Classical space-time geometry in the IKKT matrix model

Harold C. Steinacker∗
Department of Physics, University of Vienna, Boltzmannasse 5, A-1090 Vienna, Austria
E-mail: harold.steinacker@univie.ac.at

We discuss the reconstruction of generic 3+1-dimensional space-time geometries from covariant quantum spaces as backgrounds in the IKKT matrix model. An explicit recipe to realize generic classical geometries is provided. Even though this typically entails some higher-spin contributions, these do not significantly modify the physical content of the model in the weak gravity regime. This justifies the framework for emergent gravity given by the semi-classical matrix model, supplemented by an induced Einstein-Hilbert action which arises in the presence of fuzzy extra dimensions.

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∗Speaker
1. Introduction

The purpose of these notes is two-fold. The first and main purpose is to provide a justification for the geometric framework which is underlying the higher-spin gravity and gauge theory in the IKKT matrix model, as described in a series of recent papers [1–5]. We will show that generic 3+1-dimensional space-time geometries can indeed be realized as backgrounds within the IKKT matrix model, whose structure is that of covariant quantum spaces. This means that there is no explicit Poisson tensor or field on space-time which would manifestly break Lorentz invariance.

The second purpose of these notes is to summarize and discuss some further implications of emergent gravity in this framework, in particular the recent 1-loop computation leading to the Einstein-Hilbert action [5]. The underlying framework is now fully justified by the present reconstruction of generic geometries.

The main result of the paper is a recipe how to realize or reconstruct generic background geometries (with trivial topology) in the matrix model, starting from some metric on space-time. Even though this was assumed in the above works, no full justification has been given, and the statement is in fact rather subtle. Here we show that generic classical geometries can be reconstructed via suitable matrix model backgrounds, provided we restrict ourselves to the weak gravity regime. This means that the gravitational curvature length scale should be much larger than any other physical scale. Under this assumption, the reconstructed geometries are well approximated by their classical counterparts, and can be described locally in terms of linearized perturbations of flat geometry.

The matrix models under consideration have an extremely simple structure, given by

$$S_{YM} = \text{Tr} [T^\hat{a}, T^\hat{b}] [T^{\hat{a}'} , T^{\hat{b}'}] \eta_{\hat{a}\hat{a}'} \eta_{\hat{b}\hat{b}'} + \text{fermions.} \quad (1)$$

Here $T^{\hat{a}}$, $\hat{a} = 0, \ldots , D-1$ are a set of hermitian matrices which transform under a global $SO(D-1, 1)$ symmetry acting on the dotted Latin indices, and $\eta_{\hat{a}\hat{b}}$ can be interpreted as $SO(D-1, 1)$-invariant metric on target space $\mathbb{R}^{D-1,1}$. The models are invariant under gauge transformations

$$T^{\hat{a}} \rightarrow U^{-1} T^{\hat{a}} U. \quad (2)$$

It is straightforward to include fermions, which is very important for the quantization; in fact we will require maximal supersymmetry, as realized in the IKKT model [6] with $D = 10$. There is no a priori notion of space-time or differential geometry; all geometrical structures relevant for the fluctuations on some given background solution emerge dynamically within the model. We will show how generic 3+1-dimensional space-time geometries as required for gravity can be realized as deformations of the covariant cosmic background $M^{3,1}$ introduced in [1].

A general framework which allows to make geometric sense of the matrix model is that of quantized symplectic spaces. We consider any given set of matrices $T^{\hat{a}}$ as a matrix configuration. Since the action is given by the square of commutators, only “almost-commutative” matrix configurations are expected to play a significant role at low energies, i.e. matrices whose commutators are much smaller in some sense than the matrices $T^{\hat{a}}$. One can then argue on rather general grounds [7, 8] that such matrix configurations can be interpreted in terms of a quantized symplectic space $(\mathcal{M}, \omega)$, where the algebra of functions $C(\mathcal{M})$ is replaced by the operator algebra $\text{End}(\mathcal{H})$. More
precisely, this is expected to hold for some subspace of IR functions and almost-local operators; more details can be found in [7]. Such functions

\[ \Phi \in \text{End}(\mathcal{H}) \sim \phi \in C(M) \]

(3)
can be identified with their classical counterpart via some (de-) quantization map defined via quasi-coherent states. We will work mostly in the semi-classical regime indicated by \( \sim \), where commutators can be replaced by Poisson brackets

\[ [\Phi, \Psi] \sim i\{\phi, \psi\} \]

(4)
as familiar from quantum mechanics. In particular, the \( T^a \) can accordingly be viewed as quantized functions on \( M \), which thereby define an embedding of \( M \) into target space:

\[ T^a \sim t^a : M \hookrightarrow \mathbb{R}^{9,1} \].

(5)
This suggests to interpret \( M \) as a brane, very much like in string theory. However from the point of view of the physics on \( M \), the \( T^a \) and their commutators

\[ \Theta^{\dot{a}\dot{b}} := i[T^{\dot{a}}, T^{\dot{b}}] \sim \{T^a, T^b\} \]

(6)
play also another role, and can be related to geometric i.e. tensorial objects on \( M \).

The key to understand \( T^a \) and \( \Theta^{\dot{a}\dot{b}} \) is to observe that they generate Hamiltonian vector fields on \( M \):

\[ E^{\dot{a}}[\phi] := \{T^{\dot{a}}, \phi\} \]

(7)
acting on some test-function \( \phi \in C(M) \). These vector fields can be made more explicit by introducing local coordinates \( y^\mu \) on the \( n \)-dimensional manifold \( M \). Define

\[ E^{\dot{a}\mu} := \{T^{\dot{a}}, y^\mu\} \],

(9)
acting on some test-function \( \phi \in C(M) \). These vector fields can be made more explicit by introducing local coordinates \( y^\mu \) on the \( n \)-dimensional manifold \( M \). Define

\[ \Theta^{\dot{a}\dot{b}\mu} := \{\Theta^{\dot{a}\dot{b}}, y^\mu\} \];

(10)
their significance will be clarified shortly. We must carefully distinguish the different types of indices: Greek indices \( \mu, \nu = 1, \ldots, n \) will denote local coordinate indices on \( M \), which play the role of tensor indices. Dotted Latin indices \( \dot{a}, \dot{b} = 0, \ldots, 9 \) indicate frame-like indices which are unaffected by a change of coordinates \( y^\mu \), but transform under the global \( SO(1, 9) \) symmetry of the matrix model. These frame-like indices will be raised and lowered with \( \eta_{\dot{a}\dot{b}} \). In particular, the \( E^{\dot{a}\mu} \) define vector fields

\[ E^{\dot{a}} = E^{\dot{a}\mu} \partial_\mu \]

(11)
on \( M \), which play a role of a (generalized) frame on \( M \). This will allow to understand the effective geometry and the gauge theory which arises on \( M \) through the matrix model. In particular, we can recognize the infinitesimal gauge transformations in the matrix model

\[ \delta_\Lambda T^{\dot{a}} = [T^{\dot{a}}, \Lambda] \sim i\{T^{\dot{a}}, \Lambda\} = iE^{\dot{a}\mu} \partial_\mu \Lambda \]

(12)
as generators of a sub-sector of diffeomorphisms on \( M \), namely of the symplectomorphisms. Finally, the tensor \( T^{\dot{a}\dot{b}\mu} \) can be recognized as torsion of the Weitzenböck connection associated to the frame \( E^{\dot{a}} \), which is very useful to describe the non-linear regime of the matrix model in the semi-classical regime [3, 9].
Covariant quantum space-time. In the following we will focus on branes $\mathcal{M}$ which are embedded in target space along the $a, b = 0, \ldots, 3$ directions. Then the extra dotted indices will mostly be ignored, but they play a role once fuzzy extra dimensions are included. However, this assumption does not mean that $\mathcal{M}$ is a 4-dimensional manifold; if $\mathcal{M}$ is 4-dimensional, then the Poisson tensor $\theta^{\mu\nu}$ on $\mathcal{M}$ plays the role of some background tensor on space-time, which is problematic since it breaks Lorentz invariance. To avoid this we will consider a different class of covariant quantum spaces, which have the structure of a $S^2$ bundle over space or space-time

$$\mathcal{M} \equiv S^2 \times \mathcal{M}^{3,1} \text{ locally} .$$

The prototype $\tilde{\mathcal{M}}$ of such a structure [1] is obtained as a certain projection of the fuzzy hyperboloid $H^4_{\alpha}$ [10, 11], and gives rise to a quantum space-time $\tilde{\mathcal{M}}^{3,1}$ with FLRW geometry and Minkowski signature. For other examples and approaches to covariant quantum spaces\(^1\) see e.g. [11–18].

