Polar Code Moderate Deviation: Recovering the Scaling Exponent

Hsin-Po Wang and Iwan Duursma
University of Illinois at Urbana--Champaign
{hpwang2, duursma}@illinois.edu

Abstract—In 2008 Arıkan proposed polar coding [1] which we summarize as follows: (a) From the root channel \( W \) synthesize recursively a series of channels \( W_N^{(1)}, \ldots, W_N^{(N)} \). (b) Select sophisticatedly a subset \( A \) of synthetic channels. (c) Transmit information using synthetic channels indexed by \( A \) and freeze the remaining synthetic channels.

Arıkan gives each synthetic channel a “score” (called the Bhattacharyya parameter) that determines whether it should be selected or frozen. As \( N \) grows, a majority of the scores are either very high or very low, i.e., they polarize. By characterizing how fast they polarize, Arıkan showed that polar coding is able to produce a series of codes that achieve capacity on symmetric binary-input memoryless channels.

In measuring how the scores polarize the relation among block length, gap to capacity, and block error probability are studied. In particular, the error exponent regime fixes the gap to capacity and varies the other two. The scaling exponent regime fixes the block error probability and varies the other two. The moderate deviation regime varies all three factors at once.

The latest result [2, Theorem 7] in the moderate deviation regime does not imply the scaling exponent regime as a special case. We give a result that does. (See Corollary 8.)

I. INTRODUCTION

A. The Path to Capacity

Assume we want to communicate over the binary erasure channel \( W \) with erasure probability \( Z(W) \). The Shannon capacity of this channel is \( I(W) = 1 - Z(W) \).

Given a series of block codes \( B_1, B_2, \ldots \) we may calculate their block lengths \( N_1, N_2, \ldots \), code rates \( R_1, R_2, \ldots \), and block error probabilities \( P_1, P_2, \ldots \). An ideal situation is that as \( N_n \) goes to infinity, the code rate \( R_n \) approaches the channel capacity \( I(W) \) while the block error probability \( P_n \) tends to zero. This is called capacity achieving in the literature. See Fig. 1 for visualization.

Let gap to capacity \( I(W) - R_n \) be the difference between the channel capacity \( I(W) \) and the code rate \( R_n \). There are three factors that we want to understand: block length \( N_n \), gap to capacity \( I(W) - R_n \), and the block error probability \( P_n \). And there are three regimes that study the relation among these factors: error exponent regime, scaling exponent regime, and moderate derivation regime. (See also [2] Abstract for a concise summary.)

1) Error Exponent Regime: Fix the gap to capacity (more precisely, bound the code rate \( R_n \) from below) and measure how fast the block error probability \( P_n \) goes to zero. See Fig. 2 for visualization.

2) Scaling Exponent Regime: Fix the block error probability \( P_n \) and vary the gap to capacity. See Fig. 3 for visualization.

3) Moderate Deviations Regime: Fix the block error probability \( P_n \), gap to capacity \( I(W) - R_n \), and vary the block length \( N_n \). See Fig. 4 for visualization.
For classical polar codes, $\mathbb{B}_n$ is generated by some subset of rows of the tensor power $[1 \ 0 \ 1]^\otimes n$ and $N_n = 2^n$. It could be made such that $P_n$ is of order

$$O \left( 2^{-n^{1/3.6261}} \right) = O \left( 2^{-2^{n/3.6261}} \right)$$

as $n \to \infty$ for all $P_n$ and $I(W)$. We hence say that classical polar codes have error exponent $\beta$ is the average of partial distances divided by the block length.

This characterization is later refined by [7, Formula (9)] since

\[
\beta := \sup_{\mathbb{B}_n: \text{polar codes}} \liminf_{n \to \infty} \frac{\log (- \log P_n)}{\log N_n} = \frac{1}{2}.
\]

for $Q(\xi) = \text{erfc}(\xi/\sqrt{2})/2$ the Q-function in statistics.

Their argument applies to generalized polar codes that use a larger kernel other than $[1 \ 0 \ 1]$. The general formula [7, Formula (9)] suggests an obvious obstacle $\beta \leq 1$ since $\beta$ is the average of partial distances divided by the block length.

In [6, Example 32] it is given an explicit 16-by-16 kernel with error exponent 0.51828, a number larger than 1/2. They also give a general construction based on Bose–Chaudhuri–Hocquenghem codes that achieves error exponents arbitrarily close to 1, as the kernel size grows [6, Abstract and Section VI].

For an even more general scenario where the alphabet is $\mathbb{F}_q$, a similar result is given in [8]. Specifically Reed–Solomon matrices achieve error exponents arbitrarily close to 1 as the field size (and thus the kernel size) grows.

See Appendix C for comparison.

2) Scaling Exponent Regime: Fix (bound from above) the block error probability $P_n$ and measure how fast the gap to capacity $I(W) - R_n$ tends to zero. See Fig. 3 for visualization.

For classical polar codes, [9, Fig. 2–5] did extensive simulations and suggests that $I(W) - R_n$ might, at best, be of order

$$O \left( N_n^{-1/3.6261} \right) = O \left( 2^{-2^{n/3.6261}} \right)$$

as $n \to \infty$ for all $P_n$ and $I(W)$. We hence say that classical polar codes “might” have scaling exponent

$$\mu := \sup_{\mathbb{B}_n: \text{polar codes}} \limsup_{n \to \infty} - \frac{\log(N_n)}{\log I(W) - R_n} \approx 3.6261.$$

Later [10] provides $\mu \leq 6$ by a more rigorous reasoning. The idea goes as follows: (See also [10, Formula (27–33)])

Let $Z_n$ be the Bhattacharyya process as in [10, Formula (67)]. Let $a > 0$ be a number close to $0$ and $b < 1$ a number close to 1. The conservation of entropy suggests that the probability $P(a < Z_n < b)$ controls the gap to capacity. Let $g_0$ be the indicator function of the open interval $(a, b)$ then

$$\mathbb{E}g_0(Z_0) = P(a < Z_0 < b).$$

Define

$$g_t(\xi) := g_0(\xi^2) + g_0(2\xi - \xi^2)$$

then

$$\mathbb{E}g_0(Z_1) = P(a < Z_1 < b) = \frac{P(a < Z_0^2 < b) + P(a < 2Z_0 - Z_0^2 < b)}{2} = \frac{g_0(Z_0^2) + g_0(2Z_0 - Z_0^2)}{2} = g_1(Z_0).$$

Iterate this idea by defining

$$g_{n+1}(\xi) := g_n(\xi^2) + g_n(2\xi - \xi^2)$$

to get

$$\mathbb{E}g_0(Z_n) = P(a < Z_n < b) = g_n(Z_0).$$

The function $2^{n/\mu}g_n$ seems to converge numerically pointwisely for some magical choice of $\mu$ [10, Fig. 5]. If the limit $g_\infty$ does exist, then

$$P(a < Z_n < b) = g_n(Z_0) \approx 2^{-n/\mu}g_\infty(Z_0).$$

We summarize the discussion above in the bra-ket notation

$$P(a < Z_n < b) = \langle g_0 | Z_n \rangle = \langle g_0 | T^n | Z_0 \rangle$$

otherwise.

If

$$\sup_{0 < \xi < 1} \frac{h(\xi^2) + h(2\xi - \xi^2)}{2h(\xi)} < 2^{-1/\mu^*}$$

for some $\mu^* > 2$, then

$$\mu \leq \mu^*.$$

Finally the idea is formulated as following clean criterion.

**Theorem 1:** [2, Theorem 1 and Formula (15)] Let $h : [0, 1] \to [0, 1]$ be such that $h(0) = h(1) = 0$ and $h(\xi) > 0$ otherwise. If

$$\sup_{0 < \xi < 1} \frac{h(\xi^2) + h(2\xi - \xi^2)}{2h(\xi)} < 2^{-1/\mu^*}$$

for some $\mu^* > 2$, then

$$\mu \leq \mu^*.$$
have to balance our efforts between reducing gap to capacity and reducing block error probability.

