Linear equations on real algebraic surfaces

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Abstract. We prove that if a linear equation, whose coefficients are continuous rational functions on a nonsingular real algebraic surface, has a continuous solution, then it also has a continuous rational solution. This is known to fail in higher dimensions.

Key words. Linear equation, continuous rational solution, real algebraic variety.

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1 Introduction

Fefferman and Kollár [5] study the following problem. Given continuous functions $f_1, \ldots, f_r$ on $\mathbb{R}^n$, which continuous functions $\varphi$ can be written in the form

$$\varphi = \varphi_1 f_1 + \cdots + \varphi_r f_r,$$

where the $\varphi_i$ are continuous functions on $\mathbb{R}^n$? Moreover, if $\varphi$ and the $f_i$ have some regularity properties, can one choose the $\varphi_i$ to have the same (or weaker) regularity properties? In other words, the questions are about solutions of linear equations of the form

$$f_1 y_1 + \cdots + f_r y_r = \varphi.$$

The problem is hard even if $\varphi$ and the $f_i$ are polynomial functions. In [5], two different ways to solve the problem are presented: the Glaeser–Michael method and the algebraic geometry approach. Each of them consists of a rather complex procedure and it does not seem possible to give a concise answer in general.

In this note we settle the problem in a simple manner, assuming that $n = 2$ and the $f_i$ are continuous rational functions. Actually, our results are more general and settle the corresponding problem for functions defined on any nonsingular real algebraic surface.

A complex version of the problem under consideration was studied by Brenner [3], Epstein and Hochster [4], and Kollár [9].

Convention 1.1. By a function we always mean a real-valued function.

Notation 1.2. If $f_1, \ldots, f_r$ are functions defined on some set $S$, then

$$Z(f_1, \ldots, f_r) := \{x \in S \mid f_1(x) = 0, \ldots, f_r(x) = 0\}.$$
We now recall the *pointwise test* (or PT for short) introduced in [5, p. 235].

**Definition 1.3.** Let Ω be a metric space and let $f_1, \ldots, f_r$ be continuous functions on Ω. We say that a continuous function $\varphi$ on Ω satisfies the PT for the functions $f_i$ if for every point $p \in \Omega$, the following two equivalent conditions hold:

(a) The function $\varphi$ can be written as

$$\varphi = \psi_1^{(p)} f_1 + \cdots + \psi_r^{(p)} f_r,$$

where the $\psi_i^{(p)}$ are functions on Ω that are continuous at $p$.

(b) The function $\varphi$ can be written as

$$\varphi = \varphi^{(p)} + c_1^{(p)} f_1 + \cdots + c_r^{(p)} f_r,$$

where $c_i^{(p)} \in \mathbb{R}$ and the functions $A_i^{(p)}$ defined by

$$A_i^{(p)} = \frac{\varphi^{(p)} f_i}{f_1^2 + \cdots + f_r^2} \quad \text{on } \Omega \setminus Z^{(p)} \quad \text{and} \quad A_i^{(p)} = 0 \quad \text{on } Z^{(p)},$$

with $Z^{(p)} := Z(f_1, \ldots, f_r) \cup \{p\}$, are continuous at $p$.

Note that conditions (a) and (b) are indeed equivalent. If (a) holds, then so does (b) with

$$\varphi^{(p)} = (\psi_1^{(p)} - \psi_1^{(p)}(p)) f_1 + \cdots + (\psi_r^{(p)} - \psi_r^{(p)}(p)) f_r \quad \text{and} \quad c_i^{(p)} = \psi_i^{(p)}(p).$$

Conversely, (b) implies (a) with $\psi_i^{(p)} = c_i^{(p)} + A_i^{(p)}$.

Clearly, the PT is a basic necessary condition for existence of continuous functions $\varphi_1, \ldots, \varphi_r$ on Ω satisfying $\varphi = \varphi_1 f_1 + \cdots + \varphi_r f_r$.

For background on real algebraic geometry the reader may consult [2]. By a *real algebraic variety* we mean a locally ringed space isomorphic to an algebraic subset of $\mathbb{R}^n$, for some $n$, endowed with the Zariski topology and the sheaf of regular functions (such an object is called an affine real algebraic variety in [2]). Recall that any quasi-projective real algebraic variety is a real algebraic variety in the sense just defined, cf. [2, Prop. 3.2.10, Thm. 3.4.4]. Each real algebraic variety carries also the Euclidean topology, which is determined by the usual metric on $\mathbb{R}$. Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties will refer to the Euclidean topology.

