A Note on the Expected Length of the Longest Common Subsequences of two i.i.d. Random Permutations

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Submitted: Apr 17, 2017; Accepted: May 22, 2018; Published: TBD
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Abstract

We address a question and a conjecture on the expected length of the longest common subsequences of two i.i.d. random permutations of $[n] := \{1, 2, ..., n\}$. The question is resolved by showing that the minimal expectation is not attained in the uniform case. The conjecture asserts that $\sqrt{n}$ is a lower bound on this expectation, but we only obtain $3\sqrt{n}$ for it.

Mathematics Subject Classifications: 05A05, 60C05

1 Introduction

The length of the longest increasing subsequences ($LIS$) of a uniform random permutation $\sigma \in S_n$ (where $S_n$ is the symmetric group) is well studied and we refer to the monograph [5] for precise results and a comprehensive bibliography on this subject. Recently, [3] showed that for two independent random permutations $\sigma_1, \sigma_2 \in S_n$, and as long as $\sigma_1$ is uniformly distributed and regardless of the distribution of $\sigma_2$, the length of the longest common subsequences ($LCS$) of the two permutations is identical in law to the length of the $LIS$ of $\sigma_1$, i.e. $LCS(\sigma_1, \sigma_2) = \mathcal{L} LIS(\sigma_1)$. This equality ensures, in particular, that when $\sigma_1$ and $\sigma_2$ are uniformly distributed, $\mathbb{E}LCS(\sigma_1, \sigma_2)$ is upper bounded by $2\sqrt{n}$, for any $n$ (see [4]) and asymptotically of order $2\sqrt{n}$ ([5]). It is then rather natural to study the behavior of $LCS(\sigma_1, \sigma_2)$, when $\sigma_1$ and $\sigma_2$ are i.i.d. but not necessarily uniform. In this respect, Bukh and Zhou raised, in [2], two issues which can be rephrased as follows:

∗Research supported in part by the grants # 246283 and # 524678 from the Simons Foundation.
Conjecture/Question 1. Let $P$ be an arbitrary probability distribution on $S_n$. Let $\sigma_1$ and $\sigma_2$ be two i.i.d. permutations sampled from $P$. Then $\mathbb{E}_P[LCS(\sigma_1, \sigma_2)] \geq \sqrt{n}$. It might even be true that the uniform distribution $U$ on $S_n$ gives a minimizer.

Below we prove the suboptimality of the uniform distribution by explicitly building a distribution having a smaller expectation. In the next section, before presenting and proving our main result, we give a few definitions and formalize this minimizing problem as a quadratic programming one. Section 3 further explore some properties of the spectrum of the coefficient matrix of our quadratic program. In the concluding section, a quick cubic root lower bound is given along with a few pointers for future research.

2 Main Results

We begin with a few notations. Throughout, $\sigma$ and $\pi$ are, respectively, used for random and deterministic permutations. By convention, $[n] := \{1, 2, 3, ..., n\}$ and so $\{\pi_i\}_{i \in [n]} = S_n$ is a particular ordered enumeration of $S_n$. (Some other orderings of $S_n$ will be given when necessary.) Next, a random permutation $\sigma$ is said to be sampled from $P = (p_i)_{i \in [n]}$, if $\mathbb{P}_P(\sigma = \pi_i) = p_i$. The uniform distribution is therefore $U = (1/n!)_{i \in [n]}$ and, for simplification, it is denoted by $E/n!$, where $E = (1)_{i \in [n]}$ is the $n$-tuple only made up of ones. When needed, a superscript will indicate the degree of the symmetric group we are studying, e.g., $\sigma^{(n)}$ and $P^{(n)}$ are respectively a random permutation and distribution from $S_n$.

Let us now formalize the expectation as a quadratic form:

$$\mathbb{E}_P[LCS(\sigma_1, \sigma_2)] = \sum_{i,j \in [n]} p_i LCS(\pi_i, \pi_j) p_j = \sum_{i,j \in [n]} p_i \ell_{ij} p_j = P^T L^{(n)} P,$$

where $\ell_{ij} := LCS(\pi_i, \pi_j)$ and $L^{(n)} := \{\ell_{ij}\}_{(i,j) \in [n] \times [n]}$. It is clear that $\ell_{ij} = \ell_{ji}$ and that $\ell_{ii} = n$. A quick analysis of the cases $n = 2$ or $3$ shows that both $L^{(2)}$ and $L^{(3)}$ are positive semi-definite. However, this property does not hold further:

Lemma 2. For $n \geq 4$, the smallest eigenvalue $\lambda_1^{(n)}$ of $L^{(n)}$ is negative.

