Nonsmooth mappings with Lipschitz shadowing

Aleksey A. Petrov, Sergei Yu. Pilyugin

Faculty of Mathematics and Mechanics, St. Petersburg State University, University av., 28, 198504, St. Petersburg, Russia

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Abstract. We study conditions under which a piecewise affine mapping has the Lipschitz shadowing property. As an application, we show that there exists a homeomorphism with a nonisolated fixed point having the Lipschitz shadowing property.

1. Introduction. The theory of shadowing of pseudotrajectories (approximate trajectories) is now a well-developed branch of the theory of dynamical systems (see, for example, the monographs [1, 2] and the recent survey [3]).

Recently, a lot of attention has been paid to dynamical systems having special shadowing properties (Lipschitz and Hölder, see [4–6]). In particular, it was shown in [4] that a diffeomorphism having the Lipschitz shadowing property is structurally stable (thus, Lipschitz shadowing property is equivalent to structural stability). The proof in [4] essentially uses the smoothness of the considered dynamical system (the Mañé theorem [7] giving several characterizations of structural stability of diffeomorphisms is applied).

At the same time, it is possible to define the Lipschitz shadowing property for homeomorphisms (and endomorphisms) of a metric space (see below).

Of course, if a homeomorphism is topologically conjugate to a structurally stable diffeomorphism and both the conjugacy and its inverse are uniformly Lipschitz continuous, then the homeomorphism has the Lipschitz shadowing property. In this connection, it is natural to ask: Are homeomorphisms having the Lipschitz shadowing property similar (in a sense) to structurally stable diffeomorphisms?

In this short note, we give an example of a homeomorphism of the segment having the Lipschitz shadowing property and a nonisolated fixed point. This example shows that the answer to the above question is negative.

Let us give the corresponding definitions (for the case of an endomorphism; for a homeomorphism, the definition is literally the same).
Let \((M, \text{dist})\) be a metric space and let \(f : M \rightarrow M\) be a continuous mapping (we do not distinguish \(f\) and the semi-dynamical system generated by \(f\)). As usual, a sequence \(\pi = \{p_k \in M; \ k \in \mathbb{Z}\}\) is called a trajectory of \(f\) if
\[ p_{k+1} = f(p_k), \ k \in \mathbb{Z}. \]

Fix a \(d > 0\). We say that a sequence \(\xi = \{x_k \in M; \ k \in \mathbb{Z}\}\) is a \(d\)-pseudotrajectory of \(f\) if
\[ \text{dist}(x_{k+1}, f(x_k)) \leq d, \ k \in \mathbb{Z}. \] (1)

The (standard) shadowing property of \(f\) means that, given an \(\varepsilon > 0\), we can find a \(d > 0\) such that for any \(d\)-pseudotrajectory \(\xi = \{x_k\}\) of \(f\) there is a trajectory \(\pi = \{p_k\}\) satisfying the inequalities
\[ \text{dist}(x_k, p_k) \leq \varepsilon, \ k \in \mathbb{Z}. \] (2)

Finally, we say that \(f\) has the Lipschitz shadowing property if there exist \(L, d_0 > 0\) such that for any \(d\)-pseudotrajectory \(\xi\) of \(f\) with \(d \leq d_0\) there is a trajectory \(\pi\) satisfying inequalities (2) with \(\varepsilon = Ld\).

The structure of the paper is as follows. In Sec. 2, we prove a general sufficient condition under which a “piecewise affine” mapping of \(\mathbb{R}^n\) has a “conditional” Lipschitz shadowing property (this means that only pseudotrajectories satisfying some additional assumptions are shadowable). In Sec. 3, we construct the above-mentioned example of a homeomorphism of the segment having the Lipschitz shadowing property and a nonisolated fixed point (and apply to it the result of Sec. 2).

2. Conditional shadowing result. To simplify presentation, we consider a Lipschitz continuous mapping \(f : \mathbb{R}^n \rightarrow \mathbb{R}^n\) with Lipschitz constant \(L_0\) (without loss of generality, we assume that \(L_0 \geq 1\)) for which there exists a family of sets \(G_l \subset \mathbb{R}^n, l \in \Lambda\), with disjoint interiors such that the following conditions hold.

