SPECTRAL SYNTHESIS
FOR EXPONENTIALS AND LOGARITHMIC LENGTH*

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ABSTRACT
We study hereditary completeness of systems of exponentials on an interval such that the corresponding generating function $G$ is small outside of a lacunary sequence of intervals $I_k$. We show that, under some technical conditions, an exponential system is hereditarily complete if and only if the logarithmic length of the union of these intervals is infinite, i.e.,
$$\sum_k \int_{I_k} \frac{dx}{1+|x|} = \infty.$$

1. Introduction
Let $\{v_n\}_{n \in \mathbb{N}}$ be a complete and minimal system of vectors in a separable Hilbert space $\mathcal{H}$, that is, $\text{Span}\{v_n\} = \mathcal{H}$ and $\text{Span}\{v_n\}_{n \neq m} \neq \mathcal{H}$ for any $m$. For any such sequence there exists its unique biorthogonal system $\{w_n\}_{n \in \mathbb{N}}$ such that $\langle v_n, w_m \rangle = \delta_{nm}$. In general, the system $\{w_n\}_{n \in \mathbb{N}}$ needs not be complete (e.g., consider $v_n = e_1 + e_{n+1}$ in $\mathcal{H} = \ell^2(\mathbb{N})$), but even if it is complete, it is possible that for some partition $\mathbb{N} = A \cup B$, $A \cap B = \emptyset$, the “mixed” system $\{v_n\}_{n \in A} \cup \{w_n\}_{n \in B}$ is incomplete. If it is not the case for any partition $\mathbb{N} = A \cup B$, then we call the system $\{v_n\}_{n \in \mathbb{N}}$ hereditarily complete. Hereditary completeness can be understood as a weakest form of reconstruction of a vector $f$ from its generalized Fourier series
$$\sum_{n \in \mathbb{N}} \langle f, w_n \rangle v_n,$$
since it is equivalent to the fact that each vector $f \in \mathcal{H}$ can be approximated by linear combinations of the partial sums of its Fourier series. Clearly, if the Fourier series with respect to the biorthogonal pair $(v_n, w_n)$ admit a linear summation method, then the system $\{v_n\}_{n \in \mathbb{N}}$ is hereditarily complete.

Hereditarily complete systems are also known as strong $M$-bases or systems which admit spectral synthesis due to the relation of this property to the structure of invariant subspaces for certain classes of linear operators discovered by A. Markus [12]. Various geometrical aspects of abstract hereditary complete systems were considered in [8, 9] while in [1, 10] some interesting relations with operator algebras can be found.
1.1. Exponential systems. We are interested in the case when \( H = L^2(-\pi, \pi) \) and \( v_n \) is a system of exponentials, \( v_n = e^{i\lambda_n t} \) for some set \( \Lambda = \{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{C} \). R. Young [17] proved that in this case the biorthogonal system is always complete and there has been a number of papers establishing hereditary completeness and existence of a linear summation method for nonharmonic Fourier series under some additional hypothesis about the set \( \Lambda \) (see, e.g., [7] or [15, 16]). Nevertheless, in [3] an example was constructed which shows that in general hereditary completeness for exponential systems does not necessarily hold. This result was extended to other functional systems (reproducing kernels in de Branges spaces of entire functions, Gaussian Gabor systems) in [4, 5] (see also a survey paper [2]). It should be mentioned that the synthesis for exponential systems fails with one-dimensional defect only: each mixed system has codimension at most one [3].

The construction in [3] was ingenious, but it had very few “degrees of freedom”, i.e., free parameters. Therefore, the structure of such examples remained rather mysterious. Our aim is to give a larger class of examples. Moreover, under some regularity conditions we are able to arrive at a certain qualitative characterization (finite logarithmic length) which we believe is intrinsic for the phenomenon of nonhereditary completeness of exponential systems.

It is well-known that if \( \{e^{i\lambda t}\}_{\lambda \in \Lambda} \) is a complete and minimal system, then the following canonical product converges in the sense of principal value (see, e.g., [11, Lecture 18, Theorem 4]):

\[
G(z) = p.v. \prod_{\lambda \in \Lambda} \left( 1 - \frac{z}{\lambda} \right) = \lim_{R \to \infty} \prod_{\lambda \in \Lambda, \lambda < R} \left( 1 - \frac{z}{\lambda} \right).
\]

The function \( G \) is called the \textit{generating function} of the system \( \{e^{i\lambda t}\}_{\lambda \in \Lambda} \). Numerous properties of exponential systems can be expressed in terms of \( G \); see, e.g., [7, 14].

1.2. Logarithmic length. We are interested in the case when the function \( G \) is small outside some \textit{lacunary} sequence of intervals \( \{I_k\}_{k=1}^\infty \),

\[
I_k = [\rho_k - d_k, \rho_k + d_k], \quad 2\rho_k \leq \rho_{k+1}, \quad 1 < d_k \leq 0.1\rho_k.
\]

We prove that under some additional restrictions the system of exponentials (reproducing kernels of \( PW_\pi \)) is hereditarily complete if and only if the total logarithmic length of these intervals is infinite, that is,

\[
\sum_{k=1}^\infty \frac{d_k}{\rho_k} = \infty;
\]
see Theorems 2.1 and 4.1. Theorem 2.1 gives sufficient conditions for the failure of hereditary completeness, whereas Theorem 4.1 gives sufficient conditions for hereditary completeness; these two results are in a sense converse to each other.

To illustrate this we present one example which immediately follows from Theorems 2.1 and 4.1.

**Example:** Let \( \{\lambda_n\} = \Lambda \) be a locally dense real sequence, i.e., \( \sup_n |\lambda_{n+1} - \lambda_n| < \infty \) such that the generating function \( G \) is of exponential type \( \pi \), and

\[
(1.2) \quad \frac{|G(z)|}{\text{dist}(z, \Lambda)} \asymp \max_k \frac{1}{\sqrt{|I_k| (\text{dist}^2(z, I_k) + 1)}} \quad |\text{Im } z| < 1,
\]

where \( \{I_k\} \) is a lacunary system of intervals satisfying (1.1) and \( d_k/\rho_k < 1/k \). Then the system \( \{e^{i\lambda t}\}_{\lambda \in \Lambda} \) is hereditarily complete if and only if

\[
\sum_{k=1}^{\infty} \frac{d_k}{\rho_k} = \infty.
\]

The existence of sequences \( \Lambda \) satisfying (1.2) can be deduced via a standard atomization technique; see, e.g., [6]. From [11, Lecture 18, Theorem 4] it follows that the system \( \{e^{i\lambda t}\}_{\lambda \in \Lambda} \) is always complete and minimal in \( L^2(-\pi, \pi) \).

In Section 3 we apply these results to give an example of a nonhereditarily complete (i.e., complete and minimal but not hereditarily complete) exponential system which partially answers a problem posed in [2]: which perturbations of integers can produce complete and minimal systems of exponentials which are not hereditarily complete? Let \( \lambda_n \in \mathbb{R} \) and

\[
(1.3) \quad \delta = \sup_{n \in \mathbb{Z}} |\lambda_n - n|.
\]

By the results of Kadets and Ingham any sequence with \( \delta < 1/4 \) generates a Riesz basis of exponentials (see, e.g., [13, Part D, Chapter 4]). One can ask, however, for which \( \delta \) any complete and minimal system \( \{e^{i\lambda n t}\} \) satisfying (1.3) is automatically hereditarily complete.

**Question:** Find \( \delta_{\text{crit}} \) which is the infimum of \( \delta > 0 \) such that there exists a nonhereditarily complete system \( \{e^{i\lambda n t}\} \) with \( |\lambda_n - n| < \delta \).

The exact value of the **synthesis constant** \( \delta_{\text{crit}} \) is not known. Theorem 3.1 shows that such \( \delta_{\text{crit}} \) cannot exceed \( 1/2 \). Therefore,

\[
\frac{1}{4} \leq \delta_{\text{crit}} \leq \frac{1}{2}.
\]
1.3. Paley–Wiener space. The classical approach to the study of the properties of exponential systems is to consider the Fourier transform $\mathcal{F}$ of our system: in this case the Hilbert space becomes the Paley–Wiener space $PW_{\pi} = \mathcal{F}L^2(-\pi, \pi)$ (the space of all entire functions of exponential type at most $\pi$ which belong to $L^2(\mathbb{R})$) and the functions $e^{i\lambda_n t}$ are mapped to the reproducing kernels of $PW_{\pi}$ (cardinal sines)

$$K_{\lambda_n}(z) = \frac{\sin \pi(z - \lambda_n)}{\pi(z - \lambda_n)}$$

corresponding to the points $\bar{\lambda}_n$. In the case when the exponential system is complete and minimal, its biorthogonal system $\{w_n\}$ is mapped to the functions $G_n \in PW_{\pi}$ which vanish on $\{\lambda_m, m \neq n\}$. It is easy to see that

$$G_n(z) = \frac{G(z)}{G'(\lambda_n)(z - \lambda_n)}$$

where $G$ is the generating function of the system $\{e^{i\lambda_n t}\}$ (or, simply put, of the set $\Lambda$). The function $G$ vanishes on $\Lambda$ and has no other zeros, it is of exponential type $\pi$ (with the diagram $[-\pi i, \pi i]$). Clearly, $G \notin PW_{\pi}$, however $G \in L^2(\mathbb{R}, \frac{dx}{1+x^2})$. Thus, the spectral synthesis problem for exponentials is equivalent to the same problem for systems of reproducing kernels in the Paley–Wiener spaces. This equivalence will be frequently used.

