Thermodynamic bounds on coherent transport in periodically driven conductors

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We establish a family of new thermodynamic constraints on heat and particle transport in coherent multi-terminal conductors subject to slowly oscillating driving fields as well as moderate electrical and thermal biases. These bounds depend only on the number of terminals of the conductor and the base temperature of the system. Going beyond the second law of thermodynamics, they imply that every local current puts a lower limit on the mean dissipation caused by the overall transport process. As a key application of this result, we derive two novel trade-off relations restricting the performance of adiabatic quantum pumps and isothermal engines. On the technical level, our work combines Floquet scattering and linear-adiabatic-response theory with recent techniques from small-scale thermodynamics. Using this framework, we illustrate our general findings by working out two specific models describing either a quantum pump or an isothermal engine. These case studies show that our bounds are tight and provide valuable benchmarks for realistic devices.

I. INTRODUCTION

In macroscopic systems at finite temperature, transport is typically governed by the irreversible dynamics of particles constantly undergoing collisions, in which they randomly change their energy and direction of motion. When the size of the sample becomes comparable to the mean free path of the carriers, however, the situation changes drastically\textsuperscript{1}. In this regime, which is realized in mesoscopic conductors at low temperatures and can be probed accurately in experiments\textsuperscript{2–12}, the carrier dynamics is no longer stochastic but is dominated by the reversible laws of quantum mechanics. The transfer of matter and energy thus becomes a coherent process, which can be elegantly described as the elastic scattering of effectively non-interacting particles originating from thermal reservoirs. This picture applies even to systems that are subject to oscillating electric or magnetic fields.

In such dynamic conductors, however, the microscopic scattering events are no longer elastic, since carriers can absorb or emit discrete quanta of energy while passing through the sample\textsuperscript{13}. This mechanism makes it possible to realize cyclic nano-scale machines like adiabatic quantum pumps\textsuperscript{13}, motors\textsuperscript{14} or heat engines\textsuperscript{15}, which use slow periodic control fields to generate either a directed flow of particles or mechanical work, see Fig. 1.

What are the fundamental performance limits of such devices? Finding answers to this question is one of the main purposes of the present paper. In fact, as our central result, we derive the universal relation

\[ \sigma \geq \frac{m}{m-1} \hat{K}^{xx}(J_x)^2, \]  

(1)

which bounds the total dissipation \( \sigma \) caused by a coherent transport process in an \( m \)-terminal conductor in terms of every period-averaged particle \((x = \rho)\) and heat \((x = q)\) current. The factor \( \hat{K}^{xx} \) thereby effectively depends only on the equilibrium temperature of the system.

Following ultimately from the fundamental laws of particle and energy conservation, our new bound (1) is significantly stronger than the second law of thermodynamics, which only requires \( \sigma \geq 0 \). It therefore restricts not...
only the efficiency but also the practically often more important output of mesoscopic transport devices\textsuperscript{16}. For example, Eq. (1) implies a universal maximum for the current that a quantum pump can generate with given energy input. This result could provide a valuable benchmark for future technologies involving adiabatic quantum pumps, which, owing to their ability to transfer particles one-by-one with high accuracy, can be used as dynamical single-electron source\textsuperscript{2–12}.

In a broader perspective, the quest for universal constraints on the figures of merit that govern the performance of coherent nano-scale machines based on mesoscopic conductors has emerged as an active research topic over the last years\textsuperscript{17–20}. As their main new feature, the bounds derived in this paper cover both mechanical driving exerted through oscillating control fields and steady-state driving induced by thermochemical gradients. In fact, we will show that our general scheme makes it possible to recover and, to some extent, unify various earlier results by considering specific limits.

On the technical level, the basis for our work is provided by the Floquet scattering formalism\textsuperscript{13}, a dynamical extension of the conventional scattering approach to quantum transport in stationary systems; going back to the pioneering work of Landauer\textsuperscript{30}, the idea to use scattering theory for the description of transport phenomena developed into a powerful and well-tested tool of mesoscopic physics during the following decades\textsuperscript{31}. As a second ingredient for our analysis, we use a combination of adiabatic and linear response theory\textsuperscript{15}. Leading to a thermodynamically consistent perturbation scheme, these methods allow us to capture the intricate interplay between coherent transport dynamics and mechanical driving in concise and physically transparent equations. This mathematical framework enables a general approach, which covers a wide range of systems and applications and does not rely on specific models.

Our paper is organized as follows. In Sec. II, we provide a short review of Floquet scattering theory, show how this framework can be furnished with a consistent thermodynamic structure and outline the linear-adiabatic response scheme. We then derive our new bounds on coherent transport in dynamic mesoscopic conductors in Sec. III A. We also discuss how our main results are related to time-reversal symmetry and geometric aspects and argue that they are robust against phase-breaking perturbations. Moving on to practical applications, in Sec. IV, we establish our thermodynamic trade-off relations for quantum pumps and isothermal engines. Furthermore, by analyzing two specific models, we show that our bounds are generally tight and demonstrate their practical applicability. Finally, in Sec. V, we briefly summarize our work and discuss its relation with earlier studies. We conclude with a short outlook.

II. FRAMEWORK

A. Scattering Theory

We begin with a brief review of the scattering approach to quantum transport in dynamic conductors. To this end, we consider a mesoscopic sample, whose energy landscape is periodically modulated by external driving fields $\mathbf{\lambda} = \{\lambda_k\}$. The terminals of this conductor are coupled to thermochemical reservoirs with different temperatures $T_\alpha$ and chemical potentials $\mu_\alpha$, see Fig. 2. In the coherent regime, the mean free path of the carriers injected by these reservoirs exceeds the dimensions of the conductor. The emerging transport process can then be described as a continuous series of coherent scattering events involving non-interacting particles, which exchange a quantized amount of energy with the external fields while passing through the sample\textsuperscript{13}.

Once the system has reached a periodic state, the average particle and energy currents flowing in terminal $\alpha$ towards the scattering region are given by the Landauer-Büttiker-type formulas\textsuperscript{13}

\begin{equation}
J_\alpha^p = -\frac{1}{\hbar} \sigma_\alpha \int_0^\infty \! d\omega \sum_{\beta} \left( \delta_{\alpha\beta} - \sum_n |S_{E_n,E_\beta,\alpha}^\beta| \right) f_\beta^E \quad \text{and}
\end{equation}

\begin{equation}
J_\alpha^E = -\frac{1}{\hbar} \sigma_\alpha \int_0^\infty \! d\omega \sum_{\beta} \left( E \delta_{\alpha\beta} - \sum_n E_n |S_{E_n,E_\beta,\alpha}^\beta| \right) f_\beta^E
\end{equation}

respectively. Here, we have used the shorthand notation $E_\alpha \equiv E + n \hbar \omega$, where $\hbar = \hbar/2\pi$ denotes the reduced Planck constant and $\omega$ the driving frequency. Thermodynamics enters the expressions (2) via the Fermi functions of the reservoirs,

\begin{equation}
f_\beta^E \equiv \frac{1}{1 + \exp((E - \mu_\alpha)/T_\alpha)},
\end{equation}

with Boltzmann’s constant being set to 1 throughout. The properties of the sample are encoded in the Floquet scattering amplitudes $S_{E_n,E_\beta,\alpha}^\beta$. These objects describe the transmission of incoming particles with energy $E$ from the terminal $\beta$ to the terminal $\alpha$ under the absorption ($n > 0$) or emission ($n < 0$) of $n$ energy quanta of size $\hbar \omega$.

