On the Methods for Constructing Meson-Baryon Reaction Models within Relativistic Quantum Field Theory

B. Juliá-Díaz,¹,² H. Kamano,¹ T.-S. H. Lee,¹,³ A. Matsuyama,¹,⁴ T. Sato,¹,⁵ and N. Suzuki¹,⁵

¹ Excited Baryon Analysis Center (EBAC),
Thomas Jefferson National Accelerator Facility, Newport News, VA 23606, USA
² Department d’Estructura i Constituents de la Matèria and Institut de Ciències del Cosmos,
Universitat de Barcelona, E-08028 Barcelona, Spain
³ Physics Division, Argonne National Laboratory, Argonne, IL 60439, USA
⁴ Department of Physics, Shizuoka University, Shizuoka 422-8529, Japan
⁵ Department of Physics, Osaka University, Toyonaka, Osaka 560-0043, Japan

Abstract
Within the relativistic quantum field theory, we analyze the differences between the πN reaction models constructed from using (1) three-dimensional reductions of Bethe-Salpeter Equation, (2) method of unitary transformation, and (3) time-ordered perturbation theory. Their relations with the approach based on the dispersion relations of S-matrix theory are discussed.

PACS numbers: 13.75.Gx, 13.60.Le, 14.20.Gk

* Notice: Authored by Jefferson Science Associates, LLC under U.S. DOE Contract No. DE-AC05-06OR23177. The U.S. Government retains a non-exclusive, paid-up, irrevocable, world-wide license to publish or reproduce this manuscript for U.S. Government purposes.
I. INTRODUCTION

Because of the non-perturbative nature of the problem and the complexities of the reaction mechanisms, the calculations of meson-baryon reactions within the framework of relativistic quantum field theory cannot be done exactly. Thus the progress we can make now and in the foreseeable future is to construct manageable reaction models for analyzing the data. Each model involves some approximations and assumptions. For understanding the information extracted from the data, such as the electromagnetic form factors of the $N$-$N^*$ transitions, it is necessary to examine this phenomenological aspect of the employed models. This is the objective of this paper concerning the approaches based on (1) three-dimensional reductions [1], (2) method of unitary transformation [2, 3, 4], and (3) time-ordered perturbation theory [5, 6]. For simplicity, we will only consider the case of single-channel $\pi N$ scattering. This is sufficient to reveal the differences between these approaches. We will also examine the relations between these approaches and the approach based on the dispersion relations of the S-matrix theory. As is well documented [7], these two theoretical frameworks are based on very different theoretical considerations and hence there is no compelling theoretical reasons to favor one of them in developing phenomenological models to analyze the data.

The considered three methods for constructing $\pi N$ models are explained and analyzed in sections II-IV. In each section we will also address their relations with the approach based on dispersion relations. In section 5, we give a summary.

II. THREE-DIMENSIONAL REDUCTIONS

To illustrate the derivations of three-dimensional equations for $\pi N$ scattering from relativistic quantum field theory, it is sufficient to consider a simple $\pi NN$ interaction Lagrangian density

$$L_{int}(x) = \bar{\psi}(x) \Gamma_0 \psi(x) \phi(x),$$  \hspace{1cm} (1)

where $\psi(x)$ and $\phi(x)$ denote respectively the nucleon and pion fields and $\Gamma_0$ is a bare $\pi NN$ vertex, such as $\Gamma_0 = ig\gamma_5$ in the familiar pseudo-scalar coupling. By using the standard method [8], it is straightforward to derive from Eq. (1) the Bethe-Salpeter equation for $\pi N$ scattering and the one-nucleon propagator. In momentum space, the resulting Bethe-Salpeter equation can be written as

$$T(k', k; P) = B(k', k; P) + \int d^4k'' B(k', k''; P) G(k''; P) T(k'', k; P),$$ \hspace{1cm} (2)

where $k$ and $P$ are respectively the relative and total momenta defined by the nucleon momentum $p$ and pion momentum $q$

$$P = p + q,$$

$$k = \eta_\pi(y)p - \eta_N(y)q.$$  \hspace{1cm} (3)

Here $\eta_N(y)$ and $\eta_\pi(y)$ can be any function of a chosen parameter $y$ with the condition

$$\eta_\pi(y) + \eta_N(y) = 1.$$  \hspace{1cm} (3)
Obviously we have from the above definitions that

\[ p = \eta_N(y) P + k, \]
\[ q = \eta_\pi(y) P - k. \]

(4)

In analogy to the nonrelativistic form, they are often chosen as: \( \eta_N = m_N/(m_\pi + m_N) \) and \( \eta_\pi = m_\pi/(m_\pi + m_N) \). The choice of the \( \eta \)'s is irrelevant to the derivation presented below in this section provided that Eq. (3) is satisfied.

Note that \( T \) in Eq. (2) is the "amputated" invariant amplitude and is related to the \( \pi N \) S-matrix by \( S \propto \bar{u} T u \) with \( u \) denoting the nucleon spinor. The driving term \( B \) in Eq. (2) is the sum of all two-particle irreducible amplitudes, and \( G \) is the product of the pion propagator \( D_\pi(q) \) and the nucleon propagator \( S_N(p) \). In the low energy region, we neglect the dressing of pion propagator and simply set

\[ D_\pi(q) = \frac{1}{q^2 - m_\pi^2 + i\epsilon}, \]

(5)

where \( m_\pi \) is the physical pion mass. The nucleon propagator can be written as

\[ S_N(p) = \frac{1}{i\bar{\gamma} \cdot \bar{\gamma} - m_0^N - \bar{\Sigma}_N(p^2) + i\epsilon}, \]

(6)

where \( m_0^N \) is the bare nucleon mass and the nucleon self energy operator \( \bar{\Sigma}_N \) is defined by

\[ \bar{\Sigma}_N(p^2) = \int d^4 k \Gamma_0 G(k; p) \bar{\Gamma}(k; p) . \]

(7)

The dressed vertex function \( \bar{\Gamma} \) on the right hand side of Eq. (7) depends on the \( \pi N \) Bethe-Salpeter amplitude

\[ \bar{\Gamma}(k; P) = \Gamma_0 + \int d^4 k' \Gamma_0 G(k'; P) T(k'; k; P) . \]

(8)

It is only possible in practice to consider the leading term of \( B \) of Eq. (2). For the simple Lagrangian Eq. (1) the leading term consists of the direct and crossed \( N \) diagrams

\[ B(k, k'; P) = B^{(a)}(k, k'; P) + B^{(b)}(k, k'; P), \]

(9)

where

\[ B^{(a)}(k, k'; P) = \Gamma_0 \frac{1}{i\bar{\gamma} \cdot P - m_0^N + i\epsilon} \Gamma_0, \]

(10)

\[ B^{(b)}(k, k'; P) = \Gamma_0 \frac{1}{i\bar{\gamma} \cdot P - m_0^N + i\epsilon} \Gamma_0, \]

(11)

with \( \bar{P} = [\eta_N(y) - \eta_\pi(y)] P + k + k' \).

