Some work on a problem of Marco Buratti

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Abstract

Marco Buratti’s conjecture states that if $p$ is a prime and $L$ a multiset containing $p - 1$ non-zero elements from the integers modulo $p$, then there exists a Hamiltonian path in the complete graph of order $p$ with edge lengths in $L$. Say that a multiset satisfying the above conjecture is realizable. We generalize the problem for trees, show that multisets can be realized as trees with diameter at least one more than the number of distinct elements in the multiset, and affirm the conjecture for multisets of the form \{\phi_k(1)^a, \phi_k(2)^b, \phi_k(3)^c\} where $\phi_k(i) = \min\{ki \pmod{p}, -ki \pmod{p}\}$.

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1 Introduction

For basic graph theoretic notation and definition see Diestel [1]. If $T$ is a tree, we say the diameter of $T$ is the number of vertices in a path of maximum length in $T$. Let $p$ be a positive integer and let $K_p$ be the complete graph on $p$ vertices. For any labeling of the vertices of $K_p$ by $\mathbb{Z}_p$, the cyclic group of order $p$, we define the cyclic length of the edge $uv$, $u, v \in K_p$ by $d(u, v) = \min\{|u-v|, p-|u-v|\}$. For any such multiset $L$ of $p - 1$ non-zero lengths from $\mathbb{Z}_p$, we define a cyclic realization of $L$ as a Hamiltonian path on the vertices of $G$ such that the multiset of edge-lengths is $L$.

A linear realization of $L$ is a path $P$ on the set of vertices of $G$ where the length of the edge $uv$ is defined by $d(u, v) = |u - v|$ and the lengths of the edges of $P$ is $L$.

Marco Buratti formulated the following conjecture:

**Conjecture 1.1** Let $p$ be a prime and $L$ a multiset of $p - 1$ non-zero lengths from $\mathbb{Z}_p$. Given a labeling of the vertices of $G = K_p$ by $\mathbb{Z}_p$, there exists a Hamiltonian path $H$ in $G$ such that the multiset of edge-lengths in $H$ is $L$.

By applying our definitions, we restate Buratti’s claim as: If $p = 2n - 1$ is a prime and $L$ is any list of $2n$ integers from $\{1, \ldots, n\}$, then there exists a cyclic realization of $L$.

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Some known results

The problem was first popularized by P. Horak and A. Rosa \[4\] in 2009. Along with a discussion of the problem, the authors made some progress towards a solution with the following results:

**Theorem 2.1** (P. Horak and A. Rosa) Let \( p = 2n + 1 \), \( L = \{d_0^{a_0}, d_1^{a_1}, \ldots, d_k^{a_k}\}, n \geq d_1 > d_2 > \cdots > d_k \) and \( a_i \leq 2 \), for \( i = 1, \ldots, k \). If \( a_0 \geq d_1 - k + t - r \) where \( t = \max\{d_i : i > 0, a_i = 2\} \) and \( r = |\{d_i : i > 0, a_i = 2\}| \), then \( L \) is cyclically realizable.

**Theorem 2.2** (P. Horak and A. Rosa) Let \( p = 2n + 1 \), and let \( L = \{d^a, t^b\} \), where \( d \leq n \), \( t \leq n \), and \( a + b = 2n \). Then \( L \) is cyclically realizable.

**Corollary 2.3** (P. Horak and A. Rosa) Let \( L = \{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\} \), and suppose there is \( j \in \{1, 2, \ldots, n\} \) such that

\[
\sum_{1 \leq i \leq n, i \neq j} a_i \leq 4
\]

Then \( L \) has a realization.

**Theorem 2.4** (P. Horak and A. Rosa) Let \( L = \{d_1^{a_1}, \ldots, d_k^{a_k}\} \). Then there exists an \( s_0 \) so that for all \( s \geq s_0 \) the multiset \( L' = L \cup \{1^s\} \) is both, linearly and cyclically realizable.

**Theorem 2.5** (P. Horak and A. Rosa) Let \( L = \{d_1^{a_1}, \ldots, d_k^{a_k}\} \). Then there exists an \( s_0 \) such that for all \( s \geq s_0 \), the list \( L' = \{d_1^{a_1}, d_2^{a_2}, \ldots, d_k^{a_k}\} \) is cyclically realizable whenever \( 1 + s + \sum_{i=1}^{k} a_i \) is relatively prime to \( d_1 \).

