WEAK TYPE ESTIMATES FOR THE ABSOLUTE VALUE MAPPING

M. CASPERS, D. POTAPOV, F. SUKOCHEV, D. ZANIN

Abstract. We prove that if $A$ and $B$ are bounded self-adjoint operators such that $A - B$ belongs to the trace class, then $|A - |B||$ belongs to the principal ideal $L_{1,\infty}$ in the algebra $L(H)$ of all bounded operators on an infinite-dimensional Hilbert space generated by an operator whose sequence of eigenvalues is $\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Moreover, $\mu(j; |A - |B||) \leq \text{const}(1 + j)^{-1} \|A - B\|_1$. We also obtain a semifinite version of this result, as well as the corresponding commutator estimates.

1. Introduction

Let $H$ be a complex separable Hilbert space, let $K(H)$ be the $*$-algebra of all compact operators on $H$ and let $L_p$, $1 \leq p < \infty$, be the $p$-th Schatten-von Neumann class (that is the class of all operators $A$ from $K(H)$ such that $\|A\|_p := (\sum_{k=0}^\infty \mu(k; A)^p)^{1/p} < \infty$, where $\{\mu(k; A)\}_{k=0}^\infty$ is the sequence of singular numbers of the operator $A$ [15, 19]). The following result was proved by E. B. Davies [8, Theorem 8] (for its extension to semifinite von Neumann algebras, we refer to [11]).

**Theorem 1.1.** If $A$, $B$ are self-adjoint bounded operators on $H$ and if $A - B \in L_p$, $1 < p < \infty$, then $|A| - |B| \in L_p$ and

$$\| |A| - |B|| \leq c_p \|A - B\|_p.$$ 

Here, $c_p$ depends only on $p$ and $c_p = O(p)$ as $p \to \infty$ and $c_p = O((p - 1)^{-1})$ as $p \to 1$.

For various extensions and generalizations of Theorem 1.1, we refer to the papers [18], [4], [11], [12], [20] studying the Lipschitz continuity of the absolute value mapping $A \to |A|$ in the setting of symmetrically-normed ideals (and more general symmetric operator spaces). Here, we contribute to an interesting open question concerning the optimal form of Theorem 1.1 in the crucial case $p = 1$. It is well known (see [8, Section 3]) that the absolute value mapping is not Lipschitz continuous in the trace class ($L_1, \| \cdot \|_1$). It was proved by H. Kosaki [18, Theorem 12] (see also [12, Corollary 3.4]) that the absolute value mapping is Lipschitz continuous from ($L_1, \| \cdot \|_1$) into Banach ideal $(M_{1,\infty}, \| \cdot \|_{M_{1,\infty}})$, where

$$L_{1,\infty} := \{ A \in K(H) : \|A\|_{L_{1,\infty}} := \sup_{k \geq 0} (k + 1) \mu(k; A) < \infty \}.$$

The main objective of this paper is to show that the latter result holds if we replace $(M_{1,\infty}, \| \cdot \|_{M_{1,\infty}})$ with a smaller (quasi-Banach) ideal $(L_{1,\infty}, \| \cdot \|_{L_{1,\infty}})$, where

$$M_{1,\infty} := \{ A \in K(H) : \|A\|_{M_{1,\infty}} := \sup_{N \geq 0} \frac{1}{N+2} \sum_{k=0}^N \mu(k; A) < \infty \}.$$ 

Acknowledgement: The first author is supported by the grant SFB 878.
Theorem 1.2. If $A, B$ are self-adjoint bounded operators on $H$ and if $A - B \in \mathcal{L}_1$, then $|A| - |B| \in \mathcal{L}_{1,\infty}$ and
\[
\| |A| - |B| \|_{1,\infty} \leq \left( 34 + \frac{2500e}{\pi} \right) \| A - B \|_1.
\]

The strength of Theorem 1.2 is seen from the fact that it implies the result of Theorem 1.1 via a combination of methods used in [11], [8] linking Lipschitz continuity and commutator estimates with a noncommutative version of the Boyd interpolation theorem (see e.g. [9, Theorem 5.8]). We refer to Remark 6.2 for more details. Such an implication is of course not available from the results of [18, Theorem 12] and [12, Corollary 3.4]. The result of Theorem 1.2 is also sharp in the sense that the quasi-norm $\| \cdot \|_{1,\infty}$ is the largest symmetric quasi-norm on the ideal of finite rank operators for which (1.2) holds (the latter follows from the proof of [8, Lemma 10]). From a certain perspective, the result of Theorem 1.2 is not unexpected. Indeed, the proof of Theorem 1.1 in [8], as well as the proofs of its analogues and extensions from [18], [4], [11], [12] are ultimately based on the famous results due to V.I. Macaev, I. C. Gohberg and M. G. Krein (see [15], [7]), describing the behavior of (generalized) triangular truncation operators in Schatten-von Neumann classes $\mathcal{L}_p$. In the case when $p = 1$, these results yield the fact that the latter operator acts boundedly from the Banach space $(\mathcal{L}_1, \| \cdot \|_1)$ into a quasi-Banach space $(\mathcal{L}_{1,\infty}, \| \cdot \|_{1,\infty})$. However, (and here lies the major difficulty) all the proofs in the just listed papers involve certain integration processes, which render them inapplicable in the quasi-normed setting. Exactly the same obstacle also manifested itself in [20, Theorem 2.5(i)]. Indeed, that theorem yields the result of Theorem 1.2 under the restrictive assumption that $\text{rank}(A - B) = 1$ and the methods used in [20] do not seem applicable to treat the general case. To circumvent this difficulty, we employ a completely different approach coming back to a celebrated theorem of I. Schur concerning positive semidefiniteness of a Schur (or Hadamard) product of two semidefinite matrices.

Section 3 contains the proof of Theorem 1.2. In Section 4 we find a sharper result assuming the extra condition that $A$ and $B$ are compact operators. In this case it turns out that $|A| - |B|$ in fact lands in the separable part of $\mathcal{L}_{1,\infty}$, see Theorem 4.3. Section 5 contains the extension of Theorem 1.2 to the setting of semifinite von Neumann algebras. This theme has been explored already in [12], however, methods employed there (again due to the obstacle explained above) were not sufficiently strong to obtain the weak type estimate similar to (1.2). Furthermore, the setting used in [12] was restricted to the case of semifinite factors. The approach used in this paper allows us to dispense with the latter condition. In Section 6 we treat the consequences of Theorem 1.2 for commutator estimates. In the final Section we give a treatment of the consequences of Theorem 1.2 for certain Lipschitz functions $f$ belonging to a subclass of the Davies class (the case $f = | \cdot |$ being Theorem 1.2). Note that for general Lipschitz functions $f$ outside of that subclass the question whether the weak $(1,1)$ estimate holds remains open.

Acknowledgement. The authors wish to express their gratitude to Peter Dodds and the anonymous referee for several improvements of this paper.

2. Preliminaries

2.1. Singular values. Let $\mathcal{L}(H)$ be the $\ast$–algebra of all bounded operators on the Hilbert space $H$ equipped with a uniform norm $\| \cdot \|_\infty$. Every proper ideal in $\mathcal{L}(H)$ consists of compact operators. For brevity, we set $\mu(A) := \{ \mu(k, A) \}_{k \geq 0}$. If $B \in \mathcal{L}(H)$ and $A \in \mathcal{K}(H)$, then it is well known that
\[
\mu(AB) \leq \| B \|_\infty \mu(A), \quad \mu(BA) \leq \| B \|_\infty \mu(A), \quad \mu(A^*) = \mu(A).
\]
If $A, B \in \mathcal{K}(H)$, then (see e.g. Corollary 2.3.16 in [19]) we have

$$
\mu(A + B) \leq \sigma_2(\mu(A) + \mu(B)).
$$

(2.2)

Here, the dilation operator $\sigma_2 : l_\infty \to l_\infty$ (acting on the space $l_\infty$ of all complex bounded sequences) is defined as follows

$$
\sigma_2(a_0, a_1, \cdots) = (a_0, a_0, a_1, \cdots).
$$

Two self-adjoint operators $A, B \in \mathcal{K}(H)$ are called identically distributed if $\mu(A_+) = \mu(B_+)$ and $\mu(A_-) = \mu(B_-)$. Here, $A = A_+ - A_-$ is the orthogonal decomposition of a self-adjoint operator $A$ (see e.g. [5, p.36]). A self-adjoint operator $A \in \mathcal{K}(H)$ is called symmetrically distributed if $\mu(A_+) = \mu(A_-)$.

In what follows, the symbol $\text{supp}(A)$ stands for the support projection of a self-adjoint operator $A \in \mathcal{L}(H)$ (that is, the spectral projection of $A$ corresponding to the set $\mathbb{R}\backslash\{0\}$).

2.2. **Ideal $\mathcal{L}_{1,\infty}$.** Let $A_0 \in \mathcal{K}(H)$ be such that $\mu(A_0) = \{1/(k + 1)\}_{k \geq 0}$. The principal ideal generated by $A_0$ is frequently called weak-$\mathcal{L}_1$ and coincides with $\mathcal{L}_{1,\infty}$. The mapping $\| \cdot \|_{1,\infty}$ on $\mathcal{L}_{1,\infty}$ (see (1.1)) is a quasi-norm. Indeed, it follows from (2.2) that

$$
\|A + B\|_{1,\infty} \leq 2\|A\|_{1,\infty} + 2\|B\|_{1,\infty}.
$$

(2.3)

It can be shown (see e.g. [22, Theorem 2.11.32]) that the quasi-normed space $(\mathcal{L}_{1,\infty}, \| \cdot \|_{1,\infty})$ is complete and is, therefore, a quasi-Banach ideal. It is important to note that the quasi-norm $\| \cdot \|_{1,\infty}$ is not equivalent to any norm. In particular, the weak form of triangle inequality in (2.3) is the best possible.

