Small-amplitude Periodic Orbit around Sun-Earth $L_1/L_2$ Controlled by Solar Radiation Pressure

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The $L_1$ and $L_2$ points of the Sun-Earth system attract much attention for various space uses, such as observation and communication. Deploying the spacecraft just on $L_1/L_2$ is, however, not convenient, because $L_1$ always overlaps with the Sun as seen from the Earth and $L_2$ is hidden behind the shadow of the Earth. Adopting a small-amplitude periodic orbit around the $L_1/L_2$ points is one option to solve this problem. The orbit can be achieved by low continuous maneuvering. The required magnitude of acceleration is at a level that can be managed by solar radiation pressure. Utilizing solar radiation pressure has the possibility of saving maneuvering to keep the spacecraft near a $L_1/L_2$ orbit. Acceleration due to solar radiation pressure depends on the surface area of the spacecraft and thus the spacecraft should be equipped with a large flat surface. A spacecraft equipped with a solar sail is appropriate. This paper presents two station-keeping examples of solar sails in the vicinity of SE $L_1/L_2$ using acceleration resulting from solar radiation pressure. The orbit control laws are built into the linear system about the equilibrium, and we confirm that they are applicable in the non-linear system through numerical calculation.

Key Words: Small-amplitude Periodic Orbits, $L_1$ and $L_2$ of the Sun-Earth System, Solar Radiation Pressure, Solar Sail

Nomenclature

$A_x, A_y, A_z$: amplitude of orbital motion  
$A_{\psi}, A_{\phi}$: amplitude of attitude motion  
$C_{\text{spe}}$: coefficient of specular reflection  
$C_{\text{dif}}$: coefficient of diffusive reflection  
$C_{\text{abs}}$: coefficient of absorption  
$I_S$: moment of inertia around spin axis  
$I_I$: moment of inertia perpendicular to spin axis  
$M$: celestial body mass  
$P$: solar radiation pressure per unit area  
$R$: celestial body radius  
$S$: surface area  
$T$: orbit period  
$\tilde{U}$: three-body system potential  
$a$: acceleration vector  
$\alpha$, $\alpha_x$, $\alpha_y$, $\alpha_z$: acceleration  
$c$: velocity of light  
$c_2$: three-body system parameter  
$k_i$: function value  
$l$: distance between the center of mass and the point of application of force  
$m$: spacecraft mass  
$n$: surface normal unit vector  
$n$: mean motion  
$p$: function value  
$r$: position vector  
$r$: distance  
$s$: Sun direction unit vector  
$t$: time  
$v$: velocity  
$x$: state vector  
$x, y, z$: Cartesian coordinates

Symbols

$\Phi$: solar radiation flux  
$\Omega$: spin rate  
$\Omega_{\alpha}$: modified spin rate  
$\alpha$: amplitude ratio  
$\beta$: amplitude ratio  
$\gamma$: distance from Earth to $L_1/L_2$  
$\theta$: phase of oscillation  
$\lambda$: eigenvalue  
$\mu$: ratio of the Sun and Earth mass  
$\psi$: in-plane angle of surface normal  
$\phi$: out-of-plane angle of surface normal  
$\omega$: angular velocity

Abbreviations

EM: Earth-Moon  
SE: Sun-Earth  
SRP: solar radiation pressure  
nd: non-dimensional

Subscript

E: Earth  
S: Sun  
d: desired  
e: equilibrium

1. Introduction

The locations where the gravitational and centrifugal acceleration balance each other in the Sun-Earth (SE) system...
are known as the libration points. Five equilibria exist in the system, and their relative positions with respect to the Sun and Earth are always constant. Two of them, $L_1$ and $L_2$, are located near the Earth, and are thus considered good positions for a deep-space port beyond the low Earth orbit. However, deploying the spacecraft just on $L_1/L_2$ is not convenient, because $L_1$ always overlaps with the Sun as seen from the Earth and $L_2$ is hidden behind the shadow of the Earth. These may cause communication problems for the spacecraft at $L_1$ and power supply problems at $L_2$.

The motion of the spacecraft in the vicinity of $L_1/L_2$ becomes quasi-periodic when the initial conditions are restricted so as not to cause a divergence mode. This type of orbit is called a Lissajous orbit, and was discovered by Farquhar\(^3\) to improve the visibility of the spacecraft from Earth near $L_2$ of the Earth-Moon (EM) system. He also proposed an orbital control law that achieves a periodic orbit around EM $L_2$ by introducing out-of-plane maneuvers to match the period of the in-plane and out-of-plane motions.\(^2\) A natural periodic orbit can be obtained by increasing the amplitude of the motion to a region where the non-linear acceleration has a large influence.\(^3\) Breakwell and Brown\(^4\) and Howell\(^5\) conducted numerical studies on halo orbits and proved that various orbital sizes exist. Since then, halo orbits have been widely used in various missions because only near-zero maneuvers need to be maintained.\(^6\)

The amplitude of the $L_1/L_2$ natural halo in the Sun-Earth system is very large, and the position of the spacecraft with respect to the Earth dramatically changes in an orbital period. One way to avoid these conditions is to put the spacecraft on a small-amplitude orbit around these points as shown in Fig. 1. Jones and Bishop\(^7\) solved the station-keeping problem around SE $L_2$ using H\(_2\) control theory. They showed it minimized the positional error of the orbit from the target halo. Tarao and Kawaguchi\(^8\) proposed a control law using continuous acceleration, and proved that small correction maneuvers can maintain the spacecraft on the circular-shaped orbit around SE $L_1/L_2$.

