Uniqueness of $\sigma$-regular solutions of quasilinear elliptic problems

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Abstract

We study the uniqueness problem of $\sigma$-regular solution of the equation,

$$-\Delta_p u + |u|^{q-1} u = h \text{ on } \mathbb{R}^N,$$

where $q > p - 1 > 0$ and $N > p$. Other coercive type equations associated to more general differential operators are also investigated. Our uniqueness results hold for equations associated to the mean curvature type operators as well as for more general quasilinear subelliptic operators.

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1 Introduction

Nonlinear elliptic problems of coercive type is still a subject of vital interest in the PDE circles. As it is well known, coercive problems have they roots in the classical calculus of variations and precisely in the problems related to the existence of minima for convex functional.

In a celebrated paper [3], Boccardo, Galluet and Vazquez studied, among other things, the simplest canonical quasilinear problem with non regular data,

\[-\Delta_p u + |u|^{q-1} u = h \quad \text{on} \quad \mathbb{R}^N,\] (1)

where \( q > p - 1 > 0 \) and \( h \in L^1_{\text{loc}}(\mathbb{R}^N) \).

An earlier and important contribution to this problem in the case \( p = 2 \), was obtained by Brezis [5]. Indeed he proved that for any \( h \in L^1_{\text{loc}}(\mathbb{R}^N) \) the semilinear equation (1) has a unique distributional solution \( u \in D'(\mathbb{R}^N) \cap L^q_{\text{loc}}(\mathbb{R}^N) \).

For the general case \( p > 1 \) existence results have been obtained later in [3]. These authors, by using a clever approximation procedure, proved that if \( q > p - 1 \) and \( p > 2 - \frac{1}{N} \), then for any \( h \in L^1_{\text{loc}}(\mathbb{R}^N) \) the equation (1) possesses a solution belonging to the space \( X = W^{1,1}_{\text{loc}}(\mathbb{R}^N) \cap W^{1,p-1}_{\text{loc}}(\mathbb{R}^N) \cap L^q_{\text{loc}}(\mathbb{R}^N) \). No general results about uniqueness were claimed in that paper.

In this work, we shall study the uniqueness problem of solutions of (1) and related qualitative properties. We emphasize that we shall prove the uniqueness of solutions of (1) in the space \( W^{1,p}_{\text{loc}}(\mathbb{R}^N) \cap L^q_{\text{loc}}(\mathbb{R}^N) \). To this end, first we set up two essential tools which are of independent interest. Namely, the regularity of weak solutions of (1) in the space \( W^{1,p}_{\text{loc}}(\mathbb{R}^N) \cap L^q_{\text{loc}}(\mathbb{R}^N) \) and comparison results on \( \mathbb{R}^N \) for related inequalities. Further we shall derive some property of the solutions of (1).

Our approach works also when dealing with more general operators and related inequalities. In this paper for sake of simplicity, we shall limit ourselves to coercive problems in the Carnot groups framework. Clearly this setting includes as special case the Euclidean framework.

The approach we propose in this paper can be successfully applied even when the differential operator is not the \( p \)-Laplacian operator. Indeed the same uniqueness problem can be studied for equations associated to the mean curvature operator as well as extensions of it.

The main results proved in this paper are the following.

**Theorem 1.1** Let \( 1 < p < 2 \), \( q > 1 \), \( h \in L^1_{\text{loc}}(\mathbb{R}^N) \), then the problem

\[-\Delta_p u + |u|^{q-1} u = h \quad \text{on} \quad \mathbb{R}^N,\]
has at most one weak solution \( v \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \cap L^q_{\text{loc}}(\mathbb{R}^N) \). Moreover,

\[
\inf_{\mathbb{R}^N} h \leq |v|^{q-1} v \leq \sup_{\mathbb{R}^N} h.
\]

**Theorem 1.2** Let \( q > 0 \), \( h \in L^1_{\text{loc}}(\mathbb{R}^N) \) then the problem ,

\[
-\text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} + |u|^{q-1} u = h \quad \text{on} \quad \mathbb{R}^N,
\]

has at most one weak solution \( v \in W^{1,1}_{\text{loc}}(\mathbb{R}^N) \cap L^q_{\text{loc}}(\mathbb{R}^N) \). Moreover,

\[
\inf_{\mathbb{R}^N} h \leq |v|^{q-1} v \leq \sup_{\mathbb{R}^N} h.
\]

With further assumptions on \( h \) or on the solutions, we have the following results.

**Corollary 1.3** Let \( q > p - 1 > 1 \), let \( h \in L^\infty(\mathbb{R}^N) \). If

\[
2 < p < \frac{2N}{N-1},
\]

then the problem,

\[
-\Delta_p u + |u|^{q-1} u = h \quad \text{on} \quad \mathbb{R}^N,
\]

has at most one weak solution.

**Theorem 1.4** Let \( p > 2 \), \( q > p - 1 \). If \( h \in L^1_{\text{loc}}(\mathbb{R}^N) \) then the problem,

\[
-\Delta_p u + |u|^{q-1} u = h \quad \text{on} \quad \mathbb{R}^N,
\]

has at most one weak solution \( v \) in the class,

\[
\left\{ u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N) \cap L^q_{\text{loc}}(\mathbb{R}^N) : \text{there exist } \theta < \frac{1}{p-2} \text{ such that } |\nabla u(x)| \leq c |x|^{\theta} \text{ for } |x| \text{ large} \right\}.
\]

Moreover,

\[
\inf_{\mathbb{R}^N} h \leq |v|^{q-1} v \leq \sup_{\mathbb{R}^N} h.
\]

Other partial results for the case \( p > 2 \) are presented in Section 4.1.

Our uniqueness results concern solutions that belong to the class \( W^{1,1}_{\text{loc}}(\mathbb{R}^N) \cap L^q_{\text{loc}}(\mathbb{R}^N) \).

Of course, this set is contained in the space \( X \) considered in [3]. However we point out that, when dealing with uniqueness results additional regularity is usually required by several Authors. See for instance [1]. Indeed, in that paper the Authors obtain the
existence of solutions of problem (1) belonging to a certain space $T_0^{1,p}$. The uniqueness result proved in [1] concerns entropy solutions.

We also emphasize that in this paper we shall restrict our analysis to the case $q > 1$. Indeed, as it is well known, see [1], if $1 < p \leq 2 - (1/N)$ and $q = 1$, then there exists $h \in L^1_{loc}(\mathbb{R}^N)$ such that (1) has no solutions belonging to $W_0^{1,1}(\mathbb{R}^N)$.

The paper is organized as follow. In the next section we describe the setting and the notations. In Section 3 we prove some a priori estimates on the solutions of the problems. Section 4 is devoted to prove the comparison results and to derive some of their consequences. In this paper a preeminent role is played by the $M,p,C$ operators (see below for the definition). In the appendix A we prove some inequalities that guaranty that an operator is $M,p,C$. In Appendix B for the convenience of the reader, we collect some basic facts about the Carnot groups.

**Note.** The results of this paper have been announced by the second author in Rome on May 7, 2012 and during a PDEs workshop dedicated to Patrizia Pucci’s birthday in Perugia on May 30, 2012. In the latter occasion, as an outcome of several discussions with Professor James Serrin and Professor Alberto Farina, we learned that they have obtained similar results to those proved in this paper. An expanded version of this work will appear in [6].

### 2 Notations and definitions

In this paper $\nabla$ and $|\cdot|$ stand respectively for the usual gradient in $\mathbb{R}^N$ and the Euclidean norm. $\Omega \subset \mathbb{R}^N$ open.

Throughout this paper we shall use some concepts briefly described in the Appendix [4]. For further details related to Carnot groups the interested reader may refer to [4].

Let $\mu \in C(\mathbb{R}^N;\mathbb{R}^l)$ be a matrix $\mu := (\mu_{ij})$, $i = 1, \ldots, l$, $j = 1, \ldots, N$ and assume that for any $i = 1, \ldots, l$, $j = 1, \ldots, N$ the derivative $\frac{\partial}{\partial x_j} \mu_{ij} \in C(\Omega)$. For $i = 1, \ldots, l$, let $X_i$ and its formal adjoint $X_i^*$ be defined as

$$X_i := \sum_{j=1}^{N} \mu_{ij}(\xi) \frac{\partial}{\partial \xi_j}, \quad X_i^* := -\sum_{j=1}^{N} \frac{\partial}{\partial \xi_j} (\mu_{ij}(\xi) \cdot),$$

and let $\nabla_L$ be the vector field defined by

$$\nabla_L := (X_1, \ldots, X_l)^T = \mu \nabla,$$

and

$$\nabla_L^* := (X_1^*, \ldots, X_l^*)^T.$$
For any vector field $h = (h_1, \ldots, h_l)^T \in \mathcal{C}^1(\Omega, \mathbb{R}^l)$, we shall use the following notation $\text{div}_L(h) := \text{div}(\mu^T h)$, that is

$$\text{div}_L(h) = - \sum_{i=1}^l X_i^* h_i = -\nabla^* L \cdot h.$$  

Examples of vector fields, which we are interested in, are the usual gradient acting on $l(\leq N)$ variables, vector fields related to Bouendi-Grushin operator, Heisenberg-Kohn sub–Laplacian, Heisenberg-Greiner operator, sub–Laplacian on Carnot Groups (see Appendix B).

However in order to avoid cumbersome notations we shall limit ourselves to consider the Carnot group case, which includes the Euclidean case. We shall assume that the matrix $\mu$ has $\mathcal{C}^\infty$ entries and that $\mathbb{R}^N$ can be endowed with a group law $\circ$ such that the operator

$$Lu := \text{div}_L(\nabla u),$$

is a sublaplacian on $(\mathbb{R}^N, \circ)$. See Appendix B for further details.

Notice that in the Euclidean framework we have $\mu = I_N$, the identity matrix on $\mathbb{R}^N$.

In what follows we shall assume that $A: \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ is a Caratheodory function, that is for each $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^l$ the function $A(\cdot, t, \xi)$ is measurable; and for a.e. $x \in \mathbb{R}^N$, $A(x, \cdot, \cdot)$ is continuous.

We consider operators $L$ “generated” by $A$, that is

$$L(u)(x) = \text{div}_L(A(x, u(x), \nabla u(x))).$$

Our canonical model cases are the $p$-Laplacian operator, the mean curvature operator and some related generalizations. See Examples 2.3 below.

