ON THE CREPANT RESOLUTION CONJECTURE IN THE LOCAL CASE

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Abstract. In this paper we analyze four examples of birational transformations between local Calabi–Yau 3-folds: two crepant resolutions, a crepant partial resolution, and a flop. We study the effect of these transformations on genus-zero Gromov–Witten invariants, proving the Coates–Corti–Iritani–Tseng/Ruan form of the Crepant Resolution Conjecture in each case. Our results suggest that this form of the Crepant Resolution Conjecture may also hold for more general crepant birational transformations. They also suggest that Ruan’s original Crepant Resolution Conjecture should be modified, by including appropriate “quantum corrections”, and that there is no straightforward generalization of either Ruan’s original Conjecture or the Cohomological Crepant Resolution Conjecture to the case of crepant partial resolutions. Our methods are based on mirror symmetry for toric orbifolds.

1. Introduction

Suppose that \(X\) is an algebraic orbifold and that \(Y\) is an orbifold or algebraic variety which is birational to \(X\). It is natural to try to understand the relationship between the quantum cohomology of \(X\) and that of \(Y\). In this paper we analyze four examples of this situation — two crepant resolutions, a crepant partial resolution, and a flop — which together exhibit some of the range of phenomena which can occur. The spaces that we consider are local Calabi–Yau 3-folds. Our methods are based on mirror symmetry for toric orbifolds.

The small quantum cohomology \(\text{QC}(X)\) of an orbifold \(X\) is a family of algebras depending on so-called quantum parameters. It arises in string theory as the chiral ring of the topological A-model with target space \(X\); from this point of view the quantum parameters are co-ordinates on the stringy Kähler moduli space \(\mathcal{M}\) of \(X\). It is expected that the chiral rings form a family of algebras over the whole of \(\mathcal{M}\) and that this family coincides with \(\text{QC}(X)\) near the so-called large radius limit point \(0_X\) of \(\mathcal{M}\). Other target spaces \(Y\) which are birational to \(X\) are expected to correspond to other limit points \(0_Y\) of \(\mathcal{M}\); this suggests that if \(X\) and \(Y\) are birational then the relationship between \(\text{QC}(X)\) and \(\text{QC}(Y)\) should involve analytic continuation in the quantum parameters.

A precise mathematical formulation of this was given by Ruan in his influential Crepant Resolution Conjecture. To state it, we need to choose co-ordinates on (patches of) the stringy Kähler moduli space \(\mathcal{M}\). Suppose that \(Y \to X\) is a crepant resolution or partial resolution of the coarse moduli space \(X\). A choice of basis \(\varphi_1, \ldots, \varphi_r\) for \(H^2(Y; \mathbb{C})\) defines co-ordinates \(t_i, 1 \leq i \leq r\) on \(H^2(Y; \mathbb{C})\), and hence defines exponentiated flat co-ordinates \(q_i = e^{t_i}, 1 \leq i \leq r\), on a neighbourhood of \(0_Y\) in \(\mathcal{M}\). Similarly a choice of basis \(\phi_1, \ldots, \phi_s\) for \(H^2(X; \mathbb{C})\) defines exponentiated flat co-ordinates \(u_i, 1 \leq i \leq s\), near \(0_X\) in \(\mathcal{M}\). We take \(\varphi_1, \ldots, \varphi_r\) to be primitive integer vectors on the rays of the Kähler cone for \(Y\) and \(\phi_1, \ldots, \phi_s\) to be primitive integer vectors on the rays of the Kähler cone for \(X\). Because \(Y \to X\) is a (partial) resolution there is a natural embedding \(j : H^2(X; \mathbb{Q}) \to H^2(Y; \mathbb{Q})\) which identifies the Kähler cone for \(X\) with a face of the Kähler cone for \(Y\). We can therefore insist that \(j(\phi_i) = r_i \varphi_i\) for some rational numbers \(r_i, 1 \leq i \leq s\). The presence of these rational numbers reflects the fact that the embedding \(j\) does not in general identify the integer lattices in \(H^2(X; \mathbb{Q})\) and \(H^2(Y; \mathbb{Q})\).

The variables \(q_1, \ldots, q_r\) and \(u_1, \ldots, u_s\) thus defined are the quantum parameters: the parameters on which the small quantum cohomology algebras \(\text{QC}(Y)\) and respectively \(\text{QC}(X)\) depend. Ruan’s Conjecture asserts that if \(Y \to X\) is a crepant resolution then there are roots of unity \(\omega_i, 1 \leq i \leq r\), and a choice of path of analytic continuation such that \(\text{QC}(X)\) is isomorphic to the algebra obtained from \(\text{QC}(Y)\) by analytic continuation in the \(q_i\) followed by the change of variables

\[
q_i = \begin{cases} 
\omega_i u_i^{r_i} & 1 \leq i \leq s \\
\omega_i & s < i \leq r.
\end{cases}
\]

One consequence of this is the Cohomological Crepant Resolution Conjecture (CCRC) [15], which asserts that the Chen–Ruan orbifold cohomology algebra of \(X\) is isomorphic to the algebra obtained from the
small quantum cohomology of \( \mathcal{Y} \) by analytic continuation in the \( q_i \) followed by the change of variables

\[
q_i = \begin{cases} 
0 & 1 \leq i \leq s \\
\omega_i & s < i \leq r.
\end{cases}
\]

These conjectures have been verified in a number of examples \([1, 8, 10, 12, 15, 20, 25, 33, 17]\).

Recent progress in both mathematics and physics suggests, however, that Ruan’s Conjecture should be modified: that it is not an accurate reflection of the physical picture. In essence this is because when we identify the family of algebras over \( \mathcal{M} \) (i.e. the chiral rings) with \( \text{QC}(\mathcal{X}) \) and \( \text{QC}(\mathcal{Y}) \), we need to use exponentiated flat co-ordinates near \( 0_\mathcal{X} \) and \( 0_\mathcal{Y} \). And even though the family of algebras near \( 0_\mathcal{X} \) is related to the family of algebras near \( 0_\mathcal{Y} \) by analytic continuation, the analytic continuation of exponentiated flat co-ordinates near \( 0_\mathcal{X} \) \textit{will not in general give exponentiated flat co-ordinates near} \( 0_\mathcal{Y} \). Thus we need also to analyze how the two co-ordinate systems are related. In some examples this has been done by Aganagic–Bouchard–Klemm \([3]\) using \textit{ad hoc} methods and by Coates–Corti–Iritani–Tseng \([18]\) in a more systematic fashion; all of their examples satisfy the original Ruan Conjecture.

One contribution of this paper is to give the first example (Example II below) of a crepant resolution by analytic continuation in the \( \omega_i \) sense to analytically continue them, and we conjectured the existence of a linear symplectic isomorphism \( \text{QC}(\mathcal{X}) \rightarrow \text{QC}(\mathcal{Y}) \) satisfying some quite restrictive conditions such that after analytic continuation we have \( \mathcal{U}(\mathcal{L}_\mathcal{X}) = \mathcal{L}_\mathcal{Y} \). We also proved our conjecture when \( \mathcal{X} \) is one of the weighted projective spaces \( \mathbb{P}(1,1,2) \) or \( \mathbb{P}(1,1,1,3) \) and \( \mathcal{Y} \rightarrow \mathcal{X} \) is a crepant resolution.

Our conjecture has consequences for quantum cohomology: it implies both the Cohomological Crepant Resolution Conjecture and a modified version of Ruan’s Conjecture, each with the caveat that we must allow the quantities \( \omega_i \) to be arbitrary constants rather than roots of unity. (In the examples below the \( \omega_i \) turn out to be roots of unity and so the caveat disappears; Iritani has suggested an attractive conceptual reason for this to be true in general \([38]\).) The modified version of Ruan’s Conjecture has an additional hypothesis, that \( \mathcal{X} \) be \textit{semi-positive}, and replaces the change of variables \((\text{I})\) by \( q_i = f_i(u_1, \ldots, u_s) \) where

\[
f_i(u_1, \ldots, u_r) = \begin{cases} 
\omega_i u_i^{r_i} \text{ + higher order terms in } u_1, \ldots, u_r & 1 \leq i \leq s \\
\omega_i \text{ + higher order terms in } u_1, \ldots, u_r & s < i \leq r.
\end{cases}
\]

Thus we get a “quantum corrected” version of Ruan’s original conjecture.

In this paper we consider four examples:

(I) the crepant resolution of \( \mathcal{X} = \mathbb{C}^3/\mathbb{Z}_3 \), where \( \mathbb{Z}_3 \) acts with weights \((1,1,1)\);

(II) the crepant resolution of the canonical bundle \( \mathcal{X} = K_{\mathbb{P}(1,1,3)} \);

(III) the crepant partial resolution of \( \mathcal{X} = \mathbb{C}^3/\mathbb{Z}_5 \), where \( \mathbb{Z}_5 \) acts with weights \((1,1,3)\);

(IV) a toric flop with \( \mathcal{X} = \mathcal{O}_{\mathbb{P}(1,2)}(-1)^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \).

We prove the Coates–Corti–Iritani–Tseng/Ruan Crepant Resolution Conjecture in each case. This has implications as follows:

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Example} & \text{CCIT/Ruan} & \text{CCRC} & \text{original Ruan} & \text{modified Ruan} \\
\hline
\text{I} & \checkmark & \checkmark & \checkmark & \checkmark \\
\text{II} & \checkmark & \checkmark & ? & \checkmark \\
\text{III} & \checkmark & ? & ? & ? \\
\text{IV} & \checkmark & \text{n/a} & \text{n/a} & \text{n/a} \\
\hline
\end{array}
\]

I expect that wherever there is a “?” in this table, the corresponding conjecture fails to hold, so that for example the original form of Ruan’s Conjecture fails in Example II and the modified form of Ruan’s Conjecture fails in Example III. It is difficult to prove these assertions, as this would involve ruling out

\[1\text{Similar ideas occurred in unpublished work of Ruan; an expository account can be found in \([23]\).}\]
every possible choice of path of analytic continuation and all choices of roots of unity, but I know of no reason to expect these conjectures to hold.

In forthcoming work, Iritani will prove our form of the Crepant Resolution Conjecture for all crepant birational transformations between toric Deligne–Mumford stacks. His method uses the full force of the mirror Landau–Ginzburg model, the variation of semi-infinite Hodge structure [5, 8, 39] associated to it, and the mirror theorem for toric Deligne–Mumford stacks [21]. Since all of our examples are included in his discussion, it is natural to ask: “what is the point of this paper?” The discussion here has quite modest goals, and is meant to illustrate four points. Firstly, these questions are not difficult. If $X$ is a toric orbifold $\mathcal{X}$ and $Y \to X$ is a crepant resolution then the relationship between the quantum cohomology of $X$ and that of $\mathcal{Y}$ can be determined systematically, using well-understood methods from toric mirror symmetry. Secondly, our form of the Crepant Resolution Conjecture may also hold, without significant change, for more general crepant birational transformations: we see this here for two crepant partial resolutions and a flop. Thirdly, the method of proof described here also applies without change to the more general crepant toric situation. Finally, it seems likely that no naive modification of Ruan’s original conjecture holds true; we discuss this further in the next paragraph. Along the way, we will see two things which were perhaps already obvious: that Givental-style mirror theorems are well-adapted to the analysis of toric birational transformations, and that the methods of [18] are applicable to the (local) Calabi–Yau examples which are of greatest interest to physicists [8].

The original conjecture of Ruan has an attractive simplicity, and one might therefore ask whether our formulation of the Crepant Resolution Conjecture is unnecessarily complicated and whether some simpler statement holds [14]. The examples below constitute some evidence that the answer to these questions is “no”. In Example II below we see that quantum corrections to Ruan’s original conjecture are probably necessary, and in Example III the situation is even worse: there is probably not even a generalization of the Cohomological Crepant Resolution Conjecture to partial resolutions which involves only small (rather than big) quantum cohomology. This is related to the absence of a Divisor Equation for degree-two classes from the twisted sectors, and is discussed further in Section 6.

A Note on the Bryan–Graber Conjecture. Jim Bryan and Tom Graber have recently given a generalization of Ruan’s Crepant Resolution Conjecture which applies to big quantum cohomology, rather than just small quantum cohomology, under the assumption that the orbifold $\mathcal{X}$ involved satisfies a Hard Lefschetz condition on Chen–Ruan orbifold cohomology. We will not consider this here, as none of our examples satisfy the Hard Lefschetz condition. But as the conclusion of the Bryan–Graber Conjecture implies the Ruan Conjecture, Example II can be thought of as further evidence that the Bryan–Graber Conjecture probably fails to hold without the Hard Lefschetz assumption: see [18] for more on this.

Conventions. We will assume that the reader is familiar with the Gromov–Witten theory of orbifolds. This theory is constructed in [1, 2, 15, 16]: a rapid overview can be found in Section 2. We work in the algebraic category, so for us “orbifold” means “smooth algebraic Deligne–Mumford stack over $\mathbb{C}$”. All of our examples are non-compact, but they carry the action of a torus $T = \mathbb{C}^\times$ such that the $T$-fixed locus is compact. We therefore work throughout with $T$-equivariant Gromov–Witten invariants, which in this setting behave much as the Gromov–Witten invariants of compact orbifolds (see e.g. [11]), and with $T$-equivariant Chen–Ruan orbifold cohomology. We always take the product of $T$-equivariant Chen–Ruan classes using the Chen–Ruan product; when we want to emphasize this, we will write the product as $\cup_T$. The degree of a Chen–Ruan class always means its orbifold or age-shifted degree.

An expository account of our Crepant Resolution Conjecture and its consequences can be found in [21]. The reader should take care when comparing the discussion in this paper with those in [11, 21], as here we measure the degrees of orbifold curves using a basis of degree-two cohomology classes chosen as above, whereas there the authors use a so-called positive basis for $H_2$. Our choice of degree conventions fits well with toric geometry, and this will be important below, but we pay a price for our choice: the presence of the rational numbers $r_i$ described above.

Outline of the Paper. In Section 2 we fix notation and give a precise description of the conjecture which we will prove. In Section 3 we collect various preparatory results, as well as describing how our conjecture implies versions of Ruan’s Conjecture and the Cohomological Crepant Resolution Conjecture. Examples I–IV are in Sections 4–7 respectively.

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2. Statement of the Conjecture

In this section we give a precise statement of the conjecture that we will prove. Before we do so, we describe our general setup and fix notation.

**General Setup.** Let $X$ be a Gorenstein orbifold with projective coarse moduli space $X$ and let $\pi : Y \to X$ be a crepant resolution. Assume that $X$, $X$, and $Y$ carry actions of a torus $T = \mathbb{C}^\times$ such that both $\pi$ and the structure map $\pi : X \to X$ are $T$-equivariant and such that the $T$-fixed loci on $X$ and $Y$ are compact. Let $\mathbb{C}[\lambda]$ denote the $T$-equivariant cohomology of a point, where $\lambda$ is the first Chern class of the line bundle $O(1) \to \mathbb{C}P^\infty$, and let $\mathbb{C}(\lambda)$ be its field of fractions. Write $H(X) := H^*_{\text{CR, T}}(X; \mathbb{C}) \otimes \mathbb{C}(\lambda)$ for the localized $T$-equivariant Chen–Ruan orbifold cohomology of $X$, and $H(Y) := H^*_{\text{T, Y}}(Y; \mathbb{C}) \otimes \mathbb{C}(\lambda)$ for the localized $T$-equivariant cohomology of $Y$. We work throughout over the field $\mathbb{C}(\lambda)$. The $\mathbb{C}(\lambda)$-vector spaces $H(X)$ and $H(Y)$ carry non-degenerate inner products, given by

\[(\alpha, \beta)_X := \int_{\mathcal{I}X^T} i^* (\alpha \cup I^* \beta) \quad \text{and} \quad (\alpha, \beta)_Y := \int_{\mathcal{Y}^T} j^* (\alpha \cup \beta)\]

where $I$ is the canonical involution on the inertia stack $\mathcal{I}X$ of $X$; $i : \mathcal{I}X^T \to \mathcal{I}X$ and $j : \mathcal{Y}^T \to \mathcal{Y}$ are the inclusions of the $T$-fixed loci in $\mathcal{I}X$ and $\mathcal{Y}$ respectively; $N_{\mathcal{I}X^T/\mathcal{I}X}$ and $N_{\mathcal{Y}^T/\mathcal{Y}}$ are the normal bundles to the $T$-fixed loci; and $e$ is the $T$-equivariant Euler class. Note that the $T$-equivariant Euler classes are invertible over $\mathbb{C}(\lambda)$.

