Rotating Convective Core Excites Non-Radial Pulsations to Cause Rotational Modulations in Early-Type Stars

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ABSTRACT

We discuss low-frequency g modes excited by resonant couplings with weakly unstable oscillatory convective modes in the rotating convective core in early-type main-sequence stars. Our non-adiabatic pulsation analyses including the effect of Coriolis force for 2\(M_\odot\) main-sequence models show that if the convective core rotates slightly faster than the surrounding radiative layers, g modes in the radiative envelope are excited by a resonance coupling. The frequency of the excited g mode in the inertial frame is close to \(|m\Omega_c|\) with \(m\) and \(\Omega_c\) being the azimuthal order of the g mode and the rotation frequency of the convective core, respectively. These g mode frequencies are consistent with those of photometric rotational modulations and harmonics observed in many early-type main-sequence stars.

Key words: stars: rotation - stars: oscillations - stars: early-type

1 INTRODUCTION

Every early-type main-sequence star has a convective core, in which a small perturbation grows monotonically in the absence of rotation (sometimes called an unstable g\(^-\) mode (e.g., Cox 1980)). In the presence of rotation, however, the perturbation oscillates with amplitude growing exponentially (e.g, Osaki 1974). If the rotation is rapid enough, the growth time becomes much longer than the period of oscillation, to which we refer as overstable convective (OsC) mode. As shown numerically by Lee & Saio (1986) for a 10\(M_\odot\) main-sequence model, an OsC mode can resonantly couple with a g mode in the surrounding radiative layers; in other words an envelope g mode can be excited by the OsC mode. The property of such resonant couplings are further discussed in Lee & Saio (1987, 1989).

The resonant excitation of g modes can occur in all fast-rotating early-type main-sequence stars, if the convective core rotates slightly faster than the radiative envelope. Those g modes should be observed with oscillation frequencies close to \(|m|\) times the rotation frequency of the convective core, where \(m\) is the azimuthal order of the oscillation. This property may offer an explanation for the rotational modulations recently detected in many early type stars from KEPLER and TESS observations (e.g., Balona 2019).

Motivated with the importance of g modes resonantly excited by OsC modes in early type stars, we revisit the resonant excitation with improving the method of calculations. Here, we make two improvements over the previous calculations. One improvement concerns the number of expansion terms. To represent the oscillation modes of rotating stars, Lee & Saio (1986) employed series expansions of the perturbations in terms of spherical harmonic functions \(Y_{m\ell}(\theta, \phi)\), but the number of \(Y_{m\ell}\)s used for the expansions was very limited. We increase the number of expansion terms from 2 (Lee & Saio 1986) to \(\geq 10\) so that numerical results are insensitive to the number of expansion terms. Second improvement concerns the superadiabatic temperature gradient in the convective core

\[
\epsilon \equiv \nabla - \nabla_{ad},
\]

where \(\nabla = d\ln T/d\ln p\) and \(\nabla_{ad} = (\partial \ln T/\partial \ln p)_{ad}\) with \(T\) and \(p\) being the temperature and the pressure. Lee & Saio (1986) assumed \(\epsilon = 10^{-3}\) considering that a large value of \(\epsilon\) would be required to keep efficient convective fluid motions in the

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rotating core. For a non-rotating convective core mixing length theory of convection predicts $\epsilon$ to be as small as $10^{-7}$ to $10^{-6}$. For rotating stars, however, we do not have any good knowledge of the magnitudes of $\epsilon$, except that rotation would enhance by a large factor $\epsilon$ compared to that in non-rotating stars (e.g., Stevenson 1979). In this paper, we employ $\epsilon = 10^{-5}$ in most cases unless stated otherwise.

We carry out non-adiabatic calculations of unstable convective modes in the core of rotating 2$M_\odot$ main sequence stars. We assume a weak differential rotation such that the convective core rotates slightly faster that the envelope (e.g. Lee & Saio 1986). The method of calculations of non-adiabatic oscillations of differentially rotating stars is the same as that used by Lee & Saio (1993). The numerical results for 2$M_\odot$ main sequence models are given in §2. We discuss and conclude in §3 and §4.

2 Numerical Analysis

We express a pulsation mode in a rotating star by a sum of terms proportional to spherical harmonic functions $Y_l^m(\theta, \phi)$ with different $l$ for a given $m$. For example, the displacement vector $\xi$ is given as

$$\xi_\nu(r, \theta, \phi, t) = r \sum_{j=1}^{j_{\text{max}}} S_{ij}(r) Y_l^m(\theta, \phi) e^{i\omega t},$$

(2)

$$\xi_\nu(r, \theta, \phi, t) = r \sum_{j=1}^{j_{\text{max}}} \left[ H_{ij}(r) \frac{\partial}{\partial \theta} Y_l^m(\theta, \phi) + T_{ij} \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} Y_l^m(\theta, \phi) \right] e^{i\omega t},$$

