On Integrable Doebner–Goldin Equations

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Abstract

We suggest a method for integrating sub-families of a family of nonlinear Schrödinger equations proposed by H.-D. Doebner and G.A. Goldin in the 1+1 dimensional case which have exceptional Lie symmetries. Since the method of integration involves non-local transformations of dependent and independent variables, general solutions obtained include implicitly determined functions. By properly specifying one of the arbitrary functions contained in these solutions, we obtain broad classes of explicit square integrable solutions. The physical significance and some analytical properties of the solutions obtained are briefly discussed.

I. INTRODUCTION

The semi-direct product of the group of diffeomorphisms and the Abelian group of smooth functions on $\mathbb{R}^n$ may be regarded as a generalized symmetry group on $\mathbb{R}^n$. From the representation theory of this group, H.-D. Doebner and G.A. Goldin derived a family of nonlinear Schrödinger equations on $\mathbb{R}^n$ [1–4], which have been called the Doebner–Goldin (DG)–equation [6] (For recent progress in the study of these equations see the contributions in [5]):

$$\iota \hbar \partial_t \psi = \left( -\frac{\hbar^2}{2m} \Delta + V(\vec{x}) \right) \psi + i \frac{\hbar}{2} \sum_{j=1}^{5} c_j R_j[\psi] \psi + \hbar D' \sum_{j=1}^{5} c_j R_j[\psi] \psi,$$

where $R_j[\psi]$, $j = 1, \ldots, 5$ are real-valued functionals of the density $\rho := \psi \bar{\psi}$ and the current $\vec{J} = \text{Im}(\psi \bar{\nabla} \psi)$,

$$R_1[\psi] := \frac{\bar{\nabla} \cdot \vec{J}}{\rho} = \text{Im} \frac{\Delta \psi}{\psi}, \quad R_2[\psi] := \frac{\Delta \rho}{|\psi|^2}, \quad R_3[\psi] := \frac{\vec{J}^2}{\rho^2} = \left( \text{Im} \frac{\bar{\nabla} \psi}{\psi} \right)^2, \quad R_4[\psi] := \frac{\vec{J} \cdot \bar{\nabla} \rho}{\rho^2} = \text{Im} \left( \frac{\bar{\nabla} \psi}{\psi} \right)^2, \quad R_5[\psi] := \frac{(\bar{\nabla} \rho)^2}{\rho^2} = \left( \frac{\bar{\nabla} (|\psi|^2)}{|\psi|^2} \right)^2.$$

Here the real number $D$ (with the physical dimension of a diffusion constant) labels unitarily inequivalent representations of the generalized symmetry group involved in the derivation of the nonlinear equations [1]. It has been interpreted as a quantum number describing dissipative quantum systems [3,4]. The real number $D'$ (also with the physical dimension of a diffusion constant) describes the magnitude of the real non-linearity and the dimension-less constants $c_j \in \mathbb{R}$ are ‘model’ parameters.

For the purpose of this paper it is more convenient to use the parameterization that is obtained by rewriting equation [1] in terms of the real functionals $R_j[\psi]$ only, following the notation of [7–10]:

$$F(\nu, \mu) : \iota \partial_t \psi = \iota \sum_{j=1}^{2} \nu_j R_j[\psi] \psi + \sum_{j=1}^{5} \mu_j R_j[\psi] \psi + \mu_0 V \psi, \quad \nu_1 \neq 0.$$
Particular homogeneous equations of this type have also been considered in the context of quantum mechanics by other authors, e.g. [11–17].

One of the interesting features of the family of DG–equations (1) is its invariance under a certain group of transformations [7,8]

\[ N(\Lambda,\gamma)(\psi) = \psi' = \psi \frac{1}{2} (1 + \Lambda + i\gamma) \psi \frac{1}{2} (1 - \Lambda + i\gamma) = |\psi| e^{i(\gamma \ln |\psi| + \Lambda \arg \psi)}, \]

(4)
n.i.e. if \( \psi \) is a solution of \( F(\nu,\mu) \), then \( \psi' = N(\Lambda,\gamma)(\psi) \) is a solution of \( F(\nu',\mu') \), where the change of parameters under \( N(\Lambda,\gamma) \) is

\[
\begin{align*}
\nu_1' &= \frac{\nu_1}{\Lambda}, & \nu_2' &= -\frac{\gamma}{2\Lambda} \nu_1 + \nu_2, \\
\mu_1' &= -\frac{\gamma}{\Lambda} \nu_1 + \mu_1, & \mu_2' &= \frac{\gamma^2}{2\Lambda} \nu_1 - \gamma \nu_2 - \frac{\gamma}{2} \mu_1 + \Lambda \mu_2, & \mu_3' &= \frac{\mu_3}{\Lambda}, \\
\mu_4' &= -\frac{\gamma}{\Lambda} \mu_3 + \mu_4, & \mu_5' &= \frac{\gamma^2}{4\Lambda} \mu_3 - \gamma \mu_4 + \Lambda \mu_5, & \mu_0' &= \Lambda \mu_0.
\end{align*}
\]

(5)

Thus, without loss of generality we can restrict our calculations to the particular choice of parameters (a particular gauge, see below)

\[
\nu_1 = -1, \quad \nu_2 = 0.
\]

(6)

Since the transformations (4) leave the position probability invariant, i.e. \( \rho'(\vec{x},t) = \rho(\vec{x},t) \), they have been called nonlinear gauge transformations [7,8,18]. This notion is physically motivated by the fact, that in (non-relativistic) quantum mechanics we basically measure positions at different times. Furthermore, the transformations have been used to construct a consistent notion of observables in a nonlinear quantum theory [19].

It turned out that besides such important properties of the DG–equation as homogeneity, separability, and Euclidean invariance, which were ‘input’ by construction, equations (1) possess a number of other attractive properties. Among them one should emphasize the possibility of constructing explicit square integrable solutions, which is important for a physical interpretation. In particular, some stationary and non-stationary (Gaussian and traveling wave) solutions have been obtained [4,6,9,20–22].

The well-known connection between exact solutions of partial differential equations (PDEs) and their symmetry properties [25–27] as well as the necessity of classifying equations (1) in a unified way, motivated a systematic study of their Lie symmetry in [10]. As a result, one has to distinguish nine sub-families (characterized by conditions on the parameters \( \mu \) in the chosen gauge) with different maximal Lie symmetry algebras \( \text{sym}_n \). The relationship between these sub-families and their symmetries is indicated in Fig.1 (using the notation of [11]).
Schrödinger classes of real numbers (due to real homogeneity of the equations). These equations thus fit into them we find the direct sum of the (centrally extended) Schrödinger algebra and the real numbers (due to real homogeneity of the equations). These equations thus fit into the classes of Schrödinger invariant nonlinear evolution equations determined in [28–31].

