Quantum Certificate Verification:
Single versus Multiple Quantum Certificates

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2 October 2001

Abstract
The class MA consists of languages that can be efficiently verified by classical probabilistic verifiers using a single classical certificate, and the class QMA consists of languages that can be efficiently verified by quantum verifiers using a single quantum certificate. Suppose that a verifier receives not only one but multiple certificates. In the classical setting, it is obvious that a classical verifier with multiple classical certificates is essentially the same with the one with a single classical certificate. However, in the quantum setting where a quantum verifier is given a set of quantum certificates in tensor product form (i.e. each quantum certificate is not entangled with others), the situation is different, because the quantum verifier might utilize the structure of the tensor product form. This suggests a possibility of another hierarchy of complexity classes, namely the QMA hierarchy. From this point of view, we extend the definition of QMA to QMA(k) for the case quantum verifiers use k quantum certificates, and analyze the properties of QMA(k).

To compare the power of QMA(2) with that of QMA(1) = QMA, we show one interesting property of “quantum indistinguishability”. This gives a strong evidence that QMA(2) is more powerful than QMA(1). Furthermore, we show that, for any fixed positive integer $k \geq 2$, if a language $L$ has a one-sided bounded error QMA($k$) protocol with a quantum verifier using $k$ quantum certificates, $L$ necessarily has a one-sided bounded error QMA(2) protocol with a quantum verifier using only two quantum certificates.

1 Introduction
The class MA [2, 3, 4] is a randomized generalization of the class NP. A quantum analogue of MA was apparently first discussed by Knill [14], was later studied by Kitaev [12], and was named QMA by Watrous [19]. Intuitively, MA is the class of languages that can be efficiently verified by classical probabilistic verifiers using a single classical certificate, and QMA is the one that can be efficiently verified by quantum verifiers using a single quantum certificate.

Consider a situation that a verifier receives not only one but multiple certificates. In the classical setting, it is obvious that using multiple classical certificates does not increase the power of the verifier and the classical verifier with multiple classical certificates remains the same in the computational power with the one with a single classical certificate. However, in the quantum setting where a quantum verifier is given a set of quantum certificates unentangled each other, the situation is different, because the quantum verifier might utilize the fact that each certificate is not entangled with others. From this point of view, we extend the definition of QMA to QMA(k) for the case quantum verifiers use $k$ quantum certificates, and analyze the properties of QMA(k).

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This work was done while he was visiting the ERATO QCI project offices between May and August, 2001.
The most natural and important question to ask is how much stronger the quantum verifier becomes using two quantum certificates instead of one, or in other words, how much amount of help it is for the verifier to know the existence of a tensor product structure. For this question, this paper gives a strong evidence of the quantum verifier with two quantum certificates being much stronger than the one with only one quantum certificate. More precisely, we show somewhat a surprising result stated as follows.

**Theorem 1** Suppose one of the following two is true for given 2n qubits certificate \( |\Psi\rangle \):

\[
(a) \quad |\Psi\rangle\langle\Psi| \in H_0, \quad H_0 = \{|\Psi\rangle\langle\Psi| \mid \exists |\psi\rangle, |\phi\rangle : n \text{ qubits pure states}, \quad |\Psi\rangle = |\psi\rangle \otimes |\phi\rangle\}, \\
(b) \quad |\Psi\rangle\langle\Psi| \in H_1, \quad H_1 = \{|\Psi\rangle\langle\Psi| \mid \forall |\psi\rangle, |\phi\rangle : n \text{ qubits pure states}, \quad F(|\Psi\rangle\langle\Psi|, |\psi\rangle|\phi\rangle) \leq 1 - \varepsilon\}.
\]

Then, for any 0 \( \leq \varepsilon \leq 1 - 2^{-n/2} \), there is no quantum measurement better than the trivial strategy where one guesses at random without any operation at all.

Note that \( F(\cdot, \cdot) \) in the statement above represents the fidelity, and the formal definition of the fidelity is given in Section 2. This theorem holds true even if we replace \( H_1 \) by the set of maximally entangled states (see Lemma 3 in Section 3). We emphasize that this theorem is a quantum information theoretical one, and the indistinguishability stated in this theorem holds as far as we obey the law of quantum physics. Thus this theorem suggests something like “quantum indistinguishability” so to speak.

Although it is given above a strong evidence that using two quantum certificates is much different from using only one quantum certificate, this paper also points out that the situation might change in comparing QMA(3) with QMA(2). Let us say that a language \( L \) has a one-sided bounded error QMA(\( k \)) protocol if there exists a quantum polynomial-time verifier \( V \) and a polynomially bounded function \( p \geq 1 \) such that, for every input \( x \) of length \( n \), (i) if \( x \in L \), there exists a set of \( k \) quantum certificates which causes \( V \) accept \( x \) with probability 1, and (ii) if \( x \notin L \), then for any set of \( k \) quantum certificates, \( V \) accepts \( x \) with probability at most \( 1 - 1/p(n) \). In fact, we prove the following property.

**Theorem 2** Let \( L \) be a language having a one-sided bounded error QMA(3) protocol. Then \( L \) has a one-sided bounded error QMA(2) protocol.

A key idea to prove this theorem is to make use of the fact that there is no entanglement between two certificates given in QMA(2) protocols. Let \( |C_1\rangle, |C_2\rangle, \) and \( |C_3\rangle \) be three certificates given in a QMA(3) protocol. If two certificates \( |D_1\rangle \) and \( |D_2\rangle \) given in the corresponding QMA(2) protocol are of the form \( |D_1\rangle = |C_1\rangle \otimes |C_3\rangle \) and \( |D_2\rangle = |C_2\rangle \otimes |C_3\rangle \), it is obvious that we can simulate the QMA(3) protocol by the QMA(2) protocol. We use the Controlled-Swap operator and construct an efficient test for the decomposability of \( |D_1\rangle \) and \( |D_2\rangle \) into the form above. We also show that, in the one-sided error cases, this Controlled-Swap test is optimal in view of error probability to check this decomposability.

