TORELLI GROUPS AND JACOBIAN VARIETIES OF
NON-ORIENTABLE COMPACT KLEIN SURFACES

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Abstract. The Torelli group of a compact non-orientable Klein surface is the
subgroup of the modular group consisting of the mapping classes that act trivially
on the first homology group of the surface. We prove that if a surface has genus
at least 3, then the Torelli group acts fixed points free on the Teichmüller space of
the surface. That gives an embedding of the Torelli space of a Klein surface in the
Torelli space of its complex double. We also construct real tori associated to Klein
surfaces, which we call the Jacobian of the surface. We prove that this Jacobian is
isomorphic to a component of the real part of the Jacobian of the complex double.

1. Statement of results

Klein surfaces are the natural generalization of Riemann surfaces to the non-
orientable situation: one considers holomorphic and anti-holomorphic changes of
coordinates. One of the points of interest in the study of Klein surfaces is to de-

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termine which results of the theory of deformation of Riemann surfaces hold for the non-orientable case. A common approach to this problem is to consider a Klein surface $\Sigma$, as a Riemann surface $\Sigma^c$, with an anti-holomorphic involution $\sigma$, and thus one wants to find $\sigma$-invariant objects. In this paper we follow these two points of view to show two related results in the theory of Klein surfaces, that is, we will do some constructions on Klein surfaces, and then find the corresponding invariant objects related to the Riemann surface $\Sigma^c$. More precisely, we construct the Torelli space $\text{Tor}(\Sigma)$, and prove that it can be identified with the set of fixed points of an involution on $\text{Tor}(\Sigma^c)$. We also construct the Jacobian variety $J(\Sigma)$ of $\Sigma$ by integrating a basis of the space of real harmonic forms over the free part of $H_1(\Sigma, \mathbb{Z})$. We prove that $J(\Sigma)$ is isomorphic to a component of the real part of the Jacobian $J(\Sigma^c)$ of the complex double $\Sigma^c$ of $\Sigma$.

Given a compact smooth non-orientable surface $\Sigma$, the Teichmüller space $T(\Sigma)$ of $\Sigma$ is defined as $T(\Sigma) = \mathcal{M}(\Sigma)/\text{Diff}_0(\Sigma)$, where $\mathcal{M}(\Sigma)$ is the set of Klein surface structures on $\Sigma$ that agree with the given smooth structure, and $\text{Diff}_0(\Sigma)$ is the group of diffeomorphisms of $\Sigma$ homotopic to the identity [11, pg. 145]. We will use $\Sigma$ for a Klein surface, if it is clear from the context what the structure is, or we will write $(\Sigma, X)$ if we need to specify more. The modular or mapping class group, $\text{Mod}(\Sigma) = \text{Diff}(\Sigma)/\text{Diff}_0(\Sigma)$, acts on $T(\Sigma)$ by pull-back of dianalytic structures (see §2). The Torelli group $U(\Sigma)$ is the subgroup of $\text{Mod}(\Sigma)$ consisting of the mapping classes that act trivially on $H_1(\Sigma, \mathbb{Z})$. The parallel result to the following theorem is a classical fact on Riemann surfaces.

**Theorem 3.1.** Let $\Sigma$ be a compact non-orientable surface of genus $g \geq 3$. Let $[f] \in \text{Mod}(\Sigma)$, and suppose that there exists a Klein surface structure $X$ on $\Sigma$ such that $f : (\Sigma, X) \to (\Sigma, X)$ is dianalytic. Then $[f] = [\text{id}]$. Therefore, $U(\Sigma)$ acts fixed-points free on $T(\Sigma)$, and the Torelli space $\text{Tor}(\Sigma) = T(\Sigma)/U(\Sigma)$ is a smooth real manifold of dimension $3g - 6$. 
Assume now that $\Sigma$ has a fixed Klein surface structure. Then there exists an unramified double covering of $\Sigma$ by a Riemann surface $\Sigma^c$, known as the **complex double**. Moreover, $\Sigma$ is isomorphic to $\Sigma^c/\langle \sigma \rangle$, where $\sigma$ is an anti-holomorphic involution. The mapping $\sigma$ induces involutions $\sigma^*$ and $\tilde{\sigma}$ on $T(\Sigma)$ and $\text{Tor}(\Sigma)$, respectively. It is a well known fact that $T(\Sigma)$ can be identified with the set of fixed points of $\sigma^*$. A similar result holds for Torelli spaces, as the next proposition shows.

**Proposition 3.3.** The Torelli space $\text{Tor}(\Sigma)$ can be identified with the set of fixed points of $\tilde{\sigma}$ on $\text{Tor}(\Sigma^c)$.

Torelli spaces are intimately related to the Jacobian variety of a compact Riemann surface. Recall that this variety $J(\Sigma^c)$, is a $g$-dimensional complex torus ($g$ is the genus of $\Sigma^c$) given by $\mathbb{C}^g/\Gamma$, where $\Gamma$ is the lattice generated by integration of a basis of holomorphic forms on $\Sigma^c$ over a basis of $H_1(\Sigma^c, \mathbb{Z})$. We can also construct the Jacobian by considering the lattice $\Gamma'$ generated by integration of harmonic forms and then taking the quotient $\mathbb{R}^{2g}/\Gamma'$, which is a real torus. The Hodge-* operator gives a complex structure to this real torus in such a way that it becomes $J(\Sigma^c)$. This point of view can be generalized to construct a Jacobian variety $J(\Sigma)$ of a non-orientable Klein surface.

