Equivalence Classes of Full-Dimensional
0/1-Polytopes with Many Vertices

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Abstract

Let $Q_n$ denote the $n$-dimensional hypercube with the vertex set $V_n = \{0, 1\}^n$. A 0/1-polytope of $Q_n$ is defined to be the convex hull of a subset of $V_n$. This paper is concerned with the enumeration of equivalence classes of full-dimensional 0/1-polytopes under the symmetries of the hypercube. With the aid of a computer program, Aichholzer completed the enumeration of equivalence classes of full-dimensional 0/1-polytopes for $Q_4$, $Q_5$, and those of $Q_6$ up to 12 vertices. In this paper, we present a method to compute the number of equivalence classes of full-dimensional 0/1-polytopes of $Q_n$ with more than $2^n-3$ vertices. As an application, we finish the counting of equivalence classes of full-dimensional 0/1-polytopes of $Q_6$ with more than 12 vertices.

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1 Introduction

Let $Q_n$ denote the $n$-dimensional hypercube with vertex set $V_n = \{0, 1\}^n$. A 0/1-polytope of $Q_n$ is defined to be the convex hull of a subset of $V_n$. The study of 0/1-polytopes has drawn much attention from different points of view, see, for example, [7, 8, 12, 13, 14, 16, 22], see also the survey of Ziegler [21].

In this paper, we are concerned with the problem of determining the number of equivalence classes of $n$-dimensional 0/1-polytopes of $Q_n$ under the symmetries of $Q_n$, which has been considered as a difficult problem, see Ziegler [21]. It is also listed by Zong [22, Problem 5.1] as one of the fundamental problems concerning 0/1-polytopes.

An $n$-dimensional 0/1-polytope of $Q_n$ is also called a full-dimensional 0/1-polytope of $Q_n$. Two 0/1-polytopes are said to be equivalent if one can be transformed to the other by a symmetry of $Q_n$. Such an equivalence relation is also called the 0/1-equivalence relation. Figure 1 gives representatives of 0/1-equivalence classes of $Q_2$, among which (d) and (e) are full-dimensional.

Sarangarajan and Ziegler [21, Proposition 8] found an lower bound on the number of equivalence classes of full-dimensional 0/1-polytopes of $Q_n$. As far as exact enumeration is
concerned, full-dimensional 0/1-equivalence classes of $Q_4$ were counted by Alexx Below, see Ziegler [21]. With the aid of a computer program, Aichholzer [1] completed the enumeration of full-dimensional 0/1-equivalence classes of of $Q_5$, and those of $Q_6$ up to 12 vertices, see Aichholzer [3] and Ziegler [21]. The 5-dimensional hypercube $Q_5$ has been considered as the last case that one can hope for a complete solution to the enumeration of full-dimensional 0/1-equivalence classes.

The objective of this paper is to present a method to compute the number of full-dimensional 0/1-equivalence classes of $Q_n$ with more than $2^{n-3}$ vertices. As an application, we solve the enumeration problem for full-dimensional 0/1-equivalence classes of the 6-dimensional hypercube with more than 12 vertices.

To describe our approach, we introduce some notation. Denote by $A_n(k)$ (resp., $F_n(k)$) the set of (resp., full-dimensional) 0/1-equivalence classes of $Q_n$ with $k$ vertices. Let $H_n(k)$ be the set of 0/1-equivalence classes of $Q_n$ with $k$ vertices that are not full-dimensional. The cardinalities of $A_n(k)$, $F_n(k)$ and $H_n(k)$ are denoted respectively by $A_n(k)$, $F_n(k)$ and $H_n(k)$. It is clear that any full-dimensional 0/1-polytope of $Q_n$ has at least $n+1$ vertices, i.e., $F_n(k) = 0$ for $1 \leq k \leq n$.

The starting point of this paper is the following obvious relation

$$F_n(k) = A_n(k) - H_n(k).$$

The number $A_n(k)$ can be computed based on the cycle index of the hyperoctahedral group. We can deduce that $H_n(k) = 0$ for $k > 2^{n-1}$ based on a result due to Saks. For the purpose of computing $H_n(k)$ for $2^{n-2} < k \leq 2^{n-1}$, we transform the computation of $H_n(k)$ to the determination of the number of equivalence classes of 0/1-polytopes with $k$ vertices that are contained in the spanned hyperplanes of $Q_n$. To be more specific, we show that $H_n(k)$ for $2^{n-2} < k \leq 2^{n-1}$ can be decomposed into a disjoint union of equivalence classes of 0/1-polytopes that are contained in the spanned hyperplanes of $Q_n$. In particular, for $n = 6$ and $k > 16$, we obtain the number of full-dimensional 0/1-equivalence classes of $Q_6$ with $k$ vertices.

Using a similar idea as in the case $2^{n-2} < k \leq 2^{n-1}$, we can compute $H_n(k)$ for $2^{n-3} < k \leq 2^{n-2}$. For $n = 6$ and $13 \leq k \leq 16$, we obtain the number of full-dimensional 0/1-equivalence classes of $Q_6$ with $k$ vertices. Together with the computation of Aichholzer up to 12 vertices, we have completed the enumeration of full-dimensional 0/1-equivalence classes of the 6-dimensional hypercube.
2 The cycle index of the hyperoctahedral group

The group of symmetries of $Q_n$ is known as the hyperoctahedral group $B_n$. In this section, we review the cycle index of $B_n$ acting on the vertex set $V_n$. Since 0/1-equivalence classes of $Q_n$ coincide with nonisomorphic vertex colorings of $Q_n$ by using two colors, we may compute the number $A_n(k)$ from the cycle index of $B_n$.

Let $G$ be a group acting on a finite set $X$. For any $g \in G$, $g$ induces a permutation on $X$. The cycle type of a permutation is defined to be a multiset \(\{c_1, c_2, \ldots\}\), where $c_i$ is the number of cycles of length $i$ that appear in the cycle decomposition of the permutation. For $g \in G$, denote by $c(g) = \{c_1, c_2, \ldots\}$ the cycle type of the permutation on $X$ induced by $g$. Let $z = (z_1, z_2, \ldots)$ be a sequence of indeterminants, and let $z^{c(g)} = z_1^{c_1} z_2^{c_2} \cdots$.

The cycle index of $G$ is defined as follows

\[
Z_G(z) = Z_G(z_1, z_2, \ldots) = \frac{1}{|G|} \sum_{g \in G} z^{c(g)}. \tag{2.1}
\]

According to Pólya’s theorem, the cycle index in (2.1) can be applied to count nonisomorphic colorings of $X$ by using a given number of colors.

For a vertex coloring of $Q_n$ with two colors, say, black and white, the black vertices can be considered as vertices of a 0/1-polytope of $Q_n$. This establishes a one-to-one correspondence between equivalence classes of vertex colorings and 0/1-equivalence classes of $Q_n$. Let $Z_n(z)$ denote the cycle index of $B_n$ acting on the vertex set $V_n$. Then, by Pólya’s theorem

\[
A_n(k) = \left[ u_1^k u_2^{2^n-k} \right] C_n(u_1, u_2), \tag{2.2}
\]

where $C_n(u_1, u_2)$ is the polynomial obtained from $Z_n(z)$ by substituting $z_i$ with $u_1^i + u_2^i$, and $\left[ u_1^p u_2^q \right] C_n(u_1, u_2)$ denotes the coefficient of $u_1^p u_2^q$ in $C_n(u_1, u_2)$.

Clearly, the total number of 0/1-equivalence classes of $Q_n$ is given by

\[
\sum_{k=1}^{2^n} A_n(k) = C_n(1, 1). \tag{2.3}
\]

It should be noted that $C_n(1, 1)$ also equals the number of types of Boolean functions, see Chen \[10\] and references therein. This number is also related to configurations of $n$-dimensional Orthogonal Pseudo-Polytopes, see, e.g., Aguila \[5\]. The computation of $Z_n(z)$ has been studied by Chen \[10\], Harrison and High \[15\], and Pólya \[18\], etc. Explicit expressions of $Z_n(z)$ for $n \leq 6$ can be found in \[5\], and we list them bellow.

\[
Z_1(z) = z_1,
\]
\[ Z_2(z) = \frac{1}{8} \left( z_1^4 + 2z_1^2z_2 + 3z_2^2 + 2z_4 \right), \]
\[ Z_3(z) = \frac{1}{48} \left( z_1^8 + 6z_1^4z_2^2 + 13z_1^4 + 8z_2^2z_3^2 + 12z_4^2 + 8z_2z_6 \right), \]
\[ Z_4(z) = \frac{1}{384} \left( z_1^{16} + 12z_1^8z_2^4 + 12z_1^4z_2^6 + 51z_2^8 + 48z_3^2 \right), \]
\[ + 48z_1^2z_2z_3^3 + 84z_4^4 + 96z_2^2z_5^2 + 32z_4^4z_3^2 \right), \]
\[ Z_5(z) = \frac{1}{3840} \left( z_1^{32} + 20z_1^{16}z_2^8 + 60z_1^8z_2^{12} + 231z_2^{16} + 80z_1^8z_3^8 + 240z_1^4z_2^2z_4^6 \right), \]
\[ + 240z_1^2z_2^6 + 520z_4^8 + 384z_1^2z_6^8 + 160z_1^4z_2^4z_3^4z_6^4 + 720z_1^2z_6^4 \right), \]
\[ + 48z_8^4 + 384z_2z_3^3 + 320z_4^2z_6^2 \)
\[ Z_6(z) = \frac{1}{46080} \left( z_1^{64} + 30z_1^{32}z_2^{16} + 180z_1^{16}z_2^{24} + 120z_1^8z_2^{28} + 1053z_2^{32} + 160z_1^8z_3^{16} + \right) \]
\[ 640z_1^4z_2^{30} + 720z_1^8z_2^{42} + 1440z_1^4z_2^{12} + 2160z_2^{32}z_4^{12} + 4920z_4^{16} + \]
\[ 2304z_1^4z_2^{12} + 960z_1^4z_2^{32}z_3^8 + 5280z_1^8z_2^8 + 3840z_2^2z_2^8z_3^8 + 5760z_8^8 \right), \]
\[ + 1920z_2^2z_6^8 + 6912z_2^2z_10^8 + 3840z_2^4z_12^6 + 3840z_4^2z_12^8 \right). \]

The method of Chen for computing \( Z_n(z) \) is based on the cycle structure of a power of a signed permutation. Let us recall the notation of a signed permutation. A signed permutation on \( \{1, 2, \ldots, n\} \) is a permutation on \( \{1, 2, \ldots, n\} \) with a + or a − sign attached to each element 1, 2, \ldots, n. Following the notation in Chen [10] or Chen and Stanley [11], we may write a signed permutation in terms of the cycle decomposition and ignore the plus sign +. For example, \((245)(3)(16)\) represents a signed permutation, where \((245)(3)(16)\) is its underlying permutation. The action of a signed permutation \( w \) on the vertices of \( Q_n \) is defined as follows. For a vertex \( (x_1, x_2, \ldots, x_n) \) of \( Q_n \), we define \( w((x_1, x_2, \ldots, x_n)) \) to be the vertex \( (y_1, y_2, \ldots, y_n) \) as given by

\[
y_i = \begin{cases} 
  x_{\pi(i)}, & \text{if } i \text{ has the sign } +, \\
  1 - x_{\pi(i)}, & \text{if } i \text{ has the sign } -,
\end{cases}
\]  

(2.4)

where \( \pi \) is the underlying permutation of \( w \).

For the purpose of this paper, we define the cycle type of a signed permutation \( w \in B_n \) as the cycle type of its underlying permutation. For example, \((245)(3)(16)(7)\) has cycle type \( \{1^2, 2, 3\} \). We should note that the above definition of a cycle type of a signed permutation is different from the definition in terms of double partitions as in [10] because it will be shown in Section 5 that any signed permutation that fixes a spanned hyperplane of \( Q_n \) either have all positive cycles or all negative cycles.

We end this section with the following formula of Chen [10], which will be used in Section 6 to compute the cycle index of the group that fixes a spanned hyperplane of \( Q_n \).
**Theorem 2.1** Let $G$ be a group that acts on some finite set $X$. For any $g \in G$, the number of $i$-cycles of the permutation on $X$ induced by $g$ is given by

$$\frac{1}{i} \sum_{j|i} \mu(i/j) \psi(g^j),$$

where $\mu$ is the classical number-theoretic Möbius function and $\psi(g^j)$ is the number of fixed points of $g^j$ on $X$.

### 3 0/1-Polytopes with many vertices

In this section, we find an inequality concerning the dimension of a $0/1$-polytope of $Q_n$ and the number of its vertices. This inequality plays a key role in the computation of $F_n(k)$ for $k > 2^{n-3}$.

The main theorem of this section is given below.

**Theorem 3.1** Let $P$ be a $0/1$-polytope of $Q_n$ with more than $2^{n-s}$ vertices, where $1 \leq s \leq n$. Then we have

$$\dim(P) \geq n - s + 1.$$ 

The above theorem can be deduced from the following assertion.

**Theorem 3.2** For any $1 \leq s \leq n$, the intersection of $s$ hyperplanes in $\mathbb{R}^n$ with linearly independent normal vectors contains at most $2^{n-s}$ vertices of $Q_n$.

Indeed, it is not difficult to see that Theorem 3.2 implies Theorem 3.1. Let $P$ be a $0/1$-polytope of $Q_n$ with more than $2^{n-s}$ vertices. Suppose to the contrary that $\dim(P) \leq n - s$. It is known that the affine space spanned by $P$ can be expressed as the intersection of a collection of hyperplanes. Since $\dim(P) \leq n - s$, there exist $s$ hyperplanes $H_1, H_2, \ldots, H_s$ whose normal vectors are linearly independent such that the intersection of $H_1, H_2, \ldots, H_s$ contains $P$. Let $V(P)$ denote the vertex set of $P$. By Theorem 3.2, we have

$$|V(P)| \leq \left| \left( \bigcap_{i=1}^{s} H_i \right) \cap V_n \right| \leq 2^{n-s},$$

which is a contradiction to the assumption that $P$ contains more than $2^{n-s}$ vertices of $Q_n$. So we conclude that $\dim(P) \geq n - s + 1$.

