Abstract

Using colored superanalysis and $\varepsilon$-Lie superalgebras, we build the minimal Poincaré superalgebra in the case of the $\mathbb{Z}_n^3$ grading. We then build a representation of this algebra, and the corresponding Poincaré supergroup.
1 Introduction

This work is based on generalized Grassmann algebras, which are graded by an arbitrary abelian group and obey generalized commutation relations. These relations are based on a commutation factor $\varepsilon$ that is a function of the degrees of the elements it applies to:

$$ab = \varepsilon(d_a, d_b)ba$$

where $d_a$ and $d_b$ are the degrees of $a$ and $b$, which are elements of the grading group. This commutation factor obeys several very restricting rules.

It is well known that the study of common generalized graded objects, such as Lie algebras, Grassmann algebras, superspaces, etc., although it has been widely conducted, can be reduced to that of the corresponding superobjects ($\mathbb{Z}_2$-graded), through a change of the commutation factor. But this only means that any theorem that is true for superobjects is also true for the generalized, arbitrarily graded objects, which is a good thing. We claim that the commutation factor has physical relevance in itself, and thus that the generalized objects can describe objects whose properties are different from those of the analogous superobjects. The commutation properties of operators describing particles are at the origin of some of their most important features, that is, their bosonic or fermionic statistics. The objects that we will describe here have some properties of ordinary bosons and fermions, but with additional features that could be useful in the modelization of quark fields.
2 $\mathbb{Z}_3^n$-graded Grassmann algebra

Among the abelian groups that could be chosen to grade a generalized Grassmann algebra, $\mathbb{Z}_3^n$ groups seem to be very particular in being the only groups to induce a Grassmann algebra that is maximally symmetric and includes fermionic, bosonic, and other types of variables\[9\]. No abelian group composed of more than three cyclic groups is able to produce a Grassmann algebra that puts them on an equal footing, and any group composed of less than three cyclic groups gives an ordinary Grassmann algebra.

We label the three grading groups with the letters $r$, $g$ and $b$ (for red, green and blue) for reasons that will become clearer in the sequel. The commutation factor for the generalized Grassmann algebra is then

$$\varepsilon(x, y) = (-1)^{x_r y_r + x_g y_g + x_b y_b} q^{x_r y_g - y_r x_g + x_g y_b - y_g x_b + x_b y_r - y_b x_r}$$

where the degrees $x$ and $y$ are expressed by three integers representing their components on the three groups:

$$x = (x_r, x_g, x_b) \quad \text{and} \quad y = (y_r, y_g, y_b)$$

and $q$ is an $n^{th}$ root of unity.

The generalized Grassmann algebra is defined as a $\mathbb{Z}_3^n$-graded, associative and $\varepsilon$-commutative algebra.

Its generators could be limited to elements of degree 1 in only one of the three colors, but we choose to include generators of degree $(\pm 1, \pm 1, \pm 1)$, which
are fermionic (anticommuting) generators, as can be seen in the commutation rules below, as well as generators of degree $-1$ in one of the three colors. We summarize the degrees of the generators in the table below:

|       | $\theta_{A_r}$ | $\theta_{A_g}$ | $\bar{\theta}_{\bar{A}_r}$ | $\bar{\theta}_{\bar{A}_g}$ | $\eta_a$ | $\bar{\eta}_{\bar{a}}$ |
|-------|----------------|----------------|----------------------------|----------------------------|---------|-------------------|
| Rouge | 1              | 0              | $-1$                       | 0                          | 1       | $-1$              |
| Vert  | 0              | 1              | 0                          | $-1$                       | 0       | 1                 |
| Bleu  | 0              | 0              | 1                          | 0                          | $-1$    | 1                 |

Some commutation rules, entirely defined by the commutation factor, are

$$\eta_a \eta_b = -\eta_b \eta_a, \quad \bar{\eta}_{\bar{a}} \bar{\eta}_{\bar{b}} = -\bar{\eta}_{\bar{b}} \bar{\eta}_{\bar{a}}, \quad \eta_a \bar{\eta}_{\bar{b}} = -\bar{\eta}_{\bar{b}} \eta_a$$

which are the commutation relations of ordinary fermionic variables; other interesting commutation rules include:

$$\theta_{A_r} \theta_{A_g} = q \theta_{A_g} \theta_{A_r}, \quad \theta_{A_g} \theta_{A_b} = q \theta_{A_b} \theta_{A_g}, \quad \theta_{A_r} \theta_{A_r} = q \theta_{A_r} \theta_{A_r}$$

$$\bar{\theta}_{\bar{A}_r} \bar{\theta}_{\bar{A}_g} = q \bar{\theta}_{\bar{A}_g} \bar{\theta}_{\bar{A}_r}, \quad \bar{\theta}_{\bar{A}_g} \bar{\theta}_{\bar{A}_b} = q \bar{\theta}_{\bar{A}_b} \bar{\theta}_{\bar{A}_g}, \quad \bar{\theta}_{\bar{A}_r} \bar{\theta}_{\bar{A}_r} = q \bar{\theta}_{\bar{A}_r} \bar{\theta}_{\bar{A}_r}$$

In a sector of a given color, the generators anticommute:

$$\theta_{A_r} \theta_{B_r} = -\theta_{B_r} \theta_{A_r}$$

Thus, the colored generators are nilpotent of rank two: $\theta_{A_r}^2 = \theta_{A_g}^2 = \theta_{A_b}^2 = \theta_{\bar{A}_r}^2 = \theta_{\bar{A}_g}^2 = \theta_{\bar{A}_b}^2 = 0$. Similarly, colored generators commute with fermionic generators:

$$\theta_{A_r} \eta_a = -\eta_a \theta_{A_r}$$
The most significant new feature of this algebra is that it includes a purely fermionic and bosonic subalgebra, that is an ordinary Grassmann subalgebra, that contains very particular combinations of the colored generators, especially if \( q \) is not a root of unity, that is, if the grading group is \( \mathbb{Z}^3 \):

- The fermionic (anticommuting) elements are:
  - The \( \eta_a \) and \( \bar{\eta}_{\bar{a}} \)
  - The products \( \theta_{A_r} \theta_{A_g} \theta_{A_b} \) and \( \bar{\theta}_{\bar{A}_r} \bar{\theta}_{\bar{A}_g} \bar{\theta}_{\bar{A}_b} \) of three generators of distinct colors.
  - The products of a bosonic element (see below) and a fermionic element, or an odd number of the above.

- The bosonic elements (commuting with all other elements) are:
  - The products \( \theta_{A_r} \bar{\theta}_{\bar{A}_r} \), \( \theta_{A_g} \bar{\theta}_{\bar{A}_g} \) and \( \theta_{A_b} \bar{\theta}_{\bar{A}_b} \) of a generator and another of opposite color.
  - The products of an even number of fermionic elements
  - The products of bosonic elements.

