Magnetic Field Induced $A$-$B$ Phase Transition and Edge States of Superfluid $^{3}$He Confined in a Slab Geometry

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Abstract. We investigate the thermodynamics and quasiparticle structures of superfluid $^{3}$He confined to a slab in the presence of a magnetic field parallel or normal to the specular surface. Based on the quasiclассical Eilenberger theory, we calculate the quantitative phase diagram in the plane of the thickness and magnetic field. It is also demonstrated that the magnetization density and the phase diagram become isotropic on the orientation of the magnetic field, where the $\hat{n}$-vectors rotate so as to maximize the magnetization density on the surface.

1. Introduction
Much attention has recently been focused on superfluid $^{3}$He, which is a typical example of topological superconductors and superfluids [1]. The non-trivial topological charge defined in the bulk reveals the emergence of the gapless state bound on the surface of the system which is regarded as the topological phase boundary between the topologically nontrivial bulk and trivial vacuum regions. Recently, it has been proposed that the surface bound state in $^{3}$He B-phase behaves as a Majorana fermion at the limit of the zero field, which gives rise to the Ising-anisotropy of spin susceptibility on the surface [2, 3, 4, 5]. In the high field regime, however, there is a statement that the Ising-anisotropy and gapless spectrum disappear and the topological charge is not defined in this regime [6].

The aim in our work is to clarify the thermodynamics and quasiparticle structures of $^{3}$He confined to a sub-micron slab [7], based on the quasiclassical theory where the magnetic field and the Fermi liquid corrections are fully taken into account. We demonstrate that while the phase diagram composed of the first- and second-order A-B transition lines is unchanged by the orientation of the magnetic field, the $\hat{n}$-vector which reflects the spin-orbit symmetry breaking of the B-phase rotates so as to maximize the magnetization on the surface. The orientation is important in the sense that it determines the NMR resonance frequency [8]. The enhancement of the magnetization on the surface is directly associated with the gapful surface bound states.

2. Quasiclassical Eilenberger theory
We start with the quasiclassical theory, which is reliable to $^{3}$He in low pressure regime. The evolution of the quasiclassical Green’s functions $\hat{g}$ is governed by the Eilenberger equation [9],

$$
\begin{aligned}
[i\omega_n \tilde{\tau}_3 - \hat{v}(\hat{k}, r) - \hat{\sigma}(\hat{k}, r), \hat{g}(\hat{k}, r; i\omega_n)] + i\nu_F(\hat{k}) \cdot \nabla \hat{g}(\hat{k}, r; i\omega_n) &= 0,
\end{aligned}
$$

(1)
where $v_F = v_T k$ is the Fermi velocity and $\omega_n = (2n+1) \pi T$ is the Matsubara frequency with $n \in \mathbb{Z}$. Throughout this work, we use $\hbar = f = 1$. The quasiclassical Green’s functions for spin-triplet superfluids are parametrized in the particle-hole space as

\[
\tilde{g}(\mathbf{k}, r; i\omega_n) = \begin{bmatrix}
\tau_j g_j(\mathbf{k}, r; i\omega_n) & i\tau_j \hat{\nu}_j d \mu(\mathbf{k}, r) \\
\hat{\nu}_j t \mu f_\mu(\mathbf{k}, r; i\omega_n) & \hat{\nu}_j t \mu \tau_j g_j(\mathbf{k}, r; i\omega_n)
\end{bmatrix} ,
\]

(2)

where $j = 0, 1, 2, 3$, $\mu = 1, 2, 3$, and $\tau_0$ and $\tau_\mu$ denote the $2 \times 2$ unit matrix and $\mu$-th Pauli matrices. Equation (1) coupled with the normalization condition $\tilde{g}^2 = -\pi^2 \mathbb{I}$ reduces to the Riccati equation [10, 11], which is numerically solved with the fourth-order Runge-Kutta method.

