MIXED TENSOR PRODUCTS AND CAPELLI-TYPE DETERMINANTS

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Abstract. In this paper we study properties of a homomorphism $\rho$ from the universal enveloping algebra $U = U(\mathfrak{gl}(n + 1))$ to a tensor product of an algebra $D'(n)$ of differential operators and $U(\mathfrak{gl}(n))$. We find a formula for the image of the Capelli determinant of $\mathfrak{gl}(n + 1)$ under $\rho$, and, in particular, of the images under $\rho$ of the Gelfand generators of the center $Z(\mathfrak{gl}(n + 1))$ of $U$. This formula is proven by relating $\rho$ to the corresponding Harish-Chandra isomorphisms, and, alternatively, by using a purely computational approach. Furthermore, we define a homomorphism from $D'(n) \otimes U(\mathfrak{gl}(n))$ to an algebra containing $U$ as a subalgebra, so that $\sigma(\rho(u)) - u \in G_1 U$, for all $u \in U$, where $G_1 = \sum_{i=0}^{n} E_{ii}$.

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1. Introduction

Capelli-type determinants are a powerful tool in invariant theory. One fundamental result is that the Capelli determinant $C_N(T)$ corresponding (up to a diagonal shift) to the $N \times N$ matrix $E$ whose $(i, j)$th entry is the elementary matrix $E_{ij}$ is a polynomial in $T$ whose coefficients $C_k$ are central elements in the universal enveloping algebra $U(\mathfrak{gl}(N))$ of the Lie algebra $\mathfrak{gl}(N)$. Moreover, the coefficients $C_k$ are generators of the center $Z(\mathfrak{gl}(N))$ of $U(\mathfrak{gl}(N))$, and are usually referred as Capelli generators. On the other hand the elements $G_k = \text{tr}(E^k)$ for $k = 1, \ldots, N$ also form a system of generators of $Z(\mathfrak{gl}(N))$, sometimes known as the Gelfand generators. There is a nice transition formula between the Gelfand generators and the Capelli generators, \cite{11}, and this formula can be considered as a noncommutative version of the Newton identities. The applications of Capelli-type determinants extend well beyond relations and properties of elements in $Z(\mathfrak{gl}(N))$. There are direct applications to classical invariant theory (see for example \cite{6}), and a more general treatment of the subject using the theory of Yangians (see \cite{9} and the references therein).

In this paper we relate Capelli-type determinants to another classical construction in representation theory - the mixed tensor type modules. These modules are modules over tensor products of an algebra of differential operators and a universal enveloping algebra $U(\mathfrak{gl}(n))$. Modules of mixed tensor type, also known as tensor modules, or modules of Shen and Larson (see \cite{10} and \cite{8}), can be defined over the Lie algebras $\mathfrak{sl}(n + 1)$ after considering a suitable homomorphism. In this paper we define a $\mathfrak{gl}(n + 1)$-version of this homomorphism, namely a map $\rho : U(\mathfrak{gl}(n + 1)) \to D'(n) \otimes U(\mathfrak{gl}(n))$, where $D'(n)$ is the algebra of polynomial differential operators on $\mathbb{C}[t_0^{\pm 1}, t_1, \ldots, t_n]$ generated by $t_i/t_0$, $t_0 \partial_j$, $i > 0$, $j \geq 0$. One of the main results of the paper is an explicit description of the image of the Capelli determinant $C_{n+1}(T)$ of $\mathfrak{gl}(n + 1)$ under $\rho$. It turns out that the result is especially pleasant: up to a shift, $\rho(C_{n+1}(T))$ is a
product of \( C_n(T) \) and a linear factor. This leads to explicit formulas for the images \( \rho(G_k^{\mathfrak{gl}(n+1)}) \) of the Gelfand generators \( G_k^{\mathfrak{gl}(n+1)} \) for \( \mathfrak{gl}(n+1) \) under \( \rho \). As a corollary, we also obtain the mixed tensor version of the noncommutative Cayley-Hamilton identities; see Corollary 5.3. To prove the formula for \( \rho(C_{n+1}(T)) \) we relate \( \rho \) to the Harish-Chandra isomorphisms of \( \mathfrak{gl}(n) \) and \( \mathfrak{gl}(n+1) \). We expect that the explicit formulas for \( \rho(C_{n+1}(T)) \) and \( \rho(G_k^{\mathfrak{gl}(n+1)}) \) to have applications in the representation theory of the tensor modules, as one can easily and explicitly compute central characters.

Another result of the paper is finding a pseudo left inverse \( \sigma \) of \( \rho \) in the following sense. The homomorphism \( \sigma \) maps \( D'(n) \otimes U(\mathfrak{gl}(n)) \) to an algebra \( U'' \) that contains \( U = U(\mathfrak{gl}(n+1)) \) as a subalgebra is such that \( \sigma(\rho(u)) - u \in G_1 U \) for every \( u \in U \), where \( G_1 = G_1^{\mathfrak{gl}(n+1)} = \sum_{i=0}^n E_{ii} \). In particular, we obtain that the kernel of \( \rho \) is the ideal \( (G_1) \) in \( U \), while the kernel of the \( \mathfrak{sl}(n+1) \) version of \( \rho \) is trivial.

It is worth noting that the results concerning the images of the generators of \( Z(\mathfrak{gl}(n+1)) \) under \( \rho \) can be obtained with long and direct computation. For this (alternative) approach we prove some more general identities that are included in the Appendix of this paper. We believe that these identities may be of independent interest.

The organization of the paper is as follows. In Section 3 we define the homomorphism \( \rho \). In Section 4 we relate \( \rho \) with the corresponding Harish-Chandra isomorphisms. The results of Section 4 are applied in the next section where the image of the Capelli-type determinant under \( \rho \) is computed. Using the latter results we derive the formulas for the images of the Gelfand generators under \( \rho \). In Section 6 we define a pseudo-left inverse of \( \rho \) and find the kernel of \( \rho \). The Appendix contains some explicit formulas for the images under \( \rho \) of a set of homogeneous elements of \( U(\mathfrak{gl}(n+1)) \). All the Gelfand generators for \( \mathfrak{gl}(n+1) \) can be written as sums of elements from this set, so we obtain an alternative proof of the results in Sections 4 and 5.

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2. **Notation and conventions**

Our base field is \( \mathbb{C} \). All vector spaces, tensor products, and associative algebras will be considered over \( \mathbb{C} \) unless otherwise specified. By \( \delta_{kl} \) we denote the Kronecker delta function, which equals 1 if \( k = l \) and 0 otherwise. Throughout the paper we fix a positive integer \( n \).

For a Lie algebra \( \mathfrak{a} \), by \( U(\mathfrak{a}) \) we denote the universal enveloping algebra of \( \mathfrak{a} \) and by \( Z(\mathfrak{a}) \) the center of \( U(\mathfrak{a}) \). By \( \mathfrak{gl}(N) \) (respectively, \( \mathfrak{sl}(N) \)) we denote the Lie algebra of all (respectively, traceless) \( N \times N \) matrices. We write \( I_N \) for the identity matrix of \( \mathfrak{gl}(N) \). For an \( N \times N \) matrix \( A \), the entries \( A_{ij} \) will be indexed by \( 1 \leq i, j \leq n \) if \( N = n \), and by \( 0 \leq i, j \leq n \) for \( N = n+1 \). In the latter case, we will refer to the top left entry as the \((0,0)\)th entry. Similarly, the weights of \( \mathfrak{gl}(n+1) \) will be written as \((n+1)\)-tuples \((\mu_0, \ldots, \mu_n)\), while those of \( \mathfrak{gl}(n) \) as \( n \)-tuples \((\mu_1, \ldots, \mu_n)\).

We will write \( E_{ij} \) for the \((i, j)\)th elementary matrix of \( \mathfrak{gl}(N) \), and \( E_N \) for the \( N \times N \) matrix whose \((i, j)\)th entry is \( E_{ij} \). Henceforth, we fix the Borel subalgebra \( \mathfrak{b}_N \) and the Cartan subalgebra \( \mathfrak{h}_N \) of \( \mathfrak{gl}(N) \) to be the ones spanned by \( E_{ij} \) \((i \leq j)\) and \( E_{kk} \) \((all k)\), respectively. By \( n_+^{\mathfrak{gl}(N)} \)
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(respectively, \( n_{\mathfrak{gl}(N)}^- \)) we denote the nilradical (respectively, the opposite nilradical) of \( \mathfrak{b}_N \). In particular, \( \mathfrak{b}_N = \mathfrak{h} \oplus n_{\mathfrak{gl}(N)}^+ \) and \( \mathfrak{gl}(N) = \mathfrak{b}_N \oplus n_{\mathfrak{gl}(N)}^- \). We will use these conventions both for \( N = n \) and \( N = n + 1 \).

By default, determinants will be column determinants. Namely, for a \( n \times n \) matrix \( A \) with entries \( A_{ij} \) in an associative algebra,

\[
\det(A) := \sum_{\sigma \in S_n} A_{\sigma(1)1}A_{\sigma(2)2} \cdots A_{\sigma(n)n},
\]

where \( S_n \) is the \( n \)th symmetric group. The determinant of an \((n+1) \times (n+1)\) matrix is defined analogously.

For a square matrix \( A \) and variable or a constant \( v \), the expression \( A + v \) (and \( v + A \)) should be understood as the sum of \( A \) and the scalar matrix of the same size as \( A \) having \( v \) on the diagonal.

By \( \mathcal{D}(n) \) we denote the algebra of polynomial differential operators on \( \mathbb{C}^n \). The algebra \( \mathcal{D}'(n) \) is the subalgebra of differential operators on \( \mathbb{C}[t_0^{\pm 1}, t_1, \ldots, t_n] \) generated by \( \frac{\partial}{\partial t_i} \) for \( i = 1, \ldots, n \) and \( t_0 \partial_j \) for \( j = 0, \ldots, n \). Here \( \partial_a \) stands for \( \frac{\partial}{\partial a} \). We set \( \mathcal{E} := \sum_{i=0}^n t_i \partial_i \).

We finish this section with some conventions that we will use throughout the paper. We assume that \( \sum_{i=r}^s x_i = 0 \) whenever \( r > s \). For a subset \( S \) of a ring \( R \), by \( (S) \) we denote the two-sided ideal of \( R \) generated by \( S \). For associative unital algebras \( A_1 \) and \( A_2 \) and elements \( a_i \in A_i \), we often write \( a_1 \) and \( a_2 \) for the elements \( a_1 \otimes 1 \) and \( 1 \otimes a_2 \), respectively.

