Flavor Symmetries and the Description of Flavor Mixing

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Abstract

It is shown that the hierarchical structure of the quark mass terms in the standard model suggests a new description of the flavor mixing. The latter is primarily a heavy quark mixing involving the $t$ and $b$ quarks, followed by a mixing exclusively in the $u$-channel or the $d$-channel. The complex phase describing $CP$ violation arises only in the light quark sector. The Cabibbo angle is not a basic parameter, but results as a superposition of both the $u$-channel and $d$-channel mixing terms. The new description has a number of significant advantages in comparison with all descriptions previously used. It is suggested that the new description be used in all future discussions of flavor physics and $CP$ violation.
A deeper understanding of the phenomenon of flavor mixing observed in the charged-current–type weak interactions remains one of the major challenges in particle physics at present. In the standard electroweak theory it is described by a $3 \times 3$ unitary flavor mixing matrix \cite{1, 2}, which can be expressed in terms of four parameters, usually taken to be as three rotation angles and a phase angle. Within the standard model there is no way to obtain any further information about these parameters. Any attempt to do so would imply physics inputs which go beyond the standard electroweak theory.

In the standard model the mixing of quark flavors arises after the diagonalization of the up- and down-type mass matrices. Both mass matrices cannot be diagonalized by unitary transformations which commute with the charged weak generators. The result of this diagonalization-mismatch, whose dynamical origin is unknown, is the flavor mixing. However, it is implied that the mechanism which is responsible for the generation of quark masses is at the same time responsible for the mixing of flavors, i.e., any change of the eigenvalues of quark masses would in general also lead to a change of the flavor mixing parameters. In many models based on flavor symmetries which go beyond the standard electroweak theory, the flavor mixing parameters are indeed functions of the mass eigenvalues (for early works, see Refs. \cite{3, 4, 5}).

Both the observed mass spectrum of quarks and the observed values of flavor mixing parameters exhibit a striking hierarchical structure. This hierarchical structure can be understood in a natural way as the result of a specific pattern of chiral symmetries whose breaking would cause the hierarchical tower of masses to appear step by step \cite{6, 7, 8}. Such a chiral evolution of the quark mass matrices leads, as argued in particular in Ref. \cite{7}, to a rather specific way to describe the flavor mixing; in the limit $m_u = m_d = 0$ the flavor mixing is merely a rotation between the second and third families, described by one rotation angle. While the original representation of the flavor mixing, introduced in Ref. \cite{1}, is not natural in the sense of a chiral evolution of the mass matrices, the standard representation \cite{2} does respect it. Nevertheless, there still remain several possibilities to describe the flavor mixing. In particular the complex phase which describes $CP$ violation can appear in a number of different ways, and in general the relation between this phase and the phases of quark mass terms is rather complicated.

In this paper we should like to point out that there exists a parametrization of the flavor mixing which is unique in the sense that it incorporates the chiral evolution of the mass matrices in a natural way, and that the phase in the flavor mixing matrix and the phases in the mass terms are related in a very simple way. Hence the phase entering the flavor mixing matrix could have a deeper physical meaning. In addition, it turns out that the mixing parameters introduced here can be described by quantities which are easily measurable in $B$-meson decays. Related descriptions of the flavor mixing have appeared in approximate forms before \cite{3, 4, 5, 6, 7}. The arguments presented below are, in our view, strong enough in order to reconsider the way how the mixing of quark flavors should be described and parametrized.

In this paper we take the point of view that the quark mass eigenvalues are dynamical
entities, and one could change their values in order to study certain symmetry limits, as it is
done in QCD. In the standard electroweak model, in which the quark mass matrices are given
by the coupling of a scalar field to various quark fields, this can certainly be done by adjusting
the related coupling constants. Whether it is possible in reality is an open question.

It is well-known that the quark mass matrices can always be made hermitian by a suitable
transformation of the right-handed fields. Without loss of generality, we shall suppose in this
paper that the quark mass matrices are hermitian.

