Law of Large Numbers for Random Quantum Dynamical Semigroups

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We present a Law of Large Numbers principle for uniformly continuous random quantum dynamical semigroups. Random iterates of independent copies of these semigroups are shown to be Chernoff equivalent to the quantum dynamical semigroup by the average generator.

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I. INTRODUCTION

The aim of this paper is show that if \((\mathcal{L}_n)_n\) is an i.i.d. sequence of random (Lindblad) generators for quantum dynamical semigroups, each with mean \(\mathbb{T}\), then we may have a Law of Large Numbers principle
\[
e^{\frac{1}{n}\mathcal{L}_n} \circ \cdots \circ e^{\frac{1}{n}\mathcal{L}_1} \xrightarrow{LLN} e^{\mathbb{T}} \quad (n \uparrow \infty).
\]

This type of principle has been established in a recent series of papers [1]-[3], for compositions of general random semigroups. The iterations may, in fact, be independent and identically distributed or at least just asymptotically so. An essential use is made of the Chernoff Theorem to handle the asymptotic convergence, and of Chebyshev’s inequality to control convergence. In the present paper, we extend this approach to quantum open systems, where the semigroups are uniformly continuous quantum dynamical semigroups and, in particular, the generators are of Lindblad form.

We will establish Chernoff equivalence of random iterates of independent copies of the semigroups with the semigroup generated by the mean Lindblad generator. The Chebyshev inequality is established using an appropriate operator-algebra theoretic notion of the variance of random quantum dynamical maps. We mention in connection with this physical situations where random Lindblad generators have recently appeared in the Physics literature, [7, 8].

One of the most important issues in the theory of dynamical systems is the relationship between dynamical maps and a possible (instantaneous) generator. In particular, we may say that two dynamical maps are equivalent (written \(\phi \sim \psi\)) if
\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \left\| \phi_{t/n}^n - \psi_{t/n}^n \right\| = 0.
\]

Here, \(\phi_{t/n}^n\) means the \(n\)-fold composition of \(\phi_t\). Using the observation that \(\xi_t \equiv \xi_{t/n}^n\) for semigroups, we adopt the following definition of instantaneous equivalence (at \(t = 0\)).

Definition 2 Let \((\phi_t)_{t \geq 0}\) and \((\psi_t)_{t \geq 0}\) be elements of \(Y = C_{str.(\mathbb{R}_+, B(X))}\) then we say that they are Chernoff equivalent (written \(\phi \sim \psi\)) if
\[
\lim_{n \to \infty} \sup_{t \in [0,T]} \left\| \phi_{t/n}^n - \psi_{t/n}^n \right\| = 0,
\]
for all \(x \in X\) and \(T > 0\).

Remark 3 1. Chernoff equivalence gives an equivalence relation on \(Y\).
2. The statement of the Chernoff Theorem may be rephrased with (3) replaced by the relation \( \phi \sim (e^{tL})_{t \geq 0} \).

3. Crucially, each equivalence class may have at most one element that is a semigroup.

4. If \( \phi \sim \psi \), then both possess a strong derivative at \( t = 0 \) which coincide on the essential domain of the generator of a semigroup in \( Y \).

**Theorem 4** ([1, 9]) Let \( \phi \in C_{\text{str}}(\mathbb{R}_+, B(\mathcal{H})) \) with \( \|\phi_t\|_{B(\mathcal{H})} \leq e^{at} \) for some real \( a \), then the sequence \( \{\phi^{(n)}\} \) determined by \( \phi_t^{(n)} = \phi_t^{(n)} \) converges uniformly on compacts in the strong operator topology to a \( C_0 \)-semigroup.

**B. Random Evolutions**

A *random dynamics* \( \Phi \) on the Banach space \( X \) can be understood as a mapping \( \Phi : \Omega \to Y : \omega \mapsto \{(\phi_t, \omega)_{t \geq 0}\} \), where \( \Omega \) is some sample space. Here we fix a probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). We require that \( \Phi \) be measurable and this necessitates that we endow \( Y \) with a suitable choice of Borel subsets. To this end, we note that \( Y \) can be be given the topology \( \tau \) generated by the family of semi-norms \( \{\rho_x, T : x \in X, T > 0\} \) where \( \rho_x, T(\phi) = \sup_{t \in [0, T]} \|\phi_t(x)\| \). We then take \( \mathcal{B} \) to be the Borel subsets generated by the topology \( \tau \). By a similar procedure, we may associate Borel subsets to any subset of \( Y \).

