Representations of vertex operator algebras

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This paper is dedicated to Professors James Lepowsky and Robert Wilson on the occasion of their 60th birthday

Abstract

This paper is an exposition of the representation theory of vertex operator algebras in terms of associative algebras \(A_n(V)\) and their bimodules. A new result on the rationality is given. That is, a simple vertex operator algebra \(V\) is rational if and only if its Zhu algebra \(A(V)\) is a semisimple associative algebra and each irreducible admissible \(V\)-module is ordinary. 2000MSC:17B69

1 Introduction

Rational vertex operator algebras, which play a fundamental role in rational conformal field theory (see [BPZ] and [MS]), form an important class of vertex operator algebras. Most vertex operator algebras which have been studied so far are rational vertex operator algebras. Familiar examples include the moonshine module \(V^{\natural}\) ([B], [FLM], [D2]), the vertex operator algebras \(V_L\) associated with positive definite even lattices \(L\) ([B], [FLM], [D1]), the vertex operator algebras associated with integrable representations of affine Lie algebras ([DL], [FZ]), the vertex operator algebras \(L(c_{p,q},0)\) associated with irreducible highest weight representations for the discrete series of the Virasoro algebra ([DMZ], [W1]), and framed vertex operator algebras ([DGH], [M]).

The notion of rational vertex operator algebra was first introduced in [Z] to study the modular invariance of trace functions for vertex operator algebras. The rational vertex operator algebra defined in [Z] satisfies the following three conditions: (1) Any admissible module is completely reducible. (2) There are only finitely many irreducible admissible modules up to isomorphisms. (3) Each irreducible admissible module is ordinary. It was proved in [DLM2] that conditions (2) and (3) are consequences of condition (1).

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makes the notion of rationality more natural comparing to the notion of semisimplicity of a finite dimensional associative algebra.

The rationality given here is clearly an external definition. One needs to verify the complete reducibility for any admissible module. So in practice, it is very hard to check if a vertex operator algebra is rational. In the classical case for finite dimensional associative algebra, there is an internal characterization of semisimplicity. That is, the Jacobson or solvable radical of the algebra is zero. Similarly, for a finite dimensional Lie algebra, one can use the Killing form or solvable radical to define semisimple Lie algebras. A natural question is whether there is an internal definition of rationality for vertex operator algebras.

There is a close relation between the representation theory of a vertex operator algebra \( V \) and the representation theory of \( A(V) \) which is an associative algebra associated to \( V \) defined in \([Z]\). The most important result about \( A(V) \) is that there is a one to one correspondence between the equivalence classes of irreducible admissible \( V \)-modules and the equivalence classes of simple \( A(V) \)-modules. This turns \( A(V) \) into a very powerful and effective tool in the classification of irreducible modules for a given vertex operator algebra. In fact, the irreducible modules for many families of vertex operator algebras such as affine vertex operator algebras \([FZ]\), the Virasoro vertex operator algebras \([DMZ]\, [W1]\), lattice type vertex operator algebras \([DN1]\, [DN3]\, [AD]\, [TY]\), certain \( W \)-algebras \([W2]\, [DLTY]\) and some other vertex operator algebras \([A2]\, [Ad]\, [KW]\, [KMY]\) have been classified using the \( A(V) \)-theory.

Motivated by the representation theory of vertex operator algebras, a sequence of associative algebras \( A_n(V) \) for \( n \geq 0 \) were introduced in \([DLM3]\) to deal with the first \( n+1 \) homogeneous subspaces of an admissible module such that \( A_0(V) \) is exactly \( A(V) \). It is proved in \([DLM3]\) that \( V \) is rational if and only if \( A_n(V) \) are finite dimensional semisimple associative algebras for all \( n \). Since the construction of \( A_n(V) \) is very complicated, it is hard to compute \( A_n(V) \) for large \( n \). It has been suspected for a long time that the semisimplicity of \( A(V) \) is good enough to characterize the rationality of \( V \).

