Elastic hadron scattering and optical theorem

Miloš V. Lokajíček, Vojtěch Kundrát and Jiří Procházka

Institute of Physics of the AS CR, v.v.i., 18221 Prague 8, Czech Republic
CERN, Geneva, Switzerland

Abstract
In principle all contemporary phenomenological models of elastic hadronic scattering have been based on the assumption of optical theorem validity that has been overtaken from optics. It will be shown that the given theorem which has not been actually proved cannot be applied to short-ranged strong interactions in any case. The actual progress in description of collision processes might then exist only if the initial states are specified on the basis of impact parameter values of colliding particles and probability dependence on this parameter is established.

keywords: optical theorem, short-ranged hadron interaction, elastic scattering

1 Introduction
Practically all hitherto models of elastic hadronic scattering have been based on the assumption of optical theorem validity. According to this theorem the total (hadronic) cross section \( \sigma_{\text{tot}} \) is proportional to the imaginary part of elastic (hadronic) scattering amplitude \( f \) at zero scattering angle \( \theta \)

\[
\sigma_{\text{tot}} \propto \text{Im} \ f(\theta = 0).
\] (1)

The complex amplitude \( f(\theta) \) is obtained with the help of Schroedinger equation.

The given theorem used now in particle physics was overtaken from the optics where it developed from the formula for refraction index (defined on the basis of wave theory of light) which contained also the influence of extinction cross section (now denoted as total cross section); see the story described by Newton [1]. The given formula has been formulated practically only on the basis of experimental refraction data without any actual theoretical reasoning.

Different attempts to prove it theoretically have been done mainly when the collisions of fundamental particles have started to be studied and the optical theorem has been applied to also in the region of strong interactions. Some of these attempts have been interpreted as successful. However, we have demonstrated recently that fundamental discrepancy is to exist especially if the optical theorem has been applied to elastic scattering caused by the very short-ranged strong interaction [2].

Some arguments used to support its validity in strong interactions are, however, still repeated. In the following we shall attempt to provide some deeper and more systematical reasoning why this theorem cannot be applied to in any hadronic elastic scattering. We shall demonstrate it on two arguments that have been described, e.g.,

Email addresses: lokaj@fzu.cz, kundrat@fzu.cz, jiri.prochazka@cern.ch

1 On leave of absence from Institute of Physics, AS CR, v.v.i., 182 21 Prague 8, Czech Republic
in [3]: one based on the optical approach (wave interpretation) and the other on unitarity of corresponding $S$ matrix. However, in both the cases the corresponding conclusions have been based evidently on some additional assumptions that have been chosen rather arbitrarily and have not been in agreement with corresponding reality.

The argument based on $S$ matrix theory has been nearer to the theory of strong interactions. However, the corresponding conclusions have been based on the assumptions concerning the basic structure of $S$ operator acting in the Hilbert space in which the incoming and outgoing states cannot be correspondingly distinguished. We shall start, therefore, by discussing the necessary Hilbert structure formed by Schroedinger equation solutions in corresponding collision processes; see Sec. 2. The two mentioned approaches trying to prove the validity of optical theorem will be then analyzed in Sec. 3; the assumptions being in contradiction to reality will be specified.

As to the contemporary models of elastic collisions they have represented quite phenomenological mathematical description of corresponding collision processes. More detailed physical description may be obtained if the statistical distribution of impact parameter values between two colliding objects is taken into account and the dependence of collision characteristics on this parameter is established. Basic aspects of corresponding probabilistic model proposed recently will be briefly described in Sec. 4. The model clearly shows that the collision process may be interpreted on fully ontological basis (without applying optical theorem).

