$L^1$-Uniqueness of the Fokker-Planck equation on a Riemannian manifold

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Abstract

In this paper, we obtain a necessary and sufficient condition for $L^\infty$-uniqueness of Sturm-Liouville operator $a(x)\frac{d^2}{d^2x} + b(x)\frac{d}{dx} - V$ on an open interval of $\mathbb{R}$, which is equivalent to the $L^1$-uniqueness of the associated Fokker-Planck equation. For a general elliptic operator $L^V := \Delta + b \cdot \nabla - V$ on a Riemannian manifold, we obtain sharp sufficient conditions for the $L^1$-uniqueness of the Fokker-Planck equation associated with $L^V$, via comparison with a one-dimensional Sturm-Liouville operator. Furthermore the $L^1$-Liouville property is derived as a direct consequence of the $L^\infty$-uniqueness of $L^V$.

Key Words: Fokker-Planck equation, Liouville property, Sturm-Liouville operator, $L^\infty$-uniqueness of operator.

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1 Introduction

On a connected non-compact Riemannian manifold $M$ without boundary, consider the heat diffusion governed by $L^V f := \Delta f + b \cdot \nabla f - V f$ where $f \in C_0^\infty(M)$ (the space of all real infinitely differentiable functions with compact support), where $\Delta, \nabla$ are respectively the Laplace-Beltrami operator and the gradient on $M$. Here the vector field $b$ is locally Lipschitzian and represents the macroscopic velocity of the heat diffusion, $V : M \to \mathbb{R}^+$ is a locally bounded potential killing the heat.

Let $u(t, x)dx$ be the heat distribution at time $t$. It satisfies the well known Fokker-Planck equation in the distribution sense

$$
\partial_t u = (L^V)^\ast u(t, x) = \Delta u - \text{div}(ub) - V u, \ u(0, \cdot) \text{ given.}
$$

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A $L^1(M, dx)$-solution to (1.1) means that $t \to u(t) = u(t, \cdot)$ is continuous from $\mathbb{R}^+$ to $L^1(M, dx)$ and

$$\langle u(t) - u(0), f \rangle = \int_0^t \langle u(s), \mathcal{L}^V f \rangle ds, \forall t \geq 0, f \in C_0^\infty(M),$$

where $\langle f, g \rangle = \int_M f(x)g(x)dx$.

The study on this subject has a long history when $\mathcal{L}^V = \Delta$:

a) the subject was opened by S. T. Yau [31, 32]. Once if $M$ is complete and $1 < p < +\infty$, every nonnegative subharmonic functions in $L^p(M, dx)$ are constant ([31]), and the $L^p$-uniqueness of the above Fokker-Planck equation holds (due to Strichwarz [24]). In [29] it is proved that the $L^p$-Liouville property for nonnegative subharmonic functions implies the $L^p$-uniqueness of the above Fokker-Planck equation for general $\mathcal{L}^V$ instead of $\Delta$.

b) For the $L^\infty$-Liouville property, Yau [31] proved that every bounded harmonic function is constant if $M$ has nonnegative Ricci curvature. The last curvature condition is shown to be sharp, since there are infinitely many bounded harmonic functions on a simply connected manifold with sectional curvature identically $-1$. The final result in this opposite direction was obtained by Sullivan [23] and Anderson [1]: on a complete $M$ with (strongly) negative sectional curvature they identified the Martin boundary of $M$ as the sphere at infinity $S(\infty)$. See Anderson-Schoen [2], Schoen-Yau [22] for development of this subject.

c) For the $L^\infty$-uniqueness of (1.1) with $\mathcal{L}^V = \Delta$, Davies [7] proved that it is equivalent to the stochastic completeness of $M$ (i.e., the Brownian motion on $M$ does not explode). Grigor’yan [13] found sharp volume growth condition for the stochastic completeness of $M$.

d) The question of $L^1$-uniqueness for (1.1) is much more delicate. Azencott [3] and P. Li and Schoen [13] found several counter-examples for which the $L^1$-uniqueness of (1.1) fails. P. Li [17] found the following sharp sufficient condition for the $L^1$-uniqueness of (1.1) (with $\mathcal{L}^V = \Delta$) on a complete Riemannian manifold:

$$Ric_x \geq -C(1 + d(x, o)^2) \tag{1.2}$$

where $Ric_x$ is the Ricci curvature at $x$, $C > 0$ is some constant, $o$ is some fixed point and $d(x, o)$ is the Riemannian distance. Under that condition he proved that every nonnegative $L^1(M, dx)$-subharmonic function is constant.

Recently the second named author and Y. P. Zhang [29] introduced the $L^\infty$-uniqueness of $\mathcal{L}^V$ and prove that it is equivalent to the $L^1$-uniqueness of (1.1) and also to the $L^1$-uniqueness of the resolvent equation:

$$\text{if } u \in L^1(M, dx) \text{ verifies } [(\mathcal{L}^V)^* - 1]u = 0, \text{ then } u = 0. \tag{1.3}$$
Furthermore when $M = \mathbb{R}^d$ and $V = 0$, necessary and sufficient conditions are found for the $L^1$-uniqueness of (1.1) in the one-dimensional case ($d = 1$), and sharp sufficient conditions are obtained in the multi-dimensional case. Our main purpose of this work is to generalize the results of [29]. However this is not just a generalization, indeed the new difficulty is comparable to that in the classical passage from the Laplacian $\Delta$ to the Schrödinger operator $-\Delta + V$.

The $L^2$-uniqueness for (1.1) might seem to be the most important and natural. This is true from the point of view of quantum mechanics when $b = 0$ (in such case it is also equivalent to the $L^2$-uniqueness of the associated Schrödinger equation or the essential self-adjointness of $-\Delta + V$). But from the point of view of heat diffusion, the $L^1$-uniqueness is physically meaningful and it is then important: indeed in the heat diffusion interpretation, $u(t, x) \geq 0$ is the energy (= heat) density and the $L^1$-norm $\int_M |u(t, x)| dx$ is the total energy in the system at time $t$; the quantities $\int u^2(t, x) dx$ or $\int |\nabla u(t, x)|^2 dx$, though called energy in mathematical language, are not energy in the physics of heat diffusion.

Let us explain where comes the non-uniqueness of solutions to the Fokker-Planck equation (1.1) from two points of view.

1) Mathematically. When $M$ is not complete, one can impose different boundary conditions on the “boundary” $\partial M := \bar{M} \setminus M$ (which may vary and depend on different topologies) to obtain different solutions, such as Dirichlet boundary and Neumann boundary etc. Even if $M$ is complete, integrability or growth conditions will be required to assure the uniqueness of solution.

2) Physically. The non-uniqueness comes from the interchange of heat between $M$ and its “boundary”. For example the $L^\infty$-uniqueness of (1.1) with $L^V = \Delta$ is equivalent to the non-explosion of the Brownian Motion on $M$ (i.e. $M$ is stochastically complete) by [7], which means that the heat from the interior of $M$ can not reach the boundary $\partial$ (the one-point compactification of $M$). This intuitive idea is realized on a connected open domain $M$ of $\mathbb{R}^d$ for $\Delta - V$ and for the Nelson’s diffusions $\Delta - \nabla \phi \cdot \nabla$ by the second named author in [26] and [27].

There is another way of interchange of heat between $M$ and its “boundary”: the heat at the boundary can enter into the interior of $M$. Indeed for the one-dimensional Sturm-Liouville operator without killing potential (i.e., $V = 0$) on an open interval $M$ of $\mathbb{R}$, the second named author with Y. Zhang [28, 29] proved that the $L^1$-uniqueness of the associated Fokker-Planck equation is equivalent to say that the boundary is no entrance boundary in the classification of Feller, which exactly means in the probabilistic interpretation that the heat at the boundary can not enter into the interior of $M$. This is very intuitive: if the heat at the “boundary” can enter into the interior of $M$, new energy can be inserted from the “boundary” into $M$ without being perceived by the local operator $L^V$; and then destroys the $L^1$-uniqueness of (1.1).

The goal of this work is to realize the last physical intuition for general $L^V$. All results in this work are inspired by probabilistic (=physical) ideas, but for a larger audience all crucial proofs will be analytic.

This paper is organized as follows. In the next section, we introduce some pre-
liminaries and present characterizations and applications of the $L^\infty$-uniqueness of $\mathcal{L}^V$ to the $L^1$-Liouville property. Section 3 is devoted to the study of one dimensional Sturm-Liouville operators $\mathcal{L}^V = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx} - V$. We shall furnish a necessary and sufficient condition for the $L^\infty$-uniqueness of $\mathcal{L}^V$ by means of a new notion of no entrance boundary. A comparison principle is derived and several examples are presented. In Section 4, we establish a sharp sufficient condition for the $L^\infty$-uniqueness of $\mathcal{L}^V$ on Riemannian manifolds by means of comparison with a one-dimensional model. Several examples are presented.

2 $L^\infty$-uniqueness of pre-generator and $L^1$-Liouville property

Throughout this paper we assume that vector filed $b$ is locally Lipschitzian and the killing potential $V$ is nonnegative and locally bounded (measurable of course).

2.1 Background on $L^\infty$-uniqueness of pre-generator

Given the operator $\mathcal{L}^V$ acting on $C_0^\infty(M)$, let $(X_t)_{0 \leq t \leq \sigma}$ be the (stochastic) diffusion generated $\mathcal{L} = \Delta + b \cdot \nabla$, defined on $(\Omega, \mathcal{F}, (\mathbb{P}_x)_{x \in \mathcal{M}})$, where $\sigma$ is the explosion time (see Ikeda-Watanabe [15]). Then by Feynman-Kac formula,

$$P_t^V g(x) = \mathbb{E}^x 1_{t<\sigma} g(X_t) \exp\left(-\int_0^t V(X_s)ds\right)$$  \hspace{1cm} (2.1)

is one semigroup generated by $\mathcal{L}^V$, i.e., for all $t \geq 0$,

$$P_t^V f - f = \int_0^t P_s^V (\mathcal{L}^V f)ds, \ \forall f \in C_0^\infty(M).$$

But $(P_t^V)$ is not strongly continuous on $L^\infty(M, dx)$ w.r.t. the norm $\| \cdot \|_\infty$ (indeed Lotz’s theorem says that the generator of every strongly continuous (or $C_0^\infty$-) semigroup of operators on $(L^\infty, \| \cdot \|_\infty)$ is bounded). So it is impossible to define the uniqueness of $C_0-$semigroup on $L^\infty$ generated by $\mathcal{L}^V$, in the norm $\| \cdot \|_\infty$. That’s why we introduce in [29] the topology $\mathcal{C}(L^\infty, L^1)$ on $L^\infty$ of uniform convergence over compact subsets of $L^1$. It is proved in [29] that a semigroup of bounded operators on $L^\infty$ is strongly continuous on $L^\infty$ with respect to $\mathcal{C}(L^\infty, L^1)$ if and only if $(P_t) = (Q_t)$, where $(Q_t)$ is a $C_0$-semigroup on $L^1$ (w.r.t. the $L^1$-norm). Now the $L^\infty$-uniqueness of $\mathcal{L}^V$ can be defined as

**Definition 2.1.** ([29]) We call that $\mathcal{L}^V$ is $L^\infty$-unique, if the closure $\overline{\mathcal{L}^V}$ of $(\mathcal{L}^V, C_0^\infty(M))$ is the generator of $(P_t^V)$ on $(L^\infty, \mathcal{C}(L^\infty, L^1))$, in the graph topology induced by $\mathcal{C}(L^\infty, L^1)$.

