 0 \text{ for any open } V \subseteq U \}$$

and similarly

$$\mathcal{G}^\perp_{\omega'}(U) = \{ r \in \mathcal{F}(U) | \omega'(r|_V, \mathcal{G}(V)) = 0 \text{ for any open } V \subseteq U \}.$$ 

Analogously we denote by $\mathcal{F}^\perp_{\omega}$ and $\mathcal{G}^\perp_{\omega}$ the kernels of $\mathcal{F}$ and $\mathcal{G}$ respectively with respect to the $\mathcal{A}$-bilinear morphism $\omega : \mathcal{E} \oplus \mathcal{E} \to \mathcal{A}$, i.e. for every open $U \subseteq X$,

$$\mathcal{F}^\perp_{\omega}(U) = \{ r \in \mathcal{E}(U) | \omega(\mathcal{F}(V), r|_V) = 0 \text{ for any open } V \subseteq U \}$$

and

$$\mathcal{G}^\perp_{\omega}(U) = \{ r \in \mathcal{E}(U) | \omega(\mathcal{G}(V), r|_V) = 0 \text{ for any open } V \subseteq U \}.$$ 

It is obvious that $\mathcal{F}^\perp_{\omega} = \mathcal{F}^\perp_{\omega'}$ and $\mathcal{G}^\perp_{\omega} = \mathcal{G}^\perp_{\omega'}$. By hypothesis, we are given that $\mathcal{F} = \mathcal{F}^\perp_{\omega}$. Clearly, for every open $U \subseteq X$, $\mathcal{F}^\perp_{\omega'}(U) \subseteq \mathcal{F}^\perp_{\omega}(U)$ and $\mathcal{G}^\perp_{\omega'}(U) \subseteq \mathcal{G}^\perp_{\omega}(U)$.
\( G^\perp(U) \). But since \( F^\perp(U) = F(U) \) and \( F(U) \cap G(U) = 0 \), \( F^\perp(U) = 0 \). Thus, \( F^\perp = 0 \). On the other hand, let \( r \in G^\perp(U) \subseteq F(U) \cap G^\perp(U) \). As \( E(U) = F(U) \oplus G(U) \), we deduce that \( r \in \text{rad} E(U) = 0 \), therefore \( r = 0 \).

Hence, \( G^\perp = 0 \). Since \( \omega' : F \oplus G \to A \) is non-degenerate, the \( A \)-morphism \( \tilde{\omega} : F \to G^* \) such that for every open \( U \subseteq X \), and sections \( r \in F(U) \) and \( s \in G(U) \), \( \omega'(r,s) := \omega(r,s) \) is bijective.

Let us construct the sought Lagrangian complement \( H \) of \( F \) in \( E \). For every open \( U \subseteq X \), we let

\[
H(U) := \{ r + \phi(r) \mid r \in G(U) \},
\]

where \( \phi : G \to F \) is some \( A \)-morphism. It is clear that \( H \) is a sub-\( A \)-module of \( E \). For \( H \) to be Lagrangian, it takes the following: For every open \( U \subseteq X \) and sections \( r, s \in G(U) \)

\[
\omega(r + \phi(r), s + \phi(s)) = 0
\]

i.e.

\[
\omega(r, s) = \omega'(\phi(s))(r) - \omega'(\phi(r))(s). \tag{11}
\]

Let \( \phi' := \tilde{\omega} \circ \phi : G \to G^* \), so that \( (11) \) becomes

\[
\omega(r, s) = \phi'(s)(r) - \phi'(r)(s). \tag{12}
\]

Clearly, by taking \( \phi'(r) = -\frac{1}{2} \omega(r, -) \) for every \( r \in G(U) \), \( (12) \) is satisfied. By setting \( \phi := (\tilde{\omega})^{-1} \circ \phi' \), we contend that the claim holds. In fact, fix an open subset \( U \) of \( X \), and suppose that \( (r_1, \ldots, r_n) \) is a basis of \( G(U) \). If \( a_1, \ldots, a_n \in A(U) \) such that

\[
a_1(r_1 + \phi(r_1)) + \ldots + a_n(r_n + \phi(r_n)) = 0,
\]

one has that

\[
a_1r_1 + \ldots + a_nr_n = -\phi(a_1r_1 + \ldots + a_nr_n). \tag{13}
\]

