AN IMPROVED LOWER BOUND FOR MULTICOLOR RAMSEY NUMBERS AND THE HALF-MULTIPLICITY RAMSEY NUMBER PROBLEM

WILL SAWIN

ABSTRACT. The multicolor Ramsey number problem asks, for each pair of natural numbers \( \ell \) and \( t \), for the largest \( \ell \)-coloring of a complete graph with no monochromatic clique of size \( t \). Recent work of Conlon-Ferber and Wigderson have improved the longstanding lower bound for this problem. We make a further improvement by replacing an explicit graph appearing in their constructions by a random graph. In addition, we formalize the problem of producing a graph useful for this argument as the half-multiplicity Ramsey problem, and prove basic lower and upper bounds.

Let \( r(t; \ell) \) be the multicolor Ramsey number, that is, the minimum \( r \) such that every \( \ell \)-coloring of the edges of the complete graph \( K_r \) on \( r \) vertices has a monochromatic clique of size \( t \).

Building on a breakthrough of Conlon and Ferber [4], Wigderson [9] proved the lower bound

\[
r(t; \ell) \geq 2^{3(\ell-2)t/8} + \frac{1}{2^{t+o(t)}}.
\]

Wigderson’s variant of Conlon and Ferber’s construction involves an explicit graph constructed from linear algebra over \( \mathbb{F}_2 \), a series of \( \ell - 2 \) random maps from the vertex set of \( K_r \) to the vertex set of this graph, and a random coloring of edges of \( K_r \) not sent to edges of this graph by any map with the last two colors.

We give the slight improvement

\[
r(t, \ell) \geq e^{266027(\ell-2)t/8} + \frac{1}{2^{t+o(t)}},
\]

with the exponent .266027 improving over \( \frac{3}{8} \log 2 = .2599302 \). We obtain this by replacing the explicit linear algebraic graph with an Erdős-Rényi random graph, with the probability that each edge lies in the graph slightly less than \( 1/2 \). (If we took the probability to be exactly \( 1/2 \), we would recover Wigderson’s bound [9]). Thus our argument follows, in part, the suggestion of Alon and Rödl [1, p. 140] to combine random homomorphisms with random graphs for the multicolor Ramsey number problem.

We further analyze this approach to multicolor Ramsey numbers by observing that the key property of this graph is that it provides a good bound for a certain variant of the two-color Ramsey number problem, which we call the half-multiplicity Ramsey number problem.

This problem is approximately equidistant between the classical two-color Ramsey problem, which asks for the largest graph with no independent set of size \( s \) and no clique of size \( t \), and the Ramsey multiplicity problem ([6], see also [5, §2.6]), which asks us to minimize the total number of independent sets of size \( s \) and cliques of size \( t \). The half-multiplicity problem asks us to minimize the total number of independent sets of size \( s \) in a graph with no cliques of size \( t \).

For natural numbers \( s \) and \( t \), define \( \mathbb{P}(s, t) \) to be the infimum over graphs \( G \) with no clique of size \( t \) of the probability that, for \( v_1, \ldots, v_s \) vertices of \( G \) chosen independently and uniformly at random, \( \{v_1, \ldots, v_s\} \) is an independent set.
By choosing \( v_1, \ldots, v_s \) independently, we incorporate independent sets of size \( < s \) in the definition of \( P(s, t) \). This is a minor variant of what is usually done in Ramsey multiplicity, where we only count independent sets of size exactly \( s \), and seems to lead to slightly more elegant statements. For instance, we can consider graphs of different sizes on equal footings, so there is no need to consider a limit as the graph size goes to infinity. This variant is also particularly compatible with the random homomorphism approach to multicolored Ramsey problems.

We prove a number of basic results for the half-multiplicity problem that are close analogues of known results for the usual Ramsey and Ramsey multiplicity problems. As the proofs are not far from existing proofs, the purpose of this section is largely expository.

In addition, we observe, as in the Ramsey multiplicity problem, that explicit graphs provide equally good values as random graphs for certain special values of \( s \) and \( t \). For example, the Conlon-Ferber graph is as efficient as a random graph for \( s = 3t/4 \). It would be interesting to see if this is true for other ratios of \( s \) and \( t \).