**Proof of Theorem 3.2** Assume that, for $1 \leq i \leq s$,

$$H_i: a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$$

...
are \( s \) hyperplanes in \( \mathbb{R}^n \), whose normal vectors \( a_i = (a_{i1}, \ldots, a_{in}) \) are linearly independent. We aim to show that the intersection of \( H_1, H_2, \ldots, H_s \) contains at most \( 2^{n-s} \) vertices of \( Q_n \). We may express the intersection of \( H_1, H_2, \ldots, H_s \) as the solution of a system of linear equations, that is,

\[
\bigcap_{i=1}^{s} H_i = \{ x^T : Ax = b \},
\]

(3.1)

where \( A \) denotes the matrix \( (a_{ij})_{1 \leq i \leq s, 1 \leq j \leq n} \), \( x = (x_1, x_2, \ldots, x_n)^T \), and \( b = (b_1, \ldots, b_s)^T \). \( T \) denotes the transpose of a vector. Then Theorem 3.64 is equivalent to the following inequality

\[
|V_n \cap \{ x^T : Ax = b \}| \leq 2^{n-s}.
\]

(3.2)

We now proceed to prove (3.2) by induction on \( n \) and \( s \). We first consider the case \( s = 1 \). Suppose that \( H : c_1 x_1 + c_2 x_2 + \cdots + c_n x_n = c \) is a hyperplane in \( \mathbb{R}^n \). Assume that among the coefficients \( c_1, c_2, \ldots, c_n \) there are \( i \) of them that are nonzero. Without loss of generality, we may assume that \( c_1, c_2, \ldots, c_i \) are nonzero, and \( c_{i+1} = c_{i+2} = \cdots = c_n = 0 \). Clearly, \( H \) reduces to a hyperplane in the \( i \)-dimensional Euclidean space \( \mathbb{R}^i \). Such a hyperplane with nonzero coefficients is called a skew hyperplane. Now the vertices of \( Q_n \) contained in \( H \) are of the form \( (d_1, \ldots, d_i, d_{i+1}, \ldots, d_n) \) where \( (d_1, \ldots, d_i) \) are vertices of \( Q_i \) contained in the skew hyperplane \( H : c_1 x_1 + c_2 x_2 + \cdots + c_i x_i = b \). Clearly, for each vertex \( (d_1, d_2, \ldots, d_i) \) in \( H \), there are \( 2^{n-i} \) choices for \((d_{i+1}, d_{i+2}, \ldots, d_n)\) such that \((d_1, d_2, \ldots, d_n)\) is contained in \( H \). Using Sperner’s lemma (see, for example, Lubell [17]), Saks [19] Theorem 3.64] has shown that the number of vertices of \( Q_i \) contained in a skew hyperplane does not exceed \( (\frac{1}{2})^i \). Let

\[
f(n, i) = 2^{n-i} \left( \frac{i}{2} \right).
\]

Thus the number of vertices of \( Q_n \) contained in \( H \) is at most \( f(n, i) \). It is easy to check that

\[
\frac{f(n, i)}{f(n, i + 1)} = \begin{cases} \frac{i+2}{i+1}, & \text{if } i \text{ is even}, \\ 1, & \text{if } i \text{ is odd}. \end{cases}
\]

This yields \( f(n, i) \geq f(n, i + 1) \) for any \( i = 1, 2, \ldots, n - 1 \). Hence \( H \) contains at most \( f(n, 1) = 2^{n-1} \) vertices of \( Q_n \), which implies (3.2) for \( s = 1 \).

We now consider the case \( s = n \). In this case, since the normal vectors \( a_1, \ldots, a_n \) are linearly independent, the square matrix \( A \) is nonsingular. It follows that \( Ax = b \) has exactly one solution. Therefore, inequality (3.2) holds when \( s = n \).

So we are left with cases of \( n, s \) such that \( 1 < s < n \). We shall use induction to complete the proof. Suppose that (3.2) holds for \( n', s' \) such that \( n' \leq n, s' \leq s \) and \((n', s') \neq (n, s)\).

Since the normal vector \( a_1 \) is nonzero, there exists some \( j_0 \) \((1 \leq j_0 \leq n)\) such that \( a_{1j_0} \neq 0 \). Without loss of generality, we may assume \( a_{ij_0} = 0 \) for \( 2 \leq i \leq s \) since
one can apply elementary row transformations to the system of linear equations $Ax = b$ to ensure that the assumption is valid. For a vector $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$, let $v^j = (v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n) \in \mathbb{R}^{n-1}$ be the vector obtained from $v$ by deleting the $j$-th coordinate. We now have two cases.

Case 1. The vectors $a_1^{j_0}, \ldots, a_s^{j_0}$ are linearly dependent. Since $a_2, \ldots, a_s$ are linearly independent and $a_{i,j_0} = 0$ for $2 \leq i \leq s$, it is clear that $a_2^{j_0}, \ldots, a_s^{j_0}$ are linearly independent. So the vector $a_1^{j_0}$ can be expressed as a linear combination of $a_2^{j_0}, \ldots, a_s^{j_0}$. Assume that $a_1^{j_0} = \alpha_2 a_2^{j_0} + \cdots + \alpha_s a_s^{j_0}$, where $\alpha_k \in \mathbb{R}$ for $2 \leq k \leq s$. For $2 \leq k \leq s$, multiplying the $k$-th row by $\alpha_k$ and subtracting it from the first row, then the first equation $a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1$ becomes

$$a_{1j_0} x_{j_0} = b_1 - \sum_{k=2}^s \alpha_k b_k.$$  \hspace{1cm} (3.3)

Let $A' = (a_2, \ldots, a_s)^T$ and $b' = (b_2, \ldots, b_s)^T$. Note that all entries in the $j_0$-th column of $A'$ are zero since we have assumed $a_{i,j_0} = 0$ for $2 \leq i \leq s$. Let $A_{j_0}'$ be the matrix obtained from $A'$ by removing this zero column. From equation (3.3), the value $x_{j_0}$ in the $j_0$-th coordinate of the solutions of $Ax = b$ is fixed. Then solutions of $Ax = b$ can be obtained from the solutions of $A_{j_0}' x = b'$ by adding the value of $x_{j_0}$ to the $j_0$-th coordinate. Concerning the number of vertices of $Q_n$ contained in $\{x^T : Ax = b \}$, we consider the following two cases.

(1). The value $x_{j_0}$ is not equal to 0 or 1. In this case, no vertex of $Q_n$ is contained in $\{x^T : Ax = b \}$. Hence inequality (3.2) holds.

(2). The value $x_{j_0}$ is equal to 0 or 1. Since every vertex of $Q_n$ contained in $\{x^T : Ax = b \}$ is obtained from a vertex of $Q_{n-1}$ contained in $\{x^T : A'_{j_0} x = b' \}$ by adding $x_{j_0}$ in the $j_0$-th coordinate, it follows that

$$| V_n \cap \{x^T : Ax = b \} | = | V_{n-1} \cap \{x^T : A'_{j_0} x = b' \} |.$$  \hspace{1cm} (3.4)

By the induction hypothesis, we find

$$| V_{n-1} \cap \{x^T : A'_{j_0} x = b' \} | \leq 2^{(n-1)-(s-1)} = 2^{n-s}.$$  

In view of (3.4), we obtain (3.2).

Case 2. Suppose $a_1^{j_0}, \ldots, a_s^{j_0}$ are linearly independent. Assume that the value of $x_{j_0}$ in the solutions of $\{x^T : Ax = b \}$ can be taken 0 or 1. Then the vertices of $Q_n$ contained in $\{x^T : Ax = b \}$ can be decomposed into a disjoint union of the following two sets

$$S_0 = V_n \cap \{x^T : x_{j_0} = 0, \ Ax = b \}$$

and

$$S_1 = V_n \cap \{x^T : x_{j_0} = 1, \ Ax = b \}.$$
We first consider the set $S_0$. Let $A'' = (a_{10}^{j_0}, \ldots, a_{s0}^{j_0})^T$ be the matrix obtained from $A$ by deleting the $j_0$-th column. Then vertices of $Q_n$ contained in $S_0$ are obtained from the vertices of $Q_{n-1}$ contained in $\{x^T: A''x = b\}$ by adding 0 to the $j_0$-th coordinate. So we have

$$|S_0| = |V_{n-1} \cap \{x^T: A''x = b\}| \leq 2^{n-1-s},$$

where the inequality follows from the induction hypothesis. Similarly, we get $|S_1| \leq 2^{n-1-s}$. Hence

$$|V_n \cap \{x^T: Ax = b\}| = |S_0| + |S_1| \leq 2^{n-s}.$$

Combining the above two cases, inequality (3.2) is true for $1 \leq s \leq n$. This completes the proof.

Note that the upper bound $2^{n-s}$ is sharp. For example, it is easy to see the intersection of hyperplanes $x_i = 0$ ($1 \leq i \leq s$) contains exactly $2^{n-s}$ vertices of $Q_n$.

By Theorem 3.2, we see that every 0/1-polytope of $Q_n$ with more than $2^{n-1}$ vertices is full-dimensional. As a direct consequence, we obtain the following relation.

**Corollary 3.3** For $k > 2^{n-1}$, we have

$$F_n(k) = A_n(k).$$

Form Corollary 3.3 the number $F_n(k)$ for $k > 2^{n-1}$ can be computed from the cycle index of the hyperoctahedral group, that is, for $k > 2^{n-1}$

$$F_n(k) = [u_1^k u_2^{2n-k}] C_n(u_1, u_2).$$

For $n = 4, 5$ and 6, the values of $F_n(k)$ for $k > 2^{n-1}$ are given in Tables 1, 2 and 3.

| $k$ | 9   | 10  | 11  | 12  | 13  | 14  | 15  | 16  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $F_4(k)$ | 56  | 50  | 27  | 19  | 6   | 4   | 1   | 1   |

Table 1: $F_4(k)$ for $k > 8$.

| $k$ | 17  | 18  | 19  | 20  | 21  | 22  | 23  | 24  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $F_5(k)$ | 158658 | 133576 | 98804 | 65664 | 38073 | 19963 | 9013 | 3779 |
| $k$ | 25  | 26  | 27  | 28  | 29  | 30  | 31  | 32  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $F_5(k)$ | 1326 | 472 | 131 | 47  | 29  | 5   | 1   | 1   |

Table 2: $F_5(k)$ for $k > 16$. 

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Table 3: $F_6(k)$ for $k > 32$.

$$
\begin{array}{|c|c|c|}
\hline
k & F_6(k) & k & F_6(k) \\
\hline
33 & 38580161986426 & 49 & 3492397119 \\
34 & 35176482187398 & 50 & 1052201890 \\
35 & 30151914536933 & 51 & 290751447 \\
36 & 24289841497881 & 52 & 73500514 \\
37 & 18382330104696 & 53 & 16938566 \\
38 & 13061946976545 & 54 & 3561696 \\
39 & 8708686182967 & 55 & 681474 \\
40 & 5443544478011 & 56 & 120843 \\
41 & 3186944273554 & 57 & 19735 \\
42 & 1745593733454 & 58 & 3253 \\
43 & 893346071377 & 59 & 497 \\
44 & 426539774378 & 60 & 103 \\
45 & 189678764492 & 61 & 16 \\
46 & 78409442414 & 62 & 6 \\
47 & 30064448972 & 63 & 1 \\
48 & 10666911842 & 64 & 1 \\
\hline
\end{array}
$$

4 $H_n(k)$ for $2^{n-2} < k \leq 2^{n-1}$

In this section, we shall aim to compute $H_n(k)$ for $2^{n-2} < k \leq 2^{n-1}$. We shall show that in this case the number $H_n(k)$ is determined by the number of (partial) 0/1-equivalence classes of a spanned hyperplane of $Q_n$ with $k$ vertices. To this end, it is necessary to consider all possible spanned hyperplanes of $Q_n$. More precisely, we need representatives of equivalence classes of such spanned hyperplanes.

Recall that a spanned hyperplane of $Q_n$ is a hyperplane in $\mathbb{R}^n$ spanned by $n$ affinely independent vertices of $Q_n$, that is, the affine space spanned by the vertices of $Q_n$ contained in this hyperplane is of dimension $n-1$. Let

$$H: a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

be a spanned hyperplane of $Q_n$, where $|a_1|, \ldots, |a_n|, |b|$ are positive integers with greatest common divisor 1. Let

$$\text{coeff}(n) = \max\{|a_1|, \ldots, |a_n|\}.$$ 

It is clear that coeff(2) = coeff(3) = 1. The study of upper and lower bounds on the number coeff$(n)$ has drawn much attention, see, for example, [4, 6, 9, 21]. The following are known bounds on coeff$(n)$ and $|b|$, see, e.g., [21 Corollary 26] and [4 Theorem 5],

$$\frac{(n-1)^{(n-1)/2}}{2^{2n+o(n)}} \leq \text{coeff}(n) \leq \frac{n^{n/2}}{2^{n-1}} \quad \text{and} \quad |b| \leq 2^{-n}(n+1)^{\frac{n+1}{2}}.$$
Using the above bounds, Aichholzer and Aurenhammer [4] obtained the exact values of \( \text{coeff}(n) \) for \( n \leq 8 \) by computing all possible spanned hyperplanes of \( Q_n \) up to dimension 8. For example, they showed that \( \text{coeff}(4) = 2 \), \( \text{coeff}(5) = 3 \), and \( \text{coeff}(6) = 5 \).

