Regularity of Harmonic Functions
for a Class of Singular Stable-like Processes

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Abstract
We consider the system of stochastic differential equations
\[ dX_t = A(X_{t-}) dZ_t, \]
where \( Z_1^1, \ldots, Z_d^d \) are independent one-dimensional symmetric stable processes of order \( \alpha \), and the matrix-valued function \( A \) is bounded, continuous and everywhere non-degenerate. We show that bounded harmonic functions associated with \( X \) are Hölder continuous, but a Harnack inequality need not hold. The Lévy measure associated with the vector-valued process \( Z \) is highly singular.

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1 Introduction
A one-dimensional symmetric stable process of index \( \alpha \in (0, 2) \) is the Lévy process taking values in \( \mathbb{R} \) with no drift, no Gaussian part, and Lévy measure
\[ n(\mathbb{R}) = c_1/\|h\|^{1+\alpha} dh. \]
Let \( Z_t = (Z_t^1, \ldots, Z_t^d) \) be a vector of \( d \) independent one-dimensional symmetric stable processes of index \( \alpha \). Consider the system of stochastic differential equations
\[ dX_t^i = \sum_{j=1}^{d} A_{ij}(X_{t-}) dZ_t^j, \quad X_0^i = x_0^i, \quad i = 1, \ldots, d, \quad (1.1) \]
where \( x_0 = (x_0^1, \ldots, x_0^d) \in \mathbb{R}^d \) and \( A(x) \) is a bounded \( d \times d \) matrix-valued function that is continuous in \( x \) and everywhere non-degenerate, that is, the determinant \( \det(A(x)) \neq 0 \) for all \( x \). The main

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result of [2] is that under these conditions there is a unique weak solution to the system \((1.1)\) and the family \(\{X, \mathbb{P}^{x_0}, x_0 \in \mathbb{R}^d\}\) forms a strong Markov process on \(\mathbb{R}^d\). The process \(X\) may be referred to as stable-like because it possesses an approximate scaling property similar to the stable processes; see [4] and [5] for other examples where the term stable-like has been used. The system \((1.1)\) has been suggested as a possible model for a financial market with jumps in the security prices ([6]). Note that by Proposition 4.1 of [2], the infinitesimal generator of the Markov process \((1.1)\) has been suggested as a possible model for a financial market with jumps in the security processes; see [4] and [5] for other examples where the terms stable-like has been used. The system referred to as stable-like because it possesses an approximate scaling property similar to the stable 

\[ Lf(x) = \sum_{j=1}^{d} \int_{\mathbb{R}^d \setminus \{0\}} \left( f(x + a_j(x)w) - f(x) - w1_{\{|w| \leq 1\}} \nabla f(x) \cdot a_j(x) \right) \frac{c_1}{|w|^{1+\alpha}} dw, \]

where \(a_j(x)\) is the \(j\)th column of the matrix \(A(x)\). Associated with the operator \(L\) is the symbol

\[ \ell(x, u) := c_2 \sum_{j=1}^{d} |u \cdot a_j(x)|^\alpha, \quad x, u \in \mathbb{R}^d. \]

This means

\[ Lf(x) = \int_{\mathbb{R}^d} \ell(x, u)e^{-iu \cdot x} \hat{f}(u) du, \]

where \(\hat{f}\) denotes the Fourier transform of \(f\). This is an example of a pseudodifferential operator with singular state-dependent symbol.

We say that a function \(h\) that is bounded in \(\mathbb{R}^d\) is harmonic (with respect to \(X\)) in a domain \(D\) if \(h(X_{t \wedge \tau_D})\) is a martingale with respect to \(\mathbb{P}^x\) for every \(x \in D\), where \(\tau_D\) is the time of first exit from \(D\). The process \(X\) is shown to have no explosions in finite time in [2] and when \(D\) is bounded, it is easy to see from \((1.1)\) that \(\mathbb{P}^x(\tau_D < \infty) = 1\) for every \(x \in D\). So by the bounded convergence theorem and the strong Markov property of \(X\), a bounded function \(h\) on \(\mathbb{R}^d\) is harmonic in a bounded domain \(D\) if and only if

\[ h(x) = \mathbb{E}^x[h(X_{\tau_D})] \quad \text{for every } x \in D. \]

Consequently, every bounded harmonic function in a bounded domain \(D\) is the difference of two non-negative bounded harmonic functions in \(D\). It follows from Proposition 4.1 of [2] that a bounded \(C^2\) function \(u\) is harmonic in \(D\) if and only if \(\mathcal{L}u = 0\) in \(D\).

The main goal of this paper is to prove the Hölder continuity of functions which are bounded and harmonic with respect to \(X\) in a domain.

There are two reasons why the Hölder continuity is perhaps a bit unexpected. Consider the case where \(A\) is identically equal to the identity matrix, and so \(\overline{X} = Z\). Even in this case a Harnack inequality may fail; see Section [3]. Nevertheless the Hölder continuity of the harmonic functions holds. The other reason is that the process \(Z\) is quite singular. It is a Lévy process, but the support of its Lévy measure is the union of the coordinate axes. By contrast, the support of the Lévy measure for a \(d\)-dimensional (rotationally) symmetric stable process is all of \(\mathbb{R}^d\), a much more tractable situation.

The key to our method is the technique of Krylov-Safonov as given, for example, in the exposition in [1]. The most difficult step in our proof is the proof of a support theorem for \(X\); this is given in Section [2]. We remark that the current paper is the first one where the full strength of the Krylov-Safonov technique has been used in the context of pure jump processes.

For a Borel subset \(C \subset \mathbb{R}^d\), let \(T_C := \inf\{t \geq 0 : X_t \in C\}\) and \(\tau_C := \inf\{t \geq 0 : X_t \notin C\}\) be the first entrance and departure time of \(C\) by \(X\). Let \(|C|\) denote the Lebesgue measure of a Borel set.
C. The open ball of radius \( r \) centered at \( x \) will be denoted as \( B(x, r) \). The paths of \( Z_t \) are right continuous with left limits. We write

\[
Z_{t-} := \lim_{s \downarrow t, s < t} Z_s, \quad \Delta Z_t := Z_t - Z_{t-},
\]

and similarly \( X_{t-} \) and \( \Delta X_t \). The letter \( c \) with a subscript denotes a positive finite constant whose exact value is unimportant and may vary from one usage to the next. Constant \( c \) typically depends on \( \alpha \) and \( d \), but for convenience this dependence will not be explicitly mentioned throughout the paper.

