Research Article

On the Fock Kernel for the Generalized Fock Space and Generalized Hypergeometric Series

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In this paper, we compute the reproducing kernel $B(x, y)$ for the generalized Fock space $F^2_m(C)$. The usual Fock space is the case when $m = 2$. We express the reproducing kernel in terms of a suitable hypergeometric series $F_\alpha$. In particular, we show that there is a close connection between $B(z, w)$ and the error function. We also obtain the closed forms of $B_m(z, w)$ when $m = 1, 2/3, 1/2$. Finally, we also prove that $B_m(z, w) \sim e^{i\pi m}|z|^{m-2}$ as $|z| \to \infty$.

1. Introduction

For any fixed parameter $\alpha > 0$, we consider

$$d\lambda_m(z) = d\lambda_{m,a}(z) = c_{m,a} e^{-|z|^\alpha} dA(z),$$ (1)

where $dA(z)$ is the Euclidean area measure on the complex plane $C$. Here, $c_{m,a}$ is a normalizing constant so that $d\lambda_{m,a}$ is a probability measure on $C$.

We call the generalized Fock space $F^2_m(C) = \mathcal{F}^2_m(C)$ the set of all entire functions $f$ in $L^2(C, d\lambda_m(z))$. It is easy to see that $F^2_m(C)$ is a Hilbert space with the inner product:

$$\langle f, g \rangle = \int_C f(z) g(z) d\lambda_m(z).$$ (2)

Let \{\phi_j(z); j \in \mathbb{N}\} be the countable orthonormal basis for $F^2_m(C)$. Then, the generalized Fock kernel $B_m(z, w) = B_{m,a}(z, w)$ for $F^2_m(C)$ is defined by

$$B_m(z, w) = \sum_{j \in \mathbb{N}} \phi_j(z)\phi_j(w).$$ (3)

If $m = 2$, then $F^2_2(C)$ is the usual Fock space. In fact, it is well known that $B_2(z, w) = e^{i\pi w}$ for $z, w \in C$. See the detailed properties on the usual Fock space in the book [1] written by Zhu. In fact, the explicit form of $B_2(z, w)$ is very useful for studying the properties of the Fock space in [2].

In this paper, we focus on the following natural question. Question: compute the Fock kernel $B_m(z, w)$ for any positive rational number $m$.

In the theory of the Bergman kernel, it is difficult to find the closed form of the Bergman kernel for a general domain. Instead, in the case of a complex ellipsoid or similar domains, one can see the expression of the Bergman kernel in terms of the hypergeometric series in [3, 4].

The generalized hypergeometric series $\mathcal{F}_\alpha(a_1, \ldots, a_p; b_1, \ldots, b_q; x)$ is defined by

$$\mathcal{F}_\alpha(a_1, \ldots, a_p; b_1, \ldots, b_q; x) = \sum_{k=0}^\infty (a_1)_k \cdots (a_p)_k x^k (b_1)_k \cdots (b_q)_k k!,$$ (4)

where $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \begin{cases} \frac{\Gamma(a+k)}{\Gamma(a)} & \text{for } k \geq 1, \\ 1 & \text{for } k = 0. \end{cases}$$ (5)

If $p = q + 1$, then the series converges for $|x| < 1$ and...
diverges for \(|x| > 1\). If \(p < q + 1\), then the series converges for all \(x\). If \(p > q + 1\), then the series converges only at \(x = 0\).

It is well known that the Bergman kernel for the complex ellipsoid

\[
D(p_1, \cdots, p_n) = \{ (z_1, \cdots, z_n) \in \mathbb{C}^n : |z_1|^{2p_1} + \cdots + |z_n|^{2p_n} < 1 \}
\]

(6)

is closely connected with \(2F_1\) and its higher dimensional hypergeometric series (Appell hypergeometric series or Lauricella hypergeometric series). Using the theory of the integral kernel, in the space with respect to \(d\mathcal{Q}\) of the integral kernel. In [10], Cho et al. computed the Fock kernel for the space with respect to \(d\mathcal{Q}\). In [9], one can see the boundedness of the Bergman projection on the generalized Fock-Sobolev space with respect to \(d\mathcal{Q}\). But they did not obtain the explicit forms of the integral kernel. In [10], Cho et al. computed the Fock kernel for the space with respect to \(d\mathcal{Q}\). For \(p > 0\), the kernel \(K_{\alpha}(z, w)\) is represented by \(K_{\alpha}(z, w) = F_{\alpha}(m, n + \alpha ; (z, w)^m)\) for \(z, w \in \mathbb{C}^n\).