This paper is an exposition and extension of our recent results in \([DJ]\) concerning the rationality of a vertex operator algebra together with the \( A_n(V) \) theory. We obtain that if \( V \) is a simple vertex operator algebra such that each weight space of any irreducible admissible \( V \)-module is finite dimensional, then \( V \) is rational if and only if \( A(V) \) is semisimple. In particular, if \( V \) is \( C_2 \)-cofinite then the semisimplicity of \( A(V) \), rationality of \( V \) and regularity of \( V \) are all equivalent. This surely gives an internal characterization of rationality as the semisimplicity of \( A(V) \) can be defined internally. The main tool used to prove this result is the bimodule theory developed in \([DJ]\).

Since the weight spaces of an irreducible admissible \( V \)-module for all well known vertex operator algebras are finite-dimensional, the assumption on the dimensions of weight spaces is not a strong assumption. For most vertex operator algebras, the computation of \( A(V) \) is necessary to classify the irreducible modules. So in practice, this new result is very useful. For example, the rationality of \( V_L^+ \) for a rank one positive definite lattice \( L \) proved in \([A1]\) is an easy corollary of our result.

We firmly believe that a simple vertex operator algebra is rational if and only if \( A(V) \)
is semisimple without any assumption. But we cannot achieve this in this paper.

2 Preliminaries

Let $V = (V, Y, 1, \omega)$ be a vertex operator algebra (see [B] and [FLM]). We define weak, admissible and ordinary modules following [DLM1]-[DLM2]. We also define rationality, $C_2$-cofiniteness and regularity.

A weak module $M$ for $V$ is a vector space equipped with a linear map

$$v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \ (v_n \in \text{End } M) \text{ for } v \in V$$

(where for any vector space $W$, we define $W[[z^{-1}, z]]$ to be the vector space of $W$-valued formal series in $z$) satisfying the following conditions for $u, v \in V, w \in M$:

$$v_n w = 0 \text{ for } n \in \mathbb{Z} \text{ sufficiently large;}$$

$$Y_M(1, z) = 1;$$

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1) Y_M(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{z_0} \right) Y_M(v, z_2) Y_M(u, z_1)$$

$$= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(u, z_0) v, z_2).$$

This completes the definition. We denote this module by $(M, Y_M)$ (or briefly by $M$). It is proved in [DLM1] that the following are true on the weak module $M$:

$$Y_M(L(-1) u, z) = \frac{d}{dz} Y_M(u, z)$$

and

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12} \delta_{m+n,0} C$$

where $Y_M(\omega, z) = \sum_{n \in \mathbb{Z}} L(n) z^{-n-2}$.

An ordinary $V$-module is a $\mathbb{C}$-graded weak $V$-module

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$$

such that dim $M_\lambda$ is finite and $M_{\lambda+n} = 0$ for fixed $\lambda$ and $n \in \mathbb{Z}$ small enough where $M_\lambda$ is the $\lambda$-eigenspace for $L(0)$ with eigenvalue $\lambda$.

An admissible $V$-module is a weak $V$-module $M$ which carries a $\mathbb{Z}_+$-grading

$$M = \bigoplus_{n \in \mathbb{Z}_+} M(n)$$

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($\mathbb{Z}_+$ is the set of all nonnegative integers) such that if $r, m \in \mathbb{Z}, n \in \mathbb{Z}_+$ and $a \in V_r$, then $a_m M(n) \subseteq M(r + n - m - 1)$. Since the uniform degree shift gives an isomorphic admissible module we assume $M(0) \neq 0$ in many occasions. The admissible module is also called the $\mathbb{N}$-graded module in [LL]. It is easy to prove that any ordinary module is an admissible module.

Following [DLM2], a vertex operator algebra is called rational if any admissible module is a direct sum of irreducible admissible modules. As we mentioned in the introduction, the rationality given here looks weaker than the rationality defined originally in [Z], although it was proved in [DLM2] these two rationalities are the same, also see Theorem 3.1 below.