Interest has been growing in the positive use of solar radiation pressure in recent space missions. The JAXA spacecraft HAYABUSA tried and succeeded to modify its attitude so as not to cause a divergence mode. This method can be applicable to both spinning and three-axis-stabilized spacecraft. The attitude control is assumed to be managed by an attitude control system, such as reaction wheels and thrusters.

The second option utilizes attitude drift motion that is induced by solar radiation torque and can automatically point the spacecraft towards the Sun. This is applicable to the spinning spacecraft and has the advantage of being able to control the attitude of the spacecraft automatically.

The orbit control laws are built into the linear system around equilibrium and we confirm that they also work in the non-linear system through numerical calculation.

2. Preliminary Development

2.1. Circular restricted three-body problem

The orbital motion of the spacecraft is described by employing the circular restricted three-body problem. The Sun and Earth are assumed to move in circular orbits around their barycenter. The system is made non-dimensional. The unit of mass is the sum of the Sun and Earth masses. The unit of length is the distance between the Sun and Earth. The unit of time is selected so that the orbital period of the Sun and Earth becomes $2\pi$. The motion of the spacecraft is represented in the Sun-Earth fixed rotating coordinate as shown in Fig. 2. The origin is located at one of the collinear libra-

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**Fig. 1.** Small-amplitude periodic orbit and natural halo around SE $L_2$.

**Fig. 2.** SE $L_2$-origin Sun-Earth fixed rotating coordinate.
tion points (i.e., \(L_1\) or \(L_2\)) of the Sun-Earth system. The \(x\)-axis lies along the line from the Sun to the Earth and the orbital plane of the two bodies forms the \(xy\)-plane. The \(z\)-axis makes the right-hand system.

The equations of motion of the circular restricted three-body problem of the Sun-Earth system are

\[
\begin{align*}
\dot{x} - 2\dot{y} &= -\frac{\partial U}{\partial x} + a_x \\
\dot{y} + 2\dot{x} &= -\frac{\partial U}{\partial y} + a_y \\
\dot{z} &= -\frac{\partial U}{\partial z} + a_z
\end{align*}
\]

where

\[
\dot{U} = -\frac{1}{2}[(x + 1 - \mu)(\gamma)^2 + y^2] - \frac{1-\mu}{r_S} - \frac{\mu}{r_E}
\]

The mass ratio between the Sun and Earth is defined as

\[
\mu = \frac{M_E}{M_S + M_E}
\]

\(M_S\) and \(M_E\) are the masses of the Sun and Earth, respectively. \(\gamma\) is the distance from the Earth to the libration point, which is the coordinate origin. \(r_S\) and \(r_E\) are the distances from the spacecraft to the two celestial bodies expressed as

\[
\begin{align*}
\gamma &= \sqrt{(x + 1 - \mu)(\gamma)^2 + y^2 + z^2} \\
r_S &= \sqrt{(x + 1 - \gamma)^2 + y^2 + z^2} \\
r_E &= \sqrt{(x - \gamma)^2 + y^2 + z^2}
\end{align*}
\]

The upper sign corresponds to the motion about \(L_1\) and the lower sign is about \(L_2\).

To design small-amplitude periodic orbits in the vicinity of the \(L_1/L_2\) point, it is more convenient to adopt linearized equations around \(L_1/L_2\). Equation (1) is modified as

\[
\begin{align*}
\dot{x} - 2\dot{y} - (1 + c_2)x &= a_x \\
\dot{y} + 2\dot{x} + (c_2 - 1)y &= a_y \\
\dot{z} + c_2z &= a_z
\end{align*}
\]

where \(c_2\) is a system parameter determined by the mass ratio and the distance between the Earth and the libration point,

\[
c_2 = \frac{1}{\gamma^3}\sqrt{\mu + \frac{(1-\mu)y^3}{(1+\gamma)^3}}
\]

When there is no external acceleration (i.e., \(a_x = a_y = a_z = 0\)) and an appropriate initial condition is selected that avoids divergence and convergence, Eq. (5) has the solutions expressed as

\[
\begin{align*}
x &= -A_x \cos(\omega_{xy}t + \theta_{xy}) \\
y &= A_x \sin(\omega_{xy}t + \theta_{xy}) \\
z &= A_z \cos(\omega_{xy}t + \theta_{xy})
\end{align*}
\]

with

\[
\omega_{xy} = \sqrt{-c_2 + 2 + \sqrt{9c_2^2 - 8c_2}} / 2
\]

and out-of-plane frequency \(\omega_z = \sqrt{c_2}\) are generally not equal, which means the free motion of this system becomes quasi-periodic.