In the limit where the masses of the $u$ and $d$ quarks are set to zero, the quark mass matrix
$\tilde{M}$ (for both charge $+2/3$ and charge $-1/3$ sectors) can be arranged such that its elements
$\tilde{M}_{ii}$ and $\tilde{M}_{i1}$ ($i = 1, 2, 3$) are all zero [6, 7]. Thus the quark mass matrices have the form

$$\tilde{M} = \begin{pmatrix}
0 & 0 & 0 \\
0 & \tilde{C} & \tilde{B} \\
0 & \tilde{B}^* & \tilde{A}
\end{pmatrix}.$$  \hspace{1cm} (1)

The observed mass hierarchy is incorporated into this structure by denoting the entry which
is of the order of the $t$-quark or $b$-quark mass by $\tilde{A}$, with $\tilde{A} \gg \tilde{C}, |\tilde{B}|$. It can easily be seen
(see, e.g., Ref. [11]) that the complex phases in the mass matrices (1) can be rotated away
by subjecting both $\tilde{M}_u$ and $\tilde{M}_d$ to the same unitary transformation. Thus we shall take $\tilde{B}$ to
be real for both up- and down-quark sectors. As expected, $CP$ violation cannot arise at this
stage. The diagonalization of the mass matrices leads to a mixing between the second and
third families, described by an angle $\tilde{\theta}$. The flavor mixing matrix is then given by

$$\tilde{V} = \begin{pmatrix}
1 & 0 & 0 \\
0 & \tilde{c} & \tilde{s} \\
0 & -\tilde{s} & \tilde{c}
\end{pmatrix},$$  \hspace{1cm} (2)

where $\tilde{s} \equiv \sin \tilde{\theta}$ and $\tilde{c} \equiv \cos \tilde{\theta}$. In view of the fact that the limit $m_u = m_d = 0$ is not far from
reality, the angle $\tilde{\theta}$ is essentially given by the observed value of $|V_{ub}| (= 0.039 \pm 0.002$ [12, 13]);
i.e., $\tilde{\theta} = 2.24^\circ \pm 0.12^\circ$.

At the next and final stage of the chiral evolution of the mass matrices, the masses of the
$u$ and $d$ quarks are introduced. The hermitian mass matrices have in general the form:

$$M = \begin{pmatrix}
E & D & F \\
D^* & C & B \\
F^* & B^* & A
\end{pmatrix}.$$  \hspace{1cm} (3)

with $A \gg C, |B| \gg E, |D|, |F|$. By a common unitary transformation of the up- and down-
type quark fields, one can always arrange the mass matrices $M_u$ and $M_d$ in such a way that
$F_u = F_d = 0$; i.e.,

$$M = \begin{pmatrix}
E & D & 0 \\
D^* & C & B \\
0 & B^* & A
\end{pmatrix}.$$  \hspace{1cm} (4)
This can easily be seen as follows. If phases are neglected, the two symmetric mass matrices \( M_u \) and \( M_d \) can be transformed by an orthogonal transformation matrix \( O \), which can be described by three angles, such that they assume the form (4). The condition \( F_u = F_d = 0 \) gives two constraints for the three angles of \( O \). If complex phases are allowed in \( M_u \) and \( M_d \), the condition \( F_u = F_u^* = F_d = F_d^* = 0 \) imposes four constraints, which can also be fulfilled, if \( M_u \) and \( M_d \) are subjected to a common unitary transformation matrix \( U \). The latter depends on nine parameters. Three of them are not suitable for our purpose, since they are just diagonal phases; but the remaining six can be chosen such that the vanishing of \( F_u \) and \( F_d \) results.

The basis in which the mass matrices take the form (4) is a basis in the space of quark flavors, which in our view is of special interest. It is a basis in which the mass matrices exhibit two texture zeros, for both up- and down-type quark sectors. These, however, do not imply special relations among mass eigenvalues and flavor mixing parameters (as pointed out above). In this basis the mixing is of the “nearest neighbour” form, since the (1,3) and (3,1) elements of \( M_u \) and \( M_d \) vanish; no direct mixing between the heavy \( t \) (or \( b \)) quark and the light \( u \) (or \( d \)) quark is present (see also Ref. [14]). In certain models (see, e.g., Refs. [5, 9]), this basis is indeed of particular interest, but we shall proceed without relying on a special texture models for the mass matrices.

