UNIFORM LIPSCHITZ REGULARITY OF FLAT SEGREGATED INTERFACES IN A SINGULARLY PERTURBED PROBLEM

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Abstract. For the singularly perturbed system
\[ \Delta u_{i,\beta} = \beta u_{i,\beta} \sum_{j \neq i} u_{j,\beta}^2, \quad 1 \leq i \leq N, \]
we prove that flat interfaces are uniformly Lipschitz. As a byproduct of the proof we also obtain the optimal lower bound near the flat interfaces,
\[ \sum_i u_{i,\beta} \geq c\beta^{-1/4}. \]

1. Main result

This note is intended as a remark on the recent paper of Soave and Zilio [6]. We study the flat segregated interfaces of the following singularly perturbed elliptic system
\[ \Delta u_{i,\beta} = \beta u_{i,\beta} \sum_{j \neq i} u_{j,\beta}^2, \quad 1 \leq i \leq N. \] (1.1)

Assume \( u_{\beta} \) is a sequence of positive solutions to this system in \( B_1(0) \subset \mathbb{R}^n \), satisfying
\[ \sup_{B_1(0)} \sum_i u_{i,\beta} \leq 1, \quad \forall \beta > 0. \]

By [4], \( u_{i,\beta} \) are uniformly bounded in \( \text{Lip}_{\text{loc}}(B_1(0)) \). Hence we can assume \( u_{i,\beta} \) converges to \( u_i \) in \( C_{\text{loc}}(B_1(0)) \). (It also converges strongly in \( H^1_{\text{loc}}(B_1) \), see [7].) Then \( (u_i) \) satisfies the segregated condition
\[ u_i u_j \equiv 0, \quad \forall \ i \neq j. \]

It was proved in [7] (see also [4]) that the free boundary \( \cup_i \partial\{u_i > 0\} \) has Hausdorff dimension \( n - 1 \) and it can be decomposed into two parts: \( \text{Reg}(u_i) \) and \( \text{Sing}(u_i) \). \( Sing(u_i) \) is a relatively closed subset of \( \cup_i \partial\{u_i > 0\} \) of Hausdorff dimension at most \( n - 2 \), while for any \( x \in \text{Reg}(u_i) \), there exists a ball \( B_r(x) \) such that there are only two components of \( (u_i) \) nonvanishing in this ball, say \( u_1 \) and \( u_2 \). Furthermore, \( u_1 - u_2 \) is harmonic and \( \nabla(u_1 - u_2) \neq 0 \) in this ball. Hence the free boundary in this ball is exactly the nodal set of this harmonic function. In [6], it was proved that in this ball non-dominating species decay as follows:
\[ \sum_{j \neq 1,2} u_{j,\beta} \leq Ce^{-c\beta^e}. \] (1.2)

1991 Mathematics Subject Classification. 35B06, 35B08, 35B25, 35J91.
Key words and phrases. Singularly perturbed equations; phase separation; uniform regularity of interfaces.
Without loss of generality and perhaps after taking a smaller \( r \), we can assume \( x = 0 \) and \( \{ u_1 - u_2 = 0 \} \cap B_r(0) \) is represented by the graph of a Lipschitz graph in the form \( \{ x_n = h(x') \} \), for \( x' \in B_r^{n-1}(0) \).

Our main result is

**Theorem 1.1.** The segregated interface \( \{ u_{1,\beta} = u_{2,\beta} \} \cap B_r(0) \) is represented by the graph of a Lipschitz function \( x_n = h_\beta(x_1, \ldots, x_{n-1}) \), with the Lipschitz constant of \( h_\beta \) uniformly bounded. Moreover, \( h_\beta \) converges uniformly to \( h \) in \( B_r^{n-1}(0) \).

Some corollaries follow from the proof of this theorem.

**Corollary 1.2.** There exists a constant \( c_1 > 0 \) independent of \( \beta \) such that
\[
|\nabla (u_{1, \beta} - u_{2, \beta})| \geq c_1, \quad \text{in } B_r(0).
\] (1.3)

**Corollary 1.3.** There exists a constant \( c_2 > 0 \) independent of \( \beta \) such that,
\[
u_{1, \beta} + u_{2, \beta} \geq c_2 \beta^{-1/4}, \quad \text{in } B_r(0).
\]

This improves the lower bound estimate in [6, Theorem 1.6] to the optimal one. Corollary 1.2 is also optimal, in the sense that there is no further uniform regularity of \( \nabla u_{1, \beta} - \nabla u_{2, \beta} \). For example, \( u_{1, \beta} - u_{2, \beta} \) does not converge to the limit in \( C^1 \), see [6, Proposition 1.16].

The argument in this paper is similar to the proof for the regularity of flat interfaces in the Allen-Cahn equation presented in the second part of [10]. The main technical tool is the improvement of flatness estimate in [9]. In [9], this estimate is only stated for entire solutions. However, thanks to the local uniform Lipschitz estimate in [9], now we can show that it also holds for local solutions. Several new estimates from [6], especially the exponential decay of non-dominating species (1.2), also allows us to treat systems with more than two equations.

