Why ghosts don’t touch: a tale of two adventurers falling one after another into a black hole

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Abstract
The case for the utility of Kruskal–Szekeres coordinates in the classroom made by Augousti et al in this journal (2012 Eur. J. Phys. 33 1–11) is strengthened by extending their discussion beyond the event horizon of the black hole. Observations made by two adventurers following one another into a Schwarzschild black hole are examined in terms of these nonsingular coordinates. Two scenarios are considered, the first corresponding to one observer following the other closely, the second to a significant distance between the two of them, precluding the existence of a common inertial system. In particular, the concepts of distance and temporal separation near the horizon and the redshift of the first infaller’s image as seen by the second are investigated. The results show that the notion of ‘touching ghosts’ does not correspond to the local physics of two observers falling into a black hole. The story line is interesting enough and the mathematical details are sufficiently simple to use the example in a general relativity course, even at the undergraduate level.

Keywords: Black holes, Kruskal–Szekeres coordinates, redshift, general relativity

1. Introduction
A discussion of the Schwarzschild spacetime is a mandatory part of any full-fledged general relativity (GR) course. Studying this metric, we may derive predictions for the four classical tests of the theory—gravitational redshift, light deflection by gravity, the perihelion precession of planets, and the Shapiro delay. In addition, the spherically symmetric vacuum metric gives access to some strong field aspects, introducing fascinating new effects and exotic physics, such as the phenomenon of an event horizon and the ensuing existence of black holes.
Often, all of this is presented in terms of a single coordinate system, nowadays called Schwarzschild coordinates. Unfortunately, this preference gives rise to a certain amount of folklore about black holes. Examples are the idea that an observer falling into a black hole will take forever to reach the horizon or that a Schwarzschild black hole cannot actually form, because the surface of the collapsing star that should produce it takes infinite time to cross the incipient event horizon, which then will never be present. Another element of folklore holds that an observer, just before crossing the horizon, will be able to see the infinite future of the Universe. The first two statements are misconceptions based on mistaking a particular time coordinate, the Schwarzschild time, for a substitute of Newton’s absolute time, the third is simply a falsehood [1]. An observer falling towards the event horizon of a black hole does not approach the ‘end of time’ nor the ‘end of the Universe’. There are other coordinate systems such as Gullstrand–Painlevé (GP) coordinates [2] or Kruskal–Szekeres (KS) coordinates [3, 4], sporting time variables that remain finite as an infaller crosses the event horizon. The GP time differs from the Schwarzschild time only by a position dependent resynchronization transformation, whereas the KS timelike coordinate is a more complicated mixture of the Schwarzschild radial and temporal coordinates.

But what seems important then is that students are exposed to some of these different coordinate systems in order not to misconceive the meaning of Schwarzschild time, which may happen easily, if that is the only time coordinate to which they ever are exposed in discussing non-rotating black holes. Therefore, it may be fruitful to look for instances where some of the other coordinate systems are either necessary or at least beneficial.

In a recent article [5], KS coordinates were argued to be a useful pedagogical tool for discussing, in the classroom, some non-trivial results on observations near a black hole. As an example, the authors of [5] propose to consider the possibilities for (radio) signal exchange and optical monitoring between two observers falling successively into a black hole. Interestingly, they do not exploit these coordinates to their full power—they avoid discussing observations after either of the two adventurers has crossed the event horizon. While different philosophical attitudes may be taken towards the relevance of predictions made by GR about what happens inside the horizon—those events remain invisible to the outside world and inaccessible to any non-suicidal observer—GR is a deterministic theory and offers precise statements about this spacetime region. It is not a priori unscientific to consider and discuss these predictions.

Therefore, amplifying on the assertion of the pedagogical value of KS coordinates, I would like to complement the paper by Augousti et al [5] with an investigation about what happens beyond, and in particular on, the event horizon, especially in view of the authors’ own suggestion that there is grounds for further discussion. Moreover, it seems to me that a few statements made in [5] are not borne out by closer inspection. In particular, I will argue that the idea of the second infalling observer (Bob) perceiving to touch a ghostly image of the first (Alice) on approaching the horizon is incorrect and, in fact, incompatible with the equivalence principle. Also, Bob will not find that Alice passes the horizon at the same instant as he himself crosses it. While he will see her traversing the horizon the moment when he himself crosses it, he will see her at a distance (despite their Schwarzschild radial coordinate being the same) and he will conclude her to have passed the horizon before him, because the light signal took time to travel from her position to his.

Confusion about the interpretation of calculational results may arise from insufficient care in distinguishing between global and local coordinates and lack of appreciation of the fact that having similar or even the same \((r, \theta, \phi)\)-coordinates does not necessarily mean physical closeness when \(r = r_S\), the Schwarzschild radius. At that radius, the Schwarzschild coordinates become singular, and the ratio between the radial proper length element of a
coordinate stationary observer and \( dr \) tends to infinity. Neither Alice nor Bob have a finite (Schwarzschild) time coordinate at the horizon. Discussing position differences and temporal separation near the horizon in terms of Schwarzschild coordinates is challenging and not really recommendable.

Kruskal–Szekeres coordinates are clearly better; they show immediately that the horizon crossings of the two observers correspond to two different events. These are separated by a finite (null) spacetime interval. How this is locally perceived in terms of space and time intervals is best investigated by a transformation to local inertial coordinates of either observer. If the two observers start their journey sufficiently closely, there may even exist a common local inertial system near the horizon, in which the discussion becomes very simple. Beyond that, inferences have to be made using global coordinates among which the KS coordinates belong to the more useful ones. Another advantageous choice would be the GP coordinates, in which the radial and time coordinates are more easily interpretable and which are also regular at the horizon, but where the graphical representation of light cones is not as simple as in KS diagrams.