We say that a function $f$ defined on a real algebraic variety $X$ is *continuous rational* if it is continuous on $X$ and regular on some Zariski open dense subset of $X$. We denote by $P(f)$ the smallest Zariski closed subset of $X$ such that $f$ is regular on $X \setminus P(f)$. The continuous rational functions form a subring of the ring of all continuous functions on $X$. Any regular function on $X$ is continuous rational. The converse does not hold in general, even if $X$ is nonsingular.

**Example 1.4.** The function $f$ on $\mathbb{R}^2$, defined by

$$f(x, y) = \frac{x^3}{x^2 + y^2} \quad \text{for } (x, y) \neq (0, 0) \quad \text{and} \quad f(0, 0) = 0,$$

is continuous rational but it is not regular; in fact, $P(f) = \{(0, 0)\}$. 

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Recently, continuous rational functions have attracted a lot of attention, cf. \cite{1, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}. On nonsingular varieties they coincide with regulous functions introduced by Fichou, Huisman, Mangolte and Monnier \cite{6}.

Our first result, to be proved in Section 2, is the following.

**Theorem 1.5.** Let $X$ be a nonsingular real algebraic surface and let $f_1, \ldots, f_r$ be continuous rational functions on $X$. For a continuous function $\varphi$ on $X$, the following conditions are equivalent:

(a) The function $\varphi$ can be written in the form

$$\varphi = \varphi_1 f_1 + \cdots + \varphi_r f_r,$$

where the $\varphi_i$ are continuous functions on $X$.

(b) The function $\varphi$ satisfies the PT for the $f_i$.

An example of Hochster \cite[p. 236]{5}, which involves simple polynomial functions on $\mathbb{R}^3$, shows that Theorem 1.5 cannot be extended to varieties of higher dimension.

In Section 3 we prove the following.

**Theorem 1.6.** Let $X$ be a nonsingular real algebraic surface and let $f_1, \ldots, f_r$ be continuous rational functions on $X$. For a continuous rational function $\varphi$ on $X$, the following conditions are equivalent:

(a) The function $\varphi$ can be written in the form

$$\varphi = \varphi_1 f_1 + \cdots + \varphi_r f_r,$$

where the $\varphi_i$ are continuous rational functions on $X$.

(b) The function $\varphi$ satisfies the PT for the $f_i$.

Furthermore, if (b) holds, then the $\varphi_i$ in (a) can be chosen so that $P(\varphi_i)$ is a finite set contained in $Z(f_1, \ldots, f_r) \cup P(f_1) \cup \ldots \cup P(f_r) \cup P(\varphi)$.

As a straightforward consequence we get

**Corollary 1.7.** Let $X$ be a nonsingular real algebraic surface and let $f_1, \ldots, f_r$ be continuous rational functions on $X$. For a continuous rational function $\varphi$ on $X$, the following conditions are equivalent:

(a) The function $\varphi$ can be written in the form

$$\varphi = \varphi_1 f_1 + \cdots + \varphi_r f_r,$$

where the $\varphi_i$ are continuous rational functions on $X$.

(b) The function $\varphi$ can be written in the form

$$\varphi = \psi_1 f_1 + \cdots + \psi_r f_r,$$

where the $\psi_i$ are continuous functions on $X$.
Corollary 1.7 cannot be extended to varieties of higher dimension. A relevant example, involving polynomial functions on \( \mathbb{R}^3 \), is provided by Kollár and Nowak [10, Example 6]. Furthermore, the argument used in [10, Example 6] shows that Corollary 1.7 does not hold for the singular real algebraic surface \( S \subset \mathbb{R}^3 \) that appears there.

We conclude this section with an example.