Though generally dependent on the specific architecture of the system, the Floquet scattering amplitudes still obey two universal relations, which arise from fundamental principles. First, the sum rules

\begin{equation}
\sum_\alpha \sum_n |S_{E_n,E_\beta,\alpha}^\beta|^2 = \sum_\alpha \sum_n |S_{E_n,E_\beta,\alpha}|^2 = 1,
\end{equation}

which follow from the unitarity of the Floquet scattering matrix\textsuperscript{13}, ensure the conservation of probabilities in single-particle scattering events. The double sum thereby runs over all terminals $\alpha$ and all integers $n$, for which $E_n > 0$. Note that, throughout this article, we focus on single-channel conductors for simplicity. Second, the time-reversal symmetry of Schrödinger’s equation implies

\begin{equation}
S_{E_n,E_\beta,\alpha}^\beta = T_B \mathbf{\lambda} S_{E_n,E_\beta,\alpha}^\beta.
\end{equation}
here, scattering events. This phenomenon is a genuine feature whereby the external driving provides a continuous inflow of energy proportional to the photon flux $J^\omega$. The non-equilibrium thermodynamics of dynamic coherent conductors can be developed starting from the first and the second law,

$$P_{ac} + P_{cl} + \sum_{\alpha} J_{\alpha}^q = 0 \quad \text{and} \quad \sigma \equiv -\sum_{\alpha} J_{\alpha}^q/T_{\alpha} \geq 0. \quad (8)$$

Here, the average electrical power absorbed by the system and the mean heat current flowing in the terminal $\alpha$ towards the scattering region are given by

$$P_{cl} \equiv \sum_{\alpha} \mu_{\alpha} J_{\alpha}^p, \quad \text{and} \quad J_{\alpha}^q \equiv J_{\alpha}^E - \mu_{\alpha} J_{\alpha}^p, \quad (9)$$

respectively, and $\sigma$ denotes the total rate of entropy production accompanying the transport process. As we show in App. A, the relations (8) can be verified explicitly using the microscopic expressions (2) and (7) for the mean particle and energy currents and the sum rules (4). Thus, the dynamical scattering approach, which provides the basis for our work, is inherently consistent with the laws of thermodynamics.

Rewriting these laws in terms of thermodynamic fluxes and forces reveals a universal structure, which resembles the standard irreversible thermodynamics of non-equilibrium steady states. Specifically, upon introducing the affinities

$$F^\omega \equiv \hbar \omega/T, \quad F^E_\alpha \equiv (\mu_{\alpha} - \mu)/T, \quad F^q_\alpha \equiv 1/T_{\alpha} - 1/T, \quad (10)$$

where $\mu$ and $T$ are the reference chemical potential and temperature, the first law becomes

$$\sum_{\alpha} J_{\alpha}^q + T J_{\alpha}^p F^\omega + \sum_{\alpha} T J_{\alpha}^E F^E_\alpha = 0 \quad (11)$$

and the second law assumes the canonical bilinear form

$$\sigma = \langle J^\omega F^\omega + \sum_{\alpha} J_{\alpha}^p F^p_\alpha + J_{\alpha}^q F^q_\alpha \rangle \geq 0. \quad (12)$$

This result shows that, on the level of mean fluxes, the thermodynamic properties of periodic states resemble those of non-equilibrium steady states as has been observed earlier for classical stochastic systems. Quite remarkably, Eqs. (11) and (12) put the conventional thermochemical forces $F^E_\alpha$ and $F^q_\alpha$, which are determined by the external reservoirs, on equal footing with the new affinity $F^\omega$, which describes the effect of periodic driving fields on charge carriers inside the conductor. This unification of thermal and mechanical driving is, as opposed to earlier works, achieved here by identifying the frequency rather than the amplitude of the time-dependent perturbation as a thermodynamic force. Though similar in their formal structure, these two schemes describe quite different physical scenarios. In particular, the frequency-based approach, which was first suggested in Ref. 15, yields a natural generalization of Onsager’s kinetic flux-force relations in the linear-adiabatic regime, which we will discuss in the next section. By contrast, a perturbation theory in the driving strength always leads to a trivial decoupling of thermochemical currents and mechanical driving for coherent conductors as we show in App. B.

B. Thermodynamics

The non-equilibrium thermodynamics of dynamic coherent conductors can be developed starting from the...
C. Linear-Adiabatic Response

1. Approximation Scheme

Using the formulas (2) and (7), the thermodynamic fluxes can in principle be determined for any given setup. However, these expressions are notoriously hard to apply in practice and the Floquet scattering amplitudes can be obtained exactly only for a few simple models. In the following we show how linear-adiabatic response theory makes it possible to simplify the situation by invoking two key assumptions, which are typically well justified under realistic conditions; coherent quantum pumps, for example, which we will discuss further in Sec. IV, can be accurately described within the adiabatic approximation.

First, the temperature and chemical potential gradients are usually small compared to their respective thermodynamic references. Specifically, we can assume that

$$F^\rho_\alpha \ll \mu/T \quad \text{and} \quad F^\Lambda_\alpha \ll 1/T$$

The Fermi functions in (2) and (7) can thus be expanded to first order in the thermodynamic forces, i.e., we have

$$f_E^\beta \equiv f_E + h g_E \cdot (F^\rho_\beta + (E - \mu) F^\Lambda_\beta), \quad \text{and} \quad g_E \equiv 1/4h \cosh^2[(E - \mu)/2T]$$

denotes the negative derivative of the Fermi function up to a scaling factor $h/T$, which has been introduced to simplify the notation in the following.

The second corner stone of the adiabatic approach is the slow-driving criterion

$$\omega \ll \delta_E/h.$$ (15)

Here, $\delta_E$ is the typical energy scale, over which the frozen scattering amplitudes $S_{E,\lambda \rho}$ vary significantly; these objects describe the transmission of particles with energy $E$ at fixed values of the control parameters $\lambda$, i.e., in the quasi-static limit. The quantity $\delta_E/h$ can be regarded as a measure for the inverse dwell time of particles in the scattering region$^{39,40}$. The criterion (15) thus essentially requires that the energy landscape of the sample changes only slightly during the passage of individual carriers. Under this condition, the Floquet scattering amplitudes are approximately given by

$$S_{E_n,\lambda \rho}^\beta \approx 1/T \int_0^T dt \left(S_{E,\lambda \rho}^\alpha + n h \omega \partial_E S_{E,\lambda \rho}^\alpha + h \omega A_{E,\lambda \rho}^\alpha \right) e^{in\omega t}$$

with

$$T \equiv 2\pi/\omega$$ (16)

and the corrections $A_{E,\lambda \rho}^\alpha$ being required to ensure that the adiabatic approximation does not spoil the unitarity of the Floquet scattering matrix, for details see$^{41}$. Note that, at low temperatures, the function $g_E$ in (14) is sharply peaked around $\mu$. The transmission of carriers then occurs only at energies close to the Fermi edge. It is therefore typically sufficient to require (15) for $E \approx \mu$.

