CHAIN RECURRENCE FOR GENERAL SPACES

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In Memory of John Mather

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1. Introduction

Let $f$ be a continuous map on a compact metric space $(X,d)$. If $\epsilon \geq 0$ then a sequence $\{x_0, \ldots, x_n\}$ with $n \geq 1$ is an $\epsilon$ chain for $f$ if $\max_{i=1}^{n} d(f(x_{i-1}), x_i) \leq \epsilon$ and a strong $\epsilon$ chain for $f$ if $\Sigma_{i=1}^{n} d(f(x_{i-1}), x_i) \leq \epsilon$. Thus, a 0 chain is just an initial piece of an orbit sequence.

The Conley chain relation $\mathcal{C}_f$ consists of those pairs $(x, y) \in X \times X$ such that there is an $\epsilon$ chain with $x_0 = x$ and $x_n = y$ for every $\epsilon > 0$. The Easton, or Aubry-Mather, strong chain relation $\mathcal{A}_d f$ consists of those pairs $(x, y) \in X \times X$ such that there is a strong $\epsilon$ chain with $x_0 = x$ and $x_n = y$ for every $\epsilon > 0$. As the notation indicates, $\mathcal{C}_f$ is independent of the choice of metric, while $\mathcal{A}_d f$ depends on the metric. See [7] and [8].

Fathi and Pageault have studied these matters using what they call barrier functions, [14], [19] and their work has been sharpened by Wise-man [16], [17]. $M^f_d(x, y)$ is the infimum of the $\epsilon$'s such that there is an $\epsilon$ chain from $x$ to $y$ and $L^f_d(x, y)$ is the infimum of the $\epsilon$'s such that there is a strong $\epsilon$ chain from $x$ to $y$. Thus, $(x, y) \in \mathcal{C}_f$ iff $M^f_d(x, y) = 0$ and $(x, y) \in \mathcal{A}_d f$ iff $L^f_d(x, y) = 0$.

Our purpose here is to extend these results in two ways.

First, while our interest focuses upon homeomorphisms or continuous maps, it is convenient, and easy, to extend the results to relations, following [1].

A relation $f : X \rightarrow Y$ is just a subset of $X \times Y$ with $f(x) = \{y \in Y : (x, y) \in f\}$ for $x \in X$, and let $f(A) = \bigcup_{x \in A} f(x)$ for $A \subset X$. So $f$ is a mapping when $f(x)$ is a singleton set for every $x \in X$, in which case we will use the notation $f(x)$ for both the singleton set and the point contained therein. For example, the identity map on a set $X$ is $1_X = \{(x, x) : x \in X\}$. If $X$ and $Y$ are topological spaces then $f$ is a closed relation when it is a closed subset of $X \times Y$ with the product topology.

The examples $\mathcal{C}_f$ and $\mathcal{A}_d f$ illustrate how relations arise naturally in dynamics.

For a relation $f : X \rightarrow Y$ the inverse relation $f^{-1} : Y \rightarrow X$ is $\{(y, x) : (x, y) \in f\}$. Thus, for $B \subset Y$, $f^{-1}(B) = \{x : f(x) \cap B \neq \emptyset\}$. We define $f^*(B) = \{x : f(x) \subset B\}$. These are equal when $f$ is a map.

If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are relations then the composition $g \circ f : X \rightarrow Z$ is $\{(x, z) : \text{there exists } y \in Y \text{ such that } (x, y) \in f \text{ and } (y, z) \in g\}$. That is, $g \circ f$ is the image of $(f \times Z) \cap (X \times g)$ under the projection $\pi_{13} : X \times Y \times Z \rightarrow X \times Z$. As with maps, composition of relations is clearly associative.
The domain of a relation \( f : X \to Y \) is
\[
\text{Dom}(f) = \{ x : f(x) \neq \emptyset \} = f^{-1}(Y).
\]
We call a relation surjective if \( \text{Dom}(f) = X \) and \( \text{Dom}(f^{-1}) = Y \), i.e. \( f(X) = Y \) and \( f^{-1}(Y) = X \).

If \( f_1 : X_1 \to Y_1 \) and \( f_2 : X_2 \to Y_2 \) are relations, then the product relation \( f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 \) is \( \{ ((x_1, x_2), (y_1, y_2)) : (x_1, y_1) \in f_1, (x_2, y_2) \in f_2 \} \).

We call \( f \) a relation on \( X \) when \( X = Y \). In that case, we define, for \( n \geq 1 \), \( f^n = f \circ f^{n-1} = f \circ f \circ \cdots \circ f \) with \( f^1 = f \). By definition, \( f^0 = 1_X \) and \( f^{-n} = (f^{-1})^n \). If \( A \subset X \), then \( A \) is called \( f \) invariant if \( f(A) \subset A \) and \( f \) invariant if \( f(A) = A \). In general, for \( A \subset X \), the restriction to \( A \) is \( f|_A = f \cap (A \times A) \). If \( u \) is a real-valued function on \( X \) we will also write \( u|_A \) for the restriction of \( u \) to \( A \), allowing context to determine which meaning is used.

The cyclic set \( |f| \) of a relation \( f \) on \( X \) is \( \{ x \in X : (x, x) \in f \} \).

A relation \( f \) on \( X \) is reflexive if \( 1_X \subset f \), symmetric if \( f^{-1} = f \) and transitive if \( f \circ f \subset f \).

If \( d \) is a pseudo-metric on a set \( X \) and \( \epsilon > 0 \), then \( V^d_\epsilon = \{ (x, y) : d(x, y) < \epsilon \} \) and \( \bar{V}^d_\epsilon = \{ (x, y) : d(x, y) \leq \epsilon \} \). Thus, for \( x \in X \), \( V^d_\epsilon(x) \) (or \( \bar{V}^d_\epsilon(x) \)) is the open (resp. closed) ball centered at \( x \) with radius \( \epsilon \).

A pseudo-ultrametric \( \bar{d} \) on \( X \) is a pseudo-metric with the triangle inequality strengthened to \( d(x, y) \leq \max(d(x, z), d(z, y)) \) for all \( z \in X \). A pseudo-metric \( d \) is a pseudo-ultrametric iff the relations \( V^d_\epsilon \) and \( \bar{V}^d_\epsilon \) are equivalence relations for all \( \epsilon > 0 \).

If \( (X_1, d_1) \) and \( (X_2, d_2) \) are pseudo-metric spaces then the product \( (X_1 \times X_2, d_1 \times d_2) \) is defined by
\[
d_1 \times d_2((x_1, x_2), (y_1, y_2)) = \max(d_1(x_1, y_1), d_2(x_2, y_2)).
\]
Thus, \( V^{d_1 \times d_2}_\epsilon = V^{d_1}_\epsilon \times V^{d_2}_\epsilon \) and \( \bar{V}^{d_1 \times d_2}_\epsilon = \bar{V}^{d_1}_\epsilon \times \bar{V}^{d_2}_\epsilon \).

Throughout this work, all pseudo-metrics are assumed bounded. For example, on \( \mathbb{R} \) we use \( d(a, b) = \min(|a - b|, 1) \). Thus, if \( A \) is a non-empty subset of \( X \) the diameter \( \text{diam}(A) = \sup\{d(x, y) : x, y \in A\} \) is finite.

For metric computations, the following will be useful.

**Lemma 1.1.** Let \( a_1, a_2, b_1, b_2 \in \mathbb{R} \). With \( a \lor b = \max(a, b) \) and \( a \land b = \min(a, b) \):
\[
|a_1 \lor b_1 - a_2 \lor b_2|, |a_1 \land b_1 - a_2 \land b_2| \leq |a_1 - a_2| \lor |b_1 - b_2|.
\]
\[
(a_1 \lor b_1) \land (a_2 \lor b_1) \land (a_1 \lor b_2) \land (a_2 \lor b_2) = (a_1 \land a_2) \lor (b_1 \land b_2).
\]
Proof: First, we may assume without loss of generality that \( a_1 \lor b_1 \geq a_2 \lor b_2 = a_2 \) and so that \( a_2 \geq b_2 \). If \( a_1 \lor b_1 = a_1 \) then \( |a_1 \lor b_1 - a_2 \lor b_2| = a_1 - a_2 \). If \( a_1 \lor b_1 = b_1 \) then \( |a_1 \lor b_1 - a_2 \lor b_2| = b_1 - a_2 \leq b_1 - b_2 \). For the \( \land \) estimate, observe that \( a \land b = -(a) \lor (b) \).

For the second, factor out \( b_1 \) and \( b_2 \) to get \((a_1 \lor b_1) \land (a_2 \lor b_1) = (a_1 \land a_2) \lor b_1\), and \((a_1 \lor b_2) \land (a_2 \lor b_2) = (a_1 \land a_2) \lor b_2\). Then factor out \( a_1 \land a_2 \).

\[ \square \]

The other extension is to non-compact spaces. This has been looked at in the past, see [11] and [14]. However, the natural setting for the theory is that of uniform spaces as described in [12] and [5], and reviewed in Appendix B below.

A uniform structure \( U \) on a set \( X \) is a collection of relations on \( X \) which satisfy various axioms so as to generalize the notion of metric space. To be precise, a \( U \) is a uniformity when

- \( 1_X \subset U \) for all \( U \in \mathcal{U} \).
- \( U_1, U_2 \in \mathcal{U} \) implies \( U_1 \cap U_2 \in \mathcal{U} \).
- If \( U \in \mathcal{U} \) and \( W \supset U \), then \( W \in \mathcal{U} \).
- \( U \in \mathcal{U} \) implies \( U^{-1} \in \mathcal{U} \).
- If \( U \in \mathcal{U} \), then there exists \( W \in \mathcal{U} \) such that \( W \circ W \subset U \).

The first condition says that the relations are reflexive and the next two imply that they form a filter.

A uniformity \( \mathcal{U} \) is equivalently given by its gage \( \Gamma(\mathcal{U}) \), the set of pseudo-metrics \( d \) on \( X \) (bounded by stipulation) with the metric uniformity \( \mathcal{U}(d) \), generated by \( \{ V^d_\epsilon : \epsilon > 0 \} \), contained in \( \mathcal{U} \). The use of covers in [14] and continuous real-valued functions in [11] are equivalent to certain choices of uniformity. To a uniformity there is an associated topology and we say that \( \mathcal{U} \) is compatible with a topology on \( X \) if the uniform topology agrees with the given topology on \( X \). A topological space admits a compatible uniformity iff it is completely regular. A completely regular space \( X \) has a maximum uniformity \( \mathcal{U}_M \) compatible with the topology. Any continuous function from a completely regular space \( X \) to a uniform space is uniformly continuous from \((X, \mathcal{U}_M)\).

A completely regular, Hausdorff space is called a Tychonoff space. A compact Hausdorff space \( X \) has a unique uniformity consisting of all neighborhoods of the diagonal \( 1_X \).

In Section 2, we define the barrier functions \( m^f_d \) and \( \ell^f_d \) of a relation \( f \) on a set \( X \) with respect to a pseudo-metric \( d \) and we describe their elementary properties. We use a symmetric definition which allows a jump at the beginning as well as the end of a sequence. In Section
6, we show that the alternative definitions yield equivalent results in cases which include when \( f \) is a continuous map.

In Section 3, we describe the properties of the Conley relation \( \mathcal{C}_d f = \{(x, y) : m_d^f(x, y) = 0\} \) and the Aubrey-Mather relation \( \mathcal{A}_d f = \{(x, y) : \ell_d^f(x, y) = 0\} \). Following \[1\] we regard \( \mathcal{C}_d \) and \( \mathcal{A}_d \) as operators on the set of relations on \( X \). We observe that each of these operators is idempotent.

In Section 4, we consider Lyapunov functions. With the pseudometric \( d \) fixed, a Lyapunov function \( L \) for a relation \( f \) on \( X \) is a continuous map \( L : X \to \mathbb{R} \) such that \( (x, y) \in f \) implies \( L(x) \leq L(y) \), or, equivalently, \( f \subset \leq_L \) where \( \leq_L = \{(x, y) : L(x) \leq L(y)\} \). Notice that we follow \[1\] in using Lyapunov functions which increase, rather than decrease, on orbits. Following \[14\] and \[9\] we show that the barrier functions can be used to define Lyapunov functions. If \( g \) is a relation on \( X \) with \( f \subset g \) and \( z \in X \) then \( x \mapsto m_d^g(x, z) \) is a Lyapunov function for \( \mathcal{C}_d f \) and \( x \mapsto \ell_d^g(x, z) \) is a Lyapunov function for \( \mathcal{A}_d f \). Even when \( f \) is a map, it is convenient to use associated relations like \( g = f \cup 1_X \) or \( g = f \cup \{(y, y)\} \) for \( y \) a point of \( X \).

In Section 5, we turn to uniform spaces. The Conley relation \( \mathcal{C}_U f \) is the intersection of \( \{\mathcal{C}_d f : d \in \Gamma(U)\} \) and \( \mathcal{A}_U f \) is the intersection of \( \{\mathcal{A}_d f : d \in \Gamma(U)\} \). Thus, \( (x, y) \in \mathcal{C}_U f \) iff \( m_d^f(x, y) = 0 \) for all \( d \in \Gamma(U) \) and similarly \( (x, y) \in \mathcal{A}_U f \) iff \( \ell_d^f(x, y) = 0 \) for all \( d \in \Gamma(U) \). While the gage definition is convenient to use, we show that each of these relations has an equivalent description which uses the uniformity directly. Each of these is a closed, transitive relation which contains \( f \). We let \( \mathcal{G} f \) denote the smallest closed, transitive relation which contains \( f \), so that \( f \subset \mathcal{G} f \subset \mathcal{A}_U f \subset \mathcal{C}_U f \).

If \( L \) is a uniformly continuous Lyapunov function for \( f \) then it is automatically a Lyapunov function for \( \mathcal{A}_U f \). If \( X \) is Hausdorff and we let \( L \) vary over all uniformly continuous Lyapunov functions for \( f \) then \( 1_X \cup \mathcal{A}_U f = \bigcap_L \leq_L \). That is, if \( (x, y) \notin 1_X \cup \mathcal{A}_U f \), then there exists a uniformly continuous Lyapunov function \( L \) such that \( L(x) > L(y) \). If, in addition, \( X \) is second countable, then there exists a uniformly continuous Lyapunov function \( L \) such that \( 1_X \cup \mathcal{A}_U f = \leq_L \). If \( X \) is Hausdorff and we let \( L \) vary over all Lyapunov functions for \( \mathcal{C}_U f \) then \( 1_X \cup \mathcal{C}_U f = \bigcap_L \leq_L \). If, in addition, \( X \) is second countable, then there exists a Lyapunov function \( L \) such that \( 1_X \cup \mathcal{C}_U f = \leq_L \). These results use the barrier function Lyapunov functions developed in the preceding section.

For the Conley relation there are special results. A set \( A \) is called \( U \) inward for a relation \( f \) on \( (X, U) \) if for some \( U \in U \), \( (U \circ f)(A) \subset A \).
A continuous function \( L : X \to [0, 1] \) is called an \textit{elementary Lyapunov function} if \((x, y) \in f\) and \(L(x) > 0\) imply \(L(y) = 1\). For a \( \mathcal{U} \) uniformly continuous elementary Lyapunov function \( L \) the sets \( \{x : L(x) > \epsilon\} \) for \( \epsilon \geq 0 \) are open \( \mathcal{U} \) inward sets. On the other hand, if \( A \) is a \( \mathcal{U} \) inward set, then there exists a \( \mathcal{U} \) uniformly continuous elementary Lyapunov function \( L \) such that \( L = 0 \) on \( X \setminus A \) and \( L = 1 \) on \( f(A) \). Each set \( \mathcal{C}_u f(x) \) is an intersection of inward sets. If \( A \) is an open \( \mathcal{U} \) inward set then it is \( \mathcal{C}_u f \) invariant and the maximum \( \mathcal{C}_u f \) invariant subset \( A_\infty \) is called the associated \textit{attractor}.

Additional results can be obtained when the relation \( f \) satisfies various topological conditions. In Section 6, we consider \textit{upper semicontinuous} (= usc) and \textit{compactly upper semicontinuous relations} (= cusc) relations and related topological results. Regarded as a relation, a continuous map is cusc. If a Hausdorff space \( X \) is locally compact and \( \sigma \)—compact, or locally compact and paracompact with \( f \) cusc, then \( \mathcal{G} f = \mathcal{A}_{\mathcal{U} f} \). We exhibit a homeomorphism on a metric space for which the inclusion \( \mathcal{G} f \subset \mathcal{A}_{\mathcal{U} f} \) is proper.

At the end of the section we consider compactifications and the special results which hold for a compact Hausdorff space. In the Hausdorff uniform space context, one proceeds by finding a totally bounded uniformity \( \mathcal{T} \subset \mathcal{U} \) which is compatible with the topology on \( X \) and then take the uniform completion.

\textbf{Theorem 1.2.} Let \( f \) be a closed relation on a Hausdorff uniform space \((X, \mathcal{U})\) with \( X \) second countable. There exists \( \mathcal{T} \subset \mathcal{U} \) a totally bounded uniformity, with \((\bar{X}, \mathcal{T})\) the completion of \((X, \mathcal{T})\), such that the space \( \bar{X} \) is a compact Hausdorff space with its unique uniformity \( \mathcal{T} \) metrizable.

Let \( \bar{f} \) be the closure of \( f \) in \( \bar{X} \times \bar{X} \). The uniformity \( \mathcal{T} \) can be chosen so that

\begin{align}
\bar{f} \cap (X \times X) &= f, \quad 1_X \cup \mathcal{G} \bar{f} \cap (X \times X) = 1_X \cup \mathcal{A}_u \mathcal{U} f, \\
\mathcal{C} \bar{f} \cap (X \times X) &= \mathcal{C}_u f.
\end{align}

If \( f \) is cusc, e.g. a continuous map, then \( \mathcal{G} \bar{f} \cap (X \times X) = \mathcal{A}_u \mathcal{U} f \).

If \( f \) is a uniformly continuous map then, in addition, we can choose \( \mathcal{T} \) so that \( \bar{f} \) is a continuous map on \( \bar{X} \). If \( f \) is a uniform isomorphism then, in addition, we can choose \( \mathcal{T} \) so that \( \bar{f} \) is a homeomorphism on \( \bar{X} \).

If \( X \) is a compact Hausdorff space, then every closed, \( \mathcal{C} f \) invariant set \( K \) is an intersection of inward sets. If a closed set \( K \) is \( \mathcal{C} f \) invariant then it is an intersection of attractors and \( K \) is determined by \( K \cap |\mathcal{C} f| \) which we call its \textit{trace}. In fact, \( K = \mathcal{C} f(K \cap |\mathcal{C} f|) \). \( K \) is an attractor.
iff it is closed and \( C_f \) invariant and, in addition, its trace is a clopen subset of \(| C_f |\).

In Section 7, we consider totally recurrent and chain transitive relations. Let \( f \) be a relation on a uniform space \((X, U)\) and let \( d \in \Gamma(U)\).

For \( F = \mathcal{S}_f, \mathcal{A}_d f, \mathcal{A}_U f, \mathcal{C}_d f \) or \( \mathcal{C}_U f \) we will say that \( f \) is \emph{totally} \( F \) recurrent when \( F \) is an equivalence relation. If \( f \) is a uniformly continuous map then \( f \) is totally \( F \) recurrent iff \( 1_X \subset F \), i.e. \( F \) is reflexive.

If \( \mathcal{A}_U f \) is an equivalence relation then the quotient space \( X/\mathcal{A}_U f \) is \emph{completely Hausdorff}, i.e. the continuous real-valued functions distinguish points. On the other hand, there exist examples such that the quotient is not regular and so the topology is strictly finer than the weak topology generated by the continuous functions. The latter is completely regular and the barrier functions \( \ell_d^f \), when symmetrized, generate the gage of a compatible uniformity.

Similarly, if \( \mathcal{C}_U f \) is an equivalence relation then the quotient space \( X/\mathcal{C}_U f \) is \emph{totally disconnected}, i.e. the clopen sets distinguish points. Again there exist examples such that the quotient is not regular and so the topology is strictly finer than the weak topology generated by the clopen subsets, i.e. it is not \emph{zero-dimensional}. The barrier functions \( m_d^f \), when symmetrized, are pseudo-ultrametrics generating the gage of a uniformity compatible with the latter zero-dimensional topology.

The relation \( f \) is called \( U \) \emph{chain transitive} when \( \mathcal{C}_U f = X \times X \). It is called \( U \) \emph{chain-mixing} if for every pair of points \( x, y \in X \) and for every \( d \in \Gamma(U) \) and \( \epsilon > 0 \) there exists a positive integer \( N \) such that for every \( n \geq N \) there are \( \epsilon, d \) chains of length \( n \) connecting \( x \) and \( y \). A \( U \) chain-transitive relation \( f \) is not \( U \) chain-mixing iff there exists a \( U \) uniformly continuous map taking \( f \) to a non-trivial periodic cycle. It follows that \( f \) is \( U \) chain-mixing iff the product relation \( f \times f \) is \( U \) chain-transitive. If \( f \) is a \( U \) uniformly continuous map, then it is \( U \) chain-mixing iff for every positive integer \( n \) the iterate \( f^n \) is \( U \) chain-transitive.

In Section 8 we restrict to compact metrizable spaces. The relation \( \mathcal{S}_f \) is the intersection of the \( \mathcal{A}_d f \)'s as \( d \) varies over \( \tilde{\Gamma} \), the set of metrics compatible with the topology. If we take the union, which we denote \( \mathcal{W}_f \), it is not obvious that the result is closed or transitive. We prove it is both by giving a uniformity characterization. The set \(| \mathcal{W}_f |\) is referred to as the Mañé set by Fathi and Pageault. Using the uniformity characterization we give an alternative proof of their description, for a homeomorphism \( f \), \(| \mathcal{W}_f | = |f| \cup |\mathcal{C}(f)(X \setminus |f|)\)|.
2. Barrier Functions

Let $f$ be a relation on a pseudo-metric space $(X,d)$. That is, $f$ is a subset of $X \times X$ and $d$ is a pseudo-metric on the non-empty set $X$.

Let $f^n$ be the $n$–fold product of copies of $f$, i.e. the space of sequences in $f$ of length $n \geq 1$, so that an element of $f^n$ is a sequence $[a,b] = (a_1,b_1), (a_2,b_2), \ldots, (a_n,b_n)$ of pairs in $f$. If $[a,b] \in f^n$, $[c,d] \in f^{n+m}$, then the concatenation $[a,b] \cdot [c,d] \in f^{n+m}$ is the sequence of pairs $(x_i,y_i) = (a_i,b_i)$ for $i = 1, \ldots, n$ and $(x_i,y_i) = (c_{i-n},d_{i-n})$ for $i = n+1, \ldots, n+m$.

Define for $(x,y) \in X \times X$ and $[a,b] \in f^n$ the $xy$ chain-length of $[a,b]$ (with respect to $d$) to be the sum
\begin{equation}
(2.1) \quad d(x,a_1) + \Sigma_{i=1}^{n-1} d(b_i,a_{i+1}) + d(b_n,y)
\end{equation}
and the $xy$ chain-bound of $[a,b]$ (with respect to $d$) to be
\begin{equation}
(2.2) \quad \max(d(x,a_1), d(b_1,a_2), \ldots , d(b_{n-1},a_n), d(b_n,y)).
\end{equation}
That is, for the vector $(d(x,a_1), d(b_1,a_2), \ldots , d(b_{n-1},a_n), d(b_n,y))$, the chain-length is the $L^1$ norm and the chain-bound is the $L^\infty$ norm. We could proceed as below, using the $L^p$ norm for any $1 \leq p \leq \infty$.

For $(x,y) \in X \times X$, define
\begin{equation}
(2.3) \quad \ell_d^f(x,y) = \inf \{d(x,a_1) + \Sigma_{i=1}^{n-1} d(b_i,a_{i+1}) + d(b_n,y) : [a,b] \in f^n, n = 1,2,\ldots \}.
\end{equation}
\begin{equation}
(2.4) \quad m_d^f(x,y) = \inf \{\max(d(x,a_1), d(b_1,a_2), \ldots , d(b_{n-1},a_n), d(b_n,y)) : [a,b] \in f^n, n = 1,2,\ldots \}.
\end{equation}
The functions $\ell_d^f$ and $m_d^f$ are the barrier functions for $f$. Clearly, $m_d^f \leq \ell_d^f$.

Using $n = 1$, we see that for all $(a,b) \in f$
\begin{equation}
(2.4) \quad \ell_d^f(x,y) \leq d(x,a) + d(b,y),
\end{equation}
\begin{equation}
(2.5) \quad \quad m_d^f(x,y) \leq \max(d(x,a), d(b,y)).
\end{equation}
and so
\begin{equation}
(2.5) \quad (x,y) \in f \Rightarrow m_d^f(x,y) = \ell_d^f(x,y) = 0.
\end{equation}
by using $(a,b) = (x,y)$.

For the special case of $f = \emptyset$ we define
\begin{equation}
(2.6) \quad m_d^0 = \text{diam}(X), \quad \ell_d^0 = 2\text{diam}(X),
\end{equation}
the constant functions.
By using equation (2.4) with \((a, b) = (y, y)\) and the triangle inequality in (2.3) we see that

\[
\ell^1_d(x, y) = d(x, y).
\]

Define for the pseudo-metric \(d\)

\[
Z_d = \{(x, y) : d(x, y) = 0\}.
\]

Thus, \(Z_d\) is a closed equivalence relation which equals \(1_X\) exactly when \(d\) is a metric. \(Z_d\) is the closure in \(X \times X\) of the diagonal \(1_X\).

Lemma 2.1. Let \(f\) be a relation on \((X, d)\) with \(A = \text{Dom}(f) = f^{-1}(X)\). If \(f \subset Z_d\), then

\[
\ell_d^f(x, y) = \inf \{d(x, a) + d(a, y) : a \in A\} \geq d(x, y)
\]

with equality if either \(x\) or \(y\) is an element of \(A\).

If \(d\) is a pseudo-ultrametric then

\[
m_d^f(x, y) = \inf \{\max(d(x, a), d(a, y)) : a \in A\} \geq d(x, y)
\]

with equality if either \(x\) or \(y\) is an element of \(A\).

Proof: If \((a, b) \in f\) then \(d(a, b) = 0\) and so the \(xy\) chain-length of \([(a, b)]\) is \(d(x, a) + d(a, y)\). If \([a, b] \in f^{\times n}\) then \(d(a_i, b_i) = 0\) for all \(i\) implies that with \(a = a_1\) the \(xy\) chain-length of \([a, b]\) is at least \(d(x, a) + d(a, y)\) by the triangle inequality.

If \(d\) is a pseudo-ultrametric then the \(xy\) chain-bound of \([(a, b)]\) is \(\max(d(x, a), d(a, y))\) and if \([a, b] \in f^{\times n}\), then with \(a = a_1\) the \(xy\) chain-bound of \([a, b]\) is at least \(\max(d(x, a), d(a, y))\) by the ultrametric version of the triangle inequality.

\[\square\]

In particular, if \(A\) is a nonempty subset of \(X\), then

\[
\ell_d^1(x, y) = \inf \{d(x, a) + d(a, y) : a \in A\} \geq d(x, y)
\]

with equality if either \(x\) or \(y\) is an element of \(A\).

It is clear that \(f \subset g\) implies \(f^{\times n} \subset g^{\times n}\) and so

\[
f \subset g \implies \ell_d^g \leq \ell_d^f \quad \text{and} \quad m_d^g \leq m_d^f \quad \text{on} \quad X \times X.
\]

In particular, if \(A\) is a subset of \(X\), then

\[
\ell_d^f \leq \ell_d^{1|A} \quad \text{and} \quad m_d^f \leq m_d^{1|A}.
\]

The relation \(f\) is reflexive when \(1_X \subset f\). We see from (2.7)

\[
1_X \subset f \implies \ell_d^f \leq d \quad \text{on} \quad X \times X.
\]
If \([a, b] \in f^{\times n}\), then we let \([a, b]^{-1} \in (f^{-1})^{\times n}\) be \((b_n, a_n), (b_{n-1}, a_{n-1}), \ldots, (b_1, a_1)\). Using these reverse sequences we see immediately that

\[
\ell_{d}^{f}(x, y) = \ell_{d}^{f^{-1}}(y, x) \quad \text{and} \quad m_{d}^{f}(x, y) = m_{d}^{f^{-1}}(y, x)
\]

for all \(x, y \in X\).

**Proposition 2.2.** Let \(f\) be a relation on \((X, d)\). Let \(x, y, z, w \in X\).

(a) The directed triangle inequalities hold:

\[
\ell_{d}^{f}(x, y) \leq \ell_{d}^{f}(x, z) + \ell_{d}^{f}(z, y),
\]

\[
m_{d}^{f}(x, y) \leq m_{d}^{f}(x, z) + m_{d}^{f}(z, y).
\]

(b) Related to the ultrametric inequalities, we have:

\[
m_{d}^{f}(x, y) \leq \max(m_{d}^{f}(x, z) + m_{d}^{f}(z, y), m_{d}^{f}(z, y) + m_{d}^{f}(y, x)).
\]

(c) From

\[
\ell_{d}^{f}(x, y) \leq d(x, w) + \ell_{d}^{f}(w, z) + d(z, y) \quad \text{for all} \quad w, x, y, z \in X,
\]

\[
m_{d}^{f}(x, y) \leq d(x, w) + m_{d}^{f}(w, z) + d(z, y) \quad \text{for all} \quad w, x, y, z \in X
\]

we obtain that the functions \(\ell_{d}^{f}\) and \(m_{d}^{f}\) from \(X \times X\) to \(\mathbb{R}\) are Lipschitz with Lipschitz constant \(\leq 2\).

**Proof:** (a) For \(x, y, z \in X\) and \([a, b] \in f^{\times n}, [c, d] \in f^{\times m}\), we note that \(d(b_n, c_1) \leq d(b_n, z) + d(z, c_1)\). So the \(x z\) chain-length of \([a, b]\) plus the \(zy\) chain-length of \([c, d]\) is greater than or equal to the \(x y\) chain-length of \([a, b] \cdot [c, d]\). Furthermore, the \(x z\) chain-bound of \([a, b]\) plus the \(zy\) chain-bound of \([c, d]\) is greater than or equal to the \(x y\) chain-bound of \([a, b] \cdot [c, d]\). The directed triangle inequalities \([2.16]\) follow.

(b) Let \([u, v] \in f^{\times p}\). We see that \(d(b_n, u_1) \leq d(b_n, z) + d(z, u_1)\) and \(d(v_p, c_1) \leq d(v_p, z) + d(z, c_1)\). Hence, the larger of the \(x z\) chain-bound of \([a, b]\) plus the \(zy\) chain-bound of \([u, v]\) and the \(zz\) chain-bound of \([a, b] \cdot [u, v]\) bounds the \(x y\) chain-bound of \([a, b] \cdot [u, v] \cdot [c, d]\). This implies \([2.17]\).

(c) Similarly, \(d(x, a_1) \leq d(x, w) + d(w, a_1)\) and \(d(b_n, y) \leq d(b_n, z) + d(z, y)\) implies \([2.18]\) from which the Lipschitz results are clear.

\(\Box\)

If \(h\) is a map from \((X_1, d_1)\) to \((X_2, d_2)\) then \(h\) is uniformly continuous if for every \(\epsilon > 0\) there exists \(\delta > 0\) such that \(d_{1}(x, y) < \delta\) implies \(d_{2}(h(x), h(y)) < \epsilon\) for all \(x, y \in X_1\). We call \(\delta\) an \(\epsilon\) modulus of uniform continuity. The map \(h\) is Lipschitz with constant \(K\) if \(d_{2}(h(x), h(y)) \leq K d_{1}(x, y)\) for all \(x, y \in X_1\).
If $f_1$ is a relation on $X_1$ and $f_2$ is a relation on $X_2$ then we say that a function $h : X_1 \to X_2$ maps $f_1$ to $f_2$ if $(h \times h)(f_1) \subset f_2$, i.e. $(x,y) \in f$ implies $(h(x), h(y))$. Since $h$ is a map, $1_{X_1} \subset h^{-1} \circ h$ and $h \circ h^{-1} \subset 1_{X_2}$. From these it easily follows that

$$(h \times h)(f_1) = h \circ f_1 \circ h^{-1},$$

$$(h \times h)(f_1) \subset f_2 \iff h \circ f_1 \subset f_2 \circ h.$$  

If $h$ maps $f_1$ to $f_2$ then clearly $h$ maps $f_1^{-1}$ and $f_2^{-1}$ and

$$h(|f_1|) \subset |f_2|.$$  

**Proposition 2.3.** Let $f_1$ and $f_2$ be relations on $(X_1, d_1)$ and $(X_2, d_2)$, respectively. Assume $h : X_1 \to X_2$ maps $f_1$ to $f_2$.

(a) If $h$ is uniformly continuous then for $\epsilon > 0$ with $\delta > 0$ an $\epsilon$ modulus of uniform continuity, $m_{d_1}^{f_1}(x,y) < \delta$ implies $m_{d_2}^{f_2}(h(x), h(y)) < \epsilon$ for all $x, y \in X_1$.

(b) If $h$ is Lipschitz with constant $K$ then $\ell_{d_1}^{f_1}(x,y) \leq K \ell_{d_2}^{f_2}(h(x), h(y))$ for all $x, y \in X_1$.

**Proof:** If $[a,b] \in f_1^x \setminus$ then $(h \times h)^x([a,b]) \in f_2^x$. If $\delta$ is an $\epsilon$ modulus of uniform continuity then if the $xy$ chain-bound of $[a,b]$ is less then $\delta$ then the $h(x)h(y)$ chain-bound of $(h \times h)^x([a,b])$ is less than $\epsilon$. If $h$ is Lipschitz with constant $K$ then the $h(x)h(y)$ chain-length is at most $K$ times the $xy$ chain-length.

\[\Box\]

3. The Conley and Aubry-Mather Chain-Relations

For a relation $f$ on $(X, d)$, the *Conley chain relation* $\mathcal{C}_d f$ is defined by

$$\mathcal{C}_d f = \{(x, y) : m_d^f(x, y) = 0\},$$

and the *Aubry-Mather chain relation* is defined by

$$\mathcal{A}_d f = \{(x, y) : \ell_d^f(x, y) = 0\}.$$  

Because $m_d^f$ and $\ell_d^f$ are continuous, it follows that $\mathcal{C}_d f$ and $\mathcal{A}_d f$ are closed in $(X \times X, d \times d)$. From the directed triangle inequalities (2.16), it follows that $\mathcal{C}_d f$ and $\mathcal{A}_d f$ are transitive, i.e.

$$\mathcal{C}_d f \circ \mathcal{C}_d f \subset \mathcal{C}_d f,$$

$$\mathcal{A}_d f \circ \mathcal{A}_d f \subset \mathcal{A}_d f.$$  

From (2.5) we see that,
\[(3.4) \quad f \subset A_d f \subset C_d f.\]

If \( A \subset X \) with \( f \subset A \times A \) we can regard \( f \) as a relation on \((X, d)\) or as a relation on \((A, d|A \times A)\) where \( d|A \times A \) is the restriction of the pseudo-metric \( d \) to \( A \times A \). It is clear that if \( f \subset A \times A \), then
\[(3.5) \quad m_d^f|(A \times A) = m_{d|(A \times A)}^f \quad \text{and} \quad \ell_d^f|(A \times A) = \ell_{d|(A \times A)}^f.\]

and so
\[(3.6) \quad (C_d f) \cap (A \times A) = C_{d|(A \times A)} f \quad \text{and} \quad (A_d f) \cap (A \times A) = A_{d|(A \times A)} f.\]

If \( A \) is closed and \( x, y \in A \) with either \( x \notin A \) or \( y \notin A \), then \( \ell_d^f(x, y) \geq m_d^f(x, y) > 0 \) and so
\[(3.7) \quad (C_d f) = C_{d|(A \times A)} f \quad \text{and} \quad (A_d f) = A_{d|(A \times A)} f.\]

From (2.12) we get monotonicity
\[(3.8) \quad f \subset g \implies C_d f \subset C_d g \quad \text{and} \quad A_d f \subset A_d g.\]

and from (2.15)
\[(3.9) \quad C_d(f^{-1}) = (C_d f)^{-1} \quad \text{and} \quad A_d(f^{-1}) = (A_d f)^{-1},\]

and so we can omit the parentheses.

**Proposition 3.1.** Let \( f, g \) be relations on \( X \).

\[(3.10) \quad m_{d,A}^f = m_d^f \quad \text{and} \quad \ell_{d,A}^f = \ell_d^f.\]

The operators \( C_d \) and \( A_d \) on relations are idempotent. That is,
\[(3.11) \quad C_d(C_d f) = C_d f \quad \text{and} \quad A_d(A_d f) = A_d f.\]

In addition,
\[(3.12) \quad C_d(C_d f \cap C_d g) = C_d f \cap C_d g \quad \text{and} \quad A_d(A_d f \cap A_d g) = A_d f \cap A_d g.\]

**Proof:** Since \( f \subset A_d f \subset C_d f \) it follows from (2.12) that \( m_d^{C_d f} \leq m_d^f \)

and \( \ell_d^{A_d f} \leq \ell_d^f. \)

For the reverse inequality fix \( x, y \in X \) an let \( t > \ell_d^{A_d f}(x, y) \) be arbitrary. Choose \( t_1 \) with \( t > t_1 > \ell_d^{A_d f}(x, y) \). Suppose that \([a, b] \in (A_d f)^{\times n}\)
whose \( xy \) chain-length is less than \( t_1 \). Let \( \epsilon = (t-t_1)/2n \) For \( i = 1, \ldots, n \)
we can choose an element of some \( f^{\times n_i} \) whose \( a_i b_i \) chain-length is less
than \( \epsilon \). Concatenating these in order we obtain a sequence in \( f^{\times m} \)
with \( m = \Sigma_{i=1}^n n_i \) whose \( xy \) chain-length is at most \( t_1 + 2n\epsilon \leq t \). Hence,
\( \ell_d^f(x, y) \leq t \). Letting \( t \) approach \( \ell_d^{A_d f}(x, y) \) we obtain in the limit that
\( \ell_d^f(x, y) \leq \ell_d^{A_d f}(x, y). \)

The argument to show \( m_d^{C_d f}(x, y) \leq m_d^f(x, y) \) is completely similar.
It is clear that (3.10) implies (3.11).

Finally, $C_d f \cap C_d g \subset C_d (C_d f \cap C_d g) \subset C_d (C_d f) = C_d f$ and similarly, $C_d f \cap C_d g \subset C_d (C_d f \cap C_d g) \subset C_d g$. Intersect to get (3.12) for $C_d$ and the same argument yields the $A_d$ result.

\[ \blacksquare \]

**Corollary 3.2.** For a relation $f$ on $(X,d)$ let $ar{f}^d$ be the closure of $f$ in $(X \times X,d \times d)$.

\[(3.13)\]

\[
m_d^f = m_d^{\bar{f}^d} \quad \text{and} \quad \ell_d^f = \ell_d^{\bar{f}^d},
\]

\[
C_d (\bar{f}^d) = C_d f \quad \text{and} \quad A_d (\bar{f}^d) = A_d f
\]

**Proof:** This is clear from (2.12) and (3.10) because $f \subset \bar{f}^d \subset A_d f \subset C_d f$.

\[ \blacksquare \]

The *Conley set* is the cyclic set $|C_d f| = \{x : (x,x) \in C_d f\}$. Since $|C_d f|$ is the pre-image of the closed set $C_d f \subset X \times X$ via the continuous map $x \mapsto (x,x)$ it follows that $|C_d f| \subset X$ is closed. The *Aubry Set* is the cyclic set $|A_d f| \subset X$ which is similarly closed.

From (3.4) we clearly have $|A_d f| \subset |C_d f|$.

On $|C_d f|$ the relation $C_d f \cap C_d f^{-1}$ is a closed equivalence relation and on $|A_d f|$ $A_d f \cap A_d f^{-1}$ is a closed equivalence relation.

Define the symmetrized functions

\[(3.14)\]

\[
sm_d^f (x,y) = \max\{m_d^f (x,y), m_d^f (y,x)\},
\]

\[
s\ell_d^f (x,y) = \max\{\ell_d^f (x,y), \ell_d^f (y,x)\}.
\]

**Proposition 3.3.** Let $f$ be a relation on $X$. Let $x, y, z \in X$

(a) $sm_d^f (x,y) \leq s\ell_d^f (x,y)$

(b) The functions $sm_d^f$ and $s\ell_d^f$ are symmetric and satisfy the triangle inequality.

(c) The functions $sm_d^f, s\ell_d^f : X \times X \to \mathbb{R}$ are Lipschitz with Lipschitz constant less than or equal to 2.

\[(3.15)\]

\[
sm_d^f (x,y) = 0 \iff (x,y), (y,x) \in C_d f \quad \text{and so} \quad x, y \in |C_d f|,
\]

\[
s\ell_d^f (x,y) = 0 \iff (x,y), (y,x) \in A_d f \quad \text{and so} \quad x, y \in |A_d f|.
\]
(e) \[ y \in |\mathcal{C}_d f| \implies s_{\mathcal{E}}^f(x, y) \leq d(x, y), \]
(3.16) \[ y \in |\mathcal{A}_d f| \implies s_{\mathcal{A}}^f(x, y) \leq d(x, y). \]

(f) If \( z \in |\mathcal{C}_d f| \) then \( m_d^f(x, y) \leq \max(m_d^f(x, z), m_d^f(z, y)) \).

**Proof:** (a) is obvious as is symmetry in (b), i.e. \( s_{\mathcal{E}}^f(x, y) = s_{\mathcal{E}}^f(y, x) \) and \( s_{\mathcal{A}}^f(x, y) = s_{\mathcal{A}}^f(y, x) \). The triangle inequality for \( s_{\mathcal{A}}^f \) follows from

\[
\begin{align*}
& s_{\mathcal{A}}^f(x, z) + s_{\mathcal{A}}^f(z, y) \geq s_{\mathcal{A}}^f(x, z) + s_{\mathcal{A}}^f(z, y) \geq s_{\mathcal{A}}^f(x, y),
\end{align*}
\]

(3.17) \[ s_{\mathcal{A}}^f(x, z) + s_{\mathcal{A}}^f(z, y) \geq s_{\mathcal{A}}^f(x, z) + s_{\mathcal{A}}^f(z, y) \geq s_{\mathcal{A}}^f(x, y), \]

with a similar argument for for \( s_{\mathcal{E}}^f \). imply that \( s_{\mathcal{A}}^f \) satisfies the triangle inequality. 

