On the large sieve with square moduli

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Abstract: We prove an estimate for the large sieve with square moduli which improves a recent result of L. Zhao. Our method uses an idea of D. Wolke and some results from Fourier analysis.

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1 Introduction

Throughout this paper, we reserve the symbols $c_i$ ($i = 1, 2, \ldots$) for absolute positive constants, and the symbol $\varepsilon$ for an arbitrary (small) positive constant. Further, we suppose that $(a_n)$ is a sequence of complex numbers and that $Q, N \geq 1$. We set

$$S(\alpha) := \sum_{n \leq N} a_n e(n\alpha)$$

and

$$Z := \int_0^1 |S(\alpha)|^2 \, d\alpha = \sum_{n \leq N} |a_n|^2.$$

In its modern form, the large sieve is an inequality connecting a discrete and the continuous mean value $Z$ of the trigometrical polynomial $S(\alpha)$, i.e. an inequality of the form

$$\sum_{r=1}^R |S(\alpha_r)|^2 \leq f(N; \alpha_1, \ldots, \alpha_r) Z.$$

One formulation of the large sieve is as follows.

**Theorem 1:** Let $(\alpha_r)_{r \in \mathbb{N}}$ be a sequence of real numbers. Suppose that $0 < \Delta \leq 1/2$ and $R \in \mathbb{N}$. Put

$$K(\Delta) := \max_{\alpha \in R} \sum_{r=1}^R 1,$$

where

$$||\alpha_r - \alpha|| \leq \Delta.$$
where \(|x|\) denotes the distance of a real \(x\) to its closest integer. Then
\[
\sum_{r=1}^{R} |S(\alpha_r)|^2 \leq c_1 K(\Delta)(N + \Delta^{-1})Z.
\]

The above Theorem 1 is an immediate consequence of Theorem 2.11 in [7].

In many applications, the sequence \(\alpha_1, ..., \alpha_R\) consists of Farey fractions.

If \(\alpha_1, ..., \alpha_R\) is the sequence of all fractions \(a/q\) with \(1 \leq a \leq q\), \((a, q) = 1\) and \(q \leq Q\), then the above Theorem 1 implies, on choosing \(\Delta := 1/Q^2\), that
\[
\sum_{q \leq Q} \sum_{a=1 \atop (a, q) = 1}^{q} \left| S\left(\frac{a}{q}\right) \right|^2 \ll (N + Q^2)Z.
\]

This is the classical large sieve inequality of Bombieri [4].

Recently, L. Zhao [10] considered the case when the moduli \(q\) are squares. A careful investigation of the term \(K(\Delta)\) for this situation led him to the estimate
\[
\sum_{q \leq Q} \sum_{a=1 \atop (a, q) = 1}^{q^2} \left| S\left(\frac{a}{q^2}\right) \right|^2 \ll (\log 2Q) \left( Q^3 + (N \sqrt{Q} + \sqrt{NQ^2})N^\varepsilon \right) Z.
\]

In [4] we proved that the middle term \(N \sqrt{Q}\) on the right-hand side of (2) can be replaced by \(N\), which gives an improvement of (2) if \(Q \ll N^{1/3 - \varepsilon}\).

In the present paper we prove

**Theorem 2:** We have
\[
\sum_{q \leq Q} \sum_{a=1 \atop (a, q) = 1}^{q^2} \left| S\left(\frac{a}{q^2}\right) \right|^2 \ll (\log Q)N^\varepsilon \left( Q^3 + N^{5/4} \right) Z.
\]

This bound is sharper than (2) if \(N^{3/8 + \varepsilon} \ll Q \ll N^{1/2 - \varepsilon}\).

To establish Theorem 2, we combine a method of D. Wolke [9] with some standard tools from harmonic analysis, like the Poisson summation formula and bounds for exponential integrals. We also use a bound for quadratic Gauß sums.
2 Counting Farey fractions in short intervals

To prove Theorem 2, we shall use the general large sieve bound given in Theorem 1.

