NETWORKS OF REINFORCED STOCHASTIC PROCESSES: 
A COMPLETE DESCRIPTION OF THE FIRST-ORDER ASYMPTOTICS

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Abstract. We consider a finite collection of reinforced stochastic processes with a general network-based interaction among them. We provide sufficient and necessary conditions in order to have some form of almost sure asymptotic synchronization, which could be roughly defined as the almost sure long-run uniformization of the behavior of interacting processes. Specifically, we detect a regime of complete synchronization, where all the processes converge toward the same random variable, a second regime where the system almost surely converges, but there exists no form of almost sure asymptotic synchronization, and another regime where the system does not converge with a strictly positive probability. In this latter case, partitioning the system in cyclic classes according to the period of the interaction matrix, we have an almost sure asymptotic synchronization within the cyclic classes, and, with a strictly positive probability, an asymptotic periodic behavior of these classes.

Key-words: interacting random systems, network-based dynamics, reinforced stochastic processes, urn models, spectral theory, synchronization, polarization, opinion dynamics.

1. Introduction

The random evolution of systems composed by agents who interact among each other has always been of great interest in several scientific fields. For example, many studies in Biology focus on the interactions between different sub-systems. Moreover, Economics and Social sciences deal with agents that take decisions under the influence of other agents. In social life, opinions are partly transmitted by means of various forms of social interaction and are driven by the tendency of individuals to become more similar when they interact (e.g. [1, 28, 36]). Sometimes, a collective behavior, usually called “synchronization”, reflects the result of the interactions among different individuals (we refer to [8] for a detailed and well structured survey on this topic, rich of examples and references).

In particular, in mathematical literature, there exists a growing interest in systems of interacting urn models (see the subsection below devoted to a literature review). The present work is placed in the stream of this scientific literature, which is mainly motivated by the attempt of understanding the role of the reinforcement mechanism in synchronization phenomena.

Specifically, the present work deals with the class of the so-called interacting reinforced stochastic processes introduced in [2, 13]. Generally speaking, by (self-)reinforcement in a stochastic dynamics we mean any mechanism for which the probability that a given event occurs has an increasing dependence on the number of times that the same event occurred in the past. This “reinforcement mechanism”, also known as “preferential attachment rule” or “Rich get richer rule” or “Matthew effect”, is a key feature governing the dynamics of many random phenomena in different scientific areas, such as Biology, Economics and Social sciences (see e.g. [31] for a general survey). Formally, in [2], it is given the following definition.

Definition 1.1. A Reinforced Stochastic Process (RSP) is a sequence $X = (X_n)_{n \geq 1}$ of random variables with values in $\{0, 1\}$ such that the predictive mean

$$Z_n = E[X_{n+1}|Z_0, X_1, \ldots, X_n] = P(X_{n+1} = 1|Z_0, X_1, \ldots, X_n), \quad n \geq 0,$$
satisfies the dynamics
\[ Z_{n+1} = (1 - r_n)Z_n + r_nX_{n+1}, \quad \text{where } 0 < r_n < 1, \]
and \( Z_0 \) is a random variable with values in \([0, 1]\).

We can image that the process \( X \) describes a sequence of actions along the time-steps and, if at time-step \( n \), the “action 1” has taken place, i.e. \( X_n = 1 \), then for “action 1” the probability of occurrence at time-step \((n + 1)\) increases. Therefore, the larger \( Z_{n-1} \), the higher the probability of having \( X_n = 1 \), and so the higher the probability of having \( Z_n \) greater than \( Z_{n-1} \). As a consequence, the larger the number of times in which \( X_k = 1 \) with \( 1 \leq k \leq n \), the higher the probability \( Z_n \) of observing \( X_{n+1} = 1 \).

For example, if we consider the sequence \((X_n)_n\) of extractions associated to a time-dependent Pólya urn (see [30, 35]), with \( \alpha_n > 0 \) the number of balls (of the color equal to the extracted one) added to the urn at each time-step, and \( s_0 \) the initial number of balls in the urn, the proportion of balls of the color, say \( A \), associated to the value 1 for \( X_n \), is
\[ Z_n = (Z_0 + \sum_{k=1}^{n} \alpha_k X_k) / (s_0 + \sum_{k=1}^{n} \alpha_k) \]
and so
\[ Z_{n+1} = (1 - r_n)Z_n + r_nX_{n+1}, \quad \text{with } r_n = \frac{\alpha_{n+1}}{s_0 + \sum_{k=1}^{n+1} \alpha_k}. \]
(The standard Eggenberger-Pólya urn [17, 26] corresponds to \( \alpha_n = \alpha \) for each \( n \)). Therefore, \((X_n)_n\) is a RSP. But, it is also true the converse (see [5] for the details). Therefore, the two notions are equivalent from a purely mathematical aspect, although the dynamics of a RSP in Definition 1.1 has the merit to highlight the key role played by the sequence \((r_n)_n\). In the sequel, we will refer to \((r_n)_n\) as the reinforcement sequence, since it regulates the reinforcement in the dynamics of \((Z_n)_n\), that is the weight of the “new information” (i.e. \( X_{n+1} \)) used to define the new status of the process (i.e. \( Z_{n+1} \)). As we will see, the reinforcement sequence will be pivotal in the obtained results.

In the present work we are interested in a system of \( N \geq 2 \) reinforced stochastic processes that interact according to a given set of relationships among them. More precisely, we suppose to have a finite directed graph \( G = (V, E) \), with \( V = \{1, \ldots, N\} \) as the set of vertices and \( E \subseteq V \times V \) as the set of edges. Each edge \((l_1, l_2) \in E\) represents the fact that the vertex \( l_1 \) has a direct influence on the vertex \( l_2 \). We also associate a weight \( w_{l_1,l_2} \geq 0 \) to each pair \((l_1, l_2) \in E \) in order to quantify how much \( l_1 \) can influence \( l_2 \). A weight equal to zero means that the edge is not present. We define the matrix \( W \), called in the sequel interaction matrix, as \( W = [w_{l_1,l_2}]_{l_1,l_2 \in V \times V} \) and we assume the weights to be normalized so that \( \sum_{l_1=1}^{N} w_{l_1,l_2} = 1 \) for each \( l_2 \in V \). Hence, \( w_{l,l} \) represents how much the vertex \( l \) is influenced by itself and \( \sum_{l_1=1, l_1 \neq l}^{N} w_{l_1,l} \in [0, 1] \) quantifies how much the vertex \( l \) is influenced by the other vertices of the graph. Finally, we suppose to have at each vertex \( l \) a reinforced stochastic process described by \((X_{n,l})_{n \geq 1} \) such that, for each \( n \geq 0 \), the random variables \( \{X_{n+1,l} : l \in V\} \) take values in \([0, 1]\) and are conditionally independent given \( F_n \) with
\[ P(X_{n+1,l} = 1 | F_n) = \sum_{l_1=1}^{N} w_{l_1,l} Z_{n,l_1}, \]
where, for each \( l \in V \),
\[ Z_{n,l} = (1 - r_{n-1})Z_{n-1,l} + r_{n-1}X_{n,l}, \]
with \( 0 \leq r_n < 1 \), \( \{Z_{0,l} : l \in V\} \) random variables with values in \([0, 1]\) and \( F_n = \sigma(Z_{0,l} : l \in V) \cup \sigma(X_{k,l} : 1 \leq k \leq n, l \in V) \).

To express the above dynamics in a compact form, let us define the vectors \( X_n = [X_{n,1}, \ldots, X_{n,N}]^\top \)
and \( Z_n = [Z_{n,1}, \ldots, Z_{n,N}]^\top \). Hence, for \( n \geq 0 \), the dynamics described by (1) and (2) can be expressed as follows:

\[
E[X_{n+1} | F_n] = W^\top Z_n,
\]

and

\[
Z_{n+1} = (1 - r_n) Z_n + r_n X_{n+1}.
\]

Moreover, the assumption about the normalization of the matrix \( W \) can be written as \( W^\top 1 = 1 \), where \( 1 \) denotes the vector with all the entries equal to 1.

In order to fix ideas, we can imagine that \( G = (V,E) \) represents a network of \( N \geq 2 \) individuals that at each time-step have to make a choice between two possible actions \( \{0,1\} \). For any \( n \geq 1 \), the random variables \( \{X_{n,l} : l \in V\} \) describe the actions adopted by the agents of the network at time-step \( n \); while each random variable \( Z_{n,l} \) takes values in \([0,1]\) and it can be interpreted as the “personal inclination” of the agent \( l \) of adopting “action 1”. Thus, the probability that the agent \( l \) adopts “action 1” at time-step \( (n+1) \) is given by a convex combination of \( l \)'s own inclination and the inclinations of the other agents at time-step \( n \), according to the “influence-weights” \( w_{l,l} \) as in (1). Note that, from a mathematical point of view, we can have \( w_{l,l} \neq 0 \) or \( w_{l,l} = 0 \). In both cases we have a reinforcement mechanism for the personal inclinations of the agents: indeed, by (2), whenever \( X_{n,l} = 1 \), we have a strictly positive increment in the personal inclination of the agent \( l \), that is \( Z_{n,l} > Z_{n-1,l} \), provided \( Z_{n-1,l} < 1 \). However, only in the case \( w_{l,l} > 0 \) (which is the most usual in applications), this fact results in a greater probability of having \( X_{n+1,l} = 1 \) according to (1). Therefore, if \( w_{l,l} > 0 \), then we have a “true self-reinforcing” mechanism; while, in the opposite case, we have a reinforcement property only in the own inclination of the single agent, but this does not affect the probability (1) of the action taken by this agent.

In the considered setting, the main goals are:

1. To understand whether and when a (complete or partial) almost sure asymptotic synchronization (that could be roughly defined as the propensity of interacting agents to uniformize their behavior) can emerge;
2. To discover and to characterize which regimes may appear when the complete almost sure asymptotic synchronization does not hold.

All the above goals are achieved by performing a detailed analysis on the interplay between the asymptotic behavior of the system and the properties of the interaction matrix \( W \) and of the reinforcement sequence \((r_n)_n\).

First of all, the present paper provides the sufficient and necessary conditions in order to have the almost sure asymptotic synchronization of the entire system (complete almost sure asymptotic synchronization), that is the almost sure converge toward zero of all the differences \((Z_{n,l_1} - Z_{n,l_2})_n\), with \( l_1, l_2 \in V \). Firstly, we observe that, in the considered setting, the complete almost sure asymptotic synchronization of the system is equivalent to the almost sure convergence of all the RSPs \((Z_{n,l})_n\), with \( l \in V \), toward a certain common random variable \( Z_{\infty} \). Then, under the assumption that \( W \) is irreducible (i.e. \( G = (V,E) \) is a strongly connected graph) and \( P(Z_0 = 0) + P(Z_0 = 1) < 1\) (in order to exclude the trivial cases), we prove that:

\begin{enumerate}
  \item[(i)] when \( W^\top \) is aperiodic, we have complete almost sure asymptotic synchronization if and only if \( \sum_n r_n = +\infty \);
  \item[(ii)] when \( W^\top \) is periodic, we have complete almost sure asymptotic synchronization if and only if \( \sum_n r_n (1 - r_n) = +\infty \).
\end{enumerate}

\footnotetext{1}{Similarly to the notation \( 1 \) already mentioned above, the symbol \( 0 \) denotes the vector with all the entries equal to 0.}
Therefore $\sum_n r_n = +\infty$ results to be a necessary conditions on the reinforcement sequence $(r_n)_n$ for the complete almost sure asymptotic synchronization. Indeed, when $\sum_n r_n < +\infty$, all the stochastic processes $\{(Z_{n,l})_n : l \in V\}$ trivially converge almost surely, but for any pair of distinct nodes we have a strictly positive probability of non-synchronization in the limit. More interesting is the regime when $W^\top$ is periodic: $\sum_n r_n = +\infty$ and $\sum_n r_n(1 - r_n) < +\infty$ (which actually means that $r_n$ is either stays very close to 0 or very close to 1). In this case, we show that a partial almost sure asymptotic synchronization holds: indeed, the RSPs positioned at the network vertices almost surely synchronize in the limit inside each cyclic class of $W^\top$, that is the difference $(Z_{n,l_1} - Z_{n,l_2})_n$ converges almost surely to zero whenever $l_1$ and $l_2$ belong to the same cyclic class, but the convergence of $(Z_{n,l})_n$ is not guaranteed. Indeed, we prove that there exists a strictly positive probability that $(Z_n)_n$ does not converge, because of an asymptotic periodic behavior of the cyclic classes (see Theorem 3.4). Moreover, the limit set of each cyclic class is given by the barrier-set $\{0, 1\}$, that is, for a large time-step, there are some cyclic classes in which the processes are very close to one and other classes in which the processes are very close to zero (see Theorem 3.3). Further, in this regime, we have a clockwise dynamics also for the agents’ actions (i.e. for the random variables $X_{n,l}$) and their synchronization inside the cyclic classes at the clock-times (see again Theorem 3.4). The behavior we have in this last regime seems to well describe some social phenomena where the shift between different groups’ polarizations is present along time.

It is worthwhile to point out that these results have been achieved by means of a suitable decomposition of $Z_n$ in terms of three components (see Theorem 2.1): the first related to the eigenvalue $\lambda_1$ (i.e. the Perron-Frobenius or leading eigenvalue) of $W^\top$, the second related to the cyclic classes of $W^\top$, that is to the eigenvalues of $W^\top$, different from 1, but with modulus equal to one, and the third component related to the other eigenvalues of $W^\top$, that is those with modulus strictly less than one. We study the behavior of all the three components and how they affect the behavior of the system (see Theorems 2.2 and Theorem 2.4). The first-order asymptotic behavior of $(Z_n)_n$ for the different regimes considered is completely described and summarized in Table 1. We also discuss how the assumption of $W^\top \mathbf{1} = \mathbf{1}$ can be relaxed through a natural generalization of the theory presented in this work.

| $\sum_n r_n < +\infty$ | $n_{\text{per}} = 1$ | $n_{\text{per}} \geq 2$ |
|-------------------------|----------------------|----------------------|
| $\sum_n r_n(1 - r_n) = +\infty$ (and so $\sum_n r_n = +\infty$) | • a.s. convergence (neither complete nor partial) | • a.s. convergence (neither complete nor partial) |
| $\sum_n r_n = +\infty$ | • a.s. convergence complete a.s. synchro. | • a.s. convergence complete a.s. synchro. |
| $\sum_n r_n(1 - r_n) < +\infty$ | • a.s. convergence complete a.s. synchro. | • non-a.s. convergence (asym. periodic behavior of the cyclic classes) |

| Table 1. Summary of the possible first-order asymptotic behavior of $(Z_n)_n$, under the assumption $P(Z_0 = 0) + P(Z_0 = 1) < 1$. The symbol $n_{\text{per}}$ denotes the period of the matrix $W^\top$. |

We also note that, when $(Z_n)_n$ almost surely converges toward a random vector $Z_\infty$ (equal to $Z_\infty \mathbf{1}$ in the case of complete almost sure asymptotic synchronization), the average of times in
which the agents adopt “action 1”, that is the vector of the empirical means \( N_{n,l} = \frac{1}{n} \sum_{k=1}^{n} X_{k,l} \), with \( l \in V \), almost surely converges toward \( W^\top Z_\infty \) (equal again to \( Z_\infty 1 \) in the case of complete almost sure asymptotic synchronization). This result is important from an applicative point of view, because the observable processes are typically the processes \((X_{n,l})_n\) of the performed actions and not the processes \((Z_{n,l})_n\) of the inclinations. Therefore, in a statistical framework, the vector of the empirical means can be used as a strongly consistent estimator of the random vector \( W^\top Z_\infty \).

When we only have the almost sure asymptotic synchronization of the inclinations \( Z_{n,l} \) within the cyclic classes, we also have the same form of asymptotic synchronization for the empirical means \( N_{n,l} \).

Regarding the assumption of irreducibility, we observe that when \( W \) is reducible, using a suitable decomposition of \( W^\top \), we obtain a natural decomposition of the graph \( G = (V, E) \) in different sub-graphs \( \{G_s; 1 \leq s \leq m\} \cup G_f \) (where \( m \) is the multiplicity of the eigenvalue 1 of \( W^\top \)) such that:

(i) the nodes in each sub-graph \( G_s \) are not influenced by the nodes of the rest of the network, and hence the dynamics of the processes in \( G_s \) can be fully established by considering only the correspondent irreducible sub-matrix of \( W^\top \) and so by applying the results presented in the present paper to each sub-graph \( G_s \);

(ii) the nodes in the sub-graph \( G_f \) are affected by the behavior of the rest of the network and hence they have to be treated according to the specific situation considered.

Finally, when the assumption of normalization for \( W \), that is \( W^\top 1 = 1 \), is not verified, the first-order asymptotic dynamics of the system become trivial: indeed, we have the complete almost sure asymptotic synchronization toward zero, as the result of the action of a "forcing input".

**Literature review.** Regarding the previous literature, it is important to mention that there are other works concerning models of interacting urns, that consider reinforcement mechanism and/or interacting mechanisms different from the model presented in this paper. For instance, the model studied in [27] describes a system of interacting units, modeled by Pólya urns, subject to perturbations and which occasionally break down. The authors consider a system of interacting Pólya urns arranged on a \( d \)-dimensional lattice. Each urn contains initially \( b \) black balls and 1 white ball. At each time step an urn is selected and a ball is drawn from it: if the ball is white, a new white ball is added to the urn; if it is black a “fatal accident” occurs and the urn becomes unstable and it “topples” coming back to the initial configuration. The toppling mechanism involves also the nearby urns.

In [29] a class of discrete time stochastic processes generated by interacting systems of reinforced urns is introduced and its asymptotic properties analyzed. Given a countable set of urns, at each time a ball is independently sampled from every urn in the system and in each urn a random number of balls of the same color of the extracted ball is added. The interaction arises since the number of added balls depends also on the colors generated by the other urns as well as on a common random factor.

In [18] a notion of partially conditionally identically distributed (c.i.d.) sequences has been studied as a form of stochastic dependence, which is equivalent to partial exchangeability in the presence of stationarity. A natural example of partially c.i.d. construction is given by a countable collection of stochastic processes with reinforcement, and possibly infinite state space, with an interaction among them obtained by inserting in their dynamics some stocastically dependent random weights. This example contains as a special case the model in [29].

Interacting two-colors urns have been considered in [23, 24]. Their main results are proved when the probability of drawing a ball of a certain color is proportional to \( \rho^k \), where \( \rho > 1 \) and \( k \) is the number of balls of this color. The interaction is of the mean-field type. More precisely, the interacting reinforcement mechanism is the following: at each step and for each urn draw a ball
from either all the urns combined with probability $p$, or from the urn alone with probability $1 - p$, and add a new ball of the same color to the urn. The higher is the interacting parameter $p$, the larger is the memory shared between the urns. The main results can be informally stated as follows: if $p \geq 1/2$, then all the urns fixate on the same color after a finite time, and if $p < 1/2$, then some urns fixate on a unique color and others keep drawing both colors.

In [12] the authors consider a network of interacting urns displaced over a lattice. Every urn is Pólya-like and its reinforcement matrix is not only a function of time (time contagion) but also of the behavior of the neighboring urns (spatial contagion), and of a random component, which can represent either simple fate or the impact of exogenous factors. In this way a non-trivial dependence structure among the urns is built, and the given construction is used to model different phenomena characterized by cascading failures such as power grids and financial networks.

In [10, 11, 25] a graph-based model, with urns at each vertex and pair-wise interactions, is considered. Given a finite connected graph, an urn is placed at each vertex. Two urns are called a pair if they share an edge. At discrete times, a ball is added to each pair of urns. In a pair of urns, one of the urns gets the ball with probability proportional to its current number of balls raised by some fixed power $\alpha > 0$. The authors characterize the limiting behavior of the proportion of balls in the bins for different values of the parameter $\alpha$.

In [21, 20, 19] another graph-based model, with Pólya urns at each vertex, is provided in order to model the diffusion of an epidemics. Given a finite connected graph, an urn is placed at each vertex and, in order to generate spatial infection among neighboring nodes, instead of drawing solely from its own urn, each node draws from a “super urn”, whose composition is the union of the composition of its own urn and of those of its neighbors’ urns. The stochastic properties and the asymptotic behavior of the resulting network contagion process are analyzed.

