Research Article

On Comparing between Two Nonlinear Cournot Duopoly Models

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The comparison between two nonlinear duopoly models constructed based on symmetric utility function that is derived from Cobb–Douglas is investigated in this paper. The first model consists of two firms which update their outputs using gradient-based mechanism called bounded rationality. The second model contains a bounded rational firm that is competing with a firm whose outputs depend on a trade-off between market share maximization and profit maximization. For the two models, the fixed points are calculated and their conditions of stability are analyzed. The obtained results show that the second model is more stabilizing provided that the second firm adopts low weights of trade-offs. We show that the two models can be destabilized via flip bifurcation only. Furthermore, the noninvertibility of the two models that can give rise to several stable attractors is discussed.

1. Introduction

Literature has reported several works that have studied the complex dynamic characteristics of Cournot duopoly games. Such games have given rise to interesting results about the models describing them. These results have included investigations of the types of bifurcation responsible for instability of games’ equilibrium points and chaos routes. Some of those games have been formed based on popular utility function such as Cobb–Douglas [1], constant elasticity of substitution (CES) [2], and Singh and Vives utility function [3]. There are other utility production functions that have been adopted to model Cournot games like, for example, the isoelastic utility derived from Cobb–Douglas (see [4]). The current paper adopts a symmetric utility function that is derived from the well-known Cobb–Douglas one. Although the adopted symmetric utility in the current paper is simple, the two models formed based on it are complicated and provide interesting results about their dynamics as will be shown later.

In order to make the current paper self-contained, we report in the introduction some important works that have been cited in this direction of research. For instance, we cite from literature the reported works ([5–10]) which have discussed and investigated the routes of chaos in Cournot games whose models have been constructed based on naive expectations. Such works have analyzed the chaotic behaviors on those models via detecting the types of bifurcations appeared on them. In [11], a gradient-based mechanism called bounded rationality has been adopted to globally study the competition of Cournot game. In this study, a rich analysis has shown that the game’s equilibrium points can be destabilized due to coexistence of periodic cycles and chaotic behaviors. The adoption of bounded rationality approach has opened the gate to several studies on its influences on duopoly games and it has been applied to other games whose players may be homogeneous or heterogeneous. In [12–15], the authors have applied the bounded rationality and other approaches such as Puu’s approach in order to analyze Cournot games and the cooperation that might be occurred between their players. These studies have confirmed what has been reported in literature on the destabilization routes of chaos that affected the stability of the equilibrium points in those games. Other useful works have been reported in [16, 17], in which Agiza et al. have investigated oligopoly games with both linear and nonlinear inverse demand function that have been used to study those games under limited rationality. In [18], chaos has been controlled in a duopoly game of master and slave players after the game’s dynamics have been analyzed and
The current paper belongs to both homogeneous and heterogeneous duopoly games in which games’ players adopt similar and different decisional mechanisms. The main results of this paper concern with the comparison between two nonlinear duopoly games which are formed based on a symmetric utility function derived from Cobb–Douglas well-known production utility. The comparison includes investigating the routes where the equilibrium points can be destabilized, analyzing the global behaviors of the two models describing the games that include investigation of the structure of attractive basins. Our obtained results show that the equilibrium points of both models cannot preserve their stabilities due to flip bifurcation only. Moreover, they show that the model with heterogeneous players seems to be more stable if the second firm uses low trade-off weights between market share and profit maximization. The structure of basins for both models may be quite complicated as we deal with noninvertible maps.

After this introduction, we can summarize the parts of the paper as follows. Section 2 introduces the two maps describing the duopoly games in this paper. In Section 3, we analyze the properties of the first map and this includes discussing the stability conditions of its equilibrium points and proves it is a noninvertible map. As in Section 3, Section 4 discusses the second map and analyzes its complex characteristics. Some comparisons between the two maps are given in Section 5. In Section 6, we end the paper with some conclusions and future works.

2. The Game of Competition

We consider a game of two firms (or players) whose decision variables in the market are quantities. We take \( q_1 \) and \( q_2 \) as the quantities produced by each firm. We recall the Cobb–Douglas function of two variables appearing in many economic models as follows:

\[
U = q_1^a q_2^b, \quad 0 < a, b < 1. \tag{1}
\]

This function is popular as Cobb–Douglas (this function is a production function used to organize production possibilities for firms from different and various branches [31]. In the application to production, as a firm’s output is something measurable, the sum \( a + b > 1 \) makes sense as watershed between increasing and decreasing returns to scale. In our assumption in this paper, maximizing utility \( U = \sqrt{q_1 q_2} \) returns \( (p_1, p_2) = (q_2/q_1) \) or \( p_1 q_1 = p_2 q_2 \) that means the total revenue for both firms is equal and this is due to the assumed symmetry as we take \( a = b \) function which should properly be called Wicksell function since the Swedish economist Knut Wicksell was the first to introduce this production function [32]. It represents the agent’s preferences. Both \( a \) and \( b \) are constants. It has the following properties:

(i) It is a monotonic function under the transformation \( u = \ln U \). This means that \( \partial u/\partial q_1 = (a/q_1) \) and \( \partial u/\partial q_2 = (b/q_2) \) and hence the marginal utility of each quantity is positive.

