Critical Connectivity and Fastest Convergence Rates of Distributed Consensus with Switching Topologies and Additive Noises*

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Abstract- Consensus conditions and convergence speeds are crucial for distributed consensus algorithms of networked systems. Based on a basic first-order average-consensus protocol with time-varying topologies and additive noises, this paper first investigates its critical consensus condition on network topology by stochastic approximation frameworks. A new joint-connectivity condition called extensible joint-connectivity that contains a parameter $\delta$ (termed the extensible exponent) is proposed. With this and a balanced topology connectivity that contains a parameter $\delta$, we show that a critical value of $\delta$ for consensus is 1/2. Optimization on convergence rate of this protocol is further investigated. It is proved that the fastest convergence rate, which is the theoretic optimal rate among all controls, is of the order $1/\ell$ for the best topologies, and is of the order $1/\ell^{1-2\delta}$ for the worst topologies which are balanced and satisfy the extensible joint-connectivity condition. For practical implementation, certain open-loop control strategies are introduced to achieve consensus with a convergence rate of the same order as the fastest convergence rate. Furthermore, a consensus condition is derived for non stationary and strongly correlated random topologies. The algorithms and consensus conditions are applied to distributed consensus computation of mobile ad-hoc networks; and their related critical exponents are derived from relative velocities of mobile agents for guaranteeing consensus.

Keywords: Average-consensus, stochastic approximation, jointly-connected topology, multi-agent system, networked system

1 Introduction

Consensus of multi-agent systems has drawn considerable attention from various fields over the past two decades. For example, physicists investigate the synchronization phenomena of coupled oscillators, flashing fireflies, and chirping crickets [1, 2]; biologists, physicists, and computer scientists try to understand and model the flocking phenomenon of animals’ behavior [3, 5]; sociologists simulate the emergence and spread of public opinions [6, 7]. Because of the importance, effort has been devoted to the mathematical analysis of consensus of flocks [8–11]. Meanwhile consensus control algorithms have been developed for a wide range of applications, such as formation control of robots and vehicles [12, 14], attitude synchronization of rigid bodies and multiple spacecrafts [15–17], and distributed computation, filtering and resource allocations of networked systems [18, 19, 40]. The common thread in the consensus research is a group of agents with interconnected neighbor graphs trying to achieve a global coordination or collective behavior by using neighborhood information permitted by the network topologies.

Although there are many interesting consensus protocols like second-order and fractional-order consensus protocols [20], the first-order average-consensus protocol is the most basic one. This protocol usually assumes that the network topologies cannot be directly controlled and each node only knows its own and neighbors’ information, and has been investigated using different approaches to accommodate different kinds of uncertainties. For example, in wireless communication networks, channel reliability is affected by thermal noise, channel fading, and signal quantization; in formation of multiple satellites, vehicles or robots, there exist measurement noises in observations of neighbors’ states. To model random failures of communication links, some papers use deterministic network topologies but allow their switching [21–23]; and others adopt stochastic settings in which network topologies evolve according to some random distributions [24–29]. We remark that most models in the existing research effort do not contain observation noise, and as such they do not cover scenarios of measurement noise or quantization error. To overcome this deficiency, some papers consider the first-order average-consensus protocols with additive noises [30–42], among which a common feature is in using stochastic approximation methodologies.

The main idea of distributed stochastic approximation is: Each agent in the network uses a decreasing gain function acting on the information received from its neighbors to reduce the impact of communication or measurement noises. Using this idea, Huang and Manton [35] and Li and Zhang [38] considered first-order discrete-time and continuum-time consensus models with fixed topology and additive noise, respectively, and showed that the algorithms could achieve...
consensus in a probability sense if the topologies were balanced and connected. Later, this consensus condition was relaxed from fixed balanced topologies to switched balanced topologies that satisfied a uniform joint-connectivity condition \[30, 59\], namely, the union of the topology graphs over a given bounded time interval was connected uniformly in time, and further relaxed to general directed topologies, which may not be balanced, with uniform joint-connectivity \[37\]. Huang \[36\] also applied stochastic approximation methods to consensus problems for networks over lossy wireless networks containing random link gains, additive noises and Markovian lossy signal receptions. On the other hand, motivated by resource allocation problems in computing, communications, inventory, space, and power generations, Yin, Sun and Wang \[40\] introduced a stochastic approximation algorithm for constrained consensus problems of networked systems, where consensus conditions were established by assuming that the topologies were randomly switched under a Markov chain framework. This algorithm was further investigated and expanded later \[41, 42\].

Despite the existing research work on first-order average-consensus protocols, some key problems remain unsolved. \[29\] showed that first-order average-consensus protocols with deterministic topologies and no additive noises could achieve consensus if and only if the time-varying topologies satisfied an infinite joint-connectivity condition, i.e., the union of the topologies from any finite time to infinite was connected, providing all topologies had the same stationary distribution. However, under the same protocol but with additive noises the current best condition on topologies for consensus is the uniform joint-connectivity \[40, 57, 59\]. Thus, there exists a critical gap between the consensus conditions on topologies with and without additive noises. Naturally, for protocols with additive noises, an open question is: What is the critical connectivity condition on topologies for consensus?

To address this problem we propose a new condition for topologies named extensible joint-connectivity in this paper. This condition allows the length of the interval during which the union of the network topologies is connected to increase as the time grows with a growing rate, called extensible exponent and denoted by \(\delta\). This is an intermediate condition between the uniform joint-connectivity and infinite joint-connectivity. Furthermore, we use a stochastic approximation approach to attenuate noise effect and treat the gain function as the control input. Under the extensible joint-connectivity and balanced topology condition it will be shown that if \(\delta \leq 1/2\), for all topology sequences there exist open-loop controls of the gain function such that the system reaches consensus; if \(\delta > 1/2\), there exist some topology sequences such that no open-loop control of the gain function can make the system to reach consensus.

One of the most basic and important tasks on multi-agent systems is to optimize their performance. However, the current theoretical research on this topic is still at an early stage \[46, 47\]. For example, the convergence speed is an important performance of average-consensus protocols which has been considered under both fixed and switching topologies \[23, 24, 31, 40\]. However its optimization currently focuses on the case of fixed topology by iterate averaging of stochastic approximation \[41\] or optimization algorithms of static graphs \[22, 45\]. These methods cannot be used to optimize average-consensus protocols under uncontrollable time-varying network topologies, leaving an open problem on how to optimize the convergence rate of average-consensus protocols \[43\].

This paper investigates this problem for the first time, still based on the stochastic approximation methodology. Since the convergence rate depends on the uncontrollable time-varying topologies whose global information is unavailable, its optimization is difficult and the traditional optimization theory for stochastic approximation cannot cover this scenario. This paper is concentrated on the fastest convergence rate, which is the theoretical optimal convergence rate among all gain functions, with respect to the best and worst topology sequences, respectively. It will be shown that the fastest convergence rate is of the order \(1/t\), in the sense of \(\Theta(1/t)\) for the best topologies, and \(\Theta(1/t^{1-2\delta})\) for the worst topologies which are balanced and satisfy the extensible joint-connectivity condition. These results indicate that for any balanced and uniformly jointly connected topologies the fastest convergence rate is \(\Theta(1/t)\). For implementation on practical systems, this paper presents some open-loop controls for this protocol to reach consensus with a convergence rate of the same order as the fastest convergence rate.

Finally, in many practical systems, especially those involving wireless communications, network topologies of distributed consensus protocols are random networks. This kind of algorithms has been investigated in many papers. However almost all of them assume that the topologies are either an i.i.d. sequence or a stationary Markov process \[24, 28, 36, 40\]. These assumptions may not fit some practical situations such as mobile wireless sensor networks or multi robot systems whose topologies depend on the distances among agents and consequently may be non stationary. Thus, the last question studied in this paper is: for the average-consensus protocol with random topology and additive noise, can we relax the topology condition for consensus to non stationary and strongly correlated sequences?

This paper gives an answer to this question by proposing a consensus condition in which the random topology sequence can be strongly correlated and its connectivity probability can be a negative power function. To illustrate relevance of this condition, this result is applied to distributed consensus computation of a mobile ad-hoc network. In this application it is shown that to guarantee consensus the distance between mobile agents cannot grow too fast. To be specific, if the velocity difference between agents is of the order \(\frac{1}{t}\), with average-consensus protocols the mobile ad-hoc network can reach a consensus state; on the other hand, from simulations it is demonstrated that if the velocity difference is bigger than \(\frac{c}{\sqrt{t}}\), where \(c > 0\) and \(b \in (0, 1)\) are two constants, then the mobile ad-hoc network cannot reach a consensus state.

The rest of the paper is organized as follows: Section 2 introduces our consensus protocol and some basic definitions. Section 3 introduces the critical connectivity condition of network topologies for consensus. Section 4 investigates the fastest convergence rates with respect to the best and worst topologies, respectively. In Section 5 we present a consensus
condition for random network topologies and an application to mobile ad-hoc networks. Finally, we conclude this paper with discussions on the main findings of this paper.

2 Preliminaries

2.1 Definitions in Graph Theory

Let $G = \{\mathcal{V}, E, A\}$ be a weighted digraph, where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the set of nodes with node $i$ representing the $i$th agent, $E$ is the set of edges, and $A \in \mathbb{R}^{n \times n}$ is the weight matrix. An edge in $G$ is an ordered pair $(j, i)$, and $(j, i) \in E$ if and only if the $i$th agent can receive information from the $j$th agent directly. The neighborhood of the $i$th agent is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} | (j, i) \in E\}$. An element of $\mathcal{N}_i$ is called a neighbor of $i$. $G$ is called an undirected graph when all of its edges are bidirectional, which means $j \in \mathcal{N}_i$ if and only if $i \in \mathcal{N}_j$. Let $a_{ij} > 0$ denote the weight of the edge $(j, i)$. The weight matrix $A$ is defined by $A_{ij} = a_{ij}$ for $(j, i) \in E$, and $A_{ij} = 0$ otherwise.

For graph $G$, the in-degree of $i$ is defined as $\text{deg}^\text{in}_i = \sum_{j \in \mathcal{N}_i} a_{ij}$ and the out-degree of $i$ is defined as $\text{deg}^\text{out}_i = \sum_{j \in \mathcal{N}_i} a_{ji}$. If $\text{deg}^\text{in}_i = \text{deg}^\text{out}_i$ for all $1 \leq i \leq n$, we call $G$ a balanced graph. The Laplacian matrix of $G$ is defined by $L_G = D_G - A_G$, where $D_G = \text{diag}(\text{deg}_1, \ldots, \text{deg}_n)$, and $[A_G]_{ij}$ equals to $a_{ij}$ if $j \in \mathcal{N}_i$ and 0 otherwise.

A sequence $(i_1, i_2, \ldots, i_k)$ of edges is called a directed path from node $i_1$ to node $i_k$. $G$ is called a strongly connected digraph, if for any $i, j \in \mathcal{V}$, there is a directed path from $i$ to $j$. A strongly connected undirected graph is also called a connected graph. For graphs $G(t) = \{\mathcal{V}, E(t), A(t)\}$, $i \leq t < j$, their union is defined by $\bigcup_{i \leq t < j} G(t) := \{\mathcal{V}, \bigcup_{i \leq t < j} E(t), A(t)\}$. Note that there may exist multiple weighted edges from one vertex to another in $\bigcup_{i \leq t < j} G(t)$.

