Generators of local gauge transformations in the covariant canonical formalism of fields

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Abstract

We investigate generators of local gauge transformations in the covariant canonical formalism (CCF) for matter fields, gauge fields and the second order formalism of gravity. The CCF treats space and time on an equal footing regarding the differential forms as the basic variables. The conjugate forms $\pi_A$ are defined as derivatives of the Lagrangian $d$-form $L(\psi^A, d\psi^A)$ with respect to $d\psi^A$, namely $\pi_A := \partial L/\partial d\psi^A$, where $\psi^A$ are $p$-form dynamical fields. The form-canonical equations are derived from the form-Legendre transformation of the Lagrangian form $H := d\psi^A \wedge \pi_A - L$. We show that the generator of the local gauge transformation in the CCF is given by $\varepsilon^r G_r + d\varepsilon^r \wedge F_r$ where $\varepsilon^r$ are infinitesimal parameters and $G_r$ are the Noether currents which are $(d-1)$-forms. $\{G_r, G_s\} = f_{rst} G_t$ holds where $\{\bullet, \bullet\}$ is the Poisson bracket of the CCF and $f_{rst}$ are the structure constants of the gauge group. For the gauge fields and the gravity, $G_r = -\{F_r, H\}$ holds. For the matter fields, $F_r = 0$ holds.

1 Introduction

In the traditional analytical mechanics of fields, the canonical formalism gives especial weight to time. The covariant canonical formalism (CCF) [1–11] is a covariant extension of the traditional canonical formalism. The form-Legendre transformation and the form-canonical equations are derived from a Lagrangian $d$-form with $p$-form dynamical fields $\psi^A$. The conjugate forms are defined as derivatives of the Lagrangian form with respect to $d\psi^A$. One can obtain the form-canonical equations of gauge theories or those of the second order formalism of gravity without fixing a gauge nor introducing Dirac bracket nor any other artificial tricks. Although the second order formalism of gravity (of which the dynamical variable is only the frame form (vielbein)) is a non-constrained system in the CCF, the first order formalism (of which the dynamical variables are both the frame form and the connection form) is a constrained system even in the CCF. In Refs. [1–4], the CCFs of the first order formalism of gravity and supergravity have been studied. Only for $d = 4$, the CCF of the second order formalism of gravity without Dirac field [6] and with Dirac field [7] have been studied [7].

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1) In Refs. [6,7], the method to derive the form-canonical equations used special characteristics of $d = 4$. Without using those, we derive the form-canonical equations in this paper.
Poisson brackets of the CCF are proposed in Refs. [1, 10] and in Ref. [9] independently. These are equivalent. Although the form-canonical equations of the CCF are equivalent to modified De Donder-Weyl equations [8], the Poisson bracket of the CCF is not equivalent to it of the De Donder-Weyl theory proposed in Ref. [12]. Reference [10] introduced the generator of the CCF and studied it of the local Lorentz transformation of gravity in the first order formalism. The generators of the local Lorentz transformation and the supersymmetry for supergravity have been studied [11] in the first order formalism.

The structure of the paper is as follows. First, we review the covariant canonical formalism (§2). Next, we investigate generators of local gauge transformations in CCF for matter fields, gauge fields and the second order formalism of gravity (§3). The total generator is given by

\[ G = \varepsilon^r G_r + d\varepsilon^r \wedge F_r \]

where \( \varepsilon^r \) are infinitesimal parameters and \( G_r \) are the Noether currents. The Noether currents satisfy \( \{ G_r, G_s \} = f_{rs}^t G_t \) where \( \{ \cdot, \cdot \} \) is the Poisson bracket of the CCF and \( f_{rs}^t \) are the structure constants of the gauge group. \( F_r = 0 \) holds for the matter fields. For the gauge fields and the gravity, \( G_r = -\{ F_r, H \} \) holds. Here, \( H \) is the form-Legendre transformation of the Lagrangian form. In Appendix A, we review the Noether currents. In Appendix B, we review the CCF of gauge fields. In Appendix C, we apply the CCF to the second order formalism of gravity with Dirac fields for the arbitrary dimension \( (d \geq 3) \). In Appendix D, several formulas are listed.

## 2 Covariant canonical formalism

In this section, we review the covariant canonical formalism.

Let us consider \( d \) dimension space-time. Suppose a \( p \)-form \( \beta \) is described by forms \( \{ \alpha^i \}_{i=1}^k \). If there exists the form \( \omega_i \) such that \( \beta \) behaves under variations \( \delta \alpha^i \) as \( \delta \beta = \delta \alpha^i \wedge \omega_i \), we call \( \omega_i \) the derivative of \( \beta \) by \( \alpha^i \) and denote

\[ \frac{\partial \beta}{\partial \alpha^i} := \omega_i. \quad (2.1) \]

The Lagrangian \( d \)-form \( L \) is given by \( L = \mathcal{L}\eta \) where \( \mathcal{L} \) is the Lagrangian density and \( \eta = \ast 1 \) is the volume form \( \ast \) and described by \( \psi \) and \( d\psi \), \( L = \mathcal{L}(\psi, d\psi) \), where \( \psi \) is a set the forms of the dynamical fields. For simplicity, we treat \( \psi \) as single \( p \)-form in this section. The Euler-Lagrange equation is given by

\[ \frac{\partial L}{\partial \psi} - (-1)^p d \frac{\partial L}{\partial d\psi} = 0. \quad (2.2) \]

The above Euler-Lagrange equation has been used since the 1970’s [13-15].