Let us describe the structure of the covariant quantum space-time $\tilde{\mathcal{M}}$ in some detail. In the semi-classical limit $n \to \infty$, $\tilde{\mathcal{M}}$ reduces to an $SO(3, 1)$-equivariant $S^2$ bundle over $\tilde{\mathcal{M}}^{3,1}$. The functions on the 6-dimensional $\tilde{\mathcal{M}}$ are generated by generators $x^\mu$ which describe $\tilde{\mathcal{M}}^{3,1}$, and $t_\mu$ which generate the internal sphere $S^2$. Both sets of generators transform covariantly under $SO(3, 1)$, and satisfy the constraints

$$x_\mu x^\mu = -R^2 - x_4^2 = -R^2 \cosh^2(\eta) , \quad R = \frac{r}{2n}$$

$$t_\mu t^\mu = r^{-2} \cosh^2(\eta)$$

$$t_\mu x^\mu = 0$$

where indices are contracted with $\eta^{\mu\nu}$. Here $\eta \in (-\infty, \infty)$ plays the role of a FLRW time parameter, featuring a big bounce at $\eta = 0$. The space of functions decomposes into a direct sum $\text{End}(\mathcal{H}_a) = \oplus C^\alpha$ of higher spin ($\alpha$) modes on $\mathcal{M}^{3,1}$, which in the semi-classical regime can be organized in terms of totally symmetric traceless tensors

$$\phi^{(s)} = \phi_{\mu_1 \ldots \mu_s}(x) t^{\mu_1} \ldots t^{\mu_s} \phi_{\mu_1 \ldots \mu_s} x^\mu = 0 = \phi_{\mu_1 \ldots \mu_s} \eta^{\mu_1 \mu_2} .$$

$\tilde{\mathcal{M}}$ is a symplectic manifold (which is quantized in the matrix model), and the Poisson tensor $\theta^{\mu\nu} = \{x^\mu, x^\nu\}$ vanishes upon projection to space-time $\mathcal{M}^{3,1}$. This projection or averaging over $S^2$ will be denoted by $[\cdot]_0$:

$$\left[ \theta^{\mu\nu} \right]_0 \equiv \int_{S^2} \theta^{\mu\nu} = 0 .$$

The more generic covariant quantum spaces under consideration here are by definition the same symplectic bundle $\mathcal{M} \equiv \tilde{\mathcal{M}}$, realized as a background of the model through a different, perturbed embedding map $T^a \sim t^a$. More explicitly,

$$T^a = \bar{T}^a + \mathcal{A}^a \sim t^a + \mathcal{A}^a$$

\(^1\)The framework of [19] is also somewhat similar to ours, but the bundles under consideration there are vastly bigger.
where $\mathcal{A}^{i}$ are functions on $\mathcal{M}$ or equivalently $\mathfrak{h} \mathfrak{s}$-valued functions on $\mathcal{M}^{3,1}$ which can be expanded in the form (17). In particular, all these backgrounds are equivalent as symplectic spaces, and we will always use the standard coordinate functions $x^a$ and $t^a$ as for the undeformed background $\mathcal{M}$, with the same the symplectic form or Poisson structure. This is very natural since symplectic manifolds are rigid, so that any deformation is equivalent (locally, at least) to the undeformed space by some diffeomorphism.

The purpose of this short paper is to clarify if and under what conditions the higher-spin gauge theory on $\mathcal{M}^{3,1}$ can be reduced to (or is dominated by) the classical geometry i.e. the lowest spin sector on $\mathcal{M}^{3,1}$, which is supposed to play the role of physical space-time. More explicitly, we want to understand if it is consistent to restrict to fluctuations of the form

$$A^{i} \sim \dot{\omega}^{i} \ (t) \ t_{\mu} \ ,$$

dropping or neglecting higher-spin $\mathfrak{h} \mathfrak{s}$ contributions $\mathcal{A}^{i} \sim \dot{\omega}^{i} \ t_{\mu} \ ... \ t_{\nu}$. We will indeed establish that backgrounds of the structure

$$T^{i} = T^{i \mu} (x) \ t_{\mu}$$

are sufficiently rich to describe generic 3+1-dimensional space-time geometries, and provide a self-consistent class of configurations in the matrix model where higher-spin corrections are negligible in the weak gravity regime, to be discussed below.

2. **Effective metric and frame on covariant quantum space-time**

Now we establish the interpretation of $E^{i \mu}$ as frame on $\mathcal{M}^{3,1}$. As in any field theory, the effective metric governing some field or fluctuation mode is encoded in the kinetic term of the action. Consider a matrix background corresponding to some $2n$-dimensional brane $\mathcal{M} \hookrightarrow \mathbb{R}^{3,1} \subset \mathbb{R}^{9,1}$. Then the kinetic (=quadratic) term for transversal fluctuations\(^2\) in Yang-Mills matrix models has the structure

$$S[\phi] = \text{Tr}( [T^{i}, \phi] [T_{a}, \phi]) \sim -\frac{1}{(2\pi)^n} \int_{\mathcal{M}} \Omega \ \{T^{i}, \phi\} \{T_{a}, \phi\}$$

$$= -\frac{1}{(2\pi)^n} \int_{\mathcal{M}} \Omega \ \eta_{ab} E^{i \mu} E^{b \nu} \partial_{\mu} \phi \partial_{\nu} \phi$$

$$= -\frac{1}{(2\pi)^n} \int_{\mathcal{M}} \Omega \ \gamma^{\mu \nu} \partial_{\mu} \phi \partial_{\nu} \phi$$

in the semi-classical regime, recognizing (9). Here $\Omega$ is the symplectic volume form on $\mathcal{M}$, and

$$\gamma^{\mu \nu} := \eta_{ab} E^{i \mu} E^{b \nu}.$$  

This is clearly the metric determined by the frame $E^{i \mu}$; however the effective metric acquires an extra conformal factor, which arises as follows. In the case of covariant quantum spaces under

\(^2\)The case of tangential fluctuations can be analyzed similar and leads to the same metric.
consideration, we can assume that \( \mathcal{M} = \tilde{\mathcal{M}} = S^2 \times \mathcal{M}^{3,1} \), with a global \( SO(3) \) symmetry acting on \( S^2 \) and \( \mathcal{M}^{3,1} \) simultaneously. Then \( \Omega \) factorizes into the volume of the \( S^2 \) fiber times the effective density \( \rho_M \) on space-time \( \mathcal{M}^{3,1} \) [1]:

\[
\Omega = \rho_M d^4 \chi \Omega_2 , \quad \rho_M = \frac{1}{r^2 R^2 \sinh(\eta)} \sim L_{NC}^{-4} . \tag{24}
\]

Here \( S^2 \) is normalized with volume \( 4\pi \), and \( x^\mu \) are the Cartesian coordinates (14) on \( \mathcal{M}^{3,1} \) or \( \tilde{\mathcal{M}}^{3,1} \). \( L_{NC} \) characterizes the scale of noncommutativity. Then (22) can be written in a more familiar form

\[
S[\phi] \sim -\frac{1}{2\pi^2} \int \rho_M \gamma^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = -\frac{1}{2\pi^2} \int \mathcal{M}^{3,1} d^4 x \sqrt{|G_{\mu\nu}|} G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi . \tag{25}
\]

