QUANTIZED W-ALGEBRA OF $\mathfrak{sl}(2,1)$: A CONSTRUCTION FROM THE QUANTIZATION OF SCREENING OPERATORS

JINTAI DING AND BORIS FEIGIN

Abstract. Starting from bosonization, we study the operator that commute or commute up-to a total difference with of any quantized screen operator of a free field. We show that if there exists a operator in the form of a sum of two vertex operators which has the simplest correlation functions with the quantized screen operator, namely a function with one pole and one zero, then, the screen operator and this operator are uniquely determined, and this operator is the quantized virasoro algebra. For the case when the screen is a fermion, there are a family of this kind of operator, which give new algebraic structures. Similarly we study the case of two quantized screen operator, which uniquely gives us the quantized W-algebra corresponding to $\mathfrak{sl}(3)$ for the generic case, and a new algebra, which is a quantized W-algebra corresponding to $\mathfrak{sl}(2,1)$, for the case that one of the two screening operators is or both are fermions.

1. Introduction.

In [3] N. Reshetikhin and E. Frenkel introduced new Poisson algebras $W_q(g)$, which are $q$–deformations of the classical $W$–algebras. In [2] J. Shiraishi, H.Kubo, H.Awata, and S. Odake quantized the formulation in [3]. They constructed a non-commutative algebra depending on two parameters $q$ and $p$, such that when $q = p$ it becomes commutative, and is isomorphic to the Poisson algebra $W_q(\mathfrak{sl}(2))$, which denote by $W_{q,p}(\mathfrak{sl}(2))$. Shiraishi, e.a., constructed a free field realization of $W_q(\mathfrak{sl}(2))$ i.e. a homomorphism into a Heisenberg algebra. They also constructed the screening currents, i.e. operators acting on the Fock representations of the Heisenberg algebra, which commute with the action of $W_q(\mathfrak{sl}(2))$ up to a total difference. In [2], the results of Shiraishi, e.a., were further generalized to to the case of the $q$-deformed $W$-algebras. They constructed an algebra $W_{q,p}(\mathfrak{sl}(N))$ depending on $q$ and $p$, such that when $q = p$ it becomes isomorphic to the $q$–deformed classical $W$–algebra $W_q(\mathfrak{sl}(N))$ from [3]. They also construct a free field
realization of $W_{q,p}(\mathfrak{sl}(N))$, which is a deformation of the free field realization from $\mathfrak{sl}(N)$, and the screening currents. They also construct the screening currents $S_i^+(z)$ satisfying certain difference equations related to the basic generator of $W_{q,p}(\mathfrak{sl}(N))$.

In all the works above, the screening operators, which are quantization of the classical screening operators, appear as a by-product of the quantization of $W$-algebras. These screening operators are defined as operators who either commute with the quantized $W$-algebras or commute up-to a total difference.

In this paper, we will start from the opposite direction of such an approach. Namely, we will start from the quantization of screening operators. This approach is based on the idea of bosonization, namely we start everything from the Heisenberg algebras.

First we consider the case of one generic screen operator (not a fermion) $S_1^+(z)$, which is given as any kind of quantization of the classical screen operator $S(z)$. Here both operator are bosonized, namely expressed by generator of a Heisenberg algebra. Surely the screening operator $S(z)$ satisfies:

$$S(z)S(w) = (z - w)^{2\beta} : S(z)S(w) :,$$

and

$$S_1^+(z)S_1^+(w) = (z)^{2\beta} f(z, w) : S_1^+(z)S_1^+(w),$$

where $\beta \in \mathbb{C}$. By quantization, we mean that the operator $S_1^+(z)$ depends on a parameter $q$ and it degenerates into the classical screening operator when $q$ goes to 1 or

$$\lim_{q \to 1} z^{2\beta} f(z, w) = (z - w)^{2\beta}.$$

Then we try to construct an operator $l(z) = \Lambda_1(z) + \Lambda_2(z)$, a sum of two vertex operators, that will commute with the action of the screen operator up-to a total difference. It turns out that this uniquely determines the quantization and enable us to recover the quantized virasoro algebra mentioned above in an unique way, if we assume that the correlation functions between $S_1^+(z)$ and $\Lambda_i(w)$ has one pole and one zero and the ration of the two pole is our quantization parameter $q$. However, when the screen operator is a fermion, the situation becomes very different. For this case, we will obtain a family of operators that commute with the screening operator up-to a total difference, which give completely new algebraic structures. Similarly, we consider the case of two generic screen operators (neither of them a fermion), which are given as any kind of q-deformation of the classical screen operators. Then we try to construct an operator in the form of the sum of three vertex operator, such that this operator commutes with the two
screening operators. This enable us to uniquely derive the quantized W-algebra for \( \mathfrak{sl}(3) \) given by Feigin and Frenkel [2].

Then we apply the same method to the case of two screening operators such that one of the screening operators is a fermion. In this case, this leads us to derive an unique operator such that it commutes with the two screening operator up-to a total difference. We will call the algebraic structure generated by this operator the quantized W-algebra of \( \mathfrak{sl}(2,1) \). Finally we apply this method to the case that two screening operators are both fermions. This also leads us to derive an unique operator that generates the same algebraic structure, namely the quantized W-algebra of \( \mathfrak{sl}(2,1) \).

This paper is organized in the way that each section is devoted to study a case mention above and in the same order. The last sections is devoted to discussions.

2. The commutant of a screen operator and the q-deformed virasoro algebra

In this section, we will start from one quantized screen operator. From the point view of quantization, this screen operator should degenerate into the classical screen operator, when the deformation parameter \( q \) goes to 1. With such a screen operator, we will try to find current operators, which commute with the integral of this screen operator. Analog to the idea that differential operators in general are deformed into difference operators, we expect that a desired operator, which should be a kind of q-deformed virasoro algebra, is in the form of a kind of difference.

Let us first introduce the Heisenberg algebra \( H_{q,p}(1) \) be the Heisenberg algebra with generators \( a_1[n], n \in \mathbb{Z} \), and relations

\[
[a_1[n], a_1[m]] = \frac{1}{n}\delta_{n,-m},
\]

which is defined on the filed of the rational functions of \( p = q^x \) and \( q \).

For each weight \( \mu \) of the Cartan subalgebra of \( \mathfrak{sl}(2) \), let \( \pi_\mu \) be the Fock representation of \( H_{q,p}(1) \) generated by a vector \( v_\mu \), such that \( a_1[n]v_\mu = 0, n > 0 \), and \( a_1[0]v_\mu = \mu(\alpha_1^\vee)v_\mu \), where \( \alpha_1^\vee \) is the \( i \)th coroot of \( \mathfrak{sl}(2) \).

Introduce operators \( Q_1 \), which satisfy commutation relations \( [a_1[n], Q_1] = 2\beta\delta_{n,0} \). The operators \( e^{Q_1} \) act from \( \pi_\mu \) to \( \pi_{\mu+\beta\alpha_1} \).

A classical screening operator \( S(z) \) is defined as

\[
S(z) = e^{Q_1}z^{a_1^+[0]} : \exp \left( \sum_{m \neq 0} s(m)a_1[m]z^{-m} \right) :,
\]
where \( m \neq 0 \) and

\[
S(z)S(w) = (z - w)^{2\beta} : S(z)S(w) :. 
\] (3)

Now we can define a quantized screening currents as the generating function

\[
S_1^+(z) = e^{Q_1 z} s_1^+[0] : \exp \left( \sum_{m \neq 0} s_1^+(m) a_1[m] z^{-m} \right) :. 
\] (4)

where \( s_1^+[m] \) are in \( \mathbb{C}|\cdot,\cdot| \) for \( m \neq 0 \) and \( s_1^+[0] = a_1[0] \). For this screen operator we would impose the following condition that the limit of this operator when \( q \) goes to one degenerate into the classical screen operators.

For the undeformed case, the virasoro algebra defined on the Fock space give us the operators that commute with the action of the operator \( \int S_1^+(z) dz/z \). Here we will impose the assumption that once we consider such an integration, the integration contour is around the point 0 and the screening operator \( S_1^+(z) \) acts on the space \( \pi_\mu \) such that this integration is single-valued.

We, then, would try to find a construction that gives us the operators that will commute with the action of this quantized screen operator \( \int S_1^+(z) dz/z \). The simplest try would be that case that they should be a sum of vertex operators.

For the simplest case, we assume that it is sum of two vertex operators. Let us define the operator as

\[
l_1(z) = \Lambda_1(z) + \Lambda_2(z),
\]

where \( \Lambda_i(z) \) as the generating function:

\[
\Lambda_i(z) = g^{i-1} : \exp \left( \sum \lambda_i(m) a_1[m] z^{-m} \right) :. 
\] (5)

Here \( \lambda_i[m] \) are in \( \mathbb{C}|\cdot,\cdot| \) for \( i = 1, 2 \).

The commutation relations between \( S_1^+(z) \) and \( \Lambda_i(w) \) are basically decided by the correlation functions of the product of this two operators. The simplest case is that the correlation function of the operators has only one pole. Similarly, we assume that it has at most one zero. Clearly, it naturally leads to that we must have the condition that the two products \( \Lambda_1(z)S_1^+(w) \) and \( S_1^+(w)\Lambda_1(z) \) have the same correlation functions. Then we have

\[
\Lambda_1(z)S_1^+(w) = A \frac{(z - wp_1)}{(z - wp_2)} : \Lambda_1(z)S_1^+(w) :, \quad |z| \gg |w|,
\]

\[
S_1^+(w)\Lambda_1(z) = A \frac{(z - p_1w)}{(z - p_2w)} : \Lambda_1(z)S_1^+(w) :, \quad |w| \gg |z|.
\]
This implies:

**Lemma 2.1.**

\[ Ap_1/p_2 = 1. \]

Therefore \( p_1 \) can not be 0.

The correlation functions also give us

\[ \Lambda_1(z) S^+_1(w) = A \exp \left( \sum_{n>0} \lambda_1(n) s^+_1(-n)(w/z)^n \right) : \Lambda_1(z) S^+_1(w) : \]

\[ = A \exp \left( \sum_{n>0} (w/z)^n (-p_1^n + p_2^n) \right) : \Lambda_1(z) S^+_1(w) : . \]

Thus we have

\[ \lambda_1(n) s^+_1(-n) = (-p_1^n + p_2^n). \]

Similarly we have

\[ \lambda_1(-n) s^+_1(n) = (-p_1^{-n} + p_2^{-n}). \]

**Lemma 2.2.** If the operator \( l_1(z) \) commute with the the operator \( \int S^+_1(z)dz/z, \)

then \( l_1(z) \) commute with \( S^+_1 w \) up-to a total difference, the correlation functions of the products \( \Lambda_1(z) S^+_1(w) \) and \( S^+_1(w) \Lambda_1(z) \) must be equal and the correlation functions must have only one pole and one zero.

