On the microscopic response of equilibrium systems to dissipative perturbations

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The standard relationships of statistical mechanics are upended my the presence of active forces. In particular, it is no longer possible to simply write down what the stationary probability of a state of such a system will be, as can be done in ordinary statistical mechanics. Moreover, it is not immediately apparent how qualitatively different notions of dissipative processes would manifest themselves in the probability of a microscopic state, including forces with curls or coupling of interacting particles to different temperature baths. In this manuscript, we demonstrate that a single microscopic response function governs the response of the stationary probability to non-equilibrium perturbations for two qualitatively different notions of non-equilibrium in an overdamped system (differential heat bath and deterministic non-conservative forces). This response function can be computed as a path integral in the unperturbed system. An analytic expression can be computed in harmonic systems, and by analogy with quantum theory, it can be shown that the microscopic probability associated with a state is related to the total dissipation along all paths starting from that state. Simple analytic expressions can be obtained for harmonic systems, where we show that two different notions of non-equilibrium show different behavior when considered in the mode space of the original system. Finally, we discuss the extension of this work to anharmonic systems. From this, it appears a generic relationship exists where the presence of dissipative processes has its largest effects on the soft modes of the equilibrium system. We verify our theoretical predictions against simulations of active polymers.

I. INTRODUCTION

Active systems constitute a class of systems whose continual consumption of energy requires the use of new theoretical methods and ideas to describe their behavior [1]. The presence of energy-consuming processes leads to violations of the conditions necessary for the establishment of equilibrium states. Such systems are of interest as many phenomena belong to this domain; most importantly, this is a defining characteristic of biological systems [2].

Analysis of problems in these areas would be deemed to be related to non-equilibrium physics. However, the analysis of non-equilibrium systems is complicated by the fact that "non-equilibrium" denotes a variety of different systems, each of which has its own peculiarities. For example, glasses are not in equilibrium [3], though intuitively appear as a different "kind" of non-equilibrium to a living cell. Other examples of non-equilibrium could include maintaining two different parts of a system at two different temperatures, leading to continuous heat transfer from the hotter part of the system to the cooler one [4]. Lastly, "classical" active matter as described above is often thought of as using consumption of energy for directed motion [5].

Focusing on problems which have relevance for biological systems, the lack of any unifying description to such systems has necessitated a variety of methods for describing the non-equilibrium stationary states in active systems. A variety of active problems are well described through field theoretic methods, where analysis of the symmetries of the problem allow construction of functionals that can give mean-field behavior [6]. Alternatively, one can describe all the microscopic degrees of freedom, however, such a level of description usually requires coarse grained simulations, given the lack of useful expressions for the microscopic distributions over states at non-equilibrium, nevertheless, a variety of different systems can be studied in such a manner, including flocking behaviors, active Brownian motion and active chemistry [7–9]. More abstractly, stochastic thermodynamics generalizes standard thermodynamic notions towards the analysis of non-equilibrium steady states [10], which can be extended towards the study of active matter [11].

Common to all the above is that the standard Boltzmann distribution over states is no longer sufficient to describe these scenarios [12]. Notably, however, occasionally such scenarios can be described rather simply through renormalization of temperature (so called active (or effective) temperature) [13–17]. One is left with the question of whether any generalized microscopic description can tell us something about the properties of non-equilibrium steady states, and moreover how well would such a description describe some of the qualitatively different notions of non-equilibrium.

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When would such systems be able to be mapped onto a Boltzmann-like description? What are the quantities which would appear in the microscopic distribution of states?

One way to explore this issue is by treating excess forces in these systems as a perturbation[18,23]. On this tack, in a prior manuscript[24] we demonstrated that the presence of non-conservative (sometimes used interchangeably with “dissipative”) forces in a Smoluchowski equation can be incorporated via a perturbative expansion in terms of integrals of various combinations of the dissipative forces over a Green’s function. This Green’s function governs the response of the microstate probability distribution to additional (not necessarily arising from a potential) forces in the system, leading to a new probability of each microstate. Different classes of active systems can be mapped onto this framework, however, it remains to be seen how much can be said of a priori distinctions between qualitatively different classes of non-equilibrium system. For example, the coupling of particles to different temperature baths is also a non-equilibrium system, but how much does it have in common with systems where some degrees of freedom are driven deterministically by the presence of non-conservative forces? Is there anything that can be said about how “Boltzmann-like” different classes of non-equilibrium systems would appear? Moreover, as deviations from Boltzmann behavior can be characterized in terms of integrals over this Green’s function kernel, which serves then as a microscopic response function, what can we say about this function that governs microscopic responses to non-equilibrium perturbations? Response functions are a canonical way of treating non-equilibrium states[25], however they are usually thought of in terms of macroscopic rather than microscopic quantities.

In this manuscript we set out to explore how this microscopic distribution over states for different forms of non-equilibrium perturbation can be calculated. To this end, we first demonstrate that the Green’s function can be written as a path integral over all paths in the equilibrium system and give an interpretation of the expression that arises. We show that the same Green’s function arises in two different contexts, that of deterministic non-conservative forces and that of particles coupled to different temperature baths, suggesting a broad role for this microscopic response function in determining the probability of non-equilibrium systems. The similarity of this path integral to those that arise in quantum theory allows us to derive exact results for different classes of harmonic systems, leading to natural comparisons with simulations that allow us to test our expressions. Finally, transformation of the system into mode space allows one to see how the response of each mode depends on the equilibrium mode properties, allowing us to determine when the response of a system to non-conservative forces can be described through a rescaled temperature and when it cannot. By doing so, we demonstrate when a non-equilibrium system will display behavior not commensurate with an equivalent equilibrium system. It is shown that a general relationship exists where the soft modes of the equilibrium system are most likely to be modified by the presence of a dissipative drive. We discuss the implications of this result.

II. RESULTS

A. Microscopic response function

We imagine an equilibrium system by positing a Hamiltonian $H_0(x)$ describing $N$ particles which are also coupled to an external thermal bath at an inverse temperature $\beta = 1/k_bT$. For simplicity, we consider the case where each of these particles are overdamped (i.e. a colloidal system), such that a state contains only the position degrees of freedom of all the particles. In standard statistical mechanics, the probability of a state $x$ is proportional to the Boltzmann factor $P(x) \sim \exp(-\beta H_0(x))$. However, in the presence of additional forces which cannot be described through potentials $F(x)$, this probability doesn’t describe the true probability of a state probability of the system. In a previous manuscript[24], we demonstrated through the Smoluchowski equation that the effect of a non-conservative force (to first order) on the microscopic probability appears as:

$$\log(P(x)) \approx -\beta H_0(x) + \int dx' G(x,x') \rho(F(x'))$$

One could also assert[1] quite generally, though of course we would have problems in determining the forms of these functions. The Green’s function $G(x',x)$ quantifies how a perturbation at a point $x'$ affects some quantity at point $x$. The “source” $\rho(F(x))$ is a function of the non-conservative perturbation on the system. The precise forms of these functions can be calculated for whatever form of non-conservative perturbation is present. Higher order perturbations are also integrated over the kernel $G(x,x')$, reinforcing its importance in governing how the probability of a state depends on non-conservative forces. The Green’s function is also sometimes referred to as a response function.

