Tangent Cones to TT Varieties

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Abstract

As already done for the matrix case in [6, p.256], [11, Thm. 6.1, p.1872] and [10, Thm. 3.2] we give a parametrization of the Bouligand tangent cone of the variety of tensors of bounded TT rank. We discuss how the proof generalizes to any binary hierarchical format. The parametrization can be rewritten as an orthogonal sum of TT tensors. Its retraction onto the variety is particularly easy to compose. We also give an implicit description of the tangent cone as the solution of a system of polynomial equations.

1 Introduction

An algebraic variety is defined to be the set of solutions to a system of polynomial equations. See [2] for a detailed textbook on algebraic varieties. It is well known that the set of tensors of bounded TT rank is an algebraic variety. It is generated by the determinants of minors whose size is the rank of the corresponding matricizations plus 1. In smooth points of the variety the tangent cone is a linear subspace and is also called tangent plane or tangent space. Even in singular points the tangent cone is an algebraic variety itself. It can be computed using Gröbner bases as described in [2] § 9.7 p. 498 bottom]. This algorithm yields an implicitly defined tangent cone. Finding a parametrization (in the context of algebraic geometry, parametrization means by polynomials) of an algebraic variety in general and of the tangent cone in particular is a more delicate matter. Even though there is no general algorithm to determine the parametrization, there is an algorithm to determine, whether a given parametrization produces an implicitly defined variety. This process is called Implicitization. It can also be done using Gröbner bases and is discussed in the textbook [2] § 3.3, p. 128]. Even for varieties with few defining polynomials and few variables Gröbner bases tend to be very large. Calculating the tangent cone (in $C$) and determining whether our guess is the correct parametrization worked for the variety of $3 \times 3 \times 3$ TT tensors. We used Macaulay2 [3]. However other non-trivial examples beyond dimensions $4 \times 4 \times 4$ are intractible with this method as the size of the Gröbner bases produced appears to grow beyond any reasonable amount of memory. Instead of using Gröbner bases, it turns out that we can parametrize the tangent cone of TT varieties by exploiting orthogonality.
Definition 1. A tensor $A$ is an element of the tensor space $\mathbb{R}^{n_1 \times \ldots \times n_d}$ where $d$ is called the order and $n_i$ is called the dimension (in the direction) of order $i$.

Remark 2. Note that we can canonically identify the spaces $\mathbb{R}^{n \times m}$ and $\mathbb{R}^{n \times m}$ and we will do so throughout this paper without explicitly stating. We write $A^{(n_1,\ldots,n_i)}_{(n_{i+1},\ldots,n_d)}$ for the matricization (i.e. combining several indices into one using e.g. lexicographic order) of $A \in \mathbb{R}^{n_1 \times \ldots \times n_d}$ and $A_{1}^{(n_1,\ldots,n_d)}$ for the tensorization. Define the shorthand $A_L := A_{n_1 \times (n_2,\ldots,n_d)}$ and $A_R := A_{(n_1,\ldots,n_{d-1}) \times n_d}$. If it is clear from the context we will often omit the matricization notation.

Throughout this paper we will use the TT product defined below. In the matrix case it is equivalent to the matrix product and it allows us with little effort to rigorously describe tensor diagrams. Even though we do not use tensor diagrams in this work, figure 1 shall serve as a dictionary to aid those familiar with tensor diagrams.

Figure 1: tensor diagrams

$$A_1 \, A_2 \, A_3 = A_1 A_2 A_3$$

$$B_1 \, B_2 \, B_3 = ((A_1 A_2 A_3)^T B_1 B_2 B_3)^R$$

Definition 3. We define a scalar product on $\mathbb{R}^{n_1 \times \ldots \times n_d}$ as the standard scalar product on $\mathbb{R}^{n_1 \times \ldots \times n_d}$. This induces a norm and the notion of orthogonality. We denote the TT product of the two tensors $A \in \mathbb{R}^{n_1 \times \ldots \times n_i \times k}$ and $B \in \mathbb{R}^{k \times n_{i+1} \times \ldots \times n_d}$ by

$$AB := (A^R B^T)^{n_1 \times \ldots \times n_d} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$$

Its entries are

$$(AB)(j_1,\ldots,j_d) := \sum_{m=1}^k A(j_1,\ldots,j_i,m) B(m,j_{i+1},\ldots,j_d).$$

Note that the TT product is associative. It is equivalent to the matrix product if $A$ and $B$ are matrices.

Definition 4. Define the variety of TT tensors $[8]$ of order $d$ and dimensions $(n_1,\ldots,n_d)$ of rank bounded by $k = (k_1,\ldots,k_{d-1})$ as

$$\mathcal{M}^{n_1 \times \ldots \times n_d}_{\leq (k_1,\ldots,k_{d-1})} := \{ A \in \mathbb{R}^{n_1 \times \ldots \times n_d} : \forall i : \text{rank} \left( A^{(n_1,\ldots,n_i)}_{(n_{i+1},\ldots,n_d)} \right) \leq k_i \}$$

and the manifold of TT tensors of order $d$ and dimensions $(n_1,\ldots,n_d)$ of rank exactly $(k_1,\ldots,k_{d-1})$ as

$$\mathcal{M}^{n_1 \times \ldots \times n_d}_{= (k_1,\ldots,k_{d-1})} := \{ A \in \mathbb{R}^{n_1 \times \ldots \times n_d} : \forall i : \text{rank} \left( A^{(n_1,\ldots,n_i)}_{(n_{i+1},\ldots,n_d)} \right) = k_i \}.$$
Note that the variety of TT tensors of bounded rank \((k_1, \ldots, k_{d-1})\) is indeed an algebraic variety. Its defining polynomials are the determinants of \((k_i + 1) \times (k_i + 1)\)-minors of \(A^{(n_1 \ldots n_i) \times (n_{i+1} \ldots n_d)}\). A proof for the TT manifold being a manifold can be found in [11].

