Classical Electromagnetism as a Consequence of Coulomb’s Law, Special Relativity and Hamilton’s Principle and its Relationship to Quantum Electrodynamics

J.H.Field

Département de Physique Nucléaire et Corpusculaire Université de Genève . 24, quai Ernest-Ansermet CH-1211 Genève 4.

e-mail; john.field@cern.ch

Abstract

It is demonstrated how all the mechanical equations of Classical Electromagnetism (CEM) may be derived from only Coulomb’s inverse square force law, special relativity and Hamilton’s Principle. The instantaneous nature of the Coulomb force in the centre-of-mass frame of two interacting charged objects, mediated by the exchange of space-like virtual photons, is predicted by QED. The interaction Lagrangian of QED is shown to be identical, in the appropriate limit, to the potential energy term in the Lorentz-invariant Lagrangian of CEM. A comparison is made with the Feynman-Wheeler action-at-a-distance formulation of CEM.

Keywords; Special Relativity, Classical Electrodynamics.
PACS 03.30+p 03.50.De

1This paper is dedicated to the memory of Valentine Telegdi
1 Introduction

At the beginning of Book III of the Principia [1] Newton introduced four ‘Rules of Reasoning in Philosophy’. The first of them was:

We are to admit no more causes of natural things than such as are both true and sufficient to explain their appearances.

It is still a salutary exercise to apply this simple principle to any domain of science. What are fundamental and truly important in the scientific description of phenomena are those concepts that cannot be discarded without destroying the predictive power of the theory. The most powerful, the best, scientific theory is that which describes the widest possible range of natural phenomena in terms of the minimum number of essential (i.e. non-discardable) concepts. It is the aim of the present paper to apply this precept of Newton to Classical Electromagnetism (CEM). The relation of CEM to Quantum Electrodynamics (QED), in the attempt to obtain a deeper physical understanding of the former, will also be discussed.

From the work of Coulomb, Ampère and Faraday on, the basic phenomena of CEM, i.e. what are actually observed in experiments, are the forces between electric charges at rest or in motion, or the dynamical consequences of such forces. The force between two static charges is given by Coulomb’s inverse square law. This law will taken as a postulate in the following, but no other dynamical concept or theoretical construction will be introduced as an independent hypothesis in order to build up the theory. Later, it will be seen that, in QED, this law is a necessary consequence of the existence of, and exchange of, space-like virtual photons between the electric charges.

It will be assumed throughout that the system of interacting electric charges is a conservative one, in Classical Mechanics, and so may be described by a Lagrangian that is a function of the coordinates and velocities of the charges, but does not depend explicitly on the time. Calculating the Action from the Lagrangian of the system and applying Hamilton’s Principle that the Action be an extremum with respect to variation of the space-time trajectories of the charges, yields, in the well-known manner, the Lagrange equations that provide a complete dynamical description of the system [2].

It is further required that the physical description be consistent with Special Relativity. For this, the Lagrangian must be a Lorentz scalar. To introduce the method to be used to construct the Lagrangian, which is likely to be familiar only to particle physicists, I quote a passage taken from some lecture notes by R.Hagedorn [3] on relativistic kinematics dating from some four decades ago:

If a question is of such a nature that its answer will be always the same, no matter in which Lorentz system one starts, then it is possible to formulate the answer entirely with the help of those invariants which one can build with the available four vectors. One then finds the answer in a particular Lorentz system which one can choose freely and in such a way that the answer there is obvious and most easy. One looks then how the invariants appear in this particular system, expresses the answer to the
problem by these invariants and one has found at the same time already the
general answer... It is worthwhile to devote some thinking to this method
of calculation until one has completely understood that there is really no
jugglery or guesswork in it and that it is absolutely safe.

It is important to stress the last sentence in this passage in relation to the word ‘true’
in Newton’s philosophical precept quoted above. Just the method outlined above was
used to derive the Bargmann-Michel-Telegdi (BMT) equation for spin motion in arbitrary
magnetic and electric fields [4].

It will be demonstrated in the following that it is sufficient to apply Hagedorn’s pro-
gramme to the simplest possible non-trivial electrodynamical system that may be con-
sidered: two mutually interacting electric charges, in order to derive all the mechanical
equations of CEM, as well as Maxwell’s equations, with Coulomb’s inverse square law as
the only dynamical hypothesis. The ’mechanical’ equations comprise the relativistic gen-
eralisation of the Biot and Savart Law, the Lorentz force equation and those describing
electromagnetic induction effects with uniformly moving source currents and test charges

An aspect that is not touched upon in the above programme is radiation. In this case
a fundamental classical description of the phenomenon, in the sense of Newton’s precept,
is not possible and Quantum Mechanics must be invoked. In the language of QED, the
existence of real photons as well as the virtual photons responsible for the Coulomb force,
must be admitted. Indeed, extra degrees of freedom must be added to the Lagrangian to
describe the propagation of real photons and their interaction with electric charges. Also
the corresponding potentials and fields are retarded, not instantaneous. A brief comment
is made in the concluding section on the relation of Maxwell’s equations to radiation
phenomena; however, no detailed comparison with QED is attempted.

It is also assumed throughout the paper that the effects of gravitation, that is of
the curvature of space-time, on the interaction between the charged physical objects
considered, may be neglected.

2 Lorentz Invariant Lagrangian for Two Mutually In-
teracting Electrically Charged Objects

Two physical objects $O_1$ and $O_2$ of masses $m_1$ and $m_2$ and electric charges $q_1$ and $q_2$,
respectively, are assumed to be in spatial proximity, far from all other electric charges,
so that they interact electromagnetically, but are subjected to no external forces. The
spatial positions of $O_1$ and $O_2$ are specified, relative to their common center of energy, by
the vectors $\vec{r}_1$ and $\vec{r}_2$ respectively. The spatial distance separating the two objects in their
common center-of-mass (CM) frame: $r_{12} = r_{21}$ is given by the modulus of the vectors $\vec{r}_{12}$,

---

2Not included are induction effects related to AC currents, where source charges are accelerated. Although described, in an identical manner, by the Faraday-Lenz Law, as non-accelerated charges, real as well as virtual photons must be taken into account, at the fundamental level, in this case. For uniformly moving charges, no real photons are created.
The non-relativistic (NR) Lagrangian describing the motion of the objects O₁ and O₂ in their overall CM frame is [5]³
\[
L_{NR}(\vec{r}_1, \vec{v}_1; \vec{r}_2, \vec{v}_2) \equiv T_1 + T_2 - V = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 - \frac{q_1q_2}{r_{12}} \tag{2.2}
\]

where:
\[
\vec{r}_{12} = -\vec{r}_{21} = \vec{r}_1 - \vec{r}_2
\]

(2.1)

\(T_1, v_1\) \((T_2, v_2)\) are the kinetic energies and velocities, respectively of O₁ \((O_2)\) and \(V\) is the potential energy of the system. A Lorentz-invariant Lagrangian describing the system O₁, O₂ will now be constructed in such a way that it reduces to Eqn(2.2) in the non-relativistic limit. The Lagrangian must be a Lorentz scalar constructed from the six Lorentz invariants listed above. Taking the NR limit:

(2.3)

so that the Lagrangian may be written as:

(2.4)

where the coefficients \(\alpha_0-\alpha_6\) are Lorentz-scalars that may also be, in general, arbitrary functions of the six Lorentz invariants listed above. Taking the NR limit:

(2.5)

gives⁵:

\[
L(x_1, u_1; x_2, u_2) = \alpha_0 + \alpha_1(x_1-x_2)^2 + \alpha_2u_1 \cdot (x_1-x_2) + \alpha_3u_2 \cdot (x_1-x_2) + \alpha_4u_1^2 + \alpha_5u_2^2 + \alpha_6u_1 \cdot u_2
\]

(2.6)

³Gaussian electromagnetic units are used.
⁴From translational invariance, the interaction between the objects does not depend upon the absolute positions of the objects, but only on their relative spatial separation: \(|\vec{x}_1-\vec{x}_2|\). Therefore, the dependence of the Lagrangian on the independent 4-vector \(x_1 + x_2\) may be neglected.
⁵Note that the term containing \(\vec{v}_1 \cdot \vec{v}_2\) vanishes in the NR limit where terms of \(O(\beta_1\beta_2)\) are neglected:

\[
u_1 \cdot v_2 \rightarrow c^2(1 - \vec{v}_1 \cdot \vec{v}_2/c^2) = c^2 + O(\beta_1\beta_2).
\]

⁶The symmetry of the Lagrangian with respect to the labels 1,2 requires that the term \(\alpha_6u_1 \cdot u_2\) be identified with the potential energy term in (2.2).
The choice \(\alpha_0 = c^2(m_1^2 + m_2^2)/2\) satisfies the last condition in (2.6) and yields for the Lorentz-scalar Lagrangian:

\[
L(x_1, u_1; x_2, u_2) = -\frac{m_1 u_1^2}{2} - \frac{m_2 u_2^2}{2} - \frac{j_1 \cdot j_2}{c^2 r_{12}} \tag{2.7}
\]

Where the current 4-vectors: \(j_1 \equiv q_1 u_1\) and \(j_2 \equiv q_2 u_2\) have been introduced. This Lagrangian may be written in a manifestly Lorentz-invariant manner by noting that:

\[
x_1 - x_2 = (0; \vec{x}_1 - \vec{x}_2) = (0; \vec{r}_{12})
\]

so that \(r_{12} = \sqrt{-(x_1 - x_2)^2}\) and

\[
L(x_1, u_1; x_2, u_2) = -\frac{m_1 u_1^2}{2} - \frac{m_2 u_2^2}{2} - \frac{j_1 \cdot j_2}{c^2 \sqrt{-(x_1 - x_2)^2}} \tag{2.8}
\]

The Lagrangian (2.7), when substituted into the covariant Lagrange equations derived from Hamilton’s Principle [2]:

\[
\frac{d}{d\tau} \left( \frac{\partial L}{\partial u_\mu^i} \right) - \frac{\partial L}{\partial x_\mu^i} = 0 \quad (i = 1, 2; \mu = 0, 1, 2, 3) : \tag{2.9}
\]

is shown in the following Sections to enable all the concepts and equations of CEM concerning inter-charge forces, in the absence of radiation, to be derived without introducing any further postulate. Note that, since the Lagrangian (2.7) is a Lorentz scalar, it provides a description of the motion of \(O_1\) and \(O_2\) in any inertial reference frame.