The remainder of the paper is organized as follows. In section 2, the Schwarzschild metric is given both in its standard form and in KS coordinates. The latter representation is needed for the calculation of KS diagrams that will be used in section 3 to visualize the free-fall journeys. This section is the one with the highest pedagogical utility. In section 4, some simple analytical calculations are given that clarify what the situation will look like locally. This kind of material might be used in supplementary exercises of a GR course. Some conclusions summarize our results. An appendix shows how to obtain the redshift of an infalling observer, seen by another infaller, when the latter reaches the event horizon. This material could be offered to students as a homework calculation (for bonus points).

## 2. Coordinate representations

To fix the notation, we introduce the Schwarzschild metric in the form:

\[
\text{d}s^2 = \left(1 - \frac{r_\text{\tiny S}}{r}\right) c^2 \text{d}t^2 - \left(1 - \frac{r_\text{\tiny S}}{r}\right)^{-1} r^2 \left(\text{d}\theta^2 + \sin^2 \theta \, \text{d}\phi^2\right)
\]

\[
= g_{\text{tt}} c^2 \text{d}t^2 + g_{\text{rr}} \, r^2 \, \text{d}r^2 + g_{\theta\theta} \, \text{d}\theta^2 + g_{\phi\phi} \, \text{d}\phi^2
\]

where

\[
r_\text{\tiny S} = \frac{2GM}{c^2}
\]

is the Schwarzschild radius. The KS coordinates \((v, u, \theta, \phi)\) are obtained by the coordinate transformation [2]

\[
u = \sqrt{\frac{1 - \frac{r}{r_\text{\tiny S}} \, e^{r/r_\text{\tiny S}} \sinh \frac{ct}{2r_\text{\tiny S}}}{r_\text{\tiny S}}}, \quad v = \sqrt{\frac{1 - \frac{r}{r_\text{\tiny S}} \, e^{r/r_\text{\tiny S}} \cosh \frac{ct}{2r_\text{\tiny S}}}{r_\text{\tiny S}}}, \quad 0 < r < r_\text{\tiny S},
\]

\[
u = \sqrt{\frac{r}{r_\text{\tiny S}} - 1 \, e^{r/r_\text{\tiny S}} \cosh \frac{ct}{2r_\text{\tiny S}}}, \quad v = \sqrt{\frac{r}{r_\text{\tiny S}} - 1 \, e^{r/r_\text{\tiny S}} \sinh \frac{ct}{2r_\text{\tiny S}}}, \quad r \geq r_\text{\tiny S}.
\]

In these coordinates, the line element is given by

\[
\text{d}s^2 = \frac{4r_\text{\tiny S}^2}{r} e^{-r/r_\text{\tiny S}} (\text{d}v^2 - \text{d}u^2) - r^2 (\text{d}\theta^2 + \sin^2 \theta \, \text{d}\phi^2).
\]
The Jacobian of the relevant part of the transformation is [5]

\[
\frac{\partial (v, u)}{\partial (ct, r)} = \begin{bmatrix}
\frac{u}{2r_S} & \frac{v}{2r_S} \\
\frac{2r_S}{2r_S(r)} & \frac{u}{2r_S(r)}
\end{bmatrix}
\]

and its determinant reads

\[
D = \left| \frac{\partial (v, u)}{\partial (ct, r)} \right| = \frac{r}{4r_S^3} e^{r/r_S},
\]

which is just the inverse of the prefactor of \(dv^2 - du^2\) in (4). Other useful formulas are [2]

\[
\begin{align*}
\left( \frac{r}{r_S} - 1 \right) e^{r/r_S} &= u^2 - v^2, \\
ct &= 2r_S \text{artanh} \frac{v}{u}, \quad r > r_S, \\
ct &= 2r_S \text{artanh} \frac{u}{v}, \quad r < r_S.
\end{align*}
\]
where $\mathcal{W}$ is Lambert’s W function [6]. The first of these equalities shows that $r = r_S$ corresponds to $|v| = |u|$ and that other constant $r$ values are described by hyperbolas in the $uv$ plane. From the second line, we note that constant time $t$ corresponds to a fixed ratio $v/u$, i.e., a straight line emanating from the origin in the $uv$ plane. The event horizon corresponds to $r = r_S$ with $u = v > 0$, so it has the Schwarzschild time coordinate $t = \infty$, i.e., that time coordinate is rather useless for its description.

3. Kruskal–Szekeres diagrams

Having prepared the mathematical details, let us return to the tale of Alice and Bob. To avoid a situation in which our adventurers would be torn apart by tidal forces before reaching the event horizon, we assume that the black hole under consideration is supermassive, comprising a few hundred million solar masses at least. Then surface gravity at the horizon is weak and local inertial systems there are neither extremely small nor very short-lived.

Consider Alice freely falling into the black hole, followed by Bob. We may then distinguish two situations. Either Bob follows Alice so closely that both may be considered being in a single local inertial system (for some time) or he starts his journey so much later than Alice that this assumption is not satisfied anywhere near the horizon. Let us call the first situation scenario I, the second scenario II. Figure 1 depicts these cases in KS diagrams, the left panel visualizing scenario I, the right one scenario II. Alice’s trajectory is the same in both images, and all the trajectories start at $r = r_0$ at zero velocity, with only the departure times of Bob being different in the two cases. With this arrangement, we can immediately tell the difference in proper times of the two adventurers on crossing the horizon, if they synchronized their clocks before the start: since they fall on identical spatial trajectories they both need precisely the same proper time interval from the beginning of their fall to the horizon. Therefore, at the horizon their clocks will differ by the proper time difference $t_B - t_A$ of their moments of departure.

In scenario I, obviously nothing particular will happen at the horizon from the point of view of our observers. They should be able to continually exchange messages without noticing their traversing the horizon. This is granted by the equivalence principle which says that locally their physics is that of special relativity. There is no fundamental reason for the equivalence principle to fail at the horizon. Therefore, in particular, there will be no ghostly image of Alice approaching Bob, as special relativity does not envisage such an effect. The only way to escape from this conclusion would be to deny the possibility of scenario I. However, the size of a local inertial system near the horizon can be made as large as we wish, as long as we may consider arbitrarily massive black holes.

