No ghost state of Gauss-Bonnet interaction in warped backgrounds

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\textbf{Abstract}

A general solution to the Einstein field equations with Gauss-Bonnet(GB) term in the $AdS_5$ bulk background implies that the GB coupling $\alpha$ can take either sign (+ or $-$), though a positive $\alpha$ will be more meaningful. By considering linearized gravity with the GB term in the Randall-Sundrum(RS) a singular 3-brane model, we study the gravitational interactions between matter sources localized on the brane. With correctly defined boundary conditions on the brane, we find a smooth behavior of graviton propagator and hence the zero-mode solution as a 4$d$ massless graviton localized on the brane with correct momentum and tensor structures. The coupling $\alpha$ modifies the graviton propagators both on the brane and in the bulk. The issue on ghost state of the GB term is resolved, and we find that there is no real ghost (negative norm) state of the GB term in the RS single brane picture. The latter condition leads to a consistency in the coupling between the brane matter and the bulk gravity. We also elucidate about the possibilities for behavior of a test particle on the brane and in the bulk.

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1 Introduction

Recently, a considerable effort has been devoted in exploring possible phenomenological and observable consequences of the brane-world scenario in warped backgrounds. This is mainly after seemingly two alternative directions pioneered by ArkaniHamed-Dimopoulos-Dvali (ADD) [1], and Randall and Sundrum [2] in resolving the mass hierarchy problem [3] (see [4] for a new approach to hierarchy problem), and in explaining a small (or vanishing) 4d cosmological constant without relying on supersymmetric approach in theories with large (compact and non-compact) extra dimensions. Probably a more interesting avenue is the Randall-Sundrum’s (we refer RS) a singular 3-brane model with an infinite extra dimension [5] (see [6] for early proposals of the non-compact Kaluza-Klein(KK) models).

The RS proposal [5] involves a conceptually fruitful alternative (to the standard KK compactification) scheme to trap gravity on a singular 3-brane embedded in higher dimensional bulk, where the usual 4-dimensional gravity has been realized as the zero-mode spectrum of the 5-dimensional theory on the 4d boundary. A non-factorizable spacetime geometry induced by a gravitating 3-brane in the higher dimensional spacetimes has a number of novel features, for example, a probable connection of the brane-world proposal to the AdS/CFT correspondences, a description of the strongly coupled four-dimensional conformal field theory with an uv cut-off at the position of the RS brane, and a realization of static AdS domain wall solution in the pure gravitational background. The intriguing ideas behind the RS proposals [2, 5], in particular for opening a new door for thinking about gravity in extra dimensions, have also led to interesting consequences for brane world black hole [7], new realizations of brane-world KK reductions in supersymmetry and supergravity theories [8], and in stringy gravity [9]. The static AdS domain wall was previously realized as the BPS domain walls of supergravity theories [10], while the RS proposals [2, 5] have further resulted in new work on AdS gravity walls coupled to scalar interactions [11], global black p-brane [12], D3 brane [13] solutions, and embedding of the brane-world scenario into a more complete setting of supergravity theories.

The Einstein equations for the linearized perturbation, $\eta_{\mu\nu} + h_{\mu\nu}(x,y)$, induced by the matter source on the brane (or without matter) have been the subject of a number of papers, including [6, 14-22]. Basically, in the RS set-up, the Einstein gravity can be combined with a warped geometry in the bulk AdS$_5$ and a positive vacuum energy 3-brane, and the resultant 4d metric fluctuations are described by an attractive delta-function potential generated by the AdS bulk curvature($\ell$). A single bound state with zero energy in the eigen-spectrum represents a 4d massless graviton, while other massive Kaluza-Klein modes living in the 5d are observed as correction to the Newton’s law. This is achieved in the RS model with different demands than that of the underlying assumptions in the conventional compactified KK theories [23]. In particular, for an infinite fifth dimension $y$, the low-energy degrees of freedom do not restrict to zero modes, the continuum of KK spectrum can exist with no gap, and that the zero-mode wave function depends on $y$ non-trivially. The picture is also distinct in a sense that the coupling of matter to gravity has been realized through Neumann-type
boundary condition on the brane rather than from the Einstein equations on the 4d boundary \[24\]. Nevertheless, because of a fruitful realization of the 4d gravity in the extra dimensions, and also due to a natural inherence of the \(p\)-branes (which are capable of carrying matter fields) in string/M-theory, the RS brane world proposal is appealing and lively.

Given these considerations, a natural approach on a route to realize a consistent theory is to introduce the higher curvature terms to the action, for the effective 4-dimensional gravity on the 3-brane should be that of Einstein plus the higher order curvature corrections. These correction terms should arise from the low energy effective action of string theory or/and as the \(1/N\) corrections in the large \(N\) limit of some gauge theory, and one could introduce them as a ghost-free Gauss-Bonnet (GB) combination [26-32]. Explicit brane solutions for arbitrary order of higher curvature terms with or without cosmological constant were also discussed in [33]. An important aspect concerns whether the higher-curvature terms (in GB combination) in the bulk action can result in a localized 4d gravity or/and still reproduce the correct zero-mode behaviors (both the momentum and tensor structure) at long distance scales on the brane. Our answer to this is positive (see Ref. [34] for very recent results on the effective four-dimensional gravity localized on the (intersection of) \(n\) domain walls in \((4 + n)\)-dimensional space-time with higher curvature terms). It is known that the generic form of the higher curvature terms can delocalize gravity [31] on the brane, and furthermore, in warped backgrounds with finite volume non-compact extra-dimension(s), one should be cautious about such terms due to the possible excitations of unphysical scalar modes (e.g., graviscalar or tachyonic mode) and, massless or massive ghosts. But the GB term does not excite any ghosts in the brane background. It would appear that in the RS single brane model with positive brane tension 3-brane. In the latter case, a non-trivial GB coupling, however, renormalizes the 4d Newton constant on the brane and also modifies the graviton propagators both on the brane and in the bulk, unlike the case is in a flat space-time background [26].

In this paper, by introducing GB term into the Einstein-Hilbert action, we mainly work with the transverse-traceless (TT) components of the metric fluctuations in the presence of matter source on the brane. However, one should take a proper account of non-TT components of the metric fluctuations to derive the correct Neumann boundary condition (Israel junction condition and enforcing \(Z_2\) symmetry) across the brane and hence an account that of brane-shift function \(\hat{\xi}^5\) for a matter-localized 3-brane. In such a case an extra polarization due to the trace part of the perturbation is actually compensated by the brane-shift function. A detailed formalism was developed in [14, 16] and analyzed in a very similar fashion in [20] by adding GB term for the non-TT components, but in the last context, the theory suffered from a ghost/ extra polarization state of GB term. In various steps our treatments also follow parallel to [16]. Instead, here we offer some resolution (or better, reformulation) on the linearized Einstein-Gauss-Bonnet gravity, in particular the negative norm state of GB term, and find that there is no any real ghost state due to such interaction term in the RS single brane model.

In Section 2 we present some important features of the EGB theory in an \(AdS_5\). In Section 3
we analyse the 5d killing equations. Sections 4 and 5 deal with the general behaviors and stability analysis of the graviton propagators, which present the most relevant generalization of the RS gravity with the GB term at the linearized level. Section 6 elucidates upon the geodesics in the general $AdS_5$ backgrounds. Section 7 is a conclusion.

2 Effective action and general solution

We study the gravitational interactions between matter fields localized on a singular 3-brane model with the Gauss-Bonnet term by considering the following effective action defined on the $D$-dimensional space-time $(M)$, where $\partial M$ represents the $(D-1)$-dimensional boundary,

$$S = \int_M d^Dx \sqrt{-g} \left\{ \kappa^{-1}(R - 2\Lambda) + \alpha(R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}) \right\} + 2\int_{\partial M} d^d x \sqrt{-\gamma} (\mathcal{L}_{m}^{\text{bdry}} - \sigma(z_i)) + \int_M d^{d+1}x \sqrt{-g} \mathcal{L}_{m}^{\text{bulk}}. \quad (1)$$

We follow the metric signature $(-, +, \cdots +)$. Here $a, \cdots d = 0, 1, \cdots D$ and $x^a = (x^\mu, z^i)$, where $x^\mu(\mu = 0, \ldots, 3)$ are the brane coordinates and $z_i$ are the bulk coordinates transverse to the brane. $\Lambda$ is the bulk cosmological constant in $AdS_D$, $\sigma(z_i)$ are the brane tensions or vacuum energy of the branes (or an interaction thereof), and $\gamma_{\mu\nu}$ is the induced metric on the brane. As in this paper we work only in the five space-time dimensions, it is useful to define the above parameters in $D = 5$. The parameters therefore have dimensions $[\Lambda] = M^2$, $[\sigma] = M^4$, $[\alpha] = M$. Since the $D(=d+1)$-dimensional mass is defined by $\kappa_{d+1} = 16\pi G_{d+1} = M_*^{1-d}$, where $M_*$ is the $(d+1)$-dimensional fundamental mass scale, one can write $\kappa_5 = M_*^{-3}$. By the same token, since the GB coupling $\alpha$ has the mass dimension of $M^{d-3}$, in five dimensions, we define $\alpha = M_* \alpha'$, where $\alpha'$ is the effective (dimensionless) GB coupling constant.