3 The 4-vector Potential, Electric and Magnetic Fields, the Lorentz Force Equation and the Biot and Savart Law

Considering only the motion of \(O_1\), introducing the ‘4-vector potential’, \(A_2\), according to the definition:

\[
A_2 \equiv \frac{j_2}{c r_{12}} \tag{3.1}
\]

the well-known [6] Lorentz-invariant Lagrangian describing the motion of the object \(O_1\) in the ‘electromagnetic field created by the object \(O_2\)’:

\[
L(x_1, u_1) = -\frac{m_1 u_1^2}{2} - \frac{1}{c} q_1 u_1 \cdot A_2 \tag{3.2}
\]

is recovered. In the same way, the motion of \(O_2\) in the ‘electromagnetic field created by the object \(O_1\)’ is given by the invariant Lagrangian:

\[
L(x_2, u_2) = -\frac{m_2 u_2^2}{2} - \frac{1}{c} q_2 u_2 \cdot A_1 \tag{3.3}
\]

where:

\[
A_1 \equiv \frac{j_1}{c r_{12}} \tag{3.4}
\]
To now introduce the concepts of distinct ‘electric’ and ‘magnetic’ fields it is sufficient to consider only the motion of $O_1$. To simplify the equations the labels ‘1’ and ‘2’ will be dropped in Eqn(3.2) and the following notation is used for spatial partial derivatives:

$$\partial_i = -\partial^i \equiv \frac{\partial}{\partial x^i} \equiv \nabla_i \quad (i = 1, 2, 3) \quad (3.5)$$

The Lagrangian (3.2) is now introduced into the Lagrange equations (2.9). Considering the 1 spatial components of the 4-vectors, the first term on the LHS of Eqn(2.9) is:

$$\frac{d}{d\tau} \left( \frac{\partial L}{\partial u^1} \right) = \frac{d}{d\tau} (mu^1 + q c A^1) = \gamma (m \frac{du^1}{dt} + \frac{q}{c} \frac{dA^1}{dt}) \quad (3.6)$$

and the second is:

$$-\frac{\partial L}{\partial x^1} = -\frac{q}{c} u \cdot (\partial^1 A) \quad (3.7)$$

Combining Eqns(2.9), (3.6) and (3.7) and transposing:

$$\gamma m \frac{du^1}{dt} = \gamma \frac{dp^1}{dt} = \frac{q}{c} \left[ u \cdot (\partial^1 A) - \frac{\gamma}{c} \frac{dA^1}{dt} \right] \quad (3.8)$$

Substituting the Euler formula for the total time derivative $7$:

$$\frac{dA^1}{dt} = \frac{\partial A^1}{\partial t} - v^1 \partial^1 A^1 - v^2 \partial^2 A^1 - v^3 \partial^3 A^1 \quad (3.9)$$

into (3.8), writing out explicitly the 4-vector product $u \cdot (\partial^1 A)$, and cancelling a common factor $\gamma$ from each term, gives:

$$\frac{dp^1}{dt} = \frac{q}{c} \left[ c \partial^1 A^0 - \frac{\partial A^1}{\partial t} + v^2 (\partial^2 A^1 - \partial^1 A^2) - v^3 (\partial^3 A^1 - \partial^1 A^3) \right] \quad (3.10)$$

Introducing now 3-vector ‘electric’ and ‘magnetic’ fields, $E^i$ and $B^i$ respectively, according to the definitions:

$$E^i \equiv \partial^i A^0 - \frac{1}{c} \frac{\partial A^i}{\partial t} = \partial^i A^0 - \partial^0 A^i \quad (3.11)$$

and

$$B^k \equiv -\epsilon_{ijk} (\partial^j A^i - \partial^i A^j) = (\vec{\nabla} \times \vec{A})^k \quad (3.12)$$

where $\epsilon_{ijk}$ is the alternating tensor equal to $+1(-1)$ when $ijk$ is an even (odd) permutation of 123, and zero otherwise, enables Eqn(3.10) to be written as the compact expression:

$$\frac{dp^1}{dt} = q \left[ E^1 + \frac{1}{c} (\vec{v} \times \vec{B})^1 \right] \quad (3.13)$$

which is the 1 component of the Lorentz force equation. The 2 and 3 components are derived by cyclic permutations of the indices 1,2,3 in Eqn(3.10), yielding finally the 3-vector Lorentz force equation:

$$\frac{d\vec{p}}{dt} = q \left[ \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right] \quad (3.14)$$

$7$The implicit time dependence of $A^1$ in the first term on the right side of (3.9) arises from the instantaneous motion of the ‘source’ $O_2$, whereas the remaining terms describe the variation of $A^1$ due to the motion of $O_1$. 

5
The concepts of ‘electric’ and ‘magnetic’ fields have therefore appeared naturally as a means to simplify the Lorentz force equation (3.10). However, the RHS of this equation is completely defined, via Eqn(3.1), by the 4-vector current $j_2$, the spatial separation $r_{12}$ of $O_1$ and $O_2$ and the 3-velocity of $O_1$, so that the 4-vector potential $A$ may be eliminated from the Lorentz force equation. Substituting the definition of $A$ from Eqn(3.1) into Eqns(3.11) and (3.12), and restoring the labels of quantities associated with $O_2$, gives:

$$\vec{E} = \frac{j_2 \vec{r}}{c r^3} - \frac{1}{c^2 r} \frac{d j_2}{dt} - \frac{\vec{j}_2 \cdot \vec{v}_2}{c^2 r^3} \quad (3.15)$$

$$\vec{B} = \frac{q_2 \gamma_2 (\vec{v}_2 \times \vec{r})}{c r^3} = \frac{\vec{j}_2 \times \vec{r}}{c r^3} \quad (3.16)$$

where $\vec{r} \equiv \vec{r}_{12}$. Eqn(3.16) is the relativistic generalisation of the Biot and Savart Law. It differs from the usual CEM formula by a factor $\gamma_2$. Note that the electric field is, in general, non-radial. The non-radial part of the field, associated with the last term on the right side of (3.15), originates in the second term on the right side of (3.11). This is the electric field that is associated with the time variation of the magnetic field in the Faraday-Lenz Law. For the case of a source charge in uniform motion in the $x$-direction, with velocity $v_2$, the electric and magnetic fields given by (3.15) and (3.16) at the field point $\vec{r} = i \cos \psi + j \sin \psi$ are:

$$\vec{E} = \frac{q}{r^2} \left[ \frac{i \cos \psi}{\gamma_2} + \gamma_2 j \sin \psi \right] \quad (3.17)$$

$$\vec{B} = \frac{\vec{v}_2 \times \vec{E}}{c} \quad (3.18)$$

where $i$ and $j$ are unit vectors in the $x$- and $y$-directions. These equations may be compared with the pre-relativistic Heaviside [7] formulae for this case:

$$\vec{E}(H) = \frac{q \vec{r}}{r^3 \gamma_2^2 (1 - \beta_2^2 \sin^2 \psi)^2} \quad (3.19)$$

$$\vec{B}(H) = \frac{\vec{v}_2 \times \vec{E}(H)}{c} \quad (3.20)$$

The fields $\vec{E}(H)$ and $\vec{B}(H)$ are also the ‘present time’ fields as derived [8] from the retarded Liénard-Wiechert potentials [9]. By considering a simple two-charge ‘magnet’, in a particular spatial configuration, either in motion or at rest, it has been shown [10] that the radial electric field of (3.19) predicts a vanishing induction effect for a moving magnet and stationary test charge. In the same configuration (3.17) predicts the same induction force on the test charge as the Faraday-Lenz Law. The Heaviside formulae are therefore valid only to first order in $\beta$, in which case the predictions of (3.19) and (3.20) are the same as those of (3.17) and (3.18). It is interesting to recall that just this problem, of induction in different frames of reference, was discussed in the Introduction of Einstein’s 1905 special relativity paper [11].

\[8\]Note that the partial time derivative in (3.11) implies that $\vec{x}_1$ but not $\vec{x}_2$ is held constant. The implicit time variation of $A^1$ in (3.11) then has contributions from both $j_2$ and $\vec{x}_2$ which yield, respectively, the last two terms on the right side of (3.15).
Substitution of (3.15) and (3.16) into (3.14) and restoring the labels associated with \( O_1 \) yields the ‘fieldless’ Lorentz force equations\(^9\) for two, discrete, mutually electromagnetically interacting, physical objects:

\[
\frac{d\vec{p}_1}{dt} = \frac{q_1}{c} \left[ \frac{j_2^0 \vec{r} + \vec{\beta}_1 \times (j_2 \times \vec{r})}{r^3} - \frac{1}{cr} \frac{d_j^2}{dt} - \frac{\vec{r} \cdot \vec{\beta}_2}{r^3} \right] \tag{3.21}
\]

\[
\frac{d\vec{p}_2}{dt} = -\frac{q_2}{c} \left[ \frac{j_1^0 \vec{r} + \vec{\beta}_2 \times (j_1 \times \vec{r})}{r^3} + \frac{1}{cr} \frac{d_j^1}{dt} - \frac{\vec{r} \cdot \vec{\beta}_1}{r^3} \right] \tag{3.22}
\]

It may be thought that the terms \( \simeq 1/r \) should be associated with radiative processes (see Section 7 below) but they are in fact of particle-kinetic nature. Since \( \vec{j} = (q/m)\vec{p} \) the two differential equations are coupled via the \( d\vec{j}/dt \) terms on the right sides of each. The solution of these equations for the case of circular Keplerian orbits has been derived [13]. One result obtained is the relativistic generalisation of Kepler’s Third Law of planetary motion for this case:

\[
\tau^2 = \frac{(2\pi)^2 \mathcal{E}^* \left[ 1 - \frac{(q_1 q_2)^2}{m_1 m_2 c^4} \right]}{|q_1||q_2|(1 + \beta_1 \beta_2)} r^3 \tag{3.23}
\]

where

\[
\mathcal{E}^* \equiv \frac{\mathcal{E}_1^* \mathcal{E}_2^*}{\mathcal{E}_1^* + \mathcal{E}_2^*} \tag{3.24}
\]

and

\[
\mathcal{E}_1^* \equiv \frac{\gamma_1 m_1 c^2}{\gamma_2 - \frac{|q_1||q_2|}{m_2 c^2 r}} \tag{3.25}
\]

\[
\mathcal{E}_2^* \equiv \frac{\gamma_2 m_2 c^2}{\gamma_1 - \frac{|q_1||q_2|}{m_1 c^2 r}} \tag{3.26}
\]

Eqn(3.23) gives the period, \( \tau \), of two objects of mass \( m_1 \) and \( m_2 \) with (opposite) electric charges \( q_1 \) and \( q_2 \), in circular orbits around their common center of energy, separated by the distance \( r \). The \( d\vec{j}/dt \) terms in (3.21) and (3.22) give the terms \( \simeq 1/r \) in the denominators on the right sides of (3.25) and (3.26). These terms effectively modify the masses of the objects due to the electromagnetic interaction.

It is also demonstrated in Ref.[13] that stable, circular, Keplerian orbits are impossible under the retarded forces generated by Liénard-Wiechert potentials.

Considering now the time components of the 4-vectors in (2.11), the first term on the LHS is:

\[
\frac{d}{d\tau} \left( \frac{\partial L}{\partial u^0} \right) = \gamma \left( -m \frac{du^0}{dt} - \frac{q dA^0}{c dt} \right) \tag{3.27}
\]
while the second is:

$$-\frac{\partial L}{\partial x^0} = \frac{q}{c} u \cdot (\partial^0 A) = \frac{q}{c} u \cdot \left(\frac{1}{c} \frac{\partial A}{\partial t}\right)$$  \hspace{1cm} (3.28)$$

Substituting (3.27) and (3.28) into (2.9) and rearranging gives:

$$\gamma \frac{d \mathcal{E}}{dt} = \frac{q}{c} \left[ u \cdot \left(\frac{1}{c} \frac{\partial A}{\partial t}\right) - \gamma \frac{q}{c} dA^0 \right]$$  \hspace{1cm} (3.29)$$

where $\mathcal{E} \equiv m u^0 c$ is the relativistic energy of $O_1$. Using the Euler formula (3.9) to express $dA_0/dt$ in terms of partial derivatives, and writing out the different terms in the 4-vector scalar products, the terms $\partial A^0/\partial t$ are seen to cancel. Dividing out the factor $\gamma$ on both sides of the equation then gives the result:

$$\frac{d \mathcal{E}}{dt} = q \left[ v_1 (\partial^1 A^0 - \partial^0 A^1) + v_2 (\partial^2 A^0 - \partial^0 A^2) + v_3 (\partial^3 A^0 - \partial^0 A^3) \right] = q \vec{v} \cdot \vec{E}$$  \hspace{1cm} (3.30)$$

where $\vec{E}$ is the electric field defined in (3.11). Restoring now the labels of $O_1$ and $O_2$ gives the ‘fieldless’ equations for the time derivatives of their relativistic energies:

$$\frac{d \mathcal{E}_1}{dt} = q_1 \left[ j_2^0 \frac{\vec{\beta}_1 \cdot \vec{r}}{r^3} - \frac{1}{c r} \vec{\beta}_1 \cdot \frac{dj_2^1}{dt} - \frac{(\vec{\beta}_1 \cdot j_2^0) (\vec{r} \cdot \vec{\beta}_2)}{r^3} \right]$$  \hspace{1cm} (3.31)$$

$$\frac{d \mathcal{E}_2}{dt} = -q_2 \left[ j_1^0 \frac{\vec{\beta}_2 \cdot \vec{r}}{r^3} + \frac{1}{c r} \vec{\beta}_2 \cdot \frac{dj_1^1}{dt} + \frac{(\vec{\beta}_2 \cdot j_1^0) (\vec{r} \cdot \vec{\beta}_1)}{r^3} \right]$$  \hspace{1cm} (3.32)$$

The equations (3.21),(3.22) and (3.31),(3.32) give a complete description of the purely mechanical aspects of CEM (that is, neglecting radiative effects) for two massive, electrically charged, objects interacting mutually through electromagnetic forces.

