Improving security and bandwidth efficiency of NewHope using error-correction schemes *

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Abstract. Among many submissions to the NIST post-quantum cryptography (PQC) project, NewHope is a promising key encapsulation mechanism (KEM) based on the Ring-Learning with errors (Ring-LWE) problem. Since the most important factors to be considered for PQC are security and cost including bandwidth and time/space complexity, in this paper, by doing exact noise analysis and using Bose Chaudhuri Hocquenghem (BCH) codes, it is shown that the security and bandwidth efficiency of NewHope can be substantially improved. In detail, the decryption failure rate (DFR) of NewHope is recalculated by performing exact noise analysis, and it is shown that the DFR of NewHope has been too conservatively calculated. Since the recalculated DFR is much lower than the required $2^{-128}$, this DFR margin is exploited to improve the security up to 8.5 % or the bandwidth efficiency up to 5.9 % without changing the procedure of NewHope.

The additive threshold encoding (ATE) used in NewHope is a simple error correcting code (ECC) robust to side channel attack, but its error-correction capability is relatively weak compared with other ECCs. Therefore, if a proper error-correction scheme is applied to NewHope, either security or bandwidth efficiency or both can be improved. Among various ECCs, BCH code has been widely studied for its application to cryptosystems due to its advantages such as no error floor problem. In this paper, the ATE and total noise channel are regarded as a super channel from an information-theoretic viewpoint. Based on this super channel analysis, various concatenated coding schemes of ATE and BCH code for NewHope have been investigated. Through numerical analysis, it is revealed that the security and bandwidth efficiency of NewHope are substantially improved by using the proposed error-correction schemes.

Keywords: Bandwidth Efficiency · BCH Code · Decryption Failure Rate · Error Correcting Codes · NewHope · NIST · Post-Quantum Cryptography · Security

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1 Introduction

Current public-key algorithms based on integer decomposition, discrete logarithm, and elliptic curve discrete logarithm problems (e.g., RSA and elliptic curve encryption) have been unlikely to be broken by currently available technology. However, with the help of upcoming quantum computing technology such as Shor’s quantum algorithm for integer factorization, current public-key algorithms can be easily broken. For that reason, in order to avoid the security problem of future network, new public-key algorithms called post-quantum cryptography (PQC) should be developed to replace the existing public-key algorithms. Therefore, the National Institute of Standards and Technology (NIST) has recently begun a PQC project to identify and evaluate post-quantum public-key algorithms secure against quantum computing [1]. Among the various PQC candidates, lattice-based cryptosystems have become one of the most promising candidate algorithms for post-quantum key exchange. Lattice-based cryptosystems have been developed based on worst-case assumptions about lattice problems that are believed to be resistant to quantum computing. Among various lattice problems, learning with errors (LWE) problem introduced by Regev in 2005 [2] has been widely analyzed and used. Furthermore, the Ring-LWE problem presented by Linbashevsky, Peikert, and Regev in 2010 [3], which improves the computational and implementation efficiency of LWE, has also been widely used [4], [5], [6], [7], [8]. NewHope has been proposed by Alkim, Ducas, Pöppelmann, Schwabe [9], [10] which is one of the various cryptosystems based on Ring-LWE. NewHope has attracted a lot of attention [11], [12], [13] and it was verified in an experiment of Google [14]. The key reasons that NewHope attracts so much attention are the use of simple and practical noise distribution, a centered binomial distribution, and a proper choice of ring parameters for better performance and security.

NewHope is an indistinguishability (IND)-chosen ciphertext attack (CCA) secure key encapsulation mechanism (KEM) that exchanges the shared secret key based on the IND-chosen plaintext attack (CPA) secure public-key encryption (PKE). Note that the IND-CPA PKE can be transformed into the IND-CCA KEM using Fujisaki-Okamoto (FO) transform [15]. The IND-CCA secure KEM obtained by applying FO transform to IND-CPA secure PKE requires a very low decryption failure rate (DFR) because an attacker can exploit the decryption error [15]. Therefore, the DFR in NewHope should be lower than $2^{-128}$ to make sure of resilience against attacks that exploit decryption errors. As in Frodo [5] and Kyber [6], this study aims to achieve the DFR lower than $2^{-140}$ to allow enough margin in NewHope.

The DFR of NewHope is most influenced by the noise parameter $k$ of centered binomial distribution and modulus $q$, and is also closely related to an error correcting code (ECC). NewHope uses an additive threshold encoding (ATE) as an ECC, which is almost similar to the repetition code used in digital communication systems [16]. In addition, ATE has an advantage of being robust to the side channel attack, but ATE inherently has much worse error-correction capability than other ECCs such as Bose Chaudhuri Hocquenghem (BCH) code,
low-density parity-check (LDPC) codes, and turbo codes [17], [18]. In [13], it was shown that by applying more powerful ECC to NewHope instead of ATE, the DFR can be improved. It is also shown that the security can be improved and the size of ciphertext can be reduced by using the improved DFR obtained by using ECC.

**Contribution** The contributions of this paper is divided into six categories.

- **Exact Noise Analysis of NewHope** NewHope can be understood as a digital communication system. Bob and Alice are transmitter and receiver, respectively, and the 256-bit shared secret key is a message bit stream. The difference between the ATE output $v$ and the received signal $v'$ distorted by many factors can be modeled as a digital communication channel. We analyze all the noise sources of this channel and numerically calculate the exact noise distribution of NewHope.

- **Recalculation of DFR of NewHope** The DFR of NewHope is recalculated as $2^{-474}$ and $2^{-431}$ for $n = 1024$ and $n = 512$, respectively, based on the exact noise analysis and theoretical analysis of ATE. New DFR values show that the DFR of NewHope has been too conservatively calculated.

- **Improvement of Security and Bandwidth Efficiency of NewHope Using New DFR Margin** Since the recalculated DFR is much lower than the required $2^{-140}$, this DFR margin is exploited to improve the security by 8.5% or bandwidth efficiency by 5.9% without changing the procedure of NewHope. If a slight increase in time/space complexity is allowed, the bandwidth efficiency can be improved by 23.5%.

- **An Information-theoretic View of Noise in NewHope as a Super Channel** The ATE and total noise of NewHope can be regarded as a super channel from an information-theoretic viewpoint. For various concatenated coding schemes of ATE and BCH code for NewHope, super channel is defined. We perform exact analysis of super channel for each concatenated coding schemes by using exact analysis of noise and ATE.

- **Proposed Concatenated Coding Schemes** The ATE used in NewHope is simple and robust to the side channel attack. However, since ATE is based on the repetition code, it shows low error-correction capability. In order to improve the security and bandwidth efficiency, advanced ECCs should be applied to NewHope. BCH codes are suitable for PQC because they do not show an error floor problem and can be analyzed by deriving an upper bound of block error rate (BLER). Based on extensive simulations, we select four concatenated coding schemes of ATE and BCH code to combine the advantages of these two ECCs.