In [14, 16, 33] the authors consider interacting urns (precisely, [14, 16] deal with standard Pólya urns and [33] regards Friedman’s urns) in which the interaction is of the mean-field type: indeed, the urns interact among each other through the average composition of the entire system, tuned by the interaction parameter $\alpha$, and the probability of drawing a ball of a certain color is proportional to the number of balls of that color. Asymptotic synchronization and central limit theorems for the urn proportions have been proved for different values of the tuning parameter $\alpha$, providing different convergence rates and asymptotic variances. In [13] the same mean-field interaction is adopted, but the analysis has been extended to the general class of reinforced stochastic processes (in the sense of Definition 1.1), providing almost sure asymptotic synchronization of the entire system and central limit theorems, also in functional form, in the case $\lim_n n^\gamma r_n = c$ with $c > 0$ and $1/2 < \gamma \leq 1$. Differently from these works, the model proposed in [7] concerns with a system of generalized Friedman’s urns with irreducible mean replacement matrices based on a general interaction matrix. Combining the information provided by the mean replacement matrices and by the interaction matrix, first and second-order asymptotic results of the urn proportions have been established. In this framework the non-synchronization is a natural phenomenon since the mean replacement matrices are irreducible (and so not diagonal) and are allowed to be different among the nodes, hence the limits of the urn proportions are deterministic and possibly different.

The paper [2] joins the class of reinforced stochastic processes studied in [13] with the general interacting framework, driven by the interaction matrix, adopted in [7]. After proving complete almost sure asymptotic synchronization for an irreducible diagonalizable interacting matrix and for $\lim_n n^\gamma r_n = c$ with $c > 0$ and $1/2 < \gamma \leq 1$, [2] provides the rates of synchronization and the second-order asymptotic distributions, in which the asymptotic variances have been expressed as functions of the parameters governing the reinforced dynamics and the eigen-structure of the interaction matrix. These results lead to the construction of asymptotic confidence intervals for the common limit random variable of the processes $\{(Z_{n,l})_n : l \in V\}$ and to the design of statistical
tests to make inference on the topology of the interaction network given the observation of the processes \( \{ (Z_{n,l})_n : l \in V \} \). Finally, in [2] the non-synchronization phenomenon is discussed only as a consequence of the non-irreducibility of the interaction matrix, which leads the system to be decomposed in sub-systems of processes evolving with different behaviors.

The previous quoted papers focus on the asymptotic behavior of the stochastic processes of the personal inclinations \( \{ (Z_{n,l})_n : l \in V \} \) of the agents, while [3, 4] study different averages of times in which the agents adopt “action 1”, i.e. the stochastic processes of the empirical means and weighted empirical means associated with the random variables \( \{ (X_{n,l})_n : l \in V \} \).

In addition, inspired by models for coordination games, technological or opinion dynamics, the paper [15] deals with stochastic models where, even if a mean-field interaction among agents is present, the absence of asymptotic synchronization may happen due to the presence of an individual non-linear reinforcement.

Finally, in the recent paper [22], a model for interacting balanced urns is introduced and analyzed. The balanced reinforcement matrix is assumed the same for all the urns and the interaction among the urns is based on a finite directed graph (not weighted), in the sense that each urn positioned at a node of the graph reinforces all the urns in its out-neighbours. The authors show conditions on the reinforcement matrix, the topology of the graph and the initial configuration in order to have the almost sure convergence of the system and a form of almost sure asymptotic synchronization. They also provide some fluctuation theorems.

The present paper have some issues in common with [2], although at the same time several significant differences can be pointed out and, in particular, the intent of the this work is different. Indeed, here we are not only interested in providing sufficient assumptions for the complete almost sure asymptotic synchronization of the system and proving theoretical results under these assumptions, but we aim at getting a complete description of the first-order asymptotic behavior of the system, which consists in finding sufficient and also necessary conditions on the interaction matrix and on the reinforcement sequence for the (complete or partial) almost sure asymptotic synchronization and also describing the behavior of the system in the non-synchronization and non-convergence regimes. In particular, it is very important to underline that, with respect to [2], in the present paper:

1. we eliminate the technical assumption that the interaction matrix is diagonalizable (indeed, this condition is difficult to be checked in practical applications and so it may lead to consider the symmetric interaction matrices as the only class of “applicable” matrices);
2. the results provided here include not only the case when \( \lim n \gamma r_n = c \) with \( c > 0 \) and \( 1/2 < \gamma \leq 1 \) considered in the previous papers mentioned above, but also the case with \( 0 < \gamma \leq 1/2 \);
3. we do not limit ourselves to the case of reinforcement sequences \( (r_n)_n \) such that \( \lim n \gamma r_n = c \) with \( 0 < \gamma \leq 1 \) (indeed, as we will show, this assumption always implies the convergence of the system and so it excludes other possible behaviors that the system can present);
4. we describe the behavior of the system in the regimes where the complete almost sure asymptotic synchronization does not hold (this includes the regime with almost sure convergence without any form of almost sure synchronization and the regime with a periodic dynamics and a partial almost sure synchronization, specifically synchronization within the cyclic classes).

Regarding the methodology, we point out that there exists a vast literature where the asymptotics for discrete-time processes, in particular urn processes, are proven through ordinary differential equation (ODE) method and stochastic approximation theory that may seem to be applicable to the dynamics considered in this work as well (see e.g [9, 34] and references therein). However, with these techniques it would be impossible to fully characterize all the possible regimes shown
by the system. For instance, the non-convergent periodic dynamics we present in Theorem 3.4 is not contemplated in an ODE framework, which shows that this theory fails in our context.

Structure of the paper. The sequel of the paper is so structured. In Section 2 we suitably decompose the process \((Z_n)_n\) and, after characterizing the (complete or partial) almost sure asymptotic synchronization of the system in terms of the components of this decomposition, we provide sufficient and necessary conditions related with this phenomenon. In Section 3 we deal with the regimes where the complete almost sure asymptotic synchronization of the system is not guaranteed, discussing the case of almost sure convergence without any form of almost sure asymptotic synchronization and the case of non-almost sure convergence (nor complete synchronization) of the system. Section 4 contains a remark on the behavior of the (weighted) empirical means associated to the random variables \(\{(X_{n,l})_n : l \in V\}\), a discussion on how to handle the case of a reducible interaction matrix and a final comment on what happens when the assumption of normalization for \(W\), that is \(W^\top 1 = 1\), is not verified. Finally, Section 5 contains the main ideas and the sketches of the proofs of the results stated in the present work. All the details of the proofs can be found in a separate supplementary material [6], along with some recalls and auxiliary results.

2. Almost sure asymptotic synchronization

Consider a system of \(N \geq 2\) Reinforced Stochastic Processes (RSPs) with a network-based interaction as defined in (1) and (2) or, equivalently, in (3) and (4), where the matrix \(W\) has non-negative entries, is irreducible (i.e. the underlying graph \(G = (V, E)\) is strongly connected) and is such that \(W^\top 1 = 1\) (where 1 denotes the vector with all the entries equal to 1).

In this section, we use the assumed properties of \(W^\top\) in order to decompose the process \((Z_n)_n\) into three components. We will claim that the behavior of these components depends only on the summability of the sequences \((r_n)_n\) and \((r_n(1 - r_n))_n\), according to the period of \(W^\top\). To this end, we recall that the Perron-Frobenius Theorem ensures that:

1. the eigenvalue 1 of \(W^\top\) has multiplicity one and it is called the Perron-Frobenius or leading eigenvalue, as all the other eigenvalues have real part strictly less than 1. Moreover, there exists a (unique) left eigenvector \(v\) associated to the leading eigenvalue of \(W^\top\) with all the entries in \((0, +\infty)\) and such that \(v^\top 1 = 1\);

2. if \(W^\top\) is periodic with period \(n_{\text{per}} \geq 2\),
   i. all the complex \(n_{\text{per}}\)-roots of the unity are eigenvalues of \(W^\top\) with multiplicity one;
   ii. there exists an equivalence relationship \(\sim_c\) that separates the set of vertices \(V = \{1, \ldots, N\}\) (i.e. the processes \(\{(Z_{n,l})_n : l \in V\}\)) into \(n_{\text{per}}\) classes of equivalence, numbered from 0 to \(n_{\text{per}} - 1\) and called cyclic classes, such that \([W^\top]_{l_1,l_2} = 0\) when \(l_1\) belongs to the \(h\)-th cyclic class and \(l_2\) does not belong to the \((h + 1)\)-th cyclic class (the numbers of the classes being defined modulus \(n_{\text{per}}\)). They correspond to the communicating classes of the matrix \((W^\top)^{n_{\text{per}}}\);

3. except the complex \(n_{\text{per}}\)-roots of the unity, all the other eigenvalues have modulus strictly less than 1.

The first component of the decomposition of the process \((Z_n)_n\) concerns the eigenspace associated to the Perron-Frobenius eigenvalue. More precisely, the spectral non-orthogonal projection of \(W^\top\) corresponding to the leading eigenvalue is the matrix \(1v^\top\), that we apply to \(Z_n\) to obtain the process

\[
Z_n^{(1)} = 1v^\top Z_n = \tilde{Z}_n 1,
\]
where \( (\tilde{Z}_n)_n \) is the bounded martingale defined as \( \tilde{Z}_n = v^\top Z_n \) for each \( n \) and with dynamics
\[
\tilde{Z}_0 = v^\top Z_0, \quad \tilde{Z}_{n+1} = (1 - r_n)\tilde{Z}_n + r_n Y_{n+1},
\]
where \( Y_{n+1} = v^\top X_{n+1} \) takes values in \([0, 1]\) and \( E[Y_{n+1} | F_n] = v^\top W^\top Z_n = \tilde{Z}_n \). This martingale plays a central role: indeed, whenever all the stochastic processes \( \{(Z_{n,l})_n, l \in V\} \) converge almost surely to the same limit random variable, say \( Z_\infty \), we trivially have that \( Z_\infty \) coincides with the almost sure limit of \( (\tilde{Z}_n)_n \).

The second component of the decomposition of the process \( (Z_n)_n \) exists only when \( n_{\text{per}} \geq 2 \) and it concerns the cyclic classes. Specifically, the process \( Z_n^{(2)} \) is defined as
\[
Z_n^{(2)} = Z_n^{(C)} - \tilde{Z}_n 1, \quad \text{where} \quad Z_n^{(C)} = \sum_{l_1 \sim_c l_2} \frac{v_{l_1}}{\sum_{l_2 \sim_c l_2} v_{l_2}} Z_{n,l_1}, \quad l \in V.
\]
Note that the process \( Z_n^{(C)} \), and so \( Z_n^{(2)} \), is constant on each cyclic class as the terms \( \sum_{l_1 \sim_c l_2} v_{l_1} Z_{n,l_1} \) and \( \sum_{l_2 \sim_c l_2} v_{l_2} \) are the same for all the vertices \( l \) in the same cyclic class. We will prove that this part may be defined by means of the spectral non-orthogonal projection on the eigenspaces corresponding to the complex \((n_{\text{per}} - 1)\) roots of the unity different from 1 (see Theorem 2.1).

The third component of the decomposition of the process \( (Z_n)_n \) is the remaining part and it concerns the spectral non-orthogonal projection on the eigenspaces corresponding to the roots of the characteristic polynomial that have modulus strictly less than 1 (see Theorem 2.1).

The full decomposition then reads
\[
Z_n = Z_n^{(1)} + Z_n^{(2)} + Z_n^{(3)}.
\]

Analyzing separately these three components, we will obtain sufficient and necessary conditions for the (complete or partial) almost sure asymptotic synchronization of the system, in the sense of the following definition:

**Definition 2.1** (Complete or partial almost sure asymptotic synchronization). Given a system of \( N \geq 2 \) interacting RSPs as defined above, we say that we have the complete almost sure asymptotic synchronization of the system (or the entire system almost surely synchronizes in the limit) if, for each pair \((l_1, l_2)\) of indices in \( V = \{1, \ldots, N\} \), we have
\[
(Z_{n,l_2} - Z_{n,l_1}) \xrightarrow{a.s.} 0.
\]
We say that we have the almost sure asymptotic synchronization of the system within each cyclic class (and so a partial almost sure asymptotic synchronization of the system) if (9) holds true for each pair \((l_1, l_2)\) of indices such that \( l_1 \sim_c l_2 \), i.e. belonging to the same cyclic class. (Of course this partial synchronization is interesting only in the case \( n_{\text{per}} \neq N \), that is when at least one cyclic class has more than one element.)

Since, by definition, the term \( Z_n^{(3)} \) is the only one of the three parts of the decomposition that differs between the processes within the same cyclic class, we have the characterization of the almost sure asymptotic synchronization of the system inside each cyclic class as the almost sure convergence toward zero of \( (Z_n^{(3)})_n \). In the next theorem we state that the three components of the decomposition of the process \( (Z_n)_n \) are linearly independent and so we get the characterization of the complete almost sure asymptotic synchronization of the system as the almost sure convergence toward zero of \( (Z_n^{(2)})_n \) and of \( (Z_n^{(3)})_n \). The proofs of the following and the other results of this section are collected more ahead in Section 5.
Theorem 2.1. Let \( Z_n^{(1)}, Z_n^{(2)}, Z_n^{(3)} \) as in (8). Then, we have\(^2\)

\[
Z_n^{(1)} = \mathbf{1} \mathbf{v}^\top \mathbf{Z}_n \in \text{Span}\{1\}, \quad Z_n^{(2)} = \sum_{j=1}^{n_{\text{per}}-1} q_j \mathbf{v}_j^\top \mathbf{Z}_n \in \text{Span}\{q_1, \ldots, q_{n_{\text{per}}-1}\},
\]

\[
Z_n^{(3)} = \sum_{j=1}^{N-n_{\text{per}}} r_j \mathbf{p}_j^\top \mathbf{Z}_n \in \text{Span}\{r_1, \ldots, r_{N-n_{\text{per}}}\},
\]

where \( q_1, \ldots, q_{n_{\text{per}}-1} \) (resp. \( \mathbf{v}_1, \ldots, \mathbf{v}_{n_{\text{per}}-1} \)) are right (resp. left) eigenvectors of \( W^\top \) related to the eigenvalues with modulus equal to 1, but different from 1 (i.e. the complex \( n_{\text{per}} \)-roots of the unit different from 1) and \( r_1, \ldots, r_{n_{\text{per}}-1} \) (resp. \( \mathbf{p}_1, \ldots, \mathbf{p}_{N-n_{\text{per}}} \)) are right (resp. left), possibly generalized, eigenvectors of \( W^\top \) related to the eigenvalues with modulus strictly smaller than 1.

As a consequence, the three components \( Z_n^{(i)}, i = 1, 2, 3 \), are linear independent and so the complete almost sure asymptotic synchronization of the system holds if and only if both \( Z_n^{(2)} \) and \( Z_n^{(3)} \) converge almost surely toward zero.

Before analyzing the behavior of the three terms \( Z_n^{(i)}, i = 1, 2, 3 \), in the general case, let us first consider the trivial scenario in which initially all the processes \( \{Z_{n,l} : l \in V\} \) start from the same barrier (0 or 1) with probability one, that is \( P(T_0 = 1) = 1 \) with \( T_0 = \{Z_0 = 0\} \cup \{Z_0 = 1\} \), where, similarly to \( 1 \), the symbol \( 0 \) denotes the vector with all the entries equal to zero. Since \( \{Z_0 = 0\} \) implies \( \{Z_n = 0, \forall n \geq 0\} \) (and, analogously, \( \{Z_0 = 1\} \) implies \( \{Z_n = 1, \forall n \geq 0\} \)), when \( P(T_0 = 1) = 1 \), we have \( Z_n^{(2)} \equiv Z_n^{(3)} \equiv 0 \) and \( Z_n^{(1)} \equiv Z_n \equiv Z_0 \) with probability one, and trivially the process \( \{Z_n\}_n \) converges almost surely and the entire system almost surely synchronizes. For this reason, from now on we will consider the non-trivial initial condition \( P(T_0) < 1 \).

Consider now the first term, i.e. \( Z_n^{(1)} \). Since \( \{\tilde{Z}_n\}_n \) is a bounded martingale and so convergent almost surely to a random variable, from Definition 2.1 and Theorem 2.1, we get that the entire system almost surely synchronizes in the limit if and only if there exists a random variable \( Z_\infty \), taking values in \([0,1]\), such that

\[
Z_n \xrightarrow{a.s.} Z_\infty \mathbf{1}.
\]

In other words, the complete almost sure asymptotic synchronization of the system implies the almost sure convergence of all the stochastic processes \( \{(Z_{n,l})_n : l \in V\} \) to the same limit random variable \( Z_\infty \).

We now investigate the conditions that characterize the regime of (complete or partial) almost sure asymptotic synchronization. In particular, in the next results we will show that, whenever \( \sum_n r_n < +\infty \), there is a strictly positive probability that \( Z_n^{(2)} \) or \( Z_n^{(3)} \) do not vanish asymptotically. This fact, since Theorem 2.1, means that the system does not synchronize (completely nor partially) with probability one. An heuristic explanation of why this occurs when \( \sum_n r_n < +\infty \) is that, when at time \( n_0 \) the remaining series \( \sum_{n>n_0} r_n \) gets very low, the single processes \( \{(Z_{n,l})_n : l \in V\} \) cannot converge too far from their values \( \{Z_{n_0,l} : l \in V\} \), respectively, and this avoids them to converge to the same limit almost surely when their distance at time \( n_0 \) can be large with strictly positive probability.

Let us now present the result regarding the third component \( Z_n^{(3)} \).

\(^2\)For the cases \( n_{\text{per}} = 1 \) and \( n_{\text{per}} = N \), we use the usual convention \( \sum_{j=1}^{N} \cdots = 0 \).
Theorem 2.2. When \( n_{\text{per}} = N \), we have \( Z^{(3)}_n \equiv 0 \). When \( n_{\text{per}} < N \), we have \( Z^{(3)}_n \xrightarrow{a.s.} 0 \) if and only if \( \sum_n r_n = +\infty \).

As observed before, Theorem 2.2 is providing necessary and sufficient conditions for a partial almost sure asymptotic synchronization of the system, that is for the almost sure asymptotic synchronization within each cyclic class. This fact is stated in the following result.

Corollary 2.3 (Almost sure asymptotic synchronization within each cyclic class). Excluding the trivial case \( n_{\text{per}} = N \) (i.e. all the cyclic classes have only one element), we have almost sure asymptotic synchronization of the system within each cyclic class if and only if \( \sum_n r_n = +\infty \).

Regarding the above result, it is important to underline that the almost sure convergence of the single processes \( \{ (Z_{n,l})_n : l \in V \} \) is not guaranteed under the condition \( \sum_n r_n = +\infty \), as we can only affirm the almost sure convergence toward zero of the difference between two processes belonging to the same cyclic class.

Remark 1. Since Theorem 2.1, Theorem 2.2 shows that, at least when \( n_{\text{per}} < N \), the condition \( \sum_n r_n = +\infty \) is necessary for the complete almost sure asymptotic synchronization. More ahead we will see that it is actually necessary also in the case \( n_{\text{per}} = N \) in order to let \( Z^{(2)}_n \) asymptotically vanishes as well.

Condition \( \sum_n r_n = +\infty \) is not sufficient in order to guarantee the complete almost sure synchronization of the system because, by Theorem 2.1, we have to control the second component \( Z^{(2)}_n \) too. Regarding this issue, the following example will result to be useful in order to understand the sufficient and necessary conditions for the almost sure convergence toward \( 0 \) of \( Z^{(2)}_n \).