The ideals $\mathcal{L}_p$ and $\mathcal{L}_{1,\infty}$ both have the Fatou property. That is, if $A_n \in \mathcal{L}_{1,\infty}$, $\|A_n\|_{1,\infty} \leq 1$ and $A_n \to A$ in measure, then $A \in \mathcal{L}_{1,\infty}$ and $\|A\|_{1,\infty} \leq 1$. Exactly the same assertion holds for $\mathcal{L}_p$.

2.3. **Schur multiplication.** Let

$$
M_n = \{A = \{A_{k,l}\}_{k,l=0}^{n-1}\}
$$

be the $*$-algebra of all complex $n \times n$ matrices. The algebra $M_n$ is $*$-isomorphic to a subalgebra $P_n \mathcal{L}(H)P_n$ in $\mathcal{L}(H)$, where $P_n$ is a projection in $\mathcal{L}(H)$ such that $\text{Tr}(P_n) = n$, where $\text{Tr}$ is the standard trace on $\mathcal{L}(H)$. We also frequently identify $M_n$ with $\mathcal{L}(H)$ when dim$(H) = n$.

In the latter case, all previously introduced notations (e.g. $\mu(A)$, $\| \cdot \|_p$, $\| \cdot \|_{1,\infty}$, $\text{Tr}$, $\text{supp}(A)$) and terminology (e.g. identically distributed) remain unambiguously defined.

For every $A, B \in M_n$, we define their Schur product (also called Hadamard product) $A \circ B$ by setting

$$(A \circ B)_{k,l} = A_{k,l}B_{k,l}, \quad 0 \leq k, l \leq n - 1.$$

Fix $B \in M_n$. The Schur multiplication operator $M_B : M_n \to M_n$ is defined by setting $M_B(A) = A \circ B$. If $B \geq 0$, then, according to the Schur theorem, we have that $M_B(A) \geq 0$ for every $A \geq 0$. For a beautiful exposition of the latter theorem and other relevant properties of Schur multiplication, we refer the reader to [2]. For the next lemma, see [1]. We included a short proof for completeness.

**Lemma 2.1.** If $B \geq 0$, then

$$
\|M_B(A)\|_1 \leq 4\|\text{diag}(B)\|_\infty \cdot \|A\|_1, \quad A \in M_n.
$$
Proof. Suppose first that $A \geq 0$. It follows that $M_B(A) \geq 0$. Hence,

$$
\|M_B(A)\|_1 = \text{Tr}(M_B(A)) = \text{Tr}(A \circ B) = \sum_{k=0}^{n-1} A_{k,k} B_{k,k} \\
\leq (\max_{0 \leq k < n} B_{k,k}) \cdot (\sum_{k=0}^{n-1} A_{k,k}) = \|\text{diag}(B)\|_\infty \text{Tr}(A) = \|\text{diag}(B)\|_\infty \|A\|_1.
$$

Consider now the general case of an arbitrary $A \in M_n$. Using Jordan decomposition (see e.g. [5, p.216]), we write

$$
A = A_1 - A_2 + iA_3 - iA_4, \quad A_m \geq 0, \quad \|A_m\|_1 \leq \|A\|_1, \quad 1 \leq m \leq 4.
$$

Therefore,

$$
\|M_B(A)\|_1 \leq \sum_{m=1}^{4} \|M_B(A_m)\|_1 \leq \|\text{diag}(B)\|_\infty \cdot (\sum_{m=1}^{4} \|A_m\|_1).
$$

\[\square\]

3. Proof of Theorem 1.2

The following lemma can be found in [3]. Its short proof is included for convenience of the reader.

Lemma 3.1. If $\alpha_k > 0$, $0 \leq k < n$, are decreasing, then the matrix

$$
\Phi = \begin{cases}
\alpha_{\max\{k,l\}} & \\
\alpha_k + \alpha_l & \end{cases}^{n-1}_{k,l=0}
$$

is positive semidefinite.

Proof. Set

$$
\Phi_1 = \begin{cases}
1 & \\
\frac{1}{\alpha_k + \alpha_l} & \end{cases}^{n-1}_{k,l=0}.
$$

Consider (rank one) projections $p_k$, $0 \leq k < n$, given by diagonal matrix units. That is,

$$
(p_k)_{i,j} = \begin{cases}
0, & i \neq k \\
0, & j \neq k \\
1, & i = j = k
\end{cases}
$$

and set

$$
P_k := \sum_{j=0}^{k-1} p_j, \quad 0 \leq k < n.
$$
The following equality can be verified directly.\footnote{For $n = 2$, we have
\[
\begin{pmatrix}
\frac{\alpha_0}{\alpha_0 + \alpha_1} & \frac{\alpha_1}{\alpha_0 + \alpha_1} \\
\frac{\alpha_0}{\alpha_0 + \alpha_1} & \frac{\alpha_1}{\alpha_0 + \alpha_1}
\end{pmatrix} = \begin{pmatrix}
\frac{\alpha_0 - \alpha_1}{\alpha_0 + \alpha_1} & 0 \\
0 & \frac{\alpha_1}{\alpha_0 + \alpha_1}
\end{pmatrix} + \begin{pmatrix}
\frac{\alpha_0 + \alpha_1}{\alpha_0 + \alpha_1} & \frac{\alpha_1 - \alpha_0}{\alpha_0 + \alpha_1} \\
\frac{\alpha_0 + \alpha_1}{\alpha_0 + \alpha_1} & \frac{\alpha_1 - \alpha_0}{\alpha_0 + \alpha_1}
\end{pmatrix}.
\]
\]
For $n = 3$, we have
\[
\begin{pmatrix}
\frac{\alpha_0}{\alpha_0 + \alpha_1} & \frac{\alpha_1}{\alpha_0 + \alpha_1} & \frac{\alpha_2}{\alpha_0 + \alpha_1} \\
\frac{\alpha_0}{\alpha_0 + \alpha_1} & \frac{\alpha_1}{\alpha_0 + \alpha_1} & \frac{\alpha_2}{\alpha_0 + \alpha_1} \\
\frac{\alpha_0}{\alpha_0 + \alpha_1} & \frac{\alpha_1}{\alpha_0 + \alpha_1} & \frac{\alpha_2}{\alpha_0 + \alpha_1}
\end{pmatrix} = 
\begin{pmatrix}
\frac{\alpha_0 - \alpha_1}{\alpha_0 + \alpha_1} & 0 & 0 \\
0 & \frac{\alpha_1}{\alpha_0 + \alpha_1} & 0 \\
0 & 0 & \frac{\alpha_2}{\alpha_0 + \alpha_1}
\end{pmatrix} + \begin{pmatrix}
\frac{\alpha_0 + \alpha_1 + \alpha_2}{\alpha_0 + \alpha_1 + \alpha_2} & \frac{\alpha_0 + \alpha_1}{\alpha_0 + \alpha_1 + \alpha_2} & \frac{\alpha_0 + \alpha_2}{\alpha_0 + \alpha_1 + \alpha_2} \\
\frac{\alpha_0 + \alpha_1}{\alpha_0 + \alpha_1 + \alpha_2} & \frac{\alpha_1}{\alpha_0 + \alpha_1 + \alpha_2} & \frac{\alpha_2}{\alpha_0 + \alpha_1 + \alpha_2} \\
\frac{\alpha_0 + \alpha_1}{\alpha_0 + \alpha_1 + \alpha_2} & \frac{\alpha_1}{\alpha_0 + \alpha_1 + \alpha_2} & \frac{\alpha_2}{\alpha_0 + \alpha_1 + \alpha_2}
\end{pmatrix}.
\]
\]
For larger $n$, the decomposition follows exactly the same way.}
\begin{equation}
\Phi = \left( \sum_{k=0}^{n-2} (\alpha_k - \alpha_{k+1}) P_k \Phi_1 P_k \right) + \alpha_{n-1} \Phi_1.
\end{equation}
It is well known (see e.g. [2]) that the Cauchy matrix $\Phi_1$ is positive semidefinite. It is now immediate from (3.1) that the matrix $\Phi$ is also positive semidefinite. □

**Lemma 3.2.** Let $\alpha_k > 0$, $0 \leq k < n$, be decreasing. Define an operator $S : M_n \to M_n$ by setting
\[ S(A) = \sum_{k,l=0}^{n-1} \frac{\alpha_k - \alpha_l}{\alpha_k + \alpha_l} A^{k,l}, \]
where $p_k$, $0 \leq k < n$, are the pairwise orthogonal rank one projections in $M_n$. We have
\[ \|S(A)\|_{1,\infty} \leq \frac{80e}{\pi} \|A\|_1 \]
for every $A \in M_n$.