**The Symplectic Vector Space.** In what follows write $Z$ for either $X$ or $Y$, and write $Z$ for the coarse moduli space of $Z$ (i.e. for either $X$ or $Y$). Introduce the symplectic vector space

\[H_Z := H(Z) \otimes \mathbb{C}((z^{-1}))\]

the vector space

\[\Omega_Z(f, g) := \text{Res}_{z=0} (f(-z), g(z)) \quad \text{the symplectic form}\]

and set $H_Z^* := H(Z) \otimes \mathbb{C}[z]$, $H_Z^{-} := z^{-1}H(Z) \otimes \mathbb{C}[z^{-1}]$. The polarization $H_Z = H_Z^* \oplus H_Z^-$ identifies $H_Z$ with the cotangent bundle $T^*H_Z$. We regard $H_Z$ as a graded vector space where $\deg z = 2$.

**Degrees and Novikov Variables.** Fix a basis $\omega_1, \ldots, \omega_s$ for $H^2(X; \mathbb{Q})$ consisting of primitive integer vectors on the rays of the Kähler cone for $X$, and a basis $\omega'_1, \ldots, \omega'_t$ for $H^2(Y; \mathbb{Q})$ consisting of primitive integer vectors on the rays of the Kähler cone for $Y$. Note that $H^2(X; \mathbb{Q})$ is canonically isomorphic to $H^2(X; \mathbb{Z})$, so we can regard $\omega_1, \ldots, \omega_s$ as cohomology classes on $X$, and in our situation we can always insist that $\pi^* \omega_i = r_i \omega'_i$, $1 \leq i \leq s$, for some rational numbers $r_i$. We measure the degrees of orbifold curves using the bases $\omega_i$ and $\omega'_i$. Recall that a stable map $f : C \to Z$ from an orbifold curve to $Z$ has a well-defined degree in the free part

\[H_2(Z; \mathbb{Z})_{\text{free}} := H_2(Z; \mathbb{Z}) / H_2(Z; \mathbb{Z})_{\text{tors}}\]

of $H_2(Z; \mathbb{Z})$; we write $\text{Eff}(Z) \subset H_2(Z; \mathbb{Z})_{\text{free}}$ for the set of degrees of stable maps from orbifold curves to $Z$. Given an element $d \in \text{Eff}(Z)$, set $d_i = (d, \omega_i)$ if $Z = X$ and $d_i = (d, \omega'_i)$ if $Z = Y$. Note that the $d_i$ here are in general rational numbers. Define $Q^d := Q^d_1 \cdots Q^d_s$ where $d \in \text{Eff}(X)$ and $Q^d := Q^d_1 \cdots Q^d_r$ where $d' \in \text{Eff}(Y)$. Here $Q_1, Q_2, \ldots$ are formal variables called Novikov variables; the number of Novikov variables associated with $Z$ is $b_2(Z)$, the second Betti number of $Z$.

**Bases and Darboux Co-ordinates.** We fix $\mathbb{C}(\lambda)$-bases $\phi_0, \ldots, \phi_N$ and $\phi^0, \ldots, \phi^N$ for $H(X)$ such that

(a) $\phi_0$ is the identity element $1_X \in H(X)$;
(b) $\phi_1, \phi_2, \ldots, \phi_s$ are lifts to $T$-equivariant cohomology of $\omega_1, \omega_2, \ldots, \omega_s$;
(c) $\phi_i, \phi^j)_{X} = \delta_i^j$;
and $\mathbb{C}(\lambda)$-bases $\varphi_0, \ldots, \varphi_N$ and $\varphi^0, \ldots, \varphi^N$ for $H(Y)$ such that

(d) $\varphi_0$ is the identity element $1_Y \in H(Y)$;
(e) $\varphi_1, \varphi_2, \ldots, \varphi_r$ are lifts to $T$-equivariant cohomology of $\omega'_1, \omega'_2, \ldots, \omega'_r$;
(f) $\varphi_i, \varphi^j)_{Y} = \delta_i^j$. 

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Conditions (b) and (c) here will be useful below when we discuss the Divisor Equation. Write

$$
\Phi_i = \begin{cases}
\phi_i & \text{if } Z = X \\
\varphi_i & \text{if } Z = Y
\end{cases}
\text{ and } 
\Phi^i = \begin{cases}
\phi^i & \text{if } Z = X \\
\varphi^i & \text{if } Z = Y.
\end{cases}
$$

Then

$$
\sum_{k \geq 0} q_k^a \Phi_a = k^{b_k} + \sum_{l \geq 0} p_{\beta,l} \Phi^\beta (-z)^{-1-l}
$$

(2)
gives a Darboux co-ordinate system \( \{ q_{a,k}, p_{\beta,l} \} \) on \( \mathcal{H}_Z \); here and henceforth we use the summation convention on Greek indices, summing repeated Greek (but not Roman) indices over the range \( 0, 1, \ldots, N \).

**Gromov–Witten Invariants.** We use correlator notation for \( T \)-equivariant Gromov–Witten invariants of \( Z \), writing

$$
\langle \alpha_1 \psi_1^{i_1}, \ldots, \alpha_n \psi_n^{i_n} \rangle^{Z}_{0, n, d} = \int_{[Z_{0,n,d}]^{vir}} \prod_{k=1}^{n} \ev^*_k(\alpha_k) \cdot \psi^{i_k}
$$

(3)
where \( \alpha_1, \ldots, \alpha_n \) are elements of \( H(Z) \) and \( i_1, \ldots, i_n \) are non-negative integers. The cohomology classes \( \psi_1, \ldots, \psi_n \) here are the first Chern classes of the universal cotangent line bundles on the moduli space \( Z_{0,n,d} \) of genus-zero \( n \)-pointed stable maps to \( Z \) of degree \( d \in \Eff(Z) \). The integral denotes the cap product with the \( T \)-equivariant virtual fundamental class of \( Z_{0,n,d} \); we discuss this further in the next paragraph. The right-hand side of equation (3) is defined in §8.3 of [2] where it is denoted \( \langle \tau_{i_1}(\alpha_1), \ldots, \tau_{i_n}(\alpha_n) \rangle_{0, d} \); our choice of notation allows compact expressions for many important quantities, such as

$$
\left( \frac{\alpha}{z - \psi} \right)^Z_{0,1,d}
$$

for

$$
\sum_{m \geq 0} \frac{1}{z^{m+1}} \langle \alpha \psi^m \rangle^{Z}_{0,1,d},
$$
as correlators are multilinear in their entries.

**Twisted Gromov–Witten Invariants.** In most of the examples we consider below, \( Z \) will be the total space of a concave vector bundle \( \mathcal{E} \) over a compact orbifold (or manifold) \( B \), and the \( T \)-action on \( Z \) will rotate the fibers of \( \mathcal{E} \) and cover the trivial action on \( B \). That \( \mathcal{E} \) is concave means that \( H^0(\mathcal{E}, f^* \mathcal{E}) = 0 \) for all stable maps \( f : \mathcal{C} \to B \) of non-zero degree. This implies that stable maps to \( \mathcal{E} \) of non-zero degree all land in the zero section and so, for \( d \neq 0 \), the moduli space \( Z_{0,n,d} \) coincides as a scheme with \( B_{0,n,d} \). The natural obstruction theories on \( Z_{0,n,d} \) and \( B_{0,n,d} \) differ, though, and the \( T \)-equivariant virtual fundamental classes satisfy

$$
[Z_{0,n,d}]^{vir} = [B_{0,n,d}]^{vir} \cap \mathbb{e}(\text{Obs}_{0,n,d})
$$

where \( \mathbb{e} \) is the \( T \)-equivariant Euler class and \( \text{Obs}_{0,n,d} \) is the vector bundle over \( B_{0,n,d} \) with fiber at a stable map \( f : \mathcal{C} \to B \) equal to \( H^1(\mathcal{C}, f^* \mathcal{E}) \). Thus

$$
\int_{[Z_{0,n,d}]^{vir}} \langle \cdots \rangle = \int_{[B_{0,n,d}]^{vir}} \langle \cdots \rangle \cup \mathbb{e}(\text{Obs}_{0,n,d}).
$$

This means that Gromov–Witten invariants of \( Z \) coincide with twisted Gromov–Witten invariants [19][23] of \( B \) where the twisting characteristic class is the inverse \( T \)-equivariant Euler class \( \mathbb{e}^{-1} \) and the twisting bundle is \( \mathcal{E} \); this is explained in detail in [19]. Results of [19] allow us to compute these twisted Gromov–Witten invariants in terms of the ordinary Gromov–Witten invariants of \( B \), a fact which we exploit repeatedly below.

In the exceptional case \( d = 0 \), the moduli space \( Z_{0,n,d} \) is non-compact and so we need to say what we mean by the integral in (3). Since \( Z_{0,n,d} \) carries a \( T \)-action with compact fixed set, we can define the integral using the virtual localization formula of Graber–Pandharipande [34] note that we could do this in the case \( d \neq 0 \), too, and this would reproduce the definition which we just gave.

**Gromov–Witten Potentials.** The genus-zero Gromov–Witten potential \( F^0_Z \) is a generating function for certain genus-zero Gromov–Witten invariants of \( Z \). It is a formal power series in variables \( \tau^a \), \( 0 \leq a \leq N \), and the Novikov variables \( Q_i, 1 \leq i \leq b_2(Z) \), defined by

$$
F^0_Z = \sum_{n \geq 0} \sum_{d \in \Eff(Z)} \frac{Q^d}{n!} \langle \underbrace{\tau, \ldots, \tau}_{n \text{ times}} \rangle^{Z}_{0,n,d},
$$

(4)
where $\tau = \tau^a \Phi_a$. Since correlators are multilinear, the expression $\langle r, r, \ldots, r \rangle^{n, d}_{0, n, d}$ expands into a polynomial in the variables $\tau^a$. The second summation here is over the set $\text{Eff}(Z)$ of degrees of maps from orbifold curves to $Z$.

The genus-zero descendant potential $F_0^Z$ is a generating function for all genus-zero Gromov–Witten invariants of $Z$. It is a formal power series in variables $t_k^a$, $0 \leq a \leq N$, $0 \leq k < \infty$, and the Novikov variables $Q_i$, $1 \leq i \leq b_2(Z)$, defined by

$$F_0^Z = \sum_{n \geq 0} \sum_{0 \leq k_1, \ldots, k_n < \infty} \sum_{d \in \text{Eff}(Z)} \frac{Q^d}{n!} \langle t_{k_1} \psi^{k_1}, \ldots, t_{k_n} \psi^{k_n} \rangle^{n, d}_{0, n, d}$$

(5)

where $t_k = t_k^a \Phi_a$. The expression $\langle t_{k_1} \psi^{k_1}, \ldots, t_{k_n} \psi^{k_n} \rangle^{n, d}_{0, n, d}$ here expands, by multilinearity again, into a polynomial in the variables $t_k^a$.

### Analytic Continuation.

Let us call the coefficient in $F_0^Z$ of any monomial $\tau^{a_1} \ldots \tau^{a_n}$ a coefficient series of $F_0^Z$, and call the coefficient in $F_0^Z$ of any monomial $t_{k_1}^a \ldots t_{k_n}^a$ a coefficient series of $F_0^Z$. Each coefficient series is a formal power series in the Novikov variables $Q_i$, $1 \leq i \leq b_2(Z)$. All of the examples we consider below satisfy:

(A) each coefficient series of $F_0^Z$ converges in a neighbourhood of $Q_1 = Q_2 = \cdots = 0$ to an analytic function of the $Q_i$; and

(B) the coefficient series of $F_0^Z$ admit simultaneous analytic continuation to a neighbourhood of $Q_1 = Q_2 = \cdots = 1$.

Condition (A) implies that each coefficient series of $F_0^Z$ converges in a neighbourhood of $Q_1 = Q_2 = \cdots = 0$ to an analytic function of the $Q_i$, and condition (B) implies that the coefficient series of $F_0^Z$ also admit simultaneous analytic continuation to a neighbourhood of $Q_1 = Q_2 = \cdots = 1$: see [24 Appendix] for discussion of a closely-related point.

In what follows we will assume that a simultaneous analytic continuation of the coefficient series has been chosen, and will set $Q_1 = Q_2 = \cdots = 1$ throughout. Thus we regard the genus-zero Gromov–Witten potential as a formal power series

$$F_0^Z = \sum_{n \geq 0} \sum_{d \in \text{Eff}(Z)} \frac{1}{n!} \langle r, r, \ldots, r \rangle^{n, d}_{0, n, d}$$

(6)

in the variables $\tau^a$, $0 \leq a \leq N$, and we regard the genus-zero descendant potential as a formal power series

$$F_0^Z = \sum_{n \geq 0} \sum_{d \in \text{Eff}(Z)} \sum_{0 \leq k_1, \ldots, k_n < \infty} \frac{1}{n!} \langle t_{k_1} \psi^{k_1}, \ldots, t_{k_n} \psi^{k_n} \rangle^{n, d}_{0, n, d}$$

(7)

in the variables $t_k$, $0 \leq a \leq N$, $0 \leq k < \infty$.

### The Divisor Equation.

The reader might worry that by suppressing Novikov variables — i.e. by setting $Q_1 = Q_2 = \cdots = 1$ — we have lost some information about the degrees of curves. This is not the case. We will discuss this for the case $Z = Y$; the case $Z = X$ is entirely analogous. Recall that our basis $\varphi_0, \ldots, \varphi_N$ for $H(Y)$ was chosen so that $\varphi_1, \ldots, \varphi_r$ is a lift to $T$-equivariant cohomology of the basis $\omega_1', \ldots, \omega_r'$ for $H^2(Y; \mathbb{C})$ with which we measure the degrees of curves. Then, writing $\tau = \tau^a \varphi_a$,

$$\tau_{\text{rest}} = \tau^0 \varphi_0 + \tau^{r+1} \varphi_{r+1} + \tau^{r+2} \varphi_{r+2} + \cdots + \tau^N \varphi_N,$$

the Divisor Equation [2 Theorem 8.3.1] gives

$$F_0^Y = \frac{1}{6} \langle \tau \cup r, \tau \rangle_Y + \sum_{n \geq 0 \atop d \neq 0} \sum_{d \in \text{Eff}(Z)} e^{d_1 \tau^1} \cdots e^{d_r \tau^r} \frac{1}{n!} \langle \tau_{\text{rest}}, \tau_{\text{rest}}, \ldots, \tau_{\text{rest}} \rangle^{n, d}_{0, n, d}$$

and so the substitution $e^{\tau^r} \mapsto Q_i e^{\tau^r}$, $1 \leq i \leq r$, turns (6) into (7). The story for the descendant potential $F_0^Y$ is a little more complicated but the upshot is the same: the Divisor Equation allows us to recover (5) from (7).
The Lagrangian Submanifold-Germ. Following Givental we encode all genus-zero Gromov–Witten invariants of \( Z \) via the formal germ of a Lagrangian submanifold of \( \mathcal{H}_Z \), defined as follows. Regard the genus-zero descendant potential \( \mathcal{F}_Z^0 \) as the formal germ of a function on \( \mathcal{H}_Z^+ \) via the change of variables

\[
q_k^a = \begin{cases} 
\frac{\lambda_k - 1}{\lambda_k} & \text{if } (a, k) = (0, 1) \\
0 & \text{otherwise.}
\end{cases}
\]

This change of variables is called the dilaton shift. The variables \( q_k^a \) here are the Darboux co-ordinates from (2), so a general point on \( \mathcal{H}_Z^+ \) is \( \sum_{k \geq 0} q_k^a \Phi_a z^k \). The dilaton shift makes \( \mathcal{F}_Z^0 \) into the formal germ at \(-z\) of a function on \( \mathcal{H}_Z^+ \). The graph of the differential of \( \mathcal{F}_Z^0 \) therefore defines the formal germ of a submanifold of \( \mathcal{H}_Z \cong T^* \mathcal{H}_Z^+ \), defined by the equations

\[
\frac{\partial \mathcal{F}_Z^0}{\partial q_k^a} = 0 \leq a \leq N, \ 0 \leq k < \infty.
\]

We denote this Lagrangian submanifold-germ by \( \mathcal{L}_Z \).

