GEOMETRY OF CROSS RATIO
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Generalization of the cross ratio to polarizations of linear finite and infinite-dimensional spaces (in particular to Sato Grassmannian) is given and explored. This cross ratio appears to be a cocycle of the canonical (tautological) bundle over the Grassmannian with coefficients in the sheaf of its endomorphisms. Operator analog of the Schwarz differential is defined. Its connections to linear Hamiltonian systems and Riccati equations are established. These constructions aim to obtain applications to KP-hierarchy.

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Sato Grassmanian and KP-hierarchy

We shall use the following standard notations. As a field of constants \( k \) one may consider both \( \mathbb{R} \) or \( \mathbb{C} \). For a basic ring \( B = k[[x]] = \{ \sum_{i \geq 0} a_i x^i \mid a_i \in k \} \) – (the formal Taylor series) let \( E = k[[x]]((\partial^{-1})) \) – be the ring of formal pseudodifferential operators with coefficients in \( B \).

Consider the direct sum decomposition \( E = E_+ \oplus E_- \), where \( E_+ = B[\partial] \) is the ring of differential operators, and \( E_- = B[[\partial^{-1}]]\partial^{-1} \) – is the ring of operators with the negative order (Volterra type operators). So that for any \( P \in E \) one have \( P = P_+ + P_- \), where \( P_+ \) is a differential part and \( P_- \) is a Volterra part of \( P \). A decomposition of an infinite-dimensional space on a direct sum of its two infinite-dimensional subspaces shall be called polarization.

Let \( H \) be a Hilbert space, equipped with a polarization, i.e. with a decomposition on two closed orthogonal subspaces

\[
H = H_+ \oplus H_-. \tag{1}
\]

It is convenient to define this decomposition by an operator of complex structure, i.e. by the unitary operator \( J : H \to H \) which is equal to +Id on \( H_+ \) and to −Id on \( H_- \). The General restricted group \( GL_{res}(H) \) is defined as a subgroup of \( GL(H) \) consisted of operators \( A \) such that the commutator \([J, A]\) is a Hilbert-Schmidt operator. Equivalently, if \( A \in GL(E) \) is represented as \((2 \times 2)\)-matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

relative to the decomposition (1), then \( A \in GL_{res}(H) \) iff \( b \) and \( c \) are Hilbert-Shmidt operators. It follows that \( a \) and \( d \) are Fredholm operators. \( U_{res}(H) \) is a subgroup of \( GL_{res}(H) \), which consists of unitary operators.

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Sato Grassmannian $SGr(H)$ is the set of all closed subspaces $W \in H$ such that
1. The orthogonal projection $pr_+: W \to H_+$ is a Fredholm operator
2. The orthogonal projection $pr_- : W \to H_-$ is a Hilbert-Schmidt operator.

It is known [1] that the subgroup $U_{res}(H)$ of the group $GL_{res}(H)$ acts on $SGr(H)$ transitively, and the stabilizer of the subspace $H_+$ is $U(H_+) \times U(H_-)$. Hence $SGr(H)$ is represented as a homogeneous space $SGr(H) = U_{res}(H)/(U(H_+) \times U(H_-))$. $SGr(H)$ is an infinite-dimensional manifold modelled by Hilbert space $H$ represented as a homogeneous space $H$. The main result (known as Sato’s correspondence) [1], [2] is: The operator of multiplication by $z^{-n}$ (which induces $S \in \mathrm{Hom}(W, H/W)$) transfers by $d\xi(d\eta)^{-1}$ into the commutator $[L^+_n, L]$, i.e. into the right hand side of KP equation. Given $n$, the operator $z^{-n} : H \to H$ defines a fix vector field on $SGr(H)$, i.e. the right hand side of “constant coefficient” Riccati equation [3]. Thus KP-hierarchy corresponds to a countable polystystem of mutually commuting flows of operator Riccati equations on Sato Grassmannian. Analogue theory for noncommutative rings of coefficients was developed in [5], and for multidimensional $x$ in [6]. To investigate KP-hierarchy it is supposed to use the following generalization of the cross ratio [4].

**Operator cross-ratio**

Consider first the finite-dimensional case and subspaces of half dimension. Let $P_1, P_2, P_3, P_4$ be four $n$-dimensional subspaces in $\mathbb{R}^{2n}$ which corresponds to four points with matrix coordinates $P_1, P_2, P_3, P_4$ of the big cell of the manifold $Gr_n(\mathbb{R}^{2n})$. Suppose that $P_1, P_2$ and $P_3, P_4$ define polarizations of $\mathbb{R}^{2n}$, i.e. $P_1 \oplus P_2 = P_3 \oplus P_4 = \mathbb{R}^{2n}$. The polarization $P_i, P_j$ will be denoted by $\Pi_{ij}$. The class of matrices which are similar to the matrix $(P_1 - P_2)^{-1}(P_2 - P_3)(P_3 - P_4)(P_4 - P_1)$ (invers matrices are defined since $\Pi_{12}$ and $\Pi_{34}$ are polarizations) is an invariant of the ordered four points of Grassmannian relative to Möbius transformations [3]. It is called matrix cross-ratio.