Moreover, the figure demonstrates that signal exchange is not disturbed by the presence of the horizon. The numbers 1 to 5 in the left panel along the trajectory $A$ and 1 to 4 along $B$ refer to a few light or radio signals sent from Alice to Bob and vice versa. Bob’s signal 2 reaches Alice as she crosses the horizon, whereupon she sends her signal 2, reaching Bob at point 3, precisely when he is about to cross the horizon. Signal 3 from Bob catches up with Alice before she hits the singularity at $r = 0$ ($v^2 - u^2 = 1$), and she still has time to answer with signal 4. KS coordinates are nonsingular outside $r = 0$, so if there is a null trajectory connecting Alice’s and Bob’s world lines, light can travel along it.

1 Alice’s and Bob’s trajectories in figure 1 were calculated by numerical integration of (18) and (20).
2 We stay within classical physics in this paper, so we do not have to discuss recent attacks on the equivalence principle based on the firewall conjecture [7].
The authors of [5] state that no signal can be emitted at the horizon (because there could be no ‘trapped’ signal). This seems to be based on a very common misconception about the nature of the event horizon, which is to visualize the horizon as being static. It is well-known and may be directly read off a KS diagram that the horizon is a null hypersurface. Hence, for any inertial (i.e. freely falling) observer close to it the horizon moves (outward) at the speed of light! This fact becomes particularly transparent in the river model of a spherical black hole [8], where space is streaming inward, toward the center of symmetry and reaches the speed of light at $r = r_g$. Thus, in order to stay put there, one would have to travel outward as fast as a photon, which the horizon does (and can do, not being a material object).

It is then quite easy to understand how a signal from Alice sent the moment she crosses the horizon will reach Bob the very moment he does so, too, in spite of the fact that this happens later. The solution is similar to that of the puzzle whether Alice when falling through with her feet first will cease to see them for a moment. Clearly, she will have no trouble seeing her feet all the time, because the time light from her feet crossing the horizon takes to her eyes is precisely the time the horizon itself takes to travel to her eyes as well. The relative speed between observer and horizon is $c$ at the moment of passage. Bob will receive Alice’s signal when he reaches the horizon, because the horizon is moving towards him. If he fired his thrusters to get away, he might outrun the horizon, but then he would also successfully run away from Alice’s signal. And of course, he will never have the impression that Alice is going to collide with him. Such an idea would only arise from a misinterpretation of the meaning of the coordinate $r$ as an absolute measure of distance. Her (proper) distance from Bob is in fact continually increasing as will be explained below.

These qualitative statements may be made more precise using the KS diagram. Since the KS coordinates are not singular at the horizon, the fact that Alice and Bob have different coordinates $(u_A, v_A)$ and $(u_B, v_B)$ on passing the horizon means that these crossings are different events. But different events cannot correspond to both the same time and the same place. In fact, since two of the four coordinates ($\vartheta = \text{const.}$ and $\varphi = \text{const.}$) are the same for both travelers, the fact that $(u_A, v_A)$ and $(u_B, v_B)$ have null separation implies that Alice and Bob will perceive their passage of the horizon to happen neither at the same place nor at the same time. Moreover, we know that two different events may be considered happening at the same place only if they can be connected by a timelike world line (which is not the case here, the connection is null). Hence, Bob will not see a ghostlike image of Alice approaching him.

Rather, he will see her at constant distance as long as they may be considered sharing a common inertial system and she will drift away as soon as tidal forces become noticeable. Rather, he will see her at constant distance as long as they may be considered sharing a common inertial system and she will drift away as soon as tidal forces become noticeable.

An interesting question is whether Alice and Bob will agree on who hits the singularity first. Alice would receive Bob’s signal the moment she hits the singularity (had she not been spaghettified before... [9]) so Bob would not yet have hit it, because he could not pass his own radio signal. However, this argument is based on the prejudice that the singularity corresponds to a single event, which it clearly does not. Bob receives signal from Alice when hitting the singularity, so he might make up a similar argument to show that Alice will survive him for a few moments. But this is not conclusive, because Alice’s signal was sent opposite to her direction of motion. Near the singularity, Alice and Bob will not share a common local inertial system, so the answer is not unambiguous, given the fact that the singularity is spacelike.

In terms of the timelike variable $v$, it is clearly Alice who hits the singularity first.

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3. $\mathrm{d}s^2 = 0$ and equality of three (nonsingular) coordinates implies equality of all four coordinates for both events. But then the events would not correspond to different spacetime points. So if one pair of coordinates are different, there must be another pair that are different, too.

4. This can be seen from the KS diagram—the magnitude of the slope of the curve $v^2 - u^2 = 1$ is smaller than 1 everywhere.
Let us now discuss scenario II. If there is no ghostly image in scenario I, it is hard to believe that such an image should appear in scenario II, where Bob follows Alice at a larger distance. Indeed, there can never be a situation of touching ghosts for our two adventurers, as long as they remain far from each other. But if they approach one another more closely than the typical diameter of a local inertial system, we are back to scenario I, in which the equivalence principle forbids such a phenomenon and in which residual tidal forces will tend to make them drift apart rather than draw near. All that might be possible in scenario II, then, is some optical illusion.

Therefore, let us turn to the issue whether Alice’s image as perceived by Bob, if not appearing to touch him in the guise of a ghost, may at least have ghostly features. If Bob starts his journey sufficiently much later than Alice, i.e., if he remains long enough at their stationary[5] ‘mother station’[5], he will of course see her slowing down and her image redshifting as she approaches the horizon. This is the standard scenario seen by a stationary distant observer. As Bob then falls towards the horizon himself, he will pick up speed, which should compensate for part of Alice’s redshift. A full compensation seems however unlikely, given the fact that Alice is always in a more strongly accelerating field than Bob. Rather, Bob’s motion should lead to Alice’s image redshifting not quite as fast as if he had stayed behind. This is shown to be true in the appendix. If Bob actually follows her through the horizon, her image will in fact never seem to approach a complete freeze.