2 Regularity

For \( 1 \leq i \leq d \), denote by \( e_i \) the unit vector in the \( x_i \) direction in \( \mathbb{R}^d \). Let \( x_0 \in \mathbb{R}^d \) and let \( B = B(x_0, 1) \). For simplicity, we write \( \tau \) for \( \tau_B \). We will use \( A(x)^{-1} \) to denote the inverse matrix of \( A(x) \).

\textbf{Proposition 2.1} There exist positive constants \( c_1, c_2 \) that depend only on the upper bound of \( A(x) \) and \( A(x)^{-1} \) on \( B \) such that

(a) \( \mathbb{E}^x[\tau] \leq c_1 \) for all \( x \in B \);

(b) \( \mathbb{E}^x[\tau] \geq c_2 \) for all \( x \in B(x_0, \frac{1}{2}) \).

\textbf{Proof.} (a) Let \( A_0 = \inf \{ |A(x)(e_1)| : x \in \overline{B} \} \). We know \( A_0 > 0 \) because \( A(x) \) is continuous in \( x \) and nondegenerate for each \( x \). Since the \( Z^i \)'s are independent one-dimensional symmetric \( \alpha \)-stable process, no two of them make a jump at the same time. So there exists a positive constant \( c_3 \) such that

\[
\mathbb{P} \left( \exists s \leq 1 : |\Delta Z_s^1| > 3/A_0 \text{ but } \Delta Z_s^k = 0 \text{ for } k = 2, \ldots, d \right) \geq c_3.
\]

Suppose there exists \( s \in [0, 1] \) such that \( |\Delta Z_s^1| > 3/A_0 \), \( \Delta Z_s^k = 0 \) for \( k = 2, \ldots, d \), and \( X_{s-} \in B \). Then by (1.1)

\[
|\Delta X_s^1| = |\Delta Z_s^1| |A(X_{s-})e_1| > 3
\]

if \( X_{s-} \in \overline{B} \). So with probability at least \( c_3 \), \( X \) will have left \( B \) by time 1. Hence if \( x \in B \),

\[
\mathbb{P}^x(\tau > 1) \leq 1 - c_3.
\]

Let \( \{\theta_t, t > 0\} \) denotes the usual shift operators for \( X \). By the Markov property,

\[
\mathbb{P}^x(\tau > m + 1) \leq \mathbb{P}^x(\tau > m, \tau \circ \theta_m > 1)
\]

\[
= \mathbb{E}^x[\mathbb{P}^{X_m}(\tau > 1); \tau > m]
\]

\[
\leq (1 - c_3)^m \mathbb{P}^x(\tau > m).
\]

By induction,

\[
\mathbb{P}^x(\tau > m) \leq (1 - c_3)^m,
\]

and (a) follows.

(b) Let

\[
\tilde{Z}_t^i := \sum_{s \leq t} \Delta Z_s^i 1(|\Delta Z_s^i| > 1) \quad \text{and} \quad \bar{Z}_t^i := Z_t^i - \tilde{Z}_t^i.
\]
Note
\[ \mathbb{E}[\mathcal{Z}'^i, \mathcal{Z}^i_t] = t \int_{-\beta}^{\beta} x^2 \frac{c_4}{|x|^{1+\alpha}} dx = c_5 t^{2-\alpha}. \]

Let \( \mathcal{X} \) solve
\[ d\mathcal{X}_t = A(\mathcal{X}_t) d\mathcal{Z}_t. \]

Note that for each \( i = 1, \ldots, d, \mathcal{X}^i \) is a purely discontinuous square integrable martingale with \( |\Delta \mathcal{X}^i_t| \leq c_6 \sum_{j=1}^d |\Delta \mathcal{Z}^j_t| \). Hence
\[ \langle \mathcal{X}^i, \mathcal{X}^i \rangle_t \leq c_7 \sum_{j=1}^d \langle \mathcal{Z}^j, \mathcal{Z}^j \rangle_t. \]

First by Chebyshev’s inequality and then by Doob’s inequality,
\[
\mathbb{P}^x \left( \sup_{s \leq t} |\mathcal{X}_s^i - \mathcal{X}_0^i| > \frac{1}{4d} \right) \leq 16d^2 \mathbb{E} \left[ \sup_{s \leq t} |\mathcal{X}_s^i - \mathcal{X}_0^i|^2 \right] \\
\leq 64d^2 \mathbb{E} \left[ (\mathcal{X}^i_t - \mathcal{X}^i_0)^2 \right] \\
= 64d^2 \mathbb{E} \left[ \mathcal{X}^i_t, \mathcal{X}^i_t \right] \\
\leq c_8 \sum_{j=1}^d \mathbb{E} \left[ \mathcal{Z}^j_t, \mathcal{Z}^j_t \right] \\
\leq c_9 t.
\]

Choose \( t \) small so that \( c_9 t \leq 1/4 \).

We can choose \( t \) smaller if necessary so that
\[ \mathbb{P}(\mathcal{Z}_s^j \neq 0 \text{ for some } s \in [0, t]) \leq 1/(4d). \]

So there exists \( t \) such that \( \mathbb{P}(\mathcal{Z}_s \neq Z_s \text{ for some } s \in [0, t]) \leq 1/4 \), and it follows that
\[ \mathbb{P}(\mathcal{X}_s \neq X_s \text{ for some } s \in [0, t]) \leq 1/4. \]

Therefore with probability at least \( 1/2 \) we have \( \sup_{s \leq t} |X_s - X_0| \leq 1/4 \) and so in particular
\[ \mathbb{P}^x(\tau > t) \geq 1/2 \quad \text{for } x \in B(x_0, \frac{1}{2}). \]

Consequently, we have \( \mathbb{E}^x \tau \geq t \mathbb{P}^x(\tau \geq t) \geq t/2 \) for \( x \in B(x, \frac{1}{2}) \).

