Vertex Disjoint Paths in Upward Planar Graphs

Saeed Akhoondian Amiri$^1$, Ali Golshani$^2$, Stephan Kreutzer$^1$, and Sebastian Siebertz$^1$

$^1$ Technical University Berlin
saeed.akhoondianamiri,stephan.kreutzer,sebastian.siebertz@tu-berlin.de
$^2$ University of Tehran, ali.golshani@ut.ac.ir

Abstract. The $k$-vertex disjoint paths problem is one of the most studied problems in algorithmic graph theory. In 1994, Schrijver proved that the problem can be solved in polynomial time for every fixed $k$ when restricted to the class of planar digraphs and it was a long standing open question whether it is fixed-parameter tractable (with respect to parameter $k$) on this restricted class. Only recently, [13], achieved a major breakthrough and answered the question positively. Despite the importance of this result (and the brilliance of their proof), it is of rather theoretical importance. Their proof technique is both technically extremely involved and also has at least double exponential parameter dependence. Thus, it seems unrealistic that the algorithm could actually be implemented. In this paper, therefore, we study a smaller class of planar digraphs, the class of upward planar digraphs, a well studied class of planar graphs which can be drawn in a plane such that all edges are drawn upwards. We show that on the class of upward planar digraphs the problem (i) remains NP-complete and (ii) the problem is fixed-parameter tractable. While membership in FPT follows immediately from [13]'s general result, our algorithm has only single exponential parameter dependency compared to the double exponential parameter dependence for general planar digraphs. Furthermore, our algorithm can easily be implemented, in contrast to the algorithm in [13].

1 Introduction

Computing vertex or edge disjoint paths in a graph connecting given sources to sinks is one of the fundamental problems in algorithmic graph theory with applications in VLSI-design, network reliability, routing and many other areas. There are many variations of this problem which differ significantly in their computational complexity. If we are simply given a graph (directed or undirected) and two sets of vertices $S, T$ of equal cardinality, and the problem is to compute $|S|$ pairwise vertex or edge disjoint paths connecting sources in $S$ to targets in $T$, then this problem can be solved efficiently by standard network flow techniques.

A variation of this is the well-known $k$-vertex disjoint paths problem, where the sources and targets are given as lists $(s_1, \ldots, s_k)$ and $(t_1, \ldots, t_k)$ and the problem is to find $k$ vertex disjoint paths connecting each source $s_i$ to its corresponding target $t_i$. The $k$-disjoint paths problem is NP-complete in general and remains NP-complete even on planar undirected graphs (see [9]).
On undirected graphs, it can be solved in polynomial time for any fixed number \( k \) of source/target pairs. This was first proved for the 2-disjoint paths problems, for instance in [18,19,21,14], before Robertson and Seymour proved in [16] that the problem can be solved in polynomial-time for every fixed \( k \). In fact, they proved more, namely that the problem is \textit{fixed-parameter tractable} with parameter \( k \), that is, solvable in time \( f(k) \cdot |G|^c \), where \( f \) is a computable function, \( G \) is the input graph, \( k \) the number of source/target pairs and \( c \) a fixed constant (not depending on \( k \)). See e.g. [6] for an introduction to fixed-parameter tractability.

For directed graphs the situation is quite different (see [1] for a survey). Fortune et al. [7] proved that the problem is already NP-complete for \( k = 2 \) and hence the problem is not fixed-parameter tractable on directed graphs. It is not even fixed-parameter tractable on acyclic digraphs, as shown by Slivkins [20]. However, on acyclic digraphs the problem can be solved in polynomial time for any fixed \( k \) [7].

In [11], Johnson et al. introduced the concept of \textit{directed tree-width} as a directed analogue of undirected tree-width for directed graphs. They showed that on classes of digraphs of bounded directed tree-width the \( k \)-disjoint paths problem can be solved in polynomial time for any fixed \( k \). As the class of acyclic digraphs has directed tree-width 1, Slivkins’ result [20] implies that the problem is not fixed-parameter tractable on such classes.

Given the computational intractability of the directed disjoint paths problem on many classes of digraphs, determining classes of digraphs on which the problem does become at least fixed-parameter tractable is an interesting and important problem. Using colour coding techniques, the problem can be shown to become fixed-parameter tractable if the length of the disjoint paths is bounded. This has, for instance, been used to show fixed-parameter tractability of the problem on classes of bounded \textit{DAG-depth} [8]. In 1994, Schrijver [17] proved that the directed \( k \)-disjoint paths problem can be solved in polynomial time for any fixed \( k \) on planar digraphs, using a group theoretical approach and it was a long standing open question whether it is fixed-parameter tractable on this restricted class. Only recently, Cygan et al. achieved a major breakthrough and answered the question positively. Despite the importance of this result (and the brilliance of their proof), it is of rather theoretical importance. Their proof technique is based on irrelevant vertices in directed grids and both technically extremely involved and also has at least double exponential parameter dependence. Thus, it seems unrealistic that the algorithm could actually be implemented.