Equations (2)-(11) form a closed set of coupled equations for determining the dressed nucleon propagator of Eq. (6) and the \( \pi N \) Bethe-Salpeter amplitude of Eq. (2). It is important to note here that this is a drastic simplification of the original field theoretical problem defined by the Lagrangian (1). However, it is still very difficult to solve this highly nonlinear problem exactly. For practical applications further approximations are usually introduced.
The first step is to define the physical nucleon mass by imposing the condition that the dressed nucleon propagator should have the limit

$$S_N(p)|_{p^2 \to m_N^2} \to \frac{1}{i\not{p} - m_N + i\epsilon},$$

where $m_N$ is the physical nucleon mass. This means that the self-energy in the nucleon propagator Eq. (10) is constrained by the condition

$$m_N^0 + \tilde{\Sigma}(m_N^2) = m_N.$$

The next step is to assume that the $p$-dependence of the nucleon self-energy is weak and we can use the condition Eq. (13) to set $m_N^0 + \tilde{\Sigma}(p^2) \sim m_N^0 + \tilde{\Sigma}(m_N^2) = m_N$. This approximation greatly simplifies the nonlinearity of the problem, since the full $\pi N$ propagator $G$ in Eqs. (2), (7) and (8) then takes the following simple form

$$G(k; P) = \frac{1}{i\not{p} - m_N + i\epsilon} \frac{1}{q^2 - m_{\pi}^2 + i\epsilon}.$$

The next commonly used approximation is to reduce the dimensionality of the above integral equations from four to three. The resulting models will be called 3dBS models in the following discussions. There exists extensive literature on this subject, as reviewed in Ref. [1]. A rather complete numerical study of several of these 3dBS $\pi N$ models was given in Ref. [9]. We therefore will not get into these subjects. Instead we will focus on how these models are interpreted in the actual analysis of $\pi N$ data.

Let us specifically consider a 3dBS model derived from using the three-dimensional reduction method of Kadyshevsky [10]. In the c.m. system, $P = (E, \vec{0})$, we first define

$$t(\vec{k'}, \vec{k}, E) = \frac{1}{(2\pi)^3} \sqrt{\frac{m_N}{E_N(k')}} \frac{1}{\sqrt{2E_\pi(k')}} \bar{u}_{\vec{k'}} T(\vec{k'}, \vec{k}, E) u_{\vec{k}} \sqrt{\frac{m_N}{E_N(k)}} \frac{1}{\sqrt{2E_\pi(k)}},$$

$$v(\vec{k'}, \vec{k}, E) = \frac{1}{(2\pi)^3} \sqrt{\frac{m_N}{E_N(k')}} \frac{1}{\sqrt{2E_\pi(k')}} \bar{u}_{\vec{k'}} B(\vec{k'}, \vec{k}, E) u_{\vec{k}} \sqrt{\frac{m_N}{E_N(k)}} \frac{1}{\sqrt{2E_\pi(k)}},$$

where the time components of the momenta of the BS amplitude $T$ and interaction $B$ in Eq. (2) have been fixed by the chosen three-dimensional reduction, as explicitly given in Ref. [9]. The resulting three-dimensional scattering equation can then be cast into the following form

$$t(\vec{k'}, \vec{k}, E) = v(\vec{k'}, \vec{k}, E) + \int d\vec{k''} v(\vec{k'}, \vec{k'', E}) g(k'', E) t(\vec{k''}, \vec{k}, E),$$

where the propagator is

$$g(k, E) = \frac{1}{E - E_N(k) - E_\pi(k) + i\epsilon}.$$

For simplicity, we only consider the case that the interaction $v$ is derived from $s$- and $u$-channel mechanisms defined by Eqs. (9)-(11). The difference between 3dBS models whose scattering equation can be cast into the form of Eq. (17) is in the expression of the driving term $v$ in Eq. (17). It is important to note that the only common condition these 3dBS
models satisfy exactly is the unitarity condition, as explained in Ref. [1]. Thus it is not surprising that their amplitudes have different analytic structure in complex-E plane.

To simplify the presentation, we use the following operator form for Eq. (17)

\[ t(E) = v(E) + v(E)g(E)t(E) \]  

(19)

For most of the 3dBS models, one gets the following form of \( v(E) \) in the \( P_{11} \) partial wave

\[ v(E) = v^{\text{pole}}(E) + v^{\text{bg}}(E) \]  

(20)

with

\[ v^{\text{pole}}(E) = \Gamma_0^\dagger \frac{1}{E - m_0} \Gamma_0 \]  

(21)

where \( m_0^N \) is the bare mass of the starting Lagrangian. The form of the second term \( v^{\text{bg}}(E) \) of Eq. (20) is not important in the following discussions.

Substituting Eqs. (20)–(21) into Eq. (19), we then get the following well known form

\[ t(E) = t^{\text{bg}}(E) + \frac{\Gamma^\dagger(E)\Gamma(E)}{E - m_0^N - \Sigma_N(E)} \]  

(22)

where

\[ t^{\text{bg}}(E) = v^{\text{bg}}(E) + v^{\text{bg}}(E)g(E)t^{\text{bg}}(E), \]

\[ \Gamma(E) = \Gamma_0[1 + g(E)t^{\text{bg}}(E)], \]

\[ \Sigma_N(E) = \langle N_0|\Gamma_0 g(E)\Gamma^\dagger(E)|N_0 \rangle. \]  

(23)

In the above equations \(|N_0 \rangle \) is the bare one-nucleon state, \( \Sigma_N(E) \) and \( \Gamma \) are the three-dimensional forms of Eqs (7), (3), respectively. The condition Eq. (13) then becomes

\[ m_0^N + \Sigma_N(m_N) = m_N \]  

