REPRESENTATIONS OF INTEGERS BY CERTAIN POSITIVE DEFINITE BINARY QUADRATIC FORMS

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Abstract. We prove part of a conjecture of Borwein and Choi concerning an estimate on the square of the number of solutions to \( n = x^2 + N y^2 \) for a squarefree integer \( N \).

1. Introduction

We consider the positive definite quadratic form \( Q(x, y) = x^2 + N y^2 \) for a squarefree integer \( N \). Let \( r_{2,N}(n) \) denote the number of solutions to \( n = Q(x, y) \) (counting signs and order). In this note, we estimate

\[
\sum_{n \leq x} r_{2,N}(n)^2.
\]

A positive squarefree integer \( N \) is called solvable if \( x^2 + N y^2 \) has one form per genus. Note that this means the class number of the form class group of discriminant \( -4N \) equals the number of genera, \( 2^t \), where \( t \) is the number of distinct prime factors of \( N \).

Concerning \( r_{2,N}(n) \), Borwein and Choi [2] proved the following:

**Theorem 1.1.** Let \( N \) be a solvable squarefree integer. Let \( x > 1 \) and \( \epsilon > 0 \). We have

\[
\sum_{n \leq x} r_{2,N}(n)^2 = 3 \frac{N}{2N} \prod_{p \mid 2N} \left( \frac{2p}{p+1} \right) (x \log x + \alpha(N)x) + O(N^{\frac{3}{4}} + x^{\frac{3}{4}} + \epsilon)
\]

where the product is over all primes dividing \( 2N \) and

\[
\alpha(N) = -1 + 2\gamma + \sum_{p \mid 2N} \frac{\log p}{p+1} + \frac{2L'(1, \chi_{-4N})}{L(1, \chi_{-4N})} - \frac{12}{\pi^2} \zeta'(2)
\]

where \( \gamma \) is the Euler-Mascheroni constant and \( L(1, \chi_{-4N}) \) is the L-function corresponding to the quadratic character mod \(-4N\).

Based on this result, Borwein and Choi posed the following:

**Conjecture 1.2.** For any squarefree \( N \),

\[
\sum_{n \leq x} r_{2,N}(n)^2 \sim 3 \frac{N}{2N} \left( \prod_{p \mid 2N} \frac{2p}{p+1} \right) x \log x
\]

Our main result is the following.

**Theorem 1.3.** Let \( Q(x, y) = x^2 + N y^2 \) for a squarefree integer \( N \) with \(-N \not\equiv 1 \mod 4\). Let \( r_{2,N}(n) \) denote the number of solutions to \( n = Q(x, y) \) (counting signs and order). Then

\[
\sum_{n \leq x} r_{2,N}(n)^2 \sim 3 \frac{N}{2N} \left( \prod_{p \mid 2N} \frac{2p}{p+1} \right) x \log x.
\]

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2. Preliminaries

We first discuss two key estimates and a result of Kronecker on genus characters. Then using Kronecker’s result, we prove a proposition relating genus characters to poles of the Rankin-Selberg convolution of L-functions. The first estimate is a recent result of Kühleitner and Nowak [13], namely

**Theorem 2.1.** Let \(a(n)\) be an arithmetic function satisfying \(a(n) \ll n^\epsilon\) for every \(\epsilon > 0\), with a Dirichlet series

\[
F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \frac{(\zeta_K(s))^2}{(\zeta(2s))^{m_1} (\zeta_K(2s))^{m_2}} G(s)
\]

where \(\Re(s) > 1\) and \(\zeta_K(s)\) is the Dedekind zeta function of some quadratic number field \(K\), \(G(s)\) is holomorphic and bounded in some half plane \(\Re(s) \geq \theta, \theta < \frac{1}{2}\), and \(m_1, m_2\) are nonnegative integers. Then for \(x\) large,

\[
\sum_{n \leq x} a(n) = \text{Res}_{s=1} \left( F(s) \frac{x^s}{s} \right) + O(x^{\frac{1}{2}} (\log x)^3 (\log \log x)^{m_1+m_2})
\]

\[
= Ax \log x + Bx + O(a^{\frac{3}{2}} (\log x)^3 (\log \log x)^{m_1+m_2})
\]

where \(A\) and \(B\) are computable constants.

