Noncommutative Self-dual Gravity

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Abstract

Starting from a self-dual formulation of gravity, we obtain a noncommutative theory of pure Einstein theory in four dimensions. In order to do that, we use Seiberg-Witten map. It is shown that the noncommutative torsion constraint is solved by the vanishing of commutative torsion. Finally, the noncommutative corrections to the action are computed up to second order.
1. INTRODUCTION

Nowadays, there are two main candidates for a quantum theory of the gravitational field: string theory (M-theory) and loop quantum gravity. From the description of the low energy excitations of open strings, in the presence of a NS constant background $B$–field, a noncommutative effective low energy gauge action \[1, 2\] appears in a natural way. It is known, from $M$-atrix-theory that, at low energies, the coordinates of a gas of D0 branes are described by matrices, which give origin to virtual effects \[3, 4\]. Such effects give rise to supergravitational interaction in eleven dimensions. Thus, gravity seems to arise from noncommutativity.

Along these lines, noncommutative gauge theory, as a continuous deformation of the usual theory, has attracted a lot of attention. Although gravitation does not arise in the low energy limit of open string theory as a gauge theory, some interesting effects of gravity processes (like the graviton-graviton-D-brane scattering, in the presence of a constant $B$-field) can be computed \[5\]. However, a deeper study of the deformations of pure gravitational theories is still needed. Thus, the study of models of noncommutative gravity, independently of how they could arise from string or M-theory, could be important. Such models could be obtained starting from those formulations of gravitation which are based on a gauge principle. One of these formulations is the one of self-dual gravity (for a review, see \[6, 7\]), from which the hamiltonian Ashtekar’s formulation \[8\] can be obtained \[9, 10\]. The properties of this formulation have allowed the search for quantum gravity in the framework of loop quantum gravity and quantum geometry (for a review, see \[7\]). In this context, it would be interesting to explore noncommutative loop quantum gravity.

On the other hand, there are proposals for a noncommutative formulation of gravitation \[11\], motivated by the understanding of the short distance behavior of the gravitational field \[12\]. Proposals based on the recent developments are given in \[13, 14, 15, 16, 17\]. In particular, in \[13, 16, 17\] a Seiberg-Witten map for the tetrad and the Lorentz connection is given, where these fields are taken as components of a SO(4,1) connection in the first work, and of a U(2,2) connection in the others. In these works a MacDowell-Mansouri (MM) type action is considered, invariant under the subgroup U(1,1)×U(1,1), and an excess of degrees of freedom, additional to the ones of the commutative theory, is handled by means of constraints. For other recent proposals of noncommutative gravity, see \[18\].
On the other hand, in [19] it has been shown that noncommutative gauge theories, based on the Seiberg-Witten map, for any commutative theory invariant under a gauge group $G$, can be constructed. The resulting noncommutative theory can be seen as an effective theory, invariant under the noncommutative enveloping algebra transformations, and also under the commutative transformations of $G$. This results from the fact that the Seiberg-Witten map may be seen as a sort of gauge fixing, in which the degrees of freedom added by noncommutativity to the fields and to the transformation parameters, are substituted by expressions depending on the commutative fields, in such a way that in the commutative limit the original theory is obtained. In this way, a minimal version of the noncommutative standard model with the gauge group $\text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ has been proposed [20].

Following these ideas, starting from a $\text{SL}(2, \mathbb{C})$, self-dual connection, in [21] we have given a formulation for quadratic noncommutative topological gravitation, which contains the $\text{SO}(3,1)$ signature and Euler topological invariants. In fact, the noncommutative signature can be straightforwardly obtained, but the Euler invariant cannot, as it involves the same difficulty as the MM action, which contains a contraction with the Levi-Civita tensor, instead of the $\text{SO}(3,1)$ trace. However, both invariants can be combined into an expression given by the signature with a $\text{SO}(3,1)$ self-dual connection, which amounts to the $\text{SL}(2, \mathbb{C})$ signature.

In this paper, taking the same $\text{SL}(2, \mathbb{C})$ connection as in [21], considering the Plebański formulation [22], we make a proposal for a noncommutative theory of gravity, which is fully invariant under the noncommutative gauge transformations [23]. The Plebański formulation is written as a $\text{SL}(2, \mathbb{C})$ topological BF formulation, given by the trace of the two-form $B$, times the field strength [24]. The contact with Einstein gravitation is done through constraints on the $B$-field, which are solved by the square of the tetrad one-form [22]. This theory can be restated in terms of self-dual $\text{SO}(3,1)$ fields, the connection and the antisymmetric tensor $B$. After the identification of the $B$ two-form with the tetrad one-form squared, a variation of this action with respect to the connection gives the vanishing of the torsion. The resulting action contains Einstein gravitation plus an imaginary term, which vanishes due to the Bianchi identities. The noncommutative version is obtained at the level of the $\text{SL}(2, \mathbb{C})$ theory, by the application of the Moyal product and the Seiberg-Witten map. In this way, a highly nonlinear theory is obtained, which depends on the commutative $\text{SL}(2, \mathbb{C})$ fields. These fields are then written in terms of the $\text{SO}(3,1)$ self-dual connection and $B$-field, and the connection is written in terms of the tetrads. The consistency of the last step is ensured.
by the fact that the variation of the action with respect to the noncommutative \( SL(2,\mathbb{C}) \)
connection, gives an equation which is exactly solved by the vanishing of the commutative
torsion.