It can be demonstrated via analysis of the Smoluchowski equation that the Green’s function for perturbation by a non-conservative force is given by the solution to the following equation:

$$(\nabla^2 - \beta \nabla H_0(x).\nabla) G(x,x') = \delta(x-x')$$

(2)
Interestingly, for a system with particles subject to two different temperature baths, the first order perturbation to the probability distribution of particles is also given by the same Green’s function (see appendix A for a full derivation). We sidestep notions of physical meaning of this construction for now, and restrict ourselves completely to the mathematical description of such a system. In this case, it would be, for example, two different particles obeying the following stochastic equations:

\[
\begin{align*}
\gamma \frac{dx_1}{dt} &= F_1(\{x\}) + \xi_1(t) \\
\gamma \frac{dx_2}{dt} &= F_2(\{x\}) + \xi_2(t)
\end{align*}
\]

(3)

where:

\[
\begin{align*}
\langle \xi_1(t)\xi_1(t') \rangle &= 2k_B T_1 \gamma \delta(t - t') \\
\langle \xi_2(t)\xi_2(t') \rangle &= 2k_B T_2 \gamma \delta(t - t')
\end{align*}
\]

(4)

(5)

(6)

Taken together, the fact that the response of the above system is given by the same Green’s function as is present in implies that this Green’s function is the microscopic response function of a equilibrium system to a non-equilibrium perturbation, for two (apparently) qualitatively different notions of non-equilibrium perturbation.

As was mentioned previously, the Green’s function characterizes how the microscopic probability is modified through the presence of non-potential forces. However, equation [2] lacks a very clear intuitive physical interpretation of the important elements of this function. Remarkably, however, equation [2] can be solved by a path integral, a derivation which arose in a rather different context (electrostatics)[26] but appears nevertheless to apply to non-equilibrium systems.

The function \( G(x, x') \) can be written as:

\[
G(x, x') = \int_{0}^{\infty} ds \int_{r(0) = x'}^{r(s) = x} D[r(s)] e^{-\int_{0}^{s} ds' L(\frac{dr(s')}{ds'})} \gamma \beta \frac{1}{2} \frac{\partial^2}{\partial s'^2} H(r(s')) - \frac{1}{2} \partial^2 H(r(s'))
\]

(7)

\[
G(x, x') = \int_{0}^{\infty} ds \int_{r(0) = x'}^{r(s) = x} D[r(s)] e^{-\int_{0}^{s} ds' L(\frac{dr(s')}{ds'})} \gamma \beta \frac{1}{2} \frac{\partial^2}{\partial s'^2} H(r(s')) - \frac{1}{2} \partial^2 H(r(s'))
\]

(8)

Which can be rewritten as:

\[
G(x, x') = e^{\frac{\beta}{2}(H(x) - H(x'))} K(x, x')
\]

(9)

\[
K(x, x') = \int_{0}^{\infty} ds \int_{r(0) = x'}^{r(s) = x} D[r(s)] e^{-\int_{0}^{s} ds' L(\frac{dr(s')}{ds'})} \gamma \beta \frac{1}{2} \frac{\partial^2}{\partial s'^2} H(r(s')) - \frac{1}{2} \partial^2 H(r(s'))
\]

(10)

Upon the substitution \( V(r) = \frac{1}{2} \beta \partial^2 H(r(s')) - \frac{1}{2} \beta \partial^2 H(r(s')) \) it can be observed that \( K(x, x') \) is the path integral solution to the wick-rotated Schrödinger equation, integrated over all “times”, but where time in this system is equal to spatial path length. We have introduced the variable \( s \) here, which has units of \( [L]^2 \). This variable characterizes distance along a path. We make the analogy more complete by characterizing the term in the exponential as a “Lagrangian” (as defined in equation [8]).

The derivation of this expression results from mathematical considerations, interestingly however, we observe in the final expression (7) that the terms in the exponential have physical interpretation. The effective “Lagrangian” (weight of the exponential) has three terms, which can be summarized as:

\[
L = \text{total length of path} + \text{total power dissipated along path} + \text{total work done along path}
\]

(11)

This is using the fact that the total power dissipated along the path has been associated with the expression \( \nabla H \cdot \nabla H - \nabla^2 H \) in prior works [27, 28]. As the equilibrium system is conservative, the final term can be integrated out explicitly. This suggests that even in equilibrium, every two points in the state space are connected by the integral over all paths weighted by this quantity. Therefore, overly long paths have little effect, but so do paths where a large amount of work is done by the system. Interestingly however, paths which also dissipate a lot of power also have minimal effects. We note here that this is a purely equilibrium function, and it could therefore be a source of confusion to speak of “dissipation” at equilibrium. While the average of \( \nabla H \cdot \nabla H - \nabla^2 H \) can be shown to be exactly zero at equilibrium (meaning on average the system doesn’t dissipate power), the same is not true for the evaluation of this quantity along a path connecting any two points.

The presence of this path dependent, non-local term makes this a difficult path integral to evaluate in general. However, a simplification arises in the cases that the Hamiltonian is slowly varying between points \( x \) and \( x' \), in this
case $\frac{1}{2}\beta^2 \nabla H(r(s')) \cdot \nabla H(r(s')) - \frac{1}{2} \beta \nabla^2 H(r(s')) \approx 0$ we can ignore the “difficult” parts of the path integral, leading to a form:

$$G(x, x') \approx e^{\frac{2}{\beta} (H(x) - H(x'))} \left| x - x' \right|^\beta$$

(12)
as was seen previously [24]. In most situations, as the effect of a perturbation at a point $x$ declines very quickly in the total distance between two microstates $|x - x'|$, the effect of the dissipated power term will be rather small.

**B. Harmonic systems**

In the previous section, we stated that the probability of a state of any system perturbed with a non-conservative force is mediated through its Green function, which is an equilibrium property of the system. A useful example to explore our expressions occurs when the Hamiltonian is harmonic:

$$H(x) = \frac{1}{2} x \cdot A \cdot x$$

(13)
where $A$ is some matrix of couplings. The advantage of our presentation thus far is that under the transformation $V(x(s')) = \frac{1}{2} \beta^2 \nabla H(x(s')) \cdot \nabla H(x(s')) - \frac{1}{2} \beta \nabla^2 H(x(s'))$ our path integral reduces to the same path integral form as observed in the Feynmann picture of quantum mechanics. This allows us to borrow considerable mathematical machinery in the resolution of generic statistical problems with non-potential forces [29]. In particular, the path integral can be calculated explicitly for systems which are at most harmonic in their energies as in equation (13). A long calculation can follow where we can calculate what the first order effect of non-equilibrium perturbation would be. The details of this calculation are rather cumbersome, but are reproduced in the appendix B. The final result for the appearance of the log microscopic probability is given by:

$$\log(P(x)) = -\frac{1}{2} \beta^2 x \cdot A \cdot x - \frac{1}{2} \beta^2 x \cdot B \cdot x$$

(14)

We have also introduced a new matrix $B$, which results from integration over the Green’s function, it is given by:

$$B = \int_0^\infty ds \exp(-\beta A s) C \cdot \exp(-\beta A s)$$

(15)
The matrix $C$ is related to the power dissipated by the non-equilibrium perturbation. In the previous section, we discussed two qualitatively different notions of non-equilibrium, each of which would lead to slightly different forms of $C$. For the case where every particle is coupled to its own heat bath, we would have:

$$C_T = A \cdot D \cdot A$$

(16)
where the matrix $D$ is diagonal matrix of couplings to different heat baths. For example, if the system contained three particles in one dimension which were all coupled to a different heat bath $\beta_1, \beta_2, \beta_3$ the matrix $D$ would go as:

$$D = \begin{pmatrix}
\beta_1 / \beta & 0 & 0 \\
0 & \beta_2 / \beta & 0 \\
0 & 0 & \beta_3 / \beta
\end{pmatrix}$$

(17)
the matrix $D$ is a diagonal matrix that gives the extent to which the temperatures of each particle are different, on the scale $\beta_1$. On the other hand if the dissipative forces are instead deterministic non-conservative forces given by $F = M x$, then the matrix $C$ will be given by:

$$C_D = M \cdot A$$

(18)
where $M$ is an anti-symmetric matrix. Both expressions go as $\sim F^2$ and are related to the power dissipated by the introduction of the non-equilibrium process.