**Theorem 4.1.** Let $\Sigma$ be a compact non-orientable surface of genus $g \geq 3$. Then we can associate to $\Sigma$ a real torus of dimension $g - 1$, the **Jacobian variety** $J(\Sigma)$ of $\Sigma$, such that $J(\Sigma)$ is isomorphic to any component of the real part of the Jacobian $J(\Sigma^c)$ of the complex double. This last set is defined as the set of fixed points of the symmetry $\sigma_1$ of $J(\Sigma^c)$ induced by $\sigma$.

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A **Klein surface** (or **dianalytic**) structure $X$ on a surface without boundary $\Sigma$ is a covering by open sets $U_i$, and a collection of homeomorphisms $z_i : U_i \to V_i$, where $V_i \subset \mathbb{C}$ are open sets, such that $z_i \circ z_j^{-1}$ is holomorphic or anti-holomorphic, whenever $U_i \cap U_j \neq \emptyset$ [1]. Observe that a Klein surface structure on an orientable surface is just a pair of conjugate Riemann surface structures [9].

A compact non-orientable surface $\Sigma$ is homeomorphic to the connected sum of $g \geq 1$ real projective planes [3]. The number $g$ is called the **genus** of $\Sigma$. If $g = 2n + 1$, then the fundamental group of $\Sigma$ has a presentation given by generators $c, a_1, \ldots, a_n, b_1, \ldots, b_n$, satisfying $c^2 \prod_{j=1}^{n} [a_j, b_j] = 1$, where $[a, b] = aba^{-1}b^{-1}$. If the genus is even, $g = 2n + 2$, then we can choose generators $c, d, a_1, \ldots, a_n, b_1, \ldots, b_n$, satisfying the relation $c^2d^2 \prod_{j=1}^{n} [a_j, b_j] = 1$. An alternative presentation for this latter case is given by generators $\gamma, \delta, a_1, \ldots, a_n, b_1, \ldots, b_n$, and the relation $\gamma\delta\gamma^{-1}\delta \prod_{j=1}^{n} [a_j, b_j] = 1$. For the rest of this paper, we will assume that all surfaces are compact without boundary. We will further assume that non-orientable surfaces have genus $g \geq 3$, while orientable surfaces satisfy $g \geq 2$.

The **complex double** [1] of a Klein surface $\Sigma$ of genus $g$ is a triple $(\Sigma^c, \pi, \sigma)$, where:

1. $\Sigma^c$ is a Riemann surface of genus $g - 1$;
2. $\pi : \Sigma^c \to \Sigma$ is an unramified double covering;
3. there exist local coordinates $z$ and $w$ on $\Sigma^c$ and $\Sigma$, respectively, such that the function $w \circ \pi \circ z^{-1}$ is either holomorphic or anti-holomorphic (i.e. $\pi$ is a morphism of Klein surfaces);
4. $\sigma : \Sigma^c \to \Sigma^c$ is a symmetry such that $\pi \circ \sigma = \pi$.

Let $S$ be a compact orientable surface, with a fixed orientation and a smooth structure. The **Teichmüller space** $T(S)$ of $S$ is $T(S) = \mathcal{M}(S)/Diff_0(S)$, where $\mathcal{M}(S)$ is the set of Riemann surface structures on $S$ that agree with the given orientation.
and smooth structure [11]. The classical definition of $T(S)$ involves quasiconformal mappings; to see that it is equivalent to the above definition, it suffices to observe that on a compact surface, any homeomorphism is homotopic to a smooth one, and diffeomorphisms are quasiconformal. The **modular** or **mapping class** group $Mod(S)$ is the group of homotopic classes of orientation preserving diffeomorphisms of $S$, that is $Mod(S) = Diff^+(S)/Diff^0(S)$. This group acts on $T(S)$ by pull-back of complex structures: if $[f] \in Mod(S)$, and $[X] \in T(S)$, then $[f]^*([X]) = [f^*(X)]$, where $f^*(X)$ is the Riemann surface structure on $S$ that makes $f : (\Sigma, f^*(X)) \to (\Sigma, X)$ biholomorphic. However, this action has fixed points; it is therefore interesting to find subgroups of $Mod(\Sigma)$ that act without fixed points on $T(S)$. A subgroup $G$ of $Mod(S)$ has the **Hurwitz-Serre** property [8] if $G$ satisfies that for any element $[g] \in G$ such that there exists an $[X] \in M(S)$ with $g : (S, X) \to (S, X)$ biholomorphic, one has that $[g] = [id]$. A group with this property will act fixed-points free on Teichmüller space and, therefore, the quotient $T(S)/G$ will be a smooth finite dimensional complex manifold. The **Torelli group** $U(S) = \{[f] \in Mod(S); f \text{ acts trivially on } H_1(S, \mathbb{Z})\}$, is known to satisfy the Hurwitz-Serre property [6]. The quotient space $Tor(S) = T(S)/U(S)$ is called the **Torelli space** of $S$.

