Connectedness of graphs arising from the dual Steenrod algebra

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Abstract

We establish connectedness criteria for graphs associated to monomials in certain quotients of the mod 2 dual Steenrod algebra $A^*$. We also investigate questions about trees and Hamilton cycles in the context of these graphs. Finally, we improve upon a known connection between the graph theoretic interpretation of $A^*$ and its structure as a Hopf algebra.

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1 Introduction

The mod 2 Steenrod algebra $A_*$ and its dual $A^*$ act on mod 2 cohomology and homology, respectively, making them indispensable computational tools in homotopy theory at the prime 2. It is therefore desirable to study these algebras from a variety of points of view (see, e.g., [5, 7, 8, 10] or the bibliography of [11]). The purpose of this paper is to study the mod 2 dual Steenrod algebra $A^*$ from the point of view of graph theory, as first advocated by R. M. W. Wood in [11, §8] and subsequently advanced by C. Yearwood in [12].

The algebra $A^*$ has the structure of a graded polynomial algebra. More precisely, $A^* = F_2[\xi_1, \xi_2, \xi_3, \ldots]$ where $|\xi_i| = 2^i - 1$. Given an integer $n \geq 0$, we may form the truncated polynomial algebra $A^*(n) = A^*/I(n)$, where

$$I(n) = (\xi_1^{n+1}, \xi_2^n, \xi_3^{n-1}, \ldots, \xi_{n+2}^2, \xi_{n+3}, \ldots).$$

(1)

Milnor [3] dualized these finite quotients over $F_2$ to yield finite subalgebras $A_*(n) \subset A_*$ that facilitate computations of homotopy groups. For example, Adams spectral sequence computations involving Ext for modules over $A_*$ can sometimes be done instead over $A_*(n)$ for some $n$, via a change-of-rings isomorphism [4, A1.3.12] (see also [1, §2] for concrete examples of this phenomenon).

Wood gives a graph theoretic interpretation of the quotients $A^*(n)$ by associating to every monomial $x \in A^*(n)$ a graph, which by abuse of notation we shall also denote $x$, on the vertex set $\{0, 2, \ldots, 2n+1\}$. We shall describe the construction of these graphs in Section 2. Let us write $x = \xi_1^{r_1} \xi_2^{r_2} \cdots \xi_{n+1}^{r_{n+1}}$, where $0 \leq r_i \leq 2^{n+2-i} - 1$ and where each $r_i$ has the dyadic expansion

$$r_i = \sum_{m=0}^{n+1-i} a_{n+1-m,n+1-i-m} 2^{n+1-i-m}.$$ 

(2)

Our first theorem gives connectedness criteria for $x$ using only the data of its underlying monomial.

**Theorem 1.1.** The graph $x$ is connected if and only if the integers

$$C(p, q) := \sum_{t=1}^{n+1} \prod_{k=1}^{t} a_{\min(p_{k-1}, p_k), \max(p_{k-1}, p_k)}$$

are positive for all $0 \leq p < q \leq n+1$, where $T$ is the set of all $(t+1)$-tuples $(p_0, p_1, \ldots, p_t) \in \{0, \ldots, n+1\}^{t+1}$ such that $p = p_0 \neq p_1 \neq \cdots \neq p_t = q$.

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With Theorem 1.1 in hand, we may easily identify which graphs $x$ are trees. Let $\alpha(m)$ denote the number of 1s in the dyadic expansion of an integer $m$.

**Theorem 1.2.** The graph $x$ is a tree if and only if the integers (3) are positive (that is to say, $x$ is connected) and $\sum_{i=1}^{n+1} \alpha(r_i) = n + 1$.

In [12], C. Yearwood sharpens the ideas in [11] by carefully interpreting the Hopf algebra structure on $\mathcal{A}$ and its quotients $\mathcal{A}(n)$ graph theoretically. We shall have more to say about this in Section 5. One important implication of Yearwood’s work is that the graphs $x$ are perhaps most naturally viewed as *digraphs*, with edges oriented in the direction of the larger vertex (i.e., $2^a \rightarrow 2^b$ if $0 \leq a < b \leq n + 1$). With this in mind, we offer a digraph version of Theorem 1.1. Let $x^\text{dir}$ denote the graph $x$ viewed as a digraph. The appropriate connectedness property to study for $x^\text{dir}$ is that of being *unilateral* (see Section 2).

**Theorem 1.3.** The digraph $x^\text{dir}$ is unilateral if and only if the integers

$$U(p, q) := \sum_{i=1}^{n+1} \prod_{k=1}^{l} a_{p_{k-1}, p_k}$$

are positive for all $0 \leq p < q \leq n+1$, where $T'$ is the set of all $(k+1)$-tuples $(p_0, p_1, \ldots, p_l) \in \{0, \ldots, n+1\}^{l+1}$ such that $p = p_0 < p_1 < \cdots < p_l = q$.

Among the questions posed by Wood in [11] regarding his graph theoretic interpretation of $\mathcal{A}(n)$ is whether there are algebraic analogs of classical questions about Hamilton cycles. In response, we offer the following result.

**Theorem 1.4.** The graph $x \in \mathcal{A}(n)$ has a Hamilton cycle if $n > 0$ and for every vertex $2^j$ of $x$,

$$\#\{1 \leq k \leq j : a_{j, j-k} = 1\} + \#\{1 \leq k \leq n + 1 - j : a_{j+k, j} = 1\} \geq \frac{n}{2}.$$

Moreover, the digraph $x^\text{dir}$ has a directed Hamilton path if and only if $x$ is divisible by $\xi_2^{n+1}$.

The aforementioned Hopf algebra structure on $\mathcal{A}(n)$, to be described in Section 5, includes a coproduct $\Delta : \mathcal{A}(n) \otimes \mathcal{A}(n) \to \mathcal{A}(n)$ and an antipode $c : \mathcal{A}(n) \to \mathcal{A}(n)$. Our last theorem is a generalization of Lemmas 3.1.7 and 3.1.8 of [12].

