We study a fermion chain under continuous monitoring of local occupation numbers with arbitrary measurement rate. The resulting stochastic quantum trajectories preserve state pureness and Gaussianity, enabling numerical simulation of quantum trajectories with hundreds of fermions. We find that any amount of measurement becomes relevant beyond a cross-over scale, which is inverse proportional to the measurement/hopping ratio. At that scale, entanglement growth saturates and transport becomes classical-diffusive instead of coherent-ballistic. Fluctuations of entanglement are bounded and have Gaussian statistics. We provide a generalized hydrodynamics theory that agrees perfectly with numerical simulations. Implications for more general many-body systems under continuous monitoring are discussed.

Introduction – The dynamics of entanglement in many-body systems is a topic under intensive study, as entanglement is a fundamental notion of quantum physics, deeply tied to basic issues such as thermalization of closed quantum systems [1–3], holography [4–6], quantum chaos and information scrambling [7, 8]. Understanding entanglement is also crucial for designing effective quantum simulators and determining the limits thereof [9–11]. Thanks to recent experimental progress, these fundamental issues can now be addressed in a laboratory [12–14].

The subtle role played by measurement is another central feature of quantum mechanics, and has stimulated debates over interpretations for almost a century [15–17]. Stepping aside from foundational considerations, it is important to understand the interplay between measurement dynamics and unitary system dynamics in practice. Such a study is made possible with the help of weak and continuous measurements, which enable a non-destructive probing of quantum systems. Continuous measurements can now be realized experimentally, notably in superconducting circuits [18–20]. The resulting stochastic quantum trajectories (QT) have been well understood theoretically for finite dimensional Hilbert spaces, especially in the Zeno limit of strong measurement, where jumps [21] and spikes [22, 23] between measurement pointer states emerge.

Less is known about QT’s of many-body states under continuous monitoring, which has been addressed only recently [24]. While the related problem of open quantum systems has been more extensively studied [25–28], the methods, e.g., the Lindblad formalism, focus on the density matrix of a mixed state which describes for instance the average over different realizations of QT’s. However, the dynamics of individual QT’s of a continuously monitored pure state can be much richer.

In this letter, we consider a simple model of many-body QT: a 1d chain of free fermions whose local occupation numbers are continuously measured, see Fig. 1. Although the Hamiltonian is non-interacting, the presence of measurements renders the dynamics non-trivial. In particular, unitary evolution and measurements have competing effects on entanglement growth, making both the stationary regime and the transient behavior (from a quench) a priori unclear; so far, only the Zeno limit has been understood [24]. Remarkably, the continuous measurements under consideration preserve the Gaussianity of the many-body QT. This allows extensive numerical simulation in any parameter regime. As a result, any nonzero measurement rate becomes relevant at space-time scales larger than the measurement/hopping ratio. Below that scale, a free-fermion, ballistic behavior is recovered for both entanglement dynamics and transport properties. Above that scale, entanglement ceases to grow and displays an area law, while transport becomes diffusive as in the Zeno limit [24]. The sample to sample fluctuations of entanglement are always below order unity and have Gaussian distribution. As we argue below, these features are essentially due to the quasi-particle nature of the model, and are expected to hold qualitatively also for interacting integrable models. Indeed, we propose a generalized hydrodynamics approach which captures quantitatively the transport properties of the system.

FIG. 1. A free fermion chain with tight-binding hopping amplitude \( \lambda \), subject to the continuous measurement with rate \( \gamma \) of their local occupation numbers.
Model – We consider a model of free fermions on a one-dimensional lattice with Hamiltonian:

\[ H = \lambda \sum_{i=1}^{L} a_{i+1}^\dagger a_i + a_i^\dagger a_{i+1}, \]

(1)

where \(a_i^\dagger, a_i\) are the fermionic creation and annihilation operators on site \(i\); periodic boundary condition \((L + 1 \equiv 1)\) is assumed throughout. We want to add to this unitary evolution the dynamics induced by the continuous measurement of the local fermion numbers \(n_i = a_i^\dagger a_i\). For a general observable (self-adjoint operator) \(O\), the associated continuous measurement dynamics of the quantum state is described by the following stochastic Schrödinger equation (SSE) [29–31]:

\[ d|\psi_t\rangle = \left(\sqrt{\gamma} [O - \langle O \rangle_{t}] \, dW_t - \frac{\gamma}{2} [O - \langle O \rangle_{t}]^2 \, dt\right) |\psi_t\rangle, \]