Returning to (13) we note that there exists here another analogy to quantum systems, in that this looks very much like the time evolution of the expectation value of the operator $\langle x(t) | \hat{O} | x(t) \rangle = \langle x | \hat{U}^\dagger (t, t_0) \hat{O} \hat{U} (t, t_0) | x \rangle$. We again note the caveat that in our system this is a fictious time, and what we are really integrating over is path lengths. The term $A \cdot D \cdot A$ arises from the source term due to different temperatures, and the Green’s function then corresponds to the integral over all the path lengths, giving the appearance of a unitary operator going as $\exp(-\beta H s)$.
This analogy is suggestive to the physical interpretation of the expression. After having gone through much derivation, it appears as if the total probability of the state \( x \) is modified through the “time” (actually path length) average of the operator \( C \) starting at that point \( x \). The factor \( \exp(-\beta A s) \) is the operator which gives the final state of the classical path after a total path length of \( s \) beginning at \( x \). The matrix \( C \) is once more related to the dissipated power due to the coupling to different heat baths. Therefore, it appears as though the probability of state \( x \) depends on the expected value of the dissipated power of all paths starting from point \( x \).

In order to demonstrate the validity of these expressions, we introduce a particular model against which we shall benchmark our theory. The harmonic system we choose is a polymer, consisting of \( N \) monomers, where monomers \( i \) and \( i+1 \) interact via a harmonic potential

\[
V(r_i, r_{i+1}) = \frac{1}{2}k(r_i - r_{i+1})^2
\]

As we are calculating microscopic probabilities, for the moment we shall restrict our system to be only three particles in one dimension. This allows us to more easily visualize the expressions we obtain. The results are summarized in figure 2, which demonstrates excellent agreement between theory and simulation. Further details about the simulations can be found in appendix D.

C. Mode space and response in harmonic systems

Some generic features of the response of systems to non-equilibrium perturbations can be discerned if we analyze the microscopic probability in terms of the modes. This is easily done for a harmonic system. Recalling that the log probability (generically now, not due to coupling with different heat baths) in these systems goes as:

\[
\log(P(x)) = -\frac{1}{2} \beta x.A.x - \frac{1}{2} \beta^2 x. \int_0^\infty ds \exp(-\beta A s) C_e \exp(-\beta A s). x + c_1
\]

Where \( C \) is some dissipation matrix (i.e. for different heat baths this was \( A.D.A \)) and \( c_1 \) is a normalization constant.

The matrix \( A \) will have certain properties. For the system to be conservative in the absence of any non-equilibrium perturbation, the matrix \( A \) must be symmetric \( A = A^T \). A symmetric matrix can always be diagonalized \( A = Q^{-1}LQ \) where the matrix \( L \) will be a diagonal matrix with each diagonal entry being equal to one of the eigenvalues of the original matrix \( L_{ij} = \lambda_i \) and the matrix \( Q \) is the matrix of eigenvectors. The matrix exponential for diagonalizable matrices is also well known to have the simplification \( e^A = Q^{-1}e^LQ \). The matrix \( Q \) defines a linear transformation, therefore we can define a new space for our problem given by \( \tilde{x} = Qx \). This allows re-expression of eq. 29 as:

\[
\log(P(\tilde{x})) = -\frac{1}{2} \beta \tilde{x}.L_{\tilde{x}}.\tilde{x} - \frac{1}{2} \beta^2 \tilde{x}. \int_0^\infty dt \exp(-\beta L t) \tilde{C}_e \exp(-\beta L t). \tilde{x}
\]

where \( \tilde{C} = P.C.P^{-1} \). Every component of \( \tilde{x} \) is a mode of the equilibrium system. In mode space, the integral over all path lengths can be performed, leaving the final probability as:

\[
\log(P(\tilde{x})) = -\frac{1}{2} \beta \tilde{x}.L_{\tilde{x}}.\tilde{x} - \frac{1}{2} \beta^2 \tilde{x}. L_{\tilde{x}}.\tilde{x}
\]
where we have a new “dissipation” matrix $\hat{D}$ whose elements are given by:

$$\hat{D}_{ij} = \frac{\tilde{C}_{ij}}{\lambda_i + \lambda_j}$$  \hspace{1cm} (23)$$

This is a general result for harmonic systems, and not without consequences. Transforming the system into mode space is effectively defining a set of collective coordinates of our unperturbed system, each of these modes has some fluctuation spectra associated with it, that depends on its eigenvalue $\lambda_i$. Additionally, these modes are all uncorrelated with one another. The driving matrix eq. (23) that arises from dissipative processes in the system leads in the first instance to a larger degree of coupling between the different modes of the system. As our new “effective” system $L + \hat{D}$ isn’t necessarily diagonal (due to the fact non-conservative forces aren’t necessarily symmetric), it is the case that our new system cannot be decomposed in terms of $N$ independent modes as was always the case for equilibrium. In other words, it would be generally true that the presence of non-equilibrium perturbations leads to the system being more highly correlated than the corresponding equilibrium system, when considered in mode space.

At first sight of equation (23) we would appear to suggest an interesting conclusion, that modes which are the least constrained under equilibrium (soft modes) are the modes which are most strongly affected by the presence of dissipative forces. However, this would be mistaken, because the matrix $C$ (the dissipative source) will also depend on the Hamiltonian. For example, when particles are each coupled to different heat baths, the form of the dissipative matrix $D$ goes as $A \hat{D} A$. In other words, the matrix $D_{ij}$ goes as $\sim \lambda_i \lambda_j$. Therefore, the entire expression goes as $\sim O(\lambda)$, i.e., the same as for the equilibrium system.

This does however, demonstrate a key difference between different types of dissipative force, despite the fact that both have the same Green’s function. If the system were to be perturbed by a deterministic force, as opposed to coupling to different heat baths, the dissipation matrix would go as $\tilde{D} \sim M$ where $M$ is some anti symmetric matrix, there would be only one $A$ matrix in the numerator, which would mean each mode would be affected by the presence of the force independently of its eigenvalue, which is rather different to coupling to different temperature baths. When viewed in mode space, then, deterministic forces will display a greater degree of difference to the original system than coupling to different heat baths.

This allows us to consider situations in which the subsequent state of the system is given by an effective rescaled temperature. The probability distribution of a mode goes as the matrix seen in equation $23 \hat{D}$, when the system is coupled to multiple temperature baths, $D_{ii} \sim O(\lambda_i)$ so the system could be described by a rescaled temperature so long as $\hat{D}$ is a strongly diagonal matrix, where each diagonal element is proportional to the eigenvalue multiplied by some constant. This is a rare set of features to be simultaneously fulfilled, therefore in most situations effective temperatures will not be useful towards evaluation of microscopic properties. Were the perturbation due to deterministic forces, however, it would be even more difficult to fulfill these conditions as the $O(\lambda)$ prefactor isn’t included naturally.

