On Oscillating Dark Energy

Eric V. Linder
Berkeley Lab, University of California, Berkeley, CA 94720

Distance-redshift data can impose strong constraints on dark energy models even when the equation of state is oscillatory. Despite the double integral dependence of the distance on the equation of state, precision measurement of the distance-redshift relation for \( z = 0 - 2 \) is more incisive than the linear growth factor, CMB last scattering surface distance, and the age of the universe in distinguishing oscillatory behavior from an average behavior. While oscillating models might help solve the coincidence problem (since acceleration occurs periodically), next generation observations will strongly constrain such possibilities.

I. INTRODUCTION

The subtle variations in the cosmic expansion hold fundamental information on the nature of dark energy and the cosmological model. While these variations are not so extreme as to cause nonmonotonicity in the scale factor vs. time relation \( a(t) \), and even less so in its integral the distance-redshift relation, we might consider nonmonotonicity in the dark energy equation of state \( w(a) \). Having more than one period of strongly negative equation of state, and hence acceleration, could ameliorate the coincidence problem of why acceleration is happening now out of all the expansion history of the universe. Here we examine the implications of periodicity (e.g. a single Fourier mode of nonmonotonicity) in the equation of state \( w(a) \), and what constraints can be placed on it.

First, we note that many periodic or nonmonotonic potentials have been put forward for dark energy, but these will not necessarily, and indeed rarely, give rise to periodic \( w(a) \). As one well-studied example, the potential for a pseudo-Nambu Goldstone boson (PNGB) field can be written as \( V(\phi) = V_0[1 + \cos(\phi/f)] \), clearly periodic, where \( f \) is a (axion) symmetry energy scale. However, unless the field has already rolled through the minimum, the relation \( w(a) \) is monotonic and indeed can be well described by the usual \( w(a) = w_0 + w_a(1 - a) \) (see [4] for further discussion).

For a damped field, we can estimate how far the field has rolled in the history of the universe by writing

\[
\dot{\phi} \sim \sqrt{(1 + w)\rho_w} \sim \sqrt{1 + w} \cdot H M_p,
\]

where \( M_p \) is the Planck energy and \( H \) is the Hubble parameter. If \( w \) is close to \(-1\), as observations seem to indicate, then the field has only traversed

\[
\Delta \phi \sim \dot{\phi}/H \ll M_p,
\]

and may well not have had time to see nonmonotonicity in the potential. That is, if one walks in darkness only a few paces one may not realize there are hills and valleys in the terrain.

Therefore, rather than examining nonmonotonic potentials and calculating periodic behavior in \( w(a) \), and then investigating those effects on cosmological observables, we start directly with a phenomenological, periodic equation of state (see [2] for some particle physics motivation for such an equation of state and [3] for its application to crossing \( w = -1 \)). From the calculation of effects on observables we will impose constraints on the amplitude and period of dark energy oscillations.

II. LIMITS ON OSCILLATION

Our four basic tools for constraining dark energy equation of state oscillations will be the distance-redshift relation out to redshift \( z = 2 \) (as can be accurately measured by Type Ia supernovae), the distance to the cosmic microwave background last scattering surface at \( z = 1089 \), the age of the universe today, and the linear growth factor of mass perturbations. These will constrain the models sufficiently that we do not need to extrapolate to and employ early time constraints from the primordial nucleosynthesis or recombination epochs. However, such early universe constraints are important for oscillatory models that are not periodic, such as the tracking oscillating model of [4].

Note that despite the double integration leading from the equation of state to the distance-redshift relation, sufficiently precise distance measurements can provide strong constraints on periodicity in \( w(a) \). Indeed we find they are the most incisive probe.

Since we take a purely phenomenological model for \( w(a) \), we do not have information on perturbations in the dark energy. However, canonical scalar fields generally do not have significant perturbations on smaller than the Hubble scale, so we will neglect inhomogeneities in the dark energy (but see [11]). Note that coupling of dark energy to matter [3 6 7 8] can produce both inhomogeneities and nonmonotonic equations of state. We leave that scenario for future investigation, but generally it should cause significant deviations in the growth behavior.