In the sequel, suppose that $Q_0 \geq 1$, and let $\alpha_1, ..., \alpha_R$ be the sequence of Farey fractions $a/q^2$ with $Q_0 \leq q^2 \leq 2Q_0$, $1 \leq a \leq q^2$ and $(a,q) = 1$. Suppose that $\alpha \in \mathbb{R}$ and $0 < \Delta \leq 1/2$. Put

$$I(\alpha) := [\alpha - \Delta, \alpha + \Delta] \quad \text{and} \quad P(\alpha) := \sum_{Q_0 \leq q^2 \leq 2Q_0 \atop (a,q) = 1} 1.$$

Then we have

$$K(\Delta) = \max_{\alpha \in \mathbb{R}} P(\alpha),$$

where $K(\Delta)$ is defined as in (1). Therefore, the proof of Theorem 2 reduces to estimating $P(\alpha)$ for all $\alpha \in \mathbb{R}$ and choosing the parameter $\Delta$ appropriately.

To estimate $P(\alpha)$, we begin with a method of D. Wolke [9]. Let

$$\tau := \frac{1}{\sqrt{\Delta}}$$

Then, by Dirichlet’s approximation theorem, $\alpha$ can be written in the form

$$\alpha = \frac{b}{r} + z, \quad \text{where} \quad r \leq \tau, \ (b,r) = 1, \ |z| \leq \frac{1}{r\tau},$$

Thus, it suffices to estimate $P(b/r + z)$ for all $b, r, z$ satisfying (5).

We further note that we can restrict ourselves to the case when

$$z \geq \Delta.$$

If $|z| < \Delta$, then

$$P(\alpha) \leq P\left(\frac{b}{r} - \Delta\right) + P\left(\frac{b}{r} + \Delta\right).$$

Furthermore, we have

$$\Delta = \frac{1}{r^2} \leq \frac{1}{r\tau}.$$
Therefore this case can be reduced to the case $|z| = \Delta$. Moreover, as $P(\alpha) = P(-\alpha)$, we can choose $z$ positive. So we can assume (6).

Summarizing the above observations, we deduce

Lemma 1: We have

\[
K(\Delta) \leq 2 \max_{r \in \mathbb{N}} \max_{b \in \mathbb{Z}} \max_{\Delta z \leq \sqrt{\Delta}/r} P \left( \frac{b}{r} + z \right).
\]

The next lemma provides a first estimate for $P(b/r + z)$.

Lemma 2: Suppose that the conditions (4), (5) and (6) are satisfied. Suppose further that

\[
\frac{Q_0 \Delta}{z} \leq \delta \leq Q_0.
\]

Then,

\[
P \left( \frac{b}{r} + z \right) \leq c_1 \left( 1 + \frac{1}{\delta} \int_{Q_0 - c_2 \delta/\sqrt{Q_0}}^{Q_0 + c_2 \delta/\sqrt{Q_0}} \sum_{(y - 4\delta)rz \leq m \leq (y + 4\delta)rz \atop m \equiv -bq^2 \mod r \atop m \neq 0} 1 \, dy \right).
\]

Proof: By $\delta \leq Q_0$, we have

\[
P(\alpha) \leq \frac{1}{\delta} \int_{Q_0}^{2Q_0} P(\alpha, y, \delta) \, dy,
\]