**Definition 5.** Define the tangent cone (also known as Bouligand contingent cone or tangent semicone \(C^+\) in [9]) of an algebraic variety \(M \in \mathbb{R}^N\) at a (possibly singular) point \(A \subset M\) as in [10] and [9] as the set of all vectors that are limits of secants through \(A\):

\[
T_A M := \{ \xi \in \mathbb{R}^N : \exists (x_n) \subset M, (a_n) \subset \mathbb{R}^+ \text{ s.t. } x_n \to A, a_n(x_n - A) \to \xi \}.
\]

**Remark 6.** Even though this will not affect the current work, we want to remark, that in the complex setting this tangent cone is equivalent to the algebraic tangent cone. See [2, 9]. But we do not know of any proof of the corresponding statement for the real case.

A direct consequence from our parametrization will be, that in the case of TT varieties the \(a_n\) do not need to be positive. The following example is included to address a certain peculiarity. In contrast to Differential Geometry the description of the tangent cone does not need all smooth arcs, but only analytic arcs. However the set of first derivatives of analytic arcs \(\{ v \in \mathbb{R}^N : \exists \gamma : [0, \varepsilon] \to M \text{ analytic : } \gamma(0) = A, \dot{\gamma}(0) = v \}\) does not suffice. To describe the set of directions of analytic arcs we need to include the higher order derivatives.

**Example 7.** Consider the variety \(M := \{(x, y) \in \mathbb{R}^2 : x^2 = y^3\}\) and an analytic arc \(\gamma\) with values in \(M\) such that \(\gamma(0) = (0, 0)\). Then \(\dot{\gamma}(0)\) always vanishes. Verify this by plugging the analytic arc

\[
\gamma : t \mapsto (a_1 t + a_2 t^2 + \ldots, b_1 t + b_2 t^2 + \ldots)
\]

into the defining equation and compare coefficients. But the tangent cone of \(M\) at \((0, 0)\) is more than \(\{(0, 0)\}\). This example also works in the complex case. Note that e.g. \(\gamma : t \mapsto (t^2, t)\) is not analytic.

In general the tangent cone can be defined by the first non-zero derivatives

\[
\{ v \in \mathbb{R}^N : \exists n \in \mathbb{N}, \gamma : [0, \varepsilon] \to M \text{ analytic : } \\
\gamma(0) = A, \gamma^{(n)}(0) = v \text{ and } \forall i < n : \gamma^{(i)}(0) = 0 \}
\]

of analytic arcs through the singular point. See [9] for a detailed discussion. Keeping in mind that any complex variety can be rewritten as a real variety, this also works in the complex case. In example 7 second derivatives suffice. We will show in Corollary [19] that for the TT variety first derivatives produce the tangent cone.

Lemmata 8 and 9 are trivial but essential for the proof of our main result.

**Lemma 8.**

\[
M_{n_1 \times n_2 \times n_3}^{\leq (k_1, k_2)} = M_{n_1 \times (n_2 n_3)}^{\leq k_1} \cap M_{(n_1 n_2) \times n_3}^{\leq k_2}
\]
Proof. by definition.

On a subset we can only define a subset of the secants and thus a subset of
the tangents.

Lemma 9. For every \(A \in \mathcal{M}_{\leq (k_1, k_2)}^{n_1 \times n_2 \times n_3}\) we have
\[T_A \mathcal{M}_{\leq (k_1, k_2)}^{n_1 \times n_2 \times n_3} \subset T_A \mathcal{M}_{\leq k_1}^{n_1 \times (n_2 n_3)}\]
and thus
\[T_A \mathcal{M}_{\leq (k_1, k_2)}^{n_1 \times n_2 \times n_3} \subset T_A \mathcal{M}_{\leq k_1}^{n_1 \times (n_2 n_3)} \cap T_A \mathcal{M}_{\leq k_2}^{(n_1 n_2) \times n_3}\]

Proof. by definition.

Definition 10. Define the range of \(A \in \mathbb{R}^{n_1 \times \ldots \times n_i \times k}\) as
\[\text{range}(A) := \{a \in \mathbb{R}^{n_1 \times \ldots \times n_i} : \exists b \in \mathbb{R}^{k \times 1} : a = Ab\}\]

