ROBUST DECODING FROM 1-BIT COMPRESSIVE SAMPLING WITH LEAST SQUARES

JIAN HUANG∗, YULING JIAO†, XILIANG LU‡, AND LIPING ZHU§

Abstract. In 1-bit compressive sensing (1-bit CS) where target signal is coded into a binary measurement, one goal is to recover the signal from noisy and quantized samples. Mathematically, the 1-bit CS model reads: $y = \eta \odot \text{sign}(\Psi x^* + \epsilon)$, where $x^* \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\Psi \in \mathbb{R}^{m \times n}$, and $\epsilon$ is the random error before quantization and $\eta \in \mathbb{R}^n$ is a random vector modeling the sign flips. Due to the presence of nonlinearity, noise and sign flips, it is quite challenging to decode from the 1-bit CS. In this paper, we consider least squares approach under the over-determined and under-determined settings. For $m > n$, we show that, up to a constant $c$, with high probability, the least squares solution $x_{ls}$ approximates $x^*$ with precision $\delta$ as long as $m \geq \tilde{O}(\frac{1}{\delta^2})$. For $m < n$, we prove that, up to a constant $c$, with high probability, the $\ell_1$-regularized least-squares solution $x_{\ell_1}$ lies in the ball with center $x^*$ and radius $\delta$ provided that $m \geq \tilde{O}(\frac{s \log n}{\delta^2})$ and $\|x^*\|_0 := s < m$. We introduce a Newton type method, the so-called primal and dual active set (PDAS) algorithm, to solve the nonsmooth optimization problem. The PDAS possesses the property of one-step convergence. It only requires to solve a small least squares problem on the active set. Therefore, the PDAS is extremely efficient for recovering sparse signals through continuation. We propose a novel regularization parameter selection rule which does not introduce any extra computational overhead. Extensive numerical experiments are presented to illustrate the robustness of our proposed model and the efficiency of our algorithm.

Keywords: 1-bit compressive sensing, $\ell_1$-regularized least squares, primal dual active set algorithm, one step convergence, continuation

1. Introduction. Compressive sensing (CS) is an important approach to acquiring low dimension signals from noisy under-determined measurements [8, 16, 19, 20]. For storage and transmission, the infinite-precision measurements are often quantized, [6] considered recovering the signals from the 1-bit compressive sensing (1-bit CS) where measurements are coded into a single bit, i.e., their signs. The 1-bit CS is superior to the CS in terms of inexpensive hardware implementation and storage. However, it is much more challenging to decode from nonlinear, noisy and sign-flipped 1-bit measurements.

1.1. Previous work. Since the seminal work of [6], much effort has been devoted to studying the theoretical and computational challenges of the 1-bit CS. Sample complexity was analyzed for support and vector recovery with and without noise [21, 28, 40, 23, 29, 22, 23, 41, 50]. Existing works indicate that, $m > \tilde{O}(s \log n)$ is adequate for both support and vector recovery. The sample size required here has the same order as that required in the standard CS setting. These results have also been refined by adaptive sampling [22, 14, 4]. Extensions include recovering the norm of the target [32, 3] and non-Gaussian measurement settings [1]. Many first order methods [6, 34, 49, 14] and greedy methods [35, 5, 29] are developed to minimize the sparsity promoting nonconvex object function arising from either the unit sphere constraint or the nonconvex regularizers. To address

∗Department of Applied Mathematics, The Hong Kong Polytechnic University, Hong Kong 999077, P.R. China. (j.huang@polyu.edu.hk)
†School of Statistics and Mathematics, Zhongnan University of Economics and Law, Wuhan 430063, P.R. China. (yulingjiaomath@whu.edu.cn)
‡School of Mathematics and Statistics, and Hubei Key Laboratory of Computational Science, Wuhan University, Wuhan 430072, P.R. China. (xllv.math@whu.edu.cn)
§Institute of Statistics and Big Data and Center for Applied Statistics, Renmin University of China, Beijing 100872, P. R. China. (zhu.liping@ruc.edu.cn)
the nonconvex optimization problem, convex relaxation models are also proposed [50, 41, 40, 51, 42], which often yield accurate solutions efficiently with polynomial-time solvers. See, for example, [38].

1.2. 1-bit CS setting. In this paper we consider the following 1-bit CS model

\[ y = \eta \odot \text{sign}(\Psi x^\ast + \epsilon), \]

where \( y \in \mathbb{R}^m \) is the 1-bit measurement, \( x^\ast \in \mathbb{R}^n \) is an unknown signal, \( \Psi = [\psi_1; ..., \psi_m] \in \mathbb{R}^{m \times n} \) is a random matrix, \( \eta \in \mathbb{R}^m \) is a random vector modeling the sign flips of \( y \), and \( \epsilon \in \mathbb{R}^n \) is a random vector with independent and identically distributed (iid) entries modeling errors before quantization. Throughout \( \text{sign}() \) operates componentwise with \( \text{sign}(z) = 1 \) if \( z \geq 0 \) and \( \text{sign}(z) = -1 \) otherwise, and \( \odot \) is the pointwise Hardmard product. Following [40] we assume that the rows of \( \Psi \) are iid random vectors sampled from the multivariate normal distribution \( \mathcal{N}(\mathbf{0}, \Sigma) \) with an unknown covariance matrix \( \Sigma \), \( \epsilon \) is distributed as \( \mathcal{N}(\mathbf{0}, \sigma^2 \mathbb{I}_m) \) with an unknown noise level \( \sigma \), and \( \eta \in \mathbb{R}^m \) has independent coordinates \( \eta_i \) satisfying \( \mathbb{P}[\eta_i = 1] = 1 - \mathbb{P}[\eta_i = -1] = q \neq \frac{1}{2} \). We assume \( \eta_i, \epsilon_i \) and \( \psi_i \) are mutually independent. Because \( \sigma \) is known model (1.1) is invariant in the sense that \( \forall \alpha > 0, y = \eta \odot \text{sign}(\Psi x^\ast + \epsilon) = \eta \odot \text{sign}(\alpha \Psi x^\ast + \alpha \epsilon) \). This indicates that the best one can hope for is to recover \( x^\ast \) up to a scale factor. Without loss of generality we assume \( \|x^\ast\|_\Sigma = 1 \).

1.3. Contributions. We study the 1-bit CS problem in both the overdetermined setting with \( m > n \) and the underdetermined setting with \( m < n \). In the former setting we allow for dense \( x^\ast \), while in the latter, we assume that \( x^\ast \) is sparse in the sense that \( \|x^\ast\|_0 = s < m \). The basic message is that we can recover \( x^\ast \) with the ordinary least squares or the \( \ell_1 \) regularized least squares.

(1) When \( m > n \), we propose to use the least squares solution

\[ x_{ls} \in \text{arg min} \frac{1}{m} \sum_{i=1}^{m} (y_i - \psi_i^\ast x)^2 \]

to approximate \( x^\ast \). We show that, with high probability, \( x_{ls} \) estimates \( x^\ast \) accurately up to a positive scale factor \( c \) defined by (2.2) in the sense that, \( \forall \delta \in (0, 1), \|x_{ls}/c - x^\ast\| \leq \delta \) if \( m \geq \Theta(\frac{s}{\delta^2}) \). We make the following observation:

*Up to a constant c, the underlying target \( x^\ast \) can be decoded from 1-bit measurements with the ordinary least squares, as long as the probability of sign flips probability is not equal to 1/2.*

(2) When \( m < n \) and the target signal \( x^\ast \) is sparse, we consider the \( \ell_1 \)-regularized least squares solution

\[ x_{\ell_1} \in \text{arg min} \frac{1}{2m} \|y - \Psi x\|_2^2 + \lambda \|x\|_1. \]

The sparsity assumption is widely used in modern signal processing [20, 36]. We show that, with high probability the error \( \|x_{\ell_1}/c - x^\ast\| \) can be bounded by a prefixed accuracy \( \delta \in (0, 1) \) if \( m \geq \mathcal{O}(\frac{\log s}{\delta^2}) \), which is the same as the order for the standard CS methods to work. Furthermore, the support of \( x^\ast \) can be exactly recovered if the minimum signal magnitude of \( x^\ast \) is larger than \( \mathcal{O}(\sqrt{s \log n/m}) \).
When the target signal is sparse, we obtain the following conclusion:

Up to a constant $c$, the sparse signal $x^*$ can also be decoded from 1-bit measurements with the $\ell_1$-regularized least squares, as long as the probability of sign flips probability is not equal to 1/2.

(3) We introduce a fast and accurate Newton method, the so-called primal dual active set method (PDAS), to solve the $\ell_1$-regularized minimization (1.2). The PDAS possesses the property of one-step convergence. The PDAS solves a small least squares problem on the active set, is thus extremely efficient if combined with continuation. We propose a novel regularization parameter selection rule, which is incorporated with continuation procedure without additional cost. The code is available at http://faculty.zuel.edu.cn/tjyjxxy/jyl/list.htm.

The optimal solution $x_{\ell_1}$ can be computed efficiently and accurately with the PDAS, a Newton type method which converges after one iteration, even if the objective function (1.2) is nonsmooth. Continuation on $\lambda$ globalizes the PDAS. The regularization parameter can be automatically determined without additional computational cost.

1.4. Notation and organization. Throughout we denote by $\Psi_i \in \mathbb{R}^{m \times 1}$, $i = 1, \ldots, m$, and $\psi_j \in \mathbb{R}^{n \times 1}$, $j = 1, \ldots, n$ the $i$th column and $j$th row of $\Psi$, respectively. We denote a vector of 0 by $\mathbf{0}$, whose length may vary in different places. We use $[n]$ to denote the set $\{1, \ldots, n\}$, and $I_n$ to denote the identity matrix of size $n \times n$. For $A, B \subseteq [n]$ with length $|A|, |B|$, $x_A = (x_i, i \in A) \in \mathbb{R}^{|A|}$, $\Psi_A = (\Psi_i, i \in A) \in \mathbb{R}^{m \times |A|}$ and $\Psi_{AB} \in \mathbb{R}^{|A| \times |B|}$ denotes a submatrix of $\Psi$ whose rows and columns are listed in $A$ and $B$, respectively. We use $(\psi_i)_j$ to denote the $j$th entry of the vector $\psi_i$, and $|x|_{\min}$ to denote the minimum absolute value of $x$. We use $\mathcal{N}(\mathbf{0}, \Sigma)$ to denote the multivariate normal distribution, with $\Sigma$ symmetric and positive definite. Let $\gamma_{\max}(\Sigma)$ and $\gamma_{\min}(\Sigma)$ be the largest and the smallest eigenvalues of $\Sigma$, respectively, and $\kappa(\Sigma)$ be the condition number $\gamma_{\max}(\Sigma)/\gamma_{\min}(\Sigma)$ of $\Sigma$. We use $||x||_{\Sigma}$ to denote the elliptic norm of $x$ with respect to $\Sigma$, i.e., $||x||_{\Sigma} = (x^T \Sigma x)^{1/2}$. Let $||x||_p = (\sum_{i=1}^n |x_i|^p)^{1/p}, p \in [1, \infty]$, be the $\ell_p$-norm of $x$. We denote the number of nonzero elements of $x$ by $||x||_0$ and let $s = ||x^*||_0$. The symbols $||\Psi||$ and $||\Psi||_\infty$ stands for the operator norm of $\Psi$ induced by $\ell_2$ norm and the maximum pointwise absolute value of $\Psi$, respectively. We use $\mathbb{E}[-], \mathbb{E}[\cdot|\cdot], \mathbb{P}[-]$ to denote the expectation, the conditional expectation and the probability on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We use $C_1$ and $C_2$ to denote generic constants which may vary from place to place. By $\mathcal{O}(\cdot)$ and $\tilde{\mathcal{O}}(\cdot)$, we ignore some positive numerical constant and $\sqrt{\log n}$, respectively.