Example 2.1. Consider the case of \( N = 3 \) and
\[
W^\top = \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},
\]
whose eigenvalues are \( \{ 1, 0, -1 \} \) and so the period is \( n_{\text{per}} = 2 \). For simplicity, suppose \( P(Z_{0,l}) \in (0,1), \forall l \in V > 0 \) (otherwise, we have to work starting from a time-step \( n_0 \) such that \( P(Z_{n_0,l}) \in (0,1), \forall l \in V > 0 \), which always exists by the irreducibility of \( W \) and the assumption \( P(T_0) < 1 \). Set \( A_n = \{ X_{n,1} = 0 \} \cap \{ X_{n,2} = 1, X_{n,3} = 1 \} \) for \( n \) even and \( A_n = \{ X_{n,1} = 1 \} \cap \{ X_{n,2} = 0, X_{n,3} = 0 \} \) for \( n \) odd, so that we have
\[
\begin{align*}
Z_{2k,1} &= (1 - r_{2k})Z_{2k-1,1} \leq (1 - r_{2k}), \\
Z_{2k+1,1} &= (1 - r_{2k+1})Z_{2k,1} + r_{2k+1} \geq r_{2k+1}, \\
Z_{2k,2} &= (1 - r_{2k})Z_{2k-1,2} + r_{2k} \geq r_{2k}, \\
Z_{2k+1,2} &= (1 - r_{2k+1})Z_{2k,2} + r_{2k+1} \leq (1 - r_{2k+1}), \\
Z_{2k,3} &= (1 - r_{2k})Z_{2k-1,3} + r_{2k} \geq r_{2k}, \\
Z_{2k+1,3} &= (1 - r_{2k+1})Z_{2k,3} + r_{2k+1} \leq (1 - r_{2k+1}).
\end{align*}
\]

Then, we will prove that, when \( \sum_n (1 - r_n) < +\infty \), there is a strictly positive probability that \( Z_n \) does not converge. More specifically, we will show that the probability of the event \( A = \cap_{n=1}^{\infty} A_n \) is strictly positive if \( \sum_n (1 - r_n) < +\infty \). To this end, let \( \bar{A}_n = \cap_{k=0}^{n-1} A_k \) and notice that
\[
P(A) = P(\cap_{n=1}^{\infty} A_n) = \prod_{n=1}^{\infty} P(A_n | \cap_{k=0}^{n-1} A_k) = \prod_{n=1}^{\infty} P(A_n | \bar{A}_{n-1}).
\]

Then, take \( n \) even and note that
\[
P(A_n | \bar{A}_{n-1}) = P(X_{n,1} = 0 | \bar{A}_{n-1}) \cdot P(X_{n,2} = 1 | \bar{A}_{n-1}) \cdot P(X_{n,3} = 1 | \bar{A}_{n-1}).
\]
Now, consider \( P(X_{n,1} = 0|\bar{A}_{n-1}) \) and notice that, since \( \{X_{n-1,2} = 0, X_{n-1,3} = 0\} \subset \bar{A}_{n-1} \), we have on \( \bar{A}_{n-1} \),
\[
Z_{n-1,2} = (1 - r_{n-1})Z_{n-2,2} + r_{n-1}X_{n-1,2} = (1 - r_{n-1})Z_{n-2,2} \leq (1 - r_{n-1}),
\]
\[
Z_{n-1,3} = (1 - r_{n-1})Z_{n-2,3} + r_{n-1}X_{n-1,3} = (1 - r_{n-1})Z_{n-2,2} \leq (1 - r_{n-1}),
\]
which implies
\[
P(X_{n,1} = 0|\bar{A}_{n-1}) = 1 - \frac{3}{4}Z_{n-1,2} - \frac{1}{3}Z_{n-1,2} \geq r_{n-1}.
\]
Analogously, consider \( P(X_{n,2} = 1|\bar{A}_{n-1}) \) (the same is for \( X_{n,3} \)) and notice that, since \( \{X_{n-1,1} = 1\} \subset \bar{A}_{n-1} \), we have on \( \bar{A}_{n-1} \),
\[
Z_{n-1,1} = (1 - r_{n-1})Z_{n-2,1} + r_{n-1}X_{n-1,1} \geq r_{n-1},
\]
which implies
\[
P(X_{n,2} = 1|\bar{A}_{n-1}) = Z_{n-1,2} \geq r_{n-1}.
\]
Then we have \( P(A_n|\bar{A}_{n-1}) \geq r_{n-1}^3 \), which implies
\[
P(A) = \prod_{n=1}^{\infty} P(A_n|\bar{A}_{n-1}) \geq \left( \prod_{n=1}^{\infty} r_{n-1} \right)^3 = \left( \prod_{n=1}^{\infty} (1 - (1 - r_{n-1})) \right)^3 > 0,
\]
if \( \sum_n (1 - r_n) < +\infty \). Therefore, we have found a non-negligible event \( A \) on which \( Z_{2n} \to (0, 1, 1)^T \) and \( Z_{2n+1} \to (1, 0, 0)^T \), that is on \( A \) we have neither asymptotic synchronization of the entire system nor almost sure convergence of \( (Z_n)_n \). Let us now discuss the asymptotic behavior of the three components of the decomposition (8). A direct calculation shows that \( \psi^T = (1/2, 1/3, 1/6) \) is the Perron eigenvector and
\[
Z_n^{(1)} = \frac{3Z_{n,1} + 2Z_{n,2} + Z_{n,3}}{6} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad Z_n^{(2)} = Z_n^{(C)} - Z_n^{(1)}, \quad Z_n^{(3)} = \begin{pmatrix} \frac{Z_{n,1}}{Z_{n,2} + Z_{n,3}} \\ \frac{2Z_{n,2} + Z_{n,3}}{2Z_{n,2} + Z_{n,3}} \\ \frac{Z_{n,3}}{3} \end{pmatrix}.
\]
Analogously, we could obtain \( Z_n^{(1)} \), \( Z_n^{(2)} \) and \( Z_n^{(3)} \) by using the spectral representation of Theorem 2.1, which in this case is the following:
\[
1\psi^T = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 1/2 & 1/3 & 1/6 \\ 1/2 & 1/3 & 1/6 \end{pmatrix}, \quad q_1\psi_1^T = \begin{pmatrix} 1/2 & -1/3 & -1/6 \\ -1/2 & 1/3 & 1/6 \\ -1/2 & 1/3 & 1/6 \end{pmatrix},
\]
\[
r_1p_1^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/3 & -1/3 \\ 0 & -2/3 & 2/3 \end{pmatrix}.
\]
On \( A \), the first term \( Z_n^{(1)} \) almost surely converges to \( Z_{(1)}^{(1)} = \frac{1}{2} \mathbf{1} \). The process \( (Z_n^{(C)})_n \), and so \( (Z_n^{(2)})_n \), does not converge on \( A \). Indeed, on \( A \) we have \( Z_{2n}^{(C)} \to (0, 1, 1)^T \) and \( Z_{2n+1}^{(C)} \to (1, 0, 0)^T \). Finally, it is easy to check that on \( A \) the third component \( (Z_n^{(3)})_n \) almost surely converges to zero (in agreement with Theorem 2.2, since \( \sum_n (1 - r_n) < +\infty \) trivially implies \( \sum_n r_n = +\infty \)). It is interesting to note that, in this framework, we have \( Z_n = Z_n^{(1)} + Z_n^{(2)} + Z_n^{(3)} \), so \( Z_n^{(C)} \), \( Z_n^{(1)} + Z_n^{(2)} = Z_n^{(C)} = (Z_{n,1}, \frac{2Z_{n,2} + Z_{n,3}}{3}, \frac{2Z_{n,2} + Z_{n,3}}{3} + Z_{n,3})^T \), which is constant on each cyclic class and, if we consider the
\[ n_{\text{per}} \text{-dimensional process } Z_n^{(C)} = (Z_{n,1}, \frac{2Z_{n,2} + Z_{n,3}}{3})^T \text{ defined by taking a single entry of } Z_n \text{ for each cyclic class, then } Z_n^{(C)} \text{ is still not convergent on } A \text{ (since } Z_n^{(C)} \to (0, 1)^T \text{ and } Z_{2n+1}^{(C)} \to (1, 0)^T \text{), but its norm almost surely converges on } A \text{ (we have } ||Z_n^{(C)}||^2 \to 1 \text{). This is a general fact that we will prove in the sequel.}}
We will now formally provide the general theory inspired by the above Example 2.1. In particular, we notice that the non-convergence of $Z_n$ in the example has been induced in the case $\sum_n (1-r_n) < +\infty$ by the component $Z_n^{(2)}$, which is present because of the period $n_{per} \geq 2$ of $W^\top$. This suggests that, for a periodic matrix $W^\top$, a necessary condition in order to have the complete almost sure asymptotic synchronization of the system is $\sum_n (1-r_n) = +\infty$. However, this is not sufficient. Indeed, we need a stronger condition as stated in the following result.

**Theorem 2.4.** When $n_{per} = 1$, we have $Z_n^{(2)} \equiv 0$. When $n_{per} \geq 2$, we have $Z_n^{(2)} \xrightarrow{a.s.} 0$ if and only if $\sum_n r_n(1-r_n) = +\infty$.

Note that $\sum_n r_n(1-r_n) = +\infty$ implies both $\sum_n r_n = +\infty$ and $\sum_n (1-r_n) = +\infty$, because $0 < r_n < 1$. Therefore, this result, combined with Remark 1, makes $\sum_n r_n = +\infty$ a necessary condition for the complete almost sure asymptotic synchronization of the system (regardless of the period $n_{per}$ of $W^\top$).

Now that we have determined the sufficient and necessary conditions in order that $Z_n^{(2)}$ and $Z_n^{(3)}$ asymptotically vanish with probability one, we can state the following synthetic result on the complete almost sure asymptotic synchronization of the system.

**Corollary 2.5** (Complete almost sure asymptotic synchronization). When $n_{per} = 1$, we have $Z_n \xrightarrow{a.s.} Z_\infty \mathbf{1}$ if and only if $\sum_n r_n = +\infty$. When $n_{per} \geq 2$, we have $Z_n \xrightarrow{a.s.} Z_\infty \mathbf{1}$ if and only if $\sum_n r_n(1-r_n) = +\infty$.

Summing up, with the results stated in this section, the question of the complete almost sure asymptotic synchronization of the system has been completely addressed (in [5] we investigate the distribution of the common limit $Z_\infty$, pointing out when it can take the extreme values, 0 or 1, with a strictly positive probability). The next step of the present work is now to determine the asymptotic dynamics of the processes $\{(Z_{n,l})_n : l \in V\}$ when the complete almost sure asymptotic synchronization does not hold (see the following Section 3).

### 3. Regimes of non-almost sure asymptotic synchronization of the entire system

As in Section 2, we consider a system of $N \geq 2$ RSPs with a network-based interaction as defined in (1) and (2) or, equivalently, in (3) and (4), where the matrix $W$ has non-negative entries, is irreducible and is such that $W^\top \mathbf{1} = \mathbf{1}$. Moreover, in order to exclude trivial cases, we fix $P(T_0) < 1$.

In this section, we analyze the asymptotic behavior of the system when the entire process $(Z_n)_n$ does not almost surely synchronize in the limit. First, we focus on the scenario in which there is almost sure convergence without almost sure synchronization (neither complete nor partial), and then we will consider the case in which we have a partial almost sure asymptotic synchronization (that is almost sure asymptotic synchronization within the cyclic classes), but there is a strictly positive probability that the entire process $(Z_n)_n$ neither converges nor asymptotically synchronizes.

**3.1. Almost sure convergent regime.** Let us first give the following simple result regarding the trivial case $\sum_n r_n < +\infty$.

**Proposition 3.1.** When $\sum_n r_n < +\infty$, all the processes $(Z_n)_n$, $(Z_n^{(1)})_n$, $(Z_n^{(2)})_n$ and $(Z_n^{(3)})_n$ converge almost surely. Moreover, for any pair of vertices $(l_1, l_2) \in V \times V$, with $l_1 \neq l_2$, we have $P(Z_{\infty,l_1} \neq Z_{\infty,l_2}) > 0$.

This result shows that, whenever $\sum_n r_n < +\infty$, the processes $\{(Z_{n,l})_n : l \in V\}$ at the network vertices converge with probability one, but none of their limits can be almost surely equal, i.e. for
any pair of processes there exists a strictly positive probability that they do not synchronize in the limit.

An interesting consideration that can be derived from combining Theorem 2.2 with Proposition 3.1, is that, exactly as for \((Z_n^{(1)})_n\), also the process \((Z_n^{(3)})_n\) always converges almost surely, without any additional assumption on the matrix \(W\) or on the reinforcement sequence \((r_n)_n\). This will not be the case of the periodic component \((Z_n^{(2)})_n\), which will lead in some situations to the non-convergence of the process \((Z_n)_n\), as we will see in the following subsection.

3.2. Non-almost sure convergent regime. In the previous subsection we have shown that the first (i.e. \(Z_n^{(1)}\)) and the third component (i.e. \(Z_n^{(3)}\)) always converge almost surely, while regarding the second component \(Z_n^{(2)}\), combining Theorem 2.4 and Proposition 3.1, it only remains to consider the set of conditions summarized in the following assumption (which is exactly the framework considered in Example 2.1)

**Assumption 1.** Assume that all the following conditions hold true:

1. \(n_{\text{per}} \geq 2\),
2. \(\sum n r_n = +\infty\),
3. \(\sum n r_n (1 - r_n) < +\infty\).

In this section, we focus on this framework and we will show that only two events are possible, both with a strictly positive probability: either all the processes both converge and asymptotically synchronize to the same barrier, or they neither converge nor asymptotically synchronize altogether, following in this second case a periodic dynamics that can be described by the structure of the cyclic classes identified by the matrix \(W^\top\). The proofs of the following results are postponed in Section 5.

From Corollary 2.3 and Corollary 2.5, we know that, under Assumption 1, there is a partial almost sure asymptotic synchronization: indeed, we have the almost sure asymptotic synchronization within each cyclic class, but not an almost sure asymptotic synchronization across different classes, i.e. not a complete almost sure asymptotic synchronization of the system. Indeed, by Theorem 2.4, we have that \(Z_n^{(2)}\) does not converge to zero with probability one. However, we have not specified yet whether it converges or not. To clarify this point, let us identify each cyclic class \(h\) with the scalar process \(Z_n^{(c)}\) of the unique value assumed by \(Z_n^{(C)} = (Z_n^{(1)} + Z_n^{(2)})\) along that class. Formally, we can define \(Z_n^{(c)}\) as the quotient of \(Z_n^{(C)}\) with respect to \(\sim_c\). Alternately, component-wise, for any cyclic class \(h = 0, \ldots, n_{\text{per}} - 1\),

\[
Z_n^{(c)} = \sum_{l_1 \in \text{cyclic class } h} v_{l_1} Z_{n,l_1}.
\]

The asymptotic behavior of the process \((Z_n^{(c)})_n\) in general will depend on the properties of the matrix \(W\) and on the reinforcement sequence \((r_n)_n\). However, there is a very general result about the almost sure convergence of the norm of \(Z_n^{(c)}\), that always holds without any additional assumption on \(W\) or \((r_n)_n\).

**Theorem 3.2** (Convergence of the norm). The sequence \((||Z_n^{(c)}||)_n\) of the norms of \((Z_n^{(c)})_n\) almost surely converges.

Notice that, although Theorem 3.2 holds regardless of the behavior of \((r_n)_n\), the interpretation of \(||Z_n^{(c)}||\), and in general of \(Z_n^{(c)}\), is meaningful only when the single processes \(Z_{n,l}\) within each
cyclic class almost surely synchronize in the limit, that is, by Corollary 2.3, when \( \sum_n r_n = +\infty \). Indeed, under this condition, the component \( Z^{(c)}_n \) converges almost surely to zero and so we have \( Z^{(c)}_{n,l} \xrightarrow{a.s.} Z^{(c)}_{n,l} \) for each \( l \), that is \( Z^{(c)}_{n,l} \) is asymptotically closed to the element of \( Z^{(c)} \) corresponding to its cyclic class. Moreover, Theorem 3.2 becomes especially interesting when \( Z^{(c)}_n \) does not converge with probability one, that is, by Corollary 2.5, when \( \sum_n r_n(1 - r_n) < +\infty \) and \( n_{\text{per}} \geq 2 \). Combining together the above two considerations, in the next theorem we consider the scenario under Assumption 1, and we show that each process \( (Z^{(c)}_{n,l})_n, h = 0, \ldots, n_{\text{per}} - 1 \), is asymptotically close to the barrier-set \( \{0,1\} \) and the number of processes close to a given barrier almost surely converges to a random variable not concentrated in \( \{0, n_{\text{per}}\} \).

**Theorem 3.3** (Limit set for the cyclic classes). **Under Assumption 1, we have:**

(a) the limit set of each cyclic class is given by the barrier-set \( \{0,1\} \) (in other words, we have \( Z^{(c)}_{n,h}(1 - Z^{(c)}_{n,h}) \xrightarrow{a.s.} 0 \) for each \( h = 0, \ldots, n_{\text{per}} - 1 \));

(b) for any fixed \( \epsilon \in (0, 1/2) \), we have \( \text{card}\{h : Z^{(c)}_{n,h} \geq 1 - \epsilon\} \xrightarrow{a.s.} N_\infty \) and \( \text{card}\{h : Z^{(c)}_{n,h} \leq \epsilon\} \xrightarrow{a.s.} n_{\text{per}} - N_\infty \), where \( N_\infty \) is a random variable taking values in \( \{0,1, \ldots, n_{\text{per}}\} \) with \( P(N_\infty = 0) + P(N_\infty = n_{\text{per}}) < 1 \).

**Remark 2.** We highlight that condition \( n_{\text{per}} \geq 2 \) is essential to have \( P(N_\infty = 0) + P(N_\infty = n_{\text{per}}) < 1 \) in the above theorem, while it is not used to get the other affirmations (that hold also for \( n_{\text{per}} = 1 \)). However, \( P(N_\infty = 0) + P(N_\infty = n_{\text{per}}) < 1 \) is a crucial point, as \( \{N_\infty = 0\} \) and \( \{N_\infty = n_{\text{per}}\} \) are the two events on which the asymptotic synchronization of the entire system still holds. Note that, since Theorem 3.3(a), on these two events we have the almost sure *asymptotic polarization* of \( (Z_n)_n \), that is in the limit the system almost surely synchronizes toward 0 or 1.

Theorem 3.3(a) states that, under the assumed conditions for \( (r_n)_n \), when \( n \) is large, \( Z^{(c)}_{n,h} \) can only be close to 1 or 0. Roughly speaking, this means that at a large time-step there are some cyclic classes in which the agents’ personal inclinations are very close to one (“on”) and other classes in which they are very close to zero (“off”). In other words, at sufficiently large time-steps, the cyclic classes are *polarized* on the status on or off. Notice that the number of cyclic classes that are on at a large time-step \( n \) can be well represented by \( ||Z^{(c)}_n||^2 \). (Indeed, this fact and Theorem 3.2 are the keys to prove the convergence stated in point (b) of Theorem 3.3.) However, when \( N_\infty \in \{1, \ldots, n_{\text{per}} - 1\} \), it is still not known if each class converges or if it can switch from on to off and vice versa infinitely often. To this end, let us first define the processes \( (X^{(c)}_n)_n \) and \( (X^{(c)}_n)_n \) analogously as what has been done with \( (Z^{(c)}_n)_n \) and \( (Z^{(c)}_n)_n \) in (7) and (10), respectively, i.e. for each element \( l \in V = \{1, \ldots, N\} \),

\[
X^{(c)}_{n,l} = \sum_{l_1 \sim l} \sum_{l_2 \sim l} v_{l_1} v_{l_2} X^{(c)}_{n,l_1,l_2},
\]

and \( X^{(c)}_n = X^{(c)}_n / \sim_c \) as the quotient of \( X^{(c)}_n \) with respect to \( \sim_c \). Then, we can present the following asymptotic result.