Actually, Theorem 2 can be generalized to the following theorem:

**Theorem 3** For any positive integer \( k \) and any \( r \in \{0, 1, 2\} \), let \( L \) be a language having a one-sided bounded error QMA(\( 3k + r \)) protocol. Then \( L \) has a one-sided bounded error QMA(\( 2k + r \)) protocol.

Applying this theorem repeatedly, we obtain the following theorem:

**Theorem 4** For any fixed positive integer \( k \), let \( L \) be a language having a one-sided bounded error QMA(\( k \)) protocol. Then \( L \) has a one-sided bounded error QMA(2) protocol.

One important point to be mentioned concerning this theorem is that we do not know if \( L \) has an arbitrary small one-sided bounded error QMA(2) protocol, even if \( L \) has an exponentially small one-sided error QMA(\( k \)) protocol. The situations are similar even in the cases of Theorem 2 and Theorem 3. It remains open for \( k \geq 2 \) whether running polynomially many copies of the the QMA(\( k \)) protocol
in parallel (i.e. parallel repetition) reduces the error probability to be exponentially small. For this reason, there still remains the possibility, even in the one-sided bounded error cases, that each QMA\((k)\) does not collapse to the other and forms the QMA hierarchy.

The remainder of this paper is organized as follows. In Section 3 we briefly review basic notations, definitions, and properties in quantum computation and quantum information theory, which are used in this paper. In Section 4 we give a formal definition of our model. In Section 5 we show an interesting property of “quantum indistinguishability”. This gives a strong evidence that using two quantum certificates is more powerful than using only one quantum certificate. In Section 6 we show that, for any fixed positive integer \(k \geq 2\), if a language \(L\) has a one-sided bounded error QMA\((k)\) protocol with a quantum verifier using \(k\) quantum certificates, \(L\) necessarily has a one-sided bounded error QMA\((2)\) protocol with a quantum verifier using only two quantum certificates. Finally we conclude with Section 7 which summarizes our results and mentions a number of open problems related to our model.

2 Quantum Fundamentals

Here we briefly review basic notations and definitions in quantum computation and quantum information theory. Detailed descriptions are, for instance, in [10,16].

A pure state is described by a unit vector in some Hilbert space. In particular, an \(n\)-dimensional pure state is a unit vector \(|\psi\rangle\) in \(\mathbb{C}^n\). Let \(|e_1\rangle, \ldots, |e_n\rangle\) be an orthonormal basis for \(\mathbb{C}^n\). Then any pure state in \(\mathbb{C}^n\) can be described as \(\sum_{i=1}^n \alpha_i |e_i\rangle\) for some \(\alpha_1, \ldots, \alpha_n \in \mathbb{C}\), \(\sum_{i=1}^n |\alpha_i|^2 = 1\).

A mixed state is a classical probability distribution \((p_i, |\psi_i\rangle)\), \(0 \leq p_i \leq 1\), \(\sum_i p_i = 1\) over pure states \(|\psi_i\rangle\). This can be interpreted as being in the pure state \(|\psi_i\rangle\) with probability \(p_i\). A mixed state is often described in the form of a density matrix \(\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|\). Any density matrix is positive semidefinite and has trace 1.

If a unitary transformation \(U\) is applied to a state \(|\psi\rangle\), the state becomes \(U|\psi\rangle\), or in the form of density matrices, a state \(\rho\) changes to \(U \rho U^\dagger\) after \(U\) is applied.

One of the important operations to density matrices is the trace-out operation. Given a density matrix \(\rho\) over \(\mathcal{H} \otimes \mathcal{K}\), the state after tracing out \(\mathcal{K}\) is a density matrix over \(\mathcal{H}\) described by

\[
\text{tr}_{\mathcal{K}} \rho = \sum_{i=1}^n (I_{\mathcal{H}} \otimes \langle e_i|) \rho (I_{\mathcal{H}} \otimes |e_i\rangle)
\]

for any orthonormal basis \(|e_1\rangle, \ldots, |e_n\rangle\) of \(\mathcal{K}\), where \(n\) is the dimension of \(\mathcal{K}\) and \(I_{\mathcal{H}}\) is the identity operator over \(\mathcal{H}\). To perform this operation on some part of a quantum system gives a partial view of the quantum system with respect to the remaining part.

For any mixed state with its density matrix \(\rho\) over \(\mathcal{H}\), there is a pure state \(|\psi\rangle\) in \(\mathcal{H} \otimes \mathcal{K}\) for the Hilbert space \(\mathcal{K}\) of \(\dim(\mathcal{K}) = \dim(\mathcal{H})\) such that \(|\psi\rangle\) is a purification of \(\rho\), that is, \(\text{tr}_{\mathcal{K}} |\psi\rangle \langle \psi| = \rho\).

One of the important concepts in quantum physics is a measurement. Any collection of linear operators \(\{A_1, \ldots, A_k\}\) satisfying \(\sum_{i=1}^k A_i^\dagger A_i = I\) defines a measurement. If a system is in a pure state \(|\psi\rangle\), such a measurement results in \(i\) with probability \(\|A_i|\psi\rangle\|^2\), and the state becomes \(A_i|\psi\rangle/\|A_i|\psi\rangle\|\). If a system is in a mixed state with its density matrix \(\rho\), the result \(i\) is observed with probability \(\text{tr}(A_i \rho A_i^\dagger)\), and the state after the measurement is with its density matrix \(A_i \rho A_i^\dagger/\text{tr}(A_i \rho A_i^\dagger)\). Let us write \(M_i = A_i^\dagger A_i\) for each \(i\). Then the measurement \(\{M_1, \ldots, M_k\}\) on \(\rho\) results in \(i\) with probability \(\text{tr}(M_i \rho)\). Statistics of results of the measurement is decided by \(M = \{M_1, \ldots, M_k\}\), and this set is called a positive operator valued measure (POVM). Formally, a POVM is defined to be a set of operators \(\mathcal{M} = \{M_1, \ldots, M_k\}\) satisfying (i) \(M_i\) is a non-negative hermitian matrix and (ii) \(\sum_{i=1}^k M_i = I\). For any POVM \(\mathcal{M}\), there is a quantum mechanical measurement such that the probability of the measurement results in \(i\) is equal to \(\text{tr}(M_i \rho)\). Therefore we may allow a little abuse of the term “measurement” instead of using the term “POVM”. A special class of measurements are projection or von Neumann measurements where \(\{A_1, \ldots, A_k\}\) is a collection of orthonormal projections. In this scheme, an observable is a decomposition of \(\mathcal{H}\) into orthogonal subspaces \(\mathcal{H}_1, \ldots, \mathcal{H}_k\), that is, \(\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_k\).
More mathematically rigorous descriptions of quantum measurements are, for example, in [11, 17]. It is important to note that two mixed states having same density matrix cannot be distinguished at all by any measurement.