The paper is organized as follows. In section 2 we briefly overview the Seiberg-Witten
map and enveloping algebra. In section 3 we state the commutative self-dual gravity theory
from which the noncommutative one is obtained. In section 4 the noncommutative theory
is formulated and corrections are computed. Finally, section 5 contains our conclusions.

2. NONCOMMUTATIVE GAUGE SYMMETRY AND THE SEIBERG-WITTEN
MAP

In this section, a few conventions and properties of noncommutative spaces will be given
for future reference. For recent reviews see e.g. [25].

Noncommutative spaces can be understood as generalizations of the usual quantum me-
chanical commutation relations, by the introduction of noncommutative coordinates \( x^\mu \)
satisfying,

\[
[x^\mu, x^\nu] = i\theta^{\mu\nu},
\]

where \( x^\mu \) are linear operators acting on the Hilbert space \( L^2(\mathbb{R}^n) \), and \( \theta^{\mu\nu} = -\theta^{\nu\mu} \) are real
numbers. Given this linear operator algebra \( \mathcal{A} \), the Weyl-Wigner-Moyal correspondence
establishes an isomorphic relation between it and the algebra of functions on \( \mathbb{R}^n \), with
an associative and noncommutative star-product, the Moyal \( \star \)-product. Thus, the Moyal
algebra \( \mathcal{A}_* \equiv \mathbb{R}^n \) is, under certain conditions, equivalent to the Heisenberg algebra (1). The
Moyal product is given by

\[
f(x) \star g(x) \equiv \exp \left( i\frac{\theta^{\mu\nu}}{2} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} \right) f(x + \varepsilon)g(x + \eta) \bigg|_{\varepsilon=\eta=0}.
\]

Under complex conjugation it satisfies \( (f \star g)^\dagger = g^\dagger \star f^\dagger \).

As far as we will be working with a nonabelian group, we must include also matrix
multiplication, so an \( \star \)-product will be used as the external product of matrix multiplication
with \( \ast \)-product. In this case hermitian conjugation is given by \( (f \ast g)^\dagger = g^\dagger \ast f^\dagger \). Inside
integrals this product has the cyclicity property \( Tr \int f_1 \ast f_2 \ast f_3 \ast \cdots \ast f_n = Tr \int f_n \ast f_1 \ast \cdots \ast f_2 \ast f_1 \).
\textit{f}_2 \ast f_3 \ast \cdots \ast f_{n-1}. \text{ In particular } Tr \int f_1 \ast f_2 = Tr \int f_1 f_2. \text{ From now on we will understand that the multiplication of noncommutative quantities is given by this } \ast - \text{product.}

Thus, to any expression containing space-time functions, a noncommutative expression can be associated by substitution of the usual product by this \ast - product. However, this procedure has the well known ambiguity of the ordering of the resulting expression, which could be fixed by physical considerations. In particular, in the case of gauge theories, we wish to have a noncommutative theory, invariant under a suitable generalization of the gauge transformations. This generalization frequently is used to fix, to some extent, the ordering ambiguities.

Let us consider a theory, invariant under the action of the Lie group \( G \), with gauge fields \( A_\mu \), and matter fields \( \Phi \) which transform under the adjoint representation \( \text{ad} \),

\[
\delta \lambda A_\mu = \partial_\mu \lambda + i [\lambda, A_\mu],
\]
\[
\delta \lambda \Phi = i [\lambda, \Phi],
\] (3)

where \( \lambda = \lambda^I T_I \), and \( T_i \) \((i = 1, \ldots, \text{dim} G)\) are the generators of the Lie algebra \( \mathcal{G} \) of \( G \), in the adjoint representation. These transformations are generalized for the noncommutative connection \[2\], and for the adjoint representation as,

\[
\delta \lambda \tilde{A}_\mu = \partial_\mu \hat{\lambda} + i [\hat{\lambda}, \hat{A}_\mu],
\]
\[
\delta \lambda \tilde{\Phi} = i [\hat{\lambda}, \tilde{\Phi}].
\] (4)

(5)

The commutators \([A; B] \equiv A \ast B - B \ast A\) have the correct derivative properties when acting on products of noncommutative fields. Due to noncommutativity, commutators like \([\hat{\lambda}; \hat{A}_\mu]\) take values in the enveloping algebra \( \mathcal{U}(\mathcal{G}, \text{ad}) \) of the adjoint representation of \( \mathcal{G} \). Therefore, \( \hat{\lambda} \) and the gauge fields \( \hat{A}_\mu \) will also take values in this algebra. In general, for some representation \( \mathbf{R} \), we will denote it by \( \mathcal{U}(\mathcal{G}, \mathbf{R}) \) the section of the enveloping algebra \( \mathcal{U} \) of \( \mathcal{G} \), which corresponds to the representation \( \mathbf{R} \).