(ii) It is a concave utility function. This means that \( (\partial^2 u/\partial q_1^2 = (-a/q_1^2) < 0 \) and \( (\partial^2 u/\partial q_2^2 = (-b/q_2^2) < 0 \) and hence the marginal utility of each quantity is decreasing.

(iii) It is strongly additive. This means that \( (\partial^2 u/\partial q_1 \partial q_2) = 0 \) and hence the marginal of the quantity \( q_1 \) is independent of the quantity \( q_2 \). So, the quantities cannot represent substitutes or complements.

(iv) It is a homothetic function. This means that if \( x = (q_1, q_2), y = (\lambda q_1, \lambda q_2), u(x) = u(y), \) and \( \theta > 0, \) then \( u(\theta x) = u(\theta y) \).

We assume the symmetric case, \( a = b = (1/2) \), with the budget constraint \( p_1 q_1 + p_2 q_2 = 1 \) where \( p_i \) is the price of \( q_i, i = 1, 2 \). Now, we have the following maximization problem:

\[
\max U = \sqrt{q_1 q_2}, \tag{2}
\]

s.t. \( p_1 q_1 + p_2 q_2 = 1. \)

Solving (2) gives the following prices:

\[
p_1 = \frac{1}{2} \sqrt{q_1}, \tag{3}
\]

\[
p_2 = \frac{1}{2} \sqrt{q_2}.
\]

At the economic market, each competitor wants to maximize its profit given by

\[
\pi_1 = \frac{1}{2} \sqrt{q_1 q_2} - c_1 q_1^2, \tag{4}
\]

\[
\pi_2 = \frac{1}{2} \sqrt{q_1 q_2} - c_2 q_2^2,
\]

where \( C(q_i) = c_i q_i^2 \) is the cost function of each quantity. It is a quadratic function and is often met in applications. For example, in the modeling of renewable resources exploitation, such as fisheries [33], in [34], it has been considered in order to investigate the role of convexity. So, the marginal cost is not constant, \( (\partial C(q_i)/\partial q_i) = 2 c_i q_i \), and \( c_i, i = 1, 2 \) is a constant parameter. The marginal profits take the following form:

\[
\frac{\partial \pi_1}{\partial q_1} = \frac{1}{4} \sqrt{q_1} - 2 c_1 q_1, \tag{5}
\]

\[
\frac{\partial \pi_2}{\partial q_2} = \frac{1}{4} \sqrt{q_2} - 2 c_2 q_2.
\]
Here, we discuss two situations. The first situation assumes that both firms use a gradient-based mechanism such as the bounded rationality defined in the literature [4, 35–37]. Using this mechanism, we get the following discrete dynamical system:

\[
T(q_1, q_2): \begin{cases}
q_1(t + 1) = q_1(t) + k_1 q_1(t) \left( \frac{1}{4} \frac{q_2(t)}{q_1(t)} - 2c_1 q_1(t) \right), \\
q_2(t + 1) = q_2(t) + k_2 q_2(t) \left( \frac{1}{4} \frac{q_1(t)}{q_2(t)} - 2c_2 q_2(t) \right),
\end{cases}
\]

(6)

where \( k_i, i = 1, 2 \) is a speed of adjustment parameter. The second situation is a heterogeneous situation. We assume that the second firm shares the market with certain profit. It trades off between market share maximization and profit maximization. The market share means it seeks the optimum output by putting \( n_2 = 0 \) and then we get the optimum, \( \bar{q}_2 = \frac{\sqrt{c}}{2(\sqrt{c})^2} \), while the profit maximization gives \( \bar{q}_2 = (1/4) \sqrt{q_i / c_i^2} \). Since it is traded off between those, we get the following:

\[
\bar{q}_2 = \omega \bar{q}_2 + (1 - \omega)\bar{q}_2,
\]

(7)

where \( \omega \in (0, 1) \). When the second firm shares market maximization only, we take \( \omega = 1 \), while \( \omega = 0 \) means it seeks profit maximization only. Now, the second firm updates its output according to the following:

\[
q_2(t + 1) = (1 - s)q_2(t) + s \left( \frac{\omega}{\sqrt{4} + \frac{1 - \omega}{4}} \right) \sqrt{\frac{q_1(t)}{c_2}}
\]