The four remaining symmetry algebras are infinite dimensional. $sym^b_1(n)$ and $sym^c_1(n)$ contain in addition to the elements of $sym_1(n)$ infinite dimensional algebras $b$ and $c$, that depend on a pair of (real) solutions of a linear forward and backward heat equations and a (complex) solution of a linear Schrödinger equation, respectively. Actually these symmetries correspond to linearizations of these sub-families, the first to a pair of forward and backward heat equations, the latter to a Schrödinger equation [23,24]. On the contrary, the symmetry algebras $sym^a_2(n)$ and $sym^a_3(n)$ contain an infinite dimensional algebra $a$ that depends only on one real-valued function. As a consequence, there is no local transformation (i.e. a transformation that does not involve integrals or derivatives of the dependent variables) linearizing the corresponding DG–equations. Nevertheless, these equations as well as the one admitting the finite dimensional symmetry algebra $sym_3(n)$ are shown in the present paper to be integrable by a non-local transformation of dependent and independent variables in the case of one spatial variable ($n = 1$). Thus, all DG–equations with exceptional symmetries (bottom row of Fig. 1) $sym^b_1$, $sym^c_1$, $sym_3$, $sym^a_2$, $sym^a_3$ are integrable, i.e. they can be reduced to an equation which is either linear or integrable by quadratures.

Fig. 1: Lie symmetries of the DG–equation. Sub-families are characterized by their parameters and arrows indicate the subfamily–structure. The equations dealt with in this paper are in bold frames.
The principal object of study in the present paper are DG–equation in 1+1 dimensions with parameters

\[\nu_1 = -1, \ \nu_2 = 0, \ \mu_1 = \mu_2 = \mu_4 = \mu_5 = 0,\] (7)

\[\nu_1 = -1, \ \nu_2 = 0, \ \mu_2 = \mu_5 = 0, \ \mu_3 = 2, \ \mu_4 = -\mu_1 \neq 0,\] (8)
i.e. the following coupled two-dimensional PDEs:

\[i\psi_t = \left\{ -i \text{Im} \frac{\psi_{xx}}{\psi} + \mu_3 \left( \text{Im} \frac{\psi_x}{\psi} \right)^2 + \mu_0 V(x) \right\} \psi,\] (9)

\[i\psi_t = \left\{ (\mu_1 - i) \text{Im} \frac{\psi_{xx}}{\psi} + 2 \left( \text{Im} \frac{\psi_x}{\psi} \right)^2 - \mu_1 \text{Im} \left( \frac{\psi_x}{\psi} \right)^2 + \mu_0 V(x) \right\} \psi.\] (10)

One of these DG-equations is contained in the so-called Ehrenfest sub-family \([4,24]\) fulfilling the second Ehrenfest relation: equation (9) with \(\mu_3 = 1\), i.e. the DG–equation with maximal Lie symmetry \(\text{sym}_3(n)\). This equation is furthermore the only Schrödinger and therefore Galilei invariant equation among (9) and (10). Nevertheless, in the free case \((V \equiv 0)\) all of these DG-equations admit traveling (solitary) wave solutions with arbitrary shape \([9]\). These solutions are rediscovered as a particular case of the general solutions in this paper.

Using a polar decomposition

\[\psi(x, t) = \exp \left( r(x, t) + is(x, t) \right)\] (11)

we rewrite the above equations in the following way:

\[F_1: \begin{cases} r_t + s_{xx} + 2r_xx_s = 0, \\ s_t + \mu_3 s_x^2 = -\mu_0 V; \end{cases}\] (12)

\[F_2: \begin{cases} r_t + s_{xx} + 2r_xx_s = 0, \\ s_t + \mu_1 s_{xx} + 2s_x^2 = -\mu_0 V. \end{cases}\] (13)

The paper is organized as follows. In the section II we integrate the free equations, i.e. equations (12), (13) with a vanishing potential \((V \equiv 0)\). In order to integrate \(F_1\) we have to distinguish between the cases \(\mu_3 \neq 1\) and \(\mu_3 = 1\), the latter corresponding to the subfamily with the larger Lie symmetry algebra \(\text{sym}_3(n) \supset \text{sym}_2(n)\).

Section III contains some remarks on the integration of the equations with potential and two particular examples where the integration is carried out. The methods of integration of \(F_1\) and \(F_2\) in sections II and III yield their general solutions containing implicitly determined function. Consequently, these solutions are, generally speaking, implicit. Therefore, in section IV we give some explicit solutions for the free equation as well as for linear and quadratic potentials by specifying one of the arbitrary functions of the general solutions obtained in the preceding sections.
II. INTEGRATION OF FREE DG–EQUATIONS

Putting in (12), (13) $V = 0$ we obtain the following PDEs:

\[ \tilde{F}_1 : \begin{cases} r_t + s_{xx} + 2r_s s_x = 0, \\ s_t + \mu_3 s_x^2 = 0, \end{cases} \tag{14} \]

\[ \tilde{F}_2 : \begin{cases} r_t + s_{xx} + 2r_s s_x = 0, \\ s_t + \mu_1 s_{xx} + 2s_x^2 = 0. \end{cases} \tag{15} \]

Henceforth we suppose that in (15) $\mu_1 \neq 0$, since otherwise system (13) is a particular case of (14) with $\mu_3 = 2$.

A. Integration of the family $\tilde{F}_1$

First, we turn to the integration of the system of nonlinear PDEs (14). As this system admits only a finite-dimensional Lie symmetry group, there is no local transformation which linearizes it. So the only possibility to transform the system in question into an integrable form is to utilize a non-local transformation of dependent and independent variables. The choice of a desired change of variables is implied by the form of the second equation of the system (14); it is nothing but the one-dimensional Hamilton–Jacobi equation, which is known to be linearizable by the following contact transformation (called in the literature the Euler–Ampère transformation):

\[ z_0 = t, \quad z_1 = s_x, \quad u = xs_x - s, \quad u_{z_0} = -s_t, \quad u_{z_1} = x. \tag{16} \]

Let us recall that a transformation $(x, t, s(x, t)) \mapsto (z_0, z_1, u(z_0, z_1))$,

\[ z_\alpha = f_\alpha(x, t, s, s_x, s_t), \]

\[ u = g(x, t, s, s_x, s_t), \]

\[ u_{z_\alpha} = h_\alpha(x, t, s, s_x, s_t), \tag{17} \]

where $\alpha = 0, 1$, is called contact, if it preserves the first-order tangency condition

\[ ds - s_x dx - s_t dt = 0 \quad \implies \quad du - u_{z_0} dz_0 - u_{z_1} dz_1 = 0. \]

The above condition ensures that the functions $u_{z_0}, u_{z_1}$ determined by the last two formulae from (17) are really derivatives of a function $u$ determined by the third formula with respect to $z_0$ and $z_1$ determined by the first and second formula, correspondingly.