A mass matrix of the type (4) can in the absence of complex phases be diagonalized by a rotation matrix, described by two angles only. At first the off-diagonal element \( B \) is rotated away by a rotation between the second and third families (angle \( \theta_{23} \)); at the second step the element \( D \) is rotated away by a transformation of the first and second families (angle \( \theta_{12} \)). No rotation between the first and third families is required. The rotation matrix for this sequence takes the form

\[
R = R_{12} R_{23} = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{12} & s_{12} \\
0 & -s_{12} & c_{12}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
c_{12} & s_{12} c_{23} & s_{12} s_{23} \\
-s_{12} & c_{12} c_{23} & c_{12} s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix},
\]

where \( c_{12} \equiv \cos \theta_{12}, \ s_{12} \equiv \sin \theta_{12}, \) etc. The flavor mixing matrix \( V \) is the product of two such matrices, one describing the rotation among the up-type quarks, and the other describing the rotation among the down-type quarks:

\[
V = R_{12}^u R_{23}^u (R_{23}^d)^{-1} (R_{12}^d)^{-1}.
\]

The product \( R_{23}^u (R_{23}^d)^{-1} \) can be written as a rotation matrix described by a single angle \( \theta \). In the limit \( m_u = m_d = 0 \), this is just the angle \( \tilde{\theta} \) encountered in Eq. (2). The angle which describes the \( R_{12}^u \) rotation shall be denoted by \( \theta_u \); and the corresponding angle for the \( R_{12}^d \)
rotation by $\theta_d$. Thus in the absence of $CP$-violating phases the flavor mixing matrix takes the following specific form:

$$V = \begin{pmatrix} c_u & s_u & 0 \\ -s_u & c_u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \begin{pmatrix} c_d & -s_d & 0 \\ s_d & c_d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s_u s_d c + c_u c_d & s_u c_d c - c_u s_d & s_u s \\ c_u s_d c - s_u c_d & c_u c_d c + s_u s_d c & c_u s \\ -s_d s & -c_d s & c \end{pmatrix},$$

(7)

where $c_u \equiv \cos \theta_u$, $s_u \equiv \sin \theta_u$, etc.

We proceed by including the phase parameters of the quark mass matrices in Eq. (4). Each mass matrix has in general two complex phases. These phases can be dealt with in a similar way as described in Refs. [3, 4]. It can easily be seen that, by suitable rephasing of the quark fields, the flavor mixing matrix can finally be written in terms of only a single phase $\varphi$ as follows:

$$V = \begin{pmatrix} c_u & s_u & 0 \\ -s_u & c_u & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\varphi} & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \begin{pmatrix} c_d & -s_d & 0 \\ s_d & c_d & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s_u s_d c + c_u c_d e^{-i\varphi} & s_u c_d c - c_u s_d e^{-i\varphi} & s_u s \\ c_u s_d c - s_u c_d e^{-i\varphi} & c_u c_d c + s_u s_d e^{-i\varphi} & c_u s \\ -s_d s & -c_d s & c \end{pmatrix},$$

(8)

Note that the three angles $\theta_u$, $\theta_d$ and $\theta$ in Eq. (8) can all be arranged to lie in the first quadrant through a suitable redefinition of quark field phases. Consequently all $s_u$, $s_d$, $s$ and $c_u$, $c_d$, $c$ are positive. The phase $\varphi$ can in general take values from 0 to $2\pi$; and $CP$ violation is present in weak interactions if $\varphi \neq 0, \pi$ and $2\pi$.

This particular representation of the flavor mixing matrix is the main result of this paper. In comparison with all other parametrizations discussed previously [1, 2], it has a number of interesting features which in our view make it very attractive and provide strong arguments for its use in future discussions of flavor mixing phenomena, in particular, those in $B$-meson physics. We shall discuss them below.

a) The flavor mixing matrix $V$ in Eq. (8) follows directly from the chiral expansion of the mass matrices. Thus it naturally takes into account the hierarchical structure of the quark mass spectrum.

b) The complex phase describing $CP$ violation ($\varphi$) appears only in the (1,1), (1,2), (2,1) and (2,2) elements of $V$, i.e., in the elements involving only the quarks of the first and second families. This is a natural description of $CP$ violation since in our hierarchical approach $CP$ violation is not directly linked to the third family, but rather to the first and second ones, and in particular to the mass terms of the $u$ and $d$ quarks.
It is instructive to consider the special case \( s_u = s_d = s = 0 \). Then the flavor mixing matrix \( V \) takes the form

\[
V = \begin{pmatrix}
e^{-i\phi} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