Another important concept is the $C_2$-cofiniteness. The $C_2$-cofiniteness is a very important concept in the theory of vertex operator algebra. Many important results such as modular invariance of trace functions [Z], [DLM5], and that the weight one subspace of a CFT type vertex operator algebra is reductive [DMI], tensor product and Verlinde formula [H] need both rationality and $C_2$-cofiniteness. The $C_2$-cofinite condition also plays a fundamental role in the study of conformal field theory [GN], [NT]. In particular, the $C_2$-cofiniteness implies that the fusion rules are finite [Bu] and the conformal blocks are finite dimensional [NT].

A vertex operator algebra $V$ is called $C_2$-cofinite if the subspace $C_2(V)$ spanned by $u_{-2}v$ for all $u, v \in V$ has finite codimension in $V$ [Z]. It is demonstrated in [GN] that if $V$ is $C_2$-cofinite then $V$ is finitely generated with a PBW-like spanning set.

**Theorem 2.1.** Let $X = \{x^\alpha | \alpha \in I\}$ be a subset of $V$ consisting of homogeneous vectors such that $\{x^\alpha + C_2(V) | \alpha \in I\}$ form a basis of $V/C_2(V)$. Then $V$ is spanned by

$$x_{-n_1} x_{-n_2} \cdots x_{-n_k} 1$$

where $n_1 > n_2 > \cdots > n_k > 0$ and $x^i \in X$ for $1 \leq i \leq k$. In particular, $V$ is finitely generated if $V$ is $C_2$-cofinite.

This result has been generalized in [Bu] to get a PBW-like spanning set for any weak module generated by one element. We should remark that there are some other results in [KL] concerning finitely generated properties.

A vertex operator algebra is called regular if any weak module is a direct sum of irreducible ordinary modules. Clearly, a regular vertex operator algebra is rational. It has been a very challenging problem to find the relationship among rationality, regularity and $C_2$-cofiniteness. It is shown in [ABD] and [L] that rationality together with $C_2$-cofiniteness and regularity are equivalent. Although it has been suspected that rationality and $C_2$-cofiniteness are equivalent, there is a counter example in [A2] where a vertex operator algebra is $C_2$-cofinite but not rational. On the other hand, the $C_2$-cofiniteness is good enough to guarantee the integrability of the vertex operator algebra of CFT type [DMI]. That is, the vertex operator subalgebra of $V$ generated by any semisimple subalgebra of $V_1$ is integrable and $V$ is an integrable module for the corresponding affine Kac-Moody Lie algebra.

In the definition of regular vertex operator algebra, we require that any weak module is a direct sum of irreducible ordinary modules. An immediate question is the following: If
every weak module is a direct sum of irreducible weak modules, is \( V \) regular? It was proved in \([ABD]\) that if \( V \) is \( C_2 \)-cofinite then any finitely generated weak module is ordinary. So the answer to this question would be positive if one could prove that rationality implies \( C_2 \)-cofiniteness. But this again is a very difficult problem.

3 The associative algebra \( A(V) \)

In this section we review the \( A(V) \) theory following \([DLM2]\) and \([Z]\). Let \( V \) be a vertex operator algebra. For any homogeneous vectors \( a \in V \), and \( b \in V \), we define

\[
a \ast b = \left( \text{Res}_z \frac{(1 + z)^{\text{wt} a}}{z} Y(a, z) \right) b,
\]

\[
a \circ b = \left( \text{Res}_z \frac{(1 + z)^{\text{wt} a}}{z^2} Y(a, z) \right) b,
\]

and extend to \( V \) bilinearly. Denote by \( O(V) \) the linear span of \( a \circ b \) \((a, b \in V)\) and set \( A(V) = V/O(V) \).