First, for any \(l \in \Lambda\) we fix complementary orthogonal linear subspaces \(S_l\) and \(U_l\) of \(\mathbb{R}^n\) (let their dimensions be \(s_l\) and \(u_l\), respectively) with coordinates \(\xi \in S_l\) and \(\eta \in U_l\) and denote
\[ N(\Delta, p) := \{ p + (\xi, \eta) : |\xi|, |\eta| \leq \Delta \} \]
for a point \(p \in G_l\) and number \(\Delta > 0\).

Let
\[ H_l(\Delta) = \{ p : N(\Delta, p) \subset G_l \}. \]
Condition 1. There exists a constant \( \lambda \in (0, 1) \) with the following property. For any \( l \in \Lambda \) there exist \( s_l \times s_l \) and \( u_l \times u_l \) matrices \( A_l \) and \( B_l \) such that

\[
\| A_l \| \leq \lambda \quad \text{and} \quad \| (B_l)^{-1} \| \leq \lambda
\]

and if \( p \in H_l(\Delta) \) for some \( \Delta > 0 \) (so that \( p + (\xi, \eta) \in N(\Delta, p) \)), then

\[
f(p + (\xi, \eta)) = f(p) + (A_l \xi, B_l \eta).
\] (4)

Remark 1. We impose these simple conditions on the mapping \( f \) for the following two reasons:

– they allow us to make the proofs and estimates maximally “transparent” (of course, similar results are valid under more general hyperbolicity conditions on \( f \) in the sets \( G_l \));

– precisely these conditions are satisfied in our main application, Theorem 2 below.

First we note that the following statement is proved by a standard reasoning (for example, it is enough to consider images under the mapping \( f \) of the “rectangles” \( N(L_1 d, x_j), \ 0 \leq j < m \)).

Lemma 1. Let

\[
L_1 = \frac{1}{1 - \lambda}.
\] (5)

If \( \{x_k : 0 \leq k \leq m\} \), where \( m > 0 \), is a finite \( d \)-pseudotrajectory of \( f \) (this means that inequalities (1) are satisfied for \( 0 \leq k \leq m - 1 \)) for which there exists an index \( l \in \Lambda \) such that

\[
x_j \subset H_l(L_1 d), \ 0 \leq j < m,
\]

then there exists a point \( y \) such that

\[
f^j(y) \in N(L_1 d, x_j), \ 0 \leq j \leq m.
\] (6)

Now we define geometric objects which are important in what follows.

Let \( p \in G_l, \ l \in \Lambda \); introduce coordinates \((\xi, \eta)\) such that \( p \) is the origin and the coordinate subspaces are parallel to \( S_l \) and \( U_l \), respectively. Fix \( \Delta_1, \Delta_2 > 0 \). Consider a continuous function \( \Xi(\eta) \) that maps the disk

\[
\{ \eta : \eta \in U_l, \ |\eta| \leq \Delta_1 \}
\]
to $S$, and such that

$$|\Xi(\eta)| \leq \Delta_2, \ |\eta| \leq \Delta_1.$$ 

Let $D$ be the graph of $\Xi(\eta)$. Denote by $\mathcal{D}(\Delta_1, \Delta_2, p)$ the set of such disks $D$.

The following lemma is geometrically obvious.

**Lemma 2.** If $p \in H_l(\Delta)$, $f(p) \in G_l$, and $D \in \mathcal{D}(\Delta_1, \Delta_2, p)$, where $\Delta_1, \Delta_2 \leq \Delta$, then $f(D)$ contains a disk $D^*$ such that $D^* \in \mathcal{D}(\Delta_1/\lambda, \lambda \Delta_2, f(p))$.

**Remark 2.** It is easily seen that in the proof of the main result we use the statements of Lemmas 1 and 2 (Lipschitz shadowing in $G_l$ with constant $L_1$ and properties of the images of disks under $f$) plus the “transversality condition when we pass from one domain to another” (Condition 2 below). The assumed linearity of $f$ in the domains $G_l$ just allows us to make Lemmas 1 and 2 obvious.