2.2 Consensus Protocol

This paper considers a discrete-time first-order system containing $n$ agents, where agent $i$’s state $x_i(t)$ is updated by

$$x_i(t+1) = x_i(t) + u_i(t), \quad t = 1, 2, \ldots \quad (1)$$

Here $u_i(t)$ is the control input of the $i$th agent. For simplicity, we suppose $x_i(t)$ and $u_i(t)$ are scalars. As mentioned above, this paper will investigate a basic stochastic approximation algorithm for average-consensus, that is, the control $u_i(t)$ is chosen by

$$u_i(t) = a(t) \sum_{j \in \mathcal{N}_i(t)} a^{ij}_{ij} [x_j(t) - x_i(t) + w_{ji}(t)], \quad (2)$$

where $a(t) \geq 0$ is the common gain control at time $t$, $\mathcal{N}_i(t)$ is the neighbors of node $i$ at time $t$, $a^{ij}_{ij}$ is the weight of the edge $(j, i)$ at time $t$, and $w_{ji}(t)$ is the noise of agent $i$ receiving information from agent $j$ at time $t$. Throughout this paper, we assume

$$1 \leq a^{ij}_{ij} \leq a_{\max}, \quad 1 \leq i, j \leq n.$$

This kind of protocols was investigated in several papers [30, 35, 37, 39] with applications to distributed computation of wireless sensor networks and formation control of multiple satellites, vehicles, or robots.

We define the $\sigma$-algebra generated by the noises $w_{ji}(k)$, $1 \leq k \leq t$, $1 \leq i \leq n$, $j \in \mathcal{N}_i(k)$ by

$$F_t = \sigma(w_{ji}(k), 1 \leq k \leq t, 1 \leq i \leq n, j \in \mathcal{N}_i(k)).$$

The probability space of system (1)-(2) is $(\Omega, \mathcal{F}_\infty, \mathbb{P})$. $\mathcal{G}(t) = (\mathcal{V}, E(t), A(t))$ represents the topology of system (1)-(2) at time $t$, where $E(t) = \{(j, i) | j \in \mathcal{N}_i(t)\}$ is the edge set of $\mathcal{G}(t)$, and $A(t)$ is the weight matrix satisfying $A_{ij}(t) = a^t_{ij}$ if $(j, i) \in E(t)$ and $A_{ij}(t) = 0$ otherwise. The corresponding topology sequence is $\{\mathcal{G}(t)\}_{t \geq 1} = \{\{\mathcal{V}, E(t), A(t)\}\}_{t \geq 1}$. For simplicity, we use $L(t) = L_{G(t)}$ for the Laplacian matrix of $G(t)$. Let $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))'$ where $z'$ denotes the transpose of $z$. The protocol (1)-(2) can be rewritten as the following matrix form:

$$x(t+1) = [I - a(t)L(t)]x(t) + a(t)\tilde{w}(t), \quad t \geq 1, \quad (4)$$

where $\tilde{w}(t) \in \mathbb{R}^n$ whose $i$-th element is $\sum_{j \in \mathcal{N}_i(t)} a^{ij}_{ij}w_{ji}(t)$.

In this paper, we assume that there is no central controller who knows the global information of the evolution of the system, and that the so-called consensus control requires to design off-line gains $a(t)$ such that all agents achieve an agreement on their states in mean square sense, when $t \rightarrow \infty$. The mean square consensus is defined as follows.

**Definition 2.1** [39] We say the system (1)-(2) reaches mean square consensus if (i) there exists a random variable $x^*$ satisfying $E(|x^*|) < \infty$ and $\text{Var}(x^*) < \infty$ such that

$$\lim_{t \rightarrow \infty} E\|x(t) - x^*\|^2 = 0,$$

where $\mathbf{1} \in \mathbb{R}^n$ is the column vector of all 1s, and (ii) reaches unbiased mean square average-consensus if in addition, $x^*$ satisfies $E(x^*) = 1/n \sum_{i=1}^n x_i(1)$.

2.3 Standard Notation

The following standard mathematical notation will be used in this paper. Given a random variable $X$, let $E[X]$ and $\text{Var}(X)$ be its expectation and variance, respectively. For a vector $Y$, let $Y_i$ denote its $i$th entry. For a real number $x$, $\lfloor x \rfloor$ is the maximum integer less than or equal to $x$, and $\lceil x \rceil$ is the smallest integer larger than or equal to $x$. Let $\| \cdot \|$ denotes the $l_2$-norm (Euclidean norm). Given two sequences of positive numbers $g_1(t), g_2(t)$,

- $g_1(t) = O(g_2(t))$ if there exists a constant $c > 0$ and a value $t_0 > 0$ such that $g_1(t) \leq cg_2(t)$ for all $t \geq t_0$.
- $g_1(t) = \Theta(g_2(t))$ if there exist two constants $c_2 > c_1 > 0$ and a value $t_0 > 0$ such that $c_1g_2(t) \leq g_1(t) \leq c_2g_2(t)$ for all $t \geq t_0$.
- $g_1(t) = o(g_2(t))$ if $\lim_{t \rightarrow \infty} g_1(t)/g_2(t) = 0$. 
3 Critical Connectivity for Consensus

This section provides some consensus conditions for system (1)-(2). In Subsection 3.1, we will propose a new condition concerning the connectivity, while the sufficient and necessary conditions of consensus are given in Subsections 3.2 and 3.3 respectively.

3.1 Extensible Joint-connectivity

The uniform joint-connectivity of the topologies is a widely used condition in the consensus research of multi-agent systems. However, this condition is not robust for some situations. For example, if the links in a networked system have a positive probability of failure, it can be computed that with probability 1 the uniform joint-connectivity condition is not satisfied. Also, this condition cannot be satisfied in some flocking models [49]. To accommodate practical uncertainties in networked systems, we propose a new condition for topologies, called extensible joint-connectivity, as follows:

(A1) There exist constants \( \delta \geq 0 \) and \( c \geq 1 \), and an infinite sequence \( 1 = t_1 < t_2 < t_3 < \cdots \) such that \( t_k \leq t_{k-1} + c \delta \) and \( \bigcup_{t_{k-1} \leq \varepsilon < t_k} G(t) \) is strongly connected for all \( k > 1 \).

In (A1), we call \( \delta \) the extensible exponent for the joint-connectivity of the topologies. For any \( \delta \), (A1) is stronger than the infinite joint-connectivity assumption, which can be formulated by \( \bigcup_{t \geq k} G(k) \) being strongly connected for all \( k \geq 1 \). On the other hand, for any positive \( \delta \), (A1) is weaker than the uniform joint-connectivity assumption. In fact, the uniform joint-connectivity is a special case of (A1) with \( \delta = 0 \).

Remark 3.1 Compared to the uniform joint-connectivity, one advantage of the extensible joint-connectivity is that it can be used to analyze systems with random topology, even if the probability of connectivity of the topology is not stationary and decays in a negative power rate. In this case (by [59]) in Appendix D) with probability 1 there exists a finite time \( T > 0 \) such that (A1) is satisfied for all \( t \geq T \). This property has been applied to distributed consensus computation of mobile ad-hoc networks in Subsection 5.7, where the probability of successful communications between two agents depends on their distance.

3.2 Sufficient Conditions for Consensus

We first give a key lemma deduced from [59]. Before the statement of this key lemma some definitions are needed. For protocol (4), define

\[ \Phi(t, i) := [I - a(t)L(t)] \cdots [I - a(i)L(i)]. \]

Take \( \prod_{i=1}^{t} \Phi(i) := I \) when \( t < i \). For any \( x \in \mathbb{R}^n \), let \( x_{\text{ave}} := \frac{1}{n} \sum_{i=1}^{n} x_i \) be the average value of \( x \), and define

\[ V(x) := \|x - x_{\text{ave}}\|^2 = \sum_{i=1}^{n}(x_i - x_{\text{ave}})^2. \]

For an integer sequence \( \{t_k\}_{k \geq 1} \), denote

\[ k^t := \min \{k : t_k \geq t \} \quad \text{and} \quad k^l := \max \{k : t_k - 1 \leq t \}. \]

Set

\[ d_{\text{max}} := \sup_{t} \sum_{j \in \mathcal{N}(t)} a_{ij}^t \leq (n-1)d_{\text{max}}. \]

Also, following the common practice [21, 27, 30, 36, 38, 39], we focus on balanced topologies on system (1)-(2).

(A2) The topology \( \mathcal{G}(t) \) is balanced for all \( t \geq 1 \).

According to the definition of balancedness, if \( \{\mathcal{G}(t)\} \) is undirected and the weight matrix \( A(t) \) is symmetric for all \( t \geq 1 \), then \( \{\mathcal{G}(t)\} \) are all balanced.

Under (A1) and (A2) we have the following lemma, whose proof is postponed to Appendix A.

Lemma 3.1 Suppose that (A1) and (A2) are satisfied. Let \( z(t) = \Phi(t, i + 1)z(i) \) for \( t > i \). If \( a(t) \in (0, 1/d_{\text{max}}) \) then

\[ V(z(t)) \leq V(z(i)) \prod_{t=k}^{l} \left(1 - \delta_t(1 - \epsilon_t)^2\right), \]

where \( \delta_t = \min_{t \leq \varepsilon < t_{k+1}} a(t) \) and \( \epsilon_t = \min_{t \leq \varepsilon < t_{k+1}} (1 - a(t)d_{\text{max}}) \).

We also characterize robustness of protocol (1)-(2) with respect to noise. This will be accomplished by accommodating a large class of noises as specified below.

For any random variables \( X \) and \( Y \), let \( \text{Corr}(X, Y) := \frac{E(XY) - EXEY}{\sqrt{\text{Var}X \text{Var}Y}} \) denote the linear correlation coefficient between \( X \) and \( Y \). Following [50], we employ the notion of \( \rho \)-mixing sequences of random variables. Let \( \{X_i\}_{i \geq 1} \) be a random variable sequence. For any subset \( S, T \subset \mathbb{N} \), the sub \( \sigma \)-algebra \( \mathcal{F}_S := \sigma(X_i, i \in S) \) and

\[ \rho(\mathcal{F}_S, \mathcal{F}_T) := \sup \{\text{Corr}(X, Y) : X \in L_2(\mathcal{F}_S), Y \in L_2(\mathcal{F}_T)\}. \]

Define the \( \tilde{\rho} \)-mixing coefficients by

\[ \tilde{\rho}(m) := \sup \left\{ \rho(\mathcal{F}_S, \mathcal{F}_T) : \right\} \]

finite sets \( S, T \subset \mathbb{N} \) such that \( \min_{i \in S, j \in T} |i - j| \geq m \)

for any integer \( m \geq 0 \). By definition, \( 0 \leq \tilde{\rho}(m+1) \leq \tilde{\rho}(m) \leq 1 \) for all \( m \geq 0 \), and \( \tilde{\rho}(0) = 1 \) except for the special case when all \( X_i \) are degenerate.

Definition 3.1 A sequence of random variables \( \{X_i\}_{i \geq 1} \) is said to be a \( \tilde{\rho} \)-mixing sequence if there exists an integer \( m > 0 \) such that \( \tilde{\rho}(m) < 1 \).

