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VOXER LATTICES AND THE BOGOLIUBOV-DE GENNES EQUATIONS

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ABSTRACT. We consider the Bogoliubov-de Gennes equations giving an equivalent formulation of the BCS theory of superconductivity. We are interested in static solutions with the magnetic field present. We carefully formulate the equations in the basis independent form, discuss their general features and isolate key physical classes of solutions (normal and vortex lattice states) which are the candidates for the ground state. We prove existence of the normal and vortex lattice states and stability of the normal states for large temperature or magnetic fields and their instability for small temperature and small magnetic fields.

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1. INTRODUCTION

1.1. Background. The Bogoliubov-de Gennes (BdG) equations describe the remarkable quantum phenomenon of superconductivity. They present an equivalent formulation of the BCS theory of superconductivity and are among the latest additions to the family of important effective equations of mathematical physics. Together with the Hartree, Hartree-Fock (Bogoliubov), Ginzburg-Landau, Gross-Pitaevskii and Landau-Lifshitz equations, they are the quantum members of this illustrious family consisting of such luminaries as the heat, Euler, Navier-Stokes, Boltzmann and Vlasov equations.

The BdG equations describe the evolution of superconductors on nanoscopic and macroscopic scales. There are still many fundamental questions about these equations which are completely open, namely

- Derivation;
- Well-posedness;
- Existence and stability of stationary magnetic solutions.

In this paper we address the third problem. By the magnetic solutions we mean solutions with non-zero magnetic fields.

The key special solutions of the Bogoliubov-de Gennes (BdG) equations are normal, superconducting and mixed or intermediate states. The superconducting (or Meissner) states assume, by the definition, that the magnetic field is zero, while mixed ones, to have non-vanishing magnetic fields. For type II superconductors, according to experiments, the latter are (magnetic) vortex lattices. In this paper, we prove the existence of the normal states for non-vanishing magnetic fields and partial results on their stability and the existence of the vortex lattices.

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1 For some physics background, see books [12, 27] and the review papers [11, 23].

2 A formal derivation of the (time-dependent) BdG equations is discussed in Appendix E. For a different but related formal derivation (going back to Dirac and Frenkel) see [6, 26]. Presently, there are no rigorous derivations. For a recent book on rigorous derivations of the simpler Hartree-Fock equation see [2], for some more recent papers [21, 22] and for the related bosonic Hartree-Fock-Bogoliubov equations [15, 16, 28]. For the literature before 2016, we refer only to the papers [21, 22], which led to the most of the recent developments and to the paper [14], dealing with the relation between the mean-field limit and deformation quantization. Following appearance of the first version of this paper on arXiv, the existence problem in the absence of magnetic field was taken up in [6] (cf. [8] for the global existence for the related bosonic Hartree-Fock-Bogoliubov equations).
There is a considerable physics literature devoted to the BdG equations, but despite the role played by magnetic phenomena in superconductivity it deals mainly with the zero magnetic field case, with only few disjoint remarks about the case when the magnetic fields are present, the main subject of this work.\footnote{The Ginzburg-Landau equations give a good account of magnetic phenomena in superconductors but only for temperatures sufficiently close to the critical one (see e.g. \[11, 12, 33\].}

As for rigorous work, it also deals exclusively with the case of zero magnetic field. The general (variational) set-up for the static BdG equations is given in \[4\]. The next seminal works on the subject are \[19\], where the authors prove the existence of superconducting states (the existence of the normal states under the assumptions of \[19\] is trivial), to which our work is closest, and \[13\], deriving the (macroscopic) Ginzburg-Landau equations. For an excellent, recent review of the subject, with extensive references and discussion see \[20\].

In the rest of this section we introduce the BdG equations, describe their properties and the main issues and present the main results of this paper. In the remaining sections we prove these results, with technical derivations delegated to appendices. In the last appendix, following \[3\], we discuss a formal, but natural, derivation of the BdG equations.

1.2. Bogoliubov-de Gennes (BdG) equations. In the Bogoliubov-de Gennes approach states of superconductors are described by the pair of bounded operators $\gamma$ and $\alpha$, acting on the one-particle complex Hilbert space, $\mathfrak{h}$, with a complex conjugation (an anti-linear involution), and satisfying

\[ 0 \leq \gamma = \gamma^* \leq 1, \quad \alpha^* = \overline{\alpha} \quad \text{and} \quad \alpha\alpha^* \leq \gamma(1-\gamma) \tag{1.1} \]

where $\overline{\gamma} := \mathcal{C}\gamma\mathcal{C}$, with $\mathcal{C}$, the operation of complex conjugation (see Appendix E for the origin of these operators). $\gamma$ is a one-particle density operator and $\alpha$ is a two-particle coherence operator, or diagonal and off-diagonal correlations. $\gamma(x,x)$ is interpreted as the one-particle density.

The one-particle space $\mathfrak{h}$ is determined by the many-body quantum problem. For zero density (or ‘finite’) systems, it is $L^2(\mathbb{R}^d)$ and for positive density ones, $L^2(\Omega)$, where $\Omega$ is an arbitrary fundamental cell in a lattice $\mathcal{L} \subset \mathbb{R}^d$, with magnetic field dependent (twisted) boundary conditions (see Subsection 1.6 for more details). To fix ideas, we take $\mathfrak{h}$ to be the latter, specifically,

\[ \mathfrak{h} := \{ f \in L^2_{\text{loc}}(\mathbb{R}^2) : u_s^\mathcal{L} f = f \text{ for all } s \in \mathcal{L} \}, \]

where $u_s^\mathcal{L}$ are the magnetic translations given in (1.23) below, with the scalar product given by $L^2(\Omega)$ for an arbitrary fundamental cell $\Omega$ of $\mathcal{L}$. Furthermore, we understand $f$ without specifying the domain of integration as taken over $\Omega$.

The BdG equations form a system of coupled, nonlinear equations for $\gamma$ and $\alpha$. It is convenient to organize the operators $\gamma$ and $\alpha$ into the self-adjoint operator-matrix

\[ \eta := \begin{pmatrix} \gamma & \alpha \\ \alpha^* & 1 - \gamma \end{pmatrix}. \tag{1.2} \]

The definition of $\gamma$ and $\alpha$ in terms of the many-body theory (see (E.2) of Appendix E) implies that

\[ 0 \leq \eta = \eta^* \leq 1 \quad \text{and} \quad J^* \eta J = 1 - \overline{\eta}, \quad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.3} \]

These relations imply relations (1.1).

Since the BdG equations describe the phenomenon of superconductivity, they are naturally coupled to the electromagnetic field. We describe the latter by the vector potential $a$ in the gauge in which the electrostatic potential is zero so that the equations have a slightly simpler
form). In what follows, for an operator $A$, we denote by $A(x,y)$ its integral kernel. Then the time-dependent BdG equations state (see e.g. [12, 11, 27])

\[ i \partial_t \eta = [\Lambda(\eta, a), \eta], \tag{1.4} \]

\[ \Lambda(\eta, a) = (\frac{h_{\gamma a}}{v^{*}_\alpha} - \frac{v^{*}_a}{h_{\gamma a}}), \quad h_{\gamma a} = -\Delta_a + v^* \gamma - v^{*}_\gamma, \tag{1.5} \]

where, for the pair potential $v(x,y)$, the operator $v^*$ (acting on operators on $\mathfrak{h}$) is defined through the integral kernels as $(v^* \alpha)(x;y) := v(x,y)\alpha(x;y)$ and $(v^*\gamma)(x) := \int v(x,y)\rho_{\gamma}(y)dy$, with $\rho_{\gamma}(x) := \gamma(x;x)$. $v^*\gamma$ and $v^{*}_\gamma$ are the direct and exchange self-interaction potentials, and $\Delta_a := |\nabla_a|^2, \nabla_a := \nabla - ia$. (1.4) is coupled to the Maxwell equation (Ampère’s law)

\[ \partial_t^2 a = - \text{curl}^* \text{curl} a + j(\eta, a), \tag{1.6} \]

where $j(\eta, a)(x) \equiv j(\gamma, a)(x) := [-i\nabla_a, \gamma]_+(x,x)$ is the superconducting current. Here $[A,B]_+$ is the anti-commutator of $A$ and $B$, i.e. $[A,B]_+ := AB + BA$. (To see that (1.6) is Ampère’s law, one recalls that in our gauge the electric field is $E = -\partial_t a$.) In what follows, we assume that

\[ v(x;y) = v(x - y) \quad \text{and} \quad v(-x) = v(x), \tag{1.7} \]

so that $v^*\gamma = v^* \rho_{\gamma}$. We specify below additional assumptions on $v(x,y)$ and on the operators $\gamma$ and $\alpha$ so that all the terms in (1.4)-(1.6) are well defined (at least weakly).

**Connection with the BCS theory.** (1.4) can be reformulated as an equation on the Fock space involving an effective quadratic Hamiltonian (see [11, 12, 20] and [3], for the bosonic version). These are the effective BCS equations and the effective BCS Hamiltonian.

**Stationary equations.** As was mentioned above, we are interested in stationary solutions to (1.4)-(1.6). These solutions satisfy the stationary BdG equations which, as we show below (see Proposition 2.1 and after), are of the form

\[ \Lambda_{\eta a} - T g'(\eta) = 0, \tag{1.8} \]

\[ \text{curl}^* \text{curl} a - j(\eta, a) = 0, \tag{1.9} \]

\[ \Lambda_{\eta a} := \left( \frac{h_{\gamma a}}{v^*_{\alpha}} - \frac{v^*_{a}}{h_{\gamma a}} \right), \tag{1.10} \]

with $h_{\gamma a} := h_{\gamma a} - \mu$, where $h_{\gamma a}$ is given in (1.5). Here $\mu$ and $T \geq 0$ are the chemical potential and temperature parameters. The physical function $g'$ is selected by either a thermodynamic limit (Gibbs states) or by a contact with a reservoir (or imposing the maximum entropy principle) and is given by

\[ g'(\lambda) = -\ln \frac{\lambda}{1-\lambda}. \tag{1.11} \]

Its inverse, $g'^{-1}$, is the Fermi-Dirac distribution

\[ f_{\text{FD}}(h) = (1 + e^h)^{-1}. \tag{1.12} \]

By inverting $g'$ in (1.8), we can rewrite this equation as the nonlinear Gibbs equation:

\[ \eta = f_T(\Lambda_{\eta a}), \quad \text{where} \quad f_T(\lambda) := f_{\text{FD}}(\frac{1}{T}\lambda). \tag{1.13} \]

In Remark 1.15 below, we rewrite (1.8) in terms of the eigenfunctions of $\Lambda_{\eta a}$ or $\eta$, the form common in physics literature and closer in spirit to the PDEs.
1.3. Symmetries and conservation laws. \((1.4)-(1.6)\) are invariant under the time-independent gauge transformations,
\[
T_χ^{\text{gauge}} : (γ, α, a) \mapsto (e^{iχγ}e^{-iχ}, e^{iχα}e^{iχ}, a + \nabla χ),
\]
for any sufficiently regular function \(χ : \mathbb{R}^d \to \mathbb{R}\), and the translation, rotation and reflection transformations,
\[
T_χ^{\text{trans}} : (γ, α, a) \mapsto (u_χγu_χ^{-1}, u_χαu_χ^{-1}, u_χa),
\]
\[
T_χ^{\text{rot}} : (γ, α, a) \mapsto (u_χγu_χ^{-1}, u_χαu_χ^{-1}, ρu_χα),
\]
\[
T_χ^{\text{refl}} : (γ, α, a) \mapsto (uγu_χ^{-1}, ua_χ^{-1}, -ua),
\]
for any \(h \in \mathbb{R}^d\) and \(ρ \in O(d)\). Here \(u_χ \equiv u_χ^{\text{trans}}\), \(u_χ \equiv u_χ^{\text{rot}}\) and \(u \equiv u_χ^{\text{refl}}\) are the standard translation, rotation and reflection transforms \(u_χ^{\text{trans}} : f(x) \mapsto f(x + h)\), \(u_χ^{\text{rot}} : f(x) \mapsto f(ρ^{-1}x)\) and \(u_χ^{\text{refl}} : f(x) \mapsto f(-x)\).

We also keep the notation \(T_χ^{\text{trans}}\) for these operators restricted to \(η\)'s. \((1.4)-(1.6)\) conserve the energy
\[
E(η, a) := \Tr ((-Δ_a)γ) + \frac{1}{2} \Tr ((v * ρ_γ)γ) - \frac{1}{2} \Tr ((v^γγ)γ)
+ \frac{1}{2} \Tr (α^*(v^α)) + \frac{1}{2} \int (|\text{curl} a|^2 + |∂ a|^2),
\]
where the trace is taken in the one-particle space \(h\). (Recall that in our gauge, \(-∂ a\) is the electric field.) Indeed, assuming \((η, a)\) solves \((1.4)-(1.6)\), taking the time derivative of \((1.18)\), using the expression
\[
d_a \Tr ((-Δ_a)γ)a' = -\Tr (j(η, a)a')
\]
for the Gâteaux derivative in \(a\) and using the notation \(\dot{f} \equiv ∂_t f\) and \(\ddot{f} \equiv ∂_{t,t} f\) for \(f = γ, α, a\), we find
\[
∂_t E(η, a) = \Tr (h_αγ) + \Tr (\dot{α}^*(v^aα) + α^*(v^α))
+ \langle \text{curl}^* \text{curl} - j(η, a), \dot{a} \rangle + \langle \dot{a}, \dot{a} \rangle,
\]
where the inner product is understood to be in \(L^2(Ω)\). A simple computation shows that line \((1.19)\) = \(\frac{1}{2} \Tr (ηΔ(η, a))\) and therefore by \((1.20)\) is 0. Line \((1.20)\) vanishes by \((1.6)\). Hence \(∂_t E = 0\).

Furthermore, the global gauge invariance implies the evolution conserves the number of particles, \(N := \Tr γ\).

1.4. Free energy. The stationary BdG equations arise as the Euler-Lagrange equations for the free energy (BCS) functional
\[
F_T(η, a) := E(η, a) - TS(η) - μN(η),
\]
where \(S(η) = \Tr g(η)\), with \(g(λ) := -λ \ln λ - (1 - λ) \ln (1 - λ)\), an anti-derivative of \((1.11)\), is the entropy, \(N(η) := \Tr γ\) is the number of particles, and \(E(η, a)\) is the energy functional \((1.18)\) for \(η\) and \(a\) time-independent and is given by
\[
E(η, a) = \Tr ((-Δ_a)γ) + \frac{1}{2} \Tr ((v * ρ_γ)γ) - \frac{1}{2} \Tr ((v^γγ)γ)
+ \frac{1}{2} \Tr (α^*(v^α)) + \int |\text{curl} a|^2.
\]
Note surprisingly, it turns out that \( E(\eta, a) := \varphi(H_a) \), where \( \varphi \) is a quasi-free state in question (see Appendix E) and \( H_a \) is the standard many-body given in (E.5), coupled to the vector potential \( a \).

From now on, we assume that \( d = 2 \). This is a typical case considered in physics applications, where it is assumed that the solutions are independent of the third coordinate (the cylindrical geometry in the physics terminology).

Let \( a_b(x) \) be the vector potential with the constant magnetic field, \( \text{curl} a_b = b \). Below, we work in the symmetric gauge: \( a_b(x) = \frac{b}{2}(-x_2, x_1) \).

1.5. Magnetic translations. Let \( \mathcal{L} \subset \mathbb{R}^2 \) be a Bravais lattice, fixed throughout the paper. We define the magnetic translation operator

\[
u^\ell_s := u^\text{gauge}_{-\chi^\ell_s} u^\text{trans}_s,
\]

(1.23)

where \( u^\text{gauge}_\chi : \phi(x) \mapsto e^{i\chi(x)} \phi(x) \) and, recall, \( u^\text{trans}_h : \phi(x) \mapsto \phi(x + h) \) (cf. the definitions after (1.17)) and \( \chi^\ell_s : \mathcal{L} \times \mathbb{R} \rightarrow \mathbb{R} \) is given by

\[
\chi^\ell_s(x) := \frac{b}{2}(s \wedge x) + c_s,
\]

(1.24)

where \( s = k\omega_1 + \ell\omega_2 =: s' + s'', k, \ell \in \mathbb{Z} \), for any \( s \in \mathcal{L} \), with \( \{\omega_1, \omega_2\} \) a basis in \( \mathcal{L} \), and \( b \) is given by

\[
b = \frac{2\pi n}{|\mathcal{L}|}, \quad n \in \mathbb{Z}.
\]

(1.25)

Here \( |\mathcal{L}| \) is the area of a fundamental cell of \( \mathcal{L} \) (which is independent of the choice of the cell). One can easily check that the operator-family \( u^\ell_s, s \in \mathcal{L} \), is a group representation:

\[
u^\ell_s u^\ell_t = u^\ell_{s+t}.
\]

(1.26)

1.6. One-particle spaces. With translations \( u^\ell_s \) given in (1.23), we define the periodic one-particle state space

\[
\mathfrak{h} := \{ f \in L^2_{\text{loc}}(\mathbb{R}^2) : u^\ell_s f = f \text{ for all } s \in \mathcal{L} \},
\]

(1.27)

which is a Hilbert space with the scalar product defined, for an arbitrary fundamental cell \( \Omega \) of \( \mathcal{L} \), as

\[
\langle f, g \rangle_{\mathfrak{h}} := \langle f, g \rangle_{L^2(\Omega)}.
\]

(1.28)

Similarly, we consider the Sobolev space of vector potentials: let \( \tilde{H}^r \) be given by

\[
\tilde{H}^r := \{ a \in H^r_{\text{loc}}(\mathbb{R}^2; \mathbb{R}^2) : T^\text{trans}_s a = a \forall s \in \mathcal{L}, \quad \text{div} a = 0, \quad \int a = 0 \},
\]

(1.29)

with the Sobolev norm \( ||a||_{H^r} \equiv ||a||_{H^r(\Omega)} \), for some (and therefore for every) fundamental cell \( \Omega \) of \( \mathcal{L} \). (The conditions \( \text{div} a = 0 \) and \( \int a = \int_\Omega a = 0 \) make the operator \( \text{curl}^* \text{curl} \) strictly positive.) Finally, we define the affine space

\[
\tilde{\mathfrak{h}}^r := a_b + \tilde{H}^r.
\]

(1.30)

(Recall that \( a_b(x) \) is the vector potential with the constant magnetic field.)
Now, we define spaces of $\gamma$'s and $\alpha$'s used below. Let $I^r$ denote the space of bounded operators satisfying $\|A\|_{I^r} := (\text{Tr}(A^*A)^{r/2})^{1/r} < \infty$ (Schatten, or a trace ideal, or non-commutative $L^r$-space) and let $M_{\bullet}$ := $\sqrt{-\Delta_{\bullet}}$. We define Sobolev-type spaces for trace class operators by

$$I_{s,1} := \{A : h \to h : \|A\|_{I_{s,1}} := \|M_{\bullet} A M_{\bullet}\|_{I^1} < \infty\},$$

(1.31)

$$I_{s,2} := \{A : h \to h : \|A\|_{I_{s,2}} := \|AM_{\bullet}\|_{I^2} < \infty\}.$$  

(1.32)

Note that $I_{0,p} = I^p$. We will usually assume $\gamma \in I_{1,1}$ and $\alpha \in I_{1,2}$. (Strictly speaking $\alpha$ acts from the dual space $h^*$ to $h$ (see [4] for details), but for the sake of notational simplicity we will ignore this subtlety, by identifying $h^*$ with $h$.) We will use the notation $\hat{I}_s$ for the space of $\eta$'s on $h \oplus h$ with $\gamma \in I_{s,1}$ and $\alpha \in I_{s,2}$ and the norm

$$\|\eta\|_{(s)} := \|\gamma\|_{I_{s,1}} + \|\alpha\|_{I_{s,2}}.$$  

(1.33)

Due to Lemma 5.4(2) below, $\gamma \in I_{1,1} \Rightarrow \alpha \in I_{1,2}$ and $\|\eta\|_{(s)} \simeq \|\gamma\|_{I_{s,1}}$. Furthermore, we define

$$D^s_{\nu} = \{\eta \in \hat{I}_s, \eta \text{ satisfies (1.3), Tr}\gamma = \nu\}.$$  

(1.34)

1.7. Ground states of BdG equations. The static BdG equations (1.8)-(1.9) have the following key classes of solutions which are candidates for the ground states:

1. Normal state: $(\eta, a)$, with $a = 0$ (i.e. $\eta$ is diagonal).
2. Superconducting state: $(\eta, a)$, with $\alpha \neq 0$ and $a = 0$.
3. Mixed state: $(\eta, a)$, with $\alpha \neq 0$ and $a \neq 0$.

