The ternary Goldbach problem with primes from arithmetic progressions

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In 1937 Vinogradov [7] found an asymptotic formula for the number of solutions of the equation

\[ p_1 + p_2 + p_3 = N \]  \hspace{1cm} (1)

in prime numbers \( p_1, p_2, p_3 \). Suppose that \( k_i, l_i; i = 1, 2, 3 \) are integers satisfying \( (k_i, l_i) = 1 \). Using Vinogradov’s method Zulauf [8] established an asymptotic formula for the number of solutions of (1) in primes \( p_i \equiv l_i \pmod{k_i} \), \( i = 1, 2, 3 \). This formula is valid not only for fixed \( k_i \), but also for \( k_i \leq L_D \), where \( L = \log N \) and \( D > 0 \) is a constant. More precisely, in this case we have

\[ R_{k_1, l_1}(N) := \sum_{\substack{p_1 + p_2 + p_3 = N \\ p_i \equiv l_i \pmod{k_i} \hspace{0.5cm} i = 1, 2, 3}} \log p_1 \log p_2 \log p_3 = M_{k_1, l_1}(N) + O \left( N^2 L^{-A} \right), \quad (2) \]

where \( A > 0 \) is arbitrarily large constant. Here the main term is given by

\[ M_{k_1, l_1}(N) = \frac{N^2 \mathcal{S}_{k_1, l_1}(N)}{2\varphi(k_1)\varphi(k_2)\varphi(k_3)}, \]

where \( \mathcal{S}_{k_1, l_1}(N) \) is the singular series (see [2] for the explicit formula) and \( \varphi(k) \) stands for the Euler function.

One may expect that (2) is valid even if one or more of the moduli \( k_i \) grow faster than a power of \( L \) but this has not been proved so far. One may try instead to prove that (2) is true on average with respect to \( k_i \) from a certain set of positive integers. More precisely, define

\[ \Delta_{k_1, l_1}(N) = R_{k_1, l_1}(N) - M_{k_1, l_1}(N) \]

and consider the sum

\[ \mathcal{E} = \mathcal{E}(H_1, H_2, H_3) = \sum_{\substack{k_i \leq H_i \\ (l_i, k_i) = 1 \hspace{0.5cm} i = 1, 2, 3}} \max_{i = 1, 2, 3} \left| \Delta_{k_1, l_1}(N) \right| \]

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Bearing in mind Bombieri-Vinogradov’s theorem, one may expect that for any constant \( A > 0 \) there exists \( B = B(A) > 0 \) such that

\[
E(H_1, H_2, H_3) \ll N^2 L^{-A} \quad \text{for} \quad H_1, H_2, H_3 \leq \sqrt{N} L^{-B}.
\] (3)

During the last years several results that approximate this conjecture were found, and some of them were applied for studying the equation (1), or similar ternary problems, with prime variables having certain additional properties. The author [5] proved a theorem of this type, but with only one prime lying in a progression. This result was later improved by K.Halupczok [1]. A theorem with two primes from independent progressions (and with fixed \( l_i \)) was established by Peneva and the author [4]. A stronger result was recently found by K.Halupczok [2]. Theorem 1 of [2] states (essentially) that for any \( A > 0 \) there exists \( B = B(A) > 0 \) such that

\[
\sum_{k_1 \leq \sqrt{N} L^{-B}} \max_{(l_1, k_1) = 1} \sum_{k_2 \leq \sqrt{N} L^{-B}} \max_{(l_2, k_2) = 1} \left| \Delta_{(k_1, k_2, 1), (l_1, l_2, 1)}(N) \right| \ll N^2 L^{-A}.
\] (4)

