Abstract

In this work, we construct the general solution to the Heat Equation (HE) and to many tensor structures associated to the Heat Equation, such as Symmetries, Lagrangians, Poisson Brackets (PB) and Lagrange Brackets (LB), using newly devised techniques that may be applied to any linear equation (e.g., Schrödinger Equation in field theory, or the small-oscillations problem in mechanics). In particular, we improve (increase the rank of) a time-independent PB found recently \[ \square \] which defines a Hamiltonian Structure for the HE, and we construct an Action Principle for the HE. We also find a new structure, which we call a Metric Structure (MS), which may be used to define alternative anti-commutative “Hamiltonian” theories, in which the Metric- or M-Hamiltonians have to be explicitly time-dependent. Finally, we map some of these results to the Potential Burgers Equation (PBE).
1 Introduction

The Heat Equation (HE) $\omega_t(x,t) = \omega_{xx}(x,t)$ is an instructive case of an integrable equation, for it is easier to find Lagrangian Structures\(^2\) than Hamiltonian Structures\(^3\) for that equation. In this paper we construct several tensor dynamical structures related to it including a symmetric Metric Structure operator which allows for an anti-commutative Hamiltonian-like evolution representation of the system. An improved version of a Hamiltonian System for the HE\(^1\), with a seemingly regular Poisson Bracket is found. Furthermore, several regular Action Principles for the HE are presented. Finally, we map the obtained results into the Potential Burgers Equation (PBE).

In section 2 we summarize useful known results as well as some new ones. In section 3 we start by developing the Lagrangian approach\(^2\) to the HE, which is easier to deal with than the Hamiltonian one in the sense that there is no problem with time-dependent Lagrangians to define Action Principles. We will see that Action Principles may be found such that their Euler-Lagrange equations are equivalent to the HE; we think this is the first Lagrangian formalism ever found for the Heat Equation. The Lagrange Brackets (LB) for these Action Principles are explicitly time-dependent, thus there is no Hamiltonian Structure related to them\(^4\).

From the contraction\(^1\) between the LB of the Lagrangian approach just mentioned and a known Strong Symmetry for the HE\(^1\), we find a new structure, consisting of a symmetric (2,0) tensor which we call a Metric, and a time-dependent Constant of the Motion (Metric- or M-Hamiltonian); this Metric Structure (MS) defines an anti-commutative formalism. This will be performed in section 4.

Next, in section 5 we make use of the seemingly trivial general solution to the HE, to build systematically non-trivial Symmetries, Lagrangians, LBs, PBs, MSs and Strong Symmetries, in a way that generalizes the above constructions.

An interesting method to find Hamiltonian Structures is applied recently in\(^1\) for the HE, nevertheless even though the ranks of the PBs found are infinite, they are not maximal. The method needs a set of time-independent Symmetries of the HE such that the Lie Derivative\(^5\) of a Constant of the Motion (the Hamiltonian) for the HE along any Symmetry in that set is zero\(^3\). In section 5.4, a larger set is found using the general solution, and thus we increase the rank of the PB, which now seems to be maximal.

Finally, using the Cole-Hopf coordinate transformation, we map the HE and some of its related structures into the Potential Burgers Equation (PBE), thus we show new LB, PB, MS, Symmetries and Lagrangian Structures for the PBE in section 6.

We remark that all these results may be mapped directly into the Burgers Equation, and that the techniques used for the Heat Equation may be performed on any linear evolution equation, e.g., the Schrödinger Equation and the finite-dimensional small-oscillations problem.

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\(^1\)A contraction of two tensors is a sum of integral over repeated indices: for example, the contraction between the (2,0) tensor $J$ and the 1-form $U$ is the vector with components $\eta^x = J^{xy}U_y$. 

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2 Preview and Notation

2.1 Special Symbols: “⇌” and “≰”

Throughout this paper, tensor structures in infinite dimensional spaces are presented in specific coordinate systems. Taking the risk of being too sharp, we intend to clarify the different representations with an example taken from Quantum Mechanics: consider the $x$-momentum, $\hat{p}$. This operator is considered as an invariant structure, which acts upon Wavefunctions linearly. In our formalism, Wavefunction $\mathbf{W}$ is a vector and typical operators are (1,1) tensors, all of which are invariant structures on $\mathbb{C}$. When we represent these structures in a specific coordinate system, say the $\psi$-coordinate system, we get:

\[
\text{Coordinate system : } \Omega^{\psi(x)} \doteq \psi(x, t) \\
\text{Wavefunction vector : } W^{\psi(x)} \doteq \psi(x, t) \\
\text{Momentum (1,1) tensor : } \hat{p}^{\psi(x)}_{\psi(y)} \doteq -i \delta'(x - y).
\]

