A Graph Signal Processing View on Functional Brain Imaging

Weiyu Huang*, Thomas AW Bolton*, Alejandro Ribeiro, Dimitri Van De Ville, Senior Member

Abstract—Modern neuroimaging techniques provide us with unique views on brain structure and function; i.e., how the brain is wired, and where and when activity takes place. These large datasets contain increasingly rich information that can be analyzed in terms of network structure to reveal organizing principles at the systems level. Graph representations are versatile models where nodes are associated to brain regions and edges to structural or functional connections. Structural graphs model neural pathways in white matter that are the anatomical backbone between regions. Functional graphs are built based on functional connectivity, which is a pairwise measure of statistical interdependency between activity traces of regions. Therefore, most research to date has focused on analyzing these graphs reflecting structure or function.

Graph signal processing (GSP) is an emerging research theme where signals recorded at the nodes of the graph are analyzed taking into account the underlying graph structure. An increasing number of fundamental operations has been generalized to the graph setting, allowing to analyze the signals from a new viewpoint. Here, we introduce GSP for functional brain imaging and demonstrate its potential to integrate brain structure, contained in the graph itself, with brain function, residing in the graph signals. We show how brain activity can be meaningfully filtered based on concepts of spectral modes derived from brain structure. We also derive other operations such as surrogate data generation in the graph itself, with brain function, residing in the graph signals. We apply this framework to analyze structural and functional data recorded in the brain. We demonstrate its potential to integrate brain structure, contained in the graph itself, with brain function, residing in the graph signals.

Index Terms—Brain, neuroimaging, network models, graph signal processing, functional MRI

I. INTRODUCTION

Advances in neuroimaging techniques such as magnetic resonance imaging (MRI) have created amazing possibilities to measure human brain structure and function in a non-invasive way [1]. Diffusion-weighted MRI allows to measure major fiber tracts in white matter and map the structural scaffold that supports neural communication. Functional MRI (fMRI) takes a snapshot of the whole brain about each second, by means of blood-oxygenation-level-dependent (BOLD) signals that provide a slow proxy to underlying neural activity. An emerging theme in computational neuroimaging is to study the brain at the systems level with fundamental questions at hand such as how it supports features such as coordinated cognition, learning, and consciousness.

Shaped by evolution, the brain has evolved connectivity patterns that often look haphazard yet are crucial in cognitive processes. The apparent importance of these connectomes, has motivated the emergence of network neuroscience as a clearly defined field to study the influence of network structure in cognitive function [2]–[4]. The fundamental components in network neuroscience are graph models [5] where nodes are associated to brain regions and edge weights are associated with the strength of the respective connections. This connectivity structure can be measured directly by counting fiber tracts in diffusion weighted MRI or can be inferred directly from fMRI BOLD measurements. In the latter case, networks are said to be functional and represent a measure of co-activation, e.g., the pairwise Pearson correlation between the activation time series of nodes. Functional connectivity networks do not necessarily represent physical connections although it has been observed that there is a strong basis of anatomical support for functional networks [6].

Connectomes, structural and functional alike, have been successfully analyzed utilizing a variety of tools from graph theory and network science [5]. These analyses have uncovered a variety of measures that reflect organizational principles of brain networks such as the presence of communities where groups of regions are more strongly connected between them than with other communities [6], [7]. Network structure has also been related to behavioral and clinical measures either by statistical methods or machine learning tools [8] and, more recently, to development, behavior, and ability [9], [10].

As network neuroscience expands from understanding connectomes into understanding how connectomes and functional brain activity mediate development, behavior, and ability, the study of dynamics has taken center stage. In addition, there is a rise of interest in analyzing and understanding dynamics of functional signals and with them, network structure. Such changes happen at different timescales, from years – e.g., in developmental studies [11] – down to seconds within a single fMRI run of several minutes [12], or following tasks such as learning paradigms [10], [13], [14]. So far, common approaches have been to study changes of network structure (e.g., reflecting segregation and integration) [15] or to look into time-resolved measures of the underlying functional signals [16]–[18]. In the case of developmental studies the evolution of structural networks is important, but anatomical changes do not occur in the shorter time scales that are involved in behavior and ability studies. In the latter case
the notion of a dynamical network itself makes little sense and the more pertinent objects of interest are the dynamical changes in brain activity signals [13], [14]. Inasmuch as brain activity is mediated by physical connections, the underlying network structure must be taken into account when studying these signals. Tools from the emerging field of graph signal processing (GSP) are tailored-made for this purpose.