In this paper, therefore, we study a smaller class of planar digraphs, the class of \textit{upward planar digraphs}. These are graphs that have a plane embedding such that every directed edge points “upward”, i.e. each directed edge is represented by a curve that is monotone increasing in the \( y \) direction. Upward planar digraphs are very well studied in a variety of settings, in particular in graph drawing applications (see e.g. [2]). In contrast to the problem of finding a planar embedding for a planar graph, which is solvable in linear time, the problem of finding an upward planar embedding is NP-complete in general [10]. Much work
has gone into finding even more restricted classes inside the upward planar class that allow to find such embeddings in polynomial time [4,3,15].

By definition, upward planar graphs are planar graphs. Hence, by the above results, the \( k \)-vertex disjoint paths problem can be solved in polynomial time on upward planar graphs for any fixed \( k \). As a first main result in this paper we show that the problem remains NP-complete on upward planar graphs, i.e., that this cannot be improved to a general polynomial-time algorithm. Our construction even shows that the problem is NP-complete on directed grid graphs.

Our second main result is that the problem is fixed-parameter tractable with respect to parameter \( k \) on the class of upward planar digraphs if we are given an upward planar graph together with an upward planar embedding. We present a linear time algorithm that has single exponential parameter dependency.

2 Preliminaries

By \( \mathbb{N} \) we denote the set of non-negative integers and for \( n \in \mathbb{N} \), we write \( [n] \) for the set \( \{1, \ldots , n\} \). We assume familiarity with the basic concepts from (directed) graph theory, planar graphs and graph drawings and refer the reader to [1,2,5] for more details. For background on parameterized complexity theory we refer the reader to [6].

An upward planar graph is a graph that has a plane embedding such that every directed edge points “upward”, i.e. each directed edge is represented by a curve that is monotone increasing in the \( y \) direction.

The \( k \)-vertex disjoint paths problem on upward planar graphs is the following problem.

**Vertex Disjoint Paths on Upward Planar Graphs (UpPlan-VDPP)**

**Input:** An upward planar graph \( G \) together with an upward planar embedding, \((s_1,t_1),\ldots,(s_k,t_k)\).

**Problem:** Decide whether there are \( k \) pairwise internally vertex disjoint paths \( P_1,\ldots, P_k \) linking \( s_i \) to \( t_i \), for all \( i \).

3 NP-Completeness of UpPlan-VDPP

This section is dedicated to the proof of one of our main theorems:

**Theorem 3.1.** UpPlan-VDPP is NP-complete.

Before we formally prove the theorem, we give a brief and informal overview of the proof structure. The proof of NP-completeness is by a reduction from SAT, the satisfiability problem for propositional logic, which is well-known to be NP-complete [9]. On a high level, our proof method is inspired by the NP-completeness proof in [12] but the fact that we are working in a restricted class of planar digraphs requires a number of changes and additional gadgets.

Let \( \mathcal{V} = \{V_1, \ldots , V_n\} \) be a set of variables and \( \mathcal{C} = \{C_1, \ldots , C_m\} \) be a set of clauses over the variables from \( \mathcal{V} \). For \( 1 \leq i \leq m \) let \( C_i = \{L_{i,1}, L_{i,2}, \ldots , L_{i,n_i}\} \)
where each \( L_{i,t} \) is a literal, i.e., a variable or the negation thereof. We will construct an upward planar graph \( G_C = (V, E) \) together with a set of pairs of vertices in \( G_C \) such that \( G_C \) contains a set of pairwise vertex disjoint directed paths connecting each source to its corresponding target if, and only if, \( C \) is satisfiable. The graph \( G_1 \) is roughly sketched in Fig. 1.

![Fig. 1. Structure of the graph \( G_C \)](image)

We will have the source/target pairs \((V_i, V'_i) \in V^2\) for \( i \in [n] \) and \((C_j, C'_j) \in V^2\) for \( j \in [m] \), as well as some other source/target pairs inside the gadgets \( G_{i,j,t} \) that guarantee further properties. As the picture suggests, there will be two possible paths from \( V_i \) to \( V'_i \), an upper path and a lower path and our construction will ensure that these paths cannot interleave. Any interpretation of the variable \( V_i \) will thus correspond to the choice of a unique path from \( V_i \) to \( V'_i \). Furthermore, we will ensure that there is a path from \( C_j \) to \( C'_j \) if and only if some literal is interpreted such that \( C_j \) is satisfied under this interpretation.

We need some additional gadgets which we describe first to simplify the presentation of the main proof. All missing proofs can be found in the appendix.

**Routing Gadget:** The role of a routing gadget is to act as a planar routing device. It has two incoming connections, the edges \( e_t \) from the top and \( e_l \) from the left, and two outgoing connections, the edges \( e_b \) to the bottom and \( e_r \) to the right. The gadget is constructed in a way that in any solution to the disjoint paths problem it allows for only two ways of routing a path through the gadget, either using \( e_t \) and \( e_b \) or \( e_l \) and \( e_r \).