Proof. Linear algebra gives $\lambda_1^{(2)} = 1$ and $\lambda_1^{(3)} = 0$. So to prove the result, it suffices to show that $\lambda_1^{(k+1)} < \lambda_1^{(k)}$, $k \geq 1$ and this is done by induction. The base case is true, since $\lambda_1^{(2)} = 1 > 0 = \lambda_1^{(3)}$. To reveal the connection between $L^{(k+1)}$ and $L^{(k)}$, the enumeration of $S_{k+1}$ is iteratively built on that of $S_k$ by inserting the new element $(k + 1)$ into the permutations from $S_k$ in the following way: the enumeration of the $(k + 1)!$ permutations is split into $(k + 1)$ trunks of equal size $k!$. In the $ith$ trunk, the new element $(k + 1)$ is inserted behind the $(k + 1 - i)$th digit in the permutation from $S_k$. (For example, if $S_2$ is enumerated as $\{[12], [21]\}$, then the enumeration of the first trunk in $S_3$ is $\{[123], [213]\}$, the second is $\{[132], [231]\}$ and the third is $\{[312], [321]\}$. Then the overall enumeration for $S_3$ is $\{[123], [213], [132], [231], [312], [321]\}$.)
Via this enumeration, the principal minor of size $k! \times k!$ is row and column indexed by the enumeration of the permutations $\{\pi_i^{(k)}\}_{i \in [k]}$ from $S_k$ with $(k + 1)$ as the last digit, i.e., $\{[\pi_i^{(k)}(k + 1)]\}_{i \in [k]} \subseteq S_{k+1}$. Then the $(i, j)$ entry of the submatrix is

$$LCS([\pi_i(k + 1)], [\pi_j(k + 1)]) = LCS(\pi_i, \pi_j) + 1,$$

since the last digit $(k + 1)$ adds an extra element into the longest common subsequences. Hence, the $k! \times k!$ principal minor of $L^{(k+1)}$ is $L^{(k)} + E^{(k)}(E^{(k)})^T$, where $E^{(k)}$ is the vector of $\mathbb{R}^{k!}$ only made up of ones. Moreover, notice that the sum of the $\pi_i$-indexed row of $L^{(k)}$ is

$$\sum_{j \in [k!]} LCS(\pi_i, \pi_j) = \sum_{j \in [k!]} LCS(id, \pi_i^{-1}\pi_j) = \sum_{j \in [k!]} LIS(\pi_i^{-1}\pi_j),$$

since simultaneously relabeling $\pi_i$ and $\pi_j$ does not change the length of the LCSs and also since a particular relabeling to make $\pi_i$ to be the identity permutation, which is equivalent to left composition by $\pi_i^{-1}$, is applied here. Further, any $LCS$ of the identity permutation and of $\pi_i^{-1}\pi_j$ is a $LIS$ of $\pi_i^{-1}\pi_j$ and vice versa. So the row sum is equal to

$$\sum_{j \in [k!]} LIS(\pi_i^{-1}\pi_j) = \sum_{\pi \in S_k} LIS(\pi),$$

since left composition by $\pi_i^{-1}$ is a bijection from $S_k$ to $S_k$. This indicates that all the row sums of $L^{(k)}$ are equal. Hence, $E^{(k)}$ is actually a right eigenvector of $L^{(k)}$ and is associated with the row sum $\sum_{\pi \in S_k} LIS(\pi) > 0$ as its eigenvalue, which is distinct from the smallest eigenvalue $\lambda_1^{(k)} \leq 0$.

On the other hand, since $L^{(k)}$ is symmetric, the eigenvectors $R_1^{(k)}$ and $E^{(k)}$ associated with the eigenvalues $\lambda_1^{(k)}$ and $\sum_{\pi \in S_k} LIS(\pi)$ are orthogonal, i.e.,

$$(E^{(k)})^T R_1^{(k)} = 0. \quad (2)$$

Without loss of generality, let $R_1^{(k)}$ be a unit vector, then from (2),

$$\lambda_1^{(k)} = (R_1^{(k)})^T L^{(k)} (R_1^{(k)}) = (R_1^{(k)})^T (L^{(k)} + E^{(k)}(E^{(k)})^T) R_1^{(k)}. \quad (3)$$