The main theorem of this paper is the following. At first, we consider the case when \(m\) is a positive integer.

**Theorem 1.** Let \(m\) be any positive integer and let \(\xi := \alpha^{2m}z\omega\).

(i) If \(m\) is even, then

\[
B_m(z, w) = \frac{m^{2m/2}q^{m-1}}{2\pi} \sum_{r=0}^{m-1} \frac{\xi^{r}}{(2r+2)m} \cdot _{2}F_{1}\left(1; \frac{2r+2}{m}; \xi^{m/2}\right).
\]

(7)

(ii) If \(m\) is odd, then

\[
B_m(z, w) = \frac{2m^{2m/m}q^{m-1}}{2\pi} \sum_{r=0}^{m-1} \frac{\xi^{r}}{(2r+2)m} \cdot _{2}F_{1}\left(1; \frac{r+1}{m}, \frac{r+1}{m}; \xi^{m/2}\right).
\]

(8)

Now, we generalize to the case when \(m\) is a positive rational number.

**Theorem 2.** Let \(m\) be any positive rational number and let \(\xi := \alpha^{2m}z\omega\).

(i) If \(m = 2p/q\), where \(2p\) and \(q\) are relatively prime, then

\[
B_m(z, w) = \frac{m^{2m/p}q^{m-1}}{2\pi} \sum_{r=0}^{m-1} \frac{\xi^{r}}{(q(r+1)/p)} \cdot _{2}F_{1}\left(1; \frac{r+1}{p}, \frac{r+1}{q}; \frac{\xi^p}{q^p}\right).
\]

(9)

where

\[
\frac{r+1}{p} + \frac{1}{q} = \left\{ \frac{r+1}{p}, \frac{r+1}{q}, \frac{r+1}{p} + 2, \ldots, \frac{r+1}{p} + q - 1 \right\}
\]

(10)

(ii) If \(m = (2p+1)/q\), where \(2p+1\) and \(q\) are relatively prime, then

\[
B_m(z, w) = \frac{m^{2m/2}q^{m-1}}{2\pi} \sum_{r=0}^{m-1} \frac{\xi^{r}}{(2r+2)m} \cdot _{2}F_{1}\left(1; \frac{r+1}{2p+1}, \frac{r+1}{2q}; \frac{\xi^{2p+1}}{(2q)^{2p+1}}\right).
\]

(11)

where

\[
\frac{r+1}{2p+1} + \frac{1}{2q} = \left\{ \frac{r+1}{2p+1}, \frac{r+1}{2q}, \frac{r+1}{2p+1} + 2, \ldots, \frac{r+1}{2p+1} + q - 1 \right\}
\]

(12)

In particular, if \(m = 4\), then there is a close connection between \(B_4(z, w)\) and the error function.

**Theorem 3.** Let \(\alpha > 0\). Then,

\[
B_4(z, w) = \frac{2\alpha}{\pi} \sqrt{\pi} \cdot e^{-\alpha \xi^2} \cdot \left( \text{erf} (\sqrt{\alpha \xi}) + 1 \right) + \frac{2\sqrt{\alpha}}{\pi \sqrt{\pi}},
\]

where \(\text{erf} (x)\) is the error function denoted by

\[
\text{erf} (x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2} dt.
\]

(13)

In general, it is difficult to find the closed form of the generalized hypergeometric series \(pF_q\). Using the hypergeometric series in Theorems 1 and 2, we obtain the following closed forms for \(m = 1, 2/3, 1/2\).

**Theorem 4.** Let \(\alpha > 0\). Then,

(i) \(B_1(z, w) = (\alpha/2\pi)(\sinh (\alpha \xi^2)) (\xi^2)\),

(ii) \(B_{2/3}(z, w) = a^{-1/2}(\xi^2) \cdot 9 \pi (\xi^2)^{2/3} \cdot (\sin((\sqrt{3}/2) \alpha \xi^2) + (\pi/6))\),

(iii) \(B_{1/2}(z, w) = a^{-1/2} \pi (\xi^2)^{0/4} (\sinh (\alpha \xi^2)^{0/4} - \sin (\xi^2)^{0/4})\).

Finally, we discuss the asymptotic behavior of the Fock kernel. Now, we write \(A(x) \sim B(x)\) if \(A(x)/B(x)\) converges to nonzero constant as \(x\) goes to some number or infinity. Denote \(K_{\ell}(z, w)\) by the Bergman kernel for the bounded
domain $D \subset \mathbb{C}^n$. It is a well-known fact that $K_D(z, z)$ diverges to infinity under some condition. More precisely, if $d(z)$ is the distance to the boundary $bD$, then

$$K_D(z, z) \sim d(z)^{m+1},$$

as $z$ approaches the strongly pseudoconvex boundary point $p \in bD$.