This list is exhaustive, and one can note that the only possible combinations of colored variables remind strongly of the observable combinations of quarks in QCD. This property will have its analogue in the irreducible representations of the Poincaré superalgebra. It is well known that a reasonable field theory in a four dimensional minkowskian space-time cannot feature observable particles
whose creation and annihilation operators do not obey ordinary commutation and anticommutation rules $^{[10]}$. Thus, in a field theory based on the objects that we introduce here, the “colored” particles would be unobservable individually, but the above combinations into fermions and bosons could be observable. This could provide an algebraic model for the confinement of quarks. We will develop this idea further in the next article.

3 $\mathbb{Z}_n^3$-graded superspace

The $\mathbb{Z}_n^3$-graded superspaces are defined as a set $Q$ and an associated family of bijections that are homogeneous in degree, the coordinate maps, from $Q$ to the generalized Grassmann algebra$^{[8]}$. In the case of ordinary $\mathbb{Z}_2$-graded superspaces, there were only two degrees, and the question of the representation of all degrees by the coordinates was irrelevant. Here, there can or can not be coordinates of any possible degree.

One can define continuity, differentiation, analytic functions on these superspaces, and the usual theorems generalize very well$^{[8]}$. For example,

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} = \varepsilon(d_{x_i}, d_{x_j}) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i}$$

where $x_i$ and $x_j$ are two coordinates, and $d_{x_i}$ and $d_{x_j}$ are their degrees. The Leibniz rule generalizes as

$$\frac{\partial}{\partial x_i} (fg) = \frac{\partial f}{\partial x_i} g + \varepsilon(d_{x_i}, d_f) f \frac{\partial g}{\partial x_i}$$

if $f$ is a function of homogeneous degree $d_f$. 
4 $\mathbb{Z}_n^3$-graded Poincaré superalgebras

To generate a $\mathbb{Z}_n^3$-graded Poincaré superalgebra, we will add to the ordinary Poincaré algebra a collection of generators of arbitrary degrees to form an $\varepsilon$-Lie superalgebra. The generalized supercommutator will be noted here as follows

$$[A, B]_c = AB - \varepsilon(d_A, d_B)BA$$

We will focus on the smallest of these algebras that include the usual minimal Poincaré superalgebra.

If we add to the Poincaré algebra the odd generators of a usual Poincaré superalgebra, we have to give them a suitable degree in our generalized grading group. Obviously, the possible degrees are $\pm 1$ in each color, like in the Grassmann algebra. The supercommutators of two supertranslations must fall into the even part of the algebra, that is the Poincaré algebra. Thus, the non-zero supercommutation relations of the Poincaré superalgebra must be reflected here by colored supercommutation relations between generators of opposite degrees (so that the result is of degree 0). Therefore, the two component and four component formulations of Poincaré superalgebras won’t generalize equivalently.

4.1 Generalized two component formulation

First, we’ll generalize the two component formulation by adding 2 generators $Q_1, Q_2$ of degree 1 in each color (that we’ll call “white” generators) and 2 generators $\bar{Q}_1, \bar{Q}_2$ of degree $-1$ in each color (that we’ll call “antiwhite” generators).
These generators will give the odd part of the ordinary Poincaré superalgebra.
Similarly, we'll add two generators and two “antigenerators” in each color. Finally, we will not assume anything à priori on the commutation relations that don’t give a zero degree result, but we’ll try to keep the algebra as small as possible. In summary, we have the following generators:

|       | $Q_1, Q_2$ | $Q_1, \bar{Q}_2$ | $\bar{Q}_1, Q_2$ | $\bar{Q}_1, \bar{Q}_2$ | $Q_{1_+}, Q_{2_+}$ | $\bar{Q}_{1_+}, \bar{Q}_{2_+}$ | $Q_{1_0}, Q_{2_0}$ | $\bar{Q}_{1_0}, \bar{Q}_{2_0}$ |
|-------|------------|------------------|------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| red   | 1          | -1               | 1                | -1                  | 0                   | 0                   | 0                   | 0                   |
| green | 1          | -1               | 0                | 0                   | 1                   | -1                  | 0                   | 0                   |
| blue  | 1          | -1               | 0                | 0                   | 0                   | 0                   | 1                   | -1                  |

As in the case of Lie superalgebras, the elements of any sector of the algebra of a given degree must form the basis for a representation of the Poincaré algebra. We’ll choose our generators here so that the representation for positive (resp. negative) degree sectors are the left (resp. right) handed irreducible two component representations of the Lorentz algebra. The translations are trivially represented. In other words, if $\alpha > \beta$, and if $d$ is any degree among $(1,1,1), (-1,-1,-1), r = (1,0,0), \bar{r} = (-1,0,0), g, \bar{g}, b, \bar{b}$, we have, the $\sigma$ being Pauli matrices:

$$[M_{\alpha\beta}, Q_{id}]_c = -\frac{\hbar}{2i} \sum_{jd=1}^{2} (\sigma_\alpha \sigma_\beta)_{idjd} Q_{jd}$$

$$[M_{\alpha\beta}, \bar{Q}_{id}]_c = -\frac{\hbar}{2i} \sum_{jd=1}^{2} (\sigma_\alpha \sigma_\beta)^*_{idjd} \bar{Q}_{jd}$$

and if $\alpha < \beta$,

$$[M_{\alpha\beta}, Q_{id}]_c = \frac{\hbar}{2i} \sum_{jd=1}^{2} (\sigma_\beta \sigma_\alpha)_{idjd} Q_{jd}$$

(1)
\[ [M_{\alpha\beta}, \bar{Q}_{i_d}]_c = \frac{\hbar}{2i} \sum_{j_d=1}^{2} (\sigma_\beta \sigma_\alpha)_{i_d j_d}^* \bar{Q}_{j_d} \]

And of course,

\[ [P_\mu, Q_{i_d}]_c = [P_\mu, \bar{Q}_{j_d}]_c = 0 \quad (2) \]

If we want the white and anti-white generators to behave like the supertranslations, the commutator of two white—or two antiwhite—generators must be equal to zero. Similarly, we’ll suppose that the commutator of two generators of the same color is equal to zero, which will keep the size of the algebra minimal.

Let us first compute the commutation relations of two generators of opposite degrees. The result must be an element of the Poincaré algebra. The generalized Jacobi identity and the commutation relations (2) give

\[ [P_\mu, [Q_{i_d}, \bar{Q}_{j_{-d}}]]_c = 0 \]

Thus, \([Q_{i_d}, \bar{Q}_{j_{-d}}]_c\) must decompose along the translations \(P_\mu\). Another application of the Jacobi identity with the rotations, and of the relations (3) give the coefficients of this decomposition:

\[ [Q_{i_d}, \bar{Q}_{j_{-d}}]_c = \kappa_d \sum_{\mu=1}^{4} (\sigma_\mu)_{i_d j_{-d}} P_\mu \]

In supersymmetry, \(\kappa_d\) is usually fixed to the value 2, but for the moment, we’ll allow for different values of this parameter for each degree \(d \in \{(1, 1, 1), r, g, b\}\).