Since \( F(U) \cap G(U) = 0 \), it follows that

\[
\phi(a_1r_1 + \ldots + a_nr_n) = 0.
\]
As the chosen $\phi'$ is injective and $\tilde{\omega}'$ is an $A$-isomorphism, $\phi$ is injective; hence

$$a_1 r_1 + \ldots + a_n r_n = 0;$$

so that $a_1 = \ldots = a_n = 0$. Now, let us show that $F(U) \cap H(U) = 0$. For this purpose, suppose that $r \in F(U) \cap H(U)$. Then for some $s \in G(U)$

$$r = s + \phi(s).$$

It follows that

$$r - \phi(s) = s \quad \in F(U)$$

from which we deduce that $s = 0$, and hence $r = 0$. That $E(U) \cong F(U) \oplus H(U)$ is now clear. Since $U$ is arbitrary, $E \cong F \oplus H$ as desired. 

**Theorem 2.1 (Witt’s Theorem)** Let $A$ be a PID algebra sheaf, let $E$ be a free $A$-module of rank $2n$, equipped with two symplectic $A$-morphisms $\omega_0$ and $\omega_1$, and finally let $F$ be a sub-$A$-module of $E$, Lagrangian with respect to both $\omega_0$ and $\omega_1$. Then, there exists an $A$-symplectomorphism $\phi : (E, \omega_0) \rightarrow (E, \omega_1)$ such that $\phi|_F = \text{Id}_F$.

**Proof.** Let $G$ be any complement of $F$ in $E$. By Proposition 2.1, given symplectic $A$-morphisms $\omega_0$ and $\omega_1$, there exist Lagrangian complements $G_0$ and $G_1$ of $F$ respectively. Again by the proof of Proposition 2.1, the restrictions $\omega'_0$, $\omega'_1$ of $\omega_0$, $\omega_1$ to $G_0 \oplus F$ and $G_1 \oplus F$ respectively are nondegenerate and yield $A$-isomorphisms $\tilde{\omega}_0 : G_0 \rightarrow F^*$ and $\tilde{\omega}_1 : G_1 \rightarrow F^*$ respectively. Since $G_0$ and $G_1$ are free and of the same finite rank, there exists an $A$-isomorphism $\psi : G_0 \rightarrow G_1$ such that $\tilde{\omega}_1 \circ \psi = \tilde{\omega}_0$, i.e. for any sections $r \in G_0(U)$ and $s \in F(U)$

$$\omega_0(r, s) = \omega_1(\psi(r), s).$$

Let us extend $\psi$ to the rest of $E$ by setting it to be the identity on $F$:

$$\phi := \text{Id}_F \oplus \psi : F \oplus G_0 \rightarrow F \oplus G_1$$

and we have for any sections $r, r' \in G_0(U)$ and $s, s' \in F(U)$

$$\omega_1(\phi(s + r), \phi(s' + r')) = \omega_1(s + \psi(r), s' + \psi(r'))$$

$$= \omega_1(s, \psi(r')) + \omega_1(\psi(r), s')$$

$$= \omega_0(s, r') + \omega_0(r, s')$$

$$= \omega_0(s + r, s' + r').$$
3 Orthogonally convenient pairings

We introduced here a new subclass of \( \mathcal{A} \)-modules: the orthogonally convenient pairings of \( \mathcal{A} \)-modules, with the aim of achieving the characterization of singular symplectic \( \mathcal{A} \)-automorphisms of symplectic orthogonally convenient \( \mathcal{A} \)-modules of finite rank.

We now make the following two definitions.

**Definition 3.1** A pairing \([\mathcal{F}, \mathcal{E}; \mathcal{A}]\) of free \( \mathcal{A} \)-modules \( \mathcal{F} \) and \( \mathcal{E} \) into the \( \mathbb{C} \)-algebra sheaf \( \mathcal{A} \) is called an **orthogonally convenient pairing** if given free sub-\( \mathcal{A} \)-modules \( \mathcal{F}_0 \) and \( \mathcal{E}_0 \) of \( \mathcal{F} \) and \( \mathcal{E} \), respectively, their orthogonal \( \mathcal{F}_0^\perp \) and \( \mathcal{E}_0^\perp \) are free sub-\( \mathcal{A} \)-modules of \( \mathcal{E} \) and \( \mathcal{F} \), respectively.

