MATRIX ALGEBRAS CONVERGE TO THE SPHERE
FOR QUANTUM GROMOV–HAUSDORFF DISTANCE

MARC A. RIEFFEL

Abstract. On looking at the literature associated with string
theory one finds statements that a sequence of matrix algebras
converges to the 2-sphere (or to other spaces). There is often care-
ful bookkeeping with lengths, which suggests that one is dealing
with “quantum metric spaces”. We show how to make these ideas
precise by means of Berezin quantization using coherent states. We
work in the general setting of integral coadjoint orbits for compact
Lie groups.

On perusing the theoretical physics literature which deals with strin
g theory and related parts of quantum field theory, one finds in many
scattered places assertions that the complex matrix algebras, $M_n$,
converge to the two-sphere, $S^2$, (or to related spaces) as $n$ goes to infinity.
Here $S^2$ is viewed as synonymous with the algebra $C(S^2)$ of continu-
ous complex-valued functions on $S^2$ (of which $S^2$ is the maximal-ideal
space). Approximating the sphere by matrix algebras is attractive for
the following reason. In trying to carry out quantum field theory on $S^2$
it is natural to try to proceed by approximating $S^2$ by finite spaces. But
“lattice” approximations coming from choosing a finite set of points in
$S^2$ break the very important symmetry of the action of $SU(2)$ on $S^2$
(via $SO(3)$). But $SU(2)$ acts naturally on the matrix algebras, in a
way coherent with its action on $S^2$, as we will recall below. So it is
natural to use them to approximate $C(S^2)$. In this setting the matrix
algebras are often referred to as “fuzzy spheres”. (See [37], [38], [19],
[24], [26] and references therein.)

When using the approximation of $S^2$ by matrix algebras, the precise
sense of convergence is usually not explicitly specified in the literature.
Much of the literature is at a largely algebraic level, with indications
that the notion of convergence which is intended involves how struc-
ture constants and important formulas change as $n$ grows. See, for

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example, \[36\], \[20\], \[6\] and section IIC of \[56\]. There is some discussion of approximation by matrix algebras at a more analytical level within the mathematical physics literature concerned with quantization of symplectic manifolds. See the references in \[31\], \[32\]. Much of this discussion goes in the direction of showing that the matrix algebras can be combined with \(C(S^2)\) to form a continuous field of \(C^*\)-algebras with \(C(S^2)\) as limit point as \(n\) grows. To me the most satisfying version of this idea has been given by Landsman \[32\] (or \[31\]). He shows, in the more general setting of coadjoint orbits of compact Lie groups, and by means of Berezin quantization, that suitable matrix algebras form a strict quantization, as defined in \[31\], \[43\], \[44\], \[45\]. This means that not only does one have a continuous field of \(C^*\)-algebras, but also that commutators in the \(M_n\)’s converge to the Poisson bracket on \(S^2\) in a precise analytical sense (so that one has a good “semi-classical limit”). A more complicated proof, in the more general setting of compact Kähler manifolds, was given earlier in \[9\].

But if one goes back to the string-theory literature, one sees that there is much more in play than just the continuous-field aspect. Almost always there are various lengths involved, and the writers are often careful in their bookkeeping with these lengths as \(n\) grows. This suggested to me that one is dealing here with metric spaces in some quantum sense, and with the convergence of quantum metric spaces. Now the only notion of convergence of classical compact metric spaces with which I am familiar is that of Gromov–Hausdorff convergence. With this in mind, I gave in \[48\] a definition of what one might mean by a compact quantum metric space, and what can be meant by quantum Gromov–Hausdorff convergence of these compact quantum metric spaces. (“Compact” here means that we restrict attention to initial algebras.) I showed that these notions have basic properties which closely parallel the classical theory.

The purpose of the present paper is to show that when matrix algebras are equipped in a natural way with a “metric”, then they converge in quantum Gromov–Hausdorff distance to \(S^2\) with its round metric (for a given radius). In fact, we show that the corresponding fact is equally well true for any integral coadjoint orbit (corresponding to an irreducible representation) of a compact Lie group, once suitable definitions are given. (Examples in the field-theory and string-theory literature of the approximation of coadjoint orbits other than the two-sphere by matrix algebras are given in \[22\], \[57\], \[1\], \[5\], \[13\] and the references therein.) As in the work of Landsman, our principal tool is Berezin quantization.
But when one goes back yet again to the field and string theory literature, one is reminded that there is still much more structure involved than just a metric structure and possible Gromov–Hausdorff convergence. The field and string-theorists need the whole apparatus of non-commutative differential geometry — “vector bundles”, connections, “Riemannian metrics” — so that they can define action functionals such as Yang–Mills functionals. (See [21], [26] and references therein.) Furthermore they also want supersymmetry. Thus what seems to be needed is a “Gromov–Hausdorff” convergence which encompasses this whole apparatus. But so far as I am aware, such a theory does not yet exist even for ordinary spaces. But hints of what such a theory might look like can be found in the literature concerned with the collapsing of Riemannian manifolds. The limit spaces need not be manifolds. But certain parts of the metric and differential geometric apparatus persist [16], [18], [30], [49], [52], [34], [35], [55] [23]. It is an interesting challenge to give a full characterization of the parts of the apparatus which do persist, and to give an effective definition of the convergence of this apparatus; and then to find the appropriate quantum generalizations.

An interesting aspect of our theory is that if \( O \) is a coadjoint orbit for some compact Lie group, then there will be a suitable sequence \( \{n_j\} \) of dimensions such that the sequence of matrix algebras \( M_{n_j} \) converge to \( O \). But this same sequence of matrix algebras also converges to the sphere \( S^2 \), which need not be homeomorphic to \( O \). What is making the crucial difference is that the metric structures which are placed on the matrix algebras are different in the two cases. This phenomenon may perhaps have some relation to the ideas of “change of topology” which one finds in some places in the string-theory literature [39], [3], [4], [25], [10]. More specifically, it will follow from what we do that for any \( \epsilon > 0 \) there is a finite sequence of compact quantum metric spaces such that the first one is \( O \), the last one is \( S^2 \), the intermediate ones are all full matrix algebras, and the sequence is an \( \epsilon \)-chain in the sense that the quantum Gromov-Hausdorff distance between any two successive elements of the chain is no greater than \( \epsilon \).

The field and string theorists are interested in approximating many more spaces by matrix algebras than just those which occur as coadjoint orbits. For example, they consider tori ([12], [2] and references therein), surfaces of higher genus [11], [7], and higher-dimensional spheres [24], [42]. I am optimistic that the main theorem of the present paper can in some form be extended (by different methods) to more general homogeneous spaces, such as higher-dimensional spheres [42], since one still has available the action of a compact Lie group which is so heavily used here. I am also optimistic that tori will not be difficult to treat,
by using yet other methods of harmonic analysis. The substantial literature on quantization of Kähler manifolds using Berezin quantization suggests that they too may well be approximable in quantum Gromov–Hausdorff distance by matrix algebras with suitable metrics. (See [50] and the references therein.) This should be an interesting project. My doctoral student Hanfeng Li has checked that by using some of the facts in [27] one can carry out a certain number of steps parallel to those which we carry out here for coadjoint orbits. (One should also consider almost Kähler manifolds [28].) But it is far from clear to me how far the theory might extend to arbitrary compact Riemannian manifolds.

As discussed in [47] and [48], the appropriate way to specify a “metric” on a unital $C^*$-algebra, $A$, is by means of a seminorm (generally unbounded) which plays the role of the Lipschitz seminorm on the functions on an ordinary metric space. In the present paper these seminorms are defined in a quite simple way. Let $G$ be a compact Lie group, and let $\ell$ be a continuous length function on $G$ (for example, coming from the Riemannian metric on $G$ corresponding to an $\text{Ad}$-invariant inner product on the Lie algebra of $G$, especially if one wants the usual round metric on the sphere). Let $\alpha$ be an action of $G$ on $A$, and assume that the action is “ergodic” in the sense that the only $\alpha$-invariant elements of $A$ are the scalar multiples of the identity element of $A$. Then we define the corresponding Lipschitz seminorm, $L$, on $A$ by

$$L(a) = \sup\{\|\alpha_x(a) - a\|/\ell(x) : x \neq e\},$$

where $e$ denotes the identity element of $G$. (We may well have $L(a) = +\infty$, but the set of $a$’s for which $L(a) < \infty$ is a dense $*$-subalgebra.) Some of the attractive properties of this definition were discussed in [46]. One pertinent example is the action of $SO(3)$, and thus $SU(2)$, on $S^2$; but an equally pertinent example comes from considering an irreducible unitary representation, $U$, of $G$ on a finite-dimensional Hilbert space $\mathcal{H}$. Let $A$ be $B(\mathcal{H})$, the algebra of operators on $\mathcal{H}$ (a full matrix algebra), and let $\alpha$ be the action of $G$ on $A$ by conjugation by $U$. Then $(B(\mathcal{H}), L)$ is a fine example of a compact quantum metric space (and $L$ depends strongly on which group, with what representation, acts on $\mathcal{H}$). For $G = SU(2)$ we will show that $(B(\mathcal{H}), L)$ converges to $(C(S^2), L)$ for quantum Gromov–Hausdorff distance as the dimension of $\mathcal{H}$ increases, and similarly for other coadjoint orbits.

