ON DOUBLE VERONESE EMBEDDINGS IN THE GRASSMANNIAN $G(1, N)$

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Abstract. We classify all the embeddings of $\mathbb{P}^n$ in a Grassmannian of lines $G(1, N)$ such that the composition with Plücker is given by a linear system of quadrics of $\mathbb{P}^n$.

Introduction

There exists a well known Hartshorne’s Conjecture saying that a smooth $r$-dimensional variety $X \subset \mathbb{P}^n$ is a complete intersection if $n < \frac{3r}{2}$. A weaker statement says that $X \subset \mathbb{P}^n$ is a c.i. if $X$ has codimension 2 and $n \geq 7$ (see [5]). According to a theorem of Serre, every codimension 2 smooth subvariety of $\mathbb{P}^n$, for $n \geq 6$, can be given as the zero locus of a section of a rank-2 vector bundle on $\mathbb{P}^n$. Therefore the weaker Hartshorne’s conjecture is equivalent to prove that every rank-2 bundle on $\mathbb{P}^n$ splits if $n \geq 7$ (indeed it is conjectured to be true for every $n \geq 5$). Moreover, to give a rank-2 bundle $\mathcal{E}$ on $\mathbb{P}^n$ together with an epimorphism $\mathcal{O}_{\mathbb{P}^n}^{N+1} \to \mathcal{E}$ is equivalent to give a map from $\mathbb{P}^n$ to a Grassmannian $G(1, N)$. If we tensor $\mathcal{E}$ with the line bundle $\mathcal{O}_{\mathbb{P}^n}(m)$, for a suitable $m \gg 0$, we get that the map associated to the new rank-2 bundle is an embedding (see [4]). Therefore in order to prove Hartshorne’s conjecture it is enough to consider bundles giving an embedding of $\mathbb{P}^n$ in $G(1, N)$. When $\det(\mathcal{E}) = 1$, there are only two embeddings of $\mathbb{P}^n$ in the Grassmannian $G(1, N)$ such that the composition with the Plücker embedding of $G(1, N)$ gives rise to a linear space. If $n \geq 3$, the only way is to take the rank-2 bundle $\mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}$. Let us consider a further step, i.e. rank-2 vector bundles $\mathcal{E}$ that give an embedding of $\mathbb{P}^n$ in a Grassmannian $G(1, N)$ and such that $\det(\mathcal{E}) = 2$ (i.e. the composition with Plücker corresponds to the line bundle $\mathcal{O}_{\mathbb{P}^n}(2)$). In this paper we classify all such vector bundles obtaining as a corollary that if $n \geq 4$ then $\mathcal{E}$ splits.

The paper is organized as follows. In the first section we recall some definitions and results on embeddings in Grassmannians and on vector bundles on $\mathbb{P}^n$. In Section 2 we give some examples of double Veronese embeddings of $\mathbb{P}^n$ in $G(1, N)$ and we prove two lemmas (concerning the cases $n = 1$ and 2) that we will use in Section 3 in order to prove our classification theorem.

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1. Preliminaries

Notation. Let us take an element of the Grassmannian $G(1, N)$; throughout the paper we will use the following notations:

i) a small letter $l$, if we refer to it as a point of $G(1, N)$;

ii) a capital letter $L$, if we consider it as a subspace of $\mathbb{P}^N$.

Definitions. A non-degenerate variety $X \subset \mathbb{P}^N$ (i.e. not contained in a $\mathbb{P}^{N-1}$) is said to be projective linearly normal if $X$ is not projected from any non-degenerate variety contained in a bigger projective space, i.e. if $h^0(X, \mathcal{O}_X(1)) = N + 1$. 


A non-degenerate variety $X \subset G(1, N)$ (i.e. not contained in a $G(1, N - 1)$) is said to be Grassmannian linearly normal if $X$ is not projected from any non-degenerate variety contained in a bigger Grassmannian of lines, i.e. if $h^0(X, Q|X) = N + 1$ (where $Q$ is the universal quotient bundle of $G(1, N)$ and $Q|X$ denotes its restriction $Q \otimes O_X$ to $X$).

