A SURVEY ON MIXED SPIN P-FIELDS

HUAI-LIANG CHANG¹, JUN LI², WEI-PING LI³, AND CHIU-CHU MELISSA LIU⁴

1. GROMOV-WITTEN INVARIANTS OF QUINTICS

For the Fermat quintic polynomial \( W_5 = x_1^5 + \ldots + x_5^5 \), the counting of genus \( g \) curves of degree \( d \) on the quintic Calabi-Yau three-fold \( Q = \{ W_5 = 0 \} \subset \mathbb{P}^4 \) is a challenging problem in enumerative geometry. Since the seminal paper of Candelas, del Ossa, Green and Parkes [COGP], a modified version has been intensively studied in string theory as well as algebraic geometry via the stable maps of Kontsevich and virtual cycles theory developed by Li-Tian [LT] and Behrend-Fantachi [BF].

For \( d, g \in \mathbb{Z} \), the moduli space of stable maps from genus \( g \) nodal curves to \( Q \) of degree \( d \) is

\[ M_g(X, d) = \{ [f : C \to X] \mid C \text{ nodal, } g(C) = g, f_*[C] = d, \text{Aut}(f) < \infty \}. \]

The Gromov-Witten invariants are defined as

\[ N_{g,d} = \int_{[\overline{M}_g(X,d)]^{vir}} 1 \in \mathbb{Q}. \]

One of the main unsolved problems in Gromov-Witten theory is to determine \( F_g(q) := \sum_d N_{g,d} q^d \).

From the Super-String Theory side, in 1991 Candelas et al. found a closed formula for genus zero \( F_0(q) \) using \( T \)-duality and mirror symmetry ([COGP]). In 1993, Bershadsky, Cecotti, Ooguri and Vafa developed the Kodaira-Spencer theory and determined the genus one \( F_1(q) \) ([BCOV]). For higher genus, in 2009 Huang, Klemm and Quackenbush determined \( F_g(q) \) for \( g \) up to 51 ([HKQ]).

From the mathematical side, Kontsevich derived a torus localization to calculate the genus zero GW-invariants \( N_{0,d} \). Givental [Gi], Lian-Liu-Yau [LLY] determined the genus zero case \( F_0(q) \). Later on, more people worked on this topic. The genus one case \( F_1(q) \) was solved in 2000’s. The second named author and Zinger in [LZ] obtained a formula \( N_{1,d}^{\text{red}} = N_{1,d} - \frac{1}{12} N_{0,d} \) where \( N_{1,d}^{\text{red}} \) is

---

¹Partially supported by Hong Kong GRF grant 600711 and 6301515.
²Partially supported by NSF grant NSF-1104553 and DMS-1159156.
³Partially supported by by Hong Kong GRF grant 602512 and 6301515.
⁴Partially supported by NSF grant DMS-1206667 and DMS-1159416.
certain reduced GW-invariants. Using this formula and $\mathbb{C}^*$-localization, Zinger in [Zi] succeeded in determining $F_1(q)$. Gathmann [Gath] provided an algorithm for $N_{1,d}$ using the relative GW-invariant formula. For higher genus case, Maulik and Pandharipande found an algorithm [MP, Section 3.2] using the algebraic version of degeneration formula [Lideg] (see analogue formula [LR]) and used it for some theoretical applications. Despite these progress, a lot of questions on higher genus GW invariants of quintic Calabi-Yau threefolds remains open.

It remains a central problem in Gromov-Witten theory to develop new techniques to calculate all genus GW-invariants of quintic Calabi-Yau threefolds.

2. Witten’s vision and FJRW invariants

2.1. Witten’s vision. The same quintic polynomial $W_5 = x_1^5 + \ldots + x_5^5$ can also give a map $\mathbb{C}^5 \to \mathbb{C}$. The corresponding physical theory is the Landau-Ginzburg theory. In [Wi], Witten studied phase transitions involving GW theory on the quintic $Q$ and the LG-model for $W_5$. Mathematically, the set-up is as follows. Let $\mathbb{C}^*$ act on

$$C^6 = \mathbb{C}^5 \times \mathbb{C} = \{(x_1, \ldots, x_5, p)\}$$

with weights $(1, \ldots, 1, -5)$. Then the map $p \cdot (x_1^5 + \ldots + x_5^5) : C^6 \to \mathbb{C}$ is $\mathbb{C}^*$-equivariant. The quotient $[C^6/\mathbb{C}^*]$ has two GIT quotients:

$$((\mathbb{C}^5 - \{0\}) \times \mathbb{C})/\mathbb{C}^* = K_{\mathbb{P}4},$$

and

$$((\mathbb{C}^5 - \{0\}) \times \mathbb{C})/\mathbb{C}^* = [\mathbb{C}^5/\mathbb{Z}_5].$$

Here $[\mathbb{C}^5/\mathbb{Z}_5]$ represents the quotient stack. The field theory valued in $K_{\mathbb{P}4}$ is the GW theory of the quintic $Q$ and the field theory valued in $[\mathbb{C}^5/\mathbb{Z}_5]$ leads to the Witten’s spin class. The latter was generalized to quasi-homogeneous polynomials by Fan, Jarvis and Ruan [FJR1, FJR2], which is called FJRW theory. Witten’s vision is that these two theories are related via a phase transition.

2.2. P-fields treatment of GW and FJRW. The notion of P-fields was introduced by Guffin and Sharpe in [GS] for genus zero LG-theory of $(K_{\mathbb{P}4}, W_5)$. Mathematically, The first and second named authors developed the theory of P-fields for all genus GW invariants.

