Backward in time problem of a double porosity material with microtemperature

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Abstract
In the present study we consider the theory of thermoelastodynamics in the case of materials with double porosity structure and microtemperature. This study is devoted to the investigation of a backward in time problem associated with double porous thermoelastic materials with microtemperature. In the first part of the paper, in case of the bounded domains the impossibility of time localization of solutions is obtained. This study is equivalent to the uniqueness of solutions for the backward in time problem. In the second part of the paper, a Phragmen-Lindelof alternative in the case of semi-infinite cylinders is obtained.

keywords: Thermoeelasticity with double porosity Microtemperature Backward problem Impossibility of localization

1 Introduction
In the last years many authors were interested in the linear theory of elastic materials with double porosity. The first studies regarding this theory are encountered in the papers of Barenblatt, [1]. The concept of double porosity model allows for the body to have a double porous structure: a macro porosity connected to pores in the body and a micro porosity connected to fissures in the skeleton. According to Barenblatt, [2], Berryman, [3] and Khalili, [4], the particular applications of materials with double porosity are in geophysics and according to Cowin, [5] in mechanics of bone. The basic equations for elastic materials with double porosity involve the displacement vector field, a pressure associated with the pores and a pressure associated with the fissures [6-8]. We note that in the equilibrium theory the fluid pressures become independent of the displacement vector field.

The theory for the behaviour of porous solids in which the skeletal or matrix materials are elastic and the interstices are void of material was studied by Nunziato and Cowin, [9]. The intended applications of this theory are to geological materials such as rocks and soils and to manufactured porous materials such as ceramics and pressed powders. Iesan and Quintanilla, [10] used the Nunziato- Cowin theory of materials with voids to derive a theory of thermoelastic solids which have a double porosity structure. In contrast with the classical theory of elastic materials with double porosity, the porosity structure in the case of equilibrium is influenced by the displacement field. According to Quintanilla [11] is proved the impossibility of the localization in time of the solutions of the linear thermoelasticity with voids.

The study of backward in time problem is very important from the thermomechanical point of view because it offers information about the behaviour of the system in the past using the information that we have at the present time. Usually the Saint Venant’s principle is used for the spatial behavior of
the solutions for partial differential equations. The studies regarding the spatial decay estimates were obtained for elliptic [12], parabolic [13-14] and hyperbolic [15] equations. The main aim of the spatial decay estimates is to model the perturbations on a side of the boundary that are damped for the points located at some distance from this side of the boundary. For this analysis it is necessary to use a semi-infinite cylinder whose finite end is perturbed and our goal is to identify the effects when the spatial variable increases. The harmonic vibrations in thermoelastic dynamics with double porosity structure for the backward in time problem was studied by Florea, [16].

The phenomenon for which the mechanisms of dissipation are very strong, such that the solutions vanish after a finite time, is known as the localization in time of solutions. The impossibility of localization in time of solutions is an open problem because the proof of this concept exists only in some linear situations. In the particular case of the linear thermodynamics theory of visco-elastic solids with voids, the solutions decay can be controlled by some particular exponential or polynomial functions, [17-21]. The problem of the impossibility of localization of solutions was proved for the classical thermoelasticity with porous dissipation [22] and in the isotherm case with porous and elastic viscosity [23].

The aim of our paper is to show that in the case of thermoelasticity with double porosity structure and microtemperature the only solution that vanishes after a finite time is the null solution, when the mechanisms of dissipation are the double porous dissipation, the temperature and the microtemperature. Our obtaining results can be also compared with those obtained in [17-21]. In our paper we will give information regarding the upper bound for the solution decay. In the previous results, [17-20], the authors proved that after a small period of time the thermomechanical deformations are very small and they can be neglected. In our paper we will highlight that they are not null for any positive time.

The present study represents a continuation of the research regarding the impossibility of localization in thermo-porous-elasticity with microtemperatures realized by Quintanilla, [24], using the results of Florea, [25].

The present study is structured as follows: in the second section the basic equations for the backward in time problem in the case of materials with double porosity structure and microtemperature are described. Also, in this section the conditions imposed on the parameters that influence the behavior of the porous materials are presented. The impossibility of localization in time of solutions for the backward in time problem for a double porous material with microtemperature is expressed in the third section. We state here the conservation of the energy law and we highlight the main theorem of the present study. For the particular case of a semi-infinite cylinder a Phragmen-Lindelof alternative is obtained in the section 4. In the last section of the paper are drawn the conclusions of the present study.