As $L^{(k)} + E^{(k)}(E^{(k)})^T$ is the $k! \times k!$ principal minor of $L^{(k+1)}$, (3) becomes

$$\begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix}^T L^{(k+1)} \begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix} \geq \min_{R^T E = 0, \|R\|=1} R^T L^{(k+1)} R = \lambda_1^{(k+1)}, \quad (4)$$

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where $R_1^{(k)}$ is properly extended to $\begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix} \in \mathbb{R}^{(k+1)!}$ and where the above inequality holds true since $\begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix}^T E^{(k+1)} = (R_1^{(k)})^T E^{(k)} = 0$ and $\left\| \begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix} \right\| = \| R_1^{(k)} \| = 1$, where $\| \cdot \|$ denotes the corresponding Euclidean norm. Moreover, equality in (4) holds if and only if $\begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix}$ is an eigenvector of $L^{(k+1)}$ associated with $\lambda_1^{(k+1)}$. We show next, by contradiction, that this cannot be the case. Indeed, assume that $L^{(k+1)} \begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix} = \lambda_1^{(k+1)} \begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix}$.

Now, consider the $k! \times k!$ submatrix at the bottom-left corner of $L^{(k+1)}$, which is row-indexed by $\{(k+1)\pi_i\}_{i\in[k]}$ and column-indexed by $\{\pi_j(k+1)\}_{i\in[k]}$. Notice that the $(i, j)$-entry of this submatrix is $\text{LCS}([(k+1)\pi_i], [\pi_j(k+1)]) = \text{LCS}(\pi_i, \pi_j)$, since $(k+1)$ can be in some $\text{LCS}$ only if the length of this $\text{LCS}$ is 1. So this submatrix is in fact equal to $L^{(k)}$. Further, the vector consisting of the bottom $k!$ elements on the left-hand-side of (5) is $L^{(k)} R_1^{(k)} = \lambda_1^{(k)} R_1^{(k)}$, which is a non-zero vector. However, on the right-hand-side, the corresponding bottom $k!$ elements of the vector $\begin{bmatrix} R_1^{(k)} \\ 0 \end{bmatrix}$ form the zero vector. This leads to a contradiction. So,

$$\lambda_1^{(2)} = 1 > 0 = \lambda_1^{(3)} > \lambda_1^{(4)} > \lambda_1^{(5)} ...$$

The above result on the smallest negative eigenvalue, and its associated eigenvector, will help build a distribution on $S_n$, for which the $\text{LCS}$s have a smaller expectation than for the uniform one.

**Theorem 3.** Let $\sigma_1$ and $\sigma_2$ be two i.i.d. random permutations sampled from a distribution $P$ on the symmetric group $S_n$. Then, for $n \leq 3$, the uniform distribution $U$ minimizes $\mathbb{E}_P[\text{LCS}(\sigma_1, \sigma_2)]$, while, for $n \geq 4$, $U$ is sub-optimal.

**Proof.** As we have seen in (1),

$$\mathbb{E}_P[\text{LCS}(\sigma_1, \sigma_2)] = PTLP$$

$$= (P - U)^T L(P - U) + 2PTLU - U^T L U$$

$$= (P - U)^T L(P - U) + 2U^T L U - U^T L U$$

$$= (P - U)^T L(P - U) + U^T L U,$$

where $PTLU = U^T L U$, since $U$ is an eigenvector of $L$ and $PTU = 1$. 


When $n = 2, 3$, $L^{(n)}$ is positive semi-definite and therefore $(P - U)^T L (P - U) \geq 0$. So, $P^T L P \geq U^T L U$.

However, when $n \geq 4$, by Lemma 2, the smallest eigenvalue $\lambda_1^{(n)}$ is strictly negative and the associated eigenvector $R_1^{(n)}$ is such that $U^T R_1^{(n)} = 0 = E^T R_1^{(n)}$. Hence, there exists a positive constant $c$ such that $c R_1^{(n)} \geq -1/n!$, where $\geq$ stands for componentwise inequality. Let $P_0$ be such that $P_0 - U = c R_1^{(n)}$, then it is immediate that

$$E^T P_0 = E^T (U + c R_1^{(n)}) = 1 + 0 = 1,$$

and that

$$P_0 = U + c R_1^{(n)} \geq 0.$$

Therefore, $P_0$ is a well-defined distribution on $S_n$. On the other hand, by (6), the expectation under $P_0$ is such that

$$\mathbb{E}_{P_0}[\text{LCS}(\sigma_1, \sigma_2)] = (P_0 - U)^T L (P_0 - U) + U^T L U$$

$$= c^2 (R_1^{(n)})^T L (R_1^{(n)}) + U^T L U$$

$$= c^2 \lambda_1^{(n)} + U^T L U$$

$$< U^T L U. \quad (7)$$

However, the right-hand side of (7) is nothing but the expectation under the uniform distribution, namely, $\mathbb{E}_{U}[\text{LCS}(\sigma_1, \sigma_2)]$.

