Almost sure multifractal spectrum of SLE

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Abstract

Suppose that \( \eta \) is a Schramm-Loewner evolution (SLE\( \kappa \)) in a smoothly bounded simply connected domain \( D \subset \mathbb{C} \) and that \( \phi \) is a conformal map from \( D \) to a connected component of \( D \setminus \eta([0,t]) \) for some \( t > 0 \). The multifractal spectrum of \( \eta \) is the function \((-1,1) \to [0,\infty)\) which, for each \( s \in (-1,1) \), gives the Hausdorff dimension of the set of points \( x \in \partial D \) such that \( |\phi'(1-\epsilon x)| = \epsilon^{s+o(1)} \) as \( \epsilon \to 0 \).

We rigorously compute the a.s. multifractal spectrum of SLE, confirming a prediction due to Duplantier. As corollaries, we confirm a conjecture made by Beliaev and Smirnov for the a.s. bulk integral means spectrum of SLE and we obtain a new derivation of the a.s. Hausdorff dimension of the SLE curve for \( \kappa \leq 4 \). Our results also hold for the SLE\( \kappa(\rho) \) processes with general vectors of weight \( \rho \).

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1 Introduction

The Schramm-Loewner evolution (SLE_κ) is a one-parameter family of random fractal curves in a simply connected domain in \( \mathbb{C} \), indexed by \( \kappa > 0 \). SLE was introduced by Schramm in [Sch00], and has since become a central object of study in both probability theory and statistical physics. See e.g. [Wer04, Law05] for an introduction to SLE. Its importance is that it describes the scaling limit of the interfaces which arise in a number of discrete models in statistical physics, see, e.g., [LSW04, Smi10, SS05, SS09, Mil10].

Roughly speaking, the multifractal spectrum of a domain \( D \subset \mathbb{C} \) refers to one of the two functions

\[
s \mapsto \dim_H \Theta^s(D) \quad \text{or} \quad s \mapsto \dim_H \tilde{\Theta}^s(D)
\]

where \( \dim_H \) denotes the Hausdorff dimension and \( \tilde{\Theta}^s(D) \) is the set of points \( x \in \partial D \) with the property that the modulus of the derivative \( |\phi'(1 - \epsilon x)| \) of a conformal map \( \phi \) from the unit disk \( D \) into \( D \) grows like \( \epsilon^{-s} \) as \( \epsilon \to 0 \) and \( \Theta^s(D) = \phi(\tilde{\Theta}^s(D)) \). There are several more or less equivalent definitions of this concept. See Section 1.1 for the precise definition we use in this paper.

The multifractal spectrum of \( D \) is a means of quantifying the behavior of \( |\phi'| \) near \( \partial D \), even though \( \phi \) need not be differentiable on \( \partial D \). It is closely related to various other quantities associated with \( \partial D \), e.g. the Hausdorff dimension, Hölder regularity, and packing dimension of \( \partial D \); the integral means spectrum of \( D \); and the harmonic measure spectrum of the complement of a hull. See [Mak98] for some results in this direction. Such complex analytic quantities are often difficult if not impossible to compute explicitly for specific deterministic domains. However, for random domains (like the complement of an SLE curve) explicit calculations can sometimes be more tractable.

There has been substantial interest in the multifractal properties of SLE_κ (i.e. that of the domain obtained by excising the curve) in both mathematics and physics recent years. For example, it is shown by Beffara in [Bef08] that the a.s. Hausdorff dimension of the SLE_κ curve is \( 1 + \kappa/8 \) for \( \kappa \in (0, 8) \) and 2 for \( \kappa \geq 8 \). The optimal Hölder exponent for the SLE_κ curve is derived in [JVL11], building on the work of Rohde and Schramm [RS05] and Lind [Lin08].

There have also been a number of works which study various versions of the multifractal spectrum of SLE. The first such works [Dup99a, Dup99b], due to Duplantier, give non-rigorous predictions of the multifractal exponents for Brownian motion and self-avoiding random walk, which correspond to SLE_κ for \( \kappa = 6 \) and \( \kappa = 8/3 \), respectively. In [Dup00], Duplantier extends this to a non-rigorous prediction of the multifractal
spectrum of the SLE_κ curve for general values of κ > 0. Observing that the predicted multifractal spectrum for SLE_κ in [Dup00] is invariant under the replacement κ ↦→ 16/κ is what originally led Duplantier to conjecture SLE duality (c.f. [Dup00, Dup03]), which states the outer boundary of an SLE_κ curve for κ > 4 is described by a type of SLE_{16/κ} curve. Various forms of SLE duality have since been rigorously proven in [Zha08a, Zha10, Dub09a, MS12a, MS13a].

In [DB02, DB08], the authors study (non-rigorously) a notion of spectrum involving the argument, rather than just the modulus, of the derivative of the SLE maps. In [Dup03], these predictions are expanded to higher multifractal spectra, e.g. the dimension of the set of points on the curve where the behavior of the derivative on both sides of the curve is prescribed. See also [Dup04] for additional discussion of these and other multifractal-type spectra.

The first mathematical work on the multifractal spectrum of SLE is due to Beliaev and Smirnov [BS09] in which they compute the average integral means spectrum for a whole-plane SLE curve. Expanding on the results of [BS09], the authors of [DNNZ12] (see also [LY13, LY14]) use exact solutions of differential equations for the moments of the derivatives of the whole-plane SLE maps to study the integral means spectrum of certain SLE and generalized SLE processes. In [JVL12], the authors rigorously compute the multifractal spectrum at the tip of the SLE curve; this is the first work in which an almost sure result for the multifractal spectrum for SLE is obtained. Binder and Duplantier have informed the authors in private communication [BD14] of a forthcoming work in which they prove formulae for the average mixed integral means spectra (i.e. β-spectrum with complex exponent) both in the bulk and at the tip, for chordal SLE. The corresponding formulae agree after Legendre transform with the predictions from [DB02, DB08] concerning the mixed multifractal spectra for harmonic measure and rotation (equivalently, modulus and argument).

In this article, we will give the first rigorous derivation of the a.s. bulk multifractal spectrum of chordal SLE_κ (i.e. that of the complementary domain). We will also obtain the a.s. bulk integral means spectrum of SLE; the spectrum that we find confirms [BS09, Conjecture 1]. Our approach differs from those used elsewhere in the literature to prove results of this type in that we make use of various couplings of SLE processes with the Gaussian free field (GFF). In the proof of the upper bound we use a coupling of the reverse SLE Loewner flow with a free boundary GFF (sometimes called the “quantum zipper”) [She10, MS13b, DMS14]. Our proof of the lower bound will make extensive use of the coupling of SLE with a GFF with Dirichlet boundary conditions (sometimes called the “imaginary geometry” coupling) [She05, Dub09b, MS12a, MS12b, MS12c, MS13a]. This latter coupling has also been used to aid in proving lower bounds for the Hausdorff dimensions of sets associated with SLE in [MW14]. Our approach at a high level is similar in spirit to the one used in [MW14], but the technical details are rather different.

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1.1 Multifractal spectrum definition

We will now introduce the sets whose Hausdorff dimension we will compute, in the setting of general domains in the complex plane. Our definitions are similar to those in [JVL12, Section 2], but we deal with the boundary of a domain rather than the tip of a given curve.

Let D ⊂ C be a simply connected domain and let φ : D → D be a conformal map. For s ∈ R, define

\[ \tilde{\Theta}^s(D) := \left\{ x \in \partial D : \lim_{\epsilon \to 0} \frac{\log |\phi'(1-\epsilon)x|}{-\log \epsilon} = s \right\} \] (1.1)

and

\[ \Theta^s(D) := \phi(\tilde{\Theta}^s(D)). \] (1.2)
Also define
\[
\Theta^{s \leq}(D) := \left\{ x \in \partial D : \limsup_{\epsilon \to 0} \frac{\log |\phi'(1-\epsilon)x|}{-\log \epsilon} \leq s \right\}
\]
\[
\Theta^{s}(D) := \phi(\Theta^{s \leq}(D))
\]
\[
\tilde{\Theta}^{s \geq}(D) := \left\{ x \in \partial D : \limsup_{\epsilon \to 0} \frac{\log |\phi'(1-\epsilon)x|}{-\log \epsilon} \geq s \right\}
\]
\[
\Theta^{s \geq}(D) := \phi(\tilde{\Theta}^{s \geq}(D)).
\]

The *multifractal spectrum* of \( D \) can be defined as one of the two functions \( s \mapsto \dim_H \Theta^s(D) \) or \( s \mapsto \dim_H \tilde{\Theta}^s(D) \). It is easy to check that these definitions do not depend on the choice of conformal map \( \phi \). We note that although the sets \( \Theta^s(D) \) and \( \tilde{\Theta}^s(D) \) are defined for all \( s \in \mathbb{R} \), these sets are empty for \( s \notin [-1,1] \) (see Lemma 2.11 below).

### 1.2 Main results

Our main result is the following theorem.

**Theorem 1.1.** Let \( \kappa \leq 4 \). Let \( \eta \) be a chordal SLE\(_{\kappa} \) from \(-i\) to \( i\) in \( D \). Let \( D_\eta \) be the connected component of \( D \setminus \eta([0,\infty)) \) containing \( 1 \) on its boundary. Let
\[
\tilde{\xi}(s) := 1 - \frac{(4+\kappa)^2 s^2}{8\kappa (1+s)} \quad (1.3)
\]
\[
\xi(s) := \frac{8\kappa (1+s) - (4+\kappa)^2 s^2}{8\kappa (1-s^2)} \quad (1.4)
\]
\[
s_- := \frac{4\kappa - 2\sqrt{2}\sqrt{\kappa(2+\kappa)(8+\kappa)}}{(4+\kappa)^2} \quad (1.5)
\]
\[
s_+ := \frac{4\kappa + 2\sqrt{2}\sqrt{\kappa(2+\kappa)(8+\kappa)}}{(4+\kappa)^2}. \quad (1.6)
\]

For \( s \in (-1,1) \), a.s.
\[
\dim_H \tilde{\Theta}^s(D_\eta) = \dim_H \tilde{\Theta}^{s \geq}(D_\eta) = \tilde{\xi}(s), \quad 0 \leq s \leq s_+
\]
\[
\dim_H \tilde{\Theta}^s(D_\eta) = \dim_H \tilde{\Theta}^{s \leq}(D_\eta) = \xi(s), \quad s_- \leq s \leq 0
\]
\[
\dim_H \Theta^s(D_\eta) = \dim_H \Theta^{s \geq}(D_\eta) = \xi(s), \quad \frac{\kappa}{4} \leq s \leq s_+
\]
\[
\dim_H \Theta^s(D_\eta) = \dim_H \Theta^{s \leq}(D_\eta) = \xi(s), \quad s_- \leq s \leq \frac{\kappa}{4}.
\]

Moreover, we a.s. have \( \tilde{\Theta}^s(D_\eta) = \Theta^s(D_\eta) = \emptyset \) for each \( s \notin [s_-,s_+] \).

**Remark 1.2.** The significance of \( s_- \) and \( s_+ \) is that \( \tilde{\xi}(s) \geq 0 \) for \( s \in [s_-,s_+] \), and the significance of \( s = \kappa/4 \) is that it is the value which maximizes \( \xi \). Note \( s_- \in (-1,0) \) and \( s_+ \in (0,1] \) for any \( \kappa > 0 \) and \( s_+ = 1 \) if and only if \( \kappa = 4 \). We refer the reader to Remark 7.7 below for more detail regarding the case \( \kappa = 4 \), \( s = 1 \).

The SLE\(_{\kappa}(\rho) \) processes are an important variant of SLE in which one keeps track of extra marked points — so-called force points. The force points can be either on the domain boundary or in its interior and are respectively referred to as boundary and interior force points. These processes were first introduced by Lawler, Schramm, and Werner in [LSW03 Section 8.3] and, just like ordinary SLE\(_{\kappa} \), the SLE\(_{\kappa}(\rho) \) processes naturally arise in many different contexts. Since SLE\(_{\kappa}(\rho) \) for different vectors of weights \( \rho \) has the same behavior when it is not interacting with its force points, one expects an analogue of Theorem 1.1 to be true for such processes provided we exclude points near the boundary of the domain and stop the path before interacting with an interior force point. Furthermore, by SLE duality, one expects an analogue of Theorem 1.1 for \( \kappa > 4 \). Such results do indeed hold true, as described in the following corollary.
**Corollary 1.3.** Let \( D \subset \mathbb{C} \) be a smoothly bounded domain. Let \( \kappa > 0 \) and let \( \rho \) be a vector of real weights. Let \( \eta \) be a chordal SLE\(_{\kappa}(\rho) \) process in \( D \), with any choice of initial and target points and force points located anywhere in \( \overline{D} \), run up until the first time it either hits an interior force point or hits the continuation threshold (c.f. [MS12a, Section 2.1]). Fix \( s \in (-1,1) \). Almost surely, the following is true. Let \( V \) be a connected component of \( D \setminus \eta \) or a connected component of \( D \setminus \eta([0,t]) \) for any \( t > 0 \) before \( \eta \) hits an interior force point or the continuation threshold and let \( \phi : D \to V \) be a conformal map. Then

\[
\dim_H (\Theta^s(V) \setminus \phi^{-1}(\partial D)) = \dim_H (\tilde{\Theta}^{s;Z}(V) \setminus \phi^{-1}(\partial D)) = \tilde{\xi}(s), \quad 0 \leq s \leq s_+ \\
\dim_H (\Theta^s(V) \setminus \phi^{-1}(\partial D)) = \dim_H (\tilde{\Theta}^{s;Z}(V) \setminus \phi^{-1}(\partial D)) = \tilde{\xi}(s), \quad s_- \leq s \leq 0 \\
\dim_H (\Theta^s(V) \setminus \partial D) = \dim_H (\tilde{\Theta}^{s;Z}(V) \setminus \partial D) = \xi(s), \quad \frac{\kappa}{4} \leq s \leq s_+ \\
\dim_H (\Theta^s(V) \setminus \partial D) = \dim_H (\tilde{\Theta}^{s;Z}(V) \setminus \partial D) = \xi(s), \quad s_- \leq s \leq \frac{\kappa}{4}
\]

That is, the conclusion of Theorem 1.1 holds a.s. away from the domain boundary at all times simultaneously for an SLE\(_{\kappa}(\rho) \) with a general \( \kappa > 0 \) and vector of weights \( \rho \) up until the process either hits an interior force point or the continuation threshold.

**Proof.** This follows from Theorem 1.1 combined with Proposition 2.16 below. Note that the functions \( \tilde{\xi}(s) \) and \( \xi(s) \) are unaffected if we replace \( \kappa \) by \( 16/\kappa \), as one would expect from SLE duality [Zha08a, Zha10, Dub09a, MS12a, MS13a].

**Remark 1.4.** We believe that the techniques developed in this paper could also be employed to describe the multifractal behavior of the SLE\(_{\kappa}(\rho) \) processes even near their intersection points with the domain boundary and near their tip, though we will not carry this out here.

Roughly speaking, the harmonic measure spectrum of a hull \( A \subset \mathbb{H} \) gives, for each \( \alpha \in (1/2, \infty) \), the Hausdorff dimension of the set \( \Theta_{hm}^\alpha(A) \) of points \( x \in \partial A \) for which the harmonic measure from \( \infty \) of \( B_\epsilon(x) \) in \( \mathbb{H} \setminus A \) decays like \( e^\alpha \) as \( \epsilon \to 0 \) (or in the pre-image \( \tilde{\Theta}_{hm}^\alpha(A) \) of \( \Theta_{hm}^\alpha(A) \) under a conformal map \( D \to \mathbb{H} \setminus A \)). In [JVL12, Section 2.3], the authors give a rigorous treatment of the harmonic measure spectrum at the tip of a curve. A nearly identical construction works for the harmonic measure spectrum of a whole hull in \( \mathbb{H} \). Similar constructions also work for hulls in \( D \) or \( \mathbb{C} \). In particular, one has (see [JVL12 Lemma 2.3])

\[
\Theta^s(A) = \Theta_{hm}^{1-s}(\mathbb{H} \setminus A) \quad \forall s \in (-1,1).
\]
Remark 1.5. In light of the relationship between SLE_6 and Brownian motion \[LSW01a\], we see that Corollary 1.3 with \(\kappa = 6\) yields the harmonic measure spectrum for the Brownian frontier computed in \[Law96,LSW01a,LSW01b,LSW01c,LSW02\].

Remark 1.6. In \[Dup00\] (see in particular \[Dup00, Equation 6\]), Duplantier predicts that the harmonic measure spectrum for the bulk of the SLE_\(\kappa\) curve is given by

\[
f(\alpha) = \alpha + \frac{25 - c}{24} \left( 1 - \frac{1}{2} \left( 2\alpha - 1 + \frac{1}{2\alpha - 1} \right) \right),
\]

where

\[
c = \frac{(6 - \kappa)(6 - 16/\kappa)}{4}
\]

is the central charge. The exponent (1.4) is related to the exponent (1.8) by

\[
\xi(s) = f \left( \frac{1}{1 - s} \right).
\]

This is what we would expect in light of (1.7).

The dimension \(\xi(s)\) attains a unique maximum value of \(1 + \kappa/8\) on \([-1, 1]\) at \(s = \kappa/4\). This maximum value coincides with the Hausdorff dimension of the SLE_\(\kappa\) curve \[Bef08\], which suggests that near a “typical point” of \(\eta\), the modulus of the derivative of a conformal map from \(D_\eta\) to \(D\) grows like \(\text{dist}(z, \eta)\). Hence Theorem 1.1 gives an alternative proof of the following.

Corollary 1.7. Let \(\kappa \leq 4\). The Hausdorff dimension of an SLE_\(\kappa\) curve \(\eta\) is a.s. equal to \(1 + \kappa/8\).

We remark that we believe that the methods that we use to establish the lower bound in Theorem 1.1 could be employed to give an independent derivation of the lower bound of the dimension of SLE_\(\kappa\) for all \(\kappa > 0\), however we will not carry this out here.

1.3 Integral means spectrum

The integral means spectrum of a simply connected domain \(D \subset \mathbb{C}\) is the function \(\text{IMS}_D : \mathbb{R} \to \mathbb{R}\) defined by

\[
\text{IMS}_D(a) := \limsup_{\epsilon \to 0} \frac{\log \int_{\partial B_1(0)} |\phi'(z)|^a \, dz}{-\log \epsilon},
\]

where \(\phi : \mathbb{D} \to D\) is a conformal map. (There is a three parameter family of such conformal maps, but \(\text{IMS}_D(a)\) does not depend on the specific choice of \(\phi\).) The integral means spectrum is of substantial interest in complex analysis, primarily in the form of the universal integral means spectrum, which is defined by

\[
\text{IMS}^U(a) := \sup_D \text{IMS}_D(a)
\]

where the supremum is over all simply connected domains \(D \subset \mathbb{C}\). It has been conjectured by Kraetzer \[Kra96\] that \(\text{IMS}^U(t^2/4) = t^2/4\) for \(|t| \leq 2\) and \(\text{IMS}^U(|t|) = |t| - 1\) for \(|t| \geq 2\). This conjecture has several important consequences in complex analysis. See, e.g., \[Pom97, BS05, HS08, Pom92\] for more details. The integral means spectrum is often very difficult to compute in practice for deterministic domains. However, domains bounded by random fractals (e.g. the complement of an SLE_\(\kappa\) curve) are sometimes more tractable. For example, in \[BS09\] Beliaev and Smirnov give an explicit calculation of the average integral means spectrum of the complement of a whole plane SLE_\(\kappa\) curve (which is defined as in (1.9) but with \(|\phi'(z)|^a\) replaced by \(E(|\phi'(z)|^a))\).

In this paper we shall be interested in a slight refinement of the definition of the integral means spectrum for the complement of a curve which negates possible pathologies arising from unusual behavior at its endpoints or when it intersects itself or the boundary of the domain. Namely, let \(D \subset \mathbb{C}\) be a bounded simply connected domain with smooth boundary and let \(\eta : [0, T] \to \overline{D}\) be a non-self-crossing curve (we
allow $T = \infty)$. Let $V$ be a connected component of $D \setminus \eta$. Let $x_V$ be the first (equivalently last) point of $\partial V$ hit by $\eta$ and let $\phi : D \to V$ be a conformal map.

For $\zeta > 0$, let

$$I^\zeta(\phi) := \phi^{-1}(\partial V \setminus (B_\zeta(\eta(T)) \cup B_\zeta(x_V) \cup B_\zeta(\partial D))).$$

(1.10)

Let $A^\zeta(\phi)$ be the set of $z \in \partial B_{1-\epsilon}(0)$ with $z/|z| \in I^\zeta(\phi)$. The bulk integral means spectrum of $V$ is the function $IMS_V : R \to R$ defined by

$$IMS^\text{bulk}_V(a) := \sup_{\zeta > 0} \limsup_{\epsilon \to 0} \frac{\log \int_{A^\zeta(\phi)} |\phi'(z)|^a \, dz}{-\log \epsilon}. \quad (1.11)$$

One can check that the definition (2.7) does not depend on the choice of $\phi$.

We extract the following from the proof of Theorem 1.1.

**Corollary 1.8.** For $a \in R$ with $a < \frac{(4+\kappa)^2}{8\kappa}$, let

$$s_+(a) := -1 + \frac{4 + \kappa}{\sqrt{(4 + \kappa)^2 - 8a\kappa}}. \quad (1.12)$$

Also let $s_-$ and $s_+$ be as in (1.5) and (1.6) and let $a_-$ (resp. $a_+$) be the value of $a$ for which $s_+(a) = s_-$ (resp. $s_+(a) = s_+$). Set

$$IMS^*(a) := \begin{cases} -1 + s_-, & a < a_- \\ -a + \frac{(4 + \kappa)(4 + \kappa - \sqrt{(4 + \kappa)^2 - 8a\kappa})}{4\kappa}, & a \in [a_-, a_+] \\ -1 + s_+, & a > a_+. \end{cases} \quad (1.13)$$

Suppose we are in the setting of Corollary 1.3. Almost surely, the following is true. Let $a \in R$ and let $V$ be a complementary connected component of either $D \setminus \eta$ or of $D \setminus \eta'$ for any $t > 0$ (before $\eta$ hits an interior force point or the continuation threshold if it is an SLE$_\kappa(\rho)$ process). Then

$$IMS^\text{bulk}_V(a) = IMS^*(a). \quad (1.14)$$

The result of Corollary 1.8 is in agreement with the (rigorously proven) formula [4] for the average bulk integral means spectrum of whole-plane SLE in [BS09, Theorem 1] for $a \in [a_-, a_+]$, and with [BS09, Conjecture 1] for the a.s. bulk integral means spectrum for all values of $a \in R$.

**Remark 1.9.** As conjectured in [BS09], the a.s. bulk integral means spectrum of Corollary 1.8 differs from the average integral means spectrum computed in [BS09] for values of $a \notin [a_-, a_+]$. We explain why this is the case. First, as noted in [BS09], we expect the average and a.s. bulk integral means spectra to differ because the function which gives the average bulk integral means spectrum does not satisfy Makarov’s [Mak98] characterization of possible integral means spectra. At a more heuristic level, the average integral means spectrum for $a \notin [a_-, a_+]$ is distorted by the occurrence of the small (but still positive) probability event that a conformal map $\phi : D \to V$ satisfies $|\phi'(z)| \approx (1 - |z|)^{-s}$ for some $z$ close to $\partial D$ and some $s \notin [s_-, s_+]$. However, this event a.s. does not occur in the limit (c.f. Theorem 1.1) so does not effect the a.s. bulk integral means spectrum.

### 1.4 Outline

There is a systematic approach to computing Hausdorff dimensions of random fractal sets of the sort we consider here. One first gets a sharp estimate for the probability that a single point is contained in the set (the “one-point estimate”) and uses this to get an upper bound on the Hausdorff dimension. One then defines a subset of the set of interest (the “perfect points”) and obtains an estimate for the probability that any two given points are perfect (the “two-point estimate”). This enables one to define a Frostman
measure on the set of perfect points and thereby obtain a lower bound on the Hausdorff dimension of the set of interest (see [MP10] Section 4 for more on Frostman measures and their connection to Hausdorff dimension). We will follow this outline here. See, e.g., [MW14, MWW14, JVL12, MSW14] for more examples of this technique.

We will now give a moderately detailed outline of the remainder of this paper. The reader should note that this section does not constitute a precise description of all of the proofs in our paper, but rather is only a heuristic guide. For the sake of brevity, many technical details have been omitted, especially in regards to proof of the two-point estimate.

In Section 2 we will give some background on the objects which appear in our proofs, including SLE, the GFF, and the various couplings between them. We will also establish some notations and prove some elementary lemmas which we will need in the sequel.

Next we will prove our one-point estimate. This is done in two stages. In Section 3, we will establish pointwise derivative estimates for the inverse centered Loewner maps $(f_t^{-1})$ for an SLE$_\kappa$. Roughly, our estimates will take the form

$$P(|(f_t^{-1})'(z)| \approx \epsilon^{-s}, \text{ regularity conditions}) \approx \epsilon^{a(s)}, \quad \forall s \in (-1,1), \quad \forall z \in \mathbb{H} \text{ with Im } z = \epsilon.$$  \hspace{1cm} (1.15)

Here $a(s) = \frac{(4+\kappa)^2}{8\kappa(1+\kappa)}$. The precise meaning of $\approx$ as well as precise regularity conditions in our estimate are given in Section 3.4. The proof of these estimates is based on a family of non-negative martingales for the reverse Loewner flow $(g_t)$, analogous to the martingales for the forward SLE$_\kappa$ flow in [SW05, Section 5]. The reverse Loewner flow is of interest because we have $g_t = \frac{d}{dt} f_t^{-1}$ for each fixed $t$ (see, e.g., [RS05, Lemma 3.1]). For a given $z \in \mathbb{H}$ with Im $z = \epsilon$, one can find a martingale $M^z_t$ with the property that $M_t^z \approx \epsilon^{-\alpha(s)}$, where $E(z)$ denotes the event in the probability in (1.15) with $g_t$ in place of $f_t^{-1}$. We then arrive at

$$P(E(z)) \approx \epsilon^{\alpha(s)} P^z_E(E(z)),$$

where $P^z_E$ denotes the measure obtained by re-weighting the law of the original SLE$_\kappa$ process by $M$ (which will be the law of a reverse chordal SLE$_\kappa(\rho)$ for an appropriate $\rho$). Hence we just need to show $P^z_E(E(z))$ is uniformly positive, independent of $\epsilon$. This is done in two steps. First, to obtain $P^z_E(|g_t(z)| \approx \epsilon^{-s}) \rightarrow 1$ as $\epsilon \rightarrow 0$, we use a coupling of $g_t$ with a GFF together with a coordinate change argument similar in spirit to the proof of [MS13b, Theorem 8.1]. To obtain that the auxiliary regularity conditions hold with uniformly positive probability under $P^z_E$, we use a combination of stochastic calculus, forward/reverse (in the sense of Loewner flows) SLE symmetry, and GFF coupling arguments.

In Section 4 we use the estimate of Section 3 to establish pointwise derivative estimates for the “time infinity” conformal map $\Psi_\eta$ associated with an SLE$_\kappa$ process $\eta$ from $-i$ to $i$ in the unit disk $D$, defined as follows. Let $D_\eta$ be the right connected component of $D \setminus \eta$, as in Theorem 1.1. Let $\Psi_\eta : D_\eta \rightarrow D$ be the unique conformal map fixing $-i$, $i$, and 1. Our estimates for $\Psi_\eta$ take the form

$$P(\text{dist}(z, \eta) \approx \epsilon^{1-s}, |\Psi_\eta'(z)| \approx \epsilon^s, \text{ regularity conditions}) \approx \epsilon^{\gamma(s)}, \quad \forall s \in (-1,1), \quad \forall z \in D.$$  \hspace{1cm} (1.16)

where $\gamma(s) = \alpha(s) - 2s + 1$ and $\alpha(s)$ as above. The idea of the proof of (1.16) is as follows. First we observe using the Koebe quarter theorem that for each $\epsilon > 0$ and each $t > 0$, the set of points $A_\eta(t)$ in $D$ for which the analogue of the event of (1.15) with $D$ in place of $H$ occurs is (approximately) the image under $f_t$ of the set $A_\eta(t)$ of points in $D$ for which the event of (1.16) holds with $\Psi_\eta$ replaced by $f_t$ and $\eta$ replaced by $\eta(0, t)$. Hence the estimates (1.15) together with an elementary change of variables yields $E(\text{Area } A_\eta(t)) \approx \epsilon^{\gamma(s)}$. We are then left to (a) transfer this area estimate from finite time to infinite time and (b) argue that the probability of the event (1.16) does not depend too strongly on $z$. Both tasks will be accomplished by means of various conditioning arguments which rely crucially on the regularity conditions involved in the estimate (1.15).

In Section 5 we will use the estimates (1.15) and (1.16) to prove upper bounds for the Hausdorff dimensions of the sets $\Theta^\ast(D_\eta)$ and $\Theta^{\ast \ast}(D_\eta)$, where $\ast$ stands for $\geq$ or $\leq$ as well as an upper bound for the bulk integral means spectrum of $D_\eta$, as claimed in Corollary 1.8.

In Section 6 we prove our two-point estimate. The first step of the proof is a slight modification of the estimate (1.16). Namely, let $\eta$ denote the time reversal of $\eta$, which has the law of a chordal SLE$_\kappa$ from $i$ to $-i$ [Zha08a]. Let $\tau_\beta$ (resp. $\bar{\tau}_\beta$) be the first time $\eta$ (resp. $\eta$) hits the ball of radius $\epsilon^{-\beta}$ centered at the origin.
Let $\eta^{t_{\beta}} = \eta([0, \tau_{\beta}])$, $\eta^{\overline{t}_{\beta}} = \eta([0, \overline{\tau}_{\beta}])$, and let $\phi_{\beta}$ be the conformal map from $D \setminus (\eta^{t_{\beta}} \cup \eta^{\overline{t}_{\beta}})$ to $D$ which fixes $-i$, $i$, and $1$. Then we will use the one-point estimate (1.16) to show

$$
\Pr \left( |\phi'_{\beta}(z)| \approx e^{-\beta q}, \text{ regularity conditions} \right) \approx e^{-\beta \gamma^*(q)}, \quad \forall q \in (-1/2, \infty).
$$

Here $q = s/(1 - s)$ and $\gamma^*(q) = \gamma(s)/(1 - s) = (q + 1)\gamma(q)$, with $\gamma$ as in (1.16).

The estimate (1.17) allows us to break the event that $|\Psi'_{\beta}(0)| \approx e^{-n\beta}$ down into several stages and estimate each individually. Indeed, if we apply a conformal map from $D \setminus (\eta^{t_{\beta}} \cup \eta^{\overline{t}_{\beta}})$ to $D$ which fixes 0, then the rest of the curve will be mapped to another curve whose law is the same as that of $\eta$ (modulo perturbations of its endpoints, which can be dealt with by growing out a little bit more of the curve). In this manner we can construct two approximately independent events $E_{0,1}$ and $E_{0,2}$ whose intersection is contained in the event $\{ |\Psi'_{\beta}(0)| \approx e^{-2\beta} \}$. By iterating this procedure we construct a sequence of approximately independent events $E_{0,j}$ such that $|\Psi'_{\beta}(0)| \approx e^{-n\beta}$ on $E_n(0) := \bigcap_{j=1}^{n} E_{0,j}$ and $\Pr(E_{z,j}) \approx e^{-\beta \gamma^*(q)}$.

We can similarly construct events $E_{z,j}$ and $E_{z}(z)$ for any $z \in D$ by first mapping $z$ to 0.

For the lower bound on $\dim_H \Theta^s(D_{\eta})$, the perfect points will be, roughly speaking, the set of $z \in D$ for which $E_n(z)$ occurs for every $n \in \mathbb{N}$. In order to obtain a lower bound on the Hausdorff dimension of the set of perfect points, we need to estimate the probability that $E_n(z)$ and $E_n(w)$ both occur for $z, w \in D$, depending on $|z - w|$. To this end, suppose $|z - w| \approx e^{-\beta k}$. We condition on the event $E_k(z)$, corresponding to what happens before we get near $z$ and $w$. After we map out the part of the curve which is grown before the $k$th stage, $z$ and $w$ will be at constant order distance from each other. See Figure 1.2

![Figure 1.2](image.png)

Figure 1.2: If $|z - w| \approx e^{-\beta k}$, then after applying a conformal map which takes the complement of the parts of $\eta$ and $\overline{\eta}$ involved in the event $E_{0,k}^c(z)$ to $D$ and takes $z$ to 0, the images of $z$ and $w$ will be at constant order distance from each other. Note, however, that in this setting the derivatives of the stage $k + 1$-map near $z$ and $w$ are not approximately independent, since they each depend on the whole curve in the picture on the right.

We would like to say that the behaviors of the curve near $z$ and near $w$ are approximately conditionally independent given $E_k(z)$. However, the derivatives of the maps we are interested in depend on the whole curve. Hence we need to localize our events. This is accomplished using a different coupling with a GFF, namely the forward SLE/GFF coupling, or “imaginary geometry” coupling studied in [Dub09b,She10,She05,MS12a,MS12b,MS12c,MS13a].

At each stage in the construction of the events $E_n(z)$, we can add auxiliary curves, which are all flow lines (in the sense of [MS12a]; c.f. Section 2.5) of the same GFF. These auxiliary curves will form pockets surrounding $z$ with the property that the parts of $\eta$ inside different pockets are independent once we condition on the pockets, and the derivative of $\Psi_{\eta}$ at a point inside a pocket can be estimated by the derivative of a

\[\text{Note: 2Actually, we will need to increase } \beta \text{ by a little bit at each stage for technical reasons, but the basic idea of the argument is the same if we consider a fixed but large } \beta.\]
map which depends only on the behavior of \( \eta \) inside this pocket. We then define the event \( E_{z,j} \) so that it depends only on the behavior of the curve inside the \( j \)th pocket. See Figure 1.3 for an illustration.

The independence of the parts of \( \eta \) inside different pockets will eventually enable us to establish the two-point estimate needed for the proof of the lower bounds in Theorem 1.1.

In Section 7, we use our two-point estimate to prove lower bounds for the Hausdorff dimensions of the sets \( \tilde{\Theta}^s(D_\eta) \) and \( \Theta^s(D_\eta) \) as well as for the bulk integral means spectrum of \( D_\eta \).

Appendices A and B contain several technical lemmas which are needed for the proofs in Section 6.

2 Preliminaries

In this section we will establish some notations, give some background on the objects involved in the paper, and prove some elementary lemmas. We recommend that the reader familiarize themselves with Section 2.1 and Section 2.2 before reading the remainder of the paper, as the notations and results of these subsections will be used frequently in the sequel. Sections 2.3, 2.4, and 2.5 contain background on results on SLE, Gaussian free fields, and the couplings between them. Readers who are already familiar with these topics may wish to skim these subsections to acquaint themselves with the notations, and refer back to them as needed. Sections 2.6 and 2.7 contain some elementary lemmas about the sets whose Hausdorff dimensions we will compute. The results of these sections are not used extensively in the sequel, but are needed in Sections 5 and 7.

2.1 Basic notations

Given two variables \( a \) and \( b \), we say \( b = o_a(1) \) if \( b \to 0 \) as \( a \to 0 \) (or as \( a \to \infty \), depending on the context) and we say \( b = O_a(1) \) if \( b \) is bounded above by an \( a \)-independent constant for sufficiently small (or sufficiently small, depending on the context) values of \( a \). We usually allow \( o_a(1) \) and \( O_a(1) \) terms to depend on certain parameters other than \( a \), but not on others. We will describe this dependence as needed.

We say that \( a \lesssim b \) (resp. \( a \gtrsim b \)) if there is a constant \( c \) which does not depend on the main parameters of interest such that \( a \leq cb \) (resp. \( a \geq cb \)). We say \( a \asymp b \) if \( a \lesssim b \) and \( a \gtrsim b \). As in the case of \( o_a(1) \) and \( O_a(1) \) above, we usually allow the implicit constants in \( \lesssim, \gtrsim \), and \( \asymp \) to depend on certain parameters, but not on others, and we describe this dependence as needed.

For a point \( z \in \mathbb{C} \) and \( r > 0 \), we write \( B_r(z) \) for the ball of radius \( r \) centered at \( z \). More generally, for a set \( A \subset \mathbb{C} \), we write \( B_r(A) = \bigcup_{z \in A} B_r(z) \).
For a curve \( \eta : [0, T] \to \mathbf{C} \), we will often use the abbreviation
\[
\eta^t = \eta([0, t]).
\]

Furthermore, when there is no risk of ambiguity we will simply write \( \eta \) for the entire image of \( \eta \).

For a domain \( D \) and \( z \in D \), we write \( \text{hm}^z(A; D) \) for the harmonic measure from \( z \) in \( D \). That is, for \( A \subset \partial D \), \( \text{hm}^z(A; D) \) is the probability that a Brownian motion started from \( z \) exits \( D \) in \( A \).

If \( D' = D \setminus \eta \) for some non-self-crossing curve \( \eta \in \mathcal{D} \) and \( z \) is a point on \( \eta \) which is visited only once, we will write \( z^- \) (resp. \( z^+ \)) for the prime end of \( D \) corresponding to the left (resp. right) side of \( z \). When we use this notation, our curve \( \eta \) will have an obvious orientation and “left” and “right” are as viewed by someone walking along \( \eta \) in the forward direction.

We will also use the following notation.

**Notation 2.1.** Given a Jordan domain \( D \) and \( x, y \in \partial D \), we write \([x, y]_{\partial D}\) for the closed counterclockwise arc from \( x \) to \( y \) in \( \partial D \). We similarly define the open arc \((x, y)_{\partial D}\) and the half-open arcs \((x, y]_{\partial D}\) and \([x, y)_{\partial D}\).

### 2.2 Reverse continuity conditions
#### 2.2.1 In the upper half plane

Here we introduce a regularity condition which will arise frequently in the remainder of the paper.

**Definition 2.2.** We denote by \( \mathcal{M} \) the set of increasing functions \( \mu : (0, \infty) \to (0, \infty) \) with \( \lim_{\delta \to 0} \mu(\delta) = 0 \).

**Definition 2.3.** Let \( f \) be a map from a subdomain \( D \) of \( \mathbf{H} \) into \( \mathbf{H} \). For \( \mu \in \mathcal{M} \), let \( G(f, \mu) \) be the event that the following occurs. For any \( \delta > 0 \) and any \( x, y \in \mathbf{R} \cap \partial D \) with \( |x|, |y| \leq \delta^{-1} \) and \( |x - y| \geq \delta \), we have \( |f(x)|, |f(y)| \leq \mu(\delta)^{-1} \) and \( |f(x) - f(y)| \geq \mu(\delta) \).

The statement that \( G(f, \mu) \) holds is the same as the statement that \( f^{-1} \) has a certain \( \mu \)-dependent modulus of continuity on \( f(\mathbf{R} \cup \infty) \), with \( \mathbf{R} \cup \infty \) given the one-point compactification topology.

We note that
\[
G(f, \mu_1) \cap G(g, \mu_2) \Rightarrow G(g \circ f, \mu_2 \circ \mu_1). \tag{2.2}
\]

We are interested in the condition \( G(f, \mu) \) (and the analogous conditions in the next subsection) for two reasons. The first is that these conditions imply bounds on the distance from certain subsets of \( \partial D \) to certain subsets of \( \mathbf{R} \) (or \( \partial \mathbf{D} \) in the setting of the next subsection) and on its diameter (see Lemmas 2.4 and 2.8 below).

The second reason for our interest in the condition of Definition 2.3 is as follows. We will often want to study conformal maps which are normalized by specifying the images of certain marked boundary points. When composing various maps, our marked points might be mapped to somewhere other than where we want them to go. So, we will frequently need to apply a conformal automorphism (of \( D \) or \( \mathbf{H} \)) at the end of our arguments to move the marked points to their desired positions. The condition \( G(\cdot, \mu) \) ensures that the images of the marked points are not too close together, and so allows us to control the derivative of this conformal automorphism.

**Lemma 2.4.** Let \( \eta \) be a simple curve in \( \mathbf{H} \) parametrized by capacity which does not hit \( \mathbf{R} \). Write \( \eta^t := \eta([0, t]) \). Let \( f_t : \mathbf{H} \setminus \eta^t \to \mathbf{H} \) be the centered Loewner maps for \( \eta \). Fix \( T \in (0, \infty) \) and suppose that for some \( \mu \in \mathcal{M} \), we have
\[
f_T(-\delta) - f_T(0^-) \leq -\mu(\delta) \leq \mu(\delta) \leq f_T(\delta) - f_T(0^+), \quad \forall \delta > 0. \tag{2.3}
\]
Then there is a \( \mu' \in \mathcal{M} \) and a \( d > 0 \) depending only on \( \mu \) and \( T \) such that
\[
\text{diam } \eta^T \leq d \quad \text{and} \quad \forall \delta > 0, \forall z \in \eta^T \text{ with } |\text{Re } z| \geq \delta, \text{ we have Im } z \geq \mu'(\delta). \tag{2.4}
\]

**Conversely, if** \( 2.3 \) **holds for some** \( d > 0 \) **and some** \( \mu' \in \mathcal{M} \), **we can find** \( \mu \in \mathcal{M} \) **depending only on** \( d \) **and** \( \mu' \) **such that** \( G(f_T, \mu) \) **holds.**

Note that it is clear that \( G(f_T, \mu) \) implies \( 2.3 \), so Lemma 2.4 implies in particular that \( 2.4 \) holds for some \( d \) and \( \mu' \) depending only on \( \mu \) whenever \( G(f_T, \mu) \) occurs.

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Proof of Lemma \([2.4]\). Let \(hm^\infty_\eta = hm^\infty(\cdot; H \setminus \eta^T)\) denote harmonic measure from \(\infty\) in \(H \setminus \eta^T\), so for a set \(I \subset \partial(H \setminus \eta^T)\) (viewed as a collection of prime ends),

\[
hm^\infty_\eta I := \lim_{y \to \infty} y P^\eta(B_\tau \in I)
\]

for \(B\) a Brownian motion and \(\tau\) its exit time from \(H \setminus \eta^T\). It follows from conformal invariance of Brownian motion that for any \(I \subset \partial(H \setminus \eta^T)\),

\[
hm^\infty_\eta(I) = \frac{1}{\pi} \text{length } f_T(I),
\]

(2.5)

where by length we mean Lebesgue measure.

Now, assume (2.3) holds. For any \(r > 0\) and \(x \in \mathbb{R}\), the harmonic measure from \(\infty\) in \(H\) of the line segment \([x, x + ir]\) from \(x\) to \(x + ir\) is constant depending only on \(r\). For \(\delta > 0\), we can find \(r = r(\delta) > 0\) such that this constant is \(< \pi \mu(\delta)\). If \(\eta^T\) contains a point \(x + iy\) with \(x \geq \delta\) and \(y \leq r\), then \(hm^\infty_\eta([0, \delta]) \leq hm^\infty_\eta([x, x + ir]) < \pi \mu(\delta)\). This contradicts our hypothesis on (2.3) and the relation (2.5). A similar statement holds if we instead consider \(x \leq -\delta\). Hence each point of \(\eta^T\) with real part \(\geq \delta\) in absolute value has imaginary part \(\geq \delta\).

For the first part of (2.4), fix \(\delta > 0\). Denote by \(S_\delta\) the set of points in \(H\) with real part at least \(\delta\). Then we have

\[
hm^\infty_\eta(\eta^T \cap S_\delta) \leq \frac{1}{\mu'(\delta)} \lim_{y \to \infty} y \mathbb{E}^\eta(\text{Im } B_T | 1_{B_\tau, \eta^T \cap S_\delta}).
\]

(2.6)

By [Law05] Proposition 3.38] we have

\[
T = \text{hcp } \eta^T = \lim_{y \to \infty} y \mathbb{E}^\eta(\text{Im } B_\tau)
\]

(2.7)

so (2.6) is at most \(T/\mu'(\delta)\). On the other hand, (2.7) and the Beurling estimate imply that \(\sup_{z \in \partial \eta^T} \text{Im } z\) is bounded above by a constant \(C_0\) depending only on \(T\). The harmonic measure from \(\infty\) in \(H\) of \([-\delta, \delta] \times [0, C_0]\) is at most a constant \(C_1\) depending only on \(\delta\) and \(T\). Therefore

\[
hm^\infty_\eta(\eta^T) \leq T/\mu'(\delta) + C_1.
\]

By [Law05] equation 3.13, this implies \(\text{diam } \eta^T\) is bounded above by a constant depending only on \(\mu\) and \(T\).

Conversely, suppose (2.4) holds. For \(\delta > 0\), let \(U_\delta\) be the set of points in \(z \in H\) with \(|z| \leq \delta\) and either \(|\text{Re } z| \geq \delta/2\) or \(|\text{Im } z| \leq \mu'(\delta/2)\). The harmonic measure from \(\infty\) of each sub-interval of \([\delta/2, -\delta/2] \cup [-\delta, -\delta/2]\) in \(H \setminus U_\delta\) of length \(\delta/2\) is at least some constant \(\mu_0(\delta)\) depending only on \(\delta/2\) and \(\mu'(\delta/2)\). By (2.5), this implies that the length of the image of such an interval under \(f_T\) is at least \(\pi \mu_0(\delta)\). On the other hand, [Law05] Proposition 3.46 implies that we can find \(\mu_1(\delta) > 0\) depending only on \(\delta\) and \(d\) such that \(|f_T(x)| \leq \mu_1(\delta)^{-1}\) for each \(x \in [-\delta, -\delta/2]\). This proves that \(G(f_T, \mu)\) holds with \(\mu = (\pi \mu_0) \lor \mu_1\).

2.2.2 In the disk

The following is the analogue of Definition 2.3 for the unit disk \(D\).

Definition 2.5. Let \(A \subset C\) be a subdomain and let \(I \subset \partial C \cap \partial D\). Let \(f : D \to C\) be a conformal map. Let \(\mu \in \mathcal{M}\) (Definition 2.2). We say that \(G(f, \mu)\) occurs if the following is true. For each \(\delta > 0\) and each \(x, y \in I\) with \(|x - y| \geq \delta\), we have \(|f(x) - f(y)| \geq \mu(\delta)\). We abbreviate

\[
G(f, \mu) = G_{\partial D \cup \partial C}(f, \mu).
\]

We also make the following definition.

Definition 2.6. Let \(A \subset \bar{D}\) be a closed set and \(I \subset \partial \bar{D} \setminus A\). (Oftentimes we will take \(I\) to be a closed arc with endpoints in \(A\), or a finite union of such arcs.) We say that \(G^I(f, \mu)\) occurs if the following is true. For each \(\delta > 0\), \(A\) lies at distance at least \(\mu(\delta)\) from \(I \setminus B_\delta(I \cap A)\). We write

\[
G^I(f, \mu) = G_{\partial D \setminus A}(f, \mu).
\]
Remark 2.7. We will frequently find ourselves in the following situation. Suppose we are given a deterministic arc $I \subset \partial D$, a random closed subset $A \subset D$ with $I \subset \partial D \setminus A$ a.s., and a deterministic $\epsilon > 0$. In this case we can find (using monotonicity) a deterministic $\mu \in \mathcal{M}$ for which $P(G_I(A, \mu)) \geq 1 - \epsilon$ where $P$ is typically the law of SLE.

The conditions of Definitions 2.5 and 2.6 will serve as the main “global regularity” conditions in our estimates starting from Section 4. The relationship between the conditions $G(\cdot)$ and $G'(\cdot)$ is contained in the following lemma.

Lemma 2.8. Let $A \subset D$ be a closed set and $I = [x, y]_A$ be an arc contained in $\partial D \setminus A$. Let $m \in (x, y)_A$ and suppose that $|x - m|$ and $|y - m|$ are each at least $\Delta > 0$. Let $D$ be the connected component of $D \setminus A$ containing $I$ on its boundary. Let $\Phi : D \to D$ be the unique conformal map taking $x$ to $-i$, $y$ to $i$, and $m$ to $1$.

1. For each $\mu \in \mathcal{M}$, there exists $\mu' \in \mathcal{M}$ depending only on $\mu$ and $\Delta$ such that if $G_I(\Phi, \mu)$ occurs, then $G'_I(\Phi, \mu')$ occurs.

2. Conversely, suppose $I' \subset I$ and $G'_I(A, \mu)$ holds for some $\mu \in \mathcal{M}$. There is a $\mu' \in \mathcal{M}$ depending only on $\mu$ and $\Delta$ such that $G_I'(\Phi, \mu')$ holds. In fact, the following superficially stronger statement is true. For each $\delta > 0$, $\Phi$ is Lipschitz continuous on $I' \setminus (B(\delta)(x) \cup B(\delta)(y))$ and $\Phi^{-1}$ is Lipschitz continuous on $\Phi(I' \setminus (B(\delta)(x) \cup B(\delta)(y)))$ with Lipschitz constants depending only on $\mu(\delta)$, $\delta$, and $\Delta$.

Proof. The basic idea of the proof is similar to that of Lemma 2.4 but we consider harmonic measure from $m$ rather than harmonic measure from $\infty$.