The trace norm of the linear operator $A$ is defined by

$$\|A\|_{tr} = \frac{1}{2} \text{tr} \sqrt{A^\dagger A}. $$

In general, the trace norm $\|\rho - \sigma\|_{tr}$ gives an appropriate measure of distance between two density matrices $\rho$ and $\sigma$.

Another important measure between two density matrices $\rho$ and $\sigma$ is the fidelity $F(\rho, \sigma)$ defined by

$$F(\rho, \sigma) = \text{tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}. $$

For any density matrices $\rho, \sigma$, $0 \leq F(\rho, \sigma) \leq 1$ is satisfied, and $F(\rho, \sigma) = 1$ if and only if $\rho = \sigma$.

The following two are important properties on the trace norm and the fidelity.

**Theorem 5 (1)** Let $p^M = (p^M_1, \ldots, p^M_m)$, $q^M = (q^M_1, \ldots, q^M_m)$ be the probability distributions generated by a POVM $M$ on mixed states with density matrices $\rho, \sigma$, respectively. Then, for any POVM $M$, $1/2|p^M - q^M| \leq \|\rho - \sigma\|_{tr}$, where $|p^M - q^M| = \sum_{i=1}^m |p^M_i - q^M_i|$.

**Theorem 6 (2)** For any density matrices $\rho$ and $\sigma$,

$$1 - F(\rho, \sigma) \leq \|\rho - \sigma\|_{tr} \leq \sqrt{1 - (F(\rho, \sigma))^2}. $$

### 3 Definitions

#### 3.1 Polynomial-time Uniformly Generated Families of Quantum Circuits

First we review the concept of polynomial-time uniformly generated families of quantum circuits according to [13].

A family $\{Q_x\}$ of quantum circuits is said to be *polynomial-time uniformly generated* if there exists a deterministic procedure that, on each input $x$, outputs a description of $Q_x$ and runs in time polynomial in $n = |x|$. For simplicity, we assume all input strings are over the alphabet $\Sigma = \{0, 1\}$. It is assumed that the circuits in such a family are composed only of gates in the Shor basis [18]: Hadamard gates, $\sqrt{\sigma_z}$ gates, and Toffoli gates. Furthermore, it is assumed that the number of gates in any circuit is not more than the length of the description of that circuit, therefore $Q_x$ must have size polynomial in $n$. For convenience, we may identify a circuit $Q_x$ with the unitary operator it induces.

As is mentioned in [13, 1, 5], to permit non-unitary quantum circuits, in particular, to permit measurements at any timing in the computation does not change the computational power of the model. See [1] for a detailed description of the equivalence of the unitary and non-unitary quantum circuit models.

#### 3.2 Quantum Verifier with Multiple Quantum Certificates

Watrous [13] defined the class QMA in terms of quantum circuits. Here we extend this definition of QMA and define the class QMA($k$) for the case quantum verifiers use $k$ quantum certificates.

Let $k$ be the number of certificates. For each input $x \in \Sigma^*$, of length $n = |x|$, each quantum certificate $|C_i\rangle$ is a quantum pure state consists of $q_M(n)$ qubits. Without loss of generality we assume that $q_{M_1} = \cdots = q_{M_k} = q_M$ holds for some polynomially bounded function $q_M: \mathbb{Z}_+ \rightarrow \mathbb{N}$. 


Besides $kq_M(n)$ qubits for the certificates, we have $q_V(n)$ qubits called private qubits in our quantum circuit. Hence, the whole system of our quantum circuit consists of $q_V(n) + kq_M(n)$ qubits. All the private qubits are initialized to the $|0\rangle$ state, and one of the private qubits is designated as the output qubit.

A $(q_V, q_M)$-restricted quantum verifier $V$ is a polynomial-time computable mapping of the form $V: \Sigma^* \to \Sigma^*$, where $\Sigma = \{0,1\}$ is the alphabet set. For any $x$ of length $n$, $V(x)$ generates a description of a polynomial-time uniformly generated quantum circuit acting on $q_V(n) + kq_M(n)$ qubits.

The probability that $V$ accepts the input $x$ is defined to be the probability that an observation of the output qubit (in the $\{|0\rangle, |1\rangle\}$ basis) yields 1, after the circuit $V(x)$ is applied to the state $|0^{q_V(n)}\rangle \otimes |C_1\rangle \otimes \cdots \otimes |C_k\rangle$.

**Definition 7** For a positive integer $k$ and functions $a, b: \mathbb{Z}^+ \to [0,1]$, a language $L$ is in $\text{QMA}(k,a,b)$ if there exist polynomially bounded functions $q_V, q_M: \mathbb{Z}^+ \to \mathbb{N}$ and a $(q_V, q_M)$-restricted quantum verifier $V$ such that, for any $x$ of length $n$,

(i) if $x \in L$, there exists a set of quantum certificates $|C_1\rangle, \ldots, |C_k\rangle$ of $q_M(n)$ qubits such that, given $|C_1\rangle, \ldots, |C_k\rangle$, $V$ accepts $x$ with probability at least $a(n)$,

(ii) if $x \notin L$, given any set of quantum certificates $|C'_1\rangle, \ldots, |C'_k\rangle$ of $q_M(n)$ qubits, $V$ accepts $x$ with probability at most $b(n)$.

For convenience, we say that a language $L$ has a one-sided bounded error $\text{QMA}(k)$ protocol iff $L$ is in $\text{QMA}(k,1,1-1/p)$ for some polynomially bounded function $p \geq 1$. We also write $\text{QMA}(k)$ in short if it is not confusing to omit the arguments corresponding to the error probabilities.