2 Basic equations for the double porous materials with microtemperature

The equations of evolution that govern the problem of thermoelasticity with double porosity structure for the materials with microtemperature in the absence of the supply terms are, [17, 18].
\begin{align*}
t_{ji,j} &= \rho \ddot{u}_i \\
\sigma_{j,j} + \xi &= k_1 \ddot{\phi} \\
\tau_{j,j} + \zeta &= k_2 \ddot{\psi} \tag{1}
\end{align*}

where: \( \rho \) is the mass density, \( k_1, k_2 \) are the coefficients of equilibrated inertia, \( \sigma_{j,j}, \tau_{j,j} \) are the equilibrated stress vectors, \( \xi, \zeta \) are the intrinsic equilibrated body forces, \( t_{ji} \) are the stress tensors, \( u_i \) is the displacement, \( \phi, \psi \) are the volume fraction fields in the reference configuration.

The equation of energy is given in (2) and the equation of the first moment of energy is given in (3):

\begin{align*}
\rho T_0 \dot{\eta} &= Q_{j,j} \tag{2} \\
\rho \dot{\varepsilon}_i &= Q_{ji,j} + Q_i - q_i \tag{3}
\end{align*}

where \( T_0 \) is the constant absolute temperature of the body in the reference configuration, \( \eta \) is the entropy, \( Q_j \) is the heat flux, \( \varepsilon_i \) represent the first moment of energy vector and \( q_i \) is the microheat flux average, \( Q_{ji} \) is the first heat flux moment tensor.

We will consider in our study that we deal with a centrosymmetric material. In this case the constitutive equations for the linear theory are:

\begin{align*}
t_{ij} &= C_{ijkl} u_{k,l} + B_{ij} \phi + D_{ij} \psi - \beta_{ij} \theta \\
\sigma_i &= \alpha_{ij} \phi_{,j} + b_{ij} \psi_{,j} - N_{ij} T_j \\
\tau_i &= b_{ji} \phi_{,j} + \gamma_{ij} \psi_{,j} - M_{ij} T_j \\
\xi &= -B_{ij} u_{i,j} - \alpha_1 \phi - \alpha_3 \psi + \gamma_1 \theta \\
\zeta &= -D_{ij} u_{i,j} - \alpha_3 \phi - \alpha_2 \psi + \gamma_2 \theta \\
\rho \eta &= \beta_{ij} u_{i,j} + \gamma_1 \phi + \gamma_2 \psi + a \theta \\
Q_i &= \kappa_{ij} \theta_{,j} + L_{ij} T_j \\
\rho \varepsilon_i &= -N_{ij} \phi_{,j} - M_{ji} \psi_{,j} - P_{ij} T_j \\
Q_{ij} &= -A_{ijrs} T_{s,r} \\
q_i &= (L_{ij} - R_{ij}) T_j + (\kappa_{ij} - \lambda_{ij}) \theta_{,j} \tag{4}
\end{align*}

where \( C_{ijkl} \) is the elasticity tensor, \( \beta_{ij} \) is the thermal dilatation tensor, \( \kappa_{ij} \) is the heat conductivity tensor, \( \beta_{ij} \) is the tensor of thermal dilatation, \( B_{ij}, D_{ij}, \alpha_{ij}, b_{ij}, \gamma_{ij}, \alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2, a \) are typical functions in double porous theory and \( N_{ij}, M_{ij}, R_{ij}, \lambda_{ij}, A_{ijrs} \) are tensors which are usual in the theories with microtemperatures. In the constitutive equations (4) \( \theta \) represents the temperature and \( T_i \) are the microtemperatures.

Introducing the constitutive equations (4) into the evolution equations (1) the system of the field equations for the thermoelasticity with double porosity and microtemperatures is obtained:

\begin{align*}
\rho \ddot{u}_i &= (C_{ijkl} u_{k,l} + B_{ij} \phi + D_{ij} \psi - \beta_{ij} \theta)_{,j} \tag{2.5.a} \\
k_1 \ddot{\phi} &= (\alpha_{ij} \phi_{,i} + b_{ij} \psi_{,i} - N_{ij} T_j)_{,j} - B_{ij} u_{i,j} - \alpha_1 \phi - \alpha_3 \psi + \gamma_1 \theta \tag{2.5.b}
\end{align*}
\[k_2 \ddot{\psi} = (b_{ij} \phi_{,i} + \gamma_{ij} \psi_{,i} - M_{ij} T_{ij})_{,j} - D_{ij} u_{i,j} - \alpha_3 \phi - \alpha_2 \psi + \gamma_2 \theta\]  

(2.5.c)

\[a \dot{\theta} = -\beta_{ij} \dot{u}_{i,j} - \gamma_1 \psi - \gamma_2 \ddot{\psi} + \frac{1}{T_0} (\kappa_{ij} \theta_{,j} + L_{ij} T_{ij})_{,j}\]  

(2.5.d*)

\[P_{ij} \dot{T}_{j} = (A_{ijrs} T_{s,r})_{,j} - R_{ij} T_{j} - \lambda_{ij} \theta_{,j} - N_{ij} \dot{\phi}_{,j} - M_{ij} \dot{\psi}_{,j}\]  