**Theorem 3.4** (Clockwise dynamics and asymptotic periodicity). **Under Assumption 1, there exists an integer-valued increasing sequence \( (\sigma_n)_n \) such that, for any cyclic class \( h \), we have for \( (Z_n)_n \) a clockwise dynamics in \( (\sigma_n)_n \), that is

\[
Z^{(c)}_{\sigma_n,h-1} - Z^{(c)}_{\sigma_{n-1},h} \xrightarrow{a.s.} 0 \quad \forall h = 0, \ldots, n_{\text{per}} - 1,
\]
and a stationary dynamics between the clock-times, that is

\[ \sup_{m_1, m_2 \in \{\sigma_{n-1}, ..., \sigma_n-1\}} \|Z_{m_1} - Z_{m_2}\| \xrightarrow{a.s.} 0. \]

Moreover, we have

\[ P(X_{\sigma_n, l_1} = X_{\sigma_n, l_2} \text{ eventually}) = 1 \quad \forall l_1 \sim_c l_2 \]

and

\[ P(X^{(c)}_{\sigma_n, h-1} = X^{(c)}_{\sigma_n-1, h} \text{ eventually}) = 1 \quad \forall h = 0, \ldots, n_{\text{per}} - 1. \]

If, in addition \( \sum_n (1 - r_n) < +\infty \), then, eventually \( \sigma_{n+1} = \sigma_n + 1 \) so that

\[ Z_{n+n_{\text{per}}} - Z_n \xrightarrow{a.s.} 0 \quad \text{and} \quad P(X_{n+n_{\text{per}}} = X_n \text{ eventually}) = 1. \]

An important consequence of Theorem 3.3 and Theorem 3.4 is that Assumption 1 is sufficient for having a non-convergent asymptotic periodic behavior of \( Z_n \) with a strictly positive probability. In addition, from Theorem 3.3 we know that the limit set of each cyclic class is \( \{0, 1\} \). Then, when \( N_\infty \in \{1, \ldots, n_{\text{per}} - 1\} \), the asymptotic periodic dynamics of the vector \( Z^{(c)}_n \) presented in Theorem 3.4 can be seen as a cycle over \( n_{\text{per}} \) elements of \( \{0, 1\}^{n_{\text{per}}} \), obtained by rotating their elements (see e.g. Example 2.1 where, on the set \( A \), we have \( Z^{(c)}_{2n} \to (0, 1)^\top \) and \( Z^{(c)}_{2n+1} \to (1, 0)^\top \)). Moreover, we note that the a.s. asymptotic synchronization of the random variables \( X_{\sigma_n, l} \) inside the same cyclic class implies \( X_{\sigma_n, l} \xrightarrow{a.s.} X^{(c)}_{\sigma_n, l} \) for each \( l \), that is \( X_{n,l} \) is asymptotically close to \( X^{(c)}_{\sigma_n, h} \), where the index \( h \) indicates the cyclic class of \( l \). Therefore, for a large \( n \), at time-step \( \sigma_n \), all the agents of the same cyclic class perform the same action (0 or 1) and these performed actions are periodic. Further, we point out that the sequence \( (\sigma_n)_n \) is explicitly defined in terms of the values of the sequence \( (r_n)_n \) as described at the beginning of the proof of Theorem 3.4.

In order to better understand the regime identified by Theorem 3.4, we have added Figure 1 and Figure 2 representing a setting in which the assumptions of Theorem 3.4 hold and the status of the agents’ inclinations (blu = close to 0 = off, red = close to 1 = on) and the actions performed by the agents (blu = 0, red = 1) inside each cyclic class are shown at different time-steps. Specifically, we consider a network with \( N = 120 \) processes connected by an interacting matrix \( W \) (randomly) generated to have period \( n_{\text{per}} = 6 \). In the two figures the agents belonging to the same cyclic class have been positioned closed to each others, in order to visualize the a.s. (partial) synchronization within each cyclic class. The sequence \( (r_n)_n \) has been chosen such that \( \sum_{n \in (4m)_n} r_n < +\infty \) and \( \sum_{n \in (4m)_n} (1 - r_n) < +\infty \), which means that \( \sigma_{n+1} - \sigma_n = 4 \). Therefore, as described in Theorem 3.4, we can see from Figure 1 that, for a large \( n \), the agents’ inclinations are synchronized inside the cyclic classes and present a clockwise periodic behavior along the subsequence \( (\sigma_n)_n \), while they remain in the same previous status for time-steps in \( \{\sigma_{n+1} + 1, \ldots, \sigma_n - 1\} \). Moreover, in Figure 2, we can see that, for a large \( n \), also the agents’ actions at the clock-times \( \sigma_n \) are synchronized inside the cyclic classes and present a clockwise periodic behavior along the subsequence \( (\sigma_n)_n \).

4. SOME COMPLEMENTS

We here collect some complements to the above results.

4.1. Empirical means. As in [4], we can consider the weighted average of times in which the agents adopt “action 1”, i.e. the stochastic processes of the weighted empirical means \( \{(N_{n,l})_n : l \in V\} \) defined, for each \( l \in V \), as \( N_{0,l} = 0 \) and, for any \( n \geq 1 \),

\[ N_{n,l} = \sum_{k=1}^{n} q_{n,k} X_{k,l}, \quad \text{where} \quad q_{n,k} = \frac{a_k}{\sum_{\ell=1}^{n} a_\ell}, \]
Figure 1. Each panel represents the agents’ inclinations at different consecutive time-steps (large enough such that the asymptotic regime described in Theorem 3.4 can be observed). Any vertex $h \in V$ is represented by a specific point in each panel, where its color indicates the value of $Z_{n,h} \in (0, 1)$ and its symbol represents its cyclic class. The sequence $(r_n)_n$ is such that $r_n = (1 - cn^{-\gamma})$ when $n \in (4m)_m$ and $r_n = cn^{-\gamma}$ otherwise, with $c = 1$ and $\gamma = 3.7$.

Figure 2. Each panel represents the agents’ actions at different clock-times $\sigma_n$ (with $n$ large enough such that the asymptotic regime described in Theorem 3.4 can be observed). Any vertex (agent) $l \in V$ is represented by a specific point in each panel, where its color indicates the value of $X_{n,l} \in \{0, 1\}$ (action performed by the agent) and its symbol represents its cyclic class. The sequence $(r_n)_n$ is such that $r_n = (1 - cn^{-\gamma})$ when $n \in (4m)_m$ and $r_n = cn^{-\gamma}$ otherwise, with $c = 1$ and $\gamma = 3.7$.

with $(a_k)_{k \geq 1}$ a suitable sequence of strictly positive real numbers. In particular, when $a_k = 1$ for each $k$, then the processes $\{(N_{n,l})_n : l \in V\}$ simply coincide with the empirical means associated to the processes $\{(X_{n,l})_n : l \in V\}$. Instead, if, according to the principle of reinforced learning, we want to give more “weight” to the current, or more recent, experience, we can choose $(a_k)_{k \geq 1}$
increasing.

When the process \((Z_n)_n\) almost surely converges toward a random variable, say \(Z_\infty\), we have

\[
E[X_{n+1} | F_n] = W^T Z_n \xrightarrow{a.s.} W^T Z_\infty.
\]

Therefore, assuming \(q_{n,n} = \frac{c}{n^q} + O\left(\frac{1}{n^q}\right)\), with \(q > 0\) and \(0 < \nu \leq 1\) and applying Lemma B.1 in [3], with the same computations done in the proof of Theorem 3.1 in [4], we obtain that

\[
N_n = [N_{n,1}, \ldots, N_{n,N}]^T \xrightarrow{a.s.} W^T Z_\infty.
\]

When the entire system almost surely synchronizes in the limit, we have \(Z_\infty = Z_\infty 1\) and so, since \(W^T 1 = 1\), we get \(W^T Z_\infty = Z_\infty 1\), that is also the weighted empirical means of the entire system almost surely synchronizes in the limit toward \(Z_\infty\).

In the case discussed in Subsection 3.2 under Assumption 1, we can easily prove that there is the same form of partial almost sure asymptotic synchronization for the processes of the weighted empirical means. Indeed, if we take \(l_1\) and \(l_2\) in the same cyclic class \(h\), we can apply the same arguments as above to the difference \((N_{n,l_2} - N_{n,l_1})_n\), provided that \(E[X_{n+1,l_2} - X_{n+1,l_1} | F_n] = \sum_l w_{l,l_2} Z_{n,l} - \sum_l w_{l,l_1} Z_{n,l} \xrightarrow{a.s.} 0\). In order to show this fact, we note that, for each \(l_2 \in \{l_1, l_2\}\), the term \(w_{l,l_2} = [W^T]_{l,l_2}\) is not null only if \(l\) belongs to the cyclic class \((h + 1)\) (recall that \(n_{\text{per}} \geq 2\) under Assumption 1), and hence we have

\[
\sum_l w_{l,l_2} Z_{n,l} = \sum_{l \sim c} w_{l,l_2} (Z_{n,l} - Z_{n,h+1}^{(c)}) + Z_{n,h+1}^{(c)}
\]

\[
\leq \max_{l \sim c} \{|Z_{n,l} - Z_{n,h+1}^{(c)}|\} + Z_{n,h+1}^{(c)} \xrightarrow{a.s.} Z_{n,h+1}^{(c)};
\]

where the last step follows by the almost sure asymptotic synchronization of the processes \((Z_{n,l})_n\) within the same cyclic class \((h + 1)\).

Regarding the almost sure convergence of the processes \(\{(N_{n,l})_n : l \in V\}\) instead, nothing can be said in general if we consider only the conditions in Assumption 1. Indeed, we may have either regimes with almost sure convergence and regimes with non-almost sure convergence, according to the specific sequence \((r_n)_n\) considered. The following two examples show both these scenarios.

**Example 4.1.** Fix \(a \in (0, \frac{1}{2})\) and set the sequence \((r_n)_n\) as follows: \(r_n = (1 - u_n)\) if \(n \in \{a^{-k} : k \geq 1\}\) and \(r_n = u_n\) otherwise, where \((u_n)_n\) is an arbitrary sequence such that \(\sum_n u_n < +\infty\). Naturally, this implies that \(\sum_n r_n = +\infty\) and \(\sum_n (1 - r_n) = +\infty\), so that, assuming \(W\) with \(n_{\text{per}} = 2\), i.e. cyclic classes \(\{0, 1\}\), we are under Assumption 1, i.e. in the framework of Subsection 3.2 with non-almost sure convergence and only almost sure asymptotic synchronization within the cyclic classes of the processes \(\{Z_{n,l} : l \in V\}\). Then, we can apply Theorem 3.4, with \(\sigma_k = [a^{-k}]\), which ensures that there exists a set \(A\) with strictly positive probability such that, for any \(l\) in one of the two cyclic classes, say \(h_1\),

\[
\sup_{n \in \{\sigma_{2k-1} \ldots \sigma_{2k-1}\}} |1 - Z_{n,l}| \xrightarrow{a.s.} 0, \quad \text{and} \quad \sup_{n \in \{\sigma_{2k} \ldots \sigma_{2k+1-1}\}} |Z_{n,l}| \xrightarrow{a.s.} 0
\]

and, for any \(l\) in the other cyclic class, say \(h_2\),

\[
\sup_{n \in \{\sigma_{2k-1} \ldots \sigma_{2k-1}\}} |Z_{n,l}| \xrightarrow{a.s.} 0, \quad \text{and} \quad \sup_{n \in \{\sigma_{2k} \ldots \sigma_{2k+1-1}\}} |1 - Z_{n,l}| \xrightarrow{a.s.} 0.
\]

Then, consider the processes of the simple empirical means, i.e. \(\{N_{n,l} = \frac{1}{m} \sum_{n=1}^m X_{n,l} : l \in V\}\); for any \(l\) in the cyclic class \(h_2\), we can write the following two relations along the sequences of
\((\sigma_{2k})_k\) and \((\sigma_{2k+1})_k\):
\[
\frac{1}{\sigma_{2k}} \sum_{n=1}^{\sigma_{2k}} \mathbb{E}[X_{n,l}|\mathcal{F}_{n-1}] = \frac{1}{\sigma_{2k}} \sum_{n=1}^{\sigma_{2k}} |WZ_{n-1}|_l \\
\geq \left( \frac{\sigma_{2k} - \sigma_{2k-1}}{\sigma_{2k}} \right) \inf_{n \in \{\sigma_{2k-1}, \ldots, \sigma_{2k-1}\}, l \in h_1} Z_{n,l} \xrightarrow{a.s.} (1 - a),
\]
\[
\frac{1}{\sigma_{2k+1}} \sum_{n=1}^{\sigma_{2k+1}} \mathbb{E}[X_{n,l}|\mathcal{F}_{n-1}] = \frac{1}{\sigma_{2k+1}} \sum_{n=1}^{\sigma_{2k+1}} |WZ_{n-1}|_l \\
\leq \left( \frac{\sigma_{2k+1} - \sigma_{2k}}{\sigma_{2k+1}} \right) \sup_{n \in \{\sigma_{2k}, \ldots, \sigma_{2k+1-1}\}, l \in h_1} Z_{n,l} + \left( \frac{\sigma_{2k}}{\sigma_{2k+1}} \right) \xrightarrow{k \to +\infty} a,
\]
which imply that, on the set \(A\), for any \(l\) in the cyclic class \(h_2\), \(N_{\sigma_{2k,l}} \xrightarrow{} (1 - a)\) and \(N_{\sigma_{2k+1,l}} \xrightarrow{} a\). Therefore, in this example the processes \(\{N_{n,l} : l \in V\}\) do not converge almost surely.

**Example 4.2.** Consider the framework of Example 4.1 with a different reinforcement sequence \((r_n)_n\); indeed, here we set \(r_n = (1 - u_n)\) for any \(n \geq 1\), where we recall that \(\sum_n u_n < +\infty\), and hence \(\sum_n (1 - r_n) < +\infty\). Then, if we apply Theorem 3.4 we have \(\sigma_k = k\) for any \(k \geq 1\), recalling that on the set \(A\), for any \(l\) in the cyclic class \(h_2\), we have
\[
[WZ_{2n}]_l \leq \max_{i \in h_1} \{Z_{2n-1,i}\} \xrightarrow{a.s.} 0, \quad \text{and} \quad [WZ_{2n-1}]_l \geq \min_{i \in h_1} \{Z_{2n-2,i}\} \xrightarrow{a.s.} 1,
\]
and so
\[
\frac{1}{\sigma_k} \sum_{n=1}^{\sigma_k} \mathbb{E}[X_{n,l}|\mathcal{F}_{n-1}] = \frac{1}{k} \sum_{n=1}^{k} |WZ_{n-1}|_l \\
= \frac{1}{k} \sum_{n=1}^{[k/2]} \left( |WZ_{2n-1}|_l + |WZ_{2n}|_l \right) \mathbb{E}[Z_{k-1}]_l \\
= \frac{1}{k/2} \sum_{n=1}^{[k/2]} \left( \frac{|WZ_{2n-1}|_l + |WZ_{2n}|_l}{2} \right) + o(1)
\]
\[
\xrightarrow{a.s.} 1 \quad \frac{1}{k/2} \sum_{n=1}^{[k/2]} \left( \frac{1}{2} \right) \xrightarrow{a.s.} \frac{1}{2}.
\]
Therefore, in this example, the processes \(\{N_{n,l} : l \in V\}\) converge almost surely.

4.2. **Reducible interaction matrix.** We here explain how to deal with the case of a *reducible* interaction matrix \(W\), as done in [2] (and in [7] for a similar approach to systems of interacting generalized Friedman’s urns).

We recall that in Network Theory a strongly connected component of a directed graph is a maximal sub-graph in which each node may be reached by the others in that sub-graph. The set of strongly connected components forms a partition of the set of the graph nodes. The assumption of irreducibility in this context means that the entire directed graph is strongly connected. When this assumption is not verified, we can consider each connected component (related to the graph seen as undirected) and, for each of these components, we can analyze its condensation graph. We remind that the condensation graph of a connected, but not strongly connected, directed graph \(G\) is the directed acyclic graph \(C_G\), where each vertex is a strongly connected component of \(G\) and an edge in \(C_G\) is present when there are edges in \(G\) between nodes of the two strongly connected components. In our context, the strongly connected components corresponding to leaves vertices in
\( C_G \) are made by network nodes that are not influenced by the nodes of the other strongly connected components.

Moreover, there is an equivalent interpretation in the Markov chain theory, where the concept of strongly connected component is replaced by that of communicating class. Hence, the strongly connected components corresponding to leaves vertices in \( C_G \) are the recurrent communicating classes.

We employ a particular decomposition of \( W^T \) that individuates its recurrent communicating classes. The same decomposition has been applied to the interaction matrix in \([7, 2]\). More precisely, denoting by \( m \), with \( 1 \leq m \leq N \), the multiplicity of the eigenvalue 1 of \( W^T \), the reducible matrix \( W^T \) can be decomposed (up to a permutation of the nodes) as follows:

\[
W^T = \begin{pmatrix}
U_1 & 0 & \ldots & 0 & 0 \\
0 & U_2 & 0 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & U_m & 0 \\
U_{1,f} & U_{2,f} & \ldots & U_{m,f} & U_f
\end{pmatrix},
\]

where

(i) \( \{U_s; 1 \leq s \leq m\} \) are irreducible \( N_s \times N_s \)-matrices with leading eigenvalue equal to 1, that identify the recurrent communicating classes;

(ii) (if \( \sum_{s=1}^{m} N_s \leq N-1 \)) \( U_f \) is a \( N_f \times N_f \)-matrix, that contains all the transient communicating classes;

(iii) (if \( \sum_{s=1}^{m} N_s = N-1 \)) \( \{U_{s,f}; 1 \leq s \leq m\} \) are \( N_s \times N_f \)-matrices.

Obviously, when \( \sum_{s=1}^{m} N_s = N \) we have \( N_f = 0 \) and hence the elements in \( \{U_{s,f}; 1 \leq s \leq m\} \) and \( U_f \) do not exist. This occurs when all the classes are closed and recurrent (leaves vertices without parents) and hence the state space can be partitioned into irreducible and disjoint sub-spaces. In the particular case of a \( W^T \) irreducible, considered previously in this paper, there is only one closed and recurrent class and hence \( m = 1, N_1 = N \) and \( N_f = 0 \).

Summing up, the structure of \( W^T \) given in (11) leads to a natural decomposition of the graph in different sub-graphs \( \{G_s; 1 \leq s \leq m\} \) associated to the sub-matrices \( \{U_s; 1 \leq s \leq m\} \) and \( G_f \) associated to \( U_f \). In addition, from (1) and (11), we can deduce that, for each \( 1 \leq s \leq m \), the nodes in \( G_s \) are not influenced by the nodes in the rest of the network, and hence the dynamics of the processes in \( G_s \) can be fully established by considering only the correspondent irreducible sub-matrix \( U_s \). Hence, applying the results presented in this paper to each sub-graph \( G_s \), it is possible to characterize the first-order dynamics of the nodes in it.

Concerning the sub-graph \( G_f \), we first note that this is composed by the union of all the transient classes. These classes are not independent of the behavior of the rest of the network. More precisely, the condensation graph shows the conditional dependence in the reverse order: starting from the leaves \( \{G_s; 1 \leq s \leq m\} \), whose dynamics may be computed independently from the rest, the behavior of another strongly connected component is always independent of its parent vertices in the condensation graph, given the dynamics of its children vertices, while the children vertices have an effect on it. For instance, if we are in the standard scenario when \( \sum_n r_n = +\infty \) and \( \sum_n r_n^2 < +\infty \), then all the agents’ inclinations in the same \( G_s \) almost surely synchronize toward a random limit \( Z_{\infty,s} \) and each of the agents’ inclinations in \( G_f \) converges almost surely toward a suitable convex combination of the limits \( \{Z_{\infty,s}; 1 \leq s \leq m\} \) of the processes related to the vertices in \( \{G_s; 1 \leq s \leq m\} \) (see [2]). Hence, they do not necessarily synchronize.

4.3. Non-stochastic interaction matrix (the case of a non-homogeneous “forcing input” toward zero). We here assume the non-negative interaction matrix to be such that \( W^T 1 = d \neq 1 \), with \( d_l = 1 \ \forall l \in V \), as \( d_l > 1 \) could lead to \( |W^T Z_n| > 1 \), which is not compatible with the
of the system is given by the process \((\tilde{Z}_n)\) where \(Z_n \equiv Z_n^{\top} \sim d\). This, together with the fact that \((\tilde{Z}_n)\) is a non-negative martingale and so it almost surely converges toward a random variable \(Z^*\) obtained before under suitable assumptions might be replaced by the relation \(Z_n \overset{a.s.}{\sim} \tilde{Z}_n \overset{a.s.}{\sim} \sum_{k=0}^{n-1}(1-r_k(1-\lambda^*)) Z^* u \overset{a.s.}{\sim} 0\), with possibly \(u \neq 1\).

From an applicative point of view, the above behavior can be explained by the presence of a forcing input that pushes agents to action 0. When \(d_1 = 0\), we simply have \(P(X_{n+1,l} = 1|\mathcal{F}_n) = 0\) for each \(n\) (and so \(Z_{n,l} \overset{a.s.}{\sim} 0\)). Hence, we can assume \(d_1 > 0\) for each \(l\) and we can write

\[ P(X_{n+1,l} = 1|\mathcal{F}_n) = |W^\top Z_n|_{l} = d_l|W^\top Z_n|_{l} + (1-d_l)q \quad \text{with } q = 0, \]

where, for each \(l\), the column \(l\) of \(W^\top\) is obtained taking the column \(l\) of \(W\) divided by \(d_l\) so that \(W^\top 1 = 1\), and \(d_l < 1\) for at least one \(l\). Note that, when \(d = d1\) with \(d \in [0,1]\), we essentially have the model with forcing input \(q\) equal to zero described in [2,13], where the input acts homogeneously on all the agents of the network. Otherwise, when the vector \(d\) has at least two different entries, the forcing input is not homogeneous, in the sense that it does not affect the agents in the same way.