Let

$$\lambda_0 := \lim_{t \to \infty} \frac{1}{t} \log \|P_t^V 1\|_\infty = \inf_{t>0} \frac{1}{t} \log \sup_{x \in \mathcal{M}} P_t^V 1(x),$$  \hspace{1cm} (2.2)
i.e., $e^{\lambda_0 t}$ is the spectral radius of $P_t^V$ in $L^\infty(M, dx)$, which is always in the spectrum of $P_t^V$ in $L^\infty(M, dx)$. Since $V \geq 0$, we have always $\lambda_0 \leq 0$. Recall

**Theorem 2.2.** ([29] a particular case of Theorem 2.1) *The following properties are equivalent:*

(i) $\mathcal{L}^V$ is $L^\infty$-unique;

(ii) for some or equivalently for all $\lambda > \lambda_0$, if $u \in L^1(M, dx)$ verifies $[(\mathcal{L}^V)^* - \lambda]u = 0$, then $u = 0$;

(iii) the Fokker-Planck equation (1.1) has a unique $L^1(M, dx)$-solution;

(iv) $(P_t^V)$ given by (2.1) is the unique $C_0$-semigroup on $(L^\infty, C(L^\infty, L^1))$ such that its generator extends $\mathcal{L}^V$.

By the theory for elliptic partial differential equations (PDE),

$$P_t^V f(x) = \int_M p_t^V(x, y)f(y)\,dy$$

and it is known that if $0 \leq u(0) = u(0, \cdot) \in L^1(M, dx)$,

$$u(t, y) := (P_t^V)^*u(0)(y) = \int_M u(0, x)p_t^V(x, y)\,dx$$

is the minimal nonnegative solution to (1.1).

### 2.2 $L^1$-Liouville property

At first we should understand the meaning of harmonic functions related with $\mathcal{L}^V$. When $\mathcal{L}^V = \Delta$, a harmonic function $h$ (i.e., $\Delta h = 0$) is a solution independent of $t$ to (1.1) (i.e., the equilibrium distribution of heat). For $\mathcal{L} = \Delta + b \cdot \nabla$, the equilibrium distribution $h$ of heat satisfies Kolmogorov’s equation

$$\mathcal{L}^* h = \Delta h - div(hb) = 0.$$

However in presence of the killing potential $V \geq 0$, usually equilibrium distribution $h$ is zero. So some further interpretation is required. Since $p_t^V(x, y) > 0$, $dy - a.e.$ for every $x \in M$, the dimension of

$$\mathcal{I} := \{h \in L^1(M, dx); (P_t^V)^*h = e^{\lambda_0 t}h, \forall t \geq 0\}$$

is at most one (Perron-Frobenius theorem), and if its dimension is one, then it is generated by some strictly positive $h_0$ such that $\int_M h_0\,dx = 1$ (by the theory of positive operators [19]).
Definition 2.3. A function $h \in L^1_{\text{loc}}(M, dx)$ (the space of real locally $dx$-integrable functions on $M$) is said to be $(\mathcal{L}^V - \lambda)^*$-harmonic where $\lambda \in \mathbb{R}$, if

$$\langle h, (\mathcal{L}^V - \lambda)f \rangle = 0, \ \forall f \in C^\infty_0(M)$$

(recall that $\langle f, g \rangle = \int_M fg \, dx$). It is said to be $(\mathcal{L}^V - \lambda)^*$-subharmonic, if

$$\langle h, (\mathcal{L}^V - \lambda)f \rangle \geq 0, \ \forall 0 \leq f \in C^\infty_0(M).$$

We now state our result about the $L^1$-Liouville property.

Theorem 2.4. Assume that $\mathcal{L}^V$ defined on $C^\infty_0(M)$ is $L^\infty$-unique. Let $\lambda \in \mathbb{R}$. Then for $h \in L^1(M, dx)$, it is $(\mathcal{L}^V - \lambda)^*$-harmonic if and only if

$$(P^V_t)^* h = e^{\lambda t} h, \ \forall t \geq 0.$$

In particular we have the following alternatives :

(a) If $\lambda > \lambda_0$ or $\lambda = \lambda_0$ but $\dim(\mathcal{I}) = 0$, then every $(\mathcal{L}^V - \lambda)^*$-harmonic function $h$ in $L^1(M, dx)$ is zero.

(b) If $\lambda = \lambda_0$ and $\dim(\mathcal{I}) = 1$, then every $(\mathcal{L}^V - \lambda)^*$-harmonic function $h$ in $L^1(M, dx)$ is $c h_0$ where $h_0$ is the strictly positive element in $\mathcal{I}$ such that $\int_M h_0 \, dx = 1$ and $c$ is a constant.

The results above without the $L^\infty$-uniqueness of $\mathcal{L}^V$ are in general false, see Li-Schoen’s Example [4,7].

Proof. The sufficient part is obvious by differentiating on $t = 0$ (and holds true even without the $L^\infty$-uniqueness of $\mathcal{L}^V$). Let us prove the necessity. Consider the generator $\mathcal{L}^V_\infty$ of $(P^V_t)$ in $L^\infty(M, dx)$. For every $f$ belonging the domain of definition $\mathbb{D}(\mathcal{L}^V_\infty)$, there is a nest $(f_i)$ in $C^\infty_0(M)$ such that

$$f_i \to f, \ \mathcal{L}^V f_i \to \mathcal{L}^V_\infty f$$

in the topology $C(L^\infty, L^1)$ by the assumed $L^\infty$-uniqueness. Thus we obtain for all $f \in \mathbb{D}(\mathcal{L}^V_\infty)$,

$$\langle h, (\mathcal{L}^V_\infty - \lambda)f \rangle = 0$$

which implies (since $P^V_t f \in \mathbb{D}(\mathcal{L}^V_\infty)$ for all $t \geq 0$)

$$\frac{d}{dt}(e^{-\lambda t}(P^V_t)^*h, f) = \frac{d}{dt}(h, e^{-\lambda t}P^V_t f) = \langle h, (\mathcal{L}^V_\infty - \lambda)e^{-\lambda t}P^V_t f \rangle = 0, \ \forall t \geq 0$$

where it follows that $\langle e^{-\lambda t}(P^V_t)^*h, f \rangle = \langle h, f \rangle$. Since $\mathbb{D}(\mathcal{L}^V_\infty)$ is dense in $L^\infty(M, dx)$ with respect to $C(L^\infty, L^1)$ ([29], we get $e^{-\lambda t}(P^V_t)^*h = h$ for all $t \geq 0$.

When $\lambda > \lambda_0$, the Liouville property in (a) is equivalent to the $L^\infty$-uniqueness of $\mathcal{L}^V$ ([29] Theorem 0.2 or Theorem 2.1]). If $\lambda = \lambda_0$, the last part of (a) and (b) follow easily from the previous equivalence. \qed
Example 2.5. $\mathcal{L}^V = \Delta$. Assume that $\Delta$ is $L^\infty$-unique. By Theorem 2.4, we have

(i) If $M$ is not stochastically complete, then every integrable harmonic function $h$ is zero. Indeed by Theorem 2.4 we have $P_t^* h = h$ where $(P_t)$ is the Brownian motion (or heat) semigroup. Then $P_t^* |h| \geq |h|$. Since $P_t 1 < 1$ everywhere on $M$, we get that if $h \neq 0$ in $L^1(M, dx)$,

$$\langle 1, |h| \rangle > \langle P_t 1, |h| \rangle = \langle 1, P_t^* |h| \rangle \geq \langle 1, |h| \rangle$$

which is a contradiction.

(ii) If $M$ is stochastically complete and the volume of $M$ is infinite, then $\dim(\mathcal{I}) = 0$ and consequently every integrable harmonic function is zero.

Indeed, if in contrary $\dim(\mathcal{I}) = 1$, i.e., $\mathcal{I}$ is spanned by some nonnegative non-zero function $h_0 \in L^1(M, dx)$, since $\lambda_0 = 0$ by the stochastic completeness of $M$, $h_0 dx$ is an invariant probability measure of the Brownian motion semigroup $(P_t)$, which implies that the kernel $R_1 := \int_0^\infty e^{-t} P_t dt$ is positively recurrent ([20, Proposition 10.1.1]). But for such Markov kernel $R_1$, it has no other nonnegative invariant measure than $ch_0 dx$ ([20, Theorem 10.0.1]) for some constant $c > 0$. However $dx$ is an invariant measure of $R_1$, which is infinite. This contradiction yields that $\dim(\mathcal{I}) = 0$.

(iii) If $M$ is stochastically complete and the volume of $M$ is finite, then $\dim(\mathcal{I}) = 1$ and $\mathcal{I}$ coincides with $\mathbb{R}$ and consequently every integrable harmonic function is constant. The argument is as in (ii). See Example 4.7 for a stochastically complete and finite volume manifold for which $\Delta$ is not $L^\infty$-unique and the $L^1$-Liouville property is violated.

The argument in the example above leads to

Corollary 2.6. Let $\mathcal{L}^V = \mathcal{L} = \Delta + b \cdot \nabla$, i.e., $V = 0$. Assume that $\mathcal{L}$ is $L^\infty(M, dx)$-unique. Then

(a) If the diffusion $(X_t)_{0 \leq t < \sigma}$ generated by $\mathcal{L}$ is explosive, i.e., $\mathbb{P}_x(\sigma < +\infty) > 0$ for some (or equivalently for all) $x \in M$, then every $\mathcal{L}^*$-harmonic function $h$ in $L^1(M, dx)$ is zero.

(b) If the diffusion $(X_t)_{0 \leq t < \sigma}$ generated by $\mathcal{L}$ is not explosive, i.e., $\mathbb{P}_x(\sigma < +\infty) = 0$ for all $x \in M$, then either there is no non-zero $\mathcal{L}^*$-harmonic and integrable function, or there is one positive $dx$-integrable $\mathcal{L}^*$-harmonic function $h_0$ such that for every non-zero $\mathcal{L}^*$-harmonic function $h \in L^1_{loc}(M, dx)$, if $h \geq 0$ or $h \in L^1(M, dx)$, then $h$ is a constant multiple of $h_0$.