**Proof.** It is sufficient to prove the assertion for the special case of projections $p_k$ defined in the proof of the preceding lemma. Set $T$ to be the triangular truncation operator defined by setting
\[ (T(A))_{i,j} = \begin{cases} A_{i,j}, & i \geq j \\ 0, & i < j \end{cases} \]
and let $M_\Phi$ be the Schur multiplication operator with respect to $\Phi$ from Lemma 3.1. We have
\[ S = (2T - 1)(2M_\Phi - 1). \]
Indeed,
\[ (S(A))_{k,l} = \frac{\alpha_k - \alpha_l}{\alpha_k + \alpha_l} A_{k,l} = \left( \frac{2\alpha_k}{\alpha_k + \alpha_l} - 1 \right) A_{k,l} = ((2M_\Phi - 1)(A))_{k,l}, \quad k \geq l \]
and
\[ (S(A))_{k,l} = \frac{\alpha_k - \alpha_l}{\alpha_k + \alpha_l} A_{k,l} = \left( 1 - \frac{2\alpha_l}{\alpha_k + \alpha_l} \right) A_{k,l} = -((2M_\Phi - 1)(A))_{k,l}, \quad k < l. \]
It is known (see [15, Theorem IV.8.2]) that
\[ \| (2T - 1)(X) \|_{1,\infty} \leq \frac{4e}{\pi} \|X\|_1, \quad X = X^* \in M_n, \quad \text{diag}(X) = 0. \]
Thus,
\[ \|(2T - 1)(X)\|_1 \leq \frac{16e}{\pi} \|X\|_1, \quad X \in M_n, \quad \text{diag}(X) = 0. \]
By Lemma 2.1 and Lemma 3.1, we have
\[ \|2M_\Phi - 1\|_{\mathcal{L}_1(\mathcal{L}_1)} \leq 1 + 8 \|\text{diag}(\Phi)\|_\infty = 5. \]
Therefore,
\[ \|S(A)\|_1 \leq \frac{16e}{\pi} \|(2M_\Phi - 1)(A)\|_1 \leq \frac{80e}{\pi} \|A\|_1. \]

Lemma 3.3. Let \( A, B \in M_{2n} \) be identically and symmetrically distributed matrices. We have
\[ \|A - B\|_{1, \infty} \leq \left( 8 + \frac{640e}{\pi} \right) \|A - B\|_1. \]

Proof. We have
\[
A = \sum_{k=0}^{n-1} \mu(k, A_+)p_{1k} - \sum_{k=0}^{n-1} \mu(k, A_+)p_{2k}, \quad |A| = \sum_{k=0}^{n-1} \mu(k, A_+)p_{1k} + \sum_{k=0}^{n-1} \mu(k, A_+)p_{2k},
\]
\[
B = \sum_{l=0}^{n-1} \mu(l, A_+)q_{1l} - \sum_{l=0}^{n-1} \mu(l, A_+)q_{2l}, \quad |B| = \sum_{l=0}^{n-1} \mu(l, A_+)q_{1l} + \sum_{l=0}^{n-1} \mu(l, A_+)q_{2l},
\]
where all the projections \( p_{1k}, p_{2k}, 0 \leq k < n \) are pairwise orthogonal and have rank 1 (and the same holds for the projections \( q_{1l}, q_{2l}, 0 \leq l < n \)). Hence, we have
\[
|A| - |B| = \sum_{k,l=0}^{n-1} (\mu(k, A_+) - \mu(l, A_+))p_{1k}q_{1l} + \sum_{k,l=0}^{n-1} (\mu(k, A_+) - \mu(l, A_+))p_{2k}q_{2l}
\]
\[
+ \sum_{k,l=0}^{n-1} (\mu(k, A_+) - \mu(l, A_+))p_{1k}q_{2l} + \sum_{k,l=0}^{n-1} (\mu(k, A_+) - \mu(l, A_+))p_{2k}q_{1l}
\]
\[
= \sum_{k,l=0}^{n-1} p_{1k}(A - B)q_{1l} - \sum_{k,l=0}^{n-1} p_{2k}(A - B)q_{2l}
\]
\[
+ \sum_{k,l=0}^{n-1} (\mu(k, A_+) - \mu(l, A_+))p_{1k}(A - B)q_{2l}
\]
\[
- \sum_{k,l=0}^{n-1} (\mu(k, A_+) - \mu(l, A_+))p_{2k}(A - B)q_{1l}.
\]
Take unitary matrices \( U, V \in M_{2n} \) such that \( q_{2l} = Up_{1l}U^{-1} \) and \( q_{1l} = Vp_{2l}V^{-1} \) for all \( 0 \leq l < n \). It is clear that
\[
p_{1k}(A - B)q_{2l} = p_{1k}\left( \text{supp}(A_+)(A - B)U \text{supp}(A_+) \right)p_{1l} \cdot U^{-1}, \quad 0 \leq k, l < n
\]
\[
p_{2k}(A - B)q_{1l} = p_{2k}\left( \text{supp}(A_-)(A - B)V \text{supp}(A_-) \right)p_{2l} \cdot V^{-1}, \quad 0 \leq k, l < n.
\]
Consider the operators $S_1 : \text{supp}(A_+)M_{2n}\text{supp}(A_+) \to \text{supp}(A_+)M_{2n}\text{supp}(A_+)$
\[ S_1(X) := \sum_{k,l=0}^{n-1} \frac{\mu(k, A_+) - \mu(l, A_+)}{\mu(k, A_+) + \mu(l, A_+)} p_{1k} X p_{1l}, \quad X \in \text{supp}(A_+)M_{2n}\text{supp}(A_+) \]
and $S_2 : \text{supp}(A_-)M_{2n}\text{supp}(A_-) \to \text{supp}(A_-)M_{2n}\text{supp}(A_-)$
\[ S_2(X) := \sum_{k,l=0}^{n-1} \frac{\mu(k, A_+) - \mu(l, A_+)}{\mu(k, A_+) + \mu(l, A_+)} p_{2k} X p_{2l}, \quad X \in \text{supp}(A_-)M_{2n}\text{supp}(A_-). \]

Employing these notations, we obtain
\[ |A| - |B| = \text{supp}(A_+) (A - B) \text{supp}(B_+) - \text{supp}(A_-) (A - B) \text{supp}(B_-) 
+ S_1(\text{supp}(A_+) (A - B) U \text{supp}(A_+)) \cdot U^{-1} 
- S_2(\text{supp}(A_-) (A - B) V \text{supp}(A_-)) \cdot V^{-1}. \]

Since the algebras $\text{supp}(A_+)M_{2n}\text{supp}(A_+)$ and $\text{supp}(A_-)M_{2n}\text{supp}(A_-)$ are $*-$isomorphic to
the algebra $M_n$, it follows that the operators $S_1$ and $S_2$ satisfy the assumptions of Lemma 3.2.

Applying Lemma 3.2, we obtain
\[ \frac{1}{4} \||A| - |B||_{1,\infty} \leq \||\text{supp}(A_+) (A - B) \text{supp}(B_+)||_{1,\infty} + \||\text{supp}(A_-) (A - B) \text{supp}(B_-)||_{1,\infty} \]
\[ + \||S_1(\text{supp}(A_+) (A - B) U \text{supp}(A_+))||_{1,\infty} + \||S_2(\text{supp}(A_-) (A - B) V \text{supp}(A_-))||_{1,\infty} \]
and
\[ \||A| - |B||_{1,\infty} \leq (4 + 4 + 4 \cdot \frac{80e}{\pi} + 4 \cdot \frac{80e}{\pi}) \|A - B\|_1 = \left(8 + \frac{640e}{\pi}\right) \|A - B\|_1. \]

In the following lemma, we get rid of the auxiliary conditions on $A$ and $B$ imposed in Lemma 3.3.

**Lemma 3.4.** For all self-adjoint matrices $A, B \in M_{2n}$, we have
\[ \||A| - |B||_{1,\infty} \leq \left(34 + \frac{2560e}{\pi}\right) \|A - B\|_1. \]

**Proof.** Let matrices $A, B$ be symmetrically (but not necessarily identically) distributed,
\[ A = \sum_{k=0}^{n-1} \mu(k, A_+) p_{1k} - \sum_{k=0}^{n-1} \mu(k, A_+) p_{2k}, \quad B = \sum_{k=0}^{n-1} \mu(k, B_+) q_{1k} - \sum_{k=0}^{n-1} \mu(k, B_+) q_{2k}, \]
where all the projections $p_{1k}, p_{2k}, q_{1k}, q_{2k}, 0 \leq k < n$ are pairwise orthogonal and have rank 1. We introduce an auxiliary matrix
\[ C := \sum_{k=0}^{n-1} \mu(k, B_+) p_{1k} - \sum_{k=0}^{n-1} \mu(k, B_+) p_{2k}. \]

Clearly, $B$ and $C$ are identically and symmetrically distributed matrices (in particular, we have $\mu(B) = \mu(C)$). Thus, we have
\[ \||B| - |A||_{1,\infty} = \|(|B| - |C|) + (|C| - |A|)||_{1,\infty} \leq 2\||B| - |C||_{1,\infty} + 2\||A| - |C||_{1,\infty}. \]
Let now where in the last step we used the classical fact \cite[(1.22)]{25}. Combining the above inequalities values, it follows that given by

\[ A \]

Since the matrices \( A \) and \( C \) commute, it follows that

\[ \|A - C\|_1 \leq \|A - C\|_{1,\infty} \leq \|A - B\|_1 = \|\mu(A) - \mu(B)\|_1 \leq \|A - B\|_1, \]

where in the last step we used the classical fact \cite[(1.22)]{25}. Combining the above inequalities we complete the proof for the case of symmetrically distributed matrices.