**Definition 2.1** Let $A: \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}^l$ be a Caratheodory function. The function $A$ is called weakly elliptic if it generates a weakly elliptic operator $L$ i.e.

$$A(x, t, \xi) \cdot \xi \geq 0 \quad \text{for each } x \in \mathbb{R}^N, \ t \in \mathbb{R}, \ \xi \in \mathbb{R}^l,$n

(WE)

$$A(x, 0, \xi) = 0 \quad \text{or} \quad A(x, t, 0) = 0$$

Let $p \geq 1$, the function $A$ is called $W$-$p$-$C$ (weakly-$p$-coercive) (see [2]), if $A$ is (WE) and it generates a weakly-$p$-coercive operator $L$, i.e. if there exists a constant $k_2 > 0$ such that

$$(A(x, t, \xi) \cdot \xi)^{p-1} \geq k_2^p |A(x, t, \xi)|^p \quad \text{for each } x \in \mathbb{R}^N, \ t \in \mathbb{R}, \ \xi \in \mathbb{R}^l. \quad (W-p-C)$$

Let $p > 1$, the function $A$ is called $S$-$p$-$C$ (strongly-$p$-coercive) (see [7, 2, 13]), if there exist $k_1, k_2 > 0$ constants such that

$$(A(x, t, \xi) \cdot \xi) \geq k_1 |\xi|^p \geq k_2^p |A(x, t, \xi)|^{p'} \quad \text{for each } x \in \mathbb{R}^N, \ t \in \mathbb{R}, \ \xi \in \mathbb{R}^l. \quad (S-p-C)$$
Definition 2.2 Let $\Omega \subset \mathbb{R}^N$ be an open set and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^l \to \mathbb{R}$ be a Caratheodory function. Let $p \geq 1$. We say that $u \in W^{1,p}_{\text{loc}}(\Omega)$ is a weak solution of
\[
\text{div}_L (A(x, u, \nabla_L u)) \geq f(x, u, \nabla_L u) \quad \text{on} \quad \Omega,
\]
if $A(\cdot, u, \nabla u) \in L^p_{\text{loc}}(\Omega)$, $f(\cdot, u, \nabla_L u) \in L^1_{\text{loc}}(\Omega)$, and for any nonnegative $\phi \in C_0^1(\Omega)$ we have
\[
-\int_\Omega A(x, u, \nabla_L u) \cdot \nabla_L \phi \geq \int_\Omega f(x, u, \nabla_L u) \phi.
\]

Example 2.3
1. Let $p > 1$. The $p$-Laplacian operator defined on suitable functions $u$ by,
\[
\Delta_p u = \text{div}_L \left( |\nabla_L u|^{p-2} \nabla_L u \right)
\]
is an operator generated by $A(x, t, \xi) := |\xi|^{p-2} \xi$ which is $S$-$p$-$C$.

2. If $A$ is of mean curvature type, that is $A$ can be written as $A(x, t, \xi) := A(|\xi|) \xi$ with $A : \mathbb{R} \to \mathbb{R}$ a positive bounded continuous function (see Appendix A), then $A$ is $W$-$p$-$C$.

3. The mean curvature operator in non parametric form
\[
T u := \text{div} \left( \frac{\nabla L u}{\sqrt{1 + |\nabla L u|^2}} \right),
\]
is generated by $A(x, t, \xi) := \frac{\xi}{\sqrt{1 + |\xi|^2}}$. In this case $A$ is $W$-$p$-$C$ with $1 \leq p \leq 2$ and of mean curvature type but it is not $S$-$2$-$C$.

4. Let $m > 1$. The operator
\[
T_m u := \text{div} \left( \frac{|\nabla L u|^{m-2} \nabla L u}{\sqrt{1 + |\nabla L u|^m}} \right)
\]
is $W$-$p$-$C$ for $m \geq p \geq m/2$.

Definition 2.4 Let $A : \mathbb{R}^N \times \mathbb{R}^l \to \mathbb{R}^l$ be a Caratheodory function. We say that $A$ is monotone if
\[
(A(x, \xi) - A(x, \eta)) \cdot (\xi - \eta) \geq 0 \quad \text{for} \quad \xi, \eta \in \mathbb{R}^l.
\]
Let $p \geq 1$. We say that $A$ is $M$-$p$-$C$ (monotone $p$-coercive) if $A$ is monotone and if there exists $k_2 > 0$ such that
\[
((A(x, \xi) - A(x, \eta)) \cdot (\xi - \eta))^{p-1} \geq k_2 |A(x, \xi) - A(x, \eta)|^p.
\]

Example 2.5
1. Let $1 < p \leq 2$ the function $A(\xi) := |\xi|^{p-2} \xi$ is $M$-$p$-$C$ (see Appendix A for details). Therefore the following theorems apply to the $p$-Laplacian operator.

2. The mean curvature operator is $M$-$p$-$C$ with $1 \leq p \leq 2$ (see appendix A).
3 A priori estimates

The following lemma is a slight variation of a result proved in [8]. For easy reference we shall include the detailed proof.

We shall consider the following inequality,

\[ \text{div}_L (\mathscr{A}(x, v, \nabla_L v)) - f \geq \text{div}_L (\mathscr{A}(x, u, \nabla_L u)) - g \quad \text{on } \Omega. \]  

(5)

**Theorem 3.1** Let \( \mathscr{A} : \Omega \times \mathbb{R}^l \to \mathbb{R}^l \) be \( M-p\)-C. Let \( f, g \in L^1_{\text{loc}}(\Omega) \) and let \((u, v)\) be weak solution of (5). Set \( w := (v - u)^+ \) and let \( s > 0 \). If \((f - g)w \geq 0 \) and \( w^{s + p - 1} \in L^1_{\text{loc}}(\Omega) \), then

\[ (f - g)w^s, \ (\mathscr{A}(x, \nabla_L v) - \mathscr{A}(x, \nabla_L u)) \cdot \nabla_L w w^{s - 1} \in L^1_{\text{loc}}(\Omega). \]  

Moreover, for any nonnegative \( \phi \in \mathcal{C}_0^1(\Omega) \) we have,

\[ \int_{\Omega} (f - g)w^s \phi + c_1 s \int_{\Omega} (\mathscr{A}(x, \nabla_L v) - \mathscr{A}(x, \nabla_L u)) \cdot \nabla_L w w^{s - 1} \phi \leq c_2 s^{1 - p} \int_{\Omega} w^{s + p - 1} \frac{\|\nabla \phi\|^p}{\phi^{p - 1}}, \]  

(7)

where \( c_1 = 1 - \frac{p - 1}{p} \left( \frac{\epsilon}{k_2} \right)^{p - 1} > 0 \), \( c_2 = \frac{p^p}{p - 1} \) and \( \epsilon > 0 \) is sufficiently small for \( p > 1 \) and \( c_1 = 1 \) and \( c_2 = 1/k_2 \) for \( p = 1 \).

**Remark 3.2** i) Notice that from the above result it follows that if \( u, v \in W^{1, p}_{\text{loc}}(\Omega) \) is a weak solution of (5), then \((f - g)w \in L^1_{\text{loc}}(\Omega)\).

ii) The above lemma still holds if we replace the function \( f - g \in L^1_{\text{loc}}(\Omega) \) with a regular Borel measure on \( \Omega \).

iii) If \((u, v)\) is a weak solution of (5) and \( u \) is a constant i.e. \( u \equiv \text{const} \), then Theorem 3.1 still holds even for \( W^{1, p}_{\text{loc}} \) operators. See the following lemma.

**Lemma 3.3** Let \( \mathscr{A} \) be \( W^{1, p}_{\text{loc}} \). Let \( f, g \in L^1_{\text{loc}}(\Omega) \) and let \( v \in W^{1, p}_{\text{loc}}(\Omega) \) be a weak solution of

\[ \text{div}_L (\mathscr{A}(x, u, \nabla_L u)) \geq f - g, \quad \text{on } \Omega. \]  

(8)

Let \( k > 0 \) and set \( w := (v - k)^+ \) and let \( s > 0 \). If \((f - g)w \geq 0 \) and \( w^{s + p - 1} \in L^1_{\text{loc}}(\Omega) \), then

\[ (f - g)w^s, \ (\mathscr{A}(x, \nabla_L v) - \mathscr{A}(x, \nabla_L u)) \cdot \nabla_L w w^{s - 1} \in L^1_{\text{loc}}(\Omega) \]  

(9)

and for any nonnegative \( \phi \in \mathcal{C}_0^1(\Omega) \) we have,

\[ \int_{\Omega} (f - g)w^s \phi + c_1 s \int_{\Omega} (\mathscr{A}(x, v, \nabla_L v)) \cdot \nabla_L w w^{s - 1} \phi \leq c_2 s^{1 - p} \int_{\Omega} w^{s + p - 1} \frac{\|\nabla \phi\|^p}{\phi^{p - 1}}, \]  

(10)

where \( c_1 \) and \( c_2 \) are as in Theorem 3.1.

The above result lies on the following result proved in [8, Theorem 2.7].
Proof of Theorem 3.1. Let $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ be monotone Caratheodory function. Let $f, g \in L^1_{loc}(\Omega)$ and let $u, v$ be weak solution of

$$
\text{div}_L(\mathcal{A}(x, v, \nabla_L v)) - f \geq \text{div}_L(\mathcal{A}(x, u, \nabla_L u)) - g \quad \text{on } \Omega. \tag{11}
$$

Let $\gamma \in C^1(\mathbb{R})$ be such that $0 \leq \gamma(t), \gamma'(t) \leq M$, then

$$
- \int_{\Omega} (\mathcal{A}(x, v, \nabla_L v) - \mathcal{A}(x, u, \nabla_L u)) \cdot \nabla_L \phi \gamma(v-u) \geq \int_{\Omega} \gamma'(v-u) (\nabla_L v - \nabla_L u) \cdot (\mathcal{A}(x, v, \nabla_L v) - \mathcal{A}(x, u, \nabla_L u)) \phi \tag{12}
$$

$$
\geq \int_{\Omega} \gamma'(v-u) (\nabla_L v - \nabla_L u) \cdot (\mathcal{A}(x, v, \nabla_L v) - \mathcal{A}(x, u, \nabla_L u)) \phi + \int_{\Omega} \phi \gamma(v-u)(f-g) \quad \text{on } \Omega. \tag{13}
$$

Hence

$$
\text{div}_L(\gamma(v-u)(\mathcal{A}(x, v, \nabla_L v) - \mathcal{A}(x, u, \nabla_L u))) \geq \gamma(v-u)(f-g) \quad \text{on } \Omega. \tag{14}
$$

Moreover

$$
\text{div}_L(\text{sign}^+(v-u)(\mathcal{A}(x, v, \nabla_L v) - \mathcal{A}(x, u, \nabla_L u))) \geq \text{sign}^+(v-u)(f-g) \quad \text{on } \Omega. \tag{15}
$$

Proof of Theorem 3.1. Let $\gamma \in C^1(\mathbb{R})$ be a bounded nonnegative function with bounded nonnegative first derivative and let $\phi \in C^1_0(\Omega)$ be a nonnegative test function.

For simplicity we shall omit the arguments of $\mathcal{A}$. So we shall write $\mathcal{A}_u$ and $\mathcal{A}_v$ instead of $\mathcal{A}(x, \nabla_L u)$ and $\mathcal{A}(x, \nabla_L v)$ respectively.