By Proposition 2.2(c) \( m_d^f \) and \( \ell_d^f \) are Lipschitz. Then (e) follows from Lemma 1.1.

The equivalences in (d) are obvious. By transitivity, \((x, y), (y, x) \in \mathcal{C}_d f \) implies \((x, x) \in \mathcal{C}_d f \). Similarly, for \( \mathcal{A}_d f \).

(e) If \( s_{\mathcal{E}}^f(y, y) = 0 \) then \( s_{\mathcal{E}}^f(x, y) = s_{\mathcal{E}}^f(x, y) - s_{\mathcal{E}}^f(y, y) \leq d(x, y) \)

by (c). Similarly, for \( s_{\mathcal{A}}^f \).

(f) follows from Proposition 2.2(b).

□

We immediately obtain the following.

**Corollary 3.4.** The map \( s_{\mathcal{A}}^f \) restricts to define a pseudo-metric on \(|\mathcal{A}_d f|\) and induces a metric on the quotient space of \( \mathcal{A}_d f \cap \mathcal{A}_d f^{-1} \) equivalence classes. Furthermore, the projection map from \(|\mathcal{A}_d f|\) to the space of equivalence classes has Lipschitz constant at most 2 with respect to this metric.

The map \( s_{\mathcal{E}}^f \) restricts to define a pseudo-ultrametric on \(|\mathcal{C}_d f|\) and induces an ultrametric on the quotient space of \( \mathcal{C}_d f \cap \mathcal{C}_d f^{-1} \) equivalence classes. Furthermore, the projection map from \(|\mathcal{C}_d f|\) to the space of equivalence classes has Lipschitz constant at most 2 with respect to this metric.

□

Let \( f_1 \) and \( f_2 \) be relations on \( X_1 \) and \( X_2 \), respectively. Recall that \( h : X_1 \to X_2 \) maps \( f_1 \) to \( f_2 \) when \( h \circ f_1 \circ h^{-1} = (h \times h)(f_1) \subset f_2 \), i.e. if \((x, y) \in f_1 \) implies \((h(x), h(y)) \in f_2 \). It then follows that \( h \) maps \( f_1^{-1} \) to \( f_2^{-1} \).
Proposition 3.5. Let $f_1$ and $f_2$ be relations on $(X_1, d_1)$ and $(X_2, d_2)$, respectively. Assume $h : X_1 \to X_2$ maps $f_1$ to $f_2$.

(a) If $h$ is uniformly continuous, then $h$ maps $\mathcal{C}_d f_1$ to $\mathcal{C}_d f_2$ and $\mathcal{C}_d f_1 \cap \mathcal{C}_d f_1^{-1}$ to $\mathcal{C}_d f_2 \cap \mathcal{C}_d f_2^{-1}$. So $h$ maps each $\mathcal{C}_d f_1 \cap \mathcal{C}_d f_1^{-1}$ equivalence class in $|\mathcal{C}_d f_1|$ into a $\mathcal{C}_d f_2 \cap \mathcal{C}_d f_2^{-1}$ equivalence class in $|\mathcal{C}_d f_2|$.

(b) If $h$ is Lipschitz, then $h$ maps $\mathcal{A}_d f_1$ to $\mathcal{A}_d f_2$ and $\mathcal{A}_d f_1 \cap \mathcal{A}_d f_1^{-1}$ to $\mathcal{A}_d f_2 \cap \mathcal{A}_d f_2^{-1}$. So $h$ maps each $\mathcal{A}_d f_1 \cap \mathcal{A}_d f_1^{-1}$ equivalence class in $|\mathcal{A}_d f_1|$ into a $\mathcal{A}_d f_2 \cap \mathcal{A}_d f_2^{-1}$ equivalence class in $|\mathcal{A}_d f_2|$.

**Proof:** This obviously follows from Proposition 2.3.

We conclude this section with some useful computations. Recall that

$$Z_d = \{(x, y) : d(x, y) = 0\}.$$  

(3.18)

Proposition 3.6. Let $f$ be a relation on $X$ and $A$ be a nonempty, closed subset of $X$.

(a) For $x, y \in X$

$$
l^1_{d A \cup f}(x, y) = \min(l^f_d(x, y), l^A_d(x, y)),
\quad
l^1_{d X \cup f}(x, y) = \min(l^f_d(x, y), d(x, y)).
$$

(b) If $h$ is Lipschitz, then $h$ maps $\mathcal{A}_d f_1$ to $\mathcal{A}_d f_2$ and $\mathcal{A}_d f_1 \cap \mathcal{A}_d f_1^{-1}$ to $\mathcal{A}_d f_2 \cap \mathcal{A}_d f_2^{-1}$. So $h$ maps each $\mathcal{A}_d f_1 \cap \mathcal{A}_d f_1^{-1}$ equivalence class in $|\mathcal{A}_d f_1|$ into a $\mathcal{A}_d f_2 \cap \mathcal{A}_d f_2^{-1}$ equivalence class in $|\mathcal{A}_d f_2|$.

$$\mathcal{A}_d(1_A \cup f) = Z_d \cap (A \times A) \cup \mathcal{A}_d f,$$

$$\mathcal{A}_d(1_X \cup f) = Z_d \cup \mathcal{A}_d f.$$  

(3.19)\hspace{0.5cm}(3.20)

**Proof:** (a) By (2.12) $l^1_{d A \cup f} \leq \min(l^f_d, l^A_d)$. By (2.7) $l^1_{d X} = d$.

Let $[a, b] \in (1_A \cup f)^n$. If $(a_i, b_i) \in 1_A$ for all $i$ then omit all but one of the pairs to obtain an element of $1^X_A$. Otherwise, omit the pairs $(a_i, b_i) \in 1_A$ and renumber. We then obtain a sequence in $f^{\times m}$ for some $m$ with $1 \leq m \leq n$. Furthermore, in either case the $xy$ chain-length has not increased. For example, if $(a_i, b_i) \in 1_A$ for some $1 < i < n$ then
since \(a_i = b_i\) the triangle inequality implies \(d(b_{i-1}, a_{i+1}) \leq d(b_{i-1}, a_i) + d(b_i, a_{i+1})\). It follows that \(\ell^T_{d_{\cup}} \geq \min(\ell^T_d, \ell^T_B)\).

\[
\text{st}_{\cup}^T(x, y) = \max[\min(\ell^T_d(x, y), d(x, y)), \min(\ell^T_B(y, x), d(x, y))].
\]

This is \(d(x, y)\) except when \(d(x, y) > \ell^T_d(x, y)\) and \(d(x, y) > \ell^T_B(y, x)\), i.e. \(d(x, y) > \text{st}_{\cup}^T(x, y)\) in which case it is \(\text{st}_{\cup}^T(x, y)\).

(b) If \(x \in |A_d|\) then by (3.16) \(\min(\text{st}_{\cup}^T(x, y), d(x, y)) = \text{st}_{d}^T(x, y)\).

It follows \(x \in |A_d(\cap f)|\) iff \(\ell^T_d(x, y) = 0\) or \(\ell^T_B(x, y) = 0\). By (2.11) the latter is true iff \(x, y \in A\) with \(d(x, y) = 0\) since \(A\) is closed. Thus, (3.20) holds and the rest is obvious.

\(\square\)

If \(A, B\) are subsets of \(X\) then we can regard \(A \times B\) as a relation on \(X\). For any relation \(g\) on \(X\) we clearly have:

\[(A \times B) \circ g \circ (A \times B) \subseteq A \times B.\]

**Lemma 3.7.** If \(A\) and \(B\) are nonempty subsets of \((X, d)\) and \(x, y \in X\), then

\[
\begin{align*}
\text{m}_{d}^{A \times B} (x, y) &= \max(\text{d}(x, A), \text{d}(y, B)) \quad \text{and} \\
\ell_{d}^{A \times B} (x, y) &= \text{d}(x, A) + \text{d}(y, B)
\end{align*}
\]

where \(\text{d}(x, A) = \inf\{\text{d}(x, z) : z \in A\}\).

**Proof:** If \([a, b] \subseteq (A \times B)^n\) then \((a_1, b_n) \in A \times B\) with \(xy\) chain-length \(d(x, a_1) + d(y, b_n)\) no larger than the \(xy\) chain-length for \([a, b]\) and with \(xy\) chain-bound \(\max(d(x, a_1), d(y, b_n))\) no larger than the \(xy\) chain-bound for \([a, b]\). This proves (3.22).

\(\square\)

From Proposition 3.6 we immediately get

**Corollary 3.8.** If \(A\) and \(B\) are nonempty subsets of \(X\) and \(x, y \in X\) then

\[
\ell_{d}^{1_{X \cup (A \times B)}} (x, y) = \min[\text{d}(x, y), \text{d}(x, A) + \text{d}(y, B)].
\]

\(\square\)

**Remark:** If \(A = B\) then \(s_{d}^{1_{X \cup (A \times A)}} = \ell_{d}^{1_{X \cup (A \times A)}}\) is the pseudometric on \(X\) induced by the equivalence relation \(1_X \cup (A \times A)\) corresponding to smashing \(A\) to a point.
Lemma 3.9. For $x, y, z \in X$

\[(3.24)\]
\[
m_d^{f \cup \{(z, z)\}}(x, y) = \min [m_d^f(x, y), \max [\min (m_d^f(x, z), d(x, z)), \min (m_d^f(z, y), d(z, y))].
\]

In particular, with $z = y$ or $z = x$

\[(3.25)\]
\[
m_d^{f \cup \{(y, y)\}}(x, y) = m_d^{f \cup \{(x, x)\}}(x, y) = \min [m_d^f(x, y), d(x, y)].
\]

If $(z, z) \in \mathcal{C}_d f$, i.e. $z \in |\mathcal{C}_d f|$, then \[m_d^{f \cup \{(z, z)\}} = m_d^f\].

**Proof:** Since $f \subset f \cup \{(z, z)\}$ we have \[m_d^{f \cup \{(z, z)\}} \leq \min (m_d^f, m_d^{\{(z, z)\}}).\]

Let $[a, b] \in (f \cup \{(z, z)\})^\times n$. If $(z, z)$ occurs more than once in $[a, b]$ we can eliminate the repeat and all of the terms between them without increasing the $xy$ chain-bound. Thus, we may take the infimum over those $[a, b]$ in which $(z, z)$ occurs at most once.

The infimum of the $xy$ chain-bounds in $f^\times n$ is $m_d^f(x, y)$.

- The $xy$ chain-bound of $(z, z) \in (f \cup \{(z, z)\})^\times 1$ is $d(x, z), d(z, y))$.
- If $[a, b]$ varies in $(f \cup \{(z, z)\})^\times n$ with $n > 1$ and $(a_i, b_i) = (z, z)$ only for $i = 1$, then the infimum of the $xy$ chain-bounds is $\max (d(x, z), m_d^f(z, y))$.
- If $[a, b]$ varies in $(f \cup \{(z, z)\})^\times n$ with $n > 1$ and $(a_i, b_i) = (z, z)$ only for $i = n$, then the infimum of the $xy$ chain-bounds is $\max (m_d^f(x, z), d(z, y))$.
- If $[a, b]$ varies in $(f \cup \{(z, z)\})^\times n$ with $n > 2$ and $(a_i, b_i) = (z, z)$ only for some $i$ with $1 < i < n$, then the infimum of the $xy$ chain-bounds is $\max (m_d^f(x, z), m_d^f(z, y))$.

Equation \((3.24)\) then follows from Lemma 1.1.

If $(z, z) \in \mathcal{C}_d f$ then $f \subset f \cup \{(z, z)\} \subset \mathcal{C}_d f$. So $m_d^{e_{af}} \leq m_d^{f \cup \{(z, z)\}} \leq m_d^f$ by \((2.12)\) and so they are equal by \((3.10)\).
4. Lyapunov Functions

A Lyapunov function for a relation \( f \) on a pseudo-metric space \((X,d)\) is a continuous map \( L : X \to \mathbb{R} \) such that
\[
(x, y) \in f \implies L(x) \leq L(y).
\]
We follow \[\Pi\] in using functions increasing on orbits rather than decreasing.

The set of Lyapunov functions contains the constants and is closed under addition, multiplication by positive scalars, max, min and post composition with any continuous non-decreasing function on \( \mathbb{R} \). A continuous function which is a pointwise limit of Lyapunov functions is itself a Lyapunov function.

We define for a real-valued function \( L \) the relation
\[
\leq_L = \{(x, y) : L(x) \leq L(y)\}.
\]
This is clearly reflexive and transitive. By continuity of \( L \) the relation \( \leq_L \) is closed and so contains \( Z_d \).

The Lyapunov function condition (4.1) can be restated as:
\[
(4.3) \quad f \subset \leq_L.
\]
For a Lyapunov function \( L \) and \( x \in X \) we have
\[
(4.4) \quad L(z) \leq L(x) \leq L(w) \quad \text{for} \quad z \in f^{-1}(x), \ w \in f(x)
\]
The point \( x \) is called an \( f \)-regular point for \( L \) when the inequalities are strict for all \( z \in f^{-1}(x), \ w \in f(x) \). Otherwise \( x \) is called an \( f \)-critical point for \( L \). Notice, for example, that if \( f^{-1}(x) = f(x) = \emptyset \) then these conditions hold vacuously and so \( x \) is an \( f \)-regular point.

We denote by \( |L|_f \) the set of \( f \)-critical points for \( L \). Clearly,
\[
(4.5) \quad |L|_f = \pi_1(A) \cup \pi_2(A) \quad \text{where} \quad A = f \cap (L \times L)^{-1}(1_\mathbb{R}),
\]
and \( \pi_1, \pi_2 : X \times X \to X \) are the two coordinate projections.

**Definition 4.1.** Let \( F \) be a transitive relation on \((X,d)\) and let \( \mathcal{L} \) be a collection of Lyapunov functions for \( F \). We define three conditions on \( \mathcal{L} \).

**ALG** If \( L_1, L_2 \in \mathcal{L} \) and \( c \geq 0 \) then
\[
L_1 + L_2, \max(L_1, L_2), \min(L_1, L_2), cL_1, c, -c \in \mathcal{L}.
\]

**CON** For every sequence \( \{L_k\} \) of elements of \( \mathcal{L} \) there exists a summable sequence of positive real numbers \( \{a_k\} \) such that \( \sum a_k L_k \) converges uniformly to an element of \( \mathcal{L} \).

**POIN** If \((x, y) \notin Z_d \cup F \) then there exists \( L \in \mathcal{L} \) such that \( L(y) < L(x) \), i.e. \( Z_d \cup F = \bigcap_{L \in \mathcal{L}} \leq_L \).
Theorem 4.2. Assume \((X,d)\) is separable. Let \(F\) be a closed, transitive relation and \(\mathcal{L}\) be a collection of Lyapunov functions for \(F\) which satisfies ALG, CON and POIN. There exists a sequence \(\{L_k\}\) in \(\mathcal{L}\) such that
\[
\bigcap_k Z \leq L_k = Z_d \cup F.
\]
If \(\{a_k\}\) is a positive, summable sequence such that \(L = \sum_k a_k L_k \in \mathcal{L}\) then \(L\) is a Lyapunov function for \(F\) such that \(Z_d \cup F = \leq L\) and
\[
(4.7) \quad x \in F(y) \implies L(y) < L(x) \quad \text{unless } y \in F(x)
\]
In particular,
\[
(4.8) \quad |L|_F = |F|
\]
Proof: For each \((x,y) \in (X \times X) \setminus (Z_d \cup F)\) use POIN to choose \(L_{xy} \in \mathcal{L}\) such that \(L_{xy}(y) < L_{xy}(x)\) and then neighborhoods \(V_{xy}\) of \(y\) and \(U_{xy}\) of \(x\) such that \(\sup_{z \in U_{xy}} L_{xy}(z) < \inf_{z \in V_{xy}} L_{xy}(z)\) and so \(L_{xy}\) is disjoint from \(U_{xy} \times V_{xy}\). Because \((X,d)\) is separable, it is second countable and so \((X \times X) \setminus (Z_d \cup F)\) is Lindelöf. Choose a sequence of pairs \((x_k,y_k)\) so that \(\{U_{x_ky_k} \times V_{x_ky_k}\}\) covers \((X \times X) \setminus (Z_d \cup F)\) and let \(L_k = L_{x_ky_k}\).
Since \(Z_d \cup F \subseteq \leq L\) for any Lyapunov function \(L\), (4.6) holds.

Now with \(L = \sum_k a_k L_k\), (4.6) implies \(Z_d \cup F = \leq L\). If \(x \in F(y)\) and \(d(y,x) = d(x,y) = 0\) then \((y,x) \in F\) implies \((x,x),(y,y),(x,y) \in F\), because \(F\) is closed. Hence, \(y \in F(x)\). Assume \((x,y) \notin Z_d\). Since \(x \in F(y)\), \(L_k(y) \leq L_k(x)\) for all \(k\). If equality holds for all \(k\) then \((x,y) \in \bigcap_k L_k = Z_d \cup F\). Since \((x,y) \notin Z_d\) we have \(y \in F(x)\). If, instead, the inequality is strict for some \(k\) then since \(a_k > 0\), \(L(y) < L(x)\), proving (4.7).

If \(x \notin |F|\) then for \(z \in F^{-1}(x)\) and \(w \in F(x)\) we have \(x \in F(z)\) but not \(z \in F(x)\) else by transitivity \(x \in |F|\). Hence, \(L(z) < L(x)\). Similarly, \(L(x) < L(w)\). Thus, \(x \notin |L|_F\).

\[
\Box
\]

Definition 4.3. For a relation \(f\) on \((X,d)\) and \(K > 0\), a function \(L : X \to \mathbb{R}\) is called \(Kf\)-dominated if for all \(x,y \in X\)
\[
(4.9) \quad L(x) - L(y) \leq Kf_d(x,y),
\]
\(Km_d\)-dominated if for all \(x,y \in X\)
\[
(4.10) \quad L(x) - L(y) \leq Km_d(x,y).
\]
**Theorem 4.4.** Let $f$ be a relation on $(X, d)$.

(a) If $L$ is a $K\ell_d^f$ dominated function then it is a Lyapunov function for $A_d f$ and so is a Lyapunov function for $f$. If $L$ is a $Km_d^f$ dominated function then it is a $K\ell_d^f$ dominated function and is a Lyapunov function for $C_d f$.

(b) If $L$ is a Lyapunov function for $f$ which is Lipschitz with respect to $d$ with Lipschitz constant at most $K$ then it is a $K\ell_d^f$ dominated function and so is a $A_d f$ Lyapunov function.

**Proof:** (a) If $(x, y) \in A_d f$ then $\ell_d^f(x, y) = 0$ and so for a $K\ell_d^f$ dominated function $L(x) - L(y) \leq 0$. Similarly, if $(x, y) \in C_d f$ and $L$ is $Km_d^f$ dominated, then $L(x) - L(y) \leq 0$. Since $m_d^f \leq \ell_d^f$ a $Km_d^f$ dominated function is a $K\ell_d^f$ dominated function.

(b) Assume $L$ is an $f$ Lyapunov function with Lipschitz constant $K$ and $x, y \in X$. For any $[a, b] \in f^{x^n}$ we note that each $L(a_i) - L(b_i) \leq 0$ since $(a_i, b_i) \in f$ and $L$ is a Lyapunov function for $f$. Hence,

\[
(4.11) \quad L(x) - L(y) = L(x) - L(a_1) + L(a_1) - L(b_1) + L(b_1) - L(a_2) + \ldots + L(a_n) - L(b_n) + L(b_n) - L(y) \leq L(x) - L(a_1) + \sum_{i=1}^{n-1} L(b_i) - L(a_{i+1}) + L(b_n) - L(y) \leq K\ell.
\]

where $\ell$ is the $xy$ chain-length of $[a, b]$. Taking the infimum over the sequences $[a, b]$ we obtain (4.19). Hence, $L$ is a $A_d f$ Lyapunov function by part (a).

\[\square\]

**Proposition 4.5.** Let $f \subset g$ be relations on $(X, d)$. For any $z \in X$, the function defined by $x \mapsto \ell_d^g(x, z)$ is a bounded, $1\ell_d^g$ dominated function, and the function defined by $x \mapsto m_d^g(x, z)$ is a bounded, $1m_d^g$ dominated function.

**Proof:** By the directed triangle inequalities for $\ell_d^g$ and $m_d^g$ we have
\[
(4.12) \quad \ell_d^g(x, z) - \ell_d^g(y, z) \leq \ell_d^g(x, y) \quad \text{and} \quad m_d^g(x, z) - m_d^g(y, z) \leq m_d^g(x, y)
\]

Since $f \subset g$, $\ell_d^g(x, y) \leq \ell_d^f(x, y)$ and $m_d^g(x, y) \leq m_d^f(x, y)$ by (2.12).

\[\square\]

**Theorem 4.6.** For $f$ a relation on $(X, d)$ let $L_{\ell}$ be the set of bounded, continuous functions which are $K\ell_d^f$ dominated for some positive $K$. 
Each $L \in \mathcal{L}_\ell$ is a $A_{df}$ Lyapunov function and so satisfies

\[(4.13) \quad A_{df} \subset \leq_L \quad \text{and} \quad |A_{df}| \subset |L|_{A_{df}}.\]

The collection $\mathcal{L}_\ell$ satisfies the conditions ALG, CON, and POIN with respect to $F = A_{df}$.

**Proof:** Each $L$ in $\mathcal{L}_\ell$ is a $A_{df}$ Lyapunov function by Theorem 4.4, and so the first inclusion of (4.13) follows by definition. Clearly, if $(x, x) \in A_{df}$ then $x$ is a $A_{df}$ critical point.

For $\mathcal{L}_\ell$ ALG is easy to check, see, e.g., Lemma 1.1. For CON let $\{L_k\}$ be a sequence in $\mathcal{L}_\ell$ and choose for each $k$, $M_k \geq 1$ which bounds $|L_k(x)|$ for all $x \in X$ and so that $L_k$ is $M_k \ell^f_d$ dominated. If $\{b_k\}$ is any positive, summable sequence with $\sum b_k = 1$, then $a_k = b_k/M_k > 0$ is summable and $\Sigma_k a_k L_k$ converges uniformly to a function which is $1 \ell^f_d$ dominated. Thus, CON holds as well.

Now assume $(x, y) \notin Z_d \cup A_{df}$. Let $g = 1_X \cup f$. By Proposition 4.5, $L(w) = \ell^f_d(w, y)$ defines a $1 \ell^f_d$ dominated function which is a $A_{df}$ Lyapunov function by Theorem 4.4(a).

By Proposition 3.6, $L(w) = \min(\ell^f_d(w, y), d(w, y))$. Hence, $L(y) = 0$. Since $(x, y) \notin Z_d \cup A_{df}$, $L(x) > 0$. This proves POIN.

\[\Box\]

**Theorem 4.7.** For $f$ a relation on $(X, d)$ let $\mathcal{L}_m$ be the set of bounded, continuous functions which are $K m^f_d$ dominated for some positive $K$. Each $L \in \mathcal{L}_m$ is a $C_{df}$ Lyapunov function and so satisfies

\[(4.14) \quad C_{df} \subset \leq_L \quad \text{and} \quad |C_{df}| \subset |L|_{C_{df}}.\]

The collection $\mathcal{L}_m$ satisfies the conditions ALG, CON, POIN with respect to $F = C_{df}$.

**Proof:** Each $L$ in $\mathcal{L}_m$ is a $C_{df}$ Lyapunov function by Theorem 4.4, and so the first inclusion of (4.14) follows by definition. Clearly, if $(x, x) \in C_{df}$ then $x$ is a $C_{df}$ critical point.

For $\mathcal{L}_m$ ALG again follows from Lemma 1.1. For CON let $\{L_k\}$ be a sequence in $\mathcal{L}_m$ and choose for each $k$, $M_k \geq 1$ which bounds $|L_k(x)|$ for all $x \in X$ and such that $L_k$ is $M_k m^f_d$ dominated. If $\{b_k\}$ is any positive, summable sequence with $\sum b_k = 1$, then $a_k = b_k/M_k > 0$ is summable and $\Sigma_k a_k L_k$ converges uniformly to a function which is $1 m^f_d$. Thus, CON holds as well.

Now assume $(x, y) \notin Z_d \cup C_{df}$. Let $g = f \cup \{(y, y)\}$. By Proposition 4.5, $L(w) = m^\varphi_d(w, y)$ defines a $1 m^f_d$ dominated function. By Equation...
(3.25) \( L(w) = \min \left( \ell_d^f(w, y), d(w, y) \right) \). Hence, \( L(y) = 0 \). Since \((x, y) \notin Z_d \cup C_d f \), \( L(x) > 0 \). This proves POIN.

\( \square \)

5. Conley and Aubry-Mather Relations for Uniform Spaces

Let \( \mathcal{U} \) be a uniformity on \( X \) with gage \( \Gamma \), the set of all bounded pseudo-metrics \( d \) on \( X \) such that the uniformity \( \mathcal{U}(d) \) is contained in \( \mathcal{U} \).

For a relation \( f \) on \( X \) we define the Conley relation and Aubry-Mather relation associated with the uniformity.

\[
(5.1) \quad C_{\mathcal{U}} f = \bigcap_{d \in \Gamma} C_d f, \quad \text{and} \quad A_{\mathcal{U}} f = \bigcap_{d \in \Gamma} A_d f
\]

with \( |C_{\mathcal{U}} f| \) the Conley set and \( |A_{\mathcal{U}} f| \) the Aubry set.

Thus, \( C_{\mathcal{U}} f \) and \( A_{\mathcal{U}} f \) are closed, transitive relations on \( X \) which contain \( f \). We define \( \mathcal{G} f \) to be the intersection of all the closed, transitive relations which contain \( f \). Thus, \( \mathcal{G} f \) is the smallest closed, transitive relation which contains \( f \). Clearly,

\[
(5.2) \quad f \subset \mathcal{G} f \subset A_{\mathcal{U}} f \subset C_{\mathcal{U}} f.
\]

Thus, \((x, y) \in C_{\mathcal{U}} f \) if for every \( d \in \mathcal{G} \) and every \( \epsilon > 0 \) there exists \([a, b] \in f \times^n \) with \( n \geq 1 \) such that the \( xy \) chain-bound of \([a, b] \) with respect to \( d \) is less than \( \epsilon \).

If \([a, b] \in f \times^n \) with \( n \geq 1 \) and \( U \in \mathcal{U} \) we say that \([a, b] \) is an \( xy, U \) chain for \( f \) if \((x, a_1), (b_1, a_2), \ldots (b_{n-1}, a_n), (b_n, y) \) \( \in U \). Clearly, then \([a, b]^{-1} \) is a \( yx, U^{-1} \) chain for \( f^{-1} \).

Since the \( V_d \)'s for \( d \in \Gamma(\mathcal{U}) \) and \( \epsilon > 0 \) generate the uniformity, it is clear that the pair \((x, y) \in C_{\mathcal{U}} f \) iff for every \( U \in \mathcal{U} \) there exists an \( xy, U \) chain for \( f \). This provides a uniformity description of \( C_{\mathcal{U}} f \).

Similarly, \((x, y) \in A_{\mathcal{U}} f \) if for every \( d \in \mathcal{G} \) and every \( \epsilon > 0 \) there exists \([a, b] \in f \times^n \) with \( n \geq 1 \) such that the \( xy \) chain-length of \([a, b] \) with respect to \( d \) is less than \( \epsilon \).

Following [16] we obtain a uniformity description of \( A_{\mathcal{U}} f \).

If \( \xi = \{U_k : k \in \mathbb{N}\} \) is a sequence of elements of \( \mathcal{U} \) and \((x, y) \in X \times X \), we call \([a, b] \in f \times^n \) an \( \xi \) sequence chain from \( x \) to \( y \) if there is an injective map \( \sigma : \{0, \ldots, n\} \rightarrow \mathbb{N} \) such that \((b_i, a_{i+1}) \in U_{\sigma(i)} \) for \( i = 0, \ldots, n \) with \( b_0 = x, a_{n+1} = y \).
Theorem 5.1. For a relation \( f \) on a uniform space \((X, \mathcal{U})\), \((x, y) \in \mathcal{A}_\mathcal{U}f\) iff for every sequence \( \xi \) in \( \mathcal{U} \) there is a \( \xi \) sequence chain from \( x \) to \( y \).

Proof: Assume \((x, y)\) satisfies the sequence chain condition. If \( d \in \Gamma(\mathcal{U}) \) and \( \epsilon > 0 \) the chain-length with respect to \( d \) of any sequence chain with \( \xi = \{V_{\epsilon/2}^d\} \) from \( x \) to \( y \) is less than \( \epsilon \). Hence, \((x, y) \in \mathcal{A}_d f\).

As \( d \) was arbitrary, \((x, y) \in \bigcap_{d \in \Gamma} \mathcal{A}_d f = \mathcal{A}_\mathcal{U} f\).

Now let \((x, y) \in \mathcal{A}_\mathcal{U}f\) and \( \xi = \{U_k : k \in \mathbb{N}\} \) be a sequence in \( \mathcal{U} \). We must show that there is a \( \xi \) sequence chain from \( x \) to \( y \).

Let \( V_0 = X \times X \). For \( k \in \mathbb{N} \), inductively choose \( V_k = V_k^{-1} \in \mathcal{U} \) such that \( V_k \circ V_k \circ V_k \subset V_{k-1} \cap U_k \). By the Metrization Lemma [12] Lemma 6.12, there exists a pseudo-metric \( d \leq 1 \) such that \( V_k \subset V_{1/2^{k-1}} \subset V_{k-1} \) for \( k \in \mathbb{N} \). It follows that \( d \in \Gamma \) and since \( V_{1/2^k} \subset U_k \) it follows that if \( \xi' = \{V_{1/2^k}\} \) then a \( \xi' \) sequence chain is a \( \xi \) sequence chain. It suffices to show that there is a \( \xi' \) sequence chain from \( x \) to \( y \).

Lemma 5.2. Let \( \phi : \mathbb{R} \to [0, \infty) \) be given by \( \phi(0) = 0 \) and \( \phi(t) = e^{-1/t^2} \) for \( t \neq 0 \). So that \( \phi \) is a \( C^\infty \) such that

(i) For all \( t > 0 \), \( \phi'(t) > 0 \) and for all \( \sqrt{2/3} > t > 0 \), \( \phi''(t) > 0 \).

(ii) For \( \epsilon = e^{-3/2}/2 \), \( \tilde{d}(x, y) = \phi^{-1}(\min(d(x, y), \epsilon)) \) defines a pseudo-metric on \( X \) with \( \mathcal{U}(d) = \mathcal{U}(d) \subset \mathcal{U} \) and so \( d \in \Gamma \).

(iii) If \( \{\alpha_k\} \) is a finite or infinite, non-increasing sequence of non-negative numbers with \( \sum_k \alpha_k < \phi^{-1}(\epsilon) < 1 \) then \( \tilde{d}(x, y) \leq \alpha_k \) implies \( d(x, y) < 2^{-k} \), for all \( k \in \mathbb{N} \).

Proof: (i) is an easy direct computation.

(ii) Observe that if \( \psi : [0, a] \to \mathbb{R} \) is \( C^2 \) with \( \psi(0) = 0 \), \( \psi'(t) > 0 \) and \( \psi''(t) < 0 \) for \( 0 < t < a \) then for all \( t, s \leq a/2, \psi(t) + \psi(s) - \psi(t+s) \geq 0 \), because with \( t \) fixed it is true for \( s = 0 \) and the derivative with respect to \( s \) is positive for \( a-t > s > 0 \). It follows that if \( d \) is a pseudo-metric with \( d \leq a/2 \) then \( \psi(d) \) is a pseudo-metric. Clearly, \( \mathcal{U}(\psi(d)) = \mathcal{U}(d) \). For (ii) we apply this with \( \psi = \phi^{-1} \).

(iii) Observe that for all \( k \in \mathbb{N} \), \( \phi(1/k) = e^{-k^2} < 2^{-k} \). Each \( \alpha_k < \phi^{-1}(\epsilon) \) and so \( \tilde{d}(x, y) \leq \alpha_k \) iff \( d(x, y) \leq \phi(\alpha_k) \). If \( \phi(\alpha_k) \geq 2^{-k} \) then for \( 1 \leq j \leq k \phi(\alpha_j) \geq \phi(\alpha_k) \geq 2^{-k} \geq \phi(1/k) \) and so \( \alpha_j \geq 1/k \) for \( j = 1, \ldots, k \). Hence, \( \sum_j \alpha_j \geq k(1/k) = 1 > \phi^{-1}(\epsilon) \), contradicting the assumption on the sum.

\[ \square \]

Since \((x, y) \in \mathcal{A}_\mathcal{U}f\), there exists \([a, b] \in f^{\times n}\) for some \( n \geq 1 \) such that with respect to the metric \( \tilde{d} \), the \( xy \) chain-length of \([a, b]\) is less...
than $\phi^{-1}(\epsilon)$. Let $b_0 = x$ and $a_{n+1} = y$. Let $k \mapsto i(k)$ be a bijection on $\{1, \ldots, n + 1\}$ so that the sequence $\alpha_k = d(b_{i(k)-1}, a_{i(k)})$ is non-increasing. From (iii) it follows that $(b_{i(k)-1}, a_{i(k)}) \in V_{2-k}^d$ for $k = 1, \ldots, n + 1$ and so $[a, b]$ is a $\xi'$ sequence chain from $x$ to $y$ as required.

$\blacksquare$

It is clear that $(\mathcal{G} f)^{-1}$ is the smallest closed, transitive relation which contains $f^{-1}$. So from (3.9) we obtain:

\begin{align*}
(5.3) \quad \mathcal{G}(f^{-1}) &= (\mathcal{G} f)^{-1}, \quad \mathcal{A}_{\mathcal{U}}(f^{-1}) = (\mathcal{A}_{\mathcal{U}} f)^{-1}, \quad \mathcal{C}_{\mathcal{U}}(f^{-1}) = (\mathcal{C}_{\mathcal{U}} f)^{-1},
\end{align*}

and so again we may omit the parentheses.

**Proposition 5.3.** For a relation $f$ on a uniform space $(X, \mathcal{U})$, the image $f(X)$ is dense in $\mathcal{C}_{\mathcal{U}} f(X)$ and the domain $f^{-1}(X)$ is dense in $\mathcal{C}_{\mathcal{U}} f^{-1}(X)$.

**Proof:** Let $A = \overline{f(X)}$ and let $y \in \mathcal{C}_{\mathcal{U}} f(x)$. If $U \in \mathcal{U}$ and $[a, b] \in f^{\times n}$ is an $xy, U$ chain, then $b_i \in A$ for all $i$ and so $y \in U(A)$. Because $A$ is closed, it equals the intersection $\bigcap_{U \in \mathcal{U}} U(A)$. Thus, $\mathcal{C}_{\mathcal{U}} f(X) \subset A$. Replacing $f$ by $f^{-1}$ we obtain the domain result.

$\blacksquare$

From (3.8) we obtain monotonicity: If $f \subset g$ are relations on $(X, \mathcal{U})$ then

\begin{align*}
(5.4) \quad \mathcal{G} f &\subset \mathcal{G} g, \quad \mathcal{A}_{\mathcal{U}} f \subset \mathcal{A}_{\mathcal{U}} g, \quad \mathcal{C}_{\mathcal{U}} f \subset \mathcal{C}_{\mathcal{U}} g,
\end{align*}

Again the operators are idempotent.

**Proposition 5.4.**

\begin{align*}
(5.5) \quad &f \subset g \subset \mathcal{C}_{\mathcal{U}} f \quad \Rightarrow \quad \mathcal{C}_{\mathcal{U}} f = \mathcal{C}_{\mathcal{U}} g, \\
&f \subset g \subset \mathcal{A}_{\mathcal{U}} f \quad \Rightarrow \quad \mathcal{A}_{\mathcal{U}} f = \mathcal{A}_{\mathcal{U}} g, \\
&f \subset g \subset \mathcal{G} f \quad \Rightarrow \quad \mathcal{G} f = \mathcal{G} g.
\end{align*}

**Proof:** For any $d \in \Gamma$, $f \subset g \subset \mathcal{C}_{\mathcal{U}} f \subset \mathcal{C}_{\mathcal{U}} d f$ and so by (3.11) and monotonicity, $\mathcal{C}_{\mathcal{U}} d f = \mathcal{C}_{\mathcal{U}} d g$. Intersect over $d \in \Gamma$. The proof for $\mathcal{A}_{\mathcal{U}}$ is similar.

Finally, if $F$ is a closed, transitive relation then $F = \mathcal{G} F$.

$\blacksquare$
Proposition 5.5. If \( \mathcal{C}_{U}f \cap \mathcal{C}_{U}g = \mathcal{C}_{U}(\mathcal{C}_{U}f \cap \mathcal{C}_{U}g), \)

\[ (5.6) \quad A_{U}f \cap A_{U}g = A_{U}(A_{U}f \cap A_{U}g), \]

\[ \mathcal{G}f \cap \mathcal{G}g = \mathcal{G}(\mathcal{G}f \cap \mathcal{G}g). \]

If \( U_{1} \) and \( U_{2} \) are uniformities on \( X \) then

\[ (5.7) \quad U_{1} \subset U_{2} \Rightarrow \mathcal{C}_{U_{2}}f \subset \mathcal{C}_{U_{1}}f \quad \text{and} \quad A_{U_{2}}f \subset A_{U_{1}}f. \]

More generally, we have

**Proposition 5.5.** If \( h : (X_{1}, U_{1}) \to (X_{2}, U_{2}) \) is a continuous map which maps the relation \( f_{1} \) on \( X_{1} \) to \( f_{2} \) on \( X_{2} \), then \( h \) maps \( \mathcal{G}f_{1} \) to \( \mathcal{G}f_{2} \). If, in addition, \( h \) is uniformly continuous, then \( h \) maps \( \mathcal{C}_{U_{1}}f_{1} \) to \( \mathcal{C}_{U_{2}}f_{2} \), and maps \( A_{U_{1}}f_{1} \) to \( A_{U_{2}}f_{2} \).

**Proof:** If \( h \) is continuous then, \( (h \times h)^{-1}(\mathcal{G}f_{2}) \) is a closed, transitive relation which contains \( f_{1} \) and so contains \( \mathcal{G}f_{1} \).

Now assume that \( h \) is uniformly continuous. Let \( d_{2} \in \Gamma(U_{2}) \). By uniform continuity, \( d_{1} = h^{*}d_{2} \in \Gamma(U_{1}) \), where

\[ (5.8) \quad h^{*}d_{2}(x, y) = d_{2}(h(x), h(y)). \]

Thus, \( h : (X_{1}, d_{1}) \to (X_{2}, d_{2}) \) is Lipschitz. In fact, it is an isometry. By Proposition 3.5, \( h \) maps \( A_{U_{1}}f_{1} \subset A_{d_{1}}f_{1} \) into \( A_{d_{2}}f_{2} \) and similarly for \( \mathcal{C} \). Intersect over all \( d_{2} \in \Gamma(U_{2}) \).

\[ \square \]

For a relation \( f \) on \( X \) let \( f^{[1, k]} = \bigcup_{j=1}^{k} f^{j} \) for any positive integer \( k \). Let \( f^{[0, k]} = 1_{X} \cup f^{[1, k]} \). If \( d \) is a pseudo-metric on \( X \) and \( f \) is a map on \( X \) we let \( d^{k} = \max_{j=0}^{k} (f^{j})^{*}d \). Let \( d^{0} = d \).

**Corollary 5.6.** Let \( k \geq 2 \) be an integer and \( f \) be a continuous map on a uniform space \( (X, U) \).

\[ (5.9) \quad \mathcal{G}f = f^{[1, k-1]} \cup \mathcal{G}(f^{k}) \circ f^{[0, k-1]}, \]

and \( |\mathcal{G}(f^{k})| = |\mathcal{G}f| \).

If \( f \) is a uniformly continuous map, then

\[ (5.10) \quad A_{U}f = f^{[1, k-1]} \cup A_{U}(f^{k}) \circ f^{[0, k-1]}, \quad \mathcal{C}_{U}f = f^{[1, k-1]} \cup \mathcal{C}_{U}(f^{k}) \circ f^{[0, k-1]}, \]

and \( |A_{U}(f^{k})| = |A_{U}f| \), \( |\mathcal{C}_{U}(f^{k})| = |\mathcal{C}_{U}f| \).

**Proof:** If \( F \) is a closed relation on \( X \) and \( f \) is a continuous map on \( X \) then \( F \circ f \) is a closed relation. For suppose \( \{(x_{i}, y_{i})\} \) is a net in \( F \circ f \) converging to \( (x, y) \). Then \( \{f(x_{i})\} \) converges to \( f(x) \) by continuity and
{(f(x_i), y_i)} is a net in F converging to (f(x), y). Since F is closed, (f(x), y) ∈ F and (x, y) ∈ F ∩ F.

Hence, f^{[1,k-1]} ∪ G(f^k) ∩ f^{[0,k-1]} is a closed relation which contains f. Since f ∈ G, transitivity of G implies that f^k ∈ G. Hence, G(f^k) ⊆ G. Transitivity again implies f^{[1,k-1]} ∪ G(f^k) ∩ f^{[0,k-1]} ⊆ G(f).

Because f maps f^k to f^k it follows from Proposition 5.5 that it maps G(f^k) to itself. Hence, f^{[0,k-1]} ∩ G(f^k) ∩ f^{[0,k-1]} ⊆ f^{[1,k-1]} ∪ f^k ∩ f^{[0,k-1]}. It follows that f^{[1,k-1]} ∩ G(f^k) ∩ f^{[1,k-1]} is transitive and so contains G since it is closed and contains f.