Let us write for instance \( \hat{\lambda} = \hat{\lambda}^I T_I \) and \( \hat{A} = \hat{A}^I T_I \), then,

\[
[\hat{\lambda}; \hat{A}_\mu] = \frac{1}{2} \{ \hat{\lambda}^J; \hat{A}^I_\mu \} [T_I, T_J] + \frac{1}{2} [\hat{\lambda}^J; \hat{A}^I_\mu] \{ T_I, T_J \}.
\] (6)

Thus all the products of the generators \( T_I \) will be needed in order to close the algebra \( \mathcal{U}(\mathcal{G}, \text{ad}) \). Its structure can be obtained by successively computing the commutators and
anticommutators starting from the generators of $\mathcal{G}$, until it closes,

$$[T_i, T_j] = i f_{i j}^K T_K, \quad \{T_i, T_j\} = d_{i j}^K T_K.$$ 

The field strength is $\hat{F}_{\mu \nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu - i [\hat{A}_\mu \hat{A}_\nu]$, hence it also takes values in $\mathcal{U}(\mathcal{G}, \text{ad})$. From (5) it turns out that,

$$\delta_\lambda \hat{F}_{\mu \nu} = i \left( \hat{\lambda} \hat{F}_{\mu \nu} - \hat{F}_{\mu \nu} \hat{\lambda} \right). \quad (7)$$

We see that these noncommutative transformation rules can be obtained from the commutative ones, just by replacing the ordinary product of functions by the Moyal product, with a suitable product ordering. This allows to construct in a simple way invariant quantities.

If we wish to have a continuous commutative limit, the noncommutative fields $\hat{\Phi}$ must be power series expansions of the noncommutativity parameter $\theta$, starting from the commutative ones,

$$\hat{\Phi} = \Phi + \theta^{\mu \nu} \Phi^{(1)}_{\mu \nu} + \theta^{\mu \nu} \theta^{\rho \sigma} \Phi^{(2)}_{\mu \nu \rho \sigma} + \cdots. \quad (8)$$

Thus, in such an expansion, the noncommutative fields will have in general an infinity of independent components. Moreover, the noncommutative gauge fields will take values in the enveloping algebra and, unless the enveloping algebra coincides with the Lie algebra of the commutative theory, as is the case of $G = U(N)$, they will have also a bigger number of matrix components. As this is also the case for the transformation parameters, it will be possible to eliminate a lot of degrees of freedom by fixing the gauge.

However, the Seiberg-Witten map establishes a one to one correspondence among the physical degrees of freedom of the noncommutative fields and the physical degrees of freedom of the commutative fields. This fact is used in reference to construct noncommutative gauge theories, in principle for any Lie group $G$.

The main point is that the Seiberg-Witten map allows for field dependent transformations. This means that if we combine two transformations, the gauge parameters will be transformed as well. Thus, if for an infinitesimal transformation matrix we have the correspondence $\lambda \rightarrow \hat{\lambda}$, to the commutator of two transformations will not correspond simply the Moyal commutator, but

$$[\lambda, \eta] = [\hat{\lambda}, \hat{\eta}] + i \left( \delta_\lambda \hat{\eta} - \delta_\eta \hat{\lambda} \right). \quad (9)$$
If we suppose that there is an expansion like (8) for these matrices,
\[ \hat{\lambda} = \lambda + \theta^{\mu\nu} \lambda^{(1)}_{\mu\nu} + \theta^{\mu\nu} \theta^\rho\sigma \lambda^{(2)}_{\mu\nu\rho\sigma} + \cdots, \] (10)
then a solution for the coefficients can be obtained [2, 19],
\[ \hat{\lambda}(\lambda, A) = \lambda + \frac{1}{4} \theta^{\mu\nu} \{ \partial_\mu \lambda, A_\nu \} + \mathcal{O}(\theta^2). \] (11)

Further, the Seiberg-Witten map determines the \( \Phi^{(a)} \) terms in (8), from the fact that the noncommutative transformations are given by (4), (5) and consequently, (7). These functions in (8) can be expressed in terms of the commutative fields and their derivatives. For the gauge fields, one solution is given by [2],
\[ \hat{A}_\mu(A) = A_\mu - \frac{1}{4} \theta^{\rho\sigma} \{ A_\nu, \partial_\rho A_\mu + F_{\rho\mu} \} + \mathcal{O}(\theta^2), \] (12)
from which, for the field strength it turns out that,
\[ \hat{F}_{\mu\nu} = F_{\mu\nu} + \frac{1}{4} \theta^{\rho\sigma} \left( 2 \{ F_{\mu\rho}, F_{\nu\sigma} \} - \{ A_\rho, D_\sigma F_{\mu\nu} + \partial_\sigma F_{\mu\nu} \} \right) + \mathcal{O}(\theta^2). \] (13)

For fields in the adjoint representation we have got the solution,
\[ \hat{\Phi}(\Phi, A) = \Phi - \frac{1}{4} \theta^{\mu\nu} \{ A_\mu, (D_\nu + \partial_\nu)\Phi \} + \mathcal{O}(\theta^2). \] (14)

It is well known that these solutions are not unique, other terms depending on continuous parameters can be added to them. In [26] this freedom has been related to renormalizability properties. However, it can be also used in order to simplify the structure of the theory [20]. In particular, it allows to give the simple forms of (13) and (14), which have the interesting property that if the commutative fields vanish, also the first order corrections will vanish. In this case, there is a solution for which all higher order terms of the expansion (8) vanish as well.