### 2.2. Solar radiation pressure

A spacecraft in space experiences perturbation that arises from absorption, deflection, and reflection of solar radiation pressure.\(^9\)\(^10\) Its acceleration depends on the mass and surface characteristics of the spacecraft. Its magnitude is not large but it can be one of the most dominant perturbations in deep space where gravitational perturbation becomes comparatively small.

The acceleration of a spacecraft due to solar radiation pressure is expressed by

\[
a_{\text{SRP}} = -\frac{PS}{m} (s \cdot n) \left( C_{\text{abs}} + C_{\text{dif}} \right) s + \left( \frac{2}{3} C_{\text{dif}} + 2(s \cdot n) C_{\text{spe}} \right) n
\]

where \(P\) is solar radiation pressure per unit area. \(S\) denotes the area of the spacecraft surface exposed to the Sun and \(m\) is its mass. \(C_{\text{spe}}, C_{\text{dif}}\) and \(C_{\text{abs}}\) are the coefficients of the specular reflection, diffusive reflection and absorption. The unit vector \(n\) gives the normal of the spacecraft surface and \(s\) points the direction of the Sun from the spacecraft.

#### 2.3. Sun-pointing coordinate system and spin-free spacecraft-fixed coordinate system

For the following development of the formula, two coordinate systems are introduced: the Sun-pointing coordinate and the spin-free spacecraft-fixed coordinate. The spacecraft is supposed to move around the Sun.

The Sun-pointing coordinate is defined as the spacecraft-origin coordinate system, with the \(z\)-axis pointing in the direction of the Sun direction from the spacecraft and the \(y\)-axis perpendicular to the spacecraft orbital plane. The \(x\)-axis completes the right-handed system.

The spin-free spacecraft-fixed coordinate system is used to define the orientation of the attitude of a spin-stabilized spacecraft. It uses \(\psi\) for the elevation and \(\phi\) for the azimuth of the spin axis measured from the direction of the Sun, as shown in Fig. 3. The \(x\) and \(y\)-axes are determined by the transformation from the Sun-pointing coordinate and the \(z\)-axis is parallel to the spin axis of the spacecraft. The transformation matrix from the Sun-pointing rotating coordinate is shown in Appendix A.

#### 2.4. Attitude drift motion due to solar radiation pressure

The torque generated by solar radiation pressure induces motion, which works to turn the spacecraft attitude towards the Sun.\(^9\)\(^10\) This is called attitude drift motion.

The spacecraft is assumed to move around the Sun and be spinning with a constant angular velocity. The spin axis vector should point to the Sun. The motion equations of attitude drift in the spin-free coordinate are given by
Eq. (10) is written as

\[
\begin{align*}
\dot{\psi} + \Omega_n \dot{\psi} - p \psi &= 0 \\
\ddot{\psi} - \Omega_n \dot{\psi} - p \dot{\psi} &= \Omega_n n
\end{align*}
\]

with

\[
\Omega_n = \frac{I_s}{I_T} \Omega
\]

\[
p = \frac{P S I(C_{bn} + C_{dif})}{I_T}
\]

where \( \Omega \) denotes the spin rate of the spacecraft. \( I_s \) and \( I_T \) are the moments of inertia around the spin axis and perpendicular to the spin axis. \( l \) denotes the distance between the center of mass and the point of application of the pressure. \( n \) is the mean motion of the orbit. The characteristic equation of Eq. (10) is written as

\[
\lambda^2 + (-2p + \Omega_n^2) \lambda^2 + p^2 = 0
\]

and its eigenvalues are approximated as

\[
\lambda = \pm \Omega_n i, \pm \frac{p}{\Omega_n} i
\]

Since \( p \) is a function of solar radiation pressure, its value is very small, resulting in \( \Omega_n \gg p/\Omega_n \). The short period motion corresponding to the eigenvalue \( \pm \Omega_n i \) is called the nutation and the long period motion of the eigenvalue \( \pm p/\Omega_n i \) is the precession. When the nutation has small amplitude compared to the precession and is therefore neglected, the solutions of Eq. (10) are approximated as

\[
\begin{align*}
\psi &= -A_\psi \sin \left( \frac{p}{\Omega_n} t + \theta_\psi \right) \\
\varphi &= A_\varphi \cos \left( \frac{p}{\Omega_n} t + \theta_\varphi \right) - \frac{\Omega_n n}{p} \psi
\end{align*}
\]

These equations suggest the spacecraft normally moves counterclockwise in a circular motion around the equilibrium point in the \( \psi\varphi \)-plane. The equilibrium point is

\[
(\psi_e, \varphi_e) = \left( 0, -\frac{\Omega_n n}{p} \right)
\]

Equation (10) can be simplified so that it has the solution of Eq. (14)

\[
\begin{align*}
\dot{\psi} &= -\frac{p}{\Omega_n} \psi - n \\
\dot{\varphi} &= \frac{p}{\Omega_n} \varphi
\end{align*}
\]  

The details of the formulation are shown in Appendix B.