It is clear from these relations that the zero degree, the white and the anti-white generated sectors form a subalgebra that is really a Poincaré superalgebra (the colored commutator reduces in these sectors to the supercommutator).
We still have to compute the commutation relations of two generators of different and non-opposite degrees. We can reduce the dimension of these bicolored sectors as low as 4 while maintaining the Jacobi identities true, by supposing that each of them is generated by four generators $R_{a,d}$, where $d$ is any bicolor degree among $r + g$, $g + b$, $b + r$, $r + \bar{g}$, $g + \bar{b}$ and their opposites, and that the following commutation relations hold true: if $(d, d') \in \{(r, g), (g, b), (b, r), (\bar{r}, \bar{g}), (\bar{g}, \bar{b}), (\bar{b}, \bar{r})\}$,

$$[Q_{a,d}, Q_{b,d'}]_c = \frac{1 - q}{2} \sqrt{r_d r_{d'}} \sum_{a_d + d' = 1}^4 (\sigma_{a_dd'})_{a_1 b_1} R_{a_1 d_1} R_{a_2 d_2}$$

If $(d, d') \in \{(r, \bar{g}), (g, \bar{b}), (\bar{b}, \bar{r})\}$,

$$[Q_{a,d}, Q_{b,d'}]_c = \frac{1 - q^{-1}}{2} \sqrt{r_d r_{d'}} \sum_{a_d + d' = 1}^4 (\sigma_{a_dd'})_{a_1 b_1} R_{a_1 d_1} R_{a_2 d_2}$$

We also have

$$[Q_\alpha, Q_{\beta r}]_c = -\sqrt{r_1 r_r} \sum_{a_1 + b_1 = 1}^4 (\sigma_{a_1 b_1})_{\alpha_1 \beta \gamma} R_{\alpha_1 \beta \gamma}$$

$$[Q_\alpha, Q_{\beta g}]_c = -\sqrt{r_1 r_g} \sum_{a_1 + b_1 = 1}^4 (\sigma_{a_1 b_1})_{\alpha_1 \beta \gamma} R_{\alpha_1 \beta \gamma}$$

$$[Q_\alpha, Q_{\beta b}]_c = -\sqrt{r_1 r_b} \sum_{a_1 + b_1 = 1}^4 (\sigma_{a_1 b_1})_{\alpha_1 \beta \gamma} R_{\alpha_1 \beta \gamma}$$

and

$$[Q_\alpha, Q_{\beta r}]_c = -\sqrt{r_1 r_r} \sum_{a_1 + b_1 = 1}^4 (\sigma_{a_1 b_1})_{\alpha_1 \beta \gamma} R_{\alpha_1 \beta \gamma}$$

$$[Q_\alpha, Q_{\beta g}]_c = -\sqrt{r_1 r_g} \sum_{a_1 + b_1 = 1}^4 (\sigma_{a_1 b_1})_{\alpha_1 \beta \gamma} R_{\alpha_1 \beta \gamma}$$

$$[Q_\alpha, Q_{\beta b}]_c = -\sqrt{r_1 r_b} \sum_{a_1 + b_1 = 1}^4 (\sigma_{a_1 b_1})_{\alpha_1 \beta \gamma} R_{\alpha_1 \beta \gamma}$$
Finally, for $\alpha < \beta$, the Jacobi identity gives

$$[M_{\alpha \beta}, R_{a_{d+d'}}]_c = \frac{\hbar}{i}(\eta_{\alpha a_{d+d'}} R_{\beta d+d'} - \eta_{\beta a_{d+d'}} R_{\alpha d+d'})$$

The $d + d'$ index clearly does not indicate the degree of $\alpha$ and $\beta$ indices, but that of $R$.

The only other commutation relations that are not equal to zero are the usual

$$[M_{\alpha \beta}, M_{\eta \lambda}]_c = \frac{\hbar}{i}(\eta_{\alpha \eta} M_{\beta \lambda} - \eta_{\lambda \alpha} M_{\beta \eta} - \eta_{\beta \eta} M_{\alpha \lambda} + \eta_{\beta \lambda} M_{\alpha \eta})$$

$$[M_{\alpha \beta}, P_{\mu}]_c = \frac{\hbar}{i}(\eta_{\alpha \mu} P_{\beta} - \eta_{\beta \mu} P_{\alpha})$$

### 4.2 Generalized four component formulation

In the generalization of this formulation, we use four generators in each color, anticolor, as well as in white and antiwhite. The notations are basically the same as in the two component section, except that the indices run from 1 to 4 instead of from 1 to 2. We choose these generators so that the representation of the Poincaré algebra in the colored sectors is a spinorial representation where the $M_{\alpha \beta}$ are represented by

$$\frac{\hbar}{2i}(\tilde{\gamma}_\alpha \tilde{\gamma}_\beta)$$

where the tilda is the transposition operation and the $\gamma_\alpha$ are Dirac matrices. The translations are trivially represented.

Like in the case of the two component formulation, we also introduce sets of four generators $R_{\alpha_d}$ in the bicolor sectors. The Jacobi identity, and the
limitation to the minimal case where the algebra is the vector space spanned by these generators (the $M_{\alpha\beta}$, $P_\mu$, $Q_a$, and $R_{a_d}$) give us the following commutation relations for $\alpha \neq \beta$:

$$
[M_{\alpha\beta}, M_{\eta\lambda}]_c = \frac{\hbar}{i} (\eta_{\alpha\eta} M_{\beta\lambda} - \eta_{\alpha\lambda} M_{\beta\eta} - \eta_{\beta\eta} M_{\alpha\lambda} + \eta_{\beta\lambda} M_{\alpha\eta})
$$

$$
[M_{\alpha\beta}, P_\mu]_c = \frac{\hbar}{i} (\eta_{\alpha\mu} P_\beta - \eta_{\beta\mu} P_\alpha)
$$

$$
[M_{\alpha\beta}, Q_{i_d}]_c = \frac{\hbar}{2i} \sum_{b_d=1}^4 (\gamma_{\alpha} \gamma_{\beta})_{i_d b_d} Q_{b_d}
$$

$$
[Q_{i_d}, Q_{j_{-d}}]_c = -\kappa_d \sum_{\mu=1}^4 (\gamma^\mu C)_{i_d j_{-d}} P_\mu
$$

$$
[Q_{a_d}, Q_{b_d}]_c = \frac{1 - q}{2\sqrt{\kappa_d \kappa_{d'}}} \sum_{a_{d+d'}=1}^4 (\gamma^{a_{d+d'}} C)_{a_d b_{d'}} R_{a_{d+d'}}
$$

for $(d, d') \in \{(r, g), (g, b), (b, r), (\bar{r}, \bar{g}), (\bar{g}, \bar{b}), (\bar{b}, r)\}$

$$
[Q_{a_d}, Q_{b_{d'}}]_c = \frac{1 - q^{-1}}{2\sqrt{\kappa_d \kappa_{d'}}} \sum_{a_{d+d'}=1}^4 (\gamma^{a_{d+d'}} C)_{a_d b_{d'}} R_{a_{d+d'}}
$$