A quantity that can be easily calculated is Alice’s redshift as perceived by Bob when he passes the horizon. The calculation gives a finite result and is exhibited in the appendix. Together with figure 1 it implies that Alice’s image is, not surprisingly, the more redshifted on Bob’s reaching the horizon, the later Bob starts his journey. Therefore, if he follows her too late, i.e., when optical frequencies emitted by her have already shifted outside the visible spectrum, he will not see her again and her ghost less than ever.

What does he see, if either he follows early enough for her image to be still visible or if he uses appropriate infrared vision gear[6]? First, aside from redshifting and acquiring a slow-motion aspect, Alice’s image will fade quickly. Light emitted by her body, will, as far as it is sent slightly outside or inside the horizon, diverge away from the latter, escaping to infinity or falling towards the singularity, so the electromagnetic energy deposited near the horizon will diminish fast, and her image will lose intensity equally fast. Second, while in the local scenario Alice will seem to move away from Bob slowly due to small tidal forces as the approximation of a common inertial system gets worse, Alice will not seem to move away to arbitrarily large distances in scenario II, as we make the time delay between hers and Bob’s journey larger. The horizon has a finite area and her image hovering there will take a finite fraction of that area. Since we are limited, in our simple approach, to the discussion of temporal and radial aspects, we cannot describe here the distortions and size effects that Bob will see on passing the horizon some substantial time after Alice. The result of such a discussion would be that there is a maximum distance at which (a strongly redshifted) Alice will appear to Bob on entering the region interior to the event horizon, dependent on the size of the black hole. She will not seem to have receded to infinity, regardless of how much time he lets pass before following her[10].

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[5] By stationary we always mean being at rest in the \((t, r, \vartheta, \varphi)\) system, i.e., \(dr = d\vartheta = d\varphi = 0\), that is, coordinate stationary in Schwarzschild coordinates. Coordinate stationary observers in KS coordinates would be moving observers in Schwarzschild coordinates.

[6] The time scales on which this kind of equipment may be useful are pretty short—in the second range for a black hole with a million solar masses. In order to allow for interesting optical effects in the late-time scenario discussed here, either a black hole in the range above a billion solar masses is needed or we must consider a few milliseconds a long time.
Both travelers will agree in both scenarios that it is Alice who passes first and Bob who passes last. They never have the feeling of passing at the same time.

The main difference between the two scenarios is that in scenario II Bob cannot send signals to Alice anymore, once he has passed the horizon. His signal 4 in the right panel of figure 1 obviously can never reach Alice, nor can signals sent later. Note that the last signal Alice can receive from Bob (signal 3) was sent while he was still outside the horizon. Alice will never know whether Bob tried to escape the black hole after sending that signal. On the other hand, Bob will still continue to receive signals from Alice until he hits the singularity. Again the signal sent by Alice while crossing the horizon (signal 4) will reach Bob only the moment he passes the horizon, too, because it cannot outrun the outward-moving horizon.

4. Local description of scenario I

It is possible to calculate the proper distance and proper time interval of a freely falling observer at the horizon, allowing Alice and Bob to explicitly derive the spatial and temporal distance between the two events of their horizon crossings in the case of scenario I. To this end, we would like to formulate these quantities in terms of KS coordinates. There are (at least) two ways to achieve this. The pedestrian approach consists in first expressing the velocity of the infaller by the proper time element $dt$ and radial proper length element $dl$, of a stationary observer next to him. (This works only outside the horizon.) Then a local Lorentz transformation to the frame of the falling observer whose velocity is $v = dl/dt$, produces the proper time and length elements of the infaller in terms of $dt$ and $dr$, a result that may be straightforwardly transformed to KS coordinates.

The second, somewhat faster approach avoids any explicit Lorentz transformation. We simply determine the local coordinate transformation that takes the KS metric to Minkowski form in the frame of the freely falling observer. The results of both approaches are the same, of course.

Let us name the proper time element of the infaller $d\tau$ and her proper (radial) length element $dl$, then there must be a relationship (assuming $d\vartheta = d\varphi = 0$)

$$
\begin{align*}
 dv &= \alpha d\tau + \beta dl, \\
 du &= \gamma d\tau + \delta dl,
\end{align*}
$$

and we must have $\alpha/\gamma = \dot{v}/\dot{u}$ (where $\dot{v} = dv/d\tau$, $\dot{u} = du/d\tau$), because the proper time is a local coordinate that is tangent to the world line of the observer. From the line element

$$
\begin{aligned}
 ds^2 &= D^{-1}(dv^2 - du^2),
\end{aligned}
$$

we gather that

$$
\dot{v}^2 - \dot{u}^2 = Dc^2.
$$

To determine the coefficients of the transformation we require

$$
D^{-1}(dv^2 - du^2) = D^{-1}\left[ (\alpha^2 - \gamma^2)d\tau^2 + 2(\alpha\beta - \gamma\delta)d\tau dl + (\beta^2 - \delta^2)dl^2 \right]
$$

$$
\begin{aligned}
&= c^2d\tau^2 - dl^2,
\end{aligned}
$$

\[8\]
This gives us three equations
\[
\begin{align*}
\alpha^2 - \gamma^2 &= Dc^2, \\
\alpha \beta - \gamma \delta &= 0, \\
\delta^2 - \beta^2 &= D. 
\end{align*}
\] (12)

With \(\alpha = \gamma \dot{v}/\dot{u}\), the second equation yields \(\delta = \beta \dot{u}/\dot{v}\). The remaining ones give
\[
\begin{align*}
\gamma^2 \left( \frac{\dot{v}^2}{\dot{u}^2} - 1 \right) &= Dc^2 \implies \gamma^2 (\dot{v}^2 - \dot{u}^2) = Dc^2 \gamma^2 = Dc^2 u^2, \\
\beta^2 \left( \frac{\dot{v}^2}{\dot{u}^2} - 1 \right) &= D \implies \beta^2 (\dot{v}^2 - \dot{u}^2) = Dc^2 \beta^2 = D \dot{u}^2. 
\end{align*}
\] (13)

