Potential analysis and absorption cross section in the D1–D5 brane system

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We analyze the potentials which arise from the D1–D5 brane (5D black hole). In the sufficiently low energy ($\omega \ll 1$), we can derive the Schrödinger-type equation with potential $V_N$ from the linearized equations. In this case one can understand the difference between absorption cross section for a free and two fixed scalars intuitively in terms of their potentials. In the low temperature limit ($\omega \gg T_H$), one expects the logarithmic correction to the cross section of a free scalar. However, we cannot obtain the Schrödinger equation with potential for this case. Finally we comment on the stability of 5D black hole.
Recently there has been a great progress in the D1–D5 brane system with momentum along the string direction which gives us a (D-brane) 5D black hole with three charges ($Q_1, Q_5, Q_K$). The first progress was achieved in the Bekenstein-Hawking entropy [1]. Apart from the success of counting the microstates of a 5D black hole through D-brane physics, a dynamical consideration becomes an important issue [2–5]. This is so because the absorption cross section (greybody factor) for the black hole arises as a consequence of the gravitational potential barrier surrounding the horizon. That is, this is an effect of spacetime curvature. In the effective string description, their origin comes from the thermal distribution for excitations of the D1–D5 bound state. An effective CFT approach was also introduced to describe the absorption of scalars by the general black holes [6]. The 5D black hole becomes $AdS_3 \times S^3$ near horizon but with an asymptotically flat space [7]. In this case the cross section agrees with that for the semiclassical calculation of 5D black hole [8]. This means that the near horizon geometry contains the essential information about the bulk 5D black hole. Also the AdS/CFT correspondence [9] can be used to derive the cross section. This is so because the $AdS_3 \times S^3$ is an exact solution of string theory and there is an exact CFT on its boundary at spatial infinity. It turns out that the cross sections in the boundary CFT computation take the same forms as those in the semiclassical and effective string calculations [10].

The calculations of cross section for a minimally coupled scalar are straightforward in both semiclassical and effective string models. The s-wave cross section is not sensitive to the energy ($\omega$) but depends only on the area of horizon [2,3]. This couples to an operator with dimension (1,1) on the boundary. A better test of the agreement between semiclassical and effective string calculations is provided by the fixed scalars. The effective string calculation is well performed in the dilute gas limit which corresponds to the decoupling limit. But the semiclassical calculations are difficult because of a complicated mixing between fixed scalars and other fields (metric and U(1) gauge fields). One of fixed scalars ($\nu$) couples solely to an operator of dimension (2,2) on the boundary CFT. When $Q_1 = Q_5$, the effective string calculation of yields the precise agreement with the semiclassical greybody factor [1].
However, the greybody factor of the other (λ) is not in agreement for $Q_1 = Q_5$ [3]. This disagreement may be caused by the additional chiral operators with dimension (3,1) and (1,3) beyond (2,2) on the boundary. This point remains unsolved up to now.

On the other hand, it is possible to visualize any black hole as presenting an effective potential barrier (or well) to the on-coming mode [11–13]. This means that one can derive the Schrödinger-type equation for the physical mode. In this case one can also perform the stability analysis [14]. For example, in case of the 4D Schwarzschild black hole

$$d^2 s^2_{4D} = -(1 - \frac{r_o}{r})dt^2 + (1 - \frac{r_o}{r})^{-1}dr^2 + r^2d\Omega^2_2, \quad (1)$$

two graviton modes arose from the metric perturbations with $l \geq 2$. One is the Regge-Wheeler (RW) graviton mode in the axial (odd-parity) perturbation equation,

$$\frac{d^2 \Psi_{RW}}{dr^{*2}} + (\omega^2 - V_{RW})\Psi_{RW} = 0. \quad (2)$$

Here a tortoise coordinate $r^* = r + r_o \ln(r - r_o)$ is introduced, so that the horizon is at $r^* = -\infty \ (r = r_o)$. The RW potential $V_{RW}$ is given by

$$V_{RW} = \frac{2(n + 1)r - 3r_o}{r^4}(r - r_o) \quad (3)$$

with $n = (l - 1)(l + 2)/2, l \geq 2$. The other is the Zerilli mode in the polar(even-parity) equation

$$\frac{d^2 \Psi_{Z}}{dr^{*2}} + (\omega^2 - V_{Z})\Psi_{Z} = 0. \quad (4)$$

which differs only in the details of the potential

$$V_{Z} = \frac{2(n + 1)r^3 + 3r_o r^2 + 9rr_o^2/2n + 9r_o^3/4n^2}{r^4(r + 3r_o/2n)^2}(r - r_o). \quad (5)$$