We can now read off the effective metric on \( \mathcal{M}^{3,1} \):

\[
G^{\mu\nu} = \rho^{-2} \gamma^{\mu\nu} \tag{26}
\]

where \( \rho \) is the dilaton, which relates the symplectic density \( \rho_M \) to the Riemannian density via

\[
\rho^{-2} \sqrt{|G_{\mu\nu}|} = \rho_M = \rho^2 \sqrt{|\gamma_{\mu\nu}|} \tag{27}
\]

using \( \sqrt{|G_{\mu\nu}|} = \rho^4 \sqrt{|\gamma_{\mu\nu}|} \). From the string theory point of view, the metric \( G_{\mu\nu} \) can be interpreted as open-string metric on \( \mathcal{M}^{3,1} \). Noting that

\[
\sqrt{|\gamma^{\mu\nu}|} = | \det E^{\bar{a}\mu} | , \tag{28}
\]

the dilaton is determined by the frame as

\[
\rho^2 = \rho_M | \det E^{\bar{a}\mu} | . \tag{29}
\]

It is important that the frame \( E^{\bar{a}\mu} \) in the present context does not admit local \( SO(3,1) \) gauge transformations acting on \( \bar{a} \), only global \( SO(3,1) \) transformations are allowed. The frame is a physical object here which is subject to certain constraints (68), and determines not only the metric but also additional physical information, such as the dilaton \( \rho \) and also an axion \( \tilde{\rho} \) (102).

### 2.1 Cosmological FLRW solution

A special case of the above class of backgrounds is given by

\[
T^\mu = \frac{1}{R} M^{\mu 4} \sim t^\mu \tag{30}
\]

where \( M^{\alpha\beta} \) are generators of the doubleton representation \( \mathcal{H}_n \) of \( \mathfrak{so}(4,1) \subset \mathfrak{so}(4,2) \). It is easy to see that \( T^\mu \) is a solution of the matrix model in the presence of a suitable mass term; we shall simply discuss some of its properties here. \( T^\mu \) defines a matrix configuration with manifest \( SO(3,1) \) symmetry, which in the semi-classical regime reduces to a 6-dimensional background \( \tilde{\mathcal{M}} \) which is an \( S^2 \) bundle over \( \tilde{\mathcal{M}}^{3,1} \). The Cartesian coordinate functions on the base manifold \( \tilde{\mathcal{M}}^{3,1} \) arise as

\[
X^\mu = r M^{\mu 5} \sim x^\mu . \tag{31}
\]
We will focus on the semi-classical (Poisson) limit $n \to \infty$, working with commutative functions of $x^\mu$ and $t^\mu$, but keeping the Poisson structure $\{.,.\} \sim i\{.,.\}$. Then $\text{End}(\mathcal{H}_n) \sim C$ reduces to the algebra of functions on the bundle space $\mathcal{M} \cong \mathbb{C}P^{2,1}$, dropping the bar for now. The sub-algebra $C^0 \subset C$ of functions on the base space $\mathcal{M}^{3,1}$ is generated by the

$$x^\mu : \mathcal{M}^{3,1} \hookrightarrow \mathbb{R}^{3,1}$$

for $\mu = 0, \ldots, 3$, which are interpreted as Cartesian coordinate functions. The generators $x^\mu$ and $t^\mu$ satisfy the constraints (16), which arise from the special properties of $\mathcal{H}_n$. The $t^\mu$ generators describe the $S^2$ fiber over $\mathcal{M}^{3,1}$, which is space-like due to (16). Here $\eta$ plays the role of a time parameter, defined via

$$x^4 = R \sinh(\eta) .$$

Hence $\eta = \text{const}$ defines a foliation of $\mathcal{M}^{3,1}$ into space-like surfaces $H^3$; this can be related to the scale parameter of a FLRW cosmology with $k = -1$. Note that $\eta$ runs from $-\infty$ to $\infty$, and the sign of $\eta$ distinguishes the two degenerate sheets of $\mathcal{M}^{3,1}$ linked by a Big Bounce, cf. [20]. The Poisson brackets on $\bar{\mathcal{M}}$ are given explicitly by

$$\{x^\mu, x^\nu\} = \theta^{\mu\nu} = -r^2 R^2 \{t^\mu, t^\nu\} ,$$
$$\{t^\mu, x^\nu\} = \frac{x^4}{R} \eta^{\mu\nu} ,$$

where the Poisson tensor $\theta^{\mu\nu}$ satisfies the constraints

$$t_\mu \theta^{\mu\alpha} = -\sinh(\eta) x^\alpha ,$$
$$x_\mu \theta^{\mu\alpha} = -r^2 R^2 \sinh(\eta) t^\alpha ,$$
$$\eta_{\mu\nu} \theta^{\mu\alpha} \theta^{\nu\beta} = R^2 r^2 \eta^{\alpha\beta} - R^2 x^\alpha x^\beta + r^2 x^\alpha x^\beta .$$

$\theta^{\mu\nu}$ can be expressed in terms of $t^\mu$ as

$$\theta^{\mu\nu} = \frac{r^2}{\cosh^2(\eta)} \left( \sinh(\eta) (x^\mu t^\nu - x^\nu t^\mu) + \epsilon^{\mu\nu\alpha\beta} x_\alpha t_\beta \right) ,$$

and can therefore be viewed as spin 1 valued “function” on $\mathcal{M}^{3,1}$. More generally, the space of functions $C$ on $\mathcal{M}$ decomposes into a tower of higher-spin ($hs$) valued functions

$$C = \bigoplus_{s \geq 0} C^s$$

on $\mathcal{M}^{3,1}$, where $C^s$ is spanned by irreducible polynomials (17) of degree $s$ in $t^\mu$. The Poisson brackets do not respect the decomposition into $C^s$, but the following holds

$$\{C^s, x^\mu\} \subset C^{s+1} \oplus C^{s-1}$$

noting that $\theta^{\mu\nu} \in C^1$. 

Frame, metric and torsion on $\bar{M}^{3,1}$. Following the general strategy discussed above, we can extract the effective metric on $\bar{M}^{3,1}$. Frame and metric are obtained in Cartesian coordinates from (34) as

$$E^a = \{t^a, \cdot\} = E^a{}_{\dot{\mu}} \partial_{\dot{\mu}}, \quad E^a{}_{\dot{\mu}} = \eta^{a\mu} \sinh(\eta),$$

$$\gamma^{\mu\nu} = \eta_{ab} E^{a\mu} E^{b\nu} = \sinh^2(\eta)\eta^{\mu\nu}. \quad (39)$$

Recalling that $\rho_M \sim \sinh(\eta)^{-1}$, the effective metric on $\bar{M}^{3,1}$ and the dilaton are obtained as

$$G_{\mu\nu} = \sinh^3(\eta)\gamma_{\mu\nu} = \sinh(\eta)\eta_{\mu\nu},$$

$$\rho^2 = \sinh^3(\eta). \quad (40)$$

This metric is $SO(3,1)$-invariant with signature $(- + + +)$ and conformal to the induced (“closed-string”) metric $\eta_{\mu\nu}$. It can be written in standard FLRW form as follows [1]

$$ds_G^2 = G_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t)d\Sigma^2 \quad (41)$$

where $d\Sigma^2$ is the metric on $H^3$, and the FLRW time $t$ is related to the time parameter $\eta$ via

$$a(t) \sim R \sinh^{3/2}(\eta) =: L_{\text{cosm}}, \quad t \to \infty. \quad (42)$$