The Jacobian variety $J(S)$ of a compact Riemann surface is an abelian variety constructed as follows: let $\mathcal{B}^c = \{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ be a symplectic basis of $H_1(S, \mathbb{Z})$. Then we can find a dual basis for $H^0(S, \Omega^1_S)$, the space of holomorphic forms on $S$, consisting of forms $\{\omega_1, \ldots, \omega_g\}$, satisfying

$$\int_{\alpha_j} \omega_k = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Gamma^c$ be the lattice on $\mathbb{C}^g$ generated by the vectors $(f_c \omega_1, \ldots, f_c \omega_g), c \in \mathcal{B}^c$; then we define $J(S) = \mathbb{C}^g/\Gamma^c$. 
3. Torelli groups of non-orientable compact surfaces

In this section we will show that the Torelli group of a compact non-orientable Klein surface $\Sigma$ has the Hurwitz-Serre property. The main idea of the proof is to see that, if a diffeomorphism of a Klein surface $\Sigma$ acts trivially on $H_1(\Sigma, \mathbb{Z})$, its orientation preserving lift to the complex double will act trivially on $H_1(\Sigma^c, \mathbb{Z})$; then we use the fact that the Hurwitz-Serre property is satisfied for Riemann surfaces. The quotient space $Tor(\Sigma) = T(\Sigma)/U(\Sigma)$ is a smooth real manifold. We will show that $Tor(\Sigma)$ can be identified with the set of fixed points of a symmetry $\tilde{\sigma}$ on $Tor(\Sigma^c)$.

Let us start with a smooth non-orientable surface $\Sigma$, of genus $g = 2n + 1$, and a diffeomorphism $f : \Sigma \to \Sigma$ such that $f$ acts trivially on $H_1(\Sigma, \mathbb{Z})$. We can find a unique orientation preserving diffeomorphism $\tilde{f}$ of $\Sigma^c$ such that the following diagram commutes [10]:

\[
\begin{array}{ccc}
\Sigma^c & \xrightarrow{\tilde{f}} & \Sigma^c \\
\pi \downarrow & & \downarrow \pi \\
\Sigma & \xrightarrow{f} & \Sigma.
\end{array}
\]

We want to show that the mapping $\tilde{f}_\#$ induced by $\tilde{f}$ on $H_1(\Sigma^c, \mathbb{Z})$ is trivial. For that purpose we need to recall the way $\Sigma^c$ is constructed, from the topological viewpoint. The reader can find more details in [3].

By the presentation of the fundamental group of $\Sigma$, we can identify this surface with a $(4n+2)$-polygon, whose sides are labeled to satisfy the relation of the fundamental group. Then $\Sigma^c$ is given by two polygons with boundary relations:

$$c_1c_2 \prod_{j=1}^{n} [a_{j,1}, b_{j,1}] = 1 \quad \text{and} \quad c_2c_1 \prod_{j=1}^{n} [a_{j,2}, b_{j,2}] = 1.$$

To obtain a single relation, we find the value of $c_2$ on the right hand side equation and substitute it on the left hand one (equivalently, we glue the two polygons by the $c_2$ sides):

$$c_2 = (\prod_{j=1}^{n} [b_{n+1-j,2}, a_{n+1-j,2}])c_1^{-1};$$
therefore
\[ c_1\left(\prod_{j=1}^{n}[b_{n+1-j,2}, a_{n+1-j,2}]\right)c_1^{-1}\left(\prod_{j=1}^{n}[a_{j,1}, b_{j,1}]\right) = \]
\[ \left(\prod_{j=1}^{n}[c_1b_{n+1-j,2}c_1^{-1}, c_1a_{n+1-j,2}c_1^{-1}]\right)\left(\prod_{j=1}^{n}[a_{j,1}, b_{j,1}]\right) = 1. \]

From this formula we see that \( \Sigma^c \) is a compact surface of genus \( g - 1 = 2n \); we can choose the following paths as generators of the fundamental group of \( \Sigma \):
\[ \alpha_1 = c_1b_{n,2}c_1^{-1}, \ldots, \alpha_n = c_1b_{n,2}c_1^{-1}, \alpha_{n+1} = a_{1,1}, \ldots, \alpha_{2n} = a_{n,1}, \]
\[ \beta_1 = c_1a_{n,2}c_1^{-1}, \ldots, \beta_n = c_1a_{n,2}c_1^{-1}, \beta_{n+1} = b_{1,1}, \ldots, \beta_{2n} = b_{n,1}. \]

These loops satisfy \( \prod_{j=1}^{2n}[a_j, b_j] = 1 \). Let \( \mathcal{B} \) and \( \mathcal{B}^c \) denote the basis of \( H_1(\Sigma, \mathbb{Z}) \) and \( H_1(\Sigma^c, \mathbb{Z}) \) induced by the two given sets of generators of the corresponding fundamental groups. By an abuse of notation, we will use the same letters for the elements of the fundamental group and their classes in homology. We can see that \( \mathcal{B}^c \) is a symplectic basis of \( H_1(\Sigma^c, \mathbb{Z}) \); that is, the intersection matrix is given by \( J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \), where \( I \) is the identity matrix. The covering map \( \pi : \Sigma^c \to \Sigma \) induces a mapping on homology, with associated matrix
\[ \pi_{\#} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & I & K & 0 \\ K & 0 & 0 & I \end{pmatrix}, \]
with respect to \( \mathcal{B}^c \) and \( \mathcal{B} \). The matrix \( K \) is given by
\[ K = \begin{pmatrix} 0 & \cdots & \cdots & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}. \]