(3)

$$\xi_\nu(r, \theta, \phi, t) = r \sum_{j=1}^{j_{\text{max}}} \left[ H_{ij}(r) \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} Y_l^m(\theta, \phi) - T_{ij} \frac{\partial}{\partial \theta} Y_l^m(\theta, \phi) \right] e^{i\omega t},$$

(4)

where $l_j = |m| + 2(j - 1)$ and $l_j' = l_j + 1$ for even modes and $l_j = |m| + 2j - 1$ and $l_j' = l_j - 1$ for odd modes with $j = 1, 2, \ldots, j_{\text{max}}$. The parameter $j_{\text{max}}$ gives the length of expansions. Substituting these expansions into linearized basic equations, we obtain a set of linear ordinary differential equations for the expansion coefficients such as $S_l(r)$, $H_l(r)$, and so on (e.g., Lee & Saio 1986). The set of differential equations for non-adiabatic oscillation modes of differentially rotating stars are given in Lee & Saio (1993). We employ the Cowling approximation, neglecting the Euler perturbation of the gravitational potential. We also ignore the terms associated with centrifugal force, which is justified because most of the kinetic energy of the convective and $g$-modes is confined into deep interior. For expansions, we use $j_{\text{max}} = 10$ to 15, with which the frequencies and eigenfunctions become insensitive to $j_{\text{max}}$.

We use 2$M_\odot$ main sequence star models with $X_c = 0.7$ and $X_c = 0.5$ as background models for modal analyses where $X_c$ is the hydrogen abundance at the stellar centre. The models are computed by using a standard stellar evolution code with the OPAL opacity (Iglesias & Rogers 1996) starting with the initial chemical abundance ($X, Z$) = (0.7, 0.02). Fig. 1 shows the propagation diagrams of these models in which the Lamb frequency $L_\nu = \sqrt{\Gamma + 1}/c$ and the Brunt-Väisälä frequency $N = \sqrt{\Gamma_0 - 1}$ are plotted versus the fractional radius $r/R$, where $c = \sqrt{\Gamma_1 p/\rho}$ with $\Gamma_1 = (\partial \ln p/\partial \ln \rho)_{\text{iso}}$ is the adiabatic sound velocity, and $g = GM/r^2$ with $M_r = \int_0^r 4\pi r^2 \rho dr$ and $G$ the gravitational constant and $A = d \ln p/dr - L^{-1} d \ln p/dr$ is the Schwarzschild discriminant. The frequencies $L_\nu$ and $N$ in the figure are normalized by $\sigma_0 = \sqrt{GM/R^3}$ where $M$ and $R$ are the mass and radius of the stars. Note that $N^2 < 0$ in the convective layers and $N^2 > 0$ in the radiative layers. Each model has a convective core and subsurface thin convective layers in the envelope. In the slightly evolved model with $X_c = 0.5$, the convective core has shrunken from the ZAMS model and is surrounded by a $\mu$ gradient zone, which disturbs frequency spectra of $g$-modes propagating in the radiative envelope. In the following, angular frequencies such as $\omega$ and $\Omega$, normalized by $\sigma_0$ are written as $\bar{\omega}$ and $\bar{\Omega}$.

To compute unstable convective modes of rotating main sequence stars, it is essential to set a non-zero super-adiabatic temperature in the convective core. As mentioned in the previous section in most cases we employ $\epsilon = 10^{-5}$, considering that $\epsilon$ should be enhanced by a large factor from that of non-rotating stars.

As we discuss below, the oscillations in the envelope are affected by the presence of a differential rotation between the convective core and radiative envelope. We assume a differential rotation in some models using the formula given by

$$\Omega(r) = \Omega_s \left[ 1 + \frac{b - 1}{e^{a(x-x_c)} - 1} \right],$$

(5)

where $x = r/R$ and $x_c$ corresponds to the core-envelope interface, $\Omega_s$ is the rotation speed at the stellar surface, and $a$ and $b$ are parameters. Uniform rotation corresponds to $b = 1$. If $b > 1$, the core rotates faster than the envelope. In this paper we use $a = 100$, for which $\Omega(r)$ stays $\approx b \Omega_s$ for $x < x_c$, but decreases steeply to $\Omega_s$ around $x_c$ (see an example in Lee (1988)).

For oscillation modes in differentially rotating stars, we use the symbol $\sigma$ for the angular frequency (eigenfrequency) in the inertial frame. Although the inertial frame frequency $\sigma$ does not depend on $r$, the frequency $\omega = \sigma + m \Omega(r)$ in a local co-rotating frame depends on $r$ (but $\text{Im}(\omega) = \text{Im}(\sigma)$). If we let $\omega_s$ denote an oscillation frequency in the co-rotating frame of
the core, the frequency $\omega_s$ in the co-rotating frame of the envelope is given by
\[
\omega_s = \omega_c - m(\Omega_c - \Omega_s) \approx \omega_c - m\Omega_s(b - 1),
\]
where $\Omega_c(0)$. If a prograde convective modes have a frequency $\omega_c > 0$ for $m < 0$, the frequency $\omega_c$ should be shifted to $\omega_s$ in the envelope. Then the $g$-mode in resonance with $\omega_s$ in the envelope should have a radial order much lower than that of a $g$-mode having the frequency $\omega_c$ in the envelope.