One finds $a(t) \sim \frac{3}{R}t$ for late times, and $a(t) \sim t^{1/5}$ near the Big Bounce. The torsion tensor (10) is also easily computed using $\Theta^{\dot{a}\dot{b}} = \frac{1}{R} M^{\dot{a}\dot{b}}$, which gives

$$\mathcal{T}_{\dot{a}\dot{b}\mu} = \{\Theta^a{}_{\dot{b}}, x^\mu\} = \frac{1}{R^2} (\eta^{a\mu} x_{\dot{b}} - \eta^{\dot{b}\mu} x^a) \quad (43)$$

in Cartesian coordinates $x^\mu$. This can be recast as a rank 3 tensor on $M^{3,1}$ using the frame $E^{a\mu}$,

$$\mathcal{T}_{\nu\sigma}{}^\mu = \frac{1}{R^2} \left( \delta^\mu_\nu \tau_\sigma - \delta^\mu_\sigma \tau_\nu \right) \quad (44)$$

where

$$\tau_\mu = G_{\mu\nu} \tau^\nu = G_{\mu\nu} x^\nu = \sinh(\eta)\eta_{\mu\nu} x^\nu \quad (45)$$

is a global time-like $SO(3,1)$-invariant vector field on the FLRW background.

Late-time regime and noncommutativity scale. Consider the regime of late time or large $\eta$, so that $\sinh(\eta) \gg 1$. Then the Poisson tensor $\theta^{\mu\nu}$ (36) reduces to

$$\theta^{\mu\nu} \sim \frac{r^2}{\cosh(\eta)} (x^\mu t^\nu - x^\nu t^\mu), \quad \eta \to \infty. \quad (46)$$

More specifically, consider some given reference point $\xi = (x^0, 0, 0)$ on $M$. Then this reduces to

$$\theta^{0i} \xi = \frac{r^2}{\cosh^2(\eta)} \sinh(\eta)x^0 t^i \sim r^2 R t^i = O(L_{NC}^2) \quad (47)$$

$$\theta^{ij} \xi = \frac{r^2}{\cosh^2(\eta)} x^0 \epsilon^{0ij} t_k \sim \frac{1}{\sinh(\eta)} r^2 R \epsilon^{ijk} t^k = O(r R) \quad (47)$$
where

$$L_{\text{NC}}^2 = R r \cosh(\eta)$$

is the effective scale of noncommutativity on $M^{3,1}$ (cf. (24)), using $|t| \sim r^{-1} \cosh(\eta)$ (15). Even though this grows with $\eta$, it is much shorter than the cosmic curvature scale (42):

$$\frac{L_{\cos}^2}{L_{\text{NC}}^2} \sim \frac{R}{r} \cosh^2(\eta) \sim n \cosh^2(\eta).$$

(49)

Therefore there is plenty of space for interesting physics in between. In particular, $\theta^{0i} \sim r^2 R t^i \gg \theta^{ij}$ at late times $\eta \gg 1$. The space-like generators $t^i$ describe the internal fuzzy sphere $S_n^2$ with

$$\{t^i, t^j\} \equiv \frac{1}{r^2 R^2} \theta^{ij} = -\frac{1}{R \sinh(\eta)} \epsilon^{ijk} t^k$$

and generate the higher-spin algebra $\mathfrak{hs}$. Even though $t^0 \in \mathfrak{hs}$ vanishes as function at $\xi$, it is a non-trivial generator which induces local time translations via $\{t^0, \cdot\}$.

2.2 Derivations

**Fuzzy hyperboloid $H_n^4$.** The above space-time $M^{3,1}$ can be understood as a projection of the fuzzy hyperboloid $H_n^4$ [10], which can be viewed as a submanifold of $\mathbb{R}^{4,1}$ defined in terms of the 5 generators

$$X^a = r M^a s, \quad a = 0, \ldots, 4$$

(51)

(cf. (31)) which transform as vectors of $SO(4, 1)$. The underlying symplectic space is the same as for $M^{3,1}$, given by the non-compact projective space $\mathbb{C}P^{2,1}$ which is nothing but (projective) twistor space, cf. [21]. The Poisson structure on the bundle space allows to define derivations as follows

$$\delta^a \phi := -\frac{1}{r^2 R^2} \theta^{ab} \{x_b, \phi\} = \frac{1}{r^2 R^2} x_b \{\theta^{ab}, \phi\}, \quad \phi \in C.$$  

(52)

They satisfy the useful identities

$$x^a \delta_a \phi = 0,$$

$$\delta^a x^c = \eta^{ab} + \frac{1}{R^2} x^a x^b,$$

$$\delta^a (\{x_a, \phi\}) = 0$$

(53)

for any $\phi \in C$. Furthermore, we note that all (even $\mathfrak{hs}$-valued) Hamiltonian vector fields on $H_n^4$ are tangential to $H^4 \subset \mathbb{R}^{4,1}$, due to the identity

$$x^a \{x_a, \Lambda\} = 0.$$  

(54)
Derivatives on $M^{3,1}$. Since the algebra of functions $C$ for $M^{3,1}$ and $H^4$ is the same, we can use the above derivative operators to define the following derivations on $M^{3,1}$

$$\partial_\mu := \delta_\mu - x_\mu \frac{1}{x_4} \delta_4 \quad \text{on} \ C . \quad (55)$$

Using the identities (53), it is easy to show

$$\partial_\mu x^\nu = \delta_\mu^\nu ,$$

$$\partial_\mu (\rho_M \theta^{\mu \nu}) = 0 . \quad (56)$$

This will imply that all Hamiltonian vector fields on $M^{3,1}$, in particular the frame, are conserved.

3. Divergence-free vector fields on $H^4$ and $M^{3,1}$

Divergence-free vector fields will play an important role in the following. Clearly any vector field $V^\alpha$ on $H^4$ can be mapped to a vector field $V^\mu$ on $M^{3,1}$, by simply dropping the $V^4$ component (in Cartesian coordinates). This can be understood as push-forward via a projection [1]. For example, a Hamiltonian vector field $V^\alpha = \{ T, x^\alpha \}$ is mapped to $V^\mu = \{ T, x^\mu \}$ in Cartesian coordinates. Conversely, any vector field $V^\mu$ on $M^{3,1}$ can be lifted to $H^4$ by defining

$$V^4 := -\frac{1}{x_4} V_\mu V^\mu , \quad (57)$$

which defines a tangential vector field $V^\mu x_\alpha = 0$ on $H^4$. We claim that this correspondence maps divergence-free vector fields $\delta_\alpha V^\alpha = 0$ on $H^4$ to divergence-free vector fields on $M^{3,1}$, in the sense that

$$\partial_\mu (\rho_M V^\mu) = 0 . \quad (58)$$

Here $\rho_M$ is the symplectic density (24) on $M^{3,1}$, which in Cartesian coordinates is given by $\rho_M = \sinh(\eta)^{-1}$. In fact the following more general result holds:

Lemma 3.1. Let $V^\alpha$ be a (tangential) vector field on $H^4$, i.e. $V^\alpha x_\alpha = 0$. Then its reduction (or push-forward) $V^\mu$ to $M^{3,1}$ satisfies

$$\delta_\alpha V^\alpha = \sinh(\eta) \partial_\mu (\rho_M V^\mu) \quad (59)$$

Conversely, the lift of $V^\mu$ to $H^4$ defined by (57) satisfies (59). If $V^\alpha$ is divergence-free on $H^4$ i.e. $\delta_\alpha V^\alpha = 0$, then its reduction to $M^{3,1}$ satisfies

$$\partial_\mu (\rho_M V^\mu) = 0 . \quad (60)$$

In particular, all Hamiltonian vector fields on fuzzy $H^4$ and $M^{3,1}$ are conserved, in the sense

$$\delta_\alpha \{ x^\alpha, T \} = 0 , \quad \partial_\mu (\rho_M \{ x^\mu, T \}) \quad (61)$$
**Proof.** Using the definition of $\partial_\mu (55)$ on $C$, we compute

$$
\delta_\mu V^\alpha = \partial_\mu V^\mu + \delta_4 V^4 \\
= (\partial_\mu + \frac{1}{x_4} x_\mu \delta_4) V^\mu + \delta_4 V^4 \\
= \partial_\mu V^\mu + \frac{1}{x_4} \delta_4 (x_\mu V^\mu) - \frac{1}{x_4} V^\mu \delta_4 x_\mu + \delta_4 V^4 \\
= \partial_\mu V^\mu - \frac{1}{x_4} \delta_4 (x_4 V^4) - \frac{1}{R^2} x_\mu V^\mu + \delta_4 V^4 \\
= \partial_\mu V^\mu - \frac{1}{x_4} V^4 \\
= \sinh(\eta) \partial_\mu \left( \frac{1}{\sinh(\eta)} V^\mu \right). 
$$

(62)

(61) now follows using (53).

