Abstract

The general hydro-thermodynamic system of equations in 2+1 dimensions with arbitrary equations of state (Taylor series approximation) is split to eigen modes: Tollmienn-Schlichting (TS) wave and two acoustic ones. A mode definition is realized via local relation equations, extracted from the linearization of the general system over boundary layer flow. Each such connection defines the subspace and the corresponding projector. So the division is performed locally and could help in initial (or boundary) problems formulation. The general nonlinearity account determines the specific form of interaction between acoustic and vortical boundary layer perturbation fields. After the next projecting to a subspace of Orr-Sommerfeld equation solution for the TS wave and the corresponding procedure for acoustics, the equations go to one-dimensional system that describes evolution along the basic stream. A new mechanism of nonlinear resonance excitation of the TS wave by sound is proposed and modelled via four-wave interaction. Subjectclass: Primary 35Q30 ; Secondary 76F70 , 35Q72; Keywords: keywords fluid mechanics, boundary layer, projector to eigen modes, Tollmien-Schlichting waves, acoustic waves, nonlinear resonance, N-wave system.
Tollmien-Shlichting and sound waves interaction: general description and nonlinear resonances

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1 Introduction

It is well-established fact now that a free-stream and a surface disturbances strongly affect the processes in boundary layer which compose a transition to a turbulent state. This transition in turn determine such important parameters of a rigid body in fluid mechanics as skin friction and heat transfer. The classification of the free stream was done first by Kovasznay et al. [1]. It is shown, that within linear approximation a general small-amplitude unsteady disturbance in the free stream can be decomposed into three independent different types: acoustic (A), vortical and entropy modes. Only the first one relates to pressure fluctuations propagating with the sound speed, the last two don’t case any pressure perturbation. The common idea of many investigations is to pick out length and time scales of each of these disturbances that would make the possibility of the Tollmienn-Schlichting (T-S) waves nonlinear generation [2]. This way the mechanism of T-S wave generation by convecting gusts interacting with sound was justified and numerically investigated [3]. Appropriate scales of boundary roughness are proved to generate T-S waves both theoretically and experimentally [4].

In spite of abundant efforts (see the big introduction and the citations in [3]), devoted to the problem of a search of an effective control mechanism that support a cumulative direct energy exchange between T-S an acoustic modes. Let us mention three important papers about the general, local and distributed acoustic receptivity [5], [6], [7]. The results however do not look complete: we revisit the problem in the all-perturbations approach [8, 9] starting from boundary layer (BL) as a background. The perturbations are considered only over the stationary boundary layer, we do not account here (but plan to do it) the layer field as a dynamic variable. This scheme gives a possibility to study mutual interactions on the base of model integrable nonlinear evolution equation. In our approach the description is local by the construction and do not need averaging procedure [10]. It covers the known results and give new hopes for understanding related phenomena appeared in papers from [11] to
We would note that the initial stage of the process of its structure changes manifests. This effect is similar to heating or streaming generation by acoustic waves, it is a development of the stationary mode and corresponds to an initial stage of the BL reconstruction.

In this paper we concentrate our efforts on the mathematical formalism: introducing the complete set of basic modes we transform the fundamental system of standard conservation laws of fluid mechanics to a set of equivalent equations. In linear approximation the specific choice of new dependent variables split the system to the set of independent equations for the given modes, the account of nonlinearity naturally introduce the interaction by projecting the fundamental equations set in a vector form. Going to the nonlinear description, we use iterations inside the operator by the small parameter related to amplitude (Mach number for acoustics) and viscosity (Reynolds number). We also analyze the possibilities of resonant interaction of quasi-plane waves on the level of so-called N-wave systems. Being integrable such systems admit explicit solutions and plenty of conservation laws. Hence the detailed investigation of situation is possible in this approximation.

The mentioned types of waves (T-S and A) are defined by eigenvectors of the linearized system of dynamic conservation equations for the free flow. Once defined, eigenvectors (or modes) are fixed and independent on time. The process by which the free stream disturbances are internalized to generate boundary-layer instability waves is referred to as receptivity. The basic idea of this paper is to define the T-S and acoustics modes as eigenvectors of the same system for a viscous flow over a rigid boundary. The eigenvectors of the viscous flow go to the known limit in the free stream over a boundary. The eigenvectors of the viscous flow are fixed relations between specific perturbations (velocity components, density and pressure) for every mode followed from the linear equations, to construct projectors and applied them for the nonlinear dynamics investigation. Thus, in the linear dynamics the overall field may be separated by projectors to the independent modes at any moment. Fixing the relations when going to the nonlinear flow, one goes to a system of coupled evolution equations for the modes. These ideas have applied to nonlinear dynamics of exponentially stratified gas and bubbly liquid and other problems.

The principal difference between this consideration and previous ones is the necessity of variable coefficients incorporation that arise from a boundary layer structure. It means that such coordinate dependence impact the projectors structure: the projector operator should be constructed by nonabelian entries. By other words the matrix elements of the projector matrix will be operator-valued ones. In fact we revisit and develop the first attempt in this direction that had been made recently.

The T-S wave takes the place of a vortical mode in unbounded space, the difference is due to the linearization on the different background. When a stationary boundary-layer flow like Blausius one appears, a linearization should be correctly proceeded with account of the boundary-layer flow as a background, that would lead obviously to the other features of the vortical mode then that in unbounded space. An important feature of T-S mode is non-zero disturbance
of pressure already in the linear theory. In the three-dimensional flow, there exist two T-S modes, two acoustic ones (corrected by background flow), and one entropy mode as well.