The existence of negative eigenvalues contributes to the above construction and to the corresponding counterexample. So, as a next step, properties of this smallest negative eigenvalue and of the spectrum of the coefficient matrix $L^{(n)}$ are explored.

3 Further Properties of $L^{(n)}$

As we have seen, the vector $E^{(n)}$ which is made up of only ones is an eigenvector associated with the eigenvalue $\sum_{\pi \in S_n} LIS(\pi)$. It is not hard to show that this eigenvalue is, in fact, the spectral radius of $L^{(n)}$.

Proposition 4. $\sum_{\pi \in S_n} LIS(\pi)$ is the spectral radius of $L^{(n)}$.

Proof. Without loss of generality, let $(\lambda, R)$ be a pair of eigenvalue and corresponding eigenvector of $L^{(n)}$ such that $\max_{i \in [n]} |r_i| = 1$, where $R = (r_1, ..., r_n)^T$, and let $i_0$ be the index such that $|r_{i_0}| = 1$. Let us focus now on the $i_0$th element of $\lambda R$. Then, since $L^{(n)} R = \lambda R$,

$$|\lambda| = |\lambda r_{i_0}|$$

$$= \left| \sum_{j \in [n]} LCS(\pi_{i_0}, \pi_j) r_j \right|$$

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\[
\leq \sum_{j \in [n!]} LCS(\pi_{i_0}, \pi_j)
= \sum_{j \in [n!]} LIS(\pi_{i_0}^{-1} \pi_j)
= \sum_{\pi \in S_n} LIS(\pi),
\]

with equality if and only if all the \(r_j\)'s have the same sign and have absolute value equal to 1.

This gives a trivial bound on the smallest negative value \(\lambda_1^{(n)}\): namely,
\[
\lambda_1^{(n)} \geq -\sum_{\pi \in S_n} LIS(\pi).
\]

Moreover, since the expectation of the longest increasing subsequence of a uniform random permutation is asymptotically \(2\sqrt{n}\), this gives an asymptotic order of \(-2\sqrt{n}\) for the lower bound. On the other hand, we are interested in an upper bound for \(\lambda_1^{(n)}\). The next result shows that \(\lambda_1^{(n)}\) decreases at least exponentially fast, in \(n\).

**Proposition 5.** \(\lambda_1^{(n)} \leq 2^{n-4}\lambda_1^{(4)} = -2^{n-3} < 0\).

**Proof.** This is proved by showing that \(\lambda_1^{(n+1)} \leq 2\lambda_1^{(n)}\). As well known,
\[
\lambda_1^{(n+1)} = \min_{E^T R = 0} \frac{R^T L^{(n+1)} R}{R^T R}. \tag{8}
\]

Let \(\lambda_1^{(n)}\) be the smallest eigenvalues of \(L^{(n)}\) and let \(R^{(n)}\) be the corresponding eigenvector. Then, in generating \(L^{(n+1)}\) from \(L^{(n)}\) as done in the proof of Lemma 2, the \(n! \times n!\) principal minor of \(L^{(n+1)}\) is \(L^{(n)} + E E^T\), while its bottom-left \(n! \times n!\) submatrix is \(L^{(n)}\). Symmetrically, it can be proved that the top-right \(n! \times n!\) submatrix is also \(L^{(n)}\), while the bottom-right \(n! \times n!\) submatrix is \(L^{(n)} + E E^T\), i.e., \(L^{(n+1)}\) is
\[
\begin{bmatrix}
L^{(n)} + E E^T & \cdots & L^{(n)} \\
\vdots & \ddots & \vdots \\
L^{(n)} & \cdots & L^{(n)} + E E^T
\end{bmatrix}.
\]