Let now \( A \) and \( B \) be arbitrary self-adjoint matrices from \( M_{2n} \). Consider an element \( F \in M_2 \) given by

\[ F := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

and observe that

\[ |A \otimes F| - |B \otimes F| = (|A| - |B|) \otimes 1, \]

where \( 1 \) is the identity in \( M_2 \). Note that

\[ \|X \otimes 1\|_{1,\infty} = 2\|X\|_{1,\infty}, \quad \|X \otimes 1\|_1 = 2\|X\|_1 \]

Now, observing that \( A \otimes F \) and \( B \otimes F \) are symmetrically distributed matrices, we infer from the first part of the proof that

\[ \|A - B\|_{1,\infty} = \frac{1}{2} \|A \otimes F| - |B \otimes F||_{1,\infty} \leq \frac{1}{2} \cdot (34 + \frac{2560e}{\pi}) \|A \otimes F - B \otimes F\|_1 = (34 + \frac{2560e}{\pi}) \cdot \|A - B\|_1. \]

\[ \square \]

**Proof of Theorem 1.2.** Let \( p_n, n \geq 0, \) be a sequence of finite rank projections in \( \mathcal{L}(H) \) such that \( p_n \uparrow 1. \) By \cite[Corollary 1.5]{11},

\[ (|p_n A p_n| - |p_n B p_n|)E \rightarrow (|A| - |B|)E \]

uniformly for every finite rank projection \( E \). By Lemma 3.4, we have

\[ \|p_n A p_n| - |p_n B p_n||_{1,\infty} \leq \left(34 + \frac{2560e}{\pi}\right) \|p_n (A - B) p_n\|_1 \leq \left(34 + \frac{2560e}{\pi}\right) \|A - B\|_1. \]

Thus,

\[ \mu(k, (|p_n A p_n| - |p_n B p_n|)E) \leq \mu(k, |p_n A p_n| - |p_n B p_n|) \leq \left(34 + \frac{2560e}{\pi}\right) \|A - B\|_1 \cdot \frac{1}{k + 1}, \quad k \geq 0, \]

for every finite rank projection \( E \). Since uniform convergence implies the convergence of singular values, it follows that

\[ \mu(k, (|A| - |B|)E) \leq \left(34 + \frac{2560e}{\pi}\right) \|A - B\|_1 \cdot \frac{1}{k + 1}, \quad k \geq 0, \]
for every finite rank projection $E$. Hence, using judicious choice of projection $E$ (which is possible since our proof yields that $|A| - |B|$ is compact and hence $E$ may be taken to be a suitable spectral projection of this operator), we have
\[
\mu(k, |A| - |B|) \leq \left(34 + \frac{2560e}{\pi}\right)\|A - B\|_1 \cdot \frac{1}{k + 1}, \quad k \geq 0.
\]

4. **Theorem 1.2 for compact operators**

Define an ideal $(L_{1,\infty})_0$ in $L(H)$ by setting
\[
(L_{1,\infty})_0 = \{A \in K(H) : \quad k\mu(k,A) \to 0 \text{ as } k \to \infty\}.
\]
This ideal coincides with the closure of the ideal of all finite rank operators in $L_{1,\infty}$ and is commonly called the separable part of $L_{1,\infty}$.

Define a (non-linear) functional $\theta$ on $L_{1,\infty}$ by setting
\[
\theta(A) = \limsup_{k \to \infty} k\mu(k,A).
\]

**Lemma 4.1.** Let $A, B \in L_{1,\infty}$. We have
(a) If $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, then
\[
\theta(A + B) \leq \alpha \theta(A) + \beta \theta(B).
\]
(b) If $B \in L_1$, then $\theta(B) = 0$.
(c) If $B \in L_1$, then $\theta(A + B) = \theta(A)$.

**Proof.** We have
\[
\mu(k, A + B) \leq \mu(\lfloor \frac{k}{\alpha} \rfloor, A) + \mu(\lfloor \frac{k}{\beta} \rfloor, B).
\]
Hence,
\[
\theta(A + B) = \limsup_{k \to \infty} k\mu(k, A + B) \leq \limsup_{k \to \infty} k\mu(\lfloor \frac{k}{\alpha} \rfloor, A) + \limsup_{k \to \infty} k\mu(\lfloor \frac{k}{\beta} \rfloor, B).
\]
It is obvious that
\[
\limsup_{k \to \infty} k\mu(\lfloor \frac{k}{\alpha} \rfloor, A) = \alpha \theta(A), \quad \limsup_{k \to \infty} k\mu(\lfloor \frac{k}{\beta} \rfloor, B) = \beta \theta(B).
\]
This proves (a).

If $B \in L_1$ and if $p_n \uparrow 1$, then $\|B(1 - p_n)\|_1 \to 0$ (see e.g. [6]). Let $e_k$ be the eigenvector of $|B|$ corresponding to the eigenvalue $\mu(k,B)$ and set $p_k$ to be the projection on the linear span of $e_n$, $0 \leq n < k$. We have
\[
\sum_{m=k}^{\infty} \mu(m, B) = \|B(1 - p_n)\|_1 \to 0.
\]
Therefore,
\[
\frac{k}{2} \mu(k, B) \leq \sum_{n=\lfloor k/2 \rfloor}^{k} \mu(k, B) \leq \sum_{n=\lfloor k/2 \rfloor}^{\infty} \mu(k, B) \to 0.
\]
This proves (b).

If $A \in L_{1,\infty}$ and $B \in L_1$, then it follows from (a) and (b) that $\theta(A + B) \leq \alpha \theta(A)$. Since $\alpha > 1$ is arbitrary, it follows that $\theta(A + B) \leq \theta(A)$. Applying the same argument to the operators $A + B \in L_{1,\infty}$ and $-B \in L_1$, we infer that $\theta(A) \leq \theta(A + B)$. This proves (c). □
Lemma 4.2. Let $\alpha_k > 0$, $k \geq 0$, be decreasing. Define an operator $S : \mathcal{L}_2 \to \mathcal{L}_2$ by setting

$$S(A) = \sum_{k,l=0}^{\infty} \frac{\alpha_k - \alpha_l}{\alpha_k + \alpha_l} p_k A p_l,$$

where $p_k$, $k \geq 0$, are the pairwise orthogonal rank one projections in $\mathcal{L}(H)$. We have $S(A) \in (\mathcal{L}_{1,\infty})_0$ and

$$\|S(A)\|_{1,\infty} \leq \frac{80e}{\pi} \|A\|_1$$

for every $A \in \mathcal{L}_1$.

Proof. The norm estimate can be proved in exactly the same way as in Lemma 3.2. Set $P_n = \sum_{k=0}^{n-1} p_k$. We have

$$A = (1 - P_n) A (1 - P_n) + (A P_n + P_n A (1 - P_n))$$

and, therefore,

$$S(A) = S((1 - P_n) A (1 - P_n)) + S(A P_n + P_n A (1 - P_n)).$$

We have $S(A P_n + P_n A (1 - P_n)) \in \mathcal{L}_1$ since this operator has finite rank (even though its norm may be quite large). It follows from Lemma 4.1 that

$$\theta(S(A)) = \theta(S((1 - P_n) A (1 - P_n)))$$

$$\leq \|S((1 - P_n) A (1 - P_n))\|_{1,\infty} \leq \text{const} \cdot \|(1 - P_n) A (1 - P_n)\|_1.$$ 

However, $\|(1 - P_n) A (1 - P_n)\|_1 \to 0$ as $n \to \infty$. It follows that $\theta(S(A)) = 0$ and, therefore, $S(A) \in (\mathcal{L}_{1,\infty})_0$. \hfill \Box

Theorem 4.3. If $A, B \in \mathcal{L}(H)$ are compact operators such that $A - B \in \mathcal{L}_1$, then $|A| - |B| \in (\mathcal{L}_{1,\infty})_0$.

Proof. The proof follows that in Lemma 3.3 and Lemma 3.4 mutatis mutandii. \hfill \Box

5. General semifinite version of Theorem 1.2

We begin by recalling a few relevant facts and notations from the theory of noncommutative integration on semifinite von Neumann algebras. For details on von Neumann algebra theory, the reader is referred to e.g. [10], [16], [17] or [28]. General facts concerning measurable operators may be found in [21], [24] (see also [29, Chapter IX]). For the convenience of the reader, some of the basic definitions are recalled.

In what follows, let $\mathcal{M}$ be a von Neumann algebra on a separable Hilbert space $H$. A linear operator $A : \text{dom}(A) \to H$, where the domain $\text{dom}(A)$ of $A$ is a linear subspace of $H$, is said to be affiliated with $\mathcal{M}$ if it commutes with every element in $\mathcal{M}'$.

An operator $A$ affiliated with $\mathcal{M}$ is called $\tau$–finite if there exists a sequence $\{p_n\}_{n=1}^{\infty}$ of $\tau$–finite projections in $\mathcal{M}$ such that $p_n \downarrow 0$ and $(1 - p_n) (H) \subset \text{dom}(A)$ for all $n$. The collection $S(\mathcal{M}, \tau)$ of all $\tau$–finite operators is a unital $*$–algebra with respect to the strong sum and strong multiplication. It is well known that a linear operator $A$ affiliated with $\mathcal{M}$ belongs to $S(\mathcal{M}, \tau)$ if and only if there exists $\lambda > 0$ such that

$$\tau(E_{|A|}(\lambda, \infty)) < \infty.$$
Here, \( E_{|A|} \) is the spectral family of the operator \(|A|\). Alternatively, an unbounded operator \( A \) affiliated with \( \mathcal{M} \) is \( \tau \)-measurable (see [14]) if and only if
\[
\tau \left( E_{|A|}(n, \infty) \right) = o(1), \quad n \to \infty.
\]
Let a semifinite von Neumann algebra \( \mathcal{M} \) be equipped with a faithful normal semi-finite trace \( \tau \). Let \( A \in \mathcal{S}(\mathcal{M}, \tau) \). The generalized singular value function \( \mu(A) : t \to \mu(t; A) \) of the operator \( A \) is defined by setting
\[
\mu(s; A) = \inf\{\|A(1 - p)\|_{\infty} : p \in \mathcal{M} \text{ is a projection}, \tau(p) \leq s\}.
\]
There exists an equivalent definition which involves the distribution function of the operator \(|A|\). For every self-adjoint operator \( A \in \mathcal{S}(\mathcal{M}, \tau) \), setting
\[
d_A(t) = \tau(E_A(t, \infty)), \quad t > 0,
\]
we have (see e.g. [14])
\[
\mu(t; A) = \inf\{s \geq 0 : d_A(s) \leq t\}.
\]
If \( \mathcal{M} = \mathcal{L}(\mathcal{H}) \) and \( \tau \) is the standard trace \( \text{Tr} \), then it is not difficult to see that \( \mathcal{S}(\mathcal{M}, \tau) = \mathcal{M} \).
In this case, for \( A \in \mathcal{M} \), we have
\[
\mu(n; A) = \mu(t; A), \quad t \in [n, n+1), \quad n \geq 0.
\]
The sequence \( \{\mu(n; A)\}_{n \geq 0} \) is just the sequence of singular values of the operator \( A \).