for $(d, d') \in \{(r, \bar{g}), (g, \bar{b}), (b, \bar{r}), (\bar{r}, g), (\bar{g}, b), (\bar{b}, r)\}$

$$
[Q_a, Q_b]_c = -\sqrt{\kappa_1 \kappa_r} \sum_{a_{g+b}=1}^4 (\gamma^{a_{g+b}} C)_{a_{g+b}} R_{a_{g+b}}
$$

$$
[Q_a, Q_{b_{g+r}}]_c = -\sqrt{\kappa_1 \kappa_g} \sum_{a_{b+r}=1}^4 (\gamma^{a_{b+r}} C)_{a_{b+r}} R_{a_{b+r}}
$$

$$
[Q_a, Q_{b_{r+g}}]_c = -\sqrt{\kappa_1 \kappa_b} \sum_{a_{r+g}=1}^4 (\gamma^{a_{r+g}} C)_{a_{r+g}} R_{a_{r+g}}
$$

$$
[Q_{\bar{a}}, Q_{b_{r}}]_c = -\sqrt{\kappa_1 \kappa_r} \sum_{a_{g+b}=1}^4 (\gamma^{a_{g+b}} C)_{a_{g+b}} R_{a_{g+b}}
$$

$$
[Q_{\bar{a}}, Q_{b_{g+r}}]_c = -\sqrt{\kappa_1 \kappa_g} \sum_{a_{b+r}=1}^4 (\gamma^{a_{b+r}} C)_{a_{b+r}} R_{a_{b+r}}
$$

$$
[Q_{\bar{a}}, Q_{b_{r+g}}]_c = -\sqrt{\kappa_1 \kappa_b} \sum_{a_{r+g}=1}^4 (\gamma^{a_{r+g}} C)_{a_{r+g}} R_{a_{r+g}}
$$
\[ [M_{\alpha\beta}, R_{\alpha_+\beta_+}];_c = \frac{i}{\hbar_\alpha} (\eta_{\alpha_+\beta_+}R_{\beta_+\alpha_+} - \eta_{\beta_+\alpha_+}R_{\alpha_+\beta_+}) \]

In these relations, \( C \) is the charge conjugation matrix.

5 Representations of the \( \mathbb{Z}_n^3 \)-graded Poincaré superalgebras

The above study of the minimal generalized Poincaré superalgebra has led us to algebras where only 21 degrees are present: 0, 1, \( \tilde{1} \), \( r \), \( g \), \( b \), \( \tilde{r} \), \( \tilde{g} \), \( \tilde{b} \), \( r + g \), \( g + b \), \( b + r \), \( \tilde{r} + \tilde{g} \), \( \tilde{g} + \tilde{b} \), \( \tilde{b} + \tilde{r} \), \( r + \tilde{g} \), \( \tilde{g} + b \), \( b + \tilde{r} \). We have to find a block structure for the representation that reproduces the grading rules. In this section, we consider only the representation of the four-component \( \mathbb{Z}_n^3 \)-graded Poincaré superalgebra.

The diagonal blocks must represent transformations of degree 0, that is the Poincaré transformations. At least one of these block representations must be faithful. On the other hand, the commutator of a generator of any color and a generator of opposite color gives a linear combination of translations. Thus, any block line or column corresponding to a faithful representation of the Poincaré algebra must contain blocks of all colors 1, \( r \), \( g \), \( b \) and their opposites. The smallest structure meeting all requirements is \( 24 \times 24 \) by blocks, and its actual size is \( 100 \times 100 \). This block structure is

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]
where the structure of $A, B, C, D$, the degrees associated with each block, when this degree is expressed in the algebra, being:

$$A = \begin{pmatrix}
0 & r + g & g + b & b + r \\
r + \bar{g} & 0 & b + \bar{r} & \bar{g} + b \\
g + \bar{b} & \bar{b} + r & 0 & r + \bar{g} \\
\bar{b} + \bar{r} & g + \bar{b} & \bar{r} + g & 0
\end{pmatrix}$$

where the blocks are $5 \times 5$.

$$B = \begin{pmatrix}
r & g & b & \bar{r} & \bar{g} & \bar{b} & 1 & \bar{1} \\
\bar{g} & \bar{r} & \bar{1} & b & \bar{b} & g & r & 1 \\
\bar{b} & \bar{g} & \bar{1} & r & \bar{r} & b & g & 1 \\
\bar{b} & \bar{r} & \bar{1} & g & \bar{g} & r & b & 1
\end{pmatrix}$$
where each block is 5 × 4.

\[
C = \begin{pmatrix}
\bar{r} & g & b \\
\bar{g} & r & b \\
\bar{b} & g & r \\
r & 1 \\
g & 1 \\
b & 1 \\
\bar{1} & \bar{b} & \bar{r} & \bar{g} \\
1 \\
b \\
r \\
g \\
\bar{g} \\
\bar{r} \\
\bar{b} \\
\bar{g} \\
\bar{r} \\
\bar{b} \\
1 \\
\bar{1} \\
1 \\
\bar{1}
\end{pmatrix}
\]

where each block is 4 × 5. \(D\) is a square matrix constituted of 20 × 20 4 × 4.
blocks that are all equal to zero, except for the diagonal blocks, which are of degree zero. The blocks whose degrees have not been represented in these block structures are always equal to zero.

A matrix and its block structure representing an element $a$ of the generalized Poincaré algebra will be noted

$$\Gamma(a) = (\Gamma_{i,j}(a))_{0 \leq i, j \leq 23}$$

The faithful representations of the Poincaré algebra will be in the four diagonal blocks of $A$:

$$\Gamma_{0,0}(M_{\alpha\beta}) = \Gamma_{1,1}(M_{\alpha\beta}) = \Gamma_{2,2}(M_{\alpha\beta}) = \Gamma_{3,3}(M_{\alpha\beta}) = \begin{pmatrix} M_{\alpha\beta} & 0 \\ 0 & 0 \end{pmatrix}$$

where the $M_{\alpha\beta}$ matrices are defined by

$$(M_{\alpha\beta})_{\mu\nu} = \frac{\hbar}{i}(\delta_{\beta\mu}\eta_{\alpha\nu} - \delta_{\alpha\mu}\eta_{\beta\nu})$$

where $\alpha, \beta, \lambda, \mu = 1, \ldots, 4$. For the translations, $\Gamma_{0,0}(P_{\mu}) = \Gamma_{1,1}(P_{\mu}) = \Gamma_{2,2}(P_{\mu}) = \Gamma_{3,3}(P_{\mu}) = \begin{pmatrix} 0 & -\frac{\hbar}{\lambda}\delta_{\mu} \\ 0 & 0 \end{pmatrix} \equiv P_{\mu}$

where the $\delta_{\mu}$ are the four $4 \times 1$ matrices defined by

$$(\delta_{\mu})_{\alpha 1} = \delta_{\alpha\mu}$$

and $\lambda$ is a real constant with the dimensions of a length.