For instance, we denote by \( \mathcal{B}_0 \) the Borel subsets of the set of semigroups \( Y_0 \). A *random semigroup* [3, 5] may then be defined as a mapping \( \Phi \) from \( (\omega, \mathcal{F}, \mathbb{P}) \) to \( \mathcal{B}_0 \) measurable. In more detail, a random semigroup is a mapping \( \Phi : \mathbb{R}_+ \times X \times \Omega \to X \) such that \( \Phi(t, x, \omega) \) is strongly continuous in \( t \), linear in \( x \) and measurable with respect to the \( \sigma \)-algebra \( \mathcal{A} \), and we have the identity

\[
\Phi(t, \Phi(s, x, \omega), \omega) = \Phi(t + s, x, \omega). \tag{5}
\]

We will write \( \Phi(t, \omega) \) for the morphism \( \Phi(t, \cdot, \omega) \) on \( X \), in which case the identity reads as \( \Phi(t, \omega) \circ \Phi(s, \omega) = \Phi(t + s, \omega) \).

We may write

\[
\Phi(t, \cdot, \omega) = e^{tL(\omega)}, \tag{6}
\]

where \( L(\cdot, \omega) \) is the generator.

**Definition 5** Let \( \Phi \) be a random dynamics on the Banach space \( X \) with underlying probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). The mean dynamics \( \overline{\Phi} \) is understood as the Pettis integral

\[
\langle \sigma, \Phi_t(x) \rangle = \int_\Omega \langle \sigma, \phi_{t, \omega}(x) \rangle \mathbb{P}[d\omega], \tag{7}
\]

for all \( x \in X \) and all \( \sigma \) in the pre-dual \( X_* \), with \( \langle \sigma, x \rangle \) is the duality pairing.

One would hope that a random semigroup possesses a well-defined mean in \( Y \) under suitable conditions. However, we should not expect the mean itself to form a semigroup. In fact, we will next recall results in this direction for Hilbert space dynamics.

**C. CP Maps**

In quantum theory, a central role is played by completely positive morphisms on the C*-algebra \( \mathcal{B} \) of operators. To recall, given a map \( \phi : \mathcal{B} \to \mathcal{B} \) and a positive integer \( n \) we define its extension \( \phi \otimes id_n \) to the algebra \( \mathcal{B} \otimes M_n \), where \( M_n \) is the algebra of \( n \times n \) matrices, as

\[
\phi \otimes id_n \left( \begin{array}{ccc} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{n1} & \cdots & x_{nn} \end{array} \right) = \left( \begin{array}{ccc} \phi(x_{11}) & \cdots & \phi(x_{1n}) \\ \vdots & \ddots & \vdots \\ \phi(x_{n1}) & \cdots & \phi(x_{nn}) \end{array} \right).
\]

We say that \( \phi \) is \( n \)-positive if \( \phi \otimes id_n \) is positive. For instance, 2-positivity enforces the Størmer inequality \( \phi(x^* x) \geq \phi(x)^* \phi(x) \).

The map is completely positive if \( \phi \otimes id_n \) is positive for all \( n \). It is possible to give a unitary dilation for a CP map, that is we may find a second Banach space \( \mathcal{A} \) such that (for every \( x \in \mathcal{B} \) and \( \rho \in \mathcal{B}_+ \))

\[
\langle \rho, \phi(x) \rangle = \langle \rho \otimes \sigma, U^*(x \otimes I_\mathcal{A})U \rangle, \tag{8}
\]
for \( \sigma \) a positive normalized element of \( \mathcal{A} \) and \( U \) a unitary on \( \mathcal{B} \otimes \mathcal{A} \). We will typically be interpreting the Banach spaces as subspaces of operators over Hilbert spaces, in which case \( \mathcal{B} \otimes \mathcal{A} \) may be understood concretely as a tensor product. The physical interpretation is that \( \mathcal{A} \) is the environment, \( \sigma \) is its state, and \( U \) the evolution coupling the system \( \mathcal{B} \) to \( \mathcal{A} \).

