Normalized solutions for the nonlinear Schrödinger equation with potential and combined nonlinearities

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Abstract: In present paper, we study the following nonlinear Schrödinger equation with combined power nonlinearities

\[-\Delta u + V(x)u + \lambda u = |u|^{2^*-2}u + \mu|u|^{q-2}u \quad \text{in} \ \mathbb{R}^N, \ N \geq 3\]

having prescribed mass

\[
\int_{\mathbb{R}^N} u^2 dx = a^2,
\]

where \(\mu, a > 0, q \in (2, 2^*), 2^* = \frac{2N}{N-2}\) is the critical Sobolev exponent, \(V\) is an external potential vanishing at infinity, and the parameter \(\lambda \in \mathbb{R}\) appears as a Lagrange multiplier. Under some mild assumptions on \(V\), for the \(L^2\)-subcritical perturbation \(q \in (2, 2 + \frac{4}{N})\), we prove that there exists \(a_0 > 0\) such that the normalized solution with negative energy to the above problem with \(\mu > 0\) can be obtained when \(a \in (0, a_0)\); for the \(L^2\)-critical perturbation \(q = 2 + \frac{4}{N}\), by limiting the range of \(\mu\), the positive ground state normalized solution to the above problem for any \(a > 0\) is also found with the aid of the Pohožaev constraint; moreover, for the \(L^2\)-supercritical perturbation \(q \in (2 + \frac{4}{N}, 2^*)\), we get a positive ground state normalized solution for the above problem with \(a > 0\) and \(\mu > 0\) by using the Pohožaev constraint. At the same time, the exponential decay property of the positive normalized solution is established, which is important for the instability analysis of the standing waves. Furthermore, we give a description of the ground state set and obtain the strong instability of the standing waves for \(q \in [2 + \frac{4}{N}, 2^*)\). This paper can be regarded as a generalization of Soave [J. Funct. Anal. (2020)] in a sense.

Keywords: Schrödinger equation; Normalized solution; Combined nonlinearities; Vanish potential; Strong instability; Variational method

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1 Introduction and main results

In this paper, we study the existence of standing waves with prescribed mass for the nonlinear Schrödinger equation with combined power nonlinearities

\[i\phi_t + \Delta \phi - V(x)\phi + \mu|\phi|^{q-2}\phi + |\phi|^{p-2}\phi = 0 \quad \text{in} \ \mathbb{R}^N,\]

where \(N \geq 3, \phi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}, \mu > 0, 2 < q < p = 2^* = \frac{2N}{N-2}\) and \(V\) is an external potential. The case \(\mu > 0\) is the focusing case, and the case \(\mu < 0\) is referred to the defocusing case. In the past decade, the nonlinear Schrödinger equation with combined nonlinearities have attracted extensive attention,
starting from the fundamental contribution by Tao et. al [34]. After then, many scholars have done a lot of research on such problem, see for example [4, 12, 14, 23, 25, 28, 31, 32, 34]. To find stationary states, one makes the ansatz $\phi(t, x) = e^{i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ and $u : \mathbb{R}^N \to \mathbb{C}$ is a time-independent function. Hence, by simple calculation one knows that $u$ satisfies the following equation

$$-\Delta u + V(x)u + \lambda u = |u|^p - 2u + \mu |u|^{q-2}u \quad \text{in } \mathbb{R}^N. \quad (1.2)$$

If we fix $\lambda \in \mathbb{R}$ to find the solutions $u$ of equation (1.2), we call equation (1.2) the fixed frequency problem. One can use the variational method to find the critical points of the corresponding energy functional of equation (1.2)

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + (V(x) + \lambda)|u|^2 \, dx - \frac{1}{q} \int_{\mathbb{R}^N} |u|^q \, dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \, dx,$$

or some other topological methods, such as fixed point theory, bifurcation or the Lyapunov-Schmidt reduction. The fixed frequency problem has been extensively studied over the past few decades, including but not limited to the existence, non-existence, multiplicity and asymptotic behavior of the solutions (e.g., ground state solution, positive solution and sign-changing solution, and so on). Alternatively, one can search for solutions to equation (1.2) having prescribed mass

$$\int_{\mathbb{R}^N} |u|^2 \, dx = a^2. \quad (1.3)$$

The solution of equation (1.2) satisfies the prescribed mass constraint (1.3), which is called the fixed mass problem, and in much of the literature called the normalized solution. A natural approach to obtain normalized solution of equation (1.2) is to find the critical points of the corresponding energy functional under the constraint (1.3). We refer the cases $2 < q < 2 + \frac{4}{N}$, $q = 2 + \frac{4}{N}$ and $2 + \frac{4}{N} < q < 2^*$ as $L^2$-subcritical, $L^2$-critical and $L^2$-supercritical, respectively.

Recently, the question of finding normalized solutions is already interesting for scalar equations, which has attracted extensive attention from scholars. For the non-potential case, namely, $V(x) = 0$, we consider the following nonlinear Schrödinger equation

$$-\Delta u + \lambda u = |u|^p - 2u + \mu |u|^{q-2}u \quad \text{in } \mathbb{R}^N \quad (1.4)$$

satisfying the normalization constraint $\int_{\mathbb{R}^N} u^2 \, dx = a^2$, where $N \geq 1$ and $2 < q \leq p \leq 2^*$. If $p = q \neq \overline{q} := 2 + \frac{4}{N}$, one obtained that, for any $a > 0$, equation (1.4) has the positive normalized solution only if $\lambda > 0$ by scaling. Indeed, from [24], the following Schrödinger equation

$$\begin{cases}
-\Delta u + u = |u|^{q-2}u & \text{in } \mathbb{R}^N, \\
u(x) \to 0 & \text{as } |x| \to \infty
\end{cases}$$

had the unique positive radial solution $W_q$. Letting

$$W_{\lambda,q}(x) := \lambda^{\frac{1}{q-2}} W_q(\sqrt[2-q]{x}),$$

it is easy to check that $W_{\lambda,q}$ is the unique positive radial solution to equation (1.4) up to a translation. By direct computation, one shows that

$$|W_{\lambda,q}|^2 = \lambda^{\frac{4-(a-2)N}{2(\alpha(2-q))}} |W_q|^2.$$
However, when \( q \neq p \), the scaling method does not work. For the case \( 2 < q < p < 2 + \frac{4}{N} \) with \( N \geq 1 \) and the case \( 2 + \frac{4}{N} < q < p < 2^* \) with \( 2^* = \frac{2N}{N-2} \) if \( N \geq 3 \) and \( 2^* = +\infty \) if \( N = 1, 2 \), the corresponding energy functional to equation \( (1.4) \) is bounded below and unbounded below on \( S_\mu \), respectively, where

\[
S_\mu = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = a^2 \right\}.
\]

Naturally, the techniques for dealing with these cases are also different and the results for the existence of normalized solutions to these cases can be found in [20, 36–39] and references therein. Next, when it comes to combined nonlinearities, the following works deserve to be highlighted. Soave in [40] first studied the existence and nonexistence of the normalized solution for equation \( (1.4) \) with \( \mu \in \mathbb{R} \) and combined power nonlinearities \( 2 < q \leq 2 + \frac{4}{N} \leq p < 2^* \) where \( N \geq 1 \) and made pioneering work by using variational method and Pohožaev constraint. In particular, when \( 2 < q < 2 + \frac{4}{N} < p < 2^* \), he obtained the existence of two solution (local minimizer and Mountain-Pass type) for equation \( (1.4) \). Moreover, he got the orbital stability of the ground state when \( 2 + \frac{4}{N} < q < 2 + \frac{4}{N} < p < 2^* \) and the strongly instability of stand wave when \( q = 2 + \frac{4}{N} < p < 2^* \). Then, he in [41] further considered the existence and nonexistence of the normalized solution for equation \( (1.4) \) with \( \mu \in \mathbb{R} \), \( p = 2^* = \frac{2N}{N-2} \) and \( q \in (2, 2^*) \) where \( N \geq 3 \) by applying a similar technique in [40]. However, in the case of \( 2 < q < 2 + \frac{4}{N} < p = 2^* \), he obtained only the existence of local minimizer for equation \( (1.4) \). About the second normalized solution (Mountain-Pass type) for equation \( (1.4) \) with \( 2 < q < 2 + \frac{4}{N} < p = 2^* \), it was given by Jeanjean et al. [22] for \( N \geq 4 \) and Wei et al. [43] for \( N = 3 \), respectively. And then, Chen et al. [11] proposed new strategies to control the energy level in the Sobolev critical case which allow to treat, in a unified way, the dimensions \( N = 3 \) and \( N \geq 4 \), and obtained the second solution (Mountain-Pass type) for equation \( (1.4) \) with \( 2 < q < 2 + \frac{4}{N} < p = 2^* \). Besides, the generalizations and improvements for [41] was carried out in the recent paper [3, 27]. To be specific, Li [27] removed the restriction on \( \mu \) and got the ground state normalized solution for equation \( (1.4) \) when \( 2 + \frac{4}{N} < q < p = 2^* \). And then, Alves et al. [3] considered the existence of normalized solution with exponential critical growth for \( N = 2 \).

Now, a lot of scholars have focused on the following mass prescribed problem with potential

\[
-\Delta u + V(x)u + \lambda u = g(u) \quad \text{in } \mathbb{R}^N
\]  (1.5)

satisfying the normalization constraint \( \int_{\mathbb{R}^N} u^2 dx = a^2 \), where \( N \geq 1 \) and \( V \) is an external potential. Note that the techniques used in the literature mentioned above studying the non-potential case can not be applied directly. Therefore, such problem are also challenging and stimulating for researchers. When dealing with the mass sub-critical case involving potential and the general nonlinearities, the functional is bounded from below, thus one can apply the minimizing argument constraint on \( S_\mu \). The main difficulty is the compactness of the minimizing sequence. In order to solve this problem, it is important to get the strict so-called sub-additive inequality. We mentioned that Ikoma et al. [19] first made progress in this direction by applying the standard concentration compactness arguments, which is due to Lions [29, 30]. Whereafter, to obtain the strict sub-additive inequality, a new approach was proposed by Zhong and Zou [50] based on iteration. For more existence results can be obtained in [2, 35, 48] and references therein under some different assumptions of \( g \) and \( V \).

When we consider the mass super-critical case involving potential, there are few works on equation \( (1.5) \). More precisely, when \( g(u) = |u|^{q-2}u \) and \( 2 + \frac{4}{N} < q < 2^* \), the authors in [5] considered the existence of normalized solutions with high Morse index for equation \( (1.5) \) with the positive potential \( V(x) \geq 0 \) vanishing at infinity by constructing a suitable linking structure, because the mountain pass structure in Jeanjean [20] does not work in this case. Soon after, when \( g(u) = |u|^{q-2}u \) with \( 2 + \frac{4}{N} < q < 2^* \) and \( V(x) \leq 0 \) vanishing at infinity, the authors in [33] proved the existence of normalized solutions for equation \( (1.5) \) under some explicit smallness assumption on \( V(x) \). Afterwards, Ding and Zhong [13] considered the general nonlinearities \( g \) and got the existence of ground state
normalized solutions to equation (1.5) with negative potential by applying the Pohožaev constraint under an explicit smallness assumption on \( V \) and some Ambrosetti-Rabinowitz type conditions on \( g \). For more results about normalized solutions to equation (1.5) with \( V(x) = 0 \) or \( V(x) \neq 0 \), we refer readers to [1, 7, 42, 46, 47] and references therein.

Motivated by [13, 22, 41], a natural question arises as to whether the normalized solution exists if \( V(x) \neq 0 \) vanishing at infinity and \( g \) is combined nonlinearities in equation (1.5) satisfying the normalization constraint \( \int_{\mathbb{R}^N} u^2 \, dx = a^2 > 0 \). Hence, we investigate the following nonlinear Schrödinger equation with potential and combined power nonlinearities in this paper

\[
\begin{cases}
-\Delta u + V(x)u + \lambda u = |u|^{2^* - 2}u + \mu |u|^{q - 2}u & \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} u^2 \, dx = a^2, 
\end{cases}
\]

where \( N \geq 3, \mu > 0, q \in (2, 2^*), 2^* = \frac{2N}{N-2} \) is the critical Sobolev exponent, \( a > 0 \) is prescribed, and the parameter \( \lambda \in \mathbb{R} \) appears as a Lagrange multiplier. To reach the conclusions of this paper, we give the following assumptions. If \( q \in (2 + \frac{4}{N}, 2^*) \), we assume that \( V \) satisfies

(\( V_1 \)) Let \( V \in C^2(\mathbb{R}^N, \mathbb{R}) \) and \( \lim_{|x| \to \infty} V(x) = \sup_{x \in \mathbb{R}^N} V(x) = 0 \). Moreover, there exists \( 0 < \sigma_1 < \min \{ \frac{2}{q}, 1 - \frac{4}{N(q-2)} \} \) such that

\[
\left| \int_{\mathbb{R}^N} V(x)|u|^2 \, dx \right| \leq \sigma_1 |\nabla u|^2_2 \quad \text{for all } u \in H^1(\mathbb{R}^N, \mathbb{C});
\]

(\( V_2 \)) let \( W(x) = \frac{1}{2} (\nabla V(x), x) \) and \( \lim_{|x| \to \infty} W(x) = 0 \). There is \( 0 < \sigma_2 < \min \{ \frac{2}{N-2} - \frac{N}{N-2} \sigma_1, 1 - \frac{4}{N(q-2)}(1 - \sigma_1) - 1 \} \) such that

\[
\left| \int_{\mathbb{R}^N} W(x)|u|^2 \, dx \right| \leq \sigma_2 |\nabla u|^2_2 \quad \text{for all } u \in H^1(\mathbb{R}^N, \mathbb{C});
\]

(\( V_3 \)) let \( L(x) = (\nabla W(x), x) \). There exists \( 0 < \sigma_3 < \frac{2}{N-2} N(1 - \sigma_2) - 2 \) such that

\[
\left| \int_{\mathbb{R}^N} L(x)|u|^2 \, dx \right| \leq \sigma_3 |\nabla u|^2_2 \quad \text{for all } u \in H^1(\mathbb{R}^N, \mathbb{C});
\]

(\( V_4 \)) \( V(x) + W(x) \leq 0 \) a.e. on \( \mathbb{R}^N \).

If \( q = \frac{7}{3} := 2 + \frac{4}{N} \), we give the following hypothesises for \( V \).

(\( \tilde{V}_1 \)) \( V \in C^2(\mathbb{R}^N, \mathbb{R}) \) and \( \lim_{|x| \to \infty} V(x) = \sup_{x \in \mathbb{R}^N} V(x) = 0 \), and there exists \( 0 < \tilde{\sigma}_1 < \min \{ \frac{2}{N}, \frac{1}{2} \} \) such that

\[
\left| \int_{\mathbb{R}^N} V(x)|u|^2 \, dx \right| \leq \tilde{\sigma}_1 |\nabla u|^2_2 \quad \text{for all } u \in H^1(\mathbb{R}^N, \mathbb{C});
\]

(\( \tilde{V}_2 \)) let \( W(x) = \frac{1}{2} (\nabla V(x), x) \), \( \lim_{|x| \to \infty} W(x) = 0 \). There exists \( 0 < \tilde{\sigma}_2 < \min \{ \frac{2}{N-2} - \frac{N}{N-2} \tilde{\sigma}_1, \frac{2}{N} \} \) such that

\[
\left| \int_{\mathbb{R}^N} W(x)|u|^2 \, dx \right| \leq \tilde{\sigma}_2 |\nabla u|^2_2 \quad \text{for all } u \in H^1(\mathbb{R}^N, \mathbb{C});
\]

(\( \tilde{V}_3 \)) let \( L(x) = (\nabla W(x), x) \). There exists \( 0 < \tilde{\sigma}_3 < \frac{4}{N-2} - \frac{2N}{N-2} \tilde{\sigma}_2 \) such that

\[
\left| \int_{\mathbb{R}^N} L(x)|u|^2 \, dx \right| \leq \tilde{\sigma}_3 |\nabla u|^2_2 \quad \text{for all } u \in H^1(\mathbb{R}^N, \mathbb{C}).
\]

Before presenting our main results, we give some notations that will be used throughout the paper. The notation \( \mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{C}) \) (or \( \mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{R}) \)) is the usual Sobolev space with the norm

\[
\|u\|_{\mathcal{D}^{1,2}} = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx.
\]
$H^1(\mathbb{R}^N, \mathbb{C})$ (or $H^1(\mathbb{R}^N, \mathbb{R})$) denotes the usual Sobolev space with the norm
\[ ||u||_H = \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \, dx. \]

$L^p(\mathbb{R}^N, \mathbb{C})$ (or $L^p(\mathbb{R}^N, \mathbb{R})$) is the Lebesgue space endowed with the norm
\[ |u|_{L^p(\mathbb{R}^N)} = |u|_p = \left( \int_{\mathbb{R}^N} |u|^p \, dx \right)^{\frac{1}{p}}, \quad \text{where} \ p \in [1, +\infty), \]
and $|u|_{L^\infty(\mathbb{R}^N)} = |u|_\infty = \text{ess sup}_{x \in \mathbb{R}^N} |u(x)|$. $C$ and $C_i$, $i = 1, 2, \ldots$, denote positive constants possibly different in different places. $o(1)$ denotes a quantity which goes to zero as $n \to \infty$. Moreover, we denote by $\mathbb{R}^+$ the interval $(0, +\infty)$. Now, we introduce the work space in this paper
\[ E = \left\{ u \in H^1(\mathbb{R}^N, \mathbb{C}) : \int_{\mathbb{R}^N} V(x)|u|^2 \, dx < +\infty \right\} \]
endowed with the inner product and the induced norm
\[ \langle u, \varphi \rangle = \int_{\mathbb{R}^N} \nabla u \cdot \nabla \varphi + V(x)u \bar{\varphi} \, dx \quad \text{and} \quad ||u|| = \left( \int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 + V(x)|u|^2 \, dx \right)^{\frac{1}{2}} \mathcal{L}^N \]
where $\bar{\varphi}$ is the complex conjugate of $\varphi$. Moreover, under our assumptions of $V$ in present paper, we deduce that the norm $||u||$ is equivalent to the usual norm $||u||_H$. The embedding $E \hookrightarrow L^p(\mathbb{R}^N, \mathbb{C})$ is continuous with $p \in [2, 2^*)$ and $N \geq 3$. Since the energy functional $I_q : E \to \mathbb{R}$ of Eq. (P) defined by
\[ I_q(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} |u|^{2^*_q} \, dx - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q \, dx \quad (1.6) \]
is unbounded from below constraint on the $L^2$-sphere $S_a$, where
\[ S_a = \left\{ u \in E : \int_{\mathbb{R}^N} |u|^2 \, dx = a^2 \right\}, \]
thus one can not apply the minimizing argument constraint on $S_a$ any more. Therefore, we intend to consider the minimization of $I_q$ on a subsets of $S_a$ in this work. It is easy to see that if $u$ is a solution to Eq. (P), then the following Nehari identity holds
\[ \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2 \, dx + \lambda \int_{\mathbb{R}^N} |u|^2 \, dx - \int_{\mathbb{R}^N} |u|^{2^*_q} \, dx - \mu \int_{\mathbb{R}^N} |u|^q \, dx = 0. \quad (1.7) \]
Moreover, from [26, Proposition 2.1] (see also [13, Lemma 3.1]) one obtains that if $u \in E$ is a solution to Eq. (P), $u$ satisfies the following Pohožaev identity
\[ \frac{N - 2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \frac{N}{2} \int_{\mathbb{R}^N} V(x)|u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} \langle \nabla V(x), u \rangle |u|^2 \, dx + \frac{\lambda N}{2} \int_{\mathbb{R}^N} |u|^2 \, dx - \frac{N}{2} \int_{\mathbb{R}^N} |u|^{2^*_q} \, dx - \frac{\mu N}{q} \int_{\mathbb{R}^N} |u|^q \, dx = 0. \quad (1.8) \]
Hence, by (1.7) and (1.8), $u$ satisfies
\[ P_q(u) = 0, \quad (1.9) \]
where
\[ P_q(u) := \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} \langle \nabla V(x), u \rangle |u|^2 \, dx - \int_{\mathbb{R}^N} |u|^{2^*_q} \, dx - \mu \gamma_q \int_{\mathbb{R}^N} |u|^q \, dx \]
with $\gamma_q = \frac{N(q - 2)}{2q} = \frac{N}{q} - \frac{N}{q}$. (1.8) and (1.9) are also called Pohožaev identity. In particular, (1.9) is widely used in the literature when study the prescribed mass problem. Thus, if not otherwise specified,
the Pohožaev identity we used in this paper is (1.9). To find the normalized solutions of Eq. (P), we introduce the following Pohožaev constrained set

$$\mathcal{P}_{q,a} = \{ u \in E : P_q(u) = 0 \} \cap S_a.$$ 

Obviously, the set $\mathcal{P}_{q,a}$ contains all of the normalized solutions to Eq. (P). Hence, this paper mainly studies the following minimization problem

$$m_{q,a} = \inf_{u \in \mathcal{P}_{q,a}} \mathcal{I}_q(u).$$

For the limit problem to Eq. (P), namely,

$$\begin{align*}
\begin{cases}
-\Delta u + u + \lambda u = |u|^{2^* - 2}u + \mu |u|^{q-2}u & \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} u^2 dx = a^2 > 0,
\end{cases}
\end{align*}$$

where $N \geq 3$, $\mu > 0$, $q \in (2, 2^*)$ with $2^* = \frac{2N}{N-2}$ and the parameter $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier. Defined the energy functional $I_q^\infty : H^1(\mathbb{R}^N, \mathbb{C}) \to \mathbb{R}$ of Eq. (P$_\infty$)

$$I_q^\infty(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q dx,$$

and the constrained set and the least energy of the normalized solutions for Eq. (P$_\infty$) is given by

$$\mathcal{P}_{q,a}^\infty = \{ u \in H^1(\mathbb{R}^N, \mathbb{C}) : P^\infty_q(u) = 0 \} \cap S_a \quad \text{and} \quad m^\infty_{q,a} = \inf_{u \in \mathcal{P}_{q,a}^\infty} I^\infty_q(u),$$

where

$$P^\infty_q(u) = \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} |u|^{2^*} dx - \mu \gamma_q \int_{\mathbb{R}^N} |u|^q dx$$

with $\gamma_q = \frac{N(q-2)}{2q} = \frac{N}{2} - \frac{N}{q}$.