Formally, the gadget is defined as the graph displayed in Fig. 2 with source/target pairs \((i, j) \) for \( i \in [4] \). Immediately from the construction of the gadget we get the following lemma which captures the properties of routing gadgets needed in the sequel.

\(^1\) To improve readability, we draw all graphs in this paper from left to right, instead of upwards.
Lemma 3.2. Let $R$ be a routing gadget.
1. There is a solution of the disjoint paths problem in $R$.
2. Let $P_1, \ldots, P_4$ be any solution to the disjoint paths problem in $R$, where $P_i$ links vertex $i$ to $i$. Let $H := R \setminus \bigcup_{i=1}^{4} P_i$. Then $H$ does not contain a path which goes through $e_1$ to $e_r$ or through $e_1$ to $e_b$ but there are paths $P, P'$ in $H$ such that $P$ goes through $e_t$ to $e_b$ and $P'$ goes through $e_l$ to $e_r$.
3. There are no two disjoint paths $P, P'$ in $G$ such that $P$ contains $e_l$ and $e_r$ and $P'$ contains $e_t$ and $e_b$.

Crossing Gadget: A crossing gadget has two incoming connections to its left via the vertices $H^{in}$ and $L^{in}$ and two outgoing connections to its right via the vertices $H^{out}$ and $L^{out}$. Furthermore, it has one incoming connection at the top via the vertex $T$ and outgoing connection at the bottom via the vertex $B$. Intuitively, we want that in any solution to the disjoint paths problem, there is exactly one path $P$ going from left to right and exactly one path $P'$ going from top to bottom. Furthermore, if $P$ enters the gadget via $H^{in}$ then it should leave it via $H^{out}$ and if it enters the gadget via $L^{in}$ then it should leave it via $L^{out}$. Of course, in a planar graph there cannot be such disjoint paths $P, P'$ as they must cross at some point. We will have to split one of the paths, say $P$, by removing the outward source/sink pair and introducing two new source/sink pairs, one to the left of $P'$ and one to its right.

Formally, the gadget is defined as the graph displayed in Fig. 3. The following lemma follows easily from Lemma 3.2 (See Appendix A for proof).

Lemma 3.3. Let $G$ be a crossing gadget.
1. There are uniquely determined vertex disjoint paths $P_1$ from $H^{in}$ to $W$, $P_2$ from $T$ to $B$ and $P_3$ from $X$ to $Y$. Let $H := G \setminus \bigcup_{i=1}^{3} P_i$. Then $H$ contains a path from $Z$ to $H^{out}$ but it does not contain a path from $Z$ to $L^{out}$. 

Fig. 2. The routing gadget. In the following, when a routing gadget appears as a subgadget in a figure, it will be represented by a black box as shown on the left.
2. There are uniquely determined vertex disjoint paths $Q_1$ from $L^{in}$ to $W$, $Q_2$ from $T$ to $B$ and $Q_3$ from $X$ to $Y$. Let $H := G \setminus \bigcup_{i=1}^3 Q_i$. Then $H$ contains a path from $Z$ to $L^{out}$ but it does not contain a path from $Z$ to $H^{out}$.

The next lemma shows that we can connect crossing gadgets in rows in a useful way. It follows easily by induction from Lemma 3.3.

Let $G_1, \ldots, G_s$ be a sequence of crossing gadgets drawn from left to right in that order. We address the inner vertices of the gadgets by their names in the gadget equipped with corresponding subscripts, e.g., we write $H^{in}_i$ for the vertex $H^{in}$ of gadget $G_i$. For each $j \in [s-1]$, we add the edges $(H^{in}_{j}, H^{in}_{j+1})$ and $(L^{out}_{j}, L^{out}_{j+1})$ and call the resulting graph a row of crossing gadgets. We equip this graph with the source/target pairs $(X_j, Y_j), (Z_j, W_{j+1})$ for $j \in [s-1]$ to obtain an associated vertex disjoint paths problem $P_r$ (the subscript $r$ stands for row). Denote by $P^+_r$ the problem $P_r$ with additional source/target pair $(H^{in}_1, W_1)$ and by $P^-_r$ the problem $P_r$ with additional source/target pair $(L^{in}_1, W_1)$.

**Lemma 3.4.** Let $G$ be a row of crossing gadgets. Then both associated vertex disjoint paths problems $P^+_r$, $P^-_r$ have unique solutions. Each path in the solution of $P^+_r$ from $Z_i$ to $W_{i+1}$ passes through $H^{in}_{i+1}$ and each path in the solution of $P^-_r$ from $Z_i$ to $W_{i+1}$ passes through $L^{in}_{i+1}$.

**Proof** is in Appendix A. The next lemma shows that we can force a relation between rows and columns of crossing gadgets.

