A comprehensive connection between the basic results and properties derived from two kinds of topologies for a random locally convex module

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Dedicated to Professor Berthold Schweizer on his 80th birthday

Abstract

The purpose of this paper is to make a comprehensive connection between the basic results and properties derived from the two kinds of topologies (namely the \((\varepsilon, \lambda)\)–topology introduced by the author and the stronger locally \(L^0\)–convex topology recently introduced by Filipović et. al) for a random locally convex module. First, we give an extremely simple proof of the known Hahn-Banach extension theorem for \(L^0\)–linear functions as well as its continuous variants. Then we give the essential relations between the hyperplane separation theorems in [Filipović et. al, J. Funct. Anal. 256(2009)3996–4029] and a basic strict separation theorem in [Guo et. al, Nonlinear Anal. 71(2009)3794–3804]: in the process obtain a useful and surprising fact that a random locally convex module with the countable concatenation property must have the same completeness under the two topologies! Based on the relation between the two kinds of completeness, we go on to present the central part of this paper: we prove that most of the previously established deep results of random conjugate spaces of random normed modules under the \((\varepsilon, \lambda)\)–topology are still valid under the locally \(L^0\)–convex topology, which considerably enriches financial applications of random normed modules.

Keywords Random locally convex modules; The \((\varepsilon, \lambda)\)–topology; The locally \(L^0\)–convex topology; Random conjugate spaces; Hahn-Banach extension theorems; Hyperplane separation theorems; The countable concatenation property; Completeness

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References
1. Introduction, preliminaries and an outline of main results

1.0. Introduction

Random metric theory originated from the theory of probabilistic metric spaces, cf. [13,14,26]. The original definition of a random normed space was presented in [26, p.240] in which the random norm of a vector is a nonnegative random variable. The new version of a random normed space was presented in [11], in which the random norm of a vector is the equivalence class of a nonnegative random variable. Based on the new version, we further presented in [11] the elaborated definition of a random normed module, which was originally introduced in [7]. Random normed modules lead to the definitive notion of the random conjugate space of a random normed space, cf. [11], and also make the theory of random conjugate spaces of random normed modules have obtained a systematic and deep development, cf. [8,10,16,19,23]. Subsequently, random locally convex modules were presented in [14] and deeply studied in [20,24](called random seminormed modules in [14,20,24]). In particular, using the theory of random conjugate spaces we recently established a basic strict separation theorem in random locally convex modules in [22, Theorem 3.1]. It should also be pointed out that random locally convex modules, including their special case—random normed modules, in all our previous papers, are endowed with a natural topology, called the $(\varepsilon, \lambda)$—topology. The terminology “$(\varepsilon, \lambda)$—topology” was first employed by B.Schweizer and A.Sklar in 1961 in their work on topologizing a probabilistic metric space, cf. [26]. The $(\varepsilon, \lambda)$—topology is both useful and natural: for example, the $(\varepsilon, \lambda)$—topology on the algebra of equivalence classes of random variables on a probability space is exactly the topology of convergence in probability, making the algebra a topological algebra and a random locally convex module endowed with its $(\varepsilon, \lambda)$—topology a topological module over the above topological algebra.

Motivated by financial applications, Filipović, Kupper and Vogelpoth recently presented in [4] locally $L^0$—convex modules, in particular the locally $L^0$—convex topology, establishing their hyperplane separation theorems, cf. [4, Theorems 2.6 and 2.8], and some basic results on convex analysis over the modules, cf. [4, Theorems 3.7 and 3.8]. Besides, they also rediscovered the notions of random locally convex modules and random normed modules [4, 25], in particular they utilized their nice gauge function to show that the theory of Hausdorff locally $L^0$—convex modules are equivalent to that of random locally convex modules endowed with the locally $L^0$—convex topology, cf. [4, Theorem 2.4]. The results in [4, 25] enough exhibit the crucial importance of the locally $L^0$—convex topology. When the topology is applied to a random locally convex module we only need to notice that the algebra of equivalence classes of random variables is only a topological ring under its locally $L^0$—convex topology (since it is, in general, so strong that it is no longer a linear topology, as pointed out in [4, 25]) and similarly that a random locally convex module endowed with the locally $L^0$—convex topology is only a topological module over the topological ring. While the locally $L^0$—convex topology is much stronger (or, finer) than the $(\varepsilon, \lambda)$—topology, there are many attractive and exciting relations between the basic results and important properties derived from the two kinds of topologies for a random locally convex module, for example, all the random locally convex modules (in particular random normed modules) that have played important roles in both financial applications and theoretic studies have the same random conjugate spaces and completeness under the two topologies.

The central purpose of this paper is to exhibit the essential relations by making a comprehensive connection between our previous work of a random locally convex module endowed with the $(\varepsilon, \lambda)$—topology and Filipovic, Kupper and Vogelpoth’s basic work [4,25] of a locally
$L^0$-convex module (equivalently, a random locally convex module endowed with the locally $L^0$-convex topology). First, we give an extremely simple proof of the known Hahn-Banach extension theorem for $L^0$-linear functions as well as its continuous variants. Then we prove that our basic strict separation theorem implies Filipović, Kupper and Vogelpoth’s hyperplane separation theorem II but is independent of their hyperplane separation theorem I, also give a general variant of either of their separation theorems, allowing a separation with an arbitrary probability rather than the only probability one, and in the process obtain a useful and surprising fact that a random locally convex module with the countable concatenation property must have the same completeness under the two topologies! Based on the nice relation between the two kinds of completeness, observing that a random normed module has the same random conjugate space under the two kinds of topologies we further present the central part of this paper: we prove that most of the previously established deep results of random conjugate spaces of random normed modules under the $(\varepsilon, \lambda)$-topology are still valid under the locally $L^0$-convex topology; at the same time, motivated by an important example constructed by Filipović, Kupper and Vogelpoth for financial applications, in this part we may construct a surprisingly wide class of random normed modules and give a unified representation theorem of random conjugate spaces of them, in particular we can also prove that the representation is isometric, which generalizes and strengthens an important result of Kupper and Vogelpoth’s. In fact, seen from both our previous work of theoretic researches and the Filipovic, Kupper and Vogelpoth’s recent work [4,25], the theory of random normed modules together with their random conjugate spaces has been and will be the most important part of the theory of random locally convex modules together with their applications to conditional risk measures, thus Section 4 of this paper is, without doubt, the most important part of this paper, since this section has given most of the important and deep results of the theory of random conjugate spaces of random normed modules under the locally $L^0$-convex topology. Finally, the principal results of this paper enough convince people that the two kinds of topologies should be simultaneously considered in the future development of random locally convex modules together with their financial applications.

The remainder of this paper is organized as follows: In Subsections 1.1 and 1.2 of Section 1, as preliminaries we recapitulate some basic facts on the two kinds of topologies and random conjugate spaces of random locally convex modules, respectively; in the other subsections of Section 1 we give an outline of the main results of this paper; following Section 1 we state and prove our main results in Sections 2 to 4 of this paper according to the order of contents. Finally in Section 5 we conclude this paper with some further remarks explaining the reason for which the two kinds of topologies should be simultaneously considered in the future study of random locally convex modules.

1.1. The two kinds of topologies for a random locally convex module

To introduce the two kinds of topologies, let us recapitulate the related terminology and notation.

Throughout this paper, $(\Omega, \mathcal{F}, P)$ denotes a probability space, $K$ the scalar field $R$ of real numbers or $C$ of complex numbers, $\bar{R} = [-\infty, +\infty]$, $L^0(\mathcal{F}, \bar{R})$ the set of equivalence classes of extended real-valued $\mathcal{F}$-measurable random variables on $\Omega$, $L^0(\mathcal{F}, K)$ the algebra of equivalence classes of $K$-valued $\mathcal{F}$-measurable random variables on $\Omega$ under the ordinary scalar multiplication, addition and multiplication operations on equivalence classes.

It is well known from [3] that $L^0(\mathcal{F}, \bar{R})$ is a complete lattice under the ordering $\leq$: $\xi \leq \eta$ if and only if $\eta - \xi \in L^0(\mathcal{F}, \bar{R})$. The usual orderings $\leq$ and $\leq^*$ are defined by $\xi \leq \eta$ if $\xi - \eta \in L^0(\mathcal{F}, \bar{R})$ and $\xi \leq^* \eta$ if $\eta - \xi \in L^0(\mathcal{F}, \bar{R})$. The order topology $\tau^\leq$ on $L^0(\mathcal{F}, \bar{R})$ is defined by $\tau^\leq = \{ B : B \subset L^0(\mathcal{F}, \bar{R}) \text{ is a neighborhood of } 0 \}$.

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η iff \( \xi^0(\omega) \leq \eta^0(\omega) \), for \( P \)-almost all \( \omega \) in \( \Omega \) (briefly, a.s.), where \( \xi^0 \) and \( \eta^0 \) are arbitrarily chosen representatives of \( \xi \) and \( \eta \), respectively. Furthermore, every subset \( A \) of \( L^0(\mathcal{F}, \mathcal{R}) \) has a supremum, denoted by \( \vee A \), and an infimum, denoted by \( \wedge A \), and there exist a sequence \( \{a_n \mid n \in N\} \) and a sequence \( \{b_n \mid n \in N\} \) in \( A \) such that \( \forall n \geq 1 \ a_n = \vee A \) and \( \wedge n \geq 1 \ b_n = \wedge A \), where \( N \) denotes the set of positive integers. If, in addition, \( A \) is directed upwards (downwards), then the above \( \{a_n\} \) (\( \{b_n\} \)) can be chosen as nondecreasing (nonincreasing). Finally \( L^0(\mathcal{F}, \mathcal{R}) \), as a sublattice of \( L^0(\mathcal{F}, \mathcal{R}) \), is complete in the sense that every subset with an upper bound has a supremum.

Besides, throughout this paper we distinguish random variables from their equivalence classes by means of symbols: for example, \( I_A \) denotes the characteristic function of the \( \mathcal{F} \)-measurable set \( A \), then we use \( I_A \) for its equivalence class. It is necessary when we apply the theory of lifting property to random normed modules as in \([13, 14]\) and apply the theory of random normed modules to the theory of random operators, cf.\([14,18]\). Therefore, for a set \( A \) of random variables, esssup(\( A \)) and essinf(\( A \)) denote its respective essential supremum and infimum, they are still random variables, we also reserve esssup(\( \mathcal{E} \)) and essinf(\( \mathcal{E} \)) for the essential supremum and infimum of a subfamily \( \mathcal{E} \) of \( \mathcal{F} \), respectively, as in \([4]\).

Specially, \( L^0_+ = \{\xi \in L^0(\mathcal{F}, \mathcal{R}) \mid \xi \geq 0\} \), \( L^0_{++} = \{\xi \in L^0(\mathcal{F}, \mathcal{R}) \mid \xi > 0 \text{ on } \Omega\} \), where for \( A \in \mathcal{F} \), “\( \xi > \eta \) on \( A^\prime \)” means \( \xi^0(\omega) > \eta^0(\omega) \) a.s. on \( A \) for any chosen representatives \( \xi^0 \) and \( \eta^0 \) of \( \xi \) and \( \eta \), respectively. As usual, \( \xi > \eta \) means \( \xi \geq \eta \) and \( \xi \neq \eta \).

Notice that a \( K \)-valued \( P \)-measurable function is exactly an \( \mathcal{F} \)-measurable one, where \( \mathcal{F} \) denotes the completion of \( \mathcal{F} \) with respect to \( P \), then one can easily see that the symbol \( L(P, K) \) in \([11, 14, 19, 20, 22, 23, 24]\) amounts to \( L^0(\mathcal{F}, K) \), which is essentially identified with \( L^0(\mathcal{F}, K) \) as a set of equivalence classes. Besides, an \( \mathcal{F} \)-measurable function must be an \( \mathcal{F} \)-measurable one, and thus the following Definition 1.1 was employed in \([11, 14, 19, 20, 22, 23, 24]\) in a slightly general way.

In the following Definition 1.1, we adopt the terminologies “\( L^0 \)-seminorms and \( L^0 \)-norms” (they were defined as “Module-absolutely homogeneous random seminorms and random norms”, respectively, in our papers \([11, 13, 14, 19, 20, 22]\)) and notation “\( \| \cdot \| \)” from \([4, 25]\) for simplicity, but the essence of Definition 1.1 is the same as the original one used in \([11, 14, 20, 22, 24]\).

**Definition 1.1.** An ordered pair \((E, \mathcal{P})\) is called a random locally convex module over \( K \) with base \((\Omega, \mathcal{F}, \mathcal{P})\) if \( E \) is a left module over the algebra \( L^0(\mathcal{F}, K) \) and \( \mathcal{P} \) is a family of mappings from \( E \) to \( L^0_+ \) such that the following three axioms are satisfied:

(i) \( \forall \{\|x\| \mid \| \cdot \| \in \mathcal{P}\} = 0 \) iff \( x = \theta \) (the null element of \( E \));

(ii) \( \|x \cdot x\| = \|x\| \|x\|, \forall \xi \in L^0(\mathcal{F}, K), x \in E \) and \( \| \cdot \| \in \mathcal{P} \);

(iii) \( \|x + y\| \leq \|x\| + \|y\|, \forall x, y \in E \) and \( \| \cdot \| \in \mathcal{P} \).

Furthermore, a mapping \( \| \cdot \| : E \rightarrow L^0_+ \) satisfying (ii) and (iii) is called an \( L^0 \)-seminorm; in addition, if \( \|x\| = 0 \) also implies \( x = \theta \), then it is called an \( L^0 \)-norm, in which case \((E, \| \cdot \|)\) is called a random normed module (briefly, an RN module) over \( K \) with base \((\Omega, \mathcal{F}, \mathcal{P})\), which was first introduced in \([11]\), and is a special case of a random locally convex module when \( \mathcal{P} \) consists of the only one \( L^0 \)-norm \( \| \cdot \| \).

Before giving the following Remark 1.2, let us first mention the notion of an \( L^0 \)-normed module, which was introduced in \([25, \text{Definition 2.1}]\) and employed in \([4]\). An ordered pair \((E, \| \cdot \|)\) is called an \( L^0 \)-normed module if \( E \) is a left module over the ring \( L^0(\mathcal{F}, \mathcal{R}) \) and \( \| \cdot \| \) is a mapping from \( E \) to \( L^0_+ \) such that the following three axioms are satisfied:
Remark 1.2. In Definition 1.1, $E$ is, of course, a linear space over $K$, and the module multiplication is a natural extension of the scalar multiplication: $\alpha \cdot x = (\alpha \cdot 1) \cdot x$, $\forall \alpha \in K$ and $x \in E$, where 1 is the unit element of $L^0(F, K)$. Conversely, if $E$ is only a left module over the ring $L^0(F, K)$, then $E$ is again a left module over the algebra $L^0(F, K)$ if the scalar multiplication is defined by $\alpha \cdot x = (\alpha \cdot 1) \cdot x$, $\forall \alpha \in K$ and $x \in E$. Thus the notion of an $L^0$-normed module in [4, 25] is equivalent to that of an $RN$ module. One of advantages of our formulation of an $RN$ module will be reflected in Proposition 1.4 below when an $RN$ module is endowed with the $(\varepsilon, \lambda)$-topology, and this formulation means that it is an $RN$ space, and thus has more advantages: for example, we can often convert a problem of an $RN$ space to one of an $RN$ module, cf. [9,16], in particular, cf. [16, Lemma 3.2], and the theory of an $RN$ module can be applied to the theory of random linear operators and functional analysis in a direct and convenient fashion, cf. [7,18,12,15].

Example 1.3. $L^0(F, K)$ is an $RN$ module over $K$ with base $(\Omega, F, P)$ if it is endowed with the $L^0$-norm $\|x\| = |x|$, $\forall x \in L^0(F, K)$. It is well known that $L^0(F, K)$ is a topological algebra over $K$ endowed with the topology of convergence in probability $P$, a local base at $\theta$ of which is $\{N_0(\varepsilon, \lambda) | \varepsilon > 0, 0 < \lambda < 1\}$, where $N_0(\varepsilon, \lambda) = \{x \in L^0(F, K) | P(\omega \in \Omega ||x|_{(\omega)} < \varepsilon > 1 - \lambda)\}.$

Since B. Schweizer and A. Sklar introduced the $(\varepsilon, \lambda)$-topology into more abstract spaces—probabilistic metric spaces, namely they introduced “the topology of convergence in probability on probabilistic metric spaces”, in 1961, their idea is also suitable on many other occasions, cf.[26]. The following Proposition 1.4 is exactly a copy of their idea in the case of a random locally convex module.

Proposition 1.4 ([14, 20, 24, 22]). Let $(E, P)$ be a random locally convex module over $K$ with base $(\Omega, F, P)$. For any $\varepsilon > 0$, $0 < \lambda < 1$ and any finite subfamily $Q \subset P$, $\|x\|_Q$ always denotes the $L^0$-seminorm defined by $\|x\|_Q = \vee\{||x||_Q | x \in Q\}, \forall x \in E$, unless otherwise stated.
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converges in probability $P$ finite and $U_T$ are denoted by $A$ comprehensive connection between the basic results and properties derived from two kinds of topologies $\varepsilon, \lambda (\Omega \parallel \cdot \parallel \in P)$. There exists some endowed with their respective locally Hausdorff topologies $\{\parallel \cdot \parallel \in P\}$. Further, we have the following statements:

1. $L^0(\mathcal{F}, K)$ is a topological algebra over $K$ endowed with its $(\varepsilon, \lambda)$-topology, which is exactly the topology of convergence in probability $P$;

2. $E$ is a topological module over the topological algebra $L^0(\mathcal{F}, K)$ when $E$ and $L^0(\mathcal{F}, K)$ are endowed with their respective $(\varepsilon, \lambda)$-topologies;

3. A net $\{x_\alpha, \alpha \in \Lambda\}$ in $E$ converges in the $(\varepsilon, \lambda)$-topology to $x \in E$ iff $\{\|x_\alpha - x\|, \alpha \in \Lambda\}$ converges in probability $P$ to 0 for each $\| \cdot \| \in P$.