The rest of the paper is organized as follows. In Section 2 we explain why the least squares works in the 1-bit CS when $m > n$, and obtain a nonasymptotic error bound for $||x_{ls}/c - x^*||$. In Section 3 we consider the sparse decoding when $m < n$ and prove a minimax bound on $||x_{ls}/c - x^*||$. In Section 4 we introduce the PDAS algorithm to solve (1.2). We propose a new regularization parameter selection rule combined with the continuation procedure. In Section 5 we conduct simulation studies and compare with existing 1-bit CS methods. We conclude with some remarks in Section 6.

2. Least squares when $m > n$. In this section, we first explain why the least squares works in the over-determined 1-bit CS model (1.1) with $m > n$. We then prove a nonasymptotic error bound on $||x_{ls}/c - x^*||$. The following lemma inspired by [7] is our starting point.
**Lemma 2.1.** Let $\tilde{y} = \tilde{\eta} \text{sign}(\tilde{\psi} x^* + \tilde{\epsilon})$ be the 1-bit model (1.1) at the population level. $\mathbb{P}[\tilde{y} = 1] = q \neq \frac{1}{2}$, $\tilde{\psi} \sim \mathcal{N}(0, \Sigma)$, $\tilde{\epsilon} \sim \mathcal{N}(0, \sigma^2)$. It follows that,

$$\Sigma^{-1} \mathbb{E}[\tilde{y}\tilde{\psi}] / c = x^*, \quad (2.1)$$

where

$$c = (2q - 1) \sqrt{\frac{2}{\pi(\sigma^2 + 1)}}. \quad (2.2)$$

**Proof.** The proof is given in Appendix A. \[\square\]

Lemma 2.1 shows that, the target $x^*$ is proportional to $\Sigma^{-1} \mathbb{E}[\tilde{y}\tilde{\psi}]$. Note that

$$\mathbb{E}[\Psi^t \Psi / m] = \mathbb{E}[\sum_{i=1}^m \psi_i \psi_i^t] / m = \Sigma, \quad \text{and}$$

$$\mathbb{E}[\Psi^t y / m] = \mathbb{E}[\sum_{i=1}^m \psi_i y_i] / m = \mathbb{E}[\tilde{y}\tilde{\psi}]. \quad (2.4)$$

As long as $\Psi^t \Psi / m$ is invertible, it is reasonable to expect that

$$x_{ls} = (\Psi^t \Psi / m)^{-1} (\Psi^t y / m) = (\Psi^t \Psi)^{-1} (\Psi^t y)$$

can approximate $x^*$ well up to a constant $c$ even if $y$ consists of sign flips.

**Theorem 2.2.** Consider the ordinary least squares:

$$x_{ls} \in \arg \min_x \frac{1}{m} \|\Psi x - y\|^2_2. \quad (2.5)$$

If $m \geq 16C_2^2 n$, then with probability at least $1 - 4 \exp(-C_1 C_2^2 n) - \frac{2}{n}$,

$$\|x_{ls} / c - x^*\|^2_2 \leq \sqrt{\frac{n}{m}} (4C_2 \sqrt{\kappa(\Sigma) \gamma_{\max}(\Sigma)} + \frac{6(\sigma + 1)}{\sqrt{C_1^2 |2q - 1|} \sqrt{\log n}}), \quad (2.6)$$

where $C_1$ and $C_2$ are some generic constants not depending on $m$ or $n$.

**Proof.** The proof is given in Appendix C. \[\square\]

**Remark 2.1.** Theorem (2.2) shows that, $\forall \delta \in (0, 1)$ if $m \geq \tilde{O}(\frac{n}{\delta^2})$, up to a constant, the simple least squares solution can approximate $x^*$ with error of order $\delta$ even if $y$ contains very large quantization error and sign flips with probability unequal to $1/2$.

**Remark 2.2.** To the best of our knowledge, this is the first nonasymptotic error bound for the 1-bit CS in the overdetermined setting. Comparing with the estimation error of the standard least squares in the complete data model $y = \Psi x^* + \epsilon$, the error bound in Theorem 2.2 is optimal up to a logarithm factor $\sqrt{\log n}$, which is due to the loss of information with the 1-bit quantization.

3. Sparse decoding with $\ell_1$-regularized least squares.
3.1. Nonasymptotic error bound. Since images and signals are often sparsely represented under certain transforms \([36, 15]\), it suffices for the standard CS to recover the sparse signal \(x^* \in \mathbb{R}^n\) with \(m = O(s \log n)\) measurements for \(s = \|x^*\|_0\). In this section we show that in the 1-bit CS setting, similar results can be derived through the \(\ell_1\)-regularized least squares (1.2). Model (1.2) has been extensively studied when \(y\) is continuous \([44, 9, 8, 16]\). Here we use model (1.2) to recover \(x^*\) from quantized \(y\), which is rarely studied in the literature.

Next we show that, up to the constant \(c\), \(\ell_1\) is a good estimate of \(x^*\) when \(m = O(s \log n)\) even if the signal is highly noisy and corrupted by sign flips in the 1-bit CS setting.

**Theorem 3.1.** Assume \(n > m \geq \max\{4C_1 C_2^2 \log n, 64(4 \kappa(\Sigma) + 1)^2 s \log \frac{n}{s}\}\), \(s \leq \exp(-C_1^2)\). Set \(\lambda = \frac{4(1 + C_3 |c|)}{\sqrt{C_1}} \sqrt{\frac{\log n}{m}}\). Then with probability at least \(1 - \frac{2}{n^3} - \frac{6}{n^2}\), we have,

\[
\|x_{\ell_1}/c - x^*\|_2 \leq \frac{816(4\kappa(\Sigma) + 1)^2 \sigma + 1 + C_3 |q - 1/2|}{\sqrt{\gamma_{\min}(\Sigma)}} \sqrt{\frac{s \log n}{m}}.
\]

**Proof.** The proof is given in Appendix D. \(\square\)

**Remark 3.1.** Theorem 3.1 shows that, \(\forall \delta \in (0, 1)\), if \(m \geq O\left(\frac{s \log n}{\delta^2}\right)\), up to a constant \(c\), the \(\ell_1\)-regularized least squares solution can approximate \(x^*\) with precision \(\delta\).

**Remark 3.2.** The error bound in Theorem 3.1 achieves the minimax optimal order \(O(\sqrt{s \log n/m})\) in the sense that it is the optimal order that can be attained even if the signal is measured precisely without 1-bit quantization \([37]\). From Theorem 3.1 if the minimum nonzero magnitude of \(x^*\) is large enough, i.e., \(|x^*|_{\min} \geq O(\sqrt{s \log n/m})\), the support of \(x_{\ell_1}\) coincides with that of \(x^*\).

3.2. Comparison with related works. Assuming \(\|x^*\|_2 = 1\) and \(\sigma = 0\) and \(q = 1\), \([6]\) proposed to decode \(x^*\) with

\[
\min_{x \in \mathbb{R}^n} \|x\|_1 \quad \text{s.t.} \quad y \odot \Psi x \geq 0, \quad \|x\|_2 = 1.
\]

A first order algorithm was devised to solve the following Lagrangian version \([34]\), i.e.,

\[
\min_{x \in \mathbb{R}^n} \|\max\{0, -y \odot \Psi x\\}\|_2^2 + \lambda \|x\|_1 \quad \text{s.t.} \quad \|x\|_2 = 1.
\]

In the presence of noise, \([29]\) introduced

\[
\min_{x \in \mathbb{R}^n} \mathcal{L}(\max\{0, -y \odot \Psi x\}) \quad \text{s.t.} \quad \|x\|_0 \leq s, \quad \|x\|_2 = 1,
\]

where \(\mathcal{L}(\cdot) = \|\cdot\|_1\) or \(\|\cdot\|_2^2\). They used a projected sub-gradient method, the so-called binary iterative hard thresholding (BITH), to solve (3.2). Assuming that there are sign flips in the noiseless model with \(\sigma = 0\), \([14]\) considered

\[
\min_{x \in \mathbb{R}^n} \lambda \|\max\{0, \nu \mathbf{1} - y \odot \Psi x\\}\|_0 + \frac{\beta}{2} \|x\|_2^2 \quad \text{s.t.} \quad \|x\|_0 \leq s,
\]

where \(\nu = \frac{\gamma_{\min}(\Sigma)}{\sqrt{C_1}}\). They used a projected sub-gradient method, the so-called binary iterative hard thresholding (BITH), to solve (3.3). Assuming that there are sign flips in the noiseless model with \(\sigma = 0\), \([14]\) considered

\[
\min_{x \in \mathbb{R}^n} \lambda \|\max\{0, \nu \mathbf{1} - y \odot \Psi x\\}\|_0 + \frac{\beta}{2} \|x\|_2^2 \quad \text{s.t.} \quad \|x\|_0 \leq s,
\]

where \(\nu = \frac{\gamma_{\min}(\Sigma)}{\sqrt{C_1}}\). They used a projected sub-gradient method, the so-called binary iterative hard thresholding (BITH), to solve (3.3). Assuming that there are sign flips in the noiseless model with \(\sigma = 0\), \([14]\) considered

\[
\min_{x \in \mathbb{R}^n} \lambda \|\max\{0, \nu \mathbf{1} - y \odot \Psi x\\}\|_0 + \frac{\beta}{2} \|x\|_2^2 \quad \text{s.t.} \quad \|x\|_0 \leq s,
\]

where \(\nu = \frac{\gamma_{\min}(\Sigma)}{\sqrt{C_1}}\). They used a projected sub-gradient method, the so-called binary iterative hard thresholding (BITH), to solve (3.3). Assuming that there are sign flips in the noiseless model with \(\sigma = 0\), \([14]\) considered

\[
\min_{x \in \mathbb{R}^n} \lambda \|\max\{0, \nu \mathbf{1} - y \odot \Psi x\\}\|_0 + \frac{\beta}{2} \|x\|_2^2 \quad \text{s.t.} \quad \|x\|_0 \leq s,
\]

where \(\nu = \frac{\gamma_{\min}(\Sigma)}{\sqrt{C_1}}\). They used a projected sub-gradient method, the so-called binary iterative hard thresholding (BITH), to solve (3.3). Assuming that there are sign flips in the noiseless model with \(\sigma = 0\), \([14]\) considered

\[
\min_{x \in \mathbb{R}^n} \lambda \|\max\{0, \nu \mathbf{1} - y \odot \Psi x\\}\|_0 + \frac{\beta}{2} \|x\|_2^2 \quad \text{s.t.} \quad \|x\|_0 \leq s,
\]
where $\nu > 0$, $\beta > 0$ are tuning parameters. An adaptive outlier pursuit (AOP) generalization of (3.2) was proposed in [49] to recover $x^*$ and simultaneously detect the entries with sign flips by

$$
\min_{x \in \mathbb{R}^n, \Lambda \in \mathbb{R}^m} \mathcal{L}(\max\{0, -\Lambda \odot y \odot \Psi x\}) \quad \text{s.t.} \quad \Lambda_i \in \{0, 1\}, \quad \|1 - \Lambda\|_1 \leq N, \quad \|x\|_0 \leq s, \quad \|x\|_2 = 1,
$$

where $N$ is the number of sign flips. Alternating minimization on $x$ and $\Lambda$ are adopted to solve the optimization problem. [24] considered the pinball loss to deal with both the noise and the sign flips, which reads

$$
\min_{x \in \mathbb{R}^n} \mathcal{L}_\tau (\nu 1 - y \odot \Psi x) \quad \text{s.t.} \quad \|x\|_0 \leq s \quad \|x\|_2 = 1,
$$

where $\mathcal{L}_\tau(t) = \tau 1_{t \geq 0} - \tau 1_{t < 0}$. Similar to the BITH, the pinball iterative hard thresholding is utilized. With the binary stable embedding, [29] and [14] proved that with high probability, the sample complexity of (3.2) and (3.3) to guarantee estimation error smaller than $\delta$ is $m \geq \mathcal{O}(s \log \frac{n}{s})$, which echoes Theorem 3.1. However, there are no theoretical results for other models mentioned above. All the aforementioned models and algorithms are the state-of-the-art works in the 1-bit CS.