Fix $\delta > 0$. Let $x_3$ and $y_3$ be the unique points of $I$ lying at distance $\delta$ from $x$ and $y$, respectively. Let $\hat{D}$ be the radial reflection of $D$ across $I$, viewed as a subset of the Riemann sphere. Extend $\Phi$ to $\hat{D}$ by Schwarz reflection. Then $\Phi$ maps $\hat{D}$ into $C \setminus [-i, i]$ and maps $I$ to $[-i, i]$. Suppose $\delta > 0$. Let $\hat{D}_\delta = D \setminus [y_3, i]_A$. Let $\hat{y}_3 := \Phi(y_3)$. Then $\hat{y}_3$ is determined by the condition that the harmonic measure of $[y_3, i]_A$ from $m$ in $\hat{D}_\delta$ equals the harmonic measure of the side of $[y_3, i]_A$ closer to 0 from 1 in $(C \cup \infty) \setminus [y_3, -i]_A$.

If $G'_I(\Phi, \mu)$ occurs, then $\hat{y}_3$ lies at distance at least $\mu(\delta)$ from $i$, which means that the harmonic measure of $[y_3, i]_A$ from 1 in $\hat{D}_\delta$ is at least some constant $\epsilon > 0$ depending only on $\mu(\delta)$. By symmetry, the same holds for $[x_3, y_3]_A$.

By the Beurling estimate, we can find some $\zeta_0 > 0$ depending only on $\epsilon$ such that $\text{dist}(m, A) \leq \zeta_0$. We can find $\zeta_1 > 0$ such that if $z \in [x_3, y_3]_A$ lies at distance at least $\zeta_0$ from $m$, then the probability that a Brownian motion started from $m$ hits $B(\zeta_1)(z)$ before hitting $[-i, i]_A$ is at most $\epsilon$. If $\text{dist}(z, A) < \zeta_1$, then a Brownian motion started from $m$ first hit $B(\zeta_1)(z)$ before hitting either $y_3, y_3$ or $x, x_3$. Hence we must have $\text{dist}(z, A) \geq \zeta_1 \land \zeta_0$ for each $z \in [x_3, y_3]_A$. This proves assertion 1 with $\mu'(\delta) = \zeta_1 \land \zeta_0$.

Conversely, suppose $I' \subset I$ and $G'_I(A, \mu)$ holds for some $\mu \in \mathcal{M}$. For $\delta > 0$ let $x'_3$ be either $x_3$ (as defined just above) or the endpoint of $I'$ closest to $x$, whichever is closest to $x$. Define $y'_3$ similarly. A Brownian motion started from any point of $[x'_3, y'_3]_A$ as a positive probability depending only on $\delta$, $\mu(\delta)$, and $\Delta$ to stay within distance $\mu(\delta)$ of $I$ until it hits $[y'_3, y'_3]_A$ (resp. $[x, x_3]_A$). By the Beurling estimate there is a $\mu_0(\delta)$ depending only on $\mu(\delta)$, $\delta$, and $\Delta$ such that $\Phi([x'_3, y'_3]_A)$ lies at distance at least $\mu_0(\delta)$ from $[-i, i]_A$.

It remains to establish the Lipschitz continuity statement. For this, we observe that for any $z \in [x'_3, y'_3]_A$, the Koebe quarter theorem implies

$$\frac{\text{dist}(\Phi(z), [-i, i]_A)}{4 \text{dist}(z, A) \land \delta} \leq |\Phi'(z)| \leq \frac{4 \text{dist}(\Phi(z), [-i, i]_A)}{\text{dist}(z, A) \land \delta}.$$ 

Hence

$$\frac{\mu_0(\delta)}{8} \leq |\Phi'(z)| \leq \frac{8}{\mu(\delta) \land \delta}.$$ 

So, $|\Phi'|$ is bounded above and below by positive constants on $[x'_3, y'_3]_A$ depending only on $\mu(\delta)$, $\delta$, and $\Delta$ which establishes the desired Lipschitz continuity. 

$\square$
2.3 Schramm-Loewner evolution

Let $t \mapsto W_t$ be a continuous function on $[0, \infty)$. The *chordal Loewner equation* is the ordinary differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z.$$  \hspace{1cm} (2.8)

A solution to (2.8) is a family of conformal maps $\{g_t : t \geq 0\}$ from subdomains of $H$ to $H$, satisfying the hydrodynamic normalization $\lim_{t \to \infty} (g_t(z) - z) = 0$. The complements $(K_t)$ of the domains of $(g_t)$ in $H$ are an increasing family of closed subsets of $H$ called the *hulls* of the process. The *centered Loewner maps* corresponding to $(g_t)$ are defined by

$$f_t := g_t - W_t.$$  

A chordal *Schramm-Loewner evolution* with parameter $\kappa > 0$ ($\text{SLE}_\kappa$) is the random evolution obtained by solving (2.8) where the driving process $W$ is $\sqrt{\kappa}$ times a Brownian motion. It can be shown [RS05] that this Loewner evolution is generated by a curve which we typically denote by $\eta$. Chordal $\text{SLE}_\kappa$ on other domains is defined by conformal mapping. We refer the reader to [Law05] or [Wer04] for a more detailed introduction to SLE.

More generally, suppose we are given a vector of real weights $\overline{\rho} = (\rho^1, \ldots, \rho^n)$ and a collection of points $z^1, \ldots, z^n \in H$. Chordal $\text{SLE}_{\overline{\rho}}(\rho)$ is the random evolution obtained by solving (2.8) with the driving function $W$ part of the solution to the system of SDE’s

$$dW_i = \sqrt{\kappa} dB_t + \sum_{i=1}^n \text{Re} \frac{\rho^i}{W_i - V^i_t} dt, \quad dV^i_t = \frac{2}{V^i_t - W_t} dt, \quad W_0 = y, \quad V^i_0 = z^i.$$  \hspace{1cm} (2.9)

The points $z^i$ are called the *force points*. See [LSW03], [SW05], [MS12a] for more on $\text{SLE}_{\overline{\rho}}(\rho)$.

We will also need to consider the *reverse* Loewner equation. This is the ODE

$$\partial_t g_t(z) = -\frac{2}{g_t(z) - W_t}, \quad g_0(z) = z,$$  \hspace{1cm} (2.10)

whose solution is a family of conformal maps from $H$ to sub-domains of $H$. Reverse $\text{SLE}_\kappa$ is obtained by taking $W_t$ to be $\sqrt{\kappa}$ times a Brownian motion. For each time $t$, the time $t$ centered Loewner map of a reverse $\text{SLE}_\kappa$ has the same law as the inverse of the time $t$ centered Loewner map of a forward $\text{SLE}_\kappa$ [RS05 Lemma 3.1].

We will also need to consider reverse $\text{SLE}_{\overline{\rho}}(\rho)$ with force points $z^1, \ldots, z^n$, which is obtained by solving (2.10) with the driving function $W$ part of the solution to the system of SDE’s

$$dW_i = \sqrt{\kappa} dB_t + \sum_{i=1}^n \text{Re} \frac{\rho^i}{W_i - V^i_t} dt, \quad dV^i_t = -\frac{2}{V^i_t - W_t} dt, \quad W_0 = y, \quad V^i_0 = z^i.$$  

For a general $\rho$ we do not have as simple a relation between forward and reverse $\text{SLE}_{\overline{\rho}}(\rho)$ as we do for ordinary $\text{SLE}_\kappa$. However, there are various forward and reverse symmetries, some of which are discussed in [DMS14], [She10].

Throughout most of the rest of this paper we will fix $\kappa \in (0, 4]$ and we will not always make dependence on $\kappa$ explicit.

2.4 Gaussian free fields

For some of our results, we will make use of couplings of $\text{SLE}_\kappa$ with Gaussian free fields. In this section we give some basic background about the latter object.

Let $D$ be a domain in $\mathbb{C}$ with harmonically non-trivial boundary (i.e. a Brownian motion started in $D$ a.s. exits $D$ in finite time). We denote by $H(D)$ the Hilbert space completion of the subspace of $C^\infty(D)$ consisting of those smooth, real-valued functions $f$ such that

$$\int_D |\nabla f(z)|^2 \, dz < \infty, \quad \int_D f(z) \, dz = 0.$$
with respect to the Dirichlet inner product

$$(f, g)\,\nabla = \frac{1}{2\pi} \int_D \nabla f(z) \cdot \nabla g(z) \, dz. \quad (2.11)$$

A free-boundary Gaussian free field (GFF) on $D$ is a random distribution (in the sense of Schwartz) on $D$ given by the formal sum

$$h = \sum_{j=1}^{\infty} X_j f_j \quad (2.12)$$

where $\{f_j\}$ is an orthonormal basis for $H(D)$ and $(X_j)$ is a sequence of i.i.d. standard Gaussian random variables. It is defined as a pointwise function, but for each $g \in H(D)$, the formal inner product

$$(h, g)\,\nabla = \sum_{j=1}^{\infty} (f_j, g)\,\nabla$$

converges almost surely. Moreover, $(h, g)$ is a.s. defined for each fixed $g \in L^2(D)$ by the formula

$$(h, g) = (h, -\Delta^{-1} g)\,\nabla \quad (2.13)$$

where $\Delta^{-1}$ denotes the inverse Laplacian with Neumann boundary conditions. More generally, this formula makes sense if $g$ is any distribution whose inverse Laplacian is in $H(D)$.

Similarly, one can define a zero-boundary GFF on $D$ by replacing $H(D)$ with $H_0(D)$, defined as the Hilbert space completion of the space of smooth compactly supported functions on $D$ in the inner product $(2.11)$. A zero boundary GFF is defined without the need to make a choice of additive constant. A Gaussian free field with a given choice of boundary data on $\partial D$ is defined to be a zero boundary GFF plus the harmonic extension of the given boundary data to $D$.

If $V, V^\perp \subset H(D)$ are complementary orthogonal subspaces, then the formula $(2.12)$ implies that $h$ decomposes as the sum of its projections onto $V$ and $V^\perp$. In particular, we can take $V$ to be the closure $H_0(D)$ of $C^\infty_c(D)$ in the inner product $(2.11)$ and $V^\perp$ the set Harm$_D$ of functions in $H(D)$ which are harmonic in $D$. This allows us to decompose a free boundary GFF as the sum of a zero boundary Gaussian free field and a random harmonic function $h$ on $D$, the latter defined modulo additive constant. We call these distributions the zero-boundary part and harmonic part of $h$, respectively.

We refer to [She07] and the introductory sections of [SS13] and [MS13b] for more details on GFF’s.

### 2.4.1 Reverse SLE/GFF coupling

The following relation between free boundary GFFs and reverse SLE$_\kappa(\rho)$ is established in [She10, Section 4.2]. Let $(g_t)$ be the centered Loewner maps of a reverse SLE$_\kappa(\rho)$ with force points $z^1, \ldots, z^n$ as in Section 2.3. Let $h$ be a free boundary GFF on $H$, independent from $(g_t)$. For $t \geq 0$ let

$$h_t = h \circ g_t + 2 \sqrt{\kappa} \log |g_t(\cdot)| + \frac{1}{2\sqrt{\kappa}} \sum_{i=1}^{n} \rho^i G(g_t(z^i), g_t(\cdot)),$$

where

$$G(x, y) := -\log |x - y| - \log |x - y|$$

is the Green’s function on $H$ with Neumann boundary conditions. Let

$$Q = \frac{2}{\sqrt{\kappa}} + \frac{\sqrt{\kappa}}{2}. \quad (2.14)$$

Let $\tau$ be a stopping time for $\eta$ which is a.s. less than the first time $t$ that $f_i(z^i) = 0$ for some $i$. Then [She10, Theorem 4.5] implies that $h_\tau + Q \log |g_\tau(\cdot)| \overset{\text{d}}{=} h_0$, modulo additive constant.

There is also an analogue of the above coupling for a zero boundary GFF paired with a forward SLE$_{\kappa}(\rho)$, which we discuss in Section 2.5.
2.4.2 Estimates for the harmonic part

In the course of proving our one-point estimate we will need some basic analytic lemmas about the harmonic part of a free boundary GFF which we will prove here.

**Lemma 2.9.** Let \( h \) be the harmonic part of a free boundary GFF on \( D \), normalized so that \( h(0) = 0 \). Then for any \( z, w \in D \), \( h(z) \) and \( h(w) \) are jointly Gaussian with means zero and covariance
\[
E(h(z)h(w)) = -2 \log |1 - z\overline{w}|.
\]

**Proof.** For \( n \geq 1 \), let
\[
\phi_n(z) = (2/n)^{1/2} \text{Re } z^n, \quad \psi_n(z) = (2/n)^{1/2} \text{Im } z^n.
\]
Then \( \{\phi_n, \psi_n : n \geq 1\} \) is an orthonormal basis for the set of harmonic functions on \( D \) in the Dirichlet inner product. So, by definition of the free boundary GFF, we can write
\[
\sum_{n=1}^{\infty} X_n \phi_n + \sum_{n=1}^{\infty} Y_n \psi_n,
\]
where the \( X_n \)'s and \( Y_n \)'s are i.i.d. \( N(0, 1) \). From this expression, it follows that \( (h(z), h(w)) \) is centered Gaussian for each \( z, w \in D \), and
\[
E(h(z)h(w)) = \sum_{n=1}^{\infty} \phi_n(z)\phi_n(w) + \sum_{n=1}^{\infty} \psi_n(z)\psi_n(w)
= 2 \sum_{n=1}^{\infty} (\text{Re } z^n)(\text{Re } w^n) + (\text{Im } z^n)(\text{Im } w^n)
= \sum_{n=1}^{\infty} (z\overline{w})^n + (w\overline{z})^n
= - \log(1 - z\overline{w}) - \log(1 - w\overline{z})
= -2 \log |1 - z\overline{w}|.
\]

\[\Box\]

**Lemma 2.10.** Let \( h \) be a free boundary GFF on \( H \) with additive constant chosen so that its harmonic part vanishes at \( a \) for some \( a \in H \). Let \( A \subset H \) be a deterministic hull lying at positive distance from \( a \) and let \( g : H \to H \setminus A \) be the inverse centered hydrodynamic map. Let \( \tilde{h} = h \circ g \) and let \( (\tilde{h}_ \epsilon) \) be the circle average process for \( \tilde{h} \) (see [DS11, Section 3.1] for more on the circle average process of a GFF). Fix \( x \in \mathbb{R} \) and \( \xi > 1/2 \). For any \( \delta \geq \epsilon > 0 \) we have
\[
P\left(|\tilde{h}_\epsilon(x + i\delta)| > (\log \epsilon^{-1})\xi\right) = o_\epsilon(1)
\]
with the $o_r(1)$ depending only on $x$, $a$, $\text{diam} A$, $\xi$, and $\delta$, but uniform for $x$ in compact subsets of $\mathbb{R}$, $a$ in compact subsets of $\mathcal{H}$, and $\delta$ in compact subsets of $(\epsilon, \infty)$.

Proof. Write $h = h^0 + h_1$, for $h^0$ a zero boundary GFF and $h_1$ an independent harmonic function. Let $h_A$ be projection of $h^0$ onto the set of functions which are harmonic on $\mathcal{H} \setminus A$ and let $h_0^0 = h^0|_A - h_A$ be the zero-boundary part of $h^0|_A$. Then we can write

$$h|_{\mathcal{H} \setminus A} = h^0 + h_A + h|_{\mathcal{H} \setminus A},$$

(2.18)

with the three summands independent. The function $g$ increases imaginary parts, so it follows from Lemma 2.9 and a coordinate change to $\mathcal{D}$ that $h(g(x + i\delta))$ is centered Gaussian with variance $\leq 2 \log \delta^{-1} + O_r(1)$.

By the Koebe distortion theorem, $|g'(x + i\delta)|$ is at least a constant depending only on $y$ times $\delta|g'(x + iy)|$ for any $y > \delta$. By [Law05] Proposition 3.46 and the Koebe quarter theorem, for large enough $y$ (depending only on $\text{diam} A$), $|g'(x + iy)|$ is bounded above by a constant depending only on $\text{diam} A$. By another application of the Koebe quarter theorem, we therefore have

$$\text{dist}(g(x + i\delta), A) \geq \delta^2.$$  

(2.19)

It follows from [MS12a] Lemma 6.4 that $h_A(g(x + i\delta))$ is centered Gaussian with variance at most $2 \log \delta^{-1} + O_r(1)$.

By conformal invariance, $h^0_A \circ g$ has the law of a zero boundary GFF on $\mathcal{H}$. By (2.19) and [DS11] Proposition 3.1, the circle average $(h^0_A \circ g)_\epsilon(x + i\delta)$ is Gaussian with mean 0 and variance at most $2 \log \epsilon^{-1} + O_r(1)$. By (2.18),

$$\tilde{h}_\epsilon(x + i\delta) = (h^0_A \circ g)_\epsilon(x + i\delta) + h_A(g(x + i\delta)) + h(g(x + i\delta))$$

is Gaussian with mean 0 and variance at most $6 \log \epsilon^{-1} + O_r(1)$. We obtain (2.17) from the Gaussian tail bound. \hfill \square

2.5 Imaginary geometry

The proof of the lower bounds in our main theorems will make heavy use of the so-called forward coupling of $\text{SLE}_\kappa$ or $\text{SLE}_{\kappa}(\rho)$ with the GFF with Dirichlet boundary conditions. In this coupling, $\text{SLE}_{\kappa}(\rho)$ for $\kappa \in (0, 4)$ can be interpreted as the flow line of the formal vector field $e^{ih/\chi}$ where $h$ is a GFF and

$$\chi = \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}.$$  

(2.20)

For $\kappa > 4$, $\text{SLE}_{\kappa}(\rho)$ can be interpreted as a “tree” or “light-cone” of $\text{SLE}_{16/\kappa}$ flow lines [MS12a]. The case $\kappa = 4$ is somewhat degenerate (though simpler to analyze) since $\chi \to 0$ as $\kappa \to 4$. $\text{SLE}_{4}(\rho)$ has the interpretation of being a level line (rather than a flow line or light cone) of the GFF. See [WW14] for a detailed study of this case.

The coupling of $\text{SLE}_4$ with the GFF was actually the first coupling in this family to be discovered [SS13] (see also [SS09] which gives the convergence of the contours of the discrete GFF to $\text{SLE}_4$). The existence of the forward coupling in the general setting is established in [Dub09b][SS13][She05][MS12a]; see [MS12a] Theorem 1.1] for a precise statement. The theory of how different flow lines and light cones of the same GFF interact is developed in [MS12a][MS12b][MS12c][MS13a]; these works are also where the term “imaginary geometry” is coined. At this point in time, there are several places which contain short “crash courses” on imaginary geometry which are sufficient to understand its usage in this work. We refer the reader to one of [MS12b] Section 2.2, [MS13a] Section 2.3, or [MWT14] Section 2.2; [MS12a] Section 1] and [MS13a] Section 4] contain many of the main theorem statements in addition to more detailed overviews of the related literature.

2.6 Properties of the multifractal spectrum sets

In this subsection we will prove some elementary properties of the sets of Section 1.1 as well as a lemma which is relevant to the integral means spectrum. Our first lemma tells us that the sets of Section 1.1 are only non-empty in the case $s \in [-1, 1]$. 

Lemma 2.11. Let $D \subset \mathbb{C}$ be a simply connected domain and let $\phi : D \to D$ be a conformal map. For each $x \in \partial D$, there is a constant $C$ depending only on $\phi$ and $\phi(x)$ but uniform for $\phi(x)$ in compact subsets of $D$ s.t. for each sufficiently small $\epsilon > 0$, 

$$C^{-1} \epsilon \leq |\phi'((1-\epsilon)x)| \leq C\epsilon^{-1}.$$ 

Proof. By the Cauchy estimate,

$$|\phi'((1-\epsilon)x)| \leq \sup_{z \in B_\epsilon((1-\epsilon)x)} |\phi(z)|$$

which gives the upper bound. For the lower bound, we apply the Koebe distortion theorem.

Next we prove some lemmas which give that the multifractal spectrum sets are invariant under reasonable modifications of the definitions.

Lemma 2.12. Let $D \subset \mathbb{C}$ be a simply connected domain, $\phi : D \to D$ a conformal map, and fix $x \in \partial D$. Let $\gamma : [0,1] \to D$ be a simple smooth curve such that $\gamma(0) = x$, $\gamma((0,1)) \subset D$, and $\gamma'(0)$ is not tangent to $\partial D$ at $x$. Then

$$\limsup_{\epsilon \to 0} \frac{\log |\phi'((1-\epsilon)x)|}{-\log \epsilon} = \limsup_{\epsilon \to 0} \frac{\log |\phi'(\gamma(\epsilon))|}{-\log \epsilon}. \quad (2.21)$$

If one of the limsups is in fact a true limit, then the other is as well.

Proof. By Taylor’s formula, we can write

$$\gamma(\epsilon) = x + \epsilon \gamma'(0) + O_\epsilon(\epsilon^2). \quad (2.22)$$

The function $w \mapsto \phi(\epsilon w + (1-\epsilon)x)/\phi'((1-\epsilon)x)\epsilon$ is a conformal map from $D$ into $\mathbb{C}$ with unit derivative at the origin. By the Koebe distortion theorem applied to this map evaluated at $w = \epsilon^{-1}(\gamma(\epsilon) - (1-\epsilon)x)$ it follows that

$$1 - |\gamma'(0)| + x + O_\epsilon(\epsilon) \leq \frac{|\phi'(\gamma(\epsilon))|}{|\phi'((1-\epsilon)x)|} \leq \frac{1}{(1 - |\gamma'(0)| + x + O_\epsilon(\epsilon))^3}. \quad (2.23)$$

Since $\gamma'(0)$ is not tangent to $\partial D$ at $x$, there is some $c > 0$ s.t. $|\gamma'(0)| + x < 1$. It follows that we can perform a linear re-parametrization of $\gamma$ in such a way that $|\gamma'(0)| + x < 1$. Then (2.23) implies the statement of the lemma.

Lemma 2.13. Let $D$ and $D'$ be two simply connected domains in $\mathbb{C}$, bounded by curves, which share a common boundary arc $I$. Let $z$ be a prime end lying in the interior of $I$. Then for each $s \in \mathbb{R}$, we have $z \in \Theta^s(D)$ iff $z \in \Theta^s(D')$. The same holds with $\Theta^{s>\cdot}$ or $\Theta^{s<\cdot}$ in place of $\Theta^s(\cdot)$.

Proof. By comparing $D$ and $D'$ to the connected component of $D \cap D'$ with $I$ on its boundary, it suffices to consider the case where $D' \subset D$. Let $\phi : D \to D$ and $\psi : D \to D'$ be the corresponding conformal maps. We can factor $\phi = \psi \circ \xi$, where $\xi = \psi^{-1} \circ \phi$. Then

$$\phi'((1-\epsilon)\phi^{-1}(z)) = \psi'((1-\epsilon)\phi^{-1}(z))\xi'((1-\epsilon)\phi^{-1}(z)) \quad (2.24)$$

By Schwarz reflection, $\xi$ extends to be analytic in a neighborhood of $\phi^{-1}(z)$, so $|\xi'((1-\epsilon)\phi^{-1}(z))|$ is bounded above and below by positive constants for small $\epsilon$. Let $\gamma(\epsilon) = \xi((1-\epsilon)\phi^{-1}(z))$. Note that $\gamma$ is a simple curve in $D$ with $\gamma(0) = \psi^{-1}(z)$. We have

$$\gamma'(0) = -\xi'((\phi^{-1}(z))\phi^{-1}(z).$$

Since $\xi$ maps a neighborhood of $\phi^{-1}(z)$ in $\partial D$ into $\partial D$, we have that $\xi'((\phi^{-1}(z))$ is a real multiple of $\xi((\phi^{-1}(z)) = \psi^{-1}(z))$. Hence $\gamma'(0)$ is a real multiple of $\psi^{-1}(z)$. In particular $\gamma$ is not tangent to $\partial D$ at $\psi^{-1}(z)$ so the stated result follows from Lemma 2.12.
Lemma 2.14. Let $D \subset \mathbb{C}$ be a simply connected domain. Let $\phi : D \to D$ and $\psi : H \to D$ be conformal maps. For each prime end $z \in \partial D$ with $\psi^{-1}(z) \neq \infty$, one has

$$\limsup_{\epsilon \to 0} \frac{\log |\phi'((1-\epsilon)\phi^{-1}(z))|}{-\log \epsilon} = \limsup_{\epsilon \to 0} \frac{\log |\psi'((\psi^{-1}(z) + i\epsilon))|}{-\log \epsilon}.$$  \hspace{1cm} (2.25)

If one of the limsup is in fact a true limit, then the other is as well.

Proof. We can write $\psi = \phi \circ \xi$ where $\xi = \phi^{-1} \circ \psi : H \to D$ is a conformal map. Then

$$\psi'(\psi^{-1}(z) + i\epsilon) = \phi'(\xi(\psi^{-1}(z) + i\epsilon))\xi'((\psi^{-1}(z) + i\epsilon)).$$

The map $\xi'$ extends smoothly to $\partial H$, so $|\xi'(\psi^{-1}(z) + i\epsilon)|$ is bounded above and below by positive constants for small $\epsilon$. Let $\gamma(\epsilon) = \xi(\psi^{-1}(z) + i\epsilon)$. Then $\gamma'(0) = i(\phi^{-1})'(\psi^{-1}(z))/(\psi^{-1})'(z)$, which is not tangent to $\partial D$ at $\phi^{-1}(z)$. Therefore the desired result follows from Lemma 2.12.

Lemma 2.14 implies in particular that if $\psi : H \to D$ is a conformal map, then $\dim_H \Theta^*(D)$ and $\dim_H \tilde{\Theta}^*(D)$ are unaffected if we replace their definitions from Section 1.1 with

$$\tilde{\Theta}^*(D) = \left\{ x \in \mathbb{R} : \lim_{\epsilon \to 0} \frac{\log |\psi'(z + i\epsilon)|}{-\log \epsilon} = s \right\} \quad \text{and} \quad \Theta^*(D) = \psi(\tilde{\Theta}^*(D)).$$  \hspace{1cm} (2.26)

The analogous statement also holds for the sets $\Theta^{s\geq}(D), \tilde{\Theta}^{s\geq}(D), \Theta^{s\leq}(D),$ and $\tilde{\Theta}^{s\leq}(D)$.

What follows is the analogue of Lemma 2.13 for the integral means spectrum.

Lemma 2.15. Let $D$ and $D'$ be two bounded Jordan domains in $\mathbb{C}$ and suppose there exists a connected boundary arc $I$ shared by $D$ and $D'$. Let $\phi : D \to D$ and $\psi : D \to D'$ be conformal maps. Let $J'$ be a closed subset of the interior of $I$ and let $J$ be a closed subset of the interior of $J'$. For $\epsilon > 0$, let $A_{\epsilon}$ be the set of $z \in \partial B_{1-\epsilon}(0)$ with $z/|z| \in \phi^{-1}(J)$ and let $A'_{\epsilon}$ be the set of $z \in \partial B_{1-\epsilon}(0)$ with $z/|z| \in \psi^{-1}(J')$. Then we have

$$\limsup_{\epsilon \to 0} \frac{\log \int_{A_{\epsilon}} |\phi'(z)|^a \, dz}{-\log \epsilon} \leq \limsup_{\epsilon \to 0} \frac{\log \int_{A'_{\epsilon}} |\psi'(z)|^a \, dz}{-\log \epsilon}.$$  \hspace{1cm} (2.27)

Proof. Let $\xi$ be the conformal map from a subdomain of $D$ to a subdomain of $D' \cap D$ which equals $\psi^{-1} \circ \phi$ wherever the latter is defined. By Schwarz reflection $\xi$ extends to a conformal map from a neighborhood of $\phi^{-1}(J')$ to a neighborhood of $\psi^{-1}(J')$. In particular $|\xi'| \approx 1$ on a neighborhood of $\phi^{-1}(J')$, with implicit constants independent of $\epsilon$. By a change of variables, for sufficiently small $\epsilon > 0$ we have

$$\int_{A_{\epsilon}} |\phi'(z)|^a \, dz \asymp \int_{A_{\epsilon}} |\psi'(\xi(z))|^a \, dz \asymp \int_{\xi(A_{\epsilon})} |\psi'(w)| \, dw.$$  \hspace{1cm} (2.28)

Let $p_{\epsilon}$ be the radial projection from $D$ onto $\partial B_{1-\epsilon}(0)$. By the above application of Schwarz reflection (and the fact that $J$ is contained in the interior of $J'$), for sufficiently small $\epsilon > 0$, we have that $p_{\epsilon}$ restricts to a diffeomorphism from $\xi(A_{\epsilon})$ to a subset $\tilde{A}'_{\epsilon}$ of $A'_{\epsilon}$. Furthermore, since $|\xi'| \approx 1$ on a neighborhood of $\psi^{-1}(J')$, we have $|p_{\epsilon}'| \approx 1$ on $\xi(A_{\epsilon})$ for sufficiently small $\epsilon$, and by the Koebe distortion theorem we have $|\psi'(p_{\epsilon}(w))| \asymp |\psi'(w)|$ for $w \in \xi(A_{\epsilon})$ and sufficiently small $\epsilon$. Therefore, a second change of variables yields

$$\int_{\xi(A_{\epsilon})} |\psi'(w)| \, dw \asymp \int_{A_{\epsilon}} |\xi'(z)| \, dz \leq \int_{A'_{\epsilon}} |\psi'(z)| \, dz.$$  \hspace{1cm} (2.29)

We obtain (2.27) by combining (2.28) and (2.29). \hfill \Box

### 2.7 Zero-one laws

In this section we will prove that the multifractal spectrum and integral means spectrum of an SLE$_\kappa(\rho)$ curve are a.s. deterministic and do not depend on $\rho$ or on which complementary component of the curve we consider.
Proposition 2.16. Let $D \subset \mathbb{C}$ be a smoothly bounded domain. Let $\kappa > 0$ and let $\rho$ be a vector of real weights. Let $\eta$ be a chordal SLE$_{\kappa}(\rho)$ process in $D$, with any choice of initial and target points and force points located anywhere in $\overline{D}$, run up until the first time it either hits an interior force point or hits the continuation threshold after which it is no longer defined (c.f. [MS12a, Section 2.1]). Fix $s \in (-1,1)$. Almost surely, the following is true. Let $V$ be a connected component of $D \setminus \eta$ or a connected component of $D \setminus \eta([0,1])$ for any $t > 0$ and let $\phi : D \to V$ be a conformal map. The Hausdorff dimension of each of the sets $\Theta^s(V) \setminus \partial D, \tilde{\Theta}^s(V) \setminus \phi^{-1}(\partial D), \Theta^{\kappa \leq s}(V) \setminus \partial D, \tilde{\Theta}^{\kappa \leq s}(V) \setminus \partial D, \Theta^{\nu \leq s}(V) \setminus \partial D, \text{and } \tilde{\Theta}^{\nu \leq s}(V) \setminus \phi^{-1}(\partial D)$ is equal to a deterministic constant which depends only on $\kappa$ and $s$. Furthermore, the a.s. Hausdorff dimensions of the corresponding sets for $\kappa$ and $16/\kappa$ are equal.

Proof. We begin with some reductions. By Lemma 2.14 we can consider SLE$_{\kappa}$ between two arbitrary points in $D$, and we can use the alternative definition (2.26) for the multifractal spectrum maps (with $(f_t)$ the centered Loewner maps for $\eta$ and $\psi = f_t^{-1}$).

Next, we note that $D \setminus \eta^t$ a.s. has only countably many connected components for any given $t > 0$. By continuity, countable stability of Hausdorff dimension, and Lemma 2.13 it therefore suffices to prove that the statement of the proposition holds a.s. for some fixed but arbitrary choice of domain $V$ as in the statement of the proposition (chosen in some deterministic manner), rather than for all such $V$ simultaneously (it will be clear from the proof that the a.s. dimension obtained does not depend on how we choose $V$).

We will prove the result for $\Theta^s(V) \setminus R$ and $\tilde{\Theta}^s(V) \setminus f_t^{-1}(R)$; the case for the other sets is similar.

First consider the case where $\kappa \leq 4$ and $\rho = 0$, so $\eta$ is an ordinary SLE$_{\kappa}$ process. In this case, the statement of the proposition for a complementary connected component $V$ of $\mathbb{H} \setminus \eta$ follows from the statement for $V = \mathbb{H} \setminus \eta^t$ by Lemma 2.13 and countable stability of Hausdorff dimension, so it suffices to prove the statement with $V = \mathbb{H} \setminus \eta^t$ for a general choice of $t > 0$.

By scale invariance the law of each $\Theta^s(\mathbb{H} \setminus \eta^t)$ is independent of $t$. Since the derivative of the conformal map $f_{t/2}$ is bounded above and below by positive (random) constants in a neighborhood of each point of $\eta^t \setminus \eta^{t/2}$, we infer that $\Theta^s(\mathbb{H} \setminus \eta^t) | \eta^{t/2} \approx \Theta^s(\mathbb{H} \setminus f_{t/2}(\eta^t \setminus \eta^{t/2}))$.

Since conformal maps preserve Hausdorff dimension and by Lemma 2.13 we thus have that the Hausdorff dimension of each $\Theta^s(D \setminus \eta^t)$ is equal to the maximum of $\dim_{\mathbb{H}} \Theta^s(\mathbb{H} \setminus \eta^{t/2})$ and $\dim_{\mathbb{H}} \Theta^s(\mathbb{H} \setminus f_{t/2}(\eta^t \setminus \eta^{t/2}))$. These latter two sets are independent and identically distributed (by the Markov property of SLE) and their Hausdorff dimensions agree in law with that of $\Theta^s(\mathbb{H} \setminus \eta^t)$ (by the scale invariance property noted above). A random variable can be equal to the maximum of two independent random variables with the same law as itself only if it is a.s. constant.

To prove the analogous statement for $\tilde{\Theta}^s(\mathbb{H} \setminus \eta^t)$, we observe that $\dim_{\mathbb{H}} \tilde{\Theta}^s(\mathbb{H} \setminus \eta^t)$ is the maximum of $\dim_{\mathbb{H}} f_{t/2}^{-1}(\tilde{\Theta}^s(\mathbb{H} \setminus \eta^t) \cap \eta^{t/2})$ and $\dim_{\mathbb{H}} f_{t/2}^{-1}(\tilde{\Theta}^s(\mathbb{H} \setminus \eta^{t/2}) \cap \eta^{t/2})$. By smoothness of the map $f_{t/2} \circ f_{t/2}^{-1}$ on $f_{t/2}(\mathbb{H} \setminus \eta^{t/2})$ and of $f_{t/2}^{-1}$ on $\eta^t \setminus \eta^{t/2}$, respectively, these dimensions equal $\dim_{\mathbb{H}} f_{t/2}^{-1}(\tilde{\Theta}^s(\mathbb{H} \setminus \eta^{t/2}))$ and $\dim_{\mathbb{H}}(f_{t/2} \circ f_{t/2}^{-1})(\tilde{\Theta}^s(\mathbb{H} \setminus f_{t/2}(\eta^t \setminus \eta^{t/2})))$, respectively. By the Markov property these latter two quantities are i.i.d., and we conclude as above.

Next suppose that $\kappa$ is still $\leq 4$, but that $\rho$ is arbitrary. For $\delta > 0$, inductively define stopping times $\tau_j^\delta$ and $\sigma_j^\delta$ for $j \in \mathbb{N}$ as follow. Let $\tau_1^\delta$ be the first time $t > 0$ that either $T$ hits an interior force point or the continuation threshold; or $\Im \eta(t) \geq 2\delta$. Let $\sigma_1^\delta$ be the first time $s > \tau_1^\delta$ that $\eta$ hits an interior force point or the continuation threshold; or $\Im \eta(s) \leq \delta$. Inductively, if $j \geq 2$ and $(\tau_{j-1}^\delta, \sigma_{j-1}^\delta)$ has been defined, let $\tau_j^\delta$ be the first time $t > \tau_{j-1}^\delta$ that $\eta$ either hits an interior force point or the continuation threshold; or $\Im \eta(t) \geq 2\delta$ and let $\sigma_j^\delta$ be the first time $s > \tau_j^\delta$ that either $\eta$ hits the continuation threshold or $\Im \eta(s) \leq \delta$. For each $\delta$ and each $j$, the law of $f_{\tau_j^\delta}(\eta_t)_{\tau_j^\delta, \sigma_j^\delta}$ is absolutely continuous with respect to the law of an ordinary SLE$_{\kappa}$ stopped at some a.s. positive time. Therefore, Lemma 2.13 implies that if $V$ is as in the statement of the lemma and $\psi : \mathbb{H} \to V$ is a conformal map, then $\dim_{\mathbb{H}} \tilde{\Theta}^s(V) \cap \eta([\tau_j^\delta, \sigma_j^\delta])$ and $\dim_{\mathbb{H}} \Theta^s(V) \cap \psi^{-1}(\eta([\tau_j^\delta, \sigma_j^\delta]))$, respectively, a.s. agree with the a.s. values of the corresponding sets for an ordinary SLE$_{\kappa}$. We have

$$\bigcup_{\delta > 0} \bigcup_{j \in \mathbb{N}} \eta([\tau_j^\delta, \sigma_j^\delta]) = \eta \setminus \mathbb{R}$$

so by countable stability of Hausdorff dimension (restrict $\delta$ to a sequence tending to 0) and we obtain the statement of the proposition for a general SLE$_{\kappa}(\rho)$ with $\kappa \leq 4$. 

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The statement for $\kappa > 4$ follows from the statement for $16/\kappa < 4$ together with Lemma 2.13 and SLE duality (see, e.g. [Zha08a, Zha10, Dub09a, MS12a, MS13a]).

For the proof of Corollary 1.8 we will also need the analogue of Proposition 2.16 for the integral means spectrum.

**Proposition 2.17.** Suppose we are in the setting of Proposition 2.16. Fix $a \in \mathbb{R}$. Almost surely, the following is true. Let $V$ be a complementary connected component of either $D \setminus \eta$ or of $D \setminus \eta^j$ for any $t > 0$. Then $\text{IMS}^\text{bulk}_{D \setminus \eta^j}(a)$ is equal to a deterministic constant which depends only on $\kappa$ and $a$. This deterministic constant is the same if we replace $\kappa$ with $16/\kappa$.

**Proof.** The proof proceeds in a similar manner to that of Proposition 2.16. By continuity of $\eta$ together with Lemma 2.15 it is enough to prove that the statement of the proposition holds a.s. for some fixed but arbitrary choice of domain $V$ in the statement of the proposition (chosen in some deterministic manner), rather than for all such $V$ simultaneously (it will be clear from the proof that the a.s. dimension obtained does not depend on how we choose $V$).

Henceforth we fix a conformal map $\psi : D \to \mathbb{H}$ taking the initial and target points for $\eta$ to 0 and $\infty$, respectively, and assume that $\eta$ is parametrized so that the half-plane capacity of $\psi(\eta^j)$ is $t$ for each $t \geq 0$. We also let $(\bar{f}_t)$ be the centered Loewner maps for $\psi(\eta)$ and set $f_t := \psi^{-1} \circ \bar{f}_t \circ \psi$. Define the sets $F^\zeta(\cdot)$ and $A^\zeta(\cdot)$ as in Section 1.3.

First consider the case where $\kappa \leq 4$ and $\rho = 0$, so that $\eta$ is an ordinary SLE$_\kappa$. Fix $t > 0$. By scale invariance (applied to $\psi(\eta)$) the law of $\text{IMS}^\text{bulk}_{D \setminus \eta^j}(a)$ does not depend on $t$. Let $\bar{\eta} := f_{t/2}(\eta|_{t/2, \infty})$ and $\bar{f}_{t/2} := f_t \circ f_{t/2}^{-1}$, so that $\bar{\eta}$ is independent from $\eta^{t/2}$, $\bar{\eta} \equiv \eta$, and $\bar{f}_{t/2} \circ \bar{f}_{t/2} = f_{t/2}$. Since the derivative of the conformal map $f_{t/2}$ is bounded above and below by positive constants on each compact subset of $\eta^t \setminus \eta^{t/2}$, we obtain that a.s.

$$\text{IMS}^\text{bulk}_{D \setminus \eta^j}(a) := \sup \limsup_{\zeta \to 0} \sup_{\epsilon \to 0} \frac{\log \int_{\bar{A}^\zeta(\bar{f}_t^{-1})} |\phi'(z)|^a \, dz}{-\log \epsilon},$$

where $\bar{A}^\zeta(\bar{f}_t^{-1})$ is the set of $z \in \partial B_1(0)$ such that $f_t^{-1}(z/|z|) \in \eta^j \setminus \eta^{t/2}$, $|f_t^{-1}(z/|z|) - \eta(t)| \geq \zeta$, and $|f_t^{-1}(z/|z|) - \eta(t/2)| \geq \zeta$. Note that $\bar{A}^\zeta(\bar{f}_t^{-1}) \subset A^\zeta(\bar{f}_t^{-1})$. Therefore, a.s.

$$\text{IMS}^\text{bulk}_{D \setminus \eta^j}(a) \geq \max \left\{ \text{IMS}^\text{bulk}_{D \setminus \eta^j}(a), \text{IMS}^\text{bulk}_{D \setminus \eta^j}(a) \right\}. $$

All three random variables in this inequality have the same law (by the considerations above), and the two on the right are independent. We infer that all three random variables are a.s. equal to a deterministic constant depending only on $a$ and $\kappa$.

It remains to treat the two complementary connected components of the whole curve. By symmetry, the a.s. value of $\text{IMS}^\text{bulk}_{D \setminus \eta^j}(a)$ is unaffected if, in the definition of the bulk integral means spectrum, we consider only the pre-images of prime ends on the right (or left) side of $\eta^j$. By transience of SLE, for any given $\zeta > 0$, we can a.s. find $t > 0$ such that $\eta$ and $\eta^j$ agree outside $B_{\zeta}(\partial D)$. It therefore follows from Lemma 2.15 that if $V$ is one of the two connected components of $D \setminus \eta$, then $\text{IMS}^\text{bulk}_{D \setminus \eta}(a)$ is a.s. equal to the same a.s. value as $\text{IMS}^\text{bulk}_{D \setminus \eta^j}(a)$ for a finite time $t$. This completes the proof of the proposition in the case $\kappa \leq 4$ and $\rho = 0$.

Next suppose that $\kappa$ is still $\leq 4$, but that $\rho$ is arbitrary. Note that in this case $\eta$ does not intersect itself, so the definition of the sets $F^\zeta(\cdot)$ in Section 1.3 is unaffected if we ignore the points $x_V$ and $y_V$. For $\delta > 0$ and $j \in \mathbb{N}$, define the stopping times $\tau^\delta_j$ and $\sigma^\delta_j$ as in the proof of Proposition 2.16 but with $\text{Im} \eta(t)$ replaced by $\text{dist}(\eta(t), \partial D)$. For each $\zeta > 0$, only finitely many of the excursions $\eta(\tau^\zeta_\delta, \sigma^\zeta_\delta)$ intersect $D \setminus B_{\zeta}(\partial D)$. By applying Lemma 2.15 to each of these excursions (with the arc $I$ equal to the excursion and the sets $J$ and $J'$ equal to its intersections with $D \setminus B_{\zeta}(\partial D)$ and $D \setminus B_{\zeta}(\partial D)$ for appropriate $\zeta$ and $\zeta'$) and using absolute continuity as in the proof of Proposition 2.16 we obtain the statement of the proposition for $\kappa \leq 4$ and a general choice of $\rho$ in the case where $V$ is a complementary connected component of $\eta^j$ for some finite $t > 0$.

For a complementary connected component of the whole curve $\eta$, we apply the same argument as in the case $\rho = 0$ (note that transience of SLE$_\kappa(\rho)$ for general $\rho$ processes which do not hit the continuation threshold is proven in [MS12a, Theorem 1.3]).
Finally, the case $\kappa > 4$ follows from SLE duality as in the proof of Proposition 2.16 but with Lemma 2.15 used in place of Lemma 2.13.

\[\square\]

3 One point estimates for the inverse maps

In this section we will prove estimates for the inverse centered Loewner maps of a chordal SLE$_\kappa$ process (see Theorem 3.1 for the statement of the main result).

3.1 Statement of the estimates

Let $\kappa \in (0, 4]$. Let $\eta$ be a chordal SLE$_\kappa$ process from $0$ to $\infty$ in $\mathbb{H}$. Let $(f_t)$ be its centered Loewner maps. For $z \in \mathbb{H}$ with $\text{Im} \, z = \epsilon$, $u > 0$, $s \in (-1, 1)$, $c > 0$, and $r > 0$, let $E^{s,u}(z; t) = E^{s,u}(z; t, c, r)$ be the event that the following is true.

1. $e^{-1}\epsilon^{-s+u} \leq |(f_t^{-1})'(z)| \leq c \epsilon^{-s-u}$.
2. $\text{Im} \, f_t^{-1}(z) \geq r$.

**Theorem 3.1.** Let $\kappa \in (0, 4]$. Let $(f_t)$ be the centered Loewner maps of a chordal SLE$_\kappa$ process from $0$ to $\infty$ in $\mathbb{H}$. Let $z \in \mathbb{H}$ with $\text{Im} \, z = \epsilon$ and $R^{-1} \leq |\text{Re} \, z| \leq R$ for some $R > 1$. Define the event $E^{s,u}(z; t) = E^{s,u}(z; t, c, r)$ as above. Let $G(f_t, \mu)$ be the event of Definition 2.3. Let

\[\alpha(s) = \frac{(4 + \kappa)^2 s^2}{8\kappa(1 + s)}, \quad \alpha_0(s) = \frac{(4 + \kappa)^2 s(2 + s)}{8\kappa(1 + s)^2}.\] (3.1)

For each $t, c, r > 0$, each $\mu \in \mathcal{M}$, each $s \in (-1, 1)$, and each $R > 1$, we have

\[\mathbb{P}(E^{s,u}(z; t) \cap G(f_t, \mu)) \leq \epsilon^{\alpha(s) - \alpha_0(s)u}.\] (3.2)

Furthermore, for each $r > 0$, there exists $t_* > 0$, s.t. for each $t \geq t_*$, we can find $\mu \in \mathcal{M}$ s.t. for each $c > 0$, each $s \in (-1, 1)$, and each $u > 0$,

\[\mathbb{P}(E^{s,u}(z; t) \cap G(f_t, \mu)) \geq \epsilon^{\alpha(s) + \alpha_0(s)u}.\] (3.3)

In both (3.2) and (3.3), the implicit constants in $\leq$ and $\geq$ depend on the other parameters but not on $\epsilon$, and are uniform for $R^{-1} \leq |\text{Re} \, z| \leq R$.

**Remark 3.2.** There are two reasons why we include the condition $G(f_t, \mu)$ in the estimates of Theorem 3.1. The first is that it implies an upper bound on the diameter of $\eta^t$ (see Lemma 2.4), which is needed to estimate some the auxiliary terms which arise in our proof. The second is that, in the sequel, we will often normalize our conformal maps by specifying the images of three marked points on the boundary. The condition $G(f_t, \mu)$ is needed so that the derivative of a conformal automorphism which takes the images of these marked points to where we want them to be is not too large or too small.

**Remark 3.3.** Estimates similar to Theorem 3.1 can be deduced in a somewhat more efficient manner from the results in [RS05, Section 3]. In particular, [RS05, Lemma 3.3] implies the upper bound (3.2) for a restricted range of parameter values and an estimate similar to (3.3) can be deduced from [RS05, Corollary 3.5]. However, these results give optimal bounds only for a certain subset of $(\kappa, s)$ pairs, and do not include the additional regularity conditions on the events in the lower bound which we include here.

3.2 Reverse SLE martingales and upper bounds

Let $(g_t)$ be the centered Loewner maps of a reverse SLE$_\kappa$ flow, so

\[dg_t(z) = -\frac{2}{g_t(z)} \, dt - dW_t, \quad g_0(z) = z\] (3.4)
for $W_t = \sqrt{\kappa} B_t$ and $(B_t)$ a standard linear Brownian motion. Our interest in $(g_t)$ stems from the fact that if $(f_t)$ is as in Theorem 3.1 then $g_t \overset{d}{=} f_t^{-1}$ for each $t$ (see, e.g. [RS05, Lemma 3.1]).

Let $K_t = H \setminus g_t(H)$ be the hulls corresponding to $(g_t)$. Since $f_t^{-1} \overset{d}{=} g_t$ for each $t$, it is only a minor abuse of notation to replace $f_t^{-1}$ with $g_t$ in the definition of the events of Theorem 3.1 and we do so in the remainder of this section.

### 3.2.1 Reverse SLE martingales

We state here a result originally due to Lawler [Law09, Proposition 2.1], but in a form which is more convenient for our purposes.

**Lemma 3.4.** Let $\kappa > 0$. Let $(g_t)$ be as above, $\rho \in \mathbb{R}$, $z \in H$, and

$$M^z_t = |g'_t(z)| \left(\frac{(8+2\kappa-\rho)\kappa}{8\kappa}\right)^{\kappa} (\text{Im } g_t(z))^{-\frac{\rho^2}{8\kappa}} |g_t(z)|^{\rho/\kappa}. \quad (3.5)$$

Then $M^z_t$ is a martingale. Let $P^z_t$ be the law of $(g_t)$ weighted by $M^z_t$. The law of $(g_t)$ under $P^z_t$ is that of the centered Loewner maps of a reverse SLE$_\kappa(\rho)$ with a force point at $z$. That is, under the reweighted law,

$$dW_t = -\text{Re} \frac{\rho}{g_t(z)} dt + \sqrt{\kappa} dB_t^z \quad (3.6)$$

for $B_t^z$ a $P^z_t$-Brownian motion.