Let us recall some general facts about embeddings in Grassmannians of lines and Plücker embedding (for a detailed description see for instance [2]).

Giving a non-degenerate map $\varphi : X \to G(1, N)$ is equivalent to giving a rank-2 vector bundle $\mathcal{E}$ and an epimorphism $\phi : V \otimes O_X \to \mathcal{E}$, where $V$ is an $(N + 1)$-dimensional subspace of $H^0(X, \mathcal{E})$. In this situation, $\mathcal{E} \cong Q|X$. Moreover $\varphi$ is an embedding if different points of $X$ (maybe infinitely close) are mapped to different lines, i.e. if any subscheme of length two of $X$ imposes at least three conditions to $V$.

Let us consider the embedding $X \hookrightarrow \mathbb{P}^M$, where $M = \binom{N + 1}{2} - 1$, composition of the Plücker embedding and $\varphi$. If the vector space $V$ is the whole $H^0(X, \mathcal{E})$, $X$ is Grassmannian linearly normal and $\bar{\varphi}$ is given by $M + 1$ sections of the line bundle $\wedge^2 \mathcal{E}$. Note that $X$ can be very degenerate in $\mathbb{P}^M$, since these sections are not necessarily independent, so we will always consider $X$ contained in its linear span $(X) \cong \mathbb{P}^r \subset \mathbb{P}^M$, where $r + 1$ is the maximal number of independent sections of $\wedge^2 \mathcal{E}$. In general, $X$ is not necessarily projective linearly normal in $(X)$.

**Definition.** Let us take a variety $X \subset G(1, N)$, image of $\mathbb{P}^n$ via an embedding $\varphi$, such that the composition $\bar{\varphi}$ with the Plücker embedding is a (maybe degenerate and not necessarily projectively linearly normal) double Veronese embedding $v_2(\mathbb{P}^n)$ (i.e. $\wedge^2 \mathcal{E}$ coincides with $O_{\mathbb{P}^n}(2)$). Throughout the paper we will say that $X$ is a double Veronese embedding of $\mathbb{P}^n$ in $G(1, N)$.

We are now going to recall some definitions and state some known results about vector bundles on $\mathbb{P}^n$ (for a detailed description see [9]).

Let $\mathcal{E}$ be a rank $r$ bundle on $\mathbb{P}^n$. According to a theorem of Grothendieck, for every $l \in G(1, n)$ there is an $r$-tuple

$$a_{\mathcal{E}}(l) = (a_1(l), \ldots, a_r(l)) \in \mathbb{Z}^r,$$

with $a_1(l) \geq a_2(l) \geq \ldots \geq a_r(l)$, such that

$$\mathcal{E}|_L = \mathcal{E} \otimes O_L = \bigoplus_{i=1}^{r} O_{\mathbb{P}^1}(a_i(l)).$$

In this way can be defined a map

$$a_{\mathcal{E}} : G(1, n) \to \mathbb{Z}^r.$$

**Definitions.** The $r$-tuple $a_{\mathcal{E}}(l)$ is called the splitting type of $\mathcal{E}$ on $L$.

The bundle $\mathcal{E}$ is defined to be uniform if $a_{\mathcal{E}}$ is constant.

Let us now give $\mathbb{Z}^r$ the lexicographical order, i.e. $(a_1, \ldots, a_r) \leq (b_1, \ldots, b_r)$ if the first non-zero difference $b_i - a_i$ is positive. We put

$$a_\varnothing = \inf_{l \in G(1, n)} a_{\mathcal{E}}(l).$$

The $r$-tuple $a_{\mathcal{E}}$ is called the generic splitting type of $\mathcal{E}$.

A line $l \in G(1, n)$ is called a jumping line if $a_{\mathcal{E}}(l) > a_\varnothing$. The set of jumping lines turns out to be a proper closed subset of the Grassmannian $G(1, n)$ (see [9]).

