A MINKOWSKI-TYPE INEQUALITY FOR CAPILLARY HYPERSURFACES IN A HALF-SPACE

GUOFANG WANG, LIANGJUN WENG, AND CHAO XIA

Abstract. In this article, we investigate a flow of inverse mean curvature type for capillary hypersurfaces in the half-space. We establish the global existence of solutions for this flow and demonstrate that it converges smoothly to a spherical cap as time tends to infinity. As a result, we derive a new Minkowski-type inequality for star-shaped and mean convex capillary hypersurfaces for the whole range of contact angle $\theta \in (0, \pi)$.

1. Introduction

Let $\mathbb{R}^{n+1}_+ = \{ x \in \mathbb{R}^{n+1} | x_{n+1} > 0 \} \ (n \geq 2)$ be the Euclidean upper half-space and $\Sigma$ an embedded compact hypersurface in $\mathbb{R}^{n+1}_+$ with boundary $\partial \Sigma$ lying in $\partial \mathbb{R}^{n+1}_+$. Such a hypersurface is called a capillary hypersurface, if $\Sigma$ intersects with $\partial \mathbb{R}^{n+1}_+$ at a constant contact angle $\theta \in (0, \pi)$ along $\partial \Sigma$. In this paper, we consider a family of embedded capillary hypersurface $\Sigma_t := x(M, t) \subset \mathbb{R}^{n+1}_+$, where $M$ is an $n$-dimensional compact orientable smooth manifold with boundary $\partial M$, satisfies the following inverse mean curvature type flow

$$
\begin{align*}
\left\{ \begin{array}{ll}
(\partial_t x)^\perp &= \left( \frac{n(1 + \cos \theta \langle \nu, e \rangle)}{H} - \langle x, \nu \rangle \right) \nu, & \text{in } M \times [0, T), \\
\langle \nu, e \rangle &= -\cos \theta & \text{on } \partial M \times [0, T),
\end{array} \right.
\end{align*}
$$

(1.1)

where $\nu$ is the unit outward normal of $\Sigma_t$ and $e := (0, \cdots, 0, -1)$.

The introduction of flow (1.1) is motivated by the idea of Guan-Li [9, 10] (see also [2, 4, 11]) for establishing the isoperimetric-type inequality for quermassintegral and by the Minkowski formulas for capillary hypersurfaces in [25]. For more description of Guan-Li’s idea for free boundary or capillary hypersurfaces, we refer to [18, 20, 21, 25, 27] and references therein.

For a capillary hypersurface $\Sigma \subset \mathbb{R}^{n+1}_+$, we denote $\hat{\Sigma}$ the bounded domain in $\mathbb{R}^{n+1}_+$ enclosed by $\Sigma \subset \mathbb{R}^{n+1}_+$ and $\partial \mathbb{R}^{n+1}_+$. The boundary of $\hat{\Sigma}$ consists of two parts: one is $\Sigma$ and the other, which will be denoted by $\partial \hat{\Sigma}$, lies on $\partial \mathbb{R}^{n+1}_+$. Both have a common boundary, namely $\partial \Sigma$. The capillary area functional (cf. [6]) is defined by

$$
V_{1,\theta}(\hat{\Sigma}) := |\Sigma| - \cos \theta |\partial \hat{\Sigma}|.
$$

For a hypersurface with boundary (not necessarily capillary) in $\mathbb{R}^{n+1}_+$, the relative (or capillary) isoperimetric inequality says that

$$
\frac{V_{1,\theta}(\Sigma)}{|\Sigma|^{\frac{1}{n+1}}} \geq \frac{V_{1,\theta}(\hat{C}_{\theta,1})}{|\hat{C}_{\theta,1}|^{\frac{1}{n+1}}} = (n + 1) b_{\theta}^{\frac{1}{n+1}},
$$

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with equality holding if and only if $\hat{\Sigma}$ is a capillary spherical cap. Here we use $C_{\theta,r}$ to denote a capillary spherical cap lying entirely in $\mathbb{R}^{n+1}_+$ by

$$C_{\theta,r} := \left\{ x \in \mathbb{R}^{n+1}_+ \mid x - r \cos \theta e = r \right\}, \quad r \in (0, \infty),$$

which is a portion of the sphere of radius $r$ and centered at $r \cos \theta e$. For notation simplicity, we denote

$$(n + 1)b_\theta := V_{1,\theta}(\hat{C}_{\theta,1}).$$

One can check easily that $b_\theta$ is the volume of $\hat{C}_{\theta,1}$.

In [25], we introduced a family of quermassintegrals for capillary hypersurfaces in the half-space, which can be regarded as a natural generalization of quermassintegrals for closed hypersurfaces. Among them the second quermassintegral is given by

$$V_{2,\theta}(\Sigma) := \frac{1}{n} \left( \int_\Sigma H dA - \cos \theta \sin \theta |\partial \Sigma| \right).$$

The main objective of this paper is to establish an optimal isoperimetric type inequality between $V_{2,\theta}(\Sigma)$ and the capillary area functional $V_{1,\theta}(\Sigma)$ under star-shaped and mean convex assumption, which is a capillary counterpart of the classical Minkowski-type inequality for closed hypersurfaces. For the latter, we refer to Schneider’s seminal book on convex bodies [22] and Guan-Li [9].

We return to flow (1.1). A key feature of flow (1.1) is that it preserves the capillary area functional $V_{1,\theta}(\Sigma)$ while it monotonically decreases $V_{2,\theta}(\Sigma)$ (cf. [25, Proposition 3.1]). It is easy to check that a family of stationary solutions of flow (1.1) is given by capillary spherical caps $C_{\theta,r}$. Our main result in this article is the following long-time existence and convergence for flow (1.1) under the assumption of star-shapedness and mean convexity. This can be viewed as a capillary counterpart of the results of Gerhardt [7] and Urbas [24] for closed hypersurfaces.

**Theorem 1.1.** Assume the initial hypersurface $\Sigma_0$ is a star-shaped and strictly mean convex capillary hypersurface in $\mathbb{R}^{n+1}_+$ $(n \geq 2)$ with a contact angle $\theta \in (0, \pi)$. Then flow (1.1) starting from $\Sigma_0$ exists for all time with uniform $C^\infty$-estimates. Moreover, it converges smoothly to a uniquely determined capillary spherical cap $C_{\theta,r}$, as $t \to +\infty$.

As an application, we obtain the following Minkowski-type inequality for star-shaped and mean convex capillary hypersurfaces in $\mathbb{R}^{n+1}_+$ for the whole angle range $\theta \in (0, \pi)$.

**Theorem 1.2.** Let $\Sigma \subset \mathbb{R}^{n+1}_+$ $(n \geq 2)$ be a star-shaped and mean convex capillary hypersurface with a contact angle $\theta \in (0, \pi)$, then

$$\int_\Sigma H dA \geq n(n + 1)^{\frac{1}{2}} b_\theta^{\frac{2}{n}} \left( |\Sigma| - \cos \theta |\partial \Sigma| \right)^{\frac{n-1}{n}} + \sin \theta \cos \theta |\partial \Sigma|. \quad (1.4)$$

Equality holds if and only if $\Sigma$ is a capillary spherical cap.