### 3. Small-amplitude Periodic Orbit around \( SE L_1/L_2 \) Controlled by Solar Radiation Pressure

This section proposes two ways to design a small-amplitude periodic orbit around \( SE L_1/L_2 \) using solar radiation pressure. We assume the spacecraft has a solar sail with a large flat surface. In the case of a solar sail, the sail is very large compared to other parts of the spacecraft, and thus acceleration due to solar radiation pressure of Eq. (9) can be considered to depend on the characteristics of the sail. Therefore, \( S \) is approximated by the sail area and \( n \) is by the sail surface normal.

In the first method, the spacecraft actively changes its attitude so that acceleration is generated in the desired direction. This can be applicable to both spinning and three-axis-stabilized spacecraft.

The second option utilizes attitude drift motion, which is induced by solar radiation torque. It is required that the spacecraft be spinning with a constant angular velocity and with the spin axis points toward the Sun.

#### 3.1. Designing the orbit by controlling the attitude

The direction of the Sun from the spacecraft in the Sun-Earth fixed rotating coordinate is expressed as

\[
s = \frac{s'}{|s'|} = \left( \begin{array}{c} -1 + \gamma - x \\ -y \\ -z \end{array} \right)
\]

When the spacecraft moves in the vicinity of the \( L_1/L_2 \) point, the orbital plane and the ecliptic plane can be considered as identical. Thus, the Sun direction is approximated as

\[
s = \left( \begin{array}{c} -1 \\ 0 \\ 0 \end{array} \right)
\]

The surface normal is written in terms of angles \( \psi \) and \( \varphi \) as

\[
n = \left( \begin{array}{c} -\cos \psi \cos \varphi \\ -\cos \psi \sin \varphi \\ \sin \varphi \end{array} \right)
\]

When the amplitudes of \( \psi \) and \( \varphi \) are sufficiently small, it can be simplified as

\[
n = \left( \begin{array}{c} -1 \\ -\psi \\ \varphi \end{array} \right)
\]

Substituting \( s \) and \( n \) of Eqs. (18) and (20) into Eq. (9) yields acceleration due to solar radiation pressure as a function of \( \psi \) and \( \varphi \)
\[ \mathbf{a}_{\text{SRP}} = \begin{pmatrix} k_1 \\ k_2 \psi \\ -k_2 \psi \end{pmatrix} \]  \hspace{1cm} (21)

with
\[ k_1 = \frac{PS}{m} \left( C_{\text{abs}} + \frac{5}{3} C_{\text{dif}} + 2 C_{\text{spe}} \right) \]
\[ k_2 = \frac{PS}{m} \left( \frac{2}{3} C_{\text{dif}} + 2 C_{\text{spe}} \right) \]  \hspace{1cm} (22)

The coefficients \( k_1 \) and \( k_2 \) are constant parameters determined by the optical properties of the sail surface. Equation (21) indicates that solar radiation pressure generates constant acceleration in the \( x \)-direction. Then, the linear equations of motion of the spacecraft subject to solar radiation are written as
\[
\begin{align*}
\dot{x} - 2 \dot{y} - (1 + 2c_2)x &= k_1 \\
y + 2 \dot{y} + (c_2 - 1)y &= k_2 \psi \\
\dot{z} + c_2 \dot{z} &= -k_2 \psi 
\end{align*}
\]  \hspace{1cm} (23)

We assume a nominal periodic orbit that has a drifted equilibrium along the \( x \)-axis. The amplitudes of the \( y \) and \( z \)-directions are equal, so that the \( yz \)-projection of the orbit is a circle.
\[
\begin{align*}
x &= -A_x \cos(\omega t + \theta_y) + x_e \\
y &= \alpha A_x \sin(\omega t + \theta_y) \\
z &= \alpha A_x \cos(\omega t + \theta_z)
\end{align*}
\]  \hspace{1cm} (24)

\( \omega \) can be arbitrarily selected. Substituting Eq. (24) into Eq. (23) yields the constraints that \( x_e \) and \( \alpha \) must satisfy.
\[
x_e = -\frac{k_1}{1 + 2c_2} \]
\[
\alpha = \frac{-\omega^2 + 1 + 2c_2}{2\omega} \]  \hspace{1cm} (25, 26)

Then, the motion of \( \psi \) and \( \varphi \) is written as
\[
\begin{align*}
\dot{\psi} &= \frac{1}{k_2} (-\omega^2 + 2\omega) / (\alpha + c_2 - 1) \alpha A_x \sin(\omega t + \theta_y) \\
\varphi &= -\frac{1}{k_2} (-\omega^2 + c_2) \alpha A_x \cos(\omega t + \theta_z)
\end{align*}
\]  \hspace{1cm} (27)

Equation (27) requires the spacecraft to change its attitude in a periodic motion with the same frequency as the orbital motion.