The projection parallel to a subspace $P_i$ will be denoted by $\pi_i$ or by the figure $i$ above the arrow which gives the corresponding mapping. If $P_i$ and $P_j$ defines a polarization, then the image of $\pi_i$ in $P_j$ is uniquely defined.

**Theorem 1** Let $\Pi_{12}$ and $\Pi_{34}$ be polarizations. Then $DV(P_1, P_2; P_3, P_4)$ is the matrix of the composition mapping $P_1 \xrightarrow{4} P_3 \xrightarrow{2} P_1$ of the space $P_1$ on itself.
**Proof.** Let \((g, P_3g)\) be the projection of an element \((f, P_1f)\) of the space \(\mathcal{P}_1\) on \(\mathcal{P}_3\) parallel to \(\mathcal{P}_4\). This means that \((f - g, P_1f - P_3g)\) belongs to \(\mathcal{P}_4\), i.e. \(P_1f - P_3g = P_4(f - g)\). Hence, \(g = (P_1 - P_3)^{-1}(P_4 - P_1)f\). Similarly, the projection of an element \((g, P_3g)\) on \(\mathcal{P}_1\) parallel to \(\mathcal{P}_2\) is \((h, P_1h)\), where \(h = (P_2 - P_1)^{-1}(P_2 - P_3)g = (P_1 - P_2)^{-1}(P_2 - P_3)(P_3 - P_1)^{-1}(P_4 - P_1)f\). \(\square\)

**Note 1** The \(\tau\)-function in the theory of integrable systems [7] is in essence only the determinant of an operator cross-ratio; remaining invariants of the corresponding operators was not used before. Hence, in [7] was proved less general assertion concerning only about the determinant of the cross-ratio.

Our theorem allows to remove the restriction of half-dimension and to define cross-ratio for a pair of polarizations of the space \(\mathbb{R}^m\) with similar dimensions, i.e. \(\dim \mathcal{P}_1 = \dim \mathcal{P}_3 = k; \dim \mathcal{P}_2 = \dim \mathcal{P}_4 = m - k\). Moreover, this gives the possibility to define an operator cross-ratio for infinite-dimensional case. In so doing, the invariance of the operator cross-ratio relative to the Möbius group is inherited by the construction itself due to the linearity of projection operators. Note that the composition mapping is invariant but its matrix is defined up to conjugation.

Let \(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4\) be subspaces of linear space \(H\). The operator \(DV(\mathcal{P}_1, \mathcal{P}_2; \mathcal{P}_3, \mathcal{P}_4)\) defined in theorem 1 will be denoted by \(DV_{12,34}\).

We need another expression for the cross ratio. Let \(H = E_1 \oplus E_2\) be a polarization of \(H\). We shall call \(E_1\) the horizontal subspace and \(E_2\) the vertical one. We suppose that \(\mathcal{P}_1, \mathcal{P}_3 \in E_1; \mathcal{P}_2, \mathcal{P}_4 \in E_2\). Let the subspaces \(\mathcal{P}_1, \mathcal{P}_3\) have as its coordinates matrices \(P_1\) and \(P_3\) correspondingly. For the subspaces \(\mathcal{P}_2\) and \(\mathcal{P}_4\) we change the role of the vertical and the horizontal subspaces. Let in this new system of coordinates the subspaces \(\mathcal{P}_2\) and \(\mathcal{P}_4\) have as its coordinates the matrices \(P_2\) and \(P_4\) correspondingly.

**Proposition 1** Let \(\Pi_{12}\) and \(\Pi_{34}\) are polarizations and let the projections parallel to \(\mathcal{P}_2\) and \(\mathcal{P}_4\) are isomorphisms of the spaces \(\mathcal{P}_1\) and \(\mathcal{P}_3\). Then the matrix \(DV(\mathcal{P}_1, \mathcal{P}_2; \mathcal{P}_3, \mathcal{P}_4)\) of the composition mapping \(\mathcal{P}_1 \xrightarrow{4} \mathcal{P}_3 \xrightarrow{2} \mathcal{P}_1\) of the space \(\mathcal{P}_1\) on itself has the form

\[(P_2P_1 - I)^{-1}(P_2P_3 - I)(P_4P_3 - I)^{-1}(P_4P_1 - I)\]

**Proof.**

Let \((g, P_3g)\) be the projection of an element \((f, P_1f)\) of the space \(\mathcal{P}_1\) on \(\mathcal{P}_3\) parallel to \(\mathcal{P}_4\). This means that \((f - g, P_1f - P_3g)\) belongs to \(\mathcal{P}_4\). As soon as the roles of of the vertical and the horizontal subspaces interchange for \(\mathcal{P}_4\) we have \(P_4(P_1f - P_3g) = (f - g)\). The matrix \((P_1P_3 - I)\) is invertible hence we have \(g = (P_1P_3 - I)^{-1}(P_4P_1 - I)f\). Analogously the projection of the element \((g, P_3g)\) on \(\mathcal{P}_1\) parallel to \(\mathcal{P}_2\) is \((h, P_1h)\), where \(h = (P_2P_1 - I)^{-1}(P_2P_3 - I)g = (P_2P_1 - I)^{-1}(P_2P_3 - I)(P_4P_3 - I)^{-1}(P_4P_1 - I)f\). \(\square\)

**Lemma 1** Let projections parallel to \(\mathcal{P}_2\) and \(\mathcal{P}_4\) be isomorphisms of the spaces \(\mathcal{P}_1\) and \(\mathcal{P}_3\). Then \(DV_{12,34} = DV_{34,12}\).