2 Parametrization of the tangent cone

We will recall the matrix case as a guiding example and as a necessary prereq-
quisite. Along the way, we will introduce all proof ideas needed for the general
case. Consider the matrix variety
\[\mathcal{M}_{\leq k+s, s > 0}^{n \times m}\]
i.e. the set of \(n \times m\) matrices of rank at most \(k+s\). Let \(A \in \mathbb{R}^{n \times k}\) and \(B \in \mathbb{R}^{k \times m}\)
have full rank. Then \(AB\) has rank \(k\) and is a singular point of \(\mathcal{M}_{\leq k+s, s > 0}^{n \times m}\). As for
example shown in [10] (compare also to [6, p.256]), any tangent vector in the
tangent cone at \(AB\) can be decomposed as
\[X = AY + XB + UV = \begin{pmatrix} Y \\ V \\ B \end{pmatrix}\]
with \(U \in \mathbb{R}^{n \times s}\) and \(V \in \mathbb{R}^{s \times m}\). The converse is true by the following: The
analytic arc
\[\gamma : t \mapsto \begin{pmatrix} A + tX \\ tU \end{pmatrix} \begin{pmatrix} B + tY \\ V \end{pmatrix}\]
lies in \(\mathcal{M}_{\leq k+s, s > 0}^{n \times m}\) and its derivative is \(\dot{\gamma}(0) = AY + XB + UV\). Use \((\gamma(\frac{1}{m}))_{N \in \mathbb{N}}\) to see, that \(\dot{\gamma}(0)\) lies in the tangent cone. We can assume \(A^TX = 0\) (i.e.
the columns of \(X\) are orthogonal to the columns of \(A\)), \(V B^T = 0\) and either
\(A^TX = 0\) or \(Y B^T = 0\) by the following argument. \(P_A := AA^T\) is the orthogonal
projector onto \(\text{range}(A)\), where \(A^T\) denotes the Moore-Penrose Pseudoinverse.
Defining \(\dot{U} := A^1U\) and \(\dot{U} := (I - P_A)U\) we can decompose
\[U = P_AU + (I - P_A)U = AA^1U + \dot{U} = A\dot{U} + \dot{\dot{U}}\] (1)
where $\bar{U}$ is orthogonal to $A$, i.e. $A^T\bar{U} = 0$. Decomposing $V$ and $X$ in the same way, we can write $\mathcal{X} = AX + (A\dot{X} + \dot{X})B + (A\dot{U} + U)(V + VB) = A(Y + X\dot{B} + UV + U\dot{V})B + \dot{X}B + U\dot{V}$. We can furthermore assume $U$ and $V$ to have full rank by choosing them from $\mathbb{R}^{n\times \tilde{s}}$ and $\mathbb{R}^{\tilde{s}\times m}$ respectively with $\tilde{s}$ minimal. We introduce a definition for this, because we will need it in the tensor case.

**Definition 11.** Let $A \in \mathbb{R}^{n\times m}$ be a matrix of rank $k$ and $A_1 \in \mathbb{R}^{n\times k}$, $A_2 \in \mathbb{R}^{k\times m}$ be such that $A = A_1A_2$. Call for the purpose of this paper

$$A_1Y + XA_2 + UV$$

an $s$-decomposition of the matrix $\mathcal{X} \in \mathbb{R}^{n\times m}$ (not every matrix is decomposable in this way) if $U \in \mathbb{R}^{n\times s}$, $V \in \mathbb{R}^{s\times m}$, $A_1^tX = 0$, $A_1^tU = 0$, $VA_2^t = 0$ and $U$ and $V$ have full rank.

As a first step, we will prove the converse of our main result as the proof is completely analogous to the matrix case.

**Lemma 12.** Assume $A \in \mathcal{M}^{n_1\times \ldots \times n_d}_{\leq (k_1, \ldots, k_d-1)}$, i.e. there are $A_1 \in \mathbb{R}^{n_1\times k_1}$, $A_i \in \mathbb{R}^{k_{i-1}\times n_i\times k_i}$ $\forall i = 2, \ldots, d - 1$ and $A_d \in \mathbb{R}^{k_{d-1}\times n_d}$ such that $A = A_1 \ldots A_d$. If a vector $\mathcal{X}$ can be factorized as

$$
\begin{pmatrix} A_1 & U_1 & X_1 \end{pmatrix} \begin{pmatrix} A_2 & U_2 & X_2 \\ 0 & Z_2 & V_2 \end{pmatrix} \cdots \begin{pmatrix} A_{d-1} & U_{d-1} & X_{d-1} \\ 0 & Z_{d-1} & V_{d-1} \end{pmatrix} \begin{pmatrix} X_d \\ V_d \\ A_d \end{pmatrix}
$$

with block matrix dimensions $(k_i + s_i + k_i) \times (k_{i+1} + s_{i+1} + k_{i+1})$ then it is in the tangent cone of $\mathcal{M}^{n_1\times \ldots \times n_d}_{\leq (k_1+s_1, \ldots, k_{d-1}+s_{d-1})}$ at $A_1 \ldots A_d$.

**Proof.** The curve

$$\gamma : (-\varepsilon, \varepsilon) \to \mathcal{M}^{n_1\times \ldots \times n_d}_{\leq (k_1+s_1, \ldots, k_{d-1}+s_{d-1})} : t \mapsto$$

$$
\begin{pmatrix} A_1 + tX_1 & U_1 \end{pmatrix} \begin{pmatrix} A_2 + tX_2 & U_2 \\ tV_2 & Z_2 \end{pmatrix} \cdots \begin{pmatrix} A_{d-1} + tX_{d-1} & U_{d-1} \\ tV_{d-1} & Z_{d-1} \end{pmatrix} \begin{pmatrix} X_d \\ V_d \\ A_d \end{pmatrix}
$$

5
is analytic and has $\mathcal{X}$ as its first derivative. See this by differentiating $\gamma$ in $t = 0$ using the product rule. For the basic definition of tangent vector use the sequence $(\gamma(t))_{t \in \mathbb{R}}$.

What follows is a technical lemma that facilitates proving both, the case for order 3 TT varieties as well as the inductive step for arbitrary order. Its first two assumptions (equations 2 and 3) arrive from applying the matrix version to the two matricizations with respect to index 1 and 3. The idea of the proof is the following: Represent an arbitrary tangent vector as the tangent vector of the matricizations using Lemma 9. Then decompose using the result on matrix tangent cones above. Orthogonalizing with respect to $A_1$ and $A_3$ allows us to decompose the tangent vector into an orthogonal sum and compare the orthogonal components separately.

