Supersymmetric Mechanics in Superspace

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1 Introduction

These Lectures have been given at Laboratori Nazionali di Frascati in the month of March, 2005. The main idea was to provide our young colleagues, who joined us in our attempts to understand the structure of \(N\)-extended supersymmetric one-dimensional systems, with short descriptions of the methods and techniques we use. This was reflected in the choice of material and in the style of presentation. We base our treatment mainly on the superfield point of view. Moreover, we prefer to deal with \(N = 4\) and \(N = 8\) superfields. At present, the exists an extensive literature on the components approach to extended supersymmetric theories in \(d = 1\) while the manifestly supersymmetric formulation in terms of properly constrained superfields is much less known. Nevertheless, we believe that just such formulations are preferable.

In order to make these Lectures more or less self-consistent, we started from the simplest examples of one-dimensional supersymmetric theories and paid a lot of attention to the peculiarities of \(d = 1\) supersymmetry. From time to time we presented the calculations in a very detailed way. In other cases we omitted the details and gave only the final answers. In any case these Lectures cannot be considered as a textbook in any respect. They can be considered as our personal point of view on the one-dimensional superfield theories and on the methods and techniques we believe to be important.

Especially, all this concerns Section 3, were we discuss the nonlinear realizations method. We did not present any proofs in this Section. Instead we focused on the details of calculations.

Finally, we apologize for the absolutely incomplete list of References.

2 Supersymmetry in \(d = 1\)

The extended supersymmetry in one dimension has plenty of peculiarities which make it quite different from higher-dimensional analogs. Indeed, even
the basic statement of any supersymmetric theory in \( d > 1 \) – the equality of bosonic and fermionic degrees of freedom – is not valid in \( d = 1 \). As a result, there are many new supermultiplets in one dimension which have no higher-dimensional ancestors. On the other hand, the constraints describing on-shell supermultiplets, being reduced to \( d = 1 \), define off-shell multiplets! Therefore, it makes sense to start with the basic properties of one-dimensional supermultiplets and give a sort of vocabulary with all linear finite-dimensional \( N = 1, 2, 4 \) and \( N = 8 \) supermultiplets. This is the goal of the present Section.

2.1 Super-Poincaré algebra in \( d = 1 \)

In one dimension there is no Lorentz group and therefore all bosonic and fermionic fields have no space-time indices. The simplest free action for one bosonic field \( \phi \) and one fermionic field \( \psi \) reads

\[
S = \gamma \int dt \left[ \dot{\phi}^2 - \frac{i}{2} \dot{\psi} \psi \right].
\]

In what follows it will be useful to treat the scalar field as dimensionless and assign dimension \( cm^{-1/2} \) to fermions. Therefore, all our actions will contain the parameter \( \gamma \) with the dimension [\( \gamma \)] = \( cm \).

The action (1) provides the first example of a supersymmetric invariant action. Indeed, it is a rather simple exercise to check its invariance with respect to the following transformations:

\[
\delta \phi = -i \epsilon \psi, \quad \delta \psi = -\epsilon \dot{\phi}.
\]

As usual, the infinitesimal parameter \( \epsilon \) anticommutes with fermionic fields and with itself. What is really important about transformations (2) is their commutator

\[
\delta_2 \delta_1 \phi = \delta_2 (\epsilon_1 \psi) = i \epsilon_1 \epsilon_2 \dot{\phi},
\]

\[
\delta_1 \delta_2 \phi = i \epsilon_2 \epsilon_1 \dot{\phi} \Rightarrow [\delta_2, \delta_1] \phi = 2i \epsilon_1 \epsilon_2 \dot{\phi}.
\]

Thus, from (3) we may see the main property of supersymmetry transformations: they commute on translations. In our simplest one-dimensional framework this is the time translation. This property has the following form in terms of the supersymmetry generator \( Q \):

\[
\{Q, Q\} = -2P.
\]

The anticommutator (4), together with

\[
[Q, P] = 0
\]

describe \( N = 1 \) super-Poincaré algebra in \( d = 1 \). It is rather easy to guess the structure of \( N \)-extended super-Poincaré algebra: it includes \( N \) real supercharges \( Q^A, A = 1, ..., N \) with the following commutators:
\{Q^A, Q^B\} = -2\delta^{AB} P, \quad [Q^A, P] = 0. \quad (6)

Let us stress that the reality of the supercharges is very important, as well as having the same sign in the r.h.s. of \{Q^A, Q^B\} for all \(Q^A\). From time to time one can see in the literature wrong statements about the number of supersymmetries in the theories when authors forget about these absolutely needed properties.

From (2) we see that the minimal \(N = 1\) supermultiplet includes one bosonic and one fermionic field. A natural question arises: how many components we need, in order to realize the \(N\)-extended superalgebra (6)? The answer has been found in a paper by S.J.Gates and L.Rana [1]. Their idea is to mimic the transformations (2) for all \(N\) super-translations as follows:

\[
\delta \phi_i = -i\epsilon^A (L_A)^i_\dot{i} \psi_\dot{i}, \quad \delta \psi_\dot{i} = -\epsilon^A (R_A)^i_\dot{i} \dot{\phi}_i . \quad (7)
\]

Here the indices \(i = 1, \ldots, d_b\) and \(\dot{i} = 1, \ldots, d_f\) count the numbers of bosonic and fermionic components, while \((L_A)^i_\dot{i}\) and \((R_A)^i_\dot{i}\) are \(N\) arbitrary, for the time being, matrices. The additional conditions one should impose on the transformations (7) are

- They should form the \(N\)-extended superalgebra (6)
- They should leave invariant the free action constructed from the involved fields.

These conditions result in some equations on the matrices \(L_A\) and \(R_A\) which has been solved in [1]. These results for the most interesting cases are summarized in the Table (1). Here, \(d = d_b = d_f\) is the number of bosonic/fermionic components. From Table (1) we see that there are four special cases with \(N = 1, 2, 4, 8\) when \(d\) coincides with \(N\). Just these cases we will discuss in the present Lectures. When \(N > 8\) the minimal dimension of the supermultiplets rapidly increases and the analysis of the corresponding theories becomes very complicated. For many reasons, the most interesting case seems to be the \(N = 8\) supersymmetric mechanics. Being the highest \(N\) case of minimal \(N\)-extended supersymmetric mechanics admitting realization on \(N\) bosons (physical and auxiliary) and \(N\) fermions, the systems with eight supercharges are the highest \(N\) ones, among the extended supersymmetric systems, which still possess a non-trivial geometry in the bosonic sector [2]. When the number of supercharges exceeds 8, the target spaces are restricted to be symmetric

**Table 1.** Minimal supermultiplets in \(N\)-extended supersymmetry

| \(N\) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 16 |
|------|---|---|---|---|---|---|---|---|---|----|----|----|
| \(d\) | 1 | 2 | 4 | 8 | 8 | 8 | 8 | 8 | 16 | 32 | 64 | 128 |
spaces. Moreover, $\mathcal{N} = 8$ supersymmetric mechanics should be related via a proper dimensional reduction with four-dimensional $\mathcal{N} = 2$ supersymmetric field theories. So, one may hope that some interesting properties of the latter will survive after reduction.

### 2.2 Auxiliary fields

In the previous Subsection we considered the realization of $N$-extended supersymmetry on the bosonic fields of the same dimensions. Indeed, in (7) all fields appear on the same footing. It is a rather special case which occurs only in $d = 1$. In higher dimensions the appearance of the auxiliary fields is inevitable. They appear also in one dimension. Moreover, in $d = 1$ one may convert any physical field to auxiliary and vice versa [1]. In order to clarify this very important property of one dimensional theories, let us consider the simplest example of $N = 2, d = 1$ supermultiplets.

The standard definition of $N = 2, d = 1$ super-Poincaré algebra follows from (6):

$$\{Q^1, Q^1\} = \{Q^2, Q^2\} = -2P, \{Q^1, Q^2\} = 0.$$  \hspace{1cm} (8)

It is very convenient to redefine the supercharges as follows:

$$Q \equiv \frac{1}{\sqrt{2}} (Q_1 + iQ_2), \quad \overline{Q} \equiv \frac{1}{\sqrt{2}} (Q_1 - iQ_2),$$

$$\{Q, \overline{Q}\} = -2P, \quad Q^2 = \overline{Q}^2 = 0.$$  \hspace{1cm} (9)

From Table 1 we know that the minimal $N = 2$ supermultiplet contains two bosonic and two fermionic components. The supersymmetry transformations may be easily written as

$$\begin{align*}
\delta \phi &= -i \bar{\epsilon} \psi, \\
\delta \overline{\psi} &= -i \epsilon \phi, \\
\delta \psi &= -2 \epsilon \phi, \\
\delta \bar{\psi} &= -2 \bar{\epsilon} \bar{\phi}.
\end{align*}$$  \hspace{1cm} (10)

Here, $\phi$ and $\psi$ are complex bosonic and fermionic components. The relevant free action has a very simple form

$$S = \gamma \int dt \left( \dot{\phi} \dot{\phi} - \frac{i}{2} \dot{\psi} \dot{\psi} \right).$$  \hspace{1cm} (11)

Now we introduce the new bosonic variables $V$ and $A$

$$V = \phi + \bar{\phi}, \quad A = i \left( \dot{\phi} - \dot{\bar{\phi}} \right).$$  \hspace{1cm} (12)

What is really impressive is that, despite the definition of $A$ in terms of time derivatives of the initial bosonic fields, the supersymmetry transformations can be written in terms of $\{V, \psi, \bar{\psi}, A\}$ only.
\[ \delta V = -i (\bar{\epsilon} \psi + \epsilon \bar{\psi}), \quad \delta A = \bar{\epsilon} \dot{\psi} - \epsilon \dot{\bar{\psi}}, \]
\[ \delta \bar{\psi} = -\epsilon \left( \dot{V} - iA \right), \quad \delta \psi = -\bar{\epsilon} \left( \dot{\bar{V}} + iA \right). \] (13)

Thus, the fields \( V, \psi, \bar{\psi}, A \) form a \( N = 2 \) supermultiplet, but the dimension of the component \( A \) is now \( cm^{-1} \). If we rewrite the action (11) in terms of new components
\[ S = \gamma \int dt \left( \frac{1}{4} \dot{V} \dot{V} - \frac{i}{2} \psi \bar{\psi} + \frac{1}{4} A^2 \right) \] (14)
one may see that the field \( A \) appears in the action without derivatives and their equation of motion is purely algebraic
\[ A = 0. \] (15)

In principle, we can exclude this component from the Lagrangian (14) using (15). As a result we will have the Lagrangian written in terms of new components
\[ V, \psi, \bar{\psi} \] and the field \( A \) appears in the action without derivatives and their equation of motion is purely algebraic
\[ A = 0. \] (15)

The next variation of these equations give
\[ \ddot{V} = 0. \]

Such components are called auxiliary fields. In what follows we will use the notation \( (n, N, N - n) \) to describe a supermultiplet with \( n \) physical bosons, \( N \) fermions and \( N - n \) auxiliary bosons. Thus, the transformations (13) describe passing from the multiplet \( (2, 2, 0) \) to the \( (1, 2, 1) \) one. One may continue this process and so pass from the multiplet \( (1, 2, 1) \) to the \( (0, 2, 2) \) one, by introducing the new components \( B = \dot{\phi}, \bar{B} = \dot{\bar{\phi}} \). The existence of such a multiplet containing no physical bosons at all is completely impossible in higher dimensions. Let us note that the inverse procedure is also possible [1]. Therefore, the field contents of linear, finite dimensional off-shell multiplets of \( N = 2, 4, 8 \) \( d = 1 \) supersymmetry read
\[ N = 2 : (2, 2, 0), (1, 2, 1) (0, 2, 2) \]
\[ N = 4 : (4, 4, 0), (3, 4, 1), (2, 4, 2), (1, 4, 3), (0, 4, 4) \]
\[ N = 8 : (8, 8, 0), \ldots, (0, 8, 8) \] (16)

Finally, one should stress that both restrictions – linearity and finiteness – are important. There are nonlinear supermultiplets [3, 4, 5], but it is not always possible to change their number of physical/auxiliary components. In the case of \( N = 8 \) supersymmetry one may define infinite-dimensional supermultiplets, but for them the interchanging of the physical and auxiliary components is impossible. We notice, in passing, that recently for the construction of \( N = 8 \) supersymmetric mechanics [6, 5, 7] the nonlinear chiral multiplet has been used [8].
2.3 Superfields

Now we will turn to the main subject of these Lectures – Superfields in $\mathcal{N}$-extended $d = 1$ Superspace. One may ask - Why do we need superfields? There are a lot of motivations, but here we present only two of them. First of all, it follows from the previous Subsections that only a few supermultiplets from the whole "zoo" of them presented in (16), contain no auxiliary fields. Therefore, for the rest of the cases, working in terms of physical components we will deal with on-shell supersymmetry. This makes life very uncomfortable - even checking the supersymmetry invariance of the action becomes a rather complicated task, while in terms of superfields everything is manifestly invariant. Secondly, it is a rather hard problem to write the interaction terms in the component approach. Of course, the superfields approach has its own problems. One of the most serious, when dealing with extended supersymmetry, is to find the irreducibility constraints which decrease the number of components in the superfields. Nevertheless, the formulation of the theory in a manifestly supersymmetric form seems preferable, not only because of its intrinsic beauty, but also since it provides an efficient technique, in particular in quantum calculations.

The key idea of manifestly invariant formulations of supersymmetric theories is using superspace, where supersymmetry is realized geometrically by coordinate transformations. Let us start with $\mathcal{N} = 2$ supersymmetry. The natural definition of $\mathcal{N} = 2$ superspace $\mathbb{R}^{(1|2)}$ involves time $t$ and two anticommuting coordinates $\theta, \bar{\theta}$

$$\mathbb{R}^{(1|2)} = (t, \theta, \bar{\theta}).$$

(17)

In this superspace $\mathcal{N} = 2$ super-Poincaré algebra (9) can be easily realized

$$\delta \theta = \epsilon, \ \delta \bar{\theta} = \bar{\epsilon}, \ \delta t = -i \left(\epsilon \bar{\theta} + \bar{\epsilon} \theta\right).$$

(18)

$\mathcal{N} = 2$ superfields $\Phi(t, \theta, \bar{\theta})$ are defined as functions on this superspace. The simplest superfield is the scalar one, which transforms under (145) as follows:

$$\Phi'(t', \theta', \bar{\theta}') = \Phi(t, \theta, \bar{\theta}).$$

(19)

From (19) one may easily find the variation of the superfield in passive form

$$\delta \Phi \equiv \Phi'(t, \theta, \bar{\theta}) - \Phi(t, \theta, \bar{\theta}) = -\epsilon \left(\frac{\partial}{\partial \theta} - i \bar{\theta} \frac{\partial}{\partial t}\right) \Phi - \bar{\epsilon} \left(\frac{\partial}{\partial \bar{\theta}} - i \theta \frac{\partial}{\partial t}\right) \Phi$$

$$\equiv -\epsilon Q \Phi - \bar{\epsilon} \overline{Q} \Phi.$$  

(20)

Thus we get the realization of supercharges $Q, \overline{Q}$ in superspace

$$Q = \frac{\partial}{\partial \theta} - i \bar{\theta} \frac{\partial}{\partial t}, \quad \overline{Q} = \frac{\partial}{\partial \bar{\theta}} - i \theta \frac{\partial}{\partial t}, \quad \{Q, \overline{Q}\} = -2i \frac{\partial}{\partial t}.$$  

(21)

In order to construct covariant objects in superspace, we have to define covariant derivatives and covariant differentials of the coordinates. Under the
transformations (18), $d\theta$ and $d\bar{\theta}$ are invariant, but $dt$ is not. It is not too hard to find the proper covariantization of $dt$

$$dt \rightarrow \Delta t = dt - i d\bar{\theta} - i d\theta .$$

(22)

Indeed, one can check that $\delta \Delta t = 0$ under (18). Having at hand the covariant differentials, one may define the covariant derivatives

$$\left( dt \frac{\partial}{\partial t} + d\theta \frac{\partial}{\partial \theta} + d\bar{\theta} \frac{\partial}{\partial \bar{\theta}} \right) \Phi = \left( \Delta t \nabla_t + d\theta D + d\bar{\theta} \overline{D} \right) \Phi .$$

\n
As important properties of the covariant derivatives let us note that they anticommute with the supercharges (12).

