FREE LEFT AND RIGHT ADEQUATE SEMIGROUPS

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Abstract. Recent research of the author has given an explicit geometric description of free (two-sided) adequate semigroups and monoids, as sets of labelled directed trees under a natural combinatorial multiplication. In this paper we show that there are natural embeddings of each free right adequate and free left adequate semigroup or monoid into the corresponding free adequate semigroup or monoid. The corresponding classes of trees are easily described and the resulting geometric representation of free left adequate and free right adequate semigroups is even easier to understand than that in the two-sided case. We use it to establish some basic structural properties of free left and right adequate semigroups and monoids.

1. Introduction

Left adequate semigroups are an important class of semigroups in which the right cancellation properties of elements in general are reflected in the right cancellation properties of the idempotent elements. Right adequate semigroups are defined dually, while semigroups which are both left and right adequate are termed adequate. Introduced by Fountain [5], these classes of semigroups form a natural generalisation of inverse semigroups, and their study is a key focus of the York School of semigroup theory. Left [right] adequate semigroups are most naturally viewed as algebras of signature (2, 1), with the usual multiplication augmented with a unary operation taking each element to an idempotent sharing its right [left] cancellation properties. Within the category of (2, 1)-algebras the left [right] adequate semigroups form a quasivariety, from which it follows [4, Proposition VI.4.5] that there exist free left and right adequate semigroups for every cardinality of generating set.

When studying any class of algebras, it is very helpful to have an explicit description of the free objects in the class. Such a description permits one to understand which identities do and do not hold in the given class, and potentially to express any member of the class as a collection of equivalence classes of elements in a free algebra. In the case of inverse semigroups, a description of the free objects first discovered by Scheiblich [12] was developed by Munn [11] into an elegant geometric representation which has been of immense value in the subsequent development of the subject. The same

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approach has subsequently been used to describe the free objects in a number of related classes of semigroups [6, 7, 8] and categories [2]; however, for reasons discussed in [10], Munn’s approach is not applicable to adequate semigroups. In [10], we gave an explicit geometric representation of the free adequate semigroup on a given set, as a collection of (isomorphism types of) edge-labelled directed trees under a natural multiplication operation.

The focus of this paper is upon the free objects in the quasivarieties of left adequate and right adequate semigroups. We show that these embed into the corresponding free adequate semigroups in a natural way, as the (2,1)-algebras generated by the free generators under the appropriate operations; the resulting representation of these semigroups is even easier to understand than that for the free adequate semigroup. These results combine with [10] to yield a number of results concerning the structure of free left and right adequate semigroups.

An alternative approach to free left and right adequate semigroups is given by recent work of Branco, Gomes and Gould [1]. Their construction arose from the fact that free left and right adequate semigroups are proper in the sense introduced in [1].

In addition to this introduction, this article comprises three sections. In Section 2 we briefly recall the definitions and elementary properties of adequate semigroups, and the results of [10] concerning free adequate semigroups. Section 3 is devoted to the proof that certain subalgebras of the free adequate semigroup are in fact the free left adequate and free right adequate semigroups on the given generating set. Finally, in Section 4 we collect together some remarks on and corollaries of our main results.

2. Preliminaries

In this section we briefly recall the definitions and basic properties of left, right and two-sided adequate semigroups (more details of which can be found in [5]), and also some of the main definitions and results from [10] characterising the free adequate semigroup.

Recall that on any semigroup $S$, an equivalence relation $\mathcal{L}^*$ is defined by $a \mathcal{L}^* b$ if and only if we have $ax = ay \iff bx = by$ for every $x, y \in S^1$. Dually, an equivalence relation $\mathcal{R}^*$ is defined by $a \mathcal{R}^* b$ if and only if we have $xa = ya \iff xb = yb$ for every $x, y \in S^1$. A semigroup is called left adequate [right adequate] if every $\mathcal{R}^*$-class [respectively, every $\mathcal{L}^*$-class] contains an idempotent, and the idempotents commute. A semigroup is adequate if it is both left adequate and right adequate. It is easily seen that in a left [right] adequate semigroup, each $\mathcal{R}^*$-class [$\mathcal{L}^*$-class] must contain a unique idempotent. We denote by $x^+$ [respectively, $x^*$] the unique idempotent in the $\mathcal{R}^*$-class [respectively, $\mathcal{L}^*$-class] of an element $x$; this idempotent acts as a left [right] identity for $x$. The unary operations $x \mapsto x^+$ and $x \mapsto x^*$ are of such critical importance in the theory of adequate [left adequate, right adequate] semigroups that it is usual to consider these semigroups as algebras of signature $(2,1,1)$ [or $(2,1)$ for left adequate and right adequate semigroups] with these operations. In particular, one restricts attention to morphisms which preserve the $+$ and/or $*$ operations (and hence coarsen the $\mathcal{R}^*$ and $\mathcal{L}^*$ relations) as well as the multiplication. Similarly, adequate [left
or right adequate] monoids may be viewed as algebras of signature \((2, 1, 1, 0)\) \([(2, 1, 0)] with the identity a distinguished constant symbol.

The following proposition recalls some basic properties of left and right adequate semigroups; these are well-known and full proofs can be found in [10].

**Proposition 1.** Let \(S\) be a left adequate [respectively, right adequate] semigroup and let \(a, b, e, f \in S\) with \(e\) and \(f\) idempotent. Then

1. \(e^+ = e\) [\(e = e^*\)];
2. \((ab)^+ = (ab^+)^*\) \([(ab)^* = (a^*b)^*]\);
3. \(a^+a = a\) [\(aa^* = a\)];
4. \(ea^+ = (ea)^+\) \([a^*e = (ae)^*]\);
5. \(a^+(ab)^+ = (ab)^+\) \([(ab)^*a^* = (ab^*)]\);
6. If \(ef = f\) then \((ae)^+(af)^+ = (af)^+\) \([(ea)^*(fa)^* = (fa)^*]\).