In particular, the identity (56) can now be understood by noting that $V^\alpha = \{x^\nu, x^\alpha\}$ is conserved on $H^4$. We also note that the divergence constraint (60) for vector fields on $M^{3,1}$ can be written using (27) in covariant form in terms of the effective metric $G^{\mu\nu}$ on $M^{3,1}$:

$$
0 = \nabla_\mu (\rho^{-2} V^\mu) = \frac{1}{\sqrt{|G|}} \partial_\mu (\rho M V^\mu) 
$$

(63)

where $\nabla$ is the Levi-Civita connection corresponding to $G$.

4. **Generic backgrounds from deformed $M^{3,1}$**

Starting from the above FLRW background, we can obtain more generic geometries as deformations, by simply adding fluctuations of the background:

$$
T^a = \tilde{T}^a + \mathcal{A}^a 
$$

(64)

The fluctuations $\mathcal{A}^a$ are any $\hbar s$ valued gauge fields, which are governed by a Yang-Mills gauge theory. We want to focus in the following on purely geometric deformations, leaving aside the higher spin modes. We therefore focus on fluctuations of the form

$$
\mathcal{A}^a = \mathcal{A}^{a\mu}(x) t_\mu 
$$

(65)

Since we don’t want to restrict ourselves to the linearized perturbations, we simply consider generic backgrounds of the form

$$
T^a = T^{a\mu}(x) t_\mu 
$$

(66)

which include the cosmic background for $T^{a\mu}(x) = \eta^{a\mu}$. As discussed in section 2, such a background defines a frame (66)

$$
E^{a\mu} = \{T^{a}, x^\mu\} 
$$

(67)
Taking into account the above results, we conclude that any such frame satisfies the divergence constraint \[ \partial_\mu (\rho M E^\alpha_\mu) = \nabla_\mu (\rho^{-2} E^\alpha_\mu) . \] (68)

In the following we will establish the converse statement: any frame given by divergence-free vector fields can indeed be implemented as above, for a suitable background of the form (66). Moreover, the $T^a$ can be computed explicitly. This entails in general some extra $\mathfrak{g}$ valued contribution to the frame, which will be shown to be insignificant in section 5 in the weak gravity regime.

### 4.1 Reconstruction of divergence-free vector fields

We start by recalling two results given in [4], starting with the Euclidean case:

**Lemma 4.1.** Given any divergence-free tangential vector field $\delta_\alpha V^a = 0$ on $H^4$ with $V^a \in C^0$, there is a unique generator $T \in C^1$ such that

\[ V^a = \{T, x^a\}_0 . \] (69)

This $T$ is given explicitly by

\[ T := -3(\Box_H - 4r^2)^{-1}\{V^a, x_a\} =: D^+(V) \in C^1 \] (70)

where $\Box_H = \{x^a, \{x_a, \ldots\}\}$.

However, the Hamiltonian vector field $\{T, x^a\}$ generated by the above $T \in C^1$ contains in general also a spin 2 component

\[ V^{(2)a} := \{T, x^a\}_2 = D^{++}(V^a) \in C^2 \] (71)

which is also divergence-free $\delta_\alpha V^{(2)a} = 0$. Here $D^{++}$ is an $SO(4, 1)$ intertwiner given by

\[ D^{++}(\mathcal{A}) = \{D^+(\mathcal{A}), x^a\} \] (72)

which satisfies

\[ \int D^{++}(\mathcal{A}) D^{++}(\mathcal{B}) \approx 4 \int \mathcal{A}^a \mathcal{B}^a , \] \[ D^{++}(x^a \mathcal{A}) \approx x^a (D^{++} \mathcal{A}) , \] \[ D^{++}(\delta^a \mathcal{A}) \approx \delta^a (D^{++} \mathcal{A}) . \] (73) (74)

in the regime where $r^{-2} \Box \gg 1$, i.e. not in the extreme IR regime. Therefore the above reconstruction of vector fields on $H^4$ generically leads to extra $\mathfrak{g}$ components $V^{(2)a} \in C^2$, which however encode the same information as $V^a$. It remains an open question if these can be cancelled by allowing higher-spin corrections to the coordinate generators $x^a$.

We can use these results to obtain an analogous “reconstruction” statement on $M^{3,1}$ [4]:
**Lemma 4.2.** Given any $C^0$-valued divergence-free vector field $V^\mu$ on $M^{3,1}$,

\[ \partial_\mu (\rho_M V^\mu) = 0 \]  

there is a generating function $T \in C^1$ such that

\[ V^\mu = \{ T, x^\mu \}_0 . \]  

Explicitly, $T$ is given by

\[ T = -3(\Box_H - 4r^2)^{-1}(\{ V^\mu , x_\mu \} + \{ V^4 , x_4 \}) \]  

where

\[ V^4 = \frac{1}{x_4} x_\mu V^\mu . \]

This is simply obtained by lifting $V^\mu$ to a divergence-free vector field $V^a$ on $H^4$ as in Lemma 3.1. Then the result (69) on $H^4$ states that $V^a = \{ T, x^a \}_0$ for some $T \in C^1$, which implies $V^\mu = \{ T, x^\mu \}_0$. Moreover, $T$ is uniquely determined by (76). One can show that this spin 2 component vanishes only for $T \in \mathfrak{so}(4, 1)$.

To summarize, we have shown that every divergence-free vector field on $M^{3,1}$ can be realized or reconstructed as Hamiltonian vector field, i.e. $V^\mu = \{ T, x^\mu \}_0$. However, this entails the presence of a spin two sibling $V^{(2)\mu} = \{ T, x^{(2)\mu} \}_2 \in C^2$. In other words, the Hamiltonian vector field generated by $T \in C^1$ acts on a function $\phi = \phi(x) \in C^0$ via

\[ \{ T, \phi \} = \{ T, x^\mu \} \partial_\mu \phi = (V^\mu + V^{(2)\mu}) \partial_\mu \phi . \]

Both components of $V^\mu + V^{(2)\mu} \in C^0 \oplus C^2$ are isomorphic as $\mathfrak{so}(4, 1)$ modes. This applies in particular to the frame in the effective field theory on $M^{3,1}$ arising from matrix models.