**Example 1.8.** Consider the functions \( f_1(x, y) = x^3, f_2(x, y) = y^3 \), \( \varphi(x, y) = x^2y^2 \) on \( \mathbb{R}^2 \). We have

\[
\varphi = \varphi_1 f_1 + \varphi_2 f_2,
\]

where \( \varphi_1, \varphi_2 \) are continuous rational functions on \( \mathbb{R}^2 \) defined by

\[
\varphi_1(x, y) = \frac{x^5y^2}{x^6+y^6} \quad \text{for} \ (x, y) \neq (0, 0), \quad \varphi_1(0,0) = 0,
\]

\[
\varphi_2(x, y) = \frac{x^2y^5}{x^6+y^6} \quad \text{for} \ (x, y) \neq (0, 0), \quad \varphi_2(0,0) = 0.
\]

However, \( \varphi \) cannot be written as a linear combination of \( f_1 \) and \( f_2 \) with coefficients that are regular (or \( C^\infty \)) functions on \( \mathbb{R}^2 \), as can be seen by comparing the Taylor’s expansions at \((0,0)\).

## 2 Continuous solutions

We begin with some preliminary results.

**Lemma 2.1.** Let \( \Omega \) be a metric space and let \( f_1, \ldots, f_r, \varphi \) be continuous functions on \( \Omega \) such that the set \( Z(f_1, \ldots, f_r) \) is nowhere dense in \( \Omega \) and \( \varphi \) satisfies the PT for the \( f_i \). Assume that \( f_i = gg_i \), where \( g \) and the \( g_i \) are continuous functions on \( \Omega \). Then there exists a unique continuous function \( \psi \) on \( \Omega \) such that \( \varphi = g \psi \). Furthermore, \( \psi \) satisfies the PT for the \( g_i \).

**Proof.** Note that the set \( Z(g) \) is nowhere dense in \( \Omega \). To prove existence of \( \psi \) (uniqueness is then automatic) it suffices to show that for every point \( p \in \Omega \) the limit

\[
\lim_{x \to p} \frac{\varphi(x)}{g(x)}, \quad \text{where} \ x \in \Omega \setminus Z(g),
\]

exists. This readily follows since \( \varphi \) can be written as

\[
\varphi = \psi_1^{(p)} f_1 + \cdots + \psi_r^{(p)} f_r = g(\psi_1^{(p)} g_1 + \cdots + \psi_r^{(p)} g_r),
\]

where the \( \psi_i^{(p)} \) are functions on \( \Omega \) that are continuous at \( p \).

It remains to prove that \( \psi \) satisfies the PT for the \( g_i \). We set \( Z^{(p)} := Z(f_1, \ldots, f_r) \cup \{p\} \) and write \( \varphi \) in the form

\[
\varphi = \varphi^{(p)} + c_1^{(p)} f_1 + \cdots + c_r^{(p)} f_r = \varphi^{(p)} + g(c_1^{(p)} g_1 + \cdots + c_r^{(p)} g_r),
\]

where \( c_i^{(p)} \in \mathbb{R} \) and the functions \( A_i^{(p)} \), defined by

\[
A_i^{(p)} = \frac{\varphi^{(p)} f_i}{f_1^2 + \cdots + f_r^2} \quad \text{on} \ \Omega \setminus Z^{(p)} \quad \text{and} \quad A_i^{(p)} = 0 \quad \text{on} \ Z^{(p)},
\]

are continuous at \( p \). Defining \( \psi^{(p)} \) by

\[
\psi = \psi^{(p)} + c_1^{(p)} g_1 + \cdots + c_r^{(p)} g_r,
\]
we get \( \varphi^{(p)} = g\psi^{(p)} \). Consequently,
\[
\frac{\varphi^{(p)} f_i}{f_1^2 + \cdots + f_r^2} = \frac{\psi^{(p)} g_i}{g_1^2 + \cdots + g_r^2} \quad \text{on } \Omega \setminus Z(f_1, \ldots, f_r).
\]

Since the set \( Z(f_1, \ldots, f_r) \) is nowhere dense in \( \Omega \), it follows that \( \psi \) satisfies the PT for the \( g_i \).

**Lemma 2.2.** Let \( X \) be an irreducible nonsingular real algebraic variety and let \( f_1, \ldots, f_r \) be continuous rational functions on \( X \), not all identically equal to 0. Then the Zariski closure of \( Z(f_1, \ldots, f_r) \) is Zariski nowhere dense in \( X \). In particular, \( Z(f_1, \ldots, f_r) \) is Euclidean nowhere dense in \( X \).