**Proposition 2.2** There exist constants \( \eta_0 > 0, p_0 \geq 2, \) and \( c_1 \) that depend only on the upper bound of \( A(x) \) and \( A(x)^{-1} \) on \( B \) such that if the oscillation of \( A \) on \( B(x_0, 1) \) is less than \( \eta_0 \), then
\[
\mathbb{E}^x \left[ \int_0^\tau 1_C(X_s) ds \right] \leq c_1 |C|^{1/p_0}, \quad x \in B.
\]

**Proof.** Note that the process \( \{X_t, t \leq \tau\} \) is determined by the matrix \( A \) on \( B \) only. Without loss of generality, for this proof we redefine \( A \) for \( x \notin B \) so that \( A \) is continuous on \( \mathbb{R}^d \) and
\[ \eta := \sup_{x \in \mathbb{R}^d} \|A(x) - A(x_0)\| = \sup_{x \in B} \|A(x) - A(x_0)\|. \]
Let $R_\lambda$ and $\mathcal{L}_0$ be the resolvent and infinitesimal generator of the Levy process $Y_t = Y_0 + A(x_0)Z_t$, $\mathcal{L}$ the infinitesimal generator of $X$, $S_\lambda$ the resolvent of $X$, and $B := \mathcal{L} - \mathcal{L}_0$. There exist $\eta_0 > 0$ and $p_0 \geq 2$ so that the conclusion of Proposition 5.2 of [2] holds, namely, $\|BR_\lambda f\|_{p_0} \leq \frac{1}{4}\|f\|_{p_0}$. For $f \in L^{p_0}(\mathbb{R}^d)$, set $h = f - \lambda R_\lambda f$. Note that $R_\lambda f = R_0 h$ and $\|h\|_{p_0} \leq 2\|f\|_{p_0}$. Hence for $\eta < \eta_0$, by [2] Proposition 5.2

$$\|BR_\lambda f\|_{p_0} = \|BR_0 h\|_{p_0} \leq \frac{1}{4}\|h\|_{p_0} \leq \frac{1}{2}\|f\|_{p_0}.$$ 

Moreover by [2] Proposition 2.2,

$$\|R_\lambda f\|_{\infty} \leq c_2\|f\|_{p_0}.$$ 

It follows from [2] Proposition 6.1 that

$$S_\lambda f = R_\lambda \left( \sum_{i=0}^{\infty} (BR_\lambda)^i \right) f$$

for $f \in L^{p_0}$ and therefore

$$\|S_\lambda f\|_{\infty} = \left\| R_\lambda \left( \sum_{i=0}^{\infty} (BR_\lambda)^i \right) f \right\|_{\infty} \leq c_2 \left\| \left( \sum_{i=0}^{\infty} (BR_\lambda)^i \right) f \right\|_{p_0} \leq 2c_2\|f\|_{p_0}.$$ 

If we apply this to $f = 1_C$, where $C \subset B$, then

$$\mathbb{E}^x \left[ \int_0^\infty e^{-\lambda t} 1_C(X_t) \, dt \right] \leq 2c_2|C|^{1/p_0}. \quad (2.1)$$

Let $M = \sup_{x \in B} \mathbb{E}^x \left[ \int_0^\tau 1_C(X_s) \, ds \right]$. Clearly $M \leq \sup_{x \in B} \mathbb{E}^x [\tau]$, which is finite by Proposition 2.1. By taking $t_1$ sufficiently large,

$$\mathbb{P}^x(\tau \geq t_1) \leq \frac{\sup_{x \in B} \mathbb{E}^x [\tau]}{t_1} \leq \frac{1}{2}.$$ 

We then have

$$\mathbb{E}^x \left[ \int_0^\tau 1_C(X_s) \, ds \right] \leq \mathbb{E}^x \left[ \int_0^{t_1} 1_C(X_s) \, ds \right] + \mathbb{E}^x \left[ \int_{t_1}^\tau 1_C(X_s) \, ds ; \tau \geq t_1 \right]$$

$$\leq e^{\lambda t_1} S_\lambda 1_C(x) + \mathbb{E}^x \left[ \mathbb{E}^{X_{t_1}} \left[ \int_0^\tau 1_C(X_s) \, ds ; \tau \geq t_1 \right] \right]$$

$$\leq c_3|C|^{1/p_0} + M\mathbb{P}^x(\tau \geq t_1).$$

Taking the supremum over $x$, we have

$$M \leq c_3|C|^{1/p_0} + \frac{1}{2}M,$$ 

and our result follows. \hfill \Box

We now prove a support theorem for $X$. First we prove some lemmas.

**Lemma 2.3** Let $x_0 \in \mathbb{R}^d$, $1 \leq k \leq d$, $v_k = A(x_0)e_k$, $\gamma \in (0,1)$, $t_0 > 0$, and $r \in [-1,1]$. There exists $c_1$ depending only on $\gamma$, $t_0$, $r$, and the upper bounds and modulus of continuity of $A(\cdot)$ in $B(x_0,2)$ such that

$$\mathbb{P}^{x_0} \left( \text{there exists a stopping time } T \leq t_0 \text{ such that} \right. \left. \sup_{s<T} |X_s - x_0| < \gamma \text{ and} \sup_{T \leq s \leq t_0} \sup_{r < k \leq d} |X_s - (x_0 + rv_k)| < \gamma \right) \geq c_1. \quad (2.2)$$

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Proof. Let \( \|A\|_\infty := 1 \vee \left( \sum_{i,j=1}^{d} \sup_{x \in B(x_0,2)} |A_{ij}(x)| \right) \). We do the case where \( r \geq 0 \), the other case being similar. We first suppose \( r \geq \gamma/3 \). Let \( \beta \in (0, r) \) be chosen later, let

\[
\tilde{Z}_t^i = \sum_{s \leq t} \Delta Z_s^i \mathbf{1}_{\{|\Delta Z_s^i| > \beta\}}, \quad \bar{Z}_t^i = Z_t^i - \tilde{Z}_t^i,
\]

and let \( \bar{X} \) be the solution to

\[
d\bar{X}_s = A(\bar{X}_s) \, d\bar{Z}_s, \quad \bar{X}_0 = x_0.
\]

Choose \( \delta < \gamma/(6\|A\|_\infty) \) such that

\[
\sup_{i,j} \sup_{|x-x_0|<\delta} |A_{ij}(x) - A_{ij}(x_0)| < \gamma/(12d).
\]

Let

\[
C = \left\{ \sup_{s \leq t_0} |\bar{X}_s - \bar{X}_0| \leq \delta \right\},
\]

\[
D = \{ \tilde{Z}_s^i = 0 \text{ for all } s \leq t_0 \text{ and } i \neq k, \bar{Z}_k \text{ has a single jump before time } t_0 \text{ and its size is in } [r, r + \delta] \},
\]

\[
E = \{ \tilde{Z}_s^i = 0 \text{ for all } s \leq t_0 \text{ and } i = 1, \ldots, d \}.
\]