## 2 Schroedinger equation and corresponding Hilbert space

Time evolution of microscopic processes is being described with the help of the Schroedinger (linear differential) equation

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = \hat{H}\psi(x, t) \tag{2}$$

where Hamiltonian operator $\hat{H}$ is given by

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 + \hat{V}(x) \tag{3}$$

where $\hat{V}(x)$ is corresponding potential. Its basic solutions (represented by the product of space and time functions) may be expressed in the form

$$\psi_E(x, t) = \lambda_E(x)e^{-iEt/\hbar}; \tag{4}$$

being standardly normed to one: $\int dx \psi^*_E(x, t)\psi_E(x, t) = \int dx |\lambda_E(x)|^2 = 1$ (at any time $t$). The function $\lambda_E(x)$ of all space coordinates $(x)$ corresponds to the Hamiltonian eigenfunctions

$$\hat{H}\lambda_E(x) = E\lambda_E(x). \tag{5}$$

General solution $\psi(x, t)$ of Schroedinger equation (2) may be then written as a superposition of corresponding Hamiltonian eigenfunctions $\lambda_E(x)$

$$\psi(x, t) = \sum_E c_E\psi_E(x, t) \tag{6}$$
where \( c_E \) are corresponding coefficients in a linear combination of particular solutions \( \psi_E(x, t) \); fulfilling \( \sum_E |c_E|^2 = 1 \).

All possible amplitudes \( \psi(x, t) \) (functions of space coordinates at different \( t \) values) form then a complete Hilbert space. Schroedinger defined then expected values \( A(t) \) of physical quantities

\[
A(t) = \int \psi^*(x, t) \hat{A} \psi(x, t) dx
\]

(7)
corresponding to classical quantities (see [4]). For each physical quantity \( A(t) \) associated operator \( \hat{A} \) acts in the given Hilbert space. It was shown originally for inertial motion only; however, it holds practically generally. Only the set of Schroedinger solutions is smaller due to discrete quantum states in closed systems. It was shown by Ioannidou [5] and Hoyer [6] that the Schroedinger equation might be derived from statistical combination of Hamilton equation solutions (or be at least equivalent to these solutions) if their whole set was limited by a rather weak condition; see also [4, 7].

A \( t \)-dependent solution \( \psi_E(x, t) \) of Schroedinger equation (a vector in the corresponding Hilbert space) represents the evolution of motion as an open trajectory in the case of continuous energy spectrum or as a closed trajectory for discrete energy values. Any vector \( \psi_E(x, t) \) represents then instantaneous state belonging to two opposite momentum directions. To distinguish these two different cases the total Hilbert space must consist of two mutually orthogonal subspaces each being spanned on the basis of Hamiltonian eigenfunctions \( \lambda_E(x) \) as it has been shown already many years ago by Lax and Phillips [8, 9] and independently derived also by Alda et al. [10] from the requirement of exponential (purely probabilistic) decay law of unstable particles. Only in such an extended Hilbert space the elastic processes may be correspondingly described. The transition from one subspace to another is then given by the evolution operator

\[
\hat{U}_{ev}(t) = e^{-i\hat{H}t/\hbar};
\]

(8)
the opposite evolution corresponding to negative values of \( t \).

The given Hilbert structure has been, however, excluded by Bohr in 1927; he asked for the Hilbert space of any physical system to be spanned always on one basis of Hamiltonian eigenvectors. It has caused that the earlier physical interpretation of Schroedinger equation solutions has been fundamentally deformed as practically any actual time evolution has been excluded. Moreover, the given model has required the existence of immediate interaction between very distant particles, which was shown and criticized by Einstein in 1935 with the help of special coincidence Gedankenexperiment. The physical scientific community preferred, however, Bohr’s approach.

Later both the alternative were admitted and discussed. Bohr’s alternative was, however, supported again on the basis of the fact that Bell’s inequality (derived in 1964 for the coincidence experiment more specified than that of Einstein) was violated in the corresponding experiment including spin measurement and performed by Aspect et al. in 1982 [11]. It has been shown only recently that Bell’s inequality was based on the assumption that did not hold in the given more specified experiment (but only in that proposed originally by Einstein). Consequently Einstein has been fully right in the given controversy with Bohr and the Hilbert space must always consist at least of two mutually orthogonal subspaces as explained in the preceding. All necessary details may be found in [4, 12, 13] and [14].
3 Two attempts of deriving optical theorem

In the region of strong interactions the decisive study of elastic processes concerns two proton collisions where the experimental data especially for small deviations (scattering angles) represent the combination of Coulomb and hadronic interactions. The ratio of these two interactions has always being determined on the basis of some theoretical predictions. However, the contemporary approaches have started often from mistaking assumptions as it will be shown in the following.

