LI-YORKE CHAOS FOR COMPOSITION OPERATORS ON ORLICZ-LORENTZ SPACE

RAJAT SINGH, ADITI SHARMA, AND ROMESH KUMAR

ABSTRACT. In this paper, we study the Li-Yorke chaotic composition operators on Orlicz-Lorentz spaces. In fact necessary and sufficient conditions are given for Li-Yorke chaotic composition operator $C_\tau$ on $L^{\phi,h}(\mu)$. Further, we present the equivalent conditions for $C_\tau$ to be Li-Yorke chaotic. We extends the result of [8] to Orlicz-Lorentz spaces.

1. INTRODUCTION AND PRELIMINARIES

Over the last two decades various authors have explored chaotic operators intensively. The notion of “Chaos” was first introduced into mathematical literature in the context of interval map by Li and Yorke [19] and became most famous. Godefroy and Shapiro [13] were the first to introduce chaos into linear dynamics by using Devaney’s notion of chaos. Other terms related to chaos include Li-Yorke chaos, distributional chaos, specification property, etc. There are several fascinating papers of Li-Yorke chaos and distributional chaos for linear operator see ([5], [6], [8], [19]) and reference therein. Further in [12], the Li-Yorke chaos for composition operator on Orlicz space were studied. Since the Orlicz-Lorentz spaces offer a common generalization of Orlicz spaces and Lorentz spaces, so it is natural to expand the study of composition operators to a more general class.

The paper is structured as follows: Section 1 is introductory and we cite certain definitions and results which will be used throughout this paper. In Section 2, we explore the Li-Yorke chaotic composition operators on Orlicz-Lorentz spaces.

Now, we recall some basic facts about the Orlicz-Lorentz space which will be useful throughout this paper. For more details on Orlicz-Lorentz space we refer to [3], [11], [20] and [22]. Let $(X, \mathcal{A}, \mu)$ be a measure space with positive measure and $L^0$ represent the space of all equivalence classes of measurable functions on $X$ which are identified as $\mu$-a.e. We now define the distribution function $\mu_g$ of $g \in L^0$ on $(0, \infty)$ as

$$\mu_g(\lambda) = \mu\{x \in X : |g(x)| > \lambda\},$$

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and the non-increasing rearrangement of \( g \) on \((0, \infty)\) is defined as

\[
g^*(t) = \inf\{\lambda > 0 : \mu_g(\lambda) \leq t\} = \sup\{\lambda > 0 : \mu_g(\lambda) > t\}.
\]

A function \( \varphi : [0, \infty) \rightarrow [0, \infty] \) is an Orlicz function if it is convex function with \( \varphi(0) = 0 \) and \( \varphi(s) \to \infty \) as \( s \to \infty \) such that \( \varphi(s) < \infty \) for some \( 0 < s < \infty \). Let \( K = [0, \mu(X)] \). Let \( h : K \to (0, \infty) \) be locally integrable and non-increasing function with respect to Lebesgue measure which is known as weight function. For a given \( \varphi \) and \( h \), the space

\[
\mathbb{L}^{\varphi,h}(\mu) = \{g \in L^0 : I_{\varphi,h}(\lambda g) < \infty \text{ for some } \lambda > 0\},
\]

where

\[
I_{\varphi,h}(\lambda g) = \int_K \varphi(\lambda g^*(t))h(t)dt.
\]

is known as Orlicz-Lorentz space. Now, we define the norm of \( g \) as

\[
||g||_{\varphi,h} = \inf\{\lambda > 0 : I_{\varphi,h}(g/\lambda) \leq 1\}.
\]

The space \( \mathbb{L}^{\varphi,h}(\mu) \) with the above norm is a Banach space. Moreover, if \( A \in \mathcal{A} \) with \( 0 < \mu(A) < \infty \), then \( ||\chi_A||_{\varphi,h} = \frac{1}{\varphi^{-1}\left(\frac{1}{\int_{\mu(A)}^{\mu(A)}h(t)dt}\right)} \). If \( h = 1 \), then \( \mathbb{L}^{\varphi,h} \) represents the Orlicz space \( \mathbb{L}^{\varphi}(\mu) \) and when \( \varphi(t) = t \), then it represents the Lorentz space \( \mathbb{L}^w(\mu) \) see [17].