3.2. Designing the orbit by utilizing attitude drift motion

The control law using attitude drift motion due to solar radiation pressure is formulated. The key to obtaining control of the motion is to synchronize the frequency of the orbit and attitude. The sail surface normal changes around the equilibrium point of Eq. (15) and is represented as
\[
\mathbf{n} = \begin{pmatrix} -\cos(\varphi_e + \psi) \cos(\psi_e + \psi) \\ -\cos(\varphi_e + \psi) \sin(\psi_e + \psi) \\ \sin(\varphi_e + \psi) \end{pmatrix} \]  \hspace{1cm} (28)

Note that the displacements from the equilibrium are expressed with \( \psi \) and \( \varphi \). It is approximated when \( \psi \) and \( \varphi \) are small as
\[
\begin{align*}
\mathbf{n} &= \begin{pmatrix}
-\cos \varphi_e + \psi \sin \varphi_e \\
-\psi \cos \varphi_e \\
\sin \varphi_e + \psi \cos \varphi_e
\end{pmatrix}
\end{align*} \]  \hspace{1cm} (29)

The direction of the Sun is expressed as Eq. (18), and thus the cosine of the angle between \( s \) and \( \mathbf{n} \) becomes
\[
s \cdot \mathbf{n} = \cos \varphi_e - \psi \sin \varphi_e \]  \hspace{1cm} (30)

Substituting \( s \), \( \mathbf{n} \) and \( s \cdot \mathbf{n} \) into Eq. (9) yields acceleration under attitude drift motion as
\[
\mathbf{a}_{\text{SRP}} = -\frac{PS}{m} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \psi \\ k_5 \psi \\ k_6 \psi \end{pmatrix} \]  \hspace{1cm} (31)

where \( k_i \) (\( 1 \leq i \leq 6 \)) are coefficient constants expressed as
\[
\begin{align*}
k_1 &= -\frac{2}{3} C_{\text{dif}} \cos^2 \varphi_e - 2 C_{\text{spe}} \cos^3 \varphi_e - (C_{\text{abs}} + C_{\text{dif}}) \cos \varphi_e \\
k_2 &= 0 \\
k_3 &= \frac{2}{3} C_{\text{dif}} \cos \varphi_e \sin \varphi_e + 2 C_{\text{spe}} \cos^2 \varphi_e \sin \varphi_e \\
k_4 &= \frac{4}{3} C_{\text{dif}} \cos \varphi_e \sin \varphi_e + 6 C_{\text{spe}} \cos^2 \varphi_e \sin \varphi_e + (C_{\text{abs}} + C_{\text{dif}}) \sin \varphi_e \\
k_5 &= \frac{2}{3} C_{\text{dif}} (\cos^2 \varphi_e - \sin^2 \varphi_e) \\
k_6 &= \frac{2}{3} C_{\text{dif}} (\cos^2 \varphi_e - \sin^2 \varphi_e) + 2 C_{\text{spe}} (\cos^3 \varphi_e - 2 \cos \varphi_e \sin^2 \varphi_e)
\end{align*}
\]  \hspace{1cm} (32)

The linear equations of motion of the spacecraft become
\[
\begin{align*}
\dot{x} - 2 \dot{y} - (1 + 2c_2)x &= -\frac{PS}{m} (k_1 + k_4 \psi) \\
\dot{y} + 2 \dot{y} + (c_2 - 1)y &= -\frac{PS}{m} (k_2 + k_5 \psi) \\
\dot{z} + c_2 \dot{z} &= -\frac{PS}{m} (k_3 + k_6 \psi)
\end{align*} \]  \hspace{1cm} (33)

Acceleration due to solar radiation pressure has constant components in the \( x \) and \( z \)-directions, and thus the nominal trajectory is assumed to have the following form
\[
\begin{align*}
x &= -A_x \cos(\omega t + \theta_y) + x_e \\
y &= \alpha A_x \sin(\omega t + \theta_y) \\
z &= \beta A_x \cos(\omega t + \theta_z) + z_e
\end{align*} \]  \hspace{1cm} (34)

where \( x_e \) and \( z_e \) are the shifted equilibrium positions along the \( x \) and \( z \)-axes. \( \alpha \) and \( \beta \) are not always the same value. Substituting these solutions of Eq. (34) into the equations of motion of Eq. (33) yields the following relations.
\[
x_e = \frac{PS}{m} \frac{k_1}{1 + 2c_2} \]  \hspace{1cm} (35)
Equations (35)–(38) give the parameters related to the orbital motion and Eqs. (39)–(41) to the attitude motion. Equation (40) is obtained by approximating $n = 1$. The sign ‘−’ of Eq. (38) corresponds to the case of $\theta_z = \theta_y$ and the sign ‘+’ corresponds to $\theta_z = \theta_y + \pi$. The circular-viewed orbits on the $yz$-projection plane can be obtained when $\omega$ is appropriately selected so that the following condition is satisfied.

$$\alpha = \pm \beta$$  \hspace{1cm} (42)

Finally, the spin rate of the spacecraft is calculated as

$$\Omega = \frac{I_T}{I_S} \frac{p}{\omega}$$  \hspace{1cm} (43)

### 4. Numerical Verification

This section shows that the proposed laws can work in the non-linear system although they are obtained in the linear system. The periodic orbit is calculated by utilizing the non-linear equations of motion of the Sun-Earth three-body system expressed in Eq. (1) considering acceleration due to solar radiation pressure derived from Eq. (9).