**Theorem 1.5.** Let $\xi_t^{2j} \in \mathcal{A}(n)$ (whose underlying graph is the single edge connecting $2^j$ and $2^{j+t}$; see Section 2). Then the coproduct $\Delta(\xi_t^{2j}) \in \mathcal{A}(n)$ is the sum of tensors of all pairs of edges that make length 2 directed paths from $2^j$ to $2^{t+j}$, and the antipode $c(\xi_t^{2j}) \in \mathcal{A}(n)$ is the sum of all directed paths from $2^j$ to $2^{t+j}$.

As a corollary, we obtain another characterization of the unilaterality of $x^\text{dir}$.

**Corollary 1.6.** For $x \in \mathcal{A}(n)$, the graph $x^\text{dir}$ is unilateral if and only if for each $\xi_t^{2j} \in \mathcal{A}(n)$, at least one summand of $c(\xi_t^{2j})$ is a factor of $x$.

The paper is structured as follows. Section 2 establishes the necessary terminology from graph theory and describes Wood’s construction of the graphs corresponding to monomials $x \in \mathcal{A}(n)$. In Section 3 we prove our connectedness criteria, namely Theorems 1.1 and 1.3. Section 4 discusses trees and Hamilton cycles and contains the proofs of Theorems 1.2 and 1.4. Finally, in Section 5, we discuss the Hopf algebra structure of $\mathcal{A}(n)$ in the context of graph theory, prove Theorem 1.5 and Corollary 1.6, and pose some open questions.

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2 Graph theory background and Wood’s construction

The purposes of this section are to recall the necessary definitions from graph theory and to describe Wood’s construction of graphs associated to monomials in $\mathcal{A}^*(n)$. Our main graph theory references are [2] and [9].

2.1 Definitions, conventions, and notation

A graph is an ordered pair $G = (V_G, E_G)$ where $V_G$ is the set of vertices, and where $E_G$, the set of edges, is a subset of the set $V_G^2$ of unordered pairs of vertices in $V_G$. If $e = \{v_0, v_1\} \in E_G$, we say $v_0$ and $v_1$ are the ends of $e$. All graphs in this paper are finite, meaning $V_G$ and $E_G$ are finite sets, and simple, meaning no edge has identical ends and no two edges have the same pair of ends. A walk in $G$ is a finite non-empty sequence $v_0, v_1, \ldots, v_k$ of vertices such that each consecutive pair $v_i, v_{i+1}$ comprises the ends of an edge in $E_G$. If the vertices of a walk in $G$ are distinct, it is called a path in $G$. We say the walk or path $v_0, v_1, \ldots, v_k$ has length $k$. We say $G$ is connected if there exists a path connecting any two distinct vertices. If $S$ is a set of vertices, a complete graph on $S$ is a graph whose vertex set is $S$ and whose edge set contains one edge for every pair of distinct vertices in $S$.

A cycle in $G$ is a finite sequence $v_0, v_1, \ldots, v_k, v_0$ of vertices such that $v_0, v_1, \ldots, v_k$ is a path and the pair $v_k, v_0$ comprises the ends of an edge in $E_G$. We say $G$ is acyclic if it contains no cycles. A tree is an acyclic connected graph. A Hamilton cycle in $G$ is a cycle containing every vertex of $G$. The degree $\deg(v)$ of a vertex $v \in V_G$ is the number of edges in $E_G$ that have $v$ as an end.

A directed graph (or digraph) is an ordered pair $D = (V_D, E_D)$ where $V_D$ is the set of vertices, and where $E_D$, the set of edges, is a subset of the set of ordered pairs of vertices in $V_G$. If $e = (v_0, v_1) \in E_D$, we say $v_0$ is the tail of $e$ and $v_1$ is the head of $e$. Any digraph has an underlying graph by replacing each ordered pair $(v_0, v_1) \in E_D$ with the corresponding unordered pair $\{v_0, v_1\}$. All directed graphs in this paper have underlying graphs that are finite and simple. A directed walk in $D$ is a finite non-empty sequence $v_0, v_1, \ldots, v_k$ of vertices such that each for consecutive pair $v_i, v_{i+1}$, there is an edge in $E_D$ with tail $v_i$ and head $v_{i+1}$. If the vertices of a directed walk in $D$ are distinct, it is called a directed path in $D$. As with walks or paths, we say the directed walk or directed path $v_0, v_1, \ldots, v_k$ has length $k$. A digraph $D$ is unilateral (see [2, Exercise 10.2.2]) if for every pair of distinct vertices $v_i, v_j \in V_D$, there is a directed path starting at $v_i$ and ending at $v_j$ or vice versa. A Hamilton directed path in $D$ is a directed path containing every vertex of $D$. The out-degree $\deg_{\text{out}}(v)$ of a vertex $v \in V_D$ is the number of edges in $D$ that have $v$ as a tail, and the in-degree $\deg_{\text{in}}(v)$ of $v$ is the number of edges in $D$ that have $v$ as a head. Note that $\deg(v) = \deg_{\text{out}}(v) + \deg_{\text{in}}(v)$, where we interpret $\deg(v)$ as being the degree of $v$ in the graph underlying the digraph $D$.