(2.5.e*)

Proving the uniqueness of the solution of the backward in time problem, implies the impossibility of localization of the solutions of the above system. The system of equations which describes the backward in time problem is given by the same set of equations as (2.5.a)-(2.5.c) while (2.5.d*) and (2.5.e*) change into:

\[a \dot{\theta} = -\beta_{ij} \dot{u}_{i,j} - \gamma_1 \psi - \gamma_2 \ddot{\psi} - \frac{1}{T_0} (\kappa_{ij} \theta_{,j} + L_{ij} T_{ij})_{,j}\]  

(2.5.d)

\[P_{ij} \dot{T}_{j} = - (A_{ijrs} T_{s,r})_{,j} + R_{ij} T_{j} + \lambda_{ij} \theta_{,j} - N_{ij} \dot{\phi}_{,j} - M_{ij} \dot{\psi}_{,j}\]  

(2.5.e)

Because the constitutive coefficients are symmetric we have:

\[C_{ijkl} = C_{klij}; \alpha_{ij} = \alpha_{ji}; b_{ij} = b_{ji}; B_{ij} = B_{ji}, D_{ij} = D_{ji}.\]

For the case of anisotropic and homogeneous material we can draw the assumption that the tensors \(A_{ijrs}, P_{ij}, N_{ij}, L_{ij}, R_{ij}, \lambda_{ij}\) are also symmetric:

\[A_{ijkl} = A_{klji}, P_{ij} = P_{ji}, M_{ij} = M_{ji}, L_{ij} = L_{ji}, N_{ij} = N_{ji}, R_{ij} = R_{ji}, \lambda_{ij} = \lambda_{ji}.\]

In the context of theories with microtemperature as a consequence of Clausius-Duhem inequality, we have the following assumption, \[17\]:

\[\kappa_{ij} \xi_{,i} + (L_{ij} + T_0 \lambda_{ij}) \theta_{,j} T_{ij} + T_0 R_{ij} T_{ij} + T_0 A_{ijrs} T_{s,r} T_{ij} \geq 0\]  

(6)

In order to obtain the estimated results it is necessary to impose the positivity of several functions and tensors:

\[\rho(X) \geq \rho_0 > 0; \quad k_1(X) \geq k_1^0 > 0; \quad k_2(x) \geq k_2^0 > 0; \quad a(x) \geq a_0 > 0; \quad P_{ij} \xi_{i,j} \geq p_0 \xi_i \xi_j, p_0 > 0\]  

(2.5.1)

\[\kappa_{ij} \xi_{,i} + (L_{ij} + T_0 \lambda_{ij}) \xi_{j} \xi_{,i} + T_0 R_{ij} \xi_{i,j} \geq C_0 (\xi_i \xi_j + \zeta_i \zeta_j), C_0 > 0, (\forall) \xi_i \xi_j\]  

(2.5.2)

\[C_{ijkl} u_{i,j} u_{k,l} + \alpha_{ij} \phi_{,i} \phi_{,j} + \gamma_{ij} \psi_{,i} \psi_{,j} + 2b_{ij} \phi_{,i} \psi_{,j} + 2B_{ij} u_{i,j} \phi + 2D_{ij} u_{i,j} \psi + + \alpha_1 \phi^2 + \alpha_2 \psi^2 + 2 \alpha_3 \phi \psi \geq C^* (u_{i,j} u_{i,j} + \phi_{,i} \phi_{,i} + \psi_{,i} \psi_{,i} + \phi^2 + \psi^2),\]  

(2.5.3)

\[\alpha_{ij} \xi_{i,j} \geq 0; \quad b_{ij} \xi_{i,j} \geq 0, (\forall) \xi_i \]  

(2.5.4)

\[A_{ijrs} \xi_{i,j} \xi_{s,r} \geq C_1 \xi_{i,j} \xi_{i,j}, C_1 > 0, (\forall) \xi_{i,j}\]

The assumption \[2.5.1\] is related to the thermomechanical characteristics, \[2.5.2\] and \[2.5.4\] are consequences of the Clausius-Duhem inequality, \[2.5.3\] gives the information that the internal energy is positive and may be expressed based on the theory of mechanical stability.
3 Main results regarding the impossibility of localization in time

Let us consider a bounded domain $B$ with the boundary $\partial B$. The study of impossibility of localization in time for solutions of the backward in time problem is equivalent with the study of the uniqueness of solutions for the mentioned problem given by the system of equations \(2.5.a-2.5.e\). To prove the uniqueness of solutions for the backward in time problem it is sufficient to show that only the null solution satisfies our problem with null initial and boundary conditions. In the next computations we assume that the domain $B$ is smooth enough to apply the divergence theorem.

