Equilibrium states for certain partially hyperbolic attractors

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Abstract

We prove that a class of partially hyperbolic attractors introduced by Castro and Nascimento have unique equilibrium states for natural classes of potentials. We also show if the attractors are $C^2$ and have invariant stable and centre-unstable foliations, then there is a unique equilibrium state for the geometric potential and its 1-parameter family. We do this by applying general techniques developed by Climenhaga and Thompson.

Keywords: equilibrium states, attractor, partially hyperbolic

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1. Introduction

The notions of topological pressure and equilibrium states were introduced by Sinai, Ruelle, and Bowen [6, 30, 32]. These are generalizations of the notion of topological entropy and measures of maximal entropy, and they provide useful invariants for studying the properties of a dynamical system.

For a continuous map $f : X \to X$ of a compact metric space and a continuous function $\varphi : X \to \mathbb{R}$, called a potential function, the topological pressure is $P(\varphi; f) = \sup_{\mu} (h_{\mu}(f) + \int \varphi \, d\mu)$ where the supremum is taken over all $f$-invariant Borel probability measures.

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An $f$-invariant Borel probability measure that maximizes the quantity $h_\mu(f) + \int \varphi d\mu$ is an equilibrium state.

A long standing problem is to find conditions that guarantee the existence and/or uniqueness of equilibrium states. For Axiom A diffeomorphisms and Hölder continuous potential functions there is a unique equilibrium state restricted to each basic set \cite{[6]}. Outside of the hyperbolic setting there are a number of results where the class of diffeomorphisms and potential functions are restricted \cite{[1, 2, 5, 7–9, 12, 15, 17, 19, 20, 27, 28]}

Recently, new symbolic tools were obtained by Sarig \cite{[31]} for $C^{1+\alpha}$ diffeomorphisms of surfaces with positive topological entropy. This result allowed Buzzi, Crovisier, and Sarig \cite{[10]} to obtain uniqueness of maximal entropy measures for transitive $C^\infty$ surface diffeomorphisms with positive topological entropy.

For non-uniformly expanding maps there have been a number of results on the existence and uniqueness of equilibrium states \cite{[23, 24, 29, 33]}. Recently, Castro and Nascimento \cite{[11]} examined measures of maximal entropy for a class of partially hyperbolic attractors that arise from non-uniformly expanding maps.

We examine equilibrium states for potentials defined on the partially hyperbolic attractors as described in \cite{[11]}. Before stating the results we define the class of attractors we will investigate.

We first describe the non-uniformly expanding maps that will use in the definition of the attractors. Let $N$ be a connected compact Riemannian manifold and $g: N \to N$ be a topologically exact (so for each open set $U$ there exists some $N \in \mathbb{N}$ such that $g^N(U) = N$) local diffeomorphism with Lipschitz inverse branches, this means that there exists a function $L: N \to \mathbb{R}^+$ such that for all $x \in N$ there is a neighbourhood $U_x$ of $x$ where $g_x := g|_{U_x}: U_x \to g(U_x)$ is invertible and

$$d(g^{-1}(y), g^{-1}(z)) \leq L(x)d(y, z) \quad \text{for all } y, z \in g(U_x).$$

The degree of $g$ is the number of pre-images of any $x \in N$ by $g$ and is denoted by $\deg(g)$. We assume there exists a constant $\lambda_u \in (0, 1)$ and an open region $\Omega \subset N$ such that we have the following:

(H1) There exists a constant $L > 1$ such that $L(x) \leq L$ for all $x \in \Omega$ and $L(x) < \lambda_u$ for $x \not\in \Omega$; and

(H2) There exists a covering $\mathcal{P}$ of $N$ by injective domains of $g$ such that $\Omega$ can be covered by $q < \deg(g)$ elements of $\mathcal{P}$.

We now describe the attractors that arise from $g$. Let $M$ be a compact manifold and $f: M \to M$ a diffeomorphism onto its image such that there is a continuous surjection $\pi: M \to N$ where

$$\pi \circ f \equiv g \circ \pi.$$

Given $y \in N$ we set $M_y = \pi^{-1}(y)$. Therefore, $M = \bigcup_{y \in N} M_y$. Note that $f(M_y) \subset M_{g(y)}$, each $M_y$ is compact, and there is a maximum diameter for the sets $M_y$. The next assumption will ensure that the $M_y$ are local stable manifolds for points in the attractor.

(H3) Assume there exists some $\lambda_y \in (0, 1)$ such that

$$d(f(x), f(y)) \leq \lambda_y d(x, y)$$

for all $x, y \in M_y$ and all $z \in N$.

The set $\Lambda = \bigcap_{n=0}^{\infty} f^n(M)$ is a closed attractor. We also need the following fact so that the metric on $M$ is related to the metric on $N$.

$$3410$$
Given $x, y \in M$ we let $\hat{x} = \pi(x)$ and $\hat{y} = \pi(y)$. Then there exist $f$-invariant holonomies $h_{i,j}: M_i \cap \Lambda \to M_j \cap \Lambda$ and a constant $C \geq 1$ such that

$$\frac{1}{C}[d_M(\hat{x}, \hat{y}) + d_M(h_{i,j}(x), y)] \leq d_M(x, y) \leq C[d_M(\hat{x}, \hat{y}) + d_M(h_{i,j}(x), y)]$$

where $d_M$ and $d_N$ are the metrics on $M$ and $N$ respectively. Furthermore, we assume that the holonomies are invariant for $f$ so that $f(h_{i,j}(z)) = h_{f(i),f(j)}(f(z))$.

The attractor $\Lambda$ can be described as ‘solenoid-like’ as it can be shown to be topologically conjugate to the natural extension of the system $(N, g)$. The topological conjugacy can be useful in proving some of the properties we need, but we will work directly with the system $(\Lambda, f)$.

In [11] it is shown that there is a unique measure of maximal entropy for $\Lambda$. Furthermore, it is shown that this measure

- Has exponential decay of correlations for Hölder continuous functions.
- Satisfies a central limit theorem for any Hölder continuous function.
- This measure varies continuously with respect to the weak$^*$-topology on the space of measures and the $C^1$-topology on the space of maps.

