STRUCTURAL PROPERTIES OF MULTIPLE ZETA VALUES

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Abstract. We study some classical identities for multiple zeta values and show that they still hold for zeta functions built on the zeros of an arbitrary function. We introduce the complementary zeta function of a system, which naturally occurs when lifting identities for multiple zeta values to identities for quasisymmetric functions.

1. Introduction and Notation

Multiple zeta values (MZVs) are simply defined, yet ubiquitous in modern number theory and physics. The MZV of depth $k$ is defined as

$$\zeta(s_1, \ldots, s_k) := \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}},$$

and in general exists for positive integral $s_i$ with $s_1 \geq 2$ and $s_i \geq 1, 2 \leq i \leq k$. They were first explored by Euler, but have only been systematically studied for the past few decades, for example by Zagier [18] and Hoffman [14]. There are many nontrivial linear dependence relations between MZVs, and characterizing all such relations is an extremely difficult (and unsolved) question.

Our main innovation is to consider generalized multiple zeta values: given a set of complex numbers $\{z_n \neq 0\}$, ordered in increasing order of magnitude, we define an associated zeta function

$$\zeta_G(s) := \sum_{n=1}^{\infty} \frac{1}{z_n^s},$$

and associated multiple zeta function

$$\zeta_G(s_1, \ldots, s_k) := \sum_{n_1 > n_2 > \cdots > n_k \geq 1} \frac{1}{z_{n_1}^{s_1} \cdots z_{n_k}^{s_k}}.$$
There exist many identities satisfied by MZVs, due to their high degree of symmetry. Our prototype is the simple and elegant Euler identity
\[ \zeta(2,1) = \zeta(3) \, . \quad (1.1) \]

The question we address in this paper is: what identities are structural, meaning, they still hold when the Riemann MZV is replaced by a generalized MZV? This discussion includes the case where the sequence \{zn\} is finite, so that we also consider finite-type sums. A prototypical example of a structural identity for multiple zeta values would be the reflection formula introduced by Euler,
\[ \zeta(s,t) + \zeta(t,s) + \zeta(s+t) = \zeta(s) \zeta(t) \quad (1.2) \]
which naturally lifts to
\[ \zeta_G(s,t) + \zeta_G(t,s) + \zeta_G(s+t) = \zeta_G(s) \zeta_G(t) \quad (1.3) \]
after rewriting the domain of summation. Another more elaborate structural relation concerns star-MZVS, which are defined by
\[ \zeta^*(s_1,\ldots,s_k) = \sum_{n_1 \geq n_2 \geq \cdots \geq n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}} \]
and can be expressed as
\[ \zeta^*(s_1,\ldots,s_k) = \sum \zeta(s_1 \Box \cdots \Box s_k) \, , \]
where the sum is over the \(2^{k-1}\) configurations obtained by choosing \(\Box = +\) or \(\Box = -\). This identity also naturally lifts to the generalized \(\zeta\) and \(\zeta^*\) functions since it only expresses a symmetry of the domain of summation.

We can recast this question in a more algebraic framework. We have skipped many technical details in the following construction, and the reader is referred to [10, Section 4] for a complete exposition. Begin with a (possible finite) set of indeterminates \(\{z_n\}\), and let \(\lambda = (\lambda_1,\lambda_2,\cdots)\) be a composition of \(n\). Then we can define the monomial symmetric function
\[ m_\lambda := \sum_{\sigma \in S_\lambda} \zeta_{\sigma(1)}^{\lambda_1} \zeta_{\sigma(2)}^{\lambda_2} \cdots \]
and denote by \(QSym\) the graded \(\mathbb{Q}\)-algebra of \textit{quasisymmetric functions}, which is spanned by the monomial symmetric functions. Under the evaluation homomorphism \(z_k \mapsto k\) we have \(m_\lambda \mapsto \zeta(\lambda_1,\lambda_2,\ldots)\). Note that many of the classical theorems about MZVs are only true after applying this evaluation map. We wish to lift these to theorems in \(QSym\), so that by applying other evaluation maps such as \(z_k \mapsto 2k + 1\) we can effortlessly obtain results about other zeta functions.

Our overarching philosophy is that asymmetry in the indices of MZVs makes them hard to study, so that we must first symmetrize them somehow. Results for such symmetrized MZVs should then hold \textit{for all generalized multiple zeta values}. This means that it is not important that multiple zeta functions are sums over natural numbers; instead, the important factor is that depth \(k\) MZVs are sums over a certain simplex in \(\mathbb{Z}^k\).

We also introduce the study of the \textit{complementary zeta function} \(\tilde{\zeta}_G\), and stress both its importance and simplicity. Given a sequence of complex numbers \(z_k\), the complementary zeta function associated to this system is a zeta function built from the numbers
\[ \frac{1}{z_k} = \sum_{i=1}^{k-1} \frac{1}{z_i - z_k} + \sum_{i=k+1}^{N} \left( \frac{1}{z_i - z_k} - \frac{1}{z_i} \right) \, . \]
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In the case $z_k = k$, we also have $\tilde{z}_k = k$ and the complementary zeta function reduces to the Riemann case. In certain cases, such as $z_i = i^2$, we can write $\tilde{\zeta}_G$ as a linear combination of values of Riemann MZVs. We are not sure whether our theorems give nonlinear dependence relations which can be deduced from known shuffle and stuffle relations for MZVs. Systematically studying certain special cases of $\zeta_G$ may give new nontrivial linear dependence relations amongst MZVs of a given depth.

In Section 2 we introduce the complementary zeta function, and in Section 3 we show that the complementary zeta function naturally occurs when generalizing some identities about Riemann MZVs to structural MZVs. In Section 4, we show that an identity by Hirose et al. is of structural type. In Section 5 we explore some natural specializations of the complementary zeta function. In Section 6, we extend a recent result by Liu et al. about averages of MZV at even arguments. Finally, in Section 7, we propose a general version of an identity by Hoffman based on a result by Mordell on symmetric functions. This work is the first in a natural program to extend identities for multiple zeta functions to identities in $QSym$.

2. Complementary zeta function

We will now carefully state and examine the definition of a complementary zeta function and its multiple analog, as well as several special cases. For any integer $N > 1$ and $\{z_1, \ldots, z_N\}$ a set of nonzero complex numbers, define the sequence $\{\tilde{z}_k\}_{1 \leq k \leq N}$ as

$$
\frac{1}{\tilde{z}_k} = \sum_{i=1}^{k-1} \frac{1}{z_i - z_k} + \sum_{i=k+1}^{N} \left( \frac{1}{z_i - z_k} - \frac{1}{z_i} \right). \tag{2.1}
$$

Then the complementary zeta function is defined as the sum

$$
\tilde{\zeta}_G(s) = \sum_{n \geq 1} \frac{1}{\tilde{z}_n z_n^{s-1}}.
$$

When $z_k = k$ with $N \to \infty$, we have $\tilde{z}_k = k$ since

$$
\frac{1}{\tilde{z}_k} = \sum_{i=1}^{k-1} \frac{1}{i - k} + \sum_{i=k+1}^{\infty} \left( \frac{1}{i - k} - \frac{1}{i} \right) = -\sum_{i=1}^{k-1} \frac{1}{i} + \sum_{i=1}^{k} \frac{1}{i} = 1/k,
$$

and hence $\zeta_G(s) = \zeta(s)$. The complementary zeta function is therefore seen as a nonstandard, but very natural generalization of the Riemann zeta function. However, finding other closed forms for $\tilde{z}_k$ is a formidable problem. Some special cases are given in Section 5.

We then introduce a multiple analog of the complementary zeta function. Whether this is the “morally correct” generalization remains to be seen. For all $N > 1$ and $\{z_n\}$ a sequence of nonzero complex numbers, define the sequence $\{\tilde{z}_n^{(r)}\}$ as

$$
\frac{1}{\tilde{z}_n^{(r)}} = \sum_{n > n_2 > \ldots > n_r > n} \frac{1}{z_n (z_{n_2} - z_n) \ldots (z_{n_r} - z_n)} + \sum_{n_1 > n_2 > \ldots > n_r > n} \frac{z_n}{z_{n_1} (z_{n_1} - z_n) (z_{n_2} - z_n) \ldots (z_{n_r} - z_n)} + \ldots + \sum_{n_1 > n_2 > \ldots > n_{r-1} > n} \frac{z_n}{z_{n_1} (z_{n_1} - z_n) (z_{n_2} - z_n) \ldots (z_{n_{r-1}} - z_n)} \tag{2.2}
$$

1 it is assumed that in (2.1) a sum $\sum_{n}^{M}$ is equal to 0 when $M < N$. 

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Another telescoping argument gives \( \tilde{z}_n^{(r)} = n^{r-1} \). The higher order complementary zeta function is then defined as

\[
\tilde{\zeta}_G^{(r)} (k) = \sum_{n \geq 1} \frac{1}{\tilde{z}_n^{(r)} z_n^{k-1}}.
\]  

(2.3)

The discussion about lifting MZV identities to identities in \( QSym \) from the introduction suggests an important open question: express \( \tilde{\zeta}_G \) and \( \tilde{\zeta}_G^{(r)} \) as quasisymmetric functions by expanding them in terms of a suitable basis of \( QSym \), such as the monomial symmetric functions. Do these correspond to known symmetric functions?

3. Euler sums

3.1. A somewhat unlikely looking identity. In their legendary paper [3], D. Borwein et al. provide 32 different proofs of Euler’s identity

\[
\zeta (2, 1) = \zeta (3).
\]  

(3.1)

One of their proofs involves the "somewhat unlikely looking" identity

\[
\sum_{n \geq 1} \frac{1}{n (n + x)} \sum_{m = 1}^{n-1} \frac{1}{m + x} = \sum_{n \geq 1} \frac{1}{n^2 (n + x)}
\]  

(3.2)

that holds for any non negative integer \( x \), and is obtained in [2] by elementary manipulations on rational fractions. The interesting feature of this identity is that choosing \( x = 0 \) provides Euler’s identity (3.1). As we will see, this identity is at the root of many structural identities for MZVs. We first give an extended structural version of this result.