**Theorem 1.1.** Let $\mathcal{E}$ be a uniform rank 2 vector bundle on $\mathbb{P}^n$. Then either $\mathcal{E}$ splits, or $n = 2$ and $\mathcal{E}$ is a twist of the tangent bundle by some line bundle.

**Proof.** See [8].

Theorem 1.2. Let $E$ be a rank $r$ vector bundle over $\mathbb{P}^n$, $x \in \mathbb{P}^n$ a point such that $E|_L = \mathcal{O}_{\mathbb{P}^n}(a)^{\oplus r}$ for each line $L$ through $x$. Then $E = \mathcal{O}_{\mathbb{P}^n}(a)^{\oplus r}$.

Proof. We just have to apply [6, Theorem 3.2.1] to the bundle $E' = E \otimes \mathcal{O}_{\mathbb{P}^n}(-a)$. □

Theorem 1.3. A vector bundle $E$ over $\mathbb{P}^n$ splits exactly when its restriction to some plane $\Pi \subset \mathbb{P}^n$ splits.

Proof. See [6, Theorem 2.3.2]. □

2. Examples

Let us now give some examples of Grassmannian linearly normal double Veronese embeddings $\mathbb{P}^n \overset{\phi}{\rightarrow} G(1, N)$. These will be the examples appearing in the statements of our main results.

Example 1. The rank-2 bundle $E = \mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$ gives an embedding $X$ of $\mathbb{P}^n$ in the Grassmannian $G(1, 2n+1)$ as the family of lines joining the corresponding points on two disjoint $\mathbb{P}^n$’s. This is a double Veronese embedding, since $\wedge^2 E = \mathcal{O}_{\mathbb{P}^n}(2)$.

Example 2. The rank 2 bundle $E = \mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}$ gives an embedding $X$ of $\mathbb{P}^n$ in the Grassmannian $G(1, N)$, with $N = \binom{n+2}{2}$. This is the family of ruling lines of a cone over the double Veronese embedding $v_2(\mathbb{P}^n) \subset \mathbb{P}^{N-1}$, with vertex a point. Again $\wedge^2 E = \mathcal{O}_{\mathbb{P}^n}(2)$, and hence $X$ is a double Veronese embedding of $\mathbb{P}^n$.

Example 3. The family of the bisecant lines to a rational normal cubic is a double Veronese embedding of $\mathbb{P}^2$ in $G(1, 3)$ (see [3]). In this case the bundle $E$ is a Steiner bundle, i.e. it is the dual of the kernel of a map $\mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \rightarrow \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, corresponding to the choice of 4 general sections of the bundle $\mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$. From the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\oplus 4} \rightarrow E \rightarrow 0,$$

we get that $h^0(\mathbb{P}^2, E) = 4$, and hence the surface is Grassmannian linearly normal.

Example 4. The family $X$ of lines contained in a smooth hyperquadric $Q \subset \mathbb{P}^4$ is a double Veronese embedding of $\mathbb{P}^3$ in $G(1, 4)$ (see [7]). The vector bundle $E$ is the cokernel of a map $\mathcal{O}_{\mathbb{P}^3} \rightarrow \Omega_{\mathbb{P}^3}(2)$ corresponding to the choice of a general section of the twist of the cotangent bundle $\Omega_{\mathbb{P}^3}$ by $\mathcal{O}_{\mathbb{P}^3}(1)$. From the sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3} \rightarrow \Omega_{\mathbb{P}^3}(2) \rightarrow E \rightarrow 0,$$

and the Euler sequence of $\mathbb{P}^3$ we get that $h^0(\mathbb{P}^3, E) = 5$ and hence $X$ is Grassmannian linearly normal in $G(1, 4)$.

Example 5. Taking the restriction of the embedding above to a general plane in $\mathbb{P}^3$ we get a double Veronese embedding of $\mathbb{P}^2$ in $G(1, 4)$. Geometrically speaking it is the set of lines contained in $Q$ and meeting a fixed line in it. It is again Grassmannian linearly normal by the same reason of Example 2.