Under this definition, we give the following assumption for protocol (1)-(2).

(A3) For any network topology sequence \( \{\mathcal{G}(t)\}_{t \geq 1} \), the noise sequence \( \{w_{ji}(t)\}_{t \geq 1, i=1, \ldots, n, j \in \mathcal{N}(t)} \) is a zero-mean \( \tilde{\rho} \)-mixing sequence satisfying \( v := \sup_{t \geq 1} \text{Var}(w_{ji}(t)) < \infty \).

Remark 3.2 It is well known that \( \tilde{\rho} \)-mixing noises include as special cases \( \phi \)-mixing noises [22], i.d. noises and martingale difference noises, see [51].

A basic property of \( \tilde{\rho} \)-mixing sequences is cited here.
Lemma 3.3 Suppose that (A1) is satisfied with \( \delta \leq 1/2 \). Then for any constant \( c_1 > 0 \) and integer \( t^* \geq 0 \),

\[
\prod_{j=k^*}^{k-1} \left( 1 - \frac{c_1}{t_j^{1-\delta} + t^*} \right) \leq \left( \frac{t_1^{1-\delta} + 2c + t^*}{(t + 1)^{1-\delta} + t^*} \right)^{\frac{2c_c}{c_1}} \]

and

\[
\prod_{j=k^*}^{k-1} \left( 1 - \frac{c_1}{(t_j^{1-\delta} + t^*)/(\log t_j + 1 + t^*)} \right) \leq \left( \frac{\log([2c + t_1^{1-\delta} + t^*])}{\log((t + 1)^{1-\delta} + t^*)} \right) \left( \frac{2c_c}{c_1} \right) \]

where \( c \) is the same constant appearing in (A1).

The proof of Lemma 3.3 is in Appendix A. The following theorem presents a sufficient condition for consensus.

Theorem 3.1 Suppose that (A1) is satisfied with \( \delta \leq 1/2 \), and (A2) and (A3) hold. Then for any initial state \( x(1) \), there exists an open-loop control of the gain sequence \( \{a(t)\} \) such that the system (1) reaches unbiased mean square average-consensus, with a convergence rate \( E[V(x(t))] = O(1/t^{1-2\delta}) \) if \( \delta < 1/2 \), and \( E[V(x(t))] = O(1/\log t) \) if \( \delta = 1/2 \).

Proof Case I: \( \delta < 1/2 \). Choose \( a(t) = \frac{\alpha}{\pi t^* + t} \) with

\[
\alpha \geq \frac{32n(n-1)^2c}{(2n-3)^2} \quad \text{and} \quad t^* \geq 2\alpha(n-1)a_{\max}.
\]

First we get \( a(t) \leq \frac{\alpha}{2\alpha(n-1)a_{\max}} \leq \frac{1}{2a_{\max}} \), which indicates \( I - a(t)L(t) \) is a nonnegative matrix for any \( t \geq 1 \). Recall that

\[
\Phi(t, i) = [I - a(t)L(t)] \cdots [I - a(i)L(i)]
\]

and \( \prod_{j=i}^{t-1} (\cdot) = I \) for \( t < j \). Using (4) repeatedly, we get

\[
x(t + 1) = \Phi(t, 1)x(1) + \sum_{i=1}^{t} a(i)\Phi(t, i + 1)\tilde{w}(i).
\]

Take \( \pi = (\frac{1}{n}, \ldots, \frac{1}{n}) \in \mathbb{R}^n \). By (A2) we have \( \pi L(t) = 0 \), and hence \( \pi\Phi(t, i) = \pi \). Take

\[
Y(i) = a(i) [\Phi(t, i + 1)\tilde{w}(i) - (\pi\tilde{w}(i)) I] \in \mathbb{R}^n.
\]

Then by (8), we have

\[
x(t + 1) - x_{\text{ave}}(t + 1)I = x(t + 1) - (\pi x(t + 1))I = \Phi(t, 1)x(1) - (\pi x(1))I + \sum_{i=1}^{t} a(i) [\Phi(t, i + 1)\tilde{w}(i) - (\pi\tilde{w}(i)) I]
\]

\[
= \Phi(t, 1)x(1) - (\pi\Phi(t, 1)x(1)) I + \sum_{i=1}^{t} Y(i).
\]

Because \( V(x) = \|x - (\pi x)I\|^2 \), by (A3) and (10) we have

\[
E[V(x(t + 1))] = V(\Phi(t, 1)x(1)) + E\left[ \sum_{i=1}^{t} Y(i) \right]^2.
\]

With Lemma 3.3 we have

\[
E\left[ \sum_{i=1}^{t} Y(i) \right]^2 = E\left[ \sum_{j=1}^{n} \sum_{i=1}^{t} Y_j(i) \right]^2 \leq O\left( \sum_{j=1}^{t} \sum_{i=1}^{t} EY_j^2(i) \right) = O\left( E[\|Y(i)\|^2] \right)
\]

\[
\leq O\left( \sum_{i=1}^{t} \alpha^2(i)E[|V(\Phi(t, i + 1)\tilde{w}(i))|] \right).
\]

By Lemma 3.1 (7) and (5) we have for any \( x \in \mathbb{R}^n \),

\[
E[V(\Phi(t, i + 1)\tilde{w}(i))]
\]

\[
\leq E[V(\tilde{w}(i))\prod_{j=k}^{k-1} \left( 1 - \frac{2c}{(2n-1)} \right)^2 \frac{a(t_{j+1})}{2(n-1)^2}]
\]

\[
\leq E[V(\tilde{w}(i))\prod_{j=k}^{k-1} \left( 1 - \frac{4c}{(t_j^{1-\delta} + t^*)} \right)^2]
\]

\[
< E[V(\tilde{w}(i))\left( \frac{2c + t_1^{1-\delta} + t^*}{(t + 1)^{1-\delta} + t^*} \right)^2].
\]

Because \( E[V(\tilde{w}(i))] \) is bounded, from (12) and (13) we get

\[
E\left[ \sum_{i=1}^{t} Y(i) \right]^2 = O\left( \frac{1}{((t + 1)^{1-\delta} + t^*)^2} \right)
\]

\[
\cdot \sum_{i=1}^{t} \left( \frac{2c + t_1^{1-\delta} + t^*)^2}{(t_1^{1-\delta} + t^*)^2} \right) = O\left( \frac{1}{t^{1-2\delta}} \right).
\]

Also, similar to (13) we get \( V(\Phi(t, 1)x(1)) = O(1/t^{1-2\delta}) \), so taking (14) into (11) yields \( E[V(x(t))] = O(1/t^{1-2\delta}) \).

It remains to evaluate the limit of \( x_{\text{ave}}(t) \). Let

\[
x^* = x_{\text{ave}}(\infty) = \pi x(\infty) = \pi x(1) + \sum_{i=1}^{\infty} a(i)\pi\tilde{w}(i).
\]

By (15) we obtain

\[
Ex^* = \pi x(1) + \sum_{i=1}^{\infty} a(i)\pi\tilde{w}(i) = \pi x(1).
\]
Also, by Lemma 3.2 we have
\[
\text{Var}(x^*) = E \left[ \sum_{i=1}^{\infty} a(i) \pi \hat{w}(i) \right]^2 = O \left( \sum_{i=1}^{\infty} a^2(i) \right) = O \left( \sum_{i=1}^{\infty} \frac{\alpha^2}{(t^{i-\delta}+t^{*})^2} \right) < \infty,
\]
so by Definition 2.1 the system (1-2) reaches unbiased mean square average-consensus.

Case II: \( \delta = 1/2 \). Choose \( \alpha(t) = \frac{\alpha}{(1+t^{1/2} \log(t+t^{*}))} \) with
\[
\alpha \geq \frac{64n(n-1)^4 c}{(2n-3)^2} \quad \text{and} \quad t^* \geq \left\lbrack 2n(n-1)a_{\text{max}} \right\rbrack.
\]
With (6), similar to (13) we get
\[
E \left[ V(\Phi(t, i+1) \hat{w}(i)) \right] < E \left[ V(\hat{w}(i)) \right] \left( \frac{\log[2c + \sqrt{t + t^*}]}{\log[\sqrt{t + 1} + t^*]} \right)^2,
\]
so similar to (13) we have
\[
E \left[ \sum_{i=1}^{t} \left( \frac{Y(i)}{2c + \sqrt{t + t^*}} \right)^2 \right] = O \left( \frac{1}{\log t} \right).
\]

Similar to Case I, taking (13) into (11) yields \( E[V(x(t))] = O(1/\log t) \). Also, because \( \sum_{i=1}^{\infty} \frac{1}{i^{1/2} \log(i)} = \infty \) (Subsection 1.3.9 in [53]) we get \( \text{Var}(x^*) = O(\sum_{i=1}^{\infty} a^2(i)) < \infty \). With the same discussion as Case I the system (1-2) reaches unbiased mean square average-consensus. \( \square \)

3.3 Necessary Conditions for Consensus

Let \( G_1 = (V, E_1) \) be an undirected complete graph, which means each vertex can receive the information of all the others. Let \( G_2 = (V, E_2) \) be the graph which has only one undirected edge between vertexes 1 and 2 without any other edges. For both \( G_1 \) and \( G_2 \), we assume the weights of their edges are all equal to 1. Thus, the corresponding Laplacian matrices for \( G_1 \) and \( G_2 \) are
\[
L_1 = nI - I I' \in \mathbb{R}^{n \times n} \quad \text{and} \quad L_2 = \begin{pmatrix}
1 & -1 & 0 & \ldots & 0 \\
-1 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & 0
\end{pmatrix} \in \mathbb{R}^{n \times n}.
\]
respectively. Define \( v_1 := n^{-1/2} I \in \mathbb{R}^n \), \( v_2 := \frac{1}{\sqrt{2}} (1, -1, 0, \ldots, 0)' \in \mathbb{R}^n \), and
\[
v_i := (i^2 - i)^{-1/2} (1, 1, 1, i, 0, 0, \ldots, 0)' \in \mathbb{R}^n
\]
for \( i \in [3, n] \). It is easy to compute that: \( v_i^tv_j = 0 \) for \( i \neq j \); \( L_1 v_i = 0; L_1 v_j = n v_i \) for \( i \geq 2 \); \( L_2 v_2 = 2 v_2 \) and \( L_2 v_1 = 0 \) for \( i \neq 2 \). From this we have the following proposition:

**Proposition 3.1** Let \( P := (v_1, v_2, \ldots, v_n) \in \mathbb{R}^{n \times n} \). Then \( P^tP = I \), \( P \text{diag}(0, n, n, \ldots, n)P' = L_1 \), and \( P \text{diag}(0, 2, 0, \ldots, 0)P' = L_2 \).