**Proof** From the correlation function, we know that

\[ [\Lambda_1(z), \int S^+_1(w)dw/w] = A(1 - p_1/p_2)p_2 \Lambda_1(z) S^+_1(zp_2^{-1}) ::. \]

Thus

\[ [\Lambda_2(w), \int S^+_1(z)dz/z] = -A(1 - p_1/p_2)p_2 : \Lambda_1(w) S^+_1(wp_2^{-1}) ::. \]

Because of the assumption on the formulas for \( \Lambda_2(w) \) and \( S^+_1(z) \), it requires that the the correlation functions of the products \( \Lambda_1(z) S^+_1(w) \) and \( S^+_1(w) \Lambda_1(z) \) must be equal and the correlation functions must have only one pole and one zero.

Therefore \( l_1(z) \) commute with \( S^+_1 w \) up-to a total difference.

From now on, we assume that \( l_1(z) \) commute with \( \int S^+_1(z)dz/z. \) So from the lemma we have:

\[ \Lambda_2(z) S^+_1(w) = A' \frac{(z - wp'_1)}{(z - wp'_2)} : \Lambda_2(z) S^+_1(w) :, \quad |z| \gg |w|, \]

\[ S^+_1(w) \Lambda_2(z) = A' \frac{(z - wp'_1)}{(z - wp'_2)} : \Lambda_2(z) S^+_1(w) :, \quad |w| \gg |z|, \]

\[ A' p'_1/p'_2 = 1. \]

Similarly we have:

\[ \lambda_2(n) s^+_1(-n) = (-(p'_1)^n + (p'_2)^n). \]
Similarly we have
\[ \lambda_2(n)s_1^+(n) = -(p_1')^{-n} + (p_2')^{-n}. \]

**Corollary 2.3.**
\[ A(1 - p_1/p_2)p_2\Lambda_1(z)S_1^+(zp_2^{-1}) := -A'(1 - p'_1/p'_2)p'_2\Lambda_2(z)S_1^+(zp'_2^{-1}) :. \]

From the assumption on \( S_1^+(z) \), we have that the following relations for the screening currents when \( |z| \gg |w| \):
\[ S_1^+(w)S_1^+(z) = w^{2\beta}f(z/w) : S_1^+(z)S_1^+(w) :, \]
where \( f(z) \) is an analytic function in \( z \)

**Proposition 2.4.**
\[ -A \frac{(z - wp_1')}{(z - wp_2')}A'(1 - p'_1/p'_2)p'_2^{-2\beta}f(zp_2^{-1}/w)p'2 = A \frac{(z - wp_1)}{(z - wp_2)}A(1 - p_1/p_2)g_2^{-2\beta}f(zp_2^{-1}/w)p_2, \]
\[ -A \frac{(z - wp_1')}{(z - wp_2')}A'(1 - p'_1/p'_2)f(p'_2w/z) = A \frac{(z - wp_1)}{(z - wp_2)}A(1 - p_1/p_2)gf(p_2w/z). \]

Thus, from above let \( z = 0 \) or \( w = 0 \), we have
\[ (p'_2/p'_1)(1 - p'_1/p'_2)(p'_2/p_2)^{-2\beta} = -(p_2/p_1)(1 - p_1/p_2)g. \]

Thus
\[ \frac{1 - z wp_1'}{1 - z wp_2'}f(zp_2^{-1}/w) = \frac{1 - z wp_1}{1 - z wp_2}f(zp_2^{-1}/w). \]

Then from the formula above and \( f(z/w)w^{2\beta} \) would degenerate into \( (w - z)^{2\beta} \), when \( q \) goes to 1, we have

**Theorem 2.5.**
\[ f(z/w) = (1 - z/w) \frac{(z/w)p_2'/p_1, q)_\infty}{(z/w)p_2'/p'_2, q)_\infty}. \]
\[ \frac{p_1}{p'_1} = q^{2\beta - 1} \]

where we set \( p'_2/p_2 = q \) and
\[ (x|a, t)_\infty = \prod_{n=0}^{\infty} (1 - at^n). \]

This follows from that
\[ \lim_{q \to 1} \frac{(z/w|q^a, q)_\infty}{(z/w|q^b, q)_\infty} = (1 - z/w)^{b-a} \]

We can set \( p_1 \) to 1, which corresponding to shift \( l_1(z) \) by \( p_1 \).
Corollary 2.6. Let $p_1 = 1$ and $p'_2 = p$, we have

\[
\begin{align*}
p_2 &= pq^{-1} \\
p'_1 &= q^{1-2\beta} \\
g &= qq^{-2\beta}(pq^{2\beta-1} - 1) \\
    &= (1 - pq^{-1}).
\end{align*}
\]

Therefore with the assumption we impose on the screening operator and the operator $l_1(z)$, all the correlation function between the vertex operators are uniquely determined. These formulas also determine the formulas for $l_1(z)$ and $S_1^+(z)$. Now, we will proceed to present here the formulas for $l_1(z)$ and $S_1^+(z)$ and check that if this construction is possible such that the operator $l_1(z)$ commute with $S_1^+(z)$.

\[
f(z/w) = (1 - z/w) \frac{(z/w|p,q)_\infty}{(z/w|pq^{2\beta-1}, q)_\infty} = \exp \left( \sum_{n>0} \frac{1}{n} (-1 + \left( \frac{q^{n(2\beta-1)}p^n - p^n}{1 - q^n} (z/w)^n \right) \right).
\]

Thus we have

\[
s_1^+(n)s_1^+(-n) = (-1 + \left( \frac{q^{n(2\beta-1)}p^n - p^n}{1 - q^n} \right),
\]

for $n > 0$. Similarly we have that

\[
\begin{align*}
s_1^+(n)\lambda_1(-n) &= (-1 + (pq^{-1})^{-n}), \\
\lambda_1(n)s_1^+(-n) &= (-1 + (pq^{-1})^n), \\
s_1^+(n)\lambda_2(-n) &= (-q^{-n(2\beta-1)} + (p)^{-n}), \\
\lambda_2(n)s_1^+(-n) &= (-q^{n(2\beta-1)} + (p)^n),
\end{align*}
\]

for $n > 0$; and

\[
\begin{align*}
\exp -2\lambda_1(0)\beta &= pq^{-1} \\
\exp -2\lambda_2(0)\beta &= q^{2\beta-1}p.
\end{align*}
\]

From the algebraic point of view, it does not affect the algebraic structure of the operator $S_1^+(z)$, not matter what the solution we choose for the first equation for $S_1^+(n)$, because they are all equivalent up to re-scale of the Heisenberg algebra. Once they are chosen, then $\lambda_1(n)$ are automatically decided. However from Corollary 2.3, the following relations must be valid in order to make $l_1(z)$ commute with the integral of $S_1^+(z)$.

\[
\begin{align*}
\lambda_1(n) + s_1^+(n)q^n p^{-n} &= \lambda_2(n) + s_1^+(n)p^{-n}. \\
\exp \lambda_1(0)(qp^{-1}) = \exp \lambda_2(0)p^{-1}.
\end{align*}
\]
These are very strong restrictions. Then we have:

**Theorem 2.7.** For any given screen operators that has the correlation function as define by Theorem 2.5. Let the operator \( l_1(z) \) be a sum of two vertex operators such that \( l_1(z) \) commutes with the integral of the screen operator and the correlation function of product of one element of the sum has one pole and one zero. \( l_1(z) \) exists and unique up-to shift of \( z \), if and only if

\[
p = q^{1-\beta}.
\]

Now we have:

\[
\Lambda_1(z)S_1^+(w) = q^{-\beta} \frac{z-w}{z-wq^{-\beta}} : \Lambda_1(z)S_1^+(w) :, \quad |z| \gg |w|,
\]

\[
S_1^+(w)\Lambda_1(z) = q^{-\beta} \frac{z-w}{z-wq^{-\beta}} : \Lambda_1(z)S_1^+(w) :, \quad |w| \gg |z|;
\]

and

\[
\Lambda_1(zq^{1-\beta})S_1^+(w) = q^{-\beta} \frac{z-wq^{-\beta}}{z-w} : \Lambda_1(z)S_1^+(w) :, \quad |z| \gg |w|,
\]

\[
S_1^+(w)\Lambda_1(z) = q^{\beta} \frac{z-wq^{-\beta}}{z-w} : \Lambda_1(z)S_1^+(w) :, \quad |w| \gg |z|.
\]

Thus we have:

**Corollary 2.8.**

\[
\lambda_1(n) = -\lambda_2(n)q^{n(\beta-1)}.
\]

Let \( g_{ij}(z) \) for \( i, j = 1, 2 \) be the function such that

\[
\Lambda_i(z)\Lambda_j(w) = g_{ij}(w/z) : \Lambda_i(z)\Lambda_j(w) :.
\]

From Corollary 2.3, we have

**Proposition 2.9.** \( g_{ij}(w/z)/g_{ji}(z/w) \) are equal for all the \( i, j = 1, 2 \).

**Theorem 2.10.** \( l_1(z) \) satisfies the same commutation relations as the quantized virasoro algebra in \([2]\).

This follow from the corollary and proposition above. Or, if we compare these formulas with the know formulas \([2]\), etc., we know that they are exactly the same as the formulas for the bosonization of \(q\)-virasoro algebra and its screen operator with a difference of a shift of the variable for \(l_1(z)\).

These results show that the \(q\)-virasoro algebra can be derived from the point view of deformation of the screen operators and then derivation of \(q\)-deformed virasoro algebra and this deformation is also uniquely determined. This shows that the quantized virasoro algebra is indeed
a rigid structure. However, even from this point of view, there is an exception case, namely the case $\beta = 1/2$.

3. The commutant for the case $\beta = 1/2$

We will follow the same notations as in the section above. for the basic definitions. But this time, we will set the value $\beta = 1/2$. We also first assume that $p$ is $q$ are generic.

Thus we have:

$$S(z)S(w) = (z - w) : S(z)S(w) :$$

As in the section above, let us define the operator as

$$l_1(z) = \Lambda_1(z) + \Lambda_2(z),$$

where $\Lambda_i(z)$ as the generating function:

$$\Lambda_i(z) = g_i^{-1} : \exp \left( \sum \lambda_i(m) a_1[m] z^{-m} \right) : .$$

(6)

Here $\lambda_i[m]$ are in $\mathbb{C}[i, n]$ for $i = 1, 2$. We also assume that that the two products $\Lambda_1(z)S_1^+(w)$ and $S_1^+(w)\Lambda_1(z)$ have the same correlation functions. Then we have

$$\Lambda_1(z)S_1^+(w) = A \frac{(z - wp_1)}{(z - wp_2)} : \Lambda_1(z)S_1^+(w) :, \quad |z| \gg |w|,$$

$$S_1^+(w)\Lambda_1(z) = A \frac{(z - p_1 w)}{(z - p_2 w)} : \Lambda_1(z)S_1^+(w) :, \quad |w| \gg |z|,$$

$$Ap_1/p_2 = 1,$$

and $l_1(z)$ commutes with the the operator $\int S_1^+(z)dz/z$.