As will be seen, in order to compute \( H_n(k) \) for \( 2^{n-2} < k \leq 2^{n-1} \), we need to consider equivalence classes of spanned hyperplanes of \( Q_n \) under the symmetries of \( Q_n \). Note that the symmetries of \( Q_n \) can be expressed by permuting the coordinates and changing \( x_i \) to \( 1 - x_i \) for some indices \( i \). Therefore, for each equivalence class of spanned hyperplanes of \( Q_n \), we can choose a representative of the following form

\[
a_1 x_1 + a_2 x_2 + \cdots + a_t x_t = b, \tag{4.1}
\]

where \( t \leq n \) and \( 0 < a_1 \leq a_2 \leq \cdots \leq a_t \leq \text{coeff}(n) \).

A complete list of spanned hyperplanes of \( Q_n \) for \( n \leq 6 \) can been found in [2]. The following hyperplanes are representatives of equivalence classes of spanned hyperplanes of \( Q_4 \):

\[
\begin{align*}
x_1 &= 0, \\
x_1 + x_2 &= 1, \\
x_1 + x_2 + x_3 &= 1, \\
x_1 + x_2 + x_3 + x_4 &= 1 \text{ or } 2, \\
x_1 + x_2 + x_3 + 2x_4 &= 2.
\end{align*}
\]

In addition to the above hyperplanes of \( \mathbb{R}^4 \), which can also be viewed as spanned hyperplanes of \( Q_5 \), we have the following representatives of equivalence classes of spanned hyperplanes of \( Q_5 \):

\[
\begin{align*}
x_1 + x_2 + x_3 + x_4 + x_5 &= 1 \text{ or } 2, \\
x_1 + x_2 + x_3 + x_4 + 2x_5 &= 2 \text{ or } 3, \\
x_1 + x_2 + x_3 + 2x_4 + 2x_5 &= 2 \text{ or } 3, \\
x_1 + x_2 + 2x_3 + 2x_4 + 2x_5 &= 2 \text{ or } 3, \\
x_1 + x_2 + x_3 + x_4 + 3x_5 &= 3, \\
x_1 + x_2 + x_3 + 2x_4 + 3x_5 &= 3, \\
x_1 + x_2 + 2x_3 + 2x_4 + 3x_5 &= 4.
\end{align*}
\]

When \( n = 6 \), for the purpose of computing \( F_6(k) \) for \( 16 < k \leq 32 \), we need the representatives of equivalence classes of spanned hyperplanes of \( Q_6 \) containing more than 16 vertices of \( Q_6 \). There are 6 such representatives as given below:

\[
\begin{align*}
x_1 &= 0, \\
x_1 + x_2 &= 1,
\end{align*}
\]
Clearly, two spanned hyperplanes of $Q_n$ in the same equivalence class contain the same number of vertices of $Q_n$. So we may say that an equivalence class of spanned hyperplanes of $Q_n$ contains $k$ vertices of $Q_n$ if every hyperplane in this class contains $k$ vertices of $Q_n$.

To state the main result of this section, we need to define the equivalence classes of $0/1$-polytopes contained in a set of points in $\mathbb{R}^n$. Given a set $\mathcal{S} \subset \mathbb{R}^n$, consider the set of $0/1$-polytopes of $Q_n$ that are contained in $\mathcal{S}$, denoted by $\mathcal{S}(Q_n)$. Restricting the $0/1$-equivalence relation to the set $\mathcal{S}(Q_n)$ indicates a equivalence relation on $\mathcal{S}(Q_n)$. More precisely, two $0/1$-polytopes in $\mathcal{S}(Q_n)$ are equivalent if one can be transformed to the other by a symmetry of $Q_n$. We call equivalence classes of $0/1$-polytopes in $\mathcal{S}(Q_n)$ partial $0/1$-equivalence classes of $\mathcal{S}$ for the reason that any partial equivalence class of $\mathcal{S}$ is a subset of a (unique) $0/1$-equivalence class of $Q_n$. Notice that for a $0/1$-polytope $P$ contained in $\mathcal{S}$ and a symmetry $w$, $w(P)$ is not in the partial $0/1$-equivalence class of $P$ when $w(P)$ is not in $\mathcal{S}(Q_n)$. Denote by $\mathcal{P}(\mathcal{S},k)$ the set of partial $0/1$-equivalence classes of $\mathcal{S}$ with $k$ vertices. Let $N_{\mathcal{S}}(k)$ be the cardinality of $\mathcal{P}(\mathcal{S},k)$.

Let $h(n,k)$ denote the number of equivalence classes of spanned hyperplanes of $Q_n$ that contain at least $k$ vertices of $Q_n$. Assume that $H_1, H_1, \ldots, H_{h(n,k)}$ are the representatives of equivalence classes of spanned hyperplanes of $Q_n$ containing at least $k$ vertices of $Q_n$. Recall that $\mathcal{H}_n(k)$ denotes the set of $0/1$-equivalence classes of $Q_n$ with $k$ vertices that are not full-dimensional. We shall define a map, denoted by $\Phi$, from the (disjoint) union of $\mathcal{P}(H_i, k)$ for $1 \leq i \leq h(n,k)$ to $\mathcal{H}_n(k)$. Given a partial $0/1$-equivalence class $\mathcal{P}_i \in \mathcal{P}(H_i, k)$ ($1 \leq i \leq h(n,k)$), then we define $\Phi(\mathcal{P}_i)$ to be the (unique) $0/1$-equivalence class in $\mathcal{H}_n(k)$ containing $\mathcal{P}_i$. Then we have the following theorem.

**Theorem 4.1** If $2^{n-2} < k \leq 2^{n-1}$, then the map $\Phi$ is a bijection.

**Proof.** We proceed to show that $\Phi$ is injective. To this end, we shall prove that for any two distinct partial $0/1$-equivalence classes $\mathcal{P}_1$ and $\mathcal{P}_2$ with $k$ vertices, their images, denoted by $\mathcal{C}_1$ and $\mathcal{C}_2$, are distinct $0/1$-equivalence classes. Assume that $\mathcal{P}_1 \in \mathcal{P}(H_i, k)$ and $\mathcal{P}_2 \in \mathcal{P}(H_j, k)$, where $1 \leq i, j \leq h(n,k)$. Let $P_1$ (resp. $P_2$) be any $0/1$-polytope in $\mathcal{P}_1$ (resp. $\mathcal{P}_2$). Evidently, $P_1$ (resp. $P_2$) is a $0/1$-polytope in $C_1$ (resp. $C_2$). To prove that $C_1 \neq C_2$, it suffices to show that $P_1$ and $P_2$ are not in the same $0/1$-equivalence class. We have two cases.

Case 1. $i = j$. In this case, it is clear that $P_1$ and $P_2$ are not equivalent.

Case 2. $i \neq j$. Suppose to the contrary that $P_1$ and $P_2$ are in the same $0/1$-equivalence class. Then there exists a symmetry $w \in B_n$ such that $w(P_1) = P_2$. Since $2^{n-2} < k \leq 2^{n-1}$,
by Theorem 3.1 we see that $P_1$ and $P_2$ are of dimension $n - 1$. Since $P_1$ is contained in $H_i$, $H_i$ coincides with the affine space spanned by $P_1$. Similarly, $H_j$ is the affine space spanned by $P_2$. This implies that $w(H_i) = H_j$, contradicting the assumption that $H_i$ and $H_j$ belong to distinct equivalence classes of spanned hyperplanes of $Q_n$. Consequently, $P_1$ and $P_2$ are not in the same 0/1-equivalence class.

It remains to show that $\Phi$ is surjective. For any $C \in H_n(k)$, we aim to find a partial 0/1-equivalence class such that its image is $C$. Let $P$ be any 0/1-polytope in $C$. Since $P$ is not full-dimensional, we can find a spanned hyperplane $H$ of $Q_n$ such that $P$ is contained in $H$. It follows that $H$ contains at least $k$ vertices of $Q_n$. Thus there exists a representative $H_j (1 \leq j \leq h(n, k))$ such that $H$ is in the equivalence class of $H_j$. Assume that $w(H) = H_j$ for some $w \in B_n$. Then $w(P)$ is contained in $H_j$. It is easily seen that under the map $\Phi$, $C_i$ is the image of the partial 0/1-equivalence class of $H_j$ containing $w(P)$. Thus we conclude that the above map is a bijection. This completes the proof.

It should also be noted that in the above proof of Theorem 4.1, the condition $2^{n-2} < k \leq 2^{n-1}$ is required only in Case 2. When $k < 2^{n-2}$, the map $\Phi$ may be no longer an injection. For the case $2^{n-3} < k \leq 2^{n-2}$, we will consider the computation of $H_n(k)$ in Section 8.

As a direct consequence of Theorem 4.1, we obtain that for $2^{n-2} < k \leq 2^{n-1}$,

$$H_n(k) = \sum_{i=1}^{h(n,k)} N_{H_i}(k). \quad (4.2)$$

Thus, for $2^{n-2} < k \leq 2^{n-1}$ the computation of $H_n(k)$ is reduced to the determination of the number $N_{H_i}(k)$ of partial 0/1-equivalence classes of $H$ with $k$ vertices. In the rest of this section, we shall explain how to compute $N_{H}(k)$.

For $2^{n-2} < k \leq 2^{n-1}$, let $H$ be a spanned hyperplane of $Q_n$ containing at least $k$ vertices. Let $P$ and $P'$ be two distinct 0/1-polytope of $Q_n$ with $k$ vertices that are contained in $H$. Assume that $P$ and $P'$ belong to the same partial 0/1-equivalence class of $H$. Then there exists a symmetry $w \in B_n$ such that $w(P) = P'$. It is clear from Theorem 3.1 that both $P$ and $P'$ have dimension $n - 1$. Then $H$ is the affine space spanned by $P$ or $P'$. So we deduce that $w(H) = H$. Let

$$F(H) = \{w \in B_n; w(H) = H\}$$

be the stabilizer subgroup of $H$, namely, the subgroup of $B_n$ that fixes $H$. So we have shown that $P$ and $P'$ belong to the same partial 0/1-equivalence class of $H$ if and only if one can be transformed to the other by a symmetry in $F(H)$.

The above fact allows us to use Pólya’s theorem to compute the number $N_{H}(k)$ for $2^{n-2} < k \leq 2^{n-1}$. Denote by $V_n(H)$ the set of vertices of $Q_n$ that are contained in $H$. Let us consider the action of $F(H)$ on $V_n(H)$. Assume that each vertex in $V_n(H)$ is assigned one of the two colors, say, black and white. For such a 2-coloring of the
vertices in $V_n(H)$, consider the black vertices as vertices of a $0/1$-polytope contained in $H$. Clearly, for $2^{n-2} < k \leq 2^{n-1}$, this establishes a one-to-one correspondence between partial $0/1$-equivalence classes of $H$ with $k$ vertices and equivalence classes of $2$-colorings of the vertices in $V_n(H)$ with $k$ black vertices.

Write $Z_H(z)$ for the cycle index of $F(H)$, and let $C_H(u_1,u_2)$ denote the polynomial obtained from $Z_H(z)$ by substituting $z_i$ with $u_1^i + u_2^i$.

Theorem 4.2 Assume that $2^{n-2} < k \leq 2^{n-1}$, and let $H$ be a spanned hyperplane of $Q_n$ containing at least $k$ vertices of $Q_n$. Then we have

$$N_H(k) = \left[u_1^k u_2^{(V_n(H)) - k}\right] C_H(u_1,u_2).$$

We will compute the cycle index $Z_H(z)$ in Section 5 and Section 6. Section 5 is devoted to the characterization of the stabilizer $F(H)$. In Section 6, we will give an explicit expression for $Z_H(z)$.

5 The structure of the stabilizer $F(H)$

In this section, we aim to characterize the stabilizer $F(H)$ for a given spanned hyperplane $H$ of $Q_n$.

Let

$$H: a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

be a spanned hyperplane of $Q_n$. Given $w \in B_n$, let $s(w)$ be the set of entries of $w$ that are assigned the minus sign. In view of (2.4), it is easy to see that $w(H)$ is of the following form

$$\sum_{i \notin s(w)} a_{\pi(i)}x_i + \sum_{j \in s(w)} a_{\pi(j)}(1 - x_j) = b,$$

where $\pi$ is the underlying permutation of $w$. The hyperplane $w(H)$ in (5.1) can be rewritten as

$$s(w,1) \cdot a_{\pi(1)}x_1 + s(w,2) \cdot a_{\pi(2)}x_2 + \cdots + s(w,n) \cdot a_{\pi(n)}x_n = b - \sum_{j \in s(w)} a_{\pi(j)},$$

where $s(w,j) = -1$ if $j \in s(w)$ and $s(w,j) = 1$ otherwise.

As an example, let

$$H: x_1 - x_2 - x_3 + 2x_4 = 1$$

be a spanned hyperplane of $Q_4$. Upon the action of the symmetry $w = (1)(23)(4) \in B_4$, $H$ is transformed into the following hyperplane

$$x_1 + x_2 + x_3 + 2x_4 = 3.$$
As mentioned in Section 4, for every equivalence class of spanned hyperplanes of $Q_n$, we can choose a representative of the following form

$$H: a_1x_1 + a_2x_2 + \cdots + a_t x_t = b,$$

(5.3)

where $a_1 \leq a_2 \leq \cdots \leq a_t$ ($t \leq n$), and the coefficients $a_i$’s and $b$ are positive integers. Note that this observation also follows from (5.2). From now on, we shall restrict our attention only to spanned hyperplanes of $Q_n$ of the form as in (5.3). The following definition is required for the determination of $F(H)$.