More Analytic Continuation. In what follows we will need to analytically continue the submanifold-germ \( \mathcal{L}_Z \). There is nothing exotic about this, as we now explain. The germ \( \mathcal{L}_Z \) is defined by the equations (8), and to analytically continue \( \mathcal{L}_Z \) we will analytically continue each partial derivative \( \frac{\partial \mathcal{F}_Z^0}{\partial q_k^a} \) in the variables \( t_0^a, 0 \leq a \leq b_2(Z) \). The partial derivative \( \frac{\partial \mathcal{F}_Z^0}{\partial q_k^a} \) is a formal power series in the variables \( t_0^a, 0 \leq b \leq N, 0 \leq l < \infty \), so we write it in the form

\[
\sum f_l t^l, \text{ } t^l \text{ a monomial in the variables } t_0^l \text{ with } b > b_2(Z) \text{ or } l > 0,
\]

\[
f_l \text{ a formal power series in the variables } t_0^l, \ 0 \leq a \leq b_2(Z),
\]

and then analytically continue each \( f_l \).

The Crepant Resolution Conjecture. We are now ready to state the conjecture.

Conjecture 2.1. There is a degree-preserving \( \mathbb{C}(\mathbb{C}^{-1}) \)-linear symplectic isomorphism \( \mathbb{U} : \mathcal{H}_X \rightarrow \mathcal{H}_Y \) and a choice of analytic continuations of \( \mathcal{L}_X \) and \( \mathcal{L}_Y \) such that \( \mathbb{U}(\mathcal{L}_X) = \mathcal{L}_Y \). Furthermore, \( \mathbb{U} \) satisfies:

(a) \( \mathbb{U}(1_X) = 1_Y + O(z^{-1}) \);
(b) \( \mathbb{U} \circ (\rho \cup) = (\pi^* \rho \cup) \circ \mathbb{U} \) for every untwisted degree-two class \( \rho \in H^2(X; \mathbb{C}) \);
(c) \( \mathbb{U}(\mathcal{H}_X^0) \oplus \mathcal{H}_Y^{-} = \mathcal{H}_Y \).

This is a slight modification of a conjecture due to Coates, Corti, Iritani, and Tseng; very similar ideas occurred, simultaneously and independently, in unpublished work of Ruan. An expository account of the conjecture and its consequences can be found in [24].

3. General Theory

In this section we describe various aspects of Givental’s symplectic formalism which we will need below, as well as stating some consequences of Conjecture 2.1.

Big and Small J-Functions. Let \( \tau = \tau^a \Phi_a \). The big J-function of \( Z \) is

\[
J_Z^{\text{big}}(\tau, z) := z + \tau + \sum_{n \geq 0} \sum_{d \in \text{Eff}(Z)} \frac{1}{n!} \left( \tau, \tau, \ldots, \tau, \frac{\Phi}{z - \psi} \right)_{0,n+1,d}^Z \Phi.
\]

It is a formal family of elements of \( \mathcal{H}_Z \) — in other words, \( J_Z^{\text{big}} \) is a formal power series in the variables \( \tau^a, 0 \leq a \leq N \), which takes values in \( \mathcal{H}_Z \). By writing out the equations (8) defining \( \mathcal{L}_Z \), it is easy to see that \( J_Z(\tau, -z) \) is the unique family of elements of \( \mathcal{L}_Z \) of the form \(-z + \tau + O(z^{-1})\).

Take \( Z = Y \) and restrict the parameter \( \tau \) in the big J-function to the locus \( \tau = \tau^1 \varphi_1 + \ldots + \tau^r \varphi_r \). Then the Divisor Equation gives that \( J_Y^{\text{big}}(\tau^1 \varphi_1 + \ldots + \tau^r \varphi_r, z) \) is equal to

\[
z e^{\tau^1 \varphi_1 / z} \ldots e^{\tau^r \varphi_r / z} \left( 1 + \sum_{d \in \text{Eff}(Y)} e^{d_1\tau^1} \ldots e^{d_r\tau^r} \left( \frac{\varphi}{z - \psi} \right)_{0,1,d}^Y \varphi \right).
\]

\(^2\)These variables correspond to basis elements of \( H(Z) \) of degree 0 or 2.
Making the change of variables $q_i = e^{\tau_i}$, $1 \leq i \leq r$, we define the small $J$-function of $Y$ to be

$$J_Y(q, z) := z q_1^{v_1/z} \cdots q_r^{v_r/z} \left( 1 + \sum_{d \in \text{Eff}(Y)} q_1^{d_1} \cdots q_r^{d_r} \left( \frac{\varphi_c}{z(z - \psi)} \right)_0^{X} \phi^d \right).$$

(10)

In examples below we will see that this converges, in a domain where each $|q_i|$ is sufficiently small, to a multi-valued analytic function of $q_1, \ldots, q_r$ which takes values in $\mathcal{H}_Y$. The multi-valuedness comes from the factors $q_i^{v_i/z} := \exp(\varphi_c \log(q_i)/z)$. We have $J_Y(q, -z) \in \mathcal{L}_Y$ for all $q$ in the domain of convergence of $J_Y$.

Similarly, take $\mathcal{Z} = \mathcal{X}$ and restrict the parameter $\tau$ in the big $J$-function to the locus $\tau = \tau_1^1 \phi_1 + \cdots + \tau^s \phi_s$. Then the Divisor Equation gives that

$$J_\mathcal{X}^{\text{big}}(\tau^1 \phi_1 + \cdots + \tau^s \phi_s, z) = z e^{\tau^1 \phi_1/z} \cdots e^{\tau^s \phi_s/z} \left( 1 + \sum_{d \in \text{Eff}(\mathcal{X})} e^{d_1 \tau^1} \cdots e^{d_s \tau^s} \left( \frac{\phi_c}{z(z - \psi)} \right)_0^{X} \phi^d \right).$$

Making the change of variables $u_i = e^{\tau_i}$, $1 \leq i \leq s$, we define the small $J$-function of $\mathcal{X}$ to be

$$J_\mathcal{X}(u, z) := z u_1^{\phi_1/z} \cdots u_s^{\phi_s/z} \left( 1 + \sum_{d \in \text{Eff}(\mathcal{X})} u_1^{d_1} \cdots u_s^{d_s} \left( \frac{\phi_c}{z(z - \psi)} \right)_0^{X} \phi^d \right).$$

(11)

In the examples below this converges, in a domain where each $|u_i|$ is sufficiently small, to a multi-valued analytic function of $u_1, \ldots, u_s$ which takes values in $\mathcal{H}_\mathcal{X}$. We have $J_\mathcal{X}(u, -z) \in \mathcal{L}_\mathcal{X}$ for all $u$ in the domain of convergence of $J_\mathcal{X}$.

**Two Consequences of Conjecture [2, 4]**. Recall that the $T$-equivariant small quantum cohomology of $\mathcal{X}$ is a family of algebra structures on $H(\mathcal{X})$ parametrized by $u_1, \ldots, u_s$, defined by

$$\phi_\alpha \bullet \phi_\beta = \sum_{d \in \text{Eff}(\mathcal{X})} u_1^{d_1} \cdots u_s^{d_s} \langle \phi_\alpha, \phi_\beta \rangle_{\text{eff}}^{\mathcal{X}} \phi^d.$$ (12)

The $T$-equivariant small quantum cohomology of $\mathcal{Y}$ is a family of algebra structures on $H(\mathcal{Y})$ parametrized by $q_1, \ldots, q_r$, defined by

$$\varphi_\alpha \bullet \varphi_\beta = \sum_{d \in \text{Eff}(\mathcal{Y})} q_1^{d_1} \cdots q_r^{d_r} \langle \varphi_\alpha, \varphi_\beta \rangle_{\text{eff}}^{\mathcal{Y}} \varphi^d.$$ (13)

For the remainder of this subsection, assume that:
- Conjecture [2, 4] holds;
- the symplectic transformation $\mathcal{U}$ remains well-defined in the non-equivariant limit $\lambda \to 0$;
- $\mathcal{X}$ is semi-positive.

Two consequences of Conjecture [2, 4] are then as follows: these are proved\(^3\) in [24]. Define the class $c \in H(\mathcal{Y})$ by

$$\mathcal{U}(1_{\mathcal{X}}) = 1_{\mathcal{Y}} - cz^{-1} + O(z^{-2}),$$

and write

$$c = c^1 \varphi_1 + \cdots + c^r \varphi_r + d\lambda, \quad c^1, \ldots, c^r, d \in \mathbb{C};$$

(14)

such an equality exists because $c$ has degree 2. Then:

**Corollary 3.1.** The algebra obtained from the small quantum cohomology algebra of $\mathcal{Y}$ by analytic continuation\(^4\) in the parameters $q_{s+1}, \ldots, q_r$ (if necessary) followed by the substitution

$$q_i = \begin{cases} 0 & 1 \leq i \leq s \\ c e^i & s < i \leq r \end{cases}$$

\(^3\)The orbifold $\mathcal{Z}$ is semi-positive if and only if there does not exist $d \in \text{Eff}(\mathcal{Z})$ such that $3 - \dim_\mathbb{C} \mathcal{Z} \leq \langle c_1(T\mathcal{Z}), d \rangle < 0$.

\(^4\)This is not, strictly speaking, true: the $T$-equivariant version of the Crepant Resolution Conjecture is not treated in [24]. It is straightforward to check, however, that the arguments given there also prove the results stated here. The key point is that $\mathcal{U}$ has a non-equivariant limit, and so only non-negative powers of $\lambda$ can occur.

The analytic continuation of the small quantum product here is induced by the analytic continuation of $\mathcal{L}_Y$. This is explained in [24].
is isomorphic to the Chen–Ruan orbifold cohomology algebra of $\mathcal{X}$, via an isomorphism which sends
$\alpha \in H^2(\mathcal{X}; \mathbb{C}) \subset H(\mathcal{X})$ to $\pi^* \alpha \in H(Y)$.

This is a version of Ruan’s Cohomological Crepant Resolution Conjecture \[10\].

Define elements $b_\varepsilon \in H(Y)$, $0 \leq \varepsilon \leq N$, by $b_\varepsilon = 0$ if $\deg \phi_\varepsilon \leq 2$ and
$$U(\phi_\varepsilon z^{1-\frac{1}{2}\deg \phi_\varepsilon}) = b_\varepsilon + O(z^{-1})$$
otherwise. Define power series $f^1, \ldots, f^r, g \in \mathbb{C}[u_1, \ldots, u_s]$ by
$$f^1 \varphi_1 + \cdots + f^r \varphi_r + g\lambda = \sum_{d \in \text{Eff}(\mathcal{X})} \sum_{\varepsilon = r+1}^N (-1)^{\frac{1}{2}\deg \phi_\varepsilon} + 1 \left\langle \phi_\varepsilon \right\rangle_{0,1,d}^X u_1^{d_1} \cdots u_s^{d_s} b_\varepsilon; \quad (15)$$
such an equality exists because each class $b_\varepsilon$ has degree 2. Recall the definition of the rational numbers $r_i$, $1 \leq i \leq s$, from Section \[2\]. Then:

**Corollary 3.2.** The algebra obtained from the small quantum cohomology algebra of $Y$ by analytic continuation\[5\] in the parameters $q_{s+1}, \ldots, q_r$ (if necessary) followed by the substitution
$$q_i = \begin{cases} e^{u_i + f^r u_i^r} & 1 \leq i \leq s \\ e^{u_i + f^r} & s < i \leq r \end{cases}$$
is isomorphic to the small quantum cohomology algebra of $\mathcal{X}$, via an isomorphism which sends $\alpha \in H^2(\mathcal{X}; \mathbb{C}) \subset H(\mathcal{X})$ to $\pi^* \alpha \in H(Y)$.

This is a “quantum-corrected” version of Ruan’s Crepant Resolution Conjecture.

**Three Results Which We Will Need.** We next record three results which we will need below. Part (a) follows from the String Equation: this is explained in e.g. \[33\]. Part (b) is a reconstruction result for Gromov–Witten invariants — it says that all genus-zero Gromov–Witten invariants can be uniquely reconstructed from the one-point descendants $\left\langle \Phi_\alpha \psi^k \right\rangle_{0,1,d}^Z$. Part (c) is a generalization of part (b). One can prove (b) and (c) by repeated application of the WDVV equations and the Topological Recursion Relations; results along similar lines can be found in \[6, 26, 37, 40, 41, 45\].

**Proposition 3.3.**

(a) The submanifold-germ $L_Z \subset H_Z$ is closed under multiplication by $\exp(a\lambda/z)$ for any $a \in \mathbb{C}$.

(b) If $Z$ is semi-positive and the Chen–Ruan orbifold cohomology algebra of $Z$ is generated by $H^2(Z; \mathbb{C})$ then the submanifold-germ $L_Z$ can be uniquely reconstructed from the small J-function $J^Z_{\text{big}}(q, z)$.

(c) If $Z$ is semi-positive and $H^2_{\text{gen}} \subset H^2_{\text{CR}}(Z; \mathbb{C})$ is a subspace such that the Chen–Ruan orbifold cohomology algebra of $Z$ is generated by $H^2_{\text{gen}}$ then the submanifold-germ $L_Z$ can be uniquely reconstructed from the restriction of the big J-function $J^Z_{\text{big}}(\tau, z)$ to the locus $\tau \in H^2_{\text{gen}}$.

It is easy to check that in all the examples we consider below, the Chen–Ruan cohomology algebra of $Z$ is generated in degree 2.

**Computing Twisted Gromov–Witten Invariants.** As discussed above, in most of our examples $Z$ will be the total space of a concave vector bundle $E$ over a compact orbifold $B$, and the $T$-action on $Z$ will be the canonical $C^\ast$-action which rotates the fibers of $E$ and covers the trivial action on $B$. In this situation $\text{Eff}(Z)$ is canonically isomorphic to $\text{Eff}(B)$ and $H(Z)$ is canonically isomorphic to $H(B) := H^\ast_{\text{CR}}(B; \mathbb{C}) \otimes \mathbb{C}(\lambda)$. Our bases $\{\Phi_a\}$ and $\{\Phi^a\}$ for $H(Z)$ determine bases for $H(B)$, which we also denote by $\{\Phi_a\}$ and $\{\Phi^a\}$. Gromov–Witten invariants of $Z$ coincide with Gromov–Witten invariants of $Z$ twisted, in the sense of \[19, 23\], by the $T$-equivariant inverse Euler class $e^{-1}$ and the vector bundle $E$. Results in \[19\] allow the calculation of twisted Gromov–Witten invariants in a quite general setting. We will need three special cases of these results, as follows. Each of these special cases determines a family of elements $q \mapsto I_Z(q, z)$ of elements of $L_Z$; in each case this family $I_Z(q, z)$ is an appropriate hypergeometric modification of the small J-function $J_B(q, z)$ of $B$.