The initial conditions are:

\[
\begin{align*}
    u_i(X,0) &= \dot{u}_i(X,0) = \phi(X,0) = \dot{\phi}(X,0) = 0 \quad \text{(8)} \\
    \psi(X,0) &= \dot{\psi}(X,0) = \theta(X,0) = 0, \quad T_i(X,0) = 0 \quad X \in B
\end{align*}
\]

and the boundary conditions:

\[
\begin{align*}
    u_i(X,t) = \phi(X,t) = \psi(X,t) = \theta(X,t) = T_i(X,0) = 0, \quad X \in \partial B, t \geq 0 \quad \text{(9)}
\end{align*}
\]

The aim of this section is to obtain the energy relation for the double porous material with microtemperature. We will multiply \(2.5.a\) by $\dot{u}_i$, \(2.5.b\) by $\dot{\phi}$, \(2.5.c\) by $\psi$, \(2.5.d\) by $\theta$ and \(2.5.e\) by $T_j$, the obtained relations will be integrated on $[0,t]$ and they will be summed. Using the divergence theorem and taking into account the boundary conditions based on the principle of conservation of energy, we have the following relation:

\[
E_1(t) = \frac{1}{2} \int_B \left( \rho \ddot{u}_i \dot{u}_i + k_1 \dot{\phi}^2 + k_2 \psi^2 + a \theta^2 + P_{ij} T_j T_j + C_{ijkl} u_{i,j,k,l} + 2B_{ij} \phi u_{i,j} + ight. \\
+ 2D_{ij} \psi u_{i,j} + \alpha_{ij} \phi_j \psi_j + \gamma_{ij} \psi_j \psi_j + 2b_{ij} \psi_j \psi_j + \alpha_1 \phi^2 + 2 \alpha_3 \phi \psi + \alpha_2 \psi^2 \bigg) dV = \quad (10)
\]

Using the same procedure of multiplying the equations \(2.5.a\) by $\dot{u}_i$, \(2.5.b\) by $\dot{\phi}$, \(2.5.c\) by $\psi$, \(2.5.d\) by $\theta$ and \(2.5.e\) by $-T_j$, integrating on $[0,t]$ and using the divergence theorem we have the following expression:

\[
E_2(t) = \frac{1}{2} \int_B \left( \rho \ddot{u}_i \dot{u}_i + k_1 \phi^2 + k_2 \psi^2 - a \theta^2 - P_{ij} T_j T_j + C_{ijkl} u_{i,j,k,l} + 2B_{ij} \psi u_{i,j} + ight. \\
+ \alpha_{ij} \phi_j \phi_j + 2b_{ij} \psi_j \psi_j + \alpha_1 \phi^2 2 \alpha_3 \phi \psi + \gamma_{ij} \psi_j \psi_j + \alpha_2 \psi^2 \bigg) dV = \quad (11)
\]

\[
= -\int_0^t \left[ \frac{1}{T_0} \int_B \left( \kappa_{ij} \theta_j \theta_j + L_{ij} T_j \theta_j + A_{ijkl} T_{s,r} T_{j,i} + R_{ij} T_j T_j + \lambda_{ij} \theta_j T_i \right) dV \right] ds + \\
+ \int_0^t \left[ (\beta_{ij} \theta_j) \dot{u}_i - (M_{ij} T_j) \dot{\psi} - (N_{ij} T_j) \dot{\phi} + \gamma_1 \dot{\theta} + \gamma_2 \dot{\psi} \right] dVds
\]
Taking into consideration the equations (2.5.a)-(2.5.e), the initial and boundary conditions, (8), (9), the following identity is obtained:

\[
\int_B \left( \rho \dot{u}_i + k_1 \dot{\phi}^2 + k_2 \dot{\psi}^2 - a \theta^2 - P_{ij} T_j T_j \right) \, dV = \int_B \left( C_{ijkl} u_{i,j} u_{k,l} + \alpha_{ij} \phi_i \phi_j + \gamma_{ij} \psi_i \psi_j + 2b_{ij} \psi_j \phi_i + 2B_{ij} u_{i,j} \phi + 2D_{ij} u_{i,j} \psi + \alpha_1 \phi^2 + \alpha_2 \psi^2 + 2\alpha_3 \phi \psi \right) \, dV
\]

(12)

The impossibility of localization of the solutions in the theory with double porosity and microtemperature is proved in the following theorem.

**Theorem 1** Let \((u_i, \phi, \psi, \theta, T_i)\) be a solution of the backward in time problem (2.5.a)-(2.5.e) with the initial conditions (8) and the boundary conditions (9). The only solution of the mentioned problem is the null solution \(u_i = 0, \phi = 0, \psi = 0, \theta = 0, T_i = 0\).