The graviton equations derived by varying the above action with respect to $g^{ab}$ take the form

$$G_{ab} + \kappa H_{ab} = T_{ab}^{(0)} + \frac{\kappa}{2} T_{ab}^{(m)}, \quad (2)$$

where $G_{ab} = R_{ab} - g_{ab}R/2$ and $H_{ab}$, an analogue of the Einstein tensor stemmed from the GB term, is given by

$$H_{ab} = -\frac{\alpha}{2} g_{ab} (R^2 - 4R_{cd}R^{cd} + R_{cd ef}R^{cd ef})$$

$$+ 2\alpha [RR_{ab} - 2R_{acbd}R^{cd} + R_{acde}R^{de} - 2R_a^c R_{bc}]. \quad (3)$$

and,

$$T_{ab}^{(m)} = -\frac{\delta}{\sqrt{-g}} \delta g^{ab} \int d^{d+1} x \sqrt{-g} \mathcal{L}_m, \quad (4)$$

$$T_{ab}^{(0)} = -\Lambda g_{ab} - \frac{\sqrt{-g}}{\sqrt{-g}} \gamma_{\mu\nu} \delta^\mu_a \delta^\nu_b \delta(z) \sigma(z). \quad (5)$$
We are interested in the solutions with a warped metric having the form
\[ ds^2 = e^{-2A(z)}(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2) = \bar{g}_{ab}dx^a dx^b. \] (6)

The system of equations of motion given by (2) has a solution if one fine-tunes the brane tension to the bulk vacuum energy as \[ \Lambda = -6\kappa^2 / \ell^2 \] and \( \sigma(z) = 6\kappa^{-1}/\ell \), whose solution in five space-time dimensions is given by
\[ A(z) = \ln(|z|/\ell + 1) \] (7)
where,
\[ \ell^2 = \frac{4\alpha'}{M_5^2} \left[ 1 \pm \sqrt{1 + \frac{4\alpha'\Lambda}{3M_5^2}} \right]^{-1} = \ell^2_{\pm}. \] (8)

To recover the usual Einstein gravity in the RS set-up, the curvature scale \( \ell \) could be set in the order of (or larger than) the fundamental (string) scale. From the above relation, we define \( \gamma = 4\alpha\kappa \ell^{-2} = 4\alpha' \ell^{-2}M_5^{-2} \) for a future use, and pause for a while to regard whether \( \gamma \) should be large or small.

The value of \( \gamma \) can be partially fixed from the ratio of \( M_{pl} \) and \( M_* \). Specifically, in the presence of Gauss-Bonnet term, \( \ell \) admits two values implied by Eq.(8)
\[ 1 - \gamma = 1 - 4\alpha' \ell^{-2}M_5^{-2} = \mp \sqrt{1 + \frac{4\alpha'\Lambda}{3M_5^2}}, \] (9)
for \( \ell_+ \) and \( \ell_- \) solutions respectively. For a large mass hierarchy between \( M_* \) and \( M_{pl} \), one needs \( \ell M_* \gg 1 \). In the RS scenario, since \( \ell M_* \sim 1 \), \( \gamma \) would be in the order of \( \alpha' \), which is small enough. While, in the ADD picture, where the fundamental mass scale could arise in TeV range and \( \ell M_* >> 1 \) is expected, \( \gamma \) would be much smaller than unity. The \( \ell_+ \) solution may violate the weak energy condition for localized gravity, the latter condition reads as \( 2\ell^{-1}(1 - \gamma/3) > 0 \) at the brane.

Thru this condition loosely restricts \( \gamma \) as \( \gamma \ll 3 \), the actual graviton propagator analysis reveals that we also need to impose \( (1 - \gamma) > 0 \) for a definite positive contribution to the Newtonian potential from the KK kernel. Further, in order to avoid anti-gravity effect (i.e., \( G_4 < 0 \))\(^1\) one must take \( \gamma > -1 \) limit as well. Hence the effective limits for \( \gamma \) become \( 1 > \gamma > -1 \) (see below). For any values of \( \alpha' \) and \( \Lambda \), the \( \ell_+ \) solution implies that \( \gamma > 1 \), and hence the leading order correction term to the Newtonian potential may appear with a negative sign, which may excite ghost states in the background. As seen from (8), in the AdS background (\( \Lambda < 0 \)), either sign of \( \alpha' \) is allowed for \( \ell_- \) solution, but a positive \( \alpha' \) would be a better choice, for \( \alpha' < 0 \) solution can violate unitarity in some parameter space of the full bulk solution. We also note that a de-Sitter bulk solution (\( \Lambda > 0 \)) is not allowed with GB term, because in this case one always encounters either an anti-gravity effect or finds an imaginary curvature scale.

\(^1\)This is merely a reflection of the assumption that the 3-brane world volume is Minkowskian. Under the axial gauge \( h_{ai} = 0 \) and 4d transverse-traceless (TT) gauge \( h_{\mu}^{tt} = 0 = \partial^\mu h_{\mu\nu} \), one may define \( \delta T^{(0)}_{\mu\nu} = T_{\mu\lambda}h_{\nu}^{\lambda} \).

\(^2\)In fact, \( G_4 \geq 0 \) limit corresponds to \( (1 + \gamma) \geq 0 \).
negative $\gamma$ but satisfying the limit $\gamma > -1$. Indeed, the effective limits $1 > \gamma > -1$ translate, in terms of bulk parameters, to $-3/4 < \alpha' \Lambda M_5^{-2} < 9/4$, and one then has to satisfy these limits with a negative bulk cosmological constant.

3 5d diffeomorphism and gauge transformation

In this section we address the issue on the choice of brane-shift function in a warped metric background. We consider the perturbed metric in the following form

$$ds^2 = \rho(y)(g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu - dy^2 = g_{ab}dx^a dx^b,$$

(10)

in axial gauge $h_{a5} = 0$. For generality, at first we do not restrict the form of $\rho(y)$, but will take a proper account of the RS-type background solution $\rho(y) = e^{2A(y)}$, $A(y) = (c - |y|)/\ell$, where $c$ is a parameter associated with the geometry of curved solution and for a flat Minkowski 3-brane one can set $c = 0$. In terms of the metric fluctuations $h_{ab}$, the full 5-dimensional diffeomorphism read

$$\delta h_{ab} = \hat{\nabla}_a \xi_b + \hat{\nabla}_b \xi_a.$$  

(11)

Here hats represent the parameters defined in the five space-time dimensions and $\hat{\nabla}$ is the 5$d$ covariant differential operator. Then the 5$d$ killing vectors are defined by $\xi^a = (\xi^\mu e^{-2A(y)}, -\xi^5)$, and $\xi_a = (\xi_\mu, \xi_5)$, and for a non-vanishing brane tension, the 5$d$ killing equations $\hat{\nabla}_a \xi_b + \hat{\nabla}_b \xi_a = 0$ can be expressed as

$$\hat{\nabla}_\mu \xi_\nu + \hat{\nabla}_\nu \xi_\mu = 0, \hat{\nabla}_\mu \xi_5 + \hat{\nabla}_5 \xi_\mu = 0, \hat{\nabla}_5 \xi_5 = 0.$$  

(12)

The last equation of (12) implies that $\xi_5 = \omega(x^\lambda)$, which can be identified as a 4$d$ scalar (or brane shift function in the warped background), while, the second equation yields

$$\partial_\mu \xi^5(x) = -\partial_y \xi_\mu(x^\lambda, y) + (\rho'/\rho) \xi_\mu(x^\lambda, y).$$  

(13)

Using a differential operator $\nabla_\nu$ on both sides, we get

$$\partial_y \left[ \rho^{-1} (\nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu) \right] = 0,$$  

(14)

where we have made use of $(\nabla_\mu \partial_\nu - \nabla_\nu \partial_\mu) \omega(x^\lambda) = 0$. Eq. (13) defines general transformations in terms of arbitrary small functions of $x$, which take the form

$$\xi^5 = \xi^5 \hat{}(x),$$  

$$\xi^\mu(x^\lambda, y) = -\rho(y) G \partial^\mu \xi^5(x^\lambda) + \beta(x^\lambda),$$

(16)

Here to make a connection to [21] we follow the metric signature $(+, +, \cdots, -)$ and consider a more general case where the 4$d$ hypersurface could be a Minkowski, de-Sitter or Anti de-Sitter 3-brane.
where $\beta^\mu(x)$ is an arbitrary dual vector field, which preserves the remaining gauge invariance, and $G$ is defined as

$$G = \int \tilde{y} \rho(y)^{-1} \, dy.$$  \hspace{1cm} (17)

Expanding the first equation of (12)

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - g_{\mu\nu} \rho' \omega(x^\lambda) = 0,$$  \hspace{1cm} (18)

multiplying this by $\rho^{-1}$, differentiating w.r.t. $y$ and combining the result with (14), we find

$$\omega^{-1} \nabla^2 \omega = -2 \rho (\rho'/\rho)'.$$  \hspace{1cm} (19)

For the RS background solution $\rho = e^{-2|y|/\ell}$, the r.h.s. is $8 \ell^{-1} e^{2|y|/\ell} \delta(y)$, which is non-vanishing at $y = 0$. Further, since the l.h.s. of Eq. (19) is a function of $x^\lambda$ and the r.h.s. is a function of just $y$, by separation of variables, one requires

$$(\nabla^2 + k) \omega = 0, \quad k - 2 \rho (\rho'/\rho)' = 0.$$  \hspace{1cm} (20)

Here $k = -8 \ell^{-1} e^{2|y|/\ell} \delta(y)$, so that $k$ vanishes in the bulk. Since $k(y = 0) \neq 0$ and $k(y \neq 0) = 0$, it may be inconsistent in requiring a vanishing $w$ at the brane, but a non-vanishing $\omega$ in the bulk. So, in general, we require a non-vanishing $\omega$ both on the brane and in the bulk.

Identifying $\hat{\xi}^5(x) = -\Phi$ as the radion field [21] associated to the fifth coordinate transformation and using $\rho(y) = e^{2A(y)}$, we get

$$\nabla^2 \Phi = 4 \lambda \Phi,$$  \hspace{1cm} (21)

where $\lambda = -e^{2A(y)} A''$ is defined as the 4d cosmological constant and $R_{(4)} = -4\lambda$. This can be attributed to the weak energy condition (i.e., $T_{00}^0 - T_{55}^5 \geq 0$): $A'' \leq -\lambda e^{-2A}$ obtainable from the Einstein field equations, and this was first invoked in [21]. The stronger version $A'' < 0$ of the standard $c$-theorem then requires $\lambda > 0$ and hence implies an $AdS_4$ brane ($R_{(4)} < 0$) embedded in $AdS_5$ bulk (i.e., in a constant negative curvature background). This leads to locally localized gravity for a sufficiently small $AdS_4$ cosmological constant studied in [21] (see [22] for a bigravity model with two positive tension $AdS_4$ branes in $AdS_5$), though for a more meaningful four-dimensional physics we might expect a completely localized gravity on the brane, if the spin-2 graviton is strictly massless. A flat Minkowski 3-brane with a singular ($\delta$-function) source is, however, compatible with the standard $c$-theorem, viz. for a vanishing 4d cosmological constant ($\lambda = 0$), one acquires $A'' = -2 \ell^{-1} \delta(y) \leq 0$.