**Remark 3.5.** The martingale (3.5) is the reverse SLE analogue of the local martingale of [SW05, Section 5] in the case of a single force point.

### 3.2.2 Proof of the upper bound

In this subsection we will prove (3.2) of Theorem 3.1. We will actually prove something a little stronger, namely the following.

**Proposition 3.6.** Let $\kappa > 0$. Let $\alpha(s)$ be as in (3.1) and let $(g_t)$ be the centered Loewner maps of a reverse SLE$_\kappa$ as above. Let $c,t,d > 0$. For $s \in [0,1]$, let $E^{s;\infty}(z;t,c,d)$ be the event that $|g'_t(z)| \geq c^{-1} e^{-s} \text{ and } |g_t(z)| \geq d^{-1}$. For $s \in (-1,0)$, let $E^{s;\infty}(z;t,c,d)$ be the event that $|g'_t(z)| \leq c e^{-s} \text{ and } |g_t(z)| \leq d$. For any bounded stopping time $\tau$ for $(g_t)$ we have

$$P(E^{s;\infty}(z;\tau)) \leq e^{\alpha(s)}. \quad (3.7)$$

For any $R > 1$, the implicit constant in (3.7) is uniform for $z \in H$ with $R^{-1} \leq |\text{Re } z| \leq R$.

The estimate (3.2) is immediate from Proposition 3.6 in the case $s \in [0,1]$. To extract (3.2) from Proposition 3.6 in the case $s \in (-1,0)$, we observe that Lemma 2.4 implies that diam $K_t$ is bounded by a constant depending only on $t$ and $\mu$ on the event $G(g_t^{-1}, \mu)$ (c.f. the discussion following Definition 2.3). For $R^{-1} \leq |\text{Re } z| \leq R$, [Law05, eqn. 3.14] then implies that $|g_t(z)|$ is bounded by a constant depending only on $t, \mu$, and $R$ on $E^{s;\infty}(z;t) \cap G(g_t^{-1}, \mu)$. Thus we have $E^{s;\infty}(z;t) \cap G(g_t^{-1}, \mu) \subset E^{s;\infty}(z;t,c,d)$ for a suitable choice of $d$.

**Proof of Proposition 3.6.** Throughout, we fix $R > 1$ and require all implicit constants to be uniform for $z \in H$ with $R^{-1} \leq |\text{Re } z| \leq R$. Let

$$\rho = \rho(s) := \frac{(4 + \kappa)s}{1 + s}. \quad (3.8)$$

and denote by $P^z_t$ the law of $(g_t)$ re-weighted by the martingale of Lemma 3.4 with this choice of $\rho$. By the Loewner equation, $\text{Im } g_t(z)$ is bounded above by a constant depending only on the essential supremum of $\tau$. Therefore,

$$M^z_t 1_{E^{s;\infty}(z;\tau)} \geq e^{-\rho/8\kappa} M^z_t 1_{E^{s;\infty}(z;\tau)} \quad (3.9)$$
(we can replace the $\succeq$ with an $\asymp$ if we assume that $\text{Im} g_t(z)$ is bounded below and $|g_t(z)|$ is bounded above). Furthermore,

$$M_0^* \asymp e^{-\frac{\rho^2}{8\kappa}}. \tag{3.10}$$

Thus the optional stopping theorem implies

$$e^{-s(8+2\kappa-\rho)\rho} \mathbb{P}(E_{s;\infty}(z;\tau)) \asymp \mathbb{E}(M_0^* 1_{E_{s;\infty}(z;\tau)}) \leq e^{-\frac{\rho^2}{8\kappa}} \mathbb{P}^*_s(E_{s;\infty}(z;\tau)).$$

Therefore

$$\mathbb{P}(E_{s;\infty}(z;\tau)) \leq e^{s(8+2\kappa-\rho)\rho} \frac{\rho^2}{8\kappa} \mathbb{P}^*_s(E_{s;\infty}(z;\tau)). \tag{3.11}$$

The value of the exponent on the right is maximized by taking $\rho = \rho(s)$, as in (3.8). Choosing this value of $\rho$ yields the upper bound (3.7).

3.3 Reduction of the lower bound to a result for a stopping time

Now we turn our attention to the lower bound \(3.3\) in Theorem \(3.1\). We continue to assume that we have replaced $f_t^{-1}$ with $g_t$ in the definition of the events of Theorem \(3.1\) as in Section \(3.2\).

Let $T$ be the first time $t$ that $\text{Im} g_t(z) \geq r$ and fix a time $\bar{t} > 0$. Put

$$\tau = T \wedge \bar{t}. \tag{3.12}$$

We claim that to prove that (3.3) holds at time $\bar{t}$, and hence to finish the proof of Theorem \(3.1\) it is enough to prove the following statement.

**Proposition 3.7.** Let $\rho = \rho(s)$ be as in (3.8). Let $\mathbb{P}_s^*$ be the law of a reverse SLE$_\kappa(\rho)$ process $(g_t)$ with hulls $(K_t)$, with an interior force point located at $z \in H$ with $\text{Im} z = \epsilon$. Let $\tau$ be as in (3.12). Define the events $E_{s;u}(z;\tau)$ as in Section \(3.1\) but with $(g_t)$ in place of $(f_t)$ and the time $\tau$ hull $K_\tau$ for $(g_t)$ in place of $\eta^\tau$. There exists $r_\tau > 0$ such that for each $r \geq r_\tau$, we can find $t_* > 0$ such that for each $t \geq t_*$, there exists $\mu \in \mathcal{M}$ such that for any $u > 0$ and any $\bar{t} \geq t_*$, it holds for each $z \in H$ with $\text{Im} z = \epsilon$ and $R^{-1} \leq |\text{Re} z| \leq R$ that

$$\mathbb{P}_s^*(E_{s;u}(z;\tau) \cap G(g_\tau^{-1}, \mu)) \geq 1. \tag{3.13}$$

Here the implicit constant and all of the above parameters are independent of $\epsilon$ and uniform for $z$ with $R^{-1} \leq |\text{Re} z| \leq R$.

We will prove Proposition 3.7 in the subsequent subsections. In the remainder of this subsection we deduce Theorem 3.1 from Proposition 3.7.

First we note that the requirement that $r \geq r_\tau$ in Lemma 3.7 is no obstacle; indeed, it is clear that the probability of the event of Theorem 3.1 is decreasing in $r$. Observe that $|g_t(z)|$ is a.s. bounded above by a positive constant on the event $E_{s;u}(z;\tau) \cap G(g_\tau^{-1}, \mu)$ (c.f. Section 3.2). By combining this with the definition of $E_{s;u}(z;\tau)$ we see that

$$M_1^* 1_{E_{s;u}(z;\tau) \cap G(g_\tau^{-1}, \mu)} \leq e^{-\frac{s(8+2\kappa-\rho)\rho}{8\kappa}} 1_{E_{s;u}(z;\tau) \cap G(g_\tau^{-1}, \mu)}.$$

By (3.10) and our choice (3.8) of $\rho$ we then have

$$e^{o(s)} P^*_s(E_{s;u}(z;\tau) \cap G(g_\tau^{-1}, \mu)) \leq \mathbb{P} (E_{s;u}(z;\tau) \cap G(g_\tau^{-1}, \mu)). \tag{3.14}$$

Assuming that Proposition 3.7 holds, \(3.14\) implies \(3.3\) with $\tau$ in place of $t$. To get the desired bound at the deterministic time $\bar{t}$, let $\tilde{g}_t = g_{t+\bar{t}} \circ g_{\tau}^{-1}$; so that (by the Markov property) the families of conformal maps $(g_t)$ and $(\tilde{g}_t)$ are i.i.d. Let $(\tilde{K}_t)$ be the hulls for $(\tilde{g}_t)$. For $w \in C$, $\mu' \in \mathcal{M}$ and $C > 1$, let $\tilde{E}(w) = \tilde{E}(w;\tilde{\tau}, C, \mu')$ be the event that the following is true.

1. $C^{-1} \leq \text{dist}(w, \tilde{K}_\tau) \leq C$.
2. $C^{-1} \leq |\tilde{g}_t(w)| \leq C$ for each $t \leq \bar{t}$.
3. \(G(g_t^{-1}, \mu')\) occurs for each \(t \leq \tilde{t}\).

If \(C\) is chosen sufficiently large, depending on \(\tilde{t}\) but uniform for \(w\) in compact subsets of \(H\) and \(\mu' \in \mathcal{M}\) is chosen sufficiently small, then we have that \(P_E(w)\) is at least a positive constant depending uniformly on \(w\) in compact subsets of \(H\). Furthermore, since we have a bound on \(\text{diam } K_r\) on the event \(\overline{E}^{s,u}(z; \tau) \cap G(g_t^{-1}, \mu)\), it follows from the Markov property that

\[
P\left(\overline{E}(g_t(z)) \cap \overline{E}^{s,u}(z; \tau) \cap G(g_t^{-1}, \mu)\right) \geq P\left(\overline{E}^{s,u}(z; \tau) \cap G(g_t^{-1}, \mu)\right).
\]

On the other hand, the definition of \(\overline{E}(g_t(z))\) implies that

\[
\overline{E}(g_t(z)) \cap \overline{E}^{s,u}(z; \tau) \cap G(g_t^{-1}, \mu) \subset \overline{E}^{s,u}(z; \tilde{t}, c', \tau) \cap G(g_t^{-1}, \mu' \circ \mu')
\]

for some \(c' > 0\) depending on the other parameters (here we use that \(\text{Im } g_t(z)\) is increasing in \(t\) for the condition involving \(r\)). By making \(c\) sufficiently small, we can make \(c'\) as small as we like. We conclude that \(3.3\) at time \(\tau\) implies \(3.3\) at time \(\tilde{t}\).

Thus to prove Theorem 3.1 it remains to prove Proposition 3.7. The proof is separated into two major steps: first we prove that the derivative condition [1] in the definition of \(\overline{E}^{s,u}(z)\) holds at time \(\tau\) with \(P^\tau\)-probability tending to 1 as \(\epsilon = \text{Im } z \to 0\). This is done in Section 3.4 via a stochastic calculus argument. Then we prove that \(P^\tau_\ast\left(\{\{\tau < \tilde{t}\} \cap G(g_t^{-1}, \mu)\right)\) is uniformly positive for sufficiently small \(\mu\) and sufficiently large \(\tilde{t}\). This is done in Section 3.5 via a stochastic calculus argument.

### 3.4 Coupling with a GFF

Assume we are in the setting of Proposition 3.7. Let \(h\) be a free boundary GFF on \(H\) independent from \((g_t)\), normalized so that its harmonic part \(\eta\) vanishes at \(iy\) for some \(y > 0\) (which we will specify below in such a way that it depends on \(\tilde{t}\), but not \(\epsilon\)). Let \(P_h\) be the law of \(h\). For \(t \geq 0\) let

\[
h_t = h \circ g_t + \frac{2}{\sqrt{\kappa}} \log |g_t(\cdot)| + \frac{\rho}{2\sqrt{\kappa}} G(g_t(z), g_t(\cdot)),
\]

(3.15)

where

\[
G(x, y) := -\log |x - y| - \log |\tau - y|
\]

is the Green’s function on \(H\) with Neumann boundary conditions.

Let \(\tau\) be as in (3.12). By [She10, Proposition 4.1], we have \(h_\tau + Q \log |g'_t| \overset{d}{=} h_0\), modulo additive constant, where \(Q = \frac{2}{\sqrt{\kappa}} + \frac{\rho}{2\sqrt{\kappa}}\) is as in (2.14). Let \(b_\tau\) be this additive constant, so

\[
h_\tau + Q \log |g'_t| - b_\tau \overset{d}{=} h_0.
\]

(3.16)

The idea of the proof of (3.7) is to estimate the terms other than \(\log |g'_t|\) in (3.16), and thereby obtain an estimate on \(|g'_t|\).

Let

\[
h'_0 = h_\tau + Q \log |g'_t| - b_\tau
\]

so that by (3.16) we have \(h'_0 \overset{d}{=} h_0\). Rearranging the definition of \(h'_0\) gives

\[
Q \log |g'_t(w)| = h'_0 - h_\tau + b_\tau
\]

\[
= h' - h \circ g_t + \frac{2}{\sqrt{\kappa}} \log \frac{|w|}{|g_t(w)|} + \frac{\rho}{2\sqrt{\kappa}} \left(\log \frac{|g_t(w) - g_t(z)|}{|w - z|} + \log \frac{|g_t(w) - g_t(z)|}{|w - \bar{z}|}\right) + b_\tau,
\]

where here \(h'\) is a field with the same law as \(h\) and we use \(w\) instead of \(\cdot\) as a dummy variable. Since all of the non-GFF terms in (3.18) are harmonic away from \(z\), the equation still holds for \(w \neq z\) if we replace \(h'\) and \(h \circ g_t\) with the circle average processes \(h'_c\) and \((h \circ g_t)_c\) for these two fields. We will use (3.18) to estimate \(b_\tau\) and then to estimate \(|g'_t(z)|\).

Throughout this subsection, we require all implicit constants to be independent of \(\epsilon\) and uniform for \(R^{-1} \leq |\text{Re } z| \leq R\) and all \(o(1)\) terms to be uniform for \(R^{-1} \leq |\text{Re } z| \leq R\).
Lemma 3.8. Let $\xi > 1/2$. If $y$ is chosen sufficiently large (independently of $\epsilon$ and uniform for $R^{-1} \leq |\text{Re} z| \leq R$) then
\[
(P^*_z \otimes P_h) \left( \{|b_\tau| > (\log \epsilon)^{-\xi} \} \cap G(g^{-1}_\tau, \mu) \cap \{\tau < \bar{T} \} \right) = o_\epsilon(1).
\] (3.19)

Proof. If we replace the GFF terms with circle averages in (3.18) and evaluate at $w = iy$, we get
\[
Q \log |g'_r(iy)| = h'_r(iy) - (h \circ g_r)_\epsilon(iy) + \frac{2}{2\sqrt{\kappa}} \log \frac{y}{|g_r(iy)|} + \frac{\rho}{2\sqrt{\kappa}} \left( \log \frac{|g_r(iy) - g_r(z)|}{|iy - z|} + \log \frac{|g_r(iy) - g_r(z)|}{|iy - \bar{z}|} \right) + b_\tau.
\] (3.20)

By Lemma 2.4 diam $K_\tau \leq 1$ on $G(g^{-1}_\tau, \mu)$. By [Law05, Proposition 3.46] we have $\text{Im} g_r(iy) \asymp |g_r(iy)| \asymp 1$ on $G(g^{-1}_\tau, \mu)$. By the Koebe quarter theorem we also have $|g'_r(iy)| \asymp 1$ on $G(g^{-1}_\tau, \mu)$. Hence each of the terms in (3.20) except for $b_\tau$ and the two circle averages is $\asymp 1$ on $G(g^{-1}_\tau, \mu) \cap \{\tau < \bar{T} \}$ if $y$ is chosen sufficiently large, depending only on $\mu$. By Lemma 2.10, for $\xi > 1/2$ we have
\[
(P^*_z \otimes P_h) \left( |h'_r(iy) - (h \circ g_r)_\epsilon(iy)| > (\log \epsilon)^{\xi} \right) = o_\epsilon(1).
\]

Note that we took $A = \emptyset$ in that lemma to estimate $h'_r(iy)$ and we took $A = K_\tau$ and used that $K_\tau$ is independent of $h$ to estimate $(h \circ g_r)_\epsilon(iy)$. By re-arranging (3.20) we conclude. \square

Proposition 3.9. Suppose we define $\rho = \rho(s)$ as in (3.8). For any $\mu \in M$ and any $c > 0$, we have
\[
(P^*_z \setminus \{|g_r(z)| \notin [e^{-s+c}e^{-s-c}] \} \cap G(g^{-1}_\tau, \mu) \cap \{\tau < \bar{T} \}) = o_\epsilon(1).
\] (3.21)

Proof. Since the circle average process is continuous [DS11, Proposition 3.1], we can take the limit as $w \to z$ in (3.18) to get
\[
Q \log |g'_r(z)| = h'_r(z) - (h \circ g_r)_\epsilon(z) + \frac{\rho}{2\sqrt{\kappa}} \log |g'_r(z)| - \frac{\rho}{2\sqrt{\kappa}} \log \epsilon + \frac{2}{\sqrt{\kappa}} \log \frac{|z|}{|g_r(z)|} + \frac{\rho}{2\sqrt{\kappa}} \log |\text{Im} g_r(z)| + b_\tau.
\] (3.22)

Since we have a uniform upper bound on diam $K_\tau$ on the event $G(g^{-1}_\tau, \mu)$ and $\text{Im} g_r(z) = r$ on the event $\{\tau < \bar{T} \}$, the absolute value of the sum of the fifth and sixth terms in the right in (3.22) is $\leq 1$ on $G(g^{-1}_\tau, \mu) \cap \{\tau < \bar{T} \}$.

By Lemma 2.10 (applied as in the proof of Lemma 3.8), for any $\xi > 1/2$,
\[
(P^*_z \otimes P_h) \left( |h'_r(z) - (h \circ g_r)_\epsilon(z)| \geq (\log \epsilon)^{-\xi} \right) = o_\epsilon(1).
\]

By Lemma 3.8 the probability that the last term in (3.22) is $\geq (\log \epsilon)^{1/2}$ and $G(g^{-1}_\tau, \mu) \cap \{\tau < \bar{T} \}$ occurs is of order $o_\epsilon(1)$. Hence except on an event of $P^*_z \otimes P_h$-probability of order $o_\epsilon(1)$, on the event $G(g^{-1}_\tau, \mu) \cap \{\tau < \bar{T} \}$ it holds that
\[
Q \log |g'_r(z)| = \frac{\rho}{2\sqrt{\kappa}} \log |g'_r(z)| + \frac{\rho}{2\sqrt{\kappa}} \log \epsilon - 1 + o_\epsilon(\log \epsilon^{-1}).
\]

Rearranging, we get that except on an event of $P^*_z \otimes P_h$-probability of order $o_\epsilon(1)$, on the event $G(g^{-1}_\tau, \mu) \cap \{\tau < \bar{T} \}$ we have
\[
\log |g'_r(z)| = \frac{\rho}{\kappa + 4 - \rho} \log \epsilon^{-1} + o_\epsilon(\log \epsilon^{-1}).
\] (3.23)

With $\rho$ as in (3.8) we have
\[
\frac{\rho}{\kappa + 4 - \rho} = s,
\]
so integrating out $P_h$ yields (3.21). \square
3.5 Auxiliary conditions hold with positive probability

3.5.1 Setup

In light of Proposition 3.9, to prove Proposition 3.7 and hence Theorem 3.1, it remains to prove the following.

**Proposition 3.10.** Let \((g_t)\) be as in (3.4), and let \((K_t)\) be the associated hulls. Let \(z \in \mathbf{H}\). For \(r > \text{Im} z\) let \(T \) be the first time \(t\) that \(\text{Im} g_t(z) = r\). Let \(\rho \in (-\infty, \kappa/2 + 2)\) and let \(P^z_\rho\) be the law of \((g_t)\) weighted by \(M^2\), as in Lemma 3.11. For any given \(R > 1\), there exists \(r_* > 0\) such that for each \(r \geq r_*\), we can find \(\mu \in \mathcal{M}\), \(\tilde{r} > 0\), and \(p > 0\) such that for \(z \in \mathbf{H}\) with \(|\text{Re} z| \leq R\) and \(\text{Im} z\) sufficiently small, we have

\[ P^z_\rho \left( \{T \leq \tilde{r} \} \cap G(g^{-1}_T, \mu) \right) \geq p. \]

The proof of Proposition 3.10 will be completed in two stages. First, we will show that we can move the real part of the force point from \(\text{Re} z\) to 0 without any pathological behavior (Lemma 3.11). Then, we will use a forward/reverse SLE symmetry argument to rule out pathological behavior after the real part of the force point has first reached 0.

We adopt the following notation. Fix \(z \in \mathbf{H}\) with \(|\text{Re} z| \leq R\) and \(\text{Im} z = \epsilon\). Let

\[ Z_t = g_t(z) = X_t + iY_t. \]

(3.24)

By (3.6), we have that under \(P^z_\rho\),

\[ dX_t = (\rho - 2) \frac{X_t}{|Z_t|^2} dt - \sqrt{\kappa} dB_t^z, \quad dY_t = \frac{2Y_t}{|Z_t|^2} dt, \quad X_0 = \text{Re} z, \quad Y_0 = \epsilon \]

(3.25)

for \(B^z_t\) a \(P^z_\rho\)-Brownian motion.

3.5.2 Pushing the force point to the imaginary axis

We use the notation (3.24) and put

\[ S_0 := \inf \{ t \geq 0 : X_t = 0 \}. \]

(3.26)

**Lemma 3.11.** Suppose we are in the setting of Proposition 3.10. Let \(Z_t = X_t + iY_t\) be as in (3.24) and let \(S_0\) be as in (3.26). Let \(\zeta \in (0, 1)\). There exists \(d > 0\) and \(p_0 > 0\), independent of \(\epsilon\) and of \(X_0 \in [-R, R]\), such that whenever \(\epsilon \leq \zeta\) the probability of the event \(E_0 = E_0(\zeta, d)\) that

1. \(S_0 \leq \zeta\);
2. \(Y_{S_0} \leq 5 \zeta^{1/2}\);
3. \(\text{diam} K_{S_0} \leq d\);

is at least \(p_0\).

**Proof.** By symmetry we can assume without loss of generality that \(X_0 > 0\). We will treat the conditions in the definition of \(E_0\) in order.

**Condition** Let \(\nu \geq 1 + \left( \frac{2(\rho - 2)}{\kappa} + 1 \right)\).

(3.27)

Let \(\bar{X}\) be \(\sqrt{\kappa}\) times a Bessel process driven by \(-B_t^z\), started from \(X_0\), of dimension \(\nu\). We have

\[ d(\bar{X}_t - X_t) = \frac{X_t(aX_t - (\rho - 2)\bar{X}_t) + aY_t^2}{(X_t^2 + Y_t^2)\bar{X}_t} dt, \]

where \(a = \kappa(\nu - 1)/2 > 0 \wedge (\rho - 2)\). This is strictly positive whenever \(X_t > \bar{X}_t\) (since \(\bar{X}_t \geq 0\)). This implies that a.s.

\[ \bar{X}_t \geq X_t, \quad \forall t \leq S_0. \]

(3.28)

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Our choice \(\rho\) for \(\rho\) implies that \((3.27)\) holds for some Bessel dimension \(\nu \in (0, 2)\), in which case \(\tilde{X}\) hits 0 before time \(\zeta\) with uniformly positive probability [Law05 Proposition 1.21]. From this and \((3.28)\) we conclude that we can find \(p_0 > 0\) independent of \(\epsilon\) and of \(X_0 \in [-R, R]\) such that

\[
P(S_0 \leq \zeta) \geq 2p_0. 
\]

\(\text{Condition 2:}\) By \((3.25)\) we have that \(Y\) is increasing and \(\partial_t Y_t^2 \leq 4\). Hence \(Y_t \leq 4t^{1/2} + \epsilon\), so on the event \(\{S_0 \leq \zeta\}\) we have \(Y_{S_0} \leq 5\zeta^{1/2}\).

\(\text{Condition 3:}\) Let \(X\) be a Bessel process of dimension \(\nu\) started from \(X_0\) as in the proof of \(\text{Condition 1}\). Since \(X\) and \(B^\nu\) are a.s. bounded up to time \(\zeta\) and their laws do not depend on \(\epsilon\), it follows from \((3.28)\) and \((3.29)\) that we can find \(C_0 > 0\), independent of \(\epsilon\) and uniform for \(X_0 \in [-R, R]\) such that the probability of the event \(E_0^*\) that

1. \(S_0 \leq \zeta\);
2. \(Y_{S_0} \leq 5\zeta^{1/2}\);
3. \(\sup_{t \leq \zeta} |\sqrt{\nu}B^\nu_t| \leq C_0\);
4. \(\sup_{t \leq \zeta} |X_t| \leq C_0\);

is at least \(p_0\). Note that for the last condition we use \((3.28)\).

By \((3.25)\), we have for \(t \leq S_0\) that

\[
|\rho - 2| \int_0^t \frac{X_s}{X_s^2 + Y_s^2} ds \leq |X_0| + |X_t| + |\sqrt{\nu}B_t|.
\]

(3.30)

In the case \(\rho \neq 2\), it follows from \((3.30)\) that on the event \(E_0^*\),

\[
\int_0^t \frac{X_s}{X_s^2 + Y_s^2} ds \leq C_1 := \frac{R + 2C_0}{|\rho - 2|}.
\]

(3.31)

In the case \(\rho = 2\), it follows from \((3.25)\) that \(X\) is a constant times a Brownian motion, so in this case we can (using \(\text{Condition 1}\)) find a possibly larger constant \(C_1\), still independent of \(\epsilon\), such that \((3.31)\) holds with probability at least \(1 - p_0/2\). In this case we add this latter condition to the event \(E_0^*\).

Now consider some \(b \in \mathbb{R}, |b| > 1\). Let \(\delta > 0\). Let \(\tau_b\) be the first time \(t\) that \(|g_t(b)| \leq \delta\). By \((3.6)\) and the Loewner equation, we have

\[
g_t(b) = -\int_0^t \frac{2}{g_s(b)} \, ds + \rho \int_0^t \frac{X_s}{X_s^2 + Y_s^2} \, ds - \sqrt{\nu}B^\nu_t + b.
\]

So, it follows from \((3.31)\) that on \(E_0^*\) we have

\[
\inf_{t \leq S_0 \wedge \tau_b} |g_t(b)| \geq |b| - C_2,
\]

where

\[
C_2 = 2\zeta\delta^{-1} + |\rho|C_1 + C_0.
\]

Hence if we take \(|b| > 2C_2\), then we have \(\inf_{t \leq S_0 \wedge \tau_b} |g_t(b)| \geq C_2\), which implies \(\tau_b > S_0\) (provided we choose \(\delta < C_2/2\)).

In particular, if \(b > 1\) is chosen sufficiently large (independent of \(\epsilon\) and \(X_0 \in [-R, R]\)), then \(g_{S_0}(-b)\) and \(g_{S_0}(b)\) lie in \(\mathbb{R}\). Therefore the map \(g_{S_0}^{-1}\) takes \(\partial K^\nu_{-b}\) into \([-b, b]\). This implies that the harmonic measure from \(\infty\) of \(K^\nu\) in \(H \setminus K^\nu\) is at most \(2\pi b\), so by [Law05 Equation 3.14], it follows that \(\text{diam } K_{S_0}\) is bounded by a constant \(d > 0\) independent of \(\epsilon\) and \(X_0 \in [-R, R]\) on \(E_0^*\). Since \(P(E_0^*) \geq p_0\), the lemma follows. 

\[\square\]
3.5.3 Pushing the force point starting from 0

In light of the strong Markov property and Proposition [3.10] we now need to consider the behavior of the process \( \theta_t \) if we start \( X_0 = 0 \) and \( Y_0 \) from \( y \in [\epsilon, 5\epsilon^{1/2}] \) for \( \zeta \) small but fixed. For this, we first need to review some calculations from [DMS14, Section 3]. Throughout this subsection, we assume \( X_0 = 0, Y_0 = y \in [\epsilon, 5\epsilon^{1/2}] \). Let

\[
\theta_t = \arg Z_t \quad \text{and} \quad s_y = \frac{1}{2} \log y. \tag{3.32}
\]

For \( s \geq s_y \) define \( \sigma(s) \) by

\[
s = \int_0^\sigma(s) \frac{1}{|Z_u|^2} \, du + s_y, \tag{3.33}
\]

so \( d\sigma(s) = |Z_{\sigma(s)}|^2 \, ds \) and \( \sigma(s_y) = 0 \). Denote processes under the time change \( t \mapsto \sigma(s) \) by a star, so \( \theta^*_s = \theta_{\sigma(s)} \), etc. By some elementary calculations using Itô’s formula (see the proof of [DMS14, Proposition 3.8]), we have \( d\log Y^*_s = 2 \, ds \) and

\[
d\theta^*_s = \sqrt{\kappa} \sin \theta^*_s \, dB_s + \left( 2 + \frac{\kappa}{2} - \frac{\beta}{\kappa} \right) \sin(2\theta^*_s) \, ds \tag{3.34}
\]

for \( B_s \) a Brownian motion. Since \( Y^*_y = Y_0 = y \), it follows that \( Y^*_s = e^{2s} \). Furthermore, as explained in the proof of [DMS14 Proposition 3.8], there is a unique stationary distribution for the SDE (3.34) which takes the form

\[
C \sin^\beta(\theta) \, d\theta, \quad \beta = \frac{8-2\rho}{\kappa}, \tag{3.35}
\]

where \( C \) is a normalizing constant.

Let \( \tilde{\theta}^*_s \) be a stationary solution to (3.34). Let \( \tilde{Z}^*_s = e^{2s} e^{\tilde{\theta}^*_s} \) so that \( \text{Im} \tilde{Z}^*_s = e^{2s} \) and \( \arg \tilde{Z}^*_s = \tilde{\theta}^*_s \). Let \( \tilde{W}^*_s \) be determined by \( \tilde{Z}^*_s \) in the same manner that \( W^*_s \) is determined by \( Z^*_s \). Let

\[
\tilde{\sigma}(t) := \int_0^t |\tilde{Z}^*_u|^2 \, du.
\]

Denote processes under the time change \( s \mapsto \tilde{\sigma}^{-1}(t) \) by removing the star. Then we have that \( (\tilde{\theta}_t, \tilde{Z}_t, \tilde{W}_t) \) are related in the same manner as \( (\theta_t, Z_t, W_t) \). Moreover,

\[
\tilde{\sigma}(s) = \inf\{ t \in \mathbb{R} : Y_t = e^{2s} \}.
\]

Following [DMS14 Section 3], we define a reverse \( \text{SLE}_\kappa(\rho) \) process with a force point infinitesimally above 0 to be the Loewner evolution driven by \( \tilde{W} \).

**Lemma 3.12.** Let \( (\tilde{g}_t) \) be the reverse Loewner maps of a reverse \( \text{SLE}_\kappa(\rho) \) process with a force point infinitesimally above 0, with hulls \( (\tilde{K}_t) \). We adopt the notation given just above, so in particular a star denotes processes under the time change \( t \mapsto \tilde{\sigma}(s) \). For \( \pi \in \mathbb{R} \) and \( s > 0 \), let

\[
\tilde{g}^*_{s,\pi} := \tilde{g}^*_{s-\pi} \circ (\tilde{g}^*_{\pi})^{-1} \quad \text{and} \quad \tilde{K}^*_{s,\pi} := \tilde{K}^*_{s-\pi} \setminus \tilde{g}^*_{s,\pi}(\tilde{K}^*_\pi).
\tag{3.36}
\]

For \( t, d > 0 \) and \( \mu \in \mathcal{M} \), let \( F = F(s, \pi, d, t, \mu) \) be the event that the following is true.

1. \( \tilde{\sigma}(s) \leq t \).

2. For each \( \delta > 0 \), the harmonic measure from \( \infty \) of each of \([-\delta, 0] \) and of \([0, \delta]\) in \( H \left( \tilde{K}^*_s \cup \tilde{g}^*_{s,\pi}(B_\delta(0) \cap H) \right) \) is at least \( \mu(\delta) \).

For any given \( d > 0 \), \( \pi \in \mathbb{R} \), and \( p \in (0, 1) \), we can find \( s_* > 0 \) (depending only on \( d, \pi, \) and \( p) \) such that whenever \( s \geq s_* \), there exists \( t > 0 \) and \( \mu \in \mathcal{M} \) (depending only \( s, \pi, \) and \( p) \) such that

\[
P(F) \geq 1 - p.
\]
Proof. By [DMS14, Proposition 3.10], for each \( s > 0 \), the law of \( K_s^* \) given \( \tilde{Z}_s^* \) is that of a forward chordal SLE\(_{\kappa}(\rho - 8)\) hull with an interior force point at \( \tilde{Z}_s^* \) stopped at the first time it hits its force point. By [SW05, Theorem 3] this law is that same as that of a radial SLE\(_{\kappa}(\kappa + 2 - \rho)\) from 0 to \( \tilde{Z}_s^* \) with a force point at \( \infty \), run until the first time it hits \( \tilde{Z}_s^* \). Since \( \kappa + 2 - \rho > \kappa/2 - 2 \) (by our choice of \( \rho \)) [MS13a, Theorem 1.12] implies that such a process is transient (i.e., almost surely tends to its target point) and [MS13a, Lemma 2.4] implies that it a.s. does not intersect itself or hit \( \mathbb{R} \cup \{\infty\} \). In particular, \( K_s^* \) is a.s. a simple curve which does not intersect \( \mathbb{R} \) except at its starting point and has finite half-plane capacity. Therefore the same is a.s. true of \( \bar{K}^*_{\pi,s} \) for each \( s, \pi \in \mathbb{R} \).

By scale invariance (which follows from uniqueness of the stationary solution to \((3.34)\)), for each \( c \in \mathbb{R} \) we have

\[
\{ e^{2c\gamma_s^*} (e^{-2c\gamma_s^*}) : \gamma_s^* \in \mathbb{R} \} \overset{d}{=} \{ \gamma_s^* : \gamma_s^* \in \mathbb{R} \}.
\]

Hence for each \( s > \bar{s} \in \mathbb{R} \) we have

\[
e^{2c\gamma_s^*} (e^{-2c\gamma_s^*}) \overset{d}{=} \gamma_s^* (e^{2c\gamma_s^*} - e^{-2c\gamma_s^*}). \quad (3.37)
\]

Since each \( K_{\pi,s}^* \) a.s. does not intersect \( \mathbb{R} \) except at its starting point, for any given \( \pi \in \mathbb{R} \) and \( s > 0 \), there a.s. exists some \( \delta > 0 \) and \( \lambda > 0 \) (random) such that \( \text{Im} \tilde{g}_{\pi,s}^*(z) \geq \lambda \) for each \( z \in B_\delta(0) \). By (3.37), if we are given \( d > 0, \pi \in \mathbb{R} \), and \( p > 0 \) we can find \( s_d > 0 \) such that whenever \( s \geq s_d \), it holds with probability at least \( 1 - p/2 \) that \( \text{Im} \tilde{g}_{\pi,s}^*(z) \geq 1 \) for each \( z \in B_d(0) \). Since \( K_{\pi,s}^* \) a.s. does not intersect \( \mathbb{R} \) and a.s. has finite half-plane capacity, for each such \( s \) we can find \( t \) and \( \mu \) as in the statement of the lemma such that

\[
\mathbb{P}(\mathbb{F}) \geq 1 - p.
\]

By convergence of solutions of SDE’s to their stationary distribution, the above lemma will allow us to control the behavior of \( \theta_t^* \) after a sufficiently large amount of time has passed (see the proof of Lemma 3.14 below). We also need to rule out pathological behavior at short time scales, which is the purpose of the next lemma.

**Lemma 3.13.** Let \( s_y \) be as in \((3.32)\). For any \( p \in (0,1) \) and \( v > 0 \), there is a \( b > 0 \) depending on \( v, p, \) and \( \zeta \) but not \( c, \zeta, \) or the particular choice of \( y \in [\epsilon, 5\zeta^{1/2}] \) such that

\[
\mathbb{P}_x^y \left( K_{s_y + v}^* \subset B_\delta(0) \right) \geq 1 - p
\]

Here \( K_s^* = K_{\sigma(s)} \), for \( K_t \) the hulls of the reverse Loewner evolution driven by \( (W_t) \).

**Proof.** First note that \( \theta_t^* \) cannot hit 0 or \( \pi \). To see this, one observes that \( \theta_t^* \) is a time change of a constant multiple of the process of [Law05, Section 1.11] with \( a = (4 + \kappa - \rho)/\kappa > 1/2 \), so the claim follows from [Law05, Lemma 1.27].

Therefore there exists \( \delta > 0 \) depending only on \( v \) such that if \( \theta_t^* \) is started at time \( s_y \) with initial condition \( \theta_{s_y}^* = \pi/2 \) then with probability at least \( 1 - p/2 \) we have \( \theta_t^* \in (\delta, 2\pi - \delta) \) for each \( s \in [s_y, s_y + v] \). Let \( G \) be the event that this occurs.

On the event \( G \), we can find a constant \( c > 0 \) depending only on \( \delta \) such that \( X^*_s/Y^*_s \leq c \) for \( s \in [s_y, s_y + v] \). It then follows from (3.25) that on this event we have

\[
\partial_t Y_t \geq \frac{1}{c} Y_t, \quad \forall t \leq \sigma(s_y + v)
\]

for a possibly larger \( c \). This implies

\[
Y_t^2 \geq c^{-1} t + y^2 \quad (3.38)
\]

for a possibly larger constant \( c \). Therefore, \( t \leq \sigma(s_y + v) \). In particular, \( (e^{tv} - 1) y^2 = Y_t^2 - y^2 \geq c^{-1} \sigma(s_y + v) \) so for some possibly larger constant \( c \) we have

\[
\sigma(s_y + v) \leq cy^2. \quad (3.39)
\]

Let \( B_t^\xi \) be the Brownian motion of \((3.6)\). We can find a \( C > 0 \) depending only on \( \zeta \) such that with probability at least \( 1 - p/2 \), we have \( \sqrt{\sigma B_t^\xi} \leq Cy^2 \) for each \( t \in [0, Cy^2] \). Let \( G' \) be the event that this occurs and that \( G \) occurs. It follows from (3.6) and (3.39) that on \( G' \), we have

\[
\sup_{0 \leq t \leq \sigma(s_y + v)} |W_t| \leq 1.
\]
By \cite{Law05} Lemma 4.13] we then have $\text{diam } K_{\sigma(s_u + v)} \leq 1$. 

**Lemma 3.14.** Suppose we are in the setting of this subsection (so that in particular $X_0 = 0$ and $Y_0 = y$). Let $T' = \inf\{t \geq 0 : Y_t = r\} = \sigma(\frac{1}{2} \log r)$. Also let $d > 0$ and $p \in (0, 1)$. There is an $r_s > 0$ (depending on $\zeta, d, \mu$, and $p$) such that for $r \geq r_s$, there exists $t_1 > 0$ and $\mu \in \mathcal{M}$, independent of $e$ and the particular choice of $y \in [e, 5\zeta^{1/2}]$ such that the probability of the event $E_1 = E_1(r, d, t_1, \mu)$ that

1. $T' \leq t_1$;
2. For each $\delta > 0$, the harmonic measure from $\infty$ of each of $[-\delta, 0]$ and of $[0, \delta]$ in $H \setminus (K_{T'} \cup g_{T'}(B_d(0) \cap H))$ is at least $1 - p$.

**Remark 3.15.** The purpose of condition 2 is as follows. When we compose with $g_{S_0}$ on the event $E_0$ of Lemma 3.14, the part of the hull grown before time $S_0$ is “pushed” into $g_{T'}(B_d(0))$. The harmonic measure condition 2 together with Lemma 2.4 will then imply the occurrence of Lemma 3.11, the part of the hull grown before time $X_{T'}$. We adopt the notation given just above Lemma 3.12, so that $Z_t$ is the evolution of the force point and $\theta_s = \arg Z_s$ is the corresponding stationary solution to (3.34).

By convergence of the law of the solution of (3.34) to its stationary distribution, there exists $v > 0$, independent of $\epsilon, \zeta$, and $y \in [e, 5\zeta^{1/2}]$, such that the following is true. The total variation distance between the law of $\theta_{s_u + v}$, started from $\pi/2$ at time $s_u$, and the stationary distribution (3.35) is at most $p/4$. Let $\bar{\sigma} = s_u + v$, and note that if $r$ is chosen sufficiently large depending only on $v$ then we have $\bar{\sigma} \leq s_r/2$. We can couple $\theta_s$ with $\bar{\theta}_s$ in such a way that with probability at least $1 - p/3$, these two processes agree at time $\bar{\sigma}$ and (by the Markov property) at every time thereafter. Let $F_1$ be the event that $\theta_s = \bar{\theta}_s$ for each $s \geq \bar{\sigma}$.

Define the maps $g_{s, \bar{\sigma}}, Y_s, Z_s, \sigma(s), \theta_s$ as above. Let $(\bar{g}_t)$ be the reverse Loewner maps of a reverse SLE$_\rho$ process with a force point immediately above 0. We adopt the notation given just above Lemma 3.12, so that $\bar{Z}_t$ is the evolution of the force point and $\bar{\theta}_s = \arg \bar{Z}_s$ is the corresponding stationary solution to (3.34).

By Lemma 3.13 we can find a $b > 0$ depending only on $v$ such that the probability of the event

$$F_2 := \{K_{\bar{\sigma}} \subset B_d(0)\}$$

is at least $1 - p/3$. By \cite{Law05} Proposition 3.46] there is a deterministic constant $d' > 0$ depending only on $b$ and $\bar{\sigma}$ such that on the event $F_2$ we have

$$K_{\sigma} \cup g_{\sigma}(B_d(0) \cap H) \subset B_{d'}(0) \cap H.$$  \hspace{1cm} (3.41)

Let $s_u$ be chosen so that the conclusion of Lemma 3.12 holds with this choice of $d'$ in place of $d$, $p/3$ in place of $p$, and $\bar{\sigma}$ as above. Let $s \geq s_u$ and let $t$ and $\mu$ be chosen so that with $F_3 = F(s, \bar{\sigma}, t, d', \mu)$ the event of Lemma 3.12 we have $P(F_3) \geq 1 - p/3$. Then we have

$$P(F_1 \cap F_2 \cap F_3) \geq 1 - p.$$  \hspace{1cm}

We will now conclude the proof by showing that $F_1 \cap F_2 \cap F_3 \subset E_1$ for an appropriate choice of parameters. If we set $r_s = e^{2(s_u + \bar{\sigma})}$ and $r = e^{2(s + \bar{\sigma})}$, then $r$ ranges over $[r_s, \infty)$ as $s$ ranges over $[0, \infty)$. On the event $F_1 \cap F_2 \cap F_3$, we have

$$T' = \text{hcap } K_{s+r} \leq \text{hcap } K_{s+r} + \text{hcap } K_{\sigma}.$$  \hspace{1cm}
The first term is at most $t$ by condition \( \mathbf{1} \) in the definition of \( F_3 \) and (3.40). The second term is at most a finite constant depending only on \( b \). Hence we can find a \( t_1 > 0 \) as in the statement of the lemma such that on \( F_1 \cap F_2 \cap F_3 \) we have \( T_r' \leq t_1 \). Furthermore, on \( F_1 \cap F_2 \cap F_3 \),

\[
K_{T_r'} \cup g_{T_r'} (B_d(0) \cap H) = K_{T_r} \cup g_{T_r} (K_{T_r} \cup g_{T_r} (B_d(0) \cap H))
\]

\[
= K_{T_r} \cup g_{T_r} (K_{T_r} \cup g_{T_r} (B_d(0) \cap H)) \quad \text{(by (3.40))}
\]

\[
\subset K_{T_r} \cup g_{T_r} (B_d'(0) \cap H) \quad \text{(by (3.41))}.
\]

Hence it follows from condition \( \mathbf{2} \) in the definition of \( F_3 \) that we can find \( \mu \in \mathcal{M} \) (satisfying the conditions of the lemma) such that condition \( \mathbf{2} \) in the definition of \( E_1 \) holds on \( F_1 \cap F_2 \cap F_3 \).

\[
\Box
\]

### 3.5.4 Proof of the main proposition

Now we can combine the results of the previous two subsections to complete the proof of Proposition 3.10.

**Proof of Proposition 3.10.** Let \( \zeta > 0 \), \( d > 0 \), and \( p_0 > 0 \) be as in Lemma 3.11 and let \( E_0 = E_0(\zeta, d) \) be the event of that lemma, so that \( \mathbf{P}(E_0) \geq p_0 \).

Conditional on \( \{g_t : t \leq S_0\} \), the law of \( \{g_t : t \geq 0\} := \{g_{t+\infty} \circ g_{S_0}^{-1} : t \geq 0\} \) is the same as that of the process started from \( X_0 = 0 \) and \( Y_0 = Y_{S_0} \). Note that \( Y_{S_0} \in [\epsilon, 5\zeta^{1/2}] \) on \( E_0 \). Define the times \( T_r' \) and the events \( E_1 = E_1(r, t_1, d, \mu) \) as in Lemma 3.14 but with \( (g_t) \) in place of \( (g_t) \). Let \( r_s, \mu, \) and \( t_1 \) satisfy the conclusion of this lemma for \( d \) as above and \( p = 1/2 \), so that if \( r \geq r_s \) then \( \mathbf{P}(E_1|E_0) \geq 1/2 \), whence \( \mathbf{P}(E_0 \cap E_1) \geq p_0/2 \).

By condition \( \mathbf{1} \) in the definition of \( E_0 \) and condition \( \mathbf{1} \) in the definition of \( E_1 \) we have \( T \leq \zeta + t_1 \) on \( E_0 \cap E_1 \). By condition \( \mathbf{2} \) in the definition of \( E_1 \), on the event \( E_0 \cap E_1 \), the harmonic measure from \( \infty \) of each of \( [-\delta, 0] \) and \( [0, \delta] \) in \( H \setminus K_{T_r} \) is at least \( \mu(\delta) \). By Lemma 2.4 we can find \( \mu' \in \mathcal{M} \) and \( \tilde{t} > 0 \) such that

\[
E_0 \cap E_1 \subset G(g_{T_r}^{-1}, \mu') \cap \{T_r \leq \tilde{t}\}.
\]

This proves the statement of the proposition.

\[
\Box
\]

### 3.6 Results for the disk

In this sequel we will work mostly in the unit disk \( D \) rather than in the upper half plane \( H \). In this brief subsection we make some trivial remarks about how Theorem 3.1 generalizes to this setting.

Suppose \( \eta \) is a chordal \( \text{SLE}_\kappa \) from \(-i\) to \( i\) in \( D \). Let \( \psi : \overline{D} \to H \) be the conformal map taking \(-i\) to \( 0 \), \( i \) to \( \infty \), and having positive real derivative at \( 0 \). Suppose \( \eta \) is parametrized in such a way that \( \psi(\eta) \) is parametrized by half-plane capacity. For each time \( t \geq 0 \), let

\[
f_t : \overline{D} \setminus \eta^t \to D
\]

be defined so that \( \psi \circ f_t \circ \psi^{-1} \) is the time \( t \) centered Loewner map for \( \psi(\eta) \).

For \( s \in (-1, 1) \), \( u > 0 \), \( z \in D \) with \( 1 - |z| = \epsilon \) and \( t, c, d > 0 \), let \( E^{u,c}_{\epsilon}(z; t) = E^{u,c}_{\epsilon}(z; t, c, d) \) be the event that the following is true.

1. \( e^{-\epsilon u} + u \leq |(f_t^{-1}'(z)| \leq e^{-\epsilon u} \).
2. \( f_t^{-1}(z) \in B_d(0) \).

Then in this context Theorem 3.1 reads as follows.

**Corollary 3.16** (Theorem 3.1 for the disk). Suppose we are in the setting described just above. Let \( \delta > 0 \). Let \( z \in D \) with \( |z - i|, |z + i| \geq \delta \) and \( 1 - |z| = \epsilon \). Define the events \( G(\cdot) \) as in Definition 2.3. For each \( t, c, d, \delta > 0 \), each \( s \in (-1, 1) \), and each \( \mu \in \mathcal{M} \),

\[
\mathbf{P} \left( E^{u,c}_{\epsilon}(z; t) \cap G(f_t, \mu) \right) \leq \alpha(\delta - \alpha(\delta) u).
\]
Furthermore, there exists $t_*>0$ such that for each $t \geq t_*$, we can find $\mu \in \mathcal{M}$ and $d>0$ such that for each $c>0$, each $s \in (-1,1)$, and each $u>0$,

$$\mathbb{P} \left( \mathbb{E}^s_{xu}(z;t) \cap \mathcal{G}(f_1,\mu) \right) \geq c^{s(s)+\alpha_0(s)u}. \quad (3.43)$$

In both (3.42) and (3.43), the implicit constants in $\leq$ and $\geq$ depend on the other parameters but not on $\epsilon$, and are uniform for $z \in D$ with $|z-i|, |z+i| \geq \delta$.

Proof. This is immediate from Theorem 3.1 and a coordinate change. Note that we use Lemma 2.4 to obtain a $d>0$, depending on $\mu$, such that (3.43) holds. \qed

4 One point estimates for the forward maps

4.1 Statement of the estimates

In this section we transfer the estimates of Theorem 3.1 to estimates for certain “time infinity” forward Loewner maps, which we will define shortly. We work in the setting of $D$, rather than $H$, as this setting will be more convenient for our two-point estimates. We start by defining our events.