Although these have different forms, Chandrasekhar have showed that $V_{RW}$ and $V_{Z}$ are equivalent in the sense of producing the same reflection ($R$) and absorption ($A$) coefficients [12]. For a minimally coupled scalar ($\psi$), one finds

$$\frac{d^2 \psi}{dr^{*2}} + (\omega^2 - V_{\psi})\psi = 0, \quad (6)$$
where the potential is given by

\[ V_\psi = \frac{2(n + 1)r + r_o(r - r_o)}{r^4} \]

with \( l \geq 0 \) \[^{[4]}\]. As is shown in Fig. 1, the lowest allowed potentials with \( r_o = 0.01 \) take all barrier-type. Here \( n = 2(l = 2) \) for RW, Zerilli modes and \( n = -1(l = 0) \) for \( \psi \). \( V_{\text{RW}} \simeq V_{\text{Z}} \) implies the same reflection and absorption coefficients. The s-wave absorption cross section for a free scalar \( \psi \) is \( \sigma_{\psi}^{4D} = A_{4D}^{4D} = 4\pi r_o^2 \).

**FIG. 1.** Three potential graphs \((V_{\text{RW}}, V_{\text{Z}}, V_\psi)\) for 4D Schwarzschild black hole with \( r_0 = 0.01 \).

In this paper we will clarify the close relationship between the potential and absorption cross section in the D1–D5 brane system. Initially we introduce all modes around the 5D black hole background. It is pointed out that in s-wave calculation fixed scalars are physically propagating modes and other fields belong to redundant modes. The relevant modes are two fixed scalars \((\nu, \lambda)\) including a free scalar(\( \phi \)). We begin with the 5D black hole with three charges,
\[ ds_{5D}^2 = -h f^{-2/3} dt^2 + f^{1/3} (h^{-1}dr^2 + r^2 d\Omega_3^2), \] (8)

where
\[ f = f_1 f_2 f_K = (1 + \frac{r_1^2}{r^2})(1 + \frac{r_5^2}{r^2})(1 + \frac{r_K^2}{r^2}), \quad h = (1 - \frac{r_0^2}{r^2}). \] (9)

Here the radii are related to the boost parameters \((\alpha_i)\) and the charges \((Q_i)\) as
\[ r_i^2 = r_0^2 \sinh^2 \alpha_i = \sqrt{Q_i^2 + \frac{r_0^4}{4} - \frac{r_0^2}{2}}, \quad i = 1, 5, K. \] (10)

Hence the D-brane black hole depends on the four parameters \((r_1, r_5, r_K, r_0)\). The background metric (8) is just the 5D Schwarzschild one with time and space components rescaled by different powers of \(f\). The event horizon (outer horizon) is clearly at \(r = r_0\). When all three charges are nonzero, the surface \(r = 0\) becomes a smooth inner horizon (Cauchy horizon). When at least one of the charges is zero, the surface \(r = 0\) becomes singular. The extremal case corresponds to the limit of \(r_0 \to 0\) with the boost parameters \(\alpha_i \to \pm \infty\), keeping the charges \((Q_i)\) fixed. We are interested in the limit of \(r_0, r_K \ll r_1, r_5\), which is called the dilute gas region. Here we have \(Q_1 = r_1^2, Q_5 = r_5^2\), and \(r_K = r_0 \sinh \alpha_K\) with a finite \(\alpha_K\).