The superfield $\Phi$ contains the ordinary bosonic and fermionic fields as coefficients in its $\theta, \bar{\theta}$ expansion. A convenient covariant way to define these components is to define them as follows:

$$V = |\Phi|, \hspace{1em} \psi = |i \overline{D} \Phi|, \hspace{1em} \bar{\psi} = - |D \Phi|, \hspace{1em} A = \frac{1}{2} \left[D, \overline{D}\right] |\Phi|,$$

(24)

where $|$ means the restriction to $\theta = \bar{\theta} = 0$. One may check that the transformations of the components (24), which follow from (20), coincide with (13). Thus, the general bosonic $N = 2$ superfield $\Phi$ describes the $(1, 2, 1)$ supermultiplet. The last thing we need to know, in order to construct the superfield action is the rule for integration over Grassmann coordinates $\theta, \bar{\theta}$. By definition, this integration is equivalent to a differentiation

$$\int dt d\theta d\bar{\theta} \mathcal{L} \equiv \int dt D\overline{D} \mathcal{L} = \frac{1}{2} \int dt \left[D, \overline{D}\right] \mathcal{L} .$$

(25)

Now we are ready to write the free action for the $N = 2 (1, 2, 1)$ supermultiplet

$$S = \gamma \int dt d\theta d\bar{\theta} \; D\Phi \overline{D} \Phi .$$

(26)

It is a simple exercise to check that, after integration over $d\theta, d\bar{\theta}$ and passing to the components (24), the action (26) coincides with (14).

The obvious question now is how to describe in superfields the supermultiplet $(2, 2, 0)$. The latter contains two physical bosons, therefore the proper superfield should be a complex one. But without any additional conditions the complex $N = 2$ superfield $\phi, \bar{\phi}$ describes a $(2, 4, 2)$ supermultiplet. The solution is to impose the so called chirality constraints on the superfield

$$D\phi = 0 \hspace{1em} \overline{D}\bar{\phi} = 0 .$$

(27)

It is rather easy to find the independent components of the chiral superfield (27)
\[ \dot{\phi} = \phi^i, \quad \dot{\psi} = i\bar{\nabla}\phi^i, \quad \ddot{\phi} = \ddot{\phi}^i, \quad \psi = i \bar{D}\phi^i. \quad (28) \]

Let us note that, due to the constraints (27), the auxiliary components are expressed through time derivatives of the physical ones

\[ A = \frac{1}{2} \left[ D, \nabla \right] \phi^i = \frac{1}{2} D\bar{D}\phi^i = i\dot{\phi}. \quad (29) \]

The free action has a very simple form

\[ S = \gamma \int dt d\bar{\theta} d\theta \left( \phi \dot{\phi} - \phi \dot{\phi} \right). \quad (30) \]

As the immediate result of the superfields formulation, one may write the actions of \( N = 2 \) \( \sigma \)-models for both supermultiplets

\[ S_{\sigma} = \gamma \int dt d\bar{\theta} d\theta \, F_1(\Phi) D\bar{\Phi} \nabla \Phi, \]

\[ S_{\sigma} = \gamma \int dt d\bar{\theta} d\theta \, F_2(\phi, \bar{\phi}) \left( \phi \dot{\phi} - \phi \dot{\phi} \right), \quad (31) \]

where \( F_1(\Phi) \) and \( F_2(\phi, \bar{\phi}) \) are two arbitrary functions defining the metric in the target space.

Another interesting example is provided by the action of \( N = 2 \) superconformal mechanics [9]

\[ S_{\text{Conf}} = \gamma \int dt d\bar{\theta} d\theta \left[ D\Phi \bar{D}\Phi + 2m \log \Phi \right]. \quad (32) \]

The last \( N = 2 \) supermultiplet in Table (16), with content \((0, 2, 2)\), may be described by the chiral fermionic superfield \( \Psi \):

\[ D\Psi = 0, \quad \bar{D}\Psi = 0. \quad (33) \]

Thus we described in superfields all \( N = 2 \) supermultiplets. But really interesting features appear in the \( N = 4 \) supersymmetric theories which we will consider in the next subsection.

### 2.4 N=4 Supermultiplets

The \( N = 4, d = 1 \) superspace \( \mathbb{R}^{(1|4)} \) is parameterized by the coordinates

\[ \mathbb{R}^{(1|4)} = \left( t, \theta_i, \bar{\theta}^j \right), \quad (\theta_i)^\dagger = \bar{\theta}^j, \quad i, j = 1, 2. \quad (34) \]

The covariant derivatives may be defined in a full analogy with the \( N = 2 \) case as

\[ D^i = \frac{\partial}{\partial \theta^i} + i\bar{\theta}^j \frac{\partial}{\partial t}, \quad \bar{D}_j = \frac{\partial}{\partial \theta^i} + i\theta_i \frac{\partial}{\partial t}, \quad \{ D^i, \bar{D}_j \} = 2i\delta^i_j \partial_t. \quad (35) \]
Such a representation of the algebra of $N = 4, d = 1$ spinor covariant derivatives manifests an automorphism $SU(2)$ symmetry (from the full $SO(4)$ automorphism symmetry of $N = 4, d = 1$ superspace) realized on the doublet indices $i, j$. The transformations from the coset $SO(4)/SU(2)$ rotate $D^i$ and $\overline{D}^i$ through each other.

Now, we are going to describe in superfields all possible $N = 4$ supermultiplets from Table (16).

$N=4, d=1$ “hypermultiplet” – $(4, 4, 0)$.

We shall start with the most general case when the supermultiplet contains four physical bosonic components. In order to describe this supermultiplet we have to introduce four $N = 4$ superfields $q^i, \bar{q}^j$. These superfields should be properly constrained to reduce 32 bosonic and 32 fermionic components, which are present in the unconstrained $q^i, \bar{q}^j$, to 4 bosonic and 4 fermionic ones. One may show that the needed constraints read

$$D^{(i}q^{j)} = 0, \overline{D}^{(i}\bar{q}^{j)} = 0, \quad D^{(i}\bar{q}}^{j)} = 0, \overline{D}^{(i}\bar{q}}^{j)} = 0.$$  \hspace{1cm} (36)

This $N = 4, d = 1$ multiplet was considered in [10, 11, 3, 12, 4] and also was recently studied in $N = 4, d = 1$ harmonic superspace [13]. It resembles the $N = 2, d = 4$ hypermultiplet. However, in contrast to the $d = 4$ case, the constraints (125) define an off-shell multiplet in $d = 1$.

The constraints (36) leave in the $N = 4$ superfield $q^i$ just four spinor components

$$D_i q^i, \overline{D}_i q^i, D_\bar{q}_i, \overline{D}_\bar{q}_i,$$  \hspace{1cm} (37)

while all higher components in the $\theta$-expansion are expressed as time-derivatives of the lowest ones. This can be immediately seen from the following consequences of (36):

$$D^i \overline{D}_j q^j = 4i \dot{q}^i, \quad D^i D_\bar{q}_j q^j = 0.$$  

The general sigma-model action for the supermultiplet $(4, 4, 0)$ reads

$$S_\sigma = \int dt d^4\theta K(q, \bar{q}),$$  \hspace{1cm} (38)

where $K(q, \bar{q})$ is an arbitrary function on $q^i$ and $\bar{q}_j$. When expressed in components, the action (38) has the following form:

$$S_\sigma = \int dt \left[ \frac{\partial^2 K(q, \bar{q})}{\partial q^i \partial \bar{q}_j} \dot{q}^i \dot{q}_j + \text{fermions} \right].$$  \hspace{1cm} (39)

Another interesting example is the superconformally invariant superfield action[4]

$^{3}$ The standard convention for integration in $N = 4, d = 1$ superspace is $\int dt d^4\theta = \frac{1}{16} \int dt D^i \overline{D}_i \overline{D}^j \overline{D}_j$. 

$$S_{\text{Conf}} = - \int dt d^4 \theta \frac{\ln (q^i \bar{q}_i)}{q^j \bar{q}_j}. \quad (40)$$

A more detailed discussion of possible actions for the $q^i$ multiplet can be found in [13]. In particular, there exists a superpotential-type off-shell invariant which, however, does not give rise in components to any scalar potential for the physical bosons. Instead, it produces a Wess-Zumino type term of the first order in the time derivative. It can be interpreted as a coupling to a four-dimensional background abelian gauge field. The superpotential just mentioned admits a concise manifestly supersymmetric superfield formulation, as an integral over an analytic subspace of $N = 4, d = 1$ harmonic superspace [13].

Notice that the $q^i$ supermultiplet can be considered as a fundamental one, since all other $N = 4$ supermultiplets can be obtained from $q^i$ by reduction. We will consider how such reduction works in the last section of this Lectures.

**N=4, d=1 “tensor” multiplet – (3, 4, 1).**

The “tensor” multiplet includes three $N = 4$ bosonic superfields which can be combined in a $N = 4$ isovector real superfield $V_{ij}$ ($V_{ij} = V_{ji}$ and $\bar{V}^{ik} = \epsilon_{ij}^{\prime} \epsilon_{kk}^{\prime} V_{j}^{i\prime} V_{k}^{i\prime}$). The irreducibility constraints may be written in the manifestly $SU(2)$-symmetric form

$$D^i (V_{jk}) = 0, \quad \bar{D}^i (V_{jk}) = 0. \quad (41)$$

The constraints (41) could be obtained by a direct dimensional reduction from the constraints defining the $N = 2, d = 4$ tensor multiplet [14], in which one suppresses the $SL(2, C)$ spinor indices of the $d = 4$ spinor derivatives, thus keeping only the doublet indices of the $R$-symmetry $SU(2)$ group. This is the reason why we can call it $N = 4, d = 1$ “tensor” multiplet. Of course, in one dimension no differential (notoph-type) constraints arise on the components of the superfield $V_{ij}$. The constraints (41) leave in $V_{ik}$ the following independent superfield projections:

$$V^{ik}, \quad D^i V^{kl} = -\frac{1}{3} (\epsilon^{ik} \chi^l + \epsilon^{il} \chi^k), \quad \bar{D}^i V^{kl} = \frac{1}{3} (\epsilon^{ik} \bar{\chi}^l + \epsilon^{il} \bar{\chi}^k), \quad D^i \bar{D}^k V_{ik}, \quad (42)$$

where

$$\chi^k \equiv D^i V_{ik}, \quad \bar{\chi}_k = \bar{\chi}^i = \bar{D}_i V^i_k. \quad (43)$$

Thus its off-shell component field content is just $(3, 4, 1)$. The $N = 4, d = 1$ superfield $V_{ik}$ subjected to the conditions (41) was introduced in [15] and, later on, rediscovered in [16, 17, 18].

As in the case of the superfield $q^i$, the general sigma-model action for the supermultiplet $(3, 4, 1)$ may be easily written as

$$S_\sigma = \int dt d^4 \theta K(V). \quad (44)$$
where \( K(q, \bar{q}) \) is an arbitrary function on \( q^i \) and \( q_j \).

As the last remark of this subsection, let us note that the “tensor” multiplet can be constructed in terms of the “hypermultiplet”. Indeed, let us represent \( V^{ij} \) as the following composite superfield:

\[
\tilde{V}^{11} = -i\sqrt{2} q^1 \bar{q}^1, \quad \tilde{V}^{22} = -i\sqrt{2} q^2 \bar{q}^2, \quad \tilde{V}^{12} = -\frac{i}{\sqrt{2}} (q^1 \bar{q}^2 + q^2 \bar{q}^1). \quad (45)
\]

One can check that, as a consequence of the “hypermultiplet” constraints (36), the composite superfield \( \tilde{V}^{ij} \) automatically obeys (41). This is just the relation established in [13].

The expressions (45) supply a rather special solution to the “tensor” multiplet constraints. In particular, they express the auxiliary field of \( \tilde{V}^{ij} \) through the time derivative of the physical components of \( q^i \), which contains no auxiliary fields. As a consequence, the superpotential of \( \tilde{V}^{ik} \) is a particular case of the \( q^i \) superpotential, which produces no genuine scalar potential for physical bosons and gives rise for them only to a Wess-Zumino type term of the first order in the time derivative.

**N=4, d=1 chiral multiplet – (2, 4, 2).**

The chiral \( N = 4 \) supermultiplet is the simplest one. It can be described, in full analogy with the \( N = 2 \) case, by a complex superfield \( \phi \) subjected to the constraints [20, 21]

\[
D^i \phi = 0, \quad \overline{D}_j \phi = 0. \quad (46)
\]

The sigma-model type action for this multiplet

\[
S_\sigma = \int d^4\theta K(\phi, \bar{\phi}) \quad (47)
\]

may be immediately extended to include the potential terms

\[
S_{pot} = \int d^2\bar{\theta} F(\phi) + \int d^2\theta \bar{F}(\phi). \quad (48)
\]

In more details, such a supermultiplet and the corresponding actions have been considered in [22].

**The ”old tensor” multiplet – (1, 4, 3).**

The last possibility corresponds to the single bosonic superfield \( u \). In this case no linear constraints appear, since four fermionic components are expressed through four spinor derivatives of \( u \). As it was shown in [22], one should impose

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\( ^4 \) This is a nonlinear version of the phenomenon which holds in general for \( d = 1 \) supersymmetry and was discovered at the linearized level in [1, 19].
some additional, second order in spinor derivatives, irreducibility constraints on $u$

$$D^{i}D_{i}e^{u} = \overline{D}_{i}D^{i}e^{u} = [D^{i}, \overline{D}_{i}]e^{u} = 0,$$  \hspace{1em} (49)

in order to pick up in $u$ the minimal off-shell field content \((1, 4, 3)\). Once again, the detailed discussion of this case can be found in \([22]\).

In fact, we could re-derive the multiplet $u$ from the tensor multiplet $V^{ik}$ discussed in the previous subsection. Indeed, one can construct the composite superfield

$$e^{\tilde{u}} = \frac{1}{\sqrt{V^{2}}},$$  \hspace{1em} (50)

which obeys just the constraints (49) as a consequence of (41). The relation (50) is of the same type as the previously explored substitution (45) and expresses two out of the three auxiliary fields of $\tilde{u}$ via physical bosonic fields of $V^{ik}$ and time derivatives thereof. The superconformal invariant action for the superfield $u$ can be also constructed \([22]\)

$$S_{\text{Conf}} = \int dt d^{4} \theta e^{u} u. \hspace{1em} (51)$$

By this, we complete the superfield description of all linear $N = 4, d = 1$ supermultiplets. But the story about $N = 4, d = 1$ supermultiplets is not finished. There are nonlinear supermultiplets which make everything much more interesting. Let us discuss here only one example - the nonlinear chiral multiplet \([4, 23]\).

**N=4, d=1 nonlinear chiral multiplet – (2, 4, 2).**

The idea of nonlinear chiral supermultiplets comes about as follows. If the two bosonic superfields $Z$ and $\overline{Z}$ parameterize the two dimensional sphere $SU(2)/U(1)$ instead of flat space, then they transform under $SU(2)/U(1)$ generators with the parameters $a, \bar{a}$ as

$$\delta Z = a + \bar{a}Z^{2}, \quad \delta \overline{Z} = \bar{a} + a\overline{Z}^{2},$$  \hspace{1em} (52)

With respect to the same group $SU(2)$, the $N = 4$ covariant derivatives could form a doublet

$$\delta D_{i} = -a \overline{D}_{i}, \quad \delta \overline{D}_{i} = \bar{a}D_{i}.$$  \hspace{1em} (53)

One may immediately check that the ordinary chirality conditions

$$D_{i}Z = 0, \quad \overline{D}_{i}\overline{Z} = 0$$

are not invariant with respect to (52), (53) and they should be replaced, if we wish to keep the $SU(2)$ symmetry. It is rather easy to guess the proper $SU(2)$ invariant constraints

$$D_{i}Z = -\alpha \overline{Z} \overline{D}_{i}Z, \quad \overline{D}_{i}\overline{Z} = \alpha Z D_{i}Z, \quad \alpha = \text{const}.$$  \hspace{1em} (54)
So, using the constraints (54), we restore the $SU(2)$ invariance, but the price for this is just the nonlinearity of the constraints. Let us stress that $N = 4, d = 1$ supersymmetry is the minimal one where the constraints (54) may appear, because the covariant derivatives (and the supercharges) form a doublet of $SU(2)$ which cannot be real.