Organization of the paper. In Section 2 we give a sufficient condition for an exponential system to be nonhereditarily complete (Theorem 2.1, the case of finite logarithmic length). In Section 3 we apply this result to give an explicit example of a nonhereditarily complete system of exponentials whose frequencies are sufficiently small perturbations of integers. In Section 4 we prove a converse result (Theorem 4.1) establishing hereditary completeness in the case of infinite logarithmic length. Finally, in Section 5, we show that for an incomplete mixed system its exponential part must be always sufficiently irregular and, in particular, cannot be a part of a Riesz basis of exponentials with some additional regularity.

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2. Case of finite logarithmic length

In this section we will need the following assumptions on $\Lambda$ and $G$:

(a) $\text{dist}(\mathbb{Z}, \Lambda) > 0$;
(b) $\Lambda$ is locally dense on $\mathbb{R}$, i.e., there exists some $C > 0$ such that any interval $I \subset \mathbb{R}, |I| \geq C$, contains at least one element of $\Lambda$;
(c) $|G(iy)| = o(e^{\pi|y|}), |y| \to \infty$.

Recall that the family $\{K_n\}_{n \in \mathbb{Z}} = \{\sin \pi(z-n)\} \in \mathbb{Z}$ is an orthonormal basis in $PW_{\pi}$ and, therefore, $\sum_{n \in \mathbb{Z}} |g(n)|^2 = \|g\|_{L^2(\mathbb{R})}^2 < \infty$ for any $g \in PW_{\pi}$. We will often use this fact in what follows.

On the other hand, note that the series $\sum_{n \in \mathbb{Z}} |G(n)|^2$ diverges for the generating function $G$ of any complete and minimal system satisfying condition (c). Indeed, otherwise we can find a function $F \in PW_{\pi}$ such that $F(n) = G(n)$, and by the Phragmén–Lindelöf principle we have $\frac{F(z) - G(z)}{\sin \pi z} \equiv 0$. This implies that $F \equiv G$, thus $G \in PW_{\pi}$, which contradicts completeness.

For our construction of nonhereditarily complete systems to work, we just need $G(n)$ to be an $\ell^2$-sequence on the most part of $\mathbb{Z} \setminus \bigcup I_k$. But in some neighborhood of $I_k$ we want $G(n)$ to be slightly better than $\ell^2$ (see condition (ii) below).

**Theorem 2.1:** Let $G$ be the generating function of some complete and minimal system $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ satisfying (a)–(c). Let $I_k$ be a system of intervals of the form (1.1) such that $\text{dist}(\rho_k, \mathbb{Z}) \geq \frac{1}{3}$ for each $k$ and $G(\rho_k) \neq 0$. Put

$$g_k = \sum_{n \in I_k} G^2(n),$$

$$s_k = \sqrt{d_k g_k \rho_k}$$

and

$$J_k = J_k^- \cup J_k^+ = [\rho_k - d_k - 2s_k, \rho_k - d_k - s_k] \cup [\rho_k + d_k + s_k, \rho_k + d_k + 2s_k].$$

Assume that the function $G$ satisfies the following conditions:

(i) $\{G(n) : n \in \mathbb{Z} \setminus \bigcup_k I_k\} \in \ell^2$;
(ii) $\sum_k s_k \sum_{n \in J_k} G^2(n) < \infty$;
(iii) $s_k \leq 0.1 \rho_k$;
(iv) $\sum_k \frac{d_k}{\rho_k} < \infty$.

Then the system $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ is not hereditarily complete.
Proof. First, we note that to prove that the system \( \{ v_n \} = \{ e^{i\lambda_n t} \} \) is not hereditarily complete it is enough to find a partition \( N = A \cup B \) and two vectors \( \eta, \nu \in L^2(-\pi, \pi) \) such that \( \eta \perp \{ v_n \}_{n \in A} \) and \( \nu \perp \{ w_n \}_{n \in B} \), but \( \eta \) and \( \nu \) are not orthogonal. Indeed, if the system \( \{ v_n \}_{n \in A} \cup \{ w_n \}_{n \in B} \) were complete, then the vector \( \nu \) would lie in the span of \( \{ v_n \}_{n \in A} \) and \( \eta \) would lie in the span of \( \{ w_n \}_{n \in B} \) and so they would be orthogonal.

Since the Fourier transform \( \mathcal{F} \) is a unitary operator from \( L^2(-\pi, \pi) \) to \( PW_\pi \), we can pass to the equivalent problem and search for

\[
f = \mathcal{F}\eta, \quad g = \mathcal{F}\nu
\]
such that for some partition \( \Lambda = \Lambda_1 \cup \Lambda_2 \) we have

\[
f \perp \{ K_\lambda \}_{\lambda \in \Lambda_1} \quad \text{and} \quad g \perp \left\{ \frac{G(z)}{z-\lambda} \right\}_{\lambda \in \Lambda_2},
\]
but \( f \) and \( g \) are not orthogonal.

**Step 1. Construction of \( f \).** We will construct a function \( f \) as a small perturbation of the function \( G \) so that they will share most of their zeros and therefore \( f \) is orthogonal to most of \( K_{\lambda_n} \), and then construct function \( g \) by the fixed point theorem so that it is orthogonal to the remaining functions \( G_n \).

It follows from (1.1) and (iii) that all intervals \( I_k, J_k \) are pairwise disjoint. Also we can assume without loss of generality that \( s_k \geq 3C \), where \( C \) is the constant from the local density condition (b), and so there exists at least one zero of \( G \) on each \( J_k^{\pm} \). Indeed, if \( s_k < 3C < 3Cd_k \), then \( g_k \leq 9C^2 \frac{d_k}{\rho_k} \). Since the series \( \sum_{k=1}^{\infty} \frac{d_k}{\rho_k} \) converges, we can pass to a smaller collection of intervals \( I_k \) by throwing away the intervals with \( s_k < 3C \) and condition (i) still will be satisfied. Later we will throw away some extra finite set of intervals – it also will not break our assumptions.

Let \( t_k \in J_k \) be a zero of \( G \) whose choice will be specified later. Put

\[
f(z) = G(z)m(z),
\]
where

\[
m(z) = \prod_{k=1}^{\infty} \frac{1 - z/\rho_k}{1 - z/t_k}.
\]

It is easy to see from the lacunarity of \( \rho_k \) that this product converges locally uniformly on \( \mathbb{C}\setminus\{t_k\} \), and since \( G(t_k) = 0 \) we conclude that the function \( f \) is entire.
STEP 2. $f \in PW_\pi$. We will select two candidates $t_k^\pm \in 1/2J_k^\pm$ for $t_k$ (as always, by the half of the interval we mean the interval with the same center and twice smaller length). Subsequently we choose one of them in such a way that

$$0.001 \leq \prod_{k=1}^{N} \frac{t_k}{\rho_k} \leq 1000$$

for all $N$ (we can always do so by (iii)). Note that in this case we have

$$|m(z)| \asymp \frac{|z-\rho_k|}{|z-t_k|}, \quad \frac{\rho_{k-1} + \rho_k}{2} \leq |z| \leq \frac{\rho_k + \rho_{k+1}}{2}.$$ 

Therefore the function $f$ is of exponential type at most $\pi$ and $|f(iy)| = o(e^\pi|y|)$. Thus, to prove that $f \in PW_\pi$, it is enough to show that $\{f(n)\} \in \ell^2(Z)$.