The equations of interaction are derived in Sec. 3 by means of the division of the perturbation field on these subspaces and projecting the system of the basic equations on the same subspaces. This transformation that in fact is nothing but a change of variables allows to proceed in a choice of adequate approximation. Moreover we could analyze possibilities of a direct nonlinear resonance interaction account. The results are the following: two-wave and three-wave interaction does not contribute due to the structure of the interaction terms in its minimal possible order. Hence (in this order) only the four-wave interaction display a resonance structure. We derive the correspondent four-wave equations and analyze it in Sec.4.

2 Basic equations treating the equations of state in the general form.

The mass, momentum and energy conservation equations read:

\[
\frac{\partial \rho}{\partial t} + \nabla (\rho \vec{v}) = 0 \tag{1}
\]

\[
\rho \left[ \frac{\partial \vec{v}}{\partial t} + (\vec{v} \nabla) \vec{v} \right] = -\nabla p + \eta \Delta \vec{v} + \left( \zeta + \frac{\eta}{3} \right) \nabla (\nabla \vec{v}) \tag{2}
\]

\[
\rho \left[ \frac{\partial e}{\partial t} + (\vec{v} \nabla) e \right] + p \nabla \vec{v} = \chi \Delta T + \zeta \left( \nabla \vec{v} \right)^2 + \frac{\eta}{2} \left( \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} - \frac{2}{3} \delta_{ik} \frac{\partial v_l}{\partial x_l} \right)^2 \tag{3}
\]

Here, \( \rho, p \) are density and pressure, \( e, T \) - internal energy per unit mass and temperature, \( \eta, \zeta \), are shear, bulk viscosities, and \( \chi \) - thermal conductivity coefficient respectively (all supposed to be constants), \( \vec{v} \) is a velocity vector, \( x_i \) - space coordinates. Except of the dynamical equations (1,2,3), the two thermodynamic relations are necessary: \( e(p, \rho), T(p, \rho) \). To treat a wide variety of substances, let us use the general form of the caloric equation (energy) as expansion in the Taylor series:

\[
\rho_0 e = E_1 p + \frac{E_2 p_0}{\rho_0} \rho + \frac{E_3}{\rho_0^2} \rho^2 + \frac{E_4 \rho_0}{\rho_0^2} \rho^2 + \frac{E_5 \rho}{\rho_0^2} \rho^2 + \frac{E_6}{\rho_0^3} \rho^2 \rho + \frac{E_7}{\rho_0^4} \rho^3 + \frac{E_8}{\rho_0^5} \rho^3 + \ldots, \tag{4}
\]

and the thermic one

\[
T = \frac{\Theta_1}{\rho_0 C_v} p + \frac{\Theta_2 \rho_0}{\rho_0^2 C_v} \rho + \ldots. \tag{5}
\]
The background values for unperturbed medium are marked by zero, perturbations of pressure and density are denoted by the same characters (no confusion is possible since only perturbations appear below), $C_v$ means the specific heat per unit mass at constant volume, $E_1, \ldots, \Theta_1, \ldots$ are dimensionless coefficients. The two-dimensional flow in the coordinates $x$ (streamwise distance from the plate (model) leading edge), $z$ (wall-normal distance from a model surface) relates to the two-component velocity vector

$$\vec{v} = (u, w) + \vec{u}_0,$$

(6)

where $\vec{u}_0 = (U_0(z), 0)$ denotes the background streamwise velocity and $(u, w)$ stands for velocity perturbations. The system (1 - 3) with account of (4, 5, 6) yields in

$$\begin{align*}
\rho_0 \left( \frac{\partial u}{\partial t} + U_0 \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial x} - \eta \Delta u - \left( \zeta + \frac{2}{3} \right) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial x \partial z} \right) &= -\rho_0 u \frac{\partial u}{\partial x} - \rho_0 w \frac{\partial u}{\partial z} + \frac{\rho_0}{\rho_0} \frac{\partial p}{\partial x}, \\
\rho_0 \left( \frac{\partial w}{\partial t} + U_0 \frac{\partial w}{\partial x} + \frac{\partial p}{\partial x} \right) + \frac{\partial p}{\partial z} - \eta \Delta w - \left( \zeta + \frac{2}{3} \right) \left( \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial z^2} \right) &= -\rho_0 u \frac{\partial w}{\partial x} - \rho_0 w \frac{\partial w}{\partial z} + \frac{\rho_0}{\rho_0} \frac{\partial p}{\partial z}, \\
\frac{\rho_0}{\rho_0 C_v} \Delta p + \frac{\partial}{\partial x} \Delta \rho &= \left[ \frac{\partial}{\partial x} \Delta \rho \right] + \frac{\partial}{\partial z} \Delta \rho = \frac{\partial}{\partial z} \Delta \rho,
\end{align*}$$

(7)

where the constants $Z$ and $S$ are defined by

$$Z = \left( -1 + 2 \frac{1 - E_2}{E_1} E_3 + E_5 \right) / E_1,$$

$$S = \frac{1}{1 - E_2} \left( 1 + E_2 + 2 E_4 + \frac{1 - E_2}{E_1} E_5 \right).$$

(8)

The constant $c = \sqrt{\frac{\rho_0 (1 - E_2)}{\rho_0 E_1}}$ has the sense of linear sound velocity in the medium under consideration when $U_0 = 0$. The right-hand side of equations involves the quadratic nonlinear and viscous terms as well as linear ones related to thermal conductivity, no cross viscous-nonlinear terms accounted. In fact, the third equation follows from the energy balance and continuity equation. No assumptions on flow compressibility was not done yet.