In summary if $\mathcal{L}^V$ is $L^\infty$-unique, we have the $L^1$-Liouville property stated in Theorem 2.4 and Corollary 2.6. Then the main task remained to us is to check the $L^\infty$-uniqueness of $\mathcal{L}^V$.

From now on in this paper, $L^\infty$ will be endowed with the topology $\mathcal{C}(L^\infty, L^1)$, and the $L^\infty$-uniqueness of operators and $C_0$-semigroups etc. on $L^\infty$ are always w.r.t. $\mathcal{C}(L^\infty, L^1)$. 

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3 \( L^\infty \)-uniqueness of Sturm-Liouville operator

Consider the following Sturm-Liouville operator:

\[
\mathcal{L}^V f(x) = a(x) f'' + b(x) f' - V(x) f, \quad \forall f \in C^\infty_0(x_0, y_0) \tag{3.1}
\]

\((-\infty \leq x_0 < y_0 \leq +\infty).\) Assume that the coefficients \(a, \ b, \ V\) of \(\mathcal{L}^V\) in (3.1) satisfy

\[
a(x), \ b(x) \in L^\infty_{loc}(x_0, y_0) \tag{3.2}
\]

\[
a(x) > 0 \text{ } dx \text{–a.e. ; } \frac{1}{a(x)} \text{ } V(x) \in L^\infty_{loc}(x_0, y_0); \text{ } V(x) \geq 0; \tag{3.3}
\]

where \(L^\infty_{loc}(x_0, y_0)\) (resp. \(L^1_{loc}(x_0, y_0)\)) denotes the space of real measurable functions which are essentially bounded (resp. integrable) w.r.t. the Lebesgue measure \(dx\) on any compact sub-interval of \((x_0, y_0)\). Fix a point \(c \in (x_0, y_0)\) and let

\[
s'(x) = \exp\left(-\int_c^x \frac{b(t)}{a(t)} dt\right), \text{ } m'(x) = \frac{1}{a(x)} \exp\left(\int_c^x \frac{b(t)}{a(t)} dt\right). \tag{3.4}
\]

Their primitives \(s\) and \(m\) are respectively the scale and speed functions of Feller. Below \(m\) will also denote the measure \(m'(x) dx\). It is easy to see that

\[
\langle \mathcal{L}^V f, \ g \rangle_m = \langle f, \ \mathcal{L}^V g \rangle_m, \forall f, g \in C^\infty_0(x_0, y_0)
\]

where \(\langle f, \ g \rangle_m := \int_{x_0}^{y_0} f(x) g(x) m'(x) dx\). For \(f \in C^\infty_0(x_0, y_0)\), we can write \(\mathcal{L}^V\) in the following form of Feller,

\[
\mathcal{L}^V f = \frac{d}{dm} \frac{d}{ds} f - V f.
\]

Now regard \(\mathcal{L}^V\) as an operator on \(L^p(m) := L^p((x_0, y_0), m)\), \(p \in [1, +\infty]\), with domain of definition \(C^\infty_0(x_0, y_0)\). Recall that \(L^\infty(m)\) is endowed always with the topology \(\mathcal{C}(L^\infty(m), L^1(m))\). Again let \((X_t)_{0 \leq t < \sigma}\) be the diffusion in \((x_0, y_0)\) generated by \(\mathcal{L}\) with the explosion time \(\sigma\) (cf. [10]) and define \(P^V_t\) by the Feynman-Kac formula as in (2.1). \(P^V_t\) is \(m\)-symmetric, and its generator \(\mathcal{L}^V_{(p)}\) in \(L^p(m) = L^p((x_0, y_0), m)\) extends \(\mathcal{L}^V\) (defined on \(C^\infty_0(x_0, y_0)\)). The problem resides again in the uniqueness.

\(\mathcal{L}^V\) is said \(L^p(m)\)-unique \((1 \leq p \leq +\infty)\), if its closure in \(L^p(m)\) coincides with \(\mathcal{L}^V_{(p)}\). That \(L^p(m)\)-uniqueness is equivalent to the uniqueness of solution \(t \to u(t)\) (continuous from \(\mathbb{R}^+ \to L^q(m)\)) to the following integral version of Fokker Planck equation

\[
\langle u(t) - u(0), \mathcal{L}^V f \rangle_m = \int_0^t \langle u(s), \mathcal{L}^V f \rangle_m ds, \forall f \in C_0^\infty(x_0, y_0), \forall t \geq 0
\]

for every \(u(0) \in L^q(m)\) given, where \(q\) is the conjugate number of \(p \in [1, +\infty]\), i.e., \(q = \frac{p}{p-1}\). \((\text{In other words the } L^p\text{-uniqueness of } \mathcal{L}^V \text{ is equivalent to the } L^q\text{-uniqueness of the associated Fokker-Planck equation.})\) It is also equivalent to : for any \(h \in L^q(m)\),
\[(h, (L^V - 1)f)_m = 0, \forall f \in C_0^\infty(x_0, y_0)) \implies h = 0. \tag{3.5}\]

See [29] for numerous other characterizations.

The study of $L^2$-uniqueness of the Sturm-Liouville operators was born with the limit point–limit cycle theory of Weil (see [21]). In a series of pioneering works (here we mention only [11, 12]) W. Feller investigated thoroughly the different sub-Markov generator-extensions of $L^V$.

The recent study is concentrated on the case where $V = 0$. Wielens [25] obtained the characterization of $L^2$-uniqueness (or equivalently the essential self-adjointness) of $L$. Furthermore, Eberle [9] and Djellout [8] have completely characterized the $L^p$-uniqueness of $L$ for $1 < p < \infty$. The $L^1$-uniqueness, the $L^\infty$-uniqueness are characterized in [27] and [29], respectively.

In presence of the killing potential, the problem of uniqueness becomes much more difficult, just because it is hard to obtain a priori estimates about solutions of the second order ordinary differential equation with a potential. This can be seen for an example in the theory of Weil: $\Delta - c/x^2$ $(c > 0)$ acting on $C_0^\infty(0, +\infty)$ is $L^2((0, +\infty), dx)$-unique iff $c \geq 3/4$ (see Reed-Simon [21]). This simple example (but profound characterization) excludes any easy integral test criteria such as those in no killing case.

Our purpose is to find an explicit characterization of the $L^\infty$-uniqueness of $(L^V, C_0^\infty(x_0, y_0))$.

### 3.1 Main result

The main result of this section is

**Theorem 3.1.** $(L^V, C_0^\infty(x_0, y_0))$ is unique in $L^\infty(m)$ iff for some or equivalently all $\delta > 0$
\[
\int_c^{y_0} \sum_{n \geq 0} I_{n+\delta}^V(y)m'(y)dy = +\infty, \tag{3.6}\]
\[
\int_{x_0}^c \sum_{n \geq 0} J_{n+\delta}^V(y)m'(y)dy = +\infty, \tag{3.7}\]

where for all $V \geq 0$,
\[
I_0^V(y) = 1, \quad I_n^V(y) = \int_c^y s'(r)dr \int_c^r m'(t)V(t)I_{n-1}^V(t)dt, \quad y \geq c; \\
J_0^V(y) = 1, \quad J_n^V(y) = \int_y^c s'(r)dr \int_r^c m'(t)V(t)J_{n-1}^V(t)dt, \quad y \leq c.
\]

**Definition 3.2.** We say that $y_0$ (resp. $x_0$) is no entrance boundary for $L^V$ if (3.6) (resp. (3.7)) holds for some or equivalently for all $\delta > 0$.

In other words the $L^\infty$-uniqueness of $L^V$ is equivalent to say that $x_0, y_0$ are no entrance boundary in the sense of Definition 3.2. In the presence of the killing
potential $V \geq 0$, our definition of no entrance boundary is different from the classical one of Feller (see Ito-Mckean [16]), so it is a new notion. The comparison is given in Corollary 3.10 and Remarks 3.11.

**Remarks 3.3.** Denote by $I^V_n(x)$ (resp. $J^V_n(x)$) by $I^V_n(c; x)$ (resp. $J^V_n(x; c)$). One can prove that (3.6) and (3.7) do not depend on $c \in (x_0, y_0)$. Its proof is given later.

### 3.2 Proof of Theorem 3.1

Throughout this section, the dual operator $(\mathcal{L}^V)^*$ is taken w.r.t. $m$, NOT w.r.t. $dx$ unlike in other places of the paper.

We begin with a series of technical lemmas similar to Lemma 4.5, Lemma 4.6, Lemma 4.7 of [29], so we omit their proofs.

**Lemma 3.4.** Let $u, v \in L^1_{loc}((x_0, y_0), m)$ such that

$$\langle u, \mathcal{L}^V f \rangle_m = \langle v, f \rangle_m, \forall f \in C^\infty_0(x_0, y_0).$$

Then

(i) $u$ has a $C^1$-smooth $dx$-version $\tilde{u}$ such that $\tilde{u}'$ is absolutely continuous;

(ii) $g := au'' + bu' - V\tilde{u} = (1/m')(\tilde{u}'/s')' - V\tilde{u} \in L^1_{loc}(m)$.

In that case $v = g$.

**Lemma 3.5.** Suppose that $h$ is $C^1(x_0, y_0)$ such that $h'$ is absolutely continuous and $(h'/s)' = Vm'h$.

Assume that $c_1 \in (x_0, y_0)$, $h(c_1) > 0$ and $h'(c_1) > 0$ (resp. $h'(c_1) < 0$). Then $h'(y) > 0$ (resp. $h'(y) < 0$) for $\forall y \in (c_1, y_0)$ (resp. $\forall y \in (x_0, c_1)$).

**Lemma 3.6.** (essentially due to Feller [11]) Assume that $V \geq \delta > 0$, $dx$-a.e., then there exist two strictly positive $C^1$-functions $h_k, k = 1, 2$ on $(x_0, y_0)$ such that

1. For $k = 1, 2$, $h_k'$ is absolutely continuous, and $(h_k'/s')' = m'Vh_k$, a.e.;
2. $h_1' > 0$ and $h_2' < 0$ over $(x_0, y_0)$.

Our key observation is

**Proposition 3.7.** Let $h$ be any $C^1$-function on $(x_0, y_0)$ such that $h'$ is absolutely continuous and $(h'/s')' = m'Vh, dx$-a.e.