Trivially, $\{f(n)\} \in \ell^2(Z \setminus \bigcup_k (I_k \cup J_k))$ since $|m(n)| \lesssim 1$ for those $n$. For $n \in I_k$ we have $|x-\rho_k| \leq \frac{d_k}{s_k}$ and, therefore,

$$\sum_{k=1}^{\infty} \sum_{n \in I_k} |f(n)|^2 \lesssim \sum_{k=1}^{\infty} \frac{d_k^2}{s_k^2} g_k = \sum_{k=1}^{\infty} \frac{d_k}{\rho_k} < \infty.$$

For $n \in J_k$ we have $|m(n)| \asymp \frac{s_k}{n-t_k}$. Let us divide $1/2J_k^\pm$ into $r_k \asymp s_k$ intervals of length $C$ and choose one root of $G$ from each of them (there is always at least one by our assumption (b)). Denote these roots in $1/2J_k^+$ by $\lambda_j$, $j = 1, \ldots, r_k$. Since $\text{dist}(\Lambda, Z) > 0$, we have

$$\sum_{j=1}^{r_k} \sum_{n \in J_k} \frac{s_k^2 G^2(n)}{(n-\lambda_j)^2} \lesssim s_k^2 \sum_{n \in J_k} G^2(n),$$

whence there exists $\lambda_j$ such that

$$\sum_{n \in J_k} |f(n)|^2 \lesssim \sum_{n \in J_k} \frac{s_k^2 G^2(n)}{(n-\lambda_j)^2} \lesssim s_k \sum_{n \in J_k} G^2(n)$$

with constants in $\lesssim$ independent of $k$. We put $t_k^+ = \lambda_j$. Similarly, one can choose $t_k^- \in 1/2J_k^-$. By condition (ii), $\sum_k \sum_{n \in J_k} |f(n)|^2 < \infty$ regardless of which of $t_k^-$ or $t_k^+$ we choose to satisfy (2.1).

STEP 3. CONSTRUCTION OF THE FUNCTION $g$. Put $a_n = (-1)^n f(n)$. We are going to construct a real sequence $\{b_n\} \in \ell^2(Z)$ such that $\sum_{n \in Z} a_n b_n \neq 0$ and the function

$$S(z) = \sum_{n \in Z} \frac{a_n b_n}{z-n}$$

has zeros at each $\rho_k$. 
Let us show that once such a system \( \{ b_n \} \) is constructed, the functions \( f \) and 
\[
g(z) = \frac{\sin \pi z}{\pi} \sum_{n \in \mathbb{Z}} \frac{b_n}{z - n} = \sum_{n \in \mathbb{Z}} (-1)^n b_n K_n(z)
\]
will achieve our goals. By construction, \( f \) is orthogonal to all \( K_\lambda \) except for \( t_k \) and 
\[
\langle f, g \rangle_{PW_\pi} = \sum_{n \in \mathbb{Z}} a_n b_n \neq 0.
\]
It remains to prove that \( g \) is orthogonal to \( \frac{G(z)}{z-t_k} \) for all \( k \) whence the mixed system
\[
\{ K_\lambda \}_{\lambda \in \Lambda_1} \cup \{ \frac{G(z)}{z-\lambda} \}_{\lambda \in \Lambda_2}
\]
with \( \Lambda_2 = \{ t_k \}_{k \geq 1}, \Lambda_1 = \Lambda \setminus \Lambda_2 \), is incomplete. Thus, we need to prove that
\[
(2.3) \quad \langle \frac{G(z)}{z-t_k}, g \rangle_{PW_\pi} = \sum_{n \in \mathbb{Z}} \frac{(-1)^n b_n G(n)}{n-t_k} = 0.
\]
We are going to prove that
\[
\frac{G(z)S(z)}{f(z)} = \sum_{n \in \mathbb{Z}} \frac{(-1)^n b_n G(n)}{z-n}.
\]
If we do so, then substituting \( z = t_k \) we get (2.3) (note that \( f(t_k) \neq 0 \)). Consider the function
\[
H(z) = \frac{G(z)S(z)}{f(z)} - \sum_{n \in \mathbb{Z}} \frac{(-1)^n b_n G(n)}{z-n}.
\]
Trivial computation shows that its residues at \( \mathbb{Z} \) are zero and since \( GS \) vanish in all zeros of \( f \) we conclude that \( H \) is entire. On the other hand, by comparing indicator diagrams of corresponding functions we see that \( H \) is of minimal exponential type. Finally, from \(|m(iy)| \asymp 1, y \to \infty \) and the definitions of \( f \) and \( S \) we see that \(|H(iy)| = o(1), |y| \to \infty \). Therefore, by the Phragmén–Lindelöf principle, \( H \equiv 0 \).

It remains to construct a sequence \( \{ b_n \}_{n \in \mathbb{Z}} \) with the desired properties.

**Step 4. Construction of the sequence \( \{ b_n \} \).** Put \( b_0 = \frac{1}{a_0} \), \( b_n = 0 \) for \( n \in \mathbb{Z} \setminus \bigcup I_k, n \neq 0 \), and \( b_n = c_k \frac{a_n}{\rho_k - n} \) for \( n \in I_k \) for some \( c_k \). We want to construct a sequence \( c_k \) such that \( S(\rho_k) = 0 \) for all \( k \).

Consider the Banach space \( \mathcal{B} \) of sequences \( \{ c_k \}_{k \geq 1} \) with the norm
\[
||c||_\mathcal{B} = \sup_{k \geq 1} \frac{|c_k|}{d_k},
\]
and consider the following operator on it:

\[(Tc)_k = \left( \sum_{n \in I_k} \frac{a_n^2}{(\rho_k - n)^2} \right)^{-1} \left( -\frac{1}{\rho_k} - \sum_{j \neq k} \sum_{n \in I_j} \frac{a_n b_n}{\rho_k - n} \right) \]

\[= \left( \sum_{n \in I_k} \frac{a_n^2}{(\rho_k - n)^2} \right)^{-1} \left( -\frac{1}{\rho_k} - \sum_{j \neq k} c_j \sum_{n \in I_j} \frac{a_n^2}{(\rho_k - n)(\rho_j - n)} \right).\]

It is easy to see that if \(\{c_k\}\) is a fixed point of this operator, then \(S(\rho_k) = 0\) for all \(k\). Thus it remains to prove that \(T\) is contractive.

Recall that \((-1)^n a_n = f(n) = G(n)/m(n)\). We have

\[(2.4) \sum_{n \in I_k} \frac{a_n^2}{(\rho_k - n)^2} = \sum_{n \in I_k} \frac{G^2(n)}{m^2(n)(\rho_k - n)^2} \leq \sum_{n \in I_k} \frac{G^2(n)}{(t_k - n)^2} \leq \frac{g_k}{s_k^2} \leq \frac{1}{d_k\rho_k}.\]

On the other hand, for \(j \neq k\), by (2.2),

\[\sum_{n \in I_j} \frac{a_n^2}{(\rho_k - n)(\rho_j - n)} \leq \frac{1}{\rho_k} \sum_{n \in I_j} \frac{a_n^2}{|\rho_j - n|} \leq \frac{1}{\rho_k} \sum_{n \in I_j} \frac{G^2(n)|\rho_j - n|}{|t_j - n|^2} \leq \frac{g_j d_j}{\rho_k s_j^2} = \frac{1}{\rho_k \rho_j}.\]

Clearly, the operator \(T\) is the sum of a constant vector and some linear operator. Moreover, by (2.4), this vector is in \(B\). So it remains to prove that the linear part of \(T\) is contractive. Denoting it by \(T_{\text{lin}}\) we have

\[\frac{|(T_{\text{lin}}c)_k|}{d_k} \leq \frac{d_k \rho_k}{d_k} \sum_{j \neq k} \frac{|c_j|}{\rho_k \rho_j} \leq \|c\|_B \sum_{j} \frac{d_j}{\rho_j}.\]

As we mentioned at the beginning of the proof, we can safely throw away any finite number of intervals. Thus, we can assume that \(\sum_j \frac{d_j}{\rho_j}\) is as small as we like and so \(\|T_{\text{lin}}c\|_B \leq \frac{\|c\|_B}{2}\). Therefore, \(T\) has a fixed point \(c\).

It remains to prove that \(\{b_n\} \in \ell^2(\mathbb{Z})\) and \(\sum_{n \in \mathbb{Z}} a_n b_n \neq 0\). We have, by (2.4),

\[\sum_{n \in \mathbb{Z}} |b_n|^2 = |b_0|^2 + \sum_{k} |c_k|^2 \sum_{n \in I_k} \frac{a_n^2}{(\rho_k - n)^2} \lesssim |b_0|^2 + \|c\|_B^2 \sum_{k} \frac{d_k}{\rho_k} < \infty,\]
and, again using (2.4),

\[
\sum_n a_n b_n = 1 + \sum_k c_k \sum_{n \in I_k} \frac{a_n^2}{\rho_k - n} \geq 1 - \|c\|_B \sum_k d_k \sum_{n \in I_k} \frac{a_n^2}{|\rho_k - n|} \geq 1 - A \|c\|_B \sum_k d_k \rho_k
\]

for some absolute constant \(A\). We can once again throw away some intervals \(I_k\) so that the expression on the right-hand side be positive (note that since \(\|T_{\text{lin}}\| \leq 1/2\), we can give a uniform upper bound for \(\|c\|_B\) so it is enough to make \(\sum_k d_k \rho_k\) sufficiently small).