It clearly follows that |G(f^k)| ⊆ |G(f)|. Assume that x ∈ |G(f)|. From (5.9) it follows that either x ∈ f^j(x) for some j ∈ [1, k-1] or x ∈ G(f^k) ∩ f^j(x) for some j ∈ [0, k-1]. If x = f^j(x) then x = (f^j)^k(x) = (f^k)^j(x) and so x ∈ G(f^k)(x). Similarly, since f^j maps G(f^k) to itself,

\[(G(f^k) ∩ f^j)^k ⊆ (G(f^k))^j ∩ f^k \subseteq G(f^k)\]

and so x ∈ |G(f^k)| if x ∈ G(f^k) ∩ f^j(x).

Transitivity again implies f^k ∈ A_{U,f} ⊆ C_{U,f}, and so monotonicity and transitivity imply

\[A_{U,f} \supseteq f^{[1,k-1]} \cup A_{U}(f^k) ∩ f^{[0,k-1]},\]

\[C_{U,f} \supseteq f^{[1,k-1]} \cup C_{U}(f^k) ∩ f^{[0,k-1]},\]

Now assume that f is a uniformly continuous map. Notice that if [a, b] ∈ f^{x^n} then b_i = f(a_i) for i = 1, . . . , n. Observe that if x ∈ X and j ≤ k

\[d(f^j(a_1), a_{j+1}) \leq \Sigma_{i=1}^j d(f^{j-i}(a_i), f^{j-i}(a_{i+1}))\]

\[\leq \Sigma_{i=1}^{j-1} d^k(f(a_i), a_{i+1}),\]

and \[d(f^j(x), f^j(a_1)) \leq d^k(x, a_1).\]

Let (x, y) ∈ A_{U}. For α = (d, ε) ∈ Γ × (0, ∞) there exists [a, b]_α ∈ f^{n_α} with xy chain-length with respect to d less than ε.

If n_α < k frequently then for some j ∈ [1, k-1] frequently n_α = j and it follows from continuity of f that y = f^j(x).

Instead assume that eventually n_α ≥ k. If ε > 0 and d_1 ∈ Γ(1), there exists d ≥ d_1 and [a, b] ∈ f^{x^n} with n ≥ k so that the xy chain-length of [a, b] with respect to d^k is less than ε. Let n = j + qk with j ∈ [0, k-1] and q ≥ 1. The sequence

\[(a, b)^k = (a_{j+1}, f^k(a_{j+1})), (a_{j+k+1}, f^k(a_{j+k+1})) \ldots (a_{j+(q-1)k+1}, f^k(a_{j+(q-1)k+1})) ∈ (f^k)^x q,\]
and with $y = a_{n+1}$. (5.12) implies that the $f^{j}(x)y$ chain-length with respect to $d$ and so with respect to $d_1$ is less than $\epsilon$. Since $d_1$ was arbitrary it follows that $y \in \mathcal{G}(f^k) \circ f^{[0,k-1]}(x)$.

For $\mathcal{C}_U f$ we proceed as before, but use chain-bound less than $\epsilon/k$.

For $|A_U(f^k)|$ and $|\mathcal{C}_U(f^k)|$ we use the same argument as for $|\mathcal{G}(f^k)|$ above.

$\square$

If a real-valued function on $X$ is uniformly continuous with respect to some $d \in \Gamma(U)$ then it is uniformly continuous from $(X, U)$. In particular, for every $d \in \Gamma(U)$ and $f \subset X \times X$, the functions $\ell_d^f$ and $m_d^f$ are uniformly continuous from $(X \times X, U \times U)$. It follows that the sets $\mathcal{C}_U f, A_U f \subset X \times X$ and $|\mathcal{C}_U f|, |A_U f| \subset X$ are closed.

As before, a Lyapunov function for a relation $f$ on a uniform space $(X, U)$ is a continuous map $L : X \to \mathbb{R}$ such that $(x, y) \in f$ implies $L(x) \leq L(y)$. Hence, the relation $\leq_L \subset X \times X$ is closed.

As in Definition 4.1

**Definition 5.7.** Let $F$ be a closed, transitive relation on a Hausdorff uniform space $(X, U)$ and let $\mathcal{L}$ be a collection of Lyapunov functions for $F$. We define three conditions on $\mathcal{L}$.

ALG If $L_1, L_2 \in \mathcal{L}$ and $c \geq 0$ then $L_1 + L_2, \max(L_1, L_2), \min(L_1, L_2), cL_1, c, -c \in \mathcal{L}$.

CON For every sequence $\{L_k\}$ of elements of $\mathcal{L}$ there exists a summable sequence of positive real numbers $\{a_k\}$ such that $\sum_k a_k L_k$ converges uniformly to an element of $\mathcal{L}$.

POIN If $(x, y) \notin 1_X \cup F$ then there exists $L \in \mathcal{L}$ such that $L(y) < L(x)$, i.e. $1_X \cup F = \bigcap_{L \in \mathcal{L}} \leq_L$.

**Theorem 5.8.** Let $f$ be a relation on a Hausdorff uniform space $(X, U)$ with gage $\Gamma$.

(a) let $\mathcal{L}_\ell$ be the set of bounded, uniformly continuous functions which are $K\ell_d^f$ dominated for some $d \in \Gamma$ and some positive $K$. Each $L \in \mathcal{L}_\ell$ is an $A_U f$ Lyapunov function and so satisfies (5.14) $A_U f \subset \leq_L$ and $|A_U f| \subset = |L|_{A_U f}$.

The collection $\mathcal{L}_\ell$ satisfies the conditions ALG, CON, and POIN with respect to $F = A_U f$.

(b) let $\mathcal{L}_m$ be the set of bounded, uniformly continuous functions which are $Km_d^f$ dominated for some $d \in \Gamma$ and some positive $K$. Each
Theorem 5.9. Let \( f \) be a relation on a uniform space \((X, \mathcal{U})\).

If \( L \) is a Lyapunov function for \( f \), then \( L \) is a Lyapunov function for \( \mathcal{U}f \).

If \( L \) is a uniformly continuous Lyapunov function for \( f \), then \( L \) is a Lyapunov function for \( \mathcal{A}f \).

Proof: If \( L \) is a Lyapunov function for \( f \) then, by continuity of \( L \), \( \leq_L \) is a closed, transitive relation which contains \( f \) and so contains \( \mathcal{U}f \).

If \( L \) is bounded and uniformly continuous, then \( d_L(x, y) = |L(x) - L(y)| \) is a pseudo-metric in \( \Gamma(\mathcal{U}) \). Let \((x, y) \in \mathcal{A}f \) and \( \varepsilon \in (0, 1) \).

There exists \([a, b] \in f^{\times n} \) such that the \( xy \) chain-length of \([a, b] \) with respect to \( d_L \) is less than \( \varepsilon \). Since \( L \) is a Lyapunov function for \( f \), we have that \( L(a_i) \leq L(b_i) \) for \( i = 1, \ldots, n \).

\[
L(y) - L(x) = L(y) - L(b_0) + \sum_{i=1}^{n} L(b_i) - L(a_i) + \sum_{i=1}^{n-1} L(a_i) - L(b_{i+1}) + L(a_1) - L(x).
\]
The first sum is non-negative and the rest has absolute value at most the chain-length. Hence, \(L(y) - L(x) \geq -\epsilon\). Since \(\epsilon\) was arbitrary, \(L(y) - L(x) \geq 0\).

If \(L\) is unbounded then for each positive \(K\), \(L_K = \max(\min(L, K), -K)\) is a bounded, uniformly continuous Lyapunov function and so is an \(A_{U_F}\) Lyapunov function. If \((x, y) \in A_{U_F}\) then by choosing \(K\) large enough we have \(L_K(x) = L(x)\) and \(L_K(y) = L(y)\). So \(L(y) - L(x) = L_K(y) - L_K(x) \geq 0\).

\[\blacksquare\]

**Corollary 5.10.** Let \(f\) be a relation on a Tychonoff space \(X\) and let \(U_M\) be the maximum uniformity compatible with the topology. Let \(L\) be the set of all bounded, Lyapunov functions for \(f\). Each \(L \in L\) is a Lyapunov function for \(A_{U_M}f\) and

\[
1_X \cup A_{U_M}f = \bigcap_{L \in L} \leq L
\]

**Proof:** With respect to the maximum uniformity every continuous real-valued function is uniformly continuous. So every \(L \in L\) is a Lyapunov function for \(A_{U_M}f\) by Theorem 5.9. Hence \(1_X \cup A_{U_M}f \subset \bigcap_{L \in L} \leq L\). The reverse inclusion follows from POIN in Theorem 5.8 (a).

\[\blacksquare\]

**Theorem 5.11.** Let \(F\) be a closed, transitive relation on a Hausdorff uniform space \((X, U)\) whose topology is second countable. Let \(L\) be a collection of Lyapunov functions for \(F\) which satisfies ALG, CON and POIN. There exists a sequence \(\{L_k\}\) in \(L\) such that

\[
\bigcap_k \leq L_k = 1_X \cup F.
\]

If \(\{a_k\}\) is a positive, summable sequence such that \(L = \Sigma_n a_k L_k \in L\) then \(L\) is a Lyapunov function for \(F\) such that \(1_X \cup F = \leq L\) and

\[
x \in F(y) \implies L(y) < L(x) \text{ unless } y \in F(x)
\]

In particular,

\[
|L|_F = |F|
\]

**Proof:** Proceed just as in the proof of Theorem 4.2 using the fact that \((X \times X) \setminus (1_X \cup F)\) is Lindelöf.

\[\blacksquare\]
For a metrizable space $X$ we let $\Gamma_m(X)$ be the set of metrics compatible with the topology on $X$.

**Theorem 5.12.** Let $f$ a relation on a Hausdorff uniform space $(X, \mathcal{U})$ whose topology is second countable. There exist bounded, uniformly continuous Lyapunov functions $L_\ell, L_m$ for $f$ such that

\begin{equation}
1_X \cup \mathcal{A}_{\ell} f = \leq_{L_\ell}, \quad 1_X \cup \mathcal{C}_{\ell} f = \leq_{L_m} \quad \text{and,}
\end{equation}

\begin{equation}
(5.22) \quad x \in \mathcal{A}_{\ell} f(y) \implies L_\ell(y) < L_\ell(x) \quad \text{unless } y \in \mathcal{A}_{\ell} f(x),
\end{equation}

\begin{equation}
(5.23) \quad x \in \mathcal{C}_{\ell} f(y) \implies L_m(y) < L_m(x) \quad \text{unless } y \notin \mathcal{C}_{\ell} f(x)
\end{equation}

In particular,

\begin{equation}
(5.24) \quad \mathcal{A}_{\ell} f = \mathcal{A}_d f \quad \text{and} \quad \mathcal{C}_{\ell} f = \mathcal{C}_d f.
\end{equation}

**Proof:** The pseudo-metrics chosen below are all assumed bounded by 1. We can always replace $d$ by $\min(d, 1)$.

We apply Theorem 5.11 to $\mathcal{L}_\ell$ and $\mathcal{A}_{\ell} f$ and to $\mathcal{L}_m$ and $\mathcal{C}_{\ell} f$ and obtain $L_\ell \in \mathcal{L}_\ell$ and $L_m \in \mathcal{L}_m$ which satisfy (5.22) and (5.23). We may assume that each maps to $[0, 1]$. In particular, there exist $d_1, d_2 \in \Gamma(\mathcal{U})$ and positive $K_1, K_2$ so that $L_\ell$ is $K_1\ell_f$ dominated and $L_m$ is $K_2m_f$ dominated.

Let $B$ be a countable base and $D$ be a countable dense subset of $X$. For each pair $(x, U)$ with $U \in B$ and $x \in U \cap D$ there exists $d = d(x, U) \in \Gamma(\mathcal{U})$ such that $V^d(x) \subset U$.

For each $x \notin \mathcal{A}_{\ell} f$ there exists $d_{x,1} \in \Gamma(\mathcal{U})$ such that $\ell_f^{d_{x,1}}(x, x) > 0$ and for each $x \notin \mathcal{C}_{\ell} f$ there exists $d_{x,2} \in \Gamma(\mathcal{U})$ such that $m_f^{d_{x,2}}(x, x) > 0$. These are open conditions and so we can choose a sequence $\{d_3, d_4, \ldots\}$ in $\mathcal{U}$ and a positive sequence $\{a_1, a_2, \ldots\}$ with sum 1 so that $d$ defined by $d(x, y) = \frac{1}{3}[|L_\ell(x) - L_\ell(y)| + |L_m(x) - L_m(y)| + \sum_{i=1}^\infty a_id_i]$ satisfies

(i) $d \in \Gamma(\mathcal{U})$.

(ii) The $\mathcal{U}(d)$ topology is that of $X$, i.e. $d \in \Gamma_m(X)$.

(iii) $x \notin \mathcal{A}_{\ell} f$ implies $\ell_f^d(x, x) > 0$, and $x \notin \mathcal{C}_{\ell} f$ implies $m_f^d(x, x) > 0$.

(iv) There exist positive $K_\ell$ and $K_m$ so that $L_\ell$ is $K_\ell\ell_f$ dominated and $L_m$ is $K_m m_f$ dominated.

Condition (i) follows from Lemma 10.1. Condition (ii) implies that $d$ is a metric since $X$ is Hausdorff. From condition (iv) and (5.22) we
obtain
\[ 1_X \cup A_{df} \subset \leq L_\ell = 1_X \cup A_{uf}, \]
\[ 1_X \cup C_{df} \subset \leq L_m = 1_X \cup C_{uf}. \] (5.25)

On the other hand, \( d \in \Gamma(U) \) implies \( A_{uf} \subset A_{df} \) and \( C_{uf} \subset C_{df} \).
Hence, if \((x, y) \in A_{df} \setminus A_{uf}\) then \((x, y) \in 1_X\) and so \( \ell^d_f(x, x) = 0 \).
By condition (iii) this implies \( x \in |A_{uf}| \) and so \((x, y) = (x, x) \in A_{uf}\). This
contradiction proves the first equation in (5.24). The second follows similarly.

Clearly, \( L_\ell \) and \( L_m \) are Lipschitz with Lipschitz constant at most 3.
\[ \blacksquare \]

If \( U_M \) the maximum uniformity compatible with the topology for
a metrizable space \( X \), then since such a space is paracompact, \( U_M \)
consists of all neighborhoods of the diagonal. The gage \( \Gamma(U_M) \) con-
sists of all pseudo-metrics which are continuous on \( X \). In particular,
\( \Gamma_m(X) \subset \Gamma(U_M) \).

**Corollary 5.13.** Let \( f \) be a relation on a second countable Tychonoff
space \( X \) and let \( U_M \) be the maximum uniformity compatible with the
topology. There exists a metric \( d_0 \in \Gamma_m(X) \) such that
\[ A_{U_M} f = A_{d_0} f \quad \text{and} \quad C_{U_M} f = C_{d_0} f. \] (5.26)

Furthermore,
\[ A_{U_M} f = \bigcap_{d \in \Gamma_m(X)} A_{df} \quad \text{and} \quad C_{U_M} f = \bigcap_{d \in \Gamma_m(X)} C_{df}. \] (5.27)

**Proof:** A second countable Hausdorff space is metrizable, i.e. there
exists a metric \( d \) with the \( U(d) \) topology that of \( X \). Thus, \( d \in \Gamma_m(X) \subset \Gamma(U_M) \).
If \( d_0 \in \Gamma(U_M) \), then \( d = d + d_0 \) is a metric in \( \Gamma(U_M) \) and so is
continuous. Since \( d \geq d \) it follows that the \( U(d) \) topology is that of \( X \)
as well, i.e. \( d \in \Gamma_m(X) \). Furthermore,
\[ A_{U_M} f \subset A_{df} \subset A_{d_0} f \quad \text{and} \quad C_{U_M} f \subset C_{df} \subset C_{d_0} f. \] (5.28)

Hence, the intersection over \( \Gamma_m(X) \) yields the same result as intersect-
ing over the entire gage, \( \Gamma(U_M) \). Furthermore, if \( d_0 \) is a metric in \( \Gamma(U_M) \)
satisfying (5.24) then (5.24) together with (5.28) implies (5.26).
\[ \square \]

For \( d \) a metric on \( X \), \( U(d) \) is the uniformity generated by \( V^d_\epsilon \) for
all \( \epsilon > 0 \). We say that \( d \) generates the uniformity \( U(d) \) and that \( U \) is
metrizable if \( U = U(d) \) for some metric \( d \). The Metrization Theorem,
Lemma 6.12 of [12], implies that a Hausdorff uniformity is metrizable if it is countably generated. Two metrics $d_1$ and $d_2$ generate the same uniformity exactly when they are uniformly equivalent. That is, the identity maps between $(X, d_1)$ and $(X, d_2)$ are uniformly continuous. For a metrizable uniformity $U$ we let $\Gamma_m(U) = \{ d : d is a metric with \ U(d) = U \}$.

If $(X, d)$ is a metric space and the set of non-isolated points is not compact, then the maximum uniformity $U_M$ is not metrizable even if $X$ is second countable. Since a metric space is paracompact, $U_M$ consists of all neighborhoods of the diagonal. By hypothesis there is a sequence $\{ x_1, x_2, \ldots \}$ of distinct non-isolated points with no convergent subsequence and so we can choose open sets $G_i$ pairwise disjoint and with $x_i \in G_i$. We can choose $y_i \in G_i \setminus \{ x_i \}$ such that $\epsilon_i = d(x_i, y_i) \to 0$ as $i \to \infty$ and let $\epsilon_0 = 1$. Let $G_0$ be the complement of a closed neighborhood of $\{ x_i \}$ in $\bigcup_{i=1}^\infty G_i$. Thus, $\{ G_i \}$ is a locally finite open cover. Choose $\{ \phi_i \}$ a partition of unity, i.e. each $\phi_i$ is a continuous real-valued function with support in $G_i$ and with $\Sigma_i \phi_i = 1$. Define $\psi(x) = \Sigma_i \epsilon_i \phi_i(x)/2$. In particular, $\psi(x_i) = \epsilon_i/2$ for $i = 1, 2, \ldots$. Thus, $\psi$ is a continuous, positive function with infimum 0. So $U = \{ (x, y) : d(x, y) < \psi(x) \}$ is a neighborhood of the diagonal disjoint from $\{ (x_i, y_i) : i = 1, 2, \ldots \}$. But if $\epsilon_i < \epsilon$ then $(x_i, y_i) \in V_\epsilon^d$. It follows that for any metric $d$ compatible with the topology of $X$ there exists a neighborhood of the diagonal, and so an element of $U_M$, which is not in $U(d)$.

**Theorem 5.14.** Let $(X, U)$ be a uniform space with $U$ metrizable and let $f$ be a relation on $X$.

(a) For every $d \in \Gamma_m(U)$, $C_{Uf} = C_{df}$.

(b) $A_{Uf} = \bigcap_{d \in \Gamma_m(U)} A_{df}$.

**Proof:** If $\tilde{d} \in \Gamma(U)$ and $d_1 \in \Gamma_m(U)$ then $d = \tilde{d} + d_1 \in \Gamma_m(U)$ and $C_{df} \subset C_{df}$. Thus, we need only intersect over $\Gamma_m(U)$ to get $C_{Uf}$. Similarly, for $A_{Uf}$.

On the other hand, if $d_1, d_2 \in \Gamma_m(U)$ then $d_1$ and $d_2$ are uniformly equivalent metrics and so Proposition [3.5] implies that $C_{d_1 f} = C_{d_2 f}$. Hence, the intersection $C_{Uf}$ is this common set.

$\square$

There are special constructions for the Conley relations.

**Definition 5.15.** Let $f$ be a relation on a uniform space $(X, U)$. 

(a) A set $A \subset X$ is called $\mathcal{U}$ inward if there exists $U \in \mathcal{U}$ such that $U(f(A)) \subset A$, or, equivalently, if there exist $d \in \Gamma(\mathcal{U})$ and $\epsilon > 0$ such that $A$ is $(V_d^\epsilon \circ f)^+$ invariant.

(b) A $\mathcal{U}$ uniformly continuous function $L : X \to [0, 1]$ is called a $\mathcal{U}$ elementary Lyapunov function for $f$ if $(x, y) \in f$ and $L(x) > 0$ imply $L(y) = 1$.

If $\mathcal{U} = \mathcal{U}_M$ for the space $X$, then a $\mathcal{U}$ inward set $A$ for $f$ is just called an inward set for $f$. For a paracompact Hausdorff space any neighborhood of a closed set is a $\mathcal{U}_M$ uniform neighborhood and so a set $A$ is inward for a relation $f$ on such a space iff $\overline{f(A)} \subset A$. A continuous function $L : X \to [0, 1]$ is $\mathcal{U}_M$ uniformly continuous and we will call a $\mathcal{U}_M$ elementary Lyapunov function just an elementary Lyapunov function.

Observe for $L : X \to [0, 1]$ that if $L(x) = 0$ or $L(y) = 1$ then $L(y) \geq L(x)$. So an elementary Lyapunov function is a Lyapunov function. In addition, the points of $G_L = \{x : 1 > L(x) > 0\}$ are regular points for $L$ and so $|L|_f \subset L^{-1}(0) \cup L^{-1}(1)$ with equality if $f$ is a surjective relation.

If $u : X \to \mathbb{R}$ is a bounded real-valued function we define the pseudo-metric $d_u$ on $X$ by $d_u(x, y) = |u(x) - u(y)|$. If $u$ is uniformly continuous on $(X, \mathcal{U})$ then $d_u \in \Gamma(\mathcal{U})$.

**Theorem 5.16.** Let $f$ be a relation on a uniform space $(X, \mathcal{U})$.

(a) If $A$ is a $\mathcal{U}$ inward subset for $f$ then there exist $d \in \Gamma(\mathcal{U})$ and $\epsilon > 0$ such that $V_d^\epsilon(f(A)) \subset A$. In particular, $A_1 = A^\circ$ and $A_2 = V_d(f(A))$ are $\mathcal{U}$ inward with $A_1$ open, $A_2$ closed and $\overline{f(A)} \subset A_2 \subset A_1 \subset A$.

(b) Let $A$ be an open $\mathcal{U}$ inward subset for $f$. If for $d \in \Gamma(\mathcal{U})$ and $\epsilon > 0$ $V_d^\epsilon(f(A)) \subset A$, then $V_d^\epsilon(\mathcal{U}_f f(A)) \subset A$. In particular, $A$ is a $\mathcal{U}$ inward subset of $X$ for $\mathcal{U}_f f$ and is and is $(V_d^\epsilon \circ \mathcal{U}_f f)^+$ invariant.

(c) If $A$ is a $\mathcal{U}$ inward subset for $f$, then there exists $B$ a closed $\mathcal{U}$ inward subset for $f^{-1}$ such that $A^\circ \cup B^\circ = X$ and $B \cap f(A) = \emptyset = A \cap f^{-1}(B)$.

(d) If $A$ is a $\mathcal{U}$ inward subset of $X$, then there exists a $\mathcal{U}$ uniformly continuous elementary Lyapunov function $L$ for $f$ such that $L^{-1}(0) \cup A = X$ and $f(A) \subset L^{-1}(1)$. 
(c) If $L$ is a $\mathcal{U}$ elementary Lyapunov function for $f$ and $1 \geq \epsilon > 0$, then $A = \{x : L(x) > 1 - \epsilon\}$ is an open set such that

$$f(A) \subset \mathcal{E}_U f(A) \subset \mathcal{E}_d f(A) \subset L^{-1}(1),$$

(5.29) $$V^d_{\epsilon}(f(A)) \subset V^d_{\epsilon}(\mathcal{E}_U f(A)) \subset V^d_{\epsilon}(\mathcal{E}_d f(A)) \subset A.$$ 

In particular, $L$ is a $\mathcal{U}(d_L)$ elementary Lyapunov function for $\mathcal{E}_d f$ and hence is a $\mathcal{U}$ elementary Lyapunov function for $\mathcal{E}_U f$ and for $\mathcal{F}$.

(f) If $L$ is a $\mathcal{U}$ elementary Lyapunov function for $f$, then $1 - L$ is a $\mathcal{U}$ elementary Lyapunov function for $f^{-1}$.

**Proof:** (a) There exist $d \in \Gamma$ and $\epsilon > 0$ such that $V^d_{2\epsilon}(f(A))$ is contained in $A$ and so is contained in $A^c$. For a subset $B$ of $X$, $x \in \overline{B}$ implies $d(x, B) = 0$ and so $V^d_{\epsilon}(f(A)) \subset V^d_{\epsilon}(f(A))$ and $f(A) \subset V^d_{\epsilon}(f(A))$.

(b) Assume that $x \in A$ and $z \in V^\epsilon_{d}(\mathcal{E}_U f(x))$. So there exist $z_1 \in \mathcal{E}_U f(x)$ and $\epsilon_1 > 0$ such that $d(z_1, z) < \epsilon$. There exist $d_1 \in \Gamma$ and $\epsilon_1 > 0$ such that $V^d_{\epsilon_1}(x) \subset A$ and $d(z_1, z) + \epsilon_1 < \epsilon$. Let $d = d + d_1$. There exists $[a, b] \in f^{\times n}$ such that the $xz_1$ chain-bound of $[a, b]$ with respect to $d$ is less than $\epsilon_1$. Because $d_1(x, a_1) < d(x, z_1) < \epsilon_1, a_1 \in A$. Since $b_1 \in f(A)$ and $d(b_1, a_2) < d(b_1, a_2) < \epsilon$, $a_2 \in A$. Inductively, we obtain $a_i \in A$ and $b_i \in f(A)$ for $i = 1, \ldots, n$. Finally, $d(b_n, z) \leq d(b_n, z_1) + d(z_1, z) < \epsilon$. So $z \in A$.

(c) Let $d \in \Gamma$ and $\epsilon > 0$ be such that $V^d_{2\epsilon}(f(A))$ is contained in $A$ and so is contained in $A^c$. Let $B = X \setminus V^d_{\epsilon}(f(A))$ so that $B^c = X \setminus V^d_{\epsilon}(f(A))$. Thus, $B$ is closed, $A^c \cup B^c = X$ and $B \cap f(A) = \emptyset$. Assume that $(x, y) \in f$ and $z \in V^\epsilon_{d}(x)$. If $y \in B$ then $x \notin A$ and so $x \notin V^\epsilon_{d}(f(A))$ and $z \notin V^\epsilon_{d}(f(A))$. That is, $z \in B$. Thus, $V^\epsilon_{d}(f^{-1}(B)) \subset B$. Finally, if $y \in B$ then $y \notin f(A)$ and $x \notin A$. That is, $f^{-1}(B) \cap A = \emptyset$. If $(x, y) \in f$ and $L(x) > 0$ then $d(x, f(A)) < \epsilon$ and so $x \in A$. Then $y \in f(A)$ implies $L(y) = 1$.

(d) Clearly, $f(A) \subset L^{-1}(1)$. Let $\epsilon > \epsilon_1 > 0$. We show that $V^d_{\epsilon_1}(\mathcal{E}_d f(A)) \subset \{y : L(y) > 1 - \epsilon_1\}$. Assume $x \in A, y \in V^d_{\epsilon_1}(\mathcal{E}_d f(x))$. So there exists $z \in \mathcal{E}_d f(x)$ with $d_L(z, y) < \epsilon_1$. Choose $\epsilon_2 > 0$ so that $d_L(z, y) + \epsilon_2 < \epsilon_1$ and $L(x) > 1 - \epsilon + \epsilon_2$. Since $L$ is uniformly continuous, $d_L \in \Gamma(\mathcal{U})$ and so there exists $[a, b] \in f^{\times n}$ such that the $xz$ chain-bound of $[a, b]$ with respect to $d_L$ is less than $\epsilon_2$. Since $d_L(x, a_1) < \epsilon_2, a_1 \in A$. Hence, $b_1 \in L^{-1}(1)$. Inductively, $a_i \in A$ and $b_i \in L^{-1}$ for all $i = 1, \ldots, n$. Finally, $d_L(b_n, y) \leq d_L(b_n, z) + d_L(z, y) < \epsilon_1$. Since $L(b_n) = 1, L(y) > 1 - \epsilon_1$. Letting $\epsilon_1 \to 0$ we obtain $\mathcal{E}_d f(A) \subset L^{-1}(1)$. Letting $\epsilon \to \epsilon$ we obtain $V^d_{\epsilon_1}(\mathcal{E}_d f(A)) \subset \{y : L(y) > 1 - \epsilon\} = A.$
(f) The contrapositive of the definition of an elementary Lyapunov function says that if \((x, y) \in f\) with \(L(y) < 1\) then \(L(x) = 0\). It follows that \(1 - L\) is an elementary Lyapunov function for \(f^{-1}\).

\[
\square
\]

**Proposition 5.17.** Let \(f\) be a relation on a uniform space \((X, \mathcal{U})\), \(\epsilon > 0\) and \(d \in \Gamma(\mathcal{U})\). Let \(K \subset X\) be closed and compact.

(a) For \(x \in X\), the set \(\{y : \ell_d^f(x, y) < \epsilon\}\) is an open subset of \(X\) containing \(A_d f(x) \supset A_{df}(x)\). It is \(A_d f\) \(+\) invariant and so is \(A_{df}\) \(+\) invariant.

\[
A_{df}(K) = \bigcap_{d \in \Gamma, \epsilon > 0} \bigcup_{x \in K} \{y : \ell_d^f(x, y) < \epsilon\},
\]

\[
K \cup A_{df}(K) = \bigcap_{d \in \Gamma, \epsilon > 0} \bigcup_{x \in K} \{y : \min(\ell_d^f(x, y), d(x, y)) < \epsilon\}
\]

(b) For \(x \in X\), the set \(\{y : m_d^f(x, y) < \epsilon\}\) is an open subset of \(X\) containing \(V_{\epsilon} \circ C_d f \circ V_{\epsilon}^d(x) \supset V_{\epsilon} \circ C_{df} \circ V_{\epsilon}^d(x)\). It is \(V_{\epsilon} \circ C_d f\) \(+\) invariant and so is \(V_{\epsilon} \circ C_{df}\) and \(V_{\epsilon} \circ f\) \(+\) invariant. In particular, \(\{(x, y) : m_d^f(x, y) < \epsilon\}\) is a \(U\) inward set for \(f\).

\[
C_{df}(K) = \bigcap_{d \in \Gamma, \epsilon > 0} \bigcup_{x \in K} \{y : m_d^f(x, y) < \epsilon\},
\]

\[
K \cup C_{df}(K) = \bigcap_{d \in \Gamma, \epsilon > 0} \bigcup_{x \in K} \{y : \min(m_d^f(x, y), d(x, y)) < \epsilon\}
\]

**Proof:** The sets are open because \(\ell_d^f\) and \(m_d^f\) are continuous. The set in (a) clearly contains \(A_d f(x) = \{y : \ell_d^f(x, y) = 0\}\). If \((y, z) \in A_d f\) then by Proposition 2.2 \(\ell_d^f(x, z) \leq \ell_d^f(x, y) + \ell_d^f(y, z) = \ell_d^f(x, y) < \epsilon\).

If \(y \in V_{\epsilon} \circ C_d f(z)\) with \(m_d^f(x, z) < \epsilon\) then there exists \(z_1 \in C_d f(z)\) with \(d(z_1, y) < \epsilon\). Let \(\epsilon_1 > 0\) and such that \(d(z_1, y) + \epsilon_1, m_d^f(x, z) + 2\epsilon_1 < \epsilon\). There exist \([a, b] \in f^{x_m}\) and \([c, d] \in f^{x_m}\) such that with respect to \(d\) the \(xz\) chain-bound of \([a, b]\) is less than \(m_d^f(x, z) + \epsilon_1\) and the \(zz_1\) chain-bound of \([c, d]\) is less than \(\epsilon_1\). Notice that \(d(b_n, c_1) \leq d(b_n, z) + d(z, c_1) < \epsilon\) and \(d(c_m, y) \leq d(c_m, z) + d(z_1, y) < \epsilon\). Hence, the \(xy\) chain-bound of the concatenation \([a, b] \cdot [c, d]\) is less than \(\epsilon\). Thus, \(\{y : m_d^f(x, y) < \epsilon\}\) is \((V_{\epsilon} \circ C_d f)\) \(+\) invariant.

Similarly, if \(y \in V_{\epsilon} \circ C_d f(z)\) with \(d(x, z) < \epsilon\) then there exists \(z_1 \in C_d f(z)\) with \(d(z_1, y) < \epsilon\). Let \(\epsilon_1 > 0\) and such that \(d(z_1, y) + \epsilon_1 < \epsilon\).
2\varepsilon_1, d(x, z) + 2\varepsilon_1 < \epsilon$. There exists $[c, d] \in f^{Xm}$ such that with respect to $d$ the $zz_1$ chain-bound of $[c, d]$ is less than $\varepsilon_1$. Notice that $d(x, c_1) \leq d(x, z) + d(z, c_1) < \epsilon$ and $d(c_m, y) \leq d(c_m, z_1) + d(z_1, y) < \epsilon$. Hence, the $xy$ chain-bound of the concatenation $[c, d]$ is less than $\epsilon$. Thus, \( \{ y : m_d^f(x, y) < \epsilon \} \) contains \( V_{\varepsilon_1} \circ C_d \circ V_{\varepsilon_1}(x) \).

If \( Q : X \times X \to \mathbb{R} \) is a continuous function with \( Q \geq 0 \), then we let \( Q(K, y) = \inf \{ Q(x, y) : x \in K \} \). Clearly, \( Q(K, y) \leq \epsilon \) iff there exists \( x \in K \) such that \( Q(x, y) < \epsilon \). Also,

\[
(5.32) \quad \{ x : Q(K, y) = 0 \} = \bigcap_{\varepsilon > 0} \{ x : Q(K, y) < \varepsilon \}. 
\]

Furthermore, if \( K \) is compact then \( Q(K, y) = 0 \) iff there exists \( x \in K \) such that \( Q(x, y) = 0 \).

Recall from (3.19) that \( \ell_d^{f,1_X}(x, y) = \min(\ell_d^f(x, y), d(x, y)) \) and from (3.25) that \( m_d^{f,1_X(x, y)}(x, y) = \min(m_d^f(x, y), d(x, y)) \).

Let \( Q_d(x, y) = m_d^{f,1_X(x, y)}(x, y) \) so that \( Q_d(K, y) = \min(m_d^f(K, y), d(K, y)) \).

Observe that if \( d_1, d_2 \in \Gamma(U) \) and \( \varepsilon_1, \varepsilon_2 \geq 0 \) then with \( d = d_1 + d_2 \) and \( \varepsilon = \min(\varepsilon_1, \varepsilon_2) \),

\[
(5.33) \quad \{ (x, y) : Q_d(x, y) \leq \varepsilon \} \subset \{ (x, y) : Q_{d_1}(x, y) \leq \varepsilon_1 \} \cap \{ (x, y) : Q_{d_2}(x, y) \leq \varepsilon_2 \}. 
\]

So if \( K \) is compact, and \( y \in \bigcap_{d \in \Gamma, \varepsilon > 0} \bigcup_{x \in K} \{ y : \min(m_d^f(x, y), d(x, y)) < \varepsilon \} \) the collection of closed subsets \( \{ \{ x \in K : Q_d(x, y) = 0 \} : d \in \Gamma(U) \} \) satisfies the finite intersection property and so has a nonempty intersection. If \( x \in K \) is a point of the intersection, then \( y \in K \cup C_u f(x) \). This proves the second equation in (5.31). The three remaining equations in (5.30) and (5.31) follow from a similar argument with \( Q_d \) equal to \( \ell_d^f, \ell_d^{f,1_X} \) and \( m_d^f \).

Notice that as functions of \( y \) \( \ell_d^f(x, y) \) and \( m_d^f(x, y) \) are \( d \) Lipschitz with Lipschitz constant at most 1. Hence, for any \( K \subset X \), as functions of \( y \), \( \ell_d^f(K, y) \) and \( m_d^f(K, y) \) are \( d \) Lipschitz with Lipschitz constant at most 1 as are \( \min(\ell_d^f(K, y), d(K, y)) \) and \( \min(m_d^f(K, y), d(K, y)) \).

\( \square \)

**Theorem 5.18.** Let \( f \) be a relation on a uniform space \((X, U)\).

(a) If \((x, y) \notin 1_X \cup C_u f\), then there exists a \( U \) elementary Lyapunov function \( L \) such that \( L(y) = 0 \) and \( L(x) = 1 \).

(b) If \( x \notin \{ C_u f \} \), then there exists a \( U \) elementary Lyapunov function \( L \) such that \( 1 > L(x) > 0 \).

**Proof:** (a) With \( g = f \cup \{ (x, x) \} \), \( m_d^g(y) = \min(m_d^f(x, y), d(x, y)) \) by Lemma 3.9. By hypothesis, there exist \( d \in \Gamma \) and \( \varepsilon > 0 \) so that
\( m^g_d(x, y) > \varepsilon \). By Proposition 5.17 (b), the set \( A = \{ y : m^g_d(x, y) < \varepsilon \} \) is a \( U \) inward set for \( g \). By Proposition 5.16 (d) there is a \( U \) uniformly continuous elementary Lyapunov function \( L \) for \( g \) (and hence for \( f \)) so that \( L^{-1}(0) \cup A = X \) and \( g(A) \subset L^{-1}(1) \). Since \( x \in A \) and \( (x, x) \in g \), \( x \in g(A) \) and so \( L(x) = 1 \). Since \( y \notin A \), \( L(y) = 0 \).

(b) By hypothesis, there exist \( d \in \Gamma \) and \( 1 > \varepsilon > 0 \) so that \( m^f_d(x, x) > 2\varepsilon \). Let \( A_0 = V^d_\varepsilon(x) \) and \( A_1 = \{ y : m^f_d(x, y) < \varepsilon \} \). Since \( m^f_d(x, x) \leq m^f_d(x, y) + d(y, x) \), it follows that \( A_0 \) and \( A_1 \) are disjoint.

By Proposition 5.17 (b) \( V^d_\varepsilon(f(A_0 \cup A_1)) \subset A_1 \). Let \( B = f(A_0 \cup A_1) \). Define \( L(y) = \max(\gamma - d(y,B) / \varepsilon, \varepsilon - d(y,x),0) \). If \( (y_1, y_2) \in f \) and \( L(y_1) > 0 \) then \( y_1 \in A_0 \cup A_1 \) and so \( y_2 \in B \). Thus, \( L(y_2) = 1 \). Thus, \( L \) is a \( U \) elementary Lyapunov function. Since \( x \in A_0 \), \( d(x, B) > \varepsilon \). Hence, \( L(x) = \varepsilon \).

\( \square \)

**Definition 5.19.** Let \( f \) be a relation on a uniform space \( (X, U) \). We denote by \( \mathcal{L}_e \) the set of \( U \) elementary Lyapunov functions for \( f \). We say that a set \( \mathcal{L} \subset \mathcal{L}_e \) satisfies the condition POIN-E for \( \mathcal{C}_U f \) if it satisfies POIN for \( \mathcal{C}_U f \) and, in addition,

- If \( x \notin |\mathcal{C}_U f| \), then there exists \( L \in \mathcal{L}_e \) such that \( 1 > L(x) > 0 \).

By Proposition 5.18 the set \( \mathcal{L}_e \) satisfies POIN-E for \( \mathcal{C}_U f \).

**Theorem 5.20.** For \( f \) a relation on a uniform space \( (X, U) \). If \( \mathcal{L} \subset \mathcal{L}_e \) satisfies POIN-E for \( \mathcal{C}_U f \) then

\[
\mathcal{C}_U f = \bigcap_{L \in \mathcal{L}} \leq_L,
\]

\[
|\mathcal{C}_U f| = \bigcap_{L \in \mathcal{L}} [L^{-1}(0) \cup L^{-1}(1)] = \bigcap_{L \in \mathcal{L}} |L|_f.
\]

**Proof:** The first equation follows from POIN for \( \mathcal{C}_U f \).

If \( L \in \mathcal{L}_e \) then it is an elementary Lyapunov function for \( \mathcal{C}_U f \) by Proposition 5.16 (e) and \( 1 - L \) is an elementary Lyapunov function for \( \mathcal{C}_U f^{-1} \) by Proposition 5.16 (f). So with \( G_L = \{ x : 1 > L(x) > 0 \} \),

\[
\mathcal{C}_U f(G_L) \subset L^{-1}(1) \quad \text{and} \quad \mathcal{C}_U f^{-1}(G_L) \subset L^{-1}(0).
\]

Hence, \( G_L \cap |\mathcal{C}_U f| = \emptyset \), i.e. \( |\mathcal{C}_U f| \subset |L|_f \).

On the other hand, if \( x \notin |\mathcal{C}_U f| \) then by POIN-E there exists \( L \in \mathcal{L}_e \) such that \( x \in G_L \).

\( \square \)
If $A$ is a $^+$ invariant subset for a relation $f$ we denote by $f^\infty(A)$ the (possibly empty) maximum invariant subset of $A$, i.e. the union of all $f$ invariant subsets of $A$. We can obtain it by a transfinite construction

\[ A_0 = A, \quad A_{\alpha+1} = f(A_\alpha), \quad A_\alpha = \bigcap_{\beta < \alpha} A_\beta \quad \text{for } \alpha \text{ a limit ordinal.} \]

The process stabilizes at $\alpha$ when $A_{\alpha+1} = A_\alpha$ which then equals $f^\infty(A)$.

**Definition 5.21.** If $A$ is a $U$ inward set for a relation $f$ then $(C_uf)^\infty(A)$ is called the $U$ attractor associated with $A$. A $U$ attractor for $f^{-1}$ is called a $U$ repellor for $f$. If $A$ is a $U$ inward set for $f$ and $B$ is a $U$ inward set for $f^{-1}$ such that $A \cup B = X$, $f(A) \cap B = \emptyset = f^{-1}(B) \cap A$ then the pair $(A_\infty, B_\infty) = ((C_uf)^\infty(A), (C_uf^{-1})^\infty(B))$ is called a $U$ attractor-repellor pair with $B_\infty = (C_uf^{-1})^\infty(B)$ the repellor dual to $A_\infty = (C_uf)^\infty(A)$ and vice-versa.

Again, if $U = U_M$ we will drop the label $U$.

**Proposition 5.22.** Let $f$ be a relation on a uniform space $(X, U)$ and let $x, y \in X$ with $y \not\in \{x\}$. The following are equivalent.

(i) $y \in C_uf(x)$.

(ii) For every $U$ elementary Lyapunov function $L$ for $f$, $L(x) > 0$ implies $L(y) = 1$.