The $N = 4, d = 1$ nonlinear chiral supermultiplet involves one complex scalar bosonic superfield $Z$ obeying the constraints (54). If the real parameter $\alpha \neq 0$, it is always possible to pass to $\alpha = 1$ by a redefinition of the superfields $Z, \bar{Z}$. So, one has only two essential values $\alpha = 1$ and $\alpha = 0$. The latter case corresponds to the standard $N = 4, d = 1$ chiral supermultiplet. Now one can write the most general $N = 4$ supersymmetric Lagrangian in $N = 4$ superspace

$$S = \int dtd^2\theta d^2\bar{\theta} \, K(Z, \bar{Z}) + \int dtd^2\theta \, F(Z) + \int dtd^2\bar{\theta} \, \bar{F}(\bar{Z}) .$$

(55)

Here $K(Z, \bar{Z})$ is an arbitrary function of the superfields $Z$ and $\bar{Z}$, while $F(Z)$ and $\bar{F}(\bar{Z})$ are arbitrary holomorphic functions depending only on $Z$ and $\bar{Z}$, respectively. Let us stress that our superfields $Z$ and $\bar{Z}$ obey the nonlinear variant of chirality conditions (54), but nevertheless the last two terms in the action $S$ (55) are still invariant with respect to the full $N = 4$ supersymmetry. Indeed, the supersymmetry transformations of the integrand of, for example, the second term in (55) read

$$\delta F(Z) = -\epsilon^i D_i F(Z) + 2i\epsilon^i \bar{\theta}_i \bar{D}_i F(Z) + 2i\bar{\epsilon}^i \theta_i D_i F(Z) .$$

(56)

Using the constraints (54) the first term in the r.h.s. of (56) may be rewritten as

$$-\epsilon^i D_i F = -\epsilon^i F_Z D_i Z = \alpha \epsilon^i F_Z Z \, D_i Z = \alpha \epsilon^i \bar{\theta}_i \int dZ \, F_Z Z .$$

(57)

Thus, all terms in (56) either are full time derivatives or disappear after integration over $d^2\bar{\theta}$.

The irreducible component content of $Z$, implied by (54), does not depend on $\alpha$ and can be defined as

$$z = Z|, \, \bar{z} = \bar{Z}|, \, A = -iD_i \bar{D}_i Z|, \, \bar{A} = -iD_i \bar{D}_i \bar{Z}|, \, \psi^i = \bar{D}_i Z|, \, \bar{\psi}^i = -D_i \bar{Z}|,$$

(58)

where $|$ means restricting expressions to $\theta_i = \bar{\theta}_j = 0$. All higher-dimensional components are expressed as time derivatives of the irreducible ones. Thus, the $N = 4$ superfield $Z$ constrained by (54) has the same field content as the linear chiral supermultiplet.

After integrating in (55) over the Grassmann variables and eliminating the auxiliary fields $A, \bar{A}$ by their equations of motion, we get the action in terms of physical components

$$S = \int dt \left\{ g \dot{z} \dot{\bar{z}} - i\alpha \frac{\dot{z} \bar{z}}{1 + \alpha^2 z \bar{z}} F_z + i\alpha \frac{\dot{z} \bar{z}}{1 + \alpha^2 z \bar{z}} \bar{F}_z - \frac{F_z \bar{F}_z}{g(1 + \alpha^2 z \bar{z})} + \text{fermions} \right\} ,$$

(59)
where
\[ g(z, \bar{z}) = \frac{\partial^2 K(z, \bar{z})}{\partial z \partial \bar{z}}, \quad F_z = \frac{dF(z)}{dz}, \quad \bar{F}_\bar{z} = \frac{d\bar{F}(\bar{z})}{d\bar{z}}. \] (60)

From the bosonic part of the action (59) one may conclude that the system contains a nonzero magnetic field with the potential
\[ A^0 = i\alpha F_z \bar{z}dz + \alpha^2 z \bar{z} - i\alpha F_{\bar{z}} z d\bar{z} \] (61)
and
\[ B = \alpha \left( \frac{F_z + \bar{F}_\bar{z}}{1 + \alpha^2 z \bar{z}} \right). \] (62)

As for the fermionic part of the kinetic term, it can be represented as follows:
\[ S_{KinF} = \frac{i}{4} \int dt (1 + \alpha^2 z \bar{z}) g \left( \bar{\psi} \frac{D\bar{\psi}}{dt} - \psi \frac{D\psi}{dt} \right), \] (63)
where
\[ D\psi = d\psi + \Gamma \psi dz + T^+ \bar{\psi} d\bar{z}, \quad D\bar{\psi} = d\bar{\psi} + \bar{\Gamma} \bar{\psi} d\bar{z} + T^- \psi dz, \] (64)
and
\[ \Gamma = \partial_z \log (1 + \alpha^2 z \bar{z}), \quad T^\pm = \pm \frac{\alpha}{1 + \alpha^2 z \bar{z}}. \] (65)

Clearly enough, \( \Gamma, \bar{\Gamma}, T^\pm \) define the components of the connection defining the configuration superspace. The components \( \Gamma \) and \( \bar{\Gamma} \) could be identified with the components of the symmetric connection on the base space equipped with the metric \( (1 + \alpha^2 z \bar{z}) g dz d\bar{z} \), while the rest does not have a similar interpretation.

Thus, we conclude that the main differences between the \( N = 4 \) supersymmetric mechanics with nonlinear chiral supermultiplet and the standard one are the coupling of the fermionic degrees of freedom to the background, via the deformed connection, the possibility to introduce a magnetic field, and the deformation of the bosonic potential.

So far, we presented the results without explanations about how they were found. It appears to be desirable to put the construction and study of \( N \)-extended supersymmetric models on a systematic basis by working out the appropriate superfield techniques. Such a framework exists and is based on a superfield nonlinear realization of the \( d = 1 \) superconformal group. It was pioneered in [22] and recently advanced in [24, 4, 6]. Its basic merits are, firstly, that in most cases it automatically yields the irreducibility conditions for \( d=1 \) superfields and, secondly, that it directly specifies the superconformal transformation properties of these superfields. The physical bosons and fermions, together with the \( d=1 \) superspace coordinates, prove to be coset parameters associated with the appropriate generators of the superconformal group. Thus, the differences in the field content of the various supermultiplets are attributed to different choices of the coset supermanifold inside the given superconformal group.
3 Nonlinear realizations

In the previous section we considered the $N = 4, d = 1$ liner supermultiplets and constructed some actions. But the most important questions concerning the irreducibility constraints and the transformation properties of the super-fields were given as an input. In this section we are going to demonstrate that most of the constraints and all transformation properties can be obtained automatically as results of using the nonlinear realization approach. Our consideration will be mostly illustrative – we will skip presenting any proofs. Instead, we will pay attention to the ideas and technical features of this approach.

3.1 Realizations in the coset space

The key statement of the nonlinear realization approach may be formulated as follows:

**Theorem 1.** If a group $G$ acts transitively on some space, and the subgroup $H$ preserves a given point of this space, then this action of the group $G$ may be realized by left multiplications on the coset $G/H$, while the coordinates which parameterize the coset $G/H$ are just the coordinates of the space.

As the simplest example let us consider the four-dimensional Poincaré group $\{P_\mu, M_{\mu\nu}\}$. Is is clear that the transformations which preserve some point are just rotations around this points. In other words, $H = \{M_{\mu\nu}\}$. Therefore, due to our Theorem, a natural realization of the Poincaré group can be achieved in the coset $G/H$. This coset contains only the translations $P_\mu$, and it is natural to parameterized this coset as

$$G/H = e^{ix_\mu P_\mu}. \quad (66)$$

It is evident that the Poincaré group may be realized on the four coordinates $\{x^\mu\}$. What is important here is that we do not need to know how our coordinates transform under the Poincaré group. Instead, we can deduce these transformations from the representation (66).

Among different cosets there are special ones which are called orthonormal. They may be described as follows. Let us consider the group $G$ with generators $\{X_i, Y_\alpha\}$ which obey the following relations:

$$[Y_\alpha, Y_\beta] = iC_{\alpha\beta}^\gamma Y_\gamma,$$

$$[Y_\alpha, X_i] = iC_{\alpha}^\beta X_j + iC_{\alpha\beta}^\gamma Y_\gamma,$$

$$[X_i, X_j] = iC_{ij}^k X_k + iC_{ij}^\alpha Y_\alpha, \quad (67)$$

This means that the transformations from $G$ relate any two arbitrary points in the space.

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5 This means that the transformations from $G$ relate any two arbitrary points in the space.
where $C$ are structure constants. We see that the generators $Y_\alpha$ form the subgroup $H$. The coset $G/H$ is called orthonormal if $C^\beta_{\alpha i} = 0$. In other words, this property means that the generators $X_i$ transform under some representation of the stability subgroup $H$. In what follows we will consider only such a coset. A more restrictive class of cosets—the symmetric spaces—corresponds to the additional constraints $C^\alpha_i = 0$.

A detailed consideration of the cosets and their geometric properties may be found in [25].

A very important class is made by the cosets which contain space-time symmetry generators as well as the generators of internal symmetries. In order to deal with such cosets we must

- introduce the coordinates for space-time translations (or/and super-translations);
- introduce the parameters for the rest of the generators in the coset. These additional parameters are treated as fields which depend on the space-time coordinates.

The fields (superfields) which appear as parameters of the coset will have inhomogeneous transformation properties. They are known as Goldstone fields. Their appearance is very important: they are definitely needed to construct an action, which is invariant with respect to transformations from the coset $G/H$. Let us repeat this point: in the nonlinear realization approach only the $H$-symmetry is manifest. The invariance under $G/H$ transformations is achieved through the interaction of matter fields with Goldstone ones.

Finally, let us note that the number of essential Goldstone fields does not always coincide with the number of coset generators. As we will see later, some of the Goldstone fields often can be expressed through other Goldstone fields. This is the so called Inverse Higgs phenomenon [26].

Now it is time to demonstrate how all this works on the simplest examples.

### 3.2 Realizations: Examples & Technique

**N=2, d=1 Super Poincaré.**

Let us start with the simplest example of the $N = 2, d = 1$ super Poincaré algebra which contains two super-translations $Q, \bar{Q}$ which anticommute on the time-translation $P$ (9)

$$\{Q, \bar{Q}\} = -2P.$$

In this case the stability subgroup is trivial and all generators are in the coset. Therefore, one should introduce coordinates for all generators:

$$g = G/H = e^{itP}e^{i\theta Q + i\bar{\theta} \bar{Q}}. \quad (68)$$

These coordinates $\{t, \theta, \bar{\theta}\}$ span $N = 2, d = 1$ superspace. Now we are going to find the realization of $\hat{N} = 2$ superalgebra (9) in this superspace. The first
step is to find the realization of the translation $P$. So, we act on the element 
\( g_0 \) from the left by $g_0 = e^{i \alpha P}$:

\[
g_0 \cdot g = e^{i \alpha P} \cdot e^{itP} e^{\theta Q + \bar{\theta} \bar{Q}} = e^{i(t + \alpha)P} e^{\theta Q + \bar{\theta} \bar{Q}} \equiv e^{it'P} e^{\theta' Q + \bar{\theta}' \bar{Q}}.
\]

Thus, we get the standard transformations of the coordinates

\[
P : \begin{cases} 
\delta t = \alpha \\
\delta \theta = \delta \bar{\theta} = 0.
\end{cases}
\]

Something more interesting happens for the supertranslations

\[
g_1 = e^{\epsilon Q + i \bar{Q}} : g_1 \cdot g = e^{\epsilon Q + i \bar{Q}} \cdot e^{itP} e^{\theta Q + \bar{\theta} \bar{Q}} = e^{itP} e^{\epsilon Q + i \bar{Q}} e^{\theta Q + \bar{\theta} \bar{Q}}.
\]

Now we need to bring the product of the exponents in (71) to the standard form

\[
e^{itP} e^{\theta Q + \bar{\theta} \bar{Q}}.
\]

In order to do this, one should use the Campbell-Hausdorff formulae

\[
e^{A} \cdot e^{B} = \exp \left( A + B + \frac{1}{2} [A, B] + \frac{1}{12} [A, [A, B]] - \frac{1}{12} [[A, B], B] + \ldots \right)
\]

Thus we will get

\[
e^{\epsilon Q + i \bar{Q}} e^{\theta Q + \bar{\theta} \bar{Q}} = e^{(\theta + \epsilon)Q + (\bar{\theta} + i) \bar{Q} + (\epsilon \bar{\theta} + \bar{\theta} \epsilon)P}.
\]

So, the supertranslation is realized as follows:

\[
Q, \bar{Q} : \begin{cases} 
\delta t = -i(\epsilon \bar{\theta} + \bar{\epsilon} \theta) \\
\delta \theta = \epsilon, \delta \bar{\theta} = \bar{\epsilon}.
\end{cases}
\]

This is just the transformation (145) we used before. Of course, in this rather simple case we could find the answer without problems. But the lesson is that the same procedure works always, for any (super)group and any coset. Let us consider a more involved example.

**d=1 Conformal group.**

The conformal algebra in $d = 1$ contains three generators: translation $P$, dilatation $D$ and conformal boost $K$ obeying to the following relations:

\[
i [P, K] = -2D, \ i [D, P] = P, \ i [D, K] = -2K.
\]

The stability subgroup is again trivial and our coset may be parameterized as

\[
g = e^{itP} e^{iz(t)K} e^{iu(t)D}.
\]
Let us stress that we want to obtain a one dimensional realization. Therefore, we can introduce only one coordinate - time \( t \). But we have three generators in the coset. The unique solution\(^6\) is to consider the two other coordinates as functions of time. Thus, we have to introduce two fields \( z(t) \) and \( u(t) \) which are just Goldstone fields.

Let us find the realization of the conformal group in our coset (76). The translation is realized trivially, as in (70), so we will start from the dilatation
\[
g_0 = e^{iaD} : \quad g_0 \cdot g = e^{iaD} \cdot e^{itP} e^{iz(t)K} e^{iu(t)D}. \tag{77}\]

Now we have a problem - how to commute the first exponent in (77) with the remaining ones? The Campbell-Hausdorff formulae (72) do not help us too much, because the series does not terminate. A useful trick is to represent r.h.s. of (77) in the form
\[
e^{iaD} e^{itP} e^{-iaD} e^{iz(t)K} e^{iu(t)D} \tag{78}\]

In order to evaluate (78) we will use the following formulas due to Bruno Zumino [27]:
\[
e^A B e^{-A} \equiv e^{A \wedge B}, \quad e^A e^B e^{-A} \equiv e^{A \wedge B}, \tag{79}\]
where
\[
A^n \wedge B \equiv [A, [A, \ldots, B]] \ldots \tag{80}\]

Using (79) we can immediately find
\[
D : \quad \begin{cases} \delta t = a t \\ \delta u = a, \quad \delta z = -a z \end{cases} \tag{81}\]

We see that the field \( u(t) \) is shifted by the constant parameter \( a \) under dilatation. Such Goldstone field is called \textit{dilaton}.

Finally, we should find the transformations of the coordinates under conformal boost \( K \):
\[
g_1 = e^{ibK} : \quad g_1 \cdot g = e^{ibK} e^{itP} e^{iz(t)K} e^{iu(t)D}. \tag{82}\]

Here, in order to commute the first two terms we will use the following trick: let us represent the first two exponents in the r.h.s. of (82) as follows:
\[
e^{ibK} e^{itP} = e^{itP} \cdot \tilde{g} \Rightarrow \tilde{g} = e^{-itP} e^{ibK} e^{itP}. \tag{83}\]

\(^6\) Of course, we may put some of the generators \( K \) and \( D \), or even both of them, in the stability subgroup \( H \). But in this case all matter fields should realize a representation of \( H = \{K, D\} \) which never happens.
So, we again can use (79) to calculate\[^7\] \( \tilde{g} \)

\[
\tilde{g} = e^{ibK + 2ibtD + ibt^2P} \approx e^{ibK} e^{2ibtD} e^{ibt^2P}.
\]

Thus, we have

\[
K:\begin{cases}
\delta t = bt^2 \\
\delta u = 2bt, \quad \delta z = b - 2btz
\end{cases}
\]  (84)

Now we know how to find the transformation properties of the coordinates and Goldstone (super)fields for any cosets. The next important question is how to construct the invariant and/or covariant objects.

### 3.3 Cartan's forms

The Cartan's forms for the coset \( g = G/H \) are defined as follows:

\[
g^{-1} dg = i\omega^i X_i + i\omega^\alpha Y_\alpha,
\]  (85)

where the generators \( \{X_i, Y_\alpha\} \) obey to (67).

By the definition (85) the Cartan's forms are invariant with respect to left multiplication of the coset element \( g \). Let us represent the result of the left multiplication of the coset element \( g \) as follows:

\[
g_0 \cdot g = \tilde{g} \cdot h.
\]  (86)

Here, \( \tilde{g} \) belongs to the coset, while the element \( h \) lies in the stability subgroup \( H \). Now we have

\[
i\omega^i X_i + i\omega^\alpha Y_\alpha = h^{-1} \left( \tilde{g}^{-1} d\tilde{g} \right) h + h^{-1} dh.
\]  (87)

If the coset \( g = G/H \) is orthonormal, then

\[
\tilde{\omega}^i X_i = h \cdot \omega^i X_i \cdot h^{-1},
\]

\[
\tilde{\omega}^\alpha Y_\alpha = h \cdot \omega^\alpha Y_\alpha \cdot h^{-1} + idh \cdot h^{-1}.
\]  (88)

Thus we see that the forms \( \omega^i \) which belong to the coset transform homogeneously, while the forms \( \omega^\alpha \) on the stability subgroup transform like connections and can be used to construct covariant derivatives.