The natural period of the cosmic expansion is given by

\[
H^{-1} = \langle d\ln a/dt \rangle^{-1},
\]

so we examine periodicity in units of the e-folding scale \( \ln a \). That is, our ansatz is

\[
w(a) = w_0 - A \sin(B \ln a),
\]

where \( w_0 \) is the value today. We expect that as the central value, \( \langle w \rangle \approx w_0 \), becomes less negative, the acceleration decreases and it will be harder to match the dis-
tance and growth relation of, say, a cosmological constant $w = -1$ universe. For example, the undulant universe model [2], with an effective $\langle w \rangle = 0$, can quickly be ruled out on the basis of possessing only 3% of the growth by today of the cosmological constant model [2].

As the amplitude of oscillations, $A$, increases we likewise expect clear distinction from a model with constant $w$. As the frequency of oscillations $B$ increases, however, the observations, which depend on integrals of the equation of state, could have difficulty distinguishing the oscillatory model from a model with constant equation of state $\langle w \rangle$.

To generally describe a sinusoidal equation of state we would need four parameters:

$$w(a) = w_c - A \sin(B \ln a + \theta),$$

(4)

where $\theta$ gives the phase of the oscillation today ($a = 1$) and $w_c$ gives the center of the range over which $w$ oscillates (and generally the average value). In Eq. (4), and for the rest of this paper, we assume that $\theta = 0$ so that $w_c = w_0$. This should make it more difficult to distinguish an oscillating model from one with a constant equation of state $w = w_0$, and so represents a conservative approach. (Note that the slinky equation of state has $\theta = \pi/2, w_c = 0, A = 1$, while the undulant model further specializes to $B = 1$.)

We explore the three remaining parameters one by one to isolate the differing aspects of the oscillations. Of course when the amplitude $A = 0$ then the frequency $B$ is not defined as we have reduced to a constant equation of state. To actually see oscillatory behavior, we need $|B \ln a_{\text{min}}| > 2\pi$, where $z_{\text{max}} = a_{\text{min}}^{-1} - 1$ gives the upper redshift bound on the precision observations, but we will consider lower frequencies. For smaller $B$ we might simply see a monotonic trend in $w$ over the region of observations. Figure 1 plots the redshift of the first full period of oscillation. Only for $B > 5.72$ does a full period occur in the range $z = 0 - 2$.

For a given $w_0$, we can find the maximum amplitude of the oscillation consistent with interpreting the observations in terms of a constant, averaged equation of state $\langle w \rangle = w_0$. Thus, larger variations could be distinguished as a varying equation of state. As observational constraints we employ future data equivalent to 1% distance measures over the redshift range $z = 0 - 2$ (e.g. from SNAP-quality Type Ia supernovae data [10]), 0.7% distance measure to the CMB last scattering surface at $z = 1089$ (e.g. from Planck Surveyor CMB temperature anisotropy data [11]), 1% measure of the current age of the universe $H_0 t_0$, or 2% measure of the linear growth factor over $z = 0 - 2$ (e.g. from SNAP-quality weak lensing data). Note that for these probes the cosmological constant ($w = -1$) case lies respectively 2.3%, 1.0%, 1.8%, 3.1% away from the $w = -0.9$ model (keeping the matter density $\Omega_m = 0.28$). We find that supernovae distances provide the most stringent constraints on the oscillation amplitude $A$ and frequency $B$.

These results are shown in Fig. 2 for $w_0 = -0.9$ and three values of frequency $B$, corresponding to one full period (i.e. a true oscillation within the range $z = 0 - 2$), one half period, and $B = 1$. Larger amplitudes than those shown would violate the observational constraints. The constraints on $A$ are rather symmetric about zero for the small amplitudes allowed; that is, the equation of state could equally well decrease into the past by the same amount.

As we take $w_0$ less negative, the allowed amplitudes for a given frequency decrease. Basically, since the dark energy density dies off less quickly for less negative $w_0$, the oscillations have a substantial impact over a larger scale factor range, and so constraints are tighter. Upon changing $w_0$ we can also adjust the oscillation amplitude $A$ so as to keep the same minimum (or maximum) equation of state. For example, for a canonical scalar field, $w \geq -1$, so if we wish to avoid oscillations violating this bound we could require $w_0 - A \geq -1$. Figure 3 shows the relative constraints on amplitude and frequency when the minimum is set to the cosmological constant value $w = -1$. For low amplitudes, the oscillations are not discernible in the observations so any value of frequency is allowed. As the amplitude increases, the frequency must decrease in order for the oscillatory equation of state to remain looking close to a constant equation of state in the observational constraints.
FIG. 2: These equation of state curves possess the maximal amplitude $A$ allowed for specific frequencies $B$, given next generation observations constraining the magnitude-redshift relation to $\Delta m < 0.02$ relative to the constant equation of state model $w = w_0 = -0.9$, for $z = 0 - 2$. Oscillatory equations of state with a given frequency must lie closer to the dotted, constant equation of state line than the pictured curves to match the constraints. Upon flipping the sign of $A$ (initially decreasing the equation of state into the past) these curves are basically reflected about the $w = -0.9$ line.

