The ground state solution for Kirchhoff-Schrödinger equations with singular exponential nonlinearities in $\mathbb{R}^4$

Yanjun Liu$^a$,∗ Shijie Qi$^b$

$^a$School of Mathematical Sciences, Nankai University, Tianjin 300071, P. R. China

$^b$Department of Mathematical Science, Tsinghua University, Beijing 100084, P. R. China

Abstract: In this paper, we consider the following singular Kirchhoff-Schrödinger problem

$$M\left(\int_{\mathbb{R}^4} |\Delta u|^2 + V(x)u^2\,dx\right)(\Delta^2 u + V(x)u) = \frac{f(x,u)}{|x|^\eta} \quad \text{in} \quad \mathbb{R}^4,$$

where $0 < \eta < 4$, $M$ is a Kirchoff-type function and $V(x)$ is a continuous function with positive lower bound, $f(x,t)$ has a critical exponential growth behavior at infinity. Using singular Adams inequality and variational techniques, we get the existence of ground state solutions for $(P_\eta)$. Moreover, we also get the same results without the Ambrosetti-Rabinowitz (AR) condition.

Keywords: The ground state solutions; Singular exponential nonlinearities; Singular Adams inequality; Kirchhoff-Schrödinger equations

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1 Introduction and main results

The relevant problems involving powers of the Laplacian started with [1]-[2]. In conformal geometry, there has been considerable interest in the Paneitz operator which enjoys the property of conformal invariance. In $\mathbb{R}^4$, the Paneitz operator is the biharmonic operator $\Delta^2$, this can be referred to [3]. Recently, Zhang and Chen in [26] established a sharp concentration-compactness principle associated with the singular

∗ Corresponding author.

Email: liuyj@mail.nankai.edu.cn(Y. Liu); qishj2019@tsinghua.edu.cn(S. Qi).
Adams inequality on the second-order Sobolev spaces in $\mathbb{R}^4$, and moreover, they consider the following problem:

$$\Delta^2 u + V(x)u = \frac{f(x,u)}{|x|^{\eta}} \quad \text{in } \mathbb{R}^4,$$

where $V(x)$ has a positive lower bound and $0 < \eta < 4$, they got a ground state solution of (1.1) under the A-R condition. In [14], let $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary, $m$ is an integer and $N \geq 2m \geq 2$, the authors considered the following Kirchhoff problem

$$\begin{cases}
-M \left( \int_{\Omega} |\nabla^m u|^\frac{2N}{N-m} \, dx \right) \Delta^m u = \frac{f(x,u)}{|x|^{\eta}} \quad &\text{in } \Omega, \\
u = \nabla u = \nabla^2 u = \ldots \nabla^{m-1} u = 0 \quad &\text{on } \partial \Omega,
\end{cases}$$

(1.2)

where $0 \leq \eta < N$, $M$ is a Kirchhoff-type function and $b(x)$ is a continuous function with positive lower bound, $f(x,t)$ has an critical exponential growth behavior at infinity.

Since we will work with exponential critical growth, we need to review the Trudinger–Moser inequality and Adams inequality, the latter is a generalization of the former and more details are as follows: On one hand, let $\Omega$ denotes a smooth bounded domain in $\mathbb{R}^N (N \geq 2)$, N. Trudinger [4] proved that there exists $\alpha > 0$ such that $W^{1,N}_0(\Omega)$ is embedded in the Orlicz space $L_{\varphi_{\alpha}}(\Omega)$ determined by the Young function $\varphi_{\alpha}(t) = e^{\alpha |t|^\frac{N}{N-\eta}}$, it was sharpened by J. Moser [5] who found the best exponent $\alpha$. On the other hand, the Trudinger–Moser inequality was extended for unbounded domains by D. M. Cao [6] in $\mathbb{R}^2$ and for any dimension $N \geq 2$ by J. M. do Ó [7]. Moreover, J. M. do Ó et al. [8] established a sharp concentration-compactness principle associated with the singular Trudinger–Moser inequality in $\mathbb{R}^N$. For more results concerning the Trudinger-Moser inequality and its application in $N$-Laplacian equations, one can refer to [9, 11, 12, 13, 19] and the references therein.

For Adams type inequality, let $\Omega \subset \mathbb{R}^4$ be a smooth bounded domain. D. Adams [22] derives

$$\sup_{u \in W^{2,2}_0(\Omega), \int_{\Omega} |\Delta u|^2 \, dx \leq 1} \int_{\Omega} e^{32\pi^2 u^2} \, dx < \infty,$$
which was extended by C. Tarsi \cite{23}, i.e.
\[
\sup_{u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega), \int_{\Omega} |\Delta u|^2 dx \leq 1} \int_{\Omega} e^{32\pi^2 u^2} dx < \infty,
\]
B. Ruf and F. Sani \cite{24} extended the Adams inequality to $\mathbb{R}^4$, namely
\[
\sup_{u \in W^{2,2}(\mathbb{R}^4), \int_{\mathbb{R}^4} (-\Delta u + u)^2 dx \leq 1} \int_{\mathbb{R}^4} (e^{32\pi^2 u^2} - 1) dx < \infty,
\]
where $32\pi^2$ is the best constant. In order to apply this inequality to partial differential equation more reasonably, Y. Yang in \cite{25} proves the following singular Adams inequality:

**Theorem A.** Suppose $0 \leq \eta < 4$, $\tau, \sigma$ are two positive constants. Then
\[
\sup_{u \in W^{2,2}(\mathbb{R}^4), \int_{\mathbb{R}^4} (|\Delta u|^2 + \tau |\nabla u|^2 + \sigma u^2) dx \leq 1} \int_{\mathbb{R}^4} \frac{e^{\alpha u^2} - 1}{|x|^\eta} dx < \infty,
\]
where $\alpha \leq 32\pi^2(1 - \frac{\eta}{4})$ is the sharp constant. If $\alpha > 32\pi^2(1 - \frac{\eta}{4})$, then the supremum is infinite.

In \cite{15}, Li and Yang studied the following Schrödinger-Kirchhoff type equation
\[
\begin{cases}
\left( \int_{\mathbb{R}^N} (|\nabla u|^N + V(x)|u|^N) dx \right)^k (-\Delta_N u + V(x)|u|^{N-2} u) = \lambda A(x)|u|^{p-2} u + f(u), \\
u \in W^{1,N}(\mathbb{R}^N),
\end{cases}
\] (1.4)
where $\Delta_N u = \text{div}(|\nabla u|^{N-2}\nabla u)$, $k > 0$, $V : \mathbb{R}^N \to (0, \infty)$ is continuous, $\lambda > 0$ is a real parameter, $A$ is a positive function in $L^{\frac{p}{p-q}}(\mathbb{R}^N)$ and $f$ satisfies exponential growth. They derived two nontrivial solutions of (1.4) as the parameter $\lambda$ small enough. Indeed, suppose $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, the above problems is related to the stationary analogue of the equation
\[
u_{tt} - (a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = f(x, u)
\] (1.5)
proposed by Kirchhoff in \cite{17} as an extension of the classical D’Alembert’s wave equation for free vibrations of elastic strings. In \cite{18}, Lions proposed an abstract framework for the problem and after that, problem (1.4) began to receive a lot of attention. In \cite{16}, the authors studied the following Schrödinger-Kirchhoff type equation
\[
M \left( \int_{\mathbb{R}^2} |\nabla u|^2 + V(x)u^2 dx \right) (-\Delta u + V(x)u) = A(x)f(u) \quad \text{in} \quad \mathbb{R}^2
\] (1.6)
where $M$ is a Kirchoff-type function and $V(x) \geq V_0$ is a continuous function, $A$ is locally bounded and the function $f$ has critical exponential growth. Applying variational methods beside a new Trudinger-Moser type inequality, they get the existence of ground state solution. Moreover, in the local case $M \equiv 1$, they also get some relevant results.