**Remark 2.2:** We can replace condition (iii) by \(s_k \leq C \rho_k\)—just replace \(s_k\) with \(\varepsilon s_k\) for sufficiently small \(\varepsilon\).

**Remark 2.3:** Note that in the proof of Theorem 2.1 we actually need only that we have a locally dense subset of the zeros of \(G\) on \(\bigcup J_k \subset \mathbb{R} \setminus \bigcup I_k\) which has positive distance from \(\mathbb{Z}\).

We will now use Theorem 2.1 to construct a completely explicit example of a function \(G\) which gives us a nonhereditarily complete system.

**Theorem 2.4:** Let \(\Lambda\) be the set of zeros of the entire function

\[
(2.5) \quad G(x) = \cos \pi x \left( \frac{1}{x - 1/2} + \sum_{k=10}^{\infty} \left( \frac{1}{x - 2^k + 1/2} - \frac{1}{x - 2^k - 1/2} \right) \right).
\]

Then the system \(\{e^{i\lambda t}\}_{\lambda \in \Lambda}\) is not hereditarily complete.

**Proof.** It is easy to see that \(G \notin PW_\pi\), but \(G \in PW_\pi + zPW_\pi\). Moreover, \(|G(z)| \gtrsim |z|^{-1} e^{\pi|\text{Im}z|}\) for \(|z| = 2^k + 2^{k-1}\) and \(k \in \mathbb{N}\) is sufficiently large. Therefore, one cannot multiply \(G\) by an entire function and remain in \(PW_\pi\).

By [11, Lecture 18, Theorem 4] \(G\) is the generating function of some complete and minimal system of exponentials.

Put \(\rho_k = 2^k\) and \(d_k = \rho_k^{1/5}\). Conditions (a)–(c) and (iii), (iv) are easy to verify (for conditions (a), (b) see Remark 2.3). Let us now verify conditions (i) and (ii). We begin with the condition (i).

For \(|n| \in \left[\frac{\rho_{k-1} + \rho_k}{2}, \frac{\rho_k + \rho_{k+1}}{2}\right]\) we have from (2.5)

\[
(2.6) \quad |G(n)| \lesssim \frac{1}{|n - \rho_k|^2} + \frac{1}{2^k} + \frac{k}{4^k} \lesssim \frac{1}{|n - \rho_k|^2} + \frac{k}{2^k}.
\]
Therefore
\[ \sum_{n \not\in \bigcup I_k} G^2(n) \lesssim \sum_{k=1}^{\infty} \left( \frac{1}{d_k^3} + \frac{k^2}{2^k} \right) < \infty. \]

To prove (ii) note that \( g_k = \sum_{n \in I_k} G^2(n) \asymp 1 \) and so \( s_k \asymp \rho_k^{3/5} \). By the bound (2.6) we get
\[ \sum_{k=1}^{\infty} s_k \sum_{n \in J_k} G^2(n) \lesssim \sum_{k=1}^{\infty} s_k \left( \frac{1}{s_k^3} + \frac{k^2 s_k}{4^k} \right) < \infty. \]

Note that the minus sign in (2.5) is absolutely essential to verify the condition (ii); similar cancelation can be observed implicitly in the example from [3]. Moreover, if we enumerate zeros of the function \( G \) in increasing order, then for all \( n \in \mathbb{Z} \) we would have (after shifting by \( \frac{1}{2} \)) \( |\lambda_n - n| \leq 1 \) just as in [3] (one can check that all the roots of the function \( G \) are real). The drawback of these examples is that they do not use the full potential of Theorem 2.1—we could have chosen \( d_k \) as any positive power of \( \rho_k \) and the analysis would still work. In the following section we will construct a more advanced example which will break this barrier and give us \( |\lambda_n - n| < \frac{1}{2} + \varepsilon \).

3. Example of a nonhereditarily complete sequence

In this section we give another explicit example of a sequence \( \Lambda \) satisfying the conditions of Theorem 2.1. Moreover, this system will be a sufficiently small perturbation of integers.

**Theorem 3.1:** For any \( \delta > \frac{1}{2} \) there exist \( G \) and \( \Lambda \) satisfying all the conditions of Theorem 2.1 and such that \( \Lambda \) satisfies (1.3).

**Proof.** We will start with the following auxiliary function. Let \( \delta_0 \in (1/2, 1) \).

Consider the function
\[ G_0(z) = (z - 1/2) \prod_{n \in \mathbb{N}} \left( 1 - \frac{z^2}{(n + \delta_0)^2} \right). \]

It is well-known that
\[ |G_0(x)| \asymp (|x| + 1)^{-2\delta_0} \text{dist}(x, \mathcal{Z}_{G_0}). \]
We will show that

\[ |G_0(iy)| = o(e^{\pi|y|}), |y| \to \infty. \]

Now let \( \delta \in (\delta_0, 1) \). The idea is to shift a part of the zeros of \( G_0 \) which belong to some lacunary sequence of intervals \( \Delta_k = [\rho_k - d_k, \rho_k + d_k] \) back to the origin. Let \( \rho_k \) be an arbitrary sequence such that \( \rho_{k+1} > 2 \rho_k > 0 \) and choose \( d_k \) so that

\[ d_k^{\delta_0 + \delta} = \rho_k^{2 \delta_0}. \]

Of course, we may assume that \( d_k \leq \rho_k/100 \) for all \( k \). Now put

\[ G(z) = G_0(z) \prod_{k} \prod_{n \in \Delta_k} \frac{z-(n-\delta)}{z-(n+\delta_0)} = G_0(z) \prod_{k} \prod_{n \in \Delta_k} \left( 1 + \frac{\delta + \delta_0}{z-(n+\delta_0)} \right). \]

Let \( x \in \left[ \frac{\rho_{k-1} + \rho_k}{2}, \frac{\rho_k + \rho_{k+1}}{2} \right] \). Then

\[
\sum_{m \neq k} \sum_{n \in \Delta_m} \frac{1}{|x-(n+\delta_0)|} \lesssim \frac{1}{\rho_k} \sum_{m < k} d_m + \sum_{m > k} d_m.
\]

Since \( \sum_k \frac{d_k}{\rho_k} < \infty \), the corresponding product converges and, moreover,

\[
\left| \prod_{m \neq k} \prod_{n \in \Delta_m} \left( 1 + \frac{\delta + \delta_0}{x-(n+\delta_0)} \right) \right| \asymp 1, \quad x \in \left[ \frac{\rho_{k-1} + \rho_k}{2}, \frac{\rho_k + \rho_{k+1}}{2} \right].
\]

Also, let \( n_0 + \delta_0 \) and \( n_1 - \delta, n_0, n_1 \in \Delta_k \), be respectively the zeros of \( G_0 \) and \( G \) closest to \( x \). Then

\[
\log \left| \prod_{n \in \Delta_k} \left( 1 + \frac{\delta + \delta_0}{x-(n+\delta_0)} \right) \right|
= \log \left| \frac{x-(n_1-\delta)}{x-(n_0+\delta_0)} \right| + \sum_{n \in \Delta_k, n \neq n_0, n_1} \frac{\delta + \delta_0}{x-(n+\delta_0)} + O(1)
= \log \left| \frac{x-(n_1-\delta)}{x-(n_0+\delta_0)} \right| + (\delta + \delta_0) \log \left| \frac{x-(\rho_k - d_k)}{x-(\rho_k + d_k)} \right| + O(1).
\]

Thus, for \( x \in \left[ \frac{\rho_{k-1} + \rho_k}{2}, \frac{\rho_k + \rho_{k+1}}{2} \right] \), we have

\[ |G(x)| \asymp |G_0(x)| \cdot \frac{\text{dist}(x, \mathcal{Z}_G)}{\text{dist}(x, \mathcal{Z}_{G_0})} \cdot \left( \frac{|x-(\rho_k - d_k)| + 1}{|x-(\rho_k + d_k)| + 1} \right)^{\delta_0 + \delta}. \]

We will show that \( G \) satisfies all conditions of Theorem 2.1 with intervals

\[ I_k = [\rho_k - 2d_k, \rho_k + 2d_k] \]
(note that there is a small change in notation since the length of $I_k$ is $4d_k$ in place of $2d_k$).

Obviously, $G$ is an entire function of exponential type $\pi$ (with the diagram $[-\pi i, \pi i]$) and $|G(iy)| = o(e^{\pi |y|})$, $y \to \infty$, since $|G(iy)| \asymp |G_0(iy)|$. Thus, all conditions (a)–(c) are satisfied.