The necessary condition says for any gain sequence, the consensus may not be reached when the extensible exponent \( \delta \) is larger than 1/2 under the following noise condition:

**A4** For any network topology sequence \( \{G(t)\}_{t \geq 1} \), assume the noises \( \{w_{ij}(t)\} \) are zero-mean random variables satisfying: i) \( E[w_{ij}(t_1)w_{jk}(t_2)] = 0 \) for any \( t_1 \neq t_2 \), \( (j,i) \in E(t_1) \), and \( (j,k) \in E(t_2) \); ii) there exist constants \( 0 < \underline{\sigma} \leq \bar{\sigma} \) such that for any non-empty edge set \( E(t) \) and real numbers \( \{c_{ij}\} \),
\[
\frac{1}{\underline{\sigma}} \sum_{(j,i) \in E(t)} c_{ij}^2 \leq E \left[ \sum_{(j,i) \in E(t)} c_{ij} w_{ij}(t) \right]^2 \leq \bar{\sigma} \sum_{(j,i) \in E(t)} c_{ij}^2.
\]

**Theorem 3.2** Suppose the noise satisfies (A4). Then for any constant \( \delta^* > 1/2 \), any non-consensus initial state, and any gain sequence \( \{a(t)\}_{t \geq 1} \), there exists at least one topology sequence \( \{G(t)\}_{t \geq 1} = \{\{V, E(t), A(t)\}\}_{t \geq 1} \) satisfying (A1)-(A2) with \( \delta = \delta^* \), such that system (7-10) cannot achieve any mean square consensus.

**Proof** The main idea of this proof is: Choose \( t_k = t_{k-1} + \frac{\delta}{t_{k-1}} \). Let \( \{a(t)\}_{t \geq 1} \) be an arbitrary gain sequence. For any \( k \geq 1 \) and \( t_k \leq t < t_{k+1} \), select \( G(t) \) to be \( G_1 \) if \( a(t) \) is the minimal value of \( \{a(s), t_k \leq s < t_{k+1} \} \), and to be \( G_2 \) otherwise. It can be verified that our choice satisfies both (A1) and (A2). With Proposition 3.1 we conclude that system (7-10) cannot achieve consensus in mean square. The detailed proof is in Appendix B. \( \square \)

3.4 A Critical Condition for Consensus

The consensus conditions of the first-order average-consensus protocols with deterministic topologies and additive noises have been investigated recently. However, the best known condition on topology for consensus to date is the uniform joint-connectivity [30, 37, 39]. On the other hand, if this type of protocols contains no noise, they can reach consensus under a much relaxed infinite joint-connectivity condition [29]. There exists a huge gap between these two consensus conditions. This paper proposes an extensible joint-connectivity condition which is an intermediate condition between the uniform joint-connectivity and infinite joint-connectivity. Under our new condition we investigate a basic problem: what is the critical extensible exponent under which we can control the system to reach consensus? Note that there does not exist a central controller who knows the global information during the system’s evolution, and the consensus control is defined by designing off-line gains \( a(t) \) such that all the agents achieve the same final state.

Let \( \mathcal{G} \) be the set of topology sequences satisfying (A1) and (A2), and \( \mathcal{W} \) be the set of noise sequences satisfying (A3).

**Theorem 3.3** If and only if \( \delta \leq 1/2 \), where \( \delta \) is the extensible exponent appearing in (A1), there exists an open-loop control of the gain sequence \( \{a(t)\} \) such that system (7-10) reaches unbiased mean square average-consensus for any topology
sequence \( \{G(t)\} \in \mathcal{G} \), any noise sequence \( \{w_{ji}(t)\} \in \mathcal{W} \), and any non-consensus initial state.

**Proof** This follows immediately from Theorems 3.1 and 3.2 since any noise satisfying (A4) must satisfy (A3). \( \square \)

**Remark 3.3** The balancedness of network topologies can guarantee that the expectation of the final consensus value is equal to the average value of the initial states \( \frac{1}{n} \sum_{i=1}^{n} x_i(1) \). In addition, by (17), the variance of the consensus value can be arbitrarily small if we choose \( t^* \) to be large enough. Overall, we can control the final consensus value to be arbitrarily close to the average value \( \frac{1}{n} \sum_{i=1}^{n} x_i(1) \).

**Remark 3.4** Without assumption (A2), Theorem 3.3 should still hold if one replaces unbiased mean square average-consensus by mean square consensus. However, its proof is quite difficult because it is related to a well-known conjecture in the field of probability that the convergence rate of a general inhomogeneous Markov chain is a negative exponential function. This conjecture was formulated as Problem 1.1 in [34]. We remark that the reference [33] obtained the convergence under the uniformly joint-connectivity by the classical infinitesimal analysis which cannot be used to obtain convergence rates or analyze the critical connectivity condition. Currently, almost all papers concerning convergence speeds of the distributed consensus protocol with time-varying topologies assume that the topologies are undirected, or balanced, or have a common stationary distribution [27, 28, 34, 35, 36, 37, 38, 39, 44].

## 4 Fastest Convergence Rates of Consensus

This section establishes bounds on the fastest convergence rate to the unbiased mean square average-consensus among all gain functions under unknown switching topologies. Different from the noise-free systems [22, 23, 45], \( x^* \) in Definition 2.1 is a random variable whose value is uncertain. Also, if system (1) reaches consensus in mean square, it must be true that \( \lim_{t \rightarrow \infty} E[V(x(t))] = 0 \), so we use \( E[V(x(t))] \) to measure the convergence rate to consensus instead of \( E[\|x(t) - x^*\|^2] \). In this paper the fastest convergence rate of consensus at time \( t \) is the minimal value of \( E[V(x(t))] \) among all controls \( a(1) \geq 0, a(2) \geq 0, \ldots, a(t-1) \geq 0 \) for a topology sequence \( G(1), \ldots, G(t-1) \). This rate depends on the time-varying topologies, however, our protocol assumes each node only knows its local information and the global topology information is unknown. As a result, its exact value cannot be obtained. A simplified notion of convergence rate will be first defined. Let

\[
\rho_1(t) := \inf_{a(1) \geq 0, \ldots, a(t-1) \geq 0} E[V(x(t))] \tag{19}
\]

be the fastest convergence rate under the best topologies. Here we recall that \( \{G(t)\}_{t \geq 1} = \{V, E(t), A(t)\}_{t \geq 1} \) is the topology sequence and note that \( \rho_1(t) \) depends on the noises and the initial state \( x(1) \).

We also consider the fastest convergence rate under the worst topologies. Define \( \mathcal{G}_{\delta, c} \) as the set of topology sequences satisfying (A1)-(A2), where \( \delta, c, \) are the constants appearing in (A1). Let

\[
\rho_2(t) := \inf_{a(1) \geq 0, \ldots, a(t-1) \geq 0} \sup_{G(1), \ldots, G(t-1)} E[V(x(t))] \tag{20}
\]

denote the fastest convergence rate with respect to the worst topologies satisfying (A1)-(A2). We note that \( \rho_2(t) \) depends on \( \delta, c, \) the noises and the initial state.

By the definitions of \( \rho_1(t) \) and \( \rho_2(t) \) we have for any topology sequences satisfying (A1)-(A2), its corresponding fastest convergence rate will be neither faster than \( \rho_1(t) \) nor slower than \( \rho_2(t) \) provided that the initial state and noises are same. Theorem 3.1 gives a upper bound for \( \rho_2(t) \), and in the following subsection we will consider the lower bounds for \( \rho_1(t) \) and \( \rho_2(t) \).

### 4.1 Lower Bounds

In this subsection we will give lower bounds on \( \rho_1(t) \) and \( \rho_2(t) \), respectively under (A4). The lower bounds on the fastest convergence rate indicate that for any control the convergence rate will not be faster than them. Before the estimation of \( \rho_1(t) \) we need introduce the following lemma:

**Lemma 4.1** Let \( L \in \mathbb{R}^{n \times n} \) be the Laplacian matrix of any weighted directed graph. Then for any \( x \in \mathbb{R}^n \) and constant \( a > 0 \) we have \( V((I - aL)x) \geq (1 - a\lambda_{\max}(L + L'))V(x) \), where \( \lambda_{\max}(\cdot) \) denotes the largest eigenvalue.

The proof of this lemma is in Appendix A.

The following theorem gives a lower bound of \( \rho_1(t) \).

**Theorem 4.1** Suppose that the noises satisfy (A4). Then for any non-consensus initial state, under protocol (7)-(2) there exists a constant \( c' > 0 \) such that \( \rho_1(t) \geq c'/t \) for all \( t \geq 1 \).

**Proof** For any \( t > 1 \), we only need to consider the case of \( E(k) \) is not empty for all \( 1 \leq k \leq t \), since if \( E(k) \) is empty then \( x(k+1) = x(k) \), which results in the waste of the time step.

First, because \( \hat{w}_i(k) = \sum_{j \in N_i(k)} a_{ij}^k w_{ji}(k) \) with \( a_{ij}^k \geq 1 \), by (A4) there exists a constant \( c_1 > 0 \) such that

\[
E[V(\hat{w}(k))] \geq c_1. \tag{21}
\]

Also, by Gershgorin’s circle theorem we have

\[
\lambda_{\max}(L(k) + L'(k)) \leq \max_{1 \leq i \leq n} \left( 2L_{ii}(k) + \sum_{j \neq i} |L_{ji}(k) + L_{ij}(k)| \right) \leq 4(n-1)\sigma_{\max} = c_2, \quad \forall k \geq 1. \tag{22}
\]

Let \( t^* \in [1, t+1] \) be the minimum time such that if \( k \geq t^* \) then \( a(k) < 1/c_2 \). By (11) and (A4) it can be computed that

\[
E[V(x(t+1))] = E[V(\Phi(t, t^*)x(t^*))] \tag{23}
\]

\[
+ \sum_{i=t^*}^{t} a^2(i)E[V(\Phi(t, i+1)\hat{w}(i))].
\]

If \( t^* = t + 1 \), we have \( a(t) \geq 1/c_2 \). Then by (23) and (21).

\[
E[V(x(t+1))] \geq a^2(t)E[V(\hat{w}(t))] \geq c_1/c_2^2.
\]
and the result follows.