As mentioned above, our main analytical tool is Berezin quantization in terms of coherent states. What we need will be reviewed below. But, with $G = SU(2)$ and with $(U, H)$ as above, and for a choice of highest weight-vector, $\xi$, in $\mathcal{H}$, Berezin defines for each $T \in B(\mathcal{H})$
its symbol, $\sigma_T$, which is a continuous function on the coadjoint orbit, $O$, of the weight for $\xi$. When $B(\mathcal{H})$ is equipped with its Hilbert–Schmidt norm, and $C(O)$ with its $L^2(O)$-norm, $\sigma$ has an adjoint, $\tilde{\sigma}$, from $C(O)$ to $B(\mathcal{H})$. The maps $\sigma$ and $\tilde{\sigma}$ provide our principal tools for estimating the quantum Gromov–Hausdorff distance from $B(\mathcal{H})$ to $C(O)$. The composition $\sigma \circ \tilde{\sigma}$ is called the Berezin transform, and has received considerable study. Nevertheless, the main estimate which we need for the Berezin transform seems to be new, and of independent interest. We give a slightly imprecise statement of it here. Label $\mathcal{H}$ by its dimension, thus $\mathcal{H}_n$. Then $\sigma$ and $\tilde{\sigma}$ depend on $n$, and we write $(\sigma \circ \tilde{\sigma})_n$ for the corresponding transform.

**Theorem 3.4** (imprecise). There is a sequence $\{\delta_n\}$ of numbers converging to 0 such that

$$\|f - (\sigma \circ \tilde{\sigma})_n(f)\|_\infty \leq \delta_n L(f)$$

for every $f \in C(O)$ and every $n$.

I have not seen a relation such as this one between the Berezin transform and Lipschitz norms discussed in the literature.

We also need information about the composition in the opposite order, $(\tilde{\sigma} \circ \sigma)_n$, which carries $B(\mathcal{H}_n)$ into itself. I have not seen this mapping discussed in the literature at all. We will prove:

**Theorem 6.1** (imprecise). Let $\gamma_n$ be the smallest constant such that

$$\|T - (\tilde{\sigma} \circ \sigma)(T)\| \leq \gamma_n L(T)$$

for all $T \in B(\mathcal{H}_n)$. Then the sequence $\{\gamma_n\}$ converges to 0.

The proof of this theorem involves other interesting facts about Berezin symbols.

This paper is organized as follows. In Section 1 we introduce much of our notation and many of the structures which we need. In Section 2 we carry the development as far as we can for general compact groups (including finite ones). We turn to compact Lie groups in Section 3, where we state our main theorem (Theorem 3.2) concerning convergence of matrix algebras to coadjoint orbits. We also develop there the facts about Berezin covariant symbols which we need to prove Theorem 3.4 (stated above). Section 4 develops further facts about covariant symbols, which are then used in Section 5 to obtain the facts about Berezin contravariant symbols which we need. Then in Section 6 we prove Theorem 6.1 (stated above), and use it to conclude the proof of our main theorem.

Let me mention that David Kerr has recently developed a matricial version of quantum Gromov-Hausdorff distance [29], and he indicates
how it applies to the topic of the present paper. Also, my former doctoral student Hanfeng Li has worked out [33] that the main theorem (Theorem 3.2) can also be successfully approached by using the results on continuous fields of quantum metric spaces developed in [48] and the results of Landsman [32], [31] showing that Berezin quantization gives a strict quantization. Here we will not assume that the continuous-field structure is known (though Theorem 4.2 and Proposition 4.3 provide the main steps in establishing it). Thus our treatment here is more self-contained.

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1. The quantum metric spaces

Let $G$ be a compact group (perhaps even finite, at first). Let $U$ be an irreducible unitary representation of $G$ on a Hilbert space $H$. Let $B = B(H)$ denote the $C^*$-algebra of all linear operators on $H$ (a “full matrix algebra”, with its operator norm). There is a natural action, $\alpha$, of $G$ on $B$ by conjugation by $U$. That is, $\alpha_x(T) = U_x T U_x^*$ for $x \in G$ and $T \in B$. Because $U$ is irreducible, the action $\alpha$ is “ergodic”, in the sense that the only $\alpha$-invariant elements of $B$ are the scalar multiples of the identity operator. Fix a continuous length function, $\ell$, on $G$ (so $G$ must be metrizable). Thus $\ell$ is non-negative, $\ell(x) = 0$ iff $x = e$ (the identity element of $G$), $\ell(x^{-1}) = \ell(x)$, and $\ell(xy) \leq \ell(x) + \ell(y)$.

For the reasons given in Section 2 we also require a condition discussed in example 6.5 of [48], namely that $\ell(xy^{-1}) = \ell(x)$ for all $x, y \in G$. Then in terms of $\alpha$ and $\ell$ we can define a seminorm, $L_B$, on $B$ by the formula (0.1) given in the introduction. Then $(B, L_B)$, or more precisely its self-adjoint part, is an example of a compact quantum metric space, as defined in [48]. (For a quite different way of defining a Lip-norm on $B$, for which the approximation of $A$ by $B$ in the quantum Gromov–Hausdorff metric is almost a tautology, see equation 3.19 of [60].)

Let $P$ be a rank-one projection in $B(H)$ (traditionally specified by giving a non-zero vector in its range). For any $T \in B$ we define its Berezin covariant symbol [8], [41], $\sigma_T$, with respect to $P$, by

$$
\sigma_T(x) = \tau(T \alpha_x(P)),
$$
where $\tau$ denotes the usual (un-normalized) trace on $B$. (When the $\alpha_x(P)$'s are viewed as giving states on $B$ via $\tau$ as above, they form a family of "coherent states" [41].) Let $H$ denote the stability subgroup of $P$ for $\alpha$. Then it is evident that $\sigma_T$ can be viewed as a (continuous) function on $G/H$. We let $\lambda$ denote the action of $G$ on $G/H$, and so on $A = C(G/H)$, by left-translation. If we note that $\tau$ is $\alpha$-invariant, then it is easily seen that $\sigma$ is a unital, positive, norm-nonincreasing, $\alpha$-$\lambda$-equivariant map from $B$ into $A$.

Of course, from $\lambda$ and $\ell$ we obtain a seminorm, $L_A$, on $A$ by formula (0.1). It is just the restriction to $A$ of the seminorm on $C(G)$ which we get from $\ell$ when we view $C(G/H)$ as a subalgebra of $C(G)$, as we will often do when convenient. We will often not restrict $L_A$ to the Lipschitz functions, but rather permit $L_A(f) = +\infty$. From $L_A$ we obtain the usual quotient metric [58] on $G/H$ coming from the metric on $G$ for $\ell$. One can check easily that $L_A$ in turn comes from this quotient metric. Thus $(A, L_A)$ is the compact quantum metric space associated to this ordinary compact metric space.

It is reasonable to ask whether $\sigma$ might say something about the quantum Gromov–Hausdorff distance (reviewed below) between $(A, L_A)$ and $(B, L_B)$. We stress that $H$, and so $A$, depends on the choice of the projection $P$, and that even if two choices of $P$ have the same stability group $H$, the symbol maps $\sigma$ may be different, and may give quite different estimates of quantum Gromov–Hausdorff distance. (Strictly speaking we should, according to [48], be using the self-adjoint parts of $A$ and $B$. But by the comments just before definition 2.1 of [48] we can, and will, be careless about this.)

According to the definition of quantum Gromov–Hausdorff distance given in [48], we must examine seminorms, $L$, on $A \oplus B$ whose quotient seminorms on $A$ and $B$ are $L_A$ and $L_B$ respectively. Furthermore, $L$ must be a Lip-norm, as defined in [47], meaning that the null-space of $L$ is spanned by $(1_A, 1_B)$, and that when $L$ is used to define a metric, $\rho_L$, on the state space, $S(A \oplus B)$, of $A \oplus B$, by

$$\rho_L(\mu, \nu) = \sup\{|\mu(c) - \nu(c)| : c \in A \oplus B, \ L(c) \leq 1\},$$

then the topology on $S(A \oplus B)$ from $\rho_L$ coincides with the weak-* topology. The state spaces $S(A)$ and $S(B)$ can be viewed in an evident way as subsets of $S(A \oplus B)$. We can then consider the usual Hausdorff distance between them for $\rho_L$. By definition [48], the quantum Gromov–Hausdorff distance between $(A, L_A)$ and $(B, L_B)$ is the infimum of these Hausdorff distances as $L$ varies.

We thus need a way to construct Lip-norms $L$ on $A \oplus B$. As discussed in section 5 of [48], a convenient way to do this is to look for seminorms
$N$ on $A \oplus B$, called “bridges”, and then set
\[ L(a, b) = L_A(a) \vee L_B(b) \vee N(a, b), \]
where $\vee$ denotes “maximum”. The advantage of this is that $N$ can be taken to be bounded. For the present situation, we will take our bridges $N$ to be of the very simple form
\[ N(f, T) = \gamma^{-1}\|f - \sigma_T\|_\infty \]
for $f \in A$ and $T \in B$, where $\gamma$ is some positive constant. This constant must be taken large enough that the corresponding $L$ has $L_A$ and $L_B$ as quotient seminorms. But for $B$ this is no problem. Note first:

**Proposition 1.1.** For any $T \in B$ we have
\[ L_A(\sigma_T) \leq L_B(T). \]

**Proof.** Since $\sigma$ is equivariant and does not increase norms, we have
\[
\|\lambda_x(\sigma_T) - \sigma_T\|_\infty /\ell(x) = \|\sigma_{(\alpha_x(T) - T)}\|_\infty /\ell(x) \\
\leq \|\alpha_x(T) - T\| /\ell(x) \leq L_B(T)
\]
for every $x \in G$. \qed

**Corollary 1.2.** For $L$ defined from $N$ as above, the quotient of $L$ on $B$ is $L_B$, regardless of the choice of $\gamma$.