First of all, we now state some growth conditions on the \( \varphi \), the Orlicz function \( \varphi \) satisfies the \( \Delta_2 \)-condition (for large \( s \)), if there is a positive constant \( M \) (a positive constant \( M \) and \( s_o > 0 \) with \( \varphi(s_o) < \infty \)) such that

\[
\varphi(2s) \leq M \varphi(s), \text{ for all } s > 0.
\]

We suppose that \( \varphi \) is left continuous at \( b_\varphi \), where

\[
b_\varphi = \sup\{s > 0 : \varphi(s) < \infty\}.
\]

Also, define

\[
a_\varphi = \inf\{s > 0 : \varphi(s) > 0\}.
\]

Throughout this paper, we take \((X, \mathcal{A}, \mu)\) to be measure space and let \( \tau : X \to X \) be non-singular measurable transformation. Then \( \tau \) induces a operator \( C_\tau \) from \( \mathbb{L}^{\varphi,h} \) into \( L^0(X) \) defined as

\[
C_\tau g = g \circ \tau, \forall g \in \mathbb{L}^{\varphi}(\mu)
\]

The non-singularity of \( \tau \) assures that \( C_\tau \) is well defined. Now, we state the lemma which will be used in the proof of Theorem 2.3.

**Lemma 1.1.** Let \( \tau \) be measurable non-singular transformation and \( \varphi \) be an Orlicz function which satisfies the \( \Delta_2 \)-condition for all \( s > 0 \). Then the following are equivalent:
(a) There exist some $k > 0$ and $A \in \mathcal{A}$,
\[ \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau^{-n}(A))} h(t) \, dt} \right) \leq k \leq \varphi^{-1} \left( \frac{1}{\int_0^{\mu(A)} h(t) \, dt} \right). \]

(b) There exist some $k > 0$ and $A \in \mathcal{A}$,
\[ \int_0^{\mu(\tau^{-n}(A))} h(t) \, dt \leq k \int_0^{\mu(A)} h(t) \, dt. \]

The proof of the lemma follows directly from the [17, Theorem 2.1], by using the fact that $\varphi$ satisfies the $\Delta_2$-condition for large $s > 0$.

**Definition 1.1.** [4, Page 1] A continuous map $g : (M, d) \to (M, d)$ is said to be Li-Yorke chaotic if there exists an uncountable set $S \subset M$ such that each pair of distinct points $p, q \in S$ is a Li-Yorke pair for $g$ i.e.,
\[ \lim_{n \to \infty} \inf d(g^n(p), g^n(q)) = 0 \quad \text{and} \quad \lim_{n \to \infty} \sup d(g^n(p), g^n(q)) > 0, \]
where $(M, d)$ is a metric space.

We say that $g$ is densely (generically) Li-Yorke chaotic whenever $S$ can be chosen to be dense (residual) in $M$.

**Definition 1.2.** [2, Page 47]

(a) If $T$ is a linear operator and a vector $z \in X$, then we say that $z$ is an irregular vector for $T$ if
\[ \lim_{n \to \infty} \inf \|T^n z\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \sup \|T^n z\| = \infty. \]

(b) If $T$ is a linear operator and a vector $z \in X$, then we say that $z$ is semi-irregular vector for $T$ if
\[ \lim_{n \to \infty} \inf \|T^n z\| = 0 \quad \text{and} \quad \lim_{n \to \infty} \sup \|T^n z\| > 0. \]

Following result gives the equivalent condition for any continuous linear operator $T$ on any Banach space to be Li-Yorke.

**Theorem 1.3.** [6, Theorem 9] Let $X$ be a Banach space and $T : X \to X$ be a bounded linear operator. Then the following are equivalent

(i) $T$ is Li-Yorke chaotic.

(ii) $T$ admits a semi-irregular vector.

