In this paper, we construct a solution to a linear matrix interval equation of the form $X = AX + B$ (the discrete stationary Bellman equation) over partially ordered semirings, including the semiring $\mathbb{R}_+$ of nonnegative real numbers and all idempotent semirings. We also discuss the computational complexity of problems in interval idempotent linear algebra (for more detail on idempotent mathematics, see, e.g., [1, 2]). In traditional interval analysis, problems of this kind are generally $NP$-hard [3, 4]. In this paper, we consider matrix equations over positive semirings (in the sense of [5]); in this case the computational complexity of the problem is polynomial.

Idempotent and other positive semirings naturally arise in optimization problems. Many of these problems turn out to be linear over appropriate idempotent semirings [1, 2]. In this case, the system of equations $X = AX + B$ is a natural analogue of a usual linear system in traditional linear algebra over fields. Carré [6] showed that many of the well-known algorithms of discrete optimization are analogous to standard algorithms in traditional computational linear algebra.

1. Consider a semiring, i.e., a set $S$ endowed with two associative operations, addition $\oplus$ and multiplication $\odot$, such that addition is commutative, multiplication is distributive over addition from either...
side, 0 and 1 are the respective neutral elements of addition and multiplication, \(0 \odot x = x \odot 0 = 0\) for all \(x \in S\), and \(0 \neq 1\). Let the semiring \(S\) be partially ordered by a relation \(\preceq\) such that 0 is the least element and the inequality \(x \preceq y\) implies that \(x \oplus z \preceq y \oplus z\), \(x \odot z \preceq y \odot z\), and \(z \odot x \preceq z \odot y\) for all \(x, y, z \in S\); in this case the semiring \(S\) is called positive (see, e.g., [3]).

A semiring \(S\) is called idempotent if \(x \oplus x = x\) for all \(x \in S\) [1, 2, 7]. Addition \(\oplus\) defines a canonical partial order \(\preceq\) on \(S\) by the rule \(x \preceq y\) iff \(x \oplus y = y\). Any idempotent semiring is positive with respect to this order. Note also that \(x \oplus y = \sup\{x, y\}\) with respect to the canonical order. In what follows, we assume that all idempotent semirings are ordered by the canonical partial order relation.

The best known and most important examples of positive semirings are “numerical” semirings consisting of (a subset of) real numbers and ordered by the conventional linear order \(\leq\) on \(\mathbb{R}\): the semiring \(\mathbb{R}_+\) with the usual operations \(\oplus = +, \odot = \cdot\) and neutral elements \(0 = 0, 1 = 1\); the semiring \(\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}\) with the operations \(\oplus = \max, \odot = +\) and neutral elements \(0 = -\infty, 1 = 0\); the semiring \(\hat{\mathbb{R}}_{\max} = \mathbb{R}_{\max} \cup \{\infty\}\), where \(x \preceq \infty, x \oplus \infty = \infty\) for all \(x, x \odot \infty = \infty \odot x = \infty\) if \(x \neq 0\), and \(0 \odot \infty = \infty \odot 0\); and the semiring \(S_{\max,\min}^{[a,b]} = [a, b]\), where \(-\infty \leq a < b \leq +\infty\), with the operations \(\oplus = \max, \odot = \min\) and neutral elements \(0 = a, 1 = b\). The semirings \(\mathbb{R}_{\max}, \hat{\mathbb{R}}_{\max}, \text{ and } S_{\max,\min}^{[a,b]} = [a, b]\) are idempotent.

Many mathematical constructions, concepts, and results over the fields of real and complex numbers have nontrivial analogues over idempotent semirings. Idempotent semirings have recently become the subject of a new branch of mathematics, idempotent analysis [1, 2, 7].
Let a positive semiring $S$ be endowed with a partial unary closure operation $\ast$ such that $x \leq y$ implies $x^\ast \leq y^\ast$ and $x^\ast = 1 \oplus (x^\ast \odot x) = 1 \oplus (x \odot x^\ast)$ on its domain of definition. In particular, $0^\ast = 1$ by definition. These axioms imply that $x^\ast = 1 \oplus x \oplus x^2 \oplus \cdots \oplus (x^\ast \odot x^n)$ if $n \geq 1$. Thus $x^\ast$ can be considered as a ‘regularized sum’ of the series $x^\ast = 1 \oplus x \oplus x^2 \oplus \cdots$; in an idempotent semiring, by definition, $x^\ast = \sup \{1, x, x^2, \ldots\}$ if this supremum exists.