As a tool to classify oscillation modes, we compute the fraction $f_j = E_j/E_K$ of energies where $E_K$ and $E_j$ are the total and partial kinetic energies of an oscillation mode defined by
\[
E_K = \frac{1}{2} \int_0^R r^4 \omega_R^2 \sum_{j=1}^{j_{\text{max}}} \left[ |S_{lj}|^2 + l_j(l_j+1)|H_{lj}|^2 + l_j^2(l_j+1)^2|T_j|^2 \right] dr \equiv \sum_{j=1}^{j_{\text{max}}} E_j,
\]
and $\sum_j f_j = 1$. In this paper, we pick up unstable convective modes that mainly consist of components with relatively slow latitudinal variations and satisfy
\[
f_1 + f_2 + f_3 \gtrsim 0.8.
\]
The number 0.8 is somewhat arbitrary. If we use 0.9, instead of 0.8, the number of modes that satisfy the condition will be smaller.

### 2.1 Convective Modes in Uniformly Rotating Stars

We start with the case of uniformly rotating stars for which $b = 1$ and $\Omega = \Omega_c = \Omega_s$. Adiabatic convective modes in the core of rotating stars appear as a pair of modes having frequencies $\omega_{\pm}$, and the relation $\omega_- = \omega_c^*$. Note that $\omega_{\pm}$ become pure imaginary when $\Omega = 0$. If we include non-adiabatic effects, core convective modes still appear as a pair of modes having $\omega_{\pm}$, but the relation $\omega_- = \omega_c^*$ holds only approximately even if the modes are well confined in the core where non-adiabatic effects are insignificant. In this paper, we discuss only the unstable modes of which imaginary part of frequency is negative.

The results of non-adiabatic calculations of even-parity $m = -1$ unstable prograde convective modes for $\epsilon = 10^{-5}$ are shown in Fig. 2, where normalized eigenfrequency in the co-rotating frame $\Omega = \Omega_c + \Omega_s$ is plotted against $\Omega$. We have used $j_{\text{max}} = 10$ for the series expansion length. Note that to obtain unstable convective modes for $\Omega \neq 0$ we do not calculate them from $\Omega = 0$ because there occurs frequent mode crossings as $\Omega$ increases and it is very difficult for us to compute convective modes from $\Omega = 0$ keeping their mode identity. As shown in Fig. 2, the real part of the eigenfrequency, $\Omega_{\text{Re}}$, of a convective mode increases with $\Omega$ up to a maximum value and then decreases steeply. The absolute value of the imaginary part $|\Omega_{\text{Im}}|$ decreases steeply (i.e., getting stabilized) with increasing $\Omega$, and stays almost constant for $\Omega$ larger than the value at the peak of $\Omega_{\text{Re}}$. We note that the rotation speed $\Omega$ at the peak of $\Omega_{\text{Re}}$ for $B_0$ and $B_1$ modes is approximately proportional to $\sqrt{\epsilon}$. We stop calculating the modes at certain values of $\Omega$ because the condition (8) does not hold any more as $\Omega$ further increases. Note that since $|\Omega_{\text{Im}}| \ll |\Omega_{\text{Re}}|$ for convective modes, we have $\Omega_{\text{Re}} \approx -\Omega_{\text{Im}}$ in the inertial frame.

The unstable convective modes plotted in Fig. 2 correspond to $B_n$ modes discussed by Lee (2019) for rotating hot Jupiters. Lee (2019) found two different kinds of convective modes, labeled $A_n$ and $B_n$ with $n$ being the number of radial nodes of $S_1$ in the convective core. Convective modes $A_n$ were first discussed by Lee & Saio (1986) for a rotating $10M_\odot$ main sequence.
Figure 2. Complex eigenfrequency $\omega/\sigma_0$ of even-parity $m = -1$ unstable convective modes versus $\Omega/\sigma_0$ for the $2M_\odot$ ZAMS model for $b = 1$ and $\epsilon = 10^{-5}$ where $\omega \equiv \sigma + m\Omega$ is the oscillation frequency in the co-rotating frame, and $\omega_R = \text{Re}(\omega)$ and $\omega_I = \text{Im}(\omega)$. Filled red circles indicate the eigenfrequency of the mode whose displacements in the interior are shown in Fig. 3 below.