Finally, one should note that, in the exponential parametrization we are using through these lectures, the evaluation of the Cartan’s forms is based on the following identity [27]:

\[
e^{-A} de^A = \frac{1 - e^{-A}}{A} \wedge dA.
\]  (89)

Now it is time for some examples.

\[^7\] We are interested in the infinitesimal transformations and omit all terms which are higher then linear in the parameters.
Choosing the parametrization of the group element as in (68) one may immediately find the Cartan’s forms

\[ \omega_P = dt - i (d\bar{\theta} + d\theta), \quad \omega_Q = d\theta, \quad \bar{\omega}_Q = d\bar{\theta}. \]  

(90)

Thus, the Cartan’s forms (90) just coincide with the covariant differentials (22) we guessed before.

**d=1 Conformal group**

The Cartan’s forms for the coset (76) may be easily calculated using (89)

\[ \omega_P = e^{-u}, \quad \omega_D = du - 2zdt, \quad \omega_K = e^u [dz + z^2 dt]. \]  

(91)

What is the most interesting here is the structure of the form \( \omega_D \). Indeed, from the previous consideration we know that \( \omega_D \) being coset forms, transforms homogeneously. Therefore, the following condition:

\[ \omega_D = 0 \Rightarrow z = \frac{1}{2} \dot{u} \]  

(92)

is invariant with respect to the whole conformal group! This means that the Goldstone field \( z(t) \) is unessential and can be expressed in terms of the dilaton \( u(t) \). This is the simplest variant of the inverse Higgs phenomenon [26]. With the help of the Cartan forms (92) one could construct the simplest invariant action

\[ S = - \int (\omega_K + m^2 \omega_P) = \int dt \left[ \frac{1}{4} e^u \dot{u}^2 - m^2 e^{-u} \right] = \int dt \left[ \dot{\rho}^2 - \frac{m^2}{\rho^2} \right], \quad \rho \equiv e^{-u/2}, \]  

(93)

with \( m \) being a parameter of the dimension of mass. The action (93) is just the conformal mechanics action [28].

It is rather interesting that we could go a little bit further.

The basis (75) in the conformal algebra \( so(1,2) \) can naturally be called “conformal”, as it implies the standard \( d = 1 \) conformal transformations for the time \( t \). Now we pass to another basis in the same algebra

\[ \hat{K} = mK - \frac{1}{m} P, \quad \hat{D} = mD. \]  

(94)

This choice will be referred to as the “AdS basis” for a reason soon to be made clear soon.

The conformal algebra (75) in the AdS basis (94) reads
\[ i\left[ P, \hat{D} \right] = -mP, \quad i\left[ \hat{K}, \hat{D} \right] = 2P + m\hat{K}, \quad i\left[ P, \hat{K} \right] = -2\hat{D}. \] (95)

An element of \( SO(1, 2) \) in the AdS basis is defined to be
\[ g = e^{iyP}e^{i\phi(y)\hat{D}}e^{i\Omega(y)\hat{K}}. \] (96)

Now we are in a position to explain the motivation for the nomenclature “AdS basis”. The generator \( \hat{K} \) (94) can be shown to correspond to a \( SO(1,1) \) subgroup of \( SO(1,2) \). Thus the parameters \( y \) and \( \phi(y) \) in (96) parameterize the coset \( SO(1,2)/SO(1,1) \), i.e. \( \text{AdS}_2 \). The parametrization (96) of \( \text{AdS}_2 \) is a particular case of the so-called “solvable subgroup parametrization” of the AdS spaces. The \( d = 4 \) analog of this parametrization is the parametrization of the \( \text{AdS}_5 \) space in such a way that its coordinates are still parameters associated with the 4-translation and dilatation generators \( P_m, D \) of \( SO(2,4) \), while it is the subgroup \( SO(1,4) \) with the algebra \( \propto \{ P_m - K_m, so(1,3) \} \) which is chosen as the stability subgroup.

The difference in the geometric meanings of the coordinate pairs \((t, u(t))\) and \((y, \phi(y))\) is manifested in their different transformation properties under the same \( d = 1 \) conformal transformations. Left shifts of the \( SO(1,2) \) group element in the parametrization (96) induce the following transformations:
\[ \delta y = a(y) + \frac{1}{m^2} c e^{2m\phi}, \quad \delta \phi = \frac{1}{m} \dot{a}(y) = \frac{1}{m} (b + 2cy), \quad \delta \Omega = \frac{1}{m} c e^{m\phi}. \] (97)

We observe the modification of the special conformal transformation of \( y \) by a field-dependent term.

The relevant left-invariant Cartan forms are given by the following expressions:
\[ \hat{\omega}_D = \frac{1 + A^2}{1 - A^2} d\phi - 2 A \left( e^{-m\phi} dy - \frac{A}{1 - A^2} e^{-m\phi} dy \right), \]
\[ \hat{\omega}_P = \frac{1 + A^2}{1 - A^2} e^{-m\phi} dy - 2 A \left( e^{-m\phi} dy - \frac{A}{1 - A^2} e^{-m\phi} dy \right), \]
\[ \hat{\omega}_K = m \frac{A}{1 - A^2} \left( \Lambda e^{-m\phi} dy - d\phi \right) + \frac{dA}{1 - A^2}, \] (98)

where
\[ A = \tanh \Omega. \] (99)

As in the previous realization, the field \( A(y) \) can be eliminated by imposing the inverse Higgs constraint
\[ \hat{\omega}_D = 0 \Rightarrow \partial_y \phi = 2 e^{-m\phi} \frac{A}{1 + A^2}, \] (100)
whence \( A \) is expressed in terms of \( \phi \)
\[ A = \partial_y \phi e^{m\phi} \frac{1}{1 + \sqrt{1 - e^{2m\phi}(\partial_y \phi)^2}}. \] (101)
The $SO(1, 2)$ invariant distance on AdS$_2$ can be defined, prior to imposing any constraints, as

$$ds^2 = -\hat{\omega}_P^2 + \hat{\omega}_D^2 = -e^{-2m\phi} dy^2 + d\phi^2 .$$

(102)

Making the redefinition

$$U = e^{-m\phi} ,$$

it can be cast into the standard Bertotti-Robinson metrics form

$$ds^2 = -U^2 dy^2 + (1/m^2)U^{-2}dU^2 ,$$

(103)

with $1/m$ as the inverse AdS$_2$ radius,

$$\frac{1}{m} = R .$$

(104)

The invariant action can now be constructed from the new Cartan forms (98) which, after substituting the inverse Higgs expression for $\Lambda$, eq. (101), read

$$\hat{\omega}_P = e^{-m\phi} \sqrt{1 - e^{2m\phi}(\partial_y \phi)^2} dy ,$$

$$\hat{\omega}_K = -\frac{m}{2} e^{-m\phi} \left( 1 - \sqrt{1 - e^{2m\phi}(\partial_y \phi)^2} \right) dy + \text{Tot. deriv.} \times dy .$$

(105)

The invariant action reads

$$S = -\int (\hat{\mu} \hat{\omega}_P - qe^{-m\phi}) = -\int dy e^{-m\phi} \left( \hat{\mu} \sqrt{1 - e^{2m\phi}(\partial_y \phi)^2} - q \right) .$$

(106)

After the above field redefinitions it is recognized as the radial-motion part of the “new” conformal mechanics action [29]. Notice that the second term in (106) is invariant under (97), up to a total derivative in the integrand. The action can be rewritten in a manifestly invariant form (with a tensor Lagrangian) by using the explicit expression for $\hat{\omega}_K$ in (105)

$$S = \int \left[ (q - \hat{\mu}) \hat{\omega}_P - (2/m)q \hat{\omega}_K \right] .$$

(107)

Now we are approaching the major point. We see that the “old” and “new” conformal mechanics models are associated with two different nonlinear realizations of the same $d = 1$ conformal group $SO(1, 2)$ corresponding, respectively, to the two different choices (75) and (96) of the parametrization of the group element. The invariant actions in both cases can be written as integrals of linear combinations of the left-invariant Cartan forms. But the latter cannot depend on the choice of parametrization. Then the actions (93) and (106) should in fact coincide with each other, up to a redefinition of the free parameters of the actions. Thus two conformal mechanics models are equivalent modulo redefinitions of the involved time coordinate and field. This statement should be contrasted with the previous view of the “old” conformal mechanics model as a “non-relativistic” approximation of the “new” one.
3.4 Nonlinear realizations and supersymmetry

One of the interesting applications of the nonlinear realizations technique is that of establishing irreducibility constraints for superfields.

In the present section we focus on the case of $N=4, d=1$ supersymmetry (with 4 real supercharges) and propose to derive its various irreducible off-shell superfields from different nonlinear realizations of the most general $N=4, d=1$ superconformal group $D(2,1;\alpha)$. An advantage of this approach is that it simultaneously specifies the superconformal transformation properties of the superfields, though the latter can equally be used for constructing non-conformal supersymmetric models as well. As the essence of these techniques, any given irreducible $N=4, d=1$ superfield comes out as a Goldstone superfield parameterizing, together with the $N=4, d=1$ superspace coordinates, some supercoset of $D(2,1;\alpha)$. The method was already employed in the paper [13] where the off-shell multiplet $(3, 4, 1)$ was re-derived from the nonlinear realization of $D(2,1;\alpha)$ in the coset with an $SL(2, R) \times [SU(2)/U(1)]$ bosonic part (the second $SU(2) \subset D(2,1;\alpha)$ was placed into the stability subgroup).

Here we consider nonlinear realizations of the same conformal supergroup $D(2,1;\alpha)$ in its other coset superspaces. In this way we reproduce the $(4, 4, 0)$ multiplet and also derive two new nonlinear off-shell multiplets. The $(4, 4, 0)$ multiplet is represented by superfields parameterizing a supercoset with the bosonic part being $SL(2, R) \times SU(2)$, where the dilaton and the three parameters of $SU(2)$ are identified with the four physical bosonic fields. One of the new Goldstone multiplets is a $d=1$ analog of the so-called nonlinear multiplet of $N=2, d=4$ supersymmetry. It has the same off-shell contents $(3, 4, 1)$ as the multiplet employed in [13], but it obeys a different constraint and enjoys different superconformal transformation properties. It corresponds to the specific nonlinear realization of $D(2,1;\alpha)$, where the dilatation generator and one of the two $SU(2)$ subgroups are placed into the stability subgroup. One more new multiplet of a similar type is obtained by placing into the stability subgroup, along with the dilatation and three $SU(2)$ generators, also the $U(1)$ generator from the second $SU(2) \subset D(2,1;\alpha)$. It has the same field content as a chiral $N=4, d=1$ multiplet, i.e. $(2, 4, 2)$. Hence, it may be termed as the nonlinear chiral supermultiplet. It is exceptional, in the sense that no analogs for it are known in $N=2, d=4$ superspace.

Supergroup $D(2,1;\alpha)$ and its nonlinear realizations

We use the standard definition of the superalgebra $D(2,1;\alpha)$ with the notations of ref. [13]. It contains nine bosonic generators which form a direct sum of $sl(2)$ with generators $P,D,K$ and two $su(2)$ subalgebras with generators $V,\overline{V},V_3$ and $T,\overline{T},T_3$, respectively

\begin{align}
&i [D,P] = P, \quad i [D,K] = -K, \quad i [P,K] = -2D, \quad i [V_3,V] = -V, \quad i [V_3,\overline{V}] = \overline{V}, \\
&i [V,\overline{V}] = 2V_3, \quad i [T_3,T] = -T, \quad i [T_3,\overline{T}] = \overline{T}, \quad i [T,\overline{T}] = 2T_3.
\end{align} (108)
The eight fermionic generators $Q^i, \overline{Q}_i, S^i, \overline{S}_i$ are in the fundamental representations of all bosonic subalgebras (in our notation only one $su(2)$ is manifest, viz. the one with generators $V, \nabla, V_3$)

\[
i [D, Q^i] = \frac{1}{2} Q^i, \quad i [D, S^i] = -\frac{1}{2} S^i, \quad i [P, S^i] = -Q^i, \quad i [K, Q^i] = S^i, \]

\[
i [V_3, Q^1] = \frac{1}{2} Q^1, \quad i [V_3, Q^2] = -\frac{1}{2} Q^2, \quad i [V, Q^1] = Q^2, \quad i [V, \overline{Q}_2] = -\overline{Q}_1, \]

\[
i [V_3, S^1] = \frac{1}{2} S^1, \quad i [V_3, S^2] = -\frac{1}{2} S^2, \quad i [V, S^1] = S^2, \quad i [V, \overline{S}_2] = -\overline{S}_1, \]

\[
i [T_3, Q^1] = \frac{1}{2} Q^1, \quad i [T_3, S^1] = \frac{1}{2} S^1, \quad i [T, Q^1] = Q^1, \quad i [T, S^1] = S^1 \quad (109)\]

(and c.c.). The splitting of the fermionic generators into the $Q$ and $S$ sets is natural and useful, because $Q^i, \overline{Q}_i$ together with $P$ form $N = 4, d = 1$ super Poincaré subalgebra, while $S^i, \overline{S}_i$ generate superconformal translations

\[
\{Q^i, \overline{Q}_j\} = -2\delta^i_j P, \quad \{S^i, \overline{S}_j\} = -2\delta^i_j K. \quad (110)\]

The non-trivial dependence of the superalgebra $D(2, 1; \alpha)$ on the parameter $\alpha$ manifests itself only in the cross-anticommutators of the Poincaré and conformal supercharges

\[
\{Q^i, S^j\} = -2(1 + \alpha) e^{b} T, \quad \{Q^1, \overline{S}_2\} = 2\alpha \nabla, \quad \{Q^1, S_1\} = -2D - 2\alpha V_3 + 2(1 + \alpha) T_3, \quad \{Q^2, \overline{S}_1\} = -2\alpha V_3, \quad \{Q^2, S_2\} = -2D + 2\alpha V_3 + 2(1 + \alpha) T_3 \quad (111)\]

(and c.c.). The generators $P, D, K$ are chosen to be hermitian, and the remaining ones obey the following conjugation rules:

\[
(T)_i^j = T, \quad (T_3)_i^j = -T_3, \quad (V)_i^j = \nabla, \quad (V_3)_i^j = -V_3, \quad (Q^i)_i^j = \overline{Q}_j, \quad (S^i)_i^j = \overline{S}_j. \quad (112)\]

The parameter $\alpha$ is an arbitrary real number. For $\alpha = 0$ and $\alpha = -1$ one of the $su(2)$ algebras decouples and the superalgebra $su(1, 1|2) \oplus su(2)$ is recovered. The superalgebra $D(2, 1; 1)$ is isomorphic to $osp(4^*|2)$.

We will be interested in diverse nonlinear realizations of the superconformal group $D(2, 1; \alpha)$ in its coset superspaces. As a starting point we shall consider the following parametrization of the supercoset:

\[
g = e^{itP} e^{\theta_i Q^i + \overline{\theta}_i \overline{Q}_i} e^{\psi_i S^i + \overline{\psi}_i \overline{S}_i} e^{izK} e^{iuD} e^{i\psi V + i\overline{\psi} \overline{V}} e^{\phi \psi V_3}. \quad (113)\]

The coordinates $t, \theta_i, \overline{\theta}_i$ parameterize the $N = 4, d = 1$ superspace. All other supercoset parameters are Goldstone $N = 4$ superfields. The group $SU(2) \propto (V, \nabla, V_3)$ linearly acts on the doublet indices $i$ of spinor coordinates and Goldstone fermionic superfields, while the bosonic Goldstone superfields $\varphi, \overline{\varphi}, \phi$ parameterize this $SU(2)$. Another $SU(2)$, as a whole, is placed in the stability subgroup and acts only on fermionic Goldstone superfields and $\theta$'s,
mixing them with their conjugates. With our choice of the $SU(2)$ coset, we are led to assume that $\alpha \neq 0$.