Theorem 1] states that there exists a \( \mu' \) (probably much larger than \( \mu \)) such that \( \beta' = .49 \) and \( \mu' \) are achievable at the same time. We comment that this result sacrifices \( \mu' \) to achieve a pretty good \( \beta' \), just 0.01 away from the best possibility.

[2] introduces a “interpolation” result.

**Theorem 2:** [3 Theorem 7 and Formula (49)] Assume the \( h \) and \( \mu^* \) in Theorem 1. Let \( \gamma \) be a free parameter such that

\[
\frac{1}{1 + \mu^*} < \gamma < 1.
\]

Then

\[
\beta' = \gamma H_2^{-1}\left(\frac{\gamma (\mu^* + 1) - 1}{\gamma \mu^*}\right) \quad \text{and} \quad \mu' := \frac{\mu^*}{1 - \gamma}
\]

are achievable at the same time. Here \( H_2 \) is the binary entropy function.

When \( \gamma \to 1 \) this recovers the error exponent \( \beta' = \beta = 1/2 \). When \( \gamma \to 1/(1 + \mu^*) \) this recovers a weaker scaling exponent \( \mu' = 1 + \mu^* \). Our result, Theorem 6, recovers the true scaling exponent by Corollary 8.

[24] Definition 1.1–1.4] proposed weaker notions to control \( I(W) - R_n \) and \( P_n \) (where \( P_n \) is exponential in \( n \), instead of doubly exponential). That said, they derive some results based on much weaker assumptions [24 Definition 1.5 and Theorem 1.6].

See Appendix [H] for comparison.

**B. The Log-loglog Plot of the Path to Capacity**

We have seen that in the context of polar coding there exist polar codes \( B_1, B_2, \ldots \) such that the gap to capacity \( I(W) - R_n \) shrinks polynomially in \( N_n = 2^n \) as \( n \to \infty \). Thus it is appropriate to compare \( n \) to \( -\log_2(\text{Gap to capacity}) \), and the best ratio is the scaling exponent \( \mu \).

Similarly, there are \( B_1, B_2, \ldots \) such that the block error probability \( P_n \) is as small as \( 2^{-n^{\frac{1}{29}}} \). So it is appropriate to compare \( \log_2(-\log_2(\text{Gap to capacity})) \) to \( n \), and the best ratio is what we called error exponent \( \beta \).

Notice that the maximum of the bit error probabilities and the block error probability differ only by a factor of \( N_n = 2^n \). Thus \( \log_2(-\log_2(\text{bit error})) \) and \( \log_2(-\log_2(\text{Block error})) \) are of the same magnitude and we will use them interchangeably.

Consider locating \( B_1, B_2, \ldots \) on the \( -\log_2(\text{Gap}) \)-versus-\( \log_2(-\log_2(\text{Block error})) \) plane. Then the goal of coding theory is such that those points converge to \( (+\infty,+\infty) \), i.e., \( B_n \) “moves” in the direction of up-right. Or, think of a coding theorist standing at where \( B_n \) is only to construct \( B_{n+1} \) and jump to where \( B_{n+1} \) is. The one and only question is: how fast can we move in exchange for larger block length \( N_n = 2^n \)? See Fig. [8] for visualization.

---

3We emphasize again that this is not the usual definition of the error exponent.
Fig. 8. $-\log_2(Gap)$ ranges from $-\log_2(Capacity)$ (which is at least one) to $+\infty$. The larger the better. And $\log_2(-\log_2(Block\ error))$ ranges from $-\infty$ to $+\infty$. The larger the better.

Fig. 9. Error exponent regime: How fast we can move rightward while not moving too much downward.

Fig. 10. Error exponent $\beta = 1/2$ means that asymptotically each step is moving rightward by about $1/2$ units.

Fig. 11. Scaling exponent regime: How fast we can move upward while not moving too much leftward.

Fig. 12. Scaling exponent $\mu = 3.627$ means that asymptotically each step is moving upward by about $1/3.627$ units.

Fig. 13. Moderate deviations regime: How fast we can move in the direction up-right.

Fig. 14. Moderate deviations concern the joint performance of the previous two regimes. It should recover the previous two regimes as special cases.

Fig. 15. Vertical segments $x = n\beta$ are those in Fig. 10 but clipped. Horizontal segments $y = n/\mu$ are those in Fig. 12 but clipped. Curves trace points $(\beta'n, n/\mu')$ as $\gamma$ varies. The curves match vertical segments but do not match horizontal ones.

The error exponent measures how fast we can move rightward while not moving too much downward. The error exponent $\beta = 1/2$ means asymptotically $B_{n+1}$ is $1/2$ units to the right of $B_n$. See Fig. 9 and 10 for visualization.

The scaling exponent measures how fast we can move upward while not moving too much leftward. The scaling exponent $\mu = 3.627$ means asymptotically $B_{n+1}$ is $1/3.627$ unit to the top of $B_n$. See Fig. 11 and 12 for visualization.

The moderate deviation regimes concern the joint performance of the previous two regimes. Ideally it should take a free parameter $\gamma \in [0,1]$ such that

- $\gamma$ controls the “slope” of the path $B_n$.
- when $\gamma \to 1$ the path $B_n$ goes primarily rightward and recovers the error exponent regime;
- when $\gamma \to 0$ the path $B_n$ goes primarily upward and recovers the scaling exponent regime.

See Fig. 13 and 14 for visualization.

Apart from the ideal case, we have seen that Theorem 2, i.e., [2, Theorem 7 and Formula (49)],
because it comes with some intrinsic penalties: (\[3\])

\[\mu\] the sense of Theorem 1 such that longer the step length the harsher the penalty. The more the steps the softer but that at best we can only achieve \[\gamma\] accurate plot. See Fig. 15 for visualization. See Appendix H for a more detailed plot.

\[\gamma\]

\[\beta n\]

\[\gamma n\]

\[\gamma \rightarrow 1\] (\[\gamma\rightarrow 1/(1+\mu^*)\]).

If we accept the Scaling Assumption \[\text{[12, Formula (12)]}\] and the consequence that \[\mu = 3.627\], then there exists \(h\) in the sense of Theorem \[\text{[1]}\] such that \(\mu^*\) is arbitrary close to \(\mu = 3.627\). Thus the suboptimality of Theorem \[\text{[2]}\] is not that \(\mu^* \neq \mu\) but that at best we can only achieve \(\mu' = 1 + \mu,\) not \(\mu' = \mu.\)

As a result, if we plot \(\beta n\) and \(n/\mu\) and trace the points \((\beta'n, n/\mu')\), there will be a discrepancy on the left hand side. See Fig. [15] for visualization. See Appendix F for a more accurate plot.

We will improve Theorem \[\text{[2]}\] in Section III. Before that, we brief the idea of Theorem \[\text{[2]}\] in the next subsection.

C. General Moving Strategy behind Theorem \[\text{[2]}\]

Granted to move \(n\) steps, we may choose a free parameter \(\gamma \in [0, 1]\) and

- move upward \(n_0 := (1 - \gamma)n\) steps to approach the \(y\)-coordinate \(n_0/\mu;\)
- move rightward \(n_1 := \gamma n\) steps to approach the \(x\)-coordinate \(\beta n_1.\)

However, just because we can reach \((x, y) = (0, n_0/\mu)\) and \((x, y) = (\beta n_1, 0)\) separately does not mean we can approach \((x, y) = (\beta n_1, n_0/\mu)\). Moving does not follow vector addition because it comes with some intrinsic penalties:

- Moving rightward will cause moving downward a little bit. This is, to avoid error we discard bad synthetic channels, and that punishes the gap to capacity. (Fig. 16)
- Moving upward will “reset” the \(x\)-coordinate. That is, to reduce the gap we collect more synthetic channels but cannot control their error probabilities. (Fig. 17)

Therefore the only productive arrangement seems to be to move upward and then move rightward. (Fig. 18) We now detail how to move and the cause of the penalties in the next subsection. We will demonstrate how to bypass these penalties in Section II-F and Section II-G.

D. Detailed Movement: A Recruit-Train-Retain Model

Moving upward \(n_0 := (1 - \gamma)n\) steps is straightforward.