We discuss the above states in more detail.

Superconducting states. The existence of superconducting, translationally invariant solutions is proven in [19]. (See the second part of Subsection 1.1 and [20] for the references to earlier results.)

Normal states. Since we assumed that the external fields are zero, the equations (considered in $\mathbb{R}^2$) are translationally invariant. Because of the gauge invariance, it is natural to consider the simplest, gauge (magnetically) translationally invariant solutions, i.e. solutions invariant under the transformations

$$T_{bs} := (T_{\text{gauge}}^s)^{-1} T_{\text{trans}}^s,$$  

(1.35)

for any $s \in \mathbb{R}^2$ and the function $\chi^b_s(x), s, x \in \mathbb{R}^2$, given by

$$\chi^b_s(x) := \frac{b}{2} (s \wedge x).$$  

(1.36)

For $b = 0$, we can choose $a = 0$. In this case, the existence of normal translationally invariant solutions was proven in [10].

For $b \neq 0$, the simplest normal states are the magnetically translation (mt-) invariant ones, i.e. ones satisfying

$$T_{bs}(\eta, a) = (\eta, a),$$  

(1.37)

for any $s \in \mathbb{R}^2$, where $T_{bs}$ is defined in (1.35). Here the operator $\eta$ acts on $L^2_{\text{loc}}(\mathbb{R}^2) \times L^2_{\text{loc}}(\mathbb{R}^2)$. 


**Mixed states.** The main candidate for a mixed state is a *vortex lattice*, i.e. a state, \((\eta, a)\), satisfying \(\alpha \neq 0\) and the equivariance condition

\[
T_s^{\text{trans}}(\eta, a) = T^{\text{gauge}}(\eta, a) \quad \text{for every} \quad s \in \mathcal{L},
\]

where \(\chi_s^L : \mathcal{L} \times \mathbb{R}^2 \to \mathbb{R}\) are defined in (1.24), with \(b\) satisfying the quantization condition (1.25).

**Proposition 1.1** (Magnetic flux quantization). If a vector potential \(a\) satisfies (1.38), then for any fundamental cell \(\Omega\) of \(\mathcal{L}\), we have

\[
\frac{1}{2\pi} \int_{\Omega} \text{curl } a = c_1(\chi^L) \in \mathbb{Z},
\]

where \(c_1(\chi^L)\), called the first Chern number of \(\chi_s^L(x)\), is the integer \(n\) entering (1.24)-(1.25).

One can show that \(\chi^L = \chi_s^L(x)\) is a co-cycle and \(c_1(\chi^L)\) can be defined in terms of the function \(\chi^L\) using its co-cycle property and without reference to the explicit form (1.24) of \(\chi^L\) used, see Remark 1.18 for the definitions and a discussion.

Recall that \(a_b(x)\) is the magnetic potential with the constant magnetic field \(b\) (\(\text{curl } a_b = b\)). In what follows we fix the gauge in which \(a_b(x) = \frac{b}{2} \times x, \ast x := (-x_2, x_1)\). Then \(a_b\) satisfies (1.38), provided \(b\) satisfies the quantization condition (1.25).

1.8. **Results.** Let the reflection operator \(\tau^\text{refl}\) be given by conjugation by the reflections, \(u^\text{refl} : f(x) \to f(-x)\). We say that a state \((\eta, a)\) is *even* (or reflection symmetric) if and only if

\[
\tau^\text{refl} \gamma = \gamma, \quad \tau^\text{refl} \alpha = \alpha \quad \text{and} \quad u^\text{refl} a = -a.
\]

The reflection symmetry of the BdG equations implies that if an initial condition is even then so is the solution at every moment of time.

In what follows, we use the notation \(A \lessgtr B\) and \(B \gtrless A\) to signify the inequalities \(A \leq CB\) and \(B \geq cA\), where \(C\) and \(c\) are positive constants independent of the parameters involved.

**Existence of normal and vortex lattice solutions.** We say an operator \(A\) is magnetically translation invariant (*mt-invariant*, for short), if it satisfies \(\tau^{\text{refl}} A = A, \forall h \in \mathbb{R}^2\).

The existence of the mt-invariant normal states for \(b \neq 0\) is stated in the following theorem proven in Section 3.

**Theorem 1.2.** Drop the exchange term \(\psi^\# \gamma\) and let \(v\) be \(\mathcal{L}\)-periodic and either \(\int v > 0\), or \(\|v\|_\infty < \infty\). Then the BdG equations (1.8)-(1.9) on the space \(\mathcal{I}^2.1 \times \mathcal{I}^2.2 \times \mathcal{H}^2\) have a mt-invariant solution, unique on the set of even (in the sense of the definition (1.40)) pairs \((\eta, a)\), with \(\eta\) mt-invariant (i.e. satisfying (1.37)).

Moreover, this solution is normal (i.e. \(\alpha = 0\)) and is of the form \((\eta_T, a_b)\) and

\[
\eta_T := \begin{pmatrix} \gamma_T & 0 \\ 0 & 1 - \gamma_T \end{pmatrix},
\]

with \(\gamma_T\) solving the equation

\[
\gamma = f_T(\lambda \gamma_0),
\]

where \(f_T(\lambda)\) is given in (1.12).

The search for mt-invariant solutions is simplified by the following statement proven in Section 3.

**Proposition 1.3.** If \(\eta\) is mt-invariant, then \(\eta\) is normal, i.e. \(\alpha = 0\).

Recall that \(b = \frac{2\pi n}{|\xi|}\), see (1.25). For vortex lattices, we have the following result
Theorem 1.4. Drop the the exchange term $v^* \gamma = v \ast \rho_\gamma$ in the definition of $h_{\gamma a}$ in (1.5). Fix a lattice $\mathcal{L}$ and a value of the Chern number $c_1(\chi^L) = n \in \mathbb{Z}$ (see (1.39)) and assume that $v$ is $\mathcal{L}$-periodic and obeys $\|v\|_\infty < \infty$. Then

(i) for any $T \geq 0$, there exists a (weak) solution $(\eta, a) \in D^1_v \times \tilde{\mathcal{H}}^1$ of the BdG equations (1.8)–(1.9) (in particular, it satisfies $T_s^{\text{trans}}(\eta, a) = T^{\text{gauge}}_{\chi^L}(\eta, a)$), which minimizes the free energy $F_T$ (for the given $c_1(\chi^L) = n \in \mathbb{Z}$);

(ii) for $T$ and $b$ sufficiently small, $(\eta, a)$ has $\alpha \neq 0$, i.e. this solution is a vortex lattice, provided $v(x) \lesssim -(1 + |x|)^{-\kappa}, \kappa < 2$, for $x$ in a fundamental cell $\Omega$ centred at the origin. More generally, the latter holds if the operator $L_{Tb}$, given in (1.46) and defined on $I^2,2$, has a negative eigenvalue.

Statement (i) is proven in Section 5 and (ii) follows from Proposition 1.8.

Stability/instability of normal solutions. We address the question of the energetic stability of the mt-invariant states. To this end we define, in the standard way, the Hessian of the free energy in $\eta$ as

$$F''_T(\eta_\ast, a_\ast) := d_{\eta_\ast} \text{grad}_{\eta} F_T(\eta_\ast, a_\ast),$$

(1.43)

where $d_{\eta_\ast}$ is the Gâteaux derivative w.r.t. $\eta$ and $\text{grad}_{\eta}$ is the gradient w.r.t. $\eta$, defined by the equation

$$\text{Tr}(\text{grad}_{\eta} F(\eta_\ast, a_\ast) \eta') = d_{\eta} F(\eta_\ast, a_\ast) \eta'.$$

We consider $F''_T(\eta_{Tb}, a_b)$ along physically relevant perturbations of the form $\eta' = \phi(\alpha)$, where $\phi(\alpha)$ is the off-diagonal operator-matrix, defined by,

$$\phi(\alpha) := \begin{pmatrix} 0 & \alpha \\ \alpha^* & 0 \end{pmatrix},$$

(1.44)

with $\alpha$ a Hilbert-Schmidt operator on $\mathfrak{h}$. Moreover, we require that perturbations $\eta' = \phi(\alpha)$ satisfy the condition

$$\alpha \alpha^* \lesssim [\gamma_{Tb}(1 - \gamma_{Tb})]^2$$

(1.45)

(which is equivalent to (2.11) at $\eta = \eta_{Tb}$, c.f. also Lemma 5.4(1)).

For an operator $h$, let $h^L$ and $h^R$ stand for the operators acting on other operators by multiplication from the left by $h$ and from right by $\bar{h}$, respectively, and recall $v^\sharp$ is defined after (1.4).

We have the following

Proposition 1.5. For off-diagonal perturbations $\eta' = \phi(\alpha)$, $F''_T(\eta_{Tb}, a_b) \phi(\alpha) = \phi(L_{Tb} \alpha)$, where the operator $L_{Tb}$ is given by

$$L_{Tb} := K_{Tb} + v^\sharp,$$

(1.46)

$$K_{Tb} := \frac{h^L_{Tb} + h^R_{Tb}}{\tanh(h^L_{Tb}/T) + \tanh(h^R_{Tb}/T)},$$

(1.47)

on the space of Hilbert-Schmidt operators on $\mathfrak{h}$. Here $h_{Tb} := h_{\gamma_{Tb} a_\mu}$ (with $h_{\gamma a} := h_{\gamma a} - \mu$, where $h_{\gamma a}$ is given in (1.5)).

Let $\langle \alpha, \alpha' \rangle := \text{Tr}(\alpha^* \alpha')$. We say that $\eta'$ is an off-diagonal perturbation if and only if $\eta' = \phi(\alpha)$ with $\alpha$ a Hilbert-Schmidt operator satisfying (1.45). The next result generalizes that of [19] for $a = 0$:

Proposition 1.6. For off-diagonal perturbations $\eta' = \phi(\alpha)$, we have

$$F_T(\eta_{Tb} + \epsilon \eta', a_b) = F_T(\eta_{Tb}, a_b) + \epsilon^2 \langle \alpha, L_{Tb} \alpha \rangle + O(\epsilon^3).$$

(1.48)
Definition (1.22) of $E(\eta, a)$ implies that $E(\eta_T b + \epsilon \eta', a_b) = E(\eta_T b, a_b) + \epsilon^2 \text{Tr}(\bar{\alpha} \bar{v}^4 \alpha)$ for $\eta' = \phi(\alpha)$. This together with Corollary A.3 and (A.17) of Appendix A dealing with the entropy yields Propositions [1.5] and [1.6] respectively. (The absence of the linear term is due to the fact that $(\eta_T b, 0)$ is a critical point of the energy function.)

The next two propositions are proven in Section 4.

**Proposition 1.7.** $L_T b \geq \frac{1}{2} T - \|v\|_\infty$ and consequently, for $T$ sufficiently large, $L_T b > 0$ and the mt-invariant state $(\eta_T b, a_b)$ is energetically stable under off-diagonal perturbations.

On the other hand, for $T$ and $b$ sufficiently small, we have

**Proposition 1.8.** Drop the exchange term $-\bar{v}^5 \gamma$. Suppose that $v(x) \lesssim - (1 + |x|)^{-\kappa}, \kappa < 2$, for $x$ in a fundamental cell $\Omega$ centred at the origin. Then, for $T$ and $b$ sufficiently small and $T \lesssim b^\sigma, \sigma > \kappa/2$, the operator $L_T b$ acting on the set of Hilbert-Schmidt operators on $\mathfrak{h}$ has a negative eigenvalue and consequently the mt-invariant state $(\eta_T b, a_b)$ is energetically unstable under perturbations $\eta' = \phi(\alpha)$.

Note that $L_T b$ is discontinuous at $T = 0.$

Let $T_c(b)$ (resp. $T'_c(b)$) be the largest (resp. smallest) temperature s.t. the normal solution is energetically unstable (resp. stable) under off-diagonal perturbations $\eta' = \phi(\alpha)$, for $T < T_c(b)$ (resp. $T > T'_c(b)$). Clearly, $\infty \geq T_c(b) \geq T_c(b') \geq 0$. Propositions [1.7] and [1.8] imply

**Corollary 1.9.** Under the conditions of Proposition [1.8] $T_c(b) > 0$ for $b$ sufficiently small and $T'_c(b) = 0$ for $b$ sufficiently large.

We conjecture that (a) mt-invariant states coincide with normal states, (b) For $T \gg 1$, the free energy $F_T$ is minimized by normal states; (c) normal states are stable if either $T \gg 1$ or $b \gg 1$ and unstable if $T \ll 1$ and $b \ll 1$; (d) $T_c(b) = T'_c(b)$.

The next corollary provides a convenient criterion for the determination of $T_c(b)$ and $T'_c(b)$.

**Corollary 1.10.** At $T = T_c(b)$ and $T = T'_c(b)$, zero is the lowest eigenvalue of the operator $L_T b$.

A proof of energetic stability under general perturbations for either $T$ or $b$ sufficiently large is more subtle. For it, one has to use the full linearized operator, Hess $F_T(\eta_T b, a_b)$. Our computations suggest that 0 is the lowest eigenvalue of Hess $F_T(\eta_T b, a_b)$ if and only if 0 is the lowest eigenvalue of $L_T b$ and, consequently, $T_c(b)$ and $T'_c(b)$ apply also to the general perturbations.

The statement that $T_c = T_c(0) = T'_c(0) > 0$ for $a = 0$ (and therefore $b = 0$) and for a large class of potentials is proven, by the variational techniques, in [10].

In conclusion of this paragraph, we mention the proposition which follows from a simple computation:

**Proposition 1.11.** The operator $L_T b$ commutes with the magnetic translations. The same is true for the $\eta-$Hessian $F'_T(\eta_T b, a_b)$ (see (1.43)).

**Remarks.** Here we collect the comments on various aspects covered in this introduction.

**Remark 1.12 (BdG equations).** 1) For $\alpha = 0$, (1.4) becomes the time-dependent von Neumann-Hartree-Fock equation for $\gamma$ (and $a$).

2) A formal derivation of (1.4) given in Appendix E can be extended to the coupled system (1.4)–(1.6) by starting with the many-body quantum Hamiltonian (E.5) coupled to the quantized electro-magnetic field.

**Remark 1.13.** It is common in physics literature to drop the direct and exchange self-interaction terms from $h_\gamma a_b$ in (1.10). In this case, $\Lambda_{\eta a}$ becomes independent of the diagonal part, $\gamma$, of $\eta$ and equation (1.8) has always the solution

$$
\eta_T a = f_T(\Lambda_a), \ \text{where} \ \Lambda_a := \Lambda_{\eta a}|_{\eta = 0}.
$$

(1.49)
Similarly, for equation (1.32) for normal states. In this case, it always has the solution
\[ \gamma_T b = f_T(h_u), \quad \text{where} \quad h_a = -\Delta_a - \mu. \] (1.50)

**Remark 1.14** (Particle density). For a definition of \( \rho_\gamma \) not relying on the integral kernels, see (3.11).

**Remark 1.15** (Particle-hole symmetry). The evolution under the BdG equations (1.4)-(1.9) preserves the relations in (1.3), i.e. if an initial condition has one of these properties, then so does the solution. This follows from the relation
\[ J^*A J = -\bar{X}, \] (1.51)
where \( J \) is defined in (1.3). The second relation in (1.3) is called the particle-hole symmetry.

**Remark 1.16** (BdG equations in ‘coordinate’ form). In physics literature, the BdG equations are written in terms of eigenfunctions of the operator \( \eta \), or \( A_{pa} \) (assuming the operator \(-\Delta_a\) has purely discrete spectrum, see [4, 11, 12, 22, 33, 34], cf. [3]). In the case of stationary equations (1.8)-(1.9), we have
\[ A_{pa} \psi_n = \varepsilon_n \psi_n, \quad A_{pa} J \bar{\psi}_n = -\varepsilon_n J \bar{\psi}_n, \] (1.52)
with \( \varepsilon_n > 0 \) and \( J \) given in (1.3). The second equation above follows from the first and relation (1.51). By (1.13), \( \psi_n \) and \( J \bar{\psi}_n \) are also eigenfunctions of \( \eta \) with the eigenvalues \( f_n := f_T(\varepsilon_n) \) and \( 1 - f_n = f_T(-\varepsilon_n) \), respectively. As \( \{\psi_n, J \bar{\psi}_n\} \) form a complete orthonormal set, we have
\[ \eta = \sum_n \left( f_n P_{\psi_n} + (1 - f_n) P_{J \bar{\psi}_n} \right), \] (1.53)
where \( P_{\psi} \) stands for the rank-one orthogonal projection onto the subspace spanned by \( \psi \). (1.52)-(1.53) yield a nonlinear eigenvalue problem replacing (1.8). Finally, one can express \( j(\eta, a) \) in (1.9) in terms of \( \{\psi_n, J \bar{\psi}_n\} \).

**Remark 1.17** (Entropy). Due to the symmetry (1.3) of \( \eta \), we see that
\[ \text{Tr}(\eta \ln \eta) = \text{Tr}((1 - \eta) \ln(1 - \eta)) \] (1.54)
which, recalling (2.9), implies that
\[ S(\eta) := \text{Tr}(s(\eta)) = \text{Tr}(g(\eta)), \] (1.55)
\[ g(\eta) := -\eta \ln \eta - (1 - \eta) \ln(1 - \eta), \quad s(\eta) := -2\eta \ln \eta. \] (1.56)

**Remark 1.18** (Co-cycle equation). One can readily show that the operator-family \( u_{bs}, s \in \mathcal{L} \), defined in (1.23) is a group representation (with the group law given in (1.29)) if and only if the functions \( \chi^L_s(x), s \in \mathcal{L}, x \in \mathbb{R}^2 \), satisfy the co-cycle condition
\[ \chi^{L}_{s+t}(x) - \chi^{L}_{s}(x+t) - \chi^{L}_{t}(x) \in 2\pi \mathbb{Z}, \quad \forall s, t \in \mathcal{L}. \] (1.57)

The function \( \chi^L_s \) defined in (1.24) satisfies this relation, while the one in (1.36) does not.

Functions \( \chi : \mathcal{L} \times \mathbb{R}^d \rightarrow \mathbb{R} \) satisfying (1.57) are called the *summands of automorphy*, or *cocycles* (see [31] for a relevant discussion). (The map \( e^{ib} : \mathcal{L} \times \mathbb{R}^d \rightarrow \mathbb{U}(1) \), where \( \chi(x, s) \equiv \chi_s(x) \) is called the *factor of automorphy*. ) Every map \( \chi_s : \mathcal{L} \times \mathbb{R}^d \rightarrow \mathbb{R} \) satisfying (1.57) is (gauge) equivalent to the one in (1.24), with \( b \) satisfying (1.25).

With a summand of automorphy \( \chi^L_s(x), s \in \mathcal{L}, x \in \mathbb{R}^2 \), one associates the function
\[ c_1(\chi) = \frac{1}{2\pi} \left( \chi_{\nu_2}(x + \nu_1) - \chi_{\nu_2}(x) - \chi_{\nu_1}(x + \nu_2) + \chi_{\nu_1}(x) \right), \] (1.58)
for some basis \( \{\nu_1, \nu_2\} \) in \( \mathcal{L} \). As it turns out, this function is independent of \( x \) and the choice of the basis and is an integer. In physics literature, \( c_1(\chi) \) is called the Chern number.
Remark 1.19 (Operator $L_{Tb}$). 1) The question of when $L_{Tb}$ has negative spectrum for a larger range of $T$'s is a delicate one. For $T$ close to $T_c$, this depends, besides of the parameters $T$ and $b$, also on $v$ and $\mu$ determining whether the superconductor is of Type I or II. (As was discovered theoretically by A.A. Abrikosov in his study of the Ginzburg-Landau equations ([1]) and confirmed later experimentally, superconductors are divided, according to their basic properties, into two groups, Type I or II superconductors. So far, there are no results (rigorous, or not) establishing within the BdG theory these properties for certain classes of potentials $v$ and $\mu$.)

2) Since the components of magnetic translations (1.35) do not commute, the fiber decomposition of $L_{Tb}$ is somewhat subtle (see [2]).

Nomenclature. In our most important results, we drop the exchange term $v^\# \gamma$. By analogy with the Hartree-Fock equations, the resulting equations could be called the reduced BdG equations.

An important and natural modification of the BdG equations would be replacing the exchange term $v^\# \gamma$ by a density dependent exchange-correlation term $xc(\rho_\gamma)$ from the density functional theory. The resulting equations could be called the density BdG equations.