If $G$ is a graph with ordered vertex set $V_G = \{v_0, \ldots, v_{n-1}\}$, the adjacency matrix of $G$ is the $n \times n$ matrix $A_G$ with $(p, q)$th entry equal to the number of edges with ends $v_p$ and $v_q$. (Here, we use $p$ and $q$ as row and column indices, respectively, where $0 \leq p \leq n-1$ and $0 \leq q \leq n-1$.) If $D$ is a digraph with vertex set $V_D = \{v_0, \ldots, v_{n-1}\}$, the adjacency matrix of $D$ is the $n \times n$ matrix $A_D$ with $(p, q)$th entry equal to the number of edges with tail $v_p$ and head $v_q$. As a result of our conventions, all adjacency matrices $A_G$ associated to graphs $G$ in this paper will be symmetric, (binary that is, all entries either 0 or 1), and will have zeros along the main diagonal. Similarly, all adjacency matrices $A_D$ associated to digraphs $D$ in this paper will be binary and strictly upper triangular.

2.2 Wood’s construction

We now explain how Wood encodes the algebras $\mathcal{A}^*(n)$ in terms of graphs. Recall from Section 1 that the graph $x$ underlying a monomial $x \in \mathcal{A}^*(n)$ will have vertex set $V_x = \{2^0, 2^1, \ldots, 2^{n+1}\}$.

To begin, consider a monomial of the form $x_{2^i} \in \mathcal{A}^*(n)$ where $1 \leq i \leq n+1$ and $1 \leq j \leq 2^{n+2-i} - 1$. Wood’s construction declares that the graph $x_{2^i}$ has edge set consisting of a single edge with ends $2^j$ and $2^{j+i}$. 

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Example 2.1.  (a) $\xi_2 \in A^*(2)$

(b) $\xi_3^2 \in A^*(3)$

If $x = \xi_{r_1}^1 \xi_{r_2}^2 \cdots \xi_{r_{n+1}}^{n+1} \in A^*(n)$, then we may use the dyadic expansions given in (2) to write $x$ as a product of monomials of the form $\xi^2_i$, and Wood’s construction declares the graph $x$ to have edge set equal to the disjoint union of the corresponding edges.

Example 2.2.  (a) $x = \xi_1^6 \xi_2 \xi_3 = \xi_1^2 \xi_2^4 \xi_3 \in A^*(2)$

(b) $x = \xi_1^{15} \xi_3^2 = \xi_1 \xi_3^2 \xi_1^4 \xi_3^2 \in A^*(3)$

Any graph on the vertices $\{2^0, 2^1, \ldots, 2^{n+1}\}$ has a corresponding monomial in $A^*(n)$. Following the notation in [11], we shall denote the top degree class $\xi_{r_1}^{2^{n+1}-1} \xi_{r_2}^{2^{n-1}} \cdots \xi_{r_{n+1}} \in A^*(n)$ by $\Delta$. The corresponding graph $\Delta$ is a complete graph on $\{2^0, 2^1, \ldots, 2^{n+1}\}$. At the other extreme, the graph corresponding to $1 \in A^*(n)$ is the graph on $\{2^0, 2^1, \ldots, 2^{n+1}\}$ with empty edge set.

Example 2.3.  (a) $\Delta = \xi_1^{15} \xi_2^2 \xi_3^3 \xi_4 \in A^*(3)$

(b) $1 \in A^*(3)$

Remark 2.4. If $x$ and $y$ are arbitrary monomials in $A^*(n)$, we can obtain the graph $xy$ from the individual graphs $x$ and $y$ via the following procedure: (1) Overlay the graphs $x$ and $y$ on their common vertex set.
{2^0, 2^1, \ldots, 2^{n+1}}; (2) For all pairs of edges between two vertices 2^i and 2^j, delete the pair and, if possible, perform a “carry” by inserting an edge between 2^{i+1} and 2^{j+1}; (3) Repeat until no further deletions/carries are required. If \( x = \xi_1^r \xi_2^s \cdots \xi_{n+1}^r \) and \( y = \xi_1^s \xi_2^r \cdots \xi_{n+1}^r \), the carries of edges one would perform correspond precisely to the carries required when adding the dyadic expansions of \( r_i \) and \( s_i \) for \( 1 \leq i \leq n + 1 \).

In fact, it is shown in [12] that if the set of all possible graphs on \( \{2^0, 2^1, \ldots, 2^{n+1}\} \) is endowed with this multiplication, and with addition given by finite formal sums over \( \mathbb{F}_2 \), the result is an \( \mathbb{F}_2 \)-algebra (referred to as a truncated “graph algebra” in [12]) that is isomorphic to \( \mathcal{A}^* (n) \).

We noted in Section 1 that it is natural to view graphs \( x \in \mathcal{A}^* (n) \) as digraphs \( x^{\text{dir}} \) with edges oriented in the direction of the larger vertex. To do this pictorially, one can put an arrow on each edge pointing toward the head. Here is the graph \( \xi_1^6 \xi_2^3 \in \mathcal{A}^* (2) \) from Example 2.2(a) viewed as a digraph \( (\xi_1^6 \xi_2^3)^{\text{dir}} \):

![Graph Diagram]

Remark 2.5. In the graph theory literature, a directed graph \( D \) is said to be strongly connected or disconnected ([2, §10.1]) if for any ordered pair of vertices \( (v_0, v_1) \), there is a directed path starting at \( v_0 \) and ending at \( v_1 \) in \( D \). The way we orient edges in graphs \( x \in \mathcal{A}^* (n) \) makes it impossible for any corresponding \( x^{\text{dir}} \) to be strongly connected. This is why we instead opt for characterizing the notion of \( x^{\text{dir}} \) being unilateral as defined in the exercises of [2].

3 Connectedness criteria

In this section, we prove the connectedness criteria in Theorems 1.1 and 1.3 using the basic theory of adjacency matrices.

3.1 Connectedness criterion for \( x \in \mathcal{A}^* (n) \)

We now prove Theorem 1.1, which asserts that the graph \( x \in \mathcal{A}^* (n) \) is connected if and only if the integers \( C(p, q) \) defined in (3) are positive.