The symmetry \( \sigma \) maps \( a_{j1} \) (resp. \( b_{j1} \)) to \( a_{j2} \) (resp. \( b_{j2} \)); it is not difficult to see that the map \( \sigma_{\#} \) induced on \( H_1(\Sigma^c, \mathbb{Z}) \) is given by \( \sigma_{\#} = K \). Let \( \tilde{f}_{\#} = (A_{jk})^{1}_{j,k=1} \); since \( f_{\#} \) acts trivially on \( H_1(\Sigma, \mathbb{Z}) \), we have \( \pi_{\#}\tilde{f}_{\#} = \pi_{\#} \). By the uniqueness of \( \tilde{f}_{\#} \) we get
\( \tilde{f}_#\sigma_# = \sigma_#\tilde{f}_# \). Finally, \( \tilde{f}_#^t J \tilde{f}_# = J \), where \( \tilde{f}_#^t \) is the transpose of \( \tilde{f}_# \), since \( \tilde{f} \) preserves the intersection matrix (\([7, \text{ theorem N13, pg. 178}]\)). The condition \( \pi_#\tilde{f}_# = \pi_# \) is equivalent to the following set of equations:

\[
\begin{align*}
A_{21} + KA_{31} &= 0 & KA_{11} + A_{41} &= K \\
A_{22} + KA_{32} &= I & KA_{12} + A_{42} &= 0 \\
A_{23} + KA_{33} &= K & KA_{13} + A_{43} &= 0 \\
A_{24} + KA_{34} &= 0 & KA_{14} + A_{44} &= I.
\end{align*}
\]

Therefore, the matrix \( \tilde{f}_# \) can be written as

\[
\tilde{f}_# = \begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} \\
A_{21} & A_{22} & A_{23} & A_{24} \\
-KA_{21} & K - KA_{22} & I - KA_{23} & -KA_{24} \\
K - KA_{11} & -KA_{12} & -KA_{13} & I - KA_{14}
\end{pmatrix}.
\]

We now use the fact that \( \tilde{f} \) and \( \sigma \) commute, to obtain the following relations among the entries of the matrix \( \tilde{f}_# \):

\[
\begin{align*}
A_{14}K &= K(K - KA_{11}) & A_{24}K &= K(-KA_{21}) \\
A_{13}K &= K(-KA_{12}) & A_{23}K &= K(K - KA_{22}) \\
A_{12}K &= K(KA_{13}) & A_{22}K &= K(I - KA_{23}) \\
A_{11}K &= K(I - KA_{14}) & A_{21}K &= K(-KA_{24}).
\end{align*}
\]

Consider the equation \( \tilde{f}_#^t J \tilde{f}_# = J \); looking at the first row of the matrices on both sides of the equality, we get, after using (3.1) to simplify the result,

\[
\begin{align*}
A_{21}^t K - KA_{21} &= 0 \\
A_{11}^t K - KA_{22} &= 0 \\
-2I + A_{11}^t + KA_{22}K &= 0 \\
A_{21}^t + KA_{21}K &= 0
\end{align*}
\]

Solving these equations, we get \( A_{21} = 0 \) and \( A_{11} = I \), which imply that \( A_{24} = 0 \) and \( A_{14} = 0 \). Using this, we now consider the equality between the second rows of the
matrices $\tilde{f}^t J \tilde{f}$ and $J$, to obtain $A_{22} = I$, and $A_{12} = 0$. By (3.1), we get $A_{23} = 0$ and $A_{13} = 0$. Therefore, we have that $\tilde{f}^t$ is the identity matrix.

**Remark:** if we would have chosen the orientation reversing lift of $f$, say $\tilde{f}_1$, then $\tilde{f}_1 = f \sigma$, so $(\tilde{f}_1)^t = \tilde{f}^t \sigma = \sigma^t$.

If $\Sigma$ has even genus $g = 2n + 2$, we use the first of the two presentations of its fundamental group given in §2. We have that $\Sigma^c$ is given by two polygons with boundary relations:

$$c_1 c_2 d_1 d_2 \prod_{j=1}^n [a_{j,1}, b_{j,1}] = 1 \quad \text{and} \quad c_2 c_1 d_1 d_1 \prod_{j=1}^n [a_{j,2}, b_{j,2}] = 1.$$ 

From the second equation we get

$$d_2 = c_1^{-1} c_2^{-1} \left( \prod_{j=1}^n [b_{n+1-j,2}, a_{n+1-j,2}] \right) d_1^{-1},$$

which reduces the first equation to

$$c_1 c_2 d_1 c_1^{-1} c_2^{-1} \left( \prod_{j=1}^n [b_{n+1-j,2}, a_{n+1-j,2}] \right) d_1^{-1} \left( \prod_{j=1}^n [a_{j,1}, b_{j,1}] \right) =$$

$$c_1 c_2 c_1^{-1} c_2^{-1} \left( \prod_{j=1}^n [d_1 b_{n+1-j,2} d_1^{-1}, d_1 a_{n+1-j,2} d_1^{-1}] \right) \left( \prod_{j=1}^n [a_{j,1}, b_{j,1}] \right) = 1.$$