A useful part of evaluating all of these integrals is that we can apply the same rules again mechanically for the second order perturbation. In other words, given that we know the following identity:

$$\int dx' G(x,x')(x',B,x') = x. \int_0^\infty ds \exp(-\beta As).B.\exp(-\beta As).x$$  \hspace{1cm} (24)$$

we can calculate perturbations to any order, so long as we know the form of the matrix $B$, which itself can be calculated, as demonstrated in appendix $A$. By introducing a new function $f(B) = \int_0^\infty ds \exp(-\beta As).B.\exp(-\beta As)$ We can rewrite this as:

$$\int dx' G(x,x')(x',B,x') = x.f(B).x$$  \hspace{1cm} (25)$$

while the general form of these functions can get rather complicated, we can understand them in terms of mode space response by tracking how the various perturbations scale with the eigenvalue of the mode. It can be shown that to second order for a system subject to different heat baths, the modification of the probability goes as:

$$\log(P_{\text{bath}}(x)) = -\frac{1}{2} \beta x.A.x - \frac{1}{2} \beta^2 f(C_1).x - \frac{1}{2} \beta^2 f(C_2).x$$  \hspace{1cm} (26)$$

$$C_1 = A \hat{D} A$$  \hspace{1cm} (27)$$

$$C_2 = f(C_1).f(C_1) + A \hat{D} f(C_1) + f(C_1).\hat{D} A$$  \hspace{1cm} (28)$$

with the matrix $\hat{D}$ defined in the previous section. Whereas for a system modified by deterministic forces the following
holds:
\[
\log(P_{\text{det}}(x)) = -\frac{1}{2} \beta x. A_x - \frac{1}{2} \beta^2 x. f(F_1). x - \frac{1}{2} \beta^2 x. f(F_2). x + O(x^4)
\]
(29)
\[
F_1 = A_x M
\]
(30)
\[
F_2 = M f(F_1)
\]
(31)

Instead of getting lost in the weeds of full evaluation of these equations, we want to discover if there exist any interesting results related to the excitations of particular modes. Naively, the impact of function \( f \) on a matrix \( X \) is to multiply the matrix by \( A^{-1} \). This is precisely true in the event that the matrices commute \([X, A] = 0\) but will not be generally true. Nevertheless, ignoring the commutator allows us to reason effectively about what would be observed in harmonic systems under different types of perturbation. For different heat baths, we have \( f(C_1) \sim O(A) \) and, moreover, we have \( f(C_2) \sim O(A) \). In other words, even the second order perturbation to a mode \( i \) will scale with the value of the eigenvalue associated to it \( \lambda_i \), the same as would exist in the absence of any perturbation at all.

![Diagram](image1.png)

**FIG. 2.** A) Each mode in an equilibrium harmonic system constitutes an independent variable which will be normally distributed. The probability distribution of each mode will have a characteristic width that is related to their eigenvalue. Under the presence of additional forces, the probability distribution of the mode can change, for instance by becoming broader. B) A harmonic polymer (eq. (19)) with \( k = 1 \) being driven by having each bead be at a different temperature. Each mode is affected roughly the same over many different realizations of the temperatures \( T_1, T_2, \ldots \) etc. C) A harmonic polymer being driven by a deterministic force (given in appendix D with \( m = 0.5 \)). In contrast to driving by temperature differences, the softest modes of the original system are most strongly affected by the drive.

The deterministic case is different to this scenario. As was discussed previously, the first order perturbation of mode \( i \) doesn’t scale with its eigenvalue, moreover we can see that the second order perturbation goes as \( O(A^{-1}) \), suggesting that the most weakly constrained modes in equilibrium would be the most strongly affected modes by analyzing the perturbation to second order.

Thus, to second order, the impact of deterministic dissipative forces on the microscopic probability distribution can be understood through the spectrum of eigenvalues of the underlying equilibrium system. If, for example, the original system had all modes with very similar eigenvalues, the effect of generic non-conservative forces on the microscopic
probability would look very similar to simply raising or lowering the temperature, in that the response of all parts of the system would be identical. However, this would require the system to be rather specially constructed for the eigenvalue spectrum to be so narrow. On the other hand, particles coupled to different heat baths would not have such behaviors, and the presence of different heat baths can be roughly compared to an effective temperature.

We can make these conclusions concrete by studying an example of a system that conforms to the above parameters. To that end, we simulate the same polymer system as in the previous section with harmonic bonds between each of the monomers subject to different forms of non-equilibrium perturbation (figure 2). We simulate the two different forms of non-equilibrium dynamics that we have studied above. This necessitates choices over averaging for the cases where the monomers are coupled to different heat baths, as a random instantiation of the temperatures will display the inherent noise due to that particular choice. This is not the case in deterministic systems, where the same force is applied consistently. We therefore average the different temperature case over 240 different realizations of the temperature, and display what the standard deviation of the measured mode spectrum is. This tells us how each mode is affected, on average, by some random coupling to different temperatures. We choose random numbers $T_i$ such that $\sum_{i=1}^N T_i = 0$. The $T_i$ are uniformly distributed around a temperature of $T = 1$, with a minima and maxima of 0.5 and 1.5 respectively.

In figure 2 we show how the response of a polymer subject to different forms of non-equilibrium perturbation. In A) we show the multiple temperature baths system. The plot displays, how each mode is affected by the coupling to different temperatures, on average. The plot shows that each mode is affected roughly the same due to perturbations, as we anticipated from the analysis performed above. By contrast, the deterministic force (panels B) produces a roughly constant change in each mode, up to the soft modes, which are far more strongly affected than the hard modes, as we predicted in our second order perturbation.

D. Beyond harmonic systems

The advantage of the method presented in equation 7 is its generality. Though assuming that the system is at most quadratic in the energies makes evaluation easier, we can in principle reason about microscopic distributions in anharmonic systems, though the details of the calculation become much more exhaustive in this case. The Green’s function for a generic system $H(x)$ can be constructed in analogy to Quantum field theory and is given by:

$$G(x'',x') = e^{\frac{\Delta}{\hbar}(H(x'')-H(x'))} \int_0^\infty dsK_h(x'',x',s) \exp \left[ \int ds'V \left( \frac{\delta}{\delta J(s')} \right) \right] \exp(-W[J]) |_{J=0}$$

(32)

Where the function $V$ is the terms in the Lagrangian that deviate from quadratic behavior, and $W[J]$ is a functional of an arbitrary function $J$ and is given by:

$$W[J] = \int dsJ(s)x_d(s) + \frac{1}{2} \int ds \int ds'J(s)G_D(s,s').J(s')$$

(33)

where $x_d(s)$ is the classical path in the harmonic system and $G_D$ is a function (given in the appendix).

Our interest in these systems lies in whether the qualitative relationships that we derived in the previous section are modified for more realistic potentials. To that end we introduce an anharmonic system where the Hamiltonian that goes as (using the Einstein summation convention) $\sim A_{ij}x^ix^j + B_{ijkl}x^ix^jx^kx^l$, where we assume that $A^2 >> B$. In appendix C we show that the modification of the microscopic probability due to additions of this term can be seen to go as, to first order:

$$\log(P(x)) \sim -\beta A_{ij}x^ix^j - B_{ijkl}x^ix^jx^kx^l - \beta^2(a_{ij}x^ix^j + b_{ijkl}x^ix^jx^kx^l)$$

(34)

where

$$a_{ij} \sim \frac{\tilde{C}_{ijkl}}{\lambda_k + \lambda_l} + \frac{\tilde{D}_{klmij}}{(\lambda_k + \lambda_l)^2 + 2(\lambda_k + \lambda_l)\lambda_m}$$

(35)

$$b_{ijkl} \sim \frac{\tilde{F}_{ijklmnop}}{\lambda_m + \lambda_n + \lambda_o + \lambda_p}$$