It is natural to conjecture that flat interfaces are also uniformly bounded in \( C^{k, \alpha} \) for any \( k \geq 1 \) and \( \alpha \in (0, 1) \). However, this is out of the reach of arguments in this note, which does not even imply any uniform \( C^{1, \alpha} \) regularity. (In the Allen-Cahn equation, the uniform \( C^{1, \alpha} \) regularity is only achieved by combining this argument with the result in [3].)

## 2. Proof of main results

After restricting to a small ball, by a suitable translation and some scalings, we are in the following setting:

1. \( u_\beta \) is a sequence of solutions to (1.1) in \( B_2(0) \);
2. \( u_\beta \) converges to \( u := (u_1, u_2, 0, \ldots, 0) \) uniformly in \( B_2(0) \), and also strongly in \( H^1_{loc}(B_2(0)) \);
3. \( u_{1, \beta}(0) = u_{2, \beta}(0) \);
4. there exists a small universal constant \( \sigma_0 \) (to be determined later) such that, for any \( x \in B_1(0) \cap \{u_1 - u_2 = 0\} \),
\[
\frac{\int_{B_1(x)} |\nabla u_1|^2 + |\nabla u_2|^2}{\int_{\partial B_1(x)} u_1^2 + u_2^2} \leq 1 + \sigma_0.
\] (2.4)

By multiplying \( u_\beta \) and \( u \) by a positive constant, we may assume
\[
\int_{\partial B_1(0)} u_1^2 + u_2^2 = \int_{\partial B_1} x_n^2.
\] (2.5)
Because $u_1(0) - u_2(0) = 0$ and $u_1 - u_2$ is harmonic, by Almgren monotonicity formula for harmonic functions, we always have

$$
\frac{\int_{B_1(x)} |\nabla u_1|^2 + |\nabla u_2|^2}{\int_{\partial B_1(x)} u_1^2 + u_2^2} \geq 1, \quad \forall \, x \in B_1(0) \cap \{u_1 - u_2 = 0\},
$$

and (2.4) implies the existence of a unit vector $e$, which we assume to be the $n$-th coordinate direction, such that

$$
\sup_{B_1(0)} (|u_1 - u_2 - x_n| + |\nabla (u_1 - u_2 - x_n)|) \leq c(\sigma_0) < 1/16, \quad (2.6)
$$

provided $\sigma_0$ has been chosen small enough.

Some remarks are in order.

**Remark 2.1.** In the following it is always assumed that (1.2) holds in $B_2(0)$. Then because $u_{i,\beta}$ is nonnegative and subharmonic, we get

$$
\sum_{i \neq 1, 2} \int_{B_{3/2}(0)} |\nabla u_{i,\beta}|^2 \leq C e^{-c\beta^e}.
$$

The following rescaling will be used many times in the proof:

$$
u_{\lambda,\beta}(x) = \lambda^{-1} u_{i,\beta}(\lambda x), \quad \lambda > 0.
$$

Once $\lambda > \beta^{-1/4}$, (1.2) still holds for $u_{\beta} := (u_{i,\beta})$, perhaps with a larger $C$ and a smaller $c$ (but still independent of $\beta \to +\infty$).

**Remark 2.2.** Throughout this section, we assume the Lipschitz constant of $u_{i,\beta}$ is bounded by a constant independent of $\beta$. Since all of the rescalings used in this paper are in the form (2.7), any rescaling of $u_{\beta}$ has the same Lipschitz bound.

Let us first recall some known results. The first one is the Almgren monotonicity formula, see for example [2, Proposition 5.2].

**Proposition 2.3.** For any $x \in B_2(0)$,

$$
N(r; x, u_{\beta}) := r \int_{B_r(x)} \sum_i |\nabla u_{i,\beta}|^2 + \beta \sum_{i < j} u_{i,\beta}^2 u_{j,\beta}^2 \sum_{i} u_{i,\beta}^2
$$

is increasing in $r \in (0, 2 - |x|)$.

By the strong convergence of $u_{\beta}$ in $H^{1}_{loc}(B_2(0))$ and the bound (2.4), we can assume that, for all $\beta$ large and $x \in \{u_{1,\beta} = u_{2,\beta}\} \cap B_1(0)$, $N(1; x, u_{\beta}) \leq 1 + 2\sigma_0$. Then by this proposition,

$$
N(r; x, u_{\beta}) \leq 1 + 2\sigma_0, \quad \forall \, r \in (0, 1).
$$

The next one is [8, Lemma 6.1] or [6, Theorem 1.1].