The authors of \[4\] also extended Theorem 2.2 to graphs of non-prime order with the following

**Theorem 2.6** (P. Horak and A. Rosa) Let \( q, d, t \) be natural numbers, and let \( L = \{d_a^a, t_b^b\} \) be a multiset where \( a + b = q - 1, d, t \leq \frac{q}{2} \). Then \( L \) is (cyclically) realizable if and only if \( (q, d, t) = 1 \) and \( (t, q) - 1 \leq a \leq q - (d, q) \) (which is equivalent to \( (d, q) - 1 \leq b \leq q - (t, q) \)).

At the same time, J.H. Dinitz and J.R. Janiszewski \[3\] proved Theorem 2.2 independently and by different methods.

Thus the case of two lengths is solved. As for three lengths, S. Capparelli and A. Del Fra \[2\] proved Buratti’s conjecture for \( L = \{1^a, 2^b, 3^c\} \).

3 Realizable Trees

For the multiset of lengths \( L \) as in the Buratti problem and \( G = K_n \) with vertices labeled from \( \mathbb{Z}_n \), does there exists a spanning tree \( T \) in \( G \) such that the multiset of edge-lengths in \( T \) is \( L \)? Moreover, can we guarantee properties of \( T \) such as large diameter? The answer to the first question was known to J.H. Dinitz and D. Archdeacon (personal communication).
We refer to vertices by their labels. As previously, we say that a multiset of 2n lengths $L$ is tree-realizable, if we can construct a spanning tree $T$ on $2n+1$ vertices so that the multiset of cyclic lengths of the edges of $T$ is $L$. In this case, we call $T$ the tree-realization of $L$. For any length $d' \in L$ and any length $d \in [1, \ldots, n]$, we say $d'$ can be exchanged for $d$ if there exists a realization of the multiset $L' := (L \setminus \{d'\}) \cup \{d\}$ so that the realizations of $L$ and $L'$ are distinct. Note that $L$ and $L'$ may be the same multiset.

3.1 Realizing All Multisets as Trees

**Theorem 3.1** For every tree-realizable multiset $L$ and any $d' \in L$, $L$ admits at least two distance exchanges with any $d \in \{1, \ldots, n\}, d \neq d'$, and at least one distance exchange for $d = d'$.

**Proof.** Consider a tree-realization $T$ of $L$ and for a length $d' \in L$, remove an edge labeled $d'$ from $T$ to produce the labeled forest $F$ with components $T_1$ and $T_2$. We show that for any $d \in \{1, \ldots, n\}$, there exist vertices $x \in T_1$ and $y \in T_2$ with cyclic length $d$.

Let $V(T_1) = \{x_1, \ldots, x_k\}$ and $V(T_2) = \{y_1, \ldots, y_{k'}\}$ where the vertices are identified by their labels. We consider the system of congruences

\[
\begin{align*}
x_1 + d & \equiv z_1 \\
x_1 + 2d & \equiv z_2 \\
& \vdots \\
x_1 + (2n + 1)d & \equiv z_{2n+1}
\end{align*}
\]

Notice that $(z_1, \ldots, z_{2n+1})$ is a derangement of $(x_1, \ldots, x_k, y_1, \ldots, y_{k'})$. Thus, by successively constructing edges of length $d$, we produce a Hamiltonian cycle on the vertices of $T$. The removal of an edge of length $d'$ produces two disjoint trees $T_1$ and $T_2$, and by the above observation, we can find at least two edges of length $d$ from $T_1$ to $T_2$. If $d = d'$, one of these may be an edge we removed, so we can guarantee only one new edge of length $d'$.

\[\square\]

**Corollary 3.2** For any prime $p = 2n + 1$ and multiset $L$ of $2n$ integers taken from $[1, \ldots, n]$, $L$ is tree-realizable.

**Proof.** Begin with the Hamiltonian cycle tree realization of $\{1^{2n}\}$. To obtain any other multiset $L$ with $2n$ elements, we iterate the above theorem, exchanging lengths until we realize a spanning tree with $L$.

\[\square\]
3.2 Extending Tree Diameter

**Theorem 3.3** For any prime \( p = 2n + 1 \) and multiset \( L = \{d_1^{a_1}, \ldots, d_l^{a_l}\} \), \( \sum_{i=1}^l a_i = 2n \), of \( 2n \) integers taken from \( [1, \ldots, n] \), there exists a tree realization \( T \) of \( L \) so that the diameter of \( T \) is at least \( l+1 \).