Recall that an object \(F\) in a concrete category \(C\) is called free on a subset \(\Sigma \subseteq F\) if every function from \(\Sigma\) to an object \(N\) in \(C\) extends uniquely to a morphism from \(F\) to \(N\). The subset \(\Sigma\) is called a free generating set for \(F\), and its cardinality is the rank of \(F\).

It is easily seen that classes of left and right adequate semigroups form a quasivariety, and it follows from general results (see, for example, [4, Proposition VI.4.5]) that free left and right adequate semigroups and monoids exist. Branco, Gomes and Gould [1] have recently made the first significant progress in the study of these semigroups. The main aim of the present paper is to give an explicit geometric representation of them. We begin with a proposition, the essence of which is that the distinction between semigroups and monoids is unimportant. The proof is essentially the same as for the corresponding result in the (two-sided) adequate case, which can be found in [10].

**Proposition 2.** Let \(\Sigma\) be an alphabet. The free left adequate [free right adequate] monoid on \(\Sigma\) is isomorphic to the free left adequate [free right adequate] semigroup on \(\Sigma\) with a single adjoined element which is an identity for multiplication and a fixed point for + [respectively, *].

We now recall some definitions and key results from [10]: a more detailed exposition may be found in that paper. We are concerned with labelled directed trees, by which we mean edge-labelled directed graphs whose underlying undirected graphs are trees. If \(e\) is an edge in such a tree, we denote by \(\alpha(e), \omega(e)\) and \(\lambda(e)\) the vertex at which \(e\) starts, the vertex at which \(e\) ends and the label of \(e\) respectively.

Let \(\Sigma\) be an alphabet. A \(\Sigma\)-tree (or just a tree if the alphabet \(\Sigma\) is clear) is a directed tree with edges labelled by elements of \(\Sigma\), and with two distinguished vertices (the start vertex and the end vertex) such that there is a (possibly empty) directed path from the start vertex to the end vertex. Figure 1 shows some examples of \(\Sigma\)-trees where \(\Sigma = \{a, b\}\); in each tree, the start and end vertices are marked by an arrow-head and a cross respectively.

A tree with only one vertex is called trivial, while a tree with start vertex equal to its end vertex is called idempotent. A tree with a single edge and distinct start and end vertices is called a base tree; we identify each base tree with the label of its edge. In any tree, the (necessarily unique) directed
path from the start vertex to the end vertex is called the \textit{trunk} of the tree; the vertices of the graph which lie on the trunk (including the start and end vertices) are called \textit{trunk vertices} and the edges which lie on the trunk are called \textit{trunk edges}. If \( X \) is a tree we write \( \theta(X) \) for the set of trunk edges of \( X \).

A \textit{subtree} of a tree \( X \) is a subgraph of \( X \) containing the start and end vertices, the underlying undirected graph of which is connected. A \textit{morphism} \( \rho : X \rightarrow Y \) of \( \Sigma \)-trees \( X \) and \( Y \) is a map taking edges to edges and vertices to vertices, such that \( \rho(\alpha(e)) = \alpha(\rho(e)) \), \( \rho(\omega(e)) = \omega(\rho(e)) \) and \( \lambda(e) = \lambda(\rho(e)) \) for all edges \( e \) in \( X \), and which maps the start and end vertex of \( X \) to the start and end vertex of \( Y \) respectively. Morphisms have the expected properties that the composition of two morphisms (where defined) is again a morphism, while the restriction of a morphism to a subtree is also a morphism. A morphism maps the trunk edges of its domain bijectively onto the trunk edges of its image.

An \textit{isomorphism} is a morphism which is bijective on both edges and vertices. The set of all isomorphism types of \( \Sigma \)-trees is denoted \( UT^1(\Sigma) \) while the set of isomorphism types of non-trivial \( \Sigma \)-trees is denoted \( UT(\Sigma) \). The set of isomorphism types of idempotent trees is denoted \( UE^1(\Sigma) \), while the set of isomorphism types of non-trivial idempotent trees is denoted \( UE(\Sigma) \).

Much of the time we shall be formally concerned not with trees themselves but rather with isomorphism types. However, where no confusion is likely, we shall for the sake of conciseness ignore the distinction and implicitly identify trees with their respective isomorphism types.

A \textit{retraction} of a tree \( X \) is an idempotent morphism from \( X \) to \( X \); its image is a \textit{retract} of \( X \). A tree \( X \) is called \textit{pruned} if it does not admit a non-identity retraction. The set of all isomorphism types of pruned trees [respectively, non-trivial pruned trees] is denoted \( T^1(\Sigma) \) [respectively, \( T(\Sigma) \)].

Just as with morphisms, it is readily verified that a composition of retractions (where defined) is a retraction, and the restriction of a retraction to a subtree is again a retraction. A key foundational result from [10] is the following.

\textbf{Proposition 3}. [Confluence of retracts] For each tree \( X \) there is a unique (up to isomorphism) pruned tree which is a retract of \( X \).
The unique pruned retract of $X$ is called the *pruning* of $X$ and denoted $\overline{X}$.