For an arbitrary quadratic number field \(K\) with discriminant \(d_K\), let \(\mathcal{O}_K\) denote the ring of integers in \(K\), and \(r_K(n)\) the number of integral ideals \(I\) in \(\mathcal{O}_K\) of norm \(N(I) = n\). From (4.1) in [13], we have

\[
\sum_{n=1}^{\infty} \frac{(r_K(n))^2}{n^s} = \frac{(\zeta_K(s))^2}{\zeta(2s)} \prod_{p|d_K} (1 + p^{-s})^{-1}.
\]

Applying Theorem 2.1 with \(m_1 = 1\) and \(m_2 = 0\), we obtain

**Corollary 2.2.** For any quadratic field \(K\) of discriminant \(d_K\) and \(x\) large,

\[
\sum_{n \leq x} (r_K(n))^2 = A_1 x \log x + B_1 x + O(x^{\frac{3}{2}} (\log x)^3 \log \log x),
\]

with \(A_1 = \frac{6}{\pi^2} L(1, \chi_{d_K})^2 \prod_{p|d_K} \frac{p}{p+1}\) and \(B_1 = A_1 \alpha(N)\) with \(\alpha(N)\) as in Theorem 1.1.

The second estimate is a classical result of Rankin [16] and Selberg [17] which estimates the size of Fourier coefficients of a modular form. Specifically, if \(f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i nz}\) is a nonzero cusp form of weight \(k\) on \(\Gamma_0(N)\), then

\[
\sum_{n \leq x} |a(n)|^2 = \alpha(f,f)x^k + O(x^{k-\frac{1}{2}})
\]

where \(\alpha > 0\) is an absolute constant and \(\langle f, f \rangle\) is the Petersson scalar product. In particular, if \(f\) is a cusp form of weight 1, then \(\sum_{n \leq x} |a(n)|^2 = O(x)\). One can adapt their result to say the following. Given two cusp forms of weight \(k\) on a suitable congruence subgroup of \(\Gamma = SL_2(\mathbb{Z})\), say \(f(z) = \sum_{n=1}^{\infty} a(n) e^{2\pi i nz}\) and \(g(z) = \sum_{n=1}^{\infty} b(n) e^{2\pi i nz}\), then

\[
\sum_{n \leq x} a(n)\overline{b(n)} n^{1-k} = Ax + O(x^{\frac{1}{2}})
\]
where $A$ is a constant. In particular, if $f$ and $g$ are cusp forms of weight 1, then
\[ \sum_{n \leq x} a(n)b(n) = O(x). \]

We will also use a result of Kronecker on genus characters. Let us first explain some terminology. Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic field of discriminant $d_K$. $d_K$ is said to be a prime discriminant if it only has one prime factor. Thus it must be of the form: $-4$, $\pm 8$, $\pm p \equiv 1 \pmod{4}$ for an odd prime $p$. Every discriminant can be written uniquely as a product of prime discriminants, say $d_K = P_1 \ldots P_k$. Here $k$ denotes the number of distinct prime factors of $d_K$. Thus $d_K$ can be written as a product of two discriminants, say $d_K = D_1D_2$ in $2^{k-1}$ distinct ways (excluding order). Now, for any such decomposition we define a character $\chi_{D_1,D_2}$ on ideals by
\[
\chi_{D_1,D_2}(\mathfrak{p}) = \begin{cases} 
\chi_{D_1}(N\mathfrak{p}) & \text{if } \mathfrak{p} \nmid D_1 \\
\chi_{D_2}(N\mathfrak{p}) & \text{if } \mathfrak{p} \nmid D_2
\end{cases}
\]
where $\chi_d(n)$ is the Kronecker symbol. This is well defined on prime ideals because $\chi_D(N\mathfrak{a}) = 1$ if $(\mathfrak{a}, D) = 1$. $\chi_{D_1,D_2}$ extends to all fractional ideals by multiplicativity. Hence we have
\[
\chi_{D_1,D_2} : I \to \{ \pm 1 \}
\]
where $I$ is the group of nonzero fractional ideals of $\mathcal{O}_K$. Thus $\chi_{D_1,D_2}$ has order two, except for the trivial character corresponding to $d_K = d_K \cdot 1 = 1 \cdot d_K$. Every such character $\chi_{D_1,D_2}$ is called the genus character of discriminant $d_K$. As these are different for distinct factorizations of $d_K$ (into a product of two discriminants), we have $2^{k-1}$ genus characters. Kronecker’s theorem (see Theorem 12.7 in [11]) is as follows.