### 3 Modes in the linear approximation.

The basic system (1 - 3) contains four dynamic equations and therefore, there are four independent modes of the linear flow: two acoustic ones, vorticity and heat modes. This is a classification by Kovasznay who defined the acoustic modes as isentropic and irrotational flow, and refers to the two last ones as to frozen motions, the vorticity one relating to the absence of pressure and density perturbations and the heat one relating to the very density perturbation. The system (1 - 3) involves three equations indeed, therefore only the three modes may be extracted - two acoustic and vorticity ones. That is due to the structure

5
of the heat mode where the only density perturbation occurs. Strictly speaking, the term treating a thermal conductivity in the third of equations (3) includes the density perturbations and the linearized system (7) (that defines the modes as possible types of flow) is not closed. If there was no this term at all there is a simple linear relation between density and pressure perturbations for the both acoustic modes. The presence of thermal conductivity corrects this relation as it was shown in [18]. When the effects of thermal conductivity are small, the corresponding terms may be placed to the right-hand side of equation together with nonlinear ones and be accounted further. So the excluding of the dynamic equation for density serve just a simplification of a problem suitable in the view of its extraordinary complexity when the heat mode is out of the area of interest.

The main idea is to define modes accordingly to the specific relations of the basic perturbation variables following from the linearized system of dynamic equations (1-3). In general, the procedure is algorithmic and may be expressed as consequent steps: to find the dispersion relation and its roots that determine all possible modes, and to define relations between specific variables for every mode. Later, projectors follow immediately from these relations, they separate every mode from the overall field of the linear flow exactly and serve as a tool for the nonlinear dynamics investigation.

Using the left-hand part of the system (7) as a basis of modes definition and introducing the two non-dimensional functions $V_0(z) = U_0(z)/U_\infty$, $\phi(z) = V_0(z)l_0$, we rewrite it in the non-dimensional variables:

$$x_* = x/l_0, w_* = w/U_\infty, u_* = u/U_\infty, t_* = tU_\infty/l_0, p_* = p/\rho_0 U_\infty^2.$$ (9)

The value $U_\infty$ marks velocity of a flow far from the boundary, and $l_0$ - boundary layer width. In the new variables (asterisks will be later omitted) (7) with zero right side reads:

$$\frac{\partial p}{\partial t} + V_0 \frac{\partial p}{\partial x} + \epsilon^{-2} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) = 0 \quad (10)$$

$$\frac{\partial u}{\partial t} + V_0 \frac{\partial u}{\partial x} + \phi w + \frac{\partial p}{\partial x} - Re^{-1} \Delta u - R^{-1} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial x \partial z} \right) = 0 \quad (11)$$

$$\frac{\partial w}{\partial t} + V_0 \frac{\partial w}{\partial x} + \frac{\partial p}{\partial z} - Re^{-1} \Delta w - R^{-1} \left( \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial z^2} \right) = 0 \quad (12)$$

with parameters $\epsilon = U_\infty/c$ (the Mach number), $Re = U_\infty l_0 \rho_0/\eta$ is the Reynolds number base on the length scale, and $R = U_\infty l_0 / (\eta/3 + \varsigma)$.

### 3.1 The Tollmienn-Schlichting mode.

Formally, the limit of incompressible fluid ($\epsilon = 0$) corresponds to the vorticity mode. The first relation for velocity components is well-known [3]:

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$ (13)
From the equations (11, 12) an expression for the pressure perturbation follows:

\[ 2\phi \frac{\partial w}{\partial x} + \Delta p = 0 \]  

(14)

where \( \Delta = \partial^2 / \partial z^2 - k^2 \) stands for the Laplacian so far the \( \partial / \partial x \) equivalent operator (multiplier) \(-ik\) is used. Both (13) and (14) define the TS mode due to relations of the specific perturbations of pressure and velocity components. Since the geometry of the viscous flow over boundary supposes strong non-uniformity in the vertical direction, all disturbances may be thought in the basis of not plane waves but in the functions like \( \Psi(x, z) = \psi(z) \exp(\omega t - i k x) \). So, one has to leave vertical derivatives that usually are large in comparison with the horizontal ones. The result is rather obvious hence we do not introduce a special small parameter. The vector of the T-S mode may be chosen as:

\[
T = \left( \begin{array}{c} p_{TS} \\ u_{TS} \\ w_{TS} \end{array} \right) = \left( \begin{array}{c} 1 \\ -\frac{1}{2ki} \frac{\partial}{\partial z} \Delta \\ -\frac{1}{2ik} \Delta \end{array} \right) p_{TS}
\]  

(15)

Note also, that the equations (10-12) yield the well-known equation for the TS mode (17), when rewritten for the new variable such the stream function \( u = \partial \Psi / \partial z , \ w = -\partial \Psi / \partial x \), with the obvious restriction to the solenoidal velocity field (13):

\[
\Delta \partial \Psi / \partial t + V_0 \Delta \partial \Psi / \partial x - \partial \Psi / \partial x \cdot \partial \phi / \partial z - Re^{-1} \Delta^2 \Psi = 0
\]  

(16)

The well-known Orr-Sommerfeld (OS) equation follows from (16):

\[
[\psi(z) - c] [\partial^2 \psi / \partial z^2 - k^2 \psi] - \psi \partial \phi / \partial z = \frac{i}{Re k} \left[ \partial^4 \psi / \partial z^4 - 2k^2 \partial^2 \psi / \partial z^2 + k^4 \psi \right]
\]  

(17)

That equation is an initial point of the laminar flow stability theory and for every pair \((k, Re)\) determines an eigenfunction \( \psi(z) \) and complex phase velocity \( c = \omega / k = c_r + ic_i \). The sign of \( c_i \) is namely a criterion of flow stability: a negative value corresponds to the growth of perturbation and therefore to the non-stability of the flow.

