A direct approach to the electromagnetic Casimir energy in a rectangular waveguide

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Abstract

In this paper we compute the leading-order Casimir energy for the electromagnetic field (EM) in an open-ended perfectly conducting rectangular waveguide in three spatial dimensions by a direct approach. For this purpose, we first obtain the second quantized expression for the EM field with boundary conditions which would be appropriate for a waveguide. We then obtain the Casimir energy by two different procedures. Our main approach does not contain any analytic continuation techniques. The second approach involves the routine zeta function regularization along with some analytic continuation techniques. Our two approaches yield identical results. This energy has been calculated previously for the EM field in a rectangular waveguide using an indirect approach invoking analogies between EM fields and massless scalar fields, and using complicated analytic continuation techniques, and the results are identical to ours. We have also calculated the pressures on different sides and the total Casimir energy per unit length, and plotted these quantities as a function of the cross-sectional dimensions of the waveguide. We also present a physical discussion about the rather peculiar effect of the change in the sign of the pressures as a function of the shape of the cross-sectional area.

1. Introduction

The Casimir effect is the physical manifestation of the change in the zero point energy of a quantum field for different configurations. The zero point configuration refers to that in which there does not exist any on-shell physical excitation of the field. The difference in the configurations could arise from either the imposition of different boundary conditions on the fields or the presence of non-trivial spatial backgrounds (e.g. solitons). In 1948, Casimir predicted the existence of this effect as an attractive force between two infinite parallel uncharged perfectly conducting plates in vacuum [1]. This effect was subsequently observed experimentally by Sparnaay in 1958 [2] (for a general review on the Casimir effect, see [3–5]). Recently, similar measurements have been done for other geometries, and their precisions have been greatly improved [6–8]. The manifestations of the Casimir effects have been studied in many different areas of physics. For example, the magnitude of the cosmological constant has been estimated using the Casimir effect [9, 10]. The effect has also been studied within the context of string theory [11]. Recently this effect has been investigated in connection with the properties of the spacetime with extra dimensions [12]. The majority of the investigations related to the Casimir effect concern the calculation of this energy or the resulting forces for different fields in different geometries, such as parallel plates [1], cubes [13–18], cylinders [19–21] and spherical geometries [22, 23]. An interesting question is the determination of the conditions under which the forces acting on the boundaries for closed geometries are attractive or repulsive in arbitrary spatial dimensions [24–29]. We should mention that in the calculations of the Casimir energy many different regularization schemes or renormalization programmes have been used to remove the divergences, and some of these techniques have been compared with each other [30–35]. However, there are sometimes ambiguities associated with the analytic continuation techniques.

Ambjørn and Wolfram [36] were the first to calculate the Casimir energy in higher spacetime dimensions, and in particular derived an expression for the change in the vacuum...
energy due to a rectangular box with \( p \) sides in \( d \)-dimensions for a massless and a massive scalar field by summing the zero point energy of the eigenmodes. The divergences were removed by using the following regularization and analytic continuation procedures: zeta function regularization, dimensional regularization and the reflection formula. The results are given in terms of the Epstein zeta function \([37, 39]\). Then, using analogies between EM and massless scalar fields, the Casimir energy for the EM radiation was indirectly deduced for the TE and TM modes in a perfectly conducting rectangular box.

For the case of cavities, this energy has been calculated directly for both the EM field and the scalar fields. Usually two different methods are used. The more routine one involves the aforementioned regularization and analytic continuation procedures. The main ingredient of the second method is the subtraction of two comparable configurations, sometimes supplemented by some regularization procedures, such as the use of convergence factors. The latter procedure was first used by Boyer for the calculation of the EM Casimir energy in a spherical cavity \([40]\), where he subtracted the zero point energies of two concentric spheres, but with different sized inner cores. Analogous methods were used for two parallel plates \([41]\) and rectangular cavities \([17]\).