In simple words, GSP addresses the problem of analyzing and extracting information from data defined not in regular domains such as time or space, but on more irregular domains that can be conveniently represented by a graph. The fundamental GSP concepts that we utilize to analyze brain signals are the graph Fourier transform (GFT) and the corresponding notions of graph frequency components and graph filters. These concepts are generalizations of the Fourier transform, frequency components, and filters that have been used in regular domains such as time and spatial grids [19]–[21]. As such, they permit the decomposition of a graph signal into pieces that represent different levels of variability. We can define low graph frequency components representing signals that change slowly with respect to brain connectivity networks in a well defined sense, and high graph frequency components representing signals that change fast in similar sense. This is crucial because low and high temporal variability have proven important in the analysis of neurological disease and behavior [22], [23]. GFT based decompositions permit a similar analysis of variability across regions of the brain for a fixed time—a sort of spatial variability measured with respect to the connectivity pattern. We demonstrate here that such a decomposition significantly explains individual cognitive differences, as illustrated in Figure 6. The theory of GSP has been growing rapidly, including sampling theory [24], [25], stationarity [26], [27] and uncertainty [28]–[31], filtering [32]–[34], directed graph [35], and dictionary learning [36]. Applications have been spanning many areas including neuroscience [14], [37], image [38], [39], medical imaging [40], video [41], online learning [42], and rating prediction [43], [44].

In this work, we provide a broad perspective on how GSP can be applied for elegant and principled analysis of brain activity. In Sect. I, we start by constructing a graph from structural connectivity—the backbone of the brain—and considering brain activity as graph signals. Then, in Sect. II we derive the graph spectral domain by the eigendecomposition of a graph shift operator. Such eigenmodes have already been recognized as useful to represent meaningful information [45]. We introduce a number of graph signal operations that are particularly useful for processing activity time-courses that are measured at the nodes of the graph; i.e., filtering in terms of anatomically-aligned or liberal modes, randomization preserving anatomical smoothness, and localized decompositions that can incorporate additional domain knowledge. In Sect. IV, we demonstrate the relevance of these GSP tools as an integrated framework to consider structure and function; i.e., their application to experimental data leads to new results that can drive future developments in neuroimaging.

II. BRAIN GRAPHS AND BRAIN SIGNALS

Brain connectivity networks describe physical connection patterns between brain regions. These connections are mathematically described by a weighted graph \( \mathcal{G} := (\mathcal{V}, \mathbf{A}) \) where \( \mathcal{V} \) is a set of \( N \) nodes associated with specific brain regions and \( \mathbf{A} \in \mathbb{R}^{N \times N} \) is a weighted adjacency matrix with entries \( A_{i,j} \), each of which represents the strength of the physical connection between brain regions \( i \) and \( j \).

The brain regions encoded in the nodes of \( \mathcal{V} \) are macro-scale parcels of the brain that our current understanding of neuroscience deems anatomically or functionally differentiated. There are various parcellations in use in the literature that differ mostly in their level of resolution [52], [54]. As an example, the networks that we study here consist of \( N = 82 \) regions from the Desikan-Killiany anatomical atlas [49] combined with the Harvard-Oxford subcortical parcels [50]. A schematic representation of a few labeled brain regions is shown in Fig. 1 (left).

The entries \( A_{i,j} \) of the adjacency matrix \( \mathbf{A} \) measure the strength of the axonal connection between region \( i \) and region \( j \). This strength is a simple count of the number of streamlines that connect the regions, and can be estimated with diffusion spectrum imaging (DSI) [46] — see Fig. 1 for an illustration of the pipeline and Callout 1 for details on the specific techniques that are used for this purpose. In a situation of healthy development and an absence of trauma, nodes in brain graphs are the same across individuals. Inter-subject variability of structural connectivity has demonstrated clinical value as it has been reliably associated with neurological [55], [56] and psychological [57] disorders.

Besides structural connectivity, it is possible to also acquire brain activity signals \( x \in \mathbb{R}^N \) such that the value of the \( i \)-th component \( x_i \) quantifies neuronal activity in brain region \( i \). Activity is not measured directly, but is inferred from BOLD signals acquired in fMRI sessions — see Fig. 2 for an illustration of these signals and Callout 2 for details on the applied procedure. BOLD signals for all the \( N \) studied brain regions are acquired over \( T \) successive time points, and therefore, we define the matrix \( \mathbf{X} \in \mathbb{R}^{N \times T} \) such that its \( j \)-th column codifies brain activity at time \( j \). An example of such a brain signal matrix is shown in Fig. 2A, with the corresponding distribution of values for each brain region illustrated in Fig. 2B.