Let $x,y \in \partial D$ be distinct and let $m$ be the midpoint of the counterclockwise arc connecting $x$ and $y$ in $\partial D$. Suppose we are given a simple curve $\eta$ in $D$ connecting $x$ and $y$. Let $D_\eta$ be the connected component of $D \setminus \eta$ containing $m$ on its boundary. Let $\Psi_\eta : D_\eta \to D$ be the unique conformal map taking $x$ to $-i$, $y$ to $i$, and $m$ to 1. For $s \in \mathbb{R}$, $u>0$, $\epsilon>0$, $c>1$, and $z \in D$, let $\mathcal{E}_\epsilon^{s,u}(\eta, z; c)$ be the event that

1. $z \in D_\eta$;
2. $\epsilon^{-1}c^{-1-s+u} \leq \text{dist}(z, \partial D_\eta) \leq c\epsilon^{1-s-u}$;
3. $\epsilon^{-1}c^{s+u} \leq |\Psi_\eta'(z)| \leq c^{s-u}$.

For technical reasons it will also be convenient to consider the counterclockwise arc of $\partial D$ from $y$ to $x$. We denote by $m^-$ the midpoint of this arc. Let $D^-_\eta$ be the connected component of $D \setminus \eta$ containing $m^-$ on its boundary and we let $\Psi^-_\eta : D^-_\eta \to D$ be the unique conformal map taking $x$ to $i$, taking $y$ to $-i$, and taking $m^-$ to $-1$. See Figure 4.1 for an illustration.

Let $\mathcal{A}_\epsilon^{s,u}(\eta, c)$ be the set of $z \in D$ for which $\mathcal{E}_\epsilon^{s,u}(\eta, z; c)$ occurs.

**Theorem 4.1.** Let $\alpha(s)$ and $\alpha_0(s)$ be as in (3.1). Let

$$\gamma(s) := \alpha(s) - 2s + 1 = \frac{(4 + \kappa)s^2}{8\kappa(1 + s)} - 2s + 1, \quad \gamma_0(s) := 2\alpha_0(s) + 2 = \frac{2(4 + \kappa)s(2 + s)}{8\kappa(1 + s)^2} + 2. \quad (4.1)$$

Let $\kappa \leq 4$. Suppose $\eta$ is a chordal SLE$_{\kappa}$ from $x$ to $y$ in $D$ and define the maps and events as above. Also define the events $\mathcal{G}(\cdot, \mu)$ as in Definition 2.5. For each $d \in (0,1)$, each $\mu \in \mathcal{M}$, and each $z \in B_d(0)$, we have

$$\mathbb{P} \left( \mathcal{E}_\epsilon^{s,u}(\eta, z; c) \cap \mathcal{G}(\Psi_\eta, \mu) \cap \mathcal{G}(\mathcal{A}_\epsilon^{s,u}(\eta, c)) \right) \leq c^{\gamma(s)-\gamma_0(s)u}. \quad (4.2)$$
Furthermore, there exists $\mu \in \mathcal{M}$ depending only on $d$ such that s.t. for $z \in B_d(0)$,
\[
P \left( E^{s,u}_{\epsilon}(\eta; z; c) \cap \mathcal{G}(\eta, \mu) \cap \mathcal{G}_{\gamma}(\eta, \mu) \right) \geq \epsilon^{7(s)+\gamma_0(s)u}. \tag{4.3} \]

In [4.2] and [4.3] the implicit constants are independent of $\epsilon$ and uniform for $z \in B_d(0)$ and for $|x - y|$ bounded below by a positive constant.

The proof of Theorem 4.1 proceeds as follows. First we use Theorem 3.1 to prove estimates for the area of certain finite-time analogues of the sets of Theorem 4.1. This is done in Section 4.2. This subsection contains a result which allows us to extend the estimate for deterministic times to estimates for certain stopping times, which will be needed in the sequel. Then, in Section 4.3 we prove several lemmas comparing finite time and infinite time maps and use these lemmas to obtain estimates for the area of the set $A^{s,u}_{\epsilon}(\eta; c)$ of points where the events of Theorem 4.1 occur. Finally, we complete the proof of Theorem 4.1 in Section 4.4 by proving a lemma which gives that the probabilities of the events of Theorem 4.1 do not depend too strongly on $z$. In Section 4.5 we deduce an analogue of Theorem 4.1 for the curve stopped at a finite time.

### 4.2 Area estimates and stopping estimates for finite time maps

In this section we will prove estimates for the expected area of the set of points where finite-time analogues of the events of Theorem 4.1 occur. We will also prove a result which allows us to compare probabilities for events at stopping times whose difference is bounded. Suppose we are in the setting of Theorem 4.1.

**Definition 4.2.** Let $\eta$ be a chordal SLE$_{\kappa}$ from $-i$ to $i$ in $\mathbb{D}$. Define its centered Loewner maps $(f_t)$ as in Section 3.6. For $t, \epsilon, u, \delta, c > 0$, and $s \in (-1, 1)$, and $z \in \mathbb{D}$, let $E^{s,u}_{\epsilon}(\eta; z; t, \delta, c)$ be the event that the following is true.

1. $e^{-1}e^{s+u} \leq |f_t(z)| \leq ce^{s-u}$.
2. $e^{-1}e^{1-s+u} \leq \text{dist}(z, \eta') \leq ce^{1-s-u}$.
3. $|f_t(z) - i|$ and $|f_t(z) + i|$ are both at least $\delta$.

Let $A^{s,u}_{\epsilon}(\eta; t, \delta, c)$ be the set of $z \in \mathbb{D}$ for which $E^{s,u}_{\epsilon}(\eta; z; t, \delta, c)$ occurs.

**Lemma 4.3.** Suppose we are in the setting of Theorem 4.1 with $x = -i$ and $y = i$. Define the sets $A^{s,u}_{\epsilon}(\eta; t, \delta, c)$ as in Definition 4.3 and the events $\mathcal{G}(f_t, \mu)$ as in Definition 2.5. For any choice of parameters $t, \delta, c, d, \mu$ and any $d \in (0, 1)$,
\[
\begin{align*}
\mathbb{E} \left[ \text{Area}(A^{s,u}_{\epsilon}(\eta; t, \delta, c) \cap B_d(0)) \right] &\leq \epsilon^{7(s)+\gamma_0(s)u} \tag{4.4} \\
\mathbb{E} \left[ \text{Area}(A^{s,u}_{\epsilon}(\eta; t, \delta, c) \cap B_d(0)) \text{1}_{\mathcal{G}(f_t, \mu)} \right] &\geq \epsilon^{7(s)+\gamma_0(s)u}. \tag{4.5}
\end{align*}
\]

where the implicit constants are independent of $\epsilon$. Moreover, if $t$ and $\mu$ are chosen sufficiently large then we can find $d \in (0, 1)$, depending on $t$ and $\mu$ such that
\[
\begin{align*}
\mathbb{E} \left[ \text{Area}(A^{s,u}_{\epsilon}(\eta; t, \delta, c) \cap B_d(0)) \right] &\leq \epsilon^{7(s)+\gamma_0(s)u} \tag{4.6}
\end{align*}
\]

**Proof.** Let $\overline{A}^{s,u}_{\epsilon} = \overline{A}^{s,u}_{\epsilon}(\eta; t, \delta, c, d)$ be the set of $z \in \mathbb{D}$ such that

1. $e^{-1}e^{1+u} \leq 1 - |z| \leq ce^{1-u}$;
2. $|z - i|$ and $|z + i|$ are each at least $\delta$;
3. The event $E^{s,u}(z; t, c, d)$ of Section 3.6 occurs.

By (3.42) in Theorem 3.16 if the first two conditions in the definition of $\overline{A}^{s,u}_{\epsilon}$ hold for some $z \in \mathbb{D}$, then
\[
P \left( E^{s,u}(z; t, c, d) \cap \mathcal{G}(f_t, \mu) \right) \leq \epsilon^{(s) - \alpha_0(s)u}. \]

By integrating this over all such $z$, we get
\[
\mathbb{E} \left[ \text{Area}(\overline{A}^{s,u}_{\epsilon}) \text{1}_{\mathcal{G}(f_t, \mu)} \right] \leq \epsilon^{(s+1) - (\alpha_0(s)+1)u}. \tag{4.6}
\]
Similarly, suppose \( t, d, \text{ and } \mu \) are chosen so that (3.43) in Theorem 3.1 holds. Then we have

\[
E \left[ \text{Area}(A_{t}^{\infty}(\eta; t, \delta, c)) \right] \geq e^{\alpha(s)+1+(a_{0}(s)+1)\mu}.
\]  

(4.7)

By the change of variables formula we have

\[
\text{Area}(A_{t}^{\infty}(\eta; t, \delta, c) \cap B_{d}(0)) = \int_{f_{t}(A_{t}^{\infty}(\eta; t, \delta, c) \cap B_{d}(0))} |(f_{t}^{-1})'(z)|^{2} \, dz.
\]

(4.8)

The Koebe quarter theorem implies

\[
A_{t}^{s,u/2}(\eta; t, \delta, c') \subset f_{t}(A_{t}^{s,u}(\eta; t, \delta, c) \cap B_{d}(0)) \subset A_{t}^{s,2u}(\eta; t, \delta, c''),
\]

for appropriate \( c', c'' > 0 \), depending only on \( c \). Thus (4.6) implies (4.4). Similarly (4.7) implies (4.5) \( \square \).

In the remainder of this subsection we prove a result which allows us to transfer estimates between stopping times and deterministic times. We first need the following lemma.

Lemma 4.4. Let \( \eta \) be a chordal SLE\(_{\kappa} \) process from \(-i \) to \( i \) in \( \mathbb{D} \). Let \( (f_{t}) \) be its centered Loewner maps. For \( w \in \mathbb{D} \), \( \delta' > 0 \), \( C > 1 \), and \( \mu \in \mathcal{M} \) let \( H(w; t) = H(w; t, \delta', C, \mu) \) be the event that the following is true.

1. \( \text{dist}(w, w') \geq C^{-1} \).
2. \( C^{-1} \leq |f_{t}'(w)| \leq C. \)
3. \( |f_{t}(w) - i| \) and \( |f_{t}(w) + i| \) are each at least \( \delta' \).
4. The event \( G(f_{t}, \mu) \) occurs.

For \( \zeta > 0 \), let \( S_{\zeta}^{\delta} \) be the set of \( w \in \mathbb{D} \) with \( |w - i|, |w + i| \geq \delta \) and \( 1 - |w| \leq \zeta \). If \( T > 0 \) and \( \delta > 0 \), then there exists \( p, \zeta, \delta' > 0 \), \( \mu \in \mathcal{M} \), and \( C > 1 \) depending on \( T \) and \( \delta \) such that

\[
P \left( H(w; t) \ \forall t \leq T, \ \forall w \in S_{\zeta}^{\delta} \right) \geq p.
\]

(4.9)

Proof. Fix \( \delta' > 0 \), to be determined later, and let

\[
U := \{ z \in \mathbb{D} : |\Re z| \leq \delta'/2 \}.
\]

Let \( E_{0} = E_{0}(T) \) be the event that \( \eta^{T} \subset U \). By [MW14 Lemma 2.3], \( P(E_{0}) > 0 \). Thus, if \( \zeta \) is chosen sufficiently small and \( C \) is chosen sufficiently large, depending on \( \delta \) and \( \delta' \), then condition 2 holds a.s. on \( E_{0} \) for each \( w \in S_{\zeta}^{\delta} \supset S_{\zeta}^{\delta} \).

On \( E_{0} \), we have by Schwarz reflection that \( f_{t} \) a.s. extends to be conformal on a neighborhood of \( S_{\zeta}^{\delta} \) for each sufficiently small \( \zeta \) and each \( t \leq T \). It follows that for any such \( \zeta \) we can find a (possibly larger) constant \( C \) depending only on \( \delta, T, \) and \( \zeta \) such that the probability of the event \( E_{1} \) that \( E_{0} \) occurs and condition 2 holds for each \( w \in S_{\zeta}^{\delta} \) and each \( t \in [0, T) \) is at least \( P(E_{0})/2 \).

By continuity, if \( \delta' \) is chosen sufficiently small, depending only on \( \delta \) and \( T \), then the conditional probability given \( E_{1} \) of the event

\[
\left\{ f_{t}(w) \in S_{\zeta}^{\delta}, \ \forall t \leq T, \ \forall w \in S_{\zeta}^{\delta} \right\}
\]

(4.10)

tends to 1 as \( \zeta \to 0 \). In particular, we can find \( \zeta > 0 \) sufficiently small that the probability of the event \( E_{2} \) that (4.10) occurs and that \( E_{1} \) occurs is at least \( P(E_{1})/2 \).

Since \( \eta \) a.s. does not hit \( \partial \mathbb{D} \) and \( f_{t}^{-1} \) is a.s. continuous, we can find \( \mu \in \mathcal{M} \) such that \( P(G(f_{t}, \mu)|E_{2}) \geq 1/2 \). If \( E_{2} \cap G(f_{t}, \mu) \) occurs, then the event in (4.9) occurs. \( \square \)

Lemma 4.5. Let \( \eta \) be a chordal SLE\(_{\kappa} \) from \(-i \) to \( i \) in \( \mathbb{D} \) with centered Loewner maps \( (f_{t}) \). Let \( \tau, \tau' \) be stopping times for \( \eta \) and suppose there is a deterministic time \( T > 0 \) such that a.s. \( \tau \leq \tau' \leq T \). For any \( c > 1 \), \( \mu \in \mathcal{M} \), and \( \delta > 0 \), we can find \( \delta > 0 \), \( \mu' \in \mathcal{M} \) depending only on \( T \) and \( \delta \) such that for each \( u > 0 \), \( s \in (-1, 1) \), and \( z \in \mathbb{D} \),

\[
P \left( E_{c,u}^{\tau}(\eta, z; \tau, \delta, c) \cap G(f_{\tau}', \mu) \right) \leq P \left( E_{c,u}^{\tau}(\eta, z; \tau', \delta', c') \cap G(f_{\tau'}, \mu') \right),
\]

(4.11)

with the implicit constant uniform for \( z \) in compact subsets of \( \mathbb{D} \) and independent of \( c \).
Proof. Let \((F_t)\) be the filtration generated by \(\eta\). Let \(\hat{f}_t = f_{t+\tau} \circ f_{\tau}^{-1}\) and \(\hat{\eta} = f_\tau(\eta|_{[\tau,\infty)})\). Fix \(T\) and \(\delta\) and let \(p, \zeta, \delta', C\), and \(\mu'\) satisfy the conclusion of Lemma 4.4 with \(\mu'\) in place of \(\mu\). Define the events \(H(f_t(z); t, \delta', C)\) as in Lemma 4.4 with \(w = f_\tau(z)\) and \(f_t\) replaced by \(\hat{f}_t\). That lemma together with the Markov property of SLE imply that if \(\epsilon\) is chosen sufficiently small, then the conditional probability of the event
\[
\hat{H} := \{H(f_t(z); t, \delta', C, \mu') \land t \leq T\}
\]
given \(E_{\epsilon}^{\ast}u(\eta, z; \tau, \delta, c) \cap G(f_t, \mu)\) is at least \(p\). By inspection, for sufficiently small \(\epsilon\) and any \(t \leq T\), we have
\[
E_{\epsilon}^{\ast}u(\eta, z; \tau, \delta, c) \cap G(f_t, \mu) \cap H(f_t(z); t, \delta', C, \mu') \subset E_{\epsilon}^{\ast}u(\eta, z; \tau + t, \delta', c') \cap G(f_{t+\tau}, \mu' \circ \mu)
\]
and
\[
E_{\epsilon}^{\ast}u(\eta, z; \tau, \delta, c) \cap G(f_{t+\tau}, \mu' \circ \mu) \subset E^\ast u(\eta, z; \tau, \delta', c') \cap G(f_t, \mu')
\]
for some \(c' > 0\) depending only on \(C\) and \(c\). Since \(\tau' - \tau \leq T\) a.s., it follows that
\[
E_{\epsilon}^{\ast}u(\eta, z; \tau, \delta, c) \cap G(f_t, \mu) \cap \hat{H} \subset E^\ast u(\eta, z; \tau', \delta', c') \cap G(f_{\tau'}, \mu' \circ \mu)
\]
for some \(c' > 0\). Taking probabilities proves (4.11) (with \(\mu' \circ \mu\) in place of \(\mu'^\prime\)).

4.3 Comparison lemmas

In this subsection we prove several lemmas comparing probabilities of sets associated with the finite time Loewner maps to probabilities of sets associated with the infinite time Loewner maps of Theorem 4.1, and use these results to estimate the areas of the sets \(\mathcal{A}^{\ast}u(\eta; c)\) of Theorem 4.1.

Our first lemma is needed for the proof of the upper bound in Theorem 4.1.

Lemma 4.6. Suppose we are in the setting of Theorem 4.1 with \(x = -i\) and \(y = i\). For each \(d \in (0, 1)\) and each \(\mu \in \mathcal{M}\), there exists \(\mu' \in \mathcal{M}\) and a bounded stopping time \(\tau\) for \(\eta\), both independent of \(u\) and \(z \in B_d(0)\), such that for each \(z \in B_d(0)\),
\[
P\left(E_{\epsilon}^{\ast}u(\eta, z; c) \cap G(\Psi, \mu) \cap G(\Psi', \mu)\right) \leq P\left(E_{\epsilon}^{\ast}u(\eta, z; \tau, \delta, c') \cap G(f_{\tau'}, \mu')\right)
\]
with the implicit constants independent of \(\epsilon\) and uniform for \(z \in B_d(0)\).

Proof. Suppose \(E_{\epsilon}^{\ast}u(\eta, z; c) \cap G(\Psi, \mu) \cap G(\Psi', \mu)\) occurs. We will prove the lemma by growing some more of the curve out from \(-i\) and \(i\) to get a new curve \(\tilde{\eta} = \eta\) with the property that \(E_{\epsilon}^{\ast}u(\tilde{\eta}, z; \tau, \delta, c') \cap G(f_{\tau'}, \mu')\) occurs for an appropriate bounded stopping time \(\tau\) and the derivatives of the conformal maps associated with \(\tilde{\eta}\) and \(\eta\) at \(z\) are comparable.

To this end, let \(\eta_0\) be a chordal SLE\(_\kappa\) from \(-i\) to \(i\) in \(D\), independent of \(\eta\). Let \(\eta_0^\tau\) be its time reversal. Then \(\eta_0^\tau\) has the law of a chordal SLE\(_\kappa\) from \(-i\) to \(i\) [Zha08b]. Let \(\delta_0, C, \beta, \zeta, \tau, a > 0\), and \(\mu_0 \in \mathcal{M}\). Let \(P\) be the event that the following is true.

1. Let \(\bar{T}\) be the first time \(\eta_0^\tau\) gets within distance \(e^{-\beta}\) of \(z\). Then \(\bar{T} < \infty\).

2. For each \(t \geq 0\), let \(\phi_t : D \setminus (\eta_0^\tau \cup \eta_0^\tau)\) be the unique conformal map fixing \(z\) and taking \(\eta_0^\tau(\bar{T})\) to \(i\). Let \(\bar{T}\) be the first time \(t\) such that \(\phi_t(\eta_0(t)) = -i\) and \(|\eta_0(t) - z| \leq 2e^{-\beta}\). Then \(\bar{T} < \infty\) and \(\eta_0^\tau\) is disjoint from \(B_{1/2}(i)\).

3. Henceforth put \(\phi = \phi_{\bar{T}}\). We have \(C^{-1} \leq |(\phi^{-1})'(w)| \leq C\) for each \(w \in B_{1+d/2}(0)\).

4. We have \(\phi^{-1}(B_{\delta_0}(-i) \cup B_{\delta_0}(i) \cup B_{1-r}(0)) \subset B_{(1-d)/2}(z)\).

5. Let \(\bar{T}\) be the last exit time of \(\eta_0\) from \(B_{\zeta}(i)\) before time \(\bar{T}\). Then \(\eta_0^\tau \subset B_{2\zeta}(i)\).

6. Let
\[
K := \eta_0^\tau \cup \eta_0^\tau(\bar{T}) \cup B_{(1-d)/2}(z).
\]
The harmonic measure from \(i\) of each side of \(K \cap B_{(1-d)/2}(i)\) and each side of \(K \cap B_{(1-d)/2}(-i)\) in the Schwarz reflection of \(D \setminus K\) across \([-1, 1]_{\partial D}\) is at least \(a\).

7. \(G'\left(K, \mu_0\right)\) occurs (Definition 2.6).
Figure 4.2: An illustration of the event $P$ and the curve $\bar{\eta}$ used in the proof of Lemma 4.6. The red points are $-i$, $i$, and $1$ and their images under $\phi$.

See Figure 4.2 for an illustration of the event $P$. In what follows, all implicit constants are required to depend only on $\mu$, $d$, and the parameters for $P$.

First we will argue that $P(P) \geq 1$ for a suitable choice of parameters. It follows from [MW14, Lemma 2.3] and reversibility of SLE that for any given choice of $r$ and $\delta_0$, conditions 1 and 2 hold with positive probability depending only on $\beta$, $\zeta$, and $d$. By the Koebe growth theorem, if $\beta$ is chosen sufficiently large (depending on $r$ and $d$) and $\delta$ is chosen sufficiently small (depending only on $d$) then condition 3 also holds simultaneously with positive probability depending only on $\beta$, $\zeta$, $d$, $\delta$, and $r$. By choosing $\alpha$ and $C$ sufficiently large and $\mu_0$ sufficiently small (see Lemma 2.7), depending only on $d$ and the other parameters for $P$, we can arrange that the remaining conditions in the definition of $P$ hold with probability arbitrarily close to 1. Thus we have $P(P) \geq 1$.

Let $\bar{\eta} = \eta_{0,1}$ on the event that $P$ does not occur. On $P$, let $\bar{\eta} = \phi^{-1}(\eta) \cup \eta_T^F \cup \eta_T^R$, parametrized in such a way that its image under the conformal map from $D$ to $H$ taking $-i$ to $0$, $i$ to $\infty$, and $0$ to $i$ is parametrized by capacity. By the Markov property and reversibility of SLE, $\bar{\eta}$ has the same law as $\eta$. Let $(f_t)$ be the centered Loewner maps for $\bar{\eta}$. Let

$$E = E^{u,v}_{\bar{\eta}}(\eta, z; c) \cap \mathcal{G}(\Psi_{\eta}, \mu) \cap \mathcal{G}(\Psi_{\bar{\eta}}, \mu) \cap P.$$ 

Let $\tau$ be the hitting time of $B_\delta(i)$ by $\bar{\eta}$. Then $\tau$ is a bounded stopping time for $\bar{\eta}$. We claim that if the parameters for $P$ are chosen appropriately, then we have

$$\bar{E} \subset E^{u,v}_{\bar{\eta}}(\bar{\eta}, z; \tau, \delta, \bar{c}) \cap \mathcal{G}(f_{\tau}, \bar{\mu}) \tag{4.14}$$

for some $\bar{c} > 0$ and $\bar{\mu} \in \mathcal{M}$, depending only on $\mu$, $d$, and the parameters for $P$. Given the claim (4.14), our desired result (4.12) follows by taking probabilities and noting that $P$ is independent from $\eta$.

By condition 4 in the definition of $P$, on the event $\bar{E}$ we have $\bar{\eta} \subset K$, as in (4.13), provided $r$ is chosen sufficiently small, depending only on $\mu$ and $\delta_0$. By condition 7 in the definition of $P$ and Lemma 2.8 we can find $\bar{\mu} \in \mathcal{M}$ depending only on $\mu$, $d$ and the parameters for $P$ such that $E \subset \mathcal{G}(f_{\tau}, \bar{\mu})$. By condition 5 in the definition of $P$, we can find $\delta > 0$ depending only on $a$ such that $\bar{f}_{\tau}(z)/|\bar{f}_{\tau}(z)|$ lies at distance at least $\delta$ from $\pm i$ on $\bar{E}$. That is, condition 5 in the definition of $E^{u,v}_{\bar{\eta}}(\bar{\eta}, z; \tau, \delta, \bar{c})$ holds on $\bar{E}$.

By condition 8 in the definition of $P$, we have $\text{dist}(z, \bar{\eta}) \asymp \text{dist}(z, \eta)$ on $P$. It therefore follows that condition 1 in the definition of $E^{u,v}_{\bar{\eta}}(\bar{\eta}, z; \tau, \delta, \bar{c})$ holds on $\bar{E}$ for some $\bar{c} \approx 1$.

It remains to show that condition 1 in the definition of $E^{u,v}_{\bar{\eta}}(\bar{\eta}, z; \tau, \delta, \bar{c})$ holds on $\bar{E}$ provided $\bar{c} \approx 1$ is chosen sufficiently large. It is enough to show $|\bar{f}_{\tau}(z)| \asymp |\Psi_{\eta}(z)|$ on $\bar{E}$. We will do this in two stages. Let $\Psi_{\bar{\eta}}$ be as in Section 4.1 with $\bar{\eta}$ in place of $\eta$. First we will show that $|\Psi_{\eta}(z)| \asymp |\Psi_{\bar{\eta}}(z)|$, and then we will show that $|\Psi_{\eta}(z)| \asymp |\bar{f}_{\tau}(z)|$.

For the first stage, let $g$ be the conformal automorphism of $D$ taking $\Psi_{\eta}(\phi(-i^+))$ to $-i$, $\Psi_{\eta}(\phi(i^-))$ to $i$, and $\Psi_{\eta}(\phi(1))$ to 1. Then we have

$$\Psi_{\bar{\eta}} = g \circ \Psi_{\eta} \circ \phi. \tag{4.15}$$
By condition 7 in the definition of $P$, together with the definition of $\overline{E}$, we have $|y'| \simeq 1$ uniformly on $D$ on $\overline{E}$, so by condition 3 in the definition of $P$, we have $|\Psi_\eta'(z)| \simeq |\Psi_\eta'(z)|$ on $\overline{E}$.

For the second stage, let $\Psi_\eta^r$ be the conformal map from $D \setminus \eta^r$ to $D$ taking $-i^+$ to $-i$ and fixing $i$ and 1. Then $\Psi_\eta^r$ differs from $\tilde{f}_\tau$ by a conformal automorphism of $D$ taking $\tilde{f}_\tau(-i^+)$ to $-i$ and $\tilde{f}_\tau(1)$ to 1. Since $G(\tilde{f}_\tau, \tilde{\mu})$ holds on $\overline{E}$, we have

$$|\Psi_\eta^r(z)| \simeq |\tilde{f}_\tau(z)|. \quad (4.16)$$

Let $I$ be the arc of $\partial D$ of length $\zeta$ centered at 1. By condition 7 in the definition of $P$ (c.f. Remark A.2), the lengths of $\Psi_\eta(I)$ and $\Psi_\eta^r(I)$ are $\geq 1$ on $\overline{E}$. By conditions 4 and 5 in the definition of $P$ and a study of the harmonic measure from 1 in the Schwarz reflection of $D_\eta$, the distances from $\Psi_\eta(z)$ to $\Psi_\eta(I)$ and from $\Psi_\eta^r(z)$ to $\Psi_\eta^r(I)$ are $\geq 1$ on $\overline{E}$ provided $\zeta$ is chosen sufficiently small relative to $d$. By Lemma A.1, we therefore have

$$|\Psi_\eta^r(z)| \asymp \frac{\text{hm}^z(I; D_\eta)}{\text{dist}(z, \eta^r)}, \quad |\Psi_\eta^r(z)| \asymp \frac{\text{hm}^z(I; D \setminus \eta^r)}{\text{dist}(z, \eta^r)} \quad (4.17)$$
on $\overline{E}$. By conformal invariance $\text{hm}^z(I; D_\eta)$ is the same as the probability that a Brownian motion started from $\Psi_\eta^r(z)$ exits $D$ in $\Psi_\eta^r(I)$ before hitting $\Psi_\eta^r(\eta^r([\tau, \infty)))$. By conditions 4 and 5 in the definition of $P$, if $\zeta$ is chosen sufficiently small, independently of $\epsilon$, then the diameter of $\Psi_\eta^r(\eta^r([\tau, \infty)))$ is at most a constant less than 1 times its distance from $\Psi_\eta^r(I)$ on $\overline{E}$ (here we again use harmonic measure from 1). Therefore, the probability that a Brownian motion started from $\Psi_\eta^r(z)$ exits $D$ in $\Psi_\eta^r(I)$ before hitting $\Psi_\eta^r(\eta^r([\tau, \infty)))$ is proportional to the probability that a Brownian motion started from $\Psi_\eta^r(z)$ exits $D$ in $\Psi_\eta^r(I)$. That is, $\text{hm}^z(I; D_\eta) \asymp \text{hm}^z(I; D \setminus \eta^r)$ on $\overline{E}$. By combining this with (4.16) and (4.17), we conclude.

The next lemma is needed for the proof of the lower bound in Theorem 4.1.

**Lemma 4.7.** Suppose we are in the setting of Theorem 4.1 with $x = -i$ and $y = i$. Let $d \in (0, 1)$. For any $\delta > 0$ and $\mu \in \mathcal{M}$, there exists $\mu' \in \mathcal{M}$ such that for any $c > 0$, there exists $c' > 0$, such that for $z \in B_d(0)$ and sufficiently small $\epsilon > 0$,

$$\mathbb{P} \left( \mathcal{E}^E_\epsilon^0(\eta, z; c') \cap G(\Psi_\eta, \mu) \cap G(\Psi_\eta^{-}, \mu') \right) \geq \mathbb{P} \left( \mathcal{E}^E_\epsilon^0(\eta, z; t, \delta, c) \cap \{ \text{Re} \, f_t(z) \geq 0 \} \cap G(f_t, \mu) \right), \quad (4.18)$$

with implicit constants independent of $\epsilon$ and uniform on $B_d(0)$. By re-choosing $c$, we can make $c'$ as small as we like.

**Proof.** Let $(f_t)$ be the centered Loewner maps for $\eta$ as in Section 4.2. For $t \geq 0$, let $\eta_t = f_t(\eta)|_{t=\infty}$. Let $D_t$ be the connected component of $D \setminus \eta_t$ containing 1 on its boundary and let $D^r_t$ be the other connected component of $D \setminus \eta_t$. Let $\Psi_t : D_t \to D$ (resp. $\Psi_t^{-} : D_t \to D$) be the unique conformal maps fixing $-i, i, 1$ (resp. $-i, i, -1$). Let $b_i$ (resp. $b^{-}_i$) be the image of the right (resp. left) side of $-i$ under $f_t$. Finally, let $\psi_t$ (resp. $\psi^{-}_t$) be the conformal automorphism of $D$ fixing $i$, taking $\Psi_t(b_i)$ to $-i$, and taking $\Psi_t^{-}(b^{-}_i)$ to $1$ (resp. fixing $i$, taking $\Psi_t^{-}(b_i)$ to $-i$, and taking $\Psi_t^{-}(f_t(-1))$ to $-1$). Then for each $t$,

$$\Psi_\eta = \psi_t \circ \Psi_t \circ f_t, \quad \Psi_\eta^{-} = \psi_t^{-} \circ \Psi_t^{-} \circ f_t. \quad (4.19)$$

Moreover, $(\Psi_t, \Psi_t^{-})$ and $f_t$ are independent and we have $\Psi_t \overset{d}{=} \Psi_\eta$, $\Psi_t^{-} \overset{d}{=} \Psi_\eta^{-}$. See Figure 4.3 for an illustration of some of these maps.

For $C > 1$, $\mu' \in \mathcal{M}$, and $w \in D$, let $F(w) = F(w; t, C, \mu')$ be the event that the following is true.

1. $w \in D_t$.
2. $C^{-1} \leq |\Psi_t(w)| \leq C$.
3. $\text{dist}(w, \eta_t) = \text{dist}(w, \partial D)$.
4. $G(\Psi_t, \mu') \cap G(\Psi_t^{-}, \mu')$ occurs.
By [MW14 Lemma 2.3], for each $\delta > 0$, we can find $C > 1$ and $\mu' \in \mathcal{M}$ such that for each $w \in \mathcal{D}$ lying at distance at least $\delta$ from $\pm i$ with $\text{Re } w \geq 0$, we have that $\mathbb{P}(F(w)) \geq 1$, with the implicit constant independent of $\epsilon$ and uniform for $w$ satisfying the conditions above.

If we let
$$F^*(z) := E^\mu_{\epsilon}(\eta; t, \delta, c) \cap \{\text{Re } f_t(z) \geq 0\} \cap \mathcal{G}(f_t, \mu) \cap F(f_t(z)),$$
then by independence of $f_t$ and $\eta$, and our choice of parameters for $F(\cdot)$ we have
$$\mathbb{P}(F^*(z)) \preceq \mathbb{P}\left(E^\mu_{\epsilon}(\eta; t, \delta, c) \cap \{\text{Re } f_t(z) \geq 0\} \cap \mathcal{G}(f_t, \mu)\right).$$
(4.20)

By condition [4] in the definition of $F(f_t(z))$, $|\psi'_t|$ and $|\psi''_t|$ are bounded above and below by positive $\epsilon$-independent constants on the event $F^*(z)$. Hence it follows from (4.19) that $F^*(z) \subset E^\mu_{\epsilon}(\eta; c') \cap \mathcal{G}(\Psi, \mu'')$ for some $c' > 0$ and some $\mu'' \in \mathcal{M}$ which do not depend on $\epsilon$ and are uniform for $z \in B_d(0)$. By combining this with (4.20) we get (4.18) (with $\mu''$ in place of $\mu'$).

Now we can transfer our area estimates for the finite time sets to area estimates for the time infinity sets.

**Lemma 4.8.** Suppose we are in the setting of Theorem 4.1 with $x = -i$ and $y = i$. For each $d \in (0, 1)$, each $\mu \in \mathcal{M}$, and each $c > 0$,

$$\mathbb{E}\left(\text{Area}(\mathcal{A}^\mu_{\epsilon}(\eta; c) \cap B_d(0))1_{\mathcal{G}(\Psi, \mu)\cap \mathcal{G}(\Psi, \mu)}\right) \preceq \epsilon^{\gamma(s) - \gamma_0(s)}u.$$  
(4.21)

Furthermore, there exists $d \in (0, 1)$ and $\mu \in \mathcal{M}$ depending only on $d$ such that for $c > 0$,

$$\mathbb{E}\left(\text{Area}(\mathcal{A}^\mu_{\epsilon}(\eta; c) \cap B_d(0))1_{\mathcal{G}(\Psi, \mu)\cap \mathcal{G}(\Psi, \mu)}\right) \succeq \epsilon^{\gamma(s) + \gamma_0(s)}u.$$  
(4.22)

In both (4.21) and (4.22) the implicit constants depend on the other parameters but not on $\epsilon$.

**Proof.** The relation (4.21) follows by integrating the estimate from Lemma 4.6 over $B_d(0)$, applying Lemma 4.5 to replace the stopping time $\tau$ with a deterministic time, then applying (4.4) from Lemma 4.3.

For (4.22), choose parameters in such a way that (4.5) from Lemma 4.3 holds. Given $c > 0$, choose $\mu' \in \mathcal{M}$ and $c' > 0$ such that the conclusion of Lemma 4.7 holds for our choice of $t$, $\mu$, and $d$. Integrate the estimate of Lemma 4.7 over $B_d(0)$ to get

$$\mathbb{E}\left(1_{\mathcal{G}(\Psi, \mu')\cap \mathcal{G}(\Psi, \mu')}\right) \preceq \mathbb{E}\left(\text{Area}(\mathcal{A}^\mu_{\epsilon}(\eta; c') \cap B_d(0))1_{\mathcal{G}(\Psi, \mu')}\right).$$
(4.23)

Since the law of $(f_t)$ is symmetric about the imaginary axis the right side of (4.23) is at least

$$\frac{1}{2}\mathbb{E}\left(\text{Area}(\mathcal{A}^\mu_{\epsilon}(\tau, \delta, c) \cap B_d(0))1_{\mathcal{G}(\Psi, \mu)}\right).$$

By combining this with (4.5) from Lemma 4.3 and re-labelling $c$ and $\mu$, we deduce (4.22). \qed
4.4 Proof of the estimates

In this subsection we will complete the proof of Theorem 4.4.1. To get Theorem 4.4.1 from the area estimates of Lemmas 4.6 and 4.7, we need to argue that the probabilities of the events of Theorem 4.4.1 do not depend too strongly on $z$. This is accomplished in the next lemma.

Lemma 4.9. Suppose we are in the setting of Theorem 4.4.1 with $x = -i$, $y = i$. Fix $d \in (0,1)$. For any $\mu \in \mathcal{M}$ and $c > 0$, we can find $\mu' \in \mathcal{M}$ and $c' > 0$ such that for $z, w \in B_d(0)$,

$$P \left( \mathcal{E}^\infty \left( \eta, w; c \right) \cap \mathcal{G}(\Psi_{\eta, \mu}) \cap \mathcal{G}(\Psi_{\eta, \mu'}) \right) \leq P \left( \mathcal{E}^\infty \left( \eta, z; c' \right) \cap \mathcal{G}(\Psi_{\eta, \mu}) \cap \mathcal{G}(\Psi_{\eta, \mu'}) \right)$$

(4.24)

with implicit constants independent of $\epsilon$ and uniform in $B_d(0)$. By re-choosing $c$, independently of $\epsilon$, we can make $c'$ as small as we like.

Proof. The basic idea of the proof is as follows. First we apply a conformal map taking $z$ to $w$ and fixing $-i$. The image of $\eta$ under such a map will be an SLE$_\kappa$ with a new target point $b$. To compare such a curve to our original curve, we grow a carefully chosen segment of the new curve backward from $b$ in such a way that when we map back to $D$, we get a chordal SLE$_\kappa$ from $-i$ to $i$. We now commence with the details.

For $z, w \in B_d(0)$, let $\phi = \phi_{z,w} : D \to D$ be the unique conformal map fixing $-i$ and taking $z$ to $w$. Let $b := \phi(i)$ and $\eta^b = \phi(\eta)$. The law of $\eta^b$ is that of a chordal SLE$_\kappa$ process from $-i$ to $b$ in $D$.

The map $\phi$ depends continuously on $z$ in the topology of uniform convergence on compacts. It follows that for any $\mu \in \mathcal{M}$ we can find a deterministic constant $c' > 0$ depending only on $c$, $\mu$, and $d$, linearly on $c$, and a deterministic $\mu' \in \mathcal{M}$ depending only on $\mu$ and $d$ such that for $z, w \in B_d(0)$,

$$E^\infty_\kappa(\eta^b, w; c) \cap \mathcal{G}(\Psi_{\eta^b, \mu}) \cap \mathcal{G}(\Psi_{\eta^b, \mu'}) \subset E^\infty_\kappa(\eta, z; c') \cap \mathcal{G}(\Psi_{\eta, \mu}) \cap \mathcal{G}(\Psi_{\eta, \mu'}) \quad \text{(4.25)}$$

Let $\eta^b$ be the time reversal of $\eta^b$. Then $\eta^b$ is a chordal SLE$_\kappa$ from $b$ to $-i$ in $D$ [Zha08b]. We give $\eta^b$ the usual chordal parametrization, so that it is the conformal image of a chordal SLE$_\kappa$ parametrized by capacity from 0 to $\infty$ in $H$. For each $t \geq 0$, let $\eta^b(t) \in \mathcal{G}(\eta^b([0, t])) \to D$ be the unique conformal map fixing $-i$ and $w$. Let $\tau$ be the first time $t$ that $\eta^b(t) = i$.

Fix $\mu^b \in \mathcal{M}$ and let $E^\infty_\kappa$ be the event that the following occurs.

1. $\tau$ is less than or equal to the first time $t$ that $\eta^b(t)$ hits $B_{d^*}(0)$, where $d^* = 1 - \frac{1}{4} \inf_{z,w \in B_d(0)} \text{dist}(\phi_{z,w}(B_d(0)), \partial D)$.

2. $\mathcal{G}(\eta_{\tau}, \mu^b)$ occurs.

By [MW14, Lemma 2.3], if $\mu^b$ is chosen sufficiently small then $P(E^\infty_\kappa)$ is a positive constant depending only on $\mu^b$ and $B_d(0)$.

By the Markov property, conditional on $E^\infty_\kappa$, the law of $\eta_{\tau}(\eta^b|_{[\tau, \infty)})$ is that of a chordal SLE$_\kappa$ process from $i$ to $-i$ in $D$. Therefore its time reversal $\hat{\eta} := \eta_{\tau}(\eta^b|_{[\tau, \infty])}$, where $\tau^b$ is the time corresponding to $\tau$ under the time reversal, has the law of a chordal SLE$_\kappa$ from $-i$ to $i$ in $D$. In particular, $\hat{\eta} \overset{d}{=} \eta$.

Define the open sets $D_{\mu^b}, D_{\hat{\eta}}$ and the maps $\Psi_{\eta^b}, \Psi_{\hat{\eta}}$ as in Section 4.1 with $\eta^b, \hat{\eta}$, resp., in place of $\eta$. Let $\psi$ (resp. $\psi^-$) be the conformal automorphism of $D$ which fixes $-i$, takes $(\Psi_{\eta^b} \circ \eta^b)_i$ to $i$, and takes $\Psi_{\hat{\eta}}(\partial D)(1)$ to 1 (resp. fixes $-i$, takes $(\Psi_{\hat{\eta}} \circ \eta^b)_i(b^+)$ to $i$, and takes $\Psi_{\hat{\eta}}(\partial D)(-1)$ to $-1$). Then we have

$$\Psi_{\eta^b} = \psi \circ \Phi_{\eta^b} \circ \eta^b, \quad \Psi_{\hat{\eta}} = \psi^- \circ \Phi_{\hat{\eta}} \circ \eta^b.$$

See Figure 4.4 for an illustration of some of these maps.

By condition 2 in the definition of $E^\infty_\kappa$, on the event $E^\infty_\kappa \cap \mathcal{E}^\infty_\kappa(\hat{\eta}, w; c) \cap \mathcal{G}(\Psi_{\hat{\eta}}, \mu) \cap \mathcal{G}(\Psi_{\hat{\eta}^b}, \mu)$, it holds that $|\psi'|$ and $|\psi^-'|$ are bounded above and below by deterministic positive constants depending only on $\mu^b$ and $\mu$. Furthermore, we have that $\mathcal{G}(\psi_{\mu_2} \circ \eta^b) \cap \mathcal{G}(\psi^-, \mu_2)$ holds for some $\mu_2 \in \mathcal{M}$ depending on $\mu^b, \mu$. The Koebe distortion theorem and condition 1 in the definition of $E^\infty_\kappa$ imply that $|g^b_{\hat{\eta}}(w)|$ is bounded above and below by positive constants depending only on $d$ on the event $E^\infty_\kappa$. Hence for some $c' > 0$, independent of $\epsilon$ and uniform for $z, w \in B_d(0)$, we have

$$E^\infty_\kappa \cap \mathcal{E}^\infty_\kappa(\eta, w; c) \cap \mathcal{G}(\Psi_{\eta^b}, \mu) \cap \mathcal{G}(\Psi_{\eta}, \mu) \subset \mathcal{E}^\infty_\kappa(\eta^b, w; c') \cap \mathcal{G}(\Psi_{\eta^b}, \mu_2 \circ \mu \circ \mu^b) \cap \mathcal{G}(\Psi_{\eta}, \mu_2 \circ \mu \circ \mu^b). \quad \text{(4.26)}$$
We conclude by combining this with Lemma 4.8 (and re-labelling Theorem 4.10. The results in this subsection are not needed for the proof of our main result, finite time Loewner maps. In this subsection we use Theorem 4.1 and the comparison lemmas of Section 4.3 to prove estimates for the finite time estimates

By the Markov property and the fact that $P(E^b)$ is uniformly positive, we have

$$
P(E^b \cap E^{\tau_n}(\widehat{\eta}, w; c) \cap G(\widehat{\Psi}_0, \mu) \cap G(\widehat{\Psi}_0^{-1}, \mu)) = P(E^{\tau_n}(\widehat{\eta}, w; c, \lambda, \ell) \cap G(\widehat{\Psi}_0, \mu) \cap G(\widehat{\Psi}_0^{-1}, \mu)).$$

(4.27)

Since $\widehat{\eta} \approx_d \eta$, (4.24) now follows from (4.26), (4.27), and (4.25).

**Proof of Theorem 4.1.** By applying a coordinate change it is enough to consider the case $x = -i, y = i$. By Lemma 4.9 for any $z \in B_d(0)$, we have, in the notation of that lemma,

$$
P(E^{\tau_n}(\eta, z; c) \cap G(\Psi_0, \mu) \cap G(\Psi_0^{-1}, \mu)) \leq E\left(\text{Area}(A^{\tau_n}_{\Psi}(\eta, z; c) \cap B_d(0))\mathbf{1}_{G(\Psi_0, \mu) \cap G(\Psi_0^{-1}, \mu)}\right)
$$

(4.28)

$$
P(E^{\tau_n}(\eta, z; c' \cap G(\Psi_0, \mu') \cap G(\Psi_0^{-1}, \mu')) \geq E\left(\text{Area}(A^{\tau_n}_{\Psi}(\eta, z; c) \cap B_d(0))\mathbf{1}_{G(\Psi_0, \mu) \cap G(\Psi_0^{-1}, \mu)}\right).
$$

We conclude by combining this with Lemma 4.8 (and re-labelling $c'$ in the lower bound).

**4.5 Finite time estimates**

In this subsection we use Theorem 4.1 and the comparison lemmas of Section 4.3 to prove estimates for the finite time Loewner maps. The results in this subsection are not needed for the proof of our main result, and are stated only for the sake of completeness.

**Theorem 4.10.** Let $\kappa \in (0, 4)$. Let $(f_t)$ be the centered Loewner maps of a chordal $\kappa$-SLE process $\eta$ from $-i$ to $i$ in $D$. Define the events $E^{\tau_n}(z; t, \delta, c)$ as in Definition 4.2 and the sets $G(f_t, \mu)$ as in Definition 2.5. For any $\mu \in \mathcal{M}$ and any $z \in D$,

$$
P(E^{\tau_n}(\eta; t, \delta, c) \cap G(f_t, \mu) \cap \{\text{Re } f_t(z) \geq 0\}) \leq e^{\gamma(s) - \gamma_0(s)u}.
$$

(4.28)

Moreover, there exists $\tau_0 > 0$ depending on $z$ and uniform for $z$ in compacts such that for each $t \geq \tau_0$ and sufficiently small $\delta > 0$, we can find $\mu \in \mathcal{M}$ such that for each $c > 0$,

$$
P(E^{\tau_n}(\eta; t, \delta, c) \cap G(f_t, \mu)) \geq e^{\gamma(s) + \gamma_0(s)u}.
$$

(4.29)

In (4.28) and (4.29) the implicit constants are independent of $\epsilon$ and uniform for $z$ in compacts. The estimate (4.28) holds with $t$ replaced by a bounded stopping time. The estimate (4.29) holds with $t$ replaced by a bounded stopping time which is a.s. $\geq \tau_0$.

**Proof.** The statement for deterministic times follows by combining Theorem 4.1 with Lemmas 4.5, 4.6 and 4.7. The statement for stopping times follows from this and Lemma 4.5.

Theorem 4.10 implies the following corollary.
Corollary 4.11. The statement of Theorem 4.10 also holds with SLE\(_\kappa\) replaced by SLE\(_\kappa(\rho)\) with finitely many boundary force points located at positive distance from \(-i\) provided \(\rho\) is chosen so that an SLE\(_\kappa(\rho)\) process does not hit \(\partial D\).

Proof. Lemma 2.4 shows that for any \(\delta > 0\), we can find a deterministic bounded simply connected open set \(U \subset D\) with \(\partial U \cap D \subset \partial D \setminus B_\delta(-i)\) such that \(\pi^tU \subset U\) on the event \(G(f_t, \mu)\). If \(\delta\) is chosen to be smaller than half of the distance from any of the force points to the \(-i\), then the form of the Radon-Nikodym derivatives in [SW05, Section 5] implies that the laws of a chordal SLE\(_\kappa\) and a chordal SLE\(_\kappa(\rho)\) are stopped at their first exit time from \(U\) are s.m.a.c. (Definition B.1) with constants depending only on \(\rho, \kappa, U\), and the distance of the force points to \(U\). Hence the statement of the corollary follows from Theorem 4.10.

5 Upper bounds for multifractal and integral means spectra

In this section we will use the upper bounds in Theorems 3.1 and 4.1 to prove the Hausdorff dimension upper bounds in Theorem 1.1 as well (most of) the upper bound in Corollary 1.8.

5.1 Upper bound for the Hausdorff dimension of the subset of the circle

In this subsection we use Proposition 3.1 to obtain upper bounds on the Hausdorff dimension of the sets \(\tilde{\Theta}^s(D \setminus K_t)\) of Section 1.1 for the hulls \((K_t)\) of a chordal SLE\(_\kappa\) from \(-i\) to \(i\) in \(D\). In light of Lemma 2.10, Proposition 5.1 implies the upper bounds for \(\dim_H \tilde{\Theta}^s(D)\) and \(\dim_H \tilde{\Theta}^{s \cup \Sigma} (D)\) in Theorem 1.1.

Proposition 5.1. Let \(\eta\) be a chordal SLE\(_\kappa\) process from \(-i\) to \(i\) in \(D\) with centered Loewner maps \((f_t)\) (defined as in Section 3.6) and hulls \((K_t)\). Let \(\tilde{\xi}(s), s_-, \text{ and } s_+\) be as in (1.3). For each \(t > 0\) and \(s \in [-1, 1]\), a.s.

\[
\dim_H \tilde{\Theta}^s(D \setminus K_t) \leq \tilde{\xi}(s), \quad 0 \leq s \leq s_+ \\
\dim_H \tilde{\Theta}^{s \cup \Sigma}(D \setminus K_t) \leq \tilde{\xi}(s), \quad s_- \leq s \leq 0.
\]

Almost surely, for each \(s \notin [s_-, s_+]\) we have \(\tilde{\Theta}^s(D \setminus K_t) = \emptyset\).