Let’s fix $p_2'/p_2 = q$. From Proposition 2.4, we have

$$\frac{(1 - \frac{z}{wp_2})}{(1 - \frac{z}{wp_2'})} f(zp_2'^{-1}/w) = \frac{(1 - \frac{z}{wp_1})}{(1 - \frac{z}{wp_2'})} f(zp_2^{-1}/w)$$

and

$$f(z/w) = (1 - z/w) (z/w|p_2'/p_1, q)_\infty \div (z/w|p_2'/p_1, q)_\infty.$$

Therefore, we have
Theorem 3.1.

\[ f(z/w) = (1 - z/w), \]

\( S_1^+(z) \) is a fermion.

Corollary 3.2.

\[ (1 - \frac{z}{wp_1'}) = (1 - \frac{z}{wp_1}) \]

Here we can see clearly that this a very special situation compared with other \( \beta \), because the equation shows that we have one fewer restriction on the choice of the parameters that decides the poles and the zeros of the correlation function of \( \Lambda_1(z) \) and \( S_1^+(z) \). This basically tells us that, for any operator \( \Lambda_1(z) \) such that \( \Lambda_1(z) \) has the correlation functions with the fermion \( S_1^+(z) \) as defined above, we can derive uniquely another vertex operator \( \Lambda_2(z) \) such that \( l_1(z) \) commute with \( S_1^+(z) \). This construction can be explicitly given as

Theorem 3.3. Let \( \Lambda_1(z) \) be an operator such that the correlation function of \( \Lambda_1(z) \) with the fermion \( S_1^+(z) \) defined as above. The the operator

\[ l_1(z) = \Lambda(z)_1 + p_2(p_2/p_1 - 1)\Lambda(z)_1 S_1^+(zp_2^{-1})(S_1^+(z))^{-1}, \]

commute with the action of the integral of \( S_1^+(z) \).

Corollary 3.4. Let

\[ s_1^+(n) = 1, s_1^+(-n) = -1, \]

Then

\[ \lambda_1(-n) = (-p_1^{-n} + (p_2)^{-n}), \]
\[ \lambda_1(n) = (-p_1^{n} + (p_2)^{n}), \]
\[ \lambda_2(-n) = (-p_1^{-n} + (p_2q)^{-n}), \]
\[ \lambda_2(n) = (-p_1^{n} + (p_2q)^{n}), \]

for \( n > 0 \).

These automatically gives us the commutation relations between \( \Lambda_i(z) \) and \( l_1(z) \)

Theorem 3.5. On the Fock space, the matrix coefficients of \( l_1(z)l_1(w) \)
and \( l_1(w)l_1(z) \) are equal.

Next we will deal with the case, if they are two or more screen operator, which would also leads us the quantized W-algebra and other new algebras, when we choose the screening operators for other cases such as \( sl(2,1) \).
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4. THE CASE FOR TWO SCREEN OPERATORS.

In this section, we will extend the similar construction to the case of two screen operators.

Let us first introduce the Heisenberg algebra $H_{q,p}(2)$ be the Heisenberg algebra with generators $a_i[n], n \in \mathbb{Z}$, and relations

$$[a_i[n], a_j[m]] = \frac{1}{n} A_{ij}(n) \delta_{n,-m},$$

which is defined on the field of the rational functions of $p$ and $q$, two generic parameters for $|q| < 1$, $A_{i,i} = 1$ and $A_{ij}(n) = a(n)$ is a rational function of $p$ and $q$ for $i \neq j$.

For $\mu$ be a element of a two dimensional space $A_2$ generated by $\alpha$, for $i = 1, 2$. Let $\alpha_i^*$ be the generator of its dual space $A_2^*$, such that $\alpha_i^*(\alpha_i) = 2$ and $\alpha_i^*(\alpha_j) = -b/\beta_i$. Let $\pi_\mu$ be the Fock representation of $H_{q,p}(2)$ generated by a vector $v_\mu$, such that $a_i[n]v_\mu = 0, n > 0$, and $a_1[0]v_\mu = \mu(\alpha_i^*)v_\mu$. We assume here $\beta_1$ and $\beta_2$ generic.

Introduce operators $Q_1$, which satisfy commutation relations $[a_i[n], Q_1] = 2\beta_i \delta_{n,0}, [a_j[n], Q_1] = -b \delta_{n,0}$. The operators $e^{Q_1}$ act from $\pi_\mu$ to $\pi_{\mu+\beta_\alpha}$.

Now we can define two quantized screening currents as the generating function

$$S_i^+(z) = e^{Q_1z}s_i^+[0]:\exp\left(\sum_{m \neq 0} s_i^+(m)a_i[m]z^{-m}\right):,$$

where $s_i^+[m]$ are in $\mathbb{C}[i, i]$ for $m \neq 0$ and $s_i^+[0] = a_i[0]$. We assume that the limit of these operators, when $q$ goes to one, degenerate into the classical screen operators. Let $S_i(z)$ be the classical counter part of $S_i^+(z)$. Then we have that

$$S_i^+(z)S_i^+(w) = z^{2\beta_i}(1-w/z)^{2\beta_i}:S_i^+(z)S_i^+(w):,$$

$$S_2^+(z)S_1^+(w) = z^{-b}(1-w/z)^{-b}:S_1^+(z)S_2^+(w):,$$

Thus we have that

$$S_i^+(z)S_i^+(w) = z^{2\beta_i} f_{i,i}(z, w):S_i^+(z)S_i^+(w): =$$

$$z^{2\beta_i} \exp\left(\exp \sum_{m > 0} s_i^+(m)s_i^+(-m)w^m/z^m\right):S_i^+(z)S_i^+(w):$$

$$S_2^+(z)S_1^+(w) = z^{-b} f_{2,1}(z, w):S_2^+(z)S_1^+(w): =$$

$$w^{-b} \exp\left(\exp \frac{1}{n} \sum_{m > 0} A_{2,1} s_2^+(m)s_i^+(-m)w^m/z^m\right):S_2^+(z)S_1^+(w):$$

$$S_1^+(z)S_2^+(w) = z^{-b} f_{1,2}(z, w):S_1^+(z)S_2^+(w): =$$

$$z^{-b} \exp\left(\exp \sum_{m > 0} A_{1,2} s_1^+(m)s_2^+(-m)w^m/z^m\right):S_1^+(z)S_2^+(w):$$

$$S_i^+(z)S_i^+(w) = z^{2\beta_i} f_{i,i}(z, w):S_i^+(z)S_i^+(w): =$$

$$z^{2\beta_i} \exp\left(\exp \sum_{m > 0} s_i^+(m)s_i^+(-m)w^m/z^m\right):S_i^+(z)S_i^+(w):$$

$$S_2^+(z)S_1^+(w) = z^{-b} f_{2,1}(z, w):S_2^+(z)S_1^+(w): =$$

$$w^{-b} \exp\left(\exp \frac{1}{n} \sum_{m > 0} A_{2,1} s_2^+(m)s_i^+(-m)w^m/z^m\right):S_2^+(z)S_1^+(w):$$

$$S_1^+(z)S_2^+(w) = z^{-b} f_{1,2}(z, w):S_1^+(z)S_2^+(w): =$$

$$z^{-b} \exp\left(\exp \sum_{m > 0} A_{1,2} s_1^+(m)s_2^+(-m)w^m/z^m\right):S_1^+(z)S_2^+(w):$$

$$S_i^+(z)S_i^+(w) = z^{2\beta_i} f_{i,i}(z, w):S_i^+(z)S_i^+(w): =$$

$$z^{2\beta_i} \exp\left(\exp \sum_{m > 0} s_i^+(m)s_i^+(-m)w^m/z^m\right):S_i^+(z)S_i^+(w):$$

$$S_2^+(z)S_1^+(w) = z^{-b} f_{2,1}(z, w):S_2^+(z)S_1^+(w): =$$

$$w^{-b} \exp\left(\exp \frac{1}{n} \sum_{m > 0} A_{2,1} s_2^+(m)s_i^+(-m)w^m/z^m\right):S_2^+(z)S_1^+(w):$$

$$S_1^+(z)S_2^+(w) = z^{-b} f_{1,2}(z, w):S_1^+(z)S_2^+(w): =$$

$$z^{-b} \exp\left(\exp \sum_{m > 0} A_{1,2} s_1^+(m)s_2^+(-m)w^m/z^m\right):S_1^+(z)S_2^+(w):$$
We would try to find a similar construction that gives us the operators that will commute with the action of this quantized screen operator $\int S^+_1(z)dz/z$. The simplest choice again would be the same assumption as in the sections above that this operator should be a sum of two vertex operators. For this case, it is a straightforward argument as in the sections above to show that it is impossible. We leave this as an exercise.