In this paper, we consider the following singular biharmonic Kirchhoff-Schrödinger problem

\[ M \left( \int_{\mathbb{R}^4} |\Delta u|^2 + V(x)u^2 \, dx \right) (\Delta^2 u + V(x)u) = \frac{f(x,u)}{|x|^{\eta}} \quad \text{in} \quad \mathbb{R}^4, \quad (P_\eta) \]

where $0 < \eta < 4$, $M$ is a Kirchoff-type function and $V(x) \geq V_0$ is a continuous function, $f(x,t)$ has an critical exponential growth behavior at infinity. Using singular Adams inequality and variational techniques, we get the existence of ground state solutions for $(P_\eta)$.

Let $\mathfrak{M}(t) = \int_0^t M(s)ds$, we assume that $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with $M(0) = 0$, and satisfies

(M1) $M_0 = \inf_{t \geq 0} M(t) > 0$;

(M2) for any $t_1, t_2 \geq 0$, it holds

\[ \mathfrak{M}(t_1 + t_2) \geq \mathfrak{M}(t_1) + \mathfrak{M}(t_2); \]

(M3) $\frac{M(t)}{t}$ is decreasing in $(0, \infty)$;

**Remark 1.1.** By (M3), we can obtain that $2\mathfrak{M}(t) - M(t)t$ is nondecreasing for $t > 0$, In particular,

\[ 2\mathfrak{M}(t) - M(t)t \geq 0 \quad \forall t \geq 0. \tag{1.7} \]

we require that $f(x,t) = 0$ for all $(x,t) \in \mathbb{R}^4 \times (-\infty, 0]$. Furthermore, we assume the function $f$ satisfying:

(f1) $f$ is a continuous function and $f(x,t) > 0$ for all $t > 0$.

(f2) There exist constants $\alpha_0, c_1, c_2 > 0$ such that for all $(x,t) \in \mathbb{R}^4 \times \mathbb{R}^+$,

\[ f(x,t) \leq c_1|t|^3 + c_2(e^{\alpha_0 t^2} - 1), \]

(f3) There exist $R_0 > 0$ and $\mu > 4$, for any $x \in \Omega$,

\[ \mu F(x,t) \leq tf(x,t) \quad \forall |t| \geq R_0. \tag{1.8} \]
where $F(x, t) = \int_0^\infty f(x, s)ds$. This is the so-called the Ambrosetti-Rabinowitz (AR) condition.

We also give the following conditions on the potential $V(x)$:

$$(V_1)$$ $V$ is a continuous function satisfying $V(x) \geq V_0 > 0$;

Define a function space

$$E = \{ u \in W^{2,2}(\mathbb{R}^4) : \int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|\nabla u|^2) dx < \infty \},$$

which be equipped with the norm

$$\|u\|_E = \left( \int_{\mathbb{R}^4} (|\Delta u|^2 + V(x)|\nabla u|^2) dx \right)^{\frac{1}{2}},$$

then the assumption $(V_1)$ implies $E$ is a reflexive Banach space. For any $p \geq 2$, we define

$$S_p = \inf_{u \in E \setminus \{0\}} \left( \frac{\|u\|^p}{\int_{\mathbb{R}^4} |u|^p |x|^\eta dx} \right)^{\frac{1}{p}},$$

and

$$\lambda_\eta = \inf_{u \in E \setminus \{0\}} \left( \frac{\|u\|^4}{\int_{\mathbb{R}^4} |u|^4 dx} \right).$$

The continuous embedding of $E \hookrightarrow W^{2,2}(\mathbb{R}^4) \hookrightarrow L^{2p}(\mathbb{R}^4)(p \geq 2)$ and Hölder inequality implies

$$\int_{\mathbb{R}^4} \frac{|u|^{2p}}{|x|^\eta} dx$$

$$\leq \int_{\{|x| > 1\}} |u|^{2p} dx + \left( \int_{\{|x| \leq 1\}} |u|^{2p'} dx \right)^{\frac{1}{p}} \left( \int_{\{|x| \leq 1\}} \frac{1}{|x|^\eta} dx \right)^{\frac{1}{p'}}$$

\leq C\|u\|^{2p}_E.$$

where $1/t + 1/t' = 1$ and $t > 1$ such that $\eta t < 4$. Thus we have $S_p > 0$. We now introduce the following two conditions.

$$(f_4) \limsup_{t \to 0^+} \frac{2F(x, t)}{t^3} < 2\mathfrak{M}(1)\lambda_\eta \quad \text{uniformly in } \mathbb{R}^4.$$

$$(f_5) \frac{f(x, t)}{t^3} \text{ is increasing in } t > 0.$$

Our main results can be stated as follows:

**Theorem 1.1.** Suppose $V$ satisfies $(V_1)$, $f$ satisfies $(f_1) - (f_5)$. Furthermore we assume
There exist constants $p > 4$ and $C_p$ such that for all $(x, t) \in \mathbb{R}^4 \times [0, \infty)$

$$f(x, t) \geq C_p t^{p-1},$$

where

$$C_p := \inf \left\{ C > 0 : p \mathcal{M}(t^2 S_p^2) - 2Ct^p < p \mathcal{M}((1 - \frac{\eta}{4})^3 32\pi^2 \frac{\alpha_0}{(32\pi^2 \alpha_0)}) \right\}.$$ 

Then the problem $(P_{\eta})$ has a nontrivial nonnegative ground state solution in $E$.

Now instead the condition $(f_3)$, we assume that $(f'_3)$ \lim_{|t| \to +\infty} F(x, t) \frac{F(x, t)}{|t|^4} = \infty$ uniformly on $x \in \mathbb{R}^4$. We derive the results without the Ambrosetti-Rabinowitz (AR) condition.

**Theorem 1.2.** Suppose $V$ satisfies $(V_1)$, $f$ satisfies $(f_1) - (f_2)$, $(f'_3)$ and $(f_4) - (f_6)$. Then the problem $(P_{\eta})$ possesses a positive ground state solution.

This paper is organized as follows: In Section 2, we give some preliminary results. In Section 3, we study the functionals and compactness analysis. In section 4, we prove Theorem 1.1. In section 5, we study the results without the Ambrosetti-Rabinowitz (AR) condition.

## 2 Preliminaries

In this section we will give some preliminaries for our use later.

**Lemma 2.1 (see [26])** Suppose $q \geq 2$ and $0 < s < 4$. Then $E$ can be compactly embedded into $L^q(\mathbb{R}^4, |x|^{-s}dx)$.