The definition of \( A(V) \) is very natural from the representation theory of vertex operator algebra. Let \( M \) be an admissible \( V \)-module. For each homogeneous \( v \in V \), we set \( o(v) = v_{\text{wt} v - 1} \) on \( M \) and extend linearly to whole \( V \). Then it is not hard to prove that \( o(u) o(v) = o(u \ast v) \) on \( M(0) \) for all \( u, v \in V \). So the operation \( \ast \) will be the product in \( A(V) \). Since \( o((L(-1) + L(0))v) = 0 \) on \( M \) we have \( o(((L(-1) + L(0))v) \ast u) = 0 \) on \( M(0) \) for all \( u, v \in V \). So we have to modulo out all \( ((L(-1) + L(0))v) \ast u \) in order to get an effective action. One can verify that \( ((L(-1) + L(0))v) \ast u = v \circ u \). This should explain why the algebra \( A(V) \) is so defined.

We write \([a]\) for the image \( a + O(V) \) of \( a \in V \). The following theorem is due to \([Z]\) and \([DLM2]\).

**Theorem 3.1.** Assume that \( V \) is a vertex operator algebra. Then

(1) The bilinear operation \( \ast \) on \( V \) induces an associative algebra structure on \( A(V) \). The vector \([1]\) is the identity and \([\omega]\) is in the center of \( A(V) \).

(2) Let \( M = \bigoplus_{n=0}^\infty M(n) \) be an admissible \( V \)-module with \( M(0) \neq 0 \). Then the linear map

\[
o : V \to \text{End} M(0), \ a \mapsto o(a)|_{M(0)}
\]

induces an algebra homomorphism from \( A(V) \) to \( \text{End} M(0) \). Thus \( M(0) \) is a left \( A(V) \)-module.

(3) The map \( M \mapsto M(0) \) induces a bijection from the set of equivalence classes of irreducible admissible \( V \)-modules to the set of equivalence classes of irreducible \( A(V) \)-modules.

(4) If \( V \) is rational, then \( A(V) \) is a finite dimensional semisimple associative algebra.

(5) If \( V \) is rational then there are only finitely many irreducible admissible \( V \)-modules up to isomorphism and each irreducible admissible module is ordinary.
We remark that Theorem 3.1 (5) was part of the definition of rational vertex operator algebra given in [Z]. If $V$ is $C_2$-cofinite, then Theorem 3.1 (5) also holds (see [KL]). This suggests that there is a strong link between rationality and $C_2$-cofiniteness. But it is not clear what the link might be.

Theorem 3.1 is very powerful in the classification of irreducible modules for a vertex operator algebra. In order to classify the irreducible modules for a vertex operator algebra $V$, it is enough to classify the irreducible modules for $A(V)$ which is computable in many cases. The classifications of irreducible modules have been achieved for many well known vertex operator algebras (see [FZ], [W1], [W2], [KW], [DN1]-[DN3], [A3], [AD], [DLTY], [TY], [KMY], [A2]) in this way. But the structure of $A(V)$ cannot tell whether or not a vertex operator algebra $V$ is rational at this stage.

4 The associative algebras $A_n(V)$

Although $A(V)$ theory is very useful, the $A(V)$ is not good enough to determine the representation theory for $V$ completely. Motivated by the “graded structure” of admissible modules, an associative algebra $A_n(V)$ was introduced in [DLM3] for any nonnegative integer $n$, generalizing the Zhu algebra $A(V)$ which is $A_0(V)$. The role of $A_n(V)$ playing in the representation theory of vertex operator algebra will become clear after Theorem 4.1 below.

Let $O_n(V)$ be the linear span of all $u \circ_n v$ and $L(-1)u + L(0)u$ where for homogeneous $u \in V$ and $v \in V$,

$$u \circ_n v = \text{Res}_z Y(u, z) \frac{(1 + z)^{wt u + n}}{z^{n+2}}.$$ 

Define the linear space $A_n(V)$ to be the quotient $V/O_n(V)$. We also define a second product $*_n$ on $V$ for $u$ and $v$ as follows:

$$u *_n v = \sum_{m=0}^{n} (-1)^m \binom{m+n}{n} \text{Res}_z Y(u, z) \frac{(1 + z)^{wt u + n}}{z^{n+m+1}} v.$$ 

Extend linearly to obtain a bilinear product on $V$ which coincides with that of Zhu algebra [Z] if $n = 0$. We denote the product $*_0$ by $*$ in this case.