#### 4.1. Calculation of periodic orbits

Periodic orbits can be numerically obtained using the symmetry with respect to the $xz$-plane. The approximate initial condition on the $xz$-plane should be selected as

$$x(0) = (x_0 \ 0 \ z_0 \ 0 \ y_0 \ 0)^T$$  \hspace{1cm} (44)

and integrated until the orbit crosses the $xz$-plane at a half period, $T/2$. The complete periodic orbit requires crossing the plane perpendicularly and it has the form of

$$x(T/2) = (x_1 \ 0 \ z_1 \ 0 \ y_1 \ 0)^T$$  \hspace{1cm} (45)

The values obtained by the integration do not always satisfy this form and some variation of the initial condition is required. In this study, the one-dimensional search on $v_{zc} = \sqrt{\dot{x}_1 + \dot{z}_1}$ by $\Delta \dot{x}_0$ is conducted (i.e., searching the desired initial $x$-position $x_0 = x_0 + \delta x_0$ which makes $v_{zc}$ zero).

#### 4.2. Calculation condition

The values of the constants of the Sun-Earth system are shown in Table 1. $\gamma$ is the distance from the Earth to $L_2$.

### Table 1. Values of the constants in the Sun-Earth system.

| Parameter | Value |
|-----------|-------|
| $\mu$     | $3.04043 \times 10^{-6}$ |
| $c_2$     | 3.94052 |
| $\gamma$  | $1.00782 \times 10^{-2}$ |
| $n$       | 1.0   |

### Table 2. Values of the constants related to solar radiation pressure.

| Parameter | Value |
|-----------|-------|
| $\Phi_l$  | 1.367 | W/m² |
| $c$       | 299,792,458 | m/s |
| $P$       | 4.46928 $\times 10^{-6}$ | N/m² |

Solar radiation pressure at $L_2$ can be calculated by dividing the solar flux by the speed of light

$$P = \frac{\Phi}{c}$$  \hspace{1cm} (46)

The solar flux is inversely proportional to the square of the distance from the Sun, and its value at $L_2$ is obtained as

$$\Phi = \frac{\Phi_E}{(1 + \gamma)^2}$$  \hspace{1cm} (47)

The values of the constants related to solar radiation pressure are shown in Table 2.

Table 3 shows the solar sail-related parameters. The optical parameters related to the surface of the sail are determined by referring to the actual data of IKAROS, JAXA’s solar sail.\textsuperscript{11,12}

$L_2$ is located on the line connecting the Sun with the Earth and is hidden entirely behind the Earth shadow as shown in Fig. 4. To ensure a stable solar power supply, the path of the spacecraft must avoid intersecting this shadow area. The minimum orbital radius can be analytically obtained as

$$A_{\min} = \gamma(R_S + R_E) + R_E$$  \hspace{1cm} (48)

where $R_S$ and $R_E$ are the radius of the Sun and Earth. In this paper, the targeted $\gamma$ and $z$-amplitudes of the orbit are selected as 15,000 km, adding some margin to $A_{\min}$. Other orbit-related parameters are shown in Table 4.

### 4.3. Designing SE $L_2$ orbit by controlling the attitude

The orbit angular velocity $\omega$ is assumed to be 2.0 nd (non-dimensional). $A_x, A_y$, and $k_\theta$ are the orbit design parameters and we use the values shown in Table 4. Equations (25) and (26) give $x_0$ and $\alpha$. $A_x$ and $A_y$ are calculated from the relations, $A_x = A_x/\alpha$ and $A_y = A_y$. The values $k_1$ and $k_2$ are obtained from Eq. (22). Then, we can calculate the attitude control law of Eq. (27), that is, the spacecraft attitudes $\psi$ and $\varphi$ should move satisfying this equation to generate the orbit.

The initial guess of the state is approximated by Eq. (24).
Numerical integration is conducted using Eq. (1) considering acceleration due to solar radiation pressure derived from Eq. (9). The surface normal can be obtained from Eq. (19) and the Sun direction from Eq. (17). The initial state is modified through the iterative calculation mentioned in Section 4.1 until $\|v_{xz}\|$ becomes sufficiently small (e.g., $|v_{xz}| < 10^{-7}$).

Figure 5 shows the designed small-amplitude periodic orbit around $L_2$. The arrows indicate the direction of acceleration due to solar radiation pressure. The figure suggests that the control law derived successfully generates a periodic orbit. The center of the orbit is shifted toward the direction of the Sun. Table 5 shows the values of the orbit parameters resulting from the iterative calculation. The orbit amplitude of each axis and the position of the equilibrium are slightly different from the analytical prediction because of the non-linear effects. The positional error between the initial point and the final point (i.e., the point after the period), is 7.74915 km. This value is sufficiently small compared to the amplitude of the periodic orbit.