Finally, substituting Eq. (13) into Eq. (18), we find

$$\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = g_{\mu\nu} (\rho'/\rho) \omega + 2G \nabla_\mu \partial_\nu \omega.$$  \hspace{1cm} (22)

Contracting the space-time indices and using $(\nabla^2 + k)\omega = 0$, this gives

$$\omega^{-1} \nabla_\mu \beta^\mu = 2[(\rho'/\rho) - k].$$  \hspace{1cm} (23)

\hspace{1cm} 4

To be more precise we choose zero as the lower limit on the integral.
Since $\beta^\mu$ is the killing vector associated with four dimensional space-times, by separation of variables, one requires $(p'/p) - k = \text{constant}$. This demands that the warp factor $\rho(y)$ takes a form $\rho(y) = \rho(0) e^{k_1 y}$. This is also justified from the second equation of (20). At the location of the brane we choose $\rho(0) = 1$ and define $k_1 = -2/\ell$ to arrive at $\rho(y) = e^{-2|y|/\ell}$. This implies that one may set $\omega(y = 0) = 0$ only for $\rho(y) \neq \rho(0) e^{2|y|/\ell}$, but not for the RS-type background solution. In particular, with the choice $\omega(y = 0) = 0$ one cannot solve the killing equations without breaking the $D$-dimensional diffeomorphism. With $\omega = 0$, one gets the 4$d$ killing equations $\mathcal{L}_\beta g_{\mu\nu} = \nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0$.

In the RS set-up, as the coordinate systems are based on the 4$d$ hypersurface, in the presence of matter fields localized on the brane, the brane would shift to a new location or appear bent. There will, however, not be a global coordinate system which is Gaussian-normal in the latter case. Therefore, a brane bending mechanism, which may be important to study the gravitational interactions between matter sources localized on the brane, is needed to preserve the axial gauge, $h_{a5} = 0$. The latter is also useful to remove the gauge degrees of freedom completely. Though the “brane-bending” mechanism may not be so obvious when one works with different gauge [17, 19], we find still more convenient to work with Gaussian-normal conditions.

Let’s consider the ground state metric in a more relevance form

$$ds^2 = dy^2 + e^{-2|y|/\ell} (g_{\mu\nu} + h_{\mu\nu}) dx^\mu dx_\nu.$$  \hspace{1cm} (24)

When one deforms the coordinates changing the base hypersurface to a new set of coordinates $(x', y')$ as

$$y' = y + \xi^5(x), \quad x'^\mu = x^\mu + \xi^\mu(x^\lambda, y) = x^\mu - \ell e^{2|y|/\ell} \partial^\mu \xi^5(x) + \beta^\mu(x^\lambda),$$  \hspace{1cm} (25)

the graviton fluctuations $h_{\mu\nu}$, under these coordinate transformations, transform as

$$h_{\mu\nu}(x) \rightarrow h'_{\mu\nu}(x) = h_{\mu\nu}(x) + \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - 2 g_{\mu\nu} \ell^{-1} \xi^5(x).$$  \hspace{1cm} (26)

For a flat Minkowski 3-brane, $g_{\mu\nu} = \eta_{\mu\nu}$, the above transformation for the metric fluctuations reads

$$h_{\mu\nu}(x) \rightarrow h'_{\mu\nu}(x) = h_{\mu\nu}(x) - \ell^{-1} \partial_\mu \partial_\nu \xi^5(x) + e^{-2|y|/\ell} (\partial_\mu \beta_\nu + \partial_\nu \beta_\mu - 2 \eta_{\mu\nu} \ell^{-1} \xi^5(x)),$$  \hspace{1cm} (27)

where the first and second terms in the bracket are pure gauge terms in the 4-dimensional brane worldsheet, and can be gauged away by defining $\partial_\mu \partial_\nu \xi^5(x)$ appropriately.

4 Linearized equations and graviton propagators

For the action (3), the linearized equations of motion read as

$$\delta \hat{G}_{ab} + \kappa \delta^{(1)} \hat{H}_{ab} = \frac{\kappa}{2} T^{(m)}_{ab},$$  \hspace{1cm} (28)

where $\delta \hat{G}_{ab} = \delta G_{ab} - \delta T^{(0)}_{ac} h_b^c$ and $\delta \hat{H}_{ab} = \delta H_{ab} - H_{ac} h_b^c$. The metric (3) would be more convenient to simplify the analysis of gravitational fluctuations. Thus we consider the metric perturbations in
the form \( g_{ab} = e^{-2A(z)}(\eta_{ab} + h_{ab}) \). In axial gauge, \( h_{a5} = 0 \), to the first order in \( h_{ab} \), the tensor mode of the metric fluctuation \( h_{\mu\nu} \) satisfies \( 28 \)

\[
\begin{align*}
  &\left[ -(1 + 4\alpha\kappa e^{2A} A') \partial^2_{\lambda} - (1 - 4\alpha\kappa e^{2A} A'^2) \partial^2_z + 3 A' \partial_z \right] (h_{\mu\nu} - \eta_{\mu\nu} h) \\
  &+ (1 - 4\alpha\kappa e^{2A} (2A'^2 - A'')) \left( 2\partial_{(\mu}\partial_{\nu)\lambda} - \partial_{\mu}\partial_{\nu} h \right) - (1 + 4\alpha\kappa e^{2A} A'') \times \eta\mu\nu \partial_\lambda \partial_\rho h^{\lambda\rho} + 4\alpha\kappa e^{2A} A' \left( (2A'' - A'^2) \partial_z h_{\mu\nu} + (A'' - 2A'') \eta_{\mu\nu} \partial_z h \right) = \kappa T^{(m)}_{\mu\nu}.
\end{align*}
\]

One can look for solutions of the form \( h_{\mu\nu}(x, z) = \epsilon_{\mu\nu} e^{ipx} \psi(z) \). Here \( \epsilon_{\mu\nu} \) is the constant polarization tensor of the graviton wave function, \( m^2 = -p^2 \) and \( m(= \sqrt{-p \cdot p}) \) is the four-dimensional mass of the perturbation. Consider first the case without matter source on the brane. Then with the RS background solution \( A = \log(|z|/\ell + 1) \), the TT components of the metric fluctuations would imply the following expression of graviton propagator

\[
\begin{align*}
  &\left[ \left( 1 - \frac{4\alpha\kappa}{\ell^2} \text{sgn}(z)^2 + \frac{8\alpha\kappa}{\ell} \delta(z) \right) \partial^2_{\lambda} + \left( 1 - \frac{4\alpha\kappa}{\ell^2} \text{sgn}(z)^2 \right) \partial^2_z - \frac{16\alpha\kappa}{\ell^2} \text{sgn}(z) \delta(z) \partial_z \\
  &- \frac{3}{(\ell + |z|)} \left( 1 - \frac{4\alpha\kappa}{\ell^2} \text{sgn}(z)^2 \right) \text{sgn}(z) \partial_z \right] G_5(x, z; x', z') = \delta^{(4)}(x - x') \delta(z - z').
\end{align*}
\]

The graviton propagator along the brane can be decomposed into the Fourier modes

\[
G_5(x, z; x', z') = \int \frac{d^4p}{(2\pi)^4} e^{ip(x - x')} G_p(z, z'),
\]

where the Fourier components \( G_p(z, z') \) in the bulk satisfy

\[
e^{2A}(\partial_z^2 - p^2 - \frac{3}{z} \partial_z) G_p(z, z') = e^{A}(1 - \gamma)^{-1} \delta(z - z').
\]

Since \( e^{A(z)} = (|z|/\ell + 1) \), one can use \( e^{A(z)} = z/\ell \) for \( |z| >> \ell \). Writing \( G_p = (z'z/\ell^2)^2 \hat{G}_p \) and \( p^2 = -q^2 \), we arrive at

\[
(z^2 \partial_z^2 + z \partial_z + q^2 z^2 - 4) \hat{G}_p(z, z') = (1 - \gamma)^{-1} \ell z \delta(z - z').
\]

The translation invariance of \( \hat{G}_p(z, z') \) implies that this Bessel equation is generally valid after a general coordinate transformation in Gaussian-normal form. When one defines a coordinate transformation \( z = \ell e^{y/\ell} \) and works in the RS background, for \( z > \ell \) (or \( z < \ell \)) the metric is the 5-dimensional \( \text{AdS} \) metric given by

\[
ds^2 = \frac{\ell^2}{z^2}(dx^2 + dz^2).
\]

So without loss of generality, one can locate the brane at \( z = \ell \), and the brane at \( y = 0 \) is mapped to \( z = \ell \). Therefore, in \( y \)-coordinate, the TT components of the metric fluctuations satisfy

\[
\left[ \left( 1 - 4\alpha\kappa A'^2 + 4\alpha\kappa A'' \right) e^{2|y|/\ell} \partial^2_{\lambda} + \partial_y^2 - 4\alpha\kappa (A'^2 \partial_y^2 + 2A' A'' \partial_y) \\
- 4A'^2 \left( 1 - 4\alpha\kappa A'^2 \right) + 2A'' \left( 1 - 12\alpha\kappa A'^2 \right) \right] h_{\mu\nu}(x, y) = -\kappa T_{\mu\nu}(x, y).
\]

See also ref. [20].
Here the energy momentum tensor includes a contribution from matter source on the brane, \( T_{\mu\nu}(x, y) = S_{\mu\nu}(x) \delta(y) \). Now we deform the coordinates from \((x, y)\) to \((x', y')\). In the latter gauge the brane is located at \( y' = y + \xi^5(x) = 0 \), where \( \xi^5(x) \) is an arbitrary brane shift function defined in the Section 3.