We now define some *unpruned operations* on (isomorphism types of) trees. If $X, Y \in UT^1(\Sigma)$ then $X \times Y$ is (the isomorphism type of) the tree obtained by gluing together $X$ and $Y$, identifying the end vertex of $X$ with the start vertex of $Y$ and keeping all other vertices and all edges distinct. If $X \in UT^1(\Sigma)$ then $X^+$ is (the isomorphism type of) the tree with the same labelled graph and start vertex of $X$, but with end vertex the start vertex of $X$. Dually, $X^*$ is the isomorphism type of the idempotent tree with the same underlying graph and end vertex as $X$, but with start vertex the end vertex of $X$. It was shown in [10] that the unpruned multiplication operation $\times$ is a well-defined associative binary operation on $UT^1(\Sigma)$; the (isomorphism type of the) trivial tree is an identity element for this operation, and $UT^1(\Sigma)$ forms a subsemigroup. The maps $X \mapsto X^+$ and $X \mapsto X^*$ are well-defined idempotent unary operations on $UT^1(\Sigma)$, and the subsemigroup generated by their images is idempotent and commutative.

We define corresponding *pruned operations* on $T^1(\Sigma)$ by $XY = \overline{X \times Y}$, $X^* = \overline{X^*}$ and $X^+ = \overline{X^+}$. These inherit the properties noted above for unpruned operations, and have the additional property that the images of the $\ast$ and $+$ operations are composed entirely of idempotent elements. We recall some more key results from [10]

**Theorem 1.** The pruning map $UT^1(\Sigma) \to T^1(\Sigma)$, $X \mapsto \overline{X}$ is a surjective $(2,1,1,0)$-morphism from the set of isomorphism types of $\Sigma$-trees under unpruned multiplication, unpruned ($\ast$) and unpruned ($+$) with distinguished identity element to the set of isomorphism types of pruned trees under pruned multiplication, $\ast$ and $+$ with distinguished identity element.

**Theorem 2.** $T^1(\Sigma)$ is a free adequate monoid, freely generated by the set $\Sigma$ of base trees.

**Corollary 1.** Any subset of $T^1(\Sigma)$ closed under the operations of pruned multiplication and $+$ [respectively, $\ast$] forms a left adequate [respectively, right adequate] semigroup under these operations.

If $X$ is a tree and $S$ is a set of non-trunk edges and vertices of $X$ then $X \setminus S$ denotes the largest subtree of $X$ (recalling that a subtree must be connected and contain the start and end vertices, and hence the trunk) which does not contain any vertices or edges from $S$. If $s$ is a single edge or vertex we write $X \setminus s$ for $X \setminus \{s\}$. If $u$ and $v$ are vertices of $X$ such that there is a directed path from $u$ to $v$ then we shall denote by $X|_u^v$ the tree which has the same underlying labelled directed graph as $X$ but start vertex $u$ and end vertex $v$. If $X$ has start vertex $a$ and end vertex $b$ then we define $X|_a^u = X|_b^v$ and $X|_v^b = X|_a^v$ where applicable.

3. **Free Left Adequate Monoids and Semigroups**

In [10] we saw that the monoids $T^1(\Sigma)$ and semigroups $T(\Sigma)$ are precisely the free objects in the quasivarieties of adequate monoids and semigroups.
respectively. In this section, we prove the main results of the present paper by establishing a corresponding result for left adequate and right adequate monoids and semigroups. The spirit and outline of the proof are similar to that of [10], but the technical details are in places rather different.

**Definition 1** (Left and right adequate trees). A $\Sigma$-tree $X$ is called left adequate if for each vertex $v$ of $X$ there is a directed path from the start vertex to $v$, or equivalently, if every non-trunk edge in $X$ is orientated away from the trunk. The sets of isomorphism types of left adequate $\Sigma$-trees and left adequate pruned $\Sigma$-trees are denoted $\text{LUT}^1(\Sigma)$ and $\text{LT}^1(\Sigma)$ respectively.

Dually, a $\Sigma$-tree $X$ is called right adequate if for each vertex $v$ of $X$ there is a directed path from $v$ to the end vertex, or equivalently, if every non-trunk edge in $X$ is orientated towards the trunk. The sets of isomorphism types of right adequate $\Sigma$-trees and right adequate pruned $\Sigma$-trees are denoted $\text{RUT}^1(\Sigma)$ and $\text{RT}^1(\Sigma)$ respectively.

Returning to our examples in Figure 1, the left-hand and middle tree are left adequate, while the right-hand tree is not, because of the presence of the rightmost edge which is orientated towards the start vertex. None of the trees shown are right adequate.

From now on we shall work with left adequate trees and left adequate monoids, but of course duals for all of our results apply to right adequate trees and right adequate monoids.

**Proposition 4.** The set $\text{LUT}^1(\Sigma)$ of left adequate $\Sigma$-trees contains the trivial tree and the base trees, and is closed under unpruned multiplication, unpruned $+$, and taking retracts.

**Proof.** It follows immediately from the definitions that the trivial tree and base trees are left adequate.

Let $X$ and $Y$ be left adequate trees with start vertices $u$ and $v$ respectively. Then $u$ is the start vertex of $X \times Y$, and $X \times Y$ has a directed path from $u$ to $v$. Now for any vertex $w \in X \times Y$, either $w$ is a vertex of $X$ or $w$ is a vertex of $Y$. In the former case, there is a directed path from $u$ to $w$ in $X$, and hence in $X \times Y$. In the latter case, there is a directed path from $v$ to $w$ in $Y$, and hence in $X \times Y$, which composed with the path from $u$ to $v$ yields a directed path from $u$ to $w$. Thus, $X \times Y$ is left adequate.

Consider next the tree $X^{(+)}$. This has the same underlying directed graph as $X$ and the same start vertex, so it is immediate that it is left adequate.

