The Barut Second-Order Equation, Dynamical Invariants and Interactions

Valeri V. Dvoeglazov
Universidad de Zacatecas, Apartado Postal 636, Suc. UAZ Zacatecas 98062 Zac. México

Abstract. The second-order equation in the \((1/2, 0) \oplus (0, 1/2)\) representation of the Lorentz group has been proposed by A. Barut in the beginning of the 70s, ref. [1]. It permits to explain the mass splitting of leptons (\(e, \mu, \tau\)). Recently, the interest has grown to this model (see, for instance, the papers by S. Kruglov [2] and J. P. Vigier et al. [3]). We continue the research deriving the equation from the first principles, finding the dynamical invariants for this model, investigating the influence of the potential interactions.

The Barut main equation is

\[
[iv_\mu \partial_\mu - \alpha_2 \frac{\partial^2}{m^2} + \kappa] \Psi = 0. \tag{1}
\]

- It represents a theory with the conserved current that is linear in 15 generators of the 4-dimensional representation of the \(O(4, 2)\) group, \(N_{ab} = \frac{i}{2} \gamma_a \gamma_b, \gamma_a = \{\gamma_\mu, \gamma_5, i\}\).
- Instead of 4 solutions it has 8 solutions with the correct relativistic relation \(E = \pm \sqrt{p^2 + m^2}\). In fact, it describes states of different masses (the second one is \(m_\mu = m_e(1 + \frac{3}{2} \alpha)\), \(\alpha\) is the fine structure constant), provided that a certain physical condition is imposed on the \(\alpha_2\) parameter (the anomalous magnetic moment should be equal to \(\frac{4}{3} \alpha\)).
- One can also generalize the formalism to include the third state, the \(\tau\)-lepton [1b].
- Barut has indicated the possibility of including \(\gamma_5\) terms (e.g., \(\sim \gamma_5 \kappa\)).

If we present the 4-spinor as \(\Psi(p) = \text{column}(\phi_R(p)) \phi_L(p))\) then Ryder states [5] that \(\phi_R(0) = \phi_L(0)\). Similar argument has been given by Faustov [6]: “the matrix \(B\) exists such that \(B_{\lambda\lambda}(0) = u_{\lambda}(0), B^2 = I\) for any \((2J + 1)\)-component function within the Lorentz invariant theories”\(^1\). The most general form of the relation in the \((1/2, 0) \oplus (0, 1/2)\) representation has been given by Dvoeglazov [7,4a]:

\[
\phi_L^h(0) = a(-1)^{1-h} e^{i(\theta_1 + \theta_2)} \Theta_{1/2} [\phi_L^{-h}(0)]^* + be^{2i\eta} \Xi_{1/2} [\phi_L^h(0)]^*, \tag{2}
\]

with

\[
\Theta_{1/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_2, \quad \Xi_{1/2} = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}. \tag{3}
\]

\(^1\) The latter statement is more general than the Ryder one, because it admits \(B = \begin{pmatrix} 0 & e^{i\alpha} \\ e^{-i\alpha} & 0 \end{pmatrix}\), so that \(\phi_R(0) = e^{i\alpha}\phi_L(0)\).
\( \Theta J \) is the Wigner operator for spin \( J = 1/2 \), \( \varphi \) is the azimuthal angle \( p \rightarrow 0 \) of the spherical coordinate system.

Next, we use the Lorentz transformations:

\[
\Lambda_{R,L} = \exp(\pm \sigma \cdot \phi/2), \quad \cosh \phi = E_p/m, \quad \sinh \phi = |p|/m, \quad \dot{\phi} = p/|p|.
\]  

Applying the boosts and the relations between spinors in the rest frame, one can obtain:

\[
\begin{align*}
\phi^h_L(p) &= a \frac{p_0 - \sigma \cdot p}{m} \phi^h_R(p) + b(-1)^{\frac{1}{2} + h} \Theta_{1/2} \Xi_{1/2} \phi_R^{-h}(p), \\
\phi^h_R(p) &= a \frac{p_0 + \sigma \cdot p}{m} \phi^h_L(p) + b(-1)^{\frac{1}{2} + h} \Theta_{1/2} \Xi_{1/2} \phi_L^{-h}(p).
\end{align*}
\]  

