Global Existence of Strong Solution to 3D Atmospheric Thermal Equation

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Abstract: In this paper, the atmospheric thermal system (1.1) in a bounded domain with Neumann boundary condition (1.4) is considered. In order to get the global existence of strong solution, the difficulties which result from the boundary condition need to be dealt with, and the uniform estimates with respect to the time $t$ in $[0,T]$ should be shown. For this, the modified Lax-Milgram theorem is used to get the existence of linearised system, then take the method of fixed-point theorem to obtain the existence of solution to the nonlinear system.

1. Introduction

1.1 The atmosphere thermal equation
In this paper, we consider the following three-dimensional atmosphere thermal equation (ATE) in a bounded domain $\Omega \subset \mathbb{R}^3$:

$$\bar{p}(\partial_t \theta + u \cdot \nabla \theta) + p\text{div} \mathbf{u} = \kappa \Delta \theta + 2\mu_D(\bar{p}) : D(\bar{p}) + \lambda(\text{div} \mathbf{u})^2$$  

(1.1)

with fixed density $\rho$ and velocity $\mathbf{u}$ of the atmosphere. Here, $\kappa > 0$ stands for the heat conductive coefficient, and $\mu$, $\lambda$ are shear and bulk viscosity coefficients satisfying $\mu > 0, 2\mu + 3\lambda \geq 0$. The pressure has the following relationship for perfect gas

$$p = \bar{R}\bar{\rho}\bar{\theta}.$$  

(1.2)

We assume the initial data

$$\theta(t=0, x) = \theta_0(x), \quad x \in \Omega$$  

(1.3)

and Neumann boundary condition

$$\frac{\partial \theta}{\partial n} \bigg|_{\text{on } \partial \Omega} = 0.$$  

(1.4)

1.2 The current research status
The model we study is the heat equation of the compressible Navier-Stokes equations, which is the important model about atmosphere. These systems have important physical meanings and useful values, and have rich physical phenomena and mathematical challenges, so these systems are in view of physicians and mathematicians. These equations have been extensively studied. In the absence of vacuum, i.e., the case that the initial density is uniformly bounded away from zero, global well-posedness

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of strong solutions to the one dimensional compressible Navier-Stokes equations has been well-known, which can be seen that Kazhikhov–Shelukhin [18] and Kazhikhov [17]. Global existence and uniqueness of weak solutions was gotten in [37, 38, 1] if considering the initial boundary value problems. Jiang–Zlotnik [16] obtained the global existence and uniqueness of weak solution for the Cauchy problem. Li–Liang [21] led to the large time behavior of solutions to the one dimensional compressible Navier-Stokes equations with large initial data. The corresponding global well-posedness results for the multi-dimensional case were established only for small perturbed initial data around some non-vacuum equilibrium or for spherically symmetric large initial data, seeing [26–30, 32, 7, 14, 11, 19, 6, 2]. In the presence of vacuum, the density may vanish on some set or tends to zero at the far field. Global existence of weak solutions to the isentropic compressible Navier-Stokes equations was first proved by Lions [24, 25], with adiabatic constant $\gamma \geq 9/5$, and later generalized by [8] to $\gamma > 3/2$, and further by Jiang–Zhang [15] to $\gamma > 1$ for the axisymmetric solutions.

For the full compressible Navier-Stokes equations, global existence of the variational weak solutions was proved by [9, 10], where the ideal gases is not applicable. Local wellposedness of strong solutions to compressible Navier-Stokes equations, in the presence of vacuum, was proved by [5, 31, 3, 4]. In [3–5, 31], the solutions in the homogeneous Sobolev spaces are established, however one can not expect the solutions in the inhomogeneous Sobolev spaces if the initial density has compact support in [20]. Global existence of strong solutions to the compressible Navier-Stokes equations was first proved by [13] for the isentropic case with small initial data, in the presence of initial vacuum, (see also [23, 12, 34]). Due to the finite blow-up results in [35, 36], the global solutions must have infinite entropy somewhere in the vacuum region if the initial density has an isolated mass group. If the initial density is positive everywhere and tends to vacuum at the far field, global existence of solutions was proved to be with uniformly bounded entropy for the full compressible Navier-Stokes equations, see [22]. However, the global existence of multi-dimensional full compressible Navier-Stokes equations is still open if considering the large data.