**Lemma 13.** Let $A \in \mathcal{M}^{n_1 \times n_2 \times n_3}_{= (k_1, k_2)}$ be a singular point in $\mathcal{M}^{n_1 \times n_2 \times n_3 \leq (k_1 + s_1, k_2 + s_2)}$ ($s_1, s_2 \geq 0$) and let $A_1 \in \mathbb{R}^{n_1 \times k_1}$, $A_2 \in \mathbb{R}^{k_1 \times n_2 \times k_2}$ and $A_3 \in \mathbb{R}^{k_2 \times n_3}$ be three tensors such that $A_1 A_2 A_3 = A$. Assume further the orthogonality of $A_1$ and $A_2$, $A_1^T A_1 = I$, $(A_2^R)^T A_2^R = I$. Let $\mathcal{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$ be a tensor that admits the $\tilde{s}_1$-decomposition

$$\mathcal{X} = A_1 Y + X A_2 A_3 + UV$$

and the $\tilde{s}_2$-decomposition

$$\mathcal{X} = A_1 A_2 T + S A_3 + O P$$

with $\tilde{s}_1 \leq s_1$ and $\tilde{s}_2 \leq s_2$. Then $\mathcal{X}$ is decomposable as

$$\mathcal{X} = \begin{pmatrix} A_1 & U & X \end{pmatrix} \begin{pmatrix} A_2 & 0 & \dot{S} \\ 0 & Z_2 & \dot{V} \\ 0 & 0 & A_2 \end{pmatrix} \begin{pmatrix} T \\ P \\ A_3 \end{pmatrix}$$

with $\dot{O} = A_1^T \dot{O} = A_2^T \dot{S}$ and $\dot{V} = V A_3$. In particular we have the orthogonality statements $(A_2^R)^T \dot{O} = 0$, $(A_2^R)^T \dot{S} = 0$, $(\dot{V} A_3)^L ((A_2 A_3)^L)^T = 0$ and that $Z_2 P + \dot{V} A_3$ and $A_1 \dot{O} + U Z_2$ have full rank and the equivalence

$$\begin{pmatrix} A_1 & U & X \end{pmatrix} \begin{pmatrix} A_2 & \dot{O} & \dot{S} \\ 0 & Z_2 & \dot{V} \\ 0 & 0 & A_2 \end{pmatrix} = \begin{pmatrix} A_1 & S & O \end{pmatrix}.$$
where the hat-wearing variables are orthogonal to $A_1$ or $A_3$ respectively:

\[
\hat{Y} A_3^T = 0, \hat{V} A_3^T = 0, A_1^T \hat{O} = 0, A_1^T \hat{S} = 0, \hat{T} A_3^T = 0.
\]

Then we can write the tangent vector as an orthogonal sum (w.r.t. the scalar product on $\mathbb{R}^{n_1 n_2 n_3}$) in the four spaces

- range($A_1$) $\otimes \mathbb{R}^{n_2} \otimes$ range($A_3^T$),
- range($A_1$) $\otimes \mathbb{R}^{n_2} \otimes$ range($A_3^T$)$^\perp$,
- range($A_1$)$^\perp$ $\otimes \mathbb{R}^{n_2} \otimes$ range($A_3^T$),
- range($A_1$)$^\perp$ $\otimes \mathbb{R}^{n_2} \otimes$ range($A_3^T$)$^\perp$.

Rewriting equations 2 and 3 yields

\[X = A_1 \hat{Y} A_3 + A_1 \hat{Y} + (X A_2 + U \hat{V}) A_3 + U \hat{V}.\]

(5)

and

\[X = A_1 (A_2 \hat{T} + \hat{S}) A_3 + A_1 (A_2 \hat{T} + \hat{O} P) + \hat{S} A_3 + \hat{O} P.\]

(6)

respectively. Both representations need to be equal. Because they are orthogonal sums in the same four spaces, each summand has to be equal to the corresponding summand in the other sum. In particular we have

\[\hat{O} P = U \hat{V}.\]

By defining $Z_2 := U^\perp \hat{O}$, we can write

\[U \hat{V} = \hat{O} P = U Z_2 P\]

(7)

and see that $Z_2 = \hat{V} P^\perp$ (by multiplying equation 7 by the full rank matrices $U^\perp$ and $P^\perp$). Using the first and second summand of equation 6, the third summand of equation 5 and equation 7 we assemble the desired representation from equation 5

\[X = A_1 \hat{S} A_3 + A_1 A_2 T + A_1 \hat{O} P + X A_2 A_3 + U \hat{V} A_3 + U Z_2 P.
\]

with all the desired properties. See this in the following way: $A_1 \hat{O} + U Z_2 = A_1 \hat{O} + \hat{O} = \mathbf{0}$ is orthogonal to $A_1 A_2$, therefore $0 = ((A_1 A_2)^R)^T \mathbf{O}^R = ((A_1 A_2)^R)^T (A_1 \hat{O} + U Z_2)^R = ((A_1 A_2)^R)^T (A_1 \hat{O})^R = (A_2^R)^T \hat{O}^R$. And analogously for $Z_2 P + \hat{V} A_3 = \mathbf{V}$ and $A_1 \hat{S} + U \hat{V} + X A_2 = \mathbf{S}$ (by $X A_2 + U \hat{V} = \hat{S}$ from equations 5 and 6).\]