**Proof.** Consider the mappings

\[DV_{12,34} : \mathcal{P}_1 \xrightarrow{4} \mathcal{P}_3 \xrightarrow{2} \mathcal{P}_1; \]
\[DV_{34,12} : \mathcal{P}_3 \xrightarrow{2} \mathcal{P}_1 \xrightarrow{4} \mathcal{P}_3.\]
If one identifies \( \mathcal{P}_1 \) and \( \mathcal{P}_3 \) by using projection \( \pi_4 \), then both maps will coincide. \( \square \)

Thus the cross-ratio does not depend on the order of polarization pairs.

**Lemma 2** Let \( \mathcal{P}_1 \oplus \mathcal{P}_2 = \mathcal{P}_3 \oplus \mathcal{P}_4 = \mathcal{P}_1 \oplus \mathcal{P}_4 = \mathcal{P}_3 \oplus \mathcal{P}_2 = H \). Then \( DV_{12;34} = DV_{14;32}^{-1} \).

**Proof.** Consider the mappings
\[
DV_{12;34} : \mathcal{P}_1 \xrightarrow{4} \mathcal{P}_3 \xrightarrow{2} \mathcal{P}_1;
DV_{14;32} : \mathcal{P}_1 \xrightarrow{2} \mathcal{P}_3 \xrightarrow{4} \mathcal{P}_1.
\]
We have \( \mathcal{P}_1 \xrightarrow{4} \mathcal{P}_3 = (\mathcal{P}_3 \xrightarrow{4} \mathcal{P}_1)^{-1} \) \( \mathcal{P}_1 \xrightarrow{2} \mathcal{P}_3 = (\mathcal{P}_3 \xrightarrow{2} \mathcal{P}_1)^{-1} \), which follows the lemma. \( \square \)

**Lemma 3** Let \( \mathcal{P}_1 \oplus \mathcal{P}_2 = \mathcal{P}_3 \oplus \mathcal{P}_4 = H \). Then \( DV_{12;34} = \text{Id} - DV_{12;34} \).

**Proof.** Design the image of \( h_1 \in \mathcal{P}_1 \) under the mapping \( DV_{12;43} : \mathcal{P}_1 \xrightarrow{3} \mathcal{P}_4 \xrightarrow{2} \mathcal{P}_1 \). We have \( \pi_3 h_1 = h_1 + h_3 \), where \( h_3 \in \mathcal{P}_3 \), and \( h_1 + h_3 \in \mathcal{P}_4 \). Further on, \( \pi_2 (h_1 + h_3) = h_1 + h_3 + h_2 \), where \( h_2 \in \mathcal{P}_2 \), and \( h_1 + h_3 + h_2 \in \mathcal{P}_1 \). Let us rewrite the image of \( DV_{12;43} \) in the form \( h_1 + h_3 + h_2 = h_1 - (h_1 - (h_1 + h_3 + h_2)) \). Since \( h_1 + h_3 \in \mathcal{P}_4 \), and \( h_1 - (h_1 + h_3) \in \mathcal{P}_3 \), then \( h_1 - (h_1 + h_3) \) is the image of \( h_1 \) under the projection \( \pi_4 \) on \( \mathcal{P}_3 \). The subtraction of \( h_2 \) gives the image of projection \( \pi_2 \) on \( \mathcal{P}_1 \). Hence \( (h_1 - (h_1 + h_3 + h_2)) = DV_{12;34} h_1 \). Consequently, \( DV_{12;43} = \text{Id} - DV_{12;34} \). \( \square \)

Lemmas 1-3 define generators of the representation of the group of permutations for four subspaces in the group generated by identity and \( D = DV_{12;34} \). Let us formulate corollaries from these lemmas first for various composition mappings of the space \( \mathcal{P}_1 \) on itself.

\[
DV_{12;34} = \text{Id} - D; \ DV_{13;34} = DV_{14;33} = (\text{Id} - D^{-1})^{-1}, \ DV_{13;24} = (\text{Id} - D)^{-1}, \ DV_{14;32} = D^{-1}.
\]

It remains to consider the case when the images (and the preimages) of composition mappings are nonisomorphic (say have different dimensions). The corresponding operators in this case differs only by direct summands of bigger subspace which maps identically. Reduction to the case of isomorphic subspaces is realized as follows. Let \( \dim \mathcal{P}_2 > \dim \mathcal{P}_1 \), and one wishes to compare \( DV_{12;34} \) with \( DV_{21;43} \). Under the mapping \( DV_{21;43} \) the subspace \( (\mathcal{P}_2 \cap \mathcal{P}_4) \) maps on itself identically because its vectors belong both to \( \mathcal{P}_2 \) and \( \mathcal{P}_4 \). The subspace \( \mathcal{P}_{13} = (\mathcal{P}_2 \oplus \mathcal{P}_3) \) is invariant also because both mappings \( \pi_1 \) and \( \pi_3 \) are projections parallel to this subspace. \( \mathcal{P}_{13} \) is invariant relative to \( DV_{12;34} \) also because it contained both subspaces \( \mathcal{P}_1 \), and \( \mathcal{P}_3 \). Hence, the question is reduced to the structure of restriction of mappings \( DV_{12;34} \) and \( DV_{21;43} \) on \( \mathcal{P}_{13} \). In general case the intersection of subspaces \( \mathcal{P}_2 \) and \( \mathcal{P}_4 \) with \( \mathcal{P}_{13} \) have the same dimension as \( \mathcal{P}_1 \). Hence, one can apply lemmas 1-3 in the space \( \mathcal{P}_{13} \) :

