Abstract

In 1965 Dag Prawitz presented an extension of Gentzen-type systems of Natural Deduction to modal concepts, obtaining three new systems of Natural Deduction for S4. Maria da Paz Medeiros showed in 2006 that the proof of normalization for classical S4 does not hold and proposed a new proof of normalization for a logically equivalent system, the NS4. However, two problems in the proof of a lemma used by Medeiros in her proof were presented by Andou Yuuki in 2010. This paper introduces a new proof of normalization for NS4.

1 Introduction

In his doctoral dissertation, Dag Prawitz extended the Gentzen-type systems of Natural Deduction (ND) \[2\] to modal concepts, obtaining three Gentzen-type systems of ND for S4 based on classical, intuitionistic and minimal predicate logic. For this purpose, Prawitz added a modal operator for necessity (the “N” operator, here represented by $\square$) and defined the following rules for introduction and elimination of $\square$:

\[
\begin{array}{c}
A \\
\hline
\square A
\end{array}
\] $\square$-I \hspace{1cm}

\[
\begin{array}{c}
\square A \\
\hline
A
\end{array}
\] $\square$-E \hspace{1cm} (1)

Prawitz noticed that with these new rules the normalization procedure does not hold. He presented three versions of those modal systems of which only the third one would accept the Normalization Theorem.

About forty years later, Medeiros \[1\] provided a counterexample showing that the Normalization Theorem does not hold even on Prawitz’s third version of the ND system for classical S4. Medeiros also proposed a correction to Prawitz’s system.
However, recently Yuuki [3] pointed out two problems in the proof of a lemma (the critical lemma) used in the proof of Medeiros’ Normalization Theorem.

Our goal on this paper is to present a correction of Medeiros’s proof of the aforementioned lemma. In section [2] we outline the original Prawitz’s system for classical S4, and the counterexamples by Medeiros are stated at section [3]. At section [4] we discuss the two cases in which the system may not produce valid derivations on NS4 due to problems in the proof of the critical lemma. In section [5] we introduce a new proof of the critical lemma.

2 Prawitz’s S4 classical modal logic system

The difference between each one of the S4 classical modal logic systems presented by Prawitz is on the restriction of the ◯-I rule. The first version, has the restriction that ◯-I can be applied only when its premisses depend solely on modal formulas. However, the Normalization Theorem does not hold. For instance, Prawitz pointed out that the derivation

$$\begin{array}{c}
\frac{\Box A}{A} \\
\frac{A \land B}{\Box(A \land B)} \\
\frac{\Box A \land \Box B}{\Box A} \\
\frac{\Box B}{\Box (A \land B)} \\
\frac{\Box B \rightarrow \Box (A \land B)}{\Box A \rightarrow (\Box B \rightarrow \Box (A \land B))}
\end{array}$$

is a derivation in this system but its reduction

$$\begin{array}{c}
\frac{\Box A \land \Box B}{\Box A} \\
\frac{\Box A \land B}{\Box (A \land B) \ast} \\
\frac{\Box B}{\Box (A \land B)} \\
\frac{\Box (A \land B)}{\Box (A \land B) \ast}
\end{array}$$

is not. Notice that ◯-I cannot be applied in * for it depends on a formula that is not modal (◯A ∧ ◯B).

Hence Prawitz liberalized the restriction on ◯-I, also allowing it to be applied when its premisses depend only on essentially modal formulas. A
formula \( A \) is *essentially modal* when “each occurrence of a predicate parameter or predicate constant in \( A \) stands within the scope of an occurrence of” \( \Box \).

This notion clears the way to a second version of the system, in which both \((2)\) and its reduction are correct derivations. However, we can still find a derivation whose reduction is not a derivation in the system:

\[
\begin{array}{c}
\Box A \land B \\
\hline
\Box A \\
\hline
\Box \Box A \\
\hline
\Box \Box \Box A
\end{array}
\]

\[
\begin{array}{c}
\Box A \land B \\
\hline
\Box A \\
\hline
\Box \Box A \ast
\end{array}
\]

Notice that \( \Box \)-I could not be applied in \( \ast \) for \( \Box A \land B \) is not an essentially modal formula.

Finally, Prawitz’s last restriction states that, if a formula \( A \) depends on an assumption \( B \) and there exists an essentially modal formula \( F \) on the thread of \( A \) from \( B \) such that \( A \) depends on every assumption which \( F \) depends on, then \( \Box \)-I could be applied at \( A \).