5. Proofs

In this section, we describe the crucial ideas and the sketches of the proofs of the results presented in this work. All the details are collected in a separate supplementary material [6].

First, let us introduce the following notation the we will adopt in this section:

1. \(n \in \mathbb{N}\) is the time-step index;
2. \(l,l_1,l_2,\ldots \in \{1,\ldots,N\}\) are spatial indexes related to the vertices of the network;
3. the indices \(j,j_1,j_2,\ldots \in \{0,\ldots,n_{\text{per}}-1\}\) are related to the \(n_{\text{per}}\) complex roots of the unit, and they are always defined modules \(n_{\text{per}}\);
4. the indices \(h,h_1,h_2,\ldots \in \{0,\ldots,n_{\text{per}}-1\}\) are related to the cyclic classes, and they are always defined modules \(n_{\text{per}}\).

Moreover, given a complex matrix \(A\), we denote by \(Sp(A)\) the set of its eigenvalues and by \(A^*\) the conjugate transpose of \(A\).

Before discussing the structures of the proofs, we would like to point out that the following proofs can be considerably simplified in the “standard” case \(\sum r_n = +\infty\) and \(\sum r_n^2 < +\infty\). However,
our aim is not only to obtain sufficient conditions in order to obtain the complete almost sure asymptotic synchronization of the system, but to get sufficient and necessary conditions for complete or partial almost sure asymptotic synchronization. Therefore, we also include in our analysis the case \( \sum_n r_n < +\infty \) and the case \( \sum_n r_n^2 = +\infty \), which is a completely different scenario compared to the one considered in [2, 3, 4, 14, 16]. The case \( \sum_n r_n^2 = +\infty \) has been considered in [13], but only for a very specific choice of \( W \), i.e. the mean-field interaction (that is also included in the present study as a special case), and with regard to the problem of asymptotic polarization of the process \( \sum_{l=1}^N Z_{n,l}/N_n \).

5.1. Spectral representation of the components of the decomposition of \( (Z_n)_n \) (proof of Theorem 2.1). The key-point in order to prove the asymptotic results of Section 2 and Section 3 is the spectral representation of the three components of the decomposition of \( Z_n \), that is stated in Theorem 2.1. To this end, let us denote with \( P^{(2)} \) and \( P^{(3)} \) (resp. \( P^{i(2)} \) and \( P^{i(3)} \)) the matrix whose row (resp. column) vectors are the left (resp. right) eigenvectors of \( W^\top \) associated to the eigenvectors with \( |\lambda| = 1 \) (except \( \lambda = 1 \)) and \( |\lambda| < 1 \), respectively. Then, the proof of Theorem 2.1, which is detailed in the supplementary material, can be here summarized in the following points:

(i) By construction we immediately have that \( Z^{(1)}_n = 1v^\top Z_n = \tilde{Z}_n 1 \).

(ii) Regarding \( Z^{(2)}_n \), first we derive the analytic expressions of the \( n_{\text{per}} \) eigenvalues and (both left and right) eigenvectors associated with \( Sp(W^\top) = \{ \lambda : |\lambda| = 1 \} \), i.e. for \( j = 0, \ldots, n_{\text{per}} - 1 \),

(i.a) \( \lambda_{1,j} = \exp(\frac{2\pi j}{n_{\text{per}}}) \),

(i.b) \( [P^{(2)}]_{l_2,j_2} = [v_1^\top]_{l_2} v_{l_2} \lambda_{1,j}^{-h} = e^{-\frac{2\pi j}{n_{\text{per}}} h} v_{l_2} \),

(i.c) \( [P^{(2)}]_{l_2,j} = [q_j]_{l_2} \lambda_{1,j} = e^{\frac{2\pi j}{n_{\text{per}}} h} \).

(iii) Secondly, we use the expressions derived at point (ii) to show that the element \((l_1, l_2)\) of the matrix \( (1v^\top + P^{i(2)} P^{(2)}) \) is equal to \( v_{l_2} n_{\text{per}} \) if \( l_1 \) and \( l_2 \) belong to the same cyclic class, and zero otherwise. This will imply that

\[
\begin{pmatrix}
(1v^\top + P^{i(2)} P^{(2)}) Z_n \end{pmatrix}_l = n_{\text{per}} \sum_{l_2 \sim l} v_{l_2} Z_{n,l_2},
\]

(iv) Then, we focus on the leading eigenvector \( v \) and we show that the quantity \( \sum_{l \in \text{cyclic class } h} v_l \) does not depend on the choice of the cyclic class \( h \) and, in particular, \( \sum_{l \in \text{cyclic class } h} v_l = n_{\text{per}}^{-1} \). This implies by definition (7) that

\[
Z^{(1)}_{n,l} + Z^{(2)}_{n,l} = n_{\text{per}} \sum_{l_1 \sim l} v_{l_1} Z_{n,l_1},
\]

which combined with point (iii) concludes the part on \( Z^{(2)}_n \).

(v) Finally, the part on \( Z^{(3)}_n \) simply follows by the properties of a basis of eigenvectors:

\[
Z^{(3)}_n = Z_n - (Z^{(1)}_n + Z^{(2)}_n) = (I - (1v^\top + P^{i(2)} P^{(2)})) Z_n = P^{(3)} P^{i(3)} Z_n.
\]

5.2. Proofs of the asymptotic results stated in Section 2. In this section, we employ the matrices \( P^{(2)}, P^{i(2)}, P^{(3)} \) and \( P^{i(3)} \) and the representation of the three components of the decomposition of \( Z_n \) given in Theorem 2.1. Moreover, we firstly provide an equation that will be used in the sequel. By definition of \( Z_n \) in (4) we immediately obtain

\[
Z_{n+1} = Z_n - r_n (I - W^\top) Z_n + r_n \Delta M_{n+1},
\]
where $\Delta M_{n+1} = (X_{n+1} - W^\top Z_n)$ is the bounded increment of a martingale. Note that the conditional independence of the entries of $X_{n+1}$ implies that

$$E[\Delta M_{n+1,l_1}\Delta M_{n+1,l_2} | F_n] = \begin{cases} (1 - |W^\top Z_n|_{l_1})|W^\top Z_n|_{l_2} & \text{if } l_1 = l_2, \\ 0 & \text{otherwise}, \end{cases}$$

and hence, for any complex matrix $A$, consider

$$\Delta \sum A$$

and denote by $\tilde{V}_n = (1 - W^\top Z_n)^\top W^\top Z_n$. To this process $(\tilde{V}_n)_n$ is devoted the next lemma.

**Lemma 5.1.** Set $V_n = (1 - W^\top Z_n)^\top W^\top Z_n = \sum_{j=1}^N |W^\top Z_n|_j (1 - |W^\top Z_n|_j)$. Then, we have

$$\sum r_n^2 V_n < +\infty \ a.s. \quad \text{and} \quad \sum r_n^2 E[V_n] < +\infty.$$

**Proof.** Consider the dynamics (6) of the process $(\tilde{Z}_n)_n$, i.e. $\tilde{Z}_0 = v^\top Z_0$, $\tilde{Z}_{n+1} - \tilde{Z}_n = r_n (Y_{n+1} - \tilde{Z}_n)$, $Y_{n+1} = v^\top X_{n+1}$, and denote by $\langle \tilde{Z} \rangle = (\langle \tilde{Z} \rangle)_n$ the predictable compensator of the submartingale $\tilde{Z}^2 = (\tilde{Z}_n^2)_n$. Since $\tilde{Z}$ is a bounded martingale, we have that $\langle \tilde{Z} \rangle_n$ converges a.s. and its limit $\langle \tilde{Z} \rangle_\infty$ is such that $E[\langle \tilde{Z} \rangle_\infty] < +\infty$. Then, setting $F_n = \sigma(Z_0, X_1, \ldots, X_n)$, since

$$\langle \tilde{Z} \rangle_\infty = \sum_n (\langle \tilde{Z} \rangle_{n+1} - \langle \tilde{Z} \rangle_n) = \sum_n E[(\tilde{Z}_{n+1} - \tilde{Z}_n)^2 | F_n] = \sum_n r_n^2 E[(Y_{n+1} - \tilde{Z}_n)^2 | F_n],$$

the result (14) follows if we show that there exists a constant $c > 0$ such that $cE[(Y_{n+1} - \tilde{Z}_n)^2 | F_n] \geq V_n$. To this end, we observe that

$$E[(Y_{n+1} - \tilde{Z}_n)^2 | F_n] = E[(v^\top X_{n+1} - \tilde{Z}_n)^2 | F_n] = E[(v^\top X_{n+1} - v^\top W^\top Z_n)^2 | F_n]$$

$$= E[(v, X_{n+1} - W^\top Z_n)^2 | F_n] = \sum_{j=1}^N v_j^2 E[(X_{n+1,j} - |W^\top Z_n|_j)^2 | F_n]$$

$$= \sum_{j=1}^N v_j^2 |W^\top Z_n|_j (1 - |W^\top Z_n|_j) \geq v_{\min}^2 \sum_{j=1}^N |W^\top Z_n|_j (1 - |W^\top Z_n|_j)$$

where we recall that the eigenvector $v$ of $W$ associated to the eigenvalue $\lambda = 1$ has strictly positive entries and so we have $v_{\min} = \min_j \{v_j\} > 0$. This concludes the proof.

Now we can focus on the main asymptotic results on $Z_n^{(2)}$ and $Z_n^{(3)}$. In particular, we start by proving when $Z_n^{(3)}$ asymptotically vanishes with probability one.

**Proof of Theorem 2.2.** Note that when $N = n_{\text{per}}$, $Z_n^{(3)} \equiv 0$ by definition (see Theorem 2.1). Then, we assume $N > n_{\text{per}}$.

$(\Leftarrow)$ We will prove that $\sum r_n = +\infty$ implies $P^{(3)} Z_n \to 0$. To do so, let us recall that

$$P^{(3)} W^\top = J_{N-n_{\text{per}}} P^{(3)}.$$


where $J_{N-n_{\text{per}}}$ contains the Jordan blocks associated to the eigenvalues $\{\lambda \in Sp(W) : |\lambda| < 1\}$. Therefore, it is sufficient to prove that $P_\lambda Z_n \rightarrow 0$, where $P_\lambda$ is the generalized eigenspace associated to an eigenvalue $\lambda$ of $W^\top$ with $|\lambda| < 1$, i.e. $P_\lambda W^\top = J_{W,\lambda} P_\lambda$, and let us introduce as $\|A\|_{p, q}$ the $(p, q)$-operator norm of a complex matrix $A \in \mathbb{C}^{M \times N}$, whose properties are described in the supplementary material [6]. We will not prove that $P_\lambda Z_n$ tends to 0 directly, since the presence of the ones on the upper diagonal of $J_{W,\lambda}$ causes that the term $\|J_{W,\lambda}\|_{2, 2}$ is not close to $|\lambda|$, and in particular it is bigger than 1. For this reason, we need to modify the Jordan space to face this issue. Technical details on this task have been relegated in the supplementary materials. Then, from now on, we can consider to have

\begin{equation}
\|J_{W,\lambda}\|_{2, 2} \leq \frac{1 + |\lambda|}{2} < 1.
\end{equation}

**Almost sure convergence of $P_\lambda Z_n$ to zero:**

Now we apply the Jordan base $P_\lambda$ to $Z_n$, and we show that $B_n = P_\lambda Z_n$ tends to 0. By (12) we immediately obtain that

\[
B_{n+1} = P_\lambda Z_{n+1} = B_n - r_n(I - J_{W,\lambda})B_n + r_n P_\lambda \Delta M_{n+1} = ((1 - r_n)I + r_n J_{W,\lambda})B_n + r_n P_\lambda \Delta M_{n+1}.
\]

Set $\|B_{n+1}\|^2 = B_{n+1}^\top B_{n+1}$ so that, by (12), (13) and (15), we have

\[
E[\|B_{n+1}\|^2 | \mathcal{F}_n] = \|(1 - r_n)I + r_n J_{W,\lambda} B_n\|^2 + r_n^2 (1^\top - Z_n^\top W) \text{diag}(P_\lambda^* P_\lambda) W^\top Z_n
\]

\[
\leq \left((1 - r_n) + r_n \|J_{W,\lambda}\|_{2, 2}\right)^2 \|B_n\|^2 + \left(\max_{h=1,\ldots,\lambda} (|P_\lambda^* P_\lambda|_{h,h})\right) r_n^2 (1^\top - Z_n^\top W) W^\top Z_n
\]

\[
\leq \left(1 - r_n (1 - \frac{1 + |\lambda|}{2})\right)^2 \|B_n\|^2 + \vartheta r_n^2 V_n.
\]

As a consequence of Lemma 5.1, $\|B_{n+1}\|^2$ is a non-negative almost supermartingale that converges almost surely (see [32]). We are now going to prove that its almost sure limit is zero. To this purpose, since $\|B_n\|^2_n$ is uniformly bounded by a constant and so $E[a.s. - \lim_n \|B_n\|^2] = \lim_n E[\|B_n\|^2]$, it is enough to prove that $E[\|B_n\|^2]$ converges to zero. To show this last fact, we take the expected values on both sides of the above relation and we obtain

\[
E[\|B_{n+1}\|^2] \leq (1 - ar_n)^2 E[\|B_n\|^2] + \vartheta r_n^2 E[V_n],
\]

where $a = \frac{1 - |\lambda|}{2} > 0$. Hence, $y_n = E[\|B_n\|^2]$ converges to 0 by Lemma S2.2. This means that $\|B_n\|^2_n$ converges almost surely to zero and so $B_n$ converges almost surely to 0, which implies that also $P_\lambda Z_n$ converges almost surely to 0 and the proof of the $\Leftarrow$ implication is completed.

($\Rightarrow$) This part of the proof is reported in the supplementary material. The main steps are the following:

(i) first we prove that, when $P(T_0) < 1$, condition $\sum_n r_n < +\infty$ implies $P(Z, l_2 - Z, l_1 \not\rightarrow 0)$ (i.e. $P(\limsup_n |Z, l_2 - Z, l_1| > 0) > 0$, for any pair of vertices $(l_1, l_2) \in V \times V$, with $l_1 \neq l_2$;

(ii) then we observe that, if $N > n_{\text{per}}$ and we choose $l_1$ and $l_2$ within the same cyclic class, the result of Step (i) implies $P(Z_n^{(3)} \not\rightarrow 0) > 0$. 

\[\square\]
The almost sure convergence to zero of \( Z_n^{(3)} \) has as a direct consequence the almost sure asymptotic synchronization within the cyclic classes as stated in Corollary 2.3. The brief proof of this result is reported in the supplementary material.

We now are going to show when \( Z_n^{(2)} \) asymptotically vanishes with probability one.

**Proof of Theorem 2.4.** By the representation given in Theorem 2.1 we have to prove
\[
Z^{(2)} = P^{(2)}(P^{(2)} Z_n) \overset{a.s.}{\rightarrow} 0
\]
Then, by the representation the left eigenvectors of \( W^\top \) in \( P^{(2)} \), it is sufficient to show that, fixed any \( j_1 \in \{1, \ldots, n_{\text{per}} - 1 \}, \)
\[
\eta_n = v_{j_1}^\top Z_n \overset{a.s.}{\rightarrow} 0,
\]
if and only if \( \sum_n r_n(1 - r_n) = +\infty \). Since \( v_{j_1}^\top W^\top = \lambda_{1,j_1} v_{j_1}^\top \), we immediately obtain by (12)
\[
\eta_{n+1} = v_{j_1}^\top Z_{n+1} = \eta_n - r_n(1 - \lambda_{1,j_1})\eta_n + r_n v_{j_1}^\top \Delta M_{n+1}
\]
and hence
\[
E[|\eta_{n+1}|^2 | F_n] = a_{j_1,n} |\eta_n|^2 + C_{j_1,n},
\]
where
\[
\begin{cases}
a_{j_1,n} = |1 - r_n(1 - \lambda_{1,j_1})|^2, \\
C_{j_1,n} = r_n^2 E[|v_{j_1}^\top \Delta M_{n+1}|^2 | F_n] = r_n^2 E[|\Delta M_{n+1}^\top v_{j_1}^\top \Delta M_{n+1} | F_n].
\end{cases}
\]
Then, we can prove that \( a_{j_1,n} = 1 - s_{j_1,n} \), with \( 0 < s_{j_1,n} \leq 1 \), and by (13) we also have that \( 0 \leq C_{j_1,n} \leq r_n^2 V_n \) (see supplementary material for these technical details). Hence, combining these results in (16) we obtain,
\[
E[|\eta_{n+1}|^2 | F_n] \geq (1 - s_{j_1,n}) E[|\eta_n|^2],
\]
and
\[
E[|\eta_{n+1}|^2 | F_n] \leq (1 - s_{j_1,n}) |\eta_n|^2 + C_{j_1,n}.
\]
We are now ready to conclude.

\((\Leftarrow)\) Case \( \sum_n r_n(1 - r_n) = +\infty \)
Note that, since \( 0 < s_{j_1,n} \leq 1 \) and \( \sum_n C_{j_1,n} < +\infty \) almost surely by Lemma 5.1, we have from (18) that \( |\eta_n|^2 \) is a non-negative almost supermartingale that converges almost surely (see [32]). Since \( (|\eta_n|^2)_n \) is uniformly bounded by a constant and so \( E[a.s. - \lim_n |\eta_n|^2] = \lim_n E[|\eta_n|^2] \), it is enough to prove that \( E[|\eta_n|^2] \) converges to zero. To prove this fact, we take the expected values on both sides of (18), so that we obtain
\[
E[|\eta_{n+1}|^2] \leq (1 - s_{j_1,n}) E[|\eta_n|^2] + E[C_{j_1,n}],
\]
which is of the form \( y_{n+1} \leq (1 - s_n) y_n + \delta_n \). Then, the convergence \( y_n \to 0 \) follows from Lemma S2.2 once we show that the assumptions are verified. We have already checked that \( 0 < s_n \leq 1 \) and that \( \sum_n s_n = +\infty \), while \( \sum_n \delta_n < +\infty \) follows by Lemma 5.1.

\((\Rightarrow)\) Case \( \sum_n r_n(1 - r_n) < +\infty \)
Note that (17) is of the form \( y_{n+1} \geq (1 - s_n) y_n \). Since \( 0 < s_n \leq 1 \) and \( \sum_n s_n < +\infty \), by Lemma S2.1 we have \( \liminf_n y_n > 0 \) whenever \( y_0 > 0 \). Then, we do not have the \( L^2 \)-convergence of \( \eta_n \) to 0 (which, since \( (\eta_n)_n \) is uniformly bounded by a constant, is a necessary condition for the almost sure convergence of \( \eta_n \) to 0) when \( y_0 = E[|\eta_0|^2] > 0 \), that is \( P(|\eta_0| > 0) > 0 \). This last condition is
verified when \( P(T_0) < 1 \), because of the expression of the components of \( v_j \) given in Lemma S1.1 and the fact that \( v_l > 0 \) for each \( l \).

The almost sure convergence to zero of both \( Z_n^{(2)} \) (see Theorem 2.4) and \( Z_n^{(3)} \) (see Theorem 2.2) implies the complete almost sure asymptotic synchronization of the system, as stated in Corollary 2.5. The short proof of this result is reported in the supplementary material.

5.3. Proofs of the results given in Section 3. The simple proof of Proposition 3.1 regarding the case \( \sum_n r_n < +\infty \) can be found in the supplementary material.

Let us now point out in the following lemma the connection of the modulus of the process \( Z_n^{(c)} \) defined in (10) and the non-orthogonal projection of \( Z_n \) on the periodic eigenspaces of \( W^\top \), i.e. \( \text{Span}(1, q_1, \ldots, q_{n_{\text{per}}-1}) \). Also the proof of this result is reported in the supplementary material.