Finally, let $\pi: X \to Y$ be a retraction with image $Y$ a subtree of $X$. Now for any vertex $w$ in $Y$ there is a directed path from the start vertex of $X$ to $w$ in $X$; since $Y$ is a subtree it is connected and has the same start vertex as $X$, so this must also be a path in $Y$. Thus, $Y$ is left adequate.

**Proposition 5.** The set $\text{LT}^1(\Sigma)$ of pruned left adequate trees is generated as a $(2,1,0)$-algebra (with operations pruned multiplication and pruned $+$ and a distinguished identity element) by the set $\Sigma$ of base trees.

**Proof.** The proof is similar to the corresponding one in [10], so we describe it only in outline. Let $(\Sigma)$ denote the $(2,1,0)$-subalgebra of $\text{LT}^1(\Sigma)$ generated by $\Sigma$. We show that every left adequate $\Sigma$-tree is contained in $(\Sigma)$ by induction on number of edges. The tree with no edges is the identity
element of $LT^1(\Sigma)$ and so by definition is contained in $\langle \Sigma \rangle$. Now suppose for induction that $X \in LT^1(\Sigma)$ has at least one edge, and that every tree in $LT^1(\Sigma)$ with strictly fewer edges lies in $\langle \Sigma \rangle$.

If $X$ has a trunk edge then let $v_0$ be the start vertex of $X$, $e$ be the trunk edge incident with $v_0$, $a = \lambda(e)$ and $v_1 = \omega(e)$. Let $Y = X|_{v_0} \setminus e$ and $Z = X|_{v_1} \setminus e$. Then $Y$ and $Z$ are pruned trees with strictly fewer edges than $X$, and so by induction lie in $\langle \Sigma \rangle$. Now clearly from the definitions we have $Y \times a \times Z = X$, and since $X$ is pruned using Theorem 1 we have

$$Y a Z = Y \times a \times Z = X = X$$

so that $X \in \langle \Sigma \rangle$ as required.

If, on the other hand, $X$ has no trunk edges then let $e$ be any edge incident with the start vertex $v_0$, and suppose $e$ has label $a$. Since the tree is left adequate, $e$ must be orientated away from $v_0$; let $v_1 = \omega(e)$. We define $Y = X|_{v_0} \setminus e$ and $Z = X|_{v_1} \setminus e$, and a similar argument to that above shows that $X = Y(aZ)^+$ where $Y, Z \in \langle \Sigma \rangle$, so that again $X \in \langle \Sigma \rangle$. \qed

Now suppose $M$ is a left adequate monoid and $\chi : \Sigma \to M$ is a function. Our objective is to show that there is a unique $(2,1,0)$-morphism from $LT^1(\Sigma)$ to $M$ which extends $\chi$. Following the strategy of [10], we begin by defining a map $\tau$ from the set of idempotent left adequate $\Sigma$-trees to the set $E(M)$ of idempotents in the monoid $M$. Let $X$ be an idempotent left adequate $\Sigma$-tree with start vertex $v$. If $X$ has no edges then we define $\tau(X) = 1$. Otherwise, we define $\tau(X)$ recursively, in terms of the value of $\tau$ on left adequate trees with strictly fewer edges than $X$, as follows. Let $E^+(X)$ be the set of edges in $X$ which start at the start vertex $v$ and define

$$\tau(X) = \prod_{e \in E^+(X)} [\chi(\lambda(e))\tau(X_{|\omega(e)} \setminus e)]^+.$$

It is easily seen that each $X_{|\omega(e)} \setminus e$ is a left adequate tree with strictly fewer edges than $X$, so this gives a valid recursive definition of $\tau$. Moreover, the product is non-empty and because idempotents commute in the left adequate monoid $M$, its value is idempotent and independent of the order in which the factors are multiplied. Note that if the left adequate monoid $M$ is in fact adequate then the function $\tau$ defined here takes the same values on left adequate trees as the corresponding map defined in [10].

**Proposition 6.** Let $X$ be an idempotent left adequate tree with start vertex $v$, and suppose $X_1$ and $X_2$ are subtrees of $X$ such that $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \{v\}$. Then $\tau(X) = \tau(X_1)\tau(X_2)$.

**Proof.** Clearly we have $E^+(X) = E^+(X_1) \cup E^+(X_2)$, and for $i \in \{1,2\}$ and $e \in E^+(X_i)$ we have

$$\tau(X_{|\omega(e)} \setminus e) = \tau(X_i_{|\omega(e)} \setminus e)$$

so it follows that

$$[\chi(\lambda(e))\tau(X_{|\omega(e)} \setminus e)]^+ = [\chi(\lambda(e))\tau(X_i_{|\omega(e)} \setminus e)]^+.$$

The claim now follows directly from the definition of $\tau$. \qed
Corollary 2. Let $X$ be an idempotent left adequate tree with start vertex $v$, and $e$ an edge incident with $v$. Then
\[ \tau(X) = \tau(X \setminus \omega(e)) [\chi(\lambda(e))\tau(X_{\omega(e)} \setminus e)]^+. \]

Proof. Let $X_1 = X \setminus e = X \setminus \omega(e)$, let $S$ be the set of edges in $X$ which are incident with $v$ and let $X_2 = X \setminus (S \setminus \{e\})$ be the maximum subtree of $X$ containing $e$ but none of the other edges incident with $v$. Now clearly we have $E^+(X_2) = \{e\}$ so by the definition of $\tau$ we have
\[ \tau(X_2) = [\chi(\lambda(e))\tau(X_{\omega(e)} \setminus e)]^+. \]
We also have $X = X_1 \cup X_2$ and $X_1 \cap X_2 = \{v\}$ so by Proposition 6
\[ \tau(X) = \tau(X_1)\tau(X_2) = \tau(X \setminus \omega(e))[\chi(\lambda(e))\tau(X_{\omega(e)} \setminus e)]^+ \]
as required. \qed