Hence, we only need to consider the case $t^* \leq t$. By (22) and repeatedly using Lemma 4.1, we have

$$E[V(\Phi(t,t^*)x)] \geq E[V(x)] \prod_{j=1}^{t} (1 - c_2 a(j)).$$

Taking this into (23) yields

$$E[V(x(t + 1))] \geq E[V(x(t^*))] \prod_{j=1}^{t} (1 - c_2 a(j))$$

$$+ \sum_{i=t+1}^{t} a^2(i) E[V(\hat{w}(i))] \prod_{j=i+1}^{t} (1 - c_2 a(j)).$$

Let $I(\cdot)$ be the indicator function. Since

$$1 - c_2 a(j) \geq \left[ 1 - c_2 a(j) I(a(j) > \frac{1}{c_2}) \right] \cdot \left[ 1 - c_2 a(j) I(a(j) \leq \frac{1}{c_2}) \right],$$

and

$$\prod_{j=1}^{t} \left( 1 - c_2 a(j) I(a(j) \leq \frac{1}{c_2}) \right) \geq \left( 1 - \frac{1}{c_2} \right)^t \geq \frac{1}{c_2}.$$

From (24) and (21), we obtain

$$E[V(x(t + 1))] \geq \frac{1}{c_2} E[V(x(t^*))] \prod_{j=1}^{t} \left( 1 - c_2 a(j) I(a(j) > \frac{1}{c_2}) \right)$$

$$+ \frac{c_1}{c_2} \sum_{i=t+1}^{t} a^2(i) \prod_{j=i+1}^{t} \left( 1 - c_2 a(j) I(a(j) > \frac{1}{c_2}) \right).$$

It remains to discuss the value of the right side of (25). We first consider $E[V(x(t^*))]$. If $t^* = 1$ then $E[V(x(t^*))] = V(x(1))$. Otherwise, by the definition of $t^*$ we have $a(t^* - 1) \geq 1/c_2$. As a result, similar to (23) and by (21) we have

$$E[V(x(t^*))] \geq a^2(t^* - 1) E[V(\hat{w}(t^* - 1))] \geq \frac{c_2}{c_1}.$$ 

These lead to

$$E[V(x(t^*))] \geq \min \{ V(x(1)), c_1/c_2^2 \}.$$ 

(26)

Also, from $y_1 \geq y_2 I(y_1 > y_2)$ for any $y_1, y_2 \geq 0$, we get

$$\sum_{i=t+1}^{t} a^2(i) \prod_{j=i+1}^{t} \left( 1 - c_2 a(j) I(a(j) > \frac{1}{c_2}) \right)$$

$$\geq \sum_{i=t+1}^{t} \frac{a(i)}{c_2} \prod_{j=i+1}^{t} \left( 1 - c_2 a(j) I(a(j) > \frac{1}{c_2}) \right)$$

$$= \frac{1}{c_2t} \left( 1 - \prod_{j=t+1}^{t} \left( 1 - c_2 a(j) I(a(j) > \frac{1}{c_2}) \right) \right).$$

(27)

which can be obtained by induction. Here we recall that

$$\prod_{i=1}^{t} (\cdot) := 1 \text{ if } t < i.$$ 

Take $z_t = \prod_{i=1}^{t} (1 - c_2 a(j) I(a(j) > \frac{1}{c_2}))$. By substituting (27) into (25) we have

$$E[V(x(t + 1))] \geq \frac{1}{c_2} E[V(x(t^*))] z_t + \frac{c_1}{c_2t} (1 - z_t)$$

$$\geq \min \left\{ \frac{1}{c_2} E[V(x(t^*))] z_t, \frac{c_1}{c_2t} \right\}.$$ 

From this and (26) our result is obtained. □

For $\rho_2(t)$ we get the following lower bound, whose proof is in Appendix C.

**Theorem 4.2** Assume (A1) is satisfied with $\delta < 1/2$, and (A2) and (A4) hold. Then for any inconsistent initial state, under protocol (7)–(2) there exists a constant $c^* \in (0, 1)$ such that $\rho_2(t) \geq c^*/t^{1-2\delta}$.

### 4.2 Fastest Convergence Rates and Sub-optimal Open-loop Control

The convergence speed is one of the most important performances of distributed consensus algorithms for networked systems. Most existing work focuses on noise-free algorithms [21, 24, 26, 27, 43, 44] where the control gains $\{a(t)\}$ are constant. Among these, some try to maximize the convergence speed by optimizing weighted network topologies [22, 45].

There are some results considering convergence speed of distributed consensus algorithms with fixed topologies and additive noises [41, 42]. Nevertheless, it appears that our paper is the first to optimize the convergence rate of this type of protocols with time-varying network topologies and additive noises. It is noted that in our system each node only knows its own and neighbors’ information and the network topologies cannot be real-time controlled.

In this paper the fastest convergence rate of consensus at time $t$ is the minimal value of $E[V(x(t))]$ among all the gain functions $a(1) \geq 0, a(2) \geq 0, \ldots, a(t - 1) \geq 0$ which are the only controllable variables. Recall that $\rho_1(t)$ defined by (19) denotes the fastest convergence rate for the best topologies, and $\rho_2(t)$ defined by (20) denotes the fastest convergence rate for the worst topologies satisfying (A1)–(A2). With the same noise sequence it is clear that $\rho_1(t) \leq \rho_2(t)$ from their definitions.

**Theorem 4.3** Suppose that the noise satisfies (A4) and the initial state is not in consensus, then $\rho_1(t) = \Theta \left( \frac{1}{t} \right)$ under system (7)–(2).

**Proof** By the definitions of $\rho_1(t)$ and $\rho_2(t)$ we have $\rho_1(t) \leq \rho_2(t)$ with $\delta = 0$. By Theorem 4.1 we have $\rho_1(t) = O\left( \frac{1}{t} \right)$. Combining this with Theorem 4.1 yields our result. □

**Remark 4.1** We just evaluate the fastest convergence rate to the accurate order. In fact, it is conjectured that $\rho_1(t) = \frac{b_1}{t} (1 + o(1))$ under (A4), where $b_1$ is a constant depending on $n, a_{\text{max}}$ and $\nu$ only.
Theorem 4.4 Suppose that the topology sequence \( \{G(t)\} \) satisfies (A1)-(A2) with \( \delta < \frac{1}{2} \), the noises satisfy (A4), and the initial state is not in consensus. Then under system (14-22),

(i) \( \rho_2(t) = \Theta\left(\frac{1}{t}\right) \).

(ii) The unbiased mean square average-consensus will be reached with a rate \( O\left(\frac{1}{t^{\mu}}\right) \) by choosing \( \alpha(t) = \frac{\alpha}{1+t^\mu} \), where \( \alpha \) and \( t^\mu \) are two constants satisfying \( \Theta \).

Proof Since any noise satisfying (A4) must satisfy (A3), (i) follows immediately from Theorems 4.2 and 5.1 and (ii) follows immediately from the proof of Theorem 3.1. □

From Theorems 4.3 and 4.4 we get the following corollary.

Corollary 4.1 Suppose that the noise satisfies (A4) and the initial state is not in consensus. Then for system (14-22) with any balanced and uniformly jointly connected topology sequence, i) the fastest convergence rate \( \inf_{a(t) \geq 0, \ldots, a(t-1) \geq 0} E[V(x(t))] \) is \( \Theta\left(\frac{1}{t}\right) \); ii) the unbiased mean square average-consensus will be reached with a rate \( \Theta\left(\frac{1}{t}\right) \) by choosing \( \alpha(t) = \frac{\alpha}{1+t^\mu} \), where \( \alpha \) and \( t^\mu \) are two constants satisfying \( \Theta \).

Proof i) Let \( \rho_3(t) := \inf_{a(t) \geq 0, \ldots, a(t-1) \geq 0} E[V(x(t))] \) which depends on the topology sequence. First by (19) and Theorem 4.3 we can get \( \rho_3(t) \geq \rho_1(t) = \Theta\left(\frac{1}{t}\right) \). Also, because the balanced and uniformly jointly connected topology condition equals to the condition (A1)-(A2) with \( \delta = 0 \), by (20) and Theorem 4.4(i) we can get \( \rho_3(t) \leq \rho_2(t) = \Theta\left(\frac{1}{t}\right) \). Thus, we have \( \rho_2(t) = \Theta\left(\frac{1}{t}\right) \).

ii) It follows immediately from i) and Theorem 4.4(ii). □

5 Consensus under Non Stationary and Strongly Correlated Random Topologies

As mentioned in Remark 3.1, one advantage of the extensible joint-connectivity is that it can be used to analyze systems with random topologies compared to the uniform joint-connectivity condition. This is because with probability 1 there exists a finite time \( T > 0 \) such that random topologies satisfy (A1) for all \( t \geq T \), even if the topology processes are not stationary and strongly correlated. In this section we consider random network topologies \( \{G(t)\}_{t \geq 1} = \{(V, E(t), A(t))\}_{t \geq 1} \) satisfying:

(A1') There exist three constants \( K \in \mathbb{Z}^+ \), \( \mu \in (0, 1/2) \) and \( p > 0 \) such that for any \( t \geq 1 \),

\[
P\left(\bigcup_{t'=t}^{t+K-1} G(t') \text{ is strongly connected} \right)
= \prod_{i=1}^{n} \left(1 - \left(\frac{\delta}{2}\right)^{\alpha} \right) \geq pt^{-\mu} \log t.
\]

Remark 5.1 Assumption (A1') includes a wide class of non-stationary and strongly correlated random matrix sequence \( \{G(t)\} \), including as special cases the ergodic and stationary Markov processes used in [28,31,44].

Remark 5.2 The constant \( \mu \) in (A1') essentially corresponds to the extensible exponent \( \delta \) in (A1). Also, in practical applications this constant would be converted into a certain parameter of practical systems. For example, in mobile ad-hoc networks, because the probability of successful communication between two agents depends on their distance, (A1') can be translated into a limitation to the growth rate of the distance between agents, where \( \mu \) corresponds to a coefficient of this growth rate, see the following (32-33).

From Theorem 3.1 we obtain the following result for the case of random network topologies, whose proof is contained in Appendix D. An application of Theorem 5.1 is provided in the following subsection.

Theorem 5.1 Consider the system given by (14-22) with random network topologies satisfying (A1') and (A2). Assume that the noise satisfies (A3). Then from any initial state the system will reach unbiased mean square average-consensus with the convergence rate \( E[V(x(t))] = O\left(\frac{1}{t^{\mu}}\right) \) if one selects \( \alpha(t) = \frac{\alpha}{1+t^{\mu}} \), where \( \alpha \) and \( t^\mu \) are two constants satisfying \( \Theta \), and \( \mu \) is the same constant appearing in (A1').

5.1 Application to Mobile Ad-Hoc Networks

To investigate the distributed consensus protocol with random network topologies, the existing results assume that the topologies are either i.i.d. or stationary Markov processes [24, 28, 31, 40]. This assumption fits stationary wireless networks; but in mobile systems network topologies will no longer be stationary because communications between nodes depend on their distances. Different from the previous work, Theorem 5.1 treats non-stationary random network topologies and can be applied to distributed computation of mobile networks. For example, a mobile wireless sensor network or a multi-robot system needs to compute the average value of some data (such as temperature, humidity, light intensity, pressure etc.) measured by each node (or agent). Assume that the data of each agent \( i \) is encoded to a scalar \( x_i(0) \). We aim to design a distributed protocol to obtain the average value of \( x_i(0) \). Since communication packet delivery ratios between agents depend on their distances, we must take into consideration of agent movement.

We adopt the first-order average-consensus protocol as our communication protocol. Let \( x_i(t) \) be the state of agent \( i \) at time \( t \) which is initially set to be \( x_i(0) \). We assume that each agent periodically, with period \( T > 0 \), broadcasts its current state to all other agents. To reduce signal interference the agents are arranged to send information in different times. This leads to the following communication protocol.