Figure 6 shows acceleration of the spacecraft due to solar radiation pressure. The magnitude of acceleration is about $10^{-10}$ km/s$^2$. The change in the normal vector of the sail surface is shown in Fig. 7 in terms of $\psi$ and $\phi$. The vector move elliptically around the center (0, 0) in the $\psi\phi$-plane at the same angular velocity as $\omega$.

### 4.4. Designing SE $L_2$ orbit by utilizing attitude drift motion

The characteristics of this control strategy are that the amplitudes in the $x$ and other directions are determined as functions of these parameters and that they cannot be chosen freely.

The same iteration procedure as Section 4.3 is conducted. Suppose the a priori parameters of Tables 1, 2, 3 and 4 are given. First, we compute the orbit-related parameters. The orbit angular velocity $\omega$ is assumed to be 1.96645 nd. Equation (40) gives the equilibrium of $\psi$ and $\phi$. The approximate initial condition is obtained from Eq. (34). Equations (35)–(38) give $x_e$, $z_e$, $\alpha$ and $\beta$. The values $k_i$ ($1 \leq i \leq 6$) are obtained from Eq. (32). Next, the attitude-related parameters
are calculated. The amplitude of attitude drift motion is obtained from Eq. (41). The surface normal can be obtained from Eq. (28) and the Sun direction from Eq. (17). Then, the equations of orbital motion of Eq. (1) and attitude motion of Eq. (16) are simultaneously integrated.

Plots of the small periodic orbit designed using attitude drift motion are shown in Fig. 8, and its specifications are summarized in Table 6. Figure 9 shows acceleration due to solar radiation pressure in each direction. The remarkable feature of the orbit is that it has positional shift in the $x$ and $z$-directions. In this case, the modified equilibrium point is $(x_e, y_e, z_e) = (1653.83 \text{ km}, 10.1471 \text{ km}, 1464.32 \text{ km}).$ The orbit is also shifted in the $y$-direction because of the non-linear effect. The ratio of the $yz$-orbital amplitude is uniquely determined by a combination of the parameters, which results in an approximately-circular orbit in this case. Figure 10 shows the direction of the normal vector of the surface. $\psi$ and $\varphi$ make circular motion around their equilibrium point (i.e., $(\psi_e, \varphi_e) = (-3.31515 \cdot 10^{-3} \text{ deg}, -2.91385 \cdot 10 \text{ deg})$).

5. Conclusions

In this paper, we formulated two control strategies to achieve a small periodic orbit around SE $L_2$ by utilizing solar radiation pressure via a solar sail.

First, a station-keeping technique using attitude control was developed. The attitude of the spacecraft (i.e., the direction of the normal vector of the surface) was represented by two angles measured from the direction of the Sun. In order to maintain the orbit, an analysis indicated that these angles...
should be changed in accordance to Eq. (27). The equilibrium point was shifted toward the direction of the Sun as a result of the constant pressure along the x-axis, as shown in Eq. (24).

Second, a method of using attitude drift motion from solar radiation pressure was presented. This method requires that the spacecraft is spinning and the spinning axis vector points toward the Sun. The key to obtaining control of the motion was to synchronize the frequency of the orbit and attitude, toward the Sun. The control derived assumed that was to synchronize the frequency of the orbit and attitude, as expressed in Eq. (39). The control derived assumed that the attitude of the spacecraft changed in accordance to Eq. (14), and the periodic orbit was obtained by selecting the orbital parameters to satisfy Eqs. (35)–(41). In contrast to the former orbit, this orbit shifted in the x and z-directions, as shown in Eq. (34).

These strategies were simulated via numerical calculations. Figure 5 shows the orbit obtained when the first control strategy was applied, and Fig. 8 shows the results of the second strategy. The second strategy did not accept an arbitrary orbital amplitude and it depends on the angular velocity of the orbit.

Further advancements of this study would be optimization of the design parameters. An appropriate combination of parameters could reduce the size of the sail.

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Appendix A

The transformation matrix from the Sun-pointing rotating coordinate to spin-free coordinate is

\[
R_{\text{SunPointing}}^{\text{SpinFree}} = R_s(-\varphi)R_y(\psi) = \begin{pmatrix}
\cos \varphi & 0 & -\sin \varphi \\
-\sin \varphi \sin \psi & \cos \varphi - \sin \varphi \cos \psi & \\
\cos \varphi \sin \psi & \sin \varphi & \cos \varphi \cos \psi
\end{pmatrix}
\]

where \( R_s \) and \( R_y \) are the rotation matrix about the x-axis and y-axis, respectively.