Upon inserting the approximations (14) and (16), into Eqs. (2) and (7), the thermodynamic fluxes become linear functions of the corresponding affinities given by

$$J^\rho_\alpha = F^\rho_\alpha F^\rho_\beta + \sum_{\beta} \sum_{y} L^x_{xy} F^y_\beta \quad \text{and}$$

$$J^\Lambda_\alpha = F^\Lambda_\alpha F^\rho_\beta + \sum_{\beta} \sum_{x} L^x_{xy} F^\rho_\alpha$$

with $x, y = \rho, q$.

The kinetic coefficients appearing in these relations can, after some algebra, be expressed in the compact form

$$L^x_{\alpha \beta} = \int_0^\infty dE g_E \cdot \xi_E f_E \left(\delta_{\alpha \beta} - \langle|S_{E,\lambda \rho}^\beta|^2\rangle\right),$$

$$L^{\Lambda x}_{\alpha \beta} = \int_0^\infty dE g_E \cdot \xi_E \delta_{\alpha \beta} \sum_{\gamma} \Im \langle|S_{E,\lambda \rho}^\beta S_{E,\lambda \rho}^{\beta \gamma}|\rangle,$$

$$L^{\Lambda x}_{\alpha \beta} = \int_0^\infty dE g_E \cdot (\xi_E^2) \sum_{\gamma} \Im \langle|S_{E,\lambda \rho}^{\beta \gamma}|\rangle/2$$

with

$$\xi_E^E \equiv 1, \quad \xi_E^L \equiv E - \mu, \quad \omega \equiv 1/\omega,$$

dots indicating time derivatives and double brackets denoting the time average over one period of the driving. This result shows that, under linear-adiabatic-response conditions, the thermodynamic fluxes depend only on the frozen scattering amplitudes $S_{E,\lambda \rho}$, which are significantly easier accessible than the full Floquet scattering amplitudes entering the non-linear expressions (2) and (7). Note in particular that the corrections $A_{E,\lambda \rho}^\alpha$ appearing in (16) do not contribute to the response coefficients (18) as a consequence of the unitarity requirement$^{15,42}$.

2. Thermodynamic Consistency & Reciprocity Relations

After introducing the linear-adiabatic-response scheme, we are now ready to discuss the connection between thermodynamics and microdynamics in slowly driven coherent conductors. To this end, we first note that the frozen scattering amplitudes and their time derivatives have to satisfy the sum rules

$$\sum_\alpha |S_{E,\lambda \rho}^\alpha|^2 = \sum_\alpha |S_{E,\lambda \rho}^\alpha|^2 = 1, \quad \sum_\alpha S_{E,\lambda \rho}^\alpha S_{E,\lambda \rho}^{\beta \gamma} = 0.$$ (19)

The first condition thereby accounts for probability conservation in single-particle scattering events at fixed control parameters. The second one fixes the global phase of the frozen scattering amplitudes such that the particle and heat currents (17) obey the conservation laws

$$\sum_\alpha J^\rho_\alpha = 0 \quad \text{and} \quad \sum_\alpha J^\Lambda_\alpha = 0,$$ (20)

respectively. Note that, while the particle currents satisfy this constraint also in the non-linear regime, the heat currents are conserved only in linear order with respect to the thermodynamic forces (10). This result is well in agreement with the first law (11), since the mechanical and the electrical power are both quadratic in the affinities; to recover (11) exactly, we would have to include second-order corrections to the heat currents.
To validate the second law (12), we observe that, upon inserting the kinetic equations (17), the total rate of entropy production becomes a quadratic form in the affinities (10), which can be written as

$$\sigma = \int_0^\infty dE \, g_E \sum_{\alpha \beta} \left\langle \left| X_{E, \alpha \beta} \right|^2 S_{E, \alpha \beta}^{\alpha \beta} + Z \tilde{S}_{E, \alpha \beta}^{\alpha \beta} \right\rangle / 2 \geq 0$$

(21)

with

$$X_{E, \alpha \beta}^{\alpha \beta} = \sum_x \xi_x^E (F_{\alpha}^x - F_{\beta}^x) \quad \text{and} \quad Z \equiv i \xi_x^F F^{\omega}.$$

This expression follows from (18) and the sum rules (19); it shows that the rate of entropy production is an inherently positive quantity in the linear-adiabatic-response scheme, which is thus consistent with the second law.

Finally, owing to time-reversal symmetry, the frozen scattering amplitudes obey the relation

$$S_{E, \alpha \beta}^{\alpha \beta} = T_{B} S_{E, \alpha \beta}^{\alpha \beta}.$$

(22)

This constraint is substantially stronger than (5), which requires the reversal of driving protocols. It implies the reciprocity relations

$$T_B L_{\alpha \beta}^{yx} = L_{\beta \alpha}^{yx}, \quad T_B L_{\alpha \beta}^{x} = -L_{\alpha \beta}^{x}, \quad T_B T_{\alpha \beta}^{x} = L_{\alpha \beta}^{x}$$

(23)

for the adiabatic response coefficients (18), which can be regarded as extensions of the conventional Onsager-Casimir relations. As we will discuss in Sec. III B, these symmetries have profound consequences for the thermodynamics of slowly driven coherent conductors.

III. THERMODYNAMIC CONSTRAINTS

A. New Bounds

We are now ready to derive our new bounds on coherent transport in slowly driven mesoscopic conductors. To this end, we employ a mathematical technique that has proven effective to unveil thermodynamic constraints in systems subject to steady-state and periodic driving. The key idea of this approach is to introduce a generating function for the quantities of interest, which is then bounded from below and afterwards contracted to a given set of variables. Here, we use the quadratic generating function

$$G \equiv \sigma + \sum_\alpha \sum_x J_{\alpha}^x G_{\alpha}^x + \sum_\alpha \sum_{xy} K_{xy}^{xy} G_{\alpha}^x G_{\alpha}^y / 2$$

(24)

and the $G_{\alpha}^x$ being real but otherwise arbitrary variables. Upon expressing the rate of entropy production $\sigma$ and the currents $J_{\alpha}^x$ in terms of the kinetic coefficients (18) and using the sum rules (19), this object can be shown to be manifestly non-negative. That is, we have

$$G = \int_0^\infty dE \, g_E \sum_{\alpha \beta} \left\langle \left| Y_{E, \alpha \beta} \right|^2 S_{E, \alpha \beta}^{\alpha \beta} + Z \tilde{S}_{E, \alpha \beta}^{\alpha \beta} \right\rangle / 2 \geq 0$$

(25)

with

$$Y_{E, \alpha \beta}^{\alpha \beta} = \sum_x \xi_x^E (F_{\alpha}^x - F_{\beta}^x + G_{\alpha}^x).$$