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2) In this case, \( \beta \) is differentiable by \( \alpha^i \).

3) The Hodge operator \( \ast \) maps an arbitrary \( p \)-form \( \omega = \omega_{\mu_1...\mu_p} dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} \) \( (p = 0, 1, ... , d) \) to a \((d-p)\)-form as

\[ \ast \omega = \frac{1}{(d-p)!} E_{\mu_1...\mu_p;\nu_1...\nu_{d-p}} \omega_{\mu_1...\mu_p} dx^{\nu_1} \wedge ... \wedge dx^{\nu_{d-p}}. \]

Here, \( E_{\mu_1...\mu_d} \) is the complete anti-symmetric tensor such that \( E_{01...d-1} = \sqrt{-g} \) (\( g \) is the determinant of the metric \( g_{\mu\nu} \)). And \( \ast \ast \omega = -(-1)^{p(d-p)} \omega \) holds.
The conjugate form \( \pi \) is defined by
\[
\pi := \frac{\partial L}{\partial d\psi}.
\] (2.3)
This is a \( q \)-form where \( q := d - p - 1 \). The Hamilton \( d \)-form is defined by
\[
H(\psi, \pi) := d\psi \wedge \pi - L
\] (2.4)
and described by \( \psi \) and \( \pi \). The variation of \( H \) is given by
\[
\delta H = (-1)^{(p+1)q} \delta \pi \wedge d\psi - \delta \psi \wedge \frac{\partial L}{\partial \psi}.
\] (2.5)
Then, we obtain
\[
\frac{\partial H}{\partial \psi} = -\frac{\partial L}{\partial \psi}, \quad \frac{\partial H}{\partial \pi} = (-1)^{(p+1)q} d\psi.
\] (2.6)
By substituting the Euler-Lagrange equation (2.2), we obtain the canonical equations
\[
d\psi = (-1)^{(p+1)q} \frac{\partial H}{\partial \pi}, \quad d\pi = -(-1)^p \frac{\partial H}{\partial \psi}.
\] (2.7)

The Poisson bracket proposed in Ref. [9] is given by
\[
\{ F, G \} = (-1)^{(f+d+1)} \frac{\partial F}{\partial \psi} \wedge \frac{\partial G}{\partial \pi} - (-1)^{(d+p-1)(f+1)} \frac{\partial F}{\partial \pi} \wedge \frac{\partial G}{\partial \psi}.
\] (2.8)
Here, \( F \) and \( G \) are differentiable by \( \psi \) and \( \pi \), and \( F \) is a \( f \)-form. The Poisson bracket proposed in Ref. [10], denoted by \( \{ F, G \}_F \), is given by \( \{ F, G \}_F = -\{ G, F \} \). If \( F \), \( G \) and \( H \) are \( f \)-form, \( g \)-form and \( h \)-form respectively and differentiable by \( \psi \) and \( \pi \),
\[
\{ G, F \} = -(-1)^{(f+d+1)(g+d+1)} \{ F, G \},
\] (2.9)
\[
\{ F, G \wedge H \} = \{ F, G \} \wedge H + (-1)^{(f+d+1)g} G \wedge \{ F, H \},
\] (2.10)
and
\[
(-1)^{(f+d+1)(h+d+1)} \{ F, \{ G, H \} \} + (-1)^{(g+d+1)(f+d+1)} \{ G, \{ H, F \} \} + (-1)^{(h+d+1)(g+d+1)} \{ H, \{ F, G \} \} = 0
\] (2.11)
hold. The canonical equations can be written as
\[
d\psi = -\{ H, \psi \}, \quad d\pi = -\{ H, \pi \}.
\] (2.12)
The fundamental brackets are
\[
\{ \psi, \pi \} = (-1)^{pd}, \quad \{ \pi, \psi \} = -1, \quad \{ \psi, \psi \} = 0 = \{ \pi, \pi \}.
\] (2.13)
If a form \( F \) is differentiable by \( \psi \) and \( \pi \), and \( F \) does not depend positively on space-time points,
\[
dF = d\psi \wedge \frac{\partial F}{\partial \psi} + d\pi \wedge \frac{\partial F}{\partial \pi}
\]
\[
= (-1)^{(p+1)q} \frac{\partial H}{\partial \pi} \wedge \frac{\partial F}{\partial \psi} - (-1)^p \frac{\partial H}{\partial \psi} \wedge \frac{\partial F}{\partial \pi}
\]
\[
= -\{ H, F \}
\] (2.14)
3 Generators of local gauge transformations