Remark 5.2. If \(\alpha(s)\) is as in (3.1) in the statement of Theorem 3.1, we have \(\tilde{\xi}(s) = 1 - \alpha(s)\).

Proof of Proposition 5.1. For \(\delta > 0\) and \(s \in (-1, 1)\), let

\[
\tilde{\Theta}^{s \cup \Sigma}_\delta (D \setminus K_t) := \tilde{\Theta}^{s \cup \Sigma}(D \setminus K_t) \cap \{x \in \partial D : |x - i|, |x + i| \geq \delta, \quad 1 - |f_t^{-1}(x)| \geq \delta\},
\]

where \(\ast\) stands for \(\geq\) in the case \(s \geq 0\) or \(\leq\) in the case \(s < 0\). The reason for this definition is that it will allow us to apply the estimates of Proposition 3.6 after a change of coordinates from \(D\) to \(H\). By countable stability of Hausdorff dimension, to prove (5.1), it is enough to show that a.s.

\[
\mathcal{H}^\beta(\tilde{\Theta}^{s \cup \Sigma}_\delta (D \setminus K_t)) = 0 \quad \forall \beta > 0, \quad \forall \beta > \tilde{\xi}(s).
\]

Henceforth fix \(\delta, \beta, \text{ and } s\) as above. Also let \(s' \in [0, s)\) (if \(s \geq 0\)) or \(s' \in (s, 0)\) (if \(s < 0\)) be chosen in such a way that \(\tilde{\xi}(s') < \beta\).

For \(n \in \mathbb{N}\) and \(k \in \{1, \ldots, n\}\), let

\[
B_n^k := \left\{ w \in D : \frac{\pi(k - 1)}{2^{n-1}} \leq \arg w \leq \frac{\pi k}{2^{n-1}}, \quad 2^{-n} \leq 1 - |w| \leq 2^{-n+1} \right\}.
\]

Let \(E_n^k\) be the event there is a \(w \in B_n^k\) with \(1 - |f_t^{-1}(w)| \geq \delta/2\) and

\[
\begin{align*}
|f_t^{-1}(w)| &\geq 2^{n s'}, 
\quad s \geq 0 \\
|f_t^{-1}(w)| &\leq 2^{n s'}, 
\quad s < 0.
\end{align*}
\]

(5.3)
Each $B^k_n$ can be covered by at most an $(n, k)$-independent constant number of balls of radius $< 2^{-n-1}$, and each point of $B^k_n$ lies at distance at least $2^{-n}$ from $\partial \mathbf{D}$. So the Koebe distortion and growth theorems imply that for sufficiently large $n$, on the event $E^k_n$, we have that for the center $z$ of one of these balls, $|(f^{-1}_s)'(z)|$ is at least (if $s \geq 0$) or at most (if $s < 0$) an $(n, k)$-independent constant times $2^{n s'}$; |$f^{-1}_s(z)| \leq \delta$; and $1 - |f^{-1}_s(z)| \geq \delta$.

For $n \in \mathbb{N}$, let $\mathcal{K}_n$ be the set of those $k \in \{1, \ldots, n\}$ such that $\exp(\pi k/2^{n-1})$ lies at distance at least $\delta/2$ from $-i$ and $i$. By Proposition [5.9] and a change of coordinates to $\mathbf{H}$, whenever $k \in \mathcal{K}_n$, we have

$$\mathbf{P}(E^k_n) \leq 2^{-n(1 - \tilde{\xi}(s'))} \tag{5.4}$$

where the implicit constant is independent of $n$ and uniform for $k \in \mathcal{K}_n$.

For $n \in \mathbb{N}$ and $k \in \{1, \ldots, n\}$, let

$$I^k_n := \left\{ x \in \partial \mathbf{D} : \frac{\pi (k - 1)}{2^{n-1}} \leq \arg x \leq \frac{\pi k}{2^{n-1}} \right\}.$$

For $m \in \mathbb{N}$, let $\mathcal{I}_m$ be the collection of those intervals $I^k_n$ for pairs $(n, k)$ such that $n \geq m$, $k \in \mathcal{K}_n$, and $E^k_n$ occurs. We claim that for each $m \in \mathbb{N}$, $\mathcal{I}_m$ is a cover of $\tilde{\Theta}^{\xi^*_s}(\mathbf{D} \setminus K_i)$. Indeed, if $x \in \tilde{\Theta}^{\xi^*_s}(\mathbf{D} \setminus K_i)$, then for any $m \in \mathbb{N}$ we can find $n \geq m$ and $w \in \mathbf{D}$ with $1 - |w| \leq 2^{-n \beta}$, $\arg w = \arg x$, and $|(f^{-1}_s)'(w)| \geq (1 - |w|)^{-s'}$ (resp. $|(f^{-1}_s)'(w)| \leq (1 - |w|)^{-s'}$ if $s < 0$), and $1 - |f^{-1}_s(w)| \geq \delta/2$. The point $w$ lies in $B^k_n$ for some pair $(n, k)$ with $I_{n,k} \in \mathcal{I}_m$. Since $\arg w = \arg x$, we have $x \in I_{n,k}$ for this choice of $(n, k)$.

Now, observe that [5.4] implies

$$\mathbf{E} \left( \sum_{I \in \mathcal{I}_m} (\text{diam } I)^\beta \right) \leq \sum_{n=m} \sum_{k \in \mathcal{K}_n} 2^{-n\beta} \mathbf{P}(E^k_n) \leq \sum_{n=m} 2^{-n(\beta - \tilde{\xi}(s'))} \tag{5.5}$$

This tends to $0$ as $m \to \infty$ since $\beta > \tilde{\xi}(s')$ (by our choice of parameters above). Since $\mathcal{I}_m$ is a covering of $\tilde{\Theta}^{\xi^*_s}(\mathbf{D} \setminus K_i)$ by intervals of diameter tending to zero as $m \to \infty$, this proves $\mathcal{H}^\beta(\tilde{\Theta}^{\xi^*_s}(\mathbf{D} \setminus K_i)) = 0$.

If $s \in [-1, 1] \setminus \{s_-, s_+\}$, then $\tilde{\xi}(s) < 0$, so the right side of (5.5) for $\beta = 0$ decays exponentially fast in $m$. Thus the expected number of sets in $\mathcal{I}_m$ tends to zero exponentially fast, and it follows from the Borel Cantelli lemma that $\mathcal{I}_m = \emptyset$ for sufficiently large $m$. Hence a.s. $\tilde{\Theta}^{\xi^*_s}(\mathbf{D} \setminus K_i) = \emptyset$ for each $\delta > 0$. Finally, it follows from Lemma [2.11] that $\tilde{\Theta}^{\xi^*_s}(\mathbf{D} \setminus K_i) = \emptyset$ for $s \notin [-1, 1]$.

### 5.2 Upper bound for the Hausdorff dimension of the subset of the curve

In this subsection we will use Theorem 4.1.1 to give an upper bound for the Hausdorff dimension of the sets $\Theta^{s+\xi}(D)$ and $\Theta^{s-\xi}(D)$ of Section 1.1 with $D = D_n$ as in Theorem 1.1. For this purpose it will be convenient to introduce a slightly variant of the sets of Section 1.1. For a domain $D \subset \mathbb{C}$, a conformal map $\phi : \mathbf{D} \to D$, $s \in \mathbb{R}$, and $u > 0$, let

$$\Theta^{s;u}(D) := \left\{ x \in \partial D : s - u \leq \limsup_{\epsilon \to 0} \frac{\log |\phi'(1 - \epsilon)\phi^{-1}(x)|}{-\log \epsilon} \leq s + u \right\}. \tag{5.6}$$

**Lemma 5.3.** Let $\eta$ be a chordal SLE$_{\kappa}$ from $-i$ to $i$ in $\mathbf{D}$ and let $D_\eta$, $\xi(s)$, $s_-$, and $s_+$ be as in Theorem 1.1. Then a.s.

$$\dim \mathcal{H} \Theta^{s;u}(D_\eta) \leq \xi(s) + o_u(1), \tag{5.7}$$

whenever $s \in [s_-, s_+]$, and a.s. $\Theta^{s;u}(D_\eta) = \emptyset$ for sufficiently small $u$ otherwise. The $o_u(1)$ in (5.7) tends to $0$ as $u \to 0$ and can be taken to be uniform for $s$ in compact subsets of $(-1, 1)$.

**Remark 5.4.** If $\alpha(s)$ is as in (3.1), $\gamma(s)$ is as in (4.1), and $\xi(s)$ is as in (1.4), we have

$$\xi(s) = 2 - \frac{\gamma(s)}{1 - s} = \frac{1 - \alpha(s)}{1 - s}. \tag{5.8}$$
By (5.11), we also have
\[
\begin{align*}
\frac{-s-u}{1-s+u} & \leq \liminf_{k \to \infty} \frac{\log |(\phi^{-1})'(w_k)|}{-\log \text{dist}(w_k, \partial D)} \\
& \leq \limsup_{k \to \infty} \frac{\log |(\phi^{-1})'(w_k)|}{-\log \text{dist}(w_k, \partial D)} & \leq \frac{-s+u}{1-s-u},
\end{align*}
\]
and
\[
\limsup_{k \to \infty} \frac{\log |w_k - x|}{-\log \text{dist}(w_k, \partial D)} \leq \frac{1-s-u}{1-s+u}.
\]

Proof. Let \( x \in \Theta^\nu(D) \). For \( \epsilon > 0 \), put \( z_\epsilon = \phi((1-\epsilon)\phi^{-1}(x)) \). By the Koebe quarter theorem, we have
\[
\text{dist}(z_\epsilon, \partial D) \geq \epsilon |\phi'((1-\epsilon)\phi^{-1}(x))|,
\]
with proportionality constants 1/4 and 4 in each \( x \). Clearly,
\[
(\phi^{-1})'(z_\epsilon) = \frac{1}{\phi'((1-\epsilon)\phi^{-1}(x))}.
\]

Now let \( \delta > 0 \). By [JVL12, Proposition 2.7] we have
\[
\limsup_{\epsilon \to 0} \frac{\log \epsilon^{1-s-u}}{\log v(x; \epsilon)} \leq 1,
\]
where \( v(x; \epsilon) \) is the length of the image of the curve \( t \mapsto z_\epsilon \) for \( t \in [0, \epsilon] \). Consequently, for sufficiently small \( \epsilon > 0 \) we have
\[
|z_\epsilon - x| \leq \epsilon^{1-s-u-\delta}.
\]

By assumption, for sufficiently small \( \epsilon > 0 \) we have \( |\phi'((1-\epsilon)\phi^{-1}(x))| \leq \epsilon^{s-u-\delta} \), so by (5.9), \( \text{dist}(z_\epsilon, \partial D) \leq \epsilon^{1-s-u-\delta} \) and by (5.10), \( |(\phi^{-1})'(z_\epsilon)| \geq \epsilon^{s+u+\delta} \). Furthermore, for any \( k \in \mathbb{N} \), we can find \( \epsilon_k \) such that \( |\phi'((1+\epsilon_k)\phi^{-1}(x))| \geq \epsilon_k^{-s+u+\delta} \). For each such \( k \), we have \( \text{dist}(z_{\epsilon_k}, \partial D) \geq \epsilon_k^{1-s+u+4} \) and \( |(\phi^{-1})'(z_{\epsilon_k})| \leq \epsilon_k^{-s-u-\delta} \). Hence
\[
\liminf_{k \to \infty} \frac{\log |(\phi^{-1})'(z_{\epsilon_k})|}{-\log \text{dist}(z_{\epsilon_k}, \partial D)} \geq \frac{-s-u-\delta}{1-s+u+\delta}.
\]
and
\[
\limsup_{k \to \infty} \frac{\log |(\phi^{-1})'(z_{\epsilon_k})|}{-\log \text{dist}(z_{\epsilon_k}, \partial D)} \leq \frac{-s+u+\delta}{1-s-u-\delta}.
\]

By (5.11), we also have
\[
\limsup_{k \to \infty} \frac{\log |z_{\epsilon_k} - x|}{-\log \text{dist}(z_{\epsilon_k}, \partial D)} \leq \frac{1-s-u-\delta}{1-s+u+\delta}.
\]
Since \( \delta \) is arbitrary, by combining (5.12), (5.14), and (5.13) we obtain the statement of the lemma with \( w_k = z_{\epsilon_k} \).

Proof of Lemma 5.3. The statement for \( s \in [s_-, s_+] \) follows from the analogous statement in Proposition 5.1, so we henceforth assume \( s \in [s_-, s_+] \).

By countable stability of Hausdorff dimension \( \xi(s) \), to prove (5.7), it is enough to show that a.s. \( \mathcal{H}^\beta(\Theta^\nu(D_n) \cap B_0(0)) = 0 \) for each \( \beta > \xi(s) + o_\mu(1) \), and each \( d \in (0, 1) \). It is moreover enough to prove the result restricted to the event \( \mathcal{G}(\Psi_\mu, \mu) \cap \mathcal{G}(\Psi_\mu, \mu) \) (in the notation of Theorem 4.1) for an arbitrary choice of \( \mu \in \mathcal{M} \).

Let
\[
r > \frac{1-s-u}{1-s+u}.
\]
Note that we can take \( r = 1 + o_\mu(1) \). For \( n \in \mathbb{N} \) let \( D^n = 2^{-n(1-s)-4} \mathbb{Z}^2 \) be the dyadic lattice of mesh size \( 2^{-n(1-s)-4} \). For \( z \in D^n \), let \( B^n_0(z) \), \( B^n_1(z) \), \( B^n_2(z) \), and \( B^n_3(z) \) be the disks centered at \( z \) of radii \( 2^{-n(1-s)-4} \), \( 2^{-n(1-s)-2} \), \( 2^{-n(1-s)+2} \), and \( 2^{-n(1-s)+4} \), respectively.

Define \( \Psi_\mu \) as in Section 4.1. For \( z \in D \) let \( E^n(z) \) be the event that the following occurs.

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1. \( \eta \cap B^u_2(z) \neq \emptyset \).
2. \( \eta \cap B^u_1(z) = \emptyset \).
3. There is a \( w \in B^0_1(z) \) with \( 2^{-n(s+2u)} \leq |\Psi'_\eta(w)| \leq 2^{-n(s-2u)} \).

On \( E^n(z) \), we have
\[
2^{-n(1-s)} \leq \text{dist}(z, \partial D_\eta) \leq 2^{-n(1-s)}, \quad 2^{-n(s+2u)} \leq |\Psi'_\eta(z)| \leq 2^{-n(s-2u)},
\]
with constants uniform in \( B_d(0) \) (the inequality for \( |\Psi'_\eta| \) follows from the Koebe distortion theorem). So, by Proposition 5.1
\[
P(E^n(z) \cap G(\Psi_\eta, \mu) \cap G(\Psi'_\eta, \mu)) \leq 2^{-n(\gamma_{s})-2\gamma_{o}(s)u}) \quad (5.15)
\]
with constants uniform in \( B_d(0) \).

Let \( U^n \) be the set of disks \( B^u_2(z) \) for \( z \in D^n \) such that \( z \in B_d(0) \) and \( E^n(z) \) occurs. Note that the cardinality of the set of disks which can belong to \( U^n \) is of order \( 2^{2n(1-s)} \). We claim that \( \Theta^{\gamma_{s}}(D_\eta) \cap B_d(0) \subset \bigcup_{n \geq N} \bigcup_{B^u_2(z) \in U^n} B^u_2(z) \) for each \( N \in \mathbb{N} \).

Indeed, suppose \( x \in \Theta^{\gamma_{s}}(D_\eta) \cap B_d(0) \). By Lemma 5.3, for any \( \delta > 0 \), we can find a sequence \( n_k \to \infty \) and a sequence of points \( w_k \in D_\eta \) converging to \( x \) such that for each \( k \), \( 2^{-n_k(1-s)} \leq \text{dist}(w_k, \partial D_\eta) \leq 2^{-n_k(1-s)}, \quad |w_k-x| \leq 2^{-n_k(1-s)r}, \quad 2^{-n_k(s+2u)} \leq |\Psi'_\eta(w_k)| \leq 2^{-n_k(s-2u)} \).

Each \( w_k \) belongs to \( B^u_2(z) \) for some \( z \in D^n \). Our hypothesis on the distance from \( w_k \) to \( \partial D_\eta \) implies that conditions 1 and 2 in the definition of \( E^n(z) \) hold for this \( z \). Clearly, condition 3 also holds for this \( z \). Thus for such a \( z \), \( E^n(z) \) holds and we have \( x \in B^u_2(z) \) (here we use the condition on \( |w_k-x| \)). This proves our claim.

Thus, for any \( n \in \mathbb{N} \), \( \bigcup_{n \geq m} U^n \) is a cover of \( \Theta^{\gamma_{s}}(\partial D_\eta) \cap B_d(0) \). Each set in this cover has diameter \( \leq 2^{-n(1-s)r} \) and we have by (5.15) that
\[
E\left(1_{G(\Psi_\eta, \mu) \cap G(\Psi'_\eta, \mu)} \sum_{n \geq m} \sum_{U \in U_n} \left(\text{diam} B^n(z)\right)^\beta \right) \leq \sum_{n \geq m} \sum_{z \in D^n \cap B_d(0)} 2^{-n(1-s)r} \sum_{n \geq m} \sum_{U \in U_n} \left(\text{diam} B^n(z)\right)^\beta \leq \sum_{n \geq m} 2^{2n(1-s)r} 2^{-n(1-s)r} 2^{-n(\gamma_{s})-2\gamma_{o}(s)u}) \quad (5.16)
\]
This tends to 0 as \( m \to \infty \) provided
\[
\beta > 2 - \frac{\gamma_{s}}{(1-s)r} = \xi(s) + o_s(1),
\]
where the \( o_s(1) \) can be taken to be uniform for \( s \) in compact subsets of \((-1, 1)\). Since \( \mu \) is arbitrary we conclude that \( \mathcal{H}^\beta(\Theta^{\gamma_{s}}(\partial D_\eta) \cap B_d(0)) = 0 \) for any such \( \beta \). \( \square \)

From Lemma 5.3, we can deduce the upper bounds on \( \dim_{\mathcal{H}} \Theta^{\gamma_{s}}(D_\eta) \) and \( \dim_{\mathcal{H}}(\Theta^{\gamma_{s}}(D_\eta)) \) in Theorem 1.1

**Proposition 5.6.** Suppose we are in the setting of Theorem 1.1. Then a.s.
\[
\dim_{\mathcal{H}} \Theta^{\gamma_{s}}(D_\eta) \leq \xi(s), \quad \frac{K}{4} \leq s \leq s_+.
\]
\[
\dim_{\mathcal{H}} \Theta^{\gamma_{s}}(D_\eta) \leq \xi(s), \quad s_- \leq s \leq \frac{K}{4}.
\]

**Proof.** For \( s \leq \kappa/4 \) and any \( n \in \mathbb{N} \), we have
\[
\Theta^{\gamma_{s}}(D_\eta) \subset \bigcup_{j=m_0}^{m_1} \Theta^{-j/n:1/n}(D_\eta), \quad (5.17)
\]
where \( m_0 \) is the greatest integer such that \( m_0/n \) is smaller than \( s_- \) and \( m_1 \) is the least integer such that \( m_1/n \geq s \). We have that \( \xi(s) \) is increasing on \([0, \kappa/4]\) and \( \gamma_{o}(s) \) is uniformly bounded for \( s \in [s_-, s_+] \) (and for \( s \leq s_1 \) for some \( s_1 < 1 \) in the case \( \kappa = 4 \)). Our upper bound in the case \( s \leq \kappa/4 \) thus follows from Lemma 5.3 and (5.17) together with stability of Hausdorff dimension under unions. Similarly for the case \( s \geq \kappa/4 \). \( \square \)
### 5.3 Upper bound for the integral means spectrum

In this subsection we will prove the upper bound for the bulk integral means spectrum of the SLE curve in Corollary 1.8. In light of Lemma 2.17, it will be enough to prove an upper bound for the bulk integral means spectrum of $D \setminus \eta^i$ for given $t \geq 0$ in the case of an ordinary SLE$_\kappa$ from $-i$ to $i$ in $D$ for $\kappa \leq 4$.

**Proposition 5.7.** Let $\kappa \in (0, 4]$ and let $\text{IMS}^*(a)$ be defined as in Corollary 1.8. Let $\eta$ be a chordal SLE$\kappa$ from $-i$ to $i$ in $D$. For each $t > 0$ and each $a \in \mathbb{R}$, a.s. $\text{IMS}^*_{\text{bulk}}(a) \leq \text{IMS}^*(a)$.

**Proof.** Let $(f_t)$ be the centered Loewner maps for $\eta$, as defined in Section 3.6. For $\delta > 0$, let $U_t(\delta)$ be the set of $z \in D \setminus \eta^i$ with $1 - |f_t^{-1}(z)| \geq \delta$ and $|z - i|, |z + i| \geq \delta$. Also define the sets $A_k'(f_t^{-1})$ as in Section 3.6 (immediately following (1.10)). For any given $\zeta > 0$ there a.s. exists (random) $\delta > 0$ such that $A_k'(f_t^{-1}) \subset \partial B_{1-\epsilon}(0) \cap U_t(\delta)$ for sufficiently small $\epsilon$. Therefore, it is enough to show that for each $\delta > 0$ and each $\beta > \text{IMS}^*(a)$, we a.s. have

$$\limsup_{\epsilon \to 0} \frac{\log \int_{\partial B_{1-\epsilon}(0) \cap U_t(\delta)} |(f_t^{-1})'(z)|^a \, dz}{-\log \epsilon} \leq \beta.$$  

(5.18)

Fix $\delta > 0$ and $\beta > \text{IMS}^*(a)$ as above. Also fix $t > 0$ and $c > 2$ and define the events $E^{s_0^i}(z) = E^{s_0^i}(z, t, c, 1-\delta)$ as in Section 3.6 with $d = 1-\delta$. Let $s_-$ and $s_+$ be as in the statement of Theorem 1.1. For $n \in \mathbb{N}$ and $k \in \{0, \ldots, n\}$, let

$$u_n = \frac{s_+ - s_-}{n}, \quad s_k^n = s_0 + ku_n.$$

For $n \in \mathbb{N}$ and $\epsilon > 0$, let $A^n_\epsilon(-)$ (resp. $A^n_\epsilon(+)$) be the set of $z \in \partial B_{1-\epsilon}(0) \cap U_t(\delta)$ such that $|(f_t^{-1})'(z)| \leq \epsilon^{-s_- + 1/n}$ (resp. $|(f_t^{-1})'(z)| \geq \epsilon^{-s_+ - 1/n}$). For $k \in \{0, \ldots, n\}$, let $A^n_\epsilon(k)$ be the set of $z \in \partial B_{1-\epsilon}(0) \cap U_t(\delta)$ such that the $E^{s_k^n}_{\text{bulk}}(z)$ occurs. Let $\ell^n_\epsilon(k)$ be the Lebesgue measure of $A^n_\epsilon(k)$ and let $\ell^n_\epsilon(\pm)$ be the Lebesgue measure of $A^n_\epsilon(\pm)$.

In what follows, we require implicit constants to be independent of $\epsilon$, but not of $n$ or $k$, and we denote by $o_n(1)$ a term which tends to 0 as $n \to \infty$ and does not depend on $k$ or $\epsilon$.

By construction, we have $\partial B_{1-\epsilon}(0) \cap U_t(\delta) = A^n_\epsilon(-) \cup A^n_\epsilon(+) \cup \bigcup_{k=0}^n A^n_\epsilon$, whence

$$\int_{\partial B_{1-\epsilon}(0) \cap U_t(\delta)} |(f_t^{-1})'(z)|^a \, dz \leq \sum_{k=0}^n \epsilon^{-as_k^n + o_n(1)} \ell^n_\epsilon(k) + \epsilon^{-as_-} \ell^n_\epsilon(-) + \epsilon^{-as_+} \ell^n_\epsilon(+) + \epsilon^{-as_0} \ell^n_\epsilon(0).$$

The proof of Lemma 5.1 shows that for each $n \in \mathbb{N}$, there a.s. exists a random $c_0^n > 0$ such that for $\epsilon < c_0^n$, the sets $A^n_\epsilon(-)$ and $A^n_\epsilon(+)\|$ are empty. Hence for $\epsilon < c_0^n$, we have

$$\int_{\partial B_{1-\epsilon}(0) \cap U_t(\delta)} |(f_t^{-1})'(z)|^a \, dz \leq \sum_{k=0}^n \epsilon^{-as_k^n} \ell^n_\epsilon(k).$$  

(5.19)

By Corollary 3.16 for $k \in \{0, \ldots, n\}$, we have

$$\mathbb{E}(\ell^n_\epsilon(k)) \leq \epsilon^{\alpha(s_k^n) + o_n(1)}.$$  

By Chebyshev’s inequality,

$$\mathbb{P}\left(\epsilon^{-as_k^n} \ell^n_\epsilon(k) > \epsilon^{-\beta}\right) \leq \epsilon^{\alpha(s_k^n) - as_k^n + \beta + o_n(1)}.$$  

(5.20)

We have

$$\inf_{s \in [s_-, s_+]} \alpha(s_k^n) - as_k^n = -\text{IMS}^*(a).$$  

(5.21)

Note that the range $[s_-, s_+]$ in Corollary 1.8 is precisely the set of $a \in \mathbb{R}$ for which the minimizer in (5.21) is not equal to $s_-$ or $s_+$. It follows that for sufficiently large $n \in \mathbb{N}$, depending only on $\beta$, we have

$$\mathbb{P}\left(\sup_{k \in \{0, \ldots, n\}} \epsilon^{-as_k^n} \ell^n_\epsilon(k) > \epsilon^{-\beta}\right) \leq \epsilon^{\beta - \text{IMS}^*(a) + o_n(1)}.$$  

(5.22)
Since $\beta > \text{IMS}^*(a)$, if $n \in \mathbb{N}$ is chosen sufficiently large (depending only on $\beta$ and $a$), then the Borel-Cantelli lemma together with (5.19) implies that a.s.

$$
\int_{\partial B_{1-\epsilon} (0) \cap U_i (\delta)} | (f_t^{-1})' (z) |^\alpha \, dz \leq 2^{-j\beta}
$$

for sufficiently large $j \in \mathbb{N}$. By the Koebe distortion theorem, it follows that a.s.

$$
\limsup_{\epsilon \to 0} \frac{\log \int_{\partial B_{1-\epsilon} (0) \cap U_i (\delta)} | (f_t^{-1})' (z) |^\alpha \, dz}{-\log \epsilon} \leq \beta.
$$

This proves (5.18), and hence the statement of the proposition. \qed

6 Two point estimate

6.1 Event at the hitting time

In this subsection we introduce an event which will serve as the basic building block for the “perfect points” which we will use to prove our lower bounds on the Hausdorff dimensions of $\Theta^s (D_\eta)$ and $\tilde{\Theta}^s (D_\eta)$.

6.1.1 Definition of the event

Suppose $\eta : [0, \infty] \to \overline{D}$ is a simple curve in $D$ which connects distinct points $x, y \in \partial D$ and does not otherwise hit $\partial D$. We recall the notation

$$
\eta^t = \eta ([0, t]), \quad \eta = \eta ([0, \infty])
$$

from Section 2.1.

Throughout this section, for $\beta > 0$, we will write

$$
B_\beta := B_{e^{-\beta}} (0).
$$

Let $\eta$ be the time reversal of $\eta$. Fix parameters $\beta > 0, u, a, c > 0, v \geq 0, q \in (-1/2, \infty)$, and $\mu \in \mathcal{M}$.

The parameter $q$ corresponds to $s/(1 - s)$, for $s$ the parameter of Theorem 1.1.

Let $E = E^{q, u}_\beta (\eta; a, c, \mu)$ be the event that the following holds.

1. Let $\tau_\beta$ (resp. $\tau_\beta'$) be the first time that $\eta$ (resp. $\eta'$) hits $\partial B_\beta$. Then $\tau_\beta, \tau_\beta' < \infty$.

2. The harmonic measure from 0 in $D \setminus (\eta^{\tau_\beta} \cup \eta'^{\tau_\beta})$ of each of the two sides of $\eta^{\tau_\beta}$ and each of the two sides of $\eta'^{\tau_\beta}$ is at least $a$.

3. Let $\phi_\beta : D \setminus (\eta^{\tau_\beta} \cup \eta'^{\tau_\beta}) \to D$ be the unique conformal transformation which takes $x$ to $-i$, $y$ to $i$, and the midpoint $m$ of $[x, y]_{\partial D}$ to 1. Then we have that $e^{-\beta (u + v)} \leq | \phi' (0) | \leq e^{-\beta (u - v)}$.

4. $G' (\eta^{\tau_\beta} \cup \eta'^{\tau_\beta}, \mu)$ occurs (Definition 2.6).

We will be primarily interested in the case where $\eta$ is a chordal $\text{SLE}_\kappa$ process from $x$ to $y$. In this case, we write $F_\beta$ for the $\sigma$-algebra generated by $\eta|_{[0, \tau_\beta]}$ and $\eta'|_{[0, \tau_\beta]}$.

6.1.2 Setup and upper bound

We need to estimate the probability of the event $E$ of the preceding subsection. To this end we will prove the following.

**Proposition 6.1.** Let $x, y \in \partial D$. Let $\eta$ be a chordal $\text{SLE}_\kappa$ process from $x$ to $y$ in $D$. Let $E = E^{q, u}_\beta (\eta; a, c, \mu)$ be as in Section 6.1. Let $\gamma$ and $\gamma_0$ be as in (4.1) and

$$
\gamma^* (q) := (q + 1) \gamma \left( \frac{q}{1 + q} \right) = \frac{8\kappa + 8\kappa q + (4 - \kappa)^2 q^2}{8(\kappa + 2\kappa q)}.
$$

(6.1)
There exists a continuous function $\gamma_0^\beta : (-1/2, \infty) \to (0, \infty)$ (with $\gamma_0^\beta(q)$ depending only on $q$) such that the following is true for sufficiently small $u > 0$ (depending only on $q$). We have
\[
P(\mathcal{E}) \leq e^{-\beta(\gamma^*(q) + \gamma_0^\beta(q)u)}.
\] (6.2)
Moreover, there exists $\mu \in \mathcal{M}$ depending only on $z$ but uniform for $z$ in compacts such that for each $q \in (-1/2, \infty)$, each $\beta > 0$, and each $a, u, c > 0$ with $\beta u \geq 100$,
\[
P(\mathcal{E}) \geq e^{-\beta(\gamma^*(q) + \gamma_0^\beta(q)u)}.
\] (6.3)
The implicit constants are independent of $\beta$, uniform for $z$ in compacts, and uniform in $x, y$ provided $|x - y|$ is bounded below by a positive constant.

First we will prove the upper bound (6.2), which is a straightforward consequence of Theorem 4.1.

Proof of Proposition 6.1, upper bound. Let $\hat{\eta}$ be the image under $\phi_\beta$ of the part of $\eta$ lying between $\eta(\tau_\beta)$ and $\eta(\tau_\beta)$. Let $\hat{x} = \phi_\beta(\eta(\tau_\beta))$ and $\hat{y} = \phi_\beta(\eta(\tau_\beta))$, so that the conditional law of $\hat{\eta}$ given $\mathcal{F}_\beta$ is that of an SLE$_\kappa$ from $\hat{x}$ to $\hat{y}$ in $\hat{D}$. Let $C > 1$. Let $E = E(C)$ be the event that the following occurs.

1. $\hat{\eta}$ does not exit $\phi_\beta(B_1)$.
2. Define the domain $D_{\hat{\eta}}$ as in Section 4.1. Then $\phi_\beta(0) \in D_{\hat{\eta}}$ and $C^{-1}(1 - |\phi_\beta(0)|) \leq \text{dist}(\phi_\beta(0), \partial D_{\hat{\eta}}) \leq C(1 - |\phi_\beta(0)|)$.
3. Let $\Phi_{\hat{\eta}} : D_{\hat{\eta}} \to D$ be the conformal map fixing $-i, i, and 1$. Then $C^{-1} \leq |\Phi'_{\hat{\eta}}(\phi_\beta(0))| \leq C$.

It follows from condition 2 in the definition of $E$ and [MW14, Lemma 2.3] that we can find a $C > 0$ depending only on $a$ such that for sufficiently large $\beta$, $P(\hat{E}|E) \geq 1$. Thus
\[
P(E) \propto P(E \cap \hat{E}).
\] (6.4)
So, it will suffice to prove an upper bound for $P(E \cap \hat{E})$.

Let $s \in (-1, 1)$ and $\epsilon > 0$ be chosen so that
\[
\frac{s}{1 - s} = q, \quad e^{1-s} = e^{-\beta}.
\] (6.5)
Let $D_{\eta}$, $\Psi_{\eta}$, $\Psi_{\eta}^-$, and $\mathcal{E}^{\epsilon_-}_{x^0}(\eta, 0; c)$ be as in Section 4.1. It follows from Lemma 2.8 and condition 4 in the definition of $E$ that
\[
E \subset \mathcal{G}(\phi_\beta, \mu')
\] (6.6)
for some $\mu' \in \mathcal{M}$ depending only on $\mu$. By combining this with condition 1 in the definition $\hat{E}$ we see that $E \cap \hat{E} \subset \mathcal{G}(\Psi_{\eta}, \mu') \cap \mathcal{G}(\Psi_{\eta}^-, \mu')$ for some (possibly smaller) $\mu' \in \mathcal{M}$ depending only on $\mu$. We furthermore have
\[
\Psi_{\eta} = \Psi_{\hat{\eta}} \circ \phi_\beta.
\]
Hence we have
\[
E \cap \hat{E} \subset \mathcal{E}^{\epsilon_-}_{x^0}(\eta, 0; c) \cap \mathcal{G}(\Psi_{\eta}, \mu') \cap \mathcal{G}(\Psi_{\eta}^-, \mu')
\]
for suitable choice of $\mu'$ and $c$. Thus (6.2) follows from (6.4) and the upper bound in Theorem 4.1. Note that we can take the dependence on $u$ to be linear (with slope depending on $q$) since the exponent in the upper bound in Theorem 4.1 depends smoothly on $s \in (-1, 1)$ and $u > 0$ sufficiently small.
6.1.3 Lower bound

The proof of the lower bound in Proposition 6.1 will take substantially more work than the proof of the upper bound. The basic idea is to stop η and ⃗η at times t and ⃗t for which the following is true. On the event $E_\beta^{s,u}(\cdot)$ of Theorem 4.1, the conformal map from $D \setminus (\eta^0 \cup \overline{\eta^0})$ to $D$ which takes $x$ to $-i$, $y$ to $i$, and the midpoint of $[x, y]_{D}$ to $1$ has the same derivative behavior as the conformal map $\Psi_\beta : D \setminus \eta \to D$ with the same normalization; the points $\eta(t_0)$ and $\overline{\eta(t_0)}$ are at distance slightly less than $e^{-\beta}$ from $0$; and the law of the remainder of the curve given $\eta^0 \cup \overline{\eta^0}$ has the law of a chordal SLE$_{\kappa_0}$. We also need to require that $\eta(t_0)$ and $\overline{\eta(t_0)}$ are sufficiently far apart in a conformal sense, so that they do not immediately link up after times $t_0$ and $\overline{t_0}$. We then condition on $\eta^0 \cup \overline{\eta^0}$ and use standard arguments to get that the curves reach $B_\beta$ without any pathological behavior. The main difficulty in the proof is constructing the times $t_0$ and $\overline{t_0}$.

We start by inductively defining a means of growing $\eta$ and $\overline{\eta}$ simultaneously to get an increasing family of hulls $K_i \subset D$. Assume $\eta$ (resp. $\overline{\eta}$) is parametrized in such a way that its image under the conformal map $D \to H$ taking $-i$ to $0$, $i$ to $\infty$, and $0$ to $i$ (resp. the reciprocal of this conformal map) is parametrized by half plane capacity. Let $\sigma_1$ be the first time $t$ that $\lim_0^0(\eta^0; D \setminus (\eta^0 \cup \overline{\eta^0})) = 1/2$. This time is a.s. finite since a Brownian motion started from $0$ has probability at least $1/2$ to hit $\eta$ before $\partial D$. For $t \leq \sigma_1$, let $K_t = \eta^t$. Let $\sigma_1$ be the first $\tilde{t}$ that either $\lim_0^0(\eta^0; D \setminus (\eta^0 \cup \overline{\eta^0})) = 1/2$ or $\overline{\eta(\tilde{t})} = \eta(\sigma_1)$. For $t \in [\sigma_1, \sigma_1 + \sigma_1]$ let $K_t = \eta^0 \cup \overline{\eta^0}$.

Inductively, suppose $n \geq 2$ and $\sigma_{n-1}, \overline{\sigma}_{n-1}$, and $K_t$ for $t \leq \sigma_{n-1} + \overline{\sigma}_{n-1}$ have been defined. If $K_{\sigma_{n-1} + \overline{\sigma}_{n-1}} = \eta$ we let $\sigma_n = \sigma_{n-1}$ and $\overline{\sigma}_n = \overline{\sigma}_{n-1}$. Otherwise, let $\sigma_n$ be the least $t \geq \sigma_{n-1}$ such that $\lim_0^0(\eta^0; D \setminus (\eta^0 \cup \overline{\eta^0} - 1)) = 1/2$ or $\eta(t) = \eta(\overline{\sigma}_{n-1})$. Let $K_t = \eta^0 \setminus \overline{\eta^0} - 1 \cup \overline{\eta^0} - 1$ for $t \leq [\sigma_{n-1} + \overline{\sigma}_{n-1}, \sigma_n + \overline{\sigma}_n]$. Let $\overline{\sigma}_n$ be the first time $\tilde{t} \geq \sigma_{n-1}$ such that $\lim_0^0(\eta^0; D \setminus (\eta^0 \cup \overline{\eta^0})) = 1/2$ or $\overline{\eta(\tilde{t})} = \eta(\sigma_n)$. Let $K_t = \eta^0 \setminus \overline{\eta^0}$ for $t \in [\sigma_n + \overline{\sigma}_n - 1, \sigma_n + \overline{\sigma}_n]$.

For each $t \geq 0$, let $T_t$ (resp. $\overline{T}_t$) be the time such that $\eta(T_t)$ (resp. $\overline{\eta(T_t)}$) is the tip of the part of $\eta$ (resp. $\overline{\eta}$) included in $K_t$. Observe that the Markov property and reversibility of SLE imply that for each $t$, the conditional law of $\eta \setminus K_t$ given $K_t$ is that of a chordal SLE$_{\kappa_0}$ from $\eta(T_t)$ to $\overline{\eta(T_t)}$ in $D \setminus K_t$.

Lemma 6.2. Let $\sigma_\infty = \lim_{n \to \infty} \sigma_n$ and $\overline{\sigma}_\infty = \lim_{n \to \infty} \overline{\sigma}_n$ (the limits necessarily exist by monotonicity). Let $K_\infty = \eta^0 \cup \overline{\eta^0}$. Then we a.s. have
\[
\lim_{n \to \infty} \lim_{\eta^0; D \setminus (K_{\sigma_n + \overline{\sigma}_n})} = \lim_{n \to \infty} \lim_{n \to \infty} (\eta^0; D \setminus (K_{\sigma_n + \overline{\sigma}_n})) = \lim_{n \to \infty} (\eta^0; D \setminus K_\infty)
\]
and
\[
\lim_{n \to \infty} \lim_{n \to \infty} (\overline{\eta^0}; D \setminus (K_{\sigma_n + \overline{\sigma}_n})) = \lim_{n \to \infty} \lim_{n \to \infty} (\overline{\eta^0}; D \setminus K_\infty).
\]

Proof. We a.s. have $0 \notin \eta$ so it is the a.s. case that for each $\epsilon > 0$, we can find a random $\delta > 0$ such that for any $z \in \partial D$, the probability that a Brownian motion started from $0$ hits $B_\delta(z)$ before leaving $D$ is at most $\epsilon$. By a.s. continuity of $\eta$, we can a.s. find a (random) $N \in \mathbb{N}$ such that for $n \geq N$, $\eta([\sigma_n, \sigma_\infty]) \subset B_\delta(\eta(\overline{\sigma}_\infty))$. Hence with probability at least $1 - \epsilon$, a Brownian motion started from $0$ exists $D \setminus K_{\sigma_n + \overline{\sigma}_n}$ in the same place it exits $D \setminus K_\infty$. This proves the limits involving $K_{\sigma_n + \overline{\sigma}_n}$. The limits involving $K_{\sigma_n + \overline{\sigma}_n - 1}$ are proven similarly.

Lemma 6.3. We a.s. have $K_\infty = \eta$. Let $z_\infty = \eta(\overline{\sigma}_\infty) = \overline{\eta(\sigma_\infty)}$ be the meeting point. On the event that $0$ lies to the right of $\eta$ and dist$(0, \eta) \leq e^{-\beta}$, it holds a.s. that $\lim_{n \to \infty} (\eta^0; D_n)$ and $\lim_{n \to \infty} (\overline{\eta^0}; D_n)$ are each at least $1/2 - o_\beta(1)$, where $o_\beta(1)$ is a deterministic quantity which tends to $0$ as $\beta \to 0$.

Proof. First we argue that $K_\infty = \eta$. Suppose not. Almost surely, either $\lim_{n \to \infty} (\eta^0; D \setminus K_\infty)$ or $\lim_{n \to \infty} (\overline{\eta^0}; D \setminus K_\infty)$ is $< 1/2$. Suppose $\lim_{n \to \infty} (\eta^0; D \setminus K_\infty) < 1/2$. The other case is treated similarly. By Lemma 6.2 we a.s. have $\lim_{n \to \infty} (\eta^0; D \setminus (K_{\sigma_n + \overline{\sigma}_n - 1}) < 1/2$ for sufficiently large $n$. By definition of $\sigma_n$ this can be the case only if $\eta(\sigma_n) = \overline{\eta(\sigma_{n-1})}$ which implies $K_\infty = \eta$.

It is immediate from Lemma 6.2 that $\lim_{n \to \infty} (\eta^0; D_n)$ and $\lim_{n \to \infty} (\overline{\eta^0}; D_n)$ are each at most $1/2$. Furthermore, the Beurling estimate implies $\lim_{n \to \infty} (\partial D; D_\eta) = o_\beta(1)$. Hence
\[
\lim_{n \to \infty} (\eta^0; D_\eta) = 1 - \lim_{n \to \infty} (\eta^0; D_n) - \lim_{n \to \infty} (\partial D; D_\eta) \geq 1/2 - o_\beta(1)
\]
and similarly for $\overline{\eta^0}$.
we require all implicit constants to be deterministic and depend only on $\mu$. Let $\eta$ be the first time $\Psi(\eta, \mu) \cap \{\dist(0, \eta) \leq e^{-\beta}\} \cap \{0 \in D_\eta\}$, there a.s. exists a time $\tau > 0$ such that the following is true.

1. $\dist(0, K_\tau) \leq C \dist(0, \eta)$.
2. $C^{-1}|\Psi_\eta'(0)| \leq |\Phi_\tau'(0)| \leq C|\Psi_\eta'(0)|$.
3. $\Phi_\tau(\eta(T_\tau))$ and $\Phi_\tau(\eta(T_\tau))$ lie in $[i, -i]_{\partial D}$.
4. $\dist^0(\eta \setminus K_\tau, D_\eta) \geq 1/4 - \alpha_\beta(1)$. Here the $\alpha_\beta(1)$ is deterministic and depends only on $\beta$.

Proof. Throughout, we assume we are working on the event $G(\Psi_\eta, \mu) \cap \{\dist(0, \eta) \leq e^{-\beta}\} \cap \{0 \in D_\eta\}$ and we require all implicit constants to be deterministic and depend only on $\mu$.

Let $\Psi_\eta : D_\eta \to D$ be the conformal map which fixes 0 and takes $m$ to 1. If $z_\infty$ is as in Lemma 6.3 then by conformal invariance of harmonic measure we have

$$|\Psi_\eta(z_\infty) + 1| = o_\beta(1). \tag{6.7}$$

Let $\tau$ be the first time $t$ that $\Phi_\eta(\eta(T_i))$ and $\Phi_\eta(\eta(T_\tau))$ are both in $[i, -i]_{\partial D}$. By Lemma 6.3 such a $t$ necessarily exists provided $\beta$ is large enough. Let $A_\tau = [\Phi_\eta(\eta(T_{\tau})), \Phi_\eta(\eta(T_{\tau}))]_{\partial D}$. By continuity one of the two endpoints of $A_\tau$ is $-i$ or $i$ so by (6.7) we have $\dist^0(\eta, D_\eta) \geq 1/4 - o_\beta(1)$. Furthermore, the harmonic measure from 0 in $D$ of each of the two arcs connecting $A_\tau$ and 1 is at least $1/4 - o_\beta(1)$.

Let $A_\tau = \Phi_\eta^{-1}(A_\tau) = \eta \setminus K_\tau$. By conformal invariance of harmonic measure we have that $\dist^0(\eta(T_\tau); D_\eta)$, $\dist^0(\eta(T_{\tau}); D_\eta)$, and $\dist^0(\eta(T_{\tau}); D_\eta)$ are each at least $1/4 - o_\beta(1)$. By Lemma 6.4 we have $\dist(0, K_\tau) = \dist(0, \eta)$ and $|\Phi_\tau(0)| \geq |\Psi_\eta(0)|$. Since $\Phi_\eta(\eta(T_\tau))$ and $\Phi_\eta(\eta(T_{\tau}))$ lie in $[i, -i]_{\partial D}$ and removing $A_\tau$ can only increase the harmonic measure from 0 of parts of $\partial D_\eta$ outside of $A_\tau$. $\Phi_\tau(\eta(T_{\tau}))$ and $\Phi_\tau(\eta(T_{\tau}))$ must lie in $[i, -i]_{\partial D}$. Thus, the conditions of the lemma hold for this choice of $\tau$.

**Lemma 6.5.** Let $v > 0$, $\zeta > 0$, and $\mu_0 \in \mathcal{M}$. For two times $t, \tilde{t} > 0$, let $E_0(t, \tilde{t}) = E_0(t, \tilde{t}, v, \zeta, \mu_0)$ be the event that the following occurs.

---

Figure 6.1: An illustration of the argument of Lemma 6.5 in the case $\{T' < \infty\}$. The hull $K_{t_0}$ is shown in black, the curve $\eta'$ and its pre-image under $\Phi_{t_0}$ are shown in blue. The extra part of the curve which we grow after growing $K_{t_0}$ is shown in orange.
1. $32e^{-\beta} \leq \text{dist}(z, \eta^t \cup \overline{\eta^t}) \leq e^{-\beta(1-v)}$.

2. Let $\phi_{t,\overline{t}} : D \setminus (\eta^t \cup \overline{\eta^t}) \to D$ be the conformal map which takes $x^+ \to -i$, $y^- \to i$, and the midpoint $m$ of $[x, y]_{|D}$ to 1. Then $e^{-\beta(q+v)} \leq |\phi'_{t,\overline{t}}(0)| \leq e^{-\beta(q-v)}$.

3. Let $\psi_{t,\overline{t}} : D \setminus (\eta^t \cup \overline{\eta^t}) \to D$ be the conformal map which fixes 0 and takes $m$ to 1. Then $|\psi_{t,\overline{t}}(\eta(t)) - \psi_{t,\overline{t}}(\overline{\eta(t)})| \geq \zeta$.

4. $G'(\eta^t \cup \overline{\eta^t}, \mu_0)$ occurs.

There is a $\zeta > 0$, a $\mu_0 \in \mathcal{M}$, and random times $t_0, \overline{t}_0$, all independent of $v$, such that

$$P(E_0(t_0, \overline{t}_0)) \geq e^{-\beta(\gamma^*(q)+\gamma_0(q)v)},$$

with the implicit constant independent of $\beta$ and uniform in $x, y$ with $|x - y|$ bounded below and $\gamma^*(q), \gamma_0(q)$ as in Proposition 6.1. Furthermore, we can choose $t_0$ and $\overline{t}_0$ in such a way that the conditional law given $\eta^t \cup \overline{\eta^t}$ of the part of $\eta$ between $\eta(t_0)$ and $\overline{\eta(t_0)}$ on the event $E_0(t_0, \overline{t}_0)$ is that of a chordal SLE$_\kappa$ from $\eta(t_0)$ to $\overline{\eta(t_0)}$ in $D \setminus (\eta^t \cup \overline{\eta^t})$.