We therefore obtain that the fundamental group of $\Sigma^c$ is generated by the loops

$$\alpha_1 = c_1 d_1^{-1}, \quad \alpha_2 = d_1 b_{n,2} d_1^{-1}, \ldots, \alpha_{n+1} = d_1 b_{1,2} d_1^{-1}, \quad \alpha_{n+2} = a_{1,1}, \ldots, \alpha_{2n+1} = a_{n,1},$$

$$\beta_1 = d_1 c_2, \quad \beta_2 = d_1 a_{n,2} d_1^{-1}, \ldots, \beta_{n+1} = d_1 1, 2d_1^{-1}, \beta_{n+2} = a_{1,1}, \ldots, \beta_{2n+1} = a_{n,1},$$

satisfying the relation $\prod_{j=1}^{n+1} [\alpha_j, \beta_j] = 1$. The basis $\{ \alpha_1, \alpha_2, \ldots, \alpha_{2n+1}, \beta_1, \beta_2, \ldots, \beta_{2n+1} \}$ is symplectic, but computations are easier if we rearrange the basis as $B^c =$
\{\alpha_1, \beta_1, \alpha_2, \ldots, \alpha_{2n+1}, \beta_2, \ldots, \beta_{2n+1}\}$, whose intersection matrix is given by:

\[
J = \begin{pmatrix}
N & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & I \\
-I & 0 & 0 & 0 & 0 \\
0 & -I & 0 & 0 & 0
\end{pmatrix},
\]

where $N = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. With respect to $B^c$, the symmetry $\sigma$ has the following matrix representation, for its action on $H_1(\Sigma^c, \mathbb{Z})$:

\[
\sigma\# = \begin{pmatrix}
M & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & K & 0 \\
0 & 0 & K & 0 & 0 \\
0 & K & 0 & 0 & 0 \\
K & 0 & 0 & 0 & 0
\end{pmatrix},
\]

where the matrix $M = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}$. To obtain the matrix $M$, observe that in homology one has $\alpha = c_1 + d_1$ and $\beta = d_1 + c_2$, so $\sigma(\alpha) = c_2 + d_2$, $\sigma(\beta) = d_2 + c_1$. Substituting the value of $d_2$ obtained previously, we get $M$. Similarly we have that the expression for the action of the covering map $\pi$ on homology is given by:

\[
\pi\# = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & I & K & 0 \\
0 & K & 0 & 0 & I
\end{pmatrix}.
\]

Computing in a way similar to the odd genus case, we get that if $f : \Sigma \to \Sigma$ is a diffeomorphism of $\Sigma$ that acts trivially on $H_1(\Sigma, \mathbb{Z})$, and $\tilde{f}$ is a lift of $f$ to $\Sigma^c$, then the action of this last mapping on $H_1(\Sigma^c, \mathbb{Z})$ is given by either the identity or $\sigma\#$, depending on whether $\tilde{f}$ is orientation preserving or reversing.
Recall that a map \( f : \Sigma \to \Sigma \) on a Klein surface is **dianalytic** if, when expressed in local coordinates \((U, z)\), \( f \circ z^{-1} \) is either holomorphic or anti-holomorphic. The above computations show that \( U(\Sigma) \) satisfies the equivalent of the Hurwitz-Serre property.

**Theorem 3.1.** Let \( \Sigma \) be a compact non-orientable surface of genus \( g \geq 2 \). Let \([f] \in \text{Mod}(\Sigma)\), and suppose that there exists a Klein surface structure \( X \) on \( \Sigma \) such that \( f : (\Sigma, X) \to (\Sigma, X) \) is dianalytic. Then \( f \) is homotopic to the identity.

**Corollary 3.2.** The Torelli space \( \text{Tor}(\Sigma) = T(\Sigma)/U(\Sigma) \) is a smooth manifold of real dimension \( 3g - 6 \).

**Proof of the Proposition.** Since \( f \) is dianalytic on the Klein surface \((\Sigma, X)\), the orientation preserving lift \( \tilde{f} \) is biholomorphic on the Riemann surface \( \Sigma^c \). But then, since the genus of \( \Sigma^c \) is at least 2, we have that \( \tilde{f} \) is the identity, which proves the result. \( \square \)

The involution \( \sigma \) of \( \Sigma^c \) induces a symmetry \( \sigma^* \) on \( T(\sigma^c) \). The Teichmüller space \( T(\Sigma) \) can be identified with the set of fixed points of \( \sigma^* \), which proves the corollary. It is clear that \( \sigma^* \) descends to a symmetry \( \tilde{\sigma} \) on \( \text{Tor}(\Sigma^c) \).

**Proposition 3.3.** The Torelli space \( \text{Tor}(\Sigma) = T(\Sigma)/U(\Sigma) \) can be identified with the set of fixed points of \( \tilde{\sigma} \) in \( \text{Tor}(\Sigma^c) \).

**Proof.** The proof follows immediately from the definition of Torelli spaces. In fact, we have that two elements \([X_1]\) and \([X_2]\) of \( \mathcal{M}(\Sigma) \) project to the same point in \( \text{Tor}(\Sigma) \) if and only if there exists a diffeomorphism \( h \in Diff(\Sigma) \) such that \( h_\# : H_1(\Sigma, \mathbb{Z}) \to H_1(\Sigma, \mathbb{Z}) \) is the identity, and \( h : (\Sigma, X_2) \to (\Sigma, X_1) \) is dianalytic. The rest of the proof is similar to the proof that \( T(\Sigma) \) can be identified with the set of fixed points of \( \sigma^* \); see [11] for more details. \( \square \)
4. Jacobi varieties of Klein surfaces

