Generalized Intelligent States for Nonlinear Oscillators

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Abstract

The construction of Generalized Intelligent States (GIS) for the $x^4$-anharmonic oscillator is presented. These GIS families are required to minimize the Robertson-Schrödinger uncertainty relation. As a particular case, we will get the so-called Gazeau-Klauder coherent states. The properties of the latters are discussed in detail. Analytical representation is also considered and its advantage is shown in obtaining the GIS in an analytical way. Further extensions are finally proposed.
1 Introduction and motivation

The study of coherent states for a quantum mechanical system has received a lot of attention [1 – 4] and the definitions, applications, generalizations of such states have been the subject of many papers. In the recent years, they were discussed in connection with exactly solvable models and non-linear algebras [5 – 8] as well as deformed algebras [9]. They were also produced by using supersymmetric methods [10 – 11]. More recently, a new approach has been introduced defining the set of coherent states, for a general quantum system, as eigenstates of an annihilation operator which maintain all the properties known for the standard harmonic oscillator [12 – 14] (see also [15] where an illustration of such construction is given for infinite square well and Pöschl-Teller potentials).

In this paper, we study the so-called generalized intelligent states for the one-dimensional $x^4$-anharmonic oscillator. We recall that this quantum system has been extensively studied since the early 1970 (for a review see [16]). Due mainly to its equivalence to one-dimensional $\phi^4$ quantum field theory, it was hoped that a detailed study of this simplified system would shed some light on the $\phi^4$ theory in higher dimensions. Research in this direction continues today. Indeed, recently A. D. Speliotopoulos [17, and references therein] purposed a new method to study the general structure of the Hilbert space and corresponding eigenvalues for $x^4$-anharmonic oscillator. The author constructed the creation and annihilation operators which diagonalize the Hamiltonian and showed that, unlike the standard harmonic oscillator, these operators obey the noncanonical commutation relation. This commutation relation will be the starting point of our analysis to construct the generalized intelligent states (generalized squeezed and coherent states) for nonlinear oscillator. Remember that the generalized intelligent states has been discussed in the literature for the $SU(2)$ and $SU(1,1)$ simple Lie groups [18 – 20] and for the supersymmetric oscillator [21].

The main purpose of this paper is to consider some general properties of generalized intelligent states corresponding to $x^4$-nonlinear oscillator. These states generalize those defined by Gazeau and Klauder [12] and minimize the Robertson-Schrödinger uncertainty relation [22 – 24]. The paper is organized as follows: In section 2, the Gazeau-Klauder coherent states (eigenvectors of the annihilation operator) are analyzed. Their properties (continuity in the labeling, temporal stability, overcompleteness and action identity) are discussed. We show also that there exists a dynamical algebra, generated by the lowering and raising (creation and annihilation) operators, which arise in the analysis of spectral structure of the $x^4$-anharmonic oscillator. This dynamical algebra seems to be a deformation of the Weyl-Heisenberg algebra and is isomorphic to $su(1,1)$ algebra. The harmonic oscillator limit is discussed in the end of this section. Section 3 is devoted to the construction of even and odd Gazeau-Klauder coherent states for the system under consideration. The real and imaginary Schrödinger Cat states are also constructed using the result of the section 2. In section 4, by means of the Robertson-Schrödinger uncertainty relation, we construct explicitly the so-called
generalized intelligent states. We discuss the analytical representation and we give an analytical realization of the generalized intelligent states. Conclusion and perspectives are summarized in the last section.

2 Gazeau-Klauder Coherent states for $x^4$-anharmonic oscillator

2.1 Structure of $x^4$-anharmonic oscillator eigenvalues

In this section, we recall the general structure of the energy eigenvalues for one-dimensional nonlinear oscillators. To be specific, we are interested in the Hamiltonian which has the form

$$H = a^+a^- + \frac{I}{2} + \frac{\varepsilon}{4} (a^- + a^+)^4 - e_0,$$

where $a^+$ and $a^-$ are the creation and annihilation operators for the harmonic oscillator and $I$ is the identity operator. The parameter $\varepsilon$ is positive. The quantity $e_0$ is defined as follows

$$e_0 = \frac{1}{2} + \frac{3}{4} \varepsilon - \frac{21}{8} \varepsilon^2. \quad (2)$$

The Hamiltonian $H$ can be factorized in the following form [17]

$$H = A^+ A^- \quad (3)$$

in terms of the new operators $A^-$ and $A^+$ ($(A^+)\dagger = A^-$) which are defined by

$$A^- = a^- + \frac{\varepsilon}{4} f_1 (a^-, a^+) + \frac{\varepsilon^2}{2} f_2 (a^-, a^+), \quad (4)$$

where the functions $f_1$ and $f_2$ of the $a^-$ and $a^+$ are given by

$$f_1 (a^-, a^+) = -3 (a^{-2} - a^{+2}) a^- + (a^- + a^+)^3 + 3 (a^- + a^+) \quad (5)$$

and

$$f_2 (a^-, a^+) = \frac{3}{2} a^{-5} + \frac{39}{4} a^+ a^{-4} + \frac{25}{8} a^2 a^{-3} - 12 a^3 a^{-2} - \frac{3}{8} a^4 a^- + \frac{1}{4} a^5 \quad (6)$$

$$+ \frac{75}{4} a^{-3} - \frac{135}{8} a^+ a^{-2} - \frac{135}{4} a^2 a^- - \frac{3}{8} a^3 - \frac{135}{8} a^- - \frac{27}{2} a^+. \quad (7)$$

The energy levels are given by [17] (see also [16])

$$e_n = n + \frac{3}{2} \varepsilon (n^2 + n).$$

We keep terms only up to $\varepsilon$ which is the standard first-order perturbation result. The limit $\varepsilon \to 0$ give the standard harmonic oscillator. The energy levels constitute a strictly increasing sequence of positive numbers. The Hilbert space $\mathcal{H}$ for $x^4$-nonlinear oscillator is easily constructed in the same way as the standard harmonic oscillator. This space is spanned by the states.
\[
|n, \varepsilon\rangle = \frac{(A^+)^n}{\sqrt{F(n)}} e^{i\alpha n} |0, \varepsilon\rangle, \quad n \in \mathbb{N}
\] (8)

where \(|0, \varepsilon\rangle\) is the ground state and the function \(F(n)\) is defined by

\[
F(n) = \begin{cases} 
1 & \text{if } n = 0 \\
e_1e_2...e_n & \text{if } n \neq 0
\end{cases}
\] (9)

The action of the annihilation and creation operators are defined as follows:

\[
A^+ |n, \varepsilon\rangle = \sqrt{e_n+1}e^{-i\alpha(e_{n+1}-e_n)} |n+1, \varepsilon\rangle
\]
\[
A^- |n, \varepsilon\rangle = \sqrt{e_ne_i\alpha(e_n-e_{n-1})} |n-1, \varepsilon\rangle
\] (10)

where \(\alpha \in \mathbb{R}\). We define the number operator \(N\) as

\[
N |n, \varepsilon\rangle = n |n, \varepsilon\rangle.
\] (11)

The operator \(N\) is different from the product \(A^+A^- (= H)\).