For every \( \varepsilon, \delta > 0 \), we define the set
\[
V(\varepsilon, \delta) = \{x \in \mathcal{S}(\mathcal{M}, \tau) : \exists p = p^2 = p^* \in \mathcal{M} \text{ such that } \|x(1 - p)\| \leq \varepsilon, \tau(p) \leq \delta\}.
\]
The topology generated by the sets \( V(\varepsilon, \delta) \), \( \varepsilon, \delta > 0 \), is called a measure topology.
Let \( L_1(0, \infty) \) and \( L_\infty(0, \infty) \) be Lebesgue spaces on \((0, \infty)\).
We define the space
\[
\mathcal{L}_1(\mathcal{M}, \tau) = \{A \in \mathcal{S}(\mathcal{M}, \tau) : \mu(A) \in L_1(0, \infty)\}.
\]
It is well-known that the functional
\[
\| \cdot \|_1 : A \to \|\mu(A)\|_1
\]
is a Banach norm on \( \mathcal{L}_1(\mathcal{M}, \tau) \). Similarly, we say that \( A \in (\mathcal{L}_1 + \mathcal{L}_\infty)(\mathcal{M}, \tau) \) if and only if \( \mu(A) \in (L_1 + L_\infty)(0, \infty) \). Here, we identify \( \mathcal{M} \) with \( \mathcal{L}_\infty(\mathcal{M}, \tau) \). The space \( (\mathcal{L}_1 + \mathcal{L}_\infty)(\mathcal{M}, \tau) \) can be also viewed as a sum of Banach spaces \( \mathcal{L}_1(\mathcal{M}, \tau) \) and \( \mathcal{L}_\infty(\mathcal{M}, \tau) \) (the latter space is equipped with the uniform norm, which we denote simply by \( \| \cdot \|_\infty \)).
Define a linear space
\[
(\mathcal{L}_1 + \mathcal{L}_\infty)(\mathcal{M}, \tau) = \{A \in \mathcal{S}(\mathcal{M}, \tau) : \mu(A) \in L_1 + L_\infty\}.
\]
One can define the noncommutative weak \( L_1 \) space in a similar manner. Set
\[
\mathcal{L}_{1,\infty}(\mathcal{M}, \tau) = \{A \in \mathcal{S}(\mathcal{M}, \tau) : \sup_{t > 0} t\mu(t, A) < \infty\}.
\]
The mapping
\[
\| \cdot \|_{1,\infty} : A \to \sup_{t > 0} t\mu(t, A), \quad A \in \mathcal{L}_{1,\infty}(\mathcal{M}, \tau)
\]
defines a quasi-norm on \( \mathcal{L}_{1,\infty}(\mathcal{M}, \tau) \). It can be easily seen that \( \mathcal{L}_{1,\infty}(\mathcal{M}, \tau) \) equipped with the latter quasi-norm becomes a quasi-Banach space. The quasi-Banach space \( \mathcal{L}_{1,\infty}(\mathcal{M}, \tau) \) has the Fatou property: if \( A_n \in \mathcal{L}_{1,\infty}(\mathcal{M}, \tau) \), \( \|A_n\|_{1,\infty} \leq 1 \) and \( A_n \to A \) in measure, then \( A \in \mathcal{L}_{1,\infty}(\mathcal{M}, \tau) \) and \( \|A\|_{1,\infty} \leq 1 \).
Lemma 5.1. Let $\mathcal{M}$ be a semifinite von Neumann algebra. Let $\alpha_k > 0$, $0 \leq k < n$, be decreasing. Define an operator $S : \mathcal{M} \to \mathcal{M}$ by setting
\[
S(A) = \sum_{k,l=0}^{n-1} \frac{\alpha_k - \alpha_l}{\alpha_k + \alpha_l} p_k A p_l,
\]
where $p_k$, $0 \leq k < n$, are the pairwise orthogonal $\tau$-finite projections in $\mathcal{M}$. We have
\[
\|S(A)\|_{1,\infty} \leq \text{const} \cdot \|A\|_1, \quad A \in \mathcal{M}.
\]

Proof. The proof follows that of Lemma 3.2 mutatis mutandi. The reference to [15] must be replaced with the reference to Theorem 1.4 in [12].

Lemma 5.2. Let $\mathcal{M}$ be a semifinite factor. Let $A, B \in \mathcal{M}$ be identically and symmetrically distributed finitely supported operators. We have
\[
\| |A| - |B| \|_{1,\infty} \leq \text{const} \|A - B\|_1.
\]

Proof. Since the type I factors were already treated in Theorem 1.2, we may assume without loss of generality that $\mathcal{M}$ is a type II factor. Suppose first that $\mu(A)$ (and, hence, $\mu(B)$) takes finitely many values. The proof in this case follows that of Lemma 3.3 mutatis mutandi. Observe that in the last argument we have used the assumption that $\mathcal{M}$ is a factor to find unitaries $U$ and $V$ as in the proof of Lemma 3.3 (recall: any two projections in a type II factor with equal finite trace are unitarily equivalent [28, Theorem V.1.8]).

Let now $A, B \in \mathcal{M}$ be arbitrary identically and symmetrically distributed finitely supported operators. There exist projections $p_{k,s}$, $s > 0$, $1 \leq k \leq 4$, such that $\tau(p_{k,s}) = s$ and
\[
A_+ = \int_{0}^{\infty} \mu(s, A_+) dp_{1,s}, \quad A_- = \int_{0}^{\infty} \mu(s, A_-) dp_{2,s},
\]
\[
B_+ = \int_{0}^{\infty} \mu(s, A_+) dp_{3,s}, \quad B_- = \int_{0}^{\infty} \mu(s, A_-) dp_{4,s}.
\]

Fix $\varepsilon$ such that
\[
\int_{0}^{\varepsilon} \mu(s, A_+) ds \leq \|A - B\|_1
\]
and set
\[
C_+ = \sum_{k=0}^{\infty} \mu((k+1)\varepsilon, A_+) (p_{1,(k+1)\varepsilon} - p_{1,k\varepsilon}), \quad C_- = \sum_{k=0}^{\infty} \mu((k+1)\varepsilon, A_-) (p_{2,(k+1)\varepsilon} - p_{2,k\varepsilon}),
\]
\[
D_+ = \sum_{k=0}^{\infty} \mu((k+1)\varepsilon, A_+) (p_{3,(k+1)\varepsilon} - p_{3,k\varepsilon}), \quad D_- = \sum_{k=0}^{\infty} \mu((k+1)\varepsilon, A_-) (p_{4,(k+1)\varepsilon} - p_{4,k\varepsilon}).
\]

It is easy to see that
\[
\|A_+ - C_+\|_1 = \sum_{k=0}^{\infty} \int_{k\varepsilon}^{(k+1)\varepsilon} \left( \mu(s, A_+) - \mu((k+1)\varepsilon, A_+) \right) ds
\]
\[
\leq \int_{0}^{\varepsilon} \left( \mu(s, A_+) - \mu(\varepsilon, A_+) \right) ds + \sum_{k=1}^{\infty} \varepsilon \left( \mu(k\varepsilon, A_+) - \mu((k+1)\varepsilon, A_+) \right)
\]
\[
= \int_{0}^{\varepsilon} \left( \mu(s, A_+) - \mu(\varepsilon, A_+) \right) ds + \varepsilon \mu(\varepsilon, A_+) = \int_{0}^{\varepsilon} \mu(s, A_+) ds \leq \|A - B\|_1.
\]
We have
\[ \|A - |B|\|_{1,\infty} = \|(A - |C|) + (|C| - |D|) - (|B| - |D|)\|_{1,\infty} \leq \]
\[ \leq 4\|(A - |C|)_{1,\infty} + 4\|(C - |D|)_{1,\infty} + 4\|(B - |D|)_{1,\infty}. \]
Recall that \( C \) and \( D \) are symmetrically and identically distributed finitely supported operators. By construction, \( \mu(C) \) (and, hence, \( \mu(D) \)) takes only finitely many values. We infer from the previous paragraph that
\[ \|(C - |D|)_{1,\infty} \leq \text{const} \cdot \|C - D\|_1. \]
Since \( A \) and \( C \) commute, it follows that
\[ \|(A - |C|)_{1,\infty} \leq \|A - C\|_{1,\infty} \leq \|A - C\|_1 \leq 2\|A - B\|_1. \]
Similarly,
\[ \|(B - |D|)_{1,\infty} \leq 2\|A - B\|_1. \]
Also, we have
\[ \|C - D\|_1 = \|(C - A) + (A - B) + (B - D)\|_1 \leq \]
\[ \leq \|C - A\|_1 + \|A - B\|_1 + \|B - D\|_1 \leq 5\|A - B\|_1. \]
Combining these estimates, we conclude the proof. \( \square \)

The following lemma should be compared to the results on positive Schur multipliers in Section 2.3.