Note that (1.4) is equivalent to

$$\frac{V_{2,\theta}(\Sigma)}{V_{1,\theta}(\Sigma)^{\frac{n}{n-1}}} \geq \frac{V_{2,\theta}(\hat{C}_{\theta,1})}{V_{1,\theta}(\hat{C}_{\theta,1})^{\frac{n}{n-1}}} = n(n + 1)^{\frac{1}{2}} b_\theta^{\frac{1}{n}}. \quad (1.5)$$

From (1.5) one can see easily that the functional $\frac{V_{2,\theta}(\Sigma)}{V_{1,\theta}(\Sigma)^{\frac{n}{n-1}}} \hat{C}_{\theta,1}$ achieves its minimum at spherical caps.

When $\theta = \pi/2$, Theorem 1.2 follows from the result of Guan-Li [9, Theorem 2] for closed hypersurfaces by using a reflection argument. It is clear that if $\theta \neq \pi/2$ the reflection argument does not work.
Under the stronger geometric condition that $\Sigma$ is convex, Theorem 1.2 has been proved for contact angle $\theta \in (0, \pi/2]$ in [12], where the authors proved actually more general geometric inequalities, the Alexandrov-Fenchel inequalities, which were initiated and conjectured in [25]. We emphasize that the results in [25] and [12] crucially depend on the restriction $\theta \leq \pi/2$, which was used to obtain $C^2$ estimates. We observed in [19] that for convex capillary hypersurfaces, all the Alexandrov-Fenchel inequalities with the whole range of the contact angle $\theta \in (0, \pi)$ follow actually from the classical Alexandrov-Fenchel inequalities for non-smooth convex bodies and provided a proof in a smooth setting with a Robin-type boundary condition. The proof follows closely the original idea in the theory of convex geometry with necessary modifications on the boundary.

It remains to be asked if the convexity condition could be weakened for these Alexandrov-Fenchel inequalities for capillary hypersurfaces. This paper is the first one in this direction. We remark that without the star-shaped assumption, the Minkowski inequality for closed mean convex hypersurfaces remains an open problem, despite numerous efforts made to solve it, see [1, 8, 13] for instance. It is also natural to expect that (1.4) holds for general mean convex capillary hypersurfaces. Especially a capacity method used in [1] would be expected to be useful by a necessary modification. We also mention several works on the Minkowski-type inequality in warped product spaces [3, 26, 20], as well as anisotropic Minkowski-type inequality [28].

In order to prove Theorem 1.1, it is crucial to show that flow (1.1) preserves the star-shapedness and strictly mean convex. For the star-shapedness, we prove a uniform lower bound for a new test function $\bar{u} := \langle x, \nu \rangle \left(1 + \cos \theta \langle \nu, e \rangle \right)$, which is a variant of the support function $\langle x, \nu \rangle$. In fact, it can be viewed as the anisotropic support function (see e.g. [28]) with respect to $C_{\theta, 1}$. Another key point is the uniform lower and upper bounds for $H$. For this, instead of $H$ we consider $P = \bar{u}H$. To get the curvature estimate, we employ a similar method in [17] and [28] with a few modifications to take care of the Robin boundary condition, which explore the specific divergence structure of mean curvature, to directly obtain Schauder’s estimate for the radial function for star-shaped hypersurfaces. This completes the proof of Theorem 1.1. For the proof of Theorem 1.2, it follows from the monotonicity property $\mathcal{V}_{2, \theta}$ along flow (1.1) and the convergence result in Theorem 1.1.

The paper is organized as follows: In Section 2, some relevant evolution equations are collected and we prove the preservation of the star-shapedness and obtain uniform a priori estimates for flow (1.1) and the convergence result in Theorem 1.1. As a result, we conclude the proof of Theorem 1.2.

2. Mean convexity and star-shapedness

2.1. Evolution equations. Let $\Sigma_t$ be a family of smooth, embedded hypersurfaces with capillary boundary in $\mathbb{R}^{n+1}_+$, given by the embeddings $x(\cdot, t) : M \rightarrow \mathbb{R}^{n+1}_+$, which evolve by the general flow.

$$\partial_t x = f\nu + \mathcal{T},$$

with $\mathcal{T} \in T\Sigma_t$ and a general velocity function $f$. For a fixed $f$, we choose $\mathcal{T}|_{\partial \Sigma_t} = f \cot \theta \mu$ with $\mu$ being the unit outward conormal of $\partial \Sigma_t$ in $\Sigma_t$, which implies that the restriction
of \(x(\cdot,t)\) on \(\partial M\) is contained in \(\mathbb{R}^n\). We refer to [25, Section 2] for further discussions. Along flow (2.1), we have the following evolution equations for the induced metric \(g_{ij}\), the volume element \(d\mu_t\), the unit outward normal \(\nu\), the second fundamental form \((h_{ij})\), the Weingarten curvature tensor \((h^i_j)\) and the mean curvature \(H\) of the hypersurfaces \(\Sigma_t := x(M,t) \subset \mathbb{R}^{n+1}_+\). (cf. [27, Proposition 2.11] for a proof)

**Proposition 2.2** ([27]). Along flow (2.1), there holds

1. \(\partial_t g_{ij} = 2fh_{ij} + \nabla_i T_j + \nabla_j T_i\).
2. \(\partial_t d\mu_t = (fH + \text{div}(\nabla T))d\mu_t\).
3. \(\partial_t \nu = -\nabla f + h(e_i, T)e_i\).
4. \(\partial_t h_{ij} = -\nabla_i f + f h_{ik}h^k_j + \nabla_j h_{ij} + h^k_j \nabla_k T_i + h^k_i \nabla_k T_j\).
5. \(\partial_t h^i_j = -\nabla^i \nabla_j f - f h^k_j h^i_k + \nabla_j h^i_j\).
6. \(\partial_t H = -\Delta f - |h|^2 f + \langle \nabla H, T \rangle\).

In above, \(\nabla\) and \(\Delta\) are the Levi-Civita connection and Beltrami-Laplacian operator on \(\Sigma_t\) with respect to the induced metric.

### 2.2. Star-shapedness and Mean convexity.

We turn to the study of the flow. In this subsection, we show that the star-shapedness and the mean convexity are preserved along flow (1.1). In order to prove this, the key is a choice of a suitable test function.

In the following we study flow (1.1), i.e, namely we consider

\[
 f = \frac{n(1 + \cos \theta(p,e))}{H} - \langle x, \nu \rangle.
\]

For convenience, we introduce the parabolic operator with respect to flow (1.1) as

\[
 \mathcal{L} := \partial_t - \frac{n(1 + \cos \theta(p,e))}{H^2} \Delta - \left( T + x - \frac{n \cos \theta}{H} e, \nabla \right).
\]

Let \(\Sigma_0\) be an initial capillary hypersurface which is star-shaped and mean convex. The star-shapedness of \(\Sigma_0\) implies that there exist \(0 < r_1 < r_2 < \infty\) such that

\[
 \Sigma_0 \subset \overline{C_{\theta, r_2}} \setminus \overline{C_{\theta, r_1}},
\]

where \(\overline{C_{\theta, r}}\) denotes the bounded domain in \(\mathbb{R}^{n+1}_+\) enclosed by \(C_{\theta, r}\) and \(\partial \mathbb{R}^{n+1}_+\). Following the argument in [25, Proposition 4.2], we have the following height estimate, which follows from the maximum principle (or the avoidable principle) since \(C_{\theta, r}\) is a stationary solution of flow (1.1).

**Proposition 2.2.** For any \(t \in [0, T)\), along flow (1.1), there holds

\[
 \Sigma_t \subset \overline{C_{\theta, r_2}} \setminus \overline{C_{\theta, r_1}},
\]

where \(C_{\theta, r}\) defined by (1.2) and \(r_1, r_2\) only depend on \(\Sigma_0\).