In Eq. (1), $\tilde{\sigma}$ describes the quasiclassical self-energy which consists of the pair potential $d_\mu (\mu = x, y, z)$ and Fermi liquid correction $\nu_j (j = 0, x, y, z)$ as

\[
\tilde{\sigma}(\mathbf{k}, r) = \begin{bmatrix}
\tau_j \nu_j(\mathbf{k}, r) & i\tau_j \hat{\nu}_j d \mu(\mathbf{k}, r) \\
\hat{\nu}_j t \mu \tau_j d \mu(\mathbf{k}, r) & \hat{\nu}_j t \mu \nu_j(\mathbf{k}, r)
\end{bmatrix} .
\]

(3)

The diagonal and off-diagonal self-energies are associated with the quasiclassical Green’s functions as $\nu_j(\mathbf{k}, r) = \langle A_j(\mathbf{k}, \mathbf{k}^{'}) g_j(\mathbf{k}^{'}, r; i\omega_n) \rangle_{\mathbf{k}^{'}\omega_n}$ and $d_\mu(\mathbf{k}, r) = \langle V(\mathbf{k}, \mathbf{k}^{'}) f_\mu(\mathbf{k}^{'}, r; i\omega_n) \rangle_{\mathbf{k}^{'}\omega_n}$. Here, $A_j(\mathbf{k}) = A^a$ and $A_j(\mathbf{k}^{'}) = A^a$ are the quasiparticle scattering amplitude and $V(\mathbf{k}, \mathbf{k}^{'}) = 3V_F \hat{k}_x \hat{k}_y$ is the pairing interaction. The coupling constant $V_F$ is parameterized with the physical quantities, the transition temperature of the bulk B phase $T_{c0}$ and the energy cutoff $E_c$ as $V_F^{-1} = 2\pi T \sum_{\omega_n \omega_n ' E_c} \omega_n^{-1}$, where we use $E_c = 20\pi T_{c0}$. Also, $\langle \cdot \cdot \cdot \rangle_{\mathbf{k}^{'}\omega_n}$ indicates the Matsubara sum and the Fermi surface average. The amplitudes, $A_{j}^{a}$, are parameterized with the Landau’s Fermi liquid parameter $F^a(\mathbf{k}) [9]$, where $F_0^a = 9.3, F_1^a = 5.39, F_0^a = -0.695$, and $F_1^a = -0.5$.

Here, we consider superfluid $^3$He sandwiched between two specular walls normal to $\hat{z}$. The walls located at $z = 0$ and $D$ are supposed to be infinitely large in $x, y$-plane. We impose the specular boundary condition at $z = 0$ and $D$ on $\tilde{g}$, which requires matching of $\tilde{g}$ at the surface for two trajectories $(\mathbf{k}_x, k_y, \pm k_z) [12, 13, 14]$. The matrix $\hat{\sigma}$ in Eq. (1) consists of the magnetic Zeeman term, $\hat{\sigma} = \text{diag}[\mu H, \mu \nu, \mu H, \mu \nu]/(1 + F_2^2)$ and in Sec. 3, we consider the two different orientations of the magnetic field, that is, $\mathbf{H} \parallel \hat{z}$ and $\mathbf{H} \parallel \hat{x}$. In this situation, the A-B phase transition is induced by the thickness $D$, in addition to the magnetic field. For specular walls, the transition occurs at $D \approx 9.8 \xi_0$ when $T = 0.2 T_{c0}$ and $H = 0 [12]$. Here, $\xi_0 \equiv v_F / \pi T_{c0}$ is the coherence length, which is estimated as $\xi_0 \approx 80 \text{nm}$ at zero pressure.