(iii) For every open $U$ inward set $A$ for $f$, $x \in A$ implies $y \in A$.

If $x \in |C_uf|$, then these conditions are further equivalent to

(iv) For every $U$ attractor $A_\infty$ for $f$, $x \in A_\infty$ implies $y \in A_\infty$.

**Proof:**

(i) $\Rightarrow$ (ii): A $U$ elementary Lyapunov function for $f$ is a $U$ elementary Lyapunov function for $C_uf$ by Theorem 5.16(e).

(i) $\Rightarrow$ (iii): A $U$ inward set for $f$ is $C_uf$ $^+$ invariant by Theorem 5.16(b).

(ii) $\Rightarrow$ (i): Apply Theorem 5.18(a).

(iii) $\Rightarrow$ (i): By Proposition 5.17(b), with $g = f \cup \{ (x, x) \}$, \{ $y : m_d^g(x, y) < \epsilon \} = \{ y : \min(m_d^f(x, y), d(x, y)) < \epsilon \}$ is a $U$ inward set for $g$ and hence for $f$. So (5.31) implies that $\{ x \} \cup C_uf(x)$ is the intersection of $U$ inward sets.

If $x \in |C_uf|$, then $C_uf(x)$ is $C_uf$ invariant and so $x$ is contained in an inward set $A$ iff it is contained in the associated attractor. Hence (iii) $\Rightarrow$ (iv) in this case.

Notice that if $x \in |C_uf|$ then $\overline{\{ x \}}$ is contained in the closed set $C_uf(x)$.  


Proposition 5.23. If $A_\infty$ is the $\mathcal{U}$ attractor associated with the $\mathcal{U}$ inward set $A$, then $A \cap |\mathcal{C}_\mathcal{U}f| \subset A_\infty$. Furthermore,

\begin{equation}
|\mathcal{C}_\mathcal{U}f| = \bigcap \{A_\infty \cup B_\infty : (A, B) a \mathcal{U} attractor-repellor pair for f\}.
\end{equation}

If $x \in |\mathcal{C}_\mathcal{U}f|$ then the $\mathcal{C}_\mathcal{U}f \cap \mathcal{C}_\mathcal{U}f^{-1}$ equivalence class of $x$ in $|\mathcal{C}_\mathcal{U}f|$ is given by

\begin{equation}
(\mathcal{C}_\mathcal{U}f \cap \mathcal{C}_\mathcal{U}f^{-1})(x) = \bigcap \{B : B a \mathcal{U} attractor or repellor with \ x \in B\}.
\end{equation}

Proof: For any $\mathcal{C}_\mathcal{U}f^+$ invariant set $A$, if $x \in |\mathcal{C}_\mathcal{U}f|$ then $\mathcal{C}_\mathcal{U}f(x)$ is a $\mathcal{C}_\mathcal{U}f$ invariant subset of $A$ and so is contained $(\mathcal{C}_\mathcal{U}f)^\infty(A)$. So if $(A, B)$ is an attractor-repellor pair then $|\mathcal{C}_\mathcal{U}f| = |\mathcal{C}_\mathcal{U}f| \cap (A \cup B) \subset |\mathcal{C}_\mathcal{U}f| \cap (A_\infty \cup B_\infty)$.

In particular, if $L$ is a $\mathcal{U}$ elementary Lyapunov function then with $A = \{x : L(x) > 0\}$ and $B = \{x : L(x) < 1\}$, the associated attractor-repellor pair $(A_\infty, B_\infty)$ satisfies $A_\infty \subset L^{-1}(1)$, $B_\infty \subset L^{-1}(0)$ and so $|\mathcal{C}_\mathcal{U}f| \cap L^{-1}(1) = |\mathcal{C}_\mathcal{U}f| \cap A_\infty$ and $|\mathcal{C}_\mathcal{U}f| \cap L^{-1}(0) = |\mathcal{C}_\mathcal{U}f| \cap B_\infty$. Hence, (5.37) follows from (5.34).

Finally, $(\mathcal{C}_\mathcal{U}f \cap \mathcal{C}_\mathcal{U}f^{-1})(x) = \mathcal{C}_\mathcal{U}f(x) \cap \mathcal{C}_\mathcal{U}f^{-1}(x)$. By Proposition 5.22 $\mathcal{C}_\mathcal{U}f(x)$ is the intersection of the attractors containing $x$ and $\mathcal{C}_\mathcal{U}f^{-1}(x)$ is the intersection of the repellors containing $x$. 

\square
6. Upper-semicontinuous Relations and Compactifications

Up to now we have generally imposed no topological conditions on the relation $f$. Consider $f : X \to Y$ a relation with $X$ and $Y$ Tychonoff spaces, i.e. $f \subset X \times Y$. Call $f$ a closed relation when it is a closed subset of $X \times Y$. Call $f$ pointwise closed when $f(x)$ is closed for every $x \in X$. Call $f$ pointwise compact when $f(x)$ is compact for every $x \in X$. Since $f(x)$ is the pre-image of $f$ by the continuous map $y \mapsto (x, y)$ it follows that a closed relation is pointwise closed. Since $Y$ is Hausdorff a pointwise compact relation is pointwise closed.

If $f : X \to Y$ is a relation and $B \subset Y$, recall that $f^*(B) = \{x \in X : f(x) \subset B\}$. For example, $f^*(\emptyset) = \{x : f(x) = \emptyset\}$ which is the complement of the domain of $f$, $\text{Dom}(f) = f^{-1}(X)$.

We will need the properties of proper maps. These are reviewed in Appendix C.

**Theorem 6.1.** Let $f : X \to Y$ be a relation between Tychonoff spaces.

(a) If $f$ is a closed relation and $A \subset X$ is compact, then $f(A) \subset Y$ is closed.

(b) The following conditions are equivalent. When they hold we call $f$ an upper semi-continuous relation, written $f$ is usc.
   (i) If $B$ is a closed subset of $Y$, then $f^{-1}(B)$ is a closed subset of $X$.
   (ii) If $B$ is an open subset of $Y$ then $f^*(B)$ is an open subset of $X$.
   (iii) If $\{x_i : i \in I\}$ is a net in $X$ converging to $x \in X$ and $B$ is an open set containing $f(x)$ then eventually $f(x_i) \subset B$.

(c) A usc relation is closed iff it is pointwise closed.

(d) If $f$ and $f^{-1}$ are usc, then $f$ and $f^{-1}$ are closed relations.

(e) Let $\pi_1 : X \times Y \to X$ be the projection map. If the restriction $\pi_1|f : f \to X$ is a closed map, then $f$ is usc.

(f) The following conditions are equivalent. When they hold we call $f$ a compactly upper semi-continuous relation, written $f$ is cusc.
   (i) With $\pi_1 : X \times Y \to X$ the projection map, the restriction $\pi_1|f : f \to X$ is a proper map.
   (ii) The relation $f$ is pointwise compact and usc.

(g) If $f$ is cusc then $f$ is a closed relation and $A$ a compact subset of $X$, implies that $f(A)$ is a compact subset of $Y$.

(h) If $X$ is a $k$-space, $f$ is a closed relation and for every compact subset $A$ of $X$, the subset $f(A)$ of $Y$ is compact, then $f$ is cusc.

(i) If $f$ is cusc and $g \subset f$ then $g$ is cusc iff $g$ is closed.
Proof: (a) Since $A$ is compact, the trivial map of $A$ to a point is proper. Hence, $\pi_2: A \times Y \to Y$ is a closed map. If $f$ is closed then $\pi_2((A \times Y) \cap f) = f(A)$ is closed.

(b) (i) $\iff$ (iii): $f^*(B) = X \setminus f^{-1}(Y \setminus B)$.

(ii) $\iff$ (iii): If $f^*(B)$ is open then eventually $x_i \in f^*(B)$. If $f^*B$ is not open then there is a net $\{x_i\}$ in the complement which converges to a point $x \in f^*(B)$. Then $f(x) \subset B$ but never $f(x_i) \subset B$, contradicting (iii).

(c) Assume $f$ is usc and pointwise closed. Suppose $\{(x_i, y_i)\}$ is a net in $f$ converging to $(x, y)$ but with $(x, y) \not\in f$ and so $y \not\in f(x)$. Since $f(x)$ is closed and $Y$ is Tychonoff, there is are disjoint open sets $B, G$ with $f(x) \subset B$ and $y \in G$. Since $f$ is usc, eventually $f(x_i) \subset B$. In particular, eventually $y_i \in B$ and so eventually $y_i \not\in G$. This contradicts convergence of $\{y_i\}$ to $y$.

We saw above that a closed relation is always pointwise closed.

(d) If $f^{-1}$ is use then $f(x) = (f^{-1})^{-1}(x)$ is closed. Since $f$ is usc, it is closed by (c). Hence, $f^{-1}$ is closed as well.

(e) If $B$ is a closed subset of $Y$, then $f \cap (X \times B)$ is a closed subset of $f$. If $\pi_1|f$ is a closed map then $f^{-1}(B) = \pi_1(f \cap (X \times B))$ is closed.

(f) (i) $\Rightarrow$ (ii): A proper map is closed and so $f$ is use by (e). Since $\pi_1|f$ is proper, $(\pi_1|f)^{-1}(x) = \{x\} \times f(x)$ is compact by Proposition 11.2(a). Hence, $f$ is pointwise compact.

(ii) $\Rightarrow$ (i): We verify condition (iv) of Proposition 11.2(a). Let $\{(x_i, y_i)\}$ be a net in $f$ such that $\{x_i\} \text{ converges to } x \in X$. If $B$ is any open set containing $f(x)$ then eventually $f(x_i) \subset B$ because $f$ is usc. So eventually $y_i \in B$. Because $f(x)$ is compact, Lemma 9.1 implies that $f(x)$ contains a cluster point of $\{y_i\}$. That is, there is a subnet $\{y_{i'}\}$ which converges to a point $y \in f(x)$. Hence $\{(x_{i'}, y_{i'})\}$ converges to $(x, y) \in (\pi_1|f)^{-1}(x)$.

(g) A pointwise compact relation is pointwise closed and so a cusc relation is a closed relation by (b). If $A \subset X$ is compact then $(\pi_2|f)^{-1}(A)$ is compact by Proposition 11.2(c). Hence, $f(A) = \pi_2((\pi_1|f)^{-1}(A))$ is compact, where $\pi_2 : X \times Y \to Y$ is the other projection.

(h) If $A$ and $f(A)$ are compact and $f$ is closed then $(A \times f(A)) \cap f = (\pi_1|f)^{-1}(A)$ is compact. So the result follows from Proposition 11.3(a).

(i) If $g$ is cusc then it is closed by (e) and (c). If $\pi|f : f \to X$ is a proper map and $g$ is a closed subset of $f$ then $\pi|g$ is proper by Proposition 11.1(d).

\[\square\]
Remark: The condition that a pointwise compact relation be usc, and so cusc, is weaker than the demand that \( x \mapsto f(x) \) is continuous as a function from \( X \) to the space of compact subsets with the Hausdorff topology. For a comparison in the compact case, see [1] Chapter 7.

We call \( f \) a proper relation when both \( f \) and \( f^{-1} \) are cusc relations, or, equivalently when \( \pi_1|f : f \to X \) and \( \pi_2|f : f \to Y \) are both proper maps.

**Proposition 6.2.** Let \( f : X \to Y \) be a map between Tychonoff spaces. the following are equivalent:

(i) \( f \) is a continuous map.
(ii) \( f \) is a usc relation.
(iii) \( f \) is a cusc relation.

If \( f \) is continuous then \( f \) is a closed map iff \( f^{-1} \) is a usc relation, and the following are equivalent

(iv) \( f \) is a proper map.
(v) \( f^{-1} \) is a cusc relation.
(vi) \( f \) is a proper relation.
(vii) \( f \) is a closed map and \( f^{-1}(y) \) is compact for every \( y \in Y \).

**Proof:** (i) \( \iff \) (ii): Both say that \( f^{-1}(B) \) is closed when \( B \) is.

(ii) \( \iff \) (iii): because \( f \) is pointwise compact.

The relation \( f^{-1} \) is usc iff \( f(A) \) is closed when \( A \) is.

(iv) \( \iff \) (vii): by Proposition [1.2]

(v) \( \iff \) (vi): Since \( f \) is a continuous map it is a cusc relation so it is a proper relation iff \( f^{-1} \) is a cusc relation.

(v) \( \iff \) (vii): Condition (vii) says that \( f^{-1} \) is usc and pointwise compact.

\( \square \)

**Theorem 6.3.** Let \( f : X \to Y \) and \( g : Y \to Z \) be relations between Tychonoff spaces.

(a) If \( f \) and \( g \) are usc then \( g \circ f \) is usc.
(b) If \( f, g \) and \( g^{-1} \) are usc and closed, then \( g \circ f \) is usc and closed.
(c) If \( f \) and \( g \) are cusc then \( g \circ f \) is cusc.
(d) If \( f \) is cusc and \( g \) is closed then \( g \circ f \) is closed.

**Proof:** (a) If \( C \subset Z \) is closed then \( (g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C)) \) is closed.

(b) By (a) \( g \circ f \) is usc. For \( x \in X \), \( g \circ f(x) = g(f(x)) \) is closed since \( f \) is pointwise closed and \( g^{-1} \) is usc. Hence, \( g \circ f \) is pointwise closed, and so is closed by [6.1] (c).
(c) By Theorem 6.1(f) $g \circ f(x) = g(f(x))$ is compact since $f$ is pointwise compact and $g$ is cusc.

(d) Since $f$ is cusc, $\pi_{13} : f \times Z \to X \times Z$ is a closed map. Since $g$ is a closed relation, $(f \times Z) \cap (X \times g)$ is a closed subset and so its image $g \circ f \subset X \times Z$ is closed.

$\blacksquare$

**Proposition 6.4.** Let $f, g : X \to Y$ be relations between Tychonoff spaces.

(a) If $f$ and $g$ are both closed, usc or cusc then $g \cup f$ satisfies the corresponding property.

(b) If $f$ is cusc and $g$ is closed, then $g \cap f$ is cusc.

(c) Assume $Y$ is a normal space. If $f$ and $g$ are both closed and usc then $g \cap f$ is closed and usc.

**Proof:** (a) For $B \subset Y$, $(f \cup g)^{-1}(B) = (f^{-1} \cup g^{-1})(B) = f^{-1}(B) \cup g^{-1}(B)$. Since the union of two closed sets is closed it follows that $g \cup f$ is closed or usc when each of $g$ and $f$ is closed or usc. Furthermore, $(f \cup g)(x) = f(x) \cup g(x)$ and so $f \cup g$ is pointwise compact when $f$ and $g$ are.

(b) Apply Theorem 6.1(i).

(c) If $U$ is an open set containing $(g \cap f)(x) = g(x) \cap f(x)$ then since $f$ and $g$ are closed $g(x) \setminus U$ and $f(x) \setminus U$ are disjoint closed sets. Since $Y$ is normal we can choose disjoint open sets $V_1 \supset g(x) \setminus U$ and $V_2 \supset f(x) \setminus U$. Hence, $U_1 = V_1 \cup U \supset g(x)$ and $U_2 = V_2 \cup U \supset f(x)$ with $U_1 \cap U_2 = U$. Since $g$ and $f$ are usc, $g^*(U_1) \cap f^*(U_2)$ is an open set containing $x$ and contained in $(g \cap f)^*(U)$. Thus, $(g \cap f)^*(U)$ is a neighborhood of $x$. Hence, $g \cap f$ is usc.

$\blacksquare$

**Example 6.5.** For $f$ a relation on $X$ with $\pi_1|f$ the first coordinate projection, $f$ can be usc without $\pi_1|f$ being closed. Furthermore, with $g \subset f$ closed, $g$ need not be usc.

**Proof:** Let $X = \mathbb{R}$ and $f = f^{-1} = \{(t,1/t) : t \neq 0 \in \mathbb{R}\} \cup \{(t,0), (0,t) : t \in \mathbb{R}\}$. Let $g = g^{-1} = \{(t,1/t) : t \neq 0 \in \mathbb{R}\} \cup \{(0,0)\}$.

$\blacksquare$

Now we illustrate how these conditions on a relation may be applied.
Lemma 6.6. Let $F$ be a closed, reflexive, transitive relation on a normal Hausdorff space $X$ with $F$ and $F^{-1}$ usc. If $A$ is a closed, $F$ invariant set and $U$ is an open set with $A \subset U$ then there exists a closed, $F$ invariant set $B$ such that $A \subset B$ and $B \subset U$.

**Proof:** Because $F^{-1}$ is usc, $F(A)$ is closed. Since $F$ is usc, $F^*(U)$ is open and since $A$ is $F$ invariant, $A \subset F^*(U)$. Use normality to choose a closed set $B_1$ so that $A \subset B_1$ and $B_1 \subset F^*(U)$. The set $B = F(B_1) \subset U$ is closed because $F^{-1}$ is usc and $A \subset B_1 \subset B$ because $F$ is reflexive.

\[\square\]

The following is a version of [13] Theorem 2, see also [3] and [4].

Theorem 6.7. Let $F$ be a closed, transitive relation on a normal Hausdorff space $X$ with $F$ and $F^{-1}$ usc. Assume that $X_0$ is a closed subset of $X$ and $L_0 : X_0 \to [a,b]$ is a bounded, Lyapunov function for the restriction $F_0 = F \cap (X_0 \times X_0)$. There exists $L : X \to [a,b]$ a Lyapunov function for $F$ such that $L(x) = L_0(x)$ for $x \in X_0$.

**Proof:** Replacing $F$ by $F \cup 1_X$, we can assume that $F$ is reflexive as well as transitive. Without loss of generality we can assume that $[a, b] = [0, 1]$.

We mimic the proof of Urysohn’s Lemma. Let $\Lambda = \mathbb{Q} \cap [0, 1]$ counted with $\lambda_0 = 0, \lambda_1 = 1$. Let $B_0 = X, B_1 = \emptyset$. For all $\lambda \in \Lambda$ we define the closed set $B_\lambda \subset X$ so that:

(a) $F(B_\lambda) = B_\lambda$, i.e. $B_\lambda$ is $F$ invariant.
(b) $L_0^{-1}((\lambda, 1]) \subset B_\lambda$.
(c) $L_0^{-1}([0, \lambda)) \cap B_\lambda = \emptyset$.
(d) If $\lambda' < \lambda \in \Lambda$, then $B_\lambda \subset B_{\lambda'}$.

Observe that if $x$ were a point of $F(L_0^{-1}([\lambda, 1])) \cap F^{-1}(L_0^{-1}([0, \lambda]))$, then there would exist $z_1, z_2 \in X_0$ with $L_0(z_1) < \lambda \leq L_0(z_2)$ and $(z_2, x), (x, z_1) \in F$ and so $(z_2, z_1) \in F_0$ which would contradict the assumption that $L_0$ is a Lyapunov function for $F_0$.

We repeatedly apply Lemma 6.6. We will use the notation $A \subset\subset B$ to mean $\overline{A} \subset \overline{B}$. A space is normal exactly when $A \subset\subset B$ implies there exists $C$ such that $A \subset\subset C \subset\subset B$. Lemma 6.6 says that if $A$ is closed and $F$ invariant and $A \subset\subset B$ then there exists $C$ closed and $F$ invariant such that $A \subset\subset C \subset\subset B$.

Proceed inductively assuming that $B_\lambda$ has been defined for all $\lambda$ in $\Lambda_n = \{\lambda_i : i = 0, \ldots, n\}$ with $n \geq 1$. Let $\lambda = \lambda_{n+1}$ and let $\lambda' < \lambda < \lambda''$ the nearest points in $\Lambda_n$ below and above $\lambda$. 
Choose a sequence \( \{ t_n^- \} \) with \( t_0^- = \lambda' \), increasing with limit \( \lambda' \) and \( \{ t_n^+ \} \) with \( t_0^+ = \lambda'' \), decreasing with limit \( \lambda'' \).

Define \( Q_0^- = B_{\lambda'} \) and \( Q_0^+ = B_{\lambda''} \). Inductively, apply Lemma 6.6 to choose \( Q_n^+ \) and then \( Q_n^- \) for \( n = 1, 2, ... \) so that \( F(Q_n^+) = Q_n^+ \) and

\[
(6.1) \quad F(L_0^{-1}([t_n^+, 1]) \cup Q_{n-1}^+ \subset \subset Q_n^+ \subset \subset Q_{n-1}^- \setminus (F)^{-1}(L_0^{-1}([0, \lambda])),
\]

\[
F(L_0^{-1}([\lambda, 1]) \cup Q_n^+ \subset \subset Q_n^- \subset \subset Q_{n-1}^- \setminus (F)^{-1}(L_0^{-1}([0, t_n^-])).
\]

Finally, define

\[
(6.2) \quad B_\lambda = \bigcap_n Q_n^-,
\]

so that

\[
(6.3) \quad B_\lambda \supset \bigcup_n Q_n^+.
\]

It is easy to check that \( B_\lambda \) satisfies the required conditions, thus extending the definitions to \( \Lambda_{n+1} \). By induction they can be defined on the entire set \( \Lambda \).

Having defined the \( B_\lambda \)'s we proceed as in Urysohn's Lemma to define \( L(x) \) by the Dedekind cut associated with \( x \). That is,

\[
(6.4) \quad L(x) = \inf \{ \lambda : x \not\in B_\lambda \} = \sup \{ \lambda : x \in B_\lambda \}.
\]

Continuity follows as in Urysohn's Lemma. Because each \( B_\lambda \) is \( F \) invariant, \( L \) is a Lyapunov function. The additional conditions on these sets imply that if \( x \in X_0 \) then \( x \in B_\lambda \) if \( \lambda < L_0(x) \) and \( x \not\in B_\lambda \) if \( \lambda > L_0(x) \). Hence, \( L \) is an extension of \( L_0 \).

\[ \square \]

Fathi and Pageault use a slightly different, asymmetric definition of the barrier functions which yields equivalent results when \( f \) is usc.

\[
(6.5) \quad L_d^f(x, y) = \inf \{ d(x, a_1) + \sum_{i=1}^{n-1} d(b_i, a_{i+1}) + d(b_n, y) : [a, b] \in f^{\times n} \text{ with } a_1 = x, n = 1, 2, ... \},
\]
\[
M_d^f(x, y) = \inf \{ \max(d(x, a_1), d(b_1, a_2), \ldots, d(b_{n-1}, a_n), d(b_n, y)) : [a, b] \in f^{\times n} \text{ with } a_1 = x, n = 1, 2, ... \},
\]
So, of course, the first term, \(d(x, a_1) = 0\). For the case where \(x\) is not in the domain of \(f\) we use the convention
\[
(6.6) \quad f(x) = \emptyset \implies M_d^f(x, y) = \text{diam}(X), L_d^f(x, y) = 2\text{diam}(X).
\]

We have
\[
(6.7) \quad \ell_d^f \leq L_d^f \quad \text{and} \quad m_d^f \leq M_d^f,
\]
because for \(L_d^f\) and \(M_d^f\) the infimum is taken over a smaller set.

**Proposition 6.8.** Let \(f\) be a usc relation on a Hausdorff uniform space \((X, \mathcal{U})\). For every \(x \in X, d \in \Gamma(\mathcal{U})\) and \(\epsilon > 0\), there exist \(d_1 \in \Gamma(\mathcal{U})\) and \(\delta > 0\) such that for all \(y \in X\)
\[
(6.8) \quad \ell_{d_1}^f(x, y) < \delta \implies L_d^f(x, y) < \epsilon,
\]
\[
\text{and} \quad m_{d_1}^f(x, y) < \delta \implies M_d^f(x, y) < \epsilon.
\]

If \(\mathcal{U} = \mathcal{U}(d)\) for a metric \(d\) then we can choose \(d_1 = d\).

**Proof:** Because \(f\) is usc, there exists \(d_0 \in \Gamma\) and \(\epsilon/2 > \delta > 0\) so that \(f(V_{\delta d_0}^d(x)) \subset V_{\epsilon/2}^d(f(x))\). Let \(d_1 = d_0 + d\). If the metric \(d\) determines the topology on \(X\) then we can use \(d_0 = d\) and use \(d_1 = d\).

Now assume \(\ell_{d_1}^f(x, y) < \delta\). We need only consider sequences \([a, b] \in f^{\times n}\) with \(xy\) chain-length with respect to \(d_1\) less than \(\delta\). With \(m_{d_1}^f(x, y) < \delta\), consider sequences \([a, b] \in f^{\times n}\) with \(xy\) chain-bound less than \(\delta\). In either case, \(d_1(x, a_1) < \delta\) and so \(d_0(x, a_1) < \delta\). Hence, \(f(a_1) \subset V_{\epsilon/2}^d(f(x))\) and we can choose \(b_1 \in f(x)\) such that \(d(b_1, b_1) < \epsilon/2\).

Replacing the initial pair \((a_1, b_1)\) in \([a, b]\) by \((x, b_1)\) we obtain a sequence with initial point \(x\) and whose chain-length is at most \(\epsilon/2\) plus the \(xy\) chain-length of \([a, b]\) with respect to \(d\) because \(d(b_1, a_2) \leq d(b_1, b_1) + d(b_1, a_2)\), or, if \(n = 1\), the same inequality is used with \(y\) replacing \(a_2\). The \(xy\) chain-length of \([a, b]\) with respect to \(d\) is at most the \(xy\) chain-length of \([a, b]\) with respect to \(d_1\) and so at most \(\delta < \epsilon/2\).

So the revised sequence which begins with \(x\) has \(xy\) chain-length with respect to \(d\) less than \(\epsilon\). Hence, \(L_d^f(x, y) < \epsilon\).

Similarly, the new chain-bound with respect to \(d\) is less than \(\epsilon/2\) plus the \(xy\) chain-bound of \([a, b]\) with respect to \(d_1\).

Notice in passing that if \(f(x) = \emptyset\) then the chosen \(\delta\) implies \(f(a) = \emptyset\) for all \(a \in V_{\delta d_0}^d(x)\) with \(d(x, a) < \delta\). Provided that \(\delta\) has been chosen less than the \(d\) diameter of \(X\), then from the convention when \(f(x) = \emptyset\) it easily follows that then \(\ell_d^f(x, y), m_d^f(x, y) \geq \delta\) for all \(y \in X\) and so the result holds vacuously.

\(\Box\)
One advantage of the asymmetric definition $M^f_d$ is that, as Pageault points out in \cite{14}, we can sharpen (2.17) to get
\begin{equation}
M^f_d(x, y) \leq \max(M^f_d(x, z), M^f_d(z, y)) \text{ for all } z \in X.
\end{equation}

From (6.7), Proposition 6.8 and Theorem 5.14 the following is obvious.

**Corollary 6.9.** If $f$ is a usc relation on a Hausdorff uniform space $(X, \mathcal{U})$, then $A_{\mathcal{U}} f = \{(x, y) : L^d_f(x, y) = 0 \text{ for all } d \in \Gamma(\mathcal{U})\}$ and $C_{\mathcal{U}} f = \{(x, y) : M^d_f(x, y) = 0 \text{ for all } d \in \Gamma(\mathcal{U})\}$. If $d$ is a metric on $X$ with $\mathcal{U} = \mathcal{U}(d)$ then $A_d f = \{(x, y) : L^d_f(x, y) = 0\}$ and $C_d f = \{(x, y) : M^d_f(x, y) = 0\}$.\hfill $\blacksquare$

**Proposition 6.10.** Let $f$ be a relation on a Hausdorff uniform space $(X, \mathcal{U})$.

(a) If $f$ is a cusc relation, then
\begin{equation}
\mathcal{G} f = f \cup (\mathcal{G} f) \circ f,
\end{equation}
\begin{equation}
A_{\mathcal{U}} f = f \cup (A_{\mathcal{U}} f) \circ f,
\end{equation}
\begin{equation}
C_{\mathcal{U}} f = f \cup (C_{\mathcal{U}} f) \circ f,
\end{equation}
and if $d \in \Gamma(\mathcal{U})$ is a metric whose topology is that of $X$ then
\begin{equation}
A_d f = f \cup (A_d f) \circ f, \quad \text{and} \quad C_d f = f \cup (C_d f) \circ f.
\end{equation}

(b) If $f^{-1}$ is a cusc relation, then
\begin{equation}
\mathcal{G} f = f \cup f \circ (\mathcal{G} f),
\end{equation}
\begin{equation}
A_{\mathcal{U}} f = f \cup f \circ (A_{\mathcal{U}} f),
\end{equation}
\begin{equation}
C_{\mathcal{U}} f = f \cup f \circ (C_{\mathcal{U}} f),
\end{equation}
and if $d \in \Gamma(\mathcal{U})$ is a metric whose topology is that of $X$ then
\begin{equation}
A_d f = f \cup f \circ (A_d f), \quad \text{and} \quad C_d f = f \cup f \circ (C_d f).
\end{equation}

**Proof:** In general, if $f \subset F$ and $F$ is transitive, then $f \cup F \circ f, f \cup f \circ F \subset F$. Furthermore, each of these relations is transitive:
\begin{equation}
(f \cup F \circ f) \circ (f \cup F \circ f) \subset F \circ (1_X \cup F) \circ f \subset F \circ f.
\end{equation}
Similarly, for $f \cup f \circ F$. Since each of $F = \mathcal{G} f, A_{\mathcal{U}} f$ and $C_{\mathcal{U}} f$ is a transitive relation containing $f$, it suffices to prove the reverse inclusions.

(a) If $f$ is cusc, then by Theorem 6.3 (d) $f \cup \mathcal{G} f \circ f$ is a closed, transitive relation which contains $f$ and so contains $\mathcal{G} f$.\hfill $\blacksquare$
Suppose \((x, y) \in A_{df}\). For every \(\alpha = (d, \epsilon) \in \Gamma \times \mathbb{R}_+\) there is 
\([a, b]_\alpha \in f^{\times n_\alpha}\) whose \(xy\) chain-length with respect to \(d\) is less than \(\epsilon\).
Since \(d(x, (a_1)_\alpha) < \epsilon\) and \(d(y, (b_{n_\alpha})_\alpha) < \epsilon\) it follows that \(\{(a_1)_\alpha \to x\}
and \(\{(b_{n_\alpha})_\alpha \to y\}\) Since \(((a_1)_\alpha, (b_1)_\alpha) \in f\) and \(\pi_1|_f : f \to X\) is proper,
Proposition 11.2(iv) implies there is a subnet \(\{(b_1)_\alpha\}\) converging to
a point \(z\) with \((x, z) \in f\). Now if \(n_\alpha' = 1\) frequently then \(z = y\)
and \((x, y) \in f\). Otherwise we may assume all \(n_\alpha' > 1\) and define
\([a, b]_\alpha' \in f^{\times (n_\alpha'-1)}\) by omitting the first pair.

Now given \(d \in \Gamma\) and \(\epsilon > 0\) there exists \(\alpha'_1 = (d_1, \epsilon_1)\) so that \(\alpha'_1 < \alpha'\)
implies \(d((b_1)_{\alpha'}, z) < \epsilon/2\). If \(\alpha'_1 < \alpha' = (d, \epsilon)\) with \(d \leq d_1\) and \(\epsilon/2 \geq \epsilon\)
then the \(zy\) chain-length of \([a, b]_{\alpha'}\) with respect to \(d\) is bounded by 
\(d((b_1)_{\alpha'}, z) < \epsilon/2\) plus the \(xy\) chain-length of \([a, b]_{\alpha'}\) with respect to \(d\) \((< d < \epsilon/2)\). It follows that \(\ell_d(z, y) = 0\) for all \(d \in \mathcal{G}\). That is,
\((x, z) \in f\) and \((z, y) \in A_{df}\).

The proof for \(\mathcal{C}_{df}\) uses the same argument with chain-bound replacing
chain-length throughout.

If \(d\) is a metric in \(\Gamma\) with the topology that of \(X\), then we keep the
metric fixed in the arguments above to prove the results for \(A_{df}\) and
\(\mathcal{C}_{df}\).

(b) We apply the results of (a) to \(f^{-1}\) and invert both sides of the
equation using \((f \circ g)^{-1} = g^{-1} \circ f^{-1}\) and \((\mathcal{G}f)^{-1} = \mathcal{G}(f^{-1}), (A_{df})^{-1} = 
A_{df}(f^{-1})\) and the similar equation for \(\mathcal{C}_{df}\).

\(\square\)

**Proposition 6.11.** Let \(f \subset F\) be relations on a set \(X\) with \(F\) transitive.

(a) If \(A\) an \(F^+\) invariant subset of \(X\), then \(A\) is \(f^+\) invariant. If, in
addition, \(F = f \cup f \circ F\), then \(f(A) = F(A)\) for any subset \(A\) of \(X\). In
particular, \(A\) is \(f\) invariant iff it is \(F\) invariant.

(b) If \(F = f \cup F \circ f\) then \(\text{Dom}(f) = \text{Dom}(F)\) (Recall that \(\text{Dom}(f) = 
(f^{-1}(X))\)).

(c) Assume that \(F = f \cup F \circ f\) and that \(F_1\) is also a transitive relation
on \(X\) with \(F_1 = f \cup F_1 \circ f\). If \(1_X \cup F_1 = 1_X \cup F\), then \(F = F_1\).

(d) Assume that \(f \cup F \circ f = F = f \cup f \circ f\). If \(L\) is a Lyapunov
function for \(F\), then \(x\) is a regular point for \(f\) iff it is a regular point
for \(F\), i.e. \(|L|_f = |L|_F\).

(e) Assume that \(F = f \cup F \circ f\). If \(f\) is a mapping then \(f\) maps \(F\) to
itself, \(F^{-1}\) to itself and \(F \cap F^{-1}\) to itself. Hence, \(f(|F|) \subset |F|\). If, in
addition, \(F = f \cup f \circ F\), then \(f(|F|) = |F|\) and if \(E\) is any \(F \cap F^{-1}\)
equivalence class in \(|F|\), then \(f(E) = E\).
Proof: (a) $A$ is $f^+$ invariant because $f \subset F$. Also, $f(A) \subset F(A)$. Conversely, if $y \in F(A)$ then there exists $x \in A$ with $y \in F(x)$. Since $F = f \cup f \circ F$, either $y \in f(x)$ or there exists $z \in F(x)$ such that $y \in f(z)$. Since $A$ is $F^+$ invariant, $z \in A$. Hence, $y \in f(A)$.

In particular, $f(A) = A$ iff $F(A) = A$.

(b) Clearly, $X$ is $F^{-1} +$ invariant. Inverting the assumed equation, we have $F^{-1} = f^{-1} \cup f^{-1} \circ F^{-1}$ and so by (a), $f^{-1}(X) = F^{-1}(X)$.

(c) If $(y, x) \in F \cap F_1$ with $y \neq x$ then $(z, x) \in F_1$. If $(x, y) \in F$ then either $(x, x) \in f \subset F_1$ or there exists $y \neq x$ such that $(x, y) \in f \subset F_1$ and $(y, x) \in F \subset F_1$. Since $y \neq x$, $(x, x) \in F_1$. By transitivity, $(x, x) \in F_1$. Hence, $F \subset F_1$. Similarly, $F_1 \subset F$.

(d) In any case, suppose $x$ is a regular point for $F$, i.e. $L$ on $F(x)$ is greater than $L(x)$ and $L$ on $F^{-1}(x)$ is less than $L(x)$. Since $f \subset F$, $x$ is a regular point for $F$. Conversely, suppose $x$ is regular for $f$ and $y \in F(x)$. Since $f \cup F \circ f = F$ either $y \in f(x)$ and so $L(y) > L(x)$ or there exists $z \in f(x)$ such that $y \in F(z)$. Hence, $L(y) \geq L(z) > L(x)$.

The argument for $y \in F^{-1}$ is similar, using $f^{-1} \cup F^{-1} \circ f^{-1} = F^{-1}.$

(e) If $f$ is a map, then $f \circ f^{-1} \subset 1_f$. Hence,

\begin{equation}
F \circ f^{-1} = (f \cup F \circ f) \circ f^{-1} \subset 1_f \cup F,
\end{equation}

and $f \circ F^{-1} \subset 1_f \cup F^{-1},$

where the second equation follows from the first by inverting. Hence, $(f \times f)(F) = f \circ F \circ f^{-1} \subset F$. Since $f$ maps $F$ to itself, it maps $F^{-1}$ to itself and $F \cap F^{-1}$ to itself. In particular, each $F \cap F^{-1}$ equivalence class is mapped into some equivalence class. If $x$ is in the $F \cap F^{-1}$ equivalence class $E$, then, since $F = f \cup F \circ f$, either $f(x) = x$ or $(f(x), x) \in F$. Because $(x, f(x)) \in f \subset F$ it follows that $f(x) \in E$. Thus, each $E$ is mapped into itself by $f$.

Now assume that $F = f \cup f \circ F$ and that $x, y$ are in the $F \cap F^{-1}$ equivalence class $E$. Since $(x, y) \in F$, either $y = f(z)$ with $z = x$ or there exists $z$ such that $(x, z) \in F$ and $f(z) = y$. Since $(z, y) \in f \subset F$, $z \in E$. In either case, there $z \in E$ with $f(z) = y$. Thus, $f(E) = E$.

\[\square\]

Proposition 6.12. Let $f$ be a relation on a normal Hausdorff space $X$, with $U_M$ the maximum uniformity on $X$. If $\mathcal{G}f$ and $\mathcal{G}f^{-1}$ are usc relations, i.e. for every closed subset $A$ of $X$, both $\mathcal{G}f(A)$ and $\mathcal{G}f^{-1}(A)$ are closed, then $1_X \cup \mathcal{G}f = 1_X \cup A_{U_M}f$. If, in addition, $f$ is cusc, then $\mathcal{G}f = A_{U_M}f$.

Proof: In any case, $A_{U_M}f$ is a closed, transitive relation which contains $f$ and so contains $\mathcal{G}f$. \[\square\]
If \((x, y) \not\in 1_X \cup \mathcal{G} f\) then let \(X_0 = \{x, y\}\). Let \(L_0(x) = 1\) and \(L_0(y) = 0\). Since \((x, y) \not\in 1_X \cup \mathcal{G} f\), \(L_0\) is a Lyapunov function on \(X_0\). By Theorem \(6.7\) there exists a Lyapunov function \(L\) for \(\mathcal{G} f\) with \(L(x) = 1\) and \(L(y) = 0\). By Corollary \(5.10\) \(L\) is an \(A_{\mathcal{U}M} f\) Lyapunov function and so \((x, y) \not\in A_{\mathcal{U}M} f\).

If \(f\) is cusc then by Proposition \(6.10\), we can apply Proposition \(6.11\) (c) to obtain \(\mathcal{G} f = A_{\mathcal{U}M} f\).

\[\square\]

We require the following lemma from \([3]\).

**Lemma 6.13.** Let \(f\) be a proper relation on a paracompact, locally compact, Hausdorff space \(X\). There exists a clopen equivalence relation \(E_f\) on \(X\) such that \(\mathcal{C}_{\mathcal{U}M} f \cup \mathcal{C}_{\mathcal{U}M} f^{-1} \subset E_f\) and \(E_f(x)\) is a \(\sigma\) compact set for every \(x \in X\).

**Proof:** Since \(X\) is paracompact, \(\mathcal{U}_M\) consists of all neighborhoods of the diagonal and there exists an open cover \(\{U_i\}\) such that \(\{U_i\}\) is a locally finite collection of compacta. It follows that \(W = \bigcup_i U_i \times \overline{U_i}\) is a closed, symmetric element of \(\mathcal{U}_M\) with every \(W(x)\) compact, i.e. \(W\) is a pointwise compact relation. Since \(\mathcal{U}_M\) is a uniformity there exists \(V\) a closed, symmetric element of \(\mathcal{U}_M\) such that \(V \circ V \subset W\). If \(K\) is any compact subset of \(X\) then there exists \(F\) a finite subset of \(X\) such that \(\{V(x) : x \in F\}\) is a cover of \(K\). Then \(V(K) \subset \bigcup\{V \circ V(x) : x \in F\}\). Since \(W\) is pointwise compact, the set on the right is compact. Since \(V\) is closed and \(K\) is compact, \(V(K)\) is closed by \(6.11\) (a) and so is compact. Since a locally compact space is a \(\kappa\)-space, it follows from \(6.11\) (h) that \(V = V^{-1}\) is cusc and so is proper.

Since \(f\) is proper, i.e. \(f\) and \(f^{-1}\) are cusc, and \(1_X\) is proper, it follows from Proposition \(6.4\) (a) that \(F = f \cup 1_X \cup f^{-1}\) is symmetric and cusc. By Theorem \(6.3\) (c) the composition \(V_f = V \circ f \circ V \subset V\) is a cusc, symmetric element of \(\mathcal{U}_M\). Hence, \(E_f = \bigcup_{n=1}^{\infty} (V_f)^n\) is an equivalence relation. Since \(F \circ E_f \subset V_f \circ E_f \subset E_f\), and \(E_f\) is a closed, transitive relation, it follows that \(\mathcal{G} f \cup \mathcal{G} f^{-1} \subset \mathcal{G} F \subset E_f\). Since \(E_f(x) \supset V_f(x)\) is a neighborhood of \(x\), each \(E_f(x)\) is open and since the equivalence classes are disjoint, each is clopen. Hence, \(E_f = \bigcup_x E_f(x) \times E_f(x)\) is a clopen subset of \(X \times X\). Beginning with the compact set \(\{x\}\) we see, inductively, that \((V_f)^{n+1}(x) = V_f((V_f)^n(x))\) is compact because \(V_f\) is proper. Hence, each \(E_f(x)\) is \(\sigma\) compact.

Finally, since \(E_f\) is a neighborhood of the diagonal, \(E_f \in \mathcal{U}_M\). For \(x, y \in X\) let \([a, b] \in f^x\) be a \(xy, E_f\) chain. Let \(b_0 = x\), and \(a_{n+1} = y\). Hence, \((a_i, b_i) \in f \subset E_f\) for \(i = 1, \ldots, n\) and \((b_i, a_{i+1}) \in E_f\) for \(i =
0, . . . , n. By transitivity of $E_f$, $(x, y) \in E_f$. Hence, $\mathcal{C}_{U_M} f \subset E_f$ and by symmetry $\mathcal{C}_{U_M} f^{-1} \subset E_f$.

\[\square\]

**Theorem 6.14.** Let $F$ be a closed, transitive relation on a paracompact, locally compact, Hausdorff space $X$ with $U_M$, the uniformity of all neighborhoods of the diagonal. Assume that $X_0$ is a closed subset of $X$ and $L_0 : X_0 \to [a, b]$ is a bounded, Lyapunov function for the restriction $F_0 = F \cap (X_0 \times X_0)$. If either

(a) $X$ is $\sigma$-compact, or,

(b) there exists a proper relation $f$ on $X$ such that $F \subset C_{U_M} f$,

then there exists $L : X \to [a, b]$ a Lyapunov function for $F$ such that $L(x) = L_0(x)$ for $x \in X_0$.