We will be particularly interested in ground state solutions, defined as follows:

**Definition 1.1.** We say that $\tilde{u}$ is a ground state of Eq. (P) on $S_a$ if it is a solution to Eq. (P) having minimal energy among all the solutions which belongs to $S_a$:

$$dI_q|_{S_a}(\tilde{u}) = 0 \quad \text{and} \quad I_q(\tilde{u}) = \inf\{ I_q(u) : dI_q|_{S_a}(u) = 0 \text{ and } u \in S_a \}. \quad \text{(1.12)}$$

The set of the ground states will be denoted by $Z_a$.

We also recall the notion of instability that we are interested in:

**Definition 1.2.** A standing wave $e^{i\lambda t}u$ is strongly unstable if for every $\varepsilon > 0$, there exists $\varphi_0 \in H^1(\mathbb{R}^N, \mathbb{C})$ such that $\|u - \varphi_0\|_H < \varepsilon$, and $\varphi(t, \cdot)$ blows-up in finite time, where $\varphi(t, \cdot)$ denotes the solution to (1.1) with initial datum $\varphi_0$.

Now, we give the main results in this paper.

**Theorem 1.3.** Let $N \geq 3$. Assume that $q \in (2 + \frac{4}{N}, 2^*)$ and (V$_1$) -- (V$_4$) hold. Then for any $\mu > 0$ and $a > 0$, there exists a couple $(\lambda, u) \in \mathbb{R}^+ \times X$ solving Eq. (P), where $u$ is a positive real-valued function in $\mathbb{R}^N$ and $I_q(u) = m_{q,a}$.

**Theorem 1.4.** Let $N \geq 3$ and $a > 0$. Assume that $q = \overline{q} := 2 + \frac{4}{N}$ and (V$_1$) -- (V$_3$) and (V$_4$) hold. If

$$0 < \mu a^{\overline{q}} < \min \left\{ 1 - \overline{\sigma}_1, \frac{N + 2}{N} - \left( 2 + \frac{4}{N} \right) \overline{\sigma}_1, 1 - \frac{N \overline{\sigma}_1}{2}, \frac{N - 2}{2} \overline{\sigma}_2, \frac{2}{N} \overline{\sigma}_2 - \frac{\overline{\sigma}_3}{2^*} \right\} (\overline{\sigma}_N)^{\overline{q}}$$

where $\overline{\sigma}_N$ is defined in (2.2), then there exists a couple $(\lambda, u) \in \mathbb{R}^+ \times E$ that solves Eq. (P), where $u$ is a positive real-valued function in $\mathbb{R}^N$ and $I_q(u) = m_{q,a}$.
**Theorem 1.5.** Let $N \geq 3$. Assume that $2 < q < \frac{4}{N}$ and $V$ satisfies $(V_q)$ and $(\tilde{V}_1)$. Let $V \in C^1(\mathbb{R}^N, \mathbb{R})$ and $\lim_{|x| \to \infty} V(x) = \sup_{x \in \mathbb{R}^N} V(x) = 0$. Moreover, there exists $0 < \tilde{\sigma}_1 < 1$ such that

$$\left| \int_{\mathbb{R}^N} V(x)|u|^2 dx \right| \leq \tilde{\sigma}_1 |\nabla u|^2_2$$

for all $u \in H^1(\mathbb{R}^N, \mathbb{C})$.

Then, for any $\mu > 0$, there exists $a_0 = a_0(\mu) > 0$ such that Eq. $(P)$ has a solution $(\lambda, u) \in \mathbb{R}^+ \times E$ for any $a \in (0, a_0)$.

**Remark 1.6.** To our best knowledge, it seems to be the first work on the existence of normalized solution for the Schrödinger equation with potential and combined power nonlinearities. The appearance of the potential term will affect the geometry of the problem. More specifically, when $q \in (\mathbb{R}, 2^*)$, verifying the geometry of the problem becomes more complicated with respect to the case $V(x) = 0$ (see Lemmas 2.6 and 4.4). And when $q \in (2, q_0)$, it is difficult to get the geometry for the problem with respect to the case $V(x) = 0$ as in [41, Lemma 4.2].

**Remark 1.7.** Because of the potential and the Sobolev critical term, the main difficulty we encounter in proving the existence results is the lack of compactness. Compared with [41], since the embedding $E \hookrightarrow L^p(\mathbb{R}^N, \mathbb{C})$ is not compact with $p \in (2, 2^*)$ and $N \geq 3$ in this paper, we can not solve it by using a similar technique as in [41]. It must be stressed that Proposition 2.2 plays a key role in overcoming this difficulty and verifying Theorems 1.3 and 1.4. Furthermore, motivated by [21], we obtain a solution with negative energy for Eq. $(P)$ satisfying $|u|^2 = a \in (0, a_0)$ when $q \in (2, q_0)$. Since the translation invariance properties of the problem established in [21] is not valid in this paper, we cannot apply the methods in [21] to exclude the vanishing for the weak limit of the minimizing sequences on a related problem. To achieve it, we make full use of Lemma 5.6. Then we show that a minimizing sequence with nontrivial weak limit of $I_q$ is precompact with the aid of Lemma 5.5 and prove Theorem 1.5. However, we do not know whether the normalized solution obtained in Theorem 1.5 is the ground state normalized solution under the sense of Definition 1.1.

**Remark 1.8.** Using a similar discussions and techniques as above, we can obtain the existence of normalized solutions for the following problem

$$\begin{cases}
-\Delta u + V(x)u + \lambda u = |u|^{p-2}u + \mu |u|^{q-2}u & \text{in } \mathbb{R}^N, \\
\int_{\mathbb{R}^N} u^2 dx = a^2 > 0,
\end{cases}
$$

where $N \geq 3$, $\mu > 0$, $2 < q < p < 2^*$, and the parameter $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier. To be more precise, the method of Theorem 1.3 and Theorem 1.4 can be used in the case of $2 + \frac{4}{N} \leq q < p < 2^*$; when $2 < q < 2 + \frac{4}{N} < p < 2^*$, we can refer to the proof of Theorem 1.5.

**Remark 1.9.** Unlike the previous paper [11, 21, 40, 43], it seems that it is not easy to get the second solution (Mountain-Pass type) on the case of $2 < q < 2 + \frac{4}{N} < p \leq 2^*$ for problem (1.13) in this paper due to the lack of compactness. Therefore, finding the second solution (Mountain-Pass type) for problem (1.13) with $2 < q < 2 + \frac{4}{N} < p \leq 2^*$ is also an interesting problem.

In what follows, we give the exponential decay property of the positive solution for Eq. $(P)$.

**Theorem 1.10.** Suppose that $(V_1)$ and $(V_q)$ or $(\tilde{V}_1)$ and $(V_q)$ hold. Let $u \in E$ be the positive real-valued solution for Eq. $(P)$ with $\mu > 0$, $a > 0$ and $q \in [2 + \frac{4}{N}, 2^*)$. Then the corresponding Lagrange multiplier $\lambda > 0$, and there exists a constant $M > 0$ such that

$$|u(x)| \leq Me^{-\sqrt{\frac{2}{\mu}|x|}}$$

for all $x \in \mathbb{R}^N$. 

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Remark 1.11. It must be emphasized that the exponential decay of the positive solution for Eq. \((P)\) plays an important role in proving the strong instability of the stand wave for problem \((1.1)\). In \([41]\), it is known from Radial Lemma \([6]\) that the radial solution obtained by the author naturally satisfies the decay property \(|u(x)| \to 0\) as \(|x| \to \infty\). However, unlike quoted paper \([41]\), the Radial Lemma \([6]\) cannot be used in our case. So the difficulty we face is that it is hard to verify \(|u(x)| \to 0\) as \(|x| \to \infty\). To overcome this obstacle, we will use a new technique in this work, i.e., Moser iteration technique, which is mainly inspired by \([9,17]\).

When \(V(x) = 0\), the following issue was firstly studied by Tao et. al \([34]\), where they proved the occurrence of finite-time blow-up for focusing \(L^2\)-supercritical and Sobolev critical case. After then, Soave in \([41]\) gave a similar results in a different (and complementary) perspective. Inspired by the above studies, we have the following result.

Theorem 1.12. Suppose that

\((V_5)\) there exist \(\epsilon_1, \epsilon_2 > 0\) small enough such that \(|V| \frac{\kappa^2}{2N-4} < \epsilon_1\) and \(|\nabla V| \frac{\kappa^2}{2N-4} < \epsilon_2\).

(i) Under the assumptions of Theorem 1.3 or Theorem 1.4, let \(u_0 \in S_a\) be such that \(\mathcal{I}_q(u_0) < \inf_{u \in P_{q,a}} \mathcal{I}_q(u)\), and if \(|x|u_0 \in L^2(\mathbb{R}^N, C)\) and \(s_{u_0} < 0\), where \(s_{u_0}\) is defined in Lemma 2.6 or Lemma 4.4, then the solution \(\phi\) of problem \((1.1)\) with initial datum \(u_0\) blows-up in finite time.

(ii) In particular, let \(q \in (7,2^*)\) and let \((V_1) - (V_2)\) be hold. If we assume that \(\mathcal{I}_q(u_0) < 0, |x|u_0 \in L^2(\mathbb{R}^N, C)\) and \(y(0) = y_0 > 0\) defined in \((6.11)\), then the blow-up time \(T\) can be estimate by

\[
0 < T \leq \frac{|xu_0|^2}{(q\gamma \kappa - \frac{1}{q} \kappa^2 - q\gamma \kappa \sigma_1)y_0}.
\]

Remark 1.13. The assumption \((V_5)\) is only used to ensure that the initial-value problem \((1.1)\) with initial datum is local well-posed on \((-T_{min}, T_{max})\) with \(T_{min}, T_{max} > 0\). More precisely, as in the quoted paper \([21]\), our purpose is to show that the Duhamel functional

\[
\Theta(u)(t) := e^{it\Delta}u(0) + i \int_0^t e^{i(t-s)\Delta}(|u|^{2^* - 2}u + |u|^q - 2 - 2 - V(x)u)ds
\]

is a contraction on a particular complete metric space, e.g., the metric space \((B_{2\gamma_0}, T, d)\) in \([21]\). If \(V \in L^\infty(\mathbb{R}^N)\) is sufficient to get the result we want, then the assumption \((V_5)\) can be removed. Moreover, there are many functions that satisfy the assumptions \((V_1) - (V_2)\) or \((\tilde{V}_1) - (\tilde{V}_2)\) or \((\tilde{V}_1)\) in this paper, for example, \(V(x) = -\frac{\kappa}{|x|^2 + \ln(|x| + 2)}\) with \(\kappa > 0\) small enough.

Based on the results mentioned above, we further study the strong instability of the stand waves for problem \((1.1)\).

Theorem 1.14. Assume that \((V_5)\) holds. Under the assumptions of Theorem 1.3 or Theorem 1.4, \(Z_a\) has the following characteristics

\[Z_a = \{e^{i\theta}U : \theta \in \mathbb{R}, U \in U_a \text{ and } u > 0 \text{ in } \mathbb{R}^N\},\]

where \(U_a\) is defined in \((3.14)\). Moreover, if \(\tilde{u}\) is a ground state, then the associated Lagrange multiplier \(\tilde{\lambda}\) is positive, and the standing wave \(e^{i\lambda t}\tilde{u}\) is strongly unstable.

Remark 1.15. To prove it, we will make use of Theorems 1.10 and 1.12. Compared to the previous literature \([40,41]\), which strongly relies on the solution of Eq. \((P)\) being a radial function, the novelty in this paper is that the solution of Eq. \((P)\) is no longer restricted to a radial function.

We conclude this section by giving the organization of this paper. In Sec. 2, we give some preliminaries, which are important to justify our results. Then we will complete the proof of Theorem 1.3 in Sec. 3. Whereafter, we discuss the \(L^2\)-critical case and prove Theorem 1.4 in Sec. 4. The proof of Theorem 1.5 is given in Sec. 5. In Sec. 6, we consider the strong instability of the standing wave for problem \((1.1)\) and prove Theorems 1.10, 1.12 and 1.14. Throughout this paper, we assume that \(\mu > 0\) and \(a > 0\) in Eq. \((P)\).
2 Preliminaries

Firstly, let us recall a key inequality, i.e., Gagliardo-Nirenberg inequality, which can be found in [44]. Let \( N \geq 3 \) and \( 2 < q < 2^* \), then the following sharp Gagliardo-Nirenberg inequality holds for any \( u \in E \)

\[
|u|_q \leq C_{N,q}|u|_{2}^{1-\gamma}|\nabla u|_{2}^{\gamma}
\]

with \( \gamma_q = \frac{N(q-2)}{2q} = \frac{N}{2} - \frac{N}{q} \), where the sharp constant \( C_{N,q} \) is

\[
C_{N,q}^{q} = \frac{2q}{2N+(2-N)q} \left( \frac{2N+(2-N)q}{N(q-2)} \right)^{\frac{N(q-2)}{4}} \frac{1}{|W_q|^{-\frac{q-2}{2}}}
\]

and \( W_q \) is the unique positive radial solution of equation

\[-\Delta u + u = |u|^{q-2} u.\]

In the special case \( q = \overline{q} := 2 + \frac{4}{N} \), \( C_{N,\overline{q}}^{\overline{q}} = \frac{\overline{q}-1}{2|W_{\overline{q}}|^2} \), or equivalently,

\[
|W_{\overline{q}}|^2 = \left( \frac{\overline{q}}{2C_{N,\overline{q}}} \right)^N =: \overline{\sigma}. \tag{2.2}
\]

For every \( N \geq 3 \), there exists an optimal constant \( S > 0 \) depending only on \( N \) such that

\[
S|u|_{2}^q \leq |\nabla u|_{2}^q \tag{2.3}
\]

for all \( u \in D^{1,2}(\mathbb{R}^N, \mathbb{C}) \).

Recall that the limit problem \((P_\infty)\) was studied by Soave [41], see also [27], they got the following theorem.

**Theorem 2.1.** Let \( N \geq 3, 2 + \frac{4}{N} \leq q < 2^* \), and let \( a, \mu > 0 \). If \( q = \overline{q} := 2 + \frac{4}{N} \), we further assume that \( \mu a^{\frac{4}{N}} < (\overline{\sigma} N)^{\frac{4}{N}} \). Then \( I_q^\infty|_{S_a} \) has a positive real-valued ground state \( u \) satisfying \( I_q^\infty(u) = m_{q,a}^\infty \).

Now, we give an important property of the ground state energy map \( a \mapsto m_{q,a}^\infty \) with \( a > 0 \).

**Proposition 2.2.** Let \( q \in [\overline{q}, 2^*) \) and \( \mu > 0 \). If \( q = \overline{q} \), we further assume that \( \mu a^{\frac{4}{N}} < (\overline{\sigma} N)^{\frac{4}{N}} \). Then \( a \mapsto m_{q,a}^\infty \) with \( a > 0 \) is non-increasing.

**Proof.** Let \( 0 < a < b < +\infty, \theta = \frac{a}{b} < 1 \) and \( q \in [\overline{q}, 2^*) \). For any \( u \in S_a \), setting \( w = \theta^{\frac{N-2}{2}} u(\theta x) \), then, we have \( \int_{\mathbb{R}^N}|w|^2dx = b^2 \), which shows that \( w \in S_b \). Similar to [41, Lemmas 5.1 and 6.1], there exists a unique \( t_w \) such that \( t_w \ast w \in \mathcal{P}_{q,b}^\infty \). Moreover, by simple calculation, one obtains

\[
\int_{\mathbb{R}^N} |\nabla w|^2 dx = \int_{\mathbb{R}^N} |\nabla u|^2 dx \quad \text{and} \quad \int_{\mathbb{R}^N} |w|^2 dx = \int_{\mathbb{R}^N} |u|^2 dx, \tag{2.4}
\]

\[
\int_{\mathbb{R}^N} |w|^q dx = \theta^{\frac{N-2}{q-N}} \int_{\mathbb{R}^N} |u|^q dx \geq \int_{\mathbb{R}^N} |u|^q dx. \tag{2.5}
\]

Then, it follows from (2.4) and (2.5) that

\[
m_{q,b}^\infty \leq I_q^\infty(t_w \ast w) = I_q^\infty(e^{\frac{N}{2}} w(e^t_w x)) = \frac{e^{2t_w}}{2} \int_{\mathbb{R}^N} |\nabla w|^2 dx - \frac{e^{2t_w}}{2} \int_{\mathbb{R}^N} |w|^2 dx - \frac{1}{q} e^{(\frac{q}{2}-1)Nt_w} \mu \int_{\mathbb{R}^N} |w|^q dx.
\]
\[
\begin{align*}
&= \frac{e^{2s}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{e^{2s}}{2^s} \int_{\mathbb{R}^N} |u|^2 dx - \frac{1}{q} \frac{(q-1)Nw^qN^2+N-N-1}{2} \mu \int_{\mathbb{R}^N} |u|^q dx \\
&< \frac{e^{2s}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{e^{2s}}{2^s} \int_{\mathbb{R}^N} |u|^2 dx - \frac{1}{q} \frac{(q-1)Nw^qN^2+N-N-1}{2} \mu \int_{\mathbb{R}^N} |u|^q dx \\
&= \mathcal{I}_q^\infty(t \ast u) \\
&\leq \max_{t > 0} \mathcal{I}_q^\infty(t \ast u),
\end{align*}
\]
which and [41, Lemma 8.1] imply that \(m_{q,b}^\infty \leq \inf_{u \in \mathcal{S}_a} \max_{t > 0} \mathcal{I}_q^\infty(t \ast u) = m_{q,a}^\infty\) by the arbitrariness of \(u\). Thus, we complete the proof of Proposition 2.2. \qed

During the proofs, the following various expressions of \(\mathcal{I}_q(u)\) constrained on \(\mathcal{P}_{q,a}\) play an important role. In view of (1.6) and (1.9), one obtains

\[
\mathcal{I}_q(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left[V(x) + \frac{1}{q} \langle \nabla V(x), x \rangle \right] |u|^2 dx + \left(1 - \frac{1}{2} \right) \int_{\mathbb{R}^N} |u|^2 dx + \mu \left(\frac{2q}{2} - \frac{1}{q} \right) \int_{\mathbb{R}^N} |u|^q dx
\]

with \(\gamma_q = \frac{N(q-2)}{2q} = \frac{N}{2} - \frac{N}{q}\). For any \(u \in \mathcal{S}_a\), let

\[
(s \ast u)(x) := e^{\frac{N}{2} s} u(e^s x),
\]

one has \(s \ast u \in \mathcal{S}_a\). Moreover,

\[
\psi_u(s) := \mathcal{I}_q(s \ast u)
\]

\[
= \frac{e^{2s}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(e^{-s} x) |u|^2 dx - \frac{e^{2s}}{2^s} \int_{\mathbb{R}^N} |u|^2 dx - \mu \frac{q}{2^s} \int_{\mathbb{R}^N} |u|^q dx,
\]

and

\[
\mathcal{P}_q(s \ast u) = e^{2s} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} W(e^{-s} x) |u|^2 dx - e^{2s} \int_{\mathbb{R}^N} |u|^2 dx - \mu \gamma_q e^{(\frac{q}{2}) N} \int_{\mathbb{R}^N} |u|^q dx.
\]

Obviously, one knows that \(\psi_u'(s) = \mathcal{P}_q(s \ast u)\), which implies that for any \(u \in \mathcal{S}_a, s \in \mathbb{R}\) is a critical point for \(\psi_u(s)\) if and only if \(s \ast u \in \mathcal{P}_{q,a}\).