This corresponds to the near extremal black hole and its thermodynamic quantities (energy, entropy, Hawking temperature) are given by
\[
E_{next} = \frac{2\pi^2}{\kappa_5^2} \left[ r_1^2 + r_5^2 + \frac{1}{2} r_0^2 \cosh 2\alpha_K \right], \tag{11}
\]
\[
S_{next} = \frac{4\pi^3 r_0}{\kappa_5^2} r_1 r_5 \cosh \alpha_K, \tag{12}
\]
\[
\frac{1}{T_{H, next}} = \frac{2\pi}{r_0} r_1 r_5 \cosh \alpha_K, \tag{13}
\]

where \(\kappa_5^2\) is the 5D gravitational constant. The above energy and entropy are those of a gas of massless 1D particles. In this case the temperatures for left and right moving string modes are given by
\[
T_L = \frac{1}{2\pi} \left( \frac{r_0}{r_1 r_5} \right) e^{\alpha_K}, \quad T_R = \frac{1}{2\pi} \left( \frac{r_0}{r_1 r_5} \right) e^{-\alpha_K}. \tag{14}
\]

This implies that the (left and right moving) momentum modes along the string direction are excited, while the excitations of D1–anti D1 and D5–anti D5-branes are suppressed. The Hawking temperature is given by their harmonic average.
\[
\frac{2}{T_H} = \frac{1}{T_L} + \frac{1}{T_R}.
\]  
(15)

We take here \( r_1 = r_5 = R \) and \( r_0 = r_K \) for simplicity. Then the linearized equation for s-wave fixed scalar take the form \[4,5\]

\[
\left[ (hr^3 \partial_r)^2 + \omega^2 r^6 f - \frac{8hr^4 r_\pm^4}{(r^2 + r_\pm^2)^2} \left( 1 + \frac{r_0^2}{r_\pm^2} \right) \right] \tilde{\phi}_\pm = 0,
\]  
(16)

where one gets \( \delta \tilde{\nu} \), for \( r_\pm^2 = R^2 \) and \( \delta \tilde{\lambda} \), for \( r_-^2 = R^2/3 \). For a minimally coupled scalar (\( \phi \)), the equation leads to \[8\]

\[
\left[ (hr^3 \partial_r)^2 + \omega^2 r^6 f - \frac{l(l+2)h}{r^2} \right] \delta \tilde{\phi} = 0.
\]  
(17)

Considering \( \delta \tilde{N} = r^{-3/2} \delta N \), for \( N = \nu, \lambda, \phi \) and introducing a tortoise coordinate \( r^* = f (dr/h) = r + (r_0/2) \ln |(r - r_\nu)/(r + r_\nu)| \) \[2\], then the equation takes the form

\[
\frac{d^2 \delta N}{dr^*^2} + (\omega^2 - \tilde{V}_N) \delta N = 0.
\]  
(18)

Here \( \tilde{V}_N \) in the dilute gas limit is given by

\[
\tilde{V}_\nu = -\omega^2 (f - 1) + \frac{3h}{4r^2} (1 + \frac{3r_\nu^2}{r^2}) + \frac{8R^4 h}{r^2 (r^2 + R^2)^2},
\]  
(19)

\[
\tilde{V}_\lambda = -\omega^2 (f - 1) + \frac{3h}{4r^2} (1 + \frac{3r_\lambda^2}{r^2}) + \frac{8R^4 h}{r^2 (3r^2 + R^2)^2},
\]  
(20)

\[
\tilde{V}_\phi = -\omega^2 (f - 1) + \frac{3h}{4r^2} (1 + \frac{3r_\phi^2}{r^2}) + \frac{l(l+2)h}{r^2},
\]  
(21)

where

\[
f - 1 = \frac{r_\nu^2 + 2R^2}{r^2} + \frac{(2r_\nu^2 + R^2) R^2}{r^4} + \frac{r_\nu^2 R^4}{r^6}.
\]  
(22)
FIG. 2. The graph of \((f - 1)\) in 5D black hole with \(r_0 = 0.01, R = 0.3\). A peak appears near horizon \((r = r_0)\).