(24)

which defines the bare mass \( m_0^N \) from the physical mass \( m_N \). It is common [11] to use Eq. (22) at the \( E \to m_N \) position to define the dressed vertex \( \Gamma(E) \) in terms of physical coupling constant. To get this relation, we first note that at \( E \to m_N \), the self-energy \( \Sigma_N(E) \) can be expanded as

\[ \Sigma_N(E) = \Sigma_N(m_N) + (E - m_N)\Sigma_1(m_N) + \cdots \]  

(25)

where

\[ \Sigma_1(m_N) = \left. \frac{\partial \Sigma_N(E)}{\partial E} \right|_{E=m_N}. \]  

(26)

By using the above relations, Eq. (22) becomes

\[ t(E)|_{E=m_N} = t^{\text{bg}}(m_N) + \left[Z_2^{1/2}\Gamma(m_N)\right]\frac{1}{E - m_N}\left[Z_2^{1/2}\Gamma(m_N)\right] \]  

(27)
with
\[ Z_2^{-1} = 1 - \Sigma_1(m_N). \] (28)

The renormalized vertex \([Z_2^{1/2}\Gamma(m_N)]\) is then used to define the physical coupling. If the bare vertex \(\Gamma_0\) is chosen to be the usual pseudo-vector coupling \(L_I = \frac{f_{\pi NN}^{(0)}}{m_\pi} \bar{\psi} \gamma_5 \gamma_\mu \psi \partial^\mu \phi\), the above procedures relate the renormalized coupling constant \(f_{\pi NN}\) to the bare coupling constant \(f_{\pi NN}^{(0)}\):
\[ f_{\pi NN} = f_{\pi NN}^{(0)}[1 + g(m_N)t^{bg}(m_N)]Z_2^{1/2}. \] (29)

The renormalized coupling constant is identified with the empirical value \(g_{\pi NN}^2 = \frac{2m_N}{m_\pi^2} (\frac{f_{\pi NN}^2}{4\pi}) = 14.3\).

Here we point out that the above procedure implies an interpretation where the physical nucleon is made of a bare core \(N_0\) and pion cloud. To illustrate this, it is sufficient to consider the case when \(t^{(bg)} = 0\). If we set \(\langle N_0|\Gamma_0|k\rangle = \Gamma_0(k)\), Eq. (22) has the following analytic form,
\[ t(k', k, E) = \frac{1}{E - m_0^N - \Sigma_0^N(E)} \Gamma_0(k) \] (30)

where
\[ \Sigma_0^N(E) = \int k^2 dk \frac{\left|\Gamma_0(k)\right|^2}{E - E_\pi(k) - E_N(k) + i\epsilon} \] (31)
\[ \Sigma_1^N(E) = \frac{\partial \Sigma_0^N(E)}{\partial E} \]
\[ = -\int k^2 dk \frac{\left|\Gamma_0(k)\right|^2}{(E - E_\pi(k) - E_N(k))^2}. \] (32)

The nucleon pole condition Eq. (24) then becomes
\[ m_N = m_0^N + \Sigma_0^N(m_N) \] (33)
\[ \langle k'|t(E)|k\rangle |_{E\to m_N} = \tilde{\Gamma}_0^\ast(k') \frac{1}{E - m_N} \tilde{\Gamma}_0(k) \] (34)

where the renormalized vertex function is
\[ \tilde{\Gamma}_0(k) = Z_0^{1/2}\Gamma_0(k) \] (35)
with
\[ Z_0^{-1} = 1 - \Sigma_1^0(m_N) \] (36)
\[ = 1 + \int dk k^2 \frac{\left|\Gamma_0(k)\right|^2}{(m_N - E_\pi(k) - E_N(k))^2}. \] (37)

It is interesting to note that the above nucleon pole conditions can be reproduced by assuming that the structure of the nucleon can be described by the following mass operator \(h\) in a subspace spanned by the state \(|N_0\rangle\) and \(|k\rangle\) for the \(\pi N\) state
\[ h = h_0 + \Gamma_0 \] (38)
with

\[ h_0|N_0\rangle = m_N^0|N_0\rangle \]  \hspace{1cm} (39)

\[ h_0|k\rangle = (E_N(k) + E_\pi(k))|k\rangle. \]  \hspace{1cm} (40)

We assume that the physical nucleon state is defined by

\[ h|N\rangle = m_N|N\rangle, \]  \hspace{1cm} (41)

\[ |N\rangle = z_0^{1/2}\left[|N_0\rangle + \int k^2dkf(k)|k\rangle\right]. \]  \hspace{1cm} (42)

The basis states are normalized as \( \langle N_0|N_0\rangle = 1, \langle k|k'\rangle = k^2\delta(k - k') \). Projecting Eq. (41) from the left onto \(|N_0\rangle\) and \(|k\rangle\), we then get

\[ m_0 + \int k^2dkf(k)\Gamma_0(k) = m_N \]  \hspace{1cm} (43)

\[ [E_N(k) + E_\pi(k)]f(k) + \Gamma_0^*(k) = m_Nf(k) \]  \hspace{1cm} (44)

The normalization condition \( \langle N|N\rangle = 1 \) gives

\[ z_0 = 1 + \int k^2dk|f(k)|^2 \]  \hspace{1cm} (45)

From Eq. (44) we have the solution

\[ f(k) = \frac{\Gamma_0^*(k)}{m_N - E_N(k) - E_\pi(k)}. \]  \hspace{1cm} (46)

Substituting Eq. (46) into Eq. (43), we then get exactly the nucleon pole condition Eq. (33). Within the model defined by the mass operator Eq. (41), the physical \( \pi NN \) vertex can be calculated from using Eq. (42)

\[ \Gamma_N(k) = \langle N|\Gamma_0|k\rangle = z_0^{1/2}\Gamma_0(k) \]  \hspace{1cm} (47)

Substituting Eq. (46) into Eq. (45), we find that \( z_0 \) is exactly equal to \( Z_0 \) of Eq. (37) and hence \( \Gamma_N(k) \) is exactly the renormalized vertex \( \bar{\Gamma}_0 \) of Eq. (35).