From now on, for all random locally convex modules, their $(\varepsilon, \lambda)$-topologies are denoted by $T_{\varepsilon, \lambda}$ unless there is a danger of confusion.

**Proposition 1.5** ([4]). Let $(E, \mathcal{P})$ be a random locally convex module over $K$ with base $(\Omega, \mathcal{F}, P)$. For any $\varepsilon \in L^0_+$ and $Q \subset \mathcal{P}$ finite, let $B_Q(\varepsilon) = \{x \in E \mid \|x\|_Q \leq \varepsilon\}$ and $U_0 = \{B_Q(\varepsilon)|Q \subset \mathcal{P}$ finite, $\varepsilon \in L^0_+\}$. A set $G \subset E$ is called $T_{E}$-open if for every $x \in G$ there exists some $B_Q(\varepsilon) \in U_0$ such that $x + B_Q(\varepsilon) \subset G$. Let $T_{\varepsilon}$ be the family of $T_{E}$-open subsets, then $T_{\varepsilon}$ is a Hausdorff topology on $E$, called the locally $L^0$-convex topology induced by $P$. Further, the following statements are true:

1. $L^0(\mathcal{F}, K)$ is a topological ring endowed with its locally $L^0$-convex topology;

2. $E$ is a topological module over the topological ring $L^0(\mathcal{F}, K)$ when $E$ and $L^0(\mathcal{F}, K)$ are endowed with their locally $L^0$-convex topologies;

3. A net $\{x_\alpha, \alpha \in \Lambda\}$ in $E$ converges in the locally $L^0$-convex topology to $x \in E$ iff $\{\|x_\alpha - x\|, \alpha \in \Lambda\}$ converges in the locally $L^0$-convex topology of $L^0(\mathcal{F}, K)$ to $\theta$ for each $\| \cdot \| \in P$.

From now on, for all random locally convex modules, their locally $L^0$-convex topologies are denoted by $T_{\varepsilon}$ unless there is a possible confusion.

$T_{\varepsilon}$ is called locally $L^0$-convex because it has a striking local base $U_0 = \{B_Q(\varepsilon) \mid Q \subset \mathcal{P}$ finite and $\varepsilon \in L^0_+\}$, each member $U$ of which satisfies the following:

(i) $L^0$-convex: $\xi \cdot x + (1 - \xi) \cdot y \in U$ for any $x, y \in U$ and $\xi \in L^0_+$ such that $0 \leq \xi \leq 1$;

(ii) $L^0$-absorvent: there is $\xi \in L^0_+$ for each $x \in E$ such that $x \in \varepsilon \cdot U$;

(iii) $L^0$-balanced: $\xi \cdot x \in U$ for any $x \in U$ and any $\xi \in L^0(\mathcal{F}, K)$ such that $|\xi| \leq 1$.

Such an ordered pair $(E, T_{\varepsilon})$ such that $T_{\varepsilon}$ possesses the above properties (i), (ii) and (iii) is called a locally $L^0$-convex module (see [4, Definition 2.2]); conversely, for every locally $L^0$-convex module $(E, T)$, $T$ can also be induced by a family of $L^0$-seminorms on $E$ as above, see [4, Theorem 2.4] for its proof. Thus the theory of Hausdorff locally $L^0$-convex modules amounts to the theory of random locally convex modules endowed with the locally $L^0$-convex topology.

**1.2. The random conjugate spaces of a random locally convex module under $T_{\varepsilon, \lambda}$ and $T_{\varepsilon}$**

Given a random locally convex module $(E, \mathcal{P})$ over $K$ with base $(\Omega, \mathcal{F}, P)$, generally $T_{\varepsilon}$ is much stronger than $T_{\varepsilon, \lambda}$. The random conjugate spaces of $(E, \mathcal{P})$ under $T_{\varepsilon, \lambda}$ and $T_{\varepsilon}$, however,
coincide if $P$ has the countable concatenation property, in particular an $RN$ module has the same random conjugate space under $\mathcal{T}_{\epsilon,\lambda}$ and $\mathcal{T}_c$. To see this, let $E^*_\epsilon,\lambda = \{f : E \to L^0(\mathcal{F},K) \mid f$ is a continuous module homomorphism from $(E,\mathcal{T}_{\epsilon,\lambda})$ to $(L^0(\mathcal{F},K),\mathcal{T}_{\epsilon,\lambda})\}$ and $E^*_\epsilon = \{f : E \to L^0(\mathcal{F},K) \mid f$ is a continuous module homomorphism from $(E,\mathcal{T}_c)$ to $(L^0(\mathcal{F},K),\mathcal{T}_c)\}$, which are the random conjugate spaces of $(E,P)$ under $\mathcal{T}_{\epsilon,\lambda}$ and $\mathcal{T}_c$, respectively. It should be noticed that $E^*_\epsilon$ was already used in [4] in anonymous way.

Let us recall from [4] that $P$ is called having the countable concatenation property if

$$\sum_{n \geq 1} I_{A_n} \cdot \| \cdot \|_Q, \text{ still belongs to } P \text{ for any countable partition } \{A_n \mid n \in N\} \text{ of } \Omega \text{ to } \mathcal{F} \text{ and any sequence } \{Q_n \mid n \in N\} \text{ of finite subfamilies of } P.$$ 

It is easy to see that a random linear functional $f : E \to L^0(\mathcal{F},K) \in E^*_\epsilon,\lambda$ if there are $\xi \in L^0_\epsilon$ and $Q \subset P$ finite such that $|f(x)| \leq \xi \cdot \|x\|Q, \forall x \in E$. In fact, let $f \in E^*_\epsilon$, then there exists some $B_Q(\epsilon)$ as in Proposition 1.5 such that $f(B_Q(\epsilon)) \subseteq \{\xi \in L^0(\mathcal{F},K) \mid \xi \leq 1\}$, so that we can have $|f(\frac{\|x\|\xi}{\|\xi\|})| \leq 1, \forall x \in E$ and $n \in N$, which means that $|f(x)| \leq \|\xi\|\|x\|Q, \forall x \in E$, where $\xi = 1/\epsilon$, and the converse is obvious. Lemma 4.1 of [25] is a special case of this result when $(E,P)$ is an $RN$ module.

It is, however, not trivial to characterize an element in $E^*_\epsilon,\lambda$, as shown in [24]:

**Proposition 1.6** ([24]). A random linear functional $f : E \to L^0(\mathcal{F},K) \in E^*_\epsilon,\lambda$ if there are a countable partition $\{A_n \mid n \in N\}$ of $\Omega$ to $\mathcal{F}$, a sequence $\{\xi_n \mid n \in N\}$ in $L^0_\epsilon$ and a sequence $\{Q_n \mid n \in N\}$ of finite subfamilies of $P$ such that $|f(x)| \leq \sum_{n \geq 1} I_{A_n} \cdot \xi_n \cdot \|x\|_{Q_n}, \forall x \in E$, in which case if, let $\xi = \sum_{n \geq 1} I_{A_n} \cdot \xi_n$ and $\|x\| = \sum_{n \geq 1} I_{A_n} \cdot \|x\|_{Q_n}, \forall x \in E$, then $|f(x)| \leq \xi \cdot \|x\|, \forall x \in E$.

Proposition 1.6 shows that $E^*_\epsilon,\lambda \supset E^*_\epsilon$, and $E^*_\epsilon,\lambda = E^*_\epsilon$ if $P$ has the countable concatenation property, in particular $E^*_\epsilon,\lambda = E^*_\epsilon$ for any $RN$ module $(E,\| \cdot \|)$.

$E^*_\epsilon,\lambda$ and $E^*_\epsilon$ are both left modules over the algebra $L^0(\mathcal{F},K)$ by $(\xi \cdot f)(x) = \xi \cdot (f(x))$, $\forall \xi \in L^0(\mathcal{F},K), f \in E^*_\epsilon,\lambda$ or $E^*_\epsilon$, and $x \in E$, the following Definition 1.7 shows that $E^*_\epsilon,\lambda = E^*_\epsilon$ is still a complete $RN$ module for an $RN$ module $(E,\| \cdot \|)$, see Section 1.4 for completeness.

**Definition 1.7** ([11]). Let $(E,\| \cdot \|)$ be an $RN$ module over $K$ with base $(\Omega,\alpha,\mu)$ and $E^* := E^*_\epsilon,\lambda = E^*_\epsilon$. Define $\| \cdot \|^* : E^* \to L^0_\epsilon$ by $\|f\|^* = \wedge\{\xi \in L^0_\epsilon \mid \|f(x)\| \leq \xi \cdot \|x\|, \forall x \in E\}$, then $(E^*,\| \cdot \|^*)$ is an $RN$ module over $K$ with base $(\Omega,\mathcal{F},P)$, called the random conjugate space of $(E,\| \cdot \|)$, which is complete under both $\mathcal{T}_{\epsilon,\lambda}$ and $\mathcal{T}_c$ (notice: $\|f\|^* = \vee\{|f(x)| \mid x \in E \text{ and } \|x\| \leq 1\}$, cf. [19]).

### 1.3. The countable concatenation closure of a set and hyperplane separation theorems.

The hyperplane separation Theorem 2.6 of [4] is peculiar to $\mathcal{T}_c$ since the gauge function perfectly matches $\mathcal{T}_c$, it is impossible to present it under $\mathcal{T}_{\epsilon,\lambda}$, as said in [22]. The hyperplane separation Theorem 2.8 of [4] and Theorem 3.1 of [22] are both a random generalization of the famous Mazur’s theorem, it is not difficult to prove that our Theorem 3.1 of [22] implies the separation Theorem 2.8 of [4], in particular, our Theorem 3.1 of [22] allows a kind of separation with an arbitrary probability, and thus it can, like the classical Mazur’s theorem, implies that an $L^0-$convex subset (an $M-$convex subset in terms of [11, 19, 22]) is $\mathcal{T}_{\epsilon,\lambda}$-closed iff it is random weakly closed (see Corollary 3.4 of [22]). But, the separation Theorem 2.8 of [4] can not derive such a kind of result, since it was only given in a form of separation with
probability one. Though the hyperplane separation Theorems 2.6 and 2.8 of [4] have succeeded in convex analysis for locally $L^0$–convex modules, we would like to generalize them to a more general form allowing a kind of separation with an arbitrary probability. In the process, the notion of a countable concatenation closure of a set will play a key role, see Section 3 of this paper.

1.4. Completeness

$\mathcal{T}_{ε,λ}$–completeness and $\mathcal{T}_c$–completeness are very important both in convex analysis for locally $L^0$–convex modules, cf.[25], and in the deep study of random conjugate spaces, cf.[16, 18, 19, 23]. It is easy to observe that $\mathcal{T}_{ε,λ}$–completeness of a random locally convex module $(E, P)$ implies its $\mathcal{T}_c$–completeness since $\mathcal{T}_c$ has a local base $\mathcal{U}_θ = \{B_Q(ε) \mid Q \subset P$ finite and $ε \in L^α_{+}\}$ as in Proposition 1.5 such that each $B_Q(ε)$ is $\mathcal{T}_{ε,λ}$–closed, it follows immediately from this observation that $E^*$ and $L^0(F, K)$ are $\mathcal{T}_c$–complete for an RN module $(E, \|\cdot\|)$ since it is very easy to verify that they are $\mathcal{T}_{ε,λ}$–complete, cf.[9, 18]. But it is a delicate matter whether $\mathcal{T}_c$–completeness also implies $\mathcal{T}_{ε,λ}$–completeness, since one will find that it is not easy to prove this. In the final part of Section 3, we combine the very interesting Theorem 3.18 of this paper and the usual completion skill so that the following statement can be obtained: $\mathcal{T}_c$–completeness of a random locally convex module with the countable concatenation property implies $\mathcal{T}_{ε,λ}$–completeness, where the countable concatenation property is different from the one in the sense of [4], and it is easy to verify that all the random normed modules currently used in conditional risk measures, cf.[25], have the new property.

1.5. The theory of random conjugate spaces of random normed modules under $\mathcal{T}_c$

In the past ten years, the theory of random conjugate spaces of random normed modules under $\mathcal{T}_{ε,λ}$ has been quite deep and systematic, cf.[8, 10, 16, 19, 23], and it is comparatively difficult to establish the results in [8, 10, 16, 19, 23]. But the corresponding theory under $\mathcal{T}_c$ is still at the beginning stage, e.g., the representation theorems of random conjugate spaces of the only $L^0(F, K^d)$ and $L^p(F, \mathcal{E})$ were given in [25]. Based on the above discussion of completeness, in Section 4 we prove that all the results obtained in [8, 10, 19, 23] are still valid under $\mathcal{T}_c$, but those in [16] is no longer valid under $\mathcal{T}_c$ since $\mathcal{T}_c$ is too strong. Further, motivated by the idea of Filipović, kupper and vogelpoth's constructing $L^p(F, \mathcal{E})$ and representing its random conjugate space, we construct $L^p(F, \mathcal{E})$ and represent $(L^p(F, \mathcal{E}))^*$ as $L^p(F)$ in an isomorphically isometric manner, where $F \subset \mathcal{E}$, $\mathcal{F}$ and $\mathcal{E}$ are both the $σ$–algebras over $Ω$, and $F$ is an arbitrary random normed module over $K$ with base $(Ω, \mathcal{E}, P)$. Specially, take $E = L^0(F, K)$, then we have $L^p(F, \mathcal{E}) = L^p(F)$, and thus generalize and strengthen the corresponding representation theorem of [4]. Consequently the whole Section 4 has established, on a large scale, the theory of random conjugate spaces of random normed modules under $\mathcal{T}_c$, which considerably enriches financial applications of random normed modules.

By the way, although there have been several proofs of the known Hahn-Banach extension theorem for $L^0$–linear functions, cf.[2, 4, 30], each of them has to face difficulties in the existence of “one step extension”, and thus comparatively complicated. In Section 2, we first give an extremely simple proof, which avoids the above difficulties by reducing an extension problem for an $L^0$–linear function on an $L^0$–submodule to one for a random linear functional on a linear subspace. We also consider the Hahn-Banach extension theorem for a random linear functional
on a complex subspace as well as an $L^0(\mathcal{F}, C)$–linear function. Finally, we conclude Section 2 with the Hahn-Banach extension theorems for continuous $L^0$–linear functions.

2. Hahn-Banach extension theorems

2.1. Hahn-Banach extension theorems for random linear functionals

Let us first recall from [13]: let $X$ be a linear space over $K$, then a linear operator $f$ from $X$ to $L^0(\mathcal{F}, K)$ is called a random linear functional on $X$; A mapping $p : X \to L^0_+$ is called a random seminorm on $X$ if it satisfies the following:

1. $p(\alpha x) = |\alpha| p(x), \forall \alpha \in K$ and $x \in X$;
2. $p(x + y) \leq p(x) + p(y), \forall x, y \in X$.

Let $X$ be a real linear space. A mapping $p : X \to L^0(\mathcal{F}, R)$ is called a random sublinear functional if it satisfies the above (2) and the following:

3. $p(\alpha x) = \alpha p(x), \forall \alpha \geq 0$ and $x \in X$;

The main results of the subsection are stated as follows:

**Theorem 2.1.** Let $E$ be a real linear space, $M \subset E$ a linear subspace, $f : M \to L^0(\mathcal{F}, R)$ a random linear functional and $p : E \to L^0(\mathcal{F}, R)$ a random sublinear functional such that $f(x) \leq p(x), \forall x \in M$. Then there exists a random linear functional $g : E \to L^0(\mathcal{F}, R)$ such that $g$ extends $f$ and $g(x) \leq p(x), \forall x \in E$.

**Theorem 2.2.** Let $E$ be a complex linear space, $M \subset E$ a complex linear subspace, $f : M \to L^0(\mathcal{F}, \mathbb{C})$ a random linear functional and $p : E \to L^0_+$ a random seminorm such that $|f(x)| \leq p(x), \forall x \in M$. Then there exists a random linear functional $g : E \to L^0(\mathcal{F}, \mathbb{C})$ such that $g$ extends $f$ and $|g(x)| \leq p(x), \forall x \in E$.

The proofs of Theorems 2.1 and 2.2 are given in [5, 6, 7, 13] but in an indirect way, see the survey paper [13] for details. In fact, Theorem 2.1 is known in [2, 30] since $R$ is, of course, an ordered ring, and its proof is only a copy of the classical Hahn-Banach extension theorem for a real linear functional by replacing the order-completeness of $R$ with the one of $L^0(\mathcal{F}, R)$. But the proof of Theorem 2.2 is somewhat different from its classical prototype, since $E$ is not an $L^0(\mathcal{F}, C)$–module. The idea of proof comes from [5], we give a direct proof of Theorem 2.2 for the reader’s convenience.