These higher order terms can be obtained from the the Seiberg-Witten maps for which \( \frac{\partial}{\partial \theta^{\mu\nu}} \hat{\lambda}(\theta) \) and \( \frac{\partial}{\partial \theta^{\mu\nu}} \hat{\Phi}(\theta) \) are solutions, i.e. from the solutions of the equations which result from the corresponding gauge transformations, given by the \( \theta \)–derivatives of (10) and (3),
\[ \theta^{\mu\nu} \left[ \delta_\eta \frac{\partial}{\partial \theta^{\mu\nu}} \hat{\eta} - \delta_\eta \frac{\partial}{\partial \theta^{\mu\nu}} \hat{\lambda} \right] = i \theta^{\mu\nu} \left( \left[ \frac{\partial}{\partial \theta^{\mu\nu}} \hat{\lambda} ; \hat{\eta} \right] + \left[ \hat{\lambda} ; \frac{\partial}{\partial \theta^{\mu\nu}} \hat{\eta} \right] - \frac{\partial}{\partial \theta^{\mu\nu}} \left[ \hat{\lambda}, \hat{\eta} \right] + \frac{i}{2} \left\{ \partial_\mu \hat{\lambda} ; \partial_\nu \hat{\eta} \right\} \right), \] (15)
and
\[ \theta^{\mu\nu} \delta_\lambda \frac{\partial}{\partial \theta^{\mu\nu}} \Phi = i \theta^{\mu\nu} \left( \left[ \frac{\partial}{\partial \theta^{\mu\nu}} \lambda ; \Phi \right] + \left[ \lambda ; \frac{\partial}{\partial \theta^{\mu\nu}} \Phi \right] + i \frac{1}{2} \left\{ \partial_\mu \lambda ; \partial_\nu \Phi \right\} \right), \] (16)

where the \( \theta^{\mu\nu} \) factor is included in order to take into account of the antisymmetry in \( \mu \) and \( \nu \). A solution to this equation can be obtained \[2\] from the first order solution \( \Phi^{(1)}_{\mu\nu} \). Indeed, from this first order term, by substitution of the commutative fields by the noncommutative ones, with multiplication given by the \(*\)-product, full Seiberg-Witten mapped fields \( \lambda^{(1)}_{\mu\nu} \) and \( \Phi^{(1)}_{\mu\nu} \) can be constructed. Hence, from (11) and (14), we have,

\[ \theta^{\mu\nu} \lambda^{(1)}_{\mu\nu} = \frac{1}{4} \theta^{\mu\nu} \left\{ \partial_\mu \lambda ; \Lambda_\nu \right\}, \] (17)

\[ \theta^{\mu\nu} \Phi^{(1)}_{\mu\nu} = -\frac{1}{4} \theta^{\mu\nu} \left\{ \Lambda_\mu ; (D_\nu + \partial_\nu) \Phi \right\}. \] (18)

Now, after some algebra, taking into account the noncommutative gauge transformations (4), (5) and (9), we get,

\[ \theta^{\mu\nu} \left[ \delta_\lambda \eta^{(1)}_{\mu\nu} - \delta_\eta \lambda^{(1)}_{\mu\nu} \right] = i \theta^{\mu\nu} \left( \left[ \lambda^{(1)}_{\mu\nu} ; \eta \right] + \left[ \lambda ; \eta^{(1)}_{\mu\nu} \right] - \left[ \lambda, \eta^{(1)}_{\mu\nu} \right] + i \frac{1}{2} \left\{ \partial_\mu \lambda ; \partial_\nu \eta \right\} \right) \]

\[ \theta^{\mu\nu} \delta_\lambda \Phi^{(1)}_{\mu\nu} = i \theta^{\mu\nu} \left( \left[ \lambda^{(1)}_{\mu\nu} ; \Phi \right] + \left[ \lambda ; \Phi^{(1)}_{\mu\nu} \right] + i \frac{1}{2} \left\{ \partial_\mu \lambda ; \partial_\nu \Phi \right\} \right). \] (19)

These equations give solutions for (15) and (16), if the following identifications are made \[2\],

\[ \frac{\partial}{\partial \theta^{\mu\nu}} \lambda = \lambda^{(1)}_{\mu\nu}, \] (20)

\[ \frac{\partial}{\partial \theta^{\mu\nu}} \Phi = \Phi^{(1)}_{\mu\nu}, \] (21)

which at \( \theta = 0 \) are identically satisfied. In fact, (21) is more general, it is valid also for the connection \[2\], and as well for any field transforming under a linear representation. From it, together with (20), by successive derivations with respect to \( \theta \), a solution for all higher terms of the Seiberg-Witten map can be computed.