**4.2 Reconstruction of classical geometry**

Now we apply the results of the previous section to reconstruct a classical frame $e^{\hat{a}\mu}$ within the present framework. This is the basis for describing gravity through the effective metric on a suitable covariant quantum spaces. It is clear from (68) that only divergence-free frames can be realized here, but this does not restrict the possible metrics as explained in section 6. Hence for any divergence-free classical frame $e^{\hat{a}\mu}$, there is a unique $T^{\hat{a}} \in C^1$ given by

\[ T^{\hat{a}} = -3(\Box_H - 4r^2)^{-1}(\{ e^{\hat{a}\mu} , x_\mu \} + \{ e^{\hat{a}4} , x_4 \}) \]

\[ = -3(\Box_H - 4r^2)^{-1}(\{ e^{\hat{a}\mu} , x_\mu \} - \frac{1}{x_4} \{ e^{\hat{a}\mu} x_\mu , x_4 \}) \]  

such that

\[ e^{\hat{a}\mu} = \{ T^{\hat{a}} , x^\mu \}_0 . \]

E.g. for the cosmic frame $e^{\hat{a}\mu} = \sinh(\eta)\eta^{\hat{a}\mu}$ on $M^{3,1}$, this gives

\[ e^{\hat{a}4} = -\frac{x_\mu}{x_4} e^{\hat{a}\mu} = -\frac{1}{r} x^{\hat{a}} \]
and we recover the background \((30)\)
\[
T^a = -3(\Box_H - 4r^2)^{-1}\left(\{e^{\dot{a}}_{\mu}, x_{\mu}\} - \frac{1}{r}\{x^\dot{a}, x_4\}\right)
= 6(\Box_H - 4r^2)^{-1}\{\sinh(\eta), x^\dot{a}\}
= t^a
\] (83)
using \(\Box_H t^\mu = -2r^2 t^\mu\). The generator \(T^a \in \mathbb{C}^1\) is uniquely determined by \((67)\). However, the reconstructed frame will in general contain higher spin \(h\) components \(\{T^a, x^\mu\} \in \mathbb{C}^2\) due to \((38)\). Even though these drop out in the linearized theory upon averaging over \(S_2^a\), this is no longer true in the non-linear regime, and we must clarify the importance of these contributions.

5. Weak gravity regime and classical geometry

For covariant quantum spaces, the frame is in general higher-spin valued, and so is the metric. To describe real physics, we should be able to recover classical geometries in terms of backgrounds which contain no significant higher-spin contributions. This leads to the following problem: For any given classical divergence-free frame \(e^\mu_{\dot{a}} \in \mathbb{C}^0\), we would like to find generators \(T^a\) such that the (generally \(h\)-valued) reconstructed frame
\[
E^\mu_{\dot{a}} = \{T^a, x^\mu\}
\] (84)
reproduces the classical frame through its spin 0 component \(\mathbb{C}^0\):
\[
\{E^\mu_{\dot{a}}\}_0 = \{\{T^a, x^\mu\}\}_0 = e^\mu_{\dot{a}}
\] (85)
This problem of frame reconstruction is of central importance, since gravity requires to realize generic geometries in the matrix model framework. A solution of problem is given by the results in the previous sections as follows:
\[
T^a = \mathcal{D}^+(e^\dot{a}) = -3(\Box_H - 4r^2)^{-1}\left(\{e^\mu_{\dot{a}}, x_{\mu}\} + \frac{1}{x_4}\{x_4, x_{\mu} e^\mu_{\dot{a}}\}\right) \in \mathbb{C}^1
\] (86)
However in general,
\[
E^\mu_{\dot{a}} = \{T^a, x^\mu\} = e^\mu_{\dot{a}} + \mathcal{D}^{++}(e^\mu_{\dot{a}}) \in \mathbb{C}^0 \oplus \mathbb{C}^2
\] (87)
contains also \(\mathbb{C}^2\) contributions, which vanish only for very special “pure backgrounds”:

5.0.1 Pure backgrounds

Consider the class of pure backgrounds \(T^a \in \mathbb{C}^1\), which have the property that the frame is a pure function,
\[
E^\mu_{\dot{a}} = \{T^a, x^\mu\} \in \mathbb{C}^0
\] (88)
It is not hard to see that all such backgrounds are some linear combination of \(\mathfrak{so}(4, 1)\) generators
\[
T^a = e_{\dot{a}, bc} \theta^{bc} \in \mathbb{C}^1
\] (89)
This leads to the $C^0$-valued frame and torsion
\begin{align}
E^\mu_a &= e^a_{\alpha:bc} \{ \theta^b_{\mu c}, x^\mu \} = -r^2 e_{\alpha:bc} (\eta^{b\mu} x^c - \eta^{c\mu} x^b), \\
T^\mu_{a\beta} &= -\{ (T_a, T_\beta), x^\mu \} = r^2 c_{\alpha\beta:ab} (\eta^{a\mu} x^b - \eta^{b\mu} x^a)
\end{align}
where $\{ T_a, T_\beta \} = c_{\alpha\beta:ab} \theta^{ab}$. These configurations comprise the cosmic background solution, which is recovered for
\begin{equation}
T_a = t_a = \frac{1}{2R} (\delta^4_c \eta_{ab} - \delta^4_b \eta_{ac}) \theta^{bc}.
\end{equation}
In particular, any classical frame of the form
\begin{equation}
e^\mu_a = \sinh(\eta) \tilde{e}^\mu_a
\end{equation}
with constant $\tilde{e}^\mu_a \in \mathbb{R}$ is reproduced by the pure background
\begin{equation}
T_a = \tilde{e}^\mu_a t_\mu.
\end{equation}
More generally, all such frames $E^\mu_a$ (90) are in the kernel of the map $D^{++}$ (72) which links the components in $C^0$ and $C^2$, and they are closed under $SO(4,1)$ gauge transformations. They provide clean configurations which can be used as a starting point for a perturbative approach around any point $\xi \in M^{3,1}$.

5.0.2 Generic backgrounds and local linearization

In general, the reconstructed frame $E^\mu_a = \{ T^a, \cdot \}$ will contain components in $C^2$, which are obtained from the $C^0$ components via $D^{++}$. However as shown above, we can reproduce any given classical frame at some fixed point $p$ by a pure frame, which is in the kernel of $D^{++}$. Writing the classical frame in the form
\begin{equation}
e^\mu_a = \tilde{e}^\mu_a + \delta e^\mu_a
\end{equation}
where $\tilde{e}^\mu_a$ is pure and $\delta e^\mu_a$ vanishes at $p$, the reconstructed frame is given by
\begin{equation}
E^\mu_a = \tilde{e}^\mu_a + (1 + D^{++}) \delta e^\mu_a.
\end{equation}
Since $D^{++}$ is norm-preserving (73), this is well approximated or dominated by its $C^0$ component in some sufficiently small neighborhood of $p$. We can make this more quantitative by recalling that the torsion has the structure $T \sim e^{-1} d\tilde{e}$, whose scale is set by the curvature scale of gravity since $R \sim T^2$. Therefore the frame is essentially constant in a region small compared to the curvature scale of gravity, and can be approximated by its classical spin 0 component. Similarly, the metric is then well approximated by its classical component $[G_{\mu\nu}]_0 \in C^0$. This justifies the above reconstruction procedure for the frame. Since the torsion is given by derivatives of the frame and observing (74), the intertwiner $D^{++}$ applies also to the torsion
\begin{equation}
T^\mu_{a\beta} \approx (1 + D^{++}) [T^\mu_{a\beta}]_0.
\end{equation}
In particular, a gravitational action of the form (107) reduces to the classical \( C^0 \) contribution due to (73). We can therefore consider the geometrical tensors as effectively classical in the weak gravity regime, where the curvature scale \( R \sim T^2 \) is much smaller than all other physical scales.

In the strong gravity regime, these arguments are no longer justified. Nevertheless, we will show in the next section that one can always choose local normal coordinates at any given point \( p \in \mathcal{M}^{3,1} \) such that all \( h^a \) components of the frame vanish at \( p \). In this sense the metric always reduces to a classical metric, which governs the local physics near \( p \).