This result implies a whole family of bounds, each of which can be extracted by contracting (24) in a specific way. In particular, taking the minimum of $G$ with respect to the auxiliary variables $G_{\alpha}^x$ yields

$$\sigma \geq \sum_\alpha \sum_{xy} \hat{K}_{xy}^{xy} J_{\alpha}^x J_{\alpha}^y / 2,$$

(26)

where the parameters $\hat{K}_{xy}^{xy}$ are defined by the condition

$$\sum_{xy} \hat{K}_{xy}^{xy} K_{xy}^{xy} \equiv \delta_{xy}. $$

(27)

To obtain a bound that involves only a single current, we further minimize the right-hand side of (26) with respect to all but one of the flux variables $J_{\alpha}^x$, thereby taking into account the conservation laws (20) as additional constraints. This procedure yields our central result,

$$\sigma \geq m \sum_{\alpha} \sum_{xy} \hat{K}_{xy}^{xy} (J_{\alpha}^x)^2 / 2,$$

(28)

where $m$ is the number of terminals of the conductor.

This inequality provides a significantly stronger constraint than the bare second law, which requires only $\sigma \geq 0$. Specifically, (28) shows that the total rate of entropy production is bounded by the square of every individual heat and particle current times a proportionality factor, which depends only on the number of terminals $m$, the equilibrium temperature $T$ of the system and, formally, its base chemical potential $\mu$. In practice, however, the parameters $\hat{K}_{xx}$ can be estimated as

$$\hat{K}_{xx} \equiv \hbar / T \quad \text{and} \quad \hat{K}_{yy} \equiv 3 \hbar / \pi^2 T^3, $$

(29)

where equality holds in the limit $\mu / T \to \infty$, which is effectively realized in mesoscopic systems. Notably, no universal relation of the form (28) exists for the photon flux $J^{\omega}$ as we will demonstrate explicitly in Sec. IV B by working out a specific example.

B. Time-Reversal Symmetry

The bound (28) is universal in that it holds for an arbitrary potential and magnetic field landscape inside the conductor and any sufficiently slow driving protocols. As we will show in Sec. IV B, it is generally also tight. For systems with additional symmetries, however, stronger bounds can be established, as we will see in the following.

For time-symmetric driving protocols, the reciprocity relations (23) imply $L_{\alpha \beta}^{x} = L_{\beta \alpha}^{x} = 0$, that is, the thermochanical variables, $J_{\alpha}^{x}$ and $F_{\alpha}^{x}$, decouple from the mechanical ones, $J^{x}$ and $F^{x}$. The total entropy production can thus be divided into two non-negative contributions,

$$\sigma_{th} \equiv \sum_\alpha \sum_x J_{\alpha}^{x} F_{\alpha}^{x} \geq 0 \quad \text{and} \quad \sigma_{\omega} = J^{x} F^{x} \geq 0, $$

(30)

which arise solely from thermal gradients and periodic driving, respectively. Repeating essentially the steps that lead to (28), and using a convexity argument, which we provide in App. C, then leads to the refined bound

$$\sigma_{th} \geq \frac{m}{m - 1} \hat{L}_{\alpha \alpha}^{xy} (J_{\alpha}^{x})^2 / 2 \quad \text{with} \quad \sum_{xy} \hat{L}_{\alpha \alpha}^{xy} \hat{L}_{\alpha \alpha}^{x} \equiv \delta_{xy}. $$

(31)
which is stronger than (28) in two respects. First, it involves only the thermal rather than the total entropy production; these two quantities are identical if and only if the external control parameters are frozen, that is, if no time-dependent driving is applied to the system. Second, the factors $K^{xx}$, which are independent of the structural properties of the conductor, are now replaced with the inverse kinetic coefficients, $L^{xx}_{\alpha\alpha} \geq K^{xx}$.

The latter feature leads to an interesting physical interpretation of (31). According to the fluctuation-dissipation theorem, the diagonal response coefficients $L^{xx}_{\alpha\alpha}$ are proportional to the averaged equilibrium noise of the current $J^{x}_{\alpha}$. Specifically, we have

$$D^{x}_{\alpha} \equiv ||D^{x}_{\alpha,\lambda}|| = 2 L^{xx}_{\alpha\alpha},$$

(32)

where $D^{x}_{\alpha,\lambda}$ denotes the zero-frequency noise of $J^{x}_{\alpha}$ at constant temperature and chemical potential, $T$ and $\mu$, and fixed external parameters $\lambda$, for details see Ref. 13. Thus, upon noting that $L^{xx}_{\alpha\alpha} \geq 1/L^{xx}_{\alpha\alpha}$, the bound (31) implies

$$\sigma^{x}_{\alpha} \geq \sigma_{th}^{x} \geq m/(m-1), \quad \text{where} \quad \sigma^{x}_{\alpha} \equiv (J^{x}_{\alpha})^{2}/D^{x}_{\alpha}$$

(33)

denotes the relative uncertainty of the current $J^{x}_{\alpha}$. This figure can be regarded as a measure for the accuracy, at which either particles or heat are extracted from the reservoir $\alpha$. Hence, (33) entails a universal trade-off between the thermodynamic cost, i.e., the entropy that must be generated to maintain the fluxes $J^{x}_{\alpha}$, and the precision of adiabatically driven coherent transport. We will further discuss this concept and the relation of (33) with previous results at the end of this paper.

For now, we move on to fully time-reversal symmetric systems, i.e., we now assume that the driving protocols are time-symmetric and the thermochemical response coefficients are independent of the specific parameterization of $\gamma$ and vanish if the mechanical driving is exerted through a single control parameter. We then have $L^{xx}_{\alpha\beta} = L^{xx}_{\beta\alpha} = 0$ and the currents $J^{x}_{\alpha}$ obey the bounds (31), regardless of whether or not the driving protocol is time-symmetric. If, in addition, the thermochemical response coefficients are symmetric, the stronger bounds (34) apply. We conclude this discussion by noting that the expression (36) is closely connected to a geometric quantization principle and plays an important role in the theory of adiabatic quantum pumps, which we will revisit in Sec. IV.

D. Dephasing

Ideal coherent transport, which we have considered so far, is characterized by a fixed phase relation between incoming and outgoing carriers. Under realistic conditions, however, phase-breaking mechanisms such as carrier-carrier or carrier-phonon interactions are difficult to suppress completely. An elegant method to account for such effects within the scattering formalism goes back to B"uttiker. In this approach, a number of virtual reservoirs is attached to the conductor, whose temperature and chemical potential are adjusted such that they do not draw or supply any particles or heat on average. Hence, these probe terminals do not contribute to the actual transport process but rather only mimic incoherent scattering events.
On the technical level, the virtual reservoirs differ from physical ones only in that their affinities are not free parameters but implicitly fixed by the condition of zero mean currents. Since the relation (28) and (31) were derived without making any assumptions on the affinities, it is clear that they remain valid for models with arbitrarily many probe terminals. The parameter \( m \) thereby \textit{a priori} refers to the total number of physical and virtual terminals.