**Proof.** It is clear from the definition of $L$ that the quotient of $L$ on $B$ is no smaller than $L_B$. But, given $T \in B$, we can simply take $f = \sigma_T$. From the above proposition we then have $L_A(f) \leq L_B(T)$. And $N(f, T) = 0$, so that
\[
L(f, T) = L_A(f) \vee L_B(T) \vee \gamma^{-1}N(f, T) = L_B(T),
\]
as desired. \qed

Thus the difficult issue is how big $\gamma$ must be in order that the quotient of $L$ on $A$ be $L_A$. But assume that such a suitable $\gamma$ has been found. For the corresponding $L$ and $\rho_L$ we must estimate the Hausdorff distance between $S(A)$ and $S(B)$. Again, one half of this is quite simple.

**Proposition 1.3.** Let $\gamma$ be chosen such that the quotient of $L$ on $A$ is $L_A$. Then $S(A)$ is in the $\gamma$-neighborhood of $S(B)$ for $\rho_L$.

**Proof.** Let $\mu \in S(A) \subset S(A \oplus B)$. We must find $\nu \in S(B) \subset S(A \oplus B)$ such that $\rho_L(\mu, \nu) \leq \gamma$. A natural guess for $\nu$ is $\nu = \mu \circ \sigma$. Because $\sigma$ is positive and unital, $\nu$ is indeed in $S(B)$. We show that this choice
of \( \nu \) works. Let \((f, T) \in A \oplus B\) with \(L(f, T) \leq 1\). Then, in particular, 
\[ \| f - \sigma_T \|_\infty \leq \gamma. \]
Thus
\[
|\mu(f, T) - \nu(f, T)| = |\mu(f) - \nu(T)| = |\mu(f) - \mu(\sigma_T)| = |\mu(f - \sigma_T)| \leq \| f - \sigma_T \|_\infty \leq \gamma.
\]
Since this is true for all such \((f, T)\), we have \(\rho_L(\mu, \nu) \leq \gamma\), as desired.

It is thus clear that to get the best estimates, we want \(\gamma\) to be as small as possible consistent with the quotient of \(L\) on \(A\) being \(L_A\).

But we will still have the second difficult issue of showing that \(S(B)\) is in a suitably small neighborhood of \(S(A)\), so that their Hausdorff distance is small. It will take all of the discussion in Sections 4 and 5 for us to handle this issue for compact semisimple Lie groups. But we address first, in the next section, the first of these two difficult issues.

We remark that it might be interesting to see if the Stratonovich–Weyl symbols of [15] could be used in place of the Berezin symbol in the definition of the bridge \(N\). However the Stratonovich symbol does not in general carry positive operators to positive functions, and so the proof of Proposition 1.3 would not carry over directly.

\section{Choosing the bridge constant \(\gamma\)}

In this section we give one method for finding a \(\gamma\) which will work for the bridge \(N\) of the previous section. This method will be adequate for our later purposes. We continue with the same notation as in the previous section.

We choose for \(G\) the Haar measure which gives \(G\) total measure 1; and on \(G/H\) we choose the image of this Haar measure, which is a \(G\)-invariant measure giving \(G/H\) total mass 1. This is conveniently done by viewing \(C(G/H)\) as a subalgebra of \(C(G)\). We will often not distinguish between a point in \(G\) and is image in \(G/H\), with the context making clear what is intended. We will denote integration on \(G\) or \(G/H\) by \(dx\), \(dy\), etc.

As mentioned in the introduction, we put on \(A = C(G/H)\) the inner product from \(L^2(G/H)\), while on \(B = B(\mathcal{H})\) we put its Hilbert–Schmidt inner product, using now the normalized trace \(d^{-1}\tau\), where \(d\) denotes the dimension of \(\mathcal{H}\). Then the mapping \(\sigma\) from \(B\) to \(A\) has an adjoint operator, \(\hat{\sigma}\), from \(A\) to \(B\). For any \(T \in B\), a function \(f \in A\) such that \(\hat{\sigma}_F = T\) is called a Berezin contravariant symbol [8], [41] for \(T\). The mapping \(\hat{\sigma}\) is often viewed as a quantization, since it takes functions to operators.
We need a familiar formula for \( \tilde{\sigma} \). For the reader’s convenience, and to put matters into our notation, we give the derivation of this formula here. For any \( f \in A \) and \( T \in B \) we have

\[
\begin{align*}
\tau^{-1}(\tilde{\sigma}_f T^*) &= \langle \tilde{\sigma}_f, T \rangle = \langle f, \sigma_T \rangle = \int f(x)(\sigma_T(x))^-dx \\
&= \int f(x)\tau(\alpha_x(P)T^*)dx = \tau(\int f(x)a_x(P)dx T^*).
\end{align*}
\]

Since this is true for all \( T \), we obtain

\[ \tilde{\sigma}_f = d \int f(x)\alpha_x(P)dx. \]

It is well-known and easily seen that the “second orthogonality relation” for irreducible representations of groups can be written as

\[ I = d \int \alpha_x(P)dx, \]

where \( I \) is the identity operator. From this we see that \( \tilde{\sigma} \) is unital. (It was in order to make \( \tilde{\sigma} \) unital that we normalized the trace.) It is also evident that \( \tilde{\sigma} \) is positive, norm-nonincreasing, and \( \lambda-\alpha \)-equivariant. It is easy to see that \( \tilde{\sigma} \) corresponds to a “pure-state quantization” as discussed in [31].

Our aim is to find a \( \gamma \) such that the quotient on \( A \) of our \( L \), as defined in the previous section, is \( L_A \). Now from the definition of \( L \) it is clear that the quotient is never less than \( L_A \). Thus what we would like to show is that, given \( f \in C(G/H) \), we can find \( T \in B(H) \) such that \( L(f, T) = L_A(f) \). It is reasonable to try \( T = \tilde{\sigma}_f \). Now by the same argument as in the proof of Proposition 1.1, using the fact that \( \tilde{\sigma} \) is equivariant and norm-nonincreasing, we find that \( L_B(\tilde{\sigma}_f) \leq L_A(f) \). Thus it suffices to have

\[ L_A(f) \geq N(f, \tilde{\sigma}_f) = \gamma^{-1}\|f - \sigma(\tilde{\sigma}_f)\|_{\infty}. \]

That is, we seek \( \gamma \) such that

\[ \|f - \sigma(\tilde{\sigma}_f)\|_{\infty} \leq \gamma L_A(f) \]

for all \( f \in A \). Now the mapping \( f \mapsto \sigma(\tilde{\sigma}_f) \) has received substantial study, and is often referred to as the Berezin transform [50]. But I have not seen in the literature any discussion of its relation to Lipschitz norms. We need the following frequently derived formula for the Berezin transform.

\[
(\sigma(\tilde{\sigma}_f))(x) = \tau(\tilde{\sigma}_f \alpha_x(P)) = \tau \left( d \int f(y)a_x(P)dy \alpha_x(P) \right)
\]
Notation 2.1. For any rank-one projection $P$ on $\mathcal{H}$ we define $h_P \in C(G/H)$ by

$$h_P(x) = d \tau(P\alpha_x(P)).$$

It is easily checked that if $\xi$ is a vector of unit length in the range of $P$, then $h_P(x) = d|\langle U_x\xi, \xi\rangle|^2$. We note that $h_P$ is non-negative, and that

$$\int h(x)dx = \tau(Pd\int \alpha_x(P)dx) = \tau(P) = 1,$$

so that $h_P(x)dx$ is a probability measure on $G/H$. Furthermore, $h_P(x^{-1}) = h_P(x)$ in the sense that $h_P(x^{-1}H) = h_P(xH)$ for $x \in G$, because $\tau$ is $\alpha$-invariant. Also, $h_P(e) = d$, so that $h_P$ must be somewhat concentrated near $e (= eH)$ in $G/H$. The formula we need then becomes

$$(2.2) \quad (\sigma(\hat{\sigma}_f))(x) = \int f(y)h_P(y^{-1}x)dy = \int f(xy^{-1})h_P(y)dy.$$

We recognize this as an ordinary convolution (reflecting the $\lambda$-equivariance of $\sigma \circ \hat{\sigma}$), if we view the functions as defined in $G$ rather than $G/H$.

We must now bring the length function $\ell$ into the picture. We have used it to define the seminorm $L_A$ on $A = C(G/H)$, with its corresponding metric, $\rho_A$, on $G/H$. As mentioned earlier, $L_A$ is then the Lipschitz seminorm for $\rho_A$. This means that

$$|f(x) - f(y)| \leq L_A(f)\rho_A(x, y)$$

for all $f \in C(G/H)$ and $x, y \in G/H$. (We permit $L_A(f) = +\infty$.) As discussed in example 6.5 of [48], our extra requirement that $\ell(zxz^{-1}) = \ell(x)$ implies that the action of $G$ on $A$ will leave $L$ invariant. Thus $G$ acts as isometries on $G/H$ for $\rho_A$, that is, $\rho_A(zx, zy) = \rho_A(x, y)$ for all $z \in G$ and $x, y \in G/H$.