6. Realization of generic 3 + 1-dimensional geometries in matrix models

Finally, we address the question if any given metric \( G_{\mu \nu} \) can be realized in terms of a divergence-free frame. The first step is to determine the dilaton, which is obtained from (27) as

\[
\rho^2 = \rho_M^{-1} \sqrt{|G|}. \tag{97}
\]

The next step is to find some classical divergence-free frame \( e^{a \mu} \) which gives rise to (23)

\[
\rho^2 G^{\mu \nu} = \gamma^{\mu \nu} = \eta_{ab} e^{a \mu} e^{b \nu}. \tag{98}
\]

Without the constraint, there are of course many frames (in fact a 6-dimensional orbit of \( SO(3,1) \)) which achieve that. The 4 divergence constraints are fairly easy to take into account in Cartesian coordinates \( x^\mu \): for any given space-like components \( e^{ia} \), the time components \( e^{0a} \) are determined by

\[
\partial_0 (\rho_M e^{0a}) = -\partial_j (\rho_M e^{aj}) \tag{99}
\]

This can be viewed as an ordinary differential equation in \( x^0 \), which is solved by

\[
e^{0a} = -\rho^{-1}_M \int_{\xi_0}^{x^0} d\xi \partial_j (\rho_M e^{aj}) + e^{0a}(\xi_0) \tag{100}
\]

where the value \( e^{0a}(\xi_0) \) at any given time \( \xi_0 \) can be chosen as desired. This means that we can freely choose the 12 space-like \( e^{aj} \), which should allow to reproduce the 10 dof in \( \gamma^{\mu \nu} \) even if the divergence constraint is imposed.

A more systematic, iterative way to determine the frame is as follows: choose some reference point \( \tilde{x} \). After a global \( SO(3,1) \) transformation on the frame indices, we can assume that \( \gamma^{\mu \nu}|_{\tilde{x}} = c \eta^{\mu \nu} \), and we assume \( c = 1 \) for simplicity. Then choose the diagonal elements as \( e^{ia} = \eta^{ia} \), and off-diagonal frame elements which vanish at \( \tilde{x} \), such that the frame reproduces \( \gamma^{\mu \nu} \). To satisfy the divergence constraint, we define a correction of the diagonal frame elements by

\[
\delta e^{a \mu} = -\rho^{-1}_M \int_{\tilde{x}^a}^{x^a} d\xi^a \sum_{\mu \neq a} \partial_\mu (\rho_M e^{a \mu}), \tag{101}
\]

which vanishes at \( \tilde{x} \). Then the improved frame \( e^{a \mu} \rightarrow e^{a \mu} + \delta e^{a \mu} \) satisfies the divergence constraint, and reproduces \( \gamma^{\mu \nu} \) to a good approximation near \( \tilde{x} \). Now we repeat this procedure iteratively by
correcting the off-diagonal elements of the frame such that $\gamma^{\mu\nu}$ is reproduced, and correcting the diagonal elements again with (101), and so on. Since the corrections vanish at $\bar{x}$, this procedure will converge to a divergence-free frame which reproduces $\gamma^{\mu\nu}$ exactly at least in some neighborhood of $\bar{x}$. This could presumably be proved e.g. using the Banach fixed point theorem, but we leave it as a plausibility argument here and accept the statement as true.

We conclude that there are always divergence-free frames $e^{\hat{\alpha}_\mu}$ which realize (98) for any $\gamma^{\mu\nu}$. As explained in section 4.2, we can then find a corresponding matrix background which implements the frame in the weak gravity regime. Therefore generic 3+1-dimensional space-time geometries can indeed be implemented as backgrounds of the matrix model with an ansatz of the form (66), leading to a covariant quantum space-time.

Moreover, the above analysis shows that 2 of the 12 dof in $e^{\hat{\alpha}_\mu}$ remain undetermined even if the divergence constraint is imposed. They can be used to restrict the totally antisymmetric components of the torsion (112), which define a vector field via $\tilde{T}_\mu \propto T^{(AS)}_{\nu\tau\mu} e_{\nu\tau\mu}$. For example, it is plausible that the frame can be chosen such that

$$\tilde{T}_\mu = \psi^{-1} \partial_{\mu} \tilde{\rho}$$

(102)

in terms of an axion $\tilde{\rho}$; this is a consequence of the (semi-classical) matrix model equations of motion [2]. This question and its implications should be addressed elsewhere.

7. Quantization, extra dimensions and induced gravity

Even though the semi-classical matrix model action defines a dynamical theory of space-time geometry, it is expected that a (near-) realistic theory of gravity can be obtained only from the Einstein-Hilbert action. Remarkably, this arises indeed in the 1-loop effective action under certain assumptions, in the spirit of induced gravity [22, 23]. The quantization of the matrix model is defined non-perturbatively through a matrix path integral

$$Z = \int dT d\Psi e^{iS}.$$  

The oscillatory integral becomes absolutely convergent for finite-dimensional $\mathcal{H}$ upon implementing the regularization

$$S \rightarrow S + i\varepsilon \sum_\delta Y_\delta Y_{\bar{\delta}},$$

(103)

which amounts to a Feynman $i\varepsilon$ term in the noncommutative gauge theory. For recent results of numerical simulations of such models see e.g. [24, 25].

In general, the quantization of matrix models on some noncommutative background leads to highly non-local action due to UV/IR mixing, except in the maximally supersymmetric IKKT model. This phenomenon was shown first identified in [26], but it is most transparent in terms of string states $|x\rangle\langle y|\in\text{End}(\mathcal{H})$, which govern the deep quantum (or extreme UV) regime of noncommutative functions [27, 28]. These states are also extremely useful to compute the 1-loop effective action of the IKKT matrix model on generic backgrounds. It was indeed show in [5] that
the Einstein-Hilbert action arises at 1 loop, provided the transversal 6 matrices $T^k$ of the IKKT model assume some non-trivial background given by some compact fuzzy space: 

$$T^k \sim t^k : \mathcal{K} \hookrightarrow \mathbb{R}^6, \quad k = 4, \ldots, 9. \quad (104)$$

This describes a quantized compact symplectic space $\mathcal{K}$ embedded along the transversal directions, which plays the role of fuzzy extra dimensions. Together with the space-time brane $M^{3,1}$, the overall background geometry then has a product structure 

$$M^{3,1} \times \mathcal{K} \hookrightarrow \mathbb{R}^{9,1}. \quad (105)$$

The detailed structure of $\mathcal{K}$ will be irrelevant; we only require that the internal matrix Laplacian $\Box_\mathcal{K} = [T^k, [T_k, \cdot]]$ has positive spectrum,

$$\Box_\mathcal{K} Y_\Lambda = m_\Lambda^2 Y_\Lambda, \quad m_\Lambda^2 = m^2_\mu^2 \mu_\Lambda^2, \quad (106)$$

with a finite number of (Kaluza-Klein KK) eigenmodes $Y_\Lambda \in \text{End}(\mathcal{H}_\mathcal{K})$ enumerated by some label $\Lambda$. Here $m^2_\mu$ determines the radius of $\mathcal{K}$ and sets the scale of the KK modes, which will play an important role below.