Before stating our main results we need to specify some constants. Fix $\rho > 0$ sufficiently small so that $\Omega_{\rho} = \bigcup_{x \in \Omega} B_{\rho}(x)$ is still covered by $q$ elements of $\mathcal{P}$. Let

$$\alpha < \frac{\log \lambda_0}{\log \lambda_0 - \log L}. \quad (1)$$

We now define the function

$$\Psi_{\rho,\alpha}(\varphi) = \Psi(\varphi) = \alpha \sup_{x \in \Omega_{\rho}} \varphi(x) + (1 - \alpha)(\sup_{x \in \Omega_{\rho}} \varphi) + \log q + \epsilon(\alpha) + m \log L$$

where $m = \dim(N)$ and $\epsilon(\alpha)$ is a function such that $\epsilon(\alpha) \to 0$ as $\alpha \to 1$, see lemma 3.1 of [33] or [24] for a description of $\epsilon(\alpha)$.

**Theorem 1.1.** Let $f: \Lambda \to \Lambda$ be as described above and $\varphi: M \to \mathbb{R}$ be a Hölder continuous potential such that there exist constants $\rho, \alpha > 0$ where $\Psi_{\rho,\alpha}(\varphi) < P(\varphi, f|\Lambda)$, then $(\Lambda, f, \varphi)$ has a unique equilibrium state.

We mention that our results give another proof of theorem A in [11] on the existence of a unique measure of maximal entropy, but we are unable to obtain the statistical results they obtain in theorems B–D of [11].

Before stating the next theorem we point out that if $L$ is close to one, then we can fix $\alpha$ close to one in the definition of $\Psi(\varphi)$. In this case, $\Psi(\varphi)$ is close to $\sup_{\Omega_{\rho}} \varphi + \log q$. The above theorem then shows that so long as the potential is not too ‘concentrated’ in $\Omega$, then there will be a unique equilibrium state.

The next result shows that if the function $\varphi$ does not vary too much, then there is a unique equilibrium state.

**Theorem 1.2.** Let $f: \Lambda \to \Lambda$ be as described above and $\varphi: M \to \mathbb{R}$ be a Hölder continuous potential such that

$$\sup \varphi - \inf \varphi < \log \deg(g) - \log q - \epsilon(\alpha) - m \log L,$$

then $(\Lambda, f, \varphi)$ has a unique equilibrium state.
In addition to the previous hypothesis that were assumed in [11] we need some assumptions on the hyperbolic constants, and associated invariant foliations for the next result.

(H5) There exists an $f$-invariant splitting $E^s \oplus E^{cu}$ for $\Lambda$ and $\lambda_s > L^{-1}$. Also, there exists an $f$-invariant centre-unstable foliation $\mathcal{W}^{cu}$ of $\Lambda$ that is tangent to the centre-unstable subbundle $E^{cu}$ in $\Lambda$, and there exists an $f$-invariant stable foliation $\mathcal{W}^s$ tangent to the stable subbundle $E^s$ in $\Lambda$.

Let $\varphi^{geo}(x) = -\log \det(Df|_{E^{cu}(x)})$ where $E^{cu}$ is the centre-unstable subspace at $x$. This is referred to as the geometric potential. An ergodic invariant measure $\mu$ that is hyperbolic (has non-zero Lyapunov exponents) and has absolutely continuous conditionals measures on unstable (or centre-unstable) manifolds is an $SRB$ measure. The next theorem relates the equilibrium states for the geometric potentials and an $SRB$ measure.

**Theorem 1.3.** Let $f$ be $C^2$ satisfying properties (H1)–(H5) above and we assume that

$$\log q + \epsilon(\alpha) + m \log L < \min \{\log \deg g, -\sup \varphi^{geo}\},$$

then the following hold:

- $t = 1$ is the unique root of $t \mapsto P(t \varphi^{geo}; f|_{\Lambda})$.
- There is an $\epsilon > 0$ such that $t \varphi^{geo}$ has a unique equilibrium state $\mu_t$ supported on $\Lambda$ for each $t \in (-\epsilon, 1 + \epsilon)$.
- $\mu_1$ is the unique $SRB$ measure for $f$ supported on $\Lambda$.

The paper proceeds as follows. Section 2 lists background material as well as the results from [16] that we need to prove our result. In section 3, we define a decomposition of orbit segments for the partially hyperbolic attractors, and we prove properties for this decomposition that we need to apply theorem 5.5 of [16]. Section 4 contains entropy and pressure estimates we need to complete the proofs for theorems 1.1 and 1.2. Section 5 contains the proof of theorem 1.3.

We now provide an example of systems that satisfy the conditions of theorem 1.3. This example is given by a DA-type perturbation in a neighbourhood of a fixed point for a uniformly expanding endomorphism.

**Example.** Let $g_0$ be a linear expanding map of the $d$-torus with $d$ distinct positive eigenvalues. For a fixed point $p$ of $g_0$ we deform $g_0$ by a pitchfork bifurcation in a small neighbourhood of $p$ so that after the deformation $p$ is a hyperbolic saddle for the local diffeomorphism $g$. Furthermore, we assume that the perturbation is performed so that $p$ is slightly contracting in the direction associated with the weakest eigenvalue and expanding in the other directions, see
figure 1. By construction we know that there are two new fixed points created for g, and g agrees with f₀ outside of the neighbourhood of p.
For the neighbourhood of p sufficiently small and the contraction in the weak direction close to 1 we see that g satisfies the properties (H1) and (H2).

We now construct the map f. Let \( M = \mathbb{T}^d \times (D^2)^d \) where \( D^2 \) is the two-dimensional disk. Now let

\[
f(t_1, \ldots, t_d, z_1, \ldots, z_d) = (g(t_1, \ldots, t_d)), \frac{1}{2^{d+8}} z_1 + \frac{1}{2} e^{2\pi i t_1}, \ldots, \frac{1}{2^{d+8}} z_d + \frac{1}{2} e^{2\pi i t_d}.
\]

Castro [18] proves that these maps satisfy properties (H3)–(H5).

Furthermore, by the fact that the neighbourhood around p can be chosen arbitrarily small and that the contraction at p can be made arbitrarily close to 1 we see that restricted to the attractor \( \Lambda \) associated with f we see that the map f and the potential function \( \phi^{\text{geo}} \) satisfies the hypothesis of theorem 1.3.

2. Background

In this section we review basic properties of partial hyperbolicity. We also outline the results of Climenhaga and Thompson in [16].

2.1. Partial hyperbolicity.

Let \( M \) be a compact manifold. Recall that a diffeomorphism \( f: M \to M \) is (weakly) partially hyperbolic if there is a \( Df \)-invariant splitting \( TM = E^s \oplus E^c \oplus E^u \), where at least one of \( E^s \) or \( E^u \) is nontrivial, and constants \( N \in \mathbb{N}, \lambda > 1 \) such that for every \( x \in M \) and every unit vector \( v^\sigma \in E^\sigma \) for \( \sigma \in \{s, c, u\} \), we have

(a) \( \lambda \|Df^N_x v^s\| < \|Df^N_x v^c\| < \lambda^{-1} \|Df^N_x v^u\| \), and

(b) \( \|Df^N_x v^c\| < \lambda^{-1} < \lambda < \|Df^N_x v^u\| \).