\[
DV_{21;43} = \text{Id} - DV_{21;34} = \text{Id} - DV_{34;21} = DV_{34;12} = DV_{12;34}.
\]

Let us show that operator cross-ratio defines cocycle with values in operators.
Lemma 4 Let two subspaces \( \mathcal{P}_i, i = 1, 2 \) be equivalent relative to the group \( U_{res} \) and three subspaces \( \mathcal{Q}_j, j = 1, 2, 3 \) complete them to the whole space (that is \( \mathcal{P}_i \oplus \mathcal{Q}_j = H, i = 1, 2, j = 1, 2, 3 \)). Then

\[
DV(\mathcal{P}_1 \mathcal{Q}_1, \mathcal{P}_2 \mathcal{Q}_2) DV(\mathcal{P}_1 \mathcal{Q}_3, \mathcal{P}_2 \mathcal{Q}_1) DV(\mathcal{P}_1 \mathcal{Q}_2, \mathcal{P}_2 \mathcal{Q}_3) = \text{Id}, \tag{2}
\]

i.e. the product of these three operators is the identity.

**Proof.** Consider the chain of mappings

\[
\mathcal{P}_1 \xrightarrow{\mathcal{Q}_2} \mathcal{P}_2 \xrightarrow{\mathcal{Q}_1} \mathcal{P}_1 \xrightarrow{\mathcal{Q}_3} \mathcal{P}_2 \xrightarrow{\mathcal{Q}_2} \mathcal{P}_1,
\]

which defines the left hand side of the formula (2). The composition of the second and the third mappings, as well as the composition of the fourth and the fifth ones, are identities. After its reduction the remaining composition gets identity too. \( \Box \)

Let us clarify the geometrical sense of the lemma 4.

Consider the canonical (tautological) bundle \( \gamma \) on the manifold \( Gr(H) \), i.e. the bundle whose fiber at any point \( W \in Gr(H) \) is the linear space \( W \). Introduce the following trivialization of \( \gamma \). Fix a point \( W_+ \in SGr(H) \). The chart \( U_{\gamma} \) on \( \gamma \) is defined by a plane \( V \in H \) that complete \( W_+ \), i.e. \( W_+ \oplus V = H \), and besides the projecting operator \( \pi_V : H \to W_+ \) parallel to the space \( V \) is assumed to be bounded, in other words, for the decomposition \( h = w + v \), where \( h \in H, w \in W_+, v \in V \) the following estimate with a constant \( C \) has to be valid \( ||w|| \leq C||h|| \) (the space \( V \) does not have ”infinitesimally small” angles with \( W_+ \)). The coordinates of a point \( (W; x) \in \gamma \), where \( x \in W \), in the chart \( U_{\gamma} \) will be \( (W, \pi_V x) \in Gr(H) \times W_+ \). Let us calculate the transformation formulas of coordinates from a chart \( U_{\gamma_1} \) to that of \( U_{\gamma_2} \). Let coordinates of a point \( (W; x) \in \gamma \) in the chart \( U_{\gamma_1} \) be \( (W, y) \), where \( y \in W_+ \). Then \( x = \pi_{V_1}^{-1} y \), and the same point has in the chart \( U_{\gamma_2} \) coordinates \( (W, \pi_{V_2} \circ \pi_{V_1}^{-1} y) \). But \( \pi_V = \pi_{V_1}^{-1} \), since \( \pi_V \) is a projector. Hence, the transition function is defined by formula

\[
W_+ \xrightarrow{V_1} W \xrightarrow{V_2} W_+,
\]

which coincides with the cross-ratio of four subspaces \( DV(W_+, V_2; W; V_1) \). By lemma 4, the transition from \( U_{\gamma_1} \) to \( U_{\gamma_2} \), further on, to \( U_{\gamma_3} \), and finally, back to \( U_{\gamma_1} \) gives the cocycle property (2).

So, the transition from a chart \( U_{\gamma_1} \) to a chart \( U_{\gamma_2} \) is defined by the transform \( DV(W_+, V_2; W; V_1) \) which acts on coordinates \( x \in W \) as a linear fractional function from operator coordinates of a plane \( W \). These transforms are defined on intersections of charts \( U_{\gamma} \). But the set of these charts does not cover all the Grassmann manifold. In contrast with the finite-dimensional case, two isomorphic subspaces of infinite-dimensional Hilbert manifolds does not have in general a common complementary subspace \( V \). The exact necessary and sufficient conditions of existence a common complement for two given subspaces was found in ([8]). To overcome this difficulty let us exchange the cross ratio \( DV(W_+, V_2; W; V_1) \) by \( DV(W, V_1; W_+, V_2) \). In view of lemma 1 the cross ratio remains the same but now it defines a transformation of \( W \) instead of \( W_+ \). So we can change \( W_+ \) and the corresponding charts cover all Grassmann manifold.