Let \( G \) be a graph with ordered vertex set \( V_G = \{v_0, v_1, \ldots, v_{n-1}\} \) and corresponding adjacency matrix \( A_G \). Given a positive integer \( t \), the \( (p, q) \)th entry of the matrix \( (A_G)^t \) is equal to the number of distinct walks of length \( t \) in \( G \) between \( v_p \) and \( v_q \). We have the following well-known result from graph theory as a consequence.

Proposition 3.1. The graph \( G \) is connected if and only if for all \( 0 \leq p < q \leq n - 1 \), the \( (p, q) \)th entry of the matrix

\[
A_G + (A_G)^2 + \cdots + (A_G)^{n-1}
\]

is positive.

Let \( x = \xi_1^r \xi_2^s \cdots \xi_{n+1}^r \in \mathcal{A}^* (n) \) as in Section 1. Recall that the vertex set of the graph \( x \) is \( V_x = \{2^0, 2^1, \ldots, 2^{n+1}\} \). Following the notation and conventions from Subsection 2.1, let \( A_x \) denote the \( (n + 2) \times (n + 2) \) adjacency matrix of \( x \), whose rows and columns we shall index by \( p \) and \( q \), respectively, with \( 0 \leq p \leq n + 1 \) and \( 0 \leq q \leq n + 1 \). The key to our proofs of Theorems 1.1 and 1.3 is the connection between the dyadic expansions of the exponents \( r_i \) given in (2) and the entries of \( A_x \).

Lemma 3.2 ([11], or Lemma 3.2.4 of [12]). For \( 0 \leq p < q \leq n + 1 \), the \( (p, q) \)th entry of \( A_x \) is \( a_{p,q} \), where \( a_{p,q} \) is the coefficient on \( 2^i \) in the dyadic expansion of \( r_{p-q} \), as in (2).

Proof. It follows from Wood’s construction (Subsection 2.2) and the observation preceding Proposition 3.1 that the \( i \)th superdiagonal of \( A_x \), read from bottom to top, yields precisely the coefficients of the dyadic expansion of \( r_i \) in order from the largest power of 2 to the smallest. □
Let $t$ be a fixed positive integer such that $t \leq n + 1$. Since $A_x$ is symmetric, it follows from Lemma 3.2 and the definition of matrix multiplication that the integer

$$
\sum_{T} \prod_{k=1}^{t} a_{\min(p_{k-1}, p_k), \max(p_{k-1}, p_k)},
$$

where $T$ is the set of all $(t+1)$-tuples $(p_0, p_1, \ldots, p_t) \in \{0, \ldots, n+1\}^{t+1}$ such that $p = p_0 \neq p_1 \neq \cdots \neq p_t = q$, is precisely the $(p, q)$th entry of $(A_x)^t$. Therefore, Proposition 3.1 implies that $x$ is connected if and only if the integers $C(p, q)$ are positive for all $0 \leq p < q \leq n + 1$. This proves Theorem 1.1.

### 3.2 Connectedness criterion for $x^{\text{dir}}$

We now prove Theorem 1.3, the digraph analog of Theorem 1.1, which asserts that the digraph $x^{\text{dir}}$ is unilateral if and only if the integers $U(p, q)$ defined in (4) are positive. The proof will be similar in format to the proof of Theorem 1.1 given in Subsection 3.1 but with adjustments made to accommodate the directed case as needed.

Let $D$ be a digraph with ordered vertex set $V_D = \{v_0, v_1, \ldots, v_n-1\}$ and corresponding adjacency matrix $A_D$. Given a positive integer $t$, the $(p, q)$th entry of the matrix $(A_D)^t$ is equal to the number of distinct directed walks of length $t$ in $D$ starting at $v_p$ and ending at $v_q$. We therefore have the following digraph analog of Proposition 3.1.

**Proposition 3.3.** The digraph $D$ is unilateral if and only if for all $0 \leq p \neq q \leq n - 1$, either the $(p, q)$th entry or the $(q, p)$th entry of the matrix

$$A_D + (A_D)^2 + \cdots + (A_D)^{n-1}$$

is positive.

Once again, let $x = \xi_1^1 \xi_2^2 \cdots \xi_{n+1}^{n+1} \in \mathcal{O}^*(n)$ as in Section 1. Let $x^{\text{dir}}$ be the underlying digraph, and let $A_{x^{\text{dir}}}$ denote its $(n+2) \times (n+2)$ adjacency matrix. Because edges in $x^{\text{dir}}$ are always oriented toward the larger vertex, the $(p, q)$th entry of $A_{x^{\text{dir}}}$ is equal to the $(p, q)$th entry of $A_x$ if $p < q$, and is equal to zero otherwise. This implies that Lemma 3.2 holds with $A_{x^{\text{dir}}}$ in place of $A_x$. For a fixed positive integer $t \leq n + 1$, it follows from this altered version of Lemma 3.2 and the definition of matrix multiplication for strictly upper triangular matrices that the integer

$$
\sum_{T'} \prod_{k=1}^{t} a_{p_{k-1}, p_k},
$$

where $T'$ is the set of all $(k+1)$-tuples $(p_0, p_1, \ldots, p_t) \in \{0, \ldots, n+1\}^{t+1}$ such that $p = p_0 < p_1 < \cdots < p_t = q$, is precisely the $(p, q)$th entry of $(A_{x^{\text{dir}}})^t$. Therefore, Proposition 3.3 implies $x^{\text{dir}}$ is unilateral if and only if the integers $U(p, q)$ are positive for all $0 \leq p < q \leq n + 1$. This proves Theorem 1.3.