**Proof.** Setting \( f = f_1^2 + \cdots + f_r^2 \), we get \( Z(f) = Z(f_1, \ldots, f_r) \). The function \( f \) is continuous rational and satisfies
\[
Z(f) \subset Z(f|_{X \setminus P(f)}) \cup P(f),
\]
which implies both assertions.

**Lemma 2.3.** Let \( X \) be an irreducible nonsingular real algebraic surface and let \( f_1, \ldots, f_r \) be continuous rational functions on \( X \). Then, for every point \( p \in X \), there exists a Zariski open neighborhood \( X^{(p)} \subset X \) of \( p \) and there exist regular functions \( g_1, \ldots, g_r, g, h \) on \( X^{(p)} \) such that \( Z(h) \neq X^{(p)} \), \( Z(g_1, \ldots, g_r) \subset \{p\} \), and \( hf_i = gg_i \) on \( X^{(p)} \) for \( i = 1, \ldots, r \).

**Proof.** We can find regular functions \( \lambda_1, \ldots, \lambda_r, \mu \) on \( X \) such that \( Z(\mu) \neq X \) and \( f_i = \lambda_i/\mu \) on \( X \setminus Z(\mu) \) for \( i = 1, \ldots, r \). Since \( X \) is nonsingular, the local ring of regular functions at each point \( p \in X \) is a unique factorization domain. Consequently, there exists a Zariski open neighborhood \( X^{(p)} \subset X \) of \( p \) and there exist regular functions \( g_1, \ldots, g_r, g \) on \( X^{(p)} \) such that \( \lambda_i = gg_i \) on \( X^{(p)} \) and \( Z(g_1, \ldots, g_r) \subset \{p\} \). To complete the proof it suffices to set \( h := \mu|_{X^{(p)}} \).

**Lemma 2.4.** Let \( X \) be an irreducible nonsingular real algebraic surface and let \( f_1, \ldots, f_r \) be continuous rational functions on \( X \), not all identically equal to 0. Let \( \varphi \) be a continuous function on \( X \) that satisfies the PT for the \( f_i \). Then, for every point \( p \in X \), there exists a Zariski open neighborhood \( X^{(p)} \subset X \) of \( p \) and there exist continuous functions \( \alpha_1^{(p)}, \ldots, \alpha_r^{(p)} \) on \( X^{(p)} \) and real numbers \( c_1^{(p)}, \ldots, c_r^{(p)} \) such that
\[
\varphi = \alpha_1^{(p)} f_1 + \cdots + \alpha_r^{(p)} f_r \quad \text{on } X^{(p)}, \quad \text{and}
\]
\[
\alpha_i^{(p)} = c_i^{(p)} + \left( \frac{\varphi - (c_1^{(p)} f_1 + \cdots + c_r^{(p)} f_r)}{f_1^2 + \cdots + f_r^2} \right) f_i \quad \text{on } X^{(p)} \setminus Z(f_1, \ldots, f_r)
\]
for \( i = 1, \ldots, r \).

**Proof.** By Lemma 2.3, we can find a Zariski open neighborhood \( X^{(p)} \subset X \) of \( p \) and regular functions \( g_1, \ldots, g_r, g, h \) on \( X^{(p)} \) such that
\[
Z(g_1, \ldots, g_r) \subset \{p\},
\]
\[
hf_i = gg_i \quad \text{on } X^{(p)} \quad \text{for } i = 1, \ldots, r,
\]
and \( Z(h) \neq X^{(p)} \). Since \( \varphi \) satisfies the PT for the \( f_i \), it follows that \( h\varphi|_{X^{(p)}} \) satisfies the PT for the \( hf_i|_{X^{(p)}} = gg_i \). According to Lemma 2.2, the set
\[
Z(hf_1|_{X^{(p)}}, \ldots, hf_r|_{X^{(p)}}) = Z(gg_1, \ldots, gg_r)
\]

is nowhere dense in \( X(p) \). Hence, in view of Lemma \[2.1\] there exists a unique continuous function \( \psi \) on \( X(p) \) such that
\[
(3) \quad h\varphi|_{X(p)} = g\psi.
\]
Furthermore, \( \psi \) satisfies the PT for the \( g_i \). Consequently, taking [1] into account, we can write \( \psi \) in the form
\[
(4) \quad \psi = \psi(p) + c_1^{(p)} g_1 + \cdots + c_r^{(p)} g_r,
\]
where \( c_i^{(p)} \in \mathbb{R} \) and the functions \( B_i^{(p)} \) on \( X(p) \), defined by
\[
(5) \quad B_i^{(p)} = \frac{\psi(p) g_i}{g_1^2 + \cdots + g_r^2} \text{ on } X(p) \setminus \{p\} \quad \text{and} \quad B_i^{(p)}(p) = 0,
\]
are continuous at \( p \). It follows that the \( B_i^{(p)} \) are continuous on \( X(p) \).