3. The homomorphism \( \rho \)

In this section we define the homomorphism \( \rho : U(\mathfrak{gl}(n+1)) \to \mathcal{D}'(n) \otimes U(\mathfrak{gl}(n)) \) that plays important role in this paper. This homomorphism can be considered as the \( \mathfrak{gl}(n+1) \)-version of a homomorphism \( \rho_s : U(\mathfrak{sl}(n+1)) \to \mathcal{D}'(n) \otimes U(\mathfrak{gl}(n)) \), which can be defined as follows, in brief. For a finite-dimensional \( \mathfrak{gl}(n) \)-module \( V_0 \) consider the corresponding trivial vector bundle \( \mathcal{V}_0 \) on \( \mathbb{P}^n \). Then there is a natural map from \( \mathfrak{sl}(n+1) \) to the algebra of differential operators on the space of sections of \( \mathcal{V}_0 \) over the open subset \( U_0 = \{ t_0 \neq 0 \} \) of \( \mathbb{P}^n \). This map leads to the homomorphism \( \rho_s \). For details we refer the reader to, for example, §2 of [5]. The explicit formulas for \( \rho_s \) (given in local coordinates) are listed in the proof of [5, Lemma 2.3].

Introduce the following elements of \( \mathcal{D}'(n) \otimes U(\mathfrak{gl}(n)) \):

\[
R_1 = -\frac{1}{n+1} \left( \mathcal{E} \otimes 1 + 1 \otimes G_1^{\mathfrak{gl}(n)} \right), \quad R_2 = (\mathcal{E} + n) \otimes 1,
\]

where \( G_1^{\mathfrak{gl}(n)} = \sum_{i=1}^n E_{ii} \). Note that \( R_1 \) and \( R_2 \) are central in \( \mathcal{D}'(n) \otimes U(\mathfrak{gl}(n)) \).

We next introduce three natural homomorphisms: the natural embedding \( \iota_s : U(\mathfrak{sl}(n+1)) \to U(\mathfrak{gl}(n+1)) \), \( \pi_g : U(\mathfrak{gl}(n+1)) \to U(\mathfrak{sl}(n+1)) \), and \( \iota_g : U(\mathfrak{gl}(n)) \to U(\mathfrak{sl}(n+1)) \), as follows:

\[
\pi_g(B) = B - \frac{1}{n+1} \mathrm{tr}(B)I_{n+1}, \quad \iota_g(C) = C - \mathrm{tr}(C)E_{00}.
\]

Define \( \rho = \rho_s \pi_g \). The fact that \( \rho_s \) is a homomorphism and the explicit formulas for \( \rho_s \) imply the following.
Lemma 3.1. The following correspondence
\[
\begin{align*}
E_{ab} & \mapsto t_a \partial_b \otimes 1 + 1 \otimes E_{ab} + \delta_{ab} R_1 \text{ for } a, b > 0, \\
E_{a0} & \mapsto t_a \partial_0 \otimes 1 - \sum_{i > 0} \frac{t_i}{t_0} \otimes E_{ai} \text{ for } a > 0, \\
E_{0b} & \mapsto t_0 \partial_b \otimes 1 \text{ for } b > 0, \\
E_{00} & \mapsto t_0 \partial_0 \otimes 1 + R_1.
\end{align*}
\]

extends to the homomorphism \( \rho : U(\mathfrak{gl}(n + 1)) \to \mathcal{D}'(n) \otimes U(\mathfrak{gl}(n)) \) of associative unital algebras.

We note that \( \rho(G_1^{gl(n+1)}) = 0 \) and \( \rho \mu_s = \rho_s \). Let us also define
\[
\gamma : U(\mathfrak{sl}(n + 1)) \to \mathcal{D}'(n) \otimes U(\mathfrak{sl}(n + 1))
\]
by the identity \( \gamma = (1 \otimes \nu_g) \rho_s \). We finish this section by collecting the identities for the homomorphisms that we introduced in this section.

Proposition 3.2. We have \( \gamma = (1 \otimes \nu_g) \rho_s \), \( \pi_g \nu_s = \text{Id} \), and \( \rho_s \pi_g = \rho \), and all other relations that directly follow from these three; in that sense, the following diagram is commutative.

4. Images under Harish-Chandra isomorphisms

In this section we relate the restriction of \( \rho \) on the center \( Z(\mathfrak{gl}(n + 1)) \) of \( U(\mathfrak{gl}(n + 1)) \) with the Harish-Chandra isomorphisms.

We first recall the definition of the Harish-Chandra isomorphism and define an isomorphism of Harish-Chandra type with domain \( \mathbb{C}[\mathcal{E}] \otimes Z(\mathfrak{gl}(n)) \).

For a weight \( \lambda = (\lambda_1, \ldots, \lambda_n) \) of \( \mathfrak{gl}(n) \) denote by \( M_n(\lambda) \) and \( L_n(\lambda) \) the Verma module of highest weight \( \lambda \) and its unique simple quotient, respectively. We similarly define the \( \mathfrak{gl}(n+1) \)-modules \( M_{n+1}(\mu) \) and \( L_{n+1}(\mu) \) for a weight \( \mu = (\mu_0, \mu_1, \ldots, \mu_n) \) of \( \mathfrak{gl}(n + 1) \). In particular, if \( \lambda = (\lambda_1, \ldots, \lambda_n) \), then there is a (highest weight) vector \( v_0 \) of \( M_n(\lambda) \) such that \( M_n(\lambda) = U(\mathfrak{n}^-_{\mathfrak{gl}(n)}) \otimes \mathbb{C} v_0 \) as vector spaces, \( E_{ab} v_0 = 0 \) for \( 1 \leq a < b \leq n \), and \( E_i v_0 = \lambda_i v_0 \) for \( i = 1, \ldots, n \).

Henceforth we set \( \delta_N = (0, -1, \ldots, -N + 1) \). If \( (b_1, \ldots, b_N) \in \mathbb{C}^N \), then the evaluation homomorphism \( ev_{b_1, \ldots, b_N} : \mathbb{C}[x_1, \ldots, x_N] \to \mathbb{C} \) is defined by \( ev_{b_1, \ldots, b_N}(p) = p(b_1, \ldots, b_N) \). Every \( z' \in Z(\mathfrak{gl}(n)) \) acts on \( L(\lambda) \) as \( \chi_\lambda(z') \text{Id} \), where \( \chi_\lambda(z') = ev_{\lambda + \delta_n}(\chi_\lambda(z')) \) and \( \chi_\lambda : Z(\mathfrak{gl}(n)) \to \mathbb{C}[\ell_1, \ldots, \ell_n]^{S_n} \) is the Harish-Chandra isomorphism. We similarly define \( \chi_{n+1} : Z(\mathfrak{gl}(n+1)) \to \mathbb{C}[\ell_0, \ldots, \ell_n]^{S_{n+1}} \) using the action of any element of \( z \in Z(\mathfrak{gl}(n+1)) \) on a simple highest weight module of \( \mathfrak{gl}(n+1) \).

Next, define \( \chi_{0,n} : \mathbb{C}[\mathcal{E}] \otimes Z(\mathfrak{gl}(n)) \to \mathbb{C}[\ell_0] \otimes \mathbb{C}[\ell_1, \ldots, \ell_n]^{S_n} \) by \( \sum_i \ell_i \otimes \chi_n(z_i) \). Note that \( \chi_{0,n} \) is an isomorphism and that \( \ell = \chi_{0,n}(R_1) \), where \( \ell := -\frac{1}{n+1} \left( \ell_0 + \sum_{i=1}^n \ell_i + \frac{n(n-1)}{2} \right) \). The latter follows from the fact that \( \chi_n(G_1^{gl(n)}) = \sum_{i=1}^n \ell_i + \frac{n(n-1)}{2} \) (see Example 7.3.4 in [9]).
Recall that a weight $\mu$ of $\mathfrak{gl}(n+1)$ is antidominant if $\mu_i - \mu_{i+1} \notin \mathbb{Z}_{\geq 0}$ for all $i = 0, \ldots, n$. A well-known fact is that $M_{n+1}(\mu)$ is simple if and only if $\mu$ is antidominant. Also, by a Theorem of Duflo, the annihilator of a simple Verma module $M_{n+1}(\mu)$ is generated by $z - \chi_\mu(z)$, $z \in Z(\mathfrak{gl}(n+1))$; see for example §8.4.3 in [3].

Let
\[ F_a = \operatorname{Span}\{t_0^{a-k_1-k_2-k_3}t_1^{k_1} \cdots t_n^{k_n} | k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}\} \]
and consider $F_a$ as a $D'(n)$-module. Note that $F_a = D'(n)(t_0^a)$ and that $\mathcal{E} = \text{Id}$ on $F_a$.

**Lemma 4.1.** Let $\mathcal{A}W_n$ denote the set of all antidiagonal $\mathfrak{gl}(n)$-weights $\lambda$ such that $\lambda_1 \notin \mathbb{Z}$. Then the modules $\bigoplus_{a \in \mathbb{Z}} F_a$ and $\bigoplus_{\lambda \in \mathcal{A}W_n} M_n(\lambda)$ are faithful over $D'(n)$ and $U(\mathfrak{gl}(n))$, respectively.

**Proof.** The fact that $\bigoplus_{\lambda \in \mathcal{A}W_n} M_n(\lambda)$ is faithful follows from the Theorem of Duflo and the fact that the annihilator of the direct sum of modules is the intersection of their annihilators.

We now prove that the annihilator of $\bigoplus_{a \in \mathbb{Z}} F_a$ is trivial. Suppose for the sake of contradiction that $x \in D'(n)$ annihilates $\bigoplus_{a \in \mathbb{Z}} F_a$. Next, choose a monomial $x_m = t_0^{a_0} \cdots t_n^{a_n} \partial_0^{b_0} \cdots \partial_n^{b_n}$ in the sum expansion of $x$ that is $\partial$-lexicographically maximal (i.e., relative to the $\partial_0$-degree, $\partial_1$-degree, \ldots, $\partial_n$-degree) among all monomials in $x$. Then, as $a_0, \ldots, a_n$ vary in a suitable set, the coefficient of $t_0^{a_0+a_0-b_0} \cdots t_n^{a_n+a_n-b_n}$ in $x(t_0^{a_0} \cdots t_n^{a_n})$ is equal to some polynomial in $e_0, \ldots, e_n$. Furthermore, the lexicomaximal property of $t_0^{a_0} \cdots t_n^{a_n} \partial_0^{b_0} \cdots \partial_n^{b_n}$ yields that the coefficient of $e_0^{b_0} \cdots e_n^{b_n}$ in this polynomial is nonzero. Thus this polynomial is nonzero, and for suitable $e_0, \ldots, e_n$ it evaluates to a nonzero number, contradicting the fact that $x$ annihilates $D'(n)$.

