Elliptic Transverse Circulation Equations for Balanced Models
in a Generalized Vertical Coordinate

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ABSTRACT

When studying tropical cyclones using the $f$-plane, axisymmetric, gradient balanced model, there arises a second-order elliptic equation for the transverse circulation. Similarly, when studying zonally symmetric meridional circulations near the equator (the tropical Hadley cells) or the katabatically forced meridional circulation over Antarctica, there also arises a second order elliptic equation. These elliptic equations are usually derived in the pressure coordinate or the potential temperature coordinate, since the thermal wind equation has simple non-Jacobian forms in these two vertical coordinates. Because of the large variations in surface pressure that can occur in tropical cyclones and over the Antarctic ice sheet, there is interest in using other vertical coordinates, e.g., the height coordinate, the classical $\sigma$-coordinate, or some type of hybrid coordinate typically used in global numerical weather prediction or climate models. Because the thermal wind equation in these coordinates takes a Jacobian form, the derivation of the elliptic transverse circulation equation is not as simple. Here we present a method for deriving the elliptic transverse circulation equation in a generalized vertical coordinate, which allows for many particular vertical coordinates, such as height, pressure, log-pressure, potential temperature, classical $\sigma$, and most hybrid cases. Advantages and disadvantages of the various coordinates are discussed.

1. Introduction

The purpose of the present paper is to derive the elliptic transverse circulation equation for an $f$-plane gradient balanced model of a tropical cyclone and the elliptic meridional circulation equation for zonally symmetric circulations on the spherical earth. The concept of these types of balanced dynamics was first proposed by Eliassen [1951]. Since both these elliptic equations can be expressed in many different vertical coordinates, we first derive them in a generalized vertical coordinate $\eta$, and then consider five different commonly used choices of $\eta$. The paper is organized into two main parts. Part I discusses the elliptic transverse circulation equation for the $f$-plane, gradient balanced model of a tropical cyclone. The equation is first derived in section 3 using the generalized vertical coordinate $\eta$, with equation (16) being the main result. Then, five particular choices of $\eta$ are discussed in sections 4–8. Part II discusses the elliptic equation for the meridional circulation on the spherical earth, with the goal of applications...
to the tropical Hadley cells and to the forced meridional circulation over Antarctica. Again, the equation is first derived in section 10 using the generalized vertical coordinate $\eta$, with equation (34) being the main result. The five particular choices of $\eta$ are discussed in sections 11–15. A brief roadmap to the paper is given in Table 1.

**Part I: Tropical Cyclones**

2. The balanced vortex model in $(r, \eta, t)$

For simplicity, the present analysis omits frictional effects. To simplify the primitive equation model to a balanced vortex model we assume that the azimuthal flow remains in a gradient balanced state, i.e., we discard the radial equation of motion and replace it with the gradient balance condition given below in (1). A sufficient condition for the validity of this assumption is that the diabatic forcing effects have slow enough time scales that significant, azimuthal mean inertia-gravity waves are not excited, i.e., $|Du/Dr|$ remains small compared to the pressure gradient and Coriolis/centrifugal terms. We shall describe this inviscid flow using a generalized vertical coordinate $\eta$, which is as yet an unspecified function of $(z, p, p_s, \theta)$, where $z$ is the height, $p$ the pressure, $p_s$ the surface pressure, and $\theta$ the potential temperature. Under the balance condition, and using $\eta(z, p, p_s, \theta)$ as the vertical coordinate, the governing equations are

$$
\left(f + \frac{v}{r}\right) v = \frac{\partial \Phi}{\partial r} + \theta \frac{\partial \Pi}{\partial r}, \quad (1)
$$

$$
\frac{\partial v}{\partial t} + \hat{\eta} \frac{\partial v}{\partial \eta} + \left(f + \frac{\partial (rv)}{r \partial r}\right) u = 0, \quad (2)
$$

$$
\frac{\partial \Phi}{\partial \eta} + \theta \frac{\partial \Pi}{\partial \eta} = 0, \quad (3)
$$

$$
\frac{\partial m}{\partial t} + \frac{1}{r} \frac{\partial (mr\hat{u})}{\partial r} + \frac{\partial (m\hat{\eta})}{\partial \eta} = 0, \quad (4)
$$

$$
\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial r} + \hat{\eta} \frac{\partial \theta}{\partial \eta} = \hat{\theta}, \quad (5)
$$

where $f$ is the constant Coriolis parameter, $g$ the acceleration of gravity, $m = -(1/g)(\partial p/\partial \eta)$ the pseudo-density, $p$ the pressure, $\alpha = RT/p$ the specific volume, $\theta$ the potential temperature, $\Phi$ the geopotential, $\Pi = c_p(p/p_0)^{\kappa}$ the Exner function, $u$ the radial velocity component, $v$ the azimuthal velocity component, $\hat{\eta}$ the vertical velocity component, and where the partial derivatives $\partial/\partial t$ and $\partial/\partial r$ are understood to be at fixed $\eta$.

In the generalized vertical coordinate $\eta$, the radial pressure gradient force splits into two parts, as given on the right hand side of (1). We have chosen to express the second part in terms of $\Pi$ and $\theta$, but $p$ and $\alpha$ could also be used since $\theta(\partial \Pi/\partial r) = \theta(\Pi / dp)(\partial p / dr) = \alpha(\partial p / dr)$. There are several vertical coordinates for which the radial pressure gradient force can be expressed as a single term. For example, the choice $\eta = z$ corresponds to the use of the height coordinate, in which case the radial pressure gradient force is $\alpha(\partial p / dr)_z$, since $\langle \partial \Phi / \partial r \rangle_z = 0$. Similarly, the choice $\eta = p$ corresponds to the use of the pressure coordinate, in which case the radial pressure gradient force is $\langle \partial \Phi / \partial r \rangle_p$, since $\langle \partial \Pi / \partial r \rangle_p = 0$. Finally, the choice $\eta = \theta$ corresponds to the use of the potential temperature coordinate, in which case the two terms on the right hand side of (1) combine to give $\langle \partial M / \partial r \rangle_\theta$, where $M = \theta \Pi + \Phi$ is the Montgomery potential.

As we shall see in the following sections, it is the form of the radial pressure gradient force that determines the form of the thermal wind constraint and hence the mathematical intricacy in the derivation of the transverse circulation equation. For example, the simple forms of the radial pressure gradient force in the $p$-coordinate (or coordinates that are functions of $p$, such as $\ln p$) and the $\theta$-coordinate lead to simple forms of the thermal wind constraint and therefore straightforward derivations of the transverse circulation equation. This is in contrast to the $z$-coordinate, where the $\alpha$ coefficient in $\langle \partial \Phi / \partial r \rangle_z$ means that the thermal wind constraint involves a Jacobian, so the derivation of the transverse circulation equation is somewhat more complicated. With these observations in mind, the approach adopted here is to derive the transverse circulation equation in the generalized vertical coordinate (section 3), and then consider the special cases discussed in sections 4–8.