In numerical semirings, the operation $\ast$ is defined as follows: $x^\ast = (1 - x)^{-1}$ if $x < 1$ in $\mathbb{R}_+$, $x^\ast = 1$ if $x \leq 1$ in $\mathbb{R}_{\max}$ and $\hat{\mathbb{R}}_{\max}$, $x^\ast = \infty$ if $x > 1$ in $\hat{\mathbb{R}}_{\max}$, $x^\ast = 1$ for all $x$ in $S_{\max,\min}^{[a,b]}$. In all other cases, $x^\ast$ is undefined. Note that the operation $\ast$ is defined everywhere in idempotent semirings that are $a$-complete in the sense of [7] (e.g., in $\hat{\mathbb{R}}_{\max}$ or $S_{\max,\min}^{[a,b]}$). For more detail, see [5].

2. Let $S$ be a set partially ordered by a relation $\preceq$. A closed interval in $S$ is a subset of the form $x = [\underline{x}, \overline{x}] = \{x \in S \mid \underline{x} \preceq x \preceq \overline{x}\}$, where the elements $\underline{x} \preceq \overline{x}$ are called lower and upper bounds on the interval $x$. The order $\preceq$ induces a partial ordering on the set of all closed intervals in $S$: $x \preceq y$ iff $\underline{x} \preceq \underline{y}$ and $\overline{x} \preceq \overline{y}$.

A weak interval extension $I(S)$ of a positive semiring $S$ is the set of all closed intervals in $S$ endowed with the operations $\oplus$ and $\odot$ defined by $x \oplus y = [\underline{x} \oplus \underline{y}, \overline{x} \oplus \overline{y}]$, $x \odot y = [\underline{x} \odot \underline{y}, \overline{x} \odot \overline{y}]$ and with a partial order induced by the order in $S$. The closure operation in $I(S)$ is defined by $x^\ast = [\underline{x}^\ast, \overline{x}^\ast]$ (see also [3]; for interval analysis over $\mathbb{R}$, see, e.g., [8]).

**Proposition 1** The weak interval extension $I(S)$ of a positive semiring $S$ is closed under the operations $\oplus$ and $\odot$ and forms a positive semiring with a zero element $[0, 0]$ and a unit element $[1, 1]$. The interval $x \oplus y$ ($x \odot y$) contains the set $\{x \oplus y \mid x \in x, y \in y\}$ ($\{x \odot y \mid x \in x, y \in y\}$, respectively) and its bounds are elements of this set.
3. Denote by Mat\(_{mn}(S)\) a set of all matrices \(A = (a_{ij})\) with \(m\) rows and \(n\) columns, whose coefficients belong to a semiring \(S\). The sum \(A \oplus B\) of matrices \(A, B \in \text{Mat}_{mn}(S)\) and the product \(AB\) of matrices \(A \in \text{Mat}_{lm}(S)\) and \(B \in \text{Mat}_{mn}(S)\) are defined according to the usual rules of linear algebra. If the semiring \(S\) is positive, then the set \(\text{Mat}_{mn}(S)\) is ordered by the relation \(A = (a_{ij}) \preceq B = (b_{ij})\) iff \(a_{ij} \preceq b_{ij}\) in \(S\) for all \(1 \leq i \leq m, 1 \leq j \leq n\).

Matrix multiplication is consistent with the order \(\preceq\) in the following sense: if \(A, A' \in \text{Mat}_{lm}(S), B, B' \in \text{Mat}_{mn}(S)\) and \(A \preceq A', B \preceq B'\), then \(AB \preceq A'B'\) in \(\text{Mat}_{ln}(S)\). The set \(\text{Mat}_{nn}(S)\) of square matrices of order \(n\) over a (positive, idempotent) semiring \(S\) forms a (positive, idempotent) semiring with a zero element \(O = (o_{ij})\), where \(o_{ij} = 0, 1 \leq i, j \leq n\), and with a unit element \(E = (\delta_{ij})\), where \(\delta_{ij} = 1\) if \(i = j\) and \(\delta_{ij} = 0\) in the opposite case.

The closure operation in matrix semirings over a positive semiring \(S\) can be defined inductively (for another way of doing this, see [3]): \(A^* = (a_{11})^* = (a^*_{11})\) in \(\text{Mat}_{11}(S)\) and for any integer \(n > 1\) and any matrix

\[
A = \begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix},
\]

where \(A_{11} \in \text{Mat}_{kk}(S), A_{12} \in \text{Mat}_{kn-k}(S), A_{21} \in \text{Mat}_{n-kk}(S), A_{22} \in \text{Mat}_{n-kn-k}(S)\), \(1 \leq k \leq n\), by definition,

\[
A^* = \begin{pmatrix}
A^*_{11} \oplus A^*_{11}A_{12}D^*A_{21}A^*_{11} & A^*_{11}A_{12}D^* \\
D^*A_{21}A^*_{11} & D^*
\end{pmatrix},
\]

where \(D = A_{22} \oplus A_{21}A^*_{11}A_{12}\). It can be proved that this definition of \(A^*\) implies that \(A^* = A^*A \oplus E\), and, thus, \(A^*\) is a “regularized sum” of the series \(E \oplus A \oplus A^2 \oplus \ldots\).