The following lemma help us to show that \(\mathcal{I}_q\) is bounded away from 0 on \(\mathcal{P}_{q,a}\). Here the Gagliardo-Nirenberg inequality (2.1) plays a crucial role.

**Lemma 2.3.** Let \((V_2)\) be satisfied and \(q \in (\bar{q}, 2^*)\). For any \(u \in \mathcal{P}_{q,a}\), there exists \(\delta > 0\) such that \(|\nabla u|^2 \geq \delta\).

**Proof.** For any \(u \in \mathcal{P}_{q,a}\), in virtue of \((V_2), (1.9), \) Gagliardo-Nirenberg inequality (2.1) and (2.3), we know that

\[
\int_{\mathbb{R}^N} |\nabla u|^2 dx = \frac{1}{2} \int_{\mathbb{R}^N} \langle \nabla V(x), x \rangle |u|^2 dx + \int_{\mathbb{R}^N} |u|^2 dx + \mu \gamma_q \int_{\mathbb{R}^N} |u|^q dx
\]

\[
\leq \sigma_2 |\nabla u|^2 + S^{-\frac{N}{2}} |\nabla u|^2 + \mu \gamma_q C_{N,q}^q a^{(1-\gamma_q)} |\nabla u|^q,
\]

which implies that

\[
(1 - \sigma_2) |\nabla u|^2 \leq S^{-\frac{N}{2}} |\nabla u|^2 + \mu \gamma_q C_{N,q}^q a^{(1-\gamma_q)} |\nabla u|^q.
\]

Since \(q \in (\bar{q}, 2^*)\), then \(q \gamma_q > 2\). Hence, for any \(\mu > 0\), one obtains that there exists \(\delta > 0\) satisfying \(|\nabla u| \geq \delta\). So, we complete the proof. \(\square\)
Lemma 2.4. Assume that (V1) and (V2) hold. For any \( q \in (\bar{q}, 2^*) \), \( m_{q,a} = \inf_{u \in \mathcal{P}_{q,a}} \mathcal{I}_q(u) > 0 \).

Proof. By (V2) and \( P_q(u) = 0 \), one infers that

\[
(1 + \sigma_2) \int_{\mathbb{R}^N} |\nabla u|^2 dx \geq \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} \langle \nabla V(x), x \rangle |u|^2 dx
 = \int_{\mathbb{R}^N} |u|^{2^*} dx + \mu \gamma_q \int_{\mathbb{R}^N} |u|^q dx
\geq \gamma_q \int_{\mathbb{R}^N} |u|^{2^*} dx + \mu \gamma_q \int_{\mathbb{R}^N} |u|^q dx.
\]

Then, in view of (1.6), (2.12) and (V1), we have

\[
\mathcal{I}_q(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} |u|^{2^*} dx - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q dx
\geq \frac{1}{2} (1 - \sigma_1) \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} |u|^{2^*} dx - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q dx
\geq \frac{1}{2} (1 - \sigma_1) \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{q \gamma_q} (1 + \sigma_2) \int_{\mathbb{R}^N} |\nabla u|^2 dx
\geq \frac{1}{2} (1 - \sigma_1) \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{q \gamma_q} \left( \frac{1 - \sigma_1}{2} - \frac{\sigma_2}{q \gamma_q} \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx.
\]

Then, we deduce from Lemma 2.3 and (2.13) that \( m_{q,a} = \inf_{u \in \mathcal{P}_{q,a}} \mathcal{I}_q(u) > 0 \) due to the range of \( \sigma_1 \) and \( \sigma_2 \). Thus, we complete the proof. \( \square \)

Consider the decomposition of \( \mathcal{P}_{q,a} \) into the disjoint union

\[
\mathcal{P}_{q,a} = (\mathcal{P}_{q,a})^+ \cup (\mathcal{P}_{q,a})^0 \cup (\mathcal{P}_{q,a})^-,
\]

where

\[
(\mathcal{P}_{q,a})^{+(resp.\ 0,-)} = \{ u \in \mathcal{P}_{q,a} : \psi'_u(0) > (resp. \ =, <) 0 \}.
\]

Lemma 2.5. Assume that (V2) – (V3) hold and \( q \in (\bar{q}, 2^*) \). Then \( \mathcal{P}_{q,a} = (\mathcal{P}_{q,a})^- \).

Proof. For any \( u \in \mathcal{P}_{q,a} \), then \( P_q(u) = 0 \), namely,

\[
\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} \langle \nabla V(x), x \rangle |u|^2 dx - \int_{\mathbb{R}^N} |u|^{2^*} dx - \mu \gamma_q \int_{\mathbb{R}^N} |u|^q dx = 0,
\]

where \( \gamma_q = \frac{(q-2)N}{2q} \). From (2.9), one has

\[
\psi''_u(0) = 2 \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} \langle \nabla W(x), x \rangle |u|^2 dx - 2^* \int_{\mathbb{R}^N} |u|^{2^*} dx - \mu \gamma_q \frac{(q-2)N}{2} \int_{\mathbb{R}^N} |u|^q dx.
\]

Then, by (2.14), (V2) and (V3), we deduce that

\[
\psi''_u(0) = \psi''_u(0) - \frac{(q-2)N}{2} P_q(u)
 = \left( 2 - \frac{(q-2)N}{2} \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} \langle \nabla W(x), x \rangle |u|^2 dx + \frac{(q-2)N}{4} \int_{\mathbb{R}^N} |\nabla V(x), x \rangle |u|^2 dx + \frac{(q-2)N}{2} \int_{\mathbb{R}^N} |u|^{2^*} dx
\]

\[
\leq \left( 2 - \frac{(q-2)N}{2} \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} \left( \langle \nabla W(x), x \rangle + \frac{(q-2)N}{2} W(x) \right) |u|^2 dx
\]

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$$\leq \left( 2 - \frac{(q - 2)N}{2} + \frac{(q - 2)N}{2} \sigma_2 + \sigma_3 \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx$$

$$< 0,$$

which implies that $u \in (\mathcal{P}_{q,a})^-$. Hence, we deduce $\mathcal{P}_{q,a} = (\mathcal{P}_{q,a})^-$ and we complete the proof. \hfill \Box

**Lemma 2.6.** Let $(V_2)$ and $(V_3)$ be hold and $q \in (q, 2^*)$. For any $u \in S_a$, the function $\psi_u$ has a unique critical point $s_u$. In other words, there exists a unique $s_u \in \mathbb{R}$ such that $s_u * u \in \mathcal{P}_{q,a}$. Moreover, $\mathcal{I}_q(s_u * u) = \max_{s \geq 0} \mathcal{I}_q(s * u)$ and $\psi_u$ is strictly decreasing and concave on $(s_u, +\infty)$. In particular, $s_u < 0$ if and only if $P_q(u) < 0$. And the map $u \in S_a \mapsto s_u \in \mathbb{R}$ is of class $C^1$.

**Proof.** Since $\psi_u'(s) = P_q(s * u)$, then we only prove that $\psi_u'(s)$ has a unique root in $\mathbb{R}$. By (2.9) and (V2), for any $u \in S_a$, one has

$$\psi_u'(s) = e^{2s} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} W(\psi(s), e^{-s}x)|u|^2 dx - e^{2s} \int_{\mathbb{R}^N} |u|^{2^*} dx - \mu \gamma q e^{(\frac{2}{q} - 1)Ns} \int_{\mathbb{R}^N} |u|^q dx$$

$$\geq (1 - \sigma_2) e^{2s} |\nabla u|^2 - e^{2s} \int_{\mathbb{R}^N} |u|^{2^*} dx - \mu \gamma q e^{(\frac{2}{q} - 1)Ns} \int_{\mathbb{R}^N} |u|^q dx. \quad (2.16)$$

When $q \in (q, 2^*)$, we deduce that $\psi_u'(s) > 0$ for $s \to -\infty$. Hence there exists $s_0 \in \mathbb{R}$ such that $\psi_u(s)$ is increasing on $(-\infty, s_0)$. In addition, we know that $\psi_u(s) \to -\infty$ as $s \to +\infty$. Thus, there is $s_1 \in \mathbb{R}$ with $s_1 > s_0$ such that

$$\psi_u(s_1) = \max_{s > 0} \psi_u(s). \quad (2.17)$$

Therefore, $\psi_u'(s_1) = 0$, namely, $s_1 * u \in \mathcal{P}_{q,a}$. Assume that there exists $s_2 \in \mathbb{R}$ satisfying $s_2 * u \in \mathcal{P}_{q,a}$. Without loss of generality, assume that $s_2 > s_1$. From Lemma 2.5, we have $\psi_u'(s_2) < 0$. Hence, there is $s_3 \in (s_1, s_2)$ such that

$$\psi_u(s_3) = \min_{s \in (s_1, s_2)} \psi_u(s).$$

Thus, we deduce that $\psi_u'(s_3) \geq 0$ and $\psi_u'(s_3) = 0$, which implies that $s_3 * u \in \mathcal{P}_{q,a}$ and $s_3 * u \in (\mathcal{P}_{q,a})^+ \cup (\mathcal{P}_{q,a})^0$. This is a contradiction. Hence, setting $s_u = s_1$, we know that $s_u \in \mathbb{R}$ is an unique number satisfying $s_u * u \in \mathcal{P}_{q,a}$. And then it follows from (2.17) that $\mathcal{I}_q(s_u * u) = \max_{s > 0} \mathcal{I}_q(s * u)$. Furthermore,

$$\psi_u'(s) < 0 \iff s > s_u. \quad (2.18)$$

Hence, $\psi_u'(0) = P_q(u) < 0$ if and only if $s_u < 0$. In the following, we show that $\psi_u$ is strictly decreasing and concave on $(s_u, +\infty)$. Clearly, it follows from (2.18) that $\psi_u$ is strictly decreasing on $(s_u, +\infty)$. Now, we verify that $\psi_u'(s) < 0$ on $(s_u, +\infty)$. From (V3),

$$\psi_u''(s) = 2 e^{2s} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} W(e^{-s}x, e^{-s}x)|u|^2 dx - 2^* e^{2s} \int_{\mathbb{R}^N} |u|^{2^*} dx - \mu \gamma q e^{(\frac{2}{q} - 1)Ns} \int_{\mathbb{R}^N} |u|^q dx$$

$$\leq (2 + \sigma_3) e^{2s} \int_{\mathbb{R}^N} |\nabla u|^2 dx - 2^* e^{2s} \int_{\mathbb{R}^N} |u|^{2^*} dx - \mu \gamma q e^{(\frac{2}{q} - 1)Ns} \int_{\mathbb{R}^N} |u|^q dx. \quad (2.19)$$

Let

$$g_1(s) = (1 - \sigma_2) e^{2s} |\nabla u|^2 - e^{2s} \int_{\mathbb{R}^N} |u|^{2^*} dx - \mu \gamma q e^{(\frac{2}{q} - 1)Ns} \int_{\mathbb{R}^N} |u|^q dx$$

and

$$g_2(s) = (2 + \sigma_3) e^{2s} \int_{\mathbb{R}^N} |\nabla u|^2 dx - 2^* e^{2s} \int_{\mathbb{R}^N} |u|^{2^*} dx - \mu \gamma q e^{(\frac{2}{q} - 1)Ns} \int_{\mathbb{R}^N} |u|^q dx.$$
increasing on \((-\infty, \tilde{s}_1)\). Since \(\psi_u\) is strictly increasing on \((-\infty, s_u)\) and is strictly decreasing on \((s_u, +\infty)\), we infer that \(\tilde{s}_1 \leq s_u\). If \(\tilde{s}_2 \leq s_u\), then from (2.19) we have \(\psi_u''(s) < 0\) on \((s_u, +\infty)\). Thus, we only need to show that \(\tilde{s}_2 \leq \tilde{s}_1\), which is equivalent to prove \(g_2(\tilde{s}_1) \leq 0\). Then, since \(g_1(\tilde{s}_1) = 0\), namely,

\[
(1 - \sigma_2)e^{2\tilde{s}_1}|\nabla u|^2_2 - e^{2\tilde{s}_1}\int_{\mathbb{R}^N} |u|^{2*} \, dx = \mu \gamma_q e^{\frac{1}{2}q(1 - \gamma_q)} \int_{\mathbb{R}^N} |u|^q \, dx,
\]

we have

\[
g_2(\tilde{s}_1) = (2 + \sigma_3) e^{2\tilde{s}_1} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - 2 e^{2\tilde{s}_1} \int_{\mathbb{R}^N} |u|^{2*} \, dx - q \gamma_q \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + q \gamma_q e^{2\tilde{s}_1} \int_{\mathbb{R}^N} |u|^{2*} \, dx
\]

\[
< 0.
\]

So, we obtain that \(\psi_u\) is strictly concave on \((s_u, +\infty)\).

It remains to show that the map \(u \in S_a \mapsto s_u \in \mathbb{R}\) is of class \(C^1\). Since \(\pi(s,u) := \psi_u'(s)\) is of class \(C^1\), \(\pi(s_u, u) = 0\) and \(\partial_x \pi(s_u, u) = \psi_u''(s_u) < 0\), then the implicit function theorem gives that \(u \in S_a \mapsto s_u \in \mathbb{R}\) is of class \(C^1\). Thus, we complete the proof. \(\square\)

**Lemma 2.7.** Assume that (V1) and (V2) hold. For \(q \in (\bar{q}, 2^*)\), then there is a constant \(k > 0\) small enough such that

\[
0 < \sup_{\overline{A}_k} I_q < m_{q,a} \quad \text{and} \quad u \in \overline{A}_k \Rightarrow I_q(u), P_q(u) > 0,
\]

where \(A_k = \{u \in S_a : |\nabla u|^2_2 < k\}\).

**Proof.** It is follows from (V1), (2.1) and (2.3) that

\[
I_q(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2 \, dx - \frac{1}{2\sigma_1} \int_{\mathbb{R}^N} |u|^{2*} \, dx - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q \, dx
\]

\[
\geq \left(\frac{1}{2} - \sigma_1\right)|\nabla u|^2_2 - \frac{1}{2\sigma_1} S^{-\frac{N^2}{N - 2}} |\nabla u|^2 - \frac{\mu}{q} C_{N,q}^{q(1 - \gamma_q)} |\nabla u|^q_{2^*},
\]

(2.21)

When \(q \in (\bar{q}, 2^*)\), then \(q \gamma_q > 2\) and \(I_q(u) > 0\) for any \(u \in \overline{A}_k\) with \(k > 0\) small enough. Besides, by (V2), one obtains that

\[
P_q(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} (\Delta V(x), x)|u|^2 \, dx - \int_{\mathbb{R}^N} |u|^{2*} \, dx - \mu \gamma_q \int_{\mathbb{R}^N} |u|^q \, dx
\]

\[
\geq (1 - \sigma_2) \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - S^{-\frac{N^2}{N - 2}} |\nabla u|^2 - \mu \gamma_q C_{N,q}^{q(1 - \gamma_q)} |\nabla u|^q_{2^*},
\]

(2.22)

which shows that \(P_q(u) > 0\) for any \(u \in \overline{A}_k\) with \(k > 0\) small enough. Moreover, choosing \(k\) sufficient small, from Lemma 2.4 we have, for all \(u \in \overline{A}_k\),

\[
I_q(u) \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx < m_{q,a}.
\]

Hence, we complete the proof of this lemma. \(\square\)

**Lemma 2.8.** Assume that (V1) holds and \(q \in (\bar{q}, 2^*)\). If \(q = \bar{q}\), we further assume that \(\mu a^{\frac{1}{N}} < (\pi_N)^{\frac{1}{N}}\).

Then \(m_{q,a} < m_{q,a}^{\infty}\).

**Proof.** According to Theorem 2.1, there exists \(0 < u \in S_a\) such that \(I_q^\infty(u) = m_{q,a}^\infty\). Then \(u \in P_{q,a}^\infty\). Moreover, from Lemma 2.6, there exists a unique \(s_u \in \mathbb{R}\) such that \(s_u * u \in P_{q,a}\). Hence, since \(V(x) \neq 0\) and \(\sup_{x \in \mathbb{R}^N} V(x) = 0\), it is easy to know that

\[
m_{q,a} \leq I_q(s_u * u) = I_q^\infty(s_u * u) + \frac{1}{2} \int_{\mathbb{R}^N} V(e^{-s}x)|u|^2 \, dx < I_q^\infty(s_u * u) \leq I_q^\infty(u),
\]

where [41, Lemmas 5.1 and 6.1] is applied in the last inequality. Thus, \(m_{q,a} < m_{q,a}^{\infty}\). So, we complete the proof. \(\square\)
The corresponding Pohožaev identity is given by

\[ \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x), x)|u|^2 \, dx + \nu \int_{\mathbb{R}^N} (\nabla W, x)|u|^2 \, dx - \mu \gamma_q (1 + \nu \gamma_q)|u|^q = 0. \]

Moreover, since \( u \in P_{q,a} \), then \( P_q(u) = 0 \), that is,

\[ |\nabla u|^2 = \frac{1}{2} \int_{\mathbb{R}^N} (\nabla V(x), x)|u|^2 \, dx + \int_{\mathbb{R}^N} \mu |u|^2 \, dx + \mu \gamma_q \int_{\mathbb{R}^N} |u|^q \, dx. \]

Combining (2.23) with (2.24), one infers that

\[ \nu (2|\nabla u|^2 - 2^*|u|^{2*} - \mu \gamma_q^2 |u|^q) + \int_{\mathbb{R}^N} (\nabla W, x)|u|^2 \, dx = 0. \]

From Lemma 2.5, we get \( (P_{q,a})^0 = \emptyset \), namely,

\[ 2|\nabla u|^2 - 2^*|u|^{2*} - \mu \gamma_q^2 |u|^q + \int_{\mathbb{R}^N} (\nabla W, x)|u|^2 \, dx \neq 0, \]

which implies that \( \nu = 0 \). Hence, we complete the proof. \( \square \)

The following Lemma can be found in Brézis and Kato [8, Lemma 2.1] (see also [49, Lemma 2.3]), which is crucial for the \( L^\infty \) estimate for the solution of Eq. \( (P) \).