Figure 3. Real parts of the expansion coefficients for radial and horizontal displacements $xS_l$ and $xH_l$ as a function of $x = r/R$ for the $m = -1$ $B_0$ mode with $\overline{\omega} = (2.94 \times 10^{-3}, -6.59 \times 10^{-4})$ at $\overline{\Omega} = 0.046$ (indicated by filled red circles in Fig. 2). Here, the solid, dashed, and dotted lines represent the coefficients with $l = 1, 3,$ and $5$, respectively. The amplitudes are normalized by the maximum value of $|xS_l|$ in the interior.

star, for which $\epsilon = 10^{-3}$ was assumed in the convective core. The modal properties of unstable convective modes $A_n$ are reasonably well described by using an asymptotic analysis based on the traditional approximation (see, e.g., Lee & Saio 1987, 1989, 1997). For rotating hot Jupiters, Lee (2019) found another kind of convective modes, labeled $B_n$, which need much larger rotation speeds $\overline{\Omega}$ to be stabilized than $A_n$ convective modes. For $\epsilon = 10^{-5}$, unstable convective modes $A_n$ are stabilized at much smaller values of $\overline{\Omega}$ and therefore only unstable convective modes $B_n$ could survive for rapidly rotating stars.

Fig. 3 shows the first few expansion coefficients for radial and horizontal displacements, $xS_1$ and $xH_1$, of the $m = -1$ prograde convective mode with $\overline{\omega} = (2.94 \times 10^{-3}, -6.59 \times 10^{-4})$ at $\overline{\Omega} = 0.046$, which is the $B_0$ mode indicated by filled red circles in Fig. 2. The amplitudes of $S_1$ and $H_1$ are completely confined in the convective core and no amplitude penetration into the radiative envelope takes place. We note that because of the small frequency $\overline{\omega}_R \sim 3 \times 10^{-3}$, envelope $g$-modes corresponding to this frequency are extremely high radial order modes having radial nodes more than 600.

2.2 Cases with Weak Differential Rotation

In the previous section we found that no amplitude penetration of convective modes into the envelope can be expected in uniformly rotating stars. In this subsection we discuss convective modes in models with a weak differential rotation given by equation (5) with $b = 1.2$. As equation (6) indicates, if the convective core rotates slightly faster than the envelope, a $g$-mode which couples with a convective mode should have a higher frequency and hence a smaller number of nodes in the envelope. We expect such a $g$-mode to get less dissipation and to have larger amplitude in the envelope.
2.2.1 ZAMS model

For the $2M_{\odot}$ zero age main sequence (ZAMS) model with $\epsilon = 10^{-5}$ and $b = 1.2$, we have obtained unstable convective modes of even parity for $m = -1$, $-2$, and $-3$. The results are shown in Fig. 4, in which the complex frequency $\overline{\omega}_0$ is plotted versus $\Omega_c$ for $B_0$, $B_1$, and $B_2$ modes, where $\omega_c$ is the frequency in the frame co-rotating at the centre given as $\omega_c = \sigma + m\Omega_c$ with $\Omega_c = \Omega(0)$, and $\omega_{cR} = \text{Re}(\omega_c)$ and $\omega_{cI} = \text{Im}(\omega_c)$.

As in the case of uniformly rotating stars, $\overline{\omega}_{cR}$ of an unstable convective mode has a maximum and $|\overline{\omega}_0|$ decreases with increasing $\Omega_c$, and at the maximum of $\overline{\omega}_{cR}$ we find $|\text{Im}(\overline{\omega}_0)| \ll |\text{Re}(\overline{\omega}_0)|$, that is, the convective mode is stabilized to have a very small growth rate. As $\Omega_c$ further increases both $\overline{\omega}_{cR}$ and $\overline{\omega}_{cI}$ show rapid variations with small amplitudes as a function of $\Omega_c$, which are caused by resonances with envelope $g$-modes having a dense frequency spectrum. The amplitudes and separation of this resonance feature, as clearly seen for the $B_1$ modes, increase as $\Omega_c$ increases. This corresponds to the fact that the frequency $\overline{\omega}_0 \approx \overline{\omega}_e - m(\Omega_c - \Omega_0)$ in the envelope increases and hence the frequency separation of $g$-modes increases with increasing $\Omega_c$. We note that $\overline{\omega}_{cI}$ of convective modes remains negative with increasing $\Omega_c$ beyond the value of $\Omega_0$, at the $\overline{\omega}_{cR}$ maximum, except for the $m = -1$ $B_1$ mode, whose $\overline{\omega}_{cI}$ frequently changes its sign as $\Omega_c$ increases. In general, we find it difficult to obtain accurate eigenfrequencies of convective modes when both $|\overline{\omega}_{cR}|$ and $|\overline{\omega}_{cI}|$ are very small, and hence we have to stop computing convective modes at certain $\Omega_c$.

It is interesting to note that $\overline{\omega}_{cR}$ of $B_2$ modes of $m = -2$ and $-3$ become negative for $\overline{\Omega}_0 \gtrsim 0.05$. These modes are retrograde to the rotation in the convective core and have a co-rotation point at which $\overline{\omega}_{cR} = 0$ near the convective core boundary.