It is readily seen that formulae (17) determine a contact transformation preserving the tangency condition. But before applying the Euler–Ampère transformation to the system under study we must ensure its invertibility, as we may loose some solutions otherwise. It is known that transformation (16) is invertible in a domain where $s_{xx} \neq 0$. Consequently, we have to consider the cases $s_{xx} \neq 0$ and $s_{xx} \equiv 0$ separately.
Case 1. \( s_{xx} \neq 0 \)

Let us apply the transformation (13) to the system (14) having prolonged it to the second derivatives

\[
u_{z_0z_0} = \frac{s_{Lx}^2 - s_{tt}s_{xx}}{s_{xx}}, \quad u_{z_0z_1} = -\frac{s_{tx}}{s_{xx}}, \quad u_{z_1z_1} = \frac{1}{s_{xx}}. \tag{18}
\]

As a result, we get

\[
u_{z_0} = \mu_3 z_1^2, \quad r_{z_0} + \frac{2z_1 - u_{z_0z_1}}{u_{z_1z_1}} r_{z_1} = -\frac{1}{u_{z_1z_1}}. \tag{19}
\]

Now, the first equation becomes linear and is easily integrated to give the following expression for \( u(z_0, z_1) \):

\[
u = \mu_3 z_0 z_1^2 + f(z_1), \quad f \in C^2(\mathbb{R}, \mathbb{R}). \tag{20}
\]

Inserting the result into the second equation of the system (19) we get a first-order linear PDE with non-constant coefficients,

\[
r_{z_0} + \frac{2(1 - \mu_3)z_1}{2 \mu_3 z_0 + f''(z_1)} r_{z_1} = -\frac{1}{2 \mu_3 z_0 + f''(z_1)}. \tag{21}
\]

When integrating the above equation we have to distinguish two sub-cases \( \mu_3 = 1 \) and \( \mu_3 \neq 1 \). Let us recall that the DG-equation with parameters (7) under \( \mu_3 = 1 \) satisfies the Ehrenfest relation and, what is more, admits an additional symmetry operator (see Fig.1).

Sub-case 1.1. \( \mu_3 = 1 \)

In this case equation (21) takes the form

\[
r_{z_0} = -\frac{1}{2z_0 + f''(z_1)}
\]

and its general solution reads

\[
r(z_0, z_1) = -\frac{1}{2} \ln \left( f''(z_1) + 2z_0 \right) + g(z_1), \quad g \in C^2(\mathbb{R}, \mathbb{R}). \tag{22}
\]

To rewrite the result obtained in the initial variables \( (x, t, r(x, t), s(x, t)) \) we have to invert the transformation (16). From the second and third relations it follows that

\[
s = x z_1 - u = x z_1 - t z_1^2 - f(z_1).
\]

To determine the function \( z_1 = z_1(x, t) \) we make use of the last relation from (13). Substituting into it the formula (20) with \( \mu_3 = 1 \) yields
\[ x = u_{z_1} = 2z_0z_1 + f'(z_1), \]

hence

\[ 2t z_1 - x + f'(z_1) = 0. \]  

(23)

The above relation determines the function \( z_1(x,t) \) in an implicit way. Since \( s_{xx} = 2t + f''(z_1) \neq 0 \), we can always solve (23) (at least locally) with respect to \( z_1 \) thus getting an explicit form of the function \( z_1 \).

Summarizing the results we conclude that the general solution of the system of PDEs (14) with \( s_{xx} \neq 0 \), \( \mu_3 = 1 \) is of the form

\[
\begin{align*}
 r(x,t) &= -\frac{1}{2} \ln \left( f''(z_1) + 2t \right) + g(z_1), \\
 s(x,t) &= -t z_1^2 + x z_1 - f(z_1),
\end{align*}
\]

(24)

where \( f, g \in C^2(\mathbb{R}, \mathbb{R}) \) are arbitrary functions and \( z_1 = z_1(x,t) \) is determined by the relation (23).

**Sub-case 1.2.** \( \mu_3 \neq 1 \)

Using in (21) the transformation

\[ r(x,t) = \tilde{r}(x,t) - \frac{1}{2(1 - \mu_3)} \ln z_1 \]

we get

\[ \tilde{r}_{z_0} - \frac{2(1 - \mu_3)z_1}{2 \mu_3 z_0 + f''(z_1)} \tilde{r}_{z_1} = 0. \]

From the theory of the first-order PDEs it is well-known (see, e.g. [33]) that a general solution of the above equation has the form \( \tilde{r} = g(\omega(z_0, z_1)) \), where \( \omega(z_0, z_1) \) is an integral of the Euler-Lagrange system

\[
\frac{dz_0}{1} = -\frac{dz_1}{2(1 - \mu_3)z_1}. \]

The above system is rewritten as a linear first-order ordinary differential equation for a function \( z_0 = z_0(z_1) \),

\[
\frac{dz_0}{dz_1} = \frac{1}{2z_1(1 - \mu_3)} \left( 2\mu_3 z_0 + f''(z_1) \right)
\]

the general solution of which can be represented in the form
\[ C = 2(\mu_3 - 1)z_0z_1^{\frac{\mu_3}{\mu_3-1}} + \int_{z_1}^{z_2} \zeta^{\frac{1}{\mu_3-1}} f''(\zeta) d\zeta \]

with an arbitrary constant \( C \). Hence we conclude that the general solution of equation (21) is given by the following formula:

\[ r(x, t) = \frac{1}{2(\mu_3 - 1)} \ln z_1 + g \left( 2(\mu_3 - 1)z_0z_1^{\frac{\mu_3}{\mu_3-1}} + \int_{z_1}^{z_2} \zeta^{\frac{1}{\mu_3-1}} f''(\zeta) d\zeta \right), \quad (25) \]

where \( f, g \in C^2(\mathbb{R}, \mathbb{R}) \) are arbitrary functions.