Here are the main results on $A_n(V)$ obtained in [DLM3].

**Theorem 4.1.** Let $V$ be a vertex operator algebra and $n$ a nonnegative integer. Then

1. $A_n(V)$ is an associative algebra whose product is induced by $*_n$.
2. The identity map on $V$ induces an algebra epimorphism from $A_n(V)$ to $A_{n-1}(V)$.
3. Let $W$ be a weak module and set

$$\Omega_n(W) = \{w \in W | u_m w = 0, u \in V, m > wt u - 1 + n\}.$$ 

Then $\Omega_n(W)$ is an $A_n(V)$-module such that $v + O_n(V)$ acts as $o(v)$.

4. Let $M = \bigoplus_{m=0}^{\infty} M(m)$ be an admissible $V$-module. Then each $M(m)$ for $m \leq n$ is an $A_n(V)$-submodule of $\Omega_n(W)$. Furthermore, $M$ is irreducible if and only if each $M(n)$ is an irreducible $A_n(V)$-module.
(5) For any $A_n(V)$-module $U$ which cannot factor through $A_{n-1}(V)$ there is a unique Verma type admissible $V$-module $M(U)$ generated by $U$ so that $M(U)(0) \neq 0$ and $M(U)(n) = U$. Moreover, for any weak $V$-module $W$ and any $A_n(V)$-module homomorphism $f$ from $U$ to $\Omega_n(W)$ there is a unique $V$-homomorphism from $M(U)$ to $W$ which extends $f$.

(6) $V$ is rational if and only if $A_n(V)$ are finite dimensional semisimple algebras for all $n \geq 0$.

Theorem 4.4 is a generalization and extension of Theorem 3.1. In (5) the assumption that $U$ cannot factor through $A_{n-1}(V)$ is not essential. In fact, for any $A_n(V)$-module $U$ we can construct $M(U)$ but we cannot conclude that $M(U)(0) \neq 0$. The algebra $A_n(V)$ definitely carries more information than $A(V)$ on the admissible $V$-modules. For example, one cannot tell if an admissible $V$-module $M = \sum_{n \geq 0} M(n)$ is irreducible even though $M(0)$ is an irreducible $A(V)$-module. The $A_n(V)$ theory plays a fundamental role in the study of orbifold theory and dual pairs (see [DLM0], [DY], [Y], [MT]).

5 Bimodules $A_{n,m}(V)$

Let $M = \sum_{s=0}^{\infty} M(s)$ be an admissible $V$-module. Then $M(m)$ is an $A_n(V)$-module and $M(n)$ is an $A_n(V)$-module. As a result, $\text{Hom}_C(M(m), M(n))$ is an $A_n(V) - A_m(V)$-bimodule. This observation leads to the study of $A_n(V) - A_m(V)$-bimodules $A_{n,m}(V)$ in [DJ]. While the definition of $A_{n,m}(V)$ sounds complicated, the representation-theoretical meaning of $A_{n,m}(V)$ is clear.

For homogeneous $u \in V$, $v \in V$ and $m, n, p \in \mathbb{Z}_+$, define the product $*_{m,p}$ on $V$ as follows

$$u *_{m,p} v = \sum_{i=0}^{p} (-1)^i \binom{m + n - p + i}{i} \text{Res}_z \frac{(1 + z)^{wtu+m}}{z^{m+n-p+i+1}} Y(u,z)v.$$ 

In order to explain the meaning of the product $u *_{m,p} v$ we consider an admissible $V$-module $M = \sum_{s \geq 0} M(s)$. For $u \in V$ we set $o_t(u) = u_{wtu-1}$ for $t \in \mathbb{Z}$. Then $o_t(u) M(s) \subset M(s+t)$. It is proved in [DJ] that $o_{n-p}(u) o_{p-m}(v) = o_{n-m}(u *_{m,p} v)$ acting on $M(m)$.