Let $G_1, \ldots, G_t$ be a sequence of crossing gadgets drawn from top to bottom in that order. For each $i \in [t-1]$, we add the edge $(B_i, T_{i+1})$ and call the resulting
graph a column of crossing gadgets. We equip this graph with the source/target pairs \((X_i, Y_i)\) for \(i \in [l]\) and with the pair \((T_1, B_1)\) to obtain an associated vertex disjoint paths problem \(\mathcal{P}\).

**Lemma 3.5.** Let \(G\) be a column of crossing gadgets. Let \(P_1, \ldots, P_l\) be a sequence of vertex disjoint paths such that \(P_i\) connects either \(H_i^{1n}\) or \(L_i^{1n}\) to \(W_i\). Let \(H := G \setminus \bigcup_{i=1}^l P_i\).

1. The vertex disjoint paths problem \(\mathcal{P}\) on \(H\) has a solution.
2. There is a unique path \(Q\) connecting \(T_1\) to \(B_1\) which uses edge \(e^+\) in \(G_i\) if and only if \(P_i\) starts at \(H_i^{1n}\) and the edge \(e^-\) in \(G_i\) if and only if \(P_i\) starts at \(L_i^{1n}\).

Note that such paths \(P_i\) as stated in the lemma exist and they are uniquely determined by Lemma 3.3.

We are now ready to construct a vertex disjoint paths instance for any SAT instance \(C\).

**Definition 3.6.** Let \(C\) be a SAT instance over the variables \(\mathcal{V} = \{V_1, \ldots, V_n\}\) and let \(\{C_1, \ldots, C_n\}\) be its set of clauses. For \(j \in [m]\) let \(C_j = \{L_{j,1}, L_{j,2}, \ldots, L_{j,n_j}\}\), where each \(L_{j,s}\) is a literal, i.e., a variable or the negation thereof.

1. The graph \(G_C\) is defined as follows.
   - For each variable \(V \in \mathcal{V}\) we introduce two vertices \(V\) and \(V'\).
   - For each clause \(C \in \mathcal{C}\) we introduce two vertices \(C\) and \(C'\).
   - For each variable \(V_i\) and each literal \(L_{j,t}\) in clause \(j\) we introduce a crossing gadget \(G_{i,j,t}\).
   - For \(i \in [n]\) we add the edges \((V_i, H_i^{1n}_{i,1,1}), (V_i, L_i^{1n}_{i,1,1}), (H_i^{out}_{i,m,n,m}, V_i')\) and \((L_i^{out}_{i,m,n,m}, V_i')\).
   - For \(j \in [m], t \in [n_j]\) we add the edges \((C_j, T_{1,j,t})\) and \((B_{n,j,t}, C_j')\).
   - Finally, we delete the edge \(e^+\) for all \(i \in [n], j \in [m], t \in [n_j]\) in \(G_{i,j,t}\) if \(L_{j,t}\) is a variable the edge \(e^-\) if it is a negated variable.

We draw the graph \(G_C\) as shown in Fig. 1.

2. We define the following vertex disjoint paths problem \(\mathcal{P}_C\) on \(G_C\). We add all source/target pairs that are defined inside the routing gadgets. Furthermore:
   - For \(i \in [n], j \in [m], t \in [n_j - 1]\), we add the pairs
     - \((V_i, W_i^{1,1,1})\),
     - \((Z_{i,m,n,m}, V_i')\),
     - \((X_{i,j,t}, Y_{i,j,t})\) and
     - \((Z_{i,j,t}, W_{i,j,t+1})\).
   - For \(i \in [n], j \in [m - 1]\), we add the pairs \((Z_{i,j,n_j}, W_{i,j+1,1})\).
   - For \(j \in [m]\), we add the pairs \((C_j, C_j')\).

The proof of the following theorem is based on the fact that in our construction, edge \(e^+\) is present in gadget \(G_{i,j,t}\), if and only if \(C_j\) does not contain variable \(V_i\) negatively and \(e^-\) is present in gadget \(G_{i,j,t}\), if and only if \(C_j\) does not contain variable \(V_i\) positively (especially, both edges are present if the clause does not contain the variable at all). In particular, every column contains exactly one gadget where one edge is missing. Now it is easy to conclude with Lemma 3.3 and Lemma 3.5. We defer the formal proof to the appendix.
Theorem 3.7. Let $C$ be a SAT-instance and let $P_C$ be the corresponding vertex disjoint paths instance on $G_C$ as defined in Definition 3.6. Then $C$ is satisfiable if and only if $P_C$ has a solution.

It is easily seen that the presented reduction can be computed in polynomial time and this finishes the proof of Theorem 3.1.

If we replace the vertices $C_i$ and $C'_i$ with directed paths, then it is easy to convert the graph $G_C$ to a directed grid graph, i.e., a subgraph of the infinite grid. This implies that the problem is NP-complete even on upward planar graphs of maximum degree 4.