However, since only non-vanishing currents have to be considered in the contraction leading from (26) to (28)\(^6\), the constraint (28) even holds in a stronger version, where \( m \) counts only the physical terminals. Hence, our general new bound is robust against dephasing and applies even in the limit of completely incoherent transmission\(^6\). This observation further underlines the universality of our results. As we will discuss in Sec. V B 2, it also allows us to recover an earlier bound on stationary transport in multi-terminal systems. Note, however, that the same argument can not be used for the symmetric bound (31), which can be proven only with \( m \) being the total number of terminals, see App. C.

### IV. ADIABATIC QUANTUM DEVICES

#### A. Setup

We now show how our general theory can be applied to practical problems. To this end, we use our key result (28) to derive two universal trade-off relations restricting the performance of adiabatic quantum pumps and isothermal engines, respectively. To explore the quality of these bounds, we work out two specific models.

Coherent quantum pumps use a periodically modulated scattering potential to generate a directed flow of particles between two reservoirs with equal temperatures and chemical potentials\(^5\), see Fig. 1. In linear-adiabatic response, such devices are subject to the power-flux trade-off relation

\[
Q_p = \hbar (J^p)^2 / P_{ac} \leq 1, \tag{38}
\]

which provides a lower bound on the mean energy input \( P_{ac} \) that is required to sustain a given pump current \( J^p \). This result follows directly from (28) upon using (29) and noting that, for vanishing thermochemical gradients, the entropy production (11) is proportional to the mechanical power (9), i.e., \( \sigma = \hbar \omega J^o / T = P_{ac} / T \).

In order to extract useful power from a periodically driven conductor, an external bias voltage \( \Delta \mu = TF^o \) must be applied such that the pumped particle current flows uphill. The device then operates as an isothermal engine converting mechanical into electrical energy. The second law puts a universal upper limit on the thermodynamic efficiency of this process,

\[
\eta = -P_{el} / P_{ac} \leq 1. \tag{39}
\]

However, the laws of thermodynamics do not constrain the bare power output, \(-P_{el} = -\Delta \mu J^p\), due to their inherent lack of a fundamental time scale. This gap is closed by our theory as the bound (28) implies the power-efficiency trade-off relation\(^3\),\(^6\)

\[
Q_E \equiv \hbar (-P_{el} / \Delta \mu^2) / (1/T - 1) \leq 1. \tag{40}
\]

This result shows in particular that the ideal efficiency 1 can be approached only at the price of vanishing power, for further discussions of this phenomenon see Sec. V B 2.

#### B. Magnetic-Flux Device

As a first benchmark for our trade-off relations (38) and (40), we consider a technically simple model, which was introduced in\(^1\) as an example of an optimal quantum pump. This system consists of an Aharonov-Bohm loop threaded by a linearly increasing magnetic flux \( \phi = \omega t \), see Fig. 1a. The corresponding frozen scattering amplitudes and driving protocols are given by

\[
S_{E,\lambda}^{\alpha\beta} = \delta_{\alpha\beta} (\lambda_1 - i m_\alpha \lambda_2) e^{i\chi} \quad \text{and} \quad \lambda_1 = \cos[\omega t], \quad \lambda_2 = \sin[\omega t],
\]

respectively, where \( m_1 = 1, m_2 = -1 \) and the global phase \( \chi \) is determined by the circumference of the loop. Inserting these scattering amplitudes into (18) and (17) yields the mean particle and photon flux,

\[
J^p = 1 / T \quad \text{and} \quad J^o = 1 / T - \Delta \mu / \hbar. \tag{42}
\]

Note that, from here onwards, we focus on the low-temperature limit, \( \mu / T \gg 1 \), where the function \( g_E \) defined in (14) is strongly peaked around \( E = \mu \) such that the energy integrals in (18) can be carried out explicitly.

Upon evaluating the mechanical and the electrical power generated by the currents (42),

\[
P_{ac} = (\hbar / T - \Delta \mu) / T \quad \text{and} \quad -P_{el} = -\Delta \mu / T, \tag{43}
\]

we find that the trade-off relations (38) and (40) are both saturated, for \( \Delta \mu = 0 \) and \( \Delta \mu < 0 \), respectively. This results show that our bounds are tight. Furthermore, from (42), we find that the mean entropy production of the magnetic-flux engine,

\[
\sigma = F^p J^p + F^o J^o = (\hbar / T) / T^2, \tag{44}
\]

is independent of the applied bias. Thus, changing \( \Delta \mu \), in principle, makes it possible to tune \( J^o \) to any given value without altering \( \sigma \). This observation proves that the photon flux can indeed not be constrained by a universal bound of the form (28).

#### C. Tunable-Barrier Device

We now move on to a second type of periodic quantum device, which, operating in a four-stroke cycle, uses
to the path encircled by the driving protocols (48) for \( \chi \). Contour plot of the Berry curvature defined in (46) for the tunable-barrier system as adiabatic quantum device. FIG. 3. Tunable-barrier system as adiabatic quantum device. (a) Contour plot of the Berry curvature defined in (46) for the scattering amplitudes (45) and \( \chi = \pi/10 \). Black lines indicate the two symmetries of this function. The circles correspond to the path encircled by the driving protocols (48) for \( v = 1/4, \ldots, 5/4 \). (b) Performance coefficients \( Q_P \) (blue) and \( Q_E \) (red) as functions of the dimensionless current \( \mathcal{T} J^p \) and the isothermal engine efficiency \( \eta \), respectively. For both plots, we have set \( \chi = \pi/10 \) and tuned \( v \) continuously from 1/10 to 100. To evaluate \( Q_E \), the bias \( \Delta \mu \) has been determined by maximizing the efficiency (39).

a periodically modulated confinement potential to move particles one-by-one through a mesoscopic conductor, see Fig. 1b. Owing to their application as a metrological current standard such systems have attracted significant interest. In fact, it has become clear that single-electron pumps implemented with tunable-barrier quantum dots provide an experimentally accessible realization of the quantum ampere with close-to-metrological accuracy.

The behavior of these devices is determined by the conductance \( G \) of the point contacts or metallic gates separating the dot from the leads. In the Coulomb blockade regime, where \( G \ll e^2/h \) with \( e \) being the elementary charge, strong interactions lead to incoherent but quantized single-particle transport. In the coherent regime, \( G \gg e^2/h \), which is realized in large open quantum dots with nearly vanishing level spacings, quantization is connected to topological properties of the underlying scattering matrix. Furthermore, even for almost open dots, i.e., \( G \approx e^2/h \), current quantization can be achieved by approaching the resonant transmission regime, as we will show in the following.