Defining \( \varphi^{(p)} \) by
\[
(6) \quad \varphi = \varphi^{(p)} + c_1^{(p)} f_1 + \cdots + c_r^{(p)} f_r
\]
and making use of [2]–[6], we get
\[
B_i^{(p)} = \frac{\varphi^{(p)} f_i}{f_1^2 + \cdots + f_r^2} \quad \text{on } X(p) \setminus \{p\} \cup Z(g)).
\]
By continuity,
\[
(7) \quad B_i^{(p)} = \frac{\varphi^{(p)} f_i}{f_1^2 + \cdots + f_r^2} \quad \text{on } X(p) \setminus Z(f_1, \ldots, f_r).
\]
The functions \( \alpha_i^{(p)} := c_i^{(p)} + B_i^{(p)} \) are continuous on \( X(p) \) and in view of (6), (7) they satisfy
\[
\varphi = \alpha_1^{(p)} f_1 + \cdots + \alpha_r^{(p)} f_r \quad \text{on } X(p) \setminus Z(f_1, \ldots, f_r).
\]
By continuity, the last equality holds on \( X(p) \). The proof is complete. \( \Box \)

**Proof of Theorem 1.5.** By Lemma [2.4] a partition of unity argument completes the proof. \( \Box \)

Lemma [2.4] contains more information than we needed for the proof of Theorem 1.5. However, the full statement will be used to prove Theorem 1.6 in Section 3.

### 3 Continuous rational solutions

We will frequently use, not necessarily explicitly referring to it, the following fact: If \( X \) is a nonsingular real algebraic variety, \( X^0 \subset X \) a Zariski open subset, and \( U \subset X \) a Euclidean open subset, then \( X^0 \cap U \) is Euclidean dense in \( U \).

**Lemma 3.1.** Let \( X \) be a nonsingular real algebraic variety, \( \psi: X \to \mathbb{R} \) a regular function, and \( f: X \setminus Z(\psi) \to \mathbb{R} \) a continuous rational function. Then there exists an integer \( N_0 > 0 \) such that for every integer \( N \geq N_0 \), the function \( \psi^N f \), extended by 0 on \( Z(\psi) \), is continuous rational on \( X \).
Proof. According to a variant of the Łojasiewicz inequality [2 Prop. 2.6.4], it suffices to prove that $f$ is a semialgebraic function. This is straightforward since the graph of the function $f$ restricted to $(X \setminus Z(\psi)) \setminus P(f)$ is a semialgebraic subset of $(X \setminus Z(\psi)) \times \mathbb{R}$, whose closure is equal to the graph of $f$. \qed

**Lemma 3.2.** Let $X$ be a nonsingular real algebraic variety and let $\{X^1, \ldots, X^m\}$ be a Zariski open cover of $X$. Let $f_1, \ldots, f_r, \varphi$ be continuous rational functions on $X$ such that for $j = 1, \ldots, m$ the restriction $\varphi|_{X^j}$ can be written in the form

$$\varphi|_{X^j} = \sum_{i=1}^{r} \varphi_{ij} f_i|_{X^j},$$

where the $\varphi_{ij}$ are continuous rational functions on $X^j$. Then $\varphi$ can be written in the form

$$\varphi = \sum_{i=1}^{r} \varphi_i f_i,$$

where the $\varphi_i$ are continuous rational functions on $X$ with

$$P(\varphi_i) \subset \bigcup_{j=1}^{m} (P(\varphi_{ij}) \cup (X \setminus X^j)).$$