Using the properties of the incomplete gamma function, we can obtain the similar result also for the generalized Fock space.

**Theorem 5.** Let $m$ be any positive even integer. Then,

$$B_m(z, z) \sim e^{d(z)^m} |z|^{m-2} \text{ as } |z| \to \infty. \quad (16)$$

**Remark 6.** The usual Fock kernel $B_1(z, w) = e^{z\bar{w}}$ is very simple but plays an important role in the research of the function theoretic properties of the Fock space $F_{\mathbb{C}, 0}(\mathbb{C})$. Theorems 1 and 2 in this paper are the first result on the generalized Fock space $F_{m, 0}(\mathbb{C})$ for any $m \neq 2$. Also, we hope that the explicit formulas in Theorems 3 and 4 can give a clue on studying optimal pointwise estimates for $B_m(z, w)$ for some $m$.

**2. Computation of $B_m(z, w)$**

Consider $d\lambda_m(z) = \xi_m e^{-|z|^m} dA(z)$, where $\xi_m$ is a normalizing constant so that $d\lambda_m(z)$ is a probability measure on $\mathbb{C}$. In fact, we can obtain $\xi_m$ from the following lemma.

**Lemma 7.** For any nonnegative integers $k$, we have

$$\|z^k\|^2 = \frac{2\pi}{ma^{2k+2/m}} \Gamma\left(\frac{2k+2}{m}\right). \quad (17)$$

where $\Gamma(\cdot)$ is the usual gamma function. In particular, we have

$$\xi_m = \frac{ma^{2m}}{2\pi\Gamma(2/m)}. \quad (18)$$

**Proof.** Recall that the usual gamma function $\Gamma$ is defined by

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx, \ \Re(z) > 0. \quad (19)$$

Using the polar coordinate change, we have

$$\|z^k\|^2 = \int_C |z^k|^2 e^{-n|z|^m} dA(z) = 2\pi \int_0^\infty r^{2k+1} e^{-mr^2} dr. \quad (20)$$

If we can substitute $s = ar^m$, then by $\int_{1}^{\infty} e^{-s} s^{(1/m)-1} ds = \frac{2\pi}{m a^{2k+2/m}} \Gamma\left(\frac{2k+2}{m}\right). \quad (21)$

It completes the proof.

Throughout this paper, we are focusing on computing the function

$$G_m(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma((2k+2)/m)}. \quad (23)$$

Then, we have

$$B_m(z, w) = \frac{ma^{2m}}{2\pi} G_m\left(e^{2/m} z\bar{w}\right). \quad (24)$$

**Remark 8.** If $m = 2$, then $G_2(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{k!} = e^\xi$. In this case, $B_2(z, w) = \frac{\alpha}{\pi} e^{z\bar{w}}$, which is just the usual Fock kernel.

Now, we investigate the relation between $G_m(\xi)$ and generalized hypergeometric series for any positive rational number $m$.

**3. Proof of Theorem 1**

In this section, we express the Fock kernel $B_m(z, w)$ in terms of the suitable hypergeometric series $\pFq{p}{q}$ when $m$ is a positive integer. The crucial term for computing the form of $B_m(z, w)$ is $\Gamma((2k+2)/m)$.

**3.1. Proof of Theorem 1 (i).** Assume that $m$ is an even integer. Let $m = 2p$ for some $p \in \mathbb{N}$. Then, we have

$$G_m(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma((k+1)/p)}. \quad (26)$$

Theorem 1 (i) can be easily proven by the following proposition using (24).
Proposition 9. Let \( m \) be any even positive integer, and let \( \zeta := \alpha^{2m} z \). Then, we have
\[
G_m(\zeta) = \sum_{r=0}^{(m-2)/2} \frac{\zeta^r}{\Gamma((2r+2)/m)} \Phi \left( 1; \frac{2r+2}{m}; \zeta^{m/2} \right),
\]
where \( \Phi(a; b; x) = _1F_1(a; b; x) \) is the confluent hypergeometric series.