**Lemma 2.10.** Let \( \Omega \subset \mathbb{R}^N \) and \( k(x) \in L^\infty(\Omega) \) be a nonnegative function. Then, for every \( \varepsilon > 0 \), there exists a constant \( C(\varepsilon, k) > 0 \) such that

\[ \int_{\Omega} k(x)u^2 \, dx \leq \varepsilon \int_{\Omega} |\nabla u|^2 \, dx + C(\varepsilon, k) \int_{\Omega} |u|^2 \, dx \]

for all \( u \in H^1(\Omega) \).

**Lemma 2.11.** Assume that \( (V_1) \) and \( (V_4) \) or \( (\tilde{V}_1) \) and \( (V_4) \) hold. Let \( u \in S_a \) be a nonnegative real-valued solution for Eq. \( (P) \) with \( \mu, a > 0 \) and \( q \in [q, 2^*) \), then we get the associated Lagrange multiplier \( \lambda > 0 \), and there exists \( C > 0 \) such that \( |u|_\infty < C \).

**Proof.** First, we prove that if \( 0 \leq u \in S_a \) is a real-valued solution for Eq. \( (P) \), then the associated Lagrange multiplier \( \lambda > 0 \). This follows easily by testing Eq. \( (P) \) with \( u \), and using the fact that \( P_q(u) = 0 \) and \( (V_4) \):

\[ \lambda a^2 = (1 - \gamma_q)\mu |u|^q - \int_{\mathbb{R}^N} (V(x) + W(x))|u|^2 \, dx > 0 \]

because \( \gamma_q < 1 \) and \( a > 0 \). Next, we show that \( |u|_\infty < C \) for some \( C > 0 \). Let \( A_m = \{ x \in \mathbb{R}^N : |u(x)| \leq m \} \), \( B_m = \mathbb{R}^N \setminus A_m \), where \( m \in \mathbb{N} \). For any \( p > 0 \), define

\[ u_m = \begin{cases} |u|^{2p+1}, & x \in A_m \\ m^{2p}u, & x \in B_m \end{cases} \]
and
\[ w_m = \begin{cases} |u|^{p+1}, & x \in A_m \\ m^p u, & x \in B_m. \end{cases} \]

By simple calculation, one has
\[ \nabla u_m = \begin{cases} (2p + 1)|u|^{2p} \nabla u, & x \in A_m \\ m^{2p} \nabla u, & x \in B_m. \end{cases} \]

and
\[ \nabla w_m = \begin{cases} (p + 1)|u|^{p} \nabla u, & x \in A_m \\ m^{p} \nabla u, & x \in B_m. \end{cases} \]

Notice that \( u_m, w_m \in H^1(\mathbb{R}^N) \) and \( w^2_m = u u_m \leq |u|^{2(p+1)} \). Taking \( u_m \) as a test function in \( \langle I_q(u), u_m \rangle = 0 \), we obtain
\[ \int_{\mathbb{R}^N} \nabla u \cdot \nabla u_m + \lambda u u_m + V(x) u u_m dx = \int_{\mathbb{R}^N} |u|^{2* - 2} u u_m + \mu |u|^q - 2 u u_m dx, \]
which and \( \lambda > 0 \) imply that
\[ \int_{A_m} (2p + 1)|u|^{2p} |\nabla u|^2 dx + \int_{B_m} m^{2p} |\nabla u|^2 dx \leq \int_{\mathbb{R}^N} \langle V(x) |u|^{2* - 2} u u_m + \mu |u|^q - 2 u u_m dx. \]

Since
\[ \int_{\mathbb{R}^N} |\nabla w_m|^2 dx = (p + 1)^2 \int_{A_m} |u|^{2p} |\nabla u|^2 dx + \int_{B_m} m^{2p} |\nabla u|^2 dx, \]
then combining (2.26) with (2.27), by \((V_1)\) or \((\bar{V}_1)\) and Lemma 2.10, one has
\[ \frac{2p + 1}{(p + 1)^2} \int_{\mathbb{R}^N} |\nabla w_m|^2 dx = (2p + 1) \int_{A_m} |u|^{2p} |\nabla u|^2 dx + \frac{2p + 1}{(p + 1)^2} \int_{B_m} m^{2p} |\nabla u|^2 dx \]
\[ \leq \int_{\mathbb{R}^N} \langle V(x) |u|^{2* - 2} u u_m + \mu |u|^q - 2 u u_m dx \]
\[ \leq \int_{\mathbb{R}^N} (C_1 + |u|^{2* - 2} + \mu |u|^q - 2) w^2_m dx \]
\[ \leq \varepsilon \int_{\mathbb{R}^N} |\nabla w_m|^2 dx + C(\varepsilon, u, \mu) \int_{\mathbb{R}^N} w^2_m dx. \]

Hence, there exists \( C_2 \), depending on \( \varepsilon, q, u, \mu \), such that
\[ \int_{\mathbb{R}^N} |\nabla w_m|^2 dx \leq C_2 \int_{\mathbb{R}^N} w^2_m dx. \]

Furthermore, since \( w_m = |u|^{p+1} \) in \( A_m \) and \( w^2_m \leq |u|^{2(p+1)} \) on \( \mathbb{R}^N \), it follows from (2.28) that
\[ \left( \int_{A_m} |u|^{2*(p+1)} dx \right)^{\frac{1}{2*}} = \left( \int_{A_m} |w_m|^{2*} dx \right)^{\frac{1}{2*}} \leq \left( \int_{\mathbb{R}^N} |w_m|^{2*} dx \right)^{\frac{1}{2*}} \]
\[ \leq \int_{\mathbb{R}^N} |\nabla w_m|^2 dx \]
\[ \leq C_2 \int_{\mathbb{R}^N} |u|^{2(p+1)} dx. \]

Let \( m \to \infty \), we infer that
\[ \left( \int_{\mathbb{R}^N} |u|^{2*(p+1)} dx \right)^{\frac{1}{2*}} \leq C_2 \int_{\mathbb{R}^N} |u|^{2(p+1)} dx, \]
that is, for any $p \geq 2$, it holds that $u \in L^{2(p+1)}(\mathbb{R}^N)$ and $u \in L^{2^*(p+1)}(\mathbb{R}^N)$, and there exists $C_3 > 0$ satisfying
\[
|u|^{2^*(p+1)}_{2(p+1)} \leq C_3^{2(p+1)}|u|^{2(p+1)}.
\] (2.29)
Setting $\beta := \frac{2^*}{2}$, we have $\beta > 1$. Taking $p + 1 = \beta j$ for $j \in \mathbb{N}$, then (2.29) changes to
\[
|u|^{2^* \beta j}_{2^* \beta j} \leq C_3^{2^* \beta j}|u|^{2^* \beta j-1}.
\]
Further, we proceed the $j$ times iterations that
\[
|u|^{2^* \beta j}_{2^* \beta j} \leq C_3^{\sum_{i=1}^{j} \frac{1}{\beta i}}|u|^{2^*}.
\]
Let $j \to +\infty$, by Sobolev inequality and the fact that $|\nabla u|_2 \leq C_4$ and $\lim_{j \to +\infty} C_3^{\sum_{i=1}^{j} \frac{1}{\beta i}} < \infty$, there exists $C > 0$, depending on $\varepsilon, p, u, \mu$, such that
\[
|u|_{\infty} \leq C.
\]
Thus, we complete the proof. \(\square\)

3 Proof of Theorem 1.3

Now, we are going to prove the existence of the positive ground state normalized solution for Eq. (P) with $q \in (q_1, 2^*)$.

**Proof of Theorem 1.3**: For any $q \in (q_1, 2^*)$, we first show that there exists $(\lambda, u) \in \mathbb{R}^+ \times E$ that solves Eq. (P) satisfying $I_q(u) = m_{q,a}$. We are going to do it in three steps.

**Step 1**: we are looking for a couple $(\lambda, u) \in \mathbb{R}^+ \times E$ to satisfy
\[-\Delta u + V(x)u + \lambda u = |u|^{2^*-2}u + \mu|u|^{q-2}u \quad \text{in} \quad \mathbb{R}^N.
\]
The arguments are similar to [41]. For the reader’s convenience, we present the proof for it. Let $k > 0$ be defined by Lemma 2.7. Considering the augmented functional $\tilde{I}_q : \mathbb{R} \times E \to \mathbb{R}$ defined by
\[
\tilde{I}_q(s, u) := \mathcal{I}_q(s \ast u) = \frac{e^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(e^{-s}x)|u|^2 dx - \frac{e^{2^* s}}{2^{*^2}} \int_{\mathbb{R}^N} |u|^{2^*} dx - \frac{\mu}{q} e^{(\frac{q}{2}-1)Ns} \int_{\mathbb{R}^N} |u|^q dx.
\] (3.1)
Notice that $\tilde{I}_q$ is of class $C^1$. Denoting by $I^*_q$ the closed sub-level set $\{u \in S_a : I_q(u) \leq c\}$, we introduce the minimax class
\[
\Gamma := \{\gamma = (\alpha, \beta) \in C([0, 1], \mathbb{R} \times S_a) : \gamma(0) \in (0, \overline{I}_k), \gamma(1) \in (0, \overline{I}^*_q)\},
\] (3.2)
with associated minimax level
\[
\eta_{q,a} := \inf_{\gamma \in \Gamma} \max_{s \in [0, 1]} \tilde{I}_q(s, u).
\] (3.3)
Let $u \in S_a$. Since $|\nabla (s \ast u)|_2 \to 0^+$ as $s \to -\infty$ and $I_q(s \ast u) \to -\infty$ as $s \to +\infty$, then there exists $s_0 << -1$ and $s_1 >> 1$ such that
\[
\gamma_u : \tau \in [0, 1] \mapsto (0, ((1 - \tau)s_0 + \tau s_1) \ast u) \in \mathbb{R} \times S_a
\] (3.4)
is a path in $\Gamma$. Then $\eta_{q,a}$ is a real number. Now, for any $\gamma = (\alpha, \beta) \in \Gamma$, let us consider the function
\[ P_q \circ \gamma : \tau \in [0, 1] \mapsto P_q(\alpha(\tau) \ast \beta(\tau)) \in \mathbb{R}, \]
we have $P_q \circ \gamma(0) = P_q(\beta(0)) > 0$ since $\beta(0) \in \overline{A}_k$ by Lemma 2.7. Now we claim that $P_q \circ \gamma(1) = P_q(\beta(1)) < 0$. In fact, it follows from (2.9) and (2.16) that $\psi_u(s) > 0$ for $s \to -\infty$ and $\psi'_u(s) > 0$ for $s \to -\infty$. Hence, from Lemma 2.6 we obtain $\psi_{\beta(1)}(s) > 0$ for $s \leq s_{\beta(1)}$. Moreover, $\psi_{\beta(1)}(0) = I_q(\beta(1)) \leq 0$ because $\beta(1) \in T_0^n$, which implies that $s_{\beta(1)} < 0$. Thus from Lemma 2.6 we get the claim. Furthermore, the map $\tau \mapsto (\alpha(\tau) \ast \beta(\tau))$ is continuous from $[0, 1]$ to $E$, and hence we deduce that there exists $\tau_0 \in (0, 1)$ such that $P_q \circ \gamma(\tau_0) = 0$, namely $\alpha(\tau_0) \ast \beta(\tau_0) \in \cal{P}_{q,a}$; this implies that
\[ \max_{\gamma(0, 1)} \tilde{I}_q(\gamma(\tau_0)) = \tilde{I}_q(\alpha(\tau_0), \beta(\tau_0)) = I_q(\alpha(\tau_0) \ast \beta(\tau_0)) \geq \inf_{u \in \cal{P}_{q,a}} I_q = m_{q,a}. \]
Hence, $\eta_{q,a} \geq m_{q,a}$. Besides, if $u \in \cal{P}_{q,a}$, then $\gamma_u$ defined in (3.4) is a path in $\Gamma$ with
\[ I_q(u) = \max_{\gamma(0, 1)} \tilde{I}_q \geq \eta_{q,a}, \]
whence the reverse inequality $m_{q,a} \geq \eta_{q,a}$ follows. Combining this with Lemma 2.7, we infer that
\[ \eta_{q,a} = m_{q,a} > \sup_{(\overline{A}_k \cup T_0^n) \cap S_a} I_q = \sup_{((\overline{A}_k \cup T_0^n) \cap S_a) \times \mathbb{R} \times S_a} \tilde{I}_q. \]
Using the terminology in [16, Section 5], this means that $\{ \gamma([0, 1]) : \gamma \in \Gamma \}$ is a homotopy stable family of compact subsets of $\mathbb{R} \times S_a$ with extended closed boundary $(0, \overline{A}_k) \cup (0, T_0^n)$, and that the superlevel set $\{ I_q \geq \eta_{q,a} \}$ is a dual set for $\Gamma$. Therefore, applying [16, Theorem 5.2], taking any minimizing sequence $\gamma_n = (\alpha_n, \beta_n) \subset \Gamma_n$ for $\tilde{I}_q|_{\mathbb{R} \times S_a}$ at the level $\eta_{q,a}$ with the property that $\alpha_n = 0$ and $\beta_n(\tau) \geq 0$ a.e. in $\mathbb{R}^N$ for every $\tau \in [0, 1]$, there exists a sequence $\{(s_n, w_n)\} \subset \mathbb{R} \times S_a \setminus ((0, \overline{A}_k) \cup (0, T_0^n))$ such that, as $n \to \infty$,
\begin{enumerate}
  \item $\tilde{I}_q(s_n, w_n) \to \eta_{q,a}$;
  \item $\tilde{I}_q|_{\mathbb{R} \times S_a}(s_n, w_n) \to 0$;
  \item $\text{dist}((s_n, w_n), (0, \beta_u(\tau))) \to 0$.
\end{enumerate}
Let $u_n := s_n \ast w_n = e^{\frac{N\mu}{2} w_n(e^{s_n}x)}$. It follows from (i) that
\[ \lim_{n \to \infty} I_q(u_n) = \lim_{n \to \infty} I_q(s_n \ast w_n) = \lim_{n \to \infty} \tilde{I}_q(s_n, w_n) = \eta_{q,a} = m_{q,a}. \tag{3.5} \]
Moreover, from (ii) and (3.1) we know that
\[ \partial_s \tilde{I}_q(s_n, w_n) = (\tilde{I}_q(s_n, w_n), (1, 0)) \to 0 \quad \text{as } n \to \infty, \]
where
\[ \partial_s \tilde{I}_q(s_n, w_n) = e^{2s_n} \int_{\mathbb{R}^N} |\nabla w_n|^2 \, dx - \int_{\mathbb{R}^N} W(e^{s_n}x)|w_n|^2 \, dx \]
\[ - e^{2s_n} \int_{\mathbb{R}^N} |w_n|^2 \, dx - \mu \gamma q e^{(\frac{4}{N} - 1)Ns_n} \int_{\mathbb{R}^N} |w_n|^N \, dx. \]
Hence, by (2.10), one gets
\[ P_q(u_n) = P_q(s_n \ast w_n) = \partial_s \tilde{I}_q(s_n, w_n) \to 0 \quad \text{as } n \to \infty. \tag{3.6} \]
Besides, setting $h_n \in T_{u_n} := \{ z \in E : \langle z, u_n \rangle_{L^2} = \int_{\mathbb{R}^N} z u_n \, dx = 0 \}$, by simple calculate, we have
\[ \langle I_q(u_n), h_n \rangle \]
\[
\begin{align*}
\int_{\mathbb{R}^N} \nabla u_n \cdot \nabla h_n + V(x) u_n h_n dx &= \int_{\mathbb{R}^N} |u_n|^{2^*-2} u_n h_n dx - \mu \int_{\mathbb{R}^N} |u_n|^{q^*-2} u_n h_n dx \\
&= e^{2s_n} \int_{\mathbb{R}^N} e^{-\frac{N-2}{2} s_n} \nabla w_n \cdot \nabla h_n (e^{-s_n} x) dx + \int_{\mathbb{R}^N} V(e^{-s_n} x) w_n e^{-\frac{N-2}{2} s_n} h_n (e^{-s_n} x) dx \\
&\quad - e^{2s_n} \int_{\mathbb{R}^N} |w_n|^{2^*-2} w_n e^{-\frac{N-2}{2} s_n} h_n (e^{-s_n} x) dx - \mu e^{\frac{Q}{2} - 1} N s_n \int_{\mathbb{R}^N} |w_n|^{q^*-2} w_n e^{-\frac{N-2}{2} s_n} h_n (e^{-s_n} x) dx.
\end{align*}
\]
which implies that
\[
\langle I'_q(u_n), h_n \rangle = \langle I'_q(s_n, w_n), (0, \tilde{h}_n) \rangle, \tag{3.7}
\]
where \( \tilde{h}_n(x) = e^{-\frac{N-2}{2} s_n} h_n (e^{-s_n} x) \). Now, we claim that
\[
(0, \tilde{h}_n) \in \tilde{T}_{(s_n, w_n)} := \{(z_1, z_2) \in \mathbb{R} \times E : \langle w_n, z_2 \rangle_{L^2} = 0\}. \tag{3.8}
\]
In fact,
\[
(0, \tilde{h}_n) \in \tilde{T}_{(s_n, w_n)} \iff \langle w_n, \tilde{h}_n \rangle_{L^2} = 0 \\
\iff \int_{\mathbb{R}^N} w_n e^{-\frac{N-2}{2} s_n} h_n (e^{-s_n} x) dx = 0 \\
\iff \int_{\mathbb{R}^N} e^{\frac{N-2}{2} s_n} w_n (e^{s_n} x) h_n (x) dx = 0 \\
\iff \langle w_n, h_n \rangle_{L^2} = 0. \\
\iff h_n \in T_{u_n}.
\]
Therefore, in view of \((ii), (3.7)\) and \((3.8)\), one has, as \(n \to \infty\),
\[
\langle I'_q(u_n), h_n \rangle \to 0 \quad \text{for all } h_n \in T_{u_n}.
\]
That is,
\[
I'_q|_{S_n}(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.9}
\]
Furthermore, using the last item \((iii)\), it holds that \(\{s_n\}\) is bounded from above and from below. Thus, in virtue of \((3.5), (3.6)\) and \((3.9)\), we find that \(\{u_n\} \subset S_a\) is a Palais-Smale sequence for \(I_q|_{S_a}\) at the level \(\eta_{q,a} = m_{q,a}\) satisfying
\[
I_q(u_n) \to m_{q,a} \quad \text{and} \quad \mathcal{P}_q(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.10}
\]
Similar to \((2.13)\), we show that \(\{u_n\}\) is bounded in \(E\). Then, up to a subsequence, there is \(u \in E\) such that, up to a subsequence,
\[
\begin{align*}
&u_n \rightharpoonup u \quad \text{in } E; \\
&u_n \to u \quad \text{in } L^p_{\text{loc}}(\mathbb{R}^N), p \in (2, 2^*); \\
&u_n(x) \to u(x) \quad \text{a.e. on } \mathbb{R}^N.
\end{align*}
\]
Besides, by \((3.10)\) and the Lagrange multipliers rule, there exists \(\lambda_n \in \mathbb{R}\) such that
\[
Re \int_{\mathbb{R}^N} \nabla u_n \cdot \nabla \varphi + V(x) u_n \varphi + \lambda_n u_n \varphi - |u_n|^{2^*-2} u_n \varphi - \mu |u_n|^{q^*-2} u_n \varphi dx = o(1) \|
\]
for every $\varphi \in E$, where $\overline{\varphi}$ is the complex conjugate of $\varphi$ and $\text{Re}$ denotes the real part. Choosing $\varphi = u_n$ in the above equality, it provides that

$$
\lambda_n a^2 = -|\nabla u_n|^2 - \int_{\mathbb{R}^N} V(x) |u_n|^2 \, dx + |u_n|^2 + \mu |u_n|^q + o(1).
$$

(3.11)

From the boundedness of $\{u_n\}$ in $E$, we know that $\{\lambda_n\}$ is bounded in $\mathbb{R}$. Then there is $\lambda \in \mathbb{R}$ satisfying $\lambda_n \to \lambda$ as $n \to \infty$. Therefore, $(\lambda, u) \in \mathbb{R} \times E$ satisfies

$$
-\Delta u + V(x)u + \lambda u = |u|^{2^*-2}u + \mu |u|^q u \quad \text{in } \mathbb{R}^N.
$$

In the following, we prove that $\lambda > 0$. Since $P_q(u_n) \to 0$ as $n \to \infty$, namely,

$$
\int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx = \int_{\mathbb{R}^N} W(x) |u_n|^2 \, dx + \int_{\mathbb{R}^N} |u_n|^2 + \mu \gamma_q \int_{\mathbb{R}^N} |u_n|^q \, dx + o(1),
$$

which and (3.11) imply that

$$
\lambda_n a^2 = (1 - \gamma_q) \mu \int_{\mathbb{R}^N} |u_n|^q \, dx - \int_{\mathbb{R}^N} (V(x) + W(x)) |u_n|^2 \, dx + o(1),
$$

(3.12)

where $\gamma_q < 1$. Then, by $(V_4)$ and Fatou lemma we infer that

$$
\lambda a^2 \geq (1 - \gamma_q) \mu \lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^q \, dx \geq (1 - \gamma_q) \mu \int_{\mathbb{R}^N} |u|^q \, dx,
$$

which implies that $\lambda \geq 0$. Moreover, if $u \neq 0$, then we deduce that $\lambda > 0$. Next, we claim that $u \neq 0$. Assume that $u = 0$. Then it follows from $(V_1)$ and $(V_2)$ that

$$
m_{q,a} = I_q(u_n) = I_q^\infty(u_n) + o(1) \quad \text{and} \quad P_q(u_n) = P_q^\infty(u_n) + o(1),
$$

which shows that $P_q^\infty(u_n) \to 0$ as $n \to \infty$. Hence, for any $n \in \mathbb{N}$, there exists a unique $t_n > 0$ with $t_n \to 0$ as $n \to \infty$ such that $t_n * u_n \in P_q^\infty$. Then

$$
m_{q,a}^\infty \leq I_q^\infty(t_n * u_n) = m_{q,a} + o(1),
$$

which is contradiction with Lemma 2.8. Thus, $u \neq 0$. This shows that $\lambda > 0$. So $(\lambda, u) \in \mathbb{R}^+ \times E$ satisfies

$$
-\Delta u + V(x)u + \lambda u = |u|^{2^*-2}u + \mu |u|^q u \quad \text{in } \mathbb{R}^N.
$$

Consequently, $P_q(u) = 0$.