We can now state our main result for arbitrary TT varieties.
Theorem 14. Let $A \in \mathcal{M}^{n_1 \times \ldots \times n_d}$ be a singular point in $\mathcal{M}^{n_1 \times \ldots \times n_d}_{\leq (k_1+s_1, \ldots, k_d+s_d)}$ ($s_i \geq 0$) and let $A_1 \in \mathbb{R}^{n_1 \times k_1}$, $A_2 \in \mathbb{R}^{n_2 \times k_2} \ldots$ and $A_d \in \mathbb{R}^{k_{d-1} \times n_d}$ be tensors such that $A_1 \ldots A_d = A$ and $A_i^T A_i = I$, $(A_i^R)^T A_i^R = I \forall i = 2, \ldots, d-1$. Then any vector in the tangent cone of $\mathcal{M}^{n_1 \times \ldots \times n_d}_{\leq (k_1+s_1, \ldots, k_d+s_d)}$ at the point $A_1 \ldots A_d$ can be written as the TT tensor

$$
\begin{pmatrix}
A_1 & U_1 & X_1 \\
\end{pmatrix}
\begin{pmatrix}
A_2 & U_2 & X_2 \\
0 & Z_2 & V_2 \\
0 & 0 & A_2
\end{pmatrix}
\ldots
\begin{pmatrix}
A_{d-1} & U_{d-1} & X_{d-1} \\
0 & Z_{d-1} & V_{d-1} \\
0 & 0 & A_{d-1}
\end{pmatrix}
\begin{pmatrix}
X_d \\
V_d \\
A_d
\end{pmatrix}
$$

where $(A_i^R)^T U_i^R = 0 \forall i$, $(A_i^R)^T X_i^R = 0 \forall i \neq d$, $(V_i A_{i+1} \ldots A_d)^L \left((A_i \ldots A_d)^L\right)^T = 0 \forall i$.

Proof. The idea of the proof is illustrated in Figure 3. Applying the matrix version of this theorem \cite{1, 10, Thm 3.2} to the matricizations from $\mathcal{M}^{n_1 \times (n_2 \ldots n_d)}_{\leq (k_1+s_1)}$ and to $\mathcal{M}^{(n_1n_2) \times (n_3 \ldots n_d-2)}_{\leq (k_2+s_2)}$ we arrive at the assumptions of Lemma 13 and can decompose the tangent vector in the form

$$
\mathcal{X} = \begin{pmatrix}
A_1 & U_1 & X_1 \\
\end{pmatrix}
\begin{pmatrix}
A_2 & U_2 & X_2 \\
0 & Z_2 & V_2 \\
0 & 0 & A_2
\end{pmatrix}
\ldots
\begin{pmatrix}
A_{d-1} & U_{d-1} & X_{d-1} \\
0 & Z_{d-1} & V_{d-1} \\
0 & 0 & A_{d-1}
\end{pmatrix}
\begin{pmatrix}
T_3 \\
P_3 \\
A_{d} \ldots A_{d}
\end{pmatrix}
$$

with $U_1$ and $X_1$ orthogonal to $A_1$, the two matrices $U_2$ and $X_2$ orthogonal to $A_1$ and $(V_2 A_3)^L$ orthogonal to $(A_2 A_3)^L$ from the left and right respectively and $A_1 U_2 + U_1 Z_2$ having full rank. Using this as inductive basis we continue by proving the inductive step: Assume that $\mathcal{X}$ has the decomposition

$$
\mathcal{X} = \begin{pmatrix}
A_1 & U_1 & X_1 \\
\end{pmatrix}
\begin{pmatrix}
A_2 & U_2 & X_2 \\
0 & Z_2 & V_2 \\
0 & 0 & A_2
\end{pmatrix}
\ldots
\begin{pmatrix}
A_i & U_i & X_i \\
0 & Z_i & V_i \\
0 & 0 & A_i
\end{pmatrix}
\begin{pmatrix}
T_{i+1} \\
P_{i+1} \\
A_{i+1} \ldots A_{d}
\end{pmatrix}
$$

\[\rightarrow\]
Theorem 14. Combining this with equation 8 completes the inductive step and the proof of apply Lemma 13 to achieve the decomposition

The second assumption follows by the matrix version from [10]. Thus we can the second line. See [7] for a study of both, ALS and DMRG.

\[ R \]
where all summands are pairwise orthogonal in the standard scalar product on

### Remark 15
For parametrizing the tangent cone, we use the same number of parameters as in the parametrizations of the TT variety. Each block \((U_i \ X_i)\) is of size \((k_i - 1 + s_i - 1) \times (k_i + s_i)\).

### Remark 16
Evaluating the expression

\[
\begin{pmatrix} \ A_1 \ U_1 \ X_1 \end{pmatrix} \begin{pmatrix} \ A_2 & U_2 & X_2 \\ 0 & Z_2 & V_2 \\ 0 & 0 & A_2 \end{pmatrix} \ ...
\begin{pmatrix} \ \end{pmatrix} \begin{pmatrix} \ A_d & U_d & X_d \\ 0 & Z_d & V_d \\ 0 & 0 & A_d \end{pmatrix}
\]

for the tangent cone parametrization yields

\[
A_1...A_{d-1}X_d + A_1...A_{d-2}X_{d-1}A_d + ... + X_1A_2...A_d
\]

\[
+ A_1...A_{d-2}U_{d-1}V_d + A_1...A_{d-3}U_{d-2}V_{d-1}A_d + ... + U_1V_2A_3...A_d
\]

\[
+ A_1...A_{d-3}U_{d-3}Z_{d-1}V_d + ... + U_1Z_3V_4A_4...A_d
\]

\[
+ U_1Z_2...Z_{d-1}V_d
\]

where all summands are pairwise orthogonal in the standard scalar product on \(\mathbb{R}^{n_1...n_d}\). Note that an ALS algorithm only uses directions from the first line of this decomposition. The DMRG algorithm additionally uses directions from the second line. See [7] for a study of both, ALS and DMRG.
We can deduce, that in the case of TT varieties the intersection of the tangent cones is the tangent cone of the intersection.