Next we define a map $\rho : LUT^1(\Sigma) \to M$, from the set of isomorphism types of left adequate $\Sigma$-trees to the left adequate monoid $M$. Suppose a tree $X$ has trunk vertices $v_0, \ldots, v_n$ in sequence. For $1 \leq i \leq n$ let $a_i$ be the label of the edge from $v_{i-1}$ to $v_i$. For $0 \leq i \leq n$ let $X_i = X|_{v_i} \setminus \theta(X)$. We define
\[ \rho(X) = \tau(X_0) \chi(a_1) \tau(X_1) \chi(a_2) \cdots \chi(a_n-1) \tau(X_{n-1}) \chi(a_n) \tau(X_n). \]
Clearly the value of $\rho$ depends only on the isomorphism type of $X$ so $\rho$ is indeed a well-defined map from $LUT^1(\Sigma)$ to $M$. Again, if $M$ is right adequate as well as left adequate then the function $\rho$ takes the same value on left adequate trees as its counterpart in [10].

Proposition 7. Let $X$ be a left adequate tree with trunk vertices $v_0, \ldots, v_n$ in sequence, where $n \geq 1$. Let $a_1$ be the label of the edge from $v_0$ to $v_1$. Then
\[ \rho(X) = \tau(X|_{v_0} \setminus v_1)\chi(a_1)\rho(X|_{v_1} \setminus v_0) \]
Proof. Let $X_0, \ldots, X_n$ be as in the definition of $\rho$, so that
\[ \rho(X) = \tau(X_0) \chi(a_1) \tau(X_1) \chi(a_2) \cdots \chi(a_n-1) \tau(X_{n-1}) \chi(a_n) \tau(X_n). \]
It follows straight from the definition that
\[ \rho(X|_{v_1} \setminus v_0) = \tau(X_1) \chi(a_2) \cdots \chi(a_n-1) \tau(X_{n-1}) \chi(a_n) \tau(X_n) \]
so we have
\[ \rho(X) = \tau(X_0) \chi(a_1) \rho(X|_{v_1} \setminus v_0) = \tau(X|_{v_0} \setminus v_1) \chi(a_1) \rho(X|_{v_1} \setminus v_0) \]
as required. \qed

Proposition 8. The map $\rho : LUT^1(\Sigma) \to M$ is a morphism of $(2,1,0)$-algebras.

Proof. Let $X$ and $Y$ be trees, say with trunk vertices $u_0, \ldots, u_m$ and $v_0, \ldots, v_n$ in sequence respectively. For each $1 \leq i \leq m$ let $a_i$ be the label of the edge from $u_{i-1}$ to $u_i$, and for each $1 \leq i \leq n$ let $b_i$ be the label of the edge from $v_{i-1}$ to $v_i$. For each $0 \leq i \leq m$ let $X_i = X|_{v_i} \setminus \theta(X)$ and similarly for each $0 \leq i \leq n$ define $Y_i = Y|_{v_i} \setminus \theta(Y)$. 
Consider now the unpruned product $X \times Y$. It is easily seen that for $0 \leq i < m$ we have
\[(X \times Y)\left|_{u_i}ight. \setminus \theta(X \times Y) = X_i\]
while for $0 < i \leq n$ we have
\[(X \times Y)\left|_{v_i}ight. \setminus \theta(X \times Y) = Y_i.\]
Considering now the remaining trunk vertex $u_m = v_0$ of $X \times Y$ we have
\[(X \times Y)\left|_{u_m}ight. \setminus \theta(X \times Y) = (X \times Y)\left|_{v_0}ight. \setminus \theta(X \times Y) = X_m \times Y_0.\]

By Proposition 6 and the definition of unpruned multiplication we have
\[
\tau(X_m \times Y_0) = \tau(X_0) \tau(Y_0).\]
So using the definition of $\rho$ we have
\[
\rho(X \times Y) = \tau(X_0) \chi(a_1) \tau(X_1) \ldots \chi(a_m) \tau(X_m \times Y_0) \chi(b_1) \tau(Y_1) \chi(b_2) \ldots \chi(b_n) \tau(Y_n)
= \tau(X_0) \chi(a_1) \tau(X_1) \ldots \chi(a_m) \tau(X_m) \tau(Y_0) \chi(b_1) \tau(Y_1) \chi(b_2) \ldots \chi(b_n) \tau(Y_n)
= \rho(X) \rho(Y).
\]

Next we claim that $\rho(X^{(+)}) = \rho(X)^{+}$. We prove this by induction on the number of trunk edges in $X$. If $X$ has no trunk edges then $X = X^{(+)0}$ and so using the fact that $\tau(X) \in E(M)$ is fixed by the $+$ operation in $M$ we have
\[
\rho(X^{(+)}) = \tau(X) = \tau(X)^{+} = \rho(X)^{+}.
\]
Now suppose for induction that $X$ has at least one trunk edge and that the claim holds for trees with strictly fewer trunk edges. Recall that
\[X_0 = X\left|_{u_0}\right. \setminus \theta(X) = X\left|_{u_0}\right. \setminus u_1\]
and let $Z = X\left|_{u_1}\right. \setminus u_0$. Now
\[
\rho(X^{(+)}) = \tau(X^{(+)}) = \tau(X_0) \chi(a_1)^{+} \rho(Z)^{+}
= \tau(X_0) \chi(a_1)^{+} \rho(Z)^{+}
= \tau(X_0) \chi(a_1)^{+} \rho(Z)^{+}
= \rho(X)^{+}
\]
as required.