To demonstrate the main features of adiabatic single-electron pumping in the coherent regime, we here consider a one-dimensional model, where the confinement potential consists of two delta-type barriers with dimensionless strengths \( \lambda_1 \) and \( \lambda_2 \). These control parameters determine the conductances of the individual barriers according to \( G \sim e^2/h \lambda^2 \). The frozen scattering amplitudes for this system are given by

\[
S_{E,\lambda}^{12} = S_{\mu,\lambda}^{21} = \chi^2 e^{i\lambda}/Z, \\
S_{E,\lambda}^{11} = \lambda_1(\lambda_2 - i\chi)/Z - \lambda_2(\lambda_1 + i\chi)e^{2i\lambda}/Z, \\
S_{E,\lambda}^{22} = \lambda_2(\lambda_1 - i\chi)/Z - \lambda_1(\lambda_2 + i\chi)e^{2i\lambda}/Z, \\
Z \equiv \lambda_1\lambda_2 e^{2i\lambda} - (\lambda_1 - i\chi)(\lambda_2 - i\chi) \quad \text{and} \quad \chi \equiv L\sqrt{2M/E}/h.
\]

Here, \( L \) denotes the spatial distance between the two barriers and \( M \) the effective carrier mass.

The complicated structure of the scattering amplitudes (45) makes it difficult to find proper driving protocols for the tunable-barrier system. Here, we approach this problem from a geometric perspective using the framework discussed in Sec. III C. We first rewrite the particle current at zero bias and low temperatures as

\[
J^p = L\rho_\omega F^\omega = \frac{1}{2\pi T} \int_{\Sigma} dS \cdot B_{\mu,\lambda},
\]

where the total Berry curvature, \( B_{\mu,\lambda} \equiv B_{\mu,\lambda}^{11} + B_{\mu,\lambda}^{12} \), can be calculated by inserting the scattering amplitudes (45) into (35) and (37). The resulting expression is, however, quite involved. Rather than spelling it out explicitly, we here discuss the general properties of this quantity with the help of Fig. 3a.

This plot reveals three key features of \( B_{\mu,\lambda} \) as a function of the control parameters \( \lambda_1 \) and \( \lambda_2 \). First, it is symmetric with respect to the line \( \lambda_1 = \lambda_2 \) and antisymmetric with respect to \( \lambda_2 = -\lambda_1 + 2\Lambda \). Second, we have

\[
B_{\mu,\lambda} \geq 0 \quad \text{for} \quad \lambda_2 \geq -\lambda_1 + 2\Lambda \quad \text{and} \quad (47) \\
B_{\mu,\lambda} \leq 0 \quad \text{for} \quad \lambda_2 \leq -\lambda_1 + 2\Lambda \quad \text{with} \quad \Lambda = -\chi \cot(\chi)/2.
\]

Third, the function \( B_{\mu,\lambda} \) shows two isolated peaks along its symmetry axis and decays rapidly for \( \lambda_1, \lambda_2 \to \infty \); this behavior is ultimately a consequence of the resonant structure of the transmission coefficients corresponding to the scattering amplitudes (45). Thus, to generate a significant current with moderate driving amplitudes, the control path \( \gamma \) must lie in the half plane \( \lambda_2 > -\lambda_1 + 2\Lambda \) and encircle the positive peak of the Berry curvature. These two requirements can be met with the driving protocols

\[
\lambda_1 = \Lambda + v/\sqrt{2} - v\cos[\omega t], \quad \lambda_2 = \Lambda + v/\sqrt{2} - v\sin[\omega t],
\]
which correspond to harmonically modulated gate voltages. The tuning parameter \( v \), i.e., the radius of the circle parameterized by (48), thereby determines the amplitude of the driving, see Fig. 3a.

We are now ready to explore the performance of the tunable-barrier system as a coherent quantum pump and isothermal engine. To this end, we first numerically evaluate the response coefficients (18) in the low-temperature limit using the scattering amplitudes (45) and the protocols (48) for increasing values of \( v \). We then calculate the particle flux \( J^p \), the mechanical power \( P_{ac} \) and, for the isothermal engine, the electrical power \( P_{d3} \) with the bias \( \Delta \mu \) being fixed such that the efficiency (39) is maximized.

The results of this analysis are summarized in Fig. 3b, which can be understood in our geometric picture as follows. For \( \Delta \mu = 0 \), increasing \( v \) first leads to a rapidly growing particle current \( J^p \) as the contour \( \gamma \) expands into the positive peak of the Berry curvature. Once the peak is fully encircled, the generated current settles to a finite value close to \( 1/T \). Further increasing \( v \) then only leads to successively larger driving amplitudes requiring more and more power input. Thus, the coefficient \( Q_p \), which describes the performance of the system as an adiabatic quantum pump, goes to zero. A similar argument applies if the device is operated as an isothermal engine. After the pumped current is saturated, increasing \( v \) makes it necessary to apply a successively larger bias \( \Delta \mu \) to keep the efficiency at its maximum, that is, close to \( 1/4 \). The rescaled output \(-P_{d3}/\Delta \mu^2\) thus becomes smaller and smaller and the performance coefficient \( Q_E \) drops to zero.

Figure 3b shows that the tunable-barrier system can indeed produce considerable currents as a quantum pump and operate at viable efficiencies as an isothermal engine. The corresponding performance coefficients, \( Q_P \) and \( Q_E \), however, do not even come close to their upper bound of \( 1 \). In fact, these figures reach their respective maxima at \( Q_P^* \approx 0.12 \) and \( Q_E^* \approx 0.04 \). On the microscopic level, this result is a consequence of idle scattering events, in which carriers pick up energy from the external driving without contributing to the pumped current. This type of process is fully suppressed in the magnetic-flux device, where ideal beam splitters force each particle that has been accelerated inside the loop to escape in the direction of the mean current. In contrast to the tunable-barrier setup, which is well within reach of current experimental techniques, this idealized system can, however, hardly be implemented in practice. Finding strategies to test our new bounds experimentally is an interesting challenge for future investigations.

V. CONCLUDING DISCUSSION

A. Main Results

The new bounds (28) and (31) constitute the first major result of this paper. Being derived in a universal framework, they hold for any coherent multi-terminal system in adiabatic response with only (31) requiring an additional symmetry condition. In particular, both constraints (28) and (31) apply in the stationary limit, which is included in our general theory as the special case of frozen, i.e., time-independent, control parameters.

From a practical perspective, the relations (28) and (31) open a new avenue to experimentally estimate the otherwise hardly accessible total dissipation caused by a coherent transport process, a strategy known as thermodynamic interference. Specifically, to determine the total rate of entropy production, it would in principle be necessary to measure all thermodynamic fluxes in the system. Heat currents in mesoscopic system are, however, notoriously difficult to access in experiments and the photon flux is virtually out of reach for direct observation. Our bound (28) now makes it possible to obtain at least a lower bound on the entropy production through practically feasible measurements of the particle currents in the individual terminals.

Another key application of our bounds is the derivation of universal performance constraints for mesoscopic machines, as we have shown in Sec. IV A for adiabatic quantum pumps and isothermal engines. These thermodynamic trade-off relations, are our second main result. By applying them to two specific system in Secs. IV B and IV C, we have demonstrated that, first, our bounds are technically tight and, second, that they make it possible to quantitatively assess the performance of realistic mesoscopic devices. These combined insights constitute the third achievement of our work. In the following, we will briefly discuss how our results relate to ongoing developments and previous studies before concluding this paper with a brief outlook.