III. OSCILLATION AT EARLY TIMES

Oscillating equations of state can act as models contributing a substantial fraction of the total energy density at early times ($z \gg 1$). These are not really early quintessence models, though, in that the equation of state is not necessarily providing acceleration, i.e. $w < -1/3$. The energy density of the oscillating model is

$$\rho_{\text{osc}}(a) = \rho_{\text{osc}}(1) a^{-3(1+w_0)} e^{(3A/B)(1 - \cos(B \ln a))}. \quad (5)$$

For the limiting cases of Figs. 2 and 3, the contribution to the total energy density at $z = 2$ is within 20% of what a plain $w = -0.9$ model would give, and the energy density fraction at $z = 1089$ remains at the $10^{-8}$ level. The slinky model manages to have high early energy density – possibly even greater than the matter density – by taking the central value $w_c = 0$. Thus, early energy density is achieved at the price of not maintaining dark energy, i.e. acceleration.

Another caution is that early presence of substantial energy density not in the matter component can obviate the usual matter dominated epoch. While the consequences of this for structure formation are recognized in the suppression of the linear growth factor, there is a more subtle effect. The second order differential equation for the growth factor $g = (\delta \rho_m / \rho_m) / a$ requires two boundary conditions, canonically given by $g(a \ll 1) = 1$ and $dg/da(a \ll 1) = 0$ (cf. [12]). However, these arise from the usual density growth behavior $\delta \rho_m / \rho_m \sim a$ for a matter dominated universe. In numerical computation of the growth equation in an oscillating model, we find a sensitive interplay between the initial scale factor $a_i$ and the fraction of energy density at that scale factor in components other than matter, such that the boundary condition is unstable. Consequently, for slinky models, numerical instability of the growth factor arises for $B > 0.2$. This may be apparent in the nonmonotonicity of the growth shown in Fig. 11 of [3]. Proper solution in this regime requires solution of the full set of coupled matter, scalar field, and radiation perturbation equations so as to give the proper initial conditions (velocity) to the matter growth. Note that some initial velocity $dg/da > 0$ can actually help offset the suppression of the growth factor.

Models, e.g. where $w_c \sim -1$, that do have a true matter dominated epoch, do not suffer from these complications. Interestingly, if $w_c = w_0 \sim -1$, growth data alone allows two distinct regions in parameter space. For low frequencies $B$, the equation of state is nearly constant and so the growth mimics that of the constant $w$ model.
(If \( w_e = -1 \), the oscillating model in the limit \( B \to 0 \) approaches the cosmological constant.) As \( B \) increases, the growth is altered due to the change in the Hubble parameter relative to the constant \( w \) model. The tension between growth and solution of the coincidence problem (that acceleration is observed near today) is nicely highlighted by [4]. For very high frequencies \( B \), the growth once again approaches the constant \( w \) case (cf. Eq. 5). This holds even if the amplitude of oscillations \( A \approx 1 \).

A high oscillation frequency, though, may be expected to be accompanied by spatial inhomogeneities in the dark energy due to the (relatively) large effective mass of the field. This will then alter the matter perturbation growth. Regardless, use of precision distance data removes the high frequency fit region, for large amplitudes. For small amplitudes any frequency \( B \) is allowed by the distance data, as shown in Fig. 2.

IV. OTHER OSCILLATIONS

We briefly return to the issue of oscillations of the potential, or the field motion in the potential, rather than oscillations of the equation of state directly. As mentioned in [1] the PNGB model has a periodic potential. For accelerating behavior, however, the field must not have undergone many oscillations, else the equation of state would be near zero like the axion case.