Finally, it follows directly from the definition that $\rho$ maps the identity element in $LU^2(\Sigma)$ (that is, the isomorphism type of the trivial tree) to the identity of $M$, and so is a $(2, 1, 0)$-morphism. \qed

So far, we have closely followed the proof strategy from [10], but at this point it becomes necessary to diverge. This is because the arguments employed in the two-sided case involve operations on trees which do not preserve left adequacy, and hence use the $*$ operation in the monoid $M$ even when starting with left adequate trees. Instead, the following lemma about left adequate trees (which fails for general trees) allows us to follow an alternative inductive strategy.
Lemma 1. Let $\mu : X \to Y$ be a morphism of left adequate trees, let $e$ be an edge in $X$ and let $v$ be a vertex such that there is a directed path from $\omega(e)$ to $v$. Then

$$\mu(X_{\omega(e)} \setminus e) \subseteq Y_{\mu(\omega(e)) \setminus \mu(e)}.$$ 

Proof. Let $X' = X_{\omega(e)} \setminus e$ and $Y' = Y_{\mu(\omega(e)) \setminus \mu(e)}$. Notice first that the image $\mu(X')$ is connected and contains $\mu(\omega(e))$. Since the underlying undirected graph of $Y$ is a tree, this means that $\mu(X')$ is either contained in $Y'$ as required, or contains the edge $\mu(e)$; suppose for a contradiction that the latter holds, say $\mu(e) = \mu(f)$ for some edge $f$ in $X'$. Now since $X$ is left adequate, there must be a directed path from the start vertex to $\alpha(f)$. But again $e$ is orientated away from start vertex, and $\alpha(f)$ is in $X'$, which is a connected component of $X$ including $\omega(e)$ but not $e$, so this path must clearly pass through the edge $e$. Let $P$ denote the suffix of this path which leads from $\omega(e)$ to $\alpha(f)$. Then $\mu(eP)$ is a non-empty directed path in $Y$ from $\mu(\alpha(e))$ to $\mu(\alpha(f)) = \mu(\alpha(e))$, which contradicts the fact that $Y$ is a directed tree. 

Lemma 2. Suppose $\mu : X \to Y$ is a morphism of idempotent left adequate trees. Then $\tau(Y)\tau(X) = \tau(Y)$.

Proof. We use induction on the number of edges in $X$. If $X$ has no edges then we have $\tau(X) = 1$ so the result is clear. Now suppose $X$ has at least one edge and for induction that the result holds for trees $X$ with strictly fewer edges. By the definition of $\tau$ we have

$$\tau(X) = \prod_{e \in E^+(X)} [\chi(\lambda(e))\tau(X_{\omega(e)} \setminus e)]^+$$

while

$$\tau(Y) = \prod_{e \in E^+(Y)} [\chi(\lambda(e))\tau(Y_{\omega(\mu(e)) \setminus \mu(e)})]^+.$$ 

Suppose now that $e \in E^+(X)$. Then since $\mu$ is a morphism, the edge $\mu(e)$ lies in $E^+(Y)$. We claim that the factor corresponding to $e$ in the above expression for $\tau(X)$ is absorbed into the corresponding factor for $\mu(e)$ in the above expression for $\tau(Y)$.

Let $X' = X_{\omega(e)} \setminus e$ and $Y' = Y_{\omega(\mu(e)) \setminus \mu(e)}$. By Lemma 1 the morphism $\mu$ restricts to a morphism $\mu' : X' \to Y'$. Since $X'$ has strictly fewer edges than $X$, the inductive hypothesis tells us that $\tau(X')\tau(Y') = \tau(Y')$. Now by Proposition 1(vi) we have

$$[\chi(\lambda(e))\tau(X')]^+ [\chi(\lambda(e))\tau(Y')]^+ = [\chi(\lambda(e))\tau(Y')]^+.$$ 

as required.

Corollary 3. Let $X$ be a subtree of an idempotent left adequate tree $Y$. Then $\tau(Y)\tau(X) = \tau(Y)$.

Proof. The embedding of $X$ into $Y$ satisfies the conditions of Lemma 2. 

Corollary 4. Let $Y$ be a retract of an idempotent left adequate tree $X$. Then $\tau(X) = \tau(Y)$. 

Proof. 

Proof. Let $\pi : X \to X$ be a retract with image $Y$. Since $\pi$ is a morphism, Lemma 2 tells us that $\tau(X)\tau(\pi(X)) = \tau(\pi(X)) = \tau(Y)$. But since $\pi(X)$ is a subgraph of $X$, Corollary 3 yields $\tau(X)\tau(\pi(X)) = \tau(X)$. □

Lemma 3. Let $X$ be a left adequate tree with trunk vertices $v_0, \ldots, v_n$ in sequence, where $n \geq 1$. Let $a_1$ be the label of the edge from $v_0$ to $v_1$. Then

$$\rho(X) = \tau(X|_{v_0})\rho(X).$$

Proof. We use induction on the number of trunk edges in $X$. Let $X' = X|_{v_0}$. Clearly if $X$ has no trunk edges then we have $X = X'$ and from the definition of $\rho$ we have $\rho(X) = \tau(X')$, so the claim reduces to the fact that $\tau(X')$ is idempotent. Now suppose $X$ has at least one trunk edge and that the claim holds for $X$ with strictly fewer trunk edges. Let $Y = X|_{v_1} \setminus v_0$, let $Y' = Y|_{v_1}^1$, and let $X_0 = X|_{v_0} \setminus v_1$. Let $a_1$ be the label of the edge from $v_0$ to $v_1$. By Corollary 2 we have