Further, let
\[
R = \begin{bmatrix}
R_1^{(n)} \\
0 \\
\vdots \\
0 \\
R_1^{(n)}
\end{bmatrix}.
\]
Then $E^T R = E^T R_1^{(n)} + E^T R_1^{(n)} = 0$, where, by an abuse of notation, $E$ denotes the vector only made up of ones and of the appropriate dimension. Also,
\[ \| R \|^2 = R^T R = 2 \left\| R_1^{(n)} \right\|^2 = 2. \]
In (8), the corresponding numerator $R^T L^{(n+1)} R$ is
\[
\begin{bmatrix}
R_1^{(n)} \\
0 \\
\vdots \\
0
\end{bmatrix}^T
\begin{bmatrix}
L^{(n)} + E E^T & \cdots & L^{(n)} \\
\vdots & \ddots & \vdots \\
L^{(n)} & \cdots & L^{(n)} + E E^T
\end{bmatrix}
\begin{bmatrix}
R_1^{(n)} \\
0 \\
\vdots \\
0
\end{bmatrix}
= 2 \left( R_1^{(n)} \right)^T \left( L^{(n)} + E E^T \right) \left( R_1^{(n)} \right) + 2 \left( R_1^{(n)} \right)^T L^{(n)} \left( R_1^{(n)} \right)
= 4 \left( R_1^{(n)} \right)^T L^{(n)} \left( R_1^{(n)} \right) = 4 \lambda_1^{(n)}.
\]
Thus, $\lambda_1^{(n+1)} \leq 2 \lambda_1^{(n)}$.

By a very similar method, it can also be proved, as shown next, that the second largest eigenvalue $\lambda_{n!-1}^{(n)}$, which is positive, grows at least exponentially fast.

**Proposition 6.** $\lambda_{n!-1}^{(n)} \geq 2^{n-2} \lambda_1^{(2)} = 2^{n-2} > 0$.

**Proof.** Using the identity
\[
\lambda_{(n+1)!-1}^{(n+1)} = \max_{E^T R = 0} \frac{R^T L^{(n+1)} R}{R^T R},
\]
with a particular choice of
\[
R = \begin{bmatrix}
R_{n!-1}^{(n)} \\
0 \\
\vdots \\
0
\end{bmatrix},
\]
where $R_{n!-1}^{(n)}$ is the eigenvector associated with the second largest eigenvalue $\lambda_{n!-1}^{(n)}$ of $L^{(n)}$, leads to $\lambda_{(n+1)!-1}^{(n+1)} \geq 2 \lambda_{n!-1}^{(n)}$ and thus proves the result.

The above bounds for $\lambda_1^{(n)}$ and $\lambda_{n!-1}^{(n)}$ are far from tight even as far as their asymptotic orders are concerned. Numerical evidence is collected in the following table:
A reasonable conjecture will be that both the smallest and the second largest eigenvalues grow at a factorial-like speed. More precisely, we believe that

\[
\lim_{n \to +\infty} \frac{\lambda_1^{(n)}}{\lambda_1^{(n+1)}} = c_1 \geq 1,
\]

and that

\[
\lim_{n \to +\infty} \frac{\lambda_{n!-1}^{(n+1)}}{\lambda_{n!-1}^{(n)}} = c_2 \geq 1.
\]

4 Concluding Remarks

The \(\sqrt{n}\) lower-bound conjecture of Bukh and Zhou is still open and seems quite reasonable in view of the fact that \(\mathbb{E}_{\text{LCS}}(\sigma_1, \sigma_2) \sim 2\sqrt{n}\), in case \(\sigma_1\) is uniform and \(\sigma_2\) arbitrary (again, see [3]). We do not have a proof of this conjecture, but let us nevertheless present, next, a quick \(\sqrt{n}\) lower bound result.

We start with a lemma describing a balanced property among the lengths of the LCSs of pairs of any three arbitrary deterministic permutations. This result is essentially due to Beame and Huynh-Ngoc ([1]).

**Lemma 7.** For any \(\pi_i \in S_n\) \((i = 1, 2, 3)\),

\[
\text{LCS}(\pi_1, \pi_2)\text{LCS}(\pi_2, \pi_3)\text{LCS}(\pi_3, \pi_1) \geq n.
\]

**Proof.** The proof of Lemma 5.9 in [1] applies here with slight modification. We further note that this inequality is tight, since letting \(\pi_1 = \pi_2 = \text{id}\) and \(\pi_3 = \text{rev(id)}\), which is the reversal of the identity permutation gives, \(\text{LCS}(\pi_1, \pi_2)\text{LCS}(\pi_2, \pi_3)\text{LCS}(\pi_3, \pi_1) = n\).