### 3.2 Acoustic modes

The potential flow imposed two acoustic modes with

\[
\partial u / \partial z - \partial w / \partial x = 0.
\]

In the limit of \( Re^{-1} = 0, \; R^{-1} = 0 ; \; \phi = 0 \), (11, 12) naturally goes to the acoustic modes dispersion relation that is directly connected with the wave operator:

\[
\epsilon^2 \frac{\partial^2}{\partial t^2} - \Delta
\]  

(18)
We would not consider here the perturbation velocity field changes, forced by the ambient movement of the fluid. It could be account by the perturbation theory to be developed here. Then, two acoustic modes are defined with relations between specific perturbations:

\[
A_1 = \begin{pmatrix} p_{A1} \\ u_{A1} \\ w_{A1} \end{pmatrix} = \begin{pmatrix} 1 \\ -i k \Delta^{-1/2} \\ \epsilon \frac{\partial}{\partial z} \Delta^{-1/2} \end{pmatrix} p_{A1},
\]
\[
A_2 = \begin{pmatrix} p_{A2} \\ u_{A2} \\ w_{A2} \end{pmatrix} = \begin{pmatrix} 1 \\ i k \Delta^{-1/2} \\ -\epsilon \frac{\partial}{\partial z} \Delta^{-1/2} \end{pmatrix} p_{A2}
\]  

(19)

Here the square root of the operator \( \Delta \) is defined as integral operator via Fourier transform.

4 Projectors in the linear problem.

The TS and two acoustic modes are determined by relations of specific perturbations or, in the vector form (15), (19). The superposition of such disturbances appear as all possible types of flow except of the heat mode. Every mode is completely defined by one of specific perturbations - pressure or velocity components since there are strict and local relations between them. Practically, in a linear flow, the overall perturbation may be de-coupled into modes by the corresponding orthogonal projectors. The arbitrary perturbation then is a sum of modes which, taking into account (15), (19) looks:

\[
\begin{pmatrix} p \\ u \\ w \end{pmatrix} = \begin{pmatrix} p_{A1} + p_{A2} + p_{TS} \\ u_{A1} + u_{A2} + u_{TS} \\ w_{A1} + w_{A2} + w_{TS} \end{pmatrix} = \begin{pmatrix} p_{A1} + p_{A2} + p_{TS} \\ H p_{A1} - H p_{A2} + K p_{TS} \\ M p_{A1} - M p_{A2} + Q p_{TS} \end{pmatrix}
\]  

(20)

with operators

\[
H = -\epsilon \Delta^{-1/2} k, \\
M = \epsilon \Delta^{-1/2} \frac{\partial}{\partial z}, \\
K = -\frac{1}{\phi k^2} \frac{\partial}{\partial z} \Delta, \\
Q = \frac{1}{\phi \partial k} \Delta
\]

(21)

The link (20) may be considered as a one-to-one map of dynamical variables that immediately yields in the case of TS wave the projector:

\[
P_{TS} = \begin{pmatrix} 0 & -2 k^2 \Delta^{-1} \phi \partial / \partial z \Delta^{-1} & -2 i k^3 \Delta^{-1} \phi \Delta^{-1} \\ 0 & \delta^2 / \partial z^2 \Delta^{-1} & i k \partial \partial z \Delta^{-1} \\ 0 & i k \partial \partial z \Delta^{-1} & -k^2 \Delta^{-1} \end{pmatrix} 
\]  

(22)

For a right and left acoustic waves one has

\[
P_{A1} =
\]
The projectors possess all properties of orthogonal projectors and their sum is unit matrix since all eigenvectors of linear system are accounted.

\[ P_{A2} = \]

\[
\frac{\Delta^{1/2}}{2} \begin{pmatrix}
\frac{1}{\sqrt{\Delta}} & 2ik\Delta^{-1/2}\frac{\partial}{\partial z} \Delta^{-1} - \frac{i\kappa}{\sqrt{\Delta}} \Delta^{-1/2} \\
-\frac{ik}{\sqrt{\Delta}} - \frac{\sqrt{\Delta}}{2k} & \frac{1}{\sqrt{\Delta}} \sqrt{\Delta} \frac{\partial}{\partial z} \Delta^{-1} - \frac{i\kappa}{\sqrt{\Delta}} \Delta^{-1/2}
\end{pmatrix}
\]

(23)

\[
\frac{\Delta^{1/2}}{2} \begin{pmatrix}
\frac{1}{\sqrt{\Delta}} & 2ik\Delta^{-1/2}\frac{\partial}{\partial z} \Delta^{-1} - \frac{i\kappa}{\sqrt{\Delta}} \Delta^{-1/2} \\
-\frac{ik}{\sqrt{\Delta}} - \frac{\sqrt{\Delta}}{2k} & \frac{1}{\sqrt{\Delta}} \sqrt{\Delta} \frac{\partial}{\partial z} \Delta^{-1} - \frac{i\kappa}{\sqrt{\Delta}} \Delta^{-1/2}
\end{pmatrix}
\]

(24)

The projectors of a projector into account. In the linear flow, the projectors separate every mode from the overall perturbation, for example:

\[ P_{TS} \begin{pmatrix} p \\ u \\ w \end{pmatrix} = \begin{pmatrix} p_{TS} \\ u_{TS} \\ w_{TS} \end{pmatrix} \]

(25)

and so on. Moreover, acting by a projector on the basic system of dynamic equations, yields a linear evolution equation for the mode assigned to this projector. One produces three equations indeed for every specific perturbations which are essentially the same with account of relations (14), (19). For the first rightwards progressive acoustic mode the evolution equation reads:

\[
\partial p_{A1}/\partial t + V_0 \partial p_{A1}/\partial x + \epsilon^{-1} \Delta^{1/2} p_{A1} = 0
\]

(26)

The equation for the second (opposite directed) acoustic mode is produced by acting the projector \( p_{A2} \) on the basic system and differ from (24) only by the sign before the last term. Combining equations for these directed acoustic modes one arrives at the wave equation of second order

\[
(\partial^2/\partial t^2 + V_0 \partial/\partial x)^2 p - \epsilon^{-2} \Delta p = 0.
\]

This equation appears as a limit of more general one: that relates to both acoustic modes and can be found in [16].
5 Nonlinear flow: coupled dynamic equations.