**Proof.** Let $s \in (-1, 1)$, $v' \in (0, v)$, and $\epsilon > 0$ be chosen so that

$$s = \frac{q}{q+1} + o_v(1), \quad e^{1-s-2v'} \leq e^{-\beta(1-v)}, \quad e^{1-s+2v'} \geq 32e^{-\beta}.$$

Let $c > 0$ and let $E_{s,v'}(\eta, 0; c)$ be the event of Section 4.1 (with $v'$ in place of $u$). Let $\Psi_\eta : D_\eta \to D$ and $\Psi_{\overline{\eta}} : D_{\overline{\eta}} \to D$ be as in that subsection. Let $\mu' \in \mathcal{M}$ and let

$$\mathcal{E} := E_{s,v'}(\eta, 0; c) \cap G(\Psi_\eta, \mu') \cap G(\Psi_{\overline{\eta}}, \mu').$$

By Theorem 4.1 if the parameter $\mu'$ is chosen appropriately then we have

$$P(\mathcal{E}) \geq e^{-\beta(\gamma^*(q)+\gamma_0(q)v)},$$

for an appropriate choice of $\gamma_0(q)$. Lemma 2.8 implies that we can find $\mu_0 \in \mathcal{M}$ depending only on $\mu'$ such that

$$\bigcup_{t, \overline{t} \geq 0} G'(\eta^t \cup \overline{\eta^t}, \mu_0) \subset G(\Psi_\eta, \mu') \cap G(\Psi_{\overline{\eta}}, \mu').$$

Let $\tau_0$ be the first time $\tau$ that the first two conditions in the definition of $E_0(T_r, T_{\overline{r}})$ are satisfied and that $\overline{\Phi}_r(\eta(T_r))$ and $\Phi_r(\overline{\eta}(T_{\overline{r}}))$ (as defined just above Lemma 6.2) both lie in $[i, -i]_{|D}$. By Lemma 6.4 and the definition of $\mathcal{E}$, if $c$ is chosen sufficiently large then $\tau_0 < \infty$ a.s. on $\mathcal{E}$. Moreover, decreasing $\tau$ only increases $h^0(\eta \setminus K_r; D_\eta)$, so on $\mathcal{E}$ we a.s. have

$$h^0(\eta \setminus K_{\overline{\tau}}; D_\eta) \geq 1/4 - o_\beta(1).$$

Let $\eta' = \overline{\Phi}_{\tau_0}(\eta \setminus K_{\tau_0})$, with the parametrization it inherits from $\eta$. Observe that the conditional law of $\eta'$ given $K_{\overline{\tau}}$ is that of a chordal SLE$_\kappa$ from $x' := \overline{\Phi}_{\tau_0}(\eta(T_{\overline{\tau}}))$ to $y' := \overline{\Phi}_{\tau_0}(\overline{\eta}(T_{\overline{\tau}}))$ in $D$ (here we used that we made $\tau_0$ the smallest time for which our desired conditions are satisfied).

The definition of $E_0$ almost holds with $t_0 = T_{\tau_0}$ and $\overline{t}_0 = T_{\overline{\tau}}$, but $\Phi_{\tau_0}(\eta(T_{\tau_0}))$ and $\overline{\Phi}_{\tau_0}(\overline{\eta}(T_{\overline{\tau}}))$ may be too close together. To this end, we will choose slightly larger times at which the images of the tips of $\eta$ and $\overline{\eta}$ are separated. Note that (6.11) implies $\operatorname{diam} \eta' \geq \zeta_0$ on $\mathcal{E}$ for some universal constant $\zeta_0 \in (0, 1/4)$. Let $\eta'$ be the time reversal of $\eta'$, with the parametrization it inherits from $\eta$.

Let $T'$ (resp. $\overline{T}'$) be the first time $\eta'$ (resp. $\overline{\eta}'$) enters $B_{1-\zeta_0/4}(0)$. Let $T''$ be the first time $t$ that $\arg \eta'(t) \geq \arg x' + \zeta_0/8$. Let $\overline{T}''$ be the first time $t$ that $\arg \overline{\eta}'(t) \leq \arg y' - \zeta_0/8$ for some $t \leq \overline{T}$. Since $\operatorname{diam} \eta' \geq \zeta_0$ a.s. on $\mathcal{E}$, either $|x' - y'| \geq \zeta_0/8$ or one of $T', T''$ or $\overline{T}''$ is finite on this event (if not, then $\eta'$ is contained in the wedge $\{z \in D : \arg y' - \zeta_0/8 \leq \arg z \leq \arg x' + \zeta_0/8, |z| \geq 1 - \zeta_0/8\}$ and this wedge has diameter $< \zeta_0$).
Hence the intersection with \( \mathcal{E} \) of at least one of the events \{\(|x' - y'| \geq \zeta_0/8\), \{\(T' < \infty\), \{\(T'' < T'\)\}, or \{\(T'' < T'\)\} has probability at least \(\frac{1}{2}\mathbb{P}(\mathcal{E}) \geq e^{-\beta(\gamma^*(q) + \gamma^*(q)v)}\).

It is therefore enough to show that the conclusion of the lemma is true in each of the four possible cases. We will do this by choosing \(t_0\) to be one of \(T_{\tau_0}, T'\), or \(T''\) and \(\bar{t}_0\) to be one of \(\bar{T}_{\tau_0}, \bar{T}'\), or \(\bar{T}''\). It is clear that the last statement of the lemma holds for any such choice. Furthermore, for any such choice, \((\eta')^{\tau_0} \cup (\bar{\eta}')_{t_0}\) lies at uniformly positive distance from the segment \([0,1]\) on the event \(\mathcal{E}\). Hence Lemma A.1 and conformal invariance of harmonic measure imply that condition 2 in the definition of \(E_0\) holds for any such choice. Clearly, condition 4 in the definition of \(E_0\) holds a.s. on \(\mathcal{E}\) for any such choice. By (6.9), condition 4 holds for any such choice.

Therefore we just need to verify that condition 3 holds in each of the five cases.

1. If \(\mathbb{P}(\{|x' - y'| = \zeta_0/8\}, \mathcal{E}) \geq e^{-\beta(\gamma^*(q) + \gamma^*(q)v)}\) then we can just set \(t_0 = T_{\tau_0}\), \(\bar{t}_0 = \bar{T}_{\tau_0}\), and \(\zeta = \zeta_0/8\).

2. If \(\mathbb{P}(T' < \infty, \mathcal{E}) \geq e^{-\beta(\gamma^*(q) + \gamma^*(q)v)}\) then we set \(t_0 = T'\) and \(\bar{t}_0 = \bar{T}_{\tau_0}\). A Brownian motion has probability at least a constant \(\zeta > 0\) depending only on \(\zeta_0\) to exit \(B_{1-\log 16}(0)\) within distance \(\zeta_0/4\) of 1 and then make a counterclockwise loop around the origin before leaving \(D \setminus B_{1-\zeta_0/16}(0)\). In this case it necessarily exits \(D \setminus (\eta')^T\) on the left side of \((\eta')^{T'}\). See Figure 6.13 for an illustration in this case.

3. If \(\mathbb{P}(T'' < T', \mathcal{E}) \geq e^{-\beta(\gamma^*(q) + \gamma^*(q)v)}\) then we set \(t_0 = T' \wedge T''\) and \(\bar{t}_0 = \bar{T}_{\tau_0}\). A Brownian motion has probability at least a constant \(\zeta > 0\) depending only on \(\zeta_0\) to exit \(D\) before hitting any point outside of \(D \setminus B_{1-\log 16}(0)\) whose argument is not between \(\arg x'\) and \(\arg x' + \zeta_0/8\). If this is the case and \(T' \leq T''\), then a Brownian motion necessarily exits \(D \setminus (\eta')^{T' \wedge T''}\) on the left side of \((\eta')^{T' \wedge T''}\).

4. The case for \((T'' < T')\) is treated in the same manner as the case for \{(T'' < T')\}.

Thus, we have exhausted all possible cases and we conclude that the statement of the lemma holds.

**Proof of Proposition 6.1, lower bound.** Suppose \(\zeta, \mu_0, t_0\), and \(\bar{t}_0\) are chosen so that the conclusion of Lemma 6.5 holds. Let \(E_0 = E_0(t_0, \bar{t}_0, \zeta, \mu_0)\) be as in that lemma.

Let \(\beta_0 = -\log \operatorname{dist}(0, \eta_{t_0} \cup \bar{T}_{t_0})\). Note

\[
\beta(1 - v) \leq \beta_0 \leq \beta - \log 32.
\]

Fix \(r \in (\log 16, \log 32)\). Let \(\eta_1\) be the image under \(\psi_{t_0, \bar{t}_0}\) (defined as in Lemma 6.5) of the part of \(\eta\) between \(\eta(t_0)\) and \(\bar{\eta}_{\bar{t}_0}\). Let \(x_1\) and \(y_1\) be its endpoints. Let \(\tau_1^*\) (resp. \(\tau_1\)) be the first time \(\eta_1\) (resp. \(\bar{\eta}_1\)) hits \(\psi_{t_0, \bar{t}_0}(B_{\beta_0 + r})\). Let \(\mathcal{G}_1\) be the event that

1. \(|\eta_1(\tau_1^*) - \bar{\eta}_1(\tau_1)| \geq (1/32)e^{-r}\).

2. \(\eta_1^* \cup \bar{\eta}_1^* \subset \psi_{t_0, \bar{t}_0}(B_{\tau_1})\).

3. \(\eta_1^* \cup \bar{\eta}_1^*\) is disjoint from the \(\zeta/2\)-neighborhood of the segment connecting 0 and the midpoint of the shorter arc between \(x_1\) and \(y_1\).

By the Koebe quarter theorem we have

\[
B_{r + \log 16} \subset \psi_{t_0, \bar{t}_0}(B_{\beta_0 + r}) \subset B_{r - \log 16}.
\]

Hence by [MW14, Lemma 2.3], condition 3 in the definition of \(E_0\), and the last statement of Lemma 6.5 we have that \(\mathbb{P}(\mathcal{G}_1|E_0)\) is at least a \(\beta\)-independent positive constant.

For \(k = 1, 2, 3,\ldots\), let \(\bar{\psi}_k\) be the map from \(\mathbb{D} \setminus (\eta_{t_0, \bar{t}_0}(B_{\beta_0 + kr}) \cup \bar{\eta}_{t_0, \bar{t}_0}(B_{\beta_0 + kr}))\) to \(\mathbb{D}\) with \(\bar{\psi}_k(0) = 0\) and \(\bar{\psi}_k'(0) > 0\).

For \(k \geq 2\), let \(\eta_k\) be the image under \(\bar{\psi}_{k-1}\) of the part of \(\eta\) which lies between \(\eta(\tau_{\beta_0 + (k-1)r})\) and \(\bar{\eta}(\tau_{\beta_0 + (k-1)r})\). The law of \(\eta_k\) given \(\mathcal{F}_{\beta_0 + (k-1)r}\) is that of a chordal SLE\(\kappa\) from \(x_k := \psi_{t_0, \bar{t}_0}(\eta(\tau_{\beta_0 + (k-1)r}))\) to \(y_k := \psi_{t_0, \bar{t}_0}(\bar{\eta}(\tau_{\beta_0 + (k-1)r}))\). Let \(\eta_{\bar{k}}\) be the time reversal of \(\eta_k\).

Let \(\tau_k^*\) and \(\tau_k\) be the hitting times of \(\psi_{t_0, \bar{t}_0}(B_{\beta_0 + kr})\) by \(\eta_k\) and \(\eta_{\bar{k}}\), respectively. Fix \(\delta > 0\) and for \(k \geq 1\) let \(\mathcal{G}_k\) be the event that \(\eta_k^*\) (resp. \(\bar{\eta}_k^*\)) is contained in the \(\delta\)-neighborhood of the segment \([x_k, 0]\) (resp. \([y_k, 0]\)).
By the Koebe quarter theorem, whenever \( \tilde{\psi}_{k-1} \) is defined we have
\[
B_{r+\log 16} \subset \tilde{\psi}_{k-1}(B_{\beta_0+kr}) \subset B_{r-\log 16}.
\]
By conformal invariance of harmonic measure, on \( G_{k-1} \) for \( k \geq 2 \), \( |x_k - y_k| \) is at least a universal constant provided \( \delta \) is taken sufficiently small. It now follows from from [MW14, Lemma 2.3] that for each \( k \geq 2 \), \( \mathbb{P}(G_k | G_{k-1}) \geq p \) for some \( p > 0 \) which depends only on \( \delta \).

Let \( k_* \) be the least integer \( k \) such that \( kr + \beta_0 \geq \beta \). Note that \( k_* \leq \beta v/r \). Let
\[
G^* := \bigcap_{k=1}^{k_*} G_k.
\]
It is clear that on the event \( E_0 \cap G^* \), conditions 1, 2, and 4 in the definition of \( E \) hold provided we take \( \delta \) sufficiently small, depending on \( a \).

It remains to deal with condition 3. For \( k \geq 1 \), let \( \tilde{\eta}_k \) be the curve obtained by connecting \( \eta(\tau_{\beta_0+kr}) \) and \( \eta(\tau_{\beta v/k_r}) \) via the arc of \( B_{\beta_0+kr}^* \) which does not disconnect 0 from \( [x_*, y_*]_{\partial \mathbb{D}} \). Let \( \Psi_{\tilde{\eta}_k} \) be the conformal map from the connected component of \( \mathbb{D} \setminus \tilde{\eta}_k \) containing \( [x_*, y_*]_{\partial \mathbb{D}} \) on its boundary to \( \mathbb{D} \) which takes \( x_* \) to \( i \), \( y_* \) to \( i \), and the midpoint of \( [x_*, y_*]_{\partial \mathbb{D}} \) to 1. By Lemma 3.4 we have
\[
C^{-1} |\Psi'_{\tilde{\eta}_k}(0)| \leq |\phi'_{\beta}(0)| \leq C |\Psi'_{\tilde{\eta}_k}(0)|, \quad \forall \beta' \in [\beta_0 + (k-1)r, \beta_0 + kr], \quad \forall k \geq 2.
\]
on \( G^* \), for some deterministic \( C > 0 \) depending only on \( a, r, \) and \( \mu \). A similar statement holds for \( k = 1 \) provided we replace \( C_1 \) with a constant \( C_0 \) which is allowed to depend on \( \zeta \) (and hence on \( v \)), but not \( \beta \).

In particular, on \( G^* \) we have
\[
C_1^{-1} C^{-\beta v/r} e^{-\beta(q+v)} \leq |\phi'_{\beta}(0)| \leq C_1^{-1} C^{\beta v/r} e^{-\beta(q-v)}.
\]
If we choose \( v \) such that \( v \leq u/2 \) and \( C^{v/r} \leq u/v \) and \( C_1 \) sufficiently small, then condition 3 in the definition of \( E \) holds on \( E_0 \cap G^* \). By Lemma 5.3 and our choice of parameters above we then have
\[
\mathbb{P}(E) \geq \mathbb{P}(G_1 | E_0) p^{k_*-1} e^{-\beta(\gamma(q)+v)} \geq e^{-\beta(\gamma(q)+\gamma(q)u)}.
\]

\( \square \)

### 6.2 Events for the perfect points

In this subsection we will define events \( E_{z,j} \) for each \( z \in \mathbb{D} \) and \( j \in \mathbb{N} \) which we will eventually use to construct subsets of \( \Theta^*(D_p) \) and \( \tilde{\Theta}^*(D_p) \) (called the “perfect points”) whose Hausdorff dimension can be bounded below. The definition of the events \( E_{z,j} \) involves a number of auxiliary objects which we list below.

Let \( \chi = 2/\sqrt{k} - \sqrt{k}/2 \) and let \( \lambda = \pi / \sqrt{k} \) be as in (2.20). Let \( h \) be a zero boundary GFF on \( \mathbb{D} \) plus a harmonic function chosen in such a way that if \( \psi : \mathbb{H} \to \mathbb{D} \) is the conformal map taking 0 to \( -i \), \( \infty \) to \( i \), and 0 to \( i \), then \( h \circ \psi - \chi \arg \psi' \) is a GFF on \( \mathbb{H} \) with boundary data \( -\lambda \) on \( (-\infty, 0) \) and \( \lambda \) on \( [0, \infty) \). By [MS12b, Theorem 1.1] the zero-angle flow line \( \eta \) of \( h \) started from \( -i \) is a chordal SLE\(_\kappa \) from \( -i \) to \( i \) in \( \mathbb{D} \).

Fix \( \Delta > \tilde{\Delta} > 0, q \in (-1/2, \infty), \tilde{\Delta}, c, a, \theta, \delta_0, r, p, \mu_L > 0 \), and \( \mu, \mu_L, \mu_F \in \mathcal{M} \). Assume that the parameters \( a \) and \( \mu \) are chosen in such a way that the conclusion of Proposition 6.1 holds. Also fix sequences \( \beta_j \to \infty \) and \( u_j \to 0 \), which we will choose later.

For each \( z \in \mathbb{D} \) and \( j \in \mathbb{N} \), we will inductively define the following objects.

- Events \( L_{z,j}, \tilde{E}_{z,j}, F_{z,j}, E_{z,j} \).
- Points \( x_{z,j}, y_{z,j}, x_{z,j}^*, y_{z,j}^*, x_{z,j}^F, y_{z,j}^F, b_{z,j}, \tilde{E}_{z,j} \).
- Conformal maps \( \phi_{z,j}, p_{z,j}, \tilde{p}_{z,j}, \psi_{z,j}, \psi_{z,j}^F \).

\( ^3 \)In the case \( \kappa = 4 \), we replace flow lines of \( h \) with a given angle by level lines of \( h \) at a given level (see SS09, SS13, WW14). Everything that follows works identically with this replacement. In fact, since (in contrast to the situation for flow lines) the time reversal of a level line is also a level line [WW14, Theorem 1.1.3], some of the proofs are easier for \( \kappa = 4 \).
• Random times $\sigma_{z,j}, \sigma_{z,j}, \tau_{z,j}, \tau_{z,j}, \tau_{z,j}, T_{z,j}, T_{z,j}, T_{z,j}, \mathcal{T}^\ast_{z,j}, \mathcal{T}^\ast_{z,j}, t_{z,j}^\ast, \mathcal{T}^\ast_{z,j}$, and $\tilde{\eta}_{z,j}$.

• Curves $\eta_{z,j}, \tilde{\eta}_{z,j}, \eta_{z,j}^\perp$, and $\tilde{\eta}_{z,j}^\perp$.

• Domains $D_{z,j}$ and $\widehat{D}_{z,j}$.

• $\sigma$-algebras $\mathcal{F}^0_{z,j}$ and $\mathcal{F}_{z,j}$.

First consider the case $j = 1$. Let $f_{z,1}$ be the conformal automorphism of $D$ satisfying $f_{z,1}(z) = 0$ and $f_{z,1}'(0) > 0$. Let $x_{z,1} = x_{z,1}^\ast = f_{z,1}(-i)$ and $y_{z,1} = y_{z,1}^\ast = f_{z,1}(i)$. Let $\eta^0_{z,1} = f_{z,1}(\eta)$.

Let $\sigma_{z,1}$ be the first time $\eta^0_{z,1}$ hits $B_\Delta$. For $\alpha \geq 0$ let $\bar{t}_\alpha$ be the first time $\eta^0_{z,1}$ hits $B_\Delta$. Let $\psi^\alpha_{z,1}$ be the conformal map from the connected component of $D \setminus ((\eta^0_{z,1})^{\sigma_{z,1}} \cup (\mathcal{T}^\ast_{z,1}))$ containing the origin to $D$ which fixes 0 and takes $\eta^0_{z,1}(\sigma_{z,1})$ to $-i$.

Let $\sigma_{z,1} = \bar{t}_\alpha$ for the least $\alpha \geq \Delta$ such that $\psi^\alpha_{z,1}(\eta^0_{z,1}(\bar{t}_\alpha)) = i$, provided such an $\alpha$ exists, and $\sigma_{z,1} = \infty$ otherwise. Let $\psi_{z,1} = \psi^\alpha_{z,1}$ for this $\alpha$ if such an $\alpha$ exists and let $\psi_{z,1} = \text{Id}$ otherwise. Let $L_{z,1}$ be the event that the following occurs.

1. $(\eta^0_{z,1})^{\sigma_{z,1}}$ (resp. the part of $(\eta^0_{z,1})^{\sigma_{z,1}}$ traced before it hits $B_\Delta$) is contained in the $e^{-2\Delta}$-neighborhood of the segment $[x_{z,1}, 0]$ (resp. $[y_{z,1}, 0]$).
2. $\eta^0_{z,1}$ and $\eta^0_{z,1}$ do not leave $B_\Delta$ after hitting $B_\Delta/2$.
3. $\sigma_{z,1} \leq \bar{t}_{\Delta+\log 2}$ and $\eta^0_{z,1}(\bar{t}_\Delta, \sigma_{z,1}) \subset B_\Delta/2$.
4. $\psi_{z,1}^{-1}$ takes $B_{1-\mu(\delta_0)}(0) \cup B_{\delta_0}(-i) \cup B_{\delta_0}(i)$ into $B_\Delta$.
5. $e^{-\Delta+i/4} \leq |\psi'_{z,1}(0)| \leq e^{-\Delta+i/2}$.
6. $\mathcal{G}_{[x_{z,1}, y_{z,1}]}(\psi_{z,1}, \mu_L)$ occurs (Definition 2.5).
7. The conditional probability given $(\eta^0_{z,1})_{\sigma_{z,1}} \cup (\mathcal{T}^\ast_{z,1})$ that the part of $\eta^0_{z,1}$ lying between $\eta^0_{z,1}(\sigma_{z,1})$ and $\mathcal{T}^\ast_{z,1}$ never exits $B_\Delta$ is at least $p_L$.

Let $\eta_{z,1}$ be the image under $\psi_{z,1}$ of the part of $\eta^0_{z,1}$ lying between $\eta^0_{z,1}(\sigma_{z,1})$ and $\mathcal{T}^\ast_{z,1}(\sigma_{z,1})$. Let $\mathcal{F}^0_{z,1}$ be the $\sigma$-algebra generated by $\eta^0_{z,1}[0, \sigma_{z,1}]$ and $\mathcal{T}^\ast_{z,1}[0, \sigma_{z,1}]$.

**Remark 6.6.** The reason for introducing $\eta_{z,1}$ instead of working directly with $\eta^0_{z,1}$ is that we need the laws of the curves at each stage in our construction to be s.m.a.c. (Definition B.1). Otherwise we will end up with additional proportionality constants in our probability estimates. When we iterate these estimates several times, the proportionality constants will produce an exponential factor which will have a significant impact on our final estimates. The reason for condition 4 in the definition of $L_{z,1}$ is that it implies $\psi_{z,1}^{-1}(\eta_{z,1}) \subset B_{\Delta}$ provided condition 4 in the definition of $\mathcal{F}^0_{z,1}(\eta_{z,1}, \alpha, a, \mu)$ holds.

**Remark 6.7.** By [MW14, Lemma 2.3], reversibility, and the Markov property, condition 7 in the definition of $L_{z,1}$ (for sufficiently small $p_L$) follows from the other conditions in the case of an ordinary SLE. However, when we define the events $L_{z,j}$ for $j \geq 2$, the curve $\eta^0_{z,j}$ will be an SLE$\alpha(\rho^L; \rho^R)$ for certain $\rho^L, \rho^R \in (-2, 0)$, in which case condition 7 (with $j$ in place of 1) is non-trivial. The reason for introducing condition 7 is as follows. By Lemma B.4 we can estimate the law of the remainder of the middle part of $\eta^0_{z,j}$ given $(\eta^0_{z,j})^{\sigma_{z,1}} \cup (\mathcal{T}^\ast_{z,1})^{\sigma_{z,1}}$ conditioned on the event that this curve does not leave $B_\Delta$. However, we want to estimate the law restricted to the event that the curve does not leave $B_\Delta$. For this, we need the probability that it leaves $B_\Delta$ to be $\approx 1$.

**Lemma 6.8.** For any $\Delta > 0$, $\delta_0 > 0$, and $\mu \in \mathcal{M}$, and $f > 0$, we can find $\Delta > 0$, and and $\Delta > 0$ depending only on $\Delta$, $\mu_L \in \mathcal{M}$ depending only on $\mu$, and $p_L > 0$ depending only on the other parameters in the definition of $L_{z,1}$, all depending uniformly on $z$ in compacts, such that $\mathbf{P}(L_{z,1}) \geq 1$, with the implicit constant uniform for $z$ in compacts.
Figure 6.2: An illustration of the the parts of the curve \( \eta_{x,1}^0 \) (left) and \( \eta_{x,1} \) (right) associated with the events \( L_{x,1} \) and \( \tilde{E}_{x,1} \). For clarity, the disks here are shown larger than they actually are in practice. The same is true in the other figures in this section.

**Proof.** This follows from [MW14, Lemma 2.3]. Note that we apply the Koebe growth theorem to \( \psi_{x,1}^{-1} \) in order to make \( \Delta \) as large as we like.

Let

\[
\tilde{E}_{x,1} = L_{x,1} \cap E_{\beta_1}^{\eta_1}(\eta_{x,1}, c, a, \mu),
\]

with the latter event as in Section 6.1. Let \( \tau_{x,1} \) and \( \tau_{x,1} \) be the stopping times \( \tau_{x,1} \) and \( \tau_{x,1} \) from Section 6.1. Let \( \phi_{x,1} : D \setminus (\eta_{x,1,1}^+ \cup \eta_{x,1,1}^-) \to D \) be the conformal map \( \phi_{x,1} \) from Section 6.1. Let \( T_{x,1} \) be the time for \( \eta \) corresponding to \( \tau_{x,1} \).

Let \( \eta_{x,1}^+ \) and \( \eta_{x,1}^- \) be the flow lines of \( h \) started from \( \eta(T_{x,1}) \) with angles \( \theta \) and \( -\theta \), respectively. Note that the flow line with a negative sign has positive angle and vice versa. This is because a flow line with a negative angle a.s. stays to the right of \( \eta \), and a flow line with a positive angle a.s. stays to the left of \( \eta \). See [MS12a, Theorem 1.5]. Let \( \eta_{x,1,1} = (\psi_{x,1} \circ f_{x,1})(\eta_{x,1,1}^+) \).

By the results of [MS12a, Section 7], the conditional law of \( \eta_{x,1} \) (resp. \( \eta_{x,1}^- \)) given \( \eta \) is that of a SLE\(_\kappa\)(\( \rho_0; \rho_1 \)) processes from \( \eta(T_{x,1}) \) to \( \tau \) in the right (resp. left) connected component of \( D \setminus \eta \), where

\[
\rho_0 = -\frac{\theta \chi}{\lambda}, \quad \rho_1 = \frac{\theta \chi}{\lambda} - 2
\]

(6.11)

and the force points are located immediately to the left and right of \( \eta(T_{x,1}) \).

By [MS12a, Remark 5.3], \( \eta_{x,1,1}^+ \) a.s. fail to hit \( \eta_{x,1,1} \cup \partial D \) provided \( -\theta \chi / \lambda \geq \kappa / 2 - 2 \). Furthermore, \( \eta_{x,1,1}^+ \) a.s. intersect (but do not cross) each other provided \( 20 \chi / \lambda - 2 < \kappa / 2 - 2 \). Since \( \kappa \leq 4 \) we can choose a small \( \theta > 0 \) depending only on \( \kappa \) in such a way that \( \eta_{x,1,1}^+ \) a.s. do not intersect \( \partial D \) but do a.s. intersect each other.

We henceforth assume that \( \theta \) has been chosen in this manner.

Let \( D_{x,1} \) be the connected component of \( D \setminus (\eta_{x,1,1}^+ \cup \eta_{x,1,1}^-) \) which contains \( z \). Let \( D_{x,1} \) be the connected component of \( D \setminus (\eta_{x,1,1}^+ \cup \eta_{x,1,1}^-) \) which contains the origin. Let \( p_{x,1} : D_{x,1} \to D \) be the conformal map with

\[\text{Since } T_{x,1} \text{ is not a stopping time for } \eta, \text{ we define the flow lines } \eta_{x,1}^\pm \text{ by taking the limit of the flow lines of angles } \mp \theta \text{ of } h \text{ started from } \eta(t) \text{ as } t \text{ increases to } T_{x,1} \text{ along rational times. The limiting object can equivalently be defined as the left or right outer boundary of a certain counterflow line of } h \text{ started from } -i \text{ and stopped at the first time it hits } \eta(T_{x,1}) \text{ (c.f. } [\text{MS12a, Section 5}]), \text{ so in particular is a simple curve.}\]
Figure 6.3: An illustration of the curve $\eta$ grown up to its hitting time of a quasi-disk centered at $z$ on the event $E_{z,1}$, together with the flow lines $\eta_{z,1}^\pm$ (shown in green) used in the definition of the event $F_{z,1}$. The part of the curve $\eta$ associated with the event $L_{z,1}$ is shown in purple. The domain $\tilde{D}_{z,1}$ is the complementary connected component of the green flow lines which contains $z$. Also shown are the segments of $\eta$ involved in the last part of the event $E_{z,1}$ (in orange) and the stopping times $T_{z,1}^*$ and $\tilde{T}_{z,1}^*$ defined just after the definition of $E_{z,1}$.

$p_{z,1}(0) = 0$ and $p_{z,1}'(0) > 0$. Let $\hat{p}_{z,1} = p_{z,1} \circ \psi_{z,1} \circ f_{z,1}$, so that $\hat{p}_{z,1} : \tilde{D}_{z,1} \to \mathbb{D}$ and takes $z$ to 0. See Figure 6.3 for an illustration of the event $E_{z,1}$ and the flow lines $\eta_{z,1}^\pm$.

Let $t_{z,1}$ be the first time that $\eta_{z,1}^+$ hits $\eta_{z,1}^-$ after the first time it exits the disk of radius $e^{-\beta_1 - \tilde{r}}$ centered at $\eta_{z,1}(\tau_{z,1})$. Let $t_{z,1}^-$ be the time $t$ such that $\eta_{z,1}^-(t) = \eta_{z,1}^+(t_{z,1})$. Let $\bar{b}_{z,1} = \eta_{z,1}^+(t_{z,1})$ and let $b_{z,1}$ be the last intersection point of $\eta_{z,1}^\pm$ before $\bar{b}_{z,1}$. Also let $\bar{t}_{z,1}$ be the first exit times of $\eta_{z,1}^\pm$ from the annulus $B_{\beta_1 - \log 2} \setminus B_{\beta_1 + \log 2}$. Let $F_{z,1}$ be the event that the following occurs.

1. $t_{z,1}^+ \leq \bar{t}_{z,1}$ and $t_{z,1}^- \leq \bar{t}_{z,1}$.
2. $|b_{z,1}| \leq e^{-\beta_1 - \tilde{r}}$ and $\bar{b}_{z,1} \notin \eta_{z,1}^{\pm}$.
3. Let $\psi_{z,1}^F : \mathbb{D} \setminus (\eta_{z,1}^{\pm} \cup \eta_{z,1}^{\pm})$ be the conformal map with $\psi_{z,1}^F(0) = 0$ and $(\psi_{z,1}^F)'(0) > 0$. Let $x_{z,1}^F = \psi_{z,1}^F(\eta_{z,1}(\tau_{z,1}))$ and $y_{z,1}^F = \psi_{z,1}^F(\eta_{z,1}(\tau_{z,1}))$. Then $|\psi_{z,1}^F(b_{z,1}) - x_{z,1}^F|$ and $|\psi_{z,1}^F(\bar{b}_{z,1}) - y_{z,1}^F|$ are each at most $r$.
4. Each point of $\psi_{z,1}^F((\eta_{z,1}^{\pm})^{\pm,1})$ (resp. $\psi_{z,1}^F((\eta_{z,1}^{\pm})^{\mp,1})$) lies within distance $r$ of $[x_{z,1}^F, y_{z,1}^F]_{\partial \mathbb{D}}$ (resp. $[y_{z,1}^F, x_{z,1}^F]_{\partial \mathbb{D}}$).

5. $\mathcal{C}((\psi_{z,1}^F((\eta_{z,1}^{\pm})^{\pm,1}) \cup (\eta_{z,1}^{\mp})^{\mp,1}), \mu_F)$ occurs.

See Figure 6.4 for an illustration of the event $F_{z,1}$. It follows from condition 3 that $D_{z,1}$ is the “pocket” formed by $\eta_{z,1}$ between their hitting times of $b_{z,1}$ and $\bar{b}_{z,1}$ on the event $F_{z,1}$.

Remark 6.9. Our reason for introducing the auxiliary flow lines $\eta_{z,1}^{\pm}$ is as follows. The conditional law of the part of $\eta$ which lies inside $\tilde{D}_{z,1}$ is conditionally independent of the part of $\eta$ which lies outside $\tilde{D}_{z,1}$ given the flow lines $\eta_{z,1}^{\pm}$ (see Lemma 6.22 below). When applied at various scales, this fact this will eventually allow us to get the needed “near independence” of the events $E_{z,j}$ and $E_{w,j}$ for $z \neq w$. 

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Figure 6.4: An illustration of the event \( F_{z,1} \) that \( \bar{E}_{z,1} \) occurs and the flow lines \( \eta_{z,1} \) started at \( \eta_{z,1}(\tau_{z,1}) \) behave in the manner we would like. The domain \( D_{z,1} \) is that which lies between the parts of \( \eta_{z,1} \) traced between their hitting times of \( b_{z,1} \) and \( \bar{b}_{z,1} \).

**Lemma 6.10.** Given \( r > 0 \) we can choose \( \mu_F \) and \( \hat{r} \) independently of \( \beta_1 \) and \( u_1 \) and uniform for \( z \) in compacts in such a way that \( \mathbf{P}(F_{z,1} | \bar{E}_{z,1}) \asymp 1 \) with the implicit constants depending on the other parameters but not on \( \beta_1 \), \( u_1 \) or \( z \).

Proof. Let \( \eta_{z,1}^F \) be the image under \( \psi_i^F \) of the part of \( \eta_{z,1} \) between \( \eta_{z,1}(\tau_{z,1}) \) and \( \bar{E}_{z,1}(\bar{\tau}_{z,1}) \). Note that the distance between the endpoints \( x_{z,1}^F \) and \( y_{z,1}^F \) of \( \eta_{z,1}^F \) is uniformly positive on \( \bar{E}_{z,1} \) by condition 2 in the definition of \( E_{\hat{r}_i}^{\mu_F}() \). Let \( \bar{r} \in (0, r) \) and let \( U \) be the \( \bar{r} \)-neighborhood of the line segment from \( x_{z,1} \) to \( y_{z,1}^F \). Let \( \mu'_{F} \in \mathcal{M} \). Let \( S_{z,1} \) be the event that \( \eta_{z,1}^F \subset U \) and \( \mathcal{G}(\eta_{z,1}^F, \mu'_{F}) \) occurs.

By the Markov property and reversibility of SLE, the conditional law of \( \eta_{z,1}^F \) given a realization of \( F_{z,1} \) for which \( L_{z,1} \) and a realization of \( \eta_{z,1}^F \cup \eta_{z,1}^\bar{F} \) is that of a chordal SLE\( \kappa \) from \( x_{z,1}^F \) to \( y_{z,1}^F \) in \( D \). By [MW14, Lemma 2.3], we thus have \( \mathbf{P}(S_{z,1} | \bar{E}_{z,1}) \geq 1 \) provided \( \mu'_{F} \) is chosen sufficiently small, independently of \( \beta_1 \), \( u_1 \), and \( z \).

The conditional law of \( \psi_i^F(\eta_{z,1}^+) \) given \( \eta_{z,1}^0 \) is that of a SLE\( \kappa(\rho^0, \rho^1) \) process in the right connected component of \( D \setminus \eta_{z,1}^F \) from \( x_{z,1}^F \) to \( \psi_i^F(i^-) \) with force points located on either side of \( x_{z,1}^F \), where \( \rho^0 \) and \( \rho^1 \) are as in [6.11]. Our choice of \( \hat{r} \) implies that such a process a.s. does not hit \( [x_{z,1}, \psi_i^F(i^-)]_{D} \). Condition 2 in the definition of \( E_{\hat{r}_i}^{\mu_F}(\cdot) \) implies that \( \psi_i^F(\eta_{z,1}^+) \) lies at uniformly positive distance from \( x_{z,1}^F \) and \( y_{z,1}^F \) on \( E_{z,1} \). Similar statements hold with \( - \) in place of \( + \) and “left” in place of “right”. By [MW14, Lemma 2.5] (and a straightforward complex analysis argument to make a suitable choice of \( \hat{r} \)), we have \( \mathbf{P}(F_{z,1} | \bar{E}_{z,1} \cap S_{z,1}) \geq 1 \) provided \( \hat{r}, \mu_F, \) and \( \hat{r} \) are chosen sufficiently small, independently of \( \beta_1 \), \( u_1 \), and \( z \).
Let $\tau_{z,1}^*$ (resp. $\tau_{z,1}^-$) be the time that $\eta_{z,1}$ (resp. $\eta_{z,1}^+$) hits $b_{z,1}$ (resp. $b_{z,1}^\mp$). Note that these times are a.s. finite since $\eta_{z,1}$ a.s. lies between $\eta_{z,1}^\pm$. Let $E_{z,1}$ be the event that the following occurs.

1. $E_{z,1} \cap F_{z,1}$ occurs.
2. $\psi_{z,1}^\pm(\eta_{z,1}(\tau_{z,1}^*, \tau_{z,1}^-))$ (resp. $\psi_{z,1}^\pm(\eta_{z,1}^-(\tau_{z,1}^*, \tau_{z,1}^-))$) is contained in the disk of radius $2r$ centered at $x_{z,1}^F$ (resp. $y_{z,1}^F$) (notation as in condition 3 in the definition of $F_{z,1}$).

**Remark 6.11.** By [MS12a, Theorem 1.5] $\eta_{z,1}$ cannot cross $\eta_{z,1}^\pm$. By combining this with condition 4 in the definition of $L_{z,1}$, condition 4 in the definition of $E_{z,1}^\pm(u_1)$, and condition 2 in the definition of $E_{z,1}$, it follows that the whole of $\psi_{z,1}^\pm(\eta_{z,1})$ (i.e. the part of $\eta_{z,1}^0$ between $\eta_{z,1}^0(\sigma_{z,1})$ and $\eta_{z,1}^0(\tau_{z,1})$) is contained in $B\Delta_{\frac{1}{3}}$ on the event $E_{z,1}$.

Let $T_{z,1}^*$ and $T_{z,1}^\pm$, resp., be the times for $\eta$ and $\eta$, resp., corresponding to $\tau_{z,1}^*$ and $\tau_{z,1}^\pm$. Let $F_{z,1}$ be the $\sigma$-algebra generated by $\eta[D\setminus B\Delta_{\frac{1}{3}}], \eta[D\setminus B\Delta_{\frac{1}{3}}]$ and $\eta_{z,1}^\pm[0,\ell_{z,1}^\pm]$. Let $E_{z,1}$ be the event that the following occurs.

**Lemma 6.12.** We have $P(E_{z,1} | \bar{E}_{z,1} \cap F_{z,1}) \geq 1$ with the implicit constant depending on the other parameters but not on $\beta_1$ or on $z$.

**Proof.** Define the open set $U$ and the event $S_{z,1}$ as in the proof of Lemma [6.10]. By Bayes’ rule we have

$$P \left( S_{z,1} | \bar{E}_{z,1} \cap F_{z,1} \right) = \frac{P(F_{z,1} \cap S_{z,1} | \bar{E}_{z,1})}{P(F_{z,1} | \bar{E}_{z,1})}.$$

By the argument of Lemma [6.10] this is at least a positive $(\beta_1, z)$-independent constant. On the other hand we have

$$P \left( E_{z,1} | \bar{E}_{z,1} \cap S_{z,1} \cap F_{z,1} \right) \geq 1.$$

By combining Proposition [6.1], Lemma [6.8], Lemma [6.10], and Lemma [6.12] we infer

**Lemma 6.13.** Provided $\beta_1$ is chosen sufficiently large and the other parameters are chosen appropriately, independently of $\beta_1$ and $u_1$ and uniformly for $z$ in compacts, we have $e^{-\beta_1(\gamma^* + \gamma^*_u)} \leq P(E_{z,1}) \leq e^{-\beta_1(\gamma^* + \gamma^*_u)}$ with the implicit constant depending on the other parameters (including $u_1$) but not on $\beta_1$, and uniform for $z$ in compacts.

Now suppose $j \geq 2$ and the objects have been defined for all positive integers $l \leq j - 1$.

Let $\eta_{z,j}^0$ be the image under $\eta_{z,j-1}$ of the part of $\eta_{z,j-1}$ which lies in $\bar{D}_{z,j-1}$. Let $x_{z,j}$ and $y_{z,j}$ be its initial and terminal points. Define the map $\psi_{z,j}$, the times $\sigma_{z,j}$ and $\tau_{z,j}$, and the curve $\eta_{z,j}$ in the same manner as in the case $j = 1$, but with all of the $1$’s replaced by $j$’s (the auxiliary parameters remain unchanged).

Let $[x_{z,j}^*, y_{z,j}^*] \cap \partial D$ be the largest sub-arc of $[x_{z,j}, y_{z,j}] \cap \partial D$ not disconnected from the origin by $(\eta_{z,j}^0)^{\sigma_{z,j}} \cup (\eta_{z,j}^0)^{\tau_{z,j}}$ (this arc need not be equal to $[x_{z,j}, y_{z,j}] \cap \partial D$ since $\eta_{z,j}^0$ can intersect $\partial D$). Fix a constant $p_L > 0$. Define the event $L_{z,j}$ in exactly the same manner as in the case $j = 1$ but with all $1$’s replaced by $j$’s.

Let $\bar{D}_{z,j}$ be the $\sigma$-algebra generated by $\eta_{z,j}^0[0,\sigma_{z,j}]$ and $\eta_{z,j}^0[0,\tau_{z,j}]$.

Define the event $\eta_{z,j}$, the times $\tau_{z,j}$ and $\bar{D}_{z,j}$, and the map $\phi_{z,j}$ in exactly the same manner as the corresponding objects for $j = 1$, but with all the $1$’s replaced by $j$’s. Fix a constant $p_L > 0$. Define the event $E_{z,j}$ exactly the same manner as in the case $j = 1$ but with all $1$’s replaced by $j$’s.

Let $\bar{D}_{z,j}$ be the $\sigma$-algebra generated by $\eta_{z,j}^0[0,\tau_{z,j}]$ and $\eta_{z,j}^0[0,\tau_{z,j}]$.

Let $\bar{D}_{z,j}$ be the flow lines of $\eta$ started from $\eta(T_{z,j})$ with angles $\mp \theta$. Let $\bar{D}_{z,j}$ be the connected component of $D \setminus (\bar{D}_{z,j} \cup \bar{D}_{z,j})$ containing $z$. Let $\eta_{z,j}^\pm$ be the images of $\eta_{z,j}^\pm$ under the map $\psi_{z,j} \circ \bar{D}_{z,j} \circ 1$. Define the domain $D_{z,j}$, the event $F_{z,j}$, the map $\eta_{z,j}$, and the times $\tau_{z,j}$ and $\bar{D}_{z,j}$ in exactly the same manner as in the case $j = 1$ case except with all of the $1$’s replaced by $j$’s. Let $\eta_{z,j}^\pm = \psi_{z,j} \circ \bar{D}_{z,j} \circ 1$ so that $\eta_{z,j}^\pm : \bar{D}_{z,j} \to D$ takes $z$ to $0$.

Also define the event $E_{z,j}$ and the times $\tau_{z,j}^*$ and $\tau_{z,j}^\pm$ in exactly the same manner as in the case $j = 1$ case except with all of the $1$’s replaced by $j$’s. Let $\bar{T}_{z,j}$ and $\bar{D}_{z,j}$ be the times for $\eta$ and $\bar{D}$ corresponding to $\tau_{z,j}^*$ and $\tau_{z,j}^\pm$. Let $\bar{F}_{z,j}$ be the $\sigma$-algebra generated by $\eta[0,T_{z,j}], \eta[0,T_{z,j}^\pm], \eta_{z,j}^\pm[0,\ell_{z,j}^\pm]$ for $l \leq j$.

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Figure 6.5: An illustration of the curve $\eta$ and the auxiliary flow lines $\hat{\eta}_{\pm}^1$ and $\hat{\eta}_{\pm}^2$ on the event $E_{z,1} \cap E_{z,2}$. Note that only a neighborhood of $z$ is shown, not the whole disk. The parts of $\eta$ involved with the last parts of the definitions of $E_{z,1}$ and $E_{z,2}$ are shown in orange. The parts of $\eta$ involved with the event $L_{z,2}$ are shown in purple. The domain $\hat{D}_{z,1}$ (resp. $\hat{D}_{z,2}$) is the region enclosed by the outer (resp. inner) green curves.

6.3 k-perfect points

Fix sequences $\beta_j \to \infty$ and $u_j \to 0$ as above. Define the events $E_{z,j}^{m}$ in the same manner as the events $E_{z,j}$ in the preceding subsection but with the sequences $(\beta_j)$ and $(u_j)$ replaced by $(\beta_{j+m})$ and $(u_{j+m})$. Also fix $d \in (0,1)$. The estimate of Lemma 6.13 becomes

$$C_{u_{m+1}}^{-1}e^{-\beta_{m+1}(\gamma(q) - \gamma_0(q)u_{m+1})} \leq P(E_{z,j}^m) \leq C_{u_{m+1}}^{-1}e^{-\beta_{m+1}(\gamma(q) - \gamma_0(q)u_{m+1})},$$

(6.12)

where for $u > 0$, $C_u$ is a constant which is allowed to depend on $u$ and on the other parameters in the preceding subsection but not on any of the $\beta_j$’s, and is uniform for $z \in B_d(0)$. We can take $C_u$ to be decreasing in $u$. We also let

$$\beta_{m_1,m_2} = \sum_{j=m_1+1}^{m_2} \beta_j, \quad \eta_{m_1,m_2} = \sum_{j=m_1+1}^{m_2} \beta_ju_j, \quad \beta_m = \beta_{0,m}, \quad \eta_m = \eta_{0,m}.$$  (6.13)

Remark 6.14. The reason we allow $\beta$ and $u$ to vary here is that we eventually want to get a lower bound for the Hausdorff dimension of the sets $\Theta^s(D_\eta)$ and $\tilde{\Theta}^s(D_\eta)$. If we fixed $u$, we would instead get the Hausdorff dimension of the sets where the limits in the definitions of $\Theta^s(D_\eta)$ and $\tilde{\Theta}^s(D_\eta)$ are between $s-u$ and $s+u$. In order to allow $u$ to vary, we also need to allow $\beta$ to vary, for otherwise the constants $C_u$ in (6.12) would be larger than $e^3$ when $u$ is very small. The idea in Lemma 6.15 below is to let $u_j \to 0$ and $\beta_j \to \infty$ slowly enough that our estimates are not much different than they would be with fixed $\beta$ and $u$.  

Lemma 6.15. Given $d$ and the parameters from Section 6.2, one can choose sequences $(\beta_j)$ and $(u_j)$ such that the following is true.
1. $\beta_j$ increases to $\infty$ and $u_j$ decreases to 0 as $j \to \infty$.

2. For each $j \in \mathbb{N}$, $\beta_j+1 \leq \beta_j + o_1(1)$.

3. For each fixed $k \in \mathbb{N}$, $\beta_k \leq o_2(1)$.

4. For each $j \in \mathbb{N}$, $C_{u_j} \leq e^{\beta_j u_j \gamma_0(q)}$.

5. $\beta_j u_j \to \infty$ as $j \to \infty$.

Proof. Choose $\beta_0$ much larger than $\Delta + \gamma_0(q)^{-1} \log C_1$ and set $\beta_j = \log j + \beta_0$. It is clear that $\beta_j \to \infty$ as $j \to \infty$ and assertions 2 and 3 hold. We now inductively choose $(u_j)$.

Start with a sequence $(u_m)_{m \in \mathbb{N}} \subset (0, 1)$ which decreases to 0. Let $j_1$ be the least positive integer $j$ such that $C_{u_j} \leq e^{\beta_j u_j \gamma_0(q)}$. Such a $j$ exists since $\beta_j \to \infty$ as $j \to \infty$. Set $u_j = u_1^*$ for $j \leq j_1$. Inductively, suppose $m \geq 2$ and $j_1, \ldots, j_{m-1}$ and $u_j$ for $j \leq j_{m-1}$ have been defined. Let $j_m$ be the least integer $j \geq j_{m-1} + 1$ such that $C_{u_{j_m}} \leq e^{\beta_j u_{j_m} \gamma_0(q)}$. Let $u_j = u_m^*$ for $j = j_{m-1} + 1, \ldots, j_m$. It is clear that condition 4 holds for this choice of $(u_j)$. Observe that condition 4 still holds if we increase $u_j$. Hence by increasing $u_j$ we can arrange that 5 holds, and we still have $u_j \to 0$. \hfill \square

We henceforth assume the sequences $(\beta_j)$ and $(u_j)$ are chosen as in Lemma 6.15.

For $n \geq k \geq 0$, define

$$E_{k,n}^m(z) = \bigcap_{j=k+1}^n E_{z,j}^m. \quad (6.14)$$

Also let

$$E_{k,n}(z) := E_{k,n}(z), \quad E_n(z) := E_0,n(z).$$

The set of n-perfect points is

$$\mathcal{P}_n := \{ z \in \mathcal{D} : E_n(z) \text{ occurs} \} \quad (6.15)$$

6.4 Analytic properties

In this subsection we study some analytic properties of the events of Sections 6.2 and 6.3. The results of this subsection are needed to analyze the correlation structure of our events in the next subsection and to show that the perfect points are in fact contained in the sets whose Hausdorff dimension we want to compute in Section 7. The main result of this subsection is the following.

Lemma 6.16. Assume we are in the setting of Sections 6.2 and 6.3. For $n \in \mathbb{N}$ let $\Phi_{z,n}$ be the conformal map from $\mathcal{D} \setminus (\eta(T_{z,n}^+ \cup \eta(T_{z,n}^-))$ to $\mathcal{D}$ which takes $-i^+$ to $-i$, $i^-$ to $i$, and 1 to 1. The following holds a.s. on $E_n(z)$, with all implicit constants deterministic and independent of $n$ and of $z \in B_d(0)$.