**Theorem 2.3.** The $L$-function of $K$ associated with the genus character $\chi_{D_1,D_2}$ factors into the Dirichlet $L$-functions,
\[
L(s, \chi_{D_1,D_2}) = L(s, \chi_{D_1})L(s, \chi_{D_2}).
\]

Let $K = \mathbb{Q}(\sqrt{-N})$, $N$ squarefree, $I$ as above, and $P$ the subgroup of $I$ of principal ideals. For a non-zero integral ideal $\mathfrak{m}$ of $\mathcal{O}_K$, define
\[
I(\mathfrak{m}) = \{ a \in I : (a, \mathfrak{m}) = 1 \}
\]
\[
P(\mathfrak{m}) = \{ (a) \in P : a \equiv 1 \pmod{\mathfrak{m}} \}.
\]

A group homomorphism $\chi : I_\mathfrak{m} \to \mathbb{S}^1$ is an ideal class character if it is trivial on $P(\mathfrak{m})$, i.e.
\[
\chi((a)) = 1
\]
for $a \equiv 1 \pmod{\mathfrak{m}}$. Thus an ideal class character is a character on the ray class group $I(\mathfrak{m})/P(\mathfrak{m})$. Taking the trivial modulus $\mathfrak{m} = 1$, we obtain a character on the ideal class group of $K$. Note that for $K = \mathbb{Q}(\sqrt{-N})$ a genus character is an ideal class character of order at most two.

Let us also recall the notion of the Rankin-Selberg convolution of two $L$-functions. For squarefree $N$, consider two ideal class characters $\chi_1$, $\chi_2$ for $\mathbb{Q}(\sqrt{-N})$ and their associated Hecke $L$-series
\[
L(s, \chi_1) = \sum_{n=1}^{\infty} \frac{\chi_1(n)}{n^s}
\]
\[
L(s, \chi_2) = \sum_{n=1}^{\infty} \frac{\chi_2(n)}{n^s}
\]
which converge absolutely in some right half-plane. We form the convolution $L$-series by multiplying the coefficients,
\[
L(s, \chi_1 \otimes \chi_2) = \sum_{n=1}^{\infty} \frac{\chi_1(n)\chi_2(n)}{n^s}.
\]

The following result describes a relationship between genus characters \(\chi\) and the orders of poles of \(L(s, \chi \otimes \chi)\). Precisely,

**Proposition 2.4.** Let \(\chi\) be an ideal class character of \(\mathbb{Q}(\sqrt{-N})\), \(-N \not\equiv 1 \mod 4\), and \(L(s, \chi)\) the associated Hecke L-series. Then \(\chi\) is a genus character if and only if \(L(s, \chi \otimes \chi)\) has a double pole at \(s = 1\).

**Proof.** Suppose \(\chi_{D_1,D_2}\) is a genus character of discriminant \(-4N\), and \(L(s, \chi_{D_1,D_2}) = \sum_{n=1}^{\infty} \frac{b_i(n)}{n^s}\). By Theorem 2.3 and Exercise 1.2.8 in [14] (see the solution), we have

\[
\sum_{n=1}^{\infty} \frac{b_i(n)^2}{n^s} = \frac{L(s, \chi_{D_1}^2)L(s, \chi_{D_2}^2)L(s, \chi_{D_1}\chi_{D_2})^2}{L(2s, \chi_{D_1}\chi_{D_2}^2)}.
\]

Note that

\[
L(s, \chi_{D_1}^2) = \zeta(s) \cdot \prod_{p \mid D_1} (1 - p^{-s}),
\]

\[
L(s, \chi_{D_2}^2) = \zeta(s) \cdot \prod_{p \mid D_2} (1 - p^{-s}),
\]

\[
L(s, \chi_{D_1}\chi_{D_2})^2 = L(s, \chi_{-4N})^2,
\]

and

\[
L(2s, \chi_{D_1}\chi_{D_2}) = \zeta(2s) \cdot \prod_{p \mid D_1D_2} (1 - p^{-2s}).
\]

We have

\[
\sum_{n=1}^{\infty} \frac{b_i(n)^2}{n^s} = \frac{\zeta(s)^2L(s, \chi_{-4N})^2}{\zeta(2s)} \prod_{p \mid 2N} (1 + p^{-s})^{-1}
\]

and thus a double pole at \(s = 1\).