Brain activity signals carry dynamic information that is not only useful for the study of pathology [56], [58], [59], but also enables to gain precious insight into behavioural and cognitive abilities [60]–[62]. Whereas physical connectivity can be seen as a lifelong property of individuals that changes slowly over the course of years, brain activity signals display meaningful fluctuations at second or sub-second time scales that inform on how different parts of the brain exchange information in the absence of any external stimulus, or on how they are recruited upon cognitive challenge. There is increasing evidence that differences in activation patterns across individuals tightly relate to behavioural variability [14], [63]–[65].

To the extent that brain activity signals are generated on top of the physical connectivity substrate, brain graphs and...
brain signals carry complementary information and should be studied in conjunction. This has been a challenge in neuroscience due to the unavailability of appropriate methods for performing this joint analysis. Here, we advocate the use of GSP tools, as detailed in the following section.

III. GRAPH SIGNAL PROCESSING FOR NEUROIMAGING

The GSP perspective is to interpret the brain signal \( x \) as a graph signal that is supported on the brain graph \( G = (V, A) \). Here we introduce the fundamental operations that we will need for processing neuroimaging data in a meaningful way.

A. Graph Fourier Transform

The focus of GSP is not on analyzing the brain graph \( G \) per se, but to use its knowledge to analyze brain signals \( x \). For a graph with positive edge weights, we consider a graph shift operator that captures the connectivity pattern of \( G \); we can choose the adjacency matrix \( A \); we can choose the adjacency matrix

\[
A_{i,j} = \begin{cases} 
1 & \text{if there is an edge between nodes } i \text{ and } j \\
0 & \text{otherwise}
\end{cases}
\]

that factors out differences in degree and is thus only reflecting relative connectivity, or the random-walk normalized graph Laplacian: \( I_{rw} = D^{-1}L \). Generalizations of the graph Laplacian also exist for graphs with negative weights [44], [72].

Let us denote the graph shift operator as \( S \) and assume henceforth that \( S \) is diagonalizable using singular value decomposition or Jordan decomposition, so that

\[
S = V \Lambda V^{-1}
\]

where \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_N) \) contains the eigenvalues \( \lambda_k \in \mathbb{C}, k = 0, \ldots, N-1 \), and \( V = [v_0, v_1, \ldots, v_{N-1}] \). When \( \Lambda \) is symmetric we have that \( V \) is real and unitary, which implies \( V^{-1} = V^\top \). The intuition behind looking at \( S \) as an operator is to represent a transformation that characterizes exchanges between neighboring nodes. The eigendecomposition of \( S \) is then used to define the graph spectral domain.

Definition 1 Consider a signal \( x \in \mathbb{R}^N \) and a graph shift operator \( S = \mathcal{L}A \mathcal{L}^{-1} \in \mathbb{R}^{N \times N} \). Then, the vectors

\[
\tilde{x} = V^{-1}x \quad \text{and} \quad x = V \tilde{x}
\]

form a Graph Fourier Transform (GFT) pair [19], [27].

The GFT encodes the notion of variability for graph signals akin to the one that the Fourier transform encodes for temporal signals. When choosing the adjacency matrix \( A \) as shift operator for directed graphs [19], [20], [73], the eigenvalues \( \lambda_k \) can be complex; the smaller the distance between \( \lambda_k \) and \( |\lambda_{\max}(S)| \) in the complex spectrum, the lower the frequency it represents. This idea is based on defining the total variation of a graph signal \( x \) as \( \|x - Sx/\lambda_{\max}(S)\|_1 \), with smoothness being associated to small values of total variation. Then, given a \( (\lambda_k, v_k) \) pair, one has total variation with \( \|1 - \lambda_k/\lambda_{\max}(S)\|_1\|v_k\|_1 \), which provides an intuitive way to order the different frequencies. Graph frequency

\[
\|x - Sx/\lambda_{\max}(S)\|_1 \leq \|x - Sx/\lambda_k\|_1 \leq \|x - Sx/\lambda_{\min}(S)\|_1
\]