4 A Linear Time Algorithm for Fixed $k$

In this section we prove that the $k$-disjoint paths problem for upward planar digraphs can be solved in linear time for any fixed value of $k$. In other words, the problem is fixed-parameter tractable by a linear time parameterized algorithm.

Theorem 4.1. The problem UpPlan-VDPP can be solved in time $O(k! \cdot k \cdot n)$, where $n := |V(G)|$.

For the rest of the section we fix a planar upward graph $G$ together with an upward planar embedding and $k$ pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices. We will not distinguish notationally between $G$ and its upward planar embedding. Whenever we speak about a vertex $v$ on a path $P$ we mean a vertex $v \in V(G)$ which is contained in $P$. If we speak about a point on the path we mean a point $(x, y) \in \mathbb{R}^2$ which is contained in the drawing of $P$ with respect to the upward planar drawing of $G$. The algorithm is based on the concept of a path in $G$ being to the right of another path which we define next.

Definition 4.2. Let $P$ be a path in an upward planar drawing of $G$. Let $(x, y)$ and $(x', y')$ be the two endpoints of $P$ such that $y \leq y'$, i.e. $P$ starts at $(x, y)$ and ends at $(x', y')$. We define

\[
\text{right}(P) := \{(u, v) \in \mathbb{R}^2 : y \leq v \leq y' \text{ and } u' < u \text{ for all } u' \text{ such that } (u', v) \in P\}
\]

\[
\text{left}(P) := \{(u, v) \in \mathbb{R}^2 : y \leq v \leq y' \text{ and } u' > u \text{ for all } u' \text{ such that } (u', v) \in P\}.
\]

The next two lemmas follow immediately from the definition of upward planar drawings.

Lemma 4.3. Let $P$ and $Q$ be vertex disjoint paths in an upward planar drawing of $G$. Then either right($P$) $\cap$ $Q = \emptyset$ or left($P$) $\cap$ $Q = \emptyset$.

Lemma 4.4. Let $P$ be a directed path in an upward planar drawing of a di-graph $G$. For $i = 1, 2, 3$ let $p_i := (x_i, y_i)$ be distinct points in $P$ such that $y_1 < y_2 < y_3$. Then $p_1, p_2, p_3$ occur in this order on $P$.

Definition 4.5. Let $P$ and $Q$ be two vertex disjoint paths in $G$. 
1. A point \( p = (x, y) \in \mathbb{R}^2 \setminus P \) is to the right of \( P \) if \( p \in \text{right}(P) \). Analogously, we say that \( (x, y) \in \mathbb{R}^2 \setminus P \) is to the left of \( P \) if \( p \in \text{left}(P) \).

2. The path \( P \) is to the right of \( Q \), denoted by \( Q \prec P \) if there exists a point \( p \in P \) which to the right of some point \( q \in Q \). We write \( \prec^* \) for the transitive closure of \( \prec \).

3. If \( \mathcal{P} \) is a set of pairwise disjoint paths in \( G \), we write \( \prec_{\mathcal{P}} \) and \( \prec^*_{\mathcal{P}} \) for the restriction of \( \prec \) and \( \prec^* \), resp., to the paths in \( \mathcal{P} \).

We show next that for every set \( \mathcal{P} \) of pairwise vertex disjoint paths in \( G \) the relation \( \prec^* \) is a partial order on \( \mathcal{P} \). Towards this aim, we first show that \( \prec \) is irreflexive and anti-symmetric on \( \mathcal{P} \).

**Lemma 4.6.** Let \( \mathcal{P} \) be a set of pairwise disjoint paths in \( G \).

1. The relation \( \prec_{\mathcal{P}} \) is irreflexive.
2. The relation \( \prec_{\mathcal{P}} \) is anti-symmetric, i.e. if \( P_1 \prec_{\mathcal{P}} P_2 \) then \( P_2 \not\prec_{\mathcal{P}} P_1 \) for any \( P_1, P_2 \in \mathcal{P} \).

**Proof.** The first claim immediately follows from the definition of \( \prec \). Towards the second statement, suppose there are \( P_1, P_2 \in \mathcal{P} \) such that \( P_1 \prec_{\mathcal{P}} P_2 \) and \( P_2 \prec_{\mathcal{P}} P_1 \).

Hence, for \( j = 1, 2 \) and \( i = 1, 2 \), there are points \( p_j^i = (x_j^i, y_j^i) \) such that \( p_j^i \in P_i \) and \( x_1^i < x_2^i, y_1^i = y_2^i \) and \( x_2^i > x_2^i, y_2^i = y_2^i \). W.l.o.g. we assume that \( y_1^i < y_2^i \).

Let \( Q \subseteq P \) be the subpath of \( P \) from \( p_1^i \) to \( p_2^i \), including the endpoints. Let \( Q_1 := \{(x_1^i, z) : z < y_1^i\} \) and \( Q_2 := \{(x_2^i, z) : z > y_1^i\} \) be the two lines parallel to the \( y \)-axis going from \( p_1^i \) towards negative infinity and from \( p_2^i \) towards infinity.

