CHERN-SIMONS INVARIANTS OF 3-MANIFOLD GROUPS IN SL(4,R)

THILO KUESSNER

Abstract. We compute the Chern-Simons invariants for flat 4-dimensional bundles over 3-manifolds whose monodromy factors over SL(2, C), in particular for those factoring over the isomorphism PSL(2, C) = SO(3, 1), by computing their fundamental class in the extended Bloch group. We also discuss consequences for the number of connected components of SL(4, R) character varieties, and we show that there are knots with arbitrarily many components of vanishing Chern-Simons invariant in their SL(n, C) character varieties.

1. Introduction

In this paper we are going to consider finite-volume hyperbolic 3-manifolds $M$ and representations $\rho: \pi_1 M \to SL(4, \mathbb{C})$ of their fundamental groups which factor over a representation $SL(2, \mathbb{C}) \to SL(4, \mathbb{C})$. We will compute the Cheeger-Chern-Simons invariants of the associated flat bundles. The main part of the paper will be devoted to the computation in the case of the 2-fold covering $SL(2, \mathbb{C}) \to SO(3, 1)$ because this is the only $SL(2, \mathbb{C})$-representation for which the computation does not already follow easily from the results in [10].

Let us start with recalling the basic definitions. For a flat complex vector bundle $V: E \to X$ Cheeger-Simons ([3, Section 4]) define Chern characters $\hat{c}_k(V) \in H^{2k-1}(X, \mathbb{C}/4\pi^2\mathbb{Z})$. In this paper we will be interested in the character $\hat{c}_2(V)$ for flat $SL(n, \mathbb{C})$-bundles over 3-manifolds. For a closed, orientable 3-manifold $M$ and a representation $\rho: \pi_1 M \to SL(n, \mathbb{C})$ we consider its Cheeger-Chern-Simons invariant

$$CCS(M, \rho) = \int_M \hat{c}_2(V_\rho),$$

where $V_\rho$ means the flat $n$-dimensional complex vector bundle over $M$ with holonomy $\rho$. An explicit formula is

$$CCS(M, \rho) = \frac{1}{2} \int_M s^*(Tr(\theta \wedge d\theta + \frac{2}{3} \theta \wedge \theta \wedge \theta),$$

where $\theta$ is a flat connection and $s$ a section of $V_\rho$, which exists because $SL(n, \mathbb{C})$ is 2-connected.

\footnote{In their normalization the Chern character is an element of $H^{2k-1}(X, \mathbb{C}/\mathbb{Z})$.}
The universal Cheeger-Chern-Simons class $\hat{c}_2$ of flat $SL(n, \mathbb{C})$-bundles is defined in \cite{3} as an element in $H^3(SL(n, \mathbb{C})^\delta, \mathbb{C}/4\pi^2\mathbb{Z})$. To get a Cheeger-Chern-Simons invariant also for cusped manifolds one lets $N \subset SL(n, \mathbb{C})$ be the subgroup of upper triangular matrices with 1’s on the diagonals and uses the isomorphism $H^3(SL(n, \mathbb{C})^\delta, \mathbb{C}/4\pi^2\mathbb{Z}) \cong H^3(SL(n, \mathbb{C})^\delta, N^\delta \mathbb{C}/4\pi^2\mathbb{Z})$ to consider $\hat{c}_2$ as a relative class $\hat{c}_2 \in H^3(BSL(n, \mathbb{C})^\delta, BN^\delta; \mathbb{C}/4\pi^2\mathbb{Z})$, see \cite[Section 6.1]{10}. Then for a flat bundle with boundary-unipotent holonomy (i.e. the restriction of the holonomy to $\partial M$ having image in $N$) one defines the Cheeger-Chern-Simons invariant via the pullback of $\hat{c}_2$ under the classifying map $(M, \partial M) \to (BSL(n, \mathbb{C})^\delta, BN^\delta)$.

As $\mathbb{C}/4\pi^2\mathbb{Z}$ is divisible, one may consider $\hat{c}_2$ as a homomorphism $\hat{c}_2 : H^3(SL(n, \mathbb{C})^\delta, \mathbb{Z}) \to \mathbb{C}/4\pi^2\mathbb{Z}$.

The group $H^3(SL(2, \mathbb{C})^\delta, \mathbb{Z})$ has an explicit description (by the work of Neumann) as the so-called extended Bloch group $\hat{B}(\mathbb{C})$, which is a certain subgroup of the extended pre-Bloch group $\hat{P}(\mathbb{C})$ described in Section 2.1 below. Together with the Suslin-Sah isomorphism $H^3(SL(n, \mathbb{C})^\delta, \mathbb{Z}) \cong H^3(SL(2, \mathbb{C})^\delta, \mathbb{Z}) \oplus K_3^M(\mathbb{C})$ (for $n \geq 3$) one obtains a decomposition

$$H^3(SL(n, \mathbb{C})^\delta, \mathbb{Z}) \cong \hat{B}(\mathbb{C}) \oplus K_3^M(\mathbb{C})$$

and it turns out that $\hat{c}_2$ vanishes on the Milnor K-theory $K_3^M(\mathbb{C})$ (see \cite[Section 8]{10}), thus $\hat{c}_2$ depends only on its values on $\hat{B}(\mathbb{C})$. In \cite{10}, Garoufalidis, D. Thurston and Zickert associate to each flat $SL(n, \mathbb{C})$-bundle over a 3-manifold (with unipotent holonomy at the boundary) a “fundamental class”

$$(B\rho)_* [M, \partial M] \in \hat{B}(\mathbb{C})$$

and they exhibit an explicit method for computing $\hat{c}_2$ on this element. This yields a computable formula for the Cheeger-Chern-Simons invariant of the associated flat bundle. We will review this in Section 2.1 because it will be the basis for our computations.

Using the Cheeger-Chern-Simons invariant, the volume and Chern-Simons invariant of a representation are defined as follows.

**Definition 1.** (\cite[Definition 2.11]{10}): For a compact, orientable, aspherical 3-manifold $M$ and a boundary-unipotent representation $\rho : \pi_1 M \to SL(n, \mathbb{C})$ one defines the volume $\text{Vol}(\rho)$ and Chern-Simons invariant of $\rho$ by

$$-CS(\rho) + i\text{Vol}(\rho) = (\hat{c}_2, (B\rho)_* [M, \partial M]),$$

\footnote{Following \cite{8} and \cite{12} we define the extended pre-Bloch group as what Neumann in \cite{19} calls the more extended pre-Bloch group.}

\footnote{This is not the same as the volume of representations defined via pulling back the volume form of the symmetric spaces.}
where $B\rho: (M, \partial M) \to (BSL(n, \mathbb{C})^\delta, BN^\delta)$ is the classifying map of $\rho$ (i.e., of the associated flat bundle).

The motivation for this definition is Yoshida’s theorem (see [10, Theorem 2.8]) which implies that for a closed hyperbolic 3-manifold and a lift $\iota: \pi_1 M \to SL(2, \mathbb{C})$ of its geometric representation, $Vol(\iota)$ (as defined above) is the hyperbolic volume $Vol(M)$ and $CS(\iota)$ is the Chern-Simons invariant $CS(M)$ of the Levi-Civita connection for the hyperbolic metric. (The analogous result for cusped hyperbolic 3-manifolds is true modulo $\pi_2$ and is proved in [19, Corollary 14.6].)

Let now $M$ be an orientable, hyperbolic 3-manifold, then its fundamental group $\Gamma = \pi_1 M$ is a discrete subgroup of $\text{Isom}^+(H^3) = PSL(2, \mathbb{C})$. If $M$ is closed, then by [3, Corollary 2.2] it lifts to a discrete subgroup $\Gamma \subset SL(2, \mathbb{C})$. (If $M$ has cusps, then this lift is in general not boundary-unipotent, see Section 3.5 for a discussion of this case.) Let $\rho_n: SL(2, \mathbb{C}) \to SL(n, \mathbb{C})$ be the irreducible representation corresponding to the unique $\mathbb{C}$-linear $n$-dimensional representation of the Lie algebra $sl(2, \mathbb{C})$. Garoufalidis-D.Thurston-Zickert use their methods to give a short and elegant proof for the formulas

$$Vol(\rho_n \iota) = \left( \frac{n+1}{3} \right) Vol(\iota), CS(\rho_n \iota) = \left( \frac{n+1}{3} \right) CS(\iota),$$

see [10, Theorem 11.3].

The classification of representations of the Lorentz group implies that with the exception of $\rho_2 \otimes \overline{\rho}_2$ all 4-dimensional representations of $SL(2, \mathbb{C})$ are, up to conjugacy in $GL(4, \mathbb{C})$, obtained as direct sums of the $\rho_n$’s and their complex conjugates. The exceptional case $\rho_2 \otimes \overline{\rho}_2$ is equivalent to the 2-fold covering $SL(2, \mathbb{C}) \to SO(3, 1)$.

The Cheeger-Chern-Simons invariant is additive under direct sum and takes the complex conjugate upon complex conjugating the representation, see Section 2.2. From these principles and the result of Garoufalidis-D.Thurston-Zickert one can compute the CCS-invariants of $\rho \iota$ for all representations $\rho: SL(2, \mathbb{C}) \to GL(4, \mathbb{C})$ except for $\rho_2 \otimes \overline{\rho}_2$. The main result of our paper will be to compute the CCS-invariant for $\rho_2 \otimes \overline{\rho}_2$ or, what is equivalent, for the composition of $\rho$ with the isomorphism $\tau: PSL(2, \mathbb{C}) \to SO(3, 1)$.

**Corollary 1.** Let $M$ be a finite-volume, orientable, hyperbolic 3-manifold. Let $\tau \circ \iota: \pi_1 M \to SO(3, 1)$ be the representation coming from the composition of the hyperbolic monodromy $\iota: \pi_1 M \to PSL(2, \mathbb{C})$ with the isomorphism $\tau: PSL(2, \mathbb{C}) \to SO(3, 1)$. Then

$$Vol(\tau \circ \iota) = 0, CS(\tau \circ \iota) = 4CS(M).$$

More generally, if $\rho: \pi_1 M \to PSL(2, \mathbb{C})$ is any boundary-unipotent representation, then

$$Vol(\tau \circ \rho) = 0, CS(\tau \circ \rho) = 4CS(\rho)$$

---

If $M$ has cusps, then this is of course to be understood as an equality modulo $\pi^2$ because $CS(M)$ is only defined up to this ambiguity.
if \( \rho \) lifts to a boundary-unipotent representation \( \pi_1 M \to SL(2, \mathbb{C}) \) and \( Vol(\tau \circ \rho) = 0 \), \( CS(\tau \circ \rho) = 4CS(\rho) \mod \pi^2 \) otherwise.

The vanishing of the volume is no surprise since this is true for any representation to \( SL(4, \mathbb{R}) \). The interesting result is the computation of the Chern-Simons invariant which is obtained as a corollary to the following Theorem.

**Theorem 1.** Let \( M \) and \( \tau \) be as in Corollary 1 and \( \rho: \pi_1 M \to PSL(2, \mathbb{C}) \) some boundary-unipotent representation. Then 
\[
(\tau \circ \rho)_* [M, \partial M] = 2\rho_* [M, \partial M] + 2\rho_* [M, \partial M] \in \hat{B}(\mathbb{C})
\]
if \( \rho \) lifts to a boundary-unipotent representation \( \pi_1 M \to SL(2, \mathbb{C}) \) (in particular if \( M \) is closed) and 
\[
(\tau \circ \rho)_* [M, \partial M] = 2\rho_* [M, \partial M] + 2\rho_* [M, \partial M] \in \hat{B}(\mathbb{C})_{PSL}
\]
otherwise.

This will be proved in Section 3 and will (after long calculations) in the end result from some wonderful cancelations in the pre-Bloch group, using the 5-term relation and some symmetries described in Section 2.3. It would of course be interesting to find some more conceptual explanation for these cancelations.

The following table shows volume and Chern-Simons invariant of the representations \( \rho \circ \iota: \pi_1 M \to GL(4, \mathbb{C}) \), where \( \iota: \pi_1 M \to SL(2, \mathbb{C}) \) is a lift of the monodromy of a hyperbolic structure and \( \rho \) runs over all \( \mathbb{R} \)-linear representations \( \rho: SL(2, \mathbb{C}) \to GL(4, \mathbb{C}) \).

| representation \( \rho \) | Volume of \( \rho \circ \iota \) | Chern-Simons invariant of \( \rho \circ \iota \) |
|--------------------------|-------------------------------|----------------------------------|
| \( \rho_1 \)             | 10 Vol(M)                     | 10 CS(M)                         |
| \( \overline{\rho_1} \) | -10 Vol(M)                   | 10 CS(M)                         |
| \( \rho_2 \otimes \overline{\rho_2} \) | 0                             | 4 CS(M)                          |
| \( \rho_3 \oplus 1 \)    | 4 Vol(M)                      | 4 CS(M)                          |
| \( \overline{\rho_3} \oplus 1 \) | -4 Vol(M)                  | 4 CS(M)                          |
| \( \rho_2 \oplus \rho_2 \) | 2 Vol(M)                     | 2 CS(M)                          |
| \( \rho_2 \oplus \overline{\rho_2} \) | 0                             | 2 CS(M)                          |
| \( \rho_2 \oplus 1 \oplus 1 \) | Vol(M)                      | CS(M)                            |
| \( \overline{\rho_2} \oplus 1 \oplus 1 \) | -Vol(M)                     | CS(M)                            |
| \( 1 \leq 1 \)           | 0                             | 0                                |

It already follows from local rigidity results in [17] that direct sums of the \( \rho_n \) and their complex conjugates belong to different components in the \( SL(4, \mathbb{C}) \)-character variety, see Section 4.1. For a hyperbolic 3-manifold with nonvanishing Chern-Simons invariant we can then distinguish 10 components by volume and Chern-Simons invariant.

---

[For a manifold with boundary, we consider the character variety of characters of boundary-unipotent representations only, as in [17] for example.]
We recall that in the above list the three representations of volume 0 can be conjugated into \( SL(4, \mathbb{R}) \). Indeed, \( \rho_2 \otimes \overline{\rho}_2 \) is equivalent to the well-known 2-fold covering map \( SL(2, \mathbb{C}) \to SO(3, 1) \subset SL(4, \mathbb{R}) \), while \( \rho_2 \oplus \overline{\rho}_2 \) is equivalent to the embedding \( SL(2, \mathbb{C}) \to SL(4, \mathbb{R}) \) coming from \((a_1 + a_2i) \to \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix}\), see Section 4.2. This implies the following corollary.