Notice that, in the LHS, $\psi(x)$ is just a label, a discrete continuous index in the vector space in which $\mathbf{W} \equiv W^{\psi(x)} \delta_{\Omega^{\psi(x)}}$ is defined (Einstein summation convention over repeated indices is assumed). In the $\psi$-coordinate system, the vector $\hat{p} \cdot \mathbf{W}$ (which defines the momentum of the field) is written as a contraction:

\[
\hat{p}^{\psi(x)}_{\psi(y)} W^{\psi(y)} \doteq -i \int dy \delta'(x - y) \psi(y, t) = -i \frac{\partial \psi}{\partial x}(x, t),
\]

which means that the (1,1) tensor $\hat{p}$ has the following local expression as an operator:

\[
\text{Operator expression : } \hat{p}[\psi] \doteq -i \, D,
\]

where $D \equiv \frac{\partial}{\partial x}$ is the “$x$-derivative operator” $\partial_x$.

Equations of type (1) and (2) are thus local (i.e., coordinate-dependent) representations of the invariant structures. In order to see the above structures in the “$p$-representation” or $\phi$-coordinate system instead, we have to map with the transformation matrix $F^{\phi(p)}_{\psi(x)} = \frac{\delta \Omega^{\psi(x)}}{\delta \Omega^{\phi(p)}} \equiv \frac{\exp(i px)}{\sqrt{2\pi}}$ (the Fourier transform). We get:

\[
\text{Coordinate system : } \Omega^{\phi(p)} \doteq \phi(p, t) \equiv \int dx \frac{\exp(ipx)}{\sqrt{2\pi}} \psi(x, t) \\
\text{Wavefunction vector : } W^{\phi(p)} \doteq \phi(p, t) \\
\text{Momentum (1,1) tensor : } \hat{p}^{\phi(p)}_{\phi(q)} \doteq p \delta(p - q) \\
\text{Operator expression : } \hat{p}[\phi] \doteq p \mathbb{I} \\
\]

where $\mathbb{I}$ is the Identity operator. The momentum $\hat{p} \cdot \mathbf{W}$ is written in the $\phi$-coordinate system:

\[
\hat{p} \cdot \mathbf{W}^{\phi(p)} \equiv \hat{p}^{\phi(p)}_{\phi(q)} W^{\phi(q)} \doteq p \phi(p, t).
\]

We remark that an operator expression may be defined for type (2,0) and (0,2) tensors as well. This will be the case of Poisson and Lagrange Brackets, and Metrics.
2.2 The Lagrangian Structure

We are going to discuss the construction of Action Principles for field equations. Take for example the Heat Equation:

\[ \frac{\partial \omega}{\partial t}(x,t) = \frac{\partial^2 \omega}{\partial x^2}(x,t). \]

This is a first order (in time) equation for one real field (\( \omega \)), and one spatial dimension (\( x \)). Let us promote the \( x \)-coordinate to a continuous index, by means of the representation:

\[ \Omega^{\omega(x)} = \omega(x,t). \]

This is merely the definition of the coordinates \( \Omega^{\omega(x)} \) for every \( x \). Let us define the vector \( V \) by its components in the \( \omega \)-coordinate system:

\[ V^{\omega(x)} = \frac{\partial^2 \omega}{\partial x^2}(x,t). \]

We are dealing with an infinite-dimensional vector space over \( \mathbb{R} \). The Heat Equation, therefore, can be cast into the general form of an autonomous system of first-order equations of motion for one real field,

\[ \frac{d\Omega^{\psi(x)}}{dt} = V^{\psi(x)} \left[ \Omega^{\psi(y)}(t) \right], \]

where \( \psi(x) \) is a multi-label consisting of: a discrete (\( \psi \)) label, representing the coordinate system, and a set of continuous (\( x = (x_1, x_2, \ldots) \)) indices (the set of independent variables excluding time); \( V^{\psi(x)} \) are the components of the vector \( V = V^{\psi(x)} \frac{\delta}{\delta \Omega^{\psi(x)}} \) in the \( \psi \)-coordinate system. The Heat Equation is obtained setting the label \( \psi = \omega \), i.e., in the \( \omega \)-coordinate system, while the Potential Burgers Equation is obtained setting \( \psi = u \), where \( \omega(x,t) = e^{u(x,t)} \) is the change of coordinates (see section 6). We remark that this formalism may be extended straightforwardly to the case of many interacting fields.