For the other representations of the Poincaré algebra, we will choose a spinorial representation: for $i > 3$,

$$\Gamma_{i,i}(M_{\alpha\beta}) = -\frac{\hbar}{2i}\gamma_{\alpha}\gamma_{\beta}$$
\[ \Gamma_{i,i}(P_\mu) = 0 \]

The supertranslations \( Q_a \) are represented by the matrices with the following non-zero blocks:

\[
\begin{align*}
\Gamma_{0,10}(Q_a) &= \Gamma_{1,21}(Q_a) = \Gamma_{2,22}(Q_a) = \Gamma_{3,23}(Q_a) = \sqrt{\kappa_1}B_a \\
\Gamma_{11,0}(Q_a) &= \Gamma_{9,1}(Q_a) = \Gamma_{7,2}(Q_a) = \Gamma_{8,3}(Q_a) = \sqrt{\kappa_1}C_a
\end{align*}
\]

where \( B_a \) and \( C_a \) are the matrices defined by

\[
B_a = \begin{pmatrix} -\left( \frac{h}{\lambda} \right)^{1/2} e^{i\pi/4} U_a \\ 0 \end{pmatrix}
\]

\[
C_a = \begin{pmatrix} 0 & -\left( \frac{h}{\lambda} \right)^{1/2} e^{i\pi/4} \delta_a \end{pmatrix}
\]

The 0 in \( B_a \) is a 1 \( \times \) 4 zero block and the 0 in \( C_a \) is a 4 \( \times \) 4 zero block. The \( U_a \) are four 4 \( \times \) 4 matrices defined by

\[
(U_a)_{ab} = (\gamma^a c)_{ab}
\]

Similarly, the supertranslations \( Q_\hat{a} \) are represented by the 4 \( \times \) 5 blocks

\[
\begin{align*}
\Gamma_{0,11}(Q_\hat{a}) &= \Gamma_{1,9}(Q_\hat{a}) = \Gamma_{2,7}(Q_\hat{a}) = \Gamma_{3,8}(Q_\hat{a}) = \sqrt{\kappa_1}B_\hat{a} \\
\Gamma_{10,0}(Q_\hat{a}) &= \Gamma_{21,1}(Q_\hat{a}) = \Gamma_{22,2}(Q_\hat{a}) = \Gamma_{23,3}(Q_\hat{a}) = \sqrt{\kappa_1}C_\hat{a}
\end{align*}
\]

and the colored supertranslations are represented by the blocks

\[
\begin{align*}
\Gamma_{0,4}(Q_{i_r}) &= \Gamma_{1,16}(Q_{i_r}) = \Gamma_{2,10}(Q_{i_r}) = \Gamma_{3,19}(Q_{i_r}) = \sqrt{\kappa_r}B_{i_r} \\
\Gamma_{7,0}(Q_{i_r}) &= \Gamma_{5,1}(Q_{i_r}) = \Gamma_{13,2}(Q_{i_r}) = \Gamma_{6,3}(Q_{i_r}) = \sqrt{\kappa_r}C_{i_r}
\end{align*}
\]
\[ \Gamma_{0.5}(Q_{i_9}) = \Gamma_{1.15}(Q_{i_9}) = \Gamma_{2.18}(Q_{i_9}) = \Gamma_{3.10}(Q_{i_9}) = \sqrt{\kappa_0} B_{i_9} \]
\[ \Gamma_{8.0}(Q_{i_9}) = \Gamma_{4.1}(Q_{i_9}) = \Gamma_{6.2}(Q_{i_9}) = \Gamma_{14.3}(Q_{i_9}) = \sqrt{\kappa_0} C_{i_9} \]
\[ \Gamma_{6.6}(Q_{i_6}) = \Gamma_{1.10}(Q_{i_6}) = \Gamma_{2.17}(Q_{i_6}) = \Gamma_{3.20}(Q_{i_6}) = \sqrt{\kappa_0} B_{i_6} \]
\[ \Gamma_{9.0}(Q_{i_9}) = \Gamma_{12.1}(Q_{i_9}) = \Gamma_{5.2}(Q_{i_6}) = \Gamma_{4.3}(Q_{i_6}) = \sqrt{\kappa_0} C_{i_6} \]

and

\[ \Gamma_{0.7}(Q_{i_r}) = \Gamma_{1.5}(Q_{i_r}) = \Gamma_{2.13}(Q_{i_r}) = \Gamma_{3.6}(Q_{i_r}) = \sqrt{\kappa_r} B_{i_r} \]
\[ \Gamma_{4.0}(Q_{i_r}) = \Gamma_{16.1}(Q_{i_r}) = \Gamma_{10.2}(Q_{i_r}) = \Gamma_{19.3}(Q_{i_r}) = \sqrt{\kappa_r} C_{i_r} \]
\[ \Gamma_{0.8}(Q_{i_9}) = \Gamma_{1.4}(Q_{i_9}) = \Gamma_{2.6}(Q_{i_9}) = \Gamma_{3.14}(Q_{i_9}) = \sqrt{\kappa_0} B_{i_9} \]
\[ \Gamma_{5.0}(Q_{i_9}) = \Gamma_{15.1}(Q_{i_9}) = \Gamma_{18.2}(Q_{i_9}) = \Gamma_{19.3}(Q_{i_9}) = \sqrt{\kappa_0} C_{i_9} \]
\[ \Gamma_{0.9}(Q_{i_6}) = \Gamma_{1.12}(Q_{i_6}) = \Gamma_{2.5}(Q_{i_6}) = \Gamma_{3.5}(Q_{i_6}) = \sqrt{\kappa_r} B_{i_6} \]
\[ \Gamma_{6.0}(Q_{i_6}) = \Gamma_{10.1}(Q_{i_6}) = \Gamma_{17.2}(Q_{i_6}) = \Gamma_{20.3}(Q_{i_6}) = \sqrt{\kappa_r} C_{i_6} \]

Finally, the \( R_{a_{+,d^e}} \) are represented by the matrices whose non zero blocks are:

\[ \Gamma_{0.1}(R_{a_{r+g}}) = P_{a_{r+g}}; \quad \Gamma_{0.2}(R_{a_{g+b}}) = P_{a_{g+b}}; \quad \Gamma_{0.3}(R_{a_{b+r}}) = P_{a_{b+r}} \]
\[ \Gamma_{1.0}(R_{a_{r+g}}) = P_{a_{r+g}}; \quad \Gamma_{2.0}(R_{a_{g+b}}) = P_{a_{g+b}}; \quad \Gamma_{3.0}(R_{a_{b+r}}) = P_{a_{b+r}} \]
\[ \Gamma_{1.2}(R_{a_{b+r}}) = P_{a_{b+r}}; \quad \Gamma_{1.3}(R_{a_{g+b}}) = P_{a_{g+b}}; \quad \Gamma_{2.3}(R_{a_{r+g}}) = P_{a_{r+g}} \]
\[ \Gamma_{2.1}(R_{a_{b+r}}) = P_{a_{b+r}}; \quad \Gamma_{3.1}(R_{a_{g+b}}) = P_{a_{g+b}}; \quad \Gamma_{3.2}(R_{a_{r+g}}) = P_{a_{r+g}} \]