In \( y' \)-coordinate, one has
\[
\left[ (1 - 4\alpha\kappa A'{}^2 + 4\alpha\kappa A'' \ell^2 |y'\ell|^2 \delta^2 + \delta_y^2 - 4\alpha\kappa (A'{}^2 \delta_y^2 + 2A' A'' \partial_y) \right. \\
\left. - 4A''(1 - 4\alpha\kappa A'{}^2) + 2A''(1 - 12\alpha\kappa A'{}^2) \right] h'_{\mu\nu} = -\kappa \Sigma_{\mu\nu}(x') \delta(y'),
\]
where the source term \( \Sigma_{\mu\nu}(x') \) is given by (see Appendix A)
\[
\Sigma_{\mu\nu}(x') = S_{\mu\nu}(x') - \frac{1}{3}(\eta_{\mu\nu} - \partial_\mu \partial_\nu) S(x').
\]
The condition \( \Sigma_\mu = 0 \) justifies the gauge choice \( h' = 0 \), and a factor of \((1 + \gamma)\) in the second and third terms due to the Israel junction condition for \( h'_{\mu\nu} \) has been canceled by the term \((1 + \gamma)^{-1}\) coming from the brane-shift function.

Now for the TT components of the metric fluctuations, the Fourier modes \( \hat{G}_p \) of the graviton propagator satisfy
\[
\left( 1 - \frac{4\alpha\kappa}{\ell^2} \text{sgn}(y)^2 + \frac{8\alpha\kappa}{\ell^2} \delta(y) \right) e^{2|y'\ell|/\ell} \delta_y^2 \hat{G}_p + \delta_y^2 \hat{G}_p - \frac{4\alpha\kappa}{\ell^2} (\text{sgn}(y)^2 \delta_y^2 + 2\delta(y) \text{sgn}(y) \partial_y) \hat{G}_p \\
- \frac{4}{\ell^2} \text{sgn}(y)^2 \left( 1 - \frac{4\alpha\kappa}{\ell^2} \text{sgn}(y)^2 \right) \hat{G}_p + \frac{4}{\ell^2} \delta(y) \left( 1 - \frac{12\alpha\kappa}{\ell^2} \text{sgn}(y)^2 \right) \hat{G}_p = \delta(y - y').
\]
Indeed, a choice of appropriate boundary conditions (b.c.) on the brane is crucial in obtaining the correct behavior of the graviton propagators. With the GB term, in particular, there exists a subtlety in the choice of boundary condition for \( \hat{G}_p \) across the brane due to the terms involving \( \partial_y |y| \) at \( y = 0 \). But one should fix them from the requirements that one obtains a consistent low-energy limit and the theory becomes free from any unphysical (ghost) states. For this reason, one has to properly regularize the terms such as \( \text{sgn}(y)^2 \delta_y^2 \hat{G} \), \( \text{sgn}(y)^2 \delta(y) \) and \( \text{sgn}(y) \delta(y) \partial_y \hat{G} \). It should be understood that \( \partial_y \hat{G}_p(0) = 0 \), but \( \partial_y \hat{G}_p|_{y=0_+} = -\partial_y \hat{G}_p|_{y=0_-} \neq 0 \). And, just below or above \( y = 0 \), \( \delta(y) \) still dominates \( \text{sgn}(y) \), the latter shows a behavior of a step function. In other words, the function \( \text{sgn}(y) \) is defined to vanish for vanishing argument and only for \( y \neq 0 \) one can use \( \text{sgn}(y)^2 = 1 \), thus \( \text{sgn}(y) = +1 \text{ if } y > 0 \) and \( \text{sgn}(y) = -1 \text{ if } y < 0 \). The term \( \text{sgn}(y) \partial_y \hat{G}_p \) vanishes when evaluated from just below to just above the brane. We also need the following two equalities, obtained after properly regularizing the \( \delta \)-function, to simplify the above equation
\[
\delta(y) \text{sgn}(y)^2 = \delta(y)/3, \\
\left( \text{sgn}(y)^2 \partial_y^2 + 2\delta(y) \text{sgn}(y) \partial_y \right) \hat{G}_p = \partial_y^2 \hat{G}_p(|y|) + 2\delta(y) \partial_y \hat{G}_p(0_+).
\]
Given these considerations, the boundary condition at \( z = \ell \) is given by
\[
(z \partial_z + 2 + \chi q^2 z^2) \hat{G}_p(z, z')|_{z=\ell} = 0,
\]
\footnote{See also Ref. \cite{2} for a rigorous derivation of this boundary condition.}
where \( \chi = \gamma/(1 - \gamma) \). Eq. (33) also implies the following matching conditions at \( z = z' \):

\[
\hat{G}_{<}|_{z=z'} = \hat{G}_{>}|_{z=z'}, \\
\partial_z(\hat{G}_{>} - \hat{G}_{<})|_{z=z'} = (1 - \gamma)^{-1} \frac{\ell}{z'},
\]

The general solutions of the Bessel equation (33) satisfying the b.c. (39) and the matching Eqs. (40) lead to the following expression for the graviton propagator, for \( z < z' \),

\[
\hat{G}_{z<z'} = iA(z') \left[ \left( J_1(q\ell) + \chi q\ell J_2(q\ell) \right) H_2^{(1)}(qz) - \left( H_1^{(1)}(q\ell) + \chi q\ell H_2^{(1)}(q\ell) \right) J_2(qz) \right],
\]

where \( H_1^{(1)} = J_{1,2} + iY_{1,2} \) is the Hankel function of the first kind. For \( z > z' \) the use of the b.c. as \( z \to \infty \), similar to the Hartle-Hawking b.c. which requires the +ve frequency wave be ingoing to the \( AdS \) horizon \( z \to \infty \) but no re-emission, results in

\[
\hat{G}_{z>z'} = B(z') H_2^{(1)}(qz).
\]

The use of Eq. (40) would imply the following general expression for the graviton propagator (13):

\[
G_6(x, z; x', z') = (1 - \gamma)^{-1} \frac{i\pi}{2\ell^3} (zz')^2 \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} \\
\times \left[ \left( J_1(q\ell) + \chi q\ell J_2(q\ell) \right) H_2^{(1)}(qz) - \left( H_1^{(1)}(q\ell) + \chi q\ell H_2^{(1)}(q\ell) \right) J_2(qz) \right] - J_2(qz_<) H_2^{(1)}(qz_>).
\]

The second term implies a presence of extra polarization state as massless 4d scalar field in the full 5d graviton propagator. One may inquire whether this contributes to the propagator on the brane. This issue was discussed in \([35, 36, 37]\), and it has been argued in \([35]\) that this extra polarization degree of freedom in the full propagator may be cancelled by the brane-bending mode [7] and is indeed precisely correct in an appropriate gauge (see below) even in the presence of the GB interaction term.

When one of the arguments of \( G_5 \) is at \( z' = \ell \), the above graviton propagator reduces to

\[
G_5(x, z; x', \ell) = (1 - \gamma)^{-1} \frac{2z^2}{\ell^2} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} \frac{1}{q} \left[ \frac{H_2^{(1)}(qz)}{H_1^{(1)}(q\ell) + \chi q\ell H_2^{(1)}(q\ell)} \right].
\]

For both points at \( z, z' = \ell \), use of the recursion relation \( H_0^{(1)}(q\ell) + H_2^{(1)}(q\ell) = (2/\ell \ell) H_1^{(1)}(q\ell) \) would imply

\[
G_5(x, \ell; x', \ell) = (1 - \gamma)^{-1} \left[ \frac{2\ell}{\ell} \Delta_4(x, x') + \Delta_{KK}(x, x') \right],
\]

where

\[
\Delta_4(x, x') = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} \left[ \frac{H_1^{(1)}(q\ell)}{H_1^{(1)}(q\ell) + \chi q\ell H_2^{(1)}(q\ell)} \right] \\
\approx (1 + 2\chi)^{-1} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} \left[ \frac{1}{q^2} - \frac{\chi}{1 + 2\chi} \ell^2 \left( \ln(q\ell/2) + \Gamma \right) \right] + \cdots,
\]

\(^7\)However, the brane-bending effect in order to cancel this unwanted physical polarization of gravitons due to the mismatch in the tensor structure of massive and massless graviton propagators is not free of comment [37].
\[ \Delta_{KK}(x, x') = -\int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} \times \frac{1}{q} \left[ \frac{H_2(q\ell)}{H_1(q\ell) + \chi q\ell H_2(q\ell)} \right] \]

\[ \approx (1 + 2\chi)^{-1} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} \ell \ln(q\ell/2) [1 + \Gamma + \mathcal{O}((q\ell)^3)], \quad (47) \]

where \( \Gamma = 0.577216 \cdots \) is the Euler-Mascheroni constant. Clearly, for \( \chi = 0 \), there is no sub-leading term from \( \Delta_4(x, x') \) and the correction term arises only from \( \Delta_{KK}(x, x') \). However, for a non-vanishing and relatively large \( \chi \) (\( \lesssim 1 \)), the sub-leading term from \( \Delta_4 \) could be in the order of leading order term from \( \Delta_{KK} \).

In order to expect a dominant contribution from \( \Delta_{KK}(x, x') \) as a correction term of the Newtonian potential, one may require a small \( \gamma \). This indeed imposes a large mass hierarchy limit \( \ell M_s >> 1 \).

