THE EXACTLY SOLVABLE SIMPLEST
MODEL FOR QUEUE DYNAMICS

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Abstract

We present an exactly solvable model for queue dynamics. Our model is very simple but provides the essential property for such dynamics. As an example, the model has the traveling cluster solution as well as the homogeneous flow solution. The model is the limiting case of Optimal Velocity (OV) model, which is proposed for the car following model to induce traffic jam spontaneously.

The concept of queue dynamics offers simple 1-dimensional models for socio-economic and complex multi-body physical systems, such as traffic and granular transport problems. Recently, several approaches have been developed on this field [1, 2, 3, 5, 7]. We propose an exactly solvable simplest model of this kind. Our model describes the general aspects of queue dynamics, and it can be widely applicable to such systems. But the following discussion is presented with the terminology of traffic problems.

The model is

$$\ddot{x}_n = a \{ V(\Delta x_n) - \dot{x}_n \},$$

(1)

where

$$\Delta x_n = x_{n-1} - x_n,$$

(2)

for each car number \(n (n = 1, 2, \cdots)\). \(x_n\) is the position of the \(n\)-th car, \(\Delta x_n\) is the headway of that car. Dot denotes the time derivative. \(a\) is a sensitivity constant, which we set the same value for all drivers. The function \(V(\Delta x_n)\), which is called “OV-function”, is

$$V(\Delta x_n) = v_{\text{max}} \cdot \theta(\Delta x_n - d),$$

(3)

where \(\theta\) is a Heaviside function. It decides the optimal velocity (the safety velocity) according to the headway: (a) if the headway is less than \(d\), a car

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should stop; (b) while the headway is larger than \( d \), a car can accelerate to move with the maximum velocity \( v_{\text{max}} \).

For both cases, the movement of each car is easily derived as follows.

(a) In the case of \( \Delta x_n < d \)

\[
x_n(t) = x_n(t_0) + \frac{\dot{x}_n(t_0)}{a} \{ 1 - e^{-a(t-t_0)} \}, \quad (4)
\]
with the initial condition, \( t = t_0, \ x(t_0), \dot{x}(t_0) \).

(b) In the case of \( \Delta x_n \geq d \)

\[
x_n(t) = x_n(t_0) + v_{\text{max}}(t - t_0) - \frac{v_{\text{max}} - \dot{x}_n(t_0)}{a} \{ 1 - e^{-a(t-t_0)} \}, \quad (5)
\]
with the same initial condition.

For the purpose of obtaining the solution of jam flow, we should consider the two basic processes of car moving. Suppose a jam exist in the lane, a car moves from “free driving region” into jam, and another car escapes from jam to free driving region. These two processes can be expressed by the above two solutions with appropriate connection conditions. We assume the ideal case: cars stop in jam with the same distance \( \Delta x_J (< d) \), and move at the maximum velocity \( v_{\text{max}} \) with the same distance \( \Delta x_F (> d) \) in free driving region.

First, we investigate the process of car moving from jam to free driving region. When the headway of the top car in a jam cluster is \( d \), we set \( t = t_0 \). At this time the position of the car is set as \( x_0 = 0 \). The \( n \)-th car in the jam moves as the following formulas.

\[
x_n(t) = -n \cdot \Delta x_J \quad \quad (t_0 \leq t < t_n), \quad (6)
\]
and

\[
x_n(t) = -n \cdot \Delta x_J + v_{\text{max}} \{ (t - t_n) - \frac{1}{a} [1 - e^{-a(t-t_n)}] \} \quad (t_n \leq t), \quad (7)
\]
where \( t_n \) is the time when the headway of the \( n \)-th car is \( d \). After \( t_n \) the car begins to move and escapes from jam. We note \( t_0 < t_1 < \cdots < t_{n-1} < t_n < \cdots \), and \( t_n \) is defined by

\[
\Delta x_n(t_n) = d. \quad (8)
\]
From (6) and (7), the definition (8) is written as

\[
\Delta x_J + v_{\text{max}} \{ (t_n - t_{n-1}) - \frac{1}{a} [1 - e^{-a(t_n-t_{n-1})}] \} = d. \quad (9)
\]
We have derived the sequence of equations for the definitions of \( t_n (n = 1, 2, \cdots) \), which has the solution for \( t_n > t_{n-1} \). It is easily obtained by putting \( \tau = t_1 - t_0 = \cdots = t_n - t_{n-1} = \cdots (> 0) \), and that is the unique solution, which is given by the following equation

\[
\Delta x_J + v_{\text{max}} \{ \tau - \frac{1}{a} (1 - e^{-a\tau}) \} = d. \quad (10)
\]
It is easily seen the velocity of the \( n \)-th car, \( \dot{x}_n(t) \) of (7), converges to \( v_{\text{max}} \) for sufficient large time \( (t \gg t_n) \). This means the car reaches the free driving region, and the headway of this car becomes \( \Delta x_F \), which is expressed as

\[
\lim_{t \to \infty} \Delta x_n(t) = \Delta x_F ,
\]

with using (7). Thus, we have obtained the following simple relation.