**Proof:** (a) Because $X$ is locally compact and $\sigma$ compact there is an increasing sequence of compacta $\emptyset = K_0, K_1, \ldots$ with union $X$ such that $K_n \subset K_{n+1}$. Let $K_{n+\frac{1}{2}} = K_n \cup (X_0 \cap K_{n+1})$. Assume we have a Lyapunov function $L_n : X_n \to [a, b]$ for $F \cap (K_n \times K_n)$ with $L_n = L_0$ on $X_0 \cap K_n$. Extend to define $L_{n+\frac{1}{2}} : X_{n+\frac{1}{2}} \to [a, b]$ by using $L_0$ on $X_0 \cap K_{n+1}$. By Theorem [6.7] there exists a Lyapunov function $L_{n+1} : K_{n+1} \to [a, b]$ for $F \cap (K_{n+1} \times K_{n+1})$ such that $L_{n+1}$ extends $L_{n+\frac{1}{2}}$. Completing the inductive construction we define $L : X \to [a, b]$ by $L|K_n = L_n$. Since $X = \bigcup_n (K_n)$, $L$ is continuous and so is the required Lyapunov function.

(b) Let $E_f$ be a clopen equivalence relation on $X$ as given by Lemma [6.13]. Each equivalence class $E$ is $\sigma$-compact and $F \cup F^{-1}$ is invariant. Use (a) on $E$ to define $L_E : E \to [a, b]$ a Lyapunov function for $F \cap (E \times E)$ which extends $L_0|(X_0 \cap E)$. Define $L$ by $L|E = L_E$ for each equivalence class. $L$ extends $L_0$. As the equivalence classes are clopen, $L$ is continuous. Finally, $F = \bigcup_E (F \cap (E \times E))$ and so $L$ is a Lyapunov function for $F$. \[\square\]

**Corollary 6.15.** Let $f$ be a relation on a paracompact, locally compact, Hausdorff space $X$ with $U_M$, the uniformity of all neighborhoods of the diagonal.

(a) If $X$ is $\sigma$ compact, then $1_X \cup \mathcal{S} f = 1_X \cup \mathcal{A}_{U_M} f$. If, in addition, $f$ is cusc, then $\mathcal{S} f = \mathcal{A}_{U_M} f$.

(b) If $f$ is a proper relation, then $\mathcal{S} f = \mathcal{A}_{U_M} f$. 
6.14 there exists a $x, y$.

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6.11(c). On $X$ space continuous metric $X$.

$\cup L$

\begin{align*}
6.16. \text{Let } L \colon X \to [0, 1] \text{ which extends } L_0. \text{ Hence, } L : X \to [0, 1] \text{ is uniquely defined by } L|_{X_n} = L_n. \text{ Since } X = \bigcup_n (X_n)^{\circ}, L \text{ is continuous and so is the required Lyapunov function.}

\text{When } f \text{ is cusc, as in (b), we obtain } Lf = A_\Omega f \text{ from Proposition 6.11(c).}
\end{align*}

\begin{proof}
\end{proof}

Now we consider extensions to completions and compactifications.

If $X$ is a compact, Hausdorff space, then $\Omega_M$ is the unique uniformity on $X$ and we write $Cf$ for $C_{\Omega_M}f$ in the compact case. If a compact space $X$ is metrizable, then by Theorem 5.11 $Cf = A d$ for every continuous metric $d$ on $X$. Since a compact Hausdorff space is normal and every closed relation on a compact Hausdorff space is proper, it follows from Proposition 6.12 that $A_{\Omega_M}f = Lf$ when $X$ is compact.

\begin{proposition}
Let $(X, \Omega)$ be a Hausdorff uniform space with completion $(\bar{X}, \bar{\Omega})$ so that $X$ is a dense subset of $\bar{X}$ with $\Omega$ the uniformity on $X$ induced from $\Omega$. If $f$ is a closed relation on $X$ and $\bar{f}$ is the closure of $f$ in $\bar{X} \times \bar{X}$ then
\begin{align*}
\bar{f} \cap (X \times X) &= f, \quad C_{\bar{\Omega}} \bar{f} \cap (X \times X) = C_{\Omega} f, \\
A_{\bar{\Omega}} \bar{f} \cap (X \times X) &= A_{\Omega} f.
\end{align*}

If, moreover, $f$ is a uniformly continuous map on $(X, \Omega)$ then $\bar{f}$ is a uniformly continuous map on $(\bar{X}, \bar{\Omega})$.
\end{proposition}

\begin{proof}
\end{proof}

If $f$ is a uniformly continuous map and $x \in \bar{X}$ then there is a net $\{x_i : i \in I\}$ in $X$ which converges to $x$. Since $f$ is uniformly continuous, $\{f(x_i)\}$ is Cauchy and so converges to a point $y$ with $(x, y) \in \bar{f}$. If $\{x_j : j \in J\}$ is another net converging to $x$, then let $\{0, 1\}$ be directed by the relation $\{0, 1\} \times \{0, 1\}$. On $I \times J \times \{0, 1\}$ define the net by $(i, j, 0) \mapsto x_i$ and $(i, j, 1) \mapsto x_j$. This net converges to $x$ and so the limit points of $\{f(x_i)\}$ and $\{f(x_j)\}$ agree. Thus, $\bar{f}$ is a well-defined
map on $X$. Since the uniformity $\bar{U}$ is generated by the closures of $U \in \mathcal{U}$ it is easy to see that $f$ is uniformly continuous.

Let $\mathcal{B}$ be a closed subalgebra of the Banach algebra $\mathcal{B}(X, \mathcal{U})$ of bounded continuous functions on a Hausdorff uniform space. For any transitive relation $F$ on $X$, the set of those $L \in \mathcal{B}$ which are Lyapunov functions for $F$ always satisfies ALG and CON.

If $\mathcal{B}$ distinguishes points and closed sets then it generates a totally bounded uniformity $\mathcal{T}(\mathcal{B}) \subset \mathcal{U}$ with topology compatible to that of $(X, \mathcal{U})$, see Appendix B. Let $(\bar{X}, \mathcal{T}(\mathcal{B}))$ be the completion of $(X, \mathcal{T}(\mathcal{B}))$. The space $\bar{X}$ is a compact, Hausdorff space with $\mathcal{T}(\mathcal{B})$ its unique uniformity. The inclusion $(X, \mathcal{T}(\mathcal{B}))$ into $(\bar{X}, \mathcal{T}(\mathcal{B}))$ is a uniform isomorphism onto its image and so the inclusion from $(X, \mathcal{U})$ is a uniformly continuous homeomorphism.

If $h : (X_1, \mathcal{U}_1) \to (X_2, \mathcal{U}_2)$ is uniformly continuous, then $h^* : \mathcal{B}(X_2, \mathcal{U}_2) \to \mathcal{B}(X_1, \mathcal{U}_1)$ with $h^*(u) = u \circ h$ is a map of Banach algebras with norm 1. If $\mathcal{B}_1 \subset \mathcal{B}(X_1, \mathcal{U}_1)$ and $\mathcal{B}_2 \subset \mathcal{B}(X_2, \mathcal{U}_2)$ are closed subalgebras such that $h^*(\mathcal{B}_2) \subset \mathcal{B}_1$ then $h : (X_1, \mathcal{T}(\mathcal{U}_1)) \to (X_2, \mathcal{T}(\mathcal{U}_2))$ is uniformly continuous because for $u \in \mathcal{B}_2$ $h^*d_u$, that is, $d_u \circ (h \times h)$, is equal to $d_{h^*u}$.

**Lemma 6.17.** Suppose that $r : X \times X \to \mathbb{R}$ is a bounded, uniformly continuous map and let $D$ be a dense subset of $X$. For $z \in X$, the function $r^z : X \to \mathbb{R}$ is defined by $r^z(x) = r(x, z)$. If for every $z \in D$, the function $r^z$ is contained in a closed subalgebra $\mathcal{B}$ of $\mathcal{B}(X, \mathcal{U})$ then $r^z \in \mathcal{B}$ for all $z \in X$.

**Proof:** By uniform continuity, $z \mapsto r^z$ is a continuous map from $X$ to $\mathcal{B}(X, \mathcal{U})$. If the dense set $D$ is mapped into the closed subset $\mathcal{B}$ then all of $X$ is.

**Theorem 6.18.** Let $f$ be a closed relation on a Hausdorff uniform space $(X, \mathcal{U})$. There exists $\mathcal{B}$ a closed subalgebra of $\mathcal{B}(X, \mathcal{U})$ such that

- $\mathcal{B}$ distinguishes points and closed sets in $X$.
- The set of $f$ Lyapunov functions in $\mathcal{B}$ satisfies POIN for $A_{\mathcal{U}}f$.

With $\mathcal{T}(\mathcal{B}) \subset \mathcal{U}$ the totally bounded uniformity generated by $\mathcal{B}$, let $(\bar{X}, \mathcal{T}(\mathcal{B}))$ be the completion of $(X, \mathcal{T}(\mathcal{B}))$ and let $\bar{f}$ be the closure of $f$ in $\bar{X} \times \bar{X}$. The space $\bar{X}$ is a compact, Hausdorff space with $\mathcal{T}(\mathcal{B})$ its unique uniformity. Furthermore,

\begin{align*}
(6.16) \quad \bar{f} \cap (X \times X) &= f, & 1_X \cup \bar{f} \cap (X \times X) &= 1_X \cup A_{\mathcal{U}}f.
\end{align*}
If \( f \) is cusc then \( \mathcal{G} \bar{f} \cap (X \times X) = \mathcal{A}_U f \).

If \( f \) is a uniformly continuous map, and so is cusc, such that \( f^* \mathcal{B} \subset \mathcal{B} \), then \( \bar{f} \) is a continuous map on \( \bar{X} \). If \( f \) is a uniform isomorphism such that \( f^* \mathcal{B} = \mathcal{B} \), then \( \bar{f} \) is a homeomorphism on \( \bar{X} \).

**Proof:** Since \( X \) is a Tychonoff space, \( \mathcal{B} = \mathcal{B}(X, \mathcal{U}) \) distinguishes points and closed sets. The set of functions which are \( K \ell^d \) dominated for some positive \( K \) and some \( d \in \Gamma(\mathcal{U}) \) is a collection of \( \mathcal{A}_U f \) Lyapunov functions which satisfies POIN.

Now assume that \( \mathcal{B} \) is a closed subalgebra which satisfies these two conditions.

To prove (6.16) it suffices, by (6.15) to show that on \( X \) that \( 1_X \cup \mathcal{A}_T(\mathcal{B}) f = 1_X \cup \mathcal{A}_U f \) because \( \mathcal{A}_T(\mathcal{B}) \bar{f} = \mathcal{G} \bar{f} \) for the compact Hausdorff space \( \bar{X} \).

Because \( \mathcal{T}(\mathcal{B}) \subset \mathcal{U} \), \( 1_X \cup \mathcal{A}_U f \subset 1_X \cup \mathcal{A}_T(\mathcal{B}) f \). If \( (x, y) \notin 1_X \cup \mathcal{A}_U f \) then by POIN there exists \( L \in \mathcal{B} \) a Lyapunov function for \( f \) such that \( L(x) > L(y) \). Because \( L \in \mathcal{B} \) it is uniformly continuous with respect to \( \mathcal{T}(\mathcal{B}) \). By Theorem 5.9 \( L \) is an \( \mathcal{A}_T(\mathcal{B}) f \) Lyapunov function. Since \( L(x) > L(y) \), \( (x, y) \notin \mathcal{A}_T(\mathcal{B}) f \).

If \( f \) is cusc then by Proposition 6.10 \( \mathcal{A}_U f = f \cup (\mathcal{A}_U f) \circ f \) and \( \mathcal{A}_T(\mathcal{B}) f = f \cup (\mathcal{A}_T(\mathcal{B}) f) \circ f \). So by Proposition 6.11(b) we may remove \( 1_X \) from the equation.

If \( f^* \mathcal{B} \subset \mathcal{B} \) then \( f \) is uniformly continuous on \( (X, \mathcal{T}(\mathcal{B})) \) and so extends to a continuous map on the completion. If \( f \) is invertible and \( f^* \mathcal{B} = \mathcal{B} \) then the same applies to the inverse of \( f \). 

\( \square \)

**Theorem 6.19.** Let \( f \) be a closed relation on a Hausdorff uniform space \( (X, \mathcal{U}) \). There exists \( \mathcal{B} \) a closed subalgebra of \( \mathcal{B}(X, \mathcal{U}) \) such that

- \( \mathcal{B} \) distinguishes points and closed sets in \( X \).
- The set of elementary \( \mathcal{U} \) Lyapunov functions for \( f \) in \( \mathcal{B} \) satisfies POIN-E for \( \mathcal{C}_U f \).

With \( \mathcal{T}(\mathcal{B}) \subset \mathcal{U} \) the totally bounded uniformity generated by \( \mathcal{B} \), let \( (\bar{X}, \mathcal{T}(\mathcal{B})) \) be the completion of \( (X, \mathcal{T}(\mathcal{B})) \) and let \( \bar{f} \) be the closure of \( f \) in \( \bar{X} \times \bar{X} \). The space \( \bar{X} \) is a compact, Hausdorff space with \( \mathcal{T}(\mathcal{B}) \) its unique uniformity. Furthermore,

\[
(6.17) \quad \bar{f} \cap (X \times X) = f, \quad \mathcal{C}_U \bar{f} \cap (X \times X) = \mathcal{C}_U f.
\]

If \( f \) is a uniformly continuous map, and so is cusc, such that \( f^* \mathcal{B} \subset \mathcal{B} \), then \( \bar{f} \) is a continuous map on \( \bar{X} \). If \( f \) is a uniform isomorphism such that \( f^* \mathcal{B} = \mathcal{B} \), then \( \bar{f} \) is a homeomorphism on \( \bar{X} \).
Proof: Again it suffices to use \( \mathcal{B} = \mathcal{B}(X, \mathcal{U}) \) and as before it suffices to prove on \( X \) that \( \mathcal{C}_{\mathcal{F}(\mathcal{B})} f = \mathcal{C}_{\mathcal{U}} f \).

Because \( \mathcal{F}(\mathcal{U}) \subset \mathcal{U} \), \( \mathcal{C}_{\mathcal{U}} f \subset \mathcal{C}_{\mathcal{F}(\mathcal{B})} f \).

If \( (x, y) \not\in 1_X \cup \mathcal{C}_{\mathcal{U}} f \) then by POIN-E there exists \( L \in \mathcal{B} \) an elementary Lyapunov function for \( f \) such that \( L(x) > L(y) \). Because \( L \in \mathcal{B} \) it is uniformly continuous with respect to \( \mathcal{F}(\mathcal{B}) \) and so is a \( \mathcal{F}(\mathcal{B}) \) elementary Lyapunov function for \( f \). By Theorem 5.16 \( L \) is an elementary Lyapunov function for \( \mathcal{C}_{\mathcal{F}(\mathcal{B})} f \). Since \( L(x) > L(y) \), \( (x, y) \not\in \mathcal{C}_{\mathcal{F}(\mathcal{B})} f \).

In this case we can eliminate the \( 1_X \) term without assuming that \( f \) is cusc.

If \( (x, x) \not\in \mathcal{C}_{\mathcal{U}} f \), i.e. \( x \not\in |\mathcal{C}_{\mathcal{U}} f| \), then by POIN-E there exists \( L \in \mathcal{B} \) an elementary Lyapunov function for \( f \) such that \( 1 > L(x) > 0 \). As before \( L \) is an elementary Lyapunov function for \( \mathcal{C}_{\mathcal{F}(\mathcal{B})} f \). Hence, \( L = 1 \) on \( \mathcal{C}_{\mathcal{F}(\mathcal{B})} f(x) \) and so \( (x, x) \not\in \mathcal{C}_{\mathcal{F}(\mathcal{B})} f \).

The map cases are as before.

\[ \square \]

The spaces we obtain from these theorems are quite large. The conditions may well require \( \mathcal{B} = \mathcal{B}(X, \mathcal{U}) \), leading to the entire uniform version of the Stone-Čech compactification. However, in the second countable case we are able to obtain a metric compactification.

Theorem 6.20. Let \( f \) be a closed relation on a Hausdorff uniform space \( (X, \mathcal{U}) \) with \( X \) second countable. There exists \( \mathcal{B} \) a separable, closed subalgebra of \( \mathcal{B}(X, \mathcal{U}) \) such that

- \( \mathcal{B} \) distinguishes points and closed sets in \( X \).
- The set of \( f \) Lyapunov functions in \( \mathcal{B} \) satisfies POIN for \( \mathcal{A}_{\mathcal{U}} f \).
- The set of elementary \( \mathcal{U} \) Lyapunov functions for \( f \) in \( \mathcal{B} \) satisfies POIN-E for \( \mathcal{C}_{\mathcal{U}} f \).

With \( \mathcal{F}(\mathcal{B}) \subset \mathcal{U} \) the totally bounded uniformity generated by \( \mathcal{B} \), let \( (\bar{X}, \mathcal{F}(\mathcal{B})) \) be the completion of \( (X, \mathcal{F}(\mathcal{B})) \) and let \( \bar{f} \) be the closure of \( f \) in \( \bar{X} \times \bar{X} \). The space \( \bar{X} \) is a compact, metrizable Hausdorff space with its unique uniformity \( \mathcal{F}(\mathcal{B}) \) metrizable. Furthermore,

\[
\bar{f} \cap (X \times X) = f, \quad 1_X \cup \mathcal{G} \bar{f} \cap (X \times X) = 1_X \cup \mathcal{A}_{\mathcal{U}} f, \\
\mathcal{C} \bar{f} \cap (X \times X) = \mathcal{C}_{\mathcal{U}} f.
\]

If \( f \) is cusc then \( \mathcal{G} \bar{f} \cap (X \times X) = \mathcal{A}_{\mathcal{U}} f \).

If \( f \) is a uniformly continuous map then, in addition, we can choose \( \mathcal{B} \) so that \( f^* \mathcal{B} \subset \mathcal{B} \) and so \( \bar{f} \) is a continuous map on \( \bar{X} \). If \( f \) is a uniform isomorphism then, in addition, we can choose \( \mathcal{B} \) so that \( f^* \mathcal{B} = \mathcal{B} \) and so \( \bar{f} \) is a homeomorphism on \( \bar{X} \).
Proof: Apply Theorem 5.12 to obtain a metric $d \in \Gamma(\mathcal{U})$ with the topology that of $X$ and such that $A_{\mathcal{U}}f = A_d f$, $1_X \cup A_{\mathcal{U}}f = \leq L$ and $\mathcal{C}_f f = \mathcal{C}_d f$. Let $D$ be a countable dense subset of $X$.

Let $d^B(x) = d(x, z)$. Let $\ell^B(x) = \ell^{B, 1}_{X}(x, z) = \min(\ell^L(x, z), d(x, z))$. If $\mathcal{B}$ is a closed subalgebra of $\mathcal{B}(X, d) \subset \mathcal{B}(X, \mathcal{U})$ which contains $d^\mathcal{B}$ and $\ell^\mathcal{B}$ for all $z$ in $D$ then by Lemma 6.17 $d^\mathcal{B}$, $\ell^\mathcal{B} \in \mathcal{B}$ for all $z \in X$. Since $d^\mathcal{B} \in \mathcal{B}$ for all $z$, $\mathcal{B}$ distinguishes points and closed sets. Each $\ell^\mathcal{B}$ is a Lyapunov function for $A_d f$ by Theorem 4.4 and Proposition 4.5. If $(x, y) \notin A_d f$ then $\ell^y(y) = 0$ and $\ell^y(x) > 0$. So the Lyapunov functions in $\mathcal{B}$ satisfy POIN for $A_d f = A_{\mathcal{U}}f$.

Because the subspaces $X \times X \backslash (1_X \cup \mathcal{C}_d f)$ and $X \backslash |\mathcal{C}_d f|$ are Lindelöf, Theorem 5.18 implies that we can find a sequence $\{L_i\}$ of $\mathcal{U}$ elementary Lyapunov functions for $f$ such that

- For $(x, y) \in X \times X \backslash (1_X \cup \mathcal{C}_d f)$ there exists $i$ such that $L_i(x) > L_i(y)$.

- For $x \in X \backslash |\mathcal{C}_d f|$ there exists $i$ such that $1 > L_i(x) > 0$.

If $\mathcal{B}$ contains $\{L_i\}$ then the elementary Lyapunov functions in $\mathcal{B}$ satisfy POIN-E.

Thus, if $\mathcal{B}$ is the closed subalgebra generated by $\{d^\mathcal{B} : z \in D\} \cup \{\ell^\mathcal{B} : z \in D\} \cup \{L_i\}$ then $\mathcal{B}$ is a separable subalgebra of $\mathcal{B}(X, d)$ which satisfies the required properties.

If $f$ is a uniformly continuous map we extend the countable set of generators $\{u_i\}$ to include $\{(f^n)^* u_i\}$ for all positive integers $n$. If $f$ is a uniform isomorphism we use $\{(f^n)^* u_i\}$ with all integers $n$. In either case, we still have a countable set of generators and so obtain a separable algebra $\mathcal{B}$.

Since $\mathcal{B}$ is separable, the compact space $\bar{X}$ is metrizable.

The results then follow from Theorems 6.18 and 6.19.

$\square$

Let $f$ be a closed relation relation on $X$ and let $\bar{f}$ be the extension to one of the compactifications as above $X$. If the domain of $f$, $f^{-1}(X)$ is all of $X$, then it is dense in $\bar{X}$. Since the domain $\bar{f}^{-1}(\bar{X})$ is compact and contains $f^{-1}(X)$, it follows that $\bar{f}^{-1}(\bar{X}) = \bar{X}$. If $\bar{X}$ is merely a completion but $f$ is a uniformly continuous map on $X$ then $\bar{f}$ is a uniformly continuous map on $\bar{X}$ and so has domain all of $\bar{X}$. If $f$ is merely continuous, the domain of $\bar{f}$ need not be all of $\bar{X}$. For example, let $f : (0, \infty) \to (0, \infty)$ be the continuous map with $f(t) = 1/t$. With the usual metric the completion is $[0, \infty)$ and $\bar{f} = f$.

We conclude the section by considering the special results when $X$ is a compact Hausdorff space, so the $\mathcal{U}_M$ is its unique uniformity. We
need the following result which is Lemma 2.5 from [1]. Recall that a closed relation \( f \) on a compact Hausdorff space \( X \) is proper and so \( f(A) \) is closed if \( A \subset X \) is closed.

**Lemma 6.21.** Let \( F \) be a closed, transitive relation on a compact Hausdorff space \( X \) and let \( B \) be a closed subset with \( B \cap |F| = \emptyset \). There exists a positive integer \( N \) such that if \( \{a_0, \ldots, a_k\} \) is a finite sequence in \( B \) with \( (a_{i-1}, a_i) \in F \) for \( i = 1, \ldots, k \), then \( k \leq N \).

**Proof:** Since \( F \cap (B \times B) \) is disjoint from \( 1_X \), there exists an open, symmetric \( U \in \mathcal{U} \) such that \( F \cap (B \times B) \cap (U \circ U) = \emptyset \). Since \( B \) is compact, there is a subset \( \{x_0, \ldots, x_N\} \) of \( B \) such that \( \{U(x_j) : j = 0, \ldots, N\} \) covers \( B \). If \( \{a_0, \ldots, a_k\} \) is a sequence as above with \( k > N \) then by the Pigeonhole Principle there exist \( 0 \leq i_1 < i_2 \leq k \) which lie in the same \( U(x_j) \) and so \( (a_{i_1}, a_{i_2}) \in U \circ U \). By transitivity of \( F \), \( (a_{i_1}, a_{i_2}) \in F \), contradicting the choice of \( U \).

\[ \square \]

**Proposition 6.22.** Let \( f \) be a closed relation on a compact Hausdorff space \( X \) and let \( A \) be a nonempty, closed subset of \( X \).

(a) If \( A \) is \( f^+ \) invariant and \( A \subset \text{Dom}(f) \), then maximum closed \( f \) invariant subset \( f^\infty(A) \) is closed and nonempty and equals \( \bigcap_{n \in \mathbb{N}} f^n(A) \).

(b) If \( F \) is a closed, transitive relation on \( X \) such that \( F = f \cup f \circ F \) and \( A \) is \( F^+ \) invariant, then \( f^\infty(A) = F^\infty(A) \).

**Proof:** (a) Since \( \text{Dom}(f) \subset A \), \( \{f^n(A)\} \) is a non-increasing sequence of nonempty compacta and so the intersection is nonempty. If \( y \in \bigcap_{n \in \mathbb{N}} f^n(A) \) then \( \{f^{-1}(y) \cap f^n(A)\} \) is a non-increasing sequence of nonempty compacta with nonempty intersection \( f^{-1}(y) \cap \bigcap_{n \in \mathbb{N}} f^n(A) \). So \( \bigcap_{n \in \mathbb{N}} f^n(A) \) is an \( f \) invariant subset.

(b) By Proposition 6.11 and induction, \( f^n(A) = F^n(A) \) for all \( n \). Hence the intersections are equal.

\[ \square \]

**Theorem 6.23.** Let \( F \) be a closed, transitive relation on a compact Hausdorff space \( X \). If \( A \) is an \( F^+ \) invariant closed subset, then

\[
F^\infty(A) = F(A \cap |F|).
\]

If \( G \) is an open set containing \( A \) then there exists a Lyapunov function \( L : X \to [0, 1] \) for \( F \) such that \( L = 0 \) on \( X \setminus G \) and \( L = 1 \) on \( A \). In particular, the \( F^+ \) invariant open neighborhoods of \( A \) form a base for the neighborhood system of \( A \).
Proof: Since $x \in F(x)$ for $x \in |F|$, $A \cap |F| \subset F^\infty(A)$. From invariance of $F^\infty(A)$ we obtain $F(A \cap |F|) \subset F^\infty(A)$.

For $x \in A \setminus F(A \cap |F|)$, $B = \{x\} \cup F^{-1}(x)$ is disjoint from $|F|$. If $y \in B \cap F^n(A)$ then there exists a sequence $a_0, a_1, \ldots, a_n \in A$ with $a_n = y$ and with $a_i \in F(a_{i-1})$ for $i = 1, \ldots, n$. From transitivity of $F^{-1}$ it follows that $a_i \in B$ for all $i$. From Lemma 6.21 it then follows that there exists a positive integer $N$ such that $B$ is disjoint from $F^{N+1}(A)$. Hence, $x \notin F^\infty(A)$.

If $G$ is an open set containing $A$ then we let $X_0 = (X \setminus G) \cup A$. Let $L_0 = 0$ on $X \setminus G$ and $= 1$ on $A$. Since $A$ is $F$-invariant, $L_0$ is a Lyapunov function on $X_0$ for $F \cap (X_0 \times X_0)$. By Theorem 6.7 it extends to an $F$ Lyapunov function $L$ on $X$.

For any $c \in (0, 1)$ the set $\{x : L(x) > c\}$ is an $+\invariant$ neighborhood of $A$ which is contained in $G$.

$\Box$

These results apply directly to $F = \exists f = A_{\mathcal{U}_f} f$ for $f$ any closed relation on $X$, see Proposition 6.10 and Corollary 6.15. For $F = \mathcal{C} f = \mathcal{C}_{\mathcal{U}_f} f$ we obtain special results.

If $K$ is a closed $\mathcal{C} f$ invariant set, we call $K \cap |\mathcal{C} f|$ the \textit{trace} of $K$.

**Theorem 6.24.** Let $f$ be a closed relation on a compact Hausdorff space $X$. Let $K$ be a subset of $X$.

(a) Assume $K$ is closed and $\mathcal{C} f$-invariant. If $G$ is an open set which contains $K$, then there exists an open inward set $A$ with $K \subset A \subset G$ and there exists an elementary Lyapunov function $L : X \to [0, 1]$ for $f$ such that $L = 0$ on $X \setminus G$ and $L = 1$ on $K$. In particular, the open inward sets which contain a closed, $\mathcal{C} f$-invariant set form a neighborhood base of the set.

The intersection $K \cap |\mathcal{C} f|$ is a closed, $\mathcal{C} f \cap (|\mathcal{C} f| \times |\mathcal{C} f|)$ invariant subset of $|\mathcal{C} f|$.

(b) If $K$ is closed, then following conditions are equivalent.

(i) $K$ is $\mathcal{C} f$-invariant and is $f$ invariant.

(ii) $K$ is $\mathcal{C} f$ invariant.

(iii) $K = \mathcal{C} f(K \cap |\mathcal{C} f|)$.

(iv) $K$ is $\mathcal{C} f$-invariant and if $A$ is an inward set which contains $K$ then the associated attractor $A_\infty$ contains $K$.

(c) $K$ is an attractor iff $K$ is closed, $\mathcal{C} f$ invariant and $K \cap |\mathcal{C} f|$ is a clopen subset of $|\mathcal{C} f|$. Conversely, if $A_0$ is a clopen $\mathcal{C} f \cap (|\mathcal{C} f| \times |\mathcal{C} f|)$ invariant subset of $|\mathcal{C} f|$ then $K_0 = \mathcal{C} f(A_0)$ is an attractor of which $A_0$ is the trace.
Proof: (a) We apply the notation of the proof of Proposition 5.17(b). Since $K$ is assumed to be closed and $\mathcal{C}f$ + invariant, $K$ is compact and equals $K \cup \mathcal{C}f(K)$. Let $Q_d(K,y) = \min(m_d(K,y),d(K,y))$. From the Proposition we see that $K$ is the intersection of the inward sets $\{y : Q_d(K,y) < \epsilon\}$ as $(d,\epsilon)$ varies with $d \in \Gamma(U_M), \epsilon > 0$. Recall that with $d = d_1 + d_2$ and $\epsilon = \min(\epsilon_1, \epsilon_2)$, $\{y : Q_d(K,y) \leq \epsilon\} \subset \{y : Q_{d_1}(K,y) \leq \epsilon_1\} \cap \{y : Q_{d_2}(K,y) \leq \epsilon_2\}$. It follows from compactness that for some $d \in \Gamma(U_M), \epsilon > 0$, the compact set $\{y : Q_d(K,y) \leq \epsilon\}$ is contained in $G$. Hence, $A = \{y : Q_d(K,y) < \epsilon\}$ is an open inward set with $K \subset A \subset U$.

If $A$ is an open inward set containing $K$ then $X \setminus A$ and $K \cup (\mathcal{C}f(A))$ are disjoint closed sets. Since a compact Hausdorff space is normal, there exists a continuous $L : X \to [0,1]$ which $= 0$ on $X \setminus A$ and $= 1$ on $K \cup (\mathcal{C}f(A))$. Any such is clearly the required elementary Lyapunov function.

Since $x \in (\mathcal{C}f)(x)$ for $x \in (\mathcal{C}f)$ it is clear that $K \cap (\mathcal{C}f)$ is a closed, $\mathcal{C}f \cap ((\mathcal{C}f) \times (\mathcal{C}f))$ invariant subset of $|\mathcal{C}f|$.

(b) (i) $\Leftrightarrow$ (ii): By Proposition 5.17 $\mathcal{C}f = f \cup f \circ (\mathcal{C}f) = f \cup (\mathcal{C}f) \circ f$ and so $f(K) = \mathcal{C}f(K)$ by Proposition 5.17 (a).

(i) $\Leftrightarrow$ (iii): If $K$ is $\mathcal{C}f$ + invariant then $K$ is $\mathcal{C}f$ invariant iff $K = (\mathcal{C}f)^\infty(K)$ and the latter equals $\mathcal{C}f(K \cap (\mathcal{C}f))$ by (6.19).

(ii) $\Leftrightarrow$ (iv): If $A$ is $\mathcal{C}f$ + invariant and contains a $\mathcal{C}f$ invariant set $K$ then the maximum $\mathcal{C}f$ invariant set $(\mathcal{C}f)^\infty(A)$ contains $K$. If $A$ is inward then $A_\infty = (\mathcal{C}f)^\infty(A)$ is the associated attractor.

On the other hand, let $K_1 = \mathcal{C}f(K)$ and assume there exists $x \in K \setminus K_1$. Let $G = (\mathcal{C}f)^\ast(X \setminus \{x\})$. Since $K_1 \subset X \setminus \{x\}, K \subset G$. By (a) there exists $A$ an inward set with $K \subset A \subset G$. So $A_\infty \subset \mathcal{C}f(G) \subset X \setminus \{x\}$. That is, the associated attractor does not contain $K$.

(c) If $A$ is an inward set and $\hat{A}$ is a subset such that $\mathcal{C}f(A) \subset (\hat{A})^\circ$ and $\hat{A} \subset A$, then $\hat{A}$ is an inward set with $(\mathcal{C}f)^\infty(\hat{A}) = (\mathcal{C}f)^\infty(A)$, i.e. with the same associated attractor. In particular we can choose $\hat{A}$ closed and so we see that every attractor is closed. Furthermore, $|\mathcal{C}f| \cap (\mathcal{C}f)^\infty(\hat{A}) = |\mathcal{C}f| \cap A^\circ$ and so the trace of the attractor is clopen in $|\mathcal{C}f|$. It is $\mathcal{C}f \cap ((\mathcal{C}f) \times (\mathcal{C}f))$ invariant by (a).

Conversely, if $A_0$ is a clopen $\mathcal{C}f \cap ((\mathcal{C}f) \times (\mathcal{C}f))$ invariant subset of $|\mathcal{C}f|$ then $K_0 = \mathcal{C}f(A_0)$ is a $\mathcal{C}f$ invariant subset of $X$ by (b), and it is contained in the open set $G = X \setminus (|\mathcal{C}f| \setminus A_0)$ by $\mathcal{C}f \cap ((\mathcal{C}f) \times (\mathcal{C}f))$ invariance. By (a) there exists an inward set $A$ such that $K_0 \subset A \subset G$. Hence, $A \cap (\mathcal{C}f) = A_0$ and so $A_\infty = (\mathcal{C}f)^\infty(A) = \mathcal{C}f(A_0) = K_0$. That is, $K_0$ is the attractor associated with $A$ and the trace is $A_0$. \qed
Remark: Notice that while an attractor is necessarily closed, a $\mathcal{C}f$ invariant set need not be. For example, if $X$ is the Cantor set and $f = 1_X$ then $\mathcal{C}f = 1_X$ and every subset of $X$ is $\mathcal{C}f$ invariant.

7. Recurrence and Transitivity

We first consider recurrence.

Proposition 7.1. Let $f$ be a relation on a uniform space $(X, \mathcal{U})$ and let $d \in \Gamma(\mathcal{U})$. Let $F = \mathcal{S}f, \mathcal{A}_df, \mathcal{A}_uf, \mathcal{C}_df$ or $\mathcal{C}_uf$.

(a) The relation $F$ is an equivalence relation iff $f^{-1} \subset F$ and $\text{Dom}(F) = X$.

(b) If $f$ is a continuous map on $X$ then $F$ is an equivalence relation iff $1_X \subset F$.

(c) If $\mathcal{S}f$ is an equivalence relation then $\mathcal{A}_df, \mathcal{A}_uf, \mathcal{C}_df$ and $\mathcal{C}_uf$ are equivalence relations.

Proof: (a) Clearly, if $F$ is an equivalence relation on $X$ which contains $f$ then $1_X \cup f^{-1} \subset F$ and so $\text{Dom}(F) = X$.

Conversely, if $f^{-1} \subset \mathcal{S}f$ then $\mathcal{S}f^{-1} \subset \mathcal{S}f = \mathcal{S}f$ and so, inverting, $\mathcal{S}f \subset \mathcal{S}f^{-1}$. That is, $\mathcal{S}f$ is symmetric. Similarly, if $f^{-1} \subset F$ for $F = \mathcal{A}_df, \mathcal{A}_uf, \mathcal{C}_df$ or $\mathcal{C}_uf$ then $F$ is symmetric. If $F$ is symmetric and $\text{Dom}(F) = X$, then for any $x \in X$ there exists $y \in X$ such that $(x, y) \in F$. By symmetry and transitivity, $(y, x), (x, x) \in F$. So $F$ is reflexive.

(b) If $f$ is a continuous map then it is a cusc relation and so $F = f \cup \text{Dom}(f)$ by Proposition 6.10(a). For any $x \in X$ assume $f(y) = x$. Since $(y, y) \in F$, either $(y, y) \in f$, i.e. $y = f(y) = x$ and so $(x, y) = (y, y) \in F$, or $(y, x) \in f$ and $(x, y) \in F$. As $y$ was an arbitrary element of $f^{-1}(x)$ it follows that $f^{-1} \subset F$. Since $f$ is a map, $X = \text{Dom}(f) \subset \text{Dom}(F)$.

(c) If $\mathcal{S}f$ is an equivalence relation then, since it is contained in $F$ it follows that $1_X \cup f^{-1} \subset F$ and so $F$ is an equivalence relation by (a).

Definition 7.2. Let $f$ be a relation on a uniform space $(X, \mathcal{U})$ and let $d \in \Gamma(\mathcal{U})$. For $F = \mathcal{S}f, \mathcal{A}_df, \mathcal{A}_uf, \mathcal{C}_df$ or $\mathcal{C}_uf$ we will say that $f$ is totally $F$ recurrent when $F$ is an equivalence relation.
Definition 7.3. A topological space $X$ is completely Hausdorff if the Banach algebra $\mathcal{B}(X)$ of bounded, real-valued continuous functions distinguish the points of $X$.

Thus, if $X$ is completely Hausdorff and $(x, y) \in X \times X \setminus 1_X$, there exists a continuous $L_{xy} : X \to [0, 1]$ with $L_{xy}(x) = 0$ and $L_{xy}(y) = 1$. These maps define a continuous injection into a product of copies of $[0, 1]$ indexed by the points of $X \times X \setminus 1_X$. Conversely, if there is a continuous injection from $X$ to a Tychonoff space, then $X$ is completely Hausdorff.

In [6] Bing constructs a simple example of a countable, connected Hausdorff space. On such a space the only continuous real-valued functions are constants and so the space is not completely Hausdorff.

A subset $A$ of a topological space $X$ is called a zero-set if there exists $u \in \mathcal{B}(X)$ such that $A = u^{-1}(0)$. Clearly, a zero-set in $X$ is a closed, $G_\delta$ subset of $X$. The constant functions 0 and 1 show that $X$ and $\emptyset$ are zero-sets. If $u, v \in \mathcal{B}(X)$ and $w_1 = u \cdot v, w_2 = u^2 + v^2$ then $w_1^{-1}(0) = u^{-1}(0) \cup v^{-1}(0)$ and $w_2^{-1} = u^{-1}(0) \cap v^{-1}(0)$. Thus, the collection of zero-sets is closed under finite unions and finite intersections. If $(X, d)$ is a pseudo-metric space and $A$ is a closed subset of $X$ then $u(t) = d(t, A)$ is an element of $\mathcal{B}(X)$ such that $A = u^{-1}(0)$. That is, every closed subset of a pseudo-metric space is a zero-set. If $X$ is normal and $A$ is a closed, then $A$ is a zero-set if it is a $G_\delta$ set. If $h : X \to Y$ is a continuous function and $u \in \mathcal{B}(Y)$ then $h^*u = u \circ h \in \mathcal{B}(X)$ and $(h^*u)^{-1}(0) = h^{-1}(u^{-1}(0))$. That is, the continuous pre-image of a zero-set is a zero-set. It follows that if $u \in \mathcal{B}(X)$ and $K \subset \mathbb{R}$ is closed then since $\mathbb{R}$ is a metric space $K$ is a zero-set and so $u^{-1}(K)$ is a zero-set.

For a topological space $X$, we denote by $\tau X$ the set $X$ equipped with the weak topology generated by the elements of $\mathcal{B}(X)$. That is, it is the coarsest topology with respect to which every element of $\mathcal{B}(X)$ is continuous. Equivalently, if $h$ is a map to $X$ from a topological space $Y$, then $h : Y \to \tau X$ is continuous iff $h^*u \in \mathcal{B}(Y)$ for all $u \in \mathcal{B}(X)$. The set of complements of the zero-sets of $X$ forms a basis for the topology of $\tau X$. Thus, the closed sets are exactly those which are intersections of the zero-sets of $X$. Thus, the “identity map” from $X$ to $\tau X$ is continuous and $\mathcal{B}(\tau X) = \mathcal{B}(X)$.

Proposition 7.4. Let $X$ be a topological space.

(a) The following are equivalent.

(i) $X$ is completely regular.

(ii) Every closed subset of $X$ is an intersection of zero-sets.

(iii) $X = \tau X$.

(b) The following are equivalent.
(i) $X$ is completely Hausdorff.
(ii) Every point of $X$ is an intersection of zero-sets.
(iii) $X$ is a $T_1$ space and every compact subset of $X$ is an intersection of zero-sets.
(iv) $X$ is a $T_1$ space and disjoint compact subsets can be distinguished by $\mathcal{B}(X)$.
(v) $\tau X$ is a $T_1$ space.
(vi) $\tau X$ is a Tychonoff space.
(c) The space $\tau X$ is completely regular and if $h : X \to Y$ is a continuous function with $Y$ completely regular, then $h : \tau X \to Y$ is continuous.
(d) If $d$ is a pseudo-metric on $X$ then $d$ is continuous on $X \times X$ iff it is continuous on $\tau X \times \tau X$. The set of all continuous pseudo-metrics on $X$ is the gage of the maximum uniformity with topology that of $\tau X$.