This article was written while the author served as a Clay Research Fellow. I would like to thank Yuval Wigderson for many helpful comments.

1. LOWER BOUNDS AND APPLICATION TO MULTICOLOR RAMSEY NUMBERS

Our first lemma relates \( P(t, t) \) to the multicolor Ramsey numbers. It is a slight variant of [10, Lemma 3.1].

**Lemma 1.** For any \( \ell, t > 2 \), we have
\[
 r(t; \ell) \geq P(t, t)^{-\frac{\ell-2}{2} 2^{t-1}}.
\]

**Proof.** Let \( N = \lceil P(t, t)^{-\frac{\ell-2}{2}} 2^{t-1} \rceil \). It suffices to find a \( \ell \)-coloring of \( K_N \) with no monochromatic clique of size \( t \).

Fix a graph \( G \) with no clique of size \( t \) and such that the probability that \( t \) random vertices of \( G \) form an independent set is at most \( P(t, t) + \epsilon \).

We construct such an \( \ell \)-coloring of \( K_N \) as follows: We choose \( \ell - 2 \) functions \( f_1, \ldots, f_{\ell-2} \) from the vertex set of \( K_N \) to the vertex set of \( G \) at random, independently and uniformly. We color the edge between a pair of vertices \( x, y \) the \( i \)th color if \( i \) is minimal such that \( f_i(x) \) and \( f_i(y) \) are connected by an edge of \( G \). If there is no \( i \) such that \( f_i(x) \) and \( f_i(y) \) are connected by an edge of \( G \), then we randomly color this pair of vertices with either the \( \ell - 1 \)st color or \( \ell \)th color, each with probability \( \frac{1}{2} \), independently for each remaining pair.

We claim this coloring has no monochromatic cliques with positive probability.

Since \( G \) has no clique of size \( t \), there is no set of \( t \) vertices sent to a clique of size \( t \) by \( f_i \), and thus the \( i \)th color has no cliques of size \( t \). So it suffices to show that the probability that there is a clique of size \( t \) of the last two colors is less than 1. Fix a set \( S \) of \( t \) vertices of \( K_N \). For \( S \) to be a clique of the \( \ell - 1 \)st or \( \ell \)th color, it must first have no edges of the first \( \ell - 2 \) colors, so we must not have \( f_i(x) \) and \( f_i(y) \) connected by an edge of \( G \) for any \( i \). By assumption, this occurs with probability at most \( (P(t, t) + \epsilon)^{\ell-2} \). Furthermore, the randomly selected colors for the edges must be equal, which occurs with probability \( 2^{1-(t-1)/2} \).

By the union bound, the probability that there is any monochromatic clique is thus at most
\[
 \leq \binom{N}{t} (P(t, t) + \epsilon)^{t-2} 2^{1-(t-1)/2} < \frac{N^t}{t!} P(t, t)^{t-2} 2^{1-(t-1)/2} < N^t (P(t, t) + \epsilon)^{t-2} 2^{1-(t-1)/2} \leq 1
\]
by the definition of \( N \).

\( \square \)
Our next lemma gives a probabilistic upper bound for $\mathbb{P}(s, t)$.

**Lemma 2.** For any $p \in [0, 1]$, for any $s, t$, we have

\[
\mathbb{P}(s, t) \leq e^{\frac{t(4s \log(1 - p) - t \log(p)) \log(p)}{s \log(1 - p)} + s \log + O(t)}.
\]

**Proof.** Let $G$ be a random graph on $M$ vertices, with $M$ to be chosen later, where each pair of vertices is connected by an edge with probability $p$. We have

\[
\mathbb{P}(s, t) \leq \mathbb{P}(\{v_1, \ldots, v_s\} \text{ is an independent set of } G \mid v_1, \ldots, v_s \in [M] \text{ independent, uniform}) \over \mathbb{P}(G \text{ contains no clique of size } t)
\]

since this is an upper bound for the expectation of the probability that $\{v_1, \ldots, v_s\}$ is independent conditional on $G$ containing no clique of size $t$, and we can always choose a graph that attains at most the conditional expectation.