Proof. We may assume that $X$ is irreducible and the $X^j$ are all nonempty. Then each $X^j$ is Euclidean dense in $X$. We choose a regular function $\psi_j$ on $X$ with $Z(\psi_j) = X \setminus X^j$. By Lemma 3.1 there exists a positive integer $N$ such that the $\varphi_{ij}$ can be written as

$$\varphi_{ij} = \frac{a_{ij}}{\psi_j^N} \text{ on } X^j,$$

where the $a_{ij}$ are continuous rational functions on $X$. It follows that

$$\psi_j^N \varphi = \sum_{i=1}^{r} a_{ij} f_i$$

on $X^j$. By continuity, [2] holds on $X$. Multiplying both sides of (2) by $\psi_j^N$ and summing over $j$, we get

$$b \varphi = \sum_{i=1}^{r} b_i f_i,$$

where

$$b = \sum_{j=1}^{m} \psi_j^{2N} \text{ and } b_i = \sum_{j=1}^{m} a_{ij} \psi_j^N.$$

The function $b$ is regular with $Z(b) = \emptyset$, which implies that $\varphi_i := b_i/b$ is a continuous rational function on $X$. In view of (3) we have

$$\varphi = \sum_{i=1}^{r} \varphi_i f_i.$$  

By construction,

$$P(\varphi_i) \subset \bigcup_{j=1}^{m} P(a_{ij}).$$
Example 3.4. Consider the Whitney umbrella \(X\), the Euclidean closure of \(p\) of all nonsingular points of \(X\). Definition 3.3 imposes no restriction if the point \(p\) is not locally bounded on the \(-axis. The rational function \(1/(z + 1)\) is locally bounded on \(W\), but it is not locally bounded on the \(-axis.

We will consider locally bounded rational functions only on nonsingular varieties. A typical example is the following.

Example 3.5. The rational function \(xy/(x^2 + y^2)\) on \(\mathbb{R}^2\) is locally bounded (even bounded), but it cannot be extended to a continuous function on \(\mathbb{R}^2\).

Lemma 3.6. Let \(X\) be a nonsingular real algebraic variety and let \(R\) be a locally bounded rational function on \(X\). Then the polar set \(\text{pole}(R)\) is of codimension at least 2.

Proof. Using the inclusion \(\mathbb{R} \subset \mathbb{P}^1(\mathbb{R})\), we obtain a rational map \(R^* : X \dasharrow \mathbb{P}^1(\mathbb{R})\) determined by \(R\). The polar set \(\text{pole}(R^*)\) is of codimension at least 2 [8, p. 129, Thm. 2.17]. Since \(R\) is locally bounded, we have \(\text{pole}(R) = \text{pole}(R^*)\), which completes the proof.

Remark 3.7. Let \(X\) be a nonsingular real algebraic variety. Any continuous rational function \(f\) on \(X\) determines a rational function \(\tilde{f}\) on \(X\), which is represented by the regular function \(f|_{X \setminus P(f)}\). Clearly, \(P(f) = \text{pole}(\tilde{f})\). Furthermore, if \(g\) is a continuous rational function on \(X\), not identically equal to 0 on any irreducible component of \(X\), then the quotient \(\tilde{f}/\tilde{g}\) is a well defined rational function on \(X\) (see Lemma 2.2). To simplify notation, we will prefer to say “the rational function \(f\)” or “the rational function \(f/g\)” instead of writing \(\tilde{f}\) or \(\tilde{f}/\tilde{g}\), respectively. For the rational function \(f/g\), we have

\[
\text{pole}(f/g) \subset P(f) \cup P(g) \cup Z(g).
\]
Lemma 3.8. Let $X$ be an irreducible nonsingular real algebraic variety and let $\varphi, f_1, \ldots, f_r$ be continuous rational functions on $X$, where the $f_i$ are not all identically equal to 0. For $i = 1, \ldots, r$ and $c = (c_1, \ldots, c_r) \in \mathbb{R}^r$, let

$$R_{ci} := \frac{(\varphi - (c_1 f_1 + \cdots + c_r f_r)) f_i}{f_1^2 + \cdots + f_r^2}.$$ 

If $\varphi$ satisfies the PT for $f_1, \ldots, f_r$, then each rational function $R_{ci}$ is locally bounded on $X$.