If $n = p$, we denote $*_{m,p}$ by $*_{m}$, and if $m = p$, we denote $*_{m,p}$ by $*_{m}$, i.e.,

$$u *_{m} v = \sum_{i=0}^{m} (-1)^i \binom{n + i}{i} \text{Res}_z \frac{(1 + z)^{wtu+m}}{z^{n+i+1}} Y(u,z)v,$$

$$u *_{m} v = \sum_{i=0}^{n} (-1)^i \binom{m + i}{i} \text{Res}_z \frac{(1 + z)^{wtu+m}}{z^{m+i+1}} Y(u,z)v.$$ 

From the discussion above we see that $o(u) o_{n-m}(v) = o_{n-m}(u *_{m} v)$ and $o_{n-m}(u) o(v) = o_{n-m}(u *_{m} v)$ on $M(m)$. So the products $u *_{m} v$ and $u *_{m} v$ will induce the right $A_m(V)$-module and left $A_n(V)$-module structure on $A_{n,m}(V)$ which will be defined later. It is clear that if $m = n$, then $u *_{m} v$, $u *_{m} v$ and $u *_{n} v$ are equal.
Let $O'_{n,m}(V)$ be the linear span of all $u \circ_m^n v$ and $L(-1)u + (L(0) + m - n)u$, where for homogeneous $u \in V$ and $v \in V$,

$$u \circ_m^n v = \text{Res}_z \frac{(1 + z)^{wt u + m}}{z^{n+m+2}} Y(u, z)v.$$ 

Again if $m = n$, $u \circ_m^n v = u \circ_n v$ has been defined in [DLM3] (see Section 4). Then $O_n(V) = O'_{n,n}(V)$ (see [DLM3] and Section 4).

We introduce more subspaces of $V$. Let $O''_{n,m}(V)$ be the linear span of $v \in O^p_{m,p_1}((a \circ_{p_3}^p b) \circ_{m,p_1}^p c - a \circ_{m,p_2}^p (b \circ_{p_2}^p c))$, for $a, b, c, u \in V, p_1, p_2, p_3 \in \mathbb{Z}_+$, and $O'''_{n,m}(V) = \sum_{p \in \mathbb{Z}_+} (V \circ_p^n O_p(V)) \circ_{m,p}^n V$. Set

$$O_{n,m}(V) = O'_{n,m}(V) + O''_{n,m}(V) + O'''_{n,m}(V).$$

and

$$A_{n,m}(V) = V/O_{n,m}(V).$$

The following theorem about $A_{n,m}(V)$ is obtained in [DJ].

**Theorem 5.1.** Let $V$ be a vertex operator algebra and $m, n$ nonnegative integers. Then

1. $A_{n,m}(V)$ is an $A_n(V) - A_m(V)$-bimodule such that the left and right actions of $A_n(V)$ and $A_m(V)$ are given by $\circ_m^n$ and $\circ^n_m$.
2. $A_{n,m}(V)$ can be made to be an $A_n(V) - A_m(V)$-bimodule isomorphic to $A_{n,m}(V)$.
3. Let $l$ be nonnegative integers such that $m - l, n - l$ are nonnegative. Then $A_{n-l,m-l}(V)$ is an $A_n(V) - A_m(V)$-bimodule and the identity map on $V$ induces an epimorphism of $A_n(V) - A_m(V)$-bimodules from $A_{n,m}(V)$ to $A_{n-l,m-l}(V)$.
4. Define a linear map $\psi: A_{n,p}(V) \otimes_{A_p(V)} A_{p,m}(V) \rightarrow A_{n,m}(V)$ by

$$\psi(u \otimes v) = u \circ_{m,p}^n v,$$

for $u \otimes v \in A_{n,p}(V) \otimes_{A_p(V)} A_{p,m}(V)$. Then $\psi$ is an $A_n(V) - A_m(V)$-bimodule homomorphism from $A_{n,p}(V) \otimes_{A_p(V)} A_{p,m}(V)$ to $A_{n,m}(V)$.