We estimate the probabilities in both the numerator and the denominator using the union bound. For the denominator, the probability that $G$ does contain a clique of size $t$ is at most

\[
\binom{M}{t} p^{\frac{t(1-t)}{2}} \leq \frac{M^t}{t!} p^{\frac{t(1-t)}{2}} = o(1) \text{ if we take } M = \lceil p^{-t/2} \rceil. \quad \text{Since } \frac{1}{1-o(1)} = O(1), \text{ the denominator contributes an } O(1) \text{ factor.}
\]

For the numerator, taking $k$ to be the cardinality of $\{v_1, \ldots, v_s\}$, we have

\[
\mathbb{P}(\{v_1, \ldots, v_s\} \text{ is an independent set of } G \mid v_1, \ldots, v_s \in [M] \text{ independent, uniform}) \leq \sum_{k=0}^{s} \binom{M}{k} \binom{s}{k} k! \frac{1-p}{M^s} \leq \sum_{k=0}^{s} \frac{\binom{s}{k} k!}{M^s} \frac{1-p}{(1-p)^{k}}
\]

\[
\leq s! \max_{0 \leq k \leq s} \frac{(1-p)^{k}}{p^{-(s-k)t/2}} \leq s! \max_{0 \leq k \leq s} e^{k(1-p)/2 + (s-k)t \log(p)/2}
\]

where $\binom{s}{k}$ are the Stirling numbers of the second kind, since $\sum_{k=0}^{s} \binom{s}{k} \leq s!$.

Now the quadratic $k(k - 1) \log(1 - p)/2 + (s - k)t \log(p)/2$ is maximized by taking $k = \frac{t \log p}{2 \log(1 - p)} + \frac{1}{2}$ and attains a maximum value of

\[
\left( \frac{t \log p}{2 \log(1 - p)} \right)^{2} \log(1 - p) + \left( s - \frac{t \log p}{2 \log(1 - p)} \right) \frac{t \log p}{2} + O(t)
\]

\[
= \left( s - \frac{t \log p}{4 \log(1 - p)} \right) \frac{t}{2} \log p + O(t) = \frac{t(4 \log(1 - p) - t \log(p)) \log(p)}{8 \log(1 - p)} + O(t).
\]

The $s!$ term is bounded by $e^{s \log s}$ and the $O(1)$ factor can be absorbed into $e^{O(t)}$. This gives \(\square\)

In this proof, when $\frac{t \log p}{4 \log(1 - p)} + \frac{1}{2} \geq s$, we could get a better bound using the constraint $k \leq s$, but this is never needed as increasing $p$ would always give an even better bound in that range.

Specializing to $s = t$, Lemma 2 gives

\[
\mathbb{P}(t, t) \leq e^{t^2 \theta + O(t^2)}
\]

where

\[
\theta = \min_{p \in [0, 1]} \frac{4 \log(1 - p) - t \log(p)}{8 \log(1 - p)} \approx -0.266027
\]

attained for $p = .454997$
Corollary 3. We have
\[ r(t, \ell) \geq e^{266027(\ell-2)t2^{\ell/2}+o(t)}. \]

Proof. Combining Lemma 1 and Lemma 2, we have
\[ r(t, \ell) \geq \mathbb{P}(t, t) - \ell - 2t \geq e^{-( \ell-2)p(4 \log(1-p) - \log(p)) \log(p) + (\ell-2) \log t + O(\ell-2) 2^{-\ell} t^{-1}}. \]
Taking \( p = .454997 \), the constant \(- \frac{(4 \log(1-p) - \log(p)) \log(p)}{8 \log(1-p)}\) becomes .266027, and the other terms, except for \( 2^{\ell} t^{-1} \), are lower-order and may be absorbed into \( 2^{o(t)} \). \( \square \)

2. Additional estimates on \( \mathbb{P}(s, t) \)

There is a close relationship between the probabilities \( \mathbb{P}(s, t) \) and the (two-colored) Ramsey numbers \( R(s, t) \), a variant of the relationship between the usual Ramsey and Ramsey multiplicity problems established in [6, Equation (4)].

