A NOTE ON p-ADIC q-EULER MEASURE
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Abstract

In this paper, we will investigate some interesting properties of the modified q-Euler numbers and polynomials. The main purpose of this paper is to construct p-adic q-Euler measure on \( \mathbb{Z}_p \).

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1. Introduction, Definitions and Notations

Let \( p \) be a fixed odd prime. Throughout this paper \( \mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C} \) and \( \mathbb{C}_p \) will respectively denote the ring of \( p \)-adic rational integers, the field of \( p \)-adic rational numbers, the complex number field and the completion of the algebraic closure of \( \mathbb{Q}_p \). Let \( v_p \) be the normalized exponential valuation of \( \mathbb{C}_p \) with \( |p|_p = p^{-v_p(p)} = 1/p \), see [1], [2], [3]. When one talks of \( q \)-extensions, \( q \) is variously considered as an indeterminate, a complex \( q \in \mathbb{C} \), or a \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \), one normally assumes \(|q| < 1 \). If \( q \in \mathbb{C}_p \), then we assume \(|q-1| < p^{-v_p(q)} \), so that \( q^x = \exp(x \log q) \) for \(|x|_p \leq 1 \), see [4], [5], [6]. It was well known that Euler numbers are defined by

\[
\frac{2}{e^x + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad \text{see [1], [2], [3], [4], [5], [6]}. \]

In the recent paper [3], the q-extension of Euler numbers are defined inductively by
$E_{0,q} = 1$, $q(qE + 1)^n + E_{n,q} = \begin{cases} [2]_q & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$

with the usual convention of replacing $E^m$ by $E_{m,q}$. In [1], the definition of modified $q$-Euler numbers $E_{0,q}$ is introduced by

$$
E_{0,q} = \frac{[2]_q}{2}, \ (qE + 1)^k - E_{k,q} = \begin{cases} [2]_q & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}
$$

with the usual convention of replacing $E^i$ by $E_{i,q}$.

For a fixed positive integer $d$ with $(p, d) = 1$, set

$$
X_d = \lim_{N \to \infty} \mathbb{Z}/dp^N\mathbb{Z}, \quad X_1 = \mathbb{Z}_p,
$$

$$
X^* = \bigcup_{0 < a < dp} (a + dp\mathbb{Z}_p),
$$

$$
a + dp^N\mathbb{Z}_p = \{ x \in X : x \equiv a \pmod{dp^N} \},
$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$, see [6].

We say that $f$ is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $f \in UD(\mathbb{Z}_p)$, if the difference quotient

$$
F_f(x, y) = \frac{f(x) - f(y)}{x - y}
$$

has a limit $f'(a)$ as $(x, y) \to (a, a)$. For $f \in UD(\mathbb{Z}_p)$, an invariant $p$-adic $q$-integral was defined by

$$
I_q(f) = \int_{\mathbb{Z}_p} f(x)d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x)g^x, \quad \text{see [8].}
$$

The $q$-extension of $n \in \mathbb{N}$ is defined by

$$
[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + q^2 + \ldots + q^{n-1},
$$

and

$$
[n]_{-q} = \frac{1 - (-q)^n}{1 - (-q)} = 1 - q + q^2 - \ldots + (-q)^{n-1}, \quad \text{see [3], [4], [5].}
$$

The modified $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by
where \( d\mu_{-q}(x) = \lim_{q \to -q} d\mu_q(x) \). In this paper, we will investigate some interesting properties of the modified \( q \)-Euler numbers and polynomials. The purpose of this paper is to construct \( p \)-adic \( q \)-Euler measure on \( \mathbb{Z}_p \).

2. \( p \)-adic \( q \)-Euler Measure on \( \mathbb{Z}_p \)

\( q \)-Euler numbers are known by

\[
\mathcal{E}_{0,q} = \frac{[2]_q}{2}, \quad (q\mathcal{E} + 1)^n - \mathcal{E}_{n,q} = \begin{cases} [2]_q & \text{if } n = 0 \\ 0 & \text{if } n > 0 \end{cases}
\]

with the usual convention of replacing \( \mathcal{E}^i \) by \( \mathcal{E}_{i,q} \), see [1]. It was known that the \( q \)-Euler numbers can be represented by \( p \)-adic \( q \)-integrals on \( \mathbb{Z}_p \) as follows:

\[
\mathcal{E}_{n,q} = \int_{\mathbb{Z}_p} q^{-x} [x]^n d\mu_{-q}(x) = [2]_q \left( \frac{1}{1 - q} \right)^n \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{1}{1 + q^l}, \text{ see [1]}. 
\]

We now also consider the \( q \)-Euler polynomials as follows:

\[
\mathcal{E}_{n,q}(x) = \int_{\mathbb{Z}_p} q^{-x} [t + x]^n d\mu_{-q}(t) = \sum_{l=0}^{n} \binom{n}{l} q^l \mathcal{E}_{l,q} [x]^{n-l}.
\]