In this setting, it can be shown \[3\] that the equations of motion (3) are related to a Variational Principle, given by

\[ S = \int_{t_0}^{t_1} dt \left[ L_{\psi(x)} \left[ \Omega^{\psi(y)}(t), t \right] \left( \frac{d\Omega^{\psi(x)}}{dt} - V^{\psi(x)} \left[ \Omega^{\psi(y)}(t) \right] \right) \right], \]

where the integration over repeated continuous indices is assumed. Here, \( L_{\psi(x)} \left[ \Omega^{\psi(y)}(t), t \right] \) are the components of the Lagrangian 1-form, which obeys the following equation:

\[ \frac{\partial L_{\psi(x)}}{\partial t} + L_{\psi(x), \psi(y)} V^{\psi(y)} + L_{\psi(y)} V^{\psi(y), \psi(x)} = 0 , \]
\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_V \right) L = 0
\]

where \( L_{\psi(x),\psi(y)} \equiv \frac{\delta L_{\psi(x)}}{\delta \Omega_{\psi(y)}} \) is the functional derivative, \( \frac{\partial}{\partial t} \) denotes a derivative on the explicit dependence on time (i.e., under constant \( \Omega_{\psi(x)} \)) and \( \mathcal{L}_V \) is the Lie derivative along the vector \( V \).

The Euler-Lagrange equations which come from the action (4) are:

\[
\Sigma_{\psi(x)\psi(y)} \left( \frac{d\Omega_{\psi(y)}}{dt} - V_{\psi(y)} \right) = 0,
\]

where \( \Sigma_{\psi(x)\psi(y)} \equiv L_{\psi(y),\psi(x)} - L_{\psi(x),\psi(y)} \) are the components of the 2-form Lagrange Bracket (LB) \( \Sigma \equiv \delta L \); it is closed under exterior derivation \( \delta \Sigma = 0 \) by definition and it obeys the Flow-Along Equation:

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_V \right) \Sigma = 0.
\]

These structures are not unique, for there may be different Lagrangians (not differing by a total derivative) for the same equation of motion \( [\text{II}] \). We construct some of them using the Projector, which is defined in the next section. It should be noted that the LB must have no Kernel in order that the Action Principle (II) be equivalent to the equations of motion \( [\text{II}] \), otherwise we would get “deformed” equations, the deformation being related to the Kernel of the LB (this is an interesting issue, anyway).

### 2.3 The Projector

For any tensorial object we may associate its Projection to a solution of the Flow-Along Equation for some vector \( V \). We define the Projector \( \mathcal{P}_V \) as:

\[
\mathcal{P}_V \equiv \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \left( \frac{\partial}{\partial t} + \mathcal{L}_V \right)^n.
\]

It can be shown that if \( T \) is any tensor, then

\[
\left( \frac{\partial}{\partial t} + \mathcal{L}_V \right) (\mathcal{P}_V T) = 0,
\]

that is, \( \mathcal{P}_V T \) is the required solution to the Flow-Along Equation for \( V \). In this way, starting from any tensor, we can get Symmetries, Lagrangians, and other structures, if the value of the projected tensor converges. We comment finally that if two tensors solve the Flow-Along Equation, then any contraction of their indices solves it also.

\[2\text{We use the name “Flow-Along Equation” here for the first time, though this equation has been already introduced in the respective references.}\]
3 Lagrangian approach to the Heat Equation

The Heat Equation (HE), for the field $\omega(x,t)$ and the independent coordinate $x \in \mathbb{R}$:

$$\omega_t(x,t) = \omega_{xx}(x,t),$$

where $\omega_t \equiv \frac{\partial \omega}{\partial t}$ and $\omega_{xx} \equiv \frac{\partial^2 \omega}{\partial x^2}$, is represented by the vector

$$V_{\omega(x)} \doteq \omega_{xx}(x,t).$$

From here on, we drop the label $\omega$ for all the tensors in the $\omega$-coordinate system only when there is no ambiguity. In this way, the above equation reads:

$$V_x \doteq \omega_{xx}(x,t).$$

We will project a LB for $V$ starting from the 2-form “anti-derivative operator”\[6\] (which is clearly not a LB for $V$):

$$\sigma \succ D^{-1}.$$

As $\sigma$ is not explicitly time-dependent, the projection $P_V \sigma$ involves only multiple Lie derivatives along $V$. It can be shown that, for $n \geq 0$:

$$(\mathcal{L}_V^n \sigma) \succ 2^n D^{2n-1}.$$

We sum to get the projection, finding:

$$\Sigma^{[1]} \equiv P_V \sigma \succ \sum_{n=0}^{\infty} \frac{(-2t)^n}{n!} D^{2n-1},$$

or

$$\Sigma^{[1]} \succ e^{-2tD^2} D^{-1}.$$

The inverse\[4\] of this operator is a time-dependent PB (which, unfortunately, does not define a Hamiltonian Structure\[4\]):

$$J_{[1]} \equiv (\Sigma^{[1]})^{-1} \succ D e^{2tD^2}.$$

Notice that this PB has a finite dimensional Kernel, generated by the 1-form with constant components.