**Proof 1** Replacing (12) into (11) we obtain a new expression for \(E_2(t)\):

\[
E_2(t) = \int_B \left( C_{ijkl} u_{i,j} u_{k,l} + \alpha_{ij} \phi_i \phi_j + \gamma_{ij} \psi_i \psi_j + 2b_{ij} \psi_j \phi_i + 2B_{ij} u_{i,j} \phi + 2D_{ij} u_{i,j} \psi + \alpha_1 \phi^2 + \alpha_2 \psi^2 + 2\alpha_3 \phi \psi \right) \, dV
\]

\[= - \int_0^t \int_B \left( \kappa_{ij} \theta_i \theta_j + L_{ij} \theta_j T_i + T_0 A_{ijr} T_{s,r} T_{j,i} + T_0 R_{ij} T_i T_j + T_0 \lambda_{ij} \theta_j T_i \right) \, dV \, ds + \int_0^t \int_B \left[ (\beta_{ij} \theta)_j \psi - (M_{ij} T_i)_j \phi + \gamma_1 \theta \phi + \gamma_2 \theta \psi \right] \, dV \, ds
\]

The energy can be expressed under the below form, if we consider a positive constant \(\varepsilon\), small enough:

\[
E(t) = E_2(t) + \varepsilon E_1(t), \quad \varepsilon \in (0, 1)
\]

Taking into account that \(E(t)\) is a positive function we have the following form for the energy:

\[
E(t) = \frac{\varepsilon}{2} \int_B \left( \rho \dot{u}_i + k_1 \dot{\phi}^2 + k_2 \dot{\psi}^2 + a \theta^2 + P_{ij} T_j T_j \right) \, dV + \frac{2 + \varepsilon}{2} \int_B \left( C_{ijkl} u_{i,j} u_{k,l} + \alpha_{ij} \phi_i \phi_j + \gamma_{ij} \psi_i \psi_j + 2b_{ij} \psi_j \phi_i + 2B_{ij} u_{i,j} \phi + 2D_{ij} u_{i,j} \psi + \alpha_1 \phi^2 + \alpha_2 \psi^2 + 2\alpha_3 \phi \psi \right) \, dV
\]
On the other hand,

\[ E(t) = - \int_0^t \int_B \left( \frac{1}{T_0} \kappa_{ij} \theta_i \theta_j + L_{ij} \theta_j T_i + T_0 A_{ij rs} T_s T_j, i + T_0 R_{ij} T_i T_j + T_0 \lambda_{ij} T_i \theta_j \right) dV ds + \]

\[ + \int_B \left[ \int_0^t \left( (\beta_{ij} \theta), j \right) \dot{u}_i - (M_{ij} T_i), j \dot{\psi} - (N_{ij} T_i), j \dot{\phi} + \gamma_1 \theta \dot{\phi} + \gamma_2 \theta \dot{\psi} \right] dV ds + \]

\[ + \varepsilon \int_0^t \frac{1}{T_0} (\kappa_{ij} \theta_i \theta_j + L_{ij} \theta_j T_i + T_0 A_{ij rs} T_s T_j, i + T_0 R_{ij} T_i T_j + T_0 \lambda_{ij} T_i \theta_j) dV ds \]

The above relation yields, for \( \varepsilon \in (0, 1) \):

\[ E(t) = - (1 - \varepsilon) \int_0^t \int_B \left( \frac{1}{T_0} \kappa_{ij} \theta_i \theta_j + L_{ij} \theta_j T_i + T_0 A_{ij rs} T_s T_j, i + T_0 R_{ij} T_i T_j + T_0 \lambda_{ij} T_i \theta_j \right) dV ds + \]

\[ + \int_B \left[ \int_0^t \left( (\beta_{ij} \theta), j \right) \dot{u}_i - (M_{ij} T_i), j \dot{\psi} - (N_{ij} T_i), j \dot{\phi} + \gamma_1 \theta \dot{\phi} + \gamma_2 \theta \dot{\psi} \right] dV ds \]

from where:

\[ \frac{dE(t)}{dt} = - (1 - \varepsilon) \int_B \frac{1}{T_0} (\kappa_{ij} \theta_i \theta_j + L_{ij} \theta_j T_i + T_0 A_{ij rs} T_s T_j, i + T_0 R_{ij} T_i T_j + T_0 \lambda_{ij} T_i \theta_j) dV ds + \]

\[ + \int_B \left[ \int_0^t \left( (\beta_{ij} \theta), j \right) \dot{u}_i - (M_{ij} T_i), j \dot{\psi} - (N_{ij} T_i), j \dot{\phi} + \gamma_1 \theta \dot{\phi} + \gamma_2 \theta \dot{\psi} \right] dV ds \]

but,

\[ \int_B (\beta_{ij} \theta), j \dot{u}_i dV = \int_B \beta_{ij}, j \dot{u}_i dV + \int_B \beta_{ij} \theta, j \dot{u}_i dV \]