Hence, coordinate transformations of the canonical bundle \( \gamma \) are endomorphisms \( DV(W, V_1; W_+, V_2) \) that can be regarded as endomorphisms of \( \gamma \) itself. In view of lemma
the cross ratio defines on the Grassmannian a cocycle \( \{DV\} \) with coefficients in the sheaf of endomorphisms of the canonical bundle \( \gamma \), i.e.

\[
\{DV\} \in H^1(SGr(H), \text{End}(\gamma)).
\]

Following the scheme of Atijah [10] and Turin [9] consider the principal bundle \( P \) with the group \( G \) corresponding to the vector bundle \( \gamma \). Take the exact sequence of bundles over \( SGr(H) \)

\[
0 \to L \xrightarrow{f} Q \xrightarrow{g} T \to 0,
\]

(3)

where \( T \) is the tangent bundle to \( SGr(H) \); \( Q \) is the bundle of invariant tangent vector fields on \( P \); \( L \) is the bundle of Lie algebras corresponded to left invariant vector fields on \( G \) (vector fields tangent to fibers). The exact sequence (3) defines an extension of \( T \) by \( L \). Classes of equivalent extensions are in one-to-one correspondence with the elements of \( H^1(SGr(H), \text{Hom}(T, L)) \) — the one-dimensional cohomology group with coefficients in the sheaf \( \text{Hom}(T, L) \). As in the finite-dimensional case, we say that the sequence (3) is split if there exists a homomorphism \( h : T \to Q \) such that \( gh = \text{Id} : T \to T \).

A connection in the principal bundle is a splitting of the corresponding exact sequence. Let us denote by

\[
a(\gamma) \in H^1(SGr(H), \text{Hom}(T, L))
\]

an element corresponding to the extension (3). The coordinate transformations \( \varphi_{ij} = u_i^{-1}u_j \) from a chart \( u_i : U_i \times G \to P|_{U_i} \) to a chart \( u_j : U_j \times G \to P|_{U_j} \) of the bundle \( P \) are defined by the cross ratio \( \varphi_{ij} = DV(W, V_i; W, V_j) \). The chart \( u_i \) induces an isomorphism of tangent bundles and since it commute with the action of \( G \) we obtain the isomorphism \( \hat{u}_i : T_i \oplus L_i \to Q_i \). Here \( T_i, Q_i, L_i \) are the restrictions of the corresponding bundles to the neighbourhood \( U_i \). The sequence (3) splits on this neighbourhood hence there exists a lifting of the identity endomorphism of \( T \) that gives an element \( a_i : T_i \to Q_i \), namely \( a_i(t) = \hat{u}_i(t \oplus 0) \). Put \( a_{ij} = (a_j - a_i) : T_{ij} \to Q_{ij} \). Then \( \{a_{ij}\} \) represents the cocycle \( a(\gamma) \).

Let \( \Omega^1 \) be the sheaf of germs of differential 1-forms on the manifold \( SGr(H) \). We have \( \Omega^1 = \text{Hom}(T, 1) \). Hence \( H^1(SGr(H), \text{Hom}(T, L)) = H^1(SGr(H), (L \times \Omega^1)) \). Thus \( a(\gamma) \in H^1(SGr(H), (L \times \Omega^1)) \). For compact Kähler manifolds in view of isomorphism Dolbeault \( H^1(SGr(H), (L \times \Omega^1)) \) correspondes to cohomologies of \( H^{1,1} \)-type. It generates the ring of characteristic classes of the corresponding bundle. In our case (the manifold \( SGr(H) \) is noncompact) we will by definition consider \( a(P) \) as an analog of the generating Chern class of the canonical bundle \( \gamma \).

The group \( U_{res} \) transitively acts on the big cell of Sato’s Grassmannian but not double transitively, as it is known even in finite-dimensional case. It is natural to find invariants of pairs of points relative to the group \( U_{res} \). These invariants are designed as in finite-dimensional case [3] and are closely related with the operator cross-ratio.

Let \( \mathcal{A}; \mathcal{B} \in Gr_+(H) \). Its coordinates are Hilbert-Shmidt operators \( A \) and \( B \) respectively. The classe of operator \( DV_{AB} := (\text{Id} + A^*A)^{-1}(\text{Id} + A^*B)(\text{Id} + B^*B)^{-1}(\text{Id} + B^*A) \) will be called the operator angle between subspaces \( \mathcal{A} \) and \( \mathcal{B} \). Let us remark that the operator \( (\text{Id} + A^*A) \) is positive definite and invertible.
Two pairs of subspaces $S, T$ and $P, Q$ shall be called equivalent relative to $U_{res}$, if there exists an element $g \in U_{res}$, such that $gS = P$, $gT = Q$.

Two operators $V, W : H_+ \to H_-$ shall be called comparable if there exist two unitary operators $\alpha \in U(H_-)$ and $\beta \in U(H_+)$, such that $W = \alpha V \beta$.