Returning to the initial variables \( x, t, r, s \) we obtain the general solution of DG–equation (14) for the case \( s_{xx} \neq 0, \mu_3 \neq 1 \)

\[ \begin{align*}
   r(x, t) &= \frac{1}{2(\mu_3 - 1)} \ln z_1 + g \left( 2(\mu_3 - 1)t\zeta_1^{\frac{\mu_3}{\mu_3-1}} + \int_{z_1}^{z_2} \zeta^{\frac{1}{\mu_3-1}} f''(\zeta) d\zeta \right), \\
   s(x, t) &= -\mu_3 t\zeta_1^2 + xz_1 - f(z_1),
\end{align*} \quad (26) \]

where \( f, g \in C^2(\mathbb{R}, \mathbb{R}) \) are arbitrary functions and the \( z_1 = z_1(x, t) \) is determined implicitly

\[ 2\mu_3 tz_1 - x + f'(z_1) = 0. \quad (27) \]

**Case 2.** \( s_{xx} \equiv 0 \)

In this case \( s(x, t) = \alpha(t)x + \beta(t) \) with arbitrary smooth functions \( \alpha(t), \beta(t) \). Substituting this expression into the second equation from (14) we arrive at the relations:

\[ \alpha'(t)x + \beta'(t) + \mu_3 \alpha'^2(t) = 0, \quad r_t + 2\alpha(t)r_x = 0. \]

An integration of these equations gives rise to the following expressions for \( r \) and \( s \):

\[ \begin{align*}
   r(x, t) &= f(x - 2C_1 t), \\
   s(x, t) &= C_1 x - \mu_3 C_1^2 t + C_2,
\end{align*} \quad (28) \]

where \( C_1, C_2 \) are arbitrary real constants. Thus we have rediscovered the traveling (solitary) wave solutions with arbitrary shape \( \mathbf{[9]} \) as a particular case of the general solution.

We have established that any smooth solution is contained (at least locally) in one of the classes given by equations (24), (26), and (28). Summarizing we arrive at the conclusion that the general solution of the free DG–equation (3) splits into two inequivalent classes:
\[ 1. \quad \mu_3 = 1 \]
\[ \psi(x,t) = f(x-2C_1t) \exp \left\{ i(C_1x - C_1^2 t + C_2) \right\}, \]
\[ \psi(x,t) = \left( f''(z_1) + 2t \right)^{-\frac{1}{2}} g(z_1) \exp \left\{ -i\left( tz_1^2 - xz_1 + f(z_1) \right) \right\}, \]

where \( f, g \) are arbitrary sufficiently smooth functions, \( C_1, C_2 \) are arbitrary real parameters, and \( z_1 = z_1(x,t) \) is determined implicitly by formula (23);

\[ 2. \quad \mu_3 \neq 1 \]
\[ \psi(x,t) = f(x-2C_1t) \exp \left\{ i(C_1x - \mu_3 C_1^2 t + C_2) \right\}, \]
\[ \psi(x,t) = z_1^{2(\mu_3^{-1})} g \left( 2(\mu_3 - 1) t z_1^{\mu_3^{-1}} + \int_{z_1}^{z_1} \zeta^{\mu_3^{-1}} f''(\zeta) d\zeta \right) \]
\[ \times \exp \left\{ -i\left( \mu_3 tz_1^2 - xz_1 + f(z_1) \right) \right\}, \]

where \( f, g \) are arbitrary sufficiently smooth functions, \( C_1, C_2 \) are arbitrary real parameters, and \( z_1 = z_1(x,t) \) is determined implicitly by formula (27).

Although formulae (31) and (32) give the general solution of the corresponding DG–equation for all \( \mu_3 \neq 1 \), the case \( \mu_3 = 0 \) deserves a special consideration, as the system of PDEs (14) with \( \mu_3 = 0 \) is easily integrated without applying the contact transformation (16), (18). The second equation of (14) yields that \( s \) does not depend on time, so

\[ s(x,t) = f(x), \quad f \in C^2(\mathbb{R}, \mathbb{R}). \]

Inserting this into the first equation we get a first order PDE for \( r \),

\[ r_t + 2f'(x)r_x + f''(x) = 0. \]

The case \( s(x,t) \equiv const \) leads to a traveling wave solution (31) with \( \mu_3 = 0 \), \( C_1 = 0 \); if \( f'(x) \neq 0 \), then the general solution reads

\[ r(x,t) = g \left( 2t - \int_{x}^{x} \frac{d\xi}{f'(\xi)} \right) - \frac{1}{2} \ln f'(x), \]

where \( g \in C^2(\mathbb{R}, \mathbb{R}) \) is again an arbitrary function.

Consequently, the general solution of the DG–equation (8) with \( \mu_3 = 0 \) is either given by a traveling wave solution (31) with \( \mu_3 = 0 \), \( C_1 = 0 \), or by

\[ \psi(x,t) = \left( f'(x) \right)^{-\frac{1}{2}} g \left( 2t - \int_{x}^{x} \frac{d\xi}{f'(\xi)} \right) \exp \{ if(x) \}. \]
B. Integration of the family $\bar{F}_2$

Let us turn to the integration of the DG–equation (15). First, we note that the second equation is the potential Burgers equation, which is linearized by the logarithmic substitution. Furthermore, we reduce the order of spatial derivatives in the first equation by a linear transformation of the dependent variables. Thus, the transformation

$$v(x, t) = -\mu_1 r(x, t) + s(x, t), \quad u(x, t) = \exp \left( \frac{2}{\mu_1} s(x, t) \right),$$

reduces system (15) to the form

$$uv_t + \mu_1 u_x v_x = 0, \quad u_t + \mu_1 u_{xx} = 0.$$ (37)

The second equation may be taken as the integrability condition of the vector-field $(u, -\mu_1 u_x)$ on space–time, $\partial_t u = \partial_x (-\mu_1 u_x)$, so that it is the gradient of a smooth function $\varphi(x, t)$,

$$\varphi_t = -\mu_1 u_x, \quad \varphi_x = u.$$ (38)

With this remark the first equation of the system (37) is rewritten to be

$$\varphi_x v_t - \varphi_t v_x = 0$$

and is easily integrated $v(x, t) = f(\varphi(x, t))$, where $f$ is arbitrary, sufficiently smooth function.

Solving (38) with respect to $\varphi$ we get

$$\varphi(x, t) = \int_0^x u(\xi, t) d\xi - \mu_1 \int_0^t u_x(0, \tau) d\tau + C,$$ (39)

where $C$ is an arbitrary constant.

Returning to the initial variables $(r(x, t), s(x, t))$ we get the general solution of the DG–equation (10)

$$\psi(x, t) = (u(x, t))^\frac{1}{2} f \left( \int_0^x u(\xi, t) d\xi - \mu_1 \int_0^t u_x(0, \tau) d\tau \right) \exp \left\{ \frac{i\mu_1}{2} \ln u(x, t) \right\},$$ (40)

where $u(x, t)$ is an arbitrary solution of the heat equation

$$u_t + \mu_1 u_{xx} = 0,$$ (41)

and $f$ is an arbitrary smooth function.