- **Analysis of Security and Bandwidth Efficiency of NewHope with the Proposed Error-correction Schemes** The results of numerical analysis show that the proposed concatenated coding schemes of ATE and BCH code can improve the security level by 21.5% or reduce the ciphertext size by 41.5% while achieving the required DFR $2^{-140}$ for NewHope with $n = 1024$. Also, for NewHope with $n = 512$, the proposed concatenated coding schemes can
improve the security level by 22.8% or reduce the ciphertext size by 35.3% while achieving the required DFR $2^{-140}$.

The contributions of this paper differ from the contributions of [13] in three ways. First, we interpret NewHope as a communication system over noisy channel from an information-theoretic view, and hence a super channel is defined and analyzed in performing exact analysis of noise and ATE. Based on this analysis, we recalculate the exact DFR of NewHope. Second, we show that by using the recalculated DFR, the security and bandwidth efficiency of NewHope can be improved without changing the procedure of NewHope. Finally, compared with [13], we numerically evaluate more various compression rates of ciphertext using the super channel analysis. Through this evaluation, we present many effective compression rates show that better performance in reducing the size of the ciphertext and decreasing show lower DFR.

2 Preliminaries

2.1 NewHope

Parameters There are three important parameters in NewHope: $n$, $q$, and $k$. The dimension $n = 512$ or 1024 for NewHope guarantees the security properties of Ring-LWE and enables efficient number theoretic transform (NTT) [19]. The modulus $q = 12289$ is determined to support security and efficient NTT, and is closely related with the bandwidth. The noise parameter $k = 8$ is the parameter of centered binomial distribution, which determines the noise strength and hence directly affects the security and DFR [4].

Notations Let $\mathcal{R}_q = \mathbb{Z}_q[x]/(X^n + 1)$ be the ring of integer polynomials modulo $X^n + 1$ where each coefficient is reduced modulo $q$. Let $a \xleftarrow{\$} \chi$ be the sampling of $a \in \mathcal{R}_q$ following the probability distribution $\chi$ over $\mathcal{R}_q$. Let $\psi_{k}$ denote the centered binominal distribution with parameter $k$, which is practically realized by $\sum_{i=0}^{k-1}(b_i - b'_i)$, where $b_i$ and $b'_i$ are uniformly and independently sampled from $\{0, 1\}$. The variance of $\psi_{k}$ is $k/2$ [4]. $a \circ b$ denotes the coefficient-wise product of polynomials $a$ and $b$.

NewHope Protocol NewHope is a lattice-based KEM for Alice (Server) and Bob (Client) to share 256-bit secret key with each other. The protocol of NewHope is briefly expressed based on Fig. 1 as follows, where the functions are the same ones as defined in [4].

Step 1) $\text{seed} \xleftarrow{\$} \{0, 1, \ldots, 255\}^{32}$ denotes a uniformly sampling of 32 byte arrays (corresponding to 256 bits) with 32 integer elements selected between 0 and 255 by using a random number generator. Then $SHAKE256(l, d)$, a strong hash function [20], takes an integer $l$ that specifies the number of output bytes and an data byte array $d$ as inputs. In NewHope, $z \leftarrow SHAKE256(64, seed)$
Fig. 1: NewHope Protocol.

denotes that 32 byte arrays seed are hashed to generate a pseudorandom 64 byte arrays $z$ with 64 integer elements uniformly selected between 0 and 255. Then GenA expands the pseudorandom 32 byte arrays $z[0:31]$ using SHAKE128 hash function [20] to generate the polynomial $\hat{a} \in \mathcal{R}_q$ where $z[0:31]$ is the first 32 byte arrays of $z$. Since $\hat{a}$ is generated from the seed sampled from a uniform distribution, the coefficients of $\hat{a}$ also follow a uniform distribution on $[0, q - 1]$.

Step 2) Generate polynomials $(s, s', e, e', e'' \in \mathcal{R}_q)$ whose coefficients are sampled from the centered binomial distribution $\psi_k$. The polynomials $(s, s', e)$ are transformed to $(\hat{s}, \hat{t}, \hat{e})$, respectively, by applying NTT for efficient polynomial multiplication. Then Alice transforms the secret key $(\hat{s})$ into byte arrays using EncodePolynomial(), which converts the polynomial $(\hat{s})$ into 2048 byte arrays.

Step 3) Alice creates a public key $(pk)$ by converting $\hat{b} = \hat{a} \circ \hat{s} + \hat{e}$ and $z[0:31]$ into 1824 byte arrays using EncodePK(), and transmits $(pk)$ to Bob. Then Bob transforms the received public key $(pk)$ into $(\hat{b}, z[0:31])$ using DecodePK(), and creates $(\hat{a})$, which is the same $(\hat{a})$ generated in Step 1.

Step 4) A 256-bit shared secret key $(\mu)$ is created by performing ATE encoding.

Step 5) Generate a ciphertext $(\hat{u}, v')$ using the public key and $(\mu)$.

Step 6) To efficiently reduce bandwidth, compression is performed on the coefficients of $v'$ to generate the polynomial $h$, and then the ciphertext polynomials
$(\hat{u}, \hat{h})$ are transformed into the byte arrays $c$ using $\text{EncodeC}()$, and $c$ is transmitted to Alice. Alice performs decompression on $\hat{h}$ to restore $v'$. However, this restored polynomial $v'_{\text{decomp}}$ is different from $v'$ generated in Step 5, due to the compression and decompression. Alice creates $v''$ using the received ciphertext $c$ and $sk$ generated in Step 2. Each coefficient of $v''$ is a sum of the corresponding coefficient of $v$ and noise. Note that $v''$ is not a polynomial used in NewHope, for easy explanation of the results in this paper, $v''$ is added in Fig. 1.

Step 7) The 256-bit shared secret key $(\mu)$ is recovered (or decrypted) from the coefficients of $v''$ by performing ATE decoding.

### 2.2 BCH Codes

BCH codes are algebraic block codes widely used in digital communications and storage systems. Unlike other advanced ECCs such as low-density parity-check (LDPC) codes and Turbo codes, BCH codes do not show an error floor problem and thus can achieve the required DFR ($\approx 2^{-140}$) \cite{17}, \cite{18}. Also, BCH codes can be theoretically analyzed whether the required DFR is achieved or not. Therefore, BCH codes are ECCs suitable for PQC In addition, various research activities \cite{7}, \cite{13}, \cite{21}, \cite{22}, \cite{23} are currently underway to apply BCH codes to the cryptosystem.