The matrix \( \Gamma(a) \) representing an arbitrary element \( a \) of the generalized
Poincaré superalgebra can then be written

\[
\Gamma(a) = \frac{i}{\hbar} \left[ \sum_{1 \leq \alpha < \beta \leq 4} \omega^{\alpha\beta} \Gamma(M_{\alpha\beta}) + \sum_{\mu=1}^{4} t^{\mu} \Gamma(P_{\mu}) + \sum_{(d+d') \in T}^{(d+d') \in T} u^{a_{d+d'}} \Gamma(R_{a_{d+d'}}) \right] + \\
+ \hbar^{-1/2} e^{-i\pi/4} \sum_{d \in \{1,1,r,g,b,\bar{r},\bar{g},\bar{b}\}}^{4} \psi^{a_{d}} \Gamma(Q_{a_{d}})
\]

where

\[
T = \{ r + g, g + b, b + r, \bar{r} + \bar{g}, \bar{g} + \bar{b}, \bar{b} + \bar{r}, r + \bar{g}, g + \bar{b}, b + r, \bar{r} + g, \bar{g} + b, \bar{b} + \bar{r} \}
\]

\(\omega^{\alpha\beta}\) are six real dimensionless parameters, \(t^{\mu}\) and \(u^{a_{d+d'}}\) are fifty-two parameters with the dimensions of a length, and \(\psi^{a_{d}}\) are thirty-two parameters with the dimension of the square root of a length, that are real in the case of the Majorana representation of the Dirac matrices.

### 6 \(\mathbb{Z}_n^{3}\)-graded Poincaré supergroup

It is possible to rewrite the representation of an arbitrary element of the generalized Poincaré algebra as a supermatrix of degree 0, introducing Grassmann valued parameters:

\[
\Gamma(a) = \frac{i}{\hbar} \left[ \sum_{1 \leq \alpha < \beta \leq 4} \Omega^{\alpha\beta} \Gamma(M_{\alpha\beta}) + \sum_{\mu=1}^{4} T^{\mu} \Gamma(P_{\mu}) + \sum_{(d+d') \in T}^{(d+d') \in T} U^{a_{d+d'}} \# \Gamma(R_{a_{d+d'}}) \right] \\
+ \frac{i}{\hbar^{1/2}} \sum_{d \in \{1,1,r,g,b,\bar{r},\bar{g},\bar{b}\}}^{4} \zeta^{a_{d}} \# (\gamma_{4})_{a_{d}b_{d}} \Gamma(Q_{b_{d}})
\]
where $\Omega^{\alpha\beta}$ are 6 dimensionless parameters of degree 0, $T^\mu$ are 4 parameters with the dimensions of a length and of degree 0, $U^{a+d'}$ are 48 parameters with the dimensions of a length and of degrees $d + d'$, $\zeta^{a_d}$ are 32 parameters with the dimensions of the square root of a length and of degrees $d$. The $\#$ operator is the adjoint operator of the generalized Grassmann algebra, which is defined by

\[
(x, 1)^\# = x^* \cdot 1
\]

\[
(\eta_a)^\# = -i\eta_a, \quad (\bar{\eta}_\dot{a})^\# = -i\bar{\eta}_{\dot{a}}
\]

\[
(\theta_{A_d})^\# = iq\theta_{A_d}, \quad (\bar{\theta}_{\dot{A}\dot{d}})^\# = iq\bar{\theta}_{\dot{A}\dot{d}}
\]

\[
(XY)^\# = Y^\# X^\#
\]

The choice of parameters ensures that they transform in the same way as the operators they multiply under a transformation of the Dirac matrices.

A representation of the generalized Poincaré supergroup is obtained by exponentiating these matrices.

The generalized superspace parametrizing the generalized Poincaré supergroup has the following dimensions

\[
\langle \text{dim}_d; d \in \{\text{degrees}\} \rangle = (10_b, 4_1, 4_{\dot{1}}, 4_r, 4_g, 4_{\dot{g}}, 4_b, 4_{\dot{b}}, 4_r, 4_g, 4_{\dot{g}}, 4_b, 4_{\dot{b}})
\]

\[
4_{r+g}, 4_{g+b}, 4_{b+r}, 4_{r+\dot{g}}, 4_{\dot{g}+b}, 4_{b+\dot{r}},
\]

\[
4_{\dot{r}+g}, 4_{\dot{g}+b}, 4_{\dot{b}+r}, 4_{r+\dot{g}}, 4_{g+\dot{b}}, 4_{b+\dot{r}}
\]

An element of the Poincaré supergroup specified by the parameters $\Omega^{\alpha\beta}$, $T^\mu$, $\zeta^{a_d}$, and $U^{a+d'}$ will be noted $[\Lambda(\Omega) \mid T \mid \zeta \mid U]$, and its representation $\Gamma([\Lambda(\Omega) \mid T \mid \zeta \mid U])$. 20
For an element whose only nonvanishing coordinates are the $\Omega^{\alpha\beta}$, 

$$\Gamma([\Lambda(\Omega) | 0 | 0 | 0]) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where

$$A = \begin{pmatrix} \Lambda(\Omega) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Lambda(\Omega) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Lambda(\Omega) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Lambda(\Omega) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B = \begin{pmatrix} \Gamma^{\text{spin}}(\Lambda(\Omega)) & 0 & \cdots & 0 \\ 0 & \Gamma^{\text{spin}}(\Lambda(\Omega)) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Gamma^{\text{spin}}(\Lambda(\Omega)) \end{pmatrix}$$

where the $\Lambda(\Omega)$ are the Lorentz supermatrices obtained from the Lorentz matrices by replacing the real parameters $\omega^{\alpha\beta}$ with their zero degree grassmannian counterparts, the $\Omega^{\alpha\beta}$; the $\Gamma^{\text{spin}}(\Lambda(\Omega))$ are spinorial representations of the Lorentz supermatrices.