3. Transverse circulation equation in $(r, \eta, t)$

To derive the transverse circulation equation associated with (1–5), first consider the mass continuity equation (4), which can be written in the form

$$
\frac{\partial (mr\hat{u})}{\partial r} + \frac{\partial (m\hat{\eta})}{\partial \eta} = 0, \quad (6)
$$

where

$$
m\hat{\eta} = m\hat{\eta} - \frac{1}{g} \frac{\partial p}{\partial \eta}. \quad (7)
$$

Using this form of the mass conservation principle we define a streamfunction $\psi$ such that

$$
u m = \hat{\psi} = \frac{\partial (r \psi)}{\partial r}. \quad (8)
$$

In the following sections we consider only those cases for which the vertical coordinate $\eta$ increases in the upward direction, so that $m = -(1/g)\langle \partial p / \partial \eta \rangle > 0$.

The thermal wind equation, derived from (1) and (3), is

$$
\hat{f} \frac{\partial \hat{\psi}}{\partial \eta} = \frac{\partial (\Pi, \theta)}{\partial (r, \eta)}. \quad (9)
$$
where \( \hat{f} = f + 2v/r \). Taking \( \partial/\partial t \) of (9) we obtain

\[
\frac{\partial}{\partial \eta} \left( \hat{f} \frac{\partial v}{\partial t} \right) = \frac{\partial}{\partial \eta} \left( \hat{f} \frac{\partial (\Pi, \theta)}{\partial (r, \eta)} \right),
\]  

(10)

where we have made use of the identity \( \langle \partial/\partial t \rangle [\hat{f} (\partial v/\partial \eta)] = (\partial/\partial \eta)[\hat{f} (\partial v/\partial t)] \) to express the left hand side of (10) in a form that has \( (\partial v/\partial t) \) inside the \( (\partial/\partial \eta) \) operator. This sets the stage for the elimination of \( (\partial v/\partial t) \) through the use of \( \mathcal{M} \). Similarly, it is convenient to rewrite the right hand side of (10) in a form that has local time derivatives inside the \( (\partial/\partial r) \) and \( (\partial/\partial \eta) \) operators. This can be accomplished through the use of the Jacobid identity

\[
\frac{\partial}{\partial r} \left( \frac{\partial (\Pi, \theta)}{\partial (\eta, t)} \right) + \frac{\partial}{\partial \eta} \left( \frac{\partial (\Pi, \theta)}{\partial (t, r)} \right) + \frac{\partial}{\partial t} \left( \frac{\partial (\Pi, \theta)}{\partial (r, \eta)} \right) = 0,
\]

(11)

which allows (10) to be written in the form

\[
\frac{\partial}{\partial \eta} \left( \frac{\partial (\Pi, \theta)}{\partial (\eta, t)} \right) + \frac{\partial}{\partial \eta} \left( \hat{f} \frac{\partial v}{\partial t} + \frac{\partial (\Pi, \theta)}{\partial (t, r)} \right) = 0.
\]

(12)

This is the constraint that must be satisfied by the local time derivatives of the azimuthal wind field and the mass field. This constraint will lead directly to the elliptic equation for \( \psi \), after we have derived equations for the two quantities in the large parentheses of (12). The equation for the first of these is derived by multiplying (5) by \( (\partial \Pi/\partial \eta) \), while the equation for the second is derived by adding \( \hat{f} \) times (2) to \( -(\partial \Pi/\partial r) \) times (5). The results are

\[
\frac{\partial (\Pi, \theta)}{\partial (\eta, t)} = Am\hat{\eta} - Bm\hat{\psi} + \frac{\partial \Pi, \theta}{\partial \eta},
\]

(13)

\[
\hat{f} \frac{\partial v}{\partial t} + \frac{\partial (\Pi, \theta)}{\partial (t, r)} = Bm\hat{\eta} - Cm\hat{\psi} - \frac{\partial \Pi, \theta}{\partial r},
\]

(14)

where

\[
A = \frac{g\alpha}{\theta} \frac{\partial \theta}{\partial \eta}, \quad B = -\frac{g\alpha}{\theta} \frac{\partial \theta}{\partial r},
\]

(15)

\[
C = \frac{f}{m} \left( f + \frac{\partial (rv)}{\partial r} \right) - \frac{1}{m} \frac{\partial \Pi, \theta}{\partial r}.
\]

(16)

Using (13) and (14) in (12) we obtain the meridional circulation equation

\[
\frac{\partial}{\partial r} \left( A \frac{\partial (rv)}{\partial r} + B \frac{\partial \psi}{\partial \eta} \right) + \frac{\partial}{\partial \eta} \left( B \frac{\partial (rv)}{\partial r} + C \frac{\partial \psi}{\partial \eta} \right) = \frac{\partial (\Pi, \theta)}{\partial (r, \eta)}.
\]

(16)

Note that \( AC - B^2 = (go/\theta)\hat{f}P \), where

\[
P = \frac{1}{m} \left[ -\frac{\partial \psi}{\partial \eta} + \left( f + \frac{\partial (rv)}{\partial r} \right) \frac{\partial \theta}{\partial \eta} \right]
\]

(17)

is the potential vorticity. The partial differential equation (10) is elliptic if \( \hat{f}P > 0 \).

In the next five sections we consider the special cases of the height coordinate, the log-pressure coordinate, the pseudo-height coordinate, the isentropic coordinate, and a particular form of the sigma coordinate, as summarized in Table 1.
4. Transverse circulation equation in the height coordinate

As the first special case, consider the height coordinate, \( \eta = z \). In the height coordinate the pseudodensity equals the true density, i.e., \( m = -(1/g)(\partial \rho / \partial z) = \rho \). The governing equations (1)–(5) take the form

\[
(f + \frac{v}{r}) v = \theta \frac{\partial \Pi}{\partial r}, \tag{18}
\]

\[
\frac{\partial v}{\partial t} + w \frac{\partial v}{\partial z} + \left( f + \frac{\partial (rv)}{\partial r} \right) u = 0, \tag{19}
\]

\[
\theta \frac{\partial \Pi}{\partial z} = -g, \tag{20}
\]

\[
\frac{\partial p}{\partial t} + \frac{\partial (\rho u)}{\partial r} + \frac{\partial (\rho w)}{\partial z} = 0, \tag{21}
\]

\[
\frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial r} + w \frac{\partial \theta}{\partial z} = \bar{\theta}, \tag{22}
\]

where in this section the partial derivatives \( (\partial / \partial t) \) and \( (\partial / \partial r) \) are taken at fixed \( z \). The thermal wind equation (20) becomes

\[
\bar{f} \frac{\partial v}{\partial z} = \frac{\partial (\Pi, \theta)}{\partial (r, z)}. \tag{23}
\]

The transverse circulation equation (16) becomes

\[
\frac{\partial}{\partial r} \left( A \frac{\partial (rv)}{\partial r} + B \frac{\partial \psi}{\partial z} \right) + \frac{\partial}{\partial z} \left( B \frac{\partial (rv)}{\partial r} + C \frac{\partial \psi}{\partial z} \right) = \frac{\partial (\Pi, \bar{\theta})}{\partial (r, z)}, \tag{24}
\]

where, using the hydrostatic equation (21), the coefficients (15) become

\[
\rho A = \frac{g}{\theta} \frac{\partial \theta}{\partial z}, \quad \rho B = -\frac{g}{\theta} \frac{\partial \theta}{\partial r}, \quad \rho C = \bar{f} \left( f + \frac{\partial (rv)}{\partial r} \right) - \frac{\partial \Pi}{\partial r} \frac{\partial \theta}{\partial r}. \tag{25}
\]