The left-covariant Cartan one-form $\Omega$ with values in the superalgebra $D(2,1;\alpha)$ is defined by the standard relation

$$g^{-1} dg = \Omega . \quad (114)$$

In what follows we shall need the explicit structure of several important one-forms in the expansion of $\Omega$ over the generators,

$$\omega_D = idu - 2 \left( \bar{\psi}^i d\theta_i + \psi_i d\bar{\theta}^i \right) - 2izd\tilde{t} ,$$

$$\omega_V = \frac{e^{-i\phi}}{1 + \Lambda \bar{\Lambda}} \left[ id\Lambda + \hat{\omega}_V + \Lambda^2 \hat{\omega}_V - \Lambda \hat{\omega}_V \right] , \quad \omega_{V_3} = d\phi + \frac{1}{1 + \Lambda \bar{\Lambda}} \left[ i \left( d\Lambda \bar{\Lambda} - \Lambda d\bar{\Lambda} \right) + (1 - \Lambda \bar{\Lambda}) \right] \hat{\omega}_{V_3} - 2 \left( \Lambda \hat{\omega}_V - \bar{\Lambda} \hat{\omega}_V \right) . \quad (115)$$

Here

$$\hat{\omega}_V = 2\alpha \left[ \psi_2 d\bar{\theta}^1 - \psi_1 \right] \left( d\bar{\theta}_2 - \psi_2 d\bar{\theta}^1 \right) , \quad \hat{\omega}_V = 2\alpha \left[ \bar{\psi}^2 d\theta_1 - \psi_1 \right] \left( d\theta^2 - \bar{\psi}^2 d\bar{\theta}^1 \right) ,$$

$$\hat{\omega}_{V_3} = 2\alpha \left[ \psi_1 d\theta_1 - \psi_2 d\theta_2 + \psi^2 d\theta_2 + (\psi_1 \psi_1 - \bar{\psi}^2 \psi_2) d\bar{\theta} \right] , \quad (116)$$

and

$$d\tilde{t} \equiv dt + i \left( \bar{\theta}_i d\bar{\theta}^i + \bar{\theta}^i d\theta_i \right) , \quad \delta \theta_i = \bar{\theta}_i \frac{\partial}{\partial \theta_i} \right) , \quad \delta \Lambda = a + \bar{a} \Lambda , \quad \delta \phi = i \left( a \bar{\Lambda} - \bar{a} \Lambda \right) . \quad (117)$$

The semi-covariant (fully covariant only under Poincaré supersymmetry) spinor derivatives are defined by

$$D^i = \frac{\partial}{\partial \theta_i} + i \bar{\theta}^i \partial_t , \quad \bar{D}_i = \frac{\partial}{\partial \bar{\theta}^i} + i \theta_i \partial_t , \quad \{ D^i , \bar{D}_j \} = 2i \delta^i_j \partial_t . \quad (119)$$

Let us remind that the transformation properties of the $N = 4$ superspace coordinates and the basic Goldstone superfields under the transformations of the supergroup $D(2,1;\alpha)$ could be easily found, as we did in the previous sections. Here we give the explicit expressions only for the variations of our superspace coordinates and superfields, with respect to two $SU(2)$ subgroup. They are generated by the left action of the group element

$$g_0 = e^{i a V + \bar{a} \bar{V}} e^{i b T + \bar{a} \bar{T}} \quad (120)$$

and read

$$\delta \theta_1 = \bar{b} \bar{\theta}^1 - \bar{a} \theta_2 , \quad \delta \theta_2 = -\bar{b} \bar{\theta}^1 + a \theta_1 ,$$

$$\delta \Lambda = a + \bar{a} \Lambda , \quad \delta \phi = i \left( a \bar{\Lambda} - \bar{a} \Lambda \right) . \quad (121)$$
The basic idea of our method is to impose the appropriate $D(2, 1; \alpha)$ covariant constraints on the Cartan forms (114), (115), so as to end up with some minimal $N = 4, d = 1$ superfield set carrying an irreducible off-shell multiplet of $N = 4, d = 1$ supersymmetry. Due to the covariance of the constraints, the ultimate Goldstone superfields will support the corresponding nonlinear realization of the superconformal group $D(2, 1; \alpha)$.

Let us elaborate on this in some detail. It was the desire to keep $N = 4, d = 1$ Poincaré supersymmetry unbroken that led us to associate the Grassmann coordinates $\theta_i, \bar{\theta}^i$ with the Poincaré supercharges in (113) and the fermionic Goldstone superfields $\psi_i, \bar{\psi}^i$ with the remaining four supercharges which generate conformal supersymmetry. The minimal number of physical fermions in an irreducible $N = 4, d = 1$ supermultiplet is four, and it nicely matches with the number of fermionic Goldstone superfields in (113), the first components of which can so be naturally identified with the fermionic fields of the ultimate Goldstone supermultiplet. On the other hand, we can vary the number of bosonic Goldstone superfields in (113): by putting some of them equal to zero we can enlarge the stability subgroup by the corresponding generators and so switch to another coset with a smaller set of parameters. Thus, for different choices of the stability subalgebra, the coset (113) will contain a different number of bosonic superfields, but always the same number of fermionic superfields $\psi_i, \bar{\psi}^i$. Yet, the corresponding sets of bosonic and fermionic Goldstone superfields contain too many field components, and it is natural to impose on them the appropriate covariant constraints, in order to reduce the number of components, as much as possible. For preserving off-shell $N = 4$ supersymmetry these constraints must be purely kinematical, i.e. they must not imply any dynamical restriction, such as equations of motion.

Some of the constraints just mentioned above should express the Goldstone fermionic superfields in terms of spinor derivatives of the bosonic ones. On the other hand, as soon as the first components of the fermionic superfields $\psi_i, \bar{\psi}^k$ are required to be the only physical fermions, we are led to impose much the stronger condition that all spinor derivatives of all bosonic superfields be properly expressed in terms of $\psi_i, \bar{\psi}^i$. Remarkably, the latter conditions will prove to be just the irreducibility constraints picking up irreducible $N = 4$ supermultiplets.

Here we will consider in details only the most general case when the coset (113) contains all four bosonic superfields $u, \varphi, \bar{\varphi}, \phi$. Looking at the structure of the Cartan 1-forms (115), it is easy to find that the covariant constraints which express all spinor covariant derivatives of bosonic superfields in terms of the Goldstone fermions amount to setting equal to zero the spinor projections of these 1-forms (these conditions are a particular case of the inverse Higgs effect [26]). Thus, in the case at hand we impose the following constraints:

$$\omega_D = \omega_V = \bar{\omega}_V = \omega_{V_3} = 0,$$

(122)
where $|\rangle$ means restriction to spinor projections. These constraints are manifestly covariant under the whole supergroup $D(2, 1; \alpha)$. They allow one to express the Goldstone spinor superfields as the spinor derivatives of the residual bosonic Goldstone superfields $u, A, \bar{A}, \phi$ and imply some irreducibility constraints for the latter

\begin{align*}
D^1 \Lambda &= -2i\alpha (\bar{\psi}^1 + A\bar{\psi}^2), \quad D^1 \bar{\Lambda} = -2i\alpha (\bar{\psi}^2 - \bar{\Lambda}\psi^1), \quad D^1 \phi = -2\alpha (\bar{\psi}^1 + A\bar{\psi}^2), \\
D^2 \Lambda &= 2i\alpha (\bar{\psi}^1 + A\bar{\psi}^2), \quad D^2 \bar{\Lambda} = -2i\alpha (\bar{\psi}^2 - \bar{\Lambda}\psi^1), \quad D^2 \phi = 2\alpha (\bar{\psi}^2 - \bar{\Lambda}\psi^1), \\
D^1 u &= 2i\bar{\psi}^1, \quad D^2 u = 2i\bar{\psi}^2, \quad \dot{u} = 2z
\end{align*}

(123)

(and c.c.). The irreducibility conditions, in this and other cases which we shall consider further, arise due to the property that the Goldstone fermionic superfields are simultaneously expressed by (123) in terms of spinor derivatives of different bosonic superfields. Then, eliminating these spinor superfields, we end up with the relations between the spinor derivatives of bosonic Goldstone superfields. In order to make the use of these constraints the most feasible, it is advantageous to pass to the new variables

\begin{align*}
q^1 &= \frac{e^{\frac{i}{2}(au-i\phi)}}{\sqrt{1 + \Lambda A}} \Lambda, \quad q^2 = -\frac{e^{\frac{i}{2}(au-i\phi)}}{\sqrt{1 + \Lambda A}} \bar{\Lambda}, \quad \bar{q}_1 = \frac{e^{\frac{i}{2}(au+i\phi)}}{\sqrt{1 + \Lambda A}} \bar{\Lambda}, \quad \bar{q}_2 = -\frac{e^{\frac{i}{2}(au+i\phi)}}{\sqrt{1 + \Lambda A}} \Lambda.
\end{align*}

(124)

In terms of these variables the irreducibility constraints acquire the manifestly $SU(2)$ covariant form

\begin{equation}
D^{(i} q^{j)} = 0, \quad D^\dagger^{(i} q^{j)} = 0.
\end{equation}

(125)

This is just the $N = 4, d = 1$ hypermultiplet multiplet we considered in the previous section.

The rest of the supermultiplets from Table 1 may be obtained similarly. For example, the tensor $N = 4, d = 1$ supermultiplet corresponds to the coset with the $V_3$ generator in the stability subgroup and so on. The detailed discussion of all cases may be found in [4].

4 N=8 Supersymmetry

Most of the models explored in the previous sections possess $N \leq 4, d = 1$ supersymmetries. Much less is known about higher-$N$ systems. Some of them were addressed many years ago in the seminal paper [15], within an on-shell Hamiltonian approach. Some others (with $N=8$) received attention lately [30, 31].

As we stressed many times, the natural formalism for dealing with supersymmetric models is the off-shell superfield approach. Thus, for the construction of new SQM models with extended $d=1$ supersymmetry, one needs, first of all, the complete list of the corresponding off-shell $d=1$ supermultiplets and the superfields which encompass these multiplets. One of the peculiarities of
$d=1$ supersymmetry is that some of its off-shell multiplets cannot be obtained via a direct dimensional reduction from the multiplets of higher-$d$ supersymmetries with the same number of spinorial charges. Another peculiarity is that some on-shell multiplets of the latter have off-shell $d=1$ counterparts.

In the previous section we considered nonlinear realizations of the finite-dimensional $N=4$ superconformal group in $d=1$. We showed that the irreducible superfields representing one or another off-shell $N=4, d=1$ supermultiplet come out as Goldstone superfields parameterizing one or another coset manifold of the superconformal group. The superfield irreducibility constraints naturally emerge as a part of manifestly covariant inverse Higgs [26] conditions on the relevant Cartan superforms.

This method is advantageous in that it automatically specifies the superconformal properties of the involved supermultiplets, which are of importance. The application of the nonlinear realization approach to the case of $N=8, d=1$ supersymmetry was initiated in [6]. There, nonlinear realizations of the $N=8, d=1$ superconformal group $OSp(4^*|4)$ in its two different cosets were considered, and it was shown that two interesting $N=8, d=1$ multiplets, with off-shell field contents $(3, 8, 5)$ and $(5, 8, 3)$, naturally come out as the corresponding Goldstone multiplets. These supermultiplets admit a few inequivalent splittings into pairs of irreducible off-shell $N=4, d=1$ multiplets, such that different $N=4$ superconformal subgroups of $OSp(4^*|4)$, viz. $SU(1,1|2)$ and $OSp(4^*|2)$, are manifest for different splittings. Respectively, the off-shell component action of the given $N=8$ multiplet in general admits several different representations in terms of $N=4, d=1$ superfields.

Now we are going to present a superfield description of all other linear off-shell $N=8, d=1$ supermultiplets with 8 fermions, in both $N=8$ and $N=4$ superspaces.

Towards deriving an exhaustive list of off-shell $N=8$ supermultiplets and the relevant constrained $N=8, d=1$ superfields, we could proceed in the same way as in the case of $N=4$ supermultiplets, i.e. by considering nonlinear realizations of all known $N=8$ superconformal groups in their various cosets. However, this task is more complicated, as compared to the $N=4$ case, in view of the existence of many inequivalent $N=8$ superconformal groups ($OSp(4^*|4)$, $OSp(8|2)$, $F(4)$ and $SU(1,1|4)$, see e.g. [2]), with numerous coset manifolds.

In order to avoid these complications, we take advantage of two fortunate circumstances. Firstly, as we already know, the field contents of linear off-shell multiplets of $N=8, d=1$ supersymmetry with 8 physical fermions range from $(8, 8, 0)$ to $(0, 8, 8)$, with the intermediate multiplets corresponding to all possible splittings of 8 bosonic fields into physical and auxiliary ones. Thus we are aware of the full list of such multiplets, independently of the issue of their interpretation as the Goldstone ones parameterizing the proper superconformal cosets.

The second circumstance allowing us to advance without resorting to the nonlinear realizations techniques is the aforesaid existence of various splittings of $N=8$ multiplets into pairs of irreducible $N=4$ supermultiplets. We
know how to represent the latter in terms of constrained \( N=4 \) superfields, so it proves to be a matter of simple algebra to guess the form of the four extra supersymmetries mixing the \( N=4 \) superfields inside each pair and extending the manifest \( N=4 \) supersymmetry to \( N=8 \). After fixing such pairs, it is again rather easy to embed them into appropriately constrained \( N=8, d=1 \) superfields.

4.1 \( N=8, d=1 \) superspace

The maximal automorphism group of \( N=8, d=1 \) super Poincaré algebra (without central charges) is \( SO(8) \) and so eight real Grassmann coordinates of \( N=8, d=1 \) superspace \( \mathbb{R}^{(1|8)} \) can be arranged into one of three 8-dimensional real irreps of \( SO(8) \). The constraints defining the irreducible \( N=8 \) supermultiplets in general break this \( SO(8) \) symmetry. So, it is preferable to split the 8 coordinates into two real quartets

\[
\mathbb{R}^{(1|8)} = (t, \theta_{ia}, \vartheta_{\alpha A}), \quad (\theta_{ia}) = \theta^{ia}, \quad (\vartheta_{\alpha A}) = \vartheta^\alpha A, \quad i, a, \alpha, A = 1, 2, \quad (126)
\]

in terms of which only four commuting automorphism \( SU(2) \) groups will be explicit. The further symmetry breaking can be understood as the identification of some of these \( SU(2) \), while extra symmetries, if they exist, mix different \( SU(2) \) indices. The corresponding covariant derivatives are defined by

\[
D^{ia} = \frac{\partial}{\partial \theta_{ia}}, \quad i \theta^{ia} \partial_t, \quad \nabla^{\alpha A} = \frac{\partial}{\partial \vartheta_{\alpha A}}, \quad i \vartheta^\alpha A \partial_t. \quad (127)
\]

By construction, they obey the algebra

\[
\{D^{ia}, D^{jb}\} = 2i \epsilon^{ij} \epsilon^{ab} \partial_t, \quad \{\nabla^{\alpha A}, \nabla^{\beta B}\} = 2i \epsilon^{\alpha \beta} \epsilon^{AB} \partial_t. \quad (128)
\]

4.2 \( N=8, d=1 \) supermultiplets

As we already mentioned, our real strategy of deducing a superfield description of the \( N=8, d=1 \) supermultiplets consisted in selecting an appropriate pair of constrained \( N=4, d=1 \) superfields and then guessing the constrained \( N=8 \) superfield. Now, just to make the presentation more coherent, we turn the argument around and start with postulating the \( N=8, d=1 \) constraints. The \( N=4 \) superfield formulations will be deduced from the \( N=8 \) ones.

Supermultiplet \( (0, 8, 8) \)

The off-shell \( N=8, d=1 \) supermultiplet \( (0, 8, 8) \) is carried out by two real fermionic \( N=8 \) superfields \( \psi_A^{\alpha A}, \Xi^{i\alpha} \) subjected to the following constraints:

\[
D^{(ia} \Xi_{i)}^{\alpha} = 0, \quad D^{(ia} \psi_{aB}^{\beta)} = 0, \quad \nabla^{(\alpha A} \Xi_{i)}^{\beta)} = 0, \quad \nabla^{(\alpha A} \psi_{aB}^{\beta)} = 0, \quad (129)
\]

\[
\nabla^{\alpha A} \psi_{a}^{A} = D^{ia} \Xi_{i}^{\alpha}, \quad \nabla^{\alpha A} \Xi_{i}^{i} = -D^{ia} \psi_{a}^{A}. \quad (130)
\]
In order to understand the structure of this supermultiplet in terms of \( N=4 \) superfields we proceed as follows. As a first step, let us single out the \( N=4 \) subspace in the \( N=8 \) superspace \( \mathbb{R}^{(1|8)} \) as the set of coordinates
\[
\mathbb{R}^{(1|4)} = (t, \theta_{ia}) \subset \mathbb{R}^{(1|8)},
\]
and expand the \( N=8 \) superfields over the extra Grassmann coordinate \( \vartheta_{\alpha A} \). Then we observe that the constraints (130) imply that the spinor derivatives of all involved superfields with respect to \( \vartheta_{\alpha A} \) can be expressed in terms of spinor derivatives with respect to \( \theta_{ia} \). This means that the only essential \( N=4 \) superfield components of \( \Psi^{aA} \) and \( \Xi^{i\alpha} \) in their \( \vartheta \)-expansion are the first ones
\[
\psi^{aA} \equiv \Psi^{aA}|_{\vartheta=0}, \quad \xi^{i\alpha} \equiv \Xi^{i\alpha}|_{\vartheta=0}.
\] (132)

These fermionic \( N=4 \) superfields are subjected, in virtue of eqs. (129) and (130), to the irreducibility constraints in \( N=4 \) superspace
\[
D^a_\alpha (i\xi^j) = 0, \quad D^i (a\psi^b)_A = 0.
\] (133)

As it follows from Section 2, these superfields are just two fermionic \( N=4 \) hypermultiplets, each carrying \((0, 4, 4)\) independent component fields. So, being combined together, they accommodate the whole off-shell component content of the \( N=8 \) multiplet \((0, 8, 8)\), which proves that the \( N=8 \) constraints (129), (130) are the true choice.