Let us show that $G$ is the generating function of some complete and minimal system of reproducing kernels in $PW_\pi$. It is clear from (3.2) that

$$|G(x)| \gtrsim (|x| + 1)^{-K} \text{dist}(x, Z_G), \quad x \in \mathbb{R},$$

for some $K > 0$. Thus, if $GT \in PW_\pi$ for some entire function $T$ of zero exponential type, then $T$ is a polynomial. However, by (3.1), for any $n \in [\rho_k + d_k, \rho_k + d_k + 2]$ we have $|G(n)| \asymp \rho_k^{-2\delta_0} d_k^{\delta_0 + \delta} = 1$. Thus, $GT \notin PW_\pi$ for any polynomial $T$. By (3.2) we also have

$$|G(x)| \sim (|x| + 1)^{-2\delta_0} \text{dist}(x, Z_G), \quad x \notin \cup_k I_k,$$

and so \{ $G(n) : n \in \mathbb{Z} \setminus \cup_k I_k$ \} $\in \ell^2$. Also,

$$1 \lesssim g_k = \sum_{n \in I_k} |G(n)|^2 \lesssim \frac{d_k^{\delta_0 + \delta}}{\rho_k^{2\delta_0}} \sum_{n \in I_k} \frac{1}{(|n - (\rho_k + d_k)| + 1)^{\delta_0 + \delta}} \lesssim 1.$$

It follows that $\frac{G}{x - \lambda} \in PW_\pi$ for any $\lambda \in Z_G$ and so $G$ is the generating function of some complete and minimal system which satisfies (i) of Theorem 2.1.

Since $g_k = \sum_{n \in I_k} G^2(n) \asymp 1$ we have $s_k = \sqrt{2d_k g_k \rho_k} \asymp \sqrt{\rho_k t_k} < \rho_k/100$ for sufficiently large $k$. It remains to verify (ii). Let

$$J_k = [\rho_k - 2d_k - 2s_k, \rho_k - 2d_k - s_k] \cup [\rho_k + 2d_k + s_k, \rho_k + 2d_k + 2s_k].$$

Then

$$\sum_k s_k \sum_{n \in J_k} G^2(n) \lesssim \sum_k s_k^2 \rho_k^{-4\delta_0} = \sum_k \rho_k^{1 + \frac{2\delta_0}{\rho_k^{2\delta_0} - 4\delta_0}} < \sum_k \rho_k^{\frac{2\delta_0}{\rho_k^{2\delta_0} - 1}}.$$

Since $\delta > \delta_0$, we conclude that the above sum converges.

Note that the constants $\delta > \delta_0 \geq 1/2$ were arbitrary and so Theorem 3.1 is proved. It is clear from the last step of the proof that the condition $\delta_0 > 1/2$ is essential for this construction. ■
4. Case of infinite logarithmic length

Throughout this section the symbols $I_k$, $\rho_k$, $d_k$ and $g_k$ will have the same meaning as in Section 2 (note that $J_k$ will denote a different object).

For a statement which will be in a sense converse to Theorem 2.1 we need to somehow formalize the statement “$G$ is big on the intervals $I_k$”. This must include the following two ingredients: first of all we do not want, for some $k$, the sum $g_k = \sum_{n \in I_k} |G(n)|^2$ to be significantly larger than the same sum for its neighbors because otherwise we will not “feel” them in $G$ and the total logarithmic length may become finite; secondly, we do not want the main contribution to $g_k$ to be due to the values of $G$ on a small part of $I_k$ because in that case we will only “feel” this small part of $I_k$ and logarithmic length may again become finite. These two parts corresponds to the assumptions (i) and (ii) of the following theorem.

**Theorem 4.1:** Let $G$ be the generating function of some complete and minimal system $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ such that $\Lambda \subset \mathbb{C}$, $\Lambda \cap \mathbb{Z} = \emptyset$ and

$$\sum_{n \in \mathbb{Z}} \frac{|G(n)|^2 + |G(n)|}{|n| + 1} < \infty.$$ 

Assume that there exists a constant $C > 0$ such that for all $k$ we have

(i) $\sum_{n \notin I_k} \frac{|G(n)|^2}{|n - \rho_k|} \leq C \frac{g_k}{d_k}$;

(ii) $\sqrt{\frac{g_k}{d_k}} \leq C|G(x)|$, $x \in I_k$;

(iii) $\sum_k \frac{d_k}{\rho_k} = \infty$.

Then the system $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ is hereditarily complete.

**Remark 4.2:** Formally, Theorems 2.1 and 4.1 apply to different classes of functions $G$, since condition (ii) of Theorem 4.1 is incompatible with the existence of any roots of $G$ on $I_k$ (condition (b) in Theorem 2.1). However, for Theorem 2.1 we do not need condition (b) in full, but only its weaker form indicated in Remark 2.3 which is consistent with the assumptions of Theorem 4.1. Alternatively, we can weaken condition (b) and assume that for every $x \in \mathbb{R}$ there exists $\lambda \in \Lambda$ with $|x - \lambda| < C$. For example, a locally dense subset $\Lambda$ of $(\mathbb{R} + i) \cup (\mathbb{R} - i)$ is compatible with the condition (ii). Thus, there exists a class
of functions \( G \) for which hereditary completeness depends only on finiteness of the logarithmic length of the intervals \( I_k \).

For the proof of Theorem 4.1 we need the following proposition.

**Proposition 4.3:** Let \( t_k \in \mathbb{R} \) be an increasing sequence (one-sided or two-sided) which is separated, i.e., \( t_{k+1} - t_k \geq \delta \) for some \( \delta > 0 \), and let \( \mu_k > 0 \), \( \{\mu_k\} \in \ell^1(\mathbb{Z}) \). Let \( \{\gamma_n\} \) be an increasing separated sequence such that

\[
\text{dist}(\{\gamma_n\}, \{t_k\}) = d > 0 \quad \text{and} \quad \sum_n \frac{1}{\gamma_n} = \infty.
\]

Then, if for the function

\[
f(z) = \sum_k \frac{\mu_k}{z - t_k}
\]

we have \( \{f(\gamma_n)\} \in \ell^1(\mathbb{N}) \), then \( \mu_k \equiv 0 \).

**Proof of Proposition 4.3.** Let us assume that \( \sup\{t_k\} = \infty \). Otherwise,

\[
f(x) \asymp x^{-1}, \quad x \to \infty,
\]

and the statement is trivial.

Clearly, \( f \) has a unique zero \( s_k \) in each interval \( (t_k, t_{k+1}) \). It is known and not difficult to show (see [3, Proposition 5.4]) that these zeros in a sense approach the “outer” ends of the intervals \( (t_k, t_{k+1}) \), namely,

\[
\sum_{t_k > 0} \frac{t_{k+1} - s_k}{s_k} < \infty, \quad \sum_{t_k < 0} \frac{s_k - t_k}{|s_k|} < \infty.
\]

Put \( \tilde{f}(z) = f(z) - \frac{\mu_0}{2(z-t_0)} \) and denote by \( \tilde{s}_k \) the unique zero of \( \tilde{f} \) in \( (t_k, t_{k+1}) \). Then we also have

\[
(4.1) \quad \sum_{t_k > 0} \frac{t_{k+1} - \tilde{s}_k}{\tilde{s}_k} < \infty.
\]

Consider first those \( \gamma_n \) which belong to \( \bigcup_k E_k \), where \( E_k = (\tilde{s}_k, t_{k+1}) \). Since \( \{\gamma_n\} \) is separated and \( \text{dist}(\{\gamma_n\}, \{t_k\}) = d > 0 \), we conclude that for a fixed \( k \) the number of points \( \gamma_n \in E_k \) does not exceed \( C(t_{k+1} - \tilde{s}_k) \) for some \( C > 0 \) independent on \( k \). In particular, the interval \( E_k \) contains no points \( \gamma_n \) if \( t_{k+1} - \tilde{s}_k < d \). Hence,

\[
\sum_{t_k > 0} \sum_{\gamma_n \in E_k} \frac{1}{\gamma_n} \lesssim \sum_{t_k > 0} \frac{t_{k+1} - \tilde{s}_k}{\tilde{s}_k} < \infty.
\]
Thus, we may assume without loss of generality that \( \gamma_n \notin \bigcup_{t_k > 0} (t_k, \tilde{s}_k) \) for all \( n \). Since \( \tilde{f} \) is decreasing on each interval \((t_k, t_{k+1})\) we have \( \tilde{f}(\gamma_n) \geq f(\tilde{s}_k) = 0 \) whenever \( \gamma_n \in (t_k, t_{k+1}) \), and so

\[
f(\gamma_n) = \frac{\mu_0}{2(\gamma_n - t_0)} + \tilde{f}(\gamma_n) \geq \frac{1}{\gamma_n}.
\]

This contradicts the assumption that \( \{f(\gamma_n)\} \in \ell^1 \).