### 2.2 Gazeau-Klauder coherent states

It is well known that, for harmonic oscillator, there are three equivalent definitions for the coherent states. One consists of defining them as eigenstates of the annihilation operator of the system. Another possibility is to define them as the vectors resulting from the application of the unitary displacement operator on an extremal state (the ground state in general). A third definition characterizes the coherent states as minimum-uncertainty states. Recently, Gazeau and Klauder defined the coherent states, for an arbitrary quantum system, as eigenvectors of the annihilation operator. Using this definition, the coherent states for \(x^4\)-anharmonic oscillator will be constructed in what follows.

Let us denote the coherent states \(|z, \alpha\rangle\) to show explicitly their dependance on the parameter \(\alpha\) (the relevance of this parameter will be clear afterwards (see equations (15) and (31))). They are eigenstates of \(A^-\)

\[
A^- |z, \alpha\rangle = z |z, \alpha\rangle.
\] (12)

To get their explicit form, we decompose \(|z, \alpha\rangle\) in terms of the \(|n, \varepsilon\rangle\) basis

\[
|z, \alpha\rangle = \sum_{n=0}^{\infty} a_n |n, \varepsilon\rangle.
\] (13)

Substituting this expression in (12) and using (10), we get the coefficients \(a_n\)

\[
a_n = \frac{z^n e^{-i\alpha e_n}}{\sqrt{F(n)}} a_0, \quad n \in \mathbb{N}.
\] (14)

Then, the normalized coherent states are given by

\[
|z, \alpha\rangle = a_0 \sum_{n=0}^{\infty} \frac{\Gamma \left(2 + \frac{2}{3\varepsilon}\right)}{(3\varepsilon)^n \Gamma (n+1) \Gamma \left(n+2 + \frac{2}{3\varepsilon}\right)} (z\sqrt{2})^n e^{-i\alpha e_n} |n, \varepsilon\rangle
\] (15)
where
\[ a_0 \equiv a_0 (|z|) = \left[ {}_0F_1 \left( 2 + \frac{2}{3 \epsilon}, \frac{2}{3 \epsilon} |z|^2 \right) \right]^{-\frac{1}{2}} \] (16)
and \( {}_0F_1 (\beta, x) \) is the hypergeometric function defined by:
\[ {}_0F_1 (\beta, x) = \sum_{n=0}^{\infty} \frac{\Gamma (\beta)}{\Gamma (\beta + n) n!} x^n. \] (17)

We observe that the coherent states \(|z, \alpha \rangle\) are continuously labeled by \(z\) and \(\alpha\), and exist only if the radius of convergence
\[ R = \lim_{n \to \infty} \sqrt{F(n)} \] (18)
is non zero. In our case, this radius is infinite.

Let us analyze now the completeness of the set \(\{|z, \alpha \rangle, z \in \mathbb{C} \text{ and } \alpha \in \mathbb{R}\}\). We impose
\[ \int |z, \alpha \rangle \langle z, \alpha| \, d\mu (z) = I_H, \] (19)
where the measure \(d\mu (z)\) has to be determined. If we suppose that \(d\mu (z)\) depends only on \(|z|\), it can be determined as in [6]. Indeed, let us take
\[ d\mu (z) = \left[ {}_0F_1 \left( 2 + \frac{2}{3 \epsilon}, \frac{2}{3 \epsilon} |z|^2 \right) \right] h \left( r^2 \right) r dr d\phi, \quad z = re^{i \phi}. \] (20)

Then, performing the integral in the angular variable \(\phi\), we get
\[ I = \sum_{n=0}^{\infty} |n, \epsilon \rangle \langle n, \epsilon| \left[ \frac{\pi}{F(n)} \int_0^{+\infty} x^n h(x) \, dx \right]. \] (21)

In order to recover the resolution of the identity in terms of the basis \(\{|n, \epsilon \}\) we must have
\[ \int_0^{+\infty} y^{n-1} g(y) \, dy = \Gamma (n) \Gamma \left( n + \frac{2}{3 \epsilon} + 1 \right), \] (22)
where
\[ g(y) = \frac{3 \pi \epsilon}{2} h \left( \frac{3 \epsilon}{2} y \right) \Gamma \left( 2 + \frac{2}{3 \epsilon} \right). \] (23)

Then it is clear that the function we are looking for is the inverse Mellin transform of \(g(s) = \Gamma (s) \Gamma \left( s + \frac{2}{3 \epsilon} + 1 \right), s \in \mathbb{C}\).

The result is [25]
\[ h(x) = \frac{4}{3 \pi \epsilon} \left( \frac{\sqrt{2 \epsilon} x}{2} \right)^{\frac{2}{3 \epsilon} + 1} K \left( \frac{2}{3 \epsilon} \right) \left( 2 \sqrt{2 \epsilon} x \right), \] (24)
where
\[ K_v (z) = \frac{\pi}{2} \frac{I_{-v} (z) - I_v (z)}{\sin (\nu z)} \] (25)
is the modified Bessel function of the third kind and
\[ I_v (z) = \sum_{n=0}^{\infty} \frac{\left( \frac{z}{2} \right)^{v+2n}}{n! \Gamma (n + v + 1)} \] (26)
is the modified Bessel function of the first kind.

There are two main consequences arising from the former result. First, we can express any coherent state in terms of the others

\[
|z', \alpha\rangle = \int |z, \alpha\rangle \langle z, \alpha | z', \alpha\rangle \, d\mu(z). \tag{27}
\]

The kernel \(\langle z, \alpha | z', \alpha\rangle\) is

\[
\langle z, \alpha | z', \alpha\rangle = \frac{{}_0F_1 \left( 2 + \frac{2}{3\varepsilon}, \frac{2}{3\varepsilon} z' \right)}{\sqrt{\sqrt{2} |z|^2} \, {}_0F_1 \left( \frac{2}{3\varepsilon} + 2 + \frac{2}{3\varepsilon} |z|^2 \right)} \tag{28}
\]

Second, an arbitrary element \(|g, \varepsilon\rangle = \sum_{m=0}^{\infty} b_m |m, \varepsilon\rangle\), of the Hilbert space \(H\), can be written in terms of the coherent states as

\[
|g, \varepsilon\rangle = \int \langle z, \alpha | g, \varepsilon \rangle |z, \alpha\rangle \, d\mu(z), \tag{29}
\]

where

\[
\langle z, \alpha | g, \varepsilon \rangle = a_0 \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma \left( 2 + \frac{2}{3\varepsilon} \right)}{(3\varepsilon)^n \, \Gamma \left( n + 1 \right) \, \Gamma \left( n + 2 + \frac{2}{3\varepsilon} \right)}} \left( \frac{\sqrt{2}}{z} \right)^n b_n e^{i\alpha \varepsilon_n} \tag{30}
\]

determine completely the state \(|g, \varepsilon\rangle \in H\).