We are now ready to obtain the estimate which we need. Let $f \in C(G/H)$. Because $h_P$ gives a probability measure, and $h_P(y^{-1}) = h_P(y)$, we have

$$|f(x) - (\sigma(\hat{\sigma}_f))(x)| = \left| \int (f(x) - f(y))h_P(y^{-1}x)dy \right|$$

$$\leq \int |f(x) - f(y)|h_P(y^{-1}x)dy \leq L_A(f) \int \rho_A(x, y)h_P(y^{-1}x)dy$$
\[ L_A(f) \int \rho_A(x, xy) h_P(y^{-1}) dy = L_A(f) \int \rho_A(e, y) h_P(y) dy. \]

We have thus obtained:

**Theorem 2.3.** With notation as above, we have

\[ \|f - \sigma(\tilde{\sigma} f)\|_\infty \leq \gamma L_A(f) \]

for all \( f \in A = C(G/H) \) if we set

\[ \gamma = \int_{G/H} \rho_A(e, y) h_P(y) dy. \]

**Corollary 2.4.** For \( \gamma \) chosen by the formula just above, the seminorm \( L \) on \( A \oplus B \) defined by

\[ L(f, T) = L_A(f) \lor L_B(T) \lor \gamma^{-1}\|f - \sigma_T\|_\infty \]

has \( L_A \) as its quotient on \( A \).

It follows from Proposition 1.3 that \( S(A) \) is then in the \( \gamma \)-neighborhood of \( S(B) \) for \( \rho_L \). Consequently, in order to obtain an upper bound on the quantum Gromov–Hausdorff distance between \( (A, L_A) \) and \( (B, L_B) \), we must show also that \( S(B) \) is in a suitably small neighborhood of \( S(A) \). That is, for each \( \nu \in S(B) \) we must find \( \mu \in S(A) \) close to \( \nu \). In view of our success in Proposition 1.3, it is reasonable to try setting \( \mu = \nu \circ \tilde{\sigma} \). For this choice of \( \mu \) we can estimate \( \rho_L(\mu, \nu) \) as follows. Suppose that \( (f, T) \in A \oplus B \) and that \( L(f, T) \leq 1 \), so that \( \|f - \sigma_T\|_\infty \leq \gamma \). Then, because \( \tilde{\sigma} \) is norm-non-increasing, we have

\[ |\mu(f, T) - \nu(f, T)| = |\nu(\tilde{\sigma} f - T)| \leq \|\tilde{\sigma} f - T\| \leq \|\tilde{\sigma} f - \tilde{\sigma} \sigma_T\| + \|\tilde{\sigma} \sigma_T - T\| \leq \|f - \sigma_T\|_\infty + \|\tilde{\sigma} \sigma_T - T\| \leq \gamma + \|\tilde{\sigma} \sigma_T - T\|. \]

Thus any bound which we can obtain on \( \|\tilde{\sigma} \sigma_T - T\| \) for \( L_B(T) \leq 1 \) will give us a bound on the quantum Gromov–Hausdorff distance. I do not see an effective method for obtaining a good bound in general, e.g., for finite groups. But for compact semisimple Lie groups we will see how to do this in Sections 4 and 5.

It should be stressed that \( \gamma \) as above (as well as \( \sigma \) and \( \tilde{\sigma} \) and \( H \)) depends on our choice of the rank-one projection \( P \), as we will glimpse further in the next sections. It is not clear to me how to make an optimal choice of \( P \) in general, e.g., for finite groups.
There is a natural question which we did not address in the previous section. For $\sigma$ to be most useful, it is reasonable to expect that if two operators have the same symbol then they should be the same operator, that is, $\sigma$ should be faithful. But if we consider the Hilbert–Schmidt inner product on $B = B(H)$ using $\tau$, it is clear that we will have $\sigma_T = 0$ exactly if $T$ is orthogonal to the linear span of $\{\alpha_x(P) : x \in G\}$. Thus we want this linear span to be all of $B$. This will not happen in general, but I have found in the literature almost no discussion of exactly when it does happen. The only situation in which it is well understood that the span is all of $B$ seems to be that in which $G$ is a compact semisimple Lie group and $P$ corresponds to a highest weight vector. Most proofs for this case use the complex structure which one then has on $G/H$ [41]. However, Simon [53] has given a relatively elementary proof. (A more complicated but elementary proof appears in theorem 4.11 of [59], while an earlier variant of it appears as theorem 1 of [15].) Simon also gives a few examples showing that the span can be all of $B$ for some other weight vectors, but not for all. There is additional discussion of these matters in [59], [40]. Since the case of a highest weight vector is crucial for our purposes, we include a proof here, along the lines of Simon’s proof [53], but very slightly simpler. This also gives us an opportunity to introduce some of the notation which we will use extensively later on. We remark that in what follows it is not essential that $G$ be semisimple — we could permit it to be the product of a semisimple group with a torus. But this would provide no new coadjoint orbits, and would somewhat complicate our notation.

Thus, let $G$ be a connected compact semisimple Lie group. We denote its Lie algebra by $\mathfrak{g}_0$, because we will usually work with the complexification, $\mathfrak{g}$, of $\mathfrak{g}_0$. We choose a maximal torus in $G$, with corresponding Cartan subalgebra of $\mathfrak{g}$, its set of roots $\Delta$, positive roots $\Delta^+$, and elements $\{H_\beta, E_\beta, E_{-\beta} : \beta \in \Delta^+\}$ for $\mathfrak{g}$, where $[E_\beta, E_{-\beta}] = H_\beta$, etc., much as in equations VIII.56 of [54]. Let $(U, \mathcal{H})$ be an irreducible unitary representation of $G$. We let $U$ also denote the corresponding representation of $\mathfrak{g}$. Then $(U, \mathcal{H})$ will have a highest weight vector, $\xi$, of norm 1, unique up to a scalar multiple, and characterized by the fact that $U_{E_\beta}\xi = 0$ for all $\beta \in \Delta^+$.

**Theorem 3.1.** Let $P$ be the rank-one projection which has the highest weight vector $\xi$ in its range. Then

$$\text{span}\{\alpha_x(P) : x \in G\} = B(\mathcal{H}).$$
Proof. Let $S$ denote the linear span of the $\alpha_x(P)$’s. It is clear that $S$ is an $\alpha$-invariant subspace of $B(\mathcal{H})$ which contains $P$. We let $\alpha$ also denote the corresponding action of $\mathfrak{g}$ on $B(\mathcal{H})$, given by

$$\alpha_X(T) = [U_X, T] = U_X T - T U_X$$

for $X \in \mathfrak{g}$ and $T \in B(\mathcal{H})$. Then this action of $\mathfrak{g}$ carries $S$ into itself.

From equation VIII.5.6a of [54] we know that we can choose the $E_\beta$’s so that $(U E_\beta)^* = U E_{-\beta}$. For $\eta, \zeta \in \mathcal{H}$ let $\langle \eta, \zeta \rangle_\mathcal{K}$ denote the rank-one operator on $\mathcal{H}$ defined by $\langle \eta, \zeta \rangle_\mathcal{K} \theta = \langle \theta, \zeta \rangle \eta$. Then for $\beta \in \Delta^+$ we have

$$\alpha_{E_\beta}(P) = \alpha_{E_\beta}(\langle \xi, \xi \rangle_\mathcal{K})$$

$$= \langle U E_\beta \xi, \xi \rangle_\mathcal{K} + \langle \xi, U_{E_{-\beta}} \xi \rangle_\mathcal{K} = \langle \xi, U_{E_{-\beta}} \xi \rangle_\mathcal{K}.$$ 

In the same way, if we apply a product $E_{\beta_1} \ldots E_{\beta_k}$ to $P$, where $\beta_j \in \Delta^+$ for each $j$, we obtain $\langle \xi, \eta \rangle_\mathcal{K}$, where

$$\eta = U_{E_{-\beta_k}} \ldots U_{E_{-\beta_1}} \xi.$$ 

But it is known (proposition VII.2 of [51]) that the various $\eta$’s of this form span $\mathcal{H}$. In this way we see that $S$ must contain all rank-one operators of the form $\langle \xi, \eta \rangle_\mathcal{K}$ for $\eta \in \mathcal{H}$. But $S$ is invariant for the $G$-action $\alpha$. It follows that $S$ contains all rank-one operators, and so coincides with $B(\mathcal{H})$. $\square$

We will let $\text{Ad}$ denote the adjoint action of $G$ on $\mathfrak{g}_0$ and $\mathfrak{g}$, and let $\text{Ad}^*$ denote the coadjoint action of $G$ on the dual vector spaces $\mathfrak{g}_0^*$ and $\mathfrak{g}^*$. Let $P$ be the rank-one projection for the highest weight vector $\xi$. Define an element $\omega$ of $\mathfrak{g}^*$ by

$$\omega(X) = -i \tau(U_X P) = -i \langle U_X \xi, \xi \rangle.$$ 

Note that $\omega$ is real on $\mathfrak{g}_0$, so it can be viewed as an element of $\mathfrak{g}_0^*$. Now

$$(\text{Ad}_x^* \omega)(X) = \omega(\text{Ad}_{x^{-1}}(X)) = -i \tau(U_{x^{-1}} U_X U_x P) = -i \tau(U_X \alpha_x(P)).$$

From this we see that the stability subgroup for $\omega$ under $\text{Ad}^*$ coincides with the stability subgroup, $H$, for $P$ under $\alpha$, as is well-known [32], [31]. The coadjoint orbit, $O$, of $\omega$ is naturally identified with $G/H$ via $x \mapsto \text{Ad}_x^*(\omega)$. It is worth remarking that $H$ clearly contains the center of $G$, and so it is not important here whether $G$ is simply connected. But if, in fact, $G$ is simply connected, then to every coadjoint orbit which is integral in the usual algebraic sense there will correspond an irreducible representation of $G$ with highest weight vector giving that orbit. Thus if we need to apply our metric considerations to all integral coadjoint orbits, we should take $G$ to be simply connected (or work with
projective representations). Of course, the coadjoint orbit for a highest weight vector for $SU(2)$ will be a two-sphere $S^2$.