**Lemma 2.4.** For any $x \in \{u_{1,\beta} = u_{2,\beta}\} \cap B_{3/2}(0)$,

$$
u_{1,\beta}(x) = u_{2,\beta}(x) \leq C\beta^{-1/4}.
$$

The main technical result we will use is the following decay estimate, first proved in [9].
Theorem 2.5. There exist four universal constants $\theta \in (0, 1/2)$, $\varepsilon_0$ small and $K_0, C$ large such that, if $u_\beta$ is a solution of (1.1) in $B_1(0)$, satisfying

$$\sum_{i \neq 1, 2} \left[ \sup_{B_1(0)} u_{i,\beta}^2 + \int_{B_1(0)} |\nabla u_{i,\beta}|^2 \right] \leq Ce^{-c\beta\varepsilon}, \quad (2.9)$$

$$\varepsilon^2 := \int_{B_1(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e|^2 \leq \varepsilon_0^2, \quad (2.10)$$

where $e$ is a vector satisfying $|e| \geq 1/4$, and $\beta^{1/8}\varepsilon^2 \geq K_0$, then there exists another vector $\tilde{e}$, with $|\tilde{e} - e| \leq C(n)\varepsilon$, such that

$$\theta^{-n} \int_{B_1(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - \tilde{e}|^2 \leq \frac{1}{2}\varepsilon^2. \quad (2.11)$$

Proof. The proof is similar to [9, Theorem 2.2] with only three different points:

(i) Now the system (1.1) could contain more than two equations. However, with the hypothesis (2.9) the effect of $u_{i,\beta}$ ($i \neq 1, 2$) is exponentially small, hence it does not affect the final conclusion.

(ii) We do not claim [9, Lemma 3.4]. This estimate is used in [9, Eq. (5.1)]. Instead, we only provide a weaker estimate

$$\int_{B_{3/4}(0)} \beta u_{1,\beta} u_{2,\beta} + \beta u_{2,\beta} u_{1,\beta} \leq C\beta^{-1/8}. \quad (2.11)$$

This is the reason we replace the condition $\varepsilon^2 \gg \beta^{-1/4}$ in [9, Theorem 2.2] by a more restrictive one $\varepsilon^2 \gg \beta^{-1/8}$.

Note that $\beta u_{1,\beta} u_{2,\beta} \leq u_{2,\beta} \Delta u_{1,\beta}$. Thus

$$\int_{B_{3/4}(0)} \beta u_{1,\beta} u_{2,\beta} \leq \int_0^{+\infty} \left( \int_{B_{3/4}(0) \cap \{ u_{2,\beta} > t \}} \Delta u_{1,\beta} \right) dt$$

$$\leq \int_0^{L\beta^{-1/4}\log \beta} \int_{B_{3/4}(0)} \Delta u_{1,\beta} + \int_{L\beta^{-1/4}\log \beta}^{+\infty} Ce^{-c\beta^{1/4}t} dt$$

$$\leq C\beta^{-1/4}\log \beta,$$

where $L$ is a large constant (fixed to be independent of $\beta > 0$) and we have used the fact that $\Delta u_{1,\beta} \leq Ce^{-c\beta^{1/4}t}$ in $\{ u_{2,\beta} > t \}$. (2.11) follows from this estimate if $\beta$ is large enough.

(iii) It is also not known whether [9, Lemma 3.3] holds. However, in [9] this estimate is only used to derive [9, Eq. (4.6)], which will be replaced by the following weaker estimate

$$\int_{B_{3/4}(0)} |\nabla u_{1,\beta}| |\nabla u_{2,\beta}| \leq C\beta^{-1/8}. \quad (2.12)$$

For simplicity, we will take a rescaling as in (2.7) so that $\beta = 1$ in the equation and the domain is $B_R(0)$ where $R = \beta^{1/4}$. Solutions are denoted by $(u_i)$.