(iii) $T$ admits irregular vector.
2. Li-Yorke chaotic composition operators on Orlicz-Lorentz Space

In this section, we discuss Li-Yorke chaotic composition operators on Orlicz-Lorentz spaces.

**Theorem 2.1.** Let $C_\tau$ be a continuous linear operator on $L^{\varphi,h}(\mu)$. Then the composition operator on $L^{\varphi,h}(\mu)$ is Li-Yorke chaotic iff there is an increasing sequence of positive integers $\{\gamma_k\}_{k \in \mathbb{N}}$ and a countable family of non-empty measurable sets $\{A_i\}_{i \in I}$ with $0 < \mu(A_i) < \infty$ such that

(a) $\lim_{k \to \infty} \varphi^{-1}\left(\frac{1}{\int_0^{\mu(\tau^{-\gamma_k}(A_i))} h(t)dt}\right) = \infty$, $\forall i \in I$.

(b) $\sup\left\{\varphi^{-1}\left(\frac{1}{\int_0^{\mu(A_i^n)} h(t)dt}\right) : i \in I, n \in \mathbb{N}\right\} = \infty$.

**Proof.** First, we suppose that $C_\tau$ is a continuous Li-Yorke chaotic composition operator on Orlicz-Lorentz space $L^{\varphi,h}(\mu)$. Then there is an irregular vector $g \in L^{\varphi,h}(\mu)$ for $C_\tau$. Let $A_i = \{x \in X : 3^{i-1} \leq |g(x)| < 3^i\}$, $i \in \mathbb{Z}$. Then clearly, $A_i \in \mathcal{A}$, $\forall i \in \mathbb{Z}$ and suppose $I = \{i \in \mathbb{Z} : \mu(A_i) > 0\}$. As $g \in L^{\varphi,h}(\mu)$, so there is some $\lambda > 0$ such that

$$\int_K \varphi(\lambda f^*(t))h(t)dt < \infty.$$ 

Thus for all $i \in I$, we get

$$\varphi(3^{i-1}\lambda)\mu(A_i) = \int_{A_i} \varphi(|g(t)||d\mu(t)
= \int_K \varphi(\lambda g^*(t))h(t)d(t)
< \infty.$$

This implies that $0 < \mu(A_i) < \infty, \forall i \in I$. Since $g$ is an irregular vector, so there is an increasing sequence $\{\gamma_k\}_{k \in \mathbb{N}}$ of positive integers such that

$$\lim_{k \to \infty} ||C_{\tau}^{\gamma_k}g||_{\varphi,h} = 0.$$

Now, by using the fact that $h$ is decreasing, we have

$$\int_K \varphi\left(\frac{3^{i-1} \chi_{\tau^{-\gamma_k}(A_i)^*}(t)}{M||C_{\tau}^{\gamma_k}g||_{\varphi,h}}\right) h(t)dt = \int_{\tau^{-\gamma_k}(A_i)} \varphi\left(\frac{3^{i-1} \chi_{A_i} \circ \tau^{\gamma_k}(t)}{M||C_{\tau}^{\gamma_k}g||_{\varphi,h}}\right) h(t)d\mu(t)
= \int_{\tau^{-\gamma_k}(A_i)} \varphi\left(\frac{|g \circ \tau^{\gamma_k}(t)|}{M||C_{\tau}^{\gamma_k}g||_{\varphi,h}}\right) h(t)d\mu(t)
= \int_{\mu(\tau^{-\gamma_k}(A_i))} \varphi\left(\frac{(g \circ \tau^{\gamma_k})^*(t/M)}{M||C_{\tau}^{\gamma_k}g||_{\varphi,h}}\right) h(t)dt
\leq \int_K \varphi\left(\frac{(g \circ \tau^{\gamma_k})^*(u)}{||C_{\tau}^{\gamma_k}g||_{\varphi,h}}\right) h(u)du
\leq 1.$$
This means that \( \|\chi_{\gamma_k(A_i)}\|_{\varphi,h} \leq M \|C_{\gamma_k} g\|_{\varphi,h} \). By using the Equation \((\ref{eq1})\), we get

\[
\lim_{k \to \infty} \|\chi_{\gamma_k(A_i)}\|_{\varphi,h} = 0
\]

\[
\lim_{k \to \infty} \frac{1}{\int_0^{\mu(\gamma_k(A_i))} h(t)dt} = \infty, \; i \in \mathbb{Z}.
\]