Proof. Note that there exist unique integers \( \ell \) and \( r \) such that \( k = p\ell + r \) with \( 0 \leq r \leq p - 1 \). Thus, we have
\[
G_m(\zeta) = \sum_{r=0}^{p-1} \sum_{c=0}^{\infty} \frac{\zeta^r}{\Gamma((r+1)/p)} \Phi \left( 1; \frac{r+1}{p}; \zeta^{p} \right).
\]

Note that
\[
\Phi(\alpha; \beta; x) = _1F_1(\alpha; \beta; x) = \sum_{k=0}^{\infty} \frac{1}{(\beta)_k} x^k.
\]

It follows that
\[
G_m(\zeta) = \sum_{r=0}^{p-1} \zeta^r \sum_{c=0}^{\infty} \frac{1}{(r+1)/p} \Gamma\left( \frac{r+1}{p} \right) \Phi \left( 1; \frac{r+1}{p}; \zeta^{p} \right).
\]

which completes the proof. \( \square \)

3.2. Proof of Theorem 1 (ii). Assume that \( m \) is an odd integer. Let \( m = 2p+1 \) for some \( p \in \mathbb{N} \). Then,
\[
G_m(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma((2k+2)/(2p+1))}.
\]

Theorem 1 (ii) can be easily proven by the following proposition using (24).

Proposition 10. Let \( m \) be any odd positive integer, and let \( \zeta := \alpha^{2m} z \). Then,
\[
G_m(\zeta) = \sum_{r=0}^{m-1} \frac{\zeta^r}{\Gamma((2r+2)/m)} \left( _1F_1 \left( 1; \frac{r+1}{m}, \frac{r+1}{m} + \frac{1}{2}; \zeta^{m/4} \right) \right).
\]

Proof. Note that there exist unique integers \( \ell \) and \( r \) such that \( k = (2p+1)\ell + r \) with \( 0 \leq r \leq 2p \). Then,
\[
G_m(\zeta) = \sum_{r=0}^{2p} \zeta^r \sum_{c=0}^{\infty} \frac{1}{(2r+2)/m} \Gamma\left( 2\ell + ((2r+2)/(2p+1)) \right).
\]

Now, we will use the identity
\[
\Gamma(2\ell + 2t) = 2^{2\ell}(t)_{\ell} \Gamma(2t),
\]
for any nonnegative integer \( \ell \) and \( t \in \mathbb{R} \). In fact, the identity (34) can be proven by
\[
\frac{\Gamma(2\ell + 2t)}{\Gamma(2t)} = (2t)(t+1) \cdots (2t + 2\ell - 1)
\]
\[
= 2^{2\ell}(t+1)(t + \ell - 1)(t + 3/2) \cdots (t + 2\ell - 1/2)
\]
\[
= 2^{2\ell}(t)_{\ell} \left( 1 + \frac{1}{2} \right)_{\ell}.
\]

Then, by (34), we have
\[
\Gamma_m(\zeta) = \sum_{t=0}^{\infty} \frac{1}{\Gamma((2r+2)/(2p+1))} \sum_{c=0}^{\infty} \frac{1}{(r+1)/p} \left( \frac{\zeta^{2p+1}}{p} \right)^c,
\]
\[
= \sum_{r=0}^{p} \frac{\zeta^r}{\Gamma((2r+2)/(2p+1))} \left( _1F_1 \left( 1; \frac{r+1}{p}, \frac{r+1}{p} + \frac{1}{2}; \zeta^{2p+1} \right) \right).
\]

since \( _1F_1(1; \beta; x) = \sum_{k=0}^{\infty} x^k/(\beta)_k \).

\( \square \)

4. Proof of Theorem 2

In this section, we focus on computing \( G_m \) when \( m \) is a positive rational number.

4.1. Proof of Theorem 2 (i): Even Numerator. Let \( m = 2p/q \), where \( 2p \) and \( q \) are relatively prime. Then, we have
\[
G_m(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma((k+1)/q)} \sum_{r=0}^{p-1} \zeta^r \left( _1F_1 \left( 1; \frac{r+1}{q}, \frac{r+1}{q} + \frac{1}{2}; \zeta^{q} \right) \right),
\]
where \( k = p\ell + r \) with \( 0 \leq r \leq p - 1 \).

Lemma 11. The gamma function \( \Gamma \) satisfies the identity
\[
\Gamma(x)\Gamma\left( x + \frac{1}{n} \right) \cdots \Gamma\left( x + \frac{n-1}{n} \right) = (2\pi)^{n/2} \Gamma(1^{1/2}) \cdots \Gamma(nx).
\]

Using the above lemma, we can prove the following.