**Lemma 2.2.** Let $\beta > 0, 0 < \eta < 4$ and $\|u\|_E \leq T$ such that $\beta T^2 < 32\pi^2 (1 - \frac{\eta}{4})$ and $q > 2$, then

$$\int_{\mathbb{R}^4} \frac{e^{\beta|u|^2} - 1}{|x|^{\eta}} |u|^q dx \leq C(\beta) \|u\|_E^q.$$

**Proof.** Set $R(\beta, u) = e^{\beta u^2} - 1$, using the Hölder inequality, we have

$$\int_{\mathbb{R}^4} \frac{R(\beta, u)}{|x|^{\eta}} |u|^q dx \leq \int_{|u| \leq 1} \frac{R(\beta, u)}{|x|^{\eta}} |u|^q dx + \int_{|u| > 1} \frac{R(\beta, u)}{|x|^{\eta}} |u|^q dx$$

$$\leq R(\beta, 1) \int_{|u| \leq 1} |u|^q dx + \left( \int_{\mathbb{R}^4} \frac{R(p\beta, u)}{|x|^{\eta}} dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^4} \frac{|u|^{pq'}}{|x|^{\eta}} dx \right)^{\frac{1}{q'}}$$

$$\leq R(\beta, 1) \|u\|_E^q + \left( \int_{\mathbb{R}^4} \frac{R(p\beta, u)}{|x|^{\eta}} dx \right)^{\frac{1}{p}} \|u\|_E^q$$

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where the last inequality is a direct consequence of Lemma 2.1. Choosing \( p > 1 \) is sufficiently close 1 such that \( \beta p T^2 < 32\pi^2(1 - \frac{4}{p}) \), \( \frac{1}{p} + \frac{1}{p'} = 1 \), Then the result can be derived from Theorem A.

**Lemma 2.3.** If \((f_5)\) holds, then for all \( x \in \mathbb{R}^4 \), we have that \( H(x, t) = tf(x, t) - 4F(x, t) \) is increasing in \( t > 0 \).

**Proof.** Let \( 0 < t_1 < t_2 \) be fixed. It follows from \((f_5)\) that

\[
t_1 f(x, t_1) - 4F(x, t_1) < \frac{f(x, t_2)}{t_2^2} t_1^2 - 4F(x, t_2) + 4 \int_{t_1}^{t_2} f(x, s) ds. \tag{2.1}
\]

On the other hand,

\[
4 \int_{t_1}^{t_2} f(x, s) ds < 4 \frac{f(x, t_2)}{t_2^2} \int_{t_1}^{t_2} s^3 ds = \frac{f(x, t_2)}{t_2^2} (t_2^4 - t_1^4). \tag{2.2}
\]

From (2.1) and (2.2), we derive that

\[
t_1 f(x, t_1) - 4F(x, t_1) < t_2 f(x, t_2) - 4F(x, t_2).
\]

This completes the proof. \( \square \)

### 3 Mountain pass geometry and minimax estimates

We say that \( u \in E \) is a weak solution of problem \((P_\eta)\) if for all \( \phi \in E \),

\[
M(\|u\|_E^2) \int_{\mathbb{R}^4} (\Delta u \Delta \phi + V(x)u\phi) dx - \int_{\mathbb{R}^4} \frac{f(x, u)}{|x|^{\eta}} \phi dx = 0.
\]

Define the functional \( I : E \rightarrow \mathbb{R} \) by

\[
I(u) = \frac{1}{2} M(\|u\|_E^2) - \int_{\mathbb{R}^4} \frac{F(x, u)}{|x|^{\eta}} dx. \tag{3.1}
\]

where \( F(x, t) = \int_0^t f(x, s) ds \). \( I \) is well defined and \( I \in C^1(E, \mathbb{R}) \) thanks to the singular Adams inequality. A straightforward calculation shows that

\[
\langle I'(u), \phi \rangle = M(\|u\|_E^2) \int_{\mathbb{R}^4} (\Delta u \Delta \phi + V(x)u\phi) dx - \int_{\mathbb{R}^4} \frac{f(x, u)}{|x|^{\eta}} \phi dx, \tag{3.2}
\]

for all \( u, \phi \in E \), hence, a critical point of (3.2) is a weak solution of \((P)\).

**Lemma 3.1** Assume that \((f_2)\) and \((f_4)\) hold. Then there exists positive constants \( \delta \) and \( r \) such that
\[ I(u) \geq \delta \text{ for } \|u\|_E = r. \]

**Proof.** From \((f_4)\), there exist \(\sigma, \epsilon > 0\), such that if \(\|u\|_E \leq \epsilon\),
\[ F(x, u) \leq \frac{\mathfrak{M}(1)\lambda_\eta - \sigma}{2} |u|^4, \]
for all \(x \in \mathbb{R}^4\). On the other hand, using \((f_2)\) for each \(q > 4\), we have
\[ F(x, u) \leq \frac{c_1}{4}|u|^4 + c_2|u| (e^{\alpha_0 |u|^2} - 1) \leq C |u|^q (e^{\alpha_0 |u|^2} - 1) \]
for \(\|u\|_E \geq \epsilon\) and \(x \in \mathbb{R}^4\). Combining the above estimates, we obtain
\[ F(x, u) \leq \frac{\mathfrak{M}(1)\lambda_\eta - \sigma}{2} |u|^4 + C |u|^q (e^{\alpha_0 |u|^2} - 1) \]
for all \((x, u) \in \mathbb{R}^4 \times \mathbb{R}\). On the other hand, \((1.7)\) gives \(\mathfrak{M}(t) \geq \mathfrak{M}(1)t^2, t \in [0,1]\).

Fixed \(r > 0\) and \(\|u\|_E \leq r \leq 1\) such that \(\alpha_0 r^2 < 32\pi^2 (1 - \frac{r}{4})\), then Lemma 2.2 implies
\[ I(u) = \frac{1}{2} \mathfrak{M}(\|u\|^2_E) - \int_{\mathbb{R}^4} \frac{F(x, u)}{|x|^\eta} dx \geq \frac{\mathfrak{M}(1)}{2} \|u\|^4_E - \frac{\mathfrak{M}(1)\lambda_\eta - \sigma}{2} \int_{\mathbb{R}^4} \frac{|u|^4}{|x|^\eta} dx - C \int_{\mathbb{R}^4} \frac{|u|^q (e^{\alpha_0 |u|^2} - 1)}{|x|^\eta} dx \]
\[ \geq \frac{\mathfrak{M}(1)}{2} \|u\|^4_E - \frac{\mathfrak{M}(1)\lambda_\eta - \sigma}{2} \int_{\mathbb{R}^4} \frac{|u|^4}{|x|^\eta} dx - C \|u\|^q_E \]
\[ \geq \frac{\mathfrak{M}(1)}{2} \|u\|^4_E - \frac{\mathfrak{M}(1)\lambda_\eta - \sigma}{2\lambda_\eta} \|u\|^4_E - C \|u\|^q_E \]
\[ = \frac{\sigma}{2\lambda_\eta} \|u\|^4_E - C \|u\|^q_E. \]
Hence, \(I\) is bounded form below for \(\|u\|_E \leq r \leq 1\). Since \(\sigma > 0\) and \(q > 4\), we may choose sufficiently small \(r > 0\) such that
\[ \frac{\sigma}{2\lambda_\eta} r^4 - Cr^q \geq \frac{\sigma}{4\lambda_\eta} r^4, \]
we derive that
\[ I(u) \geq \frac{\sigma}{4\lambda_\eta} r^4 := \delta \text{ for } \|u\|_E = r. \]
This completes the proof. \(\square\)

**Lemma 3.2** Assume \((f_3)\) is satisfied. Then there exists \(e \in B_r^c(0)\) such that
\[ I(e) < \inf_{\|u\|_E = r} I(u), \]

where \( r \) are given in Lemma 3.1.