Remark 1. We recall that the variety given by the bundle $\mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$ can be isomorphically projected from $G(1, 2n+1)$ to $G(1, m)$, for $n+1 \leq m \leq 2n$ (see [11]). The variety given by $\mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}$ can be isomorphically projected from $G(1, N)$ to $G(1, m)$, for $2n+1 \leq m \leq N-1$, since the Veronese variety $v_2(\mathbb{P}^n)$ can be projected from $\mathbb{P}^{N-1}$ to $\mathbb{P}^{m-1}$.

Conversely, varieties of Examples 3, 4 and 5 cannot be isomorphically projected to a smaller Grassmannian. This claim is obvious for Example 3 while for Examples 4 and 5 it is enough to realise that through the general point of $\mathbb{P}^4$ there pass a 2-dimensional family of planes intersecting $Q$ along a degenerate conic, i.e. two lines. These two lines are projected to the same line, giving rise to a singularity.
We distinguish two different cases depending on the codimension of $J$.

**Lemma 1.** The only Grassmannian linearly normal double Veronese embeddings of $\mathbb{P}^1$ in a Grassmannian of lines are as in Examples 4 or 5.

**Proof.** Let us denote by $X$ the image of $\mathbb{P}^1$ via the double Veronese embedding $\varphi : \mathbb{P}^1 \to G(1, N)$, corresponding to the bundle $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^1}(\beta)$. Since $\varphi$ is an embedding, we must have $\alpha, \beta \geq 0$. Moreover, by definition it must be $\wedge^2 \mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(2)$, which implies $\alpha + \beta = 2$. Therefore the only two possibilities are either $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ (corresponding to Example 4) or $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}$ (corresponding to Example 5). We remark that in the first case $X$ is one of the rulings of a smooth quadric surface (and can be projected to $G(1, 2)$ as the tangent lines to a smooth conic). In the second case $X$ corresponds to the ruling lines of a cone over a smooth conic. □

**Definitions.** Given a surface $S \subset G(1, N)$, in the Chow ring of $G(1, N)$ we can write $[S] = a\Omega(0, 3) + b\Omega(1, 2)$, where $a = [S] \cdot \Omega(N - 3, N)$ is the order of $S$ and $b = [S] \cdot \Omega(N - 2, N - 1)$ is its class.

The pair $(a, b)$ is defined to be the bidegree of $S$.

The degree of $S$ via Plücker embedding turns out to be $a + b$.

**Proposition 2.1.** The only Grassmannian linearly normal double Veronese embeddings of $\mathbb{P}^2$ in a Grassmannian of lines are as in Examples 4, 5, 6 or 7.

**Proof.** Let us denote by $\mathcal{E}$ the rank-2 bundle giving the embedding of $X = \mathbb{P}^2$ in a Grassmannian of lines $G(1, N)$, i.e. $\mathcal{E} \cong \mathcal{Q}|_X$, and consider its restriction $\mathcal{E}|_L = \mathcal{E} \otimes \mathcal{O}_L$ to a general line $L \subset \mathbb{P}^2$. We remark that $\mathcal{E}|_L$ gives a double Veronese embedding of $L$ since it is the restriction of the embedding given by $\mathcal{E}$. Hence by Lemma 4 either $\mathcal{E}|_L = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}$ or $\mathcal{E}|_L = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$, corresponding to $a_{\mathcal{E}} = (2, 0)$ or $(1, 1)$ respectively.

If the generic splitting type $a_{\mathcal{E}}$ is $(2, 0)$, then there are no jumping lines and $\mathcal{E}$ must be uniform. Hence, by Theorem 4, either $\mathcal{E}$ splits, or $\mathcal{E}$ is the twist by a line bundle of the tangent bundle. But this last possibility cannot occur, since in this case the line bundle $\wedge^2 \mathcal{E}$ would have odd degree and hence it could not give the double Veronese embedding of $\mathbb{P}^2$. Therefore, since $\mathcal{E}$ is decomposable and $a_{\mathcal{E}} = (2, 0)$, it must be $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}$. Moreover we assume $X$ Grassmannian linearly normal, so we take $V$ to be the whole $H^0(\mathbb{P}^2, \mathcal{E})$, corresponding to the double Veronese embedding of Example 6 for $n = 2$.