Lemma 12.
\[
\Gamma(qt + qt) = q^{qt} \prod_{j=0}^{q-1} \left( t + \frac{j}{q} \right) \Gamma(qt).
\]

Proof. We will prove it in two different methods. Using the property \( \Gamma(x+1) = x\Gamma(x) \), we have
\[
\frac{\Gamma(qt + qt)}{\Gamma(qt)} = q^{qt} \prod_{i=0}^{q-1} \left( t + \frac{i}{q} \right).
\]

Then, there exists \( x, y \in \mathbb{Z} \) such that \( i = qj + y \) with \( 0 \leq j \)
where \( \leq \ell - 1 \) and \( 0 \leq \gamma \leq q - 1 \). It follows that

\[
\prod_{j=0}^{q-1} \left( t + \frac{j}{q} \right) = \prod_{j=0}^{q-1} \prod_{x=0}^{q-1} \left( t + \frac{j + x}{q} \right) = \prod_{j=0}^{q-1} \left( t + \frac{j}{q} \right). \tag{41}
\]

It can be proven also using Lemma 11. Note that \( r \) and \( q \) are relatively prime. Then, it can be proven also using Lemma 11. It follows that

\[
\Gamma(t + \frac{1}{q}) = \frac{(2n)(q-1)\cdots(q-\gamma)}{(2n)!} (t+1) \cdots \left( t + \frac{q-1}{q} \right). \tag{42}
\]

Now, we prove Theorem 2 (i) using Lemma 12.

**Theorem 13** (Theorem 2 (i) again). Let \( m = 2p/q \), where \( 2p \) and \( q \) are relatively prime. Then,

\[
G_m(\zeta) = \frac{\sum_{r=0}^{p-1}}{\Gamma(q(r+1)/p)} F_q \left( 1; \frac{r + 1}{p} + \frac{j}{q} ; \frac{\zeta^{q}}{q} \right). \tag{43}
\]

where

\[
\frac{r + 1}{p} + \frac{j}{q} = \left( \frac{r + 1}{p} + \frac{1}{q} \right) + \left( \frac{r + 1}{p} + \frac{2}{q} \right) + \cdots + \left( \frac{r + 1}{p} + \frac{q - 1}{q} \right). \tag{44}
\]

Thus, we have

\[
B_m(z, w) = \frac{\sum_{r=0}^{p-1}}{\Gamma(q(r+1)/p)} F_q \left( 1; \frac{r + 1}{p} + \frac{j}{q} ; \frac{\zeta^{q}}{q^q} \right). \tag{45}
\]

**Proof.** By Lemma 12, we have

\[
G_m(\zeta) = \frac{\sum_{r=0}^{p-1}}{\Gamma(q(r+1)/p)} \sum_{t=0}^{2p-1} \frac{1}{(r+1)/p + (j/2q))} \left( \frac{\zeta^{q^q}}{q^q} \right). \tag{46}
\]

By the definition (4), we see that

\[
\Gamma(t + \frac{1}{q}) = \frac{x^t}{(b_1)_{\ell} \cdots (b_q)_{\ell}}. \tag{47}
\]

It follows that

\[
G_m(\zeta) = \sum_{r=0}^{p-1} \frac{\zeta^r}{\Gamma(q(r+1)/p)} F_q \left( 1; \frac{r + 1}{p} + \frac{j}{q} ; \frac{\zeta^{q}}{q} \right). \tag{48}
\]

If we use (24), then it completes the proof. \( \square \)

4.2. Proof of Theorem 2 (ii): Odd Numerator. Let \( m = 2p + 1/q \), where \( 2p + 1 \) and \( q \) are relatively prime. Then,

\[
G_m(\zeta) = \sum_{k=0}^{2p} \frac{\zeta^k}{\Gamma(2qk + (2q(r+1)/(2p+1)))} \tag{49}
\]

where \( k = (2p+1)\ell + r \) with \( 0 \leq r \leq 2p \). By Lemma 12, we have

\[
\Gamma \left( \frac{2q(k + r + 1)}{2p + 1} \right) = \Gamma(2qk)(2q)^q \prod_{j=0}^{q-1} \left( t + \frac{j}{2q} \right) \tag{50}
\]

where

\[
G_m(\zeta) = \sum_{k=0}^{2p} \frac{\zeta^k}{\Gamma(2qk + (2q(r+1)/(2p+1)))} \tag{51}
\]

If we use (24), then we obtain the following.