**Proof.** From (\( M_3 \)), we have \( \mathcal{M}(t) \leq \mathcal{M}(1)t^2, t \geq 1 \). Let \( u \in E \setminus \{0\}, u \geq 0 \) with compact support \( \Omega = \text{supp}(u) \) and \( \|u\| = 1 \), by (\( f_3 \)), for \( \mu > 4 \), there exists \( C_1, C_2 > 0 \) such that for all \((x, s) \in \Omega \times \mathbb{R}^+, \)

\[ F(x, s) \geq C_1 s^{\mu} - C_2. \]

Then
\[ I(tu) \leq \frac{\mathcal{M}(1)t^4}{2} \|u\|_E^4 - C_1 t^\mu \int_{\Omega} \frac{|u|^\mu}{|x|^\eta} dx + C_2|\Omega|, \]

which implies that \( I(tu) \to -\infty \) as \( t \to \infty \). Setting \( e = tu \) with \( t \) sufficiently large, we finish the proof of the lemma. \( \square \)

From Lemma 3.1, Lemma 3.2, we get a \((PS)_c\) sequence \( \{u_n\} \subset E \), i.e.

\[ I(u_n) \to c > 0 \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as} \quad n \to \infty, \quad (3.1) \]

where

\[ c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \quad (3.2) \]

and

\[ \Gamma =: \{\gamma \in C([0,1] : E) : \gamma(0) = 0, \gamma(1) = e\}. \]

**Lemma 3.3** Suppose (\( f_6 \)) is satisfied, then the level \( c \in \left(0, \frac{1}{2} \mathcal{M}(\frac{32\pi^2}{a_0} (1 - \frac{\eta}{4}))\right) \).

**Proof.** Firstly, we claim the best constant \( S_p \) can be obtained. In fact, since

\[ S_p = \inf_{u \in E \setminus \{0\}} \frac{\|u\|_E}{\left(\int_{\mathbb{R}^4} \frac{|u|^p}{|x|^\eta} dx\right)^{\frac{1}{p}}}, \]

we can choose \( u_n \) such that

\[ \int_{\mathbb{R}^4} \frac{|u_n|^p}{|x|^\eta} dx = 1 \quad \text{and} \quad \|u_n\|_E \to S_p \quad \text{as} \quad n \to \infty, \]

so \( u_n \) is bounded in \( E \). From Lemma 2.1, there exists \( u_0 \in E \) such that up to a subsequence \( u_n \to u_0 \) in \( E, u_n \to u_0 \) in \( L^p(\mathbb{R}^4, |x|^{-\eta} dx) \) and \( u_n(x) \to u_0(x) \) almost everywhere in \( \mathbb{R}^4 \). This implies

\[ \int_{\mathbb{R}^4} \frac{|u_0|^p}{|x|^\eta} dx = \lim_{n \to \infty} \int_{\mathbb{R}^4} \frac{|u_n|^p}{|x|^\eta} dx = 1. \]
We also have \( \|u_0\|_E \leq \lim_{n \to \infty} \|u_n\|_E = S_p \), thus \( \|u_0\|_E = S_p \). From the definition of \( c \), let \( \gamma : [0, 1] \to E, \gamma(t) = tt_0u \), where \( t_0 \) is a real number which satisfies \( I(t_0u_0) < 0 \), we have \( \gamma \in \Gamma \), and therefore

\[
c \leq \max_{t \in [0,1]} I(\gamma(t)) \leq \max_{t \geq 0} I(tu_0) = \max_{t \geq 0} \left( \frac{\mathcal{M}(t^2 S_p^2)}{2} - \int_{\mathbb{R}^4} \frac{F(x,tu_0)}{|x|^q} \, dx \right),
\]

by \((f'_6)\), we have

\[
c \leq \max_{t \geq 0} I(tu) = \max_{t \geq 0} \left( \frac{\mathcal{M}(t^2 S_p^2)}{2} - \frac{t^p}{p} C_p \right) < \frac{1}{2} \mathcal{M}(\frac{32 \pi^2}{\alpha_0} (1 - \frac{\eta}{4})).
\]

The proof of the lemma is completed. \( \square \)

Consider the Nehari manifold associated to the functional \( I \), that is,

\[
\mathcal{N} := \{u \in E \setminus \{0\} : I'(u)u = 0\}
\]

and \( c^* = \inf_{u \in \mathcal{N}} I(u) \).

**Lemma 3.4** Suppose \( M \) satisfies \((M_3)\), \( f \) satisfies \((f_5)\). Then \( c \leq c^* \).

**Proof.** Let \( u \in \mathcal{N} \), we define \( h : (0, +\infty) \to \mathbb{R} \) by \( h(t) = I(tu) \). We have that \( h \) is differentiable and

\[
h'(t) = I'(tu)u = M(t^2 \|u\|^2) t \|u\|^2 - \int_{\mathbb{R}^4} \frac{f(x,tu)}{|x|^q} \, dx, \quad \forall t \geq 0.
\]

From \( I'(u)u = 0 \), we get

\[
h'(t) = I'(tu)u - t^3 I'(u)u,
\]

so

\[
h'(t) = t^3 \|u\|_E^4 \left[ \frac{M(t^2 \|u\|^2)}{t^2 \|u\|_E^2} - \frac{M(\|u\|_E^2)}{\|u\|_E^2} \right]
\]

\[
+ t^3 \int_{\mathbb{R}^4} \left( \frac{f(x,u)}{u^3} - \frac{f(x,tu)}{(tu)^3} \right) u^4 \, dx,
\]

By \((M_3), (f_5)\), we conclude that \( h'(t) > 0 \) for \( 0 < t < 1 \) and \( h'(t) < 0 \) for \( t > 1 \). Thus, \( h(1) = \max_{t \geq 0} h(t) \), which means

\[
I(u) = \max_{t \geq 0} I(tu).
\]

From the above argument, we see that \( h'(t) < 0 \) is strongly decreasing in \( t \in (1, +\infty) \), so \( h(t) \to -\infty \) as \( t \to +\infty \). Now, define \( \gamma : [0, 1] \to E, \gamma(t) = tt_0u \), where \( t_0 \) is a real number which satisfies \( I(t_0u) < 0 \), we have \( \gamma \in \Gamma \), and therefore

\[
c \leq \max_{t \in [0,1]} I(\gamma(t)) \leq \max_{t \geq 0} I(tu) = I(u).
\]

Since \( u \in \mathcal{N} \) is arbitrary, we have \( c \leq c^* \). \( \square \)
4 The ground state solution

In this section, we consider the ground state solution. We first prove the following convergence results.