A partially hyperbolic diffeomorphism \( f \) admits stable and unstable foliations \( W^s \) and \( W^u \), which are \( f \)-invariant and tangent to \( E^s \) and \( E^u \), respectively [26, theorem 4.8]. In our situation (H3) assures the existence of a stable foliation and the local stable leaves are given by the \( M_j \). There may or may not be foliations tangent to \( E^c \) and \( E^u \), respectively [26, theorem 4.8]. When these exist we denote these by \( W^c \), \( W^cs \), and \( W^{cu} \) and refer to these as the centre, centre-stable, and centre-unstable foliations respectively. For \( x \in M \), we let \( W^c(x) \) be the leaf of the foliation \( \sigma \in \{s, u, c, cs, cu\} \) containing \( x \) when this is defined. In our situation we know there are \( W^{cu} \) leaves in the attractor.

2.2. Pressure

Let \( f: X \to X \) be a continuous map on a compact metric space. We identify \( X \times \mathbb{N} \) with the space of finite orbit segments by identifying \( (x, n) \) with \((x, f(x), \ldots, f^{n-1}(x))\).

Given a continuous potential function \( \varphi: X \to \mathbb{R} \), write \( S^n_{x, \varphi}(x) = S^1_{x, \varphi}(x) = \sum_{k=0}^{n-1} \varphi(f^k x) \).

The nth Bowen metric associated to \( f \) is defined by

\[
d_n(x, y) = \max\{d(f^k x, f^k y) : 0 \leq k < n\}.
\]

Given \( x \in X, \epsilon > 0 \), and \( n \in \mathbb{N} \), the Bowen ball of order \( n \) with centre \( x \) and radius \( \epsilon \) is \( B_n(x, \epsilon) = \{y \in X : d_n(x, y) < \epsilon\} \). A set \( E \subset X \) is \((n, \epsilon)\)-separated if \( d_n(x, y) \geq \epsilon \) for all \( x, y \in E \).
Given $\mathcal{D} \subset X \times \mathbb{N}$, we interpret $\mathcal{D}$ as a collection of orbit segments. Write $\mathcal{D}_n = \{x \in X : (x, n) \in \mathcal{D}\}$ for the set of initial points of orbits of length $n$ in $\mathcal{D}$. Then we consider the partition sum

$$\Lambda^\text{sep}_n(\mathcal{D}, \varphi; \epsilon; f) = \sup \left\{ \sum_{E \in \mathcal{F}} e^{\varphi(E) + \epsilon} : E \subset \mathcal{D}_n \text{ is } (n, \epsilon) \text{ - separated} \right\}.$$ 

The pressure of $\varphi$ on $\mathcal{D}$ at scale $\epsilon$ is

$$P(\mathcal{D}, \varphi; \epsilon; f) = \lim_{n \to \infty} \frac{1}{n} \log \Lambda^\text{sep}_n(\mathcal{D}, \varphi; \epsilon),$$

and the pressure of $\varphi$ on $\mathcal{D}$ is

$$P(\mathcal{D}, \varphi; f) = \lim_{\epsilon \to 0} P(\mathcal{D}, \varphi; \epsilon).$$

Given $Z \subset X$, let $P(Z, \varphi; \epsilon; f) := P(Z \times \mathbb{N}, \varphi; f)$; observe that $P(Z, \varphi; f)$ denotes the usual upper capacity pressure $[25]$. We often write $P(\varphi; f)$ in place of $P(X, \varphi; f)$ for the pressure of the whole space.

When $\varphi = 0$, our definition gives the entropy of $\mathcal{D}$:

$$h(\mathcal{D}; \epsilon; f) = h(\mathcal{D}, \epsilon) := P(\mathcal{D}; 0, \epsilon), \quad h(\mathcal{D}) = \lim_{\epsilon \to 0} h(\mathcal{D}; \epsilon). \quad (3)$$

Write $\mathcal{M}(f)$ for the set of $f$-invariant Borel probability measures, and $\mathcal{M}_c(f)$ for the set of ergodic measures in $\mathcal{M}(f)$. The variational principle for pressure $[34, \text{ theorem } 10.4.1]$ states that

$$P(\varphi; f) = \sup_{\mu \in \mathcal{M}(f)} \left\{ \int h_\mu(f) + \int \varphi \, d\mu \right\} = \sup_{\mu \in \mathcal{M}_c(f)} \left\{ \int h_\mu(f) + \int \varphi \, d\mu \right\}.$$ 

A measure achieving the supremum is an equilibrium state.

### 2.3. Obstructions to expansivity, specification, and regularity

Bowen showed in $[6]$ that if $(X, f)$ has expansivity and specification, and $\varphi$ has a certain regularity property (now called the Bowen property), then there is a unique equilibrium state. We recall definitions and results from $[16]$, which show that non-uniform versions of Bowen’s hypotheses suffice to prove uniqueness.

Given a homeomorphism $f : X \to X$, the bi-infinite Bowen ball around $x \in X$ of size $\epsilon > 0$ is the set

$$\Gamma_x(\epsilon) := \{y \in X : d(f^k x, f^k y) < \epsilon \text{ for all } k \in \mathbb{Z}\}.$$ 

If there exists $\epsilon > 0$ for which $\Gamma_x(\epsilon) = \{x\}$ for all $x \in X$, we say $(X, f)$ is expansive.

**Definition 2.1.** For $f : X \to X$ the set of non-expansive points at scale $\epsilon$ is $\text{NE}(\epsilon) := \{x \in X : \Gamma_x(\epsilon) \neq \{x\}\}$. An $f$-invariant measure $\mu$ is almost expansive at scale $\epsilon$ if $\mu(\text{NE}(\epsilon)) = 0$.