Lemma 4. For any natural numbers \( s, t \), we have
\[ \mathbb{P}(s, t) \geq \frac{1}{R(s, t)}. \]
and for any \( a < s \), we have
\[ \mathbb{P}(s, t) \leq \left( \frac{R(a, t) - 1}{a - 1} \right) \left( \frac{a - 1}{R(a, t) - 1} \right)^s. \]

Proof. For the first inequality, there exists by definition a graph \( G \) with no cliques of size \( t \) and where the probability that \( s \) random elements form a clique is at most \( \mathbb{P}(s, t) + \epsilon \). Consider a random map from the vertex set of the complete graph of size \( R(s, t) \) to \( G \). The inverse image of the edge set of \( G \) under this map has no cliques of size \( t \), and its expected number of cliques of size \( s \) is \( \binom{R(s, t)}{s}(\mathbb{P}(s, t) + \epsilon) \), which must be \( \geq 1 \) since, by the definition of \( R(s, t) \), all such graphs have a clique of size \( s \). Taking \( \epsilon \) arbitrary small, we obtain the first inequality.

For the second inequality, fix a graph \( G \) with \( R(a, t) - 1 \) vertices containing no cliques of size \( t \) and no independent sets of size \( a \). In this graph, \( s \) elements form an independent set only if they are contained in a set of vertices of size \( a - 1 \), so the probability that \( s \) random elements form an independent set is at most the number \( \binom{R(a, t) - 1}{a - 1} \) of sets of size \( a - 1 \) times the probability \( \left( \frac{a - 1}{R(a, t) - 1} \right)^s \) that they are all contained in one such set. \( \square \)

We do not expect the bounds from Lemma 4 to be useful in many concrete cases. Instead, we should get better bounds for \( \mathbb{P}(s, t) \) by taking ideas from Ramsey theory and applying them to \( \mathbb{P}(s, t) \), but not so much better that they imply stronger bounds for the original Ramsey number problems.

Motivated by this, we prove a lower bound for \( R(s, t) \) (analogous to an upper bound for \( R(s, t) \)) using the neighborhood method):

Lemma 5. For any natural numbers \( t, s \), we have
\[ \mathbb{P}(s, t) \geq \min_{x \in [0, 1]} \left( \max(x^s \mathbb{P}(s, t - 1), (1 - x)^{s-1} \mathbb{P}(s - 1, t)) \right), \]

Compare the inequality
\[ R(s, t) \leq R(s, t - 1) + R(s - 1, t) \]
for Ramsey numbers, which can be equivalently expressed as

$$R(s, t) \leq \max_{x \in [0,1]} \left( \min(x^{-1}R(s, t - 1), (1 - x)^{-1}R(s - 1, t)) \right).$$

**Proof.** It suffices to show that for each graph $G$ lacking cliques of size $t$, the probability that $s$ uniformly random vertices form an independent set is at least $\min_{x \in [0,1]}(\max(x^s\mathbb{P}(s, t - 1), (1 - x)^{s-1}\mathbb{P}(s - 1, t)))$.

Fix such a graph $G$. Let $x$ be the maximum over vertices $v$ of $G$ of the fraction of vertices of $G$ that are connected to $v$ by an edge. (When calculating this fraction, $v$ is not counted as connected to itself).

If we let $v$ be a vertex attaining this maximum and $G_1$ the subgraph of $G$ consisting of vertices connected to $v$ by an edge, then $G_1$ has no clique of size $t - 1$, so the probability that $s$ vertices sampled uniformly at random from $G_1$ form a clique is at least $\mathbb{P}(s, t - 1)$. Since $s$ random elements of $G$ have a $x^s$ probability of all lying in $G_1$, and conditionally on lying in $G_1$, are uniformly distributed in $G_1$, it follows that the probability that $s$ random elements of $G$ form an independent set is at least $x^s\mathbb{P}(s, t - 1)$.