In this paper, we want to discuss the global existence of weak solution to the atmospheric thermal system (1.1) in a bounded domain with Neumann boundary condition (1.4). In order to get the global existence of strong solution, we need to deal with the difficulties which results from the boundary condition, and should show the uniform estimates with respect to the time $t \in [0, T]$. For this, we have to use the modified Lax-Milgram theorem to get the existence of linearised system, then take the method of fixed-point theorem to obtain the existence of solution to the nonlinear system.

In the following paper, we suppose that

$$\rho \in C(0, T; H^2), \rho_t \in C(0, T; H^1);$$
$$\bar{u} \in C(0, T; H^1) \cap L^2(0, T; H^3), \bar{u}_t \in C(0, T; H^1) \cap L^2(0, T; H^2).$$

(1.5)

And assume that there is a small constant $\alpha_0 < 1$ such that

$$\| \rho \|_{C(0, T; H^2)} + \| \rho_t \|_{C(0, T; H^1)} + \| \bar{u} \|_{C(0, T; H^1) \cap L^2(0, T; H^3)} + \| \bar{u}_t \|_{C(0, T; H^1) \cap L^2(0, T; H^2)} \leq \alpha_0. \quad (1.6)$$

In the second section, we state the four lemmas which are useful to the main results which is shown in section 3. And in section 4, we give the main proof of Theorem 3.1 and 3.2.

2. Methods

First of all, we state the useful lemma, which can be seen in [1].

**Lemma 2.1.** (Modified Lax-Milgram Theorem) Let $H$ be a Hilbert space, $V \subset H$ a dense subspace and $a(u, v)$ a bilinear form in $H \times V$ satisfying the following conditions:

1) For some constant $M > 0$, 
$$| a(u, v) | \leq M \| u \|_H \| v \|_H, \forall u \in H, v \in V;$$

2) For some constant $\delta > 0$, 
$$a(v, v) \geq \delta \| v \|_H^2, \forall v \in V.$$
Then for any bounded linear functional \( F(v) \) in \( H \), there exists \( u \in H \), such that
\[
F(u) = a(u, v), \forall v \in V.
\]

**Lemma 2.2.** (Poincare’s Inequality) Let \( 1 \leq p < +\infty \) and \( \Omega \subset R^n \) be a bounded domain.

i) If \( u \in W^{1,p}_0(\Omega) \), then
\[
\int_{\Omega} |u|^p \, dx \leq C \int_{\Omega} |Du|^p \, dx.
\]

ii) If \( \partial \Omega \) is locally Lipschitz continuous and \( u \in W^{1,p}(\Omega) \), then
\[
\int_{\Omega} |u - u_{\Omega}|^p \, dx \leq C \int_{\Omega} |Du|^p \, dx
\]
with a constant \( C \) depending only on \( n, p \) and \( \Omega \), where
\[
u_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega} u(x) \, dx
\]
and \( |\Omega| \) is the measure of \( \Omega \).

**Lemma 2.3.** (Schauder Fixed-point Theorem) If \( K \) is a nonempty convex closed subset of a Hausdorff topological vector space \( V \) and \( T \) is continuous mapping of \( K \) into itself such that \( T(K) \) is contained in a compact subset of \( K \), then \( T \) has a fixed point.

A consequence, called Schaefer’s fixed point theorem, is particularly useful for proving existence of solutions to nonlinear partial differential equations.

**Lemma 2.4.** (Schaefer Fixed-point Theorem) Let \( T \) be a continuous and compact mapping of a Banach space \( X \) into itself, such that the set
\[
\{ x \in X : x = \lambda Tx \text{ for some } 0 \leq \lambda \leq 1 \}
\]
is bounded. Then \( T \) has a fixed point.

### 3. Results

If we consider the linearized equation of the nonlinear system (1.1)
\[
\overline{\rho} \partial_t \vartheta - \kappa \Delta \vartheta = f
\]
where
\[
f := -\overline{\rho} u \cdot \nabla \tilde{\vartheta} - R\overline{\rho} \partial \text{div} u + 2 \mu D(u) : D(u) + \lambda (\text{div} u)^2,
\]
for fixed \( \tilde{\vartheta} \) satisfying
\[
\tilde{\vartheta} \in C(0,T;H^2) \cap L^2(0,T;H^3), \tilde{\vartheta}_t \in C(0,T;H^1) \cap L^2_0(0,T;H^2).
\]