The above simple model illustrated that the nucleon pole conditions Eqs (33)-(37) can be related to the substructure of the nucleon. If we write Eq. (42) as an operator form \(|N\rangle = |N_0\rangle + f|\pi N\rangle\) and iterate it, we then get

\[ |N\rangle = z_0^{1/2}[|N_0\rangle + f|\pi N_0\rangle + ff|\pi\pi N_0\rangle + fff|\pi\pi\pi N_0\rangle + \cdots. \]  \hspace{1cm} (48)

This illustrates that the usual procedure of requiring the \( \pi N \) amplitudes to have a nucleon pole implies that the physical nucleon is made of a \( N_0 \) core and meson cloud.

To be consistent, one in principle should also replace \( E_N(k) \) in the propagator Eq. (18) by an expression which is related to bare mass \( m_N^0 \) and the matrix element of the self energy \( \Sigma_N^0 \), defined in Eq. (31). But this complicates the unitarity condition for the scattering amplitude \( t \) defined by Eq. (17). To do it properly, one needs to also consider the \( \pi\pi N \) unitarity condition since the self energy \( \Sigma_N^0 \) contains \( \pi N \) intermediate state.
Let us stress that the above observation has a connection with the $\pi N$ scattering equation derived by Aaron and Amado and Young [12] (AAY) using the three-dimensional reduction of Blankenbecler and Sugar [13]. The essential assumption of their derivation is that the $\pi N$ scattering can be described from the isobar model where the pion is scattered from an $\pi N$ of Blankenbecler and Sugar [13]. The essential assumption of their derivation is that the $\pi N$ scattering can be described from the isobar model where the pion is scattered from an

$\pi$ isobar system which can decay into $\pi N$. If we identify their isobar as $N_0$ of the simple mass operator defined by Eq. (38), their equation can be schematically cast into the following from

$$t(\vec{k}', \vec{k}, E) = v^{(opex)}(\vec{k}', \vec{k}, E) + \int d\vec{k}''v^{(opex)}(\vec{k}', \vec{k}'' E)g_{AAy}(k'', E)t(\vec{k}'' \vec{k}, E)$$  \hspace{1cm} (49)

where the propagator is

$$g_{AAy}(k, E) = \frac{1}{E - E_{N_0}(k) - E_\pi(k) - \Sigma^0_N(k, E) + i\epsilon}.$$  \hspace{1cm} (50)

Here $E_{N_0}(k) = [(m_N^0)^2 + k^2]^{1/2}$ and $\Sigma^0_N(k, E)$ is determined by properly boosting the expression Eq. (31). The driving term of Eq. (19) is determined by the one-nucleon-exchange mechanism

$$v^{(opex)}(\vec{k}', \vec{k}, E) = \frac{\Gamma_0^0(k)Z_0^{1/2}}{E - E_N(\vec{k} + \vec{k}') - E_\pi(k) - E_\pi(k') + i\epsilon}.$$  \hspace{1cm} (51)

Note that $\Sigma^0_N(k, E)$ in the above equation are defined by the same vertex function $\Gamma_0(k)$. In the AAY approach, this consistent treatment of the propagator $g_{AAy}(k, E)$ and the interaction $v^{(opex)}$ is the consequence of requiring that the scattering equation satisfies the $\pi\pi N$ unitarity condition. In other words, if one replaces the propagator $g_{AAy}(k, E)$ by $g(k, E)$ of Eq. (18), the resulting amplitude from solving Eq. (19) will not satisfy the unitarity condition. With some derivations, one can also see that the solution of Eq. (19) will not have a pole at $E = m_N$. The AAY approach simply was not developed to reproduce the same analytic structure of the dispersion relations in the unphysical region $E \leq m_\pi + m_N$.

To further explore the differences with the approaches based on dispersion relations, we note that the models considered in this section as well as in the next one solve integral equations, such as Eq. (17), and require form factors to regularize the matrix elements of the potential $v$ and the vertex interaction $\Gamma_0$. These form factors can give poles to the on-shell scattering amplitudes in the unphysical region of $E \leq m_N + m_\pi$ where the nucleon pole is identified. More explicitly, if a dipole form is used to parameterize $\Gamma_0$, the on-shell matrix element of Eq. (30) becomes

$$t(k_0, k_0, E) \sim \left[\frac{\Lambda^2}{k_0^2 + \Lambda^2}\right]^2 \frac{1}{E - m_0 - \Sigma^0_N(E)} \left[\frac{\Lambda^2}{k_0^2 + \Lambda^2}\right]^2$$  \hspace{1cm} (52)

where $k_0$ is defined by $E = E_N(k_0) + E_\pi(k_0)$. Thus this amplitude can have pole in the region where we define the nucleon pole if $\Lambda \leq m_N + m_\pi$. This illustrates that the analytic structure of the dynamical models deduced from relativistic quantum field theory can not be completely consistent with that defined by the dispersion relations of the S-matrix theory. In fact, there is no compelling reason to require that they have the same analytic structure. In the very extensive literature, as thoroughly reviewed in the textbook of Goldberger and Watson [4], the widely used fixed-t dispersion relations in analyzing $\pi N$ scattering can not be derived from relativistic quantum field theory exactly. Historically, the S-matrix theory is considered as an alternative to relativistic quantum field theory to study strong interactions. There is no rigorous theoretical argument to favor one of them in developing phenomenological models to analyze the data.
The method of unitary transformation was essentially based on the same idea of the Foldy-Wouthuysen transformation developed in the study of electromagnetic interactions. Instead of considering the original Lagrangians with bare masses and bare vertex interactions, we simply ask how the strong interactions can be described with a phenomenological Lagrangian defined by the physical masses and physical coupling constants. It is understood that the application of such a phenomenological Lagrangian to calculate any amplitude should drop loops associated with one-particle states and vertices which are already absorbed in the definitions of physical masses and coupling constants. It is an non-trivial problem to justify these rules within the exact theory. But such rules are valid in practice since we will only consider leading order terms of a perturbative expansion which will be specified later. This means that we assume that we have already solved the one-particle problem within a model, such as that defined by Eqs. (35)-(42), and this problem will not be dealt with in developing reaction models. This is the main difference between the models based on three-dimensional reductions described in section II and the model based on the unitary transformation. The advantage of the latter is that the unitarity condition can be satisfied trivially; in particular in handling the multi-channel multi-resonance reactions. Of course, the price we pay is that the connection to the theory of nucleon structure is perhaps more remote than the approaches based on three dimensional reductions.