The Lagrangian (2.7) is readily generalised to describe the mutual electromagnetic interactions of an arbitrary number of charged objects:

$$L(x_1, u_1; x_2, u_2; ..., x_n, u_n) = -\frac{1}{2} \sum_{i=1}^{n} m_i u_i^2 - \frac{1}{c^2} \sum_{i>j} q_i q_j \frac{u_i \cdot u_j}{r_{ij}}$$  \hspace{1cm} (3.33)$$

Here $r_{ij} = |\vec{r}_i - \vec{r}_j|$ where $\vec{r}_i$ and $\vec{r}_j$ specify the positions of $O_i$ and $O_j$, respectively, relative to the centre-of-energy on the $n$ interacting objects. Note that, as all these distances are specified at a fixed time in the overall CM frame of the objects, the $r_{ij}$ are Lorentz invariant quantities, similar to $r_{12}$ in Eqn(2.7). See also [14] for a general discussion of such invariant length intervals. The Lagrangian describing the motion of the object $i$ ‘in the electromagnetic field of’ the remaining $n-1$ objects may be derived from Eqn(3.33):

$$L(x_i, u_i) = -\frac{m_i u_i^2}{2} - \frac{1}{c} q_i u_i \cdot A(n-1)$$  \hspace{1cm} (3.34)$$

where

$$A(n-1) \equiv \sum_{j\neq i}^{n} \frac{q_j u_j}{r_{ij}} = \sum_{j\neq i}^{n} \frac{j_j}{r_{ij}}$$  \hspace{1cm} (3.35)$$
This equation embodies the classical superposition principle for the electromagnetic 4-vector potential, and hence, via the linear equations (3.11) and (3.12), that for the electric and magnetic fields.

4 Derivation of Maxwell’s Equations

Writing out explicitly the spatial components of the quantity $\vec{\nabla} \cdot \vec{B}$ using the definition of $\vec{B}$, Eqn(3.12):

\begin{align*}
\partial^1 B^1 &= \partial^1 \partial^3 A^2 - \partial^1 \partial^2 A^3 \\
\partial^2 B^2 &= \partial^2 \partial^1 A^3 - \partial^2 \partial^3 A^1 \\
\partial^3 B^3 &= \partial^3 \partial^2 A^1 - \partial^3 \partial^1 A^2
\end{align*}

(4.1) (4.2) (4.3)

it follows, since $\partial^i \partial^j = \partial^j \partial^i$ ($i, j = 1, 2, 3$) that, on summing Eqns(4.1), (4.2) and (4.3),

$$\vec{\nabla} \cdot \vec{B} = -(\partial^1 B^1 + \partial^2 B^2 + \partial^3 B^3) = 0$$

(4.4)

which is the magnetostatic Maxwell equation. Since $\vec{B} \equiv \vec{\nabla} \times \vec{A}$, (4.4) can also be seen to follow from the 3-vector identity $\vec{a} \cdot (\vec{a} \times \vec{b}) \equiv 0$ for arbitrary $\vec{a}$ and $\vec{b}$.

The Faraday-Lenz Law follows directly from the defining equations Eqn(3.11), (3.12) of the electric and magnetic fields. Taking the curl of both sides of the 3-vector form of Eqn(3.11) with $\vec{\nabla}$ gives:

$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times (\vec{\nabla} A^0) - \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{A})$$

(4.5)

Since $\vec{\nabla} \times (\vec{\nabla} \phi) = \text{curl}(\text{div} \phi) = 0$ for an arbitrary scalar $\phi$, the first term on the RHS of Eqn(4.5) vanishes. Substituting the 3-vector form of Eqn(3.12) in the second term on the RHS of Eqn(4.5) then yields the Faraday-Lenz Law:

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

(4.6)

The electrostatic Maxwell equation:

$$\vec{\nabla} \cdot \vec{E} = 4\pi J^0$$

(4.7)

is a well-known consequence of the inverse square law for a ‘static’ electric field defined by only the first term on the RHS of Eqn(3.11) and Gauss’ theorem [15]. The 4-vector current density: $\vec{J} \equiv (c \rho; \vec{J})$, the 0 component of which appears in Eqn(4.7), is related to the currents, $j_i$, of elementary charges $q_i$ by the relation:

$$\vec{J} = \frac{1}{V_R} \sum_{i \in R} j_i$$

(4.8)

where $V_R$ is the volume of a spatial region $R$. Hence $\rho = J^0/c$ is, in the non-relativistic limit where $\gamma \approx 1$, the average spatial density of electric charge in the region $R$. Conservation of electric charge requires that:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

(4.9)
This continuity equation may be simply derived from the properties of the 4-vector product:

$$\partial \cdot j_i = \partial^0 j_i^0 - \sum_{k=1}^{3} \partial^k j_i^k = q_i \left[ \frac{\partial \gamma_i}{\partial t} + \vec{\nabla} \cdot (\gamma_i \vec{v}_i) \right]$$  \hspace{1cm} (4.10)$$

In the rest frame of the object $O_1$, $\gamma_i - 1 = |\vec{v}_i| = 0$, so that $\partial \cdot j_i = 0$. Since $\partial \cdot j_i$ is a Lorentz invariant this quantity then vanishes in all inertial reference frames. Taking the scalar product of $\partial$ and $J$ gives:

$$\partial \cdot J = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = \frac{1}{V_R} \sum_{i \in R} \partial \cdot j_i = 0$$  \hspace{1cm} (4.11)$$

Which is just the continuity equation (4.9). It can be seen that the conservation of electric charge is a consequence of its Lorentz-scalar nature, i.e. the charge $q_i$ in Eqn(4.10) does not depend on the frame in which $\vec{v}_i$ is evaluated. Indeed, the definition $j_i \equiv q_i u_i$ implies that $j_i \cdot j_i = q_i^2 u_i \cdot u_i = c^2 q_i^2$, so that $q_i^2$ is manifestly Lorentz invariant, in precise analogy with the mass of an object: $p_i \cdot p_i = m_i^2 u_i \cdot u_i = c^2 m_i^2$. Both $j_i$ and $p_i$ are proportional to the 4-vector velocity $u_i$.

A relation similar to (4.9) is:

$$\frac{1}{c} \frac{\partial A^0}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0$$  \hspace{1cm} (4.12)$$

the so-called ‘Lorenz Condition’\footnote{Not ‘Lorentz Condition’, as found in many text books. See Reference [16].}, which may also be written more simply as $\partial \cdot A = 0$. This relation is, in the present approach, not, as in conventional discussions of CEM, the result of a particular choice of gauge in the definition of $\vec{A}$, but an identity following from the definition of $A$ in Eqn(3.1). In fact as is easily shown:

$$\vec{\nabla} \cdot \vec{A} = -\frac{\vec{j} \cdot \vec{r}}{cr^3} = -\frac{1}{c} \frac{\partial A^0}{\partial t}$$  \hspace{1cm} (4.13)$$

Here the derivatives in $\vec{\nabla}$ are with respect to the ‘field point’ $\vec{x}_1$ in contrast with those in $\vec{\nabla}$ in Eqns(4.9)-(4.11), which are with respect to the spatial coordinate $\vec{x}_2$ of the object $O_2$ associated with the current $\vec{j}$. The partial time derivative in (4.12) is defined for $\vec{x}_1$ constant. The time variation of $A^0$ is then due solely to the time dependence of $\vec{x}_2$, which leads to the second member of (4.13). Eqn(4.12) shows that the 4-vector potential, like the current and energy-momentum 4-vectors corresponds to a conserved (Lorentz invariant) quantity: $A \cdot A = q^2/r^2$\footnote{$r$ is the manifestly invariant quantity $\sqrt{-(\vec{x}_1 - \vec{x}_2)^2}$ that appears in eqn(2.10) above.}. So both $j$ and $A$ differ only by Lorentz invariant multiplicative factors from $p$ and $u$:

$$c^2 = u \cdot u = \frac{p \cdot p}{m^2} = \frac{j \cdot j}{q^2} = c^2 r^2 \frac{A \cdot A}{q^2}$$  \hspace{1cm} (4.14)$$

The relation (4.12) is found to be important in an interpretation of the electrodynamic Maxwell equation, (4.20) below, as a description of radiation phenomena (creation of real photons). This point will be briefly discussed in Section 7.

The electrodynamic Maxwell equation (Ampère’s Law, including Maxwell’s ‘displacement current’) is derived immediately on writing the electrostatic Maxwell equation (4.7)
in a covariant form. The latter then appears as an equation for the 0 component of a 4-vector. The corresponding spatial components, written down simply by inspection, are Ampère’s Law. Writing Eqn(4.7) in 4-vector notation, and introducing also the ‘non-static’ component of the electric field, given by the second term on the RHS of Eqn(3.11), gives:

\[
\left(\sum_{i=1}^{3} -\partial^i \partial^i\right)A^0 - \partial^0 \left(\sum_{i=1}^{3} -\partial^i A^i\right) = 4\pi J^0
\]  

Adding to Eqn(4.15) the identity:

\[
\partial^0 \partial^0 A^0 - \partial^0 \partial^0 A^0 = 0
\]
gives:

\[
(\partial \cdot \partial) A^0 - \partial^0 (\partial \cdot A) = 4\pi J^0
\]  

Since the coefficients of \(A^0\) and \(-\partial^0\) are Lorentz scalars, the corresponding \(i\)th spatial component of the 4-vector \(J\), is from the manifest covariance of Eqn(4.16), given by the equation:

\[
(\partial \cdot \partial) A^i - \partial^i (\partial \cdot A) = 4\pi J^i
\]

This is Ampère’s Law in 4-vector notation. In order to recover the more familiar 3-vector equation, the 4-vector potential must be eliminated in favour of the electric and magnetic fields defined in Eqns(3.11) and (3.12) respectively. To do this, consider the contribution of the spatial parts (SP) of the 4-vector products on the LHS of Eqn(4.17) to \(J^1\). This gives:

\[
4\pi J^1(SP) = -\sum_{i=1}^{3} (\partial^i)^2 A^1 + \partial^1 \sum_{i=1}^{3} (\partial^i A^i) = \sum_{i=1}^{3} \partial^i (\partial^1 A^i - \partial^i A^1)
\]

\[
= (\partial^1)^2 A^1 + \partial^2 \partial^1 A^2 + \partial^3 \partial^1 A^3 - (\partial^1)^2 A^1 - (\partial^2)^2 A^1 - (\partial^3)^2 A^1
\]

\[
= -\partial^2 (\partial^2 A^1 - \partial^1 A^2) + \partial^3 (\partial^3 A^3 - \partial^3 A^1)
\]

\[
= -\partial^2 B^3 + \partial^3 B^2 = (\nabla \times \vec{B})^1
\]  

where, in the fourth line the definition, Eqn(3.12), of the magnetic field has been used. The contribution of the temporal parts (TP) of the 4-vector products on the LHS of Eqn(4.17) to \(J^1\) is:

\[
4\pi J^1(TP) = (\partial^0)^2 A^1 - \partial^1 \partial^0 A^0 = \partial^0 (\partial^0 A^1 - \partial^1 A^0) = -\frac{1}{c} \frac{\partial E^1}{\partial t}
\]  