This matrix describes a phase change in the weak transition between \( u \) and \( d \), while no phase change is present in the transitions between \( c \) and \( s \) as well as \( t \) and \( b \). Of course, this effect can be absorbed in a phase change of the \( u \)- and \( d \)-quark fields, and no \( CP \) violation is present. Once the angles \( \theta_u \), \( \theta_d \) and \( \theta \) are introduced, however, \( CP \) violation arises. It is due to a phase change in the weak transition between \( u' \) and \( d' \), where \( u' \) and \( d' \) are the rotated quark fields, obtained by applying the corresponding rotation matrices given in Eq. (8) to the quark mass eigenstates (\( u' \): mainly \( u \), small admixture of \( c \); \( d' \): mainly \( d \), small admixture of \( s \)).

Since the mixing matrix elements involving \( t \) or \( b \) quark are real in the representation (8), one can find that the phase parameter of \( B_0^q - \bar{B}_0^q \) mixing (\( q = d \) or \( s \)), dominated by the box-diagram contributions in the standard model [15], is essentially unity:

\[
(q/p)_{B_q} = V_{tb}^* V^*_{tq} = 1.
\]

In most of other parametrizations of the flavor mixing matrix, however, the two rephasing-variant quantities \((q/p)_{B_d}\) and \((q/p)_{B_s}\) take different (maybe complex) values.

c) The dynamics of flavor mixing can easily be interpreted by considering certain limiting cases in Eq. (8). In the limit \( \theta \rightarrow 0 \) (i.e., \( s \rightarrow 0 \) and \( c \rightarrow 1 \)), the flavor mixing is, of course, just a mixing between the first and second families, described by only one mixing angle (the Cabibbo angle \( \theta_C \) [16]). It is a special and essential feature of the representation (8) that the Cabibbo angle is not a basic angle, used in the parametrization. The matrix element \( V_{us} \) (or \( V_{cd} \)) is indeed a superposition of two terms including a phase. This feature arises naturally in our hierarchical approach, but it is not new. In many models of specific textures of mass matrices, it is indeed the case that the Cabibbo-type transition \( V_{us} \) (or \( V_{cd} \)) is a superposition of several terms. At first, it was obtained by one of the authors in the discussion of the two-family mixing [3].

In the limit \( \theta = 0 \) considered here, one has \(|V_{us}| = |V_{cd}| = \sin \theta_C \equiv s_C \) and

\[
s_C = |s_u c_d - c_u s_d e^{-i\phi}|.
\]

This relation describes a triangle in the complex plane, as illustrated in Fig. 1, which we shall denote as the “Cabibbo triangle”. This triangle is a feature of the mixing of the first two families (see also Ref. [3]). Explicitly one has (for \( s = 0 \)):

\[
\tan \theta_C = \sqrt{\frac{\tan^2 \theta_u + \tan^2 \theta_d - 2 \tan \theta_u \tan \theta_d \cos \varphi}{1 + \tan^2 \theta_u \tan^2 \theta_d + 2 \tan \theta_u \tan \theta_d \cos \varphi}}.
\]
Certainly the flavor mixing matrix $V$ cannot accommodate $CP$ violation in this limit. However, the existence of $\varphi$ seems necessary in order to make Eq. (12) compatible with current data, as one can see below.

d) The three mixing angles $\theta$, $\theta_u$ and $\theta_d$ have a precise physical meaning. The angle $\theta$ describes the mixing between the second and third families, which is generated by the off-diagonal terms $B_u$ and $B_d$ in the up and down mass matrices of Eq. (4). We shall refer to this mixing involving $t$ and $b$ as the “heavy quark mixing”. The angle $\theta_u$, however, solely describes the $u$-$c$ mixing, corresponding to the $D_u$ term in $M_u$. We shall denote this as the “$u$-channel mixing”. The angle $\theta_d$ solely describes the $d$-$s$ mixing, corresponding to the $D_d$ term in $M_d$; this will be denoted as the “$d$-channel mixing”. Thus there exists an asymmetry between the mixing of the first and second families and that of the second and third families, which in our view reflects interesting details of the underlying dynamics of flavor mixing. The heavy quark mixing is a combined effect, involving both charge $+2/3$ and charge $-1/3$ quarks, while the $u$- or $d$-channel mixing (described by the angle $\theta_u$ or $\theta_d$) proceeds solely in the charge $+2/3$ or charge $-1/3$ sector. Therefore an experimental determination of these two angles would allow to draw interesting conclusions about the amount and perhaps the underlying pattern of the $u$- or $d$-channel mixing.