**Theorem 3.4.** Suppose that $E \rightarrow B$ is a concave line bundle. Let $\rho$ denote the first Chern class of $E$, regarded as an element of localized $T$-equivariant Chen–Ruan cohomology $H^\ast(B)$ via the canonical inclusion $H^\ast(B; \mathbb{C}) \hookrightarrow H^\ast_{\text{CR}}(B; \mathbb{C})$, and set
$$M_E(d) := \prod_{b \in \{\rho, d\} \subset \mathbb{Z}, \frac{\text{frac}(b)}{\text{frac}(\rho, d)} \neq 0} (\lambda + \rho + b\lambda).$$
where \( d \in \text{Eff}(B) \) and \( \text{frac}(r) \) denotes the fractional part of \( r \). Let \( k = b_2(B) \), so that the small \( J \)-function of \( B \) is

\[
J_B(q, z) = z^{q_1^{\phi_1/z} \cdots q_k^{\phi_k/z}} \left( 1 + \sum_{d \in \text{Eff}(B)} q_1^{d_1} \cdots q_k^{d_k} \left( \frac{\Phi_e}{z(z - \psi)} \right)^B_{0,1,d} \Phi_e \right).
\]

Then

\[
I_Z(q, z) := z^{q_1^{\phi_1/z} \cdots q_k^{\phi_k/z}} \left( 1 + \sum_{d \in \text{Eff}(B)} q_1^{d_1} \cdots q_k^{d_k} M_E(d) \left( \frac{\Phi_e}{z(z - \psi)} \right)^B_{0,1,d} \Phi_e \right)
\]

satisfies \( I_Z(q, -z) \in \mathcal{L}_Z \) for all \( q \) in the domain of convergence of \( I_Z \).

**Proof.** Theorem 4.6 in [19] concerns a Lagrangian submanifold-germ \( L^{tw} \) which encodes twisted Gromov–Witten invariants: in our situation, \( L^{tw} = \mathcal{L}_Z \). The Theorem gives a formula for a formal family \( \tau \mapsto \mathcal{I}^{tw}(\tau, -z) \) of elements of \( L^{tw} \), as follows. Let \( I \) be a set which indexes the components of the inertia stack \( I_B \) of \( B \), and let \( 0 \in I \) be the index of the distinguished component \( B \subset I_B \). One decomposes the big \( J \)-function of \( B \) as a sum

\[
J_B^{\text{big}}(\tau, z) = \sum_{\theta \in \text{NETT}(B)} J_\theta(\tau, z)
\]

of contributions from stable maps of different topological types; here \( \text{NETT}(B) \) is the set of topological types. The topological type of a degree-\( d \) stable map \( f : C \to B \) from a genus-\( g \) orbifold curve with \( n \) marked points is the triple \((g, d, S)\), where \( S = (i_1, \ldots, i_n) \) is the ordered \( n \)-tuple of elements of \( I \) indexing the components of \( I_B \) picked out by the marked points. Then

\[
\mathcal{I}^{tw}(\tau, z) := \sum_{\theta \in \text{NETT}(B)} M_\theta(z) \cdot J_\theta(\tau, z)
\]

where \( M_\theta(z) \) is a modification factor defined in §4.2 of [19].

If we set \( \tau = \tau^1\Phi_1 + \cdots + \tau^k\Phi_k \) then \( J_\theta(\tau, z) \) vanishes unless the topological type \( \theta \) is of the form \((0, d, S)\) where \( S = (0, 0, \ldots, 0, i) \) for some \( i \in I \); this is because the classes \( \Phi_i, 1 \leq i \leq k \) are supported on the distinguished component \( B \) of \( I_B \). In this case the modification factor \( M_\theta(z) \) depends only on \( d \) and is equal to \( M_E(d) \). Also,

\[
J_B^{\text{big}}(\tau^1\Phi_1 + \cdots + \tau^k\Phi_k, z) = z^{e^{\tau^1\Phi_1} \cdots e^{\tau^k\Phi_k/z}} \left( 1 + \sum_{d \in \text{Eff}(B)} e^{d_1\tau^1} \cdots e^{d_k\tau^k} M_E(d) \left( \frac{\Phi_e}{z(z - \psi)} \right)^B_{0,1,d} \Phi_e \right)
\]

and it follows that \( \mathcal{I}^{tw}(\tau^1\Phi_1 + \cdots + \tau^k\Phi_k, z) \) is equal to

\[
z^{e^{\tau^1\Phi_1} \cdots e^{\tau^k\Phi_k/z}} \left( 1 + \sum_{d \in \text{Eff}(B)} e^{d_1\tau^1} \cdots e^{d_k\tau^k} M_E(d) \left( \frac{\Phi_e}{z(z - \psi)} \right)^B_{0,1,d} \Phi_e \right).
\]

Making the change of variables \( q_i = e^{\tau^i}, 1 \leq i \leq k \), we conclude that \( I_Z(q, -z) \in \mathcal{L}_Z \) for all \( q \) such that the series defining \( I_Z \) converges.

Exactly the same argument proves:

**Theorem 3.5.** If \( \mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n \) is the direct sum of convex line bundles,

\[
M_E(d) := \prod_{1 \leq i \leq m} M_{\mathcal{E}_i}(d),
\]

and \( I_Z(q, z) \) is defined exactly as in [16] then \( I_Z(q, -z) \in \mathcal{L}_Z \) for all \( q \) in the domain of convergence of \( I_Z \).

The final special case which we need is where \( \mathcal{Z} \) is the total space of a direct sum of line bundles \( \mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_n \) over \( B = \mathcal{B}_Z \). Components of the inertia stack of \( \mathcal{B}_Z \) are indexed by fractions \( k/n, 0 \leq k < n \); the component indexed by \( k/n \) corresponds to the element \( [k] \in \mathbb{Z}_n \). Let \( 1_k/n \in H(\mathcal{B}) \) denote the orbifold cohomology class which restricts to the unit class on the component of the inertia stack indexed by \( k/n \) and restricts to zero on the other components. The set \( \{1_k/n : 0 \leq k < n\} \) forms a basis for \( H(\mathcal{B}) \); as \( H(\mathcal{B}) \) and \( H(\mathcal{Z}) \) are canonically isomorphic it determines a basis for \( H(\mathcal{Z}) \) as well.
Theorem 3.6. Let $\mathcal{Z}$ be the total space of the direct sum of line bundles $\mathcal{E} = \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_m$ over $\mathcal{B} = B\mathbb{Z}_n$. Let $e_i$ be the integer such that $\mathcal{E}_i$ is given by the character $[k] \mapsto \exp(\frac{2\pi i e_i k}{n})$ of $\mathbb{Z}_n$ and that $0 \leq e_i < n$. Let

$$P_{i,k} := \left\{ b : \frac{b}{n} = \frac{-e_i k}{n}, -\frac{e_i k}{n} < b \leq 0 \right\}$$

and

$$I_\mathcal{Z}(x,z) := \sum_{k \geq 0} x^k \prod_{i=1}^m \prod_{b \in P_{i,k}} \left( \frac{\theta - \frac{e_i k}{n} + b z}{k!} \frac{1}{z^{k \frac{b}{n}} \exp\left(\frac{k}{n} \right)} \right).$$

Then $x \mapsto I_\mathcal{Z}(x,-z)$ is a formal family of elements of $\mathcal{L}_\mathcal{Z}$.

Proof. We argue as in the proof of Theorem 3.4. If we decompose the big $J$-function of $\mathcal{B}$ as a sum

$$J_\mathcal{B}^\text{big}(\tau, z) = \sum_{\theta \in \mathcal{NETT}(\mathcal{B})} J_\theta(\tau, z)$$

of contributions from stable maps of different topological types and set

$$I^\text{tw}(\tau, z) := \sum_{\theta \in \mathcal{NETT}(\mathcal{B})} M_\theta(z) \cdot J_\theta(\tau, z)$$

where $M_\theta(z)$ is defined in [19, §4.2] then $\tau \mapsto I^\text{tw}(\tau, z)$ defines a formal family of elements of $\mathcal{L}_{\mathcal{B}} = \mathcal{L}_\mathcal{Z}$.

Proposition 6.1 in [19] gives an explicit formula for the big $J$-function of $\mathcal{B} = B\mathbb{Z}_n$, and we see from this that if $\tau = x \frac{1}{n}$ then $J_\theta(\tau, z)$ vanishes unless the topological type $\theta$ is $(0, 0, S)$ with

$$S = \left( \frac{1}{n}, \ldots, \frac{1}{n}, \frac{1}{n}, \frac{\frac{n-k}{n}}{k \text{ times}} \right).$$

In this case,

$$J_\theta(\tau, z) = \frac{x^k}{k!} z^k \frac{1}{z^{\frac{1}{n}}} \exp\left(\frac{k}{n} \right)$$

and

$$M_\theta(z) = \prod_{i=1}^m \prod_{b \in P_{i,k}} \left( \frac{\theta - \frac{e_i k}{n} + b z}{k!} \frac{1}{z^{k \frac{b}{n}} \exp\left(\frac{k}{n} \right)} \right).$$

Thus $x \mapsto I_\mathcal{Z}(x,-z)$ is a formal family of elements of $\mathcal{L}_\mathcal{Z}$.

\[ \square \]

4. Example I: $X = [\mathbb{C}^3/\mathbb{Z}_3]$, $Y = \mathbb{K}_3$.

Let $X$ be the orbifold $[\mathbb{C}^3/\mathbb{Z}_3]$ where $\mathbb{Z}_3$ acts on $\mathbb{C}^3$ with weights $(1,1,1)$. The coarse moduli space $X$ of $X$ is the quotient singularity $\mathbb{C}^3/\mathbb{Z}_3$ and the crepant resolution $Y$ of $X$ is the canonical bundle $\mathbb{K}_3$.

Toric Geometry. Let $e_1, e_2, e_3$ denote the standard basis vectors for $\mathbb{R}^3$. The space $Y$ is the toric variety corresponding to a fan with rays

$$e_1 + e_3, e_2 + e_3, -e_1 - e_2 + e_3, e_3;$$

this fan is a cone over the picture in the plane $z = 1$ shown in Figure 1

![Figure 1: The fans for $X$ and $Y$ (respectively) are the cones over these pictures in the plane $z = 1$](image)

We can construct $Y$ as a GIT quotient, following e.g. [1], by considering the exact sequence

$$0 \longrightarrow \mathbb{Z} \stackrel{\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}}{\longrightarrow} \mathbb{Z}^4 \longrightarrow \mathbb{Z}^4 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

(18)
This shows that $Y$ is a quotient $\mathbb{C}^4/\mathbb{C}^\times$, where $\tau \in \mathbb{C}^\times$ acts on $\mathbb{C}^4$ as
\[(x, y, z, w) \mapsto (\tau x, \tau y, \tau z, \tau^{-3} w)\] (19)

Dualizing (18) gives
\[
\begin{array}{c}
0 \longrightarrow \mathbb{Z}^3 \xrightarrow{egin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix}} \mathbb{Z}^4 \xrightarrow{egin{pmatrix} 1 & 1 & 1 & -3 \end{pmatrix}} \mathbb{Z} \longrightarrow 0
\end{array}
\]

where the right-hand entry is $H^2(Y; \mathbb{Z})$ and the columns of the right-hand matrix give the four toric divisors in $Y$. If we draw this picture in $H^2(Y; \mathbb{R})$ then it gives the chamber decomposition for the GIT problem (Figure 2 below); this chamber decomposition is also known as the secondary fan for $Y$.

![Figure 2: The secondary fan for $Y = K_{P^2}$](image)

Each chamber in the secondary fan corresponds to a fan $\Sigma$ which is a triangulation of the rays (17): a cone $\sigma$ is in $\Sigma$ if and only if the co-ordinate subspace corresponding to the complement of $\sigma$ covers the chosen chamber. The fans are shown in Figure 1. For $\xi$ in the left-hand chamber the GIT quotient $\mathbb{C}^4/\mathbb{C}^\times$ gives $X$; we delete the locus $w = 0$ from $\mathbb{C}^4$ and then take the quotient by the action (19). For $\xi$ in the right-hand chamber we have $\mathbb{C}^4/\mathbb{C}^\times = Y$; we delete the locus $(x, y, z) = (0, 0, 0)$ from $\mathbb{C}^4$ and then take the quotient by (19). For $\xi = 0$ the quotient $\mathbb{C}^4/\mathbb{C}^\times$ gives the coarse moduli space $X$. Moving from the right-hand chamber into the “wall” $\xi = 0$ gives the resolution map $Y \to X$; this sends $[x, y, z, w] \in \mathbb{C}^4/\mathbb{C}^\times$ to $[xw^{1/3}, yw^{1/3}, zw^{1/3}] \in \mathbb{C}^3/\mathbb{Z}_3$ where $[A]$ denotes class of $A$ in the appropriate quotient.

The $T$-Action. Consider the action of $T = \mathbb{C}^\times$ on $\mathbb{C}^4$ such that $\alpha \in T$ acts as
\[(x, y, z, w) \mapsto (x, y, z, \alpha w)\]

This action descends to give $T$-actions on $X$, $Y$, and $Y$. The induced action on $X$ is
\[[x, y, z] \mapsto [\alpha^{1/3} x, \alpha^{1/3} y, \alpha^{1/3} z]\]

The induced action on $Y$ is the canonical $\mathbb{C}^\times$-action on the line bundle $K_{P^2} \to P^2$; it covers the trivial action on $P^2$. The diagram

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
\]

is $T$-equivariant.

Bases for Everything. We have
\[r := \text{rank } H^2(Y; \mathbb{C}) = 1, \quad s := \text{rank } H^2(X; \mathbb{C}) = 0.\]

Let $p$ be the first Chern class of the line bundle $\mathcal{O}(1) \to P^2$, pulled back to $Y = K_{P^2}$ via the projection $K_{P^2} \to P^2$. The class $p$ has a canonical lift to $T$-equivariant cohomology, which we also denote by $p$, and
\[H(Y) = \mathbb{C}(\lambda)[p]/\langle p^3 \rangle.\]

We set
\[
\begin{align*}
\varphi_0 &= 1, & \varphi_1 &= p, & \varphi_2 &= p^2, \\
\varphi_0 &= \lambda p^2, & \varphi_1 &= \lambda p - 3p^2, & \varphi_2 &= \lambda - 3p.
\end{align*}
\]
The components of the inertia stack $\mathcal{X}$ are indexed by elements of $\mathbb{Z}_3$. Let $1_{k/3} \in H(\mathcal{X})$ denote the orbifold cohomology class which restricts to the unit class on the inertia component indexed by $[k] \in \mathbb{Z}_3$ and restricts to zero on the other components. Set

\[
\begin{align*}
\phi_0 &= 1_0, \\
\phi^0 &= \sqrt[3]{y}1_0, \\
\phi_1 &= 1_{1/3}, \\
\phi^1 &= 31_{2/3}, \\
\phi_2 &= 1_{2/3}, \\
\phi^2 &= 31_{1/3}.
\end{align*}
\]

**Step 1: A Family of Elements of $\mathcal{L}_Y$.** Consider

\[
I_Y(y, z) := z \sum_{d \geq 0} \frac{\Gamma(1 + \frac{\lambda}{2})^3}{\Gamma(1 + \frac{\lambda}{2} + d)} \frac{\Gamma(1 + \frac{\lambda}{2} - 3p)}{\Gamma(1 + \frac{\lambda}{2} - 3d)} y^{d+p/z}.
\]  

(20)

This series converges in the region $\{y \in \mathbb{C} : 0 < |y| < \frac{1}{27}\}$ to a multi-valued analytic function of $y$ which takes values in $\mathcal{H}_Y$. We have

\[
I_Y(y, z) = z \sum_{d \geq 0} \frac{\prod_{-3d < m \leq 0} (\lambda - 3p + mz)}{\prod_{0 < m \leq d} (p + mz)^3} y^{d+p/z}.
\]  

(21)

**Proposition 4.1.**

$I_Y(y, -z) \in \mathcal{L}_Y$ for all $y$ such that $0 < |y| < \frac{1}{27}$.