The mass flux (8) becomes

\[
\rho u = -\frac{\partial \psi}{\partial z}, \quad \rho w = \frac{1}{\theta} \frac{\partial p}{\partial z} - \frac{\partial (rv)}{\partial r} \frac{\partial \theta}{\partial r}. \tag{26}
\]

and that \( AC - B^2 = (g \alpha/\theta) \bar{f} P \), where

\[
P = \frac{1}{\rho} \left[ -\frac{\partial v}{\partial z} \frac{\partial \theta}{\partial \eta} + \left( f + \frac{\partial (rv)}{\partial r} \right) \frac{\partial \theta}{\partial \eta} \right]. \tag{27}
\]

Pendergrass and Willoughby (2009) have presented an interesting analysis of the diabatically induced transverse circulation in tropical cyclones for both quasi-steady forcing and periodic forcing. Their formulation uses the height coordinate, which has the advantage that the lower boundary is a coordinate surface. Their mathematical analysis is generally similar to that presented here, although their derivation of the transverse circulation equation begins with the anelastic version of (21).

5. Transverse circulation equation in the log-pressure coordinate

As the second special case, consider the the log-pressure coordinate, \( \eta = z / H = \ln(p/p_0) \), where the scale height \( H = RT_0 / g \), the reference temperature \( T_0 \), and the reference pressure \( p_0 \) are constants. In this coordinate the pseudo-density becomes \( m = \rho_0 e^{-z/H} = \hat{\rho}(\hat{z}) \), where the constant reference density is \( \rho_0 = p_0 / RT_0 \). The governing equations (11)–(15) take the form

\[
(f + \frac{v}{r}) v = \frac{\partial \Phi}{\partial r}, \tag{28}
\]

\[
\frac{\partial v}{\partial t} + \frac{\partial \hat{\rho} u}{\partial r} + \left( f + \frac{\partial (rv)}{\partial r} \right) \hat{u} = 0, \tag{29}
\]

\[
\frac{\partial \Phi}{\partial t} = \frac{g}{T_0} \hat{z}, \tag{30}
\]

\[
\frac{\partial \hat{\rho} u}{\partial r} + \frac{\partial \hat{\rho} w}{\partial z} = 0, \tag{31}
\]

\[
\frac{\partial \hat{\rho} w}{\partial t} + u \frac{\partial \hat{\rho} w}{\partial r} + w \frac{\partial \hat{\rho} w}{\partial z} = \bar{\hat{\theta}}, \tag{32}
\]

where in this section the partial derivatives \( (\partial / \partial t) \) and \( (\partial / \partial r) \) are taken at fixed \( \hat{z} \). Since the pseudo-density \( \hat{\rho} \) is a function of \( \hat{z} \) only, the time derivative term in the continuity equation vanishes.

In contrast to (28), the thermal wind equation (20) simplifies to the non-Jacobian form

\[
\bar{f} \frac{\partial v}{\partial z} = \frac{g}{T_0} \frac{\partial T}{\partial r}. \tag{33}
\]

The mass flux (8) becomes

\[
\hat{\rho} u = -\frac{\partial \psi}{\partial \eta}, \quad \hat{\rho} w = \frac{\partial (rv)}{\partial r}. \tag{34}
\]

The last term in the second line of (15) vanishes and the transverse circulation equation (16) becomes

\[
\frac{\partial}{\partial r} \left( A \frac{\partial (rv)}{\partial r} + B \frac{\partial \psi}{\partial z} \right) + \frac{\partial}{\partial z} \left( B \frac{\partial (rv)}{\partial r} + C \frac{\partial \psi}{\partial z} \right) = \frac{\partial \hat{\Pi}}{\partial \eta} \frac{\partial \hat{\theta}}{\partial \eta}, \tag{35}
\]

where, using the hydrostatic equation (30), the coefficients (15) become

\[
\hat{\rho} A = \frac{g}{T_0} \frac{\partial \hat{\theta}}{\partial \eta}, \quad \hat{\rho} B = -\frac{g}{T_0} \frac{\partial \hat{\theta}}{\partial r}, \quad \hat{\rho} C = \bar{f} \left( f + \frac{\partial (rv)}{\partial r} \right). \tag{36}
\]

Note that \( AC - B^2 = (g \alpha/\theta) \bar{f} P \), where

\[
P = \frac{1}{\rho} \left[ -\frac{\partial v}{\partial \eta} \frac{\partial \hat{\theta}}{\partial \eta} + \left( f + \frac{\partial (rv)}{\partial r} \right) \frac{\partial \hat{\theta}}{\partial \eta} \right]. \tag{37}
\]
is the potential vorticity. Examples of the use of the log-pressure coordinate are \cite{Schubert2007}, who studied the distribution of subsidence in the hurricane eye, and \cite{Vigh2003} and \cite{Musgrave2012}, who studied the rapid development of the tropical cyclone warm core. An obvious disadvantage of using \( \hat{z} \) as the vertical coordinate in numerical modeling (although not so much in conceptual models) is that the lower boundary for \( \hat{z} \) is not generally a coordinate surface.

6. Transverse circulation equation in the pseudo-height coordinate

Now consider the special case of the pseudo-height coordinate, \( \eta = \hat{z} \), where

\[
m = \bar{\rho}(\hat{z}) = \frac{p_{0}}{\theta_{0}} \left( 1 - \frac{g \hat{z}}{c_{r} \theta_{0}} \right)^{(1-\kappa)/\kappa}.
\]

This coordinate has been widely used in the study of frontogenesis, e.g., see \cite{Hoskins1972}. Since \( \hat{z} = (\theta_{0}/g)(c_{p} - \Pi) \), we can regard \( \hat{z} \) as essentially an Exner function vertical coordinate, but scaled to have the unit of length and to increase in the upward direction. In this coordinate the pseudo-density is given by

\[
\frac{\partial \hat{v}}{\partial t} + \hat{w} \frac{\partial \hat{v}}{\partial \hat{z}} + \left( f + \frac{\partial (rv)}{r \partial r} \right) u = 0,
\]

and the thermal wind equation \( \Box \) becomes

\[
\frac{\partial \hat{v}}{\partial t} + \hat{w} \frac{\partial \hat{v}}{\partial \hat{z}} + \left( f + \frac{\partial (rv)}{r \partial r} \right) u = 0,
\]

where, using the hydrostatic equation \( \Box \), the coefficients \( \Box \) become

\[
\rho A = \frac{g}{\theta_{0}} \frac{\partial \theta}{\partial \hat{z}}, \quad \rho B = - \frac{g}{\theta_{0}} \frac{\partial \theta}{\partial r} = - \frac{f}{\partial \hat{z}},
\]

\[
\rho C = \hat{f} \left( f + \frac{\partial (rv)}{r \partial r} \right).
\]

Note that
\[
\rho u = - \frac{\partial \psi}{\partial \hat{z}}, \quad \rho \hat{w} = \frac{\partial (rv)}{r \partial r},
\]

and \( AC - B^{2} = \frac{\alpha}{\theta} \hat{f} P \), where

\[
P = \frac{1}{\rho} \left[ - \frac{\partial v}{\partial \hat{z}} + \frac{f + \partial (rv)}{r \partial r} \frac{\partial \theta}{\partial \hat{z}} \right].
\]

Examples of the use of the pseudo-height coordinate \( \hat{z} \) can be found in \cite{Schubert1982} and \cite{Hack1984}, who studied the nonlinear response of atmospheric vortices to heating by organized cumulus convection.