Therefore, the simplest possible choice is that they should be a sum of three vertex operators. Let us define the operator as

$$l_1(z) = \Lambda_1(z) + \Lambda_2(z) + \Lambda_3(z),$$

where $\Lambda_i(z)$ as the generating function:

$$\Lambda_i(z) = g_i : \exp \left( \sum \lambda_{ij}(m)a_j[m]z^{-m} \right) : .$$ (9)

Here $\lambda_i[m]$ are in $\mathbb{C}[i,j]$ for $i = 1, 2, 3$. As in the sections above, assumptions on the correlation functions are necessary. The possible simplest case to deal with is to have once more the simplest choice that the correlation functions between $S^+_1(z)$ and $\Lambda_i(w)$ are 1, for the two pairs $i = 1, j = 3$, and $i = 2, j = 1$, which also means that, for either pair of the operators, they commute with each other. Again, we will start from the correlation functions between $S^+_1(z)$ and $\Lambda_1(z)$. As in the section above, we picked the simplest choice that the two products $\Lambda_1(z)S^+_1(w)$ and $S^+_1(w)\Lambda_1(z)$ have the same correlation functions and

$$\Lambda_1(z)S^+_1(w) = A \frac{(z-w)}{(z-wpq^{-1})} : \Lambda_1(z)S^+_1(w) :, \quad |z| \gg |w|,$$

$$S^+_1(w)\Lambda_1(z) = A \frac{(z-w)}{(z-wpq^{-1})} : \Lambda_1(z)S^+_1(w) :, \quad |w| \gg |z|.$$

and

$$A = qp^{-1}.$$

From the results in the sections above, we have

**Proposition 4.1.** If the operator $l_1(z)$ commute with the the operator $\int S^+_1(z)dz/z$, then the correlation functions of the products $\Lambda_1(z)S^+_1(w)$ and $S^+_1(w)\Lambda_1(z)$ must be equal and the correlation functions must have only one pole and one zero.

$$\Lambda_2(z)S^+_1(w) = A \frac{(z-wp_1)}{(z-wp_2)} : \Lambda_2(z)S^+_1(w) :, \quad |z| \gg |w|,$$

$$S^+_1(w)\Lambda_2(z) = A \frac{(z-wp_1)}{(z-wp_2)} : \Lambda_2(z)S^+_1(w) :, \quad |w| \gg |z|,$$
\[
A' p'_1 / p'_2 = 1.
\]

\[
A(1-p_1/p_2)p_2 : \Lambda_1(z)S^+_1(zp_2^{-1}) := -A'(1-p'_1/p'_2)p'_2 : \Lambda_2(z)S^+_1(zp'_2^{-1}) :
\]

Let \( p'_2 = p \) and \( p'_2/p_2 = q \), then

\[
p'_1 = q^{1-2\beta}
\]

\[
g_2 = q^p q^{-2\beta} (pq^{-2\beta} - 1) \cdot
\]

This follows directly from the argument in the section above.

The proposition gives us that:

\[
A(1-p_1/p_2)p_2 : \Lambda_1(z)S^+_1(zp_2^{-1}) : S^+_2(w) = -A'(1-p'_1/p'_2)p'_2 : \Lambda_2(z)S^+_1(zp'_2^{-1}) : S^+_2(w)
\]

\[
S^+_2(w)A(1-p_1/p_2)p_2 : \Lambda_1(z)S^+_1(zp_2^{-1}) := -S^+_2(w)A'(1-p'_1/p'_2)p'_2 : \Lambda_2(z)S^+_1(zp'_2^{-1}) :
\]

Let

\[
S^+_i(z) \Lambda_j(w) = S \Lambda_{ij}(z, w) : S^+_i(z) \Lambda(w) ;,
\]

\[
\Lambda_i(z) S^+_j(w) = \Lambda S_{ij}(z, w) : S^+_i(z) \Lambda(w) ;
\]

**Corollary 4.2.**

\[
b = \beta_1,
\]

\[
B(1-p_1/p_2)q_2^{-b} f_{1,2}(zp_2^{-1}, w) = -B'(1-p'_1/p'_2)(p'_2)^{-b} g_2 f_{1,2}(zp'_2^{-1}, w) \Lambda S_{2,2}(z, w),
\]

\[
B(1-p_1/p_2) f_{2,1}(w, zp_2^{-1}) = -B'(1-p'_1/p'_2) g_2 \Lambda S_{2,2}(w, z) f_{2,1}((w, p'_2^{-1} z).
\]

We have:

\[
f_{2,1}(w, zp_2^{-1}) / f_{2,1}(w, zp'_2^{-1}) = S \Lambda_{2,2}(w, z),
\]

\[
f_{1,2}(zp_2^{-1}, w) / f_{1,2}(zp'_2^{-1}, w) = B^{-1} \Lambda S_{2,2}(z, w).
\]

\[
f_{2,1}(w, zq^\beta) / f_{2,1}(w, zq^\beta - 1) = S \Lambda_{2,2}(w, z),
\]

\[
f_{1,2}(zq^\beta, w) / f_{1,2}(zq^\beta - 1, w) = B^{-1} \Lambda S_{2,2}(z, w).
\]

Let’s assume the following condition:

\[
\Lambda_2(z) S^+_2(w) = B \left(\frac{z-wq_1}{z-wq_2}\right) : \Lambda_2(z) S^+_2(w) ;,
\]

\[
|z| \gg |w|,
\]

\[
S^+_2(w) \Lambda_2(z) = B \left(\frac{z-wq_1}{z-wq_2}\right) : \Lambda_2(z) S^+_2(w) ;,
\]

\[
|w| \gg |z|.
\]

Similarly from the results for the the case of one fermionic screening operator, we have
Proposition 4.3. If the operator $l_1(z)$ commute with the the operator $\int S_1^+(z)dz/z$, then the correlation functions of the products $\Lambda_3(z)S_2^+(w)$ and $S_2^+(w)\Lambda_3(z)$ must be equal and the correlation functions must have only one pole and one zero.

\[
\Lambda_3(z)S_2^+(w) = B'(\frac{z-wq_1}{z-wq_2}) : \Lambda_3(z)S_2^+(w) : , \quad |z| \gg |w|,
\]
\[
S_2^+(w)\Lambda_3(z) = B'(\frac{z-wq_1}{z-wq_2}) : \Lambda_3(z)S_2^+(w) : , \quad |w| \gg |z|,
\]

\[
B'(1-q_1/q_2)q_2 : \Lambda_2(z)S_2^+(zq_2^{-1}) := -B'(1-q_1/q_2')q_2' : \Lambda_3(z)S_2^+(zq_2'^{-1}) : .
\]

\[
B'q_1/q_2 = 1.
\]

Let $q_2'/q_2 = p'$.

Therefore we have:

\[
S_1^+(w)B'(1-q_1/q_2)q_2 : \Lambda_2(z)S_2^+(zq_2^{-1}) := -S_1^+(w)(1-q_1/q_2')q_2' : \Lambda_3(z)S_2^+(zq_2'^{-1}) : .
\]

\[
B'(1-q_1/q_2)q_2 : \Lambda_2(z)S_2^+(zq_2^{-1}) : S_1^+(w) = -A'(1-q_1/q_2')q_2' : \Lambda_3(z)S_2^+(zq_2'^{-1}) : S_1^+(w).
\]

Corollary 4.4.

\[
\beta_2 = b,
\]

\[
B'(1-q_1/q_2)q_2q_2^{-b}\Lambda S_2,1(z,w)f_{2,1}(zq_2^{-1},w) = -B'(1-q_1/q_2')q_2'(q_2'^{-b}g_3/g_2f_{2,1}(zq_2'^{-1},w),
\]

\[
B'(1-q_1/q_2)q_2S\Lambda_1,2(w,z)f_{1,2}(w,(zq_2^{-1})) = -B'(1-q_1/q_2')q_2'g_3/g_2f_{1,2}(w,(zq_2'^{-1})).
\]

\[
A^{-1}\Lambda S_{2,1}(z,w) = \frac{f_{2,1}(zq_2^{-1},w)}{f_{2,1}(zq_2^{-1},w)}.
\]

\[
S\Lambda_{1,2}(w,z) = \frac{f_{1,2}(w,zq_2^{-1})}{f_{1,2}(w,zq_2'^{-1})}.
\]

Because

\[
f_{2,1}(w,zp_2^{-1})/f_{2,1}(w,zp_2'^{-1}) = S\Lambda_{2,2}(w,z),
\]

We can see that, for $q'$, there are two possibilities: $q' = q$, or $q' = q^{-1}$.

If $q' = q_2'/q_2 = q_1$, we have

\[
\frac{(z-wq_1^{1-2\beta})}{(z-wq_1^{1-\beta})} = \frac{f_{2,1}(zq_2'^{-1},w)}{f_{2,1}(zq_2'^{-1},w)},
\]

\[
\frac{q_2/q_1(z-wq_1)}{(z-wq_2)} = \frac{f_{2,1}(w,zq_2^\beta)}{f_{2,1}(w,zq_2'^{-1})},
\]

\[
\frac{q_2/q_1(z-wq_2^\beta q_2)}{(z-wq_2^\beta q_1)} = \frac{f_{2,1}(w,zq_2'^{-1})}{f_{2,1}(w,zq_2'^{-1})}.
\]
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Or, if $q' = q^{-1} = q_2'/q_2$, we have

$$
\frac{q_2'}{q_1} \frac{(z - wq^\beta q_1)}{(z - wq^\beta q_2)} = \left(\frac{qw_2 - zq^{1-2\beta}}{q_2w - zq^{1-\beta}}\right)^{-1}.
$$

This is impossible.

**Theorem 4.5.** This operator $l_1(z)$ exists if and only $q' = q$, $q_2 = q_1q^{-\beta}$,

$$
\Lambda S_{2,1}(z, w) = A f_{2,1}(zq_2^{-1}, w) = f_{1,2}(w, zq_2^{-1}).
$$

$$
A^{-1} \Lambda S_{2,1}(wq_2', z) = S \Lambda_{2,2}(zq^{-\beta}, w).
$$

Here, the operator $l_1(z)$ is not uniquely determined, because we $q_i$ and $q'_i$ are determined up to one parameter. However we see that from the very beginning, we do not fix the structure of the Heisenberg algebra, rather let it depend on the parameter $A_{1,2}(n)$ and $A_{2,1}(n)$, which is decided by this extra parameter. On the other hand, it reflects nothing but the possibility to re-scale one of the two screening operators from $s^+_i(z)$, which does not change the correlation functions of $f_{i,i}(z, w)$ but the functions $f_{i,j}(z, w)$ for $i \neq j$.

Nevertheless, all the correlation functions between the vertex operators and the function $A(i, j)(n)$ are all uniquely determined, once we fix any of the parameter $q_i$ and $q'_i$. From now on, we will assume that $q'_2$ is fixed. With the formulas above, we can see that all the operators are thus uniquely determined.

On the other, this construction becomes the same as the ones given to the screen operators and the quantized Virasoro algebra for the case of $\mathfrak{sl}(3)$ in [2], if $q'_2 = p^{3/2}$.

Let’s assume that $l_1(z)$ exists. Then we have

$$
\frac{(q'_2z - wq^{1-2\beta})}{(q_2z - wq^{1-\beta})} = \frac{f_{2,1}(z, w)}{f_{2,1}(zq, w)}
$$

$$
\frac{(q'_2z - wq^{1-2\beta})}{(q_2z - wq^{1-\beta})} = \frac{f_{1,2}(w, z)}{f_{1,2}(w, zq)}.
$$

Therefore

**Proposition 4.6.**

$$
f_{2,1}(z, w) = \frac{(w/z|q'^{-1}\beta, q_2^{-2\beta}, q}_\infty}{(w/z|q'^{-1}\beta, q_2^{-2\beta}, q}_\infty),
$$
\[ f_{1,2}(w, z) = \frac{(z/w|q_2^2 q^{-1+2\beta}, q)_\infty}{(z/w|q_2^2 q^{-1+\beta}, q)_\infty}. \]

On the other, this construction becomes the same as the ones given to the screen operators and the quantized virasoro algebra for the case of \( \mathfrak{sl}(3) \) in [2], when \( q_2 = p^{3/2} \) and
\[ f_{1,2}(z, w) = f_{2,1}(z, w). \]

5. THE CASE FOR TWO SCREEN OPERATORS WITH ONE OF THEM AS A FERMION.

We use the same notations as in the section above. But here we assume that \( \beta_1 = \beta \) and \( \beta_2 = 1/2 \). We have the Heisenberg algebra \( \mathcal{H} \) and the Fock spaces. The two quantized screening currents are defined just before. and we assume that the limit of this operator when \( q \) goes to one degenerate into the classical screen operators.