Choose a $T$ large so that $u_1 u_2 < T^2$ in $B_R(0)$ (see [8, Lemma 6.1]). By this choice $\{ u_1 > T \}$ and $\{ u_2 > T \}$ are disjoint.
For any $x \in \{ u_1 < T, u_2 < T \}$, by the Lipschitz continuity of $u_1$ and $u_2$, $u_1 \leq T + C$ and $u_2 \leq T + C$ in $B_1(x)$. Then by standard gradient estimates and Harnack inequality,
\[
|\nabla u_i(x)| \leq C \sup_{B_1(x)} u_i \leq Cu_i(x), \quad \forall \ i = 1, 2.
\]
Thus by the Cauchy inequality,
\[
\int_{B_{R-1}(0) \cap \{ u_1 < T, u_2 < T \}} |\nabla u_1| |\nabla u_2| \leq C \int_{B_{R-1}(0) \cap \{ u_1 < T, u_2 < T \}} u_1 u_2 \\
\leq CR^{\frac{2}{2}} \left( \int_{B_{R-1}(0)} u_1^2 u_2^2 \right)^{1/2} \tag{2.13}
\leq CR^{n-1/2},
\]
where we have used \cite[Lemma 6.4]{8}, which implies
\[
\int_{B_{R-1}(0)} u_1^2 u_2^2 \leq CR^{n-1}.
\]
For $x \in \{ u_1 \geq T \}$, by noting that
\[
\Delta |\nabla u_2| \geq u_2^2 |\nabla u_2| - 2u_1 u_2 |\nabla u_1|,
\]
we get
\[
|\nabla u_2(x)| \leq C \sup_{B_{1/2}(x)} (u_1 u_2). \tag{2.14}
\]
Because $u_2$ is subharmonic,
\[
\sup_{B_{1/2}(x)} u_2 \leq C \int_{B_1(x)} u_2. \tag{2.15}
\]
Since $u_1(x) \geq T$, by the Lipschitz bound on $u_1$, if we have chosen $T$ sufficiently large,
\[
\frac{1}{2} \sup_{B_1(x)} u_1 \leq u_1(x) \leq \sup_{B_1(x)} u_1. \tag{2.16}
\]
Combining (2.13) with (2.16) with the Lipschitz continuity of $u_1$, we get
\[
|\nabla u_1(x)| |\nabla u_2(x)| \leq C \int_{B_1(x)} u_1 u_2, \quad \forall \ x \in \{ u_1 > T \} \cap B_{3R/4}(0).
\]
Integrating this on $\{ u_1 > T \} \cap B_{3R/4}(0)$ and using the Fubini theorem and the Cauchy inequality, we obtain
\[
\int_{\{ u_1 > T \} \cap B_{3R/4}(0)} |\nabla u_1| |\nabla u_2| \leq C \int_{B_1(0)} \int_{\{ u_1 > T \} \cap B_{3R/4}(0)} u_1(x + y) u_2(x + y) dxdy \\
\leq C \int_{B_{\frac{R}{4}}(0)} u_1 u_2 \tag{2.17} \\
\leq CR^{n-1/2},
\]
A similar estimate holds in $\{ u_2 > T \} \cap B_{3R/4}(0)$. Combining (2.13) with these we get (2.12).
The next lemma can be used to show that the condition (2.10) is always satisfied for \((u_{1,\beta}^\lambda,1)\), provided \(\lambda \gg \beta^{-1/4}\).

**Lemma 2.6.** For any \(\varepsilon > 0\), there exist two constants \(K(\varepsilon)\) and \(\delta(\varepsilon)\) so that the following holds. Suppose \(u_\beta\) is a solution of (1.1) in \(B_2(0)\), with \(\beta \geq K(\varepsilon)\), satisfying \(u_{1,\beta}(0) = u_{2,\beta}(0)\), (2.20) and

\[
\frac{2 \int_{B_2(0)} \sum_i |\nabla u_{i,\beta}|^2 + \sum_{i < j} \beta u_{i,\beta}^2 u_{j,\beta}^2}{\int_{\partial B_2(0)} \sum_i u_{i,\beta}^2} \leq 1 + \delta(\varepsilon),
\]

then there exists a vector \(\epsilon\) such that

\[
\int_{B_3(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - \epsilon|^2 \leq \varepsilon^2. \tag{2.19}
\]

**Proof.** Assume by the contrary, there exists an \(\varepsilon > 0\), a sequence of solutions \(u_\beta\) with \(\beta \to +\infty\), satisfying \(u_{1,\beta}(0) = u_{2,\beta}(0)\), (2.20) and

\[
\limsup_{\beta \to +\infty} \frac{2 \int_{B_2(0)} \sum_i |\nabla u_{i,\beta}|^2 + \sum_{i < j} \beta u_{i,\beta}^2 u_{j,\beta}^2}{\int_{\partial B_2(0)} \sum_i u_{i,\beta}^2} \leq 1, \tag{2.20}
\]

but for any vector \(\epsilon\),

\[
\int_{B_3(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - \epsilon|^2 \geq \varepsilon^2. \tag{2.21}
\]

By our assumption, the Lipschitz constant of \(u_{i,\beta}\) in \(B_{3/2}(0)\) are uniformly bounded in \(\beta\). By Lemma 2.4

\[u_{1,\beta}(0) = u_{2,\beta}(0) \leq C\beta^{-1/4}.\]

Hence \(u_{1,\beta}\) and \(u_{2,\beta}\) are also uniformly bounded in \(B_{3/2}(0)\). Assume it converges uniformly to \((u_1, u_2, 0, \cdots)\) in \(B_{3/2}(0)\). As before, \(u_1 u_2 \equiv 0\) and \(u_1 - u_2\) is a harmonic function. Moreover, \((u_{1,\beta})\) also converges to \((u_1, u_2, 0, \cdots)\) in \(H^1(B_1(0))\). Hence by Proposition 2.3 and (2.20), we obtain

\[
\frac{\int_{B_3(0)} \sum_i |\nabla u_i|^2}{\int_{\partial B_3(0)} \sum_i u_i^2} \leq 1.
\]

Then by the characterization of linear functions using Almgren monotonicity formula (noting that \(u_1(0) = u_2(0) = 0\), we get a vector \(\epsilon\) such that

\[u_1(x) - u_2(x) \equiv \epsilon \cdot x, \text{ in } B_1(0)\]