Thus (a) holds. Now, for the second suppose on the contrary that (b) will not be holds. Then for some \( r > 0 \), we have

\[
\varphi^{-1}\left(\frac{1}{\int_0^{\mu(A_i)} h(t)dt}\right) \leq r \varphi^{-1}\left(\frac{1}{\int_0^{\mu(\gamma_k(A_i))} h(t)dt}\right), \; \forall \; i \in I, \; n \in \mathbb{N}.
\]

By using the Lemma \((\ref{lem1})\) we see that

\[
\int_0^{\mu(\gamma_k(A_i))} h(t)dt \leq r \int_0^{\mu(A_i)} h(t)dt, \; \text{for some} \; r > 0.
\]

Now, for every \( n \in \mathbb{N} \),

\[
\int_K \varphi\left(\frac{(g \circ \gamma^n)(t)}{3r\|g\|_{\varphi,h}}\right) h(t)dt = \sum_{i \in I} \left[ \int_0^{\mu(\gamma_k(A_i))} \varphi\left(\frac{(g \circ \gamma^n)(t)}{3r\|g\|_{\varphi,h}}\right) h(t)dt \right]
\]

\[
= \sum_{i \in I} \left[ \int_{\gamma_k(A_i)} \varphi\left(\frac{(g \circ \gamma^n)(t)}{3r\|g\|_{\varphi,h}}\right) h(t)d\mu(t) \right]
\]

\[
< \sum_{i \in I} \varphi\left(\frac{3^i}{3r\|g\|_{\varphi,h}}\right) \int_{\gamma_k(A_i)} h(t)d\mu(t)
\]

\[
= \sum_{i \in I} \varphi\left(\frac{3^i}{3r\|g\|_{\varphi,h}}\right) \int_0^{\mu(\gamma_k(A_i))} h(t)dt
\]

\[
< \sum_{i \in I} \varphi\left(\frac{3^i}{\|g\|_{\varphi,h}}\right) \int_0^{\mu(A_i)} h(t)dt
\]

\[
= \sum_{i \in I} \int_0^{\mu(A_i)} \varphi\left(\frac{|g(t)|}{\|g\|_{\varphi,h}}\right) h(t)dt
\]

\[
= \sum_{i \in I} \int_{A_i} \varphi\left(\frac{|g(t)|}{\|g\|_{\varphi,h}}\right) h(t)d\mu(t)
\]

\[
= \sum_{i \in I} \int_0^{\mu(A_i)} \varphi\left(\frac{g^*(t)}{|g|_{\varphi,h}}\right) h(t)d(t)
\]

\[
= \int_K \varphi\left(\frac{g^*(t)}{|g|_{\varphi,h}}\right) h(t)d(t)
\]

\[
\leq 1.
\]

i.e., \( \|C_{\gamma^n} g\|_{\varphi,h} \leq 3r\|f\|_{\varphi,h} \), which is a contradiction to the fact that \( g \) is irregular and so the condition (b) holds.

Conversely, suppose that the condition (a) and (b) holds for measurable non-empty
countable family \( \{A_i\}_{i \in \mathbb{N}} \) with \( 0 < \mu(A_i) < \infty \). Let \( S \) be a linear span of \( \{\chi_{A_i} : i \in I\} \) in Orlicz-Lorentz space. Then the orbit of \( C_\tau \) is

\[ O_{C_\tau, g} = \{g, C_\tau g, ...\}. \]

By using the conditions (a), we see that the set \( S \subset C \) in Orlicz-Lorentz space. Then the orbit of \( H \) is

\[ \text{Banach-Steinhaus theorem, the set } S \text{ and hence } C_\tau \text{ is Li-Yorke chaotic as } g \in S_1 \cap S_2. \]

\[ \square \]