Applying Lemma 3.4, we obtain

$$
\int_{\Omega} (f-g)\gamma(w)\phi + \int_{\Omega} (\mathcal{A}_v - \mathcal{A}_u) \cdot \nabla_L w \gamma'(w)\phi \leq - \int_{\Omega} (\mathcal{A}_v - \mathcal{A}_u) \cdot \nabla_L \phi \gamma(w) \leq \int_{\Omega} |\mathcal{A}_v - \mathcal{A}_u| |\nabla_L| \gamma(w) \tag{16}
$$

Let $p > 1$. From (16) we have

$$
\int_{\Omega} (f-g)\gamma(w)\phi + \int_{\Omega} (\mathcal{A}_v - \mathcal{A}_u) \cdot \nabla_L w \gamma'(w)\phi \leq \left( \int_{\Omega} |\mathcal{A}_v - \mathcal{A}_u|^p |\gamma'(w)|\phi \right)^{1/p} \left( \int_{\Omega} \gamma(w)^p |\nabla_L \phi|^p \phi^{p-1} \right)^{1/p} \leq e^{p' \int \mathcal{A}_v - \mathcal{A}_u} \cdot \nabla_L w \gamma'(w)\phi + \frac{1}{pe^p \int \gamma(w)^p |\nabla_L \phi|^p}.
$$

\footnote{We recall that the function sign$^+$ is defined as sign$^+(t) := 0$ if $t \leq 0$ and sign$^+(t) := 1$ otherwise.}
where $\epsilon > 0$ and all integrals are well defined provided $\frac{\gamma(w)^p}{\gamma(w)^p - 1} \in L^1_{\text{loc}}(\Omega)$. With a suitable choice of $\epsilon > 0$, for any nonnegative $\phi \in C^1_0(\Omega)$ and $\gamma \in C^1(\mathbb{R})$ as above such that $\frac{\gamma(w)^p}{\gamma(w)^p - 1} \in L^1_{\text{loc}}(\Omega)$, it follows that,

$$
\int_{\Omega} (f - g)\gamma(w)\phi + c_1 \int_{\Omega} (A_{iv} - A_{iu}) \cdot \nabla_L w \gamma'(w)\phi \leq \frac{1}{p}\int_{\Omega} \frac{\gamma(w)^p}{\gamma(w)^p - 1} \left| \nabla \phi \right|^p \phi^{p-1}.
$$

(17)

Now for $s > 0$, $1 > \delta > 0$ and $n \geq 1$, define

$$
\gamma_n(t) := \begin{cases} 
(t + \delta)^s & \text{if } 0 \leq t < n - \delta, \\
cn^s - \frac{s}{\beta - 1}n^{\beta+s-1}(t + \delta)^{1-\beta} & \text{if } t \geq n - \delta,
\end{cases}
$$

(18)

where $c := \frac{\beta - 1 + s}{\beta - 1}$ and $\beta > 1$ will be chosen later. Clearly $\gamma_n \in C^1$,

$$
\gamma'_n(t) = \begin{cases} 
s(t + \delta)^{s-1} & \text{if } 0 \leq t < n - \delta, \\
sn^{\beta+s-1}(t + \delta)^{-\beta} & \text{if } t \geq n - \delta,
\end{cases}
$$

and $\gamma_n$, $\gamma'_n$ are nonnegative and bounded with $||\gamma_n||_\infty = cn^s$ and $||\gamma'_n||_\infty = sn^{s-1}$. Moreover

$$
\frac{\gamma_n(t)^p}{\gamma'_n(t)^{p-1}} = \begin{cases} 
s^{1-p}(t + \delta)^{s+p-1} & \text{for } t < n - \delta, \\
\theta(t, n) & \text{for } t \geq n - \delta,
\end{cases}
$$

where

$$
\theta(t, n) := \frac{(cn^s - \frac{s}{\beta - 1}n^{\beta+s-1}(t + \delta)^{1-\beta})^p}{(sn^{\beta+s-1}(t + \delta)^{-\beta})^{p-1}} \leq \frac{(cn^s)^p s^{1-p}n^{-(\beta+s-1)(p-1)}(t + \delta)^{\beta(p-1)}}{(sn^{\beta+s-1}(t + \delta)^{-\beta})^{p-1}}.
$$

Choosing $\beta := \frac{s+p-1}{p-1}$ we have $c = p$, and

$$
\theta(t, n) \leq p^p s^{1-p}n^{s-(\beta+s-1)(p-1)}(t + \delta)^{s+p-1} = p^p s^{1-p}(t + \delta)^{s+p-1}.
$$

Therefore, for $t \geq 0$ we have,

$$
\frac{\gamma_n(t)^p}{\gamma'_n(t)^{p-1}} \leq p^p s^{1-p}(t + \delta)^{s+p-1}.
$$

Since by assumption $w^{s+p-1} \in L^1_{\text{loc}}(\Omega)$, from (17) with $\gamma = \gamma_n$, it follows that

$$
\int_{\Omega} (f - g)\gamma_n(w)\phi + c_1 \int_{\Omega} (A_{iv} - A_{iu}) \cdot \nabla_L w \gamma'_n(w)\phi \leq \frac{p^p s^{1-p}}{p^{p-1}} \int_{\Omega} (w + \delta)^{s+p-1} \left| \nabla \phi \right|^p \phi^{p-1}.
$$
Now, noticing that $\gamma_n(t) \to (t+\delta)^s$ and $\gamma_n'(t) \to s(t+\delta)^{s-1}$ as $n \to +\infty$, $(f-g)\gamma_n \geq 0$ and $\mathcal{A}$ is monotone (that is $(\mathcal{A}_v - \mathcal{A}_u) \cdot \nabla_L w \geq 0$), by Beppo Levi theorem we obtain
\[
\int_{\Omega} (f-g)(w+\delta)^s \phi + c_1 s \int_{\Omega} (\mathcal{A}_v - \mathcal{A}_u) \cdot \nabla_L w (w+\delta)^{s-1} \phi \leq c_2 s^{1-p} \int_{\Omega} (w+\delta)^{s+p-1} |\nabla_L \phi|^p |\phi|^p \cdot 1.
\]
By letting $\delta \to 0$ in the above inequality, we complete the proof of the claim in the case $p > 1$.

Let $p = 1$. From (16) and the fact that $\mathcal{A}_v - \mathcal{A}_u$ is bounded, the estimate (17) holds provided we replace $p$ with 1 and $\epsilon$ with $k_2$. The remaining argument is similar to the case $p > 1$ and we omit it. $\square$

**Remark 3.5** The assumption $u^{s+p-1} \in L^1_{loc}(\Omega)$, is not needed for the statement (14). Indeed what really matters for the validity of (14) is the assumption $u^{s+p-1} \in L^1_{loc}(S)$. Here $S$ is the support of $\nabla_L \phi$. This remark will be useful when dealing with inequalities on unbounded set.

**Lemma 3.6** Let $p \geq 1$ and let $\mathcal{A}: \Omega \times \mathbb{R}^l \to \mathbb{R}^l$ be $M$-$p$-$C$. Let $f, g \in L^1_{loc}(\Omega)$ and let $(u, v)$ be weak solution of (3). Set $w = (v-u)^+$. If $(f-g)w \geq 0$ and $w^q \in L^1_{loc}(\Omega)$ for $q > p - 1$, then
\[
(f-g)w^{q-p+1} \cdot ((\mathcal{A}(x, \nabla_L v) - \mathcal{A}(x, \nabla_L u)) \cdot \nabla_L w w^{q-p} \in L^1_{loc}(\Omega),
\]
and for any $\varphi \in C_c^0(\Omega)$ such that $0 \leq \varphi \leq 1$, we have,
\[
\int (f-g)\text{sign}^+(w) \varphi^s \leq c_3 \left( \frac{1}{|S|} \int_S w^q \varphi^s \right)^{\frac{p-1}{p}} \left( \frac{1}{|S|} \int_S |\nabla_L \varphi|^s \right)^{\frac{p}{s}} |S|,
\]
where $S$ is the support of $\nabla_L \varphi$, $c_3 := \frac{\sigma}{\kappa^{(p-1)/p}} \left( \frac{c_2}{c_1} \right)^{(p-1)/p}$ with $\sigma \geq \frac{\kappa^{(p-1)/p}}{q^{p+1-s}}$, $0 < s < \min\{1, q-p+1\}$ and $c_1, c_2$ as in the above Theorem 3.1.

**Proof.** The claim (19) follows directly applying Theorem 3.1.

Let $s > 0$ be such that $q \geq s + p - 1$. From Lemma 3.1 for any nonnegative $\phi \in C_c^0(\Omega)$ we have
\[
\int (f-g)w^s \phi + c_1 s \int (\mathcal{A}_v - \mathcal{A}_u) \cdot \nabla_L w w^{s-1} \phi \leq c_2 s^{1-p} \int w^{s+p-1} |\nabla_L \phi|^p |\phi|^p \cdot 1,
\]
where, as in the proof of Theorem 3.1, we write $\mathcal{A}_v$ and $\mathcal{A}_u$ for $\mathcal{A}(x, \nabla_L v)$ and $\mathcal{A}(x, \nabla_L u)$ respectively.

Next, an application of Theorem 3.1 gives (15). That is
\[
\text{div}_L (\text{sign}^+(v-u)(\mathcal{A}(x, v, \nabla_L v) - \mathcal{A}(x, u, \nabla_L u))) \geq \text{sign}^+(v-u)(f-g) \quad \text{on } \Omega.
\]
Next consider the case $p > 1$. Let $0 < s < \min\{1, q - p + 1\}$. By definition of weak solution and Hölder’s inequality with exponent $p'$, taking into account that $\mathcal{A}$ is $\mathcal{M}$-$p$-$\mathcal{C}$ and from (21) we get,

\[
\int \text{sign}^+ w (f - g) \phi \leq \int_S |\mathcal{A}_v - \mathcal{A}_u| |\nabla \phi| \text{sign}^+ w \tag{23}
\]

\[
= \int_S |\mathcal{A}_v - \mathcal{A}_u| w^{\frac{1}{p'}} \frac{1}{p'} |\nabla \phi| w^{\frac{1}{p'}} \phi^{\frac{1}{p'}} \tag{24}
\]

\[
\leq \frac{1}{k_2} \left( \int_S (\mathcal{A}_v - \mathcal{A}_u) \cdot \nabla w w^{s-1} \phi \right)^{1/p'} \left( \int_S w^{(1-s)(p-1)} |\nabla \phi|^p \phi^{p-1} \right)^{1/p} \tag{25}
\]

\[
\leq \frac{1}{k_2} \left( \frac{c_2}{c_1 s^q} \right)^{1/p'} \left( \int_S w^{s+p-1} |\nabla \phi|^p \phi^{p-1} \right)^{1/p'} \left( \int_S w^{(1-s)(p-1)} |\nabla \phi|^p \phi^{p-1} \right)^{1/p} \tag{26}
\]

Since $q > s + p - 1$ and $q > p - 1$, applying Hölder inequality to (26) with exponents $\chi := \frac{q}{s+p-1}$ and $\gamma := \frac{q}{(1-s)(p-1)}$, we obtain

\[
\int \text{sign}^+ w (f - g) \phi \leq c_3' \left( \int_S w^\delta \phi \right)^{\frac{1}{p'}} \left( \int |\nabla \phi|^{p' \chi} \right)^{\frac{1}{p'}} \left( \int |\nabla \phi|^{p' \gamma} \right)^{\frac{1}{p'}} \tag{27}
\]

where

\[
\delta := \frac{1}{\chi p'} + \frac{1}{\gamma p'} = \frac{p - 1}{q}, \quad c_3' := \left( \frac{c_2}{c_1 s^q} \right)^{1/p'} \frac{1}{k_2}. \tag{28}
\]

Next for $\sigma \geq p\chi'$ (and hence $\sigma > py'$ since $p\chi' > py'$) we choose $\phi := \varphi^\sigma$ with $\varphi \in \mathcal{C}_0^1(\Omega)$ such that $0 \leq \varphi \leq 1$. Setting $S := \text{support}(\varphi)$, from (28) it follows that

\[
\int \text{sign}^+ w (f - g) \varphi \varphi \leq c_3' \sigma^\delta \left( \int w^\delta \varphi^\sigma \right)^{\frac{1}{\sigma}} \left( \frac{1}{|S|} \int |\nabla \varphi|^\sigma \right)^{\frac{1}{\sigma}} |S|^{1-\delta},
\]

completing the proof of (20).