We now fix the representation $(U, \mathcal{H})$, and thus the coadjoint orbit $\mathcal{O}$. We will follow a path established by Berezin [8], and then followed by many others, including Simon [53]. See [32], [31] for a nice account. For any $n$ we can form $(U \otimes^n, \mathcal{H} \otimes^n)$, the $n^{th}$ inner tensor power of $(U, \mathcal{H})$. Let $(U^n, \mathcal{H}^n)$ denote the subrepresentation generated by $\xi^n = \xi \otimes^n$. It is well-known (e.g., as a consequence of proposition VII.2 of [51]) that $(U^n, \mathcal{H}^n)$ is irreducible, with $\xi^n$ as highest weight vector. The weight for $\xi^n$ is easily calculated to be $n \omega$, thus the corresponding orbit is $n \mathcal{O}$, which we identify with $\mathcal{O}$ by dividing by $n$. In particular, the stability subgroup of $n \omega$ is clearly still $H$. We let $B_n = B(\mathcal{H}^n)$. The action of $G$ on $B^n$ by conjugation by $U^n$ will be denoted again simply by $\alpha$. We denote the corresponding Lip-norm (for $\ell$) on $B_n$ by $L_n$, and we denote the Lip-norm on $A = C(G/H)$ still by $L_A$.

We are now prepared to state the main theorem of this paper.

**Theorem 3.2.** With notation as above, the quantum metric spaces $(B^n, L_n)$ converge to $(A, L_A)$ for quantum Gromov–Hausdorff distance as $n$ goes to $\infty$.

Our proof of this theorem will extend over the remaining sections of this paper. But we take the initial steps here. We let $P_n$ denote the rank-one projection for $\xi^n$. We denote the corresponding Berezin symbol map from $B^n$ to $A = C(G/H) = C(\mathcal{O})$ by $\sigma^n$. Let $d_n$ denote the dimension of $\mathcal{H}^n$, and let $h_n = h_{P^n}$ as in Notation 2.1. A crucial fact for our purposes is:

**Lemma 3.3.** The sequence of probability measures $h_n(x)dx$ converges in the weak-* topology to the $\delta$-function at $e$; that is, $\int_{G/H} f(x)h_n(x)dx$ converges to $f(e)$ for every $f \in C(G/H)$.

A proof of this fact, but in a slightly more complicated context, is given in proposition 4b of [14]. That proof is not elementary, because it depends on facts about the rate of growth of the dimensions of representations. We give here, for our context, a very elementary proof. (We remark that this lemma corresponds to condition 2 of definition 3 of [32].)

**Proof of Lemma 3.3.** For $x \in G/H$ we have

$$h_n(x) = d_n \tau(P^n\alpha_x(P^n)) = d_n |(U^n_x \xi^n, \xi^n)|^2$$

$$= d_n |(U^n_x, \xi^2n)|^2 = d_n \tau(\alpha(P))^{n}.$$ 

Set $g(x) = \tau(P\alpha_x(P))$, so that $h_n = d_ng^n$, where $g^n$ now means the $n^{th}$ power of $g$. Note that $g(e) = 1$. Suppose that $x \in G$ and $g(x) = 1$. 
Then the Hilbert–Schmidt inner product tells us that $\alpha_x(P)$ must agree with $P$. But this says that $x \in H$. In other words, for $g$ viewed on $G/H$, the only point $x$ at which $g(x) = 1$ is $x = e$. It is clear that $0 \leq g(x) \leq 1$ for any $x$. Notice that $d_n = \|g^n\|_1^{-1}$, so that for the moment the $d_n$’s can be viewed simply as normalizing constants to obtain probability measures.

Thus let $X$ be any compact space with distinguished point $e$ and full measure $dx$. Let $g \in C(X)$ with $0 \leq g \leq 1$, and with $g(x) = 1$ exactly if $x = e$. We need to know that the probability measures $g^n/\|g^n\|_1$ converge to the delta-function at $e$. The simple argument that this is true is given in the course of the proof of theorem 8.2 of [48].

As before, we assume that we have fixed a continuous length function, $\ell$, on $G$, and that $\rho_A$ is the corresponding $G$-invariant metric on $G/H$ as discussed in the previous section. For each $n$ let $\gamma_n$ be defined as in Theorem 2.3, that is,

$$\gamma_n = \int_{G/H} \rho_A(e, y)h_n(y)dy.$$ 

Since $\rho_A$ is continuous and $\rho_A(e, e) = 0$, and since $h_n$ converges to the $\delta$-function at $e$, it is now clear that the sequence $\{\gamma_n\}$ converges to 0. Putting all of the above together with Theorem 2.3, we obtain the following fact about the Berezin transform, which seems to be new and of independent interest.

**Theorem 3.4.** Let $G$ be a connected compact semisimple Lie group with length function $\ell$, and let $\omega$ be an element of an integral coadjoint orbit for $G$. Let $H$ be the stabilizer of $\omega$, and let $L_A$ be the Lip-norm on $A = C(G/H)$ corresponding to $\ell$. For each integer $n$ let $(U^n, H^n)$ be the irreducible unitary representation of $G$ with highest weight $n\omega$, and let $B^n = B(H^n)$ with its action of $G$ by conjugation by $U^n$. Let $\sigma^n$ be the Berezin covariant symbol map from $B^n$ to $A$ using the projection on the highest weight vector for $n\omega$, and let $\tilde{\sigma}^n$ be its adjoint. Then there is a sequence of numbers, $\{\gamma_n\}$, converging to 0, such that

$$\|f - \sigma^n(\tilde{\sigma}^n(f))\|_\infty \leq \gamma_n L_A(f)$$

for all $f \in A$ and all $n$.

By the comment right after Corollary 2.4 it follows that, in terms of the metrics $\rho_L$ on $S(A \oplus B^n)$ defined there using $\sigma^n$ for the bridges for $L$, we can find for any $\varepsilon > 0$ an integer $N$ such that $S(A)$ is in the $\varepsilon$-neighborhood of $S(B^n)$ for all $n \geq N$. The next sections are devoted to showing that for $N$ large enough it is also true that $S(B^n)$ is in the $\varepsilon$-neighborhood of $S(A)$. 
4. Covariant symbols

At the end of Section 2 we indicated the importance of studying the mappings $\tilde{\sigma}_n \circ \sigma^n$ on $B^n$. This requires a more careful study of $\sigma^n$ and of $\tilde{\sigma}_n$. This section is devoted to obtaining the information which we need about $\sigma^n$.

We use the notation of the previous section. Since $\mathcal{O} \subset g_0^*$, we can view $g$ as the space of linear complex-valued polynomials on $g_0^*$, which we can then restrict to $\mathcal{O}$. For $X \in g$ we define $\Phi_X$ on $\mathcal{O}$ by

$$\Phi_X(Ad^* x \omega) = i(Ad^* x \omega)(X) = i\omega(Ad_{x^{-1}}(X)) = \langle U_X U_x \xi, U_x \xi \rangle.$$ 

We include the factor of $i$ because when the definition of $\sigma$ which we gave in Section 1 is applied here, we see that $\sigma(U_X)(e) = \langle U_X U_x \xi, U_x \xi \rangle = \Phi_X(x)$. Here we are using the identification $x \mapsto Ad^* x \omega$ of $G/H$ with $\mathcal{O}$ to view $\Phi_X$ as a function on $G/H$. Note also that $\lambda_y \Phi_X = \Phi_{Ad_y(X)}$ for $x \in G$ and $X \in g$, so that $\Phi$ is equivariant.

We let $\mathcal{P}(\mathcal{O})$ denote the (unital) algebra of functions on $\mathcal{O}$, or $G/H$, generated by all the $\Phi_X$'s. The algebra $\mathcal{P}(\mathcal{O})$ is clearly carried into itself by the action $Ad^*$ of $G$ on $\mathcal{O}$. We denote this action on $\mathcal{P}(\mathcal{O})$ again by $Ad$, or, when we view $\mathcal{P}(\mathcal{O})$ on $G/H$, by $\lambda$.

Let $T$ denote the full tensor algebra over $g$, whose homogeneous parts are the various tensor powers $g^\otimes m$. Again $G$ acts on $T$ via $Ad$ (the diagonal action). By the universal property of $T$ we extend $\Phi$ to an algebra homomorphism from $T$ onto $\mathcal{P}(\mathcal{O})$ defined on elementary tensors by the product

$$\Phi(X_1 \otimes \cdots \otimes X_k) = \prod_j \Phi_{X_j}.$$ 

It is easily seen that $\Phi$ is still $Ad$-equivariant. (All of this is related, of course, to the universal enveloping algebra of $g$, but we don’t need that structure here.)

We let $(U^n, H^n, \xi^n)$ be as defined in the previous section. Then for $X \in g$ and for any $n$ we have

$$\sigma^n(U^n_X)(e) = \langle U^n_X \xi^n, \xi^n \rangle = \langle U_X \xi \otimes \xi \cdots \otimes \xi, \xi^n \rangle + \langle \xi \otimes U_X \xi \otimes \xi \cdots, \xi^n \rangle + \cdots = n\sigma(U_X)(e).$$ 

By the equivariance of $\sigma$ it follows that

$$\sigma^n(U^n_X)(x) = n\sigma(U_X)(x)$$
for every $x \in G$, that is, $\sigma^n(U^n_X) = n\sigma(U_X)$. It is then natural to define
a linear map, $\Phi^n$, from $g$ to $B^n$ by

$$\Phi^n(X) = n^{-1}U^n_X.$$ 

**Notation 4.1.** We extend $\Phi^n$ to an (Ad-equivariant) algebra homomorphism from $\mathcal{T}$ to $B^n$, defined on elementary tensors by

$$\Phi^n(X_1 \otimes \cdots \otimes X_k) = \Phi^n(X_1)\Phi^n(X_2)\cdots \Phi^n(X_k),$$

where the order of the terms is now important.