**Corollary 2.** For finite-volume hyperbolic, orientable 3-manifolds with nonvanishing Chern-Simons invariant there are at least 3 connected components in their \( SL(4, \mathbb{R}) \) character variety.

The ptolemy module \[11\] in SnapPy \[6\] computes all \( SL(2, \mathbb{C}) \)- and \( SL(3, \mathbb{C}) \)-representations for 3-manifold groups but at the time of writing can compute \( SL(4, \mathbb{C}) \)-representations only for 3-manifolds composed of two ideal tetrahedra, that is for the figure eight knot complement. (In this case it detects only irreducible representations because the reducible ones would need more than two simplices to allow a generic decoration.) It turns out that for the figure eight knot complement the only computed irreducible \( SL(4, \mathbb{C}) \)-representations are those coming from \( \rho_4 \circ \iota, \overline{\rho}_4 \circ \iota \) and \( (\rho_2 \otimes \overline{\rho}_2) \circ \iota \). (There are however more \( PSL(4, \mathbb{C}) \)-representations, see \[10, Example 10.2\].) So in this case one should actually have no more than 3 components in the \( SL(4, \mathbb{R}) \) character varieties of the figure eight knot complement. For general knots, however, there will be more than 3 components and in Section 4 we will discuss methods for constructing some of them.

Concerning \( SL(n, \mathbb{C}) \) character varieties we use a construction of Ohtsuki-Riley-Sakuma to show the existence of knot complements having arbitrarily many components with vanishing Chern-Simons invariant in their \( SL(n, \mathbb{C}) \)-character variety. (This actually does not use Theorem 1 but the computation for the geometric representation in \[10\].)

**Corollary 3.** For any natural numbers \( N \) and \( m \) there exist 2-bridge knots whose \( SL(m, \mathbb{C}) \) character variety has more than \( N \) connected components, such that the Chern-Simons invariant on each of \( N \) components vanishes.

The paper is organised as follows. Section 2 recollects known facts, especially the results from \[10\]. Section 3 is the heart of the paper, it computes the fundamental class and hence the Chern-Simons invariant for representations of the form \( \rho \otimes \overline{\rho} \) with \( \rho : \pi_1 M \to SL(2, \mathbb{C}) \). In Section 4 we discuss some facts and conjectures about \( SL(4, \mathbb{C}) \)- and \( SL(4, \mathbb{R}) \)-character varieties and in particular the proof of Corollary 3.

I thank Matthias Görner and Sebastian Goette for answering some questions about \[11\] and \[12\], respectively, and Neil Hoffman for contributing the proof of Proposition 4 on mathoverflow. The computations in the paper have been done with the help of \[25\].
2. Recollections

Throughout the paper \( \log(z) \) will mean the branch of the logarithm of \( z \in \mathbb{C} \) with imaginary part
\[-\pi < \text{Im}(\log(z)) \leq \pi.\]

For a manifold \( M \) we will always assume to have fixed a basepoint \( m_0 \) and hence an action of \( \pi_1 M := \pi_1(M, m_0) \) on \( \hat{M} \).

2.1. Computing CCS-invariants after Garoufalidis-D.Thurston-Zickert.

**Definition 2.** The extended pre-Bloch group \( \hat{P}(\mathbb{C}) \) is the free abelian group on the set
\[ \hat{C} = \{(e, f) \in \mathbb{C}^2 : \exp(e) + \exp(f) = 1 \} \]
modulo the relations
\[ (e_0, f_0) - (e_1, f_1) + (e_2, f_2) - (e_3, f_3) + (e_4, f_4) = 0 \]
whenever the equations
\[ e_2 = e_1 - e_0, \]
\[ e_3 = e_1 - e_0 - f_1 + f_0, f_3 = f_2 - f_1, \]
\[ e_4 = f_0 - f_1, f_4 = f_2 - f_1 + e_0 \]
hold.

One should pay attention that \([z; 2p, 2q]\) in the notation of [19], [8] and [12] corresponds to
\[ (e, f) = (\log(z) + 2p\pi i, \log(1 - z) - 2q\pi i) \]
and hence to \([z; 2p, -2q]\) in the notation of [10]. (Here \( z \in \mathbb{C} \setminus \{0, 1\} \) and \( p, q \in \mathbb{Z} \).)

**Definition 3.** The extended Rogers’ dilogarithm
\[ R : \hat{P}(\mathbb{C}) \to \mathbb{C}/4\pi^2\mathbb{Z} \]
is defined on generators of \( \hat{P}(\mathbb{C}) \) by
\[ R((\log(z)+2p\pi i, \log(1-z)+2q\pi i)) := \text{Li}_2(z)+\frac{1}{2}(\log(z)+2p\pi i)(\log(1-z)−2q\pi i)−\frac{\pi^2}{6}, \]
where \( \text{Li}_2(z) \) denotes the classical dilogarithm.

It is proved in [19] and [12] that \( R \) is well-defined and a homomorphism. We will occasionally also use the name \( R \) for the Rogers’ dilogaritm itself, so we have the equality
\[ R((\log(z)+2p\pi i, \log(1-z)+2q\pi i)) = R(\log(z))+(p\log(1-z)−q\log(z))\pi i−2pqq^2. \]

**Definition 4.** A closed 3-cycle is a cell complex \( K \) obtained from a finite collection of ordered 3-simplices by order preserving simplicial gluing maps between pairs of faces. We call \( K \) a generalized ideal triangulation of a compact 3-manifold \( M \) if it is homeomorphic to the space \( \hat{M} \) obtained from \( M \) by collapsing each boundary component to one point.
Definition 5. Let $G$ be a Lie group, $N \subset G$ a subgroup and $M$ a 3-manifold. A $(G, N)$-representation is a representation $\pi_1 M \to G$ which sends each peripheral subgroup to a conjugate of $N$.

Definition 6. Let $M$ be a compact 3-manifold (possibly with boundary) and $K$ a generalized ideal triangulation. Then $L$ denotes the corresponding generalized triangulation of the space obtained from the universal covering $\tilde{M}$ by collapsing each boundary component of $\tilde{M}$ to a point.

We remark that the action of $\pi_1 M$ on $\tilde{M}$ extends to an action of $\pi_1 M$ on $L$.

Definition 7. Let $M$ be a compact 3-manifold (possibly with boundary) and $K$ a generalized ideal triangulation. A decoration of a $(G, N)$-representation $\rho: \pi_1 M \to G$ is a $\rho$-equivariant assignment $L_0 \to G/N$, i.e., associating an $N$-coset to each vertex of $L$ such that if $\alpha \in \pi_1 M$ and the coset $g_v N$ is associated to $v$, then the coset $\rho(\alpha) g_v N$ is associated to $\alpha v$.

We will say that a simplex $\sigma = (v_0, v_1, v_2, v_3)$ from $K$ is decorated by the tuple $(g_0 N, g_1 N, g_2 N, g_3 N)$.

Definition 8. Let $K$ be a generalized ideal triangulation of a compact 3-manifold $M$ and let $\rho: \pi_1 M \to G$ be a decorated $(G, N)$-representation, where $G$ is a linear group, i.e., contained in $GL(n, \mathbb{C})$ for some $n$.

The ptolemy coordinates of a decorated 3-simplex $(g_0 N, g_1 N, g_2 N, g_3 N)$ are the assignment

$$(t_0, t_1, t_2, t_3) \to \text{det}(\bigcup_{i=0}^{3} \{g_i\}_{t_i})$$

for each tuple $(t_0, t_1, t_2, t_3)$ of nonnegative integers with $t_0 + t_1 + t_2 + t_3 = n$.

Here $\{g_i\}_{t_i}$ means the (ordered) set of the first $t_i$ column vectors of $g_i \in GL(n, \mathbb{C})$ and $\bigcup_{i=0}^{3} \{g_i\}_{t_i}$ means the matrix whose (ordered) column set is composed by the $\{g_i\}_{t_i}$.

A decoration is called generic if all ptolemy coordinates are nonzero.

One can always obtain generic decorations by performing a barycentric subdivision on simplices with nongeneric decorations.

One visualizes the ptolemy coordinates (of a simplex) by fixing some identification of the simplex with

$$\Delta_3^3 := \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4: x_i \geq 0, x_0 + x_1 + x_2 + x_3 = n\}$$

and by attaching $c_t$ to the point $(t_0, t_1, t_2, t_3)$.

For each $\alpha \in \Delta_3^3 \cap \mathbb{Z}^4$
one can consider the subsimplex with vertices corresponding to

$$\alpha + e_i, \, i = 0, 1, 2, 3.$$  

Each of its edges

$$(\alpha + e_i, \alpha + e_j)$$  

has one ptolemy coordinate attached to it, namely

$$c_{\alpha_{ij}} := c_{\alpha + e_i + e_j}.$$  

In what follows $G$ will be a subgroup of $GL(n, \mathbb{C})$ and $N \subset G$ will be the subgroup of upper triangular matrices with all diagonal entries equal to 1.

**Definition 9.** Let $K = \bigcup_{k=1}^{r} T_k$ be a generalized ideal triangulation of a compact, orientable 3-manifold $M$ and let $\rho: \pi_1M \rightarrow G$ be a generic decorated $(G, N)$-representation, with ptolemy coordinates $c^k_{T}$ for each simplex $T_k$.

For each simplex $T$ and each $\alpha \in \Delta_{n-2}^{3} \cap \mathbb{Z}^4$ define

$$\lambda(\alpha) = (\log(c_{\alpha_{03}}) + \log(c_{\alpha_{12}}) - \log(c_{\alpha_{02}}) - \log(c_{\alpha_{13}}), \log(c_{\alpha_{01}}) + \log(c_{\alpha_{23}}) - \log(c_{\alpha_{02}}) - \log(c_{\alpha_{13}})) \in \hat{P}(\mathbb{C}).$$

Then define

$$\lambda(K, \rho) = \sum_{k=1}^{r} \epsilon_k \sum_{\alpha \in \Delta_{n-2}^{3} \cap \mathbb{Z}^4} \lambda(c^k_{\alpha}) \in \hat{P}(\mathbb{C}),$$

where $\epsilon_k$ is $\pm 1$ according to whether the orientation of $T_k$ agrees with the orientation of $M$ or not.

It is known (but it will play actually no role for our calculations) that $\lambda(K, \rho) \in \mathcal{B}(\mathbb{C}) := ker(\hat{\nu})$ for $\hat{\nu}(e, f) = e \wedge f \in \mathcal{C} \wedge_{\mathbb{Z}} \mathbb{C}$. The element $\lambda(K, \rho)$ does not depend on the triangulation and we will denote it by

$$\rho_* [M, \partial M] \in \mathcal{B}(\mathbb{C}).$$

In fact for $G = SL(2, \mathbb{C})$ it corresponds to the image of the fundamental class under the isomorphism $H_3(SL(2, \mathbb{C})^\delta, N^\delta; \mathbb{Z}) \cong H_3(SL(2, \mathbb{C})^\delta; \mathbb{Z}) \cong \hat{B}(\mathbb{C}).$

The following is the main result of Garoufalidis-D.Thurston-Zickert in [10]. It shows that the Cheeger-Chern-Simons invariant (see Definition 1) can be computed from $\rho_* [M, \partial M]$.

**Proposition 1.** (10 Theorem 1.3) Let $K = \bigcup_{k=1}^{r} T_k$ be a generalized ideal triangulation of a compact, orientable 3-manifold $M$ and let $\rho: \pi_1M \rightarrow G$ be a generic decorated $(G, N)$-representation, with ptolemy coordinates $c^k_{T}$ for each simplex $T_i$. Then

$$R(\lambda(K, \rho)) = -CS(\rho) + iVol(\rho) \in \mathbb{C}/4\pi^2\mathbb{Z}.$$  

The following result from [10] will be useful to avoid too many case distinctions in our arguments.
Proposition 2. (10, Proposition 7.7) Under the assumptions of Definition 9 let \( c \) be a ptolemy cochain on \( K \). For any lift \( \tilde{c} \) of \( c \) we have

\[
\lambda(K, \rho) = \sum_{k=1}^{r} c_k \sum_{\alpha \in \Delta_{n-2} \cap \mathbb{Z}^4} \tilde{\lambda}(c^k_\alpha) \in \hat{P}(\mathbb{C}).
\]

Here, a lift \( \tilde{c} \) of \( c \) means a choice of logarithm for each \( c^k_\alpha \) (i.e., of a complex number whose difference with \( \log(c^k_\alpha) \) is an integer multiple of \( 2\pi i \)) such that the choices agree whenever coordinates correspond to glued faces in \( K \), and \( \tilde{\lambda}(c_\alpha) \) is then defined as

\[
\tilde{\lambda}(c_\alpha) = (\tilde{c}_{\alpha 03} + \tilde{c}_{\alpha 12} - \tilde{c}_{\alpha 02} - \tilde{c}_{\alpha 13}, \tilde{c}_{\alpha 01} + \tilde{c}_{\alpha 23} - \tilde{c}_{\alpha 02} - \tilde{c}_{\alpha 13}) \in \hat{P}(\mathbb{C}).
\]

So, in the formula of Definition 9, one can replace \( \log \) by any choice of logarithm as long as we make the same choice on common faces or edges of different simplices.

For boundary-unipotent representations (10, Section 6.3). For boundary-unipotent representations to \( pSL(n, \mathbb{C}) = SL(n, \mathbb{C})/\{\pm 1\} \) the ptolemy coordinates are only well-defined as elements of \( \mathbb{C}^* / \{\pm 1\} \) and thus \( \lambda(c_\alpha) \) takes value in \( \hat{P}(\mathbb{C})_{PSL} \), the free abelian group over \( \{(e, f) \in \mathbb{C}^2: \pm \exp(e) \pm \exp(f) = 1\} \) modulo the 5-term relation. The extended Rogers’ dilogarithm is then well-defined modulo \( \pi^2 \mathbb{Z} \).

2.2. Properties of CCS-invariants. While the Cheeger-Chern-Simons invariant is in general not additive for direct sums of \( GL(n, \mathbb{C}) \)-bundles, additivity holds for \( SL(n, \mathbb{C}) \)-bundles.