**Theorem 2** Two pairs of points $P, Q$ and $S, T$ of Sato’s Grassmannian are equivalent iff the classes $DV_{PQ}$ and $DV_{ST}$ coincide.

First we prove the following lemma.

**Lemma 5** The operator angle $DV_{AB}$ between subspaces $A$ and $B$ is the operator of the composite mapping

$$A \xrightarrow{B^+} B \xrightarrow{A^+} A.$$  

**Proof.** Let us decompose an element $(h, Ah)$ of the subspace $A$ relative to the basis $B, B^\perp$. If $(x, Bx)$ is its projection on $B$, then $(h, Ah) - (x, Bx)$ is orthogonal to $B$, i.e. $\langle (h - x, Ah - Bx), (y, By) \rangle = 0$ for any $y$ where angle brackets stand for scalar product in $H$. It follows that $x = (\text{Id} + B^* B)^{-1}(\text{Id} + B^* A)h$. Making the decomposition relative to the basis $A, A^\perp$, one obtains the element $(z, Az)$ with $z = (\text{Id} + A^* A)^{-1}(\text{Id} + A^* B)(\text{Id} + B^* B)^{-1}(\text{Id} + B^* A)h$. It remains to apply the proposition 1.

\[ \square \]

**Proof of the Theorem 2.**

Necessity follows from the lemma 5 in view of invariance of the composite mapping $S$.

Sufficiency.

Since $U_{res}$ transitively acts on the big cell, let us transfer $S$ into $P$ and put the coordinate of the obtaining subspace equal to zero. Let the coordinate of $T$ became $W$ and the coordinate of $Q$ became $V$. Then $DV_{PQ} = (\text{Id} + V^* V)^{-1}$, $DV_{ST} = (\text{Id} + W^* W)^{-1}$. Since these operators are similar then operators $V^* V$ and $W^* W$ are similar also. These operators correspond to the quadratic forms $\langle Vx, Vx \rangle$ and $\langle Wx, Wx \rangle$ correspondingly on the space $H_+$. To continue the proof we need the following lemma.

**Lemma 6** Let $V, W : H_+ \to H_-$ are Hilbert-Shmidt operators. Two quadratic forms $\langle Vx, Vx \rangle$ and $\langle Wx, Wx \rangle$ on the space $H_+$ are similar iff operators $V$ and $W$ are comparable.

**Sufficiency.**

Let $W = \alpha V \beta$; $\alpha^* \alpha = \text{Id}$; $\beta^* \beta = \text{Id}$. Then

$$(W^* W) = \beta^* V^* \alpha^* \alpha V \beta = \beta^* (V^* V) \beta.$$  

**Necessity.**

Let $(W^* W) = \beta^* (V^* V) \beta$. Then $\text{Ker} W = \beta(\text{Ker} V)$. Multiply $V$ from the right by the unitary operator $\beta$ such that $\text{Ker} W = \text{Ker} V_1$, where $V_1 = V \beta$. In view of proved sufficiency the operators $V_1^* V_1$ and $W^* W$ are similar. Multiply $V_1$ from the left by the unitary operator $\gamma : H_- \to H_-$, which transfers the image of $\text{Im} V$ into $\text{Im} W$. Denote $\gamma V_1 = V_2$. Operators $V_2^* V_2$ and $W^* W$ remain similar.
The operator $\tilde{V} : (\text{Ker} W)^\perp \rightarrow \text{Im} W$, being restriction of the operator $V_2$, is invertible. Denote by $\tilde{W}$ the restriction of the operator $W$ to the subspace $(\text{Ker} W)^\perp$. The operators $\tilde{V}$ and $\tilde{W}$ are equivalent. Hence, there exists a unitary transform $q$ of the subspace $(\text{Ker} W)^\perp$ such that

$$\tilde{W}^* \tilde{W} = q^* \tilde{V}^* \tilde{V} q. \quad (4)$$

Take $\alpha_1 = q^*$, $\alpha_2 = \tilde{W} q^* \tilde{V}^{-1}$. Then, firstly, $\alpha_2 \tilde{V} \alpha_1 = \tilde{W}$, i.e. $\tilde{V}$ and $\tilde{W}$ are comparable, and, secondly, $\alpha_2^* \alpha_2 = (\tilde{V}^*)^{-1} q \tilde{W}^* \tilde{W} q^* \tilde{V}^{-1} = \text{Id}$, in view of (4). It follows that the operator $\alpha_2$ is unitary. If we extend $\alpha_1$ and $\alpha_2$ by identity to the kernel we obtain that $V$ and $W$ are comparable. $\square$

To complete the proof of the theorem it remains to note that the stabilizer of the point $0$ in $U_{res}$ are block diagonal operators

$$\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix},$$

transferring coordinate $V$ of the subspace $Q$ into the comparable operator. Hence, the pair of subspaces $P, Q$ transforms into the pair $S, T$.