**Theorem 2.2.** Suppose that the non-singular measurable transformation \( \tau \) is injective. The composition operator \( C_\tau \) is Li-Yorke chaotic if there is a measurable set \( A \) with \( 0 < \mu(A) < \infty \) such that

\[ (i) \lim_{n \to \infty} \sup_{t \in I} \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau^{-n}(A))} h(t) dt} \right) = \infty, \]

\[ (ii) \sup_{t, q \in I, p < q} \left\{ \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau^{-n}(A))} h(t) dt} \right) : p, q \in I, p < q \right\} = \infty, \]

where \( I = \{q \in \mathbb{Z} : 0 < \mu(\tau^q(A)) < \infty \} \).

**Proof.** Since \( \tau \) is injective. Taking \( A_i = \tau^i(A), i \in \mathbb{Z} \). Then by using the (i), it follows that the condition (a) of Theorem 2.1 will hold. Now for \( p, q \in I \), with \( p < q \), we have

\[ A_p = \tau^p(A) = \tau^{p-q}(\tau^q(A)) = \tau^{p-q}(A_q). \]

Hence,

\[ \sup_{t, q \in I, p < q} \left\{ \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau^{-n}(A))} h(t) dt} \right) : p, q \in I, p < q \right\} = \sup_{t, q \in I, p < q} \left\{ \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau^{-n}(A))} h(t) dt} \right) : p, q \in I, p < q \right\} \]

\[ = \sup_{t, q \in I, p \in I} \left\{ \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau^{-n}(A))} h(t) dt} \right) : p, q \in I, p < q \right\}. \]

The condition (ii) implies that (b) of Theorem 2.1 holds and thus \( C_\tau \) is Li-Yorke chaotic. \[ \square \]

We now provided some necessary and sufficient conditions for the Li-Yorke chaotic composition operator \( C_\tau \) on \( L^{p, h} \).
Theorem 2.3. Let $\tau$ be injective and $\varphi$ satisfies the $\Delta_2$ condition. Then the following are equivalent:

(a) The composition operator $C_\tau$ on $L^{\varphi,h}(\mu)$ is Li-Yorke chaotic.
(b) There exists measurable function $0 \neq g \in L^{\varphi,h}(\mu)$ such that
\[
\lim_{n \to \infty} \inf \{ C_\tau^n g \} \varphi = 0.
\]
(c) There exists $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$ such that
\[
\lim_{n \to \infty} \sup \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau^{-n}(A))} h(t) dt} \right) = \infty.
\]
(d) There exists a measurable set $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$ such that
\[
\lim_{n \to \infty} \sup \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau^n(A))} h(t) dt} \right) = \infty.
\]
(e) There exists a measurable set $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$ such that
\[
\lim_{n \to \infty} \sup \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau^{-n}(A))} h(t) dt} \right) = \infty \quad \text{and} \quad \lim_{n \to \infty} \sup \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau^n(A))} h(t) dt} \right) = \infty.
\]
(f) There exists a measurable set $A \in \mathcal{A}$ with $0 < \mu(A) < \infty$ such that
\[
\lim_{n \to \infty} \inf \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau^{-n}(A))} h(t) dt} \right) > 0 \quad \text{and} \quad \lim_{n \to \infty} \sup \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau^n(A))} h(t) dt} \right) = \infty.
\]
(g) There is a measurable set $A$ such that the characteristic function is a semi-irregular vector for $C_\tau$ on $L^{\varphi,h}$.