allows the acquisition of detailed brain structural information. The brain graph investigated in the present article was constructed following acquisition on a Siemens 3.0T Tim Trio whole-body scanner with a whole-head elliptical coil [46]. In total, 28 healthy volunteers had their 3D T1-weighted anatomical scan segmented using FreeSurfer [47], and parcellated using the connectome mapping toolkit [48] into \( N = 82 \) regions from the Desikan-Killiany anatomical atlas [49] combined with the Harvard-Oxford subcortical parcels [50]. These regions correspond to the nodes of the brain graph model. Second, diffusion spectrum imaging (DSI) was performed to establish structural connectivity. By combining parcellation and streamline information, we constructed subject-specific structural connectivity matrices, whose elements represent the number of streamlines connecting two different regions [51], divided by the sum of their volumes [52]. This yields the weighted adjacency matrix \( A \in \mathbb{R}^{N \times N} \) for each individual considered here.
To derive the studied brain activity signals, functional MRI (fMRI) was first acquired by means of single-shot gradient-echo T2*-weighted images (TR = 1500 ms; TE = 30 ms; flip angle = 60°; FOV = 19.2 cm, resolution 3 mm x 3 mm x 3 mm). Preprocessing was performed using FEAT [66], and included skull-stripping with BET [67] to remove non-brain material, motion correction with MCFLIRT [66], slice timing correction (interleaved), spatial smoothing with a 6 mm 3D Gaussian kernel, and high-pass temporal filtering to reduce low-frequency artifacts. Subject-specific functional images were co-registered to their corresponding high-resolution anatomical images via a Boundary Based Registration technique [68]. Each participant’s individual anatomical image was segmented into grey matter, white matter, and CSF using the same atlas as for structural analysis. At the end of this pipeline, we extracted region-averaged BOLD signals using the same atlas as for structural analysis. Finally, we extracted region-averaged BOLD signals using the same atlas as for structural analysis. At the end of this pipeline, we are thus left with a signal matrix \( \mathbf{X} \in \mathbb{R}^{N \times T} \) for each subject, reflecting the activity levels of the set of assessed brain regions over time.

![Image](image_url)

**Fig. 2. Example brain activity signals.** (A) For an indicative subject, heat map of fMRI signal activity across brain regions (vertically) and time points (horizontally). (B) For the same subject, distribution of fMRI signal values for each brain region (horizontally) across all time points.

ordering becomes more obvious for undirected graphs and thus symmetric adjacency matrices, as eigenvalues become real numbers. Specifically, the quadratic form of \( \mathbf{A} \) is given by \( \lambda_k = \mathbf{v}^T_k \mathbf{A} \mathbf{v}_k = \sum_{i \neq j} A_{i,j} |v_k[i]| |v_k[j]| \). In this setting, lower frequencies will be associated to larger eigenvalues, to represent the fact that highly connected nodes in the graph possess signals with same sign and similar values.

When using the graph Laplacian \( \mathbf{L} \) as shift operator [21] for an undirected graph, the quadratic form of \( \mathbf{L} \) is given by \( \lambda_k = \mathbf{v}^T_k \mathbf{L} \mathbf{v}_k = \sum_{i \neq j} A_{i,j} (|v_k[i]| - |v_k[j]|)^2 \). If the considered signal variations follow the graph structure, the resulting value will be low. Thus, in this setup, the eigenvectors associated to smaller eigenvalues can be regarded as the graph lowest frequencies. Further, the basis \( \mathbf{V} \) is then a common solution to several well known signal processing problems, including Laplacian embedding, where the aim is to find a mapping of the graph nodes on a line so that connected ones stay as close as possible, or in other words, to minimize \( x^T \mathbf{L} x \) under the constraints \( x^T x = 1 \) and \( x^T 1 = 0 \) [74]. Another is the classical graph cut problem [75], [76], where the goal is to partition a graph into sub-communities of nodes with as few cross-connections as possible, with a similar obtained solution upon relaxation of the \( x_i = \pm 1 \) constraint.

Notice that the classical discrete Fourier transform (DFT) can also be obtained using the graph formalism by considering cycle graphs that represent discrete periodic signals [20], [21], [24], [77]. Let us first consider the DFT basis vectors for a signal of length \( T \) that is even

\[
\mathbf{f}_l = \frac{1}{\sqrt{T}} [1, e^{j2\pi l/T}, \ldots, e^{j2\pi (T-1)/T}]^T,
\]

for all \( l = 0, \ldots, N/2 \), where \( \mathbf{f}_l \) corresponds to the frequency \( \omega_l = 2\pi l / (T-1)/2 \). Next, for the undirected graph \( \mathcal{G} \) with adjacency matrix \( \mathbf{A}_\text{cycle} \) such that \( \mathbf{A}_\text{cycle}[i,i+1 \mod T] = \mathbf{A}_\text{cycle}[i,i-1 \mod T] = 1 \), and \( \mathbf{A}_\text{cycle}[i,j] = 0 \) otherwise, the eigenvalues \( \lambda_k \) correspond to the squared DFT frequencies and the eigenvectors \( \mathbf{v}_k \) of the Laplacian matrix \( \mathbf{L} \) are equivalent to the DFT basis vectors; i.e., eigenvectors with same eigenvalue span the same space as the corresponding complex-valued DFT basis vector \( \mathbf{f}_l \). The link can also be established for the eigendecomposition of the adjacency matrix.