In each period \( [iT, (i+1)T) \), \( i \geq 0 \), at the time \( iT + \frac{(i-1)T}{n} \), \( 1 \leq i \leq n \), agent \( i \) broadcasts its current state \( x_i(I + \frac{(i-1)T}{n}) = x_i(IT) := x_i^l \) to all other agents. The more bits its sending package contains, the more difficult this package is to be successfully received by other agents. Practical wireless systems only transfer a finite number of bits per transmission. Thus, the broadcasting signal is equal to \( x_i^l \) with a quantization noise \( \xi_i^l \). Each link \( (i, j) \) has a probability which depends the distance between agents \( i \) and \( j \) for successfully receiving the information

\[
R_{ij}^l = x_i^l + \xi_i^l + \xi_{ij}^l,
\]
where $c_{ij}^l$ denotes the possible reception error of agent $j$ that the error detection code in its received package cannot detect. After agent $j$ receives the information $P_{ij}^l$, it sends an acknowledgment to agent $i$ immediately. When the response signal reaches agent $i$, agent $i$ can know the response signal coming from agent $j$ according to its carrier frequency if each agent is assigned a distinct carrier frequency. This process does not have to decode the signal so we can assume that the response can be indeed received by agent $i$. Agent $i$ collects the response only in the interval $\left((n-1)T \pm \frac{T}{n}, nT + \frac{T}{n}\right)$. Because the above sending-reception-response process contains no retransmission, its total time is very short and then agent $i$ can receive all responses to itself in $\left((n-1)T \pm \frac{T}{n}, nT + \frac{T}{n}\right)$ if we choose $T$ to be a suitable real number. This fact indicates that each agent can collect all responses to itself, and will not wrongly collect the responses to others. Thus, each agent actually knows who receives its sending state. Define

$$N_i^l := \{j : \text{Agents } i \text{ and } j \text{ receive each other’s state in the time interval } [lT, (l+1)T)\},$$

to be the neighbor set of agent $i$. At the end of every period $[lT, (l+1)T)$ each agent updates its state by

$$x_i^{l+1} := x_i(l+1)T = x_i^l + a(l) \sum_{j \in N_i^l} (R_{ij}^l - x_j^l) = x_i^l + a(l) \sum_{j \in N_i^l} (x_j^l + c_{ij}^l + \xi_j^l - x_j^l) \quad (29)$$

for all $l \geq 0$ and $1 \leq i \leq n$, where the last line uses (28).

Let $G^l := (\mathcal{V}, \mathcal{E}^l)$ where $\mathcal{E}^l := \{(i, j) : j \in N_i^l\}$. Then $G^l$ is an undirected graph. Also, it is natural to assume that $\{\xi_j^l + c_{ij}^l\}$ is a zero-mean and bounded variance $\rho$-mixing sequence. Thus, according to Theorem 3.1 if the topologies $\{G^l\}_{l \geq 1}$ satisfy (A1’), we can use the open-loop control $a(l)$ such that system (29) reaches unbiased mean square average-consensus.

Next we consider the movement restriction guaranteeing that the topologies $\{G^l\}_{l \geq 1}$ satisfy (A1’). According to the log-normal shadowing model[55, 56], the probability of agent $i$ successfully receiving a data packet from agent $j$ can be approximated by

$$P_i^l(t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\alpha_j - \beta \log d_{ij}(t)} e^{-x^2} dx, \quad (30)$$

where $S_j(t)$ is the transmission signal strength of agent $j$ at time $t$, $L_i^l(t)$ is the path loss between agents $i$ and $j$ at time $t$, $R_{th}$ is a constant depending on the size of the data package, and $\delta$ is the standard deviation of a Gaussian random variable ($S_j(t), R_{th}$, and $\delta$ are measured in dBm, $L_i^l(t)$ is measured in dB).

For simplicity, we assume that the transmission signal strength $S_j(t) = S_j$ is a constant, and the path loss is estimated by the free-space path loss (FSPL) which is

$$L_i^l(t) = 32.45 + 20 \log d_{ij}(t) + 20 \log f_j,$$

where $d_{ij}(t)$ is the distance between agents $i$ and $j$ at time $t$ measured in kilometer, and $f_j$ is the carrier frequency of agent $j$ measured in MHz. Take $\alpha_j = \frac{1}{2\delta} S_j - 32.4 - 20 \log f_j - R_{th}$ and $\beta = 10\sqrt{2}/\delta$. Then equation (30) can be rewritten as

$$P_i^l(t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\alpha_j - \beta \log d_{ij}(t)} e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\alpha_j - \beta \log d_{ij}(t)} e^{-x^2} dx > \frac{1}{\sqrt{\pi}} \exp \left(-\frac{1}{\beta} \left(\beta \log d_{ij}(t) - \alpha_j + 1\right)^2\right).$$

From this expression, there exist two positive constants $c_1 = c_1(n)$ and $c_2 = c_2(n)$ such that

$$P(G^l \text{ is connected}) > c_1 e^{-U l - \log l} \left(\frac{\log \log l}{\log l}\right)^{\bar{c}_2} + c_2 \min(\alpha_i, 1), \quad (31)$$

where $d_{max} := \max_{i < j \leq n, t \leq (l+1)T} d_{ij}(t)$ and $\alpha_i := \min_{i < j \leq n} \alpha_j$. If there exist two constants $u \in (0, 1/2)$ and $U > 0$ such that

$$d_{max} \leq \exp \left(\sqrt{\frac{U}{\beta}} \log l - \log \log l + U \frac{\log \log l}{\log l} \right)^{\alpha_i} \left(\frac{\log \log l}{\log l}\right)^{\bar{c}_2} \quad (32)$$

for any $l \geq 1$, then from (31) we have

$$P(G^l \text{ is connected}) > c_1 e^{-U l - \log l} \log l, \quad \forall l \geq 1. \quad (33)$$

We assume that the topologies $\{G^l\}$ are mutually independent. Consequently, (33) implies that the topologies satisfy (A1’). Recall that $G^l$ is undirected and $\{c_{ij}^l + \xi_j^l\}$ is assumed to be a zero-mean and bounded variance $\rho$-mixing sequence. By Theorem 3.1 system (29) reaches unbiased mean square average-consensus if we choose a suitable $a(l)$. Also, by Remark 3.3 we can control the final consensus value to be arbitrarily close to the average value of the initial state.

Inequality (32) claims that to guarantee convergence to consensus the distance between agents cannot grow too fast. In fact, (32) can be satisfied if the velocity difference $\|V_i(t) - V_j(t)\| = O(\frac{1}{\sqrt{T}})$ for any $1 \leq i, j \leq n$, where $V_i(t)$ denotes the velocity of agent $i$ at time $t$. Of course, the consensus speed depends not only on the growth rate of the distance between agents but also the initial distance. On the other hand, if there exists a constant $b \in [0, 1)$ such that $\|V_i(t) - V_j(t)\| \geq \Theta(\frac{1}{\sqrt{T}})$, then inequality (32) may not be satisfied.

### 5.2 Simulations

We now perform simulations to evaluate distributed average-consensus of mobile ad-hoc networks. Assume that nine agents are moving in the plane with velocities $V(t) + v_i(t)$, $1 \leq i \leq 9, t \geq 0$, where $V(t)$ is the velocity of the whole group at time $t$, and $v_i(t)$ is the relative velocity of agent $i$. Suppose that the initial position of agent $i$ is $\left(\cos \frac{(i-1)\pi}{8}, \sin \frac{(i-1)\pi}{8}\right)$, and the heading of the relative velocity $v_i(t)$ of agent $i$ is the constant $\frac{(i-1)\pi}{8}$. The initial positions and headings of the relative velocities of all agents are shown in Fig. 11.
Each agent $i$ contains a state $x_i(t) \in \mathbb{R}$ which is initially set to be $\frac{t-1}{8}$.

We adopt the consensus protocol in Subsection 5.1 which means that the states of all agents are updated by (29). The period length $T$ is selected to be 1. Let $\delta = 1, \beta = 10\sqrt{2}$, and $\alpha_j = 4$ for $1 \leq j \leq 9$, where $\delta, \beta$, and $\alpha_j$ are the same constants appearing in Subsection 5.1 Assume that the quantization noise $\{\xi^i_j\}$ is independent and uniformly distributed in $[-0.1, 0.1]$, and the reception error $\{\zeta^i_l\}$ obeys a Gaussian distribution whose expectation is zero and standard deviation is 0.05.

We first simulate the case where the relative velocity magnitude $|v_i(t)|$ of agent $i$ equals $\frac{1}{t+200}$ for all $1 \leq i \leq 9$ and $t \geq 0$. In this case (32) holds for any $\mu > 0$ if we choose a suitable $U$. In consideration of the consensus speed and its variance, we select $a(t) = \frac{1}{(t+200)^{0.75}}$ according to Theorem 3.1 and Remark 3.3. Fig. 2 shows a simulation result in which the states of all agents converge to a consensus value close to $0.5 = \frac{1}{9} \sum_{i=1}^{9} x_i(0)$.

From this simulation it can be seen that the final states of the agents have a gap. If the relative velocity magnitude grows to $\frac{1}{(t+200)^{0.75}}$, the gap between the agents’ final states become more significant, see Fig. 4.

From above simulations it is conjectured that if the magnitude of the relative velocity of every agent is $\frac{1}{(t+200)^{0.75}}$, the critical value of $b$ equals 1 for consensus.

6 Conclusions

Consensus behavior of multi-agent systems has drawn substantial interests over the past two decades. However, some key problems remain unsolved, including the fastest convergence speeds and critical consensus conditions of connectivity on network topologies. This paper addresses these problems based on a first-order average-consensus protocol with switching topologies and additive noises. We first propose an extensible joint-connectivity condition on topologies. Using stochastic approximation methods and under our new condition, we establish a critical consensus condition for network topologies, and provide the fastest convergence rates with respect to the best and worst topologies. Our results give a quantitative description of the relation between convergence speed and connectivity of network topologies. Also, we give
a consensus analysis for our systems with non-stationary and strongly correlated stochastic topologies, and apply it to distributed consensus of mobile ad-hoc networks.

Appendices

Appendix A

Proof[Proof of Lemma 3.1] Take \( A_t = I - a(t) L(t) \). Using (A2) and the condition of \( a(t) \in (0, 1/d_{\text{max}}) \), we have that \( A_t \) is a doubly stochastic matrix. Also, we can compute

\[
(A_t' A_t)_{ij} \geq (1 - a(t)d_{\text{max}}) [(A_t)_{ij} + (A_t)_{ji}]
\]

for all \( t \geq 1 \), and from (A1) and (35) we have

\[
\min_{\theta \subseteq S \subseteq \{1, \ldots, r\}} \sum_{i \in S, j \in S^c} \sum_{t = t_i}^{t_{i+1} - 1} [(A_t)_{ij} + (A_t)_{ji}]
\]

With these it is deduced directly from the proof of Theorem 6 in [29] that

\[
V \left( z(t_{k+1} - 1) \right) = V \left( \Phi(t_{k+1} - 1, t_{k+1} - 2) \cdots \Phi(t_{k+1} - 1, t_{k+1} - 1) \right) 
\]

for all \( t_{k+1} \geq 1 \), so (35) holds for \( j = -1 \). Also, if (35) holds for \( j \geq -1 \), then

\[
t_{k_i + j + 1} \leq c(j + 1) + i + 1 - 1
\]

By the definition of \( k_i \) we have \( t_{k_i - 1} \leq i \), so (35) holds for all \( j \geq -1 \).