Lemma 1. Let \( \rho_1, \rho_2 \) be boundary-unipotent representations from \( \pi_1 M \) to \( SL(n, \mathbb{C}) \), for a compact manifold \( M \). Then

\[
CCS(M, \rho_1 \oplus \rho_2) = CCS(M, \rho_1) + CCS(M, \rho_2)
\]

Proof: From Cheeger-Simons (3, Theorem 4.6) follows

\[
\hat{c}_2(V \oplus W) = \hat{c}_2(V) + \hat{c}_2(W) + \hat{c}_1(V) * \hat{c}_1(W)
\]

for a certain multiplication \( * \) defined in (3, page 56).

It is well-known that \( \hat{c}_1 \) vanishes for all flat \( SL(n, \mathbb{C}) \)-bundles. Indeed, \( \hat{c}_1 \in H^1(BGL(n, \mathbb{C}), \mathbb{C}/\mathbb{Z}) \) is explicitly given by \( \hat{c}_1(1, g) = \log(\det(g)) \). The claim follows.

QED

A direct consequence of the explicit formula in (10) is the compatibility of \( CCS \) with complex conjugation.
Lemma 2. For any boundary-unipotent representation $\rho: \pi_1 M \to SL(n, \mathbb{C})$ we have

$$CCS(M, \mathcal{P}) = \overline{CCS(M, \rho)}.$$ 

Proof: Going through the formulas in [10] one sees that a decoration for $\mathcal{P}$ can be obtained by applying complex conjugation to a decoration for $\rho$, and that the ptolemy coordinates of these decorations are related by complex conjugation. According to Proposition 2 the value of $CCS(M, \rho)$ does not depend on the lifts of the ptolemy coordinates, so one may choose the lifts of the ptolemy coordinates for $\mathcal{P}$ to be exactly the complex conjugates of the lifts of the ptolemy coordinates for $\rho$. (For example one may choose $\tilde{c} = \log(c)$ whenever $c \notin \mathbb{R}_{<0}$, and for all $c \in \mathbb{R}_{<0}$ one may choose $\tilde{c} = \log(c)$ for $\rho$, but $\tilde{c} = \log(c) - 2\pi i = \log(c)$ for $\mathcal{P}$.)

The formula in Definition 9 then implies $\lambda(K, \mathcal{P}) = \overline{\lambda(K, \rho)}$ and now the claim follows from Proposition 1 and the equality $R(\mathcal{P}, f) = \overline{R(\rho, f)}$ which is immediate from Definition 3. \(QED\)

An immediate consequence is that a boundary-unipotent representations which can be conjugated to a representation into $SL(n, \mathbb{R})$ must have vanishing volume. It is perhaps worth-mentioning that the ptolemy coordinates of such a representation are not necessarily real, basically because the peripheral subgroups are conjugate to $N \cap SL(n, \mathbb{R})$ inside $SL(n, \mathbb{C})$ but not necessarily inside $SL(n, \mathbb{R})$. The computations in Section 3 actually provide an example of this phenomenon.

Lemma 3. If $f: (M_1, \partial M_1) \to (M_2, \partial M_2)$ has mapping degree $\deg(f)$, then

$$CCS(M_2, \rho) = \frac{1}{\deg(f)} CCS(M_1, \rho \circ f_*).$$

for any boundary-unipotent representation $\rho: \pi_1 M_2 \to SL(n, \mathbb{C})$ and the induced homomorphism $f_*: \pi_1 M_1 \to \pi_1 M_2$.

Proof: This is immediate from

$$CCS(M_2, \rho) = \langle \hat{c}_2, (B\rho)_* [M_2, \partial M_2] \rangle$$

and

$$[M_2, \partial M_2] = \deg(f) [M_1, \partial M_1].$$

\(QED\)

2.3. Symmetries of the extended Rogers’ dilogarithm.

Definition 10. For $e \in \mathbb{C}$ define $\chi(e) \in \hat{P}(\mathbb{C})$ by

$$\chi(e) = (e, f + 2\pi i) - (e, f)$$

where $f \in \mathbb{C}$ is some complex number satisfying $\exp(e) + \exp(f) = 1$. 

Lemma 4. i) \( \chi \) is a homomorphism \( \mathbb{C}/4\pi i \mathbb{Z} \to \hat{P}(\mathbb{C}) \) with respect to the additive structures on \( \mathbb{C} \) and \( \hat{P}(\mathbb{C}) \).

ii) \( R(\chi(e)) = -\pi i e \mod 4\pi^2 \) for all \( e \in \mathbb{C} \).

iii) \( R \) is injective on the image of \( \chi \), and \( \text{im}(\chi) = \ker(\hat{P}(\mathbb{C}) \to P(\mathbb{C})) \).

IV) For \( (e, f) \in \hat{C} \) and \( p, q \in \mathbb{Z} \) we have

\[
(e + 2p\pi i, f + 2q\pi i) - (e, f) = \chi(2pq\pi i + qe - pf).
\]

Proof: i)-iii) are (in a slightly different language) proved in [12, Theorem 3.12]. (There is a different sign in ii) because of the notation difference explained in the remark before Definition 3.) Equation iv) follows from i)-iii), for general fields \( F \) it is also proved as a consequence of the 5-term relation in [28, Lemma 3.16].

QED

Lemma 5. The following relations hold whenever \( \text{Im}(z) > 0 \) and \( p, q \in \mathbb{Z} \).

i) \( (\log(\frac{1}{z}) - 2p\pi i, \log(1 - \frac{1}{z}) + 2(q - p)\pi i) \)

\[
= - (\log(z) + 2p\pi i, \log(1 - z) + 2q\pi i) + \chi(-\frac{1}{2}\log(z) + (2p^2 + p)\pi i)
\]

ii) \( (\log(1 - z) + 2q\pi i, \log(z) + 2p\pi i) \)

\[
= - (\log(z) + 2p\pi i, \log(1 - z) + 2q\pi i) + \chi(-\frac{\pi i}{6})
\]

iii) \( (\log(\frac{1}{1 - z}) - 2q\pi i, \log(\frac{1 \pm z}{1 - z}) + 2(p - q)\pi i) \)

\[
= (\log(z) + 2p\pi i, \log(1 - z) + 2q\pi i) + \chi(\frac{1}{2}\log(1 - z) + (2q^2 - q + \frac{1}{6})\pi i)
\]

iv) \( (\log(1 - \frac{1}{z}) + 2(q - p)\pi i, \log(\frac{1}{1 - z}) - 2p\pi i) \)

\[
= (\log(z) + 2p\pi i, \log(1 - z) + 2q\pi i) + \chi(\frac{1}{2}\log(z) + (2p^2 - p - \frac{1}{6})\pi i)
\]

v) \( (\log(\frac{1}{1 - z}) + 2(p - q)\pi i, \log(\frac{1}{1 - z}) - 2q\pi i) \)

\[
= - (\log(z) + 2p\pi i, \log(1 - z) + 2q\pi i) + \chi(\frac{1}{2}\log(1 - z) + (2q^2 - q - \frac{1}{3})\pi i)
\]

Proof:

i) From Definition 3 we get

\[
R(\log(z), \log(1 - z)) + R(\log(\frac{1}{z}), \log(1 - \frac{1}{z})) =
\]

\[
Li_2(z) + \frac{1}{2}\log(z) \log(1 - z) - \frac{\pi^2}{6} + Li_2(\frac{1}{z}) + \frac{1}{2}\log(\frac{1}{z}) \log(1 - \frac{1}{z}) - \frac{\pi^2}{6} \mod 4\pi^2.
\]

From [26, Section 2] we have

\[
Li_2(z) + Li_2(\frac{1}{z}) - \frac{\pi^2}{3} = -\frac{\pi^2}{2} - \frac{1}{2}\log^2(-z).
\]
Moreover we have $\log\left(\frac{1}{z}\right) = -\log(z), \log(1 - \frac{1}{z}) = \log(z - 1) - \log(z)$ because of $|\arg(z - 1) - \arg(z)| < \pi$, and for $\text{Im}(z) > 0$ we have $\log(1-z) = \log(z - 1) - \pi i$ and $\log(-z) = \log(z) - \pi i$. Thus the above expression simplifies to

$$-\frac{1}{2}(\log(z) - \pi i)^2 - \frac{\pi^2}{2} + \frac{1}{2} \log(z) \log(1-z) + \frac{1}{2}(-\log(z))(\log(1-z) - \log(z) + \pi i) \mod 4\pi^2$$

$$= \frac{1}{2} \pi i \log(z) = R\left(\frac{1}{2} \log(z)\right) \mod 4\pi^2,$$

thus

$$(\log(z), \log(1-z)) + (\log\left(\frac{1}{z}\right), \log(1 - \frac{1}{z})) = \chi(-\frac{1}{2} \log(z)).$$

Then we apply Lemma 4 iv) to get

$$(\log(z) + 2p\pi i, \log(1-z) + 2q\pi i) + (\log\left(\frac{1}{z}\right) - 2p\pi i, \log(1 - \frac{1}{z}) + (2q - 2p)\pi i) =$$

$$(\log(z), \log(1-z)) + \chi(2pq\pi i + q \log(z) - p \log(1-z)) + (\log\left(\frac{1}{z}\right), \log(1 - \frac{1}{z})) + \chi(2p(q-p)\pi i + (q-p) \log(1 - \frac{1}{z})) +$$

$$= \chi(-\frac{1}{2} \log(z)) + \chi(2p^2\pi i + (q - (q-p)) \log(z) - p \log(1-z) + p \log(1 - \frac{1}{z}))$$

We are assuming $\text{Im}(z) > 0$, which implies $\log(z - 1) - \log(1-z) = \pi i$. Moreover $z$ and $z - 1$ have positive imaginary parts, which implies that their arguments differ by less than $\pi$, so $\log\left(\frac{z-1}{z}\right) = \log(z-1) - \log(z)$. So the above sum simplifies to

$$= \chi(-\frac{1}{2} \log(z) + (2p^2 + p)\pi i)$$

ii) We have from Section 2] that

$$Li_2(z) + Li_2(1-z) = \frac{\pi^2}{6} - \log(z) \log(1-z),$$

hence

$$R(\log(z), \log(1-z)) + R(\log(1-z), \log(z)) = \frac{\pi^2}{6} - \log(z) \log(1-z) + 2 \left(\frac{1}{2}\log(z) \log(1-z) - \frac{\pi^2}{6}\right) = -\frac{\pi^2}{6},$$

so

$$(\log(z), \log(1-z)) + (\log(1-z), \log(z)) = \chi(-\frac{\pi i}{6}).$$

Then we apply Lemma 4 iv) to get

$$(\log(z) + 2p\pi i, \log(1-z) + 2q\pi i) + (\log(1-z) + 2q\pi i, \log(z) + 2p\pi i) =$$

$$(\log(z), \log(1-z)) + \chi(2pq\pi i + q \log(z) - p \log(1-z)) + (\log(1-z), \log(z)) + \chi(2pq\pi i + p \log(1-z) - q \log(z))$$

$$= (\log(z), \log(1-z)) + (\log(1-z), \log(z)) = \chi(-\frac{\pi i}{6}).$$

because of $\chi(4pq\pi i) = 0$. (For equation ii) one actually does not need the assumption $\text{Im}(z) > 0$.)
iii) Because of $\text{Im}(\frac{1}{1-z}) > 0$ one can apply i) to $\frac{1}{1-z}$ and get
\[(\log(\frac{1}{1-z}) - 2q\pi i, \log(\frac{-z}{1-z}) + 2(p-q)\pi i) = -(\log(1-z) + 2q\pi i, \log(z) + 2p\pi i) + \chi(-\frac{1}{2}\log(\frac{1}{1-z}) + (2q^2 - q)\pi i)\]
\[= (\log(z) + 2p\pi i, \log(1-z) + 2q\pi i) + \chi(\frac{1}{2}\log(1-z) + (2q^2 - q)\pi i + \frac{\pi i}{6}).\]

iv) Application of ii) to $1 - \frac{1}{z}$ yields
\[(\log(1-\frac{1}{z}) + 2(q-p)\pi i, \log(1-\frac{1}{z}) - 2p\pi i) = -(\log(\frac{1-z}{z}), \log(z) + 2p\pi i) + \chi(-\frac{\pi i}{6})\]
\[= (\log(z) + 2p\pi i, \log(1-z) + 2q\pi i) + \chi(\frac{1}{2}\log(z) + (2p^2 - p)\pi i - \frac{\pi i}{6}).\]

v) Because of $\text{Im}(1-\overline{z}) > 0$ one can apply iv) to $1 - \overline{z}$ and use $1 - \frac{1}{\overline{z}} = -\frac{1}{z}$ to get
\[(\log(-\frac{1}{z}) + 2(p-q)\pi i, \log(-\frac{1}{z}) - 2q\pi i) = (\log(1-\overline{z}) + 2q\pi i, \log(z) + 2p\pi i) + \chi(\frac{1}{2}\log(1-z) + (2q^2 - q - \frac{1}{6})\pi i),\]
which by ii) equals to
\[-(\log(z) + 2p\pi i, \log(1-z) + 2q\pi i) + \chi(\frac{1}{2}\log(1-z) + (2q^2 - q - \frac{1}{3})\pi i).\]

Remark: Other relations have been proved in [19, Proposition 13.1] and [12, Proposition 5.1], but they appear not to be correct.

We will also use some elementary facts about sums of (imaginary parts of) logarithms, i.e. sums of arguments of complex numbers. Recall that we use the convention that $-\pi < \text{arg}(z) \leq \pi$ for $z \in \mathbb{C}\setminus\{0\}$. In particular, $\log(\frac{z}{z}) = -\log(z)$ for all $z \neq 0$. Whenever $\text{Im}(z) > 0$ holds, one has the equality $\log(z) = \log(-z) + \pi i$. The following lemma collects some further elementary facts which we will be used especially in the proof of Lemma 8.