Addressing the density or original BdG equations would be an important next step.

The paper is organized as follows. In Sections 3 and 5, we prove Theorems 1.2 and 1.4, on existence of the normal and vortex lattice solutions, respectively. These are our principal results. In Section 4, we prove Propositions 1.7 and 1.8 on the stability/instability of the normal solutions. In Appendix A we study the entropy functional. Results of this appendix imply the proofs of Propositions 1.5 and 1.6 and are used in Appendix B in the proof of technical Theorem 2.2. In Appendix C, we prove an elementary technical result from Section 3. Finally, in Appendix D we prove some bounds on functions relative to magnetic Laplacian and bounds on densities (both elementary and probably well-known) and in Appendix E, we discuss the derivation of the BdG equations from the quantum many-body problem.

2. Stationary Bogoliubov-de Gennes equations

In this section we establish the connection between the time-dependent and stationary BdG equations and show that the latter are the Euler-Lagrange equations for the free energy, (1.21).

In terms of $\eta$, the gauge transformation, $T^\text{gauge}_\chi$, could be written as

$$T^\text{gauge}_\chi : \eta \rightarrow U^\text{gauge}_\chi \eta (U^\text{gauge}_\chi)^{-1}, \text{ where } U^\text{gauge}_\chi = \begin{pmatrix} e^{i\chi} & 0 \\ 0 & e^{-i\chi} \end{pmatrix}. \quad (2.1)$$

It is extended correspondingly to $(\eta, a)$ by $T^\text{gauge}(\eta, a) = (T^\text{gauge}_\chi \eta, a + \nabla \chi)$. Notice the difference in the action on the diagonal and off-diagonal elements of $\eta$.

The invariance under the gauge transformations follows from the relation

$$\Lambda(T^\text{gauge}_\chi(\eta, a)) = T^\text{gauge}_\chi (\Lambda(\eta, a)), \quad (2.2)$$

shown by using the operator calculus.

We consider stationary solutions to (1.4), with $a$ independent of $t$, of the form

$$\eta_t := T^\text{gauge}_\chi \eta = U^\text{gauge}_\chi \eta (U^\text{gauge}_\chi)^{-1}, \quad (2.3)$$

where $\eta$ is independent of $t$. We have

**Proposition 2.1.** The operator-family (2.3), with $\eta$ independent of $t$ and $\dot{\chi}$, constant, say $\dot{\chi} \equiv -\mu$, is a solution to (1.4), if and only if $\eta$ solves the equation

$$[\Lambda_{\eta a}, \eta] = 0, \quad (2.4)$$
where \( \Lambda_{\eta a} \equiv \Lambda_{\eta a \mu} \) is given explicitly in (1.10).

**Proof.** We write \( U_\chi^{\text{gauge}} \equiv U_\chi \) and use that \( \partial_t U_\chi = i\chi S U_\chi \), where

\[
S := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

and therefore \( \partial_t(U_\chi \eta U_\chi^{-1}) = i\chi[S, U_\chi \eta U_\chi^{-1}] = i\chi U_\chi[S, \eta] U_\chi^{-1} \). Plugging (2.3) into (1.4), and using (2.2), the previous relation and that \( \chi \) is independent of \( x \), we see that

\[
-\dot{\chi} U_\chi[S, \eta] U_\chi^{-1} = [\Lambda(U_\chi \eta U_\chi^{-1}, a), U_\chi \eta U_\chi^{-1}]
\]

(5.5)

\[
=[\Lambda(U_\chi \eta U_\chi^{-1}, a + \nabla \chi), U_\chi \eta U_\chi^{-1}]
\]

(5.6)

\[
=U_\chi[\Lambda(\eta, a), \eta] U_\chi^{-1}.
\]

(5.7)

Since \( \dot{\chi} \equiv -\mu \), it follows then that \([\Lambda(\eta, a) - \mu S, \eta] = 0 \). Since \( \Lambda(\eta, a) - \mu S \) is equal to (1.10), the last equation is exactly (2.4) which gives the statement of the proposition.

For any reasonable function \( f \) and time-independent \( a \), solutions of the equation

\[
\eta = f\left(\frac{1}{T}\Lambda_{\eta a}\right),
\]

(5.8)

solve (2.4) and therefore give stationary solutions of (1.4). Conversely, solutions to (2.4), s.t. the spectrum of \( \Lambda_{\eta a} \) is simple, solve (5.8). (The parameter \( T > 0 \), the temperature, is introduced here for the future reference.)

Inverting the function \( f \), one can rewrite (2.8) as \( \Lambda_{\eta a} = T f^{-1}(\eta) \). Let \( f^{-1} = g' \). Then the stationary BdG equations can be written (in the Coulomb gauge \( \text{div } a = 0 \)) as (1.5)–(1.9).

The physical function \( f \) is selected by either a thermodynamic limit (Gibbs states) or by coupling the system in question to a thermal reservoir (or imposing the maximum entropy principle). It is given by the Fermi-Dirac distribution (1.12). It follows from the equations \( g' = f^{-1} \) and (1.12) that the function \( g' \) is given by (1.11) and the function \( g \) is equal to

\[
g(\lambda) = -\lambda \ln \lambda - (1 - \lambda) \ln(1 - \lambda).
\]

(5.9)

From now on, we assume that \( f(\lambda) \) and \( g(\lambda) \) are given in (1.12) and (2.9), respectively.

Our next result shows that the BdG equations arise as the Euler-Lagrange equations for the free energy functional \( F_T \) differentiated along the perturbations (‘tangent vectors’) at \((\eta, a)\) from the following class (c.f. also Lemma 5.4 (1))

\[
\mathcal{P}_\eta := \{ (\eta', a') \in \hat{I}^1 \times \hat{H}^1 : (2.11) \text{ holds} \},
\]

(5.10)

\[
J' \eta' J = -\bar{\eta}', \ (\eta')^2 \lesssim [\eta(1-\eta)]^2, \text{ and } \text{Tr}(S_1 \eta') = 0,
\]

(2.11)

where \( S_1 = \text{diag}(1,0) \) and \( J \) is defined in (1.3). Conditions (2.11) are designed to handle a delicate problem of non-differentiability of \( s(\lambda) := 2\lambda \ln \lambda \) at \( \lambda = 0 \), while providing a sufficiently rich set, \( \mathcal{P}_\eta \), to derive the BdG equations.

**Theorem 2.2.** (a) The free energy functional \( F_T \) is well defined on the space \( \mathcal{D}_\nu^1 \times \hat{h}_1^1 \).

(b) \( F_T \) is continuously (Gâteaux or Fréchet) differentiable at \((\eta, a) \in \mathcal{D}_\nu^1 \times \hat{h}_1^1 \), with respect of perturbations \((\eta', a') \in \mathcal{P}_\eta \).

(c) Critical points of \( F_T \), which are even (in the sense of the definition (1.40)) and obey \( 0 < \eta < 1 \), satisfy the BdG equations for some \( \mu \) (determined by the constraint \( \text{Tr} \gamma = \nu \)).

(d) Minimizers of \( F_T \) over \( \mathcal{D}_\nu^1 \times \hat{h}_1^1 \) satisfy the conditions of statement (c) and therefore satisfy the BdG equations for some \( \mu \).
This theorem is proven in Appendix [13]. For the translation invariant case, it is proven in [19]. For a = 0, but γ not necessarily translation invariant, the fact that the BdG equations are the Euler-Lagrange equations of the free energy functional is used in [13], but, it seems, with no proof provided.

As a result of Theorem 2.2, we can write the BdG equations as

$$F_T^+(\eta, a) = 0,$$

(2.12)

where the map $F_T^+(\eta, a)$ is defined by the r.h.s. of the static BdG equations (1.8)+(1.9) and can be thought of as a gradient map of $F_T$.

3. THE NORMAL STATES: PROOF OF THEOREM 1.2

We begin with some results about the magnetic translations. Recall the operators $T_{bs}$ and $U_{s}^{\text{gauge}}$, defined in (1.35) and (2.11). Group properties of $T_{bs}$ are established in following

Lemma 3.1. The operators $T_{bs}$, defined in (1.35) and restricted to $\eta$'s, satisfy

$$T_{bs}T_{bt} = I_{bst}T_{bs+t},$$

(3.1)

where $I_{bst}$ acts on bounded operators on $\mathfrak{h}$ as

$$I_{bst}A := I_{bst}A^{-1}_{bst}, \quad I_{bst} := U_{s}^{\text{gauge}}\mathfrak{I}_{s\\\wedge t}.$$

(3.2)

Proof of Lemma 3.1 Let $U_{s}^{\text{trans}} := \text{diag}(u_{s}^{\text{trans}}, u_{s}^{\text{trans}})$ on $L_{\text{loc}}^{2}(\mathbb{R}^{2}) \times L_{\text{loc}}^{2}(\mathbb{R}^{2})$ and define the transformations

$$U_{bs} := (U_{s}^{\text{gauge}})^{-1}U_{s}^{\text{trans}}.$$  

(3.3)

Then $T_{bs} \eta = U_{bs} \eta U_{bs}^{-1}$. Using the definition above and the relation $U_{s}^{\text{trans}} U_{s}^{\text{gauge}} = U_{s}^{\text{gauge}} U_{s}^{\text{trans}}$, we compute

$$U_{bs} U_{bt} = (U_{s}^{\text{gauge}} \chi_{s}^{b} + u_{s}^{\text{trans}} \chi_{t}^{b})^{-1}U_{s+t}^{\text{trans}} = (U_{s}^{\text{gauge}} \chi_{s}^{b} + u_{s}^{\text{trans}} \chi_{t}^{b} - \chi_{s+t}^{b})^{-1}U_{bs+t}.$$  

By definition (1.36), we have $\chi_{s}^{b} + u_{s}^{\text{trans}} \chi_{t}^{b} - \chi_{s+t}^{b} = -b(s \wedge t)$, which implies $U_{bs} U_{bt} = I_{bst} U_{bs+t} = U_{bs+t} I_{bst}$. Hence, the result follows.

Now, Lemma 3.1 implies Proposition 1.3

Proof of Proposition 1.3 (3.1)-(3.2) and the mt-invariance, (1.37), yield $\eta = T_{bs} T_{bt} \eta = I_{bst} \eta I_{bst}^{-1}$, which implies that $\alpha = e^{i\frac{1}{2}(s \wedge t)} \alpha$ for all $s, t \in \mathbb{R}^{2}$. This yields that $\alpha = 0$.  

Note that, ignoring $a$, the fact that $T_{bs}$ maps the set of bounded diagonal operator-matrices on $\mathfrak{h} \times \mathfrak{h}$ into itself follows from (3.1)-(3.2) and the fact that $I_{bst}$ is an identity on diagonal operator-matrices.

Proposition 1.3 shows that we can restrict our search of mt-invariant solutions to normal states. For normal states, i.e. for $\alpha = 0$, the BdG equations (1.8)-(1.9) (with (1.8) replaced by (1.13)) reduce to the following equations for $\gamma$ and $a$

$$\gamma = f_{T}(h_{\tau a}),$$

(3.4)

$$\text{curl}^{*} \text{ curl}a = j(\gamma, a),$$

(3.5)

where, recall, $j(\gamma, a)(x) := \left([-i\nabla_{a}, \gamma]_{+}\right)(x, x)$. These are the coupled Hartree-Fock and Ampére equations.

Recall that $a_{b}(x)$ is the magnetic potential with the constant magnetic field $b$ ($\text{curl} a_{b} = b$) in the gauge s.t. $a_{b}(x) = \frac{b}{2} \times x, *x := (x_{2}, x_{1})$. First, we show that the second equation is
automatically satisfied for \(a = a_b\) and \(\gamma\) a magnetically translation invariant operator, which is even in the sense of (1.40).

Define the magnetic translations \(u_{bs}\) as (cf. (1.23) and (3.3))

\[
u_{bs} := u^{\text{gauge}}_{-\chi_s^b} u^{\text{trans}}_{s},
\]

(3.6)

where, recall the operators \(u^{\text{gauge}}_{\chi}\) and \(u^{\text{trans}}_{s}\) are defined as defined after (1.23) and \(\chi^b: \mathcal{L} \times \mathbb{R}^d \to \mathbb{R}\) is given in (1.36). Recall the definition of the space \(\mathfrak{h}\) in (1.27).

**Lemma 3.2.** With the definition \(\tau_{bs}(\gamma) = u_{bs} \gamma u_{bs}^{-1}\), we have (cf. (3.1))

\[
u_{bs} u_{bh} = e^{-i\hbar \chi s} u_{bh} u_{bs},
\]

(3.7)

\[
\tau_{bs} \tau_{bh} = \tau_{bh} \tau_{bs},
\]

(3.8)

If \(\gamma\) maps \(\mathfrak{h}\) into itself, then so does \(\tau_{bh} \gamma\).

**Proof.** (3.7) follows from \(u_{bs} u_{bh} = e^{-i\hbar \chi s} u_{bh} u_{bs} = e^{-i\hbar \chi s} e^{-i\hbar \chi s} u_{bs} u_{bh}\). To prove (3.8), we use that \(\tau_{b} e^{-i\chi h} = e^{-i\hbar \chi s} e^{-i\hbar \chi s} \tau_{b}\), where, recall, \(\chi_{bs}\) is defined in (1.24), and (3.7) which yield \((u_{bs} u_{bh})^{-1} = e^{i\hbar \chi s} (u_{bh} u_{bs})^{-1}\) and therefore, due to \(\tau_{bh} \gamma = u_{bh} \gamma u_{bh}^{-1}\), (3.8) follows.

Finally, \(\gamma\) maps the space \(\mathfrak{h}\) into itself if and only if \(\tau_{bs}(\gamma) = \gamma\), \(\forall s \in \mathfrak{L}\). On the other hand, (3.8) implies \(\tau_{bs} \tau_{bh}(\gamma) = \tau_{bh} \tau_{bs}(\gamma) = \tau_{bh}(\gamma), \forall s \in \mathfrak{L}, h \in \mathbb{R}^2\), so (3.9) follows. \(\square\)

Note that (3.7) shows that \(u_{bs}\) does not leave \(\mathfrak{h}\) invariant.

The generators of the magnetic translations, \(u_{bs}\), defined in (3.6), and their properties are described in the following

**Lemma 3.3** (2). Let \(p_b = -i\nabla a_b\) and \(\pi_b = -\bar{p}_b := -C p_b \mathcal{C}\), with the components \(p_{bi}\) and \(\pi_{bi}\). Then

1. \(-i \partial_{s_j}|_{s = 0} u_{bs} = \pi_{bj}\);
2. \([\pi_{bi}, p_{bj}] = 0\) and therefore \([u_{bs}, p_{bj}] = 0\).

To begin with, we define the operator \(\gamma\) on \(L^2_{\text{loc}}(\mathbb{R}^2)\) which could be further specified as \(\mathfrak{h}\). For an operator \(A\) on \(\mathfrak{h}\), we can define the integral kernel \(A'(x, y)\) by the relation

\[
\langle g \otimes \tilde{f}, A'(h) \rangle_{\mathfrak{h} \otimes \mathfrak{h}} = \langle g, A f \rangle_{\mathfrak{h}}, \quad \forall f, g \in \mathfrak{h}.
\]

(3.10)

In this section, it is convenient to use the notation \(\text{den}(A)\) for the one-particle density \(\rho_A\). For an operator \(A\) on \(\mathfrak{h}\), s.t. \(fA\) and \(Af\) are trace-class, the density \(\text{den}(A) \equiv \rho_A\) obeys the relation

\[
\int f \text{den}(A) = \text{Tr}(f A), \quad \forall f \in L^\infty.
\]

(3.11)

which can be also used as a definition of \(\text{den}(A)\). (Recall our convention \(\int \equiv \int_{\Omega}\) for an arbitrary and fixed lattice cell \(\Omega\).) With this notation, we have e.g. \(j(\gamma, a) := \text{den}([-i\nabla a, \gamma]_+)).\n
We assume that all operators below are originally defined on \(L^2_{\text{loc}}(\mathbb{R}^2)\) (or on a local Sobolev space). This allows us to define compositions and commutators of operators some of which do not leave \(\mathfrak{h}\) invariant.

Recall that we say an operator \(A\) to be magnetically translation invariant (\textit{mt-invariant}, for short), if and only if it satisfies \(\tau_{bh} A = A, \forall h \in \mathbb{R}^2\). Our key result here is the following

**Proposition 3.4.** (i) For a trace-class, mt-invariant operator \(A\) on \(\mathfrak{h}\), \(\text{den}(A)\) is constant.

(ii) If, in addition, \(\tau_{\text{refl}} A = -A\) (with the reflection operator \(\tau_{\text{refl}}\) defined before (1.40)), then \(\text{den}(A) = 0\).

Recall that \(\nabla_a := \nabla - ia\). We derive Proposition 3.4 from the following
Lemma 3.5. For any linear vector field $a$ and any integral operator $A$ on $\mathfrak{h}$, $[\nabla_a, A]$ leaves $\mathfrak{h}$ invariant (though $\mathfrak{h}$ is not invariant under $\nabla_a$) and
\[
\nabla \text{den}(A) = \text{den}([\nabla_a, A])
\] (3.12)

Proof. Since $a$ is linear, the invariance of $\mathfrak{h}$ under $[\nabla_a, A]$ is straightforward. We prove (3.12). We have
\[
\text{den}([\nabla_a, A]) = \text{den}([\nabla_a, A]) + i \text{den}([a_b - a, A]).
\] (3.13)

Since $\nabla_a$ leaves $\mathfrak{h}$ invariant, we can use the cyclicity of the trace to compute
\[
\int f \text{den}([\nabla_a, A]) = \text{Tr}_\mathfrak{h}(f[\nabla_a, A])
\]
\[
= -\text{Tr}_\mathfrak{h}(\nabla f A) = -\int \nabla f \text{den}(A).
\] (3.14)

For any (linear) vector field $c$, the integral kernel of $[c, A]$ is $(c(x) - c(y))A(x, y)$ (see (3.10)). Hence, we see that $\text{den}([c, A]) = 0$. Combining this with (3.13) and (3.14) gives $\int f \text{den}([\nabla_a, A]) = -\int \nabla f \text{den}(A)$ for any $f \in L^\infty(\mathbb{R}^3)$ and $\mathcal{L}$-periodic. Hence $\text{den}([\nabla_a, A]) = \nabla \text{den}(A)$.

Proof of Proposition 3.4. Since by Lemma 3.3, $\tau_{b\mathfrak{h}}$ is generated by the map $A \rightarrow i[\pi_b, A]$, the $\text{mt}$-invariance of $A$ implies that $[\pi_b, A] = 0$. (Though $\pi_b$ does not leave $A$ invariant, $[\pi_b, A]$ does.) This and (3.12) yield that $\nabla \text{den}(A) = \text{den}([\pi_b, A]) = 0$ and therefore $\text{den} A$ is constant.

Remark 3.6. The argument above proving (3.12) establishes the intuitive fact that the integral kernel of the operator $[\nabla_a, A]$ acting on $\mathfrak{h}$ is same as the integral kernel of this operator acting on $L^2(\Omega)$, which is $(\nabla \text{ax} + \nabla \text{ay})A'(x, y) = (\nabla \text{x} + \nabla \text{y})A'(x, y)$ and consequently $\text{den}([\nabla_a, A]) = (\nabla \text{x} + \nabla \text{y})A'(x, y)|_{x=y} = \nabla \text{den}(A)$.

Lemma 3.7. (3.4)-(3.5) have a solution $(\gamma_b, a_b)$, where $\gamma_b \geq 0$ and $\gamma_b$ is an mt-invariant operator on the space $\mathfrak{h}$, if and only if the fixed point problem
\[
\xi = (\int v) \text{den}(f_T(h_{a_b\mu} + \xi))(0),
\] (3.15)

where $h_{a_b\mu} := -\Delta_{a_b} - \mu$ and $\xi$ is a real number, has a solution. (The operator $f_T(h_{a_b} + \xi)$ is well-defined and real since $h_{a_b}$ is self-adjoint on the space $\mathfrak{h}$.)

We show in Appendix C that the fixed point problem (3.15) has a unique solution, provided the first condition of Theorem 1.2 holds.