(2)

where \(\langle \cdot \rangle_t = \langle \cdot |_{t} \cdot |_{t}\rangle\), \(W_t\) is a Wiener process (Brownian motion), and \(\gamma\) encodes the measurement strength or rate. The multiplicative noise is understood in the Itô convention [32]. The SSE (2) can be realized by homodyne detection in quantum optics [33] (see [34] for a proposal in the cold atom context), and more generally from infinitely weak and frequent interactions with ancillas which are then projectively measured [35–37].

In the present many-body setting, we monitor the occupation number of all sites. Combined with the unitary evolution (1), this leads to the following SSE:

\[ d|\psi_t\rangle = \sum_{i=1}^{L} \left( -i\lambda [a_{i+1}^\dagger a_{i} + a_i^\dagger a_{i+1}] \, dt + \sqrt{\gamma} [n_i - \langle n_i \rangle_{t}] \, dW^i_t - \frac{\gamma}{2} [n_i - \langle n_i \rangle_{t}]^2 \, dt\right) |\psi_t\rangle, \]

(3)

where \(W^i_t, i = 1, \ldots, L\) are independent Wiener processes and we set \(\hbar = 1\).

A remarkable property of Eq. (3) is that it preserves the Gaussianity of states. Indeed, since \(n_i^2 = n_i\) for fermions, the evolution generator is quadratic. Note that this property would be lost averaging over the measurement results. Now, a Gaussian state is fully characterized by its two point function \(D_{ij}(t) = \langle a_{i}^\dagger a_{j}\rangle_{t}\). It satisfies a closed equation, which we derived from Eq. (3) by Wick’s theorem and Itô’s lemma:

\[ dD = -i[h, D] \, dt - \gamma(D - D^{\text{diag}}) \, dt + \sqrt{\gamma} [D \, dW + D^{\dagger} \, d\bar{W} - 2 \, D \, dW \, d\bar{W}] . \]

(4)

Here, \([h, D] := \langle \delta_{k,1+1} + \delta_{k,1-1} \rangle, D^{\text{diag}} \rangle_{kl} := \delta_{k,l} D_{kl},\) and \([W]_{kl} := \delta_{k,l} W^l\). Eq. (4) holds for general Gaussian states preserving particle number, and extends readily to arbitrary quadratic Hamiltonians, by changing the hopping matrix \(h\). We stress that even though the state remains Gaussian, the evolution in Eq. (4) is non-linear.

We shall focus on QT’s of pure Gaussian states, for which an effective numerical scheme is devised (see Supplementary Material). Their entanglement entropy (EE) is readily computed in terms of \(D\). For the sub-system \([1, \ell]\),

\[ S = -\frac{1}{\ln 2} \text{tr} [D_{t} \log D_{t} + (1 - D_{t}) \log(1 - D_{t})] \]

(5)

where \(D_{t} = [D_{ij}]_{j=1}^{\ell}\) is the corresponding sub-matrix of \(D\).

Hydrodynamics and quasiparticle picture – The dynamics of the model depends on the competition between coherent hopping and continuous measurement, via the ratio \(\gamma/\lambda\), so it is helpful to review the two extreme cases.

When \(\gamma/\lambda = 0\), we have a free fermion chain. Starting from the Néel product state \(\prod_{j=1}^{L/2} a^\dagger_{2j} |\Omega\rangle\), the EE of any interval of length \(\ell\) grows linearly in time \(dS/dt \approx 1.25\lambda\) until saturating a volume \(S \sim \ell/a\) (\(a = 1\) is the lattice spacing) at time \(t_{s} \sim \ell/(a\lambda)\) [38]. In the long time limit, the system reaches a generalized Gibbs equilibrium uniform (GGE) in space [39] (see also [40] for a review).