In addition, the DFT can also be used to decompose \( \mathbf{X} \) along its temporal dimension. Indeed, the graph signal data matrix \( \mathbf{X} \) can be transformed in the (temporal) frequency domain as:

\[
\hat{\mathbf{X}} = \mathbf{X} \mathbf{H}^H ,
\]

where \( \cdot^H \) indicates the Hermitian transpose, \( \hat{\mathbf{X}} \in \mathbb{C}^{N \times T} \) contains \( T \) Fourier coefficients for each of the \( N \) time courses. Filtering can then be applied by multiplying with a diagonal matrix \( \mathbf{H} \) defined by the windowing function \( [\mathbf{H}][i,i] = h(\lambda_i) \). The filtered output is then given by:

\[
\mathbf{Y}_H = \mathbf{X} \mathbf{H}^H \mathbf{H}.
\]

**B. Graph Signal Filtering**

Given the above relationships, it becomes possible to manipulate the graph signals stored in the matrix \( \mathbf{X} \) by extracting...
signal components associated to different graph frequency ranges. In specific, we can define the diagonal filtering matrix \( G \), where \( [G]_{i,i} = g(\lambda_i) \) is the frequency response for the graph frequency associated with eigenvalue \( \lambda_i \), and retrieve the filtered signals as:

\[
Y_G = V G V^\top X.
\]  

(5)

Generic filtering operations can now be defined for the graph setting, such as ideal low-pass filtering, where \( g(\lambda_i) \) would be 1 for \( \lambda_i \) corresponding to low-frequency modes, and 0 otherwise.

Using the definition of the GFT pair, the effect of the filtering in Eq. (5) on the graph spectral coefficients is directly visible from \( Y_G = G X \). This also allows to generalize the convolution operation of a graph signal \( x \) by a filter defined through the spectral window \( g \) as [21]:

\[
[y_G]_{k'} = \sum_{k=0}^{N-1} [v_k]_{k'} g(\lambda_k) \hat{x}_k.
\]

It is also possible to translate the operation to the vertex domain by considering the Taylor approximation of the window function \( g(\lambda) = \sum_{m=0}^{\infty} c_m \lambda^m \):

\[
Y_G = \sum_{m=0}^{M} c_m S^m X,
\]

which uses iterated versions of the shift operator \( S \). Other operations such as translation, modulation, or dilation can be generalized in a similar way [21].

C. Generation of Graph Surrogate Signals

A pivotal aspect in any research field is to assess the significance of obtained results through statistical testing. More precisely, one aims at invalidating the so-called null hypothesis, which expresses the absence of the effect of interest. Standard parametric tests such as the well-known \( t \)-test assume independent and identically distributed Gaussian noise, which is making a weak null hypothesis for most applications. Non-parametric tests such as the permutation test provide a powerful alternative by mimicking the distribution of the empirical data. For correlated data, the Fourier phase-randomization procedure [78] has been widely applied as it
preserves autocorrelation structure under stationarity assumptions. This standard method can be applied to the temporal dimension of our graph signals:

\[ Y = X \Phi_{time}^H \Phi_{time} F, \]

where the diagonal of \( \Phi_{time} \) contains random phase factors according to the windowing function \( \Phi(\lambda_i) = \exp(j2\pi \phi_i) \), where \( \phi_i \) are realizations\(^1\) of a random variable uniformly distributed in the interval \([0, 1]\). From the surrogate signals, one can then compute a test statistic and establish its distribution under the null hypothesis by repeating the randomization procedure; i.e., the power spectrum density of the surrogate data is dictated by the empirical data.

The phase randomization procedure can be generalized to the graph setting \([29]\) by considering the GFT. In particular, the graph signal can be decomposed on the GFT basis and then the graph spectral coefficients can be randomized by flipping the signs. Assuming the random sign flips are stored on the diagonal of \( \Phi_{graph} \), we can formally write the procedure as

\[ Y = V \Phi_{graph} V^T X. \]

In the context of brain graphs, this procedure generates, for a given graph signal representing a measured activation pattern, surrogate graph signals that have the same smoothness measured on the graph.

