L-approximation of B-splines by trigonometric polynomials

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Dedicated to Professor Eleonora Storozenko on the occasion of her eighteen\textsuperscript{th} birthday

Denote by $\mathcal{T}_{2n-1}$ a space of real trigonometric polynomials

$$\tau(x) = \sum_{j=-n+1}^{n-1} \alpha_j \exp(2\pi i jx), \quad x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}, \quad \alpha_{-j} = \overline{\alpha}_j,$$

and denote by $\mathcal{T}_{2n-1}^\perp$ a subspace of real functions $f \in L_\infty = L_\infty(\mathbb{T})$ that are orthogonal to $\mathcal{T}_{2n-1}$ with respect to the “scalar product”

$$(f,g) = \int_{-1/2}^{1/2} f(u) g(u) \, du = \int_\mathbb{T} f(u) g(u) \, du.$$

It is known that the best approximation in $L = L(\mathbb{T})$ by trigonometric polynomials from $\mathcal{T}_{2n-1}$ may be calculated as follows (see [7])

$$E_n(f) := \inf_{\tau \in T_{2n-1}} \| f - \tau \|_1 = \sup_{g \in T_{2n-1}^\perp, \|g\| = 1} \int_\mathbb{T} g(u) f(u) \, du.$$

Here $\| \cdot \|_1$ and $\| \cdot \|$ denotes norms in $L$ and $L_\infty$ respectively.

This note is a continuation of our papers [1, 2], devoted to $L$-approximation of $L$-normed characteristic function

$$\chi_h(x) := \begin{cases} h^{-1}, & x \in (-h/2, h/2), \\ 0, & x \notin (-h/2, h/2), \end{cases}$$

of the interval $(-h/2, h/2)$ by trigonometric polynomials. In the paper [1] the sharp values of the best approximation for the special values of $h$ were found. In [2] we gave the complete solution of the problem for arbitrary values of $h \in (0, 1]$. In general case [2] the situation is more deep and results are not so simple as in [1]. For applications to the problem of optimal constants in the Jackson-type inequalities we need, however, results on $L$-approximation of B-splines and linear combinations of B-splines (see [6, 3]). Here we present some simple results about $L$-approximation of B-splines as well as give the proof of its sharpness for the special values of $h$. In some sense we give the appendix to the paper [1].

The B-splines are the convolutions of function $\chi_h$ with itself:

$$\chi_h^k(x) := \chi_h(x),$$

$$\chi_h^k(x) := \int_\mathbb{R} \chi_h(u) \chi_h^{k-1}(x-u) \, du = (\chi_h \ast \chi_h^{k-1})(x).$$
The $B$-splines are the functions with the $k - 1$ order smoothness and the supports $\text{supp } \chi^k_h = (-kh/2, kh/2)$, $|\text{supp } \chi^k_h| = kh$.

Particularly

\[
\chi^2_h(x) = \begin{cases} 
    h^{-1}(1 - |x|h^{-1}), & x \in (-h, h), \\
    0, & x \notin (-h, h).
\end{cases}
\]

It is easy to check that the operator of $k$-th order differentiation transforms $k$-th $B$-splines to $k$-th central differences:

\[
D^k(f \ast \chi^k_h)(x) = h^{-k}\Delta^k_h f(x),
\]

\[
\Delta^k_h f(x) := \sum_{j=0}^{k} (-1)^j \binom{k}{j} f(x + kh/2 - jh).
\]

Since

\[
\sup_x |\Delta^k_h f(x)| = \|\Delta^k_h f\| \leq \sum_{j=0}^{k} \binom{k}{j} \|f\| \leq 2^k \|f\|,
\]

then

\[
\|D^k(f \ast \chi^k_h)\| \leq (h/2)^{-k}\|f\|. \tag{1}
\]

One of the main tools in approximation theory is the classical Favard’s inequality:

\[
\|g\| \leq K_k(2\pi n)^{-k}\|D^k g\|, \quad g \in T_{2n-1}^1, \quad K_k := \frac{4}{\pi} \sum_{j=-\infty}^{\infty} \frac{1}{(4j + 1)^{k+1}}. \tag{2}
\]

The Favard’s constants $K_k$ have the following properties

\[1 = K_0 < K_2 = \pi^2/8 < \ldots < 4/\pi < \ldots < K_3 = \pi^3/24 < K_4 = \pi/2.\]

Direct consequence of (2) and (1) is

\[
\|g \ast \chi^k_h\| \leq K_k(2\pi n)^{-k}\|D^k(g \ast \chi^k_h)\| \leq K_k(\pi nh)^{-k}\|g\|, \quad g \in T_{2n-1}^1.
\]