Thus, from the \( N=4 \) superspace perspective, the \( N=8 \) supermultiplet \((0, 8, 8)\) amounts to the sum of two \( N=4 \), \( d=1 \) fermionic hypermultiplets with the off-shell component content \((0, 4, 4) \oplus (0, 4, 4)\).

The transformations of the implicit \( N=4 \) Poincaré supersymmetry, completing the manifest one to the full \( N=8 \) supersymmetry, have the following form in terms of the \( N=4 \) superfields defined above:
\[
\delta \psi^{aA} = \frac{1}{2} \eta^a D^{ia} \xi^{i\alpha}, \quad \delta \xi^{i\alpha} = -\frac{1}{2} \eta^a D^i_\alpha \psi^{aA}.
\] (134)

The invariant free action can be written as
\[
S = \int dt d^4 \theta \left[ \theta^{ia} \theta^b_\alpha \psi^{aA}_{ib} + \theta^{ia} \theta^i_\alpha \xi^{i\alpha} \right].
\] (135)

Because of the presence of explicit theta’s in the action (135), the latter is not manifestly invariant even with respect to the manifest \( N=4 \) supersymmetry. Nevertheless, one can check that (135) is invariant under this supersymmetry, which is realized on the superfields as
\[
\delta^* \psi^{aA} = -\varepsilon_j b Q^{jb} \psi^{aA}, \quad \delta^* \xi^{i\alpha} = -\varepsilon_j b Q^{jb} \xi^{i\alpha},
\] (136)
where
\[
Q^{ia} = \frac{\partial}{\partial \theta_{ia}} - i \theta^{ia} \partial_t,
\] (137)
\( \varepsilon_{ia} \) is the supertranslation parameter and * denotes the ‘active’ variation (taken at a fixed point of the \( N=4 \) superspace).
Supermultiplet (1, 8, 7)

This supermultiplet can be described by a single scalar $N=8$ superfield $U$ which obeys the following irreducibility conditions:

\begin{align}
D^i D_a^j U &= -\nabla^{\alpha j} \nabla_i U, \\
\nabla^{(\alpha i} \nabla^{j)} U &= 0, \quad D^i(a D^j b) U = 0.
\end{align}

Let us note that the constraints (138) reduce the manifest R-symmetry to $[SU(2)]^3$, due to the identification of the indices $i$ and $A$ of the covariant derivatives $D^a$ and $\nabla^A$.

This supermultiplet possesses a unique decomposition into the pair of $N=4$ supermultiplets as $(1, 8, 7) = (1, 4, 3) \oplus (0, 4, 4)$. The corresponding $N=4$ superfield projections can be defined as

\begin{align}
u &= U|_{\theta=0}, \quad \psi^{i\alpha} = \nabla^{\alpha i} U|_{\theta=0},
\end{align}

and they obey the standard constraints

\begin{align}
D^{(i} a \psi^{j)a} &= 0, \quad D^i(a D^j b) u = 0.
\end{align}

The second constraint directly follows from (139), while the first one is implied by the relation

\begin{align}
\frac{\partial}{\partial t} D^i(a \nabla_j b) U = 0,
\end{align}

which can be proven by applying the differential operator $D^k \nabla^l$ to the $N=8$ superfield constraint (138) and making use of the algebra of covariant derivatives.

The additional implicit $N=4$ supersymmetry is realized on these $N=4$ superfields as follows:

\begin{align}
\delta u &= -\eta_{i\alpha} \psi^{i\alpha}, \quad \delta \psi^{i\alpha} = -\frac{1}{2} \eta^{j\alpha} D^{i(a} D^{j)b} u.
\end{align}

The simplest way to deal with the action for this supermultiplet is to use harmonic superspace [32, 33, 34], but this approach is out of the scope of the present Lectures.

Supermultiplet (2, 8, 6)

The $N=8$ superfield formulation of this supermultiplet involves two scalar bosonic superfields $U$, $\Phi$ obeying the constraints

\begin{align}
\nabla^{(ai} \nabla^{bj)} U &= 0, \quad \nabla^{a(i} \nabla^{bj)} \Phi = 0, \\
\nabla^{ai} U &= D^a \Phi, \quad \nabla^{ai} \Phi = -D^a U
\end{align}
where we have identified the indices $i$ and $A$, $a$ and $\alpha$ of the covariant derivatives, thus retaining only two manifest SU(2) automorphism groups. From (144), (145) some useful corollaries follow:

\begin{align*}
D^i D^a U + \nabla^a \nabla^i U &= 0, \quad D^{(a} D^{b)} U = 0, \quad (146) \\
D^i D^b \Phi + \nabla^b \nabla^i \Phi &= 0, \quad D^{(a} D^{b)} \Phi = 0. \quad (147)
\end{align*}

Comparing (146), (147) and (144) with (138), (139), we observe that the $N=8$ supermultiplet with the field content $(2, 8, 6)$ can be obtained by combining two $(1, 8, 7)$ supermultiplets and imposing the additional relations (145) on the corresponding $N=8$ superfields.

In order to construct the invariant actions and prove that the above $N=8$ constraints indeed yield the multiplet $(2, 8, 6)$, we should reveal the structure of this supermultiplet in terms of $N=4$ superfields, as we did in the previous cases. However, in the case at hand, we have two different choices for splitting the $(2, 8, 6)$ supermultiplet

1. $(2, 8, 6) = (1, 4, 3) \oplus (1, 4, 3)$
2. $(2, 8, 6) = (2, 4, 2) \oplus (0, 4, 4)$

As already mentioned, the possibility to have a few different off-shell $N=4$ decompositions of the same $N=8$ multiplet is related to different choices of the manifest $N=4$ supersymmetries, as subgroups of the $N=8$ super Poincaré group. We shall treat both options.

1. $(2, 8, 6) = (1, 4, 3) \oplus (1, 4, 3)$

In order to describe the $N=8$ $(2, 8, 6)$ multiplet in terms of $N=4$ superfields, we should choose the appropriate $N=4$ superspace. The first (evident) possibility is to choose the $N=4$ superspace with coordinates $(t, \theta^i)$. In this superspace one $N=4$ Poincaré supergroup is naturally realized, while the second one mixes two irreducible $N=4$ superfields which comprise the $N=8$ $(2, 8, 6)$ supermultiplet in question. Expanding the $N=8$ superfields $U, \Phi$ in $\vartheta^i$, one finds that the constraints (144), (145) leave in $U$ and $\Phi$ as independent $N=4$ projections only those of zeroth order in $\vartheta^i$

\begin{align*}
u &= U|_{\vartheta^i=0}, \quad \phi = \Phi|_{\vartheta^i=0}. \quad (148)
\end{align*}

Each $N=4$ superfield proves to be subjected, in virtue of (144), (145), to the additional constraint

\begin{align*}
D^{(a} D^{b)} u = 0, \quad D^{(a} D^{b)} \phi = 0. \quad (149)
\end{align*}

Thus we conclude that our $N=8$ multiplet $U, \Phi$, when rewritten in terms of $N=4$ superfields, amounts to a direct sum of two $N=4$ multiplets $u$ and $\phi$, both having the same off-shell field contents $(1, 4, 3)$.

The transformations of the implicit $N=4$ Poincaré supersymmetry, completing the manifest one to the full $N=8$ Poincaré supersymmetry, have the following form in terms of these $N=4$ superfields:
\[ \delta^* u = -\eta_\alpha D^{ia} \phi, \quad \delta^* \phi = \eta_\alpha D^{ia} u. \]  

(150)

It is rather easy to construct the action in terms of \( N=4 \) superfields \( u \) and \( \phi \), such that it is invariant with respect to the implicit \( N=4 \) supersymmetry. The generic action has the form

\[ S = \int dt d^4 \theta F(u, \phi), \]

(151)

where the function \( F \) obeys the Laplace equation

\[ F_{uu} + F_{\phi \phi} = 0. \]

(152)

2. \((2, 8, 6) = (2, 4, 2) \oplus (0, 4, 4)\)

There is a more sophisticated choice of a \( N=4 \) subspace in the \( N=8, d=1 \) superspace, which gives rise to the second possible \( N=4 \) superfield splitting of the considered \( N=8 \) supermultiplet, that is into the multiplets \((2, 4, 2)\) and \((0, 4, 4)\).

First of all, let us define a new set of covariant derivatives

\[ D^{ia} = \frac{1}{\sqrt{2}}(D^{ia} - i \nabla^{ai}), \quad \overline{D}^{ia} = \frac{1}{\sqrt{2}}(D^{ia} + i \nabla^{ai}), \quad \{D^{ia}, \overline{D}^{jb}\} = 2i \epsilon^{ij} \epsilon^{ab} \partial_t, \]

(153)

and new \( N=8 \) superfields \( V, \overline{V} \) related to the original ones as

\[ V = \mathcal{U} + i \Phi, \quad \overline{V} = \mathcal{U} - i \Phi. \]

(154)

In this basis the constraints (144), (145) read

\[ D^{ia} V = 0, \quad \overline{D}^{ia} \overline{V} = 0, \]

\[ D^{(ia} D^{jb)} V + \overline{D}^{(ia} \overline{D}^{jb)} \overline{V} = 0, \quad D^{(ia} D^{jb)} \overline{V} - \overline{D}^{(ia} \overline{D}^{jb)} V = 0. \]

(155)

Now we split the complex quartet covariant derivatives (153) into two sets of the doublet \( N=4 \) ones as

\[ D^i = \mathcal{D}^1, \quad \overline{D}^i = \overline{\mathcal{D}}^2, \quad \nabla^i = \mathcal{D}^{i2}, \quad \overline{\nabla}^i = -\overline{\mathcal{D}}^{i1} \]

(156)

and cast the constraints (155) in the form

\[ D^i V = 0, \quad \nabla^i V = 0, \quad \mathcal{D}_i V = 0, \quad \overline{\nabla}^i \overline{V} = 0, \]

\[ D^i \mathcal{D}_j V - \nabla^i \nabla_j V = 0, \quad D^i \overline{\nabla}^j V - \overline{\mathcal{D}}^i \overline{\mathcal{D}}^j V = 0. \]

(157)

Next, as an alternative \( N=4 \) superspace, we choose the set of coordinates closed under the action of \( D^i, \overline{D}^i \), i.e.

\[ (t, \theta_{i1} + i \theta_{i1}, \theta_{i2} - i \theta_{i2}). \]

(158)
while the $N=8$ superfields are expanded with respect to the orthogonal combinations $\theta_{i1} - i\vartheta_{i1}, \theta_{i2} + i\vartheta_{i2}$ which are annihilated by $D^i, \bar{D}^i$.

As a consequence of the constraints (157), the quadratic action of the derivatives $\nabla^i$ and $\bar{\nabla}^i$ on every $N=8$ superfield $V, \bar{V}$ can be expressed as $D^i, \bar{D}^i$ of some other superfield. Therefore, only the zeroth and first order components of each $N=8$ superfield are independent $N=4$ superfield projections. Thus, we are left with the following set of $N=4$ superfields:

\[ v = V|, \quad \bar{v} = \bar{V}|, \quad \psi^i = \nabla^i V|, \quad \bar{\psi}^i = -\nabla^i \bar{V}|. \quad (159) \]

These $N=4$ superfields prove to be subjected to the additional constraints which also follow from (157)

\[ D^i v = 0, \quad \bar{D}^i \bar{v} = 0, \quad D^i \bar{\psi}^j = 0, \quad D^i \psi^j = -\bar{D}^i \bar{\psi}^j. \quad (160) \]

The $N=4$ superfields $v, \bar{v}$ comprise the standard $N=4, d=1$ chiral multiplet $(2, 4, 2)$, while the $N=4$ superfields $\psi^i, \bar{\psi}^j$ subjected to (160) and both having the off-shell contents $(0, 4, 4)$ are recognized as the fermionic version of the $N=4, d=1$ hypermultiplet.

The implicit $N=4$ supersymmetry is realized by the transformations

\[ \delta v = -\bar{\eta}^i \psi_i, \quad \delta \psi^j = -\frac{1}{2} \eta^i \nabla^i \bar{v}, \]
\[ \delta \bar{v} = \eta^i \bar{\psi}_i, \quad \delta \bar{\psi}^i = \frac{1}{2} \eta^i \bar{D}^i v + 2i\bar{\eta}\bar{\psi}. \quad (161) \]

The invariant free action has the following form:

\[ S_f = \int dt d^4\theta \bar{v}v - \frac{1}{2} \int dt d^2\bar{\theta} \bar{\psi}^j \psi_i - \frac{1}{2} \int dt d^2\theta \bar{\psi}_i \psi^j. \quad (162) \]

Let us note that this very simple form of the action for the $N=4$ $(0, 4, 4)$ supermultiplet $\psi_i, \bar{\psi}^j$ is related to our choice of the $N=4$ superspace. It is worthwhile to emphasize that all differently looking superspace off-shell actions of the multiplet $(0, 4, 4)$ yield the same component action for this multiplet.

**Supermultiplet $(3, 8, 5)$**

In the $N=8$ superspace this supermultiplet is described by the triplet of bosonic superfields $V^{ij}$ obeying the irreducibility constraints

\[ D_a^{(ij)\eta} = 0, \quad \nabla_a^{(ij)\eta} = 0. \quad (163) \]

So, three out of four original automorphism $SU(2)$ symmetries remain manifest in this description.

The $N=8$ supermultiplet $(3, 8, 5)$ can be decomposed into $N=4$ supermultiplets in two ways
• 1. \((3, 8, 5) = (3, 4, 1) \oplus (0, 4, 4)\)

• 2. \((3, 8, 5) = (1, 4, 3) \oplus (2, 4, 2)\)

As in the previous case, we discuss both options.

1. \((3, 8, 5) = (3, 4, 1) \oplus (0, 4, 4)\)

This splitting requires choosing the coordinate set (131) as the relevant \(N=4\) superspace. Expanding the \(N=8\) superfields \(V^{ij}\) in \(\vartheta_{i\alpha}\), one finds that the constraints (163) leave in \(V^{ij}\) the following four bosonic and four fermionic \(N=4\) projections:

\[
v^{ij} = V^{ij}, \quad \xi^i_{\alpha} \equiv \nabla_{j\alpha} V^{ij}, \quad A \equiv \nabla^i_{\alpha} \nabla_{j\alpha} V^{ij}\]

where \(|\) means the restriction to \(\vartheta_{i\alpha} = 0\). As a consequence of (163), these \(N=4\) superfields obey the constraints

\[
D^a_a \phi = 0, \quad D^a_v \nabla^a \phi = 0, \quad \nabla^a_v + D^a \phi = 0, \quad \nabla^a \bar{\phi} = 0, \quad \nabla_a \phi = 0, \quad \nabla_a v + D_a \phi = 0, \quad \bar{D}_a v - \nabla_a \bar{\phi} = 0, \quad \bar{D}_a \bar{\phi} = 0
\]

(168)

Thus, for the considered splitting, the \(N=8\) tensor multiplet superfield \(V^{ij}\) amounts to a direct sum of the \(N=4\) ‘tensor’ multiplet superfield \(v^{ij}\) with the off-shell content \((3, 4, 1)\) and a fermionic \(N=4\) hypermultiplet \(\xi^i_{\alpha}\) with the off-shell content \((0, 4, 4)\), plus a constant \(m\) of the mass dimension.