**Theorem 3.1.** Suppose (1.5) - (1.6) and (1.8) hold. Then, linearized problem (1.7) with Neumann boundary condition (1.4) has a unique global strong solution, which satisfy
\[
\vartheta \in C(0,T;H^2) \cap L^2(0,T;H^3), \vartheta_t \in C(0,T;H^1) \cap L^2_0(0,T;H^2).
\]

**Theorem 3.2.** Suppose (1.5) - (1.6) hold. Then, the atmospheric thermal equation (1.1) with Neumann boundary condition (1.4) has a unique global strong solution
\[
\theta \in C(0,T;H^2) \cap L^2(0,T;H^3), \theta_t \in C(0,T;H^1) \cap L^2_0(0,T;H^2)
\]
such that the following estimate
\[
\| \theta \|_{L^\infty(0,T;H^2) \cap L^2(0,T;H^3)} + \| \theta_t \|_{L^\infty(0,T;H^1) \cap L^2_0(0,T;H^2)} \leq C.
\]
4. Proof of Theorems

Proof of Theorem 3.1.

Let \( W_{0}^{1,2}(0,T;W^{1,2}(\Omega)) = \left\{ \theta \mid \theta \mid_{L_{2}^{0}(0,T;W^{1,2}(\Omega))} + \theta_{t} \mid_{L_{2}^{0}(0,T;W^{1,2}(\Omega))} \leq C, \nabla \theta \mid_{t=0} = \nabla \theta \mid_{t=T} = 0 \right\} \) and

\[
\theta_{[0,T] \in \Omega} = \frac{1}{|[0,T] \times \Omega|} \int_{(0,T] \in \Omega} \theta(x,t) dx dt
\]

Denote

\[
a(u, (\theta - \theta_{[0,T] \in \Omega})) = \int_{(0,T] \in \Omega} (u_{t} (\theta - \theta_{[0,T] \in \Omega}), + \nabla u \cdot \nabla (\theta - \theta_{[0,T] \in \Omega}),) e^{-\gamma t} dx dt,
\]

where \( \gamma \) is a positive constant. It is easy to see

\[
|a(u, (\theta - \theta_{[0,T] \in \Omega}))| \leq \|u \|_{W^{2,0,\omega}(\Omega)}  \| \theta - \theta_{[0,T] \in \Omega} \|_{W^{0,2,0}(\Omega)}, \theta \in W_{0}^{1,2}(0,T;W^{1,2}(\Omega)), u \in V(0,T;\Omega)
\]

On the other hand, for \( \theta \in V(0,T;\Omega) \), we get

\[
\int_{(0,T] \in \Omega} \nabla (\theta - \theta_{[0,T] \in \Omega}) \cdot \nabla (\theta - \theta_{[0,T] \in \Omega}), e^{-\gamma t} dx dt
\]

\[
= \frac{1}{2} \int_{(0,T] \in \Omega} |\nabla (\theta - \theta_{[0,T] \in \Omega})|^{2} e^{-\gamma t} dx dt
\]

\[
= \frac{1}{2} \int_{(0,T] \in \Omega} (\nabla (\theta - \theta_{[0,T] \in \Omega})^{2} e^{-\gamma t} dx dt + \frac{\gamma}{2} \int_{(0,T] \in \Omega} |\nabla (\theta - \theta_{[0,T] \in \Omega})|^{2} e^{-\gamma t} dx dt
\]

\[
= \frac{e^{\gamma T}}{2} \int_{\Omega} |\nabla \theta|^{2} (t = T, x) dx + \frac{1}{2} \int_{\Omega} |\nabla (\theta - \theta_{[0,T] \in \Omega})|^{2} (t = 0, x) dx + \frac{\gamma}{2} \int_{(0,T] \in \Omega} |\nabla (\theta - \theta_{[0,T] \in \Omega})|^{2} e^{-\gamma t} dx dt,
\]

where \( \nabla \theta \mid_{t=0} \) denotes the trace of \( \nabla \theta \mid_{t=0} \) when \( t = 0 \). Noticing that \( \theta \in V(0,T;\Omega) \) implies \( \nabla \theta \mid_{t=0} = 0 \), we have

\[
\int_{(0,T] \in \Omega} \nabla (\theta - \theta_{[0,T] \in \Omega}) \cdot \nabla (\theta - \theta_{[0,T] \in \Omega}), e^{-\gamma t} dx dt \geq \frac{e^{\gamma T}}{2} \left| \int_{(0,T] \in \Omega} |\nabla (\theta - \theta_{[0,T] \in \Omega})|^{2} dx dt,
\]