Proof. Let $Z := Z(f_1, \ldots, f_r)$ and let $S \subset X$ be an arbitrary subset. The $R_{ci}$ are well defined functions on $X \setminus Z$. Setting $R_i = R_{ci}$, where $0 = (0, \ldots, 0) \in \mathbb{R}^r$, we get

$$R_{ci} = R_i - \frac{c_1 f_i f_1 + \cdots + c_r f_i f_r}{f_1^2 + \cdots + f_r^2}.$$ 

Since

$$\left| \frac{f_i f_j}{f_1^2 + \cdots + f_r^2} \right| \leq \frac{1}{2} \text{ on } X \setminus Z,$$

it follows that $R_{ci}$ is bounded on $S \cap (X \setminus Z)$ if and only if $R_i$ is such.

Suppose that $\varphi$ satisfies the PT for $f_1, \ldots, f_r$, fix a point $p \in X$, and set $Z^{(p)} = Z \cup \{p\}$. The function $\varphi$ can be written in the form

$$\varphi = \varphi^{(p)} + c_1^{(p)} f_1 + \cdots + c_r^{(p)} f_r,$$

where $c_i^{(p)} \in \mathbb{R}$ and the functions $A_i^{(p)}$ on $X$, defined by

$$A_i^{(p)} = \frac{\varphi^{(p)} f_i}{f_1^2 + \cdots + f_r^2} \text{ on } X \setminus Z^{(p)} \text{ and } A_i^{(p)} = 0 \text{ on } Z^{(p)},$$

are continuous at $p$. In particular, the $A_i^{(p)}$ are bounded on some Euclidean open neighborhood $U_p \subset X$ of $p$. Consequently, the functions $R_{c_i^{(p)}}$, where $c_i^{(p)} = (c_1^{(p)}, \ldots, c_r^{(p)})$, are bounded on $U_p \cap (X \setminus Z)$, which in turn implies that the $R_i$ are bounded on $U_p \cap (X \setminus Z)$. This conclusion remains valid if $Z$ is replaced by its Zariski closure $V$ in $X$. The proof is complete since the subset $X \setminus V \subset X$ is Zariski open dense by Lemma 2.2. \[\square\]

Proof of Theorem 1.6. It suffices to prove that [b] implies [a] together with the extra conditions stipulated on the $c_i$. Let us suppose that [b] holds. We may assume that $X$ is irreducible and the $f_i$ are not all identically equal to 0. According to Lemma 2.4, for each point $p \in X$, we can find a Zariski open neighborhood $X^{(p)} \subset X$ of $p$ and continuous functions $\alpha_1^{(p)}, \ldots, \alpha_r^{(p)}$ on $X^{(p)}$ such that

$$\varphi = \alpha_1^{(p)} f_1 + \cdots + \alpha_r^{(p)} \text{ on } X^{(p)}$$

and

$$\alpha_i^{(p)} = c_i^{(p)} + R_i^{(p)} \text{ on } X^{(p)} \setminus Z,$$

where $Z = Z(f_1, \ldots, f_r)$, $c_i^{(p)} \in \mathbb{R}$, and

$$R_i^{(p)} = \frac{(\varphi - (c_i^{(p)} f_1 + \cdots + c_r^{(p)} f_r)) f_i}{f_1^2 + \cdots + f_r^2} \text{ on } X^{(p)} \setminus Z.$$ 

We regard the $R_i^{(p)}$ as rational functions on $X$ and set

$$X_0^{(p)} := \text{dom}(R_1^{(p)}) \cap \ldots \cap \text{dom}(R_r^{(p)}).$$
Clearly, 
\[ X \setminus X_0^{(p)} \subset \mathbb{Z}. \]
Furthermore, according to Lemmas 3.6 and 3.8, \( X \setminus X_0^{(p)} \) is a finite set. Since the set \( X^{(p)} \setminus \mathbb{Z} \) is Euclidean dense in \( X \) (see Lemma 2.2), it follows that 
\[ \alpha_i^{(p)} = R_i^{(p)} \quad \text{on} \quad X^{(p)} \cap X_0^{(p)}. \]

Consequently, we obtain a well defined continuous rational function \( \beta_i^{(p)} \) on \( X_1^{(p)} := X^{(p)} \cup X_0^{(p)} \) by setting 
\[ \beta_i^{(p)} = \alpha_i^{(p)} \quad \text{on} \quad X^{(p)} \quad \text{and} \quad \beta_i^{(p)} = c_i^{(p)} + R_i^{(p)} \quad \text{on} \quad X_0^{(p)}. \]