e) The three angles $\theta$, $\theta_u$ and $\theta_d$ are related in a very simple way to observable quantities of $B$-meson physics. For example, $\theta$ is related to the rate of the semileptonic decay $B \to D^* l \nu_l$; $\theta_u$ is associated with the ratio of the decay rate of $B \to (\pi, \rho) l \nu_l$ to that of $B \to D^* l \nu_l$; and $\theta_d$ can be determined from the ratio of the mass difference between two $B_d$ mass eigenstates to that between two $B_s$ mass eigenstates. From Eq. (8) we find the following exact relations:

$$\sin \theta = |V_{cb}| \sqrt{1 + \frac{|V_{ub}|^2}{V_{cb}^2}},$$

and

$$\tan \theta_u = \frac{V_{ub}}{V_{cb}},$$

$$\tan \theta_d = \frac{V_{td}}{V_{ts}}. \quad (14)$$

These simple results make the parametrization (8) uniquely favorable for the study of $B$-meson physics.
By use of current data on $|V_{ub}|$ and $|V_{cb}|$, i.e., $|V_{cb}| = 0.039 \pm 0.002$ \cite{12, 13} and $|V_{ub}/V_{cb}| = 0.08 \pm 0.02$ \cite{13}, we obtain $\theta_u = 4.57^\circ \pm 1.14^\circ$ and $\theta = 2.25^\circ \pm 0.12^\circ$. Taking $|V_{td}| = (8.6 \pm 2.1) \times 10^{-3}$ \cite{13}, which was obtained from the analysis of current data on $B_d^0 - \bar{B}_d^0$ mixing, we get $|V_{td}/V_{ts}| = 0.22 \pm 0.07$, i.e., $\theta_d = 12.7^\circ \pm 3.8^\circ$. Both the heavy quark mixing angle $\theta$ and the u-channel mixing angle $\theta_u$ are relatively small. The smallness of $\theta$ implies that Eqs. (11) and (12) are valid to a high degree of precision (of order $1 - c \approx 0.001$).

f) It is instructive to consider the limiting case $\theta_u \to 0$ or $\theta_d \to 0$, which can be achieved by setting $D_u \to 0$ or $D_d \to 0$ in Eq. (4). In the absence of the u-channel mixing ($\theta_u = 0$), one has $V_{ub} = 0$ \cite{7}; in the absence of the d-channel mixing, $V_{td} = 0$ appears. In both cases $CP$ violation is absent. In reality, we have $\theta_d/\theta_u \approx 2 \cdots 3$, i.e., the u-channel mixing is significantly smaller than the d-channel mixing.

In the absence of the u-channel mixing ($\theta_u = 0$), one finds
\begin{equation}
\left| \frac{V_{us}}{V_{ud}} \right| = \left| \frac{V_{td}}{V_{cs}} \right| = \left| \frac{V_{td}}{V_{ts}} \right| = \tan \theta_d ; \tag{15}
\end{equation}
and in the absence of the d-channel mixing ($\theta_d = 0$), one arrives at
\begin{equation}
\left| \frac{V_{us}}{V_{cs}} \right| = \left| \frac{V_{td}}{V_{ud}} \right| = \left| \frac{V_{ub}}{V_{cb}} \right| = \tan \theta_u . \tag{16}
\end{equation}
Certainly the last relation does not hold well, since the experimental value of $|V_{ub}/V_{cb}|$ is only about 1/3 of that of $|V_{us}/V_{cs}|$. Of course, this is due to the fact that the d-channel mixing angle $\theta_d$ is relatively large. In contrast, the relation (15) is consistent with current data. If it were exact, i.e., $\theta_u = 0$, then $\tan \theta_d$ would be determined by $|V_{us}/V_{ud}|$, which has been precisely measured. Using the experimental values $|V_{us}| = 0.2205 \pm 0.0018$ and $|V_{ud}| = 0.9736 \pm 0.0010$ \cite{15}, one would find $\theta_d = 12.76^\circ \pm 0.12^\circ$ on the basis of Eq. (15). However, since $\theta_u$ does not vanish exactly, the actual error for $\theta_d$ obtained from Eq. (15) should be as large as that given above: $\theta_d = 12.7^\circ \pm 3.8^\circ$. It is interesting, nevertheless, that the central values of $\theta_d$ obtained from the relations (14) and (15) are essentially identical.