Adding the spatial and temporal contributions to \(J^1\) from Eqns(4.18) and (4.19) gives the 1 component of the electrodynamic Maxwell equation:

\[
\nabla \times \vec{B} - \frac{1}{c} \frac{\partial \vec{E}}{\partial t} = 4\pi \vec{J}
\]

The 2 and 3 components are obtained by cyclic permutation of the indices 1,2,3 in Eqns(4.18) and (4.19). This derivation of Ampère’s Law, starting from the electrostatic Maxwell equation, (4.7) has been previously given by Schwartz [17], and, independently, by the present author in Reference [18], where it was noted that Eqn(4.17) may be derived from Eqn(4.16) using space-time exchange symmetry invariance.
5 Fundamental Concepts and Different Levels of Mathematical Abstraction

Equations (3.21), (3.22), (3.31) and (3.32) show that the dynamics of any system of mutually interacting electrically charged objects is completely specified by their masses, electric charges and 4-vector positions and velocities. Other useful and important concepts of CEM such as the 4-vector potential and electric and magnetic fields are completely specified, in terms of the geometrical and kinematical configuration of the charged objects by Eqn(3.1) for $A^\mu$, Eqns(3.1) and (3.11) for $\vec{E}$ and Eqns(3.1) and (3.12) for $\vec{B}$. Historically, of course, Faraday arrived at the concepts of electric and magnetic fields in complete ignorance of the existence of elementary electric charges or of Special Relativity. With our present-day understanding of both the existence of the former and the necessary constraints provided by the latter, it can be seen that both the 4-vector potential and electric and magnetic fields are, in fact, only convenient mathematical abstractions. The 4-vector potential is at a first level of abstraction. The phenomenologically most useful concepts of CEM, the electric and magnetic fields are, in turn, completely specified by $A^\mu$ and so are at a second level of abstraction from the fundamental and irreducible concepts (charged, interacting, physical objects) of the theory.

Indeed, there is yet a third level of abstraction, the tensor $F^{\mu\nu}$ of the electromagnetic field defined as:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$  \hspace{1cm} (5.1)

This description was introduced by Einstein in his original paper on General Relativity [19] in analogy with the tensor $G^{\mu\nu}$ of the classical gravitational field. It has the merit of enabling the electrostatic and electrodynamic Maxwell equations to be written as a single compact equation\textsuperscript{12}:

$$\partial_\nu F^{\mu\nu} = 4\pi J^\mu$$  \hspace{1cm} (5.2)

As in the case of the introduction of electric and magnetic fields into the covariant Lorentz force equation (3.10) to obtain the 3-vector version (3.14), a cumbersome equation is reduced to an elegant one, at the the cost of introducing a higher level of mathematical abstraction. Viewed, however in the light of the strict criteria of Newton’s precept, $A^\mu$, $\vec{E}$, $\vec{B}$ and $F^{\mu\nu}$, (although in the case of $\vec{E}$ and $\vec{B}$ extremely useful phenomenologically) are certainly not ‘sufficient’ to explain, in any fundamental manner, the phenomena of CEM. On the contrary, as shown above, Coulomb’s law and Special Relativity, given, of course, the a priori existence of charged physical objects, do provide such a fundamental description, in which the ‘fields’ of electromagnetism appear naturally by mathematical substitution. If all that was known of CEM was Eqn(5.2), it is hard to see any logical path to derive from it the Lorentz Force, Biot and Savart and Faraday-Lenz Laws that actually describe the results of laboratory experiments in CEM. However these laws, Eqn(5.2) and the magnetostatic Maxwell equation (4.4) are all necessary consequences of Coulomb’s Law, Special Relativity and Hamilton’s Principle. The higher the level of mathematical abstraction, the more elegant the electrodynamic formulae appear to be, but the further removed they become from the physical realities of the subject.

\textsuperscript{12}The covariant operator $\partial_\nu$ is introduced by multiplying the contravariant operator $\partial^\mu$ by the metric tensor: $\partial_\nu = g_{\nu\mu}\partial^\mu$ where $g_{\nu\mu} = 0$ for $\nu \neq \mu$ and $g_{\mu\mu} = (1,-1,-1,-1)$. Repeated upper and lower indices: $\nu, \mu$ are summed over 0,1,2,3.
Although Einstein spent some decades of his life in the unsuccessful attempt to realise a unifying synthesis between the classical field tensors $F^{\mu\nu}$ and $G^{\mu\nu}$ he still made a clear distinction between physical reality and mathematical abstraction[20]:

We have seen, indeed, that in a more complete analysis the energy tensor can be regarded only as a provisional means of representing matter. In reality, matter consists of electrically charged particles, and is to be regarded itself as a part, in fact the principle part, of the electromagnetic field.

In fact, electrically charged particles and real and virtual photons (which are also particles) are the true irreducible concepts of CEM. These are not the ‘principle part’ of the electromagnetic field, but rather replace it in the most fundamental description of the phenomena of CEM.

Since the only dynamical postulate in CEM is Coulomb’s Law, the only way to obtain a deeper physical understanding is by a deeper understanding of this Law. Indeed, as will be discussed in the following Section, this does seem to be possible by considering the particle aspects of the microscopic underlying QED process, which is basically Möller scattering: $e^- e^- \rightarrow e^- e^-$. 

6 Quantum Electrodynamical Foundations of Classical Electromagnetism

If the electrodynamical force is transmitted by particle exchange, and it is assumed that the magnitude of the force is proportional to the number of interacting particles, which are emitted isotropically by the source, the inverse square law follows from spatial geometry and conservation of the number of particles$^{13}$. However, in the Coulomb interaction the exchanged particle is a virtual, not a real, photon. This means that it cannot always be considered to move in a particular direction in space-time. It will be shown below, however, that the Fourier transform of the momentum-space virtual photon propagator does yield a space-time propagator with the $1/r$ dependence of the Coulomb potential, which corresponds, in the classical limit, to an inverse square force law. It is also shown that, in the CM frame of the interacting charged particles, this interaction is instantaneous, as assumed in the derivation of the classical Lagrangian (2.7).

According to QED, the Biot and Savart and Lorentz Force Laws are the classical limit of Möller scattering for very large numbers of electrons at very large spatial separations. Conversely, Möller scattering is the quantum limit of the Biot and Savart and Lorentz Force Laws when each current contains a single electron and the spatial separation of the currents is very small. The fundamental quantum mechanical laws governing Möller scattering were followed by Kepler in his attempts to understand the gravitational force. As, however, the agents of force were constrained to propagate in the plane of a planetary orbit, rather than in three spatial dimensions, a $1/r$ force law was predicted [21]. Also, as a consequence of Kepler’s Aristotelian understanding of dynamics, the force was conjectured to sweep the planets around the Sun in the transverse direction, rather than diverting them radially from their natural rectilinear motion, as in Newtonian dynamics.

$^{13}$A similar physical reasoning was followed by Kepler in his attempts to understand the gravitational force. As, however, the agents of force were constrained to propagate in the plane of a planetary orbit, rather than in three spatial dimensions, a $1/r$ force law was predicted [21]. Also, as a consequence of Kepler’s Aristotelian understanding of dynamics, the force was conjectured to sweep the planets around the Sun in the transverse direction, rather than diverting them radially from their natural rectilinear motion, as in Newtonian dynamics.
scattering do not change when many electrons, with macroscopic spatial separations, participate in the observed physical phenomenon. A more fundamental understanding of CEM is therefore provided, not by any kind of field concept, but by properly taking into account the existence of virtual photons, just as an analysis in terms of real photon production is mandatory for a fundamental description of the radiative processes of CEM, a subject beyond the scope of the present paper.

The invariant QED amplitude for Möller scattering by the exchange of a single virtual photon\(^{14}\) is given by the expression \(^{22}\)\(^{15}\):

\[
T_{fi} = -i \int \frac{\mathcal{J}^A(x_A) \cdot \mathcal{J}^B(x_A)}{q^2} d^4 x_A
\]  

(6.1)

The virtual photon is exchanged between the 4-vector currents \(\mathcal{J}^A\) and \(\mathcal{J}^B\) defined in terms of plane-wave solutions, \(u_i, u_f\) of the Dirac equation:

\[
\mathcal{J}^A_\mu \equiv -e u^A_f \gamma_\mu u^A_i \exp \left[ i (p^A_f - p^A_i) \cdot x_A \right]
\]  

(6.2)

where \(p^A_i\) and \(p^A_f\) are the energy-momentum 4-vectors of the incoming and scattered electron, respectively, that emit a virtual photon at the space-time point \(x_A\) and \(-e\) is the electron charge. The overall centre-of-mass frame (Fig1b) is a Breit frame for the virtual photon, i.e. the latter has vanishing energy:

\[
q^{A0} = p^{A0}_i - p^{A0}_f = -q^{B0} = p^{B0}_f - p^{B0}_i = 0
\]  

(6.3)

Thus, in this frame, the invariant amplitude may be written:

\[
T_{fi} = i \int \frac{\mathcal{J}^A(x_A) \cdot \mathcal{J}^B(x_A)}{|\vec{q}|^2} d^4 x_A
\]  

(6.4)

As shown in the Appendix, use of the Fourier transform:

\[
\frac{1}{|\vec{q}|^2} = \frac{1}{4\pi} \int d^3 x e^{i \vec{q} \cdot \vec{x}} |\vec{x}|
\]  

(6.5)

enables the invariant amplitude to be written as the space-time integral:

\[
T_{fi} = \frac{i}{4\pi} \int dt_A \int d^3 x_A \int d^3 x_B \frac{\mathcal{J}^A(\vec{x}_A, t_A) \cdot \mathcal{J}^B(\vec{x}_B, t_A)}{|\vec{x}_B - \vec{x}_A|}
\]  

(6.6)

It can be seen that the integrand in Eqn(6.6) has exactly the same \(j \cdot j/r\) structure as the potential energy term in the invariant CEM Lagrangian (2.7). Indeed this is to be expected in the Feynman Path Integral (FPI) formulation of quantum mechanics \([24]\). The physical meaning of Eqn(6.6) is that the total amplitude is given by integration over all spatial positions: \(\vec{x}_A(t_A), \vec{x}_B(t_A)\) at time \(t_A\), and all times \(t_A\), of emission and absorption

\(^{14}\)Actually there are two such amplitudes related by exchange of the identical final state electrons. In the present case, where the classical limit of CEM is under discussion, it suffices to consider only the amplitude given by (6.1) in the limit \(q^2 \to 0\). The contribution of the second amplitude is negligible in this limit.

\(^{15}\)Here units with \(\hbar = c = 1\) are assumed.
of a single virtual photon in the scattering process: $e^- e^- \rightarrow e^- e^-$. Since the virtual photon is not observed, this is just a manifestation of quantum mechanical superposition: a sum of different probability amplitudes with the same initial and final states. Notice that the virtual photon propagates with infinite velocity between the spatial positions $\vec{x}_A, \vec{x}_B$ so that the ambiguity in the direction of propagation of the space-like virtual photon (see Fig.1b and c) has no relevance. Thus QED predicts that virtual photons produce instantaneous ‘action at a distance’ in the overall centre-of-mass frame of Møller scattering. This is also implicit in the discussion of CEM in Sections 2 and 3 above, since all forces are defined at a fixed time in the CM frame of the interacting charges. The meaning of the retarded Liénard-Wiechert potentials and ‘causality’ in relation to the instantaneous forces transmitted by space-like virtual photons is discussed in the concluding section of this paper.