### 3.3 Examples

We present two examples of Theorems 1.1 and 1.3 applied to graphs in $\mathcal{O}^*(n)$. Consider first $\xi_1^6 \xi_2^3 \xi_3^1 \in \mathcal{O}^*(2)$ from Example 2.2(a). The integers $C(p, q)$ associated to the underlying graph are

$$
C(0, 1) = 2, \quad C(0, 2) = 6, \quad C(0, 3) = 5, \quad C(1, 2) = 4, \quad C(1, 3) = 2, \quad C(2, 3) = 6,
$$

while among the associated integers $U(p, q)$, one finds

$$U(0, 1) = 0. \quad (6)
$$

Since the integers in (5) are all positive, Theorem 1.1 implies $\xi_1^6 \xi_2^3 \xi_3^1$ is connected. On the other hand, (6) implies $\xi_1^6 \xi_2^3 \xi_3^1$ is not unilateral by Theorem 1.3.
Next, consider $\xi_1^{15} \xi_2^3 \in A^*(3)$ from Example 2.2(b), for which the associated integers $C(p, q)$ are

$$C(0, 1) = 4, \ C(0, 2) = 6, \ C(0, 3) = 2, \ C(0, 4) = 6, \ C(1, 2) = 6,$$
$$C(1, 3) = 12, \ C(1, 4) = 6, \ C(2, 3) = 5, \ C(2, 4) = 11, \ C(3, 4) = 5. \quad (7)$$

and the associated integers $U(p, q)$ are

$$U(0, 1) = 1, \ U(0, 2) = 1, \ U(0, 3) = 1, \ U(0, 4) = 2, \ U(1, 2) = 1,$$
$$U(1, 3) = 1, \ U(1, 4) = 2, \ U(2, 3) = 1, \ U(2, 4) = 1, \ U(3, 4) = 1. \quad (8)$$

Since the integers in (7) are all positive, Theorem 1.1 implies $\xi_1^{15} \xi_2^3$ is connected. Since the integers in (8) are all positive, Theorem 1.3 implies $\xi_1^{15} \xi_2^3$ is also unilateral.

We now prove Theorem 1.4, starting with the sufficient condition for $x \in A^*(n)$ to have a Hamilton cycle. We begin by establishing a lemma that counts the degree of a vertex in $x$. We now prove Theorem 1.4, starting with the sufficient condition for $x \in A^*(n)$ to have a Hamilton cycle. We begin by establishing a lemma that counts the degree of a vertex in $x$. We now prove Theorem 1.4, starting with the sufficient condition for $x \in A^*(n)$ to have a Hamilton cycle. We begin by establishing a lemma that counts the degree of a vertex in $x$. We now prove Theorem 1.4, starting with the sufficient condition for $x \in A^*(n)$ to have a Hamilton cycle. We begin by establishing a lemma that counts the degree of a vertex in $x$.
Lemma 4.3. The out-degree of a vertex 2^j in x^{dir} is

\[ \deg_{\text{out}}(2^j) = \#\{1 \leq k \leq n + 1 - j : a_{j+k,j} = 1\} \]

and its in-degree is

\[ \deg_{\text{in}}(2^j) = \#\{1 \leq k \leq j : a_{j,j-k} = 1\} \]

so that the degree of the vertex 2^j in x is

\[ \deg(2^j) = \#\{1 \leq k \leq n + 1 - j : a_{j+k,j} = 1\} + \#\{1 \leq k \leq j : a_{j,j-k} = 1\}. \]

Proof. Using the dyadic expansions given in (2), we may factor x as

\[ x = \prod_{i=1}^{n+1} \prod_{m=0}^{n+1-i} \xi_i^{a_{n+1-m,n+1-i-m}2^{n+1-i-m}}. \] (9)

By Wood’s construction (Subsection 2.2), the edges of the digraph x^{dir} with tail 2^j correspond precisely to factors in the product (9) of the form \( \xi_k^{2^j} \) for \( 1 \leq k \leq n+1-j \) (such an edge has tail 2^j and head 2^{j+k}). The factor \( \xi_k^{2^j} \) appears in (9) if and only if the corresponding dyadic coefficient \( a_{j+k,j} \) is equal to 1. Similarly, Wood’s construction implies the edges of x^{dir} with head 2^j correspond precisely to factors in the product (9) of the form \( \xi_k^{2^j-k} \) for \( 1 \leq k \leq j \) (such an edge has tail 2^{j-k} and head 2^j). The factor \( \xi_k^{2^j-k} \) appears in (9) if and only if the corresponding dyadic coefficient \( a_{j,j-k} \) is equal to 1. This yields the first two equations of the lemma, and the third follows from the fact that \( \deg(v) = \deg_{\text{out}}(v) + \deg_{\text{in}}(v) \) for any vertex v, as we noted in Subsection 2.1. \( \square \)

Dirac’s Theorem from graph theory gives a sufficient condition for a graph G to have a Hamilton cycle in terms of the degrees of the vertices of G.

Theorem 4.4 (Dirac’s Theorem). A graph G with at least 3 vertices has a Hamilton cycle if \( \deg(v) \geq n/2 \) for all vertices v of G.

Lemma 4.3 and Theorem 4.4 together imply that a sufficient condition for \( x \in \mathcal{A}^*(n) \) to have a Hamilton cycle is to assume that \( n > 0 \) (so that x has at least 3 vertices) and that for all vertices 2^j of x,

\[ \frac{n}{2} \leq \deg(2^j) = \deg_{\text{out}}(2^j) + \deg_{\text{in}}(2^j) = \#\{1 \leq k \leq n + 1 - j : a_{j+k,j} = 1\} + \#\{1 \leq k \leq j : a_{j,j-k} = 1\}. \]

This completes the proof of the first statement of Theorem 1.4.