As to the Coulomb interaction it has been assumed that the corresponding probability has risen to infinity for very small deviations, which has followed from the fact that the zero deviation should be obtained at infinite distance (at infinite impact parameter). However, the measured region is in the range of less than micrometers, which is not respected in the usual formula that is used for interpretation of measured data.

The similar criticism concerns, of course, the assumed behavior of strong interactions in the same region. Here, the validity of optical theorem given by Eq. (1) has been assumed practically in all theoretical studies. The optical theorem has been overtaken from optics without having been proved in the past. It will be shown that also all contemporary attempts to prove its validity in the case of strong interactions have been based on very complicated and false arguments.

As it has been already mentioned there are two main approaches that have been used in trying to derive the optical theorem for elastic scattering of two particles in the case of strong interactions. The main attempt to derive it has been done in the framework of $S$ matrix theory when some important assumptions have concerned the structure of corresponding Hilbert space as well as of $S$ matrix itself (see, e.g., [3]). In the other approach (introduced also in [3]) the ambition to derive the given theorem has been based on the wave theory. Both the approaches will be analyzed in the following.

3.1 $S$ matrix theory and transition operator

The $S$ operator has been assumed to transform initial state $|i\rangle$ directly to final one $|f\rangle$:

$$|f\rangle = S|i\rangle.$$  \hspace{1cm} (9)

The probability of corresponding transition has been given by matrix element

$$P_{i\rightarrow f} = |\langle f | S | i \rangle|^2.$$ \hspace{1cm} (10)

The given $S$ operator has been then required to fulfill the condition of unitarity

$$S^+S = SS^+ = I.$$ \hspace{1cm} (11)

Practically in all approaches attempting to derive optical theorem the $S$ matrix has been defined in the form (see, e.g., [3], p. 52)

$$S = I + iT$$ \hspace{1cm} (12)

where the introduction of unit matrix $I$ has been based necessarily on the assumption that final collision events have always been represented by the same state set as initial states. It has then followed from the unitarity of $S$ operator (Eq. (11)) and Eq. (12)

$$T^+T = i(T^+ - T).$$ \hspace{1cm} (13)
From this equation the usual optical theorem given by Eq. (1) has been derived under further additional assumptions; e.g., the final and initial states has been taken as identical. There is, however, problem with the definition of initial states being identical to the final states if the different deviations from original direction are measured in elastic short-ranged collisions while only one (singular value) of these values, i.e., $\theta = 0$, is to be attributed to the whole set of initial states. Any other initial states cannot exist under usual conditions.

More detailed analysis of equations Eqs. (10) to (12) allows us to derive the following conditions for corresponding probabilities

$$\sum_f P_{i \rightarrow f} = 1 - 2 \text{Im} T_{ii} + \sum_f |T_{if}|^2 = 1$$

or

$$\text{Im} T_{ii} = \frac{1}{2} \sum_f |T_{if}|^2$$

which should hold for any $i$ (this condition may be also derived from Eq. (4.51) of [3] for a final state being identical to the initial one). The transition $i \rightarrow i$ is to be interpreted as an event when any collision process has not existed (or has been fully negligible). The condition (15) requires then for the number of corresponding events to increase when the number of collision processes rises, which is undoubtedly a contradictory condition requiring practically $T = 0$; the given definition of $S$ matrix being admissible in perturbation approaches only. Anyway, the transition matrix added to unit matrix leads to quite unacceptable physical characteristics when the collision processes do not represent only a small perturbation. It means that in the case of strong interactions the $S$ matrix cannot be defined generally by the condition (12) and Eq. (13) has not any sense in this case.