Proof. Let $\gamma$ be an even, mt-invariant operator. Then $\tilde{\gamma} = -i\nabla \text{ax} \gamma$ is an mt-invariant operator and odd. Applying Proposition 3.4 to $\tilde{\gamma} = -i\nabla \text{ax} \gamma$ gives that $\gamma(\gamma, a_b) = 0$ and therefore, since $\text{curl}^\star \text{curl} a_b = 0$, the pair $(\gamma, a_b)$ satisfies (3.5). Hence $(\gamma, a_b)$ solves (3.4)-(3.5) if and only if $\gamma$ solves
\[
\gamma = f_T(h_{a_b\mu}).
\] (3.16)

Now, we solve (3.16) for magnetically translation invariant $\gamma$'s. We treat this equation as a fixed point problem. This problem simplifies considerably since we dropped the exchange term, as in this case, $h_{\gamma a_b\mu}$ becomes $h_{a_b\mu} + v \ast \rho$, where, recall, $h_{a_b\mu} := -\Delta_{a_b} - \mu$. Using this and applying den to (3.4) gives the equation for $\rho = \text{den}\gamma$:
\[
\rho = \text{den}(f_T(h_{a_b\mu} + v \ast \rho)).
\] (3.16)

\footnote{If both the direct and exchange self-interactions are dropped from $h_{\gamma a_b\mu}$, then the latter equation gives $\gamma $ to directly: $\gamma_{Tb} = f_T(h_{a_b\mu})$, where, recall, $h_{a_b\mu} := -\Delta_{a_b} - \mu$.}
Furthermore, by Proposition 3.4, $\rho = \text{den}\gamma$ and $\text{den}(f_T(h_{a,b} + v \ast \rho))$ are constant functions and therefore $\xi = v \ast \rho = \int v\rho(0)$ is a real constant satisfying the fixed point equation (3.15).

To summarize, we have shown that if an mt-invariant $\gamma$ solves (3.16) (i.e. (3.4) with $a = a_b$), then $\xi = v \ast \text{den}\gamma = \int v\rho(0)$ is a real constant solving (3.15).

Now, in the opposite direction, suppose that a real $\xi$ solves (3.15) and define
\[
\gamma := f_T(h_{a_b} + \xi).
\] (3.17)

Since $f_T > 0$, $h_{a_b}$ is self-adjoint and $\xi$ is real, we have that $\gamma \geq 0$. Since $\xi = v \ast \text{den}(f_T(h_{a_b} + \xi)) = v \ast \rho_\gamma$, where, recall, $\rho_\gamma \equiv \text{den}(\gamma)$, (3.17) becomes
\[
\gamma = f_T(h_{a_b} + v \ast \rho_\gamma).
\] (3.18)

Hence $\gamma$ satisfies (3.16), i.e. (3.4) with $a = a_b$.

Since, as we show in Appendix C the fixed point problem (3.15) has a unique solution, provided the potential $v$ satisfies the first condition of Theorem 1.2 we obtain an unique magnetic translation invariant solution of (3.4)- (3.5), under the same condition.

The uniqueness part of Lemma 3.7 is strengthened and extended to the second condition in the following

**Lemma 3.8.** A solution of (3.4)-(3.5) is unique among pairs $(\gamma, a)$ with $\gamma$ mt-invariant.

**Proof.** First observe that, by Proposition 3.4, $\gamma(x, x)$ is constant and the term $\text{Re}(-i\nabla_{a_b}\gamma)(x, x)$ vanishes. We decompose $a = a_b + a'$, where $a'$ is defined by this expression. Using (3.5) and $\text{curl} \ast \text{curl} a_b = 0$, we see that
\[
\text{curl} \ast \text{curl} a' = -\gamma(0,0)a'.
\] (3.19)

Since $h_{a_b} + \xi$ is bounded below and $f_T$ is strictly positive and monotonically increasing, we see that $\gamma = f_T(h_{a_b} + \xi) \geq c > 0$. Thus $\gamma(0,0) > 0$. Multiplying both sides of (3.19) by $a'$ and integrating, we find
\[
\int |\text{curl} a'|^2 + \gamma(0,0) \int |a'|^2 = 0.
\] (3.20)

Since $\gamma(0,0) > 0$, this implies that $a' = 0$. Hence $a = a_b$ and therefore (3.4) shows that $\gamma$ is a function of $-\Delta_{a_b}$ and therefore mt-invariant. Hence, we can conclude uniqueness by Lemma 3.7 and the proof is complete.

Lemmas 3.7, 3.8 and C.1 of Appendix C imply Theorem 1.2 under the first condition.

To prove Theorem 1.2 under the second condition, i.e. for $\|v\|_\infty < \infty$, we use the variational approach. Let $D^*_\nu = \{\gamma \in L^{s,1}, 0 \leq \gamma = \gamma^* \leq 1, \gamma \text{ satisfies } (1.40), \text{Tr}\gamma = \nu\}$ (see (1.1) and (1.31)). Recall, that $S(\gamma)$ is the entropy defined in (1.55)-(1.56) and define the free energy functional
\[
F_1(\gamma) = \text{Tr}(h_{a_b} \gamma) + \frac{1}{2} \int \rho_\gamma(v \ast \rho_\gamma) - TS(\gamma),
\] (3.21)
on $D^*_\nu$, setting $F(\gamma) = -\infty$, if $S(\eta) = \infty$. (To compare with (1.21)-(1.22), $\int \rho_\gamma(v \ast \rho_\gamma) = \text{Tr}(v \ast \rho_\gamma) \gamma$. ) We show below the following

**Proposition 3.9.** Assume $v$ is $L$-periodic and satisfies $\|v\|_\infty < \infty$. Then the functional $F_1(\gamma)$ on $D^*_\nu$ has a minimizer; minimizers of $F_1(\gamma)$ satisfy $0 < \gamma < 1$. It follows then, by a special case of Theorem 2.2, that minimizers of $F_1(\gamma)$ satisfy the Euler-Lagrange equation, which is equivalent to (3.16). Using the latter equation, the solution could be bootstrapped from $I^{1,1}$ to $I^{2,1}$, giving Theorem 1.2.
Proof of Proposition 3.9. First, we show that \( F_1(\gamma) \) is coercive, namely, that
\[
F_1(\gamma) \geq \frac{1}{2} \|\gamma\|_{1,1}^2 - C_{\nu,T}.
\]
Indeed, by the definition, \( \|\gamma\|_{1,1} = \text{Tr}(h_{ab}\gamma) \) and the elementary estimate
\[
| \int \rho(v * \rho) \leq \|v\|_\infty \|\rho\|_1^2,
\]
we have
\[
F_1(\gamma) \geq \frac{1}{2} \|\gamma\|_{1,1}^2 + f(\gamma) - \|v\|_\infty \|\rho\|_1^2,
\]
where \( f(\gamma) := \frac{1}{2} \|\gamma\|_{1,1}^2 - TS(\gamma) \). We minimize the functional \( f(\gamma) \) on the r.h.s. on the set
\[
I_{\nu}^{1,1} := \{ \gamma \in I_{\nu}^{1,1} : \text{Tr} \gamma = \nu \}.
\]
Since \( f(\gamma) \) is convex and the constraint \( \text{Tr} \gamma = \nu \) is linear, each solution to the standard Euler-Lagrange equation \( df(\gamma) - \mu d\text{Tr} \gamma = 0 \) (written in terms of the Gâteaux derivatives), where \( \mu \) is the Lagrange multiplier, is a global minimizer. The latter equation is computed to be \( \frac{1}{2} h_{ab} - T \ln \frac{1}{1-\gamma} - \mu 1 = 0 \). Solving this equation gives the minimizer
\[
\gamma_{\mu,T} = f_{T}(\frac{1}{2} h_{ab} - \mu),
\]
for \( \mu \) such that \( \text{Tr} \gamma_{\mu,T} = \nu \). By the inverse function theorem, the latter equation has a solution, \( \mu = \mu(T,\nu) \), for \( \mu \). This shows \(-C_{\nu,T} := \inf\{ f(\gamma) : \gamma \in I_{\nu}^{1,1} \} = f(\gamma_{\mu,T}) > -\infty \), with \( \mu = \mu(T,\nu) \), which, together with (3.24), implies (3.22).

Next, we show that \( F_1(\gamma) \) is weak lower semi-continuous. To this end, we pass from the positive, trace class operators \( \gamma \) to the Hilbert-Schmidt ones, \( \kappa := \sqrt{\gamma} \). Note that \( \gamma \in I_{\nu}^{1,1}, \gamma \geq 0 \iff \kappa := \sqrt{\gamma} \in I_{\nu}^{1,2} \). Thus we consider \( F_1(\gamma) =: F'(\kappa) \) on the space \( \mathcal{D}_{\nu} := \{ \kappa \in I_{\nu}^{1,2}, 0 \leq \kappa = \kappa^* \leq 1, \kappa \text{ satisfies } (1.40), \text{ Tr}\kappa^2 = \nu \} \). The first term on the r.h.s. of (3.21) satisfies \( \text{Tr}(h_{ab}\gamma) = \|\kappa\|_{1,2}^2 \) and is quadratic in \( \kappa \). Hence, it is \( \|\cdot\|_{1,2}-\text{weakly lower semi-continuous. For the second term, we use the inequalities } (3.24) \) and
\[
\|\rho_{\kappa'} - \rho_{\kappa} \|_1 \leq \|\kappa' - \kappa\|_{I_2}(\|\kappa'\|_{I_2} + \|\kappa\|_{I_2})
\]
(3.26)
to show that it is also lower semi-continuous. The third term on the r.h.s. of (3.21), \(-TS(\gamma)\), is lower semi-continuous, by Lemma A.4 of Appendix A. Hence \( F_1(\gamma) =: F'(\kappa) \) is lower semi-continuous.

Finally, we observe that the set \( \mathcal{D}_{\nu} \) is closed in \( I_{\nu}^{1,2} \) under the weak convergence.

With the results above, the proof of existence of a minimizer is standard. To avoid repetitions, we refer to the second paragraph after Lemma 3.5 where this is done in somewhat more complicated notation.

Now, we establish properties of minimizers \( \gamma_s = \kappa_s^2 \). By (3.22), we have \( S(\gamma_s) = \text{Tr}g(\gamma_s) < \infty \). This and the fact that \( g(\gamma) \geq 0 \) imply that \( g(\gamma_s) \) is trace class.

Finally, a simplified version of the proof of Lemma 3.6 shows that 0 and 1 are not eigenvalues of \( \gamma_s \) and therefore, \( 0 < \gamma_s < 1 \).

4. Stability/instability of the normal states for small \( T \) and \( b \): Proof of Propositions 1.7 and 1.8

Proof of Proposition 1.7. Recall that \( K_{Tb} = Tf(h_x/T, h_y/T) \), where \( f(u,v) := \frac{u+v}{\tanh(u)+\tanh(v)} \) and \( h_z \) is the operator \( h_{T\bar{b}} \), defined in Proposition 1.5 acting on the variable \( z \). By Lemma 1.1
below $f(u, v) \geq 1$. (A weaker bound $f(u, v) \geq \frac{1}{4}$ which suffices for us could be easily proved directly.) Hence

$$K_{Tb} \geq T,$$

which implies that $L_{Tb} \geq T - \|v\|_\infty$ and consequently, Proposition 1.7 follows.

Proof of Proposition 1.8. We use the Birman-Schwinger principle to show that $L_{Tb}$ has a negative eigenvalue. Set $w^2 = -v \geq 0$ so that $L_{Tb} = K_{Tb} - (w^2)^2$, where, recall, $(w^2 \alpha)(x, y) := w(x - y)\alpha(x, y)$.

By the Birman-Schwinger principle, $L_{Tb}$ has a negative eigenvalue $-E$ if and only if $G_{Tb}(E) := w^2(K_{Tb} + E)^{-1}w^2$ has the eigenvalue 1 for some $E > 0$ (see e.g. [18]). By (4.1), we have $G_{Tb}(E) \geq 0$ for all $E \geq 0$. Moreover, since $(K_{Tb} + E)^{-1}$ is continuous and monotonically decreasing in $E \geq 0$ and vanishing as $E \to \infty$, so is $G_{Tb}(E)$. Hence, it suffices to show that $G_{Tb} := G_{Tb}(0)$ satisfies the estimate $\|G_{Tb}\| > 1$, which we now prove.

Recall that $K_{Tb} = T(f(x)T_y + h_{Tb})$, where $f(s, t) := \frac{\delta + t}{\tanh(s) + \tanh(t)}$ and $h_{Tb}$ is the operator $h_{Tb}$, defined in Proposition 1.5, acting on the variable $z$. Since $h_{Tb}$ satisfies $h_{Tb} \geq -\mu'$, for some $\mu' > \mu$, it suffices to consider $f(s, t)$ for $s, t \geq -\mu'$. A simple estimate

$$f(s, t) \lesssim 1 + |s + t|,$$

for $s, t \geq -\mu'$, which follows from Lemma 4.1 below, yields $K_{Tb} \lesssim T + |h_x + h_y|$. This implies the inequality

$$G_{Tb} \geq w^2(T + |h_x + h_y|)^{-1}w^2 \geq 0.$$

Since we omit the exchange term $-v^2\gamma$, the operator $h_{Tb}$ is of the form $h_{a_b\mu} := -\Delta_{a_b} + v \star \rho_{\gamma_Tb} - \mu$. By Proposition 3.4, $\rho_{\gamma_Tb}$ is a constant function and therefore $\xi := v \star \rho = \int v\rho(0)$ is a real constant. Hence $h_{a_b\mu} := -\Delta_{a_b} + \xi - \mu$.

Since the gaps between the eigenvalues $\lambda_n = b(2n + 1)$ of $-\Delta_{a_b}$ on $\mathfrak{h}$ are equal to $b$, we can choose $m$ s.t. $|\lambda_m + \xi - \mu| \lesssim b$.

Recall that $L_{Tb}$ acts on the space of the Hilbert-Schmidt operators which can be identified through their integral kernels with $\mathfrak{h} \otimes \mathfrak{h}$. Let $\phi_m$ be the normalized eigenfunctions of $-\Delta_{a_b}$ corresponding to the eigenvalues $\lambda_m = b(2m + 1)$. We take $u := c(w^2)^{-1}(\phi_m \otimes \phi_m)$, where $c = \|(w^2)^{-1}(\phi_m \otimes \phi_m)\|^{-1}$, so that $\|u\| = 1$. By analyzing $f(s, t)$, $s, t \geq -\mu'$, separately in several domains, we obtain

$$\langle u, G_{Tb}u \rangle \gtrsim (T + |\lambda_m + \xi - \mu|)^{-1}\|(w^2)^{-1}(\phi_m \otimes \phi_m)\|^{-2}
\gtrsim (T + b)^{-1}\|(w^2)^{-1}(\phi_m \otimes \phi_m)\|^{-2}. \quad (4.3)$$

(4.3) also follows from the stronger, but more involved, Lemma 4.1 below.) Now, write $\phi_m(x) = \sqrt{b}\phi_m^0(\sqrt{b}x)$, where $\phi_m^0(x)$ is independent of $b$. Furthermore, by the assumption on $v = -w^2$, we have $w(x - y) \gtrsim (1 + |x - y|)^{-\kappa/2}, \kappa < 2$. The last two relations, together with the inequality $(a + b)^{\kappa} \leq 2^{\kappa}(a^\kappa + b^\kappa)$, imply

$$\|(w^2)^{-1}(\phi_m \otimes \phi_m)\|^2 \lesssim \int (1 + |x - y|^{\kappa}||\sqrt{b}\phi_m^0(\sqrt{b}x)\sqrt{b}\phi_m^0(\sqrt{b}y)||^2dx dy.$$  

Changing the variables of integration as $x' = \sqrt{b}x, y' = \sqrt{b}y$, we find $\|(w^2)^{-1}(\phi_m \otimes \phi_m)\| \lesssim b^{-\kappa/4}$, which in turn gives $\langle u, G_{Tb}u \rangle \gtrsim (T + b)^{-1}b^{-\kappa/2} \to \infty$ as $b \to 0$, provided $T \lesssim b^{\sigma}, \sigma > \kappa/2$.

Thus we have shown that $\|G_{Tb}\|$, or the largest eigenvalue of $G_{Tb}$, can be made arbitrarily large if $T$ and $b$ are sufficiently small, which, by the Birman-Schwinger principle, proves Proposition 1.8. \qed
Bounds on the function \( f(s,t) := \frac{s+t}{\tanh(s)+\tanh(t)} \) used in the proof above could be proved directly; they also follow from the following

**Lemma 4.1.** The function \( f(s,t) := \frac{s+t}{\tanh(s)+\tanh(t)} \) has the minimum 1 achieved at \( s = t = 0 \).

**Proof.** To find minimum of \( f \), we look for its critical points. We let \( g(s,t) = \tanh(s) + \tanh(t) \) and compute

\[
\nabla f = \frac{1}{g}(1 - f(s,t) \sech^2(s), 1 - f(s,t) \sech^2(t)).
\]

(4.4)

Setting \( \nabla f = 0 \) yields that

\[
f(s,t)^{-1} = \sech^2(s) \quad \text{and} \quad f(s,t)^{-1} = \sech^2(t).
\]

(4.5)

It follows that \( \sech^2(s) = \sech^2(t) \) and therefore either \( s = t \) or \( s = -t \). If \( s = -t \), then \( f(s,t) = \sech^{-2}(s) \), which has a single critical point - minimum - \( s = 0 \). Hence in this case we have a single critical point - minimum - at \( s = -t = 0 \) and \( f(0,0) = 1 \). If \( s = t \), then (4.5) becomes

\[
tanh(s) = s \sech^2(s), \quad \text{or equivalently} \quad \sinh(u) \cosh(s) = s.
\]

(4.6)

This is equivalent to \( \sin(2s) = 2s \) which implies that \( s = 0 \). Hence the minimum is reached at \( s = 0 \) and \( f(0,0) = 1 \). \( \square \)

### 5. The existence of the vortex lattices

In this section, we prove Theorem 1.4 on existence of the vortex lattice solutions to the BdG equations with arbitrary but fixed lattice \( \mathcal{L} \) and first Chern (vortex) number \( c_1(\chi^\mathcal{L}) = n \in \mathbb{Z} \), i.e. the integer \( n \) entering (1.25) (see (1.39) and (1.58)). Recall that \( b = \frac{2\pi n}{|L|}, \ n \in \mathbb{Z} \), see (1.25).

Recall that we drop the exchange self-interaction term \( v^2 \gamma \). We minimize the resulting energy for \( \Tr(\gamma) \) fixed. Hence we omit the term \( -\mu \Tr(\gamma) \) in (1.21). Also, in this section, we display the domain of integration \( \Omega \) which is an arbitrary but fixed fundamental cell of the lattice \( \mathcal{L} \). Thus, with the notation \( h_\alpha := -\Delta_\alpha \), the free energy functional \( F_T(\eta, a) \) in (1.21) becomes

\[
\mathcal{F}(\eta, a) = \Tr(h_\alpha) + \frac{1}{2} \int \rho_\gamma(v * \rho_\gamma)
\]

\[
+ \frac{1}{2} \Tr(\alpha^* v^2 \alpha) + \int_\Omega dx |\curl a|^2 - T S(\eta),
\]

(5.1)

where, recall, the entropy \( S(\eta) \) is defined in (1.55)-(1.56). The functional \( \mathcal{F} \) is defined on \( \mathcal{D}_\nu \times \tilde{h}_1^1 \), if \( S(\eta) < \infty \). Otherwise, we set \( \mathcal{F}(\eta, a) = \infty \).

Theorem 1.4(i) follows from Theorem 2.2 and the following

**Theorem 5.1.** Fix a lattice \( \mathcal{L} \) and a Chern number \( c_1(\chi^\mathcal{L}) = n \). Assume that \( T > 0 \) and \( ||v||_\infty < \infty \). There exists a finite energy minimizer \( (\eta_*, a_*) \in \mathcal{D}_\nu \times \tilde{h}_1^1 \) of the functional \( \mathcal{F}(\eta, a) \) on the set \( \mathcal{D}_\nu \times \tilde{h}_1^1 \). This minimizer has the equivariance and the flux quantization properties, (1.38) and (1.39), satisfies \( 0 < \eta_* < 1 \) and is s.t. \( g(\eta_*) \) (see (1.55)) is trace class. Furthermore, the minimizer \( (\eta_*, a_*) \) can be chosen to be even, i.e. satisfying (1.40).

**Proof of Theorem 5.1.** We will use standard minimization techniques proving that \( \mathcal{F}(\eta, a) \) is coercive and weakly lower semi-continuous, and \( \mathcal{D}_\nu \times \tilde{h}_1^1 \) weakly closed.