Proof. First, suppose that $C_\tau$ is Li-Yorke chaotic. Then there exists $g \in L^{\varphi,h}(\mu)$ is semi-irregular function such that
\[
\lim_{n \to \infty} \inf \{ C_\tau^n g \} \varphi = 0 \quad \text{and} \quad \lim_{n \to \infty} \inf \{ C_\tau^n g \} \varphi > 0.
\]
From the earlier inequality, it follows that $g \neq 0$ and so the condition (b) holds. Now, we prove the relation (b) $\Leftrightarrow$ (c). First assume that the condition (b) holds. That is there is a some function $0 \neq g \in L^{\varphi,h}$ such that \( \lim_{n \to \infty} \inf \{ C_\tau^n g \} \varphi = 0 \). Since $g \neq 0$, there is positive constant $\varepsilon$ such that $\mu(A = \{ x \in X : |g(x)| > \varepsilon \}) > 0$. Further,
\[
\int_K \varphi \left( \frac{\varphi(X_{\tau^{-n}(A))}^*}{\| C_\tau^n g \| \varphi, h} \right) h(t) dt = \int_0^{\mu(\tau^{-n}(A))} \varphi \left( \frac{\varphi(X_{\tau^{-n}(A))}^*}{\| C_\tau^n g \| \varphi, h} \right) h(t) dt
\]
\[
= \int_{\tau^{-n}(A)} \varphi \left( \frac{\varphi(X_{\tau^{-n}(A))}^*}{\| C_\tau^n g \| \varphi, h} \right) h(t) dp(t)
\]
\[
\leq \int_{\tau^{-n}(A)} \varphi \left( \frac{\varphi(g \circ \tau^n)}{\| C_\tau^n g \| \varphi, h} \right) h(t) dp(t)
\]
\[
= \int_0^{\mu(\tau^{-n}(A))} \varphi \left( \frac{\varphi(g \circ \tau^n)}{\| C_\tau^n g \| \varphi, h} \right) h(t) dt
\]
\[
\leq 1
\]
i.e.,
\[ \varepsilon \| \chi_{\tau-n(A)} \|_{\varphi,h} \leq \| C_{\tau}^n g \|_{\varphi,h} \]
\[ \frac{1}{\varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau-n(A))} h(t)dt} \right)} \leq \| C_{\tau}^n g \|_{\varphi,h} \]
and by using the given hypothesis, we get
\[ \varepsilon \lim_{n \to \infty} \inf \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau-n(A))} h(t)dt} \right) \leq \lim_{n \to \infty} \inf \| C_{\tau}^n g \|_{\varphi,h} = 0. \]
This implies that
\[ \lim_{n \to \infty} \sup \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau-n(A))} h(t)dt} \right) = \infty. \]
In order to prove the reverse inclusion, we take \( A \in \mathcal{A} \) with \( 0 < \mu(A) < \infty \) such that
\[ \lim_{n \to \infty} \sup \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau-n(A))} h(t)dt} \right) = \infty. \] Then by taking \( g = \chi_A \), we see that
\[ \lim_{n \to \infty} \inf \| C_{\tau}^n \circ \chi_A \|_{\varphi,h} = \lim_{n \to \infty} \inf \| \chi_{\tau-n(A)} \|_{\varphi,h} \]
\[ = \lim_{n \to \infty} \inf \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau-n(A))} h(t)dt} \right) \]
\[ = \lim_{n \to \infty} \sup \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau-n(A))} h(t)dt} \right) \]
\[ = 0. \]
This implies \((c) \implies (b).\)
\((f) \iff (g):\) By putting \( g = \chi_A \) for all \( A \in \mathcal{A} \) with \( 0 < \mu(A) < \infty \). Since
\[ \| C_{\tau}^n g \|_{\varphi,h} = \| C_{\tau}^n \chi_A \|_{\varphi,h} = \| \chi_{\tau-n(A)} \|_{\varphi,h} \]
\[ = \frac{1}{\varphi^{-1} \left( \int_0^{\mu(\tau-n(A))} h(t)dt \right)} \]
which implies that \((f)\) and \((g)\) are equivalent property.