We shall refer to one-parameter \( C_0 \)-semigroups of CP maps as quantum dynamical semigroups and denote the collection of such maps over \( \mathcal{B} \) as \( \text{QDS} (\mathcal{B}) \).

The seminal result of Lindblad \cite{10} and Gorini-Kossakowski-Sudarshan \cite{11} was the categorization of the generators of uniformly continuous one-parameter semigroups of CP maps.

**Theorem 6 (Lindblad Generators, \cite{10, 11})** The generator of a uniformly continuous QDS \( \Phi = (\omega_\omega)_{\omega \geq 0} \) on \( X = B(\mathfrak{h}) \), where \( \mathfrak{h} \) is a separable Hilbert space, takes the form

\[
\mathcal{L}(x) = \sum_k L_k^* x L_k + x K + K^* x,
\]

where \( L_k, K \in B(X) \) and

\[
\sum_k L_k^* L_k + K + K^* = 0,
\]

where the sums in \( \mathcal{L} \) and \( \mathcal{H} \) are understood to converge strongly.

### II. RANDOM CP DYNAMICS

We now fix a separable Hilbert space \( \mathfrak{h} \) and set \( X = B(\mathfrak{h}) \) and consider one-parameter families of morphisms \( (\omega_\omega)_{\omega \geq 0} \) belonging to \( Y = \text{C_{str}} (\mathbb{R}_+, B(X)) \) with continuity understood as in the uniform topology. In particular, we will restrict our attentions to the subspaces \( Y_0 \) of CP families on \( \mathcal{B} = B(\mathfrak{h}) \) and \( Y_0^\text{CP} \) of CP \( C_0 \)-semigroups. With these, we associate the Borel subsets \( \mathcal{F}_{\text{CP}} \) and \( \mathcal{F}_{\text{CP}}^0 \), respectively.

**Definition 7** A random quantum dynamical semigroup (random QDS) \( \Phi = (\mathcal{F}_\mathcal{T})_{\mathcal{T} \geq 0} \) is a measurable function taking values in \( Y_0^\text{CP} \). We take the underlying probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \) and so \( (\mathcal{F}_\mathcal{T})_{\mathcal{T} \geq 0} : \omega \mapsto (\omega_\omega)_{\omega \geq 0} \) is \( \mathcal{A} \)-measurable mapping taking values in \( Y_0^\text{CP} \).

Its average \( \overline{\mathcal{F}} \) is then defined as above, though generally it will not form a semigroup. From Theorem 6 we have that, for each \( \omega \in \Omega \), \( (\omega_\omega)_{\omega \geq 0} \) will have a generator \( \mathcal{L}_\mathcal{T} \) of the form \( \mathcal{L}_\mathcal{T} \). We refer to \( \omega \mapsto \mathcal{L}_\mathcal{T} \) as the random Lindblad generator associated with \( \Phi \).

**Definition 8** We say that a random QDS \( \Phi \) is densely strongly equicontinuous if there exists a dense linear subspace \( \mathcal{D} \subset X \) such that for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that \( \| \omega_\omega - \omega_\omega' \| < \varepsilon \), whenever \( \omega, \omega' \in \mathcal{D}, \mathcal{T}_1, \mathcal{T}_2 \geq 0 \) and \( |\mathcal{T}_1 - \mathcal{T}_2| < \delta \).

Our first result is a straightforward specialization of an argument presented in \cite{11, 12}.

**Proposition 9** Let \( \Phi \) be a random uniformly continuous QDS on \( \mathcal{B} = B(\mathfrak{h}) \). If \( \Phi \) is uniformly bounded (i.e., there exists an \( M > 0 \) such that \( \| \omega_\omega \|_{\mathcal{B}(\mathfrak{h})} < M \) for all \( \mathcal{T} \geq 0 \) and \( \omega \in \Omega \)) and densely strongly equicontinuous then \( \overline{\mathcal{F}} \) exists in \( Y_0^\text{CP} \).