**Theorem 4.2.** Let $\tau$ be the endomorphism on $\mathbb{C}[\ell_0, \ell_1, \ldots, \ell_n]$ defined by $\tau(p(\ell_0, \ell_1, \ldots, \ell_n)) = p(\ell_0 + \ell, \ell_1 + \ell - 1, \ldots, \ell_n + \ell - 1)$. Then we have that
\[ \rho|_{Z(\mathfrak{gl}(n+1))} = \chi_{0,n}^{-1} \tau \chi_{n+1}. \]

In particular, $\rho(Z(\mathfrak{gl}(n+1)))$ is a subalgebra of $\mathbb{C}[\mathcal{E}] \otimes Z(\mathfrak{gl}(n))$, and the following diagram is commutative:
\[
\begin{array}{ccc}
Z(\mathfrak{gl}(n+1)) & \xrightarrow{\rho} & \mathbb{C}[\mathcal{E}] \otimes Z(\mathfrak{gl}(n)) \\
\chi_{n+1} \downarrow & & \downarrow \chi_{0,n} \\
\mathbb{C}[\ell_0, \ell_1, \ldots, \ell_n]^{S_{n+1}} & \xrightarrow{\tau} & \mathbb{C}[\ell_0] \otimes \mathbb{C}[\ell_1, \ldots, \ell_n]^{S_n}
\end{array}
\]

**Proof.** Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be in $\mathcal{A}W_n$ and $a \in \mathbb{Z}$. Also, let $M(a, \lambda) = F_a \otimes M_n(\lambda)$. We first note that, in order to prove the identity in the theorem, it is sufficient to check that $\rho(z) = \chi_{0,n}^{-1} \tau \chi_{n+1}(z)$ for all $z \in Z(\mathfrak{gl}(n+1))$, as an identity of endomorphisms of $M(a, \lambda)$. Indeed, by Lemma 4.1 and by the fact that the tensor product of faithful modules is a faithful module (see, for example, [1]), the module $\bigoplus_{a \in \mathbb{Z}, \lambda \in \mathcal{A}W_n} M(a, \lambda)$ is a faithful module over $D'(n) \otimes U(\mathfrak{gl}(n))$.

We next observe that if $M(a, \lambda)$ is considered as a $\mathfrak{gl}(n+1)$-module through $\rho$, then $M(a, \lambda) \cong M_{n+1}(\lambda)$, where $\lambda = (a + r_1, \lambda_1 + r_1, \ldots, \lambda_n + r_1)$ and $r_1 = -\frac{1}{n+1}(a + \sum_{i=1}^n \lambda_i)$. To prove this, let us fix a highest weight vector $v_\lambda$ of $M_n(\lambda)$. Then it is straightforward to check that $E_{ab}(t_0^a \otimes v_\lambda) = 0$ for $a < b$ and that the weight of $t_0^a \otimes v_\lambda$ is $\lambda$. On the other hand, since $a \in \mathbb{Z}$ and $\lambda \in \mathcal{A}W_n$, $\lambda$ is anti-dominant (since we know $\lambda_1 \notin \mathbb{Z}$). Hence, $M_{n+1}(\lambda)$ is simple. To conclude the proof of $M(a, \lambda) \cong M_{n+1}(\lambda)$ we show that both modules have the same formal characters.
Indeed, observe that for a monomial \( u \in U(n_{\mathfrak{gl}(n)}) \), the vector \( t_0^n \cdot (t_1/t_0)^{k_1} \cdots (t_n/t_0)^{k_n} \otimes u \) in \( M(a, \lambda) \), and the vector \( E_{i_1}^{k_1} \cdots E_{i_n}^{k_n} \otimes u \) in \( M_{n+1}(\lambda) \), have the same weights. Therefore,

\[
\text{ch} (\mathcal{F}_a \otimes M_n(\lambda)) = \text{ch} (t_0^n \cdot [t_1/t_0, \ldots, t_n/t_0] \otimes U(n_{\mathfrak{gl}(n)})v_{\lambda}) = \text{ch} (U(n_{\mathfrak{gl}(n+1)})v_{\lambda}).
\]

Since \( M(a, \lambda) \cong M_{n+1}(\lambda) \), we have that every \( z \in Z(\mathfrak{gl}(n+1)) \) acts on \( M(a, \lambda) \) as \( \chi(z) \text{Id} \). Recall that by definition,

\[
\chi_{\lambda}(z') = \text{ev}_{\lambda+\delta_{\alpha}}(\chi_{\lambda}(z')) \quad \chi_{\lambda}(z) = \text{ev}_{\lambda+\delta_{\alpha+1}}(\chi_{\lambda+1}(z)).
\]

for every \( z' \in Z(\mathfrak{gl}(n)) \) and \( z \in Z(\mathfrak{gl}(n+1)) \).

On the other hand, if \( \xi = \chi_{\lambda}^{\alpha} T_{\lambda n+1} \), then \( \xi(z) \) acts on \( M(a, \lambda) \) as \( \chi_{a, \lambda}(\xi(z)) \text{Id} \), where \( \chi_{a, \lambda} = \sum_{i} \mathcal{E}_{i}^{\alpha} \otimes \chi_{\lambda}(z'_i) \), for \( z'_i \in Z(\mathfrak{gl}(n)) \). Let \( p = \chi_{\lambda+1}(z) \). It remains to prove that \( \chi_{a, \lambda}^{\alpha} T_{\lambda n+1} \) is \( \text{ev}_{\lambda+\delta_{\alpha+1}}(\chi_{\lambda+1}(z')) \) for every \( z' \in Z(\mathfrak{gl}(n)) \) and that \( \text{ev}_{a, \lambda+\delta_{\alpha}} \tau = \text{ev}_{\lambda+\delta_{\alpha+1}} \).

### 5. Capelli-type determinants

For any formal variable \( T \), we define the Capelli determinant of \( \mathfrak{gl}(n+1) \) and \( \mathfrak{gl}(n) \) as follows:

\[
C_{n+1}(T) = \det \begin{bmatrix}
E_{00} - T & E_{01} & \cdots & E_{0n} \\
E_{10} & E_{11} - T - 1 & \cdots & E_{1n} \\
\vdots & \vdots & \ddots & \vdots \\
E_{n0} & E_{n1} & \cdots & E_{nn} - T - n
\end{bmatrix}
\]

and

\[
C_n(T) = \det \begin{bmatrix}
E_{11} - T & E_{12} & \cdots & E_{1n} \\
E_{21} & E_{22} - T - 1 & \cdots & E_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
E_{n1} & E_{n2} & \cdots & E_{nn} - T - n + 1
\end{bmatrix},
\]

respectively. Note that \( T \) is a formal variable that commutes with all \( E_{ij} \), and \( C_{n}(T) \) and \( C_{n+1}(T) \) will be treated as polynomials in this formal variable. We also note that the polynomials \( C_{n+1}(T) \) appear in [9] and [11] with a slight change. Namely, the polynomial \( C_{n+1}^{M}(T) \) defined in §7.1 of [9] (also called Capelli determinant) is related to our \( C_{n+1}(T) \) via the identity \( C_{n+1}(T) = C_{n+1}^{M}(-T) \). On the other hand, the polynomial \( C_{n+1}^{U}(T) \) defined in [11] satisfies the relation \( C_{n+1}(T) = C_{n+1}^{U}(T + n) \).

Define \( C_{\rho}(T) \) via the identity \( C_{\rho}(\rho(T)) = \rho(C_{n+1}(T)) \). For convenience we will write \( T \) for \( \rho(T) \).

**Theorem 5.1.** The following identity holds:

\[
C_{\rho}(T + R_1) = (\mathcal{E} - T)C_{n}(T + 1).
\]

In particular, \( C_{\rho}(\mathcal{E} + R_1) = 0 \).

**Proof.** The identity is equivalent to the following:

\[
\rho(C_{n+1}^{M}(T)) = (\mathcal{E} + T + R_1)C_{n}^{M}(T + R_1 - 1).
\]
To prove the latter we use that
\[ \chi_n(C_n^M(T)) = (T + \ell_1) \ldots (T + \ell_n). \]
(Theorem 7.1.1 in [9] and the corresponding formula for \( \chi_{n+1}(C_{n+1}^M(T)) \)). To complete the proof we use Theorem [4.2] and that \( \chi_{0,n}(R_1) = \ell. \)

Example 5.2. In the case \( n = 2 \), the identity in Theorem 7.1 is the following:

\[
\det \begin{bmatrix}
    t_0 \partial_0 - T & t_0 \partial_1 & t_0 \partial_2 \\
    t_1 \partial_0 - \frac{t_0}{t_0} E_{11} - \frac{t_0}{t_0} E_{12} & t_1 \partial_1 + E_{11} - T - 1 & t_1 \partial_2 + E_{12} \\
    t_2 \partial_0 - \frac{t_0}{t_0} E_{21} - \frac{t_0}{t_0} E_{22} & t_2 \partial_1 + E_{21} & t_2 \partial_2 + E_{22} - T - 2
\end{bmatrix}
\]

\[= (E - T)C_2(T + 1) = (E - T) \det \begin{bmatrix}
    E_{11} - T - 1 & E_{12} \\
    E_{21} & E_{22} - T - 2
\end{bmatrix}.\]

Corollary 5.3. The following identities hold:
\[ C_\rho(E_n + R_1 - n) = 0, \quad C_\rho(E_n^t - 1 + R_1) = 0 \]

Proof. The identities follow from Theorem 5.1 and the noncommutative version of the Cayley-Hamilton theorem:
\[ C_n^M(-E_n + n - 1) = C_n^M(-E_n^t) = 0. \]
(This is Theorem 7.2.1 in [9].) \( \square \)

We conclude this section by giving explicit formulas of the images of the Gelfand invariants of \( Z(\mathfrak{gl}(n + 1)) \) under \( \rho \). To define these invariants we first introduce some special elements in \( U(\mathfrak{gl}(n + 1)) \). Set \( r_k^{\mathfrak{gl}(n+1)}(a, b) = \delta_{ab} \), and let
\[(1) \quad r_k^{\mathfrak{gl}(n+1)}(a, b) = \sum_{i_1, \ldots, i_k} E_{a_{i_1}i_1} E_{i_1i_2} \ldots E_{i_ki_k} \]
where the sum runs over all (not necessarily distinct) \( 0 \leq i_1, \ldots, i_k \leq n \). Then
\[ G_k^{\mathfrak{gl}(n+1)} = \sum_{i=0}^n r_k^{\mathfrak{gl}(n+1)}(i, i) \]
is the Gelfand invariant of degree \( k \) of \( \mathfrak{gl}(n + 1) \). In other words, \( G_k^{\mathfrak{gl}(n+1)} = \text{tr}(E_{n+1}^k) \). We define \( r_k^{\mathfrak{gl}(n)}(a, b) \) and \( G_k^{\mathfrak{gl}(n)} \) for \( \mathfrak{gl}(n) \) analogously. It is well-known fact that \( Z(\mathfrak{gl}(n + 1)) \) is a polynomial algebra in \( G_k^{\mathfrak{gl}(n+1)} \) for \( k = 1, 2, \ldots, n + 1 \).