By the strong convergence of \(u_{i,\beta}\) in \(H^1(B_1(0))\) again,

\[
\lim_{\beta \to +\infty} \int_{B_1(0)} |\nabla u_{1,\beta} - \nabla u_1|^2 + |\nabla u_{2,\beta} - \nabla u_2|^2 = 0.
\]

This is a contradiction with (2.21) and finishes the proof of this lemma. \qed

After these preliminaries now we prove

**Lemma 2.7.** For any \(\sigma > 0\), there exist two universal constants \(K_1(\sigma), K_2 (K_2\text{ independent of }\sigma)\) such that the following holds. For any \(x \in \{u_{1,\beta} = u_{2,\beta}\} \cap B_1(0)\), there exists an \(r_\beta(x) \in (K_1\beta^{-1/4}, \theta)\) such that,
• for any \( r > r_\beta(x) \), there exists a vector \( e(r, x) \), with \( |e(r, x)| \geq 1/4 \), such that

\[
   r^{-n} \int_{B_r(x)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e(r, x)|^2 \leq C r^n \int_{B_{3r/2}(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e_n|^2, \tag{2.22}
\]

where \( \alpha = \log 2/|\log \theta| \) and \( \theta \) is as in Theorem 2.3.

• for \( r \in (K_1 \beta^{-1/4}, r_\beta(x)) \), there exists a vector \( e(r, x) \), with \( |e(r, x)| \geq 1/4 \), such that

\[
   r^{-n} \int_{B_r(x)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e(r, x)|^2 \leq K_2 \beta^{-\frac{1}{2}} r^{-\frac{1}{2}}. \tag{2.23}
\]

Moreover, for any \( r \in (K_1 \beta^{-1/4}, \theta) \),

\[
   |e(r, x) - e_n| \leq \sigma + C \left( \int_{B_{3r/2}(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e_n|^2 \right)^{1/2} < 1/2, \tag{2.24}
\]

for all \( \beta \) large.

**Proof.** Without loss of generality assume \( x \) is the origin 0. For each \( k \geq 0 \), let

\[
   E_k := \min_{e \in \mathbb{R}^n} \theta^{-kn} \int_{B_{\theta^k}(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e|^2,
\]

which can be assumed to be attained by a vector \( e_k \).

By our hypothesis, in particular (2.6), \( E_0 \) is very small for all \( \beta \) large. Moreover, \( e_0 \) is close to the \( n \)-th direction. In the following we will show that \( |e_k| \geq 1/2 \) up to scales \( \theta^k \sim \beta^{-1/4} \).

**Claim 1.** For any \( k \geq 0 \), \( E_k \geq \theta^n E_{k+1} \).

This is because, for any vector \( e \),

\[
   \theta^{-kn} \int_{B_{\theta^k}(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e|^2 \geq \theta^{-kn} \int_{B_{\theta^k+1}(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e|^2.
\]

Let \( \varepsilon_0 \) be as in Theorem 2.3. Then choose \( \sigma_0 \) and \( K_1 \) according to Lemma 2.6 so that \( 2\sigma_0 \leq \delta(\varepsilon_0) \) and \( K_1 \geq K(\varepsilon_0) \). By Lemma 2.6 we obtain

**Claim 2.** If \( \beta^{1/4} \theta^k \geq K_1 \), then \( E_k \leq \varepsilon_0^2 \).

In the following we take \( k_1 \) to be the largest \( k \) satisfying \( \beta^{1/4} \theta^k \geq K_1 \). \( k_1 \) is defined to be the largest \( k \leq k_1 \) so that for any \( i \leq k \), \( |e_i| \geq 1/2 \).

**Claim 3.** For any \( 1 \leq k \leq k_1 \), if \( E_k \geq K_2 \beta^{-1/8} \theta^{-k/2} \), where \( K_2 = K_0 \theta^{-n} \), then \( E_k \leq \varepsilon_0^2 E_{k-1} \).

Let

\[
   \tilde{u}_{k,\beta}(x) := \theta^{1-k} u_{k,\beta}(\theta^{k-1} x),
\]

which satisfies (1.1) with \( \beta \) replaced by \( \beta_{k-1} := \beta \theta^{4k-4} \).

By Claim 2,

\[
   \varepsilon_{k-1}^2 := \int_{B_1(0)} |\nabla \tilde{u}_{1,\beta} - \nabla \tilde{u}_{2,\beta} - e_{k-1}|^2 = E_{k-1} \leq \varepsilon_0^2.
\]

By Claim 1, \( E_{k-1} \geq K_0 \beta_{k-1}^{-1/8} \). Thus \( \beta_{k-1}^{1/8} \varepsilon_{k-1}^2 \geq K_0 \). Moreover, by definition we also have \( |e_{k-1}| \geq 1/2 \). Hence Theorem 2.3 applies, which implies the existence of a vector \( \tilde{e}_k \) such that

\[
   \theta^{-n} \int_{B_{\tilde{e}_k}(0)} |\nabla \tilde{u}_{1,\beta} - \nabla \tilde{u}_{2,\beta} - \tilde{e}_k|^2 \leq \frac{1}{2} \varepsilon_{k}^2.
\]
Rescaling back, by the definition of $E_k$, we get \textbf{Claim 3.}