Let \( S_i(z) \) be the classical counter part of \( S_i^+(z) \). Then we have that
\[
S_1^+(z)S_2^+(w) = (z - w)S_1^+(z)S_2^+(w),
S_1^+(z)S_1^+(w) = z^{2\beta_1}(1 - w/z) S_1^+(z)S_1^+(w),
S_2^+(z)S_1^+(w) = -b(1 - w/z)^{-b} S_2^+(z)S_1^+(w),
S_1^+(z)S_2^+(w) = z^{-b}(1 - w/z)^{-b} S_1^+(z)S_2^+(w).
\]
Therefore here we have:
\[
S_2^+(z)S_2^+(w) = z f_{i,i}(w, z) : S_i^+(z)S_i^+(w) :
\]
\[
z \exp \left( \exp \sum_{m>0} s_i^+ (m) s_i^+ (-m) w^m / z^m \right) : S_i^+(z)S_i^+(w) :
\]
\[
S_1^+(z)S_1^+(w) = z^{2\beta_1} f_{i,i}(w, z) : S_i^+(z)S_i^+(w) :
\]
\[
z^{2\beta_2} \exp \left( \exp \sum_{m>0} s_i^+ (m) s_i^+ (-m) w^m / z^m \right) : S_2^+(z)S_1^+(w) :
\]
\[
S_2^+(z)S_1^+(w) = z^{-b} f_{2,1}(w, z) : S_2^+(z)S_1^+(w) :
\]
\[
w^{-b} \exp \left( \exp 1/n \sum_{m>0} A_{2,1} s_2^+ (m) s_i^+ (-m) w^m / z^m \right) : S_2^+(z)S_1^+(w) :
\]
\[
S_1^+(z)S_2^+(w) = z^{-b} f_{1,2}(w, z) : S_1^+(z)S_2^+(w) :
\]
\[
z^{-b} \exp \left( \exp \sum_{m>0} A_{1,2} (m) s_2^+ (m) s_2^+ (-m) w^m / z^m \right) : S_1^+(z)S_2^+(w) :
\]
We will proceed as in the section above:

Let
\[
l_i(z) = \Lambda_1(z) + \Lambda_2(z) + \Lambda_3(z),
\]
where \( \Lambda_i(z) \) as the generating function:
\[
\Lambda_i(z) = g_i : \exp \left( \sum \lambda_{ij}(m) a_j [m] z^{-m} \right) :.
\]
(10)

Here \( \lambda_i[m] \) are in \( \mathbb{C}[l, i] \) for \( i = 1, 2, 3 \) and \( g_1 = 1 \).
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We assume that the correlation functions between $S_i^+(z)$ and $\Lambda_j(w)$ are 1, for the two pairs $i = 1, j = 3$, and $i = 2, j = 1$, which also means that for either pair of the operators, they commute with each other: the two products $\Lambda_1(z)S_i^+(w)$ and $S_i^+(w)\Lambda_1(z)$ have the same correlation functions; and

$$\Lambda_1(z)S_i^+(w) = A \frac{(z - w)}{(z - wpq^{-1})} : \Lambda_1(z)S_i^+(w) : , |z| \gg |w|,$$

$$S_i^+(w)\Lambda_1(z) = A \frac{(z - w)}{(z - wpq^{-1})} : \Lambda_1(z)S_i^+(w) : , |w| \gg |z|.$$ 

and

$$A = pq^{-1}.$$

**Proposition 5.1.** If the operator $l_1(z)$ commute with the the operator $\int S_i^+(z)dz/z$, then the correlation functions of the products $\Lambda_1(z)S_i^+(w)$ and $S_i^+(w)\Lambda_1(z)$ must be equal and the correlation functions must have only one pole and one zero.

$$\Lambda_2(z)S_i^+(w) = A' \frac{(z - wp)}{(z - wpq)} : \Lambda_2(z)S_i^+(w) : , |z| \gg |w|,$$

$$S_i^+(w)\Lambda_2(z) = A' \frac{(z - wp)}{(z - wpq)} : \Lambda_2(z)S_i^+(w) : , |w| \gg |z|,$$

$$A' p' / p_2 = 1.$$

$$A(1 - p_1 / p_2) p_2 : \Lambda_1(z)S_i^+(zp_2^{-1}) := -A' (1 - p'_1 / p'_2) p'_2 : \Lambda_2(z)S_i^+(zp_2^{-1}) : .$$

Let $p'_2 = p$ and $p'_2 / p_2 = q$, then

$$p'_1 = q^{1 - 2\beta}$$

$$g_2 = qp^{-2\beta} (pq^{2\beta_1 - 1} - 1) / (1 - pq^{-1}).$$

Let

$$S_i^+(z)\Lambda_j(w) = SL_{ij}(z, w) : S_i^+(z)\Lambda(w) : ,$$

$$\Lambda_i(z)S_j^+(w) = LS_{ij}(z, w) : S_i^+(z)\Lambda(w) : .$$

**Corollary 5.2.**

$$B(1 - p_1 / p_2) q_2^{-b} f_{1,2} (zp_2^{-1}, w) = -B' (1 - p'_1 / p'_2) (p'_2)^{-b} g_2 f_{1,2} (zp_2^{-1}, w) \Lambda S_{2,2}(z, w),$$

$$B(1 - p_1 / p_2) f_{2,1} (w, zp_2^{-1}) = -B' (1 - p'_1 / p'_2) g_2 S\Lambda S_{2,2}(w, z) f_{2,1} ((w, p'_2^{-1}z).$$
We also assume the following condition:

\[ \Lambda_2(z)S_2^+(w) = B \frac{(z - wq_1)}{(z - wq_2)} : \Lambda_2(z)S_2^+(w) :, \quad |z| \gg |w|, \]

\[ S_2^+(w)\Lambda_2(z) = B \frac{(z - wq_1)}{(z - wq_2)} : \Lambda_2(z)S_2^+(w) :, \quad |w| \gg |z|. \]

Similarly from the results for the the case of one fermionic screening operator, we have

**Proposition 5.3.** If the operator \( l_1(z) \) commute with the the operator \( \int S_2^+(z)dz/z \), then the correlation functions of the products \( \Lambda_3(z)S_2^+(w) \) and \( S_2^+(w)\Lambda_3(z) \) must be equal and the correlation functions must have only one pole and one zero.

\[ \Lambda_3(z)S_2^+(w) = B' \frac{(z - wq_1)}{(z - wq_2)} : \Lambda_3(z)S_2^+(w) :, \quad |z| \gg |w|, \]

\[ S_2^+(w)\Lambda_3(z) = B' \frac{(z - wq_1)}{(z - wq_2)} : \Lambda_3(z)S_2^+(w) :, \quad |w| \gg |z|, \]

\( B'(1-q_1/q_2)q_2 : \Lambda_2(z)S_2^+(zq_2^{-1}) := -B'(1-q_1/q_2)'q_2' : \Lambda_3(z)S_2^+(zq_2'^{-1}) : . \)

\[ B'q_1'/q_2' = 1. \]

Let \( q_2'/q_2 = p' \).

**Corollary 5.4.**

\( B'(1-q_1/q_2)q_2q_2^{-b}A\Lambda_{1,2}(z, w)f_{2,1}(zq_2^{-1}, w) = -B'(1-q_1/q_2)'q_2'g_3/g_2f_{2,1}(zq_2'^{-1}, w), \)

\( B'(1-q_1/q_2)q_2\Lambda_{1,2}(w, z)f_{1,2}(w, (zq_2^{-1})) = -B'(1-q_1/q_2)'q_2'g_2f_{1,2}(w, (zq_2'^{-1})). \)

Similarly we have

**Theorem 5.5.** This operator \( l_1(z) \) exists if and only

\[ q' = q \]

\[ q_2 = q_1q^{-\beta} \]

\[ \beta = b \]

\[ \Lambda S_{2,1}(z, w) = A \frac{f_{2,1}(zq_2^{-1}, w)}{f_{2,1}(zq_2^{-1}, w)} = \frac{f_{1,2}(w, zq_2^{-1})}{f_{1,2}(w, zq_2^{-1})}; \]

\[ A^{-1}\Lambda S_{2,1}(wq_2', z) = S\Lambda_{2,2}(zq^{-\beta}, w). \]
Proposition 5.6.

\[
f_{2,1}(z, w) = \frac{(w/z|q_2'q_2^{-1}q_2^{2-\beta}, q)_{\infty}}{(w/z|q_2'^{-1}q_2^{2-2\beta}, q)_{\infty}}
\]

\[
f_{1,2}(w, z) = \frac{(z/w|q_2q_2'^{-1+2\beta}, q)_{\infty}}{(z/w|q_2'q_2^{-1+\beta}, q)_{\infty}}
\]

The next is to derive the commutation relations of \(l_1(z)\) with itself.

Let's set

\[
s_1^+(n) = \frac{1}{1 - q^n},
\]

\[
s_1^+(-n) = (q^n - 1 + q^n\beta - q^n(1-\beta)),
\]

\[
s_2^+(n) = 1,
\]

\[
s_2^+(-n) = -1,
\]

for \(n > 0\).