Substituting Eqs. (46, 47) into Eq. (45) or using the expansion:

\[ \frac{H_2(q\ell)}{H_1(q\ell) + \chi q\ell H_2(q\ell)} \approx \frac{1}{1 + 2\chi} \left( \frac{q\ell}{2} + \frac{2}{q\ell} \right) + \frac{q\ell}{(1 + 2\chi)^2} \left[ \ln(q\ell/2) - \chi + (\Gamma - 1/2) \right] + \mathcal{O}((q\ell)^3), \quad (48) \]

for \( |x - x'| >> \ell \), i.e., \( q\ell << 1 \), the propagator (43) is given by, to the leading order terms,

\[ G_5(x, z; x', \ell) \approx (1 - \gamma)^{-1}(1 + 2\chi)^{-1} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} \left[ 2 \frac{1}{q^2} + \frac{1}{1 + 2\chi} \ell \ln(q\ell/2) \right] \]

\[ = (1 + \gamma)^{-1} \left[ \frac{2}{\ell} G_4(x, x') + G_{KK}(x, x') \right], \quad (49) \]

where \( \chi = \gamma/(1 - \gamma) \) is substituted. In physical sense, the correction term \( (1 + 2\chi)^{-1} \) arises from the coupling between GB curvature term and the brane matter. Evidently, the zero-mode graviton propagator on the brane is given by

\[ G_4(x, x') = \int \frac{d^4p}{(2\pi)^4} \frac{1}{q^2} = \Box_4(x, x'), \quad (50) \]

while \( \Box_4(x, x') \) is the usual 4d flat space (massless) scalar propagator. Eq. (45), therefore, suggests that even with the GB interaction term, in the low-energy scale \( q\ell << 1 \), the zero-mode propagator reproduces a correct 4d massless graviton propagator. Also the zero-mode graviton propagator on the brane satisfies

\[ \partial_\mu \partial^\mu G_4(x, x') = 2 \delta^4(x, x'). \quad (51) \]

Obviously, the overall constant term \( (1 + \gamma)^{-1} \) renormalizes the 4d Newton’s constant on the brane.

The limiting behavior of the graviton propagator is given by

\[ G_4(x, x') = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} \frac{1}{q^2} \propto \frac{1}{|x - x'|^2}, \quad (52) \]

\[ G_{KK}(x, x') = (1 + 2\chi)^{-1} \ell \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} \ln(iq\ell/2) \propto \frac{(1 - \gamma)}{(1 + \gamma)} \frac{\ell}{|x - x'|^4}. \quad (53) \]

Clearly, at large distances along the brane \( |x - x'| = r >> \ell \), contribution from the KK kernel (53) would be very small compared to the zero mode piece (52), and one also finds a smooth infra-red behavior of the propagators.
The very short distance $r << \ell$ behavior of the propagator on the brane is governed by the ultra large $q$-behavior of the Fourier mode. In this case $q \ell > 1$, then using the asymptotic expansion of the Hankel functions, one finds from Eq. (45), to the leading order,

$$\Delta_4(x,x') \approx -(1 + 2\chi)^{-1} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} \frac{\Delta m}{p^2(p + \Delta m)},$$

$$\Delta_{KK}(x,x') \approx (1 + 2\chi)^{-1} \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} \frac{\Delta m}{p(p + \Delta m)},$$

where

$$\Delta m \equiv \frac{1}{\ell^4} \left[ 2 + \frac{\ell^2 M_z^2}{4\alpha'} \right], \quad \alpha' \neq 0.$$  

Even in the ultra-violet range, the graviton propagators do not blow up.

5 (In) stability of the linearized solutions

In fact, though the propagator is a good first test of the linearized approximation, it sometimes does not contain all information about the graviton spectrum. So it may be the case that one does not notice an instability in the propagator on the brane, but there are still unstable modes in the spectrum. In order to judge this we would like to see whether there is any unphysical (ghost) states in the solutions at the linearized level.

5.1 Extra polarization of GB term

Let us briefly review the results in Ref. [20], where metric fluctuation on the brane at $z = \ell$ was evaluated with the result (one has to replace $\beta$ in [20] by $\gamma/(1 - 3\gamma)$ in our notation)

$$h_{\mu\nu}(x) = -M_{pl}^{-2} \int d^4x' \Box_4(x,x') \left[ S_{\mu\nu}(x') - \left( \frac{1}{2} + \frac{\gamma}{3(1 - 3\gamma)} \right) \eta_{\mu\nu} S_\xi(x') \right] - M_{pl}^{-3} \frac{(1 - \gamma)^{-1}}{(1 - 3\gamma)} \int d^4x' G_{KK}(x,x') \left[ S_{\mu\nu}(x') - \frac{1}{3} \eta_{\mu\nu} S_\xi(x') \right].$$

In Eq. (57), in particular, $M_{pl}^2 = M_5^2 \ell (1 - \gamma)^{-2}(1 - 3\gamma)$ is to be understood. For $\gamma = 0$, contribution from the brane-bending mode changes the factor $1/3$ to $1/2$ in Eq.(57), and hence yields the usual massless $4d$ graviton propagator

$$\Box_4(x,x')_{\mu\lambda\nu\rho} = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-x')} \left( g_{\mu\nu} g_{\lambda\rho} + g_{\mu\rho} g_{\lambda\nu} - g_{\mu\lambda} g_{\nu\rho} \right).$$

However, for a non-trivial $\gamma$, the usual $4d$ gravity appears to be modified by the extra polarization factor $-\gamma/3(1 - 3\gamma)$, and the coupling $\gamma$ here induces an extra polarization/ghost state that cannot be resolved by adjusting any physical parameters. For the result (57), the leading behaviors of $h_{00}$ and $h_{ij}$ components will also be different for a non-vanishing $\gamma$. In this case, one essentially encounters

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Addendum: however, an importance of $\delta$-function regularization and some needed corrections on those results were later narrated by the authors of [20] in Erratum-ibid, Nucl. Phys. B 619 (2001), 763.
massive or massless ghosts, which delocalize gravity at the background. This implies that the zero mode of the quantum mechanical system becomes unstable due to the higher-curvature terms even in the GB combination. In other words, the extra polarization state in the graviton propagator reveals that one does not recover a correct tensor structure while admitting the brane brane bending effect in the presence of the GB term. As the brane bending effect itself results from the presence of a localized source on the brane, this might signal out a conflict of the localized source in the presence of the GB interaction.

One of the main points of this paper is to argue that none of the statements about the ghost/extra polarization state in the previous paragraph is necessarily true. As we know, the GB term is a ghost free combination independent of the dimensions and the topology of the space-time, and there is no ghost state in the RS model when $\alpha = 0$ [14]. Thus we do expect to recover a correct momentum behavior (i.e., 4d Newton’s law), and also a correct tensor structure (i.e., graviton polarization state) in the RS single brane model. This is actually what we find by correctly defining the Neumann boundary condition (Israel junction condition, and enforcing $Z_2$ symmetry) on the brane, which requires a proper regularization of the $\delta$-function. In doing so the extra polarization state simply does not arise, and hence and one find both–a correct momentum behavior and also a correct tensor structure for the massless graviton propagators on the brane.

At any rate, to visualize the negative norm (for $\gamma > 0$) state for the result (57), one can evaluate the one-particle exchange amplitude in the presence of matter source, $T^{(m)}_{\mu\nu}$. This is given by

$$A = \frac{1}{2} \int_{\partial M} \sqrt{-g} h_{\mu\nu}(x) T^{\mu\nu}_{\text{brane}}(x) \equiv \frac{8\pi G_N}{p^2} \left[ S_{\mu\nu} S^{\mu\nu} - \left( \frac{1}{2} + \frac{\gamma}{3(1-3\gamma)} \right) S^2 \right],$$

which, in the massless limit, yields

$$A_{\text{massless}} = \frac{8\pi \tilde{G}_N}{-p_0^2 + p_3^2} \left[ |S_{+2}|^2 + |S_{-}^2|^2 - \frac{\gamma}{3(1-3\gamma)} (S_{11} + S_{22})^2 \right].$$

The first two terms, $S_{\pm 2} = \frac{1}{2}(S_{11} - S_{22})$, are the contributions from the gravitons with helicities $\lambda = \pm 2$, which represent the massless spin-2 propagation, and the last term is due to the massless graviscalar. So one needs either $S_{11} + S_{22} = 0$ or $\gamma < 0$ for no negative norm state. The requirement $S_{33} = 2(S_{11} + S_{22}) = 0$ is invalid, because this implies $S_{\mu} = 0$, but $S_{11} + S_{22} \neq 0$ in general. Further, by demanding a negative $\gamma$, if one requires definite-positive norms, one will introduce an unphysical state into the theory due to the presence of a graviscalar mode and this further implies that the gauge $h_{55} = 0$ could be an insufficient gauge for EGB gravity. In particular, this may suggest that the cancellation of an unphysical scalar mode by the brane bending effect is incomplete in the presence of higher-curvature terms. But, this is indeed not the case as we see below.

It may perhaps be argued that the above residual effect of the graviscalar mode may have been due to the ignorance of the $T_{55}$ component or a choice of Gaussian normal condition $h_{55} = h_{\mu 5} = 0$, the former (a non-trivial $T_{55}$) can arise from the physics responsible for the stabilization of fifth space
we can write the transform w.r.t. the brane-world coordinates perturbation appropriate to discuss the observation in the bulk. Expressing \( h \), the first equality follows from \( T_{55} \) (as implicitly implied by the energy conservation in the bulk), and the latter is used to completely remove the gauge degrees of freedom in the RS set-up. However, even a non-trivial contribution of the \( T_{55} \) component, which generally need not vanish at the quantum level, does not appear to remedy the ghost (negative norm) state of the above nature. In fact, it can be easily seen that an apparent presence of ghost state in the above observations was simply due to an incorrect boundary condition imposed on the brane at \( y = 0 \), and for properly regularized \( \delta \) function, the theory is completely free from ghost (negative norm) state. This is the case we discuss below.