Given a potential $\varphi$, the pressure of obstructions to expansivity at scale $\epsilon$ is

$$P^\perp(\varphi, \epsilon) = \sup_{\mu \in \mathcal{M}_c(f)} \left\{ h_\mu(f) + \int \varphi \, d\mu : \mu(\text{NE}(\epsilon)) > 0 \right\} = \sup_{\mu \in \mathcal{M}_c(f)} \left\{ h_\mu(f) + \int \varphi \, d\mu : \mu(\text{NE}(\epsilon)) = 1 \right\}.$$
This is monotonic in $\epsilon$, so we can define a scale-free quantity by

$$P_{\exp}^\perp(\varphi) = \lim_{\epsilon \to 0} P_{\exp}^\perp(\varphi, \epsilon).$$

**Definition 2.2.** A collection of orbit segments $G \subset X \times \mathbb{N}$ has $(W)$-specification at scale $\epsilon$ if there exists $\tau \in \mathbb{N}$ and $k_0$ such that for every $\{(x_j, n_j) : 1 \leq j \leq k\} \subset G$ with $n_j > k_0$, there is a point $x$ in

$$\bigcap_{j=1}^{k} f^{-(m_j-1+\tau)}B_{n_j}(x_j, \epsilon),$$

where $m_0 = -\tau$ and $m_j = \left(\sum_{i=1}^{j} n_i\right) + (j-1)\tau$ for each $j \geq 1$.

The above definition says that there is some point $x$ whose trajectory shadows each of the $(x_j, n_j)$ in turn, taking a transition time of exactly $\tau$ iterates between each one. The numbers $m_j$ for $j \geq 1$ are the time taken for $x$ to shadow $(x_1, n_1)$ up to $(x_j, n_j)$.

**Definition 2.3.** Given $G \subset X \times \mathbb{N}$, a potential $\varphi$ has the Bowen property on $G$ at scale $\epsilon$ if

$$V(G, \varphi, \epsilon) := \sup\{|S_\varphi(x) - S_\varphi(y)| : (x, n) \in G, y \in B_\epsilon(x, \epsilon)\} < \infty.$$

We say $\varphi$ has the Bowen property on $G$ if there exists $\epsilon > 0$ so that $\varphi$ has the Bowen property on $G$ at scale $\epsilon$.

Note that if $G$ has the Bowen property at scale $\epsilon$, then it has it for all smaller scales.

### 2.4. General results on uniqueness of equilibrium states

Our main tool for existence and uniqueness of equilibrium states is [16, theorem 5.5].

**Definition 2.4.** A decomposition for $(X, f)$ consists of three collections $P, G, S \subset X \times (\mathbb{N} \cup \{0\})$ and three functions $p, g, s : X \times \mathbb{N} \to \mathbb{N} \cup \{0\}$ such that for every $(x, n) \in X \times \mathbb{N}$, the values $p = p(x, n), g = g(x, n), \text{and } s = s(x, n)$ satisfy $n = p + g + s$, and

$$(x, p) \in P, \quad (f^p(x), g) \in G, \quad (f^{p+g}(x), s) \in S. \quad (4)$$

Note that the symbol $(x, 0)$ denotes the empty set, and the functions $p, g, s$ are permitted to take the value zero.

**Theorem 2.5** (Theorem 5.5 of [16]). Let $X$ be a compact metric space and $f : X \to X$ a homeomorphism. Let $\varphi : X \to \mathbb{R}$ be a continuous potential function. Suppose that $P_{\exp}^\perp(\varphi) < P(\varphi)$, and that $(X, f)$ admits a decomposition $(P, G, S)$ with the following properties:

(a) $G$ has $(W)$-specification at any scale;

(b) $\varphi$ has the Bowen property on $G$;

(c) $P(\mathbb{P} \cup S, \varphi) < P(\varphi)$.

Then there is a unique equilibrium state for $\varphi$.

### 3. Decomposition, specification, the Bowen property, and nonexpansive points

In this section we establish a number of the properties needed in theorems 1.1–1.3. We leave most of the entropy and pressure estimates until the next section.
3.1. Decomposition

We first define the decomposition we will use on the orbits in order to apply theorem 2.5. As the stable direction is uniformly hyperbolic we can allow the set \( \mathcal{P} = \emptyset \) and the function \( p(x, n) = 0 \) for all \((x, n)\). An orbit segment \((x, n) \in S\) if

\[
\beta(x, n) := \frac{1}{n} \sum_{i=0}^{n-1} \chi_{\Omega_{\beta}}(\pi(f^i x)) \geq \alpha
\]

where \( \alpha \) is given by (1) and \( \chi_{\Omega_{\beta}} \) is the characteristic function for \( \Omega_{\beta} \). An orbit segment \((x, n) \in G\) if \( n = 0 \) or for all \( j \in \{1, \ldots, n\} \) we have

\[
\beta(f^j x, n-j) = \frac{1}{n} \sum_{i=0}^{n-1} \chi_{\Omega_{\beta}}(\pi(f^i x)) < \alpha.
\]

The definition of the collection of orbits \( G \) is iterates chosen so that the centre-unstable direction is uniformly contracted by \( f^{-1} \) along orbits segments from \( f^n(x) \) to \( x \). This is related to the notion of hyperbolic times introduced by Alves [3].

**Remark 3.1.** A simple computation shows the following concatenation property of \( G \) if \((x, n) \) and \((f^n(x), m) \) are in \( G \), then \((x, n + m) \in G \).

We now define the decomposition we use for the orbits segments. For an arbitrary orbit segment \((x, n)\) we let \( s \in \{0, 1, \ldots, n\} \) be the largest integer such that \((x, s) \in G\). Notice this implies that \((f^s x, n-s) \in S \). Indeed, suppose that \((f^s x, n-s) \not\in S\), we may consider \( k \in \{1, \ldots, n-s\} \), such that \( k = \min\{i \geq 1; (f^s x, i) \not\in S\} \). Observe that this imply that \((f^s x, k) \in G \) and since \((x, s) \in G\), we have that \((x, s + k) \in G \), contradicting the maximality of \( s \). Thus, \((f^s x, n-s) \in S \) and we have a decomposition.

3.2. Specification

We now show that the set \( G \) described above has the desired specification property. The proof will proceed first by proving the specification property at any scale \( \delta \) for the map \( g \). As the decomposition is defined in terms of the projection map \( \pi \) we have a canonically defined decomposition \((G, \hat{S})\) for \( g \) from the decomposition \((\hat{G}, \hat{S})\) for \( f \) described above. More specifically, for \((x, n) \in G \) we let \((\hat{x}, n) \in \hat{G} \) and \((x, n) \in \hat{S} \) implies \((\hat{x}, n) \in \hat{S} \). The decomposition for \((\hat{x}, n)\) is then defined by \((\hat{x}, s) \in \hat{G} \) and \((g^n\hat{x}, n-s) \) if \((x, n)\) decomposes to \((s, n)\) and \((f^n x, n-s)\).