**Lemma 6.** For all $z \in \mathbb{C}\setminus\mathbb{R}$ we have the following identities.

\[
\log(z) - \log(\overline{z}) = \begin{cases} 
\log(\frac{z}{\overline{z}}) & \text{Re}(z) > 0 \text{ or } (\text{Re}(z) = 0, \text{Im}(z) > 0) \\
\log(\frac{z}{\overline{z}}) + 2\pi i & \text{Re}(z) < 0, \text{Im}(z) > 0 \\
\log(\frac{z}{\overline{z}}) - 2\pi i & \text{Re}(z) \leq 0, \text{Im}(z) < 0
\end{cases}
\]

\[
\log(z) - \log(z - \overline{z}) = \log(\frac{z}{z - \overline{z}}), \log(\overline{z}) - \log(\overline{z} - z) = \log(\frac{z}{z - \overline{z}})
\]

\[
\log(1 - \overline{z}) - \log(z) = \log(\frac{1 - \overline{z}}{z}), \log(1 - z) - \log(\overline{z}) = \log(\frac{1 - z}{z})
\]

\[
\log(\overline{z} - z) - \log(\overline{z} - 1) = \log(\frac{z - \overline{z}}{1 - \overline{z}}) = \log(z) - \log(\overline{z} - 1)
\]

\[
\log(z) - \log(1 - z) = \log(\frac{z}{1 - z}), \log(\overline{z}) - \log(1 - \overline{z}) = \log(\frac{z}{1 - z})
\]

\[
\log(z) + \log(1 - \overline{z}) = \log(z(1 - \overline{z})), \log(\overline{z}) + \log(1 - z) = \log(\overline{z}(1 - z))
\]

QED
\[
\log(z(1 - \overline{z})) - \log(z - \overline{z}) = \log\left(\frac{z}{\overline{z}}\right), \quad \log(\overline{z}(1 - z)) - \log(z - \overline{z}) = \log\left(\frac{\overline{z}(1 - z)}{z - \overline{z}}\right)
\]

\[
\log(1 - z) - \log(1 - \overline{z}) = \begin{cases} 
\log \frac{1 - z}{|z|} & Re(z) < 1 \\
\log \frac{1 - \overline{z}}{|z|} + 2\pi i & Re(z) \geq 1
\end{cases}
\]

**Proof:**

i) is obvious and ii) follows from the fact that the imaginary parts of \(z\) and \(z - \overline{z}\) have the same sign, so the difference of their arguments must be smaller than \(\pi\). Similarly iii) follows because the imaginary parts of \(1 - \overline{z}\) and \(z\) have the same sign and iv) follows because the imaginary parts of \(z - \overline{z}\) and \(1 - \overline{z}\) have the same sign.

From \(|\arg(z) - \arg(-z)| = \pi\) one can easily conclude \(|\arg(z) - \arg(1 - z)| < \pi\), which implies v). Similarly from \(|\arg(z) + \arg(-\overline{z})| = \pi\) one can easily conclude \(|\arg(z) + \arg(1 - \overline{z})| < \pi\), which implies vi).

For vii), one can check by explicit computation that \(Im(z(1 - \overline{z})) = Im(z)\), hence the imaginary parts of \(z(1 - \overline{z})\) and \(z - \overline{z}\) have the same sign and the claim follows.

For viii), the difference of the arguments of \(1 - z\) and \(1 - \overline{z}\) is \(2\arg(1 - z)\), so the equality holds if \(\arg(1 - z) \leq \frac{\pi}{2}\), which is equivalent to \(Re(1 - z) \geq 0\), thus \(Re(z) \leq 1\).

**QED**

2.4. **Example: the figure eight knot complement.** As the computations in the next section will be rather lengthy it shall be helpful to have an explicit example at hand to check correctness of the calculations at each step.

Consider the figure eight knot complement with its well-known ideal triangulation by two ideal simplices, let \(L\) be the lift of this triangulation to the universal cover. A fundamental domain for the action of \(\pi_1 M\) on \(L\) has 5 vertices \(v_0, \ldots, v_4\), where \((v_1, v_2, v_4)\) is the common face and the (order-preserving) gluing sends \((v_0, v_1, v_2)\) to \((v_1, v_3, v_4)\), \((v_0, v_2, v_4)\) to \((v_1, v_2, v_3)\) and \((v_0, v_1, v_4)\) to \((v_2, v_3, v_4)\), see [10, Section 4.4.2].

An obstruction cycle is given by \(\sigma = (v_0, v_1, v_2) + (v_0, v_1, v_4)\). With [9, Example 3.1.1] and the algorithm in [10, Section 9] we obtain the \(PSL(2, \mathbb{C})/N\)-valued decoration (with the \(N\)-cosets of course depending only on the first column) given by

\[
g_{v_0}N = N \quad \text{and} \quad g_{v_1}N = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}N, \quad g_{v_2}N = \begin{pmatrix} \omega & \omega^2 \\ \omega & 0 \end{pmatrix}N, \quad g_{v_3}N = \begin{pmatrix} -1 & \overline{\omega} \\ 0 & -1 \end{pmatrix}N, \quad g_{v_4}N = \begin{pmatrix} \omega & -1 \\ 1 & 0 \end{pmatrix}N.
\]

Let \(\Delta_\sigma = (v_0, v_1, v_2, v_4)\) and \(\Delta_\omega = (v_1, v_2, v_3, v_4)\), then the fundamental class of the hyperbolic monodromy \(\rho\) (see Definition 9) is

\[
\rho \cdot [M, \partial M] = \lambda(\Delta_\sigma, \rho) - \lambda(\Delta_\omega, \rho) = (\log \overline{\omega}, \log \omega) - (\log \omega, \log \overline{\omega}) \in \hat{B}(\mathbb{C})
\]

with \(\omega = \frac{1}{\lambda} + \overline{\Delta_\sigma}\), see [19, Section 15].

The ptolemy coordinates are defined up to sign. With the given obstruction cycle \(\sigma\), we can choose the signs such that the equation \(\sigma_0 \sigma_2 \sigma_0 \sigma_2 \sigma_1 c_{12} + \sigma_0 \sigma_1 c_{01} c_{23} = \)
σ_0σ_2c_02c_13 from \[9, Definition 3.5\] is satisfied for both simplices. We obtain so for the simplex \(Δ_σ\) (with \(σ_2 = σ_3 = -1\))

\[c_{01} = 1, c_{02} = ω, c_{03} = 1, c_{12} = c_{13} = ω, c_{23} = 1,\]

and for the simplex \(Δ_ω\) (with \(σ_0 = σ_1 = -1\))

\[c_{01} = ω, c_{02} = 1, c_{03} = ω, c_{12} = c_{13} = 1, c_{23} = ω.\]

Now, even though this is only a \(PSL(2, C)/N\)-decoration, the ±1-ambiguity will disappear when we consider \(ρ ⊗ ρ\). So (denoting by abuse of notation \(N ⊗ N \subset SL(4, C)\) again by \(N\)) we obtain an \(SL(4, C)/N\)-decoration for \(ρ ⊗ ρ\) by \(g_{v_0}N = N\) and

\[
g_{v_1}N = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} N, g_{v_2}N = \begin{pmatrix} 1 & ω & 1 \\ 1 & 0 & ω & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} N, \\
g_{v_3}N = \begin{pmatrix} 1 & -ω & -1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -ω \\ 0 & 0 & 0 & 1 \end{pmatrix} N, g_{v_4}N = \begin{pmatrix} 1 & -ω & -1 & 1 \\ ω & 0 & -1 & 0 \\ ω & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} N.
\]

One can check that this is in the normal form of the next section but with \(d\) and \(e\) replaced by \(-d\) and \(-e\). (And of course for \(Δ_ω\) one has to multiply by \(g_{v_1}^{-1}\) from the left.) Then the computations in the next section yield (in analogy to the computations leading to the proof of Corollary 4):

\[
λ(Δ_σ, ρ ⊗ \overline{ρ}) = 2(\log(ω), \log(ω)) + 2(\log(ω), \log(ω)) \\
λ(Δ_ω, ρ ⊗ \overline{ρ}) = 2(\log(ω), \log(ω)) + 2(\log(ω), \log(ω))
\]

and so \((ρ ⊗ \overline{ρ})_∗ [M, \partial M] = λ(Δ_σ, ρ ⊗ \overline{ρ}) - λ(Δ_σ, ρ ⊗ \overline{ρ}) = 0\) in \(\hat{B}(C)\).

We will take up this example in Section 3.5.

3. Computations

In this section we compute the Chern-Simons invariants (and actually the fundamental class in the extended Bloch group) for representations of the form \(ρ ⊗ \overline{ρ}\) when \(ρ: π_1 M → PSL(2, C)\) is a representation of some 3-manifold group. These representations are equivalent in \(GL(4, C)\) to those coming from the composition of \(ρ\) with the isomorphism \(PSL(2, C) = SO(3, 1)\) and it might, at first glance, have seemed more natural to compute the Chern-Simons invariants directly for that representation. It turns out however that that would have been much harder because a simplex with an (in the sense of Definition 8) generic \(PSL(2, C)/N\)-decoration need not have a generic \(SO(3, 1)/N\) coordinates, in general some of the ptolemy coordinates may be zero. So further subdivision of the triangulation would be necessary to obtain generic \(SO(3, 1)/N\)-decorations. (One can check that the canonical triangulation of the figure eight knot complement from Section 2.4...
yields an instance of this phenomenon.) For this reason we will work with the representation \( \rho \otimes \overline{\rho} : \pi_1 M \to SL(4, \mathbb{C}) \).

### 3.1. Standard form for simplices in \( G/N \)

The proof of Theorem 1 will work by a simplexwise computation, so for most of this section we will consider one simplex and try to compute its contribution to the fundamental class and the Chern-Simons invariant. At first we describe a standard form for \( SL(2, \mathbb{C})/N \)-decorated simplices in the sense of Definition 7.

Consider \( G = SL(2, \mathbb{C}) \) and \( N \subset G \) the subset of upper triangular matrices with 1’s on the diagonal. There is a \( G \)-equivariant bijection \( G/N = \mathbb{C}^2 \setminus \{0\} \). A 4-tuple \( (v_0, v_1, v_2, v_3) \) of pairwise distinct elements in \( \mathbb{C}^2 \setminus \{0\} \) is in the \( SL(2, \mathbb{C}) \)-orbit of some 4-tuple with \( v_0 = (1, 0) \) and \( v_1 = (0, a), a \in \mathbb{C} \setminus \{0\} \). This implies that each decorated 3-simplex \( \Delta \) is in the \( SL(2, \mathbb{C}) \)-orbit of a decorated simplex

\[
(( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} N, \begin{pmatrix} 0 & -\bar{a} \\ a & 0 \end{pmatrix} N, \begin{pmatrix} -\frac{d}{a} & -\frac{1}{b} \\ \frac{a}{b} & 0 \end{pmatrix} N, \begin{pmatrix} -\frac{e}{a} & -\frac{1}{c} \\ \frac{a}{c} & 0 \end{pmatrix} N)).
\]

One may check that the ptolemy coordinates of the edges are

\[
c_{1000} = a, c_{0100} = b, c_{1001} = c, c_{0110} = d, c_{0101} = e, c_{0011} = f
\]

with \( af + cd = be \), and thus

\[
\lambda(c_{0000}) = (\log(e) + \log(d) - \log(b) - \log(e), \log(a) + \log(f) - \log(b) - \log(e)).
\]

When \( \Delta \subset H^3 \cup \partial_{\infty} H^3 \) is the ideal hyperbolic simplex whose ideal vertices are the projections of \( v_0, v_1, v_2, v_3 \) to \( \mathbb{C}P^1 = \partial_{\infty} H^3 \), then one can easily check that \( z = \frac{cd}{ef} \) is the cross ratio of the vertices of \( \Delta \) and it is well-known that \( \Delta \) is non-degenerate if and only if \( z \notin \mathbb{R} \) and that the ordering of \( \Delta \) agrees with the orientation of \( H^3 \) if and only if \( \text{Im}(z) > 0 \).

We remark for later use that \( \lambda(c_{0000}) = (\log(z) + 2p\pi i, \log(1-z) + 2q\pi i) \) for some integers \( p \) and \( q \).

We will see in the next subsection that our computations will only work for \( z \notin \mathbb{R} \), i.e., for non-degenerate simplices. At the time of writing it is not known whether every hyperbolic 3-manifold admits an ideal triangulation with no degenerate simplex. However the methods of [10] do not require ideal triangulations but allow interior vertices, so upon performing barycentric subdivision and suitably decorating the interior vertices we can always assume to have simplices with \( z \notin \mathbb{R} \) throughout. (See [10] Proposition 5.4.)

### 3.2. Ptolemy coordinates of the tensor product

For a representation \( \rho: \pi_1 M \to SL(2, \mathbb{C}) \) we consider its tensor product with its complex conjugate \( \rho \otimes \overline{\rho} : \pi_1 M \to SL(4, \mathbb{C}) \).

Triangular matrices with 1’s on the diagonal are sent to triangular matrices with 1’s on the diagonal.
Given an $SL(2, \mathbb{C})/N$-decoration for $\rho$, we obtain a decoration for $\rho \otimes \overline{\rho}$ in the obvious way, replacing $A$ by $A \otimes \overline{A}$ on each vertex.

So we fix again one simplex $\Delta$, then the normal form decoration from the previous [Section 3.1] is mapped to the $N$-cosets of

$$
\left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{array} \right) , \quad \left( \begin{array}{cccc}
0 & 0 & 0 & \frac{1}{c} \\
0 & 0 & -\frac{a}{b} & 0 \\
0 & -\frac{a}{b} & 0 & 0 \\
| a^2 | & 0 & 0 & 0 \\
\end{array} \right) , \quad \left( \begin{array}{cccc}
| d |^2 & \frac{d}{e} & \frac{e}{d} & \frac{1}{| c^2 |} \\
-\frac{d}{e} & 0 & -\frac{e}{d} & 0 \\
-\frac{e}{d} & 0 & 0 & 0 \\
| c^2 | & 0 & 0 & 0 \\
\end{array} \right) .
$$

From this and $af + cd = be$ we compute the ptolemy coordinates of $\Delta$ as follows

- $c_{3100} = |a|^2$, $c_{3010} = |b|^2$, $c_{3001} = |c|^2$
- $c_{2200} = a^2$, $c_{2110} = ab \overline{a}$, $c_{2101} = ace$
- $c_{2020} = b^2$, $c_{2011} = bc \overline{f}$, $c_{2002} = c^2$
- $c_{1300} = -|a|^2$, $c_{1210} = -ab \overline{d}$, $c_{1201} = -ace$
- $c_{1120} = abd$, $c_{1111} = 2Im(\overline{bcde})i$, $c_{1102} = ace$
- $c_{1030} = -|b|^2$, $c_{1021} = -\overline{b}f$, $c_{1012} = bcf$
- $c_{1003} = -|c|^2$, $c_{0310} = |d|^2$, $c_{0301} = |e|^2$
- $c_{0220} = d^2$, $c_{0211} = def$, $c_{0202} = e^2$
- $c_{0130} = -|d|^2$, $c_{0121} = -\overline{df}$, $c_{0112} = def$
- $c_{0103} = -|e|^2$, $c_{0031} = |f|^2$, $c_{0022} = f^2$, $c_{0013} = -|f|^2$

(It should have sufficed to compute $c_{3100}, c_{2200}, c_{2110}, c_{1120}$ and then proceed by symmetry, however to be safe we doublechecked all computations with [25].)