Proof: (a) (i) $\iff$ (ii): If $\{u_i\} \subset \mathcal{B}(X)$ and $A = \bigcap_i u_i^{-1}(0)$ then for every $x \not\in A$ there exists $u_i$ with $u_i(x) \neq 0$, while for all $i$ $u_i = 0$ on $A$. On the other hand, if for every $x \not\in A$ there exists a $v_x \in \mathcal{B}(X)$ with $v_x(x) \not\in v_x(A)$ then $u_x(y) = d(v_x(y), v_x(A))$ then $A \subset u_x^{-1}(0)$ and $u_x(x) \neq 0$. Thus, $A$ is the intersection of the $u_x^{-1}(0)$'s as $x$ varies over $X \setminus A$. Thus, $A$ is a zero-set iff $\mathcal{B}(X)$ distinguishes $A$ from the points of $X \setminus A$.
(ii) $\iff$ (iii): The closed sets of $\tau X$ are exactly the intersections of the zero-sets of $X$.
(c) Since $\mathcal{B}(X) = \mathcal{B}(\tau X)$ it is clear that $\tau(\tau X) = \tau X$ and so $\tau X$ is completely regular by (a). If $A$ is a closed subset of $Y$ then because $Y$ is completely regular, $A$ is an intersection of zero-sets by (a). Since $h : X \to Y$ is continuous, $h^{-1}(A)$ is an intersection of zero-sets in $X$ and so is closed in $\tau X$. Thus, $h : \tau X \to Y$ is continuous.
(b) (i) $\iff$ (ii): Just as in (a).
(ii) $\iff$ (iii): If $A$ is compact and $x \not\in A$ then for each $a \in A$ there exists $u_a \in \mathcal{B}(X)$ such that $u_a(a) = 0$ and $u_a(x) = 1$. Let $v_a = 2\max(u_a - \frac{1}{2}, 0)$. That is, $v_a(x) = 1$ and $v_a = 0$ on a neighborhood of $a$. By compactness there exists a finite subset $A_0$ of $A$ such that $u = \Pi_{a \in A_0} v_a$ is 1 at $x$ and 0 on $A$. The converse is obvious.
(iii) $\iff$ (iv): If $A$ and $B$ are disjoint compact sets then for every $x \in B$ there exists $u_x = 1$ on $A$ and has $v_x(x) = 0$. Use $u_x = 1 - u$ from the above proof. Again, let $v_x = 2\max(u_x - \frac{1}{2}, 0)$. As above, there is a finite subset $B_0$ of $B$ so that $u = \Pi_{x \in B_0} v_x$ is 1 on $A$ and 0 on $B$. Again, the converse is obvious.
(ii) $\implies$ (v): From (ii), every point is closed in $\tau X$. 


(v) ⇒ (vi): A $T_1$ completely regular space is Tychonoff.
(vi) ⇒ (i): $X$ injects into the Tychonoff space $\tau X$.

(d) If $d$ is a continuous pseudo-metric on $\tau X$ then it is a continuous pseudo-metric on $X$ since $\tau X$ is coarser than $X$. If $d$ is a continuous pseudo-metric on $X$ then $(X, d)$ is a pseudo-metric space with $X \to (X, d)$ continuous. Since a pseudo-metric space is completely regular, (c) implies that $\tau X \to (X, d)$ is continuous. Since $d$ is a continuous function on $(X, d) \times (X, d)$, it is continuous on $\tau X \times \tau X$. For a completely regular space, like $\tau X$, the collection of all continuous pseudo-metrics is the gage of the maximum uniformity.

A clopen set is clearly a zero-set. Recall that the quasi-component of a point $x \in X$ is the intersection of all the clopen sets which contain $x$. In a compact space the quasi-components are the components, but even in a locally compact space this need not be true. If $X_0 = [0, 1] \times \{0, 1/n : n \in \mathbb{N}\}$ and $X = X_0 \setminus \{(\frac{1}{2}, 0)\}$ then the quasi-component of $(0, 0)$ is $([0, 1] \setminus \{\frac{1}{2}\}) \times \{0\}$.

**Definition 7.5.** A topological space $X$ is

- totally disconnected when the quasi-components are singletons.
- zero-dimensional when the clopen sets form a basis for the topology.
- strongly zero-dimensional when the clopen sets contain a neighborhood basis for every closed subset.

Recall from Appendix B, that we call a uniformity $U$ zero-dimensional when it is generated by equivalence relations.

For a space $X$ let $B_0(X)$ consist of those $u \in B(X)$ with $u(X) \subset \{0, 1\}$, i.e. $B_0(X)$ is the set of characteristic functions of the clopen subsets. For a topological space $X$, we denote by $\tau_0 X$ the set $X$ equipped with the weak topology generated by the elements of $B_0(X)$. That is, it is the coarsest topology with respect to which every element of $B_0(X)$ is continuous. Equivalently, if $h$ is a map to $X$ from a topological space $Y$, then $h : Y \to \tau X$ is continuous iff $h^{-1}(A)$ is clopen in $Y$ whenever $A$ is a clopen subset of $X$.

**Proposition 7.6.** Let $X$ be a topological space.

(a) The following are equivalent.

(i) $X$ is zero-dimensional.
(ii) Every closed subset of $X$ is an intersection of clopen sets.
(iii) $X = \tau_0 X$.

If $X$ is zero-dimensional, then it is completely regular.
(b) The following are equivalent.
   (i) $X$ is totally disconnected.
   (ii) Every point of $X$ is an intersection of clopen sets.
   (iii) $X$ is a $T_1$ space and every compact subset of $X$ is an intersection of clopen sets.
   (iv) $X$ is a $T_1$ space and if $A, B$ are disjoint compact subsets of $X$ then there exists a clopen set $U$ with $A \subset U$ and $B \cap U = \emptyset$.
   (v) $\tau_0 X$ is a $T_1$ space.
   If $X$ is totally disconnected, then it is completely Hausdorff.

(c) The space $\tau_0 X$ is zero-dimensional and if $h : X \to Y$ is a continuous function with $Y$ zero-dimensional, then $h : \tau_0 X \to Y$ is continuous.

(d) If $d$ is a pseudo-ultrametric on $X$ then $d$ is continuous on $X \times X$ iff it is continuous on $\tau_0 X \times \tau_0 X$. The set of all continuous pseudo-ultrametrics on $X$ is the gage of the maximum zero-dimensional uniformity with topology that of $\tau_0 X$.

Proof: The proofs are completely analogous to those of Proposition 7.4. The details are left to the reader.

Proposition 7.7.  
(a) If a space is compact and totally disconnected then it is strongly zero-dimensional.
(b) If a space is locally compact and totally disconnected then it is zero-dimensional.
(c) A $T_1$ space is zero-dimensional iff it admits an embedding into a compact, totally disconnected space.
(d) A space is totally disconnected iff it admits a continuous injection into a compact, totally disconnected space.

Proof: (a) If $X$ is a compact Hausdorff space then disjoint closed sets are disjoint compact sets. So (i) $\Rightarrow$ (iv) of Proposition 7.6 (b) implies that a compact, totally disconnected space is strongly zero-dimensional.

(b) If $x \in X$ and $x$ is contained in an open set $U$ with closure $\overline{U}$ compact, then there exists a clopen set $A_0$ containing $x$ and disjoint from the compact set $\overline{U} \setminus U$. Hence, $A = A_0 \cap U = A_0 \cap \overline{U}$ is a clopen set containing $x$ and contained in $U$.

(c), (d) Using the elements of $\mathcal{B}_0(X)$ we can inject totally disconnected space $X$, or embed a $T_1$ zero-dimensional space into a product of copies of $\{0, 1\}$, which is compact and totally disconnected.
Conversely, a subspace of a zero-dimensional space is zero-dimensional and if $X$ injects into a totally disconnected space then it is totally disconnected.

\[\square\]

Questions 7.8. Does there exist a space which is completely Hausdorff and regular, but not completely regular?

Does there exist a completely regular, totally disconnected space which is not zero-dimensional? In particular, for a totally disconnected space $X$ is $\tau X = \tau_0 X$?

Call a Hausdorff space $X$ strongly $\sigma$-compact if there is a sequence $\{K_n\}$ of compacta covering $X$ such that $A \cap K_n$ closed for all $n$ implies $A$ is closed. Equivalently, by taking complements, we have that $A \cap K_n$ is open in $K_n$ for all $n$ implies $A$ is open. Consequently, if $A \cap K_n$ is clopen in $K_n$ for all $n$ then $A$ is clopen. Observe that the condition is a strengthening of the condition that $X$ be a k-space.

**Proposition 7.9.** (a) If $X$ is a locally compact, $\sigma$-compact Hausdorff space then $X$ is strongly $\sigma$-compact.

(b) If $q : X \to Y$ is a quotient map with $Y$ Hausdorff and $X$ Hausdorff and strongly $\sigma$-compact, then $Y$ is strongly $\sigma$-compact.

(c) $X$ is strongly $\sigma$-compact iff it is a Hausdorff quotient space of a locally compact, $\sigma$-compact Hausdorff space.

(d) If $X$ is a strongly $\sigma$-compact, Hausdorff space, then $X$ is normal.

(e) If $X$ is strongly $\sigma$-compact and totally disconnected, then $X$ is strongly zero-dimensional.

**Proof:** (a) If $\{K_n\}$ is an increasing sequence of compacta with $K_n \subset K_{n+1}$ and $\bigcup_n K_n = X$ then $A \cap K_n$ open in $K_n$ implies $A \cap K_n^c$ is open in $X$ and so $A = \bigcup_n A \cap K_n^c$ is open.

(b) Assume that $\{K_n\}$ is a sequence of compacta in $X$ which determine the topology. Assume that $B \subset Y$ is such that $B \cap q(K_n)$ is closed for every $n$. Then $q^{-1}(B \cap q(K_n)) \cap K_n = q^{-1}(B) \cap K_n$ is closed for every $n$. Hence, $q^{-1}(B)$ is closed since the sequence determines the topology of $X$. Since $q$ is a quotient map, $B$ is closed. Thus, $\{q(K_n)\}$ determines the topology of $Y$.

(c) If the sequence $\{K_n\}$ determines the topology of $X$ then $X$ is a quotient of the disjoint union of the $K_n$'s. The converse follows from (a) and (b).
(d) Let $Y$ be a locally compact, $\sigma$-compact, Hausdorff space and $q : Y \to X$ be a quotient map. Let $F$ be the closed equivalence relation $(q \times q)^{-1}(1_X)$ on $Y$. Let $B, \bar{B}$ be disjoint closed subsets of $X$. Let $Y_0 = q^{-1}(B \cup \bar{B})$. Define $L_0 : Y_0 \to [0, 1]$ by $L_0(x) = 0$ for $x \in q^{-1}(B)$ and $= 1$ for $x \in q^{-1}(\bar{B})$. Thus, $L_0$ is a Lyapunov function for $F \cap (Y_0 \times Y_0)$. By Theorem 6.14 there exists $L : Y \to [0, 1]$ an $F$ Lyapunov function which extends $L_0$. Since $F$ is an equivalence relation, $L$ is constant on the $F$ equivalence classes and so factors to define a continuous map on $X$ which is $0$ on $B$ and $1$ on $\bar{B}$.

(e) Replacing $K_n$ by $\bigcup_{i \leq n} K_i$, if necessary, we can assume that the determining sequence of compacta is non-decreasing. We may also assume $\emptyset = K_0$. Let $B, \bar{B}$ be disjoint closed subsets of $X$. Let $A_0 = \emptyset$. Assume inductively, that $A_n$ is a subset of $K_n \setminus \bar{B}$ clopen with respect to $K_n$ with $B \cap K_n \subset A_n$ and with with $A_{n-1} = A_n \cap K_{n-1}$. Observe that $A_n \cup (K_{n+1} \cap B)$ and $(K_n \setminus A_n) \cup (K_{n+1} \cap B)$ are disjoint compact sets in $X$. Since $X$ is totally disconnected, Proposition 7.6 implies there is a clopen subset $U$ of $X$ which contains $A_n$ and is disjoint from $(K_n \setminus A_n) \cup (K_{n+1} \cap B)$. Hence, $A_{n+1} = U \cap K_{n+1}$ is the required subset clopen in $K_{n+1}$. The set $A = \bigcup A_n$ is disjoint from $B$ and since $A \cap K_n = A_n$ for all $n$, $A$ is clopen in $X$.

\[ \square \]

Lemma 7.10. (a) For a pseudo-metric space $(X, d)$ the relation $Z_d = \{(x, y) : d(x, y) = 0\}$ is a closed equivalence relation and $d$ induces on the quotient space $X/Z_d$ a metric $\bar{d}$, so that $d = \bar{q}^*\bar{d} = \bar{d} \circ (\bar{q} \times \bar{q})$, with $\bar{q} : (X, d) \to (X/Z_d, \bar{d})$ the induced “isometry”. The map $\bar{q}$ is an open map and a closed map and so is a quotient map.

(b) Let $E$ be a closed equivalence relation on a topological space $X$ and let $q : X \to X/E$ be the quotient map. A continuous pseudo-metric $d$ on $X$ with $E \subset Z_d$ induces a continuous pseudo-metric $\bar{d}$ on $X/E$ so that $d = q^*\bar{d}$. Conversely, if $\bar{d}$ is a continuous pseudo-metric on $X/E$, then $d = q^*\bar{d}$ is a continuous pseudo-metric on $X$ with $E \subset Z_d$.

Proof: (a) A subset $A$ is closed in $(X, d)$ iff $d(x, A) = 0$ implies $x \in A$. Hence, a closed set is $Z_d$ saturated and $\bar{q}(A)$ is a closed set in $(X/Z_d, \bar{d})$. Taking complements we see that an $\bar{q}$ is an open map as well.

(b) If $d$ is a continuous pseudo-metric on $X$ with $Z_d \subset E$ then $\bar{q} : X \to X/Z_d$ factors through the projection $q$ to define a map $h : X/E \to X/Z_d$ so that $\bar{q} = h \circ q$. Since $q$ is a quotient map, $h$ is continuous. Hence, $\bar{d} = h^*\bar{d}$ is a continuous pseudo-metric on $X/E$ with $d = q^*\bar{d}$. The converse is obvious.
Theorem 7.11. Let $f$ be a relation on a Tychonoff space $X$.

(a) If $\mathcal{G}f$ is an equivalence relation, then $\mathcal{G}f$ is the smallest closed equivalence relation which contains $f$.

(b) Assume that $\mathcal{A}_{\mathcal{U}_M}f$ is an equivalence relation. The relation $\mathcal{A}_{\mathcal{U}_M}f$ is the smallest closed equivalence relation $E$ containing $f$ such that the quotient space $X/E$ is completely Hausdorff. In particular, $\mathcal{G}f = \mathcal{A}_{\mathcal{U}_M}f$ iff $\mathcal{G}f$ is an equivalence relation with the quotient space $X/\mathcal{G}f$ completely Hausdorff.

The set $\{\ell_d^f = st_d^f : d \in \Gamma(\mathcal{U}_M)\}$ projects to the gage of the maximum uniformity with topology $\tau(X/\mathcal{A}_{\mathcal{U}_M}f)$.

If $X$ is a locally compact, paracompact Hausdorff space and either $X$ is $\sigma$-compact or $f$ is a proper relation, then $1_X \cup \mathcal{G}f = \mathcal{A}_{\mathcal{U}_M}f$. The space $X/\mathcal{A}_{\mathcal{U}_M}f$ is a Hausdorff and normal and so $X/\mathcal{A}_{\mathcal{U}_M}f = \tau(X/\mathcal{A}_{\mathcal{U}_M}f)$.

(c) Assume that $\mathcal{C}_{\mathcal{U}_M}f$ is an equivalence relation. The relation $\mathcal{C}_{\mathcal{U}_M}f$ is the smallest closed equivalence relation $E$ containing $f$ such that the quotient space $X/E$ is totally disconnected.

The set $\{m_d^f = sm_d^f : d \in \Gamma(\mathcal{U}_M)\}$ projects to the gage of the maximum zero-dimensional uniformity with topology $\tau_0(X/\mathcal{A}_{\mathcal{U}_M}f)$.

If $X$ is a locally compact, paracompact Hausdorff space and either $X$ is $\sigma$-compact or $f$ is a proper relation, then $X/\mathcal{C}_{\mathcal{U}_M}f$ is a Hausdorff, strongly zero-dimensional space and so $X/\mathcal{C}_{\mathcal{U}_M}f = \tau(X/\mathcal{C}_{\mathcal{U}_M}f) = \tau_0(X/\mathcal{C}_{\mathcal{U}_M}f)$.

Proof: (a) If $E$ is a closed equivalence relation which contains $f$ then, because it is transitive, $\mathcal{G}f \subseteq E$. Because $\mathcal{G}f$ is a closed equivalence relation which contains $f$, it is the smallest such.

(b) If $d \in \Gamma(\mathcal{U}_M)$, i.e. $d$ is a continuous pseudo-metric on $X$ then since $\mathcal{A}_{\mathcal{U}_M}f$ is reflexive and symmetric, Proposition 2.2 together with Proposition 3.1 implies that $\ell_d^f = st_d^f$ is a pseudo-metric on $X$ with $\mathcal{A}_{\mathcal{U}_M}f \subseteq Z_{\ell_d^f}$. On the other hand, if $d$ is a continuous pseudo-metric on $X$ with $\mathcal{A}_{\mathcal{U}_M}f \subseteq Z_d$ then by Lemma 2.1 $\ell_d^f = d$. By Lemma 7.10 these are exactly the pullbacks via $q : X \to X/\mathcal{A}_{\mathcal{U}_M}f$ of continuous pseudo-metrics on the quotient space, i.e. the gage of the maximum uniformity with topology $\tau(X/\mathcal{A}_{\mathcal{U}_M}f)$.

If $E$ is an equivalence relation then a Lyapunov function $L$ for $E$ is exactly a continuous real-valued function which is constant on each equivalence class, i.e. $L$ factors through the projection $q : X \to X/E$ to define a continuous real-valued function on $X/E$. Hence, $-L$ is a
Lyapunov function for $E$ as well. Hence, $X/E$ is completely Hausdorff iff $E = \bigcap \leq_L$ with $L$ varying over the Lyapunov functions for $E$. So Corollary 5.10 implies that $X/\mathcal{A}_u f$ is completely Hausdorff.

On the other hand, if $E$ is a closed equivalence relation which contains $f$ and which has a completely Hausdorff quotient, then $E = \bigcap \leq_L$ with $L$ varying over the Lyapunov functions for $E$. Each such $L$ is a Lyapunov function for $f$ and so is an $\mathcal{A}_u f$ Lyapunov function by Corollary 5.10 again. Hence, $\mathcal{A}_u f \subset \leq_L$ for each such $L$. Hence, $\mathcal{A}_u f \subset E$.

Lyapunov function for $V_q$.

Lemma 2.1

Proposition 7.9 the quotient $X/\mathcal{A}_u f$ is a strongly $\sigma$-compact Hausdorff space and so it normal. As it is completely regular, it follows that $X/\mathcal{A}_u f = \tau(X/\mathcal{A}_u f)$.

If $X$ is a locally compact, $\sigma$-compact, Hausdorff space, then by Proposition 7.9 the quotient $X/\mathcal{A}_u f$ is a strongly $\sigma$-compact Hausdorff space and so it normal. As it is completely regular, it follows that $X/\mathcal{A}_u f = \tau(X/\mathcal{A}_u f)$.

On the other hand, if $\mathcal{A}_c f$ is reflexive and symmetric, Proposition 2.2 together with Proposition 3.1 implies that $m_d f = sm_d f$ is a pseudo-ultrametric on $X$ with $\mathcal{C}_u f \subset Z_m f$. On the other hand, if $d$ is a continuous pseudo-ultrametric on $X$ with $\mathcal{C}_u f \subset Z_d f$ then by Lemma 2.1 $m_d f = d$. By Lemma 7.10 these are exactly the pullbacks via $q : X \to X/\mathcal{C}_u f$ of continuous pseudo-ultrametrics on the quotient space, i.e. the gage of the maximum zero-dimensional uniformity with topology $\tau_0(X/\mathcal{C}_u f)$.

Assume that $(x, y) \notin \mathcal{C}_u f$. There exists a continuous pseudo-metric $d$ on $X$ such that $m_d(x, y) = \epsilon > 0$. Since $m_d f$ is a pseudo-ultrametric, $V_\epsilon d(x)$ is a clopen set which contains $x$ but not $y$. Furthermore, $V_\epsilon d(x)$ is $\mathcal{C}_u f$ saturated. Hence, if $q : X \to X/\mathcal{C}_u f$ is the projection, $q(\mathcal{C}_u f)$ is a clopen subset of $X/\mathcal{C}_u f$ which contains $q(x)$ but not $q(y)$. It follows that $X/\mathcal{C}_u f$ is totally disconnected.

On the other hand, let $E$ be a closed equivalence relation which contains $f$ and which has a totally disconnected quotient with quotient map $q : X \to X/E$. If $(x, y) \notin E$ then there exists a clopen set $A_1 \subset X/E$ with $q(x) \in A_1$ and $q(y) \in B_1 = (X/E) \setminus A_1$. So $A = q^{-1}(A_1)$ and $B = q^{-1}(B_1)$ form a clopen partition of $X$. Let $U = (A \times A) \cup (B \times B)$. This is a clopen equivalence relation on $X$ with $f \subset E \subset U$. It follows that if $[a, b] \in f^{\times n}$ is an $(x, z), U$ chain, then with $b_0 = x, a_{n+1} = z, (a_i, b_i) \in f \subset U$ for $i = 1, \ldots, n$ and $(b_i, a_{i+1}) \in U$ for $i = 0, \ldots, n$. 
Since $U$ is an equivalence relation $z \in U(x) = A$ and so $z \neq y$. Hence, $(x, y) \not\in \mathcal{C}_{U_M}f$. Contrapositively, $\mathcal{C}_{U_M}f \subset E$.

If $X$ is a locally compact, $\sigma$-compact, Hausdorff space, then by Proposition 7.9 the quotient $X/\mathcal{C}_{U_M}f$ is a strongly $\sigma$-compact, totally disconnected space and so it strongly zero-dimensional. As it is completely regular, it follows that $X/\mathcal{C}_{U_M}f = \tau(X/\mathcal{C}_{U_M}f)$. As it is zero-dimensional, it follows that $X/\mathcal{C}_{U_M}f = \tau_0(X/\mathcal{C}_{U_M}f)$.

If $X$ is a locally compact, paracompact Hausdorff space and $f$ is proper, then by Lemma 6.13 $X/\mathcal{C}_{U_M}f$ is a disjoint union of clopen strongly $\sigma$-compact totally disconnected subspaces and so it is strongly zero-dimensional. Again, $X/\mathcal{C}_{U_M}f = \tau(X/\mathcal{C}_{U_M}f) = \tau_0(X/\mathcal{C}_{U_M}f)$.

$\square$

**Corollary 7.12.** For a Tychonoff space $1_X$, $\mathcal{C}_{U_M}1_X$ is a closed equivalence relation with equivalence classes the quasi-components of $X$.

**Proof:** Since $1_X$ is symmetric, $\mathcal{C}_{U_M}1_X$ is a closed equivalence relation with a totally disconnected quotient via the quotient map $q : X \to X/\mathcal{C}_{U_M}1_X$ by Theorem 7.11. So if $q(x) \neq q(y)$ there is a clopen set $A \subset X/\mathcal{C}_{U_M}1_X$ with $q(x) \in A$ and $q(y) \not\in A$. Since $U = q^{-1}(A)$ is clopen with $x \in U$ and $y \not\in U$, $x$ and $y$ lie in separate quasi-components. On the other hand, if $U_1$ is a clopen subset of $X$ with $x \in X$ and $y \in U_2 = X \setminus U_1$ then $E = (U_1 \times U_1) \cup (U_2 \times U_2)$ is a clopen equivalence relation on $X$ and so $E \in U_M$. If $[a, b]_X \in 1_X^n$ defining an $xz, E$ chain then $z \in U_1$ and so $z \neq y$. Hence, $(x, y) \not\in \mathcal{C}_{U_M}1_X$.

$\square$

**Lemma 7.13.** If $f$ is a relation on a Hausdorff uniform space $(X, U)$, then $\mathcal{A}_U(1_X \cup f) = 1_X \cup A_Uf$.

**Proof:** Clearly $1_X \cup A_Uf \subset \mathcal{A}_U(1_X \cup f)$. If $(x, y) \not\in 1_X \cup A_Uf$ then because $(X, U)$ is Hausdorff there exists $d_1 \in \Gamma(U)$ such that $d_1(x, y) > 0$. Also, there exists $d_2 \in \Gamma(U)$ such that $d_2(x, y) > 0$. Hence, $d = d_1 + d_2 \in \Gamma(U)$ with $(x, y) \not\in Z_d \cup \mathcal{A}_df$. By (3.20) $(x, y) \not\in \mathcal{A}_d(1_X \cup f)$ and so is not in $\mathcal{A}_U(1_X \cup f)$.

$\square$

**Corollary 7.14.** Let $f$ be a relation on a Hausdorff uniform space $(X, U)$.

The closed equivalence relations $1_X \cup (\mathcal{A}_Uf \cap \mathcal{A}_Uf^{-1})$ and $1_X \cup (\mathcal{C}_Uf \cap \mathcal{C}_Uf^{-1})$ have completely Hausdorff quotients. On $|\mathcal{C}_Uf|$ the equivalence relation $\mathcal{C}_Uf \cap \mathcal{C}_Uf^{-1}$ has a totally disconnected quotient.
If $X$ is a locally compact, $\sigma$-compact Hausdorff space, then the quotients are Hausdorff and normal and $|C_{ul}|/[C_{ul}f \cap C_{ul}f^{-1}]$ is Hausdorff and strongly zero-dimensional.

**Proof:** $X$ is Tychonoff and so we can apply Lemma 7.13, together with monotonicity and idempotence of the operator $A_{ul}$ to get

$$1_X \cup (A_{ul}f \cap A_{ul}f^{-1}) \subset A_{ul}[1_X \cup (A_{ul}f \cap A_{ul}f^{-1})]$$

$$= 1_X \cup (A_{ul}f \cap A_{ul}f^{-1}).$$

Thus, $E = 1_X \cup (A_{ul}f \cap A_{ul}f^{-1})$ is a closed equivalence relation with $A_{ul}E = E$. Similarly, $E = 1_X \cup (C_{ul}f \cap C_{ul}f^{-1})$ is a closed equivalence relation with $A_{ul}E = E$. By Theorem 7.11 (b) each has a completely Hausdorff quotient and a normal Hausdorff quotient when $X$ is locally compact and $\sigma$-compact.

Similarly, if $E = C_{ul}f \cap C_{ul}f^{-1}$ then $C_{ul}E = E$. Since $E \subset |C_{ul}f| \times |C_{ul}f|$, we can apply Theorem 7.11 (c), replacing $X$ by $|C_{ul}f|$ on which $E$ is a closed equivalence relation. We obtain that the quotient is totally disconnected and is Hausdorff and strongly zero-dimensional when $X$ is locally compact and $\sigma$-compact.

$\square$

**Proposition 7.15.** Let $E$ be a closed equivalence relation on a Tychonoff space $X$.

(a) The relation $E$ is usc iff the quotient map $q : X \to X/E$ is a closed map.

(b) If $E$ is usc and $X$ is normal, then $X/E$ is a Hausdorff normal space.

(c) If $E$ is cusc, and $X$ is locally compact, then $X/E$ is locally compact.

(d) If $E$ is cusc, and $X$ is second countable, then $X/E$ is second countable.

**Proof:** (a) If $A \subset X$ then $q^{-1}(q(A)) = E(A)$. To say that $E$ is usc is to say that $E(A)$ is closed whenever $A$ is. To say that $q$ is closed is to say that $q^{-1}(q(A))$ is closed whenever $A$ is. So the equivalence is clear.

(b) If $A_0, A_1 \subset X$ are disjoint closed sets with $E(A_0) = A_0$ and $E(A_1) = A_1$ then let $X_0 = A_0 \cup A_1$, $L_0(x) = 0$ for $x \in A_0$ and $= 1$ for $x \in A_1$. Thus, $L_0$ is a Lyapunov function for $E|(X_0 \times X_0)$ and so by Theorem 6.4 extends to a Lyapunov function $L$ for $E$. This implies normality of $X/E$. 
(c), (d) We choose a basis $\mathcal{B}$ for $X$ which is closed under finite unions. Let $\tilde{\mathcal{B}} = \mathcal{B} = \{E^*U : U \in \mathcal{B}\}$. Since $E$ is usc, each member of $\tilde{\mathcal{B}}$ is an $E$ saturated open set. If $x \in X$ and $V$ is open with $E(x) \subset V$ then since $E(x)$ is compact, there exists $U \in \mathcal{B}$ such that $E(x) \subset U \subset V$ and so $E(x) \subset E^*(U) \subset U \subset V$. Thus, $\mathcal{B}_E = \{q(V) : V \in \tilde{\mathcal{B}}\}$ is a basis for $X/E$.

For (c) we can choose $\mathcal{B}$ so that every member has compact closure. Since $q(E^*(U)) \subset q(\overline{U})$ it follows that each $q(V)$ for $V \in \mathcal{B}_E$ has compact closure in $X/E$ and so $X/E$ is locally compact.

For (d) choose $\mathcal{B}$ countable. Then $\mathcal{B}_E$ is a countable basis for $X/E$. $\square$

A second countable space which admits a complete metric is called a Polish space. Any $G_\delta$ subset of a Polish space is a Polish space. A locally compact, second countable space is $\sigma$-compact and Polish.

**Examples 7.16.**

(a) There exists a homeomorphism $f$ on a separable metric space $X$ such that $\mathcal{G}f$ is an equivalence relation such that the quotient space $X/\mathcal{G}f$ is not Hausdorff and so $\mathcal{G}f$ is a proper subset of $A_{U_{mf}}$.

(b) There exists a homeomorphism $f$ on a locally compact space $X$ such that $\mathcal{G}f$ is an equivalence relation such that the quotient space $X/\mathcal{G}f$ is not Hausdorff and so $\mathcal{G}f$ is a proper subset of $A_{U_{mf}}$.

(c) There exists a homeomorphism $f$ on a Polish space $X$ with metric $d$, such that $\mathcal{G}f = C_{U_{mf}} = C_{mf}$ is an equivalence relation with a totally disconnected quotient which is not regular.

(d) There exists a homeomorphism $f$ on a locally compact space, such that $\mathcal{G}f = C_{U_{mf}} f$ is an equivalence relation with a totally disconnected quotient which is not regular and so is not zero-dimensional.

(e) There exists a homeomorphism $f$ on a locally compact, $\sigma$-compact, metrizable space, such that $\mathcal{G}f = C_{U_{mf}} f$ is an equivalence relation with a Hausdorff, strongly zero-dimensional quotient which is not first countable and so is not metrizable.

**Proof:** (a) The following is a variation of the example in Problem 3J of [10].

Let $g$ be a topologically transitive homeomorphism on a compact metric space $Y$ with a Cantor set $C \subset Y$ of fixed points. Such maps can be constructed with $Y$ the torus or the Cantor set itself.

Let $D$ be a countable dense subset of $C$ and $J = C \setminus D$ so that $J$ is a dense $G_\delta$ subset of $C$. Choose $e \in C$. For the homeomorphism $g \times g$
on $Y \times Y$, the compact set $Y \times \{e\}$ and the $G_{\delta}$ set $J \times Y$ are $g \times g$

invariant. The restriction of $g \times g$ to $Y \times \{e\}$ is topologically transitive

with $C \times \{e\}$ a set of fixed points. For each $j \in J$, the restriction of

$g \times g$ to $\{j\} \times Y$ is topologically transitive with $(j,e)$ a fixed point.

Let $X_0 = Y \times \{e\} \cup J \times Y$ and $f_0$ be the restriction of $g \times g$ to this

invariant set.

Mapping $Y$ to $e$ we obtain a retraction $\pi : J \times Y \to J \times \{e\}$. By

extending the definition of $\pi$ to be the identity on $Y \times \{e\}$, we define

the continuous retraction $\pi : X_0 \to Y \times \{e\}$.

Let $E_1$ denote the closed equivalence relation

$$\pi^{-1} \circ \pi = (\pi \times \pi)^{-1}(1_{(y \times \{e\}) \times (y \times \{e\}))},$$

Let $E_2 = 1_{J \times Y} \cup (Y \times \{e\}) \times (Y \times \{e\})$ which is also a closed equivalence

relation. Hence, $E_0 = E_1 \cup E_2$ is a closed, reflexive, symmetric relation

on $X_0$. It is not, however, transitive.

Let $X = X_0 \setminus (J \times \{e\})$. Because we are removing a set of fixed

points, $f_0$ restricts to a homeomorphism $f$ on $X$. Let $E$ denote the

restriction $E_0 \cap (X \times X)$, a closed, reflexive, symmetric relation on $X$.

We show that it is also transitive.

Let $x,y \in X$.

- $(x,y) \in E_1 \cap (X \times X) \setminus 1_X$ iff $x_1 = y_1 \in J$ and $x_2, y_2 \in Y \setminus \{e\}$
  with $x_2 \neq y_2$.
- $(x,y) \in E_2 \cap (X \times X) \setminus 1_X$ iff $x_2 = y_2 = e$ and $x_1, y_1 \in Y \setminus J$
  with $x_1 \neq y_1$.

Assume $(x,y), (y,z) \in E$ if $x = y$ (or $y = z$) then $(x,z) = (y,z)$ (resp.

$(x,z) = (x,y)$) and so $(x,z) \in E$. So we may assume $(x,y), (y,z) \in$

$E \setminus 1_X$.

If $(x,y) \in E_1 \cap (X \times X) \setminus 1_X$ then $y_2 \neq e$ and so $(y,z) \not\in E_2 \cap

(X \times X) \setminus 1_X$. Hence, $(y,z) \in E_1 \cap (X \times X) \setminus 1_X$ and so $(x,z) \in

E_1 \cap (X \times X) \subseteq E$. If $(x,y) \in E_2 \cap (X \times X) \setminus 1_X$ then $y_2 = e$

and so, as before, $(y,z) \in E_2 \cap (X \times X) \setminus 1_X$ and $(x,z) \in E_2 \cap (X \times X) \subseteq E$.

Thus, $E$ is transitive.

From the invariance and transitivity results, it is clear that $f \subseteq E$

and $E \subseteq \mathcal{G}f$. Since $E$ is a closed, transitive relation which contains $f$,

it contains $\mathcal{G}f$. Thus, $\mathcal{G}f = E$.

Now consider the quotient space of $X$ by the equivalence relation

$E$, with quotient map $q : X \to X/E$. We will see that $X/E$ is not

Hausdorff even though $E$ is a closed relation. In particular, this implies

that $q \times q : (X \times X) \to (X/E) \times (X/E)$ is not a quotient map since

$1_{X/E}$ not closed, because $X/E$ is not Hausdorff, but its pre-image is

the closed set $E$. 

The set \((Y \setminus J) \times \{e\}\) is mapped by \(q\) to a single point which we will call \(e^*\). Let \(G\) be a nonempty open subset of \(X/E\). Since \((Y \setminus J) \times \{e\}\) is not open in \(X\), it follows that the \(E\) saturated open set \(U = q^{-1}(G) \cap (J \times (Y \setminus \{e\}))\) is nonempty. The projection \(\pi_1 : J \times Y \to J\) is an open map and so the image \(\pi_1(U)\) is a nonempty open subset of \(J\). Since \(D = C \setminus J\) is dense in \(C\), it follows that the closure in \(C\) of \(\pi_1(U)\) meets \(D\). That is, there exists a sequence \(\{(j_n, y_n) \in U\}\) such that \(j_n \to d\) with \(d \in D\). Since \(U\) is \(E\) saturated, we can vary \(y_n\) arbitrarily in \(Y \setminus \{e\}\). Because \(g\) was topologically transitive, \(e\) is not an isolated point in \(Y\) and so we can choose \(y_n \in Y \setminus \{e\}\) converging to \(e\). It follows that \(\overline{U}\) contains the point \((d, e) \in (Y \setminus J) \times \{e\}\). Hence, \(e^* \in q(\overline{U}) \subset \overline{G}\).

It follows that every neighborhood of \(e^*\) is dense in \(X/E\).

Any Lyapunov function \(L\) for \(f\) is a Lyapunov function for \(\mathcal{G}f = E\) and so factors through \(q\) to yield a continuous real-valued function \(\tilde{L} : X/E \to \mathbb{R}\). If \(t \neq \tilde{L}(e^*)\) then we can choose disjoint open sets \(U_1, U_2 \subset \mathbb{R}\) with \(\tilde{L}(e^*) \in U_1, t \in U_2\). Thus, \(e^*\) is in the open set \((\tilde{L})^{-1}(U_1)\) which is disjoint from the open set \((\tilde{L})^{-1}(U_2)\). Since \((\tilde{L})^{-1}(U_1)\) is dense, \((\tilde{L})^{-1}(U_2)\) is empty. So \(t\) is not in the image of \(\tilde{L}\). Thus, \(L = \tilde{L} \circ q\) is constant at the value \(\tilde{L}(e^*)\).

Thus, the only Lyapunov functions for \(f\) are constant functions. It follows from Corollary 5.10 that \(1_{X \cup \mathcal{A}_{\mathcal{U}, M}}f = X \times X\). Since there are no isolated points in \(X\), \(X \times X \setminus 1_X\) is dense in \(X \times X\). Since \(\mathcal{A}_{\mathcal{U}, M}f\) is a closed relation, it follows that \(\mathcal{A}_{\mathcal{U}, M}f = X \times X\). On the other hand, \(E = \mathcal{G}f\) is a proper subset of \(X \times X\).

While \(X_0\) is a \(G_\delta\) subset of the compact metric space \(Y \times Y\), \(X\) is not. We do not know of an example of a closed equivalence relation on a Polish space with a non-Hausdorff quotient.

(b), (c), (d), (e): Let \(\omega\) and \(\Omega\) denote the first countable and first uncountable ordinal respectively. In particular, \(\omega\) is the set of non-negative integers. The ordered set \(\mathbb{R}_+ = \omega \times [0, 1]\) with the lexicographical ordering is order-isomorphic with the half-open interval \([0, \infty)\) by \((n, t) \mapsto n + t\). With the order topology this bijection is a homeomorphism. The ordered set \(L = \Omega \times [0, 1]\) with the lexicographical ordering can be similarly equipped with the order topology to obtain the Long Line. It is a non-paracompact, locally compact space and for every \(\alpha \in \Omega\) the interval \(\{(0, 0), (\alpha, 0)\}\) is order-isomorphic and thus homeomorphic with the unit interval. We double each example. Let \(\mathbb{R} = \mathbb{R}_+ \times \{+, -\}\) with each \((n, 0, +)\) identified with \((n, 0, -)\). We identify \(\omega \subset \mathbb{R}\) by \(n \mapsto (n, 0, \pm)\). Let \(\tilde{L} = L \times \{+, -\}\) with each \((\alpha, 0, +)\) identified with \((\alpha, 0, -)\). We identify \(\Omega \subset \tilde{L}\) by \(\alpha \mapsto (\alpha, 0, \pm)\).
Let \( \omega^* = \omega + 1 = \omega \cup \{\omega\} \) and \( \Omega^* = \Omega + 1 = \Omega \cup \{\Omega\} \). These are the one-point compactifications of \( \omega \) and \( \Omega \), respectively. Similarly, let \( \tilde{\mathbb{R}}^* \) and \( \tilde{L}^* \) denote the one-point compactifications with points \( \omega \) and \( \Omega \) the respective points at infinity. The product \( \Omega^* \times \omega^* \) is compact and removing the point \((\Omega, \omega)\) we obtain the locally compact Tychonoff Plank \( T \), see [12] Example 4F. As described there, the Tychonoff Plank is not normal as the closed subsets \( \Omega \times \{\omega\} \) and \( \{\Omega\} \times \omega \) cannot be separated by open sets.

On the unit interval \([0, 1]\) let \( u_+(t) = \sqrt{t} \) and \( u_-(t) = t^2 \). Each is a homeomorphism with fixed points 0 and 1. Observe that \( u_+(t) > t \) and \( u_-(t) < t \) for all \( t \in (0, 1) \). Thus, for every \( t \in (0, 1) \) the bi-infinite orbit sequence \( \{(u_-)^n(t)\} \) converges to 0 and \( n \to \infty \) and to 1 as \( n \to -\infty \). Since \( u_+ = (u_-)^{-1} \) the reverse is true for the \( u_+ \) orbit sequences. On \( \tilde{\mathbb{R}} \) define the homeomorphism \( g \) by \( g(n, t, \pm) = (n, u_+(t), \pm) \) and on \( \tilde{L} \) define the homeomorphism \( G \) by \( G(\alpha, t, \pm) = (\alpha, u_+(t), \pm) \). Observe that \( \omega \subset \tilde{\mathbb{R}} \) is the set of fixed points of \( g \) and \( \Omega \subset \tilde{L} \) is the set of fixed points of \( G \). Notice that \( \mathcal{S}g = \tilde{\mathbb{R}} \times \mathbb{R} \) and \( \mathcal{S}G = \tilde{L} \times \tilde{L} \).

We use these to construct our remaining examples.

(b) Let \( X \) equal \( T \cup \tilde{L} \cup \tilde{\mathbb{R}} \) with \((\alpha, \omega) \in T \) identified with \( \alpha \in \tilde{L} \) for all \( \alpha \in \Omega \) and with \((\Omega, n) \in T \) identified with \( n \in \tilde{\mathbb{R}} \) for all \( n \in \omega \). Thus, \( X \) is a locally compact, non-paracompact, Hausdorff space. The homeomorphism \( f \) is the homeomorphism induced from \( 1_T \cup G \cup g \) via these identifications. Thus, \( T \) is the set of fixed points of \( f \). Clearly, \( \mathcal{S}f \) is the equivalence relation \( 1_T \cup (\tilde{L} \times \tilde{L}) \cup (\tilde{\mathbb{R}} \times \tilde{\mathbb{R}}) \). The quotient space \( X/\mathcal{S}f \) is the quotient space of the Tychonoff plank \( T \) with the two closed subsets \( \Omega \times \{\omega\} \) and \( \{\Omega\} \times \omega \) each smashed to a point. Since the closed sets cannot be separated in \( T \), the quotient space is not Hausdorff.