**Proof of Theorem 2.2.** Let $f_1 : M \to L^0(\mathcal{F}, R)$ be the real part of $f$, then it is easy to see that $f(x) = f_1(x) - if_1(ix), \forall x \in M$. Then it is clear that $f_1$ satisfies the following:

$$f_1(x) \leq p(x), \forall x \in M$$

(2.1)

Regarding $M$ and $E$ as a real linear spaces, then Theorem 2.1 yields a random linear functional $g_1 : E \to L^0(\mathcal{F}, R)$ such that $g_1$ extends $f_1$ and $g_1$ satisfies:

$$g_1(x) \leq p(x), \forall x \in E$$

(2.2)

Define $g : E \to L^0(\mathcal{F}, \mathbb{C})$ by $g(x) = g_1(x) - ig_1(ix), \forall x \in E$, then it is easy to check that $g$ is a random linear functional extending $f$, we want to prove that $g$ satisfies the following:

$$|g(x)| \leq p(x), \forall x \in E$$

(2.3)
Given \( x \in E \), let \( g^0_1(x) \), \( g^0(x) \) and \( p^0(x) \) be arbitrarily chosen representatives of \( g_1(x) \), \( g(x) \) and \( p(x) \), respectively. Then we have the following relations:

\[
\begin{align*}
g^0_1(x)(\omega) & \leq g^0(x)(\omega), \ a.s., \ \forall x \in E, \\
g^0_1(x)(\omega) & = Re(g^0(x)(\omega)), \ a.s., \ \forall x \in E, \\
g^0(x+y)(\omega) & = g^0(x)(\omega) + g^0(y)(\omega), \ a.s., \ \forall x, \ y \in E, \\
g^0(\alpha x)(\omega) & = \alpha \cdot g^0(x)(\omega), \ a.s., \ \forall \alpha \in C \text{ and } x \in E, \\
p^0(\alpha x)(\omega) & = |\alpha| \cdot p^0(x)(\omega), \ a.s., \ \forall \alpha \in C \text{ and } x \in E.
\end{align*}
\] (2.4)

(2.5)

(2.6)

(2.7)

(2.8)

It follows immediately from (2.7), (2.5), (2.4) and (2.8) that \( Re(\alpha \cdot g^0(x)(\omega)) = Re(g^0(\alpha x)(\omega)) = g^0_1(\alpha x)(\omega) \leq p^0(\alpha x)(\omega) = |\alpha| \cdot p^0(x)(\omega), \ a.s., \ \forall \alpha \in C \text{ and } x \in E. \)

Now, let \( x \) be an arbitrary but fixed element of \( E \) and \( \{c_n \mid n \in N\} \) a countable subset dense in \( C \). Then there exists \( A_n(x) \in \mathcal{F} \) for each \( n \in N \) such that \( P(A_n(x)) = 1 \) and \( Re(c_n \cdot g^0(x)(\omega)) \leq |c_n| \cdot p^0(x)(\omega), \ \forall \omega \in A_n(x). \)

Let \( A(x) = \cap_{n \in N} A_n(x). \) Then \( A(x) \in \mathcal{F}, \ P(A(x)) = 1 \) and the following is also true:

\[
Re(\alpha \cdot g^0(x)(\omega)) \leq |\alpha| \cdot p^0(x)(\omega), \ \forall \omega \in A(x) \text{ and } \alpha \in C,
\] (2.9)

since \( \{c_n \mid n \in N\} \) is dense in \( C. \)

Given \( \omega \in \Omega, \) let \( \theta(\omega) \) be the principal argument of \( g^0(x)(\omega) \), then \( \theta \) is a random variable and \( |g^0(x)(\omega)| = e^{-i\theta(\omega)} \cdot g^0(x)(\omega), \ \forall \omega \in \Omega. \)

It follows immediately from (2.9) that \( |g^0(x)(\omega)| = Re(|g^0(x)(\omega)|) = Re(e^{-i\theta(\omega)} \cdot g^0(x)(\omega)) \leq |e^{-i\theta(\omega)}| \cdot p^0(x)(\omega) = p^0(x)(\omega), \ \forall \omega \in A(x). \) Thus, we have \( |g^0(x)(\omega)| \leq p^0(x)(\omega), \ a.s., \) since \( P(A(x)) = 1. \) Namely, \( |g(x)| \leq p(x), \) which comes from the definition of the ordering \( \leq. \)

Finally, since \( x \) is arbitrary, then (2.3) has been proved. \( \Box \)

**Remark 2.3.** Let \( \mathcal{L}^0(\mathcal{F}, K) \) be the linear space of \( K \)-valued \( \mathcal{F} \)-measurable random variables on \((\Omega, \mathcal{F}, P)\) under the ordinary pointwise scalar multiplication and addition operations, it is easy to see that \( L^0(\mathcal{F}, K) \) is just the quotient space of \( \mathcal{L}^0(\mathcal{F}, K) \) under the equivalence relation \( \sim: \xi \sim \eta \text{ iff } \xi(\omega) = \eta(\omega), \ a.s. \) Namely, when elements equal a.s. are identified, \( \mathcal{L}^0(\mathcal{F}, K) \) is exactly \( L^0(\mathcal{F}, K). \) But some situations do not allow us to regard elements equal a.s. in \( \mathcal{L}^0(\mathcal{F}, K) \) as identified: for example, let \( g^0: E \to \mathcal{L}^0(\mathcal{F}, K) \) (by replacing \( C \) with \( K \)) be the mapping as defined in the proof of Theorem 2.2. Conditions (2.6) and (2.7) shows that \( g^0 \) is a kind of random linear operator (precisely, a random linear functional) as defined in [27]. If for each \( \omega \in \Omega, \) \( g^0(\cdot)(\omega): E \to K \) is a linear functional, then \( g^0 \) is called sample-linear, in which case \( g^0: E \to \mathcal{L}^0(\mathcal{F}, K) \) is an ordinary linear operator! To obtain a sample-linear random functional from a random linear functional, we often employ the theory of lifting property, cf. [28, 29], in all the cases when we consider a sample—linear random functional, we can not, of course, identify \( L^0(\mathcal{F}, K) \) with \( \mathcal{L}^0(\mathcal{F}, K). \) Sometimes, the domain of \( g^0 \) is not the whole \( E \) but a random subset in \( E, \) we are forced to use measurable selection theorems of [31], cf.[18]. Since the theories of [27, 28, 31] all require the completeness of \( \mathcal{F} \) with respect to \( P, \) but \( \mathcal{F} = \) the completion of \( \mathcal{F} \) with respect to \( P \) is, of course, complete, consequently we used to employ \( L(P, K) \) in [8-24], namely \( L^0(\mathcal{F}, K), \) so that we need not assume \( \mathcal{F} \) to be complete.

**Remark 2.4.** Before 1996, we employed the original definition of a random normed space, and thus the Hahn-Banach extension theorem for a random linear functional was given for such as \( g^0 \)s in Remark 2.3, in [5, 6, 7], which leads to a kind of theory of random conjugate space of
a random normed space, see [13, Section 3]. After 1996, in particular after 1999, the author gave a new version of a random normed space in [11], we began to consider the Hahn-Banach extension theorem for a random linear functional $g$ as in Theorems 2.1 and 2.2, which leads to the current frequently used theory of a random conjugate space, cf. [13, Section 4].

2.2. Hahn-Banach extension theorems for $L^0$–linear functions

Let $E$ be a left module over the algebra $L^0(\mathcal{F}, K)$ (equivalently, the ring $L^0(\mathcal{F}, K)$, see Remark 1.2 for details), a module homomorphism $f : E \to L^0(\mathcal{F}, K)$, is called an $L^0$–sublinear function such that $f(x) \leq p(x)$, $\forall x \in M$. Then there exists an $L^0$–linear function $g : E \to L^0(\mathcal{F}, R)$ such that $g$ extends $f$ and $g(x) \leq p(x)$, $\forall x \in E$.

**Theorem 2.5.** Let $E$ be a left module over the algebra $L^0(\mathcal{F}, R)$, $M \subset E$ an $L^0(\mathcal{F}, R)$–submodule, $f : M \to L^0(\mathcal{F}, R)$ an $L^0$–linear function and $p : E \to L^0(\mathcal{F}, R)$ an $L^0$–sublinear function such that $f(x) \leq p(x)$, $\forall x \in M$. Then there exists an $L^0$–linear function $g : E \to L^0(\mathcal{F}, R)$ such that $g$ extends $f$ and $g(x) \leq p(x)$, $\forall x \in E$.

**Theorem 2.6.** Let $E$ be a left module over the algebra $L^0(\mathcal{F}, C)$, $M \subset E$ an $L^0(\mathcal{F}, C)$–submodule, $f : M \to L^0(\mathcal{F}, C)$ an $L^0$–linear function and $p : E \to L^0$ an $L^0$–seminorm such that $|f(x)| \leq p(x)$, $\forall x \in E$. Then there exists an $L^0$–linear function $g : E \to L^0(\mathcal{F}, C)$ such that $g$ extends $f$ and $|g(x)| \leq p(x)$, $\forall x \in E$.

Theorem 2.6 follows immediately from Theorem 2.5, since $E$ is a module and the method used to prove its classical prototype will still be feasible. The following simple lemma can lead to an extremely simple proof of Theorem 2.5.

**Lemma 2.7.** Let $E$ be a left module over the algebra $L^0(\mathcal{F}, R)$, $f : E \to L^0(\mathcal{F}, R)$ a random linear functional and $p : E \to L^0(\mathcal{F}, R)$ an $L^0$–sublinear function such that $f(x) \leq p(x)$, $\forall x \in E$. Then $f$ is an $L^0$–linear function. If $R$ is replaced by $C$ and $p$ is an $L^0$–seminorm such that $|f(x)| \leq p(x)$, $\forall x \in E$, then $f$ is also an $L^0$–linear function.

**Proof.** We only give the proof of the first part, since the second one is similar.

Since $f$ is linear, it suffices to prove that $f(\xi x) = \xi f(x)$, $\forall \xi \in L^0$ and $x \in E$.

Let $x \in E$ be fixed. For any $\xi \in L^0$, since there exists a sequence $\{\xi_n | n \in N\}$ of simple elements in $L^0$ such that $\{\xi_n | n \in N\}$ converges a.s. to $\xi$ in a nondecreasing way, and since $|f(\xi x) - f(\xi_n x)| = |f(\xi - \xi_n) x| \leq (\xi - \xi_n)(|p(x)| \lor |p(-x)|)$, it also suffices to prove that $f(I_A x) = I_A \cdot f(x)$, $\forall x \in E$.

Since $-(I_A x) \leq f(\tilde{I}_A x) \leq p(I_A x)$, namely $-(\tilde{I}_A p) \leq f(\tilde{I}_A x) \leq \tilde{I}_A p$, we have $I_A \cdot f(I_A x) = 0$, $\forall A \in \mathcal{F}$, where $A^c = \Omega \setminus A$. Then we have that $f(I_A x) = I_A f(I_A x) + I_A \cdot f(I_A x) = I_A f(I_A x) + I_A f(I_A x) = I_A f(I_A x + I_A x) = I_A f(x)$. $\Box$

We can now prove Theorem 2.5 in an extremely simple way.

**Proof of Theorem 2.5.** Applying Theorem 2.1 to $f$ and $p$ yields a random linear functional $g : E \to L^0(\mathcal{F}, R)$ such that $g$ extends $f$ and $f(x) \leq p(x)$, $\forall x \in E$. Since $E$ is a left module over the algebra $L^0(\mathcal{F}, R)$ and $p$ is an $L^0$–sublinear function, Lemma 2.7 shows that $g$ is again
an $L^0$–linear function. □

**Remark 2.8.** The idea of proof of Theorem 2.5 comes from [7], which is also used in [20, 24].

### 2.3. Hahn-Banach extension theorems for continuous $L^0$–linear functions in random locally convex modules under the two kinds of topologies.

**Theorem 2.9.** Let $(E, \mathcal{P})$ be a random locally convex module over $K$ with base $(\Omega, \mathcal{F}, P)$ and $M$ an $L^0(\mathcal{F}, K)$–submodule of $E$. Then we have the following:

1. Every continuous $L^0$–linear function $f$ from $(M, \mathcal{T}_c)$ to $(L^0(\mathcal{F}, K), \mathcal{T}_c)$ admits a continuous $L^0$–linear extension $g$ from $(E, \mathcal{T}_c)$ to $(L^0(\mathcal{F}, K), \mathcal{T}_c)$.

2. Every continuous $L^0$–linear function $f$ from $(M, \mathcal{T}_{c,\lambda})$ to $(L^0(\mathcal{F}, K), \mathcal{T}_{c,\lambda})$ admits a continuous $L^0$–linear extension $g$ from $(E, \mathcal{T}_{c,\lambda})$ to $(L^0(\mathcal{F}, K), \mathcal{T}_{c,\lambda})$.

3. If $(E, \mathcal{P})$ is an RN module $(E, \| \cdot \|)$, then $g$ in both (1) and (2) can be required to be such that $\|g\| = \|f\|$, namely $g$ is a random-norm preserving extension.

**Proof.** (1). Let $\mathcal{P}_M = \{ \| \cdot \|_M | \| \cdot \| \in \mathcal{P} \}$, where $\| \cdot \|_M$ is the restriction of $\| \cdot \|$ to $M$, then $(M, \mathcal{P}_M)$ is still a random locally convex module. Since $f$ is a continuous $L^0$–linear function from $(M, \mathcal{T}_c)$ to $(L^0(\mathcal{F}, K), \mathcal{T}_c)$, there exists a finite subfamily $Q$ of $\mathcal{P}$ and $\xi \in L^0_+$ such that $|f(x)| \leq \xi \|x\|_Q$, $\forall x \in M$. Then Theorem 2.5 and Theorem 2.6 jointly yield an $L^0$–linear function $g : E \to L^0(\mathcal{F}, K)$ such that $g$ extends $f$ and $|g(x)| \leq \xi \|x\|_Q$, $\forall x \in E$. Of course, $g$ satisfies the requirement of (1).

(2). It follows immediately from Proposition 1.6 that there exists $\xi \in L^0_+$, a countable partition $\{ A_n | n \in N \}$ of $\Omega$ to $\mathcal{F}$ and a sequence $\{ Q_n | n \in N \}$ of finite subfamilies of $\mathcal{P}$ such that $|f(x)| \leq \xi \|x\|$, $\forall x \in M$, where $\| \cdot \| : E \to L^0_+$ is given by $\|x\| = \sum_{n \geq 1} I_{A_n} \|x\|_{Q_n}$, $\forall x \in E$.

Obviously, $\| \cdot \|$ is an $L^0$–seminorm, then Theorems 2.5 and 2.6 jointly yield an $L^0$–linear function $g : E \to L^0(\mathcal{F}, K)$ such that $|g(x)| \leq \xi \|x\|$, $\forall x \in E$. Further, since $\sum_{n \geq 1} P(A_n) = P(\Omega) = 1$, then $P(A_n) \to 0$ as $n \to \infty$, and hence $\| \cdot \|$ is continuous from $(E, \mathcal{T}_{c,\lambda})$ to $(L^0_+, \mathcal{T}_{c,\lambda})$, which implies that $g$ is also continuous.

(3). Let $p(x) = \|f\| \|x\|$, $\forall x \in E$, then Theorems 2.5 and 2.6 again jointly complete the proof.

**Remark 2.10.** Theorem 2.9 is given to tidy up the three diverse results in it. (2) of Theorem 2.9 is already given in [24] as a corollary of Proposition 1.6 but the proof in [24] did not use Theorems 2.5 and 2.6 but used the idea of proof of Lemma 2.7; (3) of Theorem 2.9 has been known for at least 20 years, as the Hahn-Banach extension theorem for a.s. bounded random linear functionals defined on a random normed space, which was given in [5]; Since such a $L^0$–seminorm $\| \cdot \|$ as in the proof of (2) of Theorem 2.9 is always $\mathcal{T}_{c,\lambda}$–continuous, we can, without loss of generality, assume that $\mathcal{P}$ has the countable concatenation property if we only consider $\mathcal{T}_{c,\lambda}$.
3. Hyperplane separation theorems

3.1. The countable concatenation hull, the countable concatenation property and relations between hyperplane separation theorems currently available.

Filipović, Kupper and Vogelpoht introduced in [4] the two countable concatenation properties, one is relative to topology and the other is relative to the family of $L^0$—seminorms, in particular the two are essentially the same one for a random locally convex module $(E, \mathcal{P})$ endowed with $\mathcal{T}$. Thus neither of the two has anything to do with the $L^0$—module $E$ itself. The main results in Sections 3 and 4 show that there is another kind of countable concatenation property, which is concerned with the $L^0$—module $E$ itself and is very important for the theory of locally $L^0$—convex module. To introduce it, we give the following:

First, we make the following convention that all the $L^0(\mathcal{F}, K)$—modules $E$ in the sequel of this paper satisfy the property: For any two elements $x$ and $y \in E$, if there exists a countable partition $\{A_n \mid n \in N\}$ of $\Omega$ to $\mathcal{F}$ such that $\tilde{I}_{A_n}x = \tilde{I}_{A_n}y$ for each $n \in N$, then $x = y$, where $\tilde{I}_{A}$ is called the $A$—stratification of $x$ for any $A \in \mathcal{F}$ (see [16, 22, 24] for some discussions of stratification structure). Clearly, a random locally convex module $(E, \mathcal{P})$ always satisfies the convention, since $\vee\{\|x - y\| \mid \|\cdot\| \in \mathcal{P}\} = \sum_{n \geq 1} \tilde{I}_{A_n}(\vee\{\|x - y\| \mid \|\cdot\| \in \mathcal{P}\}) = \sum_{n \geq 1} (\vee\{\|\tilde{I}_{A_n}x - \tilde{I}_{A_n}y\| \mid \|\cdot\| \in \mathcal{P}\}) = 0$ if $\tilde{I}_{A_n}x = \tilde{I}_{A_n}y, \forall n \in N$, then we have $x = y$ by the definition of a random locally convex module!

Definition 3.1. Let $E$ be a left module over the algebra $L^0(\mathcal{F}, K)$. Such a formal sum $\sum_{n \geq 1} \tilde{I}_{A_n}x_n$ for some countable partition $\{A_n \mid n \in N\}$ of $\Omega$ to $\mathcal{F}$ and some sequence $\{x_n \mid n \in N\}$ in $E$, is called a countable concatenation of $\{x_n \mid n \in N\}$ with respect to $\{A_n \mid n \in N\}$. Furthermore a countable concatenation $\sum_{n \geq 1} \tilde{I}_{A_n}x_n$ is well defined or $\sum_{n \geq 1} \tilde{I}_{A_n}x_n \in E$ if there is $x \in E$ such that $\tilde{I}_{A_n}x = \tilde{I}_{A_n}x_n, \forall n \in N$ (Clearly, $x$ is unique, in which case we write $x = \sum_{n \geq 1} \tilde{I}_{A_n}x_n$). A subset $G$ of $E$ is called having the countable concatenation property if every countable concatenation $\sum_{n \geq 1} \tilde{I}_{A_n}x_n$ with $x_n \in G$ for each $n \in N$ still belongs to $G$, namely $\sum_{n \geq 1} \tilde{I}_{A_n}x_n$ is well defined and there exists $x \in G$ such that $x = \sum_{n \geq 1} \tilde{I}_{A_n}x_n$.