The condition \((c) \implies (e), (c) \implies (d)\) are directly and \((g) \implies (a)\) follows from the Theorem 1.3. Now, in order to prove the condition \((e) \implies (f)\), we assume that the \( \tau \) to be injective and the Orlicz function \( \varphi \) satisfy the \( \Delta_2 \) conditions for large \( s > 0 \). Suppose there is \( A \in \mathcal{A} \) with \( 0 < \mu(A) < \infty \) such that
\[ \lim_{n \to \infty} \sup \varphi^{-1} \left( \frac{1}{\int_0^{\mu(\tau-n(A))} h(t)dt} \right) = \infty \]
and
\[ \lim_{n \to \infty} \sup \varphi^{-1} \left( \frac{1}{\int_0^{\mu_{\tau^n}(A)} h(t)dt} \right) = \infty. \]

As \( \varphi \) is continuous, increasing and satisfy the \( \Delta_2 \) condition, we get
\[ \lim_{n \to \infty} \sup \left( \frac{1}{\int_0^{\mu_{\tau^n}(A)} h(t)dt} \right) = \infty \]
and so
\[ \lim_{n \to \infty} \int_0^{\mu_{\tau^n}(A)} h(t)dt = 0 \]
and
\[ \lim_{n \to \infty} \int_0^{\mu_{\tau^n}(A)} h(t)dt > 0, \]
Thus by using the Lemma\{1.1 and [8, corollary 1.3], we have
\[ \lim_{n \to \infty} \inf \int_0^{\mu_{\tau^n}(A)} h(t)dt = 0 \]
and so it follows that
\[ \lim_{n \to \infty} \varphi^{-1} \left( \frac{1}{\int_0^{\mu_{\tau^n}(A)} h(t)dt} \right) > 0 \]
and consequently,
\[ \lim_{n \to \infty} \inf \int_0^{\mu_{\tau^n}(A)} h(t)dt = 0. \]
Taking here \( \mu(X) < \infty \) and using the fact that \( \varphi \) is continuous and satisfies the \( \Delta_2 \) condition, we have
\[ \lim_{n \to \infty} \inf \int_0^{\mu_{\tau^n}(A)} h(t)dt = 0. \]
By Lemma\{1.1 we get
\[ \lim_{n \to \infty} \sup \varphi^{-1} \left( \frac{1}{\int_0^{\mu_{\tau^n}(A)} h(t)dt} \right) = \infty. \]

(d) \( \Rightarrow \) (e) follows on the similar line in (c) \( \Rightarrow \) (d). Thus the result holds. \( \square \)

In the following, we consider the space \( L^s(X, \mathcal{A}, \mu) \cap L^\infty(X, \mathcal{A}, \mu) \) with the norm defined as
\[ \| g \|_{\varphi \cap \infty} = \max \{ \| g \|_\varphi, \| g \|_\infty \}. \]

**Theorem 2.4.** Assume that \( (X, \mathcal{A}, \mu) \) be non-atomic infinite measure space and \( \tau \) be non-singular measurable transformation. Let \( \varphi \) be an Orlicz function which satisfies the \( \Delta_2 \)-condition for all \( s > 0 \). Then \( C_\tau \) on \( L^s(X, \mathcal{A}, \mu) \cap L^\infty(X, \mathcal{A}, \mu) \) is not Li-Yorke chaotic.
Proof. Suppose on the contrary that $C_\tau$ is Li-Yorke chaotic. Then by Theorem 1.3, it admits an irregular vector $g \in L^2(X, \mathcal{A}, \mu) \cap L^\infty(X, \mathcal{A}, \mu)$. Let $\{n_j\}$ be an increasing sequence of positive integers such that $C_{\tau}^{n_j}g \to 0$ in $L^2(X, \mathcal{A}, \mu)$. Then it converges in $L^2$ norm also. Therefore,

$$||C_{\tau}^{n_j}g||_{\varphi \cap \infty} = \max\{||C_{\tau}^{n_j}g||_{\varphi}, ||C_{\tau}^{n_j}g||_{\infty}\} \leq \max\{0, 1\} \leq 1,$$

which contradicts the fact that $g$ is an irregular vector for $C_\tau$. \qed

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Department of Mathematics, University of Jammu, Jammu 180006, INDIA.
Email address: rajat.singh.rs634@gmail.com

Department of Mathematics, University of Jammu, Jammu 180006, INDIA.
Email address: aditi.sharmao@gmail.com

Department of Mathematics, University of Jammu, Jammu 180006, INDIA.
Email address: romeshmath@gmail.com