The inequality of arithmetic and geometric means implies that:

\[ \int_B (\beta_{ij} \theta), j \dot{u}_i dV \leq C_1 \int_B \left( \rho \dot{u}_i + a \theta^2 \right) dV + \varepsilon_1 \int_B \kappa_{ij} \theta, i \theta, j dV \]

where \( \varepsilon_1 \) is small enough, \( C_1 \) is a positive constant that can be determined based on the constitutive coefficients and \( \varepsilon_1 \);

\[ \int_B (M_{ij} T_i), j \dot{\psi} dV \leq C_2 \int_B \left( k_2 \dot{\phi}^2 + P_{ij} T_i T_j \right) dV \]

where \( C_2 \) can be determined. Therefore, there is a positive constant \( C \) such that:

\[ \frac{dE}{dt} \leq C \int_B \left( \rho \dot{u}_i + k_1 \dot{\phi}^2 + k_2 \dot{\psi}^2 + a \theta^2 + P_{ij} T_i T_j \right) dV \]
which is equivalent with the estimate:

\[ \frac{dE}{dt} \leq C^*E(t) \iff \frac{dE}{E} \leq C^*dt \iff \ln E \leq C^*t + C \iff E(t) \leq Ce^{C^*t}. \]

For \( t = 0 \) we will have the estimate:

\[ E(t) \leq E(0)e^{C^*t} \]

But, the initial condition leads us to \( E(t) = 0 \) for every \( t \geq 0 \) that is equivalent with:

\[
\dot{u}_i = 0; \dot{\phi} = 0; \dot{\psi} = 0; \theta = 0; T_i(t) = 0 \iff u_i = C_1; \phi = C_2; \psi = C_3; \theta = T_i = 0
\]

taking into account the initial conditions we obtain that the solution for our problem is the null solution:

\[ u_i = 0; \phi = 0; \psi = 0; \theta = 0; T_i = 0 \]

4 Phragmen-Lindelof alternative for the solution of backward in time problem with double porosity and microtemperature

We consider a semi-infinite prismatic cylinder \( B = D \times (0, \infty) \) that is occupied by a body with a double porosity structure with micro-temperature. By \( D \) we note the cross section in the cylinder. The boundary of the section is a piece-wise continuously differentiable curve denoted by \( \partial D \) sufficiently smooth to admit application of divergence theorem. The lateral surface of the cylinder is \( \Pi = \partial D \times (0, \infty) \). The cylinder is assumed to be free of load on the lateral boundary surface.

The lateral boundary conditions are:

\[
u_i(x_1, x_2, 0, t) = \tilde{u}_i; \phi(x_1, x_2, 0, t) = \tilde{\phi}; \psi(x_1, x_2, 0, t) = \tilde{\psi}; \theta(x_1, x_2, 0, t) = \tilde{\theta}; T_i(x_1, x_2, 0, t) = \tilde{T}_i \quad (14)
\]

On the base of the cylinder the following boundary conditions are assumed:

\[
u_i(x_1, x_2, 0, t) = \tilde{u}_i; \phi(x_1, x_2, 0, t) = \tilde{\phi}; \psi(x_1, x_2, 0, t) = \tilde{\psi}; \theta(x_1, x_2, 0, t) = \tilde{\theta}; T_i(x_1, x_2, 0, t) = \tilde{T}_i \quad (14)
\]

For the solution of the problem determined by the system \( (2.5.a)-(2.5.e) \) with initial conditions \( (13) \) and boundary conditions \( (14) \) we want to obtain a Phragmen-Lindelof alternative necessary for the interpretation of the behavior of the solution of our boundary value problem. Our aim in this section is to estimate the absolute value of the defined function \( H_\omega \) from \( (15) \) by means of its spatial derivative.
We define the function:

\[ H_\omega(z, t) = \int_0^t \int_{D(z)} e^{-2\omega s} \left[ C_{i3kl} u_{k,l} + B_{i3} \phi + D_{i3} \psi - \beta_{i3} \theta \right] \dot{u}_i \, d\sigma + \int_0^t \int_{D(z)} e^{-2\omega s} \left[ \alpha_{i3} \phi, i + b_{i3} \psi, i - N_{i3} T_i \right] \dot{\phi} \, d\sigma + \int_0^t \int_{D(z)} e^{-2\omega s} \left[ b_{3i} \phi, i + \gamma_{i3} \psi, i - M_{i3} T_i \right] \dot{\psi} \, d\sigma + \int_0^t \int_{D(z)} e^{-2\omega s} \left[ b_{3i} \phi, i + \gamma_{i3} \psi, i - M_{i3} T_i \right] \dot{\psi} \, d\sigma \]