Let us now consider the dynamical evolution of the coherent states, which is quite simple. Indeed, we have

\[
U(t) |z, \alpha\rangle = e^{-itH} |z, \alpha\rangle = |z, \alpha + t\rangle. \tag{31}
\]

It is clear from equations (12), (15), (19) and (31) that the \(x^4\)-anharmonic oscillator coherent states constructed as eigenvectors of the annihilation operator \(A^-\) satisfy the following properties

(i) Continuity: \(z \in \mathbb{C}, \alpha \in \mathbb{R} \to |z, \alpha\rangle\) is continuous

(ii) The resolution of unity: \(I_H = \int |z, \alpha\rangle \langle z, \alpha| \, d\mu(z)\)

(iii) Temporal stability: \(e^{-itH} |z, \alpha\rangle = |z, \alpha + t\rangle\)

(iv) Action identity: \(\langle z, \alpha | H |z, \alpha\rangle = |z|^2\)

These properties (i) - (iv) are the set of requirements to define coherent states of an arbitrary quantum system imposed by Gazeau and Klauder [12].

2.3 The dynamical algebra: Extended Weyl-Heisenberg algebra

Behind the spectral structure of the \(x^4\)-anharmonic oscillator, there exists a dynamical algebra generated by the lowering and raising operators \(A^-\) and \(A^+\). We follow a similar approach of the references [14] and [15] in which the authors discussed the dynamical algebra \(su(1,1)\) for the infinite square well and Pöschl-Teller Hamiltonians.

For the \(x^4\)-anharmonic oscillator, we defined the operator number \(N\) by

\[
N |n, \varepsilon\rangle = n |n, \varepsilon\rangle \tag{32}
\]
which can be given, formally, in terms of the Hamiltonian $H$ by

$$N = \sqrt{\frac{2}{3\varepsilon}H + \left(\frac{1}{2} + \frac{1}{3\varepsilon}\right)^2} - \left(\frac{1}{2} + \frac{1}{3\varepsilon}\right).$$  \hspace{1cm} (33)

The creation and annihilation operators $A^+$ and $A^-$ satisfy the following commutation relations

$$[A^-, A^+] = 1 + 3\varepsilon \left( N + 1 \right), \quad [N, A^\pm] = \mp A^\pm.$$  \hspace{1cm} (34)

In the limit $\varepsilon \to 0$, we get the well known Weyl-Heisenberg algebra. The operators $A^-, A^+$ and $N$ generate an extended Weyl Heisenberg algebra.

We note that there exists a dynamical Lie algebra, which is generated by the new set of operators $\{\tilde{A}^-, \tilde{A}^+, \tilde{N}\}$ defined as

$$\tilde{A}^\pm = \sqrt{\frac{2}{3\varepsilon}} A^\pm, \quad \tilde{N} = 2(N + 1) + \frac{2}{3\varepsilon}$$  \hspace{1cm} (35)

and satisfying the following commutation relations

$$[\tilde{A}^-, \tilde{A}^+] = \tilde{N}, \quad \tilde{A}^\pm, \tilde{N} = \mp 2\tilde{A}^\pm$$  \hspace{1cm} (36)

which is isomorphic to $su(1,1) \sim sl(2,\mathbb{R}) \sim so(2,1)$. A more familiar basis for $su(1,1)$ is given by

$$J_- = \frac{1}{\sqrt{2}} \tilde{A}^-, \quad J_+ = \frac{1}{\sqrt{2}} \tilde{A}^+, \quad J_{12} = \frac{1}{2} \tilde{N},$$  \hspace{1cm} (37)

where $J_{12}$ can be seen as the generator of the compact subgroup $SO(2)$, i.e.,

$$[J_-, J_+] = J_{12}, \quad [J_\pm, J_{12}] = \mp J_\pm.$$  \hspace{1cm} (38)

It follows from the above considerations that the space $\mathcal{H}$ of states $|n,\varepsilon\rangle$ carries some representation of $su(1,1)$. The actions of the Lie algebra elements read

$$J_+ |n,\varepsilon\rangle = \frac{1}{\sqrt{2}} \sqrt{(n + 1) \left(\frac{2}{3\varepsilon} + n + 2\right)} e^{i\alpha(3\varepsilon n + 3\varepsilon + 1)} |n + 1,\varepsilon\rangle,$$  \hspace{1cm} (39)

$$J_- |n,\varepsilon\rangle = \frac{1}{\sqrt{2}} \sqrt{n \left(\frac{2}{3\varepsilon} + n + 1\right)} e^{-i\alpha(3\varepsilon n + 1)} |n - 1,\varepsilon\rangle,$$  \hspace{1cm} (40)

$$J_{12} |n,\varepsilon\rangle = \left(\frac{1}{3\varepsilon} + n + 1\right) |n,\varepsilon\rangle.$$  \hspace{1cm} (41)

The $su(1,1)$ casimir operator is

$$C = 2J_+ J_- - J_{12} (J_{12} - 1)$$  \hspace{1cm} (42)

with the following eigenvalue

$$C |n,\varepsilon\rangle = \frac{1}{3\varepsilon} \left(\frac{1}{3\varepsilon} + 1\right) |n,\varepsilon\rangle.$$  \hspace{1cm} (43)
Finally, we note that $su(1,1)$ is the only dynamical Lie algebra that can arise in such a problem.