**Lemma 5.2.** Let \( C_n = (n^{(2)}) Z_n \) be the coefficients of the non-orthogonal projection of \( Z_n \) on the left eigenvectors of \( W \) associated to the eigenvalues in \( D_{n_{\text{per}}} \), i.e. \( \{ \lambda \in \text{Sp}(W) : |\lambda| = 1 \} \). Then, \( Z_n^{(c)} = n_{\text{per}}^3 O_{n_{\text{per}}} C_n \), where \( O_{n_{\text{per}}} \) is the orthogonal matrix with \( (O_{n_{\text{per}}})_{ij} = \frac{1}{\sqrt{n_{\text{per}}}} \lambda_{i,j}^{-1} \), that is

\[
O_{n_{\text{per}}} = \frac{1}{\sqrt{n_{\text{per}}}} \begin{pmatrix}
\lambda_{0,0}^0 & \lambda_{1,0}^1 & \cdots & \lambda_{n_{\text{per}}-1,0}^1 \\
\lambda_{0,1}^0 & \lambda_{1,1}^1 & \cdots & \lambda_{n_{\text{per}}-1,1}^1 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{0,n_{\text{per}}-1,0}^0 & \lambda_{1,n_{\text{per}}-1,1}^1 & \cdots & \lambda_{n_{\text{per}}-1,n_{\text{per}}-1,1}^1
\end{pmatrix}.
\]

Therefore, we have that \( ||Z_n^{(c)}|| = n_{\text{per}}^3 ||C_n|| \), where \( ||C_n||^2 = C_{n+1}^* C_n \).

We can now present the proof of the almost sure convergence of \( ||Z_n^{(c)}|| \).

**Proof of Theorem 3.2.** By Lemma 5.2, we have that \( ||Z_n^{(c)}||^2 = n_{\text{per}}^6 ||C_n||^2 \), with \( ||C_n||^2 = C_{n+1}^* C_n \). Then, it is sufficient to prove that \( ||C_n||^2 \) almost surely converges, using the fact that

\[
\left( \frac{v^\top}{p^{(2)}} \right) W^\top = D_{n_{\text{per}}} \left( \frac{v^\top}{p^{(2)}} \right).
\]

Indeed, by (12), we immediately obtain that

\[
C_{n+1} = \left( \frac{v^\top}{p^{(2)}} \right) Z_{n+1} = C_n - n (I - D_{n_{\text{per}}}) C_n + n \left( \frac{v^\top}{p^{(2)}} \right) \Delta M_{n+1}.
\]

Since \( ||C_{n+1}||^2 = C_{n+1}^* C_{n+1} \), \( D_{n_{\text{per}}}^* D_{n_{\text{per}}} = I \) and defining \( C_{n_{\text{per}}} = D_{n_{\text{per}}} + D_{n_{\text{per}}}^* = 2 \text{diag}(\{ \cos(\frac{2 \pi}{n_{\text{per}}} h), h = 0, \ldots, n_{\text{per}} - 1 \}) \), by (13), we obtain

\[
E[||C_{n+1}||^2 | F_n] = C_n^*((1 - r_n) I + r_n D_{n_{\text{per}}}^*)(1 - r_n) I + r_n D_{n_{\text{per}}}) C_n
\]

\[
+ n^2 (1^\top - Z_n W) \text{diag}(\{ \frac{v^\top}{p^{(2)}} \}) (1^\top - Z_n W) \text{diag}(\{ \frac{v^\top}{p^{(2)}} \}) W^\top Z_n
\]

\[
= C_n^* \left( I - 2 r_n (1 - r_n) (I - C_{n_{\text{per}}}) \right) C_n + n^2 (1^\top - Z_n^\top W) \text{diag}(\{ \frac{v^\top}{p^{(2)}} \}) (1^\top - Z_n W) \text{diag}(\{ \frac{v^\top}{p^{(2)}} \}) W^\top Z_n
\]

\[
\leq C_n^* \left( I - 2 r_n (1 - r_n) (I - C_{n_{\text{per}}}) \right) C_n + \left( \max_{h=1,\ldots,N} \{ \{ \frac{v^\top}{p^{(2)}} \}^\top (\frac{v^\top}{p^{(2)}}) h \} \right) n^2 V_n.
\]

The diagonal matrix \( I - C_{n_{\text{per}}} \) has real non-negative entries between 0 and 2 and we have \( r_n (1 - r_n) \leq \frac{1}{4} \), thus

\[
0 \leq C_n^* \left( I - 2 r_n (1 - r_n) (I - C_{n_{\text{per}}}) \right) C_n \leq \frac{||C_n||^2}{4}.
\]
which, together with Lemma 5.1, shows that $||C_n||^2$ is a non-negative almost supermartingales that converges almost surely (see [32]).

The second important result to describe the periodic behavior of the system is Theorem 3.3, which is focused on the limit set of the processes within the cyclic classes. Before presenting the proof of Theorem 3.3, we need to show some technical results (proven in the supplementary material).

**Lemma 5.3.** If $V_n = (1 - W^T Z_n)^T W^T Z_n \xrightarrow{a.s.} 0$, then, for any $l \in \{1, \ldots, N\}$, the limit set of the sequences $(Z_{n,l})_n$ is $\{0,1\}$, that is $Z_{n,l} (1 - Z_{n,l}) \xrightarrow{a.s.} 0$.

**Lemma 5.4.** Assume $\sum_{i=1}^n r_i = +\infty$ and $\sum_{i=1}^n r_i (1-r_i) < +\infty$. For any $n$, set $\delta_n = 1_{\{r_n > 1/2\}}$ and $
 = \inf\{r \geq 1: \sum_{i=1}^n r_i \delta_i \geq n\}$ ($\tau_0 = 0$). Then $\sum_{n} \delta_n = +\infty$, or, equivalently, $\n < +\infty$ for any $n$. In addition, there exists a sequence $(\epsilon_n)_n$ such that $\sum_{n} \epsilon_n < +\infty$ and $\sup_{m \in (\n, \n+1)} \{ |Z_m l_1 - Z_{m+1,l_1}| \} < \epsilon_n$ for any $l_1 \in V = \{1, \ldots, N\}$.

**Proof of Theorem 3.3.** Let $(\n)_n$ as in Lemma 5.4. Now, since $r_{\n} > 1/2$, Lemma 5.1 implies that $V_{r_{\n}} \xrightarrow{a.s.} 0$, and hence Lemma 5.3 entails that, for any $l_1 \in V = \{1, \ldots, N\}$,

$$\min \{ Z_{\n,l_1}, 1 - Z_{\n,l_1} \} \xrightarrow{a.s.} 0.$$  

By Lemma 5.4, we have $\sup_{m \in (\n, \n+1)} \{ |Z_m l_1 - Z_{m+1,l_1}| \} < \epsilon_n \to 0$ and so $\min \{ Z_{\n,l_1}, 1 - Z_{\n,l_1} \} \to 0$ almost surely. This implies that $V_n \to 0$ a.s., which means that the limit set of the sequences $(|W^T Z_n|^l_2)_n$ is $\{0,1\}$, for any $l_2 \in V$. Lemma 5.3 implies that also the limit set of the sequences $(Z_{n,l_1})_n$ is $\{0,1\}$, for any $l \in V$. Then, by Theorem 2.2, for any $l_1$ in the cyclic class $h$, we have that the limit set of

$$Z^{(c)}_{n,h} = Z_{n,l_1} - Z^{(3)}_{n,l_1}$$

is $\{0,1\}$, and hence, for any $0 < \epsilon < 1$,

$$||Z^{(c)}||^2 - \#\{h: Z^{(c)}_{n,h} > 1 - \epsilon\} \xrightarrow{a.s.} 0.$$  

Then, the second part of the proof is a consequence of Theorem 3.2, once we have defined $N_\infty = a.s. - \lim_n ||Z^{(c)}||^2$. In particular, the last statement simply follows by the fact that $P(N_\infty = 0) + P(N_\infty = n_{per}) = 1$ would imply the almost sure synchronization of the entire system, which is in contradiction with Corollary 2.5 when $n_{per} \geq 2$, as here we are assuming $\sum_{i} r_i (1-r_i) < +\infty$. □

Finally, we present the basic ideas and the structure of the proof concerning the asymptotic periodic dynamics of the system. The details are reported in the supplementary material.

**Proof of Theorem 3.4.** Notation and structure of the proof
Let $(\n)_n$ as in Lemma 5.4 and define, for any $n \geq 0$, $\sigma_n = \n + 1$ and the set

$$A_n = \bigcap_{l=1,\ldots,N} \left\{ X_{\sigma_n,l} = 1_{\{1/2,1\}}(|W^T Z_{\n,l}|) \right\}.$$
Moreover, for each cyclic class \( h \in \{0, \ldots, \ell_{\text{per}} - 1 \} \) and binary index \( g \in \{0, 1\} \), let us introduce the following sets:

\[
B^h_{X,n}(g) = \bigcap_{l \in \text{cyclic class } h} \{ X_{\sigma_n,l} = g \},
\]

\[
B^h_{Z,n}(g) = \bigcap_{l \in \text{cyclic class } h} \left\{ \frac{1}{2} \right\} \left( Z_{\tau_n,l} = g \right) = g \},
\]

\[
B^h_{WZ,n}(g) = \bigcap_{l \in \text{cyclic class } h} \left\{ \frac{1}{2} \right\} \left( \left( W^T Z_{\tau_n,l} \right) g \right) = g \},
\]

Notice that the thesis \( P(X_{\sigma_n,n_1} = X_{\sigma_n,n_2} \text{ ev.}) = 1 \) for all \( l_1 \sim_c l_2 \) and \( P(X_{\sigma_n,h-1} = X_{\sigma_n-1,h} \text{ ev.}) = 1 \) can be equivalently written as

\[
P\left( \bigcap_{h=0}^{\ell_{\text{per}}-1} \{ B^h_{X,n-1}(0) \cap B^{h-1}_{X,n}(0) \} \cup \{ B^h_{X,n-1}(1) \cap B^{h-1}_{X,n}(1) \}, \text{ ev.} \right) = 1.
\]

In addition, fix an integer \( n_0 \) and, for any \( n \geq n_0 \), let us define

\[
B_n = \bigcap_{h=0}^{\ell_{\text{per}}-1} B^h_{X,n}(0) \cup B^h_{X,n}(1) = \bigcap_{(l_1,l_2): l_1 \sim_c l_2} \{ X_{\sigma_n,l_1} = X_{\sigma_n,l_2} \},
\]

\[
A_{n_0,n} = \bigcap_{m=n_0}^{n-1} A_m, \quad \text{and} \quad B_{n_0,n} = \bigcap_{m=n_0}^{n-1} B_m.
\]

The proof is structured as follows:

1. We prove that \( \sup_{m_1,m_2 \in \{\sigma_n-1, \ldots, \sigma_n-1\}} \|Z_{m_1} - Z_{m_2}\| \xrightarrow{a.s.} 0 \), i.e. the dynamics between the clock times \((\sigma_n)_n\) is stationary;
2. We show that \( (X_{\sigma_n} - Z_{\sigma_n}) \xrightarrow{a.s.} 0 \), and hence proving (19) is enough to have also that \( (Z_{\sigma_n-1,h} - Z_{\sigma_n-1,h}) \xrightarrow{a.s.} 0 \);
3. We prove that \( A_n \cap B^h_{X,n-1}(g) \subseteq B^{h-1}_{X,n}(g) \), which implies that (19) follows once we show that \( P(A_n \cap B_n, \text{ ev.}) \rightarrow 1 \); the result \( A_n \cap B^h_{X,n-1}(g) \subseteq B^{h-1}_{X,n}(g) \) is established by proving the following:
   3a. \( B^h_{X,n-1}(g) \subseteq B^h_{Z,n}(g) \);
   3b. \( B^h_{Z,n}(g) \subseteq B^h_{WZ,n}(g) \);
   3c. \( B^h_{WZ,n}(g) \cap A_n \subseteq B^h_{X,n}(g) \);
4. We prove that \( P(A_n \cap B_X, \text{ ev.}) = 1 \): since this fact is equivalent to showing that the probability
   \[
P\left( \bigcap_{m=n_0}^{\infty} (A_m \cap B_m) \right) = P(A_{n_0} \cap B_{n_0}) \prod_{m=n_0+1}^{\infty} P(A_m \cap B_m | A_{n_0,m} \cap B_{n_0,m}),
\]
   tends to one as \( n_0 \rightarrow +\infty \), we show
   4a. \( P(A_{n_0} \cap B_{n_0}) \rightarrow 1 \);
   4b. \( \prod_{m=n_0+1}^{\infty} P(A_m \cap B_m | A_{n_0,m} \cap B_{n_0,m}) \rightarrow 1 \);
5. Finally, we show that, since \( \sum_{n}(1 - r_n) < +\infty \) implies \( r_n \rightarrow 1 \), in this case it holds that \( \sigma_{n+1} = \sigma_n + 1 \).
Declaration
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Appendix S1. Details for the proofs

In this section, we will adopt the following notation in accordance with Section 5:

1. \( n \in \mathbb{N} \) is the time-step index;
2. \( l, l_1, l_2, \ldots \in \{1, \ldots, N\} \) are spatial indexes related to the vertices of the network;
3. the indices \( j, j_1, j_2, \ldots \in \{0, \ldots, n_{\text{per}} - 1\} \) are related to the \( n_{\text{per}} \) complex roots of the unit, and they are always defined modules \( n_{\text{per}} \);
4. the indices \( h, h_1, h_2, \ldots \in \{0, \ldots, n_{\text{per}} - 1\} \) are related to the cyclic classes, and they are always defined modules \( n_{\text{per}} \).

Moreover, given a complex matrix \( A \), we denote by \( \text{Sp}(A) \) the set of its eigenvalues and by \( A^* \) the conjugate transpose of \( A \), and we recall that \( (A + B)^* = A^* + B^* \), \( (zA)^* = \overline{z}A^* \), \( (AB)^* = B^*A^* \), \( (A^*)^* = A \).

S1.1. Spectral representation of the components of the decomposition of \((Z_n)_n\) (detailed proof of Theorem 2.1). In this section, we present the proof of Theorem 2.1, which characterizes the decomposition (8) in terms of the non-orthogonal projections on the spectral eigenspaces related to \( W^\top \). Let us consider the Jordan representation of \( W^\top \) in terms of its left and right generalized eigenvectors given by the Perron-Frobenious Theorem for irreducible non-negative matrices:

\[
P W^\top = J_W P, \quad W^\top P^{-1} = P^{-1} J_W,
\]

with

\[
J_W = \begin{pmatrix}
D_{n_{\text{per}}} & 0 \\
0 & J_{N-n_{\text{per}}}
\end{pmatrix},
\]

where \( D_{n_{\text{per}}} \) is the diagonal matrix with the complex \( n_{\text{per}} \)-roots of the unity \( \lambda_{1,j} = \exp\left(\frac{2\pi i}{n_{\text{per}}} j\right) \), \( j = 0, \ldots, n_{\text{per}} - 1 \):

\[
D_{n_{\text{per}}} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & e^{\frac{2\pi i}{n_{\text{per}}}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e^{\frac{2\pi i}{n_{\text{per}}}(-2)} = \lambda_{1,n_{\text{per}}-2} \\
0 & 0 & \cdots & 0 = e^{\frac{2\pi i}{n_{\text{per}}}(-1)} = \lambda_{1,n_{\text{per}}-1}
\end{pmatrix}
\]

and \( J_{N-n_{\text{per}}} \) contains the Jordan blocks associated to the eigenvalues \( \{\lambda \in \text{Sp}(W) : |\lambda| < 1\} \). We already know that we may choose the first line of \( P \) as \( v^\top \) and the first column of \( P^{-1} \) as \( 1 \). In the next lemma, we give a characterization of the left and right eigenvectors related to the complex
roots of the unity. To be more explicit, let us define $c$

$$P = \begin{pmatrix} v^\top_1 \\
\vdots \\
 v^\top_{n_{\text{per}}-1} \\
 p^\top_1 \\
\vdots \\
 p^\top_{N-n_{\text{per}}}
\end{pmatrix} = \begin{pmatrix} v^\top \\
 p^{(2)}_1 \\
\vdots \\
 p^{(3)}_N
\end{pmatrix},$$

where

$$P^{(2)} = \begin{pmatrix} v^\top_1 \\
\vdots \\
 v^\top_{n_{\text{per}}-1}
\end{pmatrix} \quad \text{and} \quad P^{(3)} = \begin{pmatrix} p^\top_1 \\
\vdots \\
 p^\top_{N-n_{\text{per}}}
\end{pmatrix},$$

and analogously

$$P^{-1} = \begin{pmatrix} 1 & q_1 & \cdots & q_{n_{\text{per}}-1} & r_1 & \cdots & r_{N-n_{\text{per}}}
\end{pmatrix} = \begin{pmatrix} 1 & P^{(2)} & P^{(3)}
\end{pmatrix},$$

with

$$P^{(2)} = \begin{pmatrix} q_1 & \cdots & q_{n_{\text{per}}-1}
\end{pmatrix} \quad \text{and} \quad P^{(3)} = \begin{pmatrix} r_1 & \cdots & r_{N-n_{\text{per}}}
\end{pmatrix}.$$
The same holds for the right eigenvectors:

\[ [W^T q_{j_1}]_{l_2} = \sum_{l \in \text{cyclic class } h+1} [W^T]_{l_2}[q_{j_1}]_l = \sum_{l \in \text{cyclic class } h+1} [W^T]_{l_2,l} 1\lambda_{l,j_1}^{h+1} = \lambda_{1,j_1}^h 1\lambda_{1,j_1}^h [W^T]_{l_2} = \lambda_{1,j_1}^h [q_{j_1}]_{l_2}. \]

For what concerns the orthonormalization condition, we note that

\[ \lambda_{1,j_1}^h v_{j_1}^T q_{j_2} = v_{j_1}^T W^T q_{j_2} = \lambda_{1,j_2}^h v_{j_1}^T q_{j_2}, \]

that means that \( v_{j_1}^T q_{j_2} = 0 \) if \( j_1 \neq j_2 \), as in this case \( \lambda_{1,j_1}^h \neq \lambda_{1,j_2}^h \). Furthermore, denoting by \( h_l \) the cyclic class that element \( l \) belongs to, we have

\[ v_{j_1}^T q_{j_1} = \sum_{l=1}^N v_l \lambda_{1,j_1}^{-h_l} \lambda_{1,j_1}^{+h_l} = \sum_{l=1}^N v_l = 1, \]

which completes the proof. \( \square \)

The following result states that \( v^T \) divides the total mass equally into each cyclic class.

**Lemma S1.2.** The quantity \( \sum_{l \in \text{cyclic class } h} v_l \) does not depend on the choice of the cyclic class \( h \). Then, for any \( l \in \{1, \ldots, N\} \),

\[ \sum_{l_2 \sim_c l} v_{l_2} = \frac{1}{n_{\text{per}}}. \]

**Proof.** By applying (S:0) multiple times, we obtain, for any cyclic class \( h \),

\[ \sum_{l_1 \in \text{cyclic class } h} v_{l_1} = \sum_{l_1 \in \text{cyclic class } h} v_{l_1}^T [1]_{l_1} = \sum_{l_1 \in \text{cyclic class } h} v_{l_1} [W^T 1]_{l_1} \]

\[ = \sum_{l_1 \in \text{cyclic class } h} v_{l_1} \sum_{l_2 \in \text{cyclic class } h+1} [W^T]_{l_1,l_2} [1]_{l_2} \]

\[ = \sum_{l_2 \in \text{cyclic class } h+1} \left( \sum_{l_1 \in \text{cyclic class } h} v_{l_1} [W^T]_{l_1,l_2} [1]_{l_2} \right) \]

\[ = \sum_{l_2 \in \text{cyclic class } h+1} [v^T W^T]_{l_2} [1]_{l_2} = \sum_{l_2 \in \text{cyclic class } h+1} v_{l_2} [1]_{l_2} \]

\[ = \sum_{l_2 \in \text{cyclic class } h+1} v_{l_2}. \]

The last part of the statement is a consequence of the normalizing condition \( v^T 1 = 1 \), because the number of cyclic classes is \( n_{\text{per}} \). \( \square \)

Finally, we are ready for presenting the proof of the main result of this subsection, i.e. Theorem 2.1.