7. Transverse circulation equation in the isentropic coordinate

Now consider the special case of the isentropic coordinate, \( \eta = \theta \). In the isentropic coordinate the pseudo-density becomes \( m = -(1/g)(\partial p/\partial \theta) \). The governing equations \( \Box \) take the form

\[
(f + \frac{v}{r}) v = \frac{\partial M}{\partial r},
\]

\[
\frac{\partial v}{\partial t} + \hat{w} \frac{\partial v}{\partial \hat{z}} + \left( f + \frac{\partial (rv)}{r \partial r} \right) u = 0,
\]

and the thermal wind equation \( \Box \) takes the non-Jacobian form

\[
\frac{\partial \hat{v}}{\partial \hat{z}} = \frac{\partial \Pi}{\partial r}.
\]

The mass flux \( \Box \) becomes

\[
\frac{\partial \hat{v}}{\partial \hat{z}} = \frac{\partial \theta}{\partial \theta}.
\]

The transverse circulation equation \( \Box \) becomes

\[
\frac{\partial}{\partial r} \left( A \frac{\partial (rv)}{r \partial r} + B \frac{\partial \psi}{\partial \hat{z}} \right) + \frac{\partial}{\partial \hat{z}} \left( B \frac{\partial (rv)}{r \partial r} + C \frac{\partial \psi}{\partial \hat{z}} \right) = \frac{g}{\theta_{0}} \frac{\partial \theta}{\partial \hat{z}}.
\]

Note that \( A = g \eta / \theta, \ B = 0 \), and the last term in the second line of \( \Box \) vanishes so that \( C = \hat{f} P \). The transverse circulation equation takes the simple form

\[
\frac{\partial}{\partial r} \left( \frac{g \eta \partial \psi}{\theta r \partial r} \right) + \frac{\partial}{\partial \hat{z}} \left( \hat{f} P \frac{\psi}{\partial \hat{z}} \right) = \frac{\partial (\Pi, \theta)}{\partial (r, \theta)},
\]

\[
\frac{\partial}{\partial r} \left( \frac{g \eta \partial \psi}{\theta r \partial r} \right) + \frac{\partial}{\partial \hat{z}} \left( \hat{f} P \frac{\psi}{\partial \hat{z}} \right) = \frac{\partial (\Pi, \theta)}{\partial (r, \theta)}.
\]
where

\[ P = \left( f + \frac{\partial (rv)}{r \partial r} \right) \left( -\frac{1}{g} \frac{\partial p}{\partial \theta} \right)^{-1} \] (58)

is the potential vorticity.

Although the transverse circulation equation (57) and the potential vorticity (58) have compact forms in the \( \theta \)-coordinate, a disadvantage of using \( \theta \) as the vertical coordinate is that the lower boundary of the atmosphere is generally not a \( \theta \)-surface. However, in \( \theta \)-coordinate model this disadvantage is sometimes overcome through the use of a massless layer, which effectively makes the lower boundary a \( \theta \)-surface. Fulton and Schubert (1991) have used this massless layer approach to study surface frontogenesis in isentropic coordinates. In the context of tropical cyclone dynamics, Hendricks and Schubert (2010) have used a generalized version of (57) to describe the azimuthal mean overturning circulation forced by the adiabatic rearrangement of PV that occurs during the instability of hollow PV towers.

8. Transverse circulation equation in the \( \sigma \)-coordinate

Now consider the special case of the sigma coordinate, \( \eta = \sigma = 1 - (p/p_s) \), where \( p_s(r, t) \) is the pressure at the earth’s surface. In the sigma coordinate the pseudo-density becomes \( m = p_s/g \), and \( p_s \hat{\sigma} = p_s \sigma - \partial p/\partial \tau \). The governing equations (1–5) take the form

\[ \left( f + \frac{\partial (rv)}{r \partial r} \right) v = \frac{\partial \Phi}{\partial r} + (1 - \sigma) \alpha \frac{\partial p_s}{\partial r}, \] (59)

\[ \frac{\partial v}{\partial t} + \sigma \frac{\partial v}{\partial \sigma} + \left( f + \frac{\partial (rv)}{r \partial r} \right) u = 0, \] (60)

\[ \frac{\partial \Phi}{\partial \sigma} = p_s \alpha, \] (61)

\[ \frac{\partial p_s}{\partial t} + \frac{\partial (p_s ru)}{\partial r} \sigma = 0, \] (62)

\[ \frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial r} + \sigma \frac{\partial \theta}{\partial \sigma} = \dot{\theta}, \] (63)

where in this section the partial derivatives (\( \partial/\partial \tau \)) and (\( \partial/\partial r \)) are taken at fixed \( \sigma \). The thermal wind equation (9) takes the Jacobian form

\[ \hat{f} \frac{\partial v}{\partial \sigma} = \frac{\partial (\Pi, \theta)}{\partial (r, \sigma)}. \] (64)

The mass flux (8) becomes

\[ (p_s/g)u = -\frac{\partial \psi}{\partial \sigma}, \quad (p_s/g)\dot{\sigma} - \frac{1}{g} \frac{\partial p}{\partial \tau} = \frac{\partial (rv)}{r \partial r}. \] (65)

The transverse circulation equation becomes

\[ \frac{\partial}{\partial r} \left( A \frac{\partial (rv)}{r \partial r} + B \frac{\partial \psi}{\partial \sigma} \right) + \frac{\partial}{\partial \sigma} \left( B \frac{\partial (rv)}{r \partial r} + C \frac{\partial \psi}{\partial \sigma} \right) = \frac{\partial (\Pi, \dot{\theta})}{\partial (r, \sigma)}. \] (66)

where, using the hydrostatic equation (61), the coefficients (15) become

\[ A = \frac{g \alpha}{\theta} \frac{\partial \theta}{\partial \sigma}, \quad B = -\frac{g \alpha}{\theta} \frac{\partial \theta}{\partial r}, \quad C = \frac{g \hat{f}}{p_s} \left( f + \frac{\partial (rv)}{r \partial r} - \frac{g}{p_s} \frac{\partial \Pi}{\partial \theta} \right). \] (67)

Note that \( AC - B^2 = (g \alpha/\theta) \hat{f} P \), where

\[ P = \frac{g}{p_s} \left[ -\frac{\partial v}{\partial \sigma} \frac{\partial \theta}{\partial r} + (f + \frac{\partial (rv)}{r \partial r}) \frac{\partial \theta}{\partial \sigma} \right]. \] (68)

An advantage of using \( \sigma \) as the vertical coordinate is that the lower boundary condition for (66) is applied on the coordinate surface \( \sigma = 0 \). To the authors’ knowledge the \( \sigma \)-coordinate form (66) has not been used in studies of tropical cyclone dynamics.