We build a Lagrangian for the HE contracting the vector $V$ with $\Sigma^{[1]}$, obtaining

$$L^{[2]}_x \equiv \Sigma^{[1]}_{xy} V^y \doteq e^{-2tD^2} \omega_x(x,t)$$

and

$$\delta L^{[2]} \succ -2D e^{-2tD^2}.$$

\[3\]We define the (up-to-the-kernel) inverse of a (0, 2) tensor $\Sigma$ as a (2, 0) tensor $J$ in terms of a contraction: $J^{xz} \Sigma_{zy} = \delta(x-y)$. 

6
This defines a new LB:

\[ \Sigma^{[2]} \equiv \delta L^{[2]} \succ -2De^{-2tD^2}. \]

Notice that this new LB has a finite dimensional Kernel, generated by the constant Symmetry, while \( \Sigma^{[1]} \) does not have a Kernel, and thus the first LB defines a good Action Principle for the HE, with Lagrangian 1-form given by:

\[ L^{[1]}_x = -\frac{1}{2}e^{-2tD^2}D^{-1}\omega(x,t) \]

and the Action is:

\[ S = \frac{1}{2} \int_{t_0}^{t_1} dt \int dx \left( e^{-2tD^2}D^{-1}\omega(x,t) \right) \left( \omega_t(x,t) - \omega_{xx}(x,t) \right). \]

The limits in the \( x \)-integration are \([-a,a]\), with \( 0 < a < \infty \) (HE in the circle) or \( a = \infty \) (HE in the real line).

It is easy to see that these two LB (\( \Sigma^{[1]} \) and \( \Sigma^{[2]} \)) for \( V \) are part of a hierarchy of LBs for the HE, for consider the Strong Symmetry \( \Lambda^{[1]} \succ D \).

This is a known Strong Symmetry for \( V \) (see [1]). If we contract it twice with our (anti-symmetric) \( \Sigma^{[1]} \) we obtain, apart from a constant factor, \( \Sigma^{[2]} \), which is also anti-symmetric and closed. Every two contractions with \( \Lambda^{[1]} \) or \( (\Lambda^{[1]})^{-1} \) will give us an anti-symmetric \((0,2)\) closed tensor, solution of the Flow-Along Equation; in other words, we get a hierarchy or succession of LBs (and therefore Action Principles) for the HE.

### 4 The Metric Structure for the Heat Equation

Consider now just one contraction to obtain a symmetric \((0,2)\) tensor, solution of the Flow-Along Equation:

\[ G^{[1]} \equiv \Sigma^{[1]} \cdot \Lambda^{[1]} \succ e^{-2tD^2}, \]

which has no Kernel. Its inverse is the \((2,0)\) tensor:

\[ G^{[1]} \succ e^{2tD^2}. \]

We define a Metric Structure (MS) as a pair of solutions of the Flow-Along Equation for \( V \): a –hopefully invertible– symmetric \((2,0)\) tensor \( G \) (Metric) and a Constant of Motion zero-form \( M \) (Metric- or M-Hamiltonian), such that:

\[ V^x = G^{xy}\delta M_y, \]

\(^4\)A Strong Symmetry is a \((1,1)\) tensor solution of the Flow-Along Equation.
where $\delta M_y \equiv M_y \equiv \frac{\delta M[\omega]}{\delta \omega(y,t)}$. 

Equation (9) is analogous to that which appears in the definition of a Hamiltonian Structure [1, 3, 4], but we see that in a MS the dynamics of any zero-form $C$ is now:

$$\frac{dC}{dt} = \frac{\partial C}{\partial t} + \{C, M\},$$

where $\{C, M\} \equiv C_{x}G^{xy}M_y$ is the anti-commutator. We remark that the M-Hamiltonian $M$ itself must be an explicitly time-dependent Constant of Motion in general, since $\frac{dM}{dt} = 0$ and thus $\frac{\partial M}{\partial t} = \frac{dM}{dt} - \{M, M\} = -\{M, M\} \neq 0$.