Theorem 5.4. The following formula holds for all positive integers \( k \):
\[
\rho(G_k^{\mathfrak{gl}(n+1)}) = \left( \sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g} \right) (E \otimes 1) + (n + 1) R_k^1 + \sum_{g=0}^{k-1} \binom{k}{g} R_1^g \left( 1 \otimes G_k^{\mathfrak{gl}(n)} \right)
\]
\[ - \sum_{m=2}^{k} \left( \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \right) \left( 1 \otimes G_m^{\mathfrak{gl}(n)} \right). \]
Example 5.5. Theorem 5.4 applied for $k = 3$ gives the following formula:

$$
\rho(G_{3}^{(n+1)}) = \left( R_{2}^{2} + 3R_{1}R_{2} + 3R_{1}^{2} \right)(E \otimes 1) + (n + 1)R_{1}^{4} + (1 \otimes G_{3}^{(n)}) + 3R_{1}(1 \otimes G_{2}^{(n)}) + 3R_{1}(1 \otimes G_{1}^{(n)}) - (1 \otimes G_{2}^{(n)}) - (R_{2} + 3R_{1})(1 \otimes G_{1}^{(n)}).
$$

6. PSEUDO LEFT INVERSE OF $\rho$

In this subsection we prove that the kernel of $\rho : U(\mathfrak{gl}(n + 1)) \to \mathcal{D}'(n) \otimes U(\mathfrak{gl}(n))$ is $(G_{1})$ and define a family of homomorphisms $\sigma$, such that $\sigma(\rho(t)) - t \in G_{1}U$ for all $t \in U$. Here and henceforth $U = U(\mathfrak{gl}(n + 1))$ and $G_{1} = G_{1}^{(n+1)} = \sum_{i=0}^{n} E_{ii}$. The map $\sigma$ is pseudo left inverse of $\rho$ in the sense that $\sigma\rho = \text{Id} \mod (G_{1})$.

We first define the domain of $\sigma$. From now on, we set for simplicity $U = U(\mathfrak{gl}(n + 1))$. Let $U'$ be the extension of $U$ defined by $U' = U\langle X \rangle / \langle [U, X], C_{n+1}(X) \rangle$. Equivalently, we define $U'$ by considering a trivial central extension $\mathfrak{g}_{n+1}^{X} = \mathfrak{gl}(n+1) \oplus \mathbb{C}X$ of $\mathfrak{gl}(n+1)$, and letting $U' = U(\mathfrak{g}_{n+1}^{X}) / (C_{n+1}(X))$. The following lemma is standard.

Lemma 6.1. Every element in $U'$ can be written uniquely in the form $u = \sum_{i=0}^{n} w_{i}X^{i}$ for some $w_{i} \in U$. In particular, $C_{n}(X)$ and $C_{n}(X + 1)$ are nonzero in $U'$.

Lemma 6.2. $U'$ is a domain.

Proof. Let $U[X] = U(\mathfrak{g}_{n+1}^{X})$. To prove that the ring $U' = U[X] / (C_{n+1}(X))$ is a domain, it is enough to show that the associated graded ring $\text{gr}U'$ is a domain. Let $I = (C_{n+1}(X))$. Then we have that $\text{gr}U' = \text{gr}U[X] / \text{gr}I$ (see for example, §2.3.10 in [3]). Since $\text{gr}U[X]$ is a polynomial ring, and hence, a unique factorization domain, it is enough to show that the polynomial $C_{n+1}(-X)$ is irreducible in $\text{gr}U[X]$.

Let $C_{n+1}(-X) = X^{n+1} + c_{1}X^{n} + \cdots + c_{n+1}$. We know that the center $Z(\mathfrak{gl}(n + 1))$ of $U(\mathfrak{gl}(n + 1))$ is the polynomial algebra $\mathbb{C}[c_{1}, \ldots, c_{n+1}]$. Assume that $C_{n+1}(-X) = (X^{k} + \cdots +
\(a_{k-1}X + a_k)(X^m + \ldots + b_{m-1}X + b_m)\) for some \(a_i, b_i\) in \(\text{gr}\ U\) of graded degree \(i\). In particular, \(a_kb_m = c_{n+1}\). Note that \(c_{n+1}\) is the determinant of \(E\) and hence the ideal generated by \(c_{n+1}\) in \(\text{gr}\ U\) is a determinantal ideal. It is well-known that any such ideal is a prime ideal (equivalently, that \(\text{gr}\ U/(c_{n+1})\) is an irreducible variety); see, for example, Proposition 1.1 in [2]. Therefore, \(c_{n+1}\) is irreducible in \(\text{gr}\ U\). We thus may assume that \(a_k\) is a constant, and that \(b_m\) is a constant multiple of \(c_{n+1}\). Then \(k = 0\) and \(m = n + 1\), and hence, \(C_{n+1}(-X)\) is irreducible. \(\square\)

Since \(U'\) is a quotient of the universal enveloping algebra of \(g_{n+1}^X\), \(U'\) is a right (and left) Noetherian domain. Then by a theorem of Goldie, [4], \(U'\) is also a right Ore domain. Let \(U''\) denote the right quotient ring of \(U'\), i.e. the right Ore localization of \(U'\) relative to all nonzero elements of \(U''\). We have natural embeddings \(\iota : U \rightarrow U'\) and \(\iota' : U' \rightarrow U''\) that will allow us to consider the elements of \(U\) as elements of \(U'\), and those of \(U'\) as elements of \(U''\). Denote by \(Y\) the left and right inverse of \(C_n(X)\) in \(U''\), i.e. \(YC_n(X) = C_n(X)Y = 1\).

**Lemma 6.3.** \(Y\) commutes with \(X\) and \(E_{ij}\) whenever \(i, j > 0\).

**Proof.** To prove the result we observe that \(Y\) commutes with all elements that commute with \(C_n(X)\). \(\square\)

In order to define the homomorphism \(\sigma : D'(n) \otimes U(\mathfrak{gl}(n)) \rightarrow U''\), we next introduce some distinguished elements of \(U''\). We treat again \(T\) as a formal variable that commute with all \(E_{ij}\). We first define the \(n \times n\) matrix \(M_n(T)\), whose \((i, j)\)th entry is

\[
M_n(T)_{ij} = \det \begin{bmatrix}
E_{11} - T - 1 & E_{12} & \cdots & 0 & \cdots & E_{1n}
E_{21} & E_{22} - T - 2 & \cdots & 0 & \cdots & E_{2n}
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots
E_{i1} & E_{i2} & \cdots & 1 & \cdots & E_{in}
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots
E_{n1} & E_{n2} & \cdots & 0 & \cdots & E_{nn} - T - n + 1
\end{bmatrix}
\]

where the entry of 1 in the determinant above is the \((i, j)\)th entry, while the \((k, k)\)th entry equals \(E_{kk} - T - k\) for \(0 < k < j\) and \(E_{kk} - T - k + 1\) for \(k > j\).

Similarly, we define

\[
M_{n+1}(T)_{ij} = \det \begin{bmatrix}
E_{00} - T - 1 & E_{01} & \cdots & 0 & \cdots & E_{0n}
E_{10} & E_{11} - T - 2 & \cdots & 0 & \cdots & E_{1n}
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots
E_{i0} & E_{i1} & \cdots & 1 & \cdots & E_{in}
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots
E_{n0} & E_{n1} & \cdots & 0 & \cdots & E_{nn} - T - n
\end{bmatrix}
\]

The next lemma is standard but for reader’s convenience we include a short proof. We will use the result both for \(\mathfrak{gl}(n)\) and \(\mathfrak{gl}(n+1)\).

**Lemma 6.4.** Let \(V\) and \(W\) be \(n \times n\) matrices with entries in \(U(\mathfrak{gl}(n))[T]\) and let \(P\) be a nonzero element of \(Z(\mathfrak{gl}(n))[T]\) such that \(VW = PI_n\). Then \(WV = PI_n\).

**Proof.** Since \(P\) is central in \(U(\mathfrak{gl}(n))[T]\), we can localize \(U(\mathfrak{gl}(n))[T]\) relative to its multiplicative subset generated by \(P\). Denote by \(L\) the corresponding localized ring. Let \(V' = P^{-1}V\). We
Lemma 6.5. The following identities hold:
\[(E_n^{t-T})M_n(T) = M_n(T)(E_n^{t-T}) = C_n(T); \quad M_n(T)^t(E_n^{t-T}) = (E_n-T-n)M_n(T)^t = C_n(T+1).\]
Also, \[(E_{n+1}^{t-T})M_{n+1}(T) = C_{n+1}(T), \text{ etc.}\]

Proof. First we show \[(E_n^{t-T})M_n(T) = C_n(T), \text{ or, equivalently, } \sum_k (E_{ki} - \delta_{ki}T)M_n(T)_{kj} = \delta_{ij}C_n(T).\] We adopt the reasoning used in Section 1 of [11]. For this we work over the algebra \(\Lambda_n \otimes U(\mathfrak{gl}(n)),\) where \(\Lambda_n\) is the exterior algebra with generators \(e_1, \ldots, e_n.\) Let \(F_m = \sum_l E_{lm}e_l.\) Then
\[M_n(T)_{kj}e_1 \ldots e_n = (F_1 - (T + 1)e_1) \ldots (F_n - (T + n - 1)e_n),\]
where \(e_k\) is the \(j\)th term in the product. We have that
\[\sum_k (E_{ki} - \delta_{ki}T)M_n(T)_{kj}e_1 \ldots e_n = (-1)^{j-1} \left( \sum_k (E_{ki} - \delta_{ki}T)e_k \right) (F_1 - (T + 1)e_1) \ldots (F_j - (T + j - 1)e_j) \ldots (F_n - (T + n - 1)e_n)
= (-1)^{j-1}(F_1 - Te_1)(F_1 - (T + 1)e_1) \ldots (F_{j-1} - (T + j - 1)e_{j-1})
(F_{j+1} - (T + j)e_j) \ldots (F_n - (T + n - 1)e_n).
We next prove an identity analogous to Lemma 1 in [11]. Specifically, we claim that
\[(F_r - (T + l)e_r)(F_s - (T + l + 1)e_s) = -(F_s - (T + l)e_s)(F_r - (T + l + 1)e_r),\]
or, equivalently,
\[\sum_{p,q} (E_{pr}e_p - (T + l)e_r)(E_{qs}e_q - (T + l + 1)e_s) + \sum_{p,q} (E_{qs}e_q - (T + l)e_s)(E_{pr}e_p - (T + l + 1)e_r) = 0.\]
The left hand side equals
\[\sum_{p,q} [E_{pr}, E_{qs}]e_p e_q + \sum_{q} E_{qs}e_r e_q + \sum_{p} E_{pr}e_s e_p
= \sum_{p,q} \delta_{qr} E_{ps} e_p e_q - \sum_{p,q} \delta_{ps} E_{qr} e_p e_q + \sum_{q} E_{qs}e_r e_q + \sum_{p} E_{pr}e_s e_p
= \left( \sum_{p} E_{ps} e_p e_r + \sum_{q} E_{qs} e_r e_q \right) + \left( \sum_{p} E_{pr}e_s e_p - \sum_{q} E_{qr}e_s e_q \right) = 0,
\]
as claimed. Now, applying (3) repeatedly, we obtain
\[\sum_k (E_{ki} - \delta_{ki}T)M_n(T)_{kj}e_1 \ldots e_n = (F_1 - Te_1) \ldots (F_i - (T + j - 1)e_i) \ldots (F_n - (T + n - 1)e_n),\]
where the \(F_i - (T + j - 1)e_i\) is the \(j\)th term. If \(j = i,\) this yields the desired. Otherwise, note that if \(r = s\) in (3) then \((F_r - (T + l)e_r)(F_r - (T + l + 1)e_r) = 0.\) Applying (3) repeatedly and
Proof. We have (6.3, \( Y_i < j \)) verify that \( q \) is defined.