**Definition 3.2** The pairing \([\mathcal{E}^*, \mathcal{E}; \mathcal{A}] \equiv [(\mathcal{E}^*, \mathcal{E}; \phi); \mathcal{A}]\), where \( \mathcal{E} \) is a free \( \mathcal{A} \)-module and such that for every open \( U \subseteq X \),

\[
\phi_U(\psi, r) := \psi_U(r)
\]

with \( \psi \in \mathcal{E}^*(U) := \text{Hom}_{\mathcal{A}|U}(\mathcal{E}|U, \mathcal{A}|U) \) and \( r \in \mathcal{E}(U) \), is called the **canonical pairing** of \( \mathcal{E}^* \) and \( \mathcal{E} \).

**Theorem 3.1** Let \( \mathcal{E} \) be a free \( \mathcal{A} \)-module of finite rank. The canonical pairing \([\mathcal{E}^*, \mathcal{E}; \phi]; \mathcal{A}\) is orthogonally convenient.

**Proof.** First, we notice by Theorem [13] that both kernels, i.e. \( (\mathcal{E}^*)^\perp \) and \( \mathcal{E}^\perp \), are 0. Let \( \mathcal{E}_0 \) be a free sub-\( \mathcal{A} \)-module of \( \mathcal{E} \), and consider the map \( (2) \) of Theorem [14] \( \mathcal{E}_0^\perp \longrightarrow (\mathcal{E}/\mathcal{E}_0)^* \). It is an \( \mathcal{A} \)-isomorphism into, and we shall show that it is onto. Fix an open set \( U \) in \( X \), and let \( \psi \in (\mathcal{E}/\mathcal{E}_0)^*(U) := \)
$\text{Hom}_{\mathcal{A}|_U}((\mathcal{E}/\mathcal{E}_0)|_U, \mathcal{A}|_U)$. Let us consider a family $\overline{\psi} \equiv (\overline{\psi}_V)_{U \supseteq V, \text{open}}$ such that

$$\overline{\psi}_V(r) := \psi_V(r + \mathcal{E}_0(V)), \quad r \in \mathcal{E}(V).$$

(13)

It is easy to see that $\overline{\psi}_V$ is $\mathcal{A}(V)$-linear for any open $V \subseteq U$. Now, let $\{\rho^V_W\}, \overline{\rho}^V_W$ and $\{\tau^V_W\}$ be the restriction maps for the (complete) presheaves of sections of $\mathcal{E}$, $\mathcal{E}/\mathcal{E}_0$ and $\mathcal{A}$, respectively. The restriction maps $\overline{\rho}^V_W$ are defined by setting

$$\overline{\rho}^V_W(r + \mathcal{E}_0(V)) := \rho^V_W(r) + \mathcal{E}_0(W), \quad r \in \mathcal{E}(V).$$

It clearly follows that

$$(\tau^V_W \circ \overline{\psi}_V)(r) = \tau^V_W(\psi_V(r + \mathcal{E}_0(V))) = \psi_W(\rho^V_W(r) + \mathcal{E}_0(W)) = \overline{\psi}_W(\rho^V_W(r)) = (\overline{\psi}_W \circ \rho^V_W)(r),$$

from which we deduce that

$$\tau^V_W \circ \overline{\psi}_V = \overline{\psi}_W \circ \rho^V_W,$$

which implies that

$$\overline{\psi} \in \text{Hom}_{\mathcal{A}|_U}(\mathcal{E}|_U, \mathcal{A}|_U) =: \mathcal{E}^*(U).$$

Suppose $r \in \mathcal{E}_0(V)$, where $V$ is open in $U$. Then

$$\overline{\psi}_V(r) = \psi_V(r + \mathcal{E}_0(V)) = \psi_V(\mathcal{E}_0(V)) = 0,$$

therefore

$$\phi_V(\overline{\psi} \big|_V, \mathcal{E}_0(V)) = \overline{\psi}_V(\mathcal{E}_0(V)) = 0,$$

i.e. $\overline{\psi} \in \mathcal{E}_0^\perp(U)$. We contend that $\overline{\psi}$ has the given $\psi$ as image under the map (2), and this will show the ontoness of (2) and that $\mathcal{E}_0^\perp$ is a free sub-$\mathcal{A}$-module of $\mathcal{E}^*$.