When \(\gamma/\lambda = +\infty\) (\(\lambda = 0\)), it is known that the QT converges to a random choice of the \(L!/N!(L-N)!)\) pointer states \(a^\dagger_{j_1} \ldots a^\dagger_{j_N} |\Omega\rangle\), \(j_1 < \cdots < j_N\) (the probabilities depend on the initial state), destroying all entanglement. Now, a small hopping term \(0 < \lambda \ll \gamma\) will generate classical stochastic jumps between pointer states when the corresponding matrix element is non-zero, leading to a symmetric simple exclusion process (SSEP) [24]. In the long-time limit, the system reaches the stationary state in which all pointer states appear with equal probability: the corresponding density matrix has infinite temperature. More generally, it is known [28] that the system heats up eventually to infinite temperature for any nonzero \(\gamma\).

When \(\gamma/\lambda = O(1)\), the dynamics of the system displays a crossover between the isolated quantum evolution at short scales \(\gamma t \ll 1\) and the SSEP behavior at large times and distances. An effective way to understand the behavior of the system along this crossover is to employ the quasiparticle picture. This approach emerged originally in applying conformal field theories to isolated quantum systems undergoing quantum quenches [41, 42]. Thereafter, it has been proved to be exact in non-interacting models [38] and more recently in Bethe-Ansatz integrable systems [43]. The basic idea is that a weakly-entangled but highly excited initial state behaves as a reservoir of quasiparticles which are entangled in pairs travelling at opposite momenta. During quantum evolution, these pairs propagate coherently but generate dephasing in the system which leads to the relaxation of local expectation values. This simplified picture is particularly useful when investigating the evolution in time of EE: the EE across a cut grows linearly as it is proportional to the larger and larger number of
pairs which are progressively separated by the cut. Considering the pure quantum evolution of EE for a semi-infinite interval \([0, \infty)\), one then obtains the simple formula \([43, 44]\)

\[
S(t) = 2t \int_{-\pi}^{\pi} \frac{dk}{2\pi} v(k)s(k)
\]

(6)

where \(k\) labels the momentum of quasiparticles, \(v(k)\) their velocity and \(s(k)\) accounts for the production rate of entropy of quasiparticles with momentum \(k\).

It is not straightforward to treat exactly the effect of the measurements within the quasiparticle picture: as one can read from Eq. (4), they are non-linear interactions, which thus couple different quasiparticles. However, we can derive a generalized hydrodynamic description (GHD) of the quasiparticle momentum distribution, by considering the average of Eq. (4) (i.e., discarding the \(dW\) term). We introduce the Wigner distribution \(n(x, k, t) := \sum_s e^{ikx}D(x - s/2, x + s/2)\), which satisfies \([45]\)

\[
\partial_t n(k, x, t) = -v(k)\partial_x n(k, x, t) - \gamma \left( n(k, x, t) - \int_{-\pi}^{\pi} \frac{dk}{2\pi} n(k, x, t) \right)
\]

(7)

where \(v(k) = 2\lambda \sin(k)\). This equation describes non-interacting classical particles which propagate with velocity \(v(k)\) [first line of (7)], and which have a probability \(\gamma dt\) in every interval \(dt\) of picking a new random \(k \in [0, 2\pi]\) with uniform distribution [second line of (7)].

**Entanglement growth** – A direct connection between the Wigner function and the EE is still a non-trivial task \([43]\). However, the semiclassical picture resulting from (7) suggests a simple qualitative description. We assume that measurements are local processes happening at a rate \(\gamma\), which have a finite probability to disentangle a travelling pair of quasiparticles. This suggests to replace (6) with

\[
S(t) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} X_t(k), \quad X_t(k) = \sum_{|j| < v(k)t} \tilde{s}_j(k).
\]

(8)

The random variable \(\tilde{s}_j(k)\) represents the contribution to EE from pairs emitted at site \(j\). In a time \(t = |j|/v(k)\), the quasiparticle reaches the cut and with probability \(e^{-\gamma t}\) it is spared by the measurements, i.e. \(\tilde{s}_j(k) = s(k)\); otherwise, it is disentangled and we set \(\tilde{s}_j(k) \simeq 0\). Averaging over QT, we therefore expect the simple expression

\[
\tilde{S}(t) = \frac{2(1 - e^{-\gamma t})}{\gamma} \int dk v(k)s(k).
\]

(9)