Let us consider that an infinitesimal transformation of dynamical fields $\psi^A$ and its conjugate forms $\pi_A$:

$$\psi^A \rightarrow \psi^A + \delta \psi^A, \quad \pi_A \rightarrow \pi_A + \delta \pi_A.$$  \hspace{1cm} (3.1)

Here, $A$ is the label of the fields. If there exists $(d - 1)$-form $G$ such that

$$\delta \psi^A = \{\psi^A, G\}, \quad \delta \pi_A = \{\pi_A, G\},$$  \hspace{1cm} (3.2)

we call $G$ the generator of the transformation \cite{10}. If a form $F$ is differentiable by $\psi^A$ and $\pi_A$, the transformation of $F$ is given by

$$\delta F = \{F, G\}.$$  \hspace{1cm} (3.3)

In this section, we find the generators of the gauge transformations for matter fields \Sect{3.1}, gauge fields \Sect{3.2} and the gravitational field within the second order formalism \Sect{3.3}.

3.1 Matter fields

Let us consider that an infinitesimal global gauge transformation of matter fields:

$$\delta \psi^A = \varepsilon^r (G_r)^A_B \psi^B, \quad \delta L_0 = 0.$$  \hspace{1cm} (3.4)

Here, $\varepsilon^r$ are infinitesimal parameters, $G_r$ are representations of the generators of a linear Lie group $G$ and $L_0(\psi^A, d\psi^A)$ is the Lagrangian form of the matter fields $\psi^A$. The matrices $G_r$ satisfy

$$[G_r, G_s] = f_t^{rs} G_t,$$  \hspace{1cm} (3.5)

where $[A, B] := AB - BA$ and $f_t^{rs}$ are the structure constants of $G$. Under the transformation \eqref{3.4}, the conjugate forms $\pi_A$ behave as

$$\delta \pi_A = -\varepsilon^r (G_r)^B_A \pi_B.$$  \hspace{1cm} (3.6)

The Noether currents \eqref{A.4} are given by

$$G_r^{(0)} := (G_r)^A_B \psi^B \wedge \pi_A.$$  \hspace{1cm} (3.7)

The Noether currents $G_r^{(0)}$ satisfy

$$\{\psi^A, G_r^{(0)}\} = (G_r)^A_B \psi^B,$$  \hspace{1cm} (3.8)

$$\{\pi_A, G_r^{(0)}\} = -(G_r)^B_A \pi_B,$$  \hspace{1cm} (3.9)

$$\{G_r^{(0)}, G_s^{(0)}\} = f_t^{rs} G_t^{(0)}.$$  \hspace{1cm} (3.10)

The generator of the transformation \eqref{3.4} is given by $\varepsilon^r G_r^{(0)}$.

To generalize \eqref{3.4} to the local gauge transformation, $L_0(\psi^A, d\psi^A)$ should be replaced by $L_0(\psi^A, (D\psi)^A)$ where $(D\psi)^A := d\psi^A + A'(G_r)^A_B \wedge \psi^B$ and $A'$ are the gauge fields. The forms $\psi^A$ and $\pi_A$ are independent from $A'$ and $\pi_r$ where $\pi_r$ is the conjugate forms of $A'$.

4
3.2 Gauge fields

Let us consider that the infinitesimal local gauge transformation of the gauge fields:

\[ \delta A^r = \varepsilon^s f_{sr}^t A^t - d\varepsilon^r, \quad \delta L_1 = 0. \]  

(3.11)

Here, \( L_1 \) is the Lagrangian form of the gauge fields. Under the transformation (3.11), \( \pi_r \) behave as

\[ \delta \pi_r = -\varepsilon^s f_{sr}^t \pi_t. \]  

(3.12)

The Noether currents \((\ref{A.4})\) are given by

\[ G^{(1)}_s := f_{sr}^t A^t \wedge \pi_r. \]  

(3.13)

The Noether currents \( G^{(1)}_r \) satisfy

\[ \{ A^s, G^{(1)}_r \} = f_{rt}^s A^t, \]  

(3.14)

\[ \{ \pi_s, G^{(1)}_r \} = -f_{rs}^t \pi_t, \]  

(3.15)

\[ \{ G^{(1)}_r, G^{(1)}_s \} = f_{rs}^t G^{(1)}_t. \]  

(3.16)

We put \( G_r := G^{(0)}_r + G^{(1)}_r \). The Noether currents \( G_r \) satisfy

\[ \{ G_r, G_s \} = f_{rs}^t G_t. \]  

(3.17)

The generator of the transformation without \( d\varepsilon^r \) is given by \( \varepsilon^r G_r \). We assume that the generator of the local gauge transformation (denoted by \( G \)) is given by

\[ G = \varepsilon^r G_r + d\varepsilon^r \wedge F_r, \]  

(3.18)

where \( F_r \) are unknown \((d - 2)\)-forms described by only \( \pi_r \). Because

\[ \{ A^s, d\varepsilon^r \wedge F_r \} = d\varepsilon^r \wedge \frac{\partial F_r}{\partial \pi_s} \]  