It is clear that the $x^4$-anharmonic oscillator tends to the harmonic oscillator Hamiltonian in the limit $\varepsilon \to 0$. Let us consider this limit in detail to see that the coherent states we have computed give the harmonic oscillator ones. Because

$$\lim_{\varepsilon \to 0} \frac{2^n \Gamma \left( 2 + \frac{2}{3 \varepsilon} \right)}{(3\varepsilon)^n \Gamma \left( n + 2 + \frac{2}{3 \varepsilon} \right)} = 1,$$

one can see that

$$\lim_{\varepsilon \to 0} [a_0 (|z|)]^{-2} = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\Gamma(n+1)} = e^{\left| z \right|^2}$$

and

$$\lim_{\varepsilon \to 0} |z, \alpha\rangle = e^{-\frac{\left| z \right|^2}{4}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} e^{-i\alpha} |n, 0\rangle,$$

where $|n, 0\rangle$ is the number state for the usual harmonic oscillator ($|n, 0\rangle \equiv |n, \varepsilon\rangle$ for $\varepsilon \to 0$).

### 3 Superpositions of Gazeau-Klauder Coherent states

#### 3.1 Even and odd coherent states

Even and odd coherent states give rise to states with non-classical properties. An important case is the superposition of the coherent states $|z, \alpha\rangle$ and $|\bar{z}, \alpha\rangle$ and the resultant states are eigenvalues of the operator $(A^-)^2$. Here, we show that this class of even and odd coherent states can be generated for the $x^4$-anharmonic oscillator.

Using the Gazeau-Klauder coherent states (15) for $x^4$-anharmonic oscillator, we define the even coherent states by the symmetric combination

$$|z, \alpha\rangle_e = \frac{N_e}{2a_0} (|z, \alpha\rangle + |\bar{z}, \alpha\rangle)$$

and odd coherent states by the antisymmetric combination

$$|z, \alpha\rangle_o = \frac{N_o}{2a_0} (|z, \alpha\rangle - |\bar{z}, \alpha\rangle),$$

where the constant $a_0$ is given by equation (16).

It is easy to derive the corresponding expansions for even and odd coherent states

$$|z, \alpha\rangle_e = N_e \sum_{k=0}^{\infty} \sqrt{\frac{\Gamma \left( 2 + \frac{2}{3 \varepsilon} \right)}{(3\varepsilon)^{2k} \Gamma \left( 2k+1 \right) \Gamma \left( 2k + 2 + \frac{2}{3 \varepsilon} \right)}} \left( z\sqrt{2} \right)^{2k} e^{-i\alpha\varepsilon_k} |2k, \varepsilon\rangle,$$

and

$$|z, \alpha\rangle_o = N_o \sum_{k=0}^{\infty} \sqrt{\frac{\Gamma \left( 2 + \frac{2}{3 \varepsilon} \right)}{(3\varepsilon)^{2k+1} \Gamma \left( 2k+2 \right) \Gamma \left( 2k + 3 + \frac{2}{3 \varepsilon} \right)}} \left( z\sqrt{2} \right)^{2k+1} e^{-i\alpha\varepsilon_{2k+1}} |2k+1, \varepsilon\rangle.$$
The normalization constant \( N_e \) and \( N_o \) can be obtained from the normalization conditions
\[
e \langle z, \alpha | z, \alpha \rangle_e = 1, \quad o \langle z, \alpha | z, \alpha \rangle_o = 1
\] (51)
which leads to
\[
N_e = N_e(|z|) = \left[ \frac{3}{(1 + 3\varepsilon)} {}_0F_3 \left( \frac{3}{2}, \frac{1}{3\varepsilon} + 2, \frac{1}{3\varepsilon} + \frac{3}{2} \right) \frac{|z|^4}{(6\varepsilon)^2} \right]^{-\frac{1}{2}}
\] (52)
and
\[
N_o = N_o(|z|) = \left[ \frac{|z|^2}{(1 + 3\varepsilon)} {}_0F_3 \left( \frac{3}{2}, \frac{1}{3\varepsilon} + 1, \frac{1}{3\varepsilon} + \frac{3}{2} \right) \frac{|z|^4}{(6\varepsilon)^2} \right]^{-\frac{1}{2}}.
\] (53)
It is straightforward to show that the even (or odd) coherent states cannot form separately a complete set. However the even coherent states together with the odd ones build an overcomplete Hilbert space. Their completeness relation takes the form
\[
\int |z, \alpha\rangle_o \langle z, \alpha| \, d\mu_o(z) + \int |z, \alpha\rangle_e \langle z, \alpha| \, d\mu_e(z) = I_H,
\] (54)
where the two weight function are given by
\[
d\mu_o(z) = [N_o]^{-2} h(r^2) r dr d\phi, \quad z = re^{i\phi}
\] (55)
and
\[
d\mu_e(z) = [N_e]^{-2} h(r^2) r dr d\phi, \quad z = re^{i\phi}
\] (56)
where \( h(r^2) \) in given by equation (24).

The even and odd coherent states have the following orthogonal relations
\[
e \langle z, \alpha | z', \alpha \rangle_e = \frac{{}_0F_3 \left( \frac{3}{2}, \frac{1}{3\varepsilon} + 1, \frac{1}{3\varepsilon} + \frac{3}{2} \right) \frac{(|z'|^2)}{(6\varepsilon)^2}}{\frac{{}_0F_3 \left( \frac{3}{2}, \frac{1}{3\varepsilon} + 1, \frac{1}{3\varepsilon} + \frac{3}{2} \right)}{\frac{(|z|^2)}{(6\varepsilon)^2}}},
\] (57)
\[
o \langle z, \alpha | z', \alpha \rangle_o = \frac{z z'}{{}_0F_3 \left( \frac{3}{2}, \frac{1}{3\varepsilon} + 1, \frac{1}{3\varepsilon} + \frac{3}{2} \right) \frac{(|z'|^2)}{(6\varepsilon)^2}} \frac{(|z|^2)}{\frac{{}_0F_3 \left( \frac{3}{2}, \frac{1}{3\varepsilon} + 1, \frac{1}{3\varepsilon} + \frac{3}{2} \right)}{\frac{(|z'|^2)}{(6\varepsilon)^2}}},
\] (58)
\[
o \langle z, \alpha | z', \alpha \rangle_e = 0
\] (59)
where \( \overline{z} \) is the complex conjugation of \( z \).

### 3.2 Probability distribution of even and odd Gazeau-Klauder coherent states

The probability distribution of even and odd Gazeau-Klauder coherent states for \( x^4 \)-anharmonic oscillator is defined as follow
\[
P_e (n, \varepsilon) = |\langle n, \varepsilon | z, \alpha \rangle_e|^2, \quad P_o (n, \varepsilon) = |\langle n, \varepsilon | z, \alpha \rangle_o|^2.
\] (60)
Using equations (49) and (50) we obtain
\[ \mathcal{P}_e(n, \varepsilon) = \mathcal{P}_e(2k, \varepsilon) = \frac{\mathcal{N}_e^2 |z|^{4k}}{F(2k)}, \quad (61) \]

and

\[ \mathcal{P}_o(n, \varepsilon) = \mathcal{P}_o(2k + 1, \varepsilon) = \frac{\mathcal{N}_o^2 |z|^{4k+2}}{F(2k + 1)}. \quad (62) \]

In the \( \varepsilon \to 0 \) limit, one obtain the probability distribution of even and odd harmonic oscillator states. In equations (61) and (62), the function \( F \) is defined by (9).