**Condition 2.** There exist numbers $K \geq L_0 + 1$ and $\delta_0 > 0$ with the following properties. If $L_2 = L_1 + L_0 + 1$, $d \leq \delta_0$, and there exist three points $p, q, r$ such that

(2.1) $p \in G_l$ and $f^2(p) \in G_m$ for some $l, m \in \Lambda$ with $l \neq m$;

(2.2) $q \in H_l(Kd)$ and $r \in H_m(Kd)$;

(2.3) $|p - q| \leq L_1 d$ and $|f^2(p) - r| \leq L_2 d$;

and

(2.4) $D \in \mathcal{D}(Kd, d, q)$,

then the image $f^2(D)$ contains a disk $D^*$ such that $D^* \in \mathcal{D}(d, Kd, r)$.

**Remark 3.** The above condition is applied in the situation where points $p$ and $f^2(p)$ belong to different sets $G_l$ and $G_m$ and we know nothing about the position of the point $f(p)$; in a sense, this condition means that the image $f^2(D)$ is “uniformly transverse” to the “stable subspace” for $f$ at a point $r$ that is close enough to $f^2(p)$.

Of course, an analog of this condition can be formulated for any pair of points $p$ and $f^m(p)$, but for our main application (see Sec. 3), the present form of Condition 2 is enough.

We prove the following “conditional” theorem on Lipschitz shadowing for a mapping $f$ satisfying the above-formulated conditions. In Theorem 1, we deal with finite $d$-pseudotrajectories $\{x_k : 0 \leq k \leq T\}$ of $f$ and show that there exist $\delta_0$ and $\mathcal{L}$ such that any such finite $d$-pseudotrajectory with $d \leq \delta_0$ is $\mathcal{L}d$-shadowed by a fragment of an exact trajectory of $f$. It is shown that $\delta_0$ and $\mathcal{L}$ depend only on $f$ and not on the length of the pseudotrajectory. It is easily seen that if the phase space of a dynamical system generated by $f$
homeomorphism is locally compact, then such a “finite Lipschitz shadowing property” implies the Lipschitz shadowing property (cf. [1, Lemma 1.1.1] and the proof of Lemma 4 below).

**Theorem 1.** Let $X = \{x_k : 0 \leq k \leq T\}$ be a finite $d$-pseudotrajectory of $f$ with $d \leq \delta_0$ (where $\delta_0$ is from Condition 2). Assume that there exist (not necessarily different) indices $l_0, l_1, \ldots, l_t \in \Lambda$ with $l_{i+1} \neq l_i$ and integers

$$0 = m_0 < n_0 < m_1 < n_1 < m_2 < n_2 < \cdots < m_t < n_t = T,$$

where $m_{j+1} = n_j + 2$, $j = 0, \ldots, t - 1$, with the following properties:

(a) $x_k \in H_{l_j}(K_1 d), \quad m_j \leq k \leq n_j, \quad j = 0, \ldots, t,$

(b) there exists a positive number $\mu$ for which the inequalities

$$\mu_j := n_j - m_i \geq \mu, \quad j = 0, \ldots, t,$$

and

$$\lambda^\mu K < 1$$

are satisfied.

Let

$$L = L_0(L_1 + 2K) + 1.$$  \hspace{1cm} (10)

Then there exists a point $z$ such that

$$|f^k(z) - x_k| \leq Ld, \quad k = 0, \ldots, T.$$ \hspace{1cm} (11)

**Remark 4.** Let us emphasize that only adjacent indices $l_{i+1}$ and $l_i$ are assumed to be different; thus, we do not exclude the situation where the pseudotrajectory “returns” to some sets $G_i$ several times.

In the proof of Theorem 1, we apply the following corollary which is a direct corollary of Lemma 2.

**Lemma 3.** Assume that for some $d > 0$ and set $G_i$ there exists a point $y$ and a number $m$ such that

$$N(Kd, f^k(y)) \subset G_i, \quad 0 \leq k \leq m,$$  \hspace{1cm} (12)

and

$$\lambda^m K < 1.$$ \hspace{1cm} (13)
Then for any disk \( D \in \mathcal{D}(d, Kd, y) \) there exists a subset \( D' \subset D \) such that
\[
 f^k(D') \subset N(Kd, f^k(y)), \quad 0 \leq k \leq m, \tag{14}
\]
and \( f^m(D') \) contains a disk \( D^* \in \mathcal{D}(Kd, d, f^m(y)) \).