Note that the definition of the class $\text{QMA}(k,a,b)$ is closely related to quantum multi-prover interactive proof systems which was introduced by Kobayashi and Matsumoto [15], the model without any prior entanglement among provers. Of particular interest are 1-message quantum $k$-prover interactive proof systems, which can be shown equivalent in view of computational power to the model of quantum verifiers with $k$ quantum certificates. In fact, the class $\text{QMA}(k,a,b)$ is equal to $\text{QMIP}(k,1,a,b)$, the class of languages having 1-message quantum $k$-prover interactive proof systems with two-sided error probability $(a,b)$, where the provers do not share any prior entanglement before the computation.

### 4 Two Quantum Certificates versus One Quantum Certificate

First we focus on the relation between $\text{QMA}(2)$ and $\text{QMA}(1)$. Obviously, $\text{QMA}(1,a,b) \subseteq \text{QMA}(2,a,b)$ is satisfied for any functions $a, b$. Concerning whether the other side of inclusion holds, it is natural to consider the simulation of the protocol with two quantum certificates by using only one quantum certificate. In this section, we show the strong implication that the simulation is impossible.

Suppose we have a quantum subroutine which answers which of (a) and (b) is true for given certificate $|\Psi\rangle \in \mathcal{H}^\otimes 2$ of $2n$ qubits, where $\mathcal{H}$ is the Hilbert space which consists of $n$ qubits:

(a) $|\Psi\rangle \langle \Psi| \in \mathcal{H}_0$, $\mathcal{H}_0 = \{|\Psi\rangle \langle \Psi| \mid \exists |\psi\rangle, |\phi\rangle : n$ qubits pure states, $|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle\}$,

(b) $|\Psi\rangle \langle \Psi| \in \mathcal{H}_1^\otimes$, $\mathcal{H}_1^\otimes = \{|\Psi\rangle \langle \Psi| \mid \forall |\psi\rangle, |\phi\rangle : n$ qubits pure states, $F(|\Psi\rangle \langle \Psi|, |\psi\rangle \langle \psi| \otimes |\phi\rangle \langle \phi|) \leq 1 - \varepsilon\}$.

As for the certificate $|\Psi\rangle$ which does not satisfy (a) nor (b), this subroutine may answer (a) or (b) at random. If the subroutine answers that $|\Psi\rangle$ satisfies (b), the verifier of the $\text{QMA}(1)$ protocol rejects. Otherwise, using the certificate $|\Psi\rangle$, the quantum verifier fulfills the same verification procedure as the one in the original $\text{QMA}(2)$ protocol. It seems to the authors that there is no other way to simulate two quantum certificates by only one quantum certificate. Therefore, the authors conjecture that $\text{QMA}(1)$ is strictly smaller than $\text{QMA}(2)$, because this kind of subroutines cannot be realized by any physical method, which can be proven as follows.
exists µ probability of distinguishment of two states where each µ

Proof. Let \( M = \{M_0, M_1\} \) be a POVM on given \( |Ψ⟩⟨Ψ| \). With \( M \) we conclude \( |Ψ⟩⟨Ψ| \in H_i \) if \( M \) results in \( i \) (\( i = 0, 1 \)). Let \( P^M_{i→j}(|Ψ⟩⟨Ψ|) \) denote the probability that \( |Ψ⟩⟨Ψ| \in H_j \) is concluded while \( |Ψ⟩⟨Ψ| \in H_i \) is true. We want to find the measurement which minimizes \( P^M_{0→1}(|Ψ⟩⟨Ψ|) \) keeping the other side of error small enough. More precisely, we want to evaluate \( E \) defined and bounded as follows.

\[
E \overset{\text{def}}{=} \min_M \left\{ \max_{\rho \in H_0} P^M_{0→1}(\rho) \left| \max_{\rho \in H_1} P^M_{1→0}(\rho) \leq \delta \right. \right\}
\]

\[
\geq \min_M \left\{ \int_{\rho \in H_0} P^M_{0→1}(\rho) \mu_0(\rho) d\rho \left| \int_{\rho \in H_1} P^M_{1→0}(\rho) \mu_1(\rho) d\rho \leq \delta \right. \right\}
\]

\[
= \min_M \left\{ P^M_{0→1} \left( \int_{\rho \in H_0} \rho \mu_0(\rho) d\rho \right) \left| P^M_{1→0} \left( \int_{\rho \in H_1} \rho \mu_1(\rho) d\rho \right) \leq \delta \right. \right\},
\]

where each \( \mu_i \) is an arbitrary probability measure in \( H_i \). This means that \( E \) is larger than the error probability of distinguishment of two states \( \int_{\rho \in H_0} \rho \mu_0(\rho) d\rho \) and \( \int_{\rho \in H_1} \rho \mu_1(\rho) d\rho \). Furthermore, there exists \( \mu_i \) such that

\[
\int_{\rho \in H_0} \rho \mu_0(\rho) d\rho = \int_{\rho \in H_1} \rho \mu_1(\rho) d\rho = \frac{1}{d^2} I_{H^{⊗2}}.
\]

Here, \( \mu_0 \) is a uniform distribution on the set \( \{|e_i⟩⟨e_i| \}^d_{i=1} \) and \( \mu_1 \) is a uniform distribution on the set \( \{|g_{n,m}⟩⟨g_{n,m}| \}^d_{n=1; m=1} \), that is, \( \mu_0(\{|e_i⟩⟨e_i| \}) = 1/d^2 \) for each \( i \), where \( \{|e_1⟩, \ldots, |e_d⟩\} \) is an orthonormal basis of \( H \). Similarly, \( \mu_1 \) is a uniform distribution on the set \( \{|g_{n,m}⟩⟨g_{n,m}| \}^d_{n=1; m=1} \), that is, \( \mu_1(\{|g_{n,m}⟩⟨g_{n,m}| \}) = 1/d^2 \) for each \( n, m \), where

\[
|g_{n,m}⟩ = \frac{1}{\sqrt{d}} \sum_j e^{2\pi i jn/d} |e_j⟩ \otimes |e_{(j+m)modd}⟩.
\]

This \( \{|g_{1,1}⟩, \ldots, |g_{d,d}⟩\} \) is an orthonormal basis of \( H^{⊗2} \). Thus we have the assertion from (1). □

From Lemma 8 we can easily show Theorem 3.