We need to show (6.4, \( M_n(X) = C_n(X) \)) for \( i \) and \( j \) such that \( M_n(X) = C_n(X) \) equals 0. In particular, \( qM_n(X)0_0 = 0 \). This implies \( q = 0 \) because \( M_n(X)0_0 = C_n(X + 1) \), because \( C_n(X + 1) \neq 0 \) from Lemma 6.5, and because \( U^n \) is a domain. This completes the proof of the Lemma. 

Lemma 6.7. If \( u_0, v_1, \ldots, v_n \) are such that \( \sum_{a=0}^{n} v_a E_{ia} = v_i X \) for \( i = 0, 1, \ldots, n \), then \( v_i = v_0 u_i \) for \( i = 0, 1, \ldots, n \).

Proof. We need to show \( [v_1, \ldots, v_n] = v_0 [u_1, \ldots, u_n] \). The identities given in the statement imply that \( [v_1, \ldots, v_n] (E_{1i} - X) = -[E_{10}, E_{20}, \ldots, E_{n0}] M_n(X) Y (E_{1i} - X) \). We multiply this matrix identity by \( M_n(X) Y \) on the right. Then, using the definition of \( u_1, \ldots, u_n \), Lemma 6.5, and the identity \( C_n(X) Y = 1 \), we obtain \( [v_1, \ldots, v_n] = v_0 [u_1, \ldots, u_n] \) as needed.

Lemma 6.8. \( [u_i, E_{jk}] = \delta_{ik}u_ju_i - \delta_{ij}u_j \) for \( i, j, k \) from 0 to \( n \).

Proof. Fix \( j \) and \( k \) and set \( v_i = [u_i, E_{jk}] + \delta_{ij}u_j \) for \( i = 0, 1, \ldots, n \). We will show that \( v_0, \ldots, v_n \) satisfy the hypothesis of Lemma 6.7. Note first that \( v_0 = [u_0, E_{jk}] + \delta_{k0}u_j = \delta_{k0}u_j \) as \( u_0 = 1 \). Now, for \( i = 0, 1, \ldots, n \), by Lemma 6.6, we have that

\[
\sum_{a=0}^{n} [u_a, E_{jk}] E_{ia} + \sum_{a=0}^{n} u_a (\delta_{aj}E_{ik} - \delta_{ji}E_{ja}) = \sum_{a=0}^{n} ([u_a, E_{jk}] E_{ia} + u_a [E_{ia}, E_{jk}])
\]

\[= \sum_{a=0}^{n} [u_a E_{ia}, E_{jk}] = \sum_{a=0}^{n} u_a E_{ia}, E_{jk}] = [u_i, E_{jk}] X.
\]
On the other hand,
\[
\sum_{a=0}^{n} u_a (\delta_{aj} E_{ik} - \delta_{ki} E_{ja}) = \sum_{a=0}^{n} u_a \delta_{aj} E_{ik} - \sum_{a=0}^{n} u_a \delta_{ki} E_{ja} = u_j E_{ik} - \delta_{ki} u_j X
\]
by Lemma 6.6. Thus \((\sum_{a=0}^{n} [u_a, E_{jk}] E_{ia}) + u_j E_{ik} - \delta_{ki} u_j X = [u_i, E_{jk}] X. Then
\[
\sum_{a=0}^{n} v_a E_{ia} = \left( \sum_{a=0}^{n} [u_a, E_{jk}] E_{ia} \right) + u_j E_{ik} = [u_i, E_{jk}] X + \delta_{ki} u_j X = v_i X.
\]

Lemma 6.7 implies that \(v_i = v_0 u_i = \delta_{k0} u_j u_i. Thus \([u_i, E_{jk}] = v_i - \delta_{ki} u_j = \delta_{k0} u_j u_i - \delta_{ki} u_j, as claimed. \)

**Lemma 6.9.** \([u_i, u_j] = 0 \) for \(0 \leq i, j \leq n.\)

**Proof.** Fix \(j\) and let \(v_i = [u_i, u_j]. In particular, v_0 = 0. Lemmas 6.6 and 6.8 imply that \(\sum_{a=0}^{n} v_a E_{ia} = v_i (X + 1) \) for all \(i. Since v_0 = 0, we can write the last set of identities in the following matrix form: \([v_1, \ldots, v_n] (E^t_n - (X + 1)) = [0, \ldots, 0]. After multiplying by \(M_n(X + 1)\) on the right and using Lemma 6.5, we obtain \(v_i C_n(X + 1) = 0. Since C_n(X + 1) \neq 0 \) (by Lemma 6.1) and \(U'' \) is a domain, \(v_i = 0 \) for all \(i = 1, \ldots, n. Thus we have the desired. \)

**Lemma 6.10.** The correspondence \(w \mapsto \pi_g(w), w \in U, X \mapsto X - \frac{1}{n+1} G_1 \) extends to a homomorphism \(\pi'_g : U' \to U'\) of associative unital algebras. Furthermore, \(\ker \pi'_g = (G_1) \) in \(U'.\)

**Proof.** We can see that this correspondence yields a well-defined homomorphism \(U(\mathfrak{g}^X_{n+1}) \to U(\mathfrak{g}^X_{n+1})\). To see that it yields a well-defined map \(U' \to U'\), note that the determinant definition of \(C_n(X)\) easily implies the identity \(\pi'_g(C_n(T)) = C_n(T) + \frac{1}{n+1} G_1\), so we will have \(\pi'_g(C_{n+1}(X)) = C_{n+1}(X)\); thus we have a well-defined homomorphism \(\pi'_g : U' \to U'\). For the kernel of \(\pi'_g\), we use that \((\sum_{i=0}^{n} w_i X^i) - \pi'_g(\sum_{i=0}^{n} w_i X^i) \in (G_1)\) for every \(\sum_{i=0}^{n} w_i X^i \in U'. \)

Let \(U'_s\) be the image of \(\pi'_s\). By Lemma 6.10 \(U'_s \simeq U'/(G_1).\) We have a natural embedding \(\iota'_s : U'_s \to U'\) such that \(\pi'_g \iota'_s = \text{Id}.\)

Recall \(C_\rho(\mathcal{E} + R_1) = 0\) by Theorem 5.1. Thus we may define a homomorphism \(\rho' : U' \to \mathcal{D}'(n) \otimes U(\mathfrak{gl}(n))\) of associative unital algebras by the identities \(\rho'(w) = \rho(w)\) for all \(w \in U(\mathfrak{gl}(n + 1))\) and \(\rho'(X) = \mathcal{E} + R_1\). Also, define \(\rho'_s : U'_s \to \mathcal{D}'(n) \otimes U(\mathfrak{gl}(n))\) by \(\rho'_s = \rho' \iota'_s. One hence has the following diagram.

![Diagram](https://example.com/diagram.png)

(We will define \(\sigma\) momentarily.) Note again that \(\rho\) and \(\rho'_s\) are obtained by taking the compositions of \(\rho'\) and \(\rho'_s\), respectively, with the natural embedding \(\iota : U \to U'.\)
Proposition 6.11. Let $S \in U''$ be such that $[S, u_i] = 0$ for $i > 0$, $[S, E_{0i}] = 0$ for $i > 0$, and $[S, E_{ab} - \delta_{ab}E_{00}] = 0$ for all $a, b > 0$. The correspondence
\[
\begin{align*}
t_i &\mapsto u_i, \quad \text{for } i > 0, \\
t_0 \partial_j &\mapsto E_{0j} - \delta_{0j} S, \quad \text{for } j \geq 0, \\
E_{ab} &\mapsto E_{ab} - u_a E_{0b} - \delta_{ab} S, \quad \text{for all } a, b > 0.
\end{align*}
\]
extends to a homomorphism $\sigma : \mathcal{D}'(n) \otimes U(\mathfrak{gl}(n)) \to U''$ of associative unital algebras. Furthermore, $\sigma \rho' = i' \pi'_g$ and $\sigma \rho = i \pi_g$.

Remark 6.12. We can see such $S$ exists and thus such $\sigma$ exists; e.g., we could take $S = 0$.

Proof. Note that $[S, E_{00} + u_i E_{0i}] = 0$ by Lemma 6.6. To check that the correspondence extends to a homomorphism, we verify that $\sigma([x, y]) = [\sigma(x), \sigma(y)]$ whenever $x, y$ equal one of the generators $t_i, t_0 \partial_j, E_{ab}$.

By Lemma 6.9, we have $\sigma([t_i, t_j]) = \sigma(0) = 0 = [u_i, u_j] = [\sigma(t_i), \sigma(t_j)]$, as desired.

By Lemma 6.8 we have $\sigma([t_0 \partial_i, t_0 \partial_j]) = \sigma(\delta_{ij} t_i - \delta_{ij}) = \delta_{ij} u_i - \delta_{ij} = [u_i, E_{0j}] = [\sigma(t_i), \sigma(t_0 \partial_j)]$ as $u_0 = 1$ and $[u_i, S] = 0$.

By Lemma 6.9 and Lemma 6.8 we have $\sigma([t_0 \partial_i, E_{ab}]) = \sigma(0) = 0 = \delta_{0i} u_a u_i - \delta_{0i} u_a - \delta_{0i} u_a u_i - \delta_{0i} = [u_i, E_{ab}] - [u_i, u_a] E_{0b} - u_a [u_i, E_{0b}] = [u_i, E_{ab}] - u_a [u_i, E_{0b}] - \delta_{ab} S = [\sigma(t_0 \partial_i), \sigma(E_{ab})]$, as $u_0 = 1$ and $[u_i, S] = 0$.

We have $\sigma([t_0 \partial_i, t_0 \partial_j]) = \sigma(\delta_{ij} t_0 \partial_i - \delta_{ij} t_0 \partial_i) = \delta_{ij} (E_{0j} - \delta_{ij} t_0 \partial_i) - \delta_{ij} E_{0j} = \delta_{ij} E_{0i} - \delta_{ij} E_{0j} = [E_{0i}, E_{0j}] = [\sigma(t_0 \partial_i), \sigma(t_0 \partial_j)]$, by the definition of $S$.

By Lemma 6.8 we have $\sigma([t_0 \partial_i, E_{ab}]) = 0 = [E_{0i} - \delta_{0i} S, E_{ab} - u_a E_{0b} - \delta_{ab} S] = [\sigma(t_0 \partial_i), \sigma(E_{ab})]$.