In Lemma 7, taking \((\pi_1, \pi_2) = (\text{id}, \text{rev(id)})\) gives, for any third permutation \(\pi_3\), \(\text{LCS}(\text{id}, \pi_3)\text{LCS}(\text{rev(id)}, \pi_3) \geq n/\text{LCS}(\text{id}, \text{rev(id)}) = n\). But, since \(\text{LCS}(\text{id}, \pi_3)\) and \(\text{LCS}(\text{rev(id)}, \pi_3)\) are respectively the lengths of the longest increasing/decreasing subsequences of \(\pi_3\), this lemma can be considered to be a generalization of a well-known classical result of Erdős and Szekeres (see [5]).

We are now ready for the cubic root lower bound.

**Proposition 8.** Let \(P\) be an arbitrary probability distribution on \(S_n\) and let \(\sigma_1\) and \(\sigma_2\) be two i.i.d. random permutations sampled from \(P\). Then, for any \(n \geq 1\), \(\mathbb{E}_P[\text{LCS}(\sigma_1, \sigma_2)] \geq \sqrt[3]{n}\).
Proof. Let $\pi_1$, $\pi_2$ and $\pi_3 \in S_n$ and set

$$L(\pi_i) := \sum_{\pi_1 \in S_n} p(\pi_1) LCS(\pi_1, \pi_i) = \sum_{\pi_1 \in S_n} LCS(\pi_1, \pi_i)p(\pi_1),$$

for $i = 2, 3$. Then,

$$L(\pi_2) + LCS(\pi_2, \pi_3) + L(\pi_3) = \sum_{\pi_1 \in S_n} p(\pi_1)(LCS(\pi_1, \pi_2) + LCS(\pi_2, \pi_3) + LCS(\pi_3, \pi_1)) = 3\sqrt{n} \sum_{\pi_1 \in S_n} p(\pi_1) = 3\sqrt{n}, \quad (9)$$

by the arithmetic mean-geometric mean inequality and the previous lemma. Further, summing over $p(\pi_2)$ in $(9)$ gives:

$$\sum_{\pi_2 \in S_n} p(\pi_2)(L(\pi_2) + LCS(\pi_2, \pi_3) + L(\pi_3)) = \sum_{\pi_2 \in S_n} p(\pi_2)L(\pi_2) + L(\pi_3) + L(\pi_3) \geq 3\sqrt{n}.$$

Repeating this last procedure but with weights over $p(\pi_3)$ leads to

$$\sum_{\pi_2 \in S_n} p(\pi_2)L(\pi_2) + 2 \sum_{\pi_3 \in S_n} p(\pi_3)L(\pi_3) = 3 \sum_{\pi \in S_n} p(\pi)L(\pi) \geq 3\sqrt{n}.$$ \quad (10)

However,

$$\mathbb{E}_P[LCS(\sigma_1, \sigma_2)] = \sum_{\pi_1 \in S_n} \sum_{\pi_2 \in S_n} p(\pi_1) LCS(\pi_1, \pi_2)p(\pi_2) = \sum_{\pi_1 \in S_n} \sum_{\pi_2 \in S_n} LCS(\pi_1, \pi_2)p(\pi_2) = \sum_{\pi \in S_n} p(\pi)L(\pi).$$

Combining this last identity with $(10)$ proves the result. 

The above proof is simple; it basically averages out each $LCS(\cdot, \cdot)$ as $\sqrt{n}$ on the summation weighted by $P$. However, in view of the original conjecture, our partial results, as well as those mentioned in the introductory section, the cubic root lower-bound is not tight. Apart from our curiosity concerning this $\sqrt{n}$ conjecture, it would be interesting to know the exact asymptotic order of the smallest eigenvalue $\lambda_{1}^{(n)}$ of $L^{(n)}$. In contrast, the largest eigenvalue $\lambda_{n}^{(n)}$ corresponding to the uniform distribution is known to be asymptotically of order $2n!\sqrt{n}$, since it is equal to the length of the $LISs$ of a uniform random permutation of $[n]$ scaled by $n!$. In this sense, the study of the length of the $LCSs$ between a pair of i.i.d. random permutations having an arbitrary distribution, or equivalently, the study of $L^{(n)}$, can be viewed as an extension of the study of the length of the $LISs$ of a uniform random permutation of $[n]$. Having a complete knowledge of the distribution of all the eigenvalues of $L^{(n)}$ would be a nice achievement.
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