where

\[
P(\alpha, y, \delta) := \sum_{y - \delta \leq q^2 \leq y + \delta \atop (a, q) = 1 \atop a/q^2 \in I(\alpha)} 1.
\]
Now, for
\[ y - \delta \leq q^2 \leq y + \delta, \quad (a, q) = 1, \quad \frac{a}{q^2} \in I(\alpha), \]
we have \( q^2(\alpha - \Delta) \leq a \leq q^2(\alpha + \Delta) \) or, by (5) and (6), \((y - \delta)r(z - \Delta) \leq ar - bq^2 \leq (y + \delta)r(z + \Delta)\). If \( ar - bq^2 = 0 \), then \( r = q^2 \) since \((a, q) = 1 = (b, r)\). Hence,
\[
\begin{align*}
(11) \quad P(\alpha, y, \delta) & \leq \nu(y) + \sum_{\sqrt{y-\delta} \leq q \leq \sqrt{y+\delta}} \sum_{\frac{r(z-\Delta)}{m} \leq m \leq \frac{(y+\delta)r(z+\Delta)}{m} \equiv \frac{-bq^2}{r} \text{ mod } r} 1, \\
\end{align*}
\]
where
\[
\nu(y) \begin{cases} 
1, & \text{if } y - \delta \leq r \leq y + \delta, \\
0, & \text{otherwise}. 
\end{cases}
\]
Whenever \( 1 \leq Q_0 \leq y \leq 2Q_0 \) and \( \delta \leq Q_0 \), we have, by Taylor’s formula,
\[
\sqrt{y - c_2 \delta / \sqrt{Q_0}} \leq \sqrt{y - \delta} \leq \sqrt{y + \delta} \leq \sqrt{y + c_2 \delta / \sqrt{Q_0}}
\]
for a suitable positive constant \( c_2 \). Furthermore, by (8), we have \((y - 4\delta)rz \leq (y - \delta)r(z - \Delta) \leq (y + \delta)r(z + \Delta) \leq (y + 4\delta)rz\). Thus, (11) implies
\[
(12) \quad P(\alpha, y, \delta) \leq \nu(y) + \sum_{\sqrt{y - c_2 \delta / \sqrt{Q_0}} \leq q \leq \sqrt{y + c_2 \delta / \sqrt{Q_0}}} \sum_{\frac{(y - 4\delta)rz}{m} \leq m \leq \frac{(y + 4\delta)rz}{m} \equiv \frac{-bq^2}{r} \text{ mod } r} 1.
\]
Combining (10) and (12), we obtain (9). ∎

3 Estimation of \( P(b/r + z) \) - first way

In this section we use some tools from harmonic analysis to establish the following bound.

**Theorem 3:** Suppose that the conditions (4), (5) and (6) are satisfied. Then,
\[
(13) \quad P\left(\frac{b}{r} + z\right) \leq c_4 \Delta^{-\varepsilon} \left( Q_0^{3/2} \Delta + Q_0^{1/2} \Delta r^{-1/2} z^{-1} + \Delta^{-1/4} \right).
\]
To derive Theorem 3 from Lemma 2, we need the following standard results from Fourier analysis.

**Lemma 3:** (Poisson summation formula, [3]) Let \( f(X) \) be a complex-valued function on the real numbers that is piecewise continuous with only finitely many discontinuities and for all real numbers \( a \) satisfies

\[
f(a) = \frac{1}{2} \left( \lim_{x \to a^-} f(x) + \lim_{x \to a^+} f(x) \right).
\]

Moreover, suppose that \( f(x) \ll c_5(1 + |x|)^{-c} \) for some \( c > 1 \). Then,

\[
\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n),
\]

where

\[
\hat{f}(x) := \int_{-\infty}^{\infty} f(y) e(xy)dy,
\]

the Fourier transform of \( f(x) \).

**Lemma 4:** (see [10], for example) For \( x \in \mathbb{R} \setminus \{0\} \) define

\[
\phi(x) := \left( \frac{\sin \pi x}{2x} \right)^2.
\]

Set

\[
\phi(0) := \lim_{x \to 0} \phi(x) = \frac{\pi^2}{4}.
\]

Then \( \phi(x) \geq 1 \) for \( |x| \leq 1/2 \), and the Fourier transform of the function \( \phi(x) \) is

\[
\hat{\phi}(s) = \frac{\pi^2}{4} \max\{1 - |s|, 0\}.
\]

**Lemma 5:** (see Lemma 3.1. in [6]) Let \( F : [a, b] \to \mathbb{R} \) be twice differentiable. Assume that \( |F'(x)| \geq u > 0 \) for all \( x \in [a, b] \). Then,

\[
\left| \int_a^b e^{iF(x)}dx \right| \ll \frac{c_6}{u}.
\]
Lemma 6: (see Lemma 4.3.1. in [2]) Let $F : [a, b] \to \mathbb{R}$ be twice continuously differentiable. Assume that $|F''(x)| \geq u > 0$ for all $x \in [a, b]$. Then,

$$\left| \int_{a}^{b} e^{iF(x)} dx \right| \leq \frac{c_7}{\sqrt{u}}.$$ 

We shall also need the following estimate for quadratic Gauss sums.