### 3.3 Real and imaginary Schrödinger Cat states

The real and imaginary Schrödinger Cat states constitute another representative of superposition states

\[ |z, \alpha\rangle_+ = \frac{\mathcal{N}_+}{2a_0} (|z, \alpha\rangle + |z, -\alpha\rangle), \quad (63) \]

and

\[ |z, \alpha\rangle_- = \frac{\mathcal{N}_-}{2ia_0} (|z, \alpha\rangle - |z, -\alpha\rangle), \]

where \( |z, \alpha\rangle \) is the coherent states given by equation (15) and \( \mathcal{N}_\pm \) are the normalization constants of real and imaginary Schrödinger Cat states. Similar superpositions were considered by Dodonov and al [26] for the usual harmonic oscillator coherent states, and Roy [27] for the nonlinear coherent states of the centre of mass of a trapped and biochromatically laser driven far from the Lamb-Dicke regime.

A direct computation leads to the following expansions for the real and imaginary Cat states

\[ |z, \alpha\rangle_+ = \mathcal{N}_+ \sum_{n=0}^{\infty} \frac{r^n \cos(n\phi)}{\sqrt{F(n)}} e^{-i\alpha n} |n, \varepsilon\rangle, \quad z = r e^{i\phi} \quad (64) \]

and

\[ |z, \alpha\rangle_- = \mathcal{N}_- \sum_{n=0}^{\infty} \frac{r^{n+1} \sin((n+1)\phi)}{\sqrt{F(n+1)}} e^{-i\alpha (n+1)} |n+1, \varepsilon\rangle. \quad (65) \]

The factors \( \mathcal{N}_+ \) and \( \mathcal{N}_- \) given by

\[ \mathcal{N}_+ = \mathcal{N}_+ (|z|) = \left( \sum_{n=0}^{\infty} \frac{r^{2n} \cos^2(n\phi)}{F(n)} \right)^{-\frac{1}{2}} \quad (66) \]

and

\[ \mathcal{N}_- = \mathcal{N}_- (|z|) = \left( \sum_{n=0}^{\infty} \frac{r^{2n+2} \sin^2((n+1)\phi)}{F(n+1)} \right)^{-\frac{1}{2}} \quad (67) \]

are the normalization constants.
3.4 Probability distribution of real and Imaginary nonlinear Schrödinger Cat states

The probability distribution of Cat states associated with $x^4$-anharmonic oscillator is defined as follow

$$P_{\pm}(n) = \left| \langle n, \varepsilon | z, \alpha \rangle \right|^2. \quad (68)$$

Using equations (64) and (65), we obtain

$$P_{\pm}(n) = \frac{r^{2n} (1 \pm \cos (2n\phi))}{A_\pm}, \quad (69)$$

where

$$A_+ = F(n) \left[ \sum_{m=0}^{\infty} \frac{r^{2m} (1 + \cos (2m\phi))}{F(m)} \right],$$

$$A_- = F(n) \left[ \sum_{m=0}^{\infty} \frac{r^{2m+2} (1 + \cos ((2m+2)\phi))}{F(m+1)} \right]. \quad (70)$$

It is clear that, in the limit $\varepsilon \to 0$, the equation (69) reproduce the probability distribution of Cat states for the harmonic oscillator.

4 Robertson-Schroedinger uncertainty relation

Using the creation and annihilation operators $A^+$ and $A^-$, we introduce two hermitian operators $X$ and $P$

$$X = \frac{1}{\sqrt{2}} \left( A^- + A^+ \right), \quad \quad P = \frac{i}{\sqrt{2}} \left( A^+ - A^- \right), \quad (71)$$

which satisfy the following commutation relation

$$[X, P] = iG(N) \equiv iG, \quad (72)$$

where the operator $G(N)$ is given by

$$G(N) = 1 + 3\varepsilon(N+1). \quad (73)$$

For $\varepsilon \neq 0$, the operator $G$ is not a multiple of the unit operator. Then, for the operators $X$ and $P$ satisfying the commutation relation (72), the variances $(\Delta X)^2$ and $(\Delta P)^2$ obey to the Robertson-Schroedinger uncertainty relation

$$(\Delta X)^2 (\Delta P)^2 \geq \frac{1}{4} \left( \langle G \rangle^2 + \langle C \rangle^2 \right), \quad (74)$$

where the operator $C$ is defined by

$$C = \{ X - \langle X \rangle, P - \langle P \rangle \} \quad (75)$$

or, in terms of the operators $A^-$ and $A^+$, by

$$C = i \left[ (2A^- - \langle A^- \rangle) \langle A^- \rangle + (-2A^+ + \langle A^+ \rangle) \langle A^+ \rangle - A'^{-2} + A'^{+2} \right]. \quad (76)$$
The symbol \{, \} stands for the anticommutator. When there is a correlation between \(X\) and \(P\) i.e., \(\langle C \rangle \neq 0\), such a relation is a generalization of the usual one (Heisenberg uncertainty relation)

\[
(\Delta X)^2 (\Delta P)^2 \geq \frac{1}{4} \langle G \rangle^2.
\]

The special form (77) is, of course, identical with the general form (74) if \(X\) and \(P\) are uncorrelated, i.e. \(\langle C \rangle = 0\).

The general uncertainty relation (74) is better suited to determine the lower bound on the product of variances in the measurement of observables corresponding to noncanonical operators. The Robertson-Schrödinger uncertainty relation give us a new understanding of what the states are coherent and squeezed for an arbitrary quantum system [28]. Indeed the so called generalized intelligent states are obtained when the equality in the Robertson-Schrödinger uncertainty relation is realized (see [22, 28]). The inequality in equation (74) become the equality for the states satisfying the following eigenvalues equation

\[
(X + i\lambda P) |\psi\rangle = z\sqrt{2} |\psi\rangle, \quad \lambda, z \in \mathbb{C}.
\]

As a consequence, we have the following relations

\[
(\Delta X)^2 = |\lambda| \Delta, \quad (\Delta P)^2 = \frac{1}{|\lambda|} \Delta,
\]

with

\[
\Delta = \frac{1}{2} \sqrt{\langle G \rangle^2 + \langle C \rangle^2}.
\]

Note that the average values \(\langle G \rangle\) and \(\langle C \rangle\) can be expressed in terms of the variances as

\[
\langle G \rangle = 2 \text{Re} (\lambda) (\Delta P)^2, \quad \langle C \rangle = 2 \text{Im} (\lambda) (\Delta P)^2.
\]

It is clear, from (79), that in the case where \(|\lambda| = 1\),

\[
(\Delta X)^2 = (\Delta P)^2,
\]

and we call the states satisfying (82), with \(|\lambda| = 1\), the generalized coherent states and for \(|\lambda| \neq 1\), the states are called generalized squeezed states.