(8)

where \( s \in (0, 1) \) is a constant weight. So, we have another model with heterogeneous players given by

\[
T_1(q_1, q_2): \begin{cases}
q_1(t + 1) = q_1(t) + k_1 q_1(t) \left( \frac{1}{4} \frac{q_2(t)}{q_1(t)} - 2c_1 q_1(t) \right), \\
q_2(t + 1) = (1 - s)q_2(t) + s \left( \frac{\omega}{\sqrt{4} + \frac{1 - \omega}{4}} \right) \sqrt{\frac{q_1(t)}{c_2}}.
\end{cases}
\]

(9)

It is obvious that if \( s = 0 \), then the second firm in (9) is naive, while at \( s = 1 \) means it updates its production based on market share maximization. Economically, profit maximization only is mainly a short-term goal and is primarily restricted to the accounting analysis of the financial situation. Furthermore, it is primarily concerned as to how the company will survive and grow in the existing competitive business environment [38]. On the other hand, market share can allow firms to improve profitability either by lowering prices, using advertising, or introducing new or different products. It can also grow the size of its market share by appealing to other audiences or demographics. In our paper, we introduce a weighted average decision-making mechanism between those two mechanisms discussed above. This includes the main contribution in this paper that is to compare between the maps (6) and (9) in order to see which model is more stable and gives large region of stability [39].

3. Properties of Map (6)

3.1. Analytical Results. The map (6) can make negative or unbounded trajectories if the initial condition \( (q_{01}, q_{02}) \) is chosen far from \( O = (0, 0) \). It is obvious that if \( q_{00} > (1/2c, k_i), i = 1, 2 \), negative values for \( q_i(t + 1) \) can be obtained.

**Proposition 1.** The map (6) has two fixed points that are \( O_1 = (0, 0) \) and \( O_2 = (1/(8c_1), 0), (1/(8c_1)) \sqrt{(c_1/c_2)} \).

**Proof.** Setting \( q_1(t + 1) = q_1(t) \) and \( q_2(t + 1) = q_2(t) \) in (6) and solving algebraically, we get the two fixed points. \( \square \)

**Proposition 2.** The fixed point \( O_1 \) is an unstable point.

**Proof.** The Jacobian matrix of map (6) at \( O_1 \) becomes

\[
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix},
\]

(10)

which contains indefinite form and then the point is unstable and the proof is completed.

**Proposition 3.** The fixed point \( O_2 \) is locally asymptotically stable provided that

\[
k_1 \geq \frac{16}{2} \frac{\sqrt{c_2}}{\sqrt{c_1}} - \frac{\sqrt{c_2}}{\sqrt{c_1}} k_2.
\]

(11)

**Proof.** At \( O_2 \), the Jacobian matrix becomes

\[
\begin{pmatrix}
1 - \frac{3k_1}{8} \frac{\sqrt{c_1}}{\sqrt{c_2}} & k_1 \frac{\sqrt{c_2}}{8} \sqrt{c_1} \\
\frac{k_2}{8} \sqrt{c_1} & 1 - \frac{3k_2}{8} \frac{\sqrt{c_2}}{\sqrt{c_1}}
\end{pmatrix},
\]

(12)

whose trace and determinant are given by

\[
\begin{align*}
\tau &= 2 - \frac{3k_1}{8} \frac{\sqrt{c_1}}{\sqrt{c_2}} - \frac{3k_2}{8} \frac{\sqrt{c_2}}{\sqrt{c_1}} \\
\delta &= 1 - \frac{3k_1}{8} \frac{\sqrt{c_1}}{\sqrt{c_2}} - \frac{3k_2}{8} \frac{\sqrt{c_2}}{\sqrt{c_1}} + \frac{1}{8} k_1 k_2.
\end{align*}
\]

(13)

Substituting (13) in Jury conditions [40] gives
where we assume

\[ \begin{align*}
\Delta_1 &= 1 - \tau + \delta = \frac{k_1k_2}{8}, \\
\Delta_2 &= 1 + \tau + \delta = 4 - \frac{3k_1}{4} \sqrt{\frac{c_1}{c_2}} - \frac{3k_2}{4} \sqrt{\frac{c_2}{c_1}} + \frac{k_1k_2}{8}, \\
\Delta_3 &= 1 - \delta = \frac{3k_1}{8} \sqrt{\frac{c_1}{c_2}} + \frac{3k_2}{8} \sqrt{\frac{c_2}{c_1}} - \frac{k_1k_2}{8}. 
\end{align*} \]  