We note that \(\tilde{V}_N\) depends on two parameters \((r_0, R)\) as well as the energy \((\omega)\). As (18) stands, it cannot be considered as the Schrödinger equation. The \(\omega\)-dependence is a matter of peculiar interest to us compared with the 4D black hole potentials \((V_{RW}, V_Z, V_{\psi})\). This makes the interpretation of \(\tilde{V}_N\) as a potential difficult. As is shown in Fig. 2, this is so because \((f - 1)\) is very large as \(10^6\) for \(r_0 = 0.01, R = 0.3\) near horizon. In order for \(\tilde{V}_N\) to be a potential, it is necessary to take the sufficiently low energy limit of \(\omega \rightarrow 0\). It is suitable to be \(10^{-3}\). And \(\omega^2(f - 1)\) is of order \(\mathcal{O}(1)\) and thus it can be ignored in comparison to the remaining ones. Hence we define a potential \(V_N\) to be \(\tilde{V}_N\) without \(\omega^2(f - 1)\). Further the last terms in (19)-(21) are important to compare each other. After the partial fraction, the last terms in (19)-(20) lead to

\[
\frac{8R^4}{r^2(r^2 + R^2)^2} = \frac{8}{r^2} \frac{8}{r^2 + R^2} - \frac{8R^2}{(r^2 + R^2)^2},
\]

(23)

\[
\frac{8R^4}{r^2(3r^2 + R^2)^2} = \frac{8}{r^2} \frac{24}{3r^2 + R^2} - \frac{24R^2}{(3r^2 + R^2)^2}.
\]

(24)
The last term in (21) for a minimally coupled scalar with \( l = 2 \) keeps the first terms in (23)-(24). Thus one finds immediately the sequence

\[
V_{\phi_0} \ll V_\lambda \leq V_\nu \leq V_{\phi_2}.
\] (25)

Here the subscript \( \phi_0 \) denotes the s-wave(\( l = 0 \)) free scalar and \( \phi_2 \) the free one with \( l = 2 \). This is confirmed from the graphs of potential in Fig.3 with \( r_0 = 0.01, R = 0.3 \). It is conjectured that \( V_\lambda \simeq V_\nu \simeq V_{\phi_0}^{l=2} \) gives us the nearly same \( \mathcal{R} \) and \( \mathcal{A} \). This implies the nearly same absorption cross section because of \( \sigma_{5D} = 4\pi \mathcal{A}/\omega^3 \).

![Potential Graphs](image)

**FIG. 3.** Four potential graphs \((V_{\phi_0}^{l=0}, V_\nu, V_\lambda, V_{\phi_2}^{l=2})\) for 5D black hole with \( r_0 = 0.01, R = 0.3 \).

On the other hand, using Eqs.(16)-(17), the low-energy absorption cross sections are calculated as

\[
\sigma_{5D}^{\phi_0} = A_{5D}^H, \quad \sigma_{5D}^{\phi_2} = \frac{3}{16} (\omega r_0)^4 = \frac{3}{4} (\omega R)^4 \frac{A_{5D}^H}{R} \left(\frac{r_0}{R}\right)^4, \quad (26)
\]

\[
\sigma_{5D}^\nu = \frac{A_{5D}^H}{4} \left(\frac{r_0}{R}\right)^4, \quad (27)
\]

\[
\sigma_{5D}^{\nu} = \frac{A_{5D}^H}{4} \left(\frac{r_0}{R}\right)^4, \quad (28)
\]
\[ \sigma_{5D}^\lambda = 9 \frac{A_{5H}^5}{4} \left( \frac{r_0}{R} \right)^4, \]  
(29)

(30)

with the area of horizon \( A_{5H}^5 = 2\pi^2 R^2 r_K \) for the 5D black hole. In deriving the above, one uses the condition of \( \omega < T_L, T_R, T_H \). Here we find a sequence of cross section

\[ \sigma_{5D}^{\phi_0} \gg \sigma_{5D}^\lambda \geq \sigma_{5D}^{\nu} \geq \sigma_{5D}^{\phi_2}. \]  
(31)

This originates from the potential sequence in (25). It is consistent with our naive expectation that the absorption cross section increases, as the height of potential decreases. Here we wish to point out the difference between a free and fixed scalar. In the dilute gas limit \((R \gg r_0)\) and the low energy limit \((\omega R \ll 1)\), the s-wave cross section for a minimally coupled scalar \(\sigma_{5D}^{\phi_0} \) goes to \( A_{5H}^5 \) \([3]\), while the s-wave cross sections for fixed scalars \((\nu, \lambda)\) including \(\phi_2\) approach zero \([4]\). This is consistent with our conjecture from (25).