Eqs. (9) and (6) imply the scaling form \(\tilde{S}\gamma = C(1 - e^{-x})\), with \(x = \gamma t\) and \(C = aS_\infty = 0\). In other words, the entanglement growth is linear, with the slope of the \(\gamma = 0\) case for an initial ellipse, until saturating at \(t \sim 1/\gamma\), reaching an order unity value. We indeed observe this behavior numerically, see Fig. 2-(a) for a few samples with different \(\gamma\) starting from the Néel product state. A data collapse in Fig. 2-(b), shows that the cross-over happens at \(t \sim 1/\gamma\), as predicted above. The scaling form Eq. (9) becomes quantitatively relevant when \(\gamma\) approaches 1, which is reasonable since the continuous measurements are replaced by projective ones.

The preceding statements hold as long as the finite-size effect does not saturate the entanglement growth before \(t \sim 1/\gamma\), i.e., when \(\gamma \lesssim 1/L\). Therefore, \(\ell = \lambda/\gamma\) should be also the entanglement saturation length in the stationary regime. A finite-size analysis, see Fig. 3, shows the effects of this length scale on the EE of an interval \([0, x]\), in the stationary regime \(t \gg 1/\gamma\); when \(x\) is below \(\ell\), one has volume law entanglement indistinguishable from the \(\gamma = 0\) case; beyond that scale, an area law \(S \sim 1/\gamma\) is observed.

We now study the fluctuations of entanglement, still from the Néel quench. By a reasoning similar to that behind Eq. (9), the quasiparticle model predicts that the standard deviation of EE \(\sigma(S) \propto t\) when \(t \lesssim 1/\gamma\). On the other hand, since the model is interacting, one might expect a Kardar-Parisi-Zhang (KPZ) statistics, as in random unitary circuits \([46]\). However, the latter possibility is ruled out by the numerical results, see Fig. 4. We observe a linear growth of standard deviations \(\sigma(S) \propto t\) which saturates at \(t \sim 1/\gamma\). Further long-time simulations show that it is possible to reach a saturation value \(\sigma(S) \lesssim 1\) for all system sizes and \(\gamma\) accessible. The probability distribution of \(S\) is also very close to a standard Gaussian, in agreement with (8). The absence of extensive entanglement fluctuation and KPZ universality indicates that models of purely quasi-particle nature are not “generic” even in the presence of noise, non-linearity and linear growth of averaged EE.
Transport – Finally, we consider the non-equilibrium particle transport, from a domain wall initial condition $|\psi_0\rangle = a_1^\dagger \cdots a_{L/2}^\dagger |\Omega\rangle$. It is known that the transport is ballistic when $\gamma = 0$ [47], and is diffusive in the Zeno limit $\gamma \to \infty$ [24]. To understand the behavior for arbitrary $\gamma$, see Fig. 5 (a) for an example, we employ the GHD description of Eq. (7). As an efficient way to obtain the Green function of Eq. (7), we simulate a single particle which moves at velocity $v(k_i) = 2\lambda \sin(k_i)$ in the interval $t \in [t_{i-1}, t_i]$, where $k_i \in [0, 2\pi]$ are uniformly distributed, $t_0 = 0$ and $0 < t_1 < t_2 < \ldots$ form a Poisson point process with rate $\gamma$ on $[0, +\infty)$. The Green function of Eq. (7) is then given by the distribution of the particle position.

It follows that, for any $\gamma > 0$, a ballistic-diffusive crossover takes place at the time scale $t \sim 1/\gamma$, i.e., the mean free time of the classical particle. We verify the above results by measuring the integrated current across the middle bond of the chain, $C(t) := \sum_{j>L/2} \langle n_j \rangle_t$. The result, Fig. 5 (b), shows that the hydrodynamic Eq. (7) reproduces exactly the transport behavior of the quantum system.

Another interesting aspect of the domain-wall quench is the EE growth. It is only logarithmic for free fermions [48], but can be enhanced by measurements. Indeed, as we can see from Eq. (7), the latter creates new quasi-particle pairs along the domain wall melting dynamics. A quantitative description of this effect is left to work in progress.