As in the proof of Proposition 2.1

\[
\mathbb{E}[\bar{X}_t^i, \bar{X}_t] \leq c_2 \sum_{j=1}^{d} \mathbb{E}[\bar{Z}_t^i, \bar{Z}_t^j] \leq c_3 t \beta^{2-\alpha},
\]

and by Chebyshev’s inequality and Doob’s inequality,

\[
\mathbb{P}\left( \sup_{s \leq t_0} |\bar{X}_s^i - \bar{X}_0^i| > \delta/\sqrt{d} \right) \leq \frac{\mathbb{E}\left[ \sup_{s \leq t_0} (\bar{X}_s^i - \bar{X}_0^i)^2 \right]}{\delta^2 / d} \leq 4 \mathbb{E}\left[ (\bar{X}_t^i - \bar{X}_0^i)^2 \right] \delta^2 / d \leq \frac{c_4 t_0 \beta^{2-\alpha}}{\delta^2}.
\]

We choose \( \beta < r \) so that

\[
c_4 t_0 \beta^{2-\alpha} \leq \delta^2/(2d),
\]

and then \( \mathbb{P}^x_0(C) \geq 1/2 \).

In order for \( \tilde{Z}_k \) to have a single jump before time \( t_0 \), and for that jump’s size to be in the interval \([r, r + \delta]\), then by time \( t_0 \), (a) \( \tilde{Z}_k \) must have no negative jumps; (b) \( \tilde{Z}_k \) must have no jumps whose size lies in \([\beta, r]\); (c) \( \tilde{Z}_k \) must have no jumps whose size lies in \((r + \delta, \infty)\); and (d) \( \tilde{Z}_k \) must have precisely one jump whose size lies in the interval \([r, r + \delta]\). The events described in (a)–(d) are independent and are the probabilities that Poisson random variables of parameters \( c_5 t_0 \beta^{-\alpha} \), \( c_5 t_0 (\beta^{-\alpha} - r^{-\alpha}) \), \( c_5 t_0 (r + \delta)^{-\alpha} \), and \( c_5 t_0 (r^{-\alpha} - (r + \delta)^{-\alpha}) \), respectively, take the values 0, 0, 0, 1, respectively. For \( j \neq k \), the probability that \( \tilde{Z}_j \) does not have a jump before time \( t_0 \) is the probability that a Poisson random variable with parameter \( 2c_5 t_0 \beta^{-\alpha} \) is equal to 0. Since the \( \tilde{Z}_j^i \), \( j = 1, \ldots, d \), are independent, we thus see that the probability of \( D \) is bounded below by a
constant depending on $r, \delta, t_0$ and $\beta$. Because the $Z_t$’s are independent of the $\tilde{Z}^j$’s, then $C$ and $D$ are independent. Therefore

$$P^{x_0}(C \cap D) \geq c_6/2. \quad (2.5)$$

A similar (but slightly easier) argument shows that

$$P^{x_0}(C \cap E) \geq c_7. \quad (2.6)$$

If $T$ is the time when $\tilde{Z}^k$ jumps, then $Z_{s-} = \bar{Z}_{s-}$ for $s \leq T$, and hence $X_{s-} = \bar{X}_{s-}$ for $s \leq T$. So up to time $T$, $X_s$ does not move more than $\delta$ away from its starting point. We have

$$\Delta X_T = A(X_{T-}) \Delta Z_T,$$

so using (2.3) we have that on $C \cap D$,

$$|X_T - (x_0 + r v_k)|$$

$$\leq |X_{T-} - x_0| + |\Delta X_T - r v_k|$$

$$= |X_{T-} - x_0| + |A(X_{T-}) \Delta Z_T - r v_k|$$

$$\leq |X_{T-} - x_0| + r |(A(X_{T-}) - A(x_0)) e_k| + |A(X_{T-}) (\Delta Z_T - r e_k)|$$

$$\leq \delta + r d \gamma / (12d) + \delta \|A\|_\infty < \gamma / 2.$$

We now apply the strong Markov property at time $T$. By (2.6), $P^{X_T}(C \cap E) \geq c_7$ and so

$$P \left( \sup_{T \leq s \leq T + t_0} |X_s - X_T| < \delta \right) \geq c_8.$$

Using the strong Markov property, we have our result with $c_1 = c_7 c_8 / 2$.

If $r < \gamma / 3$, the argument is easier. In this case we can take $T$ identically 0, and use (2.6). The details are left to the reader. \hfill \Box

**Lemma 2.4** Suppose $u, v$ are two vectors in $\mathbb{R}^d$, $\eta \in (0,1)$, and $p$ is the projection of $v$ onto $u$. If $|p| \geq \eta |v|$, then

$$|v - p| \leq \sqrt{1 - \eta^2} |v|.$$

**Proof.** Note that the vector $v - p$ is orthogonal to the vector $p$. So by the Pythagorean theorem, $|v - p|^2 = |v|^2 - |p|^2 \leq (1 - \eta^2)|v|^2$. \hfill \Box

**Lemma 2.5** Suppose the entries of $A$ and $A^{-1}$ are bounded by $\Lambda$. Let $v$ be a vector in $\mathbb{R}^d$, $u_k = Ae_k$, and $p_k$ the projection of $v$ onto $u_k$ for $k = 1, \ldots, d$. Then there exists $\rho \in (0,1)$ depending only on $\Lambda$ such that for some $k$,

$$|v - p_k| \leq \rho |v|.$$ 

**Proof.** Since the entries of $A^{-1}$ are bounded, then $|(A^T)^{-1}w| \leq c_1 |w|$. Setting $x = (A^T)^{-1}w$, we see $|A^Tx| \geq c_2 |x|$ for any vector $x$.  

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Let $b_k$ be the projection of $A^Tv$ onto $e_k$. If $|b_k| < (1/d)|A^Tv|$ for all $k$, then

$$|A^Tv| = \left| \sum_{k=1}^d b_k \right| \leq \sum_{k=1}^d |b_k| < |A^Tv|,$$

a contradiction. So for some $k$, $|b_k| \geq (1/d)|A^Tv| \geq c_3|v|$, where $c_3 = c_2/d$. We then write

$$c_3|v| \leq |b_k| = |v^TAe_k| \leq \frac{c_4}{|Ae_k|} |v^TAe_k| = c_4 \frac{|v^Tu_k|}{|u_k|} = c_4|p_k|.$$

We now apply Lemma 2.4 with $\eta = c_3/c_4$ and set $\rho = \sqrt{1 - (c_3/c_4)^2}$.