(3.19)

holds, we obtain

\[ F_r = -\pi_r. \]  

(3.20)

Then, \( G \) is given by

\[ G = \varepsilon^r G_r - d\varepsilon^r \wedge \pi_r. \]  

(3.21)

\( F_r \) does not affect to \( \psi^A, \pi_A \) and \( \pi_r \):

\[ \delta \psi^A = \{ \psi^A, G \} = \varepsilon^r (G_r)_B^A \psi^B, \]  

(3.22)

\[ \delta A^r = \{ A^r, G \} = \varepsilon^s f_{sr}^t A^t - d\varepsilon^r, \]  

(3.23)

\[ \delta \pi_A = \{ \pi_A, G \} = -\varepsilon^r (G_r)_B^A \pi_B, \]  

(3.24)

\[ \delta \pi_r = \{ \pi_r, G \} = -\varepsilon^s f_{sr}^t \pi_t. \]  

(3.25)
3.3 Gravitational field

3.3.1 Notation

We explain the notations used in this paper. Let \( g \) be the metric of which has signature \((-+\cdots+)\), and let \( \{\theta^a\}_{a=0}^{d-1} \) denote an orthonormal frame (vielbein). We have \( g = \hat{g}_{ab} \theta^a \otimes \theta^b \) with \( \hat{g}_{ab} := \text{diag}(-1,1,\cdots,1) \). All indices are lowered and raised with \( \hat{g}_{ab} \) or its inverse \( \hat{g}^{ab} \). The first structure equation is

\[
d \theta^a + \omega^a_b \wedge \theta^b = \Theta^a, \tag{3.26}
\]

where \( \omega^a_b \) is the connection form and \( \Theta^a \) is the torsion 2-form. In the following of this paper, we suppose \( \omega_{ba} = -\omega_{ab} \). We put

\[
\eta^a = *\theta^a, \quad \eta^{ab} = *(\theta^a \wedge \theta^b), \quad \eta^{abc} = *(\theta^a \wedge \theta^b \wedge \theta^c), \quad \eta^{abcd} = *(\theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d). \tag{3.27}
\]

In Appendix D, several identities about \( \theta^a \wedge \eta_1^1 \cdots a^r \) \((r = 1, 2, 3, 4)\), \( \delta \eta_1^1 \cdots a^r \) \((r = 0, 1, 2, 3)\) and \( d \eta_1 \cdots a^r \) \((r = 1, 2, 3)\) are listed.

3.3.2 Generators of local Lorentz transformation

Let us consider that an infinitesimal local Lorentz transformation

\[
\delta \theta^a = \varepsilon^a_b \theta^b. \tag{3.28}
\]

Here, \( \varepsilon^{ab} \) are infinitesimal parameters which satisfy \( \varepsilon^{ab} = -\varepsilon^{ba} \). Under the transformation, \( \omega^{ab} \) behave as

\[
\delta \omega^{ab} = \varepsilon^a_c \omega^{cb} + \varepsilon^b_c \omega^{ac} - d \varepsilon^{ab}. \tag{3.29}
\]

Using (C.10), the conjugate form of \( \theta^a \) is given by

\[
\pi_a = \frac{1}{2\kappa} \omega^{bc} \wedge \eta_{abc}. \tag{3.30}
\]

Here, \( \kappa \) is the Einstein constant. \( \pi_a \) behave as

\[
\delta \pi_a = \frac{1}{2\kappa} \delta \omega^{bc} \wedge \eta_{abc} + \frac{1}{2\kappa} \omega^{bc} \wedge \delta \theta^d \wedge \eta_{abcd}
= \varepsilon^{bc}(-\hat{g}_{alc} \pi_b) - d \varepsilon^{bc} \wedge \frac{1}{2\kappa} \eta_{abc}. \tag{3.31}
\]

Here, we used (D.3) in the first line and (D.1) in the second line. The Noether currents (A.4) are given by

\[
G_{cd} := 2\theta_{[d} \wedge \pi_{c]} = \theta_d \wedge \pi_c - \theta_c \wedge \pi_d. \tag{3.32}
\]

The Noether currents \( G_{cd} \) satisfy

\[
\{\theta^a, G_{cd}\} = 2 \theta_{[d} \delta^a_{c]}, \tag{3.33}
\]

\[
\{\pi_a, G_{cd}\} = -2 \hat{g}_{a[d} \pi_{c]}, \tag{3.34}
\]

\[
\{G_{ab}, G_{cd}\} = \hat{g}_{bc} G_{ad} - \hat{g}_{ac} G_{bd} + \hat{g}_{ad} G_{bc} - \hat{g}_{bd} G_{ac}. \tag{3.35}
\]
corresponds to the commutation relations of the generators of the Lorentz group:

\[ [G_{ab}, G_{cd}] = \delta_{bc} G_{ad} - \delta_{ac} G_{bd} + \delta_{ad} G_{bc} - \delta_{bd} G_{ac}. \] \hspace{1cm} (3.36)