Proof of Theorem 1.  Fix a \( d \leq \delta_0 \). Condition (a) allows us to apply Lemma 1 to any “fragment”
\[
 \{ x_k : m_j \leq k \leq n_j \}, \quad j = 0, \ldots, t,
\]
of the pseudotrajectory \( X \) and to find points \( y_j, j = 0, \ldots, t, \) such that
\[
 |f^k(y_j) - x_{m_j+k}| \leq L_1d, \quad 0 \leq k \leq \mu_j. \tag{15}
\]

It follows from condition (11) that analogs of inclusions (12) in Lemma 3 are satisfied for the points \( y_j, j = 0, \ldots, t \):
\[
 N(Kd, f^k(y_j)) \subset G_{ij}, \quad 0 \leq k \leq \mu_j, \quad j = 0, \ldots, t. \tag{16}
\]

Since \( \mu_0 = n_0 - m_0 \geq \mu \) (see (8)), it follows from (9) that condition (13) of Lemma 3 is satisfied for \( y = y_0 \) and \( m = \mu_0 \).

Let \((\xi, \eta)\) be coordinates with coordinate subspaces parallel to \( S_{l_0} \) and \( U_{l_0} \), respectively, for which \( y_0 \) is the origin.

Set
\[
 D_{0,0} = \{ (0, \eta) : |\eta| \leq d \}.
\]
Clearly, \( D_{0,0} \in \mathcal{D}(d, Kd, y_0) \).

Applying Lemma 3, we find a subset \( D_0 \) of \( D_{0,0} \) such that analogs of inclusions (14) are valid, i.e.,
\[
 f^k(D_0) \subset N(Kd, f^k(y_0)), \quad 0 \leq k \leq \mu_0,
\]
and \( f^{\mu_0}(D_0) \) contains a disk \( D^*_0 \in \mathcal{D}(Kd, d, f^{\mu_0}(y_0)) \).

Let us denote \( p = x_{n_0}, q = f^{\mu_0}(y_0), \) and \( r = y_1 \). It follows from (15) (with \( j = 0 \) and \( k = n_0 \)) that
\[
 |p - q| = |x_{n_0} - f^{\mu_0}(y_0)| = |x_{n_0} - f^{\mu_0}(y_0)| \leq L_1d. \tag{17}
\]

Since \( X \) is a \( d \)-pseudotrajectory,
\[
 |f^2(p) - x_{m_1}| = |f^2(x_{n_0}) - x_{n_0+2}| \leq |f^2(x_{n_0}) - f(x_{n_0+1})| + |f(x_{n_0+1}) - x_{n_0+2}| \leq (L_0 + 1)d
\]
(recall that $L_0$ is the Lipschitz constant of $f$). Now we estimate
\[ |f^2(p) - r| \leq |f^2(p) - x_m| + |x_m - y_1| \leq (L_0 + L_1 + 1)d = L_2d \] (18)
(we again refer to (15) to estimate the term $|x_m - y_1|$).

Condition 2 and estimates (17) and (18) imply that $f^2(D^*_0)$ contains a disk $D_{1,0} \in D(d, Kd, y_1)$. After that, we find a subset $D_1 \subset D_{1,0}$ that has properties similar to those of $D_0$, and so on.

As a result, we construct sets $D_j$, $j = 0, \ldots, t$, such that
\[ D_{j+1} \subset f^{\mu_j+2}(D_j), \quad j = 0, \ldots, t - 1, \]
and
\[ f^k(D_j) \subset N(Kd, f^k(y_j)), \quad 0 \leq k \leq \mu_j, \quad j = 0, \ldots, t. \] (19)

Hence, there exists a point $z \in D_0$ such that
\[ f^{m_j}(z) \in D_j, \quad j = 0, \ldots, t. \]

It follows from inclusions (19) and estimates (15) that
\[ |f^k(z) - x_k| \leq (L_1 + 2K)d < Ld, \quad m_j \leq k \leq n_j, \quad j = 0, \ldots, t. \] (20)

It remains to estimate the values $|f^k(z) - x_k|$ for $k = n_j + 1$. Let $z' = f^{n_j}(z)$. Then it follows from (15) that
\begin{align*}
|f(z') - x_{n_j+1}| &\leq |f(z') - f(x_{n_j})| + |f(x_{n_j}) - x_{n_j+1}| \\
&\leq L_0(L_1 + 2K)d + d = Ld.
\end{align*}
This completes the proof of Theorem 1. □

**Remark 5.** In parallel to the shadowing property, the so-called inverse shadowing property is also studied (see, for example, [8]). It seems interesting to obtain an analog of Theorem 1 for the Lipschitz inverse shadowing. Note that the reasoning applied above in the proof of Theorem 1 cannot be directly transferred to the case of inverse shadowing.