Therefore, we have

\[
E_n(\chi^k_h)_1 = \sup_{g \in T_{2n-1}^1, \|g\|=1} \int_T g(u)\chi^k_h(-u) \, du \\
\leq \sup_{g \in T_{2n-1}^1, \|g\|=1} |(\chi^k_h \ast g)(0)| \leq K_k(\pi nh)^{-k}. \tag{3}
\]

**Theorem.** Let $k, n \in \mathbb{N}$, $k \leq n$, $h(\alpha) = \alpha/(2n)$, $0 < \alpha \leq 2n/k$. Then

\[
E_n(\chi^k_{h(\alpha)})_1 \leq F_k\alpha^{-k}, \quad \text{where} \quad F_k := (2/\pi)^k K_k. \tag{4}
\]

For example for $k = 1, 2, 3$ we have
The inequalities (4) become equalities if \( \alpha = 2j + 1, \ j \in \mathbb{Z}_+, \ j \leq \frac{2n - k}{2k} \).

The question about the value of the best \( L \)-approximation of \( B \)-spline for arbitrary \( 0 < \alpha \leq 2n/k \) is not so simple (see [2] for the case \( k = 1 \)).

**Proof.** The estimate (4) follows from the inequality (3). We need to prove equalities for \( \alpha = 2j + 1 \) only. At first consider the case \( k = 1 \). We will use notation
\[
c_y(x) := \cos(2\pi xy), \quad y \in \mathbb{R}.
\]
The function \( \pm \text{sign}(c_n), \ n \in \mathbb{N} \) gives equality in (4). In other words, for \( k = 1 \) and \( h_j = (2j + 1)/(2n), \ j = 0, \ldots, n - 1 \) we have
\[
E_n(\chi_{h_j})_1 \geq \int_{\mathbb{R}} \chi_{h_j}(u)(-1)^j \text{sign} c_n(u) \, du = 1/(2j + 1).
\]
One can rewrite the equality in (5) as
\[
\int_{\mathbb{R}} \chi_{2j+1}(u)(-1)^j \text{sign} c_{1/2}(u) \, du = 1/(2j + 1).
\]
Note, that \( \text{sign}(c_{1/2}(x)) \equiv \mathcal{E}_0(x) \), where \( \mathcal{E}_0(x) \) is the first Euler’s spline (see [4], pp. 148–151). The Euler splines \( \mathcal{E}_k(x) \) are defined as follows:
\[
\mathcal{E}_{j+1}(t) = \gamma_j \int_{-T}^{T} \mathcal{E}_j(x + u) \, du, \quad \gamma_j^{-1} = \int_{-1/2}^{1/2} \mathcal{E}_j(u) \, du,
\]
and have the following properties:
\[
\mathcal{E}_j(x + 2) = \mathcal{E}_j(x), \quad \mathcal{E}_j(x + 1) = -\mathcal{E}_j(x),
\]
\[
\int_{-1}^{1} \mathcal{E}_j(u + x) \, du = 0,
\]
\[
\mathcal{E}_j(-x) = \mathcal{E}_j(x), \quad \mathcal{E}_j(-x - 1/2) = \mathcal{E}_j(x + 1/2),
\]
\[
\|\mathcal{E}_j\| = 1, \quad \mathcal{E}_j(\nu) = (-1)^\nu, \quad \nu \in \mathbb{N},
\]
\[
D\mathcal{E}_j(x) = \pi \mathcal{K}_{j-1}\mathcal{K}_j^{-1}\mathcal{E}_{j-1}(x + 1/2).
\]
Come back to (6). Integrating by parts (7) we get
\((-1)^j \int_{\mathbb{R}} \chi_{2j+1}(u) \mathcal{E}_0(u) \, du = \)
\[\frac{(-1)^j}{2} \int_{\mathbb{R}} \chi_{2j+1}(u) \, D\mathcal{E}_0(u - 1/2) \, du = \frac{(-1)^{j+1}}{2} \int_{\mathbb{R}} D\chi_{2j+1}(u) \, \mathcal{E}_0(u - 1/2) \, du = \]
\[\frac{(-1)^{j+1}}{2}(2j+1)^{-1} \int_{\mathbb{R}} \Delta_{2j+1} \delta(u) \, \mathcal{E}_0(u - 1/2) \, du = \frac{(-1)^{j+1}}{2}(2j+1)^{-1}(\Delta_{2j+1} \mathcal{E}_0(-1/2)) = \]
\[\frac{(-1)^{j+1}}{2}(2j+1)^{-1}(\mathcal{E}_0(-j - 1) - \mathcal{E}_0(j)) = \]
\[\frac{(-1)^{j+1}}{2}(2j+1)^{-1}((-1)^{j+1} - (-1)^j) = (2j+1)^{-1}. \]