In the proof of Theorem 4.1 we will need some auxiliary Hilbert space of meromorphic functions in \( \mathbb{C} \setminus \mathbb{Z} \). Put

\[
\mathcal{H} = \left\{ \sum_{n \in \mathbb{Z}} \frac{b_n G(n)}{z - n} : \{b_n\} \in \ell^2 \right\}, \quad \left\langle \sum_{n \in \mathbb{Z}} \frac{b_n G(n)}{z - n}, \sum_{n \in \mathbb{Z}} \frac{c_n G(n)}{z - n} \right\rangle_{\mathcal{H}} = \sum_{n \in \mathbb{Z}} b_n \overline{c_n}.
\]

Recall that \( \Lambda \cap \mathbb{Z} = \emptyset \). Therefore, \( G(n) \neq 0 \), \( n \in \mathbb{Z} \), and the inner product in \( \mathcal{H} \) is well defined. It is easy to see that \( \mathcal{H} \) is a Hilbert space whose reproducing kernel at \( \lambda \in \mathbb{C} \setminus \mathbb{Z} \) is given by \( K_{\lambda}(z) = \sum_{n \in \mathbb{Z}} \frac{|G(n)|^2}{(z - n)(\lambda - n)} \) and, in particular,

\[
\|K_{\lambda}\|_{\mathcal{H}}^2 = \sum_{n \in \mathbb{Z}} \frac{|G(n)|^2}{|\lambda - n|^2}.
\]

Next, consider the function

\[
M(t) = \sum_{n} \frac{|G(n)|^2}{n - t}.
\]

Each interval \((n, n + 1)\) contains exactly one root of the equation \( M(t) = 0 \). Pick those roots which lie in \( J_k = \frac{1}{2}I_k = [\rho_k - \frac{d_k}{2}, \rho_k + \frac{d_k}{2}] \) for some \( k \) and denote the resulting sequence by \( \{t_n\} \).

Note that, for real \( z \neq w \), we have

\[
\langle K_w, K_z \rangle = K_w(z) = \frac{M(z) - M(w)}{z - w}.
\]

Since \( M(t_n) = 0 \) for all \( n \), the functions \( K_{t_n} \) form an orthogonal system in \( \mathcal{H} \).

**Lemma 4.4:** There exist sets \( N_k \subset \mathbb{N} \cap J_k \) and a constant \( C > 0 \) (independent of \( k \)) such that \( |N_k| \geq d_k/2 \) and

\[
\|K_{n + \frac{1}{2}}\|_{\mathcal{H}}^2 \leq C \frac{g_k}{d_k}, \quad \left| M \left( n + \frac{1}{2} \right) \right| \leq C \frac{g_k}{d_k}, \quad n \in N_k.
\]
Proof. Throughout the proof, symbol \( C \) will denote different constants independent of \( k \). We have

\[
\sum_{n \in J_k} \| K_{n+\frac{1}{2}} \|_H^2 = \sum_{n \in J_k} \sum_{m \in \mathbb{Z}} \frac{|G(m)|^2}{(n-m+\frac{1}{2})^2}.
\]

By condition (i) of Theorem 4.1,

\[
\sum_{n \in J_k} \sum_{m \notin I_k} \frac{|G(m)|^2}{(n-m+\frac{1}{2})^2} \leq C g_k,
\]

while

\[
\sum_{n \in J_k} \sum_{m \in I_k} \frac{|G(m)|^2}{(n-m+\frac{1}{2})^2} = \sum_{m \in I_k} |G(m)|^2 \sum_{n \in J_k} \frac{1}{(n-m+\frac{1}{2})^2} \leq C g_k.
\]

We conclude that \( \sum_{n \in J_k} \| K_{n+\frac{1}{2}} \|_H^2 \leq C g_k \), whence, for any \( \varepsilon > 0 \), we have

\[
\| K_{n+\frac{1}{2}} \|_H^2 \leq C \varepsilon^{-1} \frac{g_k}{d_k} \quad \text{for} \quad n \in N_k,
\]

where \( N_k \) is a subset of \( \mathbb{N} \cap J_k \) with \( |N_k| \geq (1-\varepsilon)d_k \).

Estimate for \( |M(n+\frac{1}{2})| \) is more delicate. First, we split it into the sums over \( I_k \) and \( \mathbb{Z} \setminus I_k \)

\[
M(n+\frac{1}{2}) = \sum_{m \in I_k} \frac{|G(m)|^2}{m-n-\frac{1}{2}} + \sum_{m \notin I_k} \frac{|G(m)|^2}{m-n-\frac{1}{2}} \quad \text{is} \quad S_1(n) + S_2(n).
\]

For \( n \in J_k \) as above we can deduce from the assumption (i) that \( |S_2(n)| \leq C \frac{g_k}{d_k} \).

Therefore it remains to estimate \( S_1(n) \).

For a sequence \( x = (x_m) \in \ell^1(\mathbb{Z}) \) consider the operator \( T \) defined as

\[
(Tx)_n = \sum_{m \in \mathbb{Z}} \frac{x_m}{m-n-\frac{1}{2}}.
\]

It is well-known that the discrete Hilbert transform \( T \) satisfies a weak-type \((1,1)\) bound

\[
|\{ n \in \mathbb{Z} : |(Tx)_n| > \tau \}| \leq C \frac{|x|_{\ell^1}}{\tau}, \quad \tau > 0,
\]

where \( C \) is an absolute constant.

Applying this bound to the sequence \( x_n = |G(n)|^2 \chi_{I_k}(n) \) with \( \tau = 100C g_k/d_k \) we get

\[
|\{ n \in \mathbb{Z} : |S_1(n)| > 100C g_k/d_k \}| \leq \frac{d_k}{100}.
\]
Therefore for \( n \in J_k \) and outside of this exceptional set we get the desired estimate. Since \( |J_k| - \frac{1}{100}d_k = \frac{99}{100}d_k > \frac{d_k}{2} \), the lemma is proved.

**Lemma 4.5:** Let \( N_k \) be the sets from Lemma 4.4. Then there exists \( \varepsilon > 0 \) such that for any \( k \) and for any \( n \in N_k \) the zero \( t \) of the function \( M \) in the interval \((n, n + 1)\) belongs to \((n + \varepsilon, n + 1 - \varepsilon)\).

**Proof.** Assume that \( M(n + \frac{1}{2}) > 0 \). We have

\[
M'(t) = \sum_{m \in \mathbb{Z}} \frac{|G(m)|^2}{(t - m)^2} \geq \frac{|G(n)|^2}{(t - n)^2}.
\]

Since \( |G(n)|^2 \geq C \frac{dk}{d_k} \) and \( M(n + \frac{1}{2}) \leq C \frac{dk}{d_k} \) there exists \( \varepsilon > 0 \) (depending on \( C \) but not on \( k \) and \( n \)) such that

\[
M(n + \varepsilon) \leq M(n + \frac{1}{2}) - |G(n)|^2 \int_{n+\varepsilon}^{n+\frac{1}{2}} \frac{dt}{(t - n)^2} < 0.
\]

Thus, for some \( t \in (n + \varepsilon, n + 1/2) \) we have \( M(t) = 0 \).

In the case when \( M(n + \frac{1}{2}) < 0 \), one shows by the same argument that the root of \( M \) will lie in \((n + 1/2, n + 1 - \varepsilon)\).