**Step 2:** we prove that $u \in S_a$. Let $|u|_2 = r \in (0, a]$, then $u \in P_{q,b}$. Similar to Lemma 2.3 and (2.13), we get $I_q(u) > 0$. If $a 

b$, setting $c^2 = a^2 - b^2 \in (0, a^2)$ and $v_n := u_n - u \to 0$ in $E$ as $n \to \infty$, then $\|v_n\|_2 = c > 0$. By Brézis-Lieb Lemma [45] and the fact that $P_q(u) = 0$, one infers

$$
I_q(u_n) = I_q(v_n) + I_q(u) + o(1) = I_q^\infty(v_n) + I_q(u) + o(1),
$$

(3.13)

$$
P_q(u_n) = P_q(v_n) + P_q(u) + o(1) = P_q^\infty(v_n) + o(1) = P_q^\infty(v_n) + o(1),
$$

which implies $P_q^\infty(v_n) \to 0$ as $n \to \infty$. Hence, for any $n \in \mathbb{N}$, there exists a unique $\tilde{t}_n > 0$ with $\tilde{t}_n \to 0$ as $n \to \infty$ such that $\tilde{t}_n * v_n \in P_q^\infty$. So, it follows from (3.13) and $I_q(u) > 0$ that

$$
m_{q,c}^\infty \leq I_q^\infty(\tilde{t}_n * v_n) = I_q^\infty(v_n) + o(1) = m_{q,a} - I_q(u) + o(1),
$$

which and Lemma 2.8 imply that $m_{q,c}^\infty \leq m_{q,a} < m_{q,a}^\infty$. That is contradiction with Proposition 2.2. Hence, $a = b$, namely, $u \in S_a$.

**Step 3:** we show that $I_q(u) = m_{q,a}$. Since $u_n \to u$ in $L^2(\mathbb{R}^N, \mathbb{C})$ as $n \to \infty$ and $u \in S_a$, one has, as $n \to \infty$,

$$
u_n \to u \quad \text{in } L^2(\mathbb{R}^N, \mathbb{C}).$$
Then, by Gagliardo-Nirenberg inequality (2.1), we have, as $n \to \infty$,
\[ u_n \to u \quad \text{in } L^t(\mathbb{R}^N, \mathbb{C}) \]
with $t \in (2, 2^*)$. In virtue of $u \in \mathcal{P}_{q,a}$, one infers $\mathcal{I}_q(u) \geq m_{q,a}$. Hence, combining (2.7) with $P_q(u) = 0$, by Fatou lemma and the fact that $u_n \to u$ in $L^p(\mathbb{R}^N, \mathbb{C})$ with $p \in [2, 2^*)$ as $n \to \infty$, it holds that
\[
m_{q,a} \leq \mathcal{I}_q(u)
= \mathcal{I}_q(u) - \frac{1}{2^*} P_q(u)
= \frac{1}{2} - \frac{1}{2^*} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} \left[ V(x) + \frac{1}{2^*} |\nabla V, x| \right] |u|^2 dx + \mu \left( \frac{\gamma_q}{q} - \frac{1}{q} \right) \int_{\mathbb{R}^N} |u|^q dx
\leq \liminf_{n \to \infty} \left( \frac{1}{2} - \frac{1}{2^*} \int_{\mathbb{R}^N} |\nabla u_n|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} \left[ V(x) + \frac{1}{2^*} |\nabla V, x| \right] |u_n|^2 dx + \mu \left( \frac{\gamma_q}{q} - \frac{1}{q} \right) \int_{\mathbb{R}^N} |u_n|^q dx \right)
= \liminf_{n \to \infty} \left( \mathcal{I}_q(u_n) - \frac{1}{2^*} P_q(u_n) \right)
\leq \lim_{n \to \infty} \mathcal{I}_q(u_n)
= m_{q,a},
\]
which shows that $\mathcal{I}_q(u) = m_{q,a}$.

Next, let
\[ U_a := \{ u \in \mathcal{P}_{q,a} : \mathcal{I}_q(u) = m_{q,a} = \inf \mathcal{I}_q \}. \quad (3.14) \]
It is easy to see from Lemma 2.9 that $U_a = Z_a$. We claim that
\[ u \in Z_a \Rightarrow |u| \in Z_a, \quad |\nabla |u||_2 = |\nabla u||_2. \quad (3.15) \]
Indeed, observe that $|\nabla u||_2 \leq |\nabla u||_2$, then $\mathcal{I}_q(|u|) \leq \mathcal{I}_q(u)$ and $P_q(|u|) \leq P_q(u) = 0$. By Lemma 2.6, there exists $s_{|u|} \leq 0$ satisfying $s_{|u|} * |u| \in \mathcal{P}_{q,a}$. Hence, one has
\[
m_{q,a} \leq \mathcal{I}_q(s_{|u|} * |u|)
= \frac{e^{2s_{|u|}}}{2} \int_{\mathbb{R}^N} |\nabla |u||^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(e^{-s_{|u|}} |x|) |u|^2 dx - \frac{e^{2s_{|u|}}}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx - \mu \left( \frac{\gamma_q}{q} - \frac{1}{q} \right) \int_{\mathbb{R}^N} |u|^q dx
\leq \frac{e^{2s_{|u|}}}{2} \int_{\mathbb{R}^N} |\nabla |u||^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(e^{-s_{|u|}} |x|) |u|^2 dx - \frac{e^{2s_{|u|}}}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx - \mu \left( \frac{\gamma_q}{q} - \frac{1}{q} \right) \int_{\mathbb{R}^N} |u|^q dx
= \mathcal{I}_q(s_{|u|} * |u|)
\leq \mathcal{I}_q(u)
= m_{q,a},
\]
which shows that $|\nabla |u||_2 \leq |\nabla u||_2$. Thus we have $\mathcal{I}_q(|u|) = \mathcal{I}_q(u) = m_{q,a}$ and $P_q(|u|) = P_q(u) = 0$. This gives that $|u| \in Z_a$. Thus, $|u|$ is a non-negative real-valued solution to Eq. (P) for some $\lambda \in \mathbb{R}^+$. Let $u := |u|$, we claim that $u > 0$ on $\mathbb{R}^N$. Indeed, from Lemma 2.11, we know that
\[-\Delta u = -\lambda u - V(x)u + |u|^{2^*-2}u + |u|^{q-2}u \in L^p_{\text{loc}}(\mathbb{R}^N, \mathbb{R}), \quad \text{for any } p \geq 2,\]
which and Caldéron-Zygmund inequality [17] imply $u \in W^{2,p}_{\text{loc}}(\mathbb{R}^N, \mathbb{R})$. Moreover, by Sobolev embedding theorem, we have $u \in C^{1,\alpha}_{\text{loc}}(\mathbb{R}^N, \mathbb{R})$ for $0 < \alpha < 1$. Then the strong maximum principle shows that $u > 0$. Hence, we obtain that there exists a couple $(\lambda, u) \in \mathbb{R}^+ \times E$ solving Eq. (P), where $u$ is a positive real-valued function and $\mathcal{I}_q(u) = m_{q,a}$. Therefore, we complete the proof of Theorem 1.3. □
4 Proof of Theorem 1.4

In this section, we consider the $L^2$-critical perturbation case, i.e., $q = 2 + \frac{4}{N}$, and prove Theorem 1.4. Below we first present some preliminary lemmas that are necessary to prove Theorem 1.4.

Lemma 4.1. Let $(\tilde{V}_2)$ be satisfied and $q = \overline{q}$. Setting $\mu a^\frac{4}{N} < (\pi N)^\frac{4}{N} (1 - \overline{\sigma}_2)$, for any $u \in P_{\pi a}$, there exists $\bar{\delta} > 0$ such that $|\nabla u|_2 \geq \bar{\delta}$.

Proof. When $q = \overline{q}$, we know that $\overline{\pi}\pi = 2$. Thus, by $(\tilde{V}_2)$ and (2.1) and (2.3), similar to (2.11), for any $u \in P_{\pi a}$ one shows that

$$(1 - \overline{\sigma}_2 - \mu \gamma C_N^\pi \pi (1 - \pi^2)|\nabla u|_2^2 \leq S^{-\frac{N}{2N}}|\nabla u|_2^2.$$ 

By (2.2), let $\mu a^\frac{4}{N} < (\pi N)^\frac{4}{N} (1 - \overline{\sigma}_2)$, we infer that there exists $\bar{\delta} > 0$ satisfying $|\nabla u|_2 \geq \bar{\delta}$. Thus, we complete the proof of this lemma.

Lemma 4.2. Assume that $(\tilde{V}_1)$ and $(\tilde{V}_2)$ hold. Let $q = \overline{q}$ and $\mu a^\frac{4}{N} < \left(1 - \frac{N\overline{\sigma}_1}{2} - \frac{N - 2}{N} \overline{\sigma}_2\right) (\pi N)^\frac{4}{N}$, then $m_{\pi a} = \inf_{u \in P_{\pi a}} I_{\pi a}(u) > 0$.

Proof. From $(\tilde{V}_1), (\tilde{V}_2)$, Gagliardo-Nirenberg inequality (2.1) and (2.7), for any $u \in P_{\pi a}$ we deduce

$$I_{\pi a}(u) = I_{\pi}(u) - \frac{1}{2s} P_{\pi}(u)$$

$$= \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx + \frac{1}{2\overline{q}} \int_{\mathbb{R}^N} (\nabla V(x), x) |u|^2 dx + \left(\frac{\gamma q}{2} - \frac{1}{\overline{q}}\right) \mu \int_{\mathbb{R}^N} |u|^2 dx$$

$$\geq \frac{1}{N} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\overline{\sigma}_1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\overline{\sigma}_2}{2s} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\mu}{N + 2} \int_{\mathbb{R}^N} |u|^2 dx$$

$$\geq \left(\frac{1}{N} - \frac{\overline{\sigma}_1}{2} - \frac{\overline{\sigma}_2}{2s} - \frac{\mu}{N + 2} \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx,$$

and which and Lemma 4.1 imply that $m_{\pi a} = \inf_{u \in P_{\pi a}} I_{\pi a}(u) > 0$ because $\overline{\sigma}_1 < \frac{2}{N}$ and $\overline{\sigma}_2 < \frac{2}{N - 2} - \frac{N}{N - 2} \overline{\sigma}_1$ and $\mu a^\frac{4}{N} < \left(1 - \frac{N\overline{\sigma}_1}{2} - \frac{N - 2}{N} \overline{\sigma}_2\right) (\pi N)^\frac{4}{N}$. Thus, we complete the proof.

Consider the decomposition of $P_{q,a}$ into the disjoint union

$$P_{q,a} = (P_{q,a})^+ \cup (P_{q,a})^0 \cup (P_{q,a})^-,$$

where

$$(P_{q,a})^+ = \left\{ u \in P_{q,a} : \psi_u'(0) > 0 \right\}.$$ 

Lemma 4.3. Assume that $(\tilde{V}_2) - (\tilde{V}_3)$ hold and $q = \overline{q}$. Let $\mu a^\frac{4}{N} < \left(1 - \frac{N\overline{\sigma}_1}{2} - \frac{N - 2}{N} \overline{\sigma}_3\right) (\pi N)^\frac{4}{N}$, then $P_{\pi a} = (P_{\pi a})^-$. 

Proof. By (2.14) and (2.15) with $q = \overline{q}$, using Gagliardo-Nirenberg inequality (2.1) and $(\tilde{V}_2) - (\tilde{V}_3)$, for any $u \in P_{\pi a}$, one has

$$\psi_u'(0) = \psi_u''(0) - \frac{2N}{N - 2} P_{\pi}(u)$$

$$= -\frac{4}{N - 2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2\overline{q}} \int_{\mathbb{R}^N} (\nabla W(x), x) |u|^2 dx + \frac{2N}{N - 2} \int_{\mathbb{R}^N} W(x) |u|^2 dx + \mu \frac{4\gamma q}{N - 2} \int_{\mathbb{R}^N} |u|^2 dx$$

$$\leq \left(\overline{\sigma}_3 + \frac{2N}{N - 2} \overline{\sigma}_2 - \frac{4}{N - 2}\right) \int_{\mathbb{R}^N} |\nabla u|^2 dx + \mu \frac{4\gamma q}{N - 2} \int_{\mathbb{R}^N} |u|^2 dx$$

$$= \left(\overline{\sigma}_3 + \frac{2N}{N - 2} \overline{\sigma}_2 - \frac{4}{N - 2}\right) \int_{\mathbb{R}^N} |\nabla u|^2 dx + \mu \frac{4\gamma q}{N - 2} \int_{\mathbb{R}^N} |u|^2 dx.$$
\[
\leq \left( \tilde{\sigma}_3 + \frac{2N}{N - 2} \tilde{\sigma}_2 - \frac{4}{N - 2} + \mu a^\frac{4}{N} \frac{4}{N - 2} (\overline{u}_N)^{-\frac{4}{N}} \right) \int_{\mathbb{R}^N} |\nabla u|^2 dx < 0,
\]
which shows that \( u \in (\mathcal{P}_{q,a})^- \). Hence, \( \mathcal{P}_{q,a} = (\mathcal{P}_{q,a})^- \). Thus, we complete the proof of this lemma. \( \square \)

**Lemma 4.4.** Assume that \((\tilde{V}_2)\) and \((\tilde{V}_3)\) hold and \( q = \overline{q} \). Let
\[
\mu a^\frac{4}{N} < \min \left\{ 1 - \tilde{\sigma}_2, \frac{2N}{N - 2} - \tilde{\sigma}_2 - \tilde{\sigma}_3 \right\} (\overline{u}_N)^\frac{4}{N},
\]
for any \( u \in S_a \), there is a unique \( s_u > 0 \) such that \( s_u u + u \in \mathcal{P}_{q,a} \). Moreover, \( \mathcal{I}_q(s_uu) = \max_{s > 0} \mathcal{I}_q(s + u) \), and \( s_u < 0 \) if and only if \( \mathcal{I}_q(u) < 0 \), and \( \psi_u \) is strictly decreasing and concave on \((s_u, +\infty)\). And the map \( u \in S_a \mapsto s_u \in \mathbb{R} \) is of class \( C^1 \).

**Proof.** Since \( \psi_u'(s) = \mathcal{P}_q(s + u) \), then we only prove that \( \psi_u'(s) \) has a unique root in \( \mathbb{R} \). By (2.9) and (\( \tilde{V}_2 \)), for any \( u \in S_a \),
\[
\psi'_u(s) = e^{2s} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \int_{\mathbb{R}^N} W(e^{-s}x)|u|^2 dx - e^{2s} \int_{\mathbb{R}^N} |u|^2 dx - \mu \gamma a^\frac{3}{2} e^{\frac{3}{2}s} \int_{\mathbb{R}^N} |u|^3 dx \\
\quad \geq \left( 1 - \tilde{\sigma}_2 - \mu \gamma a^\frac{3}{2} e^{\frac{3}{2}s} \right) e^{2s} |\nabla u|^2 - e^{2s} \int_{\mathbb{R}^N} |u|^2 dx. \tag{4.2}
\]
Since \( \mu a^\frac{4}{N} < (1 - \tilde{\sigma}_2)\overline{u}_N^\frac{4}{N} \) with \( \tilde{\sigma}_2 < 1 \), we have \( 1 - \tilde{\sigma}_2 - \mu \gamma a^\frac{3}{2} e^{\frac{3}{2}s} \overline{u}_N^{\frac{4}{N}} > 0 \). Therefore, from (4.2), it holds that \( \psi'_u(s) > 0 \) for \( s \to -\infty \). Hence there exists \( s_0 \in \mathbb{R} \) such that \( \psi_u(s) \) is increasing on \(( -\infty, s_0) \). In addition, we know that \( \psi_u(s) \to -\infty \) as \( s \to +\infty \). Thus, there is \( s_1 \in \mathbb{R} \) with \( s_1 > s_0 \) such that
\[
\psi_u(s_1) = \max_{s > 0} \psi_u(s). \tag{4.3}
\]
Therefore, \( \psi'_u(s_1) = 0 \), namely, \( s_1 u + u \in \mathcal{P}_{q,a} \). Assume that there exists \( s_2 \in \mathbb{R} \) satisfying \( s_2 u + u \in \mathcal{P}_{q,a} \).

Without loss of generality, assume that \( s_2 > s_1 \). From Lemma 2.5, we have \( \psi'_u(s_2) < 0 \). Hence, there is \( s_3 \in (s_1, s_2) \) such that
\[
\psi_u(s_3) = \min_{s \in (s_1, s_2)} \psi_u(s).
\]

Thus, we deduce that \( \psi''_u(s_3) \geq 0 \) and \( \psi'_u(s_3) = 0 \), which implies that \( s_3 u + u \in (\mathcal{P}_{q,a})^+ \cup (\mathcal{P}_{q,a})^0 \). This is a contradiction. Hence, setting \( s_u = s_1 \), it follows from (4.3) that \( \mathcal{I}_q(s_u u) = \max_{s > 0} \mathcal{I}_q(s + u) \). Furthermore,
\[
\psi'_u(s) < 0 \iff s > s_u. \tag{4.4}
\]

Hence, \( \psi'_u(0) = \mathcal{P}_q(u) < 0 \) if and only if \( s_u < 0 \). In the following, we show that \( \psi_u \) is strictly decreasing and concave on \((s_u, +\infty)\). Clearly, it follows from (4.4) that \( \psi_u \) is strictly decreasing on \((s_u, +\infty)\).