5 k Quantum Certificates versus Two Quantum Certificates

In the last section we gave a strong evidence that using two quantum certificates is much different from using only one quantum certificate. Here we point out that the situation might change if we compare the case of using \( k \geq 2 \) quantum certificates with the case of using only two quantum certificates. The C-SWAP algorithm described below is the key idea of our claim.
5.1 Utilization of Controlled-Swap

Given a pair of \( n \) qubits mixed states \( \rho, \sigma \) of the form \( \rho \otimes \sigma \), consider the following algorithm, which we call C-SWAP algorithm. A restricted version of this algorithm with an input to be a pair of \( n \) qubits pure states is utilized in [7].

We prepare quantum registers \( B, R_1, \) and \( R_2 \). \( B \) consists of only one qubit, both of \( R_1 \) and \( R_2 \) consist of \( n \) qubits, and all the qubits in \( B, R_1, \) and \( R_2 \) are initially set to the \( |0\rangle \) state.

**C-SWAP Algorithm**

1. Set \( \rho, \sigma \) in \( R_1, R_2 \), respectively.
2. Apply the Hadamard transformation \( H \) to \( B \).
3. Apply controlled-swap operator on \( R_1 \) and \( R_2 \) with using \( B \) as a control qubit. That is, swap the contents of \( R_1 \) and \( R_2 \) if \( B \) contains 1, and do nothing if \( B \) contains 0.
4. Apply the Hadamard transformation \( H \) to \( B \) and accept if \( B \) contains 0.

**Proposition 9** The probability that the input \( \rho, \sigma \) is accepted in the C-SWAP algorithm is exactly \( 1/2 + \text{tr}(\rho\sigma)/2 \).

**Proof.** Let \( B, R_1, R_2 \) denote the Hilbert spaces corresponding to the qubits in \( B, R_1, R_2 \), respectively. Let \( \rho = \sum_i p_i |\phi_i\rangle\langle \phi_i| \) and \( \sigma = \sum_j q_j |\psi_j\rangle\langle \psi_j| \) be decomposition of \( \rho \) and \( \sigma \) with respect to the orthonormal bases \( \{|\phi_i\rangle\}, \{|\psi_j\rangle\} \) of \( R_1, R_2 \), respectively.

We introduce the Hilbert spaces \( S_1 = l_2(\Sigma^n) \) and \( S_2 = l_2(\Sigma^n) \). Then there exist purifications \( |\phi\rangle \in R_1 \otimes S_1 \) and \( |\psi\rangle \in R_2 \otimes S_2 \) of \( \rho \) and \( \sigma \), respectively, such that

\[
|\phi\rangle = \sum_i \sqrt{p_i} |\phi_i\rangle |\phi_i\rangle, \quad |\psi\rangle = \sum_j \sqrt{q_j} |\psi_j\rangle |\psi_j\rangle.
\]

Now consider the following pure state \( |\xi\rangle \in B \otimes R_1 \otimes S_1 \otimes R_2 \otimes S_2 \),

\[
|\xi\rangle = |0\rangle \otimes |\phi\rangle \otimes |\psi\rangle = \sum_{i,j} \sqrt{p_i q_j} |0\rangle |\phi_i\rangle |\psi_j\rangle |\phi_i\rangle |\psi_j\rangle.
\]

The probability that the input pair of \( \rho, \sigma \) is accepted in the C-SWAP algorithm is exactly equal to the probability of acceptance when the C-SWAP algorithm is applied to \( |\xi\rangle \) over the Hilbert space \( B \otimes R_1 \otimes R_2 \).

If the C-SWAP algorithm is applied to \( |\xi\rangle \), we can easily see that the state \( |\eta\rangle \in B \otimes R_1 \otimes S_1 \otimes R_2 \otimes S_2 \) before the final measurement of the output qubit is given by

\[
|\eta\rangle = \frac{1}{2} |0\rangle \otimes \left( \sum_{i,j} \sqrt{p_i q_j} (|\phi_i\rangle |\phi_i\rangle |\psi_j\rangle |\psi_j\rangle + |\psi_j\rangle |\phi_i\rangle |\phi_i\rangle |\psi_j\rangle) \right)
\]

\[
+ \frac{1}{2} |1\rangle \otimes \left( \sum_{i,j} \sqrt{p_i q_j} (|\phi_i\rangle |\psi_j\rangle |\phi_i\rangle |\psi_j\rangle - |\psi_j\rangle |\phi_i\rangle |\phi_i\rangle |\psi_j\rangle) \right).
\]

Thus the probability of acceptance is \( (1 + t)/2 \), where \( t \) is given by

\[
t = \sum_{i,j} p_i q_j (|\phi_i\rangle |\psi_j\rangle, |\psi_j\rangle |\phi_i\rangle) = \sum_{i,j} p_i q_j (|\phi_i\rangle |\psi_j\rangle, |\psi_j\rangle |\phi_i\rangle) = \sum_{i} p_i \text{tr}(\sigma |\phi_i\rangle \langle \phi_i|) = \text{tr}(\rho \sigma),
\]

where \((\cdot, \cdot)\) represents the inner product. This completes the proof. \( \square \)
5.2 Reducing the Number of Quantum Certificates

Now we consider reducing the number of quantum certificates $k$, given a QMA($k$) protocol of using $k$ quantum certificates. First we consider simulating one-sided bounded error QMA($3$) protocols by one-sided bounded error QMA(2) protocols.

Lemma 10 For any polynomially bounded function $p: \mathbb{Z}^+ \rightarrow \mathbb{R}^+, p \geq 1,$

$$\text{QMA}(3, 1, 1 - 1/p) \subseteq \text{QMA}(2, 1, 1 - 1/(10p^2)).$$

Proof. Let $L$ be a language in QMA($3, 1, 1 - 1/p$). Given a QMA($3, 1, 1 - 1/p$) protocol for $L$, we construct a QMA($2, 1, 1 - 1/(10p^2)$) protocol for $L$ in the following way.