Let us find the image of $\overline{\psi}$. Consider the pairing $[(\mathcal{E}_0^\perp, \mathcal{E}/\mathcal{E}_0; \Theta); \mathcal{A}]$ such that for any open $V \subseteq X$, we have

$$\Theta_V(\alpha, r + \mathcal{E}_0(V)) := \phi_V(\alpha, r) = \alpha_V(r),$$
where \( \alpha \in \mathcal{E}_0^+(V) \subseteq \mathcal{E}^*(V) \), \( r \in \mathcal{E}(V) \). Clearly, the left kernel of this new pairing is 0. For \( \alpha = \bar{\psi} \in \mathcal{E}_0^+(U) \subseteq \mathcal{E}^*(U) \), we have

\[
\Theta_U(\bar{\psi}, r + \mathcal{E}_0(U)) = \bar{\psi}(r)
\]

where \( r \in \mathcal{E}(U) \), and the map

\[
\Theta_U : \mathcal{E}_0^+(U) \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*(U)
\]

given by

\[
\bar{\psi} \longmapsto \Theta_U, \bar{\psi} \equiv ((\Theta_U, \bar{\psi})_U)_{U \supseteq V, \text{open}}
\]

and such that for any \( r \in \mathcal{E}(V) \)

\[
(\Theta_U, \bar{\psi})_V(r + \mathcal{E}_0(V)) := \Theta_V(\bar{\psi}|_V, r + \mathcal{E}_0(V)) = \bar{\psi}_V(r) = \psi_V(r + \mathcal{E}_0(V))
\]

is the image. Thus the image of \( \bar{\psi} \) is \( \psi \), hence the map \( \mathcal{E}_0^+(U) \longrightarrow (\mathcal{E}/\mathcal{E}_0)^*(U) \), derived from (2), is onto, and therefore an \( \mathcal{A}(U) \)-isomorphism. Since \( \mathcal{E}/\mathcal{E}_0 \) is free by Corollary 1.1, so are \( (\mathcal{E}/\mathcal{E}_0)^* \) and \( \mathcal{E}_0^+ \) free.

Now, let \( \mathcal{F}_0 \) be a free sub-\( \mathcal{A} \)-module of \( \mathcal{E}^* \cong \mathcal{E} \) (cf. Mallios [10, p.298, (5.2)]); on considering \( \mathcal{F}_0 \) as a free sub-\( \mathcal{A} \)-module of \( \mathcal{E} \), according to all that precedes above \( \mathcal{F}_0^+ \) is free in \( \mathcal{E}^* \cong \mathcal{E} \), and so the proof is finished.

We now make the following important observation concerning symplectic \( \mathcal{A} \)-modules of finite rank.

**Lemma 3.1** Let \((\mathcal{E}, \omega)\) be a symplectic \( \mathcal{A} \)-module of finite rank, and \( f \equiv (f_U)_{X \supseteq U, \text{open}} \) an \( \mathcal{A} \)-endomorphism of \( \mathcal{E} \). Then, if \( f \) satisfies two of the three following conditions, it satisfies all of them three:

1. \( I + f \) is an \( \mathcal{A} \)-automorphism of \( \mathcal{E} \);
2. \( f \) is \( \omega \)-skewsymmetric, i.e., for any open \( U \subseteq X \) and sections \( s, t \in \mathcal{E}(U) \),
   \[
   \omega_U(f_U(s), t) + \omega_U(s, f_U(t)) = 0;
   \]
3. \( \text{Im } f \equiv f(\mathcal{E}) \) is totally isotropic, i.e., for any open \( U \subseteq X \) and sections \( s, t \) of \( f(\mathcal{E}) \) on \( U \),
   \[
   \omega_U(s, t) = 0.
   \]
Proof. Using the equality
\[ \omega_U(s + f_U(s), t + f_U(t)) - \omega_U(s, t) = \omega_U((f_U + f^*_U)(s), t) + \omega_U(f_U(s), f_U(t)), \]
where \( U \) is any open subset of \( X \), \( s \) and \( t \) sections of \( \mathcal{E} \) over \( U \), one easily checks the implications: (1), (2) \( \Rightarrow \) (3); (1), (3) \( \Rightarrow \) (2); and (2), (3) \( \Rightarrow \) (1).

Now, we are going to introduce a new class of \( \mathcal{A} \)-modules we will be concerned with in the sequel.

**Definition 3.3** An \( \mathcal{A} \)-module \( \mathcal{E} \) is called a **locally free \( \mathcal{A} \)-module of varying finite rank** if there exist an open covering \( \mathcal{U} \equiv \{ U_\alpha \}_{\alpha \in I} \) of \( X \) and numbers \( n(\alpha) \in \mathbb{N} \) for every open set \( U_\alpha \) such that
\[ \mathcal{E}|_{U_\alpha} = \mathcal{A}^{n(\alpha)}|_{U_\alpha}. \]
The open covering \( \mathcal{U} \) is called a **local frame**.

**Example 3.1** Consider a free \( \mathcal{A} \)-module \( \mathcal{E} \), where \( \mathcal{A} \) is a PID-algebra sheaf. Then, every sub-\( \mathcal{A} \)-module of \( \mathcal{E} \) is a locally free \( \mathcal{A} \)-module of varying finite rank.

We come now to the following useful result, satisfied by free \( \mathcal{A} \)-modules of finite rank (cf. [10] p. 298, (5.2)) and vector sheaves of finite rank (cf. [10] p. 138, (6.26)). That is, one has:

**Lemma 3.2** Let \( \mathcal{E} \) be a locally free \( \mathcal{A} \)-module of varying finite rank. Then, the dual \( \mathcal{A} \)-module \( \mathcal{E}^* := \text{Hom}_\mathcal{A}(\mathcal{E}, \mathcal{A}) \) is a locally free \( \mathcal{A} \)-module of varying finite rank, and one has
\[ \mathcal{E}^* = \mathcal{E} \]
within an \( \mathcal{A} \)-isomorphism.
Proof. Let $U \equiv (U_\alpha)_{\alpha \in I}$ be a local frame of $\mathcal{E}$. Applying \cite{10} p. 137, (6.22) and (6.23), one has, for every $\alpha \in I$,

$$
\text{Hom}_A(\mathcal{E}, A)|_{U_\alpha} = \text{Hom}_{|U_\alpha}(\mathcal{E}|_{U_\alpha}, A|_{U_\alpha}) \\
= \text{Hom}_{A|_{U_\alpha}}(A^{n(\alpha)}|_{U_\alpha}, A|_{U_\alpha}) \\
= \text{Hom}_A(A^{n(\alpha)}, A)|_{U_\alpha} \\
= A^{n(\alpha)},
$$

within $A|_{U_\alpha}$-isomorphisms. 

Our objective now is to obtain a useful characterization of a symplectic $A$-automorphism of the form $I + f$ of a symplectic orthogonally convenient $A$-module $\mathcal{E}$, where $f$ is a skewsymmetric $A$-endomorphism of $\mathcal{E}$. For this purpose, we require a generalization of the following result, see \cite{13}.

**Theorem 3.2** Let $(\mathcal{E}, \phi)$ be a free $A$-module of finite rank. Then, every non-isotropic free sub-$A$-module $\mathcal{F}$ of $\mathcal{E}$ is a direct summand of $\mathcal{E}$; viz.

$$
\mathcal{E} = \mathcal{F} \perp \mathcal{F}^\perp.
$$

We generalize Theorem 3.2 as follows:

**Theorem 3.3** Let $(\mathcal{E}, \phi)$ be a free $A$-module of finite rank, with $A$ a PID-algebra sheaf. Then, every non-isotropic sub-$A$-module $\mathcal{F}$ of $\mathcal{E}$ is a direct summand of $\mathcal{E}$; viz.

$$
\mathcal{E} = \mathcal{F} \perp \mathcal{F}^\perp
$$

within an $A$-isomorphism.