Another line of research concerns convexification. The pioneering work is [40], where they considered the noiseless case without sign flips and proposed the following linear programming method

$$
x_{lp} \in \arg \min_{x \in \mathbb{R}^n} \|x\|_1 \quad \text{s.t.} \quad y \odot \Psi x \geq 0 \quad \|\Psi x\|_1 = m.
$$

As shown in [40], the estimation error is $\|x_{lp} - x^*\| \leq \mathcal{O}((s \log \frac{n}{s})^\frac{1}{2})$. The above result is improved to $\|x_{cv} - x^*\| \leq \mathcal{O}((s \log \frac{n}{s})^\frac{1}{4})$ in [41], where both the noise and the sign flips are allowed, through considering the convex problem

$$
x_{cv} \in \arg \min_{x \in \mathbb{R}^n} -(y, \Psi x)/m \quad \text{s.t.} \quad \|x\|_1 \leq s, \quad \|x\|_2 \leq 1.
$$

Comparing with our result in Theorem 3.1, the results derived in [40] and [41] are suboptimal.

In the noiseless case and assuming $\Sigma = I_n$, [50] considered the Lagrangian version of (3.4)

$$
\min_{x \in \mathbb{R}^n} -(y, \Psi x)/m + \lambda \|x\|_1 \quad \text{s.t.} \quad \|x\|_2 \leq 1.
$$

In this special case, the estimation error derived in [50] improved that of [41] and matched our results in Theorem 3.1. However, [50] did not discuss the scenario of $\Sigma \neq I_n$. Recently [42, 47], proposed a simple projected linear estimator $P_K(\Psi y/m)$, where $K = \{x \mid \|x\|_1 \leq s, \|x\|_2 \leq 1\}$, to estimate the low-dimensional structure target belonging to $K$ in high dimensions from noisy and possibly nonlinear observations. They derived the same order of estimation error as that in Theorem 3.1.

[51] proposed an $\ell_1$ regularized maximum likelihood estimate, and [24] introduced a convex model through replacing the linear loss in (3.5) with the pinball loss. However, neither studied sample complexity or estimation error.
4. Primal dual active set algorithm. Existing algorithms for (1.2) are mainly first order methods [45, 2, 12]. In this section we use primal dual active set method [18, 30], which is a generalized Newton type method, [27, 43] to solve (1.2). An important advantage of the PDAS is that it converges after one-step iteration if the initial value is good enough. We globalize it with continuation on regularization parameter. We also propose a novel regularization parameter selection rule which is incorporated along the continuation procedure without any additional computational burden.

4.1. PDAS. In this section we use $x$ to denote $x_\ell$ for simplicity. We begin with the following results [13]. Let $x$ be the minimizer of (1.2), then $x$ satisfies

$$\begin{cases}
  d = \Psi^t(y - \Psi x)/m, \\
  x = S_{\lambda}(x + d).
\end{cases}$$

(4.1)

Conversely, if $x$ and $d$ satisfy (4.1), then $x$ is a global minimizer of (1.2), where $S_{\lambda}(z)$ is the pointwise soft-thresholding operator defined by $S_{\lambda}(z_i) = \text{arg min}_{t \in \mathbb{R}} t^2 + \lambda |t|_1 = \text{sign}(z_i) \max(|z_i| - \lambda, 0)$.

Let $Z = \begin{pmatrix} x \\ d \end{pmatrix}$ and $F(Z) = \begin{pmatrix} F_1(Z) \\ F_2(Z) \end{pmatrix} : \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}^{2n}$, where $F_1(Z) = x - S_{\lambda}(x + d)$ and $F_2(Z) = \Psi^t \Psi x + md - \Psi^t y$. By (4.1), finding the minimizer $x$ of (1.2) is equivalent to finding the root of $F(Z)$. We use the following primal dual active set method (PDAS) [18, 30] to find the root of $F(Z)$.

Algorithm 1 PDAS: $x_\lambda \leftarrow \text{pdas}(y, \Psi, \lambda, x^0, \text{MaxIter})$

1: Input $y, \Psi, \lambda$, initial guess $x^0$, maximum number of iteration MaxIter. Let $d^0 = \Psi^t(y - \Psi x^0)/m$.
2: for $k = 0, 1, \ldots, \text{MaxIter}$ do
3:   Compute the active and inactive sets $A_k$ and $I_k$ respectively by $A_k = \{ i \in [n] | |x_i^k + d_i^k| > \lambda \}$ and $I_k = \overline{A_k}$.
4:   Update $x^{k+1}$ and $d^{k+1}$ by
5:   if $A_k = A_{k+1}$, stop.
6:   end for
7: Output $x_\lambda$.

Remark 4.1. We can stop when $k$ is greater than a user-predefined MaxIter. Since the PDAS converges after one iteration, a desirable property stated in Theorem 4.1, we set MaxIter = 1.

The PDAS is actually a generalized Newton method for finding roots of nonsmooth equations
If the columns of $x$ to (1.2), we need to have an initial guess $\tilde{x}$ for completeness, which is proved in [18].

The PDAS require one iteration to convergence. We state the results here for completeness. We prove this equivalency in Appendix E for completeness.

\[ J^k D^k = -F(Z^k), \quad Z^{k+1} = Z^k + D^k, \quad (4.2) \]

where

\[
J^k = \begin{pmatrix}
J_1^k & J_2^k \\
\Psi^T \Psi & m I
\end{pmatrix}, \quad J_1^k = \begin{pmatrix} 0_{A_k, A_k} & 0_{A_k, I_k} \\ 0_{I_k, A_k} & I_k, I_k \end{pmatrix} \quad \text{and} \quad J_2^k = \begin{pmatrix} -I_{A_k, A_k} & 0_{A_k, I_k} \\ 0_{I_k, A_k} & 0_{I_k, I_k} \end{pmatrix}. \quad (4.4)
\]

We prove this equivalency in Appendix E for completeness.

Local superlinear convergence has been established for generalized Newton methods for nonsmooth equations [27, 43]. The PDAS require one iteration to convergence. We state the results here for completeness, which is proved in [18].

**Theorem 4.1.** Let $x$ and $d$ satisfy (4.1). Denote $\tilde{A} = \{i \in [n] | |x_i + d_i| \geq \lambda \}$ and $\omega = \min_{i \in [n], |x_i + d_i| \neq \lambda} \{|x_i + d_i| - \lambda\}$. Let $x^0$ and $d^0 = \Psi^T(y - \Psi x^0)/m$ be initial input of Algorithm 1. If the columns of $\Psi A$ are full-rank and the initial input satisfies $\|x - x^0\|_\infty + \|d - d^0\|_\infty \leq \omega$. Then, $x^1 = x$, where $x^1$ is updated from $x^0$ after one iteration.

### 4.2. Globalization and automatic regularization parameter selection.

To apply the PDAS (Algorithm 1) to (1.2), we need to have an initial guess $x^0$ and specify a proper regularization parameter $\lambda$ in $\text{pdas}(y, \Psi, \lambda, x^0, \text{MaxIter})$. In this section, we address these two issues together with continuation. Since the PDAS is a Newton type algorithm with fast local convergence rate and $x_{k+1}$ is piecewise linear function of $\lambda$ [39], we adopt a continuation to fully exploit the fast local convergence.

In particular, this is a simple way to globalize the convergence of PDAS [18]. Observing that $x = 0$ satisfies (4.1) if $\lambda \geq \lambda_0 = \|\Psi^T y/m\|_\infty$, we define $\lambda_t = \lambda_0 \rho^t$ with $\rho \in (0, 1)$ for $t = 1, 2, \ldots$. We run Algorithm 1 on the sequence $\{\lambda_t\}$ with warmstart, i.e., using the solution $x_{\lambda_t}$ as an initial guess for the $\lambda_{t+1}$-problem. When the whole continuation is done we obtain a solution path of (1.2). For simplicity, we refer to the PDAS with continuation as PDASC described in Algorithm 2.

**Algorithm 2** PDASC: $\{x_{\lambda_t}\}_{t \in [\text{MaxGrid}]} \leftarrow \text{pdasc}(y, \Psi, \lambda_0, x_{\lambda_0}, \rho, \text{MaxGrid}, \text{MaxIter})$

1. Input, $y, \Psi, \lambda_0 = \|\Psi^T y/m\|_\infty$, $x_{\lambda_0} = 0$, $\rho \in (0, 1)$, MaxGrid, MaxIter.
2. for $t = 1, 2, \ldots, \text{MaxGrid}$ do
3. Run algorithm 1 $x_{\lambda_t} \leftarrow \text{pdas}(y, \Psi, \lambda, x^0, \text{MaxIter})$ with $\lambda = \lambda_t = \rho^t \lambda_0$, initialized with $x^0 = x_{\lambda_{t-1}}$.
4. Check the stopping criterion.
5. end for
6. Output, $\{x_{\lambda_t}\}_{t \in [\text{MaxGrid}]}$.

The regularization parameter $\lambda$ in the $\ell_1$-regularized 1-bit CS model (1.2), which compromises the tradeoff between data fidelity and the sparsity level of the solution, is important for theoretical analysis and practical computation. However, the well known regularization parameter selection rules such as discrepancy principle [17, 25], balancing principle [10, 31, 11, 26] or Bayesian information criterion [18, 33], are not applicable to the 1-bit CS problem considered here, since the model errors are not
available directly. Here we propose a novel rule to select regularization parameter automatically. We run the PDASC to yield a solution path until $\|x_{\lambda_t}\|_0 > \lfloor \frac{m}{\log n} \rfloor$ for the smallest possible $T$. Let $S_\ell = \{\lambda_t : \|x_{\lambda_t}\|_0 = \ell, t = 1, ..., T\}, \ell = 1, ..., \lfloor \frac{m}{\log n} \rfloor$ be the set of regularization parameter at which the output of PDAS has $\ell$ nonzero elements. We determine $\lambda$ by voting, i.e.,

$$\hat{\lambda} = \max \{S_\ell\} \quad \text{and} \quad \ell = \arg\max_{\ell} \{|S_\ell|\}.$$  

Therefore, our parameter selection rule is seamlessly integrated with the continuation strategy which serves as a globalization technique without any extra computational overhead.

Here we give an example to show the accuracy of our proposed regularization parameter selection rule (4.5) with data $\{m = 400, n = 10^3, s = 5, \nu = 0.5, \sigma = 0.01, q = 2.5\%\}$. Descriptions of the data can be found in Section 5. Left panel of Fig. 4.1 shows the size of active set $\|x_{\lambda_t}\|_0$ along the path of PDASC and right panel shows the underlying true signal $x^*$ and the solution $x_{\hat{\lambda}}$ selected by (4.5).