A BCH code is usually denoted by $\text{BCH}(n, k, t)$ where $n$ is the code length, $k$ is the dimension, and $t$ is the error-correction capability. The code length $n$ of the primitive $q$-ary BCH code is $q^n - 1$, $m = 3, 4, ...$, and the code length can be adjusted by applying many modification methods such as shortening and puncturing. The dimension $k$ denotes the number of message symbols in a codeword and $t$ is the maximum number of errors that are always corrected \cite{17}, \cite{18}.

### 3 Improving NewHope Based on Exact Noise Analysis and DFR Recalculation

#### 3.1 An Information-theoretic View of NewHope

In order to properly apply ECC to NewHope and facilitate analysis, it is necessary to understand the protocol of NewHope via an information-theoretic approach. For NewHope, the mapping $\mathbb{Z}_2 \rightarrow \mathbb{R}_2$ and the mapping $\mathbb{R}_2 \rightarrow \mathbb{Z}_2$ can be regarded as encoding and decoding of ECC, respectively. Also, the mapping $\mathbb{R}_2 \rightarrow \mathbb{R}_q$ and $\mathbb{R}_q \rightarrow \mathbb{R}_2$ can be regarded as modulation and demodulation, respectively. Then NewHope can be understood as a digital communication system as follows. Bob and Alice are transmitter and receiver, respectively, and the 256-bit shared secret key $(\mu)$ is a message bit stream. Also, the process of transmitting and receiving messages (Steps 4, 5, 6, and 7) can be viewed as a digital communication channel. In more detail, the transmitter (Bob) generates a 256-bit message bit stream, encodes this message, modulates the codeword bits to the symbols of $\mathbb{Z}_q$ and transmits the resulting signal (Step 4). At the
receiver (Alice), the received signal through the noisy channel is demodulated and decoded (Step 7). For NewHope, a process of adding the compression noise and the difference noise generated in Steps 5 and 6 can be regarded as noisy communication channel. This overall process in Steps 4-7 can be described as a digital communication system shown in Fig. 2.

Fig. 2: An information-theoretic view of NewHope as a digital communication system $n = 512$ or $n = 1024$.

In Fig. 2, $\mu_{\text{enc}}$ is the resulting signal after applying ATE to $\mu$, and $n_t$ represents the overall noise generated in Steps 5 and 6, which is called the total noise. Since ATE simultaneously performs demodulation and decoding, demodulation and decoding are combined into one block. After interpreting NewHope as a digital communication system, the DFR in NewHope is equivalent to the block error rate (BLER), i.e., $Pr(\mu \neq \mu')$, in a digital communication system. Therefore, in order to accurately calculate the DFR of NewHope, exact analysis of the noisy channel is required.

### 3.2 Exact Noise Analysis of NewHope and DFR Recalculation

**Difference Noise, Compression Noise, and Total Noise Analysis**

Total noise $n_t$ is defined as the noise contained in the received 256-bit shared secret key before demodulation. Total noise of the $i$th coefficient $n_{t,i}$ of the polynomial $v''$ in Step 6 is represented as follows.

$$n_{t,i} = (v'' - v)_i$$

$$= (v_{\text{decomp}} - us - v)_i$$

$$= (v' + n_c - us - v)_i$$

$$= (bs' + e'' - ass' - e's)_i + n_{c,i}$$

$$= (es' - e's + e'')_i + n_{c,i}$$

$$= n_{d,i} + n_{c,i}$$

where $(\cdot)_i$ denotes the $i$th coefficient of polynomial, $n_c \in \mathcal{R}_q$ is the compression noise polynomial, $n_{c,i}$ is the $i$th coefficient of compression noise polynomial in $v''$, and $n_{d,i}$ is the $i$th coefficient of difference noise polynomial in $v''$. 
To analyze the compression noise $n_{c,i}$, we first need to investigate the coefficient of the polynomial $v' = ass' + es' + e''$ being compressed, where the coefficients of $s$, $s'$, $e$, and $e''$ follow the predetermined centered binomial distribution. However, since the coefficients of polynomial $a$ follow a uniform distribution, the coefficient of the compressed polynomial $h$ will eventually follow a uniform distribution. A compression to $v'$ is performed by applying $⌊v' * r_{v'}/q⌋$ to the coefficients $v'_i$ of $v'$ to obtain a polynomial $h$, where $⌊·⌋$ is rounding function that rounds to the closest integer, $r_{v'}$ denotes the compression rate on $v'$, and $r_{v'} = 8$ for NewHope. Then the range of the compressed coefficients $h_i$ of $h$ is changed from $[0, q - 1]$ to $[0, r_{v'} - 1]$ so that the number of bits required to store a coefficient is reduced from 14 bits ($= ⌈\log_2 q⌉$) for $v'$ to 3 bits ($= ⌈\log_2 r_{v'}⌉$) for $h$. A decompression is performed by applying $⌊h_i * q/r_{v'}⌋$ to each of the coefficients of $h$. Then the coefficient takes the value from $0, [q/r_{v'}], [2q/r_{v'}] \ldots, [(r_{v'} - 1) \cdot q/r_{v'}]$. This compression and decompression are illustrated in Fig. 3, where the coefficients $v'_i$ of $v'$ from different patterns (or ranges) are mapped to different $v_{\text{decomp},i}$ values through compression and decompression. In the end, compression and decompression can be seen as a rounding operation. Therefore, the compression noise is inevitably generated with the maximum magnitude $[q/2r_{v'}]$ and the distribution of the compression noise is derived as follows:

$$n_{c}[x] = \begin{cases} 
\frac{q}{r_{v'}}, & 0 \leq x \leq \left\lfloor \frac{q}{2r_{v'}} \right\rfloor - 1 \\
0, & \text{otherwise} \\
\frac{q}{r_{v'}} - q - \left\lfloor \frac{q}{2r_{v'}} \right\rfloor \leq x \leq q - 1
\end{cases}$$

Fig. 3: Mapping corresponding to each of compression and decompression in NewHope.

In total noise, $n_{d,i} = (es' - e's + e'')i$, is defined as the $i$th coefficient of the difference noise polynomial where the coefficients of $e$, $e'$, $e''$, $s$, and $s'$ follow the same centered binomial distribution. In order to derive the distribution of difference noise, a number of convolution operations are required because the difference noise is a sum of many random variables, each of which is obtained by multiplying two random variables that follow the centered binomial distribution. However, since it is difficult to calculate the multiple convolutions of the above
distribution in closed form, the distribution of difference noise is numerically calculated [13].

Total noise is a sum of compression noise and difference noise which are independently generated. Thus, the distribution of total noise is obtained by performing convolution of the distributions of compression noise and difference noise as shown in Fig. 4.

\[ n_{t,i} = n_{d,i} + n_{c,i} - (n_u s)_i \]  

where \( n_u \in \mathbb{R}_q \) is the compression noise polynomial generated while the compression and decompression to \( \hat{u} \).