The primary purpose of this paper is to directly derive a closed form expression for the Casimir energy for the EM field in an open-ended rectangular geometry in three spatial dimensions, which we shall henceforth call a waveguide. We do this by directly finding the EM modes, quantizing the resulting field and calculating the zero point energy. We then obtain the Casimir energy using first the usual program which involves complicated analytic continuation techniques including the zeta function and the reflection formula and, second, a slight modification of Boyer’s method, henceforth called the box subtraction scheme (BSS). As we shall show, in the latter procedure there is no need for any use of analytic continuation techniques, and all of the divergences can be removed without any ambiguities. Both of the results turn out to be equivalent to those of \([36]\). Therefore, the secondary purpose of our work is to check the complicated analytic continuation techniques in common use. However, either of our direct approaches to the problem has the advantage of being easily extendable to higher orders in perturbation theory \([42, 43]\).

In section 2, we calculate the EM modes and the resulting zero point energy in a rectangular waveguide. In section 3, we first calculate the Casimir energy using the zeta function regularization, and then introduce the BSS to recalculate this energy. We then compare our results with those of \([36]\). We then show that our results for a rectangular waveguide of a cross-sectional area \( a_1 \times a_2 \) agree with the established results for a cavity in the appropriate limit. Then, we show that our results for the waveguide reduce to those of the two infinite parallel plates, in the appropriate limit. We then plot the Casimir energy per unit volume, its contour plot per unit volume and the Casimir energy per unit length, all as a function of the cross-sectional lengths of the waveguide. We then define and calculate the pressures on different sides and plot them. We also present a physical discussion about the rather peculiar effect of the change in the sign of one of the pressures as a function of the shape of the cross-sectional area. We summarize our results in section 4.

2. Zero point energy in a waveguide for the EM field

The Lagrangian density for the electromagnetic field is

\[
\mathcal{L} = \frac{1}{2}(E^2 - B^2).
\]

For a conducting uncharged waveguide the electric and magnetic fields can be easily written in terms of the vector potential, for example in Coulomb gauge (see \([44]\)). Therefore to derive the EM modes, it is sufficient to calculate the vector potential in the waveguide. However it is easier to find the physical fields \(E\) and \(B\), and then calculate the corresponding vector potential. This is due to the fact that it is easier to impose the boundary conditions on the physical fields. For this purpose, we only need to compute \(E_{x}\) for TM modes and \(B_{z}\) for TE modes, where \(z\) is defined to be the main axis of the waveguide. The rest of the components can be calculated using the Maxwell equations. Explicit expressions for all components of the EM fields are shown in appendix A. Now, for each mode the vector potential \(A^{\lambda}_{k}(\textbf{x}, t)\) can be written in terms of the electric field as

\[
A^{\lambda}_{k}(\textbf{x}, t) = -\frac{i e}{\omega_{k}} E^{\lambda}_{k}(\textbf{x}, t),
\]

and its conjugate momenta are defined as

\[
\Pi^{\lambda}_{k}(\textbf{x}, t) = \frac{1}{c} \frac{\partial}{\partial t} A^{\lambda}_{k}(\textbf{x}, t),
\]

where \(\lambda = [1, 2]\) indicate \(\text{TE}, \text{TM}\) modes, respectively, and

\[
\omega_{k} = c |\textbf{k}| = c \left( \frac{m \pi}{a_1} \right)^2 + \left( \frac{n \pi}{a_2} \right)^2 + k_{z}^2.
\]

Here \(a_1\) and \(a_2\) define the cross-sectional dimensions of the rectangular waveguide. Since the set of all modes in the waveguide are complete and orthonormal, one can expand the classical EM field in terms of these modes \([45]\),

\[
A(\textbf{x}, t) = \sum_{\lambda, m, n=1}^{\infty} \int \frac{L \, dk}{2\pi} \left[ C^{\lambda}_{mn}(\textbf{k}) A^{\lambda}_{mn}(\textbf{x}, t) + \text{c.c.} \right]
\]

\[
+ \sum_{m=1}^{\infty} \int \frac{L \, dk_{z}}{2\pi} \left[ C^{\text{TE}}_{m0}(\textbf{k}) A^{\text{TE}}_{m0}(\textbf{x}, t) + \text{c.c.} \right]
\]

\[
+ \sum_{n=1}^{\infty} \int \frac{L \, dk_{z}}{2\pi} \left[ C^{\text{TM}}_{m0}(\textbf{k}) A^{\text{TM}}_{m0}(\textbf{x}, t) + \text{c.c.} \right].
\]

where \(C^{\lambda}_{(k)}\) and \(C^{\lambda*}_{(k)}\) are the expansion coefficients. We use the canonical quantization method for quantizing the field in the waveguide. For this purpose, we let