The optical theorem may be, therefore, hardly valid in any case; especially, if it relates to a limit value lying in unmeasurable interval of events and derived from elastic processes only while in Eq. (15) the ratio of inelastic and elastic processes has not been respected at all. In addition to, the approach in [3] (see the passage between Eq. (4.53) to (4.55)) does not distinguish between the limit number of elastic (short-ranged) processes at zero scattering angle $\theta = 0$ and the number of non-interacting events corresponding also to “scattering” angle $\theta = 0$, even if the given processes are of quite different kinds.

The important problem concerns, however, also the definition of Hilbert space in which the corresponding $S$ matrix is to act. It has been shown in Sec. [2] that the initial and final states are to belong to mutually orthogonal subspaces if the collision processes are to be realistically interpreted. In such a case the $S$ matrix is to be represented by evolution operator transforming the state corresponding to a negative value of time $t$ to a state corresponding to a positive value (if the value $t = 0$ has been chosen to correspond to the states when colliding particles should have the smallest distance).

It is evident that the contemporary $S$ matrix theory does not provide any basis for the derivation of optical theorem validity.

### 3.2 Derivation based on wave approach

Other attempt to derive the optical theorem in strong interactions has been based on repeating the approach used in optics where a wave has been scattered by an
obstacle. The collision process has been described with the help of wave amplitudes (see, e.g., p. 16 in [3]). We shall not repeat the detailed approach here; only main points will be mentioned.

The initial collision state has been represented by a plane wave ($U_0$ is a constant)

$$U_{in}(x, y, z) = U_0 e^{ikz}. \quad (16)$$

The final state has been then expressed with the help of Fraunhofer diffraction, Babinet’s principle and Huygens-Fresnel principle in the form of sum of unscattered and scattered events

$$U_f(x, y, z) = U_{unsc}(x, y, z) + U_{scatt}(x, y, z) \quad (17)$$

The scattered wave has been expressed as

$$U_{scatt} = U_0 f(q) e^{ikz} \quad \frac{1}{r} \quad (18)$$

where $\vec{q} = \vec{k}' - \vec{k}$ is momentum transfer and $\vec{r} = (x, y, z)$ is a position vector ($|\vec{k}'| = |\vec{k}| = k$ in the case of elastic scattering); the squared modulus of $U_f(x, y, z)$ represents corresponding probability of outgoing wave, $U_{unsc}$ represents the non-interacting part of original beam. The outgoing scattered states $U_{scatt}$ are characterized by vectors $\vec{q}$.

It has been then written for the scattering amplitude

$$f(\vec{q}) = \frac{ik}{2\pi} \int d^2\vec{b} \Gamma(\vec{b}) e^{-i(\vec{q} \cdot \vec{b})} \quad (19)$$

where $\Gamma(\vec{b})$ has represented the profile of scattering center (obstacle). The intensity of the incident and of the scattered light has been taken as

$$I_{in} = |U_{in}|^2 = |U_0|^2 \quad (20)$$

and

$$I_{scatt} = |U_{scatt}|^2 = |U_0|^2 \frac{|f(\vec{q})|^2}{r^2} \quad (21)$$

respectively. These definition of intensities has been then used to define the elastic differential cross section as

$$\frac{d\sigma}{d\Omega} = \frac{I_{scatt}}{I_{in}} = |f(\vec{q})|^2 \quad (22)$$

The integrated elastic cross section has been then equal (in approximation for small scattered angles $\theta \cong \sin \theta$)

$$\sigma_{el} = \frac{1}{k^2} \int |f(\vec{q})|^2 d^2\vec{q} \cong \int d^2\vec{b} |\Gamma(\vec{b})|^2. \quad (23)$$

Now (see Eq. (2.37) in [3]) the absorption (inelastic) cross section has been defined with the help of function derived from experimental results obtained for elastic processes

$$\sigma_{abs} = \int d^2\vec{b} [2 \text{Re} \Gamma(\vec{b}) - |\Gamma(\vec{b})|^2]. \quad (24)$$

In such a case it has been then possible to write for total cross section

$$\sigma_{tot} = \sigma_{el} + \sigma_{abs} = 2 \int d^2\vec{b} \text{Re} \Gamma(\vec{b}) \quad (25)$$
and consequently (combining Eq. (25) with Eq. (19) for $\theta = 0$)

$$\sigma_{\text{tot}} = \frac{4\pi}{k} \text{Im} f(q = 0)$$  \hspace{1cm} (26)

which has been denoted as the optical theorem (see also Eq. (1)).