3 A counterexample for Prawitz’s classical S4 system

In what follows, \( \vec{B} \) is a sequence of deductions of the form \( \vec{B}_i \), where no two \( \Box B_i \)'s are equal. The reason of this restriction is explained in item 4 of the proof of the critical lemma and it does not affect the completeness of the system. We also write \( [\Box B]^k \) to indicate that each \( \Box B_i \) is discharged at \( k \). The labels may be dropped.

Medeiros [1] showed that Prawitz’s last version of \( \textbf{S4} \) would not avoid maximal formula\(^1\) by presenting the following derivation

\[
\begin{array}{c}
\neg \Box A \\
\hline
\neg A \rightarrow B \\
\hline
\bot \\
\hline
\Box A \\
\hline
C \rightarrow A \\
\hline
\Box (C \rightarrow A)
\end{array}
\]

which is valid in \( \textbf{S4} \) but whose reduction is not:

\(^1\)see some of the definitions of the terms used in this article in the appendix.
Then, Medeiros [1] presented a new version for the S4 system, the \textbf{NS4}. This new system is composed of the logical symbols $\land, \lor, \rightarrow, \bot$ and $\Box$ and its rules are the usual, except for $\Box$-I, which is as follows:

\[
\frac{[\Box B]^k}{\Box A}
\]

The restriction on $\Box$-I rule states that all the assumptions in $[\Box B]^k$ must be discharged by the application of $\Box$-I and the premiss $A$ must not depend on any assumption other than the ones in $\Box B$.

With this new rule, we have the following reduction:

\[
\frac{\sum_{i=1}^{\Sigma} \Lambda_i}{\Box A}
\]

(3)
There is also the following permutative reduction.

\[
\frac{\Sigma}{\Box \overline{A}}^i \quad \Lambda_1 \quad \Psi \quad \overline{\Box B}^j \quad \Box \overline{D}^k \quad \Lambda_2 \quad C_{k, j} \quad \vdots \\
\frac{\Box \overline{B}}^i \quad \Box \overline{D} \quad \Box C
\]

\[
(4)
\]

4 A problem in the normalization proof of NS4

The proof of the normalization for NS4 by Medeiros \[1\] requires the use of the critical lemma. This lemma states that “if \(\Pi\) is a critical simplified derivation of \(C\) from \(\Gamma\), then \(\Pi\) can be transformed into a simplified derivation \(\Pi'\) such that \(I(\Pi') < I(\Pi)\).” The proof states that a derivation \(\Pi\) of \(C\) from \(\Gamma\) can be transformed in a simplified derivation \(\Pi_0\). And, by the critical lemma, a subderivation \(\Sigma\) of \(\Pi\) can be transformed in a subderivation \(\Sigma'\) such that \(I(\Sigma') < I(\Sigma)\); then \(\Pi_1\) is the derivation resulted from the substitution of \(\Sigma'\) for \(\Sigma\) in \(\Pi_0\). Medeiros aimed to show that \(I(\Pi_1) < I(\Pi_0)\).

Yuuki \[3\] pointed out two flaws on the proof of this lemma. The first one concerns critical derivations of the form

\[
\frac{\Sigma_{0,1}}{\Pi \equiv F} \quad \frac{\Sigma \quad \overline{\Sigma}}{\Box F}^i \quad \overline{\Box F} \quad \Sigma_{0,2} \quad \vdots \\
\frac{\Box F}{\Box C} \quad \overline{\Box F} \quad \Box C \quad r
\]

where the major premiss \(F\) is the conclusion of \(\bot_C\) and \(r\) is an elimination
rule. According to Medeiros, this derivation can be transformed into

\[
\Pi' \equiv \frac{\Sigma_0,1 \sum}{r} F \quad \frac{\sum}{\Sigma_0,2} \quad \frac{\sum}{\perp}
\]

or to

\[
\Pi'' \equiv \frac{\Sigma_0,1 \sum}{r} F \quad \frac{\sum}{\Sigma_0,2} \quad \frac{-C}{i} \quad \frac{\sum}{\perp} \quad \frac{\sum}{C}
\]

depending on \( C \) being \( \perp \) or not.

Note that the assumptions of the form \( \neg F \) discharged at the rule \( i \) may occur more than once in \( \Pi \), multiplying the occurrences of the maximal premiss \( F \) in \( \Pi' \) and \( \Pi'' \). In this case the index of either \( \Pi' \) or \( \Pi'' \) may be even greater than that of \( \Pi \). Besides, one of the \( H_i \)'s in \( \tilde{H} \), say \( H_l \), may be a maximal formula of degree \( G(F) \) and, in this case, even if the \( F \) side connected with \( \neg F \) is not a maximal formula in \( \Pi' \), this \( H_l \) still is and the induction hypothesis cannot be used.