Let $V$ be the quantum verifier of the original QMA(3, 1, 1 – 1/p) protocol. For the input $x$ of length $n$, suppose that each of quantum certificates $V$ receives consists of $q_M(n)$ qubits and the number of private qubit of $V$ is $q_V(n)$. Let $U$ be the unitary transformation which the original quantum verifier $V$ applies. Our new quantum verifier $W$ of the QMA(2, 1, 1 – 1/p) protocol prepares quantum registers $R_1, R_2, S_1, S_2$ for quantum certificates and quantum registers $V, B$ for private computation. Each of $R_i$ and $S_i$ consists of $q_M(n)$ qubits, $V$ consists of $q_V(n)$ qubits, and $B$ consists of a single qubit. $W$ receives two quantum certificates $|D_1>, |D_2>$ of length $2q_M(n)$, which are expected to be of the form

$$|D_1> = |C_1> \otimes |C_3>, \quad |D_2> = |C_2> \otimes |C_3>,$$

(2)

where each $|C_i>$ is the $i$th quantum certificate which the original quantum verifier $V$ receives. Of course, each $|D_i>$ may not be of the form above and the first and the second $q_M(n)$ qubits of $|D_i>$ may be entangled. Let $V, B, R_i$, and each $S_i$ be the Hilbert spaces corresponding to the quantum registers $V, B, R_i, S_i$, respectively. $W$ runs the following protocol:

Verifier $W$’s Protocol

1. Set the contents of the first $q_M(n)$ qubits of $|D_1>$ in $R_1$, and the contents of the second $q_M(n)$ qubits of $|D_1>$ in $S_1$.
   Set the contents of the first $q_M(n)$ qubits of $|D_2>$ in $R_2$, and the contents of the second $q_M(n)$ qubits of $|D_2>$ in $S_2$.
2. Do one of the following two tests at random.

   2.1 Separability test:
   Apply the C-SWAP algorithm over $B \otimes S_1 \otimes S_2$, using quantum registers $B, S_1, S_2$.
   Accept if $B$ contains 0, otherwise reject.

   2.2 Consistency test:
   Apply $U$ over $V \otimes R_1 \otimes R_2 \otimes S_1$, using quantum registers $V, R_1, R_2, S_1$.
   Accept iff the result corresponds to the acceptance computation of the original quantum verifier.

(i) In case the input $x$ of length $n$ is in $L$:
In the original QMA(3, 1, 1 – 1/p) protocol for $L$, there exist quantum certificates $|C_1>, |C_2>, |C_3>$ which cause the original quantum verifier $V$ accept $x$ with probability 1. In the constructed QMA(2) protocol, let the quantum certificates $|D_1>, |D_2>$ be of the form $|D_1> = |C_1> \otimes |C_3>, |D_2> = |C_2> \otimes |C_3>$. Then it is obvious that the constructed quantum verifier $W$ accepts $x$ with probability 1.
(ii) In case the input $x$ of length $n$ is not in $L$:
Consider any pair of quantum certificates $|D_1^r, D_2^r\rangle$, which are set in the pairs of the quantum registers ($R_1, S_1$), ($R_2, S_2$), respectively. Let $\rho = \text{tr}_{R_1}|D_1^r\rangle\langle D_1^r|$ and $\sigma = \text{tr}_{R_2}|D_2^r\rangle\langle D_2^r|$. Let $\varepsilon = 1 - 1/p(n)$ and $\delta = (-1 + 2\varepsilon + 4\sqrt{1 + \varepsilon - \varepsilon^2})/5$. The reason why we set $\delta$ at this value will be clear later in the item b.

a. In case $\text{tr}(\rho\sigma) \leq \delta$:
In this case the probability $\alpha$ that the input $x$ is accepted in the SEPARABILITY TEST is at most

$$\alpha \leq \frac{1}{2} + \frac{\delta}{2} = \frac{2 + \varepsilon + 2\sqrt{1 + \varepsilon - \varepsilon^2}}{5} \leq \frac{4 + 2\varepsilon - \varepsilon^2}{5} = 1 - \frac{(1 - \varepsilon)^2}{5},$$

where the second inequality is from the fact $a + b \geq 2\sqrt{ab}, a \geq 0, b \geq 0$. Thus the verifier $W$ accepts the input $x$ with probability at most

$$\frac{1}{2} + \alpha \leq 1 - \frac{(1 - \varepsilon)^2}{10} = 1 - \frac{1}{10(p(n))^2}. $$

b. In case $\text{tr}(\rho\sigma) > \delta$:
$\text{tr}(\rho\sigma) > \delta$ means the maximum eigenvalue $\lambda$ of $\rho$ satisfies $\lambda > \delta$. Thus there exist pure states $|C_1^r\rangle \in R_1$ and $|C_3^r\rangle \in S_1$ such that

$$F(|C_1^r\rangle\langle C_1^r| \otimes |C_3^r\rangle\langle C_3^r|, |D_1^r\rangle\langle D_1^r|) > \sqrt{\delta},$$

since $\rho = \text{tr}_{R_1}|D_1^r\rangle\langle D_1^r|$. Similarly, the maximum eigenvalue of $\sigma$ is more than $\delta$ and there exist pure states $|C_2^r\rangle \in R_2$ and $|C_4^r\rangle \in S_2$ such that

$$F(|C_2^r\rangle\langle C_2^r| \otimes |C_4^r\rangle\langle C_4^r|, |D_2^r\rangle\langle D_2^r|) > \sqrt{\delta}.$$

Thus, letting $|\phi\rangle = |C_1^r\rangle \otimes |C_3^r\rangle \otimes |C_2^r\rangle \otimes |C_4^r\rangle$ and $|\psi\rangle = |D_1^r\rangle \otimes |D_2^r\rangle$, we have

$$F(|\phi\rangle\langle \phi|, |\psi\rangle\langle \psi|) > \delta.$$

Therefore, from Theorem 3 we have

$$\|\langle \phi| - |\psi\rangle\|_{\text{tr}} \leq \sqrt{1 - F(|\phi\rangle\langle \phi|, |\psi\rangle\langle \psi|)^2} < \sqrt{1 - \delta^2}.$$ 

With Theorem 3, this implies that, the probability $\beta$ that the input $x$ is accepted in the CONSISTENCY TEST is bounded by

$$\beta < \varepsilon + \sqrt{1 - \delta^2},$$

since given any quantum certificates $|C_1^r, C_2^r, C_3^r\rangle$ the original quantum verifier $V$ accepts the input $x$ with probability at most $\varepsilon = 1 - 1/p(n)$. Noticing that $\delta$ satisfies

$$\frac{1}{2} + \frac{\delta}{2} = \varepsilon + \sqrt{1 - \delta^2},$$

we can see that

$$\beta < 1 - \frac{(1 - \varepsilon)^2}{5}.$$ 

Thus the verifier $W$ accepts the input $x$ with probability at most

$$\frac{1}{2} + \frac{\beta}{2} < 1 - \frac{(1 - \varepsilon)^2}{10} = 1 - \frac{1}{10(p(n))^2}. $$

□
The following theorem is an immediate consequence of Lemma 10.