$\square$

**Integrals of KP-Hierarchy**

While finding $\tau$-function [1] one obtains an infinite matrix which was upper-triangular with units on the main diagonal, except for a finite-dimensional block. That was the reason for existing its determinant. It is easily seen that it is the matrix of operator cross-ratio. If we use the lemma 3 and change the order of four subspaces we obtain the infinite matrix $A$ which will be ”asymptotically nilpotent” (i.e. upper-triangular with zeros on the main diagonal) except for a finite-dimensional block. Such matrix $A$ we shall call almost asymptotically nilpotent. Almost asymptotically nilpotent matrix and all its powers have a trace which gives us the possibility to define $\zeta$-function of a cross-ratio.

The cross-ratio relates to four spaces $P_1, P_2, P_3, P_4 \in Gr_+$ the class of operators similar to $\text{DV}(P_1, P_2, P_3, P_4)$. Each subspace $P_i$ defines, due to Sato’s correspondence, a solution of KP-Hierarchy therewith the image of each $P_i$ is obtained by Möbius transform. Hence, the cross-ratio of four spaces remains invariant in the flow of KP. It means that it changes isospectrally. Invariants of almost asymptotically nilpotent matrix relative to isospectral deformations are traces of its powers. It is natural to describe these invariants by $\zeta$-function.

Let we have three solutions $W_i(t), i = 1, 2, 3$ of Riccati equation on Sato’s Grassmannian. Using Sato’s correspondence we obtain three solutions $L_i(t), i = 1, 2, 3$ of KP-Hierarchy on the space $E'$. Take any subspace $W$ and designe the cross-ratio $\text{DV}(W, W_1(t), W_2(t), W_3(t))$. We obtain the operator, spectrum of which does not depend on $t$. Its characteristic numbers gives the set of integrals and $\zeta$-function remains invariant. It is convenient for calculations to take, as some of $W_i(t)$, stationary solutions of the corresponding Riccati equations.

**Schwarz operator.**
Operator cross ratio allows us to define operator analog of Schwarz derivative for curves taking values in Grassmann manifolds [3].

Let $H$ be a Hilbert space equipped with a polarization $H = H_+ \oplus H_-$. Consider the Grassmann manifold $Gr(H)$ defined by the subspace $H_+$ and the dual Grassmannian $\hat{Gr}(H)$ consisted of complementary subspaces. We will consider the case when complementary subspaces are isomorphic to that of $Gr(H)$. In the finite-dimensional case this means that dimensions of $H_+$ and $H_-$ coincide.

Consider a smooth family of subspaces of $H$ depending on onedimensional parameter $s$ such that pairs $(z(s), z(\sigma))$ for $s > 0$ and $\sigma \leq 0$ give a polarization of the space $H$. Our nearest aim is to build an analog of the Schwarz derivative of the curve $z(s)$ at $s = 0$.

Take $s_2 < 0 < s_1 < s_3$. Then pairs of spaces $(z(s_2), z(s_1))$ and $(z(0), z(s_3))$ give polarizations of $H$. From now on we will for brevity write $z_i, z_i',...$ instead of $z(s_i), z'(s_i)...$; values of $z(0), z'(0)...$ will be denoted by $z, z'...$ (without indices). Consider the mapping $f$ depending on four parameters and giving by the cross ratio

$$f(s_2, s_1, 0, s_3) = DV(z_2, z_1; z, z_3) = (z_2 - z_1)^{-1}(z_1 - z)(z - z_3)^{-1}(z_3 - z_2).$$

Let $s_2 \to 0$. At $s_2 = 0$ the cross ratio $f$ gives the identity mapping. The derivative of $f$ at $s = 0$ is equal to

$$\frac{\partial f}{\partial s_2}(0, s_1, 0, s_3) = -(z - z_1)^{-1}z' + (z - z_3)^{-1}z'.$$

Now let $s_3 \to s_1$. At $s_3 = s_1$ the function $f$ and its derivative relative to $s_2$ are equal to zero. We have

$$\frac{\partial^2 f}{\partial s_3 \partial s_2}(0, s_1; 0, s_1) = -(z - z_1)^{-1}z_1'(z - z_1)^{-1}z'.$$ (5)

The right hand side of (5) is defined at $s_1 > 0$ and has a singularity at $s_1 = 0$. Consider the situation when subspaces of a polarization tend to be bound together. Find the asymptotic of (5) when $s_1 \to 0$.

$$z_1' = z' + s_1z'' + \frac{s_1^2z'''}{2} + O(s_1^3),$$
$$s_1 - z = s_1z'\left(Id + \frac{s_1(z')^{-1}z'' + \frac{s_1^2(z')^{-1}z''}{6} + O(s_1^2)}{2}\right),$$
$$(z_1 - z)^{-1} = s_1^{-1}\left(Id - \frac{s_1(z')^{-1}z''}{2} - \frac{s_1^2(z')^{-1}z''}{6} + \frac{s_1^3((z')^{-1}z'')^2}{4} + O(s_1^2)\right).$$

By substituting these expressions for (5) we obtain

$$\frac{\partial^2 f}{\partial s_3 \partial s_2}(0, 0; 0, 0) = (s_1)^{-2}\left(Id + \frac{s_1^2((z')^{-1}z''}{6} - \frac{s_1^3((z')^{-1}z'')^2}{4} + O(s_1^2)\right).$$

In accordance with this formula define the differential operator

$$S(z) = (z')^{-1}z'' - \frac{3}{2}((z')^{-1}z'')^2.$$ (6)

This expression is an analog of the Schwarz derivative. In the finite-dimensional case it was introduced and explored in [3]. The above given deduction shows that the expression
(6), as a cross ratio with the help of which it is built, defines the same class of conjugate operators independently of Möbius transforms in the ambient space. Let us show that (6) is closely relates with Hamiltonian systems and Riccati equations.