1. We have

$$e^{-\tilde{\beta}_n q - 2\pi n} \leq |\Phi_{z,n}(z)| \leq e^{-\tilde{\beta}_n q + 2\pi n}.$$

2. There is a constant $\lambda > 0$, independent of $n$ and $z \in B_d(0)$, such that

$$e^{-\tilde{\beta}_n - \lambda n} \leq \text{dist}(z, \eta(T_{z,n}^+ \cup \eta(T_{z,n}^-))) \leq e^{-\tilde{\beta}_n + \lambda n}.$$

3. We have

$$|\eta(T_{z,n}^+)| - z \asymp |\eta(T_{z,n}^-)| - z \asymp \text{dist}(z, \eta(T_{z,n}^+ \cup \eta(T_{z,n}^-))).$$

4. We have

$$e^{-\tilde{\beta}_n - \lambda n} \leq \text{dist}(z, \partial \tilde{D}_{z,n}) \leq \text{diam} \tilde{D}_{z,n} \leq e^{-\tilde{\beta}_n + \lambda n}.$$

To prove Lemma 6.16 we will need to compare the derivatives of several different maps. To this end, we will define the following objects.

1. Conformal maps $\phi_{z,j}^0$, $\hat{\phi}_{z,j}$, $\tilde{\phi}_{z,j}$, $f_{z,j}$, and $g_{z,j}$.
2. Random times $\sigma^*_{z,j}$, $\eta^*_{z,j}$, $\tau^*_{z,j}$, and $\tau^*_{z,j}$.
3. Points $\tilde{x}_{z,j}$, $\tilde{y}_{z,j}$, and $\tilde{m}_{z,j}$.
4. Curves $\tilde{n}_{z,j}$.

For $j \in \mathbb{N}$, let $\hat{\psi}_{z,j}$ be the conformal map from $D \setminus (\eta^{\tau*_{z,j}} \cup \eta^{\tau^*_{z,j}})$ to $D$ which fixes 0 and whose derivative at 0 has the same argument as $\Phi(0)$.

Let $\sigma^0_{z,j}$ and $\tau^0_{z,j}$ be the stopping times for $\eta^0_{z,j}$ corresponding to $\tau^*_{z,j}$ and $\tau^*_{z,j}$ (equivalently to $T^*_{z,j}$ and $T^*_{z,j}$). Let $\phi^0_{z,j}$ be the conformal map from the connected component of $D \setminus (\eta^{\tau*_{z,j}} \cup \eta^{\tau^*_{z,j}})$ containing 0 to $D$ which takes $x^*_{z,j}$ to $-i$, $y^*_{z,j}$ to $i$, and the midpoint $m^*_{z,j}$ of $[x^*_{z,j}, y^*_{z,j}]_D$ to 1.

For $j = 1$, the map $f_{z,j}$ has already been defined in Section 6.2. For $j \geq 2$, let $f_{z,j}$ be the conformal map which takes $\Phi_{z,j-1}(z)$ to $D$ with $\Phi_{z,j-1}(0) = 0$ and whose partials are independent of $\beta$ (and hence also as $\beta \to \infty$). Therefore, if $\beta^*_{z,j}$ is chosen sufficiently large, then $|\beta^*_{z,j}(0)| \approx 1$ so our desired result follows from (6.18).

Let $\tilde{x}_{z,j}$ and $\tilde{y}_{z,j}$ be the start and end points for $\tilde{n}_{z,j}$. Let $\tilde{\phi}_{z,j}$ be the conformal map from $D \setminus (\tilde{n}_{z,j} \cup \tilde{n}_{z,j})$ to $D$ which takes $\tilde{x}_{z,j}$ to $-i$, $\tilde{y}_{z,j}$ to $i$, and the midpoint $\tilde{m}_{z,j}$ of $[\tilde{x}_{z,j}, \tilde{y}_{z,j}]_D$ to 1. Let $g_{z,j} : D \to D$ be the conformal map taking $\tilde{\phi}_{z,j} \circ f_{z,j}(b)$ to $b$ for $b = -i^+, i^-, 1$. Let

$$\hat{\phi}_{z,j} := g_{z,j} \circ \tilde{\phi}_{z,j} \circ f_{z,j}.$$

Observe that (with $\Phi_{z,j}$ as in Lemma 6.16)

$$\Phi_{z,j} = \hat{\phi}_{z,j} \circ \cdots \circ \hat{\phi}_{z,1}.$$

See Figure 6.6 for an illustration of these maps in the case $j = 2$ (which has all of the features of the general case).

\section{Lemma 6.17}

If $\beta_1$ is chosen sufficiently large, then on the event $E_n(z)$, we have

$$|\tilde{p}_{z,n}(z)| \approx |\tilde{p}_{z,n}(z)|,$$

with the implicit constants independent of $n$ and uniform for $z \in B_\delta(0)$.

\section{Proof}

Assume we are working on the event $E_n(z)$. Let $p^*_{z,n} - 1$ be the conformal map from $\tilde{\psi}_{z,n}(\tilde{D}_{z,n-1})$ to $D$ with $p^*_{z,n-1}(0) = 0$ and $p^*_{z,n-1}'(0) > 0$ (in the case $n = 1$, we take $p^*_{z,n-1}$ to be the identity). Let $p^*_{z,n}$ be the conformal map from $(p^*_{z,n-1} \circ \tilde{\psi}_{z,n})(\tilde{D}_{z,n})$ to $D$ with $p^*_{z,n}(0) = 0$ and $p^*_{z,n}'(0) > 0$. We then have

$$\tilde{p}_{z,n} = p^*_{z,n} \circ p^*_{z,n-1} \circ \tilde{\psi}_{z,n}.$$

By the Beurling estimate and Exercise 2.7] the diameter of $D \setminus \tilde{\psi}_{z,n}(\tilde{D}_{z,n-1})$ tends to 0 as $\beta_1 \to \infty$. Therefore, if $\beta_1$ is chosen sufficiently large, then $|p^*_{z,n} - 1(0)| \approx 1$.

By condition 3 in the definition of $E_n(z)$, the distance from 0 to $(p^*_{z,n-1} \circ \tilde{\psi}_{z,n})(\partial \tilde{D}_{z,n})$ is proportional to the distance from 0 to $\tilde{\psi}_{z,n}(\partial D_{z,n})$, with $\psi_{z,n}$ as in the definition of $F_{z,n}$. By condition 1 in the definition of $F_{z,n}$, this distance is $\approx 1$. Consequently, $|(p^*_{z,n})'(0)| \approx 1$ so our desired result follows from (6.18).

\section{Lemma 6.18}

Let $\zeta \in (0, a/100)$. If we make the parameter $r$ in the definition of $E_{z,1}$ sufficiently small (depending on $\zeta$ and the other parameters but not on $\beta_1$ or $\alpha_1$ and uniform for $z \in B_\delta(0)$) then for any sub-arc $I$ of $[\tilde{x}_{z,n+1}, \tilde{y}_{z,n+1}]_D$ lying at distance at least $\zeta$ from $\tilde{x}_{z,n+1}$ and $\tilde{y}_{z,n+1}$, we have that $\tilde{\phi}_{z,n+1}$ is Lipschitz on $I$ and $\tilde{\phi}_{z,n+1}$ is Lipschitz on $\tilde{\phi}_{z,n+1}(I)$ on the event $E_n(z)$ with Lipschitz constants independent of $\beta_1$ and $\alpha_1$ and uniform for $z \in B_\delta(0)$. 

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Hence it is enough to prove (6.19) with $A = (D \times 1)$. Since the diameter of $A$ is large, where $A$ is nearly constant near $1$. By condition 2 in the definition of $A$, we have that $(p_{z,n})^{-1}$ decreases distances to $\partial D$. Hence the distance from $A$ to $I$ is at least a $\beta_1$-independent constant times the distance from $A^*$ to $I$ if $\beta_1$ is large, where

$$A^* = p_{z,n-1}(A) = \psi_{z,n}((\eta_{z,n}^+)_{t^*_{z,n}}).$$

Hence it is enough to prove (6.19) with $A^*$ in place of $A$.

Let $I'$ be a slightly larger arc. By condition 2 in the definition of $E_{z,n}$ the distance from $A^*$ to $I$ is $\geq$ the distance from $\psi_{z,n}((\eta_{z,n}^+)_{t^n_{z,n}})$ to $I'$ if $r$ is chosen sufficiently small, where $\psi_{z,n}^r$ is as in the definition of $F_{z,n}$. We conclude by applying condition 5 in the definition of $F_{z,n}$. 

\[
\begin{align*}
\text{Proof.} \quad \text{See Figure 6.7 for an illustration of the argument. Throughout, we work on the event $E_n(\cdot)$.

Let } A := \psi_{z,n}((\eta_{z,n}^+)_{t^*_{z,n}}). \text{ Then } A \text{ disconnects } \eta_{z,n+1} \text{ from } I \text{ in } \partial D. \text{ We claim that there is a constant } \delta > 0, \text{ independent of } \beta_1 \text{ and } u_1 \text{ and uniform for } z \in B_d(0), \text{ such that }

E_{z,n-1} \cap E_{z,n} \subset \{\text{dist}(A, I) \geq \delta\}. \quad (6.19)
\end{align*}
\]
Figure 6.7: An illustration of the maps used in the proof of Lemma 6.18. In order to control the distance from $\hat{\eta}_{z,n+1}$ to an arc on the right boundary of the disk, we compare $\eta_{z,n+1}$ to the curve $\eta^*_{z,n+1}$ and then to the curve $\tilde{\eta}_{z,n+1}$. The distance from the last curve to the right boundary is bounded below by condition 5 in the definition of $F_{z,n}$.

Lemma 6.19. We can choose the parameters in Section 6.2 independently of $n$ in such a way that on $E_n(z)$ we have

$$e^{-\beta_n(q+u_n)} \leq |\phi'(w)| \leq e^{-\beta_n(q-u_n)}$$

(6.20)

where the pair $(\phi, w)$ is any one of $(\phi_{z,n}, 0), (\phi_{z,n}^0, 0), (\phi_{z,n}, 0)$, or $(\phi_{z,n}, \Phi_{z,n-1}(z))$. The implicit constants are independent of $n$ and uniform for $z$ in compacts.

Proof. By definition of $E_{z,n}$ (in particular, of $E_{z,n}$), the statement of the lemma is true for $(\phi, w) = (\phi_{z,n}, 0)$. We will now transfer the estimate (6.20) from $E_{z,n}$ to $\phi_{z,n}^0$ to $\tilde{\phi}_{z,n}$ to $\tilde{\phi}_{z,n}$. This latter map is our primary interest, mostly because of (6.16). Throughout, we assume that $E_{z,n-1} \cap E_{z,n}$ occurs and require all implicit constants to be independent of $m$ and $n$ and uniform for $z$ in compacts.

Let $\tilde{\phi}_{z,n}$ be the conformal map from the connected component of $D \setminus (\eta_{z,n-1} \cup \eta_{z,n}^*)$ containing 0 to $D$ which takes $-i^+$ to $-i$, $i^-$ to $i$, and 1 to 1. Let $g_{z,n}^*$ be the conformal automorphism of $D$ which fixes $-i$ and $i$ and takes $(\phi_{z,n}^* \circ \psi_{z,n})(m_{z,n}^*)$ to 1. Then we have

$$\phi_{z,n}^0 = g_{z,n}^* \circ \tilde{\phi}_{z,n} \circ \psi_{z,n}$$

(6.21)

By condition 2 in the definition of $E_{z,n}$ and Lemma A.1, we have $|\phi_{z,n}^*(0)| \approx |\phi_{z,n}^0(0)|$ provided $r$ is chosen sufficiently small independently of $n$ and uniformly for $z$ in compacts. Furthermore, by Lemma 2.8 and condition 4 in the definition of $E^\beta_{z,n}(-\cdot)$, we have that $G(\phi_{z,n}^*, \mu')$ holds on $E_{z,n}$ for some $\mu' \in M$ depending only on $\mu$. By condition 1 in the definition of $L_{z,n}$ and condition 4 in the definition of $F_{z,n-1}$, $|x_{z,n}^* - y_{z,n}^*| \geq 1$ on $E_{z,n-1} \cap E_{z,n}$. By combining this with condition 6 in the definition of $L_{z,n}$ we have $|(g_{z,n}^*)(z)| \approx 1$ on all of $D$ on the event $E_{z,n}$. Hence (6.21) implies (6.20) for $\phi_{z,n}^0$.

By Lemma A.3 applied with $U = D \setminus ((\eta_{z,n})^* \cup (\eta_{z,n})^*)$, $D = \psi_{z,n-1}(D_{z,n-1}), \phi = \tilde{\phi}_{z,n}, \hat{\phi} = \phi_{z,n}^0, \text{ and } z = \bar{z} = 0$, the estimate (6.20) for $\phi_{z,n}^0$ implies the estimate (6.20) for $\tilde{\phi}_{z,n}$. Note that the conditions of Lemma A.3 (with parameters $\zeta, \Delta, \delta \geq 1$) follow easily from the definition of $E_n(z)$ together with Lemmas 6.17 and 6.18.

To get the estimate for $\phi_{z,n}$, write

$$|\tilde{\phi}_{z,n}(\Phi_{z,n-1}(z))| = |g_{z,n}^*(\tilde{\phi}_{z,n}(0))||\tilde{\phi}_{z,n}(0)||f_{z,n}'(\Phi_{z,n-1}(z))|.$$

(6.22)

By the Koebe quarter theorem, $|f_{z,n}'(\Phi_{z,n-1}(z))| \approx (1 - |\Phi_{z,n-1}(z)|)^{-1}$. In fact, we have $|f_{z,n}'(w)| \approx 1 - |\Phi_{z,n-1}(z)|$ on subsets of $D$ at positive distance from $\Phi_{z,n-1}(z)$.

By condition 2 in the definition of $E^\beta_{z,n-1}(-\cdot)$, we can find $\zeta > 0$ depending only on $a$ such that $f_{z,n}([-i, i])$ lies at distance at least $\zeta$ from $x_{z,n}$ and $y_{z,n}$ on $E_{z,n-1}$. By Lemma 6.18 on $E_{z,n-1}$, it
holds that $\tilde{\phi}_{z,n}$ distorts the distances between points in $f_{z,n}([-i, i]_{\partial D})$ by at most a constant factor. Since conformal automorphisms of $D$ depend smoothly on their parameters, it follows that

$$|g_{z,n}'| \asymp |(f_{z,n}^{-1})'|$$

(6.23)
on the whole left half of $D$. By combining this with (6.22) and the first statement of the lemma we conclude.

### Proof of Lemma 6.16
Assume $E_n(z)$ holds. Assertion 1 is immediate from Lemma 6.19 and the relation (6.21). Note that we can absorb the implicit constants in (6.20) into an additional factor of $e^{\eta_n}$ due to condition 5 of Lemma 6.15.

To prove item 2, we induct on $n$. The case $n = 1$ is immediate from the definitions of the events. Now suppose $n \geq 2$ and the result has been proven with $n$ replaced by $n - 1$. Since $\hat{\psi}_{z,n-1}$ maps $D \setminus (\eta_{T_z,n} \cup \eta^{-z,n})$ to $D \setminus (\eta_{T_z,n} \cup \eta^{-z,n})$ and fixes 0, the Koebe quarter theorem implies

$$\text{dist} \left( z, \eta_{T_z,n} \cup \eta^{-z,n} \right) \asymp |(\hat{\psi}_{z,n-1})'(0)| \text{dist} \left( 0, \eta_{z,n} \cup \eta^{-z,n} \right).$$

(6.24)

By a second application of the Koebe quarter theorem,

$$|\hat{\psi}_{z,n-1}'(0)| \asymp \text{dist} \left( z, \eta_{T_z,n-1} \cup \eta^{-z,n-1} \right).$$

(6.25)

By the inductive hypothesis,

$$e^{-\beta_{n-1} - \lambda(n-1)} \lesssim \text{dist} \left( z, \eta_{T_z,n-1} \cup \eta^{-z,n-1} \right) \lesssim e^{-\beta_{n-1} + \lambda(n-1)}.$$  

(6.26)

By Lemma 6.17 (applied with $n$ replaced by $n - 1$), we have

$$\text{dist} \left( 0, \eta_{z,n} \cup \eta^{-z,n} \right) \asymp \text{dist} \left( 0, \eta_{T_z,n}^0 \cup \eta^{-z,n}_0 \right).$$

(6.27)

By definition of $E_{z,n}$, this last distance is $\asymp e^{-\beta_n}$ on $E_{z,n}$. Provided $\lambda$ is chosen sufficiently large, independently of $n$ and $z \in B_d(0)$, we can now complete the induction by combining (6.24), (6.25), (6.26), and (6.27).

By condition 2 in the definition of the event from Section 6.1, condition 2 in the definition of $E_{z,n}$, and the Beurling estimate, the harmonic measure from $z$ of each of the two sides of $\eta_{T_z,n}$ (resp. each of the two sides of $\eta_{T_z,n}$ in $D \setminus (\eta_{T_z,n} \cup \eta^{-z,n})$) is at least some $n$-independent constant. This implies assertion 4.

For item 4, we use assertion 2 and the Koebe quarter theorem to see that there exists radii $\rho' > \rho > 0$ such that $\rho \geq e^{-\beta_{n-1} - \lambda n}$, $\rho' \leq e^{-\beta_{n-1} + \lambda n}$, $\hat{\psi}_{z,n-1}^{-1}(B_{\beta_n + \log 2} \supset B_{\rho}(z))$, and $\hat{\psi}_{z,n-1}^{-1}(B_{\beta_n - \log 2} \subset B_{\rho'}(z))$. By combining this with condition 2 in the definition of $F_{z,j}$ we see that assertion 4 holds (after possibly increasing $\lambda$).

### 6.5 Probabilistic properties

In this subsection we study the events defined above from a probabilistic perspective, and eventually prove our two-point estimate.

**Proposition 6.20.** Let $z, w \in B_d(0)$. Let $\lambda$ be as in assertion 2 of Lemma 6.16. Choose $k \in \mathbb{N}$ such that $e^{-\beta_{k+1} - \lambda(k+1)} \leq |z - w| \leq e^{-\beta_k - \lambda k}$. Then for any $n \in \mathbb{N}$ with $\beta_n - \lambda n \geq \beta_{k+1} + \lambda(k+2)$,

$$\mathbb{P}(E_n(z) \cap E_n(w)) \leq e^{-\beta_{k+1} \alpha_k(1)} \frac{\mathbb{P}(E_n(z)) \mathbb{P}(E_n(w))}{\mathbb{P}(E_k(z))}$$

(6.28)

with the implicit constants independent of $n$ and $k$, the $\alpha_k(1)$ independent of $n$, and both deterministic and uniform for $z, w \in B_d(0)$. 

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Remark 6.21. In the setting of Proposition 6.20 we have
\[ e^{-\bar{\gamma}_{k}} = |z - w|^{1 + o_{|z - w|}(1)} \]
so by Lemma 6.26 below we can rewrite the estimate (6.28) as
\[ P(E_{n}(z) \cap E_{n}(w)) \leq |z - w|^{-\gamma(\eta_{z,w}^{0})} P(E_{n}(z)) P(E_{n}(w)). \]
This is the form of the estimate we will use when we prove lower bounds for the Hausdorff dimensions of our sets.

In order to prove Proposition 6.20 we first need to introduce an additional family of auxiliary flow lines which form the same pocket as \( h_{z,1} \) on the event \( E_{n}(z) \), but whose law is easier to analyze.

To this end, let \( \eta_{z,1}^{0} = \eta_{z,1}^{0} \). Let \( h_{z,1}^{0} \) be the first time \( \eta_{z,1}^{0} \) hits \( B_{h_{z,1}^{0} + \Delta} \) (here \( \Delta \) is as in the definition of \( L_{z,1} \)). Let \( h_{z,1} = h \circ f_{z,1}^{-1} - \chi \arg(f_{z,1}^{-1})' \). Let \( \eta_{z,1}^{\pm} \) be the flow lines of \( h_{z,1} \) started from \( \tau_{z,1} \) with angles \( \pm \theta \).

Let \( \tilde{h}_{z,1} \) be the connected component of \( D \setminus (\eta_{z,1}^{\pm} \cup \eta_{z,1}^{0}) \) which contains the origin. Let \( \tilde{p}_{z,1} \) be the conformal map from \( \tilde{h}_{z,1} \) to \( D \) with \( \tilde{p}_{z,1}(0) = 0 \) and \( \tilde{p}_{z,1}'(0) > 0 \).

Inductively, suppose \( j \geq 2 \) and that \( \eta_{z,j-1}^{0}, \tau_{z,j-1}, h_{z,j-1}^{0}, \tilde{h}_{z,j-1}, \tilde{D}_{z,j-1}, \) and \( \tilde{p}_{z,j-1} \) have been defined. Let \( \eta_{z,j}^{0} := \tilde{p}_{z,j-1} \circ \eta_{z,j-1}^{0} \setminus \tilde{D}_{z,j-1} \). Let \( \tau_{z,j} := \tilde{p}_{z,j-1} \circ \tilde{h}_{z,j-1}^{0} \). Let \( \tilde{h}_{z,j} = h_{z,j-1} \circ \tilde{p}_{z,j-1}^{-1} - \chi \arg(p_{z,j-1}^{-1})' \). Let \( \eta_{z,j}^{\pm} \) be the flow lines of \( \tilde{h}_{z,j} \) started from \( \tau_{z,j} \) with angles \( \pm \theta \). Let \( \tilde{D}_{z,j} \) be the connected component of \( D \setminus (\eta_{z,j}^{\pm} \cup \eta_{z,j}^{0}) \) which contains the origin. Let \( \tilde{p}_{z,j} \) be the conformal map from \( \tilde{D}_{z,j} \) to \( D \) with \( \tilde{p}_{z,j}(0) = 0 \) and \( \tilde{p}_{z,j}'(0) > 0 \).

See Figure 6.8 for an illustration of this construction.

Let
\[ \hat{\eta}_{z,j}^{\pm} := (f_{z,j}^{-1} \circ \tilde{p}_{z,j}^{-1} \circ \cdots \circ \tilde{p}_{z,1}^{-1})(\eta_{z,j}^{\pm}) \quad \text{and} \quad \hat{D}_{z,j} := (f_{z,j}^{-1} \circ \tilde{p}_{z,j}^{-1} \circ \cdots \circ \tilde{p}_{z,1}^{-1})(\tilde{D}_{z,j}) \]
be, respectively, the flow lines for \( h \) corresponding to \( \eta_{z,j}^{\pm} \) and the pocket they form surrounding \( z \).

Our interest in the above objects stems from the following lemma.
Lemma 6.22. Define the objects $\tilde{h}_{z,j}^0$, $\hat{D}_{z,j}$, etc. as above, and retain the notation of Section 6.2. If $\beta_1$ (and hence each $\beta_j$ for $j \geq 1$) is chosen sufficiently large, depending only on the parameter $\hat{\iota}$, then the following is true.

1. The field $h_{z,j}$ is conditionally independent from $h|_{D\setminus \hat{D}_{z,j-1}}$ given $\hat{D}_{z,j}$ and $h|_{\partial \hat{D}_{z,j-1}}$.

2. Let $\hat{x}_{z,j}$ and $\hat{y}_{z,j}$ be the start and end points of $\tilde{h}_{z,j}$. Conditional on $\hat{D}_{z,j-1}$ and $h|_{D\setminus \hat{D}_{z,j-1}}$, we have that $\tilde{h}_{z,j}$ is the zero-angle flow line from $\hat{x}_{z,j}$ to $\hat{y}_{z,j}$. In particular, the law of $\tilde{h}_{z,j}$ under this conditioning is that of a chordal SLE$_\kappa(\rho^1; \rho^1)$ from $\hat{x}_{z,j}$ to $\hat{y}_{z,j}$ in $D$, with $\rho^1$ as in (6.11).

3. On the event $E_n(z)$, we a.s. have $\eta_{z,j}^0 = \tilde{h}_{z,j}$ for each $j \leq n+1$ and $\hat{D}_{z,j} = \hat{D}_{z,j}$ for each $j \leq n$.

Proof. To obtain assertion 1, we start by observing that by [MS12a] Theorem 1.1 and induction, for each $j \geq 2$, the set $\hat{A}_{z,j-1} := \eta_{z,j-1}^0 \cup \eta_{z,j-1}^+ \cup \eta_{z,j-1}^-$ is a local set for $h$ in the sense of [SS13] Section 3.3. Hence the conditional law of $h$ given $\tilde{h}_{z,j}$ and $h|_{\hat{D}_{z,j-1}}$ in each complementary connected component of $D \setminus \hat{A}_{z,j-1}$ is independently of that of a zero-boundary GFF plus a certain $(\hat{A}_{z,j-1}, h|_{\hat{D}_{z,j-1}})$-measurable harmonic function. In particular, the conditional law of $h_{z,j}$ given $\tilde{h}_{z,j}$ and $h|_{\hat{D}_{z,j-1}}$ is that of a GFF on $D$ with boundary data $\lambda - \theta \chi - \chi \cdot \text{winding on } [\hat{y}_{z,j}, \hat{x}_{z,j}]_{\partial D}$ and $-\lambda + \theta \chi$ on $\lambda - \theta \chi - \chi \cdot \text{winding on } [\hat{x}_{z,j}, \hat{y}_{z,j}]_{\partial D}$, where $\lambda$ and $\chi$ are as in Section 2.5.

By [MS12a] Theorem 1.2] and locality, $\hat{A}_{z,j-1}$ is a.s. determined by $\hat{D}_{z,j-1}$ and $h|_{D\setminus \hat{D}_{z,j-1}}$. Hence we get the same conditional law for $\hat{h}_{z,j}$ if we instead condition on $\hat{D}_{z,j-1}$ and $h|_{D\setminus \hat{D}_{z,j-1}}$. Since this law depends only on $\hat{D}_{z,j-1}$ and $h|_{\partial \hat{D}_{z,j-1}}$, we obtain assertion 1.

Assertion 2 follows immediately from our description of the conditional law of $\hat{h}_{z,j}$ and [MS12a] Theorems 1.1 and 2.4.

It remains to prove assertion 3. See Figure 6.5 for an illustration. By the Koebe distortion theorem, we have that as $\beta_j \rightarrow \infty$, $e^{\beta_j + \Delta} \text{diam } \psi_{z,j}(B_{\beta_j + \Delta})$ and $e^{\beta_j + \Delta} \text{dist}(0, \partial \psi_{z,j}(B_{\beta_j + \Delta}))$ tend uniformly to $|\psi_{z,j}(0)|^{-1}$. Thus if $\beta_1$ (and hence $\beta_j$) is chosen sufficiently large, then by condition 3 in the definition of $L_{z,j}$ we have

$$B_{\beta_j + \Delta} \subseteq \psi_{z,j}(B_{\beta_j + \Delta}) \subseteq B_{\beta_j}.$$  

(6.29)

By (6.29) and condition 2 in the definition of $F_{z,1}$, it follows that on $E_1(z)$, the starting point for the curves $\psi_{z,1}(\tilde{h}_{z,1}^0)$ is disconnected from $\partial D$ by the parts of $\eta_{z,1}^0$ traced before they hit $h_{z,1}$. Therefore, $\eta_{z,1}^0$ a.s. hit and (by [MS12a] Theorem 1.5) subsequently merge with $\psi_{z,1}(\tilde{h}_{z,1}^0)$ before reaching $h_{z,1}$. This proves assertion 3 in the case $n = 1$. The general case follows from (6.29) and induction.

Note that Lemma 6.22 implies in particular that the conditional law of $h_{z,j}^0$ given $\hat{D}_{z,n-1}$ and $h|_{D\setminus \hat{D}_{z,n-1}}$ on the event $E_{n-1}(z)$ is that of a chordal SLE$_\kappa(\rho^1; \rho^1)$ process from $x_{z,n}$ to $y_{z,n}$ in $D$. Furthermore, $h_{z,n}^0$ is equal to a flow line of $h|_{\hat{D}_{z,n-1}}$ on this event. Since the parts of $\eta$ and the auxiliary flow lines which generated $F_{z,n-1}$ all lie outside $\hat{D}_{z,n-1}$, [MS12a] Theorem 1.2] implies that $F_{z,n-1}$ is a.s. determined by $\hat{D}_{z,n-1}$ and $h|_{D\setminus \hat{D}_{z,n-1}}$ on the event $E_{n-1}(z)$.

For $n \geq 1$, define the event $\hat{L}_{z,n}$ and the curve $\hat{h}_{z,n}$ in the same manner as the event $L_{z,n}$ and the curve $\hat{h}_{z,n}$ but with $\tilde{h}_{z,n}$ in place of $\tilde{h}_{z,n}^0$. Also fix $\hat{\alpha} \in (0, 2)$ and let

$$\hat{K}_{z,n} = \hat{K}_{z,n}(\hat{\alpha}) := \{ |\hat{x}_{z,n+1} - \hat{y}_{z,n+1}| \geq \hat{\alpha} \}.$$  

(6.30)

We can now prove the analogue of Lemma 6.8 for $n \geq 2$.

Lemma 6.23. Suppose we are in the setting of Lemma 6.22. If the parameters involved in the definition of $L_{z,1}$ are chosen appropriately, independently of $n$, $\beta_n$, $u_n$, and $z \in B_\delta(0)$ then we can find a deterministic $p > 0$ which does not depend on $n$, $\beta_n$, $u_n$, and $z \in B_\delta(0)$ such that for each $n \geq 2$, we have

$$\mathbb{P}(L_{z,n} | F_{z,n-1}) 1_{E_{n-1}(z)} \geq p 1_{E_{n-1}(z)}, \quad \mathbb{P}(\hat{L}_{z,n} | \hat{D}_{z,n-1}, h|_{D\setminus \hat{D}_{z,n-1}}) 1_{K_{z,n-1}} \geq p 1_{K_{z,n-1}}.$$
Figure 6.9: An illustration of the proof of assertion 3 of Lemma 6.22 in the case \( j = 1 \). The inner dotted quasi-circle is the image of \( B_{\beta_1+\Delta} \) under \( \psi_{z,1} \). The orange flow lines \( \psi_{z,1}(\dot{\eta}_{z,1}^+) \) quickly merge with \( \dot{\eta}_{z,1}^+ \) and form the same pocket around zero as the flow lines \( \dot{\eta}_{z,1}^- \) on the event \( E_{z,1} \).

**Proof.** We prove only the statement for \( L_{z,n} \). The statement for \( \hat{L}_{z,n} \) is proven in exactly the same manner. Let \( A \) be the event that the part of \( \eta_{z,n}^0 \) lying between \( \eta_{z,n}^0(\sigma_{z,1}) \) and \( \eta_{z,n}^0(\bar{\sigma}_{z,1}) \) never exits \( B_\Delta \). Let \( B \) be the event that all of the conditions in the definition of \( L_{z,n} \) except possibly condition 7 occur.

In light of Lemma 6.22, we can apply [MW14, Lemma 2.5] to get that if the parameters are chosen appropriately, independently of \( n \), then \( \mathbb{P}(A \cap B | \mathcal{F}_{z,n-1}) \mathbf{1}_{\mathcal{E}_{n-1}(z)} \geq \mathbf{1}_{\mathcal{E}_{n-1}(z)} \) (here we use condition 4 in the definition of \( \mathcal{F}_{z,n-1} \) to get that \( x_{z,n} \) and \( y_{z,n} \) lie at uniformly positive distance from one another on \( \mathcal{E}_{n-1}(z) \)). Hence \( \mathbb{P}(A | B, \mathcal{F}_{z,n-1}) \mathbf{1}_{\mathcal{E}_{n-1}(z)} \geq 1 \). It follows that there exists a sub-event of \( B \) with uniformly positive probability and a \( p_L > 0 \) on which \( \mathbb{P}(A | (\eta_{z,n}^0)_{z,n} \cup (\bar{\eta}_{z,n}^0), \mathcal{F}_{z,n-1}) \mathbf{1}_{\mathcal{E}_{n-1}(z)} \geq p_L \mathbf{1}_{\mathcal{E}_{n-1}(z)} \). On this sub-event, \( L_{z,n} \) occurs.

For \( n \in \mathbb{N} \) and \( j \in \mathbb{N} \), define flow lines \( \eta_{z,j}^+ \), times \( \bar{t}_{z,j}^+, \bar{t}_{z,j}^- \), and events \( \bar{E}_{z,j} \) in the same manner as \( \eta_{0,j}, \bar{t}_{0,j}^+, \bar{t}_{0,j}^-, \) and events \( \bar{E}_{z,j} \) but with \( h \) replaced by \( \tilde{h}_{z,j} \); \( \eta_{0,1} = \eta \) replaced by \( \eta_{z,n}^0 \), and the sequence \((\beta_j, u_j)_{j \in \mathbb{N}} \) replaced by \((\beta_{n+j}, u_{n+j})_{j \in \mathbb{N}} \) (we retain the above definition of \( L_{z,n} \)). Also let

\[
\hat{E}_k^n(z) := \bigcap_{j=1}^k \mathcal{E}_{z,j}^n.
\]

**Lemma 6.24.** Suppose we are in the setting of Section 6.2. If the auxiliary parameters are chosen appropriately, independently of \( n \), \((\beta_j, u_j)_{j \geq 1} \), and \( z \in B_d(0) \), and if \( \beta_1 \) is chosen sufficiently large, then the following three laws are a.s. s.m.a.c. (Definition 6.21) with deterministic (i.e., independent of whatever realization we are conditioning on) constants uniform in \( n \), \((\beta_j, u_j)_{j \geq 1} \), and \( z \in B_d(0) \).

1. The conditional joint law of \( \eta_{z,n} \) and \( \left\{ \left( \eta_{z,j}^{\bar{t}_{z,j}^+, \bar{t}_{z,j}^-} \right)_{j \geq n}, \left( \eta_{z,j}^{\bar{t}_{z,j}^+, \bar{t}_{z,j}^-} \right)_{j \geq n} \right\} \) given any realization of \( \mathcal{F}_{z,n-1} \cup \mathcal{F}_{z,n} \) for which \( \mathcal{E}_{n-1}(z) \cap L_{z,n} \) occurs, restricted to the event \( \mathcal{G}(\eta_{z,n}, \mu) \).

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[7] The flow lines \( \eta_{z,j}^{\bar{t}_{z,j}^+} \) are only defined on the event \( \eta_{z,j} \) hits a certain ball centered at the origin. We take these flow lines to be equal to a “graveyard point” in our probability space on the event that \( \eta_{z,j} \) does not hit such a ball.
2. The conditional joint law of \( \eta_{z,n} \) and \( \left\{ (\eta_{z,j}^{n+})^{t_{z,j}^+}, (\eta_{z,j}^{n-})^{t_{z,j}^-} \right\} \) given any realization of \( \tilde{D}_{z,n-1} \) and \( h|_{\tilde{D}_{z,n-1}} \) for which \( K_{z,n-1} \cap L_{z,n} \) occurs, restricted to the event \( G'(\eta_{z,n}, \mu) \).

3. The conditional joint law of \( \eta_{z,1} \) and \( \left\{ (\eta_{z,1}^{+})^{t_{z,1}^+}, (\eta_{z,1}^{-})^{t_{z,1}^-} \right\} \) given any realization of \( F^0_{z,1} \) for which \( L_{z,1} \) occurs, restricted to the event \( G'(\eta_{z,1}, \mu) \), with the sequence \( (\beta_j, u_j)_{j \in \mathbb{N}} \) replaced by \( (\beta_{n+j}, u_{n+j})_{j \in \mathbb{N}} \).

**Proof.** Throughout, we assume that all implicit constants for S.m.a.c. are deterministic, independent of \( n \) and of \( (\beta_j, u_j)_{j \geq 1} \), and uniform for \( z \in B_d(0) \). We will prove only the strict mutual absolute continuity of the laws \([1] \) and \([3]\). Strict mutual absolute continuity of of the laws \([1]\) and \([3]\) is immediate from assertions \([1]\) and \([3]\) of Lemma 6.22.

Let \( \omega \) be a realization of \( F_{z,n-1} \vee F^0_z \), for which \( E_{n-1}(z) \cap L_{z,n} \) occurs. Let \( A \) be the event that the part of \( \eta^0_{z,n} \) between \( \eta^0_{z,n}(\sigma_{z,n}) \) and \( \eta^0_{z,n}(\sigma_{z,n}) \) never exits \( B^\Delta \). By Lemma 6.22, the conditional law of \( \eta^0_{z,n} \) given any realization of \( F_{z,n-1} \), for which \( E_{n-1}(z) \) occurs is that of an SLE\(_\kappa(\rho^0; \rho^2) \) process from \( x_{z,n} \) to \( y_{z,n} \) in \( D \) with force points located on either side of \( x_{z,n} \). Since the start and endpoints of \( \eta^0_{z,n} \) are equal to \(-i\) and \( i \), Lemma 6.4 implies that (provided \( \Delta \) is chosen sufficiently large) the conditional law of \( \eta^0_{z,n} \) given \( \omega \) and \( A \) is s.m.a.c. with respect to the law of a chordal SLE\(_\kappa \) from \(-i\) to \( i \) in \( D \) given (equivalently, restricted to) the event that it never leaves \( \psi_{z,n}(B^\Delta) \). By condition 7 in the definition of \( L_{z,n} \), the same is true of the conditional law of \( \eta^0_{z,n} \) given \( \omega \), restricted to the event \( A \). By condition \([4]\) in the definition of \( L_{z,n} \), we have \( G'(\eta_{z,n}, \mu) \cap L_{z,n} \subset A \), so the same is also true of the conditional law of \( \eta^0_{z,n} \) given \( \omega \), restricted to the event \( G'(\eta_{z,n}, \mu) \).

By the results of [MS12a], Section 7, the conditional law of \( \eta^+_{z,n} \) given \( \omega \) and \( \eta \) is that of a chordal SLE\(_\kappa(\rho^0; \rho^1; \rho^2) \) from \( \eta^+_{z,n}(\tau_{z,n}) \) to \( \psi_{z,n}(y_{z,n}) \) in the connected component of \( D \setminus \eta_{z,n} \) containing the right side of \(-i\) on its boundary, with \( \rho^0, \rho^1 \) as in \([6.11]\) and \( \rho^2 \) depending only on \( \theta \) and \( \kappa \). The force points corresponding to the weights \( \rho^0 \) and \( \rho^2 \) are located on either side of \( \eta^+_{z,n}(\tau_{z,n}) \). The extra force point corresponding to \( \rho^1 \) is located at \( \psi_{z,n}(x_{z,n}) \).

By [MW14], Lemma 2.8, the conditional law of \( (\eta^+_{z,n})^{t_{z,n}} \) given \( \omega \) and any realization of \( \eta_{z,n} \) for which \( G'(\eta_{z,n}, \mu) \) occurs is s.m.a.c. with respect to the law of a chordal SLE\(_\kappa(\rho^0; \rho^1) \) from \( \eta^+_{z,n}(\tau_{z,n}) \) to any given point \( v \) on the right boundary of the aforementioned connected component of \( D \setminus \eta^0_{z,n} \), stopped at an appropriate time. A similar statement holds for \( (\eta^-_{z,n})^{t_{z,n}} \).

Since \( \eta^+_n \) and \( \eta^-_n \) are conditionally independent given \( \eta^0_{z,n} \) and \( F_{z,n-1} \), it follows that the joint law of \( (\eta^+_n)^{t_{z,n}}(\eta^-_n)^{t_{z,n}} \) given \( \omega \) and \( \eta \) is a.s. S.m.a.c. with respect to the law of a pair of curves with the same description as the joint law of \( (\eta^+_{z,1})^{t_{z,1}}, (\eta^-_{z,1})^{t_{z,1}}, \) given \( \eta_{z,1} \) and \( F^0_{z,1} \), restricted to the event \( L_{z,1} \cap G'(\eta_{z,1}, \mu) \), but with \( \beta_1 \) replaced by \( \beta_n \).

It remains to deal with the conditional laws of the remaining flow lines. Let \( h_{z,n+1} = \tilde{h} \circ \tilde{p}_{z,n} - \chi \) are \( \tilde{p}_{z,n}^{-1} \).

By the results of [MS12a], Section 7, the conditional law of \( h_{z,n+1} \) given \( \{ \eta_n(\eta^+_{z,n}), (\eta^-_{z,n})^{t_{z,n}} \} \) and \( F_{z,n-1} \) is that of an independent GFF in each of the complementary connected components of \( D \setminus \eta^0_{z,n+1} \) with boundary data depending only on \( \theta \) and \( \kappa \). By [MS12a, Theorem 1.2], under this conditioning the collection of curves

\[ \left\{ (\tilde{p}_{z,n}(\tilde{\eta}^+_{z,j})(t_{z,j})), \tilde{p}_{z,n}(\tilde{\eta}^-_{z,j})(t_{z,j})) : j \geq n + 1 \right\} \]

is a.s. determined by \( h_{z,n+1}, \eta_{z,n}, (\eta^+_n)^{t_{z,n}}, \) and \( (\eta^-_n)^{t_{z,n}} \), in the same manner that the collection of curves

\[ \left\{ (\tilde{p}_{z,1}(\tilde{\eta}^+_{z,1})(t_{z,j})), \tilde{p}_{z,1}(\tilde{\eta}^-_{z,1})(t_{z,j})) : j \geq 2 \right\} \]

is determined by \( h_{z,2}, \eta_{z,1}, (\eta^+_{z,1})^{t_{z,1}}, \) and \( (\eta^-_{z,1})^{t_{z,1}} \), except with \( (\beta_j, u_j)_{j \in \mathbb{N}} \) replaced by \( (\beta_{n+j}, u_{n+j})_{j \in \mathbb{N}} \).

By combining everything, we get the statement of the lemma. \( \square \)

**Lemma 6.25.** Let \( z \in D, m \in \mathbb{N}, \) and \( k \leq n \in \mathbb{N} \). Recall the definitions of the events from Section 6.3. We have

\[ \mathbb{P} \left( E_{0,n}^m(z) \right) \leq \mathbb{E} \left( E_{0,k}^m(z) \right) \mathbb{P} \left( E_{0,n-k}^m(z) \right) \]

(6.32)
with proportionality constants independent of \(n, m,\) and \(k\) and uniform for \(z \in B_d(0)\).

**Proof.** We have

\[
P \left( E_{0,n}^m(z) \right) = P \left( E_{0,k}^m(z) \right) P \left( E_{k,n}^m(z) \mid E_{0,k}^m(z) \right).
\]

Furthermore

\[
P \left( E_{k,n}^m(z) \mid E_{0,k}^m(z) \right) = P \left( L_{z,k+1} \mid E_{0,k}^m(z) \right) P \left( E_{k,n}^m(z) \mid L_{z,k+1} \cap E_{0,k}^m(z) \right).
\] (6.33)

The first factor on the right in (6.33) is bounded below by a positive constant by Lemma 6.23. Conditional on \(\mathcal{I}_{z,k+1} \cap \mathcal{F}_{z,k}\), the event \(E_{0,n}^m(z)\) is determined by \(\eta_{z,k+1}\) and the auxiliary flow lines \((\eta_{z,k+1}^j)^{j \leq j}\) for \(j \geq k+1\). Hence that Lemma 6.24 implies \(P \left( E_{k,n}^m(z) \mid L_{z,k+1} \cap E_{0,k}^m(z) \right) \prec P \left( E_{0,n-k}^m(z) \right)\), as required. \(\square\)

**Lemma 6.26.** For each \(z \in B_d(0)\) and each \(n, m \in \mathbb{N}\), we have

\[
e^{-\beta_{m,m+n} \gamma(q) - 3\bar{\gamma}_{m,m+n} \bar{\beta}(q)} \preceq P \left( E_{0,n}^m(z) \right) \preceq e^{-\beta_{m,m+n} \gamma(q) + 3\bar{\gamma}_{m,m+n} \bar{\beta}(q)}
\]

with the implicit constants independent of \(n\) and uniform for \(z \in B_d(0)\).

**Proof.** Let \(C_s\) and \(C_*\) be the proportionality constants in (6.32). By Lemma 6.25 we have

\[
C_s^n \prod_{j=0}^{n-1} P \left( E_{z,1}^m \right) \leq P \left( E_{0,n}^m(z) \right) \leq C_s^n \prod_{j=0}^{n-1} P \left( E_{z,1}^m \right),
\]

with the implicit constant independent of \(\beta, n,\) and \(m\) and uniform for \(z\) in compacts. By (6.12) we then have

\[
e^{-\beta_{m,m+n} \gamma(q) - \bar{\gamma}_{m,m+n} \bar{\beta}(q)} C_s\prod_{j=m}^{m+n-1} C_{w_j} \leq P \left( E_{0,n}^m(z) \right) \leq e^{-\beta_{m,m+n} \gamma(q) - \bar{\gamma}_{m,m+n} \bar{\beta}(q)} C_s\prod_{j=m}^{m+n-1} C_{w_j},
\] (6.34)

By condition 5 in Lemma 6.15 we have \(C_s^n \leq e^{\bar{\gamma}_{m,m+n} \bar{\beta}(q)}\). By condition 4 in Lemma 6.15 we also have \(\prod_{j=m}^{m+n-1} C_{w_j} \leq e^{\bar{\gamma}_{m,m+n} \bar{\beta}(q)}\). By plugging these estimates into (6.34) we get the desired conclusion. \(\square\)

**Lemma 6.27.** Let \(z, w \in B_d(0)\). Let \(k\) be as in assertion 3 of Lemma 6.16. Choose \(k \in \mathbb{N}\) such that \(e^{-\beta_{k+1} - \lambda k} \leq |z - w| \leq e^{-\beta_k - \lambda k}\). Then for any \(n \in \mathbb{N}\) with \(\beta_n - \lambda n \geq \beta_{k+1} + \lambda (k+2)\), we have

\[
P \left( E_{k,n}(z) \cap E_{k,n}(w) \mid E_k(z) \cap E_k(w) \right) \leq e^{-\beta_k a_1(n)} P \left( E_{0,n-k}(z) \right) P \left( E_{0,n-k}(w) \right)
\]

with the implicit constants independent of \(n\) and \(k\), the \(o_1(k)\) independent of \(n,\) and both deterministic and uniform for \(z, w \in B_d(0)\).

**Proof.** Throughout, we require implicit constants and \(o_1(k)\) terms to satisfy the conditions of the statement of the lemma.

Let \(k'\) be the least integer such that \(\beta_{k'} - \lambda k' \geq \beta_{k+1} + \lambda (k+2)\). Note \(k' \leq n\). Let \(\hat{P}_{z,k'}\) be the event that \(\text{diam}(\hat{D}_{z,k'}) \leq \frac{1}{2} e^{-\beta_{k'} + \lambda k'}\) and the event \(K_{z,k'}\) defined in (6.30) occurs. Define \(\hat{P}_{w,k'}\) similarly. By Lemma 6.22, we have \(\hat{D}_{z,k'} = \hat{D}_{z,k'} \cap E_k(z)\), so the objects involved in the definition (6.31) of \(\hat{E}_{0,n-k'}(z)\) agree with the objects involved in the definition of \(E_{k,n-k}(z)\) on this event. Similar statements hold with \(w\) in place of \(z\). By combining this observation with Condition 3 in Subsection 6.1 and assertion 4 of Lemma 6.16 we get that if \(\hat{a}\) (as defined just above (6.30)) is chosen sufficiently small, depending only on \(a\), then

\[
E_n(z) \subset \hat{E}_{n-k'}(z) \cap \hat{P}_{z,k'} \quad \text{and} \quad E_n(w) \subset \hat{E}_{n-k'}(w) \cap \hat{P}_{w,k'}.
\]

Therefore,

\[
P \left( E_{k,n}(z) \cap E_{k,n}(w) \mid E_k(z) \cap E_k(w) \right) = P \left( E_{n}(z) \cap E_{n}(w) \mid E_k(z) \cap E_k(w) \right)
\]

\[
\leq P \left( \hat{E}_{k-n-k'}(z) \cap \hat{P}_{z,k'} \cap \hat{E}_{n-k'}(w) \cap \hat{P}_{w,k'} \mid E_k(z) \cap E_k(w) \right)
\]

\[
= P \left( \hat{E}_{n-k'}(z) \cap \hat{E}_{n-k'}(w) \mid E_k(z) \cap E_k(w) \cap \hat{P}_{z,k'} \cap \hat{P}_{w,k'} \right) P \left( \hat{E}_{z,k'} \cap \hat{E}_{w,k'} \right).
\] (6.35)
So, we need only estimate the last line of (6.35).

On the event $\hat{P}_{z,k'} \cap \hat{P}_{w,k'}$, the domains $\hat{D}_{z,k'}$ and $\hat{D}_{w,k'}$ are disjoint. By assertion 4 of Lemma 6.16 on the event $E_k(z) \cap E_k(w) \cap \hat{P}_{z,k'} \cap \hat{P}_{w,k'}$ we have $\hat{D}_{z,k} \cup \hat{D}_{w,k'} \subset \hat{D}_{z,k} \cap \hat{D}_{w,k}$. The event $E_k(z)$ is determined by $\hat{D}_{z,k}$ and $h|_{\mathcal{D} \setminus \hat{D}_{z,k'}}$. Consequently, the event $E_k(z) \cap \hat{P}_{z,k'}$ is determined by $\hat{D}_{z,k'}$ and $h|_{\mathcal{D} \setminus \hat{D}_{z,k'}}$. Similar statements hold with $w$ in place of $z$.