Let us next show that the probability that $s$ random elements $v_1, \ldots, v_s$ of $G$ form an independent set is at least $(1 - x)^{s-1}\mathbb{P}(s - 1, t)$. In fact we will show this for $v_1$ arbitrary and $v_2, \ldots, v_s$ uniformly random. Let $G_2$ be the subgraph of $G$ consisting of vertices not connected to $v_1$ by an edge, including $v_1$. By the definition of $x$, $|G_2| \geq (1 - x)|G|$ and so the probability that $v_2, \ldots, v_s$ lie in $G_2$ is at least $(1 - x)^{s-1}$. Conditionally on this, they are uniformly distributed in $G_2$, which has no clique of size $t$, so the conditional probability that they form an independent set is at least $\mathbb{P}(s - 1, t)$, and because none has an edge to $v_1$, they form an independent set with $v_1$ as well, with probability at least $(1 - x)^{s-1}\mathbb{P}(s - 1, t)$.

Thus, for some $x$, the probability that $s$ vertices in $G$ form an independent set is at least $\max(x^s\mathbb{P}(s, t - 1), (1 - x)^{s-1}\mathbb{P}(s - 1, t))$, and it follows that the probability is at least $\min_{x \in [0,1]}(\max(x^s\mathbb{P}(s, t - 1), (1 - x)^{s-1}\mathbb{P}(s - 1, t)))$, as desired.

Let $N(s, t)$ be the unique function that satisfies $N(s, t) = 1$ if $s = 1$ or $t = 2$ and

$$N(s, t) = \min_{x \in [0,1]} \left( \max(x^sN(s, t - 1), (1 - x)^{s-1}N(s - 1, t)) \right)$$

if $s > 1$ and $t > 2$.

**Lemma 6.** We have

$$\mathbb{P}(s, t) \geq N(s, t).$$

Thus $N(s, t)$ describes a lower bound for $\mathbb{P}(s, t)$ obtained by the neighborhood method. This is the analogue for half-multiplicity of [3, Theorem 4] in the Ramsey multiplicity problem.

**Proof.** This follows by induction. For the base cases $s = 1$ and $t = 2$, we observe $\mathbb{P}(1, t) = 1$ because one vertex always forms an independent set and $\mathbb{P}(s, 2) = 1$ because a graph with no cliques of size 2 has no edges and thus every set of vertices is independent.

For the induction step, using Lemma 5 and the induction hypothesis, we have

$$\mathbb{P}(s, t) \geq \min_{x \in [0,1]} \left( \max(x^s\mathbb{P}(s, t - 1), (1 - x)^{s-1}\mathbb{P}(s - 1, t)) \right)$$

$$\geq \min_{x \in [0,1]} \left( \max(x^sN(s, t - 1), (1 - x)^{s-1}N(s - 1, t)) \right) = N(s, t).$$
It is not clear if there exists a closed-form formula for $N(s, t)$, even approximately. Instead, we have the following lower bound, which shows that for $t$ a constant multiple of $s$, $N(s, t)$ is exponential in $-s^2$.

**Lemma 7.** We have

$$N(s, t) \geq \left( \frac{\left(\frac{s-1}{2}+t-2\right)^{\frac{s-1}{2}+t-2}}{\left(\frac{s-1}{2}\right)^{\frac{s-1}{2}}(t-2)^{t-2}} \right)^{-s}.$$

In particular, $N(t, t) \geq \left( \frac{3\sqrt{3}}{2} \right)^{-t^2} = e^{-0.95477...t^2}$, so our upper and lower bounds for $N(t, t)$ differ by a factor of slightly less than 4 in the exponent. This is the analogue for half-multiplicity of [3, Theorem 2]. It is likely possible to instead prove in this setting an analogue of the sharper bound [3, Theorem 1], by a similar analytic method, but we don’t pursue this here.