Let
\[
\begin{align*}
q' &= A(t)q + p \\
p' &= -B(t)q - A^*(t)p
\end{align*}
\]  
be a linear Hamiltonian system with the Hamiltonian
\[
H = \frac{1}{2}( \langle p, p \rangle + 2 \langle p, Aq \rangle + \langle Bq, q \rangle ),
\]
where \( A(t) \) and \( B(t) \) are \( n \times n \)-matrices; the matrix \( B(t) \) is symmetric; angle brackets stand for scalar product in \( \mathbb{R}^n \). Hamiltonian (8) corresponds to minimization problem of the quadratic functional
\[
\int_{t_0}^{t_1} (\langle q', q' \rangle - 2 \langle Aq, q' \rangle + \langle (A^*A - B)q, q \rangle)\,dt.
\]
The identity coefficient of the first summond is the result of reduction of variational problems with the strong Legendre condition. The system (7) is equivalent to the Euler equation for the functional (9):

\[
q'' + (A^* - A)q' + (B - A' - A^*A)q = 0.
\]

We will consider fundamental systems of solutions to (7) and hence \( p \) and \( q \) are \( n \times n \)-matrices also.

The coefficients matrix of the system (7) is symplectic thus (7) defines a flow on the Lagrange-Grassmann manifold \( \Lambda \) of the space \((p, q)\). Local coordinates of points of \( \Lambda \) are given by matrices \( W = pq^{-1} \). The evolution of coordinates \( W \) is described by Riccati equation
\[
W' = (-Bq - A^*p)q^{-1} - pq^{-1}(Aq + p)q^{-1} = -B - A^*W - WA - W^2.
\]

Let us return to our analog of Schwarz operator.

\[
S(z(t)) = [(z'(t))^{-1}z'']^{-1} - \frac{1}{3}[(z'(t))^{-1}z'']^2 =
(z'(t))^{-1}z'' - \frac{1}{3}[(z'(t))^{-1}z'']^2.
\]

It is convenient to consider \( t \) as changing on a projective line (real or complex) or on a Rimanian surface on which acts Möbius group of linear fractional mappings.

It was shown in [3] that the equivalent class of the image of Schwarz derivative \( S(z(\cdot)) \) is invariant relative to the Möbius group. Namely if one denote by \( M \) a Möbius transform \( M : z \mapsto (C_1z + C_2)(C_3z + C_4)^{-1} \), where \( C_i \) is \( (n \times n) \)-matrices, then there exists a matrix \( K(t) \) such that \( S(M(z(t))) = K(t)S(z(t))K^{-1}(t) \). In other words, Möbius transforms of preimage of the Schwarz operator lead to isospectral change of the image.

Let us describe a connection of the operator \( S \) with the Hamiltonian system (7) or (it is the same) with the Riccati equation (11). Suppose that the matrix \( A \) is symmetric. We find a connection between a solution to Riccati equation \( W \) and the function \( z \) given by the formula
\[
W = -\frac{1}{2}[(z'(t))^{-1}z''] - A.
\]
We have
\[
W' = -\frac{1}{2}(z'(t))^{-1}z'' + \frac{1}{2}(z'(t))^{-1}z''(z'(t))^{-1}z'' - A;
W^2 = \frac{1}{4}(z'(t))^{-1}z''(z'(t))^{-1}z'' + \frac{1}{2}(z'(t))^{-1}z''A + \frac{1}{2}A(z'(t))^{-1}z'' + A^2.
\]
Hence
\[
W' + W^2 = -\frac{1}{2}(z'(t))^{-1}z'' + \frac{3}{4}[(z'(t))^{-1}z'']^2(z'(t))^{-1}z'' - A' \\
\frac{1}{2}(z'(t))^{-1}z''A + \frac{1}{2}A(z'(t))^{-1}z'' + A^2
\]
In view of (13) the last formula gives
\[
W' + W^2 = -\frac{1}{2}S(z) - A' + A^2 - \frac{1}{2}(2W + A)A - \frac{1}{2}A(2W + A)
\]
or
\[
W' + W^2 + WA + AW = -\frac{1}{2}S(z) - A'.
\]
Let us call
\[
S(z(t)) = \left([(z'(t))^{-1}z''] - \frac{1}{2}[(z'(t))^{-1}z'']^2\right) = 2B(t) - A(t)\] (15)

Schwarz equation for Hamiltonian system (7) or (which is the same) for Riccati equation (11).

We proved the following.

**Theorem 3** If \( z(\cdot) \) is a solution to Schwarz equation (15), then \( W(\cdot) \), defined by (13), is a solution to Riccati equation (11).

If \( W(\cdot) \) is a solution to Riccati equation (11), then any function \( z(\cdot) \), being a solution of the linear relative to \( z(\cdot) \) equation (13), is a solution to Riccati equation (15).

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