The square of all elements $M$ of the algebra whose $\Omega$ coordinates are equal to zero vanish. Moreover, two matrices representing elements of the same degree...
(excluding zero) have a vanishing product. Thus \( \exp(M) = 1 + M \) and if we choose to note

\[
\mathcal{T} = \begin{pmatrix}
1 & \sum_{\mu=1}^{4} T^{\mu} \delta_{\mu} \\
0 & 1
\end{pmatrix}
\]

we have

\[
\Gamma([1 \mid T \mid 0 \mid 0]) = 1_{100} + \frac{i}{\hbar} \sum_{\mu=1}^{4} T^{\mu} \Gamma(P_{\mu})
\]

\[
\begin{pmatrix}
\mathcal{T} & 0 & \cdots & \cdots & 0 \\
0 & \mathcal{T} & \ddots & \vdots \\
\vdots & \ddots & \mathcal{T} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & 0 & 1_{80}
\end{pmatrix}
\]

\[
\Gamma([1 \mid 0 \mid \zeta^{d} \mid 0]) = 1_{100} + \frac{i}{\hbar} \sum_{a,b=1}^{4} \zeta^{a,d} \#(\gamma_{4})_{a,b} \Gamma(Q_{b})
\]

\[
\Gamma([1 \mid 0 \mid 0 \mid U]) = 1_{100} + \frac{i}{\hbar} \sum_{(d,d') \in T} U^{a(d+d')} \# \Gamma(R_{a+d'})
\]

An arbitrary element \([\Lambda \mid T \mid \zeta \mid U] \) of the Poincaré supergroup can be written as

\[
[\Lambda \mid T \mid \zeta \mid U] = [1 \mid 0 \mid 0 \mid U] \prod_{d \in \{1,r,g,b,1,r,g,b\}} [1 \mid 0 \mid \zeta^{d} \mid 0] \times \\
\times [1 \mid T \mid 0 \mid 0][\Lambda \mid 0 \mid 0 \mid 0]
\]

Another order convention for the product would amount to a phase change in the parameters.

The product of two elements \([\Lambda \mid T \mid \zeta \mid U] \) and \([\Lambda' \mid T' \mid \zeta' \mid U'] \) of the
The supergroup is defined by

$$
\Gamma(\Lambda | T | \zeta | U | \Lambda' | T' | \zeta' | U') = \Gamma(\Lambda | T | \zeta | U)\Gamma(\Lambda' | T' | \zeta' | U')
$$

which gives

$$
[\Lambda | T | \zeta | U | \Lambda' | T' | \zeta' | U'] = [\Lambda\Lambda' | T + \Lambda T' + \tau | \zeta + \Gamma^{\text{spin}}(\Lambda)\zeta' | U + \Lambda U' + \rho]
$$

where $T'$ is the vector whose components are the $T'^\mu$. $\Gamma^{\text{spin}}(\Lambda)\zeta'$ stands for the set of all parameters $\Gamma^{\text{spin}}(\Lambda)\zeta'^d$ for each degree $d$ in $\{1, r, g, \bar{1}, \bar{r}, \bar{g}, \bar{b}\}$, where $\zeta'^d$ is the vector whose components are the $\zeta'^{a,d}$. The same notations are used for $\Lambda U'$. $\tau$ is defined by

$$
\tau^\mu = \sum_{d \in \{1, r, g, \bar{1}, \bar{r}, \bar{g}, \bar{b}\}} i(\zeta'^d)\mathbf{\gamma}_4\gamma^\mu \Gamma^{\text{spin}}(\Lambda)\zeta'^{-d}
$$

and $\rho$ by

$$
\rho^{\alpha + \rho} = i(\bar{\zeta}^\alpha)\mathbf{\gamma}_4\gamma^{\alpha + \rho} \Gamma^{\text{spin}}(\Lambda)\zeta'^\rho + i(\bar{\zeta}^\rho)\mathbf{\gamma}_4\gamma^{\alpha + \rho} \Gamma^{\text{spin}}(\Lambda)\zeta'^\alpha
$$

$$
+ i(\bar{\zeta}^\rho)\mathbf{\gamma}_4\gamma^{\alpha + \rho} \Gamma^{\text{spin}}(\Lambda)\zeta'^{\alpha + \rho}
$$

$$
\rho^{\alpha + \rho} = i(\bar{\zeta}^\alpha)\mathbf{\gamma}_4\gamma^{\alpha + \rho} \Gamma^{\text{spin}}(\Lambda)\zeta'^\rho + i(\bar{\zeta}^\rho)\mathbf{\gamma}_4\gamma^{\alpha + \rho} \Gamma^{\text{spin}}(\Lambda)\zeta'^{\alpha + \rho}
$$

$$
+ i(\bar{\zeta}^\rho)\mathbf{\gamma}_4\gamma^{\alpha + \rho} \Gamma^{\text{spin}}(\Lambda)\zeta'^{\alpha}
$$

and the four equivalent formulas in other colors.

An immediate consequence is that

$$
[\Lambda | T | \zeta | U]^{-1} = [\Lambda^{-1} | -\Lambda^{-1} T | -\Gamma^{\text{spin}}(\Lambda^{-1}) \zeta | -\Lambda^{-1} U']
$$
7 Action of the $\mathbb{Z}_n^3$-graded Poincaré supergroup on the $\mathbb{Z}_n^3$-graded superspace

The multiplication rule of two elements of the Poincaré supergroup admits as a particular case

$$[1 \mid X \mid \Xi \mid \Omega] [\Lambda \mid 0 \mid 0 \mid 0] = [\Lambda \mid X \mid \Xi \mid \Omega]$$

Thus, all elements of a left orthochronous coset of the Poincaré supergroup with homogeneous orthochronous Lorentz transformations formed from a given element $[1 \mid X \mid \Xi \mid \Omega]$ of the Poincaré supergroup have the same translational parts specified by $X, \Xi$ et $\Omega$. The coset is thus entirely defined by $X, \Xi$ and $\Omega$.

The action of an arbitrary element $[\Lambda \mid T \mid \zeta \mid U]$ of the Poincaré supergroup on the representant $[1 \mid X \mid \Xi \mid \Omega]$ of the coset is given by

$$[\Lambda \mid T \mid \zeta \mid U][1 \mid X \mid \Xi \mid \Omega] = [\Lambda \mid \Lambda X + T + \tau \mid \Gamma^{\text{spin}}(\Lambda)\Xi + \zeta \mid \Lambda \Omega + U + \rho]$$

where

$$\tau^\mu = \sum_{d \in \{1,r,g,b,\bar{1},\bar{r},\bar{g},\bar{b}\}} i(\tilde{\zeta}^d)^\# \gamma_4 \gamma^\mu \Gamma^{\text{spin}}(\Lambda)\Xi^{-d}$$

and

$$\rho^a_{r+g} = i(\tilde{\zeta}^r)^\# \gamma_4 \gamma^{a+r+g} \Gamma^{\text{spin}}(\Lambda)\Xi^g + i(\tilde{\zeta}^g)^\# \gamma_4 \gamma^{a+r+g} \Gamma^{\text{spin}}(\Lambda)\Xi^r + i(\tilde{\zeta}^b)^\# \gamma_4 \gamma^{a+r+g} \Gamma^{\text{spin}}(\Lambda)\Xi^b$$

$$\rho^a_{\bar{r}+\bar{g}} = i(\tilde{\zeta}^\bar{r})^\# \gamma_4 \gamma^{a+\bar{r}+\bar{g}} \Gamma^{\text{spin}}(\Lambda)\Xi^{\bar{g}} + i(\tilde{\zeta}^{\bar{g}})^\# \gamma_4 \gamma^{a+\bar{r}+\bar{g}} \Gamma^{\text{spin}}(\Lambda)\Xi^{\bar{r}} + i(\tilde{\zeta}^{\bar{b}})^\# \gamma_4 \gamma^{a+\bar{r}+\bar{g}} \Gamma^{\text{spin}}(\Lambda)\Xi^{\bar{b}}$$
\[ \rho^{a+g} = i(\tilde{\zeta}) \gamma^a \gamma^{a+g} \Gamma^{spin}(\Lambda) \Xi^g + i(\tilde{\zeta}) \gamma^a \gamma^{a+g} \Gamma^{spin}(\Lambda) \Xi^g \]

\[ \rho^{a+r} = i(\tilde{\zeta}) \gamma^a \gamma^{a+r+g} \Gamma^{spin}(\Lambda) \Xi^g + i(\tilde{\zeta}) \gamma^a \gamma^{a+r+g} \Gamma^{spin}(\Lambda) \Xi^g \]