Finally, we note that the traveling wave solutions of [9] are reobtained using the particular solution

$$u(x, t) = \exp \left\{ - \frac{v}{\mu_1} (x - vt) \right\}$$

of the heat equation (11).
III. DG–EQUATION WITH NON-VANISHING POTENTIAL

Surprisingly enough, DG–equations (12), (13) are integrated in quadratures even in the case when $V(x) \neq 0$ (i.e. in the presence of a non-vanishing potential). Unfortunately, for the family $F_1$ the corresponding formulae are implicit and cumbersome. That is why we restrict ourselves to considering in detail system (12) with an additional constraint $\mu_3 = 1$ (i.e. the Ehrenfest subfamily of (12) is studied); this system was also considered in [23].

A. Integration of the family $F_1$

Choosing in (12) $\mu_3 = 1$, we obtain the following system of PDEs:

$$r_t + 2r_s x + s_{xx} = 0, \quad s_t + s^2_x + \mu_0 V(x) = 0.$$  \hspace{1cm} (42)

To linearize this system we make use of the following trick: instead of system of PDEs (42) one of its differential consequences is considered

$$r_t + 2r_s x + s_{xx} = 0, \quad s_{tx} + 2s_s x_{sx} + \mu_0 V'(x) = 0.$$  \hspace{1cm} (43)

Substituting in (43)

$$R(x,t) = r(x,t), \quad S(x,t) = s_x(x,t)$$  \hspace{1cm} (44)

we arrive at the system of first-order PDEs

$$R_t + 2SR_x + S_x = 0, \quad S_t + 2SS_x + \mu_0 V'(x) = 0.$$  \hspace{1cm} (45)

Thus, using the substitution (44) enables us to reduce the order of system of PDEs under study. Next, we apply to this system the hodograph transformation $(x,t,R(x,t),S(x,t)) \mapsto (z_0, z_1, v(z_0, z_1), u(z_0, z_1))$

$$t = u, \quad x = z_1, \quad S = z_0, \quad R = v,$$

$$R_t = \frac{u_0}{u z_0}, \quad R_x = v z_1 - \frac{u_0}{u z_0} v z_0, \quad S_t = \frac{1}{u z_0}, \quad S_x = -\frac{u_1}{u z_0}.$$  \hspace{1cm} (46)

This hodograph transformation is defined in an arbitrary domain where $S_t \neq 0$. So again we have to distinguish two cases, $S_t \neq 0$ and $S_t \equiv 0$.

Case 1. $S_t \neq 0$

Performing in (45) the change of variables (46) we get

$$\mu_0 V'(z_1) u z_0 - 2z_0 u z_1 + 1 = 0, \quad \mu_0 V'(z_1) v z_0 - 2z_0 v z_1 + \frac{u z_1}{u z_0} = 0.$$  \hspace{1cm} (47)

A further change of variables

$$\tilde{u}(z_0, z_1) = u(z_0, z_1), \quad \tilde{v}(z_0, z_1) = v(z_0, z_1) + \frac{1}{2} \ln u z_0(z_0, z_1)$$  \hspace{1cm} (47)
transforms the system to
\[ \mu_0 V'(z_1) \bar{u}_{z_0} - 2z_0 \bar{u}_{z_1} + 1 = 0, \quad \mu_0 V''(z_1) \bar{v}_{z_0} - 2z_0 \bar{v}_{z_1} = 0. \] (48)

Thus, combining local and non-local transformations of the dependent and independent variables we linearized and decoupled the differential consequence of (42). Integrating these equations yields
\[
\bar{u}(z_0, z_1) = \frac{1}{2} \int^{z_1} \left( z_0^2 + \mu_0 \left( V(z_1) - V(\zeta) \right) \right)^{-\frac{1}{2}} d\zeta + f \left( z_0^2 + \mu_0 V(z_1) \right),
\]
\[
\bar{v}(z_0, z_1) = g \left( z_0^2 + \mu_0 V(z_1) \right),
\] (49)

where \( f, g \in C^2(\mathbb{R}, \mathbb{R}) \) are arbitrary functions.

Returning to the variables \((x, t, R(x, t), S(x, t))\) we obtain the general solution of system (45) in an implicit form
\[
t = \frac{1}{2} \int^{x} \left( S^2(x, t) + \mu_0 \left( V(x) - V(\xi) \right) \right)^{-\frac{1}{2}} d\xi + f \left( S^2(x, t) + \mu_0 V(x) \right),
\]
\[
R(x, t) = \frac{1}{2} \ln S_t(x, t) + g \left( S^2(x, t) + \mu_0 V(x) \right). \] (50)

To rewrite these equations in the initial dependent variables \((r(x, t), s(x, t))\) we have to invert the transformation (44). As a result, we get
\[
r(x, t) = R(x, t), \quad s(x, t) = \int^{x} S(\tau, t) d\tau + \varphi(t),
\] (52)

where \( \varphi \in C^2(\mathbb{R}, \mathbb{R}) \) is an arbitrary function. The ambiguity arising is connected to the fact that we are not solving the initial system (12) but its differential consequence (13). The ‘extra’ function \( \varphi(t) \) is used to choose from the set of solutions of system (14) those ones which satisfy (12). Indeed, substituting formulae (52) into (12) and taking into account that the functions \( R(x, t), S(x, t) \) satisfy system of PDEs (13) we arrive at the following ordinary differential equation for a function \( \varphi(t) \):
\[ \varphi' + S^2(0, t) + \mu_0 V(0) = 0, \]

so
\[ \varphi(t) = - \int^{t} S^2(0, \tau) d\tau - \mu_0 V(0) t + C, \]

where \( C \) is an arbitrary real constant.

Summing up, we conclude that the general solution of the initial DG–equation reads
\[
\begin{aligned}
  r(x,t) &= -\frac{1}{2} \ln S_t(x,t) + g \left( S^2(x,t) + \mu_0 V(x) \right), \\
  s(x,t) &= \int_0^x S(\tau,t) d\tau - \int_0^t S^2(0,\tau) d\tau - \mu_0 V(0)t + C,
\end{aligned}
\]  

(53)

where \( S(x,t) \) is a smooth function determined implicitly by (50) and \( f, g \) are arbitrary sufficiently smooth functions.

**Case 2.** \( S_t \equiv 0 \)

With this condition the system of PDEs (45) is easily integrated to yield

\[
\begin{aligned}
  R(x,t) &= -\frac{1}{4} \ln (C_1 - \mu_0 V(x)) + g \left( 2t - \int_0^x \frac{d\xi}{\sqrt{C_1 - \mu_0 V(\xi)}} \right), \\
  S(x,t) &= \sqrt{C_1 - \mu_0 V(x)},
\end{aligned}
\]

where \( g \) is an arbitrary sufficiently smooth function and \( C_1 \) is an arbitrary real constant.