Note that in Claim 3, trivially we also have $E_{k-1} \geq E_k$. Thus we still have
\[ E_{k-1} \geq K_2 \beta^{-\frac{1}{2}} \theta^{-\frac{k}{2}} \geq K_2 \beta^{-\frac{1}{2}} \theta^{-\frac{k+1}{2}}. \]
Hence Claim 3 can be applied repeatedly. From this we deduce the existence of a $k_2$ such that, for any $k \geq k_2$, $E_k \leq K_2 \beta^{-1/2} \theta^{-k/2}$, while for any $k \leq k_2$, $E_k \geq K_2 \beta^{-1/2} \theta^{-k/2}$, and hence by Claim 3,
\[ E_k \leq 2^{-1} E_{k-1} \leq \cdots \leq 2^{-k} E_0. \]

It remains to show that $\theta^{k_1} \sim \beta^{-1/4}$. For $k \leq k_2$, because
\[
\theta^{-kn} \int_{B_{\theta^k}(0)} |e_k - e_{k-1}|^2 \leq 2 \theta^{-kn} \int_{B_{\theta^k}(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e_k|^2 \\
+ 2 \theta^{-kn} \int_{B_{\theta^k}(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e_{k-1}|^2 \\
\leq 2 E_k + 2 \theta^{-n} E_{k-1} \\
\leq C E_0 2^{-k},
\]
we get
\[ |e_k - e_{k-1}| \leq C E_0^{\frac{1}{2}} 2^{-\frac{k}{4}}. \tag{2.25} \]
Similarly, for $k \geq k_2$,
\[ |e_k - e_{k+1}| \leq C(n) K_2 \beta^{-\frac{1}{2}} \theta^{-\frac{n+k}{4}}. \tag{2.26} \]

Let $k_3$ be the largest number satisfying
\[ C(n) K_2 \beta^{-\frac{1}{2}} \theta^{-\frac{n+k_3}{4}} - \frac{\theta^{-k_3+1}}{\theta^{-1/4} - 1} \leq \sigma. \tag{2.27} \]

Note that by this choice, there exists a universal constant $C$ such that
\[ \frac{1}{C \sigma} \beta^{-\frac{1}{4}} \leq \theta^{k_3} \leq \frac{C}{\sigma} \beta^{-\frac{1}{4}}. \tag{2.28} \]

Adding \eqref{2.26} and \eqref{2.27} from $k = 0$ to $k$, we see for any $k \leq k_3$,
\[ |e_k - e_0| \leq C E_0^{\frac{1}{2}} + \sigma < 1/4. \tag{2.29} \]
In particular, $|e_k| \geq 1/2$ for all $k \leq k_3$. Thus we can choose $k_1 \geq k_3$. By \eqref{2.28},
\[ \theta^{k_1} \leq \frac{C}{\sigma} \beta^{-\frac{1}{4}}. \]

Finally, by choosing $K_1 := \max\{ \tilde{K}_1, \theta^{k_3} \beta^{1/4} \}$ and $r_\beta := \theta^{k_2}$ we finish the proof. \hfill $\square$

\textbf{Lemma 2.8.} For any $\varepsilon > 0$, there exists two constant $\tilde{d}(\varepsilon)$ and $\tilde{K}(\varepsilon)$ so that the following holds. Let $u_\beta$ be a solution of \eqref{1.1} in $B_2(0)$ with $\beta \geq \tilde{K}(\varepsilon)$, satisfying $u_{1,\beta}(0) = u_{2,\beta}(0)$, \eqref{2.9} and
\[ \int_{B_2(0)} |\nabla u_{1,\beta} - \nabla u_{2,\beta} - e|^2 \leq \tilde{d}(\varepsilon) \tag{2.30} \]
for some vector $e$ with $|e| \geq 1/4$. Then $\{u_{1,\beta} = u_{2,\beta}\} \cap B_1(0)$ belongs to the $\varepsilon$ neighborhood of $P_\varepsilon \cap B_1(0)$, where $P_\varepsilon$ is the hyperplane orthogonal to $e$. 
Proof. Assume by the contrary, there exists an \( \varepsilon > 0 \) and a sequence of solutions \( u_\beta \) in \( B_2(0) \), with \( \beta \to +\infty \), satisfying \( u_{1,\beta}(0) = u_{2,\beta}(0) \), \((2.9)\) and

\[
\lim_{\beta \to +\infty} \int_{B_2(0)} \left| \nabla u_{1,\beta} - \nabla u_{2,\beta} - e \right|^2 = 0, \tag{2.31}
\]

where \( |e| \geq 1/4 \). (At first this vector may depend on \( \beta \), but we can rotate \( (u_{1,\beta}) \) to make it the same one.) But there exists \( x_\beta \in B_1(0) \cap \{u_{1,\beta} = u_{2,\beta}\} \) such that

\[
\liminf_{\beta \to +\infty} \text{dist}(x_\beta, P_e) > 0. \tag{2.32}
\]

Hence we can assume \( x_\beta \to x_\infty \), which lies outside \( P_e \).