We fix a constant related to the contraction for good orbit segments. Let

\[
\theta_\alpha = L^n(\lambda_\alpha^{-1})^{1-\alpha} \in (0, 1).
\]

**Proposition 3.2.** Given \( \epsilon \), there exist \( \tau = \tau(\epsilon) \) such that if \( \{(\hat{x}_j, n_j)\}_{j=0}^l \subset \hat{G} \), then there exists \( \tau_j \leq \tau \) for \( j = 1, \ldots, l \) and some \( \hat{z} \in N \) such that

\[
d(g^{n_j}(\hat{x}_j), g^{n_j\tau_j-1}(\hat{z})) \leq \epsilon,
\]

for \( 0 \leq m \leq n_j \), where \( r_0 = 0 \) and \( r_j = \sum_{i=1}^j (n_i + \tau_i) \) for each \( j \geq 1 \).

**Proof.** We know there exists some \( \delta_0 > 0 \) such that if \( \hat{x} \in N \) is a point where \((\hat{x}, n) \in \hat{G} \), then the inverse branch \( g_j^{-1} \) of \( g^{n_j} \) that sends \( g^n(\hat{x}) \) to \( g^{n-1}(\hat{x}) \) is a \( (\theta_\alpha)^j \)-contraction on the ball of radius \( \delta_0 \), for \( 1 \leq j \leq n \). To see the existence of such a \( \delta_0 \) let \( \mathcal{P} \) be a finite open cover of \( N \) such that each element of \( \mathcal{P} \) is an open set on which \( g \) has an invertible branch. Now let \( \delta_1 \) be a Lebesgue number for the open cover, and choose \( \delta_0 = \min\{\delta_1, \rho\} \).
By hypothesis, we know that given $\epsilon > 0$, there exists a $\tau = \tau(\epsilon) \in \mathbb{N}$ such that for any two points $\hat{y}_1, \hat{y}_2 \in \mathcal{G}$, there exists some $j \leq \tau$ such that $B_{\epsilon}(\hat{y}_j) \subseteq g(B_{\epsilon}(\hat{y}_i))$.

Fix $\epsilon > 0$ and let $\delta > 0$ be less then $\min\{\epsilon, \delta_0\}$. Fix $\tau = \tau(\delta)$. Let $\{(x_j, n_j)\}_{j=0}^\ell \subset \mathcal{G}$. Then we know there exists a set of points $X_{\ell-1} \subset B_{\delta}(g^{n_{\ell-1}}(\hat{x}_{\ell-1}))$ and $\tau \leq \tau$ such that $g^\tau(X_{\ell-1})$ is the image of $B_{\delta}(g^{n_0}(\hat{x}_0))$ by the inverse branch $g^{-1}(\hat{x}_0)$.

Similarly, there exists a nonempty set $X_{\ell-2} \subset B_{\delta}(g^{n_{\ell-2}}(\hat{x}_{\ell-2}))$ and $\tau_{\ell-1} \leq \tau$ such that $g^{\tau_{\ell-1}}(X_{\ell-2})$ is the image of $X_{\ell-1}$ by the inverse branch $g^{-1}(\hat{x}_{\ell-1})$.

Continuing inductively we see that $X_0$ is nonempty and we pick $\hat{z} \subset X_0$. This proves that $\mathcal{G}$ has specification at any scale for $g$. \hfill $\square$

Using the above result we now show that $(\Lambda, f)$ has the specification property for sufficiently small scales for the set $\mathcal{G}$.

**Proposition 3.3.** Given $\epsilon > 0$, there exist $\tau = \tau(\epsilon) \in \mathbb{N}$ such that if $\{(x_j, n_j)\}_{j=0}^\ell \subset \mathcal{G}$, then there exists $\tau_j \leq \tau$ for $j = 1, \ldots, \ell$ and some $\hat{z} \in \mathbb{N}$ such that

$$d(f^{m}(x_j), f^{m+r_j}(z)) \leq \epsilon,$$

for $0 \leq m \leq n_j$, where $r_0 = 0$ and $r_j = \sum_{i=1}^j (n_i + \tau_i)$ for each $j \geq 1$.

**Proof.** From proposition 3.3 we know that we have specification when we project to $g$. The problem is the fibre direction. Since the fibres have a bounded diameter and uniformly contract under $f$ we know there exists some $\tau_j(\epsilon) = \tau_j$ such that for all $x \in \Lambda$ there exists some $j(x) = j \leq \tau_j$ where

$$f^{-j}(M_x) \subset B_{\delta}(f^{-j}(x)) \cap M_{f^{-j}(x)}.$$

Furthermore, for all $k \geq j$ we know that

$$f^{-k}(M_x) \subset B_{\delta}(f^{-k}(x)) \cap M_{f^{-k}(x)}.$$

We now denote $\hat{\tau}(\epsilon)$ to be the constant for specification given by proposition 3.3. We let $\tau(\epsilon) = \max\{\hat{\tau}(\epsilon), \tau_j(\epsilon)\}$. We see that this gives the desired result by combining the result from the previous proposition with the fibre properties listed above. \hfill $\square$

### 3.3. Bowen property for the partially hyperbolic attractor

From property (H4) we know that if $d_{\mathcal{G}}(x, y) < \rho / C$, then $d_{\mathcal{G}}(\hat{x}, \hat{y}) < \rho$. Also, if $\eta > 0$ is sufficiently small and $\hat{x} \in \Omega_\eta^+$ and $d_{\mathcal{G}}(\hat{x}, \hat{y}) < \eta$ then for each preimage $\hat{x_i} \in g^{-1}(\hat{x})$ there exists a unique $\hat{y}_i \in g^{-1}(\hat{y})$ such that $d(\hat{x}_i, \hat{y}_i) \leq \lambda d(\hat{x}, \hat{y})$.

**Lemma 3.4.** If $\eta > 0$ is sufficiently small, $(x, n) \in \mathcal{G}$, and $y \in B_\rho(x, \eta)$, then

$$d(f^k x, f^k y) \leq C \eta \left(\eta^{n-k} + \lambda^k\right)$$

for all $0 \leq k \leq n$.

**Proof.** Since $(x, n) \in \mathcal{G}$ and $y \in B_\rho(x, \eta)$ we know for all $j \in \{1, \ldots, n\}$ that we have

$$\frac{1}{n} \sum_{i=0}^{n-1} \chi_{\Omega}(\pi(f^i(y))) < \alpha.$$
By the $f$-invariance of the holonomy we have $h_{\hat{f}^kx, \hat{f}^ky} = f^k(h_{x,y})$. So

$$d_M(h_{\hat{f}^kx, \hat{f}^ky}, f^k(y)) < \lambda_k^C \eta$$

for all $k \in \{0, \ldots, n\}$. Also, we know that

$$d_N(\hat{f}^kx, \hat{f}^ky) \leq \theta_n^{-k} \lambda^C \eta$$

for all $k \in \{0, \ldots, n\}$.