Definition 3.2. Let $E$ be a left module over the algebra $L^0(\mathcal{F}, K)$ and $\{G_n \mid n \in N\}$ a sequence of subsets of $E$. The set of well defined countable concatenations $\sum_{n \geq 1} \tilde{I}_{A_n}x_n$ with $x_n \in G_n$ for each $n \in N$ is called the countable concatenation hull of the sequence $\{G_n \mid n \in N\}$, denoted by $H_{cc}(\{G_n \mid n \in N\})$. In particular when $G_n = G$ for each $n \in N$, $H_{cc}(G)$, denoting $H_{cc}(\{G_n \mid n \in N\})$, is called the countable concatenation hull of the subset $G$ (Clearly $H_{cc}(G) \supset G, \forall G \subset E$, and $H_{cc}(E) = E$).

Definition 3.1 leads to a notion of $E$ having the countable concatenation property as a subset of itself. To remove any possible confusions, we will reserve the terminologies of [4] in the following equivalent way:

Definition 3.3([4]). Let $(E, \mathcal{T})$ be a topological module over the topological ring $(L^0(\mathcal{F}, K), \mathcal{T}_c)$. $\mathcal{T}$ is called having the countable concatenation property if $H_{cc}(\{U_n \mid n \in N\})$ is again a neighborhood of $\theta$ (the null element of $E$) for every sequence $\{U_n \mid n \in N\}$ of neighborhoods of $\theta$. Let $(E, \mathcal{P})$ be a random locally convex module over $K$ with base $(\Omega, \mathcal{F}, P)$, $\mathcal{P}$ is called having the countable concatenation property if $\sum_{n \geq 1} \tilde{I}_{A_n} \cdot \|Q_n\$ still belongs to $\mathcal{P}$ for any countable partition $\{A_n \mid n \in N\}$ of $\Omega$ to $\mathcal{F}$ and any sequence $\{Q_n \mid n \in N\}$ of finite subfamilies of $\mathcal{P}$.

Example 3.4 below, adopted from [4], exhibits that it is very necessary to introduce the
A comprehensive connection between the basic results and properties derived from two kinds of topologies

notion of a well defined countable concatenation.

Example 3.4. For an $L^0(\mathcal{F}, K)$–module $E$, $M \subset E$ a subset of $E$, the $L^0(\mathcal{F}, K)$–module
generated by $M$ is denoted by $\text{Span}_L(M)$, namely $\text{Span}_L(M) = \left\{ \sum_{i=1}^{n} \xi_i x_i \mid \xi_i \in L^0(\mathcal{F}, K), x_i \in M, 1 \leq i \leq n \text{ and } n \in \mathbb{N} \right\}$. Take $E = L^0(\mathcal{F}, R)$, $M = \left\{ \tilde{I}_{[1-2^{-(n-1)},1-2^{-(n-2)}]} \mid n \in \mathbb{N} \right\}$ and $F = \text{Span}_L(M)$, then it is easy to see that $\sum_{n=1}^{\infty} 2^n \tilde{I}_{[1-2^{-(n-1)},1-2^{-(n-2)}]} \in E \setminus F$ and the sequence $\left( \sum_{n=1}^{k} 2^n \tilde{I}_{[1-2^{-(n-1)},1-2^{-(n-2)}]} \mid k \in \mathbb{N} \right)$ in $F$ is not a $\mathcal{T}_c$–Cauchy sequence but a $\mathcal{T}_{c,\lambda}$–Cauchy sequence which has no limit in $(F, \mathcal{T}_{c,\lambda})$. Thus the countable concatenation $\sum_{n=1}^{\infty} 2^n \tilde{I}_{[1-2^{-(n-1)},1-2^{-(n-2)}]}$ is not well defined in both $(F, \mathcal{T}_c)$ and $(F, \mathcal{T}_{c,\lambda})$ in any ways, namely neither of $(F, \mathcal{T}_c)$ and $(F, \mathcal{T}_{c,\lambda})$ has the countable concatenation property.

However, the following examples show that all the $L^0(\mathcal{F}, K)$–modules important for financial applications have the countable concatenation property.

Example 3.5. $L^p_x(\mathcal{E})$ in [4,25] and the Orlicz type of RN module $L^\phi_x(\mathcal{E})$ in [25] both have the countable concatenation property, which is easily seen by the smooth property of the conditional expectation of random variables.

Example 3.6. Let $E$ be an $L^0(\mathcal{F}, K)$–module and $E^\# = \{ f : E \to L^0(\mathcal{F}, K) \mid f \text{ is a module homomorphism} \}$, then it is easy to check that $E^\#$ has the countable concatenation property. Further, let $(E, \mathcal{P})$ be a random locally convex module over $K$ with base $(\Omega, \mathcal{F}, P)$, $E^\ast_x$ and $E^\ast_{\mathcal{T}_c,\lambda}$ the corresponding random conjugate spaces, see Section 1.2. Then $E^\ast_x$ always has the countable concatenation property by Proposition 1.6, and $E^\ast_{\mathcal{T}_c,\lambda}$ has the countable concatenation property if $\mathcal{P}$ has the countable concatenation property by the paragraph in front of Proposition 1.6.

Example 3.7 below, in principle, surveys an extremely useful skill in the history of the development of a $\mathcal{T}_{c,\lambda}$–complete random locally convex module, which played a crucial role in [8,10,16,17,19,22,23]. The central idea of this method is described as follows: To seek one desired element $x$ in a $\mathcal{T}_{c,\lambda}$–complete random locally convex module $E$, we are forced to first find out a sequence $\{ x_n \mid n \in \mathbb{N} \}$ in $E$ such that each $x_n$ is the $A_n$–stratification of $x$ and $\{ A_n \mid n \in \mathbb{N} \}$ exactly forms a countable partition of $\Omega$ to $\mathcal{F}$, since $\{ \sum_{k=1}^{n} \tilde{I}_{A_n} x_n \mid k \in \mathbb{N} \}$ is easily verified to be a $\mathcal{T}_{c,\lambda}$–Cauchy sequence since $P(A_n) \to 0$ as $n \to \infty$, whose limit is just the desired $x$!

Example 3.7. Every $\mathcal{T}_{c,\lambda}$–complete random locally convex module $(E, \mathcal{P})$ has the countable concatenation property: since for every countable concatenation $\sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n$, $\{ \sum_{n=1}^{k} \tilde{I}_{A_n} x_n \mid k \in \mathbb{N} \}$ is a $\mathcal{T}_{c,\lambda}$–Cauchy sequence, and hence convergent to some $x \in E$ so that $\tilde{I}_{A_n} x_n = \tilde{I}_{A_n} x$, $\forall x \in E$, namely $\sum_{n=1}^{\infty} \tilde{I}_{A_n} x_n$ is well defined.

Now, we will use the notions introduced above to study the relations between hyperplane separation theorems currently available. To do this, we first state the three hyperplane separation theorems, the first two in [4] and the third in [22], in the following equivalent ways.

Theorem 3.8 ([4, Theorem 2.6, Hyperplane separation I]). Let $(E, \mathcal{P})$ be a random locally convex module over $R$ with base $(\Omega, \mathcal{F}, P)$, $M$ and $G$ be $L^0$–convex subsets of $E$ such that $G$ is a nonempty $\mathcal{T}_c$–open subset. If $G$ and $M$ satisfy the following:

$$\tilde{I}_A M \cap \tilde{I}_A G = \emptyset \text{ for all } A \in \mathcal{F} \text{ with } P(A) > 0,$$

then there exists an $f \in E^\ast_x$ such that

$$f(x) < f(y) \text{ on } \Omega \text{ for all } x \in G \text{ and } y \in M.$$  

(3.10)  

(3.11)

Remark 3.9. If $R$ is replaced by $C$, then Theorem 3.8 still holds in the way: $(Ref)(x) <
(Ref)(y) on Ω for all \( x \in G \) and \( y \in M \), where \( \text{Ref} : E \to L^0(\mathcal{F}, R) \) defined by \( (\text{Ref})(x) = \text{Re}(f(x)), \forall x \in E \), is the real part of \( f \), it is easy to see that \( f(x) = (\text{Ref})(x) - i(\text{Ref})(ix), \forall x \in E \).

**Theorem 3.10** ([4, Theorem 2.8, Hyperplane separation II]). Let \((E, \mathcal{P})\) be a random locally convex module over \( R \) with base \((\Omega, \mathcal{F}, P)\) such that \( \mathcal{P} \) has the countable concatenation property, \( x \in E \) and \( G \subset E \) a nonempty \( \mathcal{T}_\varepsilon \)-closed \( L^0 \)-convex subset with the countable concatenation property. If \( x \) and \( G \) satisfy the following:

\[
\tilde{I}_A\{x\} \cap \tilde{I}_A G = \emptyset \quad \text{for all } A \in \mathcal{F} \text{ with } P(A) > 0, \tag{3.12}
\]

then there exist an \( f \in E_\varepsilon^* \) and \( \varepsilon \in L^0_{++} \) such that

\[
f(x) > f(y) + \varepsilon \quad \text{on } \Omega \text{ for all } y \in G. \tag{3.13}
\]

Furthermore, if \( R \) is replaced by \( C \), then Theorem 3.10 is still true in the way: \( (\text{Ref})(x) > (\text{Ref})(y) + \varepsilon \quad \text{on } \Omega \text{ for all } y \in G. \)

**Remark 3.11.** In [4], the original Theorem 2.8 of [4] did not require \( G \) to have the countable concatenation property, but Lemma 3.17 below plays a key role in the proof of Theorem 2.8 of [4] and the proof of Lemma 3.17 seems to ask \( G \) to have such a property. On the other hand, requiring \( G \) to have the countable concatenation property would not reduce the values of Theorem 3.10 and Lemma 3.17. Take, the important application of Theorem 3.10 to Theorem 3.8 of [4] and the important application of Lemma 3.17 to Lemma 3.10 of [4], for example, once the whole spaces \( E \) in both Lemma 3.10 and Theorem 3.8 of [4] are assumed to have the countable concatenation property, then Theorem 3.10 is applicable to \( \text{epi} f \) in the proof of Theorem 3.8 of [4] and Lemma 3.17 is applicable to \( V \) in the proof of Lemma 3.10 since in the two cases they automatically have the countable concatenation property, in particular locally \( L^0 \)-convex modules currently useful in financial applications all have such a property.

For a random locally convex module \((E, \mathcal{P})\), \( x \in E \) and \( G \subset E \), let \( d_\Omega^*(x, G) = \bigwedge \{|x - y|_\Omega \mid y \in G\} \) for any finite subfamily \( \mathcal{Q} \) of \( \mathcal{P} \) and \( d^*(x, G) = \bigvee \{d_\Omega^*(x, G) \mid \mathcal{Q} \subset \mathcal{P} \text{ finite}\} \). If \( G \) is a nonempty \( \mathcal{T}_{\varepsilon, \lambda} \)-closed \( L^0 \)-convex subset, then we proved in [22] that \( x \in G \) iff \( d^*(x, G) = 0 \).

**Theorem 3.12** ([22, Theorem 3.1]). Let \((E, \mathcal{P})\) be a random locally convex module over \( R \) with base \((\Omega, \mathcal{F}, P)\), \( x \in E \), \( G \) a nonempty \( \mathcal{T}_{\varepsilon, \lambda} \)-closed \( L^0 \)-convex subset of \( E \) such that \( x \not\in G \), and \( \xi \) a chosen representative of \( d^*(x, G) \). Then there exists an \( f \in E_{\varepsilon, \lambda}^* \) such that

\[
f(x) > \bigvee \{f(y) \mid y \in G\} \quad \text{on } [\xi > 0], \text{ where } [\xi > 0] = \{\omega \in \Omega \mid \xi(\omega) > 0\}, \tag{3.14}
\]

and

\[
f(x) > \bigvee \{f(y) \mid y \in G\}. \tag{3.14}'
\]

If \( R \) is replaced by \( C \), then Theorem 3.12 still holds in the following way:

\[
(\text{Ref})(x) > \bigvee \{(\text{Ref})(y) \mid y \in G\} \quad \text{on } [\xi > 0], \tag{3.15}
\]

and

\[
(\text{Ref})(x) > \bigvee \{(\text{Ref})(y) \mid y \in G\}. \tag{3.15}'
\]

**Theorem 3.13.** Theorem 3.12 is equivalent to Theorem 3.1 of [22].

**Proof.** Since (3.14) and (3.14)' are just (3.15) and (3.15)', respectively, in the case of a real space, we only need to check that (3.15)' and (3.15) are equivalent to (1) and (2) of Theorem 3.1 of [22], respectively.

Let \( \xi, (\text{Ref})(x) \) and \( \bigvee \{(\text{Ref})(y) \mid y \in G\} \) be the same ones as in Theorem 3.12, and let \( \eta \) and \( r \) be arbitrarily chosen representatives of \( (\text{Ref})(x) \) and \( \bigvee \{(\text{Ref})(y) \mid y \in G\} \), respectively. Then
(1) and (2) of Theorem 3.1 of [22] are equivalent to the following (3.16) and (3.17), respectively:

\[(Ref)(x) > \sqrt{(Ref)(y) \mid y \in G},\]

and \(P(\eta > r) \triangle \xi > 0) = 0,\)

where \([\eta > r] = \{ω \in Ω \mid \eta(ω) > r(ω)\}, \) and \([\eta > r] \triangle [\xi > 0] \) denotes the symmetric difference of \([\eta > r] \) and \([\xi > 0].\)

Clearly, (3.16) is just (3.15)', and thus we only need to prove that (3.17) is equivalent to (3.15) of Theorem 3.12.

(3.17) implies, of course, (3.15) of Theorem 3.12. On the other hand, (3.15) of Theorem 3.12 shows that \([\eta > r] \supset [\xi > 0], a.s., we will prove \([\eta > r] \subset [\xi > 0], a.s., as follows.

Otherwise, let \(D = [\eta > r] \setminus [\xi > 0], \) then \(P(D) > 0. \) Thus \(I_D(\omega) \cdot \xi(\omega) = 0, a.s., namely \(I_D \cdot d^*(x, G) = 0, \) so that \(d^*(I_D x, I_D G) = 0, \) but from the following Lemma 3.14, we have \(I_D x \in \tau_{\epsilon, \lambda}^{-} \) closure of \(I_D G, \) which means that \(I_D(Ref)(x) = (Ref)(I_D x) = \sqrt{(Ref)(I_D y) \mid y \in G} = I_D \cdot (\sqrt{(Ref)(y) \mid y \in G}), \) namely \(I_D(\omega)\eta(\omega) = I_D(\omega)r(\omega), a.s., \) which contradicts the fact that \(\eta(\omega) > r(\omega), \) a.s. on \(D. \) □

In the sequel of this paper, for a subset \(G \) of a random locally convex module \((E, \mathcal{P}), \tilde{G}_{\epsilon, \lambda}, \) denotes the \(\tau_{\epsilon, \lambda}^{-} \) closure of \(G, \) and \(G_{\epsilon} \) the \(\tau_{\epsilon}^{-} \) closure of \(G. \) For any \(x \in E, \) any finite subfamily \(Q \) of \(\mathcal{P} \) and \(\epsilon \in L_{q}^{0} \), let \(U_{Q, \epsilon}[x] = \{y \in E \mid \|x - y\| \leq \epsilon\}, \epsilon_{Q}^{*}(x, G) = \wedge \{\epsilon \in L_{q}^{0} \mid U_{Q, \epsilon}[x] \cap G \neq \emptyset\} \) and \(\epsilon(x, G) = \sqrt{\epsilon_{Q}^{*}(x, G) \mid Q \subset \mathcal{P} \text{ finite}.} \)

**Lemma 3.14.** Let \((E, \mathcal{P}) \) be a random locally convex module over \(K \) with base \((\Omega, \mathcal{F}, \mathcal{P}), x \in E \) and \(G \) a nonempty subset of \(E. \) then we have the following:

1. \(d^*(x, G) = \epsilon^*(x, G);\)
2. \(d^*(x, G) = d^*(x, \tilde{G}_{\epsilon, \lambda}) = d^*(x, \tilde{G}_{\epsilon});\)

If, in addition, \(G \) satisfies the following:

\[\tilde{I}_{A_{y}} + \tilde{I}_{A_{z}} \in G \quad \text{for all } A \in \mathcal{F} \text{ and all } y, z \in G, \]

then we have the following:

3. \(x \in \tilde{G}_{\epsilon, \lambda} \iff d^*(x, G) = 0. \)

**Proof.** (1). We only need to check that \(d_{Q}^{*}(x, G) = \epsilon_{Q}^{*}(x, G) \) for each \(Q \subset \mathcal{P} \text{ finite}. \) By definition, \(d_{Q}^{*}(x, G) = \wedge \{\|x - y\| \mid y \in G\} \) and \(\epsilon_{Q}^{*}(x, G) = \wedge \{\epsilon \in L_{q}^{0} \mid U_{Q, \epsilon}[x] \cap G \neq \emptyset\}. \) If \(\epsilon \in L_{q}^{0} \) is such that \(U_{Q, \epsilon}[x] \cap G \neq \emptyset, \) namely there exists \(y \in G \) such that \(\|x - y\| \leq \epsilon, \) which means that \(\epsilon \geq \wedge \{\|x - y\| \mid y \in G\} = d_{Q}^{*}(x, G), \) so that \(\epsilon_{Q}^{*}(x, G) = d_{Q}^{*}(x, G). \) In the other direction, for any \(y \in G \) and \(n \in N, \) it is clear that \(\|x - y\| \leq \|x - y\| + \frac{1}{n}, \) and \(\|x - y\| + \frac{1}{n} \in L_{q}^{0}, \) if, \(\epsilon = \|x - y\| + \frac{1}{n}, \) then we have that \(y \in U_{Q, \epsilon}[x] \cap G, \) of course, \(U_{Q, \epsilon}[x] \cap G \neq \emptyset, \) and we thus have that \(\epsilon = \|x - y\| + \frac{1}{n} \geq \wedge \{\epsilon \in L_{q}^{0} \mid U_{Q, \epsilon}[x] \cap G \neq \emptyset\}, \forall n \in N, \) so that \(\|x - y\| \geq \epsilon_{Q}^{*}(x, G), \) in turn \(d_{Q}^{*}(x, G) = \wedge \{\|x - y\| \mid y \in G\} \geq \epsilon_{Q}^{*}(x, G). \)

(2) is clear.