**Proof.** Let \( L = \{d_1^{a_1}, \ldots, d_l^{a_l}\} \) be a multiset of \( 2n \) integers from \( \{1, \ldots, n\} \) and \( T \) a tree realization of \( L \) with diameter \( d \). If \( d = l \), then we have produced the required cyclic realization. Thus, suppose that \( 1 < d < l \). Let \( A \) be a path of length \( d \) in \( T \) and define \( B \) to be the graph on the vertices \( V(T) \setminus V(A) \) induced in \( T \). Let \( C \) be the set of edges with one vertex in \( A \) and another in \( B \) and set \( B' = B \cup C \). Call the end vertices of \( A, x \) and \( y \).

**Claim 1:** For any edge \( e \in B' \) of length \( l \), if there exists \( u \in V(B) \) so that the cyclic length \( d(x,u) = l \), then there exists a tree realization of \( L \) with diameter larger than \( d \).

Since \( T \) is a tree, \( u \) is incident to an edge \( f \) that can be removed to produce two components, one of which contains \( A \), and the other \( u \). If \( f = e \), then we remove \( e \) and add the edge \((xu)\), which produces a tree realization of \( L \) with larger diameter. Otherwise, we remove \( f \) and add the edge \((xu)\). Next we remove \( e \), producing two components, say \( X \) and \( Y \). By arguing as in Theorem 3.1, we can find an edge between \( X \) and \( Y \) with the length of \( f \). This procedure produces a new tree realization of \( L \) with larger diameter.

**Claim 2:** For any edge \( e \in B' \) of length \( l \), if there exists \( u \in V(A) \) so that the cyclic length \( d(x,u) = l \), then there exists a different tree realization of \( L \) with diameter at least \( d \).

Since \( T \) is a tree, \( u \) is incident to an edge \( f \) that can be removed to produce two components, one of which contains \( u \), and the other \( x \). Call such an edge \( f \) a forced edge and the length of \( f \) the forced length. We remove \( f \) and add the edge \((xu)\). Next we remove \( e \), producing two components, \( X \) and \( Y \) so that \( X \) contains the vertices of \( A \). By arguing as in Theorem 3.1, we can find an edge between \( X \) and \( Y \) with the length of \( f \). Thus we have produced a new tree realization of \( L \) with diameter at least \( d \). Call such a procedure a swap.

For any subgraph \( H \subseteq G \) we define \( c(H) \) as the set of lengths of the edges in \( H \) and for any vertex \( x \in G \), define \( c(x,H) \) as the set of lengths on the edges incident to \( x \) and any vertex in \( H \). Observe that for every vertex \( v \) and length \( i \), there are two vertices of distance \( i \) from \( v \). By Claim 1, \( c(x,A) \geq c(B') \). Thus,

\[
|c(B')| \leq \frac{d-1}{2} \tag{3.1}
\]

Trivially,

\[
|c(A)| \leq d-1 \tag{3.2}
\]

\[
|c(A) \cup c(B')| = l \tag{3.3}
\]

Next, we define a procedure which takes tree realization \( T \) and produces a tree realization \( T' \) with rearranged edge-lengths so that the diameter of \( T' \) is at least \( d \).
We define $A, B, C, \text{ and } B'$ in $T'$ as we did in $T$. For any length $i$ and subset $H$ of $G$, let $r_H(i) = |\{e \in E(B') : i \text{ is the length of } e\}|$.