It is not known at present if the estimate (3) is true for \( H_1 = H_2 = H_3 = N^\delta \), where \( \delta > 0 \) is a constant. However using author’s method from [6] one can prove that if \( l_1, l_2, l_3 \) are fixed integers and \( \lambda_i(k), \quad i = 1, 2, 3 \), are real numbers satisfying \( |\lambda_i(k)| \leq 1 \) then for any \( A > 0 \) there exist \( B = B(A) > 0 \) such that

\[
\sum_{k_1, k_2 \leq \sqrt{N} L^{-B}, k_3 \leq N^{1/3} L^{-B}} \lambda_1(k_1) \lambda_2(k_2) \lambda_3(k_3) \Delta_{k_1}(N) \ll N^2 L^{-A}.
\]

(The upper bound for \( k_3 \) in the last formula can be increased to \( N^{4/9} L^{-B} \) in the case when \( \lambda_3(k) \) is a well-factorable function of level \( N^{4/9} L^{-B} \). This is a consequence of a theorem of Mikawa [3].)

In this paper we present a new result that improves the theorems mentioned above. We have the following:

**Theorem.** Suppose that \( l_3 \) is a fixed positive integer and \( \lambda(k) \) are real numbers satisfying \( |\lambda(k)| \leq 1 \). For any constant \( A > 0 \) there exists \( B = B(A) > 0 \) such that

\[
E^* := \sum_{k_1 \leq \sqrt{N} L^{-B}} \max_{(l_1, k_1) = 1} \sum_{k_2 \leq \sqrt{N} L^{-B}} \max_{(l_2, k_2) = 1} \left| \Delta_{k_1}(N) \right| \ll N^2 L^{-A}.
\] (5)
In particular, we find a stronger version of (4) (with the maximum over \(l_1\) inside the sum over \(k_2\); the method of [2] is not applicable for proving this). We also mention that if we apply Mikawa’s theorem from [3] then the upper bound for \(k_3\) in formula (5) can be increased to \(N^{4/9}L^{-B}\) in the case when \(\lambda(k)\) is a well-factorable function of level \(N^{4/9}L^{-B}\).

We use the common number-theoretic notations. By greek letters we denote real numbers and by small latin letters — integers. However, the letter \(p\), with or without subscripts, is reserved for primes. \(N\) is a sufficiently large odd integer and \(L = \log N\). As usual \(\tau(k)\) is the number of positive divisors of \(k\). By \((k, l)\) we denote the greatest common divisor of \(k\) and \(l\). Instead of \(m \equiv n \pmod{k}\) we write \(m \equiv n \pmod{k}\) and \(k \sim K\) is abbreviation of \(K < k \leq 2K\). We also denote \(e(\alpha) = \exp(2\pi i \alpha)\) and \(||\alpha|| = \min_{n \in \mathbb{Z}} |\alpha - n|\).

**Proof of the Theorem:** We may assume that

\[ A \geq 10^4, \quad B = 10^4 A. \]  

(6)

We apply the circle method with \(Q = L^{20A}, \tau = NQ^{-1}\) and with the sets of major arcs \(\mathcal{M}\) and minor arcs \(\mathcal{m}\) specified by

\[
\mathcal{M} = \bigcup_{q \leq Q} \bigcup_{\substack{a = 0 \\ (a,q) = 1}}^{q-1} \left( \frac{a}{q} - \frac{1}{q\tau}, \frac{a}{q} + \frac{1}{q\tau} \right), \quad \mathcal{m} = \left( \frac{1}{\tau}, 1 - \frac{1}{\tau} \right) \setminus \mathcal{M}.
\]

It is clear that

\[
\mathcal{R}_{k,1}(N) = \int_0^1 S_{k_1,t_1}(\alpha) S_{k_2,t_2}(\alpha) S_{k_3,t_3}(\alpha) e(-N\alpha) d\alpha = \mathcal{R}^{(\mathcal{M})}_{k,1} + \mathcal{R}^{(\mathcal{m})}_{k,1},
\]

where \(\mathcal{R}^{(\mathcal{M})}_{k,1}\) and \(\mathcal{R}^{(\mathcal{m})}_{k,1}\) are respectively the contributions coming from the major arcs and the minor arcs and where

\[
S_{k,l}(\alpha) = \sum_{\substack{p \leq N \\ p \equiv l (k)}} (\log p) e(\alpha p).
\]