5. Numerical simulation. In this section we showcase the performance of our proposed least square decoders (2.5) and (1.2). All the computations were performed on a four-core laptop with 2.90 GHz and 8 GB RAM using MATLAB 2015b. The MATLAB package 1-bitPDASC for reproducing all the numerical results can be found at http://faculty.zuel.edu.cn/tjyjxxy/jyl/list.htm.

5.1. Experiment setup. First we describe the data generation process and our parameter choice. In all numerical examples the underlying target signal $x^*$ with $\|x^*\|_0 = s$ is given, and the observation $y$ is generated by $y = \eta \odot \text{sign}(\Psi x^* + \epsilon)$, where the rows of $\Psi$ are iid samples from $\mathcal{N}(0, \Sigma)$ with $\Sigma_{jk} = \nu^{|j-k|}, 1 \leq j, k \leq n$. We keep the convention $0^0 = 1$. The elements of $\epsilon$ are generated from $\mathcal{N}(0, \mathbf{I}_m)$, $\eta \in \mathcal{R}^m$ has independent coordinate $\eta_i$ with $\mathbb{P}[\eta_i = 1] = 1 - \mathbb{P}[\eta_i = -1] = q$. Here, we use $\{m, n, s, \nu, \sigma, q\}$ to denote the data generated as above for short. We fix $\rho = 0.95, \text{MaxGrid} = 200, \text{MaxIter} = 1$ in our proposed PDASC algorithm and use (4.5) to determine regularization parameter $\lambda$. All the simulation results are based on 100 independent replications.

5.2. Accuracy and Robustness of $x_{ls}$ when $m > n$. Now we present numerical results to illustrate the accuracy of the least square decoder $x_{ls}$ and its robustness to the noise and the sign
flips. Fig. 5.1 shows the recovery error $\|x_{ls} - x^*\|$ on data set $\{m = 10^3, n = 10, s = 10, \nu = 0.3, \sigma = 0 : 0.05 : 0.5, q = 2.5\%\}$. Left panel of Fig. 5.2 shows the recovery error $\|x_{ls} - x^*\|$ on data set $\{m = 1000, n = 10, s = 10, \nu = 0.3, \sigma = 0.01, q = 0 : 1\% : 10\%\}$ and right panel gives recovery error $\|x_{ls} + x^*\|$ on data $\{m = 1000, n = 10, s = 10, \nu = 0.3, \sigma = 0.01, q = 90 : 1\% : 100\%\}$. It is observed that the recovery error $\|x_{ls} - x^*\|$ ($\|x_{ls} + x^*\|$) of the least square decoder is small (around 0.1) and robust to noise level $\sigma$ and sign flips probability $q$. This confirms theoretically investigations in Theorem 2.2, which states the error is of order $\tilde{O}(\sqrt{m}) = 0.1$.

![Fig. 5.1: Recovery error $\|x_{ls} - x^*\|$ v.s. $\sigma$ on $\{m = 1000, n = 10, s = 10, \nu = 0.3, \sigma = 0 : 0.05 : 0.5, q = 2.5\%\}$](image1)

![Fig. 5.2: Recovery error $\|x_{ls} - x^*\|$ v.s. $q$ on $\{m = 10^3, n = 10, s = 10, \nu = 0.3, \sigma = 0.01, q = 0 : 1\% : 10\%\}$ (left panel) and $\| - x_{ls} - x^*\|$ on $\{m = 1000, n = 10, s = 10, \nu = 0.3, \sigma = 0.01, q = 90 : 1\% : 100\%\}$ (right panel).](image2)

### 5.3. Support recovery of $x_{\ell_1}$ when $m < n$. We conduct simulations to illustrate the performance of model (1.2) PDASC algorithm. We report how the exact support recovery probability varies with the sparsity level $s$, the noise level $\sigma$ and the probability $q$ of sign flips. Fig. 5.3 indicates that, as long as the sparsity level $s$ is not large, $x_{\ell_1}$ recovers the underlying true support with high probability even if the measurement contains noise and is corrupted by sign flips. This confirms the theoretical investigations in Theorem 3.1.
5.4. Comparison with other state-of-the-art. Now we compare our proposed model (1.2) and PDASC algorithm with several state-of-the-art methods such as BIHT [28] (http://perso.uclouvain.be/laurent.jacques/index.php/Main/BIHTDemo), AOP [49] and PBAOP [24] (both AOP and PBAOP available at http://www.esat.kuleuven.be/stadius/ADB/huang/downloads/1bitCSLab.zip) and linear projection (LP) [47, 42]. BIHT, AOP, LP and PBAOP are all required to specify the true sparsity level $s$. Both AOP and PBAOP also need to required to specify the sign flips probability $q$. The PDASC does not require to specify the unknown parameter sparsity level $s$ or the probability of sign flips $q$. We use \{$m = 500, n = 1000, s = 5, \nu = 0.1, \sigma = 0, q = 0\}$, \{$m = 500, n = 1000, s = 5, \nu = 0.1, \sigma = 0.5, q = 5\%\}$, \{$m = 500, n = 1000, s = 5, \nu = 0.5, \sigma = 0.5, q = 10\%\}$, \{$m = 500, n = 1000, s = 5, \nu = 0.1, \sigma = 0.1, q = 1\%\}$, \{$m = 800, n = 2000, s = 10, \nu = 0.1, \sigma = 0.3, q = 3\%\}$, \{$m = 800, n = 2000, s = 10, \nu = 0.3, \sigma = 0.5, q = 5\%\}$, and \{$m = 5000, n = 20000, s = 50, \nu = 0, \sigma = 0.2, q = 3\%\}$, \{$m = 5000, n = 20000, s = 50, \nu = 0, \sigma = 0.1, q = 1\%\}$, \{$m = 5000, n = 20000, s = 5, \nu = 0, \sigma = 0.3, q = 5\%\}$. The average CPU time in seconds (Time (s)), the average of the $\ell_2$ error $\|x_\ell - x^\star\|$ ($\ell_2$-Err), and the probability of exactly recovering true support (PrE (%)) are reported in Table 5.1. The PDASC is comparatively very fast and the most accurate
Table 5.1: Comparison PDASC with state-of-the-art methods on CPU time in seconds (Time (s)), average $\ell_2$ error $\|x^* - x\|$ ($\ell_2$-Err), probability on exactly recovering of true support (PrE (%)).

| Method     | Time (s) | $\ell_2$-Err | PrE (%) |
|------------|----------|---------------|---------|
| BIHT       | 1.31e-1  | 5.73e-1       | 19      |
| AOP        | 3.58e-1  | 4.22e-1       | 44      |
| LP         | 8.30e-3  | 4.81e-1       | 26      |
| PBAOP      | 1.35e-1  | 4.53e-1       | 36      |
| PDASC      | 4.56e-1  | 2.21e-1       | 71      |

$\{m = 500, n = 1000, s = 5\}$

| Method     | Time (s) | $\ell_2$-Err | PrE (%) |
|------------|----------|---------------|---------|
| BIHT       | 1.31e-1  | 5.73e-1       | 19      |
| AOP        | 3.58e-1  | 4.22e-1       | 44      |
| LP         | 8.30e-3  | 4.81e-1       | 26      |
| PBAOP      | 1.35e-1  | 4.53e-1       | 36      |
| PDASC      | 4.56e-1  | 2.21e-1       | 71      |

$\{m = 800, n = 2000, s = 10\}$

| Method     | Time (s) | $\ell_2$-Err | PrE (%) |
|------------|----------|---------------|---------|
| BIHT       | 1.31e-1  | 5.73e-1       | 19      |
| AOP        | 3.58e-1  | 4.22e-1       | 44      |
| LP         | 8.30e-3  | 4.81e-1       | 26      |
| PBAOP      | 1.35e-1  | 4.53e-1       | 36      |
| PDASC      | 4.56e-1  | 2.21e-1       | 71      |

$\{m = 5000, n = 20000, s = 50, \nu = 0\}$

| Method     | Time (s) | $\ell_2$-Err | PrE (%) |
|------------|----------|---------------|---------|
| BIHT       | 1.31e-1  | 5.73e-1       | 19      |
| AOP        | 3.58e-1  | 4.22e-1       | 44      |
| LP         | 8.30e-3  | 4.81e-1       | 26      |
| PBAOP      | 1.35e-1  | 4.53e-1       | 36      |
| PDASC      | 4.56e-1  | 2.21e-1       | 71      |

Table 5.2: The CPU time in seconds and the PSNR of one dimensional signal recovery with $\{m = 2500, n = 8000, s = 36, \nu = 0, \sigma = 0.5, q = 4\%\}$.

| method     | CPU time (s) | PSNR |
|------------|--------------|------|
| BIHT       | 4.97         | 29   |
| AOP        | 4.98         | 33   |
| LP         | 0.11         | 33   |
| PBAOP      | 4.93         | 31   |
| PDASC      | 3.26         | 36   |

Table 5.1: Comparison PDASC with state-of-the-art methods on CPU time in seconds (Time (s)), average $\ell_2$ error $\|x^* - x\|$ ($\ell_2$-Err), probability on exactly recovering of true support (PrE (%)).

| Method     | Time (s) | $\ell_2$-Err | PrE (%) |
|------------|----------|---------------|---------|
| BIHT       | 1.31e-1  | 5.73e-1       | 19      |
| AOP        | 3.58e-1  | 4.22e-1       | 44      |
| LP         | 8.30e-3  | 4.81e-1       | 26      |
| PBAOP      | 1.35e-1  | 4.53e-1       | 36      |
| PDASC      | 4.56e-1  | 2.21e-1       | 71      |

$\{m = 500, n = 1000, s = 5\}$

$\{m = 800, n = 2000, s = 10\}$

$\{m = 5000, n = 20000, s = 36, \nu = 0\}$

even if the correlation $\nu$, the noise level $\sigma$ and the probability of sign flips $q$ are large.

Now we compare the PDASC with the aforementioned competitors to recover a one-dimensional signal. The true signal are sparse under wavelet basis “Db1” [36]. Thus, the matrix $\Psi$ is of size $2500 \times 8000$ and consists of random Gaussian and an inverse of one level Harr wavelet transform. The target coefficient has 36 nonzeros. We set $\sigma = 0.5$, $q = 4\%$. The recovered results are shown in Fig. 5.4 and Table 5.2. The reconstruction by the PHDAS is visually more appealing than others, as shown in Fig. 5.4. This is further confirmed by the PSNR value reported in Table 5.2, which is defined by $\text{PSNR} = 10 \cdot \log \frac{V^2}{\text{MSE}}$, where $V$ is the maximum absolute value of the true signal, and MSE is the mean squared error of the reconstruction.

Table 5.2: The CPU time in seconds and the PSNR of one dimensional signal recovery with $\{m = 2500, n = 8000, s = 36, \nu = 0, \sigma = 0.5, q = 4\%\}$.

| method     | CPU time (s) | PSNR |
|------------|--------------|------|
| BIHT       | 4.97         | 29   |
| AOP        | 4.98         | 33   |
| LP         | 0.11         | 33   |
| PBAOP      | 4.93         | 31   |
| PDASC      | 3.26         | 36   |

6. Conclusions. In this paper we consider decoding from 1-bit measurements with noise and sign flips. For $m > n$, we show that, up to a constant $c$, with high probability the least squares
solution $x_{ls}$ approximates $x^*$ with precision $\delta$ as long as $m \geq \tilde{O}\left(\frac{n}{\delta^2}\right)$. For $m < n$, we assume that the underlying target $x^*$ is $s$-sparse, and prove that up to a constant $c$, with high probability, the $l_1$-regularized least squares solution $x_{l_1}$ lies in the ball with center $x^*$ and radius $\delta$, provided that $m \geq O\left(\frac{s \log n}{\delta^2}\right)$. We introduce the one-step convergent PDAS method to minimize the nonsmooth objection function. We propose a novel tuning parameter selection rule which is seamlessly integrated with the continuation strategy without any extra computational overhead. Numerical experiments are presented to illustrate salient features of the model and the efficiency and accuracy of the algorithm.