Conversely, let \(\chi\) be an ideal class character of \(K = \mathbb{Q}(\sqrt{-N})\) and suppose \(L(s, \chi \otimes \chi)\) has a double pole at \(s = 1\). Now \(\chi\) is an automorphic form on \(GL_1(\mathbb{A}_K)\). By automorphic induction (see [1]), \(\chi\) is mapped to \(\pi\), a cuspidal automorphic representation of \(GL_2(\mathbb{A}_Q)\). Note that \(\pi\) is reducible as, otherwise, \(L(s, \pi \otimes \pi)\) has a simple pole at \(s = 1\) ([1], page 200). As \(K\) is a quadratic extension of \(\mathbb{Q}\), we must have \(\pi = \chi_1 + \chi_2\), where \(\chi_i\) are Dirichlet characters. As \(L(s, \chi_1) = L(s, \pi)\) (see [1]) and thus \(L(s, \chi_1 \otimes \chi_2) = L(s, \pi \otimes \pi)\),

\[
L(s, \pi \otimes \pi) = L(s, \chi_1 \otimes \chi_2) = \frac{L(s, \chi_1^2)L(s, \chi_2^2)L(s, \chi_1\chi_2)^2}{L(2s, \chi_1^2\chi_2^2)}.
\]

Now \(L(s, \chi_1 \otimes \chi_2)\) has a double pole at \(s = 1\) if and only if either \(\chi_1 = \overline{\chi_2}\), \(\chi_2^2 \not= 1\), and \(\chi_1 \not= 1\) or \(\chi_1^2 = 1\), \(\chi_2 = 1\), and \(\chi_1\chi_2 \not= 1\). The latter implies \(\chi\) is a genus character. We now need to show that the former also implies that \(\chi\) is a genus character. Note that

\[
L(s, \chi) = \prod_p \left(1 - \frac{\chi(p)}{N(p)^s}\right)^{-1}
\]
and
\[ L(s, \chi_1 + \chi_2) = \prod_p \left( 1 - \frac{\chi_1(p)}{p^s} \right)^{-1} \prod_p \left( 1 - \frac{\chi_2(p)}{p^s} \right)^{-1}. \]

As \( L(s, \chi) = L(s, \pi) \) and \( L(s, \pi) = L(s, \chi_1 + \chi_2) \), we compare Euler factors to get
\[ \chi_1(p) + \chi_2(p) = \begin{cases} 0 & \text{if } p \text{ is inert in } K \\ \chi(p) + \overline{\chi(p)} & \text{if } p \text{ splits in } K. \end{cases} \]

For \( p \) inert in \( K \), this yields \( \chi_1(p) = -\chi_2(p) \) and so \( \overline{\chi_2(p)} = \chi_1(p) = -\chi_2(p) \) which implies \( \chi_2(p) = -1 \) and so \( \chi_2(p) = \pm i \). Now consider the following equation whose sum sieves the inert primes
\[
\frac{1}{2} \sum_{p \leq x} \left( 1 - \left( \frac{-4N}{p} \right) \right) \chi_2(p) = -\pi(x).
\]
Here \( \pi(x) \) is the number of primes between 1 and \( x \). Thus
\[
\frac{1}{2} \sum_{p \leq x} \chi_2^2(p) - \frac{1}{2} \sum_{p \leq x} \left( \frac{-4N}{p} \right) \chi_2(p) = -\pi(x).
\]
As \( \chi_2^2 \neq 1 \), we have by the prime ideal theorem, \( \sum_{p \leq x} \chi_2^2(p) = o(\pi(x)) \) and so
\[
\sum_{p \leq x} \left( \frac{-4N}{p} \right) \chi_2^2(p) \sim \pi(x).
\]

This implies \( \left( \frac{-4N}{p} \right) \chi_2^2(p) = 1 \). If \( p \) splits in \( K \), then \( \chi_2^2(p) = 1 \) and so \( \chi_2(p) = \pm 1 \). A similar argument works for \( \chi_1(p) = \pm 1 \) if \( p \) splits in \( K \).