In other words, the action of the transformation \([\Lambda \mid T \mid \zeta \mid U]\) on the coset defined by \(X, \Xi\) and \(\Omega\) results in the coset defined by \(\Lambda X + T + \tau, \Gamma^{spin}(\Lambda) \Xi + \zeta\) and \(\Lambda \Omega + U + \rho\).

Thus, we define the action of the generalized Poincaré supergroup on a point of the superspace defined by the coordinates \(X, \Xi\) and \(\Omega\) as its action on the coset defined by the same coordinates. The dimensions of the superspace are given by:

\[ D = (4_0, 4_1, 4_1, 4_r, 4_g, 4_b, 4_r, 4_g, 4_b, 4_r + g, 4_b + r, 4_r + g, 4_b + r, 4_r + g, 4_b + r, 4_b + r, 4_b + r) \]

An element \([\Lambda \mid T \mid \zeta \mid U]\) of the Poincaré supergroup transforms the point \((X, \Xi, \Omega)\) into the point \((X', \Xi', \Omega')\), where

\[ X' = \Lambda X + T + \tau \]
\[ \Xi' = \Gamma^{spin}(\Lambda) \Xi + \zeta \]
\[ \Omega' = \Lambda \Omega + U + \rho \]

We'll also note \((X', \Xi', \Omega') = [\Lambda \mid T \mid \zeta \mid U](X, \Xi, \Omega)\).

Of course, the consecutive action of two transformations on a point is equivalent to the action of the product of the transformations.

This rule includes the following particular cases. If we apply a homogeneous
Lorentz transformation $[\Lambda \mid 0 \mid 0 \mid 0]$, 

\[ X' = \Lambda X \]

\[ \Xi' = \Gamma^{\text{spin}}(\Lambda)\Xi \]

\[ \Omega' = \Lambda \Omega \]

If we apply a translation $[1 \mid T \mid 0 \mid 0]$, 

\[ X' = X + T \]

\[ \Xi' = \Xi \]

\[ \Omega' = \Omega \]

If we apply a colored supertranslation $[1 \mid 0 \mid \zeta \mid 0]$, 

\[ X'_{\mu} = X_{\mu} + \sum_{d \in \{1, r, b, \bar{1}, \bar{r}, \bar{b}\}} i(\tilde{\zeta}_{d})\# \gamma_{4}\gamma^{\mu}\Gamma^{\text{spin}}(\Lambda)\Xi_{-d} \]

\[ \Xi' = \Xi + \zeta \]

\[ \Omega' = \Omega + \rho \]

Finally, if we apply $[1 \mid 0 \mid 0 \mid U]$, 

\[ X' = X \]

\[ \Xi' = \Xi \]

\[ \Omega' = \Omega + U \]

We’ll call scalar superfield an analytic operator-valued function $\Phi_{s}(X, \Xi, \Omega)$ on the superspace.
The transformation operators \( P([\Lambda \mid T \mid \zeta \mid U]) \) for the scalar superfields are defined by the prescription

\[
P([\Lambda \mid T \mid \zeta \mid U]) \Phi_s(X, \Xi, \Omega) \left[ P([\Lambda \mid T \mid \zeta \mid U]) \right]^{-1} = \Phi_s([\Lambda \mid T \mid \zeta \mid U](X, \Xi, \Omega))
\]

In the case of supertranslations, we'll use the notation

\[
\delta_\zeta \Phi_s(X, \Xi, \Omega) = \left[ P \left( \frac{i}{\hbar^{1/2}} \sum_{d \in \{1, \bar{1}, r, g, b, \bar{r}, \bar{g}, \bar{b}\}} \frac{\partial}{\partial X^\mu} \right) \right] \Phi_s(X, \Xi, \Omega)
\]

From the action of the supergroup on the superspace, we get

\[
[P(Q_{a_d}), \Phi_s(X, \Xi, \Omega)] = \hbar^{1/2} \left\{ \sum_{\mu=1}^{d \in \{1, \bar{1}, r, g, b, \bar{r}, \bar{g}, \bar{b}\}} (\gamma^\mu \Xi^d)_{a_d} \frac{\partial}{\partial X^\mu} + i \sum_{d \in \{1, \bar{1}, r, g, b, \bar{r}, \bar{g}, \bar{b}\}} b_d=1 C_{a_d b_d} \frac{\partial}{\partial \Xi^{a_d}} \right\} \Phi_s(X, \Xi, \Omega)
\]

\[
[P(R_{a_d+a''}), \Phi_s(X, \Xi, \Omega)] = \frac{\hbar}{i} \frac{\partial \Phi_s(X, \Xi, \Omega)}{\partial \Omega^{a_d+a''}}
\]

\[
[P(M_{a\beta}), \Phi_s(X, \Xi, \Omega)] = \frac{\hbar}{i} \left\{ X_a \frac{\partial}{\partial X^\beta} - X_\beta \frac{\partial}{\partial X^a} - \frac{1}{2} \sum_{d \in \{1, \bar{1}, r, g, b, \bar{r}, \bar{g}, \bar{b}\}} b_d=1 \Xi^{a_d} (\gamma_\alpha \gamma_\beta)_{b_d a_d} \frac{\partial}{\partial b_{d'}} \right\} \Phi_s(X, \Xi, \Omega)
\]

\[
[P(p_\mu), \Phi_s(X, \Xi, \Omega)] = -\frac{\hbar}{i} \frac{\partial \Phi_s(X, \Xi, \Omega)}{\partial X^\mu}
\]

8 Conclusion

We have constructed here a generalized Poincaré superalgebra and the corresponding supergroup based on the larger grading group \( \mathbb{Z}_n^3 \), as well as its action on the corresponding superspace. Even though these constructions can be brought back to ordinary superstructures through a change of the commutation
factor, some properties appear clearly only with the original commutation factor, which has some relevance in itself. This will be shown in more details in the next article, where we will describe the particle contents of the theory (especially their spin and statistics) through the study of the irreducible representations of the Poincaré superalgebra that has been developed here.

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