There are several avenues for further study. First, many practitioners observed that nonconvex sparse regularization often brings in additional benefit in the standard CS setting. Whether the theoretical and computational results derived in this paper can still be justified when nonconvex regularizers are used deserves further consideration. The 1-bit CS is a kind of nonlinear sampling approach. Analysis of some other nonlinear sampling methods are also of immense interest.

Acknowledgements. The research of Y. Jiao is supported by National Science Foundation of China (NSFC) No. 11501579 and National Science Foundation of Hubei Province No. 2016CFB486. The research of X. Lu is supported by NSFC Nos. 11471253 and 91630313, and the research of L.
The proof is completed by inverting exponential random variables.

Appendix A. Proof of Lemma 2.1.

Proof. Let $u = \tilde{\psi}^t x^*$. Then $u \sim \mathcal{N}(0, 1)$ due to $\tilde{\psi} \sim \mathcal{N}(0, \Sigma)$ and the assumption that $\|x^*\|_2 = 1.$

$$
E[\tilde{\psi} y] = E[\tilde{\psi} \eta \text{sign}(\tilde{\psi}^t x^* + \tilde{\epsilon})] = E[\eta] E[\text{sign}(\tilde{\psi}^t x^* + \tilde{\epsilon})]
$$
$$
= |q - (1 - q)|E[\text{sign}(\tilde{\psi}^t x^* + \tilde{\epsilon})]
$$
$$
= (2 - 1)E[\tilde{\psi} \text{sign}(\tilde{\psi}^t x^* + \tilde{\epsilon})] = (2q - 1)E[\tilde{\psi} \text{sign}(\tilde{\psi}^t x^* + \tilde{\epsilon})]
$$
$$
= (2q - 1)E[\tilde{\psi} \text{sign}(\tilde{\psi}^t x^* + \tilde{\epsilon})] E[\text{sign}(\tilde{\psi}^t x^* + \tilde{\epsilon})] E[\text{sign}(\tilde{\psi}^t x^* + \tilde{\epsilon})]
$$
$$
= (2q - 1)E[\text{sign}(\tilde{\psi}^t x^* + \tilde{\epsilon})]\tilde{\psi}^t x^* E[\text{sign}(\tilde{\psi}^t x^* + \tilde{\epsilon})]
$$
$$
= (2q - 1)E[\text{sign}(\tilde{\psi}^t x^* + \tilde{\epsilon})]\tilde{\psi}^t x^* E[\text{sign}(\tilde{\psi}^t x^* + \tilde{\epsilon})]
$$

where $c = (2q - 1)E[\text{sign}(\tilde{\psi}^t x^* + \tilde{\epsilon})]\tilde{\psi}^t x^*$. The second line follows from independence assumption and the third from law of total expectation, and the fourth and fifth lines are due to the independence between $\tilde{\epsilon}$ and $u$, and the sixth line uses the projection interpretation of conditional expectation i.e., $E[\tilde{\psi} | \tilde{\psi}^t x^*] = \frac{E[\tilde{\psi} \tilde{\psi}^t x^*] - E[\tilde{\psi}^t x^*] E[\tilde{\psi}]}{E[(\tilde{\psi}^t x^* - E[\tilde{\psi}^t x^*])^2]} + E[\tilde{\psi}] = \Sigma x^* \tilde{\psi}^t x^*$, where we use $E[\tilde{\psi}] = 0$ and $u \sim \mathcal{N}(0, 1)$. Let $f(t) = \frac{1}{\sqrt{2\pi\sigma}} \exp \left\{-\frac{t^2}{2\sigma^2}\right\}$ be the density function of $\tilde{\epsilon} \sim \mathcal{N}(0, \sigma^2)$. Integrating by parts shows that

$$
c = (2q - 1)E[\text{sign}(u + \tilde{\epsilon})]u
$$
$$
= (2q - 1)E[\left(1 - \text{P}[\tilde{\epsilon} \leq -u]\right) u]
$$
$$
= (2q - 1)E[\left(1 - \text{P}[\tilde{\epsilon} \leq -u]\right) \frac{\partial}{\partial u}\frac{2}{\pi (\sigma^2 + 1)}]
$$

The proof is completed by inverting $c\Sigma$. □

Appendix B. Preliminaries. We recall some simple properties of subgaussian and subexponential random variables.

**Lemma B.1.** (Lemma 2.7.7 of [48] and Remark 5.18 of [46].) Let $\xi_1$ and $\xi_2$ be subgaussian random variables. Then both $\xi_1 \xi_2$ and $\xi_1 \xi_2 - \mathbb{E}[\xi_1 \xi_2]$ are subexponential random variables.

Lemma B.2 states the nonasymptotic bound on the spectrums of $\Psi$ and the operator norm of $\Psi^t \Psi/m - \Sigma$ when $m \geq O(n)$.

**Lemma B.2.** Let $\Psi \in \mathbb{R}^{m \times n}$ whose rows $\psi_i^t$ are independent subgaussian vectors in $\mathbb{R}^n$ with mean 0 and covariance matrix $\Sigma$. Let $m > n$. Then for every $t > 0$ with probability at least
\(1 - 2 \exp(-C_1 t^2)\), one has

\[
(1 - \tau) \sqrt{\gamma_{\min}(\Sigma)} \leq \sqrt{\gamma_{\min}(\frac{\Psi^t \Psi}{m})} \leq \sqrt{\gamma_{\max}(\frac{\Psi^t \Psi}{m})} \leq (1 + \tau) \sqrt{\gamma_{\max}(\Sigma)}, \tag{B.1}
\]

and

\[
\|\Psi^t \Psi/m - \Sigma\| \leq \max\{\tau, \tau^2\} \gamma_{\max}(\Sigma), \tag{B.2}
\]

where \(\tau = C_2 \sqrt{\frac{n}{m}} + \frac{t}{\sqrt{m}}\), and \(C_1, C_2\) are generic positive constants depending on the maximum subgaussian norm of rows of \(\Psi\).

\textbf{Proof.} Let \(\Phi = \Psi \Sigma^{-\frac{1}{2}}\). Then the rows of \(\Phi\) are independent sub-gaussian isotropic vectors. (B.1) follows from Theorem 5.39 and Lemma 5.36 of [46] and (B.2) is a direct consequence of Remark 5.40 of [46]. \(\square\)

We state the Bernstein-type inequality for the sum of independent and mean 0 sub-exponential random variables.

\textbf{Lemma B.3.} (Corollary 5.17 of [46]) Let \(\xi_1, \ldots, \xi_m\) be independent centered sub-exponential random variables. Then for every \(t > 0\) one has

\[
\mathbb{P}[\|\sum_{i=1}^{m} \xi_i/m\| \geq t] \leq 2 \exp\left(-\min\{C_1 t^2, C_2 t\}m\right)
\]

where \(C_1\) and \(C_2\) are generic positive constants depending on the maximum subexponential norm of \(\xi_i\).

\textbf{Lemma B.4.} Let \(\Psi \in \mathcal{R}^{m \times n}\) whose rows \(\psi_i\) are independent subgaussian vectors in \(\mathcal{R}^{n \times 1}\) with mean 0 and covariance matrix \(\Sigma\). Then, with probability at least \(1 - 2 \exp\left(-C_1 C_2^2 n\right)\),

\[
\|\Psi^t \Psi/m - \mathbb{E}[\Psi^t \Psi/m]\| \leq 2C_2 \gamma_{\max}(\Sigma) \sqrt{\frac{n}{m}}, \tag{B.3}
\]

as long as \(m \geq 4C_2^2 n\). Furthermore, if \(m > \frac{4C_1 C_2^2 \log n}{t^2}\), then

\[
\|\sum_{i=1}^{m} (\mathbb{E}[\psi_i y_i] - \psi_i y_i)/m\|_\infty \leq 2 \sqrt{\frac{\log n}{C_1 m}}, \tag{B.4}
\]

holds with probability at least \(1 - \frac{2}{n^2}\), and

\[
\|\Psi^t \Psi/m - \Sigma\|_\infty \leq 2 \sqrt{\frac{\log n}{C_1 m}}, \tag{B.5}
\]

holds with probability at least \(1 - \frac{1}{n^2}\).

\textbf{Proof.} By (2.3), \(\mathbb{E}[\Psi^t \Psi/m] = \Sigma\), hence (B.3) follows from (B.2) with \(t = C_2 \sqrt{m}\) and the assumption \(m \geq 4C_2^2 n\). Define \(G_{i,j} := g_i(\psi_j) \in \mathcal{R}^1, i = 1, \ldots, m, j = 1, \ldots, n\), which is subexponential by
Lemma B.1. Therefore,

\[
P[\| \sum_{i=1}^{m} (E[\psi_i y_i] - \psi_i y_i) / m \|_\infty \geq t] = \mathbb{P}[\bigcup_{j=1}^{n} \{ \sum_{i=1}^{m} G_{i,j} / m \geq t \}]
\leq \sum_{j=1}^{n} P[\| \sum_{i=1}^{m} G_{i,j}^{(i)} / m \geq t]
\leq n \exp(-\min\{C_1 t^2, C_2 t\} m)
\leq 2n \exp(-C_1 t^2 m),
\]

where the first inequality is due to the union bound, the second follows from Lemma B.3 and the last is because of restrictions \( t \leq \frac{C_2}{C_1} \) and \( m < n \). Then (B.4) follows from our assumption that \( m > \frac{4C_1}{C_2^2} \log n \) by setting \( t = 2 \sqrt{\frac{\log n}{C_1 m}} \) and. Let \( G_{j,k} = (\psi_j)_{(i)} - \Sigma_{j,k} \in \mathcal{R}^1, i = 1, \ldots, n, j = 1, \ldots, n, \ell = 1, \ldots, n, \) which is mean 0 subexponential by Lemma B.1. Therefore,

\[
P[\| \Psi t \Psi / m - \Sigma \|_\infty \geq t] = \mathbb{P}[\max_{j,k} \{ \sum_{i=1}^{m} G_{j,k} / m \} \geq t]
= \mathbb{P}[\bigcup_{j=1,k=1}^{n,n} \{ \sum_{i=1}^{m} G_{j,k}^{(i)} / m \} \geq t]
\leq \sum_{j=1,k=1}^{n,n} P[\| \sum_{i=1}^{m} G_{j,k}^{(i)} / m \geq t]
\leq n^2 \exp(-\min\{C_1 t^2, C_2 t\} m)
\leq n^2 \exp(-C_1 t^2 m),
\]

where the first inequality is due to the union bound, and the second follows from Lemma B.3 and the last inequality is because of restricting \( t \leq \frac{C_2}{C_1} \). Then by the assumption that \( m > \frac{4C_1}{C_2^2} \log n \), Lemma B.5 follows by setting \( t = 2 \sqrt{\frac{\log n}{C_1 m}} \).