(7)

We have

\[
\mathcal{E}^* \ll \mathcal{E}_1 + \mathcal{E}_2.
\]

(8)
where

\[ E_1 = \sum_{k_1, k_2 \leq \sqrt{N \mathcal{L}^{-B}}} \max_{i=1, 2} \sum_{k_3 \leq N^{1/3} \mathcal{L}^{-B}} \lambda(k_3) \left( R_{k,1}^{(m)} - M_{k,1}(N) \right), \]

\[ E_2 = \sum_{k_1, k_2 \leq \sqrt{N \mathcal{L}^{-B}}} \max_{i=1, 2} \sum_{k_3 \leq N^{1/3} \mathcal{L}^{-B}} \lambda(k_3) |R_{k,1}^{(m)}| = \sum_{k_1, k_2 \leq \sqrt{N \mathcal{L}^{-B}}} \max_{i=1, 2} |U|, \]

say.

Working as in sections 4 and 5 of [6] (see also Theorem 3 of [2]) we find

\[ E_1 \ll \sum_{k_1 \leq \sqrt{N \mathcal{L}^{-B}}} \max_{(i, k_i) = 1} \left| R_{k,1}^{(m)} - M_{k,1}(N) \right| \ll N^2 \mathcal{L}^{-A}. \]  \hspace{1cm} (9)

It remains to estimate \( E_2 \). Obviously

\[ E_2 \ll \mathcal{L}^2 \max_{K_1, K_2 \leq \sqrt{N \mathcal{L}^{-B}}} \mathcal{V}(K_1, K_2), \]  \hspace{1cm} (10)

where

\[ \mathcal{V} = \mathcal{V}(K_1, K_2) = \sum_{k_1 \sim K_1, k_2 \sim K_2} \max_{i=1, 2} |U|. \]  \hspace{1cm} (11)

Using the definitions of \( R_{k,1}^{(m)} \) and \( U \) we get

\[ U = \int_m S_{k_1, l_1}(\alpha) S_{k_2, l_2}(\alpha) \mathcal{K}(\alpha)e(-N\alpha) d\alpha, \]

where

\[ \mathcal{K}(\alpha) = \sum_{k_3 \leq N^{1/3} \mathcal{L}^{-B}} \lambda(k_3) S_{k_3, l_3}(\alpha). \]

This sum behaves, in some sense, like \( S(\alpha) = \sum_{p \leq N} (\log p)e(\alpha p) \). More precisely, the following estimates hold:

\[ \max_{\alpha \in \mathbb{R}} |\mathcal{K}(\alpha)| \ll N \mathcal{L}^{350-4A}, \quad \int_0^1 |\mathcal{K}(\alpha)|^2 d\alpha \ll N \mathcal{L}^{20}. \]  \hspace{1cm} (12)

The proof of the first one can be obtained following the proof of Lemma 12 of [6], whiles the verification of the second is simple (see [6], page 88).
Using (7) we find

\[ |U| = \sum_{\substack{p \leq N \\ p \equiv l_2 (k_2) \ \Rightarrow \ \ L \sum_{r \leq N \ \ r \equiv l_2 (k_2)} |I_{k_1, l_1} (r)|, \]

where we have denoted

\[ I_{k, l} (r) = \int m S_{k_1, l_1} (\alpha) \mathcal{K}(\alpha) e((r - N) \alpha) d\alpha. \]  

(13)