**Definition 5.1** Let $H$ be a spanned hyperplane of the form as in (5.3). The type of $H$ is defined to be a vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, where $\alpha_i$ is the multiplicity of $i$ occurring in the set \{ $a_1, a_2, \ldots, a_t$ \}.

For example, let

$$H: x_1 + x_2 + 2x_3 + 2x_4 + 3x_5 = 4$$

(5.4)

be a spanned hyperplane of $Q_5$. Then, the type of $H$ is $\alpha = (\alpha_1, \alpha_2, \alpha_3) = (2, 2, 1)$.

For positive integers $i$ and $j$ such that $i \leq j$, let $[i,j]$ denote the interval \{ $i, i+1, \ldots, j$ \}. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ be the type of a spanned hyperplane. Under the assumption that $\alpha_0 = 0$, the following set

$$\{ [\alpha_1 + \cdots + \alpha_{i-1} + 1, \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i] : 1 \leq i \leq \ell \}$$

(5.5)

is a partition of the set \{ $1, 2, \ldots, t$ \}. For example, let $\alpha = (2, 2, 1)$. Then the corresponding partition is \{ \{1,2\}, \{3,4\}, \{5\} \}.

Since (5.5) is a partition of \{ $1, 2, \ldots, t$ \}, we can define the corresponding Young subgroup $S_\alpha$ of the permutation group on \{ $1, 2, \ldots, t$ \}, namely,

$$S_\alpha = S_{\alpha_1} \times S_{\alpha_2} \times \cdots \times S_{\alpha_\ell},$$

(5.6)

where $\times$ denotes the direct product of groups, and for $i = 1, 2, \ldots, \ell$, $S_{\alpha_i}$ is the permutation group on the interval

$$[\alpha_1 + \cdots + \alpha_{i-1} + 1, \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i].$$

(5.7)

Let

$$\overline{S}_\alpha = \overline{S}_{\alpha_1} \times \overline{S}_{\alpha_2} \times \cdots \times \overline{S}_{\alpha_\ell},$$

(5.8)

where $\overline{S}_{\alpha_i}$ is the set of signed permutations on the interval (5.7) with all elements assigned the minus sign. Define

$$P(H) = \begin{cases} S_\alpha, & \text{if } \sum_{i=1}^t a_i \neq 2b, \\ S_\alpha \cup \overline{S}_\alpha, & \text{if } \sum_{i=1}^t a_i = 2b. \end{cases}$$

(5.9)

The following theorem gives a characterization of the stabilizer of a spanned hyperplane.
Theorem 5.2 Let $H: a_1x_1 + a_2x_2 + \cdots + a_tx_t = b$ be a spanned hyperplane of $Q_n$. Then

$$F(H) = P(H) \times B_{n,t},$$

where $B_{n,t}$ is the group of all signed permutations on the interval $[t+1,n]$.

Proof. Assume that $w \in F(H)$ and $\pi$ is the underlying permutation of $w$. We aim to show that $w \in P(H) \times B_{n,t}$. Consider the expression of $w(H)$ as in (5.2), that is,

$$s(w,1) \cdot a_{\pi(1)}x_1 + s(w,2) \cdot a_{\pi(2)}x_2 + \cdots + s(w,n) \cdot a_{\pi(n)}x_n = b - \sum_{j \in s(w)} a_{\pi(j)}. \quad (5.10)$$

We claim that $s(w,j)$ are either all positive or all negative for $1 \leq j \leq t$. Suppose otherwise that there exist $1 \leq i, j \leq t$ ($i \neq j$) such that $s(w,i) > 0$ and $s(w,j) < 0$. Since the $a_i$'s are all positive, we see that the coefficients $s(w,i)a_{\pi(i)}$ and $s(w,j)a_{\pi(j)}$ for the hyperplane $w(H)$ have opposite signs. This implies that $w(H)$ and $H$ are distinct, which contradicts the assumption that $w$ fixes $H$. We now have the following two cases.

Case 1. The signs $s(w,j)$ are all positive for $1 \leq j \leq t$. In this case, since $w(H) = H$ it is clear that $w(H)$ is of the following form

$$a_{\pi(1)}x_1 + a_{\pi(2)}x_2 + \cdots + a_{\pi(t)}x_t = b,$$

where $a_{\pi(j)} = a_j$ for $1 \leq j \leq t$. Hence we deduce that, for any $1 \leq j \leq t$, $\pi(j)$ is in the interval $[\alpha_1 + \cdots + \alpha_{i-1} + 1, \alpha_1 + \cdots + \alpha_{i-1} + \alpha_i]$ that contains the element $j$. Thus we obtain that $w \in S_\alpha \times B_{n,t}.$

Case 2. The signs $s(w,j)$ are all negative for $1 \leq j \leq t$. In this case, we see that $w(H)$ is of the following form

$$-a_{\pi(1)}x_1 - a_{\pi(2)}x_2 - \cdots - a_{\pi(t)}x_t = b - (a_1 + \cdots + a_t).$$

Since $w(H) = H$, we have $a_{\pi(j)} = a_j$ for $1 \leq j \leq t$ and $b - (a_1 + \cdots + a_t) = -b$. Thus we obtain $w \in S_\alpha \times B_{n,t}$. Combining the above two cases, we conclude that $w \in P(H) \times B_{n,t}$.

On the other hand, from the expression (5.10) for $w(H)$, it is not difficult to check that every symmetry $w$ in $P(H) \times B_{n,t}$ fixes $H$. This completes the proof.  

As will been seen in Section 6, for the purpose of computing the cycle index $Z_H(z)$ with respect to a spanned hyperplane $H$ of $Q_n$, it is often necessary to consider the structure of the subgroup $P(H)$ of $F(H)$. We sometimes write a symmetry $\pi \in P(H)$ as a product form $\pi = \pi_1\pi_2\cdots\pi_\ell$, which means that for $i = 1,2,\ldots,\ell$, $\pi_i \in S_\alpha$, if $\pi \in S_\alpha$, and $\pi_i \in \overline{S_\alpha}$, if $\pi \in \overline{S_\alpha}$, where $\alpha$ is the type of $H$. We conclude this section with the following proposition, which will be required for the computation of $Z_H(z)$ in Section 6.

Proposition 5.3 Let $H$ be a spanned hyperplane of $Q_n$ of type $\alpha$. Let $\pi = \pi_1\pi_2\cdots\pi_\ell$ and $\pi' = \pi'_1\pi'_2\cdots\pi'_\ell$ be two symmetries in $P(H)$, and assume that both $\pi$ and $\pi'$ are either in $S_\alpha$ or in $\overline{S_\alpha}$. If $\pi_i$ and $\pi'_i$ have the same cycle type for $1 \leq i \leq \ell$, then $\pi$ and $\pi'$ are in the same conjugacy class of $P(H)$. 

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Proof. To prove that $\pi$ and $\pi'$ are conjugate in $P(H)$, it suffices to show that there exists a symmetry $w \in P(H)$ such that $\pi = w\pi'w^{-1}$. First, we consider the case when both $\pi$ and $\pi'$ are in $S_\alpha$. Since $\pi_i$ and $\pi'_i$ are of the same cycle type, they are in the same conjugacy class. So there is a permutation $w_i \in S_\alpha$ such that $\pi_i = w_i\pi'_iw_i^{-1}$. It follows that $\pi = (w_1\pi'_1w_1^{-1}) \cdots (w_\ell\pi'_\ell w_\ell^{-1}) = w\pi'w^{-1}$, where $w = w_1 \cdots w_\ell \in S_\alpha$. This implies that $\pi$ and $\pi'$ are conjugate in $P(H)$.

It remains to consider the case when both $\pi$ and $\pi'$ are in $\overline{S_\alpha}$. Let $\pi_0$ (resp. $\pi'_0$) be the underlying permutation of $\pi$ (resp. $\pi'$). Then there is a symmetry $w \in S_\alpha$ such that $\pi_0 = w\pi'_0w^{-1}$. We claim that $\pi = w\pi'w^{-1}$. Indeed, it is enough to show that $\pi(x_1, x_2, \ldots, x_t) = w\pi'w^{-1}(x_1, x_2, \ldots, x_t)$ for any point $(x_1, x_2, \ldots, x_t)$ in $\mathbb{R}^t$. Assume that $\pi(x_1, x_2, \ldots, x_t) = (y_1, y_2, \ldots, y_t)$ and $w\pi'w^{-1}(x_1, x_2, \ldots, x_t) = (z_1, z_2, \ldots, z_t)$. Since all elements of $\pi$ are assigned the minus sign, we obtain from (2.4) that $y_i = 1 - x_{\pi_0(i)}$ for $1 \leq i \leq t$. On the other hand, using (2.4), it is not hard to check that $z_i = 1 - x_{w^{-1}\pi_0(i)}$ for $1 \leq i \leq t$. Since $\pi_0 = w\pi'_0w^{-1}$, we deduce that $\pi_0(i) = w^{-1}\pi'_0w(i)$. Therefore, we have $y_i = z_i$ for $1 \leq i \leq t$. So the claim is justified. This completes the proof. □

6 The computation of $Z_H(z)$

In this section, we shall derive a formula for the cycle index $Z_H(z)$ for a spanned hyperplane $H$ of $Q_n$. It turns out that $Z_H(z)$ depends only on the cycle structures of the symmetries in the subgroup $P(H)$ of $F(H)$.

Let

$$H: a_1x_1 + a_2x_2 + \cdots + a_tx_t = b \quad (6.1)$$

be a spanned hyperplane of $Q_n$. Recall that $V_n(H)$ is the set of vertices of $Q_n$ contained in $H$. To compute the cycle index $Z_H(z)$, we need to determine the cycle structures of permutations on $V_n(H)$ induced by the symmetries in $F(H)$. By Theorem 5.2, each symmetry in $F(H)$ can be written uniquely as a product $\pi w$, where $\pi \in P(H)$ and $w \in B_{n,t}$. We shall define two group actions for the subgroups $P(H)$ and $B_{n,t}$, and shall derive an expression for the cycle type of the permutation on $V_n(H)$ induced by $\pi w$ in terms of the cycle types of the permutations induced by $\pi$ and $w$.

Let $H$ be a spanned hyperplane of $Q_n$ as in (6.1). However, to define the action for $P(H)$, we shall consider $H$ as a hyperplane in $\mathbb{R}^t$. Denoted by $V_t(H)$ the set of vertices of $Q_t$ that are contained in $H$, namely,

$$V_t(H) = \{(x_1, x_2, \ldots, x_t) \in V_t: a_1x_1 + a_2x_2 + \cdots + a_tx_t = b\}.$$

Since the vertices of $Q_n$ contained in $H$ span a hyperplane in $\mathbb{R}^n$, it can be seen that the vertices in $V_t(H)$ span a hyperplane in $\mathbb{R}^t$. Since $H$ is considered as a hyperplane in $\mathbb{R}^t$, we deduce that $H$ is a spanned hyperplane of $Q_t$. Setting $n = t$ in Theorem 5.2 it follows that the stabilizer of $H$ is $P(H)$. Therefore, $P(H)$ stabilizes the set $V_t(H)$. So any symmetry in $P(H)$ induces a permutation on $V_t(H)$.
We also need an action of the group $B_{n,t}$ on the set of vertices of $Q_{n-t}$. Assume that $w \in B_{n,t}$, namely, $w$ is a signed permutation on the interval $[t + 1, n]$. Subtracting each element of $w$ by $t$, we get a signed permutation on $[1, n - t]$. In this way, each signed permutation in $B_{n,t}$ corresponds to a symmetry of $Q_{n-t}$. Hence $B_{n,t}$ is isomorphic to the group $B_{n-t}$ of symmetries of $Q_{n-t}$. This leads to an action of the group $B_{n,t}$ on $V_{n-t}$.

Let $\pi \in P(H)$ and $w \in B_{n,t}$. Recall that, for an element $g$ in a group $G$ acting on a finite set $X$, $c(g)$ denotes the cycle type of the permutation on $X$ induced by $g$, which is written as a multiset $\{c_1, 2c_2, \ldots\}$. In this notation, $c(\pi)$ (resp. $c(w)$) represents the cycle type of the permutation on $V_i(H)$ (resp. $V_{n-t}$) induced by $\pi$ (resp. $w$). The following lemma gives an expression for the cycle type $c(w\pi)$ of the induced permutation of $\pi w$ on $V_n(H)$ in terms of the cycle types $c(\pi)$ and $c(w)$.

**Lemma 6.1** Let $H: a_1x_1 + a_2x_2 + \cdots + a_t x_t = b$ be a spanned hyperplane of $Q_n$, and $\pi w$ be a symmetry in $F(H)$, where $\pi \in P(H)$ and $w \in B_{n,t}$. Assume that $c(\pi) = \{1^{m_1}, 2^{m_2}, \ldots\}$ and $c(w) = \{1^{c_1}, 2^{c_2}, \ldots\}$. Then we have

$$c(\pi w) = \bigcup_{i \geq 1} \bigcup_{j \geq 1} \left\{ (\text{lcm}(i, j))^{i j m_{i,j}} c_{i,j} \right\}, \quad (6.2)$$

where $\bigcup$ denotes the disjoint union of multisets, and $\text{lcm}(i, j)$ denotes the least common multiple of integers $i$ and $j$.