To examine more closely the connection between Eqn(6.6) and the FPI formalism, consider the general FPI expression for a transition amplitude [24]:

$$ T_{fi}^{FPI} \equiv \langle \chi(t_f) | \psi(t_i) \rangle = \int_{\text{paths}} \chi^*(x_f, t_f) e^{iS} \psi(x_i, t_i) Dx \equiv \int_{\text{paths}} \langle f | e^{iS} | i \rangle Dx $$

(6.7)

where the Action, $S$, is given by the time integral of the classical Lagrangian, $L$, of the quantum system under consideration:

$$ S = \int_{t_i}^{t_f} L(x, \dot{x}) dt $$

(6.8)

(here the upper dot denotes time differentiation) and

$$ Dx \equiv \text{Lim } (\epsilon \rightarrow 0) \left( \frac{dx_0}{A} \frac{dx_1}{A} \cdots \frac{dx_{j-1}}{A} \frac{dx_j}{A} \right) $$

(6.9)

where $x_0, x_1, \ldots$ denote successive positions along the path, each separated by a small, fixed, time interval $\epsilon$. Also $dx_j \equiv x_j - x_{j-1}$. $A$ is a normalisation constant that depends upon $\epsilon$. In the case of present interest, Møller scattering, the one dimensional FPI (6.7), with a single particle, is generalised to three spatial dimensions and two particles with the label $p = A, B$, $x \rightarrow (x^1_p, x^2_p, x^3_p)$, corresponding to the two electrons which scatter from each other (see Fig.1). In this case, (6.7) is generalised to [25]:

$$ T_{fi}^{FPI} = \int_{\text{paths}} \langle f | e^{iS} | i \rangle \prod_{p=A,B} \prod_{j=1}^{3} Dx^j_p(t) $$

(6.10)

and (6.8) to

$$ S = \int_{t_i}^{t_f} L(\vec{x}_A, \dot{\vec{x}}_A; \vec{x}_B, \dot{\vec{x}}_B) dt $$

(6.11)

where $i$ and $f$ are the initial and final states of the Møller scattering process. Assuming that the transition $f \rightarrow i$ is caused by a small term $S_{int}$ in the action where $S = S_0 + S_{int}$ and $\langle f | S_0 | i \rangle = 0$, enables (6.10) to be written as:

$$ T_{fi}^{FPI} = \int_{\text{paths}} \langle f | e^{iS_{int}} | i \rangle \prod_{p=A,B} \prod_{j=1}^{3} Dx^j_p(t) $$

16Thus the simple momentum-space propagator $1/q^2$ of Eqn(6.1) is equivalent, in space-time, to the exchange of an infinity of virtual photons emitted and absorbed at different spatial positions and times. All these virtual photons however have, according to Eqn(6.6), infinite velocity.
Figure 1: a) Feynman diagram for Möller scattering: $e^+e^- \rightarrow e^+e^-$, by exchange of a single space-like virtual photon. b), c) show the possible momentum space diagrams for Möller scattering in the CM frame. In b)[c)] the virtual photon transfers momentum from the current $J^A$ [$J^B$] to $J^B$ [$J^A$]. These are equivalent descriptions. In both cases the energy of the virtual photon vanishes and it has infinite velocity.
In the last line the formal differentials $\mathcal{D}x^j_i(t)$ for arbitrary space-time paths $x^j_i(t)$ are replaced by those corresponding to the electrons A and B in the Møller scattering process that, in the classical limit, propagate along straight-line paths so that\footnote{Different choices of $\epsilon$ correspond to different values of $j$ in Eqn(6.9), for a given value of $\Delta x = x_j - x_0$. In the case of a straight line path the value of the limit in (6.9) is independent of the value of $\epsilon$. In particular, the choice $\epsilon = \Delta x/v$ is possible. This yields Eqn(6.13).}:

$$\prod_{p=A,B} \prod_{j=1}^3 \mathcal{D}x^j_i(t) \propto d^3x_A d^3x_B$$  \hspace{1cm} (6.13)

where the normalisation constant in (6.9) has been dropped, since only the proportionality of the matrix elements (6.6) and (6.12) is under investigation. Comparison of Eqns(6.6) and (6.12) gives:

$$\langle f | L_{int} | i \rangle \propto \frac{\mathcal{J}^A(x_A,t_A) \cdot \mathcal{J}^B(x_B,t_B)}{4\pi |x_B - x_A|}$$  \hspace{1cm} (6.14)

In order to compare Eqn(6.14) with the potential energy term in Eqn(2.7) which has the same 4-vector structure as Eqn(6.14), the classical limit of the QED transition currents $\mathcal{J}^A$ and $\mathcal{J}^B$, where the momentum carried by the virtual photon vanishes, must be considered. For this it is convenient to use the Gordon Identity [26] for the spinor product appearing in Eqn(6.2):

$$\tilde{J}^\mu \equiv -e\bar{\nu}_f \gamma^\mu u_i = \frac{-e}{2m} \bar{\nu}_f [(p_f + p_i)^\mu + i\sigma^{\mu\nu}(p_f - p_i)_\nu] u_i$$  \hspace{1cm} (6.15)

where

$$\sigma^{\mu\nu} \equiv \frac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu)$$

In the overall centre-of-mass frame in the limit of vanishing virtual photon momentum: $(p_f - p_i)_\nu \to 0$, $(p_f + p_i)^\mu \to 2p^\mu$, and $\bar{\nu}_fu_i \to \bar{\nu}_iu_i = 2m$ [23]. Thus, in this classical limit Eqn(6.2) gives:

$$\tilde{J}^\mu_{\text{class}} = -e2p^\mu = 2Qu^\mu = 2j^\mu$$  \hspace{1cm} (6.16)

where $Q = -e$. Therefore, up to a multiplicative constant, the classical limit of the QED transition current $\tilde{J}^\mu$ is identical to the CEM current $Qu^\mu$ introduced in Section 2 above, and also, up to a constant multiplicative factor, the classical limit of the matrix element of the QED interaction Lagrangian $\langle f | L_{int} | i \rangle$ is equal to the potential energy term in the CEM Lagrangian (2.7). There is thus a seamless transition from QED to CEM.

Hamilton’s Principle of classical mechanics is the $h \to 0$ limit of Feynman’s path integral formulation of quantum mechanics. So it may be said that the third postulate in the derivation, from first principles, of CEM presented in this paper is not really an
independent premise, but rather a prediction of quantum mechanics. To show this, it is necessary to consider the behaviour of the fundamental FPI formula (6.7) for a transition amplitude in the classical limit. The action $S$ in this formula is a functional of the different space-time paths $x(t)$. Writing explicitly the dependence on Planck’s constant, gives a multiplicative factor $\exp(iS[x(t)]/\hbar)$ in the transition amplitude. If the paths $x(t)$ are chosen such that the variation of $S$ is large in comparison to $\hbar$, this factor will exhibit rapid phase oscillations and give a negligible contribution to the transition amplitude. If, however, the paths are chosen in such a way that $S$ is near to an extremum with respect to their variation, $S$ will change only very slowly from path to path, so that the contributions of different paths have have almost the same phase, resulting in a large contribution to the scattering amplitude. The classical limit corresponds to $\hbar \to 0$, where only the path giving the extremum of $S$ contributes. This path is just the classical trajectory as defined by Hamilton’s Principle. This argument, that may be called the ‘Stationary Phase Principle’, was first given by Dirac in 1934 [27] (see also Reference [28]) and was an important motivation for Feynman’s space-time reformulation of the principles of quantum mechanics [24, 25].

At this point it can be truthfully said that there is ‘nothing left to explain’ for an understanding of the fundamental physics of CEM, given the laws of special relativity and quantum mechanics. The irreducible physical concepts are electrically charged physical objects and space-like virtual photons. Coulomb’s Law is a consequence of the exchange of the latter between the former. Hamilton’s Principle is naturally given by the classical limit of the FPI formulation of quantum mechanics.

It is interesting, in the light of this ‘complete understanding’ that quantum mechanics and relativity provide about CEM, to consider two further quotations from the Principia. The first is taken from the ‘General Scholium’ [29]. After describing the inverse-square law of the gravitational force Newton states:

But hitherto I have not been able to discover the cause of these properties of gravity from phenomena, and I frame no hypothesis, for whatever is not derived from the phenomena is to be called a hypothesis, and hypotheses, whether metaphysical or physical, whether of occult qualities or mechanical, have no place in experimental philosophy.

The second is from the ‘Author’s Preface to The Reader’ [30]

I wish we could derive the rest of the phenomena of Nature by the same kind of reasoning from mechanical principles, for I am induced by many reasons to suspect that they may all depend upon certain forces by which the particles of bodies, by some causes hitherto unknown, are either mutually impelled towards one another and cohere in regular figures or are repelled and recede from one another.

Now, at the beginning of the 21st century, Newton’s wish to understand, at a deeper level, the forces of nature, has been granted, at least for the case of electromagnetic ones. What was needed was not ‘the same kind of reasoning from mechanical principles’ that Newton considered but the discovery of relativity and quantum mechanics. The cause ‘hitherto unknown’ of the electromagnetic force is the exchange of space-like virtual
photons according to the known laws of QED.

Regrettably science is, at the time of this writing, riddled by many ‘hypotheses’ of the type referred to in the first of the above quotations. One such hypothesis, that has persisted through much of the 19th century and all of the 20th is that: ‘No physical influence can propagate faster than the speed of light’. This is contradicted by the arguments given above and, as discussed in the following section, also by the results of some recent experiments.

7 Discussion and Outlook

The starting point and aims of the present paper are very close to those of Feynman and Wheeler when they attempted, in the early 1940’s, to reformulate CEM in terms of direct inter-charge interactions without the *a priori* introduction of any electromagnetic field concept. In this way the infinite self-energy terms associated with the electric field of a point charge are eliminated. As Feynman put it [31]:

*You see then that my general plan was to first solve the classical problem, to get rid of the infinite self-energies in the classical theory, and to hope that when I made a quantum theory of it everything would be just fine.*

Feynman and Wheeler had a project to write three papers on the subject [32]. The first of these papers was to be a study of the classical limit of the quantum theory of radiation. Feynman had yet to formulate his space-time version of QED, and this paper was never written. In the remaining two papers [33, 34] it was proposed to introduce direct interparticle action by including the effects of both retarded and ‘advanced’ potentials as well as an array of ‘absorbers’. As suggested by Dirac [35] half of the difference between the retarded potential of an accelerated charge and of the ‘advanced’ potential from the absorbers correctly predicts the known radiative damping force of CEM. The second paper, [34], developed further this theory by exploiting the Fokker action principle formulation of action-at-a-distance in CEM [36]. As stated in the introduction of this paper, a description was being sought that was:

(a) well defined

(b) economical in postulates

(c) in agreement with experience

that is, in other words, in accordance with Newton’s first ‘Rule of Reasoning in Philosophy’ quoted above. However, in order to reproduce the known results of CEM by such a theory ‘advanced’ potentials had to be introduced. This immediately gives an apparent breakdown of causality and the logical distinction between ‘past’, ‘present’ and ‘future’. As concisely stated by Feynman and Wheeler themselves [34]:

*The apparent conflict with causality begins with the thought: if the present motion of a is affected by the future motion of b, then the*
observation of a attributes a certain inevitability to the motion of b. Is not this conclusion in conflict with our recognised ability to influence the future motion of b?

Feynman and Wheeler then gave a rather artificial example (which the present writer finds unconvincing) that was claimed to resolve this causal paradox.