In the dimensionless variables introduced by (3), the dynamic equations with account of nonlinear terms of the second order, look:

\[
\begin{align*}
\frac{\partial p}{\partial t} + V_0 \frac{\partial p}{\partial x} + \epsilon^{-2} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) &= \tilde{\varphi}_1 \\
\frac{\partial u}{\partial t} + V_0 \frac{\partial u}{\partial x} + \phi w + \frac{\partial p}{\partial x} - Re^{-1} \Delta u - R^{-1} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 w}{\partial x \partial z} \right) &= \tilde{\varphi}_2 \\
\frac{\partial w}{\partial t} + V_0 \frac{\partial w}{\partial x} + \frac{\partial p}{\partial z} - Re^{-1} \Delta w - R^{-1} \left( \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial z^2} \right) &= \tilde{\varphi}_3
\end{align*}
\]

(27)

with a vector of the second-order nonlinear terms \( \tilde{\psi} \) in the right-hand side (a non-dimensional value \( \rho^* = \rho/\rho_0 \) is used in the right-hand side, the asterisk will be omitted later):

\[
\tilde{\varphi} = \begin{pmatrix}
\tilde{\varphi}_1 \\
\tilde{\varphi}_2 \\
\tilde{\varphi}_3
\end{pmatrix} = \begin{pmatrix}
-u \frac{\partial p}{\partial x} - w \frac{\partial p}{\partial z} + \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) [ Z p + \epsilon^{-2} S \rho] \\
-u \frac{\partial u}{\partial x} - w \frac{\partial u}{\partial z} + \rho \frac{\partial p}{\partial x} \\
-u \frac{\partial w}{\partial x} - w \frac{\partial w}{\partial z} + \rho \frac{\partial p}{\partial z}
\end{pmatrix}
\]

(28)

One could rewrite a system (27) in another way:

\[
\frac{\partial}{\partial t} \varphi + L \varphi = \tilde{\varphi},
\]

(29)

where the vector state \( \varphi \) is a specific perturbations column (vector of the fluid state)

\[
\varphi = \begin{pmatrix}
p \\
u \\
w
\end{pmatrix}
\]

(30)

and \( L \) is a matrix operator

\[
L = \begin{pmatrix}
V_0 \partial / \partial x & \epsilon^{-2} \partial / \partial x & \epsilon^{-2} \partial / \partial z \\
\partial / \partial x & V_0 \partial / \partial x - Re^{-1} \Delta - R^{-1} \partial^2 / \partial x^2 & \phi - R^{-1} \partial^2 / \partial x \partial z \\
\partial / \partial z & -R^{-1} \partial^2 / \partial x \partial z & V_0 \partial / \partial x - Re^{-1} \Delta - R^{-1} \partial^2 / \partial z^2
\end{pmatrix}
\]

(31)

All the projectors do commute with operators \( \partial / \partial t \cdot I \) and \( L \), so one can act by projectors on the system of equations (30), (31) directly thus obtaining the evolution equation for the correspondent mode. There are three such equations for the specific perturbations \( p, u, w \) for every mode (that are equivalent) accounting the relations between these specific perturbations. So, as independent variable for every mode a single specific perturbation, such as pressure or one velocity component, may be chosen.

Due to the existing tradition it is convenient to use the stream function as a basis for the TS mode and pressure perturbations for the acoustic modes.

Acting by the \( P_{TS} \) on the both sides of (31), one gets an evolution equation:

\[
\Delta \partial \Psi / \partial t + V_0 \Delta \partial \Psi / \partial x - \partial \Psi / \partial x \cdot \partial \phi / \partial z - Re^{-1} \Delta^2 \Psi = \frac{\partial}{\partial x} \left( \right) - \frac{\partial}{\partial z} \left( \right).
\]

(32)
Next it should be noted that in the right-hand nonlinear side \( p, u, w \) are overall perturbations to be presented as a sum of specific perturbations of all modes:

\[
\begin{align*}
\frac{\partial \Psi}{\partial z} + \epsilon \Delta^{-1/2} \frac{\partial p_{A1}}{\partial x} - \epsilon \Delta^{-1/2} \frac{\partial p_{A2}}{\partial x} \\
\frac{\partial \Psi}{\partial x} + \epsilon \Delta^{-1/2} \frac{\partial p_{A1}}{\partial z} - \epsilon \Delta^{-1/2} \frac{\partial p_{A2}}{\partial z} \\
p = 2\partial^2 / \partial x^2 \Delta^{-1}(\phi \Psi) + p_{A1} + p_{A2}
\end{align*}
\]  

Indeed, there is also a density perturbation in the right-hand nonlinear side that was not involved in the left-hand linear one at all. The continuity equation reads

\[
\partial \rho / \partial t + V_0 \partial \rho / \partial x + (\partial u / \partial x + \partial w / \partial z) = -\partial (\rho u) / \partial x - \partial (\rho w) / \partial z
\]  

Comparing linear left-hand side of equation (34) with that of the first equation from (26), the obvious relations for the both acoustic modes follow:

\[
\rho_{A1} = \epsilon^2 p_{A1}, \quad \rho_{A2} = \epsilon^2 p_{A2}.
\]

A limit \( \epsilon = 0 \) yields in the TS mode for incompressible flow: \( \rho_{TS} = 0 \). Therefore, the last relation for the overall density perturbation looks

\[
\rho = \epsilon^2 p_{A1} + \epsilon^2 p_{A2}
\]  

Finally, (32) goes to

\[
\Delta \Psi + V_0 \Delta \Psi_x - \Psi_x \phi_z - R e^{-1} \Delta^2 \Psi = -\Psi_z \Delta \Psi_x + \Psi_z \Delta \Psi_z - \epsilon \Delta \Psi \Delta^{1/2}(p_{A1} - p_{A2}) + \Delta \Psi_z \Delta^{-1/2}(p_{A1} - p_{A2})_z + O(\epsilon^2).
\]  

Derivatives are marked with lower indices. The first two nonlinear terms in the right-hand side of (36) expresses the TS mode self-action, and the last ones - cross acoustic-vorticity terms responsible for the acoustic mode influence on the TS mode propagation. The structure of the quadratic nonlinear column (30) yields in the absence of quadratic acoustic terms in (36).