Now, we verify that \( \psi''_u(s) < 0 \) on \((s_u, +\infty)\). From (\( \tilde{V}_3 \)),
\[
\psi''_u(s) = 2e^{2s} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} \langle \nabla W(e^{-s}x), e^{-s}x \rangle |u|^2 dx - 2e^{2s} \int_{\mathbb{R}^N} |u|^2 dx - \mu q \gamma a^\frac{3}{2} e^{\frac{3}{2}s} \int_{\mathbb{R}^N} |u|^3 dx \\
\leq (2 + \tilde{\sigma}_3) e^{2s} \int_{\mathbb{R}^N} |\nabla u|^2 dx - 2e^{2s} \int_{\mathbb{R}^N} |u|^2 dx. \tag{4.5}
\]
Let
\[
g_1(s) = (1 - \tilde{\sigma}_2 - \mu \gamma a^\frac{3}{2} e^{\frac{3}{2}s}) e^{2s} |\nabla u|^2 - e^{2s} \int_{\mathbb{R}^N} |u|^2 dx
\]
and
\[
g_2(s) = (2 + \tilde{\sigma}_3) e^{2s} \int_{\mathbb{R}^N} |\nabla u|^2 dx - 2e^{2s} \int_{\mathbb{R}^N} |u|^2 dx.
\]
Obviously, \( g_1 \) and \( g_2 \) has a unique zero point. Assume that \( g_1(\tilde{s}_1) = 0 \) and \( g_2(\tilde{s}_2) = 0 \). It holds that \( g_1(s) > 0 \) on \( (-\infty, \tilde{s}_1) \) and \( g_1(s) < 0 \) on \( (\tilde{s}_1, +\infty) \) and \( g_2(s) > 0 \) on \( (-\infty, \tilde{s}_2) \) and \( g_2(s) < 0 \) on \( (\tilde{s}_2, +\infty) \). From above analysis, we know from (4.2) that \( \psi_u \) is strictly increasing on \( (-\infty, \tilde{s}_1) \). Moreover, since \( \psi_u \) is strictly increasing on \( (-\infty, s_u) \) and strictly decreasing on \( (s_u, +\infty) \), we have \( \tilde{s}_2 \leq s_u \). If \( \tilde{s}_2 \leq s_u \), then we have \( \psi_u''(s) < 0 \) on \( (s_u, +\infty) \). Thus, we only need to show that \( \tilde{s}_2 \leq \tilde{s}_1 \), which is equivalent to verify \( g_2(\tilde{s}_1) \leq 0 \). Since \( g_1(\tilde{s}_1) = 0 \), namely,

\[
\left(1 - \tilde{\sigma}_2 - \mu \gamma \sigma \left( N \sigma^a \right) \right) e^{2\tilde{\sigma}_1} |\nabla u|^2_2 = e^{2\tilde{\sigma}_1} \int_{\mathbb{R}^N} |u|^2 \, dx,
\]

then

\[
g_2(\tilde{s}_1) = (2 + \tilde{\sigma}_3) e^{2\tilde{\sigma}_1} \int_{\mathbb{R}^N} |\nabla u|^2_2 \, dx - 2 e^{2\tilde{\sigma}_1} \int_{\mathbb{R}^N} |u|^2 \, dx
\]

\[
= (2 + \tilde{\sigma}_3 - 2 (1 - \tilde{\sigma}_2 - \mu \gamma \sigma \left( N \sigma^a \right) )) e^{2\tilde{\sigma}_1} \int_{\mathbb{R}^N} |\nabla u|^2_2 \, dx
\]

\[
< 0
\]

(4.6)

because \( \mu \sigma^a < \left( \frac{q}{2} - \tilde{\sigma}_2 - \frac{\tilde{\sigma}_3}{2} \right) \sigma \left( N \sigma^a \right) \). So, we obtain that \( \psi_u \) is strictly concave on \( (s_u, +\infty) \). Similar to Lemma 2.6, we show that the map \( u \in S_a \mapsto s_u \in \mathbb{R} \) is of class \( C^1 \). Thus, we complete the proof. \( \Box \)

**Lemma 4.5.** Assume that \((\bar{V}_1) - (\bar{V}_2)\) hold and \( q = \bar{\gamma} \). Let

\[
\mu \sigma^a < \min \left\{ \left(1 - \frac{N}{2} \tilde{\sigma}_1 \right) \left( \frac{N}{2} \tilde{\sigma}_2 \right), 1 - \tilde{\sigma}_2, \frac{N + 2}{N}, \left(2 + \frac{4}{N}\right) \tilde{\sigma}_1 \right\} \left( \sigma \left( N \sigma^a \right) \right),
\]

then there is a constant \( k > 0 \) small enough such that

\[
0 < \sup_{A_k} \mathcal{I}_{\bar{\gamma}} < m_{\bar{\gamma}, a} \quad \text{and} \quad u \in A_k \Rightarrow \mathcal{I}_{\bar{\gamma}}(u), P_{\bar{\gamma}}(u) > 0,
\]

where \( A_k = \{ u \in S_a : |\nabla u|^2_2 < k \} \).

**Proof.** Firstly, by \((\bar{V}_1)\), similar to (2.21), setting \( \mu \sigma^a < \left( \frac{q}{2} - \tilde{\sigma}_2 \right) (2 + \frac{1}{q}) \sigma \left( N \sigma^a \right) \), we have \( \mathcal{I}_{\bar{\gamma}}(u) > 0 \) for any \( u \in A_k \) with \( k > 0 \) small enough. Then, similar to (2.22), by \((\bar{V}_2)\) and the fact that \( \mu \sigma^a < \left( 1 - \tilde{\sigma}_2 \right) \sigma \left( N \sigma^a \right) \), one gets \( P_{\bar{\gamma}}(u) > 0 \) for any \( u \in A_k \) with \( k > 0 \) small enough. Furthermore, choosing \( k \) sufficient small, by Lemma 4.2,

\[
\mathcal{I}_{\bar{\gamma}}(u) \leq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2_2 \, dx < m_{\bar{\gamma}, a}
\]

for all \( u \in A_k \). Hence, we complete the proof of this lemma. \( \Box \)

**Proof of Theorem 1.4:** Let

\[
0 < \mu \sigma^a < \min \left\{ \left(1 - \tilde{\sigma}_2, \frac{N + 2}{N} \right) \tilde{\sigma}_1, 1 - \tilde{\sigma}_2, \frac{N - 2}{2} \tilde{\sigma}_2, \frac{N}{2} \tilde{\sigma}_1, \tilde{\sigma}_2, \frac{N}{2} \tilde{\sigma}_2, \right\} \sigma \left( N \sigma^a \right),
\]

by Lemmas 2.8, 2.9, 4.1, 4.2, 4.3, 4.4, 4.5 and 2.11, and using Proposition 2.2, similar to the proof of Theorem 1.3, we infer that there exists a couple \((\lambda, u) \in \mathbb{R}^+ \times E\) solving Eq. \((\mathcal{P})\), where \( u \) is a real-valued positive function in \( \mathbb{R}^N \) and \( \mathcal{I}_{\bar{\gamma}}(u) = m_{\bar{\gamma}, a} \). So we complete the proof of Theorem 1.4. \( \Box \)

**5 Proof of Theorem 1.5**

In this section, we study the \( L^2 \)-subcritical perturbation case, i.e., \( 2 < q < \bar{\gamma} := 2 + \frac{4}{N} \). Firstly, we give some preliminary lemmas for proving Theorem 1.5.
For any \( u \in E \), we obtain from (\( \tilde{V}_1 \)), (2.1) and (2.3) that
\[
I_q(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2 dx - \frac{1}{2q} \int_{\mathbb{R}^N} |u|^q dx - \frac{\mu}{q} \int_{\mathbb{R}^N} |u|^q dx \\
\geq \frac{1}{2}(1-\bar{\sigma}_1)|\nabla u|^2 - \frac{1}{2}S^\frac{-2}{q} |\nabla u|^2 \left( 1 - \mu C_{N,q} q^{(1-\gamma_q)} |\nabla u|^{q_{\gamma_q}} \right) \\
\geq |\nabla u|^2 \left( \frac{1}{2}(1-\bar{\sigma}_1) - \frac{1}{2}S^\frac{-2}{q} |\nabla u|^2 - \frac{\mu}{q} C_{N,q} q^{(1-\gamma_q)} |\nabla u|^{q_{\gamma_q}} \right),
\]
where \( q_{\gamma_q} < 2 \). Now we consider the function \( f(a,k) \) defined by
\[
f(a,k) = \frac{1}{2}(1-\bar{\sigma}_1) - \frac{1}{2}S^\frac{-2}{q} k^{2^*-2} - \frac{\mu}{q} C_{N,q} q^{(1-\gamma_q)} k^{q_{\gamma_q} - 2}
\]
for all \((a,k) \in \mathbb{R}^+ \times \mathbb{R}^+ \).

In the following, for any \( \mu > 0 \) fixed, we have

**Lemma 5.1.** Assume that (\( \tilde{V}_1 \)) holds and \( 2 < q < q^* \). For any \( a > 0 \) fixed, then the function \( f(a,k) \) has a unique global maximum respect to \( k \) and the maximum value satisfies
\[
\begin{align*}
\max_{k>0} f(a,k) > 0 & \quad \text{if } a < a_0, \\
\max_{k>0} f(a,k) = 0 & \quad \text{if } a = a_0, \\
\max_{k>0} f(a,k) < 0 & \quad \text{if } a > a_0,
\end{align*}
\]
where
\[
a_0 := \left( \frac{1}{2K(1-\bar{\sigma}_1)} \right)^\frac{q_{\gamma_q} - 2}{q_{\gamma_q} - 2^*},
\]
with
\[
K = \frac{1}{2}S^\frac{-2}{q} \left( \mu C_{N,q} q^\frac{2^*}{2} C_{N,q} S^\frac{2^*}{2} \right)^\frac{q_{\gamma_q} - 2}{q_{\gamma_q} - 2^*} - \frac{\mu}{q} C_{N,q} q^{(1-\gamma_q)} S^\frac{2^*}{2} q_{\gamma_q} - 3.
\]

**Proof.** By definition of \( f(a,k) \), we have
\[
f'_k(a,k) = -\frac{2^*-2}{2}S^\frac{-2}{q} k^{2^*-3} - \frac{\mu}{q} C_{N,q} q_{\gamma_q} (2q_{\gamma_q} - 2) a^{q_{\gamma_q} - 3}.
\]
Hence, the equation \( f'_k(a,k) = 0 \) has a unique solution given by
\[
k_a = \left( \frac{\mu C_{N,q} q^\frac{2^*}{2} C_{N,q} S^\frac{2^*}{2}}{q(2^*-2)} \right)^\frac{1}{\frac{q_{\gamma_q} - 2}{q_{\gamma_q} - 2^*}} a^{q_{\gamma_q} - 3} > 0.
\]
Taking into account that \( f(a,k) \to -\infty \) as \( k \to 0 \) and \( f(a,k) \to -\infty \) as \( k \to +\infty \), we obtain that \( k_a \) is the unique global maximum point of the function \( f(a,k) \) with any fixed \( a > 0 \) and the maximum value is, for any \( a > 0 \) fixed,
\[
\max_{k>0} f(a,k) = f(a,k_a) = \frac{1}{2}(1-\bar{\sigma}_1) - \frac{1}{2}S^\frac{-2}{q} k_a^{2^*-2} - \frac{\mu}{q} C_{N,q} q^{(1-\gamma_q)} k_a^{q_{\gamma_q} - 2} \\
= \frac{1}{2}(1-\bar{\sigma}_1) - \frac{1}{2}S^\frac{-2}{q} \left( \mu C_{N,q} q^\frac{2^*}{2} C_{N,q} S^\frac{2^*}{2} \right)^\frac{q_{\gamma_q} - 2}{q_{\gamma_q} - 2^*} a^{q_{\gamma_q} - 3} \\
- \frac{\mu}{q} C_{N,q} q_{\gamma_q} (2q_{\gamma_q} - 2) a^{q_{\gamma_q} - 3} \\
= \frac{1}{2}(1-\bar{\sigma}_1) - K a^\frac{q_{\gamma_q} - 3}{q_{\gamma_q} - 2^*},
\]
where
\[
K = \frac{1}{2}S^\frac{-2}{q} \left( \mu C_{N,q} q^\frac{2^*}{2} C_{N,q} S^\frac{2^*}{2} \right)^\frac{q_{\gamma_q} - 2}{q_{\gamma_q} - 2^*} - \frac{\mu}{q} C_{N,q} q^{(1-\gamma_q)} S^\frac{2^*}{2} q_{\gamma_q} - 3.
\]
By the definition of \( a_0 \), we deduce that \( \max_{k>0} f(a_0,k) = 0 \), and hence the lemma follows. \( \square \)
**Lemma 5.2.** Assume that \((\hat{V}_1)\) holds and \(2 < q < 7\). Let \((a_1, k_1) \in \mathbb{R}^+ \times \mathbb{R}^+\) be such that \(f(a_1, k_1) \geq 0\). Then for any \(a_2 \in (0, a_1]\), there hold that

\[
f(a_2, k_2) \geq 0 \quad \text{if } k_2 \in \left[\frac{a_2}{a_1}; k_1\right].
\]

Proof. Since \(a \to f(\cdot, k)\) is a non-increasing function we clearly have that, for any \(a_2 \in (0, a_1]\),

\[
f(a_2, k_1) \geq f(a_1, k_1) \geq 0.
\]

By the definition of \(f(a, k)\) and by direct calculations, we get that

\[
f(a_2, \frac{a_2}{a_1} k_1) \geq f(a_1, k_1) \geq 0.
\]

We observe that if \(f(a_2, k') \geq 0\) and \(f(a_2, k'') \geq 0\), then

\[
f(a_2, k) \geq 0 \quad \text{for any } k \in [k', k''].
\]

Indeed, if \(f(a_2, k) < 0\) for some \(k \in [k', k'']\), then there exists a local minimum point on \((k', k'')\) and this contradicts the fact that the function \(f(a, k)\) has a unique global maximum by Lemma 5.1. Hence, by (5.4) and (5.5), taking \(k' = \frac{a_2}{a_1} k_1\) and \(k'' = k_1\), we get the conclusion. \(\square\)

Now let \(a_0 > 0\) be given by (5.2) and \(k_0 := k_{a_0} > 0\) being determined by (5.3). Note that by Lemmas 5.1 and 5.2, we have that \(f(a, 0) = 0\) and \(f(a, k_0) > 0\) for all \(a \in (0, a_0)\). We define

\[D_{k_0} := \{ u \in E : |\nabla u|^2 < k_0^2 \} \quad \text{and} \quad V_a := S_a \cap D_{k_0} \]

We shall now consider the following local minimization problem: for any \(a \in (0, a_0)\),

\[m(a) := \inf_{u \in V_a} \mathcal{I}_q(u).\]

**Lemma 5.3.** Assume that \((\hat{V}_1)\) holds and \(2 < q < 7\). For any \(a \in (0, a_0)\), it holds that

\[
m(a) = \inf_{u \in V_a} \mathcal{I}_q(u) < 0 < \inf_{u \in \partial V_a} \mathcal{I}_q(u).
\]

Proof. For any \(u \in S_a\), since

\[
\mathcal{I}_q(s \ast u) = \frac{\varepsilon^{2s}}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2} \int_{\mathbb{R}^N} V(\varepsilon^{-s} x)|u|^2 dx - \frac{\varepsilon^{2s}}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx - \frac{\mu}{q} \varepsilon^{(\frac{2}{q} - 1)Ns} \int_{\mathbb{R}^N} |u|^q dx
\]

\[
\leq \varepsilon^{2s} \left(\frac{1}{2} + \alpha_1\right)|\nabla u|^2 - \frac{\varepsilon^{2^*-2s}}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx - \frac{\mu}{q} \varepsilon^{(q-2s)Ns} \int_{\mathbb{R}^N} |u|^q dx
\]

and \(q_0 < 2\), then there exists \(s_0 \ll -1\) such that \(|\nabla (s_0 \ast u)|^2_2 = e^{2s_0} |\nabla u|^2_2 < k_0^2\) and \(\mathcal{I}_q(s_0 \ast u) < 0\). This implies that \(m(a) < 0\). Moreover, in view of (5.1) and the fact that \(f(a, k_0) > f(a_0, k_0) = 0\) for all \(a \in (0, a_0)\), we have \(\mathcal{I}_q(u) \geq |\nabla u|^2_2 f(|u|_2, |\nabla u|_2) = k_0^2 f(a, k_0) > 0\) for any \(u \in \partial V_a\). Hence, we complete the proof. \(\square\)

**Lemma 5.4.** Assume that \((\hat{V}_1)\) hold and \(2 < q < 7\). Let \(\{u_n\} \subset D_{k_0}\) be such that \(|u_n|_{q} \to 0\) as \(n \to \infty\). Then there exists a \(\beta_0 > 0\) such that

\[
\mathcal{I}_q(u_n) \geq \beta_0 |\nabla u_n|^2 + o(1).
\]
Proof. Let \(\{u_n\} \subset D_{k_0}\) be such that \(|u_n|_q \rightarrow 0\), we obtain from \((\hat{V}_1), (2.1)\) and \((2.3)\) that

\[
\mathcal{I}_q(u_n) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_n|^2 + V(x)|u_n|^2 \, dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx + o(1)
\]

\[
\geq \frac{1}{2} (1 - \beta_1) |\nabla u_n|^2 - \frac{1}{2^*} S^{2^*} |\nabla u_n|^{2^*} + o(1)
\]

\[
\geq |\nabla u_n|^2 \left( \frac{1}{2} (1 - \beta_1) - \frac{1}{2^*} S^{2^*} |\nabla u_n|^{2^* - 2} \right) + o(1)
\]

which and \(f(a_0, k_0) = 0\) imply that

\[
\beta_0 := \frac{1}{2} (1 - \beta_1) - \frac{1}{2^*} S^{2^*} k_0^{2^* - 2} = \frac{\mu}{q} C_q N q (1 - \gamma_q) k_0 q^{q - 2} > 0.
\]

Thus, we complete the proof. \(\square\)

Now, we collect some properties of \(m(a)\) defined in \((5.6)\).

**Lemma 5.5.** Assume that \((\hat{V}_1)\) hold and \(2 < q < q_1\). It holds that

(i) \(a \in (0, a_0) \mapsto m(a)\) is a continuous mapping.

(ii) Let \(a \in (0, a_0)\). For any \(\alpha \in (0, a)\), we have \(a^2 m(a) < m(\alpha) < 0\).