**Proof.** First, we notice that $\mathcal{F}$ is a locally free sub-$A$-module of varying finite rank. Then, let us consider for any open subset $U \subseteq X$ a section $t \in \mathcal{E}(U)$ and a section $\psi \in \mathcal{F}^*(U) := \text{Hom}_A(\mathcal{F}, A)(U) \equiv \text{Hom}_{A|_U}(\mathcal{F}|_U, A|_U)$, defined as follows: given $s \in \mathcal{F}(V)$, where $V$ is open in $U$,

$$
\psi_V(s) := \phi_V(t|_V, s).
$$
being non-isotropic, we have that the restriction $\phi|_F$ of $\phi$ on $F$ is non-degenerate, consequently, since $F^* \cong F$ (cf. Lemma 3.2), $\psi$ may be identified with a unique element (section) $p_U(t) \equiv p(t) \in F(U) \cong F^*(U)$ in such a way that
\[
\phi_V((t - p(t))|_V, s) = \phi_V(t|_V - p(t)|_V, s) = 0
\]
for all $t \in E(U)$ and $s \in F(V)$, with $V$ open in $U$, the supplementary $A(U)$-projection $q := I - p$ is such that for all $t \in E(U)$, $q(t) \equiv (I - p)(t) \in F^+(U)$, i.e. $q$ maps $E(U)$ on $F^+(U)$. Hence, every element $t \in E(U)$, where $U$ runs over the open subsets of $X$, may be written as
\[
t = p(t) + (t - p(t))
\]
with $p(t) \in F(U)$ and $t - p(t) \in F^+(U)$. Thus
\[
E(U) = F(U) \oplus F^+(U) = (F \oplus F^+)(U)
\]
within $A(U)$-isomorphisms (for the $A(U)$-isomorphism $F(U) \oplus F^+(U) = (F \oplus F^+)(U)$, cf. Mallios [10], relation (3.14), p. 122). Finally, since $F$ is non-isotropic, it follows that
\[
E(U) = (F \perp F^+)(U)
\]
for every open $U \subseteq X$. Thus, we reach the sought $A$-isomorphism $E = F \perp F^\perp$. 

In keeping with the notations of Theorem 3.3, we clearly have that: $F^{\perp \perp} \cong F$. Moreover, if $\phi$ is non-degenerate, then $F^+(U) \cong F(U)^\perp$, for every open $U \subseteq X$. Indeed, since $F^+(U) \subseteq F(U)^\perp$, then if $F(U) \cap F(U)^\perp \neq 0$, rad $E(U) \neq 0$, which contradicts the hypothesis that $E$ is non-isotropic.

**Theorem 3.4** Let $(E, \omega)$ be a symplectic orthogonally convenient $A$-module of rank $2n$, and $f$ a $A$-endomorphism of $E$. If $f$ is skewsymmetric and $I + f$ an $A$-automorphism of $E$, then
(1) \( f^2 = 0; \)

(2) \( \ker f \simeq (\operatorname{Im} f)^\perp; \)

(3) For every open subset \( U \subseteq X, \) there exists a symplectic basis of \( \mathcal{E}(U), \) whose first \( k \) elements (sections), \( k \leq n, \) form a basis of \( (\operatorname{Im} f)(U) := \operatorname{Im} f_U \equiv f_U(\mathcal{E}(U)), \) with respect to which the \( \mathcal{A}(U) \)-morphism

\[
(I + f)_U := I_U + f_U
\]
is represented by the matrix

\[
\begin{pmatrix}
I_n & H \\
0 & I_n
\end{pmatrix}
\]

with \( ^tH = H. \)

**Proof.** (1) From Lemma \[3.1\], \( \operatorname{Im} f \) is totally isotropic. Therefore, for any open subset \( U \) of \( X \) and sections \( s, t \in \mathcal{E}(U), \)

\[
\omega_U(f_U(s), f_U(t)) = 0.
\]