Assuming \(\dot{v} > 0\), we finally obtain
\[
\begin{align*}
\alpha &= \dot{v}, \quad \beta = \frac{\ddot{u}}{c}, \quad \gamma = \dot{u}, \quad \delta = \frac{\ddot{v}}{c},
\end{align*}
\] (14)

and solving (8) for \(d\tau\) and \(d\ell\), we have
\[
\begin{align*}
d\tau &= \frac{1}{Dc^2} (\dot{v}dv - \dot{u}du), \\
d\ell &= \frac{1}{Dc} (\dot{u}du - \dot{v}dv).
\end{align*}
\] (15)

Since we did not use the fact anywhere that our observer is freely falling—geodesic equations were not invoked—this form of the result is true for observers in arbitrary motion. To apply it to the case of Alice and Bob, we need their four-velocity components \(\dot{v}\) and \(\dot{u}\) in the actual free-fall situation. Instead of trying to solve for them in KS coordinates directly, we use the equations of motion from [5] in Schwarzschild coordinates and transform to KS coordinates. For an observer starting at \(r_0\) with zero velocity, the four-velocity components in the \((t, r)\) system are
\[
\begin{align*}
V^t &= c \dot{t} \equiv c \frac{dr}{d\tau} = c \sqrt{\frac{1 - \frac{\gamma_0}{r}}{1 - \frac{\gamma}{r}}}, \\
V^r &= \dot{r} \equiv \frac{dr}{d\tau} = -c \sqrt{\frac{\gamma_0}{r} - \frac{\gamma}{r_0}}.
\end{align*}
\] (16)

We then have, using (5)
\[
\begin{align*}
V^\nu &= \frac{\partial \nu}{\partial \nu} \dot{t} + \frac{\partial \nu}{\partial r} \dot{r} = \frac{c}{2 \gamma_0 r_0} \left( \frac{1}{\sqrt{1 - \frac{\gamma_0}{r}}} - \nu \sqrt{\frac{1 - \frac{\gamma_0}{r}}{1 - \frac{\gamma_0}{r_0}}} \right), \\
V^u &= \frac{\partial u}{\partial \nu} \dot{t} + \frac{\partial u}{\partial r} \dot{r} = \frac{c}{2 \gamma r_0} \left( \frac{1}{\sqrt{1 - \frac{\gamma}{r}}} - u \sqrt{\frac{1 - \frac{\gamma}{r}}{1 - \frac{\gamma}{r_0}}} \right). 
\end{align*}
\] (17)

Next, we wish to convince ourselves that \(\dot{v}\) and \(\dot{u}\) do not turn singular at the horizon \(r = r_S\), in spite of the denominator \(1 - \frac{\gamma}{r}\). A simple trick permitting us to demonstrate this analytically for \(\dot{v}\) and giving a manifestly nonsingular formula is to multiply the numerator and denominator by \(u \sqrt{1 - \frac{\gamma}{r_0}} + v \sqrt{\frac{r_S - \gamma}{r_0}}\). This produces
This formula, valid only for scenario I, gives
\[
\ell_r = \frac{c}{2r_S} \frac{u^2(1 - \frac{r}{r_0}) - v^2 \left(1 - \frac{\Delta u}{u}\right)}{u \sqrt{1 - \frac{r}{r_0} + v} \sqrt{r - \frac{r_0}{r_0}}}
\]
where in the last line we have used
\[
\ell_r = \frac{c}{2r_S} \frac{u^2 + \frac{c}{r_0} e^{\nu/r_0} \left(1 - \frac{\Delta u}{u}\right)}{u \sqrt{1 - \frac{r}{r_0} + v} \sqrt{r - \frac{r_0}{r_0}}}
\]
with the regular limit
\[
\lim_{r \to r_0} \ell_r = \frac{c}{2r_S} \frac{u^2 + v^2}{(u + v) \sqrt{1 - \frac{r}{r_0}}},
\]
which may be simplified a bit more by using that \(u = v\) on the horizon. An analogous calculation for \(\dot{u}\) yields
\[
\dot{u} = \frac{c}{2r_S} \frac{u^2 - \frac{c}{r_0} e^{\nu/r_0} \left(1 - \frac{\Delta u}{u}\right)}{v \sqrt{1 - \frac{r}{r_0} + u} \sqrt{r - \frac{r_0}{r_0}}},
\]
We then obtain for the proper length element of a freely falling observer, using (15),
\[
d\ell = 2r_S \left\{ \frac{\frac{c}{r_0} e^{-\nu/r_0} u^2 + 1 - \left(1 - \frac{\Delta u}{u}\right) v}{u \sqrt{1 - \frac{r}{r_0} + v} \sqrt{r - \frac{r_0}{r_0}}} du + \frac{\frac{c}{r_0} e^{-\nu/r_0} u^2 + 1 - \left(1 - \frac{\Delta u}{u}\right)}{v \sqrt{1 - \frac{r}{r_0} + u} \sqrt{r - \frac{r_0}{r_0}}} dv \right\}
\]
and a similar formula for \(c\,d\tau\). (It is obtained from that for \(d\ell\) by interchanging \(du\) and \(dv\), but not \(u\) and \(v\).) Taking the limit \(r \to r_0\), we find
\[
d\ell = r_S \sqrt{1 - \frac{r}{r_0}} \left( \frac{du}{u} + \frac{dv}{v} \right) + \frac{r_S}{2r_0} \frac{1}{\sqrt{1 - \frac{r}{r_0}}} (udu - vdv)
\]
\[
= r_S \sqrt{1 - \frac{r}{r_0}} \left( \frac{du}{u} + \frac{dv}{v} \right) + \frac{1}{2} \frac{\Delta u}{u} dr,
\]
where in the last line we have used \(udu - vdv = \frac{1}{2} d(u^2 - v^2)\) and (7). This is the proper length element of a freely falling observer who started her fall at \(r_0\), with zero velocity, at the moment when she reaches the horizon. If two observers fall sufficiently closely after one another, we may use this formula when one of them is at the horizon. To measure the spatial distance between the events of the two horizon crossings, we set \(v = u = u_A\) and \(dv = du = u_B - u_A = \Delta u\). Moreover, we obviously have \(dr = 0\). We then obtain for the spatial and the temporal proper distances between the two events
\[
\Delta \ell = 2r_S \sqrt{1 - \frac{r}{r_0}} \frac{\Delta u}{u}, \quad c \Delta \tau = 2r_S \frac{1 - \frac{r}{r_0}}{r_0} \frac{\Delta u}{u}.
\]
This formula, valid only for scenario I, gives finite results for the distance and time difference as measured by Alice between herself and her partner. Signs were chosen so that \(d\ell\) is positive for positive \(dr\), hence the distance to Bob is positive as is Alice’s proper time interval, meaning that Bob falls in after her and is positioned towards the direction from which she came (\(\Delta u\) is positive, as the figure shows). Bob may argue similarly, but for him we would have \(\Delta u = u_A - u_B\) negative.
The result also shows that the proper distance at the horizon between the two observers becomes larger as $r_0$ increases, which is reasonable, albeit the $r_0$ dependence can be seen to be weak. More importantly, the distance increases with increasing $\Delta u$ (at fixed $u$), and we may gather from the figure that $\Delta u$ becomes larger as the time difference $t_B - t_A$ between the starting events of Bob’s and Alice’s journeys increases. Of course, at some point the validity of the assumption that both are in a local inertial system, will break down. What can still be said is that Bob crosses the horizon after and behind Alice, because their separation is null and she can send him a signal connecting the crossing events, while he cannot send her one with the same property.