Now, we see that if the first order term \( \Phi^{(1)} \) vanishes, by construction also \( \Phi^{(1)} \) will vanish, and consequently in this case \( \Phi = 0 \) is a consistent solution for the Seiberg-Witten map.

The fact that the components of the noncommutativity parameter \( \theta \) are constant, has the important consequence that Lorentz covariance and general covariance under diffeomorphisms of the underlying manifold, are spoiled. The answer that is usually given to this question is that, at the scale where noncommutativity is relevant, it is possible that nature has not the same symmetries like in the commutative limit.
3. DESCRIPTION OF SELF-DUAL GRAVITY

One of the main features of the tetrad formalism of the theory of gravitation [27], is that it introduces local Lorentz SO(3,1) transformations. In this case, the generalized Hilbert-Palatini formulation is written as \( \int e_\mu^a e_\nu^b R_{\mu\nu}^{ab}(\omega) d^4x \), where \( e_\mu^a \) is the inverse tetrad, and \( R_{\mu\nu}^{ab}(\omega) \) is the so(3,1) valued field strength. The decomposition of the Lorentz group as SO(3,1) = SL(2,C) \( \otimes \) SL(2,C), and the geometrical structure of four dimensional space-time, makes it possible to formulate gravitation as a complex theory, as in [8, 22]. These formulations take advantage of the properties of the fundamental or spinorial representation of SL(2,C), which allows a simple separation of the action on the fields of both factors of SO(3,1), as shown in great detail in [22]. All the Lorentz Lie algebra valued quantities, in particular the connection and the field strength, decompose into the self-dual and anti-self-dual parts, in the same way as the Lie algebra so(3,1) = sl(2,C) \( \oplus \) sl(2,C). However, Lorentz vectors, like the tetrad, transform under mixed transformations of both factors and so this formulation cannot be written as a chiral SL(2,C) theory. Various proposals in this direction have been made (for a review, see [6]). In a early formulation, this problem has been solved by Plebański [22], where by means of a constrained Lie algebra valued two-form \( \Sigma \), the theory can be formulated as a chiral SL(2,C) invariant BF-theory, Tr\( \int \Sigma \wedge R(\omega) \). In this formulation \( \Sigma \) has two SL(2,C) spinorial indices, and it is symmetric on them \( \Sigma^{AB} = \Sigma^{BA} \), as any such sl(2,C) valued quantity. The constraints are given by \( \Sigma^{AB} \wedge \Sigma^{CD} = \frac{1}{3} \delta^C_B \Sigma^{EF} \wedge \Sigma_{EF} \) and, as shown in [22], their solution implies the existence of a tetrad one-form, which squared gives the two-form \( \Sigma \). In the language of SO(3,1), this two-form is a second rank antisymmetric self-dual two-form, \( \Sigma^{+ab} = \Pi^{+ab}_{\ cd} \Sigma^{cd} \), where \( \Pi^{+ab}_{\ cd} = \frac{1}{4} \left( \delta^{ab}_{\ cd} + i \epsilon^{ab}_{\ cd} \right) \). In this case, the constraints can be recast into the equivalent form \( \Sigma^{+ab} \wedge \Sigma^{+cd} = \Pi^{+abcd} \Sigma^{+ef} \wedge \Sigma^{+}_{\ ef} \), with solution \( \Sigma^{ab} = 2 e^a \wedge e^b \).

For the purpose of the noncommutative formulation, we will consider self-dual gravity in a somewhat different way as in the papers [8, 22]. In this section we will fix our notations and conventions.

Let us take the self-dual SO(3,1) BF action, defined on a \( (3+1) \)-dimensional pseudo-riemannian manifold \((X, g_{\mu\nu})\),

\[
I = i \text{Tr} \int_X \Sigma^+ \wedge R^+ = i \int_X \varepsilon^{\mu\rho\sigma} \Sigma^+_\mu^a \Sigma^+_\nu^b R^+_{\rho\sigma a b}(\omega) d^4x, \quad \quad (22)
\]
where \( R^+_{\rho\sigma ab} = \Pi_{ab}^{+cd} R_{\rho\sigma cd} \), is the self-dual SO(3,1) field strength tensor. This action can be rewritten as

\[
I = \frac{1}{2} \int_X \varepsilon^{\mu\nu\rho\sigma} \left( i \Sigma_{\mu\nu}^{ab} R_{\rho\sigma ab} + \frac{1}{2} \varepsilon_{ab}^{abcd} \Sigma_{\mu\nu}^{ab} R_{\rho\sigma cd} \right) d^4x. \tag{23}
\]

If now we take the solution of the constraints on \( \Sigma \), which we now write as

\[
\Sigma_{\mu\nu}^{ab} = e_\mu^a e_\nu^b - e_\mu^b e_\nu^a, \tag{24}
\]

then

\[
I = \int_X (\det e R + i \varepsilon^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}) d^4x. \tag{25}
\]

The real and imaginary parts of this action must be variated independently because the fields are real. The first part represents Einstein action in the Palatini formalism, from which, after variation of the Lorentz connection, a vanishing torsion \( T_{\mu\nu}^{\perp} = 0 \) turns out. As a consequence, the second term vanishes due to Bianchi identities.