**Theorem 3.1.** Consider a sequence \( \{z_k\}_{1 \leq k \leq N} \) of non-zero complex numbers. Then, with \( \tilde{z}_n \) defined by (2.1), we have

\[
\sum_{n = 1}^{N} \frac{1}{\tilde{z}_n (\tilde{z}_n + x)} \sum_{m = 1}^{n-1} \frac{1}{\tilde{z}_m + x} = \sum_{n = 1}^{N} \frac{1}{\tilde{z}_n \tilde{z}_n + x}.
\]  

(3.3)

**Corollary 3.2.** The special case \( x = 0 \) and \( N \to \infty \) in (3.3) gives, with

\[
\tilde{\zeta}_G (3) = \sum_{n \geq 1} \frac{1}{\tilde{z}_n^{2} \tilde{z}_n},
\]

the following generalization of Euler’s identity

\[
\zeta_G (2, 1) = \tilde{\zeta}_G (3).
\]  

(3.4)

**Proof of Theorem 3.1.** First consider the case when \( N \) is finite. Both sides of (3.3) are rational functions; let us show that they have the same poles and same residue at each pole. Clearly, the left-hand side has only simple poles \(-z_k\); for each pole \(-z_k\), the residue \( \alpha_k \) is computed as

\[
\alpha_k = \lim_{x \to -z_k} (x + z_k) \left( \sum_{n = 1}^{N} \frac{1}{\tilde{z}_n (\tilde{z}_n + x)} \sum_{m = 1}^{n-1} \frac{1}{\tilde{z}_m + x} \right).
\]
Assuming first $k > 1$, we have

$$\lim_{x \to -z_k} (x + z_k) \left( \sum_{n=1}^{N} \frac{1}{z_n (z_n + x)} \sum_{m=1}^{N-1} \frac{1}{z_m + x} \right)$$

$$= \lim_{x \to -z_k} (x + z_k) \left( \frac{1}{z_1 (z_1 + x)} + \frac{1}{z_2 (z_2 + x)} \left( \frac{1}{z_1 + x} \right) + \ldots + \frac{1}{z_k (z_k + x)} \left( \frac{1}{z_1 + x} + \ldots + \frac{1}{z_{k-1} + x} \right) \right)$$

$$+ \lim_{x \to -z_k} (x + z_k) \left( \frac{1}{z_{k+1} (z_{k+1} + x)} \left( \frac{1}{z_1 + x} + \ldots + \frac{1}{z_k + x} \right) + \ldots + \frac{1}{z_N (z_N + x)} \left( \frac{1}{z_1 + x} + \ldots + \frac{1}{z_N + x} \right) \right)$$

so that, for $k > 1$,

$$\alpha_k = \frac{1}{z_k} \left( \frac{1}{z_1 - z_k} + \ldots + \frac{1}{z_{k-1} - z_k} \right) + \frac{1}{z_{k+1} (z_{k+1} - z_k)} + \ldots + \frac{1}{z_N (z_N - z_k)},$$

which is the second result in (2.1). The computation of the boundary residues $\alpha_1$ and $\alpha_N$ is equally simple. Comparing to the definition (2.1) yields the result. □

In [2], the authors show that computing the $s-$th Taylor coefficient in (3.2) yields a corresponding formula for the Riemann zeta value $\zeta(s)$:

$$\zeta(s + 3) = \sum_{a+b=s \atop a,b \geq 0} \zeta(2 + a, 1 + b) \tag{3.5}$$

For example, with $s = 1$, this yields

$$\zeta(4) = \zeta(3,1) + \zeta(2,2).$$

The same approach, i.e. taking $N \to \infty$ and computing the $s-$th Taylor coefficient in (3.3), provides in our case the following extension of the formula (3.5).

**Theorem 3.3.** For any sequence $\{z_k\}_{k \geq 1}$ of nonzero complex numbers such that the following sums are convergent,

$$\tilde{\zeta}_G(s + 3) = \sum_{a+b=s \atop a,b \geq 0} \zeta_G(2 + a, 1 + b). \tag{3.6}$$

**Proof.** Since

$$\frac{d^k}{dx^k} \sum_{n>m \geq 1} \frac{1}{z_n (z_n + x) z_m + x} = \frac{d^k}{dx^k} \sum_{n\geq 1} \frac{1}{z_n z_n \cdot (z_n + x)},$$

applying Leibniz’ formula

$$\frac{d^k}{dx^k} \frac{1}{z_n + x} \frac{1}{z_m + x} = \sum_{j=0}^{k} \binom{k}{j} \left( \frac{d^j}{dx^j} \frac{1}{z_n + x} \right) \left( \frac{d^{k-j}}{dx^{k-j}} \frac{1}{z_m + x} \right),$$

and

$$\frac{d^j}{dx^j} \frac{1}{z_n + x} = (-1)^j j! \frac{1}{(z_n + x)^{j+1}},$$

we deduce

$$\sum_{j=0}^{k} \sum_{n>m \geq 1} \frac{1}{z_n (z_n + x)^{j+1}} \frac{1}{z_m + x} = \sum_{n \geq 1} \frac{1}{z_n z_n \cdot (z_n + x)^{k+1}}.$$
Evaluation at \( x = 0 \) of this identity yields
\[
\sum_{j=0}^{k} \sum_{n>m \geq 1} \frac{1}{z^{j+2}n} \frac{1}{z^{k-j+1}m} = \sum_{n \geq 1} \frac{1}{z^{n}z^{k+2}},
\]
which can be rewritten as
\[
\sum_{j=0}^{k} \zeta_G (j+2, k-j+1) = \tilde{\zeta}_G (k+3),
\]
which is the desired result. \( \square \)

### 3.2. Sum formula.

The sum formula for the ordinary MZVs is the following identity
\[
\sum_{a_i \geq 0} \zeta \left( a_1 + 2, a_2 + 1, \ldots, a_r + 1 \right) = \zeta \left( r + s + 1 \right).
\] (3.7)

As remarked in \[2\], it can be derived by identifying the \( s \)-th Taylor coefficient in the generalization
\[
\sum_{k_1 > k_2 > \cdots > k_r > 0} \frac{1}{k_1} \prod_{j=1}^{r} \frac{1}{k_j - x} = \sum_{n=1}^{\infty} \frac{1}{n^r (n-x)}
\]
of (3.2). An extension of our previous methods allows us to state a structural version of the sum formula. We first derive the general case of Theorem 3.1 as follows.

**Theorem 3.4.** With \( \tilde{z}_n^{(r)} \) defined in (2.2), we have
\[
\sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{z_{n_1} (z_{n_1} + x) \cdots (z_{n_r} + x)} = \sum_{n \geq 1} \frac{1}{\tilde{z}_n^{(r)} z_{n} + x}.
\] (3.8)

**Proof.** The left-hand side of (3.8) has poles at \( x = -z_n \), with residues \( \frac{1}{z_{n} \tilde{z}_n^{(r)}} \) so that
\[
\sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{z_{n_1} (z_{n_1} + x) \cdots (z_{n_r} + x)} = \sum_{n \geq 1} \frac{1}{\tilde{z}_n^{(r)} z_{n} + x}.
\]
The residue can be computed as
\[
\frac{1}{\tilde{z}_n^{(r)} z_{n}} = \lim_{x \to -z_n} (x + z_n) \sum_{n_1 > \cdots > n_r \geq 1} \frac{1}{z_{n_1} (z_{n_1} + x) \cdots (z_{n_r} + x)}
\]
so that
\[
\frac{1}{\tilde{z}_n^{(r)}} = \sum_{n > n_2 > n_3 \cdots > n_r} \frac{1}{z_{n_2} - z_n} \left( z_{n_r} - z_n \right) + \sum_{n_1 > n_2 > n_3 \cdots > n_r} \frac{z_n}{z_{n_2} - z_n} \left( z_{n_r} - z_n \right) + \cdots
\]
\[
+ \sum_{n_1 > n_2 > \cdots > n_{r-1} > n} \frac{z_n}{z_{n_1} - z_n} \left( z_{n_{r-1}} - z_n \right) \left( z_{n_r} - z_n \right) + \cdots
\]
which is (2.2). \( \square \)

As a consequence of Theorem 3.4, we deduce the following generalization of the sum formula (3.7).
Theorem 3.5. With \( \tilde{\zeta}_n^{(r)} \) defined as in (2.2) and the modified generalized multiple zeta value defined by (2.3), the generalized multiple zeta function satisfies the sum rule
\[
\sum_{\sum s_i = s} \zeta_G (s_1 + 2, s_2 + 1, \ldots, s_r + 1) = \tilde{\zeta}_G^{(r)} (r + s + 1).
\]

Proof. Compute the Taylor expansion of each side of (3.8) and identify the coefficient of \( x^{r+s+1} \) on both sides. \( \Box \)

3.3. Euler’s reduction formula. In [2], identity (3.5) is referenced to as an inversion of Euler’s reduction formula
\[
\zeta(s, 1) = \frac{s}{2} \zeta(s + 1) - \frac{1}{2} \sum_{k=1}^{s-2} \zeta(k + 1) \zeta(s - k), \ s > 1 \in \mathbb{Z}.
\]

The next result provides an extension of this reduction formula to the generalized MZVs, showing that it is of structural type.

Theorem 3.6. The generalized reduction formula is
\[
\zeta_G(s, 1) = \tilde{\zeta}_G(s + 1) + \left( \frac{s}{2} - 1 \right) \zeta_G(s + 1) - \frac{1}{2} \sum_{k=1}^{s-2} \zeta_G(k + 1) \zeta_G(s - k).
\]

Proof. Start from (3.6) with \( k + 2 \) replaced by \( s \) and extract the last term in the sum to obtain
\[
\zeta_G(s, 1) = \tilde{\zeta}_G(s + 1) - \sum_{j=1}^{s-2} \zeta_G(j + 1, s - j).
\]

Next apply the reflection formula (1.3) to each couple of terms \( \zeta_G(j + 1, k - j) \) and \( \zeta_G(k - j, j + 1) \) in the sum so that
\[
\zeta_G(s, 1) = \tilde{\zeta}_G(s + 1) - \frac{1}{2} \sum_{j=1}^{s-2} \zeta_G(j + 1) \zeta_G(s - j) - \zeta_G(s + 1)
\]
\[
= \tilde{\zeta}_G(s + 1) + \frac{s - 2}{2} \zeta_G(s + 1) - \frac{1}{2} \sum_{j=1}^{s-2} \zeta_G(j + 1) \zeta_G(s - j),
\]
which is the desired result. \( \Box \)

4. A generalization of the sum formula by Hirose et al.

In [11], M. Hirose et al. propose a generalization of the depth 2 case of the sum formula (3.7) to arbitrary complex values of the parameter \( s \) as follows:

Theorem 4.1. [11] Thm. 1.2] for \( \Re(s) > 1 \),
\[
\sum_{n \geq 0} (\zeta(n + 2, s - n - 2) - \zeta(s + n, -n)) = \zeta(s).
\] (4.1)

We prove a further generalization:

Theorem 4.2. We have, for \( s \) such that the following quantities exist,
\[
\sum_{n \geq 0} (\zeta_G(n + 2, s - n - 2) - \zeta_G(s + n, -n)) = \tilde{\zeta}_G(s).
\] (4.2)
Proof. The left-hand side is expanded as
\[
\sum_{n \geq 0} \sum_{0 < n_2 < n_1} \frac{1}{z_{n_1}^{n+2} z_{n_2}^{s-n-2}} - \frac{1}{z_{n_1}^{s+n-2} z_{n_2}^n} = \sum_{0 < n_2 < n_1} \sum_{n \geq 0} \left(\frac{z_{n_2}}{z_{n_1}}\right)^n \left(\frac{1}{z_{n_1}^{2z_{n_2}^{s-n-2}} - \frac{1}{z_{n_1}^{s-n}}}\right)
\]
\[
= \sum_{0 < n_2 < n_1} \sum_{n \geq 0} \frac{1}{z_{n_1}^{s-n} z_{n_2}^n} \left(\frac{1}{z_{n_1}^{s}-z_{n_2}^{s-n-2}} - \frac{1}{z_{n_1}^{s-n}}\right)
\]
\[
= \sum_{0 < n_2 < n_1} \frac{1}{z_{n_1}^{s-n} z_{n_2}^{n}} - \frac{1}{z_{n_1}^{s-n} z_{n_2}^{n}} \left(\frac{1}{z_{n_1}^{s-n} - z_{n_2}^{s-n-2}} - \frac{1}{z_{n_1}^{s-n}}\right)
\]
\[
= S_1 - S_2
\]
with
\[
S_1 = \sum_{0 < n_2 < n_1} \frac{1}{z_{n_1}^{s-2}} \sum_{n_1 > n_2} \frac{1}{z_{n_2}^{n}} \left(\frac{1}{z_{n_1}^{s-2}} - \frac{1}{z_{n_1}^{s-n}}\right)
\]
\[
= \sum_{0 < n_2} \frac{1}{z_{n_2}^{s-1}} \sum_{n_1 > n_2} \left(\frac{1}{z_{n_1}^{s-2}} - \frac{1}{z_{n_1}^{s-n}}\right)
\]
and
\[
S_2 = \sum_{0 < n_2 < n_1} \frac{1}{z_{n_1}^{s-n-2}} \sum_{n_1 > n_2} \frac{1}{z_{n_2}^{n}} \left(\frac{1}{z_{n_1}^{s-n-2}} - \frac{1}{z_{n_1}^{s-n}}\right)
\]
\[
= \sum_{n_1} \frac{1}{z_{n_1}^{s-1}} \sum_{0 < n_2 < n_1} \frac{1}{z_{n_1}^{s-n-2}} - \frac{1}{z_{n_1}^{s-n}}\right)
\]
We deduce
\[
\sum_{n \geq 0} \left[\zeta(n+2, s-n-2) - \zeta(s+n,-n)\right] = S_1 - S_2 = \sum_{0 < n_2} \frac{1}{z_{n_2}^{s-n-2}} \sum_{n_1 > n_2} \left(\frac{1}{z_{n_1}^{s-n-2}} - \frac{1}{z_{n_1}^{s-n}}\right)
\]
\[
- \sum_{n_1} \frac{1}{z_{n_1}^{s-1}} \sum_{0 < n_2 < n_1} \frac{1}{z_{n_1}^{s-n}}\right)
\]
Reindexing, this is
\[
\sum_{k \geq 0} \frac{1}{z_k^{s-1}} \left[\sum_{i \geq k} \left(\frac{1}{z_i - z_k} - \frac{1}{z_i}\right) + \sum_{0 < i < k} \frac{1}{z_i - z_k}\right] = \sum_{k \geq 0} \frac{1}{z_k^{s-1} z_k}\]
□