2. \((3, 8, 5) = (1, 4, 3) \oplus (2, 4, 2)\)

This option corresponds to another choice of \(N=4\) superspace, which amounts to dividing the \(N=8, d=1\) Grassmann coordinates into doublets, with respect to some other \(SU(2)\) indices. The relevant splitting of \(N=8\) superspace into the \(N=4\) subspace and the complement of the latter can be performed as follows. Firstly, we define the new covariant derivatives as

\[
D^a \equiv \frac{1}{\sqrt{2}} (D^{1a} + i\nabla^{a1}), \quad \bar{D}_a \equiv \frac{1}{\sqrt{2}} (D^2_a - i\nabla_a^2),
\]

\[
\nabla^a \equiv \frac{i}{\sqrt{2}} (D^{2a} + i\nabla_a^2), \quad \bar{\nabla}_a \equiv \frac{i}{\sqrt{2}} (D^1_a - i\nabla^1_a).
\]

(166)

Then we choose the set of coordinates closed under the action of \(D^a, \bar{D}_a\), i.e.

\[
(t, \theta^1_{1\alpha} - i\bar{\theta}^{11}_{1}, \theta^{1\alpha} + i\bar{\theta}^{11}_{\alpha}),
\]

(167)

while the \(N=8\) superfields are expanded with respect to the orthogonal combinations \(\theta^2_{1\alpha} - i\bar{\theta}^{22}_{1\alpha}, \theta^{2\alpha} + i\bar{\theta}^{22}_{\alpha}\) annihilated by \(D^a, \bar{D}_a\).

The basic constraints (163), being rewritten in the basis (166), take the form

\[
D^a \varphi = 0, \quad D^a v - \nabla^a \varphi = 0, \quad \nabla^a v + D^a \bar{\varphi} = 0, \quad \nabla^a \bar{\varphi} = 0, \quad \nabla_a \varphi = 0, \quad \nabla_a v + \bar{D}_a \varphi = 0, \quad \bar{D}_a v - \nabla_a \bar{\varphi} = 0, \quad \bar{D}_a \bar{\varphi} = 0
\]

(168)
where
\[ v \equiv -2i\gamma^{12}, \quad \varphi \equiv \gamma^{11}, \quad \bar{\varphi} \equiv \gamma^{22}. \]

Due to the constraints (168), the derivatives \( \nabla^a \) and \( \overline{\nabla}_a \) of every \( N=8 \) superfield in the triplet \((\gamma^{12}, \gamma^{11}, \gamma^{22})\) can be expressed as \( D^a, \overline{D}_a \) of some other superfield. Therefore, only the zeroth order (i.e. taken at \( \theta^a_2 - i\theta^a_1 = \theta^a_1 + i\theta^a_1 = 0 \) components of each \( N=8 \) superfield are independent \( N=4 \) superfield projections. These \( N=4 \) superfields are subjected to the additional constraints which also follow from (168)

\[ D^a D_a v = \overline{D}_a \overline{D}^a v = 0, \quad D^a \varphi = 0, \quad \overline{D}_a \bar{\varphi} = 0. \]

The \( N=4 \) superfields \( \varphi, \bar{\varphi} \) comprise the standard \( N=4, d=1 \) chiral multiplet \((2, 4, 2)\), while the \( N=4 \) superfield \( v \) subjected to (170) has the needed off-shell content \((1, 4, 3)\).

The implicit \( N=4 \) supersymmetry acts on the \( N=4 \) superfields \( v, \varphi, \bar{\varphi} \) as follows:

\[ \delta^a v = \eta_a D^a \bar{\varphi} + \bar{\eta}^a \overline{D}_a \varphi, \quad \delta^a \varphi = -\eta_a D^a v, \quad \delta^a \bar{\varphi} = -\bar{\eta}^a \overline{D}_a v. \]

Invariant \( N=4 \) superfield actions for both decompositions of the \( N=8 \) multiplet \((3, 8, 5)\) were presented in Ref. [6].

### 4.3 Supermultiplet \((4, 8, 4)\)

This supermultiplet can be described by a quartet of \( N=8 \) superfields \( Q^{a\alpha} \) obeying the constraints

\[ D_i^{(a} Q^{b)\alpha} = 0, \quad \nabla_i^{(\alpha} Q^{b)\alpha} = 0. \]

Let us note that the constraints (172) are manifestly covariant with respect to three \( SU(2) \) subgroups realized on the indices \( i, a \) and \( \alpha \).

From (172) some important relations follow:

\[ D^a D^b Q^{ca} = 2i\epsilon^{ij} e^{cb} \hat{Q}^{a\alpha}, \quad \nabla^i Q^{ab} = 2i\epsilon^{ij} e^{ab} \hat{Q}^{a\alpha}. \]

Using them, it is possible to show that the superfields \( Q^{a\alpha} \) contain the following independent components:

\[ Q^{a\alpha}|, \quad D^i_a Q^{a\alpha}|, \quad \nabla^i_a Q^{a\alpha}|, \quad D^i_a \nabla^j_a Q^{a\alpha}|, \]

where \( | \) means now restriction to \( \theta_{ia} = \theta_{ia} = 0 \). This directly proves that we deal with the irreducible \((4, 8, 4)\) supermultiplet.

There are three different possibilities to split this \( N=8 \) multiplet into the \( N=4 \) ones

- 1. \( (4, 8, 4) = (4, 4, 0) \oplus (0, 4, 4) \)
- 2. \( (4, 8, 4) = (3, 4, 1) \oplus (1, 4, 3) \)
• 3. \((4, 8, 4) = (2, 4, 2) \oplus (2, 4, 2)\)

Once again, we shall consider all three cases separately.

1. \((4, 8, 4) = (4, 4, 0) \oplus (0, 4, 4)\)

This case implies the choice of the \(N=4\) superspace (131). Expanding the \(N=8\) superfields \(Q^{a \alpha}\) in \(\bar{\partial}_{i \alpha}\), one may easily see that the constraints (172) leave in \(Q^{a \alpha}\) the following four bosonic and four fermionic \(N=4\) superfield projections:

\[ q^{a \alpha} = Q^{a \alpha}, \quad \psi^{i a} = \nabla_{i \alpha} Q^{a \alpha}. \quad \text{(175)} \]

Each \(N=4\) superfield is subjected, in virtue of (172), to an additional constraint

\[ D^{i (a q^{b) \alpha}} = 0, \quad D^{i (a \psi^{b) i}} = 0. \quad \text{(176)} \]

Consulting Section 2, we come to the conclusion that these are just the hypermultiplet \(q^{i \alpha}\) with the off-shell field content \((4, 4, 0)\) and a fermionic analog of the \(N=4\) hypermultiplet \(\psi^{i a}\) with the field content \((0, 4, 4)\).

The transformations of the implicit \(N=4\) Poincaré supersymmetry have the following form in terms of these \(N=4\) superfields:

\[ \delta^* q^{a \alpha} = \frac{1}{2} \eta^{i \alpha} \psi^{i a}, \quad \delta^* \psi^{i a} = -2i \eta^{i \alpha} q^{a \alpha}. \quad \text{(177)} \]

2. \((4, 8, 4) = (3, 4, 1) \oplus (1, 4, 3)\)

In order to describe this \(N=4\) superfield realization of the \(N=8\) supermultiplet \((4, 8, 4)\), we introduce the \(N=8\) superfields \(V^{ab}, \mathcal{V}\) as

\[ Q^{a \alpha} = \delta^{\alpha}_{\beta} V^{ab} - \epsilon^{a \alpha} \mathcal{V}, \quad \mathcal{V}^{ab} = \mathcal{V}^{ba}, \quad \text{(178)} \]

and use the covariant derivatives (153) to rewrite the basic constraints (172) as

\[ D^{i (a V^{b) c}} = 0, \quad \overline{D}^{i (a \mathcal{V}^{b) c}} = 0, \quad \text{(179)} \]
\[ D^{i a} \mathcal{V} = \frac{1}{2} \overline{D}^{i b} \mathcal{V}^{ab}, \quad \overline{D}^{i a} \mathcal{V} = \frac{1}{2} D^{i b} \mathcal{V}^{ab}. \quad \text{(180)} \]

The constraints (179) define \(V^{ab}\) as the \(N=8\) superfield encompassing the off-shell multiplet \((3, 8, 5)\), while, as one can deduce from (179), (180), the \(N=8\) superfield \(\mathcal{V}\) has the content \((1, 8, 7)\). Then the constraints (180) establish relations between the fermions in these two superfields and reduce the number of independent auxiliary fields to four, so that we end up, once again, with the irreducible \(N=8\) multiplet \((4, 8, 4)\).

Two sets of \(N=4\) covariant derivatives

\[ (D^{a}, \overline{D}^{a}) \equiv (D^{1a}, D^{2a}) \quad \text{and} \quad (\nabla^{a}, \overline{\nabla}^{a}) \equiv (\overline{D}^{2a}, D^{1a}) \]
are naturally realized in terms of the \( N=4 \) superspaces \((t, \theta_{1a}, \theta_{2a} - i \theta_{1a})\) and \((t, \theta_{2a} + i \theta_{2a}, \theta_{1a} - i \theta_{1a})\). In terms of the new derivatives, the constraints (179), (180) become

\[
\begin{align*}
D^{(a} V^{bc)} &= D^{(a} \overline{V}^{bc)} = \nabla^{(a} V^{bc)} = \overline{\nabla}^{(a} V^{bc)} = 0, \\
D^a V &= \frac{1}{2} \nabla_b V^{ab}, \quad \overline{D}^a V = \frac{1}{2} \overline{\nabla}_b V^{ab}.
\end{align*}
\]

Now we see that the \( \nabla, \overline{\nabla} \) derivatives of the superfields \( V, V^{ab} \) are expressed as \( D, \overline{D} \) of the superfields \( V, V^{ab} \), respectively. Thus, in the \((\theta_{2a} + i \theta_{2a}, \theta_{1a} - i \theta_{1a})\) expansion of the superfields \( V, V^{ab} \) only the first components (i.e. those of zero order in the coordinates \((\theta_{2a} + i \theta_{2a}, \theta_{1a} - i \theta_{1a})\)) are independent \( N=4 \) superfields. We denote them \( v, v^{ab} \). The hidden \( N=4 \) supersymmetry is realized on these \( N=4 \) superfields as

\[
\delta v = -\frac{1}{2} \eta_a D_b v^{ab} + \frac{1}{2} \overline{\eta}_a \overline{D}_b v^{ab}, \quad \delta v^{ab} = \frac{4}{3} \left( \eta^{(a} D^{b)} v - \overline{\eta}^{(a} \overline{D}^{b)} v \right),
\]

while the superfields themselves obey the constraints

\[
\begin{align*}
D^{(a} v^{bc)} &= \overline{D}^{(a} v^{bc)} = 0, \quad D^{(a} \overline{D}^{b)} v = 0,
\end{align*}
\]

which are remnant of the \( N=8 \) superfield constraints (181).

The invariant free action reads

\[
S = \int dt d^4 \theta \left( v^2 - \frac{3}{8} v^{ab} v_{ab} \right).
\]

3. \((4, 8, 4) = (2, 4, 2) \oplus (2, 4, 2)\)

This case is a little bit more tricky. First of all, we define the new set of \( N=8 \) superfields \( W, \Phi \) in terms of \( V^{ij}, V \) defined earlier in (178)

\[
W \equiv V^{11}, \quad \overline{W} \equiv V^{22}, \quad \Phi \equiv \frac{2}{3} \left( V + \frac{3}{2} V^{12} \right), \quad \overline{\Phi} \equiv \frac{2}{3} \left( V - \frac{3}{2} V^{12} \right)
\]

and construct two new sets of \( N=4 \) derivatives \( D^i, \nabla^i \) from those defined in (153)

\[
\begin{align*}
D^i &= \frac{1}{\sqrt{2}} \left( D^{1i} + \overline{D}^{1i} \right), \quad \overline{D}^i &= \frac{1}{\sqrt{2}} \left( D^{i2} + \overline{D}^{i2} \right), \\
\nabla^i &= \frac{1}{\sqrt{2}} \left( D^{1i} - \overline{D}^{1i} \right), \quad \overline{\nabla}^i &= -\frac{1}{\sqrt{2}} \left( D^{i2} - \overline{D}^{i2} \right).
\end{align*}
\]

The basic constraints (179), (180) can be rewritten in terms of the superfields \( W, \Phi \) and the derivatives \( D^i, \nabla^i \) as
\[ D^i \mathcal{W} = \nabla^i \mathcal{W} = 0, \quad D^i \Phi = \nabla^i \Phi = 0, \quad \nabla^i \Phi = D^i \Phi = 0, \quad D^i \Phi = D^i \Phi = 0, \quad \nabla^i \Phi = \nabla^i \Phi = 0, \quad \nabla^i \Phi = \nabla^i \Phi = 0. \] **(187)**

The proper \( N=4 \) superspace is defined as the one on which the covariant derivatives \( D^1, D^2, \nabla^1, \nabla^2 \) are naturally realized. The constraints (187) imply that the remaining set of covariant derivatives, i.e. \( D^2, D^2, \nabla^2, \nabla^1 \), when acting on every involved \( N=8 \) superfield, can be expressed as spinor derivatives from the first set acting on some other \( N=8 \) superfield. Thus the first \( N=4 \) superfield components of the \( N=8 \) superfields \( \mathcal{W}, \Phi \) are the only independent \( N=4 \) superfield projections. The transformations of the implicit \( N=4 \) Poincaré supersymmetry have the following form in terms of these \( N=4 \) superfields:

\[ \delta \mathcal{W} = \bar{\epsilon} D^1 \bar{\phi} + \bar{\eta} \nabla^1 \phi, \quad \delta \phi = -\eta \nabla^1 \mathcal{W} - \bar{\epsilon} D^1 \bar{\mathcal{W}} , \]
\[ \delta \bar{\mathcal{W}} = \epsilon D^1 \phi + \eta \nabla^1 \bar{\phi}, \quad \delta \bar{\phi} = -\bar{\epsilon} D^1 \mathcal{W} - \eta \nabla^1 \bar{\mathcal{W}} . \] **(188)**

The free invariant action reads

\[ S = \int dt d^3 \theta (\mathcal{W} \bar{\mathcal{W}} - \Phi \bar{\Phi}) . \] **(189)**

### 4.4 Supermultiplet \((5, 8, 3)\)

This supermultiplet has been considered in detail in Refs. \([30, 6]\). It was termed there the ‘\( N=8 \) vector multiplet’. Here we sketch its main properties.

In order to describe this supermultiplet, one should introduce five bosonic \( N=8 \) superfields \( V_{\alpha a}, U \) obeying the constraints

\[ D^{i\beta} V_{\alpha a} + \delta^{i\beta}_a \nabla^i \Phi = 0 , \quad \nabla^{\alpha i} V_{\alpha a} + \delta^{\alpha i}_a D^i U = 0 . \] **(190)**

It is worth noting that the constraints (190) are covariant not only under three \( SU(2) \) automorphism groups (realized on the doublet indices \( i, a \) and \( \alpha \)), but also under the \( SO(5) \) automorphisms. These \( SO(5) \) transformations mix the spinor derivatives \( D^{i\alpha} \) and \( \nabla^{\alpha i} \) in the indices \( \alpha \) and \( a \), while two \( SU(2) \) groups realized on these indices constitute \( SO(4) \subset SO(5) \). The superfields \( U, V_{\alpha a} \) form an \( SO(5) \) vector; under \( SO(5) \) transformations belonging to the coset \( SO(5)/SO(4) \) they transform as

\[ \delta V_{\alpha a} = a_{\alpha a} U , \quad \delta U = -2 a_{\alpha a} V_{\alpha a} . \] **(191)**

As in the previous cases we may consider two different splittings of the \( N=8 \) vector multiplet into irreducible \( N=4 \) superfields

- 1. \((5, 8, 3) = (1, 4, 3) \oplus (4, 4, 0)\)
- 2. \((5, 8, 3) = (3, 4, 1) \oplus (2, 4, 2)\)
Once again, they correspond to two different choices of the $N=4, d=1$ superspace as a subspace in the original $N=8, d=1$ superspace.

1. $(5, 8, 3) = (1, 4, 3) \oplus (4, 4, 0)$

The relevant $N=4$ superspace is $\mathbb{R}^{(1|4)}$ parameterized by the coordinates $(t, \theta_{\alpha})$ and defined in (131). As in the previous cases, it follows from the constraints (190) that the spinor derivatives of all involved superfields with respect to $\theta_{\alpha}$ are expressed in terms of spinor derivatives with respect to $\theta_{\alpha}$. Thus the only essential $N=4$ superfield components of $V_{\alpha a}$ and $U$ in their $\theta$-expansion are the first ones

$$v_{\alpha a} \equiv V_{\alpha a}|_{\theta=0}, \quad u \equiv U|_{\theta=0}. \quad (192)$$

They accommodate the whole off-shell component content of the $N=8$ vector multiplet. These five bosonic $N=4$ superfields are subjected, in virtue of (190), to the irreducibility constraints in $N=4$ superspace

$$D^{(a \beta)} \alpha = 0, \quad D^{(a \beta)} u = 0. \quad (193)$$

Thus, from the $N=4$ superspace standpoint, the vector $N=8$ supermultiplet is the sum of the $N=4, d=1$ hypermultiplet $v_{\alpha a}$ with the off-shell component contents $(4, 4, 0)$ and the $N=4$ 'old' tensor multiplet $u$ with the contents $(1, 4, 3)$.