\[
\begin{align*}
C^{\lambda}_{(k)} & \rightarrow \sqrt{\frac{\hbar c^2}{2a_0 L}} N^{\lambda}_{(k)} a_{\lambda}^{\dagger}(k), \\
C^{\lambda*}_{(k)} & \rightarrow \sqrt{\frac{\hbar c^2}{2a_0 L}} N^{\lambda*}_{(k)} a_{\lambda}(k),
\end{align*}
\]
where \( a^+ (k) \) and \( a^0 (k) \) are creation and annihilation operators, respectively. Now we impose the usual canonical commutation relations on the fields and their conjugate momenta:

\[
\left[ A_\ell^+ (x, t), A_j^0 (x', t) \right] = \left[ \Pi_\ell^0 (x, t), \Pi_j^+ (x', t) \right] = 0, \tag{7}
\]

and this will result in the usual canonical commutation relations between \( a^0 (k) \) and \( a^+ (k) \) [45]. Using this relation along with equation (6), we can find the normalization coefficients \( N^\ell (k) = w_k / c \).

According to equation (1), the Hamiltonian is

\[
H = \frac{1}{2} \int d^3 x \left( \left| E(x, t) \right|^2 + \left| B(x, t) \right|^2 \right). \tag{8}
\]

Integrating over the volume of the waveguide gives the energy, and the vacuum expectation value of energy inside the waveguide is

\[
\langle 0 | H | 0 \rangle = \int \frac{L}{2\pi} d k \sum_{m,n=0}^{\infty} (2 - \delta_{m0} - \delta_{n0}) \frac{\hbar o_k}{2}, \tag{9}
\]

which simply means that the electromagnetic vacuum energy inside the waveguide is the sum of the zero point energies of all possible modes. This sum and its analogues in any quantum field theory always turn out to be infinite.

3. The Casimir energy

We obtained an expression for the zero point energy for the EM field in a rectangular waveguide in the previous section. As mentioned earlier, the main purposes of this paper are first to obtain the Casimir energy using directly the EM field, and then to avoid any use of analytic continuation techniques, utilizing the BSS. Now we calculate the resulting Casimir energy by two different methods. First is the conventional method involving the usual regularization programmes and the ensuing analytic continuations, and second is our method. We shall find that these two methods yield identical results for the leading order case, which agrees with the results obtained indirectly in [36].

The total energy of the vacuum for the EM field inside the waveguide is given in equation (9). High frequency modes render these sums formally divergent. Our first procedure involves the usual zeta function regularization. That is, we shall compute the following expression:

\[
E_{\text{Cas}} = \frac{\hbar c^{d-2}}{16} \int \frac{L}{2\pi} d k \sum_{m,n=0}^{\infty} \sum_{l=0}^{\infty} \frac{\delta_{m0} - \delta_{n0} - \delta_{l0}}{2} a^0_k a^d_{k}, \tag{10}
\]

which is convergent for \( d < -1 \). By calculating the integral in equation (10) and using the definition of the Epstein zeta function [37], we obtain

\[
E_{\text{Cas}} = \frac{\hbar c^{d-2}}{16} \pi \frac{\Gamma \left( \frac{d-2}{2} \right)}{\Gamma \left( \frac{d+1}{2} \right)} \left[ Z_2 (a_1^{-1}, a_2^{-1}; 1 - d) \right. + \left. Z_2 (a_2^{-1}, a_1^{-1}; 1 - d) \right]. \tag{11}
\]

Note that the simple analytic continuation \( d \to 3 \) leads to a divergent result! However, we can use a simplified version of the reflection formula [36] applicable to this problem,

\[
\pi^{(d-2)/2} \Gamma \left( \frac{d-2}{2} \right) \frac{1 - d}{2} Z_2 (a_1^{-1}, a_2^{-1}; 1 - d) \tag{12}
\]

\[
= a_1 a_2 \pi^{(d+1)/2} \Gamma \left( \frac{d+1}{2} \right) Z_2 (a_1, a_2; d + 1). \tag{12}
\]