In the limit of \( V_0 = 0 \), \( \phi = 0 \) the only self-action give the well-known evolution equation for vorticities transition [17] follows from (36):

\[
\Delta \Psi_t - R e^{-1} \Delta^2 \Psi + \Psi_z \Delta \Psi_x - \Psi_x \Delta \Psi_z = 0.
\]

Let an acoustic field consists of only the first mode. Acting by projector \( P_{A1} \) on the system (35), we get an evolution equation for this mode:

\[
\begin{align*}
\frac{\partial p_{A1}}{\partial t} + V_0 \partial p_{A1} / \partial x + \Delta^{1/2} p_{A1} / \epsilon = \frac{1}{2}[-u \partial p / \partial x - w \partial p / \partial z + (\partial u / \partial x + \partial w / \partial z)(Zp + S \rho e^{-2})] + \\
-\partial^2 / \partial x^2 \Delta^{-1} \phi \partial / \partial z \Delta^{-1} + \partial / \partial x(1/2 \epsilon) \Delta^{-1/2}[ -u \partial u / \partial x - w \partial u / \partial z + \rho \partial p / \partial x] + \\
(\partial^3 / \partial x^3 \Delta^{-1} \phi \Delta^{-1} + (1/2 \epsilon) \Delta^{-1/2} \partial / \partial z)[-u \partial w / \partial x - w \partial w / \partial z + \rho \partial p / \partial z].
\end{align*}
\]  

(37)
Here constants $Z$ and $S$ are defined earlier by (8). The variables $p$, $u$, $w$, are overall perturbations accordingly to (33), with $p_{A2} = 0$. So, (37) goes to the final version for the directed acoustic mode

$$
\epsilon (\partial p_{A1}/\partial t + V_0 \partial p_{A1}/\partial x) + \Delta^{1/2} p_{A1} =
\frac{1}{2} [-\Psi_x \Delta^{-1/2} p_{A1x} + \epsilon \Delta^{-1} \epsilon \frac{\partial}{\partial x} (\Psi_x \Delta^{-1/2} p_{A1x}) + 
2 \Psi_x \Delta^{-1/2} p_{A1xz} + 2 \Psi_{xx} \Delta^{-1/2} p_{A1xx} + \epsilon [ -\Psi_x \Delta^{-1} (\partial \Psi_x/\partial x) + \Psi_x \Delta^{-1} (\partial \Psi_x/\partial x) ] + 
\epsilon \Delta^{-1} \phi \Delta^{-1} \partial^2/\partial x^2 [ \Psi_x \Delta \Psi_x - \Psi_x \Delta \Psi_x ] + O(\epsilon^2)
$$

(38)

Between the nonlinear terms one can recognize interaction ($A1-TS$) and generation ones ($TS-TS$). The equation for $p_{A2}$ is obtained by projecting $P_{A2}$ and looks very similar. The complete system includes this equation and (38). The system covers all possible processes description up to quadratic terms approximation. Here a multimode TS waves in the (OS) equation solutions basis could be incorporated. The long-wave limit of such disturbance leads to coupled KdV system [14].

6 Resonance interaction of acoustic and TS modes.

Equations (36), (38) form a coupled system of evolution equations for interacting acoustic and TS modes. In the case of the TS mode generation by an incoming first acoustic mode, the early stage of evolution (for small amplitudes of TS mode) is defined by a system:

$$
\Delta \Psi_t + V_0 \Delta \Psi_x - \Psi_x \cdot \phi_x - Re^{-1} \Delta^2 \Psi =
-\epsilon (\Delta \Psi \cdot \Delta^{1/2} p_{A1} + \Delta \Psi_x \cdot \Delta^{-1/2} p_{A1x} + \Delta \Psi_z \cdot \Delta^{-1/2} p_{A1z})
$$

(39)

$$
\epsilon (\partial p_{A1}/\partial t + V_0 \partial p_{A1}/\partial x) + \Delta^{1/2} p_{A1} =
\frac{1}{2} [-\Psi_x \Delta^{-1/2} p_{A1x} + \Psi_x \Delta^{-1/2} p_{A1x} + 
2 \Psi_x \Delta^{-1/2} p_{A1xz} + 2 \Psi_{xx} \Delta^{-1/2} p_{A1xx} + \epsilon [ -\Psi_x \Delta^{-1} (\partial \Psi_x/\partial x) + \Psi_x \Delta^{-1} (\partial \Psi_x/\partial x) ] + 
\epsilon \Delta^{-1} \phi \Delta^{-1} \partial^2/\partial x^2 [ \Psi_x \Delta \Psi_x - \Psi_x \Delta \Psi_x ] + O(\epsilon^2)
$$

(40)

All quadratic terms relating to $TS-TS$ interaction, are left out of account. For simplicity, we consider only the first incoming acoustic mode.