g) According to Eq. (8), as well as Eq. (11), the phase $\varphi$ is a phase difference between the contributions to $V_{us}$ (or $V_{cd}$) from the u-channel mixing and the d-channel mixing. Therefore $\varphi$ is given by the relative phase of $D_d$ and $D_u$ in the quark mass matrices (4), if the phases of $B_u$ and $B_d$ are absent or negligible.

The phase $\varphi$ is not likely to be $0^\circ$ or $180^\circ$, according to the experimental values given above, even though the measurement of $CP$ violation in $K^0 - \bar{K}^0$ mixing \cite{13} is not taken into account. For $\varphi = 0^\circ$, one finds $\tan \theta_C = 0.14 \pm 0.08$; and for $\varphi = 180^\circ$, one gets $\tan \theta_C = 0.30 \pm 0.08$. Both cases are barely consistent with the value of $\tan \theta_C$ obtained from experiments ($\tan \theta_C \approx |V_{us}/V_{ud}| \approx 0.226$).

h) The $CP$-violating phase $\varphi$ in the flavor mixing matrix $V$ can be determined from $|V_{us}| (= 0.2205 \pm 0.0018$ \cite{13}) through the following formula, obtained easily from Eq. (8):
\begin{equation}
\varphi = \arccos \left( \frac{s_u s_d c_c^2 + c_u s_d^2 - |V_{us}|^2}{2 s_u c_u s_d c_c} \right) . \tag{17}
\end{equation}
The two-fold ambiguity associated with the value of $\varphi$, coming from $\cos \varphi = \cos(2\pi - \varphi)$, is removed if one takes $\sin \varphi > 0$ into account (this is required by current data on CP violation in $K^0\bar{K}^0$ mixing (i.e., $\epsilon_K$) [14]). More precise measurements of the angles $\theta_u$ and $\theta_d$ in the forthcoming experiments of $B$ physics will remarkably reduce the uncertainty of $\varphi$ to be determined from Eq. (17). This approach is of course complementary to the direct determination of $\varphi$ from CP asymmetries in some weak $B$-meson decays into hadronic $CP$ eigenstates [18].

For illustration, we plot the allowed region of $\varphi$ as a function of $\tan \theta_d$ in Fig. 2, where the values of $|V_{us}|$, $\theta$, $\theta_u$ and $\theta_d$ are taken to vary independently within their corresponding errors. The constraint from the experimental result of $\epsilon_K$ on $\varphi$ is also included (see, e.g., Ref. [19] for relevant formulas of $\epsilon_K$). One can find that the value of $\varphi$ is most likely in the range $40^\circ$ to $120^\circ$; indeed the central values of the four inputs lead to $\varphi \approx 81^\circ$. Note that $\varphi$ is essentially independent of the angle $\theta$, due to the tiny observed value of the latter. Once $\tan \theta_d$ is precisely measured, we shall be able to fix the magnitude of $\varphi$ to a satisfactory degree of accuracy.

i) It is well-known that CP violation in the flavor mixing matrix $V$ can be rephasing-invariantly described by a universal quantity $J$ [20]:

$$\text{Im} \left(V_{il}V_{jm}^*V_{nl}^*\right) = J \sum_{k,n=1}^{3} (\epsilon_{ijk}\epsilon_{lmn}) .$$ (18)

In the parametrization (8), $J$ reads

$$J = s_\alpha c_\alpha s_\delta c_\delta s^2 c \sin \varphi .$$ (19)

Obviously $\varphi = 90^\circ$ leads to the maximal value of $J$. 