**Lemma 5.3.** Let \( M \subseteq \mathcal{L}(H) \) be a von Neumann algebra. Let \( B \in M_{n} \) and \( B \geq 0 \). Let \( p_1, \ldots, p_n \) be mutually orthogonal projections in \( M \). Consider the operator valued Schur multiplier (or double operator integral) defined by,
\[ (5.1) \quad S_B : M \to M : x \mapsto \sum_{i,j=1}^{n} B_{i,j} p_i x p_j. \]
Then \( S_B \) preserves positive operators: \( S_B(x) \geq 0 \) whenever \( x \geq 0 \).

**Proof.** The Schur multiplier extends to a map \( S_B : \mathcal{L}(H) \to \mathcal{L}(H) \) prescribed by the same formula (5.1) and hence it suffices to prove the statement for \( M = \mathcal{L}(H) \). In case each of the projections \( p_i \) are finite rank the statement is reduced to the matricial case and hence follows from Schur’s theorem, see Section 2.3. Indeed, this is true in case each \( p_i \) is one dimensional. Else, write \( p_i = \sum_{m}^n p_{i,m} \), a finite sum of mutually orthogonal rank 1 projections and apply the previous line to the set \( \{p_{i,m} \mid i, 1 \leq m \leq n_i\} \) using that \( (B_{i,j})_{(i,1 \leq m \leq n_i),(j,1 \leq k \leq n_j)} \) is again positive. The positivity of the latter matrix follows as this matrix is a corner of the Kronecker product \( C_n \otimes B \) where \( C_n \) is the \( n \times n \)-matrix with entries equal to 1. In the general case of not necessarily finite rank projections \( p_i \) one can write each \( p_i \) as a strong limit of finite rank projections \( p_{i,m} \to p_i \). Putting \( P_m = \sum_{i} p_{i,m} \) we see that \( x \mapsto P_m S_B(x) P_m \) preserves positive operators and \( P_m S_B(x) P_m \to S_B(x) \) strongly. This concludes the lemma. \( \square \)

**Lemma 5.4.** Let \( M \) be a semifinite factor. Let \( A, B \in M \) be self-adjoint finitely supported operators. We have
\[ \|(A - |B|)_{1,\infty} \leq \text{const} \cdot \|A - B\|_1. \]

**Proof.** The proof follows that of Lemma 3.4 mutatis mutandis. At the point that Schur’s theorem is used, see the proof of Lemma 2.1, Lemma 5.3 can be invoked. \( \square \)
Lemma 5.5. If $\mathcal{M}$ is a semifinite factor and if $A, B \in (L_1 + L_\infty)(\mathcal{M}, \tau)$ are such that $A - B \in L_1(\mathcal{M}, \tau)$, then $|A| - |B| \in L_1(\mathcal{M}, \tau)$ and
\[
|||A| - |B|||_{1,\infty} \leq \text{const} \cdot ||A - B||_1.
\]

Proof. Suppose first that $A, B \in \mathcal{M}$. Let $p_n, n \geq 0$, be a sequence of $\tau$–finite projections in $\mathcal{M}$ such that $p_n \uparrow 1$. By [11, Corollary 1.5],
\[
\left(|p_nAp_n| - |p_nBp_n|\right)E \to \left(|A| - |B|\right)E
\]
in measure for every $\tau$–finite projection $E$. By Lemma 5.4, we have
\[
|||p_nAp_n| - |p_nBp_n|||_{1,\infty} \leq \text{const} \cdot ||p_n(A - B)p_n||_1 \leq \text{const} \cdot ||A - B||_1.
\]
Therefore,
\[
\mu(t, (|p_nAp_n| - |p_nBp_n|)E) \leq \mu(t, |p_nAp_n| - |p_nBp_n|) \leq \frac{\text{const}}{t} ||A - B||_1, \quad t > 0,
\]
for every $\tau$–finite projection $E$. Since convergence in measure implies the (almost everywhere) convergence of singular value functions (see e.g. [26, Lemma 7]), it follows that
\[
\mu(t, (|A| - |B|)E) \leq \frac{\text{const}}{t} ||A - B||_1, \quad t > 0,
\]
for every $\tau$–finite projection $E$. Hence, using a judicious choice of the projection $E$ (namely a suitable spectral projection of $|A| - |B|$), we have
\[
\mu(t, |A| - |B|) \leq \frac{\text{const}}{t} ||A - B||_1, \quad t > 0.
\]
This proves the assertion for bounded $A$ and $B$.

Let now $A, B \in (L_1 + L_\infty)(\mathcal{M}, \tau)$. Set
\[
p_n = E_{|A|}(0, n] \bigcap E_{|B|}(0, n].
\]
Since $A, B$ are $\tau$–measurable, it follows from [30] (see also [11, Theorem 1.1]) that
\[
p_nAp_n \to A, \quad p_nBp_n \to B, \quad |p_nAp_n| \to |A|, \quad |p_nBp_n| \to |B|
\]
in measure. It follows from the above that
\[
|||p_nAp_n| - |p_nBp_n|||_{1,\infty} \leq \text{const} \cdot ||p_n(A - B)p_n||_1 \leq \text{const} \cdot ||A - B||_1.
\]
Since the quasi-norm in $L_{1,\infty}(\mathcal{M}, \tau)$ has the Fatou property, it follows that
\[
|||A| - |B|||_{1,\infty} \leq \text{const} \cdot ||A - B||_1.
\]

The following lemma shows the proper triangle inequality in $L_{1,\infty}$ for pairwise orthogonal summands. Let $A_k \in L_{1,\infty}(\mathcal{M}, \tau), k \geq 0$. We use the direct sum symbol $\bigoplus_{k=0}^{\infty} A_k$ to denote the operator on $H$ formed with respect to some arbitrary Hilbert space isomorphism $\bigoplus_{k=0}^{\infty} H \simeq H$.

Lemma 5.6. If $A_k \in L_{1,\infty}(\mathcal{M}, \tau), k \geq 0$, then
\[
||\bigoplus_{k=0}^{\infty} A_k||_{1,\infty} \leq \sum_{k=0}^{\infty} ||A_k||_{1,\infty}.
\]
Proof. Set \( x(t) = 1/t, \ t > 0 \). For simplicity of notations, denote \( \|A_k\|_{1,\infty} \) by \( \alpha_k \). We have \( \mu(t, A_k) \leq \alpha_k/t, \ t > 0 \). Using the notation \( d_{\alpha_k}x(t) \) for the classical distribution, it is immediate that

\[
d(\bigoplus_{k=0}^{\infty} A_k \| (t) = \sum_{k=0}^{\infty} d_{\alpha_k}x(t) \leq \sum_{k=0}^{\infty} \frac{\alpha_k}{t} = \sum_{k=0}^{\infty} \frac{\alpha_k}{t} = d(\bigoplus_{k=0}^{\infty} \alpha_k)x(t).
\]

Hence,

\[
\mu(\bigoplus_{k=0}^{\infty} A_k) \leq (\sum_{k=0}^{\infty} \alpha_k)x
\]
or, equivalently,

\[
\| \bigoplus_{k=0}^{\infty} A_k \|_{1,\infty} \leq (\sum_{k=0}^{\infty} \alpha_k)\|x\|_{1,\infty} = \sum_{k=0}^{\infty} \|A_k\|_{1,\infty}.
\]

The following lemma combines well-known facts from [28] and less known facts from [13].

**Lemma 5.7.** For every semifinite von Neumann algebra \((\mathcal{M}, \tau)\), there exist semifinite factors \((\mathcal{M}_k, \tau_k)\), \(k \geq 0\), and a trace preserving \(\ast\)-monomorphism of \((\mathcal{M}, \tau)\) into \((\bigoplus_{k=0}^{\infty} \mathcal{M}_k, \bigoplus_{k=0}^{\infty} \tau_k)\).

**Proof.** By Theorem V.1.19 in [28], we have \( \mathcal{M} = \mathcal{M}^1 \oplus \mathcal{M}^2 \oplus \mathcal{M}^3 \), where \(\mathcal{M}^1\) is type I, \(\mathcal{M}^2\) is type \(\Pi_1\) and \(\mathcal{M}^3\) is type \(\Pi_\infty\).

By Theorem V.1.27 in [28], there exist commutative algebras \(A_k, k \geq 0\), and Hilbert spaces \(H_k, k \geq 0\), such that

\[
\mathcal{M}^1 = \bigoplus_{k=0}^{\infty} A_k \tilde{\otimes} \mathcal{L}(H_k).
\]

Every \(A_k\) admits a trace preserving isomorphic embedding into \(L_\infty(0, \infty)\) and then to the hyperfinite \(\Pi_\infty\) factor \(R \tilde{\otimes} \mathcal{L}(H)\). Every \(\mathcal{L}(H_k)\) admits a trace preserving isomorphic embedding into \(\mathcal{L}(H)\). Thus, \(\mathcal{M}^1\) admits a trace preserving isomorphic embedding into

\[
\bigoplus_{k=0}^{\infty} R \tilde{\otimes} \mathcal{L}(H) \tilde{\otimes} \mathcal{L}(H).
\]

Equivalently, \(\mathcal{M}^1\) admits a trace preserving isomorphic embedding into \((R \tilde{\otimes} \mathcal{L}(H))^\oplus_{\infty}\).