Part II: Zonally Symmetric Meridional Circulations

The purpose of Part II is to derive the meridional circulation equation for the zonally symmetric balanced model in the generalized vertical coordinate. Two important applications are the tropical Hadley circulation and the radiatively forced meridional circulation over Antarctica. We begin by reviewing the zonally symmetric balance equations in section 9. Section 10 presents a derivation of the meridional circulation equation, with (67) being the main result. From (64) the meridional circulation equations in \( z, \hat{z}, \hat{z}, \theta \), and \( \sigma \) can be obtained as special cases.

9. The zonally symmetric balanced model in \( (\phi, \eta, t) \)

To simplify the primitive equation model to a balanced model we assume that the zonal flow remains in a nearly balanced state, i.e., we discard the meridional equation of motion and replace it with the balance condition given below in (70). A sufficient condition for the validity of this assumption is that the diabatic forcing term \( \theta \) has a slow enough time scale that significant, zonal mean inertia-gravity waves are not excited, i.e., \([Dv/Dt]\) remains small compared to the pressure gradient and Coriolis/centrifugal terms. We shall describe this inviscid flow using a generalized vertical coordinate \( \eta \), which is as yet an unspecified function of \((z, p, p_s, \theta)\). Under the balance condition, and using \( \eta(z, p, p_s, \theta) \) as the vertical coordinate, the governing equations are

\[ \frac{\partial u}{\partial t} + \eta \frac{\partial u}{\partial \eta} - \left( f - \frac{\partial (u \cos \phi)}{a \cos \phi} \frac{\partial \phi}{\partial \phi} \right) u = 0, \] (69)

\[ - \left( f + \frac{u \tan \phi}{a} \right) u = \frac{\partial \Phi}{a \partial \phi} + \theta \frac{\partial \Pi}{a \partial \phi}, \] (70)

\[ \frac{\partial \Phi}{\partial \eta} + \theta \frac{\partial \Pi}{\partial \eta} = 0, \] (71)
\[ \frac{\partial m}{\partial t} + \frac{\partial (mv \cos \phi)}{\partial \phi} + \frac{\partial (m\dot{\eta})}{\partial \eta} = 0, \quad (72) \]

\[ \frac{\partial \theta}{\partial t} + u \frac{\partial \theta}{\partial \phi} + \dot{\eta} \frac{\partial \theta}{\partial \eta} = \dot{\theta}, \quad (73) \]

where \( f = 2\Omega \sin \phi \) is the Coriolis parameter, \( m = -\frac{1}{g}(\partial p/\partial \eta) \) is the pseudopressure, \( \alpha = RT/p \) the specific volume, \( \Pi = \epsilon \rho/(p/\rho) \) the Enor function, and where the partial derivatives \( \partial/\partial t \) and \( \partial/\partial \phi \) are understood to be at fixed \( \eta \). In the following sections we consider only those cases for which the vertical coordinate \( \eta \) increases in the upward direction, so that \( m = -\frac{1}{g}(\partial p/\partial \eta) > 0 \).

10. Meridional circulation equation in \((\phi, \eta, t)\)

To derive the meridional circulation equation associated with (70)–(73), first consider the mass continuity equation (72), which can be written in the form

\[ \frac{\partial (mv \cos \phi)}{\partial \phi} + \frac{\partial (m\dot{\eta})}{\partial \eta} = 0, \quad (74) \]

where

\[ m\dot{\eta} = m\dot{\eta} - \frac{1}{g} \frac{\partial p}{\partial t}. \quad (75) \]

Using this form of the mass conservation principle we define a streamfunction \( \psi \) such that

\[ mv = -\frac{\partial \psi}{\partial \eta}, \quad m\dot{\eta} = \frac{\partial (\psi \cos \phi)}{\partial \phi}. \quad (76) \]

The thermal wind equation, derived from (70) and (71), is

\[ \dot{f} \frac{\partial u}{\partial t} + \frac{\partial (\Pi, \theta)}{\partial \eta} = 0, \quad (77) \]

where \( \dot{f} = f + 2(u/a) \tan \phi \). Taking \( \partial/\partial t \) of (77) we obtain

\[ \frac{\partial}{\partial \eta} \left( \dot{f} \frac{\partial u}{\partial t} \right) + \frac{\partial}{\partial t} \left( \frac{\partial (\Pi, \theta)}{\partial \phi, \eta} \right) = 0, \quad (78) \]

where we have made use of the identity \((\partial/\partial t)[\dot{f} \partial u/\partial \eta] = (\partial/\partial \eta)[\dot{f} \partial u/\partial t] \) to express the left hand side in a form that has \((\partial u/\partial t) \) inside the \((\partial/\partial \eta) \) operator. This sets the stage for the elimination of \((\partial u/\partial t) \) through the use of (73). Similarly, it is convenient to rewrite the second term of (78) in a form that has local time derivatives inside the \((\partial/\partial \phi) \) and \((\partial/\partial \eta) \) operators. This can be accomplished through the use of the Jacobi identity

\[ \frac{\partial}{\partial \phi} \left( \frac{\partial (\Pi, \theta)}{\partial (\eta, t)} \right) + \frac{\partial}{\partial \eta} \left( \frac{\partial (\Pi, \theta)}{\partial (\phi, \eta)} \right) + \frac{\partial}{\partial t} \left( \frac{\partial (\Pi, \theta)}{\partial (\phi, \eta)} \right) = 0, \quad (79) \]

which allows (78) to be written in the form

\[ \frac{\partial}{\partial \phi} \left( \frac{\partial (\Pi, \theta)}{\partial (\eta, t)} \right) + \frac{\partial}{\partial \eta} \left( \dot{f} \frac{\partial u}{\partial t} + \frac{\partial (\Pi, \theta)}{\partial (\phi, t)} \right) = 0. \quad (80) \]

This is the constraint that must be satisfied by the local time derivatives of the zonal wind field and the mass field. This constraint will lead directly to the partial differential equation for \( \psi \), after we have derived equations for the two quantities in the large parentheses of (70). The equation for the first of these is derived by multiplying (73) by \((\partial \Pi/\partial \eta) \), while the equation for the second is derived by adding \( \dot{f} \) times (73) to \((\partial \Pi/\partial \phi) \) times (73). The results are

\[ \frac{\partial (\Pi, \theta)}{\partial (t, \eta)} = -Am\dot{\eta} + Bmv - \frac{\partial \Pi}{\partial \eta} \dot{\theta}, \quad (81) \]

\[ \dot{f} \frac{\partial u}{\partial t} + \frac{\partial (\Pi, \theta)}{\partial \phi, t} = -Bm\dot{\eta} + Cmv + \frac{\partial \Pi}{\partial \phi} \dot{\theta}. \quad (82) \]

where

\[ A = \frac{g\alpha}{\theta} \frac{\partial \theta}{\partial \phi}, \quad B = -\frac{g\alpha}{\theta} \frac{\partial \theta}{\partial \phi}, \quad C = \frac{\dot{f}}{m} \left( \frac{f - \partial (u \cos \phi)}{\cos \phi \partial \phi} \right) - \frac{1}{m} \frac{\partial \Pi}{\partial \phi} \frac{\partial \theta}{\partial \phi}. \quad (83) \]