We now prove the second statement of Theorem 1.4, which asserts \( x^{dir} \) has a directed Hamilton path if and only if x is divisible by \( \xi_1^{n+1-1} \). The monomial \( \xi_1^{n+1-1} \) is the largest nonzero power of \( \xi_1 \) in the truncated polynomial algebra \( \mathcal{A}^*(n) \). Therefore, if x is divisible by \( \xi_1^{n+1-1} \), we may write x as

\[ x = \xi_1^{2^{n+1-1}} \xi_2^{2^{n+1-1}} \xi_3^{2^{n+1-1}} \cdots \xi_{n+1}^{2^{n+1-1}} \]

\[ = \xi_1^{1+2+4+\cdots+2^n} \xi_2^{1+2+4+\cdots+2^n} \xi_3^{1+2+4+\cdots+2^n} \cdots \xi_{n+1}^{1+2+4+\cdots+2^n}. \]

The first \( n+1 \) factors (i.e., the powers of \( \xi_1 \)) correspond precisely to the edges in the directed path

\[ 2^0 \rightarrow 2^1 \rightarrow 2^2 \rightarrow \cdots \rightarrow 2^n \rightarrow 2^{n+1} \] (10)

which is a Hamilton directed path we have just shown is contained in \( x^{dir} \). Conversely, suppose \( x^{dir} \) contains a Hamilton directed path. This Hamilton directed path must contain all the vertices of \( x^{dir} \) by definition, in particular 2^0. Since no edge in \( x^{dir} \) has 2^0 as its head, the Hamilton directed path must in fact start at 2^0. If the first edge of the path went from 2^0 to 2^j for \( j > 1 \), the vertex 2^1 could not be contained in the path, which implies the first edge must be the edge from 2^0 to 2^1, i.e., \( \xi_1 \). An analogous argument shows the next edge in the path must be the edge 2^1 to 2^2, i.e., \( \xi_2 \). Continuing in this manner shows that the Hamilton directed path in \( x^{dir} \) must in fact be the directed path (10), and that x is divisible by

\[ \xi_1 \xi_2 \xi_3^{2^n} \cdots = \xi_1^{n+1-1}. \]

This completes the proof of Theorem 1.4.
4.3 Examples

We present examples of Theorems 1.2 and 1.4 applied to graphs \( x \in \mathcal{A}^*(n) \). Consider first the monomial \( x = \xi_1\xi_2\xi_3 \in \mathcal{A}^*(2) \). For this choice of \( x \), \( r_1 = r_2 = r_3 = 1 \), which implies

\[
\sum_{i=1}^{3} \alpha(r_i) = 3 = 2 + 1
\]

and so \( \xi_1\xi_2\xi_3 \) is a tree by Theorem 1.2. A picture of the graph \( \xi_1\xi_2\xi_3 \) is given by

```
     8
    /|
   / 1
  4  2
```

and verifies that \( \xi_1\xi_2\xi_3 \) is indeed a tree. On the other hand, if we take \( x \) to be \( \xi_1^6\xi_2\xi_3 \in \mathcal{A}^*(2) \) from Example 2.2(a), then \( r_1 = 6 = 4 + 2 \) and \( r_2 = r_3 = 1 \), which implies

\[
\sum_{i=1}^{3} \alpha(r_i) = 4 \neq 2 + 1.
\]

We conclude the graph \( \xi_1^6\xi_2\xi_3 \) is not a tree by Theorem 1.2.

Next, let us take \( x = \xi_1^6\xi_2\xi_3\xi_4 \in \mathcal{A}^*(3) \). The dyadic coefficients \( a_{p,q} \) for the exponents \( r_1 = r_2 = 6 = 4 + 2 \) and \( r_3 = r_4 = 1 \) are given by

\[
\begin{align*}
  a_{4,3} &= 0, & a_{3,2} &= 1, & a_{2,1} &= 1, & a_{1,0} &= 0, \\
  a_{4,2} &= 1, & a_{3,1} &= 1, & a_{2,0} &= 0, \\
  a_{4,1} &= 0, & a_{3,0} &= 1, \\
  a_{4,0} &= 1.
\end{align*}
\]

Using these values, we can verify directly that \( \xi_1^6\xi_2\xi_3\xi_4 \) satisfies the sufficient condition for having a Hamilton cycle given in Theorem 1.4 by checking the requisite inequality at each vertex \( v \):

\[
\begin{align*}
  v &= 1 = 2^0 : & \#\{1 \leq k \leq 0 : a_{0,0-k} = 1\} + \#\{1 \leq k \leq 4 : a_{0+k,0} = 1\} &= 0 + 2 \geq 3/2, \\
  v &= 2 = 2^1 : & \#\{1 \leq k \leq 1 : a_{1,1-k} = 1\} + \#\{1 \leq k \leq 3 : a_{1+k,1} = 1\} &= 0 + 2 \geq 3/2, \\
  v &= 4 = 2^2 : & \#\{1 \leq k \leq 2 : a_{2,2-k} = 1\} + \#\{1 \leq k \leq 2 : a_{2+k,2} = 1\} &= 1 + 2 \geq 3/2, \\
  v &= 8 = 2^3 : & \#\{1 \leq k \leq 3 : a_{3,3-k} = 1\} + \#\{1 \leq k \leq 1 : a_{3+k,3} = 1\} &= 3 + 0 \geq 3/2, \\
  v &= 16 = 2^4 : & \#\{1 \leq k \leq 4 : a_{4,4-k} = 1\} + \#\{1 \leq k \leq 0 : a_{4+k,4} = 1\} &= 2 + 0 \geq 3/2.
\end{align*}
\]

A picture of the graph \( \xi_1^6\xi_2\xi_3\xi_4 \) is given by

```
    16
   /|
  / 8
 4  2
```

and corroborates the presence of a Hamilton cycle, e.g., \( 2^1, 2^2, 2^4, 2^0, 2^3, 2^1 \).