**Theorem 2** Let $L$ be a language having a one-sided bounded error QMA(3) protocol. Then $L$ has a one-sided bounded error QMA(2) protocol.

Actually, Lemma 10 can be generalized to the following lemma:

**Lemma 11** For any fixed positive integer $k$, any $r \in \{0, 1, 2\}$, and any polynomially bounded function $p: \mathbb{Z}^+ \to \mathbb{R}^+, p \geq 1$,

$$\text{QMA}(3k + r, 1, 1 - 1/p) \subseteq \text{QMA}(2k + r, 1, 1 - 1/(10p^2)).$$

**Proof.** Let $L$ be a language in QMA($3k + r, 1, 1 - 1/p$). Given a QMA($3k + r, 1, 1 - 1/p$) protocol for $L$, we construct a QMA($2k + r, 1, 1 - 1/(10p^2)$) protocol for $L$ in the following way.

Let $V$ be the quantum verifier of the original QMA($3k + r, 1, 1 - 1/p$) protocol. For the input $x$ of length $n$, suppose that each of quantum certificates $V$ receives consists of $q_M(n)$ qubits and the number of private qubit of $V$ is $q_V(n)$. Let $U$ be the unitary transformation which the original quantum verifier $V$ applies. Our new quantum verifier $W$ of the QMA($2k + r, 1, 1 - 1/(10p^2)$) protocol prepares quantum registers $R_{1, 1}, \ldots, R_{1,k}, R_{2,1}, \ldots, R_{2,k}, S_{1,1}, \ldots, S_{1,k}, S_{2,1}, \ldots, S_{2,k}, S_{3,1}, \ldots, S_{3,r}$ for quantum certificates and quantum registers $V, B$ for private computation. Each of $R_{i,j}$ and $S_{i,j}$ consist of $q_M(n)$ qubits, $V$ consists of $q_V(n)$ qubits, and $B$ consists of a single qubit. $W$ receives $2k + r$ quantum certificates $|D_{1,1}, \ldots, D_{1,k}, D_{2,1}, \ldots, D_{2,k}, D_{3,1}, \ldots, D_{3,r}\rangle$ of length $2q_M(n)$, which are expected to be of the form

$$|D_{1,j_1}\rangle = |C_{j_1}\rangle \otimes |C_{2k+j_1}\rangle,$$

$$|D_{2,j_2}\rangle = |C_{k+j_2}\rangle \otimes |C_{2k+j_2}\rangle,$$

$$|D_{3,j_3}\rangle = |C_{3k+j_3}\rangle \otimes |q_M(n)\rangle,$$

for each $1 \leq j_1 \leq k, 1 \leq j_2 \leq r$, where each $|C_i\rangle$ is the $i$th quantum certificate which the original quantum verifier $V$ receives. Let $\mathcal{V}, \mathcal{B}$, each $R_{i,j}$, and each $S_{i,j}$ be the Hilbert spaces corresponding to quantum registers $V, B, R_{i,j}, S_{i,j}$, respectively. $W$ runs the following protocol:

**Verifier $W$’s Protocol**

1. For each $i, j$, set the contents of the first $q_M(n)$ qubits of $|D_{i,j}\rangle$ in $R_{i,j}$, and the contents of the second $q_M(n)$ qubits of $|D_{i,j}\rangle$ in $S_{i,j}$.
2. For each $j$, if $S_{i,j}$ contains 1 in some qubit, reject.
3. Do one of the following two tests at random.

3.1 Separability Test:

   Apply the C-SWAP algorithm over $B \otimes (S_{1,1} \otimes \cdots \otimes S_{1,k}) \otimes (S_{2,1} \otimes \cdots \otimes S_{2,k})$, using the quantum register $B$, the $k$-tuple of quantum registers $(S_{1,1}, \ldots, S_{1,k})$, and the $k$-tuple of quantum registers $(S_{2,1}, \ldots, S_{2,k})$.

   Accept if $B$ contains 0, otherwise reject.

3.2 Consistency Test:

   Apply $U$ over $V \otimes R_{1,1} \otimes \cdots \otimes R_{1,k} \otimes R_{2,1} \otimes \cdots \otimes R_{2,k} \otimes S_{1,1} \otimes \cdots \otimes S_{1,k} \otimes S_{2,1} \otimes \cdots \otimes S_{3,k} \otimes S_{3,1} \otimes \cdots \otimes S_{3,r}$, using quantum registers $V, R_{1,1}, \ldots, R_{1,k}, R_{2,1}, \ldots, R_{2,k}, S_{1,1}, \ldots, S_{1,k}, R_{3,1}, \ldots, R_{3,r}$.

   Accept iff the result corresponds to the acceptance computation of the original quantum verifier.

With a similar argument to the proof of Lemma 10, we can show that the above protocol is actually a QMA($2k + r, 1, 1 - 1/(10p^2)$) protocol for $L$.

The following theorem is an immediate consequence of Lemma 11.
Theorem 3 For any positive integer $k$ and any $r \in \{0, 1, 2\}$, let $L$ be a language having a one-sided bounded error QMA($3k + r$) protocol. Then $L$ has a one-sided bounded error QMA($2k + r$) protocol.

Now we show that any one-sided bounded error QMA($k$) protocol can be simulated by a QMA($2$) protocol with one-sided bounded error.