However, in the given approach the absorption cross section $\sigma_{\text{abs}}$ (see Eq. (24)) has been defined quite arbitrarily. It has been defined on the basis of elastic processes only while quite different characteristics are to be responsible for absorption (inelastic) processes.

It is, therefore, evident that the optical theorem cannot be applied to the case of strong interactions, even if it might be probably admitted for infinite-range Coulomb interaction (as a rough approximation when no inelastic processes exist) since it holds practically $U_{\text{unsc}} = 0$ in this case. There is surely a qualitative difference between these two kinds of forces as to the validity of optical theorem that must be denoted as unacceptable for very short-ranged (practically contact) interactions.

4 Impact parameter description of collision processes

The hitherto models of elastic nucleon collisions have been in principle phenomenological, looking for the simple description of main scattering characteristics. However, when one is to understand better the corresponding causality the distribution of initial states must be considered. It means that the uniform statistical distribution of individual tracks in initial states around the common center-of-mass of colliding particles should be taken into account; and the dependence of elastic collision probability on impact parameter value should be established.

Such a description trying to take into account the realistic behaviour in the impact parameter space has been proposed by us in 1994 \cite{15}, see also \cite{16,17}. However, even if it has been possible to study some new characteristics of elastic collisions on the basis of impact parameter the deformation caused by assuming the optical theorem validity has remained until now. Its invalidity has been discovered fully only recently.

In such a case a quite new approach may be applied to in describing elastic collision processes. Corresponding collision model of two protons has been recently proposed by us in \cite{18}. Starting from the ontological interpretation of colliding objects and assuming that these objects are non-spherical ones (differently oriented in space) one should expect that the probability of collision processes will depend mainly on the values of mutual impact parameter $b$ which should be uniformly distributed in corresponding cross plane. For the probability of elastic collisions it may be then written:

$$P_{\text{el}}(b) = P_{\text{tot}}(b) P_{\text{rat}}(b)$$  \hspace{1cm} (27)

where $P_{\text{tot}}(b)$ is the probability of all possible (elastic and inelastic) collision processes and $P_{\text{rat}}(b)$ is the mutual ratio of elastic to total probabilities at corresponding value of impact parameter $b$.

In the case of short-ranged (contact) strong interactions one can expect further on the basis of ontological realistic approach that elastic collisions will be mainly peripheral. The functions $P_{\text{tot}}(b)$ and $P_{\text{rat}}(b)$ may be assumed to be monotonous functions of $b$: the first one diminishing with rising $b$ and the other increasing in
the same interval of $b$. Practically any other assumption need not be added if the influence of any further characteristics of proton structure will not be studied. Both the monotonous functions may be then determined from corresponding experimental differential elastic cross section.

The new collision model has been applied (in its preliminary form) to experimental data represented by measured elastic proton-proton differential cross section at energy of 52.8 GeV [13]. It was possible to demonstrate explicitly that some new possibilities of fundamental particle research have been opened on this basis; including also some preliminary new characteristics of proton in dependence on impact parameter, see [13] for more details.

5 Conclusion

The optical theorem has been applied commonly to the description of elastic collision processes even if an actual proof of its validity has not been given. It has been shown convincingly in the preceding that especially in the case of short-ranged strong interaction between colliding objects any argument that might provide its validity cannot follow from the approaches trying to derive it.

In all attempts to prove the optical theorem it has been assumed that the inelastic cross section may be derived from the measured values of elastic differential cross section, which might be hardly reasoned. It is evident that the frequency of individual processes differs rather strongly according to structures of colliding objects as well as in dependence on collision energy values.