\[
\Delta x_J + v_{\text{max}} \tau = \Delta x_F .
\]

Next, we make an analogous investigation for the process of car moving from free driving region into jam. When the headway of the car positioned just behind a jam cluster is \( d \), we set \( t = t_0 \) and the position of the car \( x_0 = 0 \). The \( n \)-th car in the free driving region moves as the following formulas.

\[
x_n(t) = -n \cdot \Delta x_F + v_{\text{max}}(t - t_0) \quad (t_0 \leq t < t_n) ,
\]

and

\[
x_n(t) = -n \cdot \Delta x_F + v_{\text{max}}\left\{ (t_n - t_0) + \frac{1}{a}[1 - e^{-a(t-t_n)}] \right\} \quad (t_n \leq t) ,
\]

where \( t_n \) is the time when the headway of the \( n \)-th car is \( d \). After \( t_n \) the car begins to decelerate and moves into jam. We note \( t_0 < t_1 < \cdots < t_n-1 < t_n < \cdots \), and \( t_n \) is defined by

\[
\Delta x_n(t_n) = d .
\]

From (13) and (14), the definition (15) is written as

\[
\Delta x_F + v_{\text{max}}\left\{ -(t_n - t_{n-1}) + \frac{1}{a}[1 - e^{-a(t-t_n)}] \right\} = d .
\]

As the same as the previous case, we can obtain the unique solution of the sequence of the equations for \( t_n \) with the conditions, \( \tau = t_1-t_0 = \cdots = t_n-t_{n-1} = \cdots (> 0) \), where \( \tau \) is given by

\[
\Delta x_F + v_{\text{max}}\left\{ -\tau + \frac{1}{a}(1 - e^{-a\tau}) \right\} = d .
\]

The velocity of the \( n \)-th car, \( \dot{x}_n(t) \) of (14), converges to 0 for sufficient large time \( (t \gg t_n) \), which means the car reaches a jam cluster and stops. So, the headway of this car becomes \( \Delta x_J \), which is expressed as

\[
\lim_{t \to \infty} \Delta x_n(t) = \Delta x_J ,
\]

with using (14). Again, we have obtained the same relation as (12), which means the value of \( \tau \) is the same for both processes we have discussed above.

Now we can solve the equations (10), (12) and (17), thus \( \Delta x_F, \Delta x_J \) and \( \tau \) can be expressed using \( a, d \) and \( v_{\text{max}} \) as follows,
\[ \Delta x_F = d + \frac{v_{\text{max}} \tau}{2}, \]  
\[ \Delta x_J = d - \frac{v_{\text{max}} \tau}{2}. \]  
(19)  
(20)

\( \tau \) is determined by the following equation

\[ a \tau = 2(1 - e^{-a \tau}), \]  
(21)

which has the solution \( a \tau \simeq 1.59 \). As the result, each car is moving just the same manner as its front moving car with the time delay \( \tau \), which is induced in the process of obtaining the jam flow solution. And it is proportional to the inverse of sensitivity \( 1/a \). The collective movement of these cars forms a jam cluster.

In Fig. 1 the moving of several successive cars are shown. Their orbits are consisted of (13), (14), (6) and (7) with the time delay \( \tau \). Fig. 2 shows the velocity of car with time development, which is easily obtained as the time derivatives of these formulas. Rigorously say, it takes infinite time for car velocity to reach \( v_{\text{max}} \) or 0. Practically, an appropriate finite time is enough to be considered as infinite, as you see in Fig. 2.