3. Example. Consider the segment
\[ I_0 = [-7/6, 4/3] \]
and a mapping $f_0 : I_0 \to I_0$ defined as follows:
\[ f_0(x) = 1 + (x - 1)/2, \quad x \in [1/3, 4/3], \]
\[ f_0(x) = 2x, \quad x \in (-1/3, 1/3). \]
\[ f_0(x) = -1 + (x + 1)/2, \quad x \in [-7/6, -1/3]. \]

Clearly, the restriction \( f^* \) of \( f_0 \) to \([-1, 1]\) is a homeomorphism of \([-1, 1]\) having three fixed points: the points \( x = \pm 1 \) are attracting and the point \( x = 0 \) is repelling (and this homeomorphism \( f^* \) is an example of the so-called “North Pole – South Pole” dynamical system; every trajectory starting at a point \( x \neq 0, \pm 1 \) tends to an attractive fixed point as time tends to \(+\infty\) and to the repelling fixed point as time tends to \(-\infty\)).

Now we define a homeomorphism \( f : [-1, 1] \to [-1, 1] \). For an integer \( n \geq 0 \), denote \( N_n = 2^{-(n+2)} \), and set
\[
f(x) = N_n f_0(N_n^{-1}(x - 3N_n)) + 3N_n, \quad x \in (2N_n, 4N_n]. \tag{21}
\]
This defines \( f \) on \((0, 1]\). Set \( f(0) = 0 \) and \( f(x) = -f(-x) \) for \( x \in [-1, 0) \).

Clearly, \( f \) is a homeomorphism with a nonisolated fixed point \( x = 0 \) (for example, every point \( x = \pm 2^{-n} \) is fixed). Let us note that in a neighborhood of any fixed point (with the exception of \( x = 0 \)), \( f \) is either linearly expanding with coefficient 2 or linearly contracting with coefficient 1/2.

**Theorem 2.** The homeomorphism \( f \) has the Lipschitz shadowing property.

Before proving Theorem 2, we prove two auxiliary lemmas (and refer to Theorem 1 in the first of them).

In what follows, we denote by \( N(d, A) \) the closed \( d \)-neighborhood of a set \( A \).

**Lemma 4.** The mapping \( f_0 \) has the Lipschitz shadowing property on \( I_0 \).

**Proof.** Let \( \xi = \{x_k \in I_0 : k \in \mathbb{Z}\} \) be a \( d \)-pseudotrajectory of \( f_0 \). In the following (very rough) estimates, we, as usual, decrease values of \( d \), if necessary; every time, the chosen value of \( d \) is not more than the previous values. First we assume that \( d \leq d_1 < 1/24 \).

Note that
\[
f_0(-7/6) = -13/12, \quad f_0(4/3) = 7/6. \]

Set
\[ I'_0 = [-27/24, 29/24]. \]
Since \( \xi \) must belong to \( N(d, f_0(I_0)) \), we conclude that
\[
\xi \subset I'_0. \tag{22}
\]
Now let us describe the possible position of \( \xi \) in \( I'_0 \).
We note that 

\[ f_0(5/12) = 17/24. \]

If there exists an index \( k \) such that \( |x_k| \geq 5/12 \), then

\[ |x_{k+i}| > 16/24 > 5/12, \quad i \geq 1. \]

Note that both \( f_0 \) and \( f^{-1}_0 \) have Lipschitz constant 2. Thus, if \( \xi \) is a \( d \)-pseudotrajectory of \( f_0 \), then \( \xi \) is a \( 2d \)-pseudotrajectory of \( f^{-1}_0 \).