**Proof.** Clearly, each vertex in $V_n(H)$ can be expressed as a vector of the following form

$$(x_1, \ldots, x_t, y_1, \ldots, y_{n-t}),$$

where $(x_1, \ldots, x_t)$ is a vertex in $V_i(H)$ and $(y_1, \ldots, y_{n-t})$ is a vertex of $Q_{n-t}$. Assume that $|V_i(H)| = n_0$. Let $V_i(H) = \{u_1, u_2, \ldots, u_{n_0}\}$ and $Q_{n-t} = \{v_1, v_2, \ldots, v_{2^{n-t}}\}$. Then each vertex in $V_n(H)$ can be expressed as an ordered pair $(u_i, v_j)$, where $1 \leq i \leq n_0$ and $1 \leq j \leq 2^{n-t}$. Let $C_i = (s_1, \ldots, s_i)$ be an $i$-cycle of the permutation on $V_i(H)$ induced by $\pi$, that is, $C_i$ maps the vertex $u_{sk}$ to the vertex $u_{sk+1}$ if $1 \leq k \leq i - 1$, and to the vertex $u_{si}$ if $k = i$. Similarly, let $C_j = (t_1, \ldots, t_j)$ be a $j$-cycle of the permutation on $V_{n-t}$ induced by $w$, that is, $C_j$ maps the vertex $v_{tm}$ to the vertex $v_{tm+1}$ if $1 \leq m \leq j - 1$, and to the vertex $v_{tj}$ if $m = j$. Define the direct product of $C_i$ and $C_j$, denoted $C_i \times C_j$, to be the permutation on the subset $\{(u_{sk}, v_{tm}) : 1 \leq k \leq i, 1 \leq m \leq j\}$ of $V_n(H)$ such that

$$C_i \times C_j (u_{sk}, v_{tm}) = (C_i(u_{sk}), C_j(v_{tm})).$$

It is not hard to check that the cycle type of $C_i \times C_j$ is

$$\left\{ (\text{lcm}(i, j))^{i j m_{i,j}} c_{i,j} \right\}.$$
Note that the induced permutation of \( \pi w \) on \( V_n(H) \) is the product of \( C_i \times C_j \), where \( C_i \) (resp. \( C_j \)) runs over the cycles of the permutation on \( V_i(H) \) (resp. \( V_n - i \)) induced by \( \pi \) (resp. \( w \)). Thus the cycle type of the induced permutation of \( \pi w \) on \( V_n(H) \) is given by (6.2). This completes the proof.

Before presenting a formula for the cycle index \( Z_H(z) \), we need to introduce some notation. Assume that \( \pi \) is a symmetry in \( P(H) \) such that the cycle type of the induced permutation of \( \pi \) is

\[
c(\pi) = \{1^{m_1}, 2^{m_2}, \ldots\}.
\]

For \( j \geq 1 \), we define

\[
f_\pi(z_j) = \prod_{i \geq 1} (z_{\text{lcm}(i,j)})^{i(jm_i) / \text{lcm}(i,j)}.
\]

Let

\[
f_\pi(z) = (f_\pi(z_1), f_\pi(z_2), \ldots).
\]

We have the following proposition.

**Proposition 6.2** Let \( H \) be a spanned hyperplane of \( Q_n \) with type \( \alpha \). Let \( \pi = \pi_1 \pi_2 \cdots \pi_\ell \) and \( \pi' = \pi'_1 \pi'_2 \cdots \pi'_\ell \) be two symmetries in \( P(H) \). Assume that both \( \pi \) and \( \pi' \) are either in \( S_\alpha \) or in \( \overline{S_\alpha} \). If \( \pi_i \) and \( \pi'_i \) have the same cycle type for \( 1 \leq i \leq \ell \), then \( f_\pi(z) = f_{\pi'}(z) \).

**Proof.** It follows from Proposition 5.3 that \( \pi \) and \( \pi' \) are conjugate in \( P(H) \). Since \( P(H) \) acts on \( V_i(H) \), the permutations on \( V_i(H) \) induced by \( \pi \) and \( \pi' \) are conjugate. So they have the same cycle type, i.e., \( c(\pi) = c(\pi') \). Since \( f_\pi(z) \) depends only on \( c(\pi) \), we see that \( f_\pi(z) = f_{\pi'}(z) \). This completes the proof.

We now give an overview of some notation related to integer partitions. We shall write a partition \( \lambda \) of a positive integer \( n \), denoted by \( \lambda \vdash n \), in the multiset form, that is, write \( \lambda = \{1^{m_1}, 2^{m_2}, \ldots\} \), where \( m_i \) is the number of parts of \( \lambda \) of size \( i \). Denote by \( \ell(\lambda) \) the number of parts of \( \lambda \), that is, \( \ell(\lambda) = m_1 + m_2 + \cdots \). For a partition \( \lambda = \{1^{m_1}, 2^{m_2}, \ldots\} \), let

\[
z_\lambda = 1^{m_1} 2^{m_2} m_2! \cdots
\]

For two partitions \( \lambda \) and \( \mu \), define \( \lambda \cup \mu \) to be the partition obtained by joining the parts of \( \lambda \) and \( \mu \) together. For example, for \( \lambda = \{1, 2\} \) and \( \mu = \{1^2, 3\} \), then \( \lambda \cup \mu = \{1^3, 2, 3\} \).

Let \( H: a_1 x_1 + a_2 x_2 + \cdots + a_\ell x_\ell = b \) be a spanned hyperplane of \( Q_n \), whose type is \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell) \). Assume that \( \mu = \mu^1 \cup \cdots \cup \mu^\ell \) is a partition of \( t \), where \( \mu^i \vdash \alpha_i \) for \( 1 \leq i \leq \ell \). We can write \( f_\mu(z) \) (resp. \( \tilde{f}_\mu(z) \)) for \( f_\pi(z) \), where \( \pi = \pi_1 \pi_2 \cdots \pi_\ell \) is any symmetry in \( S_\alpha \) (resp. \( \overline{S_\alpha} \)) such that \( \pi_i \) has cycle type \( \mu^i \) for \( 1 \leq i \leq \ell \). By Proposition 6.2 the functions \( f_\mu(z) \) and \( \tilde{f}_\mu(z) \) are well defined. We can now give a formula for the cycle index \( Z_H(z) \).
Theorem 6.3 Let $H : a_1x_1 + a_2x_2 + \cdots + a_\ell x_\ell = b$ be a spanned hyperplane of $Q_n$. Assume that $H$ has type $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$. Then we have

$$Z_H(z) = \frac{1}{2^{|H|}} \sum_{(\mu^1, \ldots, \mu^\ell)} \prod_{i=1}^\ell z_{\mu^i}^{-1} \left( Z_{n-\ell}(f_{\mu}(z)) + \delta(H)Z_{n-\ell}(\overline{f}_{\mu}(z)) \right), \quad (6.5)$$

where $\mu^i \vdash \alpha_i$, $\mu = \mu^1 \cup \cdots \cup \mu^\ell$, $\delta(H) = 1$ if $\sum_{i=1}^\ell a_i = 2b$ and $\delta(H) = 0$ otherwise.

Proof. Let $\pi \in P(H)$ and $w \in B_{n,t}$. Assume that $c(w) = \{1^{c_1}, 2^{c_2}, \ldots\}$. By Lemma 6.1 we have

$$z^{c(\pi \cdot w)} = f_\pi(z_1)^{c_1} f_\pi(z_2)^{c_2} \cdots. \quad (6.6)$$

From (2.1) and (6.6), we deduce that

$$\sum_{\pi \cdot w} z^{c(\pi \cdot w)} = \sum_w f_\pi(z_1)^{c_1} f_\pi(z_2)^{c_2} \cdots = (n-t)!2^{n-t} Z_{n-\ell}(f_\pi(z_1), f_\pi(z_2), \ldots) = (n-t)!2^{n-t} Z_{n-\ell}(f_\pi(z)),$$

where $w$ runs over the signed permutations in $B_{n,t}$. Thus

$$Z_H(z) = \frac{1}{|F(H)|} \sum_{\pi \cdot w \in F(H)} z^{c(\pi w)}$$

$$= \frac{1}{|F(H)|} \sum_{\pi \in P(H)} (n-t)!2^{n-t} Z_{n-\ell}(f_\pi(z))$$

$$= \frac{(n-t)!2^{n-t}}{|F(H)|} \left( \sum_{\pi \in S_\alpha} Z_{n-\ell}(f_\pi(z)) + \delta(H) \sum_{\pi' \in S_\alpha} Z_{n-\ell}(f_{\pi'}(z)) \right), \quad (6.7)$$

where $\delta(H) = 1$ if $\sum_{i=1}^\ell a_i = 2b$ and $\delta(H) = 0$ otherwise.

Recall that for any given partition $\nu \vdash m$, there are $\frac{m!}{z_\nu}$ permutations on $\{1, 2, \ldots, m\}$ such that their cycle type is $\nu$, see Stanley [20, Proposition 1.3.2]. So the number of symmetries $\pi = \pi_1 \pi_2 \cdots \pi_\ell$ in $S_\alpha$ (or, $\overline{S}_\alpha$) such that for $i = 1, 2, \ldots, \ell$, $\pi_i$ has cycle type $\mu^i$ is equal to

$$\prod_{i=1}^\ell \frac{\alpha_i!}{z_{\mu^i}}. \quad (6.8)$$

Combining (6.7), (6.8) and Proposition 6.2 we obtain that

$$Z_H(z) = \frac{(n-t)!2^{n-t}}{|F(H)|} \sum_{(\mu^1, \ldots, \mu^\ell)} \prod_{i=1}^\ell \frac{\alpha_i!}{z_{\mu^i}} \left( Z_{n-\ell}(f_{\mu}(z)) + \delta(H)Z_{n-\ell}(\overline{f}_{\mu}(z)) \right), \quad (6.9)$$
where \( \mu^i \vdash \alpha_i \), and \( \mu = \mu^1 \cup \cdots \cup \mu^\ell \).

Since
\[
|F(H)| = (n-t)!2^{n-t+\delta(H)} \prod_{i=1}^\ell \alpha_i!,
\]
by substituting (6.10) into (6.9), we are led to (6.5). This completes the proof. \( \blacksquare \)

By Theorem 6.3, the cycle index \( Z_H(z) \) depends only on \( f_\pi(z) \) for \( \pi \in P(H) \). In view of (6.3), we see that \( f_\pi(z) \) depends only on \( c(\pi) \). Assume that \( c(\pi) = \{1^{m_1}, 2^{m_2}, \ldots\} \). By Theorem 2.1, we have
\[
m_i = \frac{1}{i} \sum_{j|i} \mu(i/j)\psi(\pi^j),
\]
where \( \psi(\pi^j) \) is the number of vertices in \( V_i(H) \) that are fixed by \( \pi^j \). The following theorem gives a formula for \( \psi(\pi) \), from which \( \psi(\pi^j) \) is easily determined.

**Theorem 6.4** Let \( H : a_1x_1 + a_2x_2 + \cdots + a_t x_t = b \) be a spanned hyperplane of \( Q_n \). Assume that \( \pi = \pi_1\pi_2 \cdots \pi_\ell \) is a symmetry in \( P(H) \) such that \( \pi_i \) has cycle type \( \mu^i = \{1^{m_1}, 2^{m_2}, \ldots\} \) for \( i = 1, 2, \ldots, \ell \). Then
\[
\psi(\pi) = \left\{ \begin{array}{ll}
\left[ x^b \right] \prod_{i=1}^\ell \prod_{j \geq 1} (1 + x^{ij})^{m_{ij}}, & \text{if } \pi \in S_\alpha, \\
\chi(\mu)2^{\ell(\mu)}, & \text{if } \pi \in S_\alpha,
\end{array} \right.
\]
where \( \mu = \mu^1 \cup \cdots \cup \mu^\ell \), \( \chi(\mu) = 1 \) if \( \mu \) has no odd parts and \( \chi(\mu) = 0 \) otherwise.

Before we present the proof of the above theorem, we need to define 0/1-labelings of a symmetry \( \pi \in P(H) \) for the purpose of characterizing the vertices of \( Q_t \) fixed by \( \pi \). Let \( \pi \) be a symmetry in \( P(H) \). A 0/1-labeling of \( \pi \) is a labeling of the cycles of \( \pi \) such that each cycle of \( \pi \) is assigned one of the two numbers 0 and 1.

**Proof of Theorem 6.4** We first consider the case when \( \pi \) is in \( S_\alpha \). It is easy to observe that, a vertex \( v = (v_1, v_2, \ldots, v_t) \) of \( Q_t \) is a fixed point of \( \pi \), that is, \( \pi(v) = v \) if and only if, for each \( i \)-cycle \( (j_1, j_2, \ldots, j_i) \) of \( \pi \) and for any entry of \( v \) corresponding to \( (j_1, j_2, \ldots, j_i) \), we have
\[
v_{j_1} = v_{j_2} = \cdots = v_{j_i}
\]
(or, more precisely, \( v_{j_1} = v_{j_2} = \cdots = v_{j_i} = 0 \) or \( v_{j_1} = v_{j_2} = \cdots = v_{j_i} = 1 \)). The above characterization enables us to establish a one-to-one correspondence between 0/1-labelings of \( \pi \) and the vertices of \( Q_t \) fixed by \( \pi \), that is, for any given 0/1-labeling of \( \pi \), we can define a vertex \( v = (v_1, v_2, \ldots, v_t) \) of \( Q_t \) fixed by \( \pi \) such that \( v_i = 0 \) \((1 \leq i \leq t)\) if and only if the cycle of \( \pi \) containing \( i \) is assigned 0. Moreover, if the vertex \( v = (v_1, v_2, \ldots, v_t) \) corresponding to a 0/1-labeling of \( \pi \) is in \( V_t(H) \), that is, \( a_1v_1 + a_2v_2 + \cdots + a_tv_t = b \), then we have
\[
b_1 + 2b_2 + \cdots + \ell b_\ell = b,
\]
(6.13)
where $b_i$ ($1 \leq i \leq \ell$) is the sum of the lengths of cycles of $\pi_i$ which are labeled 1. It can be easily deduced that the number of 0/1-labelings of $\pi$ satisfying (6.13) is

$$\psi(\pi) = [x^b] \prod_{i=1}^{\ell} \prod_{j \geq 1} (1 + x^{ij})^{m_{ij}}.$$  

We now consider the case when $\pi$ is in $\overline{S}_n$. As in the previous case, it can be seen that a vertex $v = (v_1, v_2, \ldots, v_t)$ of $Q_\ell$ is fixed by $\pi$ if and only if, for any (signed) $i$-cycle $(j_1, j_2, \ldots, j_t)$ of $\pi$, the following relation holds

$$(v_{j_1}, v_{j_2}, \ldots, v_{j_t}) = (1 - v_{j_2}, 1 - v_{j_3}, \ldots, 1 - v_{j_t}).$$

Consequently, if a vertex $v = (v_1, v_2, \ldots, v_t)$ of $Q_\ell$ is fixed by $\pi$, then, for any (signed) $i$-cycle $(j_1, j_2, \ldots, j_t)$ of $\pi$, the vector $(v_{j_1}, v_{j_2}, \ldots, v_{j_t})$ is either $(0, 1, \ldots, 0)$ or $(1, 0, \ldots, 1)$. This implies that $\pi$ does not have any fixed point if $\pi$ has an odd cycle.