Lemma 4.1 Suppose \((V_1), (f_1) - (f_5)\) are satisfied, let \(\{u_n\}\) is an arbitrary \((PS)_c\) sequence, then there exists a subsequence of \(\{u_n\}\) (still denoted by \(\{u_n\}\)) and \(u \in E\) such that

\[
\begin{cases}
\frac{f(x,u_n)}{|x|^{\eta}} \to \frac{f(x,u)}{|x|^{\eta}} \quad \text{strongly in } L^1_{loc}(\mathbb{R}^4), \\
\frac{F(x,u_n)}{|x|^{\eta}} \to \frac{F(x,u)}{|x|^{\eta}} \quad \text{strongly in } L^1(\mathbb{R}^4),
\end{cases}
\]

Proof. Let \(\{u_n\} \subset E\) be an arbitrary \((PS)_c\) sequence of \(I\), i.e.

\[
I(u_n) \to c > 0 \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as} \quad n \to \infty.
\]  \hspace{1cm} (4.1)

We shall prove that the sequence \(\{u_n\}\) is bounded in \(E\). Indeed, since \(\mu > 4\), then

\[
c + o_n(1)\|u_n\|_E \geq I(u_n) - \frac{1}{\mu}\langle I'(u_n), u_n \rangle
\geq \frac{1}{2} \mathcal{M}(\|u_n\|_E^2) - \frac{1}{\mu} M(\|u_n\|_E^2)\|u_n\|_E^2 - \frac{1}{\mu} \int_{\mathbb{R}^4} \frac{\mu F(x,u_n) - f(x,u_n)u_n}{|x|^\eta} dx
\geq (\frac{1}{4} - \frac{1}{\mu}) M(\|u_n\|_E^2)\|u_n\|_E^2 - \frac{1}{\mu} \int_{\mathbb{R}^4} \frac{\mu F(x,u_n) - f(x,u_n)u_n}{|x|^\eta} dx
\geq (\frac{1}{4} - \frac{1}{\mu}) M_0\|u_n\|_E^2,
\]

which implies that \(\{u_n\}\) is bounded in \(E\). It then follows from (4.1) that

\[
\frac{f(x,u_n)u_n}{|x|^{\eta}} dx \leq C, \quad \frac{F(x,u_n)}{|x|^{\eta}} dx \leq C.
\]

By Lemma 2.1 of [11], we get

\[
\frac{f(x,u_n)}{|x|^{\eta}} \to \frac{f(x,u)}{|x|^{\eta}} \quad \text{strongly in } L^1_{loc}(\mathbb{R}^N). \hspace{1cm} (4.2)
\]

By \((f_2)\) and \((f_3)\), there exists \(C > 0\) such that

\[
F(x,u_n) \leq C_1|u_n|^4 + C_2 f(x,u_n).
\]

From Lemma 2.2 and generalized Lebesgue's dominated convergence theorem, arguing as Lemma 4.7 in [26], we can derive that

\[
\frac{F(x,u_n)}{|x|^{\eta}} \to \frac{F(x,u)}{|x|^{\eta}} \quad \text{strongly in } L^1(\mathbb{R}^4). \hspace{1cm} (4.3)
\]
This completes the proof of the lemma. □

**Lemma 4.2** Let \((M_1) - (M_3)\) and \((f_1) - (f_6)\) hold. Then the functional \(I\) satisfies the \((PS)_c\) condition.

**Proof.** By the process in proof of Lemma 4.1, we have that the \((PS)_c\) sequence \(\{u_n\}\) is bounded in \(E\). We claim that \(I(u) \geq 0\). Indeed, suppose by contradiction that \(I(u) < 0\). Then \(u \neq 0\), set \(r(t) := I(tu), t \geq 0\), we have \(r(0) = 0\) and \(r(1) < 0\). As the proof of Lemma 3.1, for \(t > 0\) small enough, it holds \(r(t) > 0\). So there exists \(t_0 \in (0, 1)\) such that

\[
r(t_0) = \max_{t \in [0,1]} r(t), \quad r'(t_0) = \langle I'(t_0u), u \rangle = 0,
\]

By Remark 1.1 and Lemma 2.3, we have

\[
c \leq I(t_0u) = I(t_0u) - \frac{1}{4} \langle I'(t_0u), u \rangle
\]

\[
= \frac{1}{2} \mathcal{M}(\|t_0u\|_E^2) - \frac{1}{4} M(\|t_0u\|_E^2) \|t_0u\|_E^2
\]

\[
+ \frac{1}{4} \int_{\mathbb{R}^4} \frac{f(x, t_0u)t_0u - 4F(x, t_0u)}{|x|^\gamma} \, dx
\]

\[
< \frac{1}{2} \mathcal{M}(\|u\|_E^2) - \frac{1}{4} M(\|u\|_E^2) \|u\|_E^2
\]

\[
+ \frac{1}{4} \int_{\mathbb{R}^4} \frac{f(x, u)u - 4F(x, u)}{|x|^\gamma} \, dx
\]

Furthermore, by the weak lower semicontinuity of the norm and Fatou’s Lemma, we have

\[
c < \liminf_{n \to \infty} \left( \frac{1}{2} \mathcal{M}(\|u_n\|_E^2) - \frac{1}{4} M(\|u_n\|_E^2) \|u_n\|_E^2 \right)
\]

\[
+ \frac{1}{4} \liminf_{n \to \infty} \int_{\mathbb{R}^4} \frac{f(x, u_n)u_n - 4F(x, u_n)}{|x|^\gamma} \, dx
\]

\[
\leq \liminf_{n \to \infty} \langle I'(u_n), u_n \rangle - \frac{1}{4} \langle I'(u_n), u_n \rangle = c.
\]

which is not impossible. Thus the claim is true. From the lower semi-continuity of the norm in \(E\), we have \(\|u\|_E \leq \lim_{n \to \infty} \|u_n\|_E\). Suppose, by contradiction, that \(\|u\|_E < \lim_{n \to \infty} \|u_n\|_E := \xi\). Set \(v_n := \frac{u_n \parallel u_n\|_E}{\xi}\) and \(v := \frac{u}{\xi}\), then \(v_n \rightharpoonup v\) weakly in \(E\).
\[\mathcal{M}(\xi^2) = \lim_{n \to \infty} \mathcal{M}(\|u_n\|_{L^2}) = \lim_{n \to \infty} (2I(u_n) + \int_{\mathbb{R}^4} \frac{F(x, u_n)}{|x|^{\eta}} \, dx)\]
\[= 2c + 2 \int_{\mathbb{R}^4} \frac{F(x, u)}{|x|^{\eta}} \, dx = 2c + \mathcal{M}(\|u\|^2_{L^2}) - 2I(u)\]
\[< \mathcal{M}((1 - \eta/4) \frac{32\pi^2}{\alpha_0}) + \mathcal{M}(\|u\|^2_{L^2})\]
\[\leq \mathcal{M}((1 - \eta/4) \frac{32\pi^2}{\alpha_0} + \|u\|^2_{L^2})\]