To illustrate the method of unitary transformation, we again consider the simplest phenomenological Lagrangian density

$$L(x) = L_0(x) + L_I(x)$$  (53)

where $L_0(x)$ is the usual free Lagrangians with physical masses $m_N$ for the nucleon field $\psi_N$ and $m_\pi$ for the pion field $\phi_\pi$, and

$$L_I(x) = \bar{\psi}_N(x) \Gamma_{N,\pi N} \psi_N(x) \phi_\pi(x), \tag{54}$$

Here $\Gamma_{N,\pi N}$ denotes the physical $\pi NN$ coupling ($\sim f_{\pi NN}$). It is not the bare coupling $\Gamma_0$ in Eq. (1). The Hamiltonian density $H(x)$ can be derived from Eqs. (53)-(54) by using the standard method of canonical quantization. We then define the Hamiltonian as

$$H = \int H(\vec{x}, t = 0) \, d\vec{x}. \tag{55}$$

The resulting Hamiltonian can be written as

$$H = H_0 + H_I \tag{56}$$

with

$$H_0 = \int d\vec{k} \left[ E_N(k) b^\dagger_{\vec{k}} b_{\vec{k}} + E_\pi(k) a^\dagger_{\vec{k}} a_{\vec{k}} \right]$$

$$H_I = \Gamma_{N,\pi N}$$

$$= \int d\vec{k}_1 d\vec{k}_2 d\vec{k} \delta(\vec{k} - \vec{k}_1 - \vec{k}_2) [\Gamma_{N,\pi N}(\vec{k}_1 - \vec{k}_2) b^\dagger_{\vec{k}_1} b_{\vec{k}_2} a_{\vec{k}_2} + (c.c.)]$$  (58)

where $b^\dagger$ and $a^\dagger$ ($b$ and $a$) are the creation (annihilation) operators for the nucleon and the pion, respectively. For simplicity, we drop the terms involving the anti-nucleon operator $d^\dagger$. 


and \( d \). Note that \( H \) along with the other constructed generators \( \vec{P}, \vec{K}, \) and \( \vec{J} \) define the instant-form relativistic quantum mechanical description of \( \pi N \) scattering. We will work in the center of mass frame and hence the forms of these other generators of Lorentz group are not relevant in the following derivations.

The essence of the unitary transformation method is to extract an effective Hamiltonian in a ‘few-body’ space defined by an unitary operator \( U \), such that the resulting scattering equations can be solved in practice. Instead of the original equation of motion \( H|\alpha\rangle = E_\alpha|\alpha\rangle \), we consider

\[
H'|\bar{\alpha}\rangle = E_\alpha|\bar{\alpha}\rangle \tag{59}
\]

where

\[
H' = UHU^+ \quad |\bar{\alpha}\rangle = U|\alpha\rangle \tag{60}
\]

In the approach of Sato, Kobayashi and Ohtsubo \( ^3 \) (SKO), the first step is to decompose the interaction Hamiltonian \( H_I \), Eq. (58), into two parts

\[
H_I = H^P_I + H^Q_I \tag{61}
\]

where \( H^P_I \) defines the process \( a \to bc \) with \( m_a \geq m_b + m_c \) which can take place in the free space, and \( H^Q_I \) defines the virtual process with \( m_a < m_b + m_c \). For the simple interaction Hamiltonian, Eq. (58), it is clear that \( H^P_I = 0 \) and \( H^Q_I = H_I \).

The essence of the SKO method is to eliminate the virtual processes from transformed Hamiltonian \( H' \) by choosing an appropriate unitary transformations \( U \). This can be done systematically by using a perturbative expansion of \( U \) in powers of coupling constants. As a result the effects of ‘virtual processes’ are included in the effective operators in the transformed Hamiltonian.

Defining \( U = \exp(-iS) \) by a hermitian operator \( S \) and expanding \( U = 1 - iS + ... \), the transformed Hamiltonian can be written as

\[
H' = UHU^+ \\
= U(H_0 + H^P_I + H^Q_I)U^+ \\
= H_0 + H^P_I + H^Q_I + [H_0, iS] + [H_I, iS] + \frac{1}{2!} [ [H_0, iS], iS ] + \cdots . \tag{62}
\]

To eliminate from Eq. (62) the virtual processes which are of first-order in the coupling constant, the SKO method imposes the condition that

\[
H^Q_I + [H_0, iS] = 0. \tag{63}
\]

Since \( H_0 \) is a diagonal operator in Fock-space, Eq. (63) implies that \( iS \) must have the same operator structure of \( H^Q_I \) and is of first order in the coupling constant. By using Eq. (63), Eq. (62) can be written as

\[
H' = H_0 + H'_I, \tag{64}
\]

with

\[
H'_I = H^P_I + [H^P_I, iS] + \frac{1}{2} [H^Q_I, iS] + \text{higher order terms}. \tag{65}
\]
Since \( H_I^P, H_I^Q, \) and \( S \) are all of the first order in the coupling constant, all processes included in the second and third terms of the \( H_I^f \) are of the second order in coupling constants.

We now turn to illustrating how the constructed \( H_I^f \) of Eq. (63) can be used to describe the \( \pi N \) scattering if the higher order terms are dropped. We consider the simple Hamiltonian defined by Eqs. (56)-(58) which gives \( H_I^f = 0 \) and \( H_I^Q = \Gamma_{N \rightarrow \pi N} \). Our first task is to find \( S \) by solving Eq. (63) within the Fock space spanned by the eigenstates of \( H_0 \)

\[
H_0|N\rangle = m_N|N\rangle \tag{66}
\]

\[
H_0|k, p\rangle = (E_\pi(k) + E_N(p))|k, p\rangle \tag{67}
\]

\[
H_0|k_1, k_2, p\rangle = ((E_\pi(k_1) + E_\pi(k_2) + E_N(p))|k_1, k_2, p\rangle \tag{68}
\]

\[\ldots\]

For two eigenstates \( f \) and \( i \) of \( H_0 \), the solution of Eq. (63) clearly is

\[
\langle f|(iS)|i\rangle = \frac{-\langle f|H_I^Q|i\rangle}{E_f - E_i} \tag{69}
\]