**Corollary 17.**
\[ \bigcap_{i=1,\ldots,d-1} T_{A\text{M}}^{(n_1\ldots n_i)\times(n_{i+1}\ldots n_d)} \subseteq T_{A\text{M}}^{(n_1\ldots n_i)\times n_d} \]
and thus
\[ T_{A\text{M}}^{n_1\ldots n_d} \leq (k_{i_1},\ldots,k_{d-1}) = \bigcap_{i=1,\ldots,d-1} T_{A\text{M}}^{(n_1\ldots n_i)\times(n_{i+1}\ldots n_d)} \]

**Proof.** If \( X \in \bigcap_{i=1,\ldots,d-1} T_{A\text{M}}^{(n_1\ldots n_i)\times(n_{i+1}\ldots n_d)} \leq k_i \) then by Lemma 13 works and we can find coefficient tensors such that we can write \( X \) in our parametrization. But then by Lemma 12
\[ X \in T_{A\text{M}}^{n_1\ldots n_d} \leq (k_1,\ldots,k_{d-1}) \]

This corollary was unexpected because of the following example.

**Example 18.** The tangent cone of the intersection is not always equal to the intersection of the tangent cones. Consider the plane \( \mathcal{M} := \{(x, y, z) \in \mathbb{R}^3 : x = 0\} \) and the cylinder \( \mathcal{N} := \{(x, y, z) \in \mathbb{R}^3 : (x - 1)^2 + y^2 = 1\} \) and the point \( (0, 0, 0) \in \mathcal{N} \cap \mathcal{M} \). Being the line where both varieties touch, the tangent cone \( T_{A\text{M}} \) of \( \mathcal{M} \) at \( A \) is the same as the tangent cone of \( \mathcal{N} \) at \( A \), namely the \( y\)-\( z\)-plane. However the tangent cone of \( \mathcal{M} \cap \mathcal{N} = \{(x, y, z) \in \mathbb{R}^3 : x = y = 0\} \) at \( A \) is only the \( z\)-axis.

We can show that the issue raised in example 2 is unimportant for TT varieties. Namely:

**Corollary 19.** The tangent cone to a TT variety is equivalent to the set of all first derivatives to analytic arcs.

**Proof.** By theorem 14 every tangent vector can be written in the form
\[
\begin{pmatrix}
A_1 & U_1 & X_1 \\
A_2 & U_2 & X_2 \\
0 & Z_2 & V_2 \\
0 & 0 & A_2 \\
\end{pmatrix}
\ldots
\begin{pmatrix}
A_{d-1} & U_{d-1} & X_{d-1} \\
0 & Z_{d-1} & V_{d-1} \\
0 & 0 & A_{d-1} \\
\end{pmatrix}
\begin{pmatrix}
X_d \\
V_d \\
A_d \\
\end{pmatrix}
\]
and by Lemma 12 this is the first derivative of the analytic curve
\[ \gamma : t \mapsto \begin{pmatrix}
A_1 + tX_1 & U_1 \\
A_2 + tX_2 & U_2 \\
tV_2 & Z_2 \\
\end{pmatrix}
\ldots
\begin{pmatrix}
A_{d-1} + tX_{d-1} & U_{d-1} \\
tV_{d-1} & Z_{d-1} \\
\end{pmatrix}
\begin{pmatrix}
A_d + tX_d \\
tV_d \\
\end{pmatrix} \]
The converse is trivial by using the sequence \( (\eta \left( \frac{1}{m} \right))_{\forall m \geq N} \) for an analytic curve \( \eta \).
3 Retraction onto the variety

As a retraction one can use the curve from Lemma 12. For a retraction we adopt the definition from [10]:

**Definition 20.** Let \( \mathcal{M} \) be an algebraic variety. The tangent bundle of the variety \( \mathcal{M} \) is the set \( \bigcup_{x \in \mathcal{M}} \{x\} \times T_x \mathcal{M} \). A retraction is a function \( R \) from the tangent bundle to the variety such that for any fixed \( x \in \mathcal{M} \) and \( v \in T_x \mathcal{M} \) the function \( t \mapsto R(x, tv) \) is continuous on \([0, \infty)\) and

\[
\lim_{t \searrow 0} \frac{R(x, tv) - x - tv}{t} = 0.
\]

**Lemma 21.** The function

\[
R : X = ( \begin{pmatrix} A_1 & U_1 & X_1 \end{pmatrix} \begin{pmatrix} A_2 & U_2 & X_2 \\ 0 & Z_2 & V_2 \\ 0 & 0 & A_2 \end{pmatrix} \cdots \begin{pmatrix} A_{d-1} & U_{d-1} & X_{d-1} \\ 0 & Z_{d-1} & V_{d-1} \\ 0 & 0 & A_{d-1} \end{pmatrix} \begin{pmatrix} X_d \\ V_d \\ A_d \end{pmatrix})
\]

\[
\mapsto ( \begin{pmatrix} A_1 + X_1 & U_1 \end{pmatrix} \begin{pmatrix} A_2 + X_2 & U_2 \\ V_2 \\ Z_2 \end{pmatrix} \cdots \begin{pmatrix} A_{d-1} + X_{d-1} & U_{d-1} \\ V_{d-1} \\ Z_{d-1} \end{pmatrix} \begin{pmatrix} A_d + X_d \\ V_d \\ A_d \end{pmatrix})
\]

defines a retraction in the sense of the definition above.