**Proof.** We have

$$N(s, t) = \min_{x \in [0, 1]} \left( \max(x^sN(s, t - 1), (1 - x)^{s-1}N(s - 1, t)) \right)$$

$$= \max \left( \min(x^sN(s, t - 1), (1 - x)^{s-1}N(s - 1, t)) \right)$$

since $x^sN(s, t - 1)$ is monotonically increasing in $x$ and $(1 - x)^{s-1}N(s - 1, t)$ is monotonically decreasing, so both the min-max and the max-min are attained at the unique point where they are equal. Thus, for any $y \in [0, 1]$

$$N(s, t) \geq \min(y^sN(s, t - 1), (1 - y)^{s-1}N(s - 1, t)).$$

We can now prove

$$N(s, t) \geq (1 - y)^{\frac{s(s-1)}{2}}y^{s(t-2)}$$

by induction on $s, t$, with base cases $s = 1$ because $1 \geq y^{t-2}$ and $t = 2$ because $1 \geq (1 - y)^{\frac{s(s-1)}{2}}$ and induction step

$$N(s, t) \geq \min(y^sN(s, t - 1),(1 - y)^{s-1}N(s - 1, t))$$

$$\geq \min(y^s(1 - y)^{\frac{s(s-1)}{2}}y^{s(t-3)}, (1 - y)^{s-1}(1 - y)^{\frac{s(s-1)(s-2)}{2}}y^{s-1(t-2)})$$

$$= \min((1 - y)^{\frac{s(s-1)}{2}}y^{s(t-2)}, (1 - y)^{\frac{s(s-1)}{2}}y^{s-1(t-2)}) = (1 - y)^{\frac{s(s-1)}{2}}y^{s(t-2)}.$$ 

Taking $y = \frac{t-2}{t+s-2}$, we obtain

$$N(s, t) \geq \left( \frac{\left(\frac{s-1}{2}+t-2\right)^{\frac{s-1}{2}+t-2}}{\left(\frac{s-1}{2}\right)^{\frac{s-1}{2}}(t-2)^{t-2}} \right)^{-s}.$$

The upper bound for $\mathbb{P}(s, t)$ arising from the probabilistic method is Lemma 2. In addition to the probabilistic method, we have access to an explicit construction, introduced by Conlon and Ferber [4]. The bound obtained this way is as follows.
Lemma 8. For $t$ even, we have

$$\mathbb{P}(s, t) \leq \begin{cases} O \left(2^{-\frac{s(s-1)}{2}}\right) & s \leq \frac{t}{2} - 1 \\ O \left(2^{-\frac{(4s-1)(t-2)}{s}}\right) & s \geq \frac{t}{2} - 1 \end{cases}$$

This is a reformulation of estimates from [4] and [9].

A weaker form of the small $s$ case follows by results of [2] from the fact that the graph $G$ described below is quasi-random, which is known by [3] Lemma 4.1.

Proof. Let $V$ be a vector space of dimension $t - 2$ over $\mathbb{F}_2$ endowed with a nondegenerate symplectic bilinear form $(\cdot, \cdot): V \times V \to \mathbb{F}_2$. Let $G$ be the graph whose vertex set is $V$ and with an edge between $v_1$ and $v_2$ if and only if $(v_1, v_2) = 1$.

Then $G$ contains no clique set of size $t$. To prove this, note that $t$ vectors in a clique would have to satisfy $(v_i, v_j) = 1 - \delta_{ij}$. Because $t$ is even, the matrix with entries $1 - \delta_{ij}$ has rank $t$, and thus $v_1, \ldots, v_t$ would have to be linearly independent, contradicting the fact that $V$ does not contain $t$ linearly independent vectors.

Any independent set of $G$ is contained in an isotropic subspace of $V$, since if the form vanishes on a set of vectors then it vanishes on all linear combinations. Since every isotropic subspace of $V$ of dimension $< t/2$ is contained in a subspace of dimension one greater, and every isotropic subspace of dimension $t/2 - 1$ is maximal, any independent set of size $s$ is contained in an isotropic subspace of dimension $\min(s, t/2 - 1)$.