5. Some special cases

In this section, we study special cases of the complementary multiple zeta function, which correspond to some specific choices for the sequence \(\{z_i\}\). In what follows, the Hurwitz zeta function is denoted as
\[
\zeta_H(s, z) = \sum_{n \geq 0} \frac{1}{(z+n)^s}
\]
in order to avoid the confusion with the MZV of depth 2
\[
\zeta(a, b) = \sum_{n>m} \frac{1}{n^a m^b}.
\]

5.1. The case \(z_i = i + a - 1\).
5.1.1. *Arbitrary* $a \in [0, 1]$. The choice $z_i = i + a - 1$ gives
\[
\frac{1}{\tilde{z}_k} = \sum_{i=1}^{k-1} \frac{1}{i-k} + \sum_{i=k+1}^\infty \frac{1}{i-k} - \frac{1}{i+a-1} = \psi(k+a) - \psi(k)
\]
so that we recover, for $a = 1$,
\[
\frac{1}{\tilde{z}_k} = \frac{1}{k}.
\]
The corresponding zeta function is
\[
\tilde{\zeta}_G(s) = \sum_{n \geq 1} \frac{1}{\tilde{z}_n} = \sum_{n \geq 1} \frac{\psi(n+a) - \psi(n)}{(n+a-1)^{s-1}}
\]
and can be computed as follows.

**Theorem 5.1.** *In the case* $z_i = i + a - 1$, *the zeta function is given, for* $s > 2$, *by*
\[
\tilde{\zeta}_G(s) = \zeta_H(s-1,a) [\psi(a+1) - \psi(1)] - a \sum_{l=0}^\infty \zeta_H(s-1,a+l+1) \frac{(a+l+1)(l+1)}{(a+l+1)(l+1)}.
\]

**Proof.** We use identity (B.6) in [13]:
\[
\sum_{l=0}^{k} \frac{\psi(b+l)}{(c+l)^s} = \psi(b) \zeta_H(s,c) - \psi(b+k+1) \zeta_H(s,c+k+1)
\]
\[
+ \sum_{l=0}^{k} \zeta_H(s,c+l+1) \frac{b+l}{b+l}.
\]
Choosing $b = a + 1$ and $c = a$, and next $b = 1$ and $c = a$, and substituting $s$ with $s-1$ yields
\[
\sum_{n=1}^{k+1} \frac{\psi(n+a) - \psi(n)}{(n+a-1)^{s-1}} = \zeta_H(s-1,a) [\psi(a+1) - \psi(1)]
\]
\[
- \zeta_H(s-1,a+k+1) [\psi(a+k+2) - \psi(k+2)]
\]
\[
+ \sum_{l=0}^{k} \zeta_H(s-1,a+l+1) \left[ \frac{1}{a+l+1} - \frac{1}{l+1} \right].
\]
Taking the limit $k \to \infty$ gives the desired result.  

5.1.2. *The case* $a = \frac{1}{2}$. *In order to obtain simpler results, we choose now the specialization* $a = \frac{1}{2}$. *In this case, the MZV simplifies to*
\[
\tilde{\zeta}_G(s) = \sum_{n \geq 1} \frac{\psi(n+\frac{1}{2}) - \psi(n)}{(n-\frac{1}{2})^{s-1}}.
\]

Notice that
\[
\psi\left(n + \frac{1}{2}\right) - \psi(n) = 2 \sum_{j=0}^{\infty} \frac{(-1)^j}{2n+j} = \sum_{j=0}^{\infty} \frac{1}{n+j} - \sum_{j=0}^{\infty} \frac{1}{n+j + \frac{1}{2}}
\]
so that
\[
\tilde{\zeta}_G(s) = \sum_{n \geq 1} \sum_{j \geq 0} \frac{1}{(n+j)(n-\frac{1}{2})^{s-1}} - \sum_{n \geq 1} \sum_{j \geq 0} \frac{1}{(n+j + \frac{1}{2})(n-\frac{1}{2})^{s-1}}.
\]
This zeta function can be related to the odd variant of the MZV studied by Hoffman [14],

\[ t(i_1, \ldots, i_k) = \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{i_1} \cdots n_k^{i_k}} \]

with the special case \( t(s) = (1 - 2^{-s}) \zeta(s) \).

**Theorem 5.2.** For \( s \) integer and \( z_i = i - \frac{1}{2} \), the complementary MZV \( \tilde{\zeta}_G(s) \) is

\[
\tilde{\zeta}_G(s) = 2^s \left[ t(s) + t(s - 1, 1) + \psi \left( \frac{1}{2} \right) t(s - 1) \right] - \left( -1 \right)^{s-1} \frac{1}{(s-2)!} \sum_{k=0}^{s-3} \binom{s-3}{k} \psi^{(k+1)} \left( \frac{1}{2} \right) \psi^{(s-k-3)} \left( \frac{1}{2} \right) - \frac{1}{2} \psi^{(s-1)} \left( \frac{1}{2} \right) \right]. \tag{5.2}
\]

**Proof.** The first part of the sum is computed as

\[
\sum_{n \geq 1} \frac{\psi(n + \frac{1}{2})}{(n - \frac{1}{2})^s} = \sum_{n \geq 1} \frac{1}{(n - \frac{1}{2})^s} + \sum_{n \geq 1} \frac{\psi(n - \frac{1}{2})}{(n - \frac{1}{2})^{s-1}}
\]

\[
= 2^s t(s) + 2^{s-1} \psi \left( \frac{1}{2} \right) + \sum_{n \geq 2} \frac{1}{(n - \frac{1}{2})^{s-1}} \left( \sum_{k=1}^{n-1} \frac{1}{k - \frac{1}{2}} + \psi \left( \frac{1}{2} \right) \right)
\]

\[
= 2^s t(s) + 2^s t(s-1, 1) + 2^{s-1} \psi \left( \frac{1}{2} \right) + \psi \left( \frac{1}{2} \right) (2^{s-1} t(s-1) - 2^{s-1})
\]

\[
= 2^s t(s) + 2^s t(s-1, 1) + 2^{s-1} t(s-1) \psi \left( \frac{1}{2} \right).
\]

The second term is computed using the identity from Thm 4.10 in [15],

\[
\sum_{n \geq 1} \frac{\psi(n)}{(n + q - 1)^s} = \frac{(-1)^s}{(s-1)!} \sum_{k=0}^{s-2} \binom{s-2}{k} \psi^{(k+1)}(q) \psi^{(s-k-2)}(q) - \frac{1}{2} \psi^{(s)}(q).
\]

\[ \square \]

The value of \( \tilde{\zeta}_G(2s+1) \) can also be computed without using Hoffman’s odd variants as follows.

**Theorem 5.3.** For \( s \) integer, we have

\[
\tilde{\zeta}_G(2s+1) = -\gamma \zeta(2s) \left( 2^{2s} - 1 \right) + \left( s + \frac{1}{2} \right) \zeta(2s+1) \left( 2^{2s+1} - 1 \right)
\]

\[
- \sum_{l=1}^{s-1} \left( 2^{2l} - 1 \right) \zeta(2l) \zeta(2s+1-2l)
\]

\[
+ \frac{1}{2s-1} \sum_{k=0}^{2s-2} \left( k + 1 \right) \left( 2^{k+2} - 1 \right) \zeta(k+2) \left( 2^{2s-k-1} - 1 \right) \zeta(2s-k-1).
\]
Proof. We’ll use identity C.28 in [15]
\[ \sum_{n \geq 1} \psi \left( n - \frac{1}{2} \right) \left( n - \frac{1}{2} \right)^2 = -\gamma \zeta (2s) \left( 2^{2s} - 1 \right) - \frac{1}{2} \zeta (2s + 1) \left( 2^{2s+1} - 1 \right) \]

as well as the identity from Thm 4.10 in [15],
\[ \sum_{n \geq 1} \psi(n) (n + q - 1)^s = \frac{(-1)^s}{(s-1)!} \left[ \sum_{k=0}^{s-2} \binom{s-2}{k} \psi^{(k+1)} (q) \psi^{(s-k-2)} (q) - \frac{1}{2} \psi^{(s)} (q) \right] , \]

for the second part of the sum. Since moreover
\[ \psi^{(n)} \left( \frac{1}{2} \right) = (-1)^{n+1} n! \left( 2^{n+1} - 1 \right) \zeta (n+1) , \]
we deduce
\[ \zeta_G (2s+1) = \sum_{n \geq 1} \frac{\psi \left( n + \frac{1}{2} \right) - \psi \left( n \right) (n + \frac{1}{2})^s}{(n - \frac{1}{2})^{2s}} = \sum_{n \geq 1} \frac{\psi \left( n - \frac{1}{2} \right) - \psi \left( n \right) (n - \frac{1}{2})^s}{(n - \frac{1}{2})^{2s}} + \sum_{n \geq 1} \frac{1}{(n - \frac{1}{2})^{2s+1}} . \]
The last sum is
\[ \sum_{n \geq 1} \frac{1}{(n - \frac{1}{2})^{2s+1}} = \left( 2^{2s+1} - 1 \right) \zeta (2s+1) , \]
while the first sum is
\[ \sum_{n \geq 1} \frac{\psi \left( n - \frac{1}{2} \right) - \psi \left( n \right)}{(n - \frac{1}{2})^{2s}} = -\gamma \zeta (2s) \left( 2^{2s} - 1 \right) - \frac{1}{2} \zeta (2s + 1) \left( 2^{2s+1} - 1 \right) \]
\[ - \sum_{l=1}^{s-1} \left( 2^{2l} - 1 \right) \zeta (2l) \zeta (2s + 1 - 2l) \]
\[ - \frac{1}{(2s-1)!} \left[ \sum_{k=0}^{2s-2} \binom{2s-2}{k} \psi^{(k+1)} \left( \frac{1}{2} \right) \psi^{(2s-k-2)} \left( \frac{1}{2} \right) - \frac{1}{2} \psi^{(2s)} \left( \frac{1}{2} \right) \right] . \]
We deduce
\[ \zeta_G (2s+1) = -\gamma \zeta (2s) \left( 2^{2s} - 1 \right) + \left( s + \frac{1}{2} \right) \zeta (2s + 1) \left( 2^{2s+1} - 1 \right) \]
\[ - \sum_{l=1}^{s-1} \left( 2^{2l} - 1 \right) \zeta (2l) \zeta (2s + 1 - 2l) \]
\[ + \frac{1}{2s-1} \left[ \sum_{k=0}^{2s-2} (k+1) \left( 2^{k+2} - 1 \right) \zeta (k+2) \left( 2^{2s-k-1} - 1 \right) \zeta (2s - k - 1) \right] . \]

\[ \square \]

Let us now verify Euler’s identity as follows: for \( s = 3 \),
\[ \zeta_G (3) = \sum_{n \geq 1} \frac{\psi \left( n + \frac{1}{2} \right) - \psi \left( n \right)}{(n - \frac{1}{2})^2} . \]
whereas
\[
\zeta_G(2, 1) = \sum_{n_1 > n_2} \frac{1}{(n_1 - \frac{1}{2})^2 (n_2 - \frac{1}{2})} = \sum_{n_1} \frac{1}{(n_1 - \frac{1}{2})^2} \sum_{n_2=1}^{n-1} \frac{1}{n_2 - \frac{1}{2}} = \sum_{n_1} \psi\left(n_1 - \frac{1}{2}\right) - \psi\left(\frac{1}{2}\right).
\]
Euler’s identity
\[
\tilde{\zeta}_G(3) = \zeta_G(2, 1)
\]
thus gives us the strange following identity.