**Proof of Theorem 4.1.** Assume that the system of reproducing kernels \( \{K_\lambda\}_{\lambda \in \Lambda} \) is not hereditarily complete. Then there exists a nonzero function \( f \in PW_\pi \),

\[
f(z) = \frac{\sin \pi z}{\pi} \sum_{n \in \mathbb{Z}} \frac{(-1)^n \bar{a}_n}{z - n},
\]

and a set \( \Lambda_1 \subset \Lambda \) such that \( f \) is orthogonal to \( K_\lambda \) for all \( \lambda \in \Lambda_1 \), and to \( G_\lambda = \frac{G(z)}{G'(\lambda)(z - \lambda)} \) for all \( \lambda \in \Lambda_2 = \Lambda \setminus \Lambda_1 \). Then

\[
\left\langle \frac{G(z)}{z - \lambda}, f \right\rangle_{PW_\pi} = \sum_{n \in \mathbb{Z}} \frac{a_n G(n)}{n - \lambda} = 0, \quad \lambda \in \Lambda_2.
\]

Our first step is to prove the following equality:

\[
(4.9) \quad f(z) \sum_n \frac{a_n G(n)}{z - n} = G(z) \sum_n \frac{|a_n|^2}{z - n}.
\]

Indeed, note that \( f(n) = \bar{a}_n \) and so the residues at \( \mathbb{Z} \) coincide. Since the left-hand side of (4.9) vanishes at \( \lambda \in \Lambda \), there is an entire function \( T \) such that

\[
(4.10) \quad f(z) \sum_n \frac{a_n G(n)}{z - n} - G(z) \sum_n \frac{|a_n|^2}{z - n} = G(z)T(z).
\]
It is clear that $T$ is of zero exponential type. Recall that \( \frac{G}{z-\lambda} \in PW_\pi \) for any zero \( \lambda \) of $G$. Hence, the left-hand side of (4.10) is in the class $PW_\pi + zPW_\pi$. If $T$ has at least one zero $\zeta$, we conclude that $G \cdot \frac{T}{z-\zeta} \in PW_\pi$, a contradiction to the fact that $G$ is the generating function of a complete sequence of reproducing kernels. Thus, $T = c \in \mathbb{C}$ and

$$f(z) \sum_n a_n G(n) \frac{1}{z-n} = G(z) \left(c + \sum a_n^2 \frac{1}{z-n}\right).$$

It remains to exclude the case when $c \neq 0$. Put

$$E_k = J_k \setminus \bigcup_{n \in \mathbb{Z}} (n - 1/10, n + 1/10).$$

Then, for $x \in E_k$,

$$\left| \sum_n a_n G(n) \frac{1}{x-n} \right|^2 \leq \sum_{n \notin I_k} |G(n)|^2 \frac{|a_n|^2}{|x-n|} + \sum_{n \in I_k} |G(n)|^2 \frac{|a_n|^2}{(x-n)^2} \leq g_k$$

by (i). Thus,

$$\int_{E_k} \left| f(x) \sum_n a_n G(n) \frac{1}{x-n} \right|^2 dx \lesssim g_k \int_{E_k} |f(x)|^2 dx = o(g_k),$$

as $k \to \infty$. Note also that $\sum_n \frac{a_n^2}{x-n} \to 0$ when $x \to \infty$, $x \in E_k$. Therefore, if $c \neq 0$, then

$$\int_{E_k} |G(x) \left(c + \sum a_n^2 \frac{1}{x-n}\right)|^2 dx \geq \int_{E_k} |G(x)|^2 dx \gtrsim \frac{g_k}{d_k} \cdot d_k = g_k$$

by (ii). This contradiction shows that $c = 0$ and (4.9) is proved.

Now, put

$$H(z) = \sum_n a_n G(n) \frac{1}{z-n}, \quad h(z) = \sum_n \frac{|a_n|^2}{z-n}.$$ 

Since $H \in \mathcal{H}$ and $\{K_{t_n}\}$ is an orthogonal system in $\mathcal{H}$, we have $\{\frac{H(t_n)}{\|K_{t_n}\|_{\mathcal{H}}}\} \in \ell^2$. Also, $\{f(t_n)\} \in \ell^2$ by the classical Plancherel–Pólya inequality. Equality (4.9) yields

$$h(t_n) \frac{G(t_n)}{\|K_{t_n}\|_{\mathcal{H}}} = \frac{H(t_n)}{\|K_{t_n}\|_{\mathcal{H}}} f(t_n) \in \ell^1.$$ 

Denote by $\{\tilde{t}_j\}$ the sequence of all zeros of $M$ which belong to the intervals $(n, n+1)$ for $n \in N_k$, where the sets $N_k$ are constructed in Lemma 4.4. By Lemma 4.5 we have $\text{dist}(\{\tilde{t}_j\}, \mathbb{Z}) \geq \varepsilon > 0$. Hence, if $\tilde{t}_j \in (n, n+1),$
then $\|K_{\tilde{t}_j}\|_H^2/\|K_{n+\frac{1}{2}}\|_H^2$ is bounded from above and from below by some positive constants depending on $\varepsilon$, since this is true for all the summands in their definitions (see (4.2)). By Lemma 4.4 and (iii) we have

$$|G(\tilde{t}_j)| \geq C \sqrt{g_k d_k}$$

for any $\tilde{t}_j \in J_k$. We conclude that $h(\tilde{t}_j) \in \ell^1$. Also, since $|N_k| \geq d_k/2$ we have

$$\sum_j \frac{1}{\tilde{t}_j} = \sum_k \sum_{\tilde{t}_j \in J_k} \frac{1}{\tilde{t}_j} = \sum_k \frac{|N_k|}{\rho_k} \approx \sum_k \frac{d_k}{\rho_k} = \infty.$$  

Applying Proposition 4.3 to $h(z) = \sum_{n \in \mathbb{Z}} |a_n|^2 z^{-n}$ and $\tilde{t}_j$ in place of $\gamma_n$ we conclude that $a_n \equiv 0$ for all $n$. This contradiction proves the theorem. 

**Remark 4.6:** Theorem 4.1 can be extended to a wider class of generating functions. Namely, we can allow some functions $G$ with divergent sum $\sum_{n \in \mathbb{Z}} G^2(n)$ with the following modification of the above method: we consider the function $M(x) = \sum_{n \in \mathbb{Z}} G^2(n) \left( \frac{1}{x-n} + \frac{n}{n^2+1} \right)$ and on the interval $I_k$ we will consider the points $\lambda_n \in (n, n+1)$ which are the solution to the equation

$$M(x) = \sum_{n \leq \rho_k + d_k} \frac{n G^2(n)}{n^2 + 1}.$$  

Although normalized reproducing kernels at the points $\lambda_n$ are not an orthonormal sequence anymore, they come in big groups of pairwise orthogonal kernels corresponding to one interval $I_k$. Thus, one can still prove that they form a Riesz sequence by examining the Gram matrix and a Riesz sequence is sufficient for our proof. This, in particular, allows us to consider $|G| \approx 1$ on $I_k$ and (with slight modifications) even the non-lacunary case $|G| \approx 1$ on $\mathbb{R}$. We leave the details to the interested reader.

**5. A remark on exponential parts of an incomplete mixed system**

Assume that $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ is a nonhereditary complete system of exponentials and the system $\{K_{\lambda}\}_{\lambda \in \Lambda_1} \cup \{\frac{G(z)}{z-\lambda}\}_{\lambda \in \Lambda_2}$ is incomplete for some partition $\Lambda = \Lambda_1 \cup \Lambda_2$. We have seen in Section 3 that $\Lambda$ can be a sufficiently small perturbation of integers. However, the system must also have a certain irregularity. We will show that the exponential part $\{e^{i\lambda t}\}_{\lambda \in \Lambda_1}$ cannot be a part of a Riesz basis of exponentials with some additional properties; e.g., $\Lambda_1$ cannot be a subset of $\mathbb{Z}$.  


Theorem 5.1: Let \( \{e^{i\lambda t}\}_{\lambda \in \Lambda} \) be a complete and minimal system in \( L^2(-\pi, \pi) \), \( \Lambda \subset \mathbb{R} \). Assume that \( \Lambda_1 \subset \Lambda \) and there exists \( \tilde{\Lambda}_2 \subset \mathbb{R} \) such that \( \{e^{i\lambda t}\}_{\lambda \in \Lambda_1 \cup \tilde{\Lambda}_2} \) is a Riesz basis in \( L^2(-\pi, \pi) \) whose generating function \( F \) satisfies \( |F'(\zeta)| \lesssim 1 \), \( \zeta \in \mathbb{Z}_F \). Let \( \{w_\lambda\}_{\lambda \in \Lambda} \) be the system biorthogonal to \( \{e^{i\lambda t}\}_{\lambda \in \Lambda} \). Then the system
\[
\{e^{i\lambda t}\}_{\lambda \in \Lambda_1} \cup \{w_\lambda\}_{\lambda \in \tilde{\Lambda}_2},
\]
where \( \Lambda_2 = \Lambda \setminus \Lambda_1 \), is complete in \( L^2(-\pi, \pi) \).

Proof. Since Riesz bases of exponentials are stable under small perturbations (even in Euclidean metric) we can perturb slightly \( \tilde{\Lambda}_2 \) so that \( \Lambda_2 \cap \tilde{\Lambda}_2 = \emptyset \) and still \( |F'(\zeta)| \lesssim 1 \), \( \zeta \in \mathbb{Z}_F \). We also assume in what follows that \( \Lambda_2 \) and \( \tilde{\Lambda}_2 \) are infinite (otherwise it is well-known that the corresponding mixed system is complete).