Lemma 7: (see page 93 in [6]) Let $c \in \mathbb{C}$, $k, l \in \mathbb{Z}$ with $(k, c) = 1$. Then,

$$\sum_{d=1}^{r} e \left( \frac{kd^2 + ld}{c} \right) \leq \sqrt{2c}.$$ 

Proof of Theorem 3: By Lemma 4, the double sum on the right-hand side of (9) can be estimated by

$$\sum_{\sqrt{y-c2\delta}/\sqrt{Q_0} \leq q \leq \sqrt{y+c2\delta}/\sqrt{Q_0}}^{1} \sum_{m \equiv -bq^2 \mod r \atop m \neq 0} \phi \left( \frac{m - yrz}{8\delta rz} \right) \delta z \sum_{m \in \mathbb{Z}} e \left( \frac{jbq^2}{r} + jyz \right) e(8j\delta z).$$

Using Lemma 3 after a linear change of variables, we transform the inner sum on the right-hand side of (14) into

$$\sum_{m \in \mathbb{Z}} \phi \left( \frac{m - yrz}{8\delta rz} \right) = 8\delta z \sum_{j \in \mathbb{Z}} e \left( \frac{jbq^2}{r} + jyz \right) e(8j\delta z).$$
Therefore, we get for the double sum on the right-hand side of (14)

\[
\sum_{q \in \mathbb{Z}} \phi \left( \frac{q - \sqrt{y}}{2c2\delta/\sqrt{Q_0}} \right) \sum_{m \in \mathbb{Z}} \phi \left( \frac{m - yrz}{8\delta rz} \right)
\]

\[
\quad \quad \quad m \equiv -bq^2 \mod r
\]

\[
= 8\delta z \sum_{j \in \mathbb{Z}} e(jyz) \hat{\phi}(8j\delta z) \sum_{d=1}^{r^*} e \left( \frac{j^*bd^2}{r^*} \right) \sum_{k \in \mathbb{Z}} \phi \left( \frac{k - \sqrt{y}}{2c2\delta/\sqrt{Q_0}} \right),
\]

where \( r^* := r/(r,j) \) and \( j^* := j/(r,j) \). Again using Lemma 3 after a linear change of variables, we transform the inner sum on the right-hand side of (15) into

\[
\sum_{k \in \mathbb{Z}} \phi \left( \frac{k - \sqrt{y}}{2c2\delta/\sqrt{Q_0}} \right) = \frac{2c2\delta}{r^*\sqrt{Q_0}} \sum_{l \in \mathbb{Z}} e \left( l \cdot \frac{d - \sqrt{y}}{r^*} \right) \hat{\phi} \left( \frac{2c2l\delta}{r^*\sqrt{Q_0}} \right).
\]

Suppose that \( \delta \) satisfies the condition (8). Then, from (15) and (16), we obtain

\[
\frac{1}{\delta} \int_{Q_0}^{2Q_0} \sum_{q \in \mathbb{Z}} \phi \left( \frac{q - \sqrt{y}}{2c2\delta/\sqrt{Q_0}} \right) \sum_{m \in \mathbb{Z}} \phi \left( \frac{m - yrz}{8\delta rz} \right) dy
\]

\[
\leq \frac{16c2\delta z}{\sqrt{Q_0}} \sum_{j \in \mathbb{Z}} \hat{\phi}(8j\delta z) \sum_{l \in \mathbb{Z}} \hat{\phi} \left( \frac{2c2l\delta}{r^*\sqrt{Q_0}} \right) \left| \sum_{d=1}^{r^*} e \left( \frac{j^*bd^2 + ld}{r^*} \right) \right| \left| \int_{Q_0}^{2Q_0} e \left( jyz - l \cdot \frac{\sqrt{y}}{r^*} \right) dy \right|.
\]
Applying the Lemmas 4 and 7 to the right-hand side of (17), we deduce