Again comparing the Euler factors in \( L(s, \chi) \) and \( L(s, \pi) \), the values of \( \chi(p) \) must coincide with the values of \( \chi_1(p) \) and \( \chi_2(p) \), that is, \( \chi(p) = \pm 1 \). Now \( \chi(p) = \chi([p]) \) where \([p]\) is the class of \( p \) in the ideal class group of \( K \). By the analog of Dirichlet’s theorem for ideal class characters, we know that in each ideal class \( \mathcal{C} \) there are infinitely many prime ideals which split. Thus \( \chi(\mathcal{C}) = \pm 1 \) and hence is of order 2. This implies \( \chi \) is a genus character.

\[ \square \]

**Remark 2.5.** By Proposition 2.4, if \( \chi \) is a non-genus character, then \( L(s, \chi \otimes \chi) \) has at most a simple pole at \( s = 1 \).

3. **Proof of Theorem 1.3**

**Proof.** As \(-N \equiv 1 \mod 4\), the discriminant of \( K = \mathbb{Q}(\sqrt{-N}) \) is \(-4N\). We also assume that \( t \) is the number of distinct prime factors of \( N \) and so the discriminant \(-4N\) has \( t + 1 \) distinct prime factors.

Given the quadratic form \( Q(x, y) = x^2 + Ny^2 \), we consider the associated Epstein zeta function (see \( \text{[7, 12, 18, 19]} \))
\[
\zeta_Q(s) = \sum_{x,y \neq 0} \frac{1}{(x^2 + Ny^2)^s} = \sum_{n=1}^{\infty} \frac{r_{2,N}(n)}{n^s}.
\]
for \( \Re(s) > 1 \). Now for \( K = \mathbb{Q}(\sqrt{-N}) \), we have Dedekind’s zeta function

\[
\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}
\]

where the sum is over all nonzero ideals \( \mathfrak{a} \) of \( \mathcal{O}_K \). We now split up \( \zeta_K(s) \), according to the classes \( c_i \) of the ideal class group \( C(K) \), into the partial zeta functions (see page 458 of [15])

\[
\zeta_{c_i}(s) = \sum_{\mathfrak{a} \in c_i} \frac{1}{N(\mathfrak{a})^s}
\]

so that \( \zeta_K(s) = \sum_{i=0}^{h-1} \zeta_{c_i}(s) \) where \( h \) is the class number of \( K \). In our case \( K = \mathbb{Q}(\sqrt{-N}) \) is an imaginary quadratic field and so by [8] (Theorem 7.7, page 137), we may write

\[
\zeta_K(s) = \sum_{i=0}^{h-1} \zeta_{Q_i}(s)
\]

where \( Q_i \) is a class in the form class group. Note that in this context, \( Q(x, y) \) corresponds to the trivial class \( c_0 \) in \( C(K) \) and so \( \zeta_{c_0}(s) = \zeta_{Q(x, y)}(s) \). Now let \( \chi \) be an ideal class character and consider the Hecke L-function for \( \chi \), namely

\[
L(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N(\mathfrak{a})^s}
\]

where \( \mathfrak{a} \) again runs over all nonzero ideals of \( \mathcal{O}_K \). We may now rewrite the Hecke L-function as

\[
L(s, \chi) = \sum_{i=0}^{h-1} \chi(c_i) \zeta_{c_i}(s).
\]

And so summing over all ideal class characters of \( C(K) \), we have

\[
\sum_{\chi} \overline{\chi}(c_0) L(s, \chi) = \sum_{i=0}^{h-1} \zeta_{c_i}(s) \left( \sum_{\chi} \overline{\chi}(c_0) \chi(c_i) \right).
\]