The generator of the transformation without \( d\varepsilon^{ab} \) is given by \( \frac{1}{2} \varepsilon^{ab} G_{ab} \). We assume that the generator of the local Lorentz transformation (3.28) (denoted by \( G \)) is given by

\[ G = \frac{1}{2} \varepsilon^{ab} G_{ab} + \frac{1}{2} d\varepsilon^{ab} \wedge F_{ab}, \] \hspace{1cm} (3.37)

where \( F_{ab} \) are \((d-2)\)-forms described by only \( \theta^a \). Because

\[ \{ \pi_a, \frac{1}{2} d\varepsilon^{bc} \wedge F_{bc} \} = \frac{1}{2} d\varepsilon^{bc} \wedge \frac{\partial F_{bc}}{\partial \theta^a} \] \hspace{1cm} (3.38)

holds and the right hand side of the above equation should be \( -d\varepsilon^{bc} \wedge \frac{1}{2\kappa} \eta_{abc} \), we obtain

\[ F_{bc} = -\frac{1}{\kappa} \eta_{bc}. \] \hspace{1cm} (3.39)

We used (D.3). Then, \( G \) is given by

\[ G = \frac{1}{2} \varepsilon^{ab} G_{ab} - \frac{1}{2\kappa} d\varepsilon^{ab} \wedge \eta_{ab}. \] \hspace{1cm} (3.40)

\( F_{ab} \) does not affect to \( \theta^a \):

\[ \delta \theta^a = \{ \theta^a, G \} = \varepsilon^a_{\ b} \theta^b, \] \hspace{1cm} (3.41)

\[ \delta \pi_a = \{ \pi_a, G \} = -\varepsilon^b_{\ a} \pi_b - d\varepsilon^{bc} \wedge \frac{1}{2\kappa} \eta_{abc}. \] \hspace{1cm} (3.42)

### 3.4 The relation between \( F_r \) and \( G_r \)

According to Ref. [10], if a generator is given by \( G = \varepsilon^r G_r + d\varepsilon^r \wedge F_r \) with nonzero \( F_r \),

\[ G_r = -\{ F_r, H_r \} \] \hspace{1cm} (3.43)

holds. We check this relationship for the gauge fields and the gravitational field. For the gauge fields,

\[ -\{ F_r, H_1 \} = \{ \pi_r, H_1 \} \]

\[ = -\frac{\partial H_1}{\partial A^r} \]

\[ = f^c_{\ r}\ A^b \wedge \pi_c \]

\[ = G_r^{(1)} \] \hspace{1cm} (3.44)
holds. Here, $H_1$ is the Hamilton form of the gauge fields (B.8) and we used (B.9) in the third line. For the gravitational field,\[ -\{ F_{ab}, H_G \} = \frac{1}{\kappa} \{ \eta_{ab}, H_G \} \]
\[ = -\frac{1}{\kappa} \frac{\partial \eta_{ab}}{\partial \theta^c} \wedge \frac{\partial H_G}{\partial \pi_c} \]
\[ = -\frac{1}{\kappa} \eta_{abc} \wedge (-\omega^d_c \wedge \theta^d) \]
\[ = \frac{1}{\kappa} \omega^c_d \wedge \theta^d \wedge \eta_{abc} \] (3.45)

holds. In the third line, we used (C.27) and (D.3). Because\[ \kappa G_{ab} = -\omega^{cd} \wedge \theta_{[b} \wedge \eta_{a]}cd = -2\omega^{[c}_{[b} \wedge \eta_{a]c}, \] (3.46)\[ -\kappa \{ F_{ab}, H_G \} = \omega^c_d \wedge \theta^d \wedge \eta_{abc} = -2\omega^{[c}_{[b} \wedge \eta_{a]c} \] (3.47)
hold using (D.1), we have\[ G_{ab} = -\{ F_{ab}, H_G \}. \] (3.48)

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## A Noether currents

We explain the Noether currents. For an infinitesimal transformation of $p$-form dynamical fields $\psi^A \to \psi^A + \delta \psi^A$, an identical equation
\[ \delta L \equiv \delta \psi^A \wedge \left( \frac{\partial L}{\partial \psi^A} - (-1)^p d \frac{\partial L}{\partial d \psi^A} \right) + d \left( \delta \psi^A \wedge \frac{\partial L}{\partial d \psi^A} \right) \] (A.1)

holds. Here, $\equiv$ denotes identical equation which holds without using the Euler-Lagrange equations. For a global transformation\[ \delta \psi^A = \varepsilon^r \Delta_r^A, \quad \delta L = \varepsilon^r dl_r, \] (A.2)
we have\[ dl_r \equiv \Delta_r^A \wedge [L]_A + d \left( \Delta_r^A \wedge \frac{\partial L}{\partial d \psi^A} \right) \] (A.3)

from (A.1). Here, $[L]_A := \partial L / \partial \psi^A - (-1)^p d(\partial L / \partial d \psi^A)$. Under the Euler-Lagrange equations $[L]_A = 0$, the Noether currents\[ N_r := \Delta_r^A \wedge \frac{\partial L}{\partial d \psi^A} - l_r \] (A.4)
are conserved: $dN_r = 0$. 