Here, we have used the condition (M2) in the last inequality. Since \(\mathcal{M}\) is increasing, it holds \(\xi^2 < (1 - \eta/4) \frac{32\pi^2}{\alpha_0} + \|u\|^2_{L^2}\). Notice that
\[\xi^2 = \frac{\xi^2 - \|u\|^2_{L^2}}{1 - \|v\|^2_{L^2}}.\]

Thus
\[\xi^2 = \frac{(1 - \eta/4) \frac{32\pi^2}{\alpha_0}}{1 - \|v\|^2_{L^2}}.\]

Choosing \(q > 1\) sufficiently close to 1 and \(\beta_0 > 0\) such that for large \(n\),
\[q\alpha_0 \|u_n\|^2_{L^2} \leq \beta_0 < \frac{(1 - \eta/4) \frac{32\pi^2}{\alpha_0}}{1 - \|v\|^2_{L^2}}.\]

From concentration compactness principle with singular Adams inequality, we have
\[\int_{\mathbb{R}^4} e^{q\alpha_0 u_n^2} \frac{1}{|x|^{\eta}} \, dx \leq \int_{\mathbb{R}^4} e^{\beta_0 |v|^2} \frac{1}{|x|^{\eta}} \, dx \leq C. \tag{4.4}\]

From (f2) and Hölder inequality, we have
\[\left| \int_{\mathbb{R}^4} \frac{f(x, u_n)(u_n - u)}{|x|^{\eta}} \, dx \right| \]
\[\leq c_1 \left( \int_{\mathbb{R}^4} \frac{|u_n|^4}{|x|^{\eta}} \, dx \right)^{1/4} \left( \int_{\mathbb{R}^4} \frac{|u_n - u|^4}{|x|^{\eta}} \, dx \right)^{1/4} \]
\[+ c_2 \left( \int_{\mathbb{R}^4} \frac{|u_n - u|^q}{|x|^{\eta}} \, dx \right)^{1/q} \left( \int_{\mathbb{R}^4} e^{q\alpha_0 u_n^2 - 1} \, dx \right)^{1/q}, \tag{4.5}\]

where \(\frac{1}{q} + \frac{1}{q'} = 1\). In view Lemma 2.1, combining (4.4) with (4.5), we obtain
\[\int_{\mathbb{R}^N} \frac{f(x, u_n)(u_n - u)}{|x|^{\eta}} \, dx \to 0. \tag{4.6}\]

Since \(I'(u_n)(u_n - u) \to 0\), we have
\[M(\|u_n\|^2_{L^2}) \int_{\mathbb{R}^4} (\Delta u_n \Delta (u_n - u) + V(x)u_n(u_n - u)) \, dx \to 0, \tag{4.7}\]

and \(\|v\|_{L^1} < 1\). From \(I(u) \geq 0\) and Lemma 4.1, we have
On the other hand, by \( u_n \rightarrow u \) in \( E \), we have
\[
M(||u_n||^2_E) \int_{\mathbb{R}^4} (\Delta u \Delta (u_n - u) + V(x)u(u_n - u)) \, dx \to 0. \tag{4.8}
\]

(4.7) minus (4.8), we can derive
\[
\lim_{n \to \infty} M(||u_n||^2_E)||u_n - u||^2_E = 0, \tag{4.9}
\]
which is in contradiction with the fact \( ||u||_E < \lim_{n \to \infty} ||u_n||_E := \xi \). Thus, we have
\[
||u||_E = \xi = \lim_{n \to \infty} ||u_n||_E. \]
Since \( \{u_n\} \) is bounded in \( E \), we can apply Brezis-Lieb lemma to obtain \( u_n \rightarrow u \) strongly in \( E \).

□

**The proof of Theorem 1.1.** Since \( I \in C^1(E, \mathbb{R}) \), by Lemma 4.2, we have \( I'(u) = 0 \) and \( I(u) = c \). Therefore, by the definition of \( c^* \) and \( c \leq c^* \), we know \( u \) is a ground state solution.

Next, we will show that \( u \) is nonzero. If \( u \equiv 0 \), since \( F(x, 0) = 0 \) for all \( x \in \mathbb{R}^4 \), from Lemma 3.4, we have
\[
\lim_{n \to \infty} \frac{1}{2} M(||u_n||^2_E)c < M \left( 1 - \frac{\eta}{4} \right) \frac{32\pi^2}{\alpha_0}, \tag{4.10}
\]
Thus, there exist some \( \epsilon_0 > 0 \) and \( n^* > 0 \) such that \( ||u_n||^2_E \leq (1 - \frac{\eta}{4}) \frac{32\pi^2}{\alpha_0} - \epsilon_0 \) for all \( n > n^* \). Choose \( q > 1 \) sufficiently close to 1 such that \( q\alpha_0 ||u_n||_E^2 \leq (1 - \eta/4)32\pi^2 - \epsilon_0 \alpha_0 \) for all \( n > n^* \). By \( (f_2) \), there holds
\[
|f(x, u_n)u_n| \leq c_1 ||u_n||^4 + c_2 ||u_n|| (e^{\alpha_0 u_n^2} - 1).
\]
Thus by using singular Adams inequality, we have
\[
\int_{\mathbb{R}^4} \frac{|f(x, u_n)u_n|}{|x|^\eta} \, dx \\
\leq c_1 \int_{\mathbb{R}^4} \frac{|u_n|^4}{|x|^\eta} \, dx + c_2 \int_{\mathbb{R}^4} \frac{|u_n|(e^{\alpha_0 ||u_n||^2} - 1)}{|x|^\eta} \, dx \\
\leq c_1 \int_{\mathbb{R}^4} \frac{|u_n|^4}{|x|^\eta} \, dx + c_2 \left( \int_{\mathbb{R}^4} \frac{e^{\alpha_0 ||u_n||^2} - 1}{|x|^\eta} \, dx \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^4} \frac{|u_n|^{q'}}{|x|^\eta} \, dx \right)^{\frac{1}{q'}} \\
\leq c_1 \int_{\mathbb{R}^4} \frac{|u_n|^4}{|x|^\eta} \, dx + C \left( \int_{\mathbb{R}^4} \frac{|u_n|^{q'}}{|x|^\eta} \, dx \right)^{\frac{1}{q'}} \rightarrow 0,
\]
here we have used Lemma 2.1 in the last estimate. From \( I'(u_n)u_n \to 0 \), we have
\[
\lim_{n \to \infty} M(||u_n||^2_E) ||u_n||^2_E = 0, \tag{4.11}
\]
From the condition \((M_1)\), we can get \(\|u_n\| \to 0\). Then \(I(u_n) \to 0\), which contradicts the fact that \(I(u_n) \to c > 0\), so \(u\) is nonzero. From \(I(u) = c > 0\), we know \(u\) is positive. This completes the proof of Theorem 1.1. \(\square\)

5 The ground state solution without the A-R condition

In this section, we instead the condition \((f_3)\), the nonlinear term satisfies the exponential growth but without satisfying the Ambrosetti-Rabinowitz condition, we assume that

\[
(f'_3) \quad \lim_{|t| \to +\infty} \frac{F(x,t)}{|t|^4} = \infty \text{ uniformly on } x \in \mathbb{R}^4, \text{ where } F(x,t) = \int_0^t f(x,s)ds.
\]