The analytic continuation embedded in the reflection formula eliminates all of the infinities, and the final expression for the Casimir energy in terms of the Epstein zeta function can be written as [37]

\[
E_{\text{Cas}} = \frac{-\hbar c L a_1 a_2}{32 \pi^2} \left[ Z_2 (a_1, a_2; 4) + Z_2 (a_2, a_1; 4) \right]. \tag{13}
\]

As outlined above, the first method uses complicated analytic continuation techniques. We share the point of view with some authors that in general, the use of analytic continuation techniques could lead to ambiguities [3, 38, 42, 43]. This ambiguity has already been shown up in our calculations, as explicitly stated in the sentence below equation (11). This is why we think it is worthwhile to obtain the Casimir energy by a method which does not use such techniques. As a first step towards this goal, we introduce two similar configurations, each of which consists of a waveguide enclosed in a larger one of a cross-sectional area \( R^2 \), as shown in figure 1. The Casimir energy can now be defined as

\[
E_{\text{Cas}} = \lim_{b/a, \{a_1, a_2, b_1, b_2\} \to \infty} \left[ \lim_{R/b \to \infty} (E_A - E_B) \right], \tag{14}
\]

where \( E_A (E_B) \) is the energy of configuration \( A \) (B), and \( a \equiv \max \{a_1, a_2\} \) and \( b \equiv \max \{b_1, b_2\} \). Subtraction of the zero point energy of \( B \) from \( A \) is equivalent to the work done in deforming the configuration from \( B \) to \( A \). Therefore, having chosen the same \( R \) for both configurations, we expect this quantity per unit length to be finite on physical grounds and to depend only on the dimensions of the original waveguide. To calculate the Casimir energy, it is necessary to have an expression for all the EM modes in the whole configuration. However, calculation of the EM modes in the middle regions is very cumbersome. Therefore, to simplify the task, without any loss of generality, we define an alternative set of configurations in figure 2. We can then define the Casimir energy as in equation (14), but with following replacement \( A \to A' \) and \( B \to B' \). As we shall show explicitly in appendix B, as expected, all of the infinities automatically cancel each other out upon subtracting the zero point energies of the two configurations, even for finite values of \( \{a_1, a_2, b_1, b_2, R\} \). It is important to note that for any finite value of \( R \), the Casimir energies of the two sets of configurations depicted in figures 1 and 2 will differ by a finite amount, after the automatic cancellation of all the infinities. However, as we shall show explicitly in appendix B, the difference between the remaining finite terms due to the boundary waveguides in the \( \{A', B'\} \) configuration goes to zero in the limit \( R \to \infty \). Therefore, the difference in the energies of the two sets of waveguides in the \( \{A', B'\} \) configuration in the limit \( R \to \infty \) is only due to the difference between the two inner waveguides, and this can be properly defined to be the Casimir energy.
We can solve the problem using the BSS by first using the Abel–Plana summation formula (APSF) [46] for the sums in our main equation for the zero point energy, equation (9),

\[
E_{\text{Cas}} = \lim_{b_1, b_2 \to \infty} \lim_{R \to \infty} \left( \sum_{n=1}^{\infty} g(n) \right)
\]

where \( g(n) \) is defined in appendix B. The calculations for each term in the integrand are very lengthy, and are done in appendix B. However, here we like to briefly outline the calculations, especially the cancellation of infinities. The first term in the integrand contains infinite terms, some of which, as expected, cancel each other out due to the BSS. However a few infinite terms remain. Similar cancellations occur for the second term. The remaining infinite terms of these two terms exactly cancel each other out. The third term (the branch-cut term) is finite. Therefore all the infinities cancel each other out due to the BSS, without resorting to any analytic continuation techniques, and more surprisingly, even without resorting to any regularization scheme. The Casimir energy can be easily obtained by collecting all the finite pieces from the above three terms which are all of branch-cut types and obtained explicitly in appendix B. The final expression for the Casimir energy, after taking the appropriate limit, is

\[
E_{\text{Cas}} = \frac{-\hbar c L \xi(3)}{32\pi} \left( \frac{1}{a_1^4} + \frac{1}{a_2^4} \right) - \frac{\hbar c L \pi^2 a_1 a_2}{1440} \left( \frac{1}{a_1^4} + \frac{1}{a_2^4} \right)
\]

\[\text{(15)}\]