Appendix B

The Sun direction \( s \) and the surface normal \( n \) in the Sun-pointing coordinate are

\[
s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

\[
n = \begin{pmatrix} \cos \varphi \sin \psi \\ \sin \varphi \\ \cos \varphi \cos \psi \end{pmatrix}
\]

The torque due to solar radiation pressure \( T \) is expressed as

\[
T_{\text{SunPointing}} = l \times F_{\text{SunPointing}} = P S l(C_{\text{diff}} + C_{\text{abs}}) \begin{pmatrix} -\cos \varphi \sin \varphi \cos \psi \\ \cos^2 \varphi \cos \varphi \sin \psi \\ 0 \end{pmatrix} \]

where \( l \) denotes the lever arm vector between the center of mass and the pressure. \( F \) is the force due to solar radiation pressure. Multiplying the rotation matrix yields the torque expressed in the spin-free spacecraft-fixed frame as

\[
T_{\text{SpinFree}} = R_{\text{SpinFree}}^{\text{SunPointing}} T_{\text{SunPointing}} = P S l(C_{\text{diff}} + C_{\text{abs}}) \begin{pmatrix} -\cos \varphi \sin \varphi \cos^2 \psi \\ \cos \varphi \cos \varphi \sin \psi \\ 0 \end{pmatrix}
\]

The Euler equations of motion in the spin-free spacecraft-fixed frame are

\[
\frac{dH_{S/C}}{dt} + w_{\text{SpinFree}} \times H_{S/C} = T_{\text{SpinFree}}
\]
The angular velocity \( \omega _{x} \) and \( \omega _{y} \) can be rewritten as

\[
\begin{align*}
\omega _{x} &= -\dot{\psi} \\
\omega _{y} &= (n + \dot{\psi}) \cos \phi
\end{align*}
\]

(57)

where \( n \) is the mean motion of the spacecraft around the Sun. Substituting them to Eq. (56) gives

\[
\begin{align*}
\ddot{\psi} - \Omega _{n} (n + \dot{\psi}) \cos \phi &= p \sin \phi \cos \phi \cos ^{2} \psi \\
\dot{\psi} \cos \phi - (n + \dot{\psi}) \dot{\psi} \sin \phi + \Omega _{n} \dot{\psi} &= p \cos \phi \sin \phi \cos \psi
\end{align*}
\]

Expanding Eq. (58) into Taylor series around the equilibrium and abbreviating high-order terms yields

\[
\begin{align*}
\ddot{\psi} - \frac{\Omega _{n} \sqrt{p^{2} - n^{2} \Omega _{n}^{2}}}{p} \ddot{\psi} &= \frac{p^{2} - n^{2} \Omega _{n}^{2}}{p} \ddot{\psi} = 0 \\
\dot{\psi} + \sqrt{p^{2} - n^{2} \Omega _{n}^{2}} \dot{\psi} - p \psi &= 0
\end{align*}
\]

(63)

The characteristic equation of the system is

\[
\lambda ^{4} + \left(-2p + \frac{2 \Omega _{n}^{2} n^{2}}{p} + \Omega _{n}^{2}\right) \lambda ^{2} + p^{2} - \Omega _{n}^{2} n^{2} = 0
\]

(64)

Because the orbital frequency is small, the equation of motion can be simplified

\[
\begin{align*}
\ddot{\psi} - \Omega _{n} \ddot{\psi} - n \Omega _{n} \dot{\psi} &= 0 \\
\dot{\psi} + \Omega _{n} \dot{\psi} - p \psi &= 0
\end{align*}
\]

(65)

and its characteristic equation is

\[
\lambda ^{4} + \left(-2p + \Omega _{n}^{2}\right) \lambda ^{2} + p^{2} = 0
\]

(66)

The eigenvalues are

\[
\lambda = \pm \sqrt{-\frac{\left(-\Omega _{n}^{2} - p\right)}{2} \pm \sqrt{\frac{\Omega _{n}^{4} - 4p \Omega _{n}^{2}}{4}}}
\]

\[
\approx \pm \frac{\Omega _{n}}{2} \left(1 \pm \sqrt{1 - \frac{4p}{\Omega _{n}^{2}}}\right) i
\]

(67)

Considering the relation, \( p/\Omega _{n}^{2} \ll 1 \), they are approximated as

\[
\lambda = \pm \Omega _{n} i, \; \pm \frac{p}{\Omega _{n}} i
\]

(68)

Since \( p \) is a function of solar radiation pressure, its value is very small, resulting in the relation \( \Omega _{n} \gg p/\Omega _{n} \). The short period motion corresponding to the eigenvalue \( \pm \Omega _{n} i \) is called as the nutation due to solar radiation pressure and the long period motion of the eigenvalue \( \pm p/\Omega _{n} i \) is the precession. The solutions of Eq. (65) are approximated as

\[
\begin{align*}
\psi &= -A_{p} \sin \left(\frac{p}{\Omega _{n}} t + \theta _{p}\right) - A_{N} \sin (\Omega _{n} t + \theta _{N}) \\
\varphi &= A_{p} \cos \left(\frac{p}{\Omega _{n}} t + \theta _{p}\right) + A_{N} \cos (\Omega _{n} t + \theta _{N}) - \frac{\Omega _{n}}{p}
\end{align*}
\]

(69)