If $a_{\mathcal{E}} = (1, 1)$, for a jumping line $L$ we must have $\mathcal{E}|_L = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}$, which is equivalent to say that $L$ is embedded in a $G(1, 3) \subseteq G(1, N)$ as the rulings of a quadratic cone. Let us denote by $\mathcal{J} \subset G(1, 2)$ the closed set of jumping lines of $\mathcal{E}$.

We distinguish two different cases depending on the codimension of $\mathcal{J}$.

i) $\text{codim} \mathcal{J} \geq 2$. In this case there are at most finitely many jumping lines for $\mathcal{E}$. In particular, through a general point $x \in \mathbb{P}^2$ there pass no jumping lines, which means that $\mathcal{E}|_L = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ for each $L$ through $x$. By Theorem 4, we get that $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(1) \oplus \mathcal{O}_{\mathbb{P}^2}(1)$ and hence $\mathcal{E}$ gives the double Veronese embedding of Example 4 when we consider all the sections of $H^0(\mathbb{P}^2, \mathcal{E})$.

ii) $\text{codim} \mathcal{J} = 1$. For every irreducible maximal component of $\mathcal{J} \subset G(1, 2)$, there exists a fundamental curve $C \subset \mathbb{P}^N$, i.e. a curve which is cut by all the lines of the surface. The points of $C$ are the vertices of the quadratic cones.

The number of fundamental points of $C$ contained in a general line of $X$
cannot be bigger than two by the classical trisecant lemma, so there are just two possibilities: either \( \mathbb{P}^2 \) is embedded as the family of bisecant lines to the curve \( C \subset \mathbb{P}^N \), or \( C \) is a line of \( X \) contained in all the quadratic cones corresponding to the component of \( J \) cited above.

In the first case, the bisecant lines passing through a general point \( c \in C \) give rise to a quadratic cone (since they correspond to the embedding of a jumping line). This implies that \( C \) is contained in a \( \mathbb{P}^3 \) and that its projection from a general point on it is a smooth conic, and hence \( C \) must be a rational normal cubic of \( \mathbb{P}^3 \). In this way \( X \) turns out to be the Veronese surface of Example 3. Note that \( J \subset G(1,2) \) is a conic embedded as the family of tangent lines to a smooth conic of \( \mathbb{P}^2 \). This conic of \( \mathbb{P}^2 \) is embedded in \( G(1,3) \) as the tangent developable to the rational normal cubic \( C \).

If \( C \) is a line we claim that the bidegree \( (a, b) \) of the surface \( X \) must be \((2, 2)\). In fact, the class \( b \) of \( X \) is the number of lines contained in a general hyperplane \( H \) of \( \mathbb{P}^N \). But \( H \) contains exactly two lines, corresponding to the hyperplane section of the quadratic cone passing through \( H \cap R \) and hence \( b = 2 \). Moreover, since \( X \) is a Veronese surface, its degree is \( a + b = 4 \), which proves our claim. This also implies that the 3-fold \( X \) covered by the lines of \( X \) is either a hyperquadric \( Q \subset \mathbb{P}^4 \) or a \( \mathbb{P}^3 \). The later case is not possible since there are no Veronese surfaces of bidegree \((2, 2)\) in \( G(1,3) \) (see [3]).

Let us see that \( Q \) is smooth. If \( Q \) is a quadric of rank 2 or 3, then it contains a family of planes. Since \( a = 2 \) through the general point of \( Q \) there pass just one line of \( X \), so the lines of \( X \) on such planes move on a pencil through a point. These pencils are embedded as lines via Plücker, which is absurd because the Veronese surface does not contain lines.