Next, we show that the star-shapedness is preserved along flow (1.1).

**Proposition 2.3.** Let \(\Sigma_0\) be a star-shaped and strictly mean convex hypersurface with capillary boundary in \(\mathbb{R}^{n+1}_+\) and \(\theta \in (0, \pi)\), then there exists \(c_0 > 0\) depending only on \(\Sigma_0\), such that the solution \(\Sigma_t\) of flow (1.1) satisfies

\[
 \langle x, \nu \rangle(p,t) \geq c_0,
\]

for all \((p, t) \in M \times [0, T)\).

**Proof.** A direct computation gives

\[
 \langle x, \nu \rangle_{ij} = h_{ij} + h_{ij;k}(x, e_k) - (h^2)_{ij}(x, \nu).
\]
Using Proposition 2.1, we have
$$ \partial_t \langle x, \nu \rangle = \langle f \nu + T, \nu \rangle + \langle x, -\nabla f + h(e_i, T)e_i \rangle $$
$$ = \frac{n(1 + \cos \theta \langle \nu, e \rangle)}{H} - \langle x, \nu \rangle + h(x^T, T) $$
$$ - \langle x, e_k \rangle \left( \frac{n \cos \theta \langle h_k e_i, e \rangle}{H} - (1 + \cos \theta \langle \nu, e \rangle) \frac{n H \langle k, i \rangle}{H^2} - \langle x, h_k e_i \rangle \right), $$
which yields
$$ \mathcal{L} \langle x, \nu \rangle = \left[ (1 + \cos \theta \langle \nu, e \rangle) \frac{n |h|^2}{H^2} - 1 \right] \langle x, \nu \rangle. \quad (2.4) $$
It is easy to see
$$ \langle \nu, e \rangle_{kl} = h_{kl,s} \langle e_s, e \rangle - (h^2)_{kl} \langle \nu, e \rangle. $$
Combining it with
$$ \langle e, \nabla f \rangle = n \langle e, e_i \rangle \left( \frac{1 + \cos \theta \langle \nu, e \rangle}{H} \right)_i - \langle e, e_i \rangle \langle x, \nu \rangle_i $$
$$ = \frac{n \cos \theta}{H} h(e^T, e^T) - n H^{-2} \langle \nabla H, e \rangle \langle x, \nu \rangle - h(x^T, e^T), $$
we have
$$ \partial_t \langle \nu, e \rangle = - \langle \nabla f, e \rangle + h(e_i, T) \langle e_i, e \rangle $$
$$ = - \frac{n \cos \theta}{H} h(e^T, e^T) + n H^{-2} \langle \nabla H, e \rangle \langle x, \nu \rangle - h(x^T, e^T) + h(T, e^T). $$
It follows
$$ \mathcal{L} \langle \nu, e \rangle = (1 + \cos \theta \langle \nu, e \rangle) n H^{-2} |h|^2 \langle \nu, e \rangle. \quad (2.5) $$
Now we introduce the function mentioned in the introduction
$$ \bar{u} := \frac{\langle x, \nu \rangle}{1 + \cos \theta \langle \nu, e \rangle}. \quad (2.6) $$
One can check that it satisfies
$$ \mathcal{L} \bar{u} = \frac{1}{1 + \cos \theta \langle \nu, e \rangle} \mathcal{L} u - \frac{\langle x, \nu \rangle}{(1 + \cos \theta \langle \nu, e \rangle)^2} \mathcal{L} (1 + \cos \theta \langle \nu, e \rangle) $$
$$ + \frac{2 n(1 + \cos \theta \langle \nu, e \rangle)}{H^2} \left[ \frac{\langle x, \nu \rangle_i (1 + \cos \theta \langle \nu, e \rangle)_i}{(1 + \cos \theta \langle \nu, e \rangle)^2} \right. $$
$$ \left. \left. - \langle x, \nu \rangle \frac{(\cos \theta \langle \nu, e \rangle)_i (\cos \theta \langle \nu, e \rangle)_i}{(1 + \cos \theta \langle \nu, e \rangle)^2} \right] \right]. $$
Combining with equations (2.4) and (2.5), we get
$$ \mathcal{L} \bar{u} = \frac{\langle x, \nu \rangle}{1 + \cos \theta \langle \nu, e \rangle} \left( \frac{n |h|^2}{H^2} - 1 \right) + \frac{2 n}{H^2} \bar{u}_i \langle 1 + \cos \theta \langle \nu, e \rangle \rangle_i. \quad (2.7) $$
On $\partial M$, we have
$$ \nabla \mu \langle x, \nu \rangle = \langle x, h(\mu, \mu) \rangle = \cos \theta h(\mu, \mu) \langle x, \nu \rangle = \cot \theta h(\mu, \mu) \langle x, \nu \rangle $$
and
$$ \nabla \mu (1 + \cos \theta \langle \nu, e \rangle) = - \sin \theta h(\mu, \mu) \langle \nu, e \rangle. $$
Altogether yields
$$ \nabla \mu \bar{u} = 0, \text{ on } \partial M. \quad (2.8) $$
Since
\[ n|\mathbf{h}|^2 \geq H^2, \quad (2.9) \]
(2.7) gives
\[ \mathcal{L} \bar{u} \geq 0, \mod \nabla \bar{u}, \]
which, together with (2.8), implies
\[ \bar{u} \geq \min_{M} \bar{u}(\cdot, 0) \]
by the maximum principle. Since
\[ 1 + \cos \theta \langle \nu, e \rangle \geq 1 - \cos \theta > 0, \]
then
\[ \langle x, \nu \rangle = (1 + \cos \theta \langle \nu, e \rangle) \cdot \bar{u} \geq c_0 > 0, \]
for some positive constant \( c_0 \), which depends only on \( \Sigma_0 \).
\[ \square \]
Next, we show the uniform bound of the mean curvature and the mean convexity is preserved along flow (1.1).

The evolution equation of \( H \) was computed in [25, Proposition 4.3].

**Proposition 2.4** ([25]). Among flow (1.1), there holds
\[ \mathcal{L} H = 2nH^{-2} \cos \theta \mathbf{h}(\nabla H, e^T) - 2nH^{-3}(1 + \cos \theta \langle \nu, e \rangle)|\nabla H|^2 + H \left( 1 - \frac{n|\mathbf{h}|^2}{H^2} \right), \]
and
\[ \nabla \mu H = 0, \quad \text{on } \partial M. \quad (2.10) \]

**Proposition 2.5.** If \( \Sigma_t \) solves flow (1.1), then
\[ H(p, t) \leq \max_{M} H(\cdot, 0). \quad (2.11) \]

**Proof.** Using (2.9), by (2.10), we see
\[ \mathcal{L} H = H \left( 1 - \frac{n|\mathbf{h}|^2}{H^2} \right) \leq 0, \mod \nabla H. \]
Since \( \nabla \mu H = 0 \) on \( \partial M \), the conclusion directly follows from the maximum principle. \( \square \)

From the evolution equation for \( H \) one could not obtain the lower bound of \( H \). Instead, we consider the function \( H \bar{u} \).

**Proposition 2.6.** If \( \Sigma_t \) solves flow (1.1) with an initial hypersurface \( \Sigma_0 \) being a star-shaped and strictly mean convex capillary hypersurface in \( \mathbb{R}^{n+1}_+ \), then
\[ H(p, t) \geq C, \quad \forall (p, t) \in M \times [0, T), \quad (2.12) \]
where the positive constant \( C \) depends only on the initial datum.