Then \( Q_1 \cup Q \cup Q_2 \) separates the plane into two disjoint regions \( R_1 \) and \( R_2 \) each containing a point of \( P_2 \). As \( P_1 \) and \( P_2 \) are vertex disjoint but \( p_1^i \) and \( p_2^i \) are connected by \( P_2 \), \( P_2 \) must contain a point in \( Q_1 \) or \( Q_2 \) which, on \( P_2 \) lies between \( p_1^i \) and \( p_2^i \). But the \( y \)-coordinate of any point in \( Q_1 \) is strictly smaller than \( y_1^i \) and \( y_2^i \) whereas the \( y \)-coordinate of any point in \( Q_2 \) is strictly bigger than \( y_1^i \) and \( y_2^i \). This contradicts Lemma 4.6. \( \square \)

We use the previous lemma to show that \( \prec^*_{\mathcal{P}} \) is a partial order for all sets \( \mathcal{P} \) of pairwise vertex disjoint paths.

**Lemma 4.7.** Let \( \mathcal{P} \) be a set of pairwise vertex disjoint directed paths. Then \( \prec^*_{\mathcal{P}} \) is a partial order.

**Proof.** By definition, \( \prec^*_{\mathcal{P}} \) is transitive. Hence we only need to show that it is anti-symmetric for which, by transitivity, it suffices to show that \( \prec^*_{\mathcal{P}} \) is irreflexive.

To show that \( \prec^*_{\mathcal{P}} \) is irreflexive, we prove by induction on \( k \) that if \( P_0, \ldots, P_k \in \mathcal{P} \) are paths such that \( P_0 \prec_{\mathcal{P}} \cdots \prec_{\mathcal{P}} P_k \) then \( P_k \not\prec_{\mathcal{P}} P_0 \). As for all \( P \in \mathcal{P} \), \( P \not\prec_{\mathcal{P}} P \), this proves the lemma.

Towards a contradiction, suppose the claim was false and let \( k \) be minimum such that there are paths \( P_0, \ldots, P_k \in \mathcal{P} \) with \( P_0 \prec_{\mathcal{P}} \cdots \prec_{\mathcal{P}} P_k \) and \( P_k \prec_{\mathcal{P}} P_0 \). By Lemma 4.6, \( k > 1 \).
Let \( R := \bigcup_{i=0}^{k-2} \text{right}(P_i) \). Note that \( k-2 \geq 0 \), so \( R \) is not empty. Furthermore, as for all \( P, Q \) with \( P \prec Q \), \( \text{right}(P) \cap \text{right}(Q) \neq \emptyset \), \( R \) is a connected region in \( \mathbb{R}^2 \) without holes. Let \( L := \bigcup_{i=1}^{k-1} \text{left}(P_i) \). Again, as \( k > 1 \), \( L \neq \emptyset \) and \( L \) is a connected region without holes.

As \( P_{k-2} \prec_{\mathcal{P}} P_{k-1} \), we have \( L \cap R \neq \emptyset \) and therefore \( L \cup R \) separates the plane into two unbounded regions, the upper region \( T \) and the lower region \( B \).

The minimality of \( k \) implies that \( P_i \not\prec_{\mathcal{P}} P_{k} \) for all \( i < k-1 \) and therefore \( R \cap P_k = \emptyset \). Analogously, as \( P_k \not\prec_{\mathcal{P}} P_i \) for any \( i > 0 \), we have \( L \cap P_k = \emptyset \).

Hence, either \( P_k \subseteq B \) or \( P_k \subseteq T \). W.l.o.g. we assume \( P_k \subseteq B \). We will show that \( \text{left}(P_0) \cap B = \emptyset \).

Suppose there was a point \( (x, y) \in P \) and some \( x' < x \) such that \( (x', y) \in B \). This implies that \( y < v \) for all \((u,v) \in L \). But this implies that \( B \) is bounded by \( \text{right}(P_0) \) and \( L \) contradicting the fact that \( \text{right}(P_{k-1}) \cap B \neq \emptyset \). \( \square \)

We have shown so far that \( \prec_{\mathcal{P}} \) is a partial order on every set of pairwise vertex disjoint paths.

**Remark 4.8.** Note that if two paths \( P, Q \in \mathcal{P} \) are incomparable with respect to \( \prec_{\mathcal{P}} \), then one path is strictly above the other, i.e. \( (\text{right}(P) \cup \text{left}(P)) \cap (\text{right}(Q) \cup \text{left}(Q)) = \emptyset \). This is used in the next lemma.

**Definition 4.9.** Let \( s, t \in V(G) \) be vertices in \( G \) such that there is a directed path from \( s \) to \( t \). The rightmost \( s,t \)-path in \( G \) is an \( s,t \)-path \( P \) such that for all \( s,t \)-paths \( P' \), \( P \subseteq P' \cup \text{right}(P') \).