We start with LG-theory for $K_{\mathbb{P}4}$. A field taking values in $((\mathbb{C}^5 - \{0\}) \times \mathbb{C})/\mathbb{C}^*$ is

$$\xi = (\mathcal{C}, \mathcal{L}, \varphi_1, \ldots, \varphi_5, \rho)$$

where $\mathcal{C}$ is a complete nodal curve, $\mathcal{L}$ is an invertible sheaf on $\mathcal{C}$, $\varphi_i \in H^0(\mathcal{C}, \mathcal{L})$, and $\rho \in H^0(\mathcal{L}^{\mathbb{C}} \otimes \omega_{\mathcal{C}})$. Since the weights of the action $\mathbb{C}^*$ on $\mathbb{C}^5$ and $\mathbb{C}$ are $(1, \ldots, 1)$ and $-5$ respectively, while $\varphi_i$ is a section of $\mathcal{L}$, a priori $\rho$ has to be a section of $\mathcal{L}^{\mathbb{C}}$, but we choose $\rho$ to be a section of $\mathcal{L}^{\mathbb{C}} \otimes \omega_{\mathcal{C}}$ due to a
The expression of the cosection scalar multiplication. Finally we say $\xi$ cycle, one obtains the cycle moduli space of such objects is $t$ as functions of $D$ 

Theorem 2.1 (H.L. Chang - J. Li [CL1]). The first and second named authors constructed the GW invariants of stable maps with P-fields as follows. Let $\sigma$ be a cosection $C$ to introduce the natural equivalence $(\xi, L)$ where $C = \xi$. Thus $\xi$ is stable if $\text{Aut}(\xi)$ is finite.

In fact, what we have gotten so far is a stable map to $P^4$ with a P-field. The moduli space of such objects is

$$\overline{M}_g(P^4, d)^p = \{ [f, C, \rho] \mid [f, C] \in \overline{M}_g(P^4, d), \rho \in H^0(C, f^* O(5) \otimes \omega_C) \}.$$ 

Note that the data $([f, C], \rho)$ is equivalent to the data $(C, L, \varphi_1, \ldots, \varphi_5, \rho)$ since the map $f$ is equivalent to the line bundle $L = f^* O_{P^4}(1)$ with five sections $(\varphi_1, \ldots, \varphi_5)$ of $L$.

The first and second named authors constructed the GW invariants of stable maps with P-fields as follows. The moduli stack $\overline{M}_g(P^4, d)^p$, relative to the stack $D = \{(\xi, L)\}$, has a perfect obstruction theory. At $\xi = (\xi, L, \varphi_1, \rho)$, the obstruction sheaf restricted to $\xi$ is

$$\mathcal{O}b|_{\xi} = H^1(L)^{\oplus 5} \oplus H^1(L^{\vee 5} \otimes \omega_C).$$

There exists a cosection $\sigma: \mathcal{O}b \to \mathcal{O}b_{\overline{M}_g(P^4, d)^p}$ constructed as follows. Let

$$(\varphi_1, \ldots, \varphi_5, \dot{\rho}) \in H^1(L)^{\oplus 5} \oplus H^1(L^{\vee 5} \otimes \omega_C) = \mathcal{O}b|_{\xi}.$$ 

Define $\sigma|_{\xi}(\varphi_1, \ldots, \varphi_5, \dot{\rho}) := \dot{\rho} \sum_{i=1}^5 \varphi_i^5 + \rho \sum_{i=1}^5 5 \varphi_i^4 \dot{\varphi}_i$.

The degeneracy locus $D(\sigma)$ of the cosection consists of $\xi$ such that $\sigma|_{\xi}$ is zero, i.e., $\sigma|_{\xi}(\varphi_1, \ldots, \varphi_5, \dot{\rho}) = 0$ for all $\varphi_i$ and $\dot{\rho}$. Thus

$$D(\sigma) = \{ \xi \in \overline{M}_g(P^4, d)^p \mid \rho = 0 \text{ and } \sum_{i=1}^5 \varphi_i^5 = 0 \} = \overline{M}_g(Q, d) \subset \overline{M}_g(P^4, d).$$

The expression of the cosection $\sigma$ comes from taking the derivative of $p \cdot W_5 = p(x_1^5 + \ldots + x_5^5)$ with respect to the time variable $t$ where $p$ and $x_i$ are regarded as functions of $t$ following physical notations.

Since $\rho$ is a section, the moduli space $\overline{M}_g(P^4, d)^p$ is not proper (when $g \geq 1$) and hence cannot be used to define invariants. However, Kiem and the second named author [KL] developed a theory of cosection localization virtual cycles which, applied to this case, resolves the non-proper issue. More precisely, one checks that the degeneracy locus $D(\sigma)$ is the moduli space of stable maps to the quintic $Q$ and thus proper.

**Theorem 2.1 (H.L. Chang - J. Li [CL1]).** Using the cosection localized virtual cycle, one obtains the cycle

$$[\overline{M}_g(P^4, d)^p]^{\text{vir}}_{\text{loc}} \in A_4 D(\sigma) = A_4 \overline{M}_g(Q, d).$$
Furthermore, let P-fields GW invariants $N^p_{g,d} = \int_{[\overline{M}_g(p^1,d)]_{\text{loc}}} 1 \in \mathbb{Q}$, then

$$N_{g,d} = (-1)^{d+g+1} N^p_{g,d'}.$$  

The advantage of this result is that $F_g(q) = \sum_d N_{g,d} q^d$ now becomes a topological string amplitude of a field theory valued in $K_{p^4} = ((\mathbb{C}^5 - \mathbf{0}) \times \mathbb{C})/\mathbb{C}^*$. 

Now let’s consider the field theory valued in $\mathbb{C}^5/\mathbb{Z}_5$. This theory originated from Witten’s spin class \cite{Wi}. Its algebraic constructions (in narrow case) were given by Polishchuk-Vaintrob \cite{PV} and Chiodo \cite{Chi}. The full theory was developed by Fan, Jarvis and Ruan \cite{FJR1, FJR2}, known as FJRW theory. We will touch (narrow) FJRW invariants following the construction by the first, second and third named authors \cite{CLL}.