Figure 2: The region of the CP-violating phase $\varphi$ allowed by current data.
Indeed $\varphi = 90^\circ$, a particularly interesting case for $CP$ violation, is quite consistent with current data. One can see from Fig. 2 that this possibility exists if $0.202 \leq \tan \theta_d \leq 0.222$, or $11.4^\circ \leq \theta_d \leq 12.5^\circ$. In this case the mixing term $D_d$ in Eq. (4) can be taken to be real, and the term $D_u$ to be imaginary, if $\text{Im}(B_u) = \text{Im}(B_d) = 0$ is assumed (see also Refs. [21, 11]). Since in our description of the flavor mixing the complex phase $\varphi$ is related in a simple way to the phases of the quark mass terms, the case $\varphi = 90^\circ$ is especially interesting. It can hardly be an accident, and this case should be studied further. The possibility that the phase $\varphi$ describing $CP$ violation in the standard model is given by the algebraic number $\pi/2$ should be taken seriously. It may provide a useful clue towards a deeper understanding of the origin of $CP$ violation and of the dynamical origin of the fermion masses.

In Ref. [21] the case $\varphi = 90^\circ$ has been denoted as “maximal” $CP$ violation. It implies in our framework that in the complex plane the u-channel and d-channel mixings are perpendicular to each other. In this special case (as well as $\theta \to 0$), we have

$$
\tan^2 \theta_C = \frac{\tan^2 \theta_u + \tan^2 \theta_d}{1 + \tan^2 \theta_u \tan^2 \theta_d}.
$$

To a good approximation (with the relative error $\sim 2\%$), one finds $s_C^2 \approx s_u^2 + s_d^2$.

j) At future $B$-meson factories, the study of $CP$ violation will concentrate on measurements of the unitarity triangle

$$
S_u + S_c + S_t = 0,
$$

where $S_i \equiv V_{id}V_{ib}^*$ in the complex plane (see Fig. 3(a) for illustration). The inner angles of this triangle are denoted as

$$
\alpha \equiv \arg(-S_tS_u^*),
\beta \equiv \arg(-S_cS_t^*),
\gamma \equiv \arg(-S_uS_c^*).
$$

Obviously $\alpha + \beta + \gamma = \pi$ is a trivial consequence of the above definition [22]. In terms of the

\footnote{An alternative notation for three angles of the unitarity triangle is $(\phi_1, \phi_2, \phi_3)$, equivalent to $(\beta, \alpha, \gamma)$.}

\begin{figure}[h]
  \centering
  \includegraphics[width=0.8\textwidth]{unitarity_triangle.png}
  \caption{The unitarity triangle (a) and its rescaled counterpart (b) in the complex plane.}
\end{figure}
parameters $\theta$, $\theta_u$, $\theta_d$ and $\varphi$, we obtain
\[
\sin(2\alpha) = \frac{2c_u c_d \sin \varphi (s_u s_d c + c_u c_d \cos \varphi)}{s_u^2 s_d^2 c^2 + c_u^2 c_d^2 + 2s_u c_u s_d c \cos \varphi},
\]
\[
\sin(2\beta) = \frac{2s_u c_d \sin \varphi (c_u s_d c - s_u c_d \cos \varphi)}{c_u^2 s_d^2 c^2 + s_u^2 c_d^2 - 2s_u c_u s_d c \cos \varphi}.
\] (23)

To an excellent degree of accuracy, one finds $\alpha \approx \varphi$. In order to illustrate how accurate this relation is, let us input the central values of $\theta$, $\theta_u$ and $\theta_d$ (i.e., $\theta = 2.25^\circ$, $\theta_u = 4.57^\circ$ and $\theta_d = 12.7^\circ$) to Eq. (23). Then one arrives at $\varphi - \alpha \approx 1^\circ$ as well as $\sin(2\alpha) \approx 0.34$ and $\sin(2\beta) \approx 0.65$. It is expected that $\sin(2\alpha)$ and $\sin(2\beta)$ will be directly measured from the $CP$ asymmetries in $B_d \to \pi^+\pi^-$ and $B_d \to J/\psi K_S$ modes at a $B$-meson factory.