**Proof.** Define the function
\[ P := H \bar{u}. \quad (2.13) \]
Using (2.7) and (2.10), we obtain
\[
\mathcal{L} P = \bar{u} \mathcal{L} H + H \mathcal{L} \bar{u} - 2n(1 + \cos \theta \langle \nu, e \rangle) \frac{\langle \nabla H, \nabla \bar{u} \rangle}{H^2} \\
= \frac{2n}{H^2} \left( \nabla (1 + \cos \theta \langle \nu, e \rangle) - (1 + \cos \theta \langle \nu, e \rangle) \frac{\nabla H}{H} \right) \cdot \nabla P.
\]
On \( \partial M \), (2.8) and (2.10) imply
\[ \nabla \mu P = 0. \quad (2.14) \]
By applying the maximum principle, we get
\[ P \geq \min_M H \bar{u}(\cdot, 0), \]
together with the uniform upper bound of \( \bar{u} \) (which follows from Proposition 2.2), it implies the desired uniform estimate of \( H \).

\[ \Box \]

3. Long-time existence and convergence

The short-time existence of flow (1.1) is well-known. Let \( T^* \) be the maximal time of the existence of a smooth solution to (1.1) in the class of star-shaped and strictly mean convex hypersurfaces. In order to establish the long-time existence of flow (1.1), we need to obtain the uniform height estimate, the gradient estimate, and the second order derivative estimate (or curvature estimates) for the solution of flow (1.1). Then the long-time existence and the uniform \( C^\infty \)-estimates follow from the standard nonlinear parabolic PDE theory with a strictly oblique boundary condition (cf. [5, 14, 23]).

Assume that a capillary hypersurface \( \Sigma \subset \mathbb{R}^{n+1}_+ \) is star-shaped with respect to the origin. One can reparametrize it as a graph over \( S^n_+ \). Namely, there exists a positive function \( \rho \) defined on \( S^n_+ \) such that
\[ \Sigma = \{ \rho(X)X | X \in S^n_+ \}, \]
where \( X := (X_1, \ldots, X_n) \) is a local coordinate of \( S^n_+ \).

We denote \( \nabla_0^0 \) the Levi-Civita connection on \( S^n_+ \) with respect to the standard round metric \( \sigma := g_{S^n_+}, \partial_i := \partial X_i, \sigma_{ij} := \sigma(\partial_i, \partial_j), \rho_i := \nabla_i^0 \rho, \) and \( \rho_{ij} := \nabla_i^0 \nabla_j^0 \rho \). The induced metric \( g \) on \( \Sigma \) is given by
\[ g_{ij} = \rho^2 \sigma_{ij} + \rho_i \rho_j = e^{2\varphi} (\sigma_{ij} + \varphi_i \varphi_j), \]
where \( \varphi(X) := \log \rho(X) \). Its inverse \( g^{-1} \) is given by
\[ g^{ij} = \frac{1}{\rho^2} \left( \sigma^{ij} - \frac{\rho^i \rho^j}{\rho^2 + |\nabla^0 \rho|^2} \right) = e^{-2\varphi} \left( \sigma^{ij} - \frac{\varphi^i \varphi^j}{v^2} \right), \]
where \( \rho^i := \sigma^{ij} \rho_j, \varphi^i := \sigma^{ij} \varphi_j \) and
\[ v := \sqrt{1 + |\nabla^0 \varphi|^2}. \]
The unit outward normal vector field on \( \Sigma \) is given by
\[ \nu = \frac{1}{v} (\partial_\rho - \rho^{-2} \nabla^0 \rho) = \frac{1}{v} (\partial_\rho - \rho^{-1} \nabla^0 \varphi). \]
The second fundamental form \( h := (h_{ij}) \) on \( \Sigma \) is
\[ h_{ij} = \frac{e^\varphi}{v} (\sigma_{ij} + \varphi_i \varphi_j - \varphi_{ij}), \]
and its Weingarten matrix \( (h^i_j) \) is
\[ h^i_j = g^{ik} h_{kj} = \frac{1}{e^\varphi v} \left[ \delta^i_j - \left( \sigma^{ik} - \frac{\varphi^i \varphi^k}{v^2} \right) \varphi_{kj} \right]. \]
We also have for the mean curvature,
\[ H = \frac{1}{e^\varphi v} \left[ n - \left( \sigma^{ij} - \frac{\varphi^i \varphi^j}{v^2} \right) \varphi_{ij} \right]. \]
Next, we derive the capillary boundary condition in (1.1) in terms of the radial function \( \varphi \). We use the polar coordinate in the half-space, that is, for \( x := (x', x_{n+1}) \in \mathbb{R}^n \times [0, +\infty) \) and \( X := (\beta, \xi) \in [0, \frac{\pi}{2}] \times S^{n-1} \), we have

\[
x_{n+1} = r \cos \beta, \quad |x'| = r \sin \beta.
\]

Then

\[
e_{n+1} = \partial x_{n+1} = \cos \beta \partial r - \frac{\sin \beta}{r} \partial \beta.
\]

In these coordinates the standard Euclidean metric is given by

\[
|dx|^2 = dr^2 + r^2 (d\beta^2 + \sin^2 \beta g_{n-1}).
\]

It follows that

\[
\langle \nu, e_{n+1} \rangle = \frac{1}{v} \left( \cos \beta + \sin \beta \nabla_{\beta}^0 \varphi \right).
\]

(3.2)

Along \( \partial S^+_n \), i.e., \( \beta = \frac{\pi}{2} \), it holds

\[
e = -e_{n+1} = -\frac{1}{r} \partial \beta,
\]

then

\[
-\cos \theta = \langle \nu, e \rangle = \left\langle \frac{1}{v} \left( \partial_r - r^{-1} \nabla_0^0 \varphi \right), \frac{1}{r} \partial \beta \right\rangle = -\frac{\nabla_0^0 \varphi}{v},
\]

that is,

\[
\nabla_0^0 \varphi = \cos \theta \sqrt{1 + |\nabla_0^0 \varphi|^2}.
\]

(3.3)

In the following, we derive the full curvature estimates (which in turn are the \( C^2 \) estimates of \( \varphi \)) of flow (1.1).

To prove the existence of flow (1.1) for all time, we follow the approach in [17, Section 4] and [14, Chapter V], which proves the Schauder estimates without having direct access to a second derivative estimate of \( \varphi \).

Consider flow (1.1) for radial graphs over \( S^+_n \). It is known that the scalar function \( \varphi(z, t) := \log \rho(X(z, t), t) \) of the standard Euclidean metric is given by

\[
\varphi = \frac{1}{v} \left( \cos \beta + \sin \beta \nabla_{\beta}^0 \varphi \right).
\]

(3.4)

where \( \partial \beta \) is the outward normal of \( \partial S^+_n \subset \mathbb{S}^n_+ \), \( \varphi_0 \) is the corresponding radial function of \( x_0(M) \) over \( \mathbb{S}^n_+ \) and

\[
G(\varphi_{ij}, \varphi_k) := \frac{n v^2}{n - \left( \sigma_{ij} - \frac{\varphi_{ij}}{v^2} \right) \varphi_{ij}} \left[ 1 - \frac{\cos \theta}{v} \left( \cos \beta + \sin \beta \nabla_{\beta}^0 \varphi \right) \right] - 1.
\]

Note that \( G \) does not explicitly depend on the solution \( \varphi \) itself, but only on its first and second derivatives, which is an important fact. Set

\[
G^i := \frac{\partial G}{\partial \varphi_i}, \quad G^k := \frac{\partial G}{\partial \varphi_k}.
\]

Recall that \( T^* \) is the maximal time such that there exists some \( \varphi \in C^{2,1}(\mathbb{S}^n_+ \times [0, T^*)) \cap C^\infty(\mathbb{S}^n_+ \times (0, T^*)) \) which solves (3.4). In the sequel, we will prove a priori estimates for the solution \( \varphi \) on \([0, T]\) for any \( T < T^* \).
3.1. Gradient estimates. First of all, we give the estimate of $\dot{\varphi} := \frac{\partial \varphi}{\partial t}$.