Finally, consider \( x = \xi_1^{15}\xi_2^3 \in \mathcal{A}^*(3) \) from Example 2.2(b). One can check that this choice of \( x \) does not satisfy the sufficient condition in Theorem 1.4 for containing a Hamilton cycle (and as it turns out, it does not contain a Hamilton cycle). However, Theorem 1.4 does guarantee that \( \xi_1^{15}\xi_2^3 \) contains a Hamilton directed path since \( \xi_1^{15}\xi_2^3 \) is divisible by \( \xi_1^{15} = \xi_1^{2^{15}+1-1} \). The Hamilton directed path it contains is

\[
2^0 \rightarrow 2^1 \rightarrow 2^2 \rightarrow 2^3 \rightarrow 2^4
\]

which, as we observed in Subsection 4.2, is the only possible Hamilton directed path that a digraph in \( \mathcal{A}^*(3) \) could possibly contain.
5 The Hopf algebra structure of $\mathcal{A}^*(n)$

In this section, we review the basic theory of Hopf algebras and describe the Hopf algebra structure of $\mathcal{A}^*(n)$. We then prove Theorem 1.5 and Corollary 1.6 and pose some open questions. Our general reference for the theory of Hopf algebras is [6].

5.1 Hopf algebras

Let $k$ be a field. Recall that a Hopf algebra over $k$ is a unital associative $k$-algebra $A = (A, \mu, u)$ equipped with three additional $k$-linear structure maps, namely $\Delta : A \to A \otimes A$ (the coproduct), $\varepsilon : A \to k$ (the counit), and $c : A \to A$ (the antipode), such that $\Delta$ is counital and coassociative, and such that the following diagram commutes:

\[
\begin{array}{c}
A \otimes A \xrightarrow{\Delta} A \\
\downarrow \Delta \quad \uparrow \epsilon \quad \downarrow \mu \\
A \otimes A \xrightarrow{1 \otimes \epsilon} A \otimes A
\end{array}
\]

This data amounts to saying that a Hopf algebra is a cogroup object in the category of $k$-algebras, with $\Delta$ corresponding to the group operation, $\varepsilon$ corresponding to the identity, and $c$ corresponding to inversion.

Let $A$ be a Hopf algebra over $k$. An ideal $I \subset A$ is said to be a Hopf ideal if $\Delta(I) \subset I \otimes A + A \otimes I$, $\epsilon(I) = 0$, and $c(I) \subset I$. If $I \subset A$ is a Hopf ideal, then the structure maps of $A$ descend to the quotient $A/I$, giving $A/I$ the structure of a Hopf algebra.

5.2 The dual Steenrod algebra as a Hopf algebra

The dual Steenrod algebra $\mathcal{A}^* = F_2[\xi_1, \xi_2, \xi_3, \ldots]$ has the structure of a Hopf algebra [3], with coproduct $\Delta$ defined by

\[
\Delta(\xi_i) = \sum_{k=0}^{i} \xi_{i-k}^{2^k} \otimes \xi_k,
\]

(11)

counit $\varepsilon$ defined by $\varepsilon(\xi_i) = 0$, and antipode $c$ given recursively by

\[
\sum_{k=0}^{i} \xi_{i-k}^{2^k} c(\xi_k) = 0.
\]

(12)

These structure maps extend to all of $\mathcal{A}^*$ by declaring them to be $F_2$-algebra homomorphisms. Milnor solved the recursion in (12) to obtain the formula

\[
c(\xi_i) = \sum_{\pi} \prod_{k=1}^{\ell(\pi)} \xi_{n(k)}^{2^{\sigma(k)}},
\]

(13)

where the sum is over all all ordered partitions $\pi$ of $i$, $\ell(\pi)$ is the length of $\pi$, $\pi(k)$ is the $k$th part of $\pi$, and $\sigma(k)$ is the sum of the first $k - 1$ parts of $\pi$.

One can check that the ideals $I(n) \subset \mathcal{A}^*$ defined in (1) are Hopf ideals, so that the quotients $\mathcal{A}^*/I(n)$ inherit Hopf algebra structures under the maps $\Delta$, $\epsilon$, and $c$ defined in this Subsection.

5.3 Graph theoretic interpretation of the coproduct and antipode

We now prove Theorem 1.5, starting with the graph theoretic interpretation of the coproduct. We must show that the image of $\xi_1^{2^0} \in \mathcal{A}^*/I(n)$ under $\Delta$ is the sum of tensors of all pairs of edges that make length 2
directed paths from $2^j$ to $2^{j+i}$. The coproduct formula (14) implies

$$\Delta(\xi^j) = \Delta(\xi^j)^2 = \xi^j \otimes 1 + 1 \otimes \xi^j + \sum_{k=1}^{i-1} \xi^{2^{k+i}} \otimes \xi^j. \quad (14)$$

The first two summands of the right-hand side of (14) represent degenerate length 2 directed paths from $2^j$ to $2^{j+i}$. A non-degenerate length 2 directed path from $2^j$ to $2^{j+i}$ corresponds to a choice of an intermediate vertex, which in this case would be of the form $2^{j+k}$ for some $k$, $1 \leq k \leq i-1$. Given this choice of $k$, the edge from $2^j$ to $2^{j+k}$ is $\xi^{2^{j+k}}$ and the edge from $2^{j+k}$ to $2^{j+i}$ is $\xi^{2^{j+k}}$. The terms of the sum indexed by $k$ on the far-right of (14) correspond precisely to the pairs of edges just described.