We note that these coordinates are nonzero (i.e., the decoration is generic) if and only if the ptolemy coordinates $a, \ldots, f$ of $\rho$ are nonzero and if moreover $\overline{bcde} \not\in \mathbb{R}$. The latter condition is in view of $\overline{bcde} = |b|^2 \frac{cd}{de}$ equivalent to the condition that the cross ratio $z = \frac{cd}{de}$ is not a real number and as argued in [Section 3.1] above this can always be assumed.

We plug the ptolemy coordinates of $\Delta$ into the formula from [Definition 9] and obtain the following.

$$
\lambda(c_{2000}) = (log |c|^2 + log abd - log |b|^2 - log ace, log |a|^2 + log bc \overline{a} - log |b|^2 - log ace)
$$

$$
\lambda(c_{1100}) = (log ac \overline{a} + log(-\overline{bc}) - log abd - log(-\overline{a}ce), log a^2 + log 2Im(\overline{bcde})i - log abd - log(-\overline{ace})
$$

$$
\lambda(c_{1010}) = (log be \overline{f} + log \overline{bc}d - log b^2 - log 2Im(\overline{bcde})i, log abd + log(-\overline{bc}f) - log b^2 - log 2Im(\overline{bcde})i)
$$

$$
\lambda(c_{0101}) = (log e^2 + log 2Im(\overline{bcde})i - log bc \overline{f} - log ace, log ac \overline{e} + log \overline{bc}f - log bc \overline{f} - log ace)
$$

$$
\lambda(c_{0202}) = (log(-\overline{a}ce) + log |d|^2 - log(-ab \overline{d}) - log |e|^2, log(- |d|^2 + log de \overline{f} - log(ab \overline{d}) - log |e|^2)
$$

$$
\lambda(c_{0211}) = (log 2Im(\overline{bcde})i + log d^2 - log \overline{abcd} - log de \overline{f}, log(-ab \overline{d} - log(-\overline{df}) - log \overline{abcd} - log de \overline{f})
$$

$$
\lambda(c_{0013}) = (log(\overline{ace} + log de \overline{f} - log 2Im(\overline{bcde})i - log e^2, log(-\overline{ace} + log de \overline{f} - log 2Im(\overline{bcde})i - log e^2)
$$

$$
\lambda(c_{0020}) = (log(-\overline{bf}) + log(- |d|^2) - log(- |b|^2) - log(-\overline{df}), log \overline{abcd} + log |f|^2 - log(- |b|^2) - log(-\overline{df})
$$
\[ \lambda(c_{0011}) = (\log \overline{bc}f + \log(-\overline{ad}f) - \log(-\overline{bc}f) - \log \overline{def}, \log 2Im(bcd\overline{r})i + \log f^2 - \log(-\overline{bc}f) - \log \overline{def}) \]
\[ \lambda(c_{0002}) = (\log(-|c|^2) + \log \overline{def} - \log \overline{bc}f - \log(-|e|^2), \log \overline{ac}c + \log(-|f|^2) - \log \overline{bc}f - \log(-|e|^2)) \]

By Proposition 2, we are free to choose lifts (i.e., logarithms) as long as we chose the same lift for the same numbers. To simplify the above expressions we fix once and for all the following lifts:

\[ \log(ab\overline{d}) = \log a + \log b + \log \overline{d}, \log a\overline{d} = \log a + \log \overline{b} + \log \overline{d}, \text{etc.} \]
\[ \log(-abd) = \pi i + \log a + \log b + \log \overline{d}, \text{etc.} \]
\[ \log a^2 = 2\log a, \log(-a^2) = \pi i + 2\log a, \text{etc.} \]
\[ \log |a|^2 = \log a + \log \overline{a}, \log(-|a|^2) = \pi i + \log a + \log \overline{a}, \text{etc.} \]
\[ \log(2Im(bcd\overline{r})i) = \pi i + \log b + \log \overline{c} + \log \overline{d} + \log e + \log(1 - \frac{bcd\overline{r}}{bcde}) \]

The reason for the somewhat unnatural seeming choice of \(\tilde{\lambda}(c_{1111})\) is to simplify the formulas in the following lemma. One can easily check that it is indeed a choice of logarithm, i.e., that exponentiating the right hand side gives \(2Im(bcd\overline{r})i\).

Now with this lifts we can simplify as follows.

**Lemma 7.** With the above-chosen lifts, the pre-Bloch group elements associated to the ptolemy coordinates of \(\rho \otimes \overline{p}\) for the decorated simplex \(\Delta\) from Section 3.1 are:

\[ \tilde{\lambda}(c_{0000}) = \tilde{\lambda}(c_{0200}) = \tilde{\lambda}(c_{0002}) = \]
\[ (-\log \overline{b} + \log \overline{c} + \log \overline{d} - \log \overline{f}, \log \overline{a} - \log \overline{b} - \log \overline{d} + \log \overline{f}) \]
\[ \tilde{\lambda}(c_{0020}) = (-\log \overline{b} + \log \overline{c} + \log \overline{d} - \log \overline{f}, -2\pi i + \log \overline{a} - \log \overline{b} - \log \overline{d} + \log \overline{f}) \]
\[ \tilde{\lambda}(c_{1000}) = \tilde{\lambda}(c_{0011}) = \]
\[ (\log \overline{b} - \log b + \log c - \log \overline{c} + \log d - \log \overline{d} + \log e - \log e, \log(1 - \frac{bcd\overline{r}}{bcde})) \]
\[ \tilde{\lambda}(c_{0101}) = \tilde{\lambda}(c_{0110}) = \]
\[ (-\pi i + \log \overline{a} - \log b + \log c - \log \overline{c} + \log d - \log \overline{d} + \log e + \log \overline{f} - \log(1 - \frac{bcd\overline{r}}{bcde}), \log a - \log b - \log c + \log \overline{b} - \log \overline{f} - \log(1 - \frac{bcd\overline{r}}{bcde})) \]
\[ \tilde{\lambda}(c_{1001}) = (\pi i - \log \overline{a} + \log \overline{b} + \log \overline{d} - \log \overline{f} + \log(1 - \frac{bcd\overline{r}}{bcde}), \log a - \log \overline{b} + \log \overline{b} - \log \overline{c} + \log \overline{b} - \log \overline{e} + \log \overline{f} - \log(1 - \frac{bcd\overline{r}}{bcde})) \]
\[ \tilde{\lambda}(c_{0110}) = (\pi i - \log \overline{a} + \log \overline{c} + \log \overline{d} - \log \overline{f} + \log(1 - \frac{bcd\overline{r}}{bcde}), 2\pi i + \log a - \log \overline{b} + \log \overline{d} - \log \overline{b} + \log \overline{e} - \log \overline{f} - \log(1 - \frac{bcd\overline{r}}{bcde})) \]

Remark: The symmetry-breaking formulas for \(\tilde{\lambda}(c_{0020})\) and \(\tilde{\lambda}(c_{0110})\) seem to be an artefact of our somewhat arbitrary choice of logarithms.

Next we want to express these formulas in the \((z,p,q)\)-form (in the notation of [10], compare the remark after Definition 2). The proof will use some elementary facts about complex logarithms, which for better readability had been collected in Lemma 6 before.
Lemma 8. Let \( z = \frac{cd}{ae} = 1 - \frac{af}{cd} \in \mathbb{R} \) be the cross ratio of \( \Delta \), and define the integers \( p, q \) via

\[
\log c + \log d - \log b - \log e = \log(z) + 2p\pi i,
\]
\[
\log a + \log f - \log b - \log e = \log(1 - z) + 2q\pi i.
\]

Then

\[
\tilde{\lambda}(c_{2000}) = \tilde{\lambda}(c_{0200}) = \tilde{\lambda}(c_{0002}) = (\log(z) - 2p\pi i, \log(1 - z) - 2(q + 1)\pi i)
\]
\[
\tilde{\lambda}(c_{0020}) = (\log(z) - 2p\pi i, \log(1 - z) - 2(q + 1)\pi i)
\]
\[
\tilde{\lambda}(c_{1100}) = \tilde{\lambda}(c_{0110}) = \tilde{\lambda}(c_{0101}) = \tilde{\lambda}(c_{1001}) = \tilde{\lambda}(c_{1010}) = \begin{cases}
(\log(\frac{z}{1-z}) + 2(2p - q)\pi i, \log(\frac{1-z}{z}) + 2q\pi i) & \text{Im}(z) > 0, \text{Re}(z) < 1 \\
(\log(\frac{z}{1-z}) + 2q - p)\pi i, \log(\frac{1-z}{z}) + 4q\pi i) & \text{Im}(z) > 0, \text{Re}(z) \geq 1 \\
(\log(\frac{z}{1-z}) + 2q - p + 1)\pi i, \log(\frac{1-z}{z}) + 4q\pi i) & \text{Im}(z) < 0, \text{Re}(z) < 1 \\
(\log(\frac{z}{1-z}) + 2q - p + 1)\pi i, \log(\frac{1-z}{z}) + 4q\pi i) & \text{Im}(z) < 0, \text{Re}(z) \geq 1 \\
(\log(\frac{z}{1-z}) + 2q - p + 1)\pi i, \log(\frac{1-z}{z}) + 4q\pi i) & \text{Im}(z) < 0, \text{Re}(z) < 1 \\
(\log(\frac{z}{1-z}) + 2q - p + 1)\pi i, \log(\frac{1-z}{z}) + 4q\pi i) & \text{Im}(z) < 0, \text{Re}(z) \geq 1 \\
\end{cases}
\]

Proof: a) The formulas for \( \tilde{\lambda}(c_{2000}) = \tilde{\lambda}(c_{0200}) = \tilde{\lambda}(c_{0002}) \) and \( \tilde{\lambda}(c_{0020}) \) are immediate by complex conjugation of the formulas for \( \log(z) + 2p\pi i \) and \( \log(1 - z) + 2q\pi i \).

b) Subtraction yields \( \tilde{\lambda}(c_{1100}) = \tilde{\lambda}(c_{0111}) = (\log(z) - \log(\frac{z}{1-z}) + 4p\pi i, \log(1 - \frac{z}{1-z})). \)

The formulas for \( \tilde{\lambda}(c_{1100}) = \tilde{\lambda}(c_{0111}) \) then follow from Lemma 6 i).

c) By Lemma 6 v) we have \( \log(z) - \log(1 - z) = \log(\frac{z}{1-z}) \). So subtraction yields

\[
- \log(a) + \log(c) + \log(d) - \log(f) = \log(\frac{z}{1-z}) + 2(p - q)\pi i.
\]

From this one obtains

\[
\tilde{\lambda}(c_{1010}) = \tilde{\lambda}(c_{0101}) = (-\pi i - \log(\frac{z}{1-z}) + 2(p - q)\pi i + \log(z) + 2p\pi i - \log(1 - \frac{z}{1-z}), \log(1 - z) + 2q\pi i - \log(1 - \frac{z}{1-z})).
\]

Using ii) and v) from Lemma 6 the first coordinate simplifies to \( -\pi i + \log(1 - \frac{z}{1-z}) + \log(z) - \log(\frac{z}{1-z}) + 2(2p - q)\pi i \). We observe that \( -\log(z - z) - \pi i \) is \( -\log(z - \frac{z}{1-z}) \) or \( -\log(z - \frac{1-z}{z}) - 2\pi i \) according to whether \( \text{Im}(z) > 0 \) or \( \text{Im}(z) < 0 \). Then with Lemma 6 vi) and vii) we get the claimed formula for the first coordinate. The formula for the second coordinate follows is also obtained by applying Lemma 6 vi) and vii).
d) Conjugating the first equation in the proof of c) and plugging it into the first coordinate yields

$$\tilde{\lambda}(c_{1001}) = (\pi i + \log(\frac{z}{1-z}) - 2(p-q)\pi i + \log(1 - \frac{z}{1-z}), \log(1 - z) + 2q\pi i - \log(1 - \frac{z}{1-z}) + 2q\pi i)$$

We have again $$\log(\frac{z}{1-z}) = \log(z) - \log(1 - \frac{z}{1-z})$$ and $$\log(1 - \frac{z}{1-z}) = \log(z) - \log(1 - z)$$. If $$\text{Im}(z) > 0$$, then $$\pi i + \log(z) = \log(\frac{z}{1-z})$$ and (as a direct computation shows) $$\text{Im}(\frac{z}{1-z}) < 0$$, so $$\pi i + \log(\frac{z}{1-z}) = \log(\frac{z}{1-z})$$, so the first coordinate simplifies to $$\pi i - \log(1 - z) + \log(z - 2) + 2(q-p)\pi i = -\log(\pi - 1) + \log(z - 2) + 2(q-p)\pi i = \log(\frac{z}{1-z}) + 2(q-p)\pi i$$, where the last equality uses that $$z - 2$$ and $$1 - z$$ have the same imaginary part, so the difference of their arguments is in $$(-\pi, \pi)$$. If $$\text{Im}(z) < 0$$, then one obtains by similar arguments that the first coordinate equals $$\log(\frac{z}{1-z}) + 2(q-p+1)\pi i$$. The formula for the second coordinate follows from Lemma 6 viii). This proves the formula for $$\tilde{\lambda}(c_{1001})$$ and by exactly the same arguments we obtain that for $$\tilde{\lambda}(c_{0110})$$.

QED

3.3. Wonderful cancelations: using the five-term relation.

Lemma 9. The following elements of $$\hat{P}(\mathbb{C})$$

$$(e_0, f_0) =$$

$$(\pi i + \log \frac{z}{1-z} + \log b + \log c + \log d - \log \frac{z}{1-z} + \log e + \log \frac{z}{1-z} - \log (1 - \frac{z}{1-z}), \log a + \log b - \log e + \log f - \log (1 - \frac{z}{1-z}))$$

$$(e_1, f_1) = (- \log b + \log c + \log d - \log e, \log a - \log b - \log e + \log f)$$

$$(e_2, f_2) = (\pi i - \log \frac{z}{1-z} + \log d - \log \frac{z}{1-z} + \log e + \log (1 - \frac{z}{1-z}), \log a + \log b - \log c + \log d - \log \frac{z}{1-z})$$

$$(e_3, f_3) = (\pi i - \log \frac{z}{1-z} + \log d - \log \frac{z}{1-z} + \log e + \log f - \log (1 - \frac{z}{1-z}))$$

$$(e_4, f_4) = (- \log \frac{z}{1-z} - \pi i - \log \frac{z}{1-z} - \log b + \log c - \log d - \log \frac{z}{1-z} - \log e - \log \frac{z}{1-z})$$

satisfy the relation

$$(e_0, f_0) - (e_1, f_1) + (e_2, f_2) - (e_3, f_3) + (e_4, f_4) = 0$$

in the extended pre-Bloch group $$\hat{P}(\mathbb{C})$$.