Throughout this section, $\Sigma$ will denote a fixed compact non-orientable Klein surface of genus $g \geq 3$, and $\Sigma^c$ its complex double. We can take $\Sigma^c$ to be defined by a polynomial, $p(z, w) = 0$, with real coefficients ([1] and [5]). Then the involution $\sigma$ is given by $\sigma(z, w) = (\overline{z}, \overline{w})$, and conjugation $z \mapsto \overline{z}$ in $\mathbb{C}^{g-1}$ induces an involution $\sigma_1$ on the Jacobian $J(\Sigma^c)$. The set of fixed points of $\sigma_1$, that is, the real part of $J(\Sigma^c)$, is a real manifold of dimension $g - 1$; the pair $J(\Sigma^c, \sigma_1)$ is usually considered as the Jacobian of $\Sigma$. On a Klein surface the concept of harmonic forms makes sense; it is not difficult to see that the space $H^1_{\overline{\partial}}(\Sigma)$ of such forms has dimension precisely $g - 1$. One can choose a basis of $H_1(\Sigma, \mathbb{Z})$, the free part of $H_1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2\mathbb{Z}$, and a dual basis for $H^1_{\overline{\partial}}(\Sigma)$; these two basis generate a lattice $\Gamma$ in $\mathbb{R}^{g-1}$. We will call the real torus $\mathbb{R}^{g-1}/\Gamma$ the Jacobian variety of $\Sigma$, and denote it by $J(\Sigma)$. On the other hand, the real part of a holomorphic form (on $\Sigma^c$) is a harmonic form, so one can expect some relationship between $J(\Sigma)$ and the real part of $J(\Sigma^c)$. We prove that, in fact, $J(\Sigma)$ is isomorphic to a component of the set of fixed points of $\sigma_1$ in $J(\Sigma^c)$.

A continuous function $f : W \to \mathbb{R}$ defined on an open set of a Klein or Riemann surface is called harmonic if for any local coordinate $(U, z)$, with $U \cap W \neq \emptyset$, the function $f \circ z^{-1}$ is harmonic. Since precomposition with holomorphic and anti-holomorphic functions preserves harmonicity, the above definition makes sense. Actually, a Klein surface is the most general surface in which the notion of harmonic function is well defined [1]. Similarly, a (real) form $\psi$ is harmonic if it can be written locally as $\psi = df$, where $f$ is harmonic. We will denote by $H^1_{\overline{\partial}}(\Sigma)$ the space of harmonic forms on $\Sigma$. Let $\sigma^*$ be the pull-back map induced by $\sigma$ on forms on $\Sigma^c$. If $\omega = gdz$, with $g$ holomorphic, we have that $\sigma^*(\omega) = g(\sigma)\sigma_1d\sigma^*z$, so $\sigma^*$ is anti-holomorphic. We also have that, for any holomorphic form $\omega \in H^0(\Sigma^c, \Omega^1)$, and for any cycle $c$ on $\Sigma^c$, $\int_c \sigma^*(\omega) = \int_{\sigma(c)} \omega$. Observe that this last equality agrees with [5], while it differs of [12] and [1], since these two authors define $\sigma^*$ as the conjugate of our definition (in
order to have that $\sigma^*$ preserves holomorphic forms).

To compute the dimension of $H^1_{\mathbb{R}}(\Sigma)$, it suffices to observe that $\sigma^*$ takes harmonic forms to harmonic forms; therefore, $H^1_{\mathbb{R}}(\Sigma)$ will be isomorphic to the set of fixed points of $\sigma^*$ in $H^1_{\mathbb{R}}(\Sigma^c)$. By Hodge theory, $\sigma^*$ acts like $\sigma_\# = K$; so $\dim H^1_{\mathbb{R}}(\Sigma) = g - 1$. This result agrees with [2].

In order to justify later computations, we need the following lemma.

**Lemma 4.1 (Duality Lemma).** On a non-orientable compact Klein surface $\Sigma$, of genus $g \geq 3$, the space of harmonic forms, $H^1_{\mathbb{R}}(\Sigma)$, and the dual space to the homology with real coefficients, $H_1(\Sigma, \mathbb{R})^*$, are isomorphic.

**Proof.** From Differential Topology [4, Theorem 15.8] we know that the Čech cohomology with coefficients in the constant presheaf $\mathbb{Z}$, $H^1_\mathbb{Z}(\Sigma)^*$, is isomorphic to the singular cohomology $H^1(\Sigma)$. Furthermore, by the Universal Coefficients Theorem [4, Corollary 15.14.1], we have that the space $H^1(\Sigma)$ is isomorphic to the free part of $H_1(\Sigma)$, which is just $\mathbb{Z}^{g-1}$. Tensoring with $\mathbb{R}$ we have that $H^1_{\mathbb{R}}(\Sigma)$ is isomorphic to $\mathbb{R}^{g-1}$. On the other hand, $H^1_{\mathbb{R}}(\Sigma)$ is isomorphic to the de Rham cohomology $H^1_{\text{DR}}(\Sigma)$ [4, 8.9, 9.8 and 14.28]. By the compactness of $\Sigma$, on each de Rham class there exists at most a harmonic form. A counting of dimensions shows that there exists exactly one harmonic form, which completes the proof. Therefore, $H^1_{\mathbb{R}}(\Sigma)$ and $H_{\text{DR}}(\Sigma)$ are isomorphic, and since this last space is isomorphic (by integration) to $H_1(\Sigma, \mathbb{R})$, we are done. $\square$