The action (22) can be written as

\[
I = i \int_X \varepsilon^{\mu\nu\rho\sigma} \Sigma_{\mu\nu}^{+ab} R_{\rho\sigma ab} (\omega^+) d^4x, \tag{26}
\]

where \( R_{\mu\nu}^{+ab}(\omega^+) = \partial_\mu \omega_{\nu}^+ + \partial_\nu \omega_{\mu}^+ + \omega_{\mu}^{+ac} \omega_{\nu}^{+b} - \omega_{\nu}^{+ac} \omega_{\mu}^{+b} \). From the decomposition \( \text{SO}(3,1) = \text{SL}(2,\mathbb{C}) \times \text{SL}(2,\mathbb{C}) \), it turns out that \( \omega_i^\mu = \omega_{\mu}^{+0i} \) is a \( \text{SL}(2,\mathbb{C}) \) connection. Further, if we take into account self-duality, \( \varepsilon_{cd}^{ab} \omega_{\mu}^{+cd} = 2 \omega_{\mu}^{+ab} \), we get \( \omega_{\mu}^{+ij} = -i \varepsilon_{ij}^k \omega_{\mu}^k \). Therefore,

\[
R_{\mu\nu}^{0i}(\omega^+) = \partial_\mu \omega_{\nu}^i + \partial_\nu \omega_{\mu}^i + 2i \varepsilon_{jk}^i \omega_{\mu}^j \omega_{\nu}^k = R_{\mu\nu}^{ij}(\omega) \tag{27}
\]

\[
R_{\mu\nu}^{ij}(\omega^+) = \partial_\mu \omega_{\nu}^{ij} - \partial_\nu \omega_{\mu}^{ij} + 2(\omega_{\mu}^i \omega_{\nu}^j - \omega_{\nu}^i \omega_{\mu}^j) = -i \varepsilon_{ij}^k R_{\mu\nu}^{k}(\omega), \tag{28}
\]

where \( R_{\mu\nu}^{ij} \) is the \( \text{SL}(2,\mathbb{C}) \) field strength.

Similarly, we define \( \Sigma_{\mu\nu}^i = \Sigma_{\mu\nu}^{+0i} \), which transforms in the \( \text{SL}(2,\mathbb{C}) \) adjoint representation. From it we get, \( \Sigma_{\mu\nu}^{+ij} = -i \varepsilon_{ij}^k \Sigma_{\mu\nu}^k \). Thus, the action (26) can be written as a \( \text{SL}(2,\mathbb{C}) \) BF-action

\[
I = i \int_X \varepsilon^{\mu\nu\rho\sigma} \left[ \Sigma_{\mu\nu}^{+0i} R_{\rho\sigma 0i}(\omega^+) + \Sigma_{\mu\nu}^{+ij} R_{\rho\sigma ij}(\omega^+) \right] d^4x = -4i \int_X \varepsilon^{\mu\nu\rho\sigma} \Sigma_{\mu\nu}^{+} R_{\rho\sigma}(\omega) d^4x. \tag{29}
\]

Therefore, if we choose the algebra \( \text{sl}(2,\mathbb{C}) \) to satisfy \( [T_i, T_j] = 2i \varepsilon_{ij}^k T_k \) and \( Tr(T_i T_j) = 2 \delta_{ij} \), we have that (22) can be rewritten as the self-dual action

\[
I = -2i \text{Tr} \int_X \Sigma \wedge R, \tag{29}
\]
which is SL(2, C) invariant.

If the variation of this action with respect to the SL(2, C) connection \( \omega \) is set to zero, we get the equations

\[
\Psi^\mu_i = \varepsilon^{\mu\nu\rho\sigma} D_\nu \Sigma^i_{\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma} \left( \partial_\nu \Sigma^i_{\rho\sigma} + 2i \varepsilon^{i}_{jk} \omega^j_\nu \Sigma^k_{\rho\sigma} \right) = 0. \tag{30}
\]

Taking into account separately both real and imaginary parts, we get, in terms of the SO(3,1) connection,

\[
\varepsilon^{\mu\nu\rho\sigma} D_\nu \Sigma^{ab}_{\rho\sigma} = \varepsilon^{\mu\nu\rho\sigma} \left( \partial_\nu \Sigma^{ab}_{\rho\sigma} + \omega^a_{\nu c} \Sigma^b_{\rho\sigma c} - \omega^b_{\nu c} \Sigma^a_{\rho\sigma c} \right) = 0, \tag{31}
\]

which after the identification \[24\], can be written as

\[
\varepsilon^{\mu\nu\rho\sigma} \left( \partial_\nu e^a_\rho e^b_\sigma - \partial_\nu e^a_\rho e^b_\sigma + \omega^a_{\nu c} e^b_\rho e^c_\sigma - \omega^b_{\nu c} e^a_\rho e^c_\sigma \right) = \varepsilon^{\mu\nu\rho\sigma} \left( T^a_{\nu\rho} e^b_\sigma - T^b_{\nu\rho} e^a_\sigma \right) = 0. \tag{32}
\]

From which the vanishing torsion condition once more turns out.