As the case in the P-fields treatment of GW theory, a field in $(\mathbb{C}^5 \times (\mathbb{C} - 0))/\mathbb{C}^* = [\mathbb{C}^5/\mathbb{Z}_5]$ consists of 

$$\xi = (\Sigma^e, \mathcal{C}, \mathcal{L}, \varphi_1, \ldots, \varphi_5, \rho)$$

where $(\Sigma^e, \mathcal{C})$ is a pointed twisted curve with markings $\Sigma^e$ possibly stacky, $\mathcal{L}$ is an invertible sheaf on $\mathcal{C}$, $\varphi_i \in H^0(\mathcal{L})$, and $\rho \in H^0(\mathcal{L}^{\otimes 5} \otimes \omega^\log_{\mathcal{C}})$ with $\omega^\log_{\mathcal{C}} = \omega_{\mathcal{C}}(\Sigma^e)$. Since we deleted the origin in $\mathbb{C}$, the section $\rho$ must be nowhere vanishing and hence $\mathcal{L}^{\otimes 5} \otimes \omega_{\mathcal{C}} \cong \mathcal{O}_{\mathcal{C}}$, or equivalently $\mathcal{L}^{\otimes 5} \cong \omega^\log_{\mathcal{C}}$. Therefore $(\Sigma^e, \mathcal{C}, \mathcal{L})$ is a 5-spin curve. $(\varphi_1, \ldots, \varphi_5)$ give five fields. Thus we get a moduli space of 5-spin curves with five fields:

$$\overline{M}_{g,\gamma}^{1/5,5p} = \{(\mathcal{C}, \Sigma^e, \mathcal{L}, \varphi_1, \ldots, \varphi_5, \rho) \ | \ \rho \text{ is nowhere zero}\}.$$  

Here $\gamma$ is the monodromy data: if $\Sigma_1$ is a stacky marking on $\mathcal{C}$, then $\mu_5$ acts on $\mathcal{L}|_{\Sigma_1}$ with weight $\gamma_1 = \exp(2\pi ir/5)$ where $0 \leq r \leq 4$. Narrow means $0 < r < 4$. If $\Sigma_1$ is an ordinary marking, $\gamma_1$ is taken to be 1.

Similar to GW case, the moduli stack $\overline{M}_{g,\gamma}^{1/5,5p}$, relative to the stack $\mathcal{D} = \{(\Sigma^e, \mathcal{C}, \mathcal{L})\}$, has a perfect obstruction theory. There exists a cosection $\sigma: \mathcal{O} \rightarrow \mathcal{O}_{\overline{M}_{g,\gamma}^{1/5,5p}}$ whose degeneracy locus is 

$$D(\sigma) = \{\xi \in \overline{M}_{g,\gamma}^{1/5,5p} \ | \ \varphi_i = 0 \text{ for all } i\} = \overline{M}_{g,\gamma}^{1/5} = \{(\Sigma^e, \mathcal{C}, \mathcal{L}) \ | \ \mathcal{L}^{\otimes 5} \cong \omega^\log_{\mathcal{C}}\},$$ 

which is the moduli space of 5-spin curves.

**Theorem 2.2** (H.L. Chang - J. Li - W.P. Li \cite{CLL}). The (narrow) FJRW invariants can be constructed using cosection localized virtual cycles of the moduli space of spin curves with five fields:

$$\overline{M}_{g,\gamma}^{1/5,5p}_{\text{vir}} \subset_{\text{loc}} A_* \overline{M}_{g,\gamma}^{1/5}.$$  

This construction is an algebraic geometric version of Witten’s original construction. Witten considered the moduli space of 5-spin curves $(\Sigma^e, \mathcal{C}, \mathcal{L})$ with
smooth sections. For our set-up, the corresponding Witten equations are given by

\[ \bar{\partial}_s i + \partial_x W_5(s_1, \ldots, s_5) = 0, \quad i.e., \quad \bar{\partial}_s i + 5s_i^4 = 0. \]

This is used to construct Witten’s top Chern class to define invariants on the moduli space of 5-spin curves. From Witten’s equation, the term \( \bar{\partial}_s i \) gives the obstruction class to extend a holomorphic section. Thus the left hand side of (2.1) gives a (differential) section of the obstruction sheaf of the moduli of spin curves with fields. Now substitute the complex conjugate in the Witten’s equation by the Serre duality, the LHS of (2.1) becomes the cosection.

There is an important subclass of FJRW invariants: those with the insertion \(-\frac{2}{5}\). Let \( C \) have \( k \) markings with all \( \gamma_j = \zeta^2 \) for \( 1 \leq j \leq k \) where \( \zeta = \exp(2\pi i/5) \). Define

\[ \Theta_{g,k} = \int_{[\overline{\mathcal{M}}_{g,k}]_{\text{vir}}} 1 \in \mathbb{Q}, \quad \text{for} \quad k + 2 - 2g = 0 \mod 5. \]

It is shown [CLLI2] that \( \{ \Theta_{g,k} \}_{g,k} \) determine all FJRW invariants with descendents (for the quintic singularity), where an explicit formula will be given in [twFJRW]. For this reason we call \( \{ \Theta_{g,k} \}_{g,k} \) the primary FJRW invariants.

3. Master space technique and mixed spin fields

In the previous section, we discussed the LG-field theoretic description of GW theory of the quintic and FJRW theory of \((C^5, W_5)\). Witten’s vision is to link these two theories via a phase transition with respect to some complexified parameter. The approach by the authors is to develop a field theory valued in the master space to geometrically realize the “wall-crossings” of these two field theories.

3.1. Master space technique. Now we explain the master space technique to understand the wall-crossings between \( K_{\mathbb{P}^4} \) and \([C^5/\mathbb{Z}_5]\).

Consider a \( \mathbb{C}^* \)-action on \( \mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^1 \), for \( t \in \mathbb{C}^* \),

\[ (x_1, \ldots, x_5, p, [u_1, u_2])^t = (tx_1, \ldots, tx_5, t^{-5}p, [t u_1, u_2]). \]

It has a GIT quotient

\[ W = (\mathbb{C}^5 \times \mathbb{C} \times \mathbb{P}^1 - S)/\mathbb{C}^* \quad \text{where} \quad S = \{ (x_i = 0 = u_1) \cup (p = 0 = u_2) \}. \]

Consider a \( \mathbb{C}^* \)-action on \( W \) and, to avoid confusions, we call this action \( T \)-action. For \( t \in T = \mathbb{C}^* \),

\[ (x_1, \ldots, x_5, p, [u_1, u_2])^t = (x_1, \ldots, x_5, p, [t u_1, u_2]). \]

The \( T \)-fixed locus is

\[ W^T = K_{\mathbb{P}^4} \times \{ 0 \} \prod \tilde{\omega} \times ((\mathbb{P}^1 - \{ 0, \infty \})/\mathbb{C}^*) \prod [C^5/\mathbb{Z}_5] \times \{ \infty \} \]
where $0 = [1, 0]$ and $\infty = [1, 0]$ in $\mathbb{P}^1$.