The second problem pointed out is when \( \Pi \) has degree \( G(\Box A) \) and is a critical derivation of the form

\[
\Pi \equiv \frac{[\Box B]^k}{\Sigma_0,1 \sum} \quad \frac{\Lambda_1}{\square A} \quad \frac{\neg A}{i} \quad \frac{[\neg A]^i}{\perp}
\]

\[
\frac{\neg B}{\square A} \quad \frac{\Lambda_2}{\Box B} \quad \frac{\neg B}{\ell, j} \quad \frac{[\Box A]^j [\Box C]^\ell}{\sum_0,2 \sum}
\]

\[
\Pi \equiv \frac{[\Box B]^k}{\Sigma_0,1 \sum} \quad \frac{\Lambda_1}{\square A} \quad \frac{\neg A}{i} \quad \frac{[\neg A]^i}{\perp}
\]

\[
\frac{\neg B}{\square A} \quad \frac{\Lambda_2}{\Box B} \quad \frac{\neg B}{\ell, j} \quad \frac{[\Box A]^j [\Box C]^\ell}{\sum_0,2 \sum}
\]
If □A occurs more than once as top-formula of Λ₂, by reducing Π to

\[ \Pi' \equiv \frac{\Sigma}{\Box B} \quad \frac{\Psi}{\Box C} \quad \frac{\Lambda_1}{\Box A} \quad \frac{\Lambda_2}{B} \quad \frac{[\Box C]^\ell}{\perp} \quad \frac{[\Box B]^k}{\perp 0} \quad \frac{\Box B}{\perp 0} \quad \frac{\Box B}{[\neg \Box B]^i} \]

the number of occurrences of □A as maximal formula in Π’ will be greater than in Π.

Thus, it is possible that the reduction process generates copies of maximal formulas, so the index of Π’ may be greater than that of Π.

5 Yet another proof of the critical lemma

Lemma 1 A critical derivation of the form Π ≡ \[ \Sigma_1 \quad [\neg F]^1 \quad \Sigma \quad \frac{[\neg F]^1}{\Box B} \quad \frac{\Sigma}{\Box B} \quad \frac{\Box B}{\Box C} \quad \frac{\Sigma}{\Box C} \] where F is the conclusion of \( \perp_c \), can be transformed in a derivation

\[ \Pi_1 \equiv \frac{\Sigma_{1,2}}{\perp} \quad \frac{\Sigma}{\Box F \quad \Box B} \]

where \( \Pi_1 \) is a simplified derivation without trivial formulas. Thus, the end-formula of \( \Sigma_{1,1} \) is not conclusion of \( \perp_c \).

Note that \( \Pi_1 \) has no more maximal formulas of degree equal to or higher than \( G(F) \) than Π. We will use the symbol \( \propto \) to indicate the transformation of a derivation into a simplified derivation without trivial formulas.

Proof 1 See the work of Medeiros [1].
Theorem 5.1 If $\Pi$ is a critical simplified derivation of $C$ from $\Gamma$, then $\Pi$ can be transformed into a simplified derivation $\Pi'$ such that $G(\Pi') < G(\Pi)$.

Proof 2 Suppose $\Pi$ is a critical simplified derivation with maximal premisses of degree $G(\Pi)$ which are premisses of the last inference of $\Pi$, $\#G(\Pi)$ is the number of maximal formulas of $\Pi$ with degree $G(\Pi)$ and $\ell(\Pi)$ is the length of $\Pi$. The proof is by induction on the pair $(\#G(\Pi), \ell(\Pi))$.

1. $\Pi \equiv \frac{\Sigma_1 \quad \Sigma_2}{A \quad B} \quad \frac{A \wedge B}{\Sigma_1 \quad A} \equiv \Pi'$
   
   It is easy to see that $G(\Pi') < G(\Pi)$.

2. $\Pi \equiv \frac{[A]}{\Sigma_1 \quad B} \quad \frac{\Sigma_2}{A \quad B} \quad \frac{A \rightarrow B}{\Sigma_1 \quad (A) \quad \Pi'}$
   
   It is easy to see that $G(\Pi') < G(\Pi)$.