Proof: An obvious computation shows that the equalities

$$e_2 = e_1 - e_0, e_3 = e_1 - e_0 - f_1 + f_0, f_3 = f_2 - f_1, e_4 = f_0 - f_1, f_4 = f_2 - f_1 + e_0$$

hold.

According to Definition 2 this implies the wanted 5-term relation in $$\hat{P}(\mathbb{C})$$.

QED
Lemma 10. With the notation from Lemma 8 and Lemma 9 we have

\[
(e_0, f_0) = \begin{cases}
(\log(\frac{1-\overline{z}}{z}) + 2(2p - q)\pi i, \log(\frac{1-z}{1-\overline{z}}) + 2q\pi i) & \text{Im}(z) > 0 \\
(\log(\frac{1-\overline{z}}{z}) + 2(2p - q - 1)\pi i, \log(\frac{1-z}{1-\overline{z}}) + 2q\pi i) & \text{Im}(z) < 0
\end{cases}
\]

\[
(e_1, f_1) = (\log(z) + 2p\pi i, \log(1-z) + 2q\pi i)
\]

\[
(e_2, f_2) = \begin{cases}
(\log(\frac{z}{1-\overline{z}}) + 2(q-p)\pi i, \log(\frac{1-z}{1-\overline{z}}) + 4q\pi i) & \text{Im}(z) > 0, \text{Re}(z) < 1 \\
(\log(\frac{z}{1-\overline{z}}) + 2(q-p)\pi i, \log(\frac{1-z}{1-\overline{z}}) + (4q+2)\pi i) & \text{Im}(z) > 0, \text{Re}(z) \geq 1 \\
(\log(\frac{z}{1-\overline{z}}) + 2(q-p+1)\pi i, \log(\frac{1-z}{1-\overline{z}}) + 4q\pi i) & \text{Im}(z) < 0, \text{Re}(z) < 1 \\
(\log(\frac{z}{1-\overline{z}}) + 2(q-p+1)\pi i, \log(\frac{1-z}{1-\overline{z}}) + (4q+2)\pi i) & \text{Im}(z) < 0, \text{Re}(z) \geq 1
\end{cases}
\]

\[
(e_3, f_3) = \begin{cases}
(\log(-\frac{1}{\overline{z}}) + 2(q-p)\pi i, \log(\frac{1}{1-\overline{z}}) + 2q\pi i) & \text{Im}(z) > 0 \\
(\log(-\frac{1}{\overline{z}}) + 2(q-p+1)\pi i, \log(\frac{1}{1-\overline{z}}) + 2q\pi i) & \text{Im}(z) < 0
\end{cases}
\]

\[
(e_4, f_4) = \begin{cases}
(-\log(1-\overline{z}), \log(\frac{1}{1-\overline{z}}) + 4p\pi i) & \text{Im}(z) > 0 \\
(-\log(1-\overline{z}), \log(\frac{1}{1-\overline{z}}) + (4p-2)\pi i) & \text{Im}(z) < 0
\end{cases}
\]

Proof: The first three equalities follow directly from the proof of Lemma 8. The fourth equality follows from the definitions together with Lemma 6 iii) and the equality \(\log(-\overline{z}) = \log(\overline{z}) \pm \pi i\) with sign of \(\pi i\) according to the sign of \(\text{Im}(z)\). The fifth equality uses the same computations (in a different order) as that of \(\lambda(c_{1100})\) in the proof of Lemma 8 together with Lemma 6 ii) and \(\log(z-\overline{z}) = \log(\overline{z}-z) \pm \pi i\).

QED

We had chosen the terms in Lemma 9 such that

\[
(e_0, f_0) = \lambda(c_{0110}) = \overline{\lambda(c_{0101})}
\]

\[
(e_1, f_1) = \lambda(c_{0000}) = \overline{\lambda(c_{0002})} = \lambda(c_{0002})
\]

\[
(e_2, f_2) = \overline{\lambda(c_{1001})}.
\]

Next we want to relate \((e_3, f_3)\) and \((e_4, f_4)\) to \(\lambda(c_{0020})\) and \(\overline{\lambda(c_{1100})}\), respectively.

Lemma 11. With the notation from Lemma 8 the following equalities hold for all \(z \in \mathbb{C} \setminus \mathbb{R}\):

a)

\[
\lambda(c_{0110}) = (e_2, f_2) + \chi((2q-2p+1)\pi i + \log(z) - \log(\overline{z}))
\]

b)

\[
\lambda(c_{0002}) = -(e_3, f_3) + \chi(\frac{1}{2} \log(1-\overline{z}) + (2q^2 + q - \frac{1}{3})\pi i)
\]

\[
\overline{\lambda(c_{0020})} = -(e_3, f_3) + \chi(\frac{1}{2} \log(1-\overline{z}) - \log(\overline{z}) + (2p + 2q^2 + q - \frac{1}{3})\pi i)
\]

c)

\[
\lambda(c_{1100}) = \overline{\lambda(c_{0011})} = (e_4, f_4) + \chi(\frac{1}{2} \log(1-\overline{z}) - \frac{1}{6} \pi i)
\]
Proof:  
a) Direct application of Lemma 4 iv) yields
\[ \tilde{\lambda}(c_{0110}) = (e_2, f_2) + \left\{ \begin{array}{ll}
\chi(2(q-p)\pi i + \log(\frac{-z}{1-z})) & \text{Im}(z) > 0 \\
\chi(2(q-p+1)\pi i + \log(\frac{-z}{1-z})) & \text{Im}(z) < 0
\end{array} \right\}. \]

From Lemma 6 we have \( \log(\frac{-z}{1-z}) = \log(\frac{(3-\pi)}{3}) - \log(\frac{(3-\pi)}{3}) \). For \( \text{Im}(z) > 0 \) we have \( \log(1-z) - \log(\frac{(3-\pi)}{3}) = \pi i \) and for \( \text{Im}(z) < 0 \) we have \( \log(1-z) - \log(\frac{(3-\pi)}{3}) = -\pi i \).

The claim follows.

b) If \( \text{Im}(z) > 0 \), then with Lemma 5 v) we obtain
\[ (e_3, f_3) = -\log(\frac{z}{1-z}) + 2(q-p)\pi i, \log(\frac{1}{1-z}) + 2q\pi i \]

= \(-\log(\frac{z}{1-z}) + 2q\pi i, \log(\frac{1}{1-z}) + 2(q-p+1)\pi i) + \chi(-\frac{\pi i}{6}) \)

If \( \text{Im}(z) < 0 \), then \( \text{Im}(\bar{z}) > 0 \) and we apply Lemma 5 ii) and iii) to \( z \) to obtain
\[ (e_3, f_3) = -\log(\frac{1}{1-z}) + 2q\pi i, \log(\frac{1}{1-z}) + 2(q-p+1)\pi i) + \chi(-\frac{\pi i}{6}) \)

This proves the formula for \( \tilde{\lambda}(c_{0002}) \). The formula for \( \tilde{\lambda}(c_{0020}) \) then follows from \( \tilde{\lambda}(c_{0002}) - \tilde{\lambda}(c_{0020}) = \chi(\log(\frac{z}{1-z}) - 2p\pi i). \)

c) We use that
\[-\log(1-z) = \log(\frac{1}{1-z}) \text{ and } \log(\frac{z}{z-1}) = \log(\frac{-z}{1-z}) \]

So, if \( \text{Im}(z) > 0 \), we can apply Lemma 5 iii) to get
\[ (e_4, f_4) = \left\{ \begin{array}{ll}
(\log(\frac{1}{1-z}), \log(\frac{-z}{1-z}) + 4q\pi i) & \text{Im}(z) > 0 \\
(\log(\frac{1}{1-z}), \log(\frac{-z}{1-z}) + (4p-2)\pi i) & \text{Im}(z) < 0
\end{array} \right\} \]

= \left\{ \begin{array}{ll}
(\log(\frac{1}{1-z}) + 4p\pi i, \log(1-\frac{z}{1-z}) + \pi i) & \text{Im}(z) > 0, \text{Re}(z) > 0 \\
(\log(\frac{1}{1-z}) + (4p-2)\pi i, \log(1-\frac{z}{1-z}) + \pi i) & \text{Im}(z) < 0, \text{Re}(z) < 0
\end{array} \right\} \]

= \tilde{\lambda}(c_{1100}) + \chi(\frac{1}{2} \log(1-\frac{z}{1-z}) + \pi i)

where the last two equality uses Lemma 8 b).
If $\text{Im}(\hat{z}) < 0$, then $\text{Im}(\hat{z}) > 0$ and we can (after application of Lemma 5 ii)) apply Lemma 5 v) to $\hat{z}$ to obtain

$$(e_4, f_4) = \begin{cases} (\log(\frac{1}{\chi - q\pi i}), \log(\frac{1}{\chi - q\pi i}) + 4p\pi i) & \text{Im}(z) > 0 \\ (\log(\frac{1}{\chi - p\pi i}), \log(\frac{1}{\chi - p\pi i}) + (4p - 2)p\pi i) & \text{Im}(z) < 0 \end{cases}$$

$$(e_2, f_2) = \begin{cases} -\log(\frac{1}{\chi - q\pi i}) + 4p\pi i, \log(\frac{1}{\chi - q\pi i}) \right) + (4p - 2)p\pi i & \text{Im}(z) > 0 \\ -\log(\frac{1}{\chi - p\pi i}) + (4p - 2)p\pi i, \log(\frac{1}{\chi - p\pi i}) \right) + (4p - 2)p\pi i & \text{Im}(z) < 0 \end{cases}$$

where the last equality used again Lemma 8 b).

One checks that c) also holds for $\text{Re}(z) = 0$.

**QED**

**Corollary 4.** With the notation from Lemma 8 we have

$$\hat{\lambda}(c_{2000}) + \hat{\lambda}(c_{0200}) + \hat{\lambda}(c_{0020}) + \hat{\lambda}(c_{1100}) +$$

$$+ \hat{\lambda}(c_{0011}) + \lambda(c_{1010}) + \hat{\lambda}(c_{1001}) + \hat{\lambda}(c_{0110}) = 2(\log(z) + 2p\pi i, \log(1 - z) + 2q\pi i) + 2(\log(\overline{z}) - 2p\pi i, \log(1 - \overline{z}) - 2q\pi i).$$

**Proof:** From Lemma 11 and using that $\chi$ is a homomorphism Lemma 4 i) we obtain that the sum equals

$$2(e_0, f_0) + 2(\overline{f_1}, \overline{f_1}) + 2(e_2, f_2) - 2(e_3, f_3) + 2(e_4, f_4) +$$

$$+ \lambda(2q - 2p + 1)p\pi i + (2q^2 + q - \frac{1}{3}p\pi i + (2p + 2q^2 + q - \frac{1}{3}p\pi i - \frac{\pi}{3}i),$$

i.e. all terms involving $\overline{z} - z, 1 - \overline{z}$ or $\overline{z}$ cancel out. (Here we have used $\log(1 - \overline{z}) = \overline{\log(z) - \log(\overline{z})}$, see Lemma 6 ii).)

By definition of the extended pre-Bloch group we have the relation

$$(e_0, f_0) - (e_1, f_1) + (e_2, f_2) - (e_3, f_3) + (e_4, f_4) = 0,$$

which means that the first row of the above formula equals

$$2(e_1, f_1) + 2(\overline{f_1}, \overline{f_1}).$$

But $\chi$ vanishes on $4\pi i \mathbb{Z}$ so the second row of the above formula is zero.

By its definition we have

$$(e_1, f_1) = (\log(z) + 2p\pi i, \log(1 - z) + 2q\pi i),$$

so we get the claim.

**QED**
3.4. Proof of Theorem 1. Let \( M \) be a finite-volume, orientable, hyperbolic 3-manifold. Let \( \tau : \text{PSL}(2, \mathbb{C}) \to \text{SO}(3, 1) \) be the isomorphism \( \text{PSL}(2, \mathbb{C}) \to \text{SO}(3, 1) \). Then for each boundary-unipotent representation \( \rho : \pi_1 M \to \text{PSL}(2, \mathbb{C}) \) one has
\[
(\tau \circ \rho)_* [M, \partial M] = 2\rho_* [M, \partial M] + 2\overline{\rho_* [M, \partial M]} \in \hat{B}(\mathbb{C})
\]
if \( \rho \) lifts to a boundary-unipotent representation \( \pi_1 M \to \text{SL}(2, \mathbb{C}) \) (in particular if \( M \) is closed) and
\[
(\tau \circ \rho)_* [M, \partial M] = 2\rho_* [M, \partial M] + 2\rho_* [M, \partial M] \in \hat{B}(\mathbb{C})_{\text{PSL}}
\]
otherwise.

Proof: Let us first assume that \( \rho : \pi_1 M \to \text{PSL}(2, \mathbb{C}) \) lifts to a boundary-unipotent representation \( \pi_1 M \to \text{SL}(2, \mathbb{C}) \), which abusing notation we will also denote by \( \rho \).