Using the equation (78), one can obtain some general relations for the average values and dispersions for the operators \(X\) and \(P\) in the states minimizing the Robertson-Schrödinger uncertainty relation (74). Indeed, we have [18, 21]

\[
(\Delta X)^2 = \frac{1}{2} \left( \text{Re} (\lambda) \langle G \rangle + \text{Im} (\lambda) \langle C \rangle \right),
\]

\[
(\Delta P)^2 = \frac{1}{2 |\lambda|^2} \left( \text{Re} (\lambda) \langle G \rangle + \text{Im} (\lambda) \langle C \rangle \right),
\]

\[
\text{Im} (\lambda) \langle G \rangle = \text{Re} (\lambda) \langle C \rangle.
\]

The minimization of the Robertson-Schrödinger uncertainty relation leads to generalized coherent states for \(|\lambda| = 1\) (including Gazeau-Klauder coherent states) and generalized squeezed states for \(|\lambda| \neq 1\).
4.1 Generalized Intelligent States for $x^4$-anharmonic oscillator

In order to give a complete classification of the coherent and squeezed states for $x^4$-anharmonic oscillator, we will, in what follows, solve the eigenvalues equation (78). So, using the definition of $X$ and $P$ in terms of the annihilation and creation operators $A^-$ and $A^+$, the eigenvalues equation (78) takes the following form:

$$
(1 - \lambda) A^+ + (1 + \lambda) A^- \langle \psi \rangle = 2z \langle \psi \rangle.
$$

Let us compute $\langle \psi \rangle$ explicitly using (86). We take

$$
\langle \psi \rangle = \sum_{n=0}^{\infty} c_n |n, \varepsilon\rangle.
$$

So that

$$
(1 - \lambda) c_{n-1} \sqrt{e_n} e^{-i\alpha(e_n-e_{n-1})} + (1 + \lambda) c_{n+1} \sqrt{e_{n+1}} e^{i\alpha(e_{n+1}-e_n)} = 2zc_n,
$$

$$
(1 + \lambda) \sqrt{e_1} c_1 = 2ze^{-i\alpha e_1} c_0.
$$

The recurrence formulae (88) will give a complete classification of generalized intelligent states corresponding to the system under consideration.

By taking specific values of $\lambda$ and $z$, we now analyze the solution arising from the recurrence relation (88). We will study the cases ($\lambda \neq -1, z \neq 0$), ($\lambda = 1, z \neq 0$), ($\lambda = -1, z \neq 0$), and ($\lambda \neq -1, z = 0$).

We start by examining the general case ($\lambda \neq -1, z \neq 0$). To solve the eigenvalues equation (86), we set

$$
A_{n+1} = \frac{c_{n+1}}{c_n} \sqrt{e_{n+1}} e^{i\alpha(e_{n+1}-e_n)}.
$$

The relation (88) can be written in terms of the new coefficients $A_n$ as follows

$$
A_1 = \frac{2z}{(1 + \lambda)}; \quad A_n = \frac{2z}{(1 + \lambda)} + \left(\frac{\lambda - 1}{\lambda + 1}\right) \frac{e_{n-1}}{A_{n-1}}.
$$

By some elementary manipulation, we obtain the coefficients $A_n$ as a continued fraction given by:

$$
A_n = \frac{2z}{1 + \lambda} + \frac{\left(\frac{\lambda - 1}{\lambda + 1}\right) e_{n-1}}{2z + \frac{2z}{1 + \lambda} + \frac{\left(\frac{\lambda - 1}{\lambda + 1}\right) e_{n-2}}{2z + \frac{2z}{1 + \lambda} + \ldots} + \ldots + \frac{2z}{1 + \lambda} + \left(\frac{\lambda - 1}{\lambda + 1}\right) e_1}.
$$

Using the result (91) and the equation (89), we prove that the coefficients $c_n$ are given by
\[ c_n = c_0 \frac{(2z)^n}{(1 + \lambda)^n} \sqrt{F(n)} \left[ \sum_{h=0(1)}^{n-(2h-1)} (-1)^h \frac{(1 - \lambda^2)^h}{(2z)^{2h}} \Delta(n,h) \right] e^{-i\alpha n}, \quad (92) \]

where the symbol \( \left\lfloor \frac{n}{2} \right\rfloor \) stands for the integer part of \( \frac{n}{2} \) and the function \( \Delta(n,h) \) is defined by

\[ \Delta(n,h) = \sum_{j_1=1}^{n-(2h-1)} e_{j_1} \left[ \sum_{j_2=j_1+2}^{n-(2h-3)} e_{j_2} \cdots \left[ \sum_{j_h=j_h-1+2}^{n-1} e_{j_h} \right] \right] \cdots. \quad (93) \]

The states \( \left| \psi \right> = \sum_{n=0}^{\infty} c_n \left| n, \varepsilon \right> \) obtained here can be also written as some operator acting on the ground state \( \left| 0, \varepsilon \right> \). Indeed, we have

\[ \left| \psi \right> = U(\lambda \neq -1, z \neq 0) \left| 0, \varepsilon \right>, \quad (94) \]

where the operator \( U(\lambda \neq -1, z \neq 0) \) is given by

\[ U(\lambda \neq -1, z \neq 0) = c_0 \sum_{n=0}^{\infty} \left( \frac{2z}{1+\lambda} \right) \frac{1}{H} A^+ + \left( \frac{\lambda - 1}{\lambda + 1} \right) \frac{1}{H} \left( A^+ \right)^2 \right)^n. \quad (95) \]

Taking \( \lambda = 1 \) one can recover the Gazeau-Klauder coherent states (up to normalization constant)

\[ \left| \psi \right> = \exp \left( z \frac{N}{H} A^+ \right) \left| 0, \varepsilon \right> \quad (96) \]

given here as the action of the operator \( \exp \left( z \frac{N}{H} A^+ \right) \) on the ground state \( \left| 0, \varepsilon \right> \). In the limit \( \varepsilon \to 0 \), we get the generalized intelligent states (up to normalization constant)

\[ \left| \psi \right> = \exp \left( \frac{2z}{1+\lambda} A^+ \right) \exp \left( \left( \frac{\lambda - 1}{\lambda + 1} \right) \frac{(a^+)^2}{2} \right) \left| 0 \right> \quad (97) \]

corresponding to harmonic oscillator.