1) Recruit Phase: Set a goal \(P_{\text{upper bound}}\) and collect as many synthetic channels \(W^{(j)}\) as possible such that the error probability does not exceed \(P_{\text{upper bound}}.\) Notice that the maximum and the sum of bit error probabilities differ only by a negligible factor of \(2^n < 2^n\) so we do not distinguish which one we are talking about. See Fig. 19.

Setting such a goal \(P_{\text{upper bound}}\) will pin us at the \(x\)-coordinate \(\log(-\log P_{\text{upper bound}}).\) By Theorem \[\text{[1]}\] or the estimate that \(\mu = 3.627\) \[\text{[12, Abstract]}\] we will collect so many synthetic channels such that the gap to capacity is \(O(2^{n_0}/\mu).\) This will bring us to the \(y\)-coordinate \(n_0/\mu.\)

With these \(W^{(j)}\) in our pocket, moving rightward \(n_1 := \gamma n\) steps consists of two phases.

2) Train Phase: For each \(W^{(j)}\) in our pocket, remove it and put both \(W^{(j)}_{2^{n_0}+1}\) and \(W^{(j)}_{2^{n_0}+1}\) in our pocket. Doing so will maintain the gap to capacity and double the error probability. Thus we are actually moving leftward, but not too much. Repeat this doubling process \(n_1\) times. Each \(W^{(j)}_{2^{n_0}+1}\) leads to \(2^{n_1}\) descendants of the form \(W^{(2^{n_1}+j-k)}_{2^{n_0}}\) for \(k = 0, \ldots, 2^{n_1} - 1.\) See Fig. 20.

3) Retain Phase: For each \(W^{(2^{n_1}+j-k)}_{2^{n_0}}\) in our pocket generated by \(W^{(j)}_{2^{n_0}},\) its error probability is doubled weight \(k\) times and squared \(n_1 - \text{weight}(k)\) times. Here weight \(k\) is the Hamming weight of \(k\) written in binary.

Fig. 16. Moving rightward will cause moving downward a little bit. The longer the step length the harsher the penalty. The more the steps the softer the penalty.

Fig. 17. Moving upward will “reset” the \(x\)-coordinate.

Fig. 18. The only productive arrangement seems to be to move upward and then move rightward. There is no way we can move rightward and then upward. Not to mention zigzagging. See Fig. 19 for what actually happens.

Fig. 19. What actually happens is that: recruit phase moves upward; then train phases moves slightly leftward; and finally retain phase moves rightward.
Phase. In general, they are not necessary consecutive. Therefore set a threshold \( \epsilon \) that is minor. What matters is the total number of squaring. We select a point \( \mu \) such that for \( x \) such that \( P(x) \geq \epsilon n \), the portion of unqualified synthetic channels is squared less than \( \epsilon n \) times. This will bring us closer to the upper bound.

When \( y \rightarrow 1 \), we have a lot of quota of moving rightward and interestingly the effect of moving downward is diluted and negligible. (Casually speaking, training a lot increases the retention rate.) We will see in the next subsection the obstacle to \( \mu' \to \mu \) when \( y \to 0 \).

E. The Main Obstacle to \( \mu' \to \mu \)

For example let \( y = 0.1 \), so \( n_0 = 0.9n \) and \( n_1 = 0.1n \).

- First move upward \( 0.9n \) steps. Now we are at the \( y \)-coordinate \( 0.9n/\mu \geq 0.24n \).
- For each \( W_2^{(j)} \) in our pocket, generate \( 2^{0.1n} \) descendants of the form \( W_2^{(2^{0.1n}j-k)} \) for \( k = 0, 1, \ldots, 2^{0.1n} - 1 \).
- For these \( 2^{0.1n} \) descendants we have several choices:
  - Keep all of them. Then the error probability is doubled \( 0.1n \) steps, so we are actually moving leftward. No progress is made.
  - Keep all but one. Then all errors probabilities are squared at least once, so we are moving rightward by one unit. But this means we lose \( 2^{-0.1n} \) of synthetic channels. The code rate will drop by about \( 2^{-0.1n} \). So the gap to capacity is at least \( 2^{-0.1n} \). This will “reset” our \( y \)-coordinate to \( 0.1n \) from \( 0.24n \). See Fig. 23 for visualization.
  - Discard more then one. Then we lose even more synthetic channels/rate/y-coordinate.

In this particular case we should not go to \( y = 0.24n \) in the first place. An obviously better way is to stop at \( y = 0.1n \) after \( 0.39n \) steps, and then move rightward using \( (1 - 0.39)n = 0.61n \) steps. This will bring us to an even larger, better \( x \)-coordinate \( 0.17n \) while maintaining a larger, better
We will demonstrate how to bypass this obstacle in the next subsection, in Section I-G, and finally in Theorem 6.

Fig. 25. The primary pocket (larger region) and the secondary pocket (smaller region) in the recruit phase. Notice that they do not "overlap".

Fig. 26. The descendants in the train phase. Note that synthetic channels in the primary pocket get more chances to square their erasure probabilities.

Fig. 27. The retained in the retain phase. The policy “to keep those who square once” is harsh in the secondary pocket. But the damage to the rate is ameliorated since the secondary pocket contains fewer to start with.

$$y \rightarrow \gamma$$

By some trivial calculation one can show that if we are allowed to move rightward only $n/(1 + \mu)$ steps then it is better not to redeem those moves at all. This explains why Theorem 7] recovers scaling exponent $1 + \mu$ when $\gamma \rightarrow 1/(1 + \mu)$ instead of $\mu$ for $\gamma \rightarrow 0$.

We will demonstrate how to bypass this obstacle in the next subsection, in Section I-G and finally in Theorem 6.

Fig. 28. Reaching a higher $y$-coordinate using a secondary pocket while not risking loosing too much from the primary pocket.

Fig. 29. Reaching a higher $y$-coordinate using three pockets.

\textbf{F. Bypassing Obstacle: Two-Pocket Recruit-Train-Retain}

We prepare two pockets to hold synthetic channels and apply the recruit-train-retain trick to both pockets separately. The advantage is that we can implement different policy in different pocket.

- Collect in a primary pocket synthetic channels $W_{2^0, 7n}^{(j)}$ with low error probability, i.e., move upward $0.7n$ steps to approach the $y$-coordinate $0.7n/\mu > 0.19n$. See Fig. 25
- For each $W_{2^0, 7n}^{(j)}$ generate $W_{2^n}^{(2^{0.3n}j-k)}$ and discard those whose error probability is squared less than $0.01n$ times. Notice $0.3n (1 - H_2(0.02/0.3)) > 0.19n$. We lose $2^{-0.19n}$ of synthetic channels in the primary pocket, which is satisfactorily few. Visually, we move rightward $0.3n$ steps to approach the $x$-coordinate $0.02n$ while maintaining the $y$-coordinate $0.19n$. See Fig. 26 and 27.
- At the same time, collect in a secondary pocket synthetic channels $W_{2^{0.9n}}^{(l)}$ that are not a descendant of a $W_{2^0, 7n}^{(j)}$ in the primary pocket. This secondary pocket should contain at most $2^{-0.19n}$ (the gap of the primary pocket) of synthetic channels. That is, a very thin branch approaches the $y$-coordinate $0.9n/\mu > 0.24n$. See Fig. 25
- For each $W_{2^{0.9n}}^{(l)}$ generate $W_{2^n}^{(2^{0.1n}l-m)}$ and discard those whose error probability is squared less than $0.01n$ times. Notice $0.1n (1 - H_2(0.01/0.1)) > 0.05n$. We lose $2^{-0.05n}$ of synthetic channels in the secondary pocket, approximately $2^{-0.19n-0.05n} = 2^{-0.24n}$ of all synthetic channels, satisfactorily few. See Fig. 26 and 27.
- Overall, we reach the $y$-coordinate $0.24n$ and the $x$-coordinate $0.01n$.

See Fig. 28 for visualization. This already surpasses Theorem 2. See Appendix H for comparison.
In Section I-E we see the conflict between retaining synthetic channels to maintain the gap to capacity and discarding bad performance ones to reduce the error probability. From the example above we see that by dividing synthetic channels into two pockets, each pocket may have its own retain-discard policy. This dissolves the conflict.