We have:

\[
A_{2,1}(n)s_2^+(n)s_1^+(-n) = \frac{(-q_2'^{-n}q^n(2-\beta) + q_2'^{-n}q^n(2-2\beta))}{(1 - q^n)},
\]

\[
A_{1,2}(n)s_1^+(n)s_2^+(-n) = \frac{(-q_2'^{-n}q^n(-1+2\beta) + q_2'^{-n}q^n(-1+\beta))}{(1 - q^n)}
\]

Proposition 5.7.

\[
A_{2,1}(n) = \frac{(-q_2'^{-n}q^n(2-\beta) + q_2'^{-n}q^n(2-2\beta))}{(1 - q^n)(q^n - 1 + q^n\beta - q^n(1-\beta))},
\]

\[
A_{1,2}(n) = -(-q_2'^{-n}q^n(-1+2\beta) + q_2'^{-n}q^n(-1+\beta))
\]

\[
A_{2,1}(n)A_{1,2}(n) = (q^n(-1+2\beta) - q^n(-1+\beta)) \frac{(-q^n(2-\beta) + q^n(2-2\beta))}{(1 - q^n)(q^n - 1 + q^n\beta - q^n(1-\beta))}
\]

Let \(\tilde{\Lambda}_3(z) = g_3^{-1}\Lambda_3(z)\). Because of \(\Lambda_3(z)\)'s correlation functions with \(S_1^+(z)\) \(S_1^+(z)\), we have that:

\[
s_1^+(n)(\lambda_{3,1}(-n) + \lambda_{3,2}(-n)A_{1,2}(n)) = 0,
\]

\[
s_1^+(-n)(\lambda_{3,1}(n) + \lambda_{3,2}(n)A_{2,1}(n)) = 0,
\]

\[
(A_{2,1}(n)\lambda_{3,1}(-n) + \lambda_{3,2}(-n)) = (q_2'^{-n}q^n-n\beta - q_2'^{-n}),
\]

\[
(A_{1,2}(n)\lambda_{3,1}(n) + \lambda_{3,2}(n)) = -(q_2'^{-n}q^n+n\beta - q_2'^{-n}),
\]

for \(n > 0\).

Therefore

\[
\lambda_{3,2}(-n) = \frac{q_2'^{-n}q^n-n\beta - q_2'^{-n}}{1 + A_{2,1}(n)A_{1,2}(n)}
\]
\( \lambda_{3,1}(-n) = \frac{q_2^{-n} q_1^{-(n+\beta)} - q_2^{-n}}{1 + A_{2,1}(n)A_{1,2}(n)} A_{1,2}(n), \)

\( \lambda_{3,2}(n) = \frac{-\left(q_2^{-n} q_1^{-n+\beta} - q_2^{-n}\right)}{1 + A_{2,1}(n)A_{1,2}(n)} \),

\( \lambda_{3,1}(n) = A_{2,1}(n) \frac{-\left(q_2^{-n} q_1^{-n+\beta} - q_2^{-n}\right)}{1 + A_{2,1}(n)A_{1,2}(n)}. \)

for \( n > 0. \)

Then we have

\[
\Lambda \Lambda_{3,3}(n) = \frac{-\left(q_2^{-n} q_1^{-n+\beta} - q_2^{-n}\right)\left(q_2^{-n} q_1^{-n+\beta} - q_2^{-n}\right)}{1 + \left(q^n(-1+2\beta) - q^n(-1+\beta)\right)\frac{-\left(q^n(2-\beta) + q^n(2-2\beta)\right)}{(1-q^n)(q^n-1+q^n\beta-q^n(1-\beta))}(q^n-\beta)}.
\]

Equivalently, we should have

\[
\Lambda \Lambda_{3,3}(n) = \frac{-\left(q_2^{-n} q_1^{-n+\beta} - q_2^{-n}\right)\left(q_2^{-n} q_1^{-n+\beta} - q_2^{-n}\right)}{1 + \left(q^n(-1+2\beta) - q^n(-1+\beta)\right)\frac{-\left(q^n(2-\beta) + q^n(2-2\beta)\right)}{(1-q^n)(q^n-1+q^n\beta-q^n(1-\beta))}(q^n-\beta)}.
\]

for \( n > 0. \)

From the formula, we can see that they are indeed equal.

\[
\frac{1}{1 + A_{2,1}(n)A_{1,2}(n)} = \frac{1}{1 + \left(q^n(-1+2\beta) - q^n(-1+\beta)\right)\frac{-\left(q^n(2-\beta) + q^n(2-2\beta)\right)}{(1-q^n)(q^n-1+q^n\beta-q^n(1-\beta))}} = \frac{(1-q^n)(q^n-1-q^n\beta+q^n(1-\beta))}{(1-q^n)(q^n-1+q^n\beta-q^n(1-\beta))} = \frac{q^n(1-\beta)(q^n\beta-1)^2 + (1-q^n)(q^n-1+q^n\beta-q^n(1-\beta))}{q^n(1-\beta)(q^n\beta-1)^2 + (1-q^n)(q^n-1+q^n\beta-q^n(1-\beta))} = \frac{-q^{-n}\beta(1-q^n)(q^n-1+q^n\beta-q^n(1-\beta))}{(1-q^{-n}\beta)(1-q^{2n-n}\beta)}.
\]
\[
\Lambda_{3,3}(n) = \frac{q^{n-n^\beta} (q^{-n+n^\beta} - 1)^2 q^{-n^\beta} (1 - q^n)(q^n - 1 + q^{n^\beta} - q^{n(1-\beta)})}{(1 - q^{n^\beta})(1 - q^{2n-n^\beta})}.
\]

\[
q^{n-n^\beta} (q^{-n+n^\beta} - 1)^2 q^{-n^\beta} (1 - q^n)(q^n + q^n).
\]

\[
(1 - q^{2n-n^\beta})
\]

\[
s_2^+(n)(A_{2,1}(n)\lambda_{1,1}(-n) + \lambda_{1,2}(-n)) = 0,
\]

\[
s_2^+(n)(A_{1,2}(n)\lambda_{1,1}(n) + \lambda_{1,2}(n)) = 0,
\]

\[
s_1^+(n)(\lambda_{1,1}(-n) + A_{1,2}(n)\lambda_{1,2}(-n)) = (1 + q^{n^\beta}),
\]

\[
s_1^+(n)(\lambda_{1,1}(n) + A_{2,1}(n)\lambda_{1,2}(n)) = (1 + q^{-n^\beta}),
\]

for \( n > 0 \).

\[
\lambda_{1,1}(n) = \frac{-1 + q^{n^\beta}}{s_1^+(n)(1 + A_{2,1}(n)A_{1,2}(n))}.
\]

Therefore

\[
\Lambda\Lambda_{1,1}(n) = \lambda_{1,1}(n)(\lambda_{1,1}(-n) + A_{1,2}(n)\lambda_{1,2}(-n))s_1^+(n)/s_1^+(n) = \frac{(-1 + q^{n^\beta})(-1 + q^{-n^\beta})}{s_1^+(n)s_1^+(n)(1 + A_{2,1}(n)A_{1,2}(n))}
\]

\[
= \frac{-(q^{n^\beta} - 1)(1 - q^n)^2 q^{-n^\beta}}{(1 - q^{2n-n^\beta})}.
\]

With this we have:

\[
\Lambda\Lambda(n) - \Lambda\Lambda_1(n) = (1 - q^n)(q^{-n+n^\beta} - q^{-n^\beta}) = q^{-n+n^\beta} + q^{-n^\beta} - q^n - q^{-n^\beta}.
\]

This shows that \( \Lambda_1(z) \) and \( \Lambda_3(z) \) basically have the same commutation relation up to certain poles.

Let \( v \) be an element in the Fock space and \( v^* \) an element in its dual space. Let us denote the matrix coefficient of an operator \( X \) by \( \langle v^*, Xv \rangle \).

**Proposition 5.8.**

\[
\langle v^*\Lambda(z)\Lambda_{1,1}(w)v \rangle = \langle v^*\Lambda(z)\Lambda_{1,1}(z)v \rangle \times \frac{\theta_{q^2,q^{1-\beta}}(\frac{w}{z}q^{1-\beta})^2 \theta_{q^2,q^{-\beta}}(\frac{w}{z}q^2)}{\theta_{q^2,q^{-\beta}}(\frac{w}{z}q^{1-\beta})^2 \theta_{q^2,q^{-\beta}}(\frac{w}{z}q^2)}
\]
This is proven by the following calculation.

\[
\Lambda_1(z)\Lambda_1(w) = \Lambda \Lambda_{1,1}(z, w) : \Lambda_1(z)\Lambda_1(w) : \\
\exp\left(\sum_{n>0}(w/z)^n - (q^{n\beta} - 1)(1 - q^n)^2q^{-n\beta} \right) : \Lambda_1(z)\Lambda_1(w) := \\
\exp\left(\sum_{n>0}(w/z)^n - 1 + 2q^n - q^{2n} + q^{-n\beta} - 2q^{n-n\beta} + q^{2n-n\beta} \right) : \Lambda_1(z)\Lambda_1(w) := \\
(w/z|q^{1-\beta}, q^{2-\beta})_\infty(w/z|1, q^{2-\beta})_\infty(w/z|q^2, q^{2-\beta})_\infty : \Lambda_1(z)\Lambda_1(w) : \\
(w/z|q, q^{2-\beta})_\infty(w/z|q^{-\beta}, q^{2-\beta})_\infty(w/z|q^2, q^{2-\beta})_\infty : \Lambda_1(z)\Lambda_1(w) := \\
\frac{(w/z|q^{1-\beta}, q^{2-\beta})_\infty(z/w|1, q^{2-\beta})_\infty(z/w|q^2, q^{2-\beta})_\infty \times (z/w|q, q^{2-\beta})_\infty(z/w|q^{-\beta}, q^{2-\beta})_\infty(z/w|q^2, q^{2-\beta})_\infty}{(w/z|q, q^{2-\beta})_\infty(w/z|q^{-\beta}, q^{2-\beta})_\infty(w/z|q^2, q^{2-\beta})_\infty} = \\
\frac{\theta_{q^{2-\beta}}(wz^{1-\beta})^2\theta_{q^{2-\beta}}(wz^2)}{\theta_{q^{2-\beta}}(wz^{1-\beta})^2\theta_{q^{2-\beta}}(wz^2)}. \\
\]

Therefore

Proposition 5.9.

\[
\Lambda \Lambda_{1,1}(z, w)/\Lambda \Lambda_{1,1}(w, z) = \frac{\theta_{q^{2-\beta}}(wz^{1-\beta})^2\theta_{q^{2-\beta}}(wz^2)}{\theta_{q^{2-\beta}}(wz^{1-\beta})^2\theta_{q^{2-\beta}}(wz^2)}. \\
\]

This follows from:

\[
\Lambda \Lambda_{1,1}(z, w)/\Lambda \Lambda_{1,1}(w, z) = \\
\frac{(w/z|q^{1-\beta}, q^{2-\beta})_\infty(z/w|1, q^{2-\beta})_\infty(z/w|q^2, q^{2-\beta})_\infty \times (z/w|q, q^{2-\beta})_\infty(z/w|q^{-\beta}, q^{2-\beta})_\infty(z/w|q^2, q^{2-\beta})_\infty}{(w/z|q, q^{2-\beta})_\infty(z/w|q^{-\beta}, q^{2-\beta})_\infty(z/w|q^2, q^{2-\beta})_\infty} = \\
\frac{\theta_{q^{2-\beta}}(wz^{1-\beta})^2\theta_{q^{2-\beta}}(wz^2)}{\theta_{q^{2-\beta}}(wz^{1-\beta})^2\theta_{q^{2-\beta}}(wz^2)}. \\
\]

Let

\[
l(z)l(w) = L(z, w) : l(z)l(w) : . \\
\]

Theorem 5.10.

\[
\frac{L(z, w)}{L(w, z)} = \frac{\theta_{q^{2-\beta}}(wz^{1-\beta})^2\theta_{q^{2-\beta}}(wz^2)}{\theta_{q^{2-\beta}}(wz^{1-\beta})^2\theta_{q^{2-\beta}}(wz^2)}. \tag{11} \\
\]

We call the algebra associated to the operator \(l(z)\), the quantized W-algebra of \(\mathfrak{sl}(2, 1)\). The degeneration of this operator, when \(q\) goes to 1, will give us the classical W-algebra of \(\mathfrak{sl}(2, 1)\).
QUANTIZED W-ALGEBRA OF $sl(2,1)$: A CONSTRUCTION FROM THE QUANTIZATION OF SCREENING OPERATORS

6. The Case for Two Fermion Screen Operators.

We use the same notations as in the section above. But here we assume that $\beta_1 = 1/2$ and $\beta_2 = 1/2$. We have the Heisenberg algebra $H_{q,p}(2)$ and the Fock spaces. The two quantized screening currents are defined just before. and we assume that the limit of this operator when $q$ goes to one degenerate into the classical screen operators.