5. Conclusions

Let us briefly summarize the results of our exploration into the experiences of two adventurers traveling towards the center of a black hole. The mathematics of this (frightening) journey of discovery is simple enough to be presented in classroom and demonstrates the utility of alternative coordinate systems in disentangling seemingly complex situations. The use of simple diagrams adds a visual level to the comprehension of the topic. And of course, the general theme is exciting, having even made an appearance in a recent science fiction movie.

Two basic assumptions were that the black hole is big enough so that tidal effects are negligible at the horizon and that our protagonists, Alice and Bob, move by free fall only. As we have seen, they will not make any spectacular observations down to the horizon, if they begin their journey so close to each other that they may be considered still sharing a common inertial system there. This is a direct consequence of the equivalence principle and this statement can be made without any use of coordinates. The KS coordinates are however useful in discussing how signal exchange between the two observers is unimpeded by the horizon, although no signal ever crosses it. That is possible because the horizon itself locally moves at the speed of light. In the left panel of figure 1, the KS coordinates are handy in visualizing both signal exchange and the fact that the horizon behaves like an outgoing signal.

Moreover, the KS coordinates show clearly that the horizon crossings of our two adventurers correspond to different events. In the scenario of close infallers (scenario I), the proper spatial and temporal separation that both observers will measure on traversal of the horizon can be calculated in terms of these coordinates.

This leaves only the second scenario of distant infallers (scenario II) for the possibility of dramatic observations such as the touching of Alice’s ghostlike image by Bob. Again, the KS coordinates do a good job in studying the situation. In particular, they allow us to calculate the redshift at the horizon and to compare redshifts of various observer arrangements. From figure 1 and the redshift formula (A.16) in the appendix, we conclude that Bob sees Alice the more redshifted on crossing the horizon the later he starts his journey, i.e., the larger their distance when he begins to fall. Also, she will appear to him at a nonzero distance, when he traverses the horizon. Therefore, there are no touching ghosts in this scenario either.

It might be added that from the point of view of a distant stationary observer, at least scenario II may offer a spectacle that could be described as ‘touching ghosts’. If Alice falls long before Bob, such a distant observer will see her slowing down near the horizon, while Bob is still falling fast. Then Bob will also slow down and both will seem to freeze at the horizon with the coordinate distance $\Delta r$ between them approaching zero. The idea that this phenomenon should also find a reflection in local physics may have been at the origin of the considerations of [5]. Its visibility would be brief, because both travelers will redshift out of the optical frequencies fast. Moreover, it must be emphasized that this touching of ghosts
cannot be considered more than an optical illusion. There is nothing resembling the touching of ghosts in the true local physics at the horizon.

Finally, assessing the utility of the KS coordinates for our study, they were useful in the discussion of scenario I and indispensable in the consideration of scenario II.

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Appendix. Redshift calculations

To derive the redshift \( z \) of signals sent by Alice to Bob, we use a general relativistic expression applicable to the most general cases \([5, 11]\)

\[
1 + z = \frac{\omega_A^e}{\omega_B^r} = \frac{(U^\alpha V^\alpha)_A}{(U^\alpha V^\alpha)_B},
\]  
(A.1)

Herein, the superscript \( e \) stands for emission, \( r \) for reception, and subscripts \( A \) and \( B \) tag observers (Alice and Bob). \( \omega \) is the frequency of the signal sent or received. \( U^\alpha \) denotes the tangent vector to the null geodesic connecting the emission and reception events, to be evaluated at the \( A \) or \( B \) end, respectively, and \( V^\alpha \) is Alice’s or Bob’s four velocity at emission and reception, respectively. We assume \( d\theta = d\varphi = 0 \) for the connecting geodesic, i.e., both observers move along purely radial trajectories.