(36)

where the factors with indices $\tilde{C}, \tilde{D}, \tilde{F}$ refer to some rather complex expressions which are multiplied by the eigenvectors of $A$. In line with our philosophy of ignoring complex details in favor of observing generic effects that occur due to the presence of dissipative forces, we can see that in this scenario, there is an additional modification of the
harmonic interaction with a term that goes as $\sim O(1/\lambda^2)$. This suggests that the effects that we observed in the previous section get attenuated when the system is anharmonic. This scenario is slightly more difficult to describe due to the fact that non-harmonic systems can no longer be decomposed in terms of independent modes, but we can see that the quadratic term (the “softer” part of the potential) is modified by an additional factor. However, as such systems are more highly coupled it’s hard to describe what happens in terms of the fate of a single “mode” without calculating what occurs in all the details of $\tilde{C}, \tilde{D}, \tilde{F}$. Nevertheless, we should generally expect that the softer parts of the potential are more strongly affected than the harder parts. We can test this is a simple polymer system like the ones we have already investigated, but now with a pair potential between monomers $i$ and $j = i + 1$ going as:

$$\phi(x_i, x_j) = \frac{1}{2}a(x_i - x_j)^2 + \frac{1}{4}b(x_i - x_j)^4$$

(37)

As we did before, we can analyze the modes of the system. This entails looking at the harmonic component of the Hamiltonian and decomposing the system in terms of these modes. This is purely for purposes of representation, as the system is not in general decomposable as independent modes, however we are free to analyze the features of the system in this coordinate system if we choose, and it allows for the closest comparison to the harmonic results presented in the previous section. We can compare the mode space of the perturbed vs unperturbed system again. We summarize these results in figure 3. When this figure is compared to figure 2B, it can be observed that in an anharmonic system, the effects of coupling particles to different temperature baths depends on the size of the mode (unlike in the harmonic case), where once more the softest modes are most affected by the presence of this dissipative force.

Therefore, the trends we observed in the previous section, where the soft modes are the most affected under drive, but only for the deterministic force, appear to hold more generally when anharmonic effects are taken into account.

III. DISCUSSION AND CONCLUSION

In the preceding, we have demonstrated that the microscopic probability for a system perturbed by a deterministic non-conservative force or due to coupling with different heat baths can be shown to be equal to the equilibrium system with an additional term appearing as an integral over a microscopic response (or Green’s) function. This response function is the same for both of the above non-equilibrium perturbations. The microscopic response function is a path integral in the unperturbed system. In equilibrium, any two states $x$ and $x'$ are connected by the sum of all paths between them weighted by the path length, work and dissipated power associated with each of the paths. The modification of the microscopic probability is analytically solvable for harmonic systems. Every state is modified by a quantity related to the dissipated power of all paths from that state. When transformed to mode space, special simplifications allow us to quantify how a particular mode of the system is perturbed. It was shown that, in general, soft modes of the corresponding equilibrium system are the most affected by the presence of non-equilibrium perturbations. The fact that different modes appear to have unique responses suggest the effective temperature framework is only useful for a small subset of non-equilibrium systems.
The microscopic response function was constructed through consideration of the dynamical processes involved, and yet its final form has a more thermodynamic interpretation. It would be interesting to start from the effective Lagrangian formulation as it appears in [1] and apply it to a completely different system without considering any dynamics and observe whether it can recapitulate the stationary distribution successfully. This would be suggestive as to the importance of the response function we have shown in this paper.

The path integral that represents the response function is summed over all path lengths, where the path length takes a form that looks suspiciously like time. In this paper, we have ignored dynamic effects, this can prompt us to ask the extent to which the effect of time appears as the path length representation that is already present in the over-damped system. Following on from this, we have ignored inertial effects here, but the theory could be modified to account for them.

The fact that the soft modes of the equilibrium system are most affected by the presence of non-equilibrium driving is interesting and of theoretical relevance. Previously, we demonstrated that the presence of non-conservative forces most strongly couple to surfaces [24], and to be somewhat non-specific in our terminology, surfaces are usually “softer” modes than volumes, in that modifying the surface of a region carries with it a smaller energy penalty than changing the arrangements of all the particles in the volume.

Interesting phenomena due to dissipative forces would arise in the case where the eigenspectrum of the original system was quite broad. In this case, some modes are basically unaffected by the drive, however the modes which were most strongly fluctuating in the equilibrium system are those which will be most strongly affected by the dissipative forces.

It should be noted here, that the direction in which the fluctuations of mode $i$ are shifted will depend on the details of the dissipative force, however, the magnitude will go as $\sim 1/\lambda$. Curiously, therefore, a mode which under equilibrium conditions had a rather broad probability distribution could under the presence of a dissipative force be much more highly constrained. The colloquial term “order” can be fraught, however it remains within the realm of probability that by adding dissipative forces we see some modes of the original system become much more highly “ordered” in the sense that the spectra of fluctuations becomes much narrower. Obviously, the converse could occur too, in which the broad modes of the equilibrium system become even broader. The direction will depend on the details of the non-equilibrium perturbation.

In terms of connection with experimental systems, active matter systems can be mapped onto this framework when the proper degrees of freedom the system are considered[24], to this we would add experiments that involve heating different parts of a system to different temperatures [31]. There has been recent interest in active matter with what has been deemed odd viscosity [32, 33], as these systems also have non-conservative forces they would be a natural candidate for investigating how the microscopic phenomena we have discussed here would bridge length scales to more macroscopic quantities.

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in order to account for different temperatures, we can modify the equation:  
\[ \nabla \cdot \left( \exp(-\phi) \left( -\mathbf{F} - k_b T \nabla \phi \right) \right) = 0 \]  
(A1)  
in order to account for different temperatures, we can modify the equation:  
\[ \nabla \cdot \left( \exp(-\phi) \left( -\mathbf{F} - \beta^{-1} \cdot \nabla \phi \right) \right) = 0 \]  
(A2)  
where \( \beta^{-1} \) is a diagonal matrix where each entry on the diagonal is given by a value \( k_b T_i \) corresponding to the temperature of that degree of freedom. This equation can be manipulated to give:  
\[ \nabla \cdot \left( \exp(-\phi) \beta^{-1} \left( -\mathbf{F} - \nabla \phi \right) \right) = 0 \]  
(A3)  
Splitting the vector field \( \beta \cdot \mathbf{F} = \mathbf{f}^c + \mathbf{f}^a \) where \( \mathbf{f}_c = -\nabla \phi \) is a solution if \( \nabla \cdot (e^{-\phi} \beta \mathbf{f}^a) = 0 \). Assuming that \( \mathbf{F} = -\nabla H_0 \) this leads to the following equation:  
\[ -\nabla H \cdot \nabla \phi + \nabla \phi \cdot \beta^{-1} \cdot \nabla \phi + \nabla^2 H - \nabla \cdot (\beta^{-1} \cdot \nabla \phi) = 0 \]  
(A4)
where if every entry into the matrix \( \beta \) is equivalent, this reproduces the Boltzmann distribution.

We can write the temperature matrix as a modification around a base temperature \( \beta_1^{-1} \):

\[
\beta^{-1} = \beta_1^{-1} I + \beta_2^{-1} D
\]

we redefine \( H \) as including the base temperature \( \beta_1^{-1} \)

\[
H := \beta_1 H_0
\]

We therefore have the following equation in \( \phi \):

\[-\nabla H \cdot \nabla \phi + |\nabla \phi|^2 + \alpha \nabla \phi \cdot D \cdot \nabla \phi + \nabla^2 H - \nabla^2 \phi - \alpha \nabla (D \cdot \nabla \phi)\]

where \( \alpha = \beta_1 \beta_2^{-1} \).