\[
1 \delta \int_{Q_0} \sum_{q \in \mathbb{Z}} \phi \left( \frac{q - \sqrt{y}}{c_2 \delta / \sqrt{Q_0}} \right) \sum_{m \in \mathbb{Z}, m \equiv -bq^2 \mod r} \phi \left( \frac{m - yrz}{8 \delta r z} \right) dy
\]

\[
\leq \frac{c_8 \delta^2 z}{\sqrt{Q_0}} \sum_{|j| \leq 1/(8 \delta z)} \frac{1}{\sqrt{r^*}} \sum_{|l| \leq r^*/(2c_2 \delta)} \left| \int_{Q_0} e \left( jyz - l \cdot \frac{\sqrt{y}}{r^*} \right) dy \right|.
\]

If \( j = 0 \) and \( l = 0 \), then the integral on the right-hand side of (18) is equal to \( Q_0 \). If \( j \neq 0 \) and \( l = 0 \), then

\[
\left| \int_{Q_0} e \left( jyz - l \cdot \frac{\sqrt{y}}{r^*} \right) dy \right| \leq \frac{1}{|j|^z}.
\]

If \( j = 0 \) and \( l \neq 0 \), then

\[
\left| \int_{Q_0} e \left( jyz - l \cdot \frac{\sqrt{y}}{r^*} \right) dy \right| \leq \frac{c_9 Q_0^{1/2}}{|l|}
\]

by Lemma 5 (take into account that \( r^* = 1 \) if \( j = 0 \)). If \( j \neq 0 \) and \( l \neq 0 \), then Lemma 6 yields

\[
\left| \int_{Q_0} e \left( jyz - l \cdot \frac{\sqrt{y}}{r^*} \right) dy \right| \leq \frac{c_{10} \sqrt{r^*} Q_0^{3/4}}{\sqrt{|l|}}.
\]
Therefore, the right-hand side of (18) can be estimated by

\[
\sum_{|j| \leq 1/(8\delta z)} \frac{1}{\sqrt{r^*}} \sum_{|l| \leq r^* \sqrt{Q_0}/(2\epsilon_2 \delta)} \left| \int_{Q_0} e^{(jyz - l \cdot \sqrt{y} \over r^*)} \, dy \right|
\]

\[
\leq c_{11} \delta \left( z \sqrt{Q_0} + \frac{1}{\sqrt{Q_0}} \sum_{1 \leq j \leq 1/(8\delta z)} \frac{1}{j \sqrt{r^*}} + z \sum_{1 \leq l \leq \sqrt{Q_0}/(2\epsilon_2 \delta)} \frac{1}{l} + zQ_0^{1/4} \sum_{1 \leq j \leq 1/(8\delta z)} \sum_{1 \leq l \leq r^*/Q_0/(2\epsilon_2 \delta)} \frac{1}{\sqrt{l}} \right)
\]

\[
\leq c_{12} \left( \delta z \sqrt{Q_0} + \frac{\delta}{\sqrt{Q_0}} \sum_{1 \leq j \leq 1/(8\delta z)} \frac{1}{j \sqrt{r^*}} + \delta z \Delta^{-\epsilon} + z \Delta^{1/2} \sum_{1 \leq j \leq 1/(8\delta z)} \sqrt{r^*} \right)
\]

Now, we evaluate the sums over \( j \) in the last line of (19). By the definition of \( r^* \), we have

\[
\sum_{1 \leq j \leq 1/(8\delta z)} \frac{1}{j \sqrt{r^*}} = \frac{1}{\sqrt{r}} \sum_{t|r} \frac{1}{t} \sum_{1 \leq j \leq 1/(8\delta z)} \frac{1}{j} (r, j) = t
\]

\[
\leq c_{13} \log(2 + 1/(8\delta z)) \frac{1}{\sqrt{r}} \sum_{t|r} \frac{1}{\sqrt{t}} \leq c_{14} \Delta^{-\epsilon} r^{-1/2}
\]

and

\[
\sum_{1 \leq j \leq 1/(8\delta z)} \sqrt{r^*} = \sqrt{r} \sum_{t|r} \frac{1}{\sqrt{t}} \sum_{1 \leq j \leq 1/(8\delta z)} \frac{1}{j} (r, j) = t
\]