Since \( k_1 \) and \( k_2 \) are positive speeds of adjustment parameters then \( \Delta_1 > 0 \) is always satisfied. \( \Delta_2 > 0 \) and \( \Delta_3 > 0 \) give

\[ \begin{align*}
4 - \frac{3}{4} \sqrt{\frac{c_1}{c_2}} k_1 - \frac{3}{4} \sqrt{\frac{c_2}{c_1}} k_2 + \frac{1}{8} k_1k_2 > 0, \\
\frac{3}{8} \sqrt{\frac{c_1}{c_2}} k_1 + \frac{3}{8} \sqrt{\frac{c_2}{c_1}} k_2 - \frac{1}{8} k_1k_2 > 0. 
\end{align*} \]  

Combining the above inequalities completes the proof. \( \square \)

**Proposition 4.** The fixed point \( O_2 \) can be destabilized due to flip bifurcation only.

**Proof.** The Jacobian (12) has the following eigenvalues:

\[ \lambda_\pm = 1 - 3k_1k_2 \left( 1 \pm \sqrt{\frac{c_1}{c_2}} \right) \left( 1 \pm \sqrt{\frac{c_2}{c_1}} \right) + \frac{4}{c_1c_2} k_1k_2. \]  

It is clear that \( 9 \left( \sqrt{\frac{c_1}{c_2}} k_1 - \sqrt{\frac{c_2}{c_1}} k_2 \right)^2 + 4 \sqrt{\frac{c_1}{c_2}} k_1k_2 \) is always positive, and therefore, we have only two real eigenvalues. So the point becomes unstable because of flip bifurcation. \( \square \)

### 3.2. Numerical Results

The following numerical simulation experiments validate the above theoretical results. We start our numerical experiments by assuming the following parameters’ set \( (c_1 > c_2), \ c_1 = 0.7, c_2 = 0.5 \). At this set, \( O_2 = (0.1942423762, 0.2298306788) \) and the Jacobian becomes

\[ \begin{pmatrix}
-0.22373 & 0.34475 \\
0.54388 & -0.37898
\end{pmatrix}, \]  

where we assume \( k_1 = 3 \) and \( k_2 = 4 \). The eigenvalues then become \( \lambda_\pm = (0.138563, -0.741273) \) with \( |\lambda_+| < 1 \) and then \( O_2 \) is locally asymptotically stable. Keeping the costing shift parameters \( c_1 \) and \( c_2 \) as previously, we now study the influence of \( k_1 \) or \( k_2 \) on the stability of \( O_2 \). Figures 1(a) and 1(b) show that \( O_2 \) is locally asymptotically stable for all values of \( k_1 \) and \( k_2 \); until we get the cycle of period 2, the point \( O_2 \) becomes unstable. Indeed, the simulation shows some interesting behavior of the dynamic of map (6) around the point \( O_2 \). To present that dynamics, we have to study the effects of \( k_1 \) or \( k_2 \) separately. Assuming the following parameters set, \( c_1 = 0.7, c_2 = 0.5, \) and \( k_1 = 2 \). As \( k_2 \) increases to 6.54, a period-2 cycle (represented by squares in Figure 1(c)) is born around the fixed point \( O_2 \).

It is plotted with its basins of attraction represented by the light green colors in Figure 1(c) while the grey color refers to divergent (or nonconvergent) points in the phase plane. This cycle turns into a cycle of period four as \( k_2 \) increases to 7.14 with quite complicated basins of attraction as displayed in Figure 1(d). Other basins of attraction for the cycle of period four are formed in Figure 1(e) at \( k_2 = 7.24 \) where nonconvergent points appeared in white color within the basins. The basins of attraction become more complicated as the period-8 starting to appear at \( k_2 = 7.24 \) as depicted in Figure 1(f). Increasing \( k_2 \) further gives rise to four disconnected pieces and two chaotic attractors and then one chaotic attractor. We give in Figures 1(g) and 1(h) examples of two and one chaotic attractors and their basins at \( k_2 = 7.53 \) and \( k_2 = 7.69 \), respectively. The white color in the basins denotes the nonconvergent points in the phase plane. The same discussion can be performed for the other parameter \( k_1 \) under the same values of costs. This makes us to display the 2D-bifurcation diagram in the \((k_1, k_2)\)-plane on which interesting readers can detect more about the dynamics of map (6) around \( O_2 \). Figure 2(a) presents the 2D-bifurcation in the \((k_1, k_2)\)-plane with examples of attractive basins of period-7 (Figure 2(b)), period-9 (Figure 2(c)), and period-10 (Figure 2(d)). Furthermore, the 2D-bifurcation diagram shows that the dynamics of map (6) do not permit the appearance of period-3 and period-5.