Caldwell & Linder [13] examined the dynamics and found the behavior in the \( w' - w \) phase plane to be nearly linear: \( w' \approx F(1 + w) \), where \( w' = dw/da \). The coefficient \( F \) is proportional to the inverse of the symmetry scale \( f \). They showed this was a thawing model, where the field starts frozen and hence looking like a cosmological constant \( w = -1 \), before the field begins to evolve as it comes to dominate the cosmic energy density. The solutions for the equation of state and energy density for such dynamics are

\[
\begin{align*}
\omega(a) &= -1 + (1 + w_0)a^{F} \quad (6) \\
\rho_\omega(a) &= \rho_\omega(1)e^{(3/F)(1+w_0)(1-a^{F})} \quad (7),
\end{align*}
\]

which are manifestly non-oscillatory. As mentioned in [1] this looks like the standard varying equation of state \( \omega(a) = w_0 + \omega(1-a) \) when \( F \approx 1 \), which holds for natural values \( f \sim O(M_P) \).

Another possibility is for the potential not to be periodic but the field motion is. Consider oscillations around a minimum of the potential. Turner [14] showed that for oscillation periods much smaller than the Hubble time (large frequency \( B \) in our notation), the averaged equation of state becomes

\[
\langle w \rangle = \frac{n - 2}{n + 2}, \quad (8)
\]

for potentials \( V \sim \phi^n \) with \( n \) even. For potentials looking like quadratic (quartic) potentials near their minimum, this gives \( \langle w \rangle = 0 \) (1/3) – behavior like matter (radiation). To provide acceleration, one would need \( n \ll 1 \) (conversely, very steep potentials give behavior like shear energy, \( \langle w \rangle = 1 \)). If the frequency of oscillations is comparable or larger than the Hubble time (so the energy density changes significantly during the oscillations), the equation of state does not obey Eq. 8, but in this case we would not discern oscillations in the data.

Thus, it is not easy to obtain oscillations in the equation of state without fine tuning the potential to have particular nonmonotonic periodicities, as in the slinky model. We can analyze the phase plane dynamics of our oscillatory model, Eq. 8, quite simply (also see Fig. 9 of [9]). It satisfies the equation

\[
\frac{(w - w_0)^2}{A^2} + \frac{w'^2}{A^2B^2} = 1, \quad (9)
\]

so the dynamics is given by an ellipse in the phase space with center at \( w = w_0 \), \( w' = 0 \), semimajor axis \( A \), semiminor axis \( AB \), and eccentricity \( \sqrt{1 - B^2} \) (or if \( B > 1 \), switch major and minor, and the eccentricity is \( \sqrt{1 - B^{-2}} \)). Note that \( B = 1 \), where the characteristic oscillation scale equals the Hubble e-fold scale, gives a circle.

V. CONCLUSION

Precision distance-redshift measurements over the range \( z = 0 - 2 \) possess power in discriminating oscillating equations of state from constant ones, despite the double integral relation. Other probes are not as constraining, but growth measurements can provide important limits on the early time behavior if the dark energy affects the degree of matter domination. Growth must be calculated carefully, however, since high amplitudes affect the growth boundary conditions and numerical stability, and high frequencies can lead to spatial inhomogeneities in the dark energy. While oscillating models offer one idea for solving the coincidence problem (since acceleration occurs periodically), next generation observations will be able to strongly constrain such possibilities.

[1] J.A. Frieman, C.T. Hill, A. Stebbins, I. Waga, Phys. Rev. Lett. 75, 2077 (1995) astro-ph/9505060
[2] G. Barenboim & J. Lykken, astro-ph/0504090
[3] B. Feng, M. Li, Y-S. Piao, X. Zhang, astro-ph/0407432
[4] S. Dodelson, M. Kaplinghat, E. Stewart, Phys. Rev. Lett. 85, 5276 (2000) astro-ph/0002360
[5] L. Amendola, C. Quercellini, E. Giallongo, MNRAS 357, 429 (2005) astro-ph/0404590
[6] T. Koivisto, Phys. Rev. D 72, 043516 (2005) astro-ph/0504571
[7] E.V. Linder, Phys. Rev. D 72, 043529 (2005) astro-ph/0507263
[8] L. Barnes, M.J. Francis, G.F. Lewis, E.V. Linder, PASA in press astro-ph/0510791
[9] G. Barenboim, O. Mena Requejo, C. Quigg, astro-ph/0510178
[10] SNAP – http://snap.lbl.gov ;
[11] Planck – http://planck.esa.int
[12] E.V. Linder & A. Jenkins, MNRAS 346, 573 (2003) astro-ph/0305286
[13] R.R. Caldwell & E.V. Linder, Phys. Rev. Lett. 95, 141301 (2005) astro-ph/0505494
[14] M.S. Turner, Phys. Rev. D 28, 1243 (1983)