And we can do better. In Theorem 6, we will declare a large number of pockets to minimize the conflict. Before that, we demonstrate a three-pocket trick in the next subsection.

G. One More Example: Three-Pocket Recruit-Train-Retain

Say we are granted to move \( n \) steps.

- Collect in a primary pocket synthetic channels \( W_{20.7n}^{(j)} \) with low error probability, i.e., move upward \( 0.7n \) steps to approach the \( y \)-coordinate \( 0.7n/\mu > 0.19n \).

- For each \( W_{20.7n}^{(j)} \) generate \( W_{2}^{(23.3n-j-k)} \) and discard those whose error probability is squared less than \( 0.01n \) times. Notice \( 0.3n(1 - H_{2}(0.02/0.3)) > 0.19n \). We lose \( 2^{-0.19n} \) of synthetic channels in the primary pocket, which is satisfactorily few. Visually, we move rightward \( 0.3n \) steps to approach the \( x \)-coordinate \( 0.02n \) while maintaining the \( y \)-coordinate \( 0.19n \).

- At the same time, collect in a secondary pocket synthetic channels \( W_{20.8n}^{(l)} \) that are not a descendant of some \( W_{20.7n}^{(j)} \) in the primary pocket. This secondary pocket should contain at most \( 2^{-0.19n} \) (the gap of the primary pocket) of synthetic channels. That is, a very thin branch approaches the \( y \)-coordinate \( 0.8n/\mu > 0.22n \).

- For each \( W_{20.8n}^{(l)} \) generate \( W_{2}^{(2.2n-l-m)} \) and discard those whose error probability is squared less than \( 0.02n \) times. Notice \( 0.2n(1 - H_{2}(0.02/0.2)) > 0.10n \). We lose \( 2^{-0.10n} \) of synthetic channels in the secondary pocket, approximately \( 2^{-0.19n-0.10n} = 2^{-0.29n} \) of all synthetic channels, satisfactorily few.

- At the same time, collect in a tertiary pocket synthetic channels \( W_{20.9n}^{(p)} \) that are neither a descendant of some \( W_{20.7n}^{(j)} \) in the primary pocket or a descendant of some \( W_{20.8n}^{(l)} \) in the secondary pocket. This tertiary pocket should contain at most \( 2^{-0.22n} \) (the gap of the secondary pocket) of synthetic channels. That is, an even thinner branch approaches the \( y \)-coordinate \( 0.9n/\mu > 0.24n \).

- For each \( W_{20.9n}^{(p)} \) generate \( W_{2}^{(2.1n-p-q)} \) and discard those whose error probability is squared less than \( 0.02n \) times. Notice \( 0.1n(1 - H_{2}(0.02/0.1)) > 0.02n \). We lose \( 2^{-0.02n} \) of synthetic channels in the secondary pocket, approximately \( 2^{-0.22n-0.02n} = 2^{-0.24n} \) of all synthetic channels, satisfactorily few.

- Overall, we reach the \( y \)-coordinate \( 0.24n \) and the \( x \)-coordinate \( 0.02n \).

See Fig. 22 for visualization. This surpasses Theorem 2 and the example in Section I-F. See Appendix H for comparison.

We now give a self-contained introduction of polar codes and related terminologies in the next section.

II. PRELIMINARY

A. Binary Erasure Channels

A binary erasure channel \( W \) of erasure probability \( Z(W) \) has input alphabet \( \mathbb{F}_2 \) and output alphabet \( \mathbb{F}_2 \cup \{?\} \). The properties of the channel are described by the probability mass function

\[
W(0|0) = W(0|1) = 0; \\
W(?)|0 = W(?)|1 = Z(W); \\
W(0|0) = W(1|1) = 1 - Z(W). 
\]

The capacity of this channel is \( I(W) = 1 - Z(W) \).

B. Channel Polarization

On binary erasure channels, channel polarization consists of the following pair of building blocks

\[
\begin{array}{c}
W \\
\hline
A \\
\hline
\end{array}
\]

This pair of building blocks has the ability that if we wrap up a pair of i.i.d. channels \( W \) like

\[
\begin{array}{c}
W \\
\hline
C \\
\hline
\end{array}
\]

\[
\begin{array}{c}
B \\
\hline
\end{array}
\]

then point \( A \) to point \( B \) forms a synthetic binary erasure channel \( W' \) with erasure probability \( Z(W') = 1 - (1 - Z(W))^2 \), while point \( C \) to point \( D \) forms another synthetic binary erasure channel \( W'' \) with erasure probability \( Z(W'') = Z(W)^2 \).

A crucial, novel idea in the construction of polar codes is that we may begin with four i.i.d channels \( W \) and wrap them up as

\[
\begin{array}{c}
W' \\
\hline
W' \\
\hline
W'' \\
\hline
W'' \\
\end{array}
\]

This setup is equivalent to four synthetic channels

\[
\begin{array}{c}
W' \\
\hline
W' \\
\hline
W'' \\
\hline
W'' \\
\end{array}
\]

where the two occurrences of \( W' \) are independent and the two occurrences of \( W'' \) are independent. Thus we can further wrap them

\[
\begin{array}{c}
W \\
\hline
W \\
\hline
W \\
\hline
W \\
\end{array}
\]

and obtain four synthetic channels \( (W')', (W'')', (W'')' \), \( (W'')'' \) with erasure probabilities

\[
Z((W')') = 1 - (1 - Z(W'))^2 = 1 - (1 - Z(W))^4; \\
Z((W'')') = Z(W')^2 = (1 - (1 - Z(W))^2)^2; \\
Z((W'')'') = 1 - (1 - Z(W''))^2 = 1 - (1 - Z(W)^2)^2; \\
Z((W'')'') = Z(W'')^2 = Z(W)^4. 
\]
that any synthetic channel in The associated channel $A$ maximal erasure probability differ from the sum by a scalar of this quantity is less than the sum of all erasure probabilities $p$. The error exponent of this series of codes is

$$
\beta := \liminf_{n \to \infty} \frac{-\log(P_n)}{\log N_n}.
$$

The error exponent of polar coding $\beta$ is the supremum of (equivalent) error exponents taken over all series of polar codes. See Section I-A1 for previous works.

The (equivalent) scaling exponent of this series of codes is

$$
\mu := \limsup_{n \to \infty} \frac{-\log(N_n)}{\log(I(W) - R_n)}.
$$

The scaling exponent of polar coding $\mu$ is the infimum of (equivalent) scaling exponents taken over all series of polar codes. See Section I-A2 for previous works.

The goal of this work is to understand what pair of $(\beta', \mu')$ is achievable simultaneously by a series of polar codes. In general, our solution is a trade-off between $\beta'$ and $\mu'$. See Section I-A3 for previous works.

### D. Error Exponent and Scaling Exponent

Let $A_n$ be a series of polar codes with block length $N_n = 2^n$, code rate $R_n$, and block error probability $P_n$. The (equivalent) error exponent of this series of codes is

$$
\beta := \liminf_{n \to \infty} \frac{-\log(P_n)}{\log N_n}.
$$

The error exponent of polar coding $\beta$ is the supremum of (equivalent) error exponents taken over all series of polar codes. See Section I-A1 for previous works.

The (equivalent) scaling exponent of this series of codes is

$$
\mu := \limsup_{n \to \infty} \frac{-\log(N_n)}{\log(I(W) - R_n)}.
$$

The scaling exponent of polar coding $\mu$ is the infimum of (equivalent) scaling exponents taken over all series of polar codes. See Section I-A2 for previous works.

The goal of this work is to understand what pair of $(\beta', \mu')$ is achievable simultaneously by a series of polar codes. In general, our solution is a trade-off between $\beta'$ and $\mu'$. See Section I-A3 for previous works.