One can show that this expression for the Casimir energy is identical to the previous one obtained by zeta function regularization, i.e. equation (13). One can also easily show that the Casimir energy obtained here directly from the second quantized form of the EM field for any waveguide in three spatial dimensions, using any of our two programmes, is identical with the results obtained indirectly in [36] using analogies between the massless scalar field and the EM field. As mentioned before, the Casimir problem inside a rectangular
Figure 3. The Casimir energy, per unit volume, for the EM field inside a perfectly conducting rectangular waveguide in three spatial dimensions with a cross-sectional area \(a_1 \times a_2\), in units \(\hbar c = 1\) and \(a_1 = 1\). Note that the asymptotic value is the Casimir energy for two infinite parallel plates \((-\hbar c\pi^2/720)\). When all the lengths are measured in units of mm and the energy in eV, the factor for converting the energy density to eV mm\(^{-3}\) is \(1.978 \times 10^{-4}\).

Figure 4. Contour plot of the Casimir energy, per unit volume, for the EM field inside a perfectly conducting rectangular waveguide, in units \(\hbar c = 1\).

cavity with perfectly conducting walls has been solved directly and exactly, although the final result is not in a closed form \[17\]. They have computed the limit of their expression reducing to a waveguide with a square cross section. We have computed the limit of their results for the slightly more general case of a waveguide with a rectangular cross section, and the results are identical to ours. The other extreme limit of the waveguide is when one of the sides approaches infinity, for example \(a_2\). Then, our result, equation (16), turns out to be exactly the Casimir energy for two infinite parallel plates \[1\],

\[
E_{\text{Cas}} = -\frac{\hbar c\pi^2 L a_2}{720a_1^3}.
\] (17)

The Casimir energy density is plotted in figure 3 and its contour plot is illustrated in figure 4. Note that in figure 4, the regions shaded darker correspond to lower energies.

We can define the Casimir pressure \(P_i\) on the plate of the waveguide whose perpendicular direction is \(i\), and its area is \(A_i\) by

\[
P_i = -\frac{1}{A_i} \frac{\partial E_{\text{Cas}}}{\partial a_i} \quad (i = 1, 2).
\] (18)

Figure 5. The Casimir energy, per unit length, as a function of \(a_2\) for the EM field inside a conducting rectangular waveguide, in units \(\hbar c = 1\) and \(a_1 = 1\). Note the existence of a maximum value signifies a change in the direction of the pressure. When all the lengths are measured in units of mm and the energy in eV, the factor for converting the total energy, per unit length, to eV mm\(^{-1}\) is \(1.978 \times 10^{-3}\).

In figure 5 we display the Casimir energy, per unit length, as a function of \(a_2\) for \(a_1 = 1\). Note that the existence of a maximum indicates that as \(a_2\) increases for fixed \(a_1\), the pressure \(P_2\) increases to a positive value after the critical value \(a_2 \approx 1.52\). When all the lengths are measured in mm, the factor for converting the pressure to \(\mu\)Pa is \(3.166 \times 10^{-8}\).

Figure 6. The Casimir pressure as a function of \(a_2\) for the EM field inside a conducting rectangular waveguide, in units \(\hbar c = 1\) and \(a_1 = 1\). Note \(P_2\) increases to a positive value after the critical value \(a_2 \approx 1.52\). When all the lengths are measured in mm, the factor for converting the pressure to \(\mu\)Pa is \(3.166 \times 10^{-8}\). Analogous findings have been reported for the rectangular cavity problem \[13, 17, 29\]. We have come up with the following physical reasoning for this phenomenon: as \(a_2\) increases, for fixed \(a_1\), past a critical threshold of about 1.52, the total energy starts to decrease although any two opposite pairs of sides always attract each other. That is, with any further increase in \(a_2\), the increase in the energy due to the attraction of the sides labelled \(A_2 = a_1 \times L\) is more than compensated by the reduction in the energy due to the attraction of the sides labelled \(A_1 = a_2 \times L\).
4. Conclusion

In this paper, we have obtained the Casimir energy for the EM field in a rectangular waveguide by directly finding the EM modes inside the waveguide, and summing over the zero modes of the corresponding second quantized EM field operator. We compute the sums using two completely different methods. The first is the zeta function analytic continuation method. The second is the BSS whose basis is confining the waveguide inside a large one and computing the difference in the vacuum energies in two comparable configurations. The results of these two techniques are identical. However, the latter provides a mechanism by which precise cancellations of divergences occur without using any analytic continuation or regularization schemes. Our results also turn out to be identical to those of [17, 36] in the appropriate limit. However in [36], the results were obtained indirectly by using some analogies between the EM field and massless scalar fields. The direct method has the added advantage of being easily extendable to calculations of radiative corrections to the Casimir energy. We have also computed and displayed the results for the Casimir energy, its density and pressures as a function of the cross-sectional sides of the waveguide. The physical reasons for the phenomena of the change in the sign of one of the pressures are also given.