The problem is yet more complicated as the optical theorem concerns one point ($\theta = 0$) of elastic differential cross section, lying in the interval of non-measurable deviations. The determination of the given limit is practically always burdened further by the fact that the influence of much stronger Coulomb effect is to be subtracted. The application of the optical theorem may then introduce important unphysical limitation.

To respect the ontological characteristics of elastic collision processes the initial states and final states are to be represented in two mutually orthogonal subspaces of the Hilbert space formed by the solutions of corresponding Schroedinger equation (see Sec. 2). Only then the influence of particle dimensions may be fully respected; the statistical distribution of impact parameter values for initial states and that of angle deviations for final states being represented in individual subspaces. New elastic collision model [13] based on these requirements has been shortly characterized in Sec. 4. It might open a deeper insight concerning the characteristics of hadronic collision processes and itself hadronic structure.

References

[1] R. G. Newton: Optical theorem and beyond; Am. J. Phys. 44 (1976), 639-42.

[2] M. V. Lokajiček, V. Kundrát: Optical theorem and elastic nucleon scattering; [arXiv:0906.3961 (2009) (see also Proc. of 13th Int. Conf., Blois Workshop; [arXiv:1002.3527 [hep-ph]).

[3] V. Barone, E. Predazzi: High-energy particle diffraction; Springer-Verlag (2002).
[4] M. V. Lokajíček: Einstein-Bohr controversy after 75 years, its actual solution and consequences; Some Applications of Quantum Mechanics (ed. M. R. Pahlavani), InTech Publisher (February 2012), 409-24.

[5] H. Ioannidou: A new derivation of Schrödinger equation; Lett. al Nuovo Cim. 34 (1982), 453-8.

[6] U. Hoyer: Synthetische Quantentheorie; Georg Olms Verlag, Hildesheim (2002).

[7] M. V. Lokajíček: Schrödinger equation, classical physics and Copenhagen quantum mechanics; New Advances in Physics 1, No. 1 (2007), 69-77; see also /arxiv/quant-ph/0611176.

[8] P. D. Lax, R. S. Phillips: Scattering theory (Academic Press, 1967).

[9] P. D. Lax, R. S. Phillips: Scattering theory for automorphic functions (Princeton, 1976).

[10] V. Alda, V. Kundrát, M. V. Lokajíček: Exponential decay and irreversibility of decay and collision processes; Aplikace matematiky 19 (1974), 307-15.

[11] A. Aspect, P. Grangier, G. Roger: Experimental realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: A new violation of Bell’s inequalities; Phys. Rev. Lett. 49 (1982), 91.

[12] M. V. Lokajíček, V. Kundrát: The controversy between Einstein and Bohr after 75 years, its actual solution and correspondence for the present; Phys. Scr. T151 (2012) 014007.

[13] M. V. Lokajíček, V. Kundrát, J. Prochážka: Schrödinger equation and (future) quantum physics; in Advances in Quantum Mechanics, edited by P. Bracken (InTech Publisher, April 2013), 105-32.

[14] M. V. Lokajíček, V. Kundrát, J. Prochážka: Schroedinger equation and mistaking interpretation of Bell’s inequality; (2013) /arXiv:1305.5503[quant-ph].

[15] V. Kundrát, M. V. Lokajíček: High-energy elastic scattering amplitude of unpolarized and charged hadrons; Z. Phys. C 63 (1994), 619-29.

[16] V. Kundrát, M. V. Lokajíček, D. Krupa: Impact parameter structure derived from elastic collisions; Phys. Lett. B544 (2002), 132-138

[17] J. Kašpar, V. Kundrát, M. V. Lokajíček, J. Prochážka: Phenomenological models of elastic nucleon scattering and predictions for LHC; Nucl. Phys. B843 (2011), 84-106.

[18] M. V. Lokajíček, V. Kundrát: Elastic pp scattering and the internal structure of colliding protons; (2009) /arXiv:0909.3199[hep-ph].