We now assume that $\pi$ has only even (signed) cycles. In this case, we see that the number of vertices of $Q_\ell$ fixed by $\pi$ is equal to $2^{t(\mu)}$. To prove $\psi(\pi) = 2^{t(\mu)}$, we need to demonstrate that any vertex of $Q_\ell$ fixed by $\pi$ is in $V_\ell(H)$. Let $v = (v_1, v_2, \ldots, v_t)$ be any vertex of $Q_\ell$ fixed by $\pi$. Using the fact that for each (signed) cycle $(j_1, j_2, \ldots, j_t)$ of $\pi$, the vector $(v_{j_1}, \ldots, v_{j_t})$ is $(1,0,\ldots,0,1)$ or $(0,1,\ldots,0,1)$, and applying the relation

$$a_1 + \cdots + a_t = 2b,$$

we deduce that $a_1 v_1 + a_2 v_2 + \cdots + a_t v_t = b$. Hence the vertex $v$ is in $V_\ell(H)$. This completes the proof.

By Theorem 6.4, we can compute $\psi(\pi^j)$. Let $\pi = \pi_1 \pi_2 \cdots \pi_\ell \in \mathcal{P}(H)$, where $\pi_i$ has cycle type $\mu_i = \{1^{m_{i1}}, 2^{m_{i2}}, \ldots\}$. Clearly, $\pi^j = \pi_1^j \pi_2^j \cdots \pi_\ell^j$. Let $\gcd(i,j)$ denote the greatest common divisor of $i$ and $j$. As is easily checked, the cycle type of $\pi_i^j$ ($1 \leq i \leq \ell$) is

$$\left\{1^{m_{i1}}, \gcd(2,j)^{2m_{i2}}, \gcd(3,j)^{3m_{i3}}, \ldots\right\}.$$  

To apply Theorem 6.4, it is still necessary to determine whether the symmetry $\pi^j$ belongs to $S_\alpha$ or $\overline{S}_\alpha$. It can be seen that if $\pi$ is in $S_\alpha$ or $\pi$ is in $\overline{S}_\alpha$ and $j$ is even, then $\pi^j$ belongs to $S_\alpha$. Similarly, if $\pi$ is in $\overline{S}_\alpha$ and $j$ is odd, then $\pi^j$ belongs to $\overline{S}_\alpha$.

7 $F_n(k)$ for $n = 4, 5, 6$ and $2^{n-2} < k \leq 2^{n-1}$

This section is devoted to the computation of $F_n(k)$ for $n = 4, 5, 6$ and $2^{n-2} < k \leq 2^{n-1}$. This requires the cycle indices $Z_H(z)$ for spanned hyperplanes of $Q_n$ for $n = 4, 5, 6$ that contain more than $2^{n-2}$ vertices of $Q_n$. 

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Let $H_1, H_2, \ldots, H_{h(n,k)}$ be the representatives of equivalence classes of spanned hyperplanes of $Q_n$ containing at least $k$ vertices. When $2^{n-2} < k \leq 2^n - 1$, combining relation (1.1), Theorem 4.1 and Theorem 4.2, we deduce that

$$F_n(k) = A_n(k) - H_n(k)$$

$$= A_n(k) - \sum_{i=1}^{h(n,k)} N_{H_i}(k)$$

$$= A_n(k) - \sum_{i=1}^{h(n,k)} \left[ u_k^i u_{|V_e(H_i)|-k} \right] C_{H_i}(z_1, z_2).$$

(7.1)

We start with the computation of $F_4(k)$ for $k = 5, 6, 7, 8$. Observing that $F_4(k) = 0$ for $k < 5$, this gives the enumeration of full-dimensional 0/1-equivalence classes of $Q_4$. For brevity, we use $H^t_n$ ($t \leq n$) to denote the following hyperplane in $\mathbb{R}^n$

$$x_1 + x_2 + \cdots + x_t = \lfloor t/2 \rfloor.$$

In this notation, representatives of equivalence classes of spanned hyperplanes of $Q_4$ containing more than 4 vertices of $Q_4$ are as follows

- $H_1^4$: $x_1 = 0$,
- $H_2^4$: $x_1 + x_2 = 1$,
- $H_3^4$: $x_1 + x_2 + x_3 = 1$,
- $H_4^4$: $x_1 + x_2 + x_3 + x_4 = 2$.

Employing the techniques in Section 6, we obtain the cycle indices $Z_{H_1^4}(z)$ and $Z_{H_2^4}(z)$ as given below:

$$Z_{H_1^4}(z) = Z_3(z),$$

$$Z_{H_2^4}(z) = \frac{1}{16} \left( 9z_2^4 + 4z_4^2 + 2z_1^4z_2^2 + z_1^8 \right).$$

For the remaining two hyperplanes $H = H_3^4$ and $H_4^4$, it can be checked that $N_{H}(k) = 1$ for $k = 5, 6$. Thus, from (7.1) we can determine $F_4(k)$ for $k = 5, 6, 7, 8$. These values are given in Table 4 which agree with the results computed by Aichholzer [11].

We now compute $F_5(k)$ for $8 < k \leq 16$. Representatives of equivalence classes of spanned hyperplanes of $Q_5$ containing more than 8 vertices of $Q_5$ are $H_1^5, H_2^5, H_3^5, H_4^5, H_5^5$. By utilizing the techniques in Section 6, we obtain that

$$Z_{H_1^5}(z) = Z_4(z),$$

$$Z_{H_2^5}(z) = \frac{1}{96} \left( z_1^{16} + 6z_1^8z_2^4 + 33z_2^8 + 8z_1^4z_3^4 + 24z_4^4 + 24\frac{z_2^2z_6^2}{z_4} \right),$$

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Table 4: $F_4(k)$ for $k = 5, 6, 7, 8$.

$$Z_{H_3^3}(z) = \frac{1}{48} \left( 12z_2^6 + 8z_3^3 + 2z_1^2z_2^3 + z_1^{12} + 6z_1^2z_2^5 + 3z_1^4z_2^4 + 6z_6^2 + 4z_12 + 4z_2^2z_6 + 2z_3^4 \right),$$

$$Z_{H_3^4}(z) = \frac{1}{96} \left( z_1^{12} + 27z_2^6 + 9z_1^4z_2^4 + 8z_3^4 + 24z_6^2 + 18z_2^2z_4^2 + 6z_1^4z_4^2 + 3z_1^8z_2^2 \right),$$

$$Z_{H_3^5}(z) = \frac{1}{120} \left( 24z_2^8 + 30z_2z_4^2 + 20z_1z_3z_6 + 20z_1z_3^3 + 15z_1^2z_2^2 + 10z_1^4z_2^3 + z_1^{10} \right).$$

Consequently, the values $F_5(k)$ for $8 < k \leq 16$ can be derived from (7.1), and they agree with the results of Aichholzer [1], see Table 5.

Table 5: $F_5(k)$ for $8 < k \leq 16$.

The main objective of this section is to compute $F_6(k)$ for $16 < k \leq 32$. As mentioned in Section 4, there are 6 representatives of equivalence classes of spanned hyperplanes of $Q_6$ containing more than 16 vertices of $Q_6$, i.e., $H_6^1, H_6^2, H_6^3, H_6^4, H_6^5, H_6^6$. Again, by applying the techniques in Section 6, we obtain that

$$Z_{H_6^3}(z) = Z_5(z),$$

$$Z_{H_6^4}(z) = \frac{1}{768} \left( z_1^4 + 6z_1z_2^2 + 2z_2^2z_3z_6 + 12z_1^6 + 12z_1^6z_2^2 + 127z_1^6z_2^4 + 32z_1^6z_2^8 + 48z_1^2z_2^2z_6 + 168z_1^2z_2^4z_6 + 96z_1^4z_2^6 + 48z_1^6z_2^8 \right),$$

$$Z_{H_6^5}(z) = \frac{1}{288} \left( z_1^{10} + 6z_1^6z_2^6 + 52z_1^6z_2^8 + 18z_1^8 + 48z_1^8z_2^6 + 32z_1^8z_2^8 + 3z_1^8z_2^{10} \right).$$
Based on relation (7.1), we can compute $F_6(k)$ for $16 < k \leq 32$. These values are listed in Table 6.

| $H_6^1$ | $H_6^2$ | $H_6^3$ | $H_6^4$ | $H_6^5$ | $F_6(k)$ |
|---------|---------|---------|---------|---------|---------|
| 17      | 158658  | 767103  | 1464    | 1334    | 12      | 5        | 30063520396 |
| 18      | 133576  | 642880  | 657     | 630     | 5       | 3        | 78408664654  |
| 19      | 98804   | 474635  | 220     | 216     | 1       | 1        | 189678190615  |
| 20      | 65664   | 312295  | 81      | 86      | 1       | 1        | 42653936250   |
| 21      | 38073   | 179829  | 19      | 20      |         |          | 893345853436  |
| 22      | 19963   | 92309   | 7       | 8       |         |          | 1745593621167  |
| 23      | 9013    | 40948   | 1       | 1       |         |          | 318694223591   |
| 24      | 3779    | 16335   | 1       | 1       |         |          | 544354457875   |
| 25      | 1326    | 5500    |         |         |         |          | 870868167141   |
| 26      | 472     | 1753    |         |         |         |          | 13061946974320  |
| 27      | 131     | 441     |         |         |         |          | 18382330104124  |
| 28      | 47      | 129     |         |         |         |          | 24289841497705  |
| 29      | 10      | 23      |         |         |         |          | 30151914536900  |
| 30      | 5       | 9       |         |         |         |          | 35176482187384  |
| 31      | 1       | 1       |         |         |         |          | 38580161986424  |
| 32      | 1       | 1       |         |         |         |          | 39785643746724  |

Table 6: $F_6(k)$ for $16 < k \leq 32$.

8 $H_n(k)$ for $2^{n-3} < k \leq 2^{n-2}$

In this section, we shall present an approach for computing $H_n(k)$ for $2^{n-3} < k \leq 2^{n-2}$. This enables us to determine $F_6(k)$ for $k = 13, 14, 15, 16$. Together with the computation of Aichholzer up to 12 vertices for $n = 6$, we have completed the enumeration of full-dimensional 0/1-equivalence classes of the 6-dimensional hypercube.
Let us recall the map $\Phi$ defined in Section 4, which will be used in the computation of $H_n(k)$ for $2^{n-3} < k \leq 2^{n-2}$. Let $H_1, H_2, \ldots, H_{h(n,k)}$ be the representatives of equivalence classes of spanned hyperplanes of $Q_n$ containing at least $k$ vertices. As before, denote by $\mathcal{P}(H_i, k) \ (1 \leq i \leq h(n,k))$ the set of partial 0/1-equivalence classes of $H_i$ with $k$ vertices. Let $\mathcal{P}_i$ be a partial 0/1-equivalence class in $\mathcal{P}(H_i, k) \ (1 \leq i \leq h(n,k))$. So $\Phi$ maps $\mathcal{P}_i$ to the (unique) 0/1-equivalence class in $\mathcal{H}_n(k)$ containing $\mathcal{P}_i$. When $2^{n-2} < k \leq 2^{n-1}$, it has been shown in Theorem 4.1 that $\Phi$ is a bijection. However, as pointed out after the proof of Theorem 4.1 when $k \leq 2^{n-2}$, $\Phi$ is surjective but not necessarily injective.

For the purpose of computing $H_n(k)$ for $2^{n-3} < k \leq 2^{n-2}$, we shall first derive an expression for $H_n(k)$, which is valid for general $k$. Let $1 \leq i \leq h(n,k)$, and define

$$A_i = \Phi(\mathcal{P}(H_i, k)).$$

Since $\Phi$ is surjective, we see that

$$\mathcal{H}_n(k) = A_1 \cap A_2 \cup \cdots \cup A_{h(n,k)}.$$

It follows from the principle of inclusion-exclusion that

$$H_n(k) = \sum_{1 \leq i \leq h(n,k)} |A_i| - \sum_{1 \leq i_1 < i_2 \leq h(n,k)} |A_{i_1} \cap A_{i_2}| + \sum_{1 \leq i_1 < i_2 < i_3 \leq h(n,k)} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \cdots. \quad (8.1)$$

Hence the task of computing $H_n(k)$ reduces to evaluating $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}|$ for $1 \leq i_1 < i_2 < \cdots < i_m \leq h(n,k)$.

Assume that $2^{n-3} < k \leq 2^{n-2}$. In what follows, we shall focus on the computation of the cardinalities of $A_i$ for $1 \leq i \leq h(n,k)$, and the cardinalities of $A_i \cap A_j$ for $1 \leq i < j \leq h(n,k)$. The computation for the cardinalities of $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}$ in the general case can be carried out in the same way. When $n = 6$ and $k = 13, 14, 15, 16$, the computations turn out to be quite simple.