For a given $m$, the rotation rate $\Omega_c$ at the peak of $\overline{\omega}_{cR}$ for $B_1$ mode is larger than that for $B_0$ mode, and the height of the peak for $B_1$ mode is lower than that for $B_0$ mode. If we compare the results for $m = -1$ and $m = -2$, $\overline{\Omega}_0$ at the $\overline{\omega}_{cR}$ peak for the $B_0$ mode is smaller for $m = -1$ than for $m = -2$, that is, we need more rapid rotation to stabilize unstable convective modes of $m = -2$ than those of $m = -1$.

We find that the amplitudes of unstable convective modes can penetrate deep into the envelope, even reach to the stellar surface, when the convective modes are stabilized to have sufficiently small $|\overline{\omega}_{cI}|$ (i.e., long growth time). This amplitude
penetration into the envelope takes place as a result of resonance with an envelope g-mode. Examples of such amplitude penetration of unstable convective modes are shown in Figs. 5 and 6, where the real parts of the expansion coefficients for radial and horizontal displacements $xS_l$ and $xH_l$ are plotted as a function of $x = r/R$ for the $m = -1$ $B_0$ mode with $\varpi_c = (4.27 \times 10^{-3}, -8.18 \times 10^{-3})$ at $\Pi_s = 0.055$ (indicated by filled red circles in the left panels of Fig. 4). Here, the solid, dashed, and dotted lines represent the coefficients with $l = 1$, 3, and 5, respectively.

Fig. 5. Real parts of the expansion coefficients for radial and horizontal displacements $xS_l$ and $xH_l$ as a function of $x = r/R$ for the $m = -1$ $B_0$ mode with $\varpi_c = (4.27 \times 10^{-3}, -8.18 \times 10^{-3})$ at $\Pi_s = 0.055$ (indicated by filled red circles in the left panels of Fig. 4). Here, the solid, dashed, and dotted lines represent the coefficients with $l = 1$, 3, and 5, respectively.

2.2.2 Slightly Evolved Main Sequence Model

We have also carried out non-adiabatic computation of unstable convective modes for a slightly evolved $2M_\odot$ main sequence model with $X_c = 0.5$. The convective core of the evolved model is slightly smaller compared to that of the ZAMS model and surrounded by a $\mu$-gradient zone, as indicated by the plot of $\log_H r/\varpi^2$ in Fig. 1.

Fig. 9 shows complex frequencies $\varpi_c$ of convective modes as a function of $\Pi_s$. As shown in this figure, the behavior of $\varpi_{Rl}$ as a function of $\Pi_s$ is very different from that found for the ZAMS model. Instead of increasing to a peak and decreasing, $\varpi_{Rl}$ keeps increasing with increasing $\Pi_s$ but the increasing rate jumps at a value of $\Pi_s$ where $|\omega_{rl}/\omega_{Rl}|$ becomes very small. As $\Pi_s$ further increases, $\varpi_{Rl}$ continues to increase while $-\varpi_{rl}$ continues to decrease changing its sign until the mode is completely stabilized. For the ZAMS model there exists no indication of complete stabilization of the unstable convective modes although we had to stop calculations when both $\varpi_{Rl}$ and $|\varpi_{rl}|$ become very small. We find that as $\Pi_s$ further increases after $|\varpi_{rl}|$ becomes sufficiently small, the ratio $\omega_{rl}/\Omega_s$ tends to be constant, which suggests that the unstable convective modes are stabilized to become inertial modes (see below).

The gross properties of unstable convective modes in the core are quite similar between the cases of $m = -1$, $-2$, and $-3$. The convective core of the evolved model is slightly smaller compared to that of the ZAMS model and surrounded by a $\mu$-gradient zone, as indicated by the plot of $\log_H r/\varpi^2$ in Fig. 1.

Fig. 7 is the same as Fig. 5 but for the $m = -2$ $B_1$ mode with $\varpi_c = (3.63 \times 10^{-3}, -1.49 \times 10^{-3})$ at $\Pi_s = 0.25$ (shown by filled red circles in the middle panels of Fig. 4). As seen in this figure, $S_{m=|m|}$ has one node in the core, and the coefficients $S_{m=|m|+2}$ and $S_{m=|m|+4}$ have noticeable amplitudes there. Fig. 7 shows that $B_1$ mode of $m = -2$, too, can have appreciable amplitude near the stellar surface.

We find that there exist unstable convective modes that have a co-rotation point as defined by $\omega_R(r) = 0$ in the interior. Fig. 8 plots the first few expansion coefficients for the $m = -2$ $B_2$ mode with $\varpi_c = (-4.24 \times 10^{-4}, -1.41 \times 10^{-3})$ at $\Pi_s = 0.25$ (indicated by filled red squares in the middle panels of Fig. 4). The negative $\varpi_{Rl}$ indicates that the mode is retrograde in a co-rotating frame in the convective core while it is prograde in the envelope because of the differential rotation, i.e., the mode has a co-rotation point near the boundary of the convective core. Since $|\varpi_c(r)| = |\varpi_{rl}| > \varpi_{Rl}$, the co-rotation point does not affect the eigenfunctions significantly. As indicated by the coefficients $xH_l$, the amplitudes of the convective mode could marginally reach to the stellar surface.
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Figure 6. Same as Fig. 5 but for $c = (2.99 \times 10^{-3}, -8.84 \times 10^{-5})$ at $0 = 0.06$ (indicated by filled red squares in the left panels of Fig. 4).