In fact, Feynman and Wheeler were compelled to introduce ‘advanced’ potentials because they were assuming, as did also Fokker and earlier authors attempting to formulate theories of direct interparticle action in CEM, that causality meant that no physical influence could be transmitted faster than the speed of light in vacuum. This definition of ‘causality’ seems to have been introduced into physics by C.F.Gauss in 1845 [37]. Somewhat later, C. Neumann proposed [38] that the electric potential responsible for interparticle forces should be transmitted, not at the speed of light, but instantaneously, like the gravitational force in Newton’s theory. As shown above, this is indeed how, in QED, space-like virtual photons transmit the electromagnetic force between charged objects in their common CM frame. These two hypotheses will be referred to below below as ‘Gaussian’ and ‘Neumann’ Causality. The fundamental Lagrangian of CEM describing the interaction of charged objects, in any inertial frame, is the simple expression Eqn(2.7) above, not the conjectured, and much more complicated, Fokker action that embodies Gaussian Causality. It is important to stress that the instantaneous action-at-a-distance, of Neumann Causality, which is just the limit of Gaussian Causality as $c \rightarrow \infty$, unlike an ‘advanced’ potential, poses no logical problem of the influence of the future on the present, as succinctly stated by Feynman and Wheeler in the above quotation.

Gaussian Causality has been an unstated (and unquestioned) axiom of physics since the advent of Special Relativity a century ago. The speed of light is certainly the limiting velocity of any physical object described by a time-like energy-momentum 4-vector. However Einstein at the time when he invented special relativity, and Feynman himself, at the time of his collaboration with Wheeler, were not aware of the concept of the ‘virtual’ particles. The latter, associated with the space-time propagators introduced into QED by Feynman and Stueckelberg, may be described by space-like energy-momentum 4-vectors. The instantaneous action at a distance of the virtual photons in Møller scattering described by the invariant amplitude in Eqn(6.6) above, can be simply understood from the relativistic kinematics of such virtual particles. The relativistic velocity $\beta = v/c$ of a particle in terms of its 3-momentum $\vec{p}$ and 4-momentum $p$ is, in general, given by the expression:

$$\beta = \frac{|\vec{p}|}{\sqrt{p^2 + p \cdot p}}$$

(7.1)

Thus space-like virtual particles, for which, by definition: $p \cdot p < 0$, are tachyons. For the case of the virtual photons exchanged in the center-of-mass-system of Møller scattering (Figs 1b and 1c): $p \cdot p = -\vec{p}^2$, since $p^0 = 0$, and so $\beta$ is infinite, consistent with the space-time description in Eqn(6.6).

That the Feynman propagator for a massive particle violates Gaussian Causality was pointed out by Feynman himself in his first QED paper [39] and later discussed by him in considerable detail [40]. This fact is also sometimes mentioned in books on Quantum Field Theory, that otherwise make the contradictory claim that, in general, quantum
field operators commute for space-like separations, so that, in consequence, no physical influence can propagate faster than the speed of light\textsuperscript{18}. The space-time propagator of a massive particle was shown by Feynman to be, in general, a Hankel function of the second kind \[39\]. For an on-shell particle, or a virtual particle propagating over a large proper time interval, $\Delta \tau$, the propagator has a simpler functional dependence $\simeq \exp(-im\Delta \tau)$ where $m$ is the pole mass of the particle and the proper time interval is defined by the relations:

$$
\Delta \tau \equiv \sqrt{\Delta t^2 - \Delta x^2} \quad \text{for} \quad \Delta t^2 \geq \Delta x^2 \\
\Delta \tau \equiv -i\sqrt{\Delta x^2 - \Delta t^2} \quad \text{for} \quad \Delta x^2 > \Delta t^2
$$

For space-like separations: $\Delta x^2 > \Delta t^2$ appropriate for the virtual photons mediating the Coulomb force, $\Delta \tau$ is imaginary. This would imply an exponentially damped range of the associated force for the exchange of a massive particle\textsuperscript{19}. Since, however, the pole mass of the photon vanishes, no such damping occurs for the exchange of virtual photons. The corresponding force law is then the same as for the exchange of real (‘on-shell’) particles, that is, inverse square.

It is instructive to compare Feynman’s own discussion of the virtual photon propagator in space-time \[41\] to the related one of the invariant amplitude for Møller scattering in Section 6 above. Feynman writes out explicitly the 4-vector product in Eqn(6.1) to obtain:

$$
T_{fi} = -i \int \frac{(\mathcal{J}^A_0 \mathcal{J}^B_0 - \mathcal{J}^A_1 \mathcal{J}^B_1 - \mathcal{J}^A_2 \mathcal{J}^B_2 - \mathcal{J}^A_3 \mathcal{J}^B_3)}{q^2} d^4x_A \quad (7.2)
$$

Conservation of the current $\mathcal{J}$ gives the condition:

$$
q \cdot \mathcal{J} = q^0 \mathcal{J}^0 - |\vec{q}| \mathcal{J}^3 = 0 \quad (7.3)
$$

where the 3 axis has been chosen parallel to $\vec{q}$. Use of (7.3) to eliminate $\mathcal{J}^A_3$ and $\mathcal{J}^B_3$ enables (7.2) to be written as:

$$
T_{fi} = i \int \left[ \frac{(\mathcal{J}^A_1 \mathcal{J}^B_1 + \mathcal{J}^A_2 \mathcal{J}^B_2)}{q^2} + \frac{\mathcal{J}^A_0 \mathcal{J}^B_0}{q^2} \right] d^4x_A \quad (7.4)
$$

Feynman then performs a Fourier transform of $(\vec{q})^{-2}$ using Eqn(6.5) to obtain, for the last term in the large square bracket of Eqn(7.4) an equation similar to (6.6) above, but with the replacement: $\mathcal{J}^A \cdot \mathcal{J}^B \rightarrow \mathcal{J}^A_0 \mathcal{J}^B_0$. The instantaneous nature of the Coulomb interaction in this term is noted, but it is also implied that the contribution of the transverse polarisation modes: $(\mathcal{J}^A_1 \mathcal{J}^B_1 + \mathcal{J}^A_2 \mathcal{J}^B_2)/q^2$ is not instantaneous. Feynman stated:

The total interaction which includes the interaction of transverse photons then gives rise to the retarded interaction.

This statement is not true when $T_{fi}$ is evaluated in the CM frame. In this case: $q^2 = -q^0^2$, Eqn(6.6) results and the whole interaction of the virtual photon is instantaneous.

\textsuperscript{18}For example in Reference [26] it is stated, in connection with the commutation relation for a pair of scalar fields (Eqn(3.55) of [26]) that: ‘Measurements at space time separated points do not interfere as a consequence of locality and causality’, whereas in the discussion of the Feynman propagator $G_F(x)$ in Section 1.3.1 it is stated that: ‘While the previous Green functions were zero outside the light cone this is not the case for $G_F(x)$ which has an exponential tail at negative $x^2$.’ The $G_F(x)$ discussed here is that corresponding to a classical field, but the corresponding quantum propagator, $S_F(x)$, has a similar property \[39, 40\].

\textsuperscript{19}For example the Yukawa force due to the exchange of virtual pions in nuclear physics.
Figure 2: Momentum space diagrams for Möller scattering of ultra-relativistic electrons by $\pi/4$ radians in the CM frame, as in Fig 1 b) and c), as viewed by different observers. In a) the observer is moving parallel to $\vec{p}_i$ with velocity $3c/5$ relative to the CM frame. Momentum conservation requires that in a) the virtual photon propagates from $J^A$ to $J^B$ so that $t_B > t_A$, whereas in b) the photon propagates from $J^B$ to $J^A$ and $t_A > t_B$, where $t_A$ and $t_B$ are the effective times of emission or absorption of the photon by the currents $J^A$ and $J^B$. In both cases the effective velocity $v = pc^2/E$ of the photon is superluminal: $v = 1.044c$. 

$\gamma_A$, $\gamma_B$
It is amusing to note that a faint ‘ghost’ of Wheeler and Feynman’s ‘advanced’ and ‘retarded’ potentials subsists in the momentum space diagrams Fig.1b and 1c. The two kinematically distinct situations (i) a virtual photon with momentum $\vec{q}^A$ propagates from current A to current B (Fig1b) and (ii) a virtual photon with momentum $\vec{q}^B = -\vec{q}^A$ propagates from current B to current A (Fig1c) are completely equivalent descriptions of the scattering process in the CM frame where $t_A = t_B$. As shown in Fig 2, however, this is no longer the case if the scattering process is observed in a different inertial frame. In Fig 2a the observer is moving with relativistic velocity $\beta = 3/5$ parallel to the direction of $\vec{p}_B^A$ in the CM frame. Thus in the observer’s proper frame, $|\vec{p}_B^A|$ is halved and $|\vec{p}_A^B|$ doubled. In Fig2b, the observer moves with the same velocity relative to the CM frame, parallel to $\vec{p}_B^A$. In both cases it follows from momentum conservation that there is no possible ambiguity between the momentum space configurations shown in Figs 2a and 2b. In Fig 2a the virtual photon must propagate from current A to B and so $t_B > t_A$, and in Fig 2b from current B to A so that $t_A > t_B$. Assuming that the electrons are ultrarelativistic, $E \simeq |\vec{p}|$, and that the electrons scatter through $\pi/4$ rad in the CM frame, as shown in Fig 1, the relativistic velocity of the virtual photon in the observer’s frame is $\beta = 1.044$ for both cases shown in Fig 2. Thus the causal description of scattering processes in momentum space is, in general, frame dependent, being ambiguous only in the CM frame.20 Because of this ambiguity, in the kinematical configuration of Fig 1b, the virtual photon $\gamma_A$ can be considered as the limit as $t_B - t_A \rightarrow 0$ of an ‘retarded’ interaction from A as seen by B, whereas in Fig 1c $\gamma_B$ corresponds to the limit as $t_B - t_A \rightarrow 0$ of an ‘advanced’ interaction produced by B that interacts with A 21. Since the two descriptions are equivalent, the effect is the same as the $t_B - t_A \rightarrow 0$ limit of half the sum of the retarded interaction produced by the current B and the ‘advanced’ interaction produced by the current A. This is the ‘ghost’ of Feynman and Wheeler’s advanced and retarded potentials mentioned above. The current A (B) behaves as the ‘absorber’ for the interactions of the current B (A). Unlike in Feynman and Wheeler’s formulation however there is no radiation and therefore no ‘radiation resistance’. The photons responsible for the intercharge interaction are purely virtual.

As often emphasised by Feynman [42], QED is based on only three elementary amplitudes describing, respectively, the propagation of electrons or photons from one space-time point to another and the amplitude for an electron to absorb or emit a photon. The latter is proportional to the classical electric charge of the electron. Since only kinematics, and not the coupling constant of QED, changes when virtual photons are replaced by real ones it should not be surprising if the various field concepts introduced to describe the effect of the virtual photons that generate intercharge forces should also be able to provide a description of the observed effects of the creation and absorption of real photons. As will now be shown, this is indeed the case.

20It must not be forgotten, however, that the configurations shown in Fig.2 are in momentum space, not space-time. The different time ordering of ‘events’ in different frames that seems apparent on comparing Fig.2a and Fig.2b must therefore be treated with caution. In fact, as discussed previously, there is not, in space-time, the exchange of single virtual photons with fixed 4-momenta, as seen in Figs 1 and 2, but rather the sum over an infinite number of amplitudes corresponding to exchanges of virtual photons between all space-time points occupied by the trajectories of the scattered particles, as in Eqn(6.6). In the CM frame all such photons have infinite velocity.

21This corresponds to time increasing from left to right in the momentum space diagrams of Fig.1b and 1c, in the same way as in Fig.2a or the Feynman diagram in Fig.1a.
A clear distinction should be made however, at the outset, between the fields so far discussed in the present paper, representing the effects of virtual photon exchange, and the related fields denoted here as $A_{\text{rad}}$, $E_{\text{rad}}$ and $B_{\text{rad}}$ that provide a description of physical systems comprised of large numbers of real photons\textsuperscript{22}. As shown in a recent paper by the present author\textsuperscript{43}, a complex representation of these radiation fields may be identified, in the limit of very low photon density, with the quantum wavefunction of a single real photon\textsuperscript{23}.