Therefore \( X \) is the set of lines of a smooth quadric \( Q \) meeting a line contained in it, which is the Veronese surface of Example 5. We remark that in this case \( J \subset G(1,2) \) is a pencil of lines.

\[ \square \]

3. Classification

In this section we classify all double Veronese embeddings \( X \) of \( \mathbb{P}^n \) in \( G(1,N) \). In order to do that, we first consider Grassmannian linearly normal double Veronese embeddings of \( \mathbb{P}^n \).

**Theorem 3.1.** Varieties of Examples 1, 2, 3, 4 and 5 are the only smooth, Grassmannian linearly normal, double Veronese embeddings of \( \mathbb{P}^n \) in \( G(1,N) \).

**Proof.** Let us denote by \( E \) the rank-2 bundle giving the embedding of \( \mathbb{P}^n \) in \( G(1,N) \). Arguing as we did in the proof of Proposition 2.1 we can say that \( a_E = (2, 0) \) or \((1, 1)\). Moreover, if \( a_E = (2, 0) \), we can conclude as before that \( X \) is the double Veronese embedding of Example 2. If \( a_E = (1, 1) \), for a jumping line \( L \) we must have \( E_L = O_{\mathbb{P}^1}(2) \oplus O_{\mathbb{P}^1} \). We still denote by \( J \subset G(1,n) \) the closed set of jumping lines of \( E \) and we distinguish two cases.

1) \( \text{codim} J \geq 2 \). Let us take a plane \( \Pi \subset \mathbb{P}^n \) and consider the restriction \( E' = E \otimes O_{\Pi} \). This gives an embedding of \( \Pi \) in a Grassmannian of lines as a Veronese surface. Since there are at most a finite number of jumping lines on \( \Pi \), it follows from Proposition 2.1 that \( E' \) splits. By Theorem 1.3 also \( E \) splits. We conclude that \( E = O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(1) \), which gives the double Veronese embedding of Example 1 when we consider all the sections of \( H^0(\mathbb{P}^n, E) \).
ii) \( \text{codim} \mathcal{J} = 1 \). Let us consider the following incidence diagram:

\[
\begin{array}{ccc}
I & \mathcal{J} \\
\downarrow_{p_1} & \downarrow_{p_2} \\
X & \\
\end{array}
\]

where we put \( I = \{(l, m) \in X \times \mathcal{J} \mid l \in M\} \). The general fiber of \( p_2 \) has dimension 1, and hence \( \dim I = 2n - 2 \) and \( \dim p_1^{-1}(l) = n - 2 \), for a general \( l \in X \). This is equivalent to say that the general line \( L \) is contained in an \( (n - 2) \)-dimensional family of cones whose ruling lines correspond also to points of \( X \). In particular the general line \( L \) meets an \( (n - 1) \)-dimensional family of lines of \( X \). Therefore, either there exists one point through which there pass an \( (n - 1) \)-dimensional family of lines, or through the general point of \( L \) there pass an \( (n - 2) \)-dimensional family of lines of \( X \) (with \( n \geq 3 \)).

In the first case we get that there exists a fundamental curve \( C \subset \mathbb{P}^N \) such that through the general point \( c \in C \) there pass an \( (n - 1) \)-dimensional family of lines of \( X \). We expect that through two general points \( c_1, c_2 \in C \) there pass an \( (n - 2) \)-dimensional family of lines, which implies \( n - 2 = 0 \). Therefore \( X \cong \mathbb{P}^2 \) and the classification follows from Proposition 2.1.

