ON A CONJECTURE OF BERNDT AND KIM

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Abstract. We prove a recent conjecture of Berndt and Kim regarding the positivity of the coefficients in the asymptotic expansion of a class of partial theta functions. This generalizes results found in Ramanujan’s second notebook, and recent work of Galway and Stanley.

1. Introduction

In his second notebook [1], page 324, Ramanujan claims an asymptotic expansion for the partial theta function

\[ 2 \sum_{n=0}^{\infty} (-1)^n q^{n^2 + n} \sim 1 + t + t^2 + 2t^3 + 5t^4 + \ldots \]

with \( q = \frac{1-t}{1+t} \) as \( t \to 0^+ \). It is not clear, just given the left-hand side of (1.1), that the coefficients of its asymptotic expansion (in \( t \)) are always positive integers, nor that one should expect this. Galway [3] proved this curious fact to be true using alternating permutations and relations to Euler numbers. Stanley [5], answering a question of Galway, then gave a nice combinatorial interpretation of the coefficients in the asymptotic expansion (1.1) as the number of fixed-point-free alternating involutions in the symmetric group \( S_{2n} \), providing a second proof that the coefficients are positive integers.

Berndt and Kim [2] study more general partial theta functions.

\[ f_b(t) := 2 \sum_{n=0}^{\infty} (-1)^n q^{n^2 + bn}, \]

where \( b \) is real. They show that, similarly to (1.1), \( f_b(t) \) admits an asymptotic expansion of the form

\[ f_b(t) \sim \sum_{n=0}^{\infty} a_n t^n, \]

where the \( a_n \) are given explicitly in terms of Euler numbers and Hermite polynomials [2 Theorem 1.1]. For the purposes of this paper, we do not require the explicit shape of the \( a_n \), so we do not state it here. (We point out a small typo in [2] equation (1.3): the exponent of \( (1-t)/(1+t) \) should read \( (1-2b)/4 \) rather than \( (2b-1)/4 \), as the authors correctly state in [2] equation (2.9).)

In analogy to the results established by Galway and Stanley pertaining to the coefficients of (1.1), Berndt and Kim prove that the coefficients \( a_n \) of the generalized partial theta functions defined in (1.2) are integers if \( b \in \mathbb{N} \) [2 Theorem 2.5], and make the following conjecture regarding their positivity.
Conjecture (Berndt-Kim [2]). For any positive integer $b$, for sufficiently large $n$, the coefficients $a_n$ in the asymptotic expansion (1.3), have the same sign.

The purpose of this note is to prove this conjecture of Berndt and Kim.

**Theorem 1.** The Berndt-Kim conjecture is true. More precisely, if $b \equiv 1, 2 \pmod{4}$, then $a_n > 0$ for $n \gg 0$, and if $b \equiv 0, 3 \pmod{4}$, then $a_n < 0$ for $n \gg 0$.

2. **Proof of Theorem 1**

To prove the theorem, we first observe that for integers $b \geq 2$,
\begin{equation}
    f_{b+1}(t) = -q^{-b} (f_{b-1}(q) - 2).
\end{equation}

We proceed by induction on $b$ to prove Theorem 1. The case $b = 1$ follows from [3] as mentioned in §1. To prove the case $b = 2$ we employ the fact that the $q$-series in this case is essentially a modular form. To be more precise, we have that
\[ f_2(t) = -q^{-1} \left( g \left( \frac{i\theta}{2\pi} \right) - 1 \right), \]
where we have adopted the notation $\theta = \log \left( \frac{1 + t}{1 - t} \right)$ from [2]. The function $g(\tau)$, where $\tau \in \mathbb{H}$, is the modular form defined by
\[ g(\tau) := \sum_{n \in \mathbb{Z}} (-1)^n e^{2\pi in^{2}\tau} = \frac{\eta(\tau)^2}{\eta(2\tau)^2}, \]
where $\eta(\tau) := e^{\frac{2\pi i\tau}{24}} \prod_{n \geq 1} (1 - e^{2\pi in\tau})$ is Dedekind’s eta-function, a well known modular form of weight $1/2$. Employing the modular transformation of $\eta$ (see [4] e.g.)
\[ \eta \left( -\frac{1}{\tau} \right) = \sqrt{-i\tau} \eta(\tau) \]
we obtain that
\[ g \left( \frac{i\theta}{2\pi} \right) \to 0 \]
as $\theta \to 0^+$, and thus as $t \to 0^+$. Therefore
\[ f_2(t) \sim q^{-1} = \frac{1 + t}{1 - t}, \]
which clearly has positive coefficients in its $t$-expansion.