Due to Lemma 2.2 Poincare’s inquality

\[
\int_{(0,T] \in \Omega} |\theta - \theta_{[0,T] \in \Omega}|^{2} dx dt \leq \int_{(0,T] \in \Omega} \nabla \theta \mid_{t=0} dx dt, (\theta_{[0,T] \in \Omega} = \frac{1}{|[0,T] \times \Omega|} \int_{(0,T] \in \Omega} \theta(x,t) dx dt),
\]

the above estimate can be led to

\[
\int_{(0,T] \in \Omega} \nabla (\theta - \theta_{[0,T] \in \Omega}) \cdot \nabla (\theta - \theta_{[0,T] \in \Omega}), e^{-\gamma t} dx dt
\]

\[
\geq \frac{e^{\gamma T}}{4} \int_{(0,T] \in \Omega} |\nabla (\theta - \theta_{[0,T] \in \Omega})|^{2} dx dt + \frac{\gamma e^{\gamma T}}{4} \int_{(0,T] \in \Omega} |\theta - \theta_{[0,T] \in \Omega}|^{2} dx dt.
\]

Therefore

\[
|a(\theta - \theta_{[0,T] \in \Omega}, \theta - \theta_{[0,T] \in \Omega})| \geq \delta \| \theta - \theta_{[0,T] \in \Omega} \|_{W_{0}^{2,0,\omega}(\Omega)}^{2}, \forall \theta \in V(0,T;\Omega),
\]

where

\[
\delta = \min \left\{ e^{-\gamma T}, \frac{e^{-\gamma T}}{4}, \frac{\gamma e^{\gamma T}}{4} \right\}.
\]
Choose $H = W^{1,2}_0(0,T;W^{1,2}(\Omega))$, $V = V(0,T;\Omega)$. Then, we show the conditions 1) and 2) in the modified Lax-Milgram’s theorem are satisfied. It is obvious that

$$
\int_{(0,T) \times \Omega} f u e^{-\gamma} \, dx \, dt
$$

is a bounded linear functional of $\theta$ in $H$.

Therefore there exists a $\theta \in H = W^{1,2}_0(0,T;W^{1,2}(\Omega))$, such that

$$
a(u, \theta - \theta|_{(0,T) \times \Omega}) = \int_{(0,T) \times \Omega} f u e^{-\gamma} \, dx \, dt, \forall u \in V(0,T;\Omega),
$$

that is

$$
\int_{(0,T) \times \Omega} (u, (\theta - \theta|_{(0,T) \times \Omega})), + \nabla u \cdot \nabla (\theta - \theta|_{(0,T) \times \Omega})) e^{-\gamma} \, dx \, dt = \int_{(0,T) \times \Omega} f u e^{-\gamma} \, dx \, dt, \forall u \in V(0,T;\Omega).
$$

By the Modified Lax-Milgram Theorem, we are able to get the existence of weak solution to the following problem:

$$
\overline{p\partial_t} (\theta - \theta|_{(0,T) \times \Omega}) - \kappa \Delta (\theta - \theta|_{(0,T) \times \Omega}) = f
$$

(4.1)

with initial data (1.3) and Neumann boundary condition (1.4). Here,

$$
f = -\overline{p\nabla} \cdot \nabla \tilde{\theta} - R\overline{p\partial_t} \nabla \overline{\nabla} + 2\mu \overline{D(\overline{u})} : D(\overline{u}) + \lambda (\overline{\nabla \overline{\nabla}})^2,
$$

for fixed $\tilde{\theta}$ satisfying

$$
\tilde{\theta} \in C(0,T;H^2) \cap L^2(0,T;H^1), \tilde{\partial}_t \in C(0,T;H^1) \cap L^2_0(0,T;H^2).
$$