B. Recent Developments and Earlier Work

1. Thermodynamic Uncertainty Relations

Thermodynamic uncertainty relations are inequalities of the general form

\[
\sigma \varepsilon \geq \psi \quad \text{with} \quad \varepsilon = J^2/D. \tag{49}
\]

Here, \( \sigma \) corresponds to the total dissipation caused by a stationary non-equilibrium process, \( \varepsilon \) denotes the relative uncertainty of a generated current with mean \( J \) and fluctuations \( D \) and \( \psi \) is a numerical constant.

This bound was first discovered as a universal feature of Markovian biomolecular processes, for which \( \psi = 2 \). It has since then triggered considerable research efforts seeking to extend its applicability and to explore its broader implications, see for example. For steady-state coherent transport in linear response, three different relations of the type (49) are now established; the corresponding values of \( \psi \) are

\[
\psi_P = m/(m-1), \quad \psi_B = 0.89612, \quad \psi_M = 2\sin^2[\pi/m]. \tag{50}
\]
where $m$ is the number of terminals of the conductor. The strongest of these bounds, $\psi_p$, follows from our new result (31) by freezing the external control parameters. The second one, $\psi_M$, was derived only for particle currents, under isothermal conditions and in a classical setting but without a linear-response assumption. It can be saturated far from equilibrium but covers the quantum regime only in the limit of small biases, where the current fluctuations become quasi-classical\textsuperscript{27}. Finally, $\psi_M$ provides the weakest but also the most general available constraint, which applies also to non-elementary fluxes, i.e., arbitrary linear combinations of the heat and particle currents flowing in the individual terminals; for $n > 3$ this bound cannot be saturated by a single currents\textsuperscript{28}.

2. Power-Efficiency Trade-off Relations

The first and the second law put universal limits on the efficiency of thermodynamic machines like heat engines, refrigerators or isothermal energy converters. They do, however, not constrain the output power of such systems, which in contrast to efficiency, is generally not a dimensionless figure. This observation has recently triggered an active debate on whether or not it is, at least in principle, possible to realize devices that operate reversibly while still delivering finite output, see for example\textsuperscript{19,20,25,33,36,62,72,77,78} . These investigations have uncovered a variety of thermodynamic trade-off relations that rule out the option of finite power at ideal efficiency for broad classes of systems. In Sec. IV A, we have derived such a constraint for adiabatic isothermal engines. Our bound (28) is, however not limited to this specific setup. It can, in fact, be applied to any type of device operating in linear-adiabatic response. In particular, for steady-state thermoelectric generators, it allows us to recover the power-efficiency trade-off relation

$$\eta(\eta_C - \eta) \geq (3h/\pi^2T^2)P, \quad \text{where} \quad \eta_C \equiv TF\eta$$

is the Carnot efficiency. This bound still holds when a magnetic field is applied to the system and dephasing is included through an arbitrary number of probe terminals. In this general form, it was first conjectured in\textsuperscript{25} based on numerical evidence and later proven in\textsuperscript{79}.

3. Optimal Quantum Pumps

Adiabatic quantum pumps can be elegantly described in terms of geometric concepts, as we have seen in Sec. IV C for a tunable-barrier device. In general, the geometric approach has proven remarkably useful to understand the physical principles that govern the properties of such systems, most importantly the origins of quantized charge currents, see for example\textsuperscript{53–59} . In\textsuperscript{17}, the adiabatic approximation was used to derive a universal criterion for optimal coherent pumps operating at zero temperature. This bound,

$$\mu J_\alpha^p - J_\alpha^E \geq h (J_\alpha^p)^2 / 2,$$

which involves the particle and the energy currents, $J_\alpha^p$ and $J_\alpha^E$, flowing in the terminal $\alpha = 1, 2$, is quite similar to our trade-off relation (38). In fact, (38) can be derived from (52) upon using the conservation laws $J_1^\rho + J_2^\rho = 0$ and $P_{ac} + J_1^E + J_2^E = 0$. However, (38) does not imply the stronger relation (52). This observation also explains, why our bound (38) is saturated by the magnetic-flux device discussed in Sec. IV B, which was introduced in\textsuperscript{17} as an example of an optimal quantum pump.

C. Outlook

The foregoing discussion underlines the versatile applicability of the theoretical results obtained in this paper. In fact, it shows that our general approach makes it possible to connect a whole variety of different topics ranging from fundamental questions of quantum and stochastic thermodynamics to practical problems of mesoscopic device engineering. Our work thus paves the way for further studies seeking to develop a broader unifying framework for the thermodynamic description of periodically driven mesoscopic conductors. This ambitious perspective leads to two immediate challenges for future investigations. First, it is yet an open problem to develop viable strategies that make it possible to test our bounds in experiments. Second, it remains to be understood how our results can be extended beyond the regimes of linear-adiabatic response and coherent transport.

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Appendix A: Thermodynamic Consistency

Here, we show that the Floquet scattering formalism discussed in Sec. II A and Sec. II B is consistent with the laws of thermodynamics (8). For the first law, it is sufficient to insert Eq. (9) into Eq. (8) and recall that Eq. (4) implies the conservation laws (6). The second law can be verified using a general argument put forward in\textsuperscript{80}.
To this end, we first rewrite the total entropy production defined in Eq. (8) as

$$\sigma = \frac{1}{\hbar} \int_0^\infty dE \sum_{\alpha, \beta} \sum_n (u_{E_n}^\beta - u_{E_n}^\alpha) |S_{E_n,E}^{\beta \alpha}|^2 f[u_{E_n}^\beta]$$

(A1)

with $u_{E_n}^\alpha \equiv (E - \mu_\alpha)/T_\alpha$, $f[u] \equiv 1/(1 + \exp[a])$.

Upon inserting this relation into (A1), we arrive at

$$\sigma \geq \frac{1}{\hbar} \int_0^\infty dE \sum_{\alpha, \beta} \sum_n (F[u_{E_n}^\beta] - F[u_{E_n}^\alpha]) |S_{E_n,E}^{\beta \alpha}|^2$$

(A3)

where the second line follows by shifting the integration variable in the first term and using that the Floquet scattering amplitudes vanish if one of their energy arguments becomes negative.

**Appendix B: Weak-Driving Theory**

It is instructive to compare the linear-adiabatic response theory of Sec. II C with the more common weak-driving scheme, which can be formally developed for coherent conductors as follows. We first divide the single-particle Hamiltonian that describes the dynamics inside the sample into two contributions,

$$H_i = H^0 + \varepsilon V_i.$$  

(B1)

The free Hamiltonian $H^0$ is thereby time-independent, the dynamical scattering potential $V_i = V_{i, \sigma}$ accounts for the external driving fields and the parameter $\varepsilon$ controls the strength of this perturbation. The Floquet scattering amplitudes can thus be formally expanded as

$$S_{E_n,E}^{\alpha \beta} \equiv \delta_{\alpha 0} S_{E,E}^{\beta \epsilon} + \varepsilon A_{E_n,E}^{\beta \epsilon}$$

up to second-order corrections in $\varepsilon$. The first contribution in this series describes elastic transmission and reflection events and the $\varepsilon$-dependent corrections arise from inelastic processes induced by the periodic driving.