**Lemma 4.10.** Let \( s, t \in V(G) \) be two vertices and let \( P \) be a path from \( s \) to \( t \) in an upward planar drawing of \( G \). If \( P' \) is an \( s,t \)-path such that \( P' \cap \text{right}(P) \neq \emptyset \) then there is an \( s,t \)-path \( Q \) such that \( Q \subseteq P \cup \text{right}(P) \) and \( Q \cap \text{right}(P) \neq \emptyset \).

**Proof.** If \( P' \subseteq P \cup \text{right}(P) \) we can take \( Q = P' \). Otherwise, i.e. if \( P' \cap \text{left}(P) \neq \emptyset \), then as the graph is planar this means that \( P \) and \( P' \) share internal vertices. In this case we can construct \( Q \) from \( P \cup P' \) where for subpaths of \( P \) and \( P' \) between two vertices in \( P \cap P' \) we always take the subpath to the right. \( \square \)

**Corollary 4.11.** Let \( s, t \in V(G) \) be vertices in \( G \) such that there is a directed path from \( s \) to \( t \). Then there is a unique rightmost \( s,t \)-path in \( G \).

The corollary states that between any two \( s \) and \( t \), if there is an \( s,t \) path then there is a rightmost one. The proof of **Lemma 4.10** also indicates how such a path can be computed. This is formalised in the next lemma.

**Lemma 4.12.** There is a linear time algorithm which, given an upward planar drawing of a graph \( G \) and two vertices \( s,t \in V(G) \) computes the rightmost \( s,t \)-path in \( G \), if such a path exists.

**Proof.** We first use a depth-first search starting at \( s \) to compute the set of vertices \( U \subseteq V(G) \) reachable from \( s \). Clearly, if \( t \not\in U \) then there is no \( s,t \)-path and we can stop. Otherwise we use a second, inverse depth-first search to compute the set
In this paper we showed that the $k$-vertex disjoint paths problem is NP-complete on a restricted and yet very interesting class of planar digraphs. On the other hand, we provided a fast algorithm to approach this hard problem by finding a good partial order. It is an interesting question to investigate whether the $k!$ factor in the running time of our algorithm can be improved. Another direction of research is to extend our result to more general but still restricted graph classes, such as to digraphs embedded on a torus such that all edges are monotonically increasing in the $z$-direction or to acyclic planar graphs.

5 Conclusion

In this paper we showed that the $k$-vertex disjoint paths problem is NP-complete on a restricted and yet very interesting class of planar digraphs. On the other hand, we provided a fast algorithm to approach this hard problem by finding a good partial order. It is an interesting question to investigate whether the $k!$ factor in the running time of our algorithm can be improved. Another direction of research is to extend our result to more general but still restricted graph classes, such as to digraphs embedded on a torus such that all edges are monotonically increasing in the $z$-direction or to acyclic planar graphs.
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Appendix

A NP-Completeness Proof

Proof of Lemma 3.3

1. In the first case, the only possible paths are the following.
   - $H^{in} \rightarrow b_1 \rightarrow m_2 \rightarrow b_3 \rightarrow W$
   - $X \rightarrow b_2 \rightarrow m_3 \rightarrow m_5 \rightarrow m_8 \rightarrow m_{10} \rightarrow Y$
   - $Z \rightarrow m_7 \rightarrow m_9 \rightarrow H^{out}$
   - $T \rightarrow m_1 \rightarrow b_4 \rightarrow m_0 \rightarrow b_5 \rightarrow b_6 \rightarrow m_{11} \rightarrow m_{12} \rightarrow B$

2. In the second case, the only possible paths are the following.
   - $L^{in} \rightarrow m_3 \rightarrow m_5 \rightarrow W$
   - $X \rightarrow m_2 \rightarrow m_4 \rightarrow b_4 \rightarrow m_7 \rightarrow m_9 \rightarrow b_6 \rightarrow Y$
   - $Z \rightarrow b_5 \rightarrow m_8 \rightarrow m_{10} \rightarrow m_{11} \rightarrow L^{out}$
   - $T \rightarrow m_1 \rightarrow b_1 \rightarrow b_2 \rightarrow b_3 \rightarrow m_0 \rightarrow m_6 \rightarrow m_{12} \rightarrow B$

□

Proof of Lemma 3.4 By Lemma 3.3 we know that if we use 

Proof of Theorem 3.7

Let $i \in [n]$. If $\beta(V_i) = 1$, we consider the problem $\mathcal{P}^+_r$ associated with row $i$ as defined in Lemma 3.4 and if $\beta(V_i) = 0$, we consider $\mathcal{P}^-_r$ associated with row $i$. By the lemma, either problem has a unique solution and we can easily identify this solution with a solution to the subproblem of $\mathcal{P}_C$ restricted to row $i$. At this point, we have constructed disjoint paths for all pairs but for $(C_j, C'_j), j \in [m]$.