By these assumptions and the uniform Lipschitz regularity of \( u_\beta \), they are uniformly bounded in \( \text{Lip}_{\text{loc}}(B_2(0)) \) and can be assumed to converge to a limit \( (u_i) \) in \( C_{\text{loc}}(B_2(0)) \). By \((2.9)\), \( u_i \equiv 0 \) for all \( i \neq 1, 2 \). By \((2.31)\),

\[
\int_{B_2(0)} \left| \nabla u_1 - \nabla u_2 - e \right|^2 = 0. \tag{2.33}
\]

Hence by the main result in \([7]\) and \([11]\), \( u_1 = (e \cdot x)^+ \) and \( u_2 = (e \cdot x)^- \).

Because \( u_{i,\beta} \to u_i \) uniformly in \( B_1 \), by the nondegeneracy of \( u_1 - u_2 \), we obtain a contradiction with \((2.32)\). \( \square \)

Fix an \( \varepsilon > 0 \) and then choose a sufficiently small \( \sigma \leq \hat{\delta}(\varepsilon)/2 \) and a sufficiently large \( K_3 \geq \tilde{K}(\varepsilon) \) according to this lemma. By Lemma \( \ref{lem:2.7} \) Lemma \( \ref{lem:2.8} \) applies to \( u_\beta \) in \( B_r(x) \) for \( r \geq K_3\beta^{-1/4} \) (after scaling to the unit ball), which says \( \{u_{1,\beta} = u_{2,\beta}\} \cap B_r(x) \) belongs to the \( \varepsilon r \) neighborhood of \( (x + P_{e(r,x)}) \cap B_r(x) \). Since \( |e(r,x) - e_n| \leq 2\sigma \) (for \( \beta \) sufficiently large and \( e_n \) denotes the \( n \)-th direction), this implies \( \{u_{1,\beta} = u_{2,\beta}\} \cap B_1(x) \subset \{ |\Pi_{e_n}(y - x)| \leq C\sigma |\Pi_{e_n}^\perp(y - x)| \} \) once \( |y - x| \geq K_3\beta^{1/4} \).

Roughly speaking, this is equivalent to saying that \( \{u_{1,\beta} = u_{2,\beta}\} \) is Lipschitz up to the scale \( K_3\beta^{-1/4} \) in the direction \( e_n \).

The next result shows that this Lipschitz property also holds for \( r \in (0, K_3\beta^{-1/4}) \).

\textbf{Lemma 2.9.} For any \( \delta > 0 \) (sufficiently small) and \( L > 0 \), there exists an \( R(\delta, L) \) so that the following holds. Suppose \( (u_i) \) is a solution of \((1.1)\) with \( \beta = 1 \), in a ball \( B_R(0) \) with \( R \geq R(\delta, L) \), satisfying \( u_1(0) = u_2(0) \),

\[
\sup_{B_L(0)} \sum_{i \neq 1,2} u_i \leq Ce^{-cR^\delta}, \tag{2.34}
\]

and

\[
r^{-n} \int_{B_r(0)} \left| \nabla u_1 - \nabla u_2 - e \right|^2 \leq \delta, \quad \forall \ L < r < R, \tag{2.35}
\]

where \( e \) is a unit vector. Then

\[
\sup_{B_L(0)} \left| \nabla u_1 - \nabla u_2 - e \right| \leq c(n) < 1. \tag{2.36}
\]

Moreover, \( \{u_1 = u_2\} \cap B_L(0) \) is a Lipschitz graph in the direction \( e \), with its Lipschitz constant bounded by \( c(\delta) \), which satisfies \( \lim_{\delta \to 0} c(\delta) = 0 \).

\textbf{Proof.} Assume by the contrary, there exist \( \delta \) and \( L \), and a sequence of solutions \( (u_{i,R}) \) defined in \( B_R(0) \) with \( R \to +\infty \), satisfying \((2.34)\) and \((2.35)\), but the conclusion of this lemma does not hold.
Because \( u_{1,R}(0) = u_{2,R}(0) \), by the Lipschitz bound, there exists a universal constant \( C \) such that
\[
u_{1,R} = u_{2,R}(0) \leq C.
\]
Combining this with (2.34) and the uniform Lipschitz bound on \( u_{i,R} \), we see \((u_{i,R})\) are uniformly bounded in \( \text{Lip}_{\text{loc}}(\mathbb{R}^n) \). Then using standard elliptic estimates and compactness results, we deduce that \((u_{i,R})\) converges to a limit \((u_i)\) in \( C^2_{\text{loc}}(\mathbb{R}^n) \), which is a solution of (1.1) with \( \beta = 1 \) in \( \mathbb{R}^n \).