We first compute $|A_i| \ (1 \leq i \leq h(n,k))$ for $2^{n-3} < k \leq 2^{n-2}$. Since $A_i = \Phi(\mathcal{P}(H_i, k))$, we have $|A_i| = |\mathcal{P}(H_i, k)|$. Recall that $|\mathcal{P}(H_i, k)|$ is defined as $N_{H_i}(k)$ in Section 4 and has been computed for the case $2^{n-2} < k \leq 2^{n-1}$. To compute $N_{H_i}(k)$ for $2^{n-3} < k \leq 2^{n-2}$, we need some notation.

Let $H$ be a spanned hyperplane of $Q_n$, and $\mathcal{S}$ be a subset of $H$. Recall that $\mathcal{S}(Q_n)$ is the set of 0/1-polytopes of $Q_n$ contained in $\mathcal{S}$. In Section 4, we defined the partial 0/1-equivalence relation on $\mathcal{S}(Q_n)$. Here we need introduce another equivalence relation on $\mathcal{S}(Q_n)$, that is, two 0/1-polytopes in $\mathcal{S}(Q_n)$ are said to be equivalent if one can be transformed to the other by a symmetry in $F(H)$. The associated equivalence classes in $\mathcal{S}(Q_n)$ are called local 0/1-equivalence classes of $\mathcal{S}$. Since $F(H)$ is a subgroup of $B_n$, each local 0/1-equivalence class of $\mathcal{S}$ is contained in a (unique) partial 0/1-equivalence class of $\mathcal{S}$.
Denote by $\mathcal{L}(\mathcal{S}, k)$ the set of local 0/1-equivalence classes of $\mathcal{S}$ with $k$ vertices. When $\mathcal{S} = H$, $\mathcal{L}(H, k)$ has appeared in Section 4, that is, $\mathcal{L}(H, k)$ is the set of equivalence classes of 0/1-polytopes contained in $H$ with $k$ vertices under the action of $F(H)$. So we have the following relation

$$|\mathcal{L}(H, k)| = \left[u_1^i u_2^{v_i(H)-k}\right] C_H(u_1, u_2).$$

(8.2)

In order to compute $N_H(k)$ for $2^{n-3} < k \leq 2^{n-2}$, we shall define a partition of $\mathcal{L}(H, k)$ into two subsets $\mathcal{L}_s(H, k)$ and $\mathcal{L}^*(H, k)$. This requires a property as given in Theorem 8.1.

Let $H$ be a spanned hyperplane of $Q_n$ containing at least $k$ vertices of $Q_n$. Denote by $E(H, k)$ the set of intersections $H \cap w(H)$ such that

1. The symmetry $w$ of $Q_n$ does not fix $H$, that is, $H \neq w(H)$;

2. The intersection $H \cap w(H)$ contains at least $k$ vertices of $Q_n$.

Denote by $h_1(H, k)$ the number of equivalence classes of $E(H, k)$ under the symmetries in $F(H)$. Let $E_1(H, k) = \{H \cap w_i(H) : 1 \leq i \leq h_1(H, k)\}$ be the set of representatives of these equivalence classes of $E(H, k)$.

Consider the (disjoint) union of $\mathcal{L}(H \cap w_i(H), k)$, where $1 \leq i \leq h_1(H, k)$. We shall define a map $\Phi_1$ from this union to $\mathcal{L}(H, k)$. For $1 \leq i \leq h_1(H, k)$, let $\mathcal{L}_i$ be a local 0/1-equivalence class in $\mathcal{L}(H \cap w_i(H), k)$. Evidently, there is a (unique) local 0/1-equivalence class in $\mathcal{L}(H, k)$ containing $\mathcal{L}_i$, denoted $\mathcal{L}'_i$. Define $\Phi_1(\mathcal{L}_i) = \mathcal{L}'_i$. Then we have the following property.

**Theorem 8.1** If $2^{n-3} < k \leq 2^{n-2}$, then the map $\Phi_1$ is an injection.

**Proof.** Let $\mathcal{L}$ and $\mathcal{L}'$ be two distinct local 0/1-equivalence classes with $k$ vertices. Assume that $\mathcal{L}$ (resp. $\mathcal{L}'$) is in $\mathcal{L}(H \cap w_i(H), k)$ (resp. $\mathcal{L}(H \cap w_j(H), k)$), where $1 \leq i, j \leq h_1(H, k)$. To prove that $\Phi_1$ is an injection, we need to show that $\Phi_1(\mathcal{L}) \neq \Phi_1(\mathcal{L}')$. Clearly, if $i = j$ then we see that $\Phi_1(\mathcal{L}) \neq \Phi_1(\mathcal{L}')$. We now consider the case $i \neq j$.

Assume to the contrary that $\Phi_1(\mathcal{L}) = \Phi_1(\mathcal{L}')$. Let $P$ (resp. $P'$) be any 0/1-polytope in $\mathcal{L}$ (resp. $\mathcal{L}'$). Then there is a symmetry $w \in F(H)$ such that $P = w(P')$. Since both $P$ and $P'$ have more than $2^{n-3}$ vertices of $Q_n$, we see from Theorem 3.1 that $\dim(P) = \dim(P') \geq n - 2$. Since $P$ (resp. $P'$) is contained in $H \cap w_i(H)$ (resp. $H \cap w_j(H)$), both $P$ and $P'$ are of dimension $n - 2$. This implies that $H \cap w_i(H)$ (resp. $H \cap w_j(H)$) is the affine space spanned by $P$ (resp. $P'$). Hence we deduce that $H \cap w_i(H) = w(H \cap w_j(H))$, which is contrary to the assumption that $H \cap w_i(H)$ and $H \cap w_j(H)$ are not equivalent under the symmetries in $F(H)$. This completes the proof.  

\[\square\]
We are now ready to define $L_*(H, k)$ to be the image of $\Phi_1$. More precisely, $L_*(H, k)$ is the (disjoint) union of $\Phi_1(L(H \cap w_i(H), k))$, where $1 \leq i \leq h_1(H, k)$. Let

$$L^*(H, k) = L(H, k) \setminus L_*(H, k).$$  \hfill (8.3)

From the above definition (8.3), it can be seen that, for any local 0/1-equivalence class $L \in L^*(H, k)$ and any 0/1-polytope $P \in L$, if $w \in B_n$ is a symmetry such that $w(P)$ is contained in $H$, then $w(H) = H$. This yields that $L$ is also a partial 0/1-equivalence class of $H$. Consequently, $L^*(H, k)$ is a subset of $P(H, k)$. Let

$$P_*(H, k) = P(H, k) \setminus L^*(H, k).$$  \hfill (8.4)

Combining (8.2), (8.3) and (8.4), we find that

$$N_H(k) = |P(H, k)| = |L^*(H, k)| + |P_*(H, k)|$$
$$= |L(H, k)| - |L_*(H, k)| + |P_*(H, k)|$$
$$= \left[ u_1^k u_2^{\nu_n(H) - k} \right] C_H(u_1, u_2) - |L_*(H, k)| + |P_*(H, k)|.$$  \hfill (8.5)

Therefore, for $2^{n-3} < k \leq 2^{n-2}$, $N_H(k)$ is determined by the cardinalities of $L_*(H, k)$ and $P_*(H, k)$. From Theorem 8.1, we see that for $2^{n-3} < k \leq 2^{n-2}$, $|L_*(H, k)|$ can be derived from the cardinalities of $L(H \cap w(H), k)$, where $H \cap w(H) \in E_1(H, k)$. We shall demonstrate that the computation of $|P_*(H, k)|$ for $2^{n-3} < k \leq 2^{n-2}$ can be carried out in a similar fashion.

Denote by $h_2(H, k)$ the number of equivalence classes of $E(H, k)$ under the symmetries of $Q_n$. Let

$$E_2(H, k) = \{ H \cap w_i(H) : 1 \leq i \leq h_2(H, k) \}$$

be the set of representatives of these equivalence classes of $E(H, k)$. We define a map $\Phi_2$ from the (disjoint) union of $P(H \cap w_i(H), k)$, where $1 \leq i \leq h_2(H, k)$, to $P_*(H, k)$. Let $P$ be a partial 0/1-equivalence class of $P(H \cap w_i(H), k)$ ($1 \leq i \leq h_2(H, k)$). Then the image $\Phi_2(P)$ is defined to be the (unique) partial 0/1-equivalence class of $P_*(H, k)$ that contains $P$. We reach the following assertion. The proof is similar to that of Theorem 8.1 hence it is omitted.

**Theorem 8.2** If $2^{n-3} < k \leq 2^{n-2}$, then the map $\Phi_2$ is a bijection.

So far, we see that the number $N_H(k)$ for $2^{n-3} < k \leq 2^{n-2}$ can be computed based on the cardinalities of $L(H \cap w(H), k)$ and $P(H \cap w(H), k)$, where $H \cap w(H) \in E(H, k)$. We shall illustrate how to compute $|L(H \cap w(H), k)|$ and $|P(H \cap w(H), k)|$ for $2^{n-3} < k \leq 2^{n-2}$.

Assume that $H \cap w(H) \in E(H, k)$. Let $P$ and $P'$ be any two 0/1-polytopes belonging to the same local (resp. partial) 0/1-equivalence class of $H \cap w(H)$ with $k$ vertices. Then
there exists a symmetry in $F(H)$ (resp. $B_n$) such that $w(P) = P'$. It is clear from Theorem 3.1 that both $P$ and $P'$ have dimension $n - 2$. Hence $H \cap w(H)$ is the affine space spanned by $P$, or, equivalently, by $P'$. So we deduce that $w(H \cap w(H)) = H \cap w(H)$.

This implies that for $2^{n-3} < k \leq 2^{n-2}$, we can use Pólya’s theorem to compute the number of local (resp. partial) 0/1-equivalence classes of $H \cap w(H)$ with $k$ vertices.

Let

$$F(H, w) = \{ w' \in F(H) \colon w'(H \cap w(H)) = H \cap w(H) \}$$

and

$$F(H \cap w(H)) = \{ w' \in B_n \colon w'(H \cap w(H)) = H \cap w(H) \}.$$

Denote by $V_n(H \cap w(H))$ the set of vertices of $Q_n$ contained in $H \cap w(H)$, and denote by $Z_{H, w}(z)$ (resp. $Z_{H \cap w(H)}(z)$) the cycle index of $F(H, w)$ (resp. $F(H \cap w(H))$) acting on $V_n(H \cap w(H))$. Write $C_{(H, w)}(u_1, u_2)$ (resp. $C_{H \cap w(H)}(u_1, u_2)$) for the polynomial obtained from $Z_{H, w}(z)$ (resp. $Z_{H \cap w(H)}(z)$) by substituting $z_i$ with $u_1^i + u_2^i$. Thus, for $2^{n-3} < k \leq 2^{n-2}$, we obtain that

$$|\mathcal{L}(H \cap w(H), k)| = \left[ u_1^k u_2^{|V_n(H \cap w(H))| - k} \right] C_{(H, w)}(u_1, u_2)$$

and

$$|\mathcal{P}(H \cap w(H), k)| = \left[ u_1^k u_2^{|V_n(H \cap w(H))| - k} \right] C_{H \cap w(H)}(u_1, u_2).$$

Thus, applying and Theorems 8.1 and 8.2 and plugging the above formulas (8.6) and (8.7) into (8.5), we arrive at the following relation.

**Theorem 8.3** Let $2^{n-3} < k \leq 2^{n-2}$, and $H$ be a spanned hyperplane of $Q_n$ containing at least $k$ vertices of $Q_n$. Set $q(w) = |V_n(H \cap w(H))|$. Then we have

$$N_H(k) = \left[ u_1^k u_2^{|V_n(H)| - k} \right] C_H(u_1, u_2) - \sum_{H \cap w(H) \in E_1(H, k)} \left[ u_1^k u_2^{q(w) - k} \right] C_{H \cap w(H)}(u_1, u_2)$$

$$+ \sum_{H \cap w(H) \in E_2(H, k)} \left[ u_1^k u_2^{q(w) - k} \right] C_{H \cap w(H)}(u_1, u_2).$$

Theorem 8.3 enables us to compute $N_H(k)$ for $k = 13, 14, 15, 16$, where $H$ is a spanned hyperplane of $Q_6$ containing more than 12 vertices of $Q_6$. In addition to $H_{16}^1, H_{16}^2, H_{16}^3, H_{16}^4, H_{16}^5, H_{16}^6$, we have 8 representatives of equivalence classes of spanned hyperplanes of $Q_6$ containing more than 12 vertices of $Q_6$, namely,

$$H_1 : x_1 + x_2 + x_3 + 2x_4 = 2,$$

$$H_2 : x_1 + x_2 + x_3 + x_4 = 1,$$

$$H_3 : x_1 + x_2 + x_3 + x_4 + 2x_5 = 3.$$
\[ H_4: x_1 + x_2 + x_3 + x_4 + x_5 + 2x_6 = 3, \]
\[ H_5: x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 2, \]
\[ H_6: x_1 + x_2 + x_3 + x_4 + 2x_5 = 2, \]
\[ H_7: x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 3, \]
\[ H_8: x_1 + x_2 + x_3 + x_4 + 2x_5 + 2x_6 = 4. \]