It is clear that the equation (86) give also the Gazeau-Klauder coherent states. In this case, the variances \((\Delta X)^2\) and \((\Delta P)^2\) are given by

\[ (\Delta X)^2 = (\Delta P)^2 = \frac{1}{2} \left< G \right>, \quad (98) \]

where

\[ \left< G \right> = (1 + 3\varepsilon) + 3\varepsilon |z|^2 \left. _0F_1 \left( \frac{2}{3\varepsilon} + 3, \frac{2}{3\varepsilon} |z|^2 \right) \right|_{1 + 3\varepsilon} \left. _0F_1 \left( \frac{2}{3\varepsilon} + 2, \frac{2}{3\varepsilon} |z|^2 \right) \right. \quad (99) \]

Note that (see the equation (85)) \( \left< C \right> = 0 \) when \( \lambda = 1 \). Then, the Gazeau-Klauder coherent states are minimum Heisenberg uncertainty states and verify

\[ (\Delta X)(\Delta P) = \frac{1}{2} \left[ (1 + 3\varepsilon) + 3\varepsilon |z|^2 \left. _0F_1 \left( \frac{2}{3\varepsilon} + 3, \frac{2}{3\varepsilon} |z|^2 \right) \right|_{1 + 3\varepsilon} \left. _0F_1 \left( \frac{2}{3\varepsilon} + 2, \frac{2}{3\varepsilon} |z|^2 \right) \right. \right]. \quad (100) \]
By taking the limit \( \varepsilon \to 0 \), one can see that the previous formula takes the simpler form

\[
(\Delta X)(\Delta P) = \frac{1}{2}
\]

which is a standard result.

Thus, we have found some additional information concerning the Gazeau-Klauder coherent states. The result obtained here is true for any (exact solvable model) arbitrary quantum system; see the reference [28]. Indeed, through the analysis of the states minimizing the Robertson-Schrödinger uncertainty relation, we have been able to show that the Gazeau-Klauder coherent states (the eigenvectors of the annihilation operator) are the states minimizing the Heisenberg uncertainty relation.

For \( (\lambda = -1, z \neq 0) \), we have to solve the eigenvalues equation

\[
A^+ |\psi\rangle = z |\psi\rangle.
\]

Then, the recurrence relation (88) rewrite as

\[
z c_n = c_{n-1} \sqrt{n} e^{-i\alpha(e_n - e_{n-1})}, \quad c_0 = 0.
\]

The coefficients \( c_n \) vanish and we exclude out this case of our analysis.

In the case where \( (\lambda \neq -1, z = 0) \) putting \( z = 0 \) in the equation (95), the solution is given by the action of the operator

\[
U(\lambda \neq -1, z = 0) = \exp \left( \frac{1}{2} \left( \frac{\lambda - 1}{\lambda + 1} \right) \frac{N}{H} \left( A^+ \right)^2 \right),
\]

on the ground state \( |0, \varepsilon\rangle \) where \( H \) is the Hamiltonian of our system. We rewrite the state \( |\psi\rangle \), in a compact form, as follows

\[
|\psi\rangle = U(\lambda \neq -1, z = 0) |0, \varepsilon\rangle.
\]

Note that for \( \lambda = 1 \), we have

\[
U(\lambda = 1, z = 0) = I
\]

and the state \( |\psi\rangle \) are nothing but the ground state \( |0, \varepsilon\rangle \) which is annihilated by the operator \( A^- (A^- |0, \varepsilon\rangle = 0) \).

Note that the solution \( |\psi\rangle \) is a linear combination of the states \( |2k, \varepsilon\rangle \) \((k = 0, 1, 2, \ldots)\)

\[
|\psi\rangle = \sum_{k=0}^{\infty} c_{2k} |2k, \varepsilon\rangle,
\]

where the coefficients \( c_{2k} \) are given by

\[
c_{2k} = \left( \frac{(\lambda - 1)}{2(\lambda + 1)} \right)^k \frac{1}{\Gamma(k + 1)} \sqrt{\frac{\Gamma(2k + 1)\Gamma \left( 1 + k + \frac{1}{3\varepsilon} \right) \Gamma \left( \frac{1}{3\varepsilon} + \frac{3}{2} \right)}{\Gamma(\frac{1}{3\varepsilon} + 1)\Gamma(k + \frac{1}{3\varepsilon} + \frac{3}{2})}} e^{-i\alpha e_{2k}} c_0.
\]

The coefficients \( c_0 \) is the normalization constant which can be evaluated by imposing the normalization condition \( \langle \psi |\psi\rangle = 1 \). We obtain

\[
c_0 = \left[ \sum_{k=0}^{\infty} \left( \frac{(\lambda - 1)}{2(\lambda + 1)} \right)^{2k} \frac{\Gamma(2k + 1)\Gamma \left( 1 + k + \frac{1}{3\varepsilon} \right) \Gamma \left( \frac{1}{3\varepsilon} + \frac{3}{2} \right)}{\left( \Gamma(k + 1) \right)^2 \Gamma(\frac{1}{3\varepsilon} + 1)\Gamma(k + \frac{1}{3\varepsilon} + \frac{3}{2})} \right]^{-\frac{1}{2}}.
\]
4.2 Analytical representation of Gazeau-Klauder coherent states

It is well known that the analytical representation enable one to find simpler solution of a number of problems, exploiting the theory of analytical entire functions. In this section, generalizing the pioneering work of Bargmann [29] for the usual harmonic oscillator, we will study the analytical representation of the extended Weyl-Heisenberg algebra (dynamical algebra for the $x^4$-anharmonic oscillator).

We recall that in the analytical representation of the standard harmonic oscillator, the creation operator $a^+$ is the multiplication by $z$ while the operator $a^-$ is the differentiation with respect to $z$.