8
B Covariant canonical formalism of gauge fields

In this section, we review that the covariant canonical formalism of gauge fields.

The curvatures of the gauge fields are defined by

\[ F^r := dA^r + \frac{1}{2} f^{r}_{bc} A^b \wedge A^c. \]  \hfill (B.1)

We put \( F^r := \kappa_{rs} F^s \) with the Killing form \( \kappa_{rs} := -f^a_{rb} f^b_{sa} = \kappa_{sr} \). The Lagrangian form of the gauge fields \( L_1 \) is given by

\[ L_1 = -\frac{1}{2k} F^r \wedge \ast F^r, \]  \hfill (B.2)

where \( k \) is a positive constant. \( L_1 \) is gauge invariant.

We consider the Euler-Lagrange equation of the gauge fields. The derivatives of the total Lagrangian form \( L := L_0 (\psi^A, (D\psi)^A) + L_1 \) are given by

\[ \frac{\partial L}{\partial A^a} = -\frac{1}{k} f^c_{ab} A^b \wedge \ast F^c + J_a, \quad \frac{\partial L}{\partial dA^a} = -\frac{1}{k} \ast F^a \]  \hfill (B.3)

with

\[ J_a := \frac{\partial L_0 (\psi^A, (D\psi)^A)}{\partial A^a}. \]  \hfill (B.4)

The Euler-Lagrange equation \( \partial L/\partial A^a + d(\partial L/\partial dA^a) = 0 \) is given by

\[ D \ast F^a := d \ast F^a + f^c_{ab} A^b \wedge \ast F^c = kJ_a. \]  \hfill (B.5)

\( D \ast F^a \) is the covariant derivative. The above equation is the Yang-Mills-Utiyama equation.

The conjugate form of \( A^a \) is given by

\[ \pi_a = -\frac{1}{k} \ast F^a. \]  \hfill (B.6)

The Hamilton form is given by

\[ H = H_1 - L_0 (\psi^A, (D\psi)^A), \]  \hfill (B.7)

\[ H_1 := -\frac{1}{2} f^a_{bc} A^b \wedge A^c \wedge \pi_a + \frac{k}{2} \pi_a \wedge \ast \pi^a, \]  \hfill (B.8)

with \( \pi^a := (\kappa^{-1})^{ab} \pi_b = -\frac{1}{k} \ast F^a \). The derivatives of \( H \) are given by

\[ \frac{\partial H}{\partial A^a} = -f^c_{ab} A^b \wedge \pi_c - J_a, \]  \hfill (B.9)

\[ \frac{\partial H}{\partial \pi_a} = k \ast \pi^a - \frac{1}{2} f^a_{bc} A^b \wedge A^c. \]  \hfill (B.10)
The canonical equations \( dA^a = \frac{\partial H}{\partial \pi_a} \) and \( d\pi_a = \frac{\partial H}{\partial A^a} \) become

\[
\begin{align*}
  dA^a &= k \ast \pi^a - \frac{1}{2} f_{bc}^a A^b \wedge A^c, \\
  d\pi_a &= -f_{cb}^a A^b \wedge \pi_c - J_a.
\end{align*}
\]  

(B.11)

(B.12)

The above two equations can be rewritten as

\[
\begin{align*}
  \mathcal{F}^a &= k \ast \pi^a, \\
  D\pi_a &:= d\pi_a + f_{cb}^a A^b \wedge \pi_c = -J_a.
\end{align*}
\]  

(B.13)

The former is equivalent with the definition of \( \pi_a \). The latter is equivalent with the Yang-Mills-Utiyama equation. The covariant canonical formalism does not need the gauge fixing.

C Second order formalism of gravity for \( d \geq 3 \)

In this section, we apply the CCF to the second order formalism of gravity with Dirac fields for the arbitrary dimension \( (d \geq 3) \).

C.1 Notation

The curvature 2-form \( \Omega^a_b \) is given by

\[
\Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b.
\]  

(C.1)

Expanding the curvature form as

\[
\Omega^a_b = \frac{1}{2} R^a_{bcd} \theta^c \wedge \theta^d,
\]  

(C.2)

we define

\[
R_{ab} := R^c_{acb}, \quad R := R^a_a.
\]  

(C.3)

Because of (D.2), \( R\eta \) can be written as

\[
R\eta = \Omega^{ab} \wedge \eta_{ab}.
\]  

(C.4)

C.2 Lagrange formalism

The Lagrangian form of the gravity in the second order formalism is given by

\[
L(\theta, d\theta) = L_G(\theta, d\theta) + L_{\text{mat}}(\theta, d\theta).
\]  

(C.5)

Here, \( L_G \) is the Lagrangian form for the pure gravity given by

\[
L_G(\theta, d\theta) = \frac{1}{2\kappa} N', \quad N' := R\eta - d(\omega^{ab} \wedge \eta_{ab}),
\]  

(C.6)
and $L_{\text{mat}}(\theta, d\theta) = L_{\text{mat}}(\theta, \omega(\theta, d\theta))$ is the Lagrangian form of “matters” which are scalar fields, Dirac fields and gauge fields. Here, $\kappa$ is the Einstein constant. Only the Dirac fields couple to $\omega^{ab}$.