Passing to the limit in (2.34) gives \( u_i(0) = 0 \) for all \( i \neq 1, 2 \). Since \( u_i \geq 0 \), by the strong maximum principle, \( u_i \equiv 0 \) for all \( i \neq 1, 2 \). (2.35) can also be passed to the limit, which gives
\[
r^{-n} \int_{B_r(0)} |\nabla u_1 - \nabla u_2 - \epsilon|^2 \leq \delta, \quad \forall \ r > L.
\]
In particular, because \( \epsilon \) is nonzero, \((u_1, u_2) \neq 0\).

It is clear that \((u_1, u_2)\) is a globally Lipschitz solution of the system
\[
\Delta u_1 = u_1 u_2^2, \quad \Delta u_2 = u_2 u_1^2, \quad \text{in} \ \mathbb{R}^n.
\]
Then the main result in [8] says \((u_1, u_2) = (g_1(\tilde{\epsilon} \cdot x), g_2(\tilde{\epsilon} \cdot x))\), where \( \tilde{\epsilon} \) is a vector and \((g_1, g_2)\) is the one dimensional solution of (2.38). (It is essentially unique, see [1] and [2].) Substituting this into (2.37) we get
\[
|\tilde{\epsilon} - \epsilon| \leq C\delta < 1/16,
\]
provided \( \delta \) has been chosen small enough. (Note that (2.38) has a scaling invariance, which however is fixed by the condition (2.37).)

By the implicit function theorem, for all \( R \) large, \( \{u_{1,R} = u_{2,R}\} \cap B_L(0) \) is the graph of a smooth function \( h_R \) in the direction of \( \tilde{\epsilon} \). By the convergence of \((u_{i,R})\) and the uniform lower bound on \( \inf_{B_L(0)} |\nabla u_{1,R} - \nabla u_{2,R}| \), this function converges to 0 in a smooth way. The conclusion then follows. \( \square \)

Finally, we prove the two corollaries in Section 1.

**Proof of Corollary 1.2.** Take an arbitrary point \( x_0 \). Let \( \rho := \text{dist}(x_0, \{u_{1,\beta} = u_{2,\beta}\}) \), which we assume to be attained at \( y_0 \). Choose a \( k \) so that \( \rho \in [\theta^{k+1}, \theta^k) \). (Notations as in the proof of Lemma 2.7.) Let
\[
\tilde{u}_{i,\beta}(x) := \frac{1}{\rho} u_{i,\beta}(y_0 + \rho x).
\]
If \( \rho \leq K_3 \beta^{-1/4} \), (1.3) follows from (2.38) in Lemma 2.4.
If \( \rho \geq K_3 \beta^{-1/4} \), (1.3) follows from (2.22) or (2.23) in Lemma 2.7 and standard interior elliptic estimates. (Note that in a neighborhood of \((x_0 - y_0)/\rho\) either \( \tilde{u}_{1,\beta} \) or \( \tilde{u}_{2,\beta} \) is very small compared to the other component.) \( \square \)

The proof of Corollary 1.3 is similar.

**Acknowledgments.** The author’s research was partially supported by NSF of China No. 11301522.


References

[1] H. Berestycki, T. Lin, J. Wei and C. Zhao, On phase-separation model: asymptotics and qualitative properties, Arch. Ration. Mech. Anal. 208 (2013), no.1, 163-200.
[2] H. Berestycki, S. Terracini, K. Wang and J. Wei, Existence and stability of entire solutions of an elliptic system modeling phase separation, Adv. Math. 243 (2013), 102-126.
[3] L. Caffarelli and A. Cordoba, Phase transitions: Uniform regularity of the intermediate layers, Journal fur die reine und angewandte Mathematik (Crelles Journal) 593 (2006), 209-235.
[4] E. N. Dancer, K. Wang and Z. Zhang, The limit equation for the Gross-Pitaevskii equations and S. Terracini’s conjecture, J. Funct. Anal., 262 (2012), no. 2, 1087-1131.
[5] N. Soave and A. Zilio, Uniform bounds for strongly competing systems: the optimal Lipschitz case, to appear in Archive for Rational Mechanics and Analysis.
[6] N. Soave and A. Zilio, On phase separation in systems of coupled elliptic equations: asymptotic analysis and geometric aspects, arXiv:1506.07779.
[7] H. Tavares and S. Terracini, Regularity of the nodal set of segregated critical configurations under a weak reflection law, Calculus of Variations and PDEs 45 (2012), no. 3-4, 273-317.
[8] K. Wang, On the De Giorgi type conjecture for an elliptic system modeling phase separation, Comm. PDE 39 (2014), no. 4, 696-739.
[9] K. Wang, Harmonic approximation and improvement of flatness in a singularly perturbation problem, Manuscripta Mathematica 146 (2015), no. 1-2, 281-298.
[10] K. Wang, A new proof of Savin’s theorem on Allen-Cahn equations, arXiv:1401.6480.

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