### 5.2 No ghost-state of GB term

By taking into proper accounts of the non TT components, we find that a net result that growing part of \( h \) can be eliminated by a general slice deformation in \( y \) satisfying (see the Appendix A)

\[
\partial_\mu \partial^\mu \hat{\xi}^5(x) = \frac{(1 + \gamma)^{-1}}{6 \, M_5^4} \left[ \frac{S^\lambda_\mu(x)}{2} + \ell \, T_{55}(0) - \frac{1 + \gamma}{1 - \gamma} \int_0^{y_m} dy \, \partial^\mu T_{\mu 5} \right]. \tag{61}
\]

Here \( y_m \) is the width of matter distribution transverse to the brane, where \( y < y_m \). In terms of the flat-space Green function \( \Box(x, x') \), this determines a brane-shift function of the form, with \( T_{55}(0) = \partial^\mu T_{\mu 5} = 0 \),

\[
\hat{\xi}^5(x) = \frac{(1 + \gamma)^{-1}}{12 \, M_5^2} \int d^4 x' \Box(x, x') S^\lambda_\mu(x'). \tag{62}
\]

We note that in terms of the 5d Neumann Green function \( G_5(X; x', 0) \), the perturbation that follows from (60) is given by

\[
h_{\mu \nu}'(X) = h_{\mu \nu}(X) = -\frac{1}{2 \, M_5^2} \int d^4 x' \sqrt{-g} \, G_5(X; x', 0) \left[ S_{\mu \nu}(x') - \frac{1}{3} \left( \eta_{\mu \nu} - \partial_\mu \partial_\nu \right) S(x') \right]. \tag{63}
\]

The first equality follows from \( h' = 0 \). Since the brane is located at \( y' = 0 \) (\( y = -\xi^5 \)), this is the perturbation appropriate to discuss the observation in the bulk. Expressing \( h_{\mu \nu} \) in terms of Fourier transform w.r.t. the brane-world coordinates

\[
h_{\mu \nu}(q, y) = \int d^4 x \, e^{-i q \cdot x} \, h_{\mu \nu}(x, y), \tag{64}
\]

we can write \( h_{\mu \nu}' \) (i.e., using Eq. 14) in the form, replacing \( M_5^{-3} \) by \( 16 \pi G_{4+1} \),

\[
h_{\mu \nu}'(q, y) = -8 \pi G_{4+1} \left( 1 - \gamma \right)^{-1} e^{2 \eta_{\mu \nu} / \ell} \left[ S_{\mu \nu}(q) - \frac{1}{3} \left( \eta_{\mu \nu} - \frac{q_{\mu} q_{\nu}}{q^2} \right) S(q) \right] \times \frac{1}{q} \left[ \frac{H_2^{(1)}(q \ell e^{2\eta_{\mu \nu} / \ell})}{H_1^{(1)}(q \ell) + \chi q \ell H_2^{(1)}(q \ell)} \right]. \tag{65}
\]

In fact, the last term in the first parenthesis can be eliminated using the gauge transformation induced by \( \beta_\mu(x) \) (Eq. (27)). For both arguments on the brane, the result is

\[
h_{\mu \nu}(x) = -16 \pi G_4 \int d^4 x' \Box(x, x') S_{\mu \nu}(x') - \frac{1}{2} \eta_{\mu \nu} S^\lambda_{\mu \nu}(x')
\]

\[
- 8 \pi G_{4+1} \left( 1 + \gamma \right)^{-1} \int d^4 x' G_{KK}(x, x') \left[ S_{\mu \nu}(x') - \frac{1}{3} \eta_{\mu \nu} S^\lambda_{\mu \nu}(x') \right]. \tag{66}
\]
where $G_4 = G_{4+1} \ell^{-1}(1 + \gamma)^{-1}$. This can also be decomposed into the part corresponding to the matter field and the part corresponding to the wall displacement [14]

$$h_{\mu\nu}(x) = h^{(m)}_{\mu\nu}(x) + \frac{2}{\ell} \eta_{\mu\nu} \xi^5(x),$$

(67)

where the matter contribution $h^{(m)}_{\mu\nu}$ is given by

$$h^{(m)}_{\mu\nu}(x) = -8\pi G_{4+1} \int d^4x' G_5(x, \ell; x', \ell) \left[ S_{\mu\nu}(x') - \frac{1}{3} \eta_{\mu\nu} S_{\lambda\lambda}(x') \right],$$

(68)

and the brane-shift function $\xi^5(x)$ is given by Eq. (62). With the result (66), the Eq. (59) would rise to give

$$A \equiv \frac{8\pi G_4}{p^2} \left[ S_{\mu\nu} S^{\mu\nu} - \frac{1}{2} S^2 \right].$$

(69)

This is obviously positive definite and ghost (negative norm state) free, and represents only the massless spin-2 graviton amplitude observed on the brane.

Now, to discuss observation on the brane, we can also define the metric deformation in trace-reversing form: $\tilde{h}_{\mu\nu} = h_{\mu\nu} - (1/2) \eta_{\mu\nu} h$, together with the gauge transformation induced by $\xi^5$. In this gauge, the modulo 4-dimensional gauge transformation, one finds

$$\tilde{h}_{\mu\nu}(x) = -8\pi G_{4+1} \int d^4x' \left\{ G_5(x, \ell; x', \ell) S_{\mu\nu}(x') - \eta_{\mu\nu} \left[ G_5(x, \ell; x', \ell) - (1 + \gamma)^{-1} \frac{2}{\ell} G_4(x, x') \right] \frac{S_{\lambda\lambda}(x)}{6} \right\}. $$

(70)

On using the expression [19], the zero-mode piece cancels in the term multiplying $S_{\lambda\lambda}$. Writing the results in terms of the 4-dimensional graviton and KK mode propagators, we find

$$\tilde{h}_{\mu\nu}(x) = -16\pi G_4 \int d^4x' \square_4(x, x') S_{\mu\nu}(x')$$

$$- 8\pi G_{4+1} (1 + \gamma)^{-1} \int d^4x' G_{KK}(x, x') \left[ S_{\mu\nu}(x') - \frac{1}{6} \eta_{\mu\nu} S_{\lambda\lambda}(x') \right].$$

(71)

The first term is the standard result of the four-dimensional gravity, with the Planck mass given by $M^2 = (1 + \gamma) M^2_s \ell = (16\pi G_4)^{-1}$, and the second term involves correction from the KK kernel. It appears that there is no real ghost (negative norm state) even with the GB term. Notice that, in the presence of the GB term, the true 4d graviton propagator on the brane is $\Delta_4(x, x') = (1 + 2\chi)^{-1} \square_4(x, x')$. However, the correction term $(1 + 2\chi)^{-1}$ to the flat-space 4d Green function on the brane, arisen from the gravitational interactions between the matter fields living on the brane and the higher curvature terms, when multiplied with the term $(1 - \gamma)^{-1}$ coming from matching equation would rise to give the factor $(1 + \gamma)^{-1}$. This is precisely the constant factor by which the 4d Newton’s constant on the brane is renormalized in the presence of Gauss-Bonnet term.
5.3 Static potential: On and off-brane profile

To understand the limiting behavior of the graviton propagator at long distances along the brane, one can consider a static point source at \( x' = 0 \). In the limit \( q\ell \ll 1 \) (i.e., \( |x - x'|/x >> \ell \)), the static potential due to a point source on the brane is given by

\[
U(r) = \int dt \mathcal{G}_5(r, \ell; 0, \ell) \approx (1 - \gamma)^{-1}(1 + 2\chi)^{-1} \int \frac{d^3p}{(2\pi)^3} e^{ipr} \left[ -\frac{2}{\ell} \frac{1}{p^2} + \frac{1}{1 + 2\chi} \ell \ln(ip\ell/2) \right]
\]

\[
\approx -(1 + \gamma)^{-1} \frac{1}{2\pi r\ell} \left[ 1 + \frac{1 - \gamma}{1 + \gamma} \frac{\ell^2}{2r^2} + \cdots \right]. \tag{72}
\]

One could obtain the similar result from the mode sums, see, for example, Ref. [23].

Now consider the source of gravitational field on the brane, \( z' = \ell \) (and \( x' = 0 \)). Then for the off-brane graviton propagator, \( z >> \ell \), with the assumption that \( r = |x| >> \ell \) and \( z > 1/q \gg r \), one can still make the small argument expansion of \( q\ell \) in Eq. (44) and obtain

\[
\mathcal{G}_5(x, z; 0, \ell) \approx (1 + \gamma)^{-1} \frac{z^2}{\ell^2} \int \frac{d^4p}{(2\pi)^4} e^{ipx} \frac{i\pi}{2} \ell H_2^{(1)}(qz) \times \left[ 1 + \frac{1 - \gamma}{1 + \gamma} \left( \ln(q\ell/2 + (\Gamma - 1/2) - \frac{\gamma}{1 - \gamma}) (q\ell)^2 \right) \right]. \tag{73}
\]

In the limit \( q\ell << 1 \), the static off-brane potential following from (73), by integrating over time, is

\[
U(r, z) = -(1 + \gamma)^{-1} \frac{3}{4\pi} \frac{1}{\ell z} \left( 1 + \frac{2r^2}{3z^2} \right) \left( 1 + \frac{r^2}{z^2} \right)^{-3/2} \left[ 1 + \mathcal{O}(\ell^2/z^2, \ell^2/r^2) \right]. \tag{74}
\]

Since \( U(r) \) is just proportional to the Green function, this explains the large \( r \) and large \( z \) dependence of the propagator off the brane. This implies that, for a static source and far from the brane, the perturbation still falls as \( h \sim 1/z \).

5.4 The Newtonian limit

Consider the case of a static and spherically symmetric point source of mass \( m_* \) localized on the brane, at \( \vec{x} = 0 \). The stress tensor can be defined by

\[
T_{\mu\nu} = m_* \delta_{\mu0} \delta_{\nu0} \delta^{(3)}(x) \delta(z). \tag{75}
\]

With this, Eq. (66) would rise to give

\[
h_{00}(x) = \frac{2G_4m_*}{r} \left[ 1 + \frac{2(1 - \gamma)}{3(1 + \gamma)} \frac{\ell^2}{r^2} \right], \quad h_{ij}(x) = \frac{2G_4m_*}{r} \left[ 1 + \frac{(1 - \gamma)}{3(1 + \gamma)} \frac{\ell^2}{r^2} \right] \delta_{ij}. \tag{76}
\]

The result is obvious, for the GB term we introduced as higher-curvature correction still behaves as a ghost-free combination and hence just renormalizes the Newton constant on the brane, though
the correction terms from the KK mode are different. The latter is generic to the RS type higher-dimensional brane models [14], independently whether one has introduced GB interaction term or not.