□

**Lemma 3.5.** For $\eta > 0$ sufficiently small any Hölder continuous potential has the Bowen property on $G$ at scale $\eta$.

**Proof.** We know there exists some $K > 0$ and $\beta \in (0, 1)$ such that

$$|\varphi(x) - \varphi(y)| \leq Kd(x, y)^\beta$$

for all $x, y \in \Lambda$.

Given $(x, n) \in G$ and $y \in B_n(x, \eta)$ we see

$$|S_y \varphi(x) - S_n \varphi(y)| \leq K \sum_{k=0}^{n-1} d(f^kx, f^ky)^\beta$$

$$\leq KC\eta^\beta \sum_{k=0}^{n-1} (\theta_n^{-k} + \lambda_k)^\beta$$

$$\leq 2^{\beta+1} KC\eta^\beta \sum_{j=0}^{\infty} (\max\{\theta_n, \lambda\})^\beta =: V < \infty.$$  □

3.4. **Pressure estimates of the nonexpansive points**

The diffeomorphism $f: \Lambda \to \Lambda$ satisfies the given property.

[E] there exist $\varepsilon > 0$ such that for $x \in M$, if there exists a sequence $n_k \to \infty$ with

$$\frac{1}{n_k} \sum_{j=0}^{n_k-1} \chi_{\Omega}(\pi(f^j(x))) \leq \alpha$$

then $\Gamma(x) = \{x\}$.

Then as in theorem 3.4 in [13] we know the following holds.

**Theorem 3.6.** If $f$ satisfies [E], then we have $P^\text{exp}_{f}(\varphi, \varepsilon) \leq P(S, \varphi)$.

We note that [14, lemma 5.9] or [13, lemma 6.12] can be easily modified to show that $f|_{\Lambda}$ satisfies property [E].

4. **Proof of theorems 1.1 and 1.2**

From the previous section we have a decomposition of the dynamics where the set $G$ has specification and the Bowen property, and the pressure on the set $S$ is an upper bound for the pressure of the obstructions to expansivity. To conclude the proofs for theorems 1.1 and 1.2 using theorem 2.5 we need to estimate the pressure on $S$. First, we find the entropy of $f: \Lambda \to \Lambda$.

**Proposition 4.1.** $h_{\text{top}}(f|_{\Lambda}) = h_{\text{top}}(g) = \log \deg(g)$. 3418
Proof. We know that $g : N \to N$ has entropy $\log \deg(g)$, see for instance [24, 33]. To see that $h_{\text{top}}(f|_{\Lambda}) = h_{\text{top}}(g) = \log \deg(g)$, we can apply Ledrappier–Walters formula from [22] as in [29] to see that the entropy of $f$ restricted to the attractor is the same as the entropy of $g$. This proves the result since the pre-image of the map $\pi$ is a uniformly contracting and has finite diameter.

We now estimate the entropy on $S$. Using estimates about $g$ and that the semi-conjugacy between $f$ and $g$ is contracting on each fibre, we are able to prove the next result.

Proposition 4.2. $h(S ; f) \leq \log q + \epsilon(\alpha) + m \log L$.

To prove the proposition above we make use of some results and notation in section 6 of [33]. The basic strategy is to make use of the estimates about the number of dynamical balls of $g'$ that we need to cover $\mathcal{S}_\text{lin}$, for $n$ big enough, and make use of the contraction in the fibres to obtain a similar estimate for $f'$.

Let $\mathcal{P}$ be the partition from (H2) and let $\mathcal{P} = \{Q_0, \ldots, Q_{j-1}\}$ where the first $q$ elements cover $\Omega$. Note that for $\rho > 0$ sufficiently small we have $\Omega_h$ covered by the first $q$ elements of $\mathcal{P}$. Denote by $\mathcal{P}^{(n)}$ the set of $n$-cylinders of $g$ by elements in $\mathcal{P}$, that is,

$$\mathcal{P}^{(n)} = \{Q_0 \cap g^{-1}(Q_1) \cap \cdots \cap g^{-(n-1)}(Q_{n-1}) : Q_k \in \mathcal{P}, k \in \{0, \ldots, n - 1\}\}.$$

Let $B(n, \alpha) = \{x \in N; \beta(x, n) \geq \alpha\}$.

If we denote by $\mathcal{R}^{(n)}$ the set of $n$-cylinders of $g$ by elements in $\mathcal{P}$ that intersects $B(n, \alpha - \delta)$, for every $0 < \delta < \alpha$, there exists $n_0 \geq 1$ such that for $n \geq n_0$ the set $\mathcal{S}_\text{lin}$ is covered by the union of elements of $\mathcal{R}^{(n)}$ (see [33, p 578] for a proof). By the estimates of lemma 3.1 of [33] or [24] there exists a function $\epsilon(\alpha)$ such that $\epsilon(\alpha) \to 0$ as $\alpha \to 1$ and for $\alpha$ given and sufficiently large $n$ we have

$$\# \mathcal{R}^{(n)} \leq e^{(\log q + \epsilon(\alpha))n}.$$

From lemma 6.2 in [33] if $D > 0$, then there exists a $C_0 > 0$ and a sequence of open coverings $\{\tilde{Q}_k\}$ of $N$ such that $\text{diam}(\tilde{Q}_k) \to 0$ as $k \to \infty$ and every set $E \subset N$ such that $\text{diam}(E) \leq D$ $\text{diam} \tilde{Q}_k$ intersects at most $C_0 D^m$ elements of $\tilde{Q}_k$.

Fix $l > 0$ and let $\{Q_j\}$ be the lift of the open covers $\{\tilde{Q}_j\}$. Denote the set of $n$-cylinders of $g'$ by elements in $Q_j$ by $C_{g', n} Q_j$. Let $\mathcal{V}_{l,n,k}$ be the set of cylinders in $C_{g', n} Q_k$ that intersect any element of $\mathcal{R}^{(n)}$. From claim 2 of [33] we have the following.

Lemma 4.3. Let $k \geq 1$ be large and fixed. Then, there exists some $n_0$ such that

$$\# \mathcal{V}_{l,n,k} \leq \# \tilde{Q}_k \times [C_0 L^{m+1} \times e^{(\log q + \epsilon(\alpha))n}],$$

for every large $n \geq n_0$.