The inner sum is nonzero precisely when \( i = 0 \). As \( \overline{\chi}(c_0) = 1 \) we have \( \zeta_{c_0}(s) = \frac{1}{h} \sum_{\chi} L(s, \chi) \). Thus

\[
\zeta_{c_0}(s) = \frac{1}{h} \left( L(s, \chi_0) + L(s, \chi_1) + \cdots + L(s, \chi_{h-1}) \right).
\]

As \( \chi_0 \) is the trivial character, \( L(s, \chi_0) = \zeta_K(s) \). We now compare \( n^{th} \) coefficients, yielding

\[
r_{2,N}(n) = \frac{1}{h} \left( a_n + b_1(n) + \cdots + b_{h-1}(n) \right)
\]

where \( a_n \) is the number of integral ideals of \( \mathcal{O}_K \) of norm \( n \) and the \( b_i \)'s are coefficients of weight 1 cusp forms (see the classical work of Hecke [9], [10] or [3]). From the modern perspective, this is straightforward. Each \( L(s, \chi_i), 1 \leq i \leq h-1 \), can be viewed as an automorphic L-function of \( GL_1(\mathbb{A}_K) \) and by automorphic induction (see [4]) they are essentially Mellin transforms of (holomorphic) cusp forms, in the classical sense. We now have
\[
\sum_{n \leq x} r_{2,N}(n)^2 = \frac{1}{H^2} \left( \sum_{n \leq x} a_n^2 + \sum_{n \leq x} b_i(n)^2 + 2 \sum_{n \leq x} a_n b_i(n) + \sum_{i \neq j, n \leq x} b_i(n) b_j(n) \right).
\]

By the Rankin-Selberg estimate, \(2 \sum_{i} a_n b_i(n), \sum_{i \neq j} b_i(n) b_j(n)\) are equal to \(O(x)\). By Corollary 2.2,
\[
\frac{1}{H^2} \sum_{n \leq x} a_n^2 = \frac{1}{H^2} \left( A_1 x \log x + B_1 x + O(x^{\frac{4}{3}} (\log x)^3 \log \log x) \right).
\]

We now must estimate \(\sum_{i} b_i(n)^2\). Let us now assume that the first \(2^t - 1\) terms arise from \(L\)-functions associated to genus characters. By Proposition 2.4 and Nowak’s proof of Theorem 2.1 (which uses Perron’s formula and the residue theorem), we obtain
\[
\sum_{n \leq x} b_i(n)^2 = A_1 x \log x + B_1 x + O(x)
\]
with \(A_1\) and \(B_1\) as in Corollary 2.2. As this estimate holds for each \(i\) such that \(1 \leq i \leq 2^t - 1\), the term \(A_1 x \log x\) appears \(2^t\) times in the estimate of \(\sum_{n \leq x} r_{2,N}(n)^2\). By Remark 2.5, the remaining terms \(\sum_{n \leq x} b_i(n)^2\) for \(2^t - 1 < i \leq h - 1\) are all \(O(x)\). Thus
\[
\sum_{n \leq x} r_{2,N}(n)^2 = \frac{1}{H^2} \left[ \left( \frac{2^t}{\pi^2} L(1, \chi_{-4N})^2 \prod_{p \mid 2N} \frac{p}{p + 1} \right) x \log x + O(x) \right] + O(x).
\]

By (4.11) in [8] (or equation (8), page 171 in [15]), we have \(L(1, \chi_{-4N}) = \frac{h \pi}{N}\) and so
\[
\sum_{n \leq x} r_{2,N}(n)^2 = \frac{3}{N} \left( \prod_{p \mid 2N} \frac{2p}{p + 1} \right) x \log x + O(x).
\]

The result then follows.

**Remark 3.1.** It should be possible to obtain the second term in the asymptotic formula. By a careful application of the Rankin-Selberg method, one should obtain an error term of the form \(O(x^\theta)\) with \(\theta < 1\). The remaining case \(-N \equiv 1 \mod 4\) requires more subtle analysis due to the fact that for \(K = \mathbb{Q}(\sqrt{-N}), \mathbb{Z}[\sqrt{-N}]\) is not the maximal order of \(K\). It involves the study of \(L\)-series attached to orders. Using the techniques in [11] and [12], we will take this and sharper error terms up in some detail in a forthcoming paper.

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