(\( \theta_1 = \theta_2 = 0, p_0 = E_p = \sqrt{p^2 + m^2}. \)) In the Dirac form we have:

\[
[a \frac{\hat{p}}{m} - 1] u_h(p) + ib(-1)^{\frac{1}{2} - h} \gamma_5 \tilde{C} u_h^{-h}(p) = 0,
\]  

where \( C = \begin{pmatrix} 0 & i \Theta_{1/2} \\ -i \Theta_{1/2} & 0 \end{pmatrix} \). In the QFT form we must introduce the creation/annihilation operators. Let \( b_\downarrow = -ia_\uparrow, b_\uparrow = +ia_\downarrow \), then

\[
[a \frac{i\gamma^\mu \partial_\mu}{m} + b\tilde{C}K - 1] \Psi(x^\nu) = 0.
\]  

If one applies the unitary transformation to the Majorana representation [8]

\[
U = \frac{1}{2} \begin{pmatrix} 1 - i\Theta_{1/2} & 1 + i\Theta_{1/2} \\ -1 - i\Theta_{1/2} & 1 - i\Theta_{1/2} \end{pmatrix}, \quad UCKU^{-1} = -K,
\]  

then \( \gamma \)-matrices become to be pure imaginary, and the equations are pure real.

\[
\begin{align*}
\begin{bmatrix} i \dot{\hat{p}}/m - b - 1 \end{bmatrix} \psi_1 &= 0, \\
\begin{bmatrix} i \dot{\hat{p}}/m + b - 1 \end{bmatrix} \psi_2 &= 0,
\end{align*}
\]  

where \( \psi = \psi_1 + i\psi_2 \). It appears as if the real and imaginary parts have different masses. Finally, for superpositions \( \phi = \psi_1 + \psi_2, \chi = \psi_1 - \psi_2 \), multiplying by \( b \neq 0 \) we have:

\[
[2a \frac{i\gamma^\mu \partial_\mu}{m} + 2 \frac{\partial^\mu \partial_\mu}{m^2} + b^2 - 1] \frac{\phi(x^\nu)}{\chi(x^\nu)} = 0,
\]  

If we put \( a/2m \rightarrow \alpha_2, \frac{-b^2}{2a^2} \rightarrow \kappa \) we recover the Barut equation.

How can we get the third lepton state? See the refs. [1b,4b]:

\[
M_\tau = M_\mu + \frac{3}{2} \alpha^{-1} n^4 M\nu = M_e + \frac{3}{2} \alpha^{-1} n^4 M_e + \frac{3}{2} \alpha^{-1} n^2 M_e = 1786.08 \ MeV.
\]  

The physical origin was claimed by Barut to be in the magnetic self-interaction of the electron (the factor \( n^4 \) appears due to the Bohr-Sommerfeld rule for the charge moving in circular orbits in the field of a fixed magnetic dipole \( \mu \)). One can start from (7), but, as opposed to the above-mentioned, one can write the coordinate-space equation in the form:

\[
[a \frac{i\gamma^\mu \partial_\mu}{m} + b_1\tilde{C}K - 1] \Psi(x^\nu) + b_2\gamma^5 \tilde{C}K \tilde{\Psi}(x^\nu) = 0,
\]  

(\( \theta_1 \rightarrow \theta_2 = 0, p_0 = E_p = \sqrt{p^2 + m^2}. \))
with $\Psi^{MR} = \Psi_1 + i\Psi_2$, $\tilde{\Psi}^{MR} = \Psi_3 + i\Psi_4$. As a result,

$$
\left(\frac{a i \gamma^\mu \partial_\mu}{m} - 1\right)\phi - b_1 \chi + ib_2 \gamma^5 \tilde{\phi} = 0, \\
\left(\frac{a i \gamma^\mu \partial_\mu}{m} - 1\right)\chi - b_1 \phi - ib_2 \gamma^5 \tilde{\chi} = 0. 
$$

(15)

(16)

The operator $\tilde{\Psi}$ may be linear-dependent on the states included in the $\Psi$. let us apply the most simple form $\Psi_1 = -i\gamma^5 \Psi_4$, $\Psi_2 = +i\gamma^5 \Psi_3$. Then, one can recover the 3rd order Barut-like equation [4b]:

$$
[i\gamma^\mu \partial_\mu - m \frac{1 \pm b_1 \pm b_2}{a}][i\gamma^\nu \partial_\nu + \frac{a}{2m} \partial^\nu \partial_\nu + \frac{b_1^2 - 1}{2a}]\Psi_{1,2} = 0. 
$$

(17)

Thus, we have three mass states.