The next theorem has been known, at least in part, for many special cases. It is stated as theorem 2A in [14] without proof, but with attribution to Gilmore [17]. But Gilmore does not give complete details, and the details which he does give can be simplified substantially. Since this theorem is crucial for our purposes, we include a proof here. We will let $T^m$ denote the direct sum of the homogeneous subspaces $g \otimes k$ for $k \leq m$.

**Theorem 4.2.** For any $Z \in T$ the sequence $\{\sigma^n(\Phi^n_Z)\}$ converges uniformly on $G/H$ to $\Phi_Z$. For any integer $m$ this convergence is also uniform in $Z$ as $Z$ ranges over any bounded subset of $T^m$.

**Proof.** It clearly suffices to prove the first part of the above theorem for homogeneous $Z$’s, and, in fact, for elementary tensors. So assume that $Z = X_1 \otimes X_2 \otimes \cdots \otimes X_k$. Then

$$\sigma^n(\Phi^n_Z)(e) = n^{-k}\langle U^n_{X_1} U^n_{X_2} \cdots U^n_{X_k} \xi^n, \xi^n \rangle.$$ 

Notice that

$$U^n_{X_k} \xi^n = U_{X_k} \xi \otimes \cdots \otimes \xi + \xi \otimes U_{X_k} \xi \otimes \xi \otimes \xi + \cdots.$$ 

From this we see that when $U^n_{X_1} \cdots U^n_{X_k} \xi^n$ is fully expanded into elementary tensors we will have $n^k$ terms. We now assume that $n > k$. Then the number of terms all of whose components are only of the form either $\xi$ or $U_X \xi$ is $n(n-1) \cdots (n-k+1) = n^k - p(n)$, where $p$ is a polynomial in $n$ of degree $k - 1$. Each of these terms will give a contribution toward $\sigma^n(\Phi^n_Z)(e)$ of the form $\prod \langle U_X, \xi, \xi \rangle = \Phi_Z(e)$. There will be $p(n)$ other terms. Let

$$K_Z = \max\{1, \|U_X\| : j = 1, \ldots, k\}.$$ 

Each of the $n^k$ terms has norm no bigger than $K_Z^k$. Thus

$$\begin{align*}
|\Phi_Z(e) - \sigma^n(\Phi^n_Z)(e)| \\
= |\Phi_Z(e) - n^{-k}((n^k - p(n))\Phi_Z(e) + (p(n) \text{ remaining terms}))|
\leq n^{-k}2p(n)K_Z^k.
\end{align*}$$
Now $\text{Ad}_x(Z)$ is also an elementary tensor, and $K_{\text{Ad}_x(Z)} = K_Z$. Thus the above discussion applies to $\text{Ad}_x(Z)$ as well. Since $\Phi_Z(x) = (\lambda_x \Phi_Z)(e) = \Phi_{\text{Ad}_x(Z)}(e)$, we obtain
\[
\|\Phi_Z - \sigma_n(\Phi^n_Z)\|_\infty \leq n^{-k}2p(n)K^k_Z.
\]
This clearly goes to 0 as $n \to \infty$ since $p$ is of degree $k - 1$.

By considering a basis for $T^m$ (consisting of elementary tensors if desired), the statement about the convergence being uniform on bounded subsets of $T^m$ follows easily.

**Proposition 4.3.** $\Phi(T)$ is uniformly dense in $C(O)$.

*Proof.* Already $\Phi(\mathfrak{g})$ alone separates the points of $O$, for if $x \in G$ is such that $\Phi_X(\text{Ad}_x^*(\omega)) = \Phi_X(\omega)$ for all $X \in \mathfrak{g}$, then $\text{Ad}_x^*(\omega) = \omega$, so that $x \in H$. This suffices, by $\text{Ad}$-equivariance. Next, $\Phi_X$ is pure imaginary for $X \in \mathfrak{g}_0$, so that $\Phi(T)$ is closed under complex conjugation. By definition $\Phi(T)$ contains the constant functions. We can thus apply the Stone–Weierstrass theorem. □

We now prepare for some approximations which we will make in the next sections. We let $\hat{G}$ denote as usual the set of equivalence classes of irreducible unitary representations of $G$. Let $S$ be a finite subset of $\hat{G}$, which we fix for the rest of this section. We let $A_S$ denote the direct sum of the isotypic components of $A$ for the representations in $S$ and the action $\lambda$. We call it the $S$-isotypic subspace of $A$. In the same way we will speak of the $S$-isotypic subspaces of other representations of $G$.

**Lemma 4.4.** The $S$-isotypic subspace $A_S$ is finite dimensional, and there is an integer $q (= q_S)$ such that $\Phi(T^q) \supseteq A_S$. (We fix this integer $q$, in terms of $S$, for most of the remainder of this paper.)

*Proof.* From the Peter–Weyl theorem the various isotypic components of $C(G)$ for $\lambda$ are all finite dimensional. But $A = C(G/H)$ is a $\lambda$-invariant subspace of $C(G)$, and thus $A_S$ is finite dimensional.

Because $\Phi(T)$ is dense in $A$ by Proposition 4.3, when we compose $\Phi$ with the usual projection onto the sum of the isotypic components for $S$, this composition will carry $T$ onto $A_S$. From the equivariance of $\Phi$ it follows that $\Phi(T) \supseteq A_S$. By choosing preimages for a basis for $A_S$ we find $q$ such that $\Phi(T^q) \supseteq A_S$. □

We remark that, conversely, given any $Z \in T$, it is in some $T^m$, which is finite dimensional and $\text{Ad}$-invariant; and so there is a finite $S \subseteq \hat{G}$ such that $\Phi_Z \in A_S$.

For the rest of this paper we fix an $\text{Ad}$-invariant inner product on $\mathfrak{g}$. It gives a corresponding inner product on each $\mathfrak{g}^\otimes k$, and so on $T$,.
where we take the $g^\otimes k$'s to be orthogonal to each other for different $k$'s. Norms of elements of $\mathcal{T}$, and of operators from $\mathcal{T}$, will always be defined with respect to this inner product (and, for operators, a specified norm on the target space).

**Lemma 4.5.** There is a constant, $K$, such that when we view each $\Phi^n$ as an operator from $\mathcal{T}^q$ to $B^n$ (with the operator norm on $B^n$), we have

$$\|\Phi^n\| \leq K$$

for all $n$.

**Proof.** Since $g$ is finite dimensional, there is a constant, $J$, such that $\|UX\| \leq J\|X\|$ for every $X \in g$. Suppose now that $X \in g_0$. Then $UX$ is skew-adjoint, and we can find an orthonormal basis, $\{e_j\}$, for $H$ consisting of eigenvectors for $UX$, with corresponding eigenvalues $\{\alpha_j\}$. For each $n$ we have the corresponding orthonormal basis $\{e_j \otimes \cdots \otimes e_j\}$ for $H^\otimes n$. Each $e_j \otimes \cdots \otimes e_j$ is an eigenvector for $U_X^{\otimes n}$, with eigenvalue $\sum_{k=1}^n \alpha_{jk}$. (Note that here and below $U_X^{\otimes n}$ denotes the inner tensor representation of the Lie algebra $g$, not the tensor power of the operator $U_X$.) Since $\|UX\|$ and $\|U_X^{\otimes n}\|$ are given by the largest absolute value of the eigenvalues of $UX$ and $U_X^{\otimes n}$, we see that $\|U_X^{\otimes n}\| = n \|UX\|$. For $X$ in $g$ rather than $g_0$ it follows that $\|U_X^{\otimes n}\| \leq 2n \|UX\|$. Upon restricting $U_X^{\otimes n}$ to $H^n$ we then have $\|U_X^{\otimes n}\| \leq 2n \|UX\| \leq 2nJ \|X\|$. Thus $\|\Phi^n_X\| = \|n^{-1}U_X^{\otimes n}\| \leq 2J \|X\|$, independent of $n$.

Now pick a basis for $g$. For each $k$ it gives a basis for $g^\otimes k$ consisting of elementary tensors. But for any elementary tensor $Z = X_1 \otimes \cdots \otimes X_k$ we have

$$\|\Phi^n_Z\| = \|\Phi^n_{X_1} \cdots \Phi^n_{X_k}\| \leq (2J)^k \prod_{j=1}^k \|X_j\|.$$

The right-hand side is independent of $n$. Since $\mathcal{T}^q$ is finite dimensional the desired result follows easily. \qed

We let $\|\cdot\|_2$ denote the usual Hilbert-space norm on $L^2(G/H)$. Let $\mathcal{T}^q_S$ denote the $S$-isotypic subspace of $\mathcal{T}^q$. Since $\Phi$ is equivariant, it carries $\mathcal{T}^q_S$ onto $A_S$.

**Notation 4.6.** Let $\mathcal{F}$ ($= \mathcal{F}_S$) denote the orthogonal complement of the kernel of the restriction of $\Phi$ to $\mathcal{T}^q_S$.