Now, we assume that $p = 1$. From (23), with the choice $\phi := \varphi^\sigma$, with $\varphi \in \mathcal{C}_0^1(\Omega)$ such that $0 \leq \varphi \leq 1$ and $\sigma \geq 1$, we have

\[
\int \text{sign}^+ w (f - g) \varphi \varphi \leq \frac{\sigma}{k_2} \int \nabla \varphi | \leq \frac{\sigma}{k_2} \left( \frac{1}{|S|} \int |\nabla \varphi|^\sigma \right)^{1/\sigma},
\]

which concludes the proof. \qed

Now, by specializing $f$ and $g$, we study

\[
\text{div}_L (\mathcal{A}(x, \nabla v)) - |v|^{q-1} v \geq \text{div}_L (\mathcal{A}(x, \nabla u)) - |u|^{q-1} u \quad \text{on } \Omega. \tag{29}
\]
Lemma 3.7 1. Let \( p > 1 \). Let \( a \in \mathbf{M}_p \) and let \( q > \max\{1, p-1\} \). For any \( \sigma > 0 \) large enough, there exists a constant \( c = c(\sigma, q, p, a) > 0 \) such that if \((u, v)\) is weak solution of (29) then for any nonnegative \( \varphi \in C^1_0(\Omega) \) such that \( \|\varphi\|_{\infty} \leq 1 \), we have

\[
\int \left( |v|^{q-1} v - |u|^{q-1} u \right)^{q/2} \leq c \left( \frac{1}{|S|} \int_S (v - u)^{q+q} \varphi \right)^{\frac{1}{q}} \left( \frac{1}{|S|} \int_S |\nabla L \varphi|^{q} \right)^{\frac{1}{q}} |S| \tag{30}
\]

\[
\int \left( |v|^{q-1} v - |u|^{q-1} u \right)^{q/2} \leq c |S| \left( \frac{1}{|S|} \int_S |\nabla L \varphi|^{q} \right)^{\frac{1}{q}} \tag{31}
\]

where \( S := \text{support}(\varphi) \).

In particular if \( B_{2R} \subset \subset \Omega \), then

\[
\left( \int_{B_R} \left( |v|^{q-1} v - |u|^{q-1} u \right)^{1/q} \right)^{1/q} \leq c R^{-\frac{p}{q+p+1}}. \tag{32}
\]

Moreover, for \( x \in \Omega \), set \( R = \text{dist}(x, \partial \Omega)/2 \), we have

\[
\left( \int_{B_{R}(x)} \left( |v|^{q-1} v - |u|^{q-1} u \right)^{1/q} \right)^{1/q} \leq c \text{dist}(x, \partial \Omega)^{-\frac{p}{q+p+1}}. \tag{33}
\]

2. Let \( a \in \mathbf{M}_p \) and let \( q > 0 \). For any \( \sigma > 0 \) large enough, there exists a constant \( c = c(\sigma, q, p, a) > 0 \) such that if \((u, v)\) is weak solution of (29) then for any nonnegative \( \varphi \in C^1_0(\Omega) \) such that \( \|\varphi\|_{\infty} \leq 1 \), we have

\[
\int \left( |v|^{q-1} v - |u|^{q-1} u \right)^{q/2} \leq c \left( \frac{1}{|S|} \int_S (v - u)^{q+q} \varphi \right)^{\frac{1}{q}} \left( \frac{1}{|S|} \int_S |\nabla L \varphi|^{q} \right)^{\frac{1}{q}} |S| \tag{34}
\]

where \( S := \text{support}(\varphi) \).

In particular if \( B_{2R} \subset \subset \Omega \), then

\[
\left( \int_{B_R} \left( |v|^{q-1} v - |u|^{q-1} u \right)^{1/q} \right)^{1/q} \leq c R^{-\frac{1}{q}}. \tag{35}
\]

Moreover, for \( x \in \Omega \), set \( R = \text{dist}(x, \partial \Omega)/2 \), we have

\[
\left( \int_{B_{R}(x)} \left( |v|^{q-1} v - |u|^{q-1} u \right)^{1/q} \right)^{1/q} \leq c \text{dist}(x, \partial \Omega)^{-\frac{1}{q}}. \tag{35}
\]
Proof. Let \( p > 1 \). From Lemma 3.6 we immediately obtain (30).

Reminding the well known inequality
\[
t^q - s^q \geq c_q (t - s)^q, \quad \text{for } t > s \quad (q > 1),
\]
from (30), we get (31).

In order to obtain the estimate (32) we specialize the test function \( \varphi \). Indeed, let \( \varphi \in C_0^1(\mathbb{R}) \) be such that \( 0 \leq \varphi \leq 1 \), \( \varphi(t) = 0 \) if \( |t| \geq 2 \) and \( \varphi(t) = 1 \) if \( |t| \leq 1 \). Next, we define \( \varphi_R(t) := \varphi(t/R) \). The claim follows by choosing \( \varphi(x) = \varphi_R(|x|) \).

The estimate (33) follows by choosing \( \varphi(y) = \varphi_R(|y - x|) \).

The case \( p = 1 \) follows the same argument as above so that we can leave the details to the interested reader.

\[\blacksquare\]

Lemma 3.8 Assume that either one of the following holds

1. Let \( p > 1 \), let \( A \) be \( M-p-C \) and \( q > \max\{1, p-1\} \).

2. Let \( p = 1 \), let \( A \) be \( M-p-C \) and \( q > 0 \).

Let \((u, v)\) be weak solution of (29). Then \((v - u)^+ \in L_{loc}^r(\Omega)\) for any \( r < +\infty \).

Proof. Let \((u, v)\) be a solution of (29) and set \( w := (v - u)^+ \). Now since \( w \in L_{loc}^q(\Omega) \), and inequality (35) holds we are in the position to apply Theorem 3.1 with \( s = q-p+1 \) obtaining \( w^{q_1} \in L_{loc}^1(\Omega) \) with \( q_1 := 2q - p + 1 \). Applying again Theorem 3.1 with \( s = q_1 - p + 1 \), we get \( w^{q_2} \in L_{loc}^1(\Omega) \) with \( q_2 := q_1 + q - p + 1 = q + 2(q - p + 1) \). Iterating \( j \) times we have that \( w^{q_j} \in L_{loc}^1(\Omega) \) with \( q_j := q + j(q - p + 1) \). Letting \( j \to +\infty \) we have the claim. \[\blacksquare\]

4 Comparison and Uniqueness

Theorem 4.1 Let \( p \geq 1 \) and let \( \mathcal{A} \) be \( M-p-C \). Let \((u, v)\) be a \( \sigma \)-regular solution of
\[
\text{div}_L(\mathcal{A}(x, \nabla_L v)) - |v|^{q-1} v \geq \text{div}_L(\mathcal{A}(x, \nabla_L u)) - |u|^{q-1} u \quad \text{on } \mathbb{R}^N.
\]
Assume that one of the following holds

1. Let \( p > 1 \), and \( q > \max\{1, p-1\} \).

2. Let \( p = 1 \), and \( q > 0 \).

Then \( v \leq u \) a.e. on \( \mathbb{R}^N \).
Proof. Let \((u, v)\) be a solution of (37) and set \(w := (v - u)^+\). From Lemma 3.8 we know that \(w \in L^1_{\text{loc}}(\mathbb{R}^N)\) for any \(r\), and hence we are in the position to apply Theorem 3.1 with \(s\) large enough. Thus, from (36) and (7) we get

\[ w^{q + s} \in L^1_{\text{loc}} \]

\[
\int w^{q + s} \phi \leq c(s, q, p) \int \frac{\lvert \nabla \phi \rvert^p}{\phi^{p - 1}}.
\]

Applying the Hölder inequality with exponent \(x := \frac{q + s}{s + p - 1} > 1\) we have

\[
\int w^{q + s} \phi \leq c(s, q, p) \int \frac{\lvert \nabla \phi \rvert^{px'}}{\phi^{px' - 1}}.
\]

By the same choice of \(\phi\) we made in Lemma 3.7, we have that

\[
\int_{B_R} w^{q + s} \leq cR^{N - px'} = cR^{N - p(q + s)/(q + p + 1)}.
\]

Choosing \(s\) large enough and letting \(R \to +\infty\), we have that \(w \equiv 0\) a.e. that is the claim.

\[ \square \]

Corollary 4.2 Let \(p \geq 1\), let \(\mathcal{A}\) be \(\mathcal{W}_{-p}\) such that \(\mathcal{A}(x, 0) = 0\). Let \(q > 0\) be as in Theorem 4.1. Let \(h \in L^1_{\text{loc}}(\mathbb{R}^N)\). Let \(v\) be a solution of the problem

\[
- \text{div} \left( \mathcal{A}(x, \nabla v) \right) + \lvert v \rvert^{q - 1} v = h.
\] (38)

Then,

\[
\inf_{\mathbb{R}^N} h \leq \lvert v \rvert^{q - 1} v \leq \sup_{\mathbb{R}^N} h.
\]

In particular, if \(h \geq 0\) [resp. \(\leq 0\)], then \(v \geq 0\) [resp. \(\leq 0\)] and if \(h \in L^\infty(\mathbb{R}^N)\), then \(v \in L^\infty(\mathbb{R}^N)\).

Proof. We prove one of the estimates, the remain one is similar. If \(\sup_{\mathbb{R}^N} h = +\infty\) there is nothing to prove. Let \(M := \sup_{\mathbb{R}^N} h < +\infty\) there is nothing to prove. We define \(u := \text{sign}(M) \lvert M \rvert^{1/q}\). Then

\[
\text{div} \left( \mathcal{A}(x, \nabla u) \right) - \lvert u \rvert^{q - 1} u + h = 0 \geq h - M = \text{div} \left( \mathcal{A}(x, \nabla u) \right) - \lvert u \rvert^{q - 1} u - h,
\]

that is \((u, v)\) satisfy (37) with \(u\) constant. In this case all the previous estimates still hold since in this case the operator can be seen as it were \(\mathcal{M}_{-p}\). See also Remark 3.2 and Lemma 3.3.