**Proof.** We are in the situation of Theorem 3.4 with $B = \mathbb{P}^2$ and $E = O(-3)$. Givental has proved [29] that the small $J$-function of $\mathbb{P}^2$ is

\[
J_{\mathbb{P}^2}(q, z) = z q^{p/z} \sum_{d \geq 0} \frac{q^d}{\prod_{0 < m \leq d} (p + mz)^3},
\]

and it follows (by comparing with the statement of Theorem 3.4) that

\[
\left\langle \frac{\Phi^e}{z - \psi} \right\rangle_{0,1,d} = \frac{1}{\prod_{0 < m \leq d} (p + mz)^3}
\]

whenever $d > 0$. (22)

Theorem 3.4 thus implies that $I_Y(y, -z) \in \mathcal{L}_Y$ for all $y$ in the domain of convergence of $I_Y$, as claimed. \Box

**Step 2: $I_Y$ Determines $\mathcal{L}_Y$.** We have:

**Corollary 4.2.**

\[
J_Y(q, z) = \exp f(y/z) I_Y(y, z)
\]

where

\[
q = y \exp \left(3f(y)\right), \\
f(y) = \sum_{d > 0} \frac{(3d-1)!}{(zd)!} (-y)^d.
\]

**Proof.** We have

\[
I_Y(y, z) = z + p \log y - (\lambda - 3p)f(y) + O(z^{-1}).
\]

Applying Propositions 3.3 and 4.1 we see that

\[
y \mapsto e^{-\lambda f(y)/z} I_Y(y, -z), \quad 0 < |y| < \frac{1}{27},
\]

is a family of elements of $\mathcal{L}_Y$. But

\[
e^{-\lambda f(y)/z} I_Y(y, -z) = -z + p \log q + O(z^{-1}),
\]

where $q$ is defined above, and the unique family of elements of $\mathcal{L}_Y$ of this form is $q \mapsto J_Y(q, -z)$. \Box

As $I_Y(y, z)$ is multivalued-analytic and the change of variables $y \sim q$ is analytic, we conclude that the series defining $J_Y(q, z)$ converges, when $|q|$ is sufficiently small, to a multivalued analytic function of $q$. Furthermore, as the small $J$-function $J_Y(q, z)$ determines $\mathcal{L}_Y$ (Proposition 3.3), it follows that $\mathcal{L}_Y$ is uniquely determined by the fact that $y \mapsto I_Y(y, -z)$ is a family of elements of $\mathcal{L}_Y$.  

13
Aside: Computing Gromov–Witten Invariants of $Y$. As is well-known, Corollary 4.2 determines many genus-zero Gromov–Witten invariants of $Y$. The inverse to the change of variables $y \rightsquigarrow q$ is

$$y = q + 6q^2 + 9q^3 + 56q^4 - 300q^5 + \ldots$$

Substituting this into the equality

$$z q^{p/z} \left( 1 + \sum_{d \geq 0} q^d \left( \frac{\varphi}{z} \right)^Y_{0,1,d} \right) = z \tau^{f(y) + p/z} \prod_{d \geq 0} \frac{1 - \lambda - 3q + m q^{d+3}}{1 - \lambda - 3q + m q} q^{d+3}$$

and comparing coefficients of $q$, one finds that

$$\langle \frac{\varphi^\alpha}{z} \rangle_{0,1,1} = -\frac{9 \lambda p^2}{z} + o(\lambda), \quad \langle \frac{\varphi^\alpha}{z} \rangle_{0,1,2} = -\frac{244 \lambda p^2}{z} + o(\lambda),$$

and so on, where $o(\lambda)$ denotes terms containing strictly positive powers of $\lambda$. Taking the non-equivariant limit $\lambda \to 0$ yields the local Gromov–Witten invariants $K_d$ calculated in [17, §2.2]:

$$\langle p \rangle_{0,1,1} = 3, \quad \langle p \rangle_{0,1,2} = -\frac{45}{4}, \quad \langle p \rangle_{0,1,3} = \frac{244}{3}, \quad \langle p \rangle_{0,1,4} = -\frac{12333}{16},$$

and therefore, using the Divisor Equation, we find

$$K_1 = 3, \quad K_2 = -\frac{45}{8}, \quad K_3 = \frac{244}{9}, \quad K_4 = -\frac{12333}{64}, \quad \text{etc.}$$

Step 3: A Family of Elements of $L_X$. Let

$$I_X(x, z) := z x^{-\lambda/z} \sum_{l \geq 0} \prod_{l \geq 0, b \leq \frac{1}{2}} \frac{1}{\frac{b}{z}} \Gamma \left( \frac{m}{z} + \frac{b}{z} \right) 1_{\text{frac}(\frac{b}{z})}.$$ (23)

This converges, in the region $|x| < 27$, to an analytic function which takes values in $H_X$. Theorem 3.6 and Proposition 3.3(a) imply that $I_X(x, -z) \in L_X$ for all $x$ such that $|x| < 27$.

Step 4: $I_X$ Determines $L_X$. We have:

Corollary 4.3.

$$J_X^{\text{big}}(\tau^1 1_{1/3}, z) = x^{\lambda/z} I_X(x, z) \quad \text{where} \quad \tau^1 = \sum_{m \geq 0} (-1)^m \frac{x^{3m+1}}{3m+1} \frac{\Gamma \left( m + \frac{1}{3} \right)}{\Gamma \left( \frac{1}{3} \right)}.$$ (24)

Proof. On the one hand, we know that $x^{-\lambda/z} I_X(x, -z) \in L_X$, and on the other hand we know that

$$x^{-\lambda/z} I_X(x, -z) = -z + \sum_{m \geq 0} (-1)^m \frac{x^{3m+1}}{3m+1} \frac{\Gamma \left( m + \frac{1}{3} \right)}{\Gamma \left( \frac{1}{3} \right)} 1_{1/3} + O(z^{-1}).$$

As the unique family of elements of $L_X$ of the form $-z + \tau^1 1_{1/3} + O(z^{-1})$ is $\tau^1 \mapsto J_X^{\text{big}}(\tau^1 1_{1/3}, z)$, the result follows.

Since $x^{\lambda/z} I_X(x, z) \$ and the change of variables $x \rightsquigarrow \tau^1$ are analytic, this implies that $J_X^{\text{big}}(\tau^1 1_{1/3}, z)$ depends analytically on $\tau^1$ in some region where $|\tau^1|$ is sufficiently small. It also, via Proposition 3.3(b), shows that $L_X$ is uniquely determined by the fact that $x \mapsto I_X(x, -z)$ is a family of elements of $L_X$.

Aside: Computing Gromov–Witten Invariants of $X$. Just as we did for $Y$, one can use Corollary 4.3 to compute genus-zero Gromov–Witten invariants of $X$. This calculation is carried out in [19, §6.3]; it verifies some of the predictions made by Aganagic, Bouchard, and Klemm [2].
Step 5: The B-model Moduli Space and the Picard–Fuchs System. The B-model moduli space $\mathcal{M}_B$ is the toric orbifold corresponding to the secondary fan for $Y$. It has two co-ordinate patches, one for each chamber. Let $x$ be the co-ordinate corresponding to the left-hand chamber (recall that this chamber gives rise to $\mathcal{X}$) and let $y$ be the co-ordinate corresponding to the right-hand chamber (recall that this chamber gives $Y$). The co-ordinate patches are related by

$$y = x^{-3}$$

(24)
and it follows that $\mathcal{M}_B$ is the weighted projective space $\mathbb{P}(1,3)$. The space $\mathcal{M}_B$ is called the B-model moduli space as it is the base of the Landau–Ginzburg model (“the B-model”) which corresponds to the quantum cohomology of $Y$ (“the A-model”) under mirror symmetry: see e.g. [30, 35].

We regard $I_X(x, z)$ as a function on the co-ordinate patch corresponding to $\mathcal{X}$ and $I_Y(y, z)$ as a function on the co-ordinate patch corresponding to $Y$. Writing

$$I_Y(y, z) = I^0_Y \varphi_0 + I^1_Y \varphi_1 + I^2_Y \varphi_2,$$

the components $\{I^j_Y : j = 0, 1, 2\}$, which are functions of $y, \lambda$, and $z$, form a basis of solutions to the differential equation

$$D^3_y f = y(\lambda - 3D_y)(\lambda - 3D_y - 2z)f, \quad D_y = zy \frac{\partial}{\partial y}.$$  

(25)

Writing

$$I_X(x, z) = I^0_X \phi_0 + I^1_X \phi_1 + I^2_X \phi_2,$$

the components $\{I^j_X : j = 0, 1, 2\}$, which are functions of $x, \lambda$, and $z$, form a basis of solutions to the differential equation

$$D^3_x f = -27x^{-3}(\lambda + D_x)(\lambda + D_x - 2z)f, \quad D_x = x \frac{\partial}{\partial x}.$$  

(26)

Recall that the functions $I^j_Y$ are defined in a region where $|y|$ is small. The change of variables (24) turns (25) into (26). This implies that if we analytically continue the functions $I^j_Y$ to a region where $|y|$ is large (and hence $|x|$ is small), and then write the analytic continuations $\tilde{I}^j_Y$ in terms of the co-ordinate $x$, then $\{\tilde{I}^j_Y(x, z) : j = 0, 1, 2\}$ will satisfy (26). We have a basis of solutions to (26), given by the components $\tilde{I}^j_X(x, z)$ of $I^j_X$, and so

$$\begin{pmatrix} \tilde{I}^0_X(x, z) \\ \tilde{I}^1_X(x, z) \\ \tilde{I}^2_X(x, z) \end{pmatrix} = M(\lambda, z) \begin{pmatrix} I^0_X(x, z) \\ I^1_X(x, z) \\ I^2_X(x, z) \end{pmatrix}$$

(27)

for some $3 \times 3$ matrix $M$ which is independent of $x$ and $y$ (and hence depends only on $\lambda$ and $z$). The matrix $M(\lambda, -z)$ defines the $\mathbb{C}((z^{-1}))$-linear symplectic transformation $U : \mathcal{H}_X \to \mathcal{H}_Y$ which we seek. It remains to calculate the analytic continuations and to determine the matrix $M$.

Step 6: Analytic Continuation. To compute the analytic continuation of $I_Y(y, z)$ we use the Mellin–Barnes method. Good references for this are [13, 18] Appendix; [36]. First, take the expression (20) for $I_Y$ and apply the identity $\Gamma(x)\Gamma(1-x) = \pi / \sin(\pi x)$ until each factor $\Gamma(a + bd)$ which occurs has $b > 0$:

$$I_Y(y, z) = -\Theta_Y \sum_{d \geq 0} \frac{\Gamma(3d - \frac{\lambda - 3\varphi}{z} + d)}{\Gamma(1 + \frac{\varphi}{z} + d)}(-1)^dy^{d+p/z}$$

(28)

where

$$\Theta_Y = \pi^{-1}z^{3}\Gamma(1 + \frac{\varphi}{z})^3 \Gamma(1 + \frac{\lambda - 3\varphi}{z}) \sin \left( \pi \left( \frac{\lambda - 3\varphi}{z} \right) \right).$$

Then, in view of [36] Lemma 3.3, consider the contour integral

$$\int_C \Theta_Y \frac{\Gamma(3s - \frac{\lambda - 3\varphi}{z})\Gamma(s)\Gamma(1-s)}{\Gamma(1 + \frac{\varphi}{z} + s)}q^{s+p/z}$$

(29)

The equation (25) is the Picard–Fuchs equation associated to the Landau–Ginzburg mirror to $Y$. The fact that the quantum cohomology of $Y$ can be determined from this Picard–Fuchs equation has been proved many times from many different points of view: see e.g. [17, 27, 31, 42].
Writing this in terms of the co-ordinate $x$, the matrix $\tilde{Y}$ does not have a simple form, but in the non-equivariant limit it becomes:

\[
\sum_{n \geq 0} \frac{(-1)^n}{3} \frac{\pi}{\sin(\pi \frac{\lambda-3p}{3z})} \frac{1}{\Gamma(1 + \frac{\lambda}{3} - \frac{n}{3})} y^{\lambda/3-n/3}.
\]

Writing this in terms of the co-ordinate $x$, we find that the analytic continuation $\tilde{I}_Y(x, -z)$ is equal to:

\[
-z x^{\lambda/2} \sum_{n \geq 0} \frac{(-x)^n}{3} \frac{\pi}{\sin(\pi \frac{\lambda-3p}{3z})} \frac{1}{\Gamma(1 + \frac{\lambda}{3} - \frac{n}{3})} \frac{1}{\Gamma(1 - \lambda - \frac{3p}{z})}.
\] (30)

**Step 7: Compute the Symplectic Transformation.** Our final step is to compute the linear symplectic transformation $U : \mathcal{H}_X \to \mathcal{H}_Y$ represented by the matrix $M(\lambda, -z)$. We have $U(I_X(x, -z)) = I_Y(x, -z)$, and

\[
I_X(x, -z) = -z x^{\lambda/2} \left( \mathbf{1}_0 - \frac{x}{z} \mathbf{1}_{1/3} + \frac{x^2}{2z^2} \mathbf{1}_{2/3} + O(x^3) \right).
\] (31)

As the transformation $U$ does not depend on $x$, we can compute it by equating powers of $x$ in (30) and (31):

\[
U(\mathbf{1}_0) = \frac{1}{3} \frac{\sin(\pi \frac{\lambda-3p}{3z})}{\sin(\pi \frac{\lambda-3p}{3z})} \frac{\Gamma(1 - \frac{\lambda}{3})}{\Gamma(1 + \frac{\lambda}{3} - \frac{n}{3})} \frac{\Gamma(1 - \lambda - \frac{3p}{z})}{\Gamma(1 + \frac{\lambda}{3} - \frac{n}{3})}.
\]

\[
U(\mathbf{1}_{1/3}) = \frac{z}{3} \frac{\sin(\pi \frac{\lambda-3p}{3z})}{\sin(\pi \frac{\lambda-3p}{3z} + \frac{1}{3})} \frac{\Gamma(1 - \frac{\lambda}{3})}{\Gamma(1 + \frac{\lambda}{3} - \frac{n}{3})} \frac{\Gamma(1 - \lambda - \frac{3p}{z})}{\Gamma(1 + \frac{\lambda}{3} - \frac{n}{3})}.
\]

\[
U(\mathbf{1}_{2/3}) = \frac{z^2}{3} \frac{\sin(\pi \frac{\lambda-3p}{3z})}{\sin(\pi \frac{\lambda-3p}{3z} + \frac{2}{3})} \frac{\Gamma(1 - \frac{\lambda}{3})}{\Gamma(1 + \frac{\lambda}{3} - \frac{n}{3})} \frac{\Gamma(1 - \lambda - \frac{3p}{z})}{\Gamma(1 + \frac{\lambda}{3} - \frac{n}{3})}.
\]

The matrix $M$ of $U$ does not have a simple form, but in the non-equivariant limit it becomes

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{2\pi}{\sqrt{3} \Gamma(\frac{1}{3})} & -\frac{2\pi}{\sqrt{3} \Gamma(\frac{1}{3})} \\
0 & -\frac{\sqrt{3} \pi}{\sqrt{3} \Gamma(\frac{1}{3})} & \frac{\sqrt{3} \pi}{\sqrt{3} \Gamma(\frac{1}{3})}
\end{pmatrix}.
\] (32)
From this point of view it is not obvious \textit{a priori} that $U$ is a symplectomorphism, or that it satisfies conditions (a) and (c) in Conjecture 2.1 — this is one advantage of the more sophisticated approach taken in [18,39] — but now that we have an explicit expression for $U$ it is easy to check these things.