We derive the Euler-Lagrange equation of the gravity. The variation of $L$ is given by

$$
\delta L(\theta, d\theta) = \delta \theta^c \wedge \left( \frac{1}{2\kappa} [\Omega^{ab} \wedge \eta_{abc} - d(\omega^{ab} \wedge \eta_{abc})] + T_c \right) + \delta d\theta^c \wedge \frac{1}{2\kappa} \omega^{ab} \wedge \eta_{abc} + \delta \omega^{ab}(\theta, d\theta) \wedge \left( \frac{1}{2\kappa} [d\eta_{ab} - \omega^c_a \wedge \eta_{cb} - \omega^c_b \wedge \eta_{ac}] + \frac{\partial L_{\text{mat}}}{\partial \omega^{ab}} \right),
$$

where

$$T_a := \frac{\partial L_{\text{mat}}(\theta, \omega)}{\partial \theta^a}$$

is the energy-momentum form. We suppose that

$$\frac{1}{2\kappa} [d\eta_{ab} - \omega^c_a \wedge \eta_{cb} - \omega^c_b \wedge \eta_{ac}] + \frac{\partial L_{\text{mat}}}{\partial \omega^{ab}} \equiv 0,$$

which is the same as the Euler-Lagrange equation for the connection of the first order formalism.

Under this supposition, (C.7) leads to

$$\frac{\partial L}{\partial \theta^c} = \frac{1}{2\kappa} [\Omega^{ab} \wedge \eta_{abc} - d(\omega^{ab} \wedge \eta_{abc})] + T_c, \quad \frac{\partial L}{\partial d\theta^c} = \frac{1}{2\kappa} \omega^{ab} \wedge \eta_{abc}.$$

The Euler-Lagrange equation $\partial L/\partial \theta^c + d(\partial L/\partial d\theta^c) = 0$ becomes the Einstein equation

$$-\frac{1}{2\kappa} \Omega^{ab} \wedge \eta_{abc} = T_c.$$

If we expand $T_c$ as $T_c = T^{b}_c \eta_b$, the above equation leads to

$$R^a_b - \frac{1}{2} R \delta^a_b = \kappa T^a_b.$$

### C.3 Covariant canonical formalism

Next, we consider the covariant canonical formalism.

In (C.6), $N'$ can be rewritten as

$$N' = \omega^a_c \wedge \omega^{cb} \wedge \eta_{ab} + \omega^{ab} \wedge d\eta_{ab} = N + \Theta^a \wedge \omega^{bc} \wedge \eta_{abc}$$

(C.13)

$\delta N' = \delta(\omega^a_c \wedge \omega^{cb} \wedge \eta_{ab} + \omega^{ab} \wedge d\eta_{ab})$ and

$$\omega^{ab} \wedge \delta d\eta_{ab} = \delta d\theta^c \wedge \omega^{ab} \wedge \eta_{abc} + \delta \theta^c \wedge \omega^{ab} \wedge d\eta_{abc}$$

hold. Here, we used the following formulas:

$$d\eta_{ab} = d\theta^c \wedge \eta_{abc}, \quad \delta \eta_{abc} = \delta \theta^d \wedge \eta_{abcd}, \quad d\eta_{abc} = d\theta^d \wedge \eta_{abcd}.$$

$\delta N'$ is not an Euler-Lagrange equation. $\omega^{ab}$ is determined by (C.9). The method to obtain (C.10) in this subsection is also called the 1.5 order formalism.
with
\[ N := \omega_a \wedge \omega^b \wedge \eta_{ba}. \]  
(C.14)

In the second line of (C.13), we used (D.4). Using (3.26) and (D.1), \( N \) can be rewritten as
\[ N = d\theta^a \wedge \frac{1}{2} \omega^{bc} \wedge \eta_{abc} - \Theta^a \wedge \frac{1}{2} \omega^{bc} \wedge \eta_{abc}. \]  
(C.15)

The conjugate form of \( \theta^a \) is given by (3.30). The Hamilton form is given by
\[ H(\theta, \pi) = d\theta^a \wedge \pi_a - L = H_G(\theta, \pi) - L_{\text{mat}}(\theta, \pi) \]  
(C.16)

with
\[ H_G(\theta, \pi) := \frac{N}{2\kappa}. \]  
(C.17)

Here, we used (C.13) and (C.15). Because \( C_{abc} \) is described by the Dirac fields, it is independent from \( \theta^a \) and \( \pi_a \). Then, \( \Theta^a = \frac{1}{2} C_{bc} \theta^b \wedge \theta^c \) is independent from \( \pi_a \).