**Theorem 14** (Theorem 2 (ii) again). Let \( m = (2p + 1)/q \), where \( 2p + 1 \) and \( q \) are relatively prime. Then,

\[
G_m(\zeta) = \sum_{r=0}^{2p} \frac{\zeta^r}{\Gamma((2q(r+1)/(2p+1)))} F_q \left( 1; \frac{r + 1}{2p + 1} + \frac{j}{2q} ; \frac{\zeta^{q+1}}{q+1} \right). \tag{52}
\]

where

\[
\frac{r + 1}{2p + 1} + \frac{j}{2q} = \left( \frac{r + 1}{2p + 1} + \frac{1}{2q} \right) + \cdots + \left( \frac{r + 1}{2p + 1} + \frac{q - 1}{2q} \right). \tag{53}
\]
Thus, we have
\begin{equation}
B_m(z, w) = \frac{ma^{mz}}{2\pi i} \Gamma(\frac{2q(r+1)}{2p+1}) \int e^{\frac{t}{2p+1} - \frac{t^p}{2q} \zeta^{q-1}} dt.
\end{equation}

5. Special Cases

In the last section, we express $B_m(z, w)$ in terms of the generalized hypergeometric series $\binom{1}{q}$ for a suitable $q$. However, in general, it is difficult to find the closed form of $\binom{1}{q}(1; b_1, \ldots, b_q; x)$ for any $b_1, \ldots, b_q$.

5.1. Proof of Theorem 3: The Case When $m = 4$. In this case, we will conclude that $\binom{1}{4}(1, 1/2; x)$ in terms of the error function. In fact, we will present the following identities.

By Proposition 9, we have

$$B_4(z, w) = \frac{2\pi}{\pi} \text{erf} \left( \frac{\pi}{\omega} \right) + 2 \frac{\sqrt{\pi}}{\sqrt{\pi}}. \tag{55}$$

By Proposition 15 (ii), we have

$$\Phi(b - 1; b_1, -x) = \frac{\Gamma(b)}{\Gamma(b - 1) \Gamma(1)} \int_0^x e^{-ru} u^{b-2} du$$

$$= (b - 1)x^{b-2} \int_0^x e^{-ru} du = (b - 1)x^{b-2} \text{erf}(b - 1, x). \tag{60}$$

It completes the proof.

In particular, if $m = 4$, then we can write $G_4(\zeta) = B_4(z, w)$ in a simple form using the error function. Recall that the error function $\text{erf}(x)$ is denoted by

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{61}$$

It is easy to see that $\gamma(1/2, x) = \sqrt{\pi} \text{erf}(\sqrt{x})$.

The following lemma can be proven easily by the integration parts of the integral.

Lemma 17. For any $s$, we have

$$\gamma(s + 1, x) = s \gamma(s, x) - x^s e^{-x}. \tag{62}$$

By Lemma 17, we have

$$\gamma \left( \frac{1}{2}, x \right) = -2 \gamma \left( \frac{1}{2}, x \right) - \frac{2e^{-x}}{\sqrt{x}} = -2\sqrt{\pi} \text{erf} \left( \sqrt{x} \right) - \frac{2e^{-x}}{\sqrt{x}}. \tag{63}$$

By Proposition 16, we have

$$\Phi \left( 1, \frac{1}{2}; x \right) = -\frac{1}{2} \sqrt{\pi} x^\frac{1}{2} \gamma \left( \frac{1}{2}, x \right) = \sqrt{\pi} x^\frac{1}{2} \text{erf} \left( \sqrt{x} \right) + 1. \tag{64}$$

Now, we are ready to express $G_4(\zeta)$ and $B_4(z, w)$ in terms of the error function.

Theorem 18 (Theorem 3 again). If $m = 4$, then

$$G_4(\zeta) = \zeta e^{\frac{1}{2}} \left( \text{erf} \left( \frac{\pi}{\omega} \right) + \frac{1}{\sqrt{\pi}} \right). \tag{65}$$

Thus, we have

$$B_4(z, w) = \frac{2\pi}{\pi} \text{erf} \left( \frac{\pi}{\omega} \right) + 2 \frac{\sqrt{\pi}}{\sqrt{\pi}}. \tag{66}$$

Proof. By Proposition 9, we have

$$G_4(\zeta) = \frac{1}{\sqrt{\pi}} \Phi \left( 1, \frac{1}{2}; \zeta^2 \right) + \zeta \Phi \left( 1, 1; \zeta^2 \right). \tag{67}$$

If we use (64) and the identity $\Phi(1, 1; \zeta) = e^{\zeta}$, then we
obtain (65). Since \( B_3(z, w) = (2\sqrt{\alpha/\pi}) G_4(\sqrt{\alpha z w}) \), we obtain the formula of \( B_1(z, w) \).

5.2. Proof of Theorem 4: The Case when \( m = 1, 2/3, 1/2 \). It is surprising that we can obtain the explicit forms of \( B_1(z, w) \), \( B_{2/3}(z, w) \), and \( B_{1/2}(z, w) \).