D. Wavelets and Slepian on the Graph

The wavelet transform is another fundamental tool of signal processing \([80]\) providing localized, multiscale decompositions. Several designs have been proposed to generalize this concept to graphs, such as approaches in the vertex domain \([81]–[83]\), based on diffusion processes \([84], [85]\), or using the spectral domain \([77], [86], [87]\). The latter design builds upon the GFT and has been applied for multiscale community mining \([88]\) or to investigate uncertainty principles \([28]–[31]\).

Here, we detail a more recent design of a localized decomposition for graph signals that is based on a generalization of Slepian functions \([89]\) and that can deal with additional domain knowledge, which is particularly useful in the context of neuroimaging. Let us consider the problem of retrieving a signal \( x \in \mathbb{R}^N \) that is maximally concentrated within a subset of nodes from the graph at hand, while at the same time setting a maximal bandwidth on the solution. As the global concentration of a signal is given by \( x^T x \), we end up maximizing

\[ \mu = \frac{x^T V^T M V x}{x^T x}, \]

where \( M \) is the diagonal selectivity matrix with elements \( M_{i,i} = 0 \) or 1 to respectively exclude, or include, a node into the ensemble of interest, and \( V \in \mathbb{R}^{N \times M} \) is a trimmed GFT matrix where only low-frequency basis vectors are kept. The interpretation here is that we aim at finding the linear combination of band-limited graph spectral coefficients enabling to best localize the signal within the defined graph subranges.

If we define the concentration matrix as \( C = V^T M V \), then the problem amounts to solving its eigendecomposition, and \( \{s_k\}, k = 0, 1, \ldots, M - 1 \), are the weighting coefficients obtained as solutions. We assume that they are ordered in decreasing eigenvalue amplitude, so that \( s_0 \) is the optimal (maximally concentrated) solution. From the set of coefficients, the Slepian matrix can then be retrieved as:

\[ S = V \hat{S}, \]

where \( S \in \mathbb{R}^{N \times M} \) and each column contains one of the Slepian vectors \( s_k \). Slepian vectors are not only orthonormal within the whole set of nodes \( \{s_k\} \), but also over the chosen subset \((s_k)_{l=1}^{M} \).

Now, recall from Sect. III-A that in Laplacian embedding, we attempt to minimize \( x^T L x \), which can be reformulated as \( \hat{x}^T \Lambda \hat{x} = \hat{x}^T \Lambda^{1/2} V^T V \Lambda^{1/2} \hat{x} \). This criterion can then be generalized by the introduction of the bandwidth and selectivity constraints, so that we attempt to maximize:

\[ \xi = \frac{\hat{x}^T \Lambda^{1/2} V^T M V \Lambda^{1/2} \hat{x}}{\hat{x}^T \hat{x}}. \]

In this formulation, \( \hat{A} \in \mathbb{R}^{M \times M} \) is the trimmed diagonal matrix of eigenvalues. The set of solution Slepian vectors are still orthonormal, but this time they satisfy \( s_k^T S s_l = \xi_k \delta_{k-l} \). Analogously to the classical graph Fourier transform setting, Slepian vectors of increasing eigenvalue \( \xi_k \) can thus be regarded as building blocks of increasing frequency, but within the selected node subset (localized frequency).

As a result, it becomes possible to apply similar tools as for the GFT, but for a decomposition that can be tailored in terms of localization by the subgraph selection and choice of the bandwidth. In fact, the Slepian matrix can be seen as an alternate set of basis vectors, themselves obtained as a linear combination of Laplacian eigenvectors under the localized concentration constraint. For example, the temporal signal matrix \( X \) at hand can be projected on the Slepian building blocks as \( S^T X \), and if we define the diagonal matrix \( \Gamma_L \) as a localized low-pass filter by setting \( \Gamma_L_{i,i} = 1 \) if \( \xi_i < \xi_L \) and 0 otherwise, the locally filtered output signal would be given by:

\[ Y_{\Gamma_L} = \Gamma_L S^T X. \]

IV. APPLICATIONS OF BRAIN GSP

We now show how the aforementioned GSP methods can be applied in the context of functional brain imaging. To do so, we focus on the data whose acquisition was described in Sect. III. Callouts. For each volunteer, fMRI recordings were obtained when performing a Navon switching task \([13]\), where local-global perception is assessed using classical Navon figures \([29]\). Local-global stimuli were comprised of four shapes – a circle, cross, triangle, or square – that were used to build the global and local aspects of the cues (see Fig. 4A for indicative examples).