**Appendix C. Proof of Theorem 2.2.**

**Proof.** First we show that the sample covariance matrix \( \Psi t \Psi / m \) is invertible with probability at least \( 1 - 2 \exp(-C_1 \frac{C_2^2 n}{m}) \) as long as \( m > 4C_2 n \). This follows from (B.1) in Lemma B.2 by setting \( t = C_2 \sqrt{n} \). Recall

\[
\tilde{x}^* = cx^*.
\]

Let

\[
\Delta = y - \Psi \tilde{x}^*.
\]

\[
\tilde{x}^* = cx^*.
\]

\[
\Delta = y - \Psi \tilde{x}^*,
\]

\[
\tilde{x}^* = cx^*.
\]
be the error in measuring nonlinearity, sign flips and noise in the 1-bit CS measurement. Then,
\[
\|\Psi^t \Delta/m\|_2 = \|\Psi^t (\Psi \hat{x}^* - y)/m\|_2 = \|\Psi^t \Psi \hat{x}^* - \Psi^t y/m\|_2 = \|\left(\Psi^t \Psi \hat{x}^* - \Sigma \hat{x}^*\right) + (\Sigma \hat{x}^* - \Psi^t y/m)\|_2 \\
= \|\left(\Psi^t \Psi \hat{x}^* - \mathbb{E}[\Psi^t \Psi \hat{x}^*]\right) + (\mathbb{E}[\Psi^t y/m] - \Psi^t y/m)\|_2 \\
\leq |e|\|x^*\|_2 \|\Psi^t \Psi/m - \mathbb{E}[\Psi^t \Psi/m]\| + \|\sum_{i=1}^m (\mathbb{E}[\psi_i y_i] - \psi_i y_i)/m\|_2 \\
\leq |e| \frac{1}{\sqrt{\gamma_{\min}(\Sigma)}} \|\Psi^t \Psi/m - \mathbb{E}[\Psi^t \Psi/m]\| + \sqrt{n} \|\sum_{i=1}^m (\mathbb{E}[\psi_i y_i] - \psi_i y_i)/m\|_\infty, \quad (C.3)
\]
where the fourth equality is due to (2.1), (2.3) and (2.4), the first inequality follows from the triangle inequality and the definition of \(\hat{x}^*\), and the last inequality uses the assumption \(1 = \|x^*\|_2^2 \geq \gamma_{\min}(\Sigma)\|x^*\|_2^2\) and the fact that \(\|\cdot\|_2 \leq \sqrt{n}\|\cdot\|_\infty\). Combining with (B.3) and (B.4), we deduce that, with probability at least \(1 - 2 \exp(-C_1 C_2^2 n - \frac{m}{n})\),
\[
\|\Psi^t \Delta/m\|_2 \leq \sqrt{\frac{n}{m}} \left(2 |e| C_2 \sqrt{\gamma_{\max}(\Sigma)} + \frac{\log n}{C_1}\right), \quad (C.4)
\]
Now we prove that \(\|x_{bs}/c - c x^*/2\|_2 = \tilde{O}(\sqrt{m}/c)\) with high probability.
\[
|e|\|x_{bs}/c - x^*/2\|_2 = \|x_{bs} - \tilde{x}^*/2\|_2 = \|\left(\Psi^t \Psi\right)^{-1} \Psi^t y - \tilde{x}^*/2\|_2 \\
\leq \|\left(\Psi^t \Psi/m\right)^{-1} \Psi^t (\Psi \hat{x}^* + y - \Psi \tilde{x}^*)/m - \tilde{x}^*/2\|_2 \\
= \|\left(\Psi^t \Psi/m\right)^{-1}\| \|\Psi^t \Delta/m\|_2 \\
\leq \sqrt{\frac{n}{m}} 2 |e| C_2 \sqrt{\gamma_{\max}(\Sigma)} + \sqrt{\frac{\log n}{C_1}} (1 - 2 C_2 \sqrt{\frac{n}{m}})^2 \\
\leq \sqrt{\frac{n}{m}} 4 |e| C_2 \sqrt{\gamma_{\max}(\Sigma)} + \sqrt{\frac{\log n}{C_1}},
\]
where the second inequality follows with probability at least \(1 - 4 \exp(-C_1 C_2^2 n - \frac{m}{n})\) from (C.3) and (B.1) by setting \(t = C_2 \sqrt{n}\), and the last line is due to the assumption \(m \geq 16 C_2^2 n\). Hence, the proof of Theorem 2.2 is completed by dividing \(|e|\) on both side and some algebra. \(\square\)

**Appendix D. Proof of Theorem 3.1.**

**Proof.** Our proof is based on Lemmas D.1 - D.3 below. Denote \(R = x_{t1} - \tilde{x}^*, A^* = \text{supp}(x^*)\) and \(I^* = \overline{A^*}\). The first lemma shows that \(R\) is sparse in the sense that its energy is mainly cumulated on \(A^*\) if \(\lambda\) is chosen properly.

**Lemma D.1.** Let
\[
C_{A^*} = \{z \in \mathbb{R}^n : \|z_{A^*}\|_1 \leq 3\|z_{A^*}\|_1\}, \quad (D.1)
\]
and define \(\mathcal{E} = \{\|\Psi^t \Delta/m\|_\infty \leq \lambda/2\}\). Conditioning on the event \(\mathcal{E}\), we have \(R \in C_{A^*}\).

**Proof.** The optimality of \(x_{t1}\) implies that \(\frac{1}{2m} \|y - \Psi x_{t1}\|_2^2 + \lambda \|x_{t1}\|_1 \leq \frac{1}{2m} \|y - \Psi \hat{x}^*\|_2^2 + \lambda \|\hat{x}^*\|_1\).
Recall that $y = \Psi\tilde{x}^* + \Delta$. Some algebra on the above display shows
\[
\frac{1}{2m}\|\Psi R\|^2_2 + \lambda\|R_{2^*}\|_1 \leq \langle R, \Psi^T\Delta/m \rangle + \lambda\|R_{A^*}\|_1
\]
\[
\leq \|R\|_1\|\Psi^T\Delta/m\|_\infty + \lambda\|R_{A^*}\|_1 \leq \|R\|_1\lambda/2 + \lambda\|R_{A^*}\|_1,
\]
where, we use Cauchy Schwartz inequality and the definition of $E$. The above inequality shows
\[
\frac{1}{m}\|\Psi R\|^2_2 + \lambda\|R_{2^*}\|_1 \leq \|R\|_1\|\Psi/\Delta/m\|_\infty
\]
\[
\leq \|R\|_1\|\tilde{x}^*\|_1\leq 3\lambda\|R_{A^*}\|_1,
\]
\[
\text{(D.2)}
\]
i.e., $R \in C_{A^*}$. This finishes the proof of Lemma D.1. \qed

The next Lemma gives a lower bound on $\mathbb{P}[E]$ with a proper regularization parameter $\lambda$.

**Lemma D.2.** Let $C_3 \geq \|x^*\|_1$. If $m > \frac{4C_1^2}{C_2^2} \log n$, taking $\lambda = \frac{4(1+|c|C_3)}{\sqrt{C_1}} \sqrt{\log n/m}$, then with probability at least $1 - \frac{2}{n^3} - \frac{2}{n^2}$, one has
\[
\|\Psi^T\Delta/m\|_\infty \leq \frac{\lambda}{2}.
\]
\[
\text{(D.3)}
\]

**Proof.**
\[
\|\Psi^T\Delta/m\|_\infty = \|\Psi^T(\Psi\tilde{x}^* - y)/m\|_\infty = \|\langle \Psi^T\Psi \tilde{x}^* - \Sigma\tilde{x}^* + (\Sigma\tilde{x}^* - \Psi^T y/m) \rangle\|_\infty
\]
\[
\leq \|\Sigma\tilde{x}^*\|_\infty + \|\langle \mathbb{E}[\Psi^T y/m] - \Psi^T y/m \rangle\|_\infty
\]
\[
\leq \|\Sigma\tilde{x}^*\|_\infty + \|\sum_{i=1}^m \langle \mathbb{E}[\psi_i y_i] - \psi_i y_i \rangle/m\|_\infty
\]
\[
\leq |c|C_3 \sqrt{\frac{\log n}{C_1 m}} + 2 \sqrt{\frac{\log n}{C_1 m}},
\]
\[
= 2\left(1 + |c|C_3\right) \frac{\sqrt{\log n}}{\sqrt{C_1 m}},
\]
where the first inequality is due to the triangle inequality, (2.1), (2.3) and (2.4), the second inequality follows from the definition of $\tilde{x}^*$ and Cauchy-Schwartz inequality, and the third one uses (B.4) and (B.5). The proof of Lemma D.2 is completed by setting $\lambda = \frac{4(1+|c|C_3)}{\sqrt{C_1}} \sqrt{\log n/m}$. \qed

The last Lemma shows $\Psi$ is strongly convex along the direction contained in the cone $C_{A^*}$ defined in (D.1).

**Lemma D.3.** If $s \leq \exp(1 - \frac{C_1}{2}) n$ and $m \geq \frac{64(4\kappa(\Sigma)+1)^2}{C_1} s \log \frac{m}{s}$, then with probability at least $1 - 1/n^2$, we have
\[
\|\Psi z\|^2_2/m \geq \frac{\gamma_{min}(\Sigma)}{68(4\kappa(\Sigma)+1)^2} \|z\|^2_2, \quad \forall z \in C_{A^*}.
\]
Proof. \( \forall z \in C_A = \{ v \in \mathbb{R}^n : \| v_{x^*} \|_1 \leq 3 \| v_A \|_1 \} \), we sort its entries such that

\[ |z_{k_1}| \geq |z_{k_2}| \geq \ldots \geq |z_{k_s}|. \]

Let \( A = \{k_1, \ldots, k_s\} \), and \( I = \bigcup_{t \geq 1} I_t = \{k_{st+1}, \ldots, k_{(t+1)s}\} \), where \( s = \| x^* \|_0 \). Then

\[ \| z_I \|_1 \leq 3 \| z_A \|_1. \] (D.4)

By the elementary inequality \( \| v \|_2 \leq \frac{\| v \|_1 + \sqrt{s} \| v \|_\infty}{4} \), \( \forall v \in \mathbb{R}^s \), we have

\[
\sum_{t \geq 1} \| z_{I_t} \|_2 \leq \sum_{t \geq 1} \frac{\| z_{I_t} \|_1}{\sqrt{s}} + \frac{\sqrt{s} \| z_{I_t} \|_\infty}{4} \\
= \frac{\| z \|_1}{\sqrt{s}} + \frac{\sum_{t \geq 1} \sqrt{s} \| z_{k_{st+1}} \|_\infty}{4} \\
\leq \frac{\| z \|_1}{\sqrt{s}} + \frac{\| z_A \|_1 + \| z_I \|_1}{4 \sqrt{s}} \\
\leq \frac{3\| z_A \|_1}{\sqrt{s}} + \frac{\| z_A \|_1 + 3\| z_A \|_1}{4 \sqrt{s}} \quad (using \ (D.4)) \\
= \frac{4\| z_A \|_1}{\sqrt{s}} \\
\leq 4\| z_A \|_2, \quad (D.5)
\]

which implies,

\[ \| z \|_2^2 = \| z_A \|_2^2 + \sum_{t \geq 1} \| z_{I_t} \|_2^2 \leq \| z_A \|_2^2 + (\sum_{t \geq 1} \| z_{I_t} \|_2)^2 = 17\| z_A \|_2^2. \] (D.6)

Define

\[
C_{2s}(\Psi) = \inf_{A \subset \{1, \ldots, n\}, |A| \leq 2s} \frac{\gamma_{\min}(\Psi_A^T \Psi_A)}{m},
\]

and

\[
O_s(\Psi) = \sup_{A, B \subset \{1, \ldots, n\}, A \cap B = \emptyset, |A| \leq s, |B| \leq s} \| \Psi_A^T \Psi_B / m \|.
\]
We obtain that,

$$
(\Psi_A z, \Psi z) / m = (\Psi_A z, \sum_{t \geq 1} \Psi z_t) / m \\
\leq \|z\|_2 \sum_{t \geq 1} \|\Psi_A z_t\|_2 \|z_t\|_2 \\
\leq O_s(\Psi) \|z\|_2 \sum_{t \geq 1} \|z_t\|_2 \\
\leq 4O_s(\Psi) \|z\|^2,
$$

(D.7)

where the first inequality uses Cauchy-Schwartz inequality, the second inequality follows from the definition of $O_s(\Psi)$, and the third is due to (D.5). Then, $\forall z \in C_A$, $z \neq 0$, we have

$$
\|\Psi z\|_2^2 / (m\|z\|^2) \geq \|\Psi z\|_2^2 / (17m\|z\|^2) \\
= (\|\Psi_A z\|^2 + \|\Psi z\|^2 + 2(\Psi_A z, \Psi z)) / (17m\|z\|^2) \\
\geq (\|\Psi_A z\|^2 - 8O_s(\Psi)\|z\|^2) / (17m\|z\|^2) \\
\geq (C_2(\Psi) - 8O_s(\Psi)) / 17,
$$

(D.8)

where the first inequality uses (D.6), the second inequality follows from (D.7), and the last holds due to the definition of $C_2(\Psi)$. It follows from (D.8) that, to complete the proof of this lemma it suffices to derive a lower bound on $C_2(\Psi)$ and an upper bound on $O_s(\Psi)$ with high probability, respectively.