Now we are in a position to discuss the \( AdS_3 \times S^3 \)-theory. In the near horizon, we approximate \((f - 1)\) in (19)-(21) as

\[ (f - 1) \approx (f - 1)^{AdS} = \frac{R^4}{r^4} \left( 1 + \frac{r_0^2}{r^2} \right). \]  
(32)

while the other terms remain invariant. The potential \( V_N^{AdS} \) where \( f - 1 \) is replaced by \((f - 1)^{AdS}\) corresponds to that for the \( AdS_3 \times S^3 \)-theory. In the sufficiently low energy limit of \( \omega \sim 10^{-3} \), two potential \((V_N, V_N^{AdS})\) take the same form. Thus we expect that two cross sections are same. Actually it turns out that the cross sections for the \( AdS \)-theory take the same form as (26) and (28) \([8]\)

\[ \sigma_{AdS}^{\phi_0} = A_{H}^{6D}, \]  
(33)

\[ \sigma_{AdS}^{\phi_2} = \sigma_{AdS}^{\nu} = \frac{1}{3} \frac{A_{H}^{6D}}{4} \left( \frac{r_0}{R} \right)^4 \]  
(34)

with the area of horizon \( A_{H}^{6D} = A_{H}^{5D} \times 2\pi R = 4\pi^3 R^3 r_K \) for the \( AdS_3 \times S^3 \). We note that \( \phi_2 \) and \( \nu \) give us slightly different cross sections in the D-brane black hole, whereas these do not make any distinction in the \( AdS \)-theory.
However, in the low temperature limit ($\omega \gg T_H, T_L, T_R$), $\omega^2 (f - 1)$-term plays an important role. Here we have to assume the low energy scattering with $\omega R \ll 1$. For example we choose $\omega^2 \sim 10^{-3}$ and $\omega^2 (f - 1) \sim 10^3$. Then this becomes comparable with $V_{\phi_0}$. This can be observed from the behavior of $f - 1$ in Fig. 2 and $V_{\phi_0}$ in Fig. 3. In this case one expects the cross section to behave as \[ \tilde{\sigma}^\phi_{5D} = A^5_H [1 + \mathcal{O} (\omega R)^2 \ln (\omega R)]. \] (35)

The logarithmic correction term encodes the leading order departure from the conformal limit. This means that nonrenormalizable interactions enter into the world sheet action at the subleading order. However, in the semiclassical approach, this implies that we cannot obtain the Schrödinger equation with potential. This is because the $\omega$-dependence term ($\omega^2 (f - 1)$) is included as a part of the potential and is comparable with $V_{\phi_0}$.

In conclusion, we analyze the physical potentials surrounding the D-brane black hole. There is an essential difference between a free and fixed scalars. But the distinction between two fixed scalars ($\nu, \lambda$) is not clearly understood in the semiclassical approach. We express the difference between absorption cross section for a free and two fixed scalars in terms of their potentials. As in the 4D Schwarzschild black hole, they take all barrier-types in the low energy limit. This implies that there is no exponentially growing mode and thus this black hole is stable against the s-mode perturbations \[11\,14\]. Here we note that the stability analysis should be based on the physical modes. In our case these are two fixed scalars.

In the low temperature limit, we cannot obtain the Schrödinger-type equation. Thus the meaning of potential is unclear and the analysis of stability is obscure. Also this is related to the logarithmic correction of cross section. It seems that there is a relation between the potential ($V_{N^{AdS}}$) and conformal symmetry on the boundary at the spatial infinity \[16\]. We expect that this can be understood from the AdS/CFT correspondence \[17\,18\].
ACKNOWLEDGMENTS

This work was supported in part by the Basic Science Research Institute Program, Ministry of Education, Project NO. BSRI–98–2413.
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