**Theorem 4.4** (The Crepant Resolution Conjecture for $[\mathbb{C}^3/\mathbb{Z}_3]$). Conjecture 2.1 holds for $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_3]$, $Y = K_{\mathbb{P}^2}$.

\textit{Proof.} It remains only to check that, after analytic continuation, $U$ maps $\mathcal{L}_X$ to $\mathcal{L}_Y$. But $U$ was constructed so as to map $I_X$ to the analytic continuation of $I_Y$, and $\mathcal{L}_X$ (respectively $\mathcal{L}_Y$) is uniquely determined by the fact that $x \mapsto I_X(x,-z)$ is a family of elements of $\mathcal{L}_X$ (respectively that $y \mapsto I_Y(y,-z)$ is a family of elements of $\mathcal{L}_Y$). Thus $U$ maps $\mathcal{L}_X$ to the analytic continuation of $\mathcal{L}_Y$. \qed

**Corollary 4.5** (The Cohomological Crepant Resolution Conjecture for $[\mathbb{C}^3/\mathbb{Z}_3]$). The algebra obtained from the $T$-equivariant small quantum cohomology algebra of $Y = K_{\mathbb{P}^2}$ by analytic continuation in the parameter $q_1$ followed by the specialization $q_1 = 1$ is isomorphic to the $T$-equivariant Chen–Ruan orbifold cohomology of $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_3]$.

\textit{Proof.} The quantity $c^1$ defined in (14) is zero. Now apply Corollary 3.1 \qed

**Remark.** The symplectic transformation (32) with $z = 1$ looks similar to the symplectic transformation computed by Aganagic–Bouchard–Klemm in [3], but it is not the same. It would be interesting to understand the source of the discrepancy.

5. **Example II: $\mathcal{X} = K_{\mathbb{P}(1,1,3)}$**

In this example we take $\mathcal{X} := K_{\mathbb{P}(1,1,3)}$ to be the total space of the canonical bundle of the weighted projective space $\mathbb{P}(1,1,3)$ and $Y$ to be the crepant resolution of the coarse moduli space of $\mathcal{X}$. We use essentially the same methods as before. The slight changes in method are needed to cope with the fact that, unlike all the other examples considered in this paper, the non-compact toric variety $Y$ is not presented as the total space of a vector bundle.

**Toric Geometry.** Consider the action of $(\mathbb{C}^\times)^2$ on $\mathbb{C}^5$ such that $(s,t) \in (\mathbb{C}^\times)^2$ acts as
\[
(x,y,z,u,v) \mapsto (tx,ty,sz,st,s^{-3}u,s^{-2}tv)
\]
(33)
The secondary fan is shown in Figure 5; the roman numerals there label the different chambers. There is an exact sequence:

\[
\begin{pmatrix}
0 & 1 \\
0 & 1 \\
1 & 0 \\
1 & -3 \\
-2 & 1
\end{pmatrix} \rightarrow \mathbb{Z}^2 \rightarrow \mathbb{Z}^5 \rightarrow \mathbb{Z}^3 \rightarrow 0,
\]
and so each chamber in the secondary fan corresponds to a toric orbifold with fan equal to some triangulation of the rays $e_1 + e_3$, $-e_1 + 3e_2 + e_3$, $-e_2 + e_3$, $e_2 + e_3$, $e_3$.

![Figure 5: The secondary fan for $Y$](image)

The fans are cones over the following pictures in the plane $z = 1$:

The toric orbifold corresponding to a chamber $C$ in the secondary fan is the GIT quotient $\mathbb{C}^5/\xi((\mathbb{C}^\times)^2)$, $\xi \in C$. This is produced by deleting an appropriate union of co-ordinate subspaces from $\mathbb{C}^5$ and then
taking the quotient by the action \((33)\). When \(C\) is chamber I, the corresponding toric orbifold is the resolution \(Y\); chamber II gives rise to the canonical bundle \(X = K_{\mathbb{P}(1,1,3)}\); chamber III gives the orbifold \([\mathbb{C}^3/\mathbb{Z}_5]\) where \(\mathbb{Z}_5\) acts on \(\mathbb{C}^3\) with weights \((1, 1, 3)\); and chamber IV gives the canonical bundle \(K_{\mathbb{P}(1,2,2)}\).

| chamber | locus to delete | quotient |
|---------|----------------|----------|
| I       | \{z = u = 0\} \cup \{x = y = z = 0\} \cup \{x = y = v = 0\} | \(Y\) |
| II      | \{u = 0\} \cup \{x = y = z = 0\} | \(X = K_{\mathbb{P}(1,1,3)}\) |
| III     | \{u = 0\} \cup \{v = 0\} | \([\mathbb{C}^3/\mathbb{Z}_5]\) |
| IV      | \{v = 0\} \cup \{x = y = z = 0\} | \(K_{\mathbb{P}(1,2,2)}\) |

Table 1: The different GIT quotients given by the secondary fan for \(Y\).

In this section we study the crepant resolution

\[
\begin{array}{ccc}
Y & \overset{K_{\mathbb{P}(1,1,3)}}{\longrightarrow} & X \\
\downarrow & & \downarrow \\
\mathbb{P}(1,1,3) & \overset{[\mathbb{C}^3/\mathbb{Z}_5]}{\longrightarrow} & \mathbb{C}^3/\mathbb{Z}_5
\end{array}
\]  

induced by moving from chamber I to chamber II. In the next section we consider the crepant partial resolution

\[
\begin{array}{ccc}
K_{\mathbb{P}(1,1,3)} & \overset{[\mathbb{C}^3/\mathbb{Z}_5]}{\longrightarrow} & \mathbb{C}^3/\mathbb{Z}_5 \\
\downarrow & & \\
\mathbb{P}(1,1,3) & \longrightarrow & \mathbb{C}^3/\mathbb{Z}_5
\end{array}
\]

obtained by moving from chamber II to chamber III. We will not discuss chamber IV at all.

**The \(T\)-Action.** The action of \(T = \mathbb{C}^\times\) on \(\mathbb{C}^5\) such that \(\alpha \in T\) maps

\[(x, y, z, u, v) \mapsto (x, y, z, u, \alpha v)\]

descends to give actions of \(T\) on \(X\), \(Y\), and \(X\). The induced action on \(X\) is the canonical \(\mathbb{C}^\times\)-action on the line bundle \(K_{\mathbb{P}(1,1,3)} \to \mathbb{P}(1,1,3)\); it covers the trivial action on \(\mathbb{P}(1,1,3)\). The crepant resolution \((33)\) is \(T\)-equivariant.

**Bases for Everything.** We have

\[r := \text{rank } H^2(Y; \mathbb{C}) = 2, \quad s := \text{rank } H^2(X; \mathbb{C}) = 1.\]

Let \(p_1, p_2 \in H(Y)\) denote the \(T\)-equivariant Poincaré-duals to the divisors \(\{z = 0\}\) and \(\{x = 0\}\) respectively, so that

\[
H(Y) = \mathbb{C}(\lambda)[p_1, p_2]/(p_2^2(\lambda + p_2 - 2p_1), p_1(p_1 - 3p_2), p_2^2p_2).
\]
Set 
\[ \varphi_0 = 1, \quad \varphi_1 = p_1, \quad \varphi_2 = p_2, \quad \varphi_3 = p_1 p_2, \quad \varphi_4 = p_2^2. \]

Write the inertia stack \( \mathcal{I} \) of \( \mathcal{X} \) as the disjoint union \( \mathcal{X}_0 \coprod \mathcal{X}_{1/3} \coprod \mathcal{X}_{2/3} \), where \( \mathcal{X}_f \) is the component of the inertia stack corresponding to the fixed locus of the element \( (1, e^{2\pi i}) \) \( \in \mathbb{C}^* \). We have \( \mathcal{X}_0 = K_{\mathbb{P}(1,1,3)} \) and \( \mathcal{X}_{1/3} = \mathcal{X}_{2/3} = \mathbb{Z}/3 \). Define \( I_f \) to be the class which restricts to the unit class on the component \( \mathcal{X}_f \) and restricts to zero on the other components, and let \( p \in H^2(\mathcal{X}) \) denote the first Chern class of the line bundle \( \mathcal{O}(1) \to \mathbb{P}(1,1,3) \), pulled back to \( K_{\mathbb{P}(1,1,3)} \) via the natural projection and then regarded as an element of Chen–Ruan cohomology via the inclusion \( \mathcal{X} = \mathcal{X}_0 \to \mathcal{I} \). Set
\[ \phi_0 = 1_0, \quad \phi_1 = p, \quad \phi_2 = p^2, \quad \phi_3 = 1_{1/3}, \quad \phi_4 = 1_{2/3}, \]
so that \( r_1 = \frac{1}{3} \).

**Step 1: A Family of Elements of \( \mathcal{L}_Y \).** Consider
\[
I_Y(y_1, y_2, z) := \sum_{k,l \geq 0} \frac{\Gamma(1 + \frac{p_2}{z})^2}{\Gamma(1 + \frac{p_2}{z} + l)^2} \frac{\Gamma(1 + \frac{p_2}{z} + k)}{\Gamma(1 + \frac{2}{z} + k)} \frac{\Gamma(1 + \frac{p_2 - 3p_1}{z})}{\Gamma(1 + \frac{2p_2 - 3p_1}{z} + k)} \prod_{m \leq 0} (p_1 - 3p_2 + mz) \prod_{m \leq -3l} (p_1 - 3p_2 + mz) \prod_{m \leq -2l} (\lambda + p_2 - 2p_1 + mz) \prod_{m \leq -2k} (\lambda + p_2 - 2p_1 + mz).
\]

This series converges, in a region where \( |y_1| \) and \( |y_2| \) are sufficiently small, to a multi-valued analytic function of \( (y_1, y_2) \) which takes values in \( \mathcal{L}_Y \). We have:
\[
I_Y(y_1, y_2, z) = \sum_{k,l \geq 0} \frac{y_1^{k+p_1/z} y_2^{l+p_2/z}}{\prod_{m=1}^{m=k} (p_1 + mz)^2 \prod_{m=1}^{m=k} (p_2 + mz) \prod_{m=0}^{m=k} (p_1 - 3p_2 + mz) \prod_{m=0}^{m=k-3l} (p_1 - 3p_2 + mz) \prod_{m=0}^{m=k-2l} (\lambda + p_2 - 2p_1 + mz) \prod_{m=0}^{m=k-2k} (\lambda + p_2 - 2p_1 + mz)}.
\]

Note that all but finitely many terms in the infinite products here cancel.

**Proposition 5.1.**
\[ I_Y(y_1, y_2, -z) \in \mathcal{L}_Y \quad \text{for all } (y_1, y_2) \text{ in the domain of convergence of } I_Y. \]

**Proof.** The argument which proves Theorem 0.1 in [30] also proves the claim here. Theorem 0.1 as stated only applies to compact toric varieties, but the proof works for the non-compact toric variety \( Y \) as well. The reader who would prefer not to check this should wait for the full generality of [21]. \( \square \)

**Step 2: \( I_Y \) Determines \( \mathcal{L}_Y \).** We have:

**Corollary 5.2.**
\[ J_Y(q_1, q_2, z) = e^{\lambda g(y_1, y_2)/z} I_Y(y_1, y_2, z) \]

where:
\[
q_1 = y_1 \exp \left( 2g(y_1, y_2) - f(y_1, y_2) \right), \quad q_2 = y_2 \exp \left( 3f(y_1, y_2) - g(y_1, y_2) \right),
\]
\[
f(y_1, y_2) = \sum_{0 \leq k \leq l/2} \frac{(-1)^{3l-k}(3l-k-1)!}{(l!)^2 k!(l-2k)!} y_1^k y_2^l, \quad g(y_1, y_2) = \sum_{0 \leq k \leq l/3} \frac{(-1)^{2k-l}(2k-l-1)!}{(l!)^2 k!(k-3l)!} y_1^k y_2^l.
\]

**Proof.** We argue exactly as in Corollary 4.2. Note that
\[
I_Y(y_1, y_2, z) = z + p_1 \left[ \log y_1 - f(y_1, y_2) + 2g(y_1, y_2) \right] + p_2 \left[ \log y_2 + 3f(y_1, y_2) - g(y_1, y_2) \right] - \lambda g(y_1, y_2) + O(z^{-1}).
\]

It follows from Propositions 3.3 and 5.1 that
\[ y \mapsto e^{-\lambda g(y_1, y_2)/z} I_Y(y_1, y_2, -z) \]

is a family of elements of \( \mathcal{L}_Y \). But
\[ e^{-\lambda g(y_1, y_2)/z} I_Y(y_1, y_2, -z) = -z + p_1 \log q_1 + p_2 \log q_2 + O(z^{-1}), \]

where \( q_1 \) and \( q_2 \) are defined above, and the unique family of elements of \( \mathcal{L}_Y \) of this form is \( (q_1, q_2) \mapsto J_Y(q_1, q_2, -z) \). \( \square \)
It follows, as before, that the series defining \( J_Y(q_1, q_2, z) \) converges (to a multivalued analytic function) when \( |q_1| \) and \( |q_2| \) are sufficiently small. Proposition 3.3 implies that \( \mathcal{L}_Y \) is uniquely determined by the fact that \( (y_1, y_2) \mapsto I_Y(y_1, y_2, -z) \) is a family of elements of \( \mathcal{L}_Y \).

**Aside: Computing Gromov–Witten Invariants of \( Y \).** As in the previous example, one can invert the change of variables \( (y_1, y_2) \mapsto (q_1, q_2) \) and read off genus-zero Gromov–Witten invariants of \( Y \). We will not do this.

**Step 3: A Family of Elements of \( \mathcal{L}_X \).** Let

\[
I_X(x_1, x_2, z) := z \sum_{d, e \geq 0, 3d \in \mathbb{Z}} x_1^{3d+3p/z} x_2^{3e} \frac{\prod_{b \leq 0} (p+bz)^2}{(3e)!} \frac{\prod_{b \leq -d-e} (p+bz)^2}{(3b)!} \frac{\prod_{b \leq -d-e} (\lambda - 5p + bz)}{\prod_{1 \leq m \leq 3d} (3m + mz)} 1^{\frac{1}{\text{frac}(-d)}}. \tag{36}
\]

This converges, in some open set \( \{(x_1, x_2) \in \mathbb{C}^\times \times \mathbb{C} : |x_1| \text{ and } |x_2| \text{ are sufficiently small}\} \), to a multivalued analytic function which takes values in \( \mathcal{H}_X \).

**Proposition 5.3.**

\[ I_X(x_1, x_2, -z) \in \mathcal{L}_X \quad \text{for all } (x_1, x_2) \text{ in the domain of convergence of } I_X. \]

**Proof.** We first show that

\[ I_X(x, 0, -z) \in \mathcal{L}_X \quad \text{for all } (x, 0) \text{ in the domain of convergence of } I_X. \tag{37} \]

For this we argue exactly as in Proposition 4.1 combining Theorem 3.4 with [22, Theorem 1.7]. Theorem 3.4 here tells us how to modify the small \( J \)-function of \( \mathbb{P}(1, 1, 3) \) and Theorem 1.7 in [22] tells us how to compute the small \( J \)-function of \( \mathbb{P}(1, 1, 3) \). Proposition 5.3 then follows from [11] and Iritani’s Reconstruction Theorem [37] cf. Example 4.15.