Take a $T$-equivariant form $\mu$ on $W$, then we have

$$0 = \left[ \int_W \mu \cap c_1(1_{wt=1}) \right]_0 = -\int_{K_{\mathbb{P}^4}} \mu + \int_{[\mathbb{C}^5/\mathbb{Z}_5]} \mu + \left[ \int_{\{\text{point}\} \ast} \frac{\mu}{*} \right]_0$$

where $1_{wt=1}$ is the $T$-linearized trivial line bundle with weight 1 and $[\ldots]_0$ means taking degree zero part in the equivariant parameter. Thus the wall-crossing can be expressed as

$$\int_{[\mathbb{C}^5/\mathbb{Z}_5]} \mu - \int_{K_{\mathbb{P}^4}} \mu = \text{error} = \left[ \int_{\{\text{point}\} \ast} \frac{\mu}{*} \right]_0$$

3.2. Mixed spin P-fields. Now we consider a field theory valued in $W$. Similar to the case of the field theory of GW valued in $K_{\mathbb{P}^4}$, the authors introduced the notion of mixed spin $P$-fields (MSP for short) ([CLLL]). An MSP field is

$$\xi = (\Sigma^e, \mathcal{C}, \mathcal{L}, \mathcal{N}, \varphi_1, \ldots, \varphi_5, \rho, \nu = [\nu_1, \nu_2]).$$

$(\Sigma^e, \mathcal{C})$ is a pointed twisted curve. $\mathcal{L}$ and $\mathcal{N}$ are invertible sheaves on $\mathcal{C}$. $\mathcal{L}$ is as before but $\mathcal{N}$ is new due to the extra factor $\mathbb{P}^1$ in the master space technique. $\varphi_i \in H^0(\mathcal{L})$ and $\rho \in H^0(\mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}}^{\log})$ as before. $\nu_1 \in H^0(\mathcal{L} \otimes \mathcal{N})$ and $\nu_2 \in H^0(\mathcal{N})$. $\nu = [\nu_1, \nu_2]$ is a new field. We also have a narrow condition: $\varphi_i|_{\Sigma^e} = 0$. There are combined GIT-like stability requirements: $(\varphi_1, \ldots, \varphi_5, \nu_1)$ is nowhere vanishing coming from excluding $\{(x_i = 0 = u_1)\}$ in $W$; $(\rho, \nu_2)$ is nowhere vanishing coming from excluding $\{(\rho = 0 = u_2)\}$ in $W$; and $(\nu_1, \nu_2)$ is nowhere vanishing coming for $[u_1, u_2] \in \mathbb{P}^1$. We say $\xi$ is stable if $\text{Aut}(\xi)$ is finite. For simplicity, we use $\varphi$ to represent $(\varphi_1, \ldots, \varphi_5)$.

In order to understand why the moduli space of MSP fields geometrically contains the moduli space of stable maps with $P$-fields and the moduli space of spin curves with five $P$-fields, we examine the moduli space of MSP fields in details.

Let $\xi$ be a MSP field. When $\nu_1 = 0$, since $(\varphi_1, \ldots, \varphi_5, \nu_1)$ is nowhere zero, we must have that $(\varphi_1, \ldots, \varphi_5)$ is nowhere zero. Since $(\nu_1, \nu_2)$ is nowhere zero, $\nu_2$ must be nowhere zero. Since $\nu_2$ is a section of $\mathcal{N}$, $\mathcal{N} \cong \mathcal{O}_c$. There is no restriction on $\rho$. Thus $\xi \in \overline{M}_g(\mathbb{P}^4, d)^p$ and we get GW theory of the quintic $Q$.

When $\nu_2 = 0$, since $(\rho, \nu_2)$ is nowhere zero, $\rho$ must be nowhere vanishing. Since $\rho$ is a section of $\mathcal{L}^{\vee 5} \otimes \omega_{\mathcal{C}}^{\log}$, we must have $\mathcal{L}^5 \cong \omega_{\mathcal{C}}^{\log}$. Also $\nu_1$ must be nowhere zero. Thus $\mathcal{L} \otimes \mathcal{N} \cong \mathcal{O}_c$, i.e., $\mathcal{N} \cong \mathcal{L}^{\vee}$. $\varphi_1, \ldots, \varphi_5$ can be arbitrary. Thus $\xi \in \overline{M}_{g, (5, 5\rho)}$ and we get FJRW theory.

When $\rho = 0$ and $\varphi_i = 0$ for $1 \leq i \leq 5$, $\nu_1, \nu_2$ must be nowhere zero. Thus $\mathcal{N} \cong \mathcal{O}_c$ and $\mathcal{L} \cong \mathcal{O}_c$. Hence we get stable curves.
Theorem 3.1 (CLLL). The moduli stack $W_{g,\gamma,d}$ of stable MSP fields of genus $g$, monodromy $\gamma = (\gamma_1, \ldots, \gamma_\ell)$ of $L$ along $\Sigma^C$, and degree $d = (d_0, d_\infty)$ of $L \otimes \mathcal{N}$ and $\mathcal{N}$ respectively, is a separated DM stack of locally finite type.

The moduli stack $W_{g,\gamma,d}$ admits a natural $\mathbb{C}^*$-action: for $t \in \mathbb{C}^*$,

$$(\Sigma^C, \mathcal{E}, L, N, \varphi, \rho, \nu_1, \nu_2)^t : = (\Sigma^C, \mathcal{E}, L, N, \varphi, \rho, t \nu_1, \nu_2).$$

$W_{g,\gamma,d}$ is not proper since $\varphi$ and $\rho$ are sections of invertible sheaves. Thus we cannot do integrations on this stack. However, there exists a cosection of its obstruction sheaf. Using the arguments similar to GW case and LG case, we have the following theorem.