Applying Cauchy’s inequality we get

\[ V \ll L(K_1 K_2)^{1/2} \left( \sum_{k_1 \sim K_1} \max_{l_1, l_2} \left( \sum_{\substack{r \leq N \\ r \equiv l_2 (k_2) \ \Rightarrow \ \ L \sum_{r \leq N \ \ r \equiv l_2 (k_2)} |I_{k_1, l_1} (r)| \right)^2 \right)^{1/2} \]

\[ \ll L(N K_1)^{1/2} \left( \sum_{k_1 \sim K_1} \max_{l_1, l_2} \mathcal{F} \right)^{1/2}, \]  

(14)

where

\[ \mathcal{F} = \sum_{\substack{r \leq N \\ r \equiv l_2 (k_2) \ \Rightarrow \ \ L \sum_{r \leq N \ \ r \equiv l_2 (k_2)} |I_{k_1, l_1} (r)|. \]

Consider \( \mathcal{F} \). First we use (13) and expand the square and then we insert the summation over \( r \) inside the double integral:

\[ \mathcal{F} = \sum_{\substack{r \leq N \\ r \equiv l_2 (k_2) \ \Rightarrow \ \ L \sum_{r \leq N \ \ r \equiv l_2 (k_2)} |S_{k_1, l_1} (\alpha) \mathcal{K}(\alpha) S_{k_1, l_1} (-\beta) \mathcal{K}(-\beta) e((r - N) (\alpha - \beta)) d\alpha d\beta} \]

\[ = \int_m \int_m S_{k_1, l_1} (\alpha) \mathcal{K}(\alpha) S_{k_1, l_1} (-\beta) \mathcal{K}(-\beta) \sum_{\substack{r \leq N \\ r \equiv l_2 (k_2) \ \Rightarrow \ \ L \sum_{r \leq N \ \ r \equiv l_2 (k_2)} e((r - N) (\alpha - \beta)) d\alpha d\beta. \]

Now we estimate the sum over \( r \) using the well-known bound for the linear exponential sum and then apply the inequality \( uv \leq u^2 + v^2 \) to get

\[ \mathcal{F} \ll \int_m \int_m |S_{k_1, l_1} (\alpha) \mathcal{K}(\alpha) S_{k_1, l_1} (\beta) \mathcal{K}(\beta)| \min \left( \frac{N}{K_2}, \frac{1}{|| (\alpha - \beta) k_2 ||} \right) d\alpha d\beta \]

\[ \ll \int_m \int_m |S_{k_1, l_1} (\alpha) \mathcal{K}(\beta)|^2 \min \left( \frac{N}{K_2}, \frac{1}{|| (\alpha - \beta) k_2 ||} \right) d\alpha d\beta. \]
Next we extend the integration over \( \alpha \) to the unit interval and use that

\[
|S_{k_1,l_1}(\alpha)|^2 = \sum_{\substack{|n| \leq N \\ n \equiv 0 (k_1)}} w(n) e(\alpha n),
\]

where

\[
w(n) = w(n, k_1, l_1) = \sum_{\substack{p_1, p_2 \leq N \\ p_1 - p_2 = n \\ p_1 \equiv l_1 (k_1)}} \log p_1 \log p_2 \ll L^2 NK_1^{-1}.
\]  

(15)

We obtain

\[
\mathcal{F} \ll \int_m |\mathcal{K}(\beta)|^2 \int_0^1 \sum_{\substack{|n| \leq N \\ n \equiv 0 (k_1)}} w(n) e(n\alpha) \min \left( \frac{N}{K_2}, \frac{1}{||\alpha - \beta||} \right) d\alpha d\beta
\]

\[
\ll \sum_{\substack{|n| \leq N \\ n \equiv 0 (k_1)}} |w(n)| \left| \int_m |\mathcal{K}(\beta)|^2 \int_0^1 e(n\alpha) \min \left( \frac{N}{K_2}, \frac{1}{||\alpha - \beta||} \right) d\alpha d\beta \right|.
\]