One can rewrite the proof for odd \(k\) without essential modifications:
\[\int_{\mathbb{R}} \chi_{i,j}(u)(-1)^j \text{sign } c_n(u) \, du = \int_{\mathbb{R}} \chi_{2j+1}(u)(-1)^j \text{sign } c_{1/2}(u) \, du = \]
\[\frac{(-1)^j K_k}{\pi^k} \int_{\mathbb{R}} \chi_{2j+1}(u) \, D^k \mathcal{E}(u - k/2) \, du = \frac{(-1)^{j+1} K_k}{\pi^k} \int_{\mathbb{R}} D^k \chi_{2j+1}(u) \, \mathcal{E}(u - k/2) \, du = \]
\[\frac{(-1)^{j+1} K_k}{\pi^k}(2j+1)^{-k} \int_{\mathbb{R}} \Delta_{2j+1} \delta(u) \, \mathcal{E}(u - k/2) \, du = \]
\[\frac{(-1)^{j+1} K_k}{\pi^k}(2j+1)^{-k}(\Delta_{2j+1} \mathcal{E}(u - k/2)) = \frac{(-1)^{j+1} K_k}{\pi^k}(2j+1)^{-k} 2^k = \frac{\mathcal{F}_k}{(2j+1)^k}. \]

Consider the case of even \(k\). We will use the equality
\[D^k \mathcal{E}(x) = \frac{\pi^k}{K_k} (-1)^{k/2} \mathcal{E}_0(x), \]
which implies
\[\int_{\mathbb{R}} \chi_{2j+1}(u) \, \text{sign } c_{1/2}(u) \, du = (-1)^{k/2} \frac{K_k}{\pi^k} \int_{\mathbb{R}} \chi_{2j+1}(u) \, D^k \mathcal{E}(u) \, du. \]
The integration by parts gives
\[\int_{\mathbb{R}} \chi_{2j+1}(u) \, D^k \mathcal{E}(u) \, du = \int_{\mathbb{R}} D^k \chi_{2j+1}(u) \, \mathcal{E}(u) \, du. \]
Since
\[D^k \chi_{2j+1}(u) = (2j+1)^{-k} \Delta_{2j+1} \delta(u), \]
then
\[(2j+1)^k \int_{\mathbb{R}} D^k \chi_{2j+1}(u) \mathcal{E}(u) \, du = (\Delta_{2j+1} \delta * \mathcal{E})(0) = \Delta_{2j+1} \mathcal{E}(0) = 2^k(-1)^{k/2}. \]

Remark 1. The restriction \(\alpha k \leq 2n\) in the Theorem means that we work with usual \(B\)-splines with support in \([-1/2, 1/2]\). The inequality (4) is true without this restriction, but we do not consider the sharpness of (4) for other values of \(\alpha k\).
Remark 2. The approximations of characteristic functions \((k = 1)\) were considered with details in [1, 2]. Note, that one can obtain the nontrivial approximation of step–function iff \(\alpha \geq 1\). For small \(\alpha\) the polynomial of the best approximation of the step–function is equal to 0, and the best approximation is equal to 1. As in the case of step–function we can indicate the value of parameter \(\alpha\) for the nontrivial estimates of the best approximation. But we do not know what is the critical value of \(\alpha_0\) for nontrivial approximation if \(k > 1\). For \(\alpha > k^{-1/k}\), \(k = 2, 3\), we have \(E_n(\chi_{h, (\alpha)}) < 1\) but we do not know the best approximation in the case \(\alpha = k^{-1/k}\). The minimum of the function \(x^{-1/x}\) lies in \([2, 3]\). Probably, there are some links between this fact and the optimal smoothness of the averaging operators. The averaging of the second order (convolution with the hat function \(\chi_h^2\)) often gives the most useful and sharp results.

Remark 3. This remark is close to Remark 2. We gave the Theorem in simple form. We can present here the more precise version of (4):

\[ E_n(\chi^k_{h, (\alpha)}) \leq \min \left( 1, \frac{F_k}{\alpha^k} \right). \]  

(4')

Note, that for \(\alpha \leq F_k^{1/k}\) the inequality \((4')\) gives trivial estimate. For \(\alpha > F_k^{1/k}\) (in other words, if the support of the \(k\)-th B-spline \(\chi_h^k\) is greater then \(kF_k^{1/k}/(2n)\)) the best approximation of \(\chi_h^k\) is less than 1.

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