Theorem 3.2 (CLLL). The moduli stack $W_{g,\gamma,d}$ has a $\mathbb{C}^*$-equivariant perfect obstruction theory, an equivariant cosection $\sigma$ of its obstruction sheaf, and thus carries an equivariant cosection localized virtual cycle $[W_{g,\gamma,d}]_{\text{vir}}^{\text{loc}} \in A^*_{\mathbb{C}^*} W_{g,\gamma,d}$

where $W_{g,\gamma,d}$ is the degeneracy locus of $\sigma$, i.e.,

$W_{g,\gamma,d}^{-} : = (\sigma = 0) = \{ \xi \in W_{g,\gamma,d} | \mathcal{E} = (\varphi = 0) \cup (\varphi_1^5 + \ldots + \varphi_5^5 = 0 = \rho) \}$.

In order to do integration on $W_{g,\gamma,d}^{-}$, one needs it to be proper. In fact, we have

Theorem 3.3 (CLLL). The degeneracy locus $W_{g,\gamma,d}^{-}$ is a proper $\mathbb{C}^*$-DM stack of finite type.

From the proof of the properness, we see a phenomenon which creates line bundles’ spin structures in the LG-phase via a limit of a family of P-fields in CY-phase. We give this phenomenon the name “Landau-Ginzburg transition” (or CY-to-LG transition). It is under this phenomenon that FJRW theory captures the ghosts’ contributions in GW theory (1) in the realm of MSP moduli.

Example 3.4. The graph of fixed points of $W_{1,0,(1,0)}^{-}$.

So the curves are elliptic curves without markings, $\deg L = 1$, and $\deg N = 0$. The graph type of fixed points which have contributions to the computations are of the following four types $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$:

---

1 A map $f$ from a curve $C$ to $\mathbb{P}^d$ or $Q$ is called a ghost if there are positive-genus components of $C$ that are contracted to points by $f$. Over ghosts, P-fields can be nonvanishing, and such P-fields contribute to GW invariants of the quintic as “counting ghosts”. For example, in Li-Zinger formula $N_{1,d} = N_{1,d}^{\text{red}} + \frac{1}{12} N_{0,d}$, the number 1/12 comes from the contribution by P-fields. When genus increases, such contribution is difficult to locate. MSP program provides a platform that ghosts’ contributions can be captured in another phase (LG-phase) instead.
Some explanations of the figure are in order. The bottom horizontal line corresponds to $\nu_1 = 0$. The middle horizontal line corresponds to $\varphi = 0$ and $\rho = 0$. The top horizontal line corresponds to $\nu_2 = 0$. A vertex represents a connected curve if it is stable (called a stable vertex), or a node if it has two edges, or nothing if it has only one edge attached to it. For each vertex, $g$ represents the genus of the curve. If it is not a stable vertex, nor a node, we use $g = 0$ here even though it doesn’t represent a rational curve. An edge is a rational curve. The number near an edge is the degree of $L$ on the curve.

A stable vertex on the horizontal lines $0$, $1$, $\infty$ means $\nu_1 = 0$, $\varphi = 0 = \rho$, or $\nu_2 = 0$ on the curve respectively.

To be more precise, the graph $\Gamma_1$ represents an elliptic curve with degree 1 line bundle $L$ on the curve and $\nu_1 = 0$ on the whole curve. So it corresponds to stable maps from elliptic curves to the quintic $Q$ with degree 1. The graph $\Gamma_2$ represents a union of an elliptic curve $E$ with a rational curve $E_0$ intersecting at one node. $E$ is a stable vertex on the bottom horizontal line. $E_0$ is the edge. $\deg L$ is 0 on the elliptic curve and 1 on the rational curve. The graph $\Gamma_3$ is similar to $\Gamma_2$, a union of an elliptic curve $E$ and a rational curve $E_0$ intersecting at one node. $E$ is a stable vertex on the middle horizontal line and $E_0$ is the edge. On $E$, $\varphi = 0 = \rho$ and both $L$ and $N$ are trivial. Thus it represents the moduli space of elliptic curves with one marking coming from the node. The graph $\Gamma_4$ represents a union of two rational curves $E_0$ and $E_\infty$ and an elliptic curve $E$. On the lower edge $E_0$, $\deg(L|_{E_0}) = 1$ and $\rho|_{E_0} = 0$. Note that on each irreducible component, either $\rho = 0$ or $\varphi = 0$. On $E_\infty$, $\deg(L|_{E_\infty}) = -1/5$ and $\varphi|_{E_\infty} = 0$. $E_\infty$ and $E_0$ intersect at one node. $E_\infty$ is a twisted curve intersecting the elliptic curve $E$ at a stacky point. Thus $L|_{E_\infty}$ is an invertible sheaf on the twisted curve $E_\infty$. $E$ is a spin elliptic curve with one marking from the node. Thus $\deg(L|_E) = 1/5$.

Example 3.5. The graph of fixed points of $W_{1,\emptyset,(2,0)}^{-1,\emptyset}$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{graphs for $g = 1$, $\gamma = \emptyset$, $d = (1,0)$.}
\end{figure}
Figure 2. graphs for $g = 1$, $\gamma = \emptyset$, $d = (2, 0)$.

In this case the curve is an elliptic curve without markings, $\deg \mathcal{L} = 2$ and $\deg \mathcal{N} = 0$. The graph type of fixed points which have contributions to the computations have 15 types, listed from $\Gamma_1$ to $\Gamma_{15}$.

4. Vanishing and polynomial relations

How to extract information of GW and/or FJRW invariants from the cycle $[W_g, \gamma, d]_{vir}^{loc}$? Let’s consider a less general case to illustrate key ideas. Take $\gamma = \emptyset$, i.e., no markings. Then by virtual dimension counting, we have

$$[W_g, d]_{vir}^{loc} \in H^*_{g, \mathbb{C}}(\mathbb{A}^d + d_\infty + 1 - g)(W_{g, d}^{\mathbb{C}, \emptyset}, \mathbb{Q}).$$

When $d_0 + d_\infty + 1 - g > 0$, letting $u = c_1(I_{\mathcal{M} = 1})$, i.e. $u$ is the parameter for $H^*_{g, \mathbb{C}}(pt)$, we have

$$[u^{d_0 + d_\infty + 1 - g} \cdot [W_g, d]_{vir}^{loc}]_0 = 0.$$

Here $[\cdot]_0$ is the degree zero term in the variable $u$.