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Appendix A. Electromagnetic modes in waveguides

In this appendix, we present explicit formulae for the TE and TM modes inside a rectangular waveguide with inner dimensions $a_1$ and $a_2$ [44]. For the TE waves the wave equation for $B_z$ is

$$\nabla^2 B_z(x, t) + k_0^2 B_z(x, t) = 0,$$

where $k_0$ is transverse momentum. The solution for $B_z$ satisfying the appropriate boundary conditions is

$$B_z = B_{z0} \cos \left(\frac{n\pi}{a_1}(x + \frac{a_1}{2})\right) \cos \left(\frac{m\pi}{a_2}(y + \frac{a_2}{2})\right),$$

and $k_0^2 = \left(\frac{m\pi}{a_2}\right)^2 + \left(\frac{n\pi}{a_1}\right)^2$. Now, we can use the Maxwell equations to obtain all the components of the EM fields:

$$B_{nm}(x, t) = B_{z0} \left[\frac{ik_z}{k_0^2} \left(\frac{n\pi}{a_1}(x + \frac{a_1}{2})\right) + \cos \left(\frac{m\pi}{a_2}(y + \frac{a_2}{2})\right) \hat{x} - \frac{m\pi}{a_2} \cos \left(\frac{n\pi}{a_1}(x + \frac{a_1}{2})\right) \sin \left(\frac{m\pi}{a_2}(y + \frac{a_2}{2})\right) \hat{y}\right] e^{ik_z z - i\omega t},$$

$$E_{nm}(x, t) = \frac{e^{ik_z z - i\omega t}}{k_0^2 c} \left[\frac{m\pi}{a_2} \cos \left(\frac{n\pi}{a_1}(x + \frac{a_1}{2})\right) \hat{x} + \frac{n\pi}{a_1} \sin \left(\frac{n\pi}{a_1}(x + \frac{a_1}{2})\right) \hat{y}\right] \sin \left(\frac{m\pi}{a_2}(y + \frac{a_2}{2})\right),$$

and

$$E_{z0} \sin \left(\frac{n\pi}{a_1}(x + \frac{a_1}{2})\right) \sin \left(\frac{m\pi}{a_2}(y + \frac{a_2}{2})\right) \hat{z}.$$
One of the main tools that we use to obtain the final results is the last line of equation (B.2). As we shall show explicitly below, the BSS, symbolically indicated in the last line of equation (B.2), renders that quantity finite. Equation (B.2) is manifestly symmetric in all its double arguments symbolically denoted by $x$ and $y$, and the calculations will greatly simplify if we make use of this symmetry in the following form:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} S_{m,n}(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{2} \left( S_{m,n}(x, y) + S_{m,n}(y, x) \right).$$

(B.4)

One of the main tools that we use to obtain the final results is the Abel–Plana summation formula:

$$\sum_{n=1}^{\infty} g(n) = \frac{-1}{2} g(0) + \int_{0}^{\infty} dx \, g(x) + i \int_{0}^{\infty} \frac{dt \, g(it) - g(-it)}{e^{2\pi t} - 1}. $$

(B.5)

The last term in the APSF is called the branch-cut term and always leads to finite results. Now we use the APSF for equation (B.2). The first term gives

$$\sum_{n=1}^{\infty} S_{n} = \sum_{n=1}^{\infty} \left\{ S_{n}(a_{1}) + S_{n}(a_{2}) + 2 \sum_{m=1}^{\infty} S_{m,n}(a_{1}, a_{2}) \right\} + 2 \sum_{n=1}^{\infty} \left[ S_{n} \left( \frac{R - a_{1}}{2} \right) + S_{n} \left( \frac{R + a_{2}}{2} \right) \right] + 2 \sum_{n=1}^{\infty} \left[ S_{n} \left( \frac{R + a_{1}}{2} \right) + S_{n} \left( \frac{R - a_{2}}{2} \right) \right]$$