A response (button press) to the local shape was expected from the participants in the case of white stimuli, and to the global shape for green ones. Two different block types
A B

ρ

signal concentration and switch cost (denoted by the red rectangle) is significantly larger than when computed on any of the null graph surrogate signals. This result indicates that the correlation between liberality and switch cost goes beyond what could be explained solely by structural connectivity.

In sum, we showed so far that individuals whose most liberal fMRI signals were more aligned with white matter architecture could switch attention faster. In other words, relative alignment with anatomy is associated with greater cognitive flexibility. This complements prior studies of executive function that have focused on node-level, edge-level, and module-level features of brain networks [91], [92]. The importance of this finding illustrates the usefulness of GSP tools in extracting relevant cognitive features.

So far, we have been dealing with a graph frequency decomposition considered at the level of the whole brain. However, GSP tools also allow us to independently evaluate separate nodes, or sets of nodes, from the graph at hand. In the present case, this permits a more in-depth study of which brain regions are specifically responsible for the observed association between liberality and switch cost. For this purpose, we considered 9 different, previously defined functional brain systems [46], each of which included a distinct set of regions. We assessed, separately for each system, the correlation between switch cost and liberality, and observed that the subcortical and fronto-parietal systems were the strongest contributors to this relationship (Fig. 6). Liberality within the auditory and cingulo-opercular systems also showed a slightly weaker, but nonetheless notable association. Those results highlight the ability of GSP tools to not only decompose signals in the frequency domain, but also in the graph spatial domain (looking at different nodes in the graph). Combining those two analytical axes enables to gather deeper insights when it comes to studying functional brain activity and its cognitive correlation.

B. Landmarks of Resting State

We now show how GSP tools can be applied to provide insights into brain activity fluctuations during the resting state.
signal parts. An excursion percentage of 5% (horizontal dashed line) denotes chance level; for instance, it was the case of the posterior cingulate and paracentral nodes, both in terms of aligned and liberal signal contributions. As null data realizations were generated in the graph domain, this means that those nodes never showed signal fluctuations going beyond what could be accounted for by the underlying structural connectivity.

Most regions did display very significant excursion percentages: considering alignment, occipital, parietal and temporal nodes were the strongest contributors (Fig. 7B, left box), while for liberality, key areas were temporal, subcortical or frontal (Fig. 7B, right box). In the case of both measures, a certain lateralization was also observed, as nodes from the left brain generally exhibited larger excursion percentages. The observation that the majority of brain nodes show moments of alignment and of liberality with respect to brain structure is consistent with the current resting state knowledge, since an alternation between time points with and without global similarity to the structural scaffold has previously been documented from second-order connectivity analyses [93], [94]. A GSP approach can also reveal those subtle interplays, with the extra advantage of conserving a frame-wise temporal resolution.

Further, other ingredients from the GSP pallet can as well be appended to the above pipeline, in order to further expand our understanding of the resting state brain. For example, to probe whether alignment and liberality would change along frequency, referring this time to the temporal frequency of the signal, we simply combined our null and alignment/liberality operators to the classical Fourier decomposition highlighted in Sect. III-A and computed the percentage of significant excursions for all key functional brain systems (Fig. 3A). For alignment (left graph), different systems were seen to undergo significant excursions to varying extents, with dorsal attention and auditory areas as primary contributors while subcortical and somatosensory regions stood around chance level. Interestingly, in a few cases, alignment with the structural brain scaffold appeared to be maximized at particular frequencies:

Fig. 5. Switch cost correlates with the concentration in liberal signal. (A) Switch cost does not significantly relate to the concentration of the low-frequency functional signal component (alignment). (B) A lower concentration of graph high frequency components is associated with a lower switch cost, that is, with faster attention switching. (C) The correlation between switch cost and liberal signal concentration is much stronger in the actual data than in null realizations for which statistical randomization has been performed in the graph domain. Blue data points denote the correlation coefficients obtained from surrogate signals under the null model, while the red rectangle indicates the real correlation coefficient ($\rho = 0.59$). $\rho$, partial Pearson’s correlation coefficient; $p$, p-value.

Fig. 6. Pinpointing the brain systems involved in attention switching. Separate partial correlation assessments between switch cost and liberality signal concentration on the brain areas belonging to different functional systems, using age and motion as covariates. Systems are ordered in decreasing correlation coefficient order. Liberality concentrations of subcortical and fronto-parietal systems exhibit the highest contribution for the association with switch cost.