Thus, we note that

\[
\mathcal{E}_{n,q}(x) = [d]_q \sum_{a=0}^{d-1} (-1)^a \mathcal{E}_{n,q^d} \left( \frac{n + a}{d} \right), \text{ see [1]}. \tag{2.1}
\]

Let \( \chi \) be the Dirichlet’s character with odd conductor \( d \in \mathbb{N} \), and let \( F_{\chi,q}(t) \) be the generating function of \( \mathcal{E}_{n,\chi,q} \) as follows:

\[
F_{\chi,q}(t) = [2]_q \sum_{n=0}^{\infty} (-1)^n \chi(n) t^n q^n = \sum_{n=0}^{\infty} \mathcal{E}_{n,\chi,q} \frac{t^n}{n!}, \text{ see [1]}. \tag{2.2}
\]

From (2.2) we derive

\[
\mathcal{E}_{n,\chi,q} = [d]_q \sum_{a=0}^{d-1} (-1)^a \chi(a) \mathcal{E}_{n,q^d} \left( \frac{a}{d} \right). \tag{2.3}
\]

For any positive integers \( N, k \) and \( d \) (odd), let \( \mu^*_k = \mu_{k,q;E} \) be defined by

\[
\mu^*_k(a + dp^N \mathbb{Z}_p) = (-1)^a [dp^N]^k \frac{[2]_q}{[2]_q + a} \mathcal{E}_{k,q^d p^N} \left( \frac{a}{dp^N} \right). \tag{2.4}
\]
Then we see that

\[
\sum_{i=0}^{p-1} \mu_k^*(a + idp^N + dp^{N+1}Z_p)
\]

\[
= [dp^{N+1}]_q [2]_q [2]_q dp^{N+1} \sum_{i=0}^{p-1} (-1)^{a+idp^N} \mathcal{E}_{k,q dp^N+1} \left( \frac{a + idp^N}{dp^{N+1}} \right)
\]

\[
= (-1)^a [dp^N+1]_q [2]_q \sum_{i=0}^{p-1} (-1)^i \mathcal{E}_{k,(q dp^N)p} \left( \frac{a + dp^N + i}{p} \right)
\]

\[
= (-1)^a [dp^N]_q [2]_q \mathcal{E}_{k,q dp^N} \left( \frac{a}{dp^N} \right)
\]

\[
= \mu_k^*(a + dp^N Z_p).
\]

It is easy to see that \(|\mu_k^*| \leq M\), for some constant \(M\). Therefore we obtain the following:

**Theorem 1.** For any positive integers \(N\), \(k\) and \(d\) (odd), let \(\mu_k^* = \mu_{k,q,E}^*\) be defined by

\[
\mu_k^*(a + dp^N Z_p) = (-1)^a [dp^N]_q [2]_q \mathcal{E}_{k,q dp^N} \left( \frac{a}{dp^N} \right).
\]

Then \(\mu_k^*\) is a measure on \(X\).

From the definition of \(\mu_k^*\), we derive the following:

\[
\int_X \chi(x) d\mu_k^*(x) = \lim_{N \to \infty} \sum_{a=0}^{dp^N-1} \chi(a) \mu_k^*(a + dp^N Z_p)
\]

\[
= \lim_{N \to \infty} \sum_{x=0}^{dp^N-1} (-1)^x \chi(x) [dp^N]_q [2]_q \mathcal{E}_{k,q dp^N} \left( \frac{x}{dp^N} \right)
\]

\[
= \lim_{N \to \infty} [d]_q [2]_q [2]_q \sum_{x=0}^{dp^N-1} (-1)^x \chi(x) \mathcal{E}_{k,q dp^N} \left( \frac{x}{dp^N} \right)
\]

\[
= [d]_q [2]_q \lim_{N \to \infty} \left\{ \sum_{a=0}^{d-1} \left[ \frac{2}{(q^a)^p} \right]_q \sum_{x=0}^{p^N-1} (-1)^a \chi(a) \mathcal{E}_{k,(q^a)p^N} \left( \frac{a + x}{p^N} \right) \right\}
\]

\[
= [d]_q [2]_q \lim_{N \to \infty} \left\{ \sum_{a=0}^{d-1} \left[ \frac{2}{(q^a)^p} \right]_q \sum_{x=0}^{p^N-1} (-1)^a \chi(a) \mathcal{E}_{k,(q^a)p^N} \left( \frac{a + x}{p^N} \right) \right\}
\]
and by equation (2.1), this becomes

\[ \int \chi(x) d\mu^*_k(x) = \left[ d^k \frac{[2]}{[2]_q} \sum_{a=0}^{d-1} (-1)^a \chi(a) \mathcal{E}_{n,q^a} \left( \frac{a}{d} \right) \right]. \]

By using (2.3) and (2.5), we obtain the following:

**Theorem 2.** For any positive integer \( k \), we have

\[ \int \chi(x) d\mu^*_k(x) = \mathcal{E}_{k,x,q}. \]

From Theorem 1 and equation (2.1), we note that

\[
\begin{align*}
\mu^*_k(a + dp^N\mathbb{Z}_p) & = (-1)^a \frac{[2]}{[2]_q} [dp^N]^k \mathcal{E}_{k,q^a N} \left( \frac{a}{dp^N} \right) \\
& = \left( \sum_{l=0}^{k} \binom{k}{l} q^{dp^N} \frac{dp^N}{dp^N} \mathcal{E}_{l,q^a N} \left( \frac{a}{dp^N} \right) \right) \left(-1\right)^a \frac{[2]}{[2]_q} [dp^N]^k \\
& = \mathcal{E}_{0,q^a N} \left(-1\right)^a \frac{[2]}{[2]_q} [dp^N]^k + \left(-1\right)^a \frac{[2]}{[2]_q} [dp^N]^k \sum_{l=1}^{k} \binom{k}{l} q^l \\
& = \mathcal{E}_{0,q^a N} \left(-1\right)^a \frac{[2]}{[2]_q} [dp^N]^k + \left(-1\right)^a \frac{[2]}{[2]_q} [dp^N]^k \sum_{l=1}^{k} \binom{k}{l} q^l.
\end{align*}
\]

Thus, we have

\[
\lim_{N \to \infty} \mu^*_k(a + dp^N\mathbb{Z}_p) = \frac{[2]}{2} (-1)^a [a]^k_q \\
= q^a \frac{[2]}{2} (-1)^a q^{-a} [a]^k_q \\
= q^{-a} [a]^k_q \lim_{N \to \infty} \mu_{-q} (a + dp^N\mathbb{Z}_p),
\]
where $d$ is a positive odd integer. Therefore we obtain the following:

**Theorem 3.** For any positive integer $k$, we have

$$q^{-x}[x]^k_d d\mu_{-q}(x) = d\mu^*_k(x).$$

**Corollary 1.** Let $k$ be a positive integer. Then we have

$$\mathcal{E}_{k,\chi,q} = \int_X \chi(x)d\mu^*_k(x) = \int_X \chi(x)q^{-x}[x]^k_d d\mu_{-q}(x).$$

Let $\overline{d} = (d, p)$ be the least common multiple of the conductor $d$ of $X$ and $p$, and let $\mathcal{E}_{n,\chi,q}$ denote the $n$-th generalized $q$-Euler number belonging to the character $\chi$. Then we have the $q$-analogue form of Witt’s formula in the cyclotomic field $\mathbb{Q}_p(\chi)$ as follows:

For all $n \geq 0$, we have

$$\mathcal{E}_{n,\chi,q} = \lim_{\rho \to \infty} \left[ \frac{[2]_q}{2} \sum_{1 \leq x \leq \overline{d}_p \rho} (-1)^x \chi(x)[x]^n_q \right]. \quad (2.6)$$

Herein as usual we set $\chi(x) = 0$ if $x$ is not prime to the conductor $d$. From (2.6) we derive

$$\mathcal{E}_{n,\chi,q} = \lim_{\rho \to \infty} \left[ \frac{[2]_q}{2} \sum_{1 \leq x \leq \overline{d}_p \rho} (-1)^x \chi(x)[x]^n_q \right] + \left[ \frac{[2]_q}{2} \right] \lim_{\rho \to \infty} \left[ \frac{[2]_q}{2} \sum_{1 \leq x \leq \overline{d}_p \rho} (-1)^x \chi(x)[x]^n_q \right],$$

where *means taking the sum over the rational integers prime to $p$ in the given range. Thus we have

$$\mathcal{E}_{n,\chi,q} = \lim_{\rho \to \infty} \left[ \frac{[2]_q}{2} \sum_{1 \leq x \leq \overline{d}_p \rho} (-1)^x \chi(x)[x]^n_q \right] + \left[ \frac{[2]_q}{2} \right] \chi(p)[p]_q^n \sum_{1 \leq x \leq \overline{d}_p \rho} (-1)^x \chi(x)[x]^n_q \mathcal{E}_{n,\chi,q},$$

that is,
\[ \mathcal{E}_{n, \chi, q} = \frac{[2]_q}{[2]_{qp}} \chi(p)[p]_q^n \mathcal{E}_{n, \chi, q^p} = \lim_{\rho \to \infty} \frac{[2]_q}{2} \sum_{1 \leq x \leq d_{\rho^p}} (-1)^x \chi(x)[x]_q^n \]
\[ = \int_{X^*} (-1)^x \chi w^n(x) \langle x \rangle_q^n d\mu_\chi(x), \]

where \( \langle x \rangle_q = \frac{[x]_q}{w(x)} \), and \( w(x) \) is the Teichmüller character.

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