Furthermore, from claim 1 in [33] and $n_0$ perhaps larger we know that for each $0 < \delta < \gamma$ and $n \geq n_0$ we have $B(k, \gamma) \subset B(\ln, \gamma - \delta)$ for each $\ln \leq k < (n + 1)l$.

Now, we are able to prove proposition 4.2.

Proof of proposition 4.2. Fix $\gamma > 0$ and $l$ sufficiently large. Given $n \geq 1$, denote by $E_n \subset \mathcal{S}_\text{lin}$ any maximal $(n, \gamma)$-separated set for $g'$. If $\text{diam} \tilde{Q}_k < \gamma$, by the construction, we have $\mathcal{V}_{l,n,k}$
covers $E_n$ and each element of $\mathcal{V}_{n,k}$ intersects $E_n$ in at most one point. By lemma 4.3, we have
\[
\#E_n \leq \#\mathcal{V}_{n,k} \leq \#Q_k \times [C_\delta L^{mn}] \times e^{(\log q + \epsilon(\alpha))n}.
\] (5)

Since the diameter of each fibre $M_n$ is uniformly bounded from above, there exists a natural number $m_0$ such that for each point $x \in E_n$, we may choose points $F_x = \{y_1(x), y_2(x), \ldots, y_{m_0}(x)\} \subseteq M_n$ such that $F_x$ is a $\gamma$-dense set in the fibre $M_n$. We claim that
\[
F_n := \bigcup_{x \in E_n} F_x,
\]
is a maximal $(n, K\gamma)$-separated set for $f^i$ and $S_{\mathfrak{m}}$, where $K = 2C$ and $C$ is as in (H4). First, since $(x, \ln) \in \mathcal{S}$, we know $(y_i(x), \ln) \in \mathcal{S}$ for every $i = 1, \ldots, m_0$. To check that $F_n$ is maximal, take any $(z, \ln) \in \mathcal{S}$. By definition, $\pi(z) \in \tilde{S}_{\mathfrak{m}}$ and by the maximality of $E_n$ there exists a point $x \in E_n$ such that for every $i = 0, \ldots, n - 1$ we have
\[
\delta\nu(g^h(x), g^h(\pi(z))) \leq \gamma.
\]

Consider the holonomy $h_{x, \pi(z)} : M_n \cap \Lambda \to M_n \cap \Lambda$. By the $\gamma$-density of $F_x$ in $M_n$ we have that there exists a point $y \in M_n$ such that if $z' = h_{x, \pi(z)}(z)$, then
\[
d_m(y, z') \leq \gamma.
\]

Now, we estimate $d_m(f^h(y), f^h(z))$, for $i = 0, \ldots, n - 1$:
\[
d_m(f^h(y), f^h(z)) \leq d_m(f^h(y), f^h(z')) + d_m(f^h(z'), f^h(z)) \leq 
\]

\[
\lambda^h d_m(y, z') + C_d(g^h(x), g^h(\pi(z))) \leq K\gamma.
\]

Thus, $F_n$ is a maximal $(n, K\gamma)$-separated set for $f^i$ and $S_{\mathfrak{m}}$. In order to estimate $\#F_n$, observe that $\#F_n \leq \#E_n \times m_0$ and
\[
h(S, K\gamma; f^i) \leq \limsup \frac{1}{n} \log \#E_n \leq \ln \log L + (\log q + \epsilon(\alpha))l + \log C_0
\]
by equation (5). Taking the limit when $\gamma \to 0$:
\[
h(S; f) \leq \ln \log L + (\log q + \epsilon(\alpha))l + \log C_0.
\]

Using that $h(S; f) = (1/l)h(S; f^l)$ we finish the proof.

**Proof of theorem 1.1.** The estimates in theorem 4.2 are the key to the pressure estimates of the proofs of theorems 1.1 and 1.2. Just as in the proof of theorem 3.3 in [13] we know that
\[
P(S, \varphi; g) \leq \alpha \sup_{\pi \Omega} \varphi(x) + (1 - \alpha) (\sup_{\Lambda} \varphi) + h(S)
\]
\[
\leq \alpha \sup_{\pi \Omega} \varphi(x) + (1 - \alpha) (\sup_{\Lambda} \varphi) + \log q + \epsilon(\alpha) + m \log L
\]
\[
= \Psi(\varphi) < P(\varphi; f_\Lambda).
\]

Hence, theorem 2.5 shows there exists a unique equilibrium state.

**Proof of theorem 1.2.** Now assume that $\varphi$ is Hölder continuous and that
\[
\sup \varphi - \inf \varphi < \log \deg(g) - \log q - \epsilon(\alpha) - m \log L,
\]
then \( \sup \varphi - \inf \varphi < h_{\text{top}}(f|_{\Lambda}) - h(S) \). Then \( h(S) + \sup \varphi < h_{\text{top}}(f|_{\Lambda}) + \inf \varphi \) and this implies that \( P(S, \varphi; f) < P(\varphi; f) \). In the previous section we proved that the pressure of obstructions to expansivity are bounded by the pressure on \( S \), then \( \varphi \) has the Bowen property, and \( \mathcal{G} \) has specification. Hence, theorem 2.5 shows there exists a unique equilibrium state. 

\[ \Box \]

5. Proof of theorem 1.3

In this section we prove theorem 1.3. The proof is similar to theorem C in [13] except we do not need to worry about the scale of the perturbation.

From the proof of lemma 7.1 in [13] we see that \( P(\varphi^{\text{geo}}; f) \geq 0 \). From the fact that

\[
\log q + \epsilon(\alpha) + m \log L < -\sup \varphi^{\text{geo}}
\]

we see that

\[
\Psi(\varphi^{\text{geo}}) = \alpha \sup_{\pi(x) \in \Omega_\mu} \varphi^{\text{geo}}(x) + (1 - \alpha)(\sup \varphi^{\text{geo}}) + \log q + \epsilon(\alpha) + m \log L
\]

\[
\leq \sup \varphi^{\text{geo}} + \log q + \epsilon(\alpha) + m \log L < 0 \leq P(\varphi^{\text{geo}}; f).
\]

Therefore, by theorem 1.1 we know that \( \varphi^{\text{geo}} \) has a unique equilibrium state.

To show that \( P(t\varphi^{\text{geo}}; f) \) has a unique equilibrium state for \( t \in [0, 1] \) we show that \( \Psi(t\varphi^{\text{geo}}) < P(t\varphi^{\text{geo}}; f) \) for all \( t \in [0, 1] \). Since the inequality is strict it will hold in a neighbourhood of \([0, 1]\).