+ $\left[ \{a_{1} \rightarrow b_{1}, a_{2} \rightarrow b_{2}\} \right].$

Note that each of the $k_{\pm}$ terms gives quadratically divergent terms when integrated. However, the BSS cancels all these terms between $A'$ and $B'$ configurations. Now we can use the APSF again for the remaining sums. To simplify the expression, we use the following change of variables:

$$\int_{0}^{\infty} \frac{m \pi \sqrt{x}}{\sqrt{x}} \, dt = \frac{m \pi}{\sqrt{x}} \int_{0}^{\infty} dt \sqrt{t^{2} + k^{2}}.$$ 

(B.7)

The result is

$$\int_{0}^{\infty} \frac{m \pi \sqrt{x}}{\sqrt{x}} \, dt = \frac{m \pi}{\sqrt{x}} \int_{0}^{\infty} dt \sqrt{t^{2} + k^{2}}.$$ 

(B.7)

where the branch-cut terms in this case become

$$B_{k}(x) = -2 \int_{0}^{\infty} \frac{\sqrt{\left( \frac{\pi t}{x} \right)^{2} - k^{2}}}{e^{2\pi t} - 1}.$$ 

(B.9)

As is apparent from this expression we have terms which again lead to divergent results when integrated, or are directly divergent. They include the previous $k_{\pm}$ terms along with new ones, involving square roots. Almost all of these terms cancel due to our BSS, with only the first square root term remaining. As we shall show, it will exactly cancel the analogous terms coming from the second term of the APSF, equation (B.5).

The remaining terms are

$$\int_{0}^{\infty} \frac{m \pi \sqrt{x}}{\sqrt{x}} \, dt = \frac{m \pi}{\sqrt{x}} \int_{0}^{\infty} dt \sqrt{t^{2} + k^{2}}.$$ 

(B.7)

The second term of APSF gives

$$\int_{0}^{\infty} \frac{m \pi \sqrt{x}}{\sqrt{x}} \, dt = \frac{m \pi}{\sqrt{x}} \int_{0}^{\infty} dt \sqrt{t^{2} + k^{2}}.$$ 

(B.7)

where

$$\int_{0}^{\infty} \frac{m \pi \sqrt{x}}{\sqrt{x}} \, dt = \frac{m \pi}{\sqrt{x}} \int_{0}^{\infty} dt \sqrt{t^{2} + k^{2}}.$$ 

(B.7)
The sum over \( j \) in the last equation comes from the Taylor expansion of the denominator of that particular branch-cut term, equation (B.14). An extremely important point to mention is that the remaining finite contribution to the Casimir energy coming from the outer waveguides, even after the BSS, is nonzero for finite values of the dimensions of the waveguides. However, as we shall show below, in the limit of large \( R \), there is partial cancellation between those terms, and the remaining terms go to zero in the limit \( R \to \infty \). This shows that the outer waveguides have done their job in the BSS of cancelling infinities, without leaving any finite contribution to the Casimir energy in the limit \( R \to \infty \). The final step in the calculation of the Casimir energy is a calculation of the limits in the appropriate order as indicated in equation (15). Then the final result is given in equation (16).

The finite contribution to the Casimir energy coming from the outer waveguides is

\[
E_{\text{Cas}}^{\text{Outer}} = \frac{-\hbar c L \zeta(3)}{8 \pi} - \frac{1}{(R-a_1)^2 + (R+a_2)^2}.
\]

Note that only terms of \( B \) cancel each other out. The even terms in \( R \) cancel each other out. The even terms are due to the fact that in the BSS, we have chosen the cross-sectional area of the outer waveguides to be equal for both \( A' \) and \( B' \) configurations. The third term of the APSF contains only two kinds of branch-cut terms. The first type is given in equation (B.9) and the second type is

\[
B(y, S_m(x)) = -2 \int_{\frac{\pi}{3}}^{\pi} \frac{d\theta}{\pi} \frac{\pi}{\sin(\theta)} - S_2^x(x) \frac{e^{2\pi\theta}}{e^{2\pi\theta} - 1}.
\]
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