The coefficients of the polynomial \( \hat{u} = \hat{a} \circ t + NTT(c') \) follow a uniform distribution because the coefficients of \( \hat{a} \) follow a uniform distribution. A compression to \( \hat{u} \) is performed by applying \( \lfloor \hat{a} \cdot r_{\hat{u}} / q \rfloor \) to the coefficients \( \hat{u}_i \) of \( \hat{u} \), where \( r_{\hat{u}} \) denotes the compression rate on \( \hat{u} \) and \( \hat{u} = q \) for NewHope.

Fig. 5 shows the total noise distribution for the various compression rates of \( r_{v'} \) and \( r_{\hat{u}} \). Note that the total compression noise of NewHope is obtained by \( (r_{v'} = 8, r_{\hat{u}} = 8) \). \( \hat{u} \) cannot be compressed as much as \( v' \) because the compression noise of \( \hat{u} \) is multiplied by \( s \) to affect total noise as shown in (8).
DFR Recalculation The DFR of NewHope calculated in [4] is defective because the compression noise \( n_c \) is not considered and the centered binomial distribution is approximated by Gaussian distribution. In addition, ATE is not considered, which clearly affects the DFR of NewHope, and an upper bound of the DFR is derived using the Chernoff-Cramer bound instead of doing exact calculation [4], [9]. Also, instead of calculating \( \Pr(\mu' \neq \mu) \) of Fig. 2 as DFR, \( \Pr(v_i \neq v_i'') \) of Fig. 2 was calculated. Therefore, current DFR values of NewHope, which are DFR \(< 2^{-213} \) for \( n = 512 \) and DFR \(< 2^{-216} \) for \( n = 1024 \), are not correct and hence it is necessary to recalculate accurate DFR values of NewHope based on exact noise analysis.

To recalculate the DFR of NewHope, it is assumed that the coefficients of polynomials are statistically independent to each other because it is shown by experiments in [13] that there is almost no influence on DFR by the correlation of coefficients. Since it is typical to give a margin to the target DFR, the DFR requirement is usually set to \( 2^{-140} \) instead of \( 2^{-128} \).

In the previous section, the total noise \( n_t \) is thoroughly analyzed. In NewHope, ATE is used to encode and decode a message bit \( \mu_i \), and the encoding (including modulation) and decoding (including demodulation) procedures of ATE with \( m \) repetitions are shown in Fig 6 [10]. The encoding/modulation of ATE is performed such that one message bit \( \mu_i \) is repeated \( m \) times and each of them is mapped to the element of \( \mathbb{Z}_q \) (usually either 0 or \( \lfloor \frac{q}{2} \rfloor \)) as the coefficients of \( v \). The demodulation/decoding of ATE sums up the \( m \) absolute values of the differences of \( \frac{q}{2} \) and the \( m \) received coefficient values of \( v'' \) corresponding to the \( m \)
repeated coefficients. Then, the sum $v''_s$ is compared with $m \cdot q/4$ to determine whether the message bit $\mu_i$ is 0 or 1 as follows.

$$v''_s \geq \frac{m \cdot q}{4}$$  \hspace{1cm} (9)

Compared with the repetition code, ATE has the same encoding/modulation process, but uses different demodulation/decoding which is simple and effective because the square operation is not required for demodulation/decoding. Therefore, although the error-correction capability of ATE is worse than those of other advanced ECCs, ATE still works well as an ECC for PQC such as NewHope.

**Fig. 6: Encoding and decoding of ATE for NewHope.**

$Pr(\mu'_i \neq \mu_i)$ is called bit error rate (BER) of $\mu_i$ for $i = 0, 1, \ldots, 255$ as follow. First, the error distribution $P_{v''}$ of $v''_i$ is calculate as follow.

$$P_{v''} = Pr\left(\left|v''_i - \frac{q}{2}\right| < \frac{m \cdot q}{4} \mid v_i = \lfloor \frac{q}{2} \rfloor\right)Pr\left(v_i = \lfloor \frac{q}{2} \rfloor\right) + Pr\left(\left|v''_i - \frac{q}{2}\right| \geq \frac{m \cdot q}{4} \mid v_i = 0\right)Pr\left(v_i = 0\right)$$  \hspace{1cm} (10)

Second, the error distribution $P_{\mu''}$ of $\mu_i$ is calculated as follow.

$$P_{\mu''} = P_{v''} \otimes P_{v''} \otimes \cdots \otimes P_{v''}$$  \hspace{1cm} (12)

where $\otimes$ is the convolution operation and the range of $P_{\mu''}$ is $[0, m \cdot q - 1]$. Let $BER_m$ be BER when the ATE with $m$ repetition is used. $BER_m$ is calculated as follow.
\[ BER_m = P_{\mu'}(x < \frac{m \cdot q}{4}) Pr(\mu_i = 1) + P_{\mu'}(x \geq \frac{m \cdot q}{4}) Pr(\mu_i = 0) \]  

(13)

where \( Pr(\mu_i = 0) = Pr(\mu_i = 1) = \frac{1}{2} \).

Since we assume that the coefficients of polynomial \( v'' \) are independent,
\( Pr(\mu'_i \neq \mu_i) = Pr(\mu'_j \neq \mu_j) \) for \( i, j = 0, 1, \ldots, n - 1 \). Therefore, BLER, which is in fact the DFR, of ATE with \( m \) repetitions in NewHope is calculated as \( 1 - (1 - BER_m)^{256} \).

Based on the above analysis, the DFR of NewHope is recalculated as \( 2^{-474} \) and \( 2^{-431} \) for \( n = 1024 \) and \( n = 512 \), respectively. These DFR values are much smaller than the corresponding DFR values \( 2^{-216} \) and \( 2^{-213} \) from the NewHope specification [4]. Therefore, it is possible to improve the security and bandwidth efficiency of NewHope only by utilizing this DFR margin.

### 3.3 Improved Security and Bandwidth Efficiency of NewHope Based on Recalculated DFR

**Improved Security** Since there exists a trade-off relation between security level and DFR, it is necessary to properly select the noise parameter \( k \) of centered binomial distribution such that the security level and the DFR are appropriately determined. Even if NewHope is designed to have an extremely low DFR, the security level can be more improved by using new DFR margin obtained from recalculated DFR.

| \( n \) | \( k \) | DFR | Cost of primal attack | Cost of dual attack |
|---|---|---|---|---|
| \( 1024 \) | 8 | \( \approx 2^{-474} \) | 259/235 | 257/233 |
| 15 | \( \approx 2^{-140} \) | 280/254 | 279/253 |
| \( 512 \) | 8 | \( \approx 2^{-431} \) | 112/101 | 112/101 |
| 14 | \( \approx 2^{-154} \) | 121/110 | 121/110 |

Table 1 shows the improved security levels which are calculated by assuming cost of the primal attack and dual attack [24] to NewHope. It is possible to improve the security level by 8.0 % \( (n = 1024, k = 15) \) and 8.9 % \( (n = 512, k = 14) \) while keeping the required DFR of \( 2^{-140} \) compared with the current NewHope. Note that such security level improvement does not require too much increase of time/space complexity in NewHope because it only changes the noise parameter \( k \) without any additional process.
Improved Bandwidth Efficiency The bandwidth efficiency of NewHope can also be improved by utilizing new DFR margin. An improvement of bandwidth efficiency is achieved by reducing the ciphertext size which, however, increases the compression noise resulting in the DFR degradation. However, even with such increased compression noise, both the improvement of bandwidth efficiency and the required DFR of $2^{-140}$ can be achieved by utilizing DFR margin.