**Lemma 2.6** Suppose the entries of $A(x)$ and $A(x)^{-1}$ on $B(x_0, 3)$ are bounded by $\Lambda$. Let $t_1 > 0$, $\varepsilon \in (0, 1)$, $r \in (0, \varepsilon/4)$ and $\gamma > 0$. Let $\psi : [0, t_1] \to \mathbb{R}^d$ be a line segment of length $r$ starting at $x_0$. Then there exists $c_1 > 0$ that depends only on $\Lambda$, the modulus of continuity of $A(x)$ on $B(x_0, 3)$, $t_1$, $\varepsilon$ and $\gamma$ such that

$$\mathbb{P}^{x_0}\left( \sup_{s \leq t_1} |X_s - \psi(s)| < \varepsilon \text{ and } |X_{t_1} - \psi(t_1)| < \gamma \right) \geq c_1.$$

**Proof.** Use the bounds on $A$ in $B(x_0, 2)$ and Lemma 2.5 to define $\rho \in (0, 1)$ so that the conclusion of Lemma 2.5 holds for all matrices $A = A(x)$ with $x \in B(x_0, 2)$. Take $\gamma \in (0, r \wedge \rho)$ smaller if necessary so that $\tilde{\rho} := \gamma + \rho < 1$. Choose $n \geq 2$ large so that $(\tilde{\rho})^n < \gamma$.

Let $v_0 = \psi(t_1) - \psi(t_0) = \psi(t_1) - x_0$, which has length $r$. By Lemma 2.5 there exists $k_0 \in \{1, \ldots, d\}$ such that if $p_0$ is the projection of $v_0$ onto $A(x_0)e_{k_0}$, then $|v_0 - p_0| \leq \rho|v_0|$. Note $|p_0| \leq |v_0| = r$.

Let $D_1$ denote the event that there is a stopping time $T_0 \leq t_1/n$ such that $|X_s - x_0| < \gamma^{n+1}$ for $s < T_0$ and $|X_s - (x_0 + p_0)| < \gamma^{n+1}$ for $s \in [T_0, t_1/n)$. By Lemma 2.3 there exists $c_2 > 0$ such that $\mathbb{P}^{x_0}(D_1) \geq c_2$. Note that on $D_1$, if $T_0 \leq s \leq t_1/n$,

$$|\psi(t_1) - X_s| \leq |\psi(t_1) - (x_0 + p_0)| + |(x_0 + p_0) - X_{t_1/n}| \leq \rho r + \gamma^{n+1} \leq \tilde{\rho}r. \quad (2.7)$$

Taking $s = t_1/n$, we have

$$|\psi(t_1) - X_{t_1/n}| \leq \tilde{\rho}r.$$

Since $\tilde{\rho} < 1$ and $|\psi(t_1) - x_0| = r$, then (2.7) shows that on $D_1$,

$$X_s \in B(x_0, 2r) \subset B(x_0, \varepsilon/2) \quad \text{if } T_0 \leq s \leq t_1/n.$$

If $0 \leq s < T_0$, then $|X_s - x_0| < \gamma^{n+1} < r$, and so $\{X_s, s \in [0, t_1/n]\} \subset B(x_0, 2r) \subset B(x_0, \varepsilon/2)$.

Now let $v_1 = \psi(t_1) - X_{t_1/n}$. When $X_{t_1/n} \in B(x_0, \varepsilon/2)$, by Lemma 2.5 there exists $k_1$ such that if $p_1$ is the projection of $v_1$ onto $A(X_{t_1/n})e_{k_1}$, then $|v_1 - p_1| \leq \rho|v_1|$. Let $D_2$ be the event that there exists a stopping time $T_1 \in [t_1/n, 2t_1/n]$ such that $|X_s - X_{t_1/n}| < \gamma^{n+1}$ for $t_1/n \leq s < T_1$ and $|X_s - (X_{t_1/n} + p_1)| < \gamma^{n+1}$ for $T_1 \leq s \leq 2t_1/n$. Using the Markov property at time $t_1/n$ and applying Lemma 2.3 again, there exists (the same) $c_2 > 0$ such that

$$\mathbb{P}^{x_0}(D_2 \mid \mathcal{F}_{t_1/n}) \geq c_2$$

on the event $\{X_{t_1/n} \in B(x_0, \varepsilon/2)\}$, where $\mathcal{F}_t$ is the minimal augmented filtration for $X$. So

$$\mathbb{P}^{x_0}(D_1 \cap D_2) \geq c_2 \mathbb{P}^{x_0}(D_1) \geq c_2^2.$$

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On the event $D_1 \cap D_2$, if $T_1 \leq s \leq 2t_1/n$, 
\[ |\psi(t_1) - X_s| \leq |\psi(t_1) - (X_{t_1/n} + p_1)| + |(X_{t_1/n} + p_1) - X_s| \leq \rho |v_1| + \gamma^{n+1} \leq \rho \bar{r} + \gamma^{n+1} \leq \bar{r}^2 r. \]

In particular
\[ |\psi(t_1) - X_{2t_1/n}| \leq \bar{r}^2 r \quad \text{on} \ D_1 \cap D_2. \]

If $T_1 \leq s \leq 2t_1/n$, then $|\psi(t_1) - X_s| < r$ and $|\psi(t_1) - x_0| = r$, and so $X_s \in B(x_0,2r) \subset B(x_0,\varepsilon/2)$. In particular,
\[ |X_{2t_1/n} - x_0| < \varepsilon/2 \quad \text{on} \ D_1 \cap D_2. \]

If $t_1/n \leq s < T_1$, then $|X_s - X_{t_1/n}| < r$ and $|X_{t_1/n} - x_0| < 2r$. So on $D_1 \cap D_2$, $X_s \in B(x_0,3r) \subset B(x_0,3\varepsilon/4)$.