For the considered \( H_I^Q = \Gamma_{N \rightarrow \pi N} \) we thus get the following non-vanishing matrix elements

\[
\langle \vec{k}\vec{p}|(iS)|N\rangle = \Gamma_{N,\pi N}(k) \frac{-1}{E_\pi(k) + E_N(p) - m_N} \delta(\vec{k} + \vec{p}) \tag{70}
\]

\[
\langle N|(iS)|\vec{k}'\vec{p}'\rangle = \frac{-1}{m_N - E_\pi(k') - E_N(p')} \Gamma^*_{N,\pi N}(\vec{k}') \delta(\vec{k}' + \vec{p}') \tag{71}
\]

and

\[
\langle \vec{k}_1, \vec{k}_2, \vec{p}_1|(iS)|\vec{k}'\vec{p}'\rangle = \Gamma^*_{N,\pi N}(k_1) \frac{-\delta(\vec{k}' - \vec{k}_2)\delta(\vec{p}' - \vec{k}_1 - \vec{p}_1)}{E_\pi(k_1) + E_\pi(k_2) + E_N(p_1) - E_\pi(k') - E_N(p')}
\]

\[
= \Gamma^*_{N,\pi N}(k_1) \frac{-\delta(\vec{k}' - \vec{k}_2)\delta(\vec{p}' - \vec{k}_1 - \vec{p}_1)}{E_\pi(k_1) + E_N(p_1) - E_\pi(k') - E_N(p')} \tag{73}
\]

\[
\langle \vec{k}\vec{p}|(iS)|\vec{k}_1, \vec{k}_2, \vec{p}_1\rangle = \Gamma_{N,\pi N}(k_2) \frac{-\delta(\vec{k} - \vec{k}_1)\delta(\vec{p} - \vec{k}_2 - \vec{p}_1)}{E_\pi(k) + E_N(p) - E_\pi(k_1) - E_\pi(k_2) - E_N(P_1)}
\]

\[
= \Gamma_{N,\pi N}(k_2) \frac{-\delta(\vec{k} - \vec{k}_1)\delta(\vec{p} - \vec{k}_2 - \vec{p}_1)}{E_N(p) - E_\pi(k_2) - E_N(P_1)} \tag{74}
\]

With the above matrix elements and recalling that \( H_I^P = 0 \) and \( H_I^Q = \Gamma_{N \rightarrow \pi N} \) for the considered simple case, the matrix element of the effective Hamiltonian Eq. (65) in the center of mass frame (\( \vec{p} = -\vec{k} \) and \( \vec{p}' = -\vec{k}' \)) is

\[
\langle \vec{k}|H_I^f|\vec{k}'\rangle = \frac{1}{2} \sum_I \text{Tr}[(\langle \vec{k}|\Gamma_{N \rightarrow \pi N}|I\rangle \langle I|(iS)|\vec{k}'\rangle - \langle \vec{k}|(iS)|I\rangle \langle I|\Gamma_{N \rightarrow \pi N}|\vec{k}'\rangle)] \tag{75}
\]

The only possible intermediate states are \( |I\rangle = |N\rangle + |\pi(k_1)\pi(k_2)N(P_1)\rangle \). By using Eqs. (70)-(74) we then obtain

\[
\langle \vec{k}|H_I^f|\vec{k}'\rangle = v^{(e)}(\vec{k}, \vec{k}') + v^{(u)}(\vec{k}, \vec{k}') \tag{76}
\]
where

\begin{align*}
    v^{(s)}(\vec{k}, \vec{k}') &= \frac{1}{2} \Gamma^*_{N,\pi N}(k) \left[ \frac{1}{E_\pi(k) + E_N(k) - m_N} + \frac{1}{E_\pi(k') + E_N(k') - m_N} \right] \Gamma_{N,\pi N}(k') \tag{77} \\
    v^{(u)}(\vec{k}, \vec{k}') &= \frac{1}{2} \Gamma^*_{N,\pi N}(k') \left[ \frac{1}{E_N(k) - E_\pi(k') - E_N(\vec{k} + \vec{k}')} \right] \Gamma_{N,\pi N}(k) + \frac{1}{E_N(k') - E_\pi(k) - E_N(\vec{k} + \vec{k}')} \right] \Gamma_{N,\pi N}(k) \tag{78}
\end{align*}

Note that \(v^{(s)}\) of Eq. (77) is due to the intermediate 'physical' nucleon state \(|I\rangle = |N\rangle\). Here we see an important difference between \(v^{(s)}\) and \(v^{(pole)}\) of Eq. (21) for the nucleon-pole term which is due to a bare nucleon state within the 3dBS model. There is no bare mass \(m^0_N\) and energy-dependence in \(v^{(s)}\). This is a consequence of the unitary transformation which eliminates the 'virtual' \(\pi N \leftrightarrow N\) process.

With the above derivations, the effective Hamiltonian Eq. (65) can be explicitly written as

\[ H' = H_0 + V \] (79)

where

\[ H_0 = \int d\vec{k} \left[ E_N(k) b_k^\dagger b_k + E_\pi(k) a_k^\dagger a_k \right] \] (80)

\[ V = \int d\vec{k} d\vec{k}' \left[ v^{(s)}(\vec{k}, \vec{k}') + v^{(u)}(\vec{k}, \vec{k}') \right] a_{-\vec{k}}^\dagger b_{\vec{k}}^\dagger \] (81)

To see further the difference between the models from the unitary transformation method and the 3dBS method, let us first recall how the bound states and resonances are defined in a Hamiltonian formulation. In operator form the reaction amplitude for a Hamiltonian Eq. (79) is defined by

\[ t(E) = V + \frac{1}{E - H_0 + i\epsilon} t(E) \] (82)

or

\[ t(E) = V + \frac{1}{E - H' + i\epsilon} V. \] (83)

The analytic structure of scattering amplitude can be most transparently seen by using the spectral expansion of the Low equation (83)

\[ \langle k' | t(E) | k \rangle = \langle k' | V | k \rangle + \sum_i \frac{\langle k' | V | \Phi_{\epsilon_i} \rangle \langle \Phi_{\epsilon_i} | V | k \rangle}{E - \epsilon_i} \]

\[ + \int_{E_{th}}^\infty dE' \frac{\langle k' | V | \Psi_{E'}^{(+)} \rangle \langle \Psi_{E'}^{(+)} | V | k \rangle}{E - E' + i\epsilon} \] (84)

where \(E_{th}\) is the threshold of the reaction channels, \(\Phi_{\epsilon_i}\) and \(\Psi_{E'}^{(+)}\) are the discrete bound states and the scattering states, respectively. They form a complete set and satisfy

\[ H' \Phi_{\epsilon_i} = \epsilon \Phi_{\epsilon_i} \] (85)

\[ H' \Psi_{E'}^{(+)} = E' \Psi_{E'}^{(+)} \] (86)
Of course bound state energies $\varepsilon_i$ are below the production threshold $E_{th}$. We now note that due to the two-body nature of $V$ defined by Eq. (51), Eq. (55) has the one-nucleon solution $H'|N\rangle = H_0|N\rangle = m_N|N\rangle$. But it does not contribute to the second term of Eq. (54) because $\langle \pi N|V|N\rangle = 0$. Thus the amplitude Eq. (54) does not have a nucleon pole which corresponds to a bound state with the mass of the physical nucleon and is formed by the physical $N$ and $\pi$ of the starting Lagrangian Eq. (53). This is consistent with the experiment.