Table 2: Improved bandwidth efficiency of NewHope based on new DFR margin.

| n   | $(r_{v'}, r_{\hat{u}})$ | Ciphertext reduction (%) | DFR       |
|-----|-------------------------|--------------------------|-----------|
| 1024| (8, q)                  | 0 (Current NewHope)      | $\approx 2^{-474}$ |
|     | (4, q)                  | 5.9                      | $\approx 2^{-227}$ |
|     | (8, 1024)               | 23.5                     | $\approx 2^{-199}$ |
| 512 | (8, q)                  | 0 (Current NewHope)      | $\approx 2^{-431}$ |
|     | (4, q)                  | 5.9                      | $\approx 2^{-420}$ |
|     | (4, 2048)               | 23.5                     | $\approx 2^{-155}$ |
|     | (8, 1024)               | 23.5                     | $\approx 2^{-185}$ |

Table 2 shows the improved bandwidth efficiency of NewHope achieved by additional ciphertext compression. The results in Table 2 are obtained in two ways. The first way ($r_{v'} = 4$, $r_{\hat{u}} = q$) changes only the compression rate on $v'$ from 8 (3 bits per coefficient) to 4 (2 bits per coefficient) to improve the bandwidth efficiency, which does not change the protocol of NewHope and hence does not increase the time/space complexity. The second way (the remaining results) is to improve the bandwidth efficiency by doing additional compression on both $\hat{u}$ and $v'$, which results in a slight increase of time/space complexity because compression of $\hat{u}$ is added to the NewHope protocol. However, the second way shows about four times improvement in bandwidth efficiency over the first way while keeping the target DFR.

4 Improving Security and Bandwidth Efficiency of NewHope Using Error-correction Schemes

ECCs are used to format the transmitted information so as to protect the information from the noisy channel. Such protection is obtained by adding systematic redundancy to the information, which enables the receiver to detect and possibly correct errors. Therefore, ECCs are an essential part of digital communication/storage systems.

ATE is used in NewHope as an ECC, which was proposed by Pöppelmann and Güneysu [16]. However, since ATE is based on the repetition code, it shows low error-correction capability. Therefore, by applying advanced ECC to NewHope, the security and bandwidth efficiency can be significantly improved.

There are various good-performing ECCs such as turbo codes, LDPC codes, and BCH codes. Turbo codes provide good error-correction performance close
to the Shannon limit and the implementation of encoder is rather simple. LDPC codes also show good error-correction performance and enable high-speed processing. However, it is difficult to estimate low BLER of turbo codes and LDPC codes because they commonly have an error floor problem and no analytic way to accurately estimate very low BLER. In contrast, BCH codes do not show error-correction performance close to the Shannon limit but low BLER can be analyzed by calculating an upper bound of BLER. Also, BCH codes do not show an error floor problem. Therefore, BCH codes can be a better choice for PQC because a very low DFR is required for PQC. In this section, we investigate concatenated coding schemes of ATE and BCH code to combine the advantages of ATE and BCH code. However, the use of BCH codes is accompanied by an increase of implementation complexity and a possibility of side channel attacks. Nevertheless we focus on theoretically investigating how much the security and bandwidth efficiency of NewHope can be improved by using ECCs rather than dealing with the implementation issues. Note that implementation issues such as constant-time ECC [21] and robustness to the side channel attacks are actively studied research topics.

Fig. 7: An information-theoretic view of NewHope in terms of ECC and super channel.

From the viewpoint of ECC, the output of encoder and the input into decoder should be well defined, and hence the modulation, channel, and demodulation can be regarded as a super channel between them as shown in Fig. 7. Note that this super channel is a binary symmetric channel (BSC). Then we can easily calculate the BLER (DFR) of NewHope with ECC after we obtain the crossover probability of super channel by applying the total noise analysis.

4.1 Candidates of Error-correction Schemes

After analyzing various concatenated coding schemes for NewHope, we select the following four options (or ECCs) to validate the role of ECCs in NewHope. Note that these four options do not mean the best ECCs but they can show that
NewHope or other lattice-based cryptosystems can be dramatically improved in terms of security, bandwidth, and DFR by using ECCs even if there are still some implementation problems to be solved. As the BCH codes used in four options, first, narrow-sense primitive BCH codes having the dimension equal to or similar to 256 and the code length equal to or similar to $\left\lfloor \frac{n}{m} \right\rfloor$ for $m = 1, 2,$ and 3 are considered. Second, shortened BCH codes satisfying the above conditions are considered.

For all four options, BCH code is used as an outer code outside the super channel and ATE is used as an inner code for encoding/modulation and decoding/demodulation inside the super channel as given in Fig. [7]

- Option 1
Inner code: ATE(3-Repetition) + Outer code: BCH(341,260,9)
Parameters: $C_n = 341$, $m = \left\lfloor \frac{n}{C_n} \right\rfloor = 3$
A 256-bit message $\mu$ is encoded by BCH (341, 260, 9) which is the shortened code of BCH(511,430,9). This encoder requires 260 bits as its input, while the size of message in NewHope is 256 bits. Therefore, the input into the encoder is generated by padding four zeros to the 256-bit message $\mu$. Then, the 341-bit BCH codeword becomes the input into the ATE. The 1023-bit output of ATE ($m = 3$) and the additional zero are used as 1024 coefficients to generate $v$. With Option 1, up to 9 bit errors are corrected.

- Option 2
Inner code: ATE(2-Repetition) + Outer code: BCH(511,259,30)
Parameters: $C_n = 511$, $m = \left\lfloor \frac{n}{C_n} \right\rfloor = 2$
BCH(511,259,30) is a narrow-sense primitive BCH code and by padding three zeros to the 256-bit message $\mu$, the input into the BCH encoder is generated. Also, the input of ATE encoder is generated by padding one zero to the 511-bit BCH codeword. With Option 2, up to 30 bit errors are corrected.

- Option 3
Inner code: ATE(1-Repetition) + Outer code: BCH(1023,258,106)
Parameters: $C_n = 1023$, $m = \left\lfloor \frac{n}{C_n} \right\rfloor = 1$
BCH(1023,258,106) is a narrow-sense primitive BCH code and by padding two zeros to the 256-bit message $\mu$, the input into the BCH encoder is generated. Also, the input of ATE encoder is generated by padding one zero to the 1023-bit BCH codeword. With Option 3, up to 106 bit errors are corrected.