We will use a Cerami’s Mountain Pass Theorem which was introduced in\[20, 21\]. The detail is the following:

**Definition A.** Let \((E, \|\cdot\|_E)\) be a real Banach space with its dual space \((E^*, \|\cdot\|_{E^*})\). Suppose \(I \in C^1(E, \mathbb{R})\). For \(c \in \mathbb{R}\), we say that \(\{u_n\} \subset E\) a \((C)_c\) sequence of the functional \(I\), if

\[
I(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|_E)\|I'(u_n)\|_{E^*} \to 0 \quad \text{as} \quad n \to \infty.
\]

**Proposition A.** Let \((E, \|\cdot\|_E)\) be a real Banach space, \(I \in C^1(E, \mathbb{R})\), \(I(0) = 0\) and satisfies:

(i) there exists positive constants \(\delta\) and \(r\) such that

\[
I(u) \geq \delta \quad \text{for} \quad \|u\|_E = r
\]

and

(ii) there exists \(e \in E\) with \(\|e\|_E > r\) such that

\[
I(e) \leq 0.
\]

Define \(c\) by

\[
c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),
\]

where

\[
\Gamma = \{ \gamma \in C([0,1] : E) : \gamma(0) = 0, \gamma(1) = e \}.
\]

Then \(I\) possesses a \((C)_c\) sequence.
Firstly, we check the geometry of the functional $I$ under the weak condition. Secondly, the key to establish the results in previous sections is prove that the Cerami sequence is bounded. Once we will have proved this, the remaining parts are similar.

**Lemma 5.1.** Assume that $(V_1)$, $(f_2)$-$(f_4)$ hold. Then

(i) there exists positive constants $\delta$ and $r$ such that $$I(u) \geq \delta \text{ for } \|u\|_E = r.$$ 

(ii) there exists $e \in E$ with $\|e\|_E > r$ such that $$I(e) < \inf_{\|u\|_E = r} I(u),$$

**Proof.** The proof of (i) is similar as Lemma 3.1. From $(M_3)$, we have $\mathfrak{M}(t) \leq \mathfrak{M}(1)t^2, t \geq 1$. Let $u \in E \setminus \{0\}, u \geq 0$ with compact support $\Omega = \text{supp}(u)$, by $(f_2)$, for all $L$, there exists $d$ such that for all $(x, s) \in \Omega \times \mathbb{R}^+$,

$$F(x, s) \geq Ls^4 - d.$$  

Then

$$I(tu) \leq \frac{\mathfrak{M}(1)t^4}{2} \|u\|_E^4 - Lt^4 \int_{\Omega} \frac{|u|^4}{|x|^\eta} dx + O(1)$$

$$\leq t^4 \left( \frac{\mathfrak{M}(1)}{2} \|u\|_E^4 - L \int_{\Omega} \frac{|u|^4}{|x|^\eta} dx \right) + O(1).$$

Now choosing $L > \frac{\mathfrak{M}(1)\|u\|_E^4}{N \int_{\Omega} \frac{|u|^4}{|x|^\eta} dx}$, it implies that $I(tu) \rightarrow -\infty$ as $t \rightarrow \infty$. Setting $e = tu$ with $t$ sufficiently large, the proof of (ii) is completed. \hfill \Box

From Lemma 3.1, Lemma 5.1 and Proposition A, we get a $(C)_c$ sequence $\{u_n\} \subset E$, i.e.

$$I(u_n) \rightarrow c > 0 \text{ and } (1 + \|u_n\|_E)\|I'(u_n)\|_{E^*} \rightarrow 0 \text{ as } n \rightarrow \infty, \quad (5.1)$$

**Lemma 5.2.** Let $\{u_n\} \subset E$ be an arbitrary Cerami sequence of $I$, Then $\{u_n\}$ is bounded up to a subsequence.

**Proof.** Let $\{u_n\} \subset E$ be an arbitrary Cerami sequence of $I$, i.e.

$$\frac{\mathfrak{M}(\|u_n\|_E^2)}{2} - \int_{\mathbb{R}^4} \frac{F(x, u_n)}{|x|^\eta} dx \rightarrow c \text{ as } n \rightarrow \infty, \quad (5.2)$$
and
\[(1 + \|u_n\|_E)\langle I'(u_n), \varphi \rangle \leq \tau_n \|\varphi\|_E \quad \text{for all } \varphi \in E, \quad (5.3)\]

where \(\tau_n \to 0\) as \(n \to \infty\). We shall prove that the sequence \(\{u_n\}\) is bounded in \(E\).

Indeed, suppose by contradiction that \(\|u_n\|_E \to +\infty\) and set
\[v_n = \frac{u_n}{\|u_n\|_E},\]
then \(\|v_n\| = 1\). From Lemma 2.1, we can assume that for any \(q \geq 4\), there exists \(v \in E\) such that up to a subsequence
\[
\begin{cases}
  v_n^+ \to v^+ \quad \text{in } E, \\
  v_n^+ \to v^+ \quad \text{in } L^q(\mathbb{R}^4), \\
  v_n^+ \to v^+ \quad \text{a.e. in } \mathbb{R}^4.
\end{cases}
\]

We will show that \(v^+ = 0\) a.e. in \(\mathbb{R}^4\). In fact, if \(\Lambda^+ = \{x \in \mathbb{R}^4 : v^+(x) > 0\}\) has a positive measure, then in \(\Lambda^+\), we have
\[
\lim_{n \to \infty} u_n^+ = \lim_{n \to \infty} v_n^+ \|u_n\| = +\infty.
\]

From \((f'_3)\) we have
\[
\lim_{n \to \infty} \frac{F(x, u_n^+(x))}{|x|^\eta |u_n^+(x)|^4} = +\infty \quad \text{a.e. in } \Lambda^+,
\]
and
\[
\lim_{n \to \infty} \frac{F(x, u_n^+(x))}{|x|^\eta |u_n^+(x)|^4} |v_n^+(x)|^4 = +\infty \quad \text{a.e. in } \Lambda^+.
\]

Thus
\[
\int_{\mathbb{R}^4} \liminf_{n \to \infty} \frac{F(x, u_n^+(x))}{|x|^\eta |u_n^+(x)|^4} |v_n^+(x)|^4 \, dx = +\infty.
\]

Since \(\{u_n\} \subset E\) be an arbitrary Cerami sequence of \(I\), we have
\[
\mathcal{M}(\|u_n\|_E^2) = 2c + 2 \int_{\mathbb{R}^4} \frac{F(x, u_n^+(x))}{|x|^\eta} \, dx + o_n(1)
\]
Since $\mathcal{M}$ is increasing, it holds
\[ \int_{\mathbb{R}^4} \frac{F(x, u_n^+(x))}{|x|^{\eta}} dx \to +\infty. \]

From (M$_3$), we have $\mathcal{M}(t) \leq \mathcal{M}(1)t^2$, $t \geq 1$. Thus
\[ \liminf_{n \to \infty} \int_{\mathbb{R}^4} \frac{F(x, u_n^+(x))}{|x|^\eta u_n^+} |v_n^+|^4 dx \]
\[ = \liminf_{n \to \infty} \int_{\mathbb{R}^4} \frac{F(x, u_n^+(x))}{|x|^\eta} dx \]
\[ \leq \liminf_{n \to \infty} \int_{\mathbb{R}^4} \frac{\mathcal{M}(1)F(x, u_n^+(x))}{|x|^\eta \mathcal{M}(\|u_n\|_E^2)} dx \]
\[ = \liminf_{n \to \infty} \frac{\int_{\mathbb{R}^4} \frac{F(x, u_n^+(x))}{|x|^\eta} dx}{2c + 2 \int_{\mathbb{R}^4} \frac{F(x, u_n^+(x))}{a(x)} dx + o_n(1)} \]
\[ = \frac{1}{2}. \]

This is a contradiction. Hence $v \leq 0$ a.e. and $v_n^+ \rightharpoonup 0$ in $E$.