A perturbative expansion in \( \alpha \), \( \phi = \sum_{n=0}^{\infty} \alpha^n \phi(n) \) gives for the term \( \phi(k) \):

\[-\nabla H \cdot \nabla \phi(k) + \sum_{|n+m|=k} \nabla \phi(n) \cdot \nabla \phi(m) + \sum_{|n+m+1|=k} \nabla \phi(n) \cdot D \cdot \nabla \phi(m) - \nabla^2 \phi(k) - \nabla (D \cdot \nabla \phi(k-1)) = 0\]

for \( k > 0 \) and where \( \phi(0) = H \) which leads to the first order perturbation:

\[(\nabla^2 - \nabla H \cdot \nabla) \phi(1) = \nabla (D \cdot \nabla H) + \nabla H \cdot D \cdot \nabla H\]

It can be seen through repeated application of (A8) that the Green’s function is of the form specified in the main text for every perturbation.

**Appendix B: Evaluation of harmonic path integral for non-conservative system**

We use the generic solution to a path integral with a harmonic Lagrangian for a system with a Hamiltonain given by \( H_0(x) = \frac{1}{2} \dot{x} \cdot A \cdot x \), which is given through the following formula [29]:

\[
\int_{r(0) = x'}^{r(t) = x''} D[r(t)] e^{i \int_0^1 dt' L(x, x')} = \left( \frac{1}{2\pi i \hbar} \right)^{D/2} \sqrt{\det \left( -\frac{\partial^2 S_{cl}[x'', x']}{\partial x''_i \partial x'_j} \right)} \exp \left( \frac{i}{\hbar} S_{cl}[x'', x'] \right)
\]

where we perform the wick rotation \( \hbar \rightarrow -i \) (note the lack of Boltzmann factor here, it is in fact not introduced into this analysis due to the fictitious nature of the time variable we have). Our Lagrangian is given by:

\[
L = \frac{1}{4} \dot{x}^2 + \frac{1}{4} x \cdot A \cdot x - \frac{1}{2} \text{Tr}(A)
\]

(B2)

The quantity \( S_{cl} \) is the classical action evaluation along the classical solution \( x_{cl}(0) = x' \) and \( x_{cl}(s) = x'' \)

The Lagrangian leads to the following Euler Lagrange equation:

\[
\frac{1}{2} \ddot{x} = \frac{1}{2} \dot{A} \cdot A \cdot x
\]

(B3)

which has the generic solution:

\[
x_{cl}(\tau) = e^{A \tau} c_1 + e^{-A \tau} c_2
\]

(B4)

where we are free to choose the value of \( v_0 \) to satisfy the boundary conditions. This can be done so, yielding the following classical path:

\[
x_{cl}(\tau) = \text{csch}(A \tau)(x'' \sinh(A \tau) + x' \sinh(A(s - \tau)))
\]

(B5)

The evaluation of the classical action evaluated along this path can be performed (using the fact that all the hyperbolic functions are power series in \( A \), and thus commute):

\[
S_{cl} = \frac{1}{4} x' \cdot A \coth(A \tau) \cdot x' + \frac{1}{4} x'' \cdot A \coth(A \tau) \cdot x'' - \frac{1}{2} x'' \cdot A \text{csch}(A \tau) \cdot x' - \frac{1}{2} \text{Tr}(A) s
\]

(B6)
this allows us also to evaluate the von Vleck determinant, which is given by:

\[
\sqrt{\det \left( -\frac{\partial^2 S_{cl}[\mathbf{x}'', \mathbf{x}']}{\partial x''_a \partial x'_b} \right)} = \sqrt{\det \left( \frac{1}{2} A \text{csch} \left( As \right) \right)}
\]  

(B7)

Finally, we can write the full expression for \( K(\mathbf{x}'', \mathbf{x}') \):

\[
K(\mathbf{x}'', \mathbf{x}') = \int_{0}^{\infty} ds \left( \frac{1}{2\pi} \right)^{D/2} \sqrt{\det \left( \frac{1}{2} A \text{csch} \left( As \right) \right)} \exp(-S_{cl})
\]

(B8)

Furthermore, the prefactors can be added leading to the full Green’s function:

\[
G(\mathbf{x}'', \mathbf{x}') = \int_{0}^{\infty} ds \left( \frac{1}{2\pi} \right)^{N/2} \sqrt{\det \left( \frac{1}{2} A \text{csch} \left( As \right) \right)} \times \exp \left(-S_{cl} + \frac{1}{4} \mathbf{x}''. A \mathbf{x}'' - \frac{1}{4} \mathbf{x}', A \mathbf{x}' \right)
\]

(B9)

As an example, we can use standard Gaussian integrals to find the perturbation associated with a harmonic source term (such as would arise from the action of different temperature baths) \( (\mathbf{x}', B, \mathbf{x}') \). We denote the difference of the log probability from equilibrium as \( \delta \log P \). Suppressing the prefactors for now, we have to calculate the following integral:

\[
\delta \log P \sim \int d\mathbf{x}' \exp \left( -\frac{1}{2} \mathbf{x}' . (M + \frac{1}{2} A) \mathbf{x}' - \frac{1}{2} \mathbf{x}''. (M - \frac{1}{2} A) \mathbf{x}'' + b \mathbf{x}' \right) (\mathbf{x}', B, \mathbf{x}')
\]

(B10)

where

\[
M = \frac{1}{2} A \text{coth} \left( As \right)
\]

(B11)

\[
b = \mathbf{x}'' \frac{1}{2} A \text{csch} \left( As \right)
\]

(B12)

By completing the square, we can obtain the following equivalent expression for the exponential:

\[
\delta \log P \sim \int_{0}^{\infty} ds \int d\mathbf{x}' \exp \left( -\frac{1}{2} \left( \mathbf{x}' - (M + \frac{1}{2} A)^{-1} b \right) . (M + \frac{1}{2} A) \left( \mathbf{x}' - (M + \frac{1}{2} A)^{-1} b \right) 
\]

\[
+ \frac{1}{2} b \left( (M + \frac{1}{2} A)^{-1} b - \frac{1}{2} \mathbf{x}''. (M - \frac{1}{2} A) \mathbf{x}'' \right)
\]

(B13)

The substitution \( y = \mathbf{x}' - (M + \frac{1}{2} A)^{-1} b \) allows us to compute the integral in equation \( \text{B10} \). This leads to the following expression:

\[
\delta \log P \sim \int dy \exp \left( -\frac{1}{2} y . (M + \frac{1}{2} A) y \right. + \frac{1}{2} b \left( (M + \frac{1}{2} A)^{-1} b - \frac{1}{2} \mathbf{x}''. (M - \frac{1}{2} A) \mathbf{x}'' \right)
\]

\[
\times \left( y . B . y + b \left( (M + \frac{1}{2} A)^{-1} B . (M + \frac{1}{2} A)^{-1} b \right) \right)
\]

(B14)

Where we have suppressed the terms linearly dependent on \( y \) outside the exponential, as these will integrate out to zero. This integral can be performed over \( y \) by using the standard properties of Gaussian integrals, leading to the following expression:

\[
\left( \frac{(2\pi)^N}{\det (M + \frac{1}{2} A)} \right)^{1/2} \text{Tr} \left( (M + \frac{1}{2} A)^{-1} B \right) + b \left( (M + \frac{1}{2} A)^{-1} B \right) \left( (M + \frac{1}{2} A)^{-1} b \right)
\]

\[
\times \exp \left( \frac{1}{2} b \left( (M + \frac{1}{2} A)^{-1} b - \frac{1}{2} \mathbf{x}''. (M - \frac{1}{2} A) \mathbf{x}'' \right) \right)
\]

(B15)
This expression is then multiplied by the von Vleck determinant and other prefactors whose dependence we suppressed earlier to lead to the full resultant expression for a Harmonic perturbation, once the integral over all times is performed. Though this expression appears difficult, remarkable simplifications occur. After all the expressions are inserted, the term in the exponent is actually equal to zero.