Using (81) and (82) in (80) we obtain the meridional circulation equation

\[ \frac{\partial}{\partial \phi} \left( A \frac{\partial (\psi \cos \phi)}{\partial \phi} + B \frac{\partial \psi}{\partial \eta} \right) \]

\[ + \frac{\partial}{\partial \eta} \left( B \frac{\partial (\psi \cos \phi)}{\partial \phi} + C \frac{\partial \psi}{\partial \phi} \right) = \frac{\partial (\Pi, \theta)}{\partial \phi, \eta}. \quad (84) \]

Note that \( AC - B^2 = (g\alpha/\theta) F \), where

\[ P = \frac{1}{m} \left[ \frac{\partial u}{\partial \eta} \frac{\partial \theta}{\partial \phi} + \left( \dot{f} - \frac{\partial (u \cos \phi)}{\cos \phi \partial \phi} \right) \frac{\partial \theta}{\partial \phi} \right]. \quad (85) \]

is the potential vorticity. Thus, the ellipticity condition \( AC - B^2 > 0 \) is equivalent to the condition \( \dot{f} P > 0 \). This condition is generally satisfied because both \( \dot{f} \) and \( P \) tend to be positive in the northern hemisphere, while both tend to be negative in the southern hemisphere. However, the condition can often be violated near the equator. For example, when the ITCZ is north of the equator, lower tropospheric air with negative \( P \) is forced across the equator into the northern hemisphere, while upper tropospheric air with positive \( P \) is forced across the equator into the southern hemisphere. This tends to result in small regions where \( \dot{f} P < 0 \), the regions being just north of the equator in the lower troposphere and just south of the equator in the upper troposphere. Some of the possible dynamical consequences of this non-ellipticity have been discussed by Tomas and Webster (1997). However, in the context of solving the meridional circulation equation (80), experience has shown that iterative methods generally work well even when there are small non-elliptic regions in the domain.

In the next five sections we consider the special cases of the height coordinate \((\eta = \bar{z})\), the log-pressure coordinate \((\eta = \bar{z})\), the pseudo-height coordinate \((\eta = \bar{z})\), the isentropic coordinate \((\eta = \theta)\), and the sigma coordinate \((\eta = 1 - (p/p_s))\).
11. The meridional circulation equation in the height coordinate

First consider the special case of the height coordinate, \( \eta = z \). In the height coordinate the pseudo-density equals the true density, i.e., \( m = \rho \). The governing equations (69) - (73) become

\[
\frac{\partial u}{\partial t} + \frac{\partial w}{\partial z} - \left( f - \frac{\partial (u \cos \phi)}{a \cos \phi \partial \phi} \right) v = 0, \quad (86)
\]

\[
- \left( f + \frac{u \tan \phi}{a} \right) u = \frac{\partial \Pi}{a \partial \phi}, \quad (87)
\]

\[
g + \frac{\partial \Pi}{\partial z} = 0, \quad (88)
\]

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v \cos \phi)}{a \cos \phi \partial \phi} + \frac{\partial (\rho w)}{\partial z} = 0, \quad (89)
\]

\[
\frac{\partial \theta}{\partial t} + \frac{\partial \theta}{a \partial \phi} + \frac{\partial \theta}{\partial z} = \dot{\theta}, \quad (90)
\]

and the thermal wind equation (74) becomes

\[
\frac{\dot{f}}{\partial \eta} + \frac{\partial \Pi(\theta, \phi)}{a \partial \phi} = 0. \quad (91)
\]

The meridional circulation equation (84) becomes

\[
\frac{\partial}{a \partial \phi} \left( A \frac{\partial (\psi \cos \phi)}{a \cos \phi \partial \phi} + B \frac{\partial \psi}{\partial z} \right) + \frac{\partial}{\partial z} \left( \frac{B \partial (\psi \cos \phi)}{a \cos \phi \partial \phi} + C \frac{\partial \psi}{\partial z} \right) = \frac{\partial \Pi(\theta, \phi)}{a \partial \phi}, \quad (92)
\]

where

\[
\rho A = \frac{g \partial \theta}{\partial \eta}, \quad \rho B = \frac{g \partial \theta}{a \partial \phi}, \quad \rho C = \dot{f} \left( f - \frac{\partial (u \cos \phi)}{a \cos \phi \partial \phi} \right) - \frac{\partial \Pi \partial \theta}{a \partial \phi \partial \phi}. \quad (93)
\]

The mass flux (76) becomes

\[
\rho v = -\frac{\partial \psi}{\partial z}, \quad \rho w = -\frac{1}{g} \frac{\partial p}{\partial t} = \frac{\partial (\psi \cos \phi)}{a \cos \phi \partial \phi}, \quad (94)
\]

and that \( AC - B^2 = (g \alpha / \partial) \dot{f} P \), where

\[
P = \frac{1}{\rho} \left[ \frac{\partial u}{\partial z} \frac{\partial \theta}{a \partial \phi} + \left( f - \frac{\partial (u \cos \phi)}{a \cos \phi \partial \phi} \right) \frac{\partial \theta}{\partial z} \right]. \quad (95)
\]

We are not aware of any studies of meridional circulation using the z-coordinate formulation.

12. The meridional circulation equation in the log-pressure coordinate

Now consider the special case of the log-pressure coordinate, \( \eta = \hat{z} = H \ln(\rho_0 / \rho) \), where as before the scale height \( H = RT_0 / g \), the reference temperature \( T_0 \), and the reference pressure \( \rho_0 \) are constants. In this coordinate the pseudodensity becomes \( m = \rho(\hat{z}) = \rho_0 e^{-\hat{z}/H} \), where the constant reference density is \( \rho_0 = \rho_0 / RT_0 \). The governing equations (69) - (73) become

\[
\frac{\partial u}{\partial t} + \frac{\partial w}{\partial \hat{z}} - \left( f - \frac{\partial (u \cos \phi)}{a \cos \phi \partial \phi} \right) v = 0, \quad (96)
\]

\[
- \left( f + \frac{u \tan \phi}{a} \right) u = \frac{\partial \Phi}{a \partial \phi}, \quad (97)
\]

\[
\frac{\partial \Phi}{\partial \hat{z}} = \frac{g}{T_0} \frac{\partial \theta}{\partial \eta}, \quad (98)
\]

\[
\frac{\partial (\rho v \cos \phi)}{a \cos \phi \partial \phi} + \frac{\partial (\rho w)}{\partial \hat{z}} = 0, \quad (99)
\]

\[
\frac{\partial \theta}{\partial t} + \frac{\partial \theta}{a \partial \phi} + \frac{\hat{w} \partial \theta}{\partial \hat{z}} = \dot{\theta}, \quad (100)
\]

and the thermal wind equation (77) loses its Jacobian form to become

\[
\frac{\dot{f}}{\partial \eta} + \frac{\partial \Pi(\theta, \phi)}{a \partial \phi} = 0. \quad (101)
\]

The mass flux (76) becomes

\[
\dot{\rho} v = -\frac{\partial \psi}{\partial \hat{z}}, \quad \dot{\rho} w = \frac{\partial (\psi \cos \phi)}{a \cos \phi \partial \phi}, \quad (102)
\]