(a) If $h(c) > 0, h'(c) > 0$ and $\int_c^{y_0} V(t)m'(t)dt > 0$, then there is a positive constant $C$ such that

$$h(c) \sum_{n=0}^{\infty} I^V_n(x) \leq h(x) \leq C \sum_{n=0}^{\infty} I^V_n(x), \forall x \geq c. \quad (3.8)$$
(b) If \( h(c) > 0, h'(c) < 0 \) and \( \int_{x_0}^{c} V(t)m'(t)dt > 0 \), then there is a positive constant \( C \) such that

\[
h(c) \sum_{n=0}^{\infty} J_n^V(x) \leq h(x) \leq C \sum_{n=0}^{\infty} J_n^V(x), \quad \forall x \leq c.
\] (3.9)

Proof. (a) By Lemma 3.5, \( h'(r) > 0 \) for \( r \in [c, y_0) \). Notice that

\[
h(x) = h(c) + \int_{c}^{x} h'(r)dr.
\]

\[
= h(c) + \int_{c}^{x} \left\{ \frac{h'(c)}{s'(c)} s'(r) + s'(r) \int_{c}^{r} m'(t)V(t)h(t)dt \right\} dr
\]

\[
> h(c) + \int_{c}^{x} s'(r)dr \int_{c}^{r} m'(t)V(t)h(t)dt.
\]

Thus using the above inequality recursively, we easily obtain:

\[
h(x) \geq h(c) \sum_{n=0}^{+\infty} I_n^V(x), \quad \forall x \geq c
\]

which is the first inequality in (3.8).

For the second inequality in (3.8), letting \( K(x) := \int_{c}^{x} (h'(c)/s'(c))s'(r)dr \) and fixing \( c_0 \in (c, y_0) \) such that \( \int_{c}^{c_0} V(t)m'(t)dt > 0 \), we can choose suitable positive constants \( C_1, C_2, C_3 \) such that for all \( x \geq c \),

\[
K(x) \leq C_2 + C_1 \int_{c_0}^{x} s'(r)dr
\]

\[
\leq C_2 + C_3 \int_{c}^{x} s'(r)dr \int_{c}^{r} m'(t)V(t)dt = C_2 + C_3 I_1^V(x).
\]

Setting \( C_4 = h(c) + C_2 \), we get for all \( x \geq c \):

\[
h(x) = h(c) + \int_{c}^{x} \left\{ \frac{h'(c)}{s'(c)} s'(r) + s'(r) \int_{c}^{r} m'(t)V(t)h(t)dt \right\} dr
\]

\[
\leq C_4 + C_3 I_1(x) + \int_{c}^{x} s'(r)dr \int_{c}^{r} m'(t)V(t)h(t)dt.
\]

Using it inductively we obtain for all \( x \geq c \),

\[
h(x) \leq C_4 + (C_3 + C_4) I_1(x) + C_3 I_2(x)
\]

\[
+ \int_{c}^{x} s'(r_1)dr_1 \int_{c}^{r_1} m'(t_1)V(t_1)dt_1 \int_{c}^{r_1} s'(r_2)dr_2 \int_{c}^{r_2} m'(t_2)V(t_2)h(t_2)dt_2
\]

\[
\ldots \ldots \ldots \ldots 
\]

\[
\leq C_4 + (C_3 + C_4) \sum_{n=1}^{+\infty} I_n^V(x).
\]
where the second inequality in (3.8) follows.

(b) Similar to part (a).

Let us now to the

**Proof of Remarks 3.3.** We prove here the no entrance property of $y_0$ does not depend on $c$. Denote $I_n^V(x)$ by $I_n^V(c; x)$ to emphasize the role of $c$. Let $x_0 < c < c_1 < y_0$. By Feller’s lemma 3.6, there is a strictly increasing positive $C^1$-function $h = h_1$ on $(x_0, y_0)$ such that $h'$ is absolutely continuous and

$$(h'/s)' = (V + \delta)hm'$$

a.e. on $(x_0, y_0)$. Hence $h'(x) > 0$ over $(x_0, y_0)$. By Proposition 3.7 there is a constant $C > 0$ such that for all $x \geq c_1$,

$$h(c) \sum_{n=0}^{\infty} I_n^{V+\delta} (c; x) \leq h(x) \leq C \sum_{n=0}^{\infty} I_n^{V+\delta} (c_1; x).$$

That completes the proof.

**Proof of Theorem 3.7.** Since the constant $\lambda_0$ defined in (2.2) is non-positive, according to Theorem 2.2 or more precisely [29, Theorem 2.1]), the $L^\infty (m)$-uniqueness of $L^V$ is equivalent to : for some or equivalently for all $\delta > 0$, if $h \in L^1(m)$ such that

$$\langle h, (L^V - \delta)f \rangle_m = \langle h, L^{V+\delta}f \rangle_m = 0, \forall f \in C_0^\infty (x_0, y_0)$$

then $h = 0$. By Lemma 3.4 for such $h$, we may assume that $h \in C^1(x_0, y_0)$ and $h'$ is absolutely continuous and

$$(h'/s')' = m'(V + \delta)h.$$ (3.10)

**Part “if”:** Assume (3.6) and (3.7) hold for some $\delta > 0$. Suppose in contrary that $0 \neq h \in L^1(m)$ is a solution of (3.10). We can assume that $h > 0$ on some interval $[x_1, y_1] \subset (x_0, y_0)$ where $x_1 < y_1$. Notice that $h' \not\equiv 0$ on $(x_1, y_1)$ by (3.10).

**Case (i):** $h'(c_1) > 0$ for some $c_1 \in (x_1, y_1)$. We obtain from Proposition 3.7(a):

$$\int_{c_1}^{y_0} h(y)m'(y)dy \geq h(c_1) \int_{c_1}^{y_0} \sum_{n=0}^{+\infty} I_n^{V+\delta}(y)m'(y)dy = +\infty;$$

which is a contradiction with the assumption that $h \in L^1(m)$.

**Case (ii):** $h'(c_1) < 0$ for some $c_1 \in (x_1, y_1)$. By Proposition 3.7(b), we have

$$\int_{x_0}^{c_1} m'(y)h(y)dy \geq h(c_1) \int_{x_0}^{c_1} \sum_{n=0}^{+\infty} J_n^{V+\delta}(y)m'(y)dy = +\infty.$$ 

**Part “only if”:** Let us prove that (3.7) holds for all $\delta > 0$. Indeed assume in contrary that for some $\delta > 0$,

$$\int_{x_0}^{c} m'(y) \sum_{n=0}^{+\infty} J_n^{V+\delta}(y)dy < +\infty.$$
In particular $\int_{x_0}^c m'(y)dy < \infty$. Consider a solution $h$ of (3.10) such that $h > 0$ and $h' < 0$ over $(x_0, y_0)$, whose existence is assured by Feller’s Lemma 3.6. We shall prove that $h \in L^1(m)$.

(1) Integrability near $y_0$: Let $c < (x_0, y_0)$. For $y < (c, y_0)$ we have
\[
0 \geq h'(y)/s'(y) = h'(c)/s'(c) + \int_c^y m'(t)h(t)(\delta + V(t))dt
\]
which implies that $\delta \int_c^{y_0} m'(t)h(t)dt \leq -h'(c)/s'(c) < +\infty$.

(2) Integrability near $x_0$: By Proposition 3.7(b),
\[
\int_{x_0}^c m'(t)h(t)dt \leq C \int_{x_0}^c \sum_{n=0}^{+\infty} J^{V+\delta}(y)m'(y)dy < +\infty.
\]
That completes the proof of the necessity of (3.7). For the necessity of (3.6) for all $\delta > 0$, the proof is similar: The only difference is to use a positive and increasing solution $h$ of (3.10) (whose existence is guaranteed by Lemma 3.6).

3.3 Several corollaries

Lemma 3.8. Assume that $V(x) = 0$, $\forall x \in (x_0, y_0)$. The point $y_0$ is no entrance boundary, i.e. (3.6) holds iff
\[
\int_c^{y_0} m'(y)dy \int_c^y s'(x)dx = +\infty; \quad (3.11)
\]
and $x_0$ is no entrance boundary, i.e. (3.7) holds iff
\[
\int_{x_0}^c m'(y)dy \int_y^c s'(x)dx = +\infty. \quad (3.12)
\]

Proof. We prove here only the equivalence between (3.6) and (3.11).

(3.6) $\implies$ (3.11). Let $I_n = I_n^{V+1}$ with $V = 0$. This implication is obvious because for some $c_1 \in (c, y_0)$,
\[
\int_c^{y_0} I_1(y)m'(y)dy \geq \int_c^{y_0} m'(y)dy \int_c^{y_1} s'(x)dx \cdot \int_c^{c_1} m'(y)dy = +\infty.
\]

(3.11) $\implies$ (3.6). If $m(y_0) := m[(c, y_0)] = +\infty$, then both (3.6) and (3.11) hold true. Assume then $m(y_0) < +\infty$. We have
\[
\int_c^{y_0} I_n(y)m'(y)dy = m(y_0)\left(\int_c^{y_0} m'(t)dt \int_c^t s'(r)dr \right)^n,
\]
hence
\[ \int_{c}^{y_0} \sum_{n \geq 0} I_n(y)m'(y)dy \leq m(y_0) \exp \left( \int_{c}^{y_0} m'(y)dy \int_{c}^{y} s'(x)dx \right). \]

Then (3.11) follows immediately from (3.6). \( \square \)

From the lemma above we get immediately from Theorem 3.1

**Corollary 3.9.** ([29, Theorem 4.1]) \( \mathcal{L}, C_0^\infty(x_0, y_0) \) is unique in \( L^\infty(m) \) if and only if
\[ \int_{c}^{y_0} m'(y)dy \int_{c}^{y} s'(x)dx = +\infty; \quad (3.13) \]

and
\[ \int_{x_0}^{c} m'(y)dy \int_{y}^{c} s'(x)dx = +\infty; \quad (3.14) \]

hold.

**Corollary 3.10.** If
\[ \int_{c}^{y_0} (1 + V(t))m'(t)dt \int_{c}^{t} s'(r)dr < +\infty \quad (3.15) \]

then \( y_0 \) is entrance boundary for \( \mathcal{L}^V \). Similarly if
\[ \int_{x_0}^{c} (1 + V(t))m'(t)dt \int_{t}^{c} s'(r)dr < +\infty \quad (3.16) \]

then \( x_0 \) is entrance boundary for \( \mathcal{L}^V \).

**Proof.** This is obtained by the same proof as that of Lemma 3.8 \( \square \)

**Remarks 3.11.** In the theory of Feller (see [16]), (3.13) and (3.14) are used for the definition of entrance boundary of \( y_0 \) and \( x_0 \) for \( \mathcal{L}^V \). So our definition of entrance boundary for \( \mathcal{L}^V \) is equivalent to his one if \( V = 0 \) by Corollary 3.9 but strictly weaker in the presence of a zero potential \( V \geq 0 \). For example when \( c \in (0, 2) \), 0 is entrance boundary for \( d^2/dx^2 - c/x^2 \) on \( (0, +\infty) \) in our sense, but it is not in the sense of Feller’s (3.16), see Example 3.18.

We now turn to

**Theorem 3.12.** *(comparison principle)* Let
\[ \mathcal{L}_k f(x) = a_k(x)f''(x) + b_k(x)f' - V_k(x)f, \quad \forall f \in C_0^\infty(x_0, y_0) \]

where \( (a_k, b_k, V_k), \ k = 1, 2 \) satisfy (3.3) and (3.3). Assume that for some \( c \in (x_0, y_0) \),
\[ a_1(x) \geq a_2(x), V_2(x) \geq V_1(x), \quad (3.17) \]
(a) If
\[
\frac{b_1(x)}{a_1(x)} \leq \frac{b_2(x)}{a_2(x)}, \quad x \geq c,
\]
and \(y_0\) is no entrance boundary for \(L_1\), so it is for \(L_2\).