$$\tau(X') = [\chi(a_0)\tau(Y')]^+ \tau(X_0).$$

Now by Proposition 7 we deduce that $\rho(X) = \tau(X_0)\chi(a_1)\rho(Y)$. Also, by the inductive hypothesis we have $\rho(Y) = \tau(Y')\rho(Y)$. Putting these observations together we have

$$\tau(X')\rho(X) = ([\chi(a_1)\tau(Y')]^+ \tau(X_0) \chi(a_1)\rho(Y))$$

$$= [\chi(a_1)\tau(Y')]^+ \tau(X_0) \chi(a_1)\tau(Y')\rho(Y)$$

$$= \tau(X_0) [\chi(a_1)\tau(Y')]^+ [\chi(a_1)\tau(Y')] \rho(Y)$$

$$= \tau(X_0) \chi(a_1) \tau(Y') \rho(Y)$$

$$= \tau(X_0) \chi(a_1) \rho(Y)$$

$$= \rho(X)$$

as required. □

Corollary 5. Let $X$ be a left adequate tree with trunk vertices $v_0, \ldots, v_n$ in sequence, where $n \geq 1$. Let $a_1$ be the label of the edge from $v_0$ to $v_1$. Then

$$\rho(X) = \tau(X|_{v_0})\chi(a_1)\rho(X|_{v_1} \setminus v_0) = \rho(X|_{v_1} \setminus v_n) \chi(a_n) \tau(X|_{v_0} \setminus v_{n-1}).$$

Proof. We prove the first equality, the rest of the claim being dual. We have

$$\rho(X) = \tau(X|_{v_0})\rho(X)$$

(by Lemma 3)

$$= \tau(X|_{v_0})\tau(X|_{v_1} \setminus v_0)\chi(a_1)\rho(X|_{v_1} \setminus v_0)$$

(by Proposition 7)

$$= \tau(X|_{v_0})\chi(a_1)\rho(X|_{v_1} \setminus v_0)$$

(by Corollary 3).

□

Proposition 9. Let $X$ be a left adequate tree. Then $\rho(X) = \rho(X)$. □

Proof. Let $\pi : X \to X$ be a retraction with image $X$. Suppose $X$ has trunk vertices $v_0, \ldots, v_n$. For $1 \leq i \leq n$ let $a_i$ be the label of the edge from $v_{i-1}$ to $v_i$. We prove the claim by induction on the number of trunk edges in $X$. If $X$ has no trunk edges then by the definition of $\rho$ and Corollary 4 we have

$$\rho(X) = \tau(X) = \tau(\pi(X)) = \rho(\pi(X)).$$
Next suppose that $X$ has at least one trunk edge, that is, that $n \geq 1$. Let $Z = X \mid v_1 \setminus v_0$. Then by Lemma 11 we have

$$\pi(Z) = \pi(X \mid v_1 \setminus v_0) \subseteq \pi(X) \mid v_1 \setminus v_0 = \overline{X} \mid v_1 \setminus v_0$$

and, since $\pi$ is idempotent with image $\overline{X}$, the converse inclusion also holds and we have

$$\pi(Z) = \overline{X} \mid v_1 \setminus v_0. \quad (1)$$

Moreover, by Lemma 11 again, the retraction $\pi$ restricts to a morphism $\pi' : Z \rightarrow Z$. Clearly this morphism must also be a retraction, and $Z$ has strictly fewer edges than $X$, so by the inductive hypothesis and Proposition 3 we have

$$\rho(Z) = \rho(Z) = \rho(\pi'(Z)) = \rho(\pi(Z)). \quad (2)$$

It also follows easily from definitions that

$$\pi(X \mid v_0) = \overline{X} \mid v_0 \quad (3)$$

Now

$$\rho(X) = \tau(X \mid v_0) \chi(a_1) \rho(Z) \quad \text{(by Corollary 5)}$$
$$= \tau(X \mid v_0) \chi(a_1) \rho(\pi(Z)) \quad \text{(by (2))}$$
$$= \tau(\pi(X \mid v_0)) \chi(a_1) \rho(\pi(Z)) \quad \text{(by Corollary 4)}$$
$$= \tau(\overline{X} \mid v_0) \chi(a_1) \rho(\overline{X} \mid v_0) \quad \text{(by (1) and (3))}$$
$$= \rho(\overline{X}) \quad \text{(by Corollary 5)}. \quad \Box$$

Now let $\hat{\rho} : LT^1(\Sigma) \rightarrow M$ be the restriction of $\rho$ to the set of (isomorphism types of) pruned left adequate trees.

**Corollary 6.** The function $\hat{\rho}$ is a $(2, 1, 0)$-morphism from $LT^1(\Sigma)$ (with pruned operations) to the left adequate monoid $M$.

**Proof.** For any $X, Y \in LT^1(\Sigma)$ by Theorem 11 and Propositions 8 and 9 we have

$$\hat{\rho}(XY) = \rho(XY) = \rho(\overline{X} \times Y) = \rho(X \times Y) = \rho(X) \rho(Y) = \hat{\rho}(X) \hat{\rho}(Y)$$

and similarly

$$\hat{\rho}(X^+) = \rho(X^+) = \rho(X^+) = \rho(X)^+ = \hat{\rho}(X)^+. \quad \Box$$

Finally, that $\hat{\rho}$ maps the identity of $LT^1(\Sigma)$ to the identity of $M$ is immediate from the definitions.

We are now ready to prove the main results of this paper, which give a concrete description of the free left adequate monoid and free right adequate monoid on a given generating set.