Figure 7. Real parts of the expansion coefficients for radial and horizontal displacements $xS_l$ and $xH_l$ as a function of $x = r/R$ for the $m = -2 B_1$ mode with $c = (3.63 \times 10^{-3}, -1.49 \times 10^{-3})$ at $0 = 0.25$ (indicated by filled red circles in the middle panels of Fig. 4). The solid, dashed, and dotted lines represent the coefficients with $l = 2, 4,$ and 6, respectively.

The value of $0$ at which $|c|$ of $B_0$ mode becomes sufficiently small shifts to larger values as $|m|$ increases. We also find that the $m = -3 B_2$ mode have a co-rotation point.

The reason why the behavior of $0$ as a function of $0$ is different between the ZAMS model and the slightly evolved model is not clear. Using a local analysis of waves in the short wavelength and low frequency limit, we may obtain a dispersion relation given by (e.g., Unno et al. 1989; see also Lee & Saio 1997)

$$\omega^2 \approx N^2k_H^2 + (2\Omega \cdot k)^2,$$

where $\omega$ is the wave frequency in the co-rotating frame, $\Omega$ is the vector of angular frequency of rotation, $k$ is the wave number vector with $k = |k|$, and $k_H$ is the magnitude of the horizontal component of $k$. As suggested by the dispersion relation, modal properties of waves in the rotating convective core are determined by balance between buoyant effects and rotation effects. If buoyant effects dominate, convective modes in the core behave like oscillatory convective modes $\omega^2 \sim \lambda N^2/k^2$ with $N^2 < 0$ and $\lambda < 0$ and if rotation effects dominate they behave like inertial modes with $\omega^2 \sim (2\Omega \cdot k)^2/k^2$, where we have replaced $k_H^2$ by $\lambda$ assuming the traditional approximation (e.g., Lee & Saio 1987, 1989, 1997). If we used $\epsilon = 10^{-3}$ in the core as in Lee & Saio (1986), buoyant effects represented by $N^2$ would always dominate rotation effects $(2\Omega \cdot k)^2$ so that the relation $\omega^2 \sim \lambda N^2/k^2$ would describe well the wave properties in the core. For $\epsilon = 10^{-5}$, however, the balance can be marginal and the stabilized convective modes behave like inertial modes in the slightly evolved model.

To examine this interpretation, we have computed unstable convective modes of the evolved model assuming $\epsilon = 4 \times 10^{-5}$, instead of $\epsilon = 10^{-5}$. We present the results for $m = -1$ in Fig. 10. It is interesting to note that $0$ of $B_0$ mode has a peak as a function of $0$, while $0$ of $B_1$ mode continues to increase. The difference may be explained as follows: Since mode $B_0$ is stabilized at a smaller value of $0$ compared with mode $B_1$, buoyant force is still significant for the former case while the rotation effect dominates for the latter.

For the slightly evolved model, amplitude penetration of unstable convective modes into the envelope also takes place.
Figure 8. Same as Fig. 7 but for the $m = -2$ $B_2$ mode having $\overline{\omega}_c = (-4.24 \times 10^{-3}, -4.14 \times 10^{-3})$ at $\overline{\Omega}_c = 0.25$ (indicated by filled red squares in the middle panels of Fig. 4). This mode is retrograde in the co-rotating frame of the convective core but due to the differential rotation it couples with a prograde $g$-mode in the envelope.

Figure 9. Same as Fig. 4 but for a slightly evolved main sequence star with $X_c = 0.5$. Filled red circles indicate the eigenfrequency of the mode whose displacements in the interior are shown in Fig. 11. Narrow breaks in some lines correspond to ranges of $\Omega_s$ where modes are damped; i.e., $\omega_{cl} > 0$.

if $|\omega_{cl}/\omega_{Rl}|$ is sufficiently small. Fig. 11 shows an example of amplitude penetration of the $m = -1$ $B_0$ mode with $\overline{\omega}_0 = (1.46 \times 10^{-2}, -2.27 \times 10^{-6})$ at $\overline{\Omega}_s = 0.092$ ($\epsilon = 10^{-5}$ and $b = 1.2$). The coefficient $S_{lm|m_l}$ of $B_0$ mode has no nodes in the convective core.