Given $A \subset [n], |A| \leq 2s$, we define the event $E_A = \{\sqrt{\gamma_{\min}(\Psi_A) / m} > \sqrt{\gamma_{\min}(\Sigma)(1 - C_2 \sqrt{2s / m} - \frac{1}{\sqrt{m}})}\}$. Then,

$$
P[C_{2s}(\Psi) > \gamma_{\min}(\Sigma)(1 - C_2 \sqrt{2s / m} - \frac{t}{\sqrt{m}})] = P[\bigcap_{A \in [n], |A| \leq 2s} E_A] = P[\bigcap_{A \in [n], |A| = \ell, 1 \leq \ell \leq 2s} E_A] \\
= 1 - P[\bigcup_{A \in [n], |A| = \ell, 1 \leq \ell \leq 2s} E_A] \\
\geq 1 - \sum_{\ell=1}^{2s} \sum_{A \in [n], |A| = \ell} (1 - P[E_A]) \\
\geq 1 - \sum_{\ell=1}^{2s} \sum_{A \in [n], |A| = \ell} 2 \exp(-C_1 t^2) \\
= 1 - \sum_{\ell=1}^{2s} \binom{n}{\ell} 2 \exp(-C_1 t^2) \\
\geq 1 - 2^{2s} \binom{n}{2s} \exp(-C_1 t^2),
$$

where the first inequality follows from the union bound, the second inequality follows from (B.1) by replacing $\Psi$ with $\Psi_A$, and the third inequality holds since $\sum_{\ell=1}^{2s} \binom{n}{\ell} \leq \binom{2s}{\ell} \sum_{\ell=0}^{2s} \binom{2s}{\ell} (\frac{2}{n})^\ell \leq$
\[
\frac{1}{2^n} (1 + \frac{2^n}{m})^n \leq \left(\frac{e^{1+\frac{2^n}{m}}}{2^n}\right)^n.
\]
Then, we derive with probability at least \(1 - 2(e^{1+\frac{2^n}{m}})^n\exp(-C_1t^2)\),

\[
C_{2s}(\Psi) > \gamma_{\min}(\Sigma)(1 - C_2 \sqrt{\frac{2s}{m}} - \frac{t}{\sqrt{m}})^2.
\]  \tag{D.9}

Given \(A \subset [n], B \subset [n], |A| \leq s, |B| \leq s, A \cap B = \emptyset\), we define the event \(E_{A,B} = \{\|\Psi_A \Psi_B/m\| > \gamma_{\max}(\Sigma)(C_2 \sqrt{\frac{2s}{m}} + \frac{t}{\sqrt{m}})\}\). Denote \(C = A \cup B\), \(\Phi_C = \Psi_C \Sigma^{-\frac{1}{2}}\), \(G_C = \Phi_C \Phi_C/m - \textbf{I}_{2s}\). Then each row of \(\Phi_C\) is multivariate normal random vector that is sampled from \(N(0, \textbf{I}_{2s})\). It follows from \((B.2)\) with \(\Psi\) and \(\Sigma\) replaced by \(\Phi_C\) and \(\textbf{I}_{2s}\), respectively, that

\[
P(\|G_C\| \geq C_2 \sqrt{\frac{2s}{m}} + \frac{t}{\sqrt{m}}) \leq 2 \exp(-C_1t^2).
\]

Observing \(\Sigma^{-\frac{1}{2}} \Psi_A \Psi_B \Sigma^{-\frac{1}{2}}/m\) is a sub-matrix of \(G_C\), we deduce,

\[
P[E_{A,B}] \leq 2 \exp(-C_1t^2).
\]

Then, similarly to the proof of \((D.9)\), we have

\[
P(O_s(\Psi) > \gamma_{\max}(\Sigma)(C_2 \sqrt{\frac{2s}{m}} + \frac{t}{\sqrt{m}})) = P[\bigcup_{A,B \subset [n], A \cap B = \emptyset, |A| \leq s, |B| \leq s} E_{A,B}]
\]

\[
= P[\bigcup_{A \subset [n], |A| = \ell, B \subset [n], |B| = \ell, A \cap B = \emptyset, 1 \leq \ell \leq s, 1 \leq \ell \leq s} E_{A,B}]
\]

\[
\leq \sum_{\ell=1}^{s} \sum_{t=1}^{s} 2 \exp(-C_1t^2)
\]

\[
\leq (\sum_{t=1}^{s} \left(\frac{n}{t}\right)^{2\ell} 2 \exp(-C_1t^2)
\]

\[
\leq 2(\frac{s}{s})^{2s} \exp(-C_1t^2),
\]

which implies with probability at least \(1 - 2(e^{1+\frac{2^n}{m}})^n\exp(-C_1t^2)\),

\[
O_s(\Psi) \leq \gamma_{\max}(\Sigma)(C_2 \sqrt{\frac{2s}{m}} + \frac{t}{\sqrt{m}}).
\]  \tag{D.10}

Combining \((D.9)\) and \((D.10)\) and setting \(t = \sqrt{\frac{4n}{C_1}} \log \frac{en}{s}\), we obtain that with probability at least \(1 - 4/(e^{1+\frac{2^n}{m}})^n \geq 1 - 4/n^2\)

\[
C_{2s}(\Psi) - 8O_s(\Psi) \geq \gamma_{\min}(\Sigma)f\left(\sqrt{\frac{2s}{m}} + \sqrt{\frac{4s}{mC_1}} \log \frac{en}{s}\right),
\]

where the unitary function \(f(z) = z^2 - (8\kappa(\Sigma) + 2)z + 1\) it follows from the assumption \(s \leq \exp\left(\frac{1}{C_1} \frac{2^{m/4}}{mC_1} \log \frac{en}{s}\right)\) that \(\sqrt{\frac{2s}{m}} \leq \sqrt{\frac{4s}{mC_1}} \log \frac{en}{s}\). Then some basic algebra shows that \(f\left(\sqrt{\frac{2s}{m}} + \sqrt{\frac{4s}{mC_1}} \log \frac{en}{s}\right) \geq f\left(\frac{1}{4(4\kappa(\Sigma) + 1)^2}\right)\) as long as \(m \geq \frac{64(4\kappa(\Sigma) + 1)^2}{C_1^2} \log \frac{en}{s}\). The proof of Lemma D.3 is completed.
Now we are in the place of combining the above pieces together to finish the proof of Theorem 3.1. Recall \( R = x_{\ell_1} - \tilde{x}^* \). It follows from Lemma D.1 that \( R \in \mathcal{C}_{A^*} \) and (D.2) holds by conditioning on \( \mathcal{E} \), i.e.,

\[
\frac{1}{m} \| \Psi R \|_2^2 + \lambda \| R_{\ell_1} \|_1 \leq 3 \lambda \| R_{A^*} \|_1,
\]

which together with Lemma D.3 implies that, with probability at least \( 1 - 4/n^2 \),

\[
\frac{\gamma_{min}(\Sigma)}{68(4\kappa(\Sigma) + 1)^2} \| R \|_2^2 \leq 3 \lambda \| R_{A^*} \|_1 \leq 3 \lambda \sqrt{s} \| R_{A^*} \|_2,
\]

i.e.,

\[
|c| \| x_{\ell_1} / c - x^* \|_2 = \| x_{\ell_1} - \tilde{x}^* \|_2 \leq \frac{204(4\kappa(\Sigma) + 1)^2}{\gamma_{min}(\Sigma)} \lambda \sqrt{s}.
\]

The proof of Theorem 3.1 is completed by dividing \(|c|\) on both side and using Lemma D.2, which guaranties that (D.2) holds with \( \lambda = \frac{4(1+|c|C)}{\sqrt{\kappa(\Sigma)}} \sqrt{\frac{\log m}{m}} \) with probability greater than \( 1 - 2/n^3 - 2/n^2 \).

\[\Box\]

Appendix E. Proof of the equivalency between the PDAS and (4.2) - (4.3).

Proof. Partition \( Z^k \), \( D^k \) and \( F(Z^k) \) according to \( A_k \) and \( I_k \) such that

\[
Z^k = \begin{pmatrix} x_{A_k} \\ x_{\mathcal{I}_k} \\ d_{A_k} \\ d_{\mathcal{I}_k} \end{pmatrix},
\]

\[
D^k = \begin{pmatrix} D_{A_k}^x \\ D_{\mathcal{I}_k}^x \\ D_{A_k}^d \\ D_{\mathcal{I}_k}^d \end{pmatrix},
\]

\[
F(Z^k) = \begin{pmatrix} -d_{A_k}^k + \lambda \text{sign}(x_{A_k}^k + d_{A_k}^k) \\ x_{A_k}^k \\ x_{\mathcal{I}_k}^k \\ d_{A_k}^k \\ -c_{A_k}^k \end{pmatrix},
\]

\[
\begin{pmatrix} \Psi_{A_k}^t \Psi_{A_k} x_{A_k}^k + \Psi_{A_k}^t \Psi_{\mathcal{I}_k} x_{\mathcal{I}_k}^k + n d_{A_k}^k - \Psi_{A_k}^t y \\ \Psi_{A_k}^t \Psi_{A_k} x_{A_k}^k + \Psi_{\mathcal{I}_k}^t \Psi_{\mathcal{I}_k} x_{\mathcal{I}_k}^k + n d_{\mathcal{I}_k}^k - \Psi_{\mathcal{I}_k}^t y \\ \Psi_{A_k}^t \Psi_{A_k} x_{A_k}^k + \Psi_{\mathcal{I}_k}^t \Psi_{\mathcal{I}_k} x_{\mathcal{I}_k}^k + n d_{A_k}^k - \Psi_{A_k}^t y \\ \Psi_{A_k}^t \Psi_{A_k} x_{A_k}^k + \Psi_{\mathcal{I}_k}^t \Psi_{\mathcal{I}_k} x_{\mathcal{I}_k}^k + n d_{\mathcal{I}_k}^k - \Psi_{\mathcal{I}_k}^t y \\ \end{pmatrix}.
\]
Substituting (E.1) - (E.2) and (4.4) into (4.2), we have

\[-(d_k^x + D_k^d) = -\lambda \text{sign}(x_k^x + d_k^x),\]

(E.3)

\[x_k^1 + D_k^2 = 0,\]

(E.4)

\[\Psi_{A_k}^t \Psi_{A_k}(x_k^x + D_k^d) = \Psi_{A_k}^t y - m(d_k^x + D_k^d) - \Psi_{A_k}^t \Psi_{I_k}(x_k^1 + D_k^2),\]

(E.5)

\[m(d_k^2 + D_k^d) = \Psi_{I_k}^t y - \Psi_{I_k}^t \Psi_{A_k}(x_k^x + D_k^d) - \Psi_{A_k}^t \Psi_{I_k}(x_k^1 + D_k^2).\]

(E.6)

It follows from (4.3) that

\[\begin{pmatrix}
  x_k^{x+1}_A
  x_k^1
  d_k^{x+1}
  d_k^2
\end{pmatrix}
= \begin{pmatrix}
  x_k^1 + D_k^2
  d_k^x + D_k^d
  x_k^x + D_k^d
  d_k^x + D_k^d
\end{pmatrix}.