Finally, using again Lemma 6.8 we have
\[
\begin{align*}
\sigma([E_{ab}, E_{cd}]) &\quad \sigma(\delta_{bc} E_{ad} - \delta_{ad} E_{bc}) = \delta_{bc} (E_{ad} - u_a E_{0d} - \delta_{ad} S) - \delta_{ad} (E_{cb} - u_c E_{0b} - \delta_{cb} S) \\
&\quad = \delta_{bc} E_{ad} - \delta_{ad} E_{cb} - \delta_{bc} u_a E_{0d} + \delta_{ad} u_c E_{0b} \\
&\quad = [E_{ab}, E_{cd}] + [E_{ab} - u_a E_{0b}, E_{cd}] + [-u_a E_{0b}, E_{cd}] \\
&\quad = [E_{ab}, E_{cd}] - u_a E_{0b} + E_{cd} - u_c E_{0d} + [-u_a E_{0b}, E_{cd}] \\
&\quad = [E_{ab} - u_a E_{0b}, E_{cd} - u_c E_{0d}] + [-u_a E_{0b}, E_{cd}] \\
&\quad = [\sigma(E_{ab}), \sigma(E_{cd})].
\end{align*}
\]

In the above sequence of identities we used that $[S, \delta_{cd} E_{ab} - \delta_{ab} E_{cd}] = [S, \delta_{cd} (E_{ab} - \delta_{ab} E_{0b}) - \delta_{cd} (E_{cd} - \delta_{cd} E_{00})] = 0$ and that $[E_{ab}, -u_c E_{0d}] + [-u_a E_{0b}, E_{cd}] + [-u_a E_{0b}, -u_c E_{0d}] = -\delta_{bc} u_a E_{0d} + \delta_{ad} u_c E_{0b}$.

Thus $\sigma$ is indeed a homomorphism of associative unital algebras.

Note that $\sigma(E) = X - S$ and $\sigma(R_1) = S - \frac{1}{n+1} G_1$. Using the definitions of $\sigma$ and $\rho'$, it is easy to verify that $\sigma \rho'(X) = X - \frac{1}{n+1} G_1$, $\sigma \rho'(E_{ij}) = E_{ij}$ for $i \neq j$, and $\sigma \rho'(E_{ii}) = E_{ii} - \frac{1}{n+1} G_1$. Hence, $\sigma \rho' = i' \pi'_g$. The identity $\sigma \rho' = i' \pi'_g$ follows from $\rho = \rho' t$ and $\pi_g' t = i \pi_g$. $\Box$

Theorem 6.13. We have the following:

(i) $\ker \rho' = (G_1)$ in $U'$, and $\ker \rho = (G_1)$ in $U$.

(ii) $\ker \rho'_s = (0)$ in $U'_s$, and $\ker \rho_s = (0)$ in $U_s$. 

Proof. For part (i) we first note that $\rho'(G_1) = \rho(G_1) = 0$. To complete the proof we use Lemma 6.10 along with Proposition 6.11 and the fact that the kernel of $\pi_g$ is $(G_1)$ in $U$. As for part (ii), note that $\pi'_s\pi'_s = \pi'_s$; then, if $t \in \ker \rho'_s = (G_1) \cap U_s$, then we have $t = \pi'_s(t) = 0$. Thus $\ker \rho'_s = \{0\}$. Then $\ker \rho_s \subset \ker \rho'_s$, so $\ker \rho_s = \{0\}$. \qed
APPENDIX. FORMULAS FOR THE IMAGES OF CERTAIN ELEMENTS UNDER $\rho$

In this appendix we provide explicit formulas for the images under $\rho$ of $r_k^{gl(n+1)}(a, b)$. For the definition of the latter see (1). The proofs of these formulas are independent of the results in the previous sections of the paper. In this way we have an alternative proof of Theorem 5.4. It is interesting to note that one can then go “backwards” and prove Theorem 5.1 and then Theorem 4.2 purely computationally. Thus, this appendix leads to an alternative (more computational) approach of the results established in Sections 4 and 5.

Note that $r_{k+1}^{gl(N)}(a, b) = \sum_i r_k^{gl(N)}(a, i)E_{ib}$ for all nonnegative integers $k$, for $N = n$ or $n + 1$, and for all $a, b$.

For a positive integer $m$ and for $a$ and $b$ with $0 \leq a \leq n$ and $1 \leq b \leq n$, define

$$f_m(a, b) = \sum_{i=1}^n t_a \partial_i \otimes r_{m-1}^{gl(n)}(i, b).$$

**Theorem.** For all positive integers $k$, $a$, $b$, such that $a, b \leq n$,

$$\rho(r_k^{gl(n+1)}(a, b)) = \sum_{m=1}^{k} \left( f_m(a, b) \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \right) + \sum_{g=0}^{k} \binom{k}{g} R_1^g \left( 1 \otimes r_{k-g}^{gl(n)}(a, b) \right),$$

$$\rho(r_k^{gl(n+1)}(a, 0)) = \left( \sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g} \right) (t_a \partial_0 \otimes 1) - \sum_{g=0}^{k-1} \binom{k}{g} R_1^g \sum_{j=0}^{t_0 \otimes r_{k-g}^{gl(n)}(a, j)} \partial_i t_j \otimes r_{m-1}^{gl(n)}(i, j),$$

$$\rho(r_k^{gl(n+1)}(0, b)) = \sum_{m=1}^{k} \left( f_m(0, b) \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \right),$$

$$\rho(r_k^{gl(n+1)}(0, 0)) = \left( \sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g} \right) (t_0 \partial_0 \otimes 1) + R_1^k$$

$$- \sum_{m=2}^{k} \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \left( \sum_{i, j > 0} \partial_i t_j \otimes r_{m-1}^{gl(n)}(i, j) \right).$$

**Proof.** We prove all four statements simultaneously by induction on $k$. The base case $k = 1$ follows from the definition of $\rho$. Suppose the formulas in the statement of the Theorem are true for some positive integer $k$. Let us prove them for $k + 1$.

First, we consider the value of $\rho(r_k^{gl(n+1)}(a, b))$ for $a, b > 0$. We have
\[
\rho(r^{(n+1)}_{k+1}(a, b)) = \rho(r^{(n+1)}_k(a, 0) E_{0b} + \sum_{i > 0} r^{(n+1)}_k(a, i) E_{ib}) \\
= \rho(r^{(n+1)}_k(a, 0)) \rho(E_{0b}) + \sum_{i > 0} \rho(r^{(n+1)}_k(a, i)) \rho(E_{ib}) \\
= \left( \sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g} \right) (t_a \partial_0 \otimes 1) - \left( \sum_{g=0}^{k-1} \binom{k}{g} R_1^g \sum_{i > 0} t_j \otimes r^{(n)}_{k-g}(a, j) \right) \\
- \left( \frac{t_a}{t_0 \otimes 1} \right)^k \sum_{m=2}^{k-m} \left( \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \right) \left( \sum_{i, j > 0} \partial_i t_j \otimes r^{(n)}_{m-1}(i, j) \right) (t_0 \partial_b \otimes 1) \\
+ \sum_{i > 0} \left( \sum_{m=1}^k f_m(a, i) \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \right) \\
+ \sum_{g=0}^k \binom{k}{g} R_1^g \left( 1 \otimes r^{(n)}_{k-g}(a, i) \right) (t_i \partial_b \otimes 1 + 1 \otimes E_{ib} + \delta_{ib} R_1) \\
= \left( \sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g} \right) (t_a \partial_0 t_0 \partial_b \otimes 1) - \left( \sum_{g=0}^{k-1} \binom{k}{g} R_1^g \sum_{i > 0} t_j \partial_b \otimes r^{(n)}_{k-g}(a, j) \right) \\
- \left( \frac{t_a}{t_0 \otimes 1} \right)^k \sum_{m=2}^{k-m} \left( \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \right) \left( \sum_{i, j > 0} \partial_i t_j \otimes r^{(n)}_{m-1}(i, j) \right) (t_0 \partial_b \otimes 1) \\
+ \sum_{i > 0} \sum_{m=1}^k f_m(a, i) (t_i \partial_b \otimes 1) \left( \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \right) + \sum_{i > 0} \sum_{g=0}^k \binom{k}{g} R_1^g (t_i \partial_b \otimes r^{(n)}_{k-g}(a, i)) \\
+ \sum_{i > 0} \sum_{m=1}^k f_m(a, i) (1 \otimes E_{ib}) \left( \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \right) + \sum_{i > 0} \sum_{g=0}^k \binom{k}{g} R_1^g \left( 1 \otimes r^{(n)}_{k-g}(a, i) E_{ib} \right) \\
+ \sum_{m=0}^k \left( f_m(a, b) \sum_{g=0}^{k-m} \binom{k}{g} R_1^{g+1} R_2^{k-m-g} \right) + \sum_{g=0}^k \binom{k}{g} R_1^{g+1} \left( 1 \otimes r^{(n)}_{k-g}(a, b) \right).
\]

Write the last expression in the form

\[
X_1 = X_2 = X_3 + X_4 + X_5 + X_6 + X_7 + X_8 + X_9.
\]

Then

\[
X_5 - X_2 = \sum_{i > 0} \sum_{g=0}^k \binom{k}{g} R_1^g \left( t_i \partial_b \otimes r^{(n)}_{k-g}(a, i) \right) - \sum_{k=0}^{k-1} \binom{k}{g} R_1^g \sum_{j > 0} t_j \partial_b \otimes r^{(n)}_{k-g}(a, j) = R_1^k (t_a \partial_b \otimes 1).
\]

On the other hand, \(X_4 = \sum_{i > 0} \sum_{g=0}^k \binom{k}{g} R_1^g \left( t_i \partial_b \otimes r^{(n)}_{k-g}(a, i) \right) - \sum_{g=0}^k \binom{k}{g} R_1^g \sum_{j > 0} t_j \partial_b \otimes r^{(n)}_{k-g}(a, j)). \) Also, \(X_3 = \sum_{i > 0} \sum_{g=0}^k \binom{k}{g} R_1^g \sum_{j > 0} t_j \partial_b \otimes r^{(n)}_{k-g}(a, j) \). Then

\[
X_4 - X_3 = \sum_{i > 0} \sum_{g=0}^k \binom{k}{g} R_1^g \sum_{j > 0} t_j \partial_b \otimes r^{(n)}_{k-g}(a, j) = \sum_{i > 0} \sum_{g=0}^k \binom{k}{g} R_1^g R_2^{k-1-g}.
\]

Therefore we have \((X_1 + X_4 - X_3) + (X_5 - X_2) = (t_a \partial_b \otimes 1) \sum_{g=0}^k \binom{k}{g} R_1^g R_2^{k-g}.
\]
Furthermore, since $\sum_{i>0} f_m(a, i)(1 \otimes E_{i0}) = f_{m+1}(a, b)$, we have

$$X_6 = \sum_{m=1}^{k} (f_m(a, b) \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g}) = \sum_{m=2}^{k+1} f_m(a, b) \sum_{g=0}^{k-m-1} \binom{k+1}{g} R_1^g R_2^{k-m-g}.$$