Let us reveal the connections with other models. For instance, in refs. [3, 9] the following equation has been studied:

$$
[(i\hat{\partial} - eA)(i\hat{\partial} - eA) - m^2] \Psi = [(i\partial_\mu - eA_\mu)(i\partial^\mu - eA^\mu) - \frac{1}{2} e\sigma^{\mu\nu} F_{\mu\nu} - m^2] \Psi = 0
$$

(18)

for the 4-component spinor $\Psi$. This is the Feynman-Gell-Mann equation. In the free case we have the Lagrangian (see Eq. (9) of ref. [3c]):

$$
L_0 = (i\bar{\Psi}(i\hat{\partial}) - m^2 \bar{\Psi}\Psi. 
$$

(19)

We can note:

- The Barut equation is a sum of the Dirac equation and the Feynman-Gell-Mann equation.
- Recently, it was suggested to associate an analogue of Eq. (19) with the dark matter [10], provided that $\Psi$ is composed of the self/anti-self charge conjugate spinors, and it has the dimension [energy]$^1$ in $c = h = 1$. The interaction Lagrangian is $L^H \sim g \bar{\Psi}\Phi\Psi^2$.
- The term $\sim \bar{\Psi}\sigma^{\mu\nu}\Psi F_{\mu\nu}$ will affect the photon propagation, and non-local terms will appear in higher orders.
- However, it was shown in [3b,c] that a) the Mott cross-section formula (which represents the Coulomb scattering up to the order $\sim e^2$) is still valid; b) the hydrogen spectrum is not much disturbed; if the electromagnetic field is weak the corrections are small.
- The solutions are the eigenstates of $\gamma^5$ operator.
- In general, $J_0$ is not the positive-defined quantity, since the general solution $\Psi = a\Psi_+ + b\Psi_-$, where $[i\gamma^\mu \partial_\mu \pm m] \Psi_\pm = 0$, see also [11].

The most general conserved current of the Barut-like theories is

$$
J_\mu = \alpha_1 \gamma_\mu + \alpha_2 p_\mu + \alpha_3 \sigma^{\mu\nu} q^\nu. 
$$

(20)

Let us try the Lagrangian:

$$
L = L_{Dirac} + L_{add}; 
$$

(21)

$$
L_{Dirac} = \alpha_1 [\bar{\Psi}\gamma^\mu (\partial_\mu \Psi) - (\partial_\mu \bar{\Psi})\gamma^\mu \Psi] - \alpha_4 \bar{\Psi}\Psi, \\
L_{add} = \alpha_2 (\partial_\mu \bar{\Psi})(\partial^\mu \Psi) + \alpha_3 \partial_\mu \bar{\Psi}\sigma^{\mu\nu} \partial_\nu \Psi. 
$$

(22)

(23)
Then, the equation follows:

\[ [2\alpha_1 \gamma^\mu \partial_\mu - \alpha_2 \partial_\mu \partial^\mu - \alpha_4] \Psi = 0, \]  
(24)

and its Dirac-conjugate:

\[ \bar{\Psi} [2\alpha_1 \gamma^\mu \partial_\mu + \alpha_2 \partial_\mu \partial^\mu + \alpha_4] = 0. \]  
(25)

The derivatives acts to the left in the second equation. Thus, we have the Dirac equation when \( \alpha_1 = \frac{i}{2}, \alpha_2 = 0 \), and the Barut equation when \( \alpha_2 = \frac{1}{m^{1+4\alpha/3}} \).