3. $\Pi \equiv \frac{\Sigma}{\Box \overline{B}} \quad \frac{\sum}{\Lambda_1 \quad C} \quad \frac{\sum}{\Lambda_1 \quad C} \equiv \Pi'$
   
   If there existed a $\Box B_l$ which is a maximal premiss at $\Pi'$, then it would be a maximal formula at $\Pi$ and, as $\Pi$ is a critical derivation, $G(\Box B_l) < G(\Box C)$. Thus, $G(\Pi') < G(\Pi)$.

4. $\Pi \equiv \frac{\sum}{\Box \overline{B}} \quad \frac{\sum}{\Lambda_1 \quad A} \quad \frac{\sum}{\Lambda_2 \quad \Psi} \quad \frac{\sum}{\Lambda_2 \quad \Psi} \equiv \Pi'$
   
   We have two cases to consider:
(a) there is an occurrence of \( \Box A \) which is top-formula of \( \Lambda_2 \) and major premiss of an application of \( \Box \)-E: in this case, the number of maximal formulas of degree \( G(\Box A) \) in \( \Pi_1 \) may be even greater than that of \( \Pi \), as there may be more than one occurrence of \( \Box A \) as top-formula of \( \Lambda_2 \).

There is a critical subderivation \( \Xi_1 \) of \( \Pi_1 \) of the form

\[
\begin{array}{c}
\Box \bar{B} \\
\Lambda_3 \\
\hline
A \\
\end{array}
\]

which can be reduced to \( \Xi'_1 \equiv \Box \bar{B} \) (case 3).

(b) there is an occurrence of \( \Box A \) which is top-formula of \( \Lambda_2 \) and major premiss of \( \Box \)-I: then, there is a critical subderivation \( \Xi_2 \) of the form

\[
\begin{array}{c}
\Box \bar{B} \\
\Lambda_3 \\
\hline
A \\
\end{array}
\]

The length of \( \Xi_2 \) is smaller than the length of \( \Pi \). Hence, by induction hypothesis, we can reduce \( \Xi_2 \) to a \( \Xi'_2 \) such that \( G(\Xi'_2) < G(\Pi) \).

Note that we cannot guarantee that the length of \( \Xi_2 \) is smaller than the length of \( \Pi \) if there were more than one occurrence of \( \Box A \) as top-formula of \( \Lambda_2 \) in \( \Pi_1 \), and if there were many occurrences of \( \Box A \) as major premiss in \( \Xi_2 \). That is the reason of the restriction on the beginning of the section.

Let \( \Pi_2 \) be the result of replacing each occurrence of critical subderivations of the form of \( \Xi_1 \) and the form of \( \Xi_2 \) in \( \Pi_1 \) by \( \Xi'_1 \) and \( \Xi'_2 \) respectively.

If \( \Box A \) is the only major premiss that is maximal formula in \( \Pi \), i.e., there is no member of \( \bar{H} \) which is a maximal premiss of the same degree of \( \Pi \), then \( G(\Pi_2) < G(\Pi) \) and \( \Pi_2 = \Pi \). Otherwise, i.e., if there exists a \( k \) such that \( H_k \) is a maximal formula in \( \Pi \), then \#\( G(\Pi_2) < \#G(\Pi) \) and, as \( \Pi_2 \) is a critical derivation, by induction hypothesis \( \Pi_2 \) can be transformed into a derivation \( \Pi' \) such that \( G(\Pi') < G(\Pi_2) \). Hence, as \( \Pi \) was transformed into \( \Pi_2 \), \( G(\Pi') < G(\Pi) \).
5. $\Pi \equiv \begin{array}{c}
\Sigma_1 \\
\downarrow \quad \downarrow \\
A \land B \\
A
\end{array} \quad \begin{array}{c}
\Sigma'_{1,1} \\
\downarrow \quad \downarrow \\
[\neg (A \land B)] \\
A \land B
\end{array} \quad \begin{array}{c}
\Sigma'_{1,1} \\
\downarrow \quad \downarrow \\
[\neg (A \land B)]^1 \\
A \land B
\end{array} \quad \begin{array}{c}
\Sigma'_{1,1} \\
\downarrow \quad \downarrow \\
A \\
[\neg A]^1
\end{array}$

The $\bot$ is major premiss in $\Pi$, so if the end formula of $\Sigma'_{1,1}$ is not the conclusion of an introduction rule, then the end-formula of $\Sigma'_{1,1}$ is not a maximal formula and $G(\Pi_1) < G(\Pi)$ and $\Pi_1 \equiv \Pi'$. If the end formula of $\Sigma'_{1,1}$ is the conclusion of an introduction rule, then $\Pi_1$ is of the form $\Sigma_3 \quad \Sigma_4 \\
A \quad B \\
\downarrow \quad \downarrow \\
A \land B \\
A$ which can be reduced to $\Sigma_3 \\
A \\
\downarrow \\
[\neg A]^1$ $\Sigma'_{1,2} \\
\downarrow \\
\bot$

$\equiv \Pi'$ and $G(\Pi') < G(\Pi)$.