Extension to interacting integrable models? – The measurement term in Eq. (7) will be unchanged in models with interacting fermions, including integrable systems such as 1d Fermi-Hubbard chain and the XXZ chain (via the Jordan-Wigner transform). It is therefore tempting to put forward a straightforward adaptation of GHD in the presence of continuous measurement of $\bar{n}_i (\bar{S}_i^z)$ in XXZ chain), by replacing in (7) the velocity $v(k)$ with the state-dependent dressed velocity $v[k]$ obtained from Bethe-Ansatz [49–51]. Essentially the term on the second line of Eq. (7) would induce local relaxation to a GGE state where the density of particles is the only conserved quantity. In this approach, the measure-induced term can be readily incorporated into the soliton gas picture [52], as done above, or directly into the numerical scheme of [51]. Nevertheless, the validity of GHD in this context is not guaranteed a priori and should be checked, since integrability (and Gaussian-ness) would be broken in general.

Discussion – We studied the many-body dynamics of a monitored fermion chain. Although the extensive numerical simulation method is model-specific, the arguments based on the quasi-particle phenomenological model can be extended, and suggest that the results should apply to integrable Hamiltonians in general. We thus predict that any amount of continuous monitoring of a complete commuting set of local observable will destroy
volume-law entanglement and ballistic transport in the thermodynamic limit: there is no “weak measurement phase” [53]. On the other hand, this may no longer hold for generic, strongly interacting, quantum systems under continuous monitoring. There can be a weak-strong measurement transition, pinpointing which is an interesting open problem.

The numerical study of generic many-body quantum system under continuous monitoring provides a new arena for classical simulation methods based on tensor networks [10, 11, 54, 55], since entanglement growth can be significantly limited by continuous monitoring when it is not too weak. Further analytic progress will be another challenge for the future, also aimed at verifying for truly interacting integrable systems the validity of GHD with dephasing we proposed [49–51, 56, 57]. Additionally, in order to capture entanglement evolution [43], an interesting route would be to develop a stochastic GHD, refining Eq. (7) to the QT level.

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The only approximation in (7) is in employing $\partial_x n$ instead of the lattice discrete derivative. See [58] for a discussion of this aspect.
Supplementary Material
Entanglement and Transport in a Free Fermion Chain under Continuous Monitoring

Numerical scheme

We describe here the numerical method used to simulate the QT of a pure Gaussian state with \( N \) particles. For this, we represent the state by an \( L \times N \) matrix \( U \):

\[
|\psi_t\rangle = |U\rangle := \prod_{k=1}^{N} \left( \sum_{j=1}^{L} U_{jk} a_j^\dagger \right) |\Omega\rangle,
\]

where \( |\Omega\rangle \) is the state with no fermions. We require that \( U \) be an isometry: \( U^\dagger U = 1_{N \times N} \), so that \( D = UU^\dagger \). We let \( N = L/2 \) in the simulations presented in the main text; we expect that any non-trivial filling fraction \( 0 < N/L < 1 \) gives rise to the same phenomena.

Then we Trotter-discretize Eq. (3) by alternating its unitary and measurement terms, with time step \( \delta t \):

\[
|\psi_{t+\delta t}\rangle \approx Ce^{-iH\delta t} |\psi_t\rangle,
\]

then where \( \delta W_j^t \) are independent, each with zero mean and \( \gamma \delta t \) variance, \( C \) is a normalization constant. It follows that:

\[
|\psi_{t+\delta t}\rangle \approx C|T := Me^{-ih\delta t} |\psi_t\rangle, M_{jk} = \delta_{jk} e^{\delta W_j^t + (2\langle n_j \rangle_t - 1)\gamma \delta t},
\]

where \( h \) is as in Eq. (4) and \( \langle n_j \rangle_t = \sum_k U_{jk} U_{jk}^\ast \). Finally we perform a QR decomposition \( T = QR \), and set \( U_{t+\delta t} := Q \), restoring the isometry property. The main advantage of the above method, compared to a direct integration of Eq. (4), is that the purity of the Gaussian state is conserved by construction. Therefore, the time step \( \delta t \) does not need to be very small. In practice, \( \delta t = 0.05 \) is sufficiently accurate for our purposes. We checked that the results below are not affected if \( \delta t \) is doubled or halved.