For the Metric $G_{[1]}$, the M-Hamiltonian is shown to be:

$$M^{[1]}[\omega] = -\frac{1}{2} \int dx \left( e^{-tD^2} \omega_x(x,t) \right)^2. \quad (10)$$

5 The General Solution to the Heat Equation and related tensors

If we look at equation (10), the definition of the M-Hamiltonian $M^{[1]}$ is given in terms of $e^{-tD^2}$, an operator with $x$-derivatives of infinite order, acting on the field $\omega(x,t)$. In order to deal with these infinite order derivatives, we are tempted to change our coordinates to

$$\tau(x,t) \equiv e^{-tD^2} \omega(x,t).$$

With this change of coordinates, the HE becomes for $\tau$:

$$\frac{d\tau}{dt}(x,t) = 0,$$

with the general solution

$$\tau(x,t) = \tau^{[0]}(x).$$

We note that the M-Hamiltonian of the last section is written now:

$$M^{[1]}[\tau] \equiv -\frac{1}{2} \int dx \left( \tau_x(x,t) \right)^2,$$

which is naturally a constant of the motion for the dynamics (11).

Finally, the general solution of the Heat Equation is:

$$\omega^{[0]}(x,t) = e^{tD^2} \tau^{[0]}(x).$$

All this may seem obvious, but it is very powerful. For example, if $\tau^{[0]}(x)$ is a polynomial in the variable $x$, we get immediately polynomial solutions for the HE. It works even with distributions: take $\tau^{[0]}(x) = \delta(x - s)$, and the associated solution to the HE is the known solution $\omega^{[0]}(x,t) = \frac{e^{-\frac{(x-s)^2}{4t}}}{\sqrt{4\pi t}}$.

In the next subsections, structures for the HE are presented.
5.1 Symmetries for the Heat Equation

First, we note that any Symmetry for the HE is also a solution of it (because it is a linear equation), thus we may say that

\[ \eta^x \equiv e^{tD^2} F[\tau](x), \]

with \( F \) an arbitrary functional of \( \tau \) and a function of \( x \) only (not of \( t \)), is a Symmetry for \( V \).

As an example, let us suppose that \( F[\tau](x) = \int dy f(x, y) \tau(y) \), where \( f(x, y) \) is an arbitrary function of its arguments. Then, assuming periodic boundary conditions or vanishing limiting values for the fields \( \omega(x, t) \) and \( f(z, x) \) if \( x \) is in the circle or the real line, respectively, we find that:

\[ \eta^x \equiv \int dy N(x, y, t) \omega(y, t) \]

is a linear, nonlocal, Symmetry for the Heat Equation provided

\[ (\partial_t - \partial_{xx} + \partial_{yy}) N(x, y, t) = 0, \]

and \( N(x, y, t) \) is periodic in \( y \) or has vanishing \( y \rightarrow \pm \infty \) values if \( y \) is in the circle or the real line, respectively.

We will use these Symmetries, choosing \( \partial_t N = 0 \), for the construction of the Hamiltonian Structure in section 5.4.

We can consider equation (12) as a solution of the HE, so this becomes a way to find new solutions for the HE starting from known ones.

Observe that the known local Symmetries \( \eta^{x(l)} \equiv \frac{\partial}{\partial x} \omega(x, t) \) of the Heat Equation are just special cases of the above construction, with \( N(x, y, t) = \delta^{(l)}(x - y) \).

5.2 Lagrangians for the Heat Equation

Now, recalling the definition of a Lagrangian 1-form (5), when we apply it to the HE it reads:

\[ \bar{L}_x + \partial_{xx} L_x = 0, \]

where the barred dot indicates a total time derivative taken on-shell. If we set

\[ L_x \equiv e^{-tD^2} P[\tau](x), \]

with \( P \) an arbitrary functional, then \( L \) is a Lagrangian for \( V \). Thus we see that the Lagrangians constructed in section 3 are obtained by setting \( P^{(1)}[\tau](x) = \tau_x(x), \ P^{(2)}[\tau](x) = -\frac{1}{2} \tau(x) \).

As an example, we show a construction similar to the one given for the Symmetry. Let us set \( P[\tau](x) = \int dy p(x, y) \tau(y) \), so that the Lagrangian reads:

\[ L_x \equiv \int dy Q(x, y, t) \omega(y, t), \]

(13)
where $Q$ is any solution of the $(2 + 1)$-dimensional negative-time Heat Equation:

$$ (\partial_t + \partial_{xx} + \partial_{yy}) Q(x, y, t) = 0, $$

and we impose boundary conditions on $Q(x, y, t)$ for the $y$ variable just as those for $N(x, y, t)$ in the example for Symmetries. The LB which comes from the Lagrangian \([13]\) is just

$$ \Sigma_{xy} = Q(y, x, t) - Q(x, y, t), $$

and we have then Action Principles for the HE modulo the Kernel of the tensor above.