Now, we study the case when \( c_1 < c_2 \) by assuming the set of parameters’ values, \( c_1 = 0.5, c_2 = 0.9 \). In this case, we do not want to repeat the above discussions and only we give some simulations represented by Figures 3(a) to 3(f) at this set in order to see the difference in dynamics in this case and the previous one. All these results so far show a peculiar shape of the basins of attraction and require from us to investigate other characteristics of map (6). From the above discussion and the numerical experiments, we highlight that the assumptions \( c_1 > c_2 \) and \( c_1 < c_2 \) affect the stability region obtained by the second condition of (14) for the interior equilibrium point in the \((k_1, k_2)\)-plane. We have seen that at \( c_1 > c_2 \) \((c_1 = 0.7, c_2 = 0.5)\), we get \( k_1 \in (0, 4.9) \) and \( k_2 \in (0, 5.8) \) while in the case \( c_1 < c_2 \) \((c_1 = 0.5, c_2 = 0.9)\), we have \( k_1 \in (0, 6.166) \) and \( k_2 \in (0, 4.625) \) for the same condition.

### 3.3. Critical Curves and Phase Plane Zones

The structure of the basins of attraction of map (6) at any \( q \) that may represent \( O_2 \) or periodic cycle or any chaotic attractor is constructed by some boundaries. These boundaries are calculated as follows.

**Proposition 5.** The origin point \( O = (0, 0) \) has four real preimages.

**Proof.** Setting \( q_1(t + 1) = 0 \) and \( q_2(t + 1) = 0 \) in (6) and solving the system algebraically, we get
Figure 1: Continued.
Figure 1: (a) Bifurcation diagram and LLE on varying $k_1$. (b) Bifurcation diagram and LLE on varying $k_2$. (c) The basins of attraction of period-2 cycle at $k_2 = 6.54$. (d) The basins of attraction of period-4 cycle at $k_2 = 7.14$. (e) The basins of attraction of period-4 cycle at $k_2 = 7.24$. (f) The basins of attraction of period-8 cycle at $k_2 = 7.43$. (g) The basins of attraction of two chaotic attractors at $k_2 = 7.53$. (h) The basins of attraction of a chaotic attractor at $k_2 = 7.69$. Other parameters’ values are, $c_1 = 0.7$ and $c_2 = 0.5$.

Figure 2: (a) The 2D-bifurcation in the $(k_1, k_2)$ plane. (b) The basins of attraction of period-7 cycle at $k_1 = 2.288$ and $k_2 = 7.5362$. (c) The basins of attraction of period-9 cycle at $k_1 = 2.713$ and $k_2 = 7.3140$. (d) The basins of attraction of period-10 cycle at $k_1 = 2.933$ and $k_2 = 7.263$. Other parameters’ values are, $c_1 = 0.7$ and $c_2 = 0.5$. 
Figure 3: (a) The basins of attraction of the fixed point $O_2$ at $k_1 = 3$ and $k_2 = 3$. (b) The basins of attraction of period-2 cycle at $k_1 = 6.06$ and $k_2 = 3$. (c) The basins of attraction of period-6 cycle at $k_1 = 7.06$ and $k_2 = 3$. (d) The basins of attraction of six chaotic attractors at $k_1 = 7.3$ and $k_2 = 3$. (e) The basins of attraction of two chaotic attractors at $k_1 = 7.33$ and $k_2 = 3$. (f) The basins of attraction of a chaotic attractor at $k_1 = 7.68$ and $k_2 = 3$. Other parameters’ values are $c_1 = 0.5$ and $c_2 = 0.9$. 
\[ O_{1}^{(0)} = (0, 0), \]
\[ O_{1}^{(1)} = \left( \frac{1}{2c_{1}k_{1}}, 0 \right), \]
\[ O_{1}^{(2)} = \left( 0, \frac{1}{2c_{2}k_{2}} \right), \]
\[ O_{1}^{(3)} = \left( q_{1}, q_{2} \right), \]
where \( O_{1}^{(3)} \) is obtained by solving algebraically the following system:
\[ 1 + k_{1}\left( \frac{1}{4} \frac{q_{2}}{q_{1}} - 2c_{1}q_{1} \right) = 0, \]
\[ 1 + k_{2}\left( \frac{1}{4} \frac{q_{1}}{q_{2}} - 2c_{2}q_{2} \right) = 0. \]

**Proposition 6.** Let \( \omega_{1} = O_{1}^{(0)}O_{1}^{(1)} \) and \( \omega_{2} = O_{1}^{(0)}O_{1}^{(2)} \) be the two line segments of the coordinates \( q_{1} \) and \( q_{2} \). Also, let \( \omega_{1}^{-1} \) and \( \omega_{2}^{-1} \) be their preimages. Then, the boundary of the basins of attraction of any \( \omega \) is denoted by \( F \) and is given by
\[ F = ( \bigcup_{n=0}^{\infty} T^{-n}(\omega_{1}) ) \cup ( \bigcup_{n=0}^{\infty} T^{-n}(\omega_{2}) ), \]
where \( T^{-n} \) refers to the set of all preimages of rank \( n \).