Fix some generalized ideal triangulation \( M = \bigcup_{k=1}^r T_k \) that admits generic ptolemy coordinates \( c_k^t \) for \( \rho \) in the sense of Definition 8. (Such a triangulation exists by \[10, Proposition 5.4\].) For \( k = 1, \ldots, r \) let \( \lambda(c_k^t) \) be defined as in Definition 9 (with \( \alpha = 0 \)) and define \( z_k \in \mathbb{C}, p_k, q_k \in \mathbb{Z} \) via
\[
\lambda(c_k^t) = (\log(z_k) + 2p_k \pi i, \log(1 - z_k) + 2q_k \pi i).
\]
By definition we have
\[
\rho_* [M, \partial M] = \sum_{k=1}^r \epsilon_k \lambda(c_k^t)
\]
with the sign \( \epsilon_k = \pm 1 \) depending on orientation of \( T_k \). By Corollary 4 we have
\[
(\rho \otimes \overline{\rho})_* [M, \partial M] = \sum_{k=1}^r \epsilon_k (2(\log(z_k) + 2p_k \pi i, \log(1 - z_k) + 2q_k \pi i) + 2(\log(z_k) - 2p_k \pi i, \log(1 - z_k) - 2q_k \pi i)),
\]
from which we conclude
\[
(\rho \otimes \overline{\rho})_* [M, \partial M] = 2\rho_* [M, \partial M] + 2\overline{\rho_* [M, \partial M]}.
\]
Finally it is known from the representation theory of \( \text{SL}(2, \mathbb{C}) \) that the 2-fold covering \( \text{SL}(2, \mathbb{C}) \to \text{SO}(3, 1) \) is conjugate in \( \text{GL}(4, \mathbb{C}) \) to
\[
id \otimes \overline{id} : \text{SL}(2, \mathbb{C}) \to \text{SL}(4, \mathbb{C}).
\]
This implies that \( \rho \otimes \overline{\rho} \) is conjugate to \( \tau \circ \rho \), so we obtain the wanted equality
\[
(\tau \circ \rho)_* [M, \partial M] = 2\rho_* [M, \partial M] + 2\overline{\rho_* [M, \partial M]}.
\]

If \( \rho \) does not lift to \( \text{SL}(2, \mathbb{C}) \), then the ptolemy coordinates \( a, \ldots, f \) in Section 3.1 are only defined up to sign. We can choose some sign, so that for each simplex \( T_k \) we have some equality of the kind \( \pm a_k f_k \pm c_k d_k = b_k e_k \) (with certain signs) and can then still do all the computations in \( \hat{P}(\mathbb{C})_{\text{PSL}} \) to get the equality there. QED
Remark: The proof of Theorem 1 via Corollary 4 might leave the impression that the equality from Theorem 1 already holds simplexwise, but one should be aware that this is just a (surprising) effect of the special choice of logarithms before Lemma 7. With other (perhaps more natural) choices of logarithms the wanted equality would not hold simplexwise, rather additional contributions from different simplices would cancel out.

3.5. Non-liftable \(PSL(2,\mathbb{C})\)-representations. When a representation \(\rho: \pi_1 M \to PSL(2,\mathbb{C})\) does not lift\(^6\) to a boundary-unipotent representation \(\pi_1 M \to SL(2,\mathbb{C})\), then its Chern-Simons invariant is only defined modulo \(\pi^2\) and so of course the equality in Corollary 1 can only hold modulo \(\pi^2\).

However, since \(\rho \otimes \bar{\rho}\) is well-defined as a boundary-unipotent representation to \(SL(4,\mathbb{C})\) it actually makes sense to compute its Chern-Simons invariant modulo \(4\pi^2\) and we will describe in this section how to do this calculation.

Given a boundary-unipotent representation \(\pi_1 M \to PSL(2,\mathbb{C})\) its obstruction to lifting it as a boundary-unipotent representation to \(SL(2,\mathbb{C})\) is represented by a 2-cycle \(\sigma \in Z_2(K, \partial K; \mathbb{Z}/2\mathbb{Z})\). Depending on \(\sigma\) the ptolemy coordinates have to satisfy a certain simplexwise equation. Namely if for \(i = 0, \ldots, 3\) we denote by \(\sigma_i\) the value of \(\sigma\) on the face opposite to the \(i\)-th vertex, then

\[
\sigma_0\sigma_1af + \sigma_0\sigma_3cd = \sigma_0\sigma_2be,
\]

see [9, Definition 3.5]. (Now we fix one simplex and denote its ptolemy coordinates for the \(SL(2,\mathbb{C})\)-representation again by \(a, \ldots, f\) as in Section 3.1.)

For a given \(\sigma\) one can then use the methods from [10, Section 9] to simplexwise compute decorations for a simplex with given ptolemy coordinates \(a = c_{01}, b = c_{02}, \ldots, f = c_{23}\). The result of these computations is that a decoration of a 3-simplex is given by

\[
(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} N, \begin{pmatrix} 0 & -\frac{1}{a} \\ a & 0 \end{pmatrix} N, \begin{pmatrix} \frac{\pm d}{b} & -\frac{1}{b} \\ b & 0 \end{pmatrix} N, \begin{pmatrix} \frac{\pm e}{c} & -\frac{1}{c} \\ c & 0 \end{pmatrix} N),
\]

where the sign in front of \(d\) is positive if and only if \(\sigma_3 \neq 0\), and the sign in front of \(e\) is positive if and only if \(\sigma_2 \neq 0\). (This is basically due to the fact that the diamond coordinates of a face are multiplied by a sign according to its appearance in the obstruction cycle, see [10, Definition 9.23].)

When the signs in \(\pm d\) and \(\pm e\) are chosen as above, then we have an equality \(a(\pm f) + c(\pm d) = b(\pm e)\) with the sign in front of \(f\) being positive if and only if \(\sigma_1 \neq 0\).

\(^6\)This is in particular the case for the hyperbolic monodromy of a cusped hyperbolic 3-manifold. Although such representations by [2] can always be lifted to \(SL(2,\mathbb{C})\), it is proved in [1] that boundaries of incompressible surfaces necessarily lift to parabolic elements with eigenvalue \(-1\). Since these manifolds are always Haken this implies that they can not have a boundary-unipotent lift to \(SL(2,\mathbb{C})\). See [10, Proposition 9.20].
This means that we can do the computations from Section 3.2 but with \( d, e, f \) replaced by \( \pm d, \pm e, \pm f \) according to the values of the obstruction cycle. (Note that the \( f \)-coordinate does not appear in the decoration, but it made its entrance in the calculations of Section 3.2 indirectly through the formula \( af + cd = be \). For this reason we also have to change the sign of \( f \) accordingly.)

Then one can use Lemma 7 to compute \((\rho \otimes \tilde{\tau})_* [M, \partial M]\) as an element in \( \hat{B}(\mathbb{C}) \) (rather just in \( \hat{B}(\mathbb{C})_{PSL} \)) and the Chern-Simons invariant modulo \( 4\pi^2 \) (rather just modulo \( \pi^2 \)).

As an illustration let us take up the example of the figure eight knot complement \( S^3 \setminus K(\frac{5}{2}) \). Our computation in Section 2.4 actually illustrates the general principle from this section. The result was that \((\rho \otimes \tilde{\tau})_* [M, \partial M] = 0\) holds in \( \hat{P}(\mathbb{C}) \), and not just in \( \hat{P}(\mathbb{C})_{PSL} \) as it would result from Theorem 1. Hence

\[
CS(S^3 \setminus K(\frac{5}{2}), \tau \circ \iota) = 0
\]

holds even modulo \( 4\pi^2 \) and not just modulo \( \pi^2 \) as it would be guaranteed by Corollary 1.

4. ON COMPONENTS OF CHARACTER VARIETIES

The variety of representations \( Hom(\Gamma, G) \) of a finitely generated group \( \Gamma \) into an algebraic group \( G \) is by definition the variety defined by the relations between the given generators. The character variety is its quotient \( Hom(\Gamma, G) \// G \) in the sense of geometric invariant theory. We consider these varieties with the euclidean topology (not the Zariski topology). If \( G \) is connected, then connected components of the character variety correspond to connected components of the representation variety. The connected components for themselves are unions of irreducible components. Hilbert’s Basisatz implies that the number of irreducible components (so a fortiori the number of connected components) is always finite.

For closed manifolds we will consider the variety of characters of all representations

\[
X(M) = \text{Hom}(\pi_1 M, SL(n, \mathbb{C}))/\text{SL}(n, \mathbb{C}).
\]

For manifolds with boundary we will consider only those representations whose restriction to \( \partial M \) has unipotent image:

\[
X(M) = \{\rho \in \text{Hom}(\pi_1 M, SL(n, \mathbb{C})): \pi_1(\partial M) \subset N\} \//\text{SL}(n, \mathbb{C}).
\]

4.1. Local rigidity. A representation \( \rho: \Gamma \to G \) is called locally rigid if its connected component in the representation variety consists only of representations conjugate to \( \rho \). (Hence its character is an isolated point in the character variety.) A sufficient (and for semisimple representations \( \rho \) also necessary) condition for local rigidity is infinitesimal rigidity, i.e.,

\[
H^1(\Gamma, Ad(\rho)) = 0.
\]

For a survey on local rigidity of 3-manifold groups in \( SL(n, \mathbb{C}) \) we refer to [21].
Let $M$ be a finite-volume hyperbolic 3-manifold. Recall that the geometric representation

$$\rho_n \circ \iota : \pi_1 M \to SL(n, \mathbb{C})$$

is the composition of a lift of the hyperbolic monodromy $\iota : \pi_1 M \to SL(2, \mathbb{C})$ with the irreducible representation $SL(2, \mathbb{C}) \to SL(n, \mathbb{C})$. It has been proved in [17] that the geometric representation is infinitesimally rigid for $n \geq 2$.

For the trivial representation $\nu : \Gamma \to GL(n, \mathbb{C})$ one has

$$H^1(\Gamma, Ad(\nu)) = H^1(\Gamma, \mathbb{C}^n),$$

i.e., all its deformations correspond to homomorphisms $\pi_1 M \to H_1 M \to \mathbb{C}^n$. Since one has

$$H^1(\Gamma, Ad(\rho \oplus \nu)) = H^1(\Gamma, Ad(\rho)) \oplus H^1(\Gamma, Ad(\nu))$$

one can conclude that all direct sums of geometric and trivial representations are either locally rigid or have deformations only corresponding to $H^1(M, \mathbb{C}^n)$.

This applies in particular to all representations of the form $\rho \circ \iota$ except for

$$\iota \otimes \tau \equiv (\rho \otimes \overline{\rho_2}) \circ \iota$$

which in fact is not always locally rigid ([4], [21]).

4.2. $\text{SL}(4,\mathbb{R})$-representations factoring over $\text{PSL}(2,\mathbb{C})$. Any representation of the form $\rho \otimes \overline{\rho}$ (for a representation $\rho$ into $SL(2, \mathbb{C})$) can be conjugated into $SL(4, \mathbb{R})$. Indeed it is easy to check that the matrix entries of $\rho \otimes \overline{\rho}$ with respect to the basis \{ $e_1 \otimes e_1, e_1 \otimes e_2 + e_2 \otimes e_1, i(e_1 \otimes e_2 - e_2 \otimes e_1), e_2 \otimes e_2$ \} are all real.

Even better, there is an isomorphism $\text{PSL}(2, \mathbb{C}) \to SO(3, 1)$ explicitly defined by $\tau \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = 1/2 \times$

$$\left( \begin{array}{cccc} |a|^2 + |b|^2 + |c|^2 + |d|^2 & \overline{ab} + \overline{ad} + \overline{cd} + \overline{ca} & \overline{i(ab - ad + cd - ca)} & |a|^2 - |b|^2 + |c|^2 - |d|^2 \\ a\overline{c} + c\overline{a} + \overline{bd} + \overline{db} & \overline{ad} + \overline{ac} + bc + \overline{bd} & i(\overline{ab} - ad + \overline{cd} - ca) & a\overline{ac} - bc \end{array} \right)$$

and it is known from the representation theory of the Lorentz group that the "four-vector representation" , i.e., the corresponding 2-fold covering $SL(2, \mathbb{C}) \to SO(3, 1)$, is equivalent to the representation $\rho^1_{1,1} = id \otimes id$. So for any representation $\rho$ we have $\tau \circ \rho \sim \rho \otimes \overline{\rho}$. (That was why our computations in Section 3 also implied $Vol(\tau \rho) = 0, CS(\tau \rho) = 4CS(\rho)$.)

We remark that this isomorphism is not well-behaved with respect to genericity of Ptolemy coordinates. Given a triangulation and a generic $\text{PSL}(2, \mathbb{C})$-decoration one may well get a non-generic $SO(3, 1)$-decoration after applying the isomorphism. For this reason it was more convenient to compute Chern-Simons invariants for $\rho \otimes \overline{\rho}$ as we did in Section 3 rather than trying to compute them for $\tau \circ \rho$ directly.
Besides the trivial representation and the four-vector representation there is only one more representation of $SL(2, \mathbb{C})$ in $SL(4, \mathbb{R})$ namely the representation $\kappa : SL(2, \mathbb{C}) \to SL(4, \mathbb{R}) \subset SL(4, \mathbb{C})$ defined by

$$\kappa \left( \begin{array}{cc} a_1 + a_2 i & b_1 + b_2 i \\ c_1 + c_2 i & d_1 + d_2 i \end{array} \right) = \left( \begin{array}{ccc} a_1 & a_2 & b_1 & b_2 \\ -a_2 & a_1 & -b_2 & b_1 \\ c_1 & c_2 & d_1 & d_2 \\ -c_2 & c_1 & -d_2 & d_1 \end{array} \right).$$

For hyperbolic 3-manifolds with lifted monodromy $\iota : \pi_1 M \to SL(2, \mathbb{C})$ it is proven in [14] that $\kappa \iota$ is not an Anosov representation, although it is discrete and faithful. Again, application of this representation to the standard decoration would yield a nongeneric decoration, thus for computing Chern-Simons invariants it is more practical to conjugate the representation (inside $GL(4, \mathbb{C})$) by the matrix

$$\left( \begin{array}{cccc} 1 & 0 & 1 & 0 \\ -i & 0 & -i & 0 \\ 0 & 1 & 0 & 1 \\ 0 & i & 0 & 0 \end{array} \right)$$

into $\rho_2 \oplus \overline{\rho}_2$, which then with Lemma 1 implies that

$$Vol(\kappa \iota) = 0, CS(\kappa \iota) = 2CS(M).$$

It follows already from $H^1(\Gamma, \rho_2 \oplus \overline{\rho}_2) = 0$ that $\kappa$ gives an isolated point in the character variety. To the best of my knowledge it is not known whether the characters of $\tau$ and the trivial representation necessarily belong to different components in the $SL(4, \mathbb{R})$ character variety. But for hyperbolic 3-manifolds with $CS(M) \neq 0$ we can now use $CS(\tau) = 4CS(M) \neq 0$ to obtain that $\tau$ belongs to a different component than the trivial character. Hence for hyperbolic 3-manifolds with $CS(M) \neq 0$ the $SL(4, \mathbb{R})$ character variety has at least three connected components, which proves Corollary 2.

For the figure eight knot complement, the experimental results from [11] suggest that there are not more than three components of the $SL(4, \mathbb{R})$ character variety containing characters of irreducible representations. In particular, even though $CS(M) = 0$ in this case, the experimental result seems to suggest that $\tau$ is an isolated point and so again we have 3 components.