### 5.3 Lagrange Brackets and Metrics for the Heat Equation

The $(0, 2)$ tensor defined by the operator:

$$ R \succ e^{-tD^2} \tilde{R}[\tau] e^{-tD^2}, $$

where $\tilde{R}[\tau]$ is any integro-differential operator depending on $\omega$ and time only through $\tau$, is a solution of the Flow-Along Equation for the Heat Equation. The proof is straightforward. We get a LB for the HE ($R$ anti-symmetric and closed) if $\tilde{R}[\tau]$ is closed in the $\tau$-coordinate system. We get a MS for the HE if $\tilde{R}[\tau]$ is a symmetric tensor, such that

$$ \partial_{yy} \tilde{R}[\tau]_{\tau(x)\tau(y)} + \int dz \tau_{zz} \tilde{R}[\tau]_{\tau(x)\tau(z),\tau(y)} - (x \leftrightarrow y) = 0, \quad (14) $$

so that an M-Hamiltonian can be defined in\[5\]

$$ V^x = R^{xy} \delta M_y. $$

For example, a Metric Structure is defined when $\tilde{R}[\tau]_{\tau(x)\tau(y)} = \tilde{R}^0(x, y)$ is a symmetric and $\tau$-independent solution of:

$$ (\partial_{xx} - \partial_{yy}) \tilde{R}^0(x, y) = 0, $$

and the M-Hamiltonian is

$$ M^0[\omega] = \frac{1}{2} \int dx \int dy \tilde{R}^0(x, y) \left( e^{-tD^2} \omega(x, t) \right) \left( e^{-tD^2} \omega_{yy}(y, t) \right). $$

The MS found in section \[4\] is a particular case of the above construction, with $\tilde{R}^0(x, y) = \delta(x - y)$, the Dirac Delta distribution.

\[5\]We are assuming we may write $R^{xz} R_{zy} = \delta(x - y)$, otherwise the equations would get extra terms, related to the Kernel of the operators $R^{xy}$ and $R_{xy}$. 

10
5.4 Poisson Brackets and Hamiltonian Structures for the Heat Equation

It should be clear that the \((2,0)\) tensor written as the operator

\[ J \succ e^{tD^2} \tilde{J}[\tau] e^{tD^2} \]

is a Poisson Bracket for the HE provided the \((2,0)\) tensor \(\tilde{J}[\tau]\) is anti-symmetric, satisfies the Jacobi Identity \([3, 4]\) in the \(\tau\)-coordinate system, and depends on time and \(\omega\) only through \(\tau\). Nevertheless, a Hamiltonian Structure is given only when the partial time derivative of \(J\) (under constant \(\omega\)) is zero \([4]\), in which case the Hamiltonian \(H\) is a time-independent Constant of the Motion and obeys:

\[
V^x = J^{xy} \delta H_y.
\]

We consider now an improvement of a time-independent PB for the periodic HE (i.e., periodic boundary conditions on the field \(\omega(x,t)\) at \(x = -a, a\)) recently found \([1]\). We use the technique illustrated in \([1, 4]\). It starts with a time-independent Constant of the Motion \(H^o\) (the Hamiltonian) for the HE, a time-independent Symmetry \(\eta^o\) such that \(\mathcal{L} H^o \neq 0\) and it needs a set \(\Gamma\) of time-independent Symmetries such that:

\[
\mathcal{L} H^o = 0 \quad \forall \quad \eta \in \Gamma, \\
\mathcal{L} \eta^o = 0 \quad \forall \quad \eta \in \Gamma, \\
\mathcal{L} \bar{\eta} = 0 \quad \forall \quad \eta, \bar{\eta} \in \Gamma.
\]

After we find these ingredients, we form a Poisson Bracket which, in components, reads:

\[
J^{xy} = \frac{V^x \eta^o_y - \eta^o_x V^y}{\mathcal{L} H^o} + \sum_{\eta, \bar{\eta} \in \Gamma} \eta^x \bar{\eta}^y - \eta^y \bar{\eta}^x.
\]