Let $\Gamma$ be a graph associated to fixed points of the $\mathbb{C}^*$-action of $W_{g, d}$ and $F_\Gamma$ be a connected component of $W_{g, d}^{\mathbb{C}, \emptyset}$ of the graph type $\Gamma$. Apply the cosection localized version proved in [CKL] of the virtual localization formula in [GP], we have

$$\sum_{\Gamma} [u^{d_0 + d_\infty + 1 - g} \cdot [F_\Gamma]_{vir}^{loc}]_0 e(N_{F_\Gamma}) = 0.$$

To deal with $[F_\Gamma]_{vir}^{loc}$, we need a decomposition result to be explained below.
Let, again, $\xi = (\mathcal{E}, \mathcal{L}, N, \varphi, \rho, \nu_1, \nu_2) \in (W_{g,d})^{\mathbb{C}^*}$ be a MSP field fixed by the $\mathbb{C}^*$-action. We set

1. $\mathcal{C}_0$ to be the part of $\mathcal{C}$ where $\nu_1 = 0$;
2. $\mathcal{C}_1$ to be the part of $\mathcal{C}$ where $\varphi = 0 = \rho$ and hence $\nu_1 = 1 = \nu_2$, i.e., $\nu_1$ and $\nu_2$ are nowhere zero;
3. $\mathcal{C}_\infty$ to be the part of $\mathcal{C}$ where $\nu_2 = 0$.

Thus $\xi|_{(\text{connected component of } \mathcal{C}_0)}$ is in $\overline{M}_{g,n}'(\mathbb{P}^4, d')^p$ which gives Gromov-Witten invariants. Here marked points appear coming from some nodes on $\mathcal{C}_0$. $\xi|_{(\text{connected component of } \mathcal{C}_1)}$ is in $\overline{M}_{g,n}'$ which gives Hodge integrals.

$\xi|_{(\text{connected component of } \mathcal{C}_\infty)}$ is in $\overline{M}_{g',\gamma'}_{1,5}$ which gives FJRW invariants where $\gamma'$ appears because of some stacky nodes on $\mathcal{C}_\infty$.

We have the following decomposition result:

$$[F_\Gamma] \text{vir}_{\text{loc}} = c \prod [\text{moduli of } \xi|_{\mathcal{C}_0}] \text{vir}_{\text{loc}} \cdot [\text{moduli of } \xi|_{\mathcal{C}_1}] \text{vir}_{\text{loc}} \cdot [\text{moduli of } \xi|_{\mathcal{C}_\infty}] \text{vir}_{\text{loc}}$$

where $c$ is a constant. The first factor gives GW invariants of stable maps to $\mathbb{P}^4$ with $P$-fields, i.e. $N_{g',d'}$. The second factor gives Hodge integrals on $\overline{M}_{g',\gamma'}_{1,5}$. The third factor gives FJRW invariants of insertions $-\frac{2}{5}$ (after using a vanishing). After $e(N_{F_\Gamma})$'s are calculated, using the polynomial relations (4.1), we obtain the following results about GW invariants of the quintic.

**Theorem 4.1** ([CLLL2]). Letting $d_\infty = 0$, the relations (4.1) provide an effective algorithm to evaluate GW invariants $N_{g,d}$ provided the following are known

1. $N_{g',d'}$ for $(g', d')$ such that $g' < g$, and $d' \leq d$;
2. $N_{g,d'}$ for $d' < d$;
3. $\Theta_{g',k}$ for $g' \leq g - 1$ and $k \leq 2g - 4$;
4. $\Theta_{g,k}$ for $k \leq 2g - 2$.

Recall that $\Theta_{g,k}$ is the genus $g$ FJRW invariants of insertions $-\frac{2}{5}$ and $\Theta_{g,k}$ may be non-zero only when $k + 2 - 2g \equiv 0(5)$. We can see that when $g = 2$ only $\Theta_{2,2}$ is needed, and when $g = 3$ only $\Theta_{3,4}$ is needed.

**Remark 4.2.** As we know, on using mathematical induction, upon more numerical datum the induction is, the less effective the computation will be. We can see from the Theorem that MSP induction for GW invariants is carried out on two numbers, genus and the degree only. Thus this provides a rather effective way to facilitate the induction procedure.

We can also use the vanishings (4.1) to get relations among FJRW invariants.

**Theorem 4.3** ([CLLL2]). Letting $d = (0, d_\infty)$, the vanishings (4.1) provide relations among FJRW invariants $\Theta_{g,k}$. 
These relations are effective in calculating FJRW invariants. For example, for the case of genus 2, \( \{ \Theta_{2,k} \}_k \) can be inductively derived from only two unknowns \( \Theta_{2,2} \) and \( \Theta_{2,7} \).

**Example 4.4. Computations of \( N_{1,1} \) and \( N_{1,2} \).**

In the Examples 3.4 and 3.5, we listed all the graph types of fixed locus. Using the formulae for \( c(N_F) \) and \( [F_1]_\text{vir} \) in [CLL2], we can calculate every term in the summation in (4.1).

For the genus 1 degree 1 case in Example 3.4, the contributions from four graph types are (here \( G_i \) is the contribution from the graph \( \Gamma_i \) in Figure 1):

\[
G_1 = -N_{1,1}, \quad G_2 = \frac{9625}{6}, \quad G_3 = \frac{-4087}{12}, \quad G_4 = -1024.
\]

From the equation (4.1), the sum of these four numbers should be zero. Thus we obtain \( N_{1,1} = \frac{2875}{12} \) which agrees with the known result.

For the genus 1 degree 2 case in Example 3.5, the contributions from 15 graph types are (here \( G_i \) is the contribution from the graph \( \Gamma_i \) in Figure 2):

\[
G_1 = N_{1,2}, \quad G_2 = \frac{1106875}{6}, \quad G_3 = \frac{1331125}{12}, \quad G_4 = -\frac{5334375}{2}, \quad G_5 = \frac{-17206775}{12}, \quad G_6 = 355000, \quad G_7 = -\frac{1018850}{3}, \quad G_8 = \frac{6806875}{4}, \quad G_9 = -\frac{12896875}{8}, \quad G_{10} = 782000, \quad G_{11} = \frac{2896600}{3}, \quad G_{12} = \frac{-4680000}{3}, \quad G_{13} = -\frac{9934400}{3}, \quad G_{14} = -\frac{23116864}{3}, \quad G_{15} = 12288.
\]

From the equation (4.1), the sum of the fifteen numbers above being zero leads to \( N_{1,2} = 407125/8 \). This is the mathematically verified number by Zinger in [Zi].

