Motif Patterns and Coverings of Points with Unit Disks, Part I

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Abstract

We consider a modification of Winkler’s “dots and coins” problem, where we constrain the dots to lie on a square lattice in the plane. We construct packings of “coins” (closed unit disks) using motif patterns.

1 Winkler’s problem

In his “Puzzled” column [3], Peter Winkler discusses the following problem:

What is the largest integer $k$ such that any $k$ points in the plane, no matter how they are arranged, can always be covered with disks with pairwise-disjoint interior having radius 1?

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Winkler states \cite{4} that there exists a constructive proof that gives a covering for any set of 12 points. In \cite{2}, the authors give a configuration of 53 points on a triangular lattice that cannot be covered with unit disks. Hence \(12 \leq k < 53\). This is a challenging problem, one that is likely to generate some interesting mathematics.

We might consider modifying the problem by constraining the locations of the points in some way; for instance, they could be restricted to lying on a square or triangular lattice. In this essay, we take up the question of covering the points of the square lattice:

For which \(d > 0\) is it possible to cover all the points of the square lattice with inter-point distance \(d\) (i.e., \((d\mathbb{Z}) \times (d\mathbb{Z})\)) with disks with pairwise-disjoint interior having radius 1?

Call this lattice \(L_d\). The principal result of this essay is the following:

**Theorem 1.** For all \(d \in \left[\frac{2}{\sqrt{13}}, \frac{1}{\sqrt{2}}\right] \cup \left[\frac{4}{\sqrt{26}}, \infty\right)\), \(L_d\) can be covered with unit disks with pairwise-disjoint interior.

Note: unlike in \cite{2}, we consider closed disks of unit radius.

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Figure 1: \(L_d\)
2 Definitions, Theorems, and a Method

It is easy to see that if \( d \geq 2 \), we can cover all the points of \( L_d \) by giving each point its own disk. What about \( d < 2 \)?

Consider \( d = \sqrt{2} \): a circle with unit radius circumscribes a square with side length \( \sqrt{2} \), so we can cover all the points, four at a time, with unit disks:

![Figure 2: A covering of \( L_{\sqrt{2}} \)](image)

It is not hard to see that this same strategy will work for smaller values of \( d \); the only thing we need to worry about is making sure the disks do not overlap. In fact, this configuration works for all \( 1 \leq d \leq \sqrt{2} \). This line of reasoning will be made more rigorous in Theorem 4.

Let’s try to extract the essential features of the previous example:

- a finite number of points were selected;
- all of those points were within one unit of the “center”;
- distinct “centers” were at least two units apart.

We codify this method in the form of a theorem. First we need a few definitions. The first is from [1].
Definition 2. A motif is a non-empty plane set. A motif pattern $\mathcal{P}$ with motif $M$ is a non-empty family $\{M_i : i \in I\}$ such that

(i.) $\forall i, M_i$ is congruent to $M$;

(ii.) $\forall i \neq j, M_i \cap M_j = \emptyset$;

(iii.) $\forall i, j$, there exists an isometry of the plane mapping $\mathcal{P}$ onto itself and $M_i$ onto $M_j$.

![Figure 3: A Motif Pattern](image)

Definition 3. Let $\mathcal{P}$ be a motif pattern with motif $M$. Let $p$ be a point on $M$, called the center, and let $p_i$ be the corresponding (center) point on $M_i$. Define

$$\alpha = \min \{d(p_i, p_j) : i \neq j\}$$

and

$$\beta = \max \{d(p, x) : x \in M\}.$$ 

Then the motif pattern $\mathcal{P}$ is admissible if $2\beta \leq \alpha$.

We consider only closed and bounded motif patterns. See Figure 4 for an illustration of an admissible motif pattern with $\beta$ indicated, as well as two candidates for $\alpha$. The following theorem is our “workhorse”:
Theorem 4. Let $\mathcal{P}$ be an admissible motif pattern. Suppose that $L_1 \subseteq \mathcal{P}$, i.e. $\mathcal{P}$ covers all the points of $L_1$. Then $L_d$ can be covered by unit disks with pairwise-disjoint interior for all $d \in [2/\alpha, 1/\beta]$.

Proof. Let $\mathcal{P}$ be an admissible motif pattern with motif $M$ that covers every point on $L_1$. Let $d \in [2/\alpha, 1/\beta]$. Place a disk $D_i$ of radius $\frac{1}{d}$ at the center $p_i$ of each copy $M_i$ of $M$. We have, by assumption, that

$$L_1 \subset \bigcup_i M_i.$$  

Since $\frac{1}{d} \geq \beta$,

$$L_1 \subset \bigcup_i M_i \subseteq \bigcup_i D_i.$$  

Since $\frac{2}{d} \geq \alpha$, distinct disks $D_i$ and $D_j$ have disjoint interiors. Now dilate $L_1$ and each $D_i$ by a factor of $d$. Then we have a covering of $L_d$ by units disks, as desired.

3 Motif Patterns and Coverings of $L_d$

In light of the previous theorem, what remains to be done is to find admissible motif patterns. In Figures 5-11 below, several motif patterns are given. The
centers of the motifs are not indicated, since they are right where you think they should be: at the centers of mass of the motifs.

Figure 5: A Motif Pattern for the Interval $\left[ \frac{2}{\sqrt{13}}, \sqrt{\frac{2}{3}} \right]$

Figure 6: A Motif Pattern for the Interval $\left[ \sqrt{\frac{2}{5}}, \frac{2}{3} \right]$
Figure 7: A Motif Pattern for the Interval $\left[\frac{2}{3}, \frac{1}{\sqrt{2}}\right]$

Figure 8: A Motif Pattern for the Interval $\left[\frac{4}{\sqrt{26}}, \frac{2}{\sqrt{5}}\right]$
Figure 9: A Motif Pattern for the Interval $\left[\frac{2}{\sqrt{3}}, 1\right]$

Figure 10: A Motif Pattern for the Interval $[1, \sqrt{2}]$
4 Further Questions

There is an annoying gap between $\frac{1}{\sqrt{2}}$ and $\frac{4}{\sqrt{20}}$, of width approximately 0.077, that contains $3/4$. All of the authors believe that the set of all $d$ such that $L_d$ can be covered with unit disks is an interval, but we have no proof; nor do we have a way of closing this gap (yet).

**Problem 1.** Is $L_{\frac{3}{4}}$ coverable?

There is also the question of a lower bound. In [2], the authors argue that for $d < 2(\frac{2\sqrt{3}}{3} - 1)$, $L_d$ cannot be covered with disjoint unit disks. The present authors believe the true lower bound to be closer to $1/2$.

**Problem 2.** Find a lower bound on

$$\{d > 0 : L_d \text{ can be covered with unit disks with disjoint interior}\}.$$  

Naturally, we could also consider the triangular lattice.

**Problem 3.** Let $T_d$ denote the triangular lattice with inter-point distance $d$. Determine for which $d > 0$ $T_d$ can be covered with unit disks with disjoint interior.
Now let us return to a question more in the spirit of the original problem.

**Problem 4.** What is the largest integer \( \ell \) so that for all \( d > 0 \), any set of \( \ell \) points on \( L_d \) can be covered with unit disks with disjoint interior?

By Winkler’s argument, \( \ell \geq 12 \). In [2], the authors note that their method, using the square lattice instead of the triangular lattice, gives a set of 102 points that cannot be covered. So \( 12 \leq \ell < 102 \).

We will take up some of these questions in a subsequent essay.

## 5 Acknowledgments

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## References

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