(3). Lemma 2.2 of [22] shows that \(x \in F \iff d^*(x, F) = 0 \) for every \(\tau_{\epsilon, \lambda}^{-} \) closed \(L_{q}^{0} \)-convex subset \(F \) of \(E, \) in which proof the only property (3.18) of an \(L_{q}^{0} \)-convex subset was used, and thus we have that \(x \in F \iff d^*(x, F) = 0 \) for every \(\tau_{\epsilon, \lambda}^{-} \) closed subset \(F \) with the property (3.18). It is easy to see that \(\tilde{G}_{\epsilon, \lambda} \) also has the property (3.18) if \(G \) does. Applying the result to \(\tilde{G}_{\epsilon, \lambda} \) yields that \(x \in \tilde{G}_{\epsilon, \lambda} \iff d^*(x, \tilde{G}_{\epsilon, \lambda}) = 0, \) so that (2) has implied that \(x \in \tilde{G}_{\epsilon, \lambda}, \) iff \(d^*(x, G) = 0. \) □
Lemma 3.17. Let \((G, \bar{G})\) and \(\xi\) be a random locally convex module over \(R\) with base \((\Omega, \mathcal{F}, P)\), \(x \in E\) and \(G\) an \(L^0\)-convex subset of \(E\) such that \(d^*(x, G) > 0\). Then there exists an \(f \in E^*_c,\lambda\) such that
\[
 f(x) > \vee\{ f(y) \mid y \in G \} \quad \text{on} \quad [\xi > 0],
\]
and
\[
 f(x) > \vee\{ f(y) \mid y \in G \}.
\]
Where \(\vee\) is an arbitrarily chosen representative of \(d^*(x, G)\).

Furthermore, if \(R\) is replaced by \(C\), then Theorem 3.15 still holds in the following way:
\[
 (Ref)(x) > \vee\{ (Ref)(y) \mid y \in G \} \quad \text{on} \quad [\xi > 0],
\]
and
\[
 (Ref)(x) > \vee\{ (Ref)(y) \mid y \in G \}. \tag{3.19}''
\]

Theorem 3.15. Let \((E, P)\) be a random locally convex module over \(R\) with base \((\Omega, \mathcal{F}, P)\), \(x \in E\) and \(G\) an \(L^0\)-convex subset of \(E\) such that \(d^*(x, G) > 0\). Then there exists an \(f \in E^*_c,\lambda\) such that
\[
 f(x) > \vee\{ f(y) \mid y \in G \} \quad \text{on} \quad [\xi > 0], \tag{3.19}''
\]
and
\[
 f(x) > \vee\{ f(y) \mid y \in G \}.
\]

Theorem 3.16. Theorem 3.15 (equivalently, Theorem 3.12) implies Theorem 3.10.

Proof. If \(x\) and \(G\) satisfy the conditions of Theorem 3.10, then the following Lemma 3.17 shows that \(d^*(x, G) > 0\) on \(\Omega\), namely \([\xi > 0]\) in Theorem 3.15 is just \(\Omega\), so there exists an \(f \in E^*_c,\lambda\) such that
\[
 f(x) > \vee\{ f(y) \mid y \in G \} \quad \text{on} \quad [\xi > 0],
\]
then \(\varepsilon \in L^0_{++} \) satisfies the following:
\[
 f(x) > \vee\{ f(y) \mid y \in G \} + \varepsilon, \tag{3.20}
\]
\(f\), of course, satisfies the requirement of Theorem 3.10 if one notices that \(f\) is also in \(E^*_c\) since \(E^*_c = E^*_c,\lambda\) by observing that \(P\) has the countable concatenation property. □

The following Lemma 3.17 occurred in [4, p.4015] where an outline of its idea of proof was also given, we give its proof in detail to find out the following Theorem 3.18.

Lemma 3.17. Let \((E, P)\) be a random locally convex module over \(R\) with base \((\Omega, \mathcal{F}, P)\), \(x \in E\) and \(G \subset E\) a \(\mathcal{T}_c\)-closed nonempty subset such that \(\bar{I}_A\{x\} \cap \bar{I}_A G = \emptyset\) for all \(A \in \mathcal{F}\) with \(P(A) > 0\) and such that \(G\) has the countable concatenation property. Then \(d^*(x, G) = \varepsilon^*(x, G) > 0\) on \(\Omega\), namely \(\varepsilon^*(x, G) \wedge 1 \in L^0_{++}\).

Proof. If it is not true that \(d^*(x, G) > 0\) on \(\Omega\), then there is an \(A \in \mathcal{F}\) such that \(P(A) > 0\) and \(\bar{I}_A \cdot d^*(x, G) = 0\), so that \(\bar{I}_A \cdot d^2_0(x, G) = 0\) for every finite subfamily \(Q\) of \(P\).

We can, without loss of generality, assume that \(\theta \in G\) (otherwise, by a translation). Since \(G\) has the countable concatenation property, it must have the property (3.18) so that the \(\{\|\bar{I}_A x - \bar{I}_A y\|_Q \mid y \in G\}\) is directed downwards for each \(Q \subset P\) finite, and it is also easy to see from the property (3.18) that \(\bar{I}_A G \subset G\), so that we can easily prove that \(\bar{I}_A G\) is also \(\mathcal{T}_c\)-closed.

Now, for each fixed \(Q \subset P\) finite and each fixed \(\alpha \in L^0_{++}\), we will prove that there is \(y_{Q, \alpha} \in G\) such that \(\|\bar{I}_A x - \bar{I}_A y_{Q, \alpha}\|_Q \leq \alpha\) as follows.
Since \( \hat{I}Ad_Q^2(x,G) = \wedge\{\|\hat{I}Ax - \hat{I}Ay\|_Q \mid y \in G \} = 0 \), then there exists a sequence \( \{y_n \mid n \in N \} \) in \( G \) such that \( \{\|\hat{I}Ax - \hat{I}Ay_n\|_Q \mid n \in N \} \) converges to 0 in a nonincreasing way. Let \( \varepsilon_n = \|\hat{I}Ax - \hat{I}Ay_n\|_Q \) for each \( n \in N \) and choose a representative \( \varepsilon_{n+1}^0 \) of \( \varepsilon_n \) for each \( n \in N \) such that \( \varepsilon_n^0(\omega) \geq \varepsilon_{n+1}^0(\omega) \) for each \( n \in N \) and \( \omega \in \Omega \), and a representative \( \alpha \) of \( \varepsilon_n^0(\omega) \) such that \( \varepsilon_n^0(\omega) \leq \alpha(\omega) \), denoted by \( E_n = \{\omega \in \Omega \mid \varepsilon_n^0(\omega) \leq \alpha(\omega)\} \). Again let \( A_n = E_n/E_{n-1}, n \geq 1 \), where \( E_0 = \emptyset \), then \( \{A_n \mid n \in N \} \) forms a countable partition of \( \Omega \) to \( F \), so that \( y_{Q,x} := \sum_{n \geq 1} A_n y_n \in G \) and satisfies that \( \|\hat{I}Ax - \hat{I}Ay_{Q,x}\|_Q \leq \alpha \).

Finally, let \( \Gamma = \mathcal{F}(\mathcal{P}) \times L_{+1} = \{(Q,\alpha) \mid Q \subset \mathcal{P} \text{ finite and } \alpha \in L_{+1} \} \), where \( \mathcal{F}(\mathcal{P}) \) is the set of finite subfamilies of \( \mathcal{P} \). It is easy to see that \( \Gamma \) is directed upwards by the ordering: \( (Q_1,\alpha_1) \leq (Q_2,\alpha_2) \) iff \( Q_1 \subset Q_2 \) and \( \alpha_2 \leq \alpha_1 \), so that \( \{\hat{I}A_{yQ,x} \mid (Q,\alpha) \in \Gamma \} \) is a net in \( \hat{I}AG \) which is convergent to \( \hat{I}Ax \), and hence \( \hat{I}Ax \in \hat{I}AG \), which contradicts the fact that \( \hat{I}Fx \notin \hat{I}F \) for all \( F \in \mathcal{F} \) with \( P(F) > 0 \).

From the process of proof of Lemma 3.17, we can easily see that if \( G \) has the countable concatenation property and \( d^*(x,G) = 0 \) (at which time, take \( A = \Omega \)) then \( x \in \bar{G}_c \), this yields a useful fact, namely Theorem 3.18 below, from which we have that a subset having the countable concatenation property has the same closure under \( \mathcal{T}_c \) and \( \mathcal{T}_{c,\lambda} \), in particular it is \( \mathcal{T}_{c,\lambda} \)-closed iff it is \( \mathcal{T}_c \)-closed, which also derives a surprising fact on \( \mathcal{T}_c \)-completeness, see Subsection 3.4.

**Theorem 3.18.** Let \( (E,\mathcal{P}) \) be a random locally convex module, \( x \in E \) and \( G \subset E \) a subset having the countable concatenation property. Then the following are equivalent:

1. \( x \in \bar{G}_c \);
2. \( x \in \bar{G}_{c,\lambda} \);
3. \( d^*(x,G) = 0 \).

Theorem 3.12 (namely, Theorem 3.15) has been known for many years, which was mentioned in [21] without proof, where a special case of which was proved. We may say that Theorem 3.12 is general enough to meet all our needs under \( \mathcal{T}_{c,\lambda} \), and it implies Theorem 3.10 but is independent of Theorem 3.8! To generalize Theorems 3.8 and 3.10 to meet our further needs of Subsection 3.3, we present the notion of a countable concatenation closure (see Definition 3.19 below) to give an interesting purely algebraic result as follows, which provides a geometric intuition on the conditions imposed on Theorems 3.8 and 3.10, namely \( \hat{I}AG \cap \hat{I}AM = \emptyset \) and \( \hat{I}A \{x\} \cap \hat{I}AG = \emptyset \) for all \( A \in \mathcal{F} \) with \( P(A) > 0 \), respectively.

**Definition 3.19.** Let \( E \) be a left module over the algebra \( L^0(\mathcal{F},K) \). Two countable concatenations \( \sum_{n \geq 1} \hat{I}A_n x_n \) and \( \sum_{n \geq 1} \hat{I}B_n y_n \) are called equal if \( \hat{I}A_n \cap B_j x_i = \hat{I}A_n \cap B_j y_j, \forall i,j \in N \). For any subset \( G \) of \( E \), the set \( C_{cc}(G) = \{\sum_{n \geq 1} \hat{I}A_n x_n \mid \sum_{n \geq 1} \hat{I}A_n x_n \text{ is a countable concatenation} \text{ and each } x_n \in G \} \) is called the countable concatenation closure of \( G \).

**Theorem 3.20.** Let \( E \) be a left module over the algebra \( L^0(\mathcal{F},K), M \) and \( G \) any two nonempty subsets of \( E \) such that \( \hat{I}AM \cup \hat{I}A \subset M \) and \( \hat{I}AG \cup \hat{I}A \subset G \). If \( C_{cc}(M) \cap C_{cc}(G) = \emptyset \), then there exists an \( \mathcal{F} \)-measurable subset \( H(M,G) \) unique a.s. such that the following are satisfied:

1. \( P(H(M,G)) > 0 \);
2. \( \hat{I}A M \cap \hat{I}AG = \emptyset \) for all \( A \in \mathcal{F}, A \subset H(M,G) \) with \( P(A) > 0 \);
3. \( \hat{I}AM \cap \hat{I}AG \neq \emptyset \) for all \( A \in \mathcal{F}, A \subset \Omega \backslash H(M,G) \) with \( P(A) > 0 \).

**Proof.** Let \( E = \{A \in \mathcal{F} \mid \hat{I}AM \cap \hat{I}AG \neq \emptyset \} \). Then \( E \) is directed upwards: in fact, for any \( A \) and \( B \in E \) there exist \( x_1, x_2 \in M \) and \( y_1, y_2 \in G \) such that \( \hat{I}A x_1 = \hat{I}Ay_1 \) and \( \hat{I}B x_2 = \hat{I}By_2 \). Since \( M \) and \( G \) are nonempty, take \( x_0 \in M \) and \( y_0 \in G \), and let \( x = \hat{I}A x_1 + \hat{I}B x_2 + \hat{I}(A \cup B)x_0 \) and...
y = \bar{I}_{A\cap B}y_1 + \bar{I}_{B\setminus A}y_2 + \bar{I}_{(A\cup B)^c}y_0, \text{ then } \bar{I}_{A\cup B}x = \bar{I}_{A\cup B}y \in \bar{I}_{A\cup B}M \cap \bar{I}_{A\cup B}G \text{ by noticing } x \in M \text{ and } y \in G.

Define \( H(M, G) = \Omega \setminus \text{esssup} (\mathcal{E}) \), then \( H(M, G) \), obviously, satisfies (2) and (3). We will verify that \( H(M, G) \) also has the property (1). In fact, if \( P(H(M, G)) = 0 \), then \( P(\text{esssup} (\mathcal{E})) = 1 \), let \( \{D_n \mid n \in N\} \) be a nondecreasing sequence of \( \mathcal{E} \) such that \( D_n \uparrow \Omega \), then there exist two sequences \( \{x_n \mid n \in N\} \) in \( M \) and \( \{y_n \mid n \in N\} \) in \( G \) such that \( \bar{I}_{D_n}x_n = \bar{I}_{D_n}y_n, \forall n \in N \). Let \( A_n = D_n \setminus D_{n-1}, \forall n \geq 1 \), where \( D_0 = \emptyset \), then \( \sum_{n \geq 1} \bar{I}_{A_n}x_n = \sum_{n \geq 1} \bar{I}_{A_n}y_n = \mathcal{C}_{cc}(M) \cap \mathcal{C}_{cc}(G) \), which is a contradiction. \( \square \)

**Definition 3.21.** Let \( E, M \) and \( G \) be the same as in Theorem 3.20 such that \( \mathcal{C}_{cc}(M) \cap \mathcal{C}_{cc}(G) = \emptyset \), then \( H(M, G) \) is called the hereditarily disjoint stratification of \( H \) and \( M \), and \( P(H(M, G)) \) is called the hereditarily disjoint probability of \( H \) and \( M \).

### 3.2. The hereditarily disjoint probability and more general forms of hyperplane separation theorems

First, we state the main result of this subsection—Theorems 3.22 and 3.23, whose proofs follow from Lemma 3.24.

**Theorem 3.22.** Let \((E, \mathcal{P})\) be a random locally convex module over \( R \) with base \((\Omega, \mathcal{F}, \mathcal{P})\), \( M \) and \( G \) two nonempty \( L^0 \)-convex subsets such that \( G \) is \( \mathcal{T}_c \)-open and \( \mathcal{C}_{cc}(G) \cap \mathcal{C}_{cc}(M) = \emptyset \). Then there exists an \( f \in E^*_c \) such that

\[
f(x) < f(y) \text{ on } H(M, G) \text{ for all } x \in M \text{ and } y \in G, \tag{3.21}
\]

and

\[
f(x) < f(y) \text{ for all } x \in M \text{ and } y \in G. \tag{3.21}'
\]

If \( R \) is replaced by \( C \), then Theorem 3.22 still holds in the following way:

\[
(Ref)(x) < (Ref)(y) \text{ on } H(M, G) \text{ for all } x \in G \text{ and } y \in M, \tag{3.22}
\]

and

\[
(Ref)(x) < (Ref)(y) \text{ for all } x \in G \text{ and } y \in M. \tag{3.22}'
\]

In the following Theorem 3.23, we only need to notice that \( C_{cc}(G) = G \) and \( H(\{x\}, G) \) is just \( |x| > 0 \) in Theorem 3.15.

**Theorem 3.23.** Let \((E, \mathcal{P})\) be a random locally convex module over \( R \) with base \((\Omega, \mathcal{F}, \mathcal{P})\), such that \( \mathcal{P} \) has the countable concatenation property, \( x \in E \) and \( G \) a nonempty \( \mathcal{T}_c \)-closed \( L^0 \)-convex subsets of \( E \) such that \( x \notin G \) and \( G \) has the countable concatenation property. Then there exists an \( f \in E^*_c \) such that

\[
f(x) > \vee\{f(y) \mid y \in G\} \text{ on } H(\{x\}, G), \tag{3.23}
\]

and

\[
f(x) > \vee\{f(y) \mid y \in G\}. \tag{3.23}'
\]

If \( R \) is replaced by \( C \), then Theorem 3.23 still holds in the following way:

\[
(Ref)(x) > \vee\{(Ref)(y) \mid y \in G\} \text{ on } H(\{x\}, G), \tag{3.24}
\]
and

$$(Re f)(x) > \vee \{(Re f)(y) \mid y \in G\}. \quad (3.24)'$$

**Lemma 3.24.** Let $(E, \mathcal{P})$ be a random locally convex module over $K$ with base $(\Omega, \mathcal{F}, \mathcal{P})$, $M$ a $\mathcal{T}_c$-closed subset of $E$ such that $\tilde{I}_A M + \tilde{I}_A - M \subset M$, for all $A \in \mathcal{F}$, and $G$ a $\mathcal{T}_c$-open subset of $E$ such that $\tilde{I}_A G + \tilde{I}_A - G \subset G$, for all $A \in \mathcal{F}$. Then for each $A \in \mathcal{F}$ with $P(A) > 0$, $\tilde{I}_A M$ is relatively $\mathcal{T}_c$-closed in $\tilde{I}_A E$ and $\tilde{I}_A G$ is relatively $\mathcal{T}_c$-open in $\tilde{I}_A E$.