Finally, in the trace-reversed form the metric deformation (71) gives the expressions

\[ \bar{h}_{00} = \frac{4G_4m_s}{r} \left[ 1 + \frac{5(1 - \gamma)}{12(1 + \gamma)} \frac{\ell^2}{r^2} \right], \quad \bar{h}_{ij} = \frac{4G_4m_s}{r} \left[ \mathcal{O}(0) + \frac{1 - \gamma}{12(1 + \gamma)} \frac{\ell^2}{r^2} \right] \delta_{ij}. \] (77)

Thus, only a detailed investigation of the astrophysical implications of these results could tell whether or not the RS-type brane model is compatible with the cosmological observations.

6 Geodesics in the brane background

Naively, there could be two possibilities for behavior of a test particle on the brane, i.e., the test particle is (i) free to move in the fifth dimension (a geodesic in \( AdS_5 \) space) or (ii) constrained to move along the brane by some non-gravitational mechanism (a geodesic on the brane). Some aspects of geodesics in the RS (and alternative) brane backgrounds, and in a non-compact 5d vacuum manifold have been studied in [38] and [39]. Here we shall discuss more on it in the framework of the RS 3-brane with an extra infinite dimension.

For the 5d line element of the form (10), one can write the Lagrangian, with \( \rho(y) \equiv e^{2(c-|y|)/\ell} \), as

\[ \mathcal{L} = \frac{1}{2} \left( ds/d\tau \right)^2 = \frac{1}{2} \left[ e^{2(c-|y|)/\ell} g_{\mu\nu} v^\mu v^\nu - \dot{y}^2 \right]. \] (78)

Here \( v^\mu \equiv dx^\mu/d\tau \) is a constant four-vector, \( \tau \) is an affine parameter and the dot represents differentiation w.r. to \( \tau \). For a flat Minkowski 3-brane, one can set \( c = 0 \). The solutions to Euler-Lagrange equations for the Lagrangian (78) give the following geodesic equations

\[ \ddot{x}^\mu = (\tau) a^\mu = -2 \ell^{-1} \partial_y |y| \dot{y} v^\mu, \quad \ddot{y} = -\ell^{-1} \partial_y |y| e^{2(c-|y|)/\ell} v^2, \] (79)

where \( (\tau) a^\mu \) is the fourth component of the 5-acceleration and \( v^2 = \eta_{\mu\nu} v^\mu v^\nu \). For an affine parameter \( \tau \) along the path, the first equation implies a velocity dependent force, while the second equation shows that a test particle, in general, accelerates in the space transverse to the brane. These aspects of the fifth force have been discussed in the literature [40] but in different contexts.

For a non-compact fifth space, it is conceivable that free massive particles may not move along the brane only, but in general can accelerate in the fifth dimension, which could be generic in an \( AdS \) bulk background, and also matter energy may leak from the 3-brane into the bulk. In particular, a highly energetic ordinary matter can leave the brane and propagate in the \( AdS \) bulk. A specific example of this behavior was noted in [41] by considering a decay of a particle of mass \( 2m \) residing on the brane into two particles of mass \( m \). In the RS scenario, since the massive KK gravitons weakly interact with the brane matter, a pair creation of such particles could lead to the transfer of energy from brane to the bulk. It has been argued in [41] that for a meaningful physics these particles should behave
as dark matter particles of fixed mass and exhibit the usual behaviors in gravitational interactions without violating locality and the 4d Newton’s law on the brane-world volume. The ideas were further extended in [12] by considering bulk fermions and scalars.

In fact, the RS brane is a gravitating 3d submanifold moving in some higher dimensional space-time to which ordinary matter is trapped, so that the geodesic for any massive matter particle $m \gg 1/\ell$, e.g., black hole, on the brane can co-accelerate with the brane [43]. Further, a 3-brane metric solution represents only the core region of a smooth domain wall, so that an observer sitting on the wall experiences no force. But, moving a certain distance from the wall, the observer begins to feel acceleration towards the AdS bulk. How can one reconcile these ideas with the above observation? It appears that the behavior of extra (fifth) force does not survive in a different parameterization, but rather brings the correct definition of the proper time into question in conventional 3 + 1 gravity.

In the brane-world scenario, this can be explained. The affine parameters we define in 4d and 5d are not the same, but could be related by $dt \sim e^{-(c-|y|)/\ell} d\tau$, where the brane is located at $y = 0$. Since $z = \ell e^{|y|/\ell}$, we find $t = (z/\ell) \tau$, this actually tells that the conformal time $\tau$ times the factor $z/\ell$. Furthermore, in the $t$- parameterization, one can actually show that $(t)_{a\mu} = 0$. To explain this behavior, we consider below the general solutions to the geodesic equations.

Suppose that the 5d trajectories are null, $v_a v^a = 0$ (more precisely, $v^a \nabla_a v^b = 0$ and $\lambda$ is a 5d affine parameter). With this hypothesis, the null paths imply that

$$y^2 = e^{2(c-|y|)/\ell} v^2, \quad \dot{y} = -e^{-1} \partial_y |y| \dot{y}^2,$$

whose general solution is given by

$$\dot{y} = a_0 e^{(c-|y|)/\ell},$$

where $a_0$ is an integration constant and its value can be fixed from the initial data on the brane, via

$$(\tau - \tau_0) = a_0^{-1} \int_{y_0}^y e^{[|y|/\ell} d\tilde{y},$$

where $y_0 = y(\tau_0)$. The time-translation symmetry implies that one can reparameterize using $\tau \rightarrow \tilde{\tau} = \tau_0 + \tau |a_0|^{-1}$. Since $d\tilde{\tau}/d\tau > 1$, this parameterization preserves the orientation of 5d light cones. Defining $a_0 |a_0|^{-1} = \epsilon$, the above set of equations give

$$(\tau)_{a\mu} = -2 \epsilon e^{-1} \partial_y |y| e^{-(c-|y|)/\ell} v^\mu, \quad e^{4(c-|y|)/\ell} v^2 = \epsilon^2.$$

One can normalize using $\epsilon = 0$ for $a_0 = 0$, and $\epsilon = \pm 1$ for $a_0 \neq 0$. Then from Eq. (82), $\epsilon = 0$ implies that $y = y_0$ for all $\tau$, which may suggest that for massless particles localized on the brane, e.g., photons, there is no motion in the fifth dimension.

Now we perform a parameter transformation $t \rightarrow t(\tau)$. From the first equation of (79), with $x^\mu = x^\mu(t)$, we get

$$(t)_{a\mu} = -\left(\frac{d\tau}{dt}\right)^2 \left(\frac{d^2t}{d\tau^2} + 2\epsilon^{-1} \partial_y |y| \frac{dy}{d\tau} \frac{dt}{d\tau}\right) v^\mu,$$

18
where \( u^\mu \equiv dx^\mu/dt \) and \( v^\mu = u^\mu dt/d\tau \). Eq. (84) gives \((t) a^\mu = 0\), the standard geodesic equation for the metric \( g_{\mu\nu}(x^\lambda) \), provided that \( dt/d\tau = a_1 e^{-2(\epsilon - |y|)/\ell} \). We set \( a_1 = 1 \). Then from the second equation of (83), \( u_\lambda u^\lambda = \epsilon^2 \). This implies that the 4d geodesic \( u^\mu(t) \) can be time-like \((\epsilon = \pm 1)\) or null \((\epsilon = 0)\), and the massive matter particles that escape to the bulk could follow the 5d null geodesic. It might be interesting to know whether these particles can follow the 5d null geodesic and can behave as dark matter particles of fixed mass in the bulk.

In the RS background (a flat Minkowski 3-brane), the solution to the geodesic equation reads

\[
x(t) = 0, \quad y(t) \sim \ell \log (\epsilon t/\ell) \equiv \frac{\ell}{2} \log \left(1 + t^2/\ell^2\right) ,
\]

where \( \epsilon = \pm 1 \). One then has to consider time \( t > 0 \) once the matter source is in the bulk. One can call \( t \) the proper time on the brane and \( y \) the proper distance off the brane. Eq. (83) implies that a large change in the 4d proper time \( t \) is accompanied by small changes in the bulk space \((y)\). Working on the \( y > 0 \) side of the brane, we set \( \epsilon = 1 \). A remarkable result from Eq. (83) is that a test particle can reach \( y = \infty \) at infinite time \( t \), but finite proper time \( \eta = \pi \ell/2 \) [12].

Finally, we note that transverse to the brane the metric deformation behaves as \( h_{00} \sim m/M_5^2 \times 1/z \) and the horizon size for a static black-hole grows like \( r_h \sim z_h \). Therefore, for a black hole of mass \( m \) on the brane, if the horizon size along the brane grows like \( \sim m \), the thickness transverse to the brane grows only like \( \sim \log m \), i.e.,

\[
y_h \sim \ell \log \left(\frac{m}{M_5^2 \ell}\right).
\]

This entails a pancake shape of the black hole and has been adequately discussed in the literature [9].

## 7 Conclusion

A linearized treatment of Einstein-Gauss-Bonnet (EGB) gravity in the presence of a singular positive tension 3-brane and localized matter distribution is presented. The full graviton propagator is shown to be well behaved in all distance scales and hence on the brane one can still reproduce the zero-mode solution as a localized gravity with correct momentum and tensor structures even in the presence of GB interaction term. In a linearized analysis, this paper has outlined the ghost problem (negative norm state) of the GB term in the brane background and ways to resolve it and many other interesting features of EGB gravity and gravitational potential corrections with GB interaction. It is shown that for the matter localized 3-brane in an \( AdS_5 \), a test particle on the brane can still follow the time like geodesic if it is null in five space.