Here we note that in the Hamiltonian formulation, the amplitude $\langle k'|t(E)|k\rangle$ depends on three independent variables: energy $E$ and momenta $k$ and $k'$. We can analytically continue this amplitude to complex $E$-plane for any $k$ and $k'$. In the complex $E$-plane, the bound state poles and unitarity cuts are on the real axis of the physical sheet and the resonance poles are on the unphysical sheet. These analytic properties with respect to the energy variable $E$ are independent of the momentum variables $k$ and $k'$. As mentioned at the end of section II, the on-shell matrix element $\langle k_0|t(E)|k_0\rangle$ could have poles from the form factors which are needed to regularize the potential $V$. Thus the analytic structure of $\langle k_0|t(E)|k_0\rangle$ can be different from that of the approaches based on dispersion relations.

To further see the differences with the 3dBS models and the approaches based on dispersion relations, let us solve Eq. (52) by considering only $V = v^{(s)}$. The matrix element of the scattering equation defined by Eq. (52) is identical to Eqs. (17)-(18). With the separable form Eq. (77) of $V = v^{(s)}$, the above equations can be solved explicitly. The solution is

$$T(k, k', E) = \frac{N(k, k', E)}{D(E)}$$

with

$$D(E) = (1 - D_1(E))^2 - D_0(E)D_2(E)$$

$$N(k, k', E) = \Gamma^*_{N,\pi N}(k') \left[\frac{D_0(E)}{4} \frac{1}{(E(k) - m)(E(k') - m_N)} + \frac{1}{2} \left(\frac{1}{E(k) - m} + \frac{1}{E(k') - m_N}\right) + D_2(E)\right] \Gamma_{N,\pi N}(k)$$

where $E(k) = E_\pi(k) + E_N(k)$ and

$$D_0(E) = \int_0^\infty q^2 dq \ |\Gamma_{N,\pi N}(q)|^2 \frac{1}{E - E(q) + i\epsilon}$$

$$D_1(E) = \int_0^\infty q^2 dq \ |\Gamma_{N,\pi N}(q)|^2 \frac{1}{2(E - E(q) + i\epsilon)(E(q) - m)}$$

$$D_2(E) = \int_0^\infty q^2 dq \ |\Gamma_{N,\pi N}(q)|^2 \frac{1}{4(E - E(q) + i\epsilon)(E(q) - m)^2}$$

With some inspection, one can see that the amplitude Eq. (87) does not have the nucleon pole condition $D(E = m_N) = 0$ for any $k$ and $k'$. This is what one expects from the spectral expansion Eq. (54). If we take the on-shell matrix element $E = E(k_0) = E(k) = E(k')$, one then finds

$$t(k_0, k_0, E) = \Gamma^*_{\pi NN}(k_0) \left[\frac{d_1(E)}{E - m} + \frac{d_0(E)}{4(E - m^2)} + d_2(E)\right] \Gamma_{\pi NN}(k_0)$$

13
where

\[ d_0(E) = \frac{D_0(E)}{D(E)}, \]  
\[ d_1(E) = \frac{1 - D_1(E)}{D(E)}, \]  
\[ d_2(E) = \frac{D_2(E)}{D(E)}. \]

(94) (95) (96)

The first term in the right-hand side of Eq. (93) does have a pole at \( E = M_N \) of the dispersion relations. But it has additional double poles from the second term as well as from the \( \pi NN \) form factor which is often parameterized as a dipole form \( \Gamma_{\pi NN}(k) = (\Lambda^2/(\Lambda^2 + k^2))^2 \). It should be noted that a pole corresponding to a bound state in a Hamiltonian formulation must be for arbitrary \( k \) and \( k' \). Thus the pole only from the on-shell matrix element Eq. (93) is not a \( \pi N \) bound state with mass \( m_N \). Eq. (93) shows again that a dynamical model deduced from relativistic quantum field theory does not have, and is not required to have, the same analytic structure of the amplitudes in the approaches based on dispersion relations in the S-matrix theory.

To end this section, let us mention that the unitarity condition only requires that an acceptable model must have a unitarity cut in the physical region \( E \geq m_\pi + m_N \). This is trivially satisfied in the model defined by the effective Hamiltonian Eqs. (80)-(81) since the interaction \( V \) is energy independent. This is an important advantage of applying the method of unitary transformation to develop a multi-channels multi-resonances reaction models for investigating meson-nucleon reactions in the nucleon resonance region, as developed in Ref. [4]. In a model with an energy-dependent \( V \) such as the AAY model the unitarity condition is much more difficult to satisfy, and the analytic continuation of the scattering \( t \)-matrix defined by Eqs. (84) to complex \( E \)-plane is in general much more involved.

IV. TIME-ORDERED PERTURBATION THEORY

Treating the Hamiltonian Eq. (56) in time-ordered perturbation theory [14], the matrix elements of the transition operator can be represented by a series expansion defined by all diagrams containing an incoming and outgoing pion-nucleon state

\[ \langle \pi N|t(E)|\pi N \rangle = \langle \pi N|H_I \left( \frac{1}{E - H_0 + i\epsilon} \right) H_I|\pi N \rangle \]
\[ + \langle \pi N|H_I \left( \frac{1}{E - H_0 + i\epsilon} \right) \left( \frac{1}{E - H_0 + i\epsilon} \right) H_I \left( \frac{1}{E - H_0 + i\epsilon} \right) H_I|\pi N \rangle \]
\[ + \cdots \]  
(97)