Corollary 5.4. We have
\[
\sum_{n \geq 1} \psi\left(n + \frac{1}{2}\right) - \psi\left(n - \frac{1}{2}\right) = \sum_{n \geq 1} \frac{\psi(n) - \psi \left(\frac{1}{2}\right)}{(n - \frac{1}{2})^2}.
\]

Remark 5.5. This identity can be checked directly as follows: let us first restate as
\[
\sum_{n \geq 1} \psi\left(n + \frac{1}{2}\right) - \psi\left(n - \frac{1}{2}\right) = \sum_{n \geq 1} \frac{\psi(n) - \psi \left(\frac{1}{2}\right)}{(n - \frac{1}{2})^2}.
\]
Using the fact that
\[
\psi\left(n + \frac{1}{2}\right) - \psi\left(n - \frac{1}{2}\right) = \frac{1}{n - \frac{1}{2}},
\]
the left-hand side is easily computed as
\[
\sum_{n \geq 1} \psi\left(n + \frac{1}{2}\right) - \psi\left(n - \frac{1}{2}\right) = \sum_{n \geq 1} \frac{1}{(n - \frac{1}{2})^2} = 7\zeta(3).
\]

Next, using identity E.2 in \[15\], namely
\[
\sum_{n \geq 1} \frac{\psi(n)}{(n - \frac{1}{2})^2} = \frac{\pi^2}{2} (\psi(1) - 2 \log 2) + 7\zeta(3),
\]
and the fact that
\[
\sum_{n \geq 1} \frac{\psi \left(\frac{1}{2}\right)}{(n - \frac{1}{2})^2} = \psi \left(\frac{1}{2}\right) \frac{\pi^2}{2},
\]
we deduce the right-hand side as
\[
\sum_{n \geq 1} \frac{\psi(n) - \psi \left(\frac{1}{2}\right)}{(n - \frac{1}{2})^2} = \frac{\pi^2}{2} \left(\psi(1) - \psi \left(\frac{1}{2}\right) - 2 \log 2\right) + 7\zeta(3).
\]
Since
\[
\psi(1) - \psi \left(\frac{1}{2}\right) = 2 \log 2,
\]
this proves the result.

For \(s = 4\), we check
\[
\tilde{\zeta}_G(4) = \zeta_G(3, 1) + \zeta_G(2, 2) = 2^4 \left[ t(3, 1) + t(2, 2) \right]
\]
as follows: by \[5.2\],
\[
\tilde{\zeta}_G(4) = 2^4 \left[ t(4) + t(3, 1) + \frac{1}{2} \psi \left(\frac{1}{2}\right)t(3) \right] + \frac{1}{2} \left[ \psi \left(\frac{1}{2}\right)^2 + \psi \left(\frac{1}{2}\right)\psi'' \left(\frac{1}{2}\right) - \frac{1}{2}\psi'' \left(\frac{1}{2}\right) \right].
\]
MULTIPLE ZETA VALUES

and we deduce
\[ t(2, 2) = t(4) + \frac{1}{2} \psi \left( \frac{1}{2} \right) t(3) + \frac{1}{32} \left[ \psi' \left( \frac{1}{2} \right)^2 + \psi \left( \frac{1}{2} \right) \psi'' \left( \frac{1}{2} \right) - \frac{1}{2} \psi''' \left( \frac{1}{2} \right) \right] = \frac{\pi^4}{384}. \]

Since \( t(4) = \frac{\pi^4}{96} \), we deduce
\[ t(2, 2) = \frac{1}{4} t(4), \]
a well-known identity that appears for instance in the Appendix of [14].

5.2. The case \( z_i = i^2 \).

5.2.1. Computation of the MZV.

**Theorem 5.6.** In the case \( z_i = i^2 \), we have
\[ \frac{1}{z_k} = \frac{3}{4k^2} - \psi'(k + 1). \]
As a consequence, the complementary zeta function is
\[ \tilde{\zeta}_G(s) = \frac{7}{4} \zeta(2s) - \zeta(2) \zeta(2s - 2) + \zeta(2s - 2, 2). \]

**Proof.** The computation of \( \frac{1}{z_k} \) is straightforward. Moreover,
\[ \tilde{\zeta}_G(s) = \sum_{n \geq 1} \left( \frac{3}{4n^2} - \psi'(n + 1) \right) \frac{1}{n^{2s - 2}} = \frac{3}{4} \zeta(2s) - \sum_{n \geq 1} \frac{\psi'(n + 1)}{n^{2s - 2}}. \]
The sum is computed using identity B.6 in [15]: with
\[ \sum_{l=0}^{k} \frac{\psi'(l + 2)}{(l + 1)^{2s - 2}} = \psi'(2) \zeta_H(2s - 2, 1) - \psi'(2 + k + 1) \zeta_H(2s - 2, k + 2) - \sum_{l=0}^{k} \frac{\zeta_H(2s - 2, l + 2)}{(l + 2)^2}, \]
taking the limit \( k \to \infty \) gives
\[ \tilde{\zeta}_G(s) = \frac{3}{4} \zeta(2s) - \psi'(2) \zeta_H(2s - 2, 1) + \sum_{l=0}^{+\infty} \frac{\zeta_H(2s - 2, l + 2)}{(l + 2)^2}. \]
The sum in the right-hand side is now computed as follows:
\[ \sum_{l=0}^{+\infty} \frac{\zeta_H(2s - 2, l + 2)}{(l + 2)^2} = \sum_{l,n \geq 0} \frac{1}{(l + 2)^2 (n + l + 2)^{2s - 2}} = \sum_{l \geq 1, n \geq 0} \frac{1}{(l + 1)^2 (n + l + 1)^{2s - 2}} \]
\[ = \sum_{l \geq 1, n=0}^{+\infty} \frac{1}{(l + 1)^{2s}} + \sum_{l \geq 2, n \geq 1}^{+\infty} \frac{1}{l^2 (n + l)^{2s - 2}}. \]
The first sum is \( \zeta(2s) - 1 \) while the second sum is recognized as
\[ \zeta(2s - 2, 2) - \sum_{l=1, n \geq 1}^{+\infty} \frac{1}{l^2 (l + n)^{2s - 1}} = \zeta(2s - 2) - (\zeta(2s - 2) - 1). \]
With \( \zeta_H(2s - 2, 1) = \zeta(2s - 2) \),
the final result is thus
\[
\tilde{\zeta}_G (s) = \frac{7}{4} \zeta (2s) - \zeta (2s - 2) + \left( 1 - \frac{\pi^2}{6} \right) \zeta (2s - 2) + \zeta (2s - 2, 2)
\]
\[
= \frac{7}{4} \zeta (2s) - \frac{\pi^2}{6} \zeta (2s - 2) + \zeta (2s - 2, 2).
\]

Euler’s identity
\[
\tilde{\zeta}_G (3) = \zeta_G (2, 1),
\]
with
\[
\tilde{\zeta}_G (3) = \frac{7}{4} \zeta (6) - \frac{\pi^2}{6} \zeta (4) + \zeta (4, 2)
\]
and
\[
\zeta_G (2, 1) = \sum_{i>j \geq 1} \frac{1}{i^2 z_j} = \sum_{i>j \geq 1} \frac{1}{i^4 j^2} = \zeta (4, 2)
\]
yields the well-known and uninteresting result
\[
\frac{7}{4} \zeta (6) - \frac{\pi^2}{6} \zeta (4) = 0.
\]
More generally, we can apply the sum identity
\[
\tilde{\zeta}_G (s + 3) = \sum_{a+b=s} \zeta_G (2 + a, 1 + b),
\]
with
\[
\zeta_G (2 + a, 1 + b) = \sum_{i>j \geq 1} \frac{1}{i^{2+a} j^{1+b}} = \sum_{i>j \geq 1} \frac{1}{i^{4+2a} j^{2+2b}} = \zeta (4 + 2a, 2 + 2b)
\]
and
\[
\tilde{\zeta}_G (s + 3) = \frac{7}{4} \zeta (2s + 6) - \frac{\pi^2}{6} \zeta (2s + 4) + \zeta (2s + 4, 2).
\]
This yields the specialization
\[
\frac{7}{4} \zeta (2s + 6) - \frac{\pi^2}{6} \zeta (2s + 4) + \zeta (2s + 4, 2) = \sum_{a+b=s} \zeta (4 + 2a, 2 + 2b),
\]
or equivalently
\[
\frac{7}{4} \zeta (2s + 6) - \zeta (2) \zeta (2s + 4) = \sum_{a=0}^{s-1} \zeta (4 + 2a, 2 + 2s - 2a).
\]
The case \(s = 1\) provides the more interesting identity
\[
\zeta (4, 4) = \frac{7}{4} \zeta (8) = \zeta (4, 4) = \frac{7}{4} \zeta (8) - \frac{4}{7} \zeta (2) \zeta (4) = \frac{\pi^8}{113400}.
\]
Since \(\zeta (8) = \frac{\pi^8}{9450}\), this reads
\[
\zeta (4, 4) = \frac{\zeta (8)}{12}.
\]
which is an alternate but equivalent version of
\[
\zeta (4, 4) = \frac{1}{2} (\zeta (4))^2 - \zeta (8)
\]
5.3. The case $z_i = i(i + 1)$.