4. THE NONCOMMUTATIVE ACTION

We start from the SL(2, C) invariant action \[24\]. From it, the noncommutative action can be obtained straightforwardly as

\[
\hat{I} = -i \text{Tr} \int_X \hat{\Sigma} \wedge \hat{R}. \tag{33}
\]

This action is invariant under the noncommutative SL(2, C) transformations

\[
\delta_\lambda \hat{\omega}_\mu = \partial_\mu \lambda + i [\lambda \ast \hat{\omega}_\mu], \tag{34}
\]

\[
\delta_\lambda \hat{\Sigma}_{\mu\nu} = i [\lambda \ast \hat{\Sigma}_{\mu\nu}]. \tag{35}
\]

Actually, in order to obtain the noncommutative generalization of the Einstein equation, we could consider the real part of \[33\],

\[
\hat{I}_E = -i \frac{1}{2} \text{Tr} \int_X \left[ \hat{\Sigma} \wedge \hat{R} - (\hat{\Sigma} \wedge \hat{R})^\dagger \right], \tag{36}
\]

which is also invariant under \[34\] and \[35\].

In order to obtain the corresponding to the torsion condition, a \( \omega \) variation of \[33\] must be done. Although we are considering the commutative fields as the fundamental ones, the action is written in terms of the noncommutative ones. Furthermore, the relation between
the commutative and the noncommutative physical degrees of freedom is one to one. So the equivalent to the variation of the action with respect to $\omega$ will be the variation with respect to $\hat{\omega}$. Thus we write,

$$\delta \hat{\omega} = \varepsilon^{\mu\rho\sigma} \partial_\rho \hat{\Sigma}_{\mu\nu} - i [\hat{\omega}_\rho, \hat{\Sigma}_{\mu\nu}] \ast \delta \hat{\omega}_\sigma = 0, \quad (37)$$

from which we obtain the noncommutative version of (31)

$$\hat{\Psi}^\mu = \varepsilon^{\mu\rho\sigma} \hat{D}_\nu \hat{\Sigma}_{\rho\sigma} = 0. \quad (38)$$

These equations are covariant under the noncommutative transformations (34) and (35), which means that their Seiberg-Witten expansion should be similar to the one of a matter field in the adjoint representation (14). In this case we have that, if the commutative field vanishes, also the first order term of the noncommutative one will vanish. If this happens, as shown at the end of section 2, all the higher orders would vanish as well. Thus, we could expect that a solution to Eq. (38) would be given by the solution of the commutative equation $\Psi^\mu = 0$, and this is indeed the case. If we compute the first order of the Seiberg-Witten expansion of the covariant derivative of $\Sigma$, we get,

$$\hat{D}_\mu \hat{\Sigma}_{\nu\rho} = D_\mu \Sigma_{\nu\rho} - \frac{1}{4} \theta^{\kappa\tau} \left( \{ \omega_\kappa, (D_\tau + \partial_\tau) D_\mu \Sigma_{\nu\rho} \} + 2 \{ R_{\mu\tau\rho}, D_\tau \Sigma_{\nu\rho} \} \right) + O (\theta^2), \quad (39)$$

from which it turns out that,

$$\hat{\Psi}^\mu = \Psi^\mu - \frac{1}{4} \theta_{\rho\sigma} \left( \{ \omega_\nu, (D_\rho + \partial_\rho) \Psi^\mu \} - 2 \{ R_{\nu\rho}, \Psi^\mu \} + 2 \delta_\nu^{\mu} \{ R_{\tau\rho}, \Psi^\tau \} \right) + O (\theta^2). \quad (40)$$

Hence the solutions of the commutative equations $\Psi^\mu = 0$, will be as well solutions of $\hat{\Psi}^\mu = 0$. Furthermore, $\Psi^\mu = 0$ are equivalent to set the torsion equal to zero, that is, after the substitution $\Sigma_{\mu\nu}^{ab} = e_\mu^a e_\nu^b - e_\nu^a e_\mu^b$, their solution is given by,

$$\omega_{\mu}^{ab} = \frac{1}{2} \varepsilon^{\mu\nu\rho} e^{bp} \left[ e_{\mu c} (\partial_\nu e_\rho^c - \partial_\rho e_\nu^c) - e_{\nu c} (\partial_\rho e_\mu^c - \partial_\mu e_\rho^c) - e_{\rho c} (\partial_\nu e_\mu^c - \partial_\mu e_\nu^c) \right]. \quad (41)$$