Let $t_n \in [0, 1]$ be such that
\[ I(t_n u_n) = \max_{t \in [0, 1]} I(t u_n). \]

For any given $A \in \left(0, \left(1 - \frac{\eta}{4}\right)^{\frac{32\pi^2}{\alpha_0}}\right)$, for the sake of simplicity, let
\[ \epsilon = \frac{(1 - \frac{\eta}{4})32\pi^2}{A^2} - \alpha_0 > 0. \]

In the following argument we will take $A \to \left(1 - \frac{\eta}{4}\right)^{\frac{32\pi^2}{\alpha_0}}\frac{1}{2}$ and so we have $\epsilon \to 0$.

By condition (f$_2$), there exists $C > 0$ such that
\[ F(x, t) \leq C|t|^4 + \epsilon R(\alpha_0 + \epsilon, |t|), \quad \forall (x, t) \in \mathbb{R}^4 \times \mathbb{R}^+, \quad (5.4) \]
where $R(\alpha, s) = e^{\alpha s^2} - 1$. In fact, from condition (f$_2$), there holds
\[ F(x, t) \leq \frac{C}{N}|t|^4 + |t|R(\alpha_0, |t|). \]

By using Young inequality, for $\frac{1}{p} + \frac{1}{q} = 1$, $p, q > 1$, there holds
\[ ab \leq \epsilon \frac{a^p}{p} + \epsilon^{-q/p} \frac{b^q}{q}. \]
So we have

\[ F(x, t) \leq \frac{C}{N} |t|^4 + \frac{\epsilon R(\alpha_0, |t|)^p}{p} + \epsilon^{-q/p} \frac{|t|^q}{q}. \]

Now we take \( p = \frac{\alpha_0 + \epsilon}{\alpha_0} \) and \( q = \frac{\alpha_0 + \epsilon}{\epsilon} > 4. \) One can see that near infinity \( |t|^q \) can be estimated from above by \( R(\alpha_0 + \epsilon, |t|) \), and near the origin \( |t|^q \) can be estimated from above by \( |t|^4 \), thus we obtain (5.4). We also have \( \frac{A}{\|u_n\|} \in (0, 1] \) with sufficient large \( n \), so by using (5.3), we have

\[
I(t_n u_n) \geq I\left( \frac{A}{\|u_n\|} u_n \right) = I(A v_n) = \frac{M(A^2)}{2} - \int_{\mathbb{R}^4} \frac{F(x, A v_n)}{|x|^\eta} \, dx
\]

\[
\geq \frac{M(A^2)}{N} - C A^4 \int_{\mathbb{R}^4} \frac{|v_n|^4}{|x|^\eta} \, dx - \epsilon \int_{\mathbb{R}^4} \frac{R(\alpha_0 + \epsilon, A v_n)}{|x|^\eta} \, dx
\]

\[
\geq \frac{M(A^2)}{2} - C A^4 \int_{\mathbb{R}^4} \frac{|v_n|^4}{|x|^\eta} \, dx - \epsilon \int_{\mathbb{R}^4} \frac{R((\alpha_0 + \epsilon) A^2, v_n)}{|x|^\eta} \, dx
\]

\[
\geq \frac{M(A^2)}{2} - C A^4 \int_{\mathbb{R}^4} \frac{|v_n|^4}{|x|^\eta} \, dx - \epsilon \int_{\mathbb{R}^4} \frac{R((1 - \frac{\eta}{4}) 32 \pi^2, v_n)}{|x|^\eta} \, dx.
\]

Since \( v_n^+ \to 0 \) in \( E \) and the embedding \( E \hookrightarrow L^q(\mathbb{R}^4, |x|^{-\eta} dx)(q \geq 4) \) is compact, by using the Hölder inequality, we have \( \int_{\mathbb{R}^4} \frac{|\psi|^2}{|x|^\eta} \, dx \to 0. \) By singular Trudinger-Moser inequality, \( \int_{\mathbb{R}^4} \frac{R((1 - \frac{\eta}{4}) 32 \pi^2, v_n)}{|x|^\eta} \, dx \) is bounded. When \( A \to ((1 - \frac{\eta}{4}) 32 \pi^2, \alpha_0)^{\frac{1}{2}} \), we can show

\[
\lim_{n \to \infty} I(t_n u_n) \geq \frac{1}{2} M\left( (1 - \frac{\eta}{4}) \frac{32 \pi^2}{\alpha_0} \right) > c. \tag{5.5}
\]

Since \( I(0) = 0 \) and \( I(u_n) \to c, \) we can assume \( t_n \in (0, 1), \) and so \( I'(t_n u_n) t_n u_n = 0, \) it follows from \((f_5)\),

\[
4I(t_n u_n) = 4I(t_n u_n) - I'(t_n u_n) t_n u_n
\]

\[
= 2M(\|t_n u_n\|^2) - 4 \int_{\mathbb{R}^4} \frac{F(x, t_n u_n)}{|x|^\eta} \, dx
\]

\[
- M(\|t_n u_n\|^2) \|t_n u_n\|^2 + \int_{\mathbb{R}^4} \frac{f(x, t_n u_n) t_n u_n}{|x|^\eta} \, dx
\]

\[
= 2M(\|t_n u_n\|^2) - M(\|t_n u_n\|^2) \|t_n u_n\|^2 + \int_{\mathbb{R}^4} \frac{H(x, t_n u_n)}{|x|^\eta} \, dx
\]

\[
\leq 2M(\|u_n\|^2) - M(\|u_n\|^2) \|t_n u_n\|^2 + \int_{\mathbb{R}^4} \frac{H(x, u_n)}{|x|^\eta} \, dx
\]

\[
= 4I(u_n) - I'(u_n) u_n
\]

\[
= 4I(u_n) + o_n(1) = 4c + o_n(1),
\]
which is a contradiction to (5.5). This proves that \( \{u_n\} \) is bounded in \( E \). \( \square \)

**Proof of Theorem 1.2.** From Lemma 5.2, we have that the Cerami sequence \( \{u_n\} \) is bounded in \( E \). Applying the same procedure in proof of Theorem 1.1, we will derive that \( I'(u) = 0 \) and \( I(u) = c \). Moreover, we also get that \( u \) is nonzero and \( u \) is ground state. \( \square \)

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