By construction, $e^+$ is present in gadget $G_{i,j,t}$, if and only if $C_j$ does not contain variable $V_i$ negatively and $e^-$ is present in gadget $G_{i,j,t}$, if and only if $C_j$ does not contain variable $V_i$ positively (especially, both edges are present if the clause does not contain the variable at all). Note that every column contains exactly one gadget where one edge is missing.

As $\beta$ is satisfying, for each $j \in [m]$, there is $t \in [n_j]$ such that $\beta(L_{j,t}) = 1$. Assume without loss of generality that $L_{j,t} = V_i$ for some $i \in [n]$. Consider the column which corresponds to $L_{j,t}$. With all the above paths fixed, we are in the situation of Lemma 3.5. By construction, both edges $e^+$ and $e^-$ are present in any gadgets $G_{t,j,t}$ where $V_i \neq L_{j,t}$. But also $e^+$ is present in $G_{i,j,t}$, as $V_i$ occurs positively in $C_j$. Hence by Lemma 3.5 we find a path disjoint to all of the above
paths from \(T_{1,j,t}\) to \(B_{n,j,t}\) and this path can be extended to a path from \(C_j\) to \(C_j'\).

Now assume that there is a solution to the disjoint paths problem in \(G_C\). Obviously, each path \(P_t\) connecting \(V_i\) to \(V_j'\) uses exactly one of \(H_{i,1,1}^n\) or \(L_{i,1,1}^n\). We define \(\beta : \{V_1, \ldots, V_n\} \to \{0, 1\}\) by \(\beta(V_i) = 1\) if and only if \(P_t\) uses \(H_{i,1,1}^n\). By Lemma 4.3, each path \(Q_k\) connecting \(C_i\) to \(C_j'\) can only pass through \(G_{i,j,t}\) if it is consistent with this assignment. Hence, \(C_k\) contains a variable \(V_i\) with \(\beta(V_i) = 1\).

\(\square\)

**B Main Algorithm and Correctness Proof**

*Proof of Theorem 4.1.* Let \(G\) with an upward planar drawing of \(G\) and \(k\) pairs \((s_1, t_1), \ldots, (s_k, t_k)\) be given. To decide whether there is a solution to the disjoint paths problem on this instance we proceed as follows. In the first step we compute the corresponding permutation such that \((s_1, t_1), \ldots, (s_k, t_k)\) is the ordering of \((s_1, t_1), \ldots, (s_k, t_k)\) according to \(\leq\). We set \(P_0 := \emptyset\) which obviously satisfies the condition. Suppose for some \(0 \leq i < k\), \(P_i\) has already been constructed. To obtain \(P_{i+1}\) we compute the right-most path linking \(u_{i+1}\) to \(v_{i+1}\) in the graph \(G \setminus \bigcup P_i\). By Lemma 4.12 this can be done in linear time for each such pair \((s_{i+1}, t_{i+1})\). If there is such a path \(P\) we define \(P_{i+1} := P_i \cup \{P\}\). Otherwise we reject the input. Once we reach \(P_k\) we stop and output \(P_k\) as solution.

Clearly, for every permutation \(\pi\) the algorithm can be implemented to run in time \(O(k \cdot n)\), using Lemma 4.12 so that the total running time is \(O(n! \cdot k \cdot n)\) as required.

Obviously, if the algorithm outputs a set \(P\) of disjoint paths then \(P\) is indeed a solution to the problem. What is left to show is that whenever there is a solution to the disjoint path problem, then the algorithm will find one.

So let \(P\) be a solution, i.e. a set of \(k\) paths \(P_1, \ldots, P_k\) so that \(P_i\) links \(s_i\) to \(t_i\). Let \(\leq\) be a linear order on \([1, \ldots, k]\) that extends \(\prec_P\) and let \(\pi\) be the corresponding permutation such that \((u_1, v_1), \ldots, (u_k, v_k)\) is the ordering of \((s_1, t_1), \ldots, (s_k, t_k)\) according to \(\leq\). We claim that for this permutation \(\pi\) the algorithm will find a solution. Let \(P\) be the right-most \(u_k-v_k\)-path in \(G\) computed by the algorithm. By Lemma 4.13, \(P \setminus \{P_k\} \cup P\) is also a valid solution so we can assume that \(P_k = P\). Hence, \(P_1, \ldots, P_{k-1}\) form a solution of the disjoint paths problem for \((u_1, v_1), \ldots, (u_{k-1}, v_{k-1})\) in \(G \setminus P\). By repeating this argument...
we get a solution $P' := \{P'_1, \ldots, P'_k\}$ such that each $P'_i$ links $u_i$ to $v_i$ and is the right-most $u_i$-$v_i$-path in $G \setminus \bigcup_{j>i} P'_j$. But this is exactly the solution found by the algorithm. This prove the correctness of the algorithm and concludes the proof of the theorem. □