### E. Bhattacharyya Process

To describe the error probabilities of synthetic channels better, define a discrete Markov process by letting $Z_0 := Z(W)$ and inductively

$$
Z_{n+1} := \begin{cases} 
1 - (1 - Z_n)^2 & \text{with probability } 1/2; \\
Z_n^2 & \text{with probability } 1/2.
\end{cases}
$$

This is called the Bhattacharyya process.

In other words, $Z_n$ is the erasure probability $Z(W)$ of a uniformly randomly chosen synthetic channel $W_{2^n}$ such that $j_n = 2j_{n-1} - 1$ or $j_n = 2j_{n-1}$. Making it a process simplifies some notation. For example, the fact that $1 - (1 - Z_n)^2 + Z_n^2 = 2Z_n$ is equivalent to $Z_n$ being martingale. Consequently $\mathbb{E}[Z_n] = Z_0 = Z(W) = 1 - I(W)$.

We now quote some lemmata from previous works to illustrate how $Z_n$ works in the next subsection.

### F. Lemmata From/Inspired by Previous Works

See also [2] Formula (11) for the definition of Bhattacharyya process $Z_n$.

**Lemma 3:** [2] Lemmata 6 and Formula (29)] Let $h : [0,1] \to [0,1]$ be such that $h(0) = h(1) = 0$ and $h(\xi) > 0$ otherwise. Assume

$$
\sup_{0 < \xi < 1} h(\xi^2) + h(2\xi - \xi^2) \leq 2^{-\rho_1}
$$

for some $\rho_1 \leq 1/2$. Fix an $\alpha \in (0,1)$, then for any $\delta \in (0,1)$ and $m \in \mathbb{N}$

$$
\mathbb{E}[Z_m(1 - Z_m)]^m \leq \frac{1}{\delta} \left( 2^{-\rho_1} + \frac{\sqrt{2}\xi\delta}{1 - \delta} \right)^m
$$

We emphasize the third time that this is not the usual definition of error exponent.

---

**Fig. 30.** A larger polar code construction. It generates eight synthetic channels. From top to bottom: $(W(2)^n)^{j_n}$, $(W(4)^n)^{j_n}$, $(W(8)^n)^{j_n}$, $(W(16)^n)^{j_n}$, $(W(32)^n)^{j_n}$, $(W(64)^n)^{j_n}$, $(W(128)^n)^{j_n}$, $(W(256)^n)^{j_n}$. Or equivalently: $W_8^{(1)}$, $W_8^{(2)}$, $W_8^{(3)}$, $W_8^{(4)}$, $W_8^{(5)}$, $W_8^{(6)}$, $W_8^{(7)}$, $W_8^{(8)}$.

The construction does not stop here. We may let $W_8^{(1)} := W$ and inductively construct synthetic channels

$$
W_2^{(2j-1)} := (W_8^{(j)})^*, \quad W_2^{(2j)} := (W_8^{(j)})^*.
$$

We call $W_8^{(Mj-k)}$ a descendant of $W_8^{(j)}$ if the former is obtained from the latter in this way, i.e., $0 \leq k < M$. Conversely we call $W_8^{(j)}$ an ancestor of $W_8^{(Mj-k)}$.

See Fig. 30 for a larger construction. See Appendix I for an even larger construction.

### C. Apply Polar Coding in Communication

Choose an $N_n = 2^n$ and among $N_n$ synthetic channels $W_8^{(1)}$, $W_8^{(2)}$, $W_8^{(N_n)}$ choose a subset $A_n$ of synthetic channels. To communicate, send messages through synthetic channels in $A_n$ and send predictable symbols (for instance, all zero) through synthetic channels not in $A_n$. A subset $A_n$ is understood as a polar code.

The block length $N_n$ associated to this code, equivalently to $A_n$, is $N_n = 2^n$. The associated code rate $R_n$ is $|A_n|/N_n$. The associated block error probability $P_n$ is the probability that any synthetic channel in $A_n$ erases the message. Clearly this quantity is less than the sum of all erasure probabilities $Z(W_n)$ of the synthetic channels $W_8^{(j)}$ in $A_n$, by the union bound.

On the one hand, the sum of erasure probabilities overestimates the block error probability $P_n$. On the other hand, the maximal erasure probability differs from the sum by a scalar of $N_n = 2^n$. This becomes negligible once we take the logarithm twice, so we do not expect to gain from a more precise estimate. For soundness, however, we will argue only with the sum, not the maximum. (Nevertheless, for the tightness of the union bound, see [25].)

The goal of this work is to understand the relation among block length $N_n$, code rate $R_n$, and the block error probability $P_n$ (bounded from above by the sum of erasure probabilities of synthetic channels in $A_n$), using terminologies defined in the next subsection.
for some constant $c_2$ depending on $h, \rho_1, \alpha$, but not $m, \delta$.

**Proof:** Omitted.

**Lemma 4:** Inspired by [2] Lemma 5 and Formula (29). Fix an $\alpha \in (0, 1)$, a $\rho \leq 1/2$, a $c_1 > 0$, and an $D > 1$. Assume for all $m \in \mathbb{N}$

$$\mathbb{E}[(Z_m(1 - Z_m))^n] \leq c_1 2^{-m\rho}.$$  

(38)

Then for all $m \in \mathbb{N}$

$$\mathbb{P}(Z_m \leq F_{\text{upper}}^{\text{bound}}2^{-Dm}) \geq I(W) - c_2 2^{-m(D - \alpha)}$$  

(39)

for some $c_2$ depending on $F_{\text{upper}}^{\text{bound}}\alpha, c_1, m$, but not $m$.

**Proof:** See Appendix A.

**Lemma 5:** Assume the $h, \mu^*$ in Theorem 1. Then for any fixed $D > 1$,

$$\mathbb{P}(Z_m \leq F_{\text{upper}}^{\text{bound}}2^{-Dm}) \geq I(W) - O\left(2^{-m/\mu^*}\right)$$  

(40)

as $m$ varies.

**Proof:** See Appendix B.

### III. MAIN RESULT

**Theorem 6:** Assume the $h$ and $\mu^*$ in Theorem 1. If for all $\pi \in [0, 1]$

$$\frac{1 - \pi}{\mu' - \mu^* \pi} + H_2\left(\frac{\beta' \mu'}{\mu' - \mu^* \pi}\right) < 1$$  

(41)

then $(\beta', \mu')$ is achievable. More Precisely, for $n$ large enough there exists a polar code $B_n$ of blocklength $2^n$ such that

$$P_n \leq 2^n \cdot 2^{-2^{\beta'n}}; \quad I(W) - R_n = O\left(2^{-n/\mu'}\right).$$  

(42)

**Proof:** Section IV sketches the proof. Section V details the proof. See Appendix A for comparison.

**Theorem 7:** Assume the Scaling Assumption [12, Formula (12)], and the consequence that $\mu = 3.627$. Then there exists $h$ in the sense of Theorem 1 such that $\mu^*$ is arbitrarily close to $\mu = 3.627$.

**Proof:** See Appendix C.

**Corollary 8:** Theorem 6 recovers the scaling exponent as a special case.

**Proof:** See Appendix D.

**Corollary 9:** Theorem 6 implies Theorem 2 (i.e., [2] Theorem 7) as a special case. It recovers the error exponent as a special case.

**Proof:** See Appendix E.

### IV. SKETCH OF PROOF OF THEOREM 6

The complete proof is in Section V. We will attempt to move upward $n_0 := \lceil n \mu^*/\mu' \rceil$ steps or less and to move rightward $n_1 := n - n_0$ steps or more.

#### A. Discretization: Calculate the Number of Pockets

We will be using $D$ pockets

$$A_n^{[n_0/D]}, A_n^{[2n_0/D]}, A_n^{[3n_0/D]}, \ldots, A_n^{[Dn_0/D]}.$$  

(43)

The tighter the Formula (41) the more pockets we need. Let $m$ be between $0$ and $n_0$ that indexes the pockets.

#### B. Multi-Pocket Recruit Phase

Pocket $A_n^{(m)}$ collects synthetic channels $W_{2^m}^{(j)}$ with erasure probability less than $P_{\text{upper}}^{\text{bound}}2^{-Dm}$ whose ancestors are not collected by any pocket with smaller index. We give each synthetic channel $W_{2^m}^{(j)}$ a weight of $2^{-m}$. Then pocket $A_n^{(m)}$ will weigh $2^{-m/\mu^* + n_0/D\mu'}$.