5. Let $M = \sum_{s=0} M(s)$ be an admissible $V$-module. Set $o_{n,m}(v) = v_{wt(v) - 1 + m - n}$. Then $v + O_{n,m}(V) \mapsto o_{n,m}(v)$ gives an $A_n(V) - A_m(V)$-bimodule homomorphism from $A_{n,m}(V)$ to $H^M_{\text{mod}}(M(m), M(n))$.

6. For any $n \geq 0$, the $A_n(V)$ and $A_{n,n}(V)$ are the same.
7. Let $U$ be an $A_{n,m}(V)$-module which can not factor through $A_{n-1}(V)$. Then

$$\bigoplus_{n \in \mathbb{Z}_+} A_{n,m}(V) \otimes_{A_m(V)} U$$

is an admissible $V$-module isomorphic to $M(U)$ given in Theorem 4.1 with $M(U)(n) = A_{n,m}(V) \otimes_{A_m(V)} U$.

8. If $V$ is rational and $W^j = \bigoplus_{n \geq 0} W^j(n)$ with $W^j(0) \neq 0$ for $j = 1, 2, \ldots, s$ are all the inequivalent irreducible modules of $V$, then

$$A_{n,m}(V) \cong \bigoplus_{l=0}^{\min\{m,n\}} \left( \bigoplus_{i=1}^s \text{Hom}_C(W^i(m-l), W^i(n-l)) \right).$$
By the definitions \(A_{n,n}(V)\) is a quotient of \(A_n(V)\) as \(O_n(V)\) is a subspace of \(O_{n,n}(V)\). The result (6) is highly nontrivial and is proved by using the representation theory. It is an interesting problem to find a direct proof that \(O_n(V)\) and \(O_{n,n}(V)\) are equal.

So far we have two constructions of Verma type admissible module \(M(U)\) generated by an \(A_n(V)\)-module \(U\) from Theorems 4.1 and 5.1. But the construction of \(M(U)\) given in Theorem 6.1 is a quotient module for certain Lie algebra associated to \(V\) (see [DLTM2] for the detail) so the structure is not clear. On the other hand, the construction of \(M(U)\) from Theorem 6.1 (7) is explicit and each homogeneous subspace \(M(U)(n)\) is computable. This new construction of \(M(U)\) is fundamental in our study of rationality in the next section.

6 Rationality

In this section we discuss the relation between rationality of \(V\) and semisimplicity of \(A(V)\). This part is totally new and has not appeared anywhere else.

Let \(U\) be an \(A(V)\)-module. Then \(M(U) = \bigoplus_{n \geq 0} A_{n,0}(V) \otimes A(V) U\) is the Verma type admissible \(V\)-module such that \(M(U)(0) = U\). It is easy to see that \(M(U)\) has a maximal admissible submodule \(J(U) = \sum_{n \geq 0} J(U)(n)\) such that \(J(U)(0) = 0\) where \(J(U)(n) = M(U)(n) \cap J(U)\). As in [DLTM2] we denote the quotient of \(M(U)\) modulo \(J(U)\) by \(L(U)\). Clearly, \(L(U)\) is irreducible if and only if \(U\) is irreducible. From the representation theory for the affine Kac-Moody algebras or the Virasoro algebra we know that in general \(M(U)\) and \(L(U)\) are different. So it is natural to ask when \(L(U)\) and \(M(U)\) are equal.

**Theorem 6.1.** Let \(V\) be a simple vertex operator algebra such that \(A(V)\) is semisimple. Let \(U\) be an irreducible module of \(A(V)\), then the Verma admissible \(V\)-module \(M(U) = \bigoplus_{n \in \mathbb{Z}^+} A_{n,0}(V) \otimes U\) generated by \(U\) is an irreducible admissible module of \(V\). That is, \(M(U) = L(U)\).