By Theorem 2.3 and Lemma 2.5 in [13], the algebra \(\mathcal{M}^2\) admits a trace preserving isomorphic embedding into a \(\Pi_1\) factor.

By Theorem V.1.40 in [28], there exist type \(\Pi_1\)-algebras \(N_k, k \geq 0\), such that

\[
\mathcal{M}^3 = \bigoplus_{k=0}^{\infty} N_k \tilde{\otimes} \mathcal{L}(H).
\]

Again applying Theorem 2.3 and Lemma 2.5 in [13], we embed the algebras \(N_k, k \geq 0\), into \(\Pi_1\) factors \(O_k, k \geq 0\). Since \(O_k \tilde{\otimes} \mathcal{L}(H)\) is a type \(\Pi_\infty\) factor, the assertion follows for \(\mathcal{M}^3\) and, thus, for \(\mathcal{M}\).

**Theorem 5.8.** If \(\mathcal{M}\) is a semifinite von Neumann algebra and if \(A, B \in (\mathcal{L}_1 + \mathcal{L}_\infty)(\mathcal{M}, \tau)\) are such that \(A - B \in \mathcal{L}_1(\mathcal{M}, \tau)\), then \(|A| - |B| \in \mathcal{L}_1, \infty(\mathcal{M}, \tau)\) and

\[
\||A| - |B||_{1,\infty} \leq \text{const} \cdot \|A - B\|_1.
\]
Proof. According to the Lemma 5.7, we can embed $(\mathcal{M}, \tau)$ into $(\bigoplus_{k \geq 0} \mathcal{M}_k, \bigoplus_{k \geq 0} \tau_k)$, where $(\mathcal{M}_k, \tau_k)$, $k \geq 0$, are semifinite factors.

Thus, $A = \bigoplus_{k \geq 0} A_k$ and $B = \bigoplus_{k \geq 0} B_k$, where $A_k, B_k \in (\mathcal{L}_1 + \mathcal{L}_\infty)(\mathcal{M}_k, \tau_k)$, $k \geq 0$. By Lemma 5.6, we have that

$$|||A| - |B|||_{1,\infty} = \left| \bigoplus_{k \geq 0} |A_k| - |B_k|| \right|_{1,\infty} \leq \sum_{k \geq 0} |||A_k| - |B_k|||_{1,\infty}.$$  

Since every $\mathcal{M}_k$ is a factor, it follows from Lemma 5.5 that

$$|||A_k| - |B_k|||_{1,\infty} \leq \text{const} \cdot ||A_k - B_k||_1.$$  

Therefore, we have

$$|||A| - |B|||_{1,\infty} \leq \text{const} \cdot \sum_{k \geq 0} ||A_k - B_k||_1 = \text{const} \cdot \left| \bigoplus_{k \geq 0} A_k - B_k \right|_1 = \text{const} \cdot ||A - B||_1.$$  

\[\square\]

6. Commutator estimates

The proof of the following consequence is essentially the same as the implication (i) $\Rightarrow$ (ii) of [11, Theorem 2.2]. For completeness and the fact that [11, Theorem 2.2] is not directly applicable since we are dealing with estimates between different spaces (and one of them only has a quasi-norm) we have included the proof.

**Theorem 6.1.** If $A, B \in (\mathcal{L}_1 + \mathcal{L}_\infty)(\mathcal{M}, \tau)$ are self-adjoint operators such that $[A, B] \in \mathcal{L}_1(\mathcal{M}, \tau)$, then

$$|||[A, B]|||_{1,\infty} \leq \text{const} \cdot ||[A, B]||_1.$$  

Proof. Suppose first that $B$ is bounded. Setting $C = e^{i\varepsilon B} A e^{-i\varepsilon B}$, we have $|C| = e^{i\varepsilon B} |A| e^{-i\varepsilon B}$. We infer from Theorem 5.8 that

$$|||e^{i\varepsilon B} |A|||_{1,\infty} = |||C| - |A|||_{1,\infty} \leq \text{const} \cdot ||C - A||_1 = \text{const} \cdot |||e^{i\varepsilon B} |A|||_1.$$  

However,

$$[e^{i\varepsilon B} |A|] = \sum_{k=1}^{\infty} \frac{(i\varepsilon)^k}{k!} [B^k, A]$$

where the series on the right hand side converges in $\mathcal{L}_1(\mathcal{M}, \tau)$. Indeed,

$$[B^k, A] = \sum_{m=0}^{k-1} B^m [B, A] B^{k-1-m}, \quad k \geq 1$$

and therefore,

$$|||e^{i\varepsilon B} |A|||_1 = ||\sum_{k=1}^{\infty} \sum_{m=0}^{k-1} \frac{(i\varepsilon)^k}{k!} B^m [B, A] B^{k-1-m}||_1 \leq \sum_{k=1}^{\infty} \sum_{m=0}^{k-1} \frac{\varepsilon^k}{k!} ||B^m [B, A] B^{k-1-m}||_1 \leq$$

$$\leq \sum_{k=1}^{\infty} \sum_{m=0}^{k-1} \frac{\varepsilon^k}{k!} ||B||^{k-1} ||[B, A]||_1 = \left( \sum_{k=1}^{\infty} \frac{k\varepsilon^k}{k!} ||B||^{k-1} \right) ||[A, B]||_1 = \varepsilon ||B||_\infty ||[A, B]||_1.$$  

Combining preceding estimates, we infer that

$$|||e^{i\varepsilon B} |A|||_{1,\infty} \leq \text{const} \cdot \varepsilon ||B||_\infty ||[A, B]||_1.$$
It follows from the Spectral Theorem and boundedness of $B$ that

$$\epsilon^{-1}(e^{i\epsilon B} - 1) \to iB$$

uniformly and, therefore,

$$\epsilon^{-1}[e^{i\epsilon B}, |A|] = \epsilon^{-1}(e^{i\epsilon B} - 1), |A| \to i[B, |A|]$$

in the norm of the space $(L_1 + L_\infty)(\mathcal{M}, \tau)$ and therefore in measure (see also [11]). Since the quasi-norm in $L_{1,\infty}$ has the Fatou property, it follows that

$$\|[[A, B]]_{1,\infty} \leq \text{const} \cdot \|[A, B]\|_1.$$ 

This proves the assertion for the case of bounded $B$. Consider now the general case of an arbitrary $B \in (L_1 + L_\infty)(\mathcal{M}, \tau)$. Set $p_n = E_{B}[0, n]$. We have that $p_n Ap_n \to A$ and $p_n Bp_n \to B$ in measure. Since $p_n$ commutes with $B$, it follows that

$$[p_n Ap_n, p_n Bp_n] = p_n[A, B]p_n.$$

It follows from [30] (see also [11, Theorem 1.1]) that $|p_n Ap_n| \to |A|$ in measure. Thus,

$$[[p_n Ap_n, p_n Bp_n]] \to [|A|, B]$$

in measure. It is proved in the previous paragraph that

$$\|[[p_n Ap_n, p_n Bp_n]]_{1,\infty} \leq \text{const} \cdot \|[[p_n Ap_n, p_n Bp_n]]_1 = \text{const} \cdot \|p_n[A, B]p_n\|_1 \leq \text{const} \cdot \|[A, B]\|_1.$$ 

Since the quasi-norm in $L_{1,\infty}$ has the Fatou property, it follows that

$$\|[[A, B]]_{1,\infty} \leq \text{const} \cdot \|[A, B]\|_1.$$ 

□

**Remark 6.2.** The result of Theorem 1.1 may be obtained from Theorem 6.1 as follows. Firstly, observe that we can interpolate between the weak $L_1$-space, $L_{1,\infty}$ and $L_2$ using weak type interpolation (see e.g. [9] and references therein). This immediately implies the estimates for Schatten $p$-norms, $1 < p < 2$ analogous to that of Theorem 6.1, with the case $2 < p < \infty$ following by duality. The result of Theorem 1.2 follows now from [11, Theorem 2.2].

7. **Final comments**

Davies introduced the class of functions representable in the form

$$f(t) = \int_{\mathbb{R}} |t - s|d\nu_f(s),$$

where $\nu_f$ is a signed measure with finite support. He proved that

$$\|f(A) - f(B)\|_p \leq c_{p,f}\|A - B\|_p, \quad 1 < p < \infty.$$ 

Though we cannot fully extend this result to $p = 1$, the following is possible. Define distorted variation $DV(\nu)$ as follows

$$DV(\nu) = \sup\{\inf_\pi \sum_{k \geq 0} 2^{\pi(k)}|\nu(A_k)| : A_m \cap A_n = \emptyset \text{ for all } m \neq n, \cup_{k \geq 0} A_k = \mathbb{R}\}.$$ 

Here, every $A_n$, $n \geq 0$, is an interval (or a semi-axis) and the infimum is taken over all permutations $\pi$ of $\mathbb{Z}_+$. 
Lemma 7.1. For every finitely supported measure \( \nu \) with \( DV(\nu) < \infty \), there exists a sequence \( \nu_m, m \geq 1 \), of discrete measures such that

\[
\int \lvert t - s \rvert d\nu_m(s) \to \int \lvert t - s \rvert d\nu(s)
\]

uniformly and \( DV(\nu_m) \leq DV(\nu) \) for all \( m \geq 1 \).