Thus the claim follows from Theorem 4.1.

\[ \square \]

Corollary 4.3 Let \(p \geq 1\) and let \(\mathcal{A}\) be \(\mathcal{W}_{-p}\). Let \(q > 0\) be as in Theorem 4.1. Let \(h \in L^1_{\text{loc}}(\mathbb{R}^N)\). Then the eventual solution of the problem (38) is unique.

Moreover if \(\mathcal{A}(x, 0) = 0\) and \(v\) is a solution of (38), then

\[
\inf_{\mathbb{R}^N} h \leq \lvert v \rvert^{q - 1} v \leq \sup_{\mathbb{R}^N} h.
\]
Proof. Uniqueness. Let \( u \) and \( v \) two solutions of (38). Then \((u, v)\) solves
\[
div_L (\mathcal{A}(x, \nabla_L v)) - |v|^{q-1} v = div_L (\mathcal{A}(x, \nabla_L u)) - |u|^{q-1} u \quad \text{on } \mathbb{R}^N,
\]
and applying Theorem 4.1 we conclude that \( u \equiv v \).

The remaining claim follows from Corollary 4.2. \( \square \)

4.1 Some results for non M-\( p \)-C operators

Notice that the \( p \)-Laplacian operator with \( p > 2 \) is not M-\( p \)-C. This fact it is easy to see by homogeneity consideration.

In this section we shall require that \( p > 2 \) and \( \mathcal{A} : \Omega \times \mathbb{R}^l \to \mathbb{R}^l \) for all \( \xi, \eta \in \mathbb{R}^l \setminus \{0\}, x \in \mathbb{R}^N \) satisfies

\[
(\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)) \cdot (\xi - \eta) \geq k_2 \frac{|\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)|^p}{(|\xi| + |\eta|)^{p(p-2)}}. \quad (39)
\]

**Example 4.4** Example of function \( \mathcal{A} \) satisfying (39) is \( \mathcal{A}(x, \xi) = a(x)|\xi|^{p-2}\xi \) with \( a \) a bounded nonnegative function and \( p \geq 2 \). Indeed, if \( \mathcal{A}(\xi) = |\xi|^{p-2}\xi \), the following inequalities holds

\[
|\xi^{p-2}\xi - \eta^{p-2}\eta| \leq c_1(|\xi| + |\eta|)^{p-1-\alpha}|\xi - \eta|^{\alpha} \quad (40)
\]
\[
(\xi^{p-2}\xi - \eta^{p-2}\eta) \cdot (\xi - \eta) \geq c_2(|\xi| + |\eta|)^{p-\beta}|\xi - \eta|^{\beta} \quad (41)
\]

with \( \beta \geq \max\{p, 2\} \) and \( 0 \leq \alpha \leq \min\{1, p - 1\} \). See [3].

Therefore choosing \( \beta = p \) and \( \alpha = 1 \) in (40) and (41) we have

\[
\frac{|\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)|^p}{(|\xi| + |\eta|)^{p(p-2)}} = a(x)^p \frac{|\xi^{p-2}\xi - \eta^{p-2}\eta|^p}{(|\xi| + |\eta|)^{p(p-2)}} \leq a(x)^p c_1^p |\xi - \eta|^p
\]
\[
\leq a(x)^p c_2^p (\xi^{p-2}\xi - \eta^{p-2}\eta) \cdot (\xi - \eta) = a(x)^p \frac{c_2^p}{c_2} (\mathcal{A}(x, \xi) - \mathcal{A}(x, \eta)) \cdot (\xi - \eta).
\]

Therefore, (39) is fulfilled with \( k_2 = \frac{c_2^p}{c_2} ||a||_\infty^{p-1} \).

We need of a version of Theorem 3.1 for operator satisfying (39).

**Lemma 4.5** Let \( \mathcal{A} : \Omega \times \mathbb{R}^l \to \mathbb{R}^l \) satisfy (43) with \( p > 2 \). Let \( f, g \in L^1_{\text{loc}}(\Omega) \) and let \((u, v)\) be weak solution of (2). Set \( w := (v - u)^+ \) and let \( s > 0 \). If \((f - g)w \geq 0 \) and \( w^{s+p'-1} (|\nabla_L u| + |\nabla_L v|)^{p'/(p-2)} \in L^1_{\text{loc}}(\Omega) \), then

\[
(f - g)w^s, \quad (\mathcal{A}(x, \nabla_L v) - \mathcal{A}(x, \nabla_L u)) \cdot \nabla_L w w^{s-1} \in L^1_{\text{loc}}(\Omega), \quad (42)
\]
and for any nonnegative \( \phi \in C^1_0(\Omega) \) we have,

\[
\int_{\Omega} (f - g) w^s \phi + c_1 \int_{\Omega} (\mathcal{A}(x, \nabla_L v) - \mathcal{A}(x, \nabla_L u)) \cdot \nabla_L w \ w^{s-1} \phi \leq c_2 \int_{\Omega} w^{s+p'-1} (|\nabla_L u| + |\nabla_L v|)^{(p-2)p'} \frac{|\nabla_L \phi|^{p'}}{\phi^{p'-1}}. \tag{44}
\]

where \( c_1 = c_1(s, p, k_2), c_2(s, p, k_2) > 0 \) are suitable constants independent of \( u, v \) and \( \phi \).

**Proof.** The proof is analogous to the proof of Theorem \ref{thm:3.1}. So we shall sketch it using the same notation. Applying Lemma \ref{lem:A1} we have (16) which, by using Hölder’s inequality, (39) and Young’s inequality, yields

\[
\int_{\Omega} (f - g) \gamma(w) \phi + \int_{\Omega} (\mathcal{A}_v - \mathcal{A}_u) \cdot \nabla_L \gamma(w) \phi \leq \\
\leq \left( \int_{\Omega} \frac{|\mathcal{A}_v - \mathcal{A}_u|^p}{(|\nabla_L v| + |\nabla_L u|)^{(p-2)p'}} \gamma(w) \phi \right)^{1/p'} \left( \int_{\Omega} \frac{\gamma(w)^{p'}}{\phi^{p'-1}} (|\nabla_L v| + |\nabla_L u|)^{(p-2)p'} \right)^{1/p'} \\
\leq \frac{c^p}{p^k_2} \int_{\Omega} (\mathcal{A}_v - \mathcal{A}_u) \cdot \nabla_L w \ \gamma(w) \phi + \frac{1}{p'c^p} \int_{\Omega} \gamma(w)^{p'} \frac{|\nabla_L \phi|^{p'}}{\phi^{p'-1}} (|\nabla_L v| + |\nabla_L u|)^{(p-2)p'}. \tag{47}
\]

Next, constructing a sequence of \( \gamma_n(t) \) approximating the function \( t^s \) as made in the proof of Theorem \ref{thm:4.1} we conclude the proof. \( \square \)

**Theorem 4.6** Let \( \mathcal{A}: \Omega \times \mathbb{R}^l \to \mathbb{R}^l \) satisfy (39) with \( p > 2 \). Let \( f, g \in L^1_{\text{loc}}(\Omega) \) and let \( (u, v) \) be weak solution of (1). Set \( w := (v - u)^+ \) and let \( s > 0 \). If \( (f - g)w \geq 0 \) and \( w^{s(p-1)+1} \in L^1_{\text{loc}}(\Omega) \), then

\[
(f - g)w^s, \quad (\mathcal{A}(x, \nabla_L v) - \mathcal{A}(x, \nabla_L u)) \cdot \nabla_L w \ w^{s-1} \in L^1_{\text{loc}}(\Omega) \tag{45}
\]

and for any nonnegative \( \phi \in C^1_0(\Omega) \) we have,

\[
\int_{\Omega} (f - g) w^s \phi + c_1 \int_{\Omega} (\mathcal{A}(x, \nabla_L v) - \mathcal{A}(x, \nabla_L u)) \cdot \nabla_L w \ w^{s-1} \phi \leq c_2 \left( \int_{A} w^{s(p-1)+1} \frac{|\nabla_L \phi|^p}{\phi} \right)^{1/(p-1)} \left( \int_{A} (|\nabla_L u| + |\nabla_L v|)^{(p-2)/(p-1)} \right), \tag{47}
\]

where \( A \) is the support of \( \nabla_L \phi \), and \( c_1, c_2 > 0 \) are suitable constants independent of \( u, v \) and \( \phi \).

In particular if there exist \( c > 0, q > 0 \) such that

\[
(f - g)(v - u)^+ \geq c \ ((v - u)^+) \tag{48}
\]

we have that \( w^{q+s} \in L^1_{\text{loc}}(\Omega) \). Moreover, \( w^{q+1} \in L^1_{\text{loc}}(\Omega) \) and if \( q > p - 1 \) we have

\[
w^r \in L^1_{\text{loc}}(\Omega) \quad \text{for any} \quad r \in \left[ 0, \frac{q(p-1) - 1}{p-2} \right]. \tag{49}
\]
Proof. In order to apply Lemma 4.5 we have to show that $w^{s+p'-1} (|\nabla L u| + |\nabla L v|)^p (p-2) \in L^1_{\text{loc}}(\Omega)$.

Let $\phi \in C_0^1(\Omega)$. An application of Hölder’s inequality with exponent $z := \frac{p-1}{p-2}$ implies
\[
\int_{\Omega} w^{s+p'-1} \frac{|\nabla L \phi|^p}{\phi^p} (|\nabla L u| + |\nabla L v|)^{(p-2)p'} \leq \left( \int_A w^{s(p-1)+1} \frac{|\nabla L \phi|^p}{\phi^p} \right)^{1/z'} \left( \int_A (|\nabla L u| + |\nabla L v|)^p \right)^{1/z}. \tag{50}
\]

Since $|\nabla u|, |\nabla v| \in L^p_{\text{loc}}(\Omega)$ and $w^{s(p-1)+1} \in L^1_{\text{loc}}(\Omega)$ by hypotheses, we obtain the claim. Using (50) in (44) we obtain (47).

Now assuming (48), we have that $w^{q+s} \in L^1_{\text{loc}}(\Omega)$.

In order to complete the proof we begin observing that since $w \in L^p_{\text{loc}}(\Omega)$, by choosing $s = 1$, we have $w^{q+1} \in L^1_{\text{loc}}(\Omega)$.