In Fig. 1 a jam cluster moves backward against the direction of car moving. The velocity of the cluster is defined by the moving of the front position of the cluster, which is obtained by (13) or (14) as

\[ v_{\text{jam}} = \frac{x_n(t_n) - x_{n-1}(t_{n-1})}{t_n - t_{n-1}} = \frac{-\Delta x_J}{\tau}. \]

(22)
The same result is given by the moving of the rear point of the cluster from (13) or (14) with (12). Using the values of \( \Delta x_F \) and \( \Delta x_J \), the velocity is also defined as [6]

\[
v_{\text{jam}} = \frac{\Delta x_F \cdot V(\Delta x_J) - \Delta x_J \cdot V(\Delta x_F)}{\Delta x_F - \Delta x_J} = -\frac{\Delta x_J}{\tau}.
\]  

We obtain the same result as (22). We note each cluster should have the same velocity given above, if a jam flow contains several clusters. (Actually, you can see such a case later in Fig.4.)

![Figure 2: The changing of car velocity of three successive cars are drawn. The maximum velocity is set as \( v_{\text{max}} = 2 \). We can see a kink like shape.](image)

The profile of a jam flow is clearly described as the trajectory in the phase space of headway and velocity \( (\Delta x, \dot{x}) \). In order to draw this, it is enough to check the movement of two successive cars. The movement is consisted of two basic processes: the moving from jam to free driving region and the opposite case.

First, we check the process of car moving from jam to free driving region. The moving of the \((n-1)\)-th and \(n\)-th cars is divided into three stages presented in Table 1.

| \( x_{n-1} \) | \( \Delta x_{n-1} < d \) | \( \Delta x_{n-1} > d \) | \( x_n \) | \( \Delta x_n < d \) | \( \Delta x_n > d \) |
|---|---|---|---|---|---|
| \( x_{n-1} \) | (3) | (7) | \( x_n \) | (3) | (7) |

i) \( t_0 \leq t < t_{n-1} \): The headway of the \(n\)-th car is \( \Delta x_J \) and its velocity is 0. The \(n\)-th car stays the point \((\Delta x_J,0)\) in the phase space in this period, which means the car stays in jam.
ii) $t_{n-1} \leq t < t_n$ : The velocity $\dot{x}_n$ is 0. The headway is

$$
\Delta x_n = \Delta x_J + v_{\text{max}} \left\{ (t - t_{n-1}) - \frac{1}{a} \left[ 1 - e^{-a(t-t_{n-1})} \right] \right\}.
$$

From (20) and (21), the headway changes $\Delta x_J \leq \Delta x_n < d$ for $t_{n-1} \leq t < t_n$. Thus, the trajectory is the line $(\Delta x_J, 0) - (d, 0)$.

iii) $t_n \leq t < \infty$ : In this period, the headway and velocity are

$$
\Delta x_n = \Delta x_J + v_{\text{max}} \left\{ (t_n - t - t_{n-1}) + \frac{1}{a} \left[ e^{-a(t-t_n)} + e^{-a(t-t_{n-1})} \right] \right\},
$$

$$
\dot{x}_n = v_{\text{max}} \left[ 1 - e^{-a(t-t_n)} \right].
$$

From these equations with using (20) and (21), we derive the following relation

$$
\Delta x_n = d + \frac{\tau \dot{x}_n}{2} \quad (0 \leq \dot{x}_n < v_{\text{max}}).
$$

The trajectory is the line $(d, 0) - (\Delta x_J, v_{\text{max}})$.

Next, we turn to the car moving from free driving region into jam. The process is also divided into 3 stages, which is presented in Table 2.

| $t$ | $t_0$ | $t_{n-1}$ | $t_n$ |
|-----|-------|------------|-------|
| $x_{n-1}$ | $\Delta x_{n-1} > d$ : (13) | $\Delta x_{n-1} < d$ : (14) | |
| $x_n$ | $\Delta x_n > d$ : (13) | $\Delta x_n < d$ : (14) | |

Table 2: The movement of two successive cars from free driving region into jam.

i) $t_0 \leq t < t_{n-1}$ : The headway of the $n$-th car is $\Delta x_J$ and it’s velocity is $v_{\text{max}}$. The $n$-th car stays the point $(\Delta x_J, v_{\text{max}})$, which means the car moves in free driving region.

ii) $t_{n-1} \leq t < t_n$ : The velocity $\dot{x}_n$ is $v_{\text{max}}$. The headway is

$$
\Delta x_n = \Delta x_J + v_{\text{max}} \left\{ -(t - t_{n-1}) + \frac{1}{a} \left[ 1 - e^{-a(t-t_{n-1})} \right] \right\}.
$$

From (19) and (21), the headway changes $\Delta x_J \geq \Delta x_n > d$ for $t_{n-1} \leq t < t_n$. Thus, the trajectory is the line $(\Delta x_J, v_{\text{max}}) - (d, v_{\text{max}})$.