In the Euclidean metrics the dynamical invariants are

\[ N^\Psi, \bar{\Psi} \text{ the Lorentz group generators.} \]

Then, the energy-momentum tensor is

\[ T_{\mu\nu} = -\alpha_1 [\Psi \gamma_\mu \partial_\nu \Psi - \partial_\nu \bar{\Psi} \gamma_\mu \Psi] - \alpha_2 [\partial_\mu \bar{\Psi} \partial_\nu \Psi + \partial_\nu \bar{\Psi} \partial_\mu \Psi] - \alpha_3 [\bar{\Psi} \sigma_{\mu\nu} \partial_\alpha \Psi + \bar{\Psi} \sigma_{\mu\nu} \partial_\alpha \Psi] + \alpha_1 (\Psi \gamma_\mu \partial_\nu \Psi - \partial_\nu \Psi \gamma_\mu \Psi) + \alpha_2 \partial_\mu \bar{\Psi} \partial_\nu \Psi + \alpha_3 \partial_\mu \bar{\Psi} \sigma_{\alpha\beta} \partial_\beta \Psi + \alpha_4 \bar{\Psi} \delta_{\mu\nu}. \]  
(29)

Hence, the Hamiltonian \( \hat{\mathcal{H}} = -iP_4 = - \int T_{4\mu} d^3x \) is

\[ \hat{\mathcal{H}} = \int \{ \alpha_1 [\partial_\mu \bar{\Psi} \gamma_\mu \Psi - \Psi \gamma_\mu \partial_\mu \Psi] + \alpha_2 [\partial_\mu \bar{\Psi} \partial_\nu \Psi + \partial_\nu \bar{\Psi} \partial_\mu \Psi] - \alpha_3 [\partial_\mu \bar{\Psi} \sigma_{\nu\lambda} \partial_\nu \Psi] - \alpha_4 \bar{\Psi} \delta_{\mu\nu} \} d^3x. \]  
(30)

The 4-current is

\[ J_\mu = -i \{ 2\alpha_1 \Psi \gamma_\mu \Psi + 2\alpha_2 (\partial_\mu \bar{\Psi}) \Psi - \bar{\Psi} (\partial_\mu \Psi) + \alpha_3 [\partial_\alpha \bar{\Psi} \sigma_{\mu\nu} \Psi - \bar{\Psi} \sigma_{\mu\nu} \partial_\alpha \Psi] \}. \]  
(31)

Hence, the charge operator \( \hat{Q} = -i \int J_4 d^3x \) is

\[ \hat{Q} = -i \int \{ 2\alpha_1 \Psi \gamma_4 \Psi + 2\alpha_2 (\partial_4 \bar{\Psi}) \Psi - \bar{\Psi} (\partial_4 \Psi) + \alpha_3 [\partial_\alpha \bar{\Psi} \sigma_{4\mu} \Psi - \bar{\Psi} \sigma_{4\mu} \partial_\alpha \Psi] \} d^3x. \]  
(32)

Finally, the spin tensor is

\[ S_{\mu\nu,\lambda} = \frac{i}{2} \{ \alpha_1 [\Psi \gamma_\lambda \sigma_{\mu\nu} \Psi + \bar{\Psi} \sigma_{\mu\nu} \gamma_\lambda \Psi] + \alpha_2 [\partial_\lambda \bar{\Psi} \sigma_{\mu\nu} \Psi - \bar{\Psi} \sigma_{\mu\nu} \partial_\lambda \Psi] + \alpha_3 [\partial_\alpha \bar{\Psi} \sigma_{\alpha\lambda \mu} \Psi - \bar{\Psi} \sigma_{\mu\nu} \sigma_{\alpha\lambda} \partial_\nu \Psi] \}. \]  
(33)

In the quantum case the corresponding field operators are written:

\[ \Psi(x^{m}u) = \sum_{h} \int \frac{d^3p}{(2\pi)^3} [u_{h}(p)a_{h}(p)e^{+ip\cdot x} + v_{h}(p)b_{h}^{\dagger}(p)e^{-ip\cdot x}], \]  
(34)