If there exists an index \( k_0 \) such that \( 1/4 \leq |x_{k_0}| \leq 5/12 \), then

\[ |f^{-1}_0(x_{k_0})| \in [1/8, 5/24], \]

and there exists a \( d_2 > 0 \) such that if \( d \leq d_2 \), then

\[ |x_k| \leq 5/24 + 2d, \quad k < k_0. \]  \hspace{1cm} (23)

Thus, only one of the following possibilities can be realized for \( d < d_2 \):

1. \( |x_k| \leq 1/4 \) for \( k \in \mathbb{Z} \);
2. \( 5/12 \leq |x_k| \leq 29/24 \) for \( k \in \mathbb{Z} \);
3. there exists an index \( k_0 \) such that \( 1/4 \leq |x_{k_0}| \leq 5/12 \) and inequalities (23) hold.

In cases (1) and (2), \( \xi \) belongs to a domain in which \( f_0 \) is hyperbolic (and \( \xi \) is uniformly separated from the boundaries of the domain); by Lemma 1, there exists a \( d_3 > 0 \) such that if \( d < d_3 \), then \( \xi \) is \( 2d \)-shadowed by an exact trajectory of \( f_0 \).

Consider the remaining case (3) (and let, for definiteness, \( k_0 = 1 \) and \( x_1 > 0 \); the case \( x_1 < 0 \) is treated similarly).

Denote \( p = x_0 \). Set \( G_0 = [-1/3, 1/3] \) and \( G_1 = [1/3, 29/24] \). Thus, \( p \in G_0 \).

As was mentioned, we can take \( L_0 = 2 \) and the statement of Lemma 1 holds for \( G_0 \) and \( G_1 \) with \( L_1 = 2 \).

Since \( p \in [1/8 - 2d, 5/24 + 2d] \), there exists a \( d_4 > 0 \) such that if \( d < d_4 \), then

\[ 5/24 + 4d < 1/3 \text{ and } N(5d, f^2(p)) \subset G_1. \]  \hspace{1cm} (24)

Take a point \( q \) such that

\[ |p - q| \leq L_1d = 2d. \]

In this case, it follows from (24) that \( q \in G_0 \), and, defining disks from \( D(\Delta_1, \Delta_2, q) \), we must take \( S_0 = \{0\} \) and \( U_0 = \mathbb{R} \). Thus, if \( K > 0 \), then the set \( D(Kd, d, q) \) contains precisely one disk

\[ D = [q - Kd, q + Kd]. \]
If $K > 2$, then $D$ contains the disk
\[ D' = [p - K'd, p + K'd], \]
where $K' = K - 2$.

Clearly, $f^2(D')$ contains the disk
\[ D'' = [f^2(p) - K'd/4, f^2(p) + K'd/4]. \]

If a point $r$ satisfies the inequality
\[ |f^2(p) - r| \leq L_2d = 5d, \]
it follows from the second inclusion in (24) that $r \in G_1$, and, defining disks from $D(\Delta_1, \Delta_2, r)$, we must take $U_1 = \{0\}$ and $S_1 = \mathbb{R}$. Thus, the set $D(d, Kd, r)$ consists of points $r'$ such that $|r' - r| \leq d$.

It follows that Condition 2 is satisfied if $d_0 \leq d_4$ and $K'/4 \geq 6$. Thus, it is enough to take $K = 26$.

Now, when $L_0$, $L_1$, and $K$ are fixed, it is easily seen that there exists a $d_0 > 0$ such that if $d \leq d_0$, then
\[ x_k \in H_0(K_1d) = [-1/3 + K_1d, 1/3 - K_1d], \quad k \leq 0, \quad (25) \]
and
\[ x_k \in H_1(K_1d) = [1/3 - K_1d, 29/24 + K_1d], \quad k \geq 2. \quad (26) \]

To apply Theorem 1, we fix a natural number $n$ and change indices of points of the $d$-pseudotrajectory $\xi = \{x_k\}$ to obtain a $d$-pseudotrajectory $\xi^{(n)} = \{x_k^{(n)}\}$, where
\[ x_k^{(n)} = x_{k-n}, \quad k \in \mathbb{Z}. \]

Setting $m_0 = 0$, $n_0 = n$, $m_1 = n + 2$, $m_2 = 2n + 2$, we conclude from inclusions (25) and (26) that
\[ x_k^{(n)} \in H_0(K_1d), \quad m_0 \leq k \leq n_0, \]
and
\[ x_k^{(n)} \in H_1(K_1d), \quad m_1 \leq k \leq n_1. \]
Thus, condition (a) of Theorem 1 is satisfied.