3. Phase diagram and quasiparticle excitations in the presence of magnetic field

First, we consider the phase diagram of $^3$He in the plane of the thickness $D$ and the magnetic field $H$ applied to the normal to the wall, that is, $\mathbf{H} \parallel \hat{z}$. In this situation, the order parameter of the B phase can reduce to the diagonal representation, $d_{zz}(\mathbf{k}, r) = \delta_{\mu \nu} d_{\nu \nu}(\mathbf{r}) \hat{k}_\mu$, and the magnetic field distorts the $d_{zz}$ component parallel to $\mathbf{H}$. Following the procedure in Ref. [12], the thermodynamic potential within the quasiclassical approximation is derived as $\delta \Omega[\tilde{g}] = \frac{N_F}{4} \int_0^1 d \lambda \int d r d \mathbf{r} \text{Tr}(\tilde{\sigma}[2 \tilde{g}_\lambda - \tilde{g}])_{\mathbf{k}\omega_n}$, where $\tilde{g}_\lambda$ is the quasiclassical auxiliary function derived from Eq. (1) with the replacement $\sigma \rightarrow \lambda \sigma$ and $N_F$ is the density of states on the Fermi surface.

The resulting phase diagram and pair amplitude $d_{zz}(z = D/2)$ parallel to the magnetic field are displayed in Fig. 1(a). For the large range of the thickness, e.g., $D = 40 \xi_0$, the first-order phase transition from B to A (or planar) phase occurs at $\mu H / \pi T_{c0} = 0.096$, which is estimated with $T_{c0} = 1 \text{mK}$ at zero pressure as $H = 0.36 \text{T}$. This is found to be quantitatively consistent to theoretical calculation and experiments in bulk superfluid $^3$He [15, 16]. As $D$ decreases, however, it is seen from Fig. 1(a) that the first-order line turns to the second-order transition. In the thermodynamic limit, the critical point between first- and second-order transition lines is
located at \((T^*/T_{c0}, \mu H^*/\pi T_{c0}) = (0.8, 0.05)\). The finite geometry effect shifts the critical point to the lower temperature and higher field, e.g., \((T^*/T_{c0}, \mu H^*/\pi T_{c0}) = (0.2, 0.085)\) at \(D = 11T_{c0}\). The upper panel of Fig. 1(b) shows the local density of states (LDOS) \(N(z, E) = -\frac{1}{\pi}\Im\langle \hat{g}_z(\hat{k}, r; i\omega_n \rightarrow E + i0_+)\rangle_\hat{k}\) where \(0_+ = 0.005\pi T_{c0}\). In the absence of the magnetic field, \(H = 0\), the low-energy LDOS on the surface \((z = 0)\) has the linear slope on \(E\), which implies that the surface Andreev bound states have the gapless cone spectrum in the \(k_x-k_y\) plane \([2, 3, 4, 5]\), called the Majorana cone. The magnetic field normal to the wall, however, makes a finite energy gap with \(\mu H/(1 + F_0^2)\) in the surface DOS. In the lower column of Fig. 1(b), we depict the \(E\)-resolved magnetization density \(\delta M_z(z, E) = -\frac{1}{\pi}\Im\langle \hat{g}_z(\hat{k}, r; i\omega_n \rightarrow E + i0_+)\rangle_\hat{k}\) which shows the qualitatively different profile between the surface \((z = 0)\) and \(z = D/2\). As we will see below, one finds \(\int \delta M_z(z, E)dE \propto M_z(z) - M_N\), corresponding to the local magnetization density relative to that in the normal state \(M_N\). The magnetization density at \(z = D/2\) is suppressed from \(M_N\). In contrast, it is seen from Fig. 1(b) that the surface DOS within \(|E| \lesssim 0.4\pi T_{c0}\) contributes to the magnetization density at the surface, which comes up to a positive value. This implies that the surface Andreev bound states lead to \(M_z(z = 0) - M_N > 0\).