(b) If
\[
\frac{b_1(x)}{a_1(x)} \geq \frac{b_2(x)}{a_2(x)}, \quad x \leq c,
\]
and \(x_0\) is no entrance boundary for \(L_1\), so it is for \(L_2\).

The conditions above are guided by the intuitive picture of no entrance boundary. Assume \(a_1 = a_2\) and \(y_0\) is no entrance boundary for \(L_1\). Condition \(b_2 \geq b_1\) means that the heat in the second system described by \(L_2\) goes more rapidly to the boundary \(y_0\) than in the first system, and condition \(V_2 \geq V_1\) means that the heat in the second system is killed more rapidly than in the first. Then the heat from the boundary \(y_0\) goes more difficulty into the interior in the second system than in the first one.

**Proof.** We prove here only the implication for \(y_0\). Let \(I_{n,k}(k = 1, 2; n \in \mathbb{N})\) denote \(v_n^{v+1}\) with respect to \((w.r.t.)\) \(L_k\) \((k = 1, 2)\). By conditions (3.17) and (3.18), we have
\[
m_1'(t)(1 + V_1(t)) \leq m_2'(t)(1 + V_2(t));
\]
\[
\exp \left\{ \int_r^y \frac{b_1(u)}{a_1(u)} \, du \right\} \leq \exp \left\{ \int_r^u \frac{b_2(u)}{a_2(u)} \, du \right\}, \quad y > r.
\]
Letting \(B_{n,k} := \int_r^y m_k'(y) I_{n,k}(y) \, dy\), we have
\[
B_{n,k} = \int_0^{r_0} \left[ \int_0^{r_0} \frac{1}{a_k(t_1)} e^{\int_{r_1}^{r_2} \frac{b_k(u)}{a_k(u)} \, du} \, dr_1 \right] dt_1 \int_0^{r_0} \frac{1}{a_k(t_2)} e^{\int_{r_2}^{r_0} \frac{b_k(u)}{a_k(u)} \, du} (V_k(t_2) + 1) \, dr_2 \, dt_2
\]
\[
\cdots \int_0^{r_0} \frac{1}{a_k(t_n)} e^{\int_{r_n}^{r_0} \frac{b_k(u)}{a_k(u)} \, du} (V_k(t_n) + 1) \, dr_n \, dt_n \int_0^{r_0} (V_k(t_{n+1}) + 1) m_k'(t_{n+1}) \, dt_{n+1}.
\]
Thus \(\forall n, B_{n,1} \leq B_{n,2}\). Hence the conclusion follows.

### 3.4 Dirichlet or Neumann boundary problem

Consider the Sturm-Liouville operator \(L^V\) on \([x_0, y_0]\) where \(x_0 \in \mathbb{R}\) and \(x_0 < y_0 \leq +\infty\), where \(a, b, V\) satisfy always (3.2) and (3.3) on \([x_0, y_0]\) (instead of \((x_0, y_0)\)). Consider
\[
\mathcal{D}_D := \{ f \in C_0^\infty[x_0, y_0]; \ f(x_0) = 0 \}
\]
and
\[
\mathcal{D}_N := \{ f \in C_0^\infty[x_0, y_0]; \ f'(x_0) = 0 \}.
\]
Denote by \(L^V_D\) (resp. \(L^V_N\)) the operator with domain of definition \(\mathcal{D}_D\) (resp. \(\mathcal{D}_N\)), which corresponds to the Dirichlet boundary (resp. Neumann) boundary condition at \(x_0\). One can define \(L^V_D\) and \(L^V_N\) similarly on \((x_0, y_0]\) where \(-\infty \leq x_0 < y_0 < +\infty\).

With exactly the same proof we have
Theorem 3.13. $\mathcal{L}^V_D$ (or $\mathcal{L}^V_N$) is $L^\infty([x_0, y_0], m)$-unique iff $y_0$ is no entrance boundary. $\mathcal{L}^V_D$ (or $\mathcal{L}^V_N$) is $L^\infty((x_0, y_0], m)$-unique iff $x_0$ is no entrance boundary.

3.5 About $L^p(m)$-uniqueness of $\mathcal{L}^V$

Proposition 3.14. $\mathcal{L}^V$ is $L^1(m)$-unique iff

$$I_{Y+1}^V(y_0) = \int_{c}^{y_0} s'(r) \int_{c}^{r} m'(t)(1 + V(t))dt = +\infty;$$

$$J_{Y+1}^V(x_0) = \int_{x_0}^{c} s'(r) \int_{r}^{c} m'(t)(1 + V(t))dt = +\infty.$$

With exactly the same proof as that of Theorem 3.1 we have by Proposition 3.7

Proposition 3.15. Let $p \in (1, +\infty)$. $\mathcal{L}^V$ is $L^p(m)$-unique iff for some or all $\delta > 0$,

$$\int_{c}^{y_0} \left( \sum_{n=0}^{\infty} I_{n+\delta}^V(y) \right)^q m'(y)dy = +\infty; \quad (3.22)$$

$$\int_{x_0}^{c} \left( \sum_{n=0}^{\infty} J_{n+\delta}^V(y) \right)^q m'(y)dy = +\infty. \quad (3.23)$$

Definition 3.16. If the condition (3.22) (resp. (3.23)) is verified, $y_0$ (resp. $x_0$) will be called $L^p(m)$-no entrance boundary for $\mathcal{L}^V$.

So the no entrance boundary in Definition 3.2 is $L^1(m)$-no entrance boundary. If $V = 0$, a much easier criterion is available:

Proposition 3.17. (due to Eberle [9] and Djellout [8]) Assume that $V = 0$. $\mathcal{L}$ is $L^p(m)$-unique iff

$$\int_{c}^{y_0} \left( \int_{c}^{y} s'(x)dx \right)^q m'(y)dy = +\infty; \quad (3.24)$$

and

$$\int_{x_0}^{c} \left( \int_{y}^{c} s'(x)dx \right)^q m'(y)dy = +\infty. \quad (3.25)$$

3.6 Several examples

For applications of the comparison principle in Theorem 3.12 we should have some standard examples.

Example 3.18. (combination of Weil’s example and Bessel’s diffusion) Let

$$(x_0, y_0) = (0, +\infty), \quad \mathcal{L}^V f = f'' + \frac{\gamma}{x} f' - \frac{c}{x^2}, \quad c \geq 0, \gamma \in \mathbb{R}.$$
When $\gamma = 0$, this is Weil’s example mentioned before, and when $c = 0$, it is the Bessel’s process with dimension $\gamma + 1$. For this example $s'(x) = x^{-\gamma}$, $m'(x) = x^{\gamma}$ and $V(x) = c/x^2$, $+\infty$ is no entrance boundary for $\mathcal{L}f = f'' + \frac{c}{x^2}f'$, so for $\mathcal{L}^V$ by the comparison principle in Theorem 3.12. Furthermore condition (3.24) is verified for $y_0 = +\infty$, so does (3.22).

One decreasing solution for $(\mathcal{L}^V)^*h = 0$ is given by

$$h_c(x) = x^\alpha, \quad \alpha = -\frac{(\gamma - 1) - \sqrt{(\gamma - 1)^2 + 4c}}{2}.$$ 

We have

(a). $(\mathcal{L}, C_0^\infty(0, +\infty))$ is $L^1$—unique if and only if $c > 0$ or $c = 0$ and $\gamma \geq 1$, by Proposition 3.14

(b) If $c = 0$ and $p \in (1, +\infty)$, $(\mathcal{L}, C_0^\infty(0, +\infty))$ is $L^p(m)$—unique if and only if $\gamma \leq -1$ or $\gamma \geq 2p - 1$ by Proposition 3.17

(c). Let $p \in (1, +\infty)$ and $c > 0$. $(\mathcal{L}, C_0^\infty(0, +\infty))$ is $L^p(m)$—unique if and only if $\alpha q + \gamma \leq -1$ (where $\alpha = -\frac{(\gamma - 1) - \sqrt{(\gamma - 1)^2 + 4c}}{2}$, $\frac{1}{p} + \frac{1}{q} = 1$), or equivalently

$$c \geq c_{cr}(q, \gamma) := \frac{(\gamma + 1)^2}{q} - \frac{\gamma^2 - 1}{q}.$$ 

When $p = 2, \gamma = 0$, we find Weil’s critical value $3/4$.

(d) If $\gamma = 0$ and $p \in (1, +\infty)$, $(\mathcal{L}, C_0^\infty(0, +\infty))$ is $L^p(dx)$—unique if and only if $c \geq \frac{1}{q} + \frac{1}{q}$. This is a particular case of (c).

Proof of part (c). If $\alpha q + \gamma \leq -1$, $\int_0^1 h^q(x)m'(x)dx = +\infty$. By Proposition 3.7

$$\sum_{n=0}^{\infty} J_n^V \notin L^q((0, 1], m'(x)dx), \text{ hence } \sum_{n=0}^{\infty} J_{n+1}^V \notin L^q((0, 1], m'(x)dx) \text{ (for } J_{n+1}^V \geq J_n^V).$$

Thus by Theorem 3.11 and Proposition 3.15 $\mathcal{L}^V$ is $L^p(m)$—unique.

If $\alpha q + \gamma > -1$, $\int_0^1 h^q(x)m'(x)dx < +\infty$ for some small $\varepsilon > 0$. By Proposition 3.7

$$\sum_{n=0}^{\infty} J_n^{(1+\varepsilon)V} \notin L^q((0, 1], m'(x)dx).$$

But for $x \in (0, 1]$, as $V + \delta = \frac{c}{x^2} + \delta \leq \frac{c(1+\varepsilon)}{x^2}$ for $\delta \in (0, \varepsilon)$, we have $J_n^{V+\delta} \leq J_n^{(1+\varepsilon)V}$. Therefore $\sum_{n=0}^{\infty} J_n^{V+\delta} \notin L^q((0, 1], m)$, $\mathcal{L}^V$ is not $L^p(m)$—unique by Theorem 3.11 and Proposition 3.13.