**Part 1: coercivity.** The main result of this step is the following proposition:
Proposition 5.2. Let $T > 0$, $e := a - ab$, $\gamma \in I^{1,1}$ and $\alpha \in I^{1,2}$, with $\Tr \gamma = \nu$ and $0 \leq \eta \leq 1$. Then
\[
\mathcal{F}(\eta, a + e) \geq c'[\|\gamma\|_{H^{1,1}/\nu}]^r + c\|e\|_{H^{1,1}}^2 - C \\
\geq \frac{1}{4}c'[\|\gamma\|_{H^{1,1}} + \|\alpha\|_{H^{1,2}}^2]\|\nu\|^r + c\|e\|_{H^{1,1}}^2,
\]
for any $0 < r < 1$ and for suitable constants $c', c, C > 0$, with $c', c > 0$ independent of $\nu, T, \nu$ and $\mathcal{L}$ depending on $\nu, T, \|v\|_\infty$ and $|\mathcal{L}|$.

Proof of Proposition 5.2. We begin with estimating the entropy term $-TS(\eta)$ (cf. [3]). Recall definition (1.35)-(1.36) of $S(\eta)$, which we reproduce here
\[
S(\eta) := \Tr(s(\eta)) = \Tr(g(\eta)),
\]
\[
g(\eta) := -\eta \ln \eta - (1 - \eta) \ln(1 - \eta), \quad s(\eta) := -2\eta \ln \eta.
\]
Note that since $s(\eta) \geq 0$ for $0 \leq \eta \leq 1$, we have that $S(\eta) \geq 0$. We define the relative entropy
\[
S(A|B) = \Tr(s(A|B)), \quad s(A|B) := A(\ln A - \ln B).
\]
We define the diagonal operator-matrix $\eta_0$ and recall the off-diagonal one $\phi$:
\[
\eta_0 := \begin{pmatrix} \gamma & 0 \\ 0 & 1 - \gamma \end{pmatrix}, \quad \phi(\beta) := \begin{pmatrix} 0 & \beta \\ \beta^* & 0 \end{pmatrix}.
\]

Lemma 5.3 (cf. [13]). We have for $\eta = \eta_0 + \phi(\alpha)$,
\[
S(\eta) = S(\eta_0) - S(\eta|\eta_0) \leq S(\eta_0).
\]
Proof. We note that for $\eta := \eta_0 + \phi(\alpha)$,
\[
\begin{align*}
\eta \ln \eta - \eta_0 \ln \eta_0 &= \eta \ln \eta - \eta \ln \eta_0 + \eta \ln \eta_0 - \eta_0 \ln \eta_0 \\
&= s(\eta, \eta_0) + (\eta - \eta_0) \ln \eta_0 \\
&= s(\eta, \eta_0) + \phi(\alpha) \ln \eta_0.
\end{align*}
\]
Since the last term, $\phi(\alpha) \ln \eta_0$, in (5.11) is off-diagonal and therefore has zero trace, the first equation in (5.8) follows.

The inequality in (5.8) follows from Klein’s inequality and the fact that $\Tr \eta = \Tr \eta_0$.

With the definitions (5.4)-(5.5) and (5.7), we have that (5.8) implies
\[
S(\eta) \leq S(\eta_0) = \Tr(g(\gamma)), \quad g(x) = -[x \ln x + (1 - x) \ln(1 - x)].
\]

Next, we estimate the second ($\alpha$-) term on the r.h.s. of (5.11). To this end, we bound $\alpha$ by $\gamma$ via the constraint $0 \leq \eta \leq 1$ using the following result (see [3] and references therein):

Lemma 5.4. The constraint $0 \leq \eta \leq 1$ implies that
\begin{enumerate}
\item $\alpha^* \alpha \leq \tilde{\gamma}(1 - \tilde{\gamma})$ and $\alpha \alpha^* \leq \gamma(1 - \gamma)$.
\item $\Tr(M\alpha^* M^*) \leq \|M\gamma M\|_1$ for any operator $M$.
\end{enumerate}

Proof. Since $0 \leq \eta \leq 1$, then $0 \leq 1 - \eta \leq 1$ as well and therefore
\[
0 \leq \eta(1 - \eta) \leq 1.
\]
From (1.2), we see that the 1,1-entry of $\eta(1 - \eta)$ is $0 \leq \gamma(1 - \gamma) - \alpha \alpha^*$. By considering the 2,2-entry, we have that $\alpha^* \alpha \leq \tilde{\gamma}(1 - \tilde{\gamma})$. Finally, since $1 - \gamma \leq 1$, we see that $M\alpha^* M^* \leq M\gamma M^*$, which completes the proof.
Since \( v \) is bounded, Lemma 5.4(1) gives
\[
|\text{Tr}(\alpha^* v^2 \alpha)| \leq \|v\|_\infty \text{Tr}(\alpha^* \alpha) \leq \|v\|_\infty \text{Tr}(\gamma).
\] (5.13)

Now, using estimates (5.12) and (5.13) in expression (5.11) and introducing the notation \( S(\gamma) := \text{Tr}(g(\gamma)) \), we find
\[
\mathcal{F}(\eta,a) \geq \text{Tr}(h_a \gamma) + \frac{1}{2} \int \rho_{\gamma}(v \ast \rho_{\gamma})
+ \int \text{curl } a|^2 - TS(\gamma) - \|v\|_\infty \text{Tr}(\gamma).
\] (5.14)
The r.h.s. depends only on \( \gamma \) and \( a := a_h + e \) with the only constraint \( \text{Tr}(\gamma) = \nu \) and we estimate it next.

Next, we pass from the vector potential \( a \) to \( e := a - a_h \). Using \( a = a_h + e \in \mathbb{H}^1 \), \( \text{curl } a_h = b \) and \( \int_\Omega \text{curl } e = 0 \) (see (1.29)-(1.30)), we compute
\[
\int_\Omega |\text{curl } a|^2 = \int_\Omega |\text{curl } e|^2 + b^2|\Omega|.
\] (5.15)

Since \( \int_\Omega e = 0 \) and \( \text{div } e = 0 \), the Poincaré’s inequality shows that, for some \( c > 0 \),
\[
2c\|e\|_{H^1}^2 \leq \int_\Omega |\text{curl } e|^2.
\] (5.16)

Now, we estimate \( \text{Tr}(h_a \gamma) \), where, recall, \( a = a_h + e \). Using \( \text{div } e = 0 \), we write
\[
\text{Tr}(h_a \gamma) = \text{Tr}(h_{a_h} \gamma) + 2i\text{Tr}(e \cdot \nabla_{a_h} \gamma) + \text{Tr}(|e|^2 \gamma)
\geq (1 - \epsilon)\text{Tr}(h_{a_h} \gamma) - (\epsilon^{-1} - 1)\text{Tr}(|e|^2 \gamma),
\] (5.17)
for any \( \epsilon > 0 \). For the last term, we claim, for any \( r \in (0,1) \) and some constant \( C \), the estimate
\[
0 \leq \text{Tr}(|e|^2 \gamma) \leq C\|\gamma\|_{L^r(\Omega)}(\text{Tr}(\gamma))^r\|e\|_{H^1}^2,
\] (5.18)
where we used that \( \text{Tr}(h_{a_h} \gamma) = \|\gamma\|_{L^1(\Omega)} \). We prove this estimate for \( r = 1/2 \), which suffices for us. For general \( r \in (0,1) \), see Lemma 5.3 of Appendix D. Recall the definition of \( \lambda_b := \sqrt{h_{a_h}} \).

We use relative bound (D.3) of Appendix D to find
\[
0 \leq \text{Tr}(|e|^2 \gamma) \leq \|e^2 M_b^{-s} \|_2 \|M_b^s \gamma\|_2 \leq \|e^2 \|_{H^1} \|\gamma\|_{L^2} \|\gamma\|_{L^2},
\] (5.19)
with \( s > 2(1 - \nu) \). The last estimate gives (5.18) with \( r = 1/2 \).

Let \( \xi(\gamma) := c/(C(\epsilon^{-1} - 1)\|\gamma\|_{L^1(\Omega)}^2 \nu) \), where, recall, \( \nu = \text{Tr}(\gamma) \) and \( r \in (0,1) \), for the constant \( C \) coming from (5.18), set \( c = \frac{1}{4} \). Then, (5.17) and (5.18), the inequality \( \text{Tr}(h_{a_h} \gamma) \geq \delta \text{Tr}(h_{a_h} \gamma) \), for \( \delta \leq 1 \), and the relation \( \|\gamma\|_{L^1(\Omega)} = \text{Tr}(h_{a_h} \gamma) \) imply
\[
\text{Tr}(h_{a_h} \gamma) \geq c\mathcal{F}(\eta,a) - c\xi(\gamma)^{-1}\|e\|_{H^1}^2.
\] (5.20)

Now, taking \( \delta = \xi(\gamma) \) and using that \( \xi(\gamma) \text{Tr}(h_{a_h} \gamma) = c/(3C)(\|\gamma\|_{L^1(\Omega)}^2 \nu)^r \) gives
\[
\text{Tr}(h_{a_h} \gamma) \geq 2c\|\gamma\|_{L^1(\Omega)}^2 \nu - c\|e\|_{H^1}^2,
\] where \( c' := c/(8C) \). The latter equation implies, in turn,
\[
\text{Tr}(h_{a_h} \gamma) - TS(\gamma) \geq c'E'(\gamma) + c'\|\gamma\|_{L^1(\Omega)}^2 \nu - c\|e\|_{H^1}^2;
\] (5.21)
\[
E'(\gamma) := (1\|\gamma\|_{L^1(\Omega)}^2 \nu)^r - TS(\gamma).
\] (5.22)

To estimate \( E'(\gamma) \) from below, let \( e_k \) be an orthonormal eigenbasis of \( h_{a_h} \) and \( \lambda_k \) be the corresponding eigenvalues. Using that \( \|\gamma\|_{L^1(\Omega)}^2 = \text{Tr}(h_{a_h} \gamma) = \sum k \lambda_k \langle e_k, \gamma e_k \rangle \), regarding \( \langle e_k, \gamma e_k \rangle / \text{Tr}(\gamma) \)
as a probability measure and applying Jensen’s inequality (for $0 < r < 1$ so that $x^r$ is concave), we find

$$\left(\frac{\text{Tr}(h_{a_0}\gamma)/\text{Tr}\gamma)}{\gamma}\right)^r = \left(\sum_k \lambda_k (e_{k,\gamma} e_k)/\text{Tr}\gamma\right)^r$$

(5.23)

$$\geq \sum_k \lambda_k^r (e_{k,\gamma} e_k)/\text{Tr}\gamma.$$  

(5.24)

Since $\sum_k \lambda_k (e_{k,\gamma} e_k) = \text{Tr}(h_{a_0}\gamma)$, we have

$$\left(\frac{\text{Tr}(h_{a_0}\gamma)/\text{Tr}\gamma)}{\gamma}\right)^r \geq \text{Tr}(h_{a_0}\gamma)/\text{Tr}\gamma.$$  

(5.25)

Let $I_{\nu}^{1,1} := \{\gamma \in I^{1,1} : \text{Tr}\gamma = \nu\}$. The above estimate, together with (5.22), restricted to the set $I_{\nu}^{1,1}$, gives

$$E'(\gamma) \geq \text{Tr}(h_{a_0}\gamma)/\nu - TS(\gamma) =: E_{\nu}(\gamma).$$  

(5.26)

We minimize the functional $E_{\nu}(\gamma)$ on the r.h.s. on $I_{\nu}^{1,1}$. Since $E_{\nu}(\gamma)$ is convex and the constraint $\text{Tr}\gamma = \nu$ is linear, each solution to the standard Euler-Lagrange equation $dE_{\nu}(\gamma) - \mu' d\text{Tr}\gamma = 0$ (written in terms of the Gâteaux derivatives), where $\mu'$ is the Lagrange multiplier, is a global minimizer. The latter equation is computed to be $h_{a_0}/\nu - \ln\left(\frac{\gamma_1}{\gamma}\right) - \mu' 1 = 0$. Solving this equation and setting $\mu' = \mu/\nu$ gives the minimizer

$$\gamma_{\mu,T\nu} = f_\mu((h_{a_0}/\nu)/\nu)$$  

(5.27)

for $\mu$ such that $\text{Tr}\gamma_{\mu,T\nu} = \nu$. By the implicit function theorem, the latter equation has a solution, $\mu = \mu(T,\nu)$, for $\mu$. This shows $-C_{\nu,T}/c' := \inf\{E_{\nu}(\gamma) : \gamma \in I_{\nu}^{1,1}\} = E_{\nu}(\gamma_{\mu,T\nu}) > -\infty$, with $\mu = \mu(T,\nu)$, which, together with (5.26), implies

$$\inf_{\gamma \in I_{\nu}^{1,1}} E'(\gamma) \geq \inf_{\gamma \in I_{\nu}^{1,1}} E_{\nu}(\gamma) = -C_{\nu,T}/c' > -\infty,$$  

(5.28)

which, together with (5.21), implies

$$\text{Tr}(h_{a_0}) - TS(\gamma) \geq c' \left(\|\gamma\|_{I^{1,1}}/\nu\right)^r - C_{\nu,T} - c\|e\|_{H^1}^2,$$  

(5.29)

with, recall, $c' := c/(8C)$. Estimates (5.14), (5.15), (5.16), (5.23) and (5.29), restricted to $I_{\nu}^{1,1}, \nu \geq 1$, give bound (5.2). This bound and Lemma 5.4(2) imply bound (5.3). This completes the proof of Proposition 5.2.

We continue with the proof of Theorem 5.1. Eq. (5.29) shows that it suffices to consider $\gamma$ with $S(\gamma) < \infty$ and therefore, by (5.12), $\eta$ with

$$S(\eta) < \infty.$$  

(5.30)

**Part 2: weak lower semi-continuity.** We pass from the positive trace class operator $\gamma$ to the Hilbert-Schmidt one, $\kappa := \sqrt{\gamma}$. Note that $\gamma \in I^{1,1}, \gamma \geq 0 \iff \kappa := \sqrt{\gamma} \in I^{1,2}$. Now, instead of the free energy functional (5.1), we consider the equivalent functional

$$F(\kappa, \alpha, e) := F(\eta, a_0 + e)\big|_{\gamma = \kappa^2} - \frac{1}{2} |\Omega|$$

(5.31)

$$= \text{Tr}(\kappa h_{a_0} + e) + \frac{1}{2} \int \rho_\gamma (v \ast \rho_\gamma)$$

$$+ \frac{1}{2} \text{Tr}(\alpha^* v^2 \alpha) + \int_{\Omega} |\text{curl} e|^2 - TS(\eta)\big|_{\gamma = \kappa^2}$$

(5.32)

on the space $I^{1,2} \times I^{1,2} \times \tilde{H}^1$ with the norm $\|\kappa, \alpha, e\|_{(1)} := \|\kappa\|_{I^{1,2}} + \|\alpha\|_{I^{1,2}} + \|e\|_{H^1}$ and with the side conditions $0 \leq \eta\big|_{\gamma = \kappa^2} \leq 1$ and $\text{Tr}\kappa^2 = \nu$. We will keep the notation $\mathcal{D}_\nu$ for $I^{1,2} \times I^{1,2}$ with these side conditions.
By Proposition 5.2 we find that, for $\text{Tr} \gamma = \nu$ and $0 \leq \eta \leq 1$,

$$F(\kappa, \alpha, e) \geq \frac{1}{4} \frac{\Gamma}{\min(||(\kappa, \alpha)||^2, ||(\kappa, \alpha)||^2)} + c ||e||^2_{H^1} - C,$$

(5.33)

where $|| (\kappa, \alpha)||^2 := ||\kappa||^2_{\mathcal{H}_1} + ||\alpha||^2_{\mathcal{H}_2}$, and for suitable constants $c, C > 0$, with $C$ depending on $\nu, T, ||v||_\infty$ and $|\mathcal{L}|$.

**Lemma 5.5.** The functional $F(\kappa, \alpha, e)$ is weakly lower semi-continuous in $I^{1,2} \times I^{1,2} \times T^1$.

**Proof.** We study the functional $F(\kappa, \alpha, e)$ term by term. For the first term on the r.h.s. of (5.32), with $a = a_b + e$ and $h_a := -\Delta a$, we write

$$\text{Tr}(h_a \gamma) = \text{Tr}((-\Delta a) \gamma) + 2i \text{Tr}(e \cdot \nabla a \gamma) + \text{Tr}(|e|^2 \gamma).$$

(5.34)

Since the first term on the r.h.s. of (5.34) satisfies $\text{Tr}((-\Delta a) \gamma) = ||\kappa||^2_{\mathcal{H}_1}$ and is quadratic in $\kappa$, it is $|| \cdot ||_{1,2}$-weakly lower semi-continuous.

For the second term on the r.h.s. of (5.34), we let $e, e' \in \tilde{H}^1$ and estimate the difference

$$\text{Tr}(e \cdot \nabla a \gamma) - \text{Tr}(e' \cdot \nabla a \gamma').$$

We write

$$\text{Tr}(e \cdot \nabla a \gamma) - \text{Tr}(e' \cdot \nabla a \gamma')$$

$$= \text{Tr}((e - e') \cdot \nabla a \gamma) - \text{Tr}(e' \cdot \nabla a (\gamma - \gamma')).$$

(5.35)

For the first term on the r.h.s., letting $c := e - e'$, we claim that

$$||\text{Tr}(c \cdot \nabla a \gamma)|| \lesssim ||c||_{H^s} ||\gamma||_{I^{1,1}}, s < 1.$$  

(5.36)

To prove this inequality, we recall that $M_b := \sqrt{-\Delta a_b}$ and write $\text{Tr}(c \cdot \nabla a \gamma) = \text{Tr}(M_b^{-1} c \cdot \nabla a b M^{-1} \gamma M_b)$ and use a standard trace class estimate to obtain $||\text{Tr}(c \cdot \nabla a \gamma)|| \lesssim ||M_b^{-1} c \cdot \nabla a b M^{-1} \gamma M_b||_{I^{1,1}}$. Next, we use the boundedness of $\nabla a b M^{-1}$, the relative bound $||M_b^{-1} c|| \lesssim ||c||_{H^{s}}, s < 1$ (see (1.1) of Appendix D), and the relation $||M_b \gamma M_b||_{I^{1,1}} = ||\gamma||_{I^{1,1}}$ to find (5.36).

(Recall that $||\kappa||_{I^{1,2}} = ||\kappa||_{I^{1,1}}^{1/2}$.)

For $\gamma$ and $\gamma'$ non-negative, we claim the following estimate for the second term on the r.h.s. of (5.35) with $c = e'$:

$$||\text{Tr}(c \cdot \nabla a (\gamma - \gamma'))||$$

$$\lesssim ||c||_{H^s} (||\kappa||_{I^{1,2}} + ||\kappa'||_{I^{1,2}})||\kappa - \kappa'||_{I^{s,2}}, s < 1,$$

(5.37)

where $\kappa := (\gamma)^{1/2}$ and $\kappa' := (\gamma')^{1/2}$. To prove this estimate, we write $\gamma = \kappa^2, \gamma' = \kappa'^2$ to expand

$$\gamma - \gamma' = \kappa(\kappa - \kappa') + (\kappa - \kappa')\kappa'.$$

(5.38)

Now, we use the boundedness of $\nabla a b M_b^{-1}$ and the relative bound $||M_{b}^{-s} c|| \lesssim ||c||_{H^s}, s < 1$ (see (D.1) of Appendix D), and the relations $||M_b \kappa||_{I^{1,2}} = ||\kappa M_b||_{I^{1,2}} = ||\kappa||_{I^{1,2}}$, to find

$$||\text{Tr}(c \cdot \nabla a \kappa (\kappa - \kappa'))||$$

$$\lesssim ||c||_{H^s} ||\kappa||_{I^{1,2}} ||\kappa - \kappa'||_{I^{s,2}}, s < 1.$$  

(5.39)

For the second term on the r.h.s. of (5.38), we use the relative bound $||M_b^{-1} c \cdot \nabla a b M_b^{-s} \kappa|| \lesssim ||c||_{H^s}$, for any $s < 1$ (see (D.2) of Appendix D) to find

$$||\text{Tr}(c \cdot \nabla a (\kappa - \kappa')\kappa'||$$

$$\lesssim ||c||_{H^s} ||\kappa'||_{I^{1,2}} ||\kappa - \kappa'||_{I^{s,2}}.$$  

(5.40)

The last two estimates yield (5.37).
Applying (5.36) and (5.37) to the terms on the r.h.s. of (5.35), we find, for \(3/4 < s < 1\),
\[
|\text{Tr}(c \cdot \nabla \alpha) - \text{Tr}(c' \cdot \nabla \alpha')| \lesssim |c - c'| \| \nabla \alpha \|_{L^1} + \|c'\|_{H^s} (\|\alpha\|_{L^2} + \|\alpha'\|_{L^2}) \|\kappa - \kappa'\|_{H^{s,2}}, \quad s, t < 1,
\]
(5.41)
where \(\kappa := \gamma^{1/2} \) and \(\kappa' := (\gamma')^{1/2}\).