The Chernoff Theorem provides sufficient conditions for the Chernoff equivalence of the average \( \overline{\mathcal{F}} \) and the semigroup generated by the averaged generator \( \mathcal{L} = \mathbb{E}[\Phi'] \). The next result is adapted from \cite{11, 13}.

**Proposition 10** Let \( \Phi \) be a random QDS associated with random generators : \( \omega \mapsto \mathcal{L}_\omega \). Further suppose that there exists an essential domain \( \mathcal{D} \subset \mathcal{F} \) for the \( \mathcal{L}_\omega \), with \( \int_\Omega \| \mathcal{L}_\omega x \| \, \mathbb{P}[d\omega] < \infty \) for every \( x \in \mathcal{F} \). Then

\[
\overline{\mathcal{L}}(x) = \int_\Omega \mathcal{L}_\omega(x) \, \mathbb{P}[d\omega],
\]

for all \( x \in \mathcal{D} \), and defines an essentially self-adjoint operator (we denote its closure by the same symbol). Then the averaged QDS \( \overline{\mathcal{F}} \) is Chernoff equivalent to the QDS with generator \( \mathcal{L}(\cdot) \): that is, \( \overline{\mathcal{F}} \sim (e^{\mathcal{L} t})_{t \geq 0} \).

**Proof.** It suffices to verify the conditions of the Chernoff Theorem. Since the functions \( (\theta_{\omega}(\omega, t))_{t \geq 0} \) are continuous for each \( \omega \in \Omega \) and takes values in the cone of identity-preserving CP maps on the Banach space \( \mathcal{B} = B(\mathfrak{h}) \) with \( \Phi_0 = id \), it follows that the average \( \overline{\mathcal{F}} = \int_\Omega \Phi_\omega \, \mathbb{P}[d\omega] \) inherits these properties.

For each \( \omega \in \Omega \), we have the inequality \( \phi_{\omega}(\omega, t) = id + t \mathcal{L}_\omega + r_{\omega(t)} \) where the remainder is bounded by \( \| r_{\omega(t)} \|_{\mathcal{B}} \leq C t^2 e^{\lambda t} \) where \( \lambda = \sup_{\omega \in \Omega} \| \mathcal{L}_\omega \|_{\mathcal{B}} \). This implies that \( \psi(t) = id + t \mathcal{L} + r_t \) satisfies the requirements of Chernoff’s Theorem provided that \( \| r_t \| \leq C t e^{\lambda t} \). The result then follows.
III. LAW OF LARGE NUMBERS FOR RANDOM QUANTUM DYNAMICS

A. Compositions of Random Operators

Let $L$ be a random variable with values in the Banach space $X = B(B(\mathfrak{h}))$ of bounded linear operators acting in the $C^*$-algebra $B(\mathfrak{h})$, defined as a weakly measurable mapping of the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ into $X$. The weak measurability of the mapping $L : \Omega \rightarrow X : \omega \mapsto L(\omega)$ (i.e., the measurability of the collection of numerical functions $(g, L(\omega)f)$ for any $f \in X, g \in X^*$) in the case of a finite-dimensional space $X$ is equivalent to the measurability of a mapping into a Banach space and implies the measurability of the real-valued random variable $\|L(\omega)\|_X$.

Suppose that random variable $L$ has mean value $\mathbb{E}[L] \in X$ and takes values in a ball of some radius $\lambda > 0$ of a Banach space $X$. Furthermore, suppose that $L$ possesses a finite third moment: $\int_0^\Omega \|L(\omega)\|_X^3 \mathbb{P}(d\omega) < \infty$.