Let $S_i(z)$ be the classical counterpart of $S_i^+(z)$. Then we have that

$$S_i^+(z)S_i^+(w) = (z - w)S_i^+(z)S_i^+(w) :,$$

$$S_i^+(z)S_i^+(w) = (z - w) : S_i^+(z)S_i^+(w) :,$$

$$S_i^+(z)S_i^+(w) = z^{-b}(1 - w/z)^{-b} : S_i^+(z)S_i^+(w) :,$$

$$S_i^+(z)S_i^+(w) = z^{-b}(1 - w/z)^{-b} : S_i^+(z)S_i^+(w) :.$$

Therefore here we have: We have that

$$S_i^+(z)S_i^+(w) = z f_i, i(w, z) : S_i^+(z)S_i^+(w) :=
$$

$$z \exp \left( \exp \Sigma_{m>0} s_i^+(m) s_i^+(-m) w^m / z^m \right) : S_i^+(z)S_i^+(w) :$$

$$S_i^+(z)S_i^+(w) = w^{-b} f_{2,1}(w, z) : S_i^+(z)S_i^+(w) :=
$$

$$w^{-b} \exp \left( \exp 1/n \Sigma_{m>0} A_{2,1} s_2^+(m) s_1^+(-m) w^m / z^m \right) : S_i^+(z)S_i^+(w) :$$

$$S_i^+(z)S_i^+(w) = w^{-b} f_{1,2}(w, z) : S_i^+(z)S_i^+(w) :=
$$

$$z^{-b} \exp \left( \exp \Sigma_{m>0} A_{1,2} s_2^+(m) s_2^+(-m) w^m / z^m \right) : S_i^+(z)S_i^+(w) :$$

We will define $l_i(z)$ as an operator that commutes with the action of the quantized screening operators $f S_i^+(z)dz/z$, and

$$l_i(z) = \Lambda_1(z) + \Lambda_2(z) + \Lambda_3(z),$$

where $\Lambda_i(z)$ as the generating function:

$$\Lambda_i(z) = g_i \exp \left( \sum \lambda_{ij}(m) a_j [m] z^{-m} \right) : . \quad (12)$$

Here $\lambda_i[m]$ are in $\mathbb{C}[i, ii]$ for $i = 1, 2, 3$ and $g_i = 1$. We use the same assumption that the correlation functions between $S_i^+(z)$ and $\Lambda_j(w)$ are 1, for the two pairs $i = 1, j = 3$, and $i = 2, j = 1$, which also means that for either pair of the operators, they commute with each other. We also assume that the correlation functions between $S_i^+(z)$ and $\Lambda_1(z)$, satisfy the condition that the two products $\Lambda_1(z)S_i^+(w)$ and $S_i^+(w)\Lambda_1(z)$ have the same correlation functions and

$$\Lambda_1(z)S_i^+(w) = A \frac{(z - w)}{(z - wpq^{-1})} : \Lambda_1(z)S_i^+(w) :, \quad |z| \gg |w|,$$

$$S_i^+(w)\Lambda_1(z) = A \frac{(z - w)}{(z - wpq^{-1})} : \Lambda_1(z)S_i^+(w) :, \quad |w| \gg |z|. $$
and

\[ A = pq^{-1}. \]

Similarly we have:

**Proposition 6.1.** If the operator \( l_1(z) \) commute with the the operator \( \int S_1^+(z) dz/z \), then the correlation functions of the products \( \Lambda_1(z)S_1^+(w) \) and \( S_1^+(w)\Lambda_1(z) \) must be equal and the correlation functions must have only one pole and one zero.

\[
\Lambda_2(z)S_1^+(w) = A' \left( \frac{z - wp_1'}{z - wp_2'} \right) : \Lambda_2(z)S_1^+(w) :, \quad |z| \gg |w|,
\]

\[
S_1^+(w)\Lambda_2(z) = A' \left( \frac{z - wp_1'}{z - wp_2'} \right) : \Lambda_2(z)S_1^+(w) :, \quad |w| \gg |z|,
\]

\[
A'p_1'/p_2' = 1.
\]

Let \( p_2' = p \) and \( p_2'/p_2 = q \), then

\[
p_1' = 1.
\]

\[
g_2 = pq^{-1}(pq^{-1} - 1).
\]

This follows directly from the argument in the section above.

Let

\[
S_1^+(z)\Lambda_i(z) = SA_{ij}(z, w) : S_1^+(z)\Lambda(w) :,
\]

\[
\Lambda_i(z)S_j^+(w) = AS_{ij}(z, w) : S_i^+(z)\Lambda(w) :.
\]

**Corollary 6.2.**

\[
B(1-p_1/p_2)q_2^{-b}f_{1,2}(zp_2^{-1}, w) = -B'(1-p_1'/p_2')(p_2')^{-b}g_2f_{1,2}(zp_2^{-1}, w)\Lambda S_{2,2}(z, w),
\]

\[
B(1-p_1/p_2)f_{2,1}(w, zp_2^{-1}) = -B'(1-p_1'/p_2')g_2S_{2,2}(w, z)f_{2,1}(w, zp_2^{-1}).
\]

We have:

\[
f_{2,1}(w, zp_2^{-1})/f_{2,1}(w, zp_2^{-1}) = A\Lambda_{2,2}(w, z),
\]

\[
f_{1,2}(zp_2^{-1}, w)/f_{1,2}(zp_2^{-1}, w) = B^{-1}\Lambda S_{2,2}(zp, w).
\]

\[
f_{2,1}(w, zq^\delta)/f_{2,1}(w, zq^\delta^{-1}) = A\Lambda_{2,2}(w, z),
\]

\[
f_{1,2}(zq^\delta, w)/f_{1,2}(zq^\delta^{-1}p_2^{-1}, w) = B^{-1}\Lambda S_{2,2}(z, w).
\]

Let’s assume the following condition:

\[
\Lambda_2(z)S_2^+(w) = B \left( \frac{z - wq_1}{z - wq_2} \right) : \Lambda_2(z)S_2^+(w) :, \quad |z| \gg |w|,
\]
Corollary 6.4.

We have:

**Proposition 6.3.** If the operator \( l_1(z) \) commute with the the operator \( \int S_2^+(z)dz/z \), then the correlation functions of the products \( \Lambda_3(z)S_2^+(w) \) and \( S_2^+(w)\Lambda_3(z) \) must be equal and the correlation functions must have only one pole and one zero.

\[
\Lambda_3(z)S_2^+(w) = B'\left(\frac{z-wq_1}{z-wq_2}\right) : \Lambda_3(z)S_2^+(w) :, \quad |z| \gg |w|,
\]

\[
S_2^+(w)\Lambda_3(z) = B'\left(\frac{z-wq_1}{z-wq_2}\right) : \Lambda_3(z)S_2^+(w) :, \quad |w| \gg |z|,
\]

\[
B'(1-q_1/q_2)q_2 : \Lambda_2(z)S_2^+(zq_2^{-1}) := -B'(1-q_1/q_2')q_2' : \Lambda_3(z)S_2^+(zq_2'^{-1}) :, \quad B'q_1'/q_2' = 1.
\]

Let \( q_2'/q_2 = p' \).

**Corollary 6.4.**

\[
B'(1-q_1/q_2)q_2g_{1,2}(w, zq_2^{-1}) = -B'(1-q_1/q_2')q_2'g_{1,2}(w, zq_2'^{-1}),
\]

\[
B'(1-q_1/q_2)g_{2,1}(w, (zq_2^{-1})) = -B'(1-q_1/q_2')g_{2,1}(w, (zq_2'^{-1})).
\]

\[
A^{-1}\Lambda S_{2,1}(z, w) = \frac{f_{2,1}(zq_2^{-1}, w)}{f_{2,1}(zq_2^{-1}, w)},
\]

\[
SA_{1,2}(w, z) = \frac{f_{1,2}(w, zq_2^{-1})}{f_{1,2}(w, zq_2^{-1})}.
\]

Because

\[
f_{2,1}(w, zp_2^{-1})/f_{2,1}(w, zp^{-1}) = SA_{2,2}(w, z),
\]

or

\[
f_{2,1}(w, zp^{-1}q)/f_{2,1}(w, zp^{-1}) = SA_{2,2}(w, z),
\]

We can see that, for \( q' \), there are two possibilities: \( q' = q \), or \( q' = q^{-1} \).

If \( q' = q_2'/q_2 = q \), we have

\[
\frac{(z-w)}{(z-wp)} = \frac{f_{2,1}(zq_2'^{-1}, w)}{f_{2,1}(zq_2^{-1}, w)},
\]

\[
\frac{q_2/q_1}{(z-wq_2)} = \frac{f_{2,1}(w, zp^{-1})}{f_{2,1}(w, zp^{-1})}.
\]

\[
\frac{q_2/q_1}{(z-wq_2)} = \frac{(wp_2^{-1}z)}{(q_2'w - zp)}.
\]
For this we must have:

\[ \frac{q_2}{q_1} = p^{-1}. \]

Or, if \( q' = q^{-1} = q'_2/q_2 \), we have

\[ \frac{q_2}{q_1} \frac{(z - wpq_1)}{(z - wpq_2)} = \frac{(wq_2 - z)}{(q_2w - zp)}^{-1}. \]

This is impossible.