**Proof of Theorem 2.1.** We are going to prove the following equalities:

\[ Z^{(1)}_n = v^T Z_n, \quad Z^{(2)}_n = P^{(2)} P^{(2)} Z_n = \sum_{j=1}^{n_{\text{per}}-1} q_j v_j^T Z_n, \]

\[ (S:0) \]

and \( Z^{(3)}_n = P^{(3)} P^{(3)} Z_n = \sum_{j=1}^{N-n_{\text{per}}} r_j p_j^T Z_n. \)
Since \( Z^{(1)}_n = \mathbf{1}^\top Z_n = \tilde{Z}_n \mathbf{1} \) by construction, for \( Z^{(2)}_n \) we have to prove that
\[
(\mathbf{1}^\top + P^{(2)} P^{(2)}) Z_n = Z^{(1)}_n + Z^{(2)}_n.
\]

Since by Lemma S1.2
\[
[Z^{(1)}_n + Z^{(2)}_n]_l = \sum_{l_2 \sim c} \sum_{l_1 \sim c} v_{l_2} Z_{n,l_2} = n_{\text{per}} \sum_{l_2 \sim c} v_{l_2} Z_{n,l_2},
\]
we have only to prove that
\[
[(\mathbf{1}^\top + P^{(2)} P^{(2)}) Z_n]_l = n_{\text{per}} \sum_{l_2 \sim c} v_{l_2} Z_{n,l_2}.
\]

Then, denoting by \( h_1 \) and \( h_2 \) the cyclic classes which \( l_1 \) and \( l_2 \) belong to, respectively, we can use Lemma S1.1 to obtain
\[
v_{l_2} + [P^{(2)} P^{(2)}]_{l_1,l_2} = v_{l_2} + \sum_{j=1}^{n_{\text{per}}-1} \lambda_{1,j}^h v_{l_2} \lambda_{1,j}^{-h} = v_{l_2} + v_{l_2} \sum_{j=1}^{n_{\text{per}}-1} \lambda_{1,j}^{h_1-h_2} = v_{l_2} + v_{l_2} \sum_{j=1}^{n_{\text{per}}-1} \lambda_{1,j}^{h_1-h_2} = v_{l_2} + v_{l_2} \sum_{j=1}^{n_{\text{per}}-1} \lambda_{1,j}^{h_1-h_2}.
\]

Now, \( \lambda_{1,h_1-h_2} \) is a root of the unity, we have, by Lemma S3.1,
\[
[\mathbf{1}^\top + P^{(2)} P^{(2)}]_{l_1,l_2} = \begin{cases} v_{l_2} n_{\text{per}} & \text{if } l_1 \sim c l_2, \\ 0 & \text{otherwise}, \end{cases}
\]
and hence \( [(\mathbf{1}^\top + P^{(2)} P^{(2)}) Z_n]_l = n_{\text{per}} \sum_{l_2 \sim c} v_{l_2} Z_{n,l_2} \), as required. The part of \( Z^{(3)}_n \) is now obvious, since \( I = P^{-1} P \), and hence
\[
Z^{(3)}_n = Z_n - (Z^{(1)}_n + Z^{(2)}_n) = (I - (\mathbf{1}^\top + P^{(2)} P^{(2)})) Z_n \]
\[
= \begin{pmatrix} 1 & P^{(2)} \\ P^{(3)} & P^{(3)} \end{pmatrix} \begin{pmatrix} \mathbf{1}^\top \\ P^{(2)} \end{pmatrix} - (\mathbf{1}^\top + P^{(2)} P^{(2)}) Z_n = P^{(3)} P^{(3)} Z_n.
\]

As a consequence, we obtain that the three components \( Z^{(i)}_n \), with \( i = 1, 2, 3 \) are linearly independent. Finally, from Definition 2.1, we have that the entire system almost surely synchronizes in the limit if and only if there exists a process \( (Z^n)_{n} \) of the form \( Z^n = Z^n \mathbf{1} \) such that \( (Z_n - Z^n) \) converges almost surely to \( \mathbf{0} \). Since \( Z^{(1)}_n \in \text{Span}\{\mathbf{1}\} \) and the linear independence of the three components, this fact is possible if and only if both the second and the third component converge almost surely to zero.

\[\square\]

S1.2. Details for the proofs of the results stated in Section 2.

Proof of Theorem 2.2. Modification of the Jordan space: We show how to replace the Jordan block \( J_{W,\lambda} \) with a new block \( J_{W,\beta,\lambda} \) such that \( ||J_{W,\beta,\lambda}||_{2,2} < 1. \)
To this end, for any real $\beta \neq 0$ let us define $D_\beta = \text{diag}(1, \beta, \beta^2, \ldots, \beta^{n_j-1})$. We have

$$D_\beta J_{W,\lambda} = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \beta & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & \beta^{n_j-1}
\end{pmatrix}
\begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & \lambda
\end{pmatrix}
\begin{pmatrix}
\lambda & \frac{1}{\beta} & 0 & \cdots & 0 \\
0 & \lambda & \frac{1}{\beta} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \frac{1}{\beta} \\
0 & 0 & 0 & \cdots & \lambda
\end{pmatrix}
= J_{W,\beta,\lambda} D_\beta.
$$

Then, using the above relations $P_\lambda W^T = J_{W,\lambda} P_\lambda$ and $D_\beta J_{W,\lambda} = J_{W,\beta,\lambda} D_\beta$, if we define $P_{\beta,\lambda} = D_\beta P_\lambda$, we have

$$(S:0)$$

$$P_{\beta,\lambda} W^T = D_\beta P_\lambda W^T = D_\beta J_{W,\lambda} P_\lambda = J_{W,\beta,\lambda} D_\beta P_\lambda = J_{W,\beta,\lambda} P_{\beta,\lambda}.$$ 

Roughly speaking, $P_{\beta,\lambda} W^T = J_{W,\beta,\lambda} P_{\beta,\lambda}$ means that $P_{\beta,\lambda}$ is a $\beta$-Jordan base for the $\beta$-Jordan block $J_{W,\beta,\lambda}$. Obviously, $|\frac{1}{\beta}|$ may be arbitrary small if $|\beta|$ is big enough, so that we expect $|J_{W,\beta,\lambda}|_{2,2}$ so close to $|\lambda|$ to be strictly smaller than 1. To prove this fact, set $\bar{J} = J_{W,\beta,\lambda} J_{W,\beta,\lambda}^*$, that can be easily computed:

$$\bar{J} = J_{W,\beta,\lambda} J_{W,\beta,\lambda}^* = \begin{pmatrix}
|\lambda| + \frac{1}{|\beta|^2} & \frac{\lambda}{\beta} & 0 & \cdots & 0 & 0 & 0 \\
\frac{\lambda}{\beta} & |\lambda| + \frac{1}{|\beta|^2} & \frac{\lambda}{\beta} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & |\lambda| + \frac{1}{|\beta|^2} & \frac{\lambda}{\beta} & \frac{\lambda}{\beta} \\
0 & 0 & 0 & \cdots & 0 & \lambda & |\lambda|
\end{pmatrix}.$$ 

Then, we get

$$||J_{W,\beta,\lambda}||_{2,2} \leq \max_{j=1,\ldots,N} \left\{ \sum_{i=1}^{N} |[\bar{J}]_{ij}| \right\} = (|\lambda| + \frac{1}{|\beta|})^2.$$ 

Now, let $1/\beta = \sqrt{\frac{1+|\lambda|}{2}} - |\lambda| > 0$ so that

$$||J_{W,\beta,\lambda}||_{2,2} \leq \frac{1+|\lambda|}{2} < 1.$$ 

This concludes the step of the modification of the Jordan space.

Second part of the proof ($\Rightarrow$):

Here we present the technical details related to Step (i) and Step (ii) of the second part of the proof of Theorem 2.2.

Step (i)

Take $n_0 \geq N$ such that the set $A_0 = \{0 < Z_{n_0,t_2} \leq Z_{n_0,t_1} < 1\}$ has strictly positive probability (this is possible by the irreducibility of $W$ and the hypothesis $P(T_0) < 1$). Now, take the set
\( A = A_0 \cap \{ X_{n,l_1} = 1, X_{n,l_2} = 0, \forall n_0 < n \leq n_1 \} \) (where \( n_1 \) will be determined more ahead), and notice that, on \( A \), we have

\[
Z_{n_1,l_2} = Z_{n_0,l_2} \prod_{k=n_0}^{n_1-1} (1-r_k), \quad \text{and} \quad 1 - Z_{n_1,l_1} = (1 - Z_{n_0,l_1}) \prod_{k=n_0}^{n_1-1} (1-r_k).
\]

Then, denoting \( \Delta Z_{n_1} = Z_{n_1,l_1} - Z_{n_1,l_2} \) and \( \Delta Z_{n_0} = Z_{n_0,l_1} - Z_{n_0,l_2} \), on \( A \), we have that

\[
\Delta Z_{n_1} = 1 - (1 - \Delta Z_{n_0}) \prod_{k=n_0}^{n_1-1} (1-r_k) \geq 1 - \prod_{k=n_0}^{n_1-1} (1-r_k),
\]

which is strictly positive as \( 0 < r_n < 1 \) by definition. Now, set \( 0 < \epsilon < 1 - \prod_{k=n_0}^{\infty} (1-r_k) \) and fix \( n_1 \) sufficiently large so that \( 1 - \prod_{k=n_0}^{n_1-1} (1-r_k) > \epsilon \) and \( \sum_{n \geq n_1} r_n < \epsilon/3 \). Since it can be easily proved that \( P(A) > 0 \) whenever \( r_n < 1 \), what we have shown is that \( P(\Delta Z_{n_1} > \epsilon) > 0 \). However, since for both \( l = l_1, l_2 \) it holds that \( \max_{n \geq n_1} \{|Z_{n_1,l} - Z_{n,l}|\} < \epsilon/3 \), we have that \( P(\max_{n \geq n_1} \{|\Delta Z_n|\} > \epsilon) > 0 \). Therefore, we cannot have that a.s. \( \lim_{n \to +\infty} Z_{n,l_1} - Z_{n,l_2} = 0 \) on \( A \).

\( \square \)

Step (ii)

Notice that, since \( N > n_{\text{per}} \), there exist two different states that belong to the same cyclic class, call them \( l_1 \) and \( l_2 \). Hence, applying the result of Step (i) to the pair \((l_1, l_2)\) within the same cyclic class, we have that \( Z_n^{(3)} \neq 0 \) with a strictly positive probability, since \( Z_n^{(1)} \) and \( Z_n^{(2)} \) are constant on the same cyclic class.

Proof of Corollary 2.3. Note that, if \( l_1 \sim_c l_2 \), then by definition we have \( (Z_{n,l_1}^{(1)} + Z_{n,l_1}^{(2)}) = (Z_{n,l_1}^{(1)} + Z_{n,l_2}^{(2)} \). Hence, if \( l_1 \sim_c l_2 \), we have \( Z_{n,l_1} - Z_{n,l_2} = Z_{n,l_1}^{(3)} - Z_{n,l_2}^{(3)} \), which tends to zero a.s. for all these pairs \((l_1, l_2)\) if and only if all \( Z_{n,l_1}^{(3)} \), \( Z_{n,l_2}^{(3)} \) tends to zero a.s., that by Theorem 2.2 occurs if and only if \( \sum_{n} r_n = +\infty \). This corresponds to the almost sure asymptotic synchronization within each cyclic class. Recall that it is not possible that a.s. \( \lim_{n} Z_{n,l_1}^{(3)} = \lim_{n} Z_{n,l_2}^{(3)} = 0 \) for all pairs \((l_1, l_2)\) with \( l_1 \sim_c l_2 \) as this would mean that a.s. \( \lim_{n} Z_{n}^{(3)} \) belongs to Span\{\(1, q_1, \ldots, q_{n_{\text{per}}-1}\)\}, which is impossible by Theorem 2.1.

\( \square \)

Proof of Theorem 2.4. Here we present some calculations related to the terms \( a_{j_1,n} \) and \( C_{j_1,n} \) derived in the Proof of Theorem 2.4.

(i) Term \( a_{j_1,n} \)

We have

\[
a_{j_1,n} = 1 + r_n^2 (1 - \Re(\lambda_{1,j_1}))^2 + r_n^2 \Im(\lambda_{1,j_1})^2 - 2r_n (1 - \Re(\lambda_{1,j_1}))
\]

\[
= 1 - 2r_n (1 - r_n) (1 - \Re(\lambda_{1,j_1}))
\]

\[
= 1 - s_{j_1,n},
\]

where, since \( |\lambda_{1,j_1}| = 1 \), we can write \( s_{j_1,n} = 2r_n (1 - r_n) (1 - \cos(\frac{2\pi}{n_{\text{per}} j_1})) \). Notice that, since we are considering \( n_{\text{per}} \geq 2 \) and \( 1 \leq j_1 \leq n_{\text{per}} - 1 \), we have \( -1 \leq \cos(\frac{2\pi}{n_{\text{per}} j_1}) < 1 \), which, combined with \( 0 < r_n (1 - r_n) \leq 1/4 \), implies \( 0 < s_{j_1,n} \leq 1 \).

(ii) Term \( C_{j_1,n} \)

By (13) and Lemma S1.1, we have that \( \max_{l=1,\ldots,N} \{ \langle \bar{v}_{j_1}, \bar{v}_{l,j_1}^T \rangle \} = \max_{l=1,\ldots,N} \{ v_{l,j_1}^2 \} \leq 1 \), and hence we can bound \( C_{j_1,n} \) as follows:

\[
0 \leq C_{j_1,n} \leq r_n^2 V_n.
\]

\( \square \)
Proof of Corollary 2.5. The spectral representation given in Theorem 2.1 states that any linear combination of $Z_n^{(2)}$ and $Z_n^{(3)}$ involves the base vectors given in $P^{-1}$ except the vector $1$, which is related to $Z_n^{(1)}$. Then, the complete almost sure asymptotic synchronization coincides with the asymptotic vanishing of the processes $Z_n^{(2)}$ and $Z_n^{(3)}$, and so it is a direct consequence of Theorem 2.2 and Theorem 2.4. □

S1.3. Details for the proofs of the results stated in Section 3.

Proof of Proposition 3.1. Fix $l$, the existence of the almost sure limit of $(Z_{n,l})_n$ follows by the fact that, under the assumption $\sum r_n < +\infty$, the process $(Z_{n,l})_n$ is a non-negative almost super-martingale (see [32]). Indeed, we have

$$E[Z_{n+1,l} | F_n] = Z_{n,l} - r_n Z_{n,l} + r_n E[X_{n+1,l} | F_n] \leq Z_{n,l} + \xi_{n,l}$$

where $\sum \xi_{n,l} = \sum r_n E[X_{n+1,l} | F_n] \leq \sum r_n < +\infty$. Then, since $\tilde{Z}_n$, and so $Z_n^{(1)}$ (see (5)), converges almost surely, we have that also $Z_n^{(2)}$ converges almost surely because by definition $Z_n^{(2)}$ is a function of $Z_n$ (see (7)). Finally, the almost sure convergence of $Z_n^{(3)}$ follows by (8) as $Z_n^{(3)} = Z_n - Z_n^{(1)} - Z_n^{(2)}$. The last statement of the result is proved in Step (i) of the second part ($\Leftarrow$) of the Proof of Theorem 2.2. □

We also observe that, in the scenario $\sum r_n < +\infty$, for each $l$, the limit random variable $Z_{\infty,l}$ of the $(Z_{n,l})_n$ cannot touch the barriers when the process starts in $(0, 1)$. This fact follows from Lemma S2.1 applied to $y_n = Z_{n,l}(\omega)$ and to $y_n = 1 - Z_{n,l}(\omega)$, with $\omega \in \{0 < Z_{0,l} < 1\}$. Indeed, we have $Z_{n+1,l} \geq (1 - r_n)Z_{n,l}$ and $(1 - Z_{n+1,l}) \geq (1 - r_n)(1 - Z_{n,l})$.

Now, we give the proof of Lemma 5.2, used in the proof of Theorem 3.2.

Proof of Lemma 5.2. First, we show that $O_{n_{\text{per}}}$ is an orthogonal matrix. To this end, notice that for any $j_1, j_2 \in \{0, \ldots, n_{\text{per}} - 1\}$, we can write

$$(O_{n_{\text{per}}}^* O_{n_{\text{per}}})_{j_1,j_2} = \frac{1}{n_{\text{per}}} \sum_{j=0}^{n_{\text{per}}-1} \lambda_{1,j}^{j_1} \lambda_{1,j}^{j_2} = \frac{1}{n_{\text{per}}} \sum_{j=0}^{n_{\text{per}}-1} \lambda_{1,-j}^{j_1} \lambda_{1,j}^{j_2} = \frac{1}{n_{\text{per}}} \sum_{j=0}^{n_{\text{per}}-1} \lambda_{1,j_2-j_1}^{j_1}.$$ 

Then, by Lemma S3.1 we have

$$\frac{1}{n_{\text{per}}} \sum_{j=0}^{n_{\text{per}}-1} \lambda_{1,j_2-j_1}^{j_1} = \begin{cases} 1 & \text{if } j_1 = j_2, \\ 0 & \text{otherwise}, \end{cases}$$

which concludes the proof that $O_{n_{\text{per}}}$ is orthogonal.

Now, we focus on proving that $Z_n^{(c)} = n_{\text{per}}^{1/2} O_{n_{\text{per}}} C_n$. For this purpose, first notice that by (10) and Lemma S1.2, for any $h_1 \in \{0, \ldots, n_{\text{per}} - 1\}$ we have

$$Z_n^{(c),h_1} = \sum_{l_2 \in \text{cyclic class } h_1} v_{l_2} \sum_{l \in \text{cyclic class } h_1} v_l Z_{n,l_2} = n_{\text{per}} \sum_{l_2 \in \text{cyclic class } h_1} v_{l_2} Z_{n,l_2},$$

and hence it is enough to prove that

$$\sqrt{n_{\text{per}}} [O_{n_{\text{per}}} C_n]_{(h_1+1)} = \sum_{l_2 \in \text{cyclic class } h_1} v_{l_2} Z_{n,l_2}.$$
Recalling that $C_n = (v_{i(2)})Z_n$, this is the same as proving that, for each $l_2 \in \{1, \ldots, N\}$,

$$\sqrt{n_{\text{per}}} \left[ O_{n_{\text{per}}} \left( \frac{v^T}{P(2)} \right) \right]_{(h_1+1)l_2} = \begin{cases} v_{l_2} & \text{if } h_2 = h_1, \\ 0 & \text{otherwise}, \end{cases}$$

where with $h_2$ we have denoted the cyclic class which $l_2$ belongs to. To this end, by Lemma S1.1 and Lemma S3.1 we obtain

$$\sqrt{n_{\text{per}}} \left[ O_{n_{\text{per}}} \left( \frac{v^T}{P(2)} \right) \right]_{(h_1+1)l_2} = v_{l_2} + \sum_{j_2=1}^{n_{\text{per}}-1} \left[ \sqrt{n_{\text{per}}} O_{n_{\text{per}}} \right]_{(h_1+1)(j_2+1)} (P(2))_{j_2l_2}$$

$$= v_{l_2} + \sum_{j_2=1}^{n_{\text{per}}-1} \lambda_{l_1h_1l_2} v_{l_2} \lambda_{l_1j_2}$$

$$= v_{l_2} + \sum_{j_2=1}^{n_{\text{per}}-1} \lambda_{l_1h_1l_2} \lambda_{l_1j_2}$$

$$= v_{l_2} + \sum_{j_2=0}^{n_{\text{per}}-1} \lambda_{l_1h_1l_2} = \begin{cases} v_{l_2} & \text{if } h_2 = h_1, \\ 0 & \text{otherwise}. \end{cases}$$

This concludes the proof that $Z_n^{(c)} = n_{\text{per}}^{\frac{3}{2}} O_{n_{\text{per}}} C_n$.

Finally, since $O_{n_{\text{per}}}$ is an orthonormal matrix, we immediately have that $||Z_n^{(c)}||^2 = n_{\text{per}}^{3/2} ||O_{n_{\text{per}}} C_n||^2 = n_{\text{per}}^{3/2} ||C_n||^2$, where $||C_n||^2 = C_{n+1}^* C_{n+1}$.

The following proofs concern Lemma 5.3 and Lemma 5.4, used in the proof of Theorem 3.4.