(c) Let \( C \subset [0, 1] \) be the Cantor Set and let \( A = \{a_1, a_2, \ldots\} \) with \( \{a_k\} \) a decreasing sequence in \( C \) which converges to 0. Let \( \tilde{C} \) be \( C \) with the topology obtained by including \( C \setminus A \) as an open set. The new topology is \( \{U_1 \cup (U_2 \setminus A) : U_1, U_2 \text{ open in } C\} \). Thus, if \( x \in C \) with \( x \neq 0 \) then a set is a neighborhood of \( x \) iff it contains a \( C \) open set \( U \) with \( x \in U \). A set is a neighborhood of 0 iff it contains \( U \setminus A \) with \( U \) a \( C \) open set such that \( 0 \in U \). Since the topology is finer than the original topology of \( C \), the space \( \tilde{C} \) is completely Hausdorff. Note that it has a countable base. However, it is not regular. The closure of any neighborhood of 0 meets \( A \) and so there is no closed neighborhood of 0 contained in the \( \tilde{C} \) open set \( C \setminus A \).

Observe that if \( E \) is a closed equivalence relation on a Tychonoff space \( X \) then the quotient \( X/E \) is \( T_1 \) and so is Hausdorff if it is regular.
If $X$ is a separable metric space, or, more generally, any Lindelöf space then the quotient is Lindelöf. Since a regular, Lindelöf space is normal (see [12] Lemma 4.1), it follows that if $E$ is a closed equivalence relation on a separable metric space $X$, then the quotient is Hausdorff and normal, and so completely regular, if it is regular.

Let $X_0 = C \times \tilde{\mathbb{R}}$ with $f_0 = 1_C \times g$ and let $p_0 : X_0 \to C$ be the first coordinate projection. Clearly, $\mathcal{G}f_0 = p_0^{-1} \circ p_0$. That is, $\mathcal{G}f_0$ is a closed equivalence relation with equivalence classes the fibers of $p_0$. $X_0$ is a locally compact, metrizable space.

Now let $Z_k = \{(n,t,\pm) \in \tilde{\mathbb{R}} : n < k\}$.

Let $X$ be the $G_\delta$ invariant subset $X_0 \setminus \bigcup_{k=1}^{\infty} \{a_k\} \times Z_k$ and let $p$ be the restriction of $f_0$ to $X$. Again $\mathcal{G}f = p^{-1} \circ p$ where $p$ is the restriction of $p_0$. Notice that $p^{-1}(A)$ is a closed subset of $X$. It easily follows that $p$ induces a homeomorphism of the quotient space $X/\mathcal{G}f$ onto $\hat{C}$. Thus, the quotient is not regular although it is completely Hausdorff.

Notice that since $C$ is totally disconnected, it follows that for any metric $d$ on $X$, $\mathcal{C}_d f = \mathcal{G}f$. Hence, $\mathcal{C}_U f = \mathcal{G}f$. Hence, for any uniformity $U$ compatible with the topology on $X$, the inclusions $\mathcal{G}f \subset \mathcal{A}_U f \subset \mathcal{A}_d f \subset \mathcal{C}_d f$ and $\mathcal{G}f \subset \mathcal{C}_U f \subset \mathcal{C}_d f$ imply that they are all equal. By Theorem 7.11 the quotient space is totally disconnected.

(d) We return to the Tychonoff Plank. Let $X$ equal $T \cup \tilde{L}$ with $(\alpha, \omega) \in T$ identified with $\alpha \in \tilde{L}$ for all $\alpha \in \Omega$. Again $X$ is a locally compact, non-paracompact, Hausdorff space. In addition, it is zero-dimensional but not strongly zero-dimensional since it is not normal.

The homeomorphism $f$ is the homeomorphism induced from $1_T \cup G$ via these identifications. Again $T$ is the set of fixed points of $f$. Clearly, $\mathcal{G}f$ is the equivalence relation $1_T \cup (\tilde{L} \times \tilde{L})$. The quotient space $X/\mathcal{G}f$ is the quotient space of the Tychonoff plank $T$ with the closed subset $\Omega \times \{\omega\}$ smashed to a point $e$. Because $X$ is locally compact, it is completely regular. It follows that the quotient space $X/\mathcal{G}f$ is completely Hausdorff. However, the point $e$ cannot be separated from the closed set $\{\Omega\} \times \omega$ and so the quotient is not regular.

Since $T$ is zero-dimensional we have that $\mathcal{G}f = \mathcal{C}_{U_T} f$. The quotient is totally disconnected but not zero-dimensional since it is not regular.

Notice that if we extend $f$ to the one-point compactification $X^*$ of $X$, by adjoining the point $(\Omega, \omega)$ we obtain a homeomorphism $f^* \mathcal{G}f^* = 1_T \cup (\tilde{L}^* \times \tilde{L}^*)$. The quotient space $X^*/\mathcal{G}f^*$ is a compact, Hausdorff space and the inclusion $X \to X^*$ induces a continuous bijection $X/\mathcal{G}f \to X^*/\mathcal{G}f^*$ which is not a homeomorphism because $\{\Omega\} \times \omega$ is not closed in $X^*/\mathcal{G}f^*$. 
(e) Let \( X = (\mathbb{R} \times \{0\}) \cup \mathbb{N} \times \{1/k : k \in \mathbb{N}\} \). Let \( f = g \times 1_{\{0\}} \cup 1_{\mathbb{N} \times \{1/k : k \in \mathbb{N}\}} \). Clearly, \( \mathcal{G}f = \mathcal{C}_{\mathcal{U}}f \) with quotient obtained by smashing \( \mathbb{R} \times \{0\} \) to a point \( e \). The point \( e \) does not have a countable neighborhood base. If \( \{U_n : n \in \mathbb{N}\} \) is a sequence of neighborhoods of \( \mathbb{R} \times \{0\} \) in \( X \) then for every \( n \in \mathbb{N} \) there exists \( k_n \in \mathbb{N} \) such that \( (n, 1/k_n) \in U_n \). The set \( \{(n, 1/k_n) : n \in \mathbb{N}\} \) is closed and disjoint from \( \mathbb{R} \times \{0\} \), but meets every \( U_n \).

For cases (b),(d) and (e) the relations \( \mathcal{G}f \) are usc. In general, if \( A, B \) are disjoint closed subsets of \( X \) then \( (A \times A) \cup 1_X \) and \( (A \times A) \cup (B \times B) \cup 1_X \) are closed, usc equivalence relations.

\[ \square \]

Recall that a relation \( f \) on \( X \) is surjective if \( \text{Dom}(f) = \text{Dom}(f^{-1}) = X \), i.e. \( f(X) = f^{-1}(X) = X \).

**Definition 7.17.** A relation \( f \) on a uniform space \( (X, \mathcal{U}) \) is called \( \mathcal{U} \) chain transitive when it is a surjective relation such that \( \mathcal{C}_{\mathcal{U}}f = X \times X \).

**Proposition 7.18.** Let \( f \) be a relation on a uniform space \( (X, \mathcal{U}) \).

(a) If \( f \) is \( \mathcal{U} \) chain transitive then \( f^{-1} \) is \( \mathcal{U} \) chain transitive.

(b) If \( f \) is a proper relation with \( \mathcal{C}_{\mathcal{U}}f = X \times X \) then \( f \) is a surjective relation.

(c) If \( f \) is a surjective relation then \( f \) is \( \mathcal{U} \) chain transitive iff for every \( d \in \Gamma(\mathcal{U}) \) \( M_d^f(x, y) = 0 \) for all \( x, y \in X \).

(d) If \( g \) is a surjective relation on a uniform space \( (Y, \mathcal{V}) \) and \( h : X \to Y \) is a uniformly continuous surjective map which maps \( f \) to \( g \), then \( g \) is \( \mathcal{V} \) chain transitive if \( f \) is \( \mathcal{U} \) chain transitive.

**Proof:** (a) The inverse of a surjective relation is clearly surjective and \( \mathcal{C}_{\mathcal{U}}(f^{-1}) = (\mathcal{C}_{\mathcal{U}}f)^{-1} \).

(b) By Proposition 6.10 and Proposition 6.11 \( \text{Dom}(f) = \text{Dom}(\mathcal{C}_{\mathcal{U}}f) = X \) and \( \text{Dom}(f^{-1}) = \text{Dom}(\mathcal{C}_{\mathcal{U}}f^{-1}) = X \).

(c) Since \( m_d^f \leq M_d^f \) it is clear that \( M_d^f(x, y) = 0 \) implies \( m_d^f(x, y) = 0 \).

So if for every \( d \in \Gamma(\mathcal{U}) \) \( M_d^f(x, y) = 0 \) for all \( x, y \in X \), then \( \mathcal{C}_{\mathcal{U}}f = X \times X \).

For the converse we cannot apply Proposition 6.8 because we are not assuming that \( f \) is usc. Given \( d \in \Gamma(\mathcal{U}) \), \( \epsilon > 0 \) and \( x, y \in X \) there exists \( z \in f(x) \) since \( f \) is surjective. Because \( (z, y) \in \mathcal{C}_{\mathcal{U}}f \) there exists \( [a, b] \in f^{\times n} \) with the \( zy \) chain-bound of \( [a, b] \) less than \( \epsilon \). Now define \( [a, b'] \in f^{\times n+1} \) with \( (x, z) = (a_1', b_1') \) and \( (a_i', b_i') = (a_{i-1}, b_{i-1}) \).
for $i = 2, \ldots, n + 1$. Since the $xy$ chain-bound of $[a, b]'$ equals the $zy$ chain-bound of $[a, b]$ and $x = a'_1$ it follows that $M^{xy}_d (a, y) < \epsilon$.

(d) $Y \times Y = (h \times h)(X \times X) = (h \times h)(\mathcal{C}_u f) \subset \mathcal{C}_v g$ by Proposition 5.5.

Definition 7.19. A relation $f$ on a uniform space $(X, \mathcal{U})$ is called $\mathcal{U}$ chain mixing when it is a surjective relation and for every $d \in \Gamma(\mathcal{U}), \epsilon > 0, x, y \in X$ there exists a positive integer $N$ so that for all $n \geq N$ there exists $[a, b] \in f^n$ with $a_1 = x$ and with the $xy$ chain-bound of $[a, b]$ with respect to $d$ less than $\epsilon$.

That is, for any $d, \epsilon$ and $x, y$ for sufficiently large $n$ there is a chain of length $n$ from $x$ to $y$ with initial position $x$.

Thus, $f$ is a $\mathcal{U}$ chain transitive relation iff $X \times X = \bigcup_{n=1}^{\infty} (V^d \circ f)^n$ for all $d \in \Gamma(\mathcal{U})$ and $\epsilon > 0$. The relation $f$ is chain mixing iff $X \times X = \bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} (V^d \circ f)^i$ for all $d \in \Gamma(\mathcal{U})$ and $\epsilon > 0$.

For a positive integer $k$ the $k$-cycle is the translation bijection $s(n) = n + 1$ on the cyclic group $\mathbb{Z}_k = \mathbb{Z}/k\mathbb{Z}$.

Theorem 7.20. Let $f$ be a $\mathcal{U}$ chain transitive relation on a uniform space $(X, \mathcal{U})$.

(a) The following conditions are equivalent
   (i) The relation $f$ is $\mathcal{U}$ chain mixing.
   (ii) The relation $f \times f$ on $(X \times X, \mathcal{U} \times \mathcal{U})$ is $\mathcal{U}$ chain mixing.
   (iii) The relation $f \times f$ on $(X \times X, \mathcal{U} \times \mathcal{U})$ is $\mathcal{U}$ chain transitive.
   (iv) There does not exist for any integer $k > 1$ a uniformly continuous surjection from $X$ to $\mathbb{Z}_k$ which maps $f$ to $s$.

(b) If $f$ is $\mathcal{U}$ chain mixing then $f^{-1}$ is $\mathcal{U}$ chain mixing.

(c) If for every positive integer $k$, the relation $f^k$ is $\mathcal{U}$ chain transitive, then $f$ is $\mathcal{U}$ chain mixing. Conversely, if $f$ is a uniformly continuous mapping which is $\mathcal{U}$ chain mixing, then for every positive integer $k$, the mapping $f^k$ is $\mathcal{U}$ chain mixing.

Proof: (a) (i) $\Rightarrow$ (ii): Easy to check.

(ii) $\Rightarrow$ (iii): A chain mixing relation is chain transitive.

If $h$ is uniformly continuous mapping $f$ onto a surjective relation $g$ then $h \times h$ maps $f \times f$ to $g \times g$ and $h$ maps $f^n$ to $g^n$. Observe that with $k > 1$ $s \times s$ on $\mathbb{Z}_k \times \mathbb{Z}_k$ is not chain transitive since it is the disjoint union of $k$ separate periodic orbits. Furthermore, $s^k = 1_{\mathbb{Z}_k}$ and so $s^k$ is not chain transitive. So Proposition 5.5 (d) implies (iii) $\Rightarrow$ (iv) and and if $f^k$ is chain transitive for all positive $k$ then (iv) holds.
We prove the contrapositive of (iv) ⇒ (i) following Exercise 8.22 of [1]. See also [15]. Assume \( f \) is \( U \) chain transitive but not \( U \) chain mixing. With \( d \in \Gamma(U) \) and \( \epsilon > 0 \) fixed we define for \( x, y \in X \) the set of positive integers \( N(x, y) \) by \( n \in N(x, y) \) iff there exists \([a, b] \subseteq f^{\times n} \) with \( a_1 = x \) and with the \( xy \) chain-bound of \([a, b] \) with respect to \( d \) less than \( \epsilon \). Since \( f \) is assumed to be \( U \) chain transitive, Proposition 7.18 (c) implies that \( N(x, y) \) is non-empty for every pair \( x, y \). With \( A, B \) nonempty subsets of \( \mathbb{N} \) we let \( A + B \) denote \( \{a + b : a \in A, b \in B\} \). By concatenating chains we observe that for \( x, y, z \in X \)

\[
(7.2) \quad N(x, y) + N(y, z) \subseteq N(x, z).
\]

In particular, \( N(x, x) \) is an additive sub-semigroup of \( \mathbb{N} \). Let \( k(x) \) be the greatest common divisor of the elements of \( N(x, x) \). We will need the following classic result.

**Lemma 7.21.** If \( A \) is a nonempty additive sub-semigroup of \( \mathbb{N} \) then there exists \( N \) such that \( nk \in A \) for all \( n \geq N \) where \( k \) is the greatest common divisor of \( A \).

**Proof:** \( A - A \) is a non-trivial additive subgroup of \( \mathbb{Z} \) and so equals \( k\mathbb{Z} \) where \( k \) is the smallest positive element of \( A - A \). Dividing through by \( k \) we may assume that that greatest common divisor is 1. So there exists \( m \in \mathbb{N} \) such that \( m, m + 1 \in A \). If \( n \geq m^2 \) then with \( 0 \leq r < m \) and \( q \geq m - 1, n = qm + r = (q - r)m + r(m + 1) \in A \).

\( \square \)

By assumption, we can choose \( d, \epsilon, x_0 \) and \( y_0 \) so that infinitely often \( i \nleq N(x_0, y_0) \). Since \( N(x_0, x_0) + N(x_0, y_0) \subseteq N(x_0, y_0) \) it cannot happen that eventually \( i \in N(x_0, x_0) \). That is, \( k(x_0) > 1 \). Observe that \( k(x) \) divides every element of \( N(x, y) + N(y, x) \subseteq N(x, x) \) and every element of \( N(x, y) + N(y, y) + N(y, x) \subseteq N(x, x) \). Consequently, \( k(x) \) divides every element of \( N(y, y) \) and so \( k(x)|k(y) \). Interchanging \( x \) and \( y \) we see that there is an integer \( k \geq 1 \) such that \( k(x) = k \) for all \( x \in X \). It then follows that all of the elements of \( N(x, y) \) are congruent mod \( k \) with congruence class inverse to to congruence class of the elements of \( N(y, x) \). If \( (x, y) \in f^p \) then \( p \in N(x, y) \) and so the elements of \( N(x, y) \) are congruent to \( p \) mod \( k \). Fix a base point \( x_0 \in X \). Map \( X \) to \( \mathbb{Z}_k \) by letting \( h(x) \) be the mod \( k \) congruence class of the elements of \( N(x_0, x) \). Observe that if \( (x, y) \in f \) then \( h(y) = h(x) + 1 = s(h(x)) \). Since \( f \) is surjective, \( h \) maps \( X \) onto \( \mathbb{Z}_k \) and maps \( f \) onto \( s \).

For uniform continuity, we prove that \( h \) is constant on \( V_\epsilon^d(x) \) for all \( x \).
Let \( y \in X \) with \( d(x, y) = \epsilon_1 < \epsilon \) and let \( \epsilon_2 = \epsilon - \epsilon_1 \). Since \( f \) is \( U \) chain transitive, there exists \([a, b] \subseteq f^{\times n} \) with \( a_1 = x_0 \) and \( x_0x \) chain-bound
with respect to $d$ less than $\epsilon_2$. Hence, $n \in N(x_0, x)$. Furthermore, the $x_0y$ chain-bound with respect to $d$ is less than $\epsilon$. Hence, $n \in N(x_0, y)$. Thus, $h(x) = h(y)$ is the congruence class of $n \mod k$.

(b) If $h : X \to \mathbb{Z}_k$ is a uniformly continuous surjection mapping $f^{-1}$ to $s$ then it maps $f$ to $s^{-1}$. The bijection $\text{inv} : t \mapsto -t$ maps $s^{-1}$ to $s$ and so $\text{inv} \circ h : X \to \mathbb{Z}_k$ is a uniformly continuous surjection mapping $f$ to $s$. It follows from (a) that if $f^{-1}$ is not $\mathcal{U}$ chain mixing then $f$ is not $\mathcal{U}$ chain mixing.

(c) We saw in the proof of (a) that if $f$ is not $\mathcal{U}$ chain mixing then, by (iv), there exists a positive integer such that $f^k$ is not chain transitive. Now assume that $f$ is a uniformly continuous map which is $\mathcal{U}$ chain mixing and that $k$ is a positive integer.

**Lemma 7.22.** If $f$ is a uniformly continuous map, then for every $d \in \Gamma(\mathcal{U}), \epsilon > 0$ and positive integer $k$, there exists $\bar{d} \in \Gamma(\mathcal{U}), \delta > 0$ such that $(V^d_\delta \circ f)^k \subset V^d_\epsilon \circ f^k$.

**Proof:** By induction on $k$. For $k = 1$ let $d_1 = d$ and $\delta = \epsilon$. Assume $d_1 \in \Gamma(\mathcal{U}), \delta_1 > 0$ such that $(V^d_{\delta_1} \circ f)^n \subset V^d_{\epsilon/2} \circ f^n$. By uniform continuity of $f^n$ there exists $d_2 \in \Gamma(\mathcal{U}), \delta_2 > 0$ such that $f^n \circ V^d_{\delta_2} \subset V^d_{\epsilon/2} \circ f^n$. If $\bar{d} = d_1 + d_2$ and $\bar{\delta} = \min(\delta_1, \delta_2)$, then

$$(V^d_\delta \circ f)^{n+1} \subset V^d_{\epsilon/2} \circ f^n \circ V^d_{\delta_2} \circ f \subset V^d_\epsilon \circ f^{n+1}.$$ 

$\square$

Given $d \in \Gamma(\mathcal{U}), \epsilon > 0$ and a positive integer $k$ choose $\bar{d}$ and $\delta$ as in Lemma 7.22. For $x, y \in X$ there exists $N$ so that $y \in (V^d_\delta \circ f)^n(x)$ for all $n \geq N$. Since $nk \geq N$,

$$y \in (V^d_\delta \circ f)^{nk}(x) \subset (V^d_\epsilon \circ f^k)^n(x).$$

Thus, $f^k$ is $\mathcal{U}$ chain mixing. $\square$

Assume that $T$ is a set of positive integers directed by divisibility, i.e. if $k_1, k_2 \in T$ then there exists $k_3 \in T$ with $k_1 | k_3$ and $k_2 | k_3$. If $k_1 | k_2$ we let $\pi : \mathbb{Z}_{k_2} \to \mathbb{Z}_{k_1}$ be the cyclic group surjection induced by the inclusion $k_2 \mathbb{Z} \subset k_1 \mathbb{Z}$. For the directed set $T$ we let $\mathbb{Z}_T = \{ t \in \prod_{k \in T} \mathbb{Z}_k : k_1 | k_2 \Rightarrow \pi(t(k_2)) = t(k_1) \}$. If $T$ is finite then $\mathbb{Z}_T$ is isomorphic to $\mathbb{Z}_k$ where $k$ is the maximum element of $T$. If $T$ is infinite, then $\mathbb{Z}_T$ is a compact monothetic group, i.e. if $1 \in \mathbb{Z}_T$ the unit element which projects to $1 \in \mathbb{Z}_k$ for all $k \in T$, then the cyclic group generated by $1$ is dense in $\mathbb{Z}_T$. We let $s_T$ be the translation by $1$ in $\mathbb{Z}_T$ which projects to $s_k$ on $\mathbb{Z}_k$ for all $k \in T$. When $T$ is infinite, the dynamical system consisting of
the homeomorphism $s_T$ on the compact space $\mathbb{Z}_T$ is called the *odometer* associated with $T$.

**Theorem 7.23.** Assume that $f$ is a $\mathcal{U}$ chain transitive relation on a uniform space $(X, \mathcal{U})$. Let $T$ be the set of positive integers $k$ such that there is a $\mathcal{U}$ uniformly continuous map $h_k : X \to \mathbb{Z}_k$ which maps $f$ to $s_k$.

(a) The set $T$ is directed by divisibility.
(b) If $T$ is infinite, then there exists a uniformly continuous map $h : X \to \mathbb{Z}_T$ with a dense image which maps $f$ to $s_T$.
(c) If $T$ is finite with maximum element $k$ and the uniformly continuous $h_k : X \to \mathbb{Z}_k$ maps $f$ to $s_k$ then for each $i \in \mathbb{Z}_k$, $X_i = (h_k)^{-1}(i)$ is an $f^k$ invariant subset. If, in addition, $f$ is a $\mathcal{U}$ uniformly continuous map then the restriction $f^k|X_i$ is $\mathcal{U}$ chain mixing for each $i \in \mathbb{Z}_k$.

**Proof:** Fix a base point $e \in X$. If $h_k(e) = p$ then by replacing $h_k$ by the composition $(s_k)^{-p} \circ h_k$ we can assume that $h_k(e) = 0$. For each $k \in T$ we will assume that $h_k(e) = 0$. Let $E_k = (h_k \times h_k)^{-1}(1_{\mathbb{Z}_k})$. Since $h_k$ is $\mathcal{U}$ uniformly continuous, $E_k \in \mathcal{U}$ and it is a clopen equivalence relation on $X$. If $(x, y) \in f$ and $(y, y_1) \in E_k$ then $h(y_1) = h(y) = s_k(h(x)) = h(x) + 1$. Thus, $h_k$ maps $E_k \circ f$ to $s_k$ and, since $h_k(e) = 0$, we see that

$$x \in (E_k \circ f)^n(e) \implies h(x) = n \in \mathbb{Z}_k.$$  

Because $f$ is assumed to be $\mathcal{U}$ chain transitive, every $x \in X$ lies in $(E_k \circ f)^n(e)$ for some $n \in \mathbb{Z}$.

(a) For $k_1, k_2 \in T$ let $E = E_{k_1} \cap E_{k_2}$, a clopen equivalence relation in $\mathcal{U}$. From (7.3) it follows that $(h_{k_1}, h_{k_2}) : X \to \mathbb{Z}_{k_1} \times \mathbb{Z}_{k_2}$ maps $E \circ f$ to the restriction of $s_{k_1} \times s_{k_2}$ on the cyclic subgroup generated by $(1, 1)$, which has order the least common multiple $k$ of $k_1$ and $k_2$. This restriction can be identified with $s_k$ on $\mathbb{Z}_k$. Thus, $k \in T$.

(b) If $k_1|k_2$ in $T$ and $E = E_{k_1} \cap E_{k_2}$ then for $x \in (E \circ f)^n(e)$, $h_{k_1}(x) = n \in \mathbb{Z}_{k_1}$ and $h_{k_2}(x) = n \in \mathbb{Z}_{k_2}$. Hence, with $\pi : \mathbb{Z}_{k_2} \to \mathbb{Z}_{k_1}$ the projection we see that $\pi(h_{k_2}(x)) = h_{k_1}(x)$. It follows that the product $h_T = \Pi_{k \in T} h_k$ maps $X$ to $\mathbb{Z}_T$ taking $f$ to $s_T$. Since each fact is uniformly continuous, the map $h_T$ is uniformly continuous. Since each $h_k$ is surjective, it follows that the image is dense in $\mathbb{Z}_T$.

Notice that from (7.3) it follows that the $h_k$’s and $h_T$ are uniquely determined by the condition that $e$ is mapped to 0.

(c) Let $k \in T$. If $(x, y) \in f^k$ then $h_k(x) = h_k(y)$. Since $f^k$ is a surjective relation, it follows that $X_i$ is $f^k$ invariant for each $i \in \mathbb{Z}_k$. 

Now assume that $f$ is a uniformly continuous map and that some $f^k|X_i$ is not $\mathcal{U}$ chain mixing. By changing the choice of base point and translating, we may assume that $i = 0$. We will show that $k$ is not the maximum element of $T$.

Since $f^k|X_0$ is not $\mathcal{U}$ chain mixing, there is an integer $p > 1$ and a uniformly continuous map $g_p : X_0 \to \mathbb{Z}_p$ taking $f^k|X_0$ to $s_p$. Label the congruence classes of $\mathbb{Z}_k$ by $i = 0, \ldots, k-1$, of $\mathbb{Z}_p$ by $j = 0, \ldots, p-1$ and of $\mathbb{Z}_{kp}$ by $kj + i$. Observe that if $x \in X_i$ then $f^{k-i}(x) \in X_0$. Define the map $H : X \to \mathbb{Z}_{kp}$ by

$$H(x) = kg_p(f^{k-i}(x)) + i \quad \text{if } x \in X_i,$$

We see that if $i < k - 1$ then $f(x) \in X_{i+1}$ and so $f^{k-(i+1)}(f(x)) = f^{k-i}(x)$. If $i = k - 1$ then $f(x) \in X_0$ and so $H(f(x)) = g_p(f^k(f(x))) = g_p(f(x)) + 1$ provided $g_p(f(x)) < p - 1$ and $= 0$ if $g_p(f(x)) = p - 1$. Hence,

$$H(f(x)) = \begin{cases} 
kg_p(f^{k-i}(x)) + i + 1 & \text{if } i < k - 1, \\
k(g_p(f(x)) + 1) + 0 & \text{if } i = k - 1, g_p(f(x)) < p - 1, \\
0 & \text{if } i = k - 1, g_p(f(x)) = p - 1.
\end{cases}$$

It is clear that $H$ is $\mathcal{U}$ uniformly continuous since $h_k, g_p$ and $f$ are. From (7.5) we see that $H$ maps $f$ to $s_{pk}$. Hence, $pk \in T$ and so $k$ is not the maximum element.

\[\blacksquare\]

**Remark:** Without compactness of $X$ the map $h_T$ in (b) need not be surjective. For example, let $X$ be the dense cyclic subgroup generated by $1_T$ in $\mathbb{Z}_T$, or, more generally, any proper, $s_T$ invariant subset of an odometer $\mathbb{Z}_T$ which includes 0. With the uniformity induced from $\mathbb{Z}_T$ the homeomorphism $s_T$ is a uniform isomorphism of $X$. Choose $e = 0$. Since every orbit of $s_T$ is dense, $s_T$ is $\mathcal{U}$ chain transitive on $X$. For every $k \in T$, the projection map $\mathbb{Z}_T \to \mathbb{Z}_k$ maps $s_T$ to $s_k$ and is surjective on $X$. But $h_T : X \to \mathbb{Z}_T$ is just the inclusion.

**Corollary 7.24.** Let $f$ be a surjective relation on a connected uniform space $(X, \mathcal{U})$.

The following conditions are equivalent.

(i) The relation $f$ is $\mathcal{U}$ chain mixing.

(ii) The relation $f$ is $\mathcal{U}$ chain transitive.
(iii) The relation $f$ is $\mathcal{U}$ chain recurrent, i.e. $\mathcal{C}_U f$ is an equivalence relation.

(iv) The relation $\mathcal{C}_U f$ is reflexive, i.e. $1_X \subset \mathcal{C}_U f$.

**Proof:** It is obvious that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

Since a connected space does not admit a continuous surjection onto a nontrivial finite set, (ii) $\Rightarrow$ (i) by Theorem 7.20 (a).

As in the proof of Corollary 7.12 $X/\mathcal{C}_U 1_X$ is totally disconnected, but as the continuous image of the connected space $X$ it is connected and so the quotient is a singleton. Hence, $\mathcal{C}_U 1_X = X \times X$, i.e. the identity map is $\mathcal{U}$ chain transitive. So if $1_X \subset \mathcal{C}_U f$ then $X \times X = \mathcal{C}_U 1_X \subset \mathcal{C}_U \mathcal{C}_U f = \mathcal{C}_U f$. Thus, (iv) $\Rightarrow$ (ii).

$\blacksquare$

8. The Mañé Set in the Compact, Metrizable Case

Throughout this section $X$ is a compact metrizable space. A compact space is metrizable iff it is Hausdorff and second countable. In that case, every continuous metric $d$ on $X$ is an element of $\Gamma(U)$ where $\mathcal{U}$ is the unique uniformity which consists of all neighborhoods of the diagonal. In particular, $\mathcal{U} = \mathcal{U}(d)$ for each such metric. Thus, for a compact metrizable space with unique uniformity $\mathcal{U}$, $\Gamma_m(\mathcal{U}) = \Gamma_m(X)$.

If $E$ is a closed equivalence relation on $X$ then the quotient $X/E$ is a compact metrizable space by Proposition 7.15. If the quotient is totally disconnected then it is strongly zero-dimensional by Proposition 7.9.

We let $\mathcal{E}_f$ denote $\mathcal{C}_U f$ where $\mathcal{U}$ is the unique uniformity. By Theorem 5.14 $\mathcal{C}_f = \mathcal{C}_d f$ for every $d \in \Gamma_m(X)$ and $\mathcal{G} f = \bigcap_{d \in \Gamma_m(X)} A_d f$. On the other hand, the union is not obviously closed or transitive. We prove that it is both using an idea from [16].

For $V$ a neighborhood of the diagonal $1_X \subset X \times X$ and a pair $x, y \in X$, $[a, b] \in f^{\times n}$ defines an $xy, V$ chain if $(x, a_1), (b_n, y)$ and $(b_i, a_{i+1})$ are in $V$ for $i = 1, \ldots, n - 1$. We will call $n$ the length of the chain.

**Definition 8.1.** Let $W f$ denote the set of pairs $(x, y) \in X \times X$ such that for every neighborhood $W$ of $1_X$ there exists a closed, symmetric neighborhood $V$ of $1_X$ and $n \in \mathbb{N}$ such that there is an $xy, V$ chain of length $n$ and $V^{3n} \subset W$. 
**Theorem 8.2.** For a relation \( f \) on a compact, metrizable space \( X \), the relation \( \mathcal{W}f \) is a closed, transitive relation and \( \mathcal{W}f = \bigcup_{d \in \Gamma_m(X)} A_{df} \).

**Proof:** We will prove that \( \bigcup_d A_{df} \subset \mathcal{W}f \) and \( \mathcal{W}f \cup (\mathcal{W}f)^2 \subset \bigcup_d A_{df} \).

Let \((x, y) \in A_{df}\) for some metric \(d\) on \(X\) and let \(W\) be a neighborhood of the diagonal. Choose \( \epsilon > 0 \) so that \( V_{3\epsilon}^d \subset W \). Since \((x, y) \in A_{df}\) there exists \([a, b] \in f^n\) for some \(n \in \mathbb{N}\) so that with respect to \(d\) the \(xy\) chain-length of \([a, b]\) is less than \(\epsilon\). Write \(b_0 = x\) and \(a_{n+1} = y\). Define \(\epsilon_i = d(b_i, a_{i+1})\) for \(i = 0, \ldots, n\). Thus, \(\sum_i \epsilon_i < \epsilon\).

Define

\[
V = V_{\epsilon/3^{n+1}}^d \cup \bigcup_{i=0}^{n} V_{\epsilon_i}^d(b_i) \times V_{\epsilon_i}^d(b_i).
\]

Clearly, \([a, b]\) defines an \(xy, V\).

We show that if \((w, z) \in V^{3^{n+1}}\) then \((w, z) \in V_{3\epsilon}^d\). There exists a sequence \(w = u_0, u_1, \ldots, u_N = z\) with \((u_i, u_{i+1}) \in V\) for \(i = 0, \ldots, N - 1\) and with \(N \leq 3^{n+1}\). Choose the sequence so that \(N\) is minimal. If \(u_j, u_{j+k} \in V_{\epsilon}^d(b_i)\) with \(k > 0\) then \(k = 1\) for otherwise we could eliminate the terms \(u_{j+1}, \ldots, u_{j+k-1}\) and obtain a sequence with \(N\) smaller. Thus, for each \(i\) there is at most one \(j\) such that \(u_j, u_{j+1} \in V_{\epsilon}^d(b_i)\). For the remaining \(j\)'s, \((u_j, u_{j+1}) \in V_{\epsilon/3^{n+1}}^d\). It follows that

\[
\Sigma_{i=0}^{N-1} d(u_j, u_{j+1}) \leq N \cdot (\epsilon/3^{n+1}) + 2\Sigma_i \epsilon_i \leq 3\epsilon.
\]

By the triangle inequality \(d(w, z) = d(u_0, u_N) \leq 3\epsilon\).

It follows that \(\bigcup_d A_{df} \subset \mathcal{W}f\).

Now assume that \((x, y) \in (\mathcal{W}f)^2\). We will use the Metrization Lemma for uniform spaces, \[12\] Lemma 6.12, to construct a metric \(d\) such that \((x, y) \in A_{df}\). We will then indicate how to adjust the proof to obtain the required metric when \((x, y) \in (\mathcal{W}f)^2\).

Fix some metric \(d_0\) on \(X\) which is bounded by 1.

Let \(U_0 = A_0 = X \times X = V_1^{d_0}\) and \(M_0 = 0\). Assume that, inductively, the closed symmetric neighborhood of the diagonal (= csn) \(U_{M_k} = A_k \subset V_2^{d_k}\) has been constructed. There exists \((x_k, y_k) \in \mathcal{W}f\) such that \((x, x_k), (y_k, y) \in A_k\). Hence, there exists \(n_k \in \mathbb{N}\) and a csn \(B_k\) such that there is an \(x_ky_kB_k\) chain length \(n_k\) and \((B_k)^{3^{n_k}} \subset A_k \cap V_2^{d_k} = V_2^{d_{k+1}}\).

We now interpolate powers of \(B_k\) between \(B_k = A_{k+1}\) and \(A_k\).

For \(i = 1, \ldots, n_k + 1\) let \(U_{M_k+i} = (B_k)^{3^{n_k+i-1}}\). Let \(M_{k+1} = M_k + n_k + 1\) and \(A_{k+1} = U_{M_{k+1}}\).

Thus, \([U_j]\) is a sequence of csn’s with \((U_j)_{j+1}^3 \subset U_j\) and \(B_j = A_{j+1} = U_{M_{j+1}}\) for \(j \geq 0\).
From the Metrization Lemma we obtain a metric $d$ such that $U_j \subset V_{2^{-j}}^{d} \subset U_{j-1}$ for all $j \in \mathbb{N}$.

It follows that with respect to $d$ the $x_k y_k$ length of the $B_k$ chain is bounded by

$$(n_k + 1)2^{-M_k+1} = (n_k + 1)2^{-(M_k+n_k+1)} \leq 2^{-M_k}$$

and since $(x, x_k), (y_k, y) \in A_k$ we have that the $xy$ length is bounded by $3 \cdot 2^{-M_k}$. Since $M_k \geq k$ it follows that $(x, y) \in \mathcal{A}_d f$.

If $(x, y) \in (\mathcal{W}f)^2$ then there exist $x_k, z_k, (z_k, y_k) \in \mathcal{W}f$ with $(x, x_k), (y_k, y) \in A_k$. We begin with a $n_k \in \mathbb{N}$ and a csn $B_k$ such that there is a $x_k z_k, B_k$ chain of size $n_k$ and $(B_k)^{3 n_k} \subset A_k \cap V_{2^{-k}}^{d}$. Then choose an $m_k \in \mathbb{N}$ and a csn $C_k$ such that there is a $z_k y_k, C_k$ chain of size $m_k$ and $(C_k)^{3 m_k} \subset B_k$.

This time for $i = 1, \ldots, n_k + 1$ let $U_{M_k+i} = (B_k)^{3 n_k+1-i}$ and $j = 1, \ldots, m_k + 1$ let $U_{M_k+n_k+1+j} = (C_k)^{3 m_k+1-j}$. Let $M_k+1 = M_k + n_k + m_k + 2$ and $A_{k+1} = U_{M_k+1} = C_k$. Estimate as before to get that the $xy$ length of the $B_k$ chain followed by the $C_k$ chain (with $z_k$ omitted between them) is at most $4 \cdot 2^{-M_k}$. Again $(x, y) \in \mathcal{A}_d f$.

Following Fathi and Pageault [9], we call $|\mathcal{W}f|$ the *Mañé set*. For every $d \in \Gamma_m(X)$ on the compact metrizable space $X$ we have

$$9f \subset \mathcal{A}_d f \subset \mathcal{W}f \subset \mathcal{C}f.$$  

Using Theorem 8.2 we follow [17] to prove the following extension of a theorem of Fathi and Pageault, see [9].

**Theorem 8.3.** Let $f$ be a continuous map on a compact, metrizable space $X$ such that $f^{-1}(|f|) = |f|$ and let $K = X \setminus |f|^\circ$. $\mathcal{W}f = 1_{|f|} \cup \mathcal{C}(f|K)$. Hence, $|\mathcal{W}f| = |f| \cup |\mathcal{C}(f|K)|$.

**Proof:** From $f^{-1}(|f|) = |f|$ it follows that $K$ is $f^+$ invariant. Because $f$ is a map, $f = 1_{|f|} \cup (f|K)$.

For any metric $d$, equation (3.20) implies that

$$\mathcal{A}_d f = \mathcal{A}_d (1_{|f|} \cup (f|K)) = 1_{|f|} \cup \mathcal{A}_d (f|K)$$

and so

$$\mathcal{W}f = \mathcal{W}(1_{|f|} \cup (f|K)) = 1_{|f|} \cup \mathcal{W}(f|K) \subset 1_{|f|} \cup \mathcal{C}(f|K).$$

To complete the proof we assume that $(x, y) \in \mathcal{C}(f|K)$ and show that $(x, y) \in \mathcal{W}f$. Fix a metric $d$ on $X$ and let $W$ be an arbitrary neighborhood of $1_X$. Choose $\epsilon > 0$ so that $V_{\epsilon}^{d} \subset W$. Let $\delta > 0$ be such that $\delta < \epsilon/2$ and $d(x, y) < \delta$ implies $d(f(x), f(y)) < \epsilon/2$. Choose
\[ [a, b] \in (f/K)^\times n \] of minimum size \( n \) such that the \( xy \) chain-bound is less than \( \delta \). We may perturb so that \( a_i \not\in [f] \) for \( i = 1, \ldots, n \) and so \( b_i = f(a_i) \not\in [f] \) since \( |f| = f^{-1}([f]) \) by assumption. Let \( b_0 = x \) and \( a_{n+1} = y \). If \( 1 \leq i < j \leq n + 1 \) then \( a_i \neq a_j, b_{i-1} \neq a_i \) and \( b_{i-1} \neq b_{j-1} \) for if not we could shorten the chain by removing the pairs \((a_k, b_k)\) for \( k = i, \ldots, j - 1 \) contradicting the minimality of \( n \). Now if \( a_i = b_{j-1} \) then \( j > i + 1 \) since \( a_j \not\in [f] \). Let \( i' \) be the smallest index such that \( a_{i'} = b_{j-1} \) for some \( j > i' + 1 \) and let \( j' \) be the largest such \( j \) for \( i' \). Eliminate the pairs \((a_k, b_k)\) for \( k = i' + 1, \ldots, j' \). Observe that

\[
d(b_{i'}, a_{j'+1}) \leq d(b_{i'}, b_{j'}) + d(b_{j'}, a_{j'+1}) = d(f(b_{j'-1}), f(a_{j'})) + d(b_{j'}, a_{j'+1}) \leq \epsilon. \tag{8.2}
\]

Moving right we may have to do several of these truncations, which do not overlap, and so eventually, we obtain \([a, b] \in f^{\times n}' \) with \([a_i, b_{i-1}] \cap [a_j, b_{j-1}] = \emptyset \) if \( i \neq j \).

Choose \( 0 < \delta_C < \epsilon \) small enough that the sets \( C_i = V_{\delta_C}^d ([a_i, b_{i-1}]) \) are pairwise disjoint for \( i = 1, \ldots, n' + 1 \). Let \( \epsilon > \epsilon_0 > 0 \) be smaller than the distance between \( C_i \) and \( C_j \) if \( i \neq j \). Let \( V = V_{\epsilon_0/3^n}^d \cup \bigcup_{i=1}^{n+1} C_i \times C_i \).

Clearly, \([a, b] \) defines an \( xy \) \( V \) chain.

If \( z_1, \ldots, z_M \) satisfies \( (z_i, z_{i+1}) \in V \) and \( M \leq 3^n \) then since \( \epsilon_0 \) is smaller than the distance between the \( C_i \)'s at most one pair \( \{z_k, z_{k+1}\} \) lies in some \( C_i \). Hence,

\[
d(z_1, z_M) \leq \sum_{k=1}^{M-1} d(z_k, z_{k+1}) \leq 3^n \cdot (\epsilon_0/3^n) + 2 \max \text{diam} C_i \leq \epsilon_0 + \epsilon + 2\delta_C \leq 4\epsilon. \tag{8.3}
\]

Hence, \( (z_1, z_M) \in W \).

\[ \square \]

The following extension of Corollary 5.6 is easy to check.