Computing the 1-loop effective action on such a background then leads in particular to the following term 

$$\Gamma^{K-M}_{\text{loop}} = - \frac{c^2_\mathcal{K}}{(2\pi)^4} \int_M d^4x \sqrt{G} \rho^{-2} m^2_\mathcal{K} T^\rho_{\sigma\mu} T^\sigma_\rho G^{\mu\nu} \quad (107)$$

which describes the effective interaction between $M^{3,1}$ and $\mathcal{K}$. Here 

$$c^2_\mathcal{K} = \frac{\pi^2}{8} \sum_{\Lambda,s} \frac{(2s + 1) C^2_\Lambda}{\mu_\Lambda^2 + m^2_\Lambda} > 0 \quad (108)$$

is finite, determined by the dimensionless KK masses $\mu_\Lambda$ on $\mathcal{K}$ (106) and their cousins $C^2_\Lambda$, which also depend on the structure of $\mathcal{K}$. The mass scale of the internal fields modes on $S^2_\Lambda$ is given by 

$$m^2_\Lambda = \frac{s(s - 1)}{R^2}. \quad (109)$$

Using partial integration, one can rewrite the above effective action in terms of an Einstein-Hilbert term with effective Newton constant 

$$\frac{1}{16\pi G_N} = \frac{c^2_\mathcal{K}}{14\pi^4} \rho^{-2} m^2_\mathcal{K}. \quad (110)$$

However, this requires assuming some specific behavior of $m^2_\mathcal{K}$ or $G_N$. If we assume $G_N = \text{const}$, we can use the identity [5]

$$\int d^4x \frac{\sqrt{|G|}}{G_N} R = - \int d^4x \frac{\sqrt{|G|}}{G_N} \left( \frac{7}{8} T^\mu_{\sigma\rho} T^\rho_{\mu\sigma} G^{\sigma\rho} + \frac{3}{4} \tilde{T}_\nu T_\mu G^{\mu\nu} \right) \quad (111)$$

\footnote{$\mathcal{K}$ could be a fuzzy sphere $S^2_N$, or some richer fuzzy space leading to interesting low-energy gauge theories, cf. [29].}
where $R$ is the Ricci scalar of the effective metric $G_{\mu\nu}$, and
\[ T_\mu dx^\mu = - \star \left( \frac{1}{2} G_{\nu\rho} T^\sigma_{\rho\mu} dx^\nu dx^\rho dx^\mu \right) \] (112)
is the Hodge-dual of the totally antisymmetric torsion. This gives
\[ \Gamma^{K-M}_{1\text{loop}} = \int_M d^4x \frac{\sqrt{G}}{16\pi G_N} \left( R + \frac{3}{4} T_\nu T_\mu G^{\mu\nu} \right). \] (113)

Using the eom of the matrix model, $\tilde{T}_\nu$ reduces to a gravitational axion $\tilde{\rho}$ [2]
\[ T_\mu = \rho^{-2} \partial_\mu \tilde{\rho}. \] (114)

Since $T_\mu$ vanishes exactly on the cosmic background, it is plausible that its effect is small, in which case we recover the Einstein-Hilbert action as desired.

However since $G_{\mu\nu}$ depends on $m_\mathcal{K}$ and $m_{\mathcal{K}}$, it is not evident that $G_{\mu\nu} = 0$. If we assume instead that $m_\mathcal{K} = 0$ (which is reasonable as discussed below), then one can derive an analogous identity
\[ \int d^4x \frac{\sqrt{|G|}}{G_N} R = - \int d^4x \frac{\sqrt{|G|}}{G_N} \left( \frac{1}{8} T_{\rho\sigma} T_{\mu\nu} G^{\rho\sigma\mu\nu} + \frac{1}{4} \tilde{T}_\nu T_\mu G^{\mu\nu} \right) \] (115)
based on results in [2]. This leads to a slightly modified gravitational action
\[ \Gamma^{K-M}_{1\text{loop}} = 7 \int_M d^4x \frac{\sqrt{G}}{16\pi G_N} \left( R + \frac{1}{4} \tilde{T}_\nu T_\mu G^{\mu\nu} \right) \] (116)
where the Newton constant is modified by a factor 7. The precise form of the gravitational action thus depends on the behavior of the compactification scale $m_\mathcal{K}^2$, which needs to be clarified in future work.

These results are remarkable in many ways. The first observation is that the Newton constant $G_N$ (110) is set by the compactification scale $m_\mathcal{K}$. This means that the Planck scale is related to the Kaluza-Klein scale for the fuzzy extra dimensions $\mathcal{K}$. Without the fuzzy extra-dimensional $\mathcal{K}$, no Einstein-Hilbert action is induced, and only some (rather obscure) higher-derivative action is obtained. It should be noted that no UV divergence arises in the loop computation, due to maximal supersymmetry of the matrix model and the fact that $\mathcal{K}$ supports only a finite number of modes.

We can justify the presence of $\mathcal{K}$ to some extent by studying how the 1-loop effective action depends on its radius, or equivalently on $m_\mathcal{K}$. This is obtained from the same computation as above: It turns out that (107)
\[ \Gamma^{K-M}_{1\text{loop}} = c^2 m_\mathcal{K}^2 = -V_{1\text{loop}}(m_\mathcal{K}^2) > 0 \] (117)
is positive for the covariant FLRW space-time in [1]. Combined with the bare matrix model action, the effective potential has the structure
\[ V(m_\mathcal{K}^2) = -c^2 m_\mathcal{K}^2 + \frac{d^2}{8\pi^2} m_\mathcal{K}^4 \] (118)
Classical space-time geometry in the IKKT matrix model

Harold C. Steinacker

at weak coupling. This clearly has a minimum for $m_K^2 > 0$ with $V < 0$. Since $m_K$ is essentially the radius of $K$, this strongly suggests that $K$ is stabilized by quantum effects, thus providing some justification for \((105)\).

One may worry that the effective potential for $m_K$ depends on the geometry of $M^{1,1}$, which we have assumed to be the cosmic background brane. Thus gravitational deformations of the geometry should have some influence on the Newton constant. Nevertheless, $m_K$ is expected to be constant to a very good approximation. Since $m_K$ is essentially the radius of $K$, its kinetic term $\int \partial^\mu m_K \partial_\mu m_K$ in the matrix model is huge, which would strongly suppress any local variations; note that $m_K^2 \sim \rho^2 G_N^{-1}$ is a huge energy beyond the Planck scale. Therefore $m_K$ should be almost constant, and hence governed by the large-scale cosmic background as assumed above.

On the other hand, this suggests that the Newton constant may change during the cosmic expansion. This may be a significant concern, since there are rather strong observational bounds on such a variation. Nevertheless, at this early stage such worries are presumably sub-leading, and the prime focus should be to gain a more detailed understanding of this new mechanism for gravity.

Furthermore, the above induced gravity action in $3+1$ dimensions can be interpreted as a quasi-local interaction of $K$ and $M$ via $9+1$-dimensional IIB supergravity, recalling that the 1-loop effective action is related to IIB supergravity [6, 28, 30, 31]. This provides additional confidence into the above rather formal computations, since $9+1$-dimensional supergravity is well established in string theory and expected to be recovered in the matrix model. A more detailed understanding of the relation with supergravity for backgrounds of the structure $M^{3,1} \times K \subset R^{9,1}$ would be desirable.

Note that in contrast to orthodox string theory, target space $R^{9,1}$ is not compactified here. This makes sense, since the perturbative physics on such backgrounds is restricted to the brane, and there are no bulk modes radiating off the brane at weak coupling. Hence the main problem of string theory - i.e. the need for compactification and the lack of preferred choices thereof - turns into a blessing, as there would be no induced gravity on space-time without the extra dimensions of target space.

**Vacuum energy due to $K$.** The 1-loop contribution to the vacuum energy due to $K$ is obtained using an analogous trace computation, leading to a result of the structure

$$
\Gamma^{K}_{1\text{loop}} = \frac{3i}{4} \text{Tr} \left( \frac{\nabla^4}{\sqrt{g}} \right) \sim -\frac{\pi^2}{8(2\pi)^4} \int \Omega \rho^{-2} m_K^4 \sum_{\lambda} \frac{V_{4,\lambda}}{\hbar^2}\Lambda \Omega^{119}\Lambda
$$

assuming $\frac{1}{\hbar^2} \ll m_K^2 \sim m_K^2$. Here $V_{4,\lambda}$ depends on the structure of $K$. This is typically a large vacuum energy with scale set by $m_K$ which was related to the Planck scale above, which could have either sign. However as the symplectic volume form $\Omega$ is independent of the metric, this 1-loop vacuum energy is not equivalent to a cosmological constant; its effect on the dilaton $\rho$ remains to be understood. The present framework can therefore be viewed as a realization of induced gravity in the spirit of Sakharov [22, 23], which is free of UV divergences, and appears to avoid the associated cosmological constant problem.

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