We use (15) and change the variable in the inner integral to get

\[
\mathcal{F} \ll \frac{L^2 N}{K_1} \sum_{\substack{|n| \leq N \\ n \equiv 0 (k_1)}} \left| \int_m |\mathcal{K}(\beta)|^2 \int_{-\beta}^{1-\beta} e(n(\beta + \gamma)) \min \left( \frac{N}{K_2}, \frac{1}{||\gamma||} \right) d\gamma d\beta \right|
\]

\[
= \frac{L^2 N}{K_1} \sum_{\substack{|n| \leq N \\ n \equiv 0 (k_1)}} \left| \int_m |\mathcal{K}(\beta)|^2 e(n\beta) d\beta \right| J(n, k_2),
\]

where

\[
J(n, k) = \int_0^1 e(n\gamma) \min \left( \frac{N}{K_2}, \frac{1}{||\gamma||} \right) d\gamma.
\]

To study this integral we apply the well-known decomposition

\[
\min \left( H, \frac{1}{||\alpha||} \right) = \sum_h c(h) e(h\alpha), \quad c(h) \ll \begin{cases} \log H & \text{for all } h, \\ H^2 h^{-2} & \text{for } h \neq 0, \end{cases}
\]

(with \( H = N/K_2 \)) and we find that

\[
J(n, k) = \int_0^1 e(n\gamma) \sum_h c(h) e(hk\gamma) d\gamma = \sum_h c(h) \int_0^1 e((n + h)k\gamma) d\gamma
\]

\[
= \begin{cases} c(-n/k) & \text{if } k \mid n, \\ 0 & \text{if } k \nmid n. \end{cases}
\]
Therefore
\[ F \ll L^3Nk_1^{-1} \sum_{\substack{|\nu| \leq N \\nu \equiv 0 \ (k_1) \\ n \equiv 0 \ (k_2)}} \left| \int m |K(\beta)|^2e(n\beta)d\beta \right|. \] (16)

We note that the last expression already does not depend on \( l_1 \) and \( l_2 \). Using (14) and (16) we get
\[ V \ll L^3N^{1/2}, \] (17)
say.

Now we write
\[ W = W' + W^*, \] (18)
where \( W' \) is the contribution of the terms with \( n = 0 \) and \( W^* \) comes from the other terms. From (10) and (12) we get
\[ W' \ll K_1K_2 \int_0^1 |K(\beta)|^2d\beta \ll L^20K_1K_2N \ll L^{20-2B}N^2. \] (19)

Consider \( W^* \). We have
\[ W^* = \sum_{1 \leq |\nu| \leq N} \sum_{\substack{k_1 \sim K_1 \\nu \equiv 0 \ (k_1) \\ k_2 \sim K_2 \\nu \equiv 0 \ (k_2) \\nu \equiv 0 \ (k_2)}} \left| \int m |K(\beta)|^2e(n\beta)d\beta \right| \ll \sum_{1 \leq |\nu| \leq N} \tau^2(n) \left| \int m |K(\beta)|^2e(n\beta)d\beta \right|. \]

Applying the inequalities of Cauchy and Bessel and using the well-known elementary estimate \( \sum_{n \leq N} \tau^4(n) \ll NL^{15} \) we find that
\[ W^* \ll \left( \sum_{n \leq N} \tau^4(n) \right)^{1/2} \left( \sum_{n \leq N} \left| \int m |K(\beta)|^2e(n\beta)d\beta \right|^2 \right)^{1/2} \ll L^8N^{1/2} \left( \int m |K(\beta)|^4d\beta \right)^{1/2} \ll L^8N^{1/2} \sup_{\beta \in \mathbb{R}} |K(\beta)| \left( \int_0^1 |K(\beta)|^2d\beta \right)^{1/2}. \]

We estimate the last expression using (12) to obtain
\[ W^* \ll N^2L^{100-4A}. \] (20)

From (6), (10) and (17) – (20) we get
\[ E_2 \ll N^2L^{-A} \]
and the proof of the theorem follows from this estimate, (8) and (9).
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