**Theorem 5.7.** In the case $z_i = i(i + 1)$, we have

$$
\frac{1}{z_k} = \frac{1}{k} - \frac{2}{k + 1} + \frac{1}{(2k + 1)^2},
$$

and the corresponding complementary MZV is a linear combination of values of the Riemann zeta function

$$
\zeta_G(s) = (-1)^s \left( \sum_{k=2}^{s} \mu_k^{(s)} \zeta(k) + \eta_s \right),
$$

where

$$
\mu_k^{(s)} = (-1)^k + 2 \left\{ \left( \begin{array}{c} 2s - 2 - k \\ s - 2 \end{array} \right) + (-1)^k \left( \begin{array}{c} 2s - 2 - k \\ s - 1 \end{array} \right) \right\} - (1 + (-1)^k) \beta_{k-1}^{(s-1)}
$$

with

$$
\beta_k^{(s)} = \sum_{i=0}^{s-k-1} 4^i \left( \begin{array}{c} 2s - 2i - k - 2 \\ s - i - 1 \end{array} \right)
$$

and

$$
\eta_s = \left( s - \frac{1}{2} \right) \left( \begin{array}{c} 2s - 2 \\ s - 1 \end{array} \right) - \left( \begin{array}{c} 2s - 2 \\ s \end{array} \right) - 4 \left( \begin{array}{c} 2s - 3 \\ s - 2 \end{array} \right) - \left( \frac{\pi^2}{8} - \frac{1}{2} \right) 2^{2s-2}.
$$

**Remark 5.8.** The first values are

$$
\zeta_G(2) = 0, \quad \zeta_G(3) = -7 + \frac{5\pi^2}{6} - \zeta(3), \quad \zeta_G(4) = 47 - \frac{16\pi^2}{3} + \frac{\pi^4}{30} + 2\zeta(3).
$$

Note that the values of $\mu_k^{(s)}$ for odd $k$, i.e. the coefficients of $\zeta(2k + 1)$, simplify to

$$
\mu_{2k+1}^{(s)} = \frac{2k}{s - 1} \left( \begin{array}{c} 2s - 2k - 1 \\ s - 2 \end{array} \right).
$$

Moreover, the sequence of coefficients $\beta_k^{(s)}$ coincides with the sequence A143019 read by antidiagonals: more precisely, denote as $a_n^{(q)}$ the $n$–th coefficient in the Taylor series expansion at $z = 0$ of the function

$$
\frac{1}{(1 - 4z)^{\frac{q}{2}}} \left( \frac{1 - \sqrt{1 - 4z}}{2z} \right)^q,
$$

then

$$
\beta_k^{(s)} = a_{s-k-1}^{(k)}.
$$

As a consequence of the recurrence identity $a_n^{(q)} = a_n^{(q-1)} + a_{n-1}^{(q+1)}$, the coefficients $\beta_k^{(s)}$ satisfy

$$
\beta_k^{(s+k+1)} - \beta_{k+1}^{(s+k+1)} = \beta_k^{(s+k)}.
$$

**Proof.** We compute

$$
\frac{1}{z_k} = \sum_{i=1}^{k-1} \frac{1}{i(i + 1) - k(k + 1)} + \sum_{i=k+1}^{\infty} \frac{1}{i(i + 1) - k(k + 1)} - \frac{1}{i(i + 1)}.
$$
The first sum is
\[
\sum_{i=1}^{k-1} \frac{1}{(i + \frac{1}{2})^2 - (k + \frac{1}{2})^2} = \sum_{i=1}^{k-1} \frac{1}{(i + k + 1)(i - k)} = \frac{1}{2k + 1} \left( \sum_{i=1}^{k-1} \frac{1}{i - k} - \frac{1}{i + k + 1} \right)
\]
\[
= \frac{1}{2k + 1} \left( -\psi(k) + \psi(1) - \psi(2k + 1) + \psi(k + 2) \right).
\]

The second sum telescopes as
\[
\sum_{i=k+1}^{\infty} \frac{1}{i(i + 1) - k(k + 1)} - \left( \frac{1}{i - 1} - \frac{1}{i + 1} \right) = \frac{1}{2k + 1} \left( \sum_{i=k+1}^{\infty} \frac{1}{i - k} - \frac{1}{i + k + 1} \right) - \frac{1}{k + 1}
\]
\[
= \frac{1}{2k + 1} (\psi(2k + 2) - \psi(1)) - \frac{1}{k + 1},
\]
so that
\[
\frac{1}{\xi_k} = \frac{1}{2k + 1} (-\psi(k) + \psi(2k + 2) - \psi(2k + 1) + \psi(k + 2)) - \frac{1}{k + 1}
\]
\[
= \frac{1}{2k + 1} \left( \frac{1}{2k + 1} + \frac{1}{k + 1} + \frac{1}{k} \right) - \frac{1}{k + 1} = \frac{1}{k} - \frac{2}{k + 1} + \frac{1}{(2k + 1)^2}.
\]

The complementary zeta function is computed as
\[
\zeta_G(s) = \sum_{n \geq 1} \frac{1}{n^{s-1}(n + 1)^{s-1}} \left( \frac{1}{n - \frac{2}{n + 1} + \frac{1}{(2n + 1)^2}} \right)
\]
\[
= \sum_{n \geq 1} \frac{1}{n^{s}(n + 1)^{s-1}} - 2 \sum_{n \geq 1} \frac{1}{n^{s-1}(n + 1)^{s}} + \sum_{n \geq 1} \frac{1}{n^{s-1}(n + 1)^{s-1}(2n + 1)^2}.
\]

The first sum is computed using the partial fraction decomposition
\[
\frac{1}{n^{s}(n + 1)^{s-1}} = (-1)^s \sum_{k=1}^{s} (-1)^k \left( \frac{2s - 2 - k}{s - 2} \right) \frac{1}{n^k} + (-1)^s \sum_{k=1}^{s-1} \left( \frac{2s - 2 - k}{s - 1} \right) \frac{1}{n^k},
\]
so that
\[
\sum_{n \geq 1} \frac{1}{n^{s}(n + 1)^{s-1}} = (-1)^s \sum_{k=1}^{s} (-1)^k \left( \frac{2s - 2 - k}{s - 2} \right) \zeta(k) + (-1)^s \sum_{k=1}^{s-1} \left( \frac{2s - 2 - k}{s - 1} \right) \zeta(k - 1)
\]
\[
= (-1)^s \sum_{k=2}^{s} (-1)^k \left( \frac{2s - 2 - k}{s - 2} \right) \zeta(k) + (-1)^s \sum_{k=2}^{s-1} \left( \frac{2s - 2 - k}{s - 1} \right) \zeta(k)
\]
\[
- (-1)^s \sum_{k=1}^{s-1} \left( \frac{2s - 2 - k}{s - 1} \right) \zeta(k)
\]
\[
= (-1)^s \sum_{k=2}^{s} \left( (-1)^k \left( \frac{2s - 2 - k}{s - 2} \right) + \left( \frac{2s - 2 - k}{s - 1} \right) \right) \zeta(k) + (-1)^{s+1} \left( \frac{2s - 2}{s} \right).
\]

The second sum is computed using the partial fraction decomposition
\[
\frac{1}{n^{s-1}(n + 1)^s} = (-1)^{s+1} \sum_{k=1}^{s} \left( \frac{2s - 2 - k}{s - 2} \right) \frac{1}{n^k} + (-1)^{s+1} \sum_{k=1}^{s-1} \left( -1 \right)^k \left( \frac{2s - 2 - k}{s - 1} \right) \frac{1}{n^k}.
\]
Since moreover it can be checked that we deduce

\[ \sum_{n \geq 1} \frac{1}{n^{s-1} (n+1)^s} = (-1)^{s+1} \sum_{k=1}^{s} \left( \begin{array}{c} 2s - 2 - k \\ s - 2 \end{array} \right) (\zeta(k) - 1) + (-1)^{s+1} \sum_{k=1}^{s-1} (-1)^k \left( \begin{array}{c} 2s - 2 - k \\ s - 1 \end{array} \right) \zeta(k) \]

\[ = (-1)^{s+1} \sum_{k=2}^{s} \left( \begin{array}{c} 2s - 2 - k \\ s - 2 \end{array} \right) + (-1)^k \left( \begin{array}{c} 2s - 2 - k \\ s - 1 \end{array} \right) \zeta(k) + (-1)^s 2 \left( \begin{array}{c} 2s - 3 \\ s - 2 \end{array} \right). \]

To compute the third sum, we use the partial fraction decomposition

\[ \frac{1}{n^s (n+1)^s (2n+1)^2} = \sum_{k=0}^{s} \frac{\alpha_k^{(s)}}{n^{k+1}} + (-1)^s \sum_{k=0}^{s} \frac{\beta_k^{(s)}}{(n+1)^{k+1}} + \frac{(-1)^s 2^{2s}}{(2n+1)^2}, \]

with residues

\[ \beta_k^{(s)} = \sum_{i=0}^{s-k-1} 4^i \left( \begin{array}{c} 2s - 2i - k - 2 \\ s - i - 1 \end{array} \right), \quad \alpha_k^{(s)} = (-1)^{s-k-1} \beta_k^{(s)}. \]

We deduce

\[ \sum_{n \geq 1} \frac{1}{n^s (n+1)^s (2n+1)^2} = \sum_{k=0}^{s} \sum_{n \geq 1} \frac{\alpha_k^{(s)}}{n^{k+1}} + \sum_{k=0}^{s} \sum_{n \geq 1} \frac{(-1)^s \beta_k^{(s)}}{(n+1)^{k+1}} + \sum_{n \geq 1} \frac{(-1)^s 2^{2s}}{(2n+1)^2} \]

\[ = \sum_{k=0}^{s} (-1)^{s-k-1} \beta_k^{(s)} \zeta(k+1) + (-1)^s \beta_k^{(s)} (\zeta(k+1) - 1) + (-1)^s 2^{2s} \sum_{n \geq 1} \frac{1}{(2n+1)^2} \]

\[ = (-1)^s \sum_{k=1}^{s} \left( 1 - (-1)^k \right) \beta_k^{(s)} \zeta(k+1) + (-1)^s \sum_{k=0}^{s} \beta_k^{(s)} + (-1)^s 2^{2s} \left( \frac{\pi^2}{8} - 1 \right). \]

Since moreover it can be checked that

\[ \sum_{k=0}^{s} \beta_k^{(s)} = \frac{1}{2} \left( \frac{(2s+1)!}{s! s!} - 2^{2s} \right), \]

we deduce

\[ \sum_{n \geq 1} \frac{1}{n^s (n+1)^s (2n+1)^2} = (-1)^s \sum_{k=1}^{s} \left( 1 - (-1)^k \right) \zeta(k+1) \beta_k^{(s)} \]

\[ + (-1)^s \left( -\frac{(2s+1)!}{2(s! s!)} + \left( \frac{\pi^2}{8} - 1 \right) 2^{2s} + 2^{2s-1} \right). \]

Putting the three terms together, we obtain

\[ \zeta_G(s) = \sum_{n \geq 1} \frac{1}{n^s (n+1)^{s-1}} - 2 \sum_{n \geq 1} \frac{1}{n^{s-1} (n+1)^s} + \sum_{n \geq 1} \frac{1}{n^{s-1} (n+1)^{s-1} (2n+1)^2} \]

\[ = (-1)^s \sum_{k=2}^{s} \left( (-1)^k \left( \begin{array}{c} 2s - 2 - k \\ s - 2 \end{array} \right) + \left( \begin{array}{c} 2s - 2 - k \\ s - 1 \end{array} \right) \right) \zeta(k) + (-1)^{s+1} \left( \begin{array}{c} 2s - 2 \\ s \end{array} \right) \]

\[ - 2 \left[ (-1)^{s+1} \sum_{k=2}^{s} \left( \begin{array}{c} 2s - 2 - k \\ s - 2 \end{array} \right) + (-1)^k \left( \begin{array}{c} 2s - 2 - k \\ s - 1 \end{array} \right) \right] \zeta(k) + (-1)^s 2 \left( \begin{array}{c} 2s - 3 \\ s - 2 \end{array} \right) \]

\[ + (-1)^{s-1} \sum_{k=2}^{s} \left( 1 - (-1)^{k-1} \right) \zeta(k) \beta_k^{(s-1)} + (-1)^{s-1} \left( -\frac{(2s-1)!}{2(s-1)! (s-1)!} + \left( \frac{\pi^2}{8} - 1 \right) 2^{2s-2} + 2^{2s-3} \right), \]
which can be simplified as
\[
(-1)^s \sum_{k=2}^{s} \zeta(k) \left( \left( (-1)^k + 2 \right) \left( \frac{2s - 2 - k}{s - 2} \right) + (-1)^k \left( \frac{2s - 2 - k}{s - 1} \right) \right) - \left( 1 + (-1)^k \right) \beta_{s-1}^{(s-1)} \\
+ (-1)^s \left[ \frac{(2s - 1)!}{2(s - 1)! (s - 1)!} - \frac{(2s - 2)}{s} - 4 \frac{2s - 3}{s - 2} - \left( \frac{\pi^2}{8} - 1 \right) 2^{2s-2} - 2^{2s-3} \right].
\]