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A Appendix A. Linearized equations and boundary solutions

Consider the metric fluctuations in the form $g_{ab} = e^{-2A(z)}(\eta_{ab} + h_{ab})$. Then the linearized equations for the action \cite{1} with $d + 1 = 5$, in axial gauge $h_{55} = 0$, take the form

$$\begin{align*}
-\left[(1 + 4\alpha\kappa e^{2A}A')\partial_{\lambda}^2 - (1 - 4\alpha\kappa e^{2A}A')\partial_{\nu}^2 + 3A'\partial_{z}\right] (h_{\mu\nu} - \eta_{\mu\nu}h)
+ (1 - 4\alpha\kappa e^{2A}(2A' - A'')) \left(2\partial_{(\mu}\partial^{\mu}\eta_{\nu)} - \partial_{\mu}\partial_{\nu}h\right)
- (1 + 4\alpha\kappa e^{2A}A') \times
\eta_{\mu\nu}\partial_{\lambda}\partial_{\rho}h^{\lambda\rho} + 4\alpha\kappa e^{2A}A' \left[(2A'' - A'')\partial_{z}h_{\mu\nu} + (A'' - 2A')\eta_{\mu\nu}\partial_{z}h\right]
= \kappa T_{\mu\nu}^{(m)},
\end{align*}$$

(A.1)

$$\begin{align*}
(1 - 4\alpha\kappa e^{2A}A') \partial_{z}(\partial^{\lambda}h_{\mu\lambda} - \partial_{\mu}h) = \kappa T_{5\mu}^{(m)},
\end{align*}$$

(A.2)

$$\begin{align*}
- (1 - 4\alpha\kappa e^{2A}A') \left(\partial^{\mu}\partial^{\nu}(h_{\mu\nu} - \eta_{\mu\nu}h) + 3A'\partial_{z}h\right) = \kappa T_{55}^{(m)}.
\end{align*}$$

(A.3)

With a coordinate transformation $|y| = \ell \log(|z|/\ell + 1)$, or defining the background metric in $y$-coordinate \cite{24}, the above set of equations take the form

$$\begin{align*}
(1 - 4\alpha\kappa A'' + 4\alpha\kappa A') e^{2|y|/\ell} \left[\partial_{\lambda}^2(h_{\mu\nu} - \eta_{\mu\nu}h) + \partial_{\mu}\partial_{\nu}h - 2\partial_{(\mu}\partial^{\lambda}\eta_{\nu)} + \kappa T_{\mu\nu}^{(m)}
\right]
\end{align*}$$

(A.4)

$$\begin{align*}
(1 - 4\alpha\kappa A'') \partial_{y} \left(e^{2|y|/\ell}(\partial_{\mu}h - \partial^{\lambda}h_{\lambda})\right) = \kappa T_{5\mu}^{(m)},
\end{align*}$$

(A.5)

$$\begin{align*}
(1 - 4\alpha\kappa A') e^{2|y|/\ell} \left(e^{2|y|/\ell}\partial^{\mu}\partial^{\nu}(h_{\mu\nu} - \eta_{\mu\nu}h) + 3A'\partial_{y}h + 2\ell^{-1}h\right)
= \kappa T_{55}^{(m)},
\end{align*}$$

(A.6)

where $A' = \ell^{-1}\partial_{y}|y|$, $A'' = 2\ell^{-1}\delta(y)$ and $A'A'' = (2/3\ell^2)\delta(y)$. In the presence of matter source on the brane, one has to put no restriction to the 4d non-TT components of metric fluctuations. Working on the $y > 0$ ($z > \ell$) side of the brane (i.e. $A'' = 0$), taking the trace of (A.4) and subtracting the Eq. (A.6) from the resulting expression, we get

$$\begin{align*}
\partial_{y}(2\ell^{-1} + \partial_{y}) h = \frac{(1 - \gamma)^{-1}}{3 M_*^2} \left[T_{\mu}^{\mu} - 2e^{-2y/\ell} T_{55}^{\mu}\right].
\end{align*}$$

(A.7)

Conservation of $T$ in the bulk brings this in the form

$$\begin{align*}
\partial_{y} \left[(\partial_{y} + 2\ell^{-1}) h + \frac{(1 - \gamma)^{-1}}{3 M_*^2} e^{-2y/\ell} T_{55}^{\mu}\right]
= \frac{(1 - \gamma)^{-1}}{3 M_*^2} \partial_{y} T^{\mu5}.
\end{align*}$$

(A.8)

This can be solved (integrated) with initial boundary condition of the first bracket term supplied by the trace of (A.4). In particular, with GB term, one first needs to address a subtlety in the boundary condition, because of the terms like $sgn(y)\delta(y) \partial_{y} h$ and $sgn(y)^2 \delta(y)$. However, one can unambiguously fix them by properly regularizing the $\delta$-function as we did in the main text.
Given these considerations, taking the trace of (A.4) and integrating from \( \epsilon_- \) to \( \epsilon_+ \), in the limit \( \epsilon \to 0 \), we arrive at

\[
(1 - \gamma)(\partial_y + 2 \ell^{-1}) h|_{0+} - \frac{2}{3} \gamma \ell (\partial_\lambda \partial_\rho h^{\lambda \rho} - \partial_\lambda^2 h)|_{0+} = \frac{1}{6M_s^2} S^\mu_\mu.
\]  
(A.9)

But from the (55)-component of the fluctuations, Eq. (A.6), we have

\[
(\partial_\lambda \partial_\rho h^{\lambda \rho} - \partial_\lambda^2 h)|_{0+} = -3\ell^{-1}(\partial_y + 2 \ell^{-1}) h|_{0+} - M_s^{-3} (1 - \gamma)^{-1} T_{55}(0).
\]  
(A.10)

Substituting this back into (A.9), we get

\[
(\partial_y + 2 \ell^{-1}) h|_{0+} = \left( \frac{1 + \gamma}{3M_s^2} \right) \left[ \frac{S^\mu_\mu(x)}{2} + \frac{2\gamma \ell}{1 - \gamma} T_{55}(0) \right].
\]  
(A.11)

The trace \( h \) involves a growing component transverse to the brane, and this may lead to failure of the linear approximation when \( S^\mu_\mu \neq 0 \). However, one can eliminate such growth in \( h \) from the initial condition (A.11) by a coordinate transformation of the form (27), and then we may integrate Eq. (A.8) to eliminate resultant growth in \( h \) with a gauge choice

\[
\partial_\mu \partial^\mu \tilde{\xi}^5(x) = \left( \frac{1 + \gamma}{6M_s^2} \right) \left[ \frac{S^\mu_\mu(x)}{2} + \ell T_{55}(0) - \frac{1 + \gamma}{1 - \gamma} \ell \int_0^{y_m} dy \partial^\mu T_{\mu 5} \right].
\]  
(A.12)

The gauge shift induced by (A.12) can be used to determine the boundary conditions for \( h'_{\mu \nu} \) and hence to find the form of the source term \( \Sigma_{\mu \nu}(x') \). The boundary condition on \( h_{\mu \nu} \) at the brane is readily determined by integrating Eq. (A.4) from just below to just above the brane (i.e., Israel junction condition) and enforcing symmetry under \( y \to -y \), which is read as

\[
\gamma \ell e^{2|y|/\ell} (\partial_\lambda^2 h_{\mu \nu} - \eta_{\mu \nu} \partial_\lambda^2 h + \partial_\nu \partial_{\nu} h - 2\partial_{(\mu} \partial^\lambda h_{\nu)} + \eta_{\mu \nu} \partial^\lambda \partial^\rho h^{\rho \lambda})|_{y=0+} + (1 - \gamma) (\partial_y + 2 \ell^{-1}) (h_{\mu \nu} - \eta_{\mu \nu} h)|_{y=0+} = -\frac{\kappa}{2} S_{\mu \nu}.
\]  
(A.13)

In terms of the \( h'_{\mu \nu} \), i.e., under the gauge transformation in the fluctuation \( h_{\mu \nu} \), Eq. (27), this becomes

\[
\gamma \ell e^{2|y|/\ell} \left( \partial_\lambda^2 h'_{\mu \nu} - \eta_{\mu \nu} \partial_\lambda^2 h' + \partial_\nu \partial_{\nu} h' - 2\partial_{(\mu} \partial^\lambda h'_{\nu)} + \eta_{\mu \nu} \partial^\lambda \partial^\rho h'_{\rho \lambda} \right)|_{y=0+} + (1 - \gamma) (\partial_y + 2 \ell^{-1}) (h'_{\mu \nu} - \eta_{\mu \nu} h')|_{y=0+} = -\frac{\kappa}{2} S_{\mu \nu} - 4(\partial_\mu \partial_{\nu} - \eta_{\mu \nu} \partial_\lambda^2) \tilde{\xi}^5 - 2(1 - \gamma) (\partial_\mu \partial_{\nu} - \eta_{\mu \nu} \partial_\lambda^2) \tilde{\xi}^5
\]  
\[
= -\frac{\kappa}{2} S_{\mu \nu} - 2(1 + \gamma) (\partial_\mu \partial_{\nu} - \eta_{\mu \nu} \partial_\lambda^2) \tilde{\xi}^5.
\]  
(A.14)

One obtains exactly the same boundary condition for \( \tilde{h}'_{\mu \nu} \), justifying the gauge choice \( \tilde{h}' = 0 \). The tracefree gauge \( h' = 0 \) generates an extra unphysical scalar degree of freedom which may correspond to the graviscalar mode. This extra polarization state, however, can be compensated by shifting the brane to \( y' = 0 \) (\( y = -\xi^5(x) \)). The cancellation of the unphysical scalar mode is, indeed, needed for the validity of linearized approximation. Further a choice \( \partial^\mu h'_{\mu \nu} = 0 \) ensures the conservation of the energy momentum tensor on the brane and hence the 4d general covariance of the 3-brane
world volume. With the gauge choice \( h' = \partial^\mu h'_{\mu \nu} = 0 \), (A.14) one ends up with the correct boundary condition for \( h'_{\mu \nu} \).

Now, using (A.12) and considering the case \( T_{55}(0) = \partial^\mu T_{\mu 5} = 0 \), the solution for \( h'_{\mu \nu} \), in terms of the 5d Neumann Green function, is given by

\[
h'_{\mu \nu}(X) = \bar{h}'_{\mu \nu}(X) = -\frac{1}{2 M^3} \int d^4x' \sqrt{-g} G_5(X; x', 0) \Sigma_{\mu \nu}(x'),
\]

where the source term is given by

\[
\Sigma_{\mu \nu}(x') = S_{\mu \nu}(x') - \frac{1}{3} (\eta_{\mu \nu} - \partial_\mu \partial_\nu) S(x').
\]

Obviously, \( \Sigma_{\mu \nu}(x') \) is transverse and trace-free, and includes the contribution of the brane-shift function, which does play a role of the source in the RS gauge.

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