As it follows from the discussion in the introduction, let us find a solution in the form:

$$
p_{A1}(x, z, t) = A_1(\mu_x, \mu_t) \pi_1 \exp(i(\omega_1 t - k_1 x)) + A_2(\mu_x, \mu_t) \pi_2 \exp(i(\omega_2 t - k_2 x)) + c.c.
$$

(41)

$$
\Psi(x, z, t) = B_3(\mu_x, \mu_t) \psi_3(z) \exp(i(\omega_3 t - k_3 x)) + B_4(\mu_x, \mu_t) \psi_4(z) \exp(i(\omega_4 t - k_4 x)) + c.c.
$$

(42)
where \( \Pi_1 = \pi_1(k_1, \omega, z) \exp(i(\omega_1 t - k_1 x)) \), \( \Pi_2 = \pi_2(k_2, \omega, z) \exp(i(\omega_2 t - k_2 x)) \) are planar waves. \( \Pi_1 \) satisfies the linear evolution equation

\[
\partial \Pi_1 / \partial t + V_0 \partial \Pi_1 / \partial x + \epsilon^{-1} \Delta^{1/2} \Pi_1 = 0.
\]

that leads to the equivalent equation for \( \Pi_1 \)

\[
i \omega_1 \Pi_1 - i k_1 V_0 \Pi_1 + \epsilon^{-1} \Delta^{1/2} \Pi_1 = 0.
\]

Suppose the vertical gradients of all wave functions inside the viscous layer are much bigger than horizontal ones. From experiments (e.g. \([11]\)), it is known, that wavelength of TS mode is much greater than a thickness of the boundary viscous layer for common values of Reynolds number. So, the operator \( \Delta^{1/2} \) may be evaluated as the generalized operator (Taylor) series with respect to \( \partial_z / k_1 \). Hence the operator radical in the first approximation is evaluated via Gataux derivative as

\[
\Delta^{1/2} \Pi_1 = \sqrt{-k_1^2 + \partial_z^2} \Pi_1 = ik_1 \sqrt{1 - \partial_z^2 / k_1^2} \Pi_1 \approx ik_1 (1 - \partial_z^2 / 2k_1^2) \Pi_1
\]

and, for \([44]\), one arrives to the ordinary differential equation, that we can consider as a spectral problem with the spectral parameter \( k_1 \),

\[
(1 - k_1^2 / \omega_1 V_0) \pi_1 + \frac{k_1}{\omega_1 \epsilon} (1 - \frac{1}{2k_1^2} \partial_z^2) \pi_1 = 0.
\]

(45)

The same equation obviously define \( \pi_2 \), it is enough to change indices 1 \( \rightarrow \) 2 in the operator. The functions \( \psi_3(z), \psi_4(z) \) are solutions of the OS equation \([17]\) suitable for a concrete problem. \( A_1, \ldots, B_4 \) are slowly varying functions of \( x, t \), that's why an additional small parameter \( \mu \) is introduced, generally, they are complex functions. Calculating the right-hand nonlinear expressions, we take only first term in series to avoid small terms of the higher order.

Let us discuss a possibility of four-waves resonance. Examining the algebraic relations between parameters yields the appropriate conditions:

\[
\omega_1 = \omega_2 - \omega_3, \omega_2 = \omega_1 + \omega_3, \omega_3 = \omega_1 + \omega_4, \omega_4 = \omega_3 - \omega_1
\]

(46)

Substituting the formulas \([41, 42]\) to \([43, 49]\), and picking up the resonant terms only, one goes to the further system of equations (complex conjugate values marked with asterisks, \( k_1 - k_2 + k_3 = \Delta k, k_3 - k_1 - k_2 = \Delta k' \)):

\[
\mu \left( \epsilon A_{1T} \pi_1 + A_{1X} \left( \epsilon V_0 \pi_1 - ik_1 \int_0^z \pi_1 dz \right) \right) = \\
\epsilon A_2 B_3 \left( 1.5 \left( ik_3^* \psi_3^* \pi_{2z} + ik_2 \psi_{3z} \pi_2 \right) + \int_0^z \left( ik_3^* \psi_3^* \pi_{2z} + ik_2 \psi_{3z} \pi_2 \right) dz e^{i(\Delta k)z}, \right) \\
\mu \left( \epsilon A_{2T} \pi_2 + A_{2X} \left( \epsilon V_0 \pi_2 - ik_2 \int_0^z \pi_2 dz \right) \right) = \\
\epsilon A_1 B_3 \left( 1.5 \left( -ik_3^* \psi_3^* \pi_{1z} + ik_1 \psi_{3z} \pi_1 \right) + \\
\right) \\
\right)

13
 integrals of

\[
\int_0^\pi \left( i k_3 \psi_3^* \pi_{1z} + i k_1 \psi_3^* \pi_{1z} \right) dze^{i(-\Delta k)x},
\]

\[
\mu B_3 T \psi_{3zz} + B_3 A \left( (V_0 + 4i k_3 Re^{-1}) \psi_{3zz} + (2k_3 \omega_3 - \phi_z) \psi_3 \right)
\]

\[
-\epsilon A_1 B_4 \left( \psi_{4zz} \pi_{1z} - k_1 k_4 \psi_{4zz} \int_0^{\pi} \pi_1 dz + \psi_{4zz} \pi_1 \right) e^{i(\Delta k)'} x,
\]

\[
\mu \left( B_4 T \psi_{4zz} + B_4 A \left( (V_0 + 4i k_4 Re^{-1}) \psi_{4zz} + (2k_4 \omega_4 - \phi_z) \psi_4 \right) \right)
\]

\[
-\epsilon A_4^* B_3^* \left( \psi_{3zz}^* \pi_{1z}^* + k_1^* k_3 \psi_{3zz}^* \int_0^{\pi} \pi_1^* dz + \psi_{3zz}^* \pi_1^* \right) e^{i(-\Delta k)'} x
\]

We take into account that acoustic wavenumbers are real, but wavenumbers of both T-S modes may be complex in the general case, namely the points of real values form the neutral curve [17].