Therefore we can compute the corrections of the noncommutative action (36) as follows. First we write the Seiberg-Witten expansion of the SL(2,C) fields $\hat{\Sigma}$ and $\hat{\omega}$. Furthermore the commutative SL(2,C) fields are written by means of the self-dual SO(3,1) fields, $\omega_\mu = \omega_\mu^{+0i}$ and $\Sigma_{\mu\nu}^{ab} = \Sigma_{\mu\nu}^{+0i}$. Then, decompose these self-dual fields into the real ones $\omega_{\mu}^{ab}$ and $\Sigma_{\mu\nu}^{ab}$, and then substitute $\Sigma_{\mu\nu}^{ab} = e_\mu^a e_\nu^b - e_\nu^a e_\mu^b$ and write the connection as in (41). In this case we will have a noncommutative action, which will depend only on the tetrad.
If we consider the real part, the first order correction vanishes, and, after a lengthy calculation, the second order one turns out to be, already written in terms of commutative $SO(3,1)$ fields,

$$
\hat{I}^{\theta}_{2} = 4 \int_{X} dx^{4} \det(e) \left\{ \partial_{\mu} \left[ \partial_{\nu} \left( \omega_{\nu}^{ab} R_{abcd} \right) \omega_{\tau}^{cd} \right] + \partial_{\mu} \left( \omega_{\nu}^{ab} \partial_{\nu} \omega_{\tau}^{cd} R_{abcd} \right) + 16 R^{a}_{m} \left( R^{bc}_{ab} R_{abc} - \omega^{bc}_{\nu} \partial_{\nu} R_{abc} \right) - \frac{1}{2} \partial_{m} \left( \omega^{ab}_{\nu} \left( \partial_{\nu} \omega_{ab} + R_{ab} \right) \right) R - \frac{1}{2} \partial_{m} \omega_{\nu}^{ab} \left( R^{e}_{bf} R_{bsef} - \omega^{ef}_{e} \partial_{e} R_{ab} \right) \right. \\
- \frac{1}{2} \partial_{r} \omega_{e}^{ef} \partial_{\nu} R^{cd}_{edef} - \frac{1}{2} \varepsilon_{abcd} \varepsilon^{efgh} \left[ \omega_{e}^{ef} \partial_{m} \left( R^{gh}_{g} R^{cd}_{hs} - \omega^{gh}_{r} \partial_{r} R^{cd}_{gh} \right) + \frac{1}{4} \partial_{s} \left( \omega_{e}^{ef} \partial_{r} \omega^{ab}_{m} \right) R^{cd}_{gh} \right. \\
+ \frac{1}{2} \partial_{m} \omega_{e}^{ef} \left( R^{gh}_{g} R^{cd}_{hs} - \omega^{gh}_{r} \partial_{r} R^{cd}_{gh} \right) \left. - \frac{1}{2} \partial_{s} \left( \partial_{m} \left( \omega^{ab}_{1} R^{cd}_{gh} \right) \omega^{ef}_{e} \right) \right\},
$$

(42)

where the connection $\omega^{ab}_{\mu}$ is given by (11). This correction terms could allow the explicit computation of deformed known gravitational metrics.

5. CONCLUSIONS

In this work we propose an ansatz to get a noncommutative formulation of standard four-dimensional Einstein gravitation. We start from a self-dual $SO(3,1)$ BF-action, which is equivalent to Einstein gravitation after the substitution of the B-field in terms of the tetrad. This action is reformulated as the chiral $SL(2,C)$ invariant self-dual action (29), from which the noncommutative action (33) is straightforwardly obtained. This chiral action allows us to find an alternative noncommutative gravity action in four dimensions, that generalizes the usual general relativity. As mentioned, there are other proposals already introduced in the literature [14, 15, 16]. In our proposed action the noncommutative spin connection variation gives the noncommutative ‘torsion condition’ (38), which is exactly solved by (41). This allows to introduce the tetrad at the commutative level in a consistent way (24). As a consequence, the corrections to Einstein gravitation can be, modulo a laborious algebra, straightforwardly computed.

In the process we have used the results from [19, 20], developed there to construct the noncommutative versions of standard model and GUT’s. In the present paper, the Seiberg-Witten map for matter fields in the adjoint representation of any gauge group has been constructed in order to get the Seiberg-Witten map for the $\Sigma$-field.

The physical consequences of the our noncommutative extension of standard general relativity remain to be studied. An interesting possibility seems to be the study of inflation in
this model. It is well known, in a very different physical setting, that the trace anomaly that leads to higher derivative corrections in the corresponding effective action, could produce inflation \[28, 29\].

Finally, the results presented here, can be regarded as a preliminary step for the construction of a noncommutative version of Ashtekars hamiltonian formulation through a noncommutative Legendre transformation. Moreover, the computation of noncommutative gravitational effects can be done, for instance, from the corrections to known metrics. The details of these works will be reported elsewhere. Eventually, it would be interesting to search for a quantization of the results obtained in the present paper and proceed to find the corresponding loop quantum gravity.

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