\[
\leq \frac{\sqrt{r}}{8\delta z} \sum_{t|\sqrt{r}} \frac{1}{t^{3/2}} \leq c_{15} \frac{\sqrt{r}}{\delta z}.
\]
Combining Lemma 2, (14), (18), (19), (20) and (21), we obtain

\[ P \left( \frac{b}{r} + z \right) \leq c_4 \Delta^{-\varepsilon} \left( 1 + \delta z \sqrt{Q_0} + \delta Q_0^{-1/2} r^{-1/2} + \delta^{-1/2} Q_0^{1/2} \sqrt{r} \right). \]

Choosing \( \delta := Q_0 \Delta / z \), we infer the desired estimate from (22) and (5). ✷

4 Estimation of \( P(b/r + z) \) - second way

In this section, we use elementary tools to derive the following bound for \( P(b/r + z) \) from Lemma 2.

**Theorem 4:** Suppose that the conditions (4), (5) and (6) are satisfied. Then,

\[ P \left( \frac{b}{r} + z \right) \leq c_{16} \Delta^{-\varepsilon} \left( 1 + Q_0rz + Q_0^{3/2} \Delta \right). \]

**Proof:** Rearranging the order of summation, the sum on the right-hand side of (22) can be written in the form

\[ \sum_{\sqrt{y} - c_2 \delta / \sqrt{Q_0} \leq q \leq \sqrt{y} + c_2 \delta / \sqrt{Q_0}} \frac{1}{(y - 4\delta)rz \leq m \leq (y + 4\delta)rz} \sum_{m \equiv -bq^2 \mod r} \frac{1}{m \neq 0} \]

\[ = \sum_{(y - 4\delta)rz \leq m \leq (y + 4\delta)rz} \sum_{m \neq 0} \frac{1}{\sqrt{y} - c_2 \delta / \sqrt{Q_0} \leq q \leq \sqrt{y} + c_2 \delta / \sqrt{Q_0}}. \]
where \( b \mod r \) is the multiplicative inverse of \( b \mod r \), i.e. \( bb \equiv 1 \mod r \). The double sum on the right-hand side of (24) can be split up as follows:

\[
(25) \quad \sum_{(y - 4\delta)x \leq m \leq (y + 4\delta)x \atop m \neq 0} \sum_{\sqrt{y} - c_2\delta/\sqrt{Q_0} \leq q \leq \sqrt{y} + c_2\delta/\sqrt{Q_0} \atop q^2 \equiv -b \mod r} 1
\]

\[
= \sum_{t | r} \sum_{(y - 4\delta)x/t \leq m' \leq (y + 4\delta)x/t \atop (m', r/t) = 1} \sum_{\sqrt{y} - c_2\delta/\sqrt{Q_0} \leq q \leq \sqrt{y} + c_2\delta/\sqrt{Q_0} \atop q^2 \equiv 0 \mod t} \sum_{q^2 \equiv -b \mod r/t} 1,
\]

where

\[
S_t(y) := \left\{ q^2/t : \sqrt{y} - c_2\delta/\sqrt{Q_0} \leq q \leq \sqrt{y} + c_2\delta/\sqrt{Q_0} \text{ and } q^2 \equiv 0 \mod t \right\}.
\]

In the following, we determine the structure of \( S_t(y) \).

Let \( t = p_1^{v_1} \cdots p_n^{v_n} \) be the prime number factorization of \( t \). For \( i = 1, \ldots, n \) let

\[
u_i := \begin{cases} v_i, & \text{if } v_i \text{ is even}, \\ v_i + 1, & \text{if } v_i \text{ is odd}. \end{cases}
\]

Put

\[
f_t := p_1^{u_1/2} \cdots p_n^{u_n/2}.
\]

Then \( q^2 (q \in \mathbb{N}) \) is divisible by \( t \) iff \( q \) is divisible by \( f_t \). Thus,

\[
(26) \quad S_t(y) = \left\{ q_t^2 g_t : \left( \sqrt{y} - c_2\delta/\sqrt{Q_0} \right)/f_t \leq q_t \leq \left( \sqrt{y} + c_2\delta/\sqrt{Q_0} \right)/f_t \right\},
\]

where

\[
g_t := \frac{f_t^2}{t} = p_1^{u_1-v_1} \cdots p_n^{u_n-v_n}.
\]