#### C. Multi-Pocket Train Phase

For each synthetic channel $W_{2^m}^{(j)}$ in pocket $A_n^{(m)}$, replace it with all its descendants of the form $W_{2^{m-j-k}}^{(j-k)}$.

#### D. Multi-Pocket Retain Phase

Claim a threshold $\epsilon = \beta' n / (n - m)$. For each synthetic channel $W_{2^m}^{(j-k)}$ obtained from $W_{2^m}^{(j)}$ in pocket $A_n^{(m)}$, discard it if its erasure probability is more than $2^{-\beta'n}$.

#### E. Estimate the Error Probability

By how we discard synthetic channels in the retain phase, the block error probability will be less than $2^n \cdot 2^{-\beta'n}$.

#### F. Estimate the Gap to Capacity

Pocket $A_n^{(m)}$ weighs $2^{-m/\mu^* + n_0/D\mu'}$ in the recruit phase and loses $2^{-(n-m)(1 - H_2(\epsilon))}$ of its weight in the retain phase. So it loses only $2^{-m/\mu^* + n_0/D\mu'} \cdot 2^{-(n-m)(1 - H_2(\epsilon))}$ units of weight. This quantity is $O(2^{-n/\mu'})$ by Formula (41).

#### G. Summary

Hence the union

$$A_n := A_n^{[n_0/D]} \cup A_n^{[2n_0/D]} \cup \ldots \cup A_n^{[Dn_0/D]}$$  

(44)

will have gap to capacity $O(2^{-n/\mu'})$ and block error probability less than $2^n \cdot 2^{-\beta'n}$.

The complete proof is in the next section.

### V. PROOF OF THEOREM 6

#### A. Discretization: Calculate the Number of Pockets

By continuity, there exists a positive integer $D > 0$ such that

$$\frac{1 - (\pi + \delta_1)}{\mu' - \mu^*(\pi + \delta_2)} + H_2\left(\frac{\beta' \mu'}{\mu' - \mu^*(\pi + \delta_3)}\right) < 1$$  

(45)

for all $-9/D < \delta_1, \delta_2, \delta_3 < 9/D$. We are going to use about $D$ pockets and apply Lemma 5 with $D = D$.

In the following proof, expressions like $2n_0/D$ and $\lceil n \mu^*/\mu' \rceil$ are meant to be integers that are very close to the real numbers $2n_0/D$ and $n \mu^*/\mu'$. It does not matter whether we round up or round down, as we will see later that Formula (41) permits such flexibility.
B. Multi-Pocket Recruit Phase

Let \( P_{\text{bound}}^{\text{upper}} \) be a small, but fixed number. Let \( n \) be large enough. Let \( n_0 = [n \mu'/\mu] \). We define the pockets
\[
\mathcal{A}_n^{(\{n_0/D\})}, \mathcal{A}_n^{(\{2n_0/D\})}, \mathcal{A}_n^{(\{3n_0/D\})}, \ldots, \mathcal{A}_n^{(\{Dn_0/D\})}
\]
by letting \( \mathcal{A}_n^{(m)} \) collect synthetic channels \( W_{2^m} \) with erasure probability less than \( P_{\text{bound}}^{\text{upper}} \). By \( Dm \),
\[
\mathcal{A}_n^{(m)} = \text{all \( \mathcal{A}_n^{(m)} \) with \( m \)}
\]
We give each synthetic channel \( W_{2^m} \) a weight of \( 2^{-m} \). Pocket \( \mathcal{A}_n^{(m)} \) should weigh \( I(W) - O(2^{-m/\mu'}) \) by Lemma 5 with \( D = D \).

For each pocket \( \mathcal{A}_n^{(m)} \), discard synthetic channels that has some ancestor collected in a pocket with smaller index because we do not want to double-count. Now the union \( \mathcal{A}_n^{(\{n_0/D\})} \cup \mathcal{A}_n^{(\{2n_0/D\})} \cup \ldots \cup \mathcal{A}_n^{(Dn_0/D)} \) weighs between \( I(W) - O(2^{-m/\mu'}) \) and \( I(W) + O(2^{-n_0}) \). (The upper bound comes from entropy conservation.) Thus \( \mathcal{A}_n^{(m)} \) weighs at most \( O(2^{-m/\mu'} + n_0/D\mu) \).

C. Multi-Pocket Train Phase

For each synthetic channel \( W_{2^m}^{(j)} \) in pocket \( \mathcal{A}_n^{(m)} \), replace it with all its descendants of the form \( W_{2^m}^{(2^m-n_j-k)} \). Doing this does not affect the weight of \( \mathcal{A}_n^{(m)} \).

For each descendant \( W_{2^m}^{(2^m-n_j-k)} \) obtained from \( W_{2^m}^{(j)} \) in pocket \( \mathcal{A}_n^{(m)} \), its erasure probability is doubled and squared totally \( n - m \) times.

D. Multi-Pocket Retain Phase

Claim a threshold \( \epsilon = \beta n/(n - m) \). For each synthetic channel \( W_{2^m}^{(n_j-k)} \) obtained from \( W_{2^m}^{(j)} \) in pocket \( \mathcal{A}_n^{(m)} \), discard if its erasure probability is squared less than \( \beta n = \epsilon(n-m) \) times out of \( n-m \) chances. By 10, Formula 1.59 with \( \epsilon = \epsilon \), pocket \( \mathcal{A}_n^{(m)} \) loses at most \( 2^{-(n-m)(1-H_2(\epsilon))} \) of its weight.

Furthermore, discard synthetic channels with erasure probability more than \( 2^{-n_0/\mu} \). By 11, Lemma 22 with \( x := P_{\text{bound}}^{\text{upper}} \), pocket \( \mathcal{A}_n^{(m)} \) loses
\[
O(x(1 - \log x)) = O(P_{\text{bound}}^{\text{upper}} (1 + n_0))
\]
of its weight. If \( n, n_0 \) are large enough, the weight loss is \( 2^{-n_0(1-\alpha_1)} \). This quantity is much smaller than the targeted gap to capacity \( O(2^{-n/\mu'}) = O(2^{-n_0/\mu'}) \) so we will simply ignore this.

E. Estimate the Error Probability

By how we discard synthetic channels in the retain phase (Section V-D), the synthetic channels in the union \( \mathcal{A}_n^{(\{n_0/D\})} \cup \mathcal{A}_n^{(\{2n_0/D\})} \cup \ldots \cup \mathcal{A}_n^{(Dn_0/D)} \) have their erasure probabilities less than \( 2^{n_0/\mu'} \). By union bound, the block error probability is less than \( 2^n \cdot 2^{n_0/\mu'} \).

F. Estimate the Gap to Capacity

Now we try to weigh the union \( \mathcal{A}_n^{(\{n_0/D\})} \cup \mathcal{A}_n^{(\{2n_0/D\})} \cup \ldots \cup \mathcal{A}_n^{(Dn_0/D)} \).

In the recruit phase (Section V-B), the union weighs at least \( I(W) - O(2^n/\mu') = I(W) - O(2^{-n/\mu'}) \) and each \( \mathcal{A}_n^{(m)} \) weighs at most \( O(2^{-m/\mu'} + n_0/D\mu) \).

And then in the train phase (Section V-C) the weight remains.

Finally in the retain phase (Section V-D), each \( \mathcal{A}_n^{(m)} \) loses at most \( 2^{-(n-m)(1-H_2(\epsilon))} \) of its weight. Thus it loses at most
\[
O(2^{-m/\mu'} + n_0/D\mu) \cdot 2^{-(n-m)(1-H_2(\epsilon))}
\]
units of weight. Take logarithm. Each \( \mathcal{A}_n^{(m)} \) loses 2 to the power of
\[
-m/\mu' + n_0/D\mu - (n-m) \left( 1 - H_2 \left( \frac{\beta'}{n-n_0} \right) \right) + O(1)
\]
units of weight.