**Step 4: \( I_X \) Determines \( \mathcal{L}_X \).** We have:

**Corollary 5.4.**

\[ J_X^{\text{big}}(\tau, z) \big|_{\tau = p \log q + r 1_{1/3}} = e^{\lambda g(x_1, x_2)/z} I_X(x_1, x_2, z) \]

where

\[
q = x_1^3 \exp \left( 5g(x_1, x_2) \right), \quad g(x_1, x_2) = \sum_{0 \leq e \leq d; \frac{\text{frac}(d)}{3d} \in \mathbb{Z}; (d,e) \neq (0,0)} (-1)^{d+e} \frac{(5d + e - 1)! x_1^{3d} x_2^{3e}}{(d-e)! (3d)! (3e)!},
\]

\[ r = h(x_1, x_2) \]

\[
h(x_1, x_2) = \sum_{d,e \geq 0; \frac{\text{frac}(d)}{3d} \in \mathbb{Z}; (d,e) \neq (0,0)} \frac{\Gamma \left( \frac{3}{4} \right)^3 x_1^{3d} x_2^{3e}}{\Gamma(1+d-e)^2 \Gamma(1-5d-e)(3d)!(3e)!}. \]

**Proof.** Argue exactly as in Corollary 4.2.

This implies that the series \( J_X^{\text{big}}(\tau, z) \big|_{\tau = p \log q + r 1_{1/3}} \) converges, in a region where \( |q| \) and \( |r| \) are sufficiently small, to a multivalued analytic function of \( q \) and \( r \) which takes values in \( \mathcal{H}_X \). It also implies, via Proposition 3.3, that \( \mathcal{L}_X \) is uniquely determined by the fact that \( (x_1, x_2) \mapsto I_X(x_1, x_2, -z) \) is a family of elements of \( \mathcal{L}_X \).
Aside: Computing Gromov–Witten Invariants of $X$. We can use Corollary 4.3 to calculate genus-zero Gromov–Witten invariants of $X$, by computing the first few terms of the power series inverse to the “mirror map” $(x_1, x_2) \mapsto (q, r)$. This gives:

$$\langle 1_{1/3}/0,1,1/3 \rangle = -2,$$

$$\langle 1_{1/3}/0,1,4/3 \rangle = \frac{3757}{648},$$

$$\langle 1_{1/3}, 1_{1/3}/0,1,2/3 \rangle = -\frac{13}{18},$$

$$\langle 1_{1/3}, 1_{1/3}/0,1,3 \rangle = \frac{1}{3},$$

$$\langle 1_{1/3}, 1_{1/3}, 1_{1/3}, 1_{1/3}/0,1,1/3 \rangle = -\frac{2}{27},$$

and so on.

Step 5: The $B$-model Moduli Space and the Picard–Fuchs System. The $B$-model moduli space $\mathcal{M}_B$ here is the toric orbifold corresponding to the secondary fan for $Y$ (Figure 5). Each chamber of the secondary fan gives a co-ordinate patch on $\mathcal{M}_B$: the co-ordinates $(y_1, y_2)$ coming from chamber I are dual respectively to $p_1$ and $p_2$, and the co-ordinates $(x_1, x_2)$ from chamber II are dual respectively to $p_1$ and $p_1 - 3p_2$. These co-ordinate patches are related by

$$y_1 = x_1 x_2 \quad x_1 = y_1 y_2^{1/3}$$

$$y_2 = x_2^{-3} \quad x_2 = y_2^{-1/3}.$$  

(39)

We regard $I_Y(y_1, y_2, z)$ as a function on the co-ordinate patch corresponding to chamber I and $I_X(x_1, x_2, z)$ as a function on the co-ordinate patch corresponding to chamber II. Let

$$D_{x_1} = z x_1 \frac{\partial}{\partial y_1}, \quad D_{x_2} = z x_2 \frac{\partial}{\partial y_2}, \quad D_{y_1} = z y_1 \frac{\partial}{\partial y_1}, \quad D_{y_2} = z y_2 \frac{\partial}{\partial y_2}.$$ 

The components of $I_Y(y_1, y_2, z)$, with respect to the basis $\{\varphi_\alpha\}$, form a basis of solutions to the system of differential equations:

$$D_{y_1}(D_{y_1} - 3D_{y_2})f = y_1(\lambda + D_{y_2} - 2D_{y_1})(\lambda + D_{y_2} - 2D_{y_1} - z)f$$

$$D_{y_2}^2(\lambda + D_{y_2} - 2D_{y_1})f = y_2(D_{y_1} - 3D_{y_2})(D_{y_1} - 3D_{y_2} - z)(D_{y_1} - 3D_{y_2} - 2z)f.$$  

(40)

and the components of $I_X(x_1, x_2, z)$, with respect to the basis $\{\phi_\alpha\}$, form a basis of solutions to the system of differential equations:

$$D_{x_2}(D_{x_2} - 2z)(D_{x_2} - 2z)f = x_2^3(\frac{4}{3}D_{x_1} - \frac{1}{3}D_{x_2})^2(\lambda - \frac{2}{3}D_{x_1} - \frac{1}{3}D_{x_2} - x_2^2)f$$

$$D_{x_1}D_{x_2}f = x_1 x_2(\lambda - \frac{2}{3}D_{x_1} - \frac{1}{3}D_{x_2})(\lambda - \frac{2}{3}D_{x_1} - \frac{1}{3}D_{x_2} - x_2^2)f$$

$$D_{x_1}(D_{x_1} - z)(D_{x_1} - z)(\frac{4}{3}D_{x_1} - \frac{1}{3}D_{x_2})^2f = x_1^2 \prod_{k=0}^{k=4}(\lambda - \frac{2}{3}D_{x_1} - \frac{1}{3}D_{x_2} - kz_2)f.$$ 

(41)

The change of variables (39) turns the system of differential equations (40) into the system of differential equations (41). It follows that if $I_Y$ is the analytic continuation of $I_Y$ to a neighbourhood of $(x_1, x_2) = (0, 0)$ then there exists a $\mathbb{C}(z)$-linear map $\mathcal{U} : \mathcal{H}_X \rightarrow \mathcal{H}_Y$ such that $\mathcal{U}(I_X(x_1, x_2, -z)) = I_Y(x_1, x_2, -z)$. As before, the map $\mathcal{U}$ is the linear symplectomorphism that we seek. To determine it, we first calculate the analytic continuation $I_Y$.

Step 6: Analytic Continuation. To calculate $I_Y$ we, for each $k \geq 0$, extract the coefficient of $y_1^k$ from (35) and then analytically continue it to a region where $|y_2|$ is large, using the Mellin–Barnes method described in Section II. The result is:

$$\tilde{I}_Y(x_1, x_2, z) = \sum_{k,n \geq 0} \frac{(-1)^{n+k}}{3.n!} \sin \left( \pi \left[ \frac{p_1 - 3p_2}{z} \right] \right) \frac{\Gamma(1 + \frac{p_2}{z})^2}{\sin \left( \pi \left[ \frac{p_1 - 3p_2 + k + 2}{z} \right] \right)} \frac{\Gamma(1 + \frac{p_1}{z})}{\Gamma(1 + \frac{p_1}{z} + k)} \frac{\Gamma(1 + \frac{p_1 - 3p_2}{z})}{\Gamma(1 + \frac{3p_1 - 5p_2}{z} - \frac{nk}{z} + k)} x_1^{k+p_1} x_2^n.$$  

(42)
Step 7: Compute the Symplectic Transformation. Since \( U(I_Y(x_1, x_2, -z)) = \overline{I_Y(x_1, x_2, -z)} \), we can read off the transformation \( U \) by comparing coefficients of \( x_1^i x_2^j (\log x_1)^c \) in (30) and (42). This gives:

\[
U(1_0) = \frac{\sin \left( \pi \left[ \frac{p_2 - 3p_3}{3} \right] \right)}{3 \sin \left( \pi \left[ \frac{p_2 - 3p_3}{3} \right] \right)} \frac{\Gamma(1 - p_2)}{\Gamma(1 - p_2)^2} \frac{\Gamma(1 - \frac{p_2 - 3p_3}{3})}{\Gamma(1 - \frac{3p_3}{3})}
\]

\[
U(p) = \frac{p_1}{3} U(1_0)
\]

\[
U(p^2) = \frac{p_1^2}{9} U(1_0)
\]

\[
U(1_{1/3}) = -\frac{z}{3} \sin \left( \pi \left[ \frac{p_2 - 3p_3}{3} \right] \right) \frac{\Gamma(1 - p_2)}{\Gamma(1 - p_2)^2} \frac{\Gamma(1 - \frac{p_2 - 3p_3}{3})}{\Gamma(1 - \frac{3p_3}{3})}
\]

\[
U(1_{2/3}) = \frac{z^2}{3} \sin \left( \pi \left[ \frac{p_2 - 3p_3}{3} \right] \right) \frac{\Gamma(1 - p_2)}{\Gamma(1 - p_2)^2} \frac{\Gamma(1 - \frac{p_2 - 3p_3}{3})}{\Gamma(1 - \frac{3p_3}{3})}
\]

The non-equivariant limit \( \lim_{\lambda \to 0} U \) has matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\pi^2}{9} & 0 & \frac{2\pi^2}{9} & \frac{\pi^2}{9} & \frac{2\pi^2}{9} \\
0 & 0 & 0 & \frac{2\pi^2}{9} & \frac{\pi^2}{9} & \frac{2\pi^2}{9} \\
0 & 0 & 0 & \frac{4\pi^2}{9} & \frac{2\pi^2}{9} & \frac{4\pi^2}{9} \\
0 & 0 & 0 & \frac{4\pi^2}{9} & \frac{2\pi^2}{9} & \frac{4\pi^2}{9} \\
0 & 0 & 0 & \frac{4\pi^2}{9} & \frac{2\pi^2}{9} & \frac{4\pi^2}{9}
\end{pmatrix}
\]

(43)

**Theorem 5.5** (The Crepant Resolution Conjecture for \( K_{P(1,1,3)} \)). Conjecture 2.1 holds for \( X = K_{P(1,1,3)} \) and \( Y \) its crepant resolution.

**Proof.** Argue exactly as in the proof of Theorem 4.2. \( \square \)

**Conclusions.** Having proved the Crepant Resolution Conjecture in this case, we can now extract information about small quantum cohomology using Corollary 3.2. When we do this, we find that the quantum corrections to Ruan’s conjecture do not vanish:

**Corollary 5.6.** Let \( X = K_{P(1,1,3)} \) and let \( Y \to X \) be the crepant resolution of the coarse moduli space of \( X \). There is a power series

\[
f(u) = \frac{2\pi}{\sqrt{3} \Gamma \left( \frac{2}{3} \right)^3} \left( -2u^{1/3} + \frac{3757}{648} u^{4/3} + \ldots \right)
\]

such that the algebra obtained from the small quantum cohomology algebra of \( Y \) by analytic continuation in the parameter \( q_2 \) followed by the substitution

\[
q_i = \begin{cases}
    e^{f(u)} u^{1/3} & i = 1 \\
    e^{-3f(u)} & i = 2
\end{cases}
\]

is isomorphic to the small quantum cohomology algebra of \( X \), via an isomorphism which sends \( p \in H(X) \) to \( \frac{1}{2} p_1 \in H(Y) \).

**Proof.** This is Corollary 3.2. The quantities \( c_1 \) and \( c_2 \) defined in (13) are zero, and the power series \( f(u) \) comes from equations (15) and (35). \( \square \)

6. **Example III:** \( X = \mathbb{C}^3/\mathbb{Z}_5 \), \( Y = K_{P(1,1,3)} \)

We next consider an example of a crepant partial resolution. Let \( X \) be the orbifold \( \mathbb{C}^3/\mathbb{Z}_5 \) where \( \mathbb{Z}_5 \) acts on \( \mathbb{C}^3 \) with weights \( (1, 1, 3) \). The coarse moduli space \( X \) of \( X \) is the quotient singularity \( \mathbb{C}^3/\mathbb{Z}_5 \), and a crepant partial resolution \( Y \) of \( X \) is the canonical bundle \( K_{P(1,1,3)} \). We make the obvious modifications to our general setup, replacing the vector space \( H(Y) \) with

\[
H(Y) := H_{CR,T}(Y; \mathbb{C}) \otimes \mathbb{C}(\lambda)
\]

and writing \( Y \) for the coarse moduli space of \( Y \). In this section we omit all details, as the argument is completely parallel to that in Sections 4 and 5.

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The secondary fan is shown in Figure 7 the B-model moduli space $M_B$ is $\mathbb{P}(1,5)$, and the $I$-functions are

$$I_X(x_1, x_2, z) := z \sum_{k,l \geq 0} \frac{x_1^k x_2^l}{k! l! \cdot z^{k+l}} \prod_{b:0 \leq b < \frac{\lambda}{3} - b z} \prod_{b:0 \leq b < \frac{2b+1}{3} - \frac{\lambda}{3}} \prod_{\text{frac}(b) = \frac{b}{\text{frac}(b)}} \prod_{b:0 \leq b < \frac{2b+1}{3} - \frac{\lambda}{3}} \left( \frac{\lambda}{3} - b z \right)^{\frac{\lambda}{3} - b z} 1_{\text{frac}(b) = \frac{b}{\text{frac}(b)}}$$

(c.f. [19, Theorem 4.6 and Proposition 6.1]) and

$$I_Y(y_1, y_2, z) := z \sum_{d \geq 0, e \geq 0, 3d \in \mathbb{Z}} \sum_{\frac{3d+3p}{z} = \frac{3e}{z}} \prod_{\text{frac}(b) = \frac{b}{\text{frac}(b)}} \prod_{b:0 \leq b < \frac{\lambda}{3} - b z} \prod_{b:0 \leq b < \frac{2b+1}{3} - \frac{\lambda}{3}} \left( \frac{\lambda}{3} - b z \right)^{\frac{\lambda}{3} - b z} 1_{\text{frac}(b) = \frac{b}{\text{frac}(b)}}$$

Use the bases

$$\phi_0 = 1_0, \quad \phi_1 = 1_{1/5}, \quad \phi_2 = 1_{2/5}, \quad \phi_3 = 1_{3/5}, \quad \phi_4 = 1_{4/5}$$

for $H(\mathcal{X})$ and

$$\varphi_0 = 1_0, \quad \varphi_1 = p, \quad \varphi_2 = p^2, \quad \varphi_3 = 1_{1/3}, \quad \varphi_4 = 1_{2/3}$$

for $H(\mathcal{Y})$: for the notation see Sections [4] and the discussion above Theorem 3.6. The Mellin–Barnes method produces a linear symplectomorphism $U : \mathcal{H}_Y \rightarrow \mathcal{H}_Y$ such that $U(I_X(x_1, x_2, -z)) = I_Y(x_1, x_2, -z)$ where $I_Y$ is the analytic continuation of $I_Y$. In the non-equivariant limit $\lambda \to 0$, the matrix of $U$ is given by

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\\nd(\phi) & e(\phi) & 0 & 0 \\
0 & -\frac{1}{5} & 0 & 0 \\
-\frac{1}{3} & -\frac{1}{2} & \sqrt{3} e(\phi) & \sqrt{3} e(\phi) \\
0 & 0 & \sqrt{2} e(\phi) & \sqrt{2} e(\phi) \\
0 & 0 & 0 & \sqrt{2} e(\phi)
\end{pmatrix}
$$

Thus Conjecture 2.1 holds, exactly as stated, for the crepant partial resolution $\mathcal{Y} \to X$.