**Proposition 8.4.** If \( f \) is a Lipschitz map on \((X, d)\), then

\[
\mathcal{A}_d(f^n) \subset \mathcal{A}_d f = f^{[1, n]} \cup ([\mathcal{A}_d(f^n)]) \circ f^{[0, n]},
\]

and so \( |\mathcal{A}_d(f^n)| = |\mathcal{A}_d f| \).

\[ \square \]

For a continuous map \( f \) on \((X, d)\) let \( \text{Per}(f) \) denote the set of periodic points, so that \( \text{Per}(f) = \bigcup_{n=1}^{\infty} |f^n| \). Let \( \text{Per}(f)^\infty = \bigcup_{n=1}^{\infty} |f^n|^\circ \).

**Lemma 8.5.** The open set \( \text{Per}(f)^\infty \) is dense in \( \text{Per}(f)^\circ \), the interior of the set of periodic points.
Proof: Each $|f^n|$ is closed in $X$. Let $U$ be a nonempty open subset of $\text{Per}(f)$, and so by the Baire Category Theorem at least one of these has a nonempty interior.

While, $\text{Per}(f)^{\circ}$ is contained in the interior of $\text{Per}(f)$, but might be a proper subset of it. By periodicity each $|f^n|$ is $f$-invariant, i.e. $f(|f^n|) = |f^n|$. So if $f$ is a homeomorphism each $|f^n|^o$ is invariant as well. Thus, if $f$ is a homeomorphism, $\text{Per}(f)^{\circ}$ is an open invariant set and its complement in $X$ is a closed invariant set. Notice also that if $A$ is any closed subset of $X$ which is $f^+$-invariant then it is $f^n$-invariant and $(f|A)^n = (f^n)|A$.

In the Lipschitz case we can extend the above results.

**Corollary 8.6.** Let $f$ be a homeomorphism on $(X,d)$. If $f$ is a Lipschitz map then

\[ (8.5) \quad |A_d| \subset \text{Per}(f) \cup |\mathcal{C}(f)(X \setminus \text{Per}(f)^{\circ})|. \]

**Proof:** Let $X_n = X \setminus |f^n|^o$. By Proposition 8.4 and Proposition 8.2 we have

\[ (8.6) \quad |A_d| = |A_d(f^n)| \subset |f^n| \cup |\mathcal{C}((f|X_n)^n)| \subset \text{Per}(f) \cup |\mathcal{C}(f|X_n)|. \]

Now $\{X_n\}$ is a decreasing sequence of closed invariant sets with intersection $X_\infty = X \setminus \text{Per}(f)^{\circ}$. Hence, $\{f|X_n = f \cap (X_n \times X_n)\}$ is a decreasing sequence of closed relations with intersection $f|X_\infty$. By [1] Theorem 7.23, the map $R \mapsto |\mathcal{C}R|$ is a monotone, usc function on closed sets and so

\[ (8.7) \quad \bigcap_n |\mathcal{C}(f|X_n)| = |\mathcal{C}(\bigcap_n (f|X_n))| = |\mathcal{C}(f|X_\infty)|. \]

Together with (8.6) this implies (8.5).

\[ \square \]

**Example 8.7.** Without the Lipschitz assumption the result is not true.

**Proof:** On $I = [0,1]$ let $\mu$ be a full, nonatomic probability measure concentrated on a dense countable union of Cantor sets of Lebesgue measure zero. Let $\pi : I \to I$ be the distribution function so that $\pi(t) = \mu([0,t])$. Then $\pi$ is a homeomorphism on $I$ fixing the end-points. Let $X_0 = I \times \{-1,0,1\}$ with the metric $d_0((s,a),(t,b)) = |s-t| + |a-b|$. Let $\tilde{\pi} : X_0 \to X_0$ be the homeomorphism defined by $\tilde{\pi}(t,-1) = (\pi(t),-1)$ and $\tilde{\pi}(t,a) = (t,a)$ for $a = 0,1$. Let $d$ be the metric $d_0$ pulled back
by $\tilde{\pi}$. Thus, if $s < t \in I$ then $d((s, a), (t, a)) = t - s$ if $a = 0, 1$ and
$\mu([s, t])$ if $a = -1$. Let $L = \{0\} \times \{-1, 0, 1\}, R = \{1\} \times \{-1, 0, 1\}.$
Let $E = 1_{X0} \cup (L \times L) \cup (R \times R)$ and use the metric $d = \ell_d^E$ on
the quotient space $X = X_0/E$ with quotient map $q : X_0 \to X$. For $a = -1, 0, 1$ let $I_a$
denote $q(I \times \{a\}).$ It is easy to check that each restriction $q : I \times \{a\} \to I_a$
is an isometry.

Now define the homeomorphism $f$ on $X$ by

\begin{equation}
(8.8) \quad f(t, a) = \begin{cases}
(t, -a) & \text{for } a = \pm 1 \\
(t^2, a) & \text{for } a = 0.
\end{cases}
\end{equation}

Let $Y$ be the subspace of $X$ which is the quotient of $I \times \{0, 1\} \subset X_0$,
i.e. $Y = I_1 \cup I_0 \subset X$, and define $g$ on $Y$ and $h : X \to Y$ by

\begin{equation}
(8.9) \quad g(t, a) = \begin{cases}
(t, a) & \text{for } a = 1 \\
(t^2, a) & \text{for } a = 0
\end{cases}
\end{equation}

\begin{equation}
(8.10) \quad h(t, a) = \begin{cases}
(t, 1) & \text{for } a = \pm 1 \\
(t, a) & \text{for } a = 0
\end{cases}
\end{equation}

Neither $f$ nor $h$ is Lipschitz. Because $g$ is the identity on $I_1$ and $f^2$ is the identity on
$I_1 \cup I_{-1}$ it follows from Proposition 3.6 that

\begin{equation}
\mathcal{A}_d(f^2) = 1_{(I_1 \cup I_{-1})} \cup \mathcal{A}_d((f|I_0)^2)
\end{equation}

and

\begin{equation}
\mathcal{A}_d g = 1_{I_1} \cup \mathcal{A}_d(f|I_0).
\end{equation}

For $f|I_0 = g|I_0$, $L(t, 0) = 1 - t$ is a Lipschitz Lyapunov function which is increasing on all
orbits except the -fixed- endpoints. It follows that $|\mathcal{C}(f|I_0)| = |\mathcal{C}((f|I_0)^2)| = \{q(0, 0), q(1, 0)\}.$

We will show that

\begin{equation}
(8.11) \quad \mathcal{A}_d f = X \times X
\end{equation}

Thus, for $0 < t < 1$ the point $(t, 0) \in |\mathcal{A}_d f|$ but is not in $|\mathcal{A}_d f^2|$ and
$(t, 0) = h(t, 0)$ is not in $|\mathcal{A}_d g|.$

Let $s < t$ in $I$. Because $\mu$ and Lebesgue measure $\lambda$ are mutually singular we can choose for any $\epsilon > 0$ an increasing sequence $s = u_1, \ldots, u_{2n+1} = t$ so that $\mu(\bigcup_{i=1}^n [u_{2i-1}, u_{2i}]) < \epsilon$ but $\lambda(\bigcup_{i=1}^n [u_{2i-1}, u_{2i}]) \geq 1 - \epsilon$ and so $\lambda(\bigcup_{i=1}^n [u_{2i}, u_{2i+1}] < \epsilon$. On $I_1$ the length of an interval is its
Lebesgue measure while on $I_{-1}$ the length is its $\mu$ measure. Thus, if $x = (s, 1)$ and $y = (t, 1)$ then

\begin{equation}
(8.12) \quad (u_1, 1), (u_2, -1), (u_3, 1), (u_4, -1), \ldots, (u_{2n}, -1)
\end{equation}

each paired with its image under $f$, defines a sequence in $f^{x2n}$ whose $xy$ chain-length is less than $2\epsilon$. Since $f$ is symmetric on $I_1 \cup I_{-1}$ we can reverse the sequence to get one whose $yx$ chain-length is the same.
Thus, any two elements of \( I_1 \cup L_1 \) are \( A_d f \) equivalent. On the other hand, it is easy to check that for any \( t \in (0, 1) \), \( q(t, 0) \in \mathcal{G}(f|I_0)(q(1, 0)) \) and \( q(0, 0) \in \mathcal{G}(f|I_0)(t, 0) \). It follows that any two elements of \( X \) are \( A_d f \) equivalent.

On the invariant set \( X_{\pm 1} =_{\text{def}} I_1 \cup L_1 \) the restriction of \( f \) has order 2 and so \( E = 1_{X_{\pm 1}} \cup f|X_{\pm 1} \) is a closed equivalence relation with each equivalence class having one or two points. However, the pseudo-metric \( X(8.13) \) in (8.5) the inclusion may fail if \( \text{Example 8.8.} \)

In (8.5) the inclusion may fail if \( \text{Per}(f)^{\infty} \) is replaced by \( \text{Per}(f)^{\circ} \) and it may fail if \( |\mathcal{G}(f)(X \setminus \text{Per}(f)^{\infty})| \) is replaced by \( |A_d(f)(X \setminus \text{Per}(f)^{\infty})| \).

**Proof:** Let \( S \) be the unit circle in the complex plane. Let \n
\begin{equation}
X =([-1, 1] \cup S) \times \{0\} \cup \left(\bigcup_{n=1}^{\infty} S \times \{1/n\}\right).
\end{equation}

equipped the restriction of the Euclidean metric from \( \mathbb{R}^3 \).

On \( X \) define the restriction of the Euclidean metric from \( \mathbb{R}^3 \) by \n
\begin{equation}
(8.14) f(x, t) = \begin{cases} 
(x \cdot e^{2\pi i t}, t) & \text{for } x \in S, \\
(\frac{1}{2}(x^2 + 2x - 1), 0) & \text{for } x \in [-1, 1], t = 0.
\end{cases}
\end{equation}

That is, on \([-1, 1] \times \{0\}\) the map is conjugate to \( x \mapsto x^2 \) via the homeomorphism \((x, 0) \mapsto (x + 1)/2\) from \([-1, 1] \times \{0\}\) to \([0, 1]\). \( \text{Per}(f) = S \times \{(0, 1, 1/2, \ldots)\} \) and so \( Y =_{\text{def}} X \setminus \text{Per}(f)^{\circ} \) is \([-1, 1] \times \{0\}\). For the restriction of \( f \) to this set, the only chain recurrent points are the endpoints, i.e. \( |\mathcal{G}_d(f|Y)| = \{(0, -1, 0), (1, 0)\}\).

\( X_\infty = X \setminus \text{Per}(f)^{\infty} = (S \cup [-1, 1]) \times \{0\} \). For the restriction of \( f \) to \( X_\infty \) every point is chain recurrent, i.e. \( |\mathcal{G}_d(f|X_\infty)| = X_\infty \), but from Proposition 3.6

\begin{equation}
(8.15) S \times \{0\} = |(f|X_\infty)| = |\mathcal{G}(f|X_\infty)| = |A_d(f|X_\infty)|
\end{equation}

Finally, it is easy to check that for \( f \) itself \n
\begin{equation}
(8.16) X = |\mathcal{G}f| = |A_d f|.
\end{equation}

Thus, in (8.5) the equation fails if \( \text{Per}(f)^{\infty} \) is replaced by \( \text{Per}(f)^{\circ} \) and it fails if \( |\mathcal{G}(f|(X \setminus \text{Per}(f)^{\infty}))| \) is replaced by \( |A_d(f|(X \setminus \text{Per}(f)^{\infty}))| \).

\( \Box \)
We review the theory of nets, following [12, Chapter 2].

A set $I$ is directed by a reflexive, transitive relation $\prec$ if for every $i_1, i_2 \in I$ there exists $j \in I$ such that $i_1, i_2 \prec j$. We call $I$ a directed set. If $I_1, I_2$ are directed sets then the product $I_1 \times I_2$ is directed by the product ordering $(i_1, j_1) \prec (i_2, j_2)$ when $i_1 \prec i_2$ and $j_1 \prec j_2$.

For $i \in I$ let $\prec_i = \{ j : i \prec j \}$. A set $F \subset I$ is called terminal if $F \supset \prec_i$ for some $i \in I$. $F$ is called cofinal if $F \cap \prec_i \neq \emptyset$ for all $i \in I$. In the family language of [2] these are dual families of subsets of $I$. Because the set $I$ is directed by $\prec$ it follows that the family of terminal sets is a filter. That is, a finite intersection of terminal sets is terminal. The cofinal sets satisfy the dual, Ramsey Property: If a finite union of subsets of $I$ is cofinal then at least one of them is cofinal.

For example, if $A \subset X$ then the set $N_A$ of neighborhoods of $A$ is directed by $\supset$ and a subset of $N_A$ is cofinal iff it is a neighborhood base. If $A$ is the singleton $\{ x \}$, then we write $N_x$ for $N_A$. The sets $\mathbb{Z}_+$ and $\mathbb{N}$ are directed by $\leq$ and a subset is terminal iff it is cofinite. A subset is cofinal iff it is infinite.

A net in a set $Q$ is a function from a directed set $I$ to $Q$, denoted $\{ x_i : i \in I \}$. If $A \subset Q$ we say that the net is eventually (or frequently) in $A$ if $\{ i : x_i \in A \}$ is terminal (resp. cofinal).

A map $k : I' \to I$ between directed sets is a directed set morphism if $k^{-1}(F)$ is terminal in $I'$ whenever $F$ is terminal in $I$. If $k$ is order-preserving, i.e. $i_1' \prec i_2'$ implies $k(i_1') \prec k(i_2')$, and, in addition, the image, $k(I')$, is cofinal in $I$ then $k$ is a morphism.

A map $k : I' \to I$ is a morphism iff whenever $F$ is cofinal in $I'$, then $k(F)$ is cofinal in $I$. This follows because

$$k(F) \cap A \neq \emptyset \iff F \cap k^{-1}(A) \neq \emptyset$$

and a set is cofinal iff it meets every terminal set and vice-versa.

With this definition of morphism, the class of directed sets becomes a category.

If $i \mapsto x_i$ is a net, then the composite $i' \mapsto x_{k(i')}$ is the subnet induced by the morphism $k$. We will usually suppress the mention of $k$ and just write $\{ x_{i'} : i' \in I' \}$ for the subnet.

If $x$ is a point of a topological space $X$ then a net in $X$ converges to $x$ (or has $x$ as a cluster point) if for every $U \in N_x$ the net is eventually in $U$ (resp. is frequently in $U$). Thus, if a net in $A$ has $x$ as a cluster point then $x$ is in the closure of $A$. Conversely, if $x \in \overline{A}$ then we can use $I = N_x$ and choose $x_U \in A \cap U$. We thus obtain a net in
A converging to $x$. For a net $\{x_i : i \in I\}$ in $X$ the set of cluster points is $\bigcap_{i \in I} \{x_j : i \prec j\}$. Equivalently, this is the set of limit points of convergent subnets of $\{x_i\}$.

**Lemma 9.1.** If $\{x_i : i \in I\}$ is a net in $X$ and $A$ is a compact subset of $X$, then $A$ contains a cluster point of the net iff $x_i \in U$ frequently for every open set containing $A$.

**Proof:** Clearly, if $x \in A$ is a cluster point of the net, then it frequently enters every neighborhood of $x$ and a fortiori it frequently enters every neighborhood of $A$. If for some $i$ the set $K_i = \{x_j : i \prec j\}$ is disjoint from $A$ then its complement is an open set containing $A$ which the net does not enter frequently. So if the net frequently enters every neighborhood of $A$ then $\{K_i \cap A : i \in I\}$ is a collection of closed subsets of $A$ with the finite intersection property. Hence, the intersection is nonempty by compactness.

$\square$

10. Appendix B: Uniform Spaces

We review from [12] Chapter 6 the facts we will need about uniform spaces.

A uniformity $\mathcal{U}$ on a set $X$ is a filter of reflexive relations on $X$ which satisfies

- $U \in \mathcal{U}$ implies $U^{-1} \in \mathcal{U}$.
- If $U \in \mathcal{U}$, then there exists $W \in \mathcal{U}$ such that $W \circ W \subset U$.

We say that a collection $\mathcal{U}_0$ of reflexive relations generates a uniformity when $\mathcal{U} = \{U : U \supset V \text{ for some } V \in \mathcal{U}_0\}$ is a uniformity. This requires that if $V_1, V_2 \in \mathcal{U}_0$, there exists $V_3 \in \mathcal{U}_0$ so that $V_3 \circ V_3 \subset V_1 \cap V_2^{-1}$.

For example, if $d$ is a pseudo-metric on $X$ then $V^d_\epsilon = \{(x, y) : d(x, y) \leq \epsilon\}$ with $\epsilon > 0$ generates a uniformity $\mathcal{U}(d)$ which we call the uniformity associated with $d$.

The *gage* $\Gamma$ of a uniformity $\mathcal{U}$ (or $\Gamma(\mathcal{U})$ when we need to keep track of the uniformity) is the set of all bounded pseudo-metrics $d$ such that $V^d_\epsilon \in \mathcal{U}$ for all $\epsilon > 0$, or, equivalently, $\mathcal{U}(d) \subset \mathcal{U}$. From the Metrization Lemma for uniformities, Lemma 6.12 of [12], it follows that if $U \in \mathcal{U}$ then there exists $d \in \Gamma$ such that $V^d_1 \subset U$.

A collection $\Gamma_0$ of pseudo-metrics generates a uniformity when $\bigcup_{d \in \Gamma_0} \mathcal{U}(d)$ is a uniformity. It suffices that if $d_1, d_2 \in \Gamma_0$, there exists $d_3 \in \Gamma_0$ such that $d_3 \leq K(d_1 + d_2)$ for some positive $K$. 
Since $\mathcal{U}$ is a filter, it is directed by $\supset$. If $d_1, d_2 \in \Gamma$ then $d_1 + d_2 \in \Gamma$, and so $\Gamma$ is directed by $\leq$.

**Lemma 10.1.** Let $\{d_1, d_2, \ldots\}$ be a sequence in $\Gamma$ with $d_i$ bounded by $K_i \geq 1$. If $\{a_1, a_2, \ldots\}$ is a summable sequence of positive reals then $d = \sum_{k=1}^\infty (a_i/K_i)d_i$ is a pseudo-metric in $\Gamma$.

**Proof:** Dividing by $\sum_{k=1}^\infty (a_i/K_i)$ we can assume the sum is 1. Given $\epsilon > 0$ choose $N$ so that $\sum_{k=N+1}^\infty (a_i/K_i) < \epsilon/2$. Then $\bigcap_{k=1}^N V_{\epsilon/2}^{d_k} \subset V_\epsilon^d$. \qed

Associated to a uniformity $\mathcal{U}$ is the $\mathcal{U}$ topology with $G$ open iff $x \in G$ implies $U(x) \subset G$ for some $U \in \mathcal{U}$. The topology is Hausdorff iff $1_X = \bigcap \{U : U \in \mathcal{U}\}$, in which case we call $\mathcal{U}$ a Hausdorff uniformity. If $X$ is a topological space then $\mathcal{U}$ is called compatible with the topology on $X$ if $X$ has the $\mathcal{U}$ topology.

If $(X, \mathcal{U}), (Y, \mathcal{V})$ are uniform spaces then the product uniformity $\mathcal{U} \times \mathcal{V}$ on $X \times Y$ is generated by the product relations $U \times V$ for $U \in \mathcal{U}, V \in \mathcal{V}$. Given pseudo-metrics in $\Gamma(\mathcal{U})$ and $\Gamma(\mathcal{V})$ the product pseudo-metrics on $X \times Y$ generate the gage $\Gamma(\mathcal{U} \times \mathcal{V})$. The associated topology is the product of the $\mathcal{U}$ topology on $X$ with the $\mathcal{V}$ topology on $Y$.

If $A \subset X$ then $\mathcal{U}|A$, the set of restrictions to $A$ of the relations $U \in \mathcal{U}$, is the induced uniformity on $A$ with associated topology the subspace topology. The restrictions to $A \times A$ of the pseudo-metrics in $\Gamma(\mathcal{U})$ generate the gage $\Gamma(\mathcal{U}|A)$.

Observe that if $E$ is an equivalence relation which contains the diagonal $1_X$ in its interior then every equivalence class is a neighborhood of each of its points and so is open. It follows that $E = \bigcup_{x \in X} \{E(x) \times E(x)\}$ is open in $X \times X$ and its complement $\bigcup_{(x,y) \notin E} \{E(x) \times E(y)\}$ is open as well. Thus, $E$ is a clopen equivalence relation. For a clopen equivalence relation $E$ on $X$, the characteristic function of $X \times X \setminus E$ is a continuous pseudo-ultrametric on $X$.

We call a uniformity $\mathcal{U}$ zero-dimensional when it is generated by equivalence relations. Equivalently, the gage is generated by pseudo-ultrametrics. In that case, the associated topology is zero-dimensional, i.e. the clopen subsets form a basis for the topology. Conversely, if $X$ is a zero-dimensional space then the set of all clopen equivalence relations on $X$ generates the maximum zero-dimensional uniformity compatible with the topology on $X$. We denote it by $\mathcal{U}_M$. The gage $\Gamma(\mathcal{U}_M)$ is generated by the pseudo-ultrametrics which are continuous on $X$. The class of zero-dimensional uniform spaces is closed under the operations of products and taking subspaces.
Proposition 10.2. Let $X$ be a topological space. The following conditions are equivalent.

(a) There exists a uniformity compatible with the topology on $X$.

(b) The topology on $X$ is completely regular. That is, the continuous real-valued functions distinguish points and closed sets.

If $X$ is Hausdorff, then these are equivalent to

(c) There exists a homeomorphism onto a subset of a compact Hausdorff space.

Proof: (a) $\Leftrightarrow$ (b) If $x$ is not in a closed set $A$ then there is a $d \in \Gamma$ such that $V_\epsilon^d(x) \cap A = \emptyset$ for some $\epsilon > 0$. The continuous function $y \mapsto \min(d(x, y), 1)$ is 0 at $x$ and 1 on $A$. If $X$ is completely regular then the uniformity generated by the pseudo-metrics $d_u(x, y) = |u(x) - u(y)|$, with $u$ varying over continuous real-valued functions, is compatible with the topology.

(b) $\Leftrightarrow$ (c) Using bounded real-valued continuous functions we can embed a Hausdorff, completely regular space into a product of intervals. On the other hand, by the Urysohn Lemma a compact Hausdorff space is completely regular and so any subspace is completely regular as well. $\square$

A completely regular, Hausdorff space is called a Tychonoff space. Clearly, a completely regular space $X$ is Tychonoff iff the points are closed, i.e. iff $X$ is $T_1$.

If there is a metric in the gage then the $\mathcal{U}$ topology is Hausdorff, but the gage of a Hausdorff uniformity need not contain a metric.

A map $h : X_1 \to X_2$ between uniform spaces is uniformly continuous if $U \in \mathcal{U}_2$ implies $(h \times h)^{-1}(U) \in \mathcal{U}_1$, or, equivalently, if $h^*d \in \Gamma(\mathcal{U}_1)$ for all $d \in \Gamma(\mathcal{U}_2)$ where $h^*d(x, y) = d(h(x), h(y))$. A pseudo-metric $d$ on $X$ is in the gage of $\mathcal{U}$ iff $1_X : (X, \mathcal{U}) \to (X, \mathcal{U}(d))$ is uniformly continuous. With the uniformity induced by the usual metric on $\mathbb{R}$, a pseudo-metric $d$ on $X$ is in the gage of $\mathcal{U}$ iff the map $d : (X \times X, \mathcal{U} \times \mathcal{U}) \to \mathbb{R}$ is uniformly continuous.

In general, there may be many uniformities with the same associated topology. Given a completely regular space there is a maximum uniformity $\mathcal{U}_M$ compatible with the topology. It is characterized by the condition that any continuous map from $X$ to a uniform space is uniformly continuous with respect to $\mathcal{U}_M$. If $X$ is paracompact then the set of all neighborhoods of the diagonal is a uniformity which is therefore $\mathcal{U}_M$. If $X$ is compact, then this is the unique uniformity compatible with the topology on $X$. 


A uniformity $\mathcal{U}$ on $X$ is **totally bounded** if for every $V \in \mathcal{U}$ the cover $\{V(x) : x \in X\}$ has a finite subcover, or, equivalently, if for every $d \in \Gamma(\mathcal{U})$ the pseudo-metric space $(X, d)$ is totally bounded. Let $\mathcal{B}(X, \mathcal{U})$ denote the Banach algebra of bounded, uniformly continuous, real-valued functions. If $u \in \mathcal{B}(X, \mathcal{U})$ then the pseudo-metric $d_u$ defined by

$$d_u(x, y) = |u(x) - u(y)|. \tag{10.1}$$

is a totally bounded pseudo-metric in $\Gamma(\mathcal{U})$. For $\mathcal{B} \subset \mathcal{B}(X, \mathcal{U})$ a closed subalgebra (assumed to contain the constant functions) the pseudo-metrics $d_F = \sum_{u \in F} d_u$, with $F$ a finite subset of $\mathcal{B}$, generate a totally bounded uniformity $\mathcal{F}(\mathcal{B}) \subset \mathcal{U}$. If $\mathcal{B}$ is separable then the uniformity $\mathcal{F}(\mathcal{B})$ is pseudo-metrizable. In fact, if $\{u_i\}$ is a dense sequence in the unit ball of $\mathcal{B}$ then

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i} d_{u_i} \tag{10.2}$$

is a metric such that $\mathcal{U}(d) = \mathcal{F}(\mathcal{B})$.

Recall that if $u \in \mathcal{B}$ then we can use the series expansion of the square root to show that $|u| = \sqrt{u^2} \in \mathcal{B}$. Hence, if $u_1, u_2 \in \mathcal{B}$ then $\max(u_1, u_2) = \frac{1}{2}(|u_1 - u_2| + u_1 + u_2)$ and $\min(u_1, u_2) = -\max(-u_1, -u_2)$ are in $\mathcal{B}$.

The subalgebra $\mathcal{B}$ **distinguishes points and closed sets** when for every closed subset $A$ of $X$ and any $x \in X \setminus A$ there exists $u \in \mathcal{B}$ such that $u(x) \notin \overline{u(A)}$. Notice that if $|u(x) - t| < \epsilon$ implies $t \notin u(A)$, then $v(z) = \frac{1}{\epsilon} \min(|u(x) - u(z)|, \epsilon)$ is an element of $\mathcal{B}$ with $v(x) = 0$ and $v = 1$ on $A$. In that case, the topology associated with $\mathcal{F}(\mathcal{B})$ is that of $(X, \mathcal{U})$, i.e. $\mathcal{F}(\mathcal{B})$ is compatible with the topology of $X$. The uniformity $\mathcal{F}(\mathcal{B}(X, \mathcal{U}))$ is the maximum totally bounded uniformity contained in $\mathcal{U}$ and we will denote it $\mathcal{F}(\mathcal{U})$. The gage of $\mathcal{F}(\mathcal{U})$ consists of all the totally bounded pseudo-metrics in the gage of $\mathcal{U}$.

If $\{x_i : i \in D\}$ and $\{y_j : j \in I\}$ are nets in $X$, then they are $\mathcal{U}$-asymptotic for a uniformity $\mathcal{U}$ on $X$ if the product net $\{(x_i, y_j) : (i, j) \in D \times I\}$ is eventually in $U$ for all $U \in \mathcal{U}$. The net $\{x_i\}$ converges to $x \in X$ exactly when it is $\mathcal{U}$-asymptotic to a net constant at $x$. The $\mathcal{U}$-asymptotic relation on nets on $X$ is symmetric and transitive, but not reflexive. A net $\{x_i\}$ is **Cauchy** when it is $\mathcal{U}$-asymptotic to itself. The uniform space $(X, \mathcal{U})$ is **complete** when every Cauchy net converges. For a Hausdorff uniform space $(X, \mathcal{U})$ there exists $j$ a uniform isomorphism from $(X, \mathcal{U})$ onto a dense subset of a complete, Hausdorff uniform space $(\bar{X}, \mathcal{U})$. Regarding $j$ as an inclusion, we call $(\bar{X}, \mathcal{U})$ the **completion** of $(X, \mathcal{U})$. We can regard $\bar{X}$ as the space of the $\mathcal{U}$-asymptotic equivalence classes of Cauchy nets in $X$. In general,
if \((Y, \mathcal{V})\) is a complete, Hausdorff uniform space and \(h : A \to Y\) is a uniformly continuous map on a subset \(A\) of \(X\) then \(h\) extends uniquely to \(\overline{h} : \overline{A} \to Y\) a uniformly continuous map on the closure. If \(x \in \overline{A}\) and \(\{x_i\}\) is a net in \(A\) converging to \(x\) then \(\{h(x_i)\}\) is a Cauchy net in \(Y\) and so converges to a unique point \(h(x)\). It follows that the completion of a Hausdorff uniform space is unique up to uniform isomorphism. For each \(d \in \Gamma(\mathcal{U})\), the map \(\tilde{d} : \overline{X} \times \overline{X} \to [0, M]\), where \(M = \sup d\) is a pseudo-metric on \(\overline{X}\) and these form the gage of \(\mathcal{U}\). If \(d\) is a metric with \(\mathcal{U} = \mathcal{U}(d)\) then \(\tilde{d}\) is a metric with \(\tilde{\mathcal{U}} = \mathcal{U}(\tilde{d})\). That is, the completion of a metric space is a metric space.

A uniform space is compact iff it is totally bounded and complete. So the completion of a totally bounded, Hausdorff uniform space is compact. In particular, if \((X, \mathcal{U})\) is Hausdorff and \(\mathcal{B}\) is a closed subalgebra of \(\mathcal{B}(X, \mathcal{U})\) which distinguishes points and closed sets then the completion \((\overline{X}, \overline{\mathcal{T}}(\mathcal{B}))\) is a compact Hausdorff space. If \(Y\) is a compact, Hausdorff space (with its unique uniformity) and \(h : (X, \mathcal{U}) \to Y\) is uniformly continuous then \(h : (X, \mathcal{T}(\mathcal{U})) \to Y\) is uniformly continuous and so extends uniquely to \(\overline{h} : (\overline{X}, \overline{\mathcal{T}}(\mathcal{U})) \to Y\). If \(\mathcal{B}\) is closed subalgebra of \(\mathcal{B}(X, \mathcal{U})\) which distinguishes points and closed sets, then with \(\overline{X}\) the \(\overline{\mathcal{T}}(\mathcal{B})\) completion of \(X\), the map \(u \mapsto \overline{u}\) is a Banach algebra isomorphism from \(\mathcal{B}\) onto the Banach algebra of continuous, real-valued maps on \(\overline{X}\). Thus, \(\overline{X}\) is version the compactification of \(X\) obtained from \(\mathcal{B}\) by the Gelfand space construction, see, e.g. [2] Chapter 5. In particular, if \(X\) is a Tychonoff space with \(\mathcal{U}_M\) the maximum uniformity compatible with the topology then \((X, \overline{\mathcal{T}}(\mathcal{U}_M))\) is a version of the Stone-Cech compactification of \(X\).

Finally, notice that \((X, \mathcal{U})\) has a second countable topology iff there exists a separable, closed subalgebra \(\mathcal{B}\) of \(\mathcal{B}(X, \mathcal{U})\) which distinguishes points and closed sets. In that case, there is a metric \(d\) such that \(\overline{\mathcal{T}}(\mathcal{B}) = \mathcal{U}(d)\) and the associated compactification \(\overline{X}\) is metrizable with metric \(\tilde{d}\).

11. Appendix C: Proper Maps

A proper map \(f : X \to Y\) is a continuous map such that \(f \times 1_Z : X \times Z \to Y \times Z\) is a closed map for every topological space \(Z\). Using a singleton for \(Z\) we see that a proper map is closed. We collect the elementary properties of proper maps from [5] Section 1.10.
**Proposition 11.1.**  
(a) If $f : X \to Y$ is injective, then it is proper iff it is a homeomorphism onto a closed subset of $Y$.  
(b) Assume that $f : X \to Y$ and $g : Y \to Z$ are continuous.  
(i) If $f$ and $g$ are proper, then $g \circ f$ is proper.  
(ii) If $g \circ f$ is proper and $f$ is surjective, then $g$ is proper.  
(iii) If $g \circ f$ is proper and $g$ is injective, then $f$ is proper.  
(c) If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are continuous maps with $X_1$ and $X_2$ nonempty, then $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is proper iff both $f_1$ and $f_2$ are proper.  
(d) Let $f : X \to Y$ be a proper map. If $A$ is a closed subset of $X$ then the restriction $f|A : A \to Y$ is a proper map.  
(e) If $B$ is an arbitrary subset of $Y$ then the restriction $f|B : f^{-1}(B) \to B$ is a proper map.  
(f) If $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ are proper maps with $X$ Hausdorff then the map $x \mapsto (f_1(x), f_2(x))$ is proper. In particular, its image is closed.  

**Proof:** These results are Propositions 2-5 of [5] Section 1.10.1.  
(a) An injective continuous map is a homeomorphism onto a closed subset iff it is a closed map.  
(b) If $A \subset X \times Z$ then $[(g \circ f) \times 1_Z](A) = (g \times 1_Z) \circ (f \times 1_Z)(A)$ and if $g$ is injective, $(f \times 1_Z)(A) = (g \times 1_Z)^{-1}((g \circ f) \times 1_Z)(A))$. If $f$ is surjective and $B \subset Y \times Z$ then $(g \times 1_Z)(B) = (g \circ f) \times 1_Z((f \times 1_Z)^{-1}(B)).$  
(c) $f_1 \times f_2$ is the composition $(f_1 \times 1_{Y_2}) \circ (1_{X_1} \times f_2).$  
(d) If $K$ is a closed subset of $A \times Z$ then $K$ is a closed subset of $X \times Z.$  
(e) If $A \subset f^{-1}(B) \times Z$ is closed relative to $f^{-1}(B) \times Z$ then there exists $A_1$ closed in $X \times Z$ with $A = A_1 \cap (f^{-1}(B) \times Z)$ and $(f \times 1_Z)(A) = (f \times 1_Z)(A_1) \cap B \times Z.$  
(f) Since $X$ is Hausdorff, the diagonal $1_X$ is closed in $X \times X$ and so the map $\Delta : X \to X \times X : x \mapsto (x, x)$ is a proper map from $X$ to $X \times X$ by (a). Since $f_1 \times f_2$ is proper by (c), the composition $(f_1 \times f_2) \circ \Delta$ is proper.  

The condition that $f$ be proper can be described in terms of compactness. For convenience we restrict attention to Tychonoff spaces, i.e. completely regular Hausdorff spaces.  

**Proposition 11.2.**  
(a) Assume that $f : X \to Y$ is continuous with $X$ a Tychonoff space. The following are equivalent.  
(i) The map $f$ is proper.
(ii) \( f \times 1_Z : X \times Z \to Y \times Z \) is a closed map for every compact Hausdorff space \( Z \).

(iii) The map \( f \) is closed and \( f^{-1}(y) \) is compact for every \( y \in Y \).

(iv) Whenever \( \{x_i : i \in I\} \) is a net in \( X \) such that \( \{f(x_i)\} \) converges to a point \( y \in Y \) then \( \{x_i\} \) has a cluster point in \( f^{-1}(y) \).

(b) If \( p \) is a singleton space and \( X \) is a Tychonoff space, then the map \( p : X \to p \) is proper iff \( X \) is compact.

(c) If \( f : X \to Y \) is proper with \( X, Y \) Tychonoff spaces and \( B \subseteq Y \) is compact, then \( f^{-1}(B) \subseteq X \) is compact.

Proof: These results are essentially Theorem 1 and Lemma 1 of [5] Section 1.10.2.

(a) (i) \( \Rightarrow \) (ii) Obvious.

Let \( Z \) be a compactification of \( X \), i.e. there is a continuous embedding \( k : X \to Z \) with \( Z \) a compact Hausdorff space. Because \( Z \) is Hausdorff, the map \( k \) is a closed subset of \( X \times Z \). The map \( p \times 1_Z : X \times Z \to p \times Z \) is isomorphic to the projection \( \pi_2 : X \times Z \to Z \). If the map \( p \times 1_Z \) is closed, then \( k(X) = \pi_2(k) \) is a closed subset of \( Z \) and so is compact. Since \( k \) is an embedding \( X \) is compact. In particular, this proves one direction of (b).

(ii) \( \Rightarrow \) (iii) Using \( Z \) as a singleton we see that \( f \) is closed. As in Proposition 11.1(e) we see that \( f \times 1_Z : f^{-1}(y) \times Z \to y \times Z \) is closed for any compact Hausdorff space. From the above argument it follows that \( f^{-1}(y) \) is compact.

(iii) \( \Rightarrow \) (iv) If for some \( i \in I \) the set \( A_i = \{x_j : i \prec j\} \) is disjoint from \( f^{-1}(y) \) then \( f(A_i) \) is a closed set disjoint from \( y \) and so \( \{f(x_i)\} \) does not converge to \( y \). Hence, \( \{A_i \cap f^{-1}(y)\} \) is a collection of closed sets satisfying the finite intersection property. Since \( f^{-1}(y) \) is compact, the intersection is nonempty and the intersection is the set of cluster points of \( \{x_i\} \) in \( f^{-1}(y) \).

(iv) \( \Rightarrow \) (i) Let \( A \) be a closed subset of \( X \times Z \) and \( (y, z) \) a point of the closure of \( (f \times 1_Z)(A) \). There exists a net \( \{(x_i, z_i)\} \) in \( A \) such that \( \{(f(x_i), z_i)\} \) converges to \( (y, z) \). From (iv) it follows that there exists \( x \in f^{-1}(y) \) and a subnet \( \{x_\nu\} \) which converges to \( x \). Hence, the subnet \( \{(x_\nu, z_\nu)\} \) converges to \( (x, z) \) and since \( A \) is closed \( (x, z) \in A \). So \( (y, z) = (f \times 1_Z)(x, z) \in (f \times 1_Z)(A) \). Thus \( (f \times 1_Z)(A) \) is closed.

(b) If \( X \) is compact, then \( X \to p \) satisfies condition (iii) of (a) and so is a proper map.

(c) Since \( B \) is compact, \( B \to p \) is proper. Since \( f \) is proper, the restriction \( f^{-1}(B) \to B \) is proper. Hence, the composition \( f^{-1}(B) \to p \) is proper and so \( f^{-1}(B) \) is compact.
A Hausdorff space $X$ is called a $k$-space when the topology is compactly generated. That is, $A \cap K$ compact for every compact subset $K$ of $X$ implies $A$ is closed. A locally compact space is clearly a $k$-space. Since a convergent sequence together with its limit is compact, any Hausdorff sequential space is a $k$-space, where $X$ is sequential when $x \in \overline{A}$ implies $x$ is the limit of a sequence in $A$. So any Hausdorff, first countable space is a $k$-space. In particular, a metrizable space is a $k$-space.

**Proposition 11.3.** Let $f : X \to Y$ be a continuous map with $X$ and $Y$ Tychonoff spaces.

(a) If $Y$ is a $k$-space and for every compact $B \subset Y$, the pre-image $f^{-1}(B)$ is compact, then $f$ is a proper map.

(b) If $X$ is a $k$-space and $A \subset X$ such that the restriction $f|A : A \to Y$ is proper then $A$ is a closed subset of $X$.

**Proof:** (a) From Proposition 11.2 (a)(iii) it suffices to show that $f$ is closed. Let $A \subset X$ be closed and let $K \subset Y$ be compact. By hypothesis, $f^{-1}(K)$ is compact and so $A \cap f^{-1}(K)$ is compact. It follows that $f(A) \cap K = f(A \cap f^{-1}(K))$ is compact. As $K$ was arbitrary, $f(A)$ is closed because $Y$ is a $k$-space.

(b) Let $K \subset X$ be compact so that $f(K) \subset Y$ is compact. By Proposition 11.2 (c) applied to $f|A$, $(f|A)^{-1}(f(K))$ is compact. Hence, $K \cap A = K \cap (f|A)^{-1}(f(K))$ is compact. Since $K$ was arbitrary and $X$ is a $k$-space, $A$ is closed.

**References**

1. E. Akin, *The general topology of dynamical systems*, Graduate Studies in Mathematics, 1, American Mathematical Society, Providence, RI, 1993.
2. E. Akin, *Recurrence in topological dynamical systems: Furstenberg families and Ellis actions*, Plenum Press, New York, 1997.
3. E. Akin and J. Auslander, *Compactifications of dynamical systems*, ArXiv 1004.0323v1.
4. E. Akin and J. Auslander, *Generalized recurrence, compactifications and the Lyapunov topology*, Studia Mathematica, (2010) 201:49-63.
5. N. Bourbaki, *Elements of Mathematics, General Topology, Chapters 1-4*, Springer-Verlag, Berlin, 1989.
6. R. H. Bing, *A connected, countable Hausdorff space*, Proc. AMS, (1953) 4: 474.
7. C. Conley, *Isolated invariant sets and the Morse index*, CBMS Regional Conference Series in Mathematics, 38, American Mathematical Society, Providence, RI, 1978.
8. R. Easton, *Chain transitivity and the domain of influence of an invariant set*, The structure of attractors in dynamical systems, Proc. Conf. North Dakota State University, 1978, 95-102.
9. A. Fathi and P. Pageault, *Aubry-Mather theory for homeomorphisms*, Ergod. Theo. & Dyn. Sys., (2015) 35: 1187-1207.
10. L. Gillman and M. Jerison, *Rings of Continuous Functions*, D.Van Nostrand Company, Princeton, 1960.
11. M. Hurley, *Noncompact chain recurrence and attraction*, Proc. AMS, (1992) 115: 1139-1148.
12. J. L. Kelley, *General Topology*, D.Van Nostrand Company, Princeton, 1955.
13. L. Nachbin *Topology and Order*, D. Van Nostrand Company, Princeton, 1965.
14. P. Pageault, *Conley barriers and their applications: chain recurrence and Lyapunov functions*, Topology and its Applications, (2009) 156: 2426-2442.
15. D. Richeson and J. Wiseman, *Chain recurrence rates and topological entropy*, Topology and its Applications, (2008) 156: 251-261.
16. J. Wiseman, *The generalized recurrent set and strong chain recurrence*, Ergod. Theo. & Dyn. Sys., (2016), to appear.
17. J. Wiseman, *Generalized recurrence and the nonwandering set for products*, Topology and its Applications, (2017), to appear.
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