**Theorem 19** (Theorem 4 (i) again). If \( m = 1 \), then

\[
G_1(\zeta) = \frac{\sinh \left( \sqrt{\zeta} \right)}{\sqrt{\zeta}}. \tag{68}
\]

Thus, we have

\[
B_1(z, w) = \frac{\alpha}{2\pi} \sinh \left( \frac{(\alpha z w)^{1/2}}{z^{1/2}} \right). \tag{69}
\]

**Proof.** Note that \( G_1(\zeta) = _1F_2(1; 1, 3/2; \zeta/4) \). Use the identity

\[
_1F_2(1; 1, 3/2; x) = \frac{\sin(2\sqrt{x})}{2\sqrt{x}}. \tag{70}
\]

In fact, the identity (70) can be proven as follows. Note that

\[
_1F_2(1; 1, 3/2; x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \frac{3}{2} \frac{5}{2} \cdots \frac{k+1}{2} \frac{3 \cdot 5 \cdots (2k+1)}{2^k} \frac{(2k+1)!}{4^k}. \tag{71}
\]

It follows that

\[
_1F_2(1; 1, 3/2; x) = \sum_{k=0}^{\infty} \frac{x^k}{(2k+1)!} \frac{3}{2} \frac{5}{2} \cdots \frac{k+1}{2} \frac{3 \cdot 5 \cdots (2k+1)}{2^k} \frac{(2k+1)!}{4^k}. \tag{72}
\]

In general, the explicit forms of the most hypergeometric series are unknown. But the very special following the hypergeometric series including (70) can be computed.

**Proposition 20.** For any \( x \), we have

(i) \( _1F_2(1; 1, 4/3, 5/3; x) = 2e^{-3x/2}x^{x/3}/27x^{3x/3} \left( e^{(9/2)x^{3/2}} - 2 \sin \left( \frac{3\sqrt{3}x^{3/2} + \pi/6} {x^{1/2}} \right) \right) \)

(ii) \( _1F_4(1; 1, 5/4, 6/4, 7/4; x) = 3/64x^{3x/4} \left( \sinh \left( \frac{4x^{1/4}} {4x^{1/4}} \right) \right) \)

One can find the closed forms of various hypergeometric series in [11]. In particular, one can find the closed forms of

\[
_1F_3 \left( 1; 1, 4, 5 \middle| \frac{3}{3}; \frac{3}{3}; x \right) = _1F_2 \left( 1; 1, 4, 5 \middle| \frac{3}{3}; \frac{3}{3}; x \right), \tag{73}
\]

\[
_1F_4 \left( 1; 1, 5, 6, 7 \middle| \frac{4}{4}; \frac{4}{4}; x \right) = _1F_3 \left( 1; 1, 5, 6, 7 \middle| \frac{4}{4}; \frac{4}{4}; x \right),
\]

in [12, 13], respectively.

Now, we prove Theorem 4 (ii) and (iii) as finding the closed forms of \( B_{2/3}(z, w) \) and \( B_{1/2}(z, w) \) using Proposition 20. Since we have

\[
G_{2/3}(\zeta) = \frac{1}{2} _1F_3 \left( 1; 1, 4, 5 \middle| \frac{3}{3}; \frac{3}{3}; \frac{\zeta}{27} \right), \tag{74}
\]

it follows that

\[
B_{2/3}(z, w) = \frac{\alpha^3}{3\pi} G_{2/3}(\alpha^3 z w) = \frac{\alpha^3}{6\pi} \left( 1; 1, 4, 5 \middle| \frac{3}{3}; \frac{3}{3}; \frac{\alpha^3 z w}{27} \right). \tag{75}
\]

By Proposition 20 (i), we have

\[
B_{2/3}(z, w) = \frac{\alpha e^{-a \alpha \alpha}}{9\pi (z w)^{3/3}} \left( e^{(2a \alpha \alpha)^{3/2}} - 2 \sin \left( \frac{3\sqrt{3} a (z w)^{1/2} + \pi}{6} \right) \right). \tag{76}
\]

Since we have

\[
G_{1/2}(\zeta) = \frac{1}{6} _1F_4 \left( 1; 1, 5, 6, 7 \middle| \frac{4}{4}; \frac{4}{4}; \frac{\zeta}{47} \right), \tag{77}
\]

it follows that

\[
B_{1/2}(z, w) = \frac{\alpha^4}{4\pi} G_{1/2}(\alpha^4 z w) = \frac{\alpha^4}{24\pi} \left( 1; 1, 5, 6, 7 \middle| \frac{4}{4}; \frac{4}{4}; \frac{\alpha^4 z w}{47} \right). \tag{78}
\]

By Proposition 20 (ii), we have

\[
B_{1/2}(z, w) = \frac{\alpha}{8\pi (z w)^{3/4}} \left( \sinh \left( \alpha (z w)^{1/4} \right) - \sin \left( \alpha (z w)^{1/4} \right) \right). \tag{79}
\]

It completes the proof of Theorem 4 (ii) and (iii).