\[ \bar{\Psi}(x^{m}u) = \sum_{h} \int \frac{d^3p}{(2\pi)^3} [\bar{v}_{h}(p)a_{h}^{\dagger}(p)e^{-ip\cdot x} + \bar{u}_{h}(p)b_{h}(p)e^{+ip\cdot x}], \]  
(35)
The commutation relations are
\[
\begin{align*}
[a_h(p), a^\dagger_{h'}(k)]_+ &= (2\pi)^3 \frac{m}{p_4} \delta^{(3)}(p-k) \delta_{hh'}, \\
[b_h(p), b^\dagger_{h'}(k)]_+ &= (2\pi)^3 \frac{m}{p_4} \delta^{(3)}(p-k) \delta_{hh'},
\end{align*}
\]
(37) (38)
with all other to be equal to zero. The dimensions of the \( \Psi, \bar{\Psi} \) are as usual, \(|energy|^{3/2} \). Hence, the second-quantized Hamiltonian is written
\[
\hat{H} = -\sum_h \int \frac{d^4p}{(2\pi)^3} \frac{2Ep}{m} [\alpha_1 + m\alpha_2] : [a^\dagger_h a_h - b_h b^\dagger_h] : .
\]
(39)
(Remember that \( \alpha_1 \sim \frac{1}{2} \), the commutation relations may give another \( i \), so the contribution of the first term to eigenvalues will be real. But if \( \alpha_2 \) is real, the contribution of the second term may be imaginary). The charge is
\[
\hat{Q} = -\sum_{hh'} \int \frac{d^4p}{(2\pi)^3} \frac{2Ep}{m} [(\alpha_1 + m\alpha_2) \delta_{hh'} - i\alpha_3 \bar{u}_h \sigma_{i4} p_i u_{h'} ] : [a^\dagger_h a_{h'} + b_h b^\dagger_{h'}] : .
\]
(40)
However, due to \([\Lambda_{R,L}, \sigma \cdot p]_- = 0 \) the last term with \( \alpha_3 \) does not contribute.

The conclusions are:

- We obtained the Barut-like equations of the 2nd order and 3rd order in derivatives. The Majorana representation has been used.
- We obtained dynamical invariants for the free Barut field on the classical and quantum level.
- We found relations with other models (such as the Feynman-Gell-Mann equation).
- As a result of analysis of dynamical invariants, we can state that at the free level the term \( \sim \alpha_3 \bar{\Psi} \sigma_{i4} \partial_i \Psi \) in the Lagrangian does not contribute.
- However, the interaction terms \( \sim \alpha_3 \bar{\Psi} \sigma_{i4} \partial_i \Psi A_\mu \) will contribute when we construct the Feynman diagrams and the S-matrix. In the curved space (the 4-momentum Lobachevsky space) the influence of such terms has been investigated in the Skachkov works [12]. Briefly, the contribution will be such as if the 4-potential were interact with some “renormalized” spin. Perhaps, this explains, why did Barut use the classical anomalous magnetic moment \( g \sim 4\alpha/3 \) instead of \( \frac{\alpha}{\pi} \).

The author acknowledges discussions with participants of recent conferences.

References

[1] Barut A O 1978 Phys. Lett. B73 310; 1979 Phys. Rev. Lett. 42 1251; Wilson R 1974 Nucl. Phys. B68 157
[2] Kruglov S I 2004 Preprint quant-ph/0408056; 2004 Ann. Fond. Broglie 29, No. H2 (the special issue dedicated to Yang and Mills, ed. by V. Dvoeglazov et al.)
[3] Petroni N C, Vigier J P et al 1984 Nuovo Cim. B81 243; 1984 Phys. Rev. D30 495; 1985 ibid. D31 3157
[4] Dvoeglazov V V 1998 Int. J. Theor. Phys. 37 1909; 2000 Ann. Fond Broglie 25 81
[5] Ryder L M 1985 Quantum Field Theory. (Cambridge University Press, Cambridge)
[6] Faustov R N 1974 Preprint ITF-71-117P (Kiev).
[7] Dvoeglazov V V 1995 Hadronic J. Suppl. 10 349
[8] Dvoeglazov V V 1997 Int. J. Theor. Phys. 36 635
[9] Feynman R and Gell-Mann M 1958 Phys. Rev. 109 193
[10] Ahluwalia D. V. and Grumiller D 2004 Preprint hep-th/0410192
[11] Markov M 1937 ZhETF 7 579; 1937 ibid. 603; 1964 Nucl. Phys. 55 130
[12] Skachkov N B 1975 Theor. Math. Phys. 22 149; 1976 ibid. 25 1154