We pass again to the equivalent formulation for the Paley–Wiener space \( PW_\pi \). Put \( Z = Z_F = \Lambda_1 \cup \tilde{\Lambda}_2 \). Since \( \{K_\zeta\}_{\zeta \in \mathbb{Z}} \) is a Riesz basis, its biorthogonal system \( \left\{ \frac{F(z)}{F'(\zeta)(z-\zeta)} \right\}_{\zeta \in \mathbb{Z}} \) also is a Riesz basis. Consider the Hilbert space
\[
\mathcal{H} = \left\{ f(z) = \sum_{\zeta \in \mathbb{Z}} c_\zeta \frac{F(z)}{F'(\zeta)(z-\zeta)} : (c_\zeta) \in \ell^2 \right\}
\]
with the norm \( \|f\|_{\mathcal{H}} = \|(c_\zeta)\|_{\ell^2} \). Then \( \mathcal{H} \) coincides with \( PW_\pi \) with equivalence of norms and the system
\[
\left\{ \frac{F(z)}{F'(\zeta)(z-\zeta)} \right\}_{\zeta \in \mathbb{Z}}
\]
is an orthonormal basis of reproducing kernels in \( \mathcal{H} \) (note that \( c_\zeta = f(\zeta), f \in \mathcal{H} \)).

Assume now that the system
\[
\{K_\lambda\}_{\lambda \in \Lambda_1} \cup \{G_\lambda\}_{\lambda \in \Lambda_2}, \quad G_\lambda(z) = \frac{G(z)}{G'(\lambda)(z-\lambda)},
\]
is not complete in \( PW_\pi \). This means that there exists a nonzero function \( f \in PW_\pi \) such that \( f|_{\Lambda_1} = 0 \) and
\[
f \notin \overline{\text{Span}_{PW_\pi}} \{G_\lambda : \lambda \in \Lambda_2\} = \overline{\text{Span}_{\mathcal{H}}} \{G_\lambda : \lambda \in \Lambda_2\}.
\]

Note that the system \( \{G_\lambda\}_{\lambda \in \Lambda} \) is biorthogonal also to the system of reproducing kernels \( \tilde{K}_\lambda \) of \( \mathcal{H} \). Thus, there exists a function \( f \in \mathcal{H} = PW_\pi \) such that
\[
f \perp \{\tilde{K}_\lambda\}_{\lambda \in \Lambda_1} \cup \{G_\lambda\}_{\lambda \in \Lambda_2}
\]
with respect to the inner product of $\mathcal{H}$. Let $f(z) = \sum_{\zeta \in \mathcal{Z}} c_{\zeta} \frac{F(z)}{F'(\zeta)(z-\zeta)}$. Recall that $\mathcal{Z} = \Lambda_1 \cup \tilde{\Lambda}_2$ whence $c_{\zeta} = f(\zeta) = 0$, $\zeta \in \Lambda_1$. Also,

$$
(5.1) \quad \langle G_\lambda, f \rangle_{\mathcal{H}} = \frac{1}{G'(\lambda)} \sum_{\zeta \in \tilde{\Lambda}_2} \frac{G(\zeta)\pi_\zeta}{\zeta - \lambda} = 0, \quad \lambda \in \Lambda_2,
$$

and so

$$
F(z) \sum_{\zeta \in \mathcal{Z}} \frac{G(\zeta)\pi_\zeta}{z - \zeta} = G(z)T(z)
$$

for some entire function $T$. Comparing the indicator diagrams of the left- and right-hand sides we see that $T$ is of zero exponential type. Since $\{\frac{G(\zeta)}{|\zeta|}\} \in \ell^2$, the left-hand side of the above equality belongs to $PW_\pi + zPW_\pi$. If $T$ has at least one zero, say $z_0$, then $G \frac{T}{z - z_0} \in PW_\pi$, a contradiction to the fact that $G$ is the generating function of a complete sequence. Thus, $T = c$ for some $c \in \mathbb{C}$. Comparing the values at $\zeta \in \tilde{\Lambda}_2$ we get $F'(\zeta)G(\zeta)\pi_\zeta = cG(\zeta)$ and so $|c\zeta| = |F'(\zeta)|^{-1}|c| \geq 1$, $\zeta \in \tilde{\Lambda}_2$, a contradiction. \[\blacksquare\]

Condition $|F'(\zeta)| \lesssim 1$, $\zeta \in \mathcal{Z}_F$, is essential and the result is no longer true as soon as this condition is not satisfied. Note that the following example gives yet another method to construct nonhereditarily complete systems of exponentials. As an example of a Riesz basis of exponentials with growing $|F'(\zeta)|$ one can take the system corresponding to $\mathcal{Z} = \{n - \delta \text{sign} \; n\}_{n \in \mathbb{Z}}$ with $0 < \delta < \frac{1}{4}$, which is a Riesz basis in $L^2(-\pi, \pi)$, e.g., by the Kadets 1/4 theorem [13, Part D, Chapter 4].

**Example:** Let $F$ be the generating function of an exponential Riesz basis $\{e^{i\lambda t}\}_{\lambda \in \mathcal{Z}}$ with $\mathcal{Z} \subset \mathbb{R}$ and $\sup_{\zeta \in \mathcal{Z}} |F'(\zeta)| = \infty$. We show that there exists a partition $\mathcal{Z} = \Lambda_1 \cup \tilde{\Lambda}_2$ and a set $\Lambda = \Lambda_1 \cup \Lambda_2$ such that $\{e^{i\lambda t}\}_{\lambda \in \Lambda}$ is a complete and minimal system in $L^2(-\pi, \pi)$, but the mixed system $\{e^{i\lambda t}\}_{\lambda \in \Lambda_1} \cup \{w_\lambda\}_{\lambda \in \Lambda_2}$ is incomplete.

Choose $\tilde{\Lambda}_2 \subset \mathcal{Z}$, $\tilde{\Lambda}_2 = \{\zeta_n\}_{k \in \mathbb{N}}$, such that $\{|F'(\zeta)|^{-1}\}_{\zeta \in \tilde{\Lambda}_2} \in \ell^2$. Then

$$
f(z) = \sum_{\zeta \in \tilde{\Lambda}_2} \frac{F(z)}{|F'(\zeta)|^2(z - \zeta)} \in PW_\pi\]

(in this expansion $c_{\zeta} = 1/|F'(\zeta)|$, $\zeta \in \tilde{\Lambda}_2$, and $c_{\zeta} = 0$ otherwise). Without loss of generality let $\{\zeta_n\}_{k \in \mathbb{N}}$ be positive, increasing and $\sum_k \zeta_n^{-1} < \infty$. We construct $\Lambda_2$ as a perturbation of $\tilde{\Lambda}_2$:

$$
\Lambda_2 = \{\zeta_n + \alpha_k\}_{k \in \mathbb{N}}, \quad 0 \leq \alpha_k \leq \zeta_{n+1} - \zeta_n.
$$
Let $G$ be the generating function of $\Lambda = \Lambda_1 \cup \Lambda_2$, i.e.,

$$G(z) = F(z) \prod_k \frac{1 - z/(\zeta_{n_k} + \alpha_k)}{1 - z/\zeta_{n_k}}.$$ 

Note that if $\alpha_k = 0$, then $G = F$, while for $\alpha_k = \zeta_{n_{k+1}} - \zeta_{n_k}$ we have

$$G(z) = F(z)/(z - \zeta_{n_1}) \in PW_\pi.$$ 

Therefore, by a certain “continuity” argument one can always find perturbations $\alpha_k$ such that

$$\lim_{y \to \infty} \left| \frac{G(iy)}{F(iy)} \right| = 0, \quad \sum_{\zeta \in \mathbb{Z}} |G(\zeta)|^2 = \infty, \quad \sum_{\zeta \in \mathbb{Z}} \frac{|G(\zeta)|^2}{|\zeta|^2 + 1} < \infty.$$ 

We omit an elementary but tedious proof of this fact. From the last two conditions one easily deduces that $G$ is the generating function of a complete and minimal system, while the first one guarantees that the interpolation formula (5.2)

$$\frac{G(z)}{F(z)} = \sum_{\zeta \in \mathbb{Z}} \frac{G(\zeta)}{F'(\zeta)(z - \zeta)} = \sum_{\zeta \in \tilde{\Lambda}_2} \frac{G(\zeta)}{F'(\zeta)(z - \zeta)}$$

holds (the difference of the left- and right-hand sides is an entire function of zero exponential type that tends to zero along the imaginary axis).

Now consider the Hilbert space $\mathcal{H}$ constructed from $F$ as in the proof of Theorem 5.1. By construction, $f$ is orthogonal to reproducing kernels $\{K_\lambda\}_{\lambda \in \Lambda_1}$ in $PW_\pi$ and to $\{\tilde{K}_\lambda\}_{\lambda \in \Lambda_1}$ in $\mathcal{H}$. It remains to show that $f \perp \{G_\lambda\}_{\lambda \in \Lambda_2}$ in $\mathcal{H}$, which follows immediately from (5.2) and (5.1).

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