- Option 4
Inner code: ATE(1-Repetition) + Outer code: BCH(511,259,30)
Parameters: $C_n = 511$, $m = \left\lfloor \frac{n}{C_n} \right\rfloor = 1$
Option 4 is used for NewHope with $n = 512$. By adding three zeros to the 256-bit message $\mu$, the input into the BCH encoder is generated. Also, the input of ATE encoder is generated by padding one zero to the 511-bit BCH codeword. With Option 4, up to 30 bit errors are corrected.

The repetition number $m$ of ATE affects the cross-over probability of super channel in Fig. [7] Obviously, the larger the repetition number $m$ of ATE, the lower the cross-over probability of super channel. Table [3] shows the cross-over probability of super channel according to the repetition number of ATE, which
Table 3: Cross-over probability of super channel for Options 1, 2, 3, and 4.

| Option   | ATE | Cross-over probability of super channel |
|----------|-----|----------------------------------------|
| NewHope  | $m = 4$ | $1.3277 \times 10^{-145}$ |
| Option 1 | $m = 3$ | $8.3884 \times 10^{-110}$ |
| Option 2 | $m = 2$ | $6.0045 \times 10^{-74}$ |
| Option 3 | $m = 1$ | $5.1119 \times 10^{-38}$ |
| Option 4 | $m = 1$ | $1.7993 \times 10^{-67}$ |

is calculated by using (13) with $m = 1, 2, 3,$ and $4$. However, as the repetition number $m$ of ATE increases, the code rate of BCH code also increases and hence the error-correction capability of BCH code degrades. Note that for NewHope with $m = 4$, the cross-over probability is the same as the DFR because there is no outer code. In summary, there exists a trade-off between the repetition number $m$ of ATE (i.e., the quality of super channel) and the error-correction capability of BCH code (outer code). Therefore, it is very important to determine the optimum repetition number $m$ of ATE and the error-correction capability of BCH code by investigating this trade-off.

In general, decoding of BCH codes is performed by the Berlekamp-Massey (BM) algorithm and the decoding complexity increases linearly with the error-correction capability $C_t$. Therefore, in terms of implementation complexity Option 1 is the best choice because it has the lowest decoding complexity among four options.

4.2 Bandwidth Efficiency and DFR for Various Compression Rates on $v'$ and $\hat{u}$

We consider various compression rates $(r_{v'}, r_{\hat{u}})$ on $v'$ and $\hat{u}$ as given in Table 4. The reason for choosing these compression rates is that the target DFR $2^{-140}$ for NewHope cannot be achieved only by using ATE due to the increased compression noise as can be seen from Table 4. Table 4 also shows the ciphertext size and DFR when each compression rate is applied to NewHope where the ciphertext size is calculated as $\frac{n}{2} (\lceil \log_2 r_{v'} \rceil + \lceil \log_2 r_{\hat{u}} \rceil)$. Therefore, we can check if the target DFR can be achieved by applying Options 1, 2, 3, and 4 to NewHope with these compression rates. Note that the ciphertext size reduction is related to compression noise, so it affects DFR and does not affect security.

If a BCH($C_n$, $C_k$, $C_t$) is used, the DFR (or BLER) of NewHope over the super channel (or BSC) in Fig. 7 with the cross-over probability $p$ is calculated as

$$DFR = 1 - \sum_{i=0}^{C_t} \binom{C_n}{i} p^i (1 - p)^{C_n - i}.$$  \hfill (14)

Table 5 shows the DFR and ciphertext size (or improvement of bandwidth efficiency) reduction of NewHope with $n = 1024$ when Options 1, 2, and 3 are
Table 4: Various compression rates on $v'$ and $\hat{u}$ and the corresponding size of ciphertext and DFR ($(8,q)$ is the compression rate of current NewHope).

| $n$ | $(r_{v'}, r_{\hat{u}})$ | Ciphertext size (bytes) | DFR          |
|-----|-------------------------|-------------------------|--------------|
| 1024| $(8,q)$                 | 2176                    | $\approx 2^{-47.4}$  |
|     | $(8,512)$               | 1536                    | $\approx 2^{-75}$    |
|     | $(8,256)$               | 1408                    | $\approx 2^{-26}$    |
|     | $(8,128)$               | 1280                    | $\approx 2^{-1}$     |
|     | $(4,512)$               | 1408                    | $\approx 2^{-99}$    |
|     | $(4,256)$               | 1280                    | $\approx 2^{-11}$    |
|     | $(4,128)$               | 1152                    | $\approx 2^{-1}$     |
| 512 | $(8,q)$                 | 1088                    | $\approx 2^{-43.1}$  |
|     | $(8,512)$               | 768                     | $\approx 2^{-43}$    |
|     | $(8,256)$               | 704                     | $\approx 2^{-7}$     |
|     | $(4,1024)$              | 768                     | $\approx 2^{-43}$    |
|     | $(4,512)$               | 704                     | $\approx 2^{-15}$    |

used with various compression rates given in Table 4. The DFR is calculated by using (14) and the cross-over probability of super channel in Table 3. The target DFR can be achieved with all the Options 1, 2, and 3 if the coefficients of $\hat{u}$ are compressed from 14 bits ($r_{\hat{u}} = q$) to 8 bits ($r_{\hat{u}} = 256$) when the conventional compression rate ($r_{v'} = 8$) for the coefficients of $v'$ is applied. This can reduce the ciphertext size by 35.3%. Furthermore, the compression for the coefficients of $\hat{u}$ and additional compression for the coefficients of $v'$, which is Option 2 and Option 3 used with $(4,256)$, can reduce the ciphertext size by 41.2% to improve the bandwidth efficiency while achieving the target DFR. If the goal is to reduce the ciphertext size by 35.3%, $(4,512)$ is much better than $(8,256)$ because $(4,512)$ shows a remarkably lower DFR than $(8,256)$. Especially, because Option 1 used with $(8,512)$ and $(4,512)$, Option 2 used with $(8,512)$, $(8,256)$, and $(4,512)$, and Option 3 used with $(8,512)$, $(8,256)$, $(4,512)$, and $(4,256)$ overachieve the target DFR, this DFR margin can also be exploited to improve the security of NewHope by increasing the noise parameter $k$.