The last term in the second line of (83) vanishes and the transverse circulation equation (84) becomes

\[
\frac{\partial}{a \partial \phi} \left( A \frac{\partial (\psi \cos \phi)}{a \cos \phi \partial \phi} + B \frac{\partial \psi}{\partial \hat{z}} \right) + \frac{\partial}{\partial \hat{z}} \left( \frac{B \partial (\psi \cos \phi)}{a \cos \phi \partial \phi} + C \frac{\partial \psi}{\partial \hat{z}} \right) = \frac{T}{\theta} \frac{\partial \theta}{T_0 \partial \phi}, \quad (103)
\]

where

\[
\dot{\rho} A = \frac{g \partial \theta}{\partial \eta}, \quad \dot{\rho} B = \frac{g \partial \theta}{T_0 \partial \phi}, \quad \dot{\rho} C = \dot{f} \left( f - \frac{\partial (u \cos \phi)}{a \cos \phi \partial \phi} \right). \quad (104)
\]

Note that \( AC - B^2 = (g \alpha / \partial) \dot{f} P \), where

\[
P = \frac{1}{\dot{\rho}} \left[ \frac{\partial u}{\partial \hat{z}} \frac{\partial \theta}{a \partial \phi} + \left( f - \frac{\partial (u \cos \phi)}{a \cos \phi \partial \phi} \right) \frac{\partial \theta}{\partial \hat{z}} \right]. \quad (105)
\]

is the potential vorticity. A disadvantage of using \( \hat{z} \) as the vertical coordinate is that the lower boundary condition for (103) is not applied on a coordinate surface. Recent studies of the deep and shallow tropical Hadley circulations using a simplified, equatorial \( \beta \)-plane version of (106) are those of Gonzalez and Mora Rojas (2014) and Gonzalez et al. (2017).
13. The meridional circulation equation in the pseudo-height coordinate

As in section 6, now consider the special case of the pseudo-height coordinate, \( \eta = \tilde{z} \), where \( \tilde{z} = (c_r \theta_0/g) [1 - (p/p_0)^\sigma] \). As before, in this coordinate the pseudo-density is given by

\[
m = \tilde{\rho}(\tilde{z}) = \frac{p_0}{R \theta_0} \left( 1 - \frac{g \tilde{z}}{c_r \theta_0} \right)^{(1-\kappa)/\kappa}. \tag{106}
\]

The governing equations (69)–(73) become

\[
\frac{\partial u}{\partial t} + \tilde{w} \frac{\partial u}{\partial \tilde{z}} - \left( f - \frac{\partial (u \cos \phi)}{a \cos \phi \partial \phi} \right) v = 0, \tag{107}
\]

\[
- \left( f + \frac{u \tan \phi}{a} \right) u = \frac{\partial \Phi}{\partial \theta}, \tag{108}
\]

\[
\frac{\partial \tilde{\rho} v \cos \phi}{a \cos \phi \partial \phi} + \frac{\partial (\tilde{\rho} w)}{\partial \tilde{z}} = 0, \tag{109}
\]

\[
\frac{\partial \tilde{\theta}}{\partial t} + v \frac{\partial \tilde{\theta}}{\partial \tilde{z}} + \tilde{w} \frac{\partial \tilde{\theta}}{\partial \tilde{z}} = \tilde{\theta}. \tag{111}
\]

The meridional circulation equation (84) becomes

\[
\frac{\partial}{\partial \phi} \left( \frac{A}{a \cos \phi \partial \phi} \frac{\partial (\psi \cos \phi)}{a \cos \phi \partial \phi} + B \frac{\partial \psi}{\partial \tilde{z}} + C \frac{\partial \psi}{\partial \tilde{z}} \right) + \frac{\partial}{\partial \tilde{z}} \left( B \frac{\partial (\psi \cos \phi)}{a \cos \phi \partial \phi} + C \frac{\partial \psi}{\partial \tilde{z}} \right) = \frac{g}{\theta_0} \frac{\partial \tilde{\theta}}{a \theta \partial \phi}, \tag{112}
\]

where

\[
\tilde{\rho} A = \frac{g}{\theta_0} \frac{\partial \tilde{\theta}}{\partial \tilde{z}}, \quad \tilde{\rho} B = \frac{g}{\theta_0} \frac{\partial \tilde{\theta}}{a \theta \partial \phi} = \hat{f} \frac{\partial u}{\partial \tilde{z}}, \tag{113}
\]

\[
\tilde{\rho} C = \hat{f} \left( f - \frac{\partial (u \cos \phi)}{a \cos \phi \partial \phi} \right). \tag{114}
\]

The mass flux (76) becomes

\[
\tilde{\rho} v = - \frac{\partial \psi}{\partial \tilde{z}}, \quad \tilde{\rho} w = \frac{\partial (\psi \cos \phi)}{a \cos \phi \partial \phi}, \tag{114}
\]

and \( AC - B^2 = (\alpha/\theta) \hat{f} P \), where

\[
P = \frac{1}{\tilde{\rho}} \left[ \frac{\partial u}{\partial \tilde{z}} \frac{\partial \tilde{\theta}}{a \theta \partial \phi} + \left( f - \frac{\partial (u \cos \phi)}{a \cos \phi \partial \phi} \right) \frac{\partial \tilde{\theta}}{\partial \tilde{z}} \right]. \tag{115}
\]

Hack et al. (1989) and Hack and Schubert (1990) have solved (112) for cases in which the diabatic forcing \( \tilde{\theta} \) is associated with an ITCZ that lies off the equator. This produces two Hadley cells, with the cross-equatorial cell carrying considerably more mass flux than its companion. This anisotropy is due to the spatial variation of the inertial stability coefficient \( C \), which is relatively small near the equator, thereby providing little resistance to horizontal flow across the equator.

14. The meridional circulation equation in \((\phi, \theta, t)\)

Now consider the special case of the isentropic coordinate, \( \eta = \theta \). In the isentropic coordinate the pseudodensity becomes \( m = - (1/g) (\partial p/\partial \theta) \) and \( A = \sigma \alpha / \theta \), \( B = 0 \), and the last term in the second line of (107) vanishes so that \( C = \hat{f} P \). The governing equations (69)–(73) become

\[
\frac{\partial u}{\partial t} + \tilde{\theta} \frac{\partial u}{\partial \theta} - \left( f - \frac{\partial (u \cos \phi)}{a \cos \phi \partial \phi} \right) v = 0, \tag{116}
\]

\[
- \left( f + \frac{u \tan \phi}{a} \right) u = \frac{\partial M}{\partial \theta}, \tag{117}
\]

\[
\frac{\partial M}{\partial \theta} = \Pi, \tag{118}
\]

\[
\frac{\partial m}{\partial t} + \frac{\partial (mv \cos \phi)}{a \cos \phi \partial \phi} + \frac{\partial (m \tilde{\theta})}{\partial \theta} = 0, \tag{119}
\]

and the thermal wind equation (77) has the non-Jacobian form

\[
\hat{f} \frac{\partial u}{\partial \theta} = \frac{\partial \Pi}{a \theta \partial \phi}. \tag{120}
\]