Since

\[
\omega_U((f^*)_U f_U(s), t) = \omega_U((f_U)^* f_U(s), t) = \omega_U(f_U(s), f_U(t)) = 0
\]

and \( \omega \) is symplectic, it follows that

\[
(f^*)_U f_U = (f_U^*) f_U = 0.
\]

Thus,

\[
f^* f = 0;
\]

since \( f^* = -f, \) one reaches the desired property that \( f^2 = 0. \)

(2) Fix an open set \( U \) in \( X \) and \( s \in (\ker f)(U) = \ker f_U, \) see \[22\] p. 37, Definition 3.1]. Moreover, let \( t \in \mathcal{E}(U); \) then

\[
\omega_U(s, f_U(t)) = -\omega_U(f_U(s), t) = 0.
\]
Thus,

\[ s \in (\text{Im} f)(U)^{\perp} \equiv f_U(\mathcal{E}(U))^{\perp} \]

and hence

\[ (\ker f)(U) = \ker f_U \subseteq (\text{Im} f)(U)^{\perp} = (\text{Im} f)^{\perp}(U) \]

or

\[ \ker f \subseteq (\text{Im} f)^{\perp} \equiv f(\mathcal{E})^{\perp}. \]

Conversely, let \( t \in (\text{Im} f)^{\perp}(U) = (\text{Im} f)(U)^{\perp} \). Then, for any \( s \in (\text{Im} f)(U) \equiv \text{Im} f_U := f_U(\mathcal{E}(U)) \equiv f(\mathcal{E})(U) \), one has

\[ \omega_U(t, s) = 0. \]

But \( s = f_U(r) \) for some \( r \in \mathcal{E}(U) \), therefore

\[ \omega_U(t, f_U(r)) = -\omega_U(f_U(t), r) = 0. \tag{14} \]

Since (14) is true for any \( r \in \mathcal{E}(U) \),

\[ f_U(t) = 0, \]

i.e.

\[ t \in (\ker f)(U) := \ker f_U. \]

Hence,

\[ (\text{Im} f)^{\perp}(U) \subseteq (\ker f)(U) \]

or

\[ (\text{Im} f)^{\perp} \subseteq \ker f. \]

(3) As \( \text{Im} f \subseteq \ker f = (\text{Im} f)^{\perp} \), so the sub-\( \mathcal{A} \)-module \( \text{Im} f \) is totally isotropic. Therefore, for any open \( U \subseteq X \),

\[ \text{rank}(\text{Im} f)(U) := \text{rank} \text{Im} f_U \leq n. \]

Now, let us fix an open set \( U \) in \( X \) and consider a basis \( (s_1, \ldots, s_k), k \leq n, \)

of \( (\text{Im} f)(U) \equiv \text{Im} f_U \). By [13, Lemma 7], there exists a totally isotropic sub-\( \mathcal{A}(U) \)-module \( S \) of \( \mathcal{E}(U) \), equipped with a basis, which we denote

\[ (s_{k+1}, \ldots, s_{n+k}) \]
such that
\[ \omega_U(s_i, s_{n+j}) = \delta_{ij}, \quad \text{for } i, j = 1, \ldots, k. \]

Clearly,
\[ S \cap (\text{Im } f)^\perp(U) = S \cap (\ker f)(U) = 0. \quad (15) \]