By construction, 
\[ \varphi = \beta_1^{(p)} f_1 + \cdots + \beta_r^{(p)} f_r \quad \text{on} \quad X_1^{(p)}. \]

Now it is easy to complete the proof. We choose a finite collection of points \( p_1, \ldots, p_m \) in \( X \) so that the sets \( X_j := X_1^{(p_j)} \) form a cover of \( X \). Setting \( \varphi_{ij} := \beta_i^{(p_j)} \), we get 
\[ \varphi|_{X_j} = \sum_{i=1}^r \varphi_{ij}|_{X_j}, \quad P(\varphi_{ij}) \subset (P(\varphi) \cup \mathbb{Z} \cup P(f_1) \cup \ldots \cup P(f_r)) \cap (X \setminus X_0^{(p_j)}). \]

By Lemma 3.2 there exist continuous rational functions \( \varphi_1, \ldots, \varphi_r \) on \( X \) such that 
\[ \varphi = \varphi_1 f_1 + \cdots + \varphi_r f_r \]
and 
\[ P(\varphi_i) \subset \bigcup_{j=1}^r (P(\varphi_{ij}) \cup (X \setminus X_j)). \]

The functions \( \varphi_i \) satisfy all the requirements. \( \square \)

References

[1] M. Bilski, W. Kucharz, A. Valette, and G. Valette, Vector bundles and reguluous maps, Math. Z. 275 (2013), 403–418.

[2] J. Bochnak, M. Coste, and M.-F. Roy, Real Algebraic Geometry, Ergeb. der Math. und ihrer Grenzgeb. Folge 3, vol. 36, Springer, 1998.

[3] H. Brenner, Continuous solutions to algebraic forcing equations, arXiv:0608611 [math.AC].

[4] N. Epstein and M. Hochster, Continuous closure, axes closure, and natural closure, arXiv:1106.3462v2 [math.AC].

[5] C. Fefferman and J. Kollár, Continuous solutions of linear equations, From Fourier analysis and number theory to Radon transforms and geometry, 233–282, Dev. Math. 28, Springer, 2013.

[6] G. Fichou, J. Huissmann, F. Mangolte, and J.-Ph. Monnier, Fonctions régulues, arXiv:1112.3800 [math.AG], to appear in J. Reine Angew. Math.
[7] G. Fichou, J.-Ph. Monnier, and R. Quarez, Continuous functions in the plane regular after one blowing-up, arXiv:1409.8223 [math.AG].

[8] S. Iitaka, Algebraic Geometry. An Introduction to Birational Geometry of Algebraic Varieties. Springer, 1982.

[9] J. Kollár, Continuous closure of sheaves, Michigan Math. J. 61 (2012), 475–491.

[10] J. Kollár and K. Nowak, Continuous rational functions on real and p-adic varieties, Math. Z. 279 (2015), 85–97.

[11] W. Kucharz, Rational maps in real algebraic geometry, Adv. Geom. 9 (2009), 517–539.

[12] W. Kucharz, Regular versus continuous rational maps, Topology Appl. 160 (2013), 1375–1378.

[13] W. Kucharz, Approximation by continuous rational maps into spheres, J. Eur.Math. Soc. 16 (2014), 1555–1569.

[14] W. Kucharz, Continuous rational maps into the unit 2-sphere, Arch. Math. (Basel) 102 (2014), 257–261.

[15] W. Kucharz, Continuous rational maps into spheres, arXiv:1403.5127 [math.AG].

[16] W. Kucharz and K. Kurdyka, Stratified-algebraic vector bundles, arXiv:1308.4376 [math.AG], to appear in J. Reine Angew. Math.

[17] W. Kucharz and K. Kurdyka, Curve-rational functions, arXiv:1509.05905 [math.AG].

[18] W. Kucharz and K. Kurdyka, Comparison of stratified-algebraic and topological K-theory, arXiv:1511.04238 [math.AG].

[19] K.J. Nowak, Algebraic geometry over Henselian rank one valued fields, arXiv:1410.3280 [math.AG].

[20] M. Zieliński, Homotopy properties of some real algebraic maps, to appear in Homology, Homotopy and Applications.

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