Thus $\Phi$ is a bijection from $\mathcal{F}$ onto $A_S$, and $\mathcal{F}$ is carried into itself by Ad. Our notation will not distinguish between $\Phi$ and its restriction to $\mathcal{F}$. Much as above, we let $B^n_S$ denote the $S$-isotypic subspace of $B^n$. 
Proposition 4.7. There is an integer, $N$, such that for $n \geq N$ we have

$$\sigma^n(\Phi^n(\mathcal{F})) = A_S.$$ 

In particular, $\sigma^n$ will for $n \geq N$ be a bijection from $B^n_S$ onto $A_S$, and $\Phi^n$ will be a bijection from $\mathcal{F}$ onto $B^n_S$.

Proof. Put on $A_S$ the inner product from $L^2(G/H)$, and let $\{e_j\}$ be an orthonormal basis for $A_S$. For each $j$ let $Z_j$ be the unique element of $F$ such that $\Phi Z_j = e_j$. Let $d$ denote the dimension of $A_S$. All norms on a finite-dimensional vector space are equivalent, and so from Theorem 4.2 we see that we can find an integer $N$ such that

$$\|\sigma^n(\Phi^n(\mathcal{F}_j)) - \Phi Z_j\|_2 < 1/d$$

for all $n \geq N$ and all $j$. Then for each $n \geq N$ the $f_j = \sigma^n(\Phi^n(\mathcal{F}_j))$ span $A_S$. To see this, write any $g \in A_S$ as $g = \sum \alpha_j e_j$, and set $h = \sum \alpha_j f_j$. Then $\|g - h\|_2 \leq \sum |\alpha_j|/d \leq \|g\|_2$, so that $g$ can not be orthogonal to the span of the $f_j$'s unless it is 0.

Since $\sigma^n$ is injective, it must then be bijective from $B^n_S$ for $n \geq N$. Since the dimension of $F$ is the same as that of $A_S$, it follows that $\Phi^n$ is bijective from $\mathcal{F}$ onto $B^n_S$ for $n \geq N$. \hfill $\square$

We fix $N$ as above, and revert to the $C^*$-norms, restricted to $B^n_S$ and $A_S$. Momentarily define $\Omega^n$ on $A_S$, for $n \geq N$, by $\Omega^n = \sigma^n(\Phi^n) \circ \Phi^{-1}$, where $\Phi^{-1}$ takes $A_S$ to $\mathcal{F}$. Note that each $\Omega^n$ is invertible. Now Theorem 4.2 tells us that $\Omega^n(f)$ converges to $f$ for each $f \in A_S$. It follows that $\{\Omega^n\}$ converges in operator norm to the identity operator. Then so does $\{((\Omega^n)^{-1})\}$, and so this sequence is uniformly bounded in norm. Since $\Phi$ is independent of $n$, we obtain:

Lemma 4.8. There is a constant, $r$, such that, as operators from $A_S$ to $\mathcal{F}$,

$$\| (\sigma^n \circ \Phi^n)^{-1} \| \leq r$$

for all $n \geq N$.

Corollary 4.9. There is a constant, $K'$, such that for each $n \geq N$, when we view $(\sigma^n)^{-1}$ as an operator from $A_S$ to $B_S$, we have

$$\| (\sigma^n)^{-1} \| \leq K'.$$

Proof. From the above lemma and from Lemma 4.5 we have

$$\| (\sigma^n)^{-1} \| = \| \Phi^n \circ (\sigma^n \circ \Phi^n)^{-1} \| \leq K r.$$\hfill $\square$
For $r$ as above let $B_r$ denote the closed ball of radius $r$ in $F$. It follows from Lemma 4.8 that every $f$ in the unit ball of $A_S$ is, for every $n \geq N$, of the form $\sigma^n(\Phi^n_{Z_n})$ for some $Z_n \in B_r$. Let $T \in B^n_S$ with $\|T\| \leq 1$. Then $\|\sigma^n(T)\| \leq 1$, and so if $n \geq N$ we conclude from the above observation that $T = \Phi^n_Z$ for some $Z \in B_r$. We have thus obtained (for $N$ as in Proposition 4.7):

**Proposition 4.10.** There is a closed ball, $B$, in $F$ such that $\Phi^n(B) \supseteq (\text{unit ball of } B^n_S)$ for every $n \geq N$.

5. **Contravariant symbols**

This section is devoted to obtaining the information about $\hat{\sigma}^n$ which we need. The essence of what we need is contained in theorem 2B of [14]. This attractive theorem, concerning convergence of contravariant symbols as $n$ increases, seems to have few antecedents in the literature, and seems not to have been used at all since its appearance. But it is crucial to our present purposes. However, we need tighter control over the situation than is explicitly given in Duffield’s proof. Part of this section will basically consist of rewriting Duffield’s proof so as to give this tighter control (and with simpler arguments). We will also draw some important consequences.

We continue with the notation of the previous section. Because $\hat{\sigma}^n$ is equivariant, it will carry $A_S$ into $B^n_S$. We continue to let the integer $N$ be as in Proposition 4.7, so that $\sigma^n$ is a bijection from $B^n_S$ onto $A_S$ for $n \geq N$. Then $\hat{\sigma}^n$ will be a bijection from $A_S$ onto $B^n_S$ for $n \geq N$, and $(\hat{\sigma}^n)^{-1}$ will exist from $B^n_S$ to $A_S$. When we write $(\hat{\sigma}^n)^{-1}$ in the next pages, we will always take it as defined on $B^n_S$ for $n \geq N$. Note that if $T \in B^n_S$ and if we set $f = (\hat{\sigma}^n)^{-1}(T)$, then $\hat{\sigma}^n_f = T$. Thus $(\hat{\sigma}^n)^{-1}$ provides a canonical way of choosing a contravariant symbol for $T$. Recall that for $n \geq N$ each $\Phi^n$ is a bijection from $F$ onto $B^n_S$ according to Proposition 4.7.

**Notation 5.1.** For $n \geq N$ set $\Psi^n = (\hat{\sigma}^n)^{-1} \circ \Phi^n$ on $F$, so that $\Psi^n$ is a bijection from $F$ onto $A_S$.

Thus $\Psi^n_Z$ is a contravariant symbol for $\Phi^n_Z$ for each $Z \in F$. We need to show, in essence, that for each $Z \in F$ the $\Psi^n_Z$’s converge to $\Phi^n_Z$.

For $n \geq N$ define a linear functional, $\theta_n$, on $F$ by

$$\theta_n(Z) = \Psi^n_Z(e).$$
Because $\Psi^*_n \in C(G/H)$, we have $\theta_n(\text{Ad}_s(Z)) = \theta_n(Z)$ for every $s \in H$. Because $\Psi^n$ is equivariant, we have

$$\theta_n(\text{Ad}_x(Z)) = \Psi^n_{\text{Ad}_x(Z)}(c) = (\lambda_x(\Psi^n_y))(c) = \Psi^n_y(x^{-1})$$

for every $x \in G$. Since $\Psi^n_Z$ is a contravariant symbol for $\Phi^n_Z$, we obtain, as seen early in Section 2,

$$\Phi^n_Z = d_n \int_{G/H} \theta_n(\text{Ad}_{y^{-1}}(Z))\alpha_y(P^n)dy.$$  

From formula (2.2), together with the notation $h^n = h_{ps}$ used in Section 3, we then obtain

$$\sigma^n(\Phi^n_Z)(x) = \int_{G/H} \theta_n(\text{Ad}_{y^{-1}}(Z))h^n(y)dy.$$  

Now Theorem 4.2 tells us that the left-hand side converges to $\Phi_Z(x)$. In view of Lemma 3.3, this suggests that perhaps $\theta_n(\text{Ad}_{s^{-1}}(Z))$ also converges to $\Phi_Z(x)$. But to show that this is correct we need control of the size of the $\theta_n$’s. The key fact which we will use for this is Lemma 4.5, which tells us that $\|\Phi^n\| \leq K$ for all $n$.

We now use the inner product on $\mathcal{F} \subseteq T^g$. Then for each $n \geq N$ there will be a $Z_n \in \mathcal{F}$ which represents $\theta_n$, so that $\theta_n(Z) = \langle Z, Z_n \rangle$ for all $Z \in \mathcal{F}$. Then $\|\theta_n\| = \|Z_n\|$, where we use the norm from the inner product. Because $\theta_n(\text{Ad}_s(Z)) = \theta_n(Z)$ for every $s \in H$, we see quickly that $Z_n$ is $H$-invariant (for $\text{Ad}$).

Let $\mathcal{F}^H$ denote the subspace of $H$-invariant elements of $\mathcal{F}$, so that each $Z_n \in \mathcal{F}^H$. As $Z$ ranges over $\mathcal{F}^H$ the functions $y \mapsto \text{Ad}_y(Z)$ form a finite-dimensional vector space of vector-valued functions on $G/H$. By considering a basis for this vector space, we see that any bounded collection of these functions will be equicontinuous. From Lemma 3.3 we find that for every $\varepsilon > 0$ there is an integer $M_\varepsilon \geq N$ such that

$$\left\| Z - \int_{G/H} \text{Ad}_y(Z)h^n(y)dy \right\| \leq \varepsilon \|Z\|$$

for all $Z \in \mathcal{F}^H$ and all $n \geq M_\varepsilon$. Then for $n \geq M_\varepsilon$ we have

$$\|Z_n\|^2 = \langle Z_n, Z_n - \int \text{Ad}_y(Z_n)h^n(y)dy \rangle + \langle Z_n, \int \text{Ad}_y(Z_n)h^n(y)dy \rangle$$

$$\leq \varepsilon \|Z_n\|^2 + \left| \int \theta_n(\text{Ad}_y(Z_n))h^n(y)dy \right|.$$  

Upon applying Lemma 4.5, with $K$ as given there, we then obtain

$$(1 - \varepsilon)\|Z_n\|^2 \leq |\sigma^n(\Phi^n_{Z_n})(c)| \leq \|\Phi^n\| \|Z_n\| \leq K \|Z_n\|.$$  

Since $\|Z_n\| = \|\theta_n\|$, this gives the proof of:
Lemma 5.2. For every \( r > 1 \) there is an integer \( M_r \geq N \) such that for \( n \geq M_r \) we have \( \|\theta_n\| \leq rK \).