We will take (all integrals in the rest of this subsection are assumed to have limits \([-a,a]\)):

\[
H^o_{\eta^o} \doteq \int dx \omega(x,t) \\
\eta^o_x \doteq 1,
\]

and look for Symmetries of type \([12]\) for the set \(\Gamma\). If we use time-independent functions (or distributions) \(N(x,y)\) such that:

\[
\int dz N(z,x) = 0 \\
\int dz \tilde{N}(x,z) = 0 \\
\int dz \left( N(x,z) \tilde{N}(z,y) - \tilde{N}(x,z) N(z,y) \right) = 0 \\
N(x,y) = A(x+y) + B(x-y),
\]

with \(A\) and \(B\) periodic distributions, then \(N, \tilde{N}\) define Symmetries in the set \(\Gamma\), via equation \([12]\).
It is an easy matter to show that the relations

\[
\begin{align*}
A &= A_s \\
B &= B_s \\
\int dz \left( A(z) + B(z) \right) &= 0,
\end{align*}
\]

where \( A_s(x) \equiv A(-x) \), define solutions of equations (15) and thus define Symmetries \( \eta_{AB}^x \equiv \int dy \left( A(x+y) + B(x-y) \right) \omega(y,t) \) in \( \Gamma \). The final result, which is the PB for the HE, is:

\[
J_{xy} = \frac{\omega_{xx}(x,t) - \omega_{yy}(y,t)}{2a} + \sum_{A,B,\bar{A},\bar{B}} C_{AB\bar{A}\bar{B}} \int dr \int ds \omega(r,t) \omega(s,t) W_{AB\bar{A}\bar{B}}(r,s;x,y),
\]

where \( C_{AB\bar{A}\bar{B}} \) are constants,

\[
W_{AB\bar{A}\bar{B}}(r,s;x,y) = \left[ A(x+r) + B(x-r) \right] \left[ \bar{A}(y+s) + \bar{B}(y-s) \right] - x \leftrightarrow y,
\]

and the sum runs over the space of distributions defined by equations (16), normalized in the sense that their first non-zero Fourier coefficient is equal to 1. In this way, we have got a true Hamiltonian Structure starting from the trivial general solution of the HE, and we claim it can be made of maximal rank, because the condition of time-independence for the PB, inherited by the Symmetries in the set \( \Gamma \), makes \( N(x,y) \) be independent of time, and the solutions to equations (15) seem to be exhausted by the set (16), at least in terms of the dimension of the space of solutions (for there is another type of solution of equations (15), namely (16) but with \( A_s = -A \)). Of course, we should be able to compute an invertible \( J \) as well as its inverse if our claim was right.

### 5.5 Strong Symmetries for the Heat Equation

Any \((1,1)\) tensor written as the operator

\[
\Lambda \succ e^{tD^2} \bar{\Lambda}[\tau] e^{-tD^2},
\]

where the \((1,1)\) tensor \( \bar{\Lambda}[\tau] \) depends on time and \( \omega \) only through \( \tau \), is a Strong Symmetry for the HE. For example, if we set \( \bar{\Lambda}_{[1]}[\tau] \succ D \) and \( \bar{\Lambda}_{[2]}[\tau] \succ x \) we obtain, respectively:

\[
\begin{align*}
\Lambda_{[1]} &\succ D \\
\Lambda_{[2]} &\succ 2tD + x,
\end{align*}
\]

which are known Strong Symmetries for the HE.\(^6\)

---

\(^6\)It is an improvement of the PB found in \([\text{4}]\), which is obtained by setting \( C_{AB\bar{A}\bar{B}} = 0 \) unless \( A = \bar{A} = 0 \), \( B(\alpha) = \delta^{(2\bar{n})}(\alpha) \), and \( \bar{B}(\alpha) = \delta^{(2\bar{n})}(\alpha) \), where \( n, \bar{n} \in \mathbb{N} \) and \( \delta^l(\alpha) \) is the \( l \)-th derivative of the Dirac Delta distribution.
6 The Potential Burgers Equation

If we make the change of coordinates (Cole-Hopf [8, 9] transformation):

\[ \omega(x, t) = e^{u(x, t)} , \]

we map the Heat Equation (6) into the Potential Burgers Equation (PBE):

\[ u_t(x, t) = u_{xx}(x, t) + [u_x(x, t)]^2 . \]

The structures just defined for the Heat Equation are mapped in the following way:

6.1 Symmetries for the Potential Burgers Equation

Any Symmetry of the PBE is written as:

\[ \eta^u(x) \equiv e^{-u(x, t)} e^{-tD^2} F[\tau](x) , \quad F \text{ arbitrary.} \]

Let us map the example for the HE, see equation (12), into the PBE. We end with a Symmetry for the PBE:

\[ \eta^u(x) \equiv \int dy N(x, y, t) e^{u(y, t) - u(x, t)} , \]

where \( (\partial_t - \partial_{xx} + \partial_{yy}) N(x, y, t) = 0 . \)

6.2 Lagrangians for the Potential Burgers Equation

The 1-form defined as:

\[ L_u(x) \equiv e^{u(x, t)} e^{-tD^2} P[\tau](x) , \quad P \text{ arbitrary,} \]

is a Lagrangian for the PBE.