Let $(L_n)_n$ be a sequence of independent identically distributed random variables, the distribution of each of which coincides with the distribution of the random variable $L$. Let $(\Phi_n)_n$ be a sequence of independent semigroups with each $\Phi_n$ generated by element $L_n$, then the sequence $\Psi_n$ of random operator-valued functions defined by

$$\Psi_n(t) = \Phi_n(t_n) \circ \cdots \circ \Phi_1(t_n), \quad t \geq 0, \quad n \in \mathbb{N},$$  \hfill (12)

converges in probability to the one-parameter semigroup $(e^{\mathbb{E}[L(\omega)]t})_{t \geq 0}$, in the space $C_s(\mathbb{R}_+, X)$.

For each $\omega \in \Omega$, the Mean Value Theorem of Lagrange \[12\], we have that

$$\|\exp(L(\omega)t) - I\|_X \leq t\lambda \exp(\lambda t),$$ \hfill (14)

and $\|\exp(L(\omega)t) - I\|_X \leq \frac{t^2\lambda^2}{2} \exp(\lambda t)$. From this estimate we have $\frac{d\mathbb{E}}{dt} \mathbb{E}[\Psi_n(t)]_{t=0} = \mathbb{E}[\Psi_n(t)_{t=0}] = \mathbb{E}[\Psi_n(t)].$

**Lemma 11** We have

$$\mathbb{E}[\Psi_n(t)] = \mathbb{E}[\frac{t_n}{n}]^n,$$ \hfill (15)

and if we introduce the operator-valued variance

$$\mathbb{D}[\Psi_n(t)] = \mathbb{E}[(\Psi_n(t) - (\mathbb{E}[\frac{t_n}{n}])^n) \cdot (\Psi_n(t) - (\mathbb{E}[\frac{t_n}{n}])^n)],$$ \hfill (16)

then for each $T > 0$ there exists a $C > 0$ depending on $T$ and $\lambda$ such that $\sup_{t \in [0,T]} \|\mathbb{D}[\Psi_n(t)]\| \leq \frac{C}{n}.$

**Proof.** For each $k$, we write $\Phi_k(\tau) = \Phi_k^{(0)}(\tau) + \Phi_k^{(1)}(\tau)$ where $\Phi_k^{(0)}(\tau) = \mathbb{E}[\Phi_k(\tau)] = \mathbb{E}[\Phi_k^{(1)}(\tau)] = \mathbb{E}[\Phi_k^{(1)}(\tau)_{\mathbb{E}[\tau]}]$. It follows that the random variables $\{\Phi_k^{(1)}(\tau)_{\mathbb{E}[\tau]}\}$ (the deviations from the mean) are mean zero independent variables. Furthermore, by virtue of \[14\], we have almost surely, for all $t \geq 0$, the bound

$$\|\Phi_k^{(1)}(\tau)\|_{\mathbb{E}[\tau]} \leq t\lambda e^{\lambda t}.$$ \hfill (17)

For each $n \in \mathbb{N}$, we have the identity $(t \geq 0)$

$$\Psi_n(t) = \left(\Phi_n^{(0)}(\frac{t_n}{n}) + \Phi_n^{(1)}(\frac{t_n}{n})\right) \cdots \circ \left(\Phi_1^{(0)}(\frac{t_n}{n}) + \Phi_1^{(1)}(\frac{t_n}{n})\right) = \sum_{\alpha} \Phi_n^{(\alpha_n)}(\frac{t_n}{n}) \circ \cdots \circ \Phi_1^{(\alpha_1)}(\frac{t_n}{n}),$$ \hfill (18)

where we have a sum over $\alpha = (\alpha_n, \cdots, \alpha_1) \in \{0,1\}^n$. As the deviations $\{\Phi_k^{(1)}(\tau)_{\mathbb{E}[\tau]}\}$ are mean zero independent we immediately obtain