**Proof.** We can assume that $\theta \in G$ and $\theta \in M$ (otherwise by a translation), respectively, then $\tilde{I}_A G \subset G$, and $\tilde{I}_A M \subset M$, so that $\tilde{I}_A E \cap M = \tilde{I}_A M$ and $\tilde{I}_A E \cap G = \tilde{I}_A G$. □

We can now prove Theorem 3.22.

**Proof of Theorem 3.22.** Let $\Omega' = H(M, G)$, $\mathcal{F}' = \Omega' \cap \mathcal{F} = \{\Omega' \cap F \mid F \in \mathcal{F}\}$ and $P' : F' \to [0, 1]$ be defined by $P'(\Omega' \cap F) = P(\Omega' \cap F)/P(\Omega')$, $\forall F \in \mathcal{F}$. Take $E' = \tilde{I}_M E$, $P' = \{\|f\|_E \mid f \in \mathcal{P}\}$, $M' = \tilde{I}_M M$, $G' = \tilde{I}_M G$ and consider $(E', \mathcal{P}')$ as a random locally convex module with base $(\Omega', \mathcal{F}', P')$. Then $M'$ and $G'$ satisfy the condition of Theorem 3.8, so that there exists an $f' \in (E')^*_c$ such that

$$f'(x) < f'(y) \text{ on } \Omega' \text{ for all } x \in G', \text{ and } y \in M'. \quad (3.25)$$

By Theorem 2.9 $f'$ has an extension $f'' \in E'^*_c$. Now let $f = \tilde{I}_{H(M, G)} f''$, then $f(x) = 0$, $\forall x \in \tilde{I}_{H(M, G)} E$ and $f(x) = f'(x)$, $\forall x \in \tilde{I}_{H(M, G)} E$, so that $f$ satisfies all the requirements of Theorem 3.22. □

**Proof of Theorem 3.23.** It is completely similar to the proof of Theorem 3.22 (In fact, it can also be derived directly from Theorem 3.15). □

### 3.3. Closed $L^0$-convex subsets with the countable concatenation property

**Definition 3.25.** Let $(E, \mathcal{P})$ be a random locally convex module over $K$ with base $(\Omega, \mathcal{F}, \mathcal{P})$. For each $f \in E_c^*$, $|f(\cdot)| : E \to L^0_+$ is clearly an $L^0$-seminorm, so that $(E, \{ |f(\cdot)| \mid f \in E_c^* \})$ is a random locally convex module, whose locally $L^0$-convex topology, denoted by $\sigma_c(E, E_c^*)$, called the weak locally $L^0$-convex topology of $E$. Similarly, we may have the weak $(\varepsilon, \lambda)$-topology $\sigma_{\varepsilon, \lambda}(E, E_c^*)$ of $E$. In particular when $E_c^* = E_c$, we briefly write $\sigma_c(E, E^*)$ and $\sigma_{\varepsilon, \lambda}(E, E^*)$ for $\sigma_c(E, E_c^*)$ and $\sigma_{\varepsilon, \lambda}(E, E_c^*)$, respectively.

**Remark 3.26.** For a random locally convex module $(E, \mathcal{P})$, dually, we may have the weak-star locally $L^0$-convex topology $\sigma_c(E^*, E)$ of $E^*$, and the weak-star $(\varepsilon, \lambda)$-topology $\sigma_{\varepsilon, \lambda}(E_c^*, E^*)$ of $E^*_{c, \lambda}$, which can be briefly denoted by $\sigma_c(E^*, E)$ and $\sigma_{\varepsilon, \lambda}(E^*, E)$ when $E_c^* = E^*_{c, \lambda}$, respectively.

The main result of the subsection is the following:

**Theorem 3.27.** Let $(E, \mathcal{P})$ be a random locally convex module over $K$ with base $(\Omega, \mathcal{F}, \mathcal{P})$ and $G$ an $L^0$-convex subset of $E$ such that $G$ has the countable concatenation property. Then we have the following equivalent statements:

1. $G$ is $\mathcal{T}_c$-closed;
2. $G$ is $\mathcal{T}_{\varepsilon, \lambda}$-closed;
3. $G$ is $\sigma_{\varepsilon, \lambda}(E, E^*_{c, \lambda})$-closed.

Further, if $\mathcal{P}$ has the countable concatenation property, then the above three are equivalent to the following:
(4) $G$ is $\sigma_c(E, E'_{\ast})$-closed.

**Proof.** (1) $\Leftrightarrow$ (2) has been proved in Theorem 3.18 and (2) $\Leftrightarrow$ (3) has been proved in Corollary 3.4 of [22] for any $L^0$-convex subset $G$.

If $P$ has the countable concatenation property, completely similar to the proof of the classical Mazur’s theorem, cf. [3], one can see (1) $\Leftrightarrow$ (4) by (3.24)' of Theorem 3.23. $\square$

### 3.4 Completeness

The main result of this subsection is the following:

**Theorem 3.28.** Let $(E, P)$ be a random locally convex module over $K$ with base $(\Omega, F, P)$. Then $E$ is $\mathcal{T}_c$-complete if $E$ is $\mathcal{T}_{c, \lambda}$-complete. Furthermore, if $E$ is $\mathcal{T}_c$-complete and has the countable concatenation property, then $E$ is also $\mathcal{T}_{c, \lambda}$-complete.

The first part of Theorem 3.18 is easy as pointed out in Section 1.4. However, the proof of the second part of it is a delicate matter since $\mathcal{T}_c$ is much stronger than $\mathcal{T}_{c, \lambda}$, and it seems to be not an easy work for us to give a direct proof even for the case of an RN module. However, we may give a clever proof of it by using Theorem 3.18 and the following Lemma 3.29.

First, let us recall some terminologies as follows.

Let $(E, P)$ be a random locally convex module over $K$ with base $(\Omega, F, P)$. Two $\mathcal{T}_{c, \lambda}$-Cauchy nets $\{x_\alpha, \alpha \in \Gamma\}$ and $\{y_\beta, \beta \in \Lambda\}$ in $E$ are called equivalent if $\{\|x_\alpha - y_\beta\|, (\alpha, \beta) \in \Gamma \times \Lambda\}$ converges to 0 in probability $P$ for each $\|\cdot\| \in P$. Since the sum of $\{x_\alpha, \alpha \in \Gamma\}$ and $\{y_\beta, \beta \in \Lambda\}$ is defined, as usual, to be $\{x_\alpha + y_\beta, (\alpha, \beta) \in \Gamma \times \Lambda\}$, which motivates us to do the following thing: If $E$ has the countable concatenation property, $\{A_n | n \in N\}$ is a countable partition of $\Omega$ to $F$ and $\{\{x_{\alpha n}, \alpha n \in \Gamma_n\} | n \in N\}$ is a sequence of Cauchy nets in $E$, then we naturally define their countable concatenation $\sum_{n \geq 1} I_{\Lambda_n} \{x_{\alpha n}, \alpha n \in \Gamma_n\} = \{\sum_{n \geq 1} I_{\Lambda_n} x_{\alpha n}, (\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots) \in \prod_{n \geq 1} \Gamma_n\}$, and it is easy to check that this is again a $\mathcal{T}_{c, \lambda}$-Cauchy net by noticing that $P(A_n) \to 0$ as $n \to \infty$, so that this countable concatenation is well defined. According to the same fact that $P(A_n) \to 0$ as $n \to \infty$, we can verify that $\{y_\beta_n : \beta_n \in \Lambda_n\}$ is equivalent to $\{x_{\alpha n}, \alpha n \in \Gamma_n\}$ for each $n \in N$, then $\sum_{n \geq 1} I_{\Lambda_n} x_{\alpha n}, (\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots) \in \prod_{n \geq 1} \Gamma_n\}$ is still equivalent to $\{\sum_{n \geq 1} I_{\Lambda_n} y_\beta_n, (\beta_1, \beta_2, \cdots, \beta_n, \cdots) \in \prod_{n \geq 1} \Lambda_n\}$. These observations lead to Lemma 3.29 below.

**Lemma 3.29.** Let $(E, P)$ be a random locally convex module over $K$ with base $(\Omega, F, P)$ such that $E$ has the countable concatenation property. For a $\mathcal{T}_{c, \lambda}$-Cauchy net $\{x_\alpha, \alpha \in \Gamma\}, \{x_\alpha, \alpha \in \Gamma\}$ denotes its $\mathcal{T}_{c, \lambda}$-equivalence class, the $\mathcal{T}_{c, \lambda}$-equivalence class of a constant net with value $x \in E$ is denoted by $[x]$. Let $E_{c, \lambda} = \{\{x_\alpha, \alpha \in \Gamma\} | \{x_\alpha, \alpha \in \Gamma\}$ is a $\mathcal{T}_{c, \lambda}$-Cauchy net $\}$. The module operations are defined as follows:

$$
\{\{x_\alpha, \alpha \in \Gamma\}\} + \{\{y_\beta, \beta \in \Lambda\}\} := \{\{x_\alpha + y_\beta, (\alpha, \beta) \in \Gamma \times \Lambda\}\},
$$

$$
\xi\{\{x_\alpha, \alpha \in \Gamma\}\} = \{\{\xi x_\alpha, \alpha \in \Gamma\}\}.
$$

Further, each $\|\cdot\| \in P$ induces an $L^0$-seminorm on $E_{c, \lambda}$, still denoted by $\|\cdot\|$, so that $\|\{x_\alpha, \alpha \in \Gamma\}\|$ is the limit of convergence in probability $P$ of $\{\|x_\alpha\|, \alpha \in \Gamma\}$.

Then $(E_{c, \lambda}, P)$ is a $\mathcal{T}_{c, \lambda}$-complete random locally convex module over $K$ with base $(\Omega, F, P)$ such that $E_{c, \lambda}$ still has the countable concatenation property, called the $\mathcal{T}_{c, \lambda}$-completion of $(E, P)$, further $(E, P)$ is $P$-isometrically isomorphic with a dense submodule $\{[x] | x \in E\}$ of $E_{c, \lambda}$.
Remark 3.30. Although every random locally convex module \((L, \varepsilon, \lambda)\) is \(T_{\varepsilon, \lambda}\)-complete. Further, let \(\{x_{\alpha_n}, \alpha_n \in \Gamma_n\} \mid n \in \mathbb{N}\) be a sequence of \(T_{\varepsilon, \lambda}\)-Cauchy nets in \(E\) and define \(\pi_n^\varepsilon = x_{\alpha_n}, \forall \alpha \in \prod_{n \geq 1} \Gamma_n\) and \(n \in \mathbb{N}\), then \(\{x_{\alpha_n}, \alpha_n \in \Gamma_n\}\) is equivalent to \(\{\pi_n^\varepsilon, \alpha \in \prod_{n \geq 1} \Gamma_n\}\) for each \(n \in \mathbb{N}\).

Thus for a countable partition \(\{A_n \mid n \in \mathbb{N}\}\) of \(\Omega\) to \(F\) we have that \(I_{A_m}[[\sum_{n \geq 1} I_{A_n} x_{\alpha_n}, (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) \in \prod_{n \geq 1} \Gamma_n]] = [[I_{A_m} \pi_n^\varepsilon, \alpha \in \prod_{n \geq 1} \Gamma_n]]\) (by the definition of the module multiplication)\(= I_{A_m}[[\pi_n^\varepsilon, \alpha \in \prod_{n \geq 1} \Gamma_n]] = I_{A_m}[[x_{\alpha_n}, \alpha_m \in \Gamma_m]]\) for each \(m \in \mathbb{N}\), so that \(\sum_{m \geq 1} I_{A_m}[[x_{\alpha_n}, \alpha_m \in \Gamma_m]] = [[\sum_{n \geq 1} I_{A_n} x_{\alpha_n}, (\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots) \in \prod_{n \geq 1} \Gamma_n]] \in \bar{E}_{\varepsilon, \lambda}\), namely \(\bar{E}_{\varepsilon, \lambda}\) has the countable concatenation property. \(\square\)

Remark 3.31. Although every random locally convex module \((E, \varepsilon, \lambda)\) admits a \(T_{\varepsilon}\)-completion \((\bar{E}_{\varepsilon}, \varepsilon, \lambda)\) and a \(T_{\varepsilon, \lambda}\)-completion \((\bar{E}_{\varepsilon, \lambda}, \varepsilon, \lambda)\) such that \((E, \varepsilon, \lambda)\) is \(\bar{E}_{\varepsilon, \lambda}\)-isometrically isomorphic with a dense submodule of either of the latter two, but since \(\bar{T}_{\varepsilon}\) is so strong that a countable concatenation of a sequence of \(T_{\varepsilon}\)-Cauchy nets is not necessarily well defined, so that we can not give Lemma 3.29 for \(\bar{T}_{\varepsilon}\)-topology. Lemma 3.29 is necessary since the countable concatenation property of \(E\) is reserved in \(\bar{E}_{\varepsilon, \lambda}\) in a proper way when \(E\) and \(\{[x] \mid x \in E\}\) are identified.

We can now prove Theorem 3.28.

Proof of Theorem 3.28. Let \((\bar{E}_{\varepsilon, \lambda}, \varepsilon, \lambda)\) be the \(T_{\varepsilon, \lambda}\)-completion of \((E, \varepsilon, \lambda)\) as in Lemma 3.29 and regard \(E\) as a subset of \(\bar{E}_{\varepsilon, \lambda}\), then Theorem 3.18 shows that \(\bar{E}_{\varepsilon} = \bar{E}_{\varepsilon, \lambda} = \bar{E}_{\varepsilon}\). On the other hand, since \(E\) is \(\bar{T}_{\varepsilon}\)-complete, we always have that \(\bar{E}_{\varepsilon} = E\), so that \(\bar{E}_{\varepsilon, \lambda} = E\), namely \(E\) must be \(T_{\varepsilon, \lambda}\)-complete. \(\square\)

Remark 3.31. Theorem 3.28 is a powerful result, for example, Kupper and Vogelpoth proved in [25] that \(L_p^c(\mathcal{E})\) and \(L_p^o(\mathcal{E})\) are \(T_{\varepsilon}\)-complete, then they must be \(T_{\varepsilon, \lambda}\)-complete by the second part of Theorem 3.28, since they both have the countable concatenation property. On the other hand, it is easy to verify that \(L_p^f(\mathcal{E})\) is \(T_{\varepsilon}\)-complete, as to \(L_p^o(\mathcal{E})\) when \(p > 1\) they are, obviously, \(T_{\varepsilon}\)-complete by the first part of Theorem 3.28 since they are all the random conjugate spaces of some \(RN\) modules and \(T_{\varepsilon, \lambda}\)-complete.

4. The theory of random conjugate spaces of random normed modules under the locally \(L^0\)-convex topology

Since the \((\varepsilon, \lambda)\)-topology is rarely a locally convex topology in the sense of traditional functional analysis, consequently, the theory of traditional conjugate spaces universally fails to serve for the deep development of \(RN\) modules under the \((\varepsilon, \lambda)\)-topology, see [22] for details. It is under such a background that the theory of random conjugate spaces of random normed modules has been developed and has been being centered at our previous work, and in fact it is also the most difficult and deepest part of our previous work, cf. [8, 10, 16, 19, 23].

The locally \(L^0\)-convex topology has the nice convexity, but it is too strong, it is, certainly, also rather difficult to establish the corresponding results of [8, 10, 16, 19, 23] under the locally \(L^0\)-convex topology in a direct way. Considering that an \(RN\) module has the same random conjugate space under the two kinds of topologies, even many results are independent of a special choice of the two kinds of topologies, we can now establish the corresponding \(T_{\varepsilon}\)-variants of those deep results previously established in [8, 10, 16, 19, 23] under \(T_{\varepsilon, \lambda}\), and based on Section 3.4 this has become an easy matter in an indirect manner!
It should also be pointed out that there are many results in Section 4 in which the hypothesis “Let \( (E, \| \cdot \|) \) be \( \mathcal{T}_\varepsilon \)-complete and have the countable concatenation property” occurs, the hypothesis automatically reduces to “Let \( (E, \| \cdot \|) \) be \( \mathcal{T}_{\varepsilon, \lambda} \)-complete” in their \( \mathcal{T}_{\varepsilon, \lambda} \)-prototypes, since \( \mathcal{T}_{\varepsilon, \lambda} \)-completeness has implied the countable concatenation property. Besides, the reader should bear in mind that \( E^*_\varepsilon = E^*_{\varepsilon, \lambda} \), denoted by \( E^* \), for an RN module.

4.1. Riesz’s representation theorems and the important connection between random conjugate spaces and classical conjugate spaces

In this subsection, we will give the Riesz’s type of representation theorems of random conjugate spaces of three extensive classes of RN modules. The main results are Theorem 4.3, Theorem 4.4 and Theorem 4.8.

To give the first of Riesz’s type of representation theorems, let us first recall the notion of a random inner product module, if we do not intend to mention the notion of a random inner product space, then the notion of a random inner product module introduced in [11] and already employed in [23] is exactly the following:

**Definition 4.1([11]).** An ordered pair \((E, \langle \cdot, \cdot \rangle)\) is called a random inner product module (briefly, an RIP module) over \( K \) with base \((\Omega, \mathcal{F}, P)\) if \( E \) is a left module over the algebra \( L^n(\mathcal{F}, K) \) and \( \langle \cdot, \cdot \rangle \) is a mapping from \( E \times E \) to \( L^n(\mathcal{F}, K) \) such that the following are satisfied:

1. \( \langle x, x \rangle \in L^n_+ \), and \( \langle x, x \rangle = 0 \) iff \( x = 0 \) (the null vector of \( E \));
2. \( \langle x, y \rangle = \overline{\langle y, x \rangle}, \forall x, y \in E \) where \( \langle y, x \rangle \) denotes the complex conjugate of \( \langle y, x \rangle \);
3. \( \langle \xi x, y \rangle = \xi \langle x, y \rangle, \forall \xi \in L^n(\mathcal{F}, K) \), and \( x, y \in E \);
4. \( \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \forall x, y, z \in E \).