Proof. Assume, for simplicity of notations, that \( \nu \) is supported on \([0, 1)\) and that \( |\nu([0,1))| = 1 \). Define a measure \( \nu_m \) by setting

\[
\nu_m = \sum_{k=0}^{m-1} \nu\left(\left[\frac{k}{m}, \frac{k+1}{m}\right]\right) \delta_{\left\{ \frac{k}{m} \right\}}.
\]

It is immediate that \( DV(\nu_m) \leq DV(\nu) \) (because every partition of the finite set \( \{0, 1/m, \ldots, (m-1)/m\} \) extends to a partition of \([0,1)\)).

Fix \( t \in \mathbb{R} \) and for a given \( m \in \mathbb{N} \), define the function \( g_m \) on \([0, 1)\) by setting \( g_m(s) = \lvert k/m - t \rvert \) for all \( s \in [k/m, (k+1)/m), 0 \leq k < m \). Since \( g_m \) is a step function with steps at \( \{0, 1/m, \ldots, (m-1)/m\} \), it follows that

\[
\int \mathbb{R} g_m(s) d\nu_m(s) = \int \mathbb{R} g_m(s) d\nu(s).
\]

It is clear that

\[
\left| \int \mathbb{R} |t - s| d\nu(s) - \int \mathbb{R} g_m(s) d\nu(s) \right| \leq \frac{1}{m} \cdot |\nu([0,1))|
\]

and

\[
\left| \int \mathbb{R} |t - s| d\nu_m(s) - \int \mathbb{R} g_m(s) d\nu_m(s) \right| \leq \frac{1}{m} \cdot |\nu_m([0,1))|.
\]

It follows that

\[
\left| \int \mathbb{R} |t - s| d\nu_m(s) - \int \mathbb{R} |t - s| d\nu(s) \right| \leq \frac{2}{m}.
\]

This proves the claim. \( \square \)

The following lemma is a particular case of [27, Lemma 17] (proved there for every quasi-Banach space and not just \( L_{1,\infty} \)). It serves as a replacement for the triangle inequality.

Lemma 7.2. Let \( A_k \in L_{1,\infty}, k \geq 0 \). We have

\[
\| \sum_{k=0}^{\infty} A_k \|_{1,\infty} \leq \text{const} \cdot \sum_{k=0}^{\infty} 2^k \| A_k \|_{1,\infty}.
\]

Here, the convergence of the series in the right hand side guarantees that the series in the left hand side converges in \( L_{1,\infty} \).

Theorem 7.3. If \( f \) is in the Davies class and if \( DV(\nu_f) < \infty \), then for any two bounded self-adjoint operators \( A \) and \( B \), such that \( A - B \in L_1 \), we have

\[
\| f(A) - f(B) \|_{1,\infty} \leq \text{const} \cdot DV(\nu_f) \cdot \| A - B \|_1.
\]

Proof. By Lemma 7.1, we may approximate \( \nu \) with \( DV(\nu) \leq 1 \) by the sequence of discrete measures \( \nu_m \) with \( DV(\nu_m) \leq 1 \). It follows that we can assume without loss of generality that \( \nu \) is discrete. Indeed, since \( A \) and \( B \) are bounded, we find that \( f_m(A) \to f(A) \) and \( f_m(B) \to f(B) \) uniformly, where

\[
f_m(t) := \int \mathbb{R} |t - s| d\nu_m(s), \quad t \in \mathbb{R}.
\]
Using the Fatou property of the $L_{1,\infty}$, we infer that it is suffices to prove the assertion for discrete measures with finite distorted variation. If the measure $\nu$ is discrete, then

$$f(t) = \sum_{k=0}^{\infty} \alpha_k |t - t_k|, \quad \sum_{k=0}^{\infty} 2^k |\alpha_k| < \infty.$$  

We have

$$f(A) - f(B) = \sum_{k=0}^{\infty} \alpha_k \left( |A - t_k| - |B - t_k| \right).$$

By Theorem 1.2, we have

$$\| |A - t_k| - |B - t_k| \|_{1,\infty} \leq \text{const} \cdot \|A - B\|_1.$$  

It follows from Lemma 7.2 above that

$$\|f(A) - f(B)\|_{1,\infty} \leq \sum_{k=0}^{\infty} 2^k |\alpha_k| \|A - B\|_1.$$  

□

References

[1] T. Ando, R. Horn, C. Johnson, The singular values of a Hadamard product: Basic inequalities, Linear and Multilinear Algebra, 21 (1987), 345–364.
[2] R. Bhatia, Infinitely divisible matrices, Amer. Math. Monthly 113 (2006), no. 3, 221–235.
[3] R. Bhatia, Min matrices and mean matrices, Math. Intelligencer, 33 (2011), 34–66; translation in J. Soviet Math. 63 (1993), no. 2, 129–148.
[4] O. Bratteli, D. Robinson, Operator algebras and quantum statistical mechanics. Vol. 1, Springer-Verlag, New York, 1979.
[5] V. Chilin, F. Sukochev, Weak convergence in non-commutative symmetric spaces, J. Operator Theory 31 (1994), no. 1, 35–65.
[6] K. Davidson, Nest algebras. Triangular forms for operator algebras on Hilbert space. Pitman Research Notes in Mathematics Series 191. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1988.
[7] E. Davies, Lipschitz continuity of functions of operators in the Schatten classes, J. London Math. Soc. 37 (1988), 148–157.
[8] S. Dirksen, Noncommutative and vector-valued Boyd interpolation theorems, ArXiv: 1203.1653v1.
[9] J. Dixmier, Les algèbres d’opérateurs dans l’espace hilbertien, 2 edition, Gauthier - Villars, Paris, 1969.
[10] P. Dodds, T. Dodds, B. de Pagter, F. Sukochev, Lipschitz continuity of the absolute value and Riesz projections in symmetric operator spaces, J. Funct. Anal. 148 (1997), 28-69.
[11] P. Dodds, T. Dodds, B. de Pagter, F. Sukochev, Lipschitz continuity of the absolute value in preduals of semifinite factors. Integral Equations Operator Theory 34 (1999), no. 1, 28–44.
[12] K. Dykema, Factoriality and Connes’ invariant $T(M)$ for free products of von Neumann algebras, J. Reine Angew. Math. 450 (1994), 159–180.
[13] T. Fack, H. Kosaki, Generalized $s$–numbers of $\tau$–measurable operators, Pacific J. Math. 123 (1986), no. 2, 269–300.
[14] I. Gohberg, M. Krein, Introduction to the theory of linear nonselfadjoint operators, Translations of Mathematical Monographs, Vol. 18 American Mathematical Society, Providence, R.I., 1969.
[15] R. Kadison, J. Ringrose, Fundamentals of the Theory of Operator Algebras I, Academic Press, Orlando, 1983.
[16] R. Kadison, J. Ringrose, Fundamentals of the Theory of Operator Algebras II, Academic Press, Orlando, 1986.
[18] H. Kosaki, *Unitarily invariant norms under which the map $A \to |A|$ is continuous*, Publ. RIMS, Kyoto Univ., 28 (1992), 299–313.
[19] S. Lord, F. Sukochev, D. Zanin, *Singular Traces: Theory and Applications*. Volume 46 of Studies in Mathematics, De Gruyter, 2013.
[20] F. Nazarov, V. Peller, *Lipschitz functions of perturbed operators*, C. R. Math. Acad. Sci. Paris 347 (2009), no. 15-16, 857–862.
[21] E. Nelson, *Notes on non-commutative integration*, J. Funct. Anal. 15 (1974), 103-116.
[22] A. Pietsch, *Eigenvalues and s-numbers*, Cambridge Studies in Advanced Mathematics, 13. Cambridge University Press, Cambridge, 1987.
[23] M. Reed, B. Simon, *Methods of modern mathematical physics. I. Functional analysis*, Second edition. Academic Press, Inc. New York, 1980.
[24] I. Segal, *A non-commutative extension of abstract integration*, Ann. of Math. 57 (1953), 401–457.
[25] B. Simon, *Trace ideals and their applications*, Second edition. Mathematical Surveys and Monographs, 120. American Mathematical Society, Providence, RI, 2005.
[26] F. Sukochev, *On a conjecture of A. Bikchentaev*, Proc. Sympos. Pure Math. 87, 327–339, Amer. Math. Soc., Providence, R.I., 2013.
[27] F. Sukochev, *Completeness of quasi-normed symmetric operator spaces*, Indag. Math. 25, Issue 2, (2014), 376–388.
[28] M. Takesaki, *Theory of operator algebras I*, Springer-Verlag, New York, 1979.
[29] M. Takesaki, *Theory of Operator Algebras II*, Springer-Verlag, Berlin-Heidelberg-New York, 2003.
[30] O. Tikhonov, *Continuity of operator functions in topologies connected with a trace on a von Neumann algebra* (Russian) Izv. Vyssh. Uchebn. Zaved. Mat. 1987, no. 1, 77–79.

M. CASPERS, FACHBEREICH MATHEMATIK UND INFORMATIK DER UNIVERSITÄT MÜNTER, EINSTEINSTRASSE 62, 48149 MÜNTER, GERMANY
E-mail address: martijn.caspers@uni-muenster.de

D. POTAPOV, F. SUKOCHEV, D. ZANIN, SCHOOL OF MATHEMATICS AND STATISTICS, UNSW, KENSINGTON 2052, NSW, AUSTRALIA
E-mail address: d.potapov@unsw.edu.au
E-mail address: f.sukochev@unsw.edu.au
E-mail address: d.zanin@unsw.edu.au