**Theorem 6.1.** Conjecture 2.1 holds for $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_5], \mathcal{Y} = K_{\mathbb{P}(1,1,3)}$. \qed

When we try to draw conclusions about small quantum cohomology, however, a new phenomenon emerges. For simplicity, let us discuss this in the non-equivariant limit $\lambda \to 0$, indicating this by a (⋆) following our equations. In Section 3.1 when we were considering $\mathcal{X} = [\mathbb{C}^3/\mathbb{Z}_5]$, we had

$$U(1_{[\mathbb{C}^3/\mathbb{Z}_5]}) = 1_{K_{\mathbb{P}(1,1,3)}} + O(z^{-2}) \quad \text{(⋆)}$$

and hence

$$U(J_{[\mathbb{C}^3/\mathbb{Z}_5]}(z)) = -z 1_{K_{\mathbb{P}(1,1,3)}} + O(z^{-1}). \quad \text{(⋆)}$$

We can therefore identify $U(J_{[\mathbb{C}^3/\mathbb{Z}_5]}(z))$ with

$$J_{K_{\mathbb{P}(1,1,3)}}(q, z) = -z 1_{K_{\mathbb{P}(1,1,3)}} + p \log q + O(z^{-1}) \quad \text{(⋆)}$$

by setting $q = 0$, or in other words $q = 1$. This is how the specialization of quantum parameters in the Cohomological Crepant Resolution Conjecture arises: see Corollary 3.3 and 24. In the case at hand, however, we have

$$U(1_{[\mathbb{C}^3/\mathbb{Z}_5]}) = 1_{K_{\mathbb{P}(1,1,3)}} + \frac{1}{5} \Gamma(\phi)^3 1_{1/5} + O(z^{-2}) \quad \text{(⋆)}$$

Figure 7: The secondary fan for $\mathcal{Y} = K_{\mathbb{P}(1,1,3)}$
and thus

$$U(J_{[C^3/Z_3]}(-z)) = -z 1_{K_{[C^3/Z_3]}} - \frac{1}{5} \Gamma(\frac{3}{2})^3 1_{1/3} + O(z^{-1}),$$

which is not equal to the small $J$-function $J_{K_{[C^3/Z_3]}}(q, -z)$ for any $q$ because the class $1_{1/3}$ comes from the twisted sector. We do have an equality

$$U(J_{[C^3/Z_3]}(-z)) = J_{K_{[C^3/Z_3]}}^{big}(\tau, -z)$$

where $\tau = - \frac{1}{5} \Gamma(\frac{3}{2})^3 1_{1/3},$ (*)

but it does not let us conclude anything about small quantum cohomology. This is because there is no Divisor Equation for Chen–Ruan classes from the twisted sector, so we cannot trade the shift $\tau = 0 \sim \tau = e \gamma_{1/3}$ for a specialization $q \sim e^r$ (or indeed for any other specialization of the quantum parameter).

**Conclusions.** In light of this, it seems likely that any generalization of the Cohomological Crepant Resolution Conjecture (and hence also any generalization of Ruan’s Conjecture) to crepant partial resolutions cannot be phrased in terms of small quantum cohomology alone: it must involve big quantum cohomology. It seems also that any such generalization will no longer involve only roots of unity.

### 7. Example IV: A Toric Flop

Finally, consider the action of $\mathbb{C}^\times$ on $\mathbb{C}^5$ such that $s \in \mathbb{C}^\times$ acts as

$$(x, y, z, u, v) \mapsto (sx, sy, sz, s^{-1}u, s^{-2}v)$$

The secondary fan is:

![Secondary Fan Diagram](image)

Figure 8: The secondary fan for a toric flop

For $\xi$ in the right-hand chamber of the secondary fan, the GIT quotient $Y := \mathbb{C}^5/\xi \mathbb{C}^\times$ is the total space of the vector bundle $O(-1) \oplus O(-2) \to \mathbb{P}^2$. For $\xi$ in the left-hand chamber, the GIT quotient $X := \mathbb{C}^5/\xi \mathbb{C}^\times$ is the total space of $O(-1) \oplus O(-1) \oplus O(-1) \to \mathbb{P}(1, 2)$. The birational transformation $Y \dasharrow X$ induced by moving from the right-hand chamber to the left-hand chamber is a flop [24].

To treat this example, we need to make some changes to our general setup (described in Section 2), but the required modifications are obvious and so we make them without comment. As we have not yet discussed a birational transformation of this type, we once again give some details of the calculation: the reader will see that our methods apply here too without significant change.

**Bases and $I$-Functions.** We have

$$r := \text{rank } H^2(Y; \mathbb{C}) = 1, \quad s := \text{rank } H^2(X; \mathbb{C}) = 1.$$

The action of $T = \mathbb{C}^\times$ on $\mathbb{C}^5$ such that $\alpha \in T$ acts as

$$(x, y, z, u, v) \mapsto (\alpha x, \alpha y, \alpha z, u, v)$$

induces actions of $T$ on $X$ and $Y$, and the flop $Y \dasharrow X$ is $T$-equivariant. Let $p$ be the canonical $T$-equivariant lift of the first Chern class of the line bundle $O(1) \to \mathbb{P}^2$, so that

$$H(Y) = \mathbb{C}(\lambda)[p]/(p^3).$$

We use the basis

$$\varphi_0 = 1, \quad \varphi_1 = p, \quad \varphi_2 = p^2$$

for $H(Y)$. The inertia stack of $X$ is the disjoint union $X_0 \bigsqcup X_{1/2}$, where $X_0 = X$ and $X_{1/2} = B\mathbb{Z}_2$. Let $1_f \in H(X)$ denote the class which restricts to the unit class on the component $X_f$ and restricts to zero on the other component, and let $p \in H(X)$ denote the canonical $T$-equivariant lift of the first Chern class of the line bundle $O(1) \to \mathbb{P}(1, 2)$, pulled back to $X$ via the natural projection $X \to \mathbb{P}(1, 2)$ and then regarded as an element of Chen–Ruan cohomology via the inclusion $X = X_0 \to \mathcal{I}X$. We use the basis

$$\phi_0 = 1_f, \quad \phi_1 = p, \quad \phi_2 = 1_{1/2}$$

for $H(X)$.  

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Let
\[ I_Y(y, z) = z \sum_{d \geq 0} \prod_{-2d < m \leq 0}(2\lambda - 2p + mz) \prod_{d < m \leq 0}(\lambda - p + mz) \frac{y^{d+p/z}}{\pi^d d^{d+1}} \]
and let
\[ I_X(x, z) = z x^{-\lambda/z} \sum_{d \geq 0} x^{d+p/z} \prod_{b \geq 0, b \in \mathbb{Z}} \frac{\prod_{b \geq 0, b \leq d}(\lambda - p + mz)^3}{\lambda/b = \text{frac}(b)} \frac{y^{d+p/z}}{\pi^d d^{d+1}} \frac{1}{\text{frac}(d)} \]
Arguing exactly as before yields:

**Proposition 7.1.** We have \( I_Y(y, -z) \in \mathcal{L}_Y \) for all \( y \) such that \( 0 < |y| < \frac{1}{4} \), and \( I_X(x, -z) \in \mathcal{L}_Y \) for all \( x \) such that \( |x| < 4 \). \( \square \)

Furthermore, as
\[ x^{-\lambda/z} I_X(x, -z) = -z + p \log x + O(z^{-1}) \quad \text{and} \quad I_Y(y, -z) = -z + p \log y + O(z^{-1}) \]
we conclude that:

**Corollary 7.2.**
\[ J_Y(u, z) = x^{\lambda/z} I_X(u, z) \quad \text{and} \quad J_Y(q, z) = I_Y(q, z). \]

Note that the mirror maps here are trivial. \( \square \)

It follows that the Lagrangian submanifold-germs \( \mathcal{L}_Y \) and \( \mathcal{L}_Y \) are uniquely determined by Proposition 7.1.

**The B-model Moduli Space and Analytic Continuation.** The B-model moduli space \( \mathcal{M}_B \) here is \( \mathbb{P}^1 \): it has a co-ordinate patch (with co-ordinate \( x \)) corresponding to \( \mathcal{X} \) and a co-ordinate patch (with co-ordinate \( y \)) corresponding to \( Y \), related by \( y = x^{-1} \). Regard \( I_X(x, z) \) as a function on the co-ordinate patch corresponding to \( \mathcal{X} \) and \( I_Y(y, z) \) as a function on the co-ordinate patch corresponding to \( Y \), and denote by \( \tilde{I}_Y(x, z) \) the analytic continuation of \( I_Y \) to a neighbourhood of \( x = 0 \). As before, both \( I_X \) and \( \tilde{I}_Y \) have components which form a basis of solutions to the Picard–Fuchs differential equation
\[ -x D^3 f = (\lambda + D)(2\lambda + 2D)(2\lambda + 2D - z)f \quad \text{and} \quad D = z x^{\vartheta}. \]

It follows that there exists a \( \mathbb{C}(z^{-1}) \)-linear isomorphism \( \mathcal{U} : \mathcal{H}_X \rightarrow \mathcal{H}_Y \) such that \( \mathcal{U}(I_X(x, -z)) = \tilde{I}_Y(x, -z) \). This is the linear symplectomorphism that we seek.

The Mellin–Barnes method gives
\[ \tilde{I}_Y(x, z) = z x^{-\lambda/z} \sum_{k \geq 0} \frac{x^{k+\frac{1}{2}}}{2(2k+1)!} \frac{\Gamma(1 + \frac{z}{2})^3}{\Gamma(1 + \frac{1}{2} - \frac{k}{2})} \frac{\Gamma(-k - \frac{1}{2}) \Gamma(1 + \frac{2\lambda - 2z}{z})}{\pi^d d^{d+1}} \]
\[ = z x^{-\lambda/z} \sum_{k \geq 0} \frac{x^{k+\frac{1}{2}}}{2(2k+1)!} \frac{\Gamma(1 + \frac{z}{2})^3}{\Gamma(1 + \frac{1}{2} - \frac{k}{2})} \frac{\Gamma(-k - \frac{1}{2}) \Gamma(1 + \frac{2\lambda - 2z}{z})}{\pi^d d^{d+1}} \frac{\left( H_{2k} + H_k - \frac{3z}{2} + \frac{1}{2} \log y - \frac{3}{2} \psi(1 + \frac{1}{2} - k) - \frac{z}{2} \cot \left( \frac{\lambda - p}{z} \right) \right)}{\pi^d d^{d+1}} \]
where \( \gamma \) is Euler’s constant, \( \psi(z) \) is the logarithmic derivative of \( \Gamma(z) \), and \( H_k \) is the \( k \)th harmonic number. Thus
\[ \mathcal{U}(1_0) = -\frac{\Gamma(1 + \frac{z}{2})^3}{\Gamma(1 + \frac{1}{2})^3} \frac{(\lambda - \frac{1}{2}) \sin \left( \frac{2\lambda - 2z}{z} \pi \right)}{\pi} \left( \frac{3z}{2} + \frac{3}{2} \psi(1 + \frac{1}{2} - k) - \frac{z}{2} \cot \left( \frac{\lambda - p}{z} \right) \right), \]
\[ \mathcal{U}(p) = -\frac{z}{2} \frac{\Gamma(1 + \frac{z}{2})^3}{\Gamma(1 + \frac{1}{2})^3} \frac{(\lambda - \frac{1}{2}) \sin \left( \frac{2\lambda - 2z}{z} \pi \right)}{\pi} , \]
\[ \mathcal{U}(1_{1/2}) = -\frac{z^2}{4} \frac{\Gamma(1 + \frac{z}{2})^3}{\Gamma(1 + \frac{1}{2})^3} \frac{(\lambda - \frac{1}{2}) \sin \left( \frac{2\lambda - 2z}{z} \pi \right)}{\pi} \frac{\left( \frac{4\lambda - 2z}{z} \pi \right)}{\pi} \sin \left( \frac{2\lambda - 2z}{z} \pi \right). \]
Note that
\[
\begin{align*}
\mathbb{U}(1_0) &= 1 + O(z^{-2}), \\
\mathbb{U}(p) &= (\lambda - p) + O(z^{-2}), \\
\mathbb{U}(1_{1/2}) &= (\lambda - p)^2 + O(z^{-1}).
\end{align*}
\] (45)

In the non-equivariant limit \( \lambda \to 0 \), our expressions for \( \mathbb{U} \) simplify:
\[
\mathbb{U}(1_0) \to 1 - \frac{\pi^2 p^2}{3z^2}, \quad \mathbb{U}(p) \to -p, \quad \mathbb{U}(1_{1/2}) \to p^2.
\]

**Theorem 7.3** (A “Flop Conjecture” for \( \mathcal{X} \) and \( \mathcal{Y} \)). There is a choice of analytic continuations of \( \mathcal{L}_\mathcal{X} \) and \( \mathcal{L}_\mathcal{Y} \) such that, after analytic continuation, \( \mathbb{U}(\mathcal{L}_\mathcal{X}) = \mathcal{L}_\mathcal{Y} \). Furthermore \( \mathbb{U} : \mathcal{H}_\mathcal{X} \to \mathcal{H}_\mathcal{Y} \) is a degree-preserving \( \mathbb{C}(z^{-1}) \)-linear symplectic isomorphism which satisfies
\[
\begin{align*}
(\text{a}) & \quad \mathbb{U}(1_0) = 1_Y + O(z^{-1}); \\
(\text{b}) & \quad \mathbb{U}(\mathcal{H}_\mathcal{X}) \oplus \mathcal{H}_\mathcal{Y} = \mathcal{H}_\mathcal{Y}.
\end{align*}
\]

**Proof.** Argue as in the proof of Theorem 4.3 \( \square \)

The transformation \( \mathbb{U} \) does not satisfy any condition analogous to property \( \text{(b)} \) in Conjecture 2.1 but we should not expect this. Property \( \text{(b)} \) arises from the fact that \( \mathbb{U} \) intertwines certain monodromies (let us call them the relevant monodromies) of the system of Picard–Fuchs equations coming from mirror symmetry: see [18, Proposition 4.7]. In the case of toric crepant resolutions the relevant monodromies generate \( H^2(\mathcal{X}) \), but for general toric crepant birational transformations this is not the case. The Mellin–Barnes method will always produce a transformation \( \mathbb{U} \) which intertwines the relevant monodromies, but in the case at hand this is vacuously true as the set of relevant monodromies is empty. For a general flop

\[
\begin{array}{c}
\mathcal{X} \\
\downarrow p_1 \\
Z \\
\uparrow \downarrow \nonumber \\
\downarrow p_2 \\
\mathcal{Y}
\end{array}
\]

it is reasonable to expect that property \( \text{(b)} \) should be replaced by the assertion
\[
\mathbb{U} \circ \left( p_1^* \alpha \cup \mathcal{C}_\mathcal{R} \right) = \left( p_2^* \alpha \cup \mathcal{C}_\mathcal{R} \right) \circ \mathbb{U}
\]
for all \( \alpha \in H^2(Z; \mathbb{C}) \); this condition is also vacuous here.

**Corollary 7.4** (A Ruan/Bryan–Graber-style Flop Conjecture). The \( \mathbb{C}(\lambda) \)-linear map \( \mathbb{U}_\infty : H(\mathcal{X}) \to H(\mathcal{Y}) \) given by
\[
\begin{align*}
\mathbb{U}_\infty(1_0) &= 1, \\
\mathbb{U}_\infty(p) &= (\lambda - p), \\
\mathbb{U}_\infty(1_{1/2}) &= (\lambda - p)^2,
\end{align*}
\]
induces an algebra isomorphism between the small quantum cohomology of \( \mathcal{X} \) and the algebra obtained from the small quantum cohomology of \( \mathcal{Y} \) by analytic continuation in the quantum parameter \( q \) followed by the substitution \( u = q^{-1} \).

**Proof.** Look at equation (45), and then apply the discussion in [24 §9]. \( \square \)

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