Notice that

\[
P(t\varphi^{\text{geo}}; f) \geq h_{\text{top}}(f|_{\Lambda}) + t \inf \varphi^{\text{geo}} = \log(\deg g) + t \inf \varphi^{\text{geo}} =: l_1(t).
\]

Also, we have

\[
\Psi(t\varphi^{\text{geo}}) \leq t \sup \varphi^{\text{geo}} + \log q + \epsilon(\alpha) + m \log L =: l_2(t).
\]

Let

\[
t_0 = -\left(\frac{\log q + \epsilon(\alpha) + m \log L}{\sup \varphi^{\text{geo}}}\right).
\]

We know that \( t_0 \in (0, 1) \). For \( t \in (t_0, 1) \) we have \( \Psi(t\varphi^{\text{geo}}) < 0 \leq P(t\varphi^{\text{geo}}; f|_{\Lambda}) \leq P(t\varphi^{\text{geo}}; f|_{\Lambda}). \)

For \( t = 0 \) we see that \( l_1(0) > l_2(0) \) by (2). We know that the root of \( l_2 \) is \( t_0 \). The root of \( l_1 \) is \(-\log(\deg g) / \inf \varphi^{\text{geo}}\). By (2) we know that \( t_0 < -\log(\deg g) / \inf \varphi^{\text{geo}} \) and so \( l_1(t_0) > l_2(t_0) \).

This implies that \( P(t\varphi^{\text{geo}}; f|_{\Lambda}) \geq l_1(t) > l_2(t) \geq \Psi(t\varphi^{\text{geo}}) \) for all \( t \in [0, t_0] \).

Now assume that \( f \) is a \( C^2 \) diffeomorphism and let \( \mathcal{M}_e(f) \) be the collection of \( f \)-invariant ergodic measures. For \( \mu \in \mathcal{M}_e(f) \), let \( \lambda_1 < \cdots < \lambda_t \) be the Lyapunov exponents of \( \mu \), and let \( d_i \) be the multiplicity of \( \lambda_i \), so that \( d_i = \dim E_i \), where for a Lyapunov regular point \( x \) for \( \mu \) we have

\[
E_i(x) = \{0\} \cup \{ v \in T_x M : \lim_{n \to \pm \infty} \frac{1}{n} \log \|Df^n_x(v)\| = \lambda_i \} \subset T_x M.
\]

Let \( k = k(\mu) = \max\{1 \leq i \leq \ell(\mu) : \lambda_i \leq 0\} \), and let \( \lambda^+(\mu) = \sum_{i > k} d_i(\mu) \lambda_i(\mu) \) be the sum of the positive Lyapunov exponents, counted with multiplicity.
The Margulis–Ruelle inequality [4, theorem 10.2.1] gives $h_\mu(f) \leq \lambda^+(\mu)$. Ledrappier and Young [21] proved that equality holds if and only if $\mu$ has absolutely continuous conditionals on unstable manifolds. This implies that for any ergodic invariant measure $\mu$, we have

$$h_\mu(f) - \lambda^+(\mu) \leq 0,$$

with equality if and only if $\mu$ is absolutely continuous on unstable manifolds. So an ergodic measure $\mu$ is an SRB measure if and only if it is hyperbolic and equality holds in (8).

We now show that $P(\phi^{\text{geo}}; f|_{\Lambda}) \leq 0$. Combining this with the arguments above we see that $P(\phi^{\text{geo}}; f|_{\Lambda}) = 0$. We know that $\phi^{\text{geo}}$ has a unique equilibrium state $\mu$; to show that $\mu$ is the SRB measure, we need to show that $-\lambda^+(\mu) = \int \phi^{\text{geo}} d\mu$.

Since the splitting on $\Lambda$ is partially hyperbolic and the stable direction is uniformly contracting we know that $\int \phi^{\text{geo}} d\mu \geq -\lambda^+(\mu)$ with equality if and only if all Lyapunov exponents associated with $E^\text{in}$ are nonnegative. From this we know that $h_\mu(f) - \lambda^+(\mu) \leq h_\mu(f) + \int \phi^{\text{geo}} d\mu$.

Now suppose that there is a nonpositive exponent associated with $E^\text{in}$. Then there exists some set $Z \subset \Lambda$ such that $\mu(Z) = 1$ and $v \in E^\text{in}(z)$ such that $\lim_{n \to \infty} \frac{1}{n} \log \|D f^n_z(v)\| \leq 0$. For $z \in Z$ if we define

$$\Lambda^- = \left\{ x : \exists K(x), \frac{1}{n} \sum_{i=0}^{n-1} \chi_{\Omega}(\pi(f^n x)) \geq \alpha, \forall n \geq K(x) \right\},$$

then we have $z \in \Lambda^-$. To see this notice that if $z \notin \Lambda^-$ then by property [E] that there exists a sequence $n_k$ such that $\frac{1}{n_k} \sum_{i=0}^{n_k-1} \chi_{\Omega}(\pi(f^n x)) < \alpha$ and so

$$\lim_{n_k \to \infty} \frac{1}{n_k} \log \|D f_{z_k}^{n_k}\| < 0.$$ 

Then $\mu(\Lambda^-) = 1$ and this implies that

$$h_\mu(f) - \lambda^+(\mu) \leq h_\mu(f) + \int \phi^{\text{geo}} d\mu \leq P(\mathcal{S}, \phi^{\text{geo}}) \leq \Psi(\phi^{\text{geo}}) < 0.$$ 

So this implies that $P(\phi^{\text{geo}}; f|_{\Lambda}) \leq 0$ and so $P(\phi^{\text{geo}}; f|_{\Lambda}) = 0$.

By (2) we know that $\sup \phi^{\text{geo}} < 0$ and so $t \mapsto P(t\phi^{\text{geo}}; f|_{\Lambda})$ is a convex strictly decreasing function from $\mathbb{R} \to \mathbb{R}$ and hence 1 is the unique root.

Hence, $h_\mu(f) - \lambda^+(\mu) = 0$ and $\mu$ is an SRB measure. Assume there is a different SRB measure $\nu$ that is ergodic. Then $h_\nu(f) - \lambda^+(\nu) \leq h_\nu(f) + \int \phi^{\text{geo}} d\nu < P(\phi^{\text{geo}}; f|_{\Lambda}) = 0$ since $\mu$ is a unique equilibrium state. This completes the proof of theorem 1.3.

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