Proof[Proof of Lemma 3.3] First, we use induction to prove that for all integer \( j \geq 1 \),

\[
t_{k_i + 1} \leq c(j + 1) + i + 1 - 1
\]

From the definition of \( \tilde{k} \),

\[
k \leq 1 \frac{(t+1)^{1-\delta} - 1}{c} - 1 \leq \tilde{k}.
\]

This, together with (35) and the fact that \( \log(1 - x) < -x/2 \) for \( x \in (0, 1) \), implies

\[
\prod_{j=k_i}^{k_{i+1}-1} \left( 1 - \frac{c_1}{j+1 + t^*} \right) 
\]

Because for any \( a > 0 \) and integer \( b > 0 \),

\[
\sum_{k=0}^{b} \frac{1}{k + a} < \int_{0}^{b+1} dx = \log(b + 1 + a) - \log a,
\]

(36) is followed by

\[
\prod_{j=k_i}^{k_{i+1}-1} \left( 1 - \frac{c_1}{j+1 + t^*} \right) < \exp \left\{ \frac{-c_1}{2c} \left[ \log \frac{(t+1)^{1-\delta} + t^*}{c} - \log \frac{2 + i + 1 - \delta + t^*}{c} \right] \right\} 
\]

(37)

With the similar process from (35) to (37) we get (36).

Proof[Proof of Lemma 4.1] let \( y := (I - aL)x \), \( \bar{y} := y - y_{\text{ave}} \), \( \bar{x} = x - x_{\text{ave}} \) and \( \pi = \frac{1}{n} 1' \). Then

\[
\bar{y} = y - (\pi y) 1 = (I - aL)x - [\pi(I - aL)x] 1 = x - (\pi x) 1 - a [Lx - (\pi Lx) 1] = \bar{x} - a [L\bar{x} - (\pi L\bar{x}) 1],
\]

Combining this with \( x_{\text{ave}} 1 = 0 \) we get

\[
V(y) = \|\bar{y}\|^2 = \bar{y}' \bar{y} \geq \|\bar{z}\|^2 - a\bar{x}' [L\bar{x} - (\pi L\bar{x}) 1] - a [L\bar{x} - (\pi L\bar{x}) 1]' \bar{x} = \|\bar{z}\|^2 - a\bar{x}' (L + L') \bar{x} \geq \|\bar{z}\|^2 - a\lambda_{\text{max}}(L + L') \|\bar{z}\|^2 = (1 - a\lambda_{\text{max}}(L + L')) V(x).
\]

□
Appendix B  proof of Theorem 3.2

We will show system (1-2) cannot reach consensus in mean square by contradiction. Because to reach consensus \(a(t)\) must converge to 0, there exists an integer \(k_1\) such that \(a(t) < \frac{1}{2^n}\) for all \(t \geq t_{k_1}\). We take \(t^* = t_{k_1}\). Similar to (8), we get

\[
x(t + 1) = \Phi(t, t^*)x(t^*) + \sum_{i = t^*}^{t} a(i)\Phi(t, i + 1)\hat{w}(i).
\]  

(38)

We choose \(t_k = t_{k-1} + c_1 k^\delta\) and select

\[
G(t) := \begin{cases} 
G_1, & \text{if } t \in \cup_{k=1}^{\infty}(t^*_k), \\
G_2, & \text{otherwise},
\end{cases}
\]

where \(t^*_k := \arg\min_{t \leq t_{k+1}} a(t)\). Here if there are more than one time reach \(a(t)\) then we randomly pick one as \(t^*_k\). Let \(\pi = \frac{n}{n^*}\), then we can compute \(\pi L(t) = 0\). So our choice satisfies both (A1) and (A2).

Set \(S := \cup_{k=1}^{\infty} \{t^*_k\}\) and let

\[
b^j_t := \prod_{j \in S \cap \lbrack t, t^* \rbrack} (1 - na(j)), c^j_t := \prod_{j \in S^c \cap \lbrack t, t^* \rbrack} (1 - 2a(j)).
\]  

(39)

By Proposition 3.1 \(\Phi(t, i) = P\text{diag}(1, b_1^c, b_1^t, \ldots, b^t_t)P^t\). Set

\[
\tilde{\Phi}(t, i) := P\text{diag}(0, b_1^c, b_1^t, \ldots, b^t_t)P^t
\]

(40)

and \(x_{\text{ave}}(t) := \frac{1}{n} \sum_{i=1}^{n} x_i(t)\) be the average value of \(x_i(t)\). By (38), we get

\[
x(t + 1) - x_{\text{ave}}(t + 1)\mathbb{I} = x(t + 1) - (\pi x(t + 1))\mathbb{I} = \Phi(t, t^*)x(t^*) - (\pi x(t^*))\mathbb{I} + a(t)[\hat{w}(t) - (\pi \hat{w}(t))\mathbb{I}]
\]

\[
= \tilde{\Phi}(t, t^*)x(t^*) + \sum_{i = t^*}^{t-1} a(i)\tilde{\Phi}(t, i + 1)\hat{w}(i)
\]

\[
+ a(t)[\hat{w}(t) - (\pi \hat{w}(t))\mathbb{I}].
\]  

(41)

Because the noises \(\{w_j(t)\}\) satisfy (A4),

\[
E[V(x(t + 1))] = \sum_{i=t^*}^{t-1} a^2(i)E[\|\tilde{\Phi}(t, i + 1)\hat{w}(i)\|^2]
\]

\[
+ \frac{E[\|\tilde{\Phi}(t, t^*)x(t^*)\|^2]}{2} + a^2(t)E[\|\hat{w}(t) - (\pi \hat{w}(t))\mathbb{I}\|^2]
\]

\[
\geq \sum_{i=t^*}^{t-1} a^2(i)E[\|\tilde{\Phi}(t, i + 1)\hat{w}(i)\|^2].
\]  

(42)

Take \(y = \tilde{\Phi}(t, i + 1)\hat{w}(i)\). We can compute

\[
y_1 = b^t_{i+1} \hat{w}_1 - \frac{1}{2} \left[ \frac{\hat{w}_1 - \hat{w}_2}{2} c^t_{i+1} + \hat{w}_1 + \hat{w}_2 - \pi \hat{w}(i) \right],
\]

so by (A4) there exists a constant \(c' > 0\) such that

\[
E[y_1^2] \geq E[y_1^2] = c'(b^t_{i+1})^2.
\]  

(43)

Substituting this into (42), we get

\[
E[V(x(t + 1))] \geq c' \sum_{i=t^*}^{t-1} a^2(i)(b^t_{i+1})^2
\]

(44)

To reach consensus the last line of (44) should converge to 0. We show that this is impossible.

First, because the initial state \(x(1)\) is not consistent, to reach consensus we must choose some positive \(a(t)\). Considering the afferent of noises, we must choose positive \(a(t)\) infinite times to guarantee consensus in mean square. Thus, we can pick \(t' \geq t^*\) such that \(a(t') > 0\). If the last line of (44) converge to 0, we have

\[
\prod_{k=k_1}^{\infty} \left(1 - na(t^*_k)\right) \rightarrow \lim_{t \to \infty} b^t_{t'} \leq \lim_{t \to \infty} b^t_{t'+1} = 0.
\]  

(45)

Also, we recall that \(t^*_k \leq t + \delta\), so

\[
\sum_{i=t^*}^{t} a^2(i)(b^t_{i+1})^2 \geq \sum_{k=k_1}^{\infty} \sum_{i=t_k}^{t_{k+1}-1} a^2(i)(b^t_{i+1})^2
\]

\[
\geq \sum_{k=k_1}^{\infty} \sum_{i=t_k}^{t_{k+1}-1} a^2(i) \prod_{j=k}^{t_{k+1}-1} \left(1 - na(t^*_j)\right)^2
\]

\[
\geq \sum_{k=k_1}^{\infty} (t_{k+1} - t_k)a^2(t_k) \prod_{j=k}^{t_{k+1}-1} \left(1 - na(t^*_j)\right)^2
\]

\[
> \frac{1}{16} \sum_{k=k_1}^{\infty} |c^t_{k+1}| a^2(t_k) \prod_{j=k+1}^{t_{k+1}-1} \left(1 - 2na(t^*_j)\right).
\]  

(46)

Let \(I_{\{1\}}\) be the indicator function, then

\[
1 - 2na(t^*_j) = \left[1 - 2na(t^*_j)I_{\{a(t^*_j) > \delta_j \}}\right] \cdot \left[1 - 2na(t^*_j)I_{\{a(t^*_j) \leq \delta_j \}}\right].
\]  

(47)

Because \(c \geq 1\) and \(\delta > 1/2\), by the choice of \(t_k\), we have

\[
t_k = t_{k-1} + c_1 k^\delta > t_{k-1} + \frac{\sqrt{k-1}}{2},
\]

then by induction we get \(t_k > \frac{1}{2^n}k^2\). So

\[
\sum_{j=k_1}^{\infty} a(t^*_k)I_{\{a(t^*_j) \leq \delta_j \}} \leq \sum_{j=k_1}^{\infty} t_{k+1}^{-\delta}
\]

\[
< \sum_{j=k_1}^{\infty} 2^{\delta} j^{-2\delta} < \infty,
\]

(48)

which indicates

\[
c_1 := \prod_{j=k_1}^{\infty} \left[1 - 2na(t^*_j)I_{\{a(t^*_j) \leq \delta_j \}}\right] > 0.
\]

Substituting this and (47) into (46) and taking

\[
d''_k = \prod_{j=k+1}^{t_{k+1}-1} [1 - 2na(t^*_j)I_{\{a(t^*_j) > \delta_j \}}]
\]

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we have
\[
\sum_{i=1}^{t} a^2(i)(b_{t+1}^i)^2 \geq \frac{1}{16} \sum_{k=k_1}^{k_1-1} |c t_k^*|^2 \sum_{j=k_1}^{k_1-1} \sum_{j=k_1}^{k_1-1} \left[ 1 - 2na(t_j^*) I_{\{0(t_j^*) \leq \tau_j^{-1}\}} \right] \\
\geq \frac{C_1}{16} \sum_{k=k_1}^{k_1-1} \sum_{j=k_1}^{k_1-1} |c t_k^*|^2 \sum_{j=k_1}^{k_1-1} \sum_{j=k_1}^{k_1-1} \left[ 1 - 2na(t_j^*) I_{\{0(t_j^*) \leq \tau_j^{-1}\}} \right] \\
\geq \frac{C_1}{16} \sum_{k=k_1}^{k_1-1} \sum_{j=k_1}^{k_1-1} |c t_k^*|^2 \sum_{j=k_1}^{k_1-1} \sum_{j=k_1}^{k_1-1} \left[ 1 - 2na(t_j^*) I_{\{0(t_j^*) \leq \tau_j^{-1}\}} \right] \\
\geq \frac{C_1 c}{32} \sum_{k=k_1}^{k_1-1} \sum_{j=k_1}^{k_1-1} \sum_{j=k_1}^{k_1-1} \left[ 1 - 2na(t_j^*) I_{\{0(t_j^*) \leq \tau_j^{-1}\}} \right].
\]
(49)

Also, by (48) and (49) we have \(\sum_{j=k_1}^{k_1-1} a(t_j^*) I_{\{0(t_j^*) \leq \tau_j^{-1}\}} = \infty\), so by (49) we get
\[
\lim_{t \to \infty} \sum_{i=t}^{t} a^2(i)(b_{t+1}^i)^2 \geq \frac{C_1 c}{64n}.
\]

Combining this with (44) we see the system cannot reach consensus in mean square.