**Proposition 3.1.** If $\varphi$ solves (3.4) and $\theta \in (0, \pi)$, then

$$\min_{S^n} \frac{nv_0}{e^\varphi H_0} \left[ 1 - \frac{\cos \theta}{v} \left( \cos \beta + \sin \beta \nabla_{\partial \beta} \varphi_0 \right) \right] \leq \dot{\varphi} + 1$$

$$\leq \max_{S^n} \frac{nv_0}{e^\varphi H_0} \left[ 1 - \frac{\cos \theta}{v} \left( \cos \beta + \sin \beta \nabla_{\partial \beta} \varphi_0 \right) \right], \text{ in } S^n_+ \times [0, T],$$

where $H_0 := H(\cdot, 0)$ and $v_0 := v(\cdot, 0)$.

**Proof.** Assume $\dot{\varphi}$ admits the maximum value at some point $(p_0, t_0) \in \overline{S^n_+} \times [0, T]$, that is $\dot{\varphi}(p_0, t_0) = \max_{\overline{S^n_+} \times [0, T]} \dot{\varphi}$.

We claim that $t_0 = 0$. If not, we have $t_0 > 0$. A direct computation gives

$$\frac{\partial \dot{\varphi}}{\partial t} = \sum_{i,j=1}^n G^{ij} \dot{\varphi}_{ij} + \sum_{k=1}^n G^k \dot{\varphi}_k \text{ in } \overline{S^n_+} \times [0, T].$$

The maximum principle yields that $p_0 \in \partial S^n_+$. Choosing an orthonormal frame around $p_0$ such that $\{e_\alpha\}_{\alpha=1}^{n-1}$ are the tangential vector fields on $\partial S^n_+$ and $e_n := \partial \beta$ the unit normal along $\partial S^n_+ \subset \overline{S^n_+}$. Then we have $(\dot{\varphi})_n(p_0, t_0) = 0$. Combining it with the boundary equation in (3.4) yields that

$$(\dot{\varphi})_n(p_0, t_0) = \frac{d}{dt} \left( \cos \theta \sqrt{1 + |\nabla^0 \varphi|^2} \right) = \sum_{i=1}^n \frac{\varphi_i(p_0)}{\sqrt{1 + |\nabla^0 \varphi|^2}} \cos \theta = \frac{\varphi_n(p_0)}{\sqrt{1 + |\nabla^0 \varphi|^2}} \cos \theta = \cos^2 \theta (\dot{\varphi})_n,$$

Since $|\cos \theta| < 1$, we obtain $(\dot{\varphi})_n(p_0, t_0) = 0$, which is a contradiction to the Hopf boundary lemma. Therefore $t_0 = 0$, as a result,

$$\dot{\varphi} \leq \max_{\overline{S^n_+}} \dot{\varphi}.$$

Similarly, we have

$$\dot{\varphi} \geq \min_{\overline{S^n_+}} \dot{\varphi}.$$

Both, together with

$$G = \frac{nv}{e^\varphi H} \left[ 1 - \frac{\cos \theta}{v} \left( \cos \beta + \sin \beta \nabla_{\partial \beta} \varphi \right) \right] - 1,$$

give the desired estimate. Hence we complete the proof. \(\square\)

Proposition 2.3 says that the star-shapedness is preserved along flow (1.1). In particular, combining with Proposition 2.2, Proposition 3.1 yields the uniform gradient estimate of $\varphi$ in (3.4). That is

**Proposition 3.2.** If $\varphi$ solves (3.4), then

$$\|\varphi\|_{C^1(S^n_+ \times [0, T])} \leq C,$$

where $C > 0$ depends only on the initial datum.
3.2. Schauder’s estimate and long-time existence. In this subsection, we follow the argument in [17] (see also [28]) to obtain the higher regularity of flow (1.1), i.e. (3.4). First, we estimate the Hölder norm of the spatial derivative of $\varphi$.

**Proposition 3.3.** If $\varphi$ solves (3.4), then there exists some $\gamma > 0$ such that

$$\left[\nabla^0 \varphi\right]_{x,\gamma} + \left[\nabla^0 \varphi\right]_{t,\frac{\gamma}{2}} \leq C,$$

where $[\psi]_{z,\gamma}$ denotes the $\gamma$-Hölder semi-norm of $\psi$ in $\mathbb{S}^n_+ \times [0,T]$ with respect to the $z$-variable and $C := C(\|\varphi_0\|_{C^{2+\alpha}(\mathbb{S}^n_+)}, n, \gamma)$.

**Proof.** The a priori estimates of $|\nabla^0 \varphi|$ and $|\partial_t \varphi|$ imply the bound for $[\varphi]_{x,\gamma}$ and $[\varphi]_{t,\frac{\gamma}{2}}$, due to Proposition 3.1 and 3.2. Together with [14, Chapter II, Lemma 3.1] one can give the bound for $\left[\nabla^0 \varphi\right]_{t,\frac{\gamma}{2}}$ provided that we have a bound for $\left[\nabla^0 \varphi\right]_{x,\gamma}$. Hence it suffices to bound $\left[\nabla^0 \varphi\right]_{x,\gamma}$ in the following. In fact, we fix $t$ and rewrite (3.4) as a quasilinear elliptic PDE with a strictly oblique boundary value condition

$$\text{div}_{\sigma} \left( \frac{\nabla^0 \varphi}{\sqrt{1 + |\nabla^0 \varphi|^2}} \right) = \frac{n}{v} - \frac{n}{1 + \varphi} \left[ v - \cos \theta \left( \cos \beta + \sin \beta \nabla^0 \partial_\beta \varphi \right) \right],$$

(3.5)

and on $\partial \mathcal{S}^n_+$,

$$\nabla^0_{\partial_\beta} \varphi = \cos \theta \sqrt{1 + |\nabla^0 \varphi|^2}.$$

The right-hand side of (3.5) is a bounded function in $x$, due to Proposition 3.1 and 3.2. Thus for any $\Omega' \subset \subset \mathcal{S}^n_+$, using [15, Chapter 3, Theorem 14.1], one can prove an interior estimate of form

$$\left[\nabla^0 \varphi\right]_{\gamma, \Omega'} \leq C \left( \text{dist}_{\sigma}(\Omega', \partial \mathcal{S}^n_+), |\nabla^0 \varphi|, |\varphi| \right),$$

for some $\gamma > 0$. And the boundary Hölder estimate of $\nabla^0 \varphi$ can be obtained by adapting the similar procedure as in [15, Chapter 10, Section 2]. Hence we prove the estimate for $\left[\nabla^0 \varphi\right]_{x,\gamma}$, which yields the conclusion. □

Next, we estimate the Hölder norm of the time derivative of $\varphi$.