Recall that \( 0 < m \leq n_0 \). Let \( \pi \) be such that \( m = n/\pi \mu'/\mu' \), so \( m = n_0\pi + O(1) \) and \( n - m = n(\mu'/\pi\mu')/\mu' \). If \( n, n_0 \) are large enough then \( \pi \in [-1/D, 1 + 1/D] \). Now the main term of the logarithm becomes
\[
-m/\mu' + n_0/D\mu - (n-m) \left( 1 - H_2 \left( \frac{\beta'}{\mu' - \pi\mu} \right) \right)
\]
By Formula 45 this quantity is less than
\[
-m/\mu' + n_0/D\mu - (n-m) \frac{1}{\mu'} - \frac{1}{\mu'} \frac{\pi}{\mu'}
\]
which, up to constants, is equal to
\[
-n(\pi - 1/\mu') - n - n(\pi - 1/\mu') = -n/\mu'.
\]
Hence each \( \mathcal{A}_n^{(m)} \) loses at most \( O(2^{-n/\mu'}) \) units of weight. Hence the union loses at most \( DO(2^{-n/\mu'}) \) units of weight, which still weighs
\[
I(W) - O(2^{-n/\mu'})
\]
units.

G. Summary

To summarize, the union
\[
\mathcal{A}_n := \mathcal{A}_n^{(\{n_0/D\})} \cup \mathcal{A}_n^{(\{2n_0/D\})} \cup \ldots \cup \mathcal{A}_n^{(Dn_0/D)}
\]
has gap to capacity
\[
O(2^{-n/\mu'})
\]
and block error probability less than
\[
2^n \cdot 2^{n_0/\mu'}
\]
This completes the proof of Theorem 6.
VI. Future Works

A. Regarding Eigenfunction

Lemma 4 plays the same role in proving Theorem 6 as that Lemma 2 plays in proving Theorem 3 and that Lemma 5 plays in proving Theorem 7. The three lemmata provide some “initial boost” before applying the “doubling-or-squaring” argument (i.e., the train phase and retain phase).

They are, in contrast to the “doubling-squaring” argument, a pretty weak starting point. But from the main theorem (Theorem 6 and Appendix H) we know that the initial boost could be doubly exponential in n. A potential proof will be to consult function h in Theorem 1 its behavior near ξ = 0.

In particular: Is there h, µ such that

\[ h(\xi^2) + h(2\xi - \xi^2) = 2^{-1/\mu}. \]  (57)

(Equivalently [12] Formula (12)).) If so, could

\[ h(\xi) \propto (-\log \xi)^{-\theta} \text{ for } \theta := -\log_2 \left(2^{1-1/\mu} - 1\right). \]  (58)

One may notice that when µ = 3.627, the number 1/µθ ≈ A4609 in Appendix [1] is that to say, such infinitesimal behavior of h implies the straight segment from (0, 1/µ) to (1/µθ, 0).

B. Regarding Convex Hull

The next question is whether moving upward and moving rightward follow vector addition. If so, then it trivially implies the straight segment from (0, 1/µ) to (β, 0). Moreover, do there exist achievable points beyond that segment?

C. Regarding General Channels

We have not said anything about binary symmetric memoryless channels but we are confident that there are similar results. The reasons are that the scaling exponent is well-defined for other channels and that the “doubling-squaring” phenomenon is simply omnipresent.

VII. Concluding Remarks

We investigate the trading-off between block length, code rate, and block error probability in constructing classical polar codes.

Our result, Theorem 6 specializes to the result that the error exponent is β = 1/2 [3, Theorem 1] by Corollary 9 and to the result that the scaling exponent is µ = 3.627 [12, Abstract] by Corollary 8.

Moreover, our result implies all known trading-off results: namely [23, Theorem 1] and [2, Theorem 7] by Corollary 9.

It remains open whether there is room for improvement or not.

References

[1] E. Arian, “Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels,” IEEE Transactions on Information Theory, vol. 55, no. 7, pp. 3051–3073, July 2009.

[2] M. Mondelli, S. H. Hassani, and R. L. Urbanke, “Unified scaling of polar codes: Error exponent, scaling exponent, moderate deviations, and error floors,” IEEE Transactions on Information Theory, vol. 62, no. 12, pp. 6698–6712, Dec 2016.

[3] E. Arikan and E. Telatar, “On the rate of channel polarization,” in 2009 IEEE International Symposium on Information Theory, June 2009, pp. 1493–1495.

[4] R. Gallager, “A simple derivation of the coding theorem and some applications,” IEEE Transactions on Information Theory, vol. 11, no. 1, pp. 3–18, January 1965.

[5] G. Gallager, Information Theory and Reliable Communication. New York, NY, USA: John Wiley & Sons Inc., 1968.

[6] S. B. Korada, E. Sasoglu, and R. Urbanke, “Polar codes: Characterization of exponent, bounds, and constructions,” IEEE Transactions on Information Theory, vol. 56, no. 12, pp. 6253–6264, Dec 2010.

[7] S. H. Hassani, R. Mori, T. Tanaka, and R. L. Urbanke, “Rate-dependent analysis of the asymptotic behavior of channel polarization,” IEEE Transactions on Information Theory, vol. 59, no. 4, pp. 2267–2276, April 2013.

[8] R. Mori and T. Tanaka, “Source and channel polarization over finite fields and reed-solomon matrices,” IEEE Transactions on Information Theory, vol. 60, no. 5, pp. 2720–2736, May 2014.

[9] S. B. Korada, A. Montanari, E. Telatar, and R. Urbanke, “An empirical scaling law for polar codes,” in 2010 IEEE International Symposium on Information Theory, June 2010, pp. 884–888.

[10] S. H. Hassani, K. Alishahi, and R. L. Urbanke, “Finite-length scaling for polar codes,” IEEE Transactions on Information Theory, vol. 60, no. 10, pp. 5875–5898, Oct 2014.

[11] D. Gollnitz and D. Burshtein, “Improved bounds on the finite length scaling of polar codes,” IEEE Transactions on Information Theory, vol. 60, no. 11, pp. 6966–6978, Nov 2014.

[12] A. Fazeli and A. Vardy, “On the scaling exponent of binary polarization kernels,” in 2014 52nd Annual Allerton Conference on Communication, Control, and Computing (Allerton), Sept 2014, pp. 797–804.

[13] H. D. Pfister and R. Urbanke, “Near-optimal finite-length scaling for polar codes over large alphabets,” in 2016 IEEE International Symposium on Information Theory (ISIT), July 2016, pp. 215–219.

[14] R. L. Dobrushin, “Mathematical problems in the shannon theory of optimal coding of information,” in Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics. Berkeley, Calif.: University of California Press, 1961, pp. 211–252. [Online]. Available: https://projecteuclid.org/euclid.bsmsp/1200512168

[15] V. Strassen, “Asymptotische abschätzungen in shannons informationstheorie,” in Transactions of the Third Prague Conference on Information Theory, Publishing House of the Czechoslovak Academy of Sciences, 1962, pp. 689–723. [Online]. Available: https://www.math.cornell.edu/~pmlut/strassen.pdf

[16] J-P. Tillich and G. Zémor, “Discrete isoperimetric inequalities and the probability of a decoding error,” Combin. Probab. Comput., vol. 9, no. 5, pp. 465–479, 2000. [Online]. Available: https://doi.org/10.1017/S0963548300004466

[17] A. Montanari, “Finite-size scaling and metastable states of good codes,” in Proceedings of the Allerton Conference on Communication, Control and Computing, Oct 2001. [Online]. Available: https://web.stanford.edu/~montanar/RESEARCHFILEPAP/allerton01.pdf

[18] M. Hayashi, “Information spectrum approach to second-order coding rate in channel coding,” IEEE Transactions on Information Theory, vol. 55, no. 11, pp. 4947–4966, Nov 2009.

[19] Y. Polyanskiy, H. V. Poor, and S. Verdu, “Channel coding rate in the finite blocklength regime,” IEEE Transactions on Information Theory, vol. 56, no. 5, pp. 2307–2359, May 2010.