If $\frac{q(p-1)-1}{p-2} > q + 1$, that is if $q > p - 1$, we shall use a bootstrap argument. If $w^r \in L^1_{\text{loc}}(\Omega)$, by the first part of the theorem, we have that $w^{h(r)} \in L^1_{\text{loc}}(\Omega)$ with $h(r) := q + \frac{r-1}{p-1}$. Therefore, setting $r_0 = q$ and $r_{n+1} := h(r_n)$ we easily verify that the sequence $(r_n)$ is increasing and it converges to $\frac{q(p-1)-1}{p-2}$. This concludes the proof. \[\square\]

**Theorem 4.7** Let $\mathcal{A}$ satisfy (33) with $p > 2$. Let $(u, v)$ be a solution of (37) with $q > 1$.

Assume that
\[
\left( \int_{A_R} |\nabla L v|^p \right)^{1/p}, \left( \int_{A_R} |\nabla L u|^p \right)^{1/p} \leq c R^\theta \quad \text{for} \quad R \text{large}, \tag{51}
\]

with
\[
\theta < \frac{1}{p-2} - \frac{Q}{p}. \tag{52}
\]

Then $v \leq u$.

**Proof.** Set $w := (v - u)^+$. From (36) we get that (48) is satisfied, and hence (39) and (47) hold. Therefore, fixing $0 < s < \frac{q-1}{p-2}$ we have that
\[
\frac{q(p-1)-1}{p-2} > q + s > s(p-1) + 1,
\]
and hence $w^{s(p-1)+1} \in L^1_{\text{loc}}(\mathbb{R}^N)$. From (47) we have
\[
\int_{\mathbb{R}^N} w^{q+s} \phi \leq c \left( \int_A w^{s(p-1)+1} \frac{|\nabla L \phi|^p}{\phi} \right)^{1/(p-1)} \left( \int_A (|\nabla L u| + |\nabla L v|)^p \right)^{(p-2)/(p-1)}.
\]
By Hölder’s inequality with exponent \( x := \frac{q+s}{s(p-1)+1} \), and choosing \( \phi = \phi_R \) with \( \phi_R \) as in the proof of Lemma 3.7, we have

\[
\left( \int_{B_R} u^{q+s} \right)^{1 - \frac{1}{x(p-1)}} \leq \left( \int_A |\nabla_L \phi|^{px'} \phi^{x'} \right)^{\frac{1}{x'(p-1)}} \left( \int_A (|\nabla_L u| + |\nabla_L v|)^p \right)^{(p-2)/(p-1)}
\]

\[
\leq c R^{\frac{x'}{p-1}} R^{\frac{p-2}{p-1}} \left( \int_A (|\nabla_L u| + |\nabla_L v|)^p \right)^{(p-2)/(p-1)}
\]

\[
\leq c R^t. \tag{53}
\]

Here the last inequality follows from (55), where

\[
t := \frac{Q - px'}{x'(p-1)} + Qp - 2 + \theta p - 2 = \frac{Q}{x'(p-1)} + p - 2 \left( \theta + \frac{Q}{p} - \frac{1}{p-2} \right).
\]

Since

\[
\lim_{s \to \frac{q-1}{p-2}^-} x' = \lim_{s \to \frac{q-1}{p-2}^-} \frac{q+s}{q-1} = +\infty,
\]

we can choose \( s \) so that \( x' \) is large enough and \( t < 0 \). By this choice, letting \( R \to +\infty \) in (53), we obtain that \( w \equiv 0 \). This proves our claim. \( \square \)

**Remark 4.8** Assumptions (51) and (52) are obviously satisfied when looking for compact supported solutions.

Further examples when the growth condition (51) holds are stated in the following.

**Proposition 4.9** Let \( q > p - 1 > 0 \) and \( \mathcal{A} \) be \( S-p-C \). Let \( h \in L^1_{\text{loc}}(\mathbb{R}^N) \) and let \( u \in L^q_{\text{loc}}(\mathbb{R}^N) \cap W^{1,p}_{\text{loc}}(\mathbb{R}^N) \) be a weak solution of

\[
\text{div}(\mathcal{A}(x, \nabla u)) = |u|^{q-1} u + h \quad \text{on} \quad \mathbb{R}^N. \tag{54}
\]

Assume that \( u, h \in L^1_{\text{loc}}(\mathbb{R}^N) \). Then \( u^{q+1} \in L^1_{\text{loc}}(\mathbb{R}^N) \). Furthermore if \( h \in L^{1+1/q}_{\text{loc}}(\mathbb{R}^N) \), then for any \( R > 0 \) we have

\[
\int_{B_R} |u|^{q+1} + \int_{B_R} |\nabla_L u|^p \phi \leq c_4 R^{Q-\frac{p+1}{q}+1} + c_5 \int_{B_{2R}} |h|^{\frac{q+1}{q}}.
\]

In particular, if there exists \( \sigma \in \mathbb{R} \) such that

\[
\left( \int_{B_R} |h|^{1+1/q} \right)^{q/(q+1)} \leq c R^\sigma \quad \text{for} \quad R \text{ large}, \tag{55}
\]

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then
\[
\left( \int_{B_R} |\nabla L u|^p \right)^{1/p} \leq c R^\theta \quad \text{for } R \text{ large},
\]
where
\[
\theta := \max \left\{ \frac{q + 1}{qp}, -\frac{q + 1}{q + 1 - p} \right\}.
\]

**Proof.** We can use \(u \phi\) as test function in (54). This follows by the fact that \(u h \in L^1_{loc}(\mathbb{R}^N)\). To see this we argue as in Theorem 3.1 (see also [8]) obtaining
\[
\int |u|^{q+1} \phi + \int A(x, \nabla u) \cdot \nabla u \phi \leq \int |A(x, \nabla u)||\nabla \phi||u| + \int |h||u| \phi.
\]
By using the fact that \(A\) is S-p-C, and by Young’s inequality we get
\[
\int |u|^{q+1} \phi + c_1 \int |\nabla u|^p \phi \leq c_2 \int |\nabla \phi|^p |u|^p + \int |h||u| \phi.
\]
This implies that \(u^{q+1} \in L^1_{loc}(\mathbb{R}^N)\).

Next, assume that \(h \in L^{1+1/q}_{loc}(\mathbb{R}^N)\). By Young’s inequality with exponents \(x := \frac{q+1}{p}\) and \(y := q + 1\), it follows that
\[
c_3 \int |u|^{q+1} \phi + c_1 \int |\nabla u|^p \phi \leq c_4 \int |\nabla \phi|^{p\sigma} + c_5 \int |h|^{y'} \phi.
\]
Next by choosing \(\phi = \phi_R\) as in Lemma 3.7, from the assumption on \(h\) we get
\[
\int_{B_R} |\nabla u|^p \leq c(R^{-p\sigma} + R^{\frac{q+1}{q\sigma}}) \leq cR^{\max\{-p\sigma, \frac{q+1}{q\sigma}\}}.
\]
\[\Box\]

**Remark 4.10** The assumption \(u h \in L^1_{loc}(\mathbb{R}^N)\) is obviously satisfied for instance if \(u \in L^{q+1}_{loc}(\mathbb{R}^N)\) and \(h \in L^{1+1/q}_{loc}(\mathbb{R}^N)\).

**Corollary 4.11** Let \(q > p-1 > 1\). Let \(A\) be S-p-C satisfying (53) and let \(h \in L^{1+1/q}_{loc}(\mathbb{R}^N)\) be such that (55) holds for \(\sigma \in \mathbb{R}\) and
\[
\max \left\{ \frac{q + 1}{qp}, -\frac{q + 1}{q + 1 - p} \right\} < \frac{1}{p - 2} - \frac{N}{p}.
\]
Then problem (54) has at most one solution on the class \(L^{q+1}_{loc}(\mathbb{R}^N) \cap W^{1,p}_{loc}(\mathbb{R}^N)\).

**Remark 4.12** The possible weak solutions belong to the space \(L^{q+1}_{loc}(\mathbb{R}^N)\) in the following cases.
1. Since $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ then, by Sobolev’s embedding, $u \in L^{q+1}_{\text{loc}}(\mathbb{R}^N)$ provided $q \leq \frac{N(p-1)+p}{N-p}$.

2. If $p \geq N$.

3. If $h \in L^\infty_{\text{loc}}(\mathbb{R}^N)$ then $uh \in L^1_{\text{loc}}(\mathbb{R}^N)$. By Proposition 4.9 it follows that $u \in L^{q+1}_{\text{loc}}(\mathbb{R}^N)$.

4. If $h \in L^\infty(\mathbb{R}^N)$ then from Theorem 4.2 it follows that $u \in L^\infty(\mathbb{R}^N)$. In particular Corollary 4.13 holds.

**Corollary 4.13** Let $q > p - 1 > 1$. Let $\mathcal{A}$ be S-$p$-$C$ satisfying (37) and $h \in L^\infty(\mathbb{R}^N)$. If

$$2 < p < \frac{2Q}{Q-1},$$

then problem (54) has at most one weak solution.

**Corollary 4.14** Let $h \in L^1_{\text{loc}}(\mathbb{R}^N)$, $q > p - 1$ and $p > 2$. Then the problem

$$-\Delta_{L_p} u + |u|^{q-1} u = h,$$

has at most one weak solution $u$ satisfying, $\int_{\mathbb{R}^N} |\nabla L^p u| < +\infty$.

**Proof.** It is enough to choose $\theta := -Q/p$ in Theorem 4.7. □

Now requiring stronger assumptions on the behavior of the gradient of the solutions, we have the following.

**Theorem 4.15** Assume that $\mathcal{A}$ satisfies condition (37) with $p > 2$. Let $(u, v)$ be a solution of (37) with $q > 1$.

Let $\theta < \frac{1}{p-2}$ and assume that there exists $\alpha > \frac{N(p-2)}{1-\theta(p-2)}$ such that

$$\alpha q(p-1) - 1 > qp,$$ (57)

$$((v - u)^+)^{\alpha \frac{2(p-1)}{p(p-2)}} \in L^1(A_R) \text{ for } R \text{ large},$$ (58)

$$\left( \int_{A_R} |\nabla L v|^\alpha \right)^{\frac{1}{\alpha}}, \left( \int_{A_R} |\nabla L u|^\alpha \right)^{\frac{1}{\alpha}} \leq cR^\theta.$$

Then $v \leq u$ a.e. on $\mathbb{R}^N$.
Proof. Let \( w := (v - u)^+ \). By (36) and Lemma 4.5, applying Hölder’s inequality to (44) with exponent \( x := \frac{q + s}{s - p' + 1} \) where \( s > 0 \), we have

\[
\int_\Omega w^{q + s} \phi \leq c_2 \left( \int_\Omega w^{q + s} \phi \right)^{1/x} \left( \int_\Omega (|\nabla_L u| + |\nabla_L v|)^{p'x'} \frac{|\nabla_L \phi|^{p'x'}}{\phi^{p'x' - 1}} \right)^{1/x'}.
\]

(60)

By choosing \( \phi = \phi_R \) as in the proof of Lemma 3.7, it follows that

\[
\int_{B_R} w^{q + s} \leq c_2 \phi_1 R^{Q - p'x'} \int_{A_R} (|\nabla_L u| + |\nabla_L v|)^{(p - 2)p'x'}.
\]

(61)

Next, we observe that we must choose \( s > 0 \) such that all the integrals in (60) are well defined. Indeed, by choosing \( s = \alpha \frac{p'}{p'(p - 2)}(q - p' + 1) - q \), from (57) it follows that \( s > 0 \).