**Proof of Lemma 5.3.** Since $V_n = (1 - W^T Z_n^T W^T Z_n) = (1^T - Z_n^T W^T W^T Z_n)$, it is enough to prove that, given a vector $x$, with $x_1 \in [0, 1]$, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$(1^T - x^T W) W^T x < \delta \Rightarrow (1^T - x^T) x < \epsilon.$$ 

In order to prove this fact, first notice that

$$(1^T - x^T W) W^T x = \sum_{l_2=1}^N (1 - [x^T W]_{l_2}) [W^T x]_{l_2} \geq \max_{l_2=1, \ldots, N} \left\{ (1 - [x^T W]_{l_2}) [W^T x]_{l_2} \right\},$$

and since $t(1-t) \geq \frac{1}{2} \min\{t, 1-t\}$ for $t \in [0, 1]$,

$$\max_{l_2=1, \ldots, N} \left\{ (1 - [x^T W]_{l_2}) [W^T x]_{l_2} \right\} \geq \frac{1}{2} \max_{l_2=1, \ldots, N} \left\{ \min\{ (1 - [x^T W]_{l_2}), [W^T x]_{l_2} \} \right\},$$

which implies that, for any $\delta > 0$,

$$(1^T - x^T W) W^T x < \delta \quad \Rightarrow \quad \min\{ 1 - [x^T W]_{l_2}, [W^T x]_{l_2} \} < 2\delta \forall l_2.$$ 

Finally, setting $w_{\text{min}} = \min\{ [W]_{l_1l_2} : w_{l_1l_2} \not= 0 \} > 0$ and $2\delta = w_{\text{min}} \epsilon$, we need to prove that,

$$\min\left\{ 1 - [x^T W]_{l_2}, [W^T x]_{l_2} \right\} < 2\delta \forall l_2 \quad \Rightarrow \quad \min\{ 1 - x_{l_1}, x_{l_1} \} < \epsilon \forall l_1.$$
which follows by showing its contrapositive as follows:

\[ x_{l_1} \geq \epsilon \quad \implies \quad |W^T x|_{l_2} = \sum_{l_1} w_{l_1,l_2} x_{l_1} \geq w_{\min} \epsilon = 2\delta, \]

\[ 1 - x_{l_1} \geq \epsilon \quad \implies \quad 1 - |W^T x|_{l_2} = \sum_{l_1} w_{l_1,l_2} (1 - x_{l_1}) \geq w_{\min} \epsilon = 2\delta, \]

for all \( l_2 \) such that \( w_{l_1,l_2} > 0 \). This concludes the proof. \( \square \)

**Proof of Lemma 5.4.** First, we will prove that \( \sum \delta_n = +\infty \). To this end, assume by contradiction that \( \sum \delta_n = M < +\infty \), which implies \( r_n \leq 1/2 \) for any \( n > \tau_M \). Hence, we can write

\[ \sum_n r_n (1 - r_n) = \sum_{n \leq \tau_M} r_n (1 - r_n) + \sum_{n > \tau_M} r_n (1 - r_n) \geq \sum_{n \leq \tau_M} r_n (1 - r_n) + \frac{1}{2} \sum_{n > \tau_M} r_n = +\infty, \]

which contradicts the assumption \( \sum_n r_n (1 - r_n) < +\infty \). This concludes the first part of the proof.

We can now define the sequence \((\epsilon_n)_n\) as follows: for \( n \geq 0 \), set

\[ \epsilon_n = \begin{cases} 0 & \text{if } \tau_n + 1 = \tau_n + 1, \\ \sum_{p = \tau_n + 1}^{\tau_n + 1} r_p & \text{if } \tau_n + 1 > \tau_n + 1, \end{cases} \]

which has the property that

\[ \sum_n \epsilon_n = \sum_n \mathbb{1}_{\{\tau_n + 1 > \tau_n + 1\}} \sum_{p = \tau_n + 1}^{\tau_n + 1} r_p = \sum_n r_n \mathbb{1}_{\{\delta_n = 0\}}. \]

First we will show that \( \sum \epsilon_n < +\infty \). Note that, when \( \delta_n = 0 \), \( r_n \leq 1/2 \) and hence \( 2(1 - r_n) \geq 1 \), which can be used to get the following

\[ \sum \epsilon_n = \sum_n \mathbb{1}_{\{\delta_n = 0\}} r_n \leq \sum_n \mathbb{1}_{\{\delta_n = 0\}} r_n 2(1 - r_n) \leq 2 \sum_n r_n (1 - r_n) < +\infty. \]

Finally, we will show that \( \sup_{m \in \{\tau_n, \tau_n + 1\}} \{ ||Z_m,l_1 - Z_{\tau_n + 1,l_1}|| \} < \epsilon_n \) for any \( l_1 \in \{1, \ldots, N\} \). To this end, by (2) and the triangular inequality, we get for \( m \in \{\tau_n + 1, \ldots, \tau_n + 1\} \) that

\[ ||Z_m,l_1 - Z_{\tau_n + 1,l_1}|| \leq \sum_{p = \tau_n + 1}^{\tau_n + 1} ||Z_p,l_1 - Z_{p - 1,l_1}|| \leq \sum_{p = \tau_n + 1}^{\tau_n + 1} r_{p - 1} ||X_{p,l_1} - Z_{p - 1,l_1}|| \]

\[ \leq \sum_{p = \tau_n + 2}^{\tau_n + 1} r_{p - 1} = \epsilon_n. \]

Finally, we present the details of the proof of Theorem 3.4, whose notation and structure was described in the main article.

**Proof of Theorem 3.4. Step (1)**

Here we prove the stationary dynamics of \((Z_n)_n\) between the clock times \((\sigma_n)_n\), that is

\[ \sup_{m_1,m_2 \in \{\sigma_{n - 1}, \ldots, \sigma_{n - 1}\}} \{ ||Z_{m_1} - Z_{m_2}|| \} \xrightarrow{\text{a.s.}} 0. \]

This simply follows by Lemma 5.4, as we have

\[ \sup_{m_1,m_2 \in \{\sigma_{n - 1}, \ldots, \sigma_{n - 1}\}} \{ ||Z_{m_1} - Z_{m_2}|| \} \leq \sup_{m_1,m_2 \in \{\tau_{n - 1} + 1, \ldots, \tau_n\}} \{ ||Z_{m_1} - Z_{\tau_n}|| + ||Z_{m_2} - Z_{\tau_n}|| \} \leq 2\sqrt{N} \epsilon_n, \]
that goes to 0 almost surely.

**Step (2)**
Now, \( r_{\tau_n} \to 1 \) by Lemma 5.4 and this, together with the definition of \( \mathbf{Z}_n \) in (4), immediately implies

\[
X_{\sigma_n} - \mathbf{Z}_{\sigma_n} = (1 - r_{\tau_n})(X_{\tau_n+1} - \mathbf{Z}_{\tau_n}) \xrightarrow{a.s.} 0.
\]

**Step (3a)**
Let \((\epsilon_n)_n\) as in Lemma 5.4, so that

\[
|Z_{\tau_{n+1}} - Z_{\sigma_n}| \leq \epsilon_n.
\]

Since \( r_{\tau_n} \to 1 \), take \( n_* \) so that \( m \geq n_* \) implies \( r_{\tau_{m-1}} - \epsilon_{m-1} > 1/2 \). Then, when \( m \geq n_* \), by (4) and (S0), we have

\[
\begin{align*}
X_{\sigma_{m-1},l_1} = 1 & \iff Z_{\sigma_{m-1},l_1} \geq r_{\tau_{m-1}} & \iff Z_{\tau_{m},l_1} \geq r_{\tau_{m-1}} - \epsilon_{m-1}, \\
X_{\sigma_{m-1},l_1} = 0 & \iff Z_{\sigma_{m-1},l_1} \leq 1 - r_{\tau_{m-1}} & \iff Z_{\tau_{m},l_1} \leq 1 - r_{\tau_{m-1}} + \epsilon_{m-1},
\end{align*}
\]

and hence, for each cyclic class \( h \),

\[
\forall m \geq n_*, \quad B_{X,m-1}(0) \subseteq B_{Z,m}^h(0) \quad \text{and} \quad B_{X,m-1}(1) \subseteq B_{Z,m}^h(1).
\]

**Step (3b)**
Note that, by (S0) and (S0), if \( l \) belongs to the \((h-1)\)-th cyclic class, then, on \( B_{m-1} \) and \( m \geq n_* \), we have that \([W^TZ_{\tau_m}]_l\) is equal to

\[
\sum_{l_1 \in \text{cyclic class } h} [W^T]_{l,l_1} Z_{\tau_{m},l_1} \begin{cases}
\geq r_{\tau_{m-1}} - \epsilon_{m-1} & \text{if } Z_{\sigma_{m-1},l_1} \geq r_{\tau_{m-1}}, \\
\leq 1 - r_{\tau_{m-1}} + \epsilon_{m-1} & \text{if } Z_{\sigma_{m-1},l_1} \leq 1 - r_{\tau_{m-1}},
\end{cases}
\]

and hence, for each cyclic class \( h \),

\[
\forall m \geq n_*, \quad B_{Z,m}^h(0) \subseteq B_{WZ,m}^{h-1}(0) \quad \text{and} \quad B_{Z,m}^h(1) \subseteq B_{WZ,m}^{h-1}(1).
\]

**Step (3c)**
By the definition of \( A_m \), it is immediate to see that, on \( A_m \), we have \( B_{WZ,m}^{h-1}(0) \equiv B_{X,m}^{h-1}(0) \) and \( B_{WZ,m}^{h-1}(1) \equiv B_{X,m}^{h-1}(1) \) for any cyclic class \( h \).

**Step (4a)**
The definition of \( V_{\tau_{n_0}} \), together with the relation \( 2t(1-t) \geq \min\{t,1-t\} \) for \( t \in [0,1] \), implies that, for any \( l = 1, \ldots, N \), we have

\[
\min \{ [W^TZ_{\tau_{n_0}}]_l, (1 - [W^TZ_{\tau_{n_0}}]_l) \} \leq 2[W^TZ_{\tau_{n_0}}]_l(1 - [W^TZ_{\tau_{n_0}}]_l) \leq 2V_{\tau_{n_0}},
\]

and hence

\[
\max \{ [W^TZ_{\tau_{n_0}}]_l, (1 - [W^TZ_{\tau_{n_0}}]_l) \} \geq (1 - 2V_{\tau_{n_0}}).
\]

The definition of \( A_{n_0} \) reads

\[
P(A_{n_0}) = \prod_{l=1}^{N} \begin{cases}
[W^TZ_{\tau_{n_0}}]_l & \text{if } [W^TZ_{\tau_{n_0}}]_l \geq \frac{1}{2}, \\
1 - [W^TZ_{\tau_{n_0}}]_l & \text{if } [W^TZ_{\tau_{n_0}}]_l < \frac{1}{2},
\end{cases}
\]

and hence by (S0), \( P(A_{n_0}) \geq (1 - 2V_{\tau_{n_0}})^N \) that goes to 1 by Theorem 3.3.

Since \( P(A_{n_0} \cap B_n) = P(B_n|A_{n_0}) P(A_{n_0}) \), for what concerns the remaining term \( P(B_{n_0}|A_{n_0}) \), we will prove the sufficient condition \( P(B_{n_0} \cap \infty) \to 1 \). Recall the definition of \( \liminf \) for a sequence of sets, that reads \( \liminf_{n \to \infty} B_n = \bigcup_{n \geq n_0} B_n \), so that the thesis becomes

\[
\lim_{n \to +\infty} P(B_{n_0} \cap \infty) = P(\bigcup_{m=1}^{\infty} B_{m,\infty}) = P(\liminf B_n) = 1.
\]
Now, using (S:0) in Step 2 and Theorem 2.2, we have
\[ X_{\sigma_n} - Z^{(C)}_{\sigma_n} = \left( Z_{\sigma_n} - (Z^{(1)}_{\sigma_n} + Z^{(2)}_{\sigma_n}) \right) + (X_{\sigma_n} - Z_{\sigma_n}) \xrightarrow{a.s.} 0. \]

Now, \((Z^{(C)}_{\sigma_n})_n\) is constant on each cyclic class, then the same holds asymptotically for \((X_{\sigma_n})_n\), that assumes values in \(\{0,1\}\). In other words
\[ P(\liminf_{n} B_n) = P(\{B_n, \text{eventually}\}) = 1. \]

**Step (4b)**
Notice that the relation \(A_m \cap B^{h-1}_{X,m-1}(g) \subseteq B^{h-1}_m(g)\) proved in Step 3 implies also that \(A_m \cap B_{m-1} \subseteq B_m\). As a consequence, for any \(m > \max\{n_*, n_0\}\), we have
\[ P(A_m \cap B_m | A_{n_0,m} \cap B_{n_0,m}) = P(A_m | A_{n_0,m} \cap B_{n_0,m}). \]

Since \(A_{n_0,m} \cap B_{n_0,m} \subseteq B_{m-1}\), the definition of \(A_m\), together with (S:0) and (S:0), implies
\[ P(A_m | A_{n_0,m} \cap B_{n_0,m}) \geq (r_{\tau_{m-1} - \epsilon_{m-1}})^N = \left(1 - ((1 - r_{\tau_{m-1}}) + \epsilon_{m-1})\right)^N. \]

Summing up, for \(n_0 \geq n_*\),
\[ \prod_{m=n_0+1}^{\infty} P(A_m \cap B_m | A_{n_0,m} \cap B_{n_0,m}) = \prod_{m=n_0+1}^{\infty} P(A_m | A_{n_0,m} \cap B_{n_0,m}) \geq \left( \prod_{m=n_0+1}^{\infty} [1 - ((1 - r_{\tau_{m-1}}) + \epsilon_{m-1})]\right)^N, \]
that goes to 1 when \(n_0 \to +\infty\), because \(\sum_{m}((1 - r_{\tau_{m-1}}) + \epsilon_{m-1}) < +\infty\) by Lemma 5.4.

**Step (5)**
Under the further request that \(\sum_{n}(1 - r_{n}) < +\infty\), we have that \(r_{n} \to 1\). The definition of \((\tau_n)_{n}\) given in Lemma 5.4 implies that, eventually, \(\sigma_{n+1} - 1 = \tau_{n+1} = \tau_{n} + 1 = \sigma_{n}\). Then, the stationary part of the dynamics disappears and the thesis follows. \(\square\)

**Appendix S2. Some auxiliary results**

We here collect some technical results.

**Lemma S2.1.** If \(y_{n+1} \geq (1 - a_{n})y_{n}\) with \(0 \leq a_{n} < 1\) and \(\sum_{n} a_{n} < +\infty\), then \(\liminf_{n} y_{n} > 0\), provided \(y_{0} > 0\).

**Proof.** If \(\sum_{n} a_{n} < +\infty\), then \(\prod_{k=0}^{n}(1 - a_{k})\) converges to a constant \(L \in (0,1]\). Since we have \(y_{n} \geq y_{0}\prod_{k=0}^{n-1}(1 - a_{k})\), we get \(\liminf_{n} y_{n} \geq y_{0}L > 0\). \(\square\)

The following result is a slight generalization of [13, Lemma A.1].

**Lemma S2.2.** If \(a_{n} \geq 0, a_{n} \leq 1\) for \(n\) large enough, \(\sum_{n} a_{n} = +\infty, \delta_{n} \geq 0, \sum_{n} \delta_{n} < +\infty, b > 0, y_{n} \geq 0\) and \(y_{n+1} \leq (1 - a_{n})^{b}y_{n} + \delta_{n}\), then \(\lim_{n} y_{n} = 0\).

**Proof.** Let \(l\) be such that \(a_{n} \leq 1\) for all \(n \geq l\). It holds
\[ y_{n} \leq y_{l}\left( \prod_{i=l}^{n-1}(1 - a_{i}) \right)^{b} + \sum_{i=l}^{n-1} \delta_{i}\left( \prod_{j=i+1}^{n-1}(1 - a_{j}) \right)^{b}. \]
Using the fact that \( \sum_{n} a_n = +\infty \), it follows that \( \prod_{i=1}^{n-1} (1 - a_i) \to 0 \). Moreover, for every \( m \geq l \),

\[
\sum_{i=l}^{n-1} \delta_i \left( \prod_{j=i+1}^{n-1} (1 - a_j) \right)^b = \sum_{i=l}^{m-1} \delta_i \left( \prod_{j=i+1}^{n-1} (1 - a_j) \right)^b + \sum_{i=m}^{n-1} \delta_i \left( \prod_{j=i+1}^{n-1} (1 - a_j) \right)^b \\
\leq \left( \prod_{j=m}^{n-1} (1 - a_j) \right)^b \sum_{i=l}^{m-1} \delta_i + \sum_{i=m}^{\infty} \delta_i.
\]

Using the fact that \( \prod_{j=m}^{n-1} (1 - a_j) \to 0 \) and that \( \sum_{n} \delta_n < +\infty \), letting first \( n \to +\infty \) and then \( m \to +\infty \) in the above formula, the conclusion follows.

\[\blacksquare\]

**Appendix S3. Complex roots of the unit and norms of complex matrices**

**Lemma S3.1.** For \( n_{\text{per}} \geq 1 \) and \( z \in \mathbb{Z} \), let \( \lambda_z = \exp(\frac{2\pi i}{n_{\text{per}}} z) \) be a \( n_{\text{per}} \)-root of the unity. Then, we have

\[
\sum_{h=0}^{n_{\text{per}}-1} \lambda_z^h = \begin{cases} 
n_{\text{per}} & \text{if } \text{mod} (z, n_{\text{per}}) = 0; \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** Let \( \tilde{z} = \text{mod} (z, n_{\text{per}}) \), so that \( \tilde{z} \in \{0, \ldots, n_{\text{per}} - 1\} \) and \( \lambda_z = \exp(\frac{2\pi i}{n_{\text{per}}} z) = \exp(\frac{2\pi i}{n_{\text{per}}} \tilde{z}) \). For \( \tilde{z} = 0 \), it is trivial that \( \lambda_z = 1 \) and so \( \sum_{h=0}^{n_{\text{per}}-1} \lambda_z^h = n_{\text{per}} \). For \( \tilde{z} \geq 1 \), since \( \lambda_z \neq 1 \) and \( \lambda_z^{n_{\text{per}}} = 1 \), from the formula of the geometric we have that

\[
\sum_{h=0}^{n_{\text{per}}-1} \lambda_z^h = \frac{1 - \lambda_z^{n_{\text{per}}}}{1 - \lambda_z} = 0.
\]

\[\blacksquare\]

Now, recall that the \((p, q)\)-operator norm \( ||A||_{p,q} \) of a complex matrix \( A \in \mathbb{C}^{M \times N} \) is defined as (note that, in the present paper, we write \( || \cdot ||_2 \) instead of \( || \cdot ||_2 \))

\[
||A||_{p,q} = \sup_{x \neq 0} \frac{||Ax||_q}{||x||_p}.
\]

We underline the following property of \( ||A||_{2,2} \):

- for any \( a_1, a_2 \in \mathbb{C} \) and \( A_1, A_2 \in \mathbb{C}^{M \times N} \), \( ||a_1 A_1 + a_2 A_2||_{2,2} \leq ||a_1||_1 ||A_1||_{2,2} + ||a_2||_1 ||A_2||_{2,2} \);
- by definition, \( ||A||_{2,2} \) is the spectral norm of \( A \), that is well known to be the square root of the largest eigenvalue of the matrix \( AA^* \) (or \( A^*A \)), where \( A^* \) is the conjugate transpose of \( A \):

\[
||A||_{2,2} = \sigma_{\text{max}}(A) = \sqrt{\lambda_{\text{max}}(AA^*)} = \sqrt{\lambda_{\text{max}}(A^*A)};
\]

- the Hölder’s inequality for matrices reads

\[
||A||_{2,2}^2 \leq ||A||_{1,1} ||A||_{\infty, \infty} = \left( \max_{j=1,\ldots,M} \left\{ \sum_{i=1}^{N} |a_{ij}| \right\} \right) \left( \max_{i=1,\ldots,N} \left\{ \sum_{j=1}^{M} |a_{ij}| \right\} \right).
\]

Note that, for a self-adjoint square matrix \( \bar{A} \), the singular values are the absolute values of the eigenvalues. Then, if \( A = AA^* \), we have

\[
||A||_{2,2}^2 = \lambda_{\text{max}}(AA^*) = \sigma_{\text{max}}(\bar{A}) = ||\bar{A}||_{2,2}^2 \\
\leq \left( \max_{j=1,\ldots,N} \left\{ \sum_{i=1}^{N} |\bar{a}_{ij}| \right\} \right) \left( \max_{i=1,\ldots,M} \left\{ \sum_{j=1}^{M} |\bar{a}_{ij}| \right\} \right) = \left( \max_{j=1,\ldots,N} \left\{ \sum_{i=1}^{N} |\bar{a}_{ij}| \right\} \right)^2,
\]
that implies $\|A\|_{2,2} \leq \max_{j=1,...,N} \{ \sum_{i=1}^{N} |\bar{a}_{ij}| \}$.

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