iii) $t_n \leq t < \infty$ : In this period, the headway and velocity are

$$
\Delta x_n = \Delta x_J + v_{\text{max}} \left\{ -(t_n - t - t_{n-1}) - \frac{1}{a} \left[ e^{-a(t-t_n)} + e^{-a(t-t_{n-1})} \right] \right\},
$$

$$
\dot{x}_n = v_{\text{max}} e^{-a(t-t_n)}.
$$

From these equations with (19) and (21), we derive the following relation

$$
\Delta x_n = \Delta x_J + \frac{\tau \dot{x}_n}{2} \quad (v_{\text{max}} \geq \dot{x}_n > 0).
$$
The trajectory is the line \((d, v_{\text{max}}) - (\Delta x_J, 0)\). We summarize the above results to Fig.3. The car movement in jam flow solution is represented as the square shaped closed loop in the phase space. All cars are moving along this loop in stable, which can be understood as a limit cycle.

In our previous work, we proposed the car following model (OV model) \([5]\), whose OV function is hyperbolic tangent instead of \((3)\). We found by the numerical simulation the model organized the limit cycle figure called “hysteresis loop”, which is attractive in the whole phase space \([4]\). The difference between our simple model and the previous case is only the shape of hysteresis loop. We are convinced that the essential property for the dynamics of jam flow solution of OV type model is clearly understood in our simple model.

The size of hysteresis loop \(\Delta x_F - \Delta x_J = v_{\text{max}} \tau\), which determine the amplitude of jam cluster, is characterized by the induced time delay \(\tau\). It is proportional to the inverse of sensitivity \(1/a\). The loop shrinks to the vertical line \(\Delta x = d\), \((0 < \dot{x} \leq v_{\text{max}})\) as \(a \to \infty\), in which case jam does not appear.

We should note the all above results do not depend on the total number of cars and the length of lane. We do not need to set the periodic boundary condition on the lane. If we set them as some conditions, it is easy to calculate the total number of cars in jam (the total sum of cluster size) or how long a car stays in jam.

Finally, we check the above results against the simulation data. The aim is to know how well the analytic solution is realized, which we have obtained by the infinite time approximation. We set the parameters of the model as \(a = 1\), \(d = 2\) and \(v_{\text{max}} = 2\). In this case \(\Delta x_F \simeq 3.59\) and \(\Delta x_J \simeq 0.41\) are derived from \((19)\) and \((20)\). The simulation is performed with putting cars on the circuit, whose length \(L = 200\), and the total number of cars \(N = 100\).

Fig.4 is the plot of the position of all cars with time development. The initial condition is set as all cars are uniformly distributed with the distance...
\[ \Delta x = L/N = 2(= d) \] and move with the same velocity \( \dot{x} = 1. \) In this case, the initial movement is highly unstable. We can observe the growth of jam clusters, and they are moving backward in stable with the same velocity, which value is in agreement with the analytic result (22).

The numerical result of \( N_J \) is just the same as that of the analytic prediction.

We should discuss about the stability of the jam flow solution as well as the homogeneous flow solutions, which can be considered as “free driving flow” with no jam. The stable cases of the homogeneous flow are trivial. If all cars stop with the headway less than \( d \), this situation is stable. While, if all cars are moving...
with the same velocity $v_{\text{max}}$ and with the headway larger than $d$, this flow is also stable. In contrast to the above two cases, if the headway of a car is more than $d$ while the others are less than $d$, or the opposite case, such flows are unstable and the jam flow is organized with time development. And, we can observe the jam flow is stable, as an example in Fig.4.

To summarize, we have proposed the simplest model for queue dynamics, which has the travelling cluster solution (jam flow) derived analytically. In the solution, cars are moving just the same manner with the induced time delay. The amplitude of the cluster is characterized by this time delay, which is related to the response sensitivity. The cluster has the profile of a limit cycle dynamics. This shows the “delay of changing motion” of each car, which means the balance of car moving in and out of a cluster, and this microscopic balance results the formation of the macroscopic cluster. Our simple model can provide the general understanding for queue dynamics by comparing with another approaches [9].

We thanks K. Nagel and M. Schreckenberg for inspiring us the idea of this work through the discussion during the one of the author(Y.S) was staying in Duisburg. And we should comment that T. Nagatani has the work concerned with the similar simplest cellular automaton model based on the OV model [9].

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Figure 6: The simulation data of the car points of jam flow in the phase space together with the hysteresis loop (Fig.3) and OV function. The distribution of car points is roughly corresponding to the rapidity of the movement of car.

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