[20] S. H. Hassani, “Polarization and spatial coupling: Two techniques to boost performance,” Ecole Polytechnique Federale de Lausanne, no. 5706, 2013. [Online]. Available: https://infoscience.epfl.ch/record/188275
Thus, provided that $P\lesssim 1/\mu^\ast$ such that Formula (36) in Lemma 3 holds. So Formula (37), the conclusion of Lemma 3 holds.

Choose a $\rho$ that lies between $\rho_1$ and $1/\mu^\ast$. Choose a small $\alpha$ such that $\rho - D\alpha \gtrsim 1/\mu^\ast$. Choose a small $\delta$ such that the right hand side of Formula (37) is $O(2^{-m\rho})$, i.e.,

$$E \left[ \left( Z_m (1 - Z_m) \right)^\alpha \right] \lesssim O(2^{-m\rho})$$

(71)

hold. Now apply Lemma 4 with $(\alpha, \rho) = (\alpha, \rho)$. Its conclusion implies

$$\mathbb{P} \left( Z_m \leq F_{\text{upper}} 2^{-Dm} \right) \gtrsim I(W) - O \left( 2^{-m/\mu^\ast} \right).$$

(72)

This completes the proof.

C. Proof of Theorem 4

The assumption says that there exists $f(z, a, b)$ such that

$$f(\xi, a, b) = \lim_{n \to \infty} 2^{n/\mu} f_n(\xi, a, b)$$

(73)

where

$$f_{n+1}(\xi, a, b) = f_n(\xi^2, a, b) + f_n(2\xi - \xi^2, a, b) \frac{2}{2} \text{.}$$

(74)

To prove the theorem, take whatever $a < b$ and let $h(\xi) := f(\xi, a, b)$. Then $2^{-1/\mu} h(\xi)$

$$= 2^{-1/\mu} \lim_{n \to \infty} 2^{(n+1)/\mu} f_{n+1}(\xi, a, b)$$

(75)

$$= \lim_{n \to \infty} 2^{n/\mu} f_n(\xi^2, a, b) + f_n(2\xi - \xi^2, a, b)$$

(76)

$$= \frac{2}{2} \lim_{n \to \infty} 2^{n/\mu} f_n(\xi^2, a, b) + \lim_{n \to \infty} 2^{n/\mu} f_n(2\xi - \xi^2, a, b)$$

(77)

$$\leq \frac{h(\xi^2) + h(2\xi - \xi^2)}{2} \text{.}$$

(78)

That is, Formula (17) is equality for $\mu^\ast = \mu$. Thus the inequality holds for $\mu^\ast$ arbitrarily close to $\mu$.

D. Proof of Corollary 8

Apply Theorem 6 with

$$\mu' := \mu^\ast \frac{1}{1 - \gamma}, \quad \beta' := \beta_x \gamma$$

(80)

for some fixed $\beta_x \leq 0.4469$ and $\mu^\ast > \mu$ and all $\gamma \in [0, 1]$. In detail: Choose, for instance, $\beta_x := 0.4469$. It is easy (numerical) to verify that

$$\frac{1 - \xi}{\mu} + H_2(\beta_x \xi) < 1 \quad \forall \xi \in [0, 1] \text{ and } \mu = 3.627 \text{.}$$

(81)

Consequently

$$\frac{1 - \xi}{\mu^\ast} + H_2(\beta_x \xi) < 1 \quad \forall \xi \in [0, 1] \text{ and } \mu = 3.627$$

(82)
for $\mu^* > \mu$. With $\xi := \gamma/(1 - \pi + \pi \gamma)$ this becomes

$$1 - \pi - \gamma - \pi \gamma + H_2 \left( \frac{\beta \gamma}{1 - \pi + \pi \gamma} \right) < 1$$

This is exactly Formula (41)

$$\frac{1 - \pi}{\mu' - \mu^* \pi} + H_2 \left( \frac{\beta' \mu'}{\mu' - \mu^* \pi} \right) < 1$$

with the corresponding $\mu'$ and $\beta'$.

Now apply Theorem 7 with $\gamma \rightarrow 0$, $\mu^* \rightarrow \mu$. (84)

E. Proof of Corollary 9

To recover Theorem 2, plug Formula (20)

$$\beta' := \gamma H_2^{-1} \left( \frac{\gamma (\mu^* + 1) - 1}{\gamma \mu^*} \right) \quad \text{and} \quad \mu' := \frac{\mu^*}{1 - \gamma}$$

in Formula (41)

$$\frac{1 - \pi}{\mu' - \mu^* \pi} + H_2 \left( \frac{\beta' \mu'}{\mu' - \mu^* \pi} \right) < 1$$

and verify it.

In detail: First with $0 \leq \pi \leq 1$, the first term of the inequality is

$$\frac{1 - \pi}{\mu' - \mu^* \pi} \leq \frac{1 - \gamma}{\mu^*} = \frac{\gamma - \gamma^2}{\gamma \mu^*}.$$  (85)

Again with $0 \leq \pi \leq 1$ and $H_2, H_2^{-1}$ monotonically increasing, the second term of the inequality is

$$H_2 \left( \frac{\beta' \mu'}{\mu' - \mu^* \pi} \right) \leq H_2 \left( \frac{\beta' \mu'}{\mu' - \mu^* \pi} \right) = H_2 \left( \frac{\beta'}{\gamma} \right)$$

$$= H_2 \left( \frac{\gamma (\mu^* + 1) - 1}{\gamma \mu^*} \right)$$

$$= \frac{\gamma (\mu^* + 1) - 1}{\gamma \mu^*}.$$  (87)

So the left hand side of the inequality is

$$\gamma - \frac{\gamma^2}{\gamma \mu^*} + \frac{\gamma (\mu^* + 1) - 1}{\gamma \mu^*} = \frac{\gamma \mu^* - (1 - \gamma)^2}{\gamma \mu^*}.$$  (89)

P.S. We know it works because multi-pocket trick implies one-pocket trick. And the one-pocket trick is a generalization of [3, Proposition 3]. (See [3, Formula (31)].)

Now Theorem 2 recovers the error exponent as a special case by driving $\gamma \rightarrow 1$.

The rest of this page is intensionally left black.
F. Visualization of $\mu$

Unless otherwise stated, we assume binary alphabet, classical kernel, and binary erasure channel. See Section I-A2 for details.

→ 2: [13] larger kernels over larger alphabets achieve optimal exponent

→ 2: [20] conjectures that larger kernels over binary alphabet suffice.

→ 2: [21] larger (random) kernels suffice

= 3.356: [12] a larger kernel of size 16

= 3.577: [12] a larger kernel of size 8

≥ 3.579: [10] for general channels

≈ 3.6261: [9] empirically

≈ 3.627: [10] conjectures this value

= 3.627: [12]

≤ 3.639: [2]

≤ 4.714: [2] for general channels

≤ 5.702: [11] for general channels

≤ 6: [10] for general channels
G. Visualization of $\beta$
See Section I-A1 for details.

$= .5$: [3] second order term given
$= .55$: [7] a 16-by-16 kernel
$= .51828$: [6] a 30-by-30 kernel
$= .52643$: [6] a 31-by-31 kernel
$\rightarrow 1$: [6] larger kernel
$\rightarrow 1$: [8] larger alphabet and larger kernel

H. Visualization of Moderate Deviation
The following plot assumes the Scaling Assumption [12] Formula (12) and $\mu = 3.627 = 3627/1000$ [12 Abstract].
See Section I-A3 for details.

$$y\text{-coordinate} = \liminf_{n \to \infty} \frac{-\log(\text{Gap to capacity})}{\log(\text{Block Length})}$$

$$\frac{1}{\mu} \approx .2757$$

$$\frac{1}{1 + \mu} \approx .2161$$

$$x\text{-coordinate} = \liminf_{n \to \infty} \frac{\log(-\log(\text{Block error probability}))}{\log(\text{Block Length})}$$

$\approx .4469 \beta = .5$

Theorem 2 [2, Theorem 7]

$O(\log n/n), O(1)$ [24, Theorem 1.6]
I. Visualization of Polar Code Construction

See also Section II-B.