This resulting equation may be considered as 4-wave resonance equation but without synchronism condition [23]. The coefficients of the equation depend on \(z\), it is due to our choice of the only one transverse mode for each "horizontal" one. Following the lines of [8], that have a connection of Galerkin numerical method,

We continue the projecting procedure considering the transverse modes as a basis. In such problems we naturally arrive at two bases. One arises from TSW theory and its origin is from OS equation (denoted by ). Other is from sound problem. Of course, such bases are not orthogonal. Hence we could only multiply each equation by its own basic vector and integrate across the boundary layer. The result is written in "back"-re-scaled variables: we put \(\mu = 1\).

\[
A_1 T + c a_1 A_1 X = n a_1 A_2 B_3^* e^{i(\Delta k)'} x,
\]

\[
A_2 T + c a_2 A_2 X = n a_2 A_1 B_3^* e^{i(-\Delta k)'} x,
\]

\[
B_3 T + c T S_1 B_3 X = -c n T S_1 A_1 B_4 e^{i(\Delta k') x},
\]

\[
B_4 T + c T S_2 B_4 X = -c n T S_1 A_1^* B_3^* e^{i(-\Delta k') x}
\]

where the group velocities and nonlinear constants are expressed via the inte-

\[
(47)
\]
grals across the boundary layer with a width $\delta$:

$$
\begin{align*}
\hat{c}_{a1} &= \frac{\int_{0}^{\delta} \pi_{1}^2 - \varepsilon^{-1} i k_{3} \pi_{1} \int_{0}^{\delta} \pi_{1} dz'}{\int_{0}^{\delta} \pi_{1} dz} \\
\frac{1}{\pi_{2}^2 - \varepsilon^{-1} i k_{2} \pi_{2} \int_{0}^{\delta} \pi_{2} dz'} \int_{0}^{\delta} \pi_{2} dz
\end{align*}
$$

$$
\begin{align*}
n_{a1} &= \frac{\int_{0}^{\delta} \pi_{2} [(1.5(-i k_{3} \psi_{i} \pi_{1} + i k_{4} \psi_{j} \pi_{1} + i k_{4} \psi_{j} \pi_{1} + i k_{4} \psi_{j} \pi_{1}) + \int_{0}^{\delta} \pi_{1} (-i k_{2} \psi_{i} \pi_{1} + i k_{2} \psi_{i} \pi_{1} + i k_{2} \psi_{i} \pi_{1} + i k_{2} \psi_{i} \pi_{1})]) dz'}{\int_{0}^{\delta} \pi_{1} dz} \\
\frac{\int_{0}^{\delta} \pi_{2} dz'}{\int_{0}^{\delta} \pi_{2} dz} \int_{0}^{\delta} \pi_{2} dz
\end{align*}
$$

$$
\begin{align*}
c_{TS1} &= \frac{\int_{0}^{\delta} \pi_{3} [(V_{0} + i k_{4} \psi_{i} \pi_{1} + i k_{4} \psi_{j} \pi_{1} + i k_{4} \psi_{i} \pi_{1} + i k_{4} \psi_{i} \pi_{1})] dz'}{\int_{0}^{\delta} \pi_{3} dz} \\
\frac{\int_{0}^{\delta} \pi_{3} dz'}{\int_{0}^{\delta} \pi_{3} dz} \int_{0}^{\delta} \pi_{3} dz
\end{align*}
$$

$$
\begin{align*}
n_{TS1} &= \frac{\int_{0}^{\delta} \pi_{3} [(V_{0} + i k_{4} \psi_{i} \pi_{1} + i k_{4} \psi_{j} \pi_{1} + i k_{4} \psi_{i} \pi_{1} + i k_{4} \psi_{i} \pi_{1})] dz'}{\int_{0}^{\delta} \pi_{3} dz} \\
\frac{\int_{0}^{\delta} \pi_{3} dz'}{\int_{0}^{\delta} \pi_{3} dz} \int_{0}^{\delta} \pi_{3} dz
\end{align*}
$$

$$
\begin{align*}
c_{TS2} &= \frac{\int_{0}^{\delta} \pi_{3} [(V_{0} + i k_{4} \psi_{i} \pi_{1} + i k_{4} \psi_{j} \pi_{1} + i k_{4} \psi_{i} \pi_{1} + i k_{4} \psi_{i} \pi_{1})] dz'}{\int_{0}^{\delta} \pi_{3} dz} \\
\frac{\int_{0}^{\delta} \pi_{3} dz'}{\int_{0}^{\delta} \pi_{3} dz} \int_{0}^{\delta} \pi_{3} dz
\end{align*}
$$

$$
\begin{align*}
n_{TS2} &= \frac{\int_{0}^{\delta} \pi_{3} [(V_{0} + i k_{4} \psi_{i} \pi_{1} + i k_{4} \psi_{j} \pi_{1} + i k_{4} \psi_{i} \pi_{1} + i k_{4} \psi_{i} \pi_{1})] dz'}{\int_{0}^{\delta} \pi_{3} dz} \\
\frac{\int_{0}^{\delta} \pi_{3} dz'}{\int_{0}^{\delta} \pi_{3} dz} \int_{0}^{\delta} \pi_{3} dz
\end{align*}
$$

A structure of the obtained equations is the particular case of general N-wave system, that may be solved by special technics valid for integrable equations [23]. The 4-wave approximation may give rise to such solutions that exhibit effective energy exchange between modes. The form of the nonlinearity is typical for a 3-wave systems and even small fluctuations of a TS field could initiate a rapid growth of both components if the acoustic field is big enough. It is pure nonlinear instability that may be supported by linear stability curve shift [15]. The numerical evaluation of the integrals in constants [48] need rather compulsory calculations. The solution of the system [47] pose also the separate problem. We plan to present the results in a next paper.

### 7 Conclusion

The resulting system (32),(38) and the equation of the opposite directed A-mode could be considered as a basic one for all-perturbations over a BL description. We also would note that the boundary layer width may depend on $x$. A slow dependence is usually accepted and do not change the general structure of the expressions. As the resonant as a non-resonant processes may be studied with the acoustic waves separated. The modes have the different scales, hence a numerical modelling of the mutual generation and control also could be more effective.

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