Now let us move to the situation where \(\mathbf{H} \parallel \hat{z}\). Since in this situation the order parameter is not necessarily diagonal, we start with the general form, \(d_{\mu}(\hat{k}, r) = R_{\mu\nu}(\hat{n}, \varphi)d_{\nu\nu}(r)\hat{k}_{\nu}\) where \(R_{\mu\nu}(\hat{n}, \varphi)\) is the matrix for a rotation around the \(\hat{n}\)-axis by the angle \(\varphi\), reflecting the spin-orbit symmetry breaking in the \(B\)-phase manifold. The microscopic determination of \(\hat{n}\) and \(\varphi\) requires the dipole interaction to be taken account into the gap equation. However, since the dipole energy scale \(\sim 2\mu\mathrm{T}\) is much smaller than the magnetic field energy scale \(O(0.1\mathrm{T})\) in the vicinity of the \(A-B\) transition line, we here remove the dipole energy from our formalism. We also mention that the \(\hat{n}\)-vector is spatially uniform over the slab geometry, because the spatial variation is of an order of the dipole coherence length \(\sim 10\mu\mathrm{m}\).

Here, we demonstrate that the thermodynamics and the quasiparticle structure in the case of \(\mathbf{H} \perp \hat{z}\) strongly depend on the orientation of the \(\hat{n}\)-vector. To clarify this, we display in Fig. 2(a) the magnetization density for a given \(\hat{n}\)-vector, defined as \(M_\mu(r) = M_N[\frac{H_\mu}{H} + \frac{1}{\mu H}(g_\mu(\hat{k}, r; i\omega_n)\hat{k}_{\mu})]_{k,\omega_n}\) \([9]\). Here, \(M_N = 2\mu^2N_F\mu H/(1 + F_0^2)\) indicates the magnetization in the normal state. For \(\hat{n} \parallel \hat{z}\), that is, the diagonal representation \(d_{\mu} = \delta_{\mu\nu}d_{\nu\nu}\hat{k}_{\nu}\), the magnetization density on the surface is rather suppressed from \(M_N\). This reflects the fact that the magnetic field parallel to the surface does not make an energy gap in the surface Andreev bound state and thus, the magnetization density is unchanged from that in the bulk region \(M_z(z = D/2) \approx M_z(z = 0)\).
As \( \hat{n} \) is tilted from the \( \hat{z} \)-axis, however, \( M_z(z) \) is considerably enhanced in the surface region. In particular, as seen in Fig. 2(b), the state with \( \hat{n} \parallel \hat{y} \) has lower energy than the diagonal representation in the whole \( H \) region. It is emphasized that the critical field beyond which the A phase becomes stable is found to be \( \mu H/\pi T_{c0} = 0.096 \). Hence, the resulting phase diagram for \( H \parallel \hat{x} \) is unchanged from Fig. 1(a), except for the rotation of \( \hat{n} \).

4. Conclusions
Here, we have calculated the quantitative phase diagram and magnetization densities of superfluid \(^3\)He confined to a slab geometry, based on the quasiclassical theory. We have found that the first-order line of the A-B transition turns to the second-order transition at the magnetic field \( \mu H/\pi T_{c0} = 0.085 \), which is larger than 0.05 in the thermodynamic limit [15]. We have also demonstrated that the magnetization density on the surface is considerably enhanced by the surface Andreev bound state when \( H \) is applied to the \( \hat{z} \)-axis normal to the surface. For \( H \) parallel to the surface, however, the magnetization density on the surface is sensitive to the orientation of the \( \hat{n} \)-vector, which is enhanced (suppressed) on the surface when \( \hat{n} \parallel \hat{y} \left( \hat{n} \parallel \hat{z} \right) \). The most stable texture of \( \hat{n} \)-vectors is found to maximize the magnetization density on the surface. For the weak field regime, however, the dipole interaction energy, which is neglected in this work, becomes competitive to the magnetic field energy. Since the dipole interaction in the Ginzburg-Landau regime is known to favor the configuration of \( \hat{n} \parallel \hat{z} \) on the surface [8], the energetics in this regime might be nontrivial. This will be studied in future.

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