For other knots one may have more than three components and the following two subsections will shortly discuss two approaches to construct some of them.

### 4.3. Using Galois actions.

One can frequently get more components by applying Galois actions. Namely, for each finite-volume hyperbolic 3-manifold $M$, the image of the hyperbolic monodromy $\rho : \pi_1 M \to PSL(2, \mathbb{C})$ is contained in $SL(2, \mathbb{C})$. It is perhaps worth mentioning that $\tau$ is not always locally rigid even in $SL(4, \mathbb{R})$. In fact, [4] shows that local rigidity in $SL(4, \mathbb{R})$ does not hold for exactly 52 of the first 4500 closed, orientable, hyperbolic 3-manifolds with 2-generator fundamental group in the Hodgson-Weeks census.
$PSL(2, K)$ for some number field and then any element of the Galois group $\sigma \in Gal(K : \mathbb{Q})$ provides a (non-discrete) representation $\rho^\sigma : \pi_1 M \to PSL(2, K) \subset PSL(2, \mathbb{C})$.

Composition with representations $PSL(2, \mathbb{C}) \to SL(m, \mathbb{C})$ then produces more $SL(m, \mathbb{C})$-representations of $\pi_1 M$. One should also note that local rigidity results from [17] and [21] carry over to the Galois conjugate representations because the group cohomology $H^1(\Gamma, Ad(\rho))$ is compatible with Galois conjugations.

As an illustration let us look at the 2-bridge knot $K(\frac{7}{3})$, which is the knot considered in [10, Example 10.1]. Its hyperbolic monodromy has image in $PSL(2, \mathbb{Q}(x))$ with $x$ the unique root of positive imaginary part of $x^3 - x^2 + 1 = 0$.

Letting $x'$ be the real root, the Galois automorphism $\mathbb{Q}(x) \to \mathbb{Q}(x')$ yields a representation $\rho : \Gamma \to PSL(2, \mathbb{R})$, which according to [27, Example 6.16] has Chern-Simons invariant $-1.1134 \ldots$. While the hyperbolic monodromy and its complex conjugate have Chern-Simons invariants $\pm 3.0241 \ldots$ Then $\rho \otimes \mathbb{7}$ is a representation to $SL(4, \mathbb{R})$ of Chern-Simons invariant $-4.453 \ldots$ which consequently does not belong to any of the three other components.

4.4. Using epimorphisms. In [20], Ohtsuki, Riley and Sakuma proved that there are 2-bridge link character varieties with arbitrarily large number of irreducible components. Namely they constructed sequence of hyperbolic links $K_n$ with non-injective epimorphisms

$$f_n : \pi_1(S^3 \setminus K_n) \to \pi_1(S^3 \setminus K_{n-1}).$$

Let $C_n$ be the irreducible component containing the defining representation for the hyperbolic metric of $S^3 \setminus K_n$. Then $f_n^* C_n$ is an irreducible component consisting of non-faithful representations and thus does not agree with the irreducible component containing the defining representation for the hyperbolic metric of $S^3 \setminus K_{n-1}$. Iterating this argument they get at least $n$ irreducible components for the character variety of $K_n$.

Since we are considering the euclidean topology, a connected component may consist of several irreducible components. So having many irreducible components does, a priori, say nothing about the number of path components of the character variety. The following argument shall show, however, that the Ohtsuki-Riley-Sakuma construction actually can yield 2-bridge link character varieties with an arbitrarily large number of connected components.

---

8I do not know whether it has some meaning that this value coincides with the Chern-Simons invariants of one of the $SL(3, \mathbb{R})$-representations in [10, Example 10.1].

9They stated this theorem for $SL(2, \mathbb{C})$, but their argument also applies to $SL(m, \mathbb{C})$ character varieties for $m > 2$. One just has to consider the geometric representation (that is, the composition of $\rho_0$, the monodromy of the hyperbolic structure, with the irreducible representation $SL(2, \mathbb{C}) \to SL(m, \mathbb{C})$) in place of $\rho_0$ to adapt their proof.
Proposition 3. There are 2-bridge links with $SL(m, \mathbb{C})$ character variety having arbitrarily large number of connected components.

Proof: Recall that 2-bridge links $K(p/q)$ are described by the coefficients $b_1, b_2, \ldots, b_k$ in the continued fraction expansion

$$
\frac{p}{q} = [b_1, b_2, \ldots, b_k] = b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \ldots}}
$$

They are hyperbolic for $q > 1$. As a special instance of the construction in [20] let us consider the sequence $L_n = K\left(\frac{p_n}{q_n}\right)$ of 2-bridge links such that $\frac{p_n}{q_n}$ has a continued fraction expansion

$$
\frac{p_n}{q_n} = [2, 2, 2, \ldots, 2, 2]
$$

with $4n - 2$ coefficients all equal to 2. So $L_1 = K(\frac{2}{2})$ is the figure eight knot, $L_2 = K(\frac{169}{70})$ and so on. Then [20] constructs boundary-preserving maps $f_n: S^3 \setminus L_n \to S^3 \setminus L_{n-1}$.

We remark that $f_n$ can be chosen to have $\text{deg}(f_n) = 1$. This is because, if we let $a = (2, 2)$, then in the notation of [20] Proposition 5.1 also $a^{-1} = (2, 2)$, so we can think of $K_n$ as obtained from $2n - 1$ boxes corresponding alternatingly to $a$ and $a^{-1}$ as in the picture in [20] Proposition 5.2]. Then the existence of $f_n$ with

$$
\text{deg}(f_n) = 1 - 1 + 1 - \ldots + 1 = 1
$$

is given by [20] Proposition 6.2.

The number of twist regions for the canonical diagram of $L_n$ is $tw(L_n) = 2n - 1$. There are linear bounds

$$
C_1tw(L) - C_2 \leq \text{vol}(S^3 \setminus L) \leq C_3tw(L) + C_4
$$

(with explicit constants $C_1, C_2, C_3, C_4$) for the hyperbolic volume in terms of the twist number of a prime alternating diagram, see [15], [13]. In particular, in our situation $\text{vol}(S^3 - L_n)$ will go to infinity approximatively linear.

Let $\rho_n: \pi_1(S^3 \setminus L_n) \to SL(m, \mathbb{C})$ be the geometric representation. By [10] we have

$$
\text{vol}(\rho_n) = \frac{(m + 1)m(m - 1)}{6} \text{Vol}(S^3 \setminus L_n),
$$

so also $\text{vol}(\rho_n)$ grows approximatively linear in $n$. In particular, there are natural numbers $N_1, N_2$ such that for all $N \geq N_1$ there are at least $N - N_2$ distinct values among the volumes

$$
\text{vol}(\rho_n), n = 1, \ldots, N.
$$

Let $n \leq N$. From $\text{deg}(f_N) = \ldots = \text{deg}(f_{n+1}) = 1$ we obtain

$$
\text{vol}(\rho_{n, f_{n+1}, \ldots, f_{N, s}}) = \text{vol}(\rho_n),
$$
see [Lemma 3]. Hence there are also at least \( N - N_2 \) distinct values among the volumes

\[
\text{vol}(\rho_n f_{n+1,*} \cdots f_{N,*}), \quad n = 1, \ldots, N
\]

for \( N \geq N_1 \). In particular the \( SL(m, \mathbb{C}) \) character variety of \( S^3 \setminus L_N \) has at least \( N - N_2 \) connected components.

QED

With the help of the following proposition we can even show that the above constructed components correspond to representations of trivial Chern-Simons invariant.

**Proposition 4.** The complement of a 2-bridge knot \( K(\frac{p}{q}) \) has vanishing \( PSL(2, \mathbb{C}) \)-Chern-Simons invariant if the continued fraction expansion \( \frac{p}{q} = [a_1, \ldots, a_k] \) is symmetric in the sense that

\[
a_1 = a_k, a_2 = a_{k-1}, a_3 = a_{k-2}, \ldots
\]

**Proof:** (\[29\]) The symmetry of the continued fraction expansion is equivalent to \( q^2 \equiv -1 \mod p \), see \[23\], or \[24\] for a more general statement. If \( K(\frac{p}{q}) \) is a knot, then \( p \) is odd (and \( q \) can be assumed to be odd) and we obtain \( q^2 \equiv -1 \mod 2p \). From the "Korollar zu Satz 4" in [22] this is equivalent to \( K(\frac{p}{q}) \) being amphichiral. According to [18] this implies \( CS(S^3 \setminus K(\frac{p}{q})) = 0 \). QED

This shows that all the examples in the proof of Proposition 3 will have vanishing Chern-Simons invariant and so Proposition 3 actually produces examples of \( SL(m, \mathbb{C}) \) character varieties having arbitrarily large number of path components of vanishing Chern-Simons invariants. So we get Corollary 3.

The argument for Proposition 3 does not apply to \( SL(m, \mathbb{R}) \) character varieties because the volume of \( SL(m, \mathbb{R}) \) representations is always zero. Instead one should use Chern-Simons invariants, but of course the above argument using approximatively linear growth of volumes does not adapt because no such statement can be true for Chern-Simons invariants. Still one can use explicitly computed values to construct distinct components in specific examples. E.g. for the 2-bridge knot \( K(\frac{p}{q}) \) one can use the degree 2-map

\[
S^3 \setminus K(\frac{51}{24}) \to S^3 \setminus K(\frac{7}{3})
\]

from [20] and use that

\[
CS(S^3 \setminus K(\frac{51}{24})) \neq 2CS(S^3 \setminus K(\frac{7}{3}))
\]

to obtain an additional component in the \( SL(4, \mathbb{R}) \)-character variety of the 2-bridge knot \( K(\frac{51}{24}) \).

\[10\] Experimental evidence from the knot table [2] suggests that also the converse of this proposition might be true.
REFERENCES

[1] D. Calegari: 'Real places and torus bundles', Geom. Dedicata 118 (2006), 209-227.
[2] J. C. Cha, C. Livingston: 'KnotInfo', http://www.indiana.edu/~knotinfo
[3] J. Cheeger, J. Simons: 'Differential characters and geometric invariants', Geometry and topology (College Park, Md., 1983/84), 50-80, Lecture Notes in Math., 1167, Springer, Berlin, 1985.
[4] D. Cooper, D. Long, M. Thistlethwaite: 'Computing varieties of representations of hyperbolic 3-manifolds into SL(4,R)', Experiment. Math. 15 (2006), 291-305.
[5] M. Culler: 'Lifting representations to covering groups', Adv. in Math. 59 (1986), 64-70.
[6] M. Culler, N. Dunfield, J. Weeks. 'SnapPy, a Computer Program for Studying the Geometry and Topology of 3-Manifolds.' http://snappy.computop.org/
[7] J. Dupont, H. Sah: 'Scissors Congruences II', J. Pure and Appl. Algebra 25 (1982), 159-195.
[8] J. Dupont, C. Zickert: 'A dilogarithmic formula for the Cheeger-Chern-Simons class', Geom. Topol. 10 (2006), 1347-1372.
[9] S. Garoufalidis, M. Görner, C. Zickert: 'The Ptolemy field of 3-manifold-representations', preprint, http://arxiv.org/abs/1401.5542
[10] S. Garoufalidis, D. Thurston, C. Zickert: 'Complex volume of SL(n,C)-representations of 3-manifolds', preprint, http://arxiv.org/abs/1111.2828
[11] M. Görner, 'Ptolemy module'. http://ptolemy.unhyperbolic.org/
[12] S. Goette, C. Zickert: 'The extended Bloch group and the Cheeger-Chern-Simons class', Geom. Topol. 11 (2007), 1623-1635.
[13] H. Gueritaud: 'On canonical triangulations of once-punctured torus bundles and two-bridge link complements. With an appendix by D. Futer.' Geom. Topol. 10 (2006), 1239-1284.
[14] Sungwoon Kim, Gyeson Lee, A. Wienhard: 'On discrete faithful representations of compact hyperbolic 3-manifolds into SL(4,R)', in preparation.
[15] M. Lackenby: 'The volume of hyperbolic alternating link complements. With an appendix by I. Agol and D. Thurston.' Proc. London Math. Soc. (3) 88 (2004), 204-224.
[16] C. Maclachlan, A. W. Reid: 'The arithmetic of hyperbolic 3-manifolds.' Graduate Texts in Mathematics, 219. Springer-Verlag, New York (2003).
[17] P. Menal-Ferrer, J. Porti: "Local coordinates for SL(n,C)-character varieties of finite-volume hyperbolic 3-manifolds." Ann. Math. Blaise Pascal 19 (2012), 107122.
[18] R. Meyerhoff, M. Ouyang: 'The η-invariants of cusped hyperbolic 3-manifolds.' Canad. Math. Bull. 40 (1997), 204213.
[19] W. Neumann: 'Extended Bloch group and the Cheeger-Chern-Simons class', Geom. Topol. 8 (2004), 413-474.
[20] T. Ohtsuki, R. Riley, M. Sakuma: 'Epimorphisms between 2-bridge link groups' The Zieschang Gedenkschrift, Geom. Topol. Monogr., 14, Geom. Topol. Publ., Coventry (2008), 417-450.
[21] J. Porti: 'Local and infinitesimal rigidity of representations of hyperbolic three manifolds', RIMS Kôkyûroku, Kyoto University Vol 1836 (2013), 154-177.
[22] H. Schubert: 'Knoten mit zwei Brücken', Math. Z. 65 (1956), 133170.
[23] J.-A. Serret: 'Sur un théorème relatif aux nombres entières', J. Math. Pures Appl. (1848), 12-14.
[24] B. Smith: 'End-symmetric continued fractions and quadratic congruences', preprint, http://arxiv.org/pdf/1406.7571.pdf
[25] W. A. Stein et al., Sage Mathematics Software (Version 4.8), The Sage Development Team, http://www.sagemath.org
[26] D. Zagier: 'The dilogarithm function', Frontiers in Number Theory, Physics, and Geometry II (2007), 3-65.
[27] C. Zickert: 'The volume and Chern-Simons invariant of a representation' Duke Math. J. 150 (2009), 489-532.
[28] C. Zickert: 'Algebraic K-theory and the extended Bloch group', preprint, http://arxiv.org/abs/0910.4005
[29] 'Chern-Simons invariants of 2-bridge knots”, http://mathoverflow.net/questions/200530/chern-simons-invariants-of-2-bridge-knots

School of Mathematics, KIAS, Hoegi-ro 85, Dongdaemun-gu, Seoul, 130-722, Republic of Korea
E-mail address: kuessner@kias.re.kr