Thus $X_6 + \left( (X_1 + X_4 - X_3) + (X_5 - X_2) \right) = \sum_{m=1}^{k+1} f_m(a, b) \sum_{g=0}^{k-m-1} \binom{k+1}{g} R_1^g R_2^{k-m-g}.$

But $X_8 = \sum_{m=1}^{k+1} f_m(a, b) \sum_{g=0}^{k-m+1} \binom{k+1}{g} R_1^g R_2^{k-m-g+1}$. Hence,

$$(X_6 + \left( (X_1 + X_4 - X_3) + (X_5 - X_2) \right) + X_8 = \sum_{m=1}^{k+1} f_m(a, b) \sum_{g=0}^{k-m+1} \binom{k+1}{g} R_1^g R_2^{k-m-g}.$$

Now using that $\sum_{i>0} r_{k-g}^{g}(a, i) E_{i0} = r_{k+1-g}^{g}(a, b)$, we find $X_7 = \sum_{g=0}^{k+1} \binom{k+1}{g} R_1^g (1 \otimes r_{k+1-g}^{g}(a, b)).$

Also $X_9 = \sum_{g=1}^{k+1} \binom{k+1}{g} R_1^g (1 \otimes r_{k+1-g}^{g}(a, b))$ implies that $X_7 + X_9 = \sum_{m=1}^{k+1} f_m(a, b) \sum_{g=0}^{k-m+1} \binom{k+1}{g} R_1^g R_2^{k-m-g} + \sum_{g=0}^{k+1} \binom{k+1}{g} R_1^g (1 \otimes r_{k+1-g}^{g}(a, b)).$ This completes the proof of the inductive step for $\rho(r_{k+1}^{g}(a, b))$ for $a, b > 0$.

Next, we consider the value of $\rho(r_{k+1}^{g}(a, 0))$ for $a > 0$. We have

$$\rho(r_{k+1}^{g}(a, 0)) = \rho(r_{k}^{g}(a, 0)) E_{00} + \sum_{i>0} r_{k}^{g}(a, i) E_{i0}$$

$$= \rho(r_{k}^{g}(a, 0)) \rho(E_{00}) + \sum_{i>0} \rho(r_{k}^{g}(a, i)) \rho(E_{i0})$$

$$= \left( \left( \sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g} \right) \left( t_0 \partial_0 \otimes 1 \right) - \sum_{g=0}^{k-1} \binom{k}{g} R_1^g \sum_{j>0} t_j \otimes r_{k-g}^{g}(a, j) \right)$$

$$+ \left( \frac{t_0}{t_0} \otimes 1 \right) \sum_{m=2}^{k} \left( \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \right) \left( \sum_{i,j>0} \partial_i \partial_j \otimes r_{m-1}^{g}(a, i, j) \right) \left( t_0 \partial_0 \otimes 1 + R_1 \right)$$

$$= \left( \sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g} \right) \left( t_0 \partial_0 t_0 \partial_0 \otimes 1 \right) - \sum_{g=0}^{k-1} \binom{k}{g} R_1^g \sum_{j>0} t_j \partial_0 \otimes r_{k-g}^{g}(a, j)$$

$$- \sum_{m=2}^{k} \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \left( \sum_{i,j>0} t_0 \partial_i \partial_j \partial_0 \otimes r_{m-1}^{g}(a, i, j) \right)$$

$$+ \frac{t_0}{t_0} \otimes 1 \sum_{m=2}^{k} \sum_{g=0}^{k-m} \binom{k}{g} R_1^{g+1} R_2^{k-1-g} \left( t_0 \partial_0 \otimes 1 \right) - \sum_{g=0}^{k-1} \binom{k}{g} R_1^{g+1} \sum_{j>0} t_j \otimes r_{k-g}^{g}(a, j)$$

$$- \frac{t}{t_0} \otimes 1 \sum_{m=2}^{k} \sum_{g=0}^{k-m} \binom{k}{g} R_1^{g+1} R_2^{k-1-g} \left( \sum_{i,j>0} \partial_i \partial_j \otimes r_{m-1}^{g}(a, i, j) \right).$$
\[
\sum_{m=1}^{k} \left( \sum_{i>0} f_m(a, i)(t_i \partial_0 \otimes 1) \right) \left( \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \right) \\
+ \sum_{g=0}^{k} \binom{k}{g} R_1^g \left( \sum_{i>0} t_i \partial_0 \otimes r_{g}^{t_{k-g}}(a, i) \right) \\
- \sum_{m=1}^{k} \left( \sum_{i,j>0} f_m(a, i) \left( \frac{t_j}{t_0} \otimes E_{ij} \right) \right) \left( \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \right) \\
- \sum_{g=0}^{k} \binom{k}{g} R_1^g \left( \sum_{i,j>0} \frac{t_j}{t_0} \otimes r_{g}^{t_{k-g}}(a, i) E_{ij} \right).
\]

We write the last expression as \(X_1 - X_2 - X_3 + X_4 - X_5 - X_6 + X_7 + X_8 - X_9 - X_{10}\). Then \(X_8 - X_2 = \sum_{g=0}^{k} \binom{k}{g} R_1^g \left( \sum_{i>0} t_i \partial_0 \otimes r_{g}^{t_{k-g}}(a, i) \right) - \sum_{g=0}^{k-1} \binom{k}{g} R_1^g \left( \sum_{i>0} t_i \partial_0 \otimes r_{g}^{t_{k-g+1}}(a, i) \right) = R_1^k (t_a \partial_0 \otimes 1).

Also, \(X_5 + X_{10} = \left( \sum_{g=0}^{k} \binom{k}{g} R_1^g \sum_{j>0} t_j \frac{t_j}{t_0} \otimes \delta^{t_{k-j-1}}(a, j) \right) + \left( \sum_{g=0}^{k} \binom{k}{g} R_1^g \sum_{j>0} t_j \frac{t_j}{t_0} \otimes \delta^{t_{k-j-1}}(a, j) \right) = \sum_{g=0}^{k} \binom{k+1}{g} R_1^g \sum_{j>0} t_j \frac{t_j}{t_0} \otimes \delta^{t_{k-j-1}}(a, j).\)

We next have \(X_9 = \sum_{m=1}^{k} \left( \sum_{i,j>0} f_m(a, i) \left( \frac{t_j}{t_0} \otimes E_{ij} \right) \right) \left( \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \right).\) Now, for any positive integer \(m\), \(\sum_{i,j>0} f_m(a, i) \left( \frac{t_j}{t_0} \otimes E_{ij} \right) = \sum_{i,j>0} \left( \sum_{t>0} t_a \partial t \otimes \delta^{t_{m-1}}(\ell, i) \right) \left( \frac{t_j}{t_0} \otimes E_{ij} \right) = \sum_{j,t>0} t_a \partial t \frac{t_j}{t_0} \otimes \delta^{t_{m-1}}(\ell, j)\), which is the same as \(\sum_{i,j>0} \partial t_j \frac{t_j}{t_0} \otimes \delta^{t_{m-1}}(i, j)\).

Then \(X_9 = \sum_{m=2}^{k} \sum_{g=0}^{k-m} \binom{k+1-m}{g} R_1^g R_2^{k+1-m-g} \left( \sum_{i,j>0} \partial t_j \otimes \delta^{t_{m-1}}(i, j) \right)\). Also,
\[X_6 = \sum_{m=2}^{k} \sum_{g=0}^{k-m} \binom{k+1-m}{g} R_1^g R_2^{k+1-m-g} \left( \sum_{i,j>0} \partial t_j \otimes \delta^{t_{m-1}}(i, j) \right)\] Also, \(X_6 + X_9 = \sum_{m=1}^{k} \sum_{g=0}^{k-m} \binom{k+1-m}{g} R_1^g R_2^{k+1-m-g} \left( \sum_{i,j>0} \partial t_j \otimes \delta^{t_{m-1}}(i, j) \right)\)

Next, for any positive integer \(m\), \(\sum_{i,j>0} f_m(a, i)(t_i \partial_0 \otimes 1) = \sum_{i,j>0} t_a \partial t_i \partial_0 \otimes \delta^{t_{m-1}}(j, i)\). Then \(X_7 = \sum_{m=1}^{k} \left( \sum_{i,j>0} t_a \partial t_i \partial_0 \otimes \delta^{t_{m-1}}(i, j) \right) \left( \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \right)\). On the other hand, \(X_7 - X_3 = \sum_{i,j>0} \partial t_i \partial_0 \otimes \delta_{ij} \left( \sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g} \right) = \sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g} \left( t_a \partial_0 \otimes 1 \right) \left( R_2 - (t_a \partial_0 \otimes 1) \right)\). Thus \(X_7 - X_3 = \sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g} \left( t_a \partial_0 \otimes 1 \right) \left( R_2 - (t_a \partial_0 \otimes 1) \right)\), and subsequently \(X_7 - X_3 + X_1 = \sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g} \left( t_a \partial_0 \otimes 1 \right) \left( R_2 - (t_a \partial_0 \otimes 1) \right) \left( R_2 - (t_a \partial_0 \otimes 1) \right)\). Therefore, \(X_7 - X_3 + X_1 = \sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g} \left( t_a \partial_0 \otimes 1 \right) \left( R_2 - (t_a \partial_0 \otimes 1) \right)\). Since \(X_4 = \sum_{g=0}^{k} \binom{k}{g} R_1^g R_2^{k-g} \left( t_a \partial_0 \otimes 1 \right)\),
\(X_7 - X_3 + X_1 + X_8 - X_2 + X_4 = \sum_{g=0}^{k} \binom{k+1}{g} R_1^g R_2^{k-g} \left( t_a \partial_0 \otimes 1 \right)\).
Hence, $\rho(r_{k+1}^{(n+1)}(a,0)) = (X_7 - X_3 + X_1 + X_8 - X_2 + X_4) - (X_5 + X_{10}) - (X_6 + X_9) = (\sum_{g=0}^{k}(k^{(1)})_g R_1^g R_2^{k-g})(t_a \partial_0 \otimes 1) - (\sum_{g=0}^{k}(k^{(1)})_g R_1^g \sum_{j>0} t_j \otimes r_{k+1-g}^{(n)}(a,j)) - \left(\frac{t_0}{t_0} \otimes 1\right) \sum_{m=2}^{k+1} \left(\sum_{g=0}^{k+1-m}(k^{(1)})_g R_1^g R_2^{k+1-m-g}\right) \left(\sum_{j>0} \partial_j t_j \otimes r_{m-1}^{(n)}(i,j)\right)$, as desired. This completes the inductive step for the value of $\rho(r_{k+1}^{(n+1)}(a,0))$.