**Proof.** Similarly as in \([21, \text{Lemma 2.6}]\), by \((\hat{V}_1)\) we can prove \((i)\). It remains to show \((ii)\). Let \(\kappa > 1\) such that \(a = \kappa \alpha\). Let \(\{u_n\} \subset V_\alpha\) be a minimizing sequence with respect to \(m(\alpha)\), that is,

\[
\mathcal{I}_q(u_n) \rightarrow m(\alpha) \quad \text{as} \ n \rightarrow \infty.
\]

By Lemma 5.3, we get \(m(\alpha) < 0\). Letting \(v_n = \kappa u_n\), we have \(|v_n|^2 = a^2\). Moreover, since \(f(a, k_0) \geq f(a_0, k_0) \geq 0\), from Lemma 5.2 we deduce that

\[
f(\alpha, k) > 0 \quad \text{for} \ k \in \left[\frac{\alpha}{\kappa} k_0, k_0\right].
\]

Then, by \((5.1)\) we get \(f(\alpha, |\nabla u_n|^2) < 0\), which implies that

\[
|\nabla u_n|^2 < \frac{\alpha}{\kappa} k_0.
\]

Then, one infers \(|\nabla v_n|^2 = \kappa^2 |\nabla u_n|^2 < k_0^2\), which and \(|v_n|^2 = a^2\) show that \(v_n \in V_\alpha\). Thus,

\[
m(a) \leq \mathcal{I}_q(v_n) = \kappa^2 \mathcal{I}_q(u_n) + \frac{(\kappa^2 - \kappa^q)}{q} \mu \int_{\mathbb{R}^N} |u_n|^q \, dx + \frac{(\kappa^2 - \kappa^2)}{2^*} \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx.
\]

Now, we claim that there exists a positive constant \(C > 0\) and \(n_0 \in \mathbb{N}\) such that

\[
\int_{\mathbb{R}^N} |u_n|^q \, dx > C
\]

for all \(n \geq n_0\). Indeed, otherwise we have \(|u_n|_q \rightarrow 0\) as \(n \rightarrow \infty\). Then it follows from Lemma 5.4 that

\[
0 > m(\alpha) = \lim_{n \rightarrow \infty} \mathcal{I}_q(u_n) \geq \beta_0 \lim_{n \rightarrow \infty} |\nabla u_n|^2 \geq 0,
\]

which is a contradiction. Hence \(\int_{\mathbb{R}^N} |u_n|^q \, dx > C\) for all \(n \geq n_0\). Then, in view of \(\kappa > 1\), one infers

\[
m(a) \leq \mathcal{I}_q(v_n) = \kappa^2 \mathcal{I}_q(u_n) + \frac{(\kappa^2 - \kappa^q)}{q} \mu \int_{\mathbb{R}^N} |u_n|^q \, dx + \frac{(\kappa^2 - \kappa^2)}{2^*} \int_{\mathbb{R}^N} |u_n|^{2^*} \, dx
\]

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\[
\leq \kappa^2 \mathcal{I}_q(u_n) + \frac{(\kappa^2 - \kappa^q)}{q} \mu C \\
\leq \kappa^2 m(\alpha) + \frac{(\kappa^2 - \kappa^q)}{q} \mu C + a_n(1),
\]

which shows that, as \( n \to \infty \),

\[
m(a) < \kappa^2 m(\alpha) = \frac{a^2}{\alpha^2} m(\alpha).
\]

We complete the proof of this lemma. \hfill \square

**Lemma 5.6.** Assume that \( \hat{(V)}_1 \) holds and \( q \in (2, 7) \). Let \( m^\infty(a) := \inf_{u \in \mathcal{V}} \mathcal{I}_q^\infty(u) \), it holds that \( m(a) < m^\infty(a) \) for any \( a \in (0, a_0) \).

**Proof.** According to [21, Theorem 1.2], we know that there exists \( 0 < u_0 \in \mathcal{V}_a \) satisfying \( \mathcal{I}_q^\infty(u_0) = m^\infty(a) \) for any \( a \in (0, a_0) \). Hence, since \( V(x) \neq 0 \) and \( \sup_{x \in \mathbb{R}^N} V(x) = 0 \), we see that

\[
m(a) \leq \mathcal{I}_q(u_0) = \mathcal{I}_q^\infty(u_0) + \int_{\mathbb{R}^N} V(x)|u_0|^2 \, dx < \mathcal{I}_q^\infty(u_0) = m^\infty(a).
\]

Thus, we get \( m(a) < m^\infty(a) \) for any \( a \in (0, a_0) \). The proof of this lemma is finished. \hfill \square

**Lemma 5.7.** Assume that \( \hat{(V)}_1 \) hold and \( 2 < q < 7 \). Let \( \{u_n\} \subset D_{k_0} \) and \( |u_n|_2 \to a \) be a minimizing sequence with respect to \( m(a) \) with \( u_n \to u \) in \( E \), \( u_n(x) \to u(x) \) a.e. in \( \mathbb{R}^N \) and \( u \neq 0 \). Then, \( u \in \mathcal{V}_a \), \( \mathcal{I}_q(u) = m(a) \) and \( u_n \to u \) in \( E \).

**Proof.** Let \( \{u_n\} \subset D_{k_0} \) and \( |u_n|_2 \to a \) be a minimizing sequence with respect to \( m(a) \) with \( u_n \to u \) in \( E \), \( u_n(x) \to u(x) \) a.e. in \( \mathbb{R}^N \). Setting \( w_n := u_n - u \to 0 \) in \( E \). The aim is to prove \( w_n \to 0 \) in \( E \). By the Brézis-Lieb lemma [16], we get

\[
|\nabla u_n|^2 = |\nabla w_n|^2 + |\nabla u|^2 + o(1); \tag{5.7}
|u_n|^2 = |w_n|^2 + |u|^2 + o(1); \tag{5.8}
|u_n|^2 = |w_n|^2 + |u|^2 + o(1),
\]

where \( t \in (2, 2^*) \). Then we have

\[
\mathcal{I}_q(u_n) = \mathcal{I}_q(w_n) + \mathcal{I}_q(u) + o(1). \tag{5.9}
\]

Now we claim that, as \( n \to \infty \),

\[
|w_n|_2 \to 0. \tag{5.10}
\]

Indeed, let \( |u|_2 = a_1 \). Since \( u \neq 0 \), (5.7) and (5.8), we have \( a_1 \in (0, a] \) and \( u \in \mathcal{V}_{a_1} \). If \( a_1 = a \), then, by (5.8) we are done. If \( a_1 \in (0, a) \), then \( |w_n|_2 < a^2 \) and \( |\nabla w_n|^2 \leq |\nabla u_n|^2 < k_0^2 \) for \( n \) large enough. Hence we know that \( w_n \in \mathcal{V}_{|w_n|_2} \) and \( \mathcal{I}_q(w_n) \geq m(|w_n|_2) \) for \( n \) large enough. Hence from (5.9), Lemma 5.5 and the fact that \( u \in \mathcal{V}_{a_1} \) one has, for \( n \) large enough,

\[
m(a) + o(1) = \mathcal{I}_q(u_n) = \mathcal{I}_q(w_n) + \mathcal{I}_q(u) + o(1)
\geq m(|w_n|_2) + m(a_1) + o(1)
\geq \frac{|w_n|_2^2}{a^2} m(a) + m(a_1) + o(1).
\]

Then, \( n \to \infty \),

\[
m(a) \geq \lim_{n \to \infty} \frac{|w_n|_2^2}{a^2} m(a) + m(a_1) > \lim_{n \to \infty} \frac{|w_n|_2^2}{a^2} m(a) + \frac{a^2}{a^2} m(a) = m(a),
\]

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which is a contradiction. Thus, \(|u|_2 = a\) and then \(u \in V_a\). This implies that (5.10) holds. In what follows, we prove that

\[ |\nabla w_n|_2 \to 0 \quad \text{as } n \to \infty. \]  

(5.11)

By (5.7), \(|\nabla w_n|_2^2 \leq |\nabla u_n|_2^2 < k_0^2\) for \(n\) large enough. Thus, \(\{w_n\} \subset D_{k_0}\) and \(\{w_n\}\) is bounded in \(E\) for \(n\) large enough. In view of Gagliardo-Nirenberg inequality (2.1) and (5.10), we deduce that

\[ |w_n|_t \to 0 \quad \text{as } n \to \infty, \]

(5.12)

where \(t \in (2, 2^*)\). Then it follows from Lemma 5.4 that

\[ I_q(w_n) \geq \beta_0 |\nabla w_n|_2^2 + o(1). \]

(5.13)

Moreover, since \(u \in V_a\), by (5.9) one gets

\[ m(a) + o(1) = I_q(u_n) = I_q(w_n) + I_q(u) + o(1) \geq I_q(w_n) + m(a) + o(1), \]

which and (5.13) show that \(|\nabla w_n|_2 \to 0\) as \(n \to \infty\). Hence, we have \(w_n \to 0\) in \(E\) as \(n \to \infty\). That is, \(u_n \to u\) in \(E\) as \(n \to \infty\). Furthermore, we see from (5.9), (5.11) and (5.12) that \(I_q(u) = m(a)\). Thus, we complete the proof.

Proof of Theorem 1.5: Let \(\{u_n\} \subset D_{k_0}\) and \(|u_n|_2 \to a\) be a minimizing sequence for \(m(a)\) with \(a \in (0, a_0)\). Then \(\{u_n\}\) is bounded in \(E\). Hence, there exists \(u \in E\) such that, up to a subsequence,

\[
\begin{align*}
 u_n &\rightharpoonup u &\quad \text{in } E; \\
u_n &\to u &\quad \text{in } L^p_{\text{loc}}(\mathbb{R}^N) \text{ with } p \in [1, 2^*); \\
u_n(x) &\to u(x) &\quad \text{a.e. on } \mathbb{R}^N.
\end{align*}
\]

Now we show that \(u \neq 0\). Assume by contradiction that \(u = 0\). Then

\[ m(a) + o(1) = I_q(u_n) = I_q^\infty(u_n) + \int_{\mathbb{R}^N} V(x)|u_n|^2 dx + o(1) \geq m^\infty(a) + o(1), \]

which gives that \(m(a) \geq m^\infty(a)\) as \(n \to \infty\). It contradicts to Lemma 5.6. So we have \(u \neq 0\). Therefore, we deduce from Lemmas 5.3 and 5.7 that \(u_n \to u\) in \(E\) as \(n \to \infty\) and \(u \notin \partial V_a\) is a minimizer for \(I_q\) on \(V_a\), that is, \(I_q(u) = m(a)\). Then, by the Lagrange multiplier, there exists \(\lambda_a \in \mathbb{R}\) such that

\[ -\Delta u + V(x)u + \lambda_a u = |u|^{2^*-2}u + \mu |u|^{q-2}u \quad \text{in } \mathbb{R}^N. \]

(5.14)

If assume that (V4) holds, similar to (2.25) we know that \(\lambda_a > 0\). Thus we obtain that there exists a couple \((\lambda, u) \in \mathbb{R}^+ \times E\) solving Eq. (P) with \(q \in (2, 2 + \frac{4}{N})\), where \(I_q(u) = m(a) < 0\). So we complete the proof of Theorem 1.5.

6 Proof of Theorems 1.10, 1.12 and 1.14

In this section, we focus on the properties of the normalized solution for Eq. (P) with \(q \in [2 + \frac{4}{N}, 2^*)\). Firstly, we give the exponential decay property of the positive normalized solution of Eq. (P) and prove Theorem 1.10. Then we verify that the solution of the initial-value problem (1.1) with initial datum blows-up in finite time in Theorem 1.12 under some suitable assumptions. Finally, we study the strong instability of the standing waves for problem (1.1) in Theorem 1.14.

Proof of Theorem 1.10: Let \(u \in E\) be the positive real-valued solution for Eq. (P) with \(\mu, a > 0\) and \(q \in [2 + \frac{4}{N}, 2^*)\), then it follows (V4) and (2.25) that the corresponding Lagrange multiplier \(\lambda > 0\),
Now, we claim that the positive real-valued normalized solution $u$ of Eq. (P) decays exponentially. In fact, we first show that $|u(x)| \to 0$ as $|x| \to \infty$. For any $k > 0$, define

$$u_k = \begin{cases} u, & |u(x)| \leq k, \\ \pm k, & |u(x)| > k. \end{cases}$$

Let $\vartheta \in C^\infty(\mathbb{R}^N, [0, 1])$ such that $|\nabla \vartheta|^2 \leq \frac{4}{\text{Vol}^2}$ and

$$\vartheta = \begin{cases} 1, & |x| \geq R, \\ 0, & |x| \leq r, \end{cases}$$

where $R > 2$ and $1 < r < \frac{R}{2}$. For all $i \geq 1$, Multiplying both sides of Eq. (P) by $\varphi_k := \vartheta^2|u_k|^2u \in E$ and integrate on $\mathbb{R}^N$, one obtains

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla \varphi_k + \lambda u \varphi_k + V(x)u \varphi_k \, dx = \int_{\mathbb{R}^N} |u|^{2s-2}u \varphi_k + \mu |u|^{q-2}u \varphi_k \, dx. \quad (6.1)$$

Since $\lambda > 0$ and

$$\nabla \varphi_k = 2\vartheta^2|u_k|^{2s-2}u_k \nabla u_k + \vartheta^2|u_k|^{2s}u + 2\vartheta|u_k|^{2s}u \nabla \vartheta,$$

then by (6.1) we get

$$\int_{\mathbb{R}^N} 2\vartheta^2|u_k|^{2s-2}u_k \nabla u_k \cdot \nabla u \, dx + \int_{\mathbb{R}^N} \vartheta^2|u_k|^{2s} |\nabla u|^2 \, dx \\
\leq \int_{\mathbb{R}^N} 2\vartheta|u_k|^{2s} u \nabla \vartheta \cdot \nabla u \, dx + \int_{\mathbb{R}^N} |V(x)|u \varphi_k \, dx + \int_{\mathbb{R}^N} |u|^{2s-2}u \varphi_k \, dx + \mu \int_{\mathbb{R}^N} |u|^{q-2}u \varphi_k \, dx \\
:= I_1 + I_2 + I_3 + I_4, \quad (6.2)$$

where

$$I_1 = \int_{\mathbb{R}^N} 2\vartheta|u_k|^{2s} u \nabla \vartheta \cdot \nabla u \, dx; \quad I_2 = \int_{\mathbb{R}^N} |V(x)|u \varphi_k \, dx;$$

$$I_3 = \int_{\mathbb{R}^N} |u|^{2s-2}u \varphi_k \, dx; \quad I_4 = \mu \int_{\mathbb{R}^N} |u|^{q-2}u \varphi_k \, dx.$$

For $I_1$, by Young inequality,

$$I_1 = \int_{\mathbb{R}^N} 2\vartheta|u_k|^{2s} u \nabla \vartheta \cdot \nabla u \, dx = \int_{\mathbb{R}^N} \vartheta|u_k|^{2s} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} 2|u_k|^s \vartheta |\nabla \vartheta|^2 \, dx \\
\leq \frac{1}{2} \int_{\mathbb{R}^N} \vartheta^2|u_k|^{2s} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} |u_k|^{2s} |\nabla \vartheta|^2 \, dx \\
\leq \frac{1}{2} \int_{\mathbb{R}^N} \vartheta^2|u_k|^{2s} |\nabla u|^2 \, dx + C \int_{|x| \geq r} |u_k|^{2s} u^2 \, dx.$$

For $I_2$, since $V \in C(\mathbb{R}^N, \mathbb{R})$ and $\lim_{|x| \to \infty} V(x) = 0$, one gets

$$I_2 = \int_{\mathbb{R}^N} |V(x)|u \varphi_k \, dx \leq C \int_{|x| \geq r} |u_k|^{2s} u^2 \, dx.$$

For $I_3$, by Lemma 2.11,

$$I_3 = \int_{\mathbb{R}^N} |u|^{2s-2}u \varphi_k \, dx \leq C \int_{|x| \geq r} |u_k|^{2s} u^2 \, dx.$$

Similarly,

$$I_4 = \mu \int_{\mathbb{R}^N} |u|^{q-2}u \varphi_k \, dx \leq C(\mu, q) \int_{|x| \geq r} |u_k|^{2s} u^2 \, dx.$$
Then, from (6.2), there exists positive constant $C$, depending on $\mu, u, q$, such that

$$
\int_{\mathbb{R}^N} 2\vartheta^2 \iota_n |u_k|^{2n-2} u_k u \nabla u_k \cdot \nabla u \, dx + \int_{\mathbb{R}^N} \vartheta^2 |u_k|^{2n} |\nabla u|^2 \, dx \leq C \int_{|x| \geq r} |u_k|^{2n} u^2 \, dx. \tag{6.3}
$$

Moreover, for all $\iota \geq 1$,

$$
\left( \int_{|x| \geq R} (|u_k|^\iota u)^2 \, dx \right)^{\frac{2}{\iota}} \leq \left( \int_{\mathbb{R}^N} (\vartheta |u_k|^\iota u)^2 \, dx \right)^{\frac{2}{\iota}} \leq \int_{\mathbb{R}^N} |\nabla (\vartheta |u_k|^\iota u)|^2 \, dx
$$

$$
\leq \int_{\mathbb{R}^N} |u_k|^{2n} u^2 |\nabla \vartheta + \iota \vartheta u| |u_k|^{2n-2} u_k \nabla u_k + \vartheta |u_k|^\iota \nabla u|^2 \, dx
$$

$$
\leq \int_{\mathbb{R}^N} |u_k|^{2n} u^2 |\nabla \vartheta|^2 + \iota^2 \vartheta^2 u^2 |u_k|^{2n-2} |\nabla u_k|^2 + \vartheta^2 |u_k|^{2n} |\nabla u|^2 \, dx
$$

$$
\leq C \int_{|x| \geq r} |u_k|^{2n} u^2 \, dx + \iota \left( 2 \int_{\mathbb{R}^N} \vartheta^2 |u_k|^{2n-2} |\nabla u_k|^2 \, dx + \int_{\mathbb{R}^N} \vartheta^2 |u_k|^{2n} |\nabla u|^2 \, dx \right)
$$

$$
\leq C (1 + \iota) \int_{|x| \geq r} |u_k|^{2n} u^2 \, dx, \tag{6.4}
$$

where (6.3) and the definition of $u_k$ are applied to the last inequality. Let $k \to \infty$, (6.4) implies that

$$
|u|_{L^{2n(1+\iota)}(|x| \geq r)} \leq \left[ C (1 + \iota) \right]^\frac{1}{2n(1+\iota)} |u|_{L^{2(1+\iota)}(|x| \geq r)}. \tag{6.5}
$$

In order to use the Moser iteration, let

$$
\iota_1 = \frac{2n}{2} - 1 > 0, \quad (1 + \iota_{n+1}) = \frac{2n}{2} (1 + \iota_n), \quad R_n = R_n + 1 = R - (R - r) \left( \frac{R - r}{R} \right)^n,
$$

then, by (6.5),

$$
|u|_{L^{2n(1+\iota_{n+1})}(|x| \geq R_{n+1})} \leq [C (1 + \iota_{n+1})]^{\frac{1}{2n(1+\iota_{n+1})}} |u|_{L^{2(1+\iota_{n+1})}(|x| \geq R_{n+1})}
$$

$$
= [C (1 + \iota_n)]^{\frac{1}{2n(1+\iota_n)}} |u|_{L^{2n}(|x| \geq R_n)}
$$

$$
\leq [C (1 + \iota_{n+1})]^{\frac{1}{2n(1+\iota_{n+1})}} [C (1 + \iota_n)]^{\frac{1}{2n}} |u|_{L^{2(1+\iota)}(|x| \geq R_n)}
$$

$$
\vdots
$$

$$
\leq [C]^{\sum_{i=1}^{n+1} \frac{1}{2n(1+\iota_i)}} \prod_{i=1}^{n+1} (1 + \iota_i)^\frac{1}{2n(1+\iota_i)} |u|_{L^{2n}(|x| \geq r_1)}. \tag{6.6}
$$

Since $\iota_1 = \frac{2n}{2} - 1$ and $1 + \iota_{n+1} = \frac{2n}{2} (1 + \iota_n)$, one can see that $1 + \iota_n = \left( \frac{2n}{2} \right)^n$ for all $n \in \mathbb{N}$. Then,

$$
[C]^{\sum_{i=1}^{\infty} \frac{1}{2n(1+\iota_i)}} = [C]^{\sum_{i=1}^{\infty} \left( \frac{2n}{2} \right)^i} < +\infty
$$

and

$$
\prod_{i=1}^{\infty} (1 + \iota_i)^\frac{1}{2n(1+\iota_i)} = \prod_{i=1}^{\infty} \left[ \left( \frac{2n}{2} \right)^i \right]^{\frac{1}{2n}} = \left( \frac{2n}{2} \right)^{\frac{1}{2n}} \sum_{i=1}^{\infty} \left( \frac{2n}{2} \right)^i < +\infty.
$$

Therefore, letting $n \to \infty$ in (6.6), we can conclude that

$$
|u|_{L^{\infty}(|x| \geq R)} \leq C |u|_{L^{2n}(|x| \geq r)}.
$$
Hence, for any $\epsilon > 0$ fixed, choosing $r > 1$ large enough one infers $|u|_{L^\infty(|x| \geq R)} \leq \epsilon$. This shows that

$$|u(x)| \to 0 \quad \text{as} \quad |x| \to \infty. \quad (6.7)$$

Next, for any $q \in [2 + \frac{4}{N}, 2^\star)$ and $\mu > 0$, by (6.7) and the fact that $\lim_{|x| \to \infty} V(x) = 0$, there exists $\tilde{R} > 0$ large enough such that

$$-\Delta u = (-V(x) - \lambda + |u|^{2^\star - 2} + \mu|u|^{q - 2})u \leq \frac{1}{2} u \quad \text{for all} \quad |x| > \tilde{R}.$$ 

Let $\xi(x) = M_1 e^{-\sqrt{\frac{r^2}{2}}|x|}$, where $M_1$ satisfies

$$M_1 e^{-\sqrt{\frac{r^2}{2}} \tilde{R}} \geq u(x) \quad \text{for} \quad |x| = \tilde{R}.$$ 

By simple calculation, we get $\Delta \xi \leq \frac{1}{2} \xi$ for all $x \neq 0$. Set $\zeta = \xi - u$, one has

$$\begin{cases} 
-\Delta \zeta + \frac{1}{2} \zeta \geq 0, & |x| > \tilde{R}, \\
\zeta(x) \geq 0, & |x| = \tilde{R}, \\
\lim_{|x| \to \infty} \zeta(x) = 0.
\end{cases}$$

Hence, it follows from maximum principle [17] that $\zeta \geq 0$. That is,

$$|u(x)| \leq M_1 e^{-\sqrt{\frac{r^2}{2}}|x|} \quad \text{for all} \quad |x| \geq \tilde{R}.$$ 

Moreover, since $u$ is continuous function, there exists $M_2 > 0$ such that

$$|u(x)| e^{-\sqrt{\frac{r^2}{2}}|x|} \leq M_2 \quad \text{for all} \quad |x| \leq \tilde{R}.$$ 

Therefore, choosing $M = \max\{M_1, M_2\}$, we have

$$|u(x)| \leq Me^{-\sqrt{\frac{r^2}{2}}|x|} \quad \text{for all} \quad x \in \mathbb{R}^N.$$ 

Thus, we complete the proof of Theorem 1.10. \qed

Now, we show that the phenomenon of finite-time blow-up occurs for the solution of the initial-value problem (1.1) with initial datum under the assumptions of Theorem 1.12.