These boundaries are displayed in Figure 4. Furthermore, the figure shows the critical curve \( LC_{-1} \) that is calculated by vanishing the determinant of the Jacobian of (6). It is represented by the following curve:
\[
\begin{cases} 
32(4c_{1}k_{2}q_{2} - 1)c_{1}k_{1}q_{1}^{3/2}q_{2}^{1/2} - 4(c_{1}q_{1}^{2} + c_{2}q_{2}^{2})k_{1}k_{2} + 
+ 8(1 - 4c_{1}k_{2}q_{2})q_{1}^{1/2}q_{2}^{1/2} + k_{1}q_{2} + k_{2}q_{1} = 0.
\end{cases}
\]

It is represented by red color in Figure 4 as it contains two parts, \( LC_{-1}^{(a)} \) and \( LC_{-1}^{(b)} \). It is hard to plot or to get an analytical form for \( LC \) in order to identify the zones in the phase plane; however, it is simple to see that \( O_{2} \) has two real preimages of rank-1 and hence it belongs to \( Z_{2} \). Furthermore, both \( \omega_{1}^{-1} \) and \( \omega_{2}^{-1} \) have no preimages and they belong to \( Z_{0} \). With the preimages of the origin, we detect that the phase plane of map (6) is divided into three zones, \( Z_{4}, Z_{2}, \) and \( Z_{0} \). Therefore, map (6) is a non-invertible map.

4. **Properties of Map (9)**

4.1. **Analytical Results**

**Proposition 7.** The map (9) has only two fixed points that are \( \alpha_{1} = (0, 0) \) and \( \alpha_{2} = (\tilde{q}_{1}, \tilde{q}_{2}) \) where
\[
\tilde{q}_{1} = \frac{1}{4^{(3/8)}c_{1}^{(3/4)}c_{2}^{(1/4)}} \left( \frac{\omega}{\sqrt{4} + \frac{1 - \omega}{4}} \right)^{(3/8)},
\]
\[
\tilde{q}_{2} = \frac{1}{4^{(3/8)}c_{1}^{(3/4)}c_{2}^{(1/4)}} \left( \frac{\omega}{\sqrt{4} + \frac{1 - \omega}{4}} \right)^{(9/8)}.
\]

**Proof.** Setting \( q_{1}(t + 1) = q_{1}(t) \) and \( q_{2}(t + 1) = q_{2}(t) \) in (9) and solving the system algebraically completes the proof. \( \square \)

**Proposition 8.** The fixed point \( \alpha_{1} = (0, 0) \) is unstable.

**Proof.** The proof is the same as Proposition 2. \( \square \)

**Proposition 9.** The fixed point \( \alpha_{2} = (\tilde{q}_{1}, \tilde{q}_{2}) \) is locally asymptotically stable provided that \( k < (4 - 2s)/(45 - 22s) \) where
\[
h = 15^{(5/8)} \left( \frac{c_{1}}{24} \sqrt{c_{2}} \left( (2\sqrt{2} - 1)(1 + 2\sqrt{2} + 4\sqrt{4} + 15\omega) \right)^{(5/8)} \right.
\]

**Proof.** The Jacobian at \( \alpha_{2} \) becomes
\[
\begin{pmatrix}
J_{11} & J_{12} \\
J_{21} & J_{22}
\end{pmatrix},
\]
where
\[
J_{11} = 1 - 45h,
\]
\[
J_{12} = \frac{c_{1}}{2c_{2}} \left( 1 - \omega + 2\sqrt{2}\omega \right),
\]
\[
J_{21} = \frac{s}{6} \sqrt{\frac{c_{1}}{c_{2}}} \left( 1 + 2\sqrt{2}\omega \right),
\]
\[
J_{22} = 1 - s,
\]
and the trace and determinant take the following form:
\[
\tau = 1 - s + \frac{k}{2} \left( 44s - 45 \right),
\]
\[
\delta = 2 - s + \frac{45}{2} kh,
\]
and then,
\[
\Delta_{1} = \frac{11sk}{12} \sqrt{\frac{c_{1}}{c_{2}}} \left( \frac{\sqrt{2} (1 - \omega) + \sqrt{32}\omega}{(1 - \omega + 2\sqrt{2}\omega)^{(3/8)}} \right),
\]
\[
\Delta_{2} = 4 - 2s + kh \left( 22s - 45 \right),
\]
\[
\Delta_{3} = s + \frac{kh}{2} \left( 45 - 44s \right).
\]
It is clear that $\Delta_1 > 0$ and $\Delta_2 > 0$ since $s \in (0, 1)$ and $h$ is always positive. While $\Delta_2 > 0$ is attained, if $k < (4 - 2s/(45 - 22s)h)$. The proof is completed.