Next, we consider the value of $\rho(r_{k}^{(n+1)}(0,b))$ for $b > 0$.

\[
\rho(r_{k+1}^{(n+1)}(0,b)) = \rho(r_{k}^{(n+1)}(0,0))E_{0b} + \sum_{i>0} r_{k}^{(n+1)}(0,i)E_{ib}
\]

\[
= \rho(r_{k}^{(n+1)}(0,0))\rho(E_{0b}) + \sum_{i>0} \rho(r_{k}^{(n+1)}(0,i))\rho(E_{ib})
\]

\[
= \left(\left(\sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g}\right)(t_0 \partial_0 \otimes 1) + R_1^k\right)
\]

\[
- \sum_{m=2}^{k} \left(\sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g}\right) \left(\sum_{i,j>0} \partial_i t_j \otimes r_{m-1}^{(n)}(i,j)\right)\left(t_0 \partial_0 \otimes 1\right)
\]

\[
+ \sum_{i>0} \left(\sum_{m=1}^{k} \left(f_m(0,i) \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g}\right)\right)\left(t_i \partial_i \otimes 1 + 1 \otimes E_{ib} + \delta_{ib} R_1\right)
\]

\[
= \sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g}(t_0 \partial_0 t_0 \partial_0 \otimes 1) + R_1^k(t_0 \partial_0 \otimes 1)
\]

\[
- \sum_{m=2}^{k} \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g} \left(\sum_{i,j>0} \partial_i t_j \otimes r_{m-1}^{(n)}(i,j)\right)\left(t_0 \partial_0 \otimes 1\right)
\]

\[
+ \sum_{i>0} \sum_{m=1}^{k} f_m(0,i)(t_i \partial_i \otimes 1) \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g}
\]

\[
+ \sum_{i>0} \sum_{m=1}^{k} f_m(0,i)(1 \otimes E_{ib}) \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g}
\]

\[
+ \sum_{m=1}^{k} f_m(0,b) \sum_{g=0}^{k-m} \binom{k}{g} R_1^{g+1} R_2^{k-m-g}
\].

We write the last expression as $X_1 + X_2 - X_3 + X_4 + X_5 + X_6$.

Now $X_4 = \sum_{m=1}^{k}((\sum_{j,i>0} t_0 \partial_j t_i \partial_b \otimes r_{m-1}^{(n)}(j,i))(\sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g}))$. Also, $X_3 = \sum_{m=2}^{k}((\sum_{j,i>0} t_0 \partial_j t_i \partial_b \otimes r_{m-1}^{(n)}(j,i))(\sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g}))$. Then $X_4 - X_3 = (\sum_{j,i>0} t_0 \partial_j t_i \partial_b \otimes r_{0}^{(n)}(j,i))(\sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g}) = (\sum_{j,i>0} t_0 \partial_j t_i \partial_b \otimes 1)(\sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g})$, 

\[
\sum_{i>0} \sum_{m=1}^{k} f_m(0,i)(t_i \partial_i \otimes 1) \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g}
\]

\[
+ \sum_{i>0} \sum_{m=1}^{k} f_m(0,i)(1 \otimes E_{ib}) \sum_{g=0}^{k-m} \binom{k}{g} R_1^g R_2^{k-m-g}
\]

\[
+ \sum_{m=1}^{k} f_m(0,b) \sum_{g=0}^{k-m} \binom{k}{g} R_1^{g+1} R_2^{k-m-g}
\].
Finally, we consider the value of $\sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g}$. As $
abla_i \otimes \frac{1}{R_0} = (\nabla_i \otimes 1) R_2$, then $X_2 + X_1 + X_4 - X_3 = (\nabla_0 \otimes 1) (\sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g})$.

Since $
abla_i f_m(0, i)(1 \otimes E_{i0}) = f_{m+1}(0, b)$. $X_5 = \sum_{m=1}^{k-1} f_{m+1}(0, b) (\sum_{g=0}^{k-m} \binom{k-m}{g} R_1^g R_2^{k-m-g}) = \sum_{m=1}^{k-1} f_{m}(0, b) (\sum_{g=0}^{k-m} \binom{k-m}{g} R_1^g R_2^{k-m-g})$. Then $X_5 + (X_2 + X_1 + X_4 - X_3) = \sum_{m=1}^{k-1} f_{m}(0, b) (\sum_{g=0}^{k-m} \binom{k-m}{g} R_1^g R_2^{k-m-g})$. We also have $X_6 = \sum_{m=1}^{k} f_{m}(0, b) (\sum_{g=0}^{k-m} \binom{k-m}{g} R_1^g R_2^{k-m-g})$. Therefore, $\rho(r^{(n+1)}_{k+1}(0, b)) = (X_5 + (X_2 + X_1 + X_4 - X_3) + X_6 = \sum_{m=1}^{k} f_{m}(0, b) (\sum_{g=0}^{k-m} \binom{k-m}{g} R_1^g R_2^{k-m-g})$, which completes the inductive step for the value of $\rho(r^{(n+1)}_{k+1}(0, b))$.

Finally, we consider the value of $\rho(r^{(n+1)}_{k+1}(0, 0))$.

$$\rho(r^{(n+1)}_{k+1}(0, 0)) = \rho(r^{(n+1)}_{k+1}(0, 0) E_{00} + \sum_{i>0} t_r^{(n+1)}(0, i) E_{i0})$$

$$= \rho(r^{(n+1)}_{k+1}(0, 0)) \rho(E_{00}) + \sum_{i>0} \rho(r^{(n+1)}_{k+1}(0, i)) \rho(E_{i0})$$

$$= \left( \sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g} \right) (t_0 \partial_0 \otimes 1) + R_1^k$$

$$- \sum_{m=2}^{k} \left( \sum_{g=0}^{k-m} \binom{k-m}{g} R_1^g R_2^{k-m-g} \right) \left( \sum_{i,j>0} \partial_i t_j \otimes r^{(n-1)}_{m-1}(i, j) \right) (t_0 \partial_0 \otimes 1 + R_1)$$

$$+ \left( \sum_{m=1}^{k} f_{m}(0, i) \sum_{g=0}^{k-m} \binom{k-m}{g} R_1^g R_2^{k-m-g} \right) (t_0 \partial_0 \otimes 1 - \sum_{j>0} t_j \otimes E_{ij})$$

$$= \left( \sum_{g=0}^{k-1} \binom{k}{g} R_1^g R_2^{k-1-g} \right) (t_0 \partial_0 \partial_0 \otimes 1) + R_1^k (t_0 \partial_0 \otimes 1)$$

$$- \sum_{m=2}^{k} \left( \sum_{g=0}^{k-m} \binom{k-m}{g} R_1^g R_2^{k-m-g} \right) \left( \sum_{i,j>0} \partial_i t_j \otimes r^{(n-1)}_{m-1}(i, j) \right) (t_0 \partial_0 \otimes 1)$$

$$+ \left( \sum_{g=0}^{k} \binom{k-1}{g} R_1^{g+1} R_2^{k-1-g} \right) (t_0 \partial_0 \otimes 1) + R_1^{k+1}$$

$$- \sum_{m=2}^{k} \left( \sum_{g=0}^{k-m} \binom{k-m}{g} R_1^{g+1} R_2^{k-m-g} \right) \left( \sum_{i,j>0} \partial_i t_j \otimes r^{(n-1)}_{m-1}(i, j) \right)$$

$$+ \left( \sum_{m=1}^{k} f_{m}(0, i) \sum_{g=0}^{k-m} \binom{k-m}{g} R_1^g R_2^{k-m-g} \right) (t_0 \partial_0 \otimes 1)$$

$$- \sum_{m=1}^{k} \left( \sum_{g=0}^{k-m} \binom{k-m}{g} R_1^g R_2^{k-m-g} \right) \left( \sum_{j>0} t_j \otimes E_{ij} \right).$$
We write the last expression as \( X_1 + X_2 - X_3 + X_4 + X_5 - X_6 + X_7 - X_8. \)

Since \( X_7 = \sum_{m=1}^{k} \left( \sum_{g=0}^{k-m} (k) R^g_1 R^{k-m-g}_2 \right) \left( \sum_{i,j>0} \partial t_i \otimes r_{m-1}^g(j,i) \right) (t_0 \partial_0 \otimes 1), \)

\[ X_7 - X_3 = \left( \sum_{g=0}^{k-1} (k) R^g_1 R^{k-1-g}_2 \right) \left( \sum_{i>0} \partial t_i \otimes 1 \right) (t_0 \partial_0 \otimes 1) \]

Then \( X_7 - X_3 + X_1 + X_2 = \left( \sum_{g=0}^{k} (k) R^g_1 R^{k-g}_2 \right) (t_0 \partial_0 \otimes 1). \) But since \( X_4 = \left( \sum_{g=1}^{k} (k) R^g_1 R^{k-g}_2 \right) \left( t_0 \partial_0 \otimes 1 \right). \)

We next have \( X_8 = \sum_{m=1}^{k} \left( \sum_{i,j>0} f_m(0,i) \left( \frac{t_i}{t_0} \otimes E_{ij} \right) \right) \left( \sum_{g=0}^{k-m} (k) R^g_1 R^{k-m-g}_2 \right). \) For any positive integer \( m, \) \( \sum_{i,j>0} f_m(0,i) \left( \frac{t_i}{t_0} \otimes E_{ij} \right) = \sum_{i,j>0} \left( \sum_{\ell>0} t_0 \partial_\ell \otimes r_{m-1}^g(\ell, i) \right) \left( \frac{t_i}{t_0} \otimes E_{ij} \right). \)

\[ X_8 = \sum_{m=2}^{k+1} \sum_{g=0}^{k-m \ell} (k) R^g_1 R^{k-1-m-g}_2 \left( \sum_{i,j>0} \partial t_j \otimes r_{m-1}^g(i,j) \right) \]

Also,

\[ X_6 = \sum_{m=2}^{k+1} \sum_{g=0}^{k-m \ell} (k) R^g_1 R^{k-1-m-g}_2 \left( \sum_{i,j>0} \partial t_j \otimes r_{m-1}^g(i,j) \right) \text{ and therefore} \]

\[ X_6 + X_8 = \sum_{m=2}^{k+1} \sum_{g=0}^{k-m \ell} (k) R^g_1 R^{k-1-m-g}_2 \left( \sum_{i,j>0} \partial t_j \otimes r_{m-1}^g(i,j) \right). \]

Combining the above, \( \rho \left( r_{k+1}^{g(n+1)}(0,0) \right) = \left( X_7 - X_3 + X_1 + X_2 + X_4 \right) + X_5 - \left( X_6 + X_8 \right) = \left( \sum_{g=0}^{k} (k) R^g_1 R^{k-g}_2 \right) \left( t_0 \partial_0 \otimes 1 \right) + R^{k+1}_1 - \sum_{m=2}^{k+1} \sum_{g=0}^{k-m \ell} (k) R^g_2 R^{k-1-m-g}_2 \left( \sum_{i,j>0} \partial t_j \otimes r_{m-1}^g(i,j) \right). \)

This completes the induction step for \( \rho \left( r_{k+1}^{g(n+1)}(0,0) \right) \) and hence the proof of the theorem. \( \square \)

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