The meridional circulation equation (84) becomes

\[
\frac{\partial}{\partial \phi} \left( \frac{g}{a \cos \phi \partial \phi} \frac{\partial (\psi \cos \phi)}{a \cos \phi \partial \phi} + \frac{\partial \psi}{\partial \tilde{z}} \right) + \frac{\partial}{\partial \tilde{z}} \left( \frac{g}{a \cos \phi \partial \phi} \frac{\partial \psi}{\partial \tilde{z}} + \frac{\partial \psi}{\partial \tilde{z}} \right) = \frac{\partial \Pi}{a \theta \partial \phi}, \tag{121}
\]

where

\[
P = \left( f - \frac{\partial (u \cos \phi)}{a \cos \phi \partial \phi} \right) \left( - \frac{1}{a \cos \phi \partial \phi} \right)^{-1} \tag{122}
\]

is the potential vorticity. The mass flux (76) becomes

\[
-(1/g) (\partial p/\partial \theta) \tilde{\psi} = - \frac{\partial \psi}{\partial \theta}, \tag{123}
\]

\[
-(1/g) (\partial p/\partial \theta) \tilde{\psi} - \frac{1}{g} \frac{\partial \psi}{\partial \theta} = \frac{\partial (\psi \cos \phi)}{a \cos \phi \partial \phi}. \tag{123}
\]

The advantages of using \( \theta \) as the vertical coordinate for stratospheric studies have been discussed by Andrews (1983), Tung (1986), and Yang et al. (1990). Although (121) and (122) have compact forms, a disadvantage of using \( \tilde{\theta} \) as the vertical coordinate for tropospheric studies is that the lower boundary condition for (121) is not applied on a coordinate surface. As discussed in section 7, this disadvantage is sometimes overcome through the use of a massless layer. Fulton et al. (2017) have applied the concept of a massless layer in solving the PV invertibility principle for the topographically bound low-level jet surrounding Antarctica.

15. The meridional circulation equation in \((\phi, \sigma, t)\)

Now consider the special case of the sigma coordinate, \( \eta = \sigma = 1 - (p/p_0) \), where \( p_0(\phi, t) \) is the pressure at the earth’s surface. In the sigma coordinate the pseudodensity
becomes \( m = p_s / g \), and \( p_s \dot{\sigma} = p_s \dot{\sigma} - \partial p / \partial t \). The governing equations (69)–(73) become

\[
\frac{\partial u}{\partial t} + \dot{\sigma} \frac{\partial u}{\partial \sigma} - \left( f - \frac{\partial (u \cos \phi)}{a \cos \phi \partial \phi} \right) v = 0, \tag{124}
\]

\[
- \left( f + \frac{u \tan \phi}{a} \right) u = \frac{\partial \Phi}{a \partial \phi} + \theta \frac{\partial \Pi}{a \partial \phi}, \tag{125}
\]

\[
\frac{\partial \Phi}{\partial \sigma} + \theta \frac{\partial \Phi}{\partial \sigma} = 0, \tag{126}
\]

\[
\frac{\partial n}{\partial t} + \frac{\partial (m \cos \phi)}{a \cos \phi \partial \phi} + \frac{\partial (m \dot{\sigma})}{\partial \sigma} = 0, \tag{127}
\]

\[
\frac{\partial \theta}{\partial t} + v \frac{\partial \theta}{a \partial \phi} + \dot{\sigma} \frac{\partial \theta}{\partial \sigma} = \theta, \tag{128}
\]

and the thermal wind equation (77) has the Jacobian form

\[
\int \frac{\partial \theta}{\partial \sigma} = \frac{\partial (\Pi, \theta)}{a \partial (\phi, \sigma)}, \tag{129}
\]

The meridional circulation equation (84) becomes

\[
\frac{\partial}{a \partial \phi} \left( A \frac{\partial (\psi \cos \phi)}{a \cos \phi \partial \phi} + B \frac{\partial \psi}{a \partial \phi} \right) + \frac{\partial}{\partial \sigma} \left( B \frac{\partial (\psi \cos \phi)}{a \cos \phi \partial \phi} + C \frac{\partial \psi}{\partial \sigma} \right) = \frac{\partial (\Pi, \theta)}{a \partial (\phi, \sigma)}, \tag{130}
\]

where

\[
A = \frac{g_\alpha}{\theta} \frac{\partial \theta}{\partial \sigma}, \quad B = \frac{g_\alpha}{\theta} \frac{\partial \theta}{a \partial \phi}, \quad C = \frac{g_\alpha}{p_s} \left( f - \frac{\partial (u \cos \phi)}{a \cos \phi \partial \phi} \right) \frac{\partial \ln p_s}{p_s} \frac{\partial \ln \theta}{a \partial \phi} \frac{\partial \phi}{a \partial \phi}. \tag{131}
\]

The mass flux (70) becomes

\[
(p_s / g)v = - \frac{\partial \psi}{\partial \sigma} - \frac{1}{g} \frac{\partial p_s}{g \partial t} = \frac{\partial (\psi \cos \phi)}{a \cos \phi \partial \phi}, \tag{132}
\]

and that \( AC - B^2 = (g_\alpha / \theta) \hat{f} P \), where

\[
P = \frac{g}{p_s} \left[ \frac{\partial u}{a \partial \phi} + \left( f - \frac{\partial (u \cos \phi)}{a \cos \phi \partial \phi} \right) \frac{\partial \theta}{\partial \sigma} \right]. \tag{133}
\]

An advantage of using \( \sigma \) as the vertical coordinate is that the lower boundary condition for (130) is applied on the coordinate surface \( \sigma = 0 \). Thus, another interesting application of (130) would be to the katabatically forced meridional circulation over Antarctica.

16. Concluding Remarks

Using the generalized vertical coordinate \( \eta(z, p, p_s, \theta) \), we have derived the transverse circulation equation (16) for tropical cyclones and the meridional circulation equation (84) for zonally symmetric flows. These derivations involve the time derivative of the thermal wind equation, which provides a constraint on the tendencies of the rotational wind field and the mass field. A crucial mathematical step in these derivations is to obtain the constraints in the forms (12) and (80), i.e., with spatial derivatives on the outside and time derivatives on the inside. This relies on the use of the Jacobi identities (11) and (29), which can thus be considered the key step in producing the final partial differential equations (10) and (34).

The solutions of the elliptic problems discussed here depend on both the forcing and the boundary conditions. For simplicity we have not discussed frictional forcing and boundary conditions. In many applications, the elliptic equations we have derived would be applied to an inviscid interior flow, with a separate frictional boundary layer dynamics providing the lower boundary condition for the inviscid interior. The boundary layer dynamics could involve simple, local, slab Ekman theory over a flat surface for the tropical cyclone case or over a sloping ice surface for the Antarctic case. In any event, the formulation of the lower boundary condition is beyond the scope of the present analysis. For an interesting discussion of some lower boundary effects the reader is referred to the interesting study by Haynes and Shepherd (1989), who discussed the importance of surface pressure changes in the response of the atmosphere to zonally-symmetric thermal and mechanical forcing.

For simplicity we have also omitted discussion of hybrid vertical coordinates, such as those proposed by Zilit et al. (1992) and Konor and Arakawa (1997). However, since these hybrid coordinates are also special cases of the generalized vertical coordinate \( \eta(z, p, p_s, \theta) \), transverse circulation equations can easily be obtained from the generalized forms (16) and (34).

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