Let us consider now the second case, i.e. through a general point of \( L \) there pass an \( (n - 2) \)-dimensional family of lines of \( X \). We denote by \( \overline{X} \subset \mathbb{P}^N \) the union of the lines of \( X \) and consider the incidence diagram

\[
\begin{array}{ccc}
W & \overline{X} \\
\downarrow_{q_1} & \downarrow_{q_2} \\
X & \\
\end{array}
\]

where we put \( W = \{(l, y) \mid y \in L\} \). Looking at the first projection we get \( \dim(W) = n + 1 \) and, since through a general point \( y \in \overline{X} \) there pass an \( (n - 2) \)-dimensional family of lines of \( X \), \( \dim \overline{X} = n + 1 - (n - 2) = 3 \). In particular \( \overline{X} \) is a 3-dimensional projective variety containing a 3-dimensional family of lines (if \( \overline{X} \) contains a bigger family of lines then \( \overline{X} = \mathbb{P}^3 \) and \( X \subset G(1, 3) \), but this is not possible since \( v_2(\mathbb{P}^3) \) cannot be embedded in \( \mathbb{P}^5 \)). Then either \( \overline{X} \) is swept out by a 1-dimensional family of planes, or it is a hyperquadric of \( \mathbb{P}^4 \). The former is not possible since in this case the 3-dimensional family of lines contained in \( \overline{X} \) would be a scroll of planes. Conversely, the later corresponds to the double Veronese embedding of Example 4. Note that \( \mathcal{J} \subset G(1, 3) \) is a hyperplane section of \( G(1, 3) \). Moreover, jumping lines \( L \in \mathcal{J} \) correspond to tangent spaces to \( Q \), since they intersect \( Q \) along quadric cones. The rulings of the cone give the embedding of \( L \).

\[\square\]

Remark 2. All the double Veronese embeddings we classified above are projective linearly normal in \( \langle X \rangle \) via Plücker embedding. This is obvious for Examples 1, 2 and 3. For Example 4 see [7] and, since Example 5 is the restriction to a plane of Example 4 it is also projective linearly normal.

In order to complete the classification we are now going to consider double Veronese embeddings that can be projected to a smaller Grassmannian. In particular we study when the Grassmannian projection gives rise to a variety which is not projective linearly normal in \( \langle X \rangle \). In this way we also classify embeddings of \( \mathbb{P}^n \) in a
Let us take the embedding of $\mathbb{P}^n$ in the Grassmannian $G(1,N)$, with $N = \binom{n+2}{2}$, given by the bundle $\mathcal{O}_{\mathbb{P}^n}(2) \oplus \mathcal{O}_{\mathbb{P}^n}$. The image $X \subset G(1,N)$ can be described as the family of ruling lines of a cone over the double Veronese embedding $\nu_2(\mathbb{P}^n) \subset H \cong \mathbb{P}^{N-1}$, with vertex a point $v \notin H$. Let us take a linear space $L \subset H$, of dimension $k$, with $0 \leq k \leq \binom{n+1}{2}(n-2)$, which does not intersect the secant variety of $\nu_2(\mathbb{P}^n)$. We put $m = N - k - 1$. The projection $\pi_L : G(1,N) \to G(1,m)$, restricted to $X$, is an isomorphism. The image $X'$ can be described as the family of ruling lines of a cone over the projection of $\nu_2(\mathbb{P}^n)$ to $\mathbb{P}^{m-1}$. Therefore the composition $\varphi$ with Plücker embedding is given by a subspace of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$ of dimension $m$ and $X'$ is not projective linearly normal.