Let us first have a look at the redshift as long as Alice and Bob are both outside the horizon, so we can do the calculation in Schwarzschild coordinates. As before, we have the infallers start at \( r_0 \) and follow the same radial path, which means that Bob’s spacetime trajectory is, once he begins falling, just a time-shifted copy of Alice’s. The four-velocity components \( V^t \) and \( V^r \) are given as a function of \( r \) in \([16]\). Null tangent vectors can be calculated essentially the same way as four velocities from a simple metric-induced Lagrangian for geodesic motion, using an affine parameter \( \lambda \) instead of the proper time. More specifics on the method are exhibited below in some detail, using KS coordinates. For Schwarzschild coordinates, we just give the result

\[
U^t = c \frac{dt}{d\lambda} = \frac{\tilde{K}}{1 - \frac{\tilde{r}}{r}}, \quad U^r = \frac{dr}{d\lambda} = \tilde{K},
\]  
(A.2)

where \( \tilde{K} \) is a positive constant that in principle can be determined from the emission frequency at Alice’s end but is not needed in the calculation of the ratio (A.1), because it is the same in the numerator and the denominator. \( U^t = \tilde{K} \) describes an outgoing light ray; an ingoing ray has \( U^t = -\tilde{K} \). The frequency ratio then evaluates to

\[
\frac{\omega_A^e}{\omega_B^r} = \frac{g_{\theta\theta}(r_A) U^\theta(r_A) V^\theta_A + g_{\varphi\varphi}(r_A) U^\varphi(r_A) V^\varphi_A}{g_{\theta\theta}(r_B) U^\theta(r_B) V^\theta_B + g_{\varphi\varphi}(r_B) U^\varphi(r_B) V^\varphi_B} = \frac{1 - \frac{\tilde{r}_A}{r_A}}{\sqrt{1 - \frac{\tilde{r}_B}{r_B} - \frac{\tilde{r}_A}{r_A}}},
\]  
(A.3)

Herein, \( r_A \) is Alice’s position on sending and \( r_B \) Bob’s position on receiving the signal. Introducing, as was done in \([5]\), the local velocity of a falling observer with respect to a stationary observer at the same position, which is given by
\[ v = -\frac{1}{g_{\mu}(r)} \frac{dr}{dt} = c \sqrt{\frac{r_{A} - r_{a}}{1 - \frac{r_{a}}{r_{o}}}}, \]  
\[ (A.4) \]

we may reformulate \((A.3)\)

\[ \frac{\omega^c_A}{\omega^c_B} = 1 - \frac{\nu_{A}}{V_{r_{A}}} \]  
\[ (A.5) \]

a result that by construction holds outside the horizon. The first thing to note is that if we keep Bob stationary, setting \(\nu_{B} = 0\), the formula reduces to the expression given in [5] for Alice’s redshift as seen by the mother station. From this we may conclude that the fact that Bob is actually moving in Alice’s direction will reduce her redshift in comparison with that seen by a fixed observer. This corresponds to expectations.

Taking the limit \(r_{0} \to \infty\) in \((A.3)\) to simplify the formula a bit, we find

\[ \frac{\omega^c_A}{\omega^c_B} = \frac{1 - \frac{\nu_{A}}{V_{r_{A}}}}{1 - \frac{\nu_{A}}{c}} \]  
\[ (A.6) \]

and this can be immediately compared with the redshift between two stationary observers at \(r_{A}\) and \(r_{B}\), i.e., the momentary positions of Alice and Bob, respectively. Let us call these Amanda \((A')\) and Brian \((B')\). A light or radio signal sent from Amanda to Brian will be redshifted, in the Schwarzschild metric, according to

\[ \frac{\omega^c_{A'}}{\omega^c_{B'}} = \frac{1 - \frac{\nu_{A'}}{V_{r_{A'}}}}{1 - \frac{\nu_{A'}}{c}} \]  
\[ (A.7) \]

The following inequalities are then evident

\[ \frac{\omega^c_{A'}}{\omega^c_{B'}} = \left( \frac{1 - \frac{\nu_{A'}}{V_{r_{A'}}}}{1 - \frac{\nu_{A'}}{c}} \right)^{1/2} \left( 1 + \frac{\nu_{A'}}{V_{r_{A'}}} \right) < \frac{\omega^c_A}{\omega^c_B} < \frac{\omega^c_A}{\omega^c_B}, \]  
\[ (A.8) \]

because \(r_{A} < r_{B}\) implies that \(\left( 1 + \frac{\nu_{A'}}{V_{r_{A'}}} \right) \left( 1 + \frac{\nu_{A'}}{c} \right) < 1\) and, due to \(\omega^c_A/\omega^c_B > 1\), the square root of the ratio is smaller than the ratio itself. The result means that the redshift between Alice and Bob is larger than the one between two stationary observers at their momentary positions on emission and reception of the signal. This also corresponds to our intuition, as Alice is ahead and always falling at a larger local velocity than Bob, so on top of the gravitational redshift there should be a redshift by the Doppler effect. Moreover, in a Newtonian universe, the redshift between Alice and Bob should increase during the fall, as she always experiences a stronger acceleration. However, the world is not Newtonian and our intuition may not be too helpful in assessing what happens near the horizon, because the local velocities of both travelers approach \(c\) there. Formulas \((A.3)\) and \((A.5)\) become indefinite. The idea that the ratio approaches 1 at the horizon and therefore Bob will see Alice without redshift when he crosses the horizon, is too simplistic. If it were true, we would really have a ghostly image of Alice that reappears back in the optical spectrum after having been redshifted out of it before. This would still not be a touching ghost but at least an interesting optical effect. Alas, it is not so.
Clearly, our analytical redshift formulas are incomplete in that they do not give us Alice’s redshift as observed by Bob as a function of her or his proper time. To obtain that, we would need to have Bob’s position on signal reception for each of Alice’s emission positions, i.e. \( r_B(r_A(\tau)) \), which could be calculated, in principle, by intersecting the world line of Bob with Alice’s future light cone. This is not analytically trivial, to say the least. Once we had \( r_B(r_A) \), we might take the limit \( \tau \to r_B \) in (A.3), using de l’Hospital’s rule. Without this information, our Schwarzschild coordinate expressions are not useful in performing the limit.