5.3. Proof of Theorem 5. In this section, \( A(x) \sim B(x) \) means that \( A(x)/B(x) \) converges to a nonzero constant as \( x \) goes to some number or infinity.

**Theorem 21** (Theorem 5 again). Let \( m \) be any positive even integer. Then,

\[
B_m(z, w) \sim e^{\pi |z|} |z|^{m-2} \alpha z \longrightarrow \infty. \tag{80}
\]
We call the (generalized) Fock space entire functions. We can conjecture that (80) holds for any $m > 0$.

6. Concluding Remarks

In fact, we can consider the more generalized Fock space. Let $d\lambda_{g}(z) = c_{g}e^{-\phi(z)}dA(z)$, where $dA(z)$ is the Euclidean area measure on the complex plane $\mathbb{C}$. We assume that $\phi(r)$ is radial and increasing on $[0, \infty)$ with $\lim_{r \to \infty}\phi(r) = \infty$. We call the (generalized) Fock space $F_{\phi}(\mathbb{C})$ as the set of all entire functions $f$ in $L^{2}(\mathbb{C}, d\lambda_{g})$. Another simple example is $\phi(r) = 1/r$. In this case, we can show that the Fock kernel can be written in terms of the Meijer-G function. It will be interesting that one finds the relation between the other hypergeometric series and the new Fock kernel with respect to $\phi$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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Proof. Let $m = 2p$. Then, by Theorem 1 (i),

\[
B_{2p}(z, w) = \frac{pa^{1/p}p^{-1}}{2\pi} \sum_{m=0}^{\infty} r^{m} \frac{\frac{\alpha}{2} + \frac{m}{2}}{\frac{\alpha - m}{2}} \Phi\left(1, \frac{r + 1}{p}; \frac{\xi^{p}}{r^{p}}\right) - \frac{pa^{1/p}}{2\pi} \sum_{m=0}^{\infty} r^{m} \frac{\frac{\alpha}{2} + \frac{m}{2}}{\frac{\alpha - m}{2}} \Phi\left(1, \frac{r + 1}{p}; \frac{\xi^{p}}{r^{p}}\right).
\]

If $0 \leq r \leq p - 2$, then by Proposition 16,

\[
\Phi\left(1, \frac{r + 1}{p}; \frac{\xi^{p}}{r^{p}}\right) = \left(\frac{r + 1}{p} - 1\right) e^{\frac{r + 1}{p} \xi^{p} - r - 1} \Phi\left(1, \frac{r + 1}{p} - 1, \frac{\xi^{p}}{r^{p}}\right).
\]

and $\Phi(1, 1; \frac{\xi^{p}}{r^{p}}) = e^{\frac{\xi^{p}}{r^{p}}}$. It follows that

\[
B_{2p}(z, w) = \frac{pa^{1/p}}{2\pi} \left\{ \sum_{m=0}^{p-2} r^{m} e^{\frac{r + 1}{p} \xi^{p} - r - 1} \right\} + \frac{\xi^{p-1} e^{\frac{\xi^{p}}{r^{p}}}}{r^{p}}.
\]

Since $\Gamma((r + 1)/p - 1, x) \to \Gamma((r + 1)/p)$ as $x \to \infty$, it completes the proof. 

In fact, it is easily checked that (80) holds also when $m = 1, 3/2, 1/2$ using the explicit forms in Theorem 4. We can conjecture that (80) holds for any $m > 0$. 

6. Concluding Remarks

In fact, we can consider the more general Fock space. Let $d\lambda_{g}(z) = c_{g}e^{-\phi(z)}dA(z)$, where $dA(z)$ is the Euclidean area measure on the complex plane $\mathbb{C}$. We assume that $\phi(r)$ is radial and increasing on $[0, \infty)$ with $\lim_{r \to \infty}\phi(r) = \infty$. We call the (generalized) Fock space $F_{\phi}(\mathbb{C})$ as the set of all entire functions $f$ in $L^{2}(\mathbb{C}, d\lambda_{g})$. Another simple example is $\phi(r) = 1/r$. In this case, we can show that the Fock kernel can be written in terms of the Meijer-G function. It will be interesting that one finds the relation between the other hypergeometric series and the new Fock kernel with respect to $\phi$. 

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