**Proposition 3.4.** If $\varphi$ solves (3.4), then there exists some $\gamma > 0$ such that

$$[\partial_t \varphi]_{x,\gamma} + [\partial_t \varphi]_{t,\frac{\gamma}{2}} \leq C,$$

where $[\psi]_{z,\gamma}$ denotes the $\gamma$-Hölder semi-norm of $\psi$ in $\mathbb{S}^n_+ \times [0,T]$ with respect to the $z$-variable and $C := C(\|\varphi_0\|_{C^{2+\alpha}(\mathbb{S}^n_+)}, n, \gamma)$.

**Proof.** Taking into account (3.1) and (3.2), the first equation in (3.4) is equivalent to

$$\dot{\varphi} = \frac{n(1 + \cos \langle \nu, e \rangle)}{u H} - 1,$$

where

$$u := \langle x, \nu \rangle = \frac{e^\varphi}{v}.$$

Combining it with (2.10), (2.4) and (2.5), and through some tedious calculations, we know that the evolution equation of

$$\Phi := \dot{\varphi}$$
has a nice divergence structure with respect to the induced metric \( g \), which is uniformly controlled (due to the uniform estimates for \( \varphi \) and \( \nabla^0 \varphi \), see Proposition 3.2). Indeed,

\[
\partial_t \Phi = \text{div}_g \left( \frac{n(1 + \cos \Theta(\nu, e)) \nabla \Phi}{H^2} \right) - \frac{2n(1 + \cos \Theta(\nu, e))}{H^2(\Phi + 1)} |\nabla \Phi|^2 \tag{3.6}
\]

\[
+ \left( T + x - \frac{n \cos \Theta}{H} e, \nabla \Phi \right) + \frac{n}{H^2} \langle \nabla (1 + \cos \Theta(\nu, e)), \nabla \Phi \rangle.
\]

Moreover by using (2.8) and (2.10), we know that \( \Phi \) satisfies the Neumann boundary value condition, as

\[
\nabla \mu \Phi = \nabla \mu \left[ \frac{n(1 + \cos \Theta(\nu, e))}{u_H} - 1 \right] = 0.
\]

In order to adopt the approach as in [14, Chapter V], we use the weak formulation of the quasi-linear parabolic equation (3.6), that is

\[
\int_{t_0}^{t_1} \int_M \eta \partial_t \Phi d\mu_t dt = \int_{t_0}^{t_1} \int_M \left[ - \frac{n(1 + \cos \Theta(\nu, e))}{H^2} (\nabla \eta, \nabla \Phi) - \frac{2n(1 + \cos \Theta(\nu, e))}{H^2(\Phi + 1)} |\nabla \Phi|^2 \eta + \frac{n}{H^2} \langle \nabla (1 + \cos \Theta(\nu, e)), \nabla \Phi \rangle \eta 
\right. 
\]

\[
+ \eta \left( T + x - \frac{n \cos \Theta}{H} e, \nabla \Phi \right) \bigg] d\mu_t dt, \tag{3.7}
\]

for the test function \( \eta \) and \( 0 < t_0 < t_1 < T \).

Choose a specific test function \( \eta \) defined by

\[
\eta := \xi^2 (1 + \cos \Theta(\nu, e))(\Phi + 1), \tag{3.8}
\]

where \( \xi \in C_c^\infty(B_\rho \times [0, T]) \) is an arbitrary smooth cut-off function with values in \([0, 1]\) for some small ball \( B_\rho \subset M \). Then the left-hand side of (3.7) is equal to

\[
\int_{t_0}^{t_1} \int_M \eta \partial_t \Phi d\mu_t dt = \frac{1}{2} \int_M \xi^2 (\Phi + 1)^2 (1 + \cos \Theta(\nu, e)) d\mu_t \bigg|_{t_0}^{t_1}
\]

\[- \int_{t_0}^{t_1} \int_M (\Phi + 1)^2 \frac{\partial \xi}{\partial t} (1 + \cos \Theta(\nu, e)) d\mu_t dt
\]

\[- \frac{1}{2} \int_{t_0}^{t_1} \int_M \xi^2 (\Phi + 1)^2 \frac{\partial }{\partial t} \left[ (1 + \cos \Theta(\nu, e)) d\mu_t \right] dt. \tag{3.9}
\]

The sum of the first three terms on the right-hand side in (3.7) equals

\[
\int_{t_0}^{t_1} \int_M \left[ - \frac{2n(1 + \cos \Theta(\nu, e))^2}{H^2} (\Phi + 1) \xi \langle \nabla \xi, \nabla \Phi \rangle - \frac{3n(1 + \cos \Theta(\nu, e))^2}{H^2} \xi^2 |\nabla \Phi|^2 \right] d\mu_t dt.
\]

Plugging these back into (3.7) yields that

\[
\frac{1}{2} \int_M \xi^2 (\Phi + 1)^2 (1 + \cos \Theta(\nu, e)) d\mu_t \bigg|_{t_0}^{t_1} + \int_{t_0}^{t_1} \int_M \frac{3n(1 + \cos \Theta(\nu, e))^2}{H^2} \xi^2 |\nabla \Phi|^2 d\mu_t dt
\]

\[
= \int_{t_0}^{t_1} \int_M \left[ (\Phi + 1) \frac{\partial \xi}{\partial t} + \xi \left( T + x - \frac{n \cos \Theta}{H} e, \nabla \Phi \right) - \frac{2n(1 + \cos \Theta(\nu, e))}{H^2} \langle \nabla \xi, \nabla \Phi \rangle \right] \cdot
\]

\[
\xi (\Phi + 1)(1 + \cos \Theta(\nu, e)) d\mu_t dt + \frac{1}{2} \int_{t_0}^{t_1} \int_M \xi^2 (\Phi + 1)^2 \frac{\partial }{\partial t} \left[ (1 + \cos \Theta(\nu, e)) d\mu_t \right] dt. \tag{3.10}
\]

Notice that Proposition 2.5 implies that

\[
\frac{(1 + \cos \Theta(\nu, e))^2 \xi^2 |\nabla \Phi|^2}{H^2} \geq \frac{(1 + \cos \Theta(\nu, e))^2 \xi^2 |\nabla \Phi|^2}{\max H^2}.
\]
For any $\varepsilon > 0$, again using Proposition 3.2, Proposition 2.6, Proposition 2.5 and Young’s inequality, we have

$$
\begin{align*}
\int_{t_0}^{t_1} \int_M \xi^2 \left( T + x - \frac{n \cos \theta}{H} e, \nabla \Phi \right) (\Phi + 1) (1 + \cos \theta \langle \nu, e \rangle) d\mu_t dt \\
\leq \int_{t_0}^{t_1} \int_M \left[ \varepsilon \xi^2 |\nabla \Phi|^2 (1 + \cos \theta \langle \nu, e \rangle)^2 + \frac{1}{4\varepsilon^2} \xi^2 \left| T + x - \frac{n \cos \theta}{H} e \right|^2 (\Phi + 1)^2 \right] d\mu_t dt \\
= \int_{t_0}^{t_1} \int_M \left[ \varepsilon \xi^2 |\nabla \Phi|^2 (1 + \cos \theta \langle \nu, e \rangle)^2 + C_{\varepsilon} \xi^2 (\Phi + 1)^2 \right] d\mu_t dt
\end{align*}
$$