6. $\Pi \equiv \begin{array}{c}
\Sigma_1 \\
\downarrow \\
A \rightarrow B \\
B
\end{array} \quad \begin{array}{c}
\Sigma_2 \\
\downarrow \\
A \\
A \rightarrow B
\end{array} \quad \begin{array}{c}
\Sigma'_{1,1} \\
\downarrow \\
[\neg (A \rightarrow B)] \\
A \rightarrow B
\end{array} \quad \begin{array}{c}
\Sigma'_{1,1} \\
\downarrow \\
[\neg (A \rightarrow B)]^1 \\
A \rightarrow B
\end{array} \quad \begin{array}{c}
\Sigma'_{1,1} \\
\downarrow \\
A \\
A \rightarrow B
\end{array}$
\[ \begin{align*}
&\Sigma'_1,1 & \Sigma_2 \\
&\frac{A \to B}{B} & A \\
&\frac{\perp}{\Sigma'_{1,2}} & \frac{[-B]}{\perp} \\
\end{align*} \]
\[ \equiv \Pi_1 \]

If the end formula of \( \Sigma'_{1,1} \) is not the conclusion of an introduction rule, then \( G(\Pi_1) < G(\Pi) \) and \( \Pi_1 \equiv \Pi' \).

If the end formula of \( \Sigma'_{1,1} \) is the conclusion of an introduction rule,

\[ \begin{align*}
&\begin{array}{c}
[\begin{array}{c}
\Sigma_3 \\
\end{array}
\end{array} & \begin{array}{c}
[\begin{array}{c}
\Sigma_4 \\
\end{array}
\end{array} \\
&\begin{array}{c}
\frac{B}{A \to B} & A \\
\frac{\perp}{\Sigma'_{1,2}} & \frac{[-B]}{\perp} \\
\end{array} \\
\end{align*} \]

then \( \Pi_1 \) is of the form \( \frac{B}{\Sigma_4} \) which can be reduced to

\[ \begin{align*}
&\begin{array}{c}
[\begin{array}{c}
\Sigma_3 \\
\end{array}
\end{array} & \begin{array}{c}
\Sigma_4 \\
\end{array} \\
&\begin{array}{c}
\frac{A}{\perp} & \frac{B}{\perp} \\
\frac{\perp}{\Sigma'_{1,2}} & \frac{\perp}{\perp} \\
\end{array} \\
\end{align*} \]

\[ \equiv \Pi' \text{ and } G(\Pi') < G(\Pi). \]

7. \( \Pi \equiv \frac{[\neg \Box A]}{\Sigma_1} \frac{\Sigma'_{1,1}}{\Sigma_1} \frac{\neg \Box A}{\perp} \frac{\Sigma'_{1,2}}{\perp} \frac{\neg A}{\perp} \frac{\Sigma_1,2}{\perp} \frac{\neg A}{\perp} \equiv \Pi_1 \)

If the end formula of \( \Sigma'_{1,1} \) is not the conclusion of an introduction rule, then \( G(\Pi_1) < G(\Pi) \) and \( \Pi_1 \equiv \Pi' \). If the end formula of \( \Sigma'_{1,1} \) is the
If one of the $\Box B_i$ were a maximal formula in $\Pi_2$, it would be a maximal formula in $\Pi$ and, as $\Pi$ is a critical derivation, $G(\Box B_i) < G(\Box A)$. Thus, $G(\Pi_2) < G(\Pi)$ and $\Pi_2 \equiv \Pi$.

The conclusion of an introduction rule, then $\Pi_1$ is of the form
Note that $\Sigma_{1,1}'$ is a subderivation of $\Sigma_1$. Hence, if the subderivation $\Lambda \equiv [\Box A], [\vec{H}]^j$
$\Sigma_{1,1}' \quad \vec{\Psi} \quad \Sigma_2 \quad C, i_j$
of $\Pi_1$ is a critical derivation, its length is smaller than the length of $\Pi$. Thus, by the induction hypothesis, $\Lambda$ can be reduced to a derivation $\Lambda'$ such that $G(\Lambda') < G(\Pi)$. The result of replacing each occurrence of $\Lambda$ in $\Pi_1$ by $\Lambda'$ is a derivation $\Pi'$ such that $G(\Pi') < G(\Pi)$.