The von vleck determinant multiplied by the determinant arising from the integral is given by:

$$\left(\frac{1}{2\pi}\right)^{N/2} \sqrt{\det\left(\frac{1}{2}A\cosh(A)\right)} \ast \left(\frac{2\pi}{\det(M + \frac{1}{2}A)}\right)^{1/2}$$  \hspace{1cm} (B16)

The hyperbolic expressions can again be simplified, leading to:

$$\sqrt{\det(\exp(-As))} = \exp(-\text{Tr}(A)s/2)$$  \hspace{1cm} (B17)

Where we use the property that the determinant of the matrix exponential being equal to the exponential of the trace. We can observe that this term cancels out with the term in the Lagrangian proportional to $\text{Tr}(A)$

Further simplifications exist:

$$b.(M + \frac{1}{2}A)^{-1}B.(M + \frac{1}{2}A)^{-1}b = x'' \exp(-As)B \exp(-As)x''$$  \hspace{1cm} (B18)

The term $\text{Tr}((M + \frac{1}{2}A)^{-1}B)$ appearing in equation B15 doesn’t have any dependence on $x''$. In other words, it would change the probability uniformly up or down, however, this dependence disappears when we divide through by the partition function.

We therefore have the following expression:

$$\int_0^\infty dsx'' \exp(-As)B \exp(-As)x''$$  \hspace{1cm} (B19)

for the modification in the probability. As seen in the main text.

**Appendix C: Anharmonic path integral for non-conservative system**

In order to calculate the effect of arbitrary perturbations, we must include the effect of anharmonicity, much of this discussion bears similarity with similar problems arising in quantum field theory[30], we recall that the definition of the Green’s function is given by:

$$G(x'', x') = \int_0^\infty ds \int_{r(0)=x'}^{\text{r}(s)=x''} D[r(s)] e^{-\int_0^s ds'L(\dot{x}, x)}$$

$$= e^{\frac{\mu}{2}(H(x)-H(x'))} \int_0^\infty ds \int_{r(0)=x'}^{\text{r}(s)=x''} D[r(s)] e^{-\int_0^s ds'L(\dot{x}, x)}$$  \hspace{1cm} (C1)

Our difficulty lies in evaluation of the path integral in the presence of anharmonic conservative forces in $L$. One way beyond this difficult is splitting the action into a baseline contribution which we can evaluate, and another contribution which we cannot: $\int_0^s ds'L(\dot{x}, x) = S_h + S_a$. We choose the baseline to be the harmonic path integral which we have just evaluated in the previous section. Then, conducting a Taylor series in $S_a$ we obtain:

$$G(x'', x') = e^{\frac{\mu}{2}(H(x)-H(x'))} \int_0^\infty ds \sum_{n=0}^\infty \frac{1}{n!} \int_{r(0)=x'}^{\text{r}(s)=x''} D[r(s)] e^{-S_h - (S_a)^n}$$  \hspace{1cm} (C2)

In order to evaluate the path integrals in eq. (C3) we introduce the following Lagrangian:

$$L_j = \int_0^s ds' \frac{1}{4} x'(s')^2 + \frac{1}{4} x(s').A.A x(s') - \frac{1}{2} \text{Tr}(A) + J(s').x(s')$$  \hspace{1cm} (C4)

By taking repeated functional integrals with respect to $J$, we can reproduce the terms appearing in eq. (C3) Crucially, with the Lagrangian given in (C4) the path integral can be evaluated analytically, giving:

$$K(x'', x', s) = \int_{r(0)=x'}^{\text{r}(s)=x''} D[r(s)] e^{-\int_0^s ds'L_j(\dot{x}, x)} = K_h(x'', x', s) \exp(-W[J])$$  \hspace{1cm} (C5)
Where we have introduced the Green’s function neglecting endpoint Hamiltonian values:

\[
K_h(x'', x', s) = \left( \frac{1}{2\pi} \right)^{D/2} \sqrt{\det \left( \frac{1}{2} A \text{csch}(A s) \right)} \exp(-S_{cl})
\]  

(C6)

where \( S_{cl} \) is the same classical action appearing in the previous section, that of a \( J = 0 \) harmonic system evaluated along the classical path \( x_{cl}(s') \). The Schwinger functional, which accounts for the terms that depend on \( J \) is given by:

\[
W[J] = \int ds J(s) x_{cl}(s) + \frac{1}{2} \int ds \int ds' J(s) G_D(s, s') J(s')
\]  

(C7)

Where \( G_D \) is another Green’s function, which for a harmonic system is given by:

\[
G_D(s, s') = \frac{1}{2} A^{-1} \text{csch}(A s) \left( \theta (s - s') (\cosh (A (s - s') - \cosh (A (s + s' - s)))
\right.

\[
+ \theta (s' - s) (\cosh (A (s - s') - \cosh (A (s - s' + s)))
\]  

(C8)

We can calculate an arbitrary perturbation accounting for a potential term going as \( V \) according to the formula:

\[
G(x'', x') = e^\frac{\pi}{2} (H(x'') - H(x')) \int_0^\infty ds K_h(x'', x', s) \exp \left[ \int ds' V \left( \frac{\delta}{\delta J(s')} \right) \right] \exp(-W[J]) |_{J=0}
\]  

(C9)

Let us imagine an initial correction going as \( V(x) \sim x^4 \), and introducing the functional equivalent of the operator \( \nabla \cdot \nabla J = \left( \frac{\delta}{\delta J(x')} \nabla, \frac{\delta}{\delta J(x')} \nabla, \ldots, \frac{\delta}{\delta J(x')} \nabla \right) \). The first order perturbation will be given by:

\[
(-\nabla^2 J W[J] + \nabla J W[J] \cdot \nabla J W[J])^2 |_{J=0}
\]  

(C10)

where higher order terms cancel out due to the fact that the functional \( W \) is at most quadratic in \( J \). The functional derivatives can be calculated, yielding:

\[
= (-\text{Tr}(G_D(s', s')) + x_{cl}(s') x_{cl}(s'))^2
\]  

(C11)

To make our example specific, we shall calculate what happens to a system with a Hamiltonian given by:

\[
H = \frac{1}{2} A_{ij} x^i x^j + \frac{1}{4} B_{ijkl} x^i x^j x^k x^l
\]  

(C12)

where we have employed the Einstein summation convention. This Hamiltonian leads to the effective Lagrangian density given by:

\[
\mathcal{L} = \frac{1}{4} x^2 + \frac{1}{4} x^i A^i A^j x^j + \frac{1}{4} x^i B^i x - \frac{1}{2} \text{Tr}(A) + V(x)
\]  

(C13)

where \( V(x) \) characterizes deviations from harmonic terms.