The electrodynamic Maxwell equation (4.20) as written above therefore describes only the effects of virtual photon exchange. All fields and currents are defined at some unique time in the CM frame of the interacting charges. The solutions of this equation, $E$, $B$ are given by Eqns(3.11), (3.12) respectively and (3.1). To arrive at a description of real photons it is convenient to express the electrodynamic Ampère Law of Eqn(4.20) uniquely in terms of the 3-vector potential by using the Lorenz Condition (4.12) to eliminate the scalar potential $A^0$. The result of this simple exercise in 3-vector algebra, which may be found in any text-book on CEM, is:

$$-\nabla^2 A_{\text{rad}} + \frac{1}{c^2} \frac{\partial^2 A_{\text{rad}}}{\partial t^2} = 4\pi \vec{j}_{\text{rad}} \tag{7.5}$$

The ‘radiation’ suffix has been added to $\vec{A}$ and $\vec{j}$ to distinguish them from the quantities $\vec{A}$ and $\vec{j}$ defined in Eqns(3.1) and (4.8) since the latter are not solutions of Eqn(7.5) unless $c$ is infinite. Similarly by using the Lorenz condition to eliminate $\vec{A}$ in favour of $A^0$ the inhomogeneous D’Alembert equation for the scalar potential may be derived:

$$-\nabla^2 A_{\text{rad}}^0 + \frac{1}{c^2} \frac{\partial^2 A_{\text{rad}}^0}{\partial t^2} = 4\pi j_{\text{rad}}^0 \tag{7.6}$$

As shown for example in Reference [46], the solutions of Eqn(7.5) and (7.6) are similar to (3.1) except that they are retarded in time:

$$A_{\text{rad}}(t) = \left\lbrace \frac{\vec{j}}{c(r - \frac{\vec{v} \cdot \vec{r}}{c})} \right\rbrace_{t - \frac{r}{c}} \tag{7.7}$$

$$A_{\text{rad}}^0(t) = \left\lbrace \frac{j^0}{c(r - \frac{\vec{v} \cdot \vec{r}}{c})} \right\rbrace_{t - \frac{r}{c}} \tag{7.8}$$

where the large curly bracket indicates that $\vec{j}$ and $r$ are evaluated at the retarded time $t - r/c$. It follows that $A_{\text{rad}}(t)$ and the associated electromagnetic fields $E_{\text{rad}}(t)$ and $B_{\text{rad}}(t)$ describe some physical effect produced by the source current at time $t - r/c$, i.e. that propagates from the source to the point of observation with velocity $c$. In reality, the energy-momentum flux, associated with the corresponding ‘electromagnetic wave’ produced by the source, consists of a very large number of real photons whose energy distribution depends on the acceleration of the source at their moment of emission. Thus the solutions (7.7) and (7.8) imply the existence of massless physical objects (‘photons’)\textsuperscript{47, 48}, created by the source current. As discussed in Reference [43], comparison

\textsuperscript{22}This distinction is usually not made in text books on CEM

\textsuperscript{23}This wavefunction occurs for example, in the construction of invariant amplitudes of all processes in which real photons are created or destroyed. The related problem of ‘non localisability’ of photons is also discussed in Reference [43].
of the known properties of both photons and the classical electromagnetic waves associated with the fields $A_{rad}$, $E_{rad}$ and $B_{rad}$ enables many fundamental concepts of quantum mechanics to be understood in a simple way.

Text books and papers on CEM do not usually make the above distinction between the fields $\vec{E}$ and $\vec{B}$, describing the mechanical forces acting on charges, and $\vec{E}_{rad}$ and $\vec{B}_{rad}$ that provide the classical description of radiation phenomena, employing identical symbols for both types of fields. An important exception to this is the work of Reference [44]. In this paper, the instantaneous nature of the interactions mediated by the $\vec{E}$ and $\vec{B}$ fields, derived in the previous section from QED, is conjectured. These fields are solutions of the Maxwell equations: (4.4), (4.6), (4.7) and (4.20). Different, retarded, fields, solutions of the D’Alembert equation, equivalent to $\vec{E}_{rad}$ and $\vec{B}_{rad}$, denoted as $\vec{E}^*$ and $\vec{B}^*$ were also introduced. The application of the Poynting vector and spatial energy density formulae uniquely to the fields $\vec{E}^*$ and $\vec{B}^*$ was pointed out. However, instead of the formulae (7.7) and (7.8) above, only ‘sourceless’ solutions of the homogeneous D’Alembert equation were considered. Also it was proposed, instead of the formulae (3.15) and (3.16) above, to define $\vec{E}$ and $\vec{B}$ as the standard ‘present time’ Liénard and Wichert formulae24 which, for a uniformly moving charge, are actually equivalent to retarded fields. The discussion of Reference [44]. was carried out entirely at the level of classical fields, considered as solutions of partial differential equations with certain boundary conditions. No identification of $\vec{E}$ and $\vec{B}$ with the exchange of virtual photons and $\vec{E}^*$ and $\vec{B}^*$ as the classical description of real photons was made. The suggestion that $\vec{E}$ and $\vec{B}$ should be associated with exchange of virtual photons ‘not subject to causal limitations’ has, however, been made in a recent paper [45].

The electric and magnetic fields derived from the Liénard and Wiechert potentials (7.7) and (7.8) contain terms with both $1/r^2$ and $1/r$ dependencies. Both fields are retarded, but conventionally only the latter are associated with radiative effects (the fields $\vec{E}_{rad}$ and $B_{rad}$) in CEM. It is interesting to note that there is now mounting experimental evidence [49, 50], that the fields $\approx 1/r^2$ are instantaneous and not retarded, and so should be associated with the force fields $\vec{E}$ and $\vec{B}$ mediated by virtual photon exchange. Particularly convincing are the results shown in Reference [50] where the temporal dependence of near- and far-magnetic fields were investigated by measuring electromagnetic induction at different distances from a circular antenna. Figure 8 of [50]apparently shows clear evidence for the instantaneous nature of the $1/r^2$ ‘bound fields’ (i.e. fields associated with virtual photon exchange). This suggests that the retarded $1/r^2$ solutions of (7.5) and (7.6) should be discarded as unphysical, whereas the retarded $1/r$ solutions describing correctly the ‘far-field’ in the experiment [50] do give the correct classical description of the radiation of real photons. There seems now to be therefore experimental evidence for electromagnetic fields respecting both Neumann causality (the force fields $\vec{E}$ and $\vec{B}$) as well as Gaussian causality (the radiation fields $\vec{E}_{rad}$ and $\vec{B}_{rad}$).

Maxwell’s original discovery of electromagnetic waves [51] was based on an equation similar to (7.5) for components of the electromagnetic fields, but without any source term, which is just the well-known classical Wave Equation in three spatial dimensions Although this procedure leads, in a heuristic manner, to the concept of ‘electromagnetic waves’

24See, for example, Reference [8].
propagating at speed \( c \), with vast practical, political and sociological consequences, it can be seen, with hindsight, to have been a mistake from the viewpoint of fundamental physics. In fact, if the current vanishes, so, by definition, do all the fields whether instantaneous as in Eqn(3.1) or retarded as in Eqn(7.5). If all the fields vanish there can evidently be no ‘waves’. The result of this mistake was many decades of fruitless work by Maxwell and others to invent a medium (the luminiferous aether) in which such ‘sourceless’ waves might propagate and whose properties would predict the value of \( c \). Now it is understood that the energy density \( (\vec{E}_\text{rad}^2 + \vec{B}_\text{rad}^2)/8\pi \) of a plane ‘electromagnetic wave’ is simply that of the beam of real photons of which it actually consists [43].

The existence of photons, massless particles with constant velocity \( c \), is predicted by Eqn(7.5) that necessarily follows from Eqns(4.12) and (4.20). These in turn may be derived from the Lagrangian (2.7) and Hamilton’s Principle. It is then interesting to ask where the constant ‘\( c \)’ was introduced into the derivation. The answer is Eqn(2.2), the definition of 4-vector velocity. The same formula contains, implicitly, the information that a massless particle has the constant velocity, \( c \), that is used to identify the ‘electromagnetic wave’, with velocity \( c \) predicted by Eqns(7.5) and (7.6), with the propagation of the massless real photons produced by the source.

The only dynamical assumption in the derivation of CEM presented above is Coulomb’s Law. If it is explained in QED as an effect due to virtual photon exchange, it also seems to require via Eqns(7.5) and (7.6), the existence of real, massless, photons. Although clearly of interest, the further study of the relationship between CEM and QED for radiative processes is, as stated earlier, beyond the scope of the present paper.

In conclusion, the results obtained in the present paper are compared with those of the similarly motivated project of Feynman and Wheeler. The latter made the following general comments on their approach [34]:

(1) There is no such concept as ‘‘the’’ field, an independent entity with degrees of freedom of its own.

(2) There is no action of an elementary charge upon itself and consequently no problem of an infinity in the energy of the electromagnetic field.

(3) The symmetry between past and future in the prescription of the fields not a mere logical possibility, as in the usual theory, but a postulational requirement.

The statements (1) and (2) remain true in the approach described in Sections 2-4 above. However the writer’s opinion is that the ‘infinite self energy’ problem of CEM is really an artifact of the possibly unphysical concept of a ‘point charge’ rather than a shortcoming of the classical electromagnetic field concept per se. That being said, it remains true that the virtual photons interacting with a given charge are produced by other charges so there is no way for the charge to ‘interact with itself’. If the energy of the ‘electromagnetic field’ is identified with that of the exchanged virtual photons in the CM frame, it vanishes, so, there is, as in (2) above, certainly no self energy problem. However, the statements (1)and (2) are only applicable to the ‘force’ fields introduced
in Eqns(3.1), (3.10) and (3.12) above, that may be denoted as \( A_{\text{for}} \), \( \vec{E}_{\text{for}} \) and \( \vec{B}_{\text{for}} \) to
distinguish them from the ‘radiation’ fields describing real photons. It is important to
reiterate that the definitions and physical meanings of these two types of fields are quite
distinct. The quantity: \( (\vec{E}_{\text{for}}^2 + \vec{B}_{\text{for}}^2)/8\pi \) does not correctly describe the energy density
of the electromagnetic field associated with virtual photons, and, in contradiction to (1),
extra degrees of freedom must be added to the Lagrangian to correctly describe real
photons. No distinction was made between real and virtual photons by Feynman and
Wheeler. In the approach of the present paper, point (3) with its introduction of acausal
‘advanced’ potentials is no longer valid. It was a consequence of Feynman and Wheeler’s
taking Gaussian Causality as an axiom. The latter is true, as shown by Eqn(7.7) and
(7.8), for any interaction transmitted by real photons (i.e. for the fields \( A_{\text{rad}}, \vec{E}_{\text{rad}} \) and
\( \vec{B}_{\text{rad}} \)) but not, as shown in Section 5 above, for the force fields describing the effects of
the exchange of space-like virtual photons. These are always tachyonic (as in Fig.2) and
may be instantaneous (as in Fig1b and c) but do not, unlike ‘advanced potentials’, violate
causality. Feynman and Wheeler’s mistake, the same as that of many previous authors,
was to try to describe the physical effects of virtual photon exchange by fields respecting
Gaussian, instead of Neumann, Causality.

It is instructive to compare the discussion of CEM in the present paper with that
of Reference [52], which also takes as fundamental physical assumptions, in constructing
the theory, special relativity and Hamilton’s Principle. However in Reference [52], the
existence of the 4-vector potential \( A \) and the relativistic Lagrangian equivalent to (3.2)
above are both postulated \textit{a priori}. This procedure is justified by the statement [53]:

The assertions which follow should be regarded as being, to a certain
extent, the consequence of experimental data. The form of the action for
a particle in an electromagnetic field cannot be fixed on the basis of
general considerations alone (such as, for example the requirement of
relativistic invariance).