**Proof.** The image under \( R \) of the tangent vector multiplied by \( t \), \( R(tX) \) is

\[
( \begin{pmatrix} A_1 + tX_1 & U_1 \end{pmatrix} \begin{pmatrix} A_2 + tX_2 & U_2 \\ tV_2 \\ Z_2 \end{pmatrix} \cdots \begin{pmatrix} A_{d-1} + tX_{d-1} & U_{d-1} \\ tV_{d-1} \\ Z_{d-1} \end{pmatrix} \begin{pmatrix} A_d + tX_d \\ tV_d \\ A_d \end{pmatrix}).
\]

We calculate

\[
\lim_{t \searrow 0} \frac{R(x, tv) - x - tv}{t} = \lim_{t \searrow 0} \frac{t^2 (\text{polynomial in } t)}{t} = \lim_{t \searrow 0} (\text{polynomial in } t) = 0
\]

\[\blacksquare\]

Note that this retraction is particularly easy to calculate if the tangent vectors are given in the described format.

4 The hierarchical format

All of the above generalises in a straight-forward way to the hierarchical and Tucker format. However the notation is difficult. Therefore we will omit some details. See [4] or [5] for the definition and a detailed study of the hierarchical tensor format. We will only give the equivalent of the technical Lemma 13 for the Tucker format with order 3. This will allow us to use the same inductive step as in theorem 14 to prove the parametrization for any binary tree. In further generalizing the technical lemma to arbitrary Tucker formats, one could prove the theorem for arbitrary tree formats.
Let \( A_1 \in \mathbb{R}^{n_1 \times k_1}, A_2 \in \mathbb{R}^{k_2 \times n_2}, A_3 \in \mathbb{R}^{n_3 \times k_3} \), and \( A_4 \in \mathbb{R}^{k_1 \times k_2 \times k_3} \) with \( A_1, A_2, A_3 \) having full rank. For writing simple tensor tree diagrams, we can use the Kronecker product. Sorting the indices \( k_1 \) and \( k_3 \) lexicographically, we can identify the tree diagram and the term depicted in figure 4.

**Figure 4: Kronecker product notation for tensor trees**

\[
\begin{align*}
A_1 & \quad k_1 \quad A_4 \quad k_2 \quad A_3 \\
& \quad A_2
\end{align*}
\]

We can write this in the following three ways:

\[
\begin{align*}
&\left( (A_1 \otimes A_3) A_4^{(k_1 k_3) \times k_2} \right) \cdot A_2 \\
= &\ A_1 \cdot \left( A_4^{k_1 \times (k_2 k_3)} (A_3 \otimes A_2) \right) \\
= &\ A_3 \cdot \left( A_4^{k_3 \times (k_1 k_2)} (A_1 \otimes A_2) \right)
\end{align*}
\]

Now any tangent vector from a Tucker variety \( \mathcal{M}_{\leq}^{n_1 \times n_2 \times n_3} \) (we use the obvious generalization of the symbols defined for the TT varieties) parametrized by \( A_1, A_2, A_3 \) and \( A_4 \) can be decomposed in the \( \tilde{s}_2 \)-decomposition

\[
\left( (A_1 \otimes A_3) A_4^{(k_2 k_3) \times k_2} \right) Y_2 + X_2 A_2 + U_2 V_2,
\]

in the \( \tilde{s}_1 \)-decomposition

\[
A_1 Y_1 + X_1 \left( A_4^{k_1 \times (k_2 k_3)} (A_2 \otimes A_3) \right) + U_1 V_1
\]

and the \( \tilde{s}_3 \)-decomposition

\[
A_3 Y_3 + X_3 \left( A_4^{k_3 \times (k_1 k_2)} (A_1 \otimes A_2) \right) + U_3 V_3
\]

with \( \tilde{s}_1 \leq s_1, \tilde{s}_2 \leq s_2 \) and \( \tilde{s}_3 \leq s_3 \). We can further decompose each of the three into the 8 orthogonal subspaces

\[
\begin{align*}
\text{range}(A_1) \otimes \text{range}(A_2^T) \otimes \text{range}(A_3), &\quad \text{range}(A_1) \otimes \text{range}(A_2^T) \otimes \text{range}(A_3)^{\perp}, \\
\text{range}(A_1)^{\perp} \otimes \text{range}(A_2^T) \otimes \text{range}(A_3), &\quad \text{range}(A_1)^{\perp} \otimes \text{range}(A_2^T) \otimes \text{range}(A_3)^{\perp}, \\
\text{range}(A_1) \otimes \text{range}(A_2^T)^{\perp} \otimes \text{range}(A_3), &\quad \text{range}(A_1) \otimes \text{range}(A_2^T)^{\perp} \otimes \text{range}(A_3)^{\perp}, \\
\text{range}(A_1)^{\perp} \otimes \text{range}(A_2^T)^{\perp} \otimes \text{range}(A_3), &\quad \text{range}(A_1)^{\perp} \otimes \text{range}(A_2^T)^{\perp} \otimes \text{range}(A_3)^{\perp},
\end{align*}
\]