To make matters more clear, we will construct $J(\Sigma)$ on the cases of genus 3 and 4. The general case will follow easily from our examples. Let us start with a Klein surface $\Sigma$ of genus $g = 3$. Then from §2 we have that $B = \{a, b\}$ is a basis of $H_1(\Sigma, \mathbb{Z})$. The loops $\alpha_1, \alpha_2, \beta_1, \beta_2$ of §3 give a basis of $H_1(\Sigma^c, \mathbb{Z})$; but for computational purposes, it is better to choose the basis

$$B^c = \{\gamma_1 = - (\alpha_2 + \beta_1), \gamma_2 = - (\alpha_1 + \beta_2), \delta_1 = \alpha_1 + \alpha_2 + \beta_1, \delta_2 = \alpha_1 + \alpha_2 + \beta_2\}.$$
Observe that the change of basis in $H_1(\Sigma^c, \mathbb{Z})$ is given by the matrix

$$C = \begin{pmatrix}
0 & -1 & 1 & 1 \\
-1 & 0 & 1 & 1 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{pmatrix},$$

which has determinant equal to 1, and satisfies $C^tJC = J$. Therefore $B^c$ is a symplectic basis. The mapping $\pi_#$ gives $\pi_#(\gamma_1) = -2b, \pi_#(\gamma_2) = -2a$. This suggests that we should take a basis $B_h = \{\phi_1, \phi_2\}$ of $\mathcal{H}_{2\Omega}(\Sigma)$ normalized by $\int_a \phi_1 = \int_b \phi_2 = 0$, and $\int_a \phi_1 = \int_b \phi_2 = -1/2$. Observe that this normalization is possible because of the Duality Lemma, and the fact that $a$ and $b$ are not torsion classes in $H_1(\Sigma, \mathbb{Z})$. We can use the pull-back mapping $\pi^*$ induced by $\pi$ to get forms $\psi_j = \pi^*(\phi_j)$ on $\Sigma^c$. These forms are real harmonic, so $\omega_j = \psi_j + i \ast \phi_j$ are holomorphic forms (where $\ast$ stands for the Hodge-$\ast$ operator). By the expression of $\sigma$ and the formula relating $\sigma^*$ with integrals, we have that

$$\int_{\gamma_j} \bar{\omega}_k = \int_{\gamma_j} \sigma^*(\omega_k) = \int_{\sigma(\gamma_j)} \omega_k = \int_{\gamma_j} \omega_k.$$

In particular, we see that $\int_{\gamma_j} \omega_k$ is real. But since by [1, Theorem 1.0.7, pg. 74] $\text{Re} \int_{\gamma_j} \omega_k = \int_{\gamma_j} \psi_k = \int_{\sigma(\gamma_j)} \phi_k$, we have that $B^* = \{\omega_1, \omega_2\}$ is normalized with respect to $B^c$. Let $P$ denote the corresponding period matrix, that is the entries of this matrix are given by $p_{jk} = \int_{\delta_k} \omega_j$. The mapping $\sigma_# : H_1(\Sigma^c, \mathbb{Z}) \to H_1(\Sigma^c, \mathbb{Z})$ has the following expression with respect to the basis $B^c$:

$$\sigma_# = \begin{pmatrix}
1 & 0 & -2 & -1 \\
0 & 1 & -1 & -2 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.$$

This results agrees with the one obtained by Natanzon in [5] except in that he gets all sign positive. Nevertheless, the computation of the real part of $J(\Sigma^c)$ yields the same
result. We have that \( P \) satisfies \( P = A - P \), where \( A = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \). The Jacobian variety of \( \Sigma^c \) is then given by \( J(\Sigma^c) = \mathbb{C}^2/\Gamma^c \), where \( \Gamma^c = \mathbb{Z}^2 + P\mathbb{Z}^2 \). To compute the real part of \( J(\Sigma^c) \), we write, for any \( z \in \mathbb{C}^2 \), \( z = P\alpha + \beta \), where \( \alpha, \beta \in \mathbb{R}^2 \). Then \( \sigma_1(z) = \overline{z} \equiv z \) is equivalent to \( P\alpha + \beta = \overline{P}\alpha + \beta + Pn + m \), for some \( n, m \in \mathbb{Z}^2 \). The imaginary part of this equation gives \( (\text{Im}P)\alpha = -(\text{Im}P)\alpha + (\text{Im}P)n \). By Riemann bilinear relations ([6]; see also [13] for a nice introduction to Jacobians and of Riemann surfaces from an algebro-geometric approach) we have that \( (\text{Im}P) \) is invertible, so we obtain \( \alpha = n/2 \). On the other hand, taking real parts in the above equation we get \( (\text{Re}P)\alpha + \beta = (A - \text{Re}P)\alpha + \beta + (\text{Re}P)n + m \). Since \( P = A - P \), we have that \( (\text{Re}P) = A - (\text{Re}P) \), so this equation reduces to \( 0 = \frac{1}{2}An + m \), or equivalently

\[
\begin{cases}
 n_1 + \frac{n_2}{2} + m_1 = 0 \\
 \frac{n_1}{2} + n_2 + m_2 = 0,
\end{cases}
\]

where notation the should be clear. This implies that \( n_j \in 2\mathbb{Z} \), so \( \alpha \in \mathbb{Z}^2 \). Therefore, the set of fixed points of \( \sigma_1 \) is given by the real torus \( \text{Re}(J(\Sigma^c)) = \{ P\mathbb{Z}^2 + \beta; \beta \in \mathbb{R}^2 \}/\Gamma^c \). which agrees with the results obtained by Silhol [12, pgs. 349 and 359] and Natanzon [9]. By the form of the lattice \( \Gamma^c \), it is clear that \( \text{Re}(J(\Sigma^c)) \cong (\mathbb{R}/\mathbb{Z})^2 \).