Example 3.19. Let $(x_0, y_0) = (0, +\infty)$, $c \geq 0, \gamma, \kappa \in \mathbb{R}, \kappa \neq 0$ and

$$\mathcal{L}^V f = x^\kappa \left( f'' + \frac{\gamma}{x} f' - \frac{c}{x^2} \right), \quad f \in C_0^\infty(0, +\infty).$$

We have $s'(x) = x^{-\gamma}, m'(x) = x^{-\kappa+\gamma}$. Let $h_c$ be given as in the previous example, which is again $(\mathcal{L}^V)^*$—harmonic function. By the same proof as above, we have: $0$ is $L^q(m)$—no entrance boundary iff $\alpha q + \gamma - \kappa \leq -1$, where $\alpha = -\frac{(\gamma - 1) - \sqrt{(\gamma - 1)^2 + 4c}}{2}$ and $q \in [1, +\infty)$. 

\[17\]
Example 3.20. Let \((x_0, y_0) = \mathbb{R}, a(x) = 1\) and \(b(x) = \gamma(|x|^\alpha)'\) for \(|x| > 1\) and continuous on \(\mathbb{R}\), and \(V(x) = c|x|^\beta\) for \(|x| > 1\) and continuous and nonnegative on \(\mathbb{R}\). Here \(\gamma \in \mathbb{R}\) and \(\alpha, \beta, c \geq 0\). In this example \(m'(x) = e^{\gamma|x|^\alpha}, s'(x) = e^{-\gamma|x|^\alpha}\) for \(|x| > 1\) (for simplicity we have forgotten a constant factor in \(m'\) and \(s'\), which plays no role in our history). The operator \(\mathcal{L}^V\) is given by

\[
\mathcal{L}^V f = f'' + \gamma \text{sgn}(x)|x|^\alpha - 1 f' - c|x|^\beta, \quad |x| > 1.
\]

\[1\). For all \(1 < p < \infty\), \(\mathcal{L}^V\) is \(L^p(m)\)-unique by applying Proposition 3.17 to the case \(V = 0\) and then Proposition 3.15.

\[2\). \(\mathcal{L}^V\) is \(L^1(m)\)-unique iff \(\gamma \leq 0\) or \((\gamma > 0\) and \(1_{c>0}\beta \geq \alpha - 2\)), by Proposition 3.14.

\[3\) \(\mathcal{L}\) is \(L^\infty(m)\)-unique iff \(\gamma \geq 0\) or \(\cdot \gamma < 0\) and \(\alpha < 2\)\) (by [29, Example 4.10]). In such case \(\mathcal{L}^V\) is \(L^\infty(m)\)-unique by the comparison principle in Theorem 3.12.

Let \(\gamma < 0\) and \(\alpha > 2\) below. Then \(\mathcal{L}\) is not \(L^\infty(m)\)-unique, and our purpose is to find the critical potential \(V = c|x|^\beta\) so that \(\mathcal{L}^V\) is \(L^\infty(m)\)-unique or equivalently \(+\infty\) is no entrance boundary.

Claim : Let \(\gamma < 0\) and \(\alpha > 2\) and \(\beta = \alpha - 2\). Set \(c_{cr} = |\gamma|\alpha(\alpha - 2)\). If \(c > c_{cr}, \mathcal{L}^V\) is \(L^\infty(m)\)-unique; if \(c < c_{cr}, \mathcal{L}^V\) is not \(L^\infty(m)\)-unique.

By the symmetry we have only to regard if \(+\infty\) is no entrance boundary. For two positive functions \(f, g\), we write \(f \sim g\) (at \(+\infty\), if \(\lim_{x \to +\infty} f(x)/g(x) = 1\); and \(f \ll g\) (at \(+\infty\), if there are two positive constants \(C_1, C_2\) such that \(C_1 f(x) \leq g(x) \leq C_2 f(x)\) for all \(x\) large enough (say \(x \gg 1\)).

To prove the claim, consider \(h = s(\log s)^\kappa\), where \(s(1) = e\) and \(\kappa > 0\). We have \(\mathcal{L} h = h \mathcal{V}\) where

\[
\mathcal{V} = \frac{s'^2}{s^2} \left( \frac{\kappa}{\log s} + \frac{\kappa(\kappa - 1)}{(\log s)^2} \right) \sim |\gamma|\alpha \kappa x^{\alpha - 2}
\]

by using \(s(x) \sim e^{\gamma|x|^\alpha}\). Note that \(s(\log x)^\kappa m\) \(1_{x^{\alpha - 1 - \alpha}}\). Then \(h \in L^1([1, +\infty), m)\) iff \(\kappa < \kappa_0 = (\alpha - 2)/\alpha\). For \(\kappa = \kappa_0, \mathcal{V} \sim c_{cr} x^{\alpha - 2}\). Now one can conclude the claim by means of Proposition 3.7 and Theorem 3.1.

4 Riemannian manifold case: comparison with one-dimensional case

In this section, let \((M, g)\) be a connected oriented non-compact Riemannian manifold of dimension \(d \geq 2\) with metric \(g\) without boundary, but not necessarily complete. Throughout this section we denote by \(dx\) the volume element, given in local coordinates by \(dx|_U = \sqrt{G} dx_1 dx_2 \cdots dx_d\), where \(G = det(g_{ij})\). Let \(TM\) be the tangent bundle on \(M\). Let \(L^p_{loc}(M)\) \((p \in [1, +\infty])\) be the space of all real measurable functions \(f\) such that \(f 1_K \in L^p(M) := L^p(M, dx)\) for every compact subset \(K\) of \(M\). Let \(H^{1,2}(M)\) (resp. \(H^{1,\infty}_{loc}(M)\)) be the space of those functions \(f \in L^2(M)\) (resp. \(f \in L^2_{loc}(M)\)) such that \(\nabla f \in L^2(M)\) (resp. \(|\nabla f| \in L^2_{loc}(M)\)) where the gradient \(\nabla f\) is taken in the distribution sense, and \(|\cdot|\) is the Riemannian metric.
Let us consider the following operator:

\[ \mathcal{L}^V f(x) = \Delta f(x) + b(x) \cdot \nabla f(x) - V(x)f(x), \quad f \in C_0^\infty(M) \]  

where \( \Delta, \nabla \) are respectively the Laplace-Beltrami operator and the gradient on \( M \), and \( b \) is a locally Lipschitzian vector field, \( 0 \leq V \in L^\infty_{\text{loc}}(M) \) (assumed throughout this section). We write \( \mathcal{L} \) instead of \( \mathcal{L}^V \) if \( V = 0 \).

Now regard \( \mathcal{L}^V \) as an operator on \( L^\infty(M) \) which is endowed with the topology \( \mathcal{C}(L^\infty(M), L^1(M)) \), with domain of definition \( C_0^\infty(M) \). Our purpose is to find some sharp sufficient condition for the \( L^\infty \)-uniqueness of \( (\mathcal{L}^V, C_0^\infty(M)) \).

**Assumption (A)**

1. \( \rho : M \rightarrow [x_0, y_0) \) is surjective, where \( 0 \leq x_0 < y_0 \leq +\infty \), such that \( \rho^{-1}([x_0, l]) \) is compact subset for all \( l \in [x_0, y_0) \), and there is some \( c \in [x_0, y_0) \) such that \( \rho \) is \( C^2 \)-smooth and \( |\nabla \rho| > 0 \) on \( [\rho > c] \);

2. there exist \( \alpha(r), \beta(r), q(r) \in L^\infty_{\text{loc}}([x_0, y_0), dr), q(r) \geq 0, \alpha > 0, 1/\alpha(r) \in L^\infty_{\text{loc}}([x_0, y_0), dr) \) and \( c \in [x_0, y_0) \) such that \( dx \)-a.e. on \( [\rho > c] \),

\[
|\nabla \rho(x)|^2 \leq \alpha(\rho(x)); \tag{4.2}
\]

\[
\mathcal{L} \rho(x) \geq \frac{\beta(\rho(x))}{\alpha(\rho(x))} |\nabla \rho(x)|^2; \tag{4.3}
\]

\[
V(x) \geq q(\rho(x)). \tag{4.4}
\]

**Theorem 4.1.** Under the assumption \( (\text{A}) \), if \( y_0 \) is no entrance boundary for \( \mathcal{L}^{1,q} = \alpha(r) \frac{d^2}{dr^2} + \beta(r) \frac{d}{dr} - q(r) \), then \( (\mathcal{L}^V, C_0^\infty(M)) \) is \( L^\infty(M, dx) \)-unique.

**Remarks 4.2.** Our assumption \( (\text{A}) \) is inspired from the comparison theorems in the theory of stochastic differential equations ([1], Chap. VI, Sections 4 and 5). Assume that \( |\nabla \rho|^2 = \alpha(\rho) \). Let \( (X_t)_{0 \leq t < \sigma} \) be the diffusion generated by \( \mathcal{L} \) and \( \eta_t \) by \( \alpha(r) \frac{d^2}{dr^2} + \beta(r) \frac{d}{dr} \) with \( \rho(X_0) = \eta_0 > c \). If \( [4.2] \) holds, one can realize \( X_t \) and \( \eta_t \) on the same probability space so that \( \rho(X_t) \geq \eta_t \) before returning to \( c \). In other words \( \rho(X_t) \) goes to \( y_0 \) (i.e., \( X_t \) goes to infinity) more rapidly than \( \eta_t \). Then under the assumption \( (\text{A}) \), if \( y_0 \) is no entrance boundary for \( \mathcal{L}^{1,q} \), the heat from “boundary” of \( M \) is again more difficult to enter into \( M \) : it should be “no entrance boundary”. The result above justifies this intuition.

Let us begin with a Kato type inequality.

**Lemma 4.3.** If \( u \in L^1(M, dx) \) satisfies

\[
\int_M u(\mathcal{L}^V - 1) f dx = 0, \quad \forall f \in C_0^\infty(M) \tag{4.5}
\]
Then \( u \in C^1(M) \) (more precisely one version of \( u \) is \( C^1 \)-smooth) and

\[
- \int_M \nabla|u| \cdot \nabla f \, dx + \int_M b \cdot \nabla f \cdot |u| \, dx \geq \int_M (V + 1)|u| \, dx \quad (4.6)
\]

for all positive, compactly supported functions \( f \in H^{1,2}_{\text{loc}}(M) \).

Proof. By [4, Theorem 1], we know \( u \in H^{1,2}_{\text{loc}}(M) \cap L^\infty(\text{loc})(M) \). Using \( \Delta u = \text{div}(ub) + (V + 1)u \) and Sobolev’s embedding theorems recursively, \( u \in C^1(M) \). The remained proof can follow word-by-word Eberle [9, Theorem 2.5 step 2] (in “\( V = 0 \)” case), so omitted.

Lemma 4.4. If \( u \in H^{1,2}_{\text{loc}}(M) \) satisfies

\[
\langle u, \mathcal{L}^V f \rangle = 0, \ \forall f \in C^\infty_0(M)
\]

and \( u = 0 \) \( dx \) – a.e. outside of some compact subset \( K \) of \( M \), then \( u = 0 \).

This is contained in the folklore of elliptic PDE, so we omit its proof.

Proof of Theorem 4.1. According to Theorem 2.2, we have only to show that the equation

\[
\int_M u(\mathcal{L}^V - 1)f \, dx = 0, \ \forall f \in C^\infty_0(M) \quad (4.7)
\]

has no non-trivial \( L^1(M, dx) \) solution. Assume by absurd that there is some non-zero \( u \in L^1(M) \) satisfying (4.7). By Lemma 4.3, \( u \in C^1(M) \) and (4.6) holds.