3 OBSERVATIONAL CONSEQUENCES

In order to see when convective modes have large amplitudes in the envelope, we have computed the ratio of amplitudes $A_{env}$ to $A_{core}$ as a function of $\overline{\Omega}_s$ where $A_{env}$ is the maximum amplitude of $|xH_l|$ in the envelope and $A_{core}$ is that of $|xS_l|$ in
the core for \( j = 1, \cdots, j_{\text{max}} \). The results are shown in Fig. 12 for the ZAMS model and in Fig. 13 for the evolved model. Comparing these figures with Figs. 4 and 9, we find that amplitude penetration of convective modes into the envelope, that is, \( A_{\text{env}}/A_{\text{core}} \gtrsim 1 \) takes place when \( |\omega_{c}| \) is much smaller than \( |\omega_{R}| \) and \( |\omega_{I}| \). Note that the maximum amplitude of \( |x_{H_{1}}| \) occurs at the surface when \( A_{\text{env}}/A_{\text{core}} \gtrsim 1 \). The ratio \( A_{\text{env}}/A_{\text{core}} \) for \( B_{0} \) mode, for example, is smaller for larger \( |m| \) because amplitude confinement into the core is stronger for larger \( |m| \). This means that prograde dipole modes with \( m = -1 \) will be most visible. The ratios for the evolved model are in general smaller than those for the ZAMS model. For both main sequence models, the ratios for \( B_{0} \) and \( B_{1} \) modes fluctuate with small amplitudes as a function of \( \Omega_{s} \). These fluctuations are a manifestation of resonances between convective modes and envelope \( g \)-modes.

In Fig. 14, we plot cyclic frequency \( \sigma_{R}/2\pi \), in the inertial frame, of unstable convective modes that are likely observable, satisfying the condition \( A_{\text{env}}/A_{\text{core}} \gtrsim 1 \). The left and right panels of this figure are for the ZAMS model and the evolved model, respectively. Since the frequency \( \omega_{c} \) in the convective core is much smaller than \( |m\Omega_{c}| \), we have \( \sigma_{R} \sim |m\Omega_{c}| \) in the inertial frame for all convective modes having large amplitudes in the envelope. Note that for the ZAMS model the wide gaps along the loci of \( m = -2 \) and \( -3 \) convective modes are due to the fact that we had to stop calculating convective modes when both \( \omega_{R} \) and \( \omega_{I} \) become very small. On the other hand, the wide gaps for the evolved model appear because the convective modes are completely stabilized (i.e., damped oscillation with \( \omega_{I} > 0 \)) when \( \Omega_{s} \) gets sufficiently large. Besides the wide gaps there appear many narrow gaps along the locus, for example, of the \( m = -1 \) \( B_{1} \) mode for the ZAMS model. These narrow gaps are caused by frequent stability changes with increasing \( \Omega_{s} \). This kind of narrow gaps are also found for the evolved model. Fig. 14 indicates that we would expect rotational modulations and their harmonics in most early-type main-sequence stars, corresponding \( |m| \) times rotation frequency of the convective core which rotates slightly faster that the envelope.
**Figure 12.** Envelope-core amplitude ratios $A_{env}/A_{core}$ against $\Omega_s/\sigma_0$ for prograde unstable convective modes of $m = -1, -2, \text{ and } -3$ in the $2M_\odot$ ZAMS model.

**Figure 13.** Same as Fig.12 but for the slightly evolved main sequence model. Breaks of sequences indicate the ranges of $\Omega_s$ where convective modes are damped; i.e., $\omega_I > 0$.

**Figure 14.** Cyclic frequency in the inertial frame $\sigma_{R}/2\pi$ of envelope $g$-modes excited by core convective modes for $m = -1, -2, \text{ and } -3$ versus surface rotation frequency $\Omega_s/2\pi$ for the $2M_\odot$ models with $X_c = 0.7$ (left panel) and $X_c = 0.5$ (right panel) where only unstable convective modes with $A_{env}/A_{core} \geq 1$ are plotted for $\Pi_s \leq 0.3$. A weak differential rotation of $b = 1.2$ is assumed.
4 CONCLUSION

We have computed unstable convective modes in 2\(M_\odot\) main-sequence models, assuming uniform rotation and a weak differential rotation, in which the convective core is rotating slightly faster than the radiative envelope. Convective modes get very weakly unstable with very small growth rates (i.e., \(|\omega A/\omega_{\text{rad}}| \ll 1\) if the rotation speed is sufficiently high. Then, in the models with the differential rotation, the amplitudes of unstable prograde convective modes in the core can penetrate into the radiative envelope of the stars as a result of resonance with high radial order \(g\)-modes in the envelope. In other words, high radial order \(g\)-modes in the radiative envelope are resonantly excited by weakly unstable convective modes generated in the core which rotates slightly faster than the envelope. We find such amplitude penetration of the modes to occur if rotation frequency is higher than about 0.3\(d^{-1}\) in our 2\(M_\odot\) models. For differentially rotating stars, we have also found that some unstable convective modes have a co-rotating point at which \(\text{Re}(\omega) = 0\) in the interior. However, their eigenfunctions are not strongly affected by the co-rotation point because \(-\sigma_{\text{en}} \gg \sigma_{\text{en}}\).