Furthermore, by the regularity of standard heat equation, we have

$$
\theta \in C(0,T;H^2) \cap L^2(0,T;H^1), \partial_t \in C(0,T;H^1) \cap L^2_0(0,T;H^2)
$$

if $\tilde{\theta} \in C(0,T;H^2) \cap L^2(0,T;H^1), \tilde{\partial}_t \in C(0,T;H^1) \cap L^2_0(0,T;H^2)$. So, we can obtain the global existence of weak solution to the equation

$$
\overline{p\partial_t} \theta - \kappa \Delta \theta = f
$$

where $f = -\overline{p\nabla} \cdot \nabla \tilde{\theta} - R\overline{p\partial_t} \nabla \overline{\nabla} + 2\mu \overline{D(\overline{u})} : D(\overline{u}) + \lambda (\overline{\nabla \overline{\nabla}})^2$, and satisfying

$$
\theta \in C(0,T;H^2) \cap L^2(0,T;H^1), \partial_t \in C(0,T;H^1) \cap L^2_0(0,T;H^2).
$$

So, we arrive at the global existence of strong solution of linearized problem (3.1) and (1.3) – (1.4).

**Proof of Theorem 3.2.**

Let

$$
K = \left\{ \theta \mid \|\theta\|_{C(0,T;H^2) \cap L^2(0,T;H^1)} + \|\partial_t \theta\|_{C(0,T;H^1) \cap L^2_0(0,T;H^2)} \leq C \right\}
$$

Let $T$ maps $K$ to $K$:

$$
T: \tilde{\theta} \mapsto \theta.
$$

And we need show that $T$ is a continuous and compact mapping.

Suppose $\theta_1 = T\tilde{\theta}_1$ and $\theta_2 = T\tilde{\theta}_2$, then we have

$$
\overline{p\partial_t} (\theta_1 - \theta_2) - \kappa \Delta (\theta_1 - \theta_2) = -\overline{p\nabla} \cdot \nabla (\tilde{\theta}_1 - \tilde{\theta}_2) - R\overline{p\partial_t} (\tilde{\theta}_1 - \tilde{\theta}_2) \nabla \overline{\nabla}
$$

(4.2)
By multiplying (1.11) with $\theta_1 - \theta_2$, and integrating the resulting identity with the spatial variables on $\Omega$, we have
\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \mathcal{P} (\theta_1 - \theta_2)^2 \, dx + \kappa \int_{\Omega} \nabla (\theta_1 - \theta_2)^2 \, dx \\
= \int_{\Omega} - \mathcal{P} \nabla \cdot \nabla (\theta_1 - \theta_2) - R \mathcal{P} (\theta_1 - \theta_2) \nabla \mathcal{P} (\theta_1 - \theta_2) \, dx \\
\leq \frac{\kappa}{2} \int_{\Omega} \nabla (\theta_1 - \theta_2)^2 \, dx + C (\| \mathcal{P} \|_{L^\infty}, \| \nabla \mathcal{P} \|_{L^2}, \| \nabla \mathcal{P} \|_{L^\infty}) \int_{\Omega} \nabla (\theta_1 - \theta_2)^2 \, dx.
\]

Here, we have used the Poincaré’s Inequality
\[
\int_{\Omega} |\theta_1 - \theta_2|^2 \, dx \leq C \int_{\Omega} |\nabla (\theta_1 - \theta_2)|^2 \, dx.
\]

That is,
\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \mathcal{P} (\theta_1 - \theta_2)^2 \, dx + \kappa \int_{\Omega} \nabla (\theta_1 - \theta_2)^2 \, dx \\
\leq C (\| \mathcal{P} \|_{L^\infty}, \| \nabla \mathcal{P} \|_{L^2}, \| \nabla \mathcal{P} \|_{L^\infty}) \int_{\Omega} \nabla (\theta_1 - \theta_2)^2 \, dx.
\]

Thus, we obtain that $T$ is a continuous and compact mapping, due to the assumption (1.6).

For Lemma 2.4 Schaefer’s Fixed-point Theorem, we are able to get $T$ has a fixed point $\theta$, such that
\[
T \theta = \theta.
\]

Now, we show then uniqueness of strong solution. In fact, if there are two strong solutions $\theta_1, \theta_2$ satisfying the atmospheric thermal equations, we have $\theta_1 = T \theta_1$ and $\theta_2 = T \theta_2$, then we have
\[
\mathcal{P} \partial_t \theta_1 - \kappa \Delta \theta_1 = - \mathcal{P} \nabla \cdot \nabla \theta_1 - R \mathcal{P} \partial_t \mathcal{P} \theta_1 \nabla \mathcal{P} + 2 \mu D(\mathcal{P}) + D(\mathcal{P}) + \lambda (\nabla \mathcal{P})
\]
and
\[
\mathcal{P} \partial_t \theta_2 - \kappa \Delta \theta_2 = - \mathcal{P} \nabla \cdot \nabla \theta_2 - R \mathcal{P} \partial_t \mathcal{P} \theta_2 \nabla \mathcal{P} + 2 \mu D(\mathcal{P}) + D(\mathcal{P}) + \lambda (\nabla \mathcal{P})
\]