Where \( \langle x, y \rangle \) is called the random inner product between \( x \) and \( y \); If \( \langle x, y \rangle = 0 \), then \( x \) and \( y \) are called orthogonal, denoted by \( x \perp y \), furthermore \( M^\perp = \{ y \in E | \langle x, y \rangle = 0, \forall x \in M \} \) is called the orthogonal complement of \( M \).

The Schwartz inequality: \( |\langle x, y \rangle| \leq \|x\| \cdot \|y\|, \forall x, y \in E \), was proved in [11], where \( \| \cdot \| : E \to L^n_0 \) defined by \( \|x\| = \sqrt{\langle x, x \rangle}, \forall x \in E \), is thus an \( L^n_0 \)-norm such that \( (E, \| \cdot \|) \) is an RN module, called the RN module derived form \((E, \langle \cdot, \cdot \rangle)\).

Let us first recall from [13] and its references there the notions of random elements and random variables.

**Example 4.2.** Let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert space over \( K \). A mapping \( V : \Omega \to H \) is called an \( \mathcal{F} \)-random element if \( V^{-1}(B) := \{ \omega \in \Omega | V(\omega) \in B \} \in \mathcal{F} \) for each open subset \( B \) of \( H \), further an \( \mathcal{F} \)-random element with its range finite is called an \( \mathcal{F} \)-simple random element. A mapping \( V : \Omega \to H \) is called an \( \mathcal{F} \)-random variable if there exists a sequence \( \{ V_n | n \in N \} \) of \( \mathcal{F} \)-simple random elements such that \( \| V_n(\omega) - V(\omega) \| \to 0 \) as \( n \to \infty \) for each \( \omega \in \Omega \). Denote by \( L^n(\mathcal{F}, H) \) the linear space of equivalence classes of \( H \)-valued \( \mathcal{F} \)-random variables on \( \Omega \), which is a left module over the algebra \( L^n(\mathcal{F}, K) \) under the module multiplication \( \xi x := \{ \xi x(\omega) \}_{\omega \in \Omega} \), defined by \( \langle \xi x(\omega), y(\omega) \rangle = \xi \langle x(\omega), y(\omega) \rangle, \forall \omega \in \Omega \), where \( \xi \) and \( x \) are arbitrarily chosen representatives of \( \xi \in L^n(\mathcal{F}, K) \) and \( x \in L^n(\mathcal{F}, H) \).

\( L^n(\mathcal{F}, H) \) becomes an RIP module over \( K \) with base \((\Omega, \mathcal{F}, P)\) under the random inner product induced from the inner product \( \langle \cdot, \cdot \rangle \), still denoted by \( \langle \cdot, \cdot \rangle \). Namely, for any \( x, y \in L^n(\mathcal{F}, H) \) with respective representatives \( x^0, y^0 \), we have \( \langle x, y \rangle = \langle x^0, y^0 \rangle(\omega) \), \( \forall \omega \in \Omega \).
It is easy to see that \((L^0(\mathcal{F}, H), \| \cdot \|)\) is \(\mathcal{T}_{\omega, \lambda}\)-complete, so that \(L^0(\mathcal{F}, H)\) is \(\mathcal{T}_{\omega}\)-complete and obviously has the countable concatenation property.

**Theorem 4.3.** Let \((E, \langle \cdot, \cdot \rangle)\) be a \(\mathcal{T}_{\omega}\)-complete RIP module over \(K\) with base \((\Omega, \mathcal{F}, \mathcal{P})\) such that \(E\) has the countable concatenation property. Then for every \(f \in E^*\) there exists a unique \(\pi(f) \in E\) such that \(f(x) = \langle x, \pi(f) \rangle, \forall x \in E\) and such that \(\|f\| = \|\pi(f)\|\). Finally the induced mapping \(\pi : E^* \to E\) is a surjective conjugate isomorphism, namely \(\pi(x + y) = \xi \pi(f) + \eta \pi(g), \forall x, y \in L^0(\mathcal{F}, K)\) and \(f, g \in E^*\).

**Proof.** By Theorem 3.28, we have that \(E\) is \(\mathcal{T}_{\omega, \lambda}\)-complete. It follows immediately that the main result of [23] is just what we want to prove. \(\square\)

Before giving Theorem 4.4, let us first recall two important examples of \(R\)N modules: Let \(L^0(\mathcal{F}, B)\) the \(R\)N module of equivalence classes of random variables from \((\Omega, \mathcal{F}, \mathcal{P})\) to a normed space \((B, \| \cdot \|)\) over \(K\), its construction is similar to Example 4.2, also see [19] for details.

Let \(B'\) be the classical conjugate space of \(B\). Then a mapping \(q : \Omega \to B'\) is called a w*-random variable if the composite function \(\langle b, q \rangle(\omega) = \langle b, q(\omega) \rangle, \forall \omega \in \Omega\), is a \(K\)-valued random variable for each fixed \(b \in B\), where \(\langle \cdot, \cdot \rangle : B \times B' \to K\) denotes the natural pairing between \(B\) and \(B'\). Two w*-random variables \(q_1\) and \(q_2\) are called w*-equivalent if \(\langle b, q_1 \rangle\) and \(\langle b, q_2 \rangle\) are equivalent for each fixed \(b \in B\). For each w*-random variable \(q\), since \(\|b, q(\omega)\| \leq \|q(\omega)\|\) for each \(\omega \in \Omega\) and \(b \in B\) such that \(\|b\| \leq 1\), esssup \(\{\|b, q\| \mid b \in B\) and \(\|b\| \leq 1\}\) is a nonnegative real-valued random variable.

Let \(L^0(\mathcal{F}, B', w^*)\) be the linear space of w*-equivalence classes of \(B'\)-valued w*-random variables on \(\Omega\). Like \(L^0(\mathcal{F}, H)\) in Example 4.2, \(L^0(\mathcal{F}, B', w^*)\) can naturally becomes a left module over the algebra \(L^0(\mathcal{F}, K)\). Finally, for each \(x \in L^0(\mathcal{F}, B', w^*)\), if we define its random norm \(\|x\|\) by \(\|x\| = \|\text{esssup } \{\|x, b\| \mid b \in B\} \leq 1\}\), where \(x^0\) is a representative of \(x\), then \(L^0(\mathcal{F}, B', w^*)\) is an \(R\)N module over \(K\) with base \((\Omega, \mathcal{F}, \mathcal{P})\). Finally, for any \(x \in L^0(\mathcal{F}, B)\) and \(y \in L^0(\mathcal{F}, B', w^*)\), as usual, the natural pairing \(\langle x, y \rangle\) between \(x\) and \(y\) can be defined as the equivalence class of the natural pairing between their respective representatives.

**Theorem 4.4.** Let \((\Omega, \mathcal{F}, \mathcal{P})\) be a complete probability space. Then \(L^0(\mathcal{F}, B', w^*)\) is isomorphic with the random conjugate space of \(L^0(\mathcal{F}, B)\), denoted by \((L^0(\mathcal{F}, B))^*\), in a random-norm preserving manner under the canonical mapping \(T : L^0(\mathcal{F}, B', w^*) \to (L^0(\mathcal{F}, B))^*\) defined as follows. For each \(f \in L^0(\mathcal{F}, B', w^*)\), \(T_f\), denoting \(T(f) : L^0(\mathcal{F}, B) \to L^0(\mathcal{F}, K)\) is given by \(T_f(g) = \langle g, f \rangle, \forall g \in L^0(\mathcal{F}, B)\). Further, if \(L^0(\mathcal{F}, B', w^*)\) is replaced by \(L^0(\mathcal{F}, B')\), then a sufficient and necessary condition for \(T : L^0(\mathcal{F}, B') \to (L^0(\mathcal{F}, B))^*\) to be again an isometric isomorphism is that \(B'\) has the Radon-Nikodým property with respect to \((\Omega, \mathcal{F}, \mathcal{P})\).

**Proof.** Since \((\Omega, \mathcal{F}, \mathcal{P})\) is complete, \(L^0(\mathcal{F}, B', w^*), L^0(\mathcal{F}, B)\) and \(L^0(\mathcal{F}, B')\) are equivalent to \(L(P, B', w^*), L(P, B)\) and \(L(P, B')\) in [8], so that our desired results follow immediately from the main results of [8]. \(\square\)

**Remark 4.5.** Proof of the first part of Theorem 4.4 needs the theory of lifting property in [27, 28], and hence also the completeness of \((\Omega, \mathcal{F}, \mathcal{P})\). The second part of it is most important and need not assume that \((\Omega, \mathcal{F}, \mathcal{P})\) is complete, since \(L^0(\mathcal{F}, B)\) and \(L^0(\mathcal{F}, B)\) as well as \(L^0(\mathcal{F}, B')\) and \(L^0(\mathcal{F}, B')\) can be identified, so that we can first prove that \(L^0(\mathcal{F}, B') \cong (L^0(\mathcal{F}, B))^*\) if \(B'\) has the Radon-Nikodým property with respect to \((\Omega, \mathcal{F}, \mathcal{P})\), and then return to our desired result.

To give Theorem 4.8 below, we first give Theorem 4.6. Let \(1 \leq p \leq \infty\) and \((S, \| \cdot \|)\) an \(R\)N module over \(K\) with base \((\Omega, \mathcal{F}, \mathcal{P})\). Define \(\| \cdot \|_p : S \to [0, +\infty]\) by \(\|x\|_p = (\int_\Omega \|x\|^p d\mathcal{P})^{1/p}\)
for \( p \in [1, +\infty) \) and \( \| \cdot \|_\infty = \) the essential supremum of \( \|x\| \), \( \forall x \in S \), and denote by \( L^p(S) = \{ x \in S \mid \|x\|_p < +\infty \} \), then \( (L^p(S), \| \cdot \|_p) \) is a normed space, for all \( q, 1 \leq q \leq +\infty \) we can also have \( (L^q(S^*), \| \cdot \|_q) \) in a similar way. The following Theorem 4.6 is essentially independent of a special choice of \( T_c \), and \( T_{c, \lambda} \), which was proved in [10] under \( T_{c, \lambda} \), whose proof in English was given in [14].

**Theorem 4.6** [10, 14]. Let \( 1 \leq p < +\infty \) and \( 1 < q \leq +\infty \) be a pair of Hölder conjugate numbers. Then \( (L^q(S^*), \| \cdot \|_q) \) is isometrically isomorphic with the classical conjugate space of \( (L^p(S), \| \cdot \|_p) \), denoted by \( (L^p(S))^* \), under the canonical mapping \( T : L^q(S^*) \rightarrow (L^p(S))^* \) defined as follows. For each \( f \in L^q(S^*) \), \( T \), denoting \( T(f) : L^p(S) \rightarrow K \) is defined by \( T_\ell(g) = \int_{\Omega} f(g)d\mathcal{P} \) for all \( g \in L^p(S) \).

Theorem 4.6 gives all representation theorems of the dual of Lebesgue-Bochner function spaces by taking \( S = L^0(\mathcal{F}, B) \) (at which time \( L^p(S) \) is just \( L^p(\mathcal{F}, B) \), the classical Lebesgue-Bochner function spaces, see [12] for more details). On the other hand, it provides the connection between the random conjugate space \( S^* \) and the classical conjugate space \( (L^p(S))^* \), and thus a powerful tool for the theory of random conjugate spaces, cf.[10, 19, 22]. In this paper, we will still employ it in the proof of Theorem 4.8. In particular, combining the ideas of constructing \( L^p_p(E) \) in [4,25] and \( L^p(S) \) as above at once leads us to the following:

**Example 4.7.** Let \( (S, \| \cdot \|) \) be an RN module over \( K \) with base \( (\Omega, \mathcal{E}, \mathcal{P}) \) and \( \mathcal{F} \) a sub-\( \sigma \)-algebra of \( \mathcal{E} \). For each \( 1 \leq p \leq +\infty \), define \( \| \cdot \|_p : S \rightarrow L^p_p(\mathcal{F}, R) = \{ \xi \in L^0(\mathcal{F}, R) \mid \xi \geq 0 \} \) as follows: for all \( x \in S \),

\[
\| \|x\|_p = \begin{cases} 
\left( E[\|x\|^p \mid \mathcal{F}] \right)^\frac{1}{p}, & \text{if } p \in [1, +\infty) \\
\wedge\{ \xi \in L^p_p(\mathcal{F}, R) \mid \xi \geq \|x\| \}, & \text{if } p = +\infty 
\end{cases}
\]

where \( E[\cdot \mid \mathcal{F}] \) denotes the conditional expectation, cf. [4,25].

Denote \( L^p_p(S) = \{ x \in E \| \|x\|_p \in L^p_p(\mathcal{F}, R) \} \), then \( (L^p_p(S), \| \cdot \|_p) \) is an RN module over \( K \) with base \( (\Omega, \mathcal{F}, \mathcal{P}) \). Similarly, we can also have \( L^q_q(S^*) \) for all \( q \in [1, +\infty) \). When \( S = L^0(\mathcal{E}, R) \), \( L^p_p(S) \) is exactly \( L^p_p(\mathcal{E}) \) of [4, 25].

**Theorem 4.8.** Let \( 1 \leq p < +\infty \) and \( 1 < q \leq +\infty \) be a pair of Hölder conjugate numbers. The canonical mapping \( T : L^p_p(S^*) \rightarrow (L^p_p(S))^* \) is surjective and random-norm preserving, where for each \( f \in L^p_p(S^*) \), \( T(f) \) (denoting \( T(f) : L^p_p(S) \rightarrow L^0(\mathcal{F}, K) \) is defined by \( T_\ell(g) = E[f(g) \mid \mathcal{F}] \) for all \( g \in L^p_p(S) \), and \( L^p_p(S^*) \) and \( L^p_p(S) \) are the same as in Example 4.7.

For the sake of clearness, the proof of Theorem 4.8 is divided into the following two Lemmas 4.9 and 4.10, Lemma 4.9 shows that \( T \) is well defined and isometric (namely random-norm preserving) and Lemma 4.10 shows that \( T \) is surjective. Specially, we need to remind the readers of noticing that \( \| \cdot \|_p \) and \( \| \cdot \|_q \) are the \( L^0 \)-norms on \( L^p_p(S) \) and on \( L^p_p(S^*) \), respectively, whereas \( \| \cdot \|_p \) and \( \| \cdot \|_q \) are norms.

**Lemma 4.9.** \( T \) is well defined and isometric.

**Proof.** For any fixed \( f \in L^p_p(S^*) \), we will first prove that \( T_\ell \in (L^p_p(S))^* \) and \( \|T_\ell\| = \|f\|_q \) when \( p > 1 \) as follows:
For any \( g \in L_p^p(S) \), \( T_f(g) = E[f(g) \mid \mathcal{F}] \), we have the following:

\[
|T_f(g)| \leq E||f(g)||^p \mid \mathcal{F} \\
\leq E||f||^p \cdot ||g|| \mid \mathcal{F} \\
\leq |||f|||^q \cdot (E|||g||^p \mid \mathcal{F})^{\frac{1}{q}} \\
= |||f|||^q \cdot |||g|||^p \tag{4.26}
\]

This shows that \( T_f \in (L_p^p(S))^* \) and \( ||T_f|| \leq |||f|||^q \), namely \( T \) is well defined. We remain to prove \( ||T_f|| = |||f|||^q \) when \( p > 1 \).

Let \( \xi \) be an arbitrary representative of \( |||f|||^q \) and \( A_n = \{ \omega \in \Omega \mid n - 1 \leq \xi(\omega) < n \} \) for each \( n \in N \). Then \( \{A_n \mid n \in N\} \) forms a countable partition of \( \Omega \) to \( \mathcal{F} \). Observing

\[
\int_{\Omega} ||g||^p dP = \int_{\Omega} E[||g||^p \mid \mathcal{F}] dP = \int_{\Omega} ||g||^p dP, \quad \forall g \in L_p^p(S),
\]

thus we have the following relation:

\[
L^p(L_p^p(S)) = L^p(S) \tag{4.27}
\]

Since \( p > 1, 1 < q < +\infty \), we also have the relation:

\[
L^q(L_p^p(S^*)) = L^q(S^*) \tag{4.28}
\]

Now, we fix \( n \) and prove

\[
I_{A_n} |||T_f|||^q = I_{A_n} |||g|||^q \tag{4.29}
\]

Since \( I_{A_n} T_f(g) = E[I_{A_n} f(g) \mid \mathcal{F}] \) for all \( g \in L_p^p(S) \), we have, of course, that \( I_{A_n} T_f(g) = E[I_{A_n} f(g) \mid \mathcal{F}] \) for all \( g \in L^p(L_p^p(S)) = L^p(S) \). Further, since \( I_A L^p(L_p^p(S)) = I_A L^p(S) \subset L^p(L_p^p(S^*)) = L^p(S) \) for all \( A \in \mathcal{F} \), we can have the following important relation:

\[
I_{A_n} I_A T_f(g) = E[I_{A_n} I_A f(g) \mid \mathcal{F}], \tag{4.30}
\]

for all \( A \in \mathcal{F} \) and all \( g \in L^p(L_p^p(S)) = L^p(S) \).

Obviously, from (4.30), for all \( A \in \mathcal{F} \) we can have the following relation:

\[
\int_{\Omega} (I_{A_n} I_A T_f) (g)(dP = \int_{\Omega} (I_{A_n} I_A f) (g) dP, \tag{4.31}
\]

for all \( g \in L^p(L_p^p(S)) = L^p(S) \).

For each fixed \( A \in \mathcal{F} \), the left side of (4.31) defines a bounded linear functional on \( L^p(L_p^p(S)) \), whose norm is equal to \( \{\int_{\Omega} ||g||^q dP \}^{\frac{1}{q}} = (\int_{\Omega} ||I_{A_n} ||^q dP)^{\frac{1}{q}} \) by applying Theorem 4.6 to \( L^p(L_p^p(S)) \). The same bounded linear functional is also a bounded linear functional on \( L^q(S^*) \) defined by the right side of (4.31), then whose norm is also equal to \( (\int_{\Omega} ||I_{A_n} ||^q dP)^{\frac{1}{q}} \).