and

$$
\begin{align*}
\int_{t_0}^{t_1} \int_M 2n \frac{1 + \cos \theta \langle \nu, e \rangle}{H^2} (\nabla \xi, \nabla \Phi) (\Phi + 1) (1 + \cos \theta \langle \nu, e \rangle) d\mu_t dt \\
\leq \int_{t_0}^{t_1} \int_M \left[ \varepsilon \xi^2 |\nabla \Phi|^2 (1 + \cos \theta \langle \nu, e \rangle)^2 + \frac{n^2}{\varepsilon H^4} |\nabla \xi|^2 (\Phi + 1)^2 (1 + \cos \theta \langle \nu, e \rangle)^2 \right] d\mu_t dt \\
= \int_{t_0}^{t_1} \int_M \left[ \varepsilon \xi^2 |\nabla \Phi|^2 (1 + \cos \theta \langle \nu, e \rangle)^2 + C_{\varepsilon} |\nabla \xi|^2 (\Phi + 1)^2 \right] d\mu_t dt,
\end{align*}
$$

where $C_{\varepsilon} > 0$ is a uniform constant. Next, we handle the last term in (3.10). From Proposition 2.1 (2) (3), we have

$$
\begin{align*}
\frac{\partial}{\partial t} [(1 + \cos \theta \langle \nu, e \rangle)] d\mu_t \\
= [\cos \theta (-\nabla f + h(e_i, T)e_i, e) + (1 + \cos \theta \langle \nu, e \rangle) (f H + \text{div}(T))] d\mu_t \\
= [-\cos \theta (\nabla f, e) + (1 + \cos \theta \langle \nu, e \rangle) f H + \text{div}((1 + \cos \theta \langle \nu, e \rangle) T)] d\mu_t,
\end{align*}
$$

where we have used the simple fact

$$
\cos \theta \langle h(e_i, T)e_i, e \rangle = \langle \nabla (1 + \cos \theta \langle \nu, e \rangle), T \rangle.
$$

Noticing that

$$
f = \frac{n (1 + \cos \theta \langle \nu, e \rangle)}{H} - \langle x, \nu \rangle = u \Phi,
$$

and

$$
(1 + \cos \theta \langle \nu, e \rangle) f H = n (1 + \cos \theta \langle \nu, e \rangle)^2 \frac{\Phi}{\Phi + 1},
$$

thus the last term in (3.10) equals to

$$
\begin{align*}
&- \frac{\cos \theta}{2} \int_{t_0}^{t_1} \int_M \xi^2 (\Phi + 1)^2 \nabla e \tau (u \Phi) d\mu_t dt + \frac{n}{2} \int_{t_0}^{t_1} \int_M (1 + \cos \theta \langle \nu, e \rangle)^2 \xi^2 (\Phi + 1) \Phi d\mu_t dt \\
&+ \frac{1}{2} \int_{t_0}^{t_1} \int_M \xi^2 (\Phi + 1)^2 \text{div}((1 + \cos \theta \langle \nu, e \rangle) T) d\mu_t dt.
\end{align*}
$$

(3.12)
By integration by parts and Young’s inequality, for any \( \varepsilon_1 > 0 \), the third term in (3.12) satisfies

\[
\left| \frac{1}{2} \int_{t_0}^{t_1} \int_M \xi^2 (\Phi + 1)^2 \text{div}((1 + \cos \theta \langle \nu, e \rangle) T) \mu dt \right| \\
= \left| \int_{t_0}^{t_1} \int_M (1 + \cos \theta \langle \nu, e \rangle) \xi (\Phi + 1)^2 (\nabla \xi, T) \mu dt \right| \\
+ \left| \int_{t_0}^{t_1} \int_M (1 + \cos \theta \langle \nu, e \rangle) \xi^2 (\Phi + 1)^2 (\nabla \Phi, T) \mu dt \right| \\
\leq \frac{\varepsilon_1}{2} \int_{t_0}^{t_1} \int_M |\nabla \Phi|^2 \mu dt + C \varepsilon_1 \int_{t_0}^{t_1} \int_M (\xi^2 + |\xi| \Phi) (\Phi + 1)^2 \mu dt.
\]

Similarly, using again integration by part and Young’s inequality, the first term in (3.12) satisfies

\[
\left| -\frac{\cos \theta}{2} \int_{t_0}^{t_1} \int_M \xi^2 (\Phi + 1)^2 \nabla \xi \cdot (u \Phi) \mu dt \right| \\
= \left| \int_{t_0}^{t_1} \int_M \cos \theta \left[ \xi (\nabla \xi \cdot \Phi) u (\Phi + 1)^2 \Phi + \xi^2 u (\Phi + 1) (\nabla \xi \cdot \Phi) \right] \mu dt \right| \\
\leq \frac{\varepsilon_1}{2} \int_{t_0}^{t_1} \int_M |\nabla \Phi|^2 \mu dt + C \varepsilon_1 \int_{t_0}^{t_1} \int_M (\xi^2 + |\xi| \Phi) (\Phi + 1)^2 \mu dt,
\]

where we have used the fact that \( u \) and \( \Phi \) are uniformly bounded in the last inequality. Now by substituting the above two terms back into (3.12), we conclude that

\[
(3.12) \leq \varepsilon_1 \int_{t_0}^{t_1} \int_M |\nabla \Phi|^2 \mu dt + C \varepsilon_1 \int_{t_0}^{t_1} \int_M (\xi^2 + |\xi| \Phi) (\Phi + 1)^2 \mu dt,
\]

which gives an estimate for the last term in (3.10). By choosing \( \varepsilon, \varepsilon_1 > 0 \) small enough, say \( \varepsilon := \frac{n}{\max H^2} \) and \( \varepsilon_1 := \frac{n(1 - \cos \theta)^2}{2 \max H^2} \) and noticing \( 2 \geq 1 + \cos \theta \langle \nu, e \rangle \geq 1 - \cos \theta > 0 \), we conclude that

\[
\frac{1}{2} (1 - \cos \theta) \int_{t_0}^{t_1} \int_M \xi^2 (\Phi + 1)^2 \mu dt \bigg|_{t_0}^{t_1} + \frac{n(1 - \cos \theta)^2}{2 \max H^2} \int_{t_0}^{t_1} \int_M \xi^2 |\nabla \Phi|^2 \mu dt \\
\leq C \int_{t_0}^{t_1} \int_M (\Phi + 1)^2 \left[ |\nabla \Phi|^2 + \xi^2 + \xi \left| \frac{\partial \xi}{\partial \theta} \right| (1 + \cos \theta \langle \nu, e \rangle) \right] \mu dt.
\]

This inequality has the same type of the one in [14, Chapter V, §1, Eq.(1.13)], therefore, similar to [14, Chapter V, §1] (interior estimate) and [14, Chapter V, §7, Page 478] (boundary estimate) or [16, Chapter XIII, section 6], the bound for \([\Phi]_{x, \gamma} = [\tilde{\varphi}]_{x, \gamma} \) and \([\tilde{\varphi}]_{\gamma, T} = [\varphi]_{\gamma, T} \) can be obtained. Moreover, all local interior and boundary estimates are independent of \( T \). The global estimates follow from the local results and a covering argument, which also does not depend on \( T \).

The Hölder estimate of gradient \( \varphi \) implies the Hölder estimate for the mean curvature.