Rewriting the above expressions in the initial variables \( (r(x,t), s(x,t)) \) we get

\[
\begin{aligned}
  r(x,t) &= -\frac{1}{4} \ln (C_1 - \mu_0 V(x)) + g \left( 2t - \int_0^x \frac{d\xi}{\sqrt{C_1 - \mu_0 V(\xi)}} \right), \\
  s(x,t) &= \int_0^x \sqrt{C_1 - \mu_0(\xi)} d\xi - C_1 t + C_2,
\end{aligned}
\]

(54)

where \( C_2 \) is an arbitrary constant.

Thus, we have established that the general solution of DG–equation (9) with \( \mu_3 = 1 \) splits into the following two classes:

1. \( S_t \neq 0 \)

\[
\psi(x,t) = \left( S_t(x,t) \right)^{\frac{1}{4}} g \left( S^2(x,t) + \mu_0 V(x) \right) \\
\times \exp \left\{ i \left( \int_0^x S(\xi,t) d\xi - \int_0^t S^2(0,\tau) d\tau - \mu_0 V(0)t + C \right) \right\},
\]

(55)

where \( g \in C^2(\mathbb{R}, \mathbb{R}), C \in \mathbb{R} \), and \( S(x,t) \) is determined implicitly by (50).

2. \( S_t \equiv 0 \)

\[
\psi(x,t) = \left( C_1 - \mu_0 V(x) \right)^{-\frac{1}{4}} g \left( 2t - \int_0^x \frac{d\xi}{\sqrt{C_1 - \mu_0 V(\xi)}} \right) \\
\times \exp \left\{ i \left( \int_0^x \sqrt{C_1 - \mu_0 V(\xi)} d\xi - C_1 t + C_2 \right) \right\},
\]

(56)

where \( g \in C^2(\mathbb{R}, \mathbb{R}) \) and \( C_1, C_2 \in \mathbb{R} \).
B. Integration of the family $F_2$

Integrating system (13) with potentials is similar to integrating the free system ($V(x) = 0$) in section II B. Using the change of variables (36) for (13) we arrive at the following system of PDEs for new functions $u(x,t)$ and $v(x,t)$:

$$
\begin{align*}
  u_t + \mu_1 u_{xx} + \frac{2\mu_0}{\mu_1} V(x) u &= 0, \\
  uv_t + \mu_1 u_x v_x + \mu_0 V(x) u &= 0.
\end{align*}
$$

(57)

Now, given a potential $V(x)$ and an arbitrary solution $u(x,t)$ of the first equation of this system, one can construct a general solution $v(x,t)$ of the second equation which leads to a general solution of the initial system:

$$
\psi(x,t) = \left( u(x,t) \right)^{\frac{1}{2}} \exp \left( -\mu_1^{-1} v(x,t) + i\frac{\mu_1}{2} \ln u(x,t) \right).
$$

(58)

IV. EXPLICIT SOLUTIONS

As mentioned before some of the solutions of DG–equation obtained are local in a sense that they are not determined on the whole plane $\mathbb{R}^2$. But for physical applications one needs global solutions, and what is more, they should be square integrable, i.e. the integral

$$
p = \int_{-\infty}^{\infty} \bar{\psi}(x,t)\psi(x,t)dx
$$

(59)

is to be finite. If it is, the quantity $\rho(x,t) = \frac{1}{p} \bar{\psi}(x,t)\psi(x,t) \geq 0$ is treated as a probability density of a distribution of the wave function $\psi$ in space at a given time.

A. Explicit solutions of the family $F_1$

Evidently, the traveling wave solutions of DG–equation (9) given by (29), (31) are defined on the whole plane and, consequently, are global. To ensure square integrability of these solutions one has to restrict the choice of the arbitrary function $f$ to square integrable ones,

$$
p = \int_{-\infty}^{\infty} f^2(\tau)d\tau < \infty.
$$

Thus, the traveling wave solutions are square integrable provided $f$ is.

Solutions (30), (32) are, generally speaking, local, since the function $z_1(x,t)$ contained in these solutions is determined implicitly by formulae (23) and (27), correspondingly, and the existence of solution is only guaranteed locally by the implicit function theorem. In order to obtain explicit expressions for global and strictly local solutions we consider solutions of (30) with quadratic and cubic functions $f$, respectively.

For quadratic functions $f$,

$$
f(z_1) = \mu_2 \alpha z_1^2,
$$

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the implicit equation (27) (resp. (23)) for $z_1$ can be solved globally and we get $z_1(x,t) = \frac{x}{2\mu_3(t-\alpha)}$. Thus, we arrive at the following class of explicit solutions of the DG–equation (9) containing an arbitrary smooth function $g$:

1. $\mu_3 = 1$

$$\psi(x,t) = (2t)^{-\frac{1}{2}} g \left( \frac{x}{2(t-\alpha)} \right) \exp \left\{ \frac{i x^2}{4(t-\alpha)} \right\},$$  \hspace{1cm} (60)

2. $\mu_3 \neq 1$

$$\psi(x,t) = (t-\alpha)^{2(1-\mu_3)} \frac{1}{x^{2(\mu_3-1)}} g \left( (t-\alpha)^{\frac{1}{\mu_3-1}} x^{\frac{1}{\mu_3-1}} \right) \exp \left\{ \frac{i x^2}{2\mu_3(t-\alpha)} \right\}.$$  \hspace{1cm} (61)

These solutions are square integrable, provided $g$ is, and are well defined on the whole plane $\mathbb{R}^2$ with a possible exception of the line $t = \alpha$, where they converge to a $\delta$-function. In particular for $g(z) = \exp(-z^2)$ solutions (60) coincide with the GAUSSian wave solutions of (21).

Cubic functions $f$,

$$f(z_1) = \frac{1}{3} z_1^3,$$

give rise to strictly local solutions of the DG–equation. Indeed, inserting them into (23) we obtain a quadratic equation with respect to $z_1$

$$z_1^2 + 2\mu_3 t z_1 - x = 0.$$  \hspace{1cm} (62)

It has real solutions in the case $x + \mu_3^2 t^2 \geq 0$ only, i.e. $z_1(x,t)$ is not defined inside the parabola $x + \mu_3^2 t^2 = 0$. Solving (62) yields $z_1(x,t) = -t \pm \sqrt{x + \mu_3^2 t^2}$; according to the general solutions (31) and (32) we have to choose the positive sign since in (31) $2t + f''(z_1)$ and in (32) $z_1$ have to be positive. Hence, we arrive at the following class of strictly local explicit solutions of the family $F_1$ (9) containing an arbitrary smooth function $g$:

1. $\mu_3 = 1$

$$\psi(x,t) = (x+t^2)^{-\frac{1}{2}} g \left( -t + \sqrt{x + t^2} \right) \exp \left\{ -i \left( \frac{2}{3} t^3 + tx - \frac{2}{3} (x+t^2)^{\frac{3}{2}} \right) \right\},$$  \hspace{1cm} (63)

2. $\mu_3 \neq 1$

$$\psi(x,t) = \left( \sqrt{x + \mu_3^2 t^2} - \mu_3 \right)^{\frac{1}{2(\mu_3-1)}} g \left( \frac{2(\mu_3-1)}{2\mu_3-1} \left( \sqrt{x + \mu_3^2 t^2} - \mu_3 \right)^{\frac{\mu_3}{\mu_3-1}} \times \left( (\mu_3 - 1)t + \sqrt{x + \mu_3^2 t^2} \right) \right) \exp \left\{ \frac{i}{3} \left( 2(x + \mu_3^2 t^2)(-\mu_3 + \sqrt{x + \mu_3^2 t^2}) + \mu_3 t x \right) \right\}.$$  \hspace{1cm} (64)
The domain of definition of these solutions is the set \( \{(x, t) : x + \mu_3 t^2 > 0\} \). Furthermore, as the function \( \sqrt{x + \mu_3^2 t^2} \) is not defined for \( x \) at \( -\infty \) at any given time \( t \), \( |t| < \infty \), the solution (33) is not square integrable.

In this context let us remark that in general solutions (30) are square integrable at a given time \( t \) provided

- the (possibly infinite) limits \( a_\pm = \lim_{x \to \pm\infty} z_1(x, t) \) exist and
- \( g \) is square integrable on the interval \([a_-, a_+]\).

This statement follows from a change of the integration variable \( x \to z_1(x, t) \).

Before turning to DG–equations with non-vanishing potentials we examine solution (35) of the particular case \( \mu_3 = 0 \) of the family \( F_1 \). If the first derivative of \( f \) has no zeros, then the solution given by (35) is certainly global. Again, a change of the integration variable shows that the solution is square integrable, provided that

- the (possibly infinite) limits \( a_\pm = \lim_{\tau \to \pm\infty} \int_0^x \frac{d\tau}{f'({\tau})} \) exist and
- \( g \) is square integrable on the interval \([a_-, a_+]\).

In case of non-vanishing potentials we concentrate on the following specific potentials:

1. the linear potential
   \[
   V(x) = \frac{a}{\mu_0} x, \quad a \in \mathbb{R}; \tag{65}
   \]

2. the harmonic oscillator potential
   \[
   V(x) = \frac{a^2}{\mu_0} x^2, \quad a \in \mathbb{R}; \tag{66}
   \]

3. the anti-harmonic oscillator potential
   \[
   V(x) = -\frac{a^2}{\mu_0} x^2, \quad a \in \mathbb{R}; \tag{67}
   \]

First we consider Ehrenfest case \( \mu_3 = 1 \), the integration of which has been studied in detail in section IIIA.

1. For linear potentials (65) the implicit equation (50) reads
   \[
   t = \frac{1}{a} S(x, t) + f(S^2(x, t) + ax). \tag{68}
   \]

If we choose \( f \equiv 0 \), then \( S(x, t) = -at \). Thus, we get a class of explicit solutions from (55):
\[ \psi(x, t) = g(x + at^2) \exp\left\{ -i \left( atx + \frac{a^2}{3} t^3 - iC \right) \right\}. \] (69)

These solutions are defined on the whole plane \( \mathbb{R}^2 \) and square integrable, provided \( g \) is.

Another class of explicit solutions is obtained directly by means of formula (56):

\[ \psi(x, t) = (C_1 - ax)^{-\frac{1}{4}} g \left( at + \sqrt{C_1 - ax} \right) \exp \left\{ i \left( -\frac{2}{3a} (C_1 - ax)^{\frac{3}{2}} - C_1 t + C_2 \right) \right\}, \] (70)

where \( g \) is an arbitrary twice continuously differentiable function, \( C_1, C_2 \) are arbitrary parameters. These solutions are defined on the half-plane \( \{ (x, t) : x < \frac{C_1}{a} \} \).

Analogously we construct explicit solutions for the (anti-)harmonic oscillator potentials. We give these without derivation. \( (f \) is an arbitrary sufficiently smooth function, \( C_1, C_2 \) are arbitrary parameters.)

2. harmonic oscillator potential (66)

\[ \psi(x, t) = x^{-\frac{1}{2}} f \left( x^{-1} \sin 2at \right) \exp \left\{ i \left( \frac{a}{2} x^2 \cot 2at + C_1 \right) \right\}, \] (71)

\[ \psi(x, t) = (C_1^2 - a^2 x^2)^{-\frac{1}{4}} f \left( \sqrt{C_1^2 - a^2 x^2} \sin 2at - ax \cos 2at \right) \times \exp \left\{ i \left( \frac{x}{2} \sqrt{C_1^2 - a^2 x^2} + \frac{C_1^2}{2a} \arcsin \frac{ax}{C_1} - C_1 t + C_2 \right) \right\}; \] (72)

3. anti-harmonic oscillator potential (67):

\[ \psi(x, t) = x^{-\frac{1}{2}} f \left( x^{-1} \sinh 2at \right) \exp \left\{ i \left( \frac{a}{2} x^2 \coth 2at + C_1 \right) \right\}, \] (73)

\[ \psi(x, t) = (C_1 + a^2 x^2)^{-\frac{1}{4}} f \left( \left( ax + \sqrt{C_1 + a^2 x^2} \right) e^{-2at} \right) \times \exp \left\{ i \left( \frac{x}{2} \sqrt{C_1 + a^2 x^2} + \frac{C_1}{2a} \ln \left| ax + \sqrt{C_1 + a^2 x^2} \right| - C_1 t + C_2 \right) \right\}. \] (74)

As mentioned in section \( \text{III A} \), general solutions of the family \( F_1 \) (4) with \( \mu_3 \neq 1 \) are given by cumbersome implicit formulae. But with the particular choice of the potentials above it has explicit solutions containing one arbitrary function:

I. \( \mu_3 = 0 \)

1. linear potential (53):

\[ \psi(x, t) = g(x + at^2) \exp \{ -iatx \}; \] (75)
2. harmonic oscillator potential (66):
\[ \psi(x, t) = e^{a^2t^2} g(x e^{2a^2t^2}) \exp\{-ia^2tx^2\}; \quad (76) \]

3. anti-harmonic oscillator potential (67):
\[ \psi(x, t) = e^{-a^2t^2} g(x e^{-2a^2t^2}) \exp\{ia^2tx^2\}; \quad (77) \]