Next, we prove the portion of Theorem 1.5 concerning the graph theoretic interpretation of the antipode. The claim here is that the image of $\xi^j$ under $c$ is the sum of all directed paths from $2^j$ to $2^{j+i}$. Our proof models that of [12, Lemma 3.1.8]. Milnor’s formula (13) for the antipode implies

$$c(\xi^j) = c(\xi^j)^j = \sum_{\pi} \prod_{k=1}^{\ell(\pi)} \xi^{\pi(k)+j} \quad (15)$$

where $\pi$, $\ell(\pi)$, $\pi(k)$, and $\sigma(k)$ are defined as in Subsection 5.2. A directed path from $2^j$ to $2^{j+i}$ corresponds to a choice of intermediate vertices, say

$$2^{j+a_1}, 2^{j+a_2}, \ldots, 2^{j+a_m}$$

for $0 = a_0 < a_1 < a_2 < \cdots < a_m < a_{m+1} = i$. The successive differences yield a unique ordered partition $\pi$ of $i$, namely

$$i = (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_m - a_{m-1}) + (a_{m+1} - a_m)$$

for which $\ell(\pi) = m+1$, $\pi(k) = a_k - a_{k-1}$, and $\sigma(k) = (a_1 - 0) + (a_2 - a_1) + \cdots + (a_k - a_{k-1}) = a_k - a_{k-1}$.

The monomial corresponding to this directed path is therefore

$$\xi^{a_1+\cdots+a_j} \xi_j^{a_2+\cdots+a_{j+1}} \cdots \xi_j^{a_{m+1}+\cdots+a_i} \xi_j^{a_{m+1}+\cdots+a_i} = \xi^{a_1+\cdots+a_j} \xi_j^{a_2+\cdots+a_{j+1}} \cdots \xi_j^{a_{m+1}+\cdots+a_i} \xi_{m+1}^{a_{m+1}+\cdots+a_i} = \xi_{\pi(1)}^{a_1} \xi_{\pi(2)}^{a_2} \cdots \xi_{\pi(m)}^{a_m}$$

which is precisely the general term of the summation in (13) indexed by $\pi$. This completes the proof of Theorem 1.5.

Recall from Section 1 that Theorem 1.5 yields an alternate characterization of unilaterality of a directed graph $x^\text{dir}$ underlying $x \in \mathcal{A}^*(n)$, namely Corollary 1.6. This corollary asserts $x^\text{dir}$ is unilateral if and only if for each $x^\text{dir} \in \mathcal{A}^*(n)$, at least one summand of $c(\xi^j)$ is a factor of $x$. To obtain the corollary, note that the digraph $x^\text{dir}$ underlying $x \in \mathcal{A}^*(n)$ is unilateral if and only if there is a directed path connecting any two of its vertices, say $2^j$ and $2^{j+i}$. Theorem 1.5 shows this is equivalent to the demand that at least one summand of $c(\xi^j)$ appears as a factor of $x$.

### 5.4 Open questions

We record some outstanding questions related to Wood’s encoding of the algebras $\mathcal{A}^*(n)$ in terms of graphs.

1. What is the analog of Wood’s construction for the mod $p$ dual Steenrod algebra for odd primes $p$? What characterizations of connectedness, trees, etc., are there in these odd primary situations?

2. In [11, §8], Wood points out that the mod 2 Steenrod algebra $\mathcal{A}_2$ (as opposed to its dual, which has been the sole focus of this paper) and some of its subalgebras can be interpreted graph theoretically. What would the results in [12] or in this paper look like in that setting?
3. Given the Hopf algebra structure on $A^*(n)$, including the coproduct $\Delta$ and antipode $c$, what is the graph theoretic meaning of $\Delta(x)$ and $c(x)$ for an arbitrary monomial $x \in A^*(n)$?

4. In [11, §5], Wood describes two procedures one can perform in the mod 2 Steenrod algebra $A_*$, called stripping and strapping, that together allow one to derive all of the Adem relations from the single relation $Sq^1Sq^1 = 0$. A step in the process of recovering the Adem relations involves assigning to each monomial $\xi^2_i \in A^*$ a “stripping operator” $\omega$ which is analogous to how Wood’s construction discussed in this paper assigns an edge of a graph to each $\xi^2_i$. How can this analogy be leveraged to obtain further results about Wood’s graph theoretic interpretation of $A^*(n)$?

5. How can Wood’s encoding of $A^*(n)$ help with calculations in homotopy theory, particularly with the Adams spectral sequence at the prime 2?

References

[1] Scott M. Bailey. On the spectrum $bo \wedge tmf$. J. Pure Appl. Algebra, 214(4):392–401, 2010.
[2] J. A. Bondy and U. S. R. Murty. Graph theory with applications. North-Holland, 1982.
[3] John Milnor. The Steenrod algebra and its dual. Ann. of Math. (2), 67:150–171, 1958.
[4] Douglas C. Ravenel. Complex cobordism and stable homotopy groups of spheres, volume 121 of Pure and Applied Mathematics. Academic Press, Inc., Orlando, FL, 1986.
[5] Larry Smith. An algebraic introduction to the Steenrod algebra. In Proceedings of the School and Conference in Algebraic Topology, volume 11 of Geom. Topol. Monogr., pages 327–348. Geom. Topol. Publ., Coventry, 2007.
[6] M. E. Sweedler. Hopf algebras. Benjamin, 1969.
[7] Grant Walker and Reginald M. W. Wood. Polynomials and the mod 2 Steenrod algebra. Vol. 1. The Peterson hit problem, volume 441 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2018.
[8] Grant Walker and Reginald M. W. Wood. Polynomials and the mod 2 Steenrod algebra. Vol. 2. Representations of $GL(n, \mathbb{F}_2)$, volume 442 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 2018.
[9] Douglas B. West. Introduction to graph theory. Prentice Hall, Inc., Upper Saddle River, NJ, 1996.
[10] R. M. W. Wood. Differential operators and the Steenrod algebra. Proc. London Math. Soc. (3), 75(1):194–220, 1997.
[11] R. M. W. Wood. Problems in the Steenrod algebra. Bull. London Math. Soc., 30(5):449–517, 1998.
[12] C. Yearwood. The Dual Steenrod Algebra and Graph Theory. Senior thesis under the supervision of K. Ormsby, Reed College, May 2019.