The example for the HE, equation (13), is mapped on a Lagrangian for the PBE:

\[ L_u(x) \equiv \int dy Q(x, y, t) e^{u(y, t) + u(x, t)} , \]

where

\[ (\partial_t + \partial_{xx} + \partial_{yy}) Q(x, y, t) = 0 . \]

Finally, we map the example for the HE, equations (7) and (8), into the PBE:

\[ L_{u}^{[1]}(x) \equiv -\frac{1}{2} e^{u(x, t)} e^{-2tD^2} D^{-1} e^{u(x, t)} , \]

and the Action is:

\[ S = \frac{1}{2} \int_0^{\tau_1} dt \int dx \left( e^{u(x, t)} e^{-2tD^2} D^{-1} e^{u(x, t)} \right) \left( u_t(x, t) - u_{xx}(x, t) + [u_x(x, t)]^2 \right) . \]

The Euler-Lagrange equations that come from this Action Principle are equivalent to the PBE.
6.3 Lagrange Brackets and Metric Structures for the Potential Burgers Equation

The general Solution of the PBE’s Flow-Along Equation for a \((0, 2)\) tensor is the operator

\[
R[u] \succ e^{\tau} e^{-tD^2} \tilde{R}[\tau] e^{-tD^2} e^{\tau},
\]

where \(\tilde{R}[\tau]\) is a \((0, 2)\) tensor defined in the \(\tau\) coordinate system. We have a Lagrange Bracket if \(\tilde{R}[\tau]\) is anti-symmetric and closed in the \(\tau\) coordinate system.

The analysis for the MS for the PBE is similar to that given in section 5.3: when \(\tilde{R}[\tau]\) is a symmetric solution of equation (14), an M-Hamiltonian \(M[u]\) can be defined in:

\[
V^u(x) = R^{u(x)u(y)} \delta M_{u(y)}.
\]

As it is shown in section 5.3, the most simple solution to equation (14) we get is when \(\tilde{R}^{\tau(x)\tau(y)} = \tilde{R}^0(x, y)\) is a symmetric and \(\tau\)-independent solution of:

\[
(\partial_{xx} - \partial_{yy}) \tilde{R}^0(x, y) = 0,
\]

and the M-Hamiltonian is mapped into the \(u\)-coordinate system (PBE):

\[
M^0[u] = \frac{1}{2} \int dx \int dy \tilde{R}^0(x, y) \left( e^{-tD^2} e^{u(x, t)} \right) \left( e^{-tD^2} e^{u(y, t)} \left( u_{yy}(y, t) + [u_y(y, t)]^2 \right) \right).
\]

7 Conclusions

We have obtained, for the Heat Equation (Potential Burgers Equation), the general solution of the Flow-Along Equation for many important tensors, such as Symmetries, Lagrangians, Poisson and Lagrange Brackets. We think these results, far from being trivial, as it could seem at first sight for a linear equation, have brought a better understanding of the structures related to these evolution equations. In addition to these, the main contributions in this work are: the Projector, an operator that takes any tensor and gives back a solution of the Flow-Along Equation; the construction of valid Action Principles for the Heat Equation (Potential Burgers Equation); the definition of a new structure, namely the Metric Structure, which bears some resemblance with the Hamiltonian Structure, but is defined in terms of a symmetric tensor: the dynamics of functions are anti-commutative and are written in terms of a time-dependent Constant of the Motion (the Metric Hamiltonian); finally, the construction of a seemingly maximal-rank Hamiltonian Structure for the Heat Equation, as an improvement of a recently found result.

The above results (e.g., Hamiltonian Structure) may be extended to the Burgers Equation and others just by mapping the corresponding tensors under the coordinate transformations. Also, the results for the Heat Equation may be generalized in a direct way to every linear equation, such as the Schrödinger Equation, and to finite-dimensional linear systems, like the small-oscillations problem with or without friction and/or gyroscopic term.
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[6] The derivative operator $D$ is represented by the derivative of the Dirac Delta distribution: $Df(x) = \int dy \delta'(x - y)f(y)$ and the anti-derivative operator $D^{-1}$ performs an integration and is represented by the anti-symmetric Heaviside Step distribution: $D^{-1}f(x) = \int dy \epsilon(x - y)f(y)$.

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