We should highlight here that if $k > (4 - 2s/(45 - 22s)h)$, then $o_2$ can be destabilized due to flip bifurcation only. □

4.2. Numerical Results. We assume the following $(c_1 > c_2)$, $c_1 = 0.7, c_2 = 0.5$ with $\omega = 0.2$. At this set, $o_2 = (0.6068957946, 0.4381254240)$ and the Jacobian becomes, at taking $k = 1.2$ and $s = 0.2$,

$$\left(\begin{array}{cc} -0.91172 & 0.17654 \\ 0.048127 & 0.8 \end{array}\right),$$

and the eigenvalues are $\lambda_1 = (-0.916669, 0.804949)$ with $|\lambda_1| < 1$ and then $o_2$ is locally asymptotically stable. Increasing $k$ above the value $(4 - 2s/(45 - 22s)h)$ gives rise to a period-doubling bifurcation, as shown in Figure 5(a). It is clear that the fixed point $o_2$ is stable for all the values of $k$ until $k$ reaches the value $(4 - 2s/(45 - 22s)h)$ where the first period-2 cycle is born. As discussed in map (6), we give here some complex behaviors of map (9) at different values of the bifurcation parameter $k$. For instance, at $k = 8.026$, a cycle of period-2 is emerged and is plotted with its basins of attraction in Figure 5(b). At $k = 8.44$, a period-4 cycle is coexisted with its attractive basins in Figure 5(c). It is clear that they are quite complicated basins. The basins become more complicated when the period-8 cycle appears as given in Figure 5(d) at $k = 8.58$. After that higher periodic cycles exist with quite complicated basins and then the map enters the chaos region at $k = 8.83$ and $k = 8.9$ where two chaotic attractors and then one chaotic attractor are emerged and displayed in Figures 5(e) and 5(f), respectively.

The numerical simulation shows that increasing $\omega$ which means the second firm gives more interesting to the market share maximization than the profit maximization reduces the stability interval of $o_2$. The same observation when the firm gives more interesting to the traded-offs between the market share maximization and profit maximization is obtained. It is noticed that low values of the traded-off parameter $s$ keep the stability interval of $o_2$ larger.

Furthermore, when we take values of $c_1$ and $c_2$ in the form $c_1 < c_2$, the dynamics of map (9) do not give any new complex dynamics and hence the dynamic is still governed by the change that may be happened on both parameters $\omega$ and $s$.

4.3. Critical Curves and Phase Plane Zones.

$$2c_1q_1^{(4/3)} - q_1^{(1/3)} - k \frac{s}{4} \left( \frac{\omega}{\sqrt[4]{c_2}} + \frac{1 - \omega}{4} \right) \sqrt{\frac{q_1}{c_2}} = 0,$$

which has no real solutions as the expression under the square root is negative. Therefore, $o$ belongs to $Z_1$ zone.

**Proposition 10.** The origin point $o = (0, 0)$ has only one real preimage that is $o_{-1}^{(0)} = (0, 0)$.

**Proof.** Setting $q_1(t + 1) = 0$ and $q_2(t + 1) = 0$ in (9), we get

$$q_1 + kq_1 \left( \frac{1}{4} \sqrt[4]{q_1} - 2c_1q_1 \right) = 0,$$

$$(1 - s)q_2 + s \left( \frac{\omega}{\sqrt[4]{c_2}} + \frac{1 - \omega}{4} \right) \sqrt{\frac{q_1}{c_2}} = 0.$$

The first equation in (29) gives $q_1 = 0$ or $1 + k((1/4)\sqrt[4]{(q_2/q_1)} - 2c_1q_1) = 0$. Substituting $q_1 = 0$ in the second equation of (29) gives $q_2 = 0$, then we get $o_{-1}^{(0)} = (0, 0)$. Now, we have the following case:

$$1 + k \left( \frac{1}{4} \sqrt[4]{q_1} - 2c_1q_1 \right) = 0,$$

$$(1 - s)q_2 + s \left( \frac{\omega}{\sqrt[4]{c_2}} + \frac{1 - \omega}{4} \right) \sqrt{\frac{q_1}{c_2}} = 0.$$

Solving (30) gives □
Figure 5: (a) Bifurcation diagram and LLE on varying $k$. (b) The basins of attraction of period-2 cycle at $k = 8.026$. (c) The basins of attraction of period-4 cycle at $k = 8.44$. (d) The basins of attraction of period-8 cycle at $k = 8.58$. (e) The basins of attraction of two chaotic attractors at $k = 8.83$. (f) The basins of attraction of a chaotic attractor at $k = 8.9$. Other parameters’ values are $c_1 = 0.7, c_2 = 0.5, \omega = 0.2$, and $\varepsilon = 0.2$. 