Let us take now the embedding $X \subset \mathbb{P}^n$ in $G(1,2n+1)$ given by the bundle $\mathcal{O}_{\mathbb{P}^n}(1) \oplus \mathcal{O}_{\mathbb{P}^n}(1)$. We prove that if we project $X$ to $G(1,m)$, with $n+1 \leq m \leq 2n$ and consider the composition with Plücker embedding, we always get the map associated to the complete linear system of quadrics $|\mathcal{O}_{\mathbb{P}^n}(2)|$. It is enough to prove it for the projection to $G(1,n+1)$. We recall that $X$ can be seen in $G(1,2n+1)$ as the family of lines joining corresponding points on two disjoint $\mathbb{P}^n$’s. The same geometric description holds after projecting to $G(1,n+1)$, but here the two $\mathbb{P}^n$’s, say $V_0$ and $W_0$, intersect in a $\mathbb{P}^{n-1}$. Let us fix coordinates $(x_0 : \ldots : x_{n+1})$ for $\mathbb{P}^{n+1}$ and let us denote by $\phi : V_0 \to W_0$ the correspondence. By induction we construct the linear spaces $W_i := \phi(V_i)$ and $V_{i+1} := V_i \cap W_i$, for $i = 0, \ldots , n$. Since the projection is an isomorphism, we have that no line of $X$ is contracted, and this implies that $V_{i+1} \subset V_i$, or $\dim(V_i) = \dim(W_i) = n - i$. Changing coordinates we can suppose that $V_0 = \{x_{n+1} = 0\}$, $V_i = \{x_{n+1} = x_0 = \ldots = x_{i-1} = 0\}$ for $i = 1, \ldots , n$ and $W_i = \{x_0 = \ldots = x_i = 0\}$. Under these assumptions $X$ can be described as the family of lines spanned by the rows of the matrix

$$
\begin{pmatrix}
    t_0 & t_1 & \cdots & t_n & 0 \\
    0 & l_0 & \cdots & l_{n-1} & l_n 
\end{pmatrix},
$$

where $t_0, t_1, \ldots , t_n$ are homogeneous coordinates of $\mathbb{P}^n$ and $l_i = a_{i,0} t_0 + a_{i,1} t_1 + \ldots a_{i,n} t_n$ is a linear form involving only the last $n-i-1$ variables. We can view $\phi$ as the map from $\mathbb{P}^n$ to $\mathbb{P}^n$, sending $t_i$ to $l_i$, and hence it is represented by a lower triangular matrix $T$, whose determinant is $\prod_{i=0}^n a_{i,i}$. We remark that, since $\phi$ is an isomorphism, $a_{i,i} \neq 0$ for $i = 0, \ldots , n$. Let us substitute to $x_1$, a suitable linear combination of $x_1, \ldots , x_{n+1}$, in order to get $t_0$ in the second place of the second row of the matrix above. The corresponding element on the first row is now a linear combination $t'_1$ of $t_1, \ldots , t_n$. Let us write now $l_1$ with respect to $t'_1, t'_2, \ldots , t'_n$. As before we can send $x_2$ to a suitable combination of $x_2, \ldots , x_{n+1}$ in order to get $t'_1$ in the third place of the second row of the matrix and so on. In this way we get a base change of $\mathbb{P}^{n+1}$ and of $V_0$, since the corresponding matrices are triangular and the elements on the diagonal are products of some $a_{i,i}^{-1}$. With the new bases the variety $X$ can be described by

$$
\begin{pmatrix}
    t_0 & t'_1 & \cdots & t'_n & 0 \\
    0 & l_0 & \cdots & l'_{n-1} & l'_n 
\end{pmatrix},
$$

Proposition 3.2. If $X \subset G(1,N)$ is a double Veronese embedding of $\mathbb{P}^n$ and it is not projective linearly normal, then $X$ is a projection of the variety of Example 2.

Proof. By Remark 1, varieties of Examples 1 and 2 are the only double Veronese embeddings that can be isomorphically projected to a smaller Grassmannian of lines. In order to prove the proposition we show that when we project, only in the first case we can obtain a non-projective linearly normal variety.
whose minors give a basis for the space of degree 2 polynomials in $t_0, t_1', \ldots, t_n'$.

**Remark 3.** In [7], H. Tango classified embeddings of $\mathbb{P}^n$ in $G(1, n+1)$. There are just 4 possibilities, namely, the *star of lines*, Examples [8] and [8], and the projection to $G(1, n+1)$ of Example [8]. As a corollary of our classification we get that all double Veronese embeddings except the cone case of Example [8] and the Veronese surface of Example [8] fit in a $G(1, n+1)$.

Finally, we state the result in connection with Hartshorne’s Conjecture quoted in the introduction.

**Corollary 3.3.** Rank-2 vector bundles over $\mathbb{P}^n$ giving a double Veronese embedding split if $n \geq 4$.

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