It is here, where the power of KS coordinates shows up again, permitting an additional step. To evaluate the redshift formula, we first calculate the null geodesics of radial light signals in KS coordinates. After dropping the angular coordinates (which are constant), the appropriate Lagrangian is given by

\[
L = \frac{4r^3}{r} e^{-r/s}(\dot{v}^2 - \dot{u}^2)
\]

where \( \dot{v} = dv/d\lambda, \dot{u} = du/d\lambda \) and \( \lambda \) is an affine parameter again. The Lagrangian equation of motion for \( v \) then reads

\[
\frac{d}{d\lambda} \frac{\partial L}{\partial \dot{v}} - \frac{\partial L}{\partial v} = \frac{d}{d\lambda} \left( \frac{8s^3}{r} e^{-r/s}\dot{v} \right) - \frac{\partial}{\partial v} \left( \frac{4s^3}{r} e^{-r/s} \right) [\dot{v}^2 - \dot{u}^2] = 0.
\]

The second equation of motion is, as usual, more easily obtained using the Lagrangian itself, which is zero here (null geodesics!), so that (A.9) gives us

\[
\dot{v}^2 = \dot{u}^2,
\]

which for an outgoing light ray implies \( \ddot{v} = \ddot{u} \). But then the second term in (A.10) is zero and we find

\[
\frac{8s^3}{r} e^{-r/s} \dot{v} = 2D^{-1} \dot{v} = \text{const}.
\]

Hence, we may write

\[
U^v = \dot{v} = DK, \quad U^u = \dot{u} = DK
\]

with some constant \( K \). At the horizon, \( D = \frac{\epsilon}{4\epsilon^3} \), therefore \( U^v = U^u = \text{const} \). The four velocity of an infalling observer in KS coordinates was calculated in (17). We find

\[
g_{v v}(r)U^vU^v + g_{u u}(r)U^uU^u = \frac{1}{D} DK (V^v - V^u)
\]

\[
= \frac{K c}{2rs} \frac{1}{1 - \frac{r}{r_0}} \left( u - v \right) \left( \frac{1 - \frac{r_0}{r}}{r_0} + \frac{r_0}{\sqrt{r - r_0}} \right)
\]

\[
= \frac{K c r}{2sr_0 r_S} \frac{1 - \frac{r}{r_0}}{u + v}
\]

and this leads to a redshift formula,

\[
\frac{\omega_A}{\omega_B} = \frac{u_B + v_B r_A \sqrt{r_A - r_0}}{u_A + v_A r_B} \sqrt{1 - \frac{r_A}{r_0} + \frac{r_S - r_A}{r_0}}
\]

\[
= \frac{\sqrt{1 - \frac{r_B}{r_0} + \frac{r_S - r_B}{r_0}}}{\sqrt{1 - \frac{r_A}{r_0} + \frac{r_S - r_A}{r_0}}},
\]

\[
= \frac{\omega_A}{\omega_B}.
\]
that is manifestly regular at the horizon. Surprising as it may seem, (A.3) and (A.15) are the same result; a small calculation benefitting from the simple light cone representation in the KS coordinates shows (A.3) to follow from (A.15) (and vice versa).

By taking the limit \( r_A, r_B \to r_S \), we get Alice’s redshift as seen by Bob on crossing the horizon

\[
1 + z = \frac{\omega_A}{\omega_B} \bigg|_{v = v_A} = \frac{u_B + v_B}{u_A + v_A} = \frac{v_B}{v_A} = \frac{u_B}{u_A}.
\]  

This is finite and \( z \) is different from zero. In fact, comparing the pictures from scenario I and scenario II, we see that the ratio \( \frac{d}{r} \) of the points where the trajectories cross the horizon becomes larger, the later Bob starts his journey. If he starts very much later than Alice, he will not see her at all on crossing the horizon, because her image will have shifted out of the optical spectrum. Since the quantity \( u + v \) is constant along ingoing light rays, we may also infer from the figure that at least in the right panel the ratio \( \frac{u_B}{u_A} - \frac{r_A + r_S}{r_A - r_S} \) increases with Alice’s proper time for events on Alice’s and Bob’s trajectories that are connected by an outgoing light ray. (The length of outgoing rays for Alice’s signals 1 through 5 increases with their ordinal number, faster than \( u_A + v_A \) does.)

Outside the horizon, the factor \( \frac{\Delta \omega/r_A}{r_A} \) is smaller than 1 and may first decrease but approaches 1 close to the horizon. The last factor of (A.15) increases as \( r_A \) decreases, and also approaches 1 near the horizon, where everything is dominated by the first factor.

Inside the horizon, \( r_B \) is smaller than \( r_A \) for events on the two trajectories connected by an outgoing light ray, as may be seen easily by (mentally) constructing a hyperbola \( v^2 - u^2 = \text{const.} \) through the endpoint of Alice’s signal 4 in the left or 5 in the right panel. Hence all three leading factors of (A.15) grow during the fall towards the singularity and this cannot be compensated for by the last factor that is decreasing more slowly. Note that the formula seems to predict infinite blueshift as Alice approaches the singularity (\( r_A \to 0 \)). However, \( r_B \) becomes zero first (at signal 5 in the left panel of the figure), and there is, along all of Alice’s and Bob’s trajectories, no pair \( (r_A, r_B) \) connected by an outgoing light ray that satisfies \( r_A = 0 \). Rather, it is Alice’s redshift that becomes infinite for Bob the moment he hits the singularity.

So the overall result seems to be that Alice’s signals arrive at Bob’s positions at consecutive (proper) times with monotonously increasing redshift, outside the horizon as well as inside. What has been shown rigorously here is that there is a nonzero redshift at the horizon and that it increases with Bob’s time delay in following Alice. Inside the horizon, there can be little doubt, given the structure of the redshift expression, that the redshift continues to grow towards the singularity. Outside the horizon, the situation is also clear far away from it, where Newtonian approximations apply, indicating that the redshift between Alice and Bob must increase as a function of time. Finally, there is no good reason to expect non-monotonous behavior at intermediate distances to the horizon.

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