By making use of (1.12) minus (1.13), we get
\[
\mathcal{P} \partial_t (\theta_1 - \theta_2) - \kappa \Delta (\theta_1 - \theta_2) = - \mathcal{P} \nabla \cdot \nabla (\theta_1 - \theta_2) - R \mathcal{P} \partial_t (\theta_1 - \theta_2) \nabla \mathcal{P}
\]

Multiplying (1.14) with $\theta_1 - \theta_2$, and integrating the resulting identity with the spatial variables on $\Omega$, we have
\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \mathcal{P} (\theta_1 - \theta_2)^2 \, dx + \kappa \int_{\Omega} \nabla (\theta_1 - \theta_2)^2 \, dx \\
= \int_{\Omega} - \mathcal{P} \nabla \cdot \nabla (\theta_1 - \theta_2) - R \mathcal{P} (\theta_1 - \theta_2) \nabla \mathcal{P} (\theta_1 - \theta_2) \, dx \\
\leq \frac{1}{2} \int_{\Omega} |\theta_1 - \theta_2|^2 \, dx - \int_{\Omega} R \nabla \mathcal{P} (\theta_1 - \theta_2) \, dx.
\]

Here, we have used the Gronwall’s Inequality
\[
\int_{\Omega} |\theta_1 - \theta_2|^2 \, dx \leq C \int_{\Omega} |\theta_1 (0, x) - \theta_2 (0, x)|^2 \, dx \exp\left(\| \nabla (\mathcal{P} \mathcal{P}) \|_{L^\infty} + R \| \nabla \mathcal{P} \mathcal{P} \|_{L^\infty} \right).
\]

For $\theta_1, \theta_2$ have the same initial data, we have
\[
\theta_1 = \theta_2.
\]
Thus, we show the global existence and uniqueness of the strong solution to atmospheric thermal problem \((1.1)\) with \((1.3) – (1.4)\) due to the regularity of the system. This is the end of the proof.

5. Conclusion

The atmospheric thermal system \((1.1)\) in a bounded domain with Neumann boundary condition \((1.4)\) is considered. In order to get the global existence of strong solution, the difficulties which result from the boundary condition need to be dealt with, and the uniform estimates with respect to the time \(t\) in \([0,T]\) should be shown. For this, the modified Lax-Milgram theorem is used to get the existence of linearised system, then take the method of fixed-point theorem to obtain the existence of solution to the nonlinear system.

Note 1: As for the full compressible Navier-Stokes equations, if we can get the higher order regularity of density and velocity

\[
\bar{\rho} \in C(0,T; H^2), \bar{p}_j \in C(0,T; H^1);
\]

\[
\bar{u} \in C(0,T; H^3) \cap L^2(0,T; H^4), \bar{u}_i \in C(0,T; H^1) \cap L^2(0,T; H^2).
\]

And there is a small constant \(\alpha_0 < 1\) such that

\[
\| \bar{\rho} \|_{C(0,T; H^2)} + \| \bar{p}_j \|_{C(0,T; H^1)} + \| \bar{u} \|_{C(0,T; H^3) \cap L^2(0,T; H^4)} + \| \bar{u}_i \|_{C(0,T; H^1) \cap L^2(0,T; H^2)} \leq \alpha_0.
\]

we can obtain the existence of strong solution to the multi-dimensional full compressible Navier-Stokes equations in a bounded domain.

Note 2: If we consider the full compressible Navier-Stokes equations, the mass equation of density and velocity would be used to decrease the regularity of velocity such that

\[
\bar{u} \in C(0,T; H^2) \cap L^2(0,T; H^3), \bar{u}_i \in C(0,T; H^1) \cap L^2(0,T; H^2).
\]

Acknowledgements

This research was financially supported by National Natural Science Foundation of China (Grant No. 11471334,11671273) and Natural Science Foundation of Beijing (Grant No. 1182007).

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