If low \(|m|\) \(g\)-modes in the envelope of rotating early-type stars are resonantly excited by weakly unstable convective modes in the core, they will be observed as low frequency variabilities having frequencies \(\omega \approx -m\Omega_c\), where \(\Omega_c\) is the rotation frequency of the convective core. Because a single frequency is associated with each \(m\) (see Fig. 14), these \(g\)-mode variations would be observed as a rotational modulation and its harmonics. This property of resonantly excited \(g\)-mode oscillations nicely explains the rotational modulations and their harmonics detected in many early type main sequence stars (e.g., Balona 2019; Balona et al. 2019). Since this \(g\)-mode excitation mechanism by convective modes should work for rotating stars that have a convective core and a radiative envelope, \(g\)-modes excited by convective modes can also explain similar low frequency variabilities observed in \(B\)-type stars (e.g., Balona & Ozuyar 2020). The presence of magnetic fields are often invoked to explain rotational modulations. However, as Balona (2019) discusses, it is unlikely for strong magnetic fields to be produced in radiative envelopes of early type stars. Our explanation by \(g\)-mode oscillations resonantly excited by convective core mode is free from such difficulty.

Let us discuss amplitude penetration of convective modes using an asymptotic method. We assume that the frequency \(\sigma_{\text{en}}\) of convective modes is determined within the convective core and that the convective modes with the shifted frequency \(\sigma_{\text{en}} = \omega_c + m(\Omega_c - \Omega_I)\) resonantly excite gravity waves in the envelope. As an asymptotic treatment of non-radial oscillations suggests, the phase of such gravity waves having a frequency \(\sigma_{\text{en}}\) in the envelope may be given by

\[
\Phi^R(\omega_c) = \int_{r_c}^r k_r(\omega_c)dr \approx \Phi^R_0 + i\Phi^I_0,
\]

where \(r_c\) is the radius of the convective core and for low frequency \(g\)-modes in the envelope

\[
k_r(\omega) \approx \sqrt{\frac{rA}{\sigma_{\text{en}}^2}} \frac{\lambda}{c_s^2} = \frac{\sqrt{\lambda N}}{r} \omega_c, \quad \Phi^R_0 = \int_{r_c}^r k_r(\omega_{\text{en}})dr, \quad \Phi^I_0 = -i\frac{\omega}{\omega_{\text{en}}} \Phi^R_0,
\]

and we have substituted \(\lambda\), instead of \(l(l + 1)\), for rotating stars under the traditional approximation (e.g., Lee & Saio 1997). The quantity \(\int_{r_c}^r k_r dr/\pi\) may be regarded as the number of radial nodes in the envelope. Since the term \((\omega A/\omega_{\text{en}}) \int_{r_c}^r k_r dr\) describes decays of wave amplitudes with wave propagation in the envelope, we may assume that gravity waves excited by core convective modes can reach to the stellar surface if

\[
\left|\Phi^R_0\right| \lesssim 1.
\]

Fig. 15 plots \(\Phi^R_0\) as a function of \(\sigma_{\text{en}}\) for low \(|m|\) unstable convective modes in the ZAMS model for a weak differential rotation with \(b = 1.2\). As shown in the figure, at rapid rotation rates, \(\Phi^R_0\) decreases to become \(\sim 1\), suggesting that amplitude...
penetration of the convective modes into the envelope takes place. Comparing Figs. 15 and 12, we find that predictions by the condition $|\Phi_R^1| \lesssim 1$ for amplitude penetration are consistent with those by the condition $A_{\text{env}}/A_{\text{core}} \gtrsim 1$. Note that in the case of uniform rotation with $b = 1$, the minimal values of $\Phi_R^1$ are of order of $\sim 10^3$, and no amplitude penetration into the envelope can be expected. If we write $\Phi_R^1$ as

$$\Phi_R^1 = -\frac{\omega_s I}{\omega_s R} \sqrt{\lambda} \int_{r_c}^R \frac{dr}{r}, \quad (13)$$

we understand from this equation that even-parity and prograde sectoral $g$-modes, for which $\lambda$ can be smallest for a given $m$ (e.g., Lee & Saio 1997), are most probable modes excited by unstable convective modes to have significant amplitudes in the envelope. This may justify our analysis which has been focused on even-parity prograde convective modes.

For B-type stars there exists a longstanding unsolved problem, that is, formation mechanism of discs around Be stars (e.g., Rivinius et al. 2013). Although Be stars are rapidly rotating B-type stars, their rotation velocities are not necessarily very close to the critical ones, above which mass shedding takes place from the equatorial regions of the stars. Certain mechanisms are necessary to accelerate the surface rotation beyond the critical velocities from subcritical ones. We suggest that angular momentum transported from the deep interior to the surface by low frequency $g$-modes excited by unstable convective modes can be such a help for disc formation around rapidly rotating B stars (e.g., Aerts et al. 2019). Efficiency of angular momentum deposition, however, depend on the amplitudes of the excited $g$-modes, which is another difficult problem to solve.

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