Complexity
**Figure 6:** The critical curve and zones of map (9) at the parameters' values, $c_1 = 0.7, c_2 = 0.5, k = 2, \omega = 0.2,$ and $s = 0.2.$

**Figure 7:** Continued.
Figure 7: Continued.
Proposition 11. The fixed point $a_2$ has no real preimages.

Proposition 12. Any point in the form $(0, q), q \neq 0$ has two real preimages.

Proposition 13. Any point in the form $(q, 0), q \neq 0$ has no real preimages.

Now, $LC_{-1}$ for map (9) is given by the following curve:

\[
\begin{align*}
96 c_1 k q_1^{(13/6)} q_2^{(1/2)} - 96 c_1 k q_1^{(13/6)} q_2^{(1/2)} - 24 s q_1^{(7/6)} q_2^{(1/2)} + 24 q_1^{7/6} q_2^{(1/2)} - 3 k q_1^{(2/3)} q_2^{(2/3)} + \\
+ 3 k s q_1^{(2/3)} - k s \left(\frac{\omega}{\sqrt{c_2^2}} + \frac{1 - \omega}{4}\right) q_1 = 0.
\end{align*}
\]

Figure 7: (a) Bifurcation diagram on varying $k$ at $c_1 = 0.7, c_2 = 0.5, \omega = 0.5,$ and $s = 0.5.$ (b) Bifurcation diagram on varying $k$ at $c_1 = 0.7, c_2 = 0.5, \omega = 0.7,$ and $s = 0.5.$ (c) Bifurcation diagram on varying $s$ at $c_1 = 0.7, c_2 = 0.5, k = 5,$ and different values for $\omega.$ (d) Bifurcation diagram on varying $\omega$ at $c_1 = 0.7, c_2 = 0.5, k = 5,$ and different values for $s.$

5. Comparison between the Maps $T$ and $T_1$

The above analysis and discussion about the two maps show that the model described by map (9) is quite complicated than those of map (6); however, map (9) gives a large stability region for the fixed point under the same set of parameters. This is clear when comparing the 1D bifurcation diagram for the two maps. This means that the firm adopting the trade-offs between the market share maximization and the profit maximization will be more stable in the market competition than those using a gradient-based mechanism such as the bounded rationality. It is also observed from the numerical experiments that when the second firm updates its outputs based on the average of market share and profit maximization, the stability region of the fixed point reduces and becomes almost like those provided by map (6) under the same set of parameters’ values. Figure 7(a) shows the bifurcation diagram at $c_1 = 0.7, c_2 = 0.5, \omega = 0.5,$ and $s = 0.5.$ Increasing $\omega$ further to 0.7 and keeping the other parameters’ values fixed, the region of stability region of the
fixed point reduces, as shown in Figure 7(b). Furthermore, Figures 7(c) and 7(d) show the influences of the parameters \( s \) and \( \omega \) on the dynamics of map (9). From an economic perspective, one can see that low values of the parameter \( \omega \) means the second firm pays more attention for the profit maximization (just as the first firm decision-making mechanism) than market share and hence the interior equilibrium point becomes stable for any value of \( s \) in the interval \([0, 1] \). As \( \omega \) increases further, the stability interval of the equilibrium point with respect to \( s \) is reduced, as shown in Figure 7(c). Similar discussion is for the influences of the parameter \( \omega \) given in Figure 7(d).

6. Conclusion

In this manuscript, we have studied two nonlinear duopoly models whose players adopted gradient-based mechanism and trade-off mechanism. We have shown that the fixed points in both models become unstable due to flip bifurcation only. The duopoly in the first model has been compared with the duopoly of the second model whose second firm uses trade-off between market share maximization and profit maximization. Our contributions showed that the second model was more stabilizing than the first model. It has been investigated that the trade-off mechanism can give better stability region for the fixed point. We found that when the second firm in the second model adopts the average between market share and profit maximization, the region of stability was reduced. Furthermore, using higher weights of the trade-off makes the stability region to shrink and hence the second firm should adopt low weights to become more stable in the market. We have also investigated that both maps are noninvertible, and for this reason, the basins of attraction of any dynamic behavior of the maps were quite complicated. In future study, we intend to apply the trade-off mechanism to oligopoly models with more than two firms.

Data Availability

The datasets generated during the current study are available from the corresponding author on reasonable request.

Conflicts of Interest

The author declares no conflicts of interest.

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