5.4. The Bessel function case. The Bessel function of the first kind with parameter \( \nu \) is defined as
\[
J_\nu(z) = \frac{z^\nu}{2\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu + k + 1)} \left( \frac{z^2}{4} \right)^k.
\]

Its normalized version
\[
\tilde{j}_\nu(x) = 2^\nu \Gamma(\nu + 1) \frac{J_\nu(x)}{x^\nu}
\]
has Weierstrass factorization \([9, (8.544, Page 942)]\)
\[
\tilde{j}_\nu(z) = \prod_{k=1}^{\infty} \left( 1 - \frac{z^2}{x_{\nu,k}^2} \right),
\]
where \( x_{\nu,k} \) are the real zeros of \( j_\nu \), ordered by increasing absolute values. We build from these zeros the Bessel zeta function by choosing \( z_k = x_{\nu,k}^2 \), so that
\[
\zeta_{B,\nu}(2s) := \sum_{k=1}^{\infty} \frac{1}{z_k^s} = \sum_{k=1}^{\infty} \frac{1}{x_{\nu,k}^{2s}}.
\]

Note that this recovers several important special cases: when \( \nu = \frac{1}{2} \) we have \( x_{\nu,k} = k\pi \) and we recover Riemann MZVs. When \( \nu = -\frac{1}{2} \), we have \( x_{\nu,k} = (k + \frac{1}{2}) \pi \) and we recover multiple \( t \)-values. It turns out that in the Bessel case, the complementary zeta function can be explicitly computed as a linear combination of the Bessel zeta function and of a depth 2 Bessel MZV.

Lemma 5.9. The Bessel complementary zeta function is equal to
\[
\tilde{\zeta}_{B,\nu}(s) = \frac{\nu + 1}{2} \tilde{\zeta}_{B,\nu}(2s) - \zeta_{B,\nu}(2, 2s - 2).
\]

Proof. The complementary zeros \( \tilde{z}_k \) defined by \((2.1)\) are computed as
\[
\frac{1}{\tilde{z}_k} = \sum_{i=1}^{k-1} \frac{1}{x_{\nu,i}^2 - x_{\nu,k}^2} + \sum_{i=k+1}^{\infty} \frac{1}{x_{\nu,i}^2 - x_{\nu,k}^2} = \sum_{i=1}^{\infty} \frac{1}{x_{\nu,i}^2 - x_{\nu,k}^2} - \sum_{i=k+1}^{\infty} \frac{1}{x_{\nu,i}^2}
\]
since both sums converge separately. The first sum can be expressed using the formula by Calogero \([7]\)
\[
\sum_{i=1}^{\infty} \frac{1}{x_{\nu,i}^2 - x_{\nu,k}^2} = \frac{\nu + 1}{2} \frac{1}{x_{\nu,k}^2}.
\]
The complementary zeta function is then obtained as
\[ \tilde{\zeta}_{B,\nu}(s) = \sum_{k \geq 1} \frac{1}{\tilde{z}_{k}^{s-1}} = \sum_{k \geq 1} \frac{1}{z_{k}^{s-1}} \left( \frac{\nu + 1}{2} - \sum_{i \geq k+1} \frac{1}{z_{i}} \right) = \frac{\nu + 1}{2} \zeta_{B,\nu}(2s) - \zeta_{B,\nu}(2, 2s - 2). \]

This result allows us to express some non-elementary identities about Bessel MZVs as follows.

**Theorem 5.10.** The Bessel MZV \((5.3)\) satisfies the identity
\[ \zeta_{B,\nu}(2s + 6) = \frac{2}{\nu + 1} \sum_{a+b=s, a \geq -1, b \geq 0} \zeta_{B,\nu}(4 + 2a, 2 + 2b), \ s \geq 0. \]

**Proof.** Apply the generalization of Euler’s identity \((3.6)\) and \((5.4)\) and some elementary algebra to obtain the first identity. \(\square\)

As a corollary of this result,

**Corollary 5.11.** We have
\[ \zeta_{B,\nu}(6) = \frac{2}{\nu + 3} \zeta_{B,\nu}(2) \zeta_{B,\nu}(4). \]

**Proof.** Choosing \(s = 0\) in Theorem \([5.10] \) yields
\[ \zeta_{B,\nu}(6) = \frac{2}{\nu + 1} \left( \zeta_{B,\nu}(2, 4) + \zeta_{B,\nu}(4, 2) \right). \]
Since moreover
\[ \zeta_{B,\nu}(2, 4) + \zeta_{B,\nu}(4, 2) = \zeta_{B,\nu}(2) \zeta_{B,\nu}(4) - \zeta_{B,\nu}(6), \]
the result follows after simple algebra. \(\square\)

This result can be checked directly since the zeta values involved have respective explicit expressions
\[ \zeta_{B,\nu}(2) = \frac{1}{4(\nu + 1)}, \ zeta_{B,\nu}(4) = \frac{1}{16(1 + \nu)^2 (2 + \nu)} \]
and
\[ \zeta_{B,\nu}(6) = \frac{1}{32 (1 + \nu)^3 (2 + \nu) (3 + \nu)}. \]

5.5. **The Bessel polynomial case.** We conclude this series of examples with a case of a zeta function built on a finite sequence of numbers \(\{z_i\}\) chosen as the sequence of the zeros \(\{z_{\nu,j}\} \leq j \leq n\) of the Bessel polynomial \(\theta_n(z)\) of degree \(n\). This polynomial is obtained from the modified Bessel function \(K_{\nu}\) with \(\nu = n + \frac{1}{2}\), as
\[ \theta_n(z) = \sqrt{\frac{2}{\pi}} e^{z} z^{n+\frac{1}{2}} K_{n+\frac{1}{2}}(z) \]
or equivalently as
\[ \theta_n(z) = \sum_{m=0}^{n} \frac{(n + m)!}{2^m (n - m)! m!} z^{n-m}. \]
For example,

\[ \theta_0(z) = 1, \quad \theta_1(z) = 1 + z, \quad \theta_2(z) = 3 + z + z^2. \]

These roots are complex conjugated and satisfy the identity \[1, 2.10b\]

\[ \sum_{k=1}^{n} \frac{1}{z_{\nu,k} - z_{\nu,j}} = -1 - \nu \frac{1}{z_{\nu,j}}. \]

We state the following theorem and omit its proof since it follows the same steps as the previous one.

**Theorem 5.12.** In the case where \(\{z_i\}\) are chosen as the zeros of the Bessel polynomial of degree \(n\), and with \(\nu = n + \frac{1}{2}\), the sequence of complementary zeros is

\[ \tilde{z}_{\nu,k} = -\frac{1}{\nu - k} - \sum_{j \geq k+1} \frac{1}{z_j}. \]

and the complementary MZV is

\[ \tilde{\zeta}_G(s) = \left(\frac{1}{2} - \nu\right) \zeta_G(s) - \zeta_G(s-1) - \zeta_G(1, s-1). \]

As a consequence,

\[ \zeta_G(s+3) = \frac{2}{1 - 2\nu} \left[ \zeta_G(s+2) + \sum_{a+b=s} \zeta(2+a, 1+b) \right], \]

and

\[ \zeta_G(2) + \zeta_G(1) \zeta_G(2) = \left(\frac{3}{2} - \nu\right) \zeta_G(3). \]

6. **A Result by Liu et al about Sums of MZV**

Let us first introduce the multiple zeta functions, the functional versions of the multiple zeta values, as

\[ \zeta(j_1, \ldots, j_k; \alpha) = \sum_{n_1 > n_2 > \ldots > n_k} \frac{1}{(\alpha + n_1)^{j_1} \ldots (\alpha + n_k)^{j_k}}, \quad (6.1) \]

and their generalized versions as

\[ \zeta_G(j_1, \ldots, j_k; \alpha) = \sum_{n_1 > n_2 > \ldots > n_k} \frac{1}{(\alpha + z_{n_1})^{j_1} \ldots (\alpha + z_{n_k})^{j_k}}. \quad (6.2) \]

Specific values of MZVs are usually difficult to compute, but their averages over a symmetric enough set of indices are sometimes explicitly computable. One of these cases appeared recently in [6], where Liu et al. provide, for the the sums

\[ E(2n, k; \alpha) = \sum_{j_1 + \ldots + j_k = n} \zeta(2j_1, \ldots, 2j_k; \alpha), \]

the explicit expression

\[ E(2n, k; \alpha) = \sum_{N \geq 0} \frac{d_{k-1}^{(N)}}{(N + \alpha)^{2n-2k+2}}, \quad (6.3) \]
where $a^{(N)}_k$ are the coefficients in the series expansion of the infinite product
\[
\prod_{\substack{r \geq 0 \\ r \neq N}} \left( 1 + \frac{x}{(r + \alpha)^2 - (N + \alpha)^2} \right) = \sum_{k \geq 0} a^{(N)}_k x^k. \tag{6.4}
\]

Let us provide now an extension of this result to an arbitrary generalized multiple zeta function as defined by (6.2). The idea is that Liu’s result is one of these structural identities for the MZV values.

We begin with a structural identity; for a detailed exposition and a proof, see [17].

**Theorem 6.1.** Let
\[
E_G(2n, k) = \sum_{j_1 + \ldots + j_k = n} \zeta_G(2j_1, \ldots, 2j_k).
\]

Then we have the generating product
\[
\prod_{k=1}^{\infty} \left( 1 + \frac{(y-1)t}{z_k^2} \right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} E_G(sn, k) y^k t^n.
\]

We can now state a generalization of Liu’s statement as follows.

**Theorem 6.2.** The sums
\[
E_G(2n, k; \alpha) = \sum_{j_1 + \ldots + j_k = n} \zeta_G(2j_1, \ldots, 2j_k; \alpha)
\]
can be expressed as
\[
E_G(2n, k; \alpha) = \sum_{N \geq 0} \frac{a^{(N)}_{k-1}}{(z_N + \alpha)^{2n-2k+2}}
\]
where the $a^{(N)}_k$ are the coefficients in the series expansion
\[
\prod_{\substack{r \geq 0 \\ r \neq N}} \left( 1 + \frac{x}{(z_r + \alpha)^2 - (z_N + \alpha)^2} \right) = \sum_{k \geq 0} a^{(N)}_k x^k.
\]

**Proof.** Start with the generating product from Theorem 6.1 and substitute $s = 2$ and $t \mapsto t^2$ to obtain
\[
\prod_{k=1}^{\infty} \left( 1 + \frac{(y-1)t^2}{z_k^2} \right) = \prod_{k=1}^{\infty} \left( 1 + \frac{yt^2}{z_{n_k}^2 - t^2} \right).
\]

We wish to extract the coefficient of $y^k t^{2n}$ in the series expansion of this product. We first expand in powers of $y$ and collect the coefficient of $y^k$, which is
\[
\sum_{n_1 > n_2 > \ldots > n_k} t^{2k} \frac{z_{n_1}^2 - t^2}{z_{n_1}^2 - t^2} \ldots \frac{z_{n_k}^2 - t^2}{z_{n_k}^2 - t^2}.
\]

A partial fraction decomposition gives
\[
\frac{t^{2k}}{(z_{n_1}^2 - t^2) \ldots (z_{n_k}^2 - t^2)} = \sum_{j=1}^{k} \frac{\beta_j}{z_{n_j}^2 - t^2}.
\]
where the residues are
\[ \beta_j = \prod_{m \neq j} \frac{1}{z_{n_m}^2 - z_{n_j}^2}. \]