It is clear that if $2^{-n-1}K < 1$, then condition (b) of Theorem 1 is satisfied as well.

By Theorem 1, there exists a point $z^{(n)}$ such that
\[ |f^k(z^{(n)}) - x_k^{(n)}| \leq Ld, \quad 0 \leq k \leq 2n + 2. \]
Hence, if $\zeta^{(n)} = f^n(z^{(n)})$, then

$$|f^k(\zeta^{(n)}) - x_k| \leq Ld, \quad -n \leq k \leq n + 2. \quad (27)$$

Let $\zeta$ be a limit point of the sequence $\zeta^{(n)}$. Passing to the limit as $n \to \infty$ in (26) and taking into account that $f$ is a homeomorphism (so that any $f^k$ is continuous), we see that

$$|f^k(\zeta) - x_k| \leq Ld, \quad k \in \mathbb{Z}.$$  

□

The following statement is almost obvious.

**Lemma 5.** Let $g$ be a mapping of a segment $J$ and let numbers $M > 0$ and $m$ be given. Consider the mapping

$$g'(y) = M^{-1}g(M(y - m)) + m$$

on the set

$$J' = \{y : M(y - m) \in J\}.$$  

If $g$ has the Lipschitz shadowing property with constants $L, d_0$, then $g'$ has the Lipschitz shadowing property with constants $L, M^{-1}d_0$.

**Proof.** First we note that if $\{y_k\}$ is a $d$-pseudotrajectory of $g'$ with $d \leq d_0/M$ and $x_k = M(y_k - m)$, then

$$g(x_k) - x_{k+1} = M(g'(y_k) - y_{k+1}).$$

Hence, $\{x_k\}$ is an $Md$-pseudotrajectory of $g$.

Since $Md \leq d_0$, there exists a point $p$ such that

$$|g^k(p) - x_k| \leq LMd.$$  

Set $p' = M^{-1}p + m$. Then, obviously,

$$|(g')^k(p') - y_k| = M^{-1}|g^k(p) - x_k| \leq Ld.$$  

□

Let us prove Theorem 2.

For a natural $n$, define the segment

$$I_n = [\alpha_n, \beta_n] = [11N_n/6, 13N_n/3]$$

and note that formula (21) defining $f$ for $x \in (2N_n, 4N_n]$ is, in fact, valid for $x \in I_n$. 

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It follows from the equalities
\[ f(\alpha_n) = 23N_n/12, \quad f(\beta_n) = 25N_n/6 \]
that \( f(I_n) \subset N(N_n/12, I_n) \). Thus, if \( d < \delta(n) = N_n/12 \) and \( \{x_k\} \) is a \( d \)-pseudotrajectory of \( f \) that intersects \( I_n \), then \( \{x_k\} \subset I_n \).

Let \( d_0 \) and \( \mathcal{L} \) be the constants of Lipschitz shadowing for \( f_0 \) given by Lemma 4. Since \( d_0 < 1/12 \), it follows from Lemma 5 that if \( \{x_k\} \) is a \( d \)-pseudotrajectory of \( f \) that intersects \( I_n \) for some \( n > 0 \), then \( \{x_k\} \) is \( \mathcal{L}d \)-shadowed. Of course, a similar statement holds for the segments \( I'_n = [-\beta_n, -\alpha_n] \).

To complete the proof, consider a \( d \)-pseudotrajectory \( \xi = \{x_k\} \) of \( f \) with \( d \leq d_0 \) and find the maximal \( n_0 \) for which \( d < \delta(n_0) \). Note that then
\[ d \geq N_{n_0+1}/12. \]

If \( \xi \) intersects one of the segments \( I_n \) or \( I'_n \) with \( n \leq n_0 \), then everything is proved.

Otherwise,
\[ |x_k| \leq \alpha(n_0) = 11N_{n_0+1}/3 \leq 44d, \]
and \( \xi \) is \( 44d \)-shadowed by the rest point \( x = 0 \). \( \square \)

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