In a similar way to the construction of \( J(\Sigma^c) \), we can form a lattice in \( \mathbb{R}^2 \) using the basis \( \mathcal{B}_h = \{ \phi_1, \phi_2 \} \) and \( \mathcal{B} \) of \( \mathcal{H}_1^1(\Sigma) \) and \( \mathcal{H}_1(\Sigma, \mathbb{Z})f \), respectively. Let us denote this lattice by \( \Gamma \). We define the Jacobian variety of \( \Sigma \) as the quotient \( J(\Sigma) = \mathbb{R}^2/\Gamma \). It is clear that \( [z] \mapsto [-\frac{1}{2}z] \) induces an isomorphism between \( \text{Re}(J(\Sigma^c)) \) and \( J(\Sigma) \), which proves our result for \( g = 3 \). The general case of a surface of odd genus is done in a similar way.

To see the even genus case, we take a surface with \( g = 4 \), and we choose the second of the two presentations of the fundamental group of \( \Sigma \) given in §2; i.e. the generators are the loops \( c, d, a \) and \( b \), and the relation is \( cdc^{-1}d[ab] = 1 \). To construct the complex double we proceed as in §3; we do not include the computations here,
since they are done as in §3. We get that the fundamental group of $\Sigma^c$ is generated by the loops

$$\alpha_1 = c_2c_1, \quad \alpha_2 = (d_1^{-1}c_2)b_1(d_1^{-1}b_1c_2)^{-1}, \quad \alpha_3 = a_2,$$

$$\beta_1 = d_1^{-1}, \quad \beta_2 = (d_1^{-1}c_2)a_1(d_1^{-1}b_1c_2)^{-1}, \quad \beta_3 = b_2,$$

We again change our basis of $H_1(\Sigma^c, \mathbb{Z})$ to

$$B^c = \{ \gamma_1 = \alpha_1, \quad \gamma_2 = -(\alpha_3 + \beta_2), \quad \gamma_3 = -(\alpha_2 + \beta_3), \quad \delta_1 = \alpha_1 + \beta_1, \quad \delta_2 = \alpha_2 + \alpha_3 + \beta_2, \quad \delta_3 = \alpha_2 + \alpha_3 + \beta_3 \}. $$

It is not hard to see that $B^c$ is symplectic; one simply has to check that the matrix

$$C = \begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 1 \\
0 & -1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 \\
\end{pmatrix},$$

satisfies $C^t JC = J$. The projection $\pi$ acts on these loops by $\pi_#(\gamma_1) = 2c$, $\pi_#(\gamma_2) = -2a$, $\pi_#(\gamma_3) = -2b$; so we take a basis $\{ \phi_1, \phi_2, \phi_3 \}$ of $H^1_{\mathbb{R}}(\Sigma)$, and normalize it by requiring

$$\int_c \phi_1 = -\int_a \phi_2 = -\int_b \phi_3 = \frac{1}{2},$$

and the other integrals equal to 0. As in the previous situation, we have that if $\psi_j = \pi^*(\phi_j)$, then $B^* = \{ \omega_j = \psi_j + i * \psi_j; \quad j = 1, 2, 3 \}$ is a basis of holomorphic 1-forms dual to $B^c$. It is not hard to see that $\sigma_#$ is given by the following matrix,
when computed with respect to $B^c$: 

$$
\sigma_# = \begin{pmatrix}
1 & 0 & 0 & 2 & 0 & 0 \\
0 & 1 & 0 & 0 & -2 & -1 \\
0 & 0 & 1 & 0 & -1 & -2 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 \\
\end{pmatrix},
$$

and the period matrix $P$ satisfies $P = A - P$, where 

$$
A = \begin{pmatrix}
2 & 0 & 0 \\
0 & -2 & -1 \\
0 & -1 & -2 \\
\end{pmatrix}.
$$

The above matrix of $\sigma_#$ is different from the one given in [5]; we have not been able to obtain the matrix of that reference, but nevertheless, we obtain similar result in the computation of the real part of $J(\Sigma^c)$. In this case we have that the real part of $J(\Sigma^c)$ (which is found in the same way that the $g = 3$ case) has two components, namely $T_1 = \{\mathbb{Z}^3 \alpha + \beta; \beta \in \mathbb{R}^3, \alpha = (n_1, n_2, n_3), n_1, n_2, n_3 \in \mathbb{Z}\}$ and $T_2 = T_1 + \mathbb{Z}^3 (\frac{1}{2}, 0, 0)^t$.

We again obtained the results of [12] and [9]. An isomorphism similar to the previous case holds in this situation, except that we have $J(\Sigma)$ is isomorphic to any of the two sets $T_1$ or $T_2$.

**Remark:** the results about the fixed points of $\sigma_1$ on $J(\Sigma^c)$ can also be obtained from the expression of the period matrix given in [12, Proposition 4, pg. 351].

The above results can be put together in the following theorem:

**Theorem 4.2.** Let $\Sigma$ be a compact non-orientable surface of genus $g \geq 3$. Then we can associate to $\Sigma$ a real torus of dimension $g - 1$, the **Jacobian variety** $J(\Sigma)$ of $\Sigma$, such that $J(\Sigma)$ is isomorphic to any component of the real part of the Jacobian $J(\Sigma^c)$ of the complex double. This last set is defined as the set of fixed points of the symmetry $\sigma_1$ of $J(\Sigma^c)$ induced by $\sigma$. 
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