We observe that since \((p - 2)p'x' = \alpha \) and \( w^{q + s} = w^{q + \alpha(\frac{p'}{p'(p - 2)}(q - p' + 1))} \in L^1_{\text{loc}}(\mathbb{R}^N)\), the integrals in (61) and (60) are well defined. Next, from (61) and (59), we obtain

\[
\int_{B_R} w^{q + s} \leq cR^\gamma \quad \text{where} \quad \gamma = Q - p'x' + \theta(p - 2)p'x'.
\]

(62)

Finally, we observe that since

\[
\gamma = Q - \alpha \frac{1}{p - 2} + \theta \alpha < Q - \left( \frac{1}{p - 2} - \theta \right) \frac{Q(p - 2)}{1 - \theta(p - 2)} = 0,
\]

by letting \( R \to +\infty \) in (62), the claim follows.

As a final remark we note that the knowledge of a pointwise estimate on the gradient of the solutions on an exterior domain of the type

\[
|\nabla u(x)| \leq c |x|^\theta \quad \text{for} \quad |x| \quad \text{large},
\]

where \( \theta < 1/(p - 2) \), then Theorem 1.15 applies and yields Theorem 1.4.

Proof of Theorem 1.4. The growth assumption on \( \nabla u \) implies that (50) holds for any \( \alpha > 0 \). By the Sobolev embedding theorem \( u \in L^r(A_R) \) for any \( r > 0 \) and for \( R \) large. Therefore by choosing \( \alpha \) large enough the claim follows from Theorem 1.15.

\[\square\]

A Inequalities and M-p-C Operators

Here, we shall prove some fundamental elementary inequalities that we use throughout the paper. Very likely these inequalities are well known, nevertheless for completeness we shall include their proof here.

In what follows we shall assume that \( \mathcal{A} \) has the form

\[
\mathcal{A}(x, \xi) = A(|\xi|)\xi,
\]

where \( \mathcal{A} : \mathbb{R}_+ \to \mathbb{R} \). We set \( \phi(t) := A(t)t \).
Theorem A.1 Let $A$ be nonincreasing and bounded function such that
\[ \phi(0) = 0, \quad \phi(t) > 0 \quad \text{for} \ t > 0, \ \phi \ \text{is nondecreasing}. \] (63)
Then $\mathcal{A}$ is $M$-$p$-$C$ with $p = 2$.

Theorem A.2 Let $1 < p \leq 2$. Let $\phi$ be increasing, concave function satisfying (63) and such that there exist positive constants $c_p, c_\phi > 0$ such that
\[ \phi(t) \leq c_p t^{p-1} \] (64)
and
\[ \phi'(s) s \leq c_\phi \phi(s). \] (65)
Then $\mathcal{A}$ is $M$-$p$-$C$.

Remark A.3 We notice that (64) is necessary condition for $\mathcal{A}$ to be an $M$-$p$-$C$ operator. Indeed, if $\mathcal{A}$ is $M$-$p$-$C$, by taking $\eta = 0$, then it follows that $\mathcal{A}$ is $W$-$p$-$C$, and (64) holds by Hölder inequality.

We set
\[ I := (\mathcal{A}(\xi) - \mathcal{A}(\eta)) \cdot (\xi - \eta), \quad J := |\mathcal{A}(\xi) - \mathcal{A}(\eta)|. \]

Our goal is to prove that there exists a constant $c > 0$ such that $I^{p-1} \geq c J^p$.

We set $t := |\xi|, \ s := |\eta|$ and let $\theta$ be such that $\xi \cdot \eta = \theta |\xi| |\eta| = ts$. Hence $\theta \in [-1.1]$, $t, s > 0$. Moreover by symmetry we can assume that $s \geq t$.

We rewrite $I$ and $J$ as
\[ I = A(|\xi|)|\xi|^2 + A(|\eta|)|\eta|^2 - A(|\xi|)|\xi| - A(|\eta|)|\eta| \cdot \xi \cdot \eta = \phi(t)t + \phi(s)s - \phi(t)s\theta - \phi(s)t\theta, \]
\[ J^2 = \phi^2(t) + \phi^2(s) - 2\phi(t)\phi(s)\theta. \]

Remark A.4 From (63) we deduce that: if $I = 0$ then $\phi(t) = \phi(s), \ \theta = 1$, and $J = 0$. Indeed, assuming $s \geq t$
\[ I = \phi(t)(t - s\theta) + \phi(s)(s - t\theta) \geq \phi(t)(t - s\theta) + \phi(t)(s - t\theta) = \phi(t)(1 - \theta)(t + s) \geq 0 \]
Therefore, if $I = 0$, then $\theta = 1$ or $\phi(t) = 0$. If $\phi(t) = 0$ then $t = 0$ and hence (since $I = 0$) also $s = 0$. If $\theta = 1$, then we have $0 = I = (\phi(s) - \phi(t))(s - t)$ and hence the claim follows.

We notice that if $\phi$ is increasing, then $I = 0$ implies also that $t = s$.

Therefore, in order to prove that $\mathcal{A}$ is $M$-$p$-$C$, we restrict ourselves to the case $s > t > 0$. 

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Proof of Theorem A.1. Set

\[ I_1 := \frac{I}{\phi(t)t} = 1 + \frac{\phi(s)s}{\phi(t)t} - \theta \frac{\phi(s)}{\phi(t)} - \theta \frac{s}{t} \]

\[ J_1 := \frac{J^2}{\phi^2(t)} = 1 + \frac{\phi^2(s)}{\phi^2(t)} - 2\theta \frac{\phi(s)}{\phi(t)} \]

We have

\[
I_1 - J_1 = \frac{\phi(s)s}{\phi(t)t} - \frac{\phi^2(s)}{\phi^2(t)} + \theta \frac{\phi(s)}{\phi(t)} - \theta \frac{s}{t} = \frac{\phi(s)}{\phi(t)} \left( \frac{s - \phi(s)}{t} - \frac{\phi(s)}{\phi(t)} \right) - \theta \left( \frac{s - \phi(s)}{t} \right) \\
= \left( \frac{s}{t} - \frac{\phi(s)}{\phi(t)} \right) \left( \frac{\phi(s)}{\phi(t)} - \theta \right) = \frac{s}{t} \left( 1 - \frac{A(s)}{A(t)} \right) \left( \frac{\phi(s)}{\phi(t)} - \theta \right).
\]

Since \( s > t > 0 \), \( \phi \) is nondecreasing and \( A \) is nonincreasing it follows that \( I_1 - J_1 \geq 0 \). Therefore

\[ I = \phi(t)ti_1 \geq \phi(t)J_1 = \frac{t}{\phi(t)}J^2 = \frac{1}{A(t)}J^2 \geq \frac{1}{||A||_\infty}J^2 = cJ^2, \]

that is the claim. \( \square \)

Proof of Theorem A.2. Our goal is to show that \( I_p^{-1}/J_p \geq \text{const} > 0 \).

We have

\[
\frac{I_p^{-1}}{J^p} = \frac{s^{p-1}}{\phi(s)} \left( \frac{t}{\phi(s)s} \right)^{p-1} = \frac{s^{p-1}}{\phi(s)} \left( 1 + \frac{\phi(t)t}{\phi(s)s} - \theta \frac{\phi(t)}{\phi(s)} - \theta \frac{t}{s} \right)^{p-1} \geq cF(t, s, \theta),
\]

where

\[
F(t, s, \theta) := \frac{\left( 1 + \frac{\phi(t)t}{\phi(s)s} - \theta \frac{\phi(t)}{\phi(s)} - \theta \frac{t}{s} \right)^{p-1}}{\left( 1 + \frac{\phi^2(t)}{\phi^2(s)} - 2\theta \frac{\phi(t)}{\phi(s)} \right)^{p/2}}.
\]

In order to prove the claim it is enough to show that \( F \) is uniformly positive for \( \theta \in [-1, 1] \) and \( s > t > 0 \). Since \( \phi \) is nondecreasing, setting

\[ \alpha := \frac{\phi(t)}{\phi(s)}, \quad z := \frac{t}{s}, \]

it is enough to prove that

\[
G(\alpha, z, \theta) := \frac{(1 + \alpha z - \alpha \theta - z\theta)^{p-1}}{(1 + \alpha^2 - 2\theta z)^{p/2}}
\]
is uniformly positive for \((\alpha, z, \theta) \in D := [0, 1] \times [0, 1] \times [-1, 1] \setminus \{1, 1, 1\}\). The function \(G\) is well defined in \(D\). Indeed the denominator vanishes if and only if \(J = 0\) that is \(\phi(t) = \phi(s)\) and \(\theta = 1\) that is \(\alpha = z = \theta = 1\). On the other hand the numerator of \(G\) vanishes if \(I = 0\), that is if \(\alpha = z = \theta = 1\). Therefore, \(G\) is strictly positive on \(D\).

Moreover taking into account that \(\phi\) is concave we have
\[
\frac{\phi(s) - \phi(t)}{s - t} \leq \phi'(s),
\]
which, together with (65), yields
\[
\frac{\phi(s) - \phi(t)}{s - t} \leq \phi'(s) \frac{s}{\phi(s)} \leq c_\phi.
\]
That is
\[
1 - \alpha \leq c_\phi(1 - z).
\]

Now the claim will follows by proving that
\[
\liminf_{\alpha, z, \theta \to 1} G(\alpha, z, \theta) > 0.
\]
Here the lim inf is computed for \((\alpha, z, \theta) \in D\) and under the constrained (66). Introducing
\[
H(a, b, e) := \left(\frac{ab + 2e - ae - be}{a^2 + 2e - 2ae}\right)^{p/2}
\]
and setting \(a := 1 - \alpha, b := 1 - z, e := 1 - \theta\), (67) is equivalent to
\[
\liminf_{a, b, e \to 0^+} H(a, b, e) > 0
\]
with \(a \leq c_\phi b\).

We shall argue by contradiction. Let \(a_n, b_n, e_n\) be three infinitesimal sequences such that \(H(a_n, b_n, e_n) \to 0\). Since \(a_ne_n\) and \(b_ne_n\) are infinitesimal sequences of order greater then \(e_n\), we have that
\[
0 = \lim_n H(a_n, b_n, e_n) = \lim_n \left(\frac{a_nb_n + 2e_n}{a_n^2 + 2e_n}\right)^{p/2}.
\]
Taking into account that \(a_n \leq c_\phi b_n\), we have
\[
0 = \lim_n \left(\frac{a_nb_n + 2e_n}{a_n^2 + 2e_n}\right)^{p/2} \geq \liminf_n \left(\frac{a_n^2/\phi + 2e_n}{a_n^2 + 2e_n}\right)^{p/2} > 0,
\]
because \(a_n^2/c_\phi + 2e_n\) and \(a_n^2 + 2e_n\) are infinitesimal of the same order and \(p \leq 2\). This contradiction concludes the proof. \(\square\)

**Remark A.5** If \(A\) has the form
\[
A(x, \xi) = a(x)A(|\xi|)\xi,
\]
then the above Theorems A.1 and A.2 hold provided \(a \in L^\infty(\mathbb{R}^N)\) and it is positive a.e..
B Carnot Groups

We quote some facts on Carnot groups and refer the interested reader to [4] for more detailed information on this subject.