The canonical equations are given by
\[ d\theta^a = \frac{\partial H_G}{\partial \pi_a} - \frac{\partial L_{\text{mat}}}{\partial \pi_a}, \]  
(C.18)
\[ d\pi_a = \frac{\partial H_G}{\partial \theta^a} - \frac{\partial L_{\text{mat}}(\theta, \pi)}{\partial \theta^a}. \]  
(C.19)

In the right hand side of (C.18), the second term can be rewritten as
\[ - \frac{\partial L_{\text{mat}}}{\partial \pi_a} = \Theta^a \]  
(C.20)
using (C.9). Then, (C.18) becomes
\[ d\theta^a = \frac{\partial H_G}{\partial \pi_a} + \Theta^a. \]  
(C.21)

To calculate \( \partial H_G / \partial \pi_a, \partial H_G / \partial \theta^a \), we represent \( N \) by \( \theta^a \) and \( \pi_a \). Using (D.2), \( N \) can be rewritten as
\[ N = (\omega_{abc} \omega^{bca} + \omega_{a} \omega^{a}) \eta. \]  
(C.22)

Here, we expand \( \omega_{ab} \) as \( \omega_{ab} = \omega_{abc} \theta^c \) and put \( \omega_a := \omega_{ab}^b \). We can represent \( \omega_{abc} \) by \( \theta^a \) and \( \pi_a \) as
\[ \omega_{abc} = \kappa \left[ v_{c,ab} + \frac{1}{d-2} (\partial_a v_b - \partial_b v_a) \right], \]  
(C.23)
\[ v_{c,ab} := - \ast V_{c,ab}, \quad V_{c,ab} := \pi_c \wedge \theta_a \wedge \theta_b \]  
(C.24)
with \( v_a := v_{a \mu}^b. \)
Next, we derive the canonical equations. For an arbitrary $d$-form $\xi$, 
\[
\delta v_{c,ab} \xi = -\delta v_{c,ab} \xi * \eta = -\delta v_{c,ab} \eta \star \xi = (-\delta [v_{c,ab} \eta] + v_{c,ab} \delta \eta) \star \xi = (-\delta V_{c,ab} + v_{c,ab} \delta \eta) \star \xi 
\]
(C.25)
holds [6]. Then, we have
\[
\delta v_{c,ab} \eta = \delta V_{c,ab} - v_{c,ab} \delta \eta. 
\]
(C.26) Using this, we have
\[
\frac{\partial H_G}{\partial \pi_c} = -\omega_a \wedge \theta^a, 
\]
(C.27) \[
\frac{\partial H_G}{\partial \theta^d} = \frac{1}{2\kappa} (\omega_d \wedge \omega^{ab} \wedge \eta_{abc} + \omega_c \wedge \omega^{bc} \wedge \eta_{bad}). 
\]
(C.28) Substituting (C.27) into (C.18), we have
\[
d\theta^a = -\omega_b \wedge \theta^b + \Theta^a, 
\]
(C.29) which is equivalent to the first structure equation (3.26). By the way, using (C.9), we can show that
\[
-\frac{\partial L_{\text{mat}}(\theta, \pi)}{\partial \theta^c} = -T_c - \frac{1}{2\kappa} \omega^{ab} \wedge \Theta^d \wedge \eta_{abcd}. 
\]
(C.30) Substituting (C.28) and (C.30) into (C.19), we have
\[
d\pi_c = \frac{1}{2\kappa} (\omega^b \wedge \omega^{ab} \wedge \eta_{abc} + \omega_c \wedge \omega^{bc} \wedge \eta_{bad}) - T_c. 
\]
(C.31)

## D Formulas

Several useful formulas are listed. For $\theta^a \wedge \eta_{a_1\ldots a_r} (r = 1, 2, 3, 4)$,
\[
\theta^b \wedge \eta_{a_1\ldots a_r} = (-1)^{r-1} R^b_{a_1\ldots a_r}, 
\]
(D.1) hold [15]. Using this, we have
\[
\theta^a \wedge \theta^b \wedge \eta_{cd} = (\delta^a_c \delta^b_d - \delta^a_d \delta^b_c) \eta. 
\]
(D.2) For $\delta \eta_{a_1\ldots a_r} (r = 0, 1, 2, 3)$,
\[
\delta \eta_{a_1\ldots a_r} = \delta \theta^b \wedge \eta_{a_1\ldots a_r} 
\]
(D.3) hold. For $d\eta_{a_1\ldots a_r} (r = 1, 2, 3)$,
\[
d\eta_{a_1\ldots a_r} = (r + 1) \omega^b_{[a_1} \wedge \eta_{a_2\ldots a_r]} + \Theta^b \wedge \eta_{a_1\ldots a_r} 
\]
(D.4) hold. $\eta_{a_1\ldots a_r} (r = 1, 2, 3, 4)$ can be written as [15]
\[
\eta_{a_1\ldots a_r} = e_{a_r} \lbrack \eta_{a_1\ldots a_{r-1}}. 
\]
(D.5) Here, $\lbrack$ is the interior product and $\{e_a\}$ is the dual basis of $\{\theta^a\}$ ($e_a \star \theta^b = \delta^b_a$).
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