$$\mathbb{E}[\Psi_n(t)] = \Phi_n^{(0)}(\frac{t_n}{n}) \circ \cdots \circ \Phi_1^{(0)}(\frac{t_n}{n}) = \mathbb{E}[\frac{t_n}{n}]^n.$$ \hfill (19)
The operator-valued variance may be written as
\[ \mathbb{D}_{\Psi_n}(t) = E[\Psi_n(t)^* \circ \Psi_n(t)] - (\mathbb{F}(\frac{t}{n}))^n \]  
and we may similarly expand this as \( \mathbb{D}_{\Psi_n}(t) = \sum_{\beta, \alpha} \mathbb{D}^{\beta, \alpha}(t) \) where
\[ \mathbb{D}^{\beta, \alpha}(t) = E\left[ \Phi_1^{(\beta_1)}(\frac{t}{n})^* \circ \cdots \circ \Phi_n^{(\beta_n)}(\frac{t}{n})^* \circ \Phi_1^{(\alpha_1)}(\frac{t}{n}) \circ \cdots \circ \Phi_n^{(\alpha_n)}(\frac{t}{n}) \right], \]  
with the exception of \( \mathbb{D}^{0,0}(t) \) which will vanish identically.

We also have that \( \mathbb{D}^{\beta, \alpha}(t) \) will vanish identically if \( \beta \neq \alpha \) since we otherwise will end up averaging a single \( \Phi_k^{(1)} \) map which will yield a zero. Therefore we have \( \mathbb{D}_{\Psi_n}(t) = \sum_{m=0}^n \mathbb{D}^m_n(t) \) where \( \mathbb{D}^m_n(t) \) is the contribution from the sum of those \( \mathbb{D}^{\alpha, \alpha}(t) \) where there are exactly \( m \) of the \( n \) labels \( \alpha_1, \cdots, \alpha_n \), take on the value 1. We have the bound \( \| \mathbb{D}^m_n(t) \| \leq \binom{n}{m} e^{\lambda t} \left( \frac{1}{n} \right)^m \) leading to the estimate
\[ \| \mathbb{D}_{\Psi_n}(t) \| \leq \| \mathbb{D}^1_n(t) \| + e^{\lambda t} \left[ (1 + \frac{t}{n})^n - 1 - t \lambda \right] \]  
According to Taylor’s Theorem, there is a number \( s \in (0,1) \) such that
\[ \| \mathbb{D}_{\Psi_n}(t) \| \leq \| \mathbb{D}^1_n(t) \| + \frac{t^2}{2n} e^{\lambda t} (1 + s \frac{t}{n})^n \leq \| \mathbb{D}^1_n(t) \| + \frac{t^2}{2n} e^{2\lambda t}. \]  
Since \( \| E[\mathbb{F}] \| \leq e^{\lambda t} \) and by virtue of the bound (17), we have \( \| \mathbb{D}^1_n(t) \| \leq n^2 e^{2\lambda t} (\frac{1}{n})^2 \).

Therefore, for each \( T > 0 \), there exists a \( C = C(T, \lambda) > 0 \) such that \( \sup_{t \in [0,T]} \| \mathbb{D}_{\Psi_n}(t) \| < \frac{C}{n} \) for all \( n \in \mathbb{N} \).

We may now use the Chebyshev inequality for operator valued measures, see Lemma 1 in [3], to complete the result.

**Theorem 12** Let \( \phi \) be a random quantum dynamical semigroup on \( \mathcal{B} = B(\mathfrak{h}) \) whose generators take values in ball of radius \( \lambda < \infty \) in the Banach space \( B(\mathcal{B}) \). If \( \{ \Phi_n \} \) is an independent sequence of random semigroups, each with the same distribution as \( \Phi \), then the sequence \( \{ \Psi_n \} \) given by (13) satisfies the Law of Large Numbers
\[ \lim_{n \to \infty} P \sup_{t \in [0,T]} \| \Psi_n(t)X - \mathbb{F}(\Psi_n(t)X) \|_{\mathcal{B}} > \varepsilon = 0, \]  
for all \( X \in \mathcal{B}, T \geq 0 \) and \( \varepsilon > 0 \).

An immediate corollary is the following.

**Corollary 13** Under the conditions of Theorem 12 is that the sequence \( \{ \Psi_n \} \) converges in probability to the quantum dynamical semigroup with average generator \( \lambda = E[\mathbb{F}] \).