The decomposition (B1) determines the dynamical perturbation only up to time-independent shift. Therefore, $V_i$ can always be chosen such that the inelastic term in (B) obeys

$$A_{E_n,E}^{\alpha \beta} \big|_{n=0} = 0.$$  

(B2)

Consequently, we have

$$|S_{E_n,E}^{\alpha \beta}|^2 \geq \delta_{\alpha 0} |S_E^{\beta \epsilon}|$$

(B3)

up to second-order corrections in the driving strength. Inserting this result into Eq. (2) and Eq. (7) shows that the thermochemical variables, $J^\epsilon$ and $F^\epsilon$, and the mechanical ones, $J^\nu$ and $F^\nu$, become independent of each other in the weak-driving limit, that is, in linear order with respect to $\varepsilon$. By contrast, these quantities are generally not decoupled in the linear-adiabatic scheme, since the slow-driving scattering amplitudes (16) remain finite for $n \neq 0$, even in lowest order with respect to the effective expansion parameter $\hbar \omega$.

**Appendix C: Symmetric New Bound**

We derive the bound (31). To this end, we proceed in four steps. First, we observe that (19) implies

$$\sum_\alpha \|S_{E,\alpha}^{\beta \epsilon}\|^2 = \sum_\alpha \|S_{E,\alpha}^{\beta \epsilon}\|^2 = 1.$$  

(C1)

Hence, the period averaged transmission coefficients form a bistochastic matrix. According to the Birkhoff-von Neumann theorem, they can thus be expressed as

$$\|S_{E,\alpha}^{\beta \epsilon}\|^2 = \sum_\nu \rho_{E}^{\nu} P_{E}^{\nu \beta \epsilon},$$  

(C2)

where the positive coefficients $\rho_{E}^{\nu}$ add up to 1 and the $P_{E}^{\nu \beta \epsilon}$ are the elements of a permutation matrix. Consequently, we can decompose the thermal entropy production, the thermochemical currents and the diagonal thermochemical response coefficients as

$$\sigma_{th} = \sum_\nu \sum_{\alpha, \beta} (\delta_{\alpha 0} - P_{\nu}^{\alpha \beta}) M_{\nu}^{\beta \epsilon} F_{\alpha}^{\nu} F_{\beta}^{\nu} \equiv \sum_\nu \sigma_{th}^{\nu},$$

$$J^\epsilon = \sum_\nu \sum_{\alpha, \beta} (\delta_{\alpha 0} - P_{\nu}^{\alpha \beta}) M_{\nu}^{\beta \epsilon} F_{\alpha}^{\nu} \equiv \sum_\nu J_{\nu}^{\beta \epsilon},$$

$$L_{\alpha \alpha}^{\nu \nu} = \sum_{\nu} (1 - P_{\nu}^{\alpha \alpha}) M_{\nu}^{\beta \epsilon} \equiv \sum_{\nu} L_{\nu}^{\beta \epsilon \nu \nu}$$

with

$$M_{\nu}^{\beta \epsilon} \equiv \int_0^\infty dE g_{E} \rho_{E}^{\nu} F_{E}^{\nu \epsilon} E_{\beta},$$  

(C3)

respectively. Note that, in the second line, we have used that the coupling coefficients $L_{\alpha \alpha}^{\nu \nu}$ and $L_{\alpha \alpha}^{\nu \nu}$ vanish for time-reversal symmetric driving.

Second, we define the detailed generating function

$$G^{\nu} \equiv \sigma_{th}^{\nu} + \sum_\alpha \sum_{\alpha, \beta} J_{\nu}^{\beta \epsilon} G_{\alpha}^{\nu} + \sum_\alpha \sum_{\nu, \beta} L_{\nu}^{\beta \epsilon \nu \nu} G_{\alpha}^{\nu} G_{\alpha}^{\nu}/2$$

$$= \int_0^\infty dE g_{E} \sum_{\alpha, \beta} \rho_{E}^{\nu} (E_{\beta}^{\nu \epsilon})^2 P_{E}^{\alpha \beta}/2 \geq 0,$$  

(C4)

where we have used the definition (25) for the variables $Y_{E}^{\nu \beta}$. We now observe that the detailed currents $J_{\nu}^{\beta \epsilon}$ vanish if $P_{\nu}^{\alpha \alpha} = 1$ for given $\alpha$. Therefore, (C4) can be rewritten as

$$G^{\nu} = \sigma_{th}^{\nu} + \sum_\alpha (1 - P_{\nu}^{\alpha \alpha}) \sum_{\beta} J_{\nu}^{\beta \epsilon} G_{\alpha}^{\nu} + \sum_\alpha (1 - P_{\nu}^{\alpha \alpha}) \sum_{\nu, \beta} L_{\nu}^{\beta \epsilon \nu \nu} G_{\alpha}^{\nu} / 2$$  

(C5)

Taking the minimum of this expression with respect to the auxiliary variables $G_{\alpha}^{\nu}$ and recalling that $G^{\nu} \geq 0$ yields

$$\sigma_{th}^{\nu} \geq \sum_\alpha (1 - P_{\nu}^{\alpha \alpha}) \sum_{\beta} J_{\nu}^{\beta \epsilon} J_{\nu}^{\beta \epsilon} G_{\alpha}^{\nu} / 2$$  

with

$$\sum_{\nu} M_{\nu}^{\beta \epsilon} M_{\nu}^{\beta \epsilon} \equiv \delta_{\beta \epsilon \beta \epsilon}.$$
Third, we note that the detailed currents satisfy the conservation laws
\[ \sum_{\alpha} J^x_{\alpha} = \sum_{\alpha} (1 - P^\alpha_{\nu}) J^x_{\alpha} = 0. \]  
(C7)

Minimizing (C6) with respect to all detailed currents except for $J^x_{\alpha}$ while taking into account these constraints leads to the bound
\[ \sigma_{\text{th}} \geq \frac{m - m_{\nu}}{m - 1} \frac{(1 - P^\alpha_{\nu}) \hat{M}^{xx}_{\nu} (J^x_{\alpha})^2}{2} \]  
(C8)

where $m_{\nu}$ denotes the number of fixed points of the permutation corresponding to $P^\alpha_{\nu}$.

Fourth, summing (C8) over $\nu$ and applying Jensen’s inequality leaves us with
\[ \sigma_{\text{th}} \geq \frac{m}{m - 1} N \sum_{\alpha} (1 - P^\alpha_{\nu}) \hat{M}^{xx}_{\nu} (J^x_{\alpha})^2 / 2 N_{\alpha} \]  
(C9)

Finally, by convexity of the matrix inverse, we have
\[ \sum_{\alpha} (1 - P^\alpha_{\nu}) \hat{M}^{xx}_{\nu} / N_{\alpha} \geq N_{\alpha} J^{xx}_{\alpha}. \]  
(C10)

Inserting this inequality into (C9) completes our proof.

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