(i.e., in the absence of any external task or stimulus). For every subject, we generated 1000 null signal matrices using the strategy outlined in Sect. III-C. We combined this operator ($\Phi_\text{graph}$) with the alignment/liberality filtering operations, to generate null data with the same graph frequency content as the real set. Formally, we thus computed a null realization as $Y = V\Phi_\text{graph}\Psi_\text{Al}V^\top X$ or $Y = V\Phi_\text{graph}\Psi_\text{Lib}V^\top X$, respectively. At an $\alpha$-level of 5%, we then used the generated null data to threshold the processed (low-pass or high-pass filtered) functional time courses, and thus locate significant signal excursions. In doing so, we considered absolute valued time courses.

Fig. 7A highlights the percentage of time points showing significant excursions for the aligned (light blue and dark blue box plots) and liberal (red and orange box plots) functional
for instance, the dorsal attention, ventral attention and auditory systems showed enhanced excursions in the 0.15 – 0.2Hz range.

Regarding liberality (right graph), almost all systems showed similar excursion percentages, to the exception of the default mode network (gray line), whose regions appeared to more rarely part away from the activation patterns expected from structural connectivity. In addition, excursions further decreased close to chance level in the 0.15 – 0.2Hz range, while at the same time, positive peaks could be seen, amongst others, for the fronto-parietal and cingulo-opercular systems. This antagonistic relationship between those functional brain systems could be the reflection of one of the resting state hallmark features: the anti-correlation between the default systems could be the reflection of one of the resting state hallmark features: the anti-correlation between the default and task-positive systems [95]. The GSP approach enables, however, a more accurate characterization in terms of both temporal and graph networks. The GSP approach enables, however, a more accurate characterization in terms of both temporal and graph networks.

Finally, another way to dig deeper into the functional signals is to consider liberality at a local scale, rather than at the whole-brain level. For this purpose, we computed a basis of Slepian vectors through the process detailed in Sect. III-D. For this step, we started from the eigendecomposition of the Laplacian matrix, instead of the adjacency matrix, and iteratively focused the analysis on a subset of nodes being part of only one given functional brain system. Every time, we derived $M = 10$ (low bandwidth) or $M = 80$ (high bandwidth) Slepian vectors, and used this new basis instead of the traditional graph basis $\mathbf{V}$ to derive the part of the functional signals in line with cognitive systems, generate null data, and quantify significant excursions.

As can be seen in Fig. 8B, some nodes stand out as having particularly significant excursions within their respective systems. For low bandwidth (top), the interactions of the system with the global graph are still taken into account, while for high bandwidth (bottom), the system is considered more on its own. In particular, significant excursions for both bandwidth settings are observed for the bilateral lateral occipital cortices (visual system), the right rostral anterior cingulate cortex (cingulo-opercular system), or the right superior parietal cortex (dorsal attention system). Those regions are thus particularly prone to part away from the activity pattern that they would be expected to follow according to structural brain connectivity, at the local scale of their respective functional systems.
V. CONCLUSION & PERSPECTIVES

The GSP framework enables the analysis of brain activity on top of the structural brain graph. In particular, we have analyzed anatomically-aligned or -liberal organization of brain activity, and in the context of an attention switching task, shown that signals aligned with anatomical connectivity are the most variable over time in cingulo-opercular and fronto-parietal systems. These results reinforce similar findings that were based on functional graphs [14]. In addition, deploying GSP tools on resting state data, we have also shed light on the diversity of functional activity patterns that exist across temporal frequencies, graph frequencies, and brain regions.

A number of intriguing connections of GSP with other approaches could be explored. For instance, the GSP methodology allows to incorporate models of linear diffusion by selecting the spectral window function \( g(\lambda) \) as the so-called diffusion kernel [5]. Therefore, graph filtering can correspond to diffusion operations of graph signals on the structural graph. A diffusion kernel puts large weights to low-frequency modes (i.e., structurally aligned in our terminology) and decreasing weights as the frequencies increase (i.e., anatomically liberal). Such network diffusion model on a structural graph has already been used to model disease progression in dementia [96] or relate structural graphs to functional ones [97]. The link with computational and simulation-based neurosciences is another topic for future interest [98]; e.g., how eigenmodes capture...
There is also a clear tendency to refine the granularity of the brain graphs, either by considering finer parcellation schemes, or by using voxel-wise approaches through explicit or implicit representations of the adjacency matrix. The availability of large data from neuroimaging initiatives such as the Human Connectome Project contributes significantly to establish these refined representations.

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