From this we see that for \( n \geq N \) and any \( Z \in F \) we have
\[
|\Psi^n_Z(x)| = |\theta_n(\text{Ad}_{x^{-1}}(Z))| \leq rK \|Z\|
\]
for all \( x \), from which we obtain:

Proposition 5.3. For every \( r > 1 \) there is an integer \( M_r \geq N \) such that if \( n \geq M_r \) then \( \|\Psi^n\| \leq rK \) (where \( \Psi^n \) is defined on \( F \), and both \( F \) and \( K \) depend on \( S \)).

The following theorem is our version of theorem 2B of \([14]\).

Theorem 5.4. For any \( \varepsilon > 0 \) there is an integer \( N_\varepsilon \geq N \) such that if \( n \geq N_\varepsilon \) then
\[
\|\Psi^n_Z - \sigma^n(\Phi^n_Z)\|_\infty \leq \varepsilon \|Z\|
\]
for all \( Z \in F \) (\( = F_S \)). In particular, \( \Psi^n_Z \) converges to \( \Phi_Z \).

Proof. For any \( Z \in F \) we have, much as above,
\[
\Psi^n_Z(e) - \sigma^n(\Phi^n_Z)(e) = \int_{G/H} \theta_n(Z - \text{Ad}_{y^{-1}}(Z))h^n(y)dy.
\]
As \( Z \) ranges over \( F \) the functions \( y \mapsto Z - \text{Ad}_{y^{-1}}(Z) \) form a finite-dimensional vector space of vector-valued functions on \( G \) (not \( G/H \)), so, as before, any bounded subset will be equicontinuous. We know from Lemma 5.2 that \( \{\|\theta_n\| : n \geq N\} \) is bounded. Thus the set of functions \( y \mapsto \theta_n(Z - \text{Ad}_{y^{-1}}(Z)) \), now on \( G/H \), is equicontinuous for \( \|Z\| \leq 1 \) and for all \( n \geq N \). Let \( \varepsilon > 0 \) be given. It follows from Lemma 3.3 that we can find an integer, \( N_\varepsilon \), such that
\[
|\Psi^n_Z(e) - \sigma^n(\Phi^n_Z)(e)| \leq \varepsilon \|Z\|
\]
for all \( n \geq N_\varepsilon \) and all \( Z \in F \). But from the equivariance of \( \Psi^n \) and \( \sigma^n \circ \Phi^n \), and the fact that \( \text{Ad} \) is unitary on \( F \), it follows that
\[
\|\Phi^n_Z - \sigma^n(\Phi^n_Z)\|_\infty \leq \varepsilon \|Z\|
\]
for \( n \geq N_\varepsilon \) and \( Z \in F \). \( \Box \)

6. Conclusion of the proof of Theorem 3.2

We continue with the notation of the previous sections. Our first objective is to prove the following fact, which seems to be new and of independent interest.
Theorem 6.1. Let the general hypotheses be the same as those for Theorem 3.4. For each \( n \geq 1 \) let \( \gamma_n \) be the smallest constant such that
\[
\| T - \check{\sigma}_T^n(\sigma_T^n) \| \leq \gamma_n L_n(T)
\]
for all \( T \in B^n \). Then the sequence \( \{\gamma_n\} \) converges to 0.

We remark that our wording here is somewhat different from that of Theorem 3.4 because in the present finite-dimensional situation it is clear that the constants \( \gamma_n \) exist, whereas in Theorem 3.4 it is not immediately clear that (finite) constants \( \gamma_n \) exist.

Proof of Theorem 6.1. Let \( \varepsilon > 0 \) be given. By theorem 8.2 and lemma 8.3 of [48] we can find a finite subset \( S \subseteq \hat{G} \) and a positive linear combination, \( \varphi \), of the characters of the elements of \( S \), with the following properties. The set \( S \) contains the trivial representation 1 and is closed under taking contragradient representations; and for any ergodic action \( \alpha \) of \( G \) on a unital \( C^* \)-algebra \( C \) the integrated operator \( \alpha \varphi \) is a completely positive unital equivariant map of \( C \) onto its \( S \)-isotypic component such that
\[
\| c - \alpha \varphi (c) \| \leq (\varepsilon/3)L(c)
\]
for all \( c \in C \). Then for every \( T \in B^n \) we have
\[
\| T - \check{\sigma}_T^n(\sigma_T^n) \| \leq (\varepsilon/3)L_n(T) + \| \alpha \varphi (T) - \check{\sigma}_T^n(\sigma_{\alpha \varphi (T)}^n) \| + (\varepsilon/3)L_n(T).
\]
Thus we see that it suffices to prove that, for \( S \) fixed as above, there is an integer \( N \) such that for all \( T \in B^n_S \) we have
\[
\| T - \check{\sigma}_T^n(\sigma_T^n) \| \leq (\varepsilon/3)L_n(T)
\]
for all \( n \geq N \).

Now by lemma 2.4 of [46] all of the algebras \( B^n \) will have radius no larger than \( r = \int_G \ell(x)dx \), in the sense that \( \| T \|_\sim \leq rL(T) \) for all \( T \), where \( \| \cdot \|_\sim \) denotes the quotient of \( \| \cdot \| \) on \( B^n/CI \). It then suffices to prove:

Lemma 6.2. For \( S \) fixed as above, there is an integer \( N \) such that
\[
\| T - \check{\sigma}_T^n(\sigma_T^n) \| \leq (\varepsilon/3r)\| T \|
\]
for all \( n \geq N \) and all \( T \in B^n_S \).

To see that this is sufficient, note that if it holds, then it holds equally well for \( T + tI \) where \( t \in \mathbb{C} \) is chosen so that \( \| T \|_\sim = \| T + tI \| \), and that the left side of the inequality will be unchanged.

Proof of Lemma 6.2. We use Lemma 4.4 and Notation 4.6 to choose \( q \) and \( F \subset T^q \) for our fixed \( S \). We then apply Proposition 4.10 to choose an integer \( N \) such that there is a closed ball \( B \) in \( F \) such that \( \Phi^n(B) \supseteq (\text{unit ball of } B^n) \) for all \( n \geq N \). Let \( R \) denote the radius of
$\mathcal{B}$, and let $\varepsilon' = \varepsilon/3rR$. We apply Theorem 5.4 to choose a yet larger $N$ such that $\Psi^n_Z$ is defined for every $n \geq N$, and for all $Z \in \mathcal{F}$ we have

$$\|\Psi^n_Z - \sigma^n(\Phi^n_Z)\| \leq \varepsilon'\|Z\|.$$  

Suppose now that $n \geq N$ and that $T \in B^n_S$ with $\|T\| \leq 1$. By our choice of $N$ there is a $Z \in \mathcal{B}$ such that $\Phi^n_Z = T$, and furthermore $T = \tilde{\sigma}^n(\Psi^n_Z)$. Then

$$\|T - \tilde{\sigma}^n(\sigma^n_T)\| = \|\tilde{\sigma}^n(\Psi^n_Z) - \tilde{\sigma}^n(\sigma^n_T)\| \leq \|\Psi^n_Z - \sigma^n(\Phi^n_Z)\| \leq \varepsilon'\|Z\| \leq \varepsilon/3r.$$  

Since this is true for all $T$ with $\|T\| \leq 1$, we obtain the desired result.

We have thus concluded the proof of Theorem 6.1.

We now complete the proof of our main theorem, Theorem 3.2. Let $\varepsilon > 0$ be given. By Theorem 3.4 we can find an integer $N$ such that for $n \geq N$ we have

$$\|f - \sigma^n(\tilde{\sigma}^n_f)\| \leq (\varepsilon/2)L_A(f)$$

for all $f \in \mathfrak{A}$. By Proposition 1.3 and Corollary 2.4, it follows that $S(\mathfrak{A})$ is in the $(\varepsilon/2)$-neighborhood of $S(B^n)$ for $\rho_L$, for each $n \geq N$.

Now according to Theorem 6.1 we can find $N$ still larger such that for $n \geq N$ we have

$$\|T - \tilde{\sigma}^n(\sigma^n_T)\| \leq (\varepsilon/2)L_n(T)$$

for $T \in B^n$. Then by the calculation done in the next-to-last paragraph of Section 2 (where the $\gamma$ there is the $\varepsilon/2$ here), it follows that each $S(B^n)$ is in the $\varepsilon$-neighborhood of $S(\mathfrak{A})$ for $\rho_L$. Accordingly, for $n \geq N$

$$\text{dist}_q((B^n, L_n), (\mathfrak{A}, L_A)) \leq \varepsilon,$$

as desired.

We remark that when one examines the steps in the proof of our main theorem, one sees that for any given coadjoint orbit and any given length function $\ell$ one can with careful bookkeeping obtain, for any $\varepsilon > 0$, a fairly explicit choice of an $N$ for which

$$\text{dist}_q((B^n, L_n), (\mathfrak{A}, L_A)) \leq \varepsilon$$

for $n \geq N$.  

\[\square\]
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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720-3840

E-mail address: rieffel@math.berkeley.edu