(E.7)

Substituting (E.7) into (E.3) - (E.6), we get the iteration procedure of PDAS in Algorithm 1. This completes the proof. □

REFERENCES

[1] Albert Ai, Alex Lapanowski, Yaniv Plan, and Roman Vershynin, One-bit compressed sensing with non-gaussian measurements, Linear Algebra and its Applications, 441 (2014), pp. 222–239.

[2] Francis Bach, Rodolphe Jenatton, Julien Mairal, and Guillaume Obozinski, Optimization with Sparsity-Inducing Penalties, Found. Trend. Mach. Learn., 4 (2012), pp. 1–106.

[3] Rich Baraniuk, Simon Foucart, Deanna Needell, Yaniv Plan, and Mary Wootters, One-bit compressive sensing of dictionary-sparse signals, arXiv preprint arXiv:1606.07531, (2016).

[4] Richard G Baraniuk, Simon Foucart, Deanna Needell, Yaniv Plan, and Mary Wootters, Exponential decay of reconstruction error from binary measurements of sparse signals, IEEE Transactions on Information Theory, 63 (2017), pp. 3368–3385.

[5] Petros T Boufounos, Greedy sparse signal reconstruction from sign measurements, in Signals, Systems and Computers, 2009 Conference Record of the Forty-Third Asilomar Conference on, IEEE, 2009, pp. 1305–1309.

[6] Petros T Boufounos and Richard G Baraniuk, 1-bit compressive sensing, in Information Sciences and Systems, 2008. CISS 2008. 42nd Annual Conference on, IEEE, 2008, pp. 16–21.

[7] David R Brillinger, A generalized linear model with gaussian regressor variables, in Selected Works of David Brillinger, Springer, 2012, pp. 589–606.

[8] Emmanuel J. Candès, Justin Romberg, and Terence Tao, Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information, IEEE Trans. Inform. Theory, 52 (2006), pp. 489–509.

[9] Scott Shaobing Chen, David L Donoho, and Michael A Saunders, Atomic decomposition by basis pursuit, SIAM J. Sci. Comput., 20 (1998), pp. 33–61.

[10] Christian Clason, Bangti Jin, and Karl Kunisch, A duality-based splitting method for l1-tv image restoration with automatic regularization parameter choice, SIAM Journal on Scientific Computing, 32 (2010), pp. 1484–1505.

[11] ———, A semismooth newton method for l1 data fitting with automatic choice of regularization parameters and noise calibration, SIAM Journal on Imaging Sciences, 3 (2010), pp. 199–231.

[12] P.L. Combettes and J.C. Pesquet, Proximal splitting methods in signal processing, in Fixed-Point Algorithms for Inverse Problems in Science and Engineering, Heinz H. Bauschke, Regina S. Burachik, Patrick L Combettes, Veit Elser, D. Russell Luke, and Henry Wolkowicz, eds., Springer, Berlin, 2011, pp. 185–212.

[13] Patrick L Combettes and Valérie R Wajs, Signal recovery by proximal forward-backward splitting, Multiscale
Modeling & Simulation, 4 (2005), pp. 1168–1200.

[14] Dao-Qing Dai, Lixin Shen, Yuexing Xu, and Na Zhang, Noisy 1-bit compressive sensing: models and algorithms, Applied and Computational Harmonic Analysis, 40 (2016), pp. 1–32.

[15] Bin Dong, Zuowei Shen, et al., MRA based wavelet frames and applications, IAS Lecture Notes Series, Summer Program on The Mathematics of Image Processing, Park City Mathematics Institute, (2010).

[16] David L. Donoho, Compressed sensing, IEEE Trans. Inform. Theory, 52 (2006), pp. 1289–1306.

[17] Heinz Werner Engl, Martin Hanke, and Andreas Neubauer, Regularization of inverse problems, vol. 375, Springer Science & Business Media, 1996.

[18] Qibin Fan, Yuling Jiao, and Xiliang Lu, A primal dual active set with continuation for compressed sensing, IEEE Trans. Signal Proc., 62 (2014), pp. 6276–6285.

[19] M. Fazel, E. Candes, B. Recht, and P. Parrilo, Compressed sensing and robust recovery of low rank matrices, in Signals, Systems and Computers, 2008 42nd Asilomar Conference on, IEEE, 2008, pp. 1043–1047.

[20] Simon Foucart and Holger Rauhut, A mathematical introduction to compressive sensing, vol. 1, Birkhäuser Basel, 2013.

[21] Sivakant Gopi, Praneeth Netrapalli, Prateek Jain, and Aditya Nori, One-bit compressed sensing: Provable support and vector recovery, in International Conference on Machine Learning, 2013, pp. 154–162.

[22] Ankit Gupta, Robert Nowak, and Benjamin Recht, Sample complexity for 1-bit compressed sensing and sparse classification, in Information Theory Proceedings (ISIT), 2010 IEEE International Symposium on, IEEE, 2010, pp. 1553–1557.

[23] Jarvis Haupt and Richard Baraniuk, Robust support recovery using sparse compressive sensing matrices, in Information Sciences and Systems (CISS), 2011 45th Annual Conference on, IEEE, 2011, pp. 1–6.

[24] Laurent Jacques, Kevin Degraux, and Christophe De Vleeschouwer, Quantized iterative hard thresholding: Bridging 1-bit and high-resolution quantized compressed sensing, arXiv preprint arXiv:1305.1786, (2013).

[25] Kazufumi Ito and Bangti Jin, Inverse Problems: Tikhonov Theory and Algorithms, vol. 22 of Series on Applied Mathematics, World Scientific, NJ, 2014.

[26] Kazufumi Ito, Bangti Jin, and Tomoya Takeuchi, A regularization parameter for nonsmooth tikhonov regularization, SIAM Journal on Scientific Computing, 33 (2011), pp. 1415–1438.

[27] Kazufumi Ito and Karl Kunisch, Lagrange multiplier approach to variational problems and applications, SIAM, 2008.

[28] Laurent Jacques, Kevin Degraux, and Christophe De Vleeschouwer, Quantized iterative hard thresholding: Bridging 1-bit and high-resolution quantized compressed sensing, arXiv preprint arXiv:1505.03898, (2015).

[29] Laurent Jacques, Jason N Laska, Petros T Boufounos, and Richard G Baraniuk, Robust 1-bit compressive sensing via binary stable embeddings of sparse vectors, IEEE Transactions on Information Theory, 59 (2013), pp. 2082–2102.

[30] Yuling Jiao, Bangti Jin, and Xiliang Lu, A primal dual active set with continuation algorithm for the \( \ell_0 \)-regularized optimization problem, Applied and Computational Harmonic Analysis, 39 (2015), pp. 400–426.

[31] Bangti Jin, Yubo Zhao, and Jun Zou, Iterative parameter choice by discrepancy principle, IMA Journal of Numerical Analysis, 32 (2012), pp. 1714–1732.

[32] Karin Knudson, Rayan Saab, and Rachel Ward, One-bit compressive sensing with norm estimation, IEEE Transactions on Information Theory, 62 (2016), pp. 2748–2758.

[33] Sadanori Konishi and Genshiro Kitagawa, Information criteria and statistical modeling, Springer Science & Business Media, 2008.

[34] Jason N Laska, Zaiwen Wen, Wotao Yin, and Richard G Baraniuk, Trust, but verify: Fast and accurate signal recovery from 1-bit compressive measurements, IEEE Transactions on Signal Processing, 59 (2011), pp. 5289–5301.

[35] Wenhui Liu, Da Gong, and Zhiquiang Xu, One-bit compressed sensing by greedy algorithms, Numerical Mathematics: Theory, Methods and Applications, 9 (2016), pp. 169–184.

[36] Stephane Mallat, A wavelet tour of signal processing: the sparse way, Academic press, 2008.

[37] Sainand N Negahban, Pradeep Ravikumar, Martin J Wainwright, Bin Yu, et al., A unified framework for high-dimensional analysis of \( m \)-estimators with decomposable regularizers, Statistical Science, 27 (2012), pp. 5289–5301.
pp. 538–557.

[38] **Yurii Nesterov**, *Introductory lectures on convex optimization: A basic course*, vol. 87, Springer Science & Business Media, 2013.

[39] **Michael R Osborne**, **Brett Presnell**, and **Berwin A Turlach**, *A new approach to variable selection in least squares problems*, IMA journal of numerical analysis, 20 (2000), pp. 389–403.

[40] **Yaniv Plan** and **Roman Vershynin**, *One-bit compressed sensing by linear programming*, Communications on Pure and Applied Mathematics, 66 (2013), pp. 1275–1297.

[41] ——*, Robust 1-bit compressed sensing and sparse logistic regression: A convex programming approach*, IEEE Transactions on Information Theory, 59 (2013), pp. 482–494.

[42] **Yaniv Plan**, **Roman Vershynin**, and **Elena Yudovina**, *High-dimensional estimation with geometric constraints*, Information and Inference: A Journal of the IMA, 6 (2017), pp. 1–40.

[43] **Liqun Qi** and **Jie Sun**, *A nonsmooth version of newton’s method*, Mathematical programming, 58 (1993), pp. 353–367.

[44] **Robert Tibshirani**, *Regression shrinkage and selection via the lasso*, J. Roy. Statist. Soc. Ser. B, 58 (1996), pp. 267–288.

[45] **J.A. Tropp** and **S.J. Wright**, *Computational methods for sparse solution of linear inverse problems*, Proc. IEEE, 98 (2010), pp. 948–958.

[46] **Roman Vershynin**, *Introduction to the non-asymptotic analysis of random matrices*, arXiv preprint arXiv:1011.3027, (2010).

[47] ——*, *Estimation in high dimensions: a geometric perspective*, in Sampling theory, a renaissance, Springer, 2015, pp. 3–66.

[48] ——*, *High Dimensional Probability*, 2017.

[49] **Ming Yan**, **Yi Yang**, and **Stanley Osher**, *Robust 1-bit compressive sensing using adaptive outlier pursuit*, IEEE Transactions on Signal Processing, 60 (2012), pp. 3868–3875.

[50] **Lijun Zhang**, **Jinfeng Yi**, and **Rong Jin**, *Efficient algorithms for robust one-bit compressive sensing*, in International Conference on Machine Learning, 2014, pp. 820–828.

[51] **Argyrios Zymnis**, **Stephen Boyd**, and **Emmanuel Candes**, *Compressed sensing with quantized measurements*, IEEE Signal Processing Letters, 17 (2010), pp. 149–152.