Lemma 12 For any fixed positive integer $k$ and any polynomially bounded function $p_1 : \mathbb{Z}^+ \rightarrow \mathbb{R}^+, p_1 \geq 1$, there exists a polynomially bounded function $p_2 : \mathbb{Z}^+ \rightarrow \mathbb{R}^+, p_2 \geq 1$ such that

$$QMA(k, 1, 1 - 1/p_1) \subseteq QMA(2, 1, 1 - 1/p_2).$$

Proof. By applying Lemma 11 $c = O(\log_{3/2} k)$ times repeatedly, we can easily obtain

$$QMA(k, 1, 1 - 1/p_1) \subseteq QMA(2, 1, 1 - 1/(10^{2^c} - p_1^{2^c})),,$$

for some constant $c$. Taking the polynomially bounded function $p_2 = 10^{2^c} - p_1^{2^c}$ completes the proof. \hfill $\square$

Thus we obtain the following theorem.

Theorem 4 For any fixed positive integer $k$, let $L$ be a language having a one-sided bounded error QMA($k$) protocol. Then $L$ has a one-sided bounded error QMA($2$) protocol.

5.3 Optimality of C-SWAP Algorithm

In the previous subsection, we showed how to simulate a one-sided bounded error QMA($3$) protocol by a one-sided bounded error QMA($2$) protocol using the C-SWAP algorithm. One might suspect that there is a better simulation than ours to avoid the increase of the error probability in the simulation. In this subsection, we show that the C-SWAP algorithm is optimal to check the decomposability of \textup{(B)} with one-sided error probability.

Let $M = \{M_0, M_1\}$ be a POVM. If the result of $M$ is $1$, the certificate $|\Psi\rangle$ is judged that $|\Psi\rangle\langle\Psi|$ is in $H_0$, where

$$H_0 = \{|\Psi\rangle\langle\Psi| \mid |\Psi\rangle = |C_1\rangle|C_2\rangle|C_3\rangle|C_3\rangle, \ |C_i\rangle \in \mathcal{H}\},$$

for the Hilbert space $\mathcal{H}$ of dimension $d$. Our problem is to derive the optimal measurement for judging whether $|\Psi\rangle\langle\Psi| \in H_0$ or not. Here we only consider one-sided error cases, hence, the measurement must conclude $|\Psi\rangle\langle\Psi| \in H_0$ with probability $1$ if $|\Psi\rangle\langle\Psi| \in H_0$ is true.

Define $P_{\text{sym}}$ to be a projection operator in $\mathcal{H} \otimes \mathcal{H}$ whose image is

$$\text{span}\{|e_i\rangle|e_i\rangle + |e_j\rangle|e_i\rangle \mid 1 \leq i \leq d, 1 \leq j \leq d\},$$

where $\{|e_1\rangle, \ldots, |e_d\rangle\}$ is an orthonormal basis of $\mathcal{H}$.

Lemma 13 In the one-sided error cases, the optimal measurement $M = \{M_0, M_1\}$ to judge whether $|\Psi\rangle\langle\Psi| \in H_0$ or not is given by

$$M_0 = I_{\mathcal{H} \otimes \mathcal{H}} \otimes P_{\text{sym}}, \quad M_1 = 1 - M_0. \quad (3)$$

Proof. Since one-sided error is assumed, if $|\Psi\rangle\langle\Psi| \in H_0$ is true, the result of the measurement $M$ must be always $1$, therefore, for all $|\Psi\rangle\langle\Psi| \in H_0$, $\text{tr}(M_0|\Psi\rangle\langle\Psi|) = 1$ is satisfied. Thus, for every $|C_1\rangle, |C_2\rangle, |C_3\rangle \in \mathcal{H}$,

$$M_0|C_1\rangle|C_2\rangle|C_3\rangle|C_3\rangle = |C_1\rangle|C_2\rangle|C_3\rangle|C_3\rangle.$$
Hence, we have

\[ M_0 \geq I_{\mathcal{H} \otimes 2} \otimes P_{\text{sym}}, \]

which implies that, for any density matrix \( \rho \) over the Hilbert space \( \mathcal{H} \otimes 2 \),

\[ \text{tr}(\rho M_0) \geq \text{tr}(\rho I_{\mathcal{H} \otimes 2} \otimes P_{\text{sym}}). \]

This means that the measurement (3) minimizes the probability of concluding \( |\Psi\rangle\langle\Psi| \in \mathcal{H}_0 \) when \( |\Psi\rangle \not\in \mathcal{H}_0 \).

It is easy to check that our C-SWAP algorithm realizes the optimal POVM (3), and we have the following theorem.

**Theorem 14** In the one-sided error cases, the C-SWAP algorithm is optimal in view of error probability to check the decomposability of (2).

6 Conclusion and Open Problems

This paper introduced the class QMA\((k)\) in which the quantum verifier uses \( k \) quantum certificates. This suggests a possibility of another hierarchy of complexity classes, namely the QMA hierarchy. It was given a strong evidence that QMA\((2)\) differs from QMA\((1)\), and was also shown that, for any fixed positive integer \( k \geq 2 \), if a language \( L \) has a one-sided bounded error QMA\((k)\) protocol, \( L \) necessarily has a one-sided bounded error QMA\((2)\) protocol.

A number of interesting problems remain open in this paper.

- In the case of QMA(1) protocols, we can easily see that a parallel repetition of the protocol works well [13, 19]. For \( k \geq 2 \), does a parallel repetition of polynomially many times of the QMA\((k)\) protocol reduce the error probability to be exponentially small?

- Kitaev and Watrous [13, 19, 20] showed the PP upper bound for the class QMA(1). Is QMA\((k)\) also contained in PP for \( k \geq 2 \)?

- Can a two-sided bounded error QMA\((k)\) protocol be modified to a one-sided bounded error one?

- Does QMA\((k)\) collapse to the other for some \( k \), or do they form the QMA hierarchy?

- Suppose that the quantum certificates be prepared by the provers isolated each other, but the provers share prior entanglement (cf. [8]). Then how the situation changes?

**Acknowledgement**

The authors would like to thank John Watrous for providing the proof of QMA \( \subseteq \text{PP} \). The authors also thank Richard Cleve and Lance Fortnow for their helpful comments on writing this paper.

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