Note that this recurrence relation coincides with the formulas of the escalator method for matrix inversion in traditional linear
algebra over the field of real or complex numbers up to the algebraic operations used. Hence this algorithm of matrix closure is polynomial in $n$.

Let $S$ be a positive semiring and $A = (a_{ij}) \in \text{Mat}_{mn}(I(S))$ be a matrix whose coefficients are closed intervals in $S$. The matrices $L(A) = (\underline{a}_{ij}), U(A) = (\overline{a}_{ij}) \in \text{Mat}_{mn}(S)$ are called the lower and the upper matrices of the interval matrix $A$. Evidently, $L(A) \preceq U(A)$ in $\text{Mat}_{mn}(S)$.

Since, for any positive semiring $S$, the sets $I(S)$ and $\text{Mat}_{nn}(S)$ form positive semirings, the sets $I(\text{Mat}_{nn}(S))$ and $\text{Mat}_{nn}(I(S))$ form positive semirings with respect to the operations defined above.

**Proposition 2** The semirings $I(\text{Mat}_{nn}(S))$ and $\text{Mat}_{nn}(I(S))$ are isomorphic to each other, and the isomorphism is defined by $A \in \text{Mat}_{nn}(I(S)) \mapsto [L(A), U(A)] \in I(\text{Mat}_{nn}(S))$.

By definition, the addition and multiplication of matrix intervals in $I(\text{Mat}_{nn}(S))$ are reduced to separate matrix addition and multiplication of their lower and upper matrices. An analogous statement for lower and upper matrices in $\text{Mat}_{nn}(I(S))$ follows from the last proposition.

4. Let $S$ be a positive semiring. The discrete stationary Bellman equation has the form

$$X = AX \oplus B,$$

where $A \in \text{Mat}_{nn}(S)$, $X, B \in \text{Mat}_{ns}(S)$, and the matrix $X$ is unknown. Let $A^*$ be the closure of the matrix $A$. It follows from the identity $A^* = A^* A \oplus E$ that the matrix $A^* B$ satisfies this equation; moreover, it can be proved that, in idempotent semirings, this solution is the least in the set of solutions to equation (*) with respect to the partial order in $\text{Mat}_{ns}(S)$. 

5
Let \( A = A \in \text{Mat}_{nn}(I(S)) \), \( B = B \in \text{Mat}_{ns}(I(S)) \). The unified (least) solution set of equation \( (*) \) is the set \( \Sigma(A, B) = \{ A^*B \mid A \in A, B \in B \} \). The interval \( X = A^*B \in \text{Mat}_{ns}(I(S)) \) that satisfies equation \( (*) \) in the algebraic sense is called the (least) algebraic solution to this equation. Other definitions of a solution set of a matrix interval linear equation can be found, for example, in [4].

Our main result is the following theorem.

**Theorem**  The closed interval \([L(A^*B), U(A^*B)]\) in \( \text{Mat}_{ns}(S) \) that corresponds to an algebraic solution to equation \( (*) \) contains the unified solution set \( \Sigma(A, B) \) of equation \( (*) \), and the bounds of this interval belong to \( \Sigma(A, B) \). The algebraic solution \( A^*B \) can be constructed in a polynomial number of operations.

The proof follows from the fact that matrix multiplication and closure are consistent with the partial order in a matrix semiring. The lower and the upper matrices of an algebraic solution \( A^*B \) to equation \( (*) \) satisfy the point equations \( X = AX \oplus B \) and \( X = A^*X \oplus B \), and algebraic solutions to these equations can be constructed by the matrix closure algorithm described in section 3, which is polynomial in \( n \).

Note that this theorem was proved in the paper [9] (see also [10], Theorem 12.2) in the case of interval linear algebra over the semiring \( \mathbb{R}_+ \) of nonnegative real numbers.

Under some natural additional conditions on the operations \( \oplus, \circ \), and \( * \) a stronger equality \( [L(A^*B), U(A^*B)] = \Sigma(A, B) \) holds in the case of an idempotent semiring \( S \). Moreover, as far as we know, there are no \( NP \)-hard computational problems in interval linear algebra in this case. This is consistent with the general observation that idempotent analogues of constructions in traditional mathematics over numerical fields are considerably simpler than
their prototypes [1, 2, 7].

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