The transformations of the implicit $N=4$ Poincaré supersymmetry read

$$\delta v_{\alpha a} = \eta_{\alpha a} D_a u, \quad \delta u = \frac{1}{2} \eta_{\alpha a} D_{\alpha} v_{\alpha a}. \quad (194)$$

2. $(5, 8, 3) = (3, 4, 1) \oplus (2, 4, 1)$

Another interesting $N=4$ superfield splitting of the $N=8$ vector multiplet can be achieved by passing to the complex parametrization of the $N=8$ superspace as

$$(t, \Theta_{\alpha} = \theta_{\alpha} + i \theta_{\alpha}, \bar{\Theta}^{\bar{\alpha}} = \theta^{\bar{\alpha}} - i \theta^{\bar{\alpha}})$$

where we have identified the indices $a$ and $\alpha$, thus reducing the number of manifest $SU(2)$ automorphism symmetries to just two. In this superspace the covariant derivatives $D^{\alpha a}, \bar{D}^{\bar{\alpha}}$ defined in (153) (with the identification of indices just mentioned) are naturally realized. We are also led to define new superfields

$$V \equiv -\epsilon_{a a} V^{a a}, \quad W^{a \beta} \equiv V^{(a \beta)} = \frac{1}{2} \left( V^{a \beta} + V^{b \alpha} \right), \quad W \equiv V + iU, \quad \bar{W} \equiv V - iU$$

In this basis of $N=8$ superspace the original constraints (190) amount to
\[ D^\alpha \mathcal{W}^{\beta \gamma} = -\frac{1}{4} (\epsilon^{\beta \alpha} \mathcal{D}^{\gamma} \mathcal{W} + \epsilon^{\gamma \alpha} \mathcal{D}^{\beta} \mathcal{W}) , \quad \overline{D}^{\alpha} \mathcal{W}^{\beta \gamma} = -\frac{1}{4} (\epsilon^{\beta \alpha} \mathcal{D}^{\gamma} \mathcal{W} + \epsilon^{\gamma \alpha} \mathcal{D}^{\beta} \mathcal{W}) , \]
\[ \mathcal{D}^a \mathcal{W} = 0 , \quad \overline{\mathcal{D}}^a \mathcal{W} = 0 , \quad (\mathcal{D}^{ka} \mathcal{D}_k^l) \mathcal{W} = (\overline{\mathcal{D}}_k^a \overline{\mathcal{D}}^k \mathcal{W}) . \]

(196)

Next, we single out the \( N=4, d=1 \) superspace as \((t, \theta_{\alpha} \equiv \Theta^1 \alpha, \overline{\theta}^\alpha)\) and split our \( N=8 \) superfields into the \( N=4 \) ones in the standard way. As in all previous cases, the spinor derivatives of each \( N=8 \) superfield with respect to \( \Theta^2 \alpha \) and \( \overline{\Theta}^2 \alpha \), as a consequence of the constraints (196), are expressed as derivatives of some other superfields with respect to \( \overline{\theta}^\alpha \) and \( \theta^\alpha \). Therefore, only the first (i.e. taken at \( \Theta_2 \alpha = 0 \) and \( \overline{\Theta}_2 \alpha = 0 \)) \( N=4 \) superfield components of the \( N=8 \) superfields really matter. They accommodate the entire off-shell field content of the multiplet. These \( N=4 \) superfields are defined as

\[ \phi \equiv \mathcal{W} | , \quad \overline{\phi} \equiv \overline{\mathcal{W}} | , \quad w^{\alpha \beta} \equiv \mathcal{W}^{\alpha \beta} | \]

(197)

and satisfy the constraints following from (196)

\[ \mathcal{D}^\alpha \overline{\phi} = 0 , \quad \overline{\mathcal{D}}_\alpha \phi = 0 , \quad \mathcal{D}^{(\alpha} w^{\beta \gamma)} = \overline{\mathcal{D}}^{(\alpha} w^{\beta \gamma)} = 0 , \quad \mathcal{D}^\alpha = \mathcal{D}^{1 \alpha} , \quad \overline{\mathcal{D}}_\alpha = \overline{\mathcal{D}}_{1 \alpha} . \]

(198)

They tell us that the \( N=4 \) superfields \( \phi \) and \( \overline{\phi} \) form the standard \( N=4 \) chiral multiplet \((2, 4, 2)\), while the \( N=4 \) superfield \( w^{\alpha \beta} \) represents the \( N=4 \) tensor multiplet \((3, 4, 1)\).

The implicit \( N=4 \) supersymmetry is realized on \( w^{\alpha \beta} , \phi \) and \( \overline{\phi} \) as

\[ \delta w^{\alpha \beta} = \frac{1}{2} \left( \eta^{(\alpha} \mathcal{D}^{\beta)} \overline{\phi} - \overline{\eta}^{(\alpha} \mathcal{D}^{\beta)} \phi \right) , \quad \delta \phi = \frac{4}{3} \eta_\alpha \mathcal{D}^\beta w^{\alpha \beta} , \quad \delta \overline{\phi} = -\frac{4}{3} \overline{\eta}^\alpha \mathcal{D}_\beta w^{\alpha \beta} . \]

(199)

An analysis of \( N=8 \) supersymmetric actions for the \( N=8 \) vector multiplet may be found in [6].

### 4.5 Supermultiplet \((6, 8, 2)\)

This supermultiplet can be described by two \( N=8 \) tensor multiplets \( \mathcal{V}^{ij} \) and \( \mathcal{W}^{ab} \),

\[ D^a_i \mathcal{V}^{jk} = 0 , \quad \nabla^a_i \mathcal{V}^{jk} = 0 , \quad D^{(\alpha} \mathcal{W}^{\beta c)} = 0 , \quad \nabla^{(\alpha} \mathcal{W}^{\beta c)} = 0 , \]

(200)

with the additional constraints

\[ D^a_i \mathcal{V}^{ij} = \nabla^a_i \mathcal{W}^{i c} , \quad \nabla^{a i} \mathcal{V}^{ij} = -D^a_i \mathcal{W}^{ab} . \]

(201)

The role of the latter constraints is to identify the eight fermions, which are present in \( \mathcal{V}^{ij} \), with the fermions from \( \mathcal{W}^{ab} \), and to reduce the number of independent auxiliary fields in both superfields to two

\[ F_1 = D^a_i D_{\alpha j} \mathcal{V}^{ij} , \quad F_2 = D^a_i D_{\alpha \beta} \mathcal{W}^{ab} | , \]

(202)

where \( | \) means here restriction to \( \theta_{ia} = \overline{\theta}_{ia} = 0 \).

There are two different possibilities to split this \( N=8 \) multiplet into \( N=4 \) ones.
1.  \((6, 8, 2) = (3, 4, 1) \oplus (3, 4, 1)\)

The corresponding \(N=4\) supersubspace is \((131)\). The \(N=8\) constraints imply that the only essential \(N=4\) superfield components of \(V^{ij}\) and \(W^{ab}\) in their \(\theta\)-expansion are the first ones
\[
v^{ij} \equiv V^{ij}|, \quad w^{ab} \equiv W^{ab}|. \tag{203}
\]
These six bosonic \(N=4\) superfields are subjected, in virtue of (200), (201), to the irreducibility constraints in \(N=4\) superspace
\[
D^{a(i}v^{jk)} = 0, \quad D^{i(a}w^{bc)} = 0. \tag{204}
\]
Thus, the \(N=8\) supermultiplet \((6, 8, 2)\) amounts to the sum of two \(N=4, d=1\) tensor multiplets \(v^{ij}, w^{ab}\) with the off-shell field contents \((3, 4, 1) \oplus (3, 4, 1)\).

The transformations of the implicit \(N=4\) Poincaré supersymmetry are
\[
\delta v^{ij} = -\frac{2}{3} \eta^{(i}_a D^{j)} w^{ab}, \quad \delta w^{ab} = \frac{2}{3} \eta^{(a}_i D^{b)} v^{ij}. \tag{205}\]

The free \(N=8\) supersymmetric action has the following form:
\[
S = \int dt d^4 \theta \left(v^2 - w^2\right). \tag{206}\]

2.  \((6, 8, 2) = (4, 4, 0) \oplus (2, 4, 2)\)

In this case, to describe the \((6, 8, 2)\) multiplet, we combine two \(N=4\) superfields, i.e. the chiral superfield
\[
D^i \phi = \overline{D} \phi = 0 \tag{207}
\]
and the hypermultiplet \(q^{ia}\)
\[
D^{(i}q^{j)a} = \overline{D}^{(i}q^{j)a} = 0. \tag{208}\]

The transformations of the implicit \(N=4\) supersymmetry read
\[
\delta q^{ia} = e^a D^{i} \phi + e^a \overline{D} \phi, \quad \delta \phi = \frac{1}{2} e^a D^{i} q_{ia}, \quad \delta \bar{\phi} = -\frac{1}{2} e^a \overline{D} q_{ia}. \tag{209}\]

The invariant free action reads
\[
S_{\text{free}} = \int dt d^4 \theta \left(q^2 - 4 \phi \bar{\phi}\right). \tag{210}\]
4.6 Supermultiplet \((7, 8, 1)\)

This supermultiplet has a natural description in terms of two \(N=8\) superfields \(V_{ij}\) and \(Q_{a\alpha}\) satisfying the constraints

\[
D^{(ia} V^{jk)} = 0, \quad \nabla^{(ia} V^{jk)} = 0, \quad D^{(a} Q^{b)} = 0, \quad \nabla^{(a} Q^{b)} = 0, \quad (211)
\]
\[
D_a^{ij} V^{jk} = \eta^{ab} Q^{b\alpha}, \quad \nabla_a^{ij} V^{jk} = -i D_a^{ij} Q^{a\alpha}. \quad (212)
\]

The constraints (211) leave in the superfields \(V_{ij}\) and \(Q_{a\alpha}\) the sets \((3, 8, 5)\) and \((4, 8, 4)\) of irreducible components, respectively. The role of the constraints (212) is to identify the fermions in the superfields \(V_{ij}\) and \(Q_{a\alpha}\) and reduce the total number of independent auxiliary components in both superfields to just one.

For this supermultiplet there is a unique splitting into \(N=4\) superfields as

\[
(7, 8, 1) = (3, 4, 1) \oplus (4, 4, 0).
\]

The proper \(N=4\) superspace is parameterized by the coordinates \((t, \theta_{ia})\). The constraints (211), (212) imply that the only essential \(N=4\) superfield components in the \(\theta\)-expansion of \(V_{ij}\) and \(Q_{a\alpha}\) are the first ones

\[
v_{ij} \equiv V_{ij}|_{\theta=0}, \quad q_{a\alpha} \equiv Q_{a\alpha}|_{\theta=0}.
\]

These seven bosonic \(N=4\) superfields are subjected, as a corollary of (211), (212), to the irreducibility constraints in \(N=4\) superspace

\[
D_a^{(i} v^{jk)} = 0, \quad D_i^{(a} q^{b)} = 0. \quad (214)
\]

Thus the \(N=8\) supermultiplet \((7, 8, 1)\) amounts to the sum of the \(N=4, d=1\) hypermultiplet \(q_{a\alpha}\) with the \((4, 4, 0)\) off-shell field content and the \(N=4\) tensor multiplet \(v_{ij}\) with the \((3, 4, 1)\) content.

The implicit \(N=4\) Poincaré supersymmetry is realized by the transformations

\[
\delta v_{ij} = -\frac{2i}{3} \eta_{ia} D_a^{ij} q^{a\alpha}, \quad \delta q_{a\alpha} = -i \eta^{ij} D^{ia} v_{ij}. \quad (215)
\]

The free action can be also easily written

\[
S = \int dt d^4 \theta \left[ v^2 - \frac{4}{3} q^2 \right]. \quad (216)
\]

4.7 Supermultiplet \((8, 8, 0)\)

This supermultiplet is analogous to the supermultiplet \((0, 8, 8)\): they differ in their overall Grassmann parity. It is described by the two real bosonic \(N=8\) superfields \(Q^{aA}, \Phi^{i\alpha}\) subjected to the constraints
\[ D^{(ia)\Phi}_j \alpha = 0, \quad D^{(aQ)bA} = 0, \quad \nabla^{(aA}\Phi^b) = 0, \quad \nabla^{a(AQb)} = 0, \quad (217) \]

\[ \nabla^{\alpha A}Q^{aA}_i = -D^{ia}\Phi^\alpha_i, \quad \nabla^{\alpha A}\Phi ^i = D^{ia}Q^{aA}_i. \quad (218) \]

Analogously to the case of the supermultiplet \((8, 8, 0)\), from the constraints \((218)\) it follows that the spinor derivatives of all involved superfields with respect to \(\theta^\alpha\) are expressed in terms of spinor derivatives with respect to \(\vartheta\). Thus the only essential \(N=4\) superfield components in the \(\vartheta\)-expansion of \(Q^{aA}\) and \(\Phi^i\) are the first ones

\[ q^{aA} \equiv Q^{aA}|_{\vartheta=0}, \quad \phi^i \equiv \Phi^i|_{\vartheta=0}. \quad (219) \]

They accommodate the whole off-shell component content of the multiplet \((8, 8, 0)\). These bosonic \(N=4\) superfields are subjected, as a consequence of \((217), (218)\), to the irreducibility constraints in \(N=4\) superspace

\[ D^{a(\vartheta b)\alpha} = 0, \quad D^{ia}q^{bA} = 0. \quad (220) \]

Thus the \(N=8\) supermultiplet \((8, 8, 0)\) can be represented as the sum of two \(N=4, d=1\) hypermultiplets with the off-shell component contents \((4, 4, 0) \oplus (4, 4, 0)\).

The transformations of the implicit \(N=4\) Poincaré supersymmetry in this last case are as follows:

\[ \delta q^{aA} = -\frac{1}{2} \eta^a A D^{ia}\phi^i, \quad \delta \phi^i = \frac{1}{2} \eta^a A D^{ia}q^{aA}. \quad (221) \]

The invariant free action reads

\[ S = \int dt d^4 \theta \left[ q^2 - \phi^2 \right]. \quad (222) \]

The most general action still respecting four \(SU(2)\) automorphism symmetries has the following form:

\[ S = \int dt d^4 \theta F(q^2, \phi^2), \quad (223) \]

where, as a necessary condition of \(N=8\) supersymmetry, the function \(F(q^2, \phi^2)\) should obey the equation

\[ \frac{\partial^2}{\partial q^2 \partial q^2} \left( F(q^2, \phi^2) \right) = 0. \quad (224) \]

Thus, we presented superfield formulations of the full amount of off-shell \(N=8, d=1\) supermultiplets with 8 physical fermions, both in \(N=8\) and \(N=4\) superspaces. We listed all possible \(N=4\) superfield splittings of these multiplets.
5 Summary and conclusions

In these Lectures we reviewed the superfield approach to extended supersymmetric one-dimensional models. We presented superfield formulations of the full amount of off-shell $N = 4$ and $N = 8, d = 1$ supermultiplets with 4 and 8 physical fermions, respectively. We also demonstrated how to reproduce $N = 4$ supermultiplets from nonlinear realizations of the $N = 4, d = 1$ superconformal group.

It should be pointed out that here we addressed only those multiplets which satisfy linear constraints in superspace. As we know, there exist $N = 4, d = 1$ multiplets with nonlinear defining constraints (e.g. nonlinear versions of the chiral $(2, 4, 2)$ multiplet, as well as of the hypermultiplet $(4, 4, 0)$). It would be interesting to construct analogous nonlinear versions of some $N = 8$ multiplets from the above set. Moreover, for all our linear supermultiplets the bosonic metrics of the general sigma-model type actions are proven to be conformally flat. This immediately raises the question - how to describe $N = 4$ and $N = 8, d = 1$ sigma models with hyper-Kähler metrics in the target space? For the $N = 8$ cases it seems the unique possibility is to use infinite dimensional supermultiplets, as in the case of $N = 2, d = 4$ supersymmetry [32, 33]. But for $N = 4, d = 1$ supersymmetric models infinite dimensional supermultiplets do not exist! Therefore, the unique possibility in this case is to use some nonlinear supermultiplets. In this respect, the harmonic superspace approach seems to yield the most relevant framework. So, all results we discussed should be regarded as preparatory for a more detailed study of $N = 4, 8 d = 1$ supersymmetric models.

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