As a result of (15), the sum \( S + \text{Im } f_U \) is direct and \( S \oplus \text{Im } f_U \) is non-isotropic; therefore, by Theorem 3.2, one has
\[ \mathcal{E}(U) = (S \oplus \text{Im } f_U)^\perp F \]
for some sub-\( \mathcal{A}(U) \)-module \( F \) of \( \mathcal{E}(U) \), (cf. [13, Theorem 1]). Since \( F = (S \oplus \text{Im } f_U)^\perp \), \( F \) is contained in \( (\text{Im } f_U)^\perp(U) = (\text{Im } f)(U) \) and
\[ F^\perp = (\text{Im } f)(U) := \text{Im } f_U; \]
i.e. \( F \) is an orthogonal supplementary of \( (\text{Im } f)(U) \) in \( (\ker f)(U) \). Since \( F \) is free, non-isotropic and of rank \( 2n - 2k \), it can be equipped with a symplectic basis, say \( (s_{k+1}, \ldots, s_n, s_{n+k+1}, \ldots, s_{2n}) \), see [17]. As \( s_1, \ldots, s_n \in (\ker f)(U) \), it follows that
\[ (I_U + f_U)(s_j) = s_j, \quad j = 1, \ldots, n. \]
Therefore, if \( H \) is the matrix representing \( f_U \), \( I_U + f_U \) is represented by the matrix
\[ \left( \begin{array}{cc} I_n & H \\ 0 & I_n \end{array} \right), \]
and this is a symplectic matrix if and only if \( H^t = H \), i.e. \( H \) is symmetric. ■

References

[1] W.A. Adkins, S.H. Weintraub: Algebra. An Approach via Module Theory. Springer-Verlag New York, 1992.

[2] E. Artin: Geometric Algebra. Interscience Publishers, New York, 1988.

[3] T.S. Blyth: Module Theory. An Approach to Linear Algebra. Second edition. Oxford Science Publications, Clarendon Press, Oxford, 1990.
[4] R. Berndt: *An Introduction to Symplectic Geometry*. American Mathematical Society, Providence, Rhode Island, 2001.

[5] L. Chambadal, J.L. Ovaert: *Algèbre Linéaire et Algèbre Tensorielle*. Dunod, Paris, 1968.

[6] A. Crumeyrolle: *Orthogonal and Symplectic Clifford Algebras. Spinor Structures*. Kluwer Academic Publishers, Dordrecht, 1990.

[7] R. Deheuvels: *Formes quadratiques et groupes classiques*. Presses Universitaires de France, 1981.

[8] J. Dieudonné: *Sur les groupes classiques*. Hermann, 1948.

[9] S. Lang: *Algebra. Revised Third Edition*. Springer, 2002.

[10] A. Mallios: *Geometry of Vector Sheaves. An Axiomatic Approach to Differential Geometry. Volume I: Vector Sheaves. General Theory*. Kluwer Academic Publishers, Dordrecht, 1998.

[11] A. Mallios: *Modern Differential Geometry in Gauge Theories: Maxwell Fields, vol. I. Yang-Mills Fields, vol. II*. Birkhäuser, Boston, 2006/2007.

[12] A. Mallios, P.P. Ntumba: *Pairings of sheaves of A-modules*. Quaestiones Mathematicae **31**(2008), 397-414.

[13] A. Mallios, P.P. Ntumba: *On a sheaf-theoretic version of the Witt’s decomposition theorem. A Lagrangian perspective*. Rend. Circ. Mat. Palermo **58**(2009), 155–168.

[14] A. Mallios, P.P. Ntumba: *Fundamentals for symplectic A-modules. Affine Darboux theorem*. Rend. Circ. Mat. Palermo **58**(2009), 169–198.

[15] A. Mallios, P.P. Ntumba: *On the Second Isomorphism Theorem for A-Modules, and . . . “all that”* (under refereeing)

[16] P. P. Ntumba: *Cartan-Dieudonné Theorem for A-Modules* (to appear in Mediterr. J. Math. **7**(2010))

[17] P. P. Ntumba: *Abstract geometric algebra. Orthogonal and symplectic geometries*. (under refereeing)
[18] P.P. Ntumba: *A note on Orthogonally Convenient Pairings of A-Modules* (under refereeing)

[19] P.P. Ntumba: *Witt’s Theorem for A-Modules. A sheaf-theoretic context* (under refereeing)

[20] O.T. O’Meara: *Symplectic Groups*. American Mathematical Society, Providence, Rhode Island, 1978.

[21] A.C. da Silva: *Lectures on Symplectic Geometry*. Springer Berlin, 2001.

[22] B. R. Tennison: *Sheaf Theory*. Cambridge University Press, London, 1975.

PP Ntumba  
Department of Mathematics and Applied Mathematics  
University of Pretoria  
Hatfield 0002, Republic of South Africa  
Email: patrice.ntumba@up.ac.za