This is true, as far as it goes, but fails to take account of either the constructive
principle put forward in the quotation from Hagedorn cited above, or the known essential
physics of the problem embodied in the inverse-square force law between charges in the
static limit. As demonstrated in Section 2 above, the assumption of this law, together with
the classical definition of potential energy and relativistic invariance is in fact sufficient to
derive just the Lagrangian that is assumed \textit{a priori} in [52]. The derivations of the Lorentz
force equation and the covariant definitions of electric and magnetic fields (3.11) and
(3.12) given in [52] are identical to those presented above, as are also the derivations of
the magnetostatic Maxwell equation and the Faraday-Lenz law. In Chapter 3 of [52] there
is a lengthy discussion of the motion of particles in magnetic fields. However at this point
the magnetic field is a purely abstract mathematical concept. How it may be obtained
from its sources — charges in motion — has still not been even mentioned! Only after the
electrostatic and electrodynamic Maxwell equations have been derived in Chapter 4 from
the principle of least action, by treating the electromagnetic fields as ‘co-ordinates’, is the
relation between fields and their sources established. Coulomb’s law is then derived at the
beginning of Chapter 5 (page 100!) from the Poisson equation. In contrast, in the present
paper, Coulomb’s law (and hence the Poisson equation) is assumed at the outset, and the
electromagnetic Maxwell equation is derived, simply by inspection, from the covariant
form of Poisson’s equation. At this point identical results have been obtained from the same essential input (the Lagrangian (3.2)) by the present paper and [52]. However the present writer feels that there are enormous pedagogical advantages, (especially in view of the crucial role of Coulomb’s law in QED, discussed above) to start the discussion with the vital experimental fact – the inverse-square force law – rather than to derive it after 100 pages of complicated mathematics, as is done in [52]. Also, in [52] no distinction is made between $A_{for}$, $E_{for}$, $B_{for}$ and $A_{rad}$, $E_{rad}$, $B_{rad}$. All fields are assumed to be derived from the same, non-relativistic, retarded, Liénard and Wichert potentials.

Finally the approach of the present paper may be compared with that of another recent paper by the present author [14] in which the Lorentz Force Law, magnetic field concept and the Faraday-Lenz Law are derived from a different set of postulates. The electrostatic definition of the electric field $\vec{E}_{\text{stat}} = -\nabla V$ is first generalised to the covariant form of Eqn(3.11) above by imposing space-time exchange symmetry invariance [18]. The magnetic field concept and the Lorentz Force Law are then shown to follow from the covariance of Eqn(3.11), and the derivation of the Faraday-Lenz law is identical to that given above. Neither Coulomb’s Law nor Hamilton’s Principle were invoked in this case, demonstrating the robustness of some essential formulae of CEM to the choice of axioms for their derivation. Another example of this is provided by Reference [52] where Coulomb’s law is derived from the principle of least action and the relativistic Lagrangian (3.2), as initial postulates.

Acknowledgement

I thank B.Echenard and P.Enders for their comments on this paper. Pertinent and constructive critical comments by an anonymous referee have enabled me to simplify, or improve, the presentation in several places. They are gratefully acknowledged.
Appendix

Factoring out the space-time dependent factor in the transition current $J_\mu$ according to the definition

$$J_\mu = -e\pi\gamma_\mu u \exp[i(p_f - p_i) \cdot x]$$

$$\equiv \tilde{J}_\mu \exp[i(p_f - p_i) \cdot x] \quad (A1)$$

enables the invariant amplitude $T_{fi}$ of Eqn(6.4) to be written as:

$$T_{fi} = i \int dt_A \int d^3x_A \frac{\tilde{J}^A e^{i(p_f^A - p_i^A) \cdot x_A} \cdot \tilde{J}^B e^{i(p_f^B - p_i^B) \cdot x_A}}{|q^2|}$$

$$= i \int dt_A \int d^3x_A \frac{\tilde{J}^A e^{i(p_f^A - p_i^A) \cdot t_A} \cdot \tilde{J}^B e^{i(p_f^B - p_i^B) \cdot t_A}}{|q^2|} \quad (A2)$$

since, from momentum conservation:

$$p_f^A - p_i^A = -(p_f^B - p_i^B) \quad (A3)$$

Using now Eqn(6.5) gives

$$T_{fi} = i \int dt_A \int d^3x_A \tilde{J}^A e^{i(p_f^A - p_i^A) \cdot t_A} \cdot \tilde{J}^B e^{i(p_f^B - p_i^B) \cdot t_A} \int e^{i\vec{q} \cdot \vec{x}} d^3x \quad (A4)$$

Making the change of variables:

$$\vec{x} = \vec{x}_B - \vec{x}_A, \quad d^3x = d^3x_B$$

and noting that $\vec{q} = \vec{p}_i^B - \vec{p}_f^A$ gives, from Eqn(A4):

$$T_{fi} = i \int dt_A \int d^3x_A \tilde{J}^A e^{i(p_f^A - p_i^A) t_A} \cdot \tilde{J}^B e^{i(p_f^B - p_i^B) t_A} \int \frac{e^{i\vec{q} \cdot \vec{x}} d^3x}{4\pi|\vec{x}|} \quad (A5)$$

Now

$$(\vec{p}_f^A - \vec{p}_i^A) \cdot (\vec{x}_B - \vec{x}_A) = -(\vec{p}_f^B - \vec{p}_i^B) \cdot \vec{x}_B - (\vec{p}_f^A - \vec{p}_i^A) \cdot \vec{x}_A \quad (A6)$$

where Eqn(A3) has been used. Substituting (A6) into (A5), yields Eqn(6.6) of the text.
References

[1] I.Newton, ‘Philosophiae naturalis principia mathematica’, 1687. English translation by A.Motte in ‘On the Shoulders of Giants’, Ed S.W.Hawking, (Running Press, Philadelphia, 2002) P1038.

[2] H.Goldstein, ‘Classical Mechanics’, (Addison-Wesley, Massachusetts, 1959) Chapter 2.

[3] R.Hagedorn, ‘Selected Topics in Scattering Theory, Part I, Relativistic Kinematics and Precession of Polarisation ’ CERN Yellow Report: CERN 62-18 1962, P27.

[4] V.Bargmann, L.Michel and V.Telegdi Phys. Rev. Lett. 2 435 (1959).

[5] Reference [2] above, Section 3.1, P58.

[6] Reference [2] above. Eqn(6-57).

[7] O.Heaviside, The Electrician, 22 1477 (1888).

[8] W.H.Panofsky and M.Phillips, ‘Classical Electricity and Magnetism’ (Addison-Wesley, Cambridge Mass, 1955) Ch 18. Eqns(18-14) and (18-15).

[9] A.Liénard, L’Eclairage Electrique, 16 pp5, 53, 106 (1898), E.Wiechert, Archives Néerland (2) 5 459 (1900).

[10] J.H. Field, ‘Inter-charge forces in relativistic classical electrodynamics: electromagnetic induction in different reference frames’, http://xxx.lanl.gov/abs physics/0511014.

[11] A.Einstein, Annalen der Physik 17, 891 (1905).

[12] F.Wilczek, Physics Today, October 2004 P11, December 2004 P10.

[13] J.H.Field, ‘Forces between electric charges in motion: Rutherford scattering, circular Keplerian orbits, action-at-a-distance and Newton’s third law in relativistic classical electrodynamics’, http://xxx.lanl.gov/abs/physics/0507150.

[14] J.H.Field, Phys. Scr. 73 639 (2006).

[15] J.D.Jackson, ‘Classical Electrodynamics’, (John Wiley and Sons, New York, 1975), Chapter 1, Sections 1.3 and 1.4, P30.

[16] J.D.Jackson and L.B.Okun, Rev. Mod. Phys. 73 663 (2001).

[17] M.Schwartz, ‘Principles of Electrodynamics’, (McGraw-Hill, New York, 1972). Chapter 3.

[18] J.H.Field, Am. J. Phys. 69 569 (2001).

[19] A.Einstein, Annalen der Physik, 49 769 (1916).

[20] A.Einstein, ‘The Meaning of Relativity’, (Princeton University Press 1956) P82. Also partially quoted as an introduction to Reference [34].
[21] G.Holton, Am. J. Phys. 24 340 (1956).

[22] F.Halzen and A.D.Martin, ‘Quarks and Leptons: an Introductory Course in Modern Particle Physics’, (John Wiley and Sons, New York, 1984) Eqn(6.96), P140.

[23] Reference [22] above, Eqn(5.46) P110.

[24] R.P.Feynman, Rev. Mod. Phys. 20 367 (1948), Eqn(38).

[25] R.P.Feynman and A.R.Hibbs, ‘Quantum Mechanics and Path Integrals’, (McGraw Hill, New York, 1965). Section 3-7, P65.

[26] G.Itzykson and J.B.Zuber, ‘Quantum Field Theory’, (McGraw-Hill, New York, 1984) Eqn(2-54) P60.

[27] P.A.M.Dirac, Physikalische Zeitschrift der Sowjetunion, Heft 1 (1933). Reprinted in ‘Selected Papers on Quantum Electrodynamics’, Ed J.Schwinger, (Dover, New York, 1958) P312.

[28] P.A.M.Dirac, The Principles of Quantum Mechanics’, Fourth Edition (O.U.P., London, 1958) Chapter V, Section 32.

[29] See, for example I.B.Cohen and R.S.Westfall, ‘Newton’, (W.W.Norton Company, New York, 1995) p118.

[30] Reference [29] above, P224.

[31] ‘Selected Papers of Richard Feynman’, Ed L.M.Brown, (World Scientific, Singapore, 2000) P11.

[32] Reference [31] P33.

[33] J.A.Wheeler and R.P.Feynman, Rev. Mod. Phys. 17 157 (1945).

[34] J.A.Wheeler and R.P.Feynman, Rev. Mod. Phys. 21 425 (1949).

[35] P.A.M.Dirac, Proc. Roy. Soc. A 167 148 (1938).

[36] A.D.Fokker, Zeit. Phys. 58 386 (1929): Physica 9 33 (1925) 12 145 (1932).

[37] C.F.Gauss, Werke 5, 629 (1867), quoted at length in Reference [34].

[38] C.Neumann, ‘Principles of Electrolysis’, Tübingen 1863; Mathematishe Annalen i.317.

[39] R.P.Feynman, Phys. Rev. 76 749 (1949).

[40] R.P.Feynman, ‘Quantum Electrodynamics’, (W.A.Benjamin, New York, 1962) P85.

[41] R.P.Feynman, ‘Theory of Fundamental Processes’, (W.A.Benjamin, New York, 1962) Ch 20.

[42] R.P.Feynman, ‘QED The Strange Theory of Light and Matter’, (Princeton University Press, 1985) P85.

[43] J.H.Field, Eur J.Phys. 25 385 (2004).
[44] A.E.Chubykalo and R.Smirnov-Rueda, Phys. Rev. E53 5373 (1996).

[45] A.L.Kholmteskii, 'On momentum and energy of a non-radiating electromagnetic field', arXiv pre-print: physics/0501148.

[46] Reference [15] above, Section 6.6, P223.

[47] J.M.Levy-Leblond, Am. J. Phys. 44 271 (1975).

[48] J.H.Field, Helv. Phys. Acta. 70 542 (1997).

[49] W.D.Walker, ‘Superluminal Electromagnetic and Gravitational Fields Generated in the Nearfield of Dipole Sources’, arXiv pre-print: physics/0603240, and references therein.

[50] A.L.Kholmetskii et al. ‘Experimental Existence on Non-Applicability of the Standard Retarding Condition to Bound Magnetic Fields and on New Generalised Biot-Savart Law’, arXiv pre-print: physics/0601084.

[51] J.C.Maxwell, ‘A Treatise on Electricity and Magnetism’, 1891. (Dover Publications Inc, New York, 1954) Vol II, Section 784.

[52] L.D.Landau and E.M.Lifshitz ‘Classical Theory of Fields’, Translated by M.Hamermesh, (Pergamon Press, Oxford, 1962).

[53] Reference [52] above, Ch 3, Section 16, P49.