We wish to evaluate the following integral:

\[
\delta(\log P) = \int dx' G(x, x') \rho(x')
\]  

(C14)

The function \( \rho \) follows the same form as the Hamiltonian, however with different prefactors:

\[
\rho(x) = \frac{1}{2} C_{ij} x^i x^j + \frac{1}{4} D_{ijkl} x^i x^j x^k x^l
\]  

(C15)

the precise form of these prefactors \( D \) and \( C \) will depend on the nature of the perturbation to the system. Let us calculate the lowest order perturbation, which will be given by the following function:
\[ G(x'', x') = \int dx' e^{\frac{\beta}{2}(H(x'') - H(x'))} \int_0^\infty ds K_h(x'', x', s) \rho(x') \]  

(C16)

In the prefactor \( e^{\frac{\beta}{2}(H(x) - H(x'))} \), we ignore the terms going as \( x^4 \). We justify this by noting that the Green’s function decays very quickly as a function of distance \( |x - x'| \) and for those short distances relevant to the calculation the terms going as \( \sim x^2 \) will dominate over the terms going as \( \sim x^4 \). We therefore have a similar integral to those in the previous section. The action \( S_{cl} \) evaluated along the path is given by:

\[ S_{cl} = \frac{1}{4} x'.A'.\coth(A'.s).x' + \frac{1}{4} x''.A'.\coth(A'.s).x'' - \frac{1}{2} x'.A'.\csch(A'.s).x' - \frac{1}{2} \text{Tr}(A)s \]  

(C17)

where now \( A' = \sqrt{A^2 + B} \).

We then have the following integral, where we use \( M' = \frac{1}{2} A'.\coth(A'.s) \) and \( b' = x''.\frac{1}{2} A'.\csch(A'.s) \):

\[ \int ds \frac{e^{rac{1}{2} \text{Tr}(A)s}}{(2\pi)^n/2} \exp \left( -\frac{\beta}{2} x''.(M' - \frac{1}{2} A).x'' \right) \sqrt{\det \left( \frac{1}{2} A'.\csch(A'.s) \right)} \times \int dx' \exp \left( -\frac{\beta}{2} x'.(M' + \frac{1}{2} A).x' + b'.x' \right) \rho(x') \]

(C18)

the integral over \( x' \) is equal to:

\[ \left( \sqrt{\frac{(2\pi)}{\det(M' + \frac{1}{2} A)}} \right)^n \exp \left( \frac{1}{2} b'.(M' + \frac{1}{2} A)^{-1} b' \right) \times \exp \left( -\frac{1}{2} (M' + \frac{1}{2} A)_i j \frac{\partial}{\partial y_i} \frac{\partial}{\partial y_j} \right) \rho \left( y + (M' + \frac{1}{2} A)^{-1} b' \right) \bigg|_{y=0} \]

(C19)

which, as the function \( \rho \) is at most quartic in \( x' \):

\[ \left( \sqrt{\frac{(2\pi)}{\det(M' + \frac{1}{2} A)}} \right)^n \exp \left( \frac{1}{2} b'.(M' + \frac{1}{2} A)^{-1} b' \right) \times \left( \rho \left( (M' + \frac{1}{2} A)^{-1} b' \right) - \frac{1}{2} \partial \rho \left( y + (M' + \frac{1}{2} A)^{-1} b' \right) + \frac{1}{8} \partial^2 \rho \left( y + (M' + \frac{1}{2} A)^{-1} b' \right) \right) \]

(C20)

where we have introduced the operator \( \partial \rho(y) \left. \right|_{y=0} \). As can be observed, this is a rather complicated expression. However, some degree of the complication arises because the presence of the Laplacian of \( B_{ijkl} x^i x^j x^k x^l \). In the absence of this term, a lot of the complicated terms would cancel away. Therefore, the small \( B \) expansion is useful in trying to extract generalities. This corresponds to \( A^2 \gg B \). As has already been observed in the previous section, this leads to the cancellation of the terms in the exponential and the determinant, and furthermore leads to the identity:

\[ (M' + \frac{1}{2} A)^{-1} b' = \exp(-A s)x'' = c(s) \]  

(C21)

where we introduced the new vector \( c(s) \). Therefore the modification will be given simply by:

\[ \delta \log P = \int_0^\infty ds \left( \rho(c(s)) - \frac{1}{2} \partial \rho(y + c(s)) + \frac{1}{8} \partial^2 \rho(y + c(s)) \right) \]

(C22)

Therefore, how the stationary probability is modified will depend entirely on the repeated application of operator \( \partial \) to \( \rho \). We can therefore obtain the modifications of the probability as:

\[ \rho(c(s)) = \frac{1}{2} C_{ij} c_i(s) c_j(s) + \frac{1}{4} D_{ijkl} c_i(s) c_j(s) c_k(s) c_l(s) \]

(C23)

\[ \partial \rho(y + c(s)) = \sum_{\text{Permutations}} (M + \frac{1}{2} A)_{ij}^l D_{ijkl} c_k(s) c_l(s) + \text{constant} \]

(C24)

\[ \partial^2 \rho(y + c(s)) = \text{constant} \]

(C25)
Where in equation [C24] the permutations we are summing over are over all unique combinations of indices in the factor $D$. Once again, we can ignore the terms that are constants as not modifying the relative probability between any two states. Where we can see that the terms going as $x^2$ is modified. Precise enumeration of the prefactors $C$ and $D$ would be required to specify the precise effect, but, once more using the fact that the matrix $A$ is symmetric allows the integrals over $s$ to be performed. This leads to:

$$\int_0^\infty ds \rho(c(s)) \sim \frac{1}{2} \frac{\tilde{C}_{ijkl}}{\lambda_k + \lambda_l} \tilde{x}_i'' \tilde{x}_j'' + \frac{1}{4} \frac{\tilde{D}_{ijklmnop}}{(\lambda_m + \lambda_n + \lambda_o + \lambda_p)^2} \tilde{x}_m'' \tilde{x}_n'' \tilde{x}_k'' \tilde{x}_l''$$ \quad (C26)

$$\int_0^\infty ds \tilde{O}(\rho(y + c(s))) = \sum_{\text{Permutations}} \frac{\tilde{D}_{ijklmn}}{(\lambda_k + \lambda_l)^2 + 2(\lambda_k + \lambda_l)\lambda_j \tilde{x}_m'' \tilde{x}_n''}$$ \quad (C27)

Where we use $\tilde{C}$ and $\tilde{D}$ to indicate that the original prefactors $C$ and $D$ have been modified by the matrix of eigenvectors of the matrix $A$, denoted previously as $Q$. Where we can see the additional modification going as $O(1/\lambda^2)$ once more.

**Appendix D: Simulation Details**

The simulated harmonic system of $N$ particles where the form of potential between particles $i$ and $j$ is given by:

$$V(x_i, x_j) = \frac{1}{2} k (x_i - x_j)^2$$ \quad (D1)

where $k = 1$ is the spring constant. For simplicity, we consider a system where particles are only bound to their 1D neighbors, leading to a polymer. The system is thermalized through the Langevin thermostat, where the noise is uncorrelated in time:

$$\langle \xi_i(t) \xi_i(t') \rangle = 2 k_b T_i \delta(t - t')$$ \quad (D2)

where $k_b T_i$ is the temperature of particle $i$, all particles have the same dissipative term $\gamma$. The equations of motion are integrated with the BBK integrator. We take the baseline temperature to be $k_b T = 1$ and we define a uniformly distributed random number $R$ between $-1$ and $1$ for each particle to correspond to particles coupled to different thermostats leading to a new temperature $k_b T_i = k_b T + \sigma R$ where $\sigma$ is some scale we choose.

For the cases where an anharmonic potential is introduced, we modify the relationship in (D3) to

$$V(x_i, x_j) = \frac{1}{2} k (x_i - x_j)^2 + \frac{1}{4} \epsilon (x_i - x_j)^4$$ \quad (D3)

When a deterministic non-conservative force is included in the system, we use the following force between neighboring monomers $i$ and $j$ with position vectors $r_i = (x_i, y_i, z_i)$ and $r_j = (x_j, y_j, z_j)$:

$$F(r_i, r_j) = (z_j - z_i, x_j - x_i, y_j - y_i)$$ \quad (D4)