4.1. **Speculations.** Let us look at Theorem 4.1 from a different aspect. Inductively we may suppose all GW/FJRW invariants for genus less than \( g \) are known. Then for genus \( g \), Theorem 4.1 reduces the problem of determining the infinitely many GW invariants \( \{ N_{g,d} \}_d \) to two finite sets of initial datum

\[
\{ N_{g,1}, \cdots, N_{g,g-1} \} \quad \text{and} \quad \{ \Theta_{g,k} \}_{k \leq 2g-2}.
\]

We formulate the following speculation:

*By suitable choice of positive \( d_0 \) and \( d_\infty \), the relations (4.1) provide an effective algorithm to determine the first set of initial data \( \{ N_{g,1}, \cdots, N_{g,g-1} \} \).*

If this is true, then one is left to determine the second set of initial data \( \{ \Theta_{g,k} \}_{k \leq 2g-2} \). We propose another conjecture about fully determining all FJRW invariants for the quintic,
**Conjecture 4.5.** The equations (4.1) using $d_0 = 0$ and nonempty $\gamma$’s (i.e. with markings) give relations that, together with Theorem 4.3, effectively evaluate all $\Theta_{g,k}$.

We have verified this conjecture for the case $\Theta_{2,2}$. Recall that for the case of genus 2, this is the only undetermined invariant in Theorem 4.1.

4.2. Other approaches. The other approach to Witten’s proposal is the recent work of Fan, Jarvis and Ruan [FJR3]. They worked on more general context of gauged linear sigma model, where more general groups $G$ were involved. In [FJR3 Ex.4.2.23], they took $G = \mathbb{C}^* \times \mathbb{C}^*$ and combined the quasi-map technique with the P-fields theory to set up the moduli space. Incidentally, a closed point of the moduli also consists of a pointed twisted curve $\Sigma \subset \mathbb{C}$, two line bundles $L$ and $N$, and a collection of sections. We point out that despite the similarity, the approach of [FJR3] is different from ours.

The theory in [FJR3] uses the concept of $\epsilon$-stability, dependent on the real parameter $\epsilon$, similar to the case of stable quotient [MOP]. The moduli for $\epsilon = 0^+$ was constructed in [FJR3]; the case for GW-theory is when $\epsilon = +\infty$, which is yet to be constructed. For $\epsilon = 0^+$ moduli space, the stability (on a point $(\mathbb{C}, L, N, \cdots)$) requires that $L^{-e_1} \otimes N^{-e_2}$ is ample on those components of $\mathbb{C}$ for which $\omega^\log_{\mathbb{C}}$ has degree zero, where $0 < e_1 < e_2$. Coming back to Example 3.5, we see that in $\Gamma_3$ of Figure 2, for the edge $E$ connecting a genus 1 curve with a genus zero curve, we have $\deg(\omega^\log_L) = 0$, $N|_E \cong 0$ and $\deg(L|_E) = 1$. Thus $\deg((L^{-e_1} \otimes N^{-e_2})|_E) < 0$. So curves with the graph type $\Gamma_3$ in Figure 2 will not be in $\epsilon = 0^+$ moduli space of [FJR3 Ex.4.2.23].

The $\theta$-parameter in [FJR3] may resemble the $r$-parameter in Witten’s vision of CY/LQ correspondence. In our approach, we introduced the new field $\nu = [\nu_1, \nu_2]$ in order to “quantize” the Witten’s parameter in his phase transition between Calabi-Yau and Landau-Ginzberg theories. We believe that MSP field theory will provide a mathematical theory to realize the vision of Witten. We hope that both approaches will be useful for eventual understanding of CY/LQ correspondence in realizing Witten’s vision that “along a suitable path, there may well be a sharply defined phase transition.”

Another approach is by Choi and Kiem. In [ChK], they introduced the moduli spaces of $\epsilon$-stable quasi-maps with P-fields similar to [FJR3] and introduced additional $\delta$-stability to make each wall-crossing more manageable.

**References**

[ACV] D. Abramovich, A. Corti and A. Vistoli, *Twisted bundles and admissible covers*, Special issue in honor of Steven L. Kleiman. Comm. Algebra 31, no. 8, 3547-3618 (2003)

[AF] D. Abramovich and B. Fantechi, *Orbifold techniques in degeneration formulas*, preprint, math.AG. arXiv:1103.5152v2

[AGV] D. Abramovich; T. Graber; A. Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks*. Amer. J. Math. 130, no. 5, 1337-1398 (2008)
[AJ] D. Abramovich, T. J. Jarvis, Moduli of twisted spin curves, Proc. Amer. Math. Soc. 131, no. 3, 685-699 (2003)

[BF] K. Behrend and B. Fantechi, The intrinsic normal cone, Invent. Math. 128, no. 1, 45-88 (1997)

[BCOV] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Holomorphic Anomalies in Topological Field Theories, Nucl.Phys. B 405 279-304 (1993); Kodaira-Spencer Theory of Gravity and Exact Results for Quantum String Amplitudes, Comm. Math. Phys. Volume 165, no. 2, 311-427 (1994)

[COGP] P. Candelas, X. dela Ossa, P. Green, and L. Parkes, A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory, Nucl. Phys. B 359 21-74 (1991)

[CKL] H.L. Chang, Y.H. Kiem and J. Li, Torus localization and wall crossing for cosection localized virtual cycles, math.AG. arXiv:1502.00078

[CL1] H.-L. Chang and J. Li, Gromov-Witten invariants of stable maps with fields, Int. Math. Res. Not. 2012, 18, 4163–4217 (2012)

[CL2] H.-L. Chang and J. Li, A vanishing for localizing MSP moduli of quintic, in preparation

[CLL] H.-L. Chang, J. Li, W.-P. Li, Witten's top Chern classes via cosection localization, Inventiones mathematicae, 200, no. 3, 3022-3051 (2015)

[CLLL] H.-L. Chang, J. Li, W.-P. Li, C.-C. Melissa Liu, Mixed-Spin-P fields of Fermat quintic polynomials, math.AG. arXiv:1505.07532

[CLLL2] H.-L. Chang, J. Li, W.-P. Li, C.-C. Melissa Liu, Toward an effective theory of GW invariants of quintic threefolds,

[twFJRW] H.-L. Chang, J. Li, W.-P. Li, C.-C. Melissa Liu, Dual twisted FJRW invariants of quintic singularity via floating MSP fields

[ChK] J-W. Choi and Y-H. Kiem, Landau-Ginzburg/Calabi-Yau correspondence via quasi-maps, I, preprint.