A Carnot group is a connected, simply connected, nilpotent Lie group $G$ of dimension $N$ with graded Lie algebra $G = V_1 \oplus \cdots \oplus V_r$ such that $[V_i, V_j] = V_{i+j}$ for $i = 1, \ldots, r - 1$ and $[V_1, V_r] = 0$. Such an integer $r$ is called the step of the group. We set $l = n_1 = \dim V_1$, $n_2 = \dim V_2, \ldots, n_r = \dim V_r$. A Carnot group $G$ of dimension $N$ can be identified, up to an isomorphism, with the structure of a homogeneous Carnot Group $(\mathbb{R}^N, \circ, \delta_R)$ defined as follows; we identify $G$ with $\mathbb{R}^N$ endowed with a Lie group law $\circ$. We consider $\mathbb{R}^N$ split in $r$ subspaces $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_r}$ with $n_1 + n_2 + \cdots + n_r = N$ and $\xi = (\xi^{(1)}, \ldots, \xi^{(r)})$ with $\xi^{(i)} \in \mathbb{R}^{n_i}$. We shall assume that for any $R > 0$ the dilation $\delta_R(\xi) = (R\xi^{(1)}, R^2\xi^{(2)}, \ldots, R^r\xi^{(r)})$ is a Lie group automorphism. The Lie algebra of left-invariant vector fields on $(\mathbb{R}^N, \circ)$ is $G$. For $i = 1, \ldots, n_1 = l$ let $X_i$ be the unique vector field in $G$ that coincides with $\partial/\partial \xi^{(1)}_i$ at the origin. We require that the Lie algebra generated by $X_1, \ldots, X_l$ is the whole $G$.

We denote with $\nabla_l$ the vector field $\nabla_l := (X_1, \ldots, X_l)^T$ and we call it horizontal vector field and by $\text{div}_L$ the formal adjoint on $\nabla_l$, that is (69). Moreover, the vector fields $X_1, \ldots, X_l$ are homogeneous of degree 1 with respect to $\delta_R$ and in this case $Q = \sum_{i=1}^l i n_i = \sum_{i=1}^r i \dim V_i$ is called the homogeneous dimension of $G$. The canonical sub-Laplacian on $G$ is the second order differential operator defined by

$$\Delta_G = \sum_{i=1}^l X_i^2 = \text{div}_L(\nabla_l \cdot)$$

and for $p > 1$ the $p$-sub-Laplacian operator is

$$\Delta_{G,p} u := \sum_{i=1}^l X_i (|\nabla_l u|^{p-2} X_i u) = \text{div}_L (|\nabla_l u|^{p-2} \nabla_l u).$$

Since $X_1, \ldots, X_l$ generate the whole $G$, the sub-Laplacian $\Delta_G$ satisfies the Hörmander hypoellipticity condition.

In this paper $\nabla$ and $| \cdot |$ stand respectively for the usual gradient in $\mathbb{R}^N$ and the Euclidean norm.

Let $\mu \in \mathcal{C}(\mathbb{R}^N; \mathbb{R}^l)$ be a matrix $\mu := (\mu_{ij}), i = 1, \ldots, l, j = 1, \ldots, N$. For $i = 1, \ldots, l$, let $X_i$ and its formal adjoint $X_i^*$ be defined as

$$X_i := \sum_{j=1}^N \mu_{ij}(\xi) \frac{\partial}{\partial \xi_j}, \quad X_i^* := -\sum_{j=1}^N \frac{\partial}{\partial \xi_j}(\mu_{ij}(\xi) \cdot),$$

and let $\nabla_L$ be the vector field defined by $\nabla_L := (X_1, \ldots, X_l)^T = \mu \nabla$ and $\nabla_{L}^* := (X_1^*, \ldots, X_l^*)^T$. 25
For any vector field \( h = (h_1, \ldots, h_l)^T \in C^1(\Omega, \mathbb{R}^l) \), we shall use the following notation \( \text{div}_L(h) := \text{div}(\mu^T h) \), that is

\[
\text{div}_L(h) = - \sum_{i=1}^l X_i^* h_i = -\nabla^*_L \cdot h.
\]

An assumption that we shall made (which actually is an assumption on the matrix \( \mu \)) is that the operator

\[
\Delta_G u = \text{div}_L(\nabla_L u)
\]

is a canonical sub-Laplacian on a Carnot group (see below for a more precise meaning). The reader, which is not acquainted with these structures, can think to the special case of \( \mu = I \), the identity matrix in \( \mathbb{R}^N \), that is the usual Laplace operator in Euclidean setting.

A nonnegative continuous function \( S : \mathbb{R}^N \rightarrow \mathbb{R}_+ \) is called a homogeneous norm on \( \mathbb{G} \), if \( S(\xi^{-1}) = S(\xi) \), \( S(\xi) = 0 \) if and only if \( \xi = 0 \), and it is homogeneous of degree 1 with respect to \( \delta_R \) (i.e. \( S(\delta_R(\xi)) = RS(\xi) \)). A homogeneous norm \( S \) defines on \( \mathbb{G} \) a pseudo-distance defined as \( d(\xi, \eta) := S(\xi^{-1}\eta) \), which in general is not a distance. If \( S \) and \( \hat{S} \) are two homogeneous norms, then they are equivalent, that is, there exists a constant \( C > 0 \) such that \( C^{-1}S(\xi) \leq \hat{S}(\xi) \leq CS(\xi) \). Let \( S \) be a homogeneous norm, then there exists a constant \( C > 0 \) such that \( C^{-1}|\xi| \leq \hat{S}(\xi) \leq C|\xi|^{1/r} \), for \( S(\xi) \leq 1 \). An example of homogeneous norm is \( S(\xi) := \left( \sum_{i=1}^r |\xi|^{2r/i} \right)^{1/2r} \).

Notice that if \( S \) is a homogeneous norm differentiable a.e., then \( |\nabla_L S| \) is homogeneous of degree 0 with respect to \( \delta_R \); hence \( |\nabla_L S| \) is bounded.

We notice that in a Carnot group, the Haar measure coincides with the Lebesgue measure.

Special examples of Carnot groups are the Euclidean spaces \( \mathbb{R}^Q \). Moreover, if \( Q \leq 3 \) then any Carnot group is the ordinary Euclidean space \( \mathbb{R}^Q \).

The simplest nontrivial example of a Carnot group is the Heisenberg group \( \mathbb{H}^1 = \mathbb{R}^3 \). For an integer \( n \geq 1 \), the Heisenberg group \( \mathbb{H}^n \) is defined as follows: let \( \xi = (\xi^{(1)}, \xi^{(2)}) \) with \( \xi^{(1)} := (x_1, \ldots, x_n, y_1, \ldots, y_n) \) and \( \xi^{(2)} := t \). We endow \( \mathbb{R}^{2n+1} \) with the group law \( \hat{\xi} \circ \xi := (\hat{x} + \ddot{x}, \ddot{y}, \ddot{t} + t + 2 \sum_{i=1}^n (\ddot{x}_i - \ddot{y}_i)) \). We consider the vector fields

\[
X_i := \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i := \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t}, \quad \text{for } i = 1, \ldots, n,
\]

and the associated Heisenberg gradient \( \nabla_H := (X_1, \ldots, X_n, Y_1, \ldots, Y_n)^T \). The Kohn Laplacian \( \Delta_H \) is then the operator defined by \( \Delta_H := \sum_{i=1}^n X_i^2 + Y_i^2 \). The family of dilations is given by \( \delta_R(\xi) := (Rx, Ry, R^2t) \) with homogeneous dimension \( Q = 2n + 2 \). In \( \mathbb{H}^n \) a canonical homogeneous norm is defined as \( |\xi|_H := \left( (\sum_{i=1}^n x_i^2 + y_i^2)^2 + t^2 \right)^{1/4} \).
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References

[1] Ph. Benilan, L. Boccardo, T. Galluet, R. Gariepy, M. Pierre and J.L. Vazquez, An $L^1$ - theory of existence and uniqueness of solutions of nonlinear elliptic equations, *Annali della scuola Normale di Pisa* 22 (1995), 241–273.

[2] M.F. Bidaut-Véron and S.I. Pohozaev, Nonexistence results and estimates for some nonlinear elliptic problems, *J. Anal. Math.* 84 (2001), 1–49.

[3] L. Boccardo, T. Galluet and J.L. Vazquez, Nonlinear Elliptic Equations in $\mathbb{R}^n$ without restriction on the data, *J. Differential Equations* 105 (1993), 334–363.

[4] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, Stratified Lie groups and potential theory for their sub-Laplacians. *Springer Monographs in Mathematics*. Springer, Berlin (2007).

[5] H. Brezis, Semilinear equations in $\mathbb{R}^n$ without condition at infinity, *Appl. Math. Optimization* 12 (1984) 271–282.

[6] L. D’Ambrosio, A. Farina, E. Mitidieri and J. Serrin, Comparison principles, uniqueness and symmetry results of solutions of quasilinear elliptic equations and inequalities, preprint 2012.

[7] L. D’Ambrosio and E. Mitidieri, A priori estimates, positivity results, and nonexistence theorems for quasilinear degenerate elliptic inequalities, *Adv. Math.* 224 (2010), 967–1020 .

[8] L. D’Ambrosio and E. Mitidieri, *A Priori Estimates and Reduction Principles for Quasilinear Elliptic Problems and Applications* in print in Advances in Differential Equations, pp. 1-68 (2012).

[9] A. Farina and J. Serrin, Entire solutions of completely coercive quasilinear elliptic equations, *J. Differential Equations* 250 (2011), 4367–4408.

[10] A. Farina and J. Serrin, Entire solutions of completely coercive quasilinear elliptic equations II, *J. Differential Equations* 250 (2011), 4409–4436.

[11] J. Maly and W.P. Ziemer, Fine Regularity of Solutions of Elliptic Differential Equations, Mathematical Surveys and Monographs, AMS 1997.
[12] E. Mitidieri and S.I. Pohozaev, Non Existence of Positive Solutions for Quasi-linear Elliptic Problems on $\mathbb{R}^N$, *Tr. Mat. Inst. Steklova* **227** (1999), 192–222.

[13] E. Mitidieri and S.I. Pohozaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities, *Tr. Mat. Inst. Steklova* **234** (2001), 1–384.

[14] P. Pucci and J. Serrin, The Strong Maximum Principle, "*Progress in Nonlinear Differential Equations and their Applications*", **73**, Birkhauser Publ., Switzerland, 2007, X, 234 pages.

[15] J. Serrin, Local behavior of solutions of quasi–linear equations, *Acta Math.* **111** (1964) 247–302.