For all \( r_1, r_2 \) such that \( x_0 \leq c < r_1 < r_2 < y_0 \), put \( h(r) := \min\{r_2 - r_1, (r_2 - |r|)^+\} \) and \( f := h(\rho(x)) \). Plugging such \( f \) into (4.6) we obtain:

\[
\int_{\{r_1 \leq \rho(x) \leq r_2\}} \nabla|u| \cdot \nabla \rho \, dx - \int_{\{r_1 \leq \rho(x) \leq r_2\}} b \cdot \nabla \rho \cdot |u| \, dx \geq \int_M (V + 1)|u|h(\rho) \, dx. \quad (4.8)
\]

Since \( \nabla|u| \cdot \nabla \rho = \text{div}(|u|\nabla \rho) - |u|\Delta \rho \), we have

\[
\int_{\{r_1 \leq \rho(x) \leq r_2\}} \nabla|u| \cdot \nabla \rho \, dx = \int_{\{r_1 \leq \rho(x) \leq r_2\}} [\text{div}(|u|\nabla \rho) - |u|\Delta \rho] \, dx
\]

\[
= (i) \int_{\{\rho = r_2\}} |u| \cdot |\nabla \rho| d\sigma_M - \int_{\{\rho = r_1\}} |u| \cdot |\nabla \rho| d\sigma_M - \int_{\{r_1 \leq \rho(x) \leq r_2\}} |u|\Delta \rho \, dx
\]

where (i) follows from the Divergence Theorem, \( \sigma_M \) is the \((d-1)\)-dimensional surface measure on the \( C^2\)-smooth \( \{\rho = r\} \) induced by the volume measure \( dx \). Using the
the previous inequality is read as:

\[
\int_{\{\rho = r_2\}} |u| \cdot |\nabla \rho| d\sigma_M - \int_{\{\rho = r_1\}} |u| \cdot |\nabla \rho| d\sigma_M - \int_{\{r_1 \leq \rho(x) \leq r_2\}} |u| \cdot |\nabla \rho|^2 \frac{\beta(\rho)}{\alpha(\rho)} dx
\]

\[
\geq \int_{\{\rho = r_2\}} |u| \cdot |\nabla \rho| d\sigma_M - \int_{\{\rho = r_1\}} |u| \cdot |\nabla \rho| d\sigma_M - \int_{\{r_1 \leq \rho(x) \leq r_2\}} |u| \cdot \mathcal{L} \rho dx
\]

\[
= \int_{\{r_1 \leq \rho(x) \leq r_2\}} \nabla |u| \cdot \nabla \rho dx + \int_{\{r_1 \leq \rho(x) \leq r_2\}} |u| \Delta \rho dx - \int_{\{r_1 \leq \rho(x) \leq r_2\}} |u| \mathcal{L} \rho dx
\]

\[
\geq \int_M (V + 1) |u| h(\rho) dx \geq \int_M \frac{V + 1}{\alpha(\rho)} |u| \cdot \nabla \rho^2 h(\rho) dx
\]

\[
\geq \int_M \frac{q(\rho) + 1}{\alpha(\rho)} |u| \cdot \nabla \rho^2 h(\rho) dx
\]

where (i) follows from condition (4.13), (ii) follows from (4.8) and (4.2), (iii) follows from (4.4).

Now let \( G(r) = \int_{\{\rho(x) \leq r\}} |\nabla \rho|^2 \cdot |u| dx \). By the Co-area formula (Federer v[10] Theorem 3.2.12): for \( r_2 > r_1 > c \),

\[
\int_{\{\rho(x) \in [r_1, r_2]\}} |\nabla \rho(x)| f(x) dx = \int_{r_1}^{r_2} dr \int_{\{\rho(x) = r\}} f(x) d\sigma_M, \quad (4.9)
\]

\( G \) is absolutely continuous on \( r \in (c, y_0) \) and

\[
G'(r) = \int_{\{\rho(x) = r\}} |\nabla \rho| \cdot |u| d\sigma_M, \quad dr - a.e. \; r > c.
\]

From now on we fix \( G'(r) \) as the right hand side above. By the Co-area formula we also have

\[
\int_{\{r_1 \leq \rho(x) \leq r_2\}} |u| \cdot |\nabla \rho|^2 \frac{\beta(\rho)}{\alpha(\rho)} dx = \int_{r_1}^{r_2} G'(r) \frac{\beta(r)}{\alpha(r)} dr
\]

the previous inequality is read as:

\[
G'(r_2) - G'(r_1) - \int_{r_1}^{r_2} G'(r) \frac{\beta(r)}{\alpha(r)} dr \geq \int_{c}^{y_0} \frac{q(r) + 1}{\alpha(r)} G'(r) h(r) dr \quad (4.10)
\]

for \( c < r_1 < r_2 \). Since

\[
\int_{c}^{y_0} \frac{q(r) + 1}{\alpha(r)} G'(r) h(r) dr = \int_{r_1}^{r_2} \frac{q(r) + 1}{\alpha(r)} (r_2 - r) G'(r) dr + \int_{c}^{r_1} (r_2 - r_1) G'(r) \frac{q(r) + 1}{\alpha(r)} dr
\]

\[
= \int_{r_1}^{r_2} \frac{q(t) + 1}{\alpha(t)} G'(t) \int_{t}^{r_2} ds dt + (r_2 - r_1) \int_{c}^{r_1} G'(r) \frac{q(t) + 1}{\alpha(t)} dt
\]

\[
= \int_{r_1}^{r_2} ds \int_{t}^{s} \frac{q(t) + 1}{\alpha(t)} G'(t) dt + \int_{r_1}^{r_2} ds \int_{c}^{t} G'(t) \frac{q(t) + 1}{\alpha(t)} dt
\]

\[
= \int_{r_1}^{r_2} ds \int_{c}^{s} \frac{q(t) + 1}{\alpha(t)} G'(t) dt.
\]
Substituting this into (4.10), we obtain
\[ - \int_{r_1}^{r_2} G'(r) \frac{\beta(r)}{\alpha(r)} dr + G'(r_2) - G'(r_1) \geq \int_{r_1}^{r_2} ds \int_c^s \frac{q(t) + 1}{\alpha(t)} G'(t) dt \]
for \( c < r_1 < r_2 \). It can be rewritten as a differential equality:
\[ \mathcal{L} G := G''(r) - \frac{\beta(r)}{\alpha(r)} G'(r) = \int_c^r \frac{q(t) + 1}{\alpha(t)} G'(t) dt + H' \quad (4.11) \]
in distribution on \((c, y_0)\), where \( H : (c, y_0) \to \mathbb{R} \) is nondecreasing and right continuous. Then \( G' \) admits a right continuous version. We shall assume \( G' \) is itself right continuous below.

Consider \( \tilde{n}'(r) = \exp\left( \int_c^r \frac{\beta(t)}{\alpha(t)} dt \right) \), \( s'(r) = \exp\left( - \int_c^r \frac{\beta(t)}{\alpha(t)} dt \right) \), which are respectively the derivative of the scale and speed function associated with \( \mathcal{L}^- \). Then \( m'(r) := \frac{\tilde{n}'(r)}{s'(r)} \), \( s'(r) \) are respectively the derivative of the speed and scale function associated with \( \mathcal{L}^{1,q} \). Hence we can write (4.11) in the Feller’s form,
\[ (G'/\tilde{n})' \geq s' \int_c^r \frac{q(t) + 1}{\alpha(t)} G'(t) dt. \]

Let us prove now that \( \exists \rho_0 \in (c, y_0) \), \( G'(\rho_0) > 0 \). Indeed, if in contrary \( G'(r) = 0, \forall r > c \), then \( u = 0, dx - a.e. \) on \( [\rho > c] \), i.e., outside of the compact set \( K = \rho^{-1}[x_0, c] \). Thus \( u = 0 \) on \( M \) by Lemma 4.4, a contradiction with our assumption.

The above inequality in distribution implies that for \( dr - a.e. \ r > r_0, r_0 \in (x_0, y_0) \)
\[ \frac{dG}{dm}(r) \geq \frac{dG}{dm}(r_0) + \int_{r_0}^r s'(u) du \int_{r_0}^u \frac{q(t) + 1}{\alpha(t)} G'(t) dt \]
\[ = \frac{dG}{dm}(r_0) + \int_{r_0}^r s'(u) du \int_{r_0}^u (q(t) + 1)m'(t) \frac{dG}{dm}(t) dt. \]

Using the above inequality by induction as in the proof of Proposition 3.7, we get
\[ \frac{G'}{m'}(y) \geq C \sum_{n=0}^{+\infty} I_{n+1}^{q+1}(y) \]
where \( I_0^{q+1} = 1 \), \( I_{n+1}^{q+1}(y) = \int_{r_0}^y s'(r) dr \int_{r_0}^r (q(t) + 1)m'(t)I_{n-1}^{q+1}(t) dt, C = \frac{dG}{dm}(r_0) > 0 \). Using Co-area formula and our assumption, we obtain:
\[ \int_{\{r_0 \leq \rho \}} |u| dx \geq \int_{\{r_0 \leq \rho \}} \frac{|u| \cdot |\nabla \rho|^2}{\alpha(\rho)} dx \]
\[ = \int_{r_0}^{y_0} \frac{G'(r)}{\alpha(r)} dr = \int_{r_0}^{y_0} m'(r) \frac{dG}{dm}(r) dr \]
\[ \geq C \sum_{n=0}^{+\infty} \int_{r_0}^{y_0} m'(r)I_{n+1}^{q+1}(r) dr. \]

Thus \( \int_{\{r_0 \leq \rho \}} |u| dx = \infty \) by our assumption that \( y_0 \) is no entrance boundary for \( \mathcal{L}^{1,q} \).
This is in contradiction with the assumption that \( u \in L^1(M, dx) \). \( \square \)
With the same proof we have the two sides’ version of Theorem 4.1:

**Theorem 4.5.** We suppose

(1) \( \rho : M \rightarrow (x_0, y_0) \) is surjective, where \( -\infty \leq x_0 < y_0 \leq +\infty \), such that \( \rho^{-1}([x_1, x_2]) \) is compact subset for all \( x_1 < x_2 \) in \((x_0, y_0)\), and there are \( c_1 < c_2 \) in \((x_0, y_0)\) such that \( \rho \) is \( C^2\)-smooth, \( |\nabla \rho| > 0 \) on \([\rho < c_1] \cup [\rho > c_2]\);

(2) there exist \( \alpha(r), \beta(r), q(r) \in L_{\text{loc}}^\infty((x_0, y_0), dr) \), \( q(r) \geq 0 \), \( \alpha > 0 \), \( 1/\alpha(r) \in L_{\text{loc}}^\infty((x_0, y_0)) \) and \( c_1 < c_2 \) in \((x_0, y_0)\) such that \( dx - \)a.e. on \([\rho < c_1] \cup [\rho > c_2]\),

\[
|\nabla \rho|^2 \leq \alpha(\rho), \text{ } dx - \text{a.e. on } [\rho < c_1] \cup [\rho > c_2]; \tag{4.12}
\]

\[
\mathcal{L} \rho \geq |\nabla \rho|^2 \frac{\beta(\rho)}{\alpha(\rho)}, \text{ } dx - \text{a.e. on } [\rho > c_2]; \tag{4.13}
\]

\[
\mathcal{L} \rho \leq |\nabla \rho|^2 \frac{\beta(\rho)}{\alpha(\rho)}, \text{ } dx - \text{a.e. on } [\rho < c_1]; \tag{4.14}
\]

\[
V(x) \geq \rho(q(\rho(x))), \text{ } dx - \text{a.e. on } [\rho < c_1] \cup [\rho > c_2]; \tag{4.15}
\]

If \( x_0, y_0 \) are no entrance boundaries for \( \mathcal{L}^{1,q} = \alpha(r) \frac{d^2}{dr^2} + \beta(r) \frac{d}{dr} - q(r) \), then \((\mathcal{L}^V, C_0^\infty(M))\) is \( L^\infty(M, dx)\)-unique.