**Proof of Theorem 1.12:** (i) Let $q \in [7, 2^\star)$ and let $u_0 \in S_a$ be such that $I_q(u_0) < \inf_{u \in \mathcal{P}_q, x} I_q(u)$. Suppose that $(V_5)$ holds, according to [34, Section 3] and [10, 21], the initial-value problem (1.1) with initial datum $u_0$ is local well-posed on $(-T_{\text{min}}, T_{\text{max}})$ with $T_{\text{min}}, T_{\text{max}} > 0$. Furthermore, the solution to problem (1.1) with initial datum $u_0$ has conservation of mass and energy. That is, let $\phi(t, x)$ be the solution of the initial-value problem (1.1) with initial datum $u_0$ on $(-T_{\text{min}}, T_{\text{max}})$, it holds that

$$\|u_0\|_2 = \|\phi\|_2 \quad \text{and} \quad I_q(u_0) = I_q(\phi). \quad (6.8)$$

By the fact that $|x|u_0 \in L^2(\mathbb{R}^N, \mathbb{C})$ and [10, Proposition 6.5.1], we get

$$H(t) := \int_{\mathbb{R}^N} |x|^2 |\phi(t, x)|^2 \, dx < +\infty \quad \text{for all} \quad t \in (-T_{\text{min}}, T_{\text{max}}). \quad (6.9)$$

Moreover, the function $H \in C^2(-T_{\text{min}}, T_{\text{max}})$ and the following Virial identity holds:

$$H'(t) = -4y(t) \quad \text{and} \quad H''(t) = -4y'(t) = 8P_q(\phi), \quad (6.10)$$

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where

\[
y(t) = -\text{Im} \int_{\mathbb{R}^N} \overline{\phi}(x) \cdot \nabla \phi(x) \, dx;
\]

\[
y'(t) = -2 \int_{\mathbb{R}^N} |\nabla \phi|^2 \, dx + \int_{\mathbb{R}^N} \langle \nabla V(x), x \rangle |\phi|^2 \, dx + 2 \int_{\mathbb{R}^N} |\phi|^2 \, dx + 2 \mu \gamma_q \int_{\mathbb{R}^N} |\phi|^q \, dx.
\]  

(6.11) (6.12)

From Lemma 2.6 or Lemma 4.4, for any \( u \in S_a \), the function \( \varphi_u(s) := \psi_u(\log s) \) with \( s > 0 \) has a unique global maximum point \( \hat{s}_u = e^{s_u} \) and \( \varphi_u(s) \) is strictly decreasing and concave on \((\hat{s}_u, +\infty)\). According to the assumption \( s_u < 0 \), one obtains \( \hat{s}_u \in (0, 1) \). We claim that

if \( u \in S_a \) and \( \hat{s}_u \in (0, 1) \), then \( P_q(u) \leq I_q(u) - \inf_{P_{q,a}} I_q \). (6.13)

In fact, since \( \hat{s}_u \in (0, 1) \) and \( \varphi_u(s) \) is strictly decreasing and concave on \((\hat{s}_u, +\infty)\), we infer that \( P_q(u) < 0 \) and

\[
I_q(u) = \psi_u(0) = \varphi_u(1) \geq \varphi_u(\hat{s}_u) - \varphi'_u(1)(\hat{s}_u - 1) = I_q(s_u * u) - |P_q(u)|(1 - \hat{s}_u) \geq \inf_{P_{q,a}} I_q + P_q(u),
\]

which completes the claim. Now, let us consider the solution \( \phi \) for the initial-value problem (1.1) with initial datum \( u_0 \). Since by assumption \( s_u < 0 \), and the map \( u \mapsto s_u \) is continuous, we deduce that \( s_u(t) < 0 \) for every \( |t| < \overline{t} \) with \( \overline{t} > 0 \) small enough. Then \( \hat{s}_u(t) \in (0, 1) \) for \( |t| < \overline{t} \). By (6.13) and recalling the assumption \( I_q(u_0) < \inf_{P_{q,a}} I_q \), we deduce from (6.8) that

\[
P_q(\phi(t)) \leq I_q(\phi(t)) - \inf_{P_{q,a}} I_q = I_q(u_0) - \inf_{P_{q,a}} I_q := -\eta < 0,
\]

for every such \( |t| < \overline{t} \). Next, we show that

\[
P_q(\phi(t)) \leq -\eta \quad \text{for any} \quad t \in (-T_{\text{min}}, T_{\text{max}}).
\]

(6.14)

Assume that there is \( t_0 \in (-T_{\text{min}}, T_{\text{max}}) \) satisfying \( P_q(\phi(t_0)) = 0 \). It holds from (6.8) that \( \phi(t_0) \in S_a \) and

\[
I_q(\phi(t_0)) \geq m_{q,a} = \inf_{P_{q,a}} I_q > I_q(u_0) = I_q(\phi(t_0)),
\]

which is a contradiction. This shows that \( P_q(\phi(t_0)) \neq 0 \) for any \( t_0 \in (-T_{\text{min}}, T_{\text{max}}) \). Then, since \( P_q(\phi(t)) < 0 \) for every \( |t| < \overline{t} \), we obtain \( P_q(\phi(t)) < 0 \) for any \( t \in (-T_{\text{min}}, T_{\text{max}}) \) if at some \( t \in (-T_{\text{min}}, T_{\text{max}}) \), \( P_q(\phi(t)) > 0 \). By continuity, we have \( P_q(\phi(t)) = 0 \) for some \( t \in (-T_{\text{min}}, T_{\text{max}}) \), a contradiction. Since \( P_q(\phi(t)) < 0 \) for any \( t \in (-T_{\text{min}}, T_{\text{max}}) \), by Lemma 2.6, \( s_u(t) < 0 \). That is, \( \hat{s}_u(t) \in (0, 1) \). Therefore, (6.13) holds and the above arguments yield

\[
P_q(\phi(t)) \leq -\eta \quad \text{for any} \quad t \in (-T_{\text{min}}, T_{\text{max}}).
\]

Thus, we deduce from (6.9), (6.10) and (6.14) that

\[
0 \leq H(t) \leq H(0) + H'(0)t + \frac{1}{2} H''(0)t^2 \leq H(0) + H'(0)t - 4\eta t^2 \quad \text{for any} \quad t \in (-T_{\text{min}}, T_{\text{max}}),
\]

which implies that \( T_{\text{max}} \) has an upper bound since the right hand side becomes negative for \( t \) large. Hence, there exists \( T \in (0, T_{\text{max}}) \) such that \( \lim_{t \to T} H(t) = 0 \). Since

\[
\int_{\mathbb{R}^N} |\phi|^2 \, dx \leq C \left( \int_{\mathbb{R}^N} |x|^2 |\phi|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} \frac{1}{|x|^2} |\phi|^2 \, dx \right)^{\frac{1}{2}} \leq C \left( \int_{\mathbb{R}^N} |x|^2 |\phi|^2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^N} |\nabla \phi|^2 \, dx \right)^{\frac{1}{2}},
\]

where the Hölder and Hardy inequality are applied, we get from (6.8) that \( |\nabla \phi|_2 \to +\infty \) as \( t \to T \). That is, the solution \( \phi \) of the initial-value problem (1.1) with initial datum \( u_0 \) blows-up in finite time.

(ii) Let \( q \in (7, 2^*) \). Suppose that \( I_q(u_0) < 0 \), \( |x|u_0 \in L^2(\mathbb{R}^N, \mathbb{C}) \) and \( y(0) = y_0 > 0 \) defined in (6.11). We know that (6.9)–(6.12) still hold. To prove the phenomenon of blow-up under the above
assumptions, we will follow the convexity method of Glassey [15]. More precisely, we show that the variance function $H(t)$ defined in (6.9) is decreasing and concave for $t > 0$, which suggests the existence of a blowup time $T$ at which $H(T) = 0$. Moreover, we will give the estimate of the blow-up time $T$. In view of (V$_1$), (V$_2$), (6.8), (6.12) and the fact that $\mathcal{I}_q(u_0) < 0$, one infers
\[
y'(t) = -2 \int_{\mathbb{R}^N} |\nabla \phi|^2 dx + \int_{\mathbb{R}^N} \langle \nabla V(x), x \rangle |\phi|^2 dx + 2 \int_{\mathbb{R}^N} |\phi|^2 dx + 2\mu q \int_{\mathbb{R}^N} |\phi|^q dx
\]
\[
= (q\gamma_q - 2) \int_{\mathbb{R}^N} |\nabla \phi|^2 dx + \int_{\mathbb{R}^N} \langle \nabla V(x), x \rangle |\phi|^2 dx + q\gamma_q \int_{\mathbb{R}^N} V(x) |\phi|^2 dx
\]
\[
+ (2 - \frac{2q\gamma_q}{2^*}) \int_{\mathbb{R}^N} |\phi|^2 dx - 2q\gamma_q T_q(\phi)
\]
\[
\geq (q\gamma_q - 2 - q\gamma_q \sigma_1 - \frac{\sigma_2}{2}) \int_{\mathbb{R}^N} |\nabla \phi|^2 dx,
\]
(6.15)
where $q\gamma_q - 2 - q\gamma_q \sigma_1 - \frac{\sigma_2}{2} > 0$. Hence, we have
\[
y'(t) > 0 \quad \text{for all } t \in (-T_{\min}, T_{\max}) \quad \text{and} \quad y(t) > y(0) = y_0 > 0 \quad \text{for all } t > 0,
\]
which shows that $H'(t) < 0$ for $t > 0$ and $H''(t) < 0$ for $t \in (-T_{\min}, T_{\max})$. Then we obtain the function $H$ is decreasing for any $t > 0$ and the function $H$ is concave for all $t \in (-T_{\min}, T_{\max})$. This shows the existence of a blowup time $T$ at which $H(T) = 0$. Moreover, by Hölder inequality, for all $t \in (-T_{\min}, T_{\max})$, we deduce
\[
y(t) \leq |x\phi|_2 |\nabla \phi|_2,
\]
which and the monotonicity and concavity of the function $H(t)$ imply that
\[
|\nabla \phi|_2 \geq \frac{y(t)}{|x\phi|_2} \geq \frac{y(t)}{|xu_0|_2}.
\]
(6.16)
In what follows, combining (6.15) with (6.16), we get the following ODE for $y$,
\[
\begin{align*}
y'(t) & \geq (q\gamma_q - 2 - \frac{1}{2}\sigma_2 - q\gamma_q \sigma_1) \frac{|y(t)|_2}{|xu_0|_2}, \\
y(0) & = y_0 > 0,
\end{align*}
\]
which implies that there exists
\[
0 < T \leq \frac{|xu_0|_2^2}{(q\gamma_q - 2 - \frac{1}{2}\sigma_2 - q\gamma_q \sigma_1)y_0}
\]
satisfying $\lim_{t \to T^-} |\nabla \phi| = +\infty$. Thus, we complete the proof of Theorem 1.12.

In the following, we are going to prove the strong instability of the standing wave for problem (1.1) by using Theorems 1.10 and 1.12.

**Proof of Theorem 1.14:** Under the assumptions of Theorem 1.3 or Theorem 1.4, we first describe the characteristic of $Z_a$ as
\[
Z_a = \{e^{i\theta}u : \theta \in \mathbb{R}, u \in U_a \text{ and } u > 0 \text{ in } \mathbb{R}^N\},
\]
(6.17)
where $U_a$ is defined in (3.14). For any $z \in E$, let $z(x) = (v(x), w(x)) = v(x) + iw(x)$, where $v, w \in E$ are real-valued functions and
\[
||z||^2 = |\nabla z|^2 + |z|^2, \quad |z|^2 = |v|^2 + |w|^2 \quad \text{and} \quad |\nabla z|^2 = |\nabla v|^2 + |\nabla w|^2.
\]
Taking $z = (v, w) \in Z_a$, we see from (3.15) that $|z| \in Z_a$ and $|\nabla|z||^2 = |\nabla z|^2$. Then, by the fact that $|\nabla|z|^2 - |\nabla z|^2 = 0$ one obtains
\[
\int_{\mathbb{R}^N} \sum_{i=1}^{N} \frac{(v \partial_i w - w \partial_i v)^2}{v^2 + w^2} dx = 0. \tag{6.18}
\]
Hence, from [18, Theorem 4.1], we know that

(i) either $v \equiv 0$ or $v(x) \neq 0$ for all $x \in \mathbb{R}^N$;

(ii) either $w \equiv 0$ or $w(x) \neq 0$ for all $x \in \mathbb{R}^N$.

Now we turn to the characterization of $Z_a$. On the one hand, let $v = e^{i\theta}u$ with $\theta \in \mathbb{R}$, $u \in U_a$ and $u > 0$, then we have $|v|^2 = a^2$ and
\[
\mathcal{I}_q(v) = \mathcal{I}_q(u) = m_{q,a} \quad \text{and} \quad P_q(v) = P_q(u) = 0,
\]
which implies that
\[
\{ e^{i\theta}u : \theta \in \mathbb{R}, u \in U_a \text{ and } u > 0 \text{ in } \mathbb{R}^N \} \subseteq U_a = Z_a.
\]

On the other hand, let $z = (v, w) \in Z_a$, we have $u := |z| \in Z_a$. If $w \equiv 0$, then we deduce that $u := |z| = |v| > 0$ on $\mathbb{R}^N$ and $z = e^{i\theta}u$ where $\theta = 0$ if $v > 0$ and $\theta = \pi$ if $v < 0$ on $\mathbb{R}^N$. Otherwise, it follows from (ii) that $w(x) \neq 0$ for all $x \in \mathbb{R}^N$. Since
\[
\frac{(v \partial_i w - w \partial_i v)^2}{v^2 + w^2} = \left[ \partial_i \left( \frac{v}{w} \right) \right]^2 \frac{w^2}{v^2 + w^2} \quad \text{where } i = 1, 2, \ldots, N,
\]
for all $x \in \mathbb{R}^N$, in view of (6.18) we get
\[
\nabla \left( \frac{v}{w} \right) = 0 \quad \text{on } \mathbb{R}^N.
\]
Therefore, there exists $C \in \mathbb{R}$ such that $v = Cw$ on $\mathbb{R}^N$. Then we have
\[
z = (v, w) = v + iw = (C + i)w \quad \text{and} \quad |z| = |C + i||w|. \tag{6.19}
\]
Let $\theta_1 \in R$ be such that $C + i = |C + i|e^{i\theta_1}$ and let $w = |w|e^{i\theta_2}$ with
\[
\theta_2 = \begin{cases} 0, & \text{if } w > 0; \\ \pi, & \text{if } w < 0. \end{cases}
\]
Then we can see from (6.19) that $z = (C + i)w = |C + i||w|e^{i(\theta_1 + \theta_2)} = |z|e^{i(\theta_1 + \theta_2)}$. Setting $\theta = \theta_1 + \theta_2$ and $u := |z|$, then $0 < u \in U_a$ and $z = e^{i\theta}u$. Thus,
\[
Z_a = \{ e^{i\theta}u : \theta \in \mathbb{R}, u \in U_a \text{ and } u > 0 \text{ on } \mathbb{R}^N \}.
\]
Thus, we know that if $\hat{u} \in Z_a$, we have $\hat{u} = e^{i\theta}u$ for $\theta \in \mathbb{R}$ and $0 < u \in U_a$, and then, similar to (2.25), the associated Lagrange multiplier $\hat{\lambda} > 0$.

**Strong instability of the stand wave** $e^{i\lambda \hat{u}}$. For any $q \in [\overline{q}, 2^*]$, let $\phi_q(x, t)$ be the solution to the initial-value problem (1.1) with initial datum $\hat{u}_q$, where $\hat{u}_q = q \ast \hat{u}$ with $q > 0$. We have $\hat{u}_q \rightarrow \hat{u}$ in $E$ as $q \rightarrow 0^+$, and hence it is sufficient to prove that $\phi_q(x, t)$ blows-up in finite time. Let $s_{\hat{u}_q} \in \mathbb{R}$ be defined in Lemma 2.6 or Lemma 4.4. Clearly $s_{\hat{u}_q} = -\varrho < 0$, and
\[
\mathcal{I}_q(\hat{u}_q) = \mathcal{I}_q(\varrho \ast \hat{u}) < \mathcal{I}_q(\hat{u}) = \inf_{P_{q,a}} \mathcal{I}_q.
\]
Furthermore, from (3.15) and (6.17) we have $0 < |\hat{u}| \in Z_a$. Since $|\hat{u}|$ decays exponentially according to Theorem 1.10, we deduce
\[
\int_{\mathbb{R}^N} |x|^2 |\hat{u}_\rho|^2 dx = e^{-2\rho} \int_{\mathbb{R}^N} |x|^2 |\hat{u}|^2 dx < +\infty,
\]
which shows that $|x|\hat{u}_\rho \in L^2(\mathbb{R}^N, \mathbb{C})$. Hence, it follows from Theorem 1.12-(i) that the solution $\phi_\rho(x,t)$ blows-up in finite time. Thus, by Definition 1.2, the standing wave $e^{i\lambda t}\hat{u}$ is strongly unstable. We complete the proof of Theorem 1.14. □

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