The proof of Theorem 6.1 is highly nontrivial and will be given in another paper. The main idea in the proof is to use the bimodule theory developed in [DL] to show that \(M(U)(n) = A_{n,0}(V) \otimes U\) is an irreducible \(A_n(V)\)-module for all \(n\).

It is proved in [DLTY] that if \(M(U) = L(U)\) for any simple \(A(V)\)-module, \(A(V)\) is semisimple and \(V\) is \(C_2\)-cofinite then \(V\) is rational. The same proof works here with \(C_2\)-cofiniteness replaced by the assumption that each irreducible admissible \(V\)-module is ordinary. So using Theorem 6.1 we have the following theorem.

**Theorem 6.2.** Let \(V\) be a simple vertex operator algebra. Then \(V\) is rational if and only if \(A(V)\) is semisimple and each irreducible admissible \(V\)-module is ordinary.

The assumption that any irreducible admissible module is ordinary in Theorem 6.2 is equivalent to that each homogeneous subspace of any irreducible admissible module is finite dimensional. It is worthy to point out that this is not a very strong assumption which is true for all known simple vertex operator algebras.

**Corollary 6.3.** Let \(V\) be a \(C_2\)-cofinite simple vertex operator algebra. Then the following are equivalent:
(1) $V$ is regular,
(2) $V$ is rational,
(3) $A(V)$ is semisimple.

The corollary follows immediately from the fact that any finitely generated admissible module is ordinary (see [KL], [Bu]) if $V$ is $C_2$-cofinite. We have already mentioned that rationality together with the $C_2$-cofiniteness and regularity are equivalent. In the proof of modular invariance of trace functions in [Z], the semisimplicity of $A(V)$ (not the rationality of $V$) and $C_2$-cofiniteness were used. But under the assumption of $C_2$-cofiniteness, the rationality of $V$ and semisimplicity of $A(V)$ are equivalent. So the rationality of $V$ is, in fact, used in the proof of modularity in [Z].

Recall from [KL] that $V$ is called $C_1$-cofinite if $V = \sum_{n \geq 0} V_n$ with $V_0 = C1$ such that $C_1(V)$ has finite codimension in $V$ where $C_1(V)$ is spanned by $u_{-1}v$ and $L(-1)u$ for all $u, v \in \sum_{n > 0} V_n$. It is proved in [KL] that if $V$ is $C_1$-cofinite then any irreducible admissible $V$-module is ordinary. So we have another important corollary:

**Corollary 6.4.** Let $V$ be a $C_1$-cofinite simple vertex operator algebra. Then $V$ is rational if and only if $A(V)$ is semisimple.

Now we can easily conclude the rationality of many vertex operator algebras which were very hard theorems. Here is an example. Let $L$ be a positive definite even lattice and $V_L$ the corresponding vertex operator algebra (see [B] and [FLM]). Then $V_L$ has a canonical automorphism $\theta$ of order 2 induced from the $-1$ isometry of the lattice and the fixed point subspace $V_L^+\theta$ is a simple vertex operator algebra. The irreducible admissible modules for $V_L^+\theta$ have been classified in [DN2] and [AD]. Moreover, each irreducible admissible module is ordinary (see [DN2], [AD]).

**Corollary 6.5.** $A(V_L^+\theta)$ is semisimple and $V_L^+\theta$ is rational for any positive definite even lattice $L$.

We remark that if the rank of $L$ is one, the semisimplicity of $A(V_L^+\theta)$ has been proved in [DN2] and the rationality of $V_L^+\theta$ has been established in [A1] using a different method.

We end this paper with the following conjecture.

**Conjecture 6.6.** Let $V$ be a simple vertex operator algebra. Then $V$ is rational if and only if $A(V)$ is semisimple.

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