\[ + \int_0^t \int_{D(z)} e^{-2\omega s} \left[ b_{3i} \phi, i + \gamma_{i3} \psi, i - M_{i3} T_i \right] \dot{\psi} \, d\sigma + \int_0^t \int_{D(z)} e^{-2\omega s} \left[ b_{3i} \phi, i + \gamma_{i3} \psi, i - M_{i3} T_i \right] \dot{\psi} \, d\sigma + \int_0^t \int_{D(z)} e^{-2\omega s} \left[ b_{3i} \phi, i + \gamma_{i3} \psi, i - M_{i3} T_i \right] \dot{\psi} \, d\sigma \]

\[ + \int_0^t \int_{D(z)} e^{-2\omega s} \left[ b_{3i} \phi, i + \gamma_{i3} \psi, i - M_{i3} T_i \right] \dot{\psi} \, d\sigma + \int_0^t \int_{D(z)} e^{-2\omega s} \left[ b_{3i} \phi, i + \gamma_{i3} \psi, i - M_{i3} T_i \right] \dot{\psi} \, d\sigma + \int_0^t \int_{D(z)} e^{-2\omega s} \left[ b_{3i} \phi, i + \gamma_{i3} \psi, i - M_{i3} T_i \right] \dot{\psi} \, d\sigma \]

Here we have \( D(z) = \{ X \in B | x_3 = z \} \) that denotes the cross section of the cylinder at a distance \( z \) from the base. Through means of the divergence theorem and employing the field equations, boundary and initial conditions we obtain:

\[ H_\omega(z + h, t) - H_\omega(z, t) = \frac{1}{2} \int_{R(z+h,z)} \chi_\omega(t) \, dV, (\forall) \]

where \( R(z + h, z) = \{ X \in B | z < x_3 < z + h \} \).

The internal energy is:

\[ \Phi = \rho \dot{u}_i \dot{u}_i + k_1 \phi^2 + k_2 \psi^2 + a \theta^2 + P_{ij} T_j T_j + C_{i3kl} u_{k,l} + + 2B_{ij} u_{i,j} \phi + 2D_{ij} u_{i,j} \psi + \]

\[ + \alpha_{i,j} \phi, i \phi, j + \gamma_{ij} \psi, i \psi, j + 2b_{ij} \phi, i \phi, j + \alpha_1 \phi^2 + \alpha_2 \psi^2 + 2\alpha_3 \phi \psi \]

such that:

\[ \chi_\omega(t) = e^{-2\omega t} \Phi(t) + \int_0^t e^{-2\omega s} \left[ 2\omega \Phi(s) + 2 \frac{\kappa_{ij}}{T_0} \theta, i(s) \theta, j(s) + 2 \frac{L_{ij}}{T_0} \theta, i T_j(s) \right] \, ds \]

\[ + 2A_{i3rs} T_{s,r}(s) T_{i,j}(s) + 2R_{i3} T_i(s) T_j(s) + 2\lambda_{ij} \theta, i T_j(s) \right] \, ds \]

From \( \Box \) we have:

\[ \frac{\partial H_\omega}{\partial z} = \frac{1}{2} \int_{D(z)} \chi_\omega(t) \, dz \]

that leads to the following relation:

\[ \frac{\partial H_\omega}{\partial z} = \frac{1}{2} \int_{D(z)} \Phi(t) \, dz + \int_0^t \int_{D(z)} e^{-2\omega s} \left[ \omega \Phi(s) + W \right] \, d\sigma \]
where,
\[
W = \frac{\kappa_{ij} T_0 \theta_i \theta_j}{T_0} + \frac{L_{ij} T_i T_j}{T_0} + A_{ijrs} T_s T_{i,j} + R_{ij} T_i T_j + 2 \lambda_{ij} T_i T_j
\]

Further, we want to estimate the absolute value of \( H_\omega \) in terms of spatial derivatives, in order to get a differential inequality, such that:
\[
|H_\omega| \leq C_\omega \frac{\partial H_\omega}{\partial z} \quad (\forall) z \geq 0 \tag{20}
\]
The above inequality is known in the literature of specialty regarding the spatial estimate as Phragmén-Lindelöf alternative.