Note that the three sides of the unitarity triangle (21) can be rescaled by $|V_{cb}|$. In a very good approximation (with the relative error $\sim 2\%$), one arrives at
\[
|S_u| : |S_c| : |S_t| \approx s_u c_d : s_C : s_d.
\] (24)

Equivalently, one can obtain
\[
s_\alpha : s_\beta : s_\gamma \approx s_C : s_u c_d : s_d,
\] (25)
where $s_\alpha \equiv \sin \alpha$, etc. The rescaled unitarity triangle is shown in Fig. 3(b). Comparing this triangle with the Cabibbo triangle in Fig. 1, we find that they are indeed congruent with each other to a high degree of accuracy. The congruent relation between these two triangles is particularly interesting, since the Cabibbo triangle is essentially a feature of the physics of the first two quark families, while the unitarity triangle is linked to all three families. In this connection it is of special interest to note that in models which specify the textures of the mass matrices the Cabibbo triangle and hence three inner angles of the unitarity triangle can be fixed by the spectrum of the light quark masses and the $CP$-violating phase $\varphi$ (see, e.g., Ref. [21]).

k) It is worth pointing out that the u-channel and d-channel mixing angles are related to the so-called Wolfenstein parameters [23] in a simple way:
\[
\tan \theta_u = \frac{|V_{ub}|}{|V_{cb}|} \approx \lambda \sqrt{\rho^2 + \eta^2},
\]
\[
\tan \theta_d = \frac{|V_{td}|}{|V_{ts}|} \approx \lambda \sqrt{(1-\rho)^2 + \eta^2},
\] (26)
where $\lambda \approx s_C$ measures the magnitude of $V_{us}$. Note that the $CP$-violating parameter $\eta$ is linked to $\varphi$ through
\[
\sin \varphi \approx \frac{\eta}{\sqrt{\rho^2 + \eta^2} \sqrt{(1-\rho)^2 + \eta^2}}
\] (27)
in the lowest-order approximation. Then $\varphi = 90^\circ$ implies $\eta^2 \approx \rho(1-\rho)$, on the condition $0 < \rho < 1$. In this interesting case, of course, the flavor mixing matrix can fully be described in terms of only three independent parameters.
1) Compared with the standard parametrization of the flavor mixing matrix $V$ advocated in Ref. [2], the parametrization (8) has an additional advantage: the renormalization-group evolution of $V$, from the weak scale to an arbitrary high energy scale, is to a very good approximation associated only with the angle $\theta$. This can easily be seen if one keeps the $t$ and $b$ Yukawa couplings only and neglects possible threshold effect in the one-loop renormalization-group equations of the Yukawa matrices [21]. Thus the parameters $\theta_u$, $\theta_d$ and $\phi$ are essentially independent of the energy scale, while $\theta$ does depend on it and will change if the underlying scale is shifted, say from the weak scale ($\sim 10^2$ GeV) to the grand unified theory scale ($\sim 10^{16}$ GeV). In short, the heavy quark mixing is subject to renormalization-group effects; but the $u$- and $d$-channel mixings are not, likewise the phase $\phi$ describing $CP$ violation.

In this paper we have presented a new description of the flavor mixing phenomenon, which is based on the phenomenological fact that the quark mass spectrum exhibits a clear hierarchy pattern. This leads uniquely to the interpretation of the flavor mixing in terms of a heavy quark mixing, followed by the $u$-channel and $d$-channel mixings. The complex phase $\phi$, describing the relative orientation of the $u$-channel mixing and the $d$-channel mixing in the complex plane, signifies $CP$ violation, which is a phenomenon primarily linked to the physics of the first two families. The Cabibbo angle is not a basic mixing parameter, but given by a superposition of two terms involving the complex phase $\phi$. The experimental data suggest that the phase $\phi$, which is directly linked to the phases of the quark mass terms, is close to $90^\circ$. This opens the possibility to interpret $CP$ violation as a maximal effect, in a similar way as parity violation.

Our description of flavor mixing has many clear advantages compared with other descriptions. We propose that it should be used in the future description of flavor mixing and $CP$ violation, in particular, for the studies of quark mass matrices and $B$-meson physics.

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