We define the analytic space as a space of functions which are holomorphic on a ring $D$ of the complex plane. The scalar product is written with an integral of the form

$$\langle f \mid g \rangle = \int f(z)g(z) \, d\mu(z),$$

where $d\mu(z)$ is the measure defined above (see eq. (20)). Let $|f\rangle$ be an arbitrary state of the system under study

$$|f\rangle = \sum_{n=0}^{\infty} f_n |n,\varepsilon\rangle \quad \text{with} \quad \sum_{n=0}^{\infty} |f_n|^2 < \infty \quad (111)$$

Following the construction of [29], any state $|f\rangle$ is represented, in the analytical representation, by a function of the complex variable $z$ (using the coherent states associated with $x^4$-anharmonic oscillator)

$$f(z) = \langle z, -\alpha \mid f \rangle = \sum_{n=0}^{\infty} \frac{z^n e^{i\alpha n}}{\sqrt{F(n)}} f_n, \quad (112)$$

where the variable $z$ belong to the domain $D = \mathbb{C}$ of definition of the eigenvalues of $A^-$ (annihilation operator). In particular to the basis vectors $|n,\varepsilon\rangle$, there correspond the monomials

$$\langle z, -\alpha \mid n,\varepsilon \rangle = \frac{z^n e^{i\alpha n}}{\sqrt{F(n)}}. \quad (113)$$

Using the above considerations, we represent in the analytical representation the annihilation operator $A^-$ by

$$A^- = (1 + 3\varepsilon) \frac{d}{dz} + \frac{3\varepsilon}{2} z \frac{d^2}{dz^2},$$

the creation operator $A^+$

$$A^+ = z, \quad (115)$$

and the operator number $N$ by

$$N = z \frac{d}{dz}. \quad (116)$$

Remark that, in the limit $\varepsilon \to 0$, we have the standard analytical representation of the Weyl-Heisenberg algebra.
\[ A^- \equiv a^- = \frac{d}{dz}, \quad A^+ \equiv a^+ = z, \quad N = a^+ a^- = z \frac{d}{dz}. \]

The analytical representation exists if we have a measure such that

\[ \int |z, \alpha \rangle \langle z, \alpha| d\mu(z) = I_H. \] (117)

The existence of the measure ensures that scalar product of the representation takes the form (110).

## 4.3 Analytical representation of Generalized Intelligent States for \(x^4\)-anharmonic oscillator

Let us now construct and discuss, using the analytical representation, the \(x^4\)-anharmonic oscillator generalized intelligent states. We constructed the \(x^4\)-oscillator equal variance \(|z, \alpha\rangle \equiv |z, \lambda = 1, \alpha\rangle\), the Gazeau-Klauder coherent states or the eigenstates of the annihilation operator \(A^-\). These states form an overcomplete family of states (resolving the unity by integration with respect to measure given by equation (24)) and provide a representation of any state \(|\psi\rangle\) by an entire function \(\langle z, -\alpha | \psi \rangle = \psi(z)\). In what follows we will use the analytical representation discussed previously to convert the eigenvalues equation (86) into homogenous differential equation. The latter permits the construction of the generalized intelligent states, corresponding to the system under consideration, in an analytical way. Indeed, using the analytic realization of the creation \(A^+\) and annihilation \(A^-\) operators, the equation (we denote for a while the eigenvalue by \(z'\))

\[ \left( (1 + \lambda) A^- + (1 - \lambda) A^+ \right) |z', \lambda, \alpha\rangle = 2z' |z', \lambda, \alpha\rangle, \] (118)

now reads

\[ \left\{ \left( (1 + \lambda) \frac{3\varepsilon}{2} \right) \left[ \frac{2}{3\varepsilon} (1 + 3\varepsilon) \frac{d}{dz} + z \frac{d^2}{dz^2} \right] + (1 - \lambda) z \right\} \Phi(z', \lambda)(z) = 2z' \Phi(z', \lambda)(z). \] (119)

By means of simple substitutions the above equation is reduced to the Kummer equation for the confluent hypergeometric function \(_1F_1 (a, b; z)\) [30]. So that we have the following solution

\[ \Phi(z', \lambda)(z) = \exp(cz) \ _1F_1 (a, b, -2cz), \] (120)

where

\[ a = 1 + \frac{1}{3\varepsilon} - \frac{z'}{2\mu c}, \quad b = \frac{2}{3\varepsilon} + 2, \quad c^2 = -\frac{v}{\mu}, \]

\[ \mu = (1 + \lambda) \frac{3\varepsilon}{2}, \quad v = (1 - \lambda). \] (121)

Thus, we obtain the \(x^4\)-anharmonic oscillator intelligent states in the coherent states representation in the form (up to the normalization constant)

\[ \langle z', \lambda, \alpha | z, \alpha \rangle = \exp(c^*z) \ _1F_1 (a^*, b, -2c^*z), \] (122)
where the parameters $a$, $b$ and $c$ are defined in the formulae (121). Using the power series of the confluent hypergeometric function $\,_{1}F_{1}(a, b, z)$, we get at $\lambda = 1$ the Gazeau-Klauder coherent states discussed in the section 2 (up to normalization constant)

$$\langle z', \lambda = 1, \alpha \mid z, \alpha \rangle = \,_{0}F_{1} \left( 2 + \frac{2}{3\varepsilon}, \frac{2}{3\varepsilon} z \, z' \right).$$

(123)

We note also that the generalized intelligent states for the harmonic oscillator can be obtained from the equation (122) in the limit $\varepsilon \rightarrow 0$ (or from the differential equation (119) by setting $\varepsilon = 0$). Thus, we have

$$\Phi_{(z', \lambda)}(z) = \Phi_{(z', \lambda)}(0) \exp \left( \frac{2z'}{1+\lambda} z + \left( \frac{\lambda - 1}{\lambda + 1} \right) z^2 \right).$$

(124)

A solution which coincides with the result obtained in the section 4 (see the equation (97)).

5 Conclusion

In this work, we have constructed the Generalized Intelligent (Squeezed $|\lambda| \neq 1$ and Coherent $|\lambda| = 1$) States (GIS) for the $x^4$-anharmonic oscillator. This construction is based on the minimization of the Robertson-Schrödinger uncertainty relation. The operators generating the (GIS) families, by their actions on the ground state of the quantum system under consideration, are introduced. In the case where the parameter $\lambda = 1$, we recover the Gazeau-Klauder coherent states corresponding to the $x^4$-anharmonic oscillator. Their properties (continuity, temporal stability, overcompleteness and action identity) are studied. We also shown that there is a dynamical algebra, generated by raising and lowering operators, which may be seen as an extended Weyl-Heisenberg algebra and is isomorphic to $su(1, 1)$. The limit of the standard harmonic oscillator is discussed. In order to get the Generalized Intelligent States in an analytical way, we considered the analytical representation. The advantage of this representation is clearly illustrated. It should be noted that this work constitute another application of the recent developments in the construction of the Generalized Intelligent States obtained previously in [28] where a first application was made for the infinite square well and Pöschl-Teller potentials. A further extension concern the construction of the Perelomov coherent states type for an arbitrary quantum system and compare them with Gazeau-Klauder ones. This matter will be considered in a forthcoming work.

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