[Chi] A. Chiodo, Towards an enumerative geometry of the moduli space of twisted curves and r-th roots, Com- pos. Math. 144, no. 6, 1461-1496 (2008)

[CR] A. Chiodo and Y.-B Ruan, Landau-Ginzburg/Calabi-Yau correspondence for quintic three-folds via symplectic transformations, Invent. Math. 182, no. 1, 117-165 (2010)

[CK] I. Ciocan-Fontanine and B. Kim, Moduli stacks of stable toric quasimaps, Advances in Math. 225, no. 6, 3022-3051 (2010)

[Cad] C. Cadman, Using stacks to impose tangency conditions on curves. Amer. J. Math. 129, no. 2, 405-427 (2007)

[Ch] A. Chiodo, The Witten top Chern class via K-theory, J. Algebraic Geom. 15, no. 4, 681-707 (2006)

[CZ] A. Chiodo and D. Zvonkine, Twisted r-spin potential and Givental's quantization, Advances in Theoretical and Mathematical Physics 13, no. 5, 1335-1369 (2009)

[FJR1] H.-J. Fan, T. J. Jarvis, Y.-B Ruan, The Witten equation, mirror symmetry, and quantum singularity theory, Ann. of Math (2) 178, no. 1, 1-106 (2013)

[FJR2] H-J Fan, T. J. Jarvis and Y-B Ruan, The Witten equation and its virtual fundamental cycle, math.AG. arXiv:0712.4025

[FJR3] H-J Fan, T. J. Jarvis and Y-B Ruan, A Mathematical Theory of the Gauged Linear Sigma Model, math.AG. arXiv:1506.02109

[Gath] A. Gathmnn, Absolute and relative Gromov-Witten invariants of very ample hypersurfaces, Duke, 115, no. 2, 171-203 (2002)

[Gi] A. Givental, Equivariant Gromov-Witten invariants, Internat. Math. Res. Notices 1996, no. 13, 613-663 (1996)

[GP] T. Graber, R. Pandharipande, Localization of virtual classes, Invent. Math. 13, no. 2, 487-518 (1999)

[GS] J. Guffin and E. Sharpe, A-twisted Landau-Ginzburg models, hep-th/arXiv:0801.3836
[HKQ] M.X. Huang, A. Klemm, and S. Quackenbush, Topological String Theory on Compact Calabi-Yau: Modularity and Boundary Conditions. Lecture Notes in Phys. 757, 45-102 (2009)

[Huy] D. Huybrechts, Fourier-Mukai transforms in algebraic geometry. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford (2006)

[JK] T. Jarvis and T. Kimura, Orbifold quantum cohomology of the classifying space of a finite group, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math. 310, 123-134 Amer. Math. Soc., Providence, RI, (2002)

[KKP] B. Kim, A. Kresch and T. Pantev, Functoriality in intersection theory and a conjecture of Cox, Katz, and Lee, J. Pure Appl. Algebra 179, no. 1-2, 127-136 (2003)

[KL] Y.H. Kiem and J. Li, Localized virtual cycle by cosections, J. Amer. Math. Soc. 26, no. 4, 1025-1050 (2013)

[Kr2] A. Kresch, Cycle groups for Artin stacks, Invent. Math. 138, no. 3, 495-536 (1999)

[LM] G. Laumon and L. Moret-Bailly, Champs algébriques. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 39, Berlin: Springer-Verlag, (2000)

[LR] A.M. Li, and Y.B. Ruan, Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds, Invent. Math. 145, no. 1, 151-218 (2001)

[Lideg] J. Li, A Degeneration Formula of GW-Invariants, J. of Differential Geom. 60, no. 2, 177-354 (2002).

[LT] J. Li and G. Tian, Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties, J. Amer. Math. Soc. 11, no. 1, 119-174 (1998)

[LZ] J. Li and A. Zinger, On the Genus-One Gromov-Witten Invariants of Complete Intersections, J. of Differential Geom. 82, no. 3, 641-690 (2009)

[LLY] B. Lian, K.F. Liu and S.T. Yau, Mirror principle. I , Asian J. Math. 1, no. 4, 729-763 (1997)

[MOP] A. Marian, D. Oprea and R. Pandharipande. The moduli space of stable quotients, Geom. Topol. 15, no. 3, 1651-1706 (2011)

[MP] D. Maulik, and R. Pandharipande, A topological view of Gromov-Witten theory, Topology 45, no. 5, 887-918 (2006)

[PV] A. Polishchuk and A. Vaintrob, Algebraic construction of Witten’s top Chern class. Advances in algebraic geometry motivated by physics (Lowell, MA, 2000), Contemp. Math. 276, 229-249, Amer. Math. Soc., Providence, RI, (2001)

[Wi] E. Witten, Phases of N = 2 theories in two dimensions, Nuclear Physics B 403, no. 1-2, 159-222 (1993)

[Zi] A. Zinger, Standard versus reduced genus-one Gromov-Witten invariants, Geom. Topol. 12, no. 2, 1203-124 (2008)

[Zi2] A. Zinger, The reduced genus 1 Gromov-Witten invariants of Calabi-Yau hypersurfaces., J. Amer. Math. Soc. 22, no. 3, 691-737 (2009)
Department of Mathematics, Hong Kong University of Science and Technology, Hong Kong
E-mail address: mahlchang@ust.hk

Shanghai Center for Mathematical Sciences, Fudan University, China; Department of Mathematics, Stanford University, USA
E-mail address: jli@math.stanford.edu

Department of Mathematics, Hong Kong University of Science and Technology, Hong Kong
E-mail address: mawpli@ust.hk

Mathematics Department, Columbia University
E-mail address: ccliu@math.columbia.edu