Let $v_2 = \psi(t_1) - X_{2t_1/n}$, and proceed as above to get events $D_3, \ldots, D_k$. At the $k$th stage, we have
\[ \mathbb{P}^{x_0}(D_k | \mathcal{F}_{(k-1)t_1/n}) \geq c_2 \]
and so $\mathbb{P}^{x_0}(\cap_{j=1}^k D_j) \geq c_2^k$. On the event $\cap_{j=1}^k D_j$, if $kt_1/n \leq s \leq (k+1)t_1/n$, then
\[ |\psi(t_1) - X_s| \leq \bar{r}^{k+1} r < r; \]
since $|\psi(t_1) - x_0| = r$, then $X_s \in B(x_0,2r) \subset B(x_0,\varepsilon/2)$. At the $k$th stage, on the event $\cap_{j=1}^k D_j$, 
\[ |X_{kt_1/n} - x_0| < \varepsilon/2 \]
and if $kt_1/n \leq s < T_k$, then
\[ |X_s - x_0| \leq |X_s - X_{kt_1/n}| + |X_{kt_1/n} - \psi(t_1)| + |\psi(t_1) - x_0| < \gamma^{n+1} + 2r + r < 3r, \]
and so $X_s \in B(x_0,3r) \subset B(x_0,3\varepsilon/4)$.

We continue this procedure $n$ times to get events $D_1, \ldots, D_n$ so that on $\cap_{k=1}^n D_k$, we have $X_s \in B(x_0,3\varepsilon/4)$ for $s \leq t_1$, $|X_{t_1} - \psi(t_1)| < \gamma$, and $\mathbb{P}^{x_0}(\cap_{k=1}^n D_k) \geq c_2^n$. Consequently, on $\cap_{k=1}^n D_k$,
\[ |X_s - \psi(s)| \leq |X_s - x_0| + |x_0 - \psi(s)| < 3\varepsilon/4 + r < \varepsilon \quad \text{for} \ s \in [0,t_1]. \]

This completes the proof of the lemma.

\[ \square \]

**Theorem 2.7** Suppose the entries of $A(x)$ and $A(x)^{-1}$ on $B(x_0,3)$ are bounded by $\Lambda$. Let $\varphi : [0,t_0] \to \mathbb{R}^d$ be continuous with $\varphi(0) = x_0$ and the image of $\varphi$ contained in $B(0,1)$. Let $\varepsilon > 0$. There exists $c_1 > 0$ depending on $\Lambda$, the modulus of continuity of $A(x)$ on $B(x_0,3)$, $\varphi, \varepsilon$, and $t_0$ such that 
\[ \mathbb{P}^{x_0} \left( \sup_{s \leq t_0} |X_s - \varphi(s)| < \varepsilon \right) > c_1. \]

**Proof.** We may approximate $\varphi$ to within $\varepsilon/2$ by a polygonal path, so by changing $\varepsilon$ to $\varepsilon/2$, we may without loss of generality assume $\varphi$ is polygonal. Let us now choose $n$ large and subdivide $[0,t_0]$ into $n$ equal subintervals so that over each subinterval $[kt_0/n, (k+1)t_0/n]$ the image of $\varphi$ is a line segment of length less than $\varepsilon/4$. We then use Lemma 2.6 and the strong Markov property $n$ times to show that, with probability at least $c_1 > 0$, on each time interval $[kt_0/n, (k+1)t_0/n]$, $X_t$ follows within $\varepsilon/2$ the line segment from $X_{kt_0/n}$ to $\varphi((k+1)t_0/n)$ and is at most $\varepsilon/(4\sqrt{n})$ away from $\varphi((k+1)t_0/n)$.

\[ \square \]
Corollary 2.8 Let \( \varepsilon, \delta \in (0, 1/4) \). Suppose \( Q \) represents either the unit ball or the unit cube, centered at \( x_0 \in \mathbb{R}^d \). Suppose the entries of \( A \) and \( A^{-1} \) on \( Q \) are bounded by \( \Lambda \). Let \( Q' \) be the ball (resp., cube) with radius (resp., side length) \( 1 - \varepsilon \) with the same center. Let \( R \) be a ball (resp., cube) of radius (resp., side length) \( \delta \) contained in \( Q' \). Then there exists \( c_1 > 0 \) depending on \( \Lambda \), the modulus of continuity of \( A(x) \) on \( Q' \), \( \varepsilon \) and \( \delta \) such that

\[
\mathbb{P}^x(T_R < \tau_Q) \geq c_1, \quad x \in Q'.
\]

Proof. Note that the above probability is determined by the values of the matrix \( A(x) \) only on \( Q \) so we can redefine \( A(x) \) outside of \( Q \) if necessary to make the entries of \( A \) and \( A^{-1} \) on \( \mathbb{R}^d \) bounded by \( \Lambda \), and the modulus of continuity of \( A(x) \) on \( \mathbb{R}^d \) be the same as that on \( Q \). To prove the corollary, we need only observe that the estimates in Theorem 2.7 can be made to hold uniformly over every line segment from \( x \) to \( y \), with \( x \in Q' \) and \( y\) being the center of \( R \).

A scaling argument shows that for \( \lambda > 0 \), \( \{ \hat{X}_t := \lambda X_t/\lambda^\alpha, \ t \geq 0 \} \) is a process of the same type as \( X \). More precisely, one can show that there exist \( d \) independent one-dimensional symmetric stable processes \( \tilde{Z}^i \) of index \( \alpha \) such that \( \hat{X} \) satisfies

\[
d\hat{X}_t^i = \sum_{j=1}^d \hat{A}_{ij}(\hat{X}_t) \, d\tilde{Z}_t^j, \quad \hat{X}_0^i = \lambda x_0^i,
\]

where \( \hat{A}_{ij}(x) = A_{ij}(x/\lambda) \). Note in particular that when \( \lambda \geq 1 \), the oscillation of \( \hat{A} \) will be no more than the oscillation of \( A \). A consequence is that the analogues of Propositions 2.1 and 2.2 and Theorem 2.7 hold in balls \( B(x_1, r) \) with the same constants provided \( r < 1 \) (so that \( \lambda = 1/r > 1 \)).

We now have what is needed to prove our main theorem.