Hence,

\[
(27) \quad |S_t(y)| \leq 1 + \frac{2c_2\delta}{f_t\sqrt{Q_0}}.
\]
By (26), we get

\[ 1 = \sum_{q' \in S_t(y) \quad q' \equiv -\overline{b_0} \mod \frac{r}{t}} 1, \]

(28)

\[ \sum_{\substack{(\sqrt{y} - c_2\sqrt{Q_0})/ft \leq q \leq (\sqrt{y} + c_2\sqrt{Q_0})/ft \quad q^2g_t \equiv -\overline{b_0} \mod \frac{r}{t}}} 1, \]

and from (27) it follows that

\[ \leq c_{17}s(r, t, m') \left(1 + \frac{t\delta}{rf_t\sqrt{Q_0}}\right), \]

(29)

where \( s(r, t, m') \) is the number of solutions mod \( r/t \) of the congruence

\[ g_t x^2 \equiv -\overline{b_0} \mod \frac{r}{t}. \]

(30)

Next, we derive a bound for \( s(t, r, m') \). In the sequel, we suppose that

\[ (m', r/t) = 1, \]

(31)

as in (25). If \( (g_t, r/t) > 1 \), then \( s(t, r, m') = 0 \) by (31) and \( (\overline{b}, r/t) = 1 \). Therefore, we can assume that \( (g_t, t/r) = 1 \). Let \( \overline{g_t} \mod r/t \) be the multiplicative inverse of \( g_t \mod r/t \), i.e. \( \overline{g_t}g_t \equiv 1 \mod r/t \). Put \( l := -\overline{g_t}bm' \). Then (30) is equivalent to

\[ x^2 \equiv l \mod k, \]

(32)

where \( k := r/t \). Taking into account that \( (l, k) = 1 \), and using some elementary facts on the number of solutions of polynomial congruences modulo prime powers (see [3], for example), we see that (32) has at most 2 solutions if \( k \) is a power of an odd prime and at most 4 solutions if \( k \) is a power of 2. From this it follows that for all \( k \in \mathbb{N} \) and \( l \in \mathbb{Z} \) with \( (l, k) = 1 \) there exist at most \( 2^\omega(k)+1 \) solutions mod \( k \) to the congruence (32), where \( \omega(k) \) is the number of distinct prime divisors of \( k \). Further, we have \( 2^\omega(k) \ll k^{c/2} \) (see [5]). Thus, we obtain

\[ s(r, t, m') \leq c_{18}r^{c/2}. \]

(33)
Suppose that the condition (8) is satisfied. Then, combining (24), (25), (28), (29) and (33), we obtain

\[
1 \delta 2 Q_0 \int_Q_0 \sum_{\sqrt{y-c_2 \delta/\sqrt{Q_0} \leq m \leq \sqrt{y+c_2 \delta/\sqrt{Q_0}}}} \sum_{m \equiv -b \delta^2 \mod r \atop m \neq 0} 1 \ dy
\]

\[
\leq c_{19} r^{\varepsilon/2} \sum_{t \mid r} \left(1 + \frac{t \delta}{r f_t \sqrt{Q_0}}\right) \cdot \frac{1}{\delta} \int_{Q_0} \sum_{(y-4 \delta) r z/t \leq m' \leq (y+4 \delta) r z/t \atop m' \neq 0} 1 \ dy.
\]

We estimate the integral on the right-hand side by

\[
\int_{Q_0} \sum_{(y-4 \delta) r z/t \leq m' \leq (y+4 \delta) r z/t \atop m' \neq 0} 1 \ dy
\]

\[
= \sum_{(Q_0 - 4 \delta) r z/t \leq m' \leq (2Q_0 + 4 \delta) r z/t \atop m' \neq 0} \min \{2Q_0, tm'/r z + 4 \delta\} \left(1 \ dy \int_{Q_0} 1 \ dy \right)
\]

\[
\leq 8 \delta \sum_{(Q_0 - 4 \delta) r z/t \leq m' \leq (2Q_0 + 4 \delta) r z/t \atop m' \neq 0} 1
\]

\[
\leq \frac{72 \delta Q_0 r z}{t}.
\]