**Proposition 3.5.** If \( \varphi \) solves (3.4), then there exists some \( \gamma > 0 \) such that

\[
[H]_{x, \gamma} + [H]_{y, \gamma} \leq C,
\]

where \([\psi]_{z, \gamma}\) denotes the \( \gamma \)-Hölder semi-norm of \( \psi \) in \( \mathbb{R}^n \times [0, T] \) with respect to the \( z \)-variable and \( C := C(\|\varphi_0\|_{C^{2,\alpha}([0,T]), \gamma}) \).

**Proof.** From (3.4), we know

\[
e^\varphi H = \frac{n}{1 + \varphi} \left[ \sqrt{1 + |\nabla \varphi|^2} - \cos \beta \left( \cos \beta + \sin \beta \nabla_{\partial \beta} \varphi \right) \right].
\]
Hence the conclusion \((3.14)\) follows from the Hölder estimates for \(|\nabla^0\varphi|, \dot{\varphi}\) and \(\varphi\) in Proposition \(3.3\) and Proposition \(3.4\), while the Hölder estimate for \(\varphi\) follows trivially from Proposition \(3.1\) and Proposition \(3.2\).

With the preparation above, we obtain the full second-order and high-order derivative estimates for \(\varphi\).

**Proposition 3.6.** If \(\varphi\) solves \((3.4)\), then for every \(t_0 \in (0, T)\), there exists some \(\gamma > 0\) such that

\[
\|\varphi\|_{C^{2+\gamma,l+2}(\overline{\Omega} \times [0,T])} \leq C(\|\varphi_0\|_{C^{2+\alpha}(\overline{\Omega})}, n, \gamma),
\]

(3.15)

and

\[
\|\varphi\|_{C^{2l+\gamma,l+2}(\overline{\Omega} \times [0,T])} \leq C(\|\varphi(\cdot,t_0)\|_{C^{2l+\alpha}(\overline{\Omega})}, n, \gamma),
\]

(3.16)

for any \(2 \leq l \in \mathbb{N}\).

**Proof.** To get the Schauder estimates \((3.15)\), we rewrite the evolution equation for \(\varphi\) with respect to the induced metric \(g\), which is uniformly controlled (due to the uniform estimate for \(\varphi\) and \(\nabla^0\varphi\), see Proposition \(3.2\)). Noticing that \(v := \sqrt{1+|\nabla^0\varphi|^2}\) and

\[
e^{\varphi} v H = n - \left(\sigma^{ij} - \frac{\varphi^i \varphi^j}{v^2}\right) \varphi_{ij} = n - e^{2\varphi} \Delta_g \varphi,
\]

we have

\[
\partial_t \varphi = \frac{nv}{e^{2\varphi}} H \left[1 - \frac{\cos \theta}{v} \left(\cos \beta + \sin \beta \nabla_\theta^0 \varphi\right)\right] - 1
\]

\[
= -\frac{n}{e^{2\varphi} H^2} (n - e^{2\varphi} \Delta_g \varphi) \left[1 - \frac{\cos \theta}{v} \left(\cos \beta + \sin \beta \nabla_\theta^0 \varphi\right)\right]
\]

\[
+ \frac{2nv}{e^{2\varphi} H} \left[1 - \frac{\cos \theta}{v} \left(\cos \beta + \sin \beta \nabla_\theta^0 \varphi\right)\right] - 1.
\]

It follows

\[
\partial_t \varphi = \frac{n}{H^2} \left[1 - \frac{\cos \theta}{v} \left(\cos \beta + \sin \beta \nabla_\theta^0 \varphi\right)\right] \Delta_g \varphi
\]

(3.17)

\[
+ \left(\frac{2nv}{e^{2\varphi} H} - \frac{n^2}{e^{2\varphi} H^2}\right) \left[1 - \frac{\cos \theta}{v} \left(\cos \beta + \sin \beta \nabla_\theta^0 \varphi\right)\right] - 1,
\]

which is a linear, uniformly parabolic equation with Hölder coefficients, due to Proposition \(3.5\) and Proposition \(3.3\). Moreover, the Neumann boundary value condition in \((3.4)\) satisfies the strictly oblique property (due to \(|\cos \theta| < 1\) and its coefficient has uniform Hölder bounds (due to again Proposition \(3.3\)). Hence \((3.15)\) follows from the linear parabolic PDE theory with a strictly oblique boundary value problem (cf. \([16, \text{Chapter V, Theorem 4.9 and Theorem 4.23}],\) see also \([14, \text{Chapter IV, Theorem 5.3}]\)).

Next, we show \((3.16)\). By differentiating both sides of \((3.17)\) with respect to \(t\) and \(\partial_j\), \(1 \leq j \leq n\), respectively, one can easily get evolution equations of \(\dot{\varphi}\) and \(\nabla_j^0 \varphi\) respectively, which, using the estimate \((3.15)\), can be treated as uniformly parabolic PDEs on the time interval \([t_0, T]\). At the initial time \(t_0\), all compatibility conditions are satisfied and the initial function \(\varphi(\cdot, t_0)\) is smooth, which implies the \(C^{3+\gamma,(3+\gamma)/2}\)-estimate for \(\nabla_j^0 \varphi\) and the \(C^{2+\gamma,(2+\gamma)/2}\)-estimate for \(\dot{\varphi}\). So, we have the \(C^{4+\gamma,(4+\gamma)/2}\)-estimate for \(\varphi\). From \([16, \text{Chapter 4, Theorem 4.3, Exercise 4.5}]\) and the above argument, it is not difficult to know that the constants are independent of the time \(T\). Higher regularity of \((3.16)\) can be proven by induction over \(2 \leq l \in \mathbb{N}\). Hence we complete the proof. \(\square\)
With the estimates given above, the long-time existence of inverse mean curvature type flow (3.4) or (1.1) follows directly. That is.

**Corollary 3.7.** If $\Sigma_0$ is star-shaped and strictly mean convex capillary hypersurface with a contact angle $\theta \in (0, \pi)$, then the solution of flow (1.1) exists for all time with a uniform $C^\infty$-estimates.

We are now ready to prove Theorem 1.1 by using the above uniform a priori estimates.

**Proof of Theorem 1.1.** Following the parabolic theory with a strictly oblique boundary value condition (cf. [5, 16, 23]), we know that the solution of equation (1.1) exists for all time with a uniform $C^\infty$-estimate. And the convergence result of Theorem 1.1 can be proved in a similar way as [25, Proposition 4.12], by adopting the monotonicity property of quermassintegrals along flow (1.1).

Finally, by applying the convergence result of flow (1.1) and the monotonicity of $V_{2,\theta}$, i.e., Theorem 1.1 and [25, Proposition 3.1], one can complete the proof of Theorem 1.2 for the strictly mean convex case. When $\Sigma$ is only mean convex, we use the approximation argument, as described in [9] to prove Theorem 1.2. The equality characterization in Theorem 1.2 can be established using the same approach as in [21, 25]. For the sake of conciseness, we will omit it here and leave it to the interested reader.

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(G. Wang) Mathematisches Institut, Albert-Ludwigs-Universität Freiburg, Freiburg im Breisgau, 79104, Germany
Email address: guofang.wang@math.uni-freiburg.de

(L. Weng) School of Mathematical Sciences, Anhui University, Hefei, 230601, P. R. China
Dipartimento Di Matematica, Università degli Studi di Roma "Tor Vergata", Via della Ricerca Scientifica 1, 00133, Roma, Italy
Email address: ljweng08@mail.ustc.edu.cn

(C. Xia) School of Mathematical Sciences, Xiamen University, Xiamen, 361005, P. R. China
Email address: chaoxia@xmu.edu.cn