In the approach of the Julich group [5, 6], the partial sum of this series is written as a three-dimensional integral equation which takes the same form of Eqs. (17)-(18). In the simple model defined by Eq. (57)-(58), the resulting potential can be schematically written as (dropping anti-nucleon terms)

\[ v(\vec{k}, \vec{k'}, E) = v^{(s)}(\vec{k}, \vec{k'}, E) + v^{(u)}(\vec{k}, \vec{k'}, E) \]  
(98)
where

\[
\begin{align*}
    v^{(s)}(\vec{k}, \vec{k}', E) &= \Gamma_{N, \pi N}^{*}(k) \frac{1}{E - m_{N}^{0}} \Gamma_{N, \pi N}(k') \\
    v^{(a)}(\vec{k}, \vec{k}', E) &= \Gamma_{N, \pi N}^{*}(k') \left[ \frac{1}{E - E_{N}(k + k') - E_{\pi}(k) - E_{\pi}(k')} \right] \Gamma_{N, \pi N}(k)
\end{align*}
\]

(99) (100)

As can be readily seen the term \(v^{a}(E)\) has the singularity of the \(\pi \pi N\) cut, and they depart from the starting Hamiltonian Eq. (57)-(58) by using the bare mass \(m_{N}^{0}\) to define the \(s\)-channel term \(v^{(s)}\). The above potential has the same form of the matrix element of that defined by Eqs. (20)-(21). Thus the interpretation of their analysis of nucleon pole term is similar to what described in section II.

V. SUMMARY

In this paper, we have examined three methods for constructing meson-nucleon reaction models from relativistic quantum field theory. For the models based on the three-dimensional reductions of Bethe-Salpeter equation and the time-ordered perturbation theory, the driving terms of the resulting three-dimensional scattering equations in general contain a nucleon pole term determined by a bare nucleon \(N_{0}\). We show that the commonly used procedure of imposing the nucleon pole condition to fix the bare nucleon parameters is related to the assumption that the nucleon is a bound state made of a bare core \(N_{0}\) and meson cloud. To correctly implement this nucleon substructure into the scattering equation, it is necessary to consider \(\pi \pi N\) unitarity condition as achieved within the model of Aaron, Amado and Young [12].

We have given a pedagogical and explicit explanation of the method of unitary transformation which has been applied in recent years to investigate meson-nucleon reactions [2, 4, 15, 16, 17]. Since only physical nucleons and pions are the basic degrees of freedom of the derived effective Hamiltonian, the resulting \(\pi N\) amplitude does not have a nucleon pole at \(E = M_{N}\) in the complex \(E\)-plane. This is due to the fact that the one-nucleon problem is decoupled from the two-particle problem by the unitary transformation and the resulting \(\pi N\) potential is energy independent. There is no \(\pi N\) interaction due to the propagation of a bare nucleon in this formulation. We explain how this can be understood from the general principles of a Hamiltonian formulation of reactions.

We also show that the scattering amplitudes from these three methods do not have the same analytic structure of the amplitude from the approach based on the dispersion relations of the S-matrix theory. Even one imposes the nucleon pole condition to make the connection to the S-matrix theory, the constructed models can have poles from the form factors which are needed to regularize the potentials for solving the resulting scattering equations. We emphasize that there is no compelling and rigorous theoretical argument to request that the constructed models should have the analytic structure of the dispersion relations. There exits no rigorous derivation of the dispersion relations of S-matrix theory from relativistic quantum field theory [7]. Historically, the S-matrix theory is considered as an alternative to relativistic quantum field theory to study strong interactions. Either one of them is a good starting point for developing phenomenological models for analyzing the data. For investigating multi-channels and multi-resonances reactions, the models deduced from relativistic field theory appear to be more practical.
Acknowledgments

We thank C. Hanhart for bringing our attention to the double pole of the on-shell amplitude Eq. (93) in the model based on the method of unitary transformation. This work is supported by the U.S. Department of Energy, Office of Nuclear Physics Division, under contract No. DE-AC02-06CH11357, and Contract No. DE-AC05-060R23177 under which Jefferson Science Associates operates Jefferson Lab, and by the Japan Society for the Promotion of Science, Grant-in-Aid for Scientific Research(c) 20540270. This work is also partially supported by Grant No. FIS2008-01661/FIS from MCIIN and CPAN CSD2007-00042 Consolider Ingenio 2010 (Spain).

[1] As reviewed and analyzed by A. Klein and T.-S. H. Lee, Phys. Rev. D 10, 4308 (1974).
[2] T. Sato and T.-S. H. Lee, Phys. Rev. C 54, 2660 (1996).
[3] T. Sato, M. Kobayashi, and H. Ohtsubo, Prog. Theor. Phys. 68, 840 (1982); 98, 927 (1997).
[4] A. Matsuyama, T. Sato, T.-S. H. Lee Phys. Reports 439, 193 (2007).
[5] D. Lohse, J.W. Durso, K. Holinde and J. Speth, Nucl. Phys. A516,513 (1990).
[6] O. Krehl, C. Hanhart, S. Krewald, and J. Speth, Phys. Rev. C 60, 055206 (1999); C 62, 025207 (2000).
[7] M.L. Goldberger and K. Watson, Collision Theory (Robert E. Krieger Publishing Company, 1975), chapter 10.
[8] C. Itzykson and J.B. Zuber, Quantum Field Theory, (McGraw-Hill, New York, 1980), Chapter 10.
[9] C.T. Hung, S.N. Yang, T.-S. H. Lee, Phys. Rev. C64, 034309 (2001).
[10] V.G. Kadyshevsky, Nucl. Phys. B6, 125 (1968).
[11] B.C. Pearce and I.R. Afnan, Phys. Rev. C 34, 991 (1986).
[12] As reviewed by R. D. Amado and R. Aaron, in Modern Three-Hadron Physics, ed. A. W. Thomas, Topics in Current Physics (Springer-Verlag, 1977).
[13] R. Blankenbecler and R. Sugar, Phys. Rev. 142, 1051 (1966).
[14] S. S. Schweber, An Introduction to Relativistic Quantum Field Theory (Haper and Row, New York, 1962).
[15] B. Julia-Diaz, T.-S. H. Lee, A. Matsuyama, T. Sato, Phys.Rev. C76, 065201 (2007).
[16] B. Julia-Diaz, T.-S. H. Lee, A. Matsuyama, T. Sato, L.C. Smith, Phys.Rev. C77, 045205 (2008).
[17] H. Kamano, B. Julia-Diaz, T.-S. H. Lee, A. Matsuyama, T. Sato, submitted to Phys. Rev. C (2008).