Theorem 2.9 Let \( r \in (0, 1] \) and \( \gamma > 1 \). Suppose \( h \) is harmonic in \( B(x_0, \gamma r) \) with respect to \( X \) and \( h \) is bounded in \( \mathbb{R}^d \). There exists positive constants \( c_1 \) and \( \beta \) that depend on \( \gamma \), the upper bound of \( A(x) \) and \( A(x)^{-1} \) on \( B(x_0, \gamma r) \), and the modulus of continuity of \( A(x) \) on \( B(x_0, \gamma r) \) but otherwise is independent of \( h \) and \( r \) such that

\[
|h(x) - h(y)| \leq c_1 \left( \frac{|x - y|}{r} \right) \beta \sup_{\mathbb{R}^d} |h(z)|
\]

Proof. If one examines the proof of Krylov-Safonov carefully (see, e.g., the presentation in [1], Theorem V.7.4), one sees that one needs the support theorem and Corollary 2.8, a result such as Proposition 2.2 and estimates such as Proposition 2.1 and that with these ingredients, one can conclude that if \( Q \) is a cube of side length \( r \leq 1 \), \( A \subset Q \subset B(x_0, r) \), and \( Q' \) is a cube with the same center as \( Q \) but side length half as long, then

\[
\mathbb{P}^x(T_A < \tau_Q) \geq \varphi(|A|/|Q|) \quad \text{for } x \in Q',
\]

where \( \varphi \) is a strictly increasing function with \( \varphi(0) = 0 \).

Now let \( B = B(y, s) \) be a ball contained in \( B(x_0, r) \) and suppose \( A \subset B \) with \( |A|/|B| \geq 1/3 \). Let \( B' = B(y, (1 - \varepsilon)s) \), where \( \varepsilon \) is chosen so that \( |B \setminus B'|/|B| = 1/6 \). Then \( |A \cap B'|/|B| \geq 1/6 \). Cover \( B' \) with \( N \) equally sized cubes whose interiors are disjoint and each contained in \( B \). We may choose \( N \) independent of \( s \). For at least one cube, say, \( Q \), we must have \( |A \cap B' \cap Q|/|Q| \geq 1/6 \). Let \( Q' \) be the cube with the same center as \( Q \) but side length half as long. By the support theorem, if
If \( x \in B(y, s/2) \), there is probability at least \( c_2 \) such that \( \mathbb{P}^x(T_{Q'} < \tau_B) \geq c_2 \). Applying (2.8) and the strong Markov property, we have

\[
\mathbb{P}^x(T_A < \tau_B) \geq c_3 > 0 \quad \text{for } x \in B(y, s/2).
\] (2.9)

Applying (2.9) and Proposition 2.1, the result now follows exactly as the proof in Theorem 4.1 of [3]. (We remark that line 15 on page 386 of [3] should read instead

\[
(b_{k-1} - a_{k-1})\mathbb{P}^y(\tau_k < T_A) \leq \frac{1}{\gamma} (b_k - a_k)(1 - \mathbb{P}^y(T_A < \tau_k)).
\]

With suitable modifications to the definition of \( \gamma \) and \( \rho \), the proof of Theorem 4.1 in [3] is valid.) \( \square \)

3 A counterexample to the Harnack inequality

We now show that one cannot expect a Harnack inequality to hold, even when \( A(x) \equiv I \), the identity matrix. We will describe \( \varepsilon \) in a moment. Write points in \( \mathbb{R}^3 \) as \( w = (x, y, z) \) and let \( w_0 = (0, \frac{1}{2}, 0) \). Write \( B \) for \( B(0, 1) \), \( \tau \) for \( \tau_B \), and let \( F_\varepsilon = (-\varepsilon, \varepsilon)^2 \subset \mathbb{R}^2 \), \( C_\varepsilon = (\mathbb{R} 	imes F_\varepsilon) \cap B \), and \( E_\varepsilon = (2, 4) \times F_\varepsilon \). Let \( X_t, Y_t \) and \( Z_t \) be independent one-dimensional symmetric \( \alpha \)-stable processes and set \( W_t = (X_t, Y_t, Z_t) \). Define \( h_\varepsilon(w) = \mathbb{P}^w(W_{\tau} \in E_\varepsilon) \). We will show that \( h_\varepsilon(0)/h_\varepsilon(w_0) \to \infty \) as \( \varepsilon \to 0 \); this implies that a Harnack inequality is not possible.

The Lévy measure \( n(w, d\tilde{w}) \) of \( W \) is

\[
n(w, d\tilde{w}) = c \sum_{k=1}^{3} |w_k - \tilde{w}_k|^{-1-\alpha} d\tilde{w}_k \prod_{j \neq k} \delta_{w_j}(d\tilde{w}_j)
\]

where \( \delta_a \) denotes the Dirac measure at the point \( a \). Since all jumps of \( W \) are in directions parallel to the coordinate axes, the only way \( W_{\tau} \) can be in \( E_\varepsilon \) is if \( W_{\tau} = (X_{\tau}, Y_{\tau}, Z_{\tau}) \). Define \( h_\varepsilon(w) = \mathbb{P}^w(W_{\tau} \in E_\varepsilon) \). We will show that \( h_\varepsilon(0)/h_\varepsilon(w_0) \to \infty \) as \( \varepsilon \to 0 \); this implies that a Harnack inequality is not possible.

The Lévy measure \( n(w, d\tilde{w}) \) of \( W \) is

\[
n(w, d\tilde{w}) = c \sum_{k=1}^{3} |w_k - \tilde{w}_k|^{-1-\alpha} d\tilde{w}_k \prod_{j \neq k} \delta_{w_j}(d\tilde{w}_j)
\]

where \( \delta_a \) denotes the Dirac measure at the point \( a \). Since all jumps of \( W \) are in directions parallel to the coordinate axes, the only way \( W_{\tau} \) can be in \( E_\varepsilon \) is if \( W_{\tau} = (X_{\tau}, Y_{\tau}, Z_{\tau}) \). This is the key observation.