Now, we use a standard result for Sobolev spaces, \(\tilde{H}^s\) is compactly embedded in \(\tilde{H}^s\), for any \(s' > s\), and, perhaps, a less standard one, that \(I^{s,2}\) is compactly embedded in \(I^{s,2}\), for any \(s' > s\). (One shows the latter fact by passing to the integral kernel \(s\) and using a standard Sobolev embedding result.)

Now, let \(\{(\kappa_n, \alpha_n, e_n)\}\) be a weakly convergent sequence in \(I^{1.2} \times I^{1.2} \times \tilde{h}^1\) and denote its limit by \((\kappa_*, \alpha_*, a_*)\). Then, by above, it converges strongly in \(D_0' \times \tilde{h}^s\), \(s < 1\). Hence, we have by (5.41),
\[
|\text{Tr}(e_n \cdot \nabla \alpha_n) - \text{Tr}(e_\tau \cdot \nabla \alpha_\tau)| \to 0, \quad n \to \infty,
\]
(5.42)
where, as usual, \(\alpha_n = \kappa_n^2\) and \(\alpha_\tau = \kappa_\tau^2\).

Finally, consider the difference \(\text{Tr}(|e|^2) \gamma - \text{Tr}(|e'|^2) \gamma'\) due to the last term in (5.44). We decompose
\[
\text{Tr}(|e|^2) \gamma - \text{Tr}(|e'|^2) \gamma = \text{Tr}(|e|^2 (\gamma - \gamma')) + \text{Tr}(|e| - |e'|) |e| \gamma' - \gamma).
\]
(5.43)
For the first term on the r.h.s. we claim the following estimate
\[
|\text{Tr}(|e|^2 (\gamma - \gamma'))| \leq \|e\|_{H^s} (\|\kappa\|_{L^2} + \|\kappa'\|_{L^2}) \|\kappa - \kappa'\|_{H^{s,2}}, \quad s < 1,
\]
(5.44)
where \(\kappa := \gamma^{1/2} \) and \(\kappa' := (\gamma')^{1/2}\). We use again (5.36) and the relative bound \(\|M_b^{-s} |e|^2 M_b^{-\tau}|\| \leq \|e\|_{H^s}, \quad s + t > 2(1 - r)\), (see (D.3) of Appendix D) to find, similarly to (5.39) and (5.40),
\[
|\text{Tr}(|e|^2 (\gamma - \gamma'))| = |\text{Tr}(|e|^2 (\kappa' - \kappa) + (\kappa - \kappa') \kappa')|
\]
\[
\leq |\text{Tr}(M_b^{-s} |e|^2 M_b^{-1} M_b (\kappa - \kappa') M_b)|
\]
\[
+ |\text{Tr}(M_b^{-1} |e|^2 M_b^{-s} M_b (\kappa - \kappa') \kappa')|
\]
\[
\lesssim \|e\|_{H^s} (\|\kappa\|_{L^2} + \|\kappa'\|_{L^2}) \|\kappa - \kappa'\|_{H^{s,2}}, \quad s < 1,
\]
(5.45)
which gives (5.44).

Finally, similarly to (5.36), we find for the second term on the r.h.s. of (5.43),
\[
|\text{Tr}(|e|^2 - |e'|^2) \gamma'\gamma| \lesssim (\|e\|_{H^s} + \|e'\|_{H^s}) |e - e'| \|H^s\| \|\gamma\|_{H^{1,1}}, \quad s < 1.
\]
(5.47)
Now, Eqs (5.43), (5.44) and (5.47) imply
\[
|\text{Tr}(e_n |\gamma| - \text{Tr}(e_\tau |\gamma_\tau|)| \lesssim \|e_n\|_{H^s} (\|\kappa_n\|_{L^2} + \|\kappa_\tau\|_{L^2}) \|\kappa_n - \kappa_\tau\|_{H^{s,2}}
\]
\[
+ |\gamma_n|_{H^{1,1}} (\|e_n\|_{H^s} + \|e_\tau\|_{H^s}) \|e_n - e_\tau\|_{H^s},
\]
(5.48)
for \(s < 1\), and therefore, as above the r.h.s. converges to 0. Hence, the first term on the r.h.s. of (5.32) is is weakly lower semi-continuous.

For the second term on the r.h.s. of (5.32), as in the proof of Proposition 3.9, we use the inequalities (5.23) and (5.26) to show that it is weakly lower semi-continuous in \(I^{1.2}\). The third term on the r.h.s. of (5.32) is quadratic in \(\alpha\) and therefore it is continuous in \(I^{1.2} \times I^{1.2} \times \tilde{h}^1\) since \(v \in L^\infty\). It follows that it is weakly lower semi-continuous in \(I^{1.2} \times I^{1.2} \times \tilde{h}^1\).

The third term on the r.h.s. of (5.32), \(\int_\Omega |\text{curl} \, e|^2\), is clearly convex. So its norm lower semi-continuity is equivalent to weak semi-continuity. Since it is clearly \(\tilde{h}^1\)-norm continuous, it is \(\tilde{h}^1\)-weakly lower semi-continuous.
Hence all the terms on the r.h.s. of the expression (5.32) for $F(\kappa, \alpha, e)$, save $-TS(\eta)$, are lower semi-continuous under the convergence indicated. The lower semi-continuity of the latter term is proven in Lemma A.1 of Appendix A. Hence $F(\kappa, \alpha, e)$ is lower semi-continuous, which completes the proof of Lemma 5.5.

Finally, we observe that the set $D^1_\nu \times \mathbb{H}$ is closed in $I^{1.2} \times I^{1.2} \times \mathbb{H}$ under the weak convergence. We continue with the proof of Theorem 5.1. With the results above, the proof of existence of a minimizer is standard. Let $\{ (\kappa_n, \alpha_n, e_n) \}$ be a weakly convergent sequence in $D^1_\nu \times \mathbb{H}$ which is a minimizing sequence for $F(\kappa, \alpha, e)$. By (5.33), the norm $\| (\kappa_n, \alpha_n, e_n) \| = \| \kappa_n \|_{1.2} + \| \alpha_n \|_{1.2} + \| e_n \|_{\mathbb{H}}$ (see the line after (5.32)) is bounded uniformly in $n$. By Sobolev-type embedding theorems, $(\kappa_n, \alpha_n, e_n)$ converges strongly in $D^1_\nu \times \mathbb{H}$ for any $s < 1$ and by the Banach-Alaoglu theorem, $(\kappa_n, \alpha_n, e_n)$ converges weakly in $D^1_\nu \times \mathbb{H}$. Denote the limit by $(\kappa_\ast, \alpha_\ast, e_\ast)$. Since, by Lemma 5.5, $F$ is lower semi-continuous, we have

$$\liminf_{n \to \infty} F(\kappa_n, \alpha_n, e_n) \geq F(\kappa_\ast, \alpha_\ast, e_\ast).$$

(5.49)

Hence, $(\kappa_\ast, \alpha_\ast, e_\ast)$ is indeed a minimizer. This proves the existence of a minimizer of the functional (5.31) and therefore of $F(\eta, a)$.

Now, we establish properties of the minimizer $(\kappa_\ast, \alpha_\ast, e_\ast)$. By (5.30), we have $\operatorname{Tr}(\eta_\ast) < \infty$, where $\eta_\ast$ corresponds to $(\kappa_\ast, \alpha_\ast)$. The statement, that $g(\eta_\ast)$ is trace class, follows from (5.30) and the fact that $g(\eta) \geq 0$.

Furthermore, if we restrict ourselves to even $(\kappa, \alpha, e)$, i.e. to $(\kappa, \alpha, e)$ satisfying (1.30), then the minimizer $(\kappa_\ast, \alpha_\ast, e_\ast)$ is also even.

Now, since a minimizing sequence $e_n$ converges to $e_\ast$ strongly in $\mathbb{H}$ for any $s < 1$, we have, by the magnetic flux quantization (1.30) for $e_n$, the convergence of $e_n$ to $e_\ast$ and by the Stokes theorem, that $\frac{1}{\nu} \int_\Omega \text{curl} a_n = c_1(\rho) \in \mathbb{Z}$, where, recall, $a_n = a_\ast + e_n$.

Finally, the last property of the minimizer $(\kappa_\ast, \alpha_\ast, e_\ast)$ is shown in the following

**Lemma 5.6.** 0 and 1 are not eigenvalues of $\eta_\ast$. Consequently, $0 < \eta_\ast < 1$.

*Proof.* We assume for the sake of contradiction that $\eta_\ast$ has the eigenvalue 0. Hence 1 is also an eigenvalue, since, if $\eta_\ast x = 0$, then (1.3) implies $\eta_\ast J \bar{x} = J \bar{x}$. Let $P_x$ and $P_{J \bar{x}}$ be the orthogonal projections onto the subspaces spanned by $x$ and $J \bar{x}$. Define

$$\eta' := P_x - J P_x J^* = P_x - P_{J \bar{x}}.$$  

(5.50)

Let $\eta_\epsilon := (1 + 2\epsilon/\nu)^{-1}(\eta_\ast + \epsilon \eta')$. Since $x$ and $J \bar{x}$ are orthogonal, it is straightforward to see that $\operatorname{Tr}(\eta_\epsilon) = \nu$, $\eta_\epsilon \in D^1_\nu$, and $0 \leq \eta_\epsilon \leq 1$.

Below, we write $F(\eta')$ for $F(\kappa, \alpha, e)$ and compute $F(\eta_\ast')$. By (5.32), we have

$$F(\eta_\ast) - F(\eta') = -T(S(\eta_\ast) - S(\eta_\ast')) + O(\epsilon).$$

(5.51)

Using that $S(\eta) = \operatorname{Tr}(s(\eta))$, with $s(\eta) = -2\eta \ln \eta$, and writing $\operatorname{Tr}(s(\eta))$ in an orthonormal eigen-basis which includes the eigenvectors $x$ and $J \bar{x}$, we find

$$S(\eta_\ast) - S(\eta_\ast') = \langle x, s(\eta_\ast) x \rangle + \langle J \bar{x}, s(\eta_\ast) J \bar{x} \rangle.$$  

Then we use Jensen’s inequality to find

$$S(\eta_\ast) - S(\eta_\ast') \geq s(\langle x, \eta_\ast x \rangle) + s(\langle J \bar{x}, \eta_\ast J \bar{x} \rangle).$$

(5.52)

Now, using the definition $\eta_\ast := (1 + 2\epsilon/\nu)^{-1}(\eta_\ast + \epsilon \eta')$ and the relations $\eta_\ast x = 0$ and $\eta_\ast J \bar{x} = J \bar{x}$, we compute $\langle x, \eta_\ast x \rangle = (1 + 2\epsilon/\nu)^{-1} \epsilon$ and $\langle J \bar{x}, \eta_\ast J \bar{x} \rangle = (1 + 2\epsilon/\nu)^{-1}(1 + \epsilon)$. This, together with
and the definition $s(\lambda) := -2\lambda \ln \lambda$, yields

$$S(\eta) - S(\eta) \geq s((1 + 2\epsilon/\nu)^{-1}\epsilon) + s((1 + 2\epsilon/\nu)^{-1}(1 + \epsilon))$$

$$= -2\epsilon \ln \epsilon + O(\epsilon).$$

(The nonlinear $\epsilon \ln \epsilon$ term in (5.53) is due to the fact that $x \ln(x)$ is not differentiable at 0.) This, together with (5.51), implies

$$F(\eta) \leq F(\eta) + 2T\epsilon \ln \epsilon + O(\epsilon).$$

Since $\ln \epsilon < 0$ and $|\epsilon \ln \epsilon| \gg \epsilon$, we conclude that $F(\eta) < F(\epsilon, \eta)$ which contradicts minimality of $\eta_*$. We conclude that $\eta_*$ has a trivial kernel; hence, it has a trivial 1-eigenspace. Consequently, $0 < \eta_* < 1$. □

This completes the proof of Theorem 5.1. □

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Appendix A. Entropy

In this appendix we prove the differentiability and expansion of the entropy functional, which we recall here

$$S(\eta) := \text{Tr}(s(\eta)) = \text{Tr}(g(\eta)), \quad g(\eta) := -\eta \ln \eta - (1 - \eta) \ln(1 - \eta), \quad s(\eta) := -2\eta \ln \eta.$$

This is used in the next two appendices in order to prove Theorem 2.2 and Propositions 1.5 and 1.6. Let $dS(\eta)\eta' := \partial_\epsilon S(\eta + \epsilon \eta') |_{\epsilon = 0}$. We have

**Proposition A.1.** Let $\eta \in \mathcal{D}_\nu^1$ be such that $g(\eta) := -\eta \ln \eta - (1 - \eta) \ln(1 - \eta)$ is trace class and $\eta'$ satisfy (2.11). Then $S$ is $C^1$ and its derivative is given by

$$dS(\eta)\eta' = \text{Tr}(g'(\eta')\eta) = \text{Tr}(s'(\eta)\eta').$$

**Proof.** By (A.1), it suffices for us to prove the proposition for $s(\eta) = -\eta \ln(\eta)$. Denote $\eta'' := \eta + \epsilon \eta'$. We write

$$S(\eta'') - S(\eta) = -\text{Tr}(\eta (\ln \eta'' - \ln \eta) - \epsilon \eta' (\ln \eta'' - \ln \eta) - \epsilon \eta' \ln \eta)$$

$$=: A + B - \epsilon \text{Tr}(\eta' \ln \eta).$$

Proof. By (A.1), it suffices for us to prove the proposition for $s(\eta) = -\eta \ln(\eta)$. Denote $\eta'' := \eta + \epsilon \eta'$. We write

$$S(\eta''') = S(\eta'') + \epsilon \text{Tr}(\eta' \ln \eta) = S(\eta''') - S(\eta)$$

$$= -\text{Tr}(\eta (\ln \eta'' - \ln \eta) - \epsilon \eta' (\ln \eta'' - \ln \eta) - \epsilon \eta' \ln \eta)$$

$$=: A + B - \epsilon \text{Tr}(\eta' \ln \eta).$$
Using the formula $\ln a - \ln b = \int_0^\infty [(b + t)^{-1} - (a + t)^{-1}] dt$ and the second resolvent equation, we compute
\[
A := -\text{Tr}(\eta'' - \ln \eta))
\]
\[
= \int_0^\infty \text{Tr}\{\eta'' + (\eta + t)^{-1} - (\eta + t)^{-1}\} dt
\]
\[
= -\int_0^\infty \text{Tr}\{\eta + (\eta + t)^{-1} \eta''(\eta + t)^{-1}\} dt
\]
\[
= -\int_0^\infty \text{Tr}\{\eta'' + (\eta + t)^{-1} \eta''(\eta + t)^{-1}\} dt.
\]  \hspace{1cm} (A.6)

Similarly, we have
\[
B := -\text{Tr}(\epsilon \eta'' - \ln \eta))
\]
\[
= \int_0^\infty \text{Tr}\{\epsilon \eta'' + (\eta + t)^{-1} - (\eta + t)^{-1}\} dt
\]
\[
= -\int_0^\infty \text{Tr}\{\epsilon \eta''(\eta + t)^{-1} \eta''(\eta + t)^{-1}\} dt.
\]  \hspace{1cm} (A.7)

Combining the last two relations with (A.5), we find
\[
S(\eta + \epsilon \eta') - S(\eta) = \epsilon S_1 + \epsilon^2 R_2,
\]  \hspace{1cm} (A.8)

\[
S_1 := -\text{Tr}\eta' \ln \eta - \int_0^\infty \text{Tr}\{\eta(\eta + t)^{-1} \eta'(\eta + t)^{-1}\} dt,
\]  \hspace{1cm} (A.9)

\[
R_2 := \int_0^\infty \text{Tr}\{\eta(\eta + t)^{-1} \eta'(\eta + t)^{-1} \eta''(\eta + t)^{-1}
\]
\[
- \eta'(\eta + t)^{-1} \eta''(\eta + t)^{-1}\} dt.
\]  \hspace{1cm} (A.10)

The estimates below show that the integrals on the r.h.s. converge. Computing the integral
\[
\int_0^\infty \text{Tr}\{\eta(\eta + t)^{-1} \eta'(\eta + t)^{-1}\} dt = \int_0^\infty \text{Tr}\{\eta + t)^{-2} \eta'\} dt = \text{Tr}\eta' \text{ in the expression for } S_1 \text{ and transforming the expression for } R_2, \text{ we obtain}
\]
\[
S_1 := -\text{Tr}\{\eta' \ln \eta + \eta'\},
\]  \hspace{1cm} (A.11)

\[
R_2 := -\int_0^\infty \text{Tr}\{t(\eta + t)^{-1} \eta'(\eta + t)^{-1} \eta''(\eta + t)^{-1}
\]
\[
- \eta'(\eta + t)^{-1} \eta''(\eta + t)^{-1}\} dt.
\]  \hspace{1cm} (A.12)

The proofs of convergence of (A.9) and (A.12) are similar. We consider the case of (A.12). We estimate the integrand on the r.h.s. of (A.12). We have
\[
|\text{Tr}\{\eta'(\eta + t)^{-1} \eta''(\eta + t)^{-1}\}|
\]
\[
\leq \|\eta'(\eta + t)^{-1}\|_{L^2} \|\eta''(\eta + t)^{-1}\|_{L^2}
\]  \hspace{1cm} (A.13)

Now, we show that the factors on the r.h.s. are $L^2(dt)$. By the second condition in (2.11) on $\eta'$, we have
\[
\|\eta'(\eta^# + t)^{-1}\|_{L^2} \leq \|\eta(1 - \eta^#)(\eta^# + t)^{-1}\|_{L^2}
\]
\[
\leq \|\xi^#(\xi^# + t)^{-1}\|_{L^2},
\]  \hspace{1cm} (A.14)
where $\eta^#$ is either $\eta$ or $\eta''$ and $\xi^# := \eta^# (1 - \eta^#)$. Let $\mu_n$ be the eigenvalues of the operator $\xi^# := \eta^# (1 - \eta^#)$. Then we have
\[ \|\xi^#(\xi^# + t)^{-1}\|_{L^2}^2 = \sum_n \mu_n^2 (\mu_n + t)^{-2}, \] (A.15)
and therefore
\[ \int_0^\infty \|\xi^#(\xi^# + t)^{-1}\|_{L^2}^2 dt = \int_0^\infty \sum_n \mu_n^2 (\mu_n + t)^{-2} dt = \sum_n \mu_n = \text{Tr} \xi^#. \] (A.16)

Since $\eta(1 - \eta)$ and $\eta''(1 - \eta'')$ are trace class operators, this proves the claim and, with it, the convergence of the integral on the l.h.s. Similarly, one shows the convergence of the other integrals.

To sum up, we proved the expansion (A.8) with $S_1$ given by (A.11), which is the same as (A.3), and $R_0$ bounded as $|R_0| \lesssim 1$. In particular, this implies that $S$ is $C^1$ and its derivative is given by (A.3).

**Proposition A.2.** $S(\eta) := \text{Tr}(g(\eta))$ is $C^3$ at $\eta_{Tb}$ w.r.t. perturbations $\eta'$ satisfying (2.11). Moreover, we have
\[ S(\eta_{Tb} + \epsilon \eta') = S(\eta_{Tb}) + \epsilon S'(\eta_{Tb}) \eta' + \frac{1}{2} \epsilon^2 S''(\eta', \eta') + O(\epsilon^3), \] (A.17)
where $S'(\eta_{Tb}) \eta' := \text{Tr}(g'(\eta_{Tb}) \eta')$, $S''(\eta', \eta')$ is a quadratic form given by
\[ S''(\eta', \eta') = \frac{1}{2} \int_0^\infty \text{Tr}((\eta + t)^{-1} \eta'(\eta + t)^{-1} \eta') dt \] (A.18)
and the error term is uniform in $\eta'$ and is bounded by $\epsilon^3 \text{Tr}(\eta_{Tb}(1 - \eta_{Tb}))$. For $\eta' = \phi(\alpha)$, the quadratic term becomes
\[ S''(\eta', \eta') = -\text{Tr}(\tilde{\alpha} K_{Tb} \alpha), \] (A.19)
\[ K_{Tb} := \frac{1}{T} \tanh(h_{Tb}/T) + \text{tanh}(h_{Tb}/T), \] (A.20)
where $h_{Tb} := h_{Tb}^L - \mu$, with $h_{Tb}^L$ defined in (1.3).