It is easily checked that \( E(H, k) = \emptyset \) for \( k = 13, 14, 15, 16 \), except for the two spanned hyperplanes \( H_6^1 \) and \( H_6^2 \). Therefore, we can deduce from Theorem 8.3 that

\[
N_H(k) = \left[ u_1^k u_2^{V_n(H) - k} \right] C_H(u_1, u_2),
\]

where \( H = H_6^3 - H_6^6, H_1 - H_8 \). The cycle indices for \( H = H_6^3 - H_6^6 \) have been given in Section 7. For \( H = H_6, H_7 \) and \( H_8 \), it is easily verified that \( N_H(13) = 2 \) and \( N_H(14) = 1 \). Using the techniques in Section 6, we can derive the cycle indices for \( H_1 - H_5 \) as shown below:

\[
Z_{H_1}(z) = \frac{1}{48} \left( z_1^{16} + 4z_1 z_4 + 4z_2^2 z_6 z_2 z_2 + 2z_3^4 z_1 + 12z_2^8 + 8z_4^4 + 6z_1^4 z_6^2 + 5z_1^3 z_2 + 6z_6^2 z_2^2 \right),
\]
\[
Z_{H_2}(z) = \frac{1}{192} \left( z_1^{16} + 68z_1^4 + 24z_2^2 z_6 z_2 + 16z_1 z_4 + 8z_3^4 z_1 + 39z_2^8 + 12z_1^2 z_6 + 8z_1^4 z_4 + 16z_1^3 z_6 z_2 z_2 \right),
\]
\[
Z_{H_3}(z) = \frac{1}{96} \left( z_1^{16} + 24z_1^2 z_2^2 + 8z_3^4 z_1^4 + 33z_2^8 + 6z_1^4 z_2^4 + 24z_4^4 \right),
\]
\[
Z_{H_4}(z) = \frac{1}{120} \left( z_1^{15} + 24z_3^3 + 30z_1 z_2 z_6 z_6 + 20z_1 z_2 z_6 z_2 + 20z_3^4 z_4 + 15z_1^3 z_2^6 + 10z_1^3 z_2^2 \right),
\]
\[
Z_{H_5}(z) = \frac{1}{720} \left( z_1^{15} + 120z_3 z_6^2 + 144z_3^3 + 40z_5^3 + 180z_1 z_2 z_4^3 + 40z_1 z_3^3 + 60z_1 z_2 z_6^2 + 15z_1 z_2 z_6^2 + 120z_1 z_2 z_6^2 \right).
\]

It remains to compute \( N_H(k) \) for \( H = H_6^1 \) and \( H_6^2 \) for \( k = 13, 14, 15, 16 \). For \( H_6^1: x_1 = 0 \) and \( k = 13, 14, 15, 16 \), it is routine to check that

\[
E_1(H_6^1, k) = E_2(H_6^1, k) = \{ H_6^1 \cap w(H_6^1): w = (1, 2)(3)(4)(5)(6) \},
\]

that is,

\[
\{(x_1, \ldots, x_6): x_1 = 0, x_2 = 0 \}.
\]

Thus, for \( k = 13, 14, 15, 16 \), it is clear that both the numbers of local and partial 0/1-equivalence classes of \( H_6^1 \cap w(H_6^1) \) with \( k \) vertices are given by

\[
\left[ u_1^k u_2^{16-k} \right] C_4(u_1, u_2).
\]
Therefore, for \(k = 13, 14, 15, 16\), by Theorem 8.3 we find that
\[
N_{H^4}(k) = \left[u_1^k u_2^{32-k}\right] C_{H^6}(u_1, u_2).  \tag{8.10}
\]

Finally, we come to the computation of \(N_{H^2}(k)\) for \(k = 13, 14, 15, 16\). In this case, it is easy to check that
\[
E_1(H^2_6, k) = E_2(H^2_6, k) = \{H^2_6 \cap w_1(H^2_6), H^2_6 \cap w_2(H^2_6)\},
\]
where \(w_1 = (1, 3, 2)(4)(5)(6)\) and \(w_2 = (1, 3)(2, 4)(5)(6)\). Since
\[
V_6(H^2_6 \cap w_1(H^2_6)) = \{(1, 0, 1, x_4, x_5, x_6) : x_i = 0 \text{ or } 1 \text{ for } i = 4, 5, 6\} \cup \\
\{(0, 1, 0, x_4, x_5, x_6) : x_i = 0 \text{ or } 1 \text{ for } i = 4, 5, 6\},
\]
it can be easily checked that \(L(H^2_6 \cap w_1(H^2_6), k) = P(H^2_6 \cap w_1(H^2_6), k)\) for \(k = 13, 14, 15, 16\). By Theorem 8.3, we obtain that for \(k = 13, 14, 15, 16\),
\[
N_{H^2_6} = \left[u_1^k u_2^{32-k}\right] C_{H^6}(u_1, u_2) - \left[u_1^k u_2^{16-k}\right] C_{H^6,w_2}(u_1, u_2)
+ \left[u_1^k u_2^{16-k}\right] C_{H^6}(u_1, u_2). \tag{8.11}
\]

Next, we proceed to demonstrate how to compute \(|A_i \cap A_j|\) for \(1 \leq i < j \leq h(n, k)\). Let \(E(H_i, H_j, k)\) be the set of intersections \(H_i \cap w(H_j) (w \in P_n)\) that contain at least \(k\) vertices of \(Q_n\). Denote by \(h(H_i, H_j, k)\) the number of equivalence classes of \(E(H_i, H_j, k)\) under the symmetries of \(Q_n\). Let \(m = h(H_i, H_j, k)\). Assume that \(E_1(H_i, H_j) = \{H_i \cap w_1(H_j), \ldots, H_i \cap w_m(H_j)\}\) is the set of representatives of equivalence classes in \(E(H_i, H_j, k)\). We define a map \(\Phi_3\) from the union of \(P(H_i \cap w_s(H_j), k)\), where \(1 \leq s \leq h(H_i, H_j, k)\), to \(A_i \cap A_j\). Let \(P_s\) be a partial 0/1-equivalence class in \(P(H_i \cap w_s(H_j), k)\). Clearly, there is a (unique) partial 0/1-equivalence class in \(A_i \cap A_j\) containing \(P_s\), which will be denoted by \(P'_s\). Define \(\Phi_3(P_s) = P'_s\). We have the following conclusion. We omit the proof since it is similar to that of Theorem 8.1.

**Theorem 8.4** If \(2^{n-3} < k \leq 2^{n-2}\), then the map \(\Phi_3\) is a bijection.

As a consequence of Theorem 8.4 for \(2^{n-3} < k \leq 2^{n-2}\), we have
\[
|A_i \cap A_j| = \sum_{s=1}^{h(H_i, H_j, k)} |P(H_i \cap w_s(H_j), k)|.
\]

The computation for \(|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}| (m \geq 3)\) in the general case can be done in a similar fashion. In fact, it will be shown that for \(2^{n-3} < k \leq 2^{n-2}\), the computation can be reduced to the case \(m = 2\).
Let $2^{n-3} < k \leq 2^{n-2}$, and $E(H_1, \ldots, H_m, k)$ be the set of intersections $H_i \cap w_2(H_i) \cap \cdots \cap w_m(H_i)$, where $w_i$ for $2 \leq i \leq m$ are symmetries of $Q_n$, that contain at least $k$ vertices of $Q_n$. Denote by $E_1(H_1, \ldots, H_m, k)$ the set of representatives of equivalence classes of $E(H_1, \ldots, H_m, k)$ under the symmetries of $Q_n$. We define a map $\Phi_m$ from the (disjoint) union of $P(H_i \cap w_2(H_i) \cap \cdots \cap w_m(H_i), k)$, where

$$H_i \cap w_2(H_i) \cap \cdots \cap w_m(H_i) \in E(H_1, \ldots, H_m, k),$$

to the set $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}$. Let $P \in P(H_i \cap w_2(H_i) \cap \cdots \cap w_m(H_i), k)$. The image $\Phi_m(P)$ is defined to be the unique partial 0/1-equivalence class in $A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}$ containing $P$. Similarly, we can prove that if $2^{n-3} < k \leq 2^{n-2}$, then $\Phi_m$ is a bijection. Thus we deduce that for $2^{n-3} < k \leq 2^{n-2}$,

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_m}| = \sum |P(H_i \cap w_2(H_i) \cap \cdots \cap w_m(H_i), k)|,$$

where the sum ranges over the representatives of $E_1(H_1, \ldots, H_m, k)$.

We further claim for $2^{n-3} < k \leq 2^{n-2}$, $E(H_1, \ldots, H_m, k)$ is a subset of $E(H_1, E_{i_2})$. This can be proved as follows. Assume that $2^{n-3} < k \leq 2^{n-2}$, and that $H_i \cap w_2(H_i) \cap \cdots \cap w_m(H_i)$ is in $E(H_1, \ldots, H_m, k)$. From Theorem [3.1] it can be seen that the dimension of $H_i \cap w_2(H_i) \cap \cdots \cap w_m(H_i)$ is at least $n-2$, since it contains more than $2^{n-3}$ vertices of $Q_n$. On the other hand, it is clear that $H_i \cap w_2(H_i) \cap \cdots \cap w_m(H_i)$ has dimension at most $n-2$. Hence, when $2^{n-3} < k \leq 2^{n-2}$, we conclude that $H_i \cap w_2(H_i) \cap \cdots \cap w_m(H_i)$ is of dimension $n-2$. Hence we obtain that $H_i \cap w_2(H_i) \cap \cdots \cap w_m(H_i) = H_i \cap w_2(H_i)$. Therefore, $E(H_1, \ldots, H_m, k)$ is a subset of $E(H_1, E_{i_2})$. This implies that for $2^{n-3} < k \leq 2^{n-2}$, the computation for $E(H_1, \ldots, H_m, k)$ can be reduced to the case $m = 2$. More specifically, for $2^{n-3} < k \leq 2^{n-2}$, an intersection $H_i \cap w_2(H_i) \cap \cdots \cap w_m(H_i)$ belongs to $E(H_1, \ldots, H_m, k)$ whenever (possibly after the action of some symmetry of $Q_n$) it belongs to $E(H_{i_1}, H_{i_2})$ for $1 \leq j_1 < j_2 \leq m$.

We now turn to the case when $n = 6$ and $k = 13, 14, 15, 16$. All possible pairs $\{H_i, H_j\}$ such that $E(H_i, H_j, k)$ is nonempty are listed below.

1. $\{H_6^1, H_6^2\}$. In this case, it can be easily checked that

$$E_1(H_6^1, H_6^2, k) = \{H_6^1 \cap H_6^2 \cap w(H_6^2) = \{w \in \{1, 3, 2\}(4)(5)(6)\} = \{H_6^1 \cap H_6^2\} \cup \{H_6^3 \cap H_6^4\}. \quad (8.12)$$

2. $\{H_6^1, H_6^3\}$ and $\{H_6^2, H_6^4\}$. In these two cases, we have

$$E_1(H_6^1, H_6^3, k) = E_1(H_6^2, H_6^4, k) = \{H_6^1 \cap H_6^3\}. \quad (8.13)$$

3. $\{H_6^2, H_6^4\}$. In this case, it can be verified that

$$E_1(H_6^2, H_6^4, k) = \{H_6^2 \cap H_6^4\}. \quad (8.14)$$
From the above, we see that \( H_1^6, H_2^6 \) and \( H_3^6 \) are the only hyperplanes such that for \( k = 13, 14, 15, 16 \), \( E(H_1, H_2, H_3, k) \) is nonempty. Moreover, for \( k = 13, 14, 15, 16 \) we have

\[
E_1(H_1^6, H_2^6, H_3^6, k) = \{ H_1^6 \cap H_3^6 \}. \tag{8.15}
\]

For \( k = 13, 14, 15, 16 \), it is easy to see that

\[
|P(H_1^6 \cap H_2^6, k)| = \left[ u_1^k u_2^{16-k} \right] C_4(u_1, u_2),
\]

\[
|P(H_1^6 \cap H_3^6, k)| = \left[ u_1^k u_2^{16-k} \right] C_H^2(u_1, u_2),
\tag{8.16}
\]

\[
|P(H_2^6 \cap H_4^6, k)| = \left[ u_1^k u_2^{16-k} \right] C_{H_6^2 \cap w(H_1^6)}(u_1, u_2),
\]

where \( w = (1, 3)(2, 4)(5)(6) \).

From (8.14) and the relations (8.9)–(8.16), we deduce that for \( n = 6 \) and \( k = 13, 14, 15, 16 \),

\[
H_6(k) = \sum_{i=1}^{6} \left[ u_1^k u_2^{\lfloor H_6^i \rfloor - k} \right] C_{H_6^i}(u_1, u_2) + \sum_{i=1}^{8} \left[ u_1^k u_2^{\lfloor H_6^i \rfloor - k} \right] C_{H_6^i}(u_1, u_2),
\tag{8.17}
\]

\[
- \left[ u_1^k u_2^{16-k} \right] C_4(u_1, u_2) - 2 \left[ u_1^k u_2^{16-k} \right] C_{H_6^2}(u_1, u_2) - \left[ u_1^k u_2^{16-k} \right] C_{H_6^2 \cap w(H_1^6)}(u_1, u_2),
\]

where \( w = (1, 3)(2, 4)(5)(6) \). Using the argument in Section 6, for \( w = (1, 3)(2, 4)(5)(6) \) we obtain that

\[
Z_{(H_6^2 \cap w)} = \frac{1}{32} \left( z_1^{16} + 21 z_2^8 + 8 z_4^4 + 2 z_4^8 \right). \tag{8.18}
\]

Hence, from (8.17) and (8.18) we obtain the values of \( H_6(k) \) for \( k = 13, 14, 15, 16 \). Utilizing the relation \( F_6(k) = A_6(k) - H_6(k) \), we deduce \( F_6(k) \) for \( k = 13, 14, 15, 16 \) as given in Table 7.

| \( k \) | 13 | 14 | 15 | 16 |
|---|---|---|---|---|
| \( F_6(k) \) | 290159817 | 1051410747 | 3491461629 | 10665920350 |

Table 7: \( F_6(k) \) for \( k = 13, 14, 15, 16 \).

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