If $s \leq \frac{t}{2} - 1$, note that the number of isotropic subspaces in $V$ of rank $s$ is

$$\frac{\prod_{i=0}^{s-1}(2^{t-2-2i} - 1)}{\prod_{i=0}^{s-1}(2^{s-1} - 1)} \leq O \left(2^{-\frac{s(s-1)}{2}}\right)$$

since in a basis for a maximal isotropic subspace we have $2^{t-2} - 1$ choices for the first vector, $2^{t-3} - 2$ choices for the second, $2^{t-4} - 4$ choices for the third, and so on, and the number of isotropic subspaces is the number of such bases divided by the number $(2^s-1)(2^s-2)\ldots(2^s-2^{s-1})$ of bases of a single isotropic subspace.

The probability that $s$ vectors will be contained in a given isotropic subspace of rank $s$ is $2^{-(t-2-s)s}$, so the total probability that $s$ vectors will form an independent set is $O \left(2^{-\frac{s(s-1)}{2}}\right)$ if $s \leq \frac{t}{2} - 1$.

In particular, the number of $\frac{t}{2} - 1$-dimensional isotropic subspaces is $O \left(2^{\frac{s(t-s)}{2}}\right)$, and the probability that $s$ vectors will be contained in one is $2^{-(t-2)s/2}$, so the total probability that $s$ vectors will form an independent set is $O \left(2^{-\frac{(4s-1)(t-2)}{s}}\right)$ if $s \geq \frac{t}{2} - 1$. \hspace{1cm} \square

To compare to earlier work, note that the bilinear form $v_i \cdot v_j$ on the space of vectors in $\mathbb{F}_2^t$ with Hamming weight even is symplectic, but not nondegenerate, since the all-ones vector has zero dot product with any vector. Taking the quotient by this vector produces a vector space of rank $t - 2$ with a nondegenerate symplectic bilinear form.

Specializing the random construction of Lemma [2] to $p = 1/2$, we obtain

$$\mathbb{P}(s, t) \leq e^{-\frac{t(t-s)}{8} \log(2) + s \log s + O(t)} = 2^{-\frac{t(t-s)}{8} + O(s \log(s) + t)},$$

which matches up to lower-order terms the bound obtained from Lemma 8.
In Lemma 2, the value of $p$ is optimal if it minimizes $\frac{t(4s\log(1-p) - t\log(p))\log(p)}{s\log(1-p)}$, which happens if

$$-\frac{4s}{1-p} - \frac{t}{p} + (4s\log(1-p) - t\log(p))\left(\frac{1}{p\log(p)} + \frac{1}{(1-p)\log(1-p)}\right) = 0.$$ 

For $p = 1/2$, this specializes to

$$-4s - t + 2(4s - t) = 0$$

or $s = 3t/4$.

Thus, when $s = 3t/4$, the explicit Lemma 8 gives the same bound as the random Lemma 2 up to lower-order terms. It would be interesting to determine if explicit constructions can match Lemma 2 for other values of $s/t$. It is possible that the graph, also studied by Conlon and Ferber [4], with vertex set the set of vectors in $v \in \mathbb{F}_q^n$ with $v \cdot v = 0$ and an edge connecting two vertices $v, w$ if $v \cdot w = 1$, for a finite field $\mathbb{F}_q$, would give the same bound as Lemma 2 for $p = 1/q$.

To further investigate the balance between structure and randomness for this problem, it might be worthwhile to calculate $P(s, t)$ for small $s, t$ and find the optimal graphs. This is a potentially subtler problem than the analogue for Ramsey numbers, since it does not a priori reduce to a finite calculation, but at least a few cases can be done.

**Lemma 9.** We have $P(2, t) = \frac{1}{t-1}$, with the optimal value attained by the complete graph $K_{t-1}$.

**Proof.** For $G$ a graph with $n$ vertices and $m$ edges, the probability that two random vertices form an independent set is $1 - \frac{2m}{n}$. The graph $K_{t-1}$ has no clique of size $t$ and has $n = (t-1)m = (t-1)(t-2)/2$, so indeed $P(s, t) \leq \frac{1}{t-1}$. To check $P(s, t) = \frac{1}{t-1}$, we apply Turán’s theorem that every graph $G$ on $n$ vertices with no clique of size $t$ has $\leq \frac{(t-2)n^2}{2(t-1)}$ edges and thus the probability that two random vertices form an independent set is at least $\frac{1}{t-1}$. \hfill \Box

**Lemma 10.** We have $P(3, 3) = \frac{1}{14}$, with the optimal value obtained by the complete graph $K_2$.

The lower bound on $P(3, 3)$ can also be deduced from the work of [7].