We first get an upper bound on \( h_\varepsilon \). It is well known that if \( p_t(u, v) \) is the transition density for a one-dimensional symmetric stable process of index \( \alpha \), then \( p_t \) is everywhere strictly positive, is jointly continuous, \( p_t(u, v) = t^{-1/\alpha} p_1(u/t^{1/\alpha}, v/t^{1/\alpha}) \), and \( p_1(u, v) \sim c_1 |u - v|^{-\alpha-1} \) for \( |u - v| \) large. An integration gives

\[
\mathbb{E}^{(y,z)} \left[ \int_0^\infty 1_{(-1,1)^2}(Y_s, Z_s) \, ds \right] \leq 1 + \int_1^\infty \left( \int_{-1}^1 p_t(y, u) \, du \right) \left( \int_{-1}^1 p_t(z, v) \, dv \right) \, ds < \infty.
\]

By scaling,

\[
\mathbb{E}^{(y,z)} \left[ \int_0^\infty 1_{F_\varepsilon}(Y_s, Z_s) \, ds \right] < c_2 \varepsilon^\alpha.
\]
By the Lévy system formula (see [3] or [5]),
\[
\mathbb{E}^w \left[ \sum_{s \leq t \wedge \tau} 1_{(W_s \in C_\varepsilon, W_s \in E_\varepsilon)} \right] = \mathbb{E}^w \left[ \int_0^{t \wedge \tau} 1_{C_\varepsilon}(W_s)n(W_s, E_\varepsilon) \, ds \right] \\
\leq c_3 \mathbb{E}^w \left[ \int_0^{\infty} 1_{C_\varepsilon}(W_s) \, ds \right] \\
\leq c_3 \mathbb{E}^{(y,z)} \left[ \int_0^{\infty} 1_{F_\varepsilon}(Y_s, Z_s) \, ds \right] \\
\leq c_2 c_3 \varepsilon^{\alpha}. \tag{3.1}
\]

Letting \( t \to \infty \), we obtain
\[
h_\varepsilon(w) = \mathbb{P}^w(W_\tau \in E_\varepsilon) \leq c_4 \varepsilon^{\alpha}. \tag{3.2}
\]

Next we get a lower bound on \( h_\varepsilon(0) \). Let
\[
C'_\varepsilon = C_\varepsilon \cap \{|x| < 1/2\}. \text{ By the Lévy system formula we have }
\]
\[
h_\varepsilon(0) \geq \mathbb{E}^0 \left[ \sum_{s \leq t \wedge \tau} 1_{(W_s \in C'_\varepsilon, W_s \in E_\varepsilon)} \right] \\
= \mathbb{E}^0 \left[ \int_0^{t \wedge \tau} 1_{C'_\varepsilon}(W_s)n(W_s, E_\varepsilon) \, ds \right] \\
\geq c_5 \mathbb{E}^0 \left[ \int_0^{t \wedge \tau} 1_{C'_\varepsilon}(W_s) \, ds \right].
\]

Letting \( t \to \infty \),
\[
h_\varepsilon(0) \geq c_5 \mathbb{E}^0 \left[ \int_0^{\tau} 1_{C'_\varepsilon}(W_s) \, ds \right].
\]

By the scaling property of \( \alpha \)-stable processes, if \( V \) is a one-dimensional symmetric \( \alpha \)-stable process starting from 0 killed on exiting \([-1/4, 1/4]\), then \( \varepsilon^{-1} V \) has the same distribution as \( U_{\varepsilon t/\varepsilon^\alpha} \), where \( U \) is a one-dimensional symmetric \( \alpha \)-stable process starting from 0 killed on exiting \([-1/(4 \varepsilon), 1/(4 \varepsilon)]\).

Hence there is a positive constant \( c_6 > 0 \) such that for every \( \varepsilon \in (0, \alpha) \) and \( t \in (0, \varepsilon^\alpha) \),
\[
\mathbb{P}^0(\nabla t \in [-\varepsilon, \varepsilon]) = \mathbb{P}^{(x)}(U_{\varepsilon t/\varepsilon^\alpha} \in [-1, 1]) \geq c_6.
\]

Consequently,
\[
\mathbb{E}^0 \left[ \int_0^{\infty} 1_{C'_\varepsilon}(W_s) \, ds \right] \geq \mathbb{E}^0 \left[ \int_0^{\varepsilon^\alpha} 1_{C'_\varepsilon}(W_s) \, ds \right] \geq c_7 \varepsilon^{\alpha},
\]

where \( \nabla \) is the process \( W \) killed when any of \( X, Y, \) or \( Z \) exceeds 1/4 in absolute value. Therefore
\[
h_\varepsilon(0) \geq c_8 \varepsilon^{\alpha}. \tag{3.3}
\]

Let \( G = (-1, 1)^2 \subset \mathbb{R}^2 \), write \( \hat{w} \) for \( (y, z) \), and \( \hat{W}_t = (Y_t, Z_t) \). By the estimates on the transition densities, we see that
\[
u(\hat{w}) := \mathbb{E}^{\hat{w}} \left[ \int_0^\infty 1_G(\hat{W}_s) \, ds \right]
\]
is bounded and
\[
u(\hat{w}) \leq \int_0^\infty \mathbb{P}^{\hat{w}}(|Y_s| < 1)\mathbb{P}^{\hat{w}}(|Z_s| < 1) \, ds \to 0 \tag{3.4}
\]
as $|\hat{w}| \to \infty$. Similarly, for $\hat{w} \in G$,

$$u(\hat{w}) \geq \int_1^2 \mathbb{P}^y(|Y_s| < 1)\mathbb{P}^z(|Z_s| < 1) \, ds \geq c_9.$$ \hfill (3.1)

Now $u(\hat{W}_{t\wedge T_B})$ is a bounded supermartingale, so by optional stopping

$$u(\hat{w}) \geq \mathbb{E}[u(\hat{W}_{T_G}); T_G < \infty] \geq c_9 \mathbb{P}^w(T_G < \infty),$$

and (3.1) then implies that $\mathbb{P}^{\hat{w}}(T_G < \infty) \to 0$ as $\hat{w} \to \infty$. Scaling then shows that

$$\mathbb{P}^{(1/2,0)}(T_{F_\varepsilon} < \infty) \to 0 \quad \text{as } \varepsilon \to 0,$$

and hence

$$\mathbb{P}^{a_0}(T_{C_\varepsilon} < \infty) \to 0 \quad \text{as } \varepsilon \to 0. \quad \text{(3.5)}$$

Therefore by (3.1)-(3.2),

$$h_\varepsilon(w_0) = \mathbb{E}^{w_0}[h_\varepsilon(W_{T_{C_\varepsilon}}); T_{C_\varepsilon} < \tau]$$

$$\leq c_{10} \varepsilon \mathbb{P}^{a_0}(T_{C_\varepsilon} < \tau)$$

$$\leq c_{11} h_\varepsilon(0) \mathbb{P}^{a_0}(T_{C_\varepsilon} < \infty).$$

This and (3.5) shows that $h_\varepsilon(0)/h_\varepsilon(w_0)$ can be made as large as we like by taking $\varepsilon$ small enough and so a Harnack inequality for $W$ is not possible.

**Remark.** When $\alpha < 1$, we can construct a two-dimensional example along the same lines.

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