Consequently, \( \int_{\Omega} ||I_{A_n} T_f||^q dP = \int_{\Omega} (I_{A_n} ||f||)^q dP \) for all \( A \in \mathcal{F} \). Since \( ||I_{A_n} T_f||^q \in L^q(Q,R) \), we have \( ||I_{A_n} T_f||^q = E[||I_{A_n} ||^q \mid \mathcal{F}] \). Again noticing \( A_n \in \mathcal{F} \), we can have \( I_{A_n} T_f = I_{A_n} (E[||f||^q \mid \mathcal{F}])^{\frac{1}{q}} \), which is just (4.29).

Since \( \sum_{n \geq 1} A_n = \Omega, ||T_f|| = |||f|||^q \).

Finally, we consider the case of \( p = 1 \), for the sake of clearness, we use \( | \cdot |_\infty \) for the usual \( L^\infty \)-norm on the Banach space \( L^\infty(\mathcal{E}, K) \) of equivalence classes of essentially bounded \( \mathcal{E} \)-measurable \( K \)-valued functions on \( (\Omega, \mathcal{E}, P) \). Then it is easy to see that \( L^\infty(L_\infty(S^*)) = L^\infty(S^*) \), in particular that \( ||f||_\infty = |||f|||^q ||f||_\infty \) for all \( f \in L^\infty(S^*) \).

Since, we can, similarly to the case \( p > 1 \), have , by noticing \( I_{A_n} f \in L^\infty(S^*) \), the following relation: \( ||I_A(I_{A_n} ||f||)||_\infty = ||I_{A_n} I_A f||_\infty \) for all \( A \in \mathcal{F} \), but as stated above, the latter is just equal to \( ||I_A|| ||I_{A_n} \mid \mid f||_\infty ||_\infty \). Since \( I_{A_n} ||T_f|| \) and \( ||I_{A_n} ||f||_\infty \) (namely \( I_{A_n} ||f||_\infty \)) are both in
Let \( T \) be an arbitrary element of \((L^p_\mathcal{F}(S))^*\). We want to prove that there exists an \( f \in L^q_\mathcal{F}(S^*) \) such that \( F = T_f \).

Since \( \|F\| \in L^0_\mathcal{F}(\mathcal{F}, R) \), letting \( \xi \) be a chosen representative of \( \|F\| \) and \( A_n = \{ \omega \in \Omega \mid n - 1 \leq \xi(\omega) < n \} \) for each \( n \in \mathbb{N} \), then \( \{A_n \mid n \in \mathbb{N} \} \) forms a countable partition of \( \Omega \) to \( \mathcal{F} \). Since \( |F(g)| \leq \|F\||g||p = \|F\|(E[|g|^p|\mathcal{F}])^{\frac{1}{p}} \), \( T_{f_n}(g) \leq n(E[|g|^p|\mathcal{F}])^{\frac{1}{p}}, \forall g \in L^p_\mathcal{F}(S) \) and \( n \in \mathbb{N} \).

Now, we fix \( n \) and notice \( \int_\Omega (\hat{I}_{A_n} F)(g)dP \leq n \int_\Omega (E[|g|^p|\mathcal{F}])^{\frac{1}{p}}dP \leq n(\int_\Omega \|g\|^p dP)^{\frac{1}{p}} \) for all \( g \in L^p(S) \) by the Hölder inequality, then by Theorem 4.6 there exists \( f_n \in L^q(S^*) \) such that \( \int_\Omega (\hat{I}_{A_n} F)(g)dP = \int_\Omega f_n(g)dP \) for all \( g \in L^p(S) \).

Since \( \hat{I}_{A_n} L^p(S) \subseteq L^p(S) \) for all \( A \in \mathcal{F} \), we have that \( \int_A (\hat{I}_{A_n} F)(g)dP = \int_\Omega (\hat{I}_{A_n} F)(\hat{I}_{A_n} g)dP = \int_\Omega f_n(g)dP = \int_A f_n(g)dP \) for all \( g \in L^p(S) \) and all \( A \in \mathcal{F} \), which yields the following important relation (by noticing \( \hat{I}_{A_n} F(g) \in L^0(\mathcal{F}, K) \)):

\[
\hat{I}_{A_n} F(g) = E[f_n(g) \mid \mathcal{F}],
\] (4.32)

for all \( g \in L^p(S) \equiv L^p(L^p_\mathcal{F}(S)). \)

Since \( f_n \in L^q(S^*) \subseteq L^q_\mathcal{F}(S^*) \), then Lemma 4.9 shows that \( T_{f_n} \in (L^p_\mathcal{F}(S))^* \), and (4.32) just shows that \( \hat{I}_{A_n} F \) and \( T_{f_n} \) are equal on \( L^p(L^p_\mathcal{F}(S)) \). Since \( \hat{I}_{A_n} F \) and \( T_{f_n} \) are both in \((L^p_\mathcal{F}(S))^* = (L^p_\mathcal{F}(S))^{\ast}_{\mathcal{T}_{\varepsilon, \lambda}} \), namely they are both continuous module homomorphism from \((L^p_\mathcal{F}(S), \mathcal{T}_{\varepsilon, \lambda})\) to \((L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon, \lambda})\), and \( L^p(L^p_\mathcal{F}(S)) \) is \( \mathcal{T}_{\varepsilon, \lambda} \)-dense in \( L^p_\mathcal{F}(S) \) (cf.[19,22]), \( \hat{I}_{A_n} F = T_{f_n} \), namely we have the following relation:

\[
\hat{I}_{A_n} F(g) = E[f_n(g) \mid \mathcal{F}],
\] (4.33)

for all \( g \in L^p_\mathcal{F}(S) \).

We have from (4.33) the following relation:

\[
\hat{I}_{A_n} F(g) = E[\hat{I}_{A_n} f_n(g) \mid \mathcal{F}],
\] (4.34)

for all \( g \in L^p_\mathcal{F}(S) \).

Let \( f = \sum_{n \geq 1} \hat{I}_{A_n} f_n \). Since \( L^p_\mathcal{F}(S^*) \) has the countable concatenation property (or we can directly define \( f(g) = \sum_{n \geq 1} \hat{I}_{A_n} f_n(g) \), \( \forall g \in L^p_\mathcal{F}(S) \) and verify that \( f \) is first in \( S^* \) and then \( f \in L^p_\mathcal{F}(S^*) \), we have that \( f \in L^p_\mathcal{F}(S^*) \).

Finally, (4.34) shows that \( F(g) = E[f(g) \mid \mathcal{F}] = T_f(g), \forall g \in L^p_\mathcal{F}(S) \). □

**Remark 4.11.** Let \( K^d \) be the \( d \)-dimensional Euclidean space over \( K \). Then Theorem 4.3 and the second part of Theorem 4.4 both imply that \((L^0(\mathcal{F}, K^d))^* = L^0(\mathcal{F}, K^d) \), which is just Proposition 4.2 of [25]. Since \((L^p(\mathcal{F}, K))^* = L^0(\mathcal{F}, K) \), if we take \( S = L^0(\mathcal{F}, K) \) in Theorem 4.8, then we have \((L^p_\mathcal{F}(E))^* = L^p_\mathcal{F}(E) \), which is just Theorem 4.5 of [25], so our Theorem 4.8 is surprisingly general. Besides, the proof of the isometric property of \( T \) of Theorem 4.8 is completely new since Theorem 4.5 of [25] did not involve any isometric arguments. Further, we can also have \( L^p_\mathcal{F}(S) = L^0(\mathcal{F}, K) \cdot L^p(S) \).
4.2. Random reflexivity and the James theorem

Let \((E, \| \cdot \|)\) be an RN module, \(E^{**}\) denotes \((E^*)^*\), the canonical embedding mapping \(J : E \rightarrow E^{**}\) defined by \((Jx)(f) = f(x), \forall x \in E\) and \(f \in E^*\), is random–norm preserving. If \(J\) is surjective, then \(E\) is called random reflexive.

Clearly, if \(E\) is random reflexive, then \(E\) is complete under both \(\mathcal{T}_c\) and \(\mathcal{T}_{c,\lambda}\), and has the countable concatenation property since \(E^{**}\) has these properties, and the main results of this subsection are essentially independent of a special choice of \(\mathcal{T}_c\) and \(\mathcal{T}_{c,\lambda}\), which were established under \(\mathcal{T}_c\). Since they are still valid under \(\mathcal{T}_c\), we only state them without proofs except Theorem 4.13 and without mention of topologies.

**Theorem 4.12** ([8]). \(L^0(\mathcal{F}, B)\) (see Theorem 4.4) is random reflexive iff \(B\) is a reflexive Banach space.

**Theorem 4.13.** Let \((S, \| \cdot \|)\) be an RN module over \(K\) with base \((\Omega, \mathcal{E}, P), 1 < p < +\infty\) and \(\mathcal{F}\) a sub-\(\sigma\)–algebra of \(\mathcal{E}\). Then the following statements are equivalent:

1. \((S, \| \cdot \|)\) is random reflexive;
2. \(L^p(S)\) is a reflexive Banach space;
3. \(L^p_{\mathcal{F}}(S)\) is random reflexive.

**Proof.** (1) \(\Leftrightarrow\) (2) was proved in [10], see also [14] for its proof in English. (2) \(\Leftrightarrow\) (3) has been implied by (1) \(\Leftrightarrow\) (2) by noticing \(L^p(L^p_{\mathcal{F}}(S)) = L^p(S)\)!

**Theorem 4.14** [19]. A complete RN module \((S, \| \cdot \|)\) is random reflexive iff their exists \(p \in S(1)\) for each \(f \in E^*\) such that \(f(p) = \|f\|\), where \(S(1) = \{p \in S \mid \|p\| \leq 1\}\).

4.3. Banach-Alaoglu Theorem and Banach-Bourbaki-Kakutani-Šmuleian Theorem

Let \((E, \| \cdot \|)\) be an RN module over \(K\) with base \((\Omega, \mathcal{F}, P)\). Let \(\xi = \vee\{\|x\| \mid x \in E\}\) and let \(\xi^0\) be a representative of \(\xi\), the set \(\{\omega \in \Omega \mid \xi^0(\omega) > 0\}\) is called the support of \(E\) (unique a.s.), if \(\Omega\) is the support, then \(E\) is called having full support, in this subsection \((E, \| \cdot \|)\) is always assumed to have full support. \(A \in \mathcal{F}\) is called a \(P\)–atom if \(P(A) > 0\), and if \(B \in \mathcal{F}\) and \(B \subset A\) must imply either \(P(B) = 0\) or \(P(A \setminus B) = 0\). \((\Omega, \mathcal{F}, P)\) is said to be essentially purely \(P\)–atomic if there is a sequence \(\{A_n \mid n \in N\}\) of disjoint \(P\)–atoms such that \(\sum_{n \geq 1} A_n = \Omega\) and \(\mathcal{F} \subset \sigma\{A_n \mid n \in N\}^p\), where \(\sigma\{A_n \mid n \in N\}^p\) denotes the \(P\)–completion of the \(\sigma\)–algebra generated by \(\{A_n \mid n \in N\}\).

The two new results of the subsection are Theorems 4.16 and 4.18, and Theorems 4.15 and 4.17 are stated in order to contrast with the former two.

For the four topologies \(\sigma_{c,\lambda}(E, E^*), \sigma_{c}(E, E^*), \sigma_{c,\lambda}(E^*, E)\) and \(\sigma_{c}(E^*, E)\), we refer to Definition 3.25.

Classical Banach-Alaoglu theorem says the closed unit ball \(B'(1)\) of the conjugate space \(B'\) of a normed space \(B\) is always \(\sigma(B', B)\)–compact, namely \(w^*\)–compact, but for an RN module \((E, \| \cdot \|)\) with base \((\Omega, \mathcal{F}, P)\) we have the following:

**Theorem 4.15** ([16]). \(E^*(1) = \{f \in E^* \mid \|f\| \leq 1\}\) is \(\sigma_{c,\lambda}(E^*, E)\)–compact iff \(\mathcal{F}\) is essentially purely \(P\)–atomic.
Theorem 4.16. If $E^*(1)$ is $\sigma_c(E^*, E)$–compact then $\mathcal{F}$ is essentially purely $P$–atomic. But the converse is not true.

**Proof.** Since $\sigma_c(E^*, E)$ is stronger than $\sigma_{\varepsilon, \lambda}(E^*, E)$, $E^*(1)$ is $\sigma_{\varepsilon, \lambda}(E^*, E)$–compact, namely $\mathcal{F}$ is essentially purely $P$–atomic if $E^*(1)$ is $\sigma_{\varepsilon}(E^*, E)$–compact.

From the proof of Theorem 4.15 given in [16] we can similarly prove that $E^*(1)$ is $\sigma_{\varepsilon}(E^*, E)$–compact iff \( \{ x \in L^0(\mathcal{F}, K) \mid |x| \leq 1 \} \) is $\mathcal{T}_{\varepsilon}$–compact. We can construct the following example to show that even if $(\Omega, \mathcal{F}, P)$ is essentially purely $P$–atomic \( \{ x \in L^0(\mathcal{F}, K) \mid |x| \leq 1 \} \) is not $\mathcal{T}_{\varepsilon}$–compact, either. Take $\Omega = N, F = 2^N$ and $P(A) = \sum_{i \in A} \frac{1}{2^i}$ for all $A \in \mathcal{F}$, then $(\Omega, \mathcal{F}, P)$ is purely atomic, but at this time \( \{ x \in L^0(\mathcal{F}, K) \mid |x| \leq 1 \} \) is exactly the closed unit ball of the Banach space $l^\infty$ of bounded sequences in $K$, it is well known that it is not norm–compact in the Banach space, and hence it is not $\mathcal{T}_{\varepsilon}$–compact, either, since $\mathcal{T}_{\varepsilon}$–topology is stronger than the norm–topology on the closed unit ball, so that $E^*(1)$ is not $\sigma_{\varepsilon}(E^*, E)$–compact. □

Classical Banach-Bourbaki-Kakutani-Šmulian theorem says that a Banach space is reflexive if its closed unit ball is weakly compact, in [16] we prove the following:

**Theorem 4.17**([16]). $E(1) = \{ x \in E \mid ||x|| \leq 1 \}$ is $\sigma_{\varepsilon, \lambda}(E, E^*)$–compact iff both $(E, || \cdot ||)$ is random reflexive and $\mathcal{F}$ is essentially purely $P$–atomic.

Just as the norm–topology and weak one on $K^d$ are the same, it is easy to check that $\mathcal{T}_{\varepsilon, \lambda} = \sigma_{\varepsilon, \lambda}(L^0(\mathcal{F}, K^d), L^0(\mathcal{F}, K^d))$ and $\mathcal{T}_{\varepsilon} = \sigma_{\varepsilon}(L^0(\mathcal{F}, K^d), L^0(\mathcal{F}, K^d))$ on $L^0(\mathcal{F}, K^d)$. From this it is easy to see the proof of the second part of Theorem 4.18 below.

**Theorem 4.18.** If $E(1)$ is $\sigma_c(E, E^*)$–compact then $(E, || \cdot ||)$ is random reflexive and $\mathcal{F}$ is essentially purely $P$–atomic. But the converse is not true.

**Proof.** The proof is similar to the one of Theorem 4.16., in particular, if we take $E = L^0(\mathcal{F}, K)$, then that counterexample in the proof of Theorem 4.6 can also serve for the proof of the converse of the theorem. □

5. Some further remarks on the $(\varepsilon, \lambda)$–topology and the locally $L^0$–convex topology

The topology of convergence in probability is one of the most useful topologies on the space of random variables. The $(\varepsilon, \lambda)$–topology, as a natural generalization of the former, makes the theory of $RN$ and $RIP$ modules naturally applicable to many topics in probability theory, for example, our recent work [18] provides some interesting applications of $RIP$ modules to random linear operators on Hilbert spaces (cf.[27]). Further it has many advantages itself. For example, it admits a countable concatenation skill in $\mathcal{T}_{\varepsilon, \lambda}$–complete random locally convex modules (see Example 3.7), and has many nice properties, for example, $L^p(S)$ is $\mathcal{T}_{\varepsilon, \lambda}$–dense in $S$ for an $RN$ module $S$, which produces the useful connection of random conjugate spaces with classical conjugate spaces [see Theorem 4.6 and the process of proof of Theorem 4.8]. The $(\varepsilon, \lambda)$–topology is in harmony with the module structure, the family of $L^0$–seminorms and the order structure on $L^0(\mathcal{F}, R)$ of a random locally convex module so that a random locally convex module and its random conjugate space can be deeply developed under the framework of topological modules over the topological algebra $(L^0(\mathcal{F}, K), \mathcal{T}_{\varepsilon, \lambda})$. However, the $(\varepsilon, \lambda)$–topology on the linear spaces is rarely a locally convex topology, which makes us not establish such results as Theorem 2.6 and Lemma 3.10 of [4], in particular the $(\varepsilon, \lambda)$–topology can neither perfectly
A comprehensive connection between the basic results and properties derived from two kinds of topologies match the notion of a gauge function introduced in [4].

The locally $L^0$–convex topology has the nice $L^0$–convexity and perfectly matches the gauge functions, which admits Theorem 2.6 and Lemma 3.10 of [4] and thus has played a crucial role in convex analysis on $L^0$–modules. We can predict that the locally $L^0$–convex topology will also develop its power in non-$L^0$–linear analysis. However, it is too strong to make the previous frequently used skills reserved, for example, the countable concatenation skill often fails, it is impossible that $L^p(S)$ is $T_c$–dense in $S$, and in particular it is also impossible to establish Theorem 3.12 under $T_c$.

Comprehensively speaking, the two kinds of topologies can be both applied to mathematical finance, cf.[1,4] and the references therein, and the principal results of this paper enough convince people that the two kinds of topologies should be, simultaneously rather than in a single way, considered in the future study of random locally convex modules together with their financial applications.

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