**IV. EXPLICIT EXAMPLES**

In the special case where \( \mathfrak{h} = \mathbb{C}^N \) so that \( \mathcal{B} = M_N \), we have that the generator takes the form
\[ L(x) = \frac{1}{2} \sum_{n,m=1}^{N-1} k_{nm} ([a_n^*, x] a_m + a_n^* [x, a_m]) - i [x, H] \]  
where \( \{ a_n : n = 1, \cdots, N^2 \} \), with \( a_{N^2} = I \), is a set of operators that are Hilbert-Schmidt orthonormal (that is, \( tr \{ a_n^* a_m \} = \delta_{nm} \)) forms a basis of \( M_N \), \( k = (k_{nm}) \) is positive semidefinite and \( H = H^* \) is Hermitean.

The matrix \( k \) (called the Kossakowski matrix) and the operator \( H \) (Hamiltonian) determine the generator.

In the case where \( \mathfrak{h} = \mathbb{C}^N \), then we have the straightforward result that if the generator \( L_{\omega} \) is determined from the Kossakowski matrix \( k(\omega) \) and Hamiltonian \( H(\omega) \). The mean generator from Proposition 10 is then \( L \) will have matrix \( \tilde{k} \) with entries \( \int_{\Omega} k_{nm}(\omega) \mathbb{P}[d\omega] \) and \( \tilde{H} = \int_{\Omega} H(\omega) \mathbb{P}[d\omega] \). Note that \( \tilde{k} \) will again be positive semidefinite and \( H \) hermitean.

[1] Y.N. Orlov, V.Zh. Sakbaev, and O.G. Smolyanov, “Feynman formulas as a method of averaging random Hamiltonians,” Proc. Steklov Inst. Math. 285, 222–232 (2014) [transl. from Tr. Mat. Inst. Steklova 285, 232–243 (2014)].
[2] L.S. Efremova and V.Zh. Sakbaev, “Notion of blowup of the solution set of differential equations and averaging of random semigroups,” Theoret. and Math. Phys. 185 1582–98 (2015).

[3] Y.N. Orlov, V.Zh. Sakbaev, and O.G. Smolyanov, “Unbounded random operators and Feynman formulae,” Izv. Math. 80 (6), 1131–1158 (2016) [transl. from Izv. Ross. Akad. Nauk, Ser. Mat. 80 (6), 141–172 (2016)]

[4] Y.N. Orlov, V.Zh. Sakbaev and O.G. Smolyanov, "Feynman Formulas and the Law of Large Numbers for Random One-Parameter Semigroups,” Proceedings of the Steklov Institute of Mathematics, Vol. 306, pp. 210–226 (2021).

[5] V.Zh. Sakbaev, "Averaging of Random Flows of Linear and Nonlinear Maps,” IOP Conf. Series: Journal of Physics: Conf. Series 990, 012012, (2018).

[6] J.E. Gough, Orlov Yu. N., Sakbaev V. Zh. and Smolyanov O. G. (2021), “Randomized Quantum Hamiltonian Systems”, Dokl. RAN. Math. Inf. Proc. upr., 498:1, 31–36 (2021).

[7] W. Tarnovski, I. Yusipov, T. Laptyeva, S. Denosov, D. Chruscinski, K. Zyczkowski. “Random generators of Markovian evolution: quantum-classical transition by superdecoherence,” Phys. Rev. E 104, 034118 (2021).

[8] S. Bonaccorci, F. Cottini, D. Mugnolo. “Random evolution equation: well-posedness, asymptotics and application to graphs,” Appl Math Optim (2021). https://doi.org/10.1007/s00245-020-09732-w

[9] A.Y. Neklyudov. “Inversion of Chernoff’s theorem,” Mathematical Notes 83 530–8, (2008).

[10] G. Lindblad, “On the generators of quantum dynamical semigroups,” Commun. Math. Phys. 48, 19 (1976).

[11] V. Gorini, A. Kossakowski, E.C.G. Sudarshan, “Completely positive dynamical semigroups of N-level systems”, J. Math. Phys. 17 (5): 821 (1976)

[12] V.I. Bogachev and O.G. Smolyanov, Real and Functional Analysis (Springer 2020)