**Theorem 3.** Let $\Sigma$ be a set. Then $LT^1(\Sigma)$ [RT^1(\Sigma)] is a free object in the quasivariety of left [right] adequate monoids, freely generated by the set $\Sigma$ of base trees.
Proof. We prove the claim in the left adequate case, the right adequate case being dual. By Corollary 1, $LT^1(\Sigma)$ is a left adequate monoid. Now for any left adequate monoid $M$ and function $\chi: \Sigma \to M$, define $\hat{\rho}: LT^1(\Sigma) \to M$ as above. By Corollary 6, $\hat{\rho}$ is a $(2,1,0)$-morphism, and it is immediate from the definitions that $\hat{\rho}(a) = \chi(a)$ for every $a \in \Sigma$, so that $\hat{\rho}$ extends $\chi$. Finally, by Proposition 5, $\Sigma$ is a $(2,1,0)$-algebra generating set for $LT^1(\Sigma)$; it follows that the morphism $\hat{\rho}$ is uniquely determined by its restriction to the set $\Sigma$ of base trees, and hence is the unique morphism with the claimed properties.

Combining with Proposition 2 we also obtain immediately a description of the free left adequate and free right adequate semigroups.

Theorem 4. Let $\Sigma$ be a set. Then the $LT(\Sigma) [RT(\Sigma)]$ is a free object in the quasivariety of left [right] adequate semigroups, freely generated by the set $\Sigma$ of base trees.

We also have the following relationship between free adequate, free left adequate and free right adequate semigroups and monoids.

Theorem 5. Let $\Sigma$ be a set. The free left adequate semigroup [monoid] on $\Sigma$ and free right adequate semigroup [monoid] on $\Sigma$ embed into the free adequate semigroup [monoid] on $\Sigma$ as the $(2,1)$-subalgebras $(2,1,0)$-subalgebras generated by the free generators under the appropriate operations. Their intersection is the free semigroup [monoid] on $\Sigma$.

4. Remarks and Consequences

In this section we collect together some remarks on and consequences of the results in Section 3 and their proofs.

In a left adequate tree, the requirement that there be a path from the start vertex to every other vertex uniquely determines the orientation on every edge in the tree. Conversely, every edge-labelled undirected tree with given start and end vertex admits an orientation on the edges which makes it left adequate. It might superficially seem attractive, then, to identify elements of $LU^1(\Sigma)$ with undirected edge-labelled trees with distinguished start and end vertices. However, the reader may easily convince herself that not every retraction of such a tree defines a retraction of the corresponding directed tree. So in order to define pruning and multiplication it would be necessary to reinstate the orientation on the edges, which negates any advantage in dropping the orientation in the first place.

The construction in Section 3 of a morphism from $LT^1(\Sigma)$ to a monoid $M$ depends only on the facts that $M$ is associative with commuting idempotents, and that the $+$ operation is idempotent with idempotent and commutative image and satisfies the six properties given in the case of left adequate semigroups by Proposition 1. So a free left adequate semigroup is also free in any class of $(2,1,0)$-algebras which contains it and satisfies these conditions. This includes in particular the class of left Ehresmann semigroups.

As observed in [10], the classes of monoids we have studied can be generalised to give corresponding classes of small categories. A natural extension of our methods can be used to describe the free left adequate and free
right adequate category generated by a given directed graph. Just as in
the previous remark, the free left adequate category will also be the free
left Ehresmann category. Left Ehresmann categories are generalisations of
the restriction categories studied by Cockett and Lack [3], which in the ter-
mminology of semigroup theory are weakly left E-ample categories [9]. The
generalisation of our results to categories thus relates to our main results in
the same way that the description of the free restriction category on a graph
given in [2] relates to the descriptions of free left ample monoids given by
Fountain, Gomes and Gould [7, 8].

To conclude, we note some properties of free left and right adequate semi-
groups and monoids, which are obtained by combining Theorem 5 with re-
sults about free adequate semigroups and monoids which were obtained in
[10]. First of all, since each finitely generated free left adequate or free right
adequate semigroup embeds into a finitely generated adequate semigroup
we have the following.

**Theorem 6.** The word problem for any finitely generated free left or right
adequate semigroup or monoid is decidable.

As in the two-sided case, the exact computational complexity of the word
problem remains unclear and deserves further study.

Recall that an equivalence relation \(\mathcal{J}\) is defined on any semigroup by
\(a \mathcal{J} b\) if and only if \(a\) and \(b\) generate the same principal two-sided ideal. A
semigroup is called \(\mathcal{J}\)-trivial if no two elements generate the same principal
two-sided ideal.

**Theorem 7.** Every free left adequate or free right adequate semigroup or
monoid is \(\mathcal{J}\)-trivial.

*Proof.* If distinct left [right] adequate \(\Sigma\)-trees \(X\) and \(Y\) are \(\mathcal{J}\)-related in
\(LT^1(\Sigma)\) [\(RT^1(\Sigma)\)] then they are \(\mathcal{J}\)-related in the free adequate monoid
\(T^1(\Sigma)\); but we saw in [10] that \(T^1(\Sigma)\) is \(\mathcal{J}\)-trivial. □

**Theorem 8.** No free left adequate or free right adequate semigroup or
monoid on a non-empty set is finitely generated as a semigroup or monoid.

*Proof.* We saw in [10] that finite subsets of \(T^1(\Sigma)\) generate subsemigroups
whose trees have a bound on the maximum distance of any vertex from the
trunk. Since \(LT^1(\Sigma)\) and \(RT^1(\Sigma)\) are subsemigroups containing trees with
vertices arbitrarily far from the trunk, it follows that they cannot even be
contained in finitely generated subsemigroups of \(T^1(\Sigma)\), let alone themselves
be finitely generated. □

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