Similarly we have

**Theorem 6.5.** This operator \( l_1(z) \) exists if and only if

\[ q' = q \]

\[ q_2 = q_1 p, \]

\[ \Lambda S_{2,1}(z, w) = A \frac{f_{2,1}(zq_2^{-1}, w)}{f_{2,1}(zq_2^{-1}, w)} = \frac{f_{1,2}(w, zq_2^{-1})}{f_{1,2}(w, zq_2^{-1})}, \]

\[ A^{-1} \Lambda S_{2,1}(wq_2', z) = S \Lambda_{2,2}(z p^{-1}, w). \]

**Proposition 6.6.**

\[ f_{2,1}(z, w) = \frac{(w/z|q_2^{-1}pq, q)_{\infty}}{(w/z|q'_2^{-1}q, q)_{\infty}}, \]

\[ f_{1,2}(w, z) = \frac{(z/w|q_2^{-1}p, q)_{\infty}}{(z/w|q'_2, q)_{\infty}} \]

The next is to derive the commutation relations of \( l_1(z) \) with itself.

Let’s set

\[ s_1^+(n) = 1, \]

\[ s_1^+(-n) = -1, \]

\[ s_2^+(n) = 1, \]

\[ s_2^+(-n) = -1, \]

for \( n > 0 \).

We have:

\[ A_{2,1}(n)s_2^+(n)s_1^+(-n) = \frac{(-q_2^{-n}q^n p^n + q'_2^{-n}q^n)}{(1 - q^n)} \]

\[ A_{1,2}(n)s_1^+(n)s_2^+(-n) = \frac{(q_2^{-n} - q'_2^{-n} p^{-n})}{(1 - q^n)} \]

Therefore
Proposition 6.7.

\[ A_{2,1}(n) = -\frac{(-q_2'^-n q^n p^n + q_2'^-n q^n)}{(1 - q^n)} \]

\[ A_{1,2}(n) = -\frac{(q_2'^n - q_2'^n p^{-n})}{(1 - q^n)} \]

\[ A_{2,1}(n)A_{12}(n) = \frac{q^n(1 - p^n)(1 - p^{-n})}{(1 - q^n)(1 - q^n)} \]

Because of \( \Lambda_3(z) \)'s correlation functions with \( S_1^+(z) \), we have that:

\[ s_1^+(n)(\lambda_{3,1}(-n) + \lambda_{3,2}(-n)A_{1,2}(n)) = 0, \]

\[ s_1^-(n)(\lambda_{3,1}(n) + \lambda_{3,2}(n)A_{2,1}(n)) = 0, \]

\[ (A_{2,1}(n)\lambda_{3,1}(-n) + \lambda_{3,2}(-n)) = -(q_2'^n p^n q^{-n} - q_2'^{-n}), \]

\[ (A_{1,2}(n)\lambda_{3,1}(n) + \lambda_{3,2}(n)) = (q_2'^{-n} q^n p^{-n} - q_2'^{-n}), \]

for \( n > 0 \).

Thus

\[ \lambda_{3,2}(-n) = \frac{(-q_2'^n p^n q^{-n} - q_2'^{-n})}{1 + A_{2,1}(n)A_{1,2}(n)}, \]

\[ \lambda_{3,1}(-n) = \frac{-q_2'^n p^n q^{-n} - q_2'^{-n}}{1 + A_{2,1}(n)A_{1,2}(n)}A_{1,2}(n), \]

\[ \lambda_{3,2}(n) = \frac{(q_2'^{-n} q^n p^{-n} - q_2'^{-n})}{1 + A_{2,1}(n)A_{1,2}(n)}, \]

\[ \lambda_{3,1}(n) = \frac{(q_2'^{-n} q^n p^{-n} - q_2'^{-n})}{1 + A_{2,1}(n)A_{1,2}(n)}A_{2,1}(n), \]

for \( n > 0 \).

Then we have

\[ \Lambda \Lambda_3(n) = \]

\[ \lambda_{3,2}(n)(A_{2,1}(n)\lambda_{3,1}(-n) + \lambda_{3,2}(-n)) = \]

\[ -(q_2'^{-n} q^n p^{-n} - q_2'^{-n})(q_2'^n p^n q^{-n} - q_2'^{-n}) + A_{2,1}(n)A_{1,2}(n), \]

Equivalently, we should have

\[ \Lambda \Lambda_3(n) = \]

\[ \lambda_{3,2}(-n)(A_{1,2}(n)\lambda_{3,1}(n) + \lambda_{3,2}(n)) = \]

\[ -(q_2'^{-n} q^n p^{-n} - q_2'^{-n})(q_2'^n p^n q^{-n} - q_2'^{-n}) + A_{2,1}(n)A_{1,2}(n), \]

These two are equal.
\[
\frac{1}{1 + A_{2,1}(n)A_{1,2}(n)} = \frac{1}{1 + (1 - p^n)(1 - p^{-n})/(1 - q^n)^2 q^{-n}} = \frac{(1 - q^n)^2}{p^n + p^{-n} - q^n - q^{-n}} = \frac{q^{-n}(1 - q^n)^2}{(p^n - q^{-n})(1 - p^{-n}q^n)}.
\]

Thus
\[
\Lambda \Lambda_{3,3}(n) = \frac{q^{-n}(1 - q^n p^{-n})(p^n q^{-n} - 1)(1 - q^n)^2}{(p^n - q^{-n})(1 - p^{-n}q^n)} = -(1 - q^n)^2(p^n q^{-n} - 1)/(1 - p^n q^n).
\]

Let \( p = q^{1-\beta} \), we see that

**Proposition 6.8.**

\[
\Lambda \Lambda_{3,3}(z, w)/\Lambda \Lambda_{3,3}(w, z) = \frac{\theta_{q^{2-\beta}}(\frac{w}{z} q^{1-\beta})^2 \theta_{q^{2-\beta}}(\frac{w}{z} q^2)}{\theta_{q^{2-\beta}}(\frac{w}{z} q^{1-\beta})^2 \theta_{q^{2-\beta}}(\frac{w}{z} q^2)}.
\]

Let
\[
l(z)l(w) = L(z, w) : l(z)l(w) :
\]

**Theorem 6.9.**

\[
\frac{L(z, w)}{L(w, z)} = \frac{\theta_{q^{2-\beta}}(\frac{w}{z} q^{1-\beta})^2 \theta_{q^{2-\beta}}(\frac{w}{z} q^2)}{\theta_{q^{2-\beta}}(\frac{w}{z} q^{1-\beta})^2 \theta_{q^{2-\beta}}(\frac{w}{z} q^2)}.
\]

(13)

In this case, we can show through calculation that the operator \( l(z) \) and the operator \( l(z) \) defined in Section 5 are the same in the sense of bosonization. Therefore we again derive the quantized W-algebra of \( \mathfrak{sl}(2,1) \).
7. Discussions

At this stage, what we actually have derived is only the the bosonization formula for the the quantized W-algebra of \( \mathfrak{sl}(2,1) \). It is important that we can describe this algebra in an abstract way. Hopefully this problem can be solved when we check more carefully the commutation relation of the operator \( \mathcal{l}(z) \). On the other hand, from all above, it is clear that we can extend our construction to the case of several generic screening operator and several fermions or the case of only several fermions, which will give us the quantized W-algebras associated with super-algebras \( \mathfrak{sl}(m,n) \). This will be given in subsequent paper. The classical W-algebra of \( \mathfrak{sl}(2,1) \) can be derive from the two parafermions of affine Lie algebra \( \hat{\mathfrak{sl}}(2) \). Similarly we manage to establish the connection of the quantized W-algebra of \( \mathfrak{sl}(2,1) \) with the quantized parafermions\[5]\[6]\[1] coming from affine quantum group algebra \( \mathcal{U}_q(\hat{\mathfrak{sl}}(2)) \) \[6\]. We also notice, in some way, the correlation function between screen operators and the vertex operator components of \( \mathcal{l}(z) \) may be related to finite dimensional representations of the corresponding affine quantum groups. The situation will become clear, once we start to look at operator like \( \mathcal{l}(z) \) such that the correlation functions have multiple poles and zeros.

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References

[1] J. Ding, B. Feigin, Quantum current operators - II: Difference equations of quantum current operators and quantum parafermion construction Publ. RIMS, 33 (1997), 285-300

[2] E. Frenkel, B. Feigin, Quantum W-algebras and Elliptic Algebras Commun. Math. Phys. 178 (1996)

[3] E. Frenkel, N. Reshetikhin, Quantum affine algebras and deformations of the Virasoro and \( W \)-algebras, Preprint \[q-alg/9505025\]

[4] J. Shiraishi, Phys. Lett. A171 (1992) 243-248; H. Awata, S. Odake, J. Shiraishi, Comm. Math. Phys. 162 (1994) 61-83.

[5] N. Jing Higher level representations of quantum algebra \( \mathcal{U}_q(\hat{\mathfrak{sl}}(2)) \), Jour. Alg., 187 (1996), 448-468

[6] A. H. Bougourzi, L. Vinet On a boson-parafermionic realization of \( \mathcal{U}_q(\hat{\mathfrak{sl}}(2)) \), CRM-2201, hep-th/9407062

[7] R.J. Baxter, Exactly Solved Models of Statistical Mechanics, Academic Press 1982;

[8] B. Feigin, E. Frenkel, N. Reshetikhin, Comm. Math. Phys. 166 (1994) 27-62.
[9] J. Shiraishi, H. Kubo, H. Awata, S. Odake, A quantum deformation of the Virasoro algebra and Macdonald symmetric functions, Preprint q-alg-9507034.
[10] H. Awata, S. Odake, J. Shiraishi, Integral representations of the Macdonald symmetric functions, Preprint q-alg/9506006.
[11] A. Bilal, J.-L. Gervais, Nucl. Phys. 318 (1989) 579.
[12] P. Bouwknegt, J. McCarthy, K. Pilch, Comm. Math. Phys. 131 (1990) 125-156.
[13] R. Borcherds, Proc. Natl. Acad. Sci. USA, 83 (1986), 3068-3071.
[14] I. Frenkel, J. Lepowsky, A. Meurman, Vertex Operator Algebras and the Monster, Academic Press, New York 1988.
[15] B. Feigin, E. Frenkel, Integrals of motion and quantum groups, Preprint hep-th/9310022, to appear in Proceedings of the Summer School Integrable Systems and Quantum Groups, Montecatini Terme, Italy, June 1993, Lect. Notes in Math., Springer Verlag.
[16] J. Ding, B. Feigin, in preparation.

JINTAI DING, RIMS, Kyoto University

LANDAU INSTITUTE FOR THEORETICAL PHYSICS, MOSCOW; RUSSIA AND RIMS, KYOTO UNIVERSITY