**Proof.** Certainly $K_2$ has no clique of size 3 and the probability that 3 vertices chosen at random form an independent set is the probability that they are all equal, which is $\frac{1}{4}$. It remains to check that in every triangle-free graph $G$, the probability that three random vertices $v_1, v_2, v_3$ form an independent set is at least $\frac{1}{4}$. Let $E_{ij}$ be the event that $v_i$ and $v_j$ are connected by an edge. Then

$$P(E_{12}) = P(E_{23}) = P(E_{13}) = \frac{1}{|G|} \sum_{v \in G} d_v.$$

By Cauchy-Schwarz,

$$P(E_{ij} \cap E_{jk}) = \frac{1}{|G|} \sum_{v \in G} \frac{d_v^2}{|G|^2} \geq \left(\frac{1}{|G|} \sum_{v \in G} \frac{d_v}{|G|}\right)^2 = P(E_{12})^2$$

so by inclusion-exclusion

$$P(v_1, v_2, v_3 \text{ independent}) = 1 - P(E_{12}) - P(E_{23}) - P(E_{13}) + P(E_{12} \cap E_{13}) + P(E_{12} \cap E_{23}) + P(E_{13} \cap E_{23}) - P(E_{12} \cap E_{13} \cap E_{23}) \geq 1 - 3P(E_{12}) + 3P(E_{12})^2 - 0 \geq 1 - \frac{3}{4} = \frac{1}{4}.$$

\hfill \Box
References

[1] Noga Alon and Vojtěch Rödl. Sharp bounds for some multicolor Ramsey numbers. *Combinatorica*, 25(2):125–141, March 2005. 10.1007/S00493-005-0011-9

[2] F. R. K. Chung, R. L. Graham, and R. M. Wilson. Quasi-random graphs. *Combinatorica*, 9(4):345–362, December 1989. [doi:10.1007/BF02125347](https://doi.org/10.1007/BF02125347)

[3] David Conlon. On the Ramsey multiplicity of complete graphs. arXiv:0711.4999, 2007.

[4] David Conlon and Asaf Ferber. Lower bounds for multicolor Ramsey numbers. *Advances in Mathematics*, 378:107528, February 2021. [doi:10.1016/j.aim.2020.107528](https://doi.org/10.1016/j.aim.2020.107528)

[5] David Conlon, Jacob Fox, and Benny Sudakov. Recent developments in graph Ramsey theory. arXiv:1501.02474, 2015.

[6] P. Erdős. On the number of complete subgraphs contained in certain graphs. *Publ. Math. Inst. Hungar. Acad. Sci.*, 7:459–464, 1962. [https://users.renyi.hu/~p_erdos/1962-14.pdf](https://users.renyi.hu/~p_erdos/1962-14.pdf)

[7] A. W. Goodman. On sets of acquaintances and strangers at any party. *American Mathematical Monthly*, 66:778–783, 1959.

[8] Pavel Pudlák, Vojtěch Rödl, and Petr Savický. Graph complexity. *Acta Informatica*, 25:515–535, 1988. [doi:10.1007/BF00279952](https://doi.org/10.1007/BF00279952)

[9] Yuval Wigderson. An improved lower bound on multicolor Ramsey numbers. arXiv:2009.12020, 2020.

[10] Yuval Wigderson. Lecture notes on multicolor Ramsey numbers. [http://web.stanford.edu/~yuvalwig/math/teaching/MulticolorRamsey.pdf](https://web.stanford.edu/~yuvalwig/math/teaching/MulticolorRamsey.pdf), 2020.