Table 6 shows the DFR and ciphertext reduction of NewHope with $n = 512$ when Options 4 is used with various compression rates given in Table 4. $(4,512)$ is better than the $(8,256)$ because $(4,512)$ and $(8,256)$ reduce the ciphertext size equally, but $(4,512)$ shows a significantly lower DFR than $(8,256)$. Because Option 4 used with $(8,512)$, $(8,256)$, $(4,1024)$, and $(4,512)$ overachieves the target DFR, this DFR margin can be exploited to improve the security of NewHope by increasing the noise parameter $k$. We will call the options that overachieve the target DFR the excellent ECC candidates to be used in the next section.
Table 5: DFR and ciphertext size of NewHope when Options 1, 2, and 3 are used with various compression rate when \( n = 1024 \).

| \((r_{w'}, r_u)\) | ECC Option | DFR | Ciphertext size (reduction rate) |
|----------------|-------------|-----|---------------------------------|
| (8,512) | Only ATE (NewHope) | \(2^{-135}\) | 1536 bytes (29.4%) |
| Option 1 | \(2^{-105}\) | | |
| Option 2 | \(2^{-111.7}\) | | |
| Option 3 | \(2^{-233.9}\) | | |
| (8,256) | Only ATE (NewHope) | \(2^{-20}\) | 1408 bytes (35.3%) |
| Option 1 | \(2^{-131}\) | | |
| Option 2 | \(2^{-317}\) | | |
| Option 3 | \(2^{-367}\) | | |
| (8,128) | Only ATE (NewHope) | \(2^{-7}\) | 1280 bytes (41.2%) |
| Option 1 | \(2^{-8}\) | | |
| Option 2 | \(2^{-2}\) | | |
| Option 3 | \(2^{-2}\) | | |
| (4,512) | Only ATE (NewHope) | \(2^{-40}\) | 1408 bytes (35.3%) |
| Option 1 | \(2^{-102}\) | | |
| Option 2 | \(2^{-122}\) | | |
| Option 3 | \(2^{-161.6}\) | | |
| (4,256) | Only ATE (NewHope) | \(2^{-11}\) | 1280 bytes (41.2%) |
| Option 1 | \(2^{-88}\) | | |
| Option 2 | \(2^{-118.8}\) | | |
| Option 3 | \(2^{-222}\) | | |
| (4,128) | Only ATE (NewHope) | \(2^{-4}\) | 1152 bytes (47.1%) |
| Option 1 | \(2^{-4}\) | | |
| Option 2 | \(2^{-4}\) | | |
| Option 3 | \(2^{-4}\) | | |

### 4.3 Security Analysis for The Excellent ECC Candidates

The excellent ECC candidates obtained in section 4.2 show excessive DFR performance owing to the properly chosen compression rate and ECC. Such over-achieved DFR can be exploited to improve the security level of NewHope. Tables 7 shows the DFR, ciphertext size, and security level, which is estimated at the cost of primal attack and dual attack of NewHope with \( n = 1024 \). Compared with current NewHope with \( n = 1024 \), Option 3 used with \((8,512)\) can improve the security level by 21.5 % and reduce the ciphertext size by 29.4% while achieving the target DFR \(2^{-140}\). If we focus on improving the bandwidth efficiency, Option 3 used with \((4,256)\) improves the security level by 3 % as well as reduces the ciphertext size by 41.5 % while achieving the target DFR. Option 3 used with \((4,512)\) and Option 3 used with \((8,256)\) improve the bandwidth
Table 6: DFR and ciphertext size of NewHope when Option 4 is used with various compression rate when \( n = 512. \)

| \((r_v', r_u')\) | ECC Option                        | DFR          | Ciphertext size (reduction rate) |
|-----------------|-----------------------------------|--------------|----------------------------------|
| (8,512)         | Only ATE (NewHope) \( \approx 2^{-33} \) | \( \approx 2^{-11.01} \) | 768 bytes (29.4 %)               |
|                 | Option 4                           | \( \approx 2^{-7} \) | 704 bytes (35.3 %)               |
| (8,256)         | Only ATE (NewHope) \( \approx 2^{-1} \) | \( \approx 2^{-3.59} \) | 640 bytes (41.2 %)               |
| (8,128)         | Only ATE (NewHope) \( \approx 2^{-1} \) | \( \approx 2^{-3.59} \) | 640 bytes (41.2 %)               |
| (4,1024)        | Only ATE (NewHope) \( \approx 2^{-1} \) | \( \approx 2^{-3.59} \) | 640 bytes (41.2 %)               |
| (4,512)         | Only ATE (NewHope) \( \approx 2^{-1} \) | \( \approx 2^{-3.59} \) | 640 bytes (41.2 %)               |
| (4,256)         | Only ATE (NewHope) \( \approx 2^{-1} \) | \( \approx 2^{-3.59} \) | 640 bytes (41.2 %)               |

efficiency equally, but the former one is better because it improves the security level by 9.4 % more than latter one.

Tables shows the DFR, ciphertext size, and security level of NewHope with \( n = 512. \) Compared with the current NewHope with \( n = 512, \) Option 4 used with (8,512) can improve the security level by 22.8 % and reduce the ciphertext size by 29.4 % while achieving the target DFR. Additionally, Option 4 used with (4,512) can improve the security level by 15.8 % and reduce the ciphertext size by 35.3 % while achieving the target DFR. The complexity of the decoding algorithm (BM algorithm) of BCH codes increases linearly with the number of correctable errors \( C_t. \) Thus, if the decoding complexity of BCH code is important, it is better to choose Option 1 used with (8,512) or Option 1 used with (4,512).

### 4.4 Closeness of Centered Binomial Distribution and the Corresponding Rounded Gaussian Distribution for Various \( k \)

The properties of rounded Gaussian distribution \( \xi \) are key to the worst-case to average-case reduction for Ring-LWE. However, since a very high-precision sampling is required for the rounded Gaussian distribution, NewHope uses the centered binomial distribution \( \psi_k \) for practical sampling without having rigorous security proof. It is generally accepted that as the centered binomial distribution and the rounded Gaussian distribution are closer to each other, NewHope is regarded as more secure. The closeness of two distribution can be measured through many methods. Among them, Rényi divergence is a well-known method, which is parameterized by a real \( a > 1 \) and defined for two distributions \( P \) and \( Q \) as follows.
Table 7: Improved security level, DFR, and ciphertext size for excellent ECC candidates for NewHope with $n = 1024$.