We now assume Theorem 1 holds for some $b - 1 \geq 1$, and prove that it also holds for $b + 1$. (The first two inductive cases $b - 1 = 1, 2$ are proven above.) We use (2.1) and split
\[ q^{-b} = (1 + t)^b (1 - t)^{-b}. \]
Since the $t$-expansion of $(1 + t)^b$ is finite and contains only positive coefficients, it suffices to prove that the $t$-coefficients in the asymptotic expansion of
\[ -(1 - t)^{-b} (f_{b-1}(q) - 2) \]
eventually all have the same sign. We will address the fact that the sign is dependent on the residue class of $b \pmod{4}$ as stated in Theorem 1 later.
It is easy to show that

$$(1 - t)^{-b} = \sum_{j=0}^{\infty} \binom{b-1 + j}{b-1} t^j.$$  

By induction, we may assume without loss of generality that

$$f_{b-1}(q) - 2 \sim \sum_{n \geq 0} \alpha_n t^n,$$

as $t \to 0^+$, where $\alpha_n > 0$ for $n \geq L$, for some $L \geq 0$. Set

$$\beta_n := \binom{b-1 + n}{b-1}.$$  

It suffices to show that

$$(2.2) \sum_{0 \leq n \leq m} \alpha_n \beta_{m-n}$$

is positive for $m \gg 0$. We break the sum (2.2) into two parts

$$\Sigma_1 := \sum_{0 \leq n \leq L-1} \alpha_n \beta_{m-n}, \quad \Sigma_2 := \sum_{L \leq n \leq m} \alpha_n \beta_{m-n},$$

where we assume $m \gg 0$ is sufficiently large to ensure $m \geq L$.

Since $L$ is independent of $m$ and the $\beta_j$’s are monotonically decreasing (in $j$, for fixed $b$), we may bound

$$\Sigma_1 \ll \beta_m \ll m^{b-1},$$

as $m \to \infty$. Similarly, $\Sigma_2$ may be estimated from below by

$$\Sigma_2 \gg \sum_{L \leq n \leq m} \beta_{m-n} = \sum_{0 \leq n \leq m-L} \beta_n \gg \sum_{1 \leq n \leq m} n^{b-1} \gg \int_0^m x^{b-1} \, dx \gg m^b,$$

as $m \to \infty$. Since clearly $m^b \gg m^{b-1}$, the positivity of $\Sigma_2$ dominates, and we have that the coefficients in the asymptotic $t$-expansion of $f_{b+1}(q)$ have for sufficiently large $m$ the same sign as claimed.

The more precise claim that the coefficients in the asymptotic $t$-expansion for $f_b(q)$ are eventually positive for $b \equiv 1, 2 \pmod{4}$ and eventually negative for $b \equiv 0, 3 \pmod{4}$ follows easily by induction using (2.1), and the previously established facts that the coefficients in the asymptotic $t$-expansion of $f_1(q)$ and $f_2(q)$ are eventually positive.

References

[1] B. Berndt, Ramanujan’s Notebooks, Part V, Springer Verlag, New York, (1998).
[2] B. Berndt and B. Kim, Asymptotic expansions of certain partial theta functions, Proc. Amer. Math. Soc. 139 (2011), 3779-3788.
[3] W. Galway, An asymptotic expansion of Ramanujan, Number Theory (Ottawa, ON, 1996), CRM Proc. Lecture Notes 19, Amer. Math. Soc., Providence, RI, (1999), 107-110.
[4] H. Rademacher, Topics in analytic number theory, Die Grundlehren der math. Wiss., Band 169, Springer-Verlag, Berlin (1973).
[5] R. Stanley, A survey of alternating permutations, Contemp. Math. 531, Amer. Math. Soc. (2010), 165-196.