Under the assumption (a.3) the internal energy from (17) leads us to the following inequality:
\[
\Phi \geq \rho \dot{u}_i \dot{u}_i + k_1 \phi^2 + k_2 \psi^2 + a \theta^2 + P_{ij} T_i T_j + C^* \left( u_{i,j} u_{i,i} + \phi, i \phi, i + \psi, i \psi, i + \phi^2 + \psi^2 \right)
\]

Therefore the relation (16) yields:
\[
H_\omega (z, t) - H_\omega (z, t) \geq \frac{1}{2} \int_R e^{-2\omega s} \left[ \rho \dot{u}_i \dot{u}_i + k_1 \phi^2 + k_2 \psi^2 + a \theta^2 + P_{ij} T_i T_j + C^* \left( u_{i,j} u_{i,i} + \phi, i \phi, i + \psi, i \psi, i + \phi^2 + \psi^2 \right) \right] da + \\
+ \int_0^t \int_R e^{-2\omega s} \left\{ \omega \left[ k_1 \phi^2 + k_2 \psi^2 + a \theta^2 + P_{ij} T_i T_j + C^* \left( u_{i,j} u_{i,i} + \phi, i \phi, i + \psi, i \psi, i + \phi^2 + \psi^2 \right) \right] + \kappa_{ij} \theta_i \theta_j \right\} dads
\]
Based on the inequality of arithmetic and geometric means and also the Cauchy-Schwarz inequality we obtain:
\[
|H_\omega (z, t)| \leq C_\omega \left\{ \frac{e^{-2\omega t}}{2} \int_{D(z)} \Phi(t) dz + \int_0^t \int_{D(z)} \omega e^{-2\omega s} \Phi(s) dads + \\
+ \int_0^t \int_{D(z)} e^{-2\omega s} \left[ \frac{1}{T_0} \left( \kappa_{ij} \theta_i \theta_j + L_{ij} T_i T_j \right) + A_{ijrs} T_i T_{r,i} + R_{ij} T_i T_j + \lambda_{ij} T_i T_j \right] dads \right\}
\]
Thus the alternative (20) was proved.

From the inequality (20) we can extract the following two inequalities:
\[
- \frac{\partial H_\omega}{\partial z} \leq \frac{1}{C_\omega} H_\omega \quad \text{and} \quad \frac{\partial H_\omega}{\partial z} \geq \frac{1}{C_\omega} H_\omega \tag{21}
\]
Taking into consideration the computations from Flavin, \[15\] we obtain two estimates:
\[
H_\omega (z, t) \geq H_\omega (z_0, t) e^{\frac{\omega (z-z_0)}{C_\omega}} \tag{22}
\]
\((\forall) z \geq z_0, z_0 > 0 \) and \(H_\omega(z_0, t) > 0\) that lead to: 
\[
\lim_{z \to \infty} e^{-\frac{z}{\zeta}} \int_{R(z)} \chi_\omega(t) dv > 0 \quad \text{and} \quad -H_\omega(z, t) \leq H_\omega(z_0, t) e^{-\frac{z}{\zeta}}
\]  
(23)

\((\forall) z \geq 0 \) and \(H_\omega(z, t) \leq 0\). From (23) it is obvious that \(H_\omega(z, t) \to 0\) for \(z \to \infty\).

Let us introduce the following estimate:
\[
E_\omega(z, t) = \frac{e^{-2\omega t}}{2} \int_{R(z)} \Phi(t) dz + \int_0^t \int_{R(z)} e^{-2\omega s} \left[ \omega \Phi(s) + \frac{1}{T_0} (\kappa_{ij} \theta_i \theta_j + L_{ij} T_i T_j) \right] \\
+ A_{ijrs} T_{ij} T_{rs} + R_{ij} T_i T_j + \lambda_{ij} T_i \theta_j \right] ds ds
\]
where \(R(z) = \{ X \in B \mid z < x_3 \}\). Based on (23) we observe that:
\[
E_\omega(z, t) \leq E_\omega(0, t) e^{-\frac{z}{\zeta}}, z \geq 0
\]  
(25)

Now, we can draw the following conclusions: if \((u_i, \phi, \psi, \theta, T_i)\) is a solution of the backward in time problem defined by the system (2.5.a)-(2.5.e) with the null initial conditions (8) and boundary conditions (9) there are two situations: the solution satisfies the asymptotic condition: 
\[
\lim_{z \to \infty} e^{-\frac{z}{\zeta}} \int_{R(z)} \chi_\omega(t) dv > 0
\]
pr it satisfies the decay estimate (23). This study can continue with obtaining of the upper bound for the amplitude \(E_\omega(0, t)\) in terms of the boundary conditions, but this analysis will be the subject of another paper.

5 Conclusions

In the present paper it was studied the impossibility of localization in time for the solutions of the boundary value problem associated with the linear thermoelastic materials with double porosity structure and microtemperature. The uniqueness of the solutions for the backward in time problem in case of the materials with double porosity structure with microtemperature was proved. We can draw the conclusion that for the backward in time problem the only solution that vanishes is the null solution for every \(t > 0\). In the case of linear thermoelastic theories this results can not certify that the thermo-mechanical deformations from double porous bodies with microtemperature vanish after a finite time. In this situation it is necessary that the time should be unbounded to guarantee that the fraction of volumes becomes the same as the reference configuration. We obtained a function that defines a measure on the solutions and we deduced the usual exponential type alternative for the solutions of the problem defined in a semi-infinite cylinder.

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