Exemplarily we further decompose the \( \tilde{s}_1 \)-decomposition. For this purpose we need to write \( Y_1 \) as the orthogonal sum

\[
Y_1^{k_1 \times (n_2 n_3)} = (I \otimes A_3) Y_1 A_2 + Y_1^3 A_2 + (I \otimes A_3) Y_1^2 + Y_1^{2,3}
\]
such that \((I \otimes A_3)^T Y_1^3 = 0, Y_1^2 A_2^T = 0, (I \otimes A_3)^T Y_1^{2,3} = 0\) and \(Y_1^{2,3} A_2^T = 0\) (use pseudo inverses for this purpose as in equation 1). Analogously we rewrite \(V_1\) as

\[
V_1^{k_1 \times (n_2 n_3)} = (I \otimes A_3) \tilde{V}_1 A_2 + V_1^3 A_2 + (I \otimes A_3) V_1^2 + V_1^{2,3}
\]
such that the \(\tilde{s}_1\)-decomposition can be rewritten as the orthogonal sum

\[
(A_1 \otimes A_3) \tilde{Y}_1 A_2 \hfill (9) \\
+ (A_1 \otimes I) Y_1^3 A_2 \hfill (10) \\
+ (A_1 \otimes A_3) Y_1^2 \hfill (11) \\
+ (A_1 \otimes I) Y_1^{2,3} \hfill (12) \\
+ ((U_1 \otimes A_3) \tilde{V}_1 + (X_1 \otimes A_3) A_4) A_2 \hfill (13) \\
+ U_1 V_1^3 A_2 \hfill (14) \\
+ (U_1 \otimes A_3) V_1^2 \hfill (15) \\
+ U_1 V_1^{2,3} \hfill (16)
\]

Comparing coefficients with the orthogonal decompositions of the \(\tilde{s}_2\)- and \(\tilde{s}_3\)-decompositions, we arrive at the representation

\[
\mathcal{X} = \left( ( A_1 \ U_1 \ X_1 ) \otimes ( A_3 \ U_3 \ X_3 ) \right) \mathbf{C} \begin{pmatrix} Y_2 \\ V_2 \\ A_2 \end{pmatrix}
\]
with \(\mathbf{C} \in \mathbb{R}^{(k_1 + \tilde{s}_1 + k_1) \times (k_3 + \tilde{s}_3 + k_3) \times (k_2 + \tilde{s}_2 + k_2)}\) having the form depicted in figure 5

Figure 5: Coefficient tensor of tangent cone parametrization for order 3 Tucker
The coefficients of the block tensor $C$ are
\[
X_4 = (A_1^1 \otimes A_3^1)X_2, \quad Z_2 = (U_1^1 \otimes U_3^1)U_2^{1,3}, \quad U_4 = \tilde{V}_1, \\
W_4 = \tilde{U}_2, \quad V_4 = \tilde{V}_3, \quad \tilde{V}_4 = V_2^2V_2^1 = (U_2^T \otimes I)U_1^2, \\
W_2 = (U_1^1 \otimes U_3^1)X_2^{1,3}, \quad \tilde{U}_4 = (I \otimes U_3^3)U_2^3.
\]
The inductive step works because by
\[
\left((A_1 \otimes A_3)A_4^{(k_1,k_3)\times k_2}, U_4, X_4\right) = ((A_1 \ U_1 \ X_1) \otimes (A_3 \ U_3 \ X_3))C
\]
we can reduce the parametrization to the matrix case and reproduce $U_4$ and $X_4$.

5 Implicit description of the tangent cone

The tangent cone for the matrix case can be implicitly defined as the variety
\[
\left\{\lambda \in \mathbb{R}^{n \times m} : \text{rank} \left((I - A_1A_1^1)\lambda(I - A_2^1A_2)\right) \leq s_1\right\}
\]
where the rank can be bounded by a set of determinants of minors. Since we have shown in Corollary 17 that the tangent cone of a tensor variety is the intersection of tangent cones of matrix varieties, the set of defining equations of the tensor variety is the union of defining equations of matrix varieties of the appropriate matricizations.

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References

[1] T. P. Cason, P.-A. Absil, and P. Van Dooren. Iterative methods for low rank approximation of graph similarity matrices. *Lin. Alg. Appl.*, 438(4):1863–1882, 2013.

[2] D. Cox, J. Little, and D. O’Shea. *Ideals, Varieties and Algorithms*. Springer, 3rd edition, 2006.

[3] D. R. Grayson and M. E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at [http://www.math.uiuc.edu/Macaulay2/](http://www.math.uiuc.edu/Macaulay2/)

[4] W. Hackbusch. *Tensor Spaces and Numerical Tensor Calculus*. Springer, 3rd edition, 2012.

[5] W. Hackbusch and S. Kühn. A new scheme for the tensor representation. *J. Fourier Anal. Appl.*, 15:706–722, 2009.

[6] J. Harris. *Algebraic Geometry - A First Course*. Springer, 3rd corrected printing edition, 1992.

[7] S. Holtz, Th. Rohwedder, and R. Schneider. The alternating linear scheme for tensor optimization in the tensor train format. *SIAM J. Sci. Comput.*, 43(2):A683–A713, 2012.

[8] I. V. Oseledets. Tensor-train decomposition. *SIAM J. Sci. Comput.*, 33(5):2295–2317, 2011.

[9] D.B. O’Shea and L.C. Wilson. Limits of tangent spaces to real surfaces. *Amer. J. Math.*, 126:951–980, 2004.

[10] R. Schneider and A. Uschmajew. Convergence results for projected line-search methods on varieties of low-rank matrices via Łojasiewicz inequality. *SIAM J. Optim.*, 25(1):622–646, 2015.

[11] A. Uschmajew and B. Vandereycken. The geometry of algorithms using hierarchical tensors. *Lin. Alg. Appl.*, 439(1):133–166, 2013.