For $b \in c(B'), b \notin c(A),$ and $r_{B'}(b) \geq 2$, we perform a swap with forced length $a$. If $r_G(a) = 1$, then the swap reduces $|c(A)|$ by 1 with the removal of $a$, and increases $|c(A) \cap c(B')|$ by 1 with the inclusion of $b$. If $r_G(a) > 1$, then the swap either

1. reduces $|c(A)|$ and leaves $|c(A) \cap c(B')|$ unchanged
2. leaves $|c(A)|$ unchanged and increases $|c(A) \cap c(B')|$  
3. if $r_A(a) > 1$ and $a \in c(A) \cap c(B')$, leaves $|c(A)|$ unchanged and leaves $|c(A) \cap c(B')|$ unchanged

Let $D = \{b \in c(B') : b \notin c(A) \text{ and } r_{B'}(b) > 2\}$. For every $b \in D$, either we can find a forced length $a$ such that $a \in c(A) \cap c(B')$, or performing a swap increases the size of $c(A) \cap c(B')$ or decreases the size of $c(A)$. Performing a swap for all elements of $D$ with forced lengths $a \notin c(A) \cap c(B')$, we produce a tree realization $T'$ so that

$$|c(A) \cap c(B')| \geq s \quad (3.4)$$
$$|c(A)| \leq d - 1 - t \quad (3.5)$$

where $s, t \geq 0, s + t = |D|$. By applying the principle of inclusion-exclusion to $3.1, 3.4$, and $3.5$ we obtain

$$d - 1 \geq \frac{2(l + |D|)}{3} \quad (3.6)$$

By $3.1$

$$d - 1 \geq 2|c(B')| \geq 2|D| \quad (3.7)$$

Equating $3.6$ and $3.7$ produces $d - 1 \geq l$. \hfill \Box

### 3.3 Permuting Lengths

We apply an automorphism on all possible lengths, which preserves the tree structure of any realized tree. Let $[n] = \{1, \ldots, n\}$. For any $k \in \{1, \ldots, n\}$ define the map $\phi_k : [n] \rightarrow [n]$ by $\phi_k(i) = \min\{ki \mod p, -ki \mod p\}$.

**Theorem 3.4** For any prime $p$, $p = 2n + 1$ and any tree realizable multiset of lengths $L, |L| = 2n$, with tree realization $T$, there exist at least $n - 1$ multisets $L'$ with tree realizations isomorphic to $T$.

**Proof.** We see that $ki \equiv -kj \Rightarrow k(i + j) \equiv 0 \mod p$, so $\phi_k$ is injective. Clearly, $\phi_k(i) \leq n$ and $\phi_k(i) \neq 0$, which means the range of $\phi_k$ is bounded between 1 and $n$ and hence $\phi_k$ is surjective.

Say $L = \{d_1^1, \ldots, d_n^m\}$. We relabel the vertices of $G$ by applying the action of the automorphism $\sigma(i) = ki \mod p$ for any vertex labeled $i$ from $\mathbb{Z}_p$ to itself. Notice that for any vertices labeled $i$ and $j$

$$d(\sigma(i), \sigma(j)) = \min\{|\sigma(i) - \sigma(j)|, p - \sigma(i) - \sigma(j)|\} =$$
\[
\min\{|k(i - j) \mod p|, \ p - |k(i - j) \mod p|\} = \phi(i - j)
\]

Such a map is consistent with applying \(\phi_k\) to the lengths of \(T\), so that the tree \(T\) is preserved but with relabeled edges. Thus, we have produced the required \(n - 1\) multisets for \(n - 1\) choices of \(k\), and by applying \(\phi_k\) to the lengths of \(L\), we obtain the tree realized multiset \(L' = \{\phi_k(d_1)^{a_1}, \ldots, \phi_k(d_l)^{a_l}\}\) for tree \(T'\) isomorphic to \(T\).

\[\Box\]

**Corollary 3.5** If \(L = \{d_1^{a_1}, d_2^{a_2}, \ldots, d_l^{a_l}\}\) is a cyclically realizable multiset of edge-lengths, then for any integer \(k, 2 \leq k \leq n\), the multiset \(L = \{(\phi_k(d_1))^{a_1}, (\phi_k(d_2))^{a_2}, \ldots, (\phi_k(d_l))^{a_l}\}\) is cyclically realizable.

**Corollary 3.6** For any prime \(p = 2n + 1\), integers \(1 \leq k \leq n\) and \(a, b, c\) where \(a + b + c = 2n\) the multiset \(\{\phi_k(1)^a, \phi_k(2)^b, \phi_k(3)^c\}\) is cyclically realizable.

**Proof.** We apply Corollary 3.5 to the multiset \(\{1^a, 2^b, 3^c\}\) which was shown realizable by Capparelli and Del Fra in [2]. \[\Box\]

**References**

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