**Proof.** For the duration of the proof we omit the subindex $Tb$ in $\eta_{Tb}$. Recall (A.5)-(A.7) and continuing computing $A$ and $B$ in (A.6)-(A.7) in the same fashion as in the derivation of these equations, we find
\[ A = \int_0^\infty \text{Tr}((\eta + t)^{-1} \epsilon \eta'(\eta + t)^{-1}) dt \]
\[ -\int_0^\infty \text{Tr}((\eta + t)^{-1} \epsilon \eta'(\eta + t)^{-1} \epsilon \eta'(\eta + t)^{-1}) dt \]
\[ + \int_0^\infty \text{Tr}((\eta + t)^{-1} \epsilon \eta'(\eta + t)^{-1} \epsilon \eta'(\eta + t)^{-1} \epsilon \eta''(\eta + t)^{-1}) dt, \] (A.21)
and
\[ B = \int_0^\infty \text{Tr}((\epsilon \eta' + t)^{-1} \epsilon \eta'(\eta + t)^{-1}) dt \]
\[ -\int_0^\infty \text{Tr}((\epsilon \eta' + t)^{-1} \epsilon \eta'(\eta + t)^{-1} \epsilon \eta'(\eta + t)^{-1}) dt. \] (A.22)
Combining the last two relations with (A.5) and recalling the computation of $S_1$, we find
\begin{equation}
S(\eta + c_1') - S(\eta) = cS_1 + c^2S_2 + c^3R_3
\tag{A.23}
\end{equation}

\begin{equation}
S_2 := -\int_0^{\infty} \text{Tr}\{\eta(\eta + t)^{-1}\eta'(\eta + t)^{-1}\eta'(\eta + t)^{-1}\}dt
- \eta'(\eta + t)^{-1}\eta'(\eta + t)^{-1}\text{dt},
\tag{A.25}
\end{equation}

\begin{equation}
R_3 := \int_0^{\infty} \text{Tr}\{\eta(\eta + t)^{-1}\eta'(\eta + t)^{-1}\eta'(\eta + t)^{-1}\eta''(\eta + t)^{-1}\
- \eta'(\eta + t)^{-1}\eta'(\eta + t)^{-1}\eta'(\eta'' + t)^{-1}\}dt.
\tag{A.26}
\end{equation}

Transforming the expressions for $S_2$ and $R_3$, we obtain
\begin{equation}
S_2 = \int_0^{\infty} \text{Tr}\{t(\eta + t)^{-1}\eta'(\eta + t)^{-1}\eta'(\eta + t)^{-1}\}dt,
\tag{A.25}
\end{equation}

\begin{equation}
R_3 = -\int_0^{\infty} \text{Tr}\{t(\eta + t)^{-1}\eta'(\eta + t)^{-1}\eta'(\eta + t)^{-1}\eta''(\eta + t)^{-1}\}dt.
\tag{A.26}
\end{equation}

Estimates similar to those done after (A.12) show that the integrals on the r.h.s. converge. This proves the expansion (A.23) with $S_1$ and $S_2$ given by (A.11), which is the same as (A.3), and (A.25) and $R_3$ bounded as $|R_3| \leq 1$. Identifying the quadratic form $S_2$ with $S''(\eta', \eta')$, we arrive at the expansion (A.17).

Before computing $S_2 \equiv S''(\eta', \eta')$, we find a simpler representation for it. Integrating the r.h.s. of (A.25) by parts, we find
\begin{equation}
S'' = \int_0^{\infty} \text{Tr}\{(\eta + t)^{-1}\eta'(\eta + t)^{-1}\eta'}dt.
\tag{A.27}
\end{equation}

But by the cyclicity of the trace the last integral is equal to the first one and therefore we have (A.18). Eq (1.54) gives
\begin{equation}
S''(\eta', \eta') = \frac{1}{4} \int_0^{\infty} \text{Tr}\{[(\eta + t)^{-1}\eta'(\eta + t)^{-1}]
+ (1 - \eta + t)^{-1}\eta'(1 - \eta + t)^{-1}\}dt.
\tag{A.28}
\end{equation}

Now, we use (A.28) to compute to $S''$ for $\eta' = \phi(\alpha)$. First, we recall that $\eta = \eta_{TB}$ and observe that for $\eta' = \phi(\alpha)$,
\begin{equation}
\text{Tr}(\eta_{TB} + t)^{-1}\eta'(\eta_{TB} + t)^{-1}\eta' = 2\text{Tr}(\gamma_{TB} + t)^{-1}\alpha(1 - \gamma_{TB} + t)^{-1}\alpha)
= 2\text{Tr}((x + t)^{-1}(y + t)^{-1}\alpha)\alpha)
\tag{A.29}
\end{equation}

where $x$ and $y$ are regarded as operators acting on $\alpha$ from the left by multiplying by $\gamma_{TB}$ and from the right, by $1 - \gamma_{TB}$. Putting this together with a similar expression for the second term on the r.h.s. of (A.28) and performing the integral in $t$, we obtain
\begin{equation}
S''(\eta', \eta') = -\text{Tr}[\bar{\alpha}K(\alpha)],
\tag{A.30}
\end{equation}

\begin{equation}
K := \frac{\log(x) - \log(y)}{x - y} + \frac{\log(1 - x) - \log(1 - y)}{(1 - x) - (1 - y)}.
\tag{A.31}
\end{equation}
with \( x \) acting on the left and \( y \) acting on the right. (A.31) can be written as

\[
K = - \frac{\log(x^{-1} - 1) - \log(y^{-1} - 1)}{x - y}.
\]  

(A.32)

Recalling that \( \gamma_{TB} = f_T(h_{TB}) = (1 + e^{2h_{TB}/T})^{-1} \) and therefore \( x^{-1} - 1 = e^{2h_{TB}/T} \) and \( y^{-1} - 1 = e^{-2h_{TB}/T} \), we see that

\[
K = \frac{1}{T} \frac{h^L_{TB} + h^R_{TB}}{(1 + e^{h_{TB}/T})^{-1} + (1 + e^{-h_{TB}/T})^{-1}},
\]  

(A.33)

which, together the hyperbolic functions identities, \( (1 + e^h)^{-1} = \frac{1}{2}(1 - \tanh h) \) and \((1 + e^{-h})^{-1} = \frac{1}{2}(1 + \tanh h) \), gives (A.19). □

By the definition of the Gâteaux derive and the Hessian and Proposition A.2 we have

**Corollary A.3.** We have

\[
dS(\eta_T^\prime) := \text{Tr}(g'(\eta_T)\eta'),
\]

(A.34)

and, for \( \eta' = \phi(\alpha) \) and with \( h_{TB} := h_{\gamma_{TB}, \alpha_b} \),

\[
S''(\eta_T^\prime)\phi(\alpha) = -\phi(K_{TB}\alpha), \quad K_{TB} := \frac{1}{T} \frac{h^L_{TB} + h^R_{TB}}{\tanh(h^L_{TB}/T) + \tanh(h^R_{TB}/T)}.
\]

(A.35)

Our next result on the entropy is the following

**Lemma A.4.** The functional \(-S(\eta)\) is weakly lower semi-continuous in \( D^1_1 \).

*Proof.* We use an idea from [25] which allows to reduce the problem to a finite-dimensional one. We use (5.8), to pass from \(-S(\eta)\) to the relative entropy, \( S(\eta|\eta_0) \), defined in (5.6), with \( \eta_0 \) of the form (5.7), with \( \text{Tr}\gamma_{\eta_0} < \infty \) and s.t. \( S(\eta_0) < \infty \). By (5.8), \( S(\eta|\eta_0) \geq 0 \). Moreover,

\[
S(\eta) = S(\eta_0) - S(\eta|\eta_0) - \text{Tr}[(\eta - \eta_0) \ln \eta_0].
\]

(A.36)

We choose \( \eta_0 \) so that \( (\eta - \eta_0) \ln \eta_0 \) is trace class and the term \( \text{Tr}[(\eta - \eta_0) \ln \eta_0] \) is weakly lower semi-continuous. We take

\[
\eta_0 = f_T(M), \quad M := \text{diag}(\sqrt{-\Delta_{a_b}}, -\sqrt{-\Delta_{a_b}}).
\]

(A.37)

Since \( f_T(h) = (e^{h/T} + 1)^{-1} \), we see that that

\[
0 \leq - \ln \eta_0 = \ln(1 + e^{M/T}) \lesssim 1 + M/T.
\]

(A.38)

This estimate and (A.37) show that \( \text{Tr}[(\eta - \eta_0) \ln \eta_0] \) is \( \hat{I}^1 \)-norm continuous (\( \hat{I}^1 \)-norm is defined in (1.33)). Indeed, writing \( \ln \eta_0 = (1 + M/T)(1 + M/T)^{-1} \ln \eta_0 = \text{Tr}[(\eta - \eta_0) \ln \eta_0] \) and using (A.38), we find

\[
\|[(\eta - \eta_0) \ln \eta_0]\|_{\hat{I}^1} \leq \|(\eta - \eta_0)\|_{1.1} \|((1 + M/T)^{-1} \ln \eta_0)\|_{1.1} \lesssim \|\eta - \eta_0\|_{1.1}.
\]

(A.39)

This completes the proof of the claim that \( \text{Tr}[(\eta - \eta_0) \ln \eta_0] \) is \( \hat{I}^1 \)-norm continuous.

Furthermore, since this term is affine in \( \eta \), it is convex. Thus it is weakly lower semicontinuous. Now, following [25], let \( s_\lambda(A|B) = \lambda^{-1}(s(\lambda A + (1 - \lambda)B) - \lambda s(A) - (1 - \lambda)s(B)) \) and write

\[
S(\eta|\eta_0) + \text{Tr}(\eta_0 - \eta_0) = \sup_{\lambda \in (0,1)} \text{Tr}(s_\lambda(\eta_0|\eta_0)).
\]

(A.40)
Since the entropy function $s$ is concave, $s_\lambda(A|B) \geq 0$ for any $A, B$. For any non-negative operator $T$ on $L^2(\Omega)$, $\text{Tr}_{L^2(\Omega)} T = \sup_P \text{Tr}_{L^2(\Omega)} PT$ where the sup is taken over all finite rank projections. Hence, we may write

$$S(\eta_n|\eta_0) + \text{Tr}(\eta_0 - \eta_n) = \sup_{\lambda \in (0,1)} \sup_P \text{Tr}(Ps_\lambda(\eta_n|\eta_0))$$

(A.41)

where the sup$_P$ is taken over all finite rank projections $P$. It follows that for any $\lambda \in (0,1)$ and any finite rank projection $P$,

$$S(\eta_n|\eta_0) + \text{Tr}(\eta_0 - \eta_n) \geq \text{Tr}(Ps_\lambda(\eta_n|\eta_0)).$$

(A.42)

Since $\eta_n \to \eta_\bullet$ in $\| \cdot \|_0$ (hence in operator norm) and $-x \ln x$ is continuous on $[0,1]$, we see that $s_\lambda(\eta_n|\eta_0) \to s_\lambda(\eta_\bullet|\eta_0)$

(A.43)

in the operator norm. In particular, for any finite dimensional projection $P$,

$$\text{Tr}(Ps_\lambda(\eta_n|\eta_0)) \to \text{Tr}(Ps_\lambda(\eta_\bullet|\eta_0)).$$

(A.44)

Consequently,

$$\liminf_{n \to \infty} S(\eta_n|\eta_0) + \text{Tr}(\eta_0 - \eta_n) \geq \text{Tr}(Ps_\lambda(\eta_\bullet|\eta_0)).$$

(A.45)

Now taking sup$_{\lambda \in (0,1)}$ and sup$_P$ and using that $\text{Tr}(\eta_0 - \eta_n) = 0$, by condition (2.11), we see that

$$\liminf_{n \to \infty} S(\eta_n|\eta_0) \geq \text{Tr}(s(\eta_\bullet|\eta_0)),$$

(A.46)

which implies the desired statement.

As an aside not used in this paper, we compute the Hessian, $\partial_{\gamma'} S(\gamma Tb)$, of $S$ w.r.t. diagonal perturbations,

$$d(\gamma') := \begin{pmatrix} \gamma' & 0 \\ 0 & -\gamma' \end{pmatrix}.$$  

(A.47)

$\partial_{\gamma'} S(\gamma Tb)$ is defined by

$$[(\gamma', \partial_{\gamma'} S(\gamma Tb))\gamma'] := \partial^2 S(\eta Tb) + \epsilon d(\gamma')|_{\epsilon = 0}.$$  

(A.48)

We have

**Proposition A.5.** The hessian operator $\partial_{\gamma'} S(\gamma Tb)$ is given by

$$\partial_{\gamma'} S(\gamma Tb) = \frac{1}{T} \frac{h^L - h^R}{\tanh(h^L_{Tb}/T) - \tanh(h^R_{Tb}/T)}.$$  

(A.49)

**Proof.** Our starting point in the formula (A.18) and hence we begin with the computation of the term $\int_0^\infty \text{Tr}[(\eta + t)^{-1}\eta'\eta + t)^{-1}\eta']dt$, with $\eta' = d(\gamma')$, where $d(\gamma')$ denotes the perturbation in $\gamma$ given by

$$d(\gamma') := \begin{pmatrix} \gamma' & 0 \\ 0 & -\gamma' \end{pmatrix}.$$  

(A.50)

First, we recall that $\eta = \eta Tb$ and observe that for $\eta' = d(\gamma')$,

$$\text{Tr}((\eta Tb + t)^{-1}\eta(\eta Tb + t)^{-1}\eta') = \text{Tr}((\gamma Tb + t)^{-1}\gamma'(\gamma Tb + t)^{-1}\gamma')$$

(A.51)

$$+ (1 - \gamma Tb + t)^{-1}\gamma'(1 - \gamma Tb + t)^{-1}\gamma')$$

(A.52)

$$= \text{Tr}((x + t)^{-1}(x' + t)^{-1} + (1 - x + t)^{-1}(1 - x' + t)^{-1}1\gamma')\gamma'),$$

(A.53)
where the last follows from Tr(A) = Tr(Å) for self-adjoint operators, and x and x' are regarded as operators acting on γ' from the left by multiplying by γ_{TB} and from the right, by γ_{TB}. Performing the integral in t, we obtain $S''(\eta', \eta') = -\text{Tr} \left[ \gamma' K'(\gamma') \right]$, where the operator $K'$ is given by

$$K' := \frac{\log(x) - \log(x')}{x - x'} + \frac{\log(1 - x) - \log(1 - x')}{(1 - x) - (1 - x')},$$

(A.54)

with x acting on the left and x' acting on the right. Clearly, $K'$ is identified with $\partial_{\gamma_1} S(\gamma_{TB})$. Rewrite the operator $K'$ as

$$K' = -\frac{\log(x^{-1} - 1) - \log(x'^{-1} - 1)}{x - x'}.$$

(A.55)

Recalling that $\gamma_{TB} = g^* (h_{TB}/T) = (1 + e^{2h_{TB}/T})^{-1}$, where $h_{TB} := h_{\gamma_{TB}, ab}$, we see that

$$K = \frac{1}{T} \frac{h^L_{TB} - h^R_{TB}}{(1 + e^{h_{TB}/T})^{-1} - (1 + e^{-h_{TB}/T})^{-1}}.$$

(A.56)

which, together with (A.55) and the hyperbolic functions identities, $(1 + e^h)^{-1} = \frac{1}{2} (1 - \tanh h)$ and $(1 + e^{-h})^{-1} = \frac{1}{2} (1 + \tanh h)$, gives (A.49). □

APPENDIX B. ENERGY FUNCTIONAL: PROOF OF THEOREM 2.2

The proof of Theorem 2.2 consists of three parts: 1) differentiability of $F_T$, 2) identification of the BdG equations with the Euler-Lagrange equation of $F_T$, and 3) showing minimizers of $F_T$ among the set $\mathcal{D}_\rho \times \mathcal{H}_0^1$ are critical points.

**Part 1: differentiability.** We consider first the variation $\eta + \epsilon \eta'$ for $\epsilon > 0$ small and perturbations satisfying (2.11). Note that such $\eta'$ satisfies, for $\epsilon$ small enough,

$$0 \leq \eta + \epsilon \eta' \leq 1.$$

(B.1)

Let $d_\eta F_T(\eta, a) \eta' := \partial_\epsilon F_T(\eta + \epsilon \eta', a)|_{\epsilon = 0}$, if the r.h.s. exists. From (1.22), it is easy to see that $E(\eta, a)$ is Fréchet differentiable and

$$d_\eta E(\eta, a) \eta' = \text{Tr}(\Lambda(\eta, a) \eta').$$

(B.2)

Hence it suffices to prove the Fréchet differentiability of $S(\eta)$. This is done in Appendix A above.

Differentiability of $F_T$ with respect to $a$ is standard and can be easily done. The only two terms in $F_T$ that depend on $a$ are $\text{Tr}((\Delta a) \gamma)$ and $\frac{1}{2} \int |\text{curl} a|^2$. The first term can be differentiated by using $-\Delta a + a' = (-\Delta a)^2 - 2a'(-i \nabla - a) + |a'|^2$ while the second term is differentiable by standard variational calculus. Hence the differentiability of $F_T$ follows from (1.21), (B.2) and Proposition A.1.

**Part 2: Euler-Lagrange equation.**

Now, we show that if $0 < \eta < 1$ and $d_\eta F_T(\eta, a) \eta' = 0$ and $d_a F_T(\eta, a) a' = 0$ for all $\eta'$ on $\mathfrak{g} \times \mathfrak{h}$ satisfying (2.11) and $a' \in \mathfrak{h}^1$, then $(\eta, a)$ satisfies the BdG equations (1.4)-(1.6).

We start with $d_\eta F_T \eta' = 0$ for all $\eta'$ satisfying (2.11). First, we construct explicitly a dense subset of perturbations $\eta'$ satisfying (2.11). For a critical point $(\eta, a), 0 < \eta < 1$, we define a reference unit vector $v_0 = (1, 0)^T \in \mathfrak{h} \times \mathfrak{h}$. We note that the difference in norm of $v_0$'s two components is simply

$$0 \neq 1 = \|(v_0)_1\|_b^2 - \|(v_0)_2\|_h^2 = \langle v_0, Sv_0 \rangle = \text{Tr}(SP_{v_0}).$$

(B.3)

For simplicity and without loss of generality, we assume that $v_0$ is in the image of $\eta(1 - \eta)$ since $0 < \eta < 1$ (i.e. its range is dense). We define $V \subset \mathfrak{h} \times \mathfrak{h}$ as

$$V = \{v : \|v\|_2 = 1, v = \eta(1 - \eta)\xi, \xi \in \mathfrak{h} \times \mathfrak{h} \}.$$
For each \( v \in V \), we define
\[
\eta'_v = (P_v - P_{Jv}) - \frac{\text{Tr}SP_v}{\text{Tr}SP_{v_0}} (P_{v_0} - P_{Jv_0}),
\]  
(B.5)
where \( P_x \) is the orthogonal projection onto \( x \) and \( J \) is the complex structure in (1.3).

**Lemma B.1.** \( \eta'_v \) satisfies (2.11).

**Proof.** To prove the first condition of (2.11), we only prove it for \( P_v - P_{Jv} = P_v - JP_vJ^* \) since this condition is real linear. (Note that \( S \) is self-adjoint, so \( \langle v, Sv \rangle = \text{Tr}SP_v \) is real for all \( v \).) We note that \( J^* = J^{-1} = -J \) and has only real number components. Let \( C \) denote the complex conjugation. It follows then that
\[
J^*(P_v - JP_vJ^*)J = J^*P_vJ - P_v
\]  
(B.6)
\[
= -C(P_v - J^*P_v)JC
\]  
(B.7)
\[
= -C(P_v - JP_vJ^*)C.
\]  
(B.8)
This proves the first condition in (2.11).

To prove the second condition in (2.11), it suffices to show that \( P_v \) satisfies this condition for every \( v \in h \times h \) since \( J \) is unitary. For any \( v = \eta(1 - \eta)\xi, x \in h \times h \), we note that
\[
\|P_vx\| = |\langle v, x \rangle| = |\langle \eta(1 - \eta)\xi, x \rangle| \lesssim \|\xi\|_2 \|\eta(1 - \eta)x\|_2.
\]  
(B.9)
This shows that \( (\eta')^2 \leq C[\eta(1 - \eta)]^2 \). This proves the second condition in (2.11).