II. \( \mu_3 = \lambda^2 > 0 \)

1. linear potential (63):
\[ \psi(x, t) = g(x + at^2) \exp\left\{-i\left(atx + \frac{a^2\lambda^2}{3}t^3 - C\right)\right\}; \quad (78) \]

2. harmonic oscillator potential (66):
\[ \psi(x, t) = x^{-\frac{1}{2}} g\left(x^{-\lambda^2} \sin 2a\lambda t\right) \exp\left\{i\left(\frac{a}{2\lambda^2}x^2 \cot 2a\lambda t + C\right)\right\}; \quad (79) \]

3. anti-harmonic oscillator potential (67):
\[ \psi(x, t) = x^{-\frac{1}{2}} g\left(x^{-\lambda^2} \sinh 2a\lambda t\right) \exp\left\{i\left(\frac{a}{2\lambda^2}x^2 \coth 2a\lambda t + C\right)\right\}; \quad (80) \]

III. \( \mu_3 = -\lambda^2 < 0 \)

1. linear potential (63):
\[ \psi(x, t) = g(x + at^2) \exp\left\{-i\left(atx - \frac{a^2\lambda^2}{3}t^3 - C\right)\right\}; \quad (81) \]

2. harmonic oscillator potential (66):
\[ \psi(x, t) = x^{-\frac{1}{2}} g\left(x^{\lambda^2} \cosh 2a\lambda t\right) \exp\left\{i\left(\frac{a}{2\lambda^2}x^2 \tanh 2a\lambda t + C\right)\right\}; \quad (82) \]

3. anti-harmonic oscillator potential (67):
\[ \psi(x, t) = x^{-\frac{1}{2}} g\left(x^{\lambda^2} \cos 2a\lambda t\right) \exp\left\{i\left(\frac{a}{2\lambda^2}x^2 \tan 2a\lambda t + C\right)\right\}. \quad (83) \]

In all these cases \( f \in C^2(\mathbb{R}, \mathbb{R}) \) and \( C \in \mathbb{R} \).
B. Explicit solutions of the family $F_2$

Clearly, if the function $u(x,t)$ is a global solution of the heat equation (41), then the formula (40) gives a global solution of DG–equation (10). And what is more, it is square integrable provided

- the (possibly infinite) limits $a_{\pm} = \lim_{\pm\infty} \left( \int_0^x u(\tau, t) d\tau - \mu_1 \int_0^t u_x(0, \tau) d\tau \right)$ exist and
- $f$ is square integrable on the interval $[a_-, a_+]$.

In order to construct explicit solutions of the family $F_2$ (13) for linear and quadratic potentials (65)–(67) we have to solve equations (57). After some tedious calculations we obtain the following solutions:

1. linear potential (65):
   $$\psi(x,t) = f(x + at^2) \exp \left\{ -iatx - \frac{2ia}{3} t^3 + iC \right\}; \quad (84)$$

2. harmonic oscillator potential (66):
   $$\psi(x,t) = (\cos 4at)^{-\frac{3}{4}} f \left( x^{-2} \cos 4at \right) \times \exp \left\{ -i \left( \frac{a}{2} x^2 \tan 4at + \frac{\mu_3}{4} \ln \cos 4at - C \right) \right\}; \quad (85)$$

3. anti-harmonic oscillator potential (67):
   $$\psi(x,t) = (\cosh 4at)^{-\frac{3}{4}} f \left( x^{-2} \cosh 4at \right) \times \exp \left\{ i \left( \frac{a}{2} x^2 \tan 4at - \frac{\mu_3}{4} \ln \cosh 4at + iC \right) \right\}. \quad (86)$$

Here $f$ is an arbitrary twice continuously differentiable function and $C$ is an arbitrary constant.

V. CONCLUSION

As mentioned above Lie symmetries of DG–equations considered are not extensive enough to provide their linearizability by means of local transformations. PDEs (3), (10) prove to be integrable because of infinite non-local symmetries admitted. Take, as an example, system (12). It has been decoupled into a system of two linear first-order PDEs (48) by means of non-local transformations of dependent and independent variables (44), (46), (47). It is well-known (see, e.g. [25]) that any linear first-order PDE admits an infinite parameter Lie transformation group. Consequently, system (48) possesses an infinite local symmetry. But after being rewritten in the initial variables $(x, t, r(x, t), s(x, t))$ it becomes non-local and can not be found by using the infinitesimal Lie algorithm.
In the case involved an existence of non-local symmetry was indicated by a change of local symmetry of the DG–equation when the parameters were specified to be (9), (10). These additional local symmetries form the top of the ‘iceberg’, the main part of which consists of non-local symmetries enabling us to integrate the corresponding DG–equations.

Since we have the formulae for general solutions of systems of PDEs (9), (10), it is not but natural to apply these to analyze the initial value problem for these systems, which is important for a physical interpretation of the equations. For example, using formula (40) it is not difficult to prove that the initial value problem

\[ i\psi_t = \left\{ (\mu_1 - i)\text{Im} \frac{\psi_{xx}}{\psi} + 2 \left( \text{Im} \frac{\psi_x}{\psi} \right)^2 - \mu_1 \text{Im} \left( \frac{\psi_x}{\psi} \right)^2 \right\} \psi, \]

\[ \psi(x, 0) = r_0(x) \exp\{is_0(x)\}, \]

where \( r_0, s_0 \in C^\infty(\mathbb{R}, \mathbb{R}) \) are arbitrary functions such that \( \mu_1 s_0(x) \neq -\infty \), has a unique solution given by the formula (40), where \( u(x,t) \) is a solution of the initial value problem for the heat equation

\[ u_t + \mu_1 u_{xx} = 0, \quad u(x, 0) = \exp\left\{ \frac{1}{\mu_1} s_0(x) \right\} \]

and the function \( f(y) \) reads

\[ f(y) = R(h(y)) \exp\left\{ -\frac{1}{\mu_1} S(h(y)) \right\}. \]

Here \( h(y) \) is determined implicitly by the relation

\[ y = \int_0^{h(y)} \exp\left\{ \frac{2}{\mu_1} s_0(\tau) \right\} d\tau. \]

But for the DG–equation (3) an analysis of the initial value problem is complicated due to the complex structure of its general solution.

The method of integration of DG–equations developed in the present paper for the case of one space variable can be extended to a physically more interesting case of three spatial dimensions. A principal idea of such an extension is a utilization of generalized Euler–Ampère transformations of the space \((t, \vec{x}, u(t, \vec{x}), u_t(t, \vec{x}), \text{grad } u(t, \vec{x}))\) suggested in [32]. The above transformations were used to study compatibility and to construct a general solution of the four-dimensional nonlinear D’ALEMBERT–eikonal system. This problem is under investigation now and will be a topic of our future publications.

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