Expanding each term of this decomposition gives
\[
\sum_{j=1}^{k} \beta_j \frac{t^{2k}}{z_{n_j}^2 - t^2} = \sum_{j=1}^{k} \beta_j \frac{t^{2k}}{z_{n_j}^2} \left( 1 + \frac{t^2}{z_{n_j}^2} + \ldots \right),
\]
so that the coefficient of \( t^{2n} \), which is the coefficient we are looking for, is
\[
\sum_{j=1}^{k} \frac{\beta_j}{z_{n_j}^{2k+2}}.
\]

We deduce
\[
E_G(2n, k) = \sum_{n_1 > n_2 > \ldots > n_k} \sum_{j=1}^{k} \frac{\beta_j}{z_{n_j}^{2n-2k+2}}.
\]

Now consider one arbitrary term of the inner sum, indexed \( j \), and rename \( n_j \) to \( N \). We look for
\[
\sum_{n_1 > \ldots > n_j-1 > n_j > n_{j+1} \ldots > n_k} \frac{\beta_j}{z_{n_j}^{2n-2k+2}}
\]
\[
= \sum_{N \geq 0} \frac{1}{z_{n_j}^{2n-2k+2}} \sum_{n_1 > \ldots > n_j-1 > n_j > n_{j+1} \ldots > n_k} \prod_{m \neq N} \frac{1}{z_{n_m}^2 - z_{n_j}^2}
\]
\[
= \sum_{N \geq 0} \frac{1}{z_{n_j}^{2n-2k+2}} \sum_{n_1 > \ldots > n_j-1 > n_j > n_{j+1} \ldots > n_k} \prod_{m \neq N} \frac{1}{z_{n_m}^2 - z_{n_j}^2}.
\]

Now map \( z_n \mapsto z_n + \alpha \), since we want to consider Hurwitz style MZVs. We recognize the inner multiple sum as an *elementary symmetric function*, which is by inspection the coefficient \( a^{(N)}_{k-1} \) in the series expansion of
\[
\prod_{r \neq N} \left( 1 + \frac{x}{(\alpha + z_r)^2 - (\alpha + z_N)^2} \right) = a^{(N)}_0 + a^{(N)}_1 x + a^{(N)}_2 x^2 + \ldots.
\]

We therefore deduce the final result. \( \square \)

It turns out that in the case of the usual MZV function, with \( z_k = k \), the product in (6.4) can be computed explicitly in terms of Gamma functions; Liu et al obtain
\[
\prod_{r \geq 0, r \neq N} \left( 1 + \frac{x}{(r + \alpha)^2 - (N + \alpha)^2} \right) = 2 (-1)^N (N + \alpha) \Gamma (2 \alpha + N) \frac{1}{N!}
\]
\[
\times \frac{1}{x \Gamma \left( \alpha + \sqrt{(N + \alpha)^2 - x} \right) \Gamma \left( \alpha - \sqrt{(N + \alpha)^2 - x} \right)}.
\]

A similar simplification does not seem to occur in the general case.
MULTIPLE ZETA VALUES

7. A generalization of an identity by Hoffman

In this section, we derive a generalization of an identity due to Hoffman [14, Thm 4.4]: for $h, k \geq 1$,

$$\zeta\left(h + 1, \{1\}^{k-1}\right) = \zeta\left(k + 1, \{1\}^{h-1}\right).$$

7.1. A result on symmetric functions. Assume $f(z_1, z_2)$ is a function symmetric in the variables $z_1$ and $z_2$ and consider the sum

$$\sum_{n_1, n_2 \geq 1} \frac{f(z_1, z_2)}{z_1 z_2}.$$

Since

$$\frac{1}{z_1 (z_1 + z_2)} = \frac{1}{z_2} \left(1 - \frac{1}{z_1 + z_2}\right),$$

we deduce that

$$\sum_{n_1, n_2 \geq 1} \left(\frac{f(z_1, z_2)}{z_1 (z_1 + z_2)} + \frac{f(z_1, z_2)}{z_2 (z_2 + z_1)}\right) = 2 \sum_{n_1, n_2 \geq 1} \frac{f(z_1, z_2)}{z_1 (z_1 + z_2)} = \sum_{n_1, n_2 \geq 1} \frac{f(z_1, z_2)}{z_1 z_2}.$$\(7.1\)

The next lemma is a generalization to the case of $k$ variables of this result, and also a generalization of Lemma 3 in [14], which is the case $z_n = n$.

**Lemma 7.1.** Consider a function $f$ symmetric in its $k$ variables. Then

$$\sum_{n_1, \ldots, n_k \geq 1} \frac{k! f(z_{n_1}, \ldots, z_{n_k})}{z_{n_1} (z_{n_1} + z_{n_2}) \ldots (z_{n_1} + z_{n_2} + \cdots + z_{n_k})} = \sum_{n_1, \ldots, n_k \geq 1} \frac{f(z_{n_1}, \ldots, z_{n_k})}{z_{n_1} z_{n_2} \cdots z_{n_k}}. \quad (7.1)$$

**Proof.** We will induct on the number $k$ of summations. The base case $k = 1$ obviously holds. Assume the statement holds for arbitrary $k$ and consider

$$\sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{(k + 1)! f(z_{n_1}, \ldots, z_{n_{k+1}})}{z_{n_1} (z_{n_1} + z_{n_2}) \ldots (z_{n_1} + z_{n_2} + \cdots + z_{n_{k+1}})} = \sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{k!}{z_{n_1} (z_{n_1} + z_{n_2}) \ldots (z_{n_1} + z_{n_2} + \cdots + z_{n_{k+1}})} \times \sum_{n_{k+1} \geq 1} \frac{(k + 1) f(z_{n_1}, \ldots, z_{n_{k+1}})}{z_{n_1} z_{n_2} + \cdots + z_{n_{k+1}}}.$$\(7.1\)

Since

$$\sum_{n_{k+1} \geq 1} \frac{(k + 1)! f(z_{n_1}, \ldots, z_{n_{k+1}})}{z_{n_1} + z_{n_2} + \cdots + z_{n_{k+1}}}$$

is symmetric in the variables $z_{n_1}, \ldots, z_{n_{k+1}}$, we apply the induction hypothesis to transform

$$\sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{k!}{z_{n_1} (z_{n_1} + z_{n_2}) \ldots (z_{n_1} + z_{n_2} + \cdots + z_{n_{k+1}})} = \sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{1}{z_{n_1} z_{n_2} \cdots z_{n_{k+1}}},$$

so that the right-hand side is

$$\sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{(k + 1)! f(z_{n_1}, \ldots, z_{n_{k+1}})}{z_{n_1} + z_{n_2} + \cdots + z_{n_{k+1}}}.$$

Define $S$ as this double sum divided by $(k + 1)$:

$$S := \sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{1}{z_{n_1} z_{n_2} \cdots z_{n_{k+1}}} \sum_{n_{k+1} \geq 1} \frac{f(z_{n_1}, \ldots, z_{n_{k+1}})}{z_{n_1} + z_{n_2} + \cdots + z_{n_{k+1}}}.$$
and compute
\[
\sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{f(z_{n_1}, \ldots, z_{n_{k+1}})}{z_{n_1} z_{n_2} \ldots z_{n_{k+1}}} - S = \sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{f(z_{n_1}, \ldots, z_{n_{k+1}})}{z_{n_1} z_{n_2} \ldots z_{n_{k+1}}} - \sum_{n_1, \ldots, n_{k} \geq 1} \frac{1}{z_{n_1} z_{n_2} \ldots z_{n_{k}}} \sum_{n_{k+1} \geq 1} \frac{f(z_{n_1}, \ldots, z_{n_{k+1}})}{z_{n_1} z_{n_2} \ldots z_{n_{k+1}}} \\
= \sum_{n_1, \ldots, n_{k} \geq 1} \frac{1}{z_{n_1} z_{n_2} \ldots z_{n_{k}}} \sum_{n_{k+1} \geq 1} f(z_{n_1}, \ldots, z_{n_k}) \left[ \frac{1}{z_{n_{k+1}}} - \frac{1}{z_{n_1} + z_{n_2} + \ldots + z_{n_{k+1}}} \right] \\
= \sum_{n_1, \ldots, n_{k} \geq 1} \frac{1}{z_{n_1} z_{n_2} \ldots z_{n_{k}}} \sum_{n_{k+1} \geq 1} f(z_{n_1}, \ldots, z_{n_{k}}) \frac{z_{n_{k+1}}}{z_{n_1} + \ldots + z_{n_{k+1}}} \\
= \sum_{j=1}^{k} \sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{f(z_{n_1}, \ldots, z_{n_{k+1}})}{z_{n_1} z_{n_2} \ldots z_{n_{k+1}} (z_{n_1} + \ldots + z_{n_{k+1}})}.
\]

Each of the \( k \) summands is equal to \( S \) so that
\[
\sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{f(z_{n_1}, \ldots, z_{n_{k+1}})}{z_{n_1} z_{n_2} \ldots z_{n_{k+1}}} - S = kS
\]

and
\[
\sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{f(z_{n_1}, \ldots, z_{n_{k+1}})}{z_{n_1} z_{n_2} \ldots z_{n_{k+1}}} = (k + 1) S \\
= \sum_{n_1, \ldots, n_{k+1} \geq 1} \frac{(k + 1)! f(z_{n_1}, \ldots, z_{n_{k+1}})}{z_{n_1} (z_{n_1} + z_{n_2}) \ldots (z_{n_1} + z_{n_2} + \ldots + z_{n_{k+1}})}.
\]

7.2. A result by Mordell. Hoffman uses the previous result together with a result of Mordell [16, Theorem 1], which we state here with its original proof.

Theorem 7.2. For \( a > -k \),
\[
\sum_{n_1, \ldots, n_k \geq 1} \frac{1}{n_1 n_2 \ldots n_k (n_1 + \ldots + n_k + a)} = k! \sum_{i \geq 0} \frac{(-1)^i}{(i + 1)^{k+1}} \binom{a - 1}{i}.
\]
Proof. The proof essentially just manipulates an integral representation for the zeta function on the left-hand side:

$$\sum_{n_1, \ldots, n_k \geq 1} \frac{1}{n_1 n_2 \ldots n_k (n_1 + \cdots + n_k + a)} = \sum_{n_1, \ldots, n_k \geq 1} \frac{1}{n_1 n_2 \ldots n_k} \int_0^1 x^{n_1 + \cdots + n_k + a - 1} dx$$

$$= \int_0^1 \sum_{n_1, \ldots, n_k \geq 1} x^{n_1 + \cdots + n_k} n_1 n_2 \ldots n_k x^{a - 1} dx$$

$$= \int_0^1 \left( \sum_{n \geq 1} \frac{x^n}{n} \right)^k x^{a - 1} dx = \int_0^1 (- \log (1 - x))^k x^{a - 1} dx.$$

The change of variable $z = - \log (1 - x)$, $x = 1 - e^{-z}$ gives

$$\int_0^\infty z^k (1 - e^{-z})^{a - 1} e^{-z} dz.$$

that can be expanded as

$$\int_0^\infty z^k (1 - e^{-z})^{a - 1} e^{-z} dz = \sum_{l \geq 0} (-1)^l \binom{a - 1}{l} \int_0^\infty z^k e^{-(l+1)z} dz$$

$$= \sum_{l \geq 0} (-1)^l \binom{a - 1}{l} \frac{k!}{(l + 1)^{k + 1}}.$$

□

The appropriate generalization of Mordell’s theorem is as follows:

**Theorem 7.3.** The extension of identity (7.2) is as follows: for any sequence $\{z_n\}$,

$$\sum_{n_1, \ldots, n_k \geq 1} \frac{1}{z_{n_1} \cdots z_{n_k} (z_{n_1} + \cdots + z_{n_k} + a)} = \sum_{l \geq 0} (-1)^l \binom{a - 1}{l} \alpha_{k,l},$$

(7.3)

with the coefficients

$$\alpha_{k,l} = \int_0^\infty \varphi^k (e^{-z}) e^{-(l+1)z} dz$$

with

$$\varphi (x) = \sum_{n \geq 1} \frac{(1 - x)^{z_n}}{z_n}.$$

**Proof.** Start from

$$\sum_{n_1, \ldots, n_k \geq 1} \frac{1}{z_{n_1} \cdots z_{n_k} (z_{n_1} + \cdots + z_{n_k} + a)} = \sum_{n_1, \ldots, n_k \geq 1} \frac{1}{z_{n_1} z_{n_2} \cdots z_{n_k}} \int_0^1 x^{z_{n_1} + \cdots + z_{n_k} + a - 1} dx$$

$$= \int_0^1 \sum_{n_1, \ldots, n_k \geq 1} x^{z_{n_1} + \cdots + z_{n_k}} z_{n_1} z_{n_2} \cdots z_{n_k} x^{a - 1} dx$$

$$= \int_0^1 \left( \sum_{n \geq 1} \frac{x^{z_n}}{z_n} \right)^k x^{a - 1} dx.$$
Denote
\[ \varphi(x) = \sum_{n \geq 1} \frac{(1-x)^z}{z_n}, \]
so that we obtain the multiple sum as the integral
\[ \int_0^1 (\varphi(1-x))^k x^{a-1} dx. \]
The change of variable \( x = 1 - e^{-z} \) gives
\[ \int_0^\infty \varphi^k(e^{-z})(1 - e^{-z})^{a-1} e^{-z} dz, \]
which can be expanded as
\[ \int_0^\infty \varphi^k(e^{-z}) \left( \sum_{l \geq 0} (-1)^l \binom{a-1}{l} e^{-(l+1)z} \right) dz, \]
with
\[ \alpha_{k,l} = \int_0^\infty \varphi^k(e^{-z}) e^{-(l+1)z} dz, \]
this is the claimed result. \( \square \)

**Remark 7.4.** The usual case is
\[ \varphi(x) = -\log x \]
and
\[ \varphi(x) = \frac{k!}{(l+1)^{k+1}}, \]
which recovers Mordell’s result.

### 7.3. Final generalization.

Putting these two results together, Hoffman obtains the following result [14, Thm 4.4]. We replicate here the short proof to highlight how the structural version is essentially identical.

**Theorem 7.5.** For integers \( h, k \geq 1 \),
\[ \zeta \left( h + 1, \{1\}^{k-1} \right) = \zeta \left( k + 1, \{1\}^{h-1} \right). \]

**Proof.** First compute the derivative \( \frac{\partial^{h-1}}{da^{h-1}} \) of (7.2): the left-hand side is
\[ \sum_{n_1, \ldots, n_k \geq 1} \frac{(-1)^{h-1} (h-1)!}{n_1 n_2 \ldots n_k (n_1 + \cdots + n_k + a)^h} \]
and, by the symmetry identity (7.1) with the choice \( f(n_1, \ldots, n_k) = (n_1 + \cdots + n_k)^{h-1} \), is equal for \( a = 0 \) to
\[ (-1)^{h-1} (h-1)! k! \zeta \left( k + 1, \{1\}^{h-1} \right). \]
Next, the right-hand side can be computed (see below) as
\[ k! \sum_{i \geq h} \frac{(-1)^{i-h}}{(i+1)^{k+1}} \sum_{1 \leq l_1 < \cdots < l_h \leq i} \frac{1}{(a-l_1) \cdots (a-l_h)} \binom{a-1}{i}. \]
Taking the value \( a = 0 \) gives the desired result. \( \square \)
MULTIPLE ZETA VALUES

Using the generalization (7.3) of Mordell’s theorem and the generalization of the symmetry lemma (7.1), we now derive a generalization of Hoffman’s identity by taking the \( p \)-th derivative of (7.3):

**Theorem 7.6.** We have

\[
\zeta_G(h + 1, \{1\}^{k-1}) = \tilde{\zeta}(k + 1, \{1\}^{h-1})
\]

where the “exotic” multiple zeta value of depth 2 \( \tilde{\zeta} \) is defined as

\[
\tilde{\zeta}(k + 1, \{1\}^{h-1}) = \sum_{i_0 > i_1 > \cdots > i_h \geq 1} \beta_{k,i_0} \frac{1}{i_0 \cdots i_h}
\]

with the coefficients

\[
\beta_{k,i} = \frac{(l + 1)^{k+1}}{k!} \alpha_{k,l} = \frac{\int_0^\infty \varphi^k(e^{-z}) e^{-(l+1)z} dz}{\int_0^\infty z^k e^{-(l+1)z} dz}
\]

**Proof.** For the left-hand side, we compute the \( p \)-th order derivative as

\[
\frac{d^p}{da^p} \sum_{n_1, \ldots, n_k \geq 1} \frac{1}{z_{n_1} \cdots z_{n_k} (z_{n_1} + \cdots + z_{n_k} + a)^p} = \sum_{n_1, \ldots, n_k \geq 1} \frac{(-1)^p}{z_{n_1} \cdots z_{n_k} (z_{n_1} + \cdots + z_{n_k} + a)^p+1}
\]

and for the right-hand side

\[
\frac{d^p}{da^p} \sum_{l \geq 0} (-1)^l \binom{a-1}{l} \alpha_{k,l} = \sum_{l \geq 0} (-1)^l \alpha_{k,l} \frac{d^p}{da^p} \binom{a-1}{l},
\]

with

\[
\frac{d^p}{da^p} \binom{a-1}{l} = \frac{d^p}{da^p} \frac{(a-1) \cdots (a-l)}{l!} = \left( a-1 \right) \sum_{l > i_1 > \cdots > i_p \geq 1} \frac{p!}{(a-i_1) \cdots (a-i_p)}
\]

Now choose \( a = 0 \): the right-hand side is, with \( \binom{-1}{l} = (-1)^l \),

\[
\sum_{l \geq 0} (-1)^l \alpha_{k,l} \frac{d^p}{da^p} \binom{a-1}{l} = p! \sum_{l \geq 0} (-1)^l \alpha_{k,l} \binom{-1}{l} \sum_{l > i_1 > \cdots > i_p \geq 1} \frac{1}{(a-i_1) \cdots (a-i_p)}
\]

\[
= p! \sum_{l \geq 0} (-1)^l \alpha_{k,l} \binom{-1}{l} \sum_{l > i_1 > \cdots > i_p \geq 1} \frac{1}{i_1 \cdots i_p},
\]

so that we obtain

\[
\sum_{n_1, \ldots, n_k \geq 1} \frac{1}{z_{n_1} \cdots z_{n_k} (z_{n_1} + \cdots + z_{n_k})^{p+1}} = \sum_{l \geq 0} \alpha_{k,l} \sum_{l > i_1 > \cdots > i_k \geq 1} \frac{1}{i_1 \cdots i_k}.
\]

Denote

\[
\frac{(l + 1)^{k+1}}{k!} \alpha_{k,l} = \beta_{k,l}
\]

so that

\[
\sum_{l \geq 0} \alpha_{k,l} \sum_{l > i_1 > \cdots > i_k \geq 1} \frac{1}{i_1 \cdots i_k} = k! \sum_{l \geq 0} \frac{\beta_{k,l}}{(l + 1)^{k+1}} \sum_{l > i_1 > \cdots > i_p \geq 1} \frac{1}{i_1 \cdots i_p}
\]

\[
= k! \sum_{i_0 > i_1 > \cdots > i_p \geq 1} \frac{\beta_{k,i_0}}{(k+1)!} \frac{1}{i_1 \cdots i_p},
\]
which gives the result.

**Remark 7.7.** In the usual case, $\beta_{k,l} = 1$ and we recover Hoffman’s identity

$$
\zeta \left( h + 1, \{1\}^k \right) = \zeta \left( k + 1, \{1\}^h \right).
$$

**Example 7.8.** We provide here a simple, although non elementary, example of application of the previous formula. Choosing $z_n = 2n - 1$ in Thm. 7.3 yields

$$
\varphi(x) = \sum_{n \geq 1} \frac{(1-x)^{2n-1}}{2n-1} = \frac{1}{2} \log \left( \frac{2-x}{x} \right)
$$

so that the coefficients $\alpha_{k,l}$ are computed as

$$
\alpha_{k,l} = \int_{0}^{+\infty} \varphi^k (e^{-z}) e^{-(l+1)z} dz = \frac{1}{2k} \int_{0}^{+\infty} \log^k (2e^{-z} - 1) e^{-(l+1)z} dz
$$

after a suitable change of variables. The integral

$$
I_{k,l} = \int_{1}^{+\infty} \frac{\log^k u}{(1 + u)^{l+2}} du,
$$

is evaluated as

$$
I_{k,l} = \frac{k!}{l+1} \sum_{n \geq 0} \binom{n+l}{l} \frac{(-1)^n}{(n+l+1)^k}
$$

so that the coefficients $\beta_{k,l}$ are

$$
\beta_{k,l} = \frac{(l+1)^k}{2^{k-l-1}} \sum_{n \geq 0} \binom{n+l}{l} \frac{(-1)^n}{(n+l+1)^k}
$$

We deduce

$$
\zeta_G \left( h + 1, \{1\}^{k-1} \right) = \sum_{i_1 > \ldots > i_k \geq 1} \frac{1}{(2i_1 - 1)^{h+1}(2i_2 - 1) \ldots (2i_k - 1)} = t \left( h + 1, \{1\}^{-} \right)
$$

and

$$
\tilde{\zeta} \left( k + 1, \{1\}^{h-1} \right) = \sum_{i_0 > \ldots > i_{h-1} \geq 1} \frac{\beta_{k,i_0}}{i_1^{k+1}i_1 \ldots i_{h-1}}.
$$

For example, with $k = 2$,

$$
\zeta_G \left( h + 1, \{1\}^{k-1} \right) = \zeta_G (h + 1, 1) = \sum_{n \geq m \geq 1} \frac{1}{(2n - 1)^{h+1}(2m - 1)} = t (h + 1, 1)
$$

and

$$
\tilde{\zeta} \left( 3, \{1\}^{h-1} \right) = \sum_{i_0 > \ldots > i_{h-1} \geq 1} \frac{\beta_{2,i_0}}{i_1^{2+1}i_1 \ldots i_{h-1}}.
$$

so that

$$
t (h + 1, 1) = \sum_{i_0 > \ldots > i_{h-1} \geq 1} \frac{\beta_{2,i_0}}{i_1^{2+1}i_1 \ldots i_{h-1}}.
$$
8. Conclusion

We have shown that several usual identities for MZVs naturally are of structural type, i.e. can be extended to MZVs built from an arbitrary sequence of numbers. The price to pay is the use of the alternate sequence of numbers defined by (2.1) or (2.2) in the multivariate case. However, the natural appearance of the complementary zeta function suggests that it is somehow fundamental to the ring of quasisymmetric functions; it is still an open question to characterize this in terms of known bases for the ring of quasisymmetric functions.

Additionally, exploring further special cases of $\tilde{z}_k$ appears to lead to many new sum relations. The case $z_k = k^{2n}$ for positive integral $n$ appears particularly promising. Many of our results are also crying out for $q$-analogs; we believe that with the correct choice of $z_k$, we will have $\tilde{\zeta}_G$ reduce to some known $q$-multiple zeta function. Finally, we are still unsure of what the “correct” higher order analog of $\zeta_G$ is, since it appears to be the structural analog of a depth 2 MZV.

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