Finally, we prove the last condition in (2.11). For any unit norm \( v \in h \times h \),
\[
\text{Tr}S_1(P_v - JP_vJ^*) = \text{Tr}S_1P_v - \text{Tr}S_1JP_vJ^*
\]  
(B.10)
\[
= \text{Tr}SP_v - \text{Tr}J^*SJP_v.
\]  
(B.11)
We note that \( J^*S_1J = S_2 := \text{diag}(0, 1) \). Hence,
\[
\text{Tr}S_1(P_v - JP_vJ^*) = \text{Tr}SP_v.
\]  
(B.12)
It follows that
\[
\text{Tr}S_1\eta'_v = \text{Tr}SP_v - \frac{\text{Tr}SP_v}{\text{Tr}SP_{v_0}} \text{Tr}SP_{v_0} = 0.
\]  
(B.13)
This proves that \( \eta'_v \) satisfies the last condition in (2.11). \( \square \)

We show that \( 0 < \eta < 1 \) and \( d_\eta F_T(\eta, a)\eta' = 0 \) for all \( \eta' \) satisfying (2.11) imply \( \Lambda(\eta, a) - \mu S - Tg'(\eta) = 0 \) for some \( \mu \) and \( S = \text{diag}(1, -1) \) (see Proposition 2.1). First note that (1.21), (B.2), (A.3) and (1.54) yield that
\[
d_\eta F_T(\eta, a)\eta' = \text{Tr} [A\eta'],
\]  
(B.14)
where \( A := \Lambda(\eta, a) - Tg'(\eta) \). If \( (\eta, a) \) is a critical point, then, for all \( v \in V \), it satisfies
\[
\text{Tr}(A\eta'_v) = 0.
\]  
(B.15)
Since \( A \) is in the tangent space of all the \( \eta \) such that \( J^* \eta J = 1 - \bar{\eta} \). We note that \( A \) also satisfies the first condition in (2.11). It follows that

\[
0 = \text{Tr}(A \eta') = \text{Tr}(AP_v) - \text{Tr}(AJ_Pv J^*)
\]

for all \( v \in V \). We note that \( \mu \) is real since \( A \) and \( S \) are self-adjoint. Since \( 0 < \eta < 1 \), the linear space spanned \( V \) is dense. We conclude that \( A \) is a multiple of \( S \), which we denote by \( \mu S \). This shows that

\[
0 = A - \mu S.
\]
where, recall, $\Delta a$ solves the BdG equation, we have that $[\Lambda(\eta,a), \eta] = 0$. Taking the upper left component of this operator-valued matrix equation, we see that

$$[h_{\gamma,a}, \gamma] + (v^s a)\bar{\alpha} - \alpha(v^s a) = 0. \quad \text{(B.28)}$$

Since $v(x) = v(-x)$, we conclude that the integral kernel of $(v^s a)\bar{\alpha} - \alpha(v^s a)$,

$$\int (v(x - z) - v(z - y))\alpha(x, z)\bar{\alpha}(z, y) dz,$$

(B.29)

is zero on the diagonal. Thus, the same conclusion holds for $[\gamma, h_{\gamma,a}]$. Consequently, $\text{Tr}([\gamma, h_{\gamma,a}] \chi) = 0$ and we conclude, by (B.27), that

$$0 = -\text{Tr}(\text{Re}(2i\nabla a_\gamma) \cdot \nabla \chi) = \int_{\Omega} j(\gamma, a) \cdot \nabla \chi = 0. \quad \text{(B.30)}$$

Since this is true for every $\chi \in H^1_{\text{loc}}$ which are $\mathcal{L}$-periodic, it follows that $\text{div} j(\gamma, a) = 0$.

To show that $j(\gamma, a)$ is mean zero, we use, that by our assumptions, $\gamma$ is even and $a$ is odd. Since $(v_{\gamma,a})^\dagger$ and $\text{den}[A] = \text{den}[A^\text{eff}]$ where $(v_{\gamma,a}^\text{eff})(x) = f(-x)$, this shows that $j(\gamma, a)$ is odd. Hence so is $v := \text{curl}^* \text{curl} a - j(\gamma, a)$ and therefore $v := \text{curl}^* \text{curl} a = 0$.

Since $\text{div} a = 0$, we may replace $\text{curl}^* \text{curl}$ by $-\Delta$. Hence, the elliptic regularity theory shows that $a \in H^1$. This completes the proof. \(\square\)

**Part 3: minimizers are critical points.** For a minimizer $(\eta, a)$, we have that $d_\eta F_T(\eta, a)\eta'$, $d_a F_T(\eta, a) a' \geq 0$. Since $\tilde{H}^1$ is linear, $a' \in \tilde{H}^1$ if and only if $-a' \in \tilde{H}^1$. So $d_a F_T(\eta, a) a'(0) = 0$ for all $a \in \tilde{H}^1$. Similarly, we note that $\eta'$ satisfies the assumptions $[2.11]$ if and only if $-\eta'$ satisfies the same requirement. Hence we conclude that $0 = dF_T(\eta, a)\eta'$, which completes the proof. \(\square\)

**Appendix C. Proof of the existence of solution to (3.15)**

**Lemma C.1.** Assume $\int v \geq 0$. Then, for each $T > 0$ and $b = \frac{2\pi n}{|\Omega|}$, the fixed point problem (3.15) has a unique solution.

**Proof.** Let $\lambda := \int v$. We define the real function $g_T : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g_T(\xi) := \lambda \text{den}(f_T(-\Delta_{a_b} - \mu + \xi))(0), \quad \text{(C.1)}$$

where, recall, $\Delta_{a_b}$ acts on the space $\mathfrak{h}$ and is self-adjoint. Then (3.15) can be rewritten as

$$\xi = g_T(\xi). \quad \text{(C.2)}$$

First, we derive a more convenient formula for the function $g_T$. Note that, since, recall $f_T(s) = (1 + e^{s/T})^{-1}$ and, for $b = \frac{2\pi n}{|\Omega|}$, the operator $-\Delta_{a_b}$ on $\mathfrak{h}$ has the eigenvalues $b(2m+1)$, $m = 0, 1, \ldots$, of the same multiplicity $n$, the trace $\text{Tr} f_T(-\Delta_{a_b} - \mu + \xi)$ is finite and is smooth in $\xi$. Since, by Proposition [3.4], $\text{den}[f_T(-\Delta_{a_b} - \mu + \xi)]$ is constant and since $\int_{\Omega} \text{den} A = \text{Tr} A$, we have that

$$g_T(\xi) = \lambda \frac{1}{|\Omega|} \text{Tr} f_T(-\Delta_{a_b} - \mu + \xi) \quad \text{(C.3)}$$

and that $g_T$ is a smooth function.

Since $f_T(s) = (1 + e^{s/T})^{-1}$ is positive and $f_T'(s) = -T^{-1} e^{s/T} (1 + e^{s/T})^{-2}$ is negative, (C.3) shows that $\pm g_T(\xi) > 0$ and $\pm g_T'(\xi) < 0$, if $\pm \lambda > 0$. Hence, for $\lambda > 0$, (C.2) has a unique solution for every $T > 0$ and $b = \frac{2\pi n}{|\Omega|}$. \(\square\)
Appendix D. Relative bounds and estimates on density

In this appendix we prove bounds on functions relative to the operator $M_b$ and estimates on density $\rho_\gamma$. Our first result is the following

**Lemma D.1.** We have the following Sobolev-type inequalities

\begin{align}
\|M_b^{-s}cM_b^{-t}\| &\lesssim \|c\|_{H^r}, s + t > 1 - r, \\
\|M_b^{-s}c \cdot \nabla a_b M_b^{-t}\| &\lesssim \|c\|_{H^r}, t > 1 - r, \\
\|M_b^{-s}|c|^2M_b^{-t}\| &\lesssim \|c\|^2_{H^r}, s + t > 2(1 - r).
\end{align}

*(D.1) (D.2) (D.3)*

where in the second estimate we assumed $\text{div } c = 0$.

**Proof.** We use the diamagnetic inequality $|M_b^{-s}f| \leq M_b^{-s_0}|f|$ (see [2]) to reduce the problem to the $b = 0$ case. To estimate the r.h.s. we write $M_b^{-s}$ as the convolution, $M_b^{-s}u = G_s * u$, where $G_s(x)$ is the Fourier transform of $(1 + |k|^2)^{-s/2}$, and use that $G_s(x)$ decays exponentially at infinity and has the singularity $\propto |x|^{-2+s}$ at the origin. Hence $G_s \in L^q(\mathbb{R}^d)$, $2 < (2 - s)/q$ and we can estimate by the Young inequality $\|G_s*u\|_{L^q} \lesssim \|G_s\|_{L^1}\|u\|_{L^q}$, $1 + 1/k = 1/t + 1/q$, $t < 2/(2-s)$, to obtain

\[ \|M_b^{-s}f\|_{L^k} \lesssim \|f\|_{L^r}, 1/k + s/2 > 1/r, s < 2. \]  

*(D.4)*

Now, to prove (D.1) and (D.3), we apply (D.4) twice and the Young inequality to obtain

\[ \|M_b^{-s}cM_b^{-t}f\|_{L^2} \lesssim \|cM_b^{-t}f\|_{L^q} \lesssim \|c\|_{L^p}\|M_b^{-t}f\|_{L^s} \lesssim \|c\|_{L^p}\|f\|_{L^q}, \]  

*(D.5)*

with $1/2 + s/2 > 1/q = 1/p + 1/k, 1/k + t/2 = 1/2$, which implies $s + t > 2/p$. To obtain (D.1) and (D.3), we use the Sobolev inequalities $\|c\|_{L^p} \lesssim \|c\|_{H^r}, r > 1 - 2/p$ and $\|c\|^2_{L^2p} \lesssim \|c\|^2_{H^r}, r > 1 - 1/p$, respectively.

To prove (D.2), we use, in addition, $M_b^{-s}c \cdot \nabla a_b M_b^{-s} = M_b^{-1}\nabla a_b \cdot cM_b^{-s} + M_b^{-1}(\nabla c)M_b^{-s}$ and $\text{div } c = 0$ to reduce the problem to (D.1) with $s = 0$.

**Lemma D.2.** Let $\gamma$ be a trace-class and positive operator and let $\kappa := \sqrt{\gamma}$. Then

\begin{align}
\|\rho_\gamma\|_{W^{s,1}} &\lesssim \|\kappa\|_{H^{s,2}}\|\kappa\|_{H^{0,2}}, \\
\|\rho_\gamma\|_{W^{1,1}} &\lesssim \|\gamma\|_{H^{1,1}}\text{Tr}\gamma)^{1/2}, \\
\|\rho_\gamma\|_{L^q} &\lesssim \|\gamma\|_{H^{1,1}}\text{Tr}\gamma)^{r}, \forall r \in (0, 1), \\
\|\rho_\gamma\|_{L^q} &\lesssim \|\gamma\|_{H^{s,1}}, s > 2(1 - 1/q).
\end{align}

*(D.6) (D.7) (D.8) (D.9)*

**Proof.** We use (3.11) and $\partial \rho_\gamma = \rho[\partial a_b, \gamma]$ to obtain

\[ \|\partial \rho_\gamma\|_{L^1} = \sup_{\|f\|_{\infty} = 1} \int |f|\partial \rho_\gamma| = \sup_{\|f\|_{\infty} = 1} |\text{Tr}(f[\partial a_b, \gamma])| \lesssim \|\partial a_b, \gamma\|_{H^{s,1}}. \]

Now, writing $\gamma = \kappa^2$ and combining $\partial a_b$ with one of the $\kappa$’s, we estimate furthermore $\|\partial a_b, \gamma\|_{H^{s,1}} \lesssim \|\kappa\|_{H^{1,2}}\|\kappa\|_{H^{0,2}}$. Then we interpolates between $s = 0$ and $s = 1$ to get the first inequality.

For the second inequality, we let $\gamma = \kappa^2$ and write $\rho_\gamma(x) = \int \kappa(x, y)\kappa(y, x)$. It is not hard to see that

\[ \|\rho_\gamma\|_{W^{s,1}} \lesssim \|\kappa\|_{H^{s,2}}\|\kappa\|_{H^{0,2}} = \|\kappa\|_{H^{1,1}}\text{Tr}\gamma)^{1/2}. \]

The second inequality, together with $\int_\Omega \rho^v \leq \int_\Omega \rho^{(q-v)/(1-v)} 1^{1-v}(\int_\Omega \rho)^v, v < 1 < q$, and a Sobolev inequality $\|\rho_\gamma\|_{L^p} \lesssim \|\rho\|_{W^{s,1}}^{1-v/q}, s > 2(1 - 1/p)$, implies (D.8).

The first inequality, together with the Sobolev inequality, $\|\rho_\gamma\|_{L^{2q/2}} \lesssim \|\rho_\gamma\|_{W^{s,1}}, s > 2(1 - 1/q)$, gives (D.9).
Lemma D.3. We have for any $r \in (0, 1)$,
\[ 0 \leq \text{Tr}(|\psi|^2) \lesssim \|\psi\|_{L^r}^r (\text{Tr} \gamma)^r \|\psi\|_{H^1}^2. \] (D.10)

Proof. We write $\text{Tr}(|\psi|^2) = \int_\Omega |\psi|^2 \rho$ and apply to this the Hölder and Sobolev inequalities to obtain
\[ 0 \leq \text{Tr}(|\psi|^2) \lesssim \left( \int \Omega |\psi|^{2p} \right)^{1/p} \left( \int \Omega \rho^q \right)^{1/q} \lesssim \|\psi\|_{H^1}^2 \|\rho\|_{L^q}, \]
where $1/p + 1/q = 1$. This inequality together with inequality (D.8) above gives (D.10) with $r \in (0, 1)$. \qed

We present another way to prove (D.10) assuming the non-abelian interpolation inequality
\[ \|\kappa\|_{I^{1,2}} \lesssim \|\kappa\|_{I^{1,2}}^s \|\kappa\|_{I^{1,2}}^{1-2s}. \] (D.11)

We use $\gamma = \kappa\kappa$ and write, for any $s, t > 0$,
\[ 0 \leq \text{Tr}(|\gamma|^2) = \text{Tr}(M_b^{-s} |\gamma|^2 M_b^{-t} \kappa \kappa M_b^{s}) \lesssim \|M_b^{-s} |\gamma|^2 M_b^{-t} \kappa \kappa M_b^{s}\|_{I^2} \|\kappa\|_{I^2} \|\kappa\|_{I^2}. \] (D.12)

The last inequality, together with (D.12) and relative bound (D.3), gives (D.10) with $r \in (0, 1)$.

Appendix E. Quasifree Reduction

In general, a many-body evolution can be defined on states (i.e. positive linear (‘expectation’) functionals) on the CAR or Weyl CCR algebra $\mathfrak{W}$ over, say, Schwartz space $\mathcal{S}(\mathbb{R}^d, \mathbb{C}^2)$. Elements of this algebra are operators acting on the fermionic/bosonic Fock space $\mathcal{F}$.\footnote{For a more detailed description, see [3] and for the background [4].}

To fix ideas we concentrate on spin 1/2 fermions. Details for bosons could be found in [3]. Let $\psi(x)$ and $\psi^*(x)$, where $x := (x, \alpha)$, $x \in \mathbb{R}^d, \alpha \in \{\frac{1}{2}, -\frac{1}{2}\}$, the spatial and spin variables, be the annihilation and creation operators satisfying the canonical anticommutation relations. Given a quantum Hamiltonian $H$ on $\mathcal{F}$, the evolution of states is given by the von Neumann-Landau equation
\[ i\partial_t \omega_t(A) = \omega_t([A, H]), \quad \forall A \in \mathfrak{W}, \] (E.1)
where $\omega_t$ is the state at time $t$. (We leave out technical questions such as a definition of $\omega_t([A, H])$ as $[A, H]$ is not in $\mathfrak{W}$.)

Let $N := \int d\mathbf{x} \psi^*(\mathbf{x}) \psi(\mathbf{x})$, where $\int d\mathbf{x} := \sum_\alpha \int dx$, be the particle number operator. We distinguish between (a) confined systems with $\omega(N) < \infty$ and (b) thermodynamic systems with $\omega(N) = \infty$. In the former case the states are given by density operators on $\mathcal{F}$, i.e. $\omega(A) = \text{Tr}(AD)$, where $D$ is a positive, trace-class operator on $\mathcal{F}$ with unit trace (see e.g. [4], Lemma 2.4).

As the evolution (E.1) is practically intractable, one is interested in manageable approximations. The natural and most commonly used ones are one-body ones, which trade the number of degrees of freedom for a nonlinearity.

The most general one-body approximation is given in terms of quasifree states. A quasifree state $\varphi$ determines and is determined by the expectations to the second order (for fermions, we may assume that $\varphi(\psi(x)) = 0$):
\[
\begin{align*}
\gamma(x, y) := \varphi(\psi^*(y) \psi(x)), \\
\alpha(x, y) := \varphi(\psi(x) \psi(y)).
\end{align*}
\] (E.2)
Namely, with the short-hand notation $\psi_j := \psi^j(x)$, where $\psi^j(x)$ is either $\psi(x)$ or $\psi^*(x)$, the $n$-point expectations, $\varphi(\psi_1 \cdots \psi_n)$, are given by the Wick theorem as $\varphi(\psi_1 \cdots \psi_n) = 0$ for $n$ odd and, for $n$ even, as

$$\varphi(\psi_1 \cdots \psi_n) = \sum_{P_n} \varepsilon(P_n) \prod_{J \in P_n} \varphi(\psi_{i_1}, \psi_{i_2}),$$

(E.3)

where the $P_n$ are partitions of the ordered set $\{1, \ldots, n\}$ into ordered subsets, $J$, of two elements and $\varepsilon(P_n)$ is $+1$ or $-1$ depending on whether the permutation $\{1, \ldots, n\} \rightarrow (J_1, \ldots, J_{n/2})$ is even or odd.

Assuming $\varphi$ is $SU(2)$ invariant, the spatial and spin variables separate as $\gamma(x, y) = \gamma(x, y)$ and $\alpha(x, y) = \alpha(x, y) \chi(\alpha, \tau)$, where $\alpha(x, y)$ is symmetric under the interchange of $x$ and $y$ and $\chi(\alpha, \tau)$ is antisymmetric under the interchange of $\alpha$ and $\tau$ (see [20]).

Let $\gamma$ and $\alpha$ denote the operators with the integral kernels $\gamma(x, y)$ and $\alpha(x, y)$. One can now verify readily that they satisfy (1.1).

However, the property of being quasifree is not preserved by the dynamics (E.1) and the main question here is how to project the true quantum evolution onto the class of quasifree states.

One elegant way was proposed by Dirac and Frenkel (see [26] for a book exposition and references and [6], for a recent treatment). Another one is due to [3]. Following [3], we define self-consistent approximation as the restriction of the many-body dynamics to quasifree states.

More precisely, we map the solution $\omega_t$ of (E.1), with an initial state $\omega_0$, to the family $\varphi_t$ of quasifree states satisfying

$$i \partial_t \varphi_t(A) = \varphi_t([A, H])$$

(E.4)

for all observables $A$, which are at most quadratic in the creation and annihilation operators, with an initial state $\varphi_0$, which is the quasifree projection of $\omega_0$. We call this map the quasifree reduction of equation (E.1).

Of course, we cannot expect $\varphi_t$ to be a good approximation of $\omega_t$, if $\omega_0$ is far from the manifold of quasifree states.

Evaluating (E.4) on monomials $A \in \{\psi(x), \psi^*(x)\psi(y), \psi(x)\psi(y)\}$ yields a system of coupled nonlinear PDE’s for $(\psi_t, \gamma_t, \alpha_t)$. For the standard many-body Hamiltonian,

$$H = \int dxdy v(x, y)\psi^*(x)\psi^*(y)\psi(x)\psi(y),$$

(E.5)

with $h := -\Delta + V(x)$ acting on the variable $x$ and $v$ a pair potential of the particle interaction, defined on Fock space, $F$, these give the (time-dependent) Bogoliubov-De Gennes (BDG) equations. (In the case of bosons, we arrive at the (time-dependent) Hartree-Fock-Bogoliubov (HFB) equations.)

This is a straightforward, but non-rigorous, derivation of the important effective equations. To prove error bounds is another matter. There was a concerted effort in the last years with important progress and extensive literature. For a recent book and references, see [3, 7, 8, 12, 16, 17, 24, 28, 29, 30].

Finally, we note that according to the BCS theory, Hamiltonian (E.5) describes Cooper pairs of electrons with non-local, attracting interaction, $v(x, y)$, due to exchange of phonons.

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