Appendix C  proof of Theorem 4.2

Without loss of generality, we assume
\[
\sum_{i=3}^{n} \left[ x_i(1) - x_{\text{ave}}(1) \right]^2 \geq \frac{n-2}{n} V(x(1)) > 0.
\]
(50)

Choose \(t_k\) and \(G(t)\) as same as the proof of Theorem 4.2 and take \(\pi = \frac{1}{n} t_1\). Also, similar to (35) we can prove that there exists a constant \(c_1 := c_1(c, \delta) > 0\) such that
\[
c_1 k^{1/(1-\delta)} \leq t_k \leq (ck)^{1/(1-\delta)}, \quad \forall k \geq 1.
\]
(51)

For the time \(t+2\) with \(t \geq 0\), we consider all the choices of \(\{a(i)\}_{i=1}^{t+1}\) to get a lower bound of \(E \|x(t+2) - x_{\text{ave}}(t+2)\|^2\).
Let \(t^* \in [1, t+2]\) be the minimum time such that if \(k \geq t^*\), then \(a(k) < \frac{1}{3n}\). We see if \(t^* \geq 2\) then \(a(t^* - 1) \geq \frac{1}{3n}\). By (42) we have
\[
E [V(x(t+2))] = E [\tilde{\Phi}(t+1, t^*)x(t^*)]^2 \\
+ \sum_{i=t^*}^{t} a^2(i) E [\tilde{\Phi}(t+1, i+1) \tilde{\omega}(i)]^2 \\
+ a^2(t+1) E [\tilde{\omega}(t+1) - (\pi \tilde{\omega}(t+1)) \|^2.
\]
(52)
If \(t^* = t + 2\) which means \(t^* - 1 = t + 1\), by (52) and (21)
\[
E [V(x(t+2))] \geq a^2(t^* - 1) V(\tilde{\omega}(t^* - 1)) \geq \frac{c'}{9n^2},
\]
which is followed by our result directly.

It remains to consider the case of \(t^* < t + 2\). Recall that \(b_{t}^i := \prod_{j \in S \cap [i, t]} (1 - na(j)) \) defined in (39). For any \(y = \tilde{\Phi}(t, i)x\) we can compute \(y_j = (x_j - \pi x) b_{t}^i\) for any \(3 \leq j \leq n\), so if \(t^* = 1\) then
\[
E\|\tilde{\Phi}(t+1, t^*)x(t^*)\|^2 \geq (b_{t}^i)^2 \sum_{j=3}^{n} (x_j(1) - x_{\text{ave}}(1))^2 \\
\geq (n-2) (b_{t}^i)^2 V(x(1))/n,
\]
(53)

where the last inequality uses (50). Otherwise, by the choice of \(t^*\) we have \(a(t^* - 1) \geq \frac{1}{3n}\), so
\[
E\|\tilde{\Phi}(t+1, t^*)x(t^*)\|^2 \geq a^2(t^* - 1) \\
E\|\tilde{\Phi}(t+1, t^*)\tilde{\omega}(t^*)\|^2 \geq \frac{c'}{9n^2} (b_{t}^i)^2,
\]
(54)

where the last inequality uses (43). Set \(k_1 := k^{* - 1}\), then we have \(t_k \geq t^*\) but \(t_{k_1 - 1} < t^*\). With the similar process from (43) to (46), there exists a constant \(c_2 > 0\) such that
\[
\sum_{i=t^*}^{t} a^2(i) E [\tilde{\Phi}(t+1, i+1) \tilde{\omega}(i)]^2 \geq c' \sum_{i=t^*}^{t} a^2(i) (b_{t+1}^i)^2 \\
\geq c_2 \sum_{k=k_1}^{k_1-1} \sum_{j=k_1}^{k_1-1} \sum_{j=k_1}^{k_1-1} \left[ 1 - 2na(t_j^*) \right].
\]
(55)

Similar to (47) we have
\[
1 - 2na(t_j^*) = \left[ 1 - 2na(t_j^*) I_{\{0(t_j^*) > (12s - 1)/|c t_j^*|\}} \right] \\
\cdot \left[ 1 - 2na(t_j^*) I_{\{0(t_j^*) \leq (12s - 1)/|c t_j^*|\}} \right].
\]

According to (51) and the fact \(t_{k^*} \leq t + 1\), we get
\[
\sum_{j=1}^{k^* - 1} \sum_{j=1}^{k^* - 1} \sum_{j=1}^{k^* - 1} \sum_{j=1}^{k^* - 1} \left[ 1 - 2na(t_j^*) I_{\{0(t_j^*) > (12s - 1)/|c t_j^*|\}} \right] \\
\leq \sum_{j=1}^{k^* - 1} \sum_{j=1}^{k^* - 1} \sum_{j=1}^{k^* - 1} \sum_{j=1}^{k^* - 1} \left[ 1 - 2na(t_j^*) I_{\{0(t_j^*) \leq (12s - 1)/|c t_j^*|\}} \right] \\
\leq t^{2s-1} O \left( \frac{k^{1/(1-\delta)}}{\pi} \right) = O \left( \frac{2^{s-1}(k^{1/(1-\delta)})}{\pi} \right) \\
= O \left( \frac{\pi^{2s-1}}{\pi^{2s-1}} \right) = O(1),
\]
so with the similar process from (48) to (49), there exists a
constant $c_3 > 0$ such that
\begin{align*}
&\sum_{k=k_1}^{k^*-1} \left[ c t_k^a (t_k^*)^2 \right] \prod_{j=k+1}^{k^*-1} \left[ 1 - 2na(t_j^*) \right] \\
&\geq \frac{c_3}{t_k^{1-2a}} \sum_{k=k_1}^{k^*-1} a(t_k^*) I_{\{a(t_k^*) > t_2^{s-1}/|c t_k^a|\}} \\
&\cdot \prod_{j=k+1}^{k^*-1} \left[ 1 - 2na(t_j^*) I_{\{a(t_j^*) > t_2^{s-1}/|c t_j^a|\}} \right] \\
&= \frac{c_3}{2na^{1-2a}} \left( 1 - \prod_{j=k_1}^{k^*-1} \left[ 1 - 2na(t_j^*) I_{\{a(t_j^*) > t_2^{s-1}/|c t_j^a|\}} \right] \right) \cdot \prod_{j=k_1}^{k^*-1} \left[ 1 - 2na(t_j^*) I_{\{a(t_j^*) > t_2^{s-1}/|c t_j^a|\}} \right] \geq \frac{c_4}{2},
\end{align*}
If
\begin{align*}
\prod_{j=k_1}^{k^*-1} \left[ 1 - 2na(t_j^*) I_{\{a(t_j^*) > t_2^{s-1}/|c t_j^a|\}} \right] \leq \frac{1}{2},
\end{align*}
then together (57), (55) and (52) our result is obtained. Otherwise, by the definition of $b_i^{t_1}$ and (56) there exists a constant $c_4 > 0$ such that
\begin{align*}
b_i^{t_1+1} \geq \frac{c_4}{2}.
\end{align*}

Substituting this into (53) and (54) we get $E[\tilde{\Phi}(t + 1, t^*)] = x(t^*)]^2$ is bigger than a positive constant, then by (52) the desired result follows.

Appendix D: Proof of Theorem 5.1

Set $t_k = 1$ and $t_{k+1} = t_k + |ct_k^a|$, where $c_i$ is a large constant. Let $E_{t_k, t_{k+1}}$ be the event of $\bigcup_{k=1}^{t_{k+1}} G(k)$ is strongly connected. Then for any given topologies $G(l)$, $1 \leq l \leq t_k - 1$,
\begin{align*}
P \left( E_{t_k, t_{k+1}} \mid \{G(l)\}_{l=1}^{t_k-1} \right)
&\geq P \left( \bigcap_{i=1}^{t_k-1} E_{t_k+(i-1)K, t_k+iK-1} \mid \{G(l)\}_{l=1}^{t_k-1} \right) \\
&= 1 - P \left( \bigcup_{i=1}^{t_k-1} E_{t_k+(i-1)K, t_k+iK-1} \mid \{G(l)\}_{l=1}^{t_k-1} \right) \\
&= 1 - P \left( E_{t_k, t_{k+1}} - \sum_{i=1}^{t_k-1} E_{t_k+(i-1)K, t_k+iK-1} \mid \{G(l)\}_{l=1}^{t_k-1} \right) \\
&\geq 1 - \prod_{i=2}^{t_k-1} P \left( E_{t_k+(i-1)K, t_k+iK-1} \mid \{G(l)\}_{l=1}^{t_k-1} \right) \\
&\cdot \prod_{i=1}^{t_k-1} \left( E_{t_k+(i-1)K, t_k+iK-1} \right) \\
&\cdot \prod_{j=1}^{t_k-1} \left( E_{t_k+(j-1)K, t_k+jK-1} \right) \\
&\geq 1 - \left( 1 - \frac{1}{t_k^{1-2a}} \right)^{t_k-1},
\end{align*}
where the last line uses the assumption (A1).

Let $k^*$ be the minimal time such that $\bigcap_{k=k_1}^{k^*} E_{t_k, t_{k+1}}$ happens, which implies the event $E_{t_k, t_{k+1}}$ happens. Then for any given topologies $G(l)$, $1 \leq l \leq t_k - 1$,
\begin{align*}
P \left( k^* = k \right) < P \left( E_{t_k-1, t_k} \right) < \frac{1}{t_k^{1-2a}} \left( t_k^a \right)^{t_k-1} \leq \frac{c_4}{2},
\end{align*}
for large $k$, where the last inequality uses the fact of $t_k + 1 - t_k = |ct_k^a|$ and $\mu < 1/2$. By the total probability theorem,
\begin{align*}
E[V(x(t))] = \sum_{k=1}^{\infty} P(k^* = k) E[V(x(t))] k^* = k.
\end{align*}
We need to consider the value of $E[V(x(t))] k^* = k$. Similar to (11) and (12) we get
\begin{align*}
E[V(x(t+1)) k^* = k] = E[V(\Phi_t(1)x(1)) k^* = k] \\
+ O \left( \sum_{i=1}^{t} a^2(i) E[V(\Phi(t, i+1) \tilde{w}(i)) k^* = k] \right),
\end{align*}
and similar to (13) for any $x \in \mathbb{R}^n$ we obtain
\begin{align*}
E[V(\Phi_t(1)x) k^* = k] = \frac{k^*}{2e + (\max\{t, k - 1\})^{2-\mu} + t^*}.
\end{align*}
where the last line uses the fact if $k^x = y$ then $x \leq t_y - 1$. Since $E[V(\tilde{w}(i)) k = k^*]$ is bounded under (A3), taking (62) into (61) we can obtain
\begin{align*}
E[V(x(t))] k^* = k \\
= O \left( \sum_{i=1}^{t_k-1} \frac{a^2(i) t_k^{2(1-\mu)}}{t_k^{2(1-\mu)}} + \sum_{i=t_k}^{t} \frac{a^2(i) t_k^{2(1-\mu)}}{t_k^{2(1-\mu)}} \right) \\
= O \left( \frac{t_k^{2(1-\mu)}}{t_{k-1}^{2(1-\mu)}} + \frac{1}{t_{k-1}^{2(1-\mu)}} \right).
\end{align*}
Substitute this and (59) into (60) we have $E[V(x(t))] = O(1/t_k^{2-\mu})$ when $c_i$ is large enough.

Finally, let $x^*$ be the same value defined by (15). As same as (16) and (17) we can obtain $E[x^*] = \pi(x(1))$ and $\text{Var}(x^*) < \infty$.

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