**Corollary 4.6.** Suppose that \( M \) is a Cartan-Hadamard manifold with dimension \( d \geq 2 \) (i.e. complete, simply connected with non-positive sectional curvature). Let \( d(x) \) be the distance from some fixed point \( o \) to \( x \). Then

(a) \( \Delta \) is \( L^\infty(M, dx)\)-unique. In particular the \( L^1\)-Liouville property holds: every \( dx\)-integrable \((\Delta-)\)harmonic function is constant.

(b) Assume that \( b \) verifies \( b(x) \cdot \nabla d(x) \geq -L[1 + d^2(x)], \text{ } x \neq o \) for some constant \( L > 0 \). Then \( \mathcal{L}^V = \Delta + b \cdot \nabla - V \) is \( L^\infty(M, dx)\)-unique. In particular the \( L^1\)-Liouville property in Theorem 2.4 and Corollary 2.6 holds true.

(c) Assume that \( b \) verifies \( b(x) \cdot \nabla d(x) \geq -L[1 + d^\alpha(x)] \) for some constants \( L > 0 \) and \( \alpha > 2 \). If \( V(x) \geq cd(x)^{\alpha-2} \) with \( c > L\alpha(\alpha - 2) \), then \( \mathcal{L}^V = \Delta + b \cdot \nabla - V \) is \( L^\infty(M, dx)\)-unique. In particular the \( L^1\)-Liouville property in Theorem 2.4 holds.

**Proof.** (a) The \( L^\infty\)-uniqueness of \( \Delta \) is a particular case of part (b). Then the \( L^1\)-Liouville theorem follows from Example 2.5.

(b) Recall that on the Cartan-Hadamard manifold, the exponential map \( \exp : T_o M \rightarrow M \) is a diffeomorphism. The distance function \( d(x) \) is \( C^\infty\)-smooth on \( M \setminus \{o\} \), and the Laplacian comparison theorem says that \((6, 22)\)

\[
\Delta d(x) \geq \frac{d - 1}{d(x)}, \text{ } x \neq o.
\]
Hence the assumption (A) holds with \( \rho(x) = d(x) \), \( [x_0, y_0) = \mathbb{R}^+ \), \( \alpha(r) = 1 \), \( q(r) = 0 \) and \( \beta(r) = -(L + 1)r^2 \) (for some point \( c \in \mathbb{R}^+ \) large enough in (A)). By Example 3.20, \( +\infty \) is no entrance boundary for \( \mathcal{L}^1, q \), thus \( \mathcal{L}^V \) is \( L^\infty \)-unique by Theorem 4.1.

(c) Take \( \rho(x) = d(x) \) as above and \( \mathcal{L}^1, q \) as follows: \( [x_0, y_0) = \mathbb{R}^+ \), \( \alpha(r) = 1 \), \( q(r) = cr^\alpha \) and \( \beta(r) = -(L + \varepsilon)r^\alpha \), where \( \varepsilon > 0 \) is small enough so that \( c > (L + \varepsilon)\alpha(\alpha - 2) \). The assumption (A) is satisfied. Again by Example 3.20, \( +\infty \) is no entrance boundary for \( \mathcal{L}^1, q \), therefore \( \mathcal{L}^V \) is \( L^\infty \)-unique by Theorem 4.1.

Example 4.7. (the first example of [18]) Let \( M \) be a compact surface with arbitrary genus. Assume the metric on \( M \) around some point \( o \in M \) is flat. Hence locally around \( o \) we can write the metric in polar coordinates as

\[
d s_o^2 = dr^2 + r^2 d\theta^2
\]

we choose the new metric to be

\[
d s^2 = \varrho^2 d s_o^2.
\]

Choose \( \varrho \) to be arbitrary outside a neighborhood of \( o \) (say \( r \geq \delta \)), and for \( r \in (0, \delta] \),

\[
\varrho(\theta, r) = \varrho(r) = r^{-1}(\log r)^{-1} (\log(\log r))^{-\alpha},
\]

where \( 0 < \delta < e^{-2} \) and \( 0 < \alpha \leq 1 \) (It is assumed in [18] that \( \frac{1}{2} < \alpha \leq 1 \)). \((M \backslash \{o\}, d s^2)\) is (metrically) complete, stochastically complete and its volume is finite. The Green’s function on \((M, d s_o^2)\) (with the pole at \( o \)) \( G(o, x) = f(x) \) is a positive harmonic on \( M \backslash \{o\} \) w.r.t. \( d s^2_o \), then w.r.t. \( d s^2 \). Let \( \Delta, \nabla, | \cdot | \) be respectively the Laplacian operator, the gradient and the Riemannian norm in the metric \( d s^2 \). Note \( \Delta r = \frac{1}{r \varrho(r)^2} \), \( \varrho \), \( | \nabla | = \frac{1}{\varrho(r)} \). Consider the Sturm-Liouville operator \( \mathcal{L}^1 := \frac{1}{\varrho^2(r)} \frac{d^2}{d r^2} + \frac{d}{r \varrho^2(r)} \frac{d}{d r} \), the derivative of speed function and that of scale function of \( \mathcal{L} \) are respectively \( m'(r) = \varrho^2(r) \exp \left( \int_1^r \frac{1}{t} dt \right) = \varrho^2(r)r, \ s'(r) = \exp \left( - \int_1^r \frac{1}{t} dt \right) = \frac{1}{r} \).

(i). Let \( \alpha \in (0, \frac{1}{2}] \). Since

\[
\int_0^\delta m'(r) dr \int_r^\delta s'(t) dt = \int_0^\delta \varrho^2(r) r \log \left( \frac{1}{r} \right) dr
\]

\[
= \int_0^\delta \frac{1}{r \log(\frac{1}{r}) (\log \log(\frac{1}{r}))}^{2\alpha} dr
\]

\[
= +\infty,
\]

it follows that \( 0 \) is no entrance boundary for \( \mathcal{L}^1 \). By Theorem 4.1 \( (\Delta, C^\infty_0(M \backslash \{o\})) \) is \( L^\infty(M \backslash \{o\}) \)-unique. Then the \( L^1 \)-Liouville property holds true on \((M \backslash \{o\}, d s^2)\) by Corollary 2.6.

(ii). If \( \alpha \in (\frac{1}{2}, 1] \). The Green function \( f(x) = G(o, x) \) with respect to the old metric \( d s^2_0 \) is \( \Delta \)-harmonic and \( dx \)-integrable as observed in [18]. \( \Delta \) can not be
$L^\infty$-unique on $M\setminus\{o\}$ by Example 2.5 (or Corollary 2.6). On the other hand, we have

$$
\int_0^1 m'(r)dr \int_r^1 s'(t)dt = \int_0^1 g^2(r)r \log(\frac{1}{r})dr < +\infty.
$$

That shows the sharpness of Theorem 4.1.

Example 4.8. Let $D$ be the unit open ball centered at the origin of $\mathbb{R}^d(d \geq 2)$. Let $\mathcal{L}^V := \Delta - V(x)$ be defined on $C^\infty_0(D)$. Let $\rho(x) = |x|$ ($|x|$ denotes the Euclidian metric). For this example, $|\nabla \rho(x)| = 1$ and $\Delta \rho(x) = \frac{d-1}{r} \geq 0$ where $r = r(x) = |x|$. If $V(x) \geq \frac{c}{1-|x|^2}$, where $c$ is a constant such that $c \geq 2$. The assumption (A) is satisfied for $\mathcal{L}^{1,q} := \frac{d^2}{dr^2} - \frac{c}{(1-r)^2}$. By Example 3.18, 1 is no entrance boundary for $\mathcal{L}^{1,q}$, then $(\mathcal{L}^V, C^\infty_0(D))$ is $L^\infty$-unique by Theorem 4.1.

Example 4.9. Let us consider $\mathcal{L}^V := \Delta - V$ defined on $C^\infty_0(D)$ where $D := \mathbb{R}^d \setminus 0(d \geq 2)$. Let $\rho(x) = r(x) := |x|$, then $|\nabla \rho(x)| = 1$, $\Delta \rho(x) = \frac{d-1}{r(x)}$. If $V(x) \geq \frac{c}{r^2(x)}$ for $x$ close to 0 (say $|x| < \delta$), where the constant $c$ satisfies

$$
c \geq d^2 - (d - 1)^2 + 1 = 2d
$$

since 0 and $+\infty$ is no entrance boundary for $\mathcal{L}^{1,q} := \frac{d^2}{dr^2} + \frac{d-1}{r} \frac{d}{dr} - \frac{c}{r^2} \mathcal{L}^{1,q}(0,\delta)(r)$ by Example 3.18 $(\mathcal{L}^V, C^\infty_0(D))$ is $L^\infty$-unique by Theorem 4.1.

In contrary if $V = c/|x|^2$ with $0 \leq c < 2d$, as $\mathcal{L}^{1,q}$ is not $L^\infty(m)$-unique (again by Example 3.18), there is some $h \in L^1((0, +\infty), m) \cap C^1(0, +\infty)$ such that $h'$ is absolutely continuous and $\frac{dh}{dx} + \frac{d-1}{r} \frac{dh}{dr} - \frac{c}{r^2} h = h$ (such $h$ is indeed $C^\infty$-smooth). Let $\tilde{h}(x) = h(r(x))$, we see readily that $\tilde{h} \in L^1(D, dx)$ and $\Delta \tilde{h} - V\tilde{h} = \tilde{h}$ over $D = \mathbb{R}^d \setminus \{0\}$. Thus $\Delta - V$ defined on $C^\infty_0(D)$ is not $L^\infty(D, dx)$-unique.

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