9. $\Pi \equiv \Sigma_{1,1}'$

\[ \begin{array}{c}
\Sigma_1 F \quad \Sigma_{1,2} \quad \vec{\Psi} \\
\hline
F \quad \vec{H} \quad \vec{H}
\end{array} \]
$\Downarrow$

The critical subderivation $\Lambda \equiv \Sigma_{1,1}' \quad \vec{\Psi} \quad \Gamma$ of $\Pi_1$ is smaller than $\Pi$.

Thus, by the induction hypothesis, $\Lambda$ can be reduced to a derivation $\Lambda'$ such that $G(\Lambda') < G(\Pi)$. By replacing each occurrence of $\Lambda$ in $\Pi_1$ by $\Lambda'$ we achieve the desired derivation.

10. $\Pi \equiv \Sigma'$

\[ \begin{array}{c}
\Sigma F \rightarrow C \\
\hline
C \quad \Sigma
\end{array} \]
$\Downarrow$

$\Pi_1 \equiv \Sigma$ 

\[ \begin{array}{c}
\Sigma \quad B \rightarrow \neg B^2 \\
\hline
\neg B \quad \Sigma
\end{array} \]

If the $\Downarrow$ is a minor premiss of an application of $E_{\rightarrow}$, then $G(\Pi) < G(\Pi_1)$
A Definitions

Some definitions found here are based on the work of Medeiros [1].

Definition 1 The premisses \((A \rightarrow B)\) of the rule \((E\rightarrow)\), \((A \land B)\) of \((E\land)\), \((A \lor B)\) of \((E\lor)\), \((\Box A)\) of \((E\Box)\), and the premisses \(\Box B_1, \ldots, \Box B_n\) of \((I\Box)\) are called major premisses and the others minor premisses.

Definition 2 A segment in a derivation is a sequence \(A_1, \ldots, A_n\) of occurrences of the same formula in a branch such that \(A_1\) is not the conclusion of an application of \((E\lor)\) nor a discharged assumption through an application of \((I\Box)\), and \(A_n\) is not a minor premiss of \((E\lor)\) nor a major premiss of \((I\Box)\).

Definition 3 The length of a segment is the number of formula occurrences in this segment.

Definition 4 A maximal segment in a derivation is a segment \(A_1, \ldots, A_n\) such that \(A_1\) is the conclusion of an application of an introduction rule or \((\bot_c)\), and \(A_n\) is a major premiss of an application of an elimination rule.

Definition 5 A maximal formula is a maximal segment whose length is 1(one). A premiss is called maximal premiss if it belongs to some maximal segment.

Definition 6 A formula \(A\) is a trivial formula if \(A\) is the conclusion of an application of \((\bot_c)\) and the minor premiss of an application of \((E\rightarrow)\) whose major premiss is the assumption \(\neg A\).

Definition 7 A derivation is a simplified derivation if the minor premisses of every application of \((E\lor)\) is the formula \(\bot\).

Definition 8 The degree of a formula \(A\), \(G(A)\), is the number of occurrences in \(A\) of logical symbols different from \(\bot\). The degree of a segment is the degree of the formula that belongs to this segment.

Definition 9 The degree of a derivation \(\Pi\), \(G(\Pi)\) is the highest degree of a maximal segment of \(\Pi\). If \(\Pi\) does not have maximal segments, then \(G(\Pi) = 0\).

Definition 10 A critical derivation is a derivation \(\Pi\) such that, if \(G(\Pi) = d\), then the last inference of \(\Pi\) has a maximal premiss with degree \(d\), and for every subderivation \(\Sigma\) of \(\Pi\), \(G(\Sigma) < G(\Pi)\).
Definition 11 The index of a derivation $\Pi$ is $I(\Pi) = \langle d, s \rangle$, where $s$ is the sum of the lengths of the maximal segments of $\Pi$ whose degree is $d$. If $\Pi$ does not have maximal segments, then $I(\Pi) = \langle 0, 0 \rangle$.

Definition 12 A derivation $\Pi$ is a normal derivation if $\Pi$ does not have maximal segments.

References

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[3] A. Yuuki. A note on modal logic S4 in natural deduction. *Hosei University Repository*, pages 15–18, 2010.