To obtain the last line of the above inequality, we use (8) and the summation
condition $m' \neq 0$. From (34) and (35), we deduce

\begin{equation}
\frac{1}{\delta} \int_{Q_0}^{2Q_0} \sum_{\sqrt{Q_0} - c_2 \delta / \sqrt{Q_0} \leq q \leq \sqrt{Q_0} + c_2 \delta / \sqrt{Q_0}} \sum_{(y - 4\delta)rz \leq m \leq (y + 4\delta)rz \atop m \equiv -bq^2 \mod r} 1 \, dy
\end{equation}

\[ \leq c_{20} r^{\epsilon/2} \sum_{t|r} \left( 1 + \frac{\delta t}{r_f / \sqrt{Q_0}} \right) \frac{Q_0 rz}{t} \]

\[ \leq c_{21} r^{\epsilon} \left( Q_0 rz + \delta z \sqrt{Q_0} \right). \]

Finally, we choose

\begin{equation}
\delta := \frac{Q_0 \Delta}{z},
\end{equation}

which is in consistency with the condition (8). Combining (36), (37) and Lemma 2, we obtain the result of Theorem 4. $\square$

## 5 Proof of Theorem 2

Suppose that the conditions (4), (5) and (6) are satisfied. Then the combination of the Theorems 3 and 4 yields

\begin{equation}
P \left( \frac{b}{r} + z \right) \leq c_{22} \Delta^{-\epsilon} \left( Q_0^{3/2} \Delta + \min \left\{ Q_0 rz, Q_0^{1/2} \Delta r^{-1/2} z^{-1} \right\} + \Delta^{-1/4} \right).
\end{equation}

If

\[ z \leq \Delta^{1/2} Q_0^{-1/4} r^{-3/4}, \]

then

\[ \min \left\{ Q_0 rz, Q_0^{1/2} \Delta r^{-1/2} z^{-1} \right\} = Q_0 rz \leq Q_0^{3/4} \Delta^{1/2} r^{1/4}. \]

If

\[ z > \Delta^{1/2} Q_0^{-1/4} r^{-3/4}, \]

then

\[ \min \left\{ Q_0 rz, Q_0^{1/2} \Delta r^{-1/2} z^{-1} \right\} = Q_0^{1/2} \Delta r^{-1/2} z^{-1} \leq Q_0^{3/4} \Delta^{1/2} r^{1/4}. \]
From the above inequalities and (5), we deduce

\[ \min \left\{ Q_0 r z, Q_0^{1/2} \Delta r^{-1/2} z^{-1} \right\} \leq Q_0^{3/4} \Delta^{3/8}. \]

Furthermore,

\[ Q_0^{3/4} \Delta^{3/8} = \sqrt{(Q_0^{3/2} \Delta) \cdot \Delta^{-1/4}} \leq Q_0^{3/2} \Delta + \Delta^{-1/4}. \]

Combining (38), (39) and (40), we get

\[ P \left( \frac{b}{r} + z \right) \leq c_{23} \Delta^{-\varepsilon} \left( Q_0^{3/2} \Delta + \Delta^{-1/4} \right). \]

We now choose \( \Delta := 1/N \). Then from Theorem 1, Lemma 1 and (41) it follows that

\[ \sum_{\sqrt{Q_0} \leq q \leq \sqrt{2Q_0}} \sum_{\substack{a = 1 \\ (a, q) = 1}} q^2 \left| S \left( \frac{a}{q^2} \right) \right|^2 \ll N^\varepsilon \left( Q_0^{3/2} + N^{5/4} \right) Z. \]

We can divide the interval \([1, Q]\) into \( O(\log Q) \) subintervals of the form \( [\sqrt{Q_0}, \sqrt{2Q_0}] \), where \( 1 \leq Q_0 \leq Q^2 \). Hence, the result of Theorem 2 follows from (42). □

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