| $(r'_v, r'_{u})$ | ECC Option | Ciphertext size (reduction rate) | Classical/Quantum | DFR |
|------------------|-------------|-------------------------------|-------------------|-----|
|                  |             |                               | $k$               |     |
|                  |             |                               | Primal [bits]     | Dual [bits] |
| $(8, q)$         | NewHope     | 2176 bytes                     | 8                 | 259/235       | 257/233       | $\approx 2^{-474}$ |
|                  | Option 1    | 1536 bytes (29.4 %)           | 23                | 296/268       | 294/267       | $\approx 2^{-147}$ |
|                  | Option 2    | 1408 bytes (35.3 %)           | 13                | 275/249       | 274/248       | $\approx 2^{-145}$ |
|                  | Option 3    | 1280 bytes (41.2 %)           | 9                 | 262/238       | 261/237       | $\approx 2^{-144}$ |
| $(8, 512)$       | Option 1    | 1536 bytes (29.4 %)           | 33                | 310/281       | 309/280       | $\approx 2^{-144}$ |
|                  | Option 2    | 1408 bytes (35.3 %)           | 36                | 314/285       | 312/283       | $\approx 2^{-144}$ |
|                  | Option 3    | 1280 bytes (41.2 %)           | 14                | 278/252       | 276/250       | $\approx 2^{-144}$ |
| $(8, 256)$       | Option 1    | 1408 bytes (35.3 %)           | 15                | 280/254       | 279/253       | $\approx 2^{-144}$ |
|                  | Option 2    | 1304 bytes (29.4 %)           | 25                | 299/271       | 298/270       | $\approx 2^{-144}$ |
|                  | Option 3    | 1280 bytes (41.2 %)           | 27                | 302/274       | 301/273       | $\approx 2^{-144}$ |
| $(4, 512)$       | Option 1    | 1408 bytes (35.3 %)           | 15                | 280/254       | 279/253       | $\approx 2^{-144}$ |
|                  | Option 2    | 1304 bytes (29.4 %)           | 25                | 299/271       | 298/270       | $\approx 2^{-144}$ |
|                  | Option 3    | 1280 bytes (41.2 %)           | 27                | 302/274       | 301/273       | $\approx 2^{-144}$ |
| $(4, 256)$       | Option 2    | 1280 bytes (41.2 %)           | 9                 | 262/238       | 261/237       | $\approx 2^{-144}$ |
|                  | Option 3    | 1280 bytes (41.2 %)           | 10                | 266/241       | 265/240       | $\approx 2^{-144}$ |

Table 8: Improved security level, DFR, and ciphertext size for excellent ECC candidates for NewHope with $n = 512$.

| $(r'_v, r'_{u})$ | ECC Option | Ciphertext size (reduction rate) | Classical/Quantum | DFR |
|------------------|-------------|-------------------------------|-------------------|-----|
|                  |             |                               | $k$               |     |
|                  |             |                               | Primal [bits]     | Dual [bits] |
| $(8, q)$         | NewHope     | 1088 bytes                     | 8                 | 112/101       | 112/101       | $\approx 2^{-431}$ |
| $(8, 512)$       | Option 4    | 708 bytes (29.4 % reduction)   | 33                | 137/124       | 137/124       | $\approx 2^{-140}$ |
| $(8, 256)$       | Option 4    | 704 bytes (35.3 % reduction)   | 13                | 120/109       | 119/108       | $\approx 2^{-151}$ |
| $(4, 512)$       | Option 4    | 704 bytes (35.3 % reduction)   | 22                | 129/117       | 129/117       | $\approx 2^{-154}$ |

$$R_a(P||Q) = \left( \sum_{x \in \sup(P)} \frac{P(x)^a}{Q(x)^{a-1}} \right)^{\frac{1}{a-1}}$$ (15)

where $\sup(P)$ represents the support of $P$ and $Q(x) \neq 0$ for $x \in \sup(P)$.

We define $\xi_k$ to be the support of $P$ and $Q$ of the centered binomial distribution $\psi_k$ and rounded Gaussian distribution $\xi_k$ with the same variance $k/2$ decreases as $k$ increases. Therefore, an increase in noise parameter $k$ can quantitatively and qualitatively improve the security of NewHope although the time complexity increases due to the complexity increase of $\sum_{i=0}^{k-1} (b_i - b'_i)$. 

Fig. 8 shows that the Rényi divergence ($a = 9$ is used as in [4]) of the centered binomial distribution $\psi_k$ and rounded Gaussian distribution $\xi_k$ with the same variance $k/2$ decreases as $k$ increases. Therefore, an increase in noise parameter $k$ can quantitatively and qualitatively improve the security of NewHope although the time complexity increases due to the complexity increase of $\sum_{i=0}^{k-1} (b_i - b'_i)$. 

$$R_a(P||Q) = \left( \sum_{x \in \sup(P)} \frac{P(x)^a}{Q(x)^{a-1}} \right)^{\frac{1}{a-1}}$$ (15)
For NewHope with $k = 8$

Fig. 8: Rényi divergence of centered binomial distribution $\psi_k$ and rounded Gaussian distribution $\xi_k$ with the same variance $k/2$ according to $k$ ($a = 9$).

By applying concatenated coding schemes to NewHope, the target DFR can be achieved even by using a large noise parameter $k$, which therefore improves the security of NewHope due to improved closeness of centered binomial distribution and the corresponding rounded Gaussian distribution.

5 Conclusions and Future Works

In this paper, it is shown that the security and bandwidth efficiency of NewHope can be substantially improved by doing exact noise analysis and adopting proper concatenated error-correction schemes of ATE and BCH code. In detail, the DFR of NewHope is recalculated as $2^{-474}$ and $2^{-431}$ for $n = 1024$ and $n = 512$ through exact noise analysis, and hence it is shown that the DFR of NewHope has been too conservatively calculated. Since the recalculated DFR is much lower than the required $2^{-140}$, this DFR margin is exploited to improve the security level of NewHope by 8.5% with little increase in time/space complexity. Also, this margin is utilized to improve the bandwidth efficiency by 5.9% with keeping the time/space complexity of NewHope. Furthermore, by allowing a slight increase in time/space complexity, the bandwidth efficiency can be improved by 23.5% using an additional compression on $\hat{u}$ while achieving the required DFR.

The ATE of NewHope is simple and robust to the side channel attack, but its error-correction capability is relatively weak compared with other ECCs. Therefore, we propose various concatenated coding schemes of ATE and BCH code to combine the advantages of these two ECCs. From an information-theoretic viewpoint, ATE and total noise channel can be regarded as a super channel,
and the security and bandwidth efficiency improvement can be analyzed based on super channel analysis. By applying selected concatenated coding schemes to NewHope, the security level can be improved up to 21.5 % and the bandwidth efficiency can be improved up to 41.2 % by reducing ciphertext size, compared with the current NewHope with \( n = 1024 \). Likewise, the security level can be improved up to 22.8 % and the bandwidth efficiency can be improved up to 35.3 % by reducing ciphertext size, compared with the current NewHope with \( n = 512 \). Furthermore, by applying concatenated coding schemes to NewHope, the security of NewHope can be enhanced by improved closeness of centered binomial distribution and the corresponding rounded Gaussian distribution the target DFR can be achieved even by using a large noise parameter \( k \), which therefore improves However, BCH codes are less robust to the side channel attack than ATE and have a high implementation complexity. For this reason, several improvements are required for BCH codes to be practically used in NewHope or other lattice-based cryptosystems. Moreover, optimizing ECC parameters will improve the security and bandwidth efficiency of NewHope.

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