Let $\mathcal{H}$ be the $\sigma$-algebra generated by $\hat{D}_{z,k'}$, $\hat{D}_{w,k'}$, $h|_{\mathcal{D} \setminus \hat{D}_{z,k'}}$, and $h|_{\mathcal{D} \setminus \hat{D}_{w,k'}}$. The above considerations together with Lemma 6.22 imply that the events $\hat{E}_{n-k'}(z)$ and $\hat{E}_{n-k'}(w)$ are conditionally independent given $E_k(z) \cap E_k(w)$ and $\mathcal{H}$ on the event $\hat{P}_{z,k'} \cap \hat{P}_{w,k'}$. We thus have

$$P \left( \hat{E}_{n-k'}(z) \cap \hat{E}_{n-k'}(w) \mid \mathcal{H}, E_k(z) \cap E_k(w) \right) 1_{\hat{P}_{z,k'} \cap \hat{P}_{w,k'}}$$

$$= P \left( \hat{E}_{n-k'}(z) \mid \mathcal{H}, E_k(z) \cap E_k(w) \right) P \left( \hat{E}_{n-k'}(w) \mid \mathcal{H}, E_k(z) \cap E_k(w) \right) 1_{\hat{P}_{z,k'} \cap \hat{P}_{w,k'}}.$$  

(6.36)

By Lemmas 6.23 and 6.24 we have

$$P \left( \hat{E}_{n-k'}(z) \mid \mathcal{H}, E_k(z) \cap E_k(w) \right) 1_{\hat{P}_{z,k'} \cap \hat{P}_{w,k'}} \approx P \left( E_{0,n-k'}^k(z) \right) 1_{\hat{P}_{z,k'} \cap \hat{P}_{w,k'}}.$$  

(6.37)

and similarly with $z$ and $w$ interchanged. By (6.35), (6.36), and (6.37),

$$P \left( E_k^n(z) \cap E_k^n(w) \mid E_k(z) \cap E_k(w) \right) \leq P \left( E_{0,n-k'}^k(z) \right) P \left( E_{0,n-k'}^k(w) \right).$$

(6.38)

By Lemma 6.25 we have

$$P \left( E_{0,n-k'}^k(z) \right) \approx \frac{P \left( E_{0,n-k'}^k(z) \right)}{P \left( E_{0,k'-k}^k(z) \right)}.$$  

(6.39)

By Lemma 6.26

$$P \left( E_{0,k'-k}^k(z) \right) \geq e^{-\beta_{k',k} \gamma^*(q) - 3\pi_{k',k} \gamma_0(q)}.$$  

By our choice of $k'$ we have $\beta_{k',k} \leq \lambda(k + k' + 1) + \beta_{k'}$. Since $\beta_j$ is increasing in $j$ we have $k' - k = o_k(1)\beta_k$ and $\lambda(k + k' + 1) = o_k(1)\beta_k$. By condition 3 in Lemma 6.15 we have $\beta_{k'} \leq \beta_{k + o_k(1)} \leq \beta_k, o_k(1)$. Therefore

$$P \left( E_{0,k'-k}^k(z) \right) = e^{k \alpha_k(1)\beta_k}. $$

Hence (6.39) implies

$$P \left( E_{0,n-k'}^k(z) \right) \leq e^\beta_{k, k'} o_k(1) P \left( E_{0,n-k}^k(z) \right).$$

A similar assertion holds with $w$ in place of $z$. We conclude by combining this with (6.38). □

**Proof of Proposition 6.20** We have

$$P \left( E_n(z) \cap E_n(w) \right) = P \left( E_{k,n}^k(z) \cap E_{k,n}^k(w) \cap E_k(z) \cap E_k(w) \right) \quad \text{(by definition)}$$

$$\leq P \left( E_{k,n}^k(z) \cap E_{k,n}^k(w) \mid E_k(z) \cap E_k(w) \right) P \left( E_k(z) \cap E_k(w) \right)$$

$$\leq e^{\beta_{k, k'} o_k(1)} P \left( E_{0,n-k}^k(z) \right) P \left( E_{0,n-k}^k(w) \right) P \left( E_k(z) \cap E_k(w) \right) \quad \text{(by Lemma 6.27)}.$$  

By Lemma 6.25 we have

$$P \left( E_{0,n-k}^k(z) \right) \approx \frac{P \left( E_n(w) \right)}{P \left( E_k(w) \right)}, \quad P \left( E_{0,n-k}^k(z) \right) \approx P \left( E_n(z) \right)$$

By combining the above relations we get (6.28). □
7 Lower bounds for multifractal and integral means spectra

7.1 Setup

Let \( \eta \) be a chordal SLE\(_{\kappa} \) from \(-i\) to \(i\) in \( \mathbb{D} \). Let \( D_\eta \) be as in Theorem 1.1 and define the sets \( \tilde{\Theta}^*(D_\eta) \) and \( \Theta^*(D_\eta) \) as in Section 1.1. The goal of this section is to obtain lower bounds on \( \dim_\eta \tilde{\Theta}^*(D_\eta) \) and \( \dim_\eta \Theta^*(D_\eta) \), and thereby complete the proof of Theorem 1.1. We accomplish this using the estimates of Section 6.

Throughout this section we use the notation defined in Sections 6.2 and 6.3 with

\[ q = \frac{s}{1-s}. \]

In particular, we recall the definition (6.15) of the \( n \)-perfect points and the definition of the exponents \( \gamma^*(q) \) and \( \gamma_0^*(q) \) from (6.1). In the next two sections we will use the \( n \)-perfect points to define various notions of “perfect points” which are contained in the sets we are interested in and which will allow us to obtain lower bounds on their Hausdorff dimensions.

In order to prove that the perfect points are contained in our sets of interest, we will need the following technical lemma.

Lemma 7.1. Let \( \Psi_\eta : D_\eta \to \mathbb{D} \) be as in Section 4.2. Suppose \( z \in \mathcal{P}_k \cap D_\eta \). For \( n \leq k-1 \) let \( I_{z,n} \) be the image under \( \Psi_\eta \) of \( \eta \cap \tilde{D}_{z,n} \). If the parameters for the events of Section 6.2 are chosen appropriately, independently of \( n \) and of \( z \in \mathcal{B}_D(0) \), we have the following.

1. We have \( e^{-\tilde{\eta}_n(q+1)-3\pi n} \leq \text{length } I_{z,n} \leq e^{-\tilde{\eta}_n(q+1)+3\pi n} \).

2. If \( n \leq k-2 \) the distance from \( I_{z,n+1} \) to \( \partial I_{z,n} \) is proportional to the length of \( I_{z,n} \).

3. If \( x \in I_{z,n} \) then there exists \( \delta_n > 0 \) such that \( |(\eta_\eta^{-1})(1-\delta_n) x| > \delta_n + o_n(1) \) and \( \delta_n \propto e^{-\tilde{\eta}_n(q+1+o_n(1))} \).

The implicit constants are independent of \( n \) and both the \( o_n(1) \) and the implicit constants are deterministic and independent of \( k, x, \) and \( z \in \mathcal{B}_D(0) \).

Proof. Fix such a \( z, n, \) and \( k \) as in the statement of the lemma. Throughout the proof we assume \( E_k(z) \) occurs and require all constants (either referred to as such or implicit in \( \varepsilon, \) etc.) to be independent of \( n, k, x, \) and \( z \in \mathcal{B}_D(0) \).

Recall the pocket \( \tilde{D}_{z,n} \) formed by the auxiliary flow lines \( \hat{\eta}_{z,n}^\pm \) from Section 6.2. The map \( \hat{\eta}_{z,n} : \tilde{D}_{z,n} \to \mathbb{D} \) defined in Section 6.2 takes \( z \) to \( 0 \) and \( \eta \cap \tilde{D}_{z,n} \) to the curve \( \eta_{z,n+1}^0 \), whose endpoints are \( x_{z,n+1} \) and \( y_{z,n+1} \). Note that condition 2 in the definition of \( E_{\tilde{x}_n}^\omega(\cdot) \) implies lower bound on \( |x_{z,n+1} - y_{z,n+1}| \), depending only on the parameter \( a \).

By conditions 1 and 4 in the definition of \( L_{z,n+1} \), there is a unique arc \( A_0 \) of \( \partial B_\Delta/2 \) which disconnects \( \eta_{z,n+1}^0 \cap \mathcal{B}_D \) from \( [x_{z,n+1}, y_{z,n+1}] \partial \mathbb{D} \) in \( \mathbb{D} \setminus \eta_{z,n+1}^0 \) (c.f. Remark 6.11). Let \( w^0 \) be the point of \( A_0 \) closest to the midpoint of \( [x_{z,n+1}, y_{z,n+1}] \partial \mathbb{D} \). Let \( D_0 \) be the connected component of \( \mathbb{D} \setminus \eta_{z,n+1}^0 \) containing \( [x_{z,n+1}, y_{z,n+1}] \partial \mathbb{D} \) on its boundary.

Observe that the harmonic measure from \( w^0 \) in \( D_0 \) of any sub-arc of \( [x_{z,n+1}, y_{z,n+1}] \partial \mathbb{D} \) is proportional to the length of that sub-arc. Furthermore, \( \text{hm}^w(\eta_{z,n+1}^0; D_0) \approx 1 \). Define \( \psi_{\eta_{z,n}^n} \) as in the definition of \( F_{z,n} \).

By condition 2 in the definition of \( E_{\tilde{x}_n}^\omega(\cdot) \) and condition 2 in the definition of \( E_{\tilde{x}_n}^\omega(\cdot) \) that the harmonic measure from \( \psi_{\eta_{z,n}^n} \) in \( D_0 \) of \( [x_{z,n+1}, y_{z,n+1}] \partial \mathbb{D} \) is the image of the right side of \( \eta_{z,n}^n \) (resp. the left side of \( \eta_{z,n}^n \)) under \( \psi_{\eta_{z,n}^n} \) has length \( \approx 1 \). By conformal invariance of Brownian motion and condition 2 in the definition of \( F_{z,n} \), provided \( r \) is chosen sufficiently small relative to \( \Delta \), the harmonic measure from \( (\psi_{\eta_{z,n}^n}(\eta_{z,n}^n))(w^0) \) in the right connected component of \( \mathbb{D} \setminus \psi_{\eta_{z,n}^n}(\eta_{z,n}^n) \) of each of these two sub-arcs is \( \approx 1 \).

Let \( w = \hat{\eta}_{z,n}^{-1}(w^0) \). It follows from the above considerations and conformal invariance of Brownian motion that (notation as in Section 2.1)

\[ \text{hm}^w(\eta_{z,n}^n; D_\eta) \propto \text{hm}^w(\eta_{z,n}^n; D_\eta) \approx 1, \quad \text{hm}^w(\eta \cap \tilde{D}_{z,n}; D_\eta) \approx 1. \]
In this subsection we will prove a lower bound on the Hausdorff dimension of the sets $\Theta^s(D_\eta)$. Assume we are in the setting of Section 7.1.
We define the set $P$ of perfect points as follows. Let $\lambda$ be the constant from Proposition 6.20. For $n \in \mathbb{N}$, let $n'$ be the greatest integer such that $\beta_n - \lambda n \geq \beta_{n'} + 1 + \lambda(n' + 2)$. Let

$$
\epsilon_n := e^{-7_n + \lambda(n' + 2)},
$$

(7.6)

Note that Lemma 6.15 implies $e^{-7_n} = e^{1+o_n(1)}$. Our reason for choosing this value of $\epsilon_n$ is that the pockets $D_{\hat{z},n}$ and $D_{w,n}$ are disjoint on $E_n(z) \cap E_n(w)$ provided $|z - w| \geq \epsilon_n$ (see Lemma 6.16).

Choose a collection $C_n$ of $\epsilon_n$ points in $B_d(0)$, no two of which lie within distance $\epsilon_n$ of each other. For $z \in C_n$ let $B_n(z)$ be the ball of radius $\epsilon_n$ centered at $z$. Let $C_n' = C_n \cap P_n$ be the set of $z \in C_n$ for which $E_n(z)$ occurs. Let

$$
P := \bigcap_{n \geq 1} \bigcup_{k \geq 0} B_k(z).
$$

(7.7)

**Lemma 7.2.** Define $P$ as in (7.7). We have $P \subset \Theta^s(D_\eta)$ for $s = q/(q + 1)$. In fact, if $w \in P$, then for $\epsilon > 0$ we have

$$
||\Psi_\eta^{-1}'((1 - \epsilon)\Psi_\eta(w))|| \cong \epsilon^{-s+o(1)},
$$

(7.8)

with the implicit constants and the $o(1)$ deterministic and uniform for $w \in B_d(0)$.

**Proof.** Fix $w \in P$. Since $\eta$ is closed, it is clear that $w \in \eta$. It remains to prove (7.8). By definition of $P$, if we are given $n \in \mathbb{N}$, then we can find $k \geq n + 1$ such that $|z - w| \leq e^{-27_n+1}$. By Lemma 6.16 we have $w \in D_{\hat{z},n}$, so $\Psi_\eta(w) \in I_{\hat{z},n}$, as defined in Lemma 7.1. Let $\delta_n$ be as in that lemma with $x = \Psi_\eta(w)$.

By the Koebe distortion theorem, for $\epsilon \in [\delta_{n+1}, \delta_n]$ we have

$$
1 - (\delta_n - \delta_{n+1})/\delta_n \leq \frac{||\Psi_\eta^{-1}'((1 - \epsilon)\Psi_\eta(w))||}{(1 + (\delta_n - \delta_{n+1})/\delta_n)^3} \leq 1 + (\delta_n - \delta_{n+1})/\delta_n.
$$

(7.9)

By Lemma 7.1 we have

$$
1 - (\delta_n - \delta_{n+1})/\delta_n \cong e^{-\beta_n + \lambda o_n(1)},
$$

which is proportional to $e^{-7_n o_n(1)} \cong e^{o(1)}$ by Lemma 6.15. We furthermore have $\delta_n = e^{1+o(1)}$. Hence (7.9) and Lemma 7.1 imply $||\Psi_\eta^{-1}'((1 - \epsilon)\Psi_\eta(w))|| \cong \epsilon^{-s+o(1)}$, as required.

**Proposition 7.3.** Let $s_- , s_+$ be as in Theorem 1.1. For each $s \in (s_-, s_+)$, we a.s. have

$$
\dim_H \Theta^s(D_\eta) \geq \xi(s),
$$

where $\xi(s)$ is as in 1.4.

**Proof.** For a Borel measure $\nu$ on a metric space $X$ and $\alpha > 0$, write

$$
I_\alpha(\nu) = \int_X \int_X \frac{d\nu(z) d\nu(w)}{|z - w|^\alpha}
$$

(7.10)

for the $\alpha$-energy of $\nu$. By standard results for Hausdorff dimension (see [MP10] Theorem 4.27]) a metric space which admits a positive finite measure with finite $\alpha$-energy has Hausdorff dimension at least $\alpha$. In view of Lemma 7.2 we are lead to construct such a measure $\nu$ on $P$ for each $\alpha < \xi(s)$. We do this via the usual argument (see, e.g. [MW14], [HMP10], [Bef08]).

Define the events $E_n(z)$ as in Section 6.3 and the sets of points $C_n$ and $C_n'$ and the balls $B_n(z)$ as in the definition of $P$ (right above (7.7)). Let $\epsilon_n$ be as in (7.6).

For each $n \in \mathbb{N}$, define a measure $\nu_n$ on $D$ by

$$
d\nu_n(x) = \sum_{z \in C_n} \frac{1_{E_n(z)}}{P(E_n(z))} 1_{(x \in B_n(z))} dx.
$$

Then $E(\nu_n(D)) \cong 1$. Moreover,

$$
E(\nu_n(D)^2) \leq \epsilon_n^4 \sum_{z \not= w \in C_n} \frac{P(E_n(z) \cap E_n(w))}{P(E_n(z)) P(E_n(w))} + \epsilon_n^4 \sum_{z \in C_n} \frac{1}{P(E_n(z))}.
$$

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By Lemma 6.26 and Proposition 6.20 this is bounded by an \( n \)-independent constant times
\[
\epsilon_n^4 \sum_{z \neq w \in C_n} |z - w|^{-\gamma(q) + o_{1-z-w}(1) + o_n(1)} + \epsilon_n^4 \sum_{z \in C_n} \epsilon_n^{-\gamma(q) + o_n(1)},
\]
with the \( o_{1-z-w}(1) \) independent of \( n \). For \( s \in (s_-, s_+) \) we have \( \gamma(q) = \gamma(s)/(1 - s) < 2 \). Therefore, for sufficiently large \( n \), \( \mathbb{E}(\nu_n(D)^2) \) is bounded above by a finite, \( n \)-independent constant. By the Vitali convergence theorem, we can a.s. find a subsequence of the measures \( \nu_n \) which converges weakly to a measure \( \nu \) whose total mass is bounded above by some deterministic constant, and whose expected mass is positive.

On the other hand, we have
\[
\mathbb{E}(I_\alpha(\nu_n)) = \sum_{z,w \in C_n} \frac{\mathbb{P}(E_n(z) \cap E_n(w))}{\mathbb{P}(E_n(z)) \mathbb{P}(E_n(w))} \int_{B^n(z) \times B^n(w)} \frac{1}{|x - y|^{\alpha}} \, dx \, dy
\]
\[
= \sum_{z \neq w \in C_n} \frac{1}{\mathbb{P}(E_n(z)) \mathbb{P}(E_n(w))} \int_{B^n(z) \times B^n(w)} \frac{1}{|x - y|^{\alpha}} \, dx \, dy + \sum_{z \in C_n} \frac{\epsilon_n^4}{\mathbb{P}(E_n(z))}
\]
\[
\leq \sum_{z \neq w \in C_n} |z - w|^{-\gamma(q) + o_{1-z-w}(1) + o_n(1)} \epsilon_n^4 + \epsilon_n^2 \, \delta_n^{2-\gamma(q)+o_n(1)}.\]

We have \( \gamma(q) + \alpha < 2 \) for \( s \in (s_-, s_+) \) and \( \alpha < \xi(s) \), so the above expression is \( \leq 1 \). We conclude that any subsequential limit \( \nu \) of the measures \( \nu_n \) satisfies \( I_\alpha(\nu) < \infty \). Our definition (7.7) of \( \mathcal{P} \) is such that \( \nu \) is necessarily supported on \( \mathcal{P} \). Hence [MP10] Theorem 4.27 and Lemma 7.2 imply that with positive probability, we have \( \dim_H \Theta^*(D_n) \geq \xi(s) \). Proposition 2.16 implies that this in fact a.s. holds.

7.3 Lower bound for the Hausdorff dimension of the subset of the circle

In order to get a lower bound on the Hausdorff dimension of \( \Theta^*(D_n) \), we will need a different set of perfect points. Define \( \epsilon_n \), the sets \( C_n \), \( C_n' \) as in the definition (7.7) of \( \mathcal{P} \). For \( z \in C_n' \), let \( I_{z,n-1} \) be as in the statement of Lemma 7.1. Let \( x_{z,n} \) be the midpoint of \( I_{z,n-1} \) and let \( I_{z,n}' \) be the arc of length \( \epsilon_n^{k+1} \) centered at \( x_{z,n} \). By Lemma 7.1 we have \( \text{length}(I_{z,n}') = \text{length}(I_{z,n-1})^{1 + o_n(1)} \). Let
\[
\mathcal{P} := \bigcap_{n \geq 1} \bigcup_{k \geq n} \bigcup_{z \in C_n'} I_{z,k-1}'.
\]

**Lemma 7.4.** Define \( \mathcal{P} \) as in (7.11). If \( \Delta \) and \( \beta_1 \) are chosen sufficiently large then \( \mathcal{P} \subset \Theta^*(D_n) \) for \( s = q/(q + 1) \). In fact, if \( x \in \mathcal{P} \), then for \( \epsilon > 0 \) we have
\[
|((\Psi_n^{-1})'(1 - \epsilon)) x| \asymp \epsilon^{-s + o_n(1)}\]
with the implicit constants and the \( o_n(1) \) deterministic and uniform in \( x \).

**Proof.** If \( x \in \mathcal{P} \) then for any given \( n \in \mathbb{N} \) we can find \( k \geq n \) and \( z \in C_n' \) such that \( x \) lies within distance \( \text{length}(I_{z,k})^2 \) of \( I_{z,k}' \). If \( k \) is chosen sufficiently large, depending on \( n \), then by assertions [1] and [2] of Lemma 7.1 we have \( x \in I_{z,n} \). We then conclude as in the proof of Lemma 7.2.

**Lemma 7.5.** For each \( n \) there is an integer \( m_n \leq n \) such that the following is true. We have \( \overline{\beta_n} - \underline{\beta_n} = \overline{\beta_n} o_n(1) \) and if \( z,w \in C_n \) with \( |z - w| \geq e^{-\overline{\beta_n} - m_n-1} \) then we have \( \text{dist}(I_{z,m_n}', I_{w,m_n}') \geq |z - w|^{q + 1 + o_{1-z-w}(1)} \), with the \( o_{1-z-w}(1) \) and implicit constants deterministic, independent of \( n \), and uniform for \( z,w \in C_n \).
Proposition 7.6. Let \( s_- , s_+ \) be as in Theorem 1.1. For each \( s \in (s_- , s_+) \) we a.s.
\[
\dim_{\mathcal{H}} \Theta^*(D_n) \geq \tilde{\xi}(s),
\]
where \( \tilde{\xi}(s) \) is as in (1.3).

Proof. We argue as in the proof of Proposition 7.3. In particular, for any given \( \alpha < \tilde{\xi}(s) \), we will construct a positive finite measure \( \tilde{\nu} \) on \( \tilde{\mathcal{D}} \) (as defined in (1.11)) with finite \( \alpha \)-energy, as defined in (7.10).

Define \( \epsilon_n \) as in (7.6). We require all implicit constants and \( o_{|z-w|}(1) \) terms to be independent of \( n \) and uniform for \( z,w \in \mathcal{C}_n \). For \( n \in \mathbb{N} \), define a measure \( \tilde{\nu}_n \) on \( \partial \mathcal{D} \) by
\[
d\tilde{\nu}_n(x) = \epsilon_n^{-q} \sum_{z \in \mathcal{C}_n} \frac{1}{P(E_n(z))} \mathbf{1}_{(x \in I_n')} \, dx.
\]

Then we have \( \mathbb{E}(\tilde{\nu}_n(\partial \mathcal{D})) \asymp 1 \).

As in the proof of Proposition 7.3 we have
\[
\mathbb{E}(\tilde{\nu}_n(\partial \mathcal{D})^2) \leq \epsilon_n^4 \sum_{z \neq w \in \mathcal{C}_n} \frac{P(E_n(z) \cap E_n(w))}{P(E_n(z))P(E_n(w))} + \epsilon_n \sum_{z \in \mathcal{C}_n} \epsilon_n^{-\gamma(q)+o_n(1)} \ll 1.
\]

Let \( m_n \) be as in Lemma 7.5 and let \( \mathcal{K}_n \) be the set of pairs \((z,w) \in \mathcal{C}_n \times \mathcal{C}_n\) with \( |z-w| \leq e^{-\tilde{\beta}m_n} \) and \( z \neq w \). By Lemma 7.5 we have \#\mathcal{K}_n \leq \epsilon_n^{-2-o_n(1)}.

By Lemma 6.26 and Proposition 6.20 and Lemma 7.5 we have
\[
\mathbb{E}(I_\alpha(\tilde{\nu}_n)) = \epsilon_n^{-2\alpha} \sum_{(z,w) \in \mathcal{C}_n \times \mathcal{C}_n} \frac{P(E_n(z) \cap E_n(w))}{P(E_n(z))P(E_n(w))} \iint_{I_n \times I_n'} \frac{1}{|x-y|^\alpha} \, dx \, dy
\]
\[
\leq \epsilon_n \sum_{z \neq w \in \mathcal{K}_n, z \neq w} |z-w|^{-\gamma(q)+o_{|z-w|}(1)} |x_{z,n} - x_{w,n}|^{-\alpha} \epsilon_n^{2(1+q)+2-2q}
\]
\[
+ \epsilon_n \sum_{(z,w) \in \mathcal{K}_n} |z-w|^{-\gamma(q)+o_{|z-w|}(1)} \epsilon_n^{2(1+q)+2-2q+o_n(1)}
\]
\[
+ \epsilon_n \sum_{z \in \mathcal{C}_n} \epsilon_n^{2(1+q)+2-2q - \gamma(q)+o_n(1)}
\]
\[
\leq \epsilon_n^4 \sum_{z \neq w \in \mathcal{C}_n} |z-w|^{-\gamma(q)+o_{|z-w|}(1)} + \epsilon_n^{2(1+q)+2-2q - \gamma(q)+o_n(1)} + \epsilon_n^{2(1+q)+2-2q - \gamma(q)+o_n(1)} + \epsilon_n^{2(1+q)+2-2q - \gamma(q)+o_n(1)}.
\]

Note that for the middle term we used \( |z-w| \geq \epsilon_n \) and \#\mathcal{K}_n \leq \epsilon_n^{-2-o_n(1)} \). If \( s \in (s_- , s_+) \) and \( q = s/(1-s) \) we have \( \gamma(q) + \alpha(q+1) < 2 \) and \( (2-\alpha)(q+1) - 2q - \gamma(q) > 0 \) for \( \alpha < \tilde{\xi}(s) \). It follows that we can a.s.
find a subsequence of the measures \( (\tilde{\nu}_n) \) which converges weakly a.s. to a finite positive limiting measure supported on \( \tilde{\mathcal{D}} \) with finite \( \alpha \)-energy. We then conclude using [MP10, Theorem 4.27], Lemma 7.4 and Proposition 2.16. □
7.4 Proof of Theorem 1.1

This follows by combining Propositions 5.1, 5.9, 7.3 and 7.6.

Remark 7.7. In the case \( \kappa = 4 \), we have \( s_+ = 1 \), so the sets \( \Theta^1(D_\eta) \) and \( \Theta^1(D_n) \) for \( \kappa = 4 \) can be non-empty. We do not explicitly mention these sets in Theorem 1.1 because the statement of our results in this case is slightly different. We are now going to explain how to obtain analogs of the statements of Theorem 1.1 in the case \( \kappa = 4, s_+ = 1 \). We prove in Proposition 5.1 that a.s. \( \dim_H \Theta^1(D_\eta) = 0 \). Since \( \xi(1) = \dim_H(\eta) = 3/2 \) for \( \kappa = 4 \), the upper bound for \( \dim_H \Theta^1(D_n) \) in the case \( \kappa = 4 \) is trivial. We can obtain a lower bound for \( \dim_H \Theta^1(D_n) \) provided we replace the limit in (1.1) with a \( \limsup \), as follows. By Frostman’s lemma [MP10, Theorem 4.30] there exists a finite positive measure \( \nu \) on \( \eta \) and a constant \( C > 0 \) such that \( \nu(A) \leq C \text{diam}(A)^{3/2} \) for each \( A \subset \eta \). By the upper bound for \( \dim_H \Theta^s:H \) in Theorem 1.1 for \( s < 1 \) and [MP10, Theorem 4.19], we have \( \nu(\Theta^{s:H}(D_\eta)) = 0 \) for each \( s < 1 \). Therefore \( \nu \) assigns zero mass to the set of points \( z \in \eta \) for which

\[
\limsup_{\epsilon \to 0} \frac{\log |(\Psi_\eta^{-1})'(1 - \epsilon)\Psi_\eta(x)|}{-\log \epsilon} < 1.
\]

By combining this with Lemma 2.11 we obtain that \( \nu \) a.s. assigns full mass to the set of \( z \in \eta \) for which

\[
\limsup_{\epsilon \to 0} \frac{\log |(\Psi_\eta^{-1})'(1 - \epsilon)\Psi_\eta(x)|}{-\log \epsilon} = 1.
\]

Hence this set a.s. has Hausdorff dimension \( \geq 3/2 \).

7.5 Lower bound for the integral means spectrum

In this subsection we prove our lower bound for the bulk integral means spectrum of the SLE curve and thereby complete the proof of Corollary 1.8.

Proof of Corollary 1.8. Throughout, we consider a fixed realization and allow implicit constants to be random (but independent of the parameters of interest).

Fix \( s \in [s_-, s_+] \) (as defined in 1.5 and 1.6) to be chosen later, and let \( \bar{\mathcal{P}} \) be the set of perfect points defined in (7.11), with \( d \) replaced by \( d/2 \). By the proof of Proposition 7.6, the probability of the event

\[
E := \{ \dim_H \bar{\mathcal{P}} \geq \bar{\xi}(s) \}
\]

is positive. Moreover, it is clear from the definition that we have \( \bar{\mathcal{P}} \subset \Psi_\eta^{-1}(\eta \cap B_d(0)) \).

Let \( \alpha < \bar{\xi}(s) \). For \( n \in \mathbb{N} \) put

\[
\epsilon_n = 2^{-n}.
\]

Let \( \mathcal{I}_n \) be the collection of arcs \( [e^{2\pi i(k-1)x_n}, e^{2\pi ikx_n}]_{\partial D} \) for \( k = 1, ..., 2^n \). Let \( \mathcal{I}'_n \) be the set of those arcs \( I \in \mathcal{I}_n \) which intersect \( \bar{\mathcal{P}} \). Then \( \mathcal{I}'_n \) is a cover of \( \bar{\mathcal{P}} \) consisting of sets of diameter \( \leq O_n(\epsilon_n) \). Hence on \( E \) we have

\[
|\mathcal{I}_n|e^{\alpha} \geq \mathcal{H}^\alpha(\bar{\mathcal{P}}) \geq 1,
\]

so

\[
|\mathcal{I}'_n| \geq e^{-\alpha}.
\]

For \( I \in \mathcal{I}'_n \) choose \( x_I \in I \cap \bar{\mathcal{P}} \). Let \( z_I = (1 - \epsilon_n)x_I \). By Lemma 7.4 we have \( |(\Psi_\eta^{-1})'(z_I)| \geq \epsilon_n^{-a+o_n(1)} \).

Here the \( o_n(1) \) and the implicit constants are uniform in \( z \).

Let \( J_I \) be the arc of \( \partial B_{1-\epsilon_n}(0) \) centered at \( z_I \) of length \( 1 + \delta_n \), where \( \delta_n \to 0 \) slower than the \( o_n(1) \) above. By the Koebe distortion theorem we have \( |(\Psi_\eta^{-1})'(w)| \geq \epsilon_n^{-a+o_n(1)} \) for each \( w \in J_I \). Since we replaced \( d \) by \( d/2 \) above, by continuity we have \( J_I \subset A_I \) for each \( I \) for sufficiently small \( \epsilon \). Hence on \( E \),

\[
\int_{A_{J_I}} |(\Psi_\eta^{-1})'(w)|^a \ dv \geq \int_{J_I} |(\Psi_\eta^{-1})'(w)|^a \ dv \geq \epsilon_n^{-a-as+a+o_n(1)}.
\]
Therefore, for any $a \in \mathbb{R}$, on $E$ it holds that

$$\limsup_{\epsilon \to 0} \frac{\log \int_{A_{\epsilon m}} |(\Psi^{-1}(w)|^a \, dw}{- \log \epsilon} \geq \alpha + as - 1.$$ 

Since $\alpha < \xi(s)$ is arbitrary we get

$$\limsup_{\epsilon \to 0} \frac{\log \int_{A_{\epsilon}} |(\Psi^{-1}(w)|^a \, dw}{- \log \epsilon} \geq \xi(s) + as - 1. \quad (7.12)$$

In the notation of Corollary 1.8 this quantity is maximized over all $s \in [s_-, s_+]$ by taking $s = s_+(a)$ if $a \in [a_-, a_+]$; $s = s_-$ if $a < a_-$; and $s = s_+$ if $a > a_+$. Choosing this value of $s$ in (7.12) gives us that the lower bound in (1.14) holds with positive probability for each fixed $a \in \mathbb{R}$ in the case $\kappa \leq 4$, $\rho = 0$, and $V = D$. By Proposition 5.7 this lower bound in fact holds a.s. for each choice of $\kappa > 0$, vector of weights $\rho$, and complementary connected component $V$. By combining this with Proposition 5.7, we get that (1.14) holds a.s. for each fixed $a \in \mathbb{R}$ for each choice of $\kappa > 0$, vector of weights $\rho$, and complementary connected component $V$. By Hölder’s inequality, it follows that the bulk integral means spectrum is a convex, hence continuous, function of $a$ (c.f. [Mak98, Theorem 5.2] for a related, but much stronger, statement for the ordinary integral means spectrum). It follows that in fact (1.14) holds a.s. for all $a \in \mathbb{R}$ simultaneously. \qed

A Comparisons of derivatives using harmonic measure

In this section we will prove some technical lemmas which allow us to compare conformal maps defined on different domains. The results of this section are needed primarily for the proof of the two-point estimate in Section 6. We start with a simple geometric description of the derivative of a certain conformal map defined on a subdomain of $D$.

**Lemma A.1.** Let $U \subset D$ be a simply connected subdomain with $[x, y]_{\partial D} \subset \partial U$ and $m \in (x, y)_{\partial D}$. Let $\Phi : U \to D$ be the conformal map taking $x$ to $-i$, $y$ to $i$, and $m$ to 1. Let $z \in U$ and let $I$ be a sub-arc of $[x, y]_{\partial D}$ and suppose that for some $\delta > 0$, the distance from $\Phi(z)$ to $\Phi(I)$ and the length of $\Phi(I)$ are each at least $\delta$.

$$\text{hm}^z(I; U) \asymp \text{dist}(z, \partial U)|\Phi'(z)|$$

with the implicit constants depending only $\delta$ and $z$ in a manner which is uniform for $z$ in compacts.

**Proof.** By conformal invariance of harmonic measure, we have $\text{hm}^z(I; U) = \text{hm}^{\Phi(z)}(\Phi(I); U)$. By our hypotheses on $\Phi(I)$ we have $\text{hm}^{\Phi(z)}(\Phi(I); U) \asymp \text{dist}(\Phi(z), \partial D)$, with the implicit constant depending only on $\delta$. By the Koebe quarter theorem, we have $\text{dist}(\Phi(z), \partial D) \asymp \text{dist}(z, \partial U)|\Phi'(z)|$ with a universal implicit constant. \qed

**Remark A.2.** We note some circumstances under which the hypotheses of Lemma A.1 are satisfied. Let $\hat{U}$ denote the Schwarz reflection of $U$ across $[x, y]_{\partial D}$. Suppose $I \subset (x, y)_{\partial D}$ with $m \in I$ and the distance from $\partial U \setminus \partial D$ to $I$ is at least a constant $\zeta > 0$. If $z$ lies at distance at least a constant $\zeta' > 0$ from $\partial D$ and is sufficiently close to $\partial U$, then by considering harmonic measure from $m$ in $\hat{U}$ (c.f. the proof of Lemma 2.8), we get that the hypotheses of Lemma A.1 are satisfied with $\delta$ depending only on $\zeta, \zeta'$ and the length of $I$. In particular, if the event $G_{[x, y]_{\partial D}}(\Phi, \mu)$ of Section 2.2.2 occurs, then Lemma 2.8 implies that, under the same hypotheses on $z$, the hypotheses of Lemma A.1 are satisfied with $\delta$ depending only on $\mu, \zeta'$, and the length of $I$.

From Lemma A.1 we can deduce some additional lemmas which allow us to compare the derivatives of conformal maps on different domains.

**Lemma A.3.** Let $U \subset D$ be a simply connected subdomain with $[x, y]_{\partial D} \subset \partial U$. Fix constants $\zeta, \delta, \Delta > 0$. Let $m \in (x, y)_{\partial D}$ with $|x - m|, |y - m| \geq \Delta > 0$. Let $\phi : U \to D$ be the conformal map taking $x$ to $-i$, $y$ to $i$, and $m$ to 1.

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Let $z \in U$ and $\tilde{z} \in D$ with $1 - |\tilde{z}| \geq \zeta$. Let $D \subset \mathbf{D}$ be another subdomain containing $z$ with $\text{dist}(z, \partial D) \geq \zeta$. Let $\psi : D \to \mathbf{D}$ be a conformal map which takes $z$ to $\tilde{z}$. Let $\tilde{U}$ be the connected component of $\psi(U \cap D)$ containing $\tilde{z}$. Suppose there is a connected arc $A$ of $\partial D$ which disconnects $z$ from $[x, y] \partial D$ in $U$. Let $[\tilde{x}, \tilde{y}]_{\partial \mathbf{D}} = \psi(A)$. Suppose there exists $\hat{m} \in (\tilde{x}, \tilde{y}]_{\partial \mathbf{D}}$ with $|\tilde{x} - \hat{m}|, |\tilde{y} - \hat{m}| \geq \Delta$. Let $\hat{\phi} : \hat{U} \to \mathbf{D}$ be the conformal map taking $\hat{m}$ to $1$.

Let $A'$ be an arc of $\partial U$ contained in $A$ and let $I$ be an arc of $[x, y]_{\partial \mathbf{D}}$. Suppose $\delta > 0$ and $\mu \in \mathcal{M}$ are such that the following hold.

1. The length of $\phi(I)$ and the distance from $\phi(z)$ to $\phi(I)$ are each at least $\delta$.
2. The distance from $\hat{\phi}(\tilde{z})$ to $[-i, \hat{\nu}]_{\partial \mathbf{D}}$ is at least $\delta$.
3. $\text{hm}^z(A'; U) \geq \delta$.
4. The probability that a Brownian motion started from any point of $A'$ exits $U$ in $I$ is at least $\delta$.
5. $\mathcal{G}_{[\tilde{x}, \tilde{y}]_{\partial \mathbf{D}}}(\hat{\phi}, \mu)$ occurs (Definition 2.3).

Then we have

$$|\phi'(z)| \asymp |\hat{\phi}'(\tilde{z})|,$$

with implicit constants depending only on $\zeta, \delta, \mu, \Delta, z$, and $\tilde{z}$ and uniform for $z$ and $\tilde{z}$ in compact subsets of $\mathbf{D}$.

See Figure A.1 for an illustration of the setup.

Proof. Throughout, we assume that all implicit constants depend only on $\zeta, \delta, \Delta, z$, and $\tilde{z}$ and are uniform for $z$ and $\tilde{z}$ in compact subsets of $\mathbf{D}$.

By Lemma A.1 and assumptions 1 and 2 we have

$$|\phi'(z)| \asymp \frac{\text{hm}^z(I; U)}{\text{dist}(z; \partial U)}, \quad |\hat{\phi}'(\tilde{z})| \asymp \frac{\text{hm}^\tilde{z}([\tilde{x}, \tilde{y}]_{\partial \mathbf{D}}; \hat{U})}{\text{dist}(\tilde{z}; \partial \hat{U})}. \quad (A.2)$$

By the Koebe quarter theorem,

$$1 \asymp \frac{\text{dist}(\tilde{z}, \partial \mathbf{D})}{\text{dist}(z, \partial U)} \asymp |\psi'(z)| \asymp \frac{\text{dist}(\tilde{z}, \partial \hat{U})}{\text{dist}(z, \partial \hat{U})}. \quad (A.1)$$
Figure A.2: An illustration of the proof of Lemma A.4. The probability that a Brownian motion started from \(z\) exits \(D_0^\eta\) in the red arc \(I\) is bounded by the supremum of the harmonic measure of \(I\) in \(D_0^\eta\) from any point of the green crosscut \(A\). This, in turn, is bounded by a constant times the supremum of the harmonic measure of \(I\) in \(D_\eta\) from any point of \(A\), which is bounded by the harmonic measure of \(I\) from \(z\) in \(D_\eta\) by our choice of \(\tilde{A}\).

Thus
\[
\text{dist}(z, \partial U) \simeq \text{dist}(\hat{z}, \partial \hat{U}).
\] (A.3)

By conformal invariance of harmonic measure, we have
\[
\text{hm}^z([\hat{x}, \hat{y}]_{\partial D}; \hat{U}) = \text{hm}^z(A; \psi^{-1}(\hat{U})).
\] (A.4)

By our assumption on \(A\), a Brownian motion started from \(z\) must exit \(\psi^{-1}(\hat{U})\) in \(A\) before leaving \(\partial U\) in \([x, y]_{\partial D}\). Hence
\[
\text{hm}^z(I, U) \leq \text{hm}^z(A; \psi^{-1}(\hat{U})).
\]

By combining (A.2), (A.3), and (A.4) we get
\[
|\hat{\phi}'(z)| \leq |\hat{\phi}'(\bar{z})|.
\]

For the reverse inequality, let \(\bar{I} = \psi(A)\). By assumptions 3 and 5, the length of \(\hat{\phi}(\bar{I})\) is \(\simeq 1\). By assumption 2 and Lemma A.1 we then get
\[
|\hat{\phi}'(\bar{z})| \simeq \frac{\text{hm}^\bar{z}(\bar{I}; \hat{U})}{\text{dist}(\bar{z}, \partial \hat{U})}.
\]

By conformal invariance of harmonic measure and (A.3), this is proportional to
\[
\frac{\text{hm}^z(A'; \psi^{-1}(\hat{U}))}{\text{dist}(z, U)}.
\]

By assumption \(1\) and the first proportionality in (A.2) we get that this last quantity is \(\leq |\phi'(z)|\). □

**Lemma A.4.** Let \(x, y \in \partial D\). Let \(\eta : [0, \infty] \to D\) be a simple curve which does not intersect \((x, y)_{\partial D}\). Let \(m \in (x, y)_{\partial D}\) with \(|x - m|, |y - m| \geq \Delta > 0\). Let \(D_\eta\) be the connected component of \(D \setminus \eta\) containing \([x, y]_{\partial D}\) on its boundary. Let \(\Phi : D_\eta \to D\) be the conformal map taking \(x\) to \(-i\), \(y\) to \(i\), and \(m\) to \(1\). Let \(t_2 > t_1 \geq 0\). Let \(D_0^\eta = D \setminus (\eta([0, t_1]) \cup \eta([t_2, \infty]))\). Let \(\tilde{\phi} : D_0^\eta \to D\) be the conformal map taking \(x^+\) to \(-i\), \(y^-\) to \(i\), and \(m\) to \(1\). Let \(I \subset [x, y]_{\partial D}\) be an arc. Let \(z \in D_\eta\). Suppose that there is some \(\ell > 0\) and \(\delta > 0\) such that the following is true.
1. $\text{hm}^z(\eta([0,t_1]); D_\eta)$ and $\text{hm}^z(\eta([t_2,\infty]); D_\eta)$ are each at least $\ell$.

2. The length of $\Phi(I)$ and the distance from $\Phi(z)$ to $\Phi(I)$ are each at least $\delta$.

3. The length of $\phi(I)$ and the distance from $\phi(z)$ to $\phi(I)$ are each at least $\delta$.

Then $|\phi'(z)| \approx |\Phi'(z)|$ and $\text{dist}(z, \partial D_\eta) \approx \text{dist}(z, \partial D_\eta')$ with implicit constants depending only on $\delta$, $\ell$, and $z$, but uniform for $z$ in compact subsets of $D$.

**Proof.** See Figure A.2 for an illustration of the proof.

By Lemma A.1, $|\phi'(z)| \approx \frac{\text{hm}^z(I; D_\eta^0)}{\text{dist}(z, \partial D_\eta^0)}$, $|\Phi'(z)| \approx \frac{\text{hm}^z(I; D_\eta)}{\text{dist}(z, \partial D_\eta)}$

with the implicit constants depending only on $\delta$. We clearly have $\text{hm}^z(I; D_\eta^0) \geq \text{hm}^z(I; D_\eta)$. By the Beurling estimate, $\text{hm}^z(\eta \cap B_{r \text{dist}(z, \eta)}(z); D_\eta) \to 1$ as $r \to \infty$, at a rate which does not depend on $\eta$ or $\text{dist}(z, \eta)$. So, our hypothesis implies that $\text{dist}(z, \partial D_\eta) \approx \text{dist}(z, \partial D_\eta')$. Therefore it is enough to prove

$$\text{hm}^z(I; D_\eta^0) \leq \text{hm}^z(I; D_\eta)$$

(A.5)

with the implicit constant depending only on $\ell$.

Let $\bar{\Phi}: D_\eta \to D$ be the conformal map taking $z$ to 0 and $m$ to 1. By conformal invariance of harmonic measure and our hypothesis, the distance from each of $\bar{\Phi}(\eta(t_1))$ and $\bar{\Phi}(\eta(t_2))$ to $\bar{\Phi}(I)$ is at least $2\pi \ell$. Hence we can choose a crosscut $A$ in $D$ which disconnects 0 from $\bar{\Phi}(I)$ such that each point of $A$ lies at distance at least $\ell$ from $\bar{\Phi}(I)$ and from $[\bar{\Phi}(\eta(t_2)), \bar{\Phi}(\eta(t_1))]_{\partial D}$. The harmonic measure of $\bar{\Phi}(I)$ from each point of $A$ in $D$ is bounded above by a constant depending only on $\ell$ times the length of $\bar{\Phi}(I)$, which in turn is proportional to $\text{hm}^z(I; D_\eta)$. Furthermore, the harmonic measure of the arc $[\bar{\Phi}(\eta(t_2)), \bar{\Phi}(\eta(t_1))]_{\partial D}$ from each point of $A$ in $D$ is bounded above by a constant $a < 1$ depending only on $\ell$.

Let $A = \bar{\Phi}^{-1}(A)$. Then we have

$$\text{hm}^w(I; D_\eta) \leq \text{hm}^z(I; D_\eta), \quad \text{hm}^w(\eta([t_1, t_2]); D_\eta) \leq a \quad \forall w \in A$$

(A.6)

with the implicit constant depending only on $\ell$.

A Brownian motion started from $z$ must hit $A$ before exiting $D_\eta^0$ in $I$. Therefore,

$$\text{hm}^z(I; D_\eta^0) \leq \sup_{w \in A} \text{hm}^w(I; D_\eta^0).$$

(A.7)

For $w \in A$, we can decompose the event that a Brownian motion $B$ started at $w$ exits $D_\eta^0$ in $I$ as the union of the event that $B$ hits $I$ before $\eta([t_1, t_2])$ and the event that $B$ hits $\eta([t_1, t_2])$ and then $I$. By (A.6) the former event has probability at most a constant $C$ (depending only on $\ell$) times $\text{hm}^z(I; D_\eta)$. By the Markov property the latter event has probability at most

$$\sup_{w \in A} \text{hm}^w(\eta([t_1, t_2]); D_\eta) \sup_{v \in \eta([t_1, t_2])} \text{hm}^w(I; D_\eta^0).$$

Since $A$ disconnects $\eta([t_1, t_2])$ from $I$ in $D_\eta^0$ we have $\sup_{v \in \eta([t_1, t_2])} \text{hm}^w(I; D_\eta^0) \leq \sup_{w \in A} \text{hm}^w(I; D_\eta^0)$. By combining this with (A.6) we get

$$\sup_{w \in A} \text{hm}^w(I; D_\eta^0) \leq C \text{hm}^z(I; D_\eta) + a \sup_{w \in A} \text{hm}^w(I; D_\eta^0).$$

(A.8)

Since $a < 1$, we can re-arrange the estimate (A.8) to get

$$\sup_{w \in A} \text{hm}^w(I; D_\eta^0) \leq \text{hm}^z(I; D_\eta),$$

which together with (A.7) yields (A.5). □
B Strict mutual absolute continuity for SLE

**Definition B.1.** We say that a measure \( \mu \) is strictly mutually absolutely continuous (s.m.a.c.) with respect to a measure \( \nu \) if \( \mu \) and \( \nu \) are mutually absolutely continuous with Radon-Nikodym derivative a.e. bounded above and below by finite and positive constants.

In this appendix we will prove a lemma which gives that the conditional law of the “middle part” of an SLE\(_\kappa\)(\( \rho^L; \rho^R \)) curve, given a suitable realization of its initial and terminal segments, is s.m.a.c. with respect to the law of the middle part of an ordinary SLE\(_\kappa\) curve (see Lemma 3.4 below for an exact statement). This result is needed in the proof of our two-point estimate. We will deduce our desired result from MW14 Lemma 2.8 (which gives a similar strict mutual absolute continuity statement for SLE\(_\kappa\)(\( \rho \)) curves in domains which agree in a neighborhood of the starting point) together with the coupling results of MS12a described in Section 2.5.

Before we can prove this result, we need to define what we mean by a “suitable realization of the initial and terminal segments of the path.” Let \( x, y \in \partial D \) be distinct. Let \( \eta \) be a random curve from \( x \) to \( y \) in \( D \), with time reversal \( \bar{\eta} \). In what follows, we write \( B_\beta = B_{\tau_\beta}(0) \) and let \( \tau_\beta \) (resp. \( \tau_\beta \)) be the first time \( \eta \) (resp. \( \bar{\eta} \)) hits \( B_\beta \), as in Section 6.1.

Fix \( \Delta > \Delta' > \bar{\Delta} > 0 \). Suppose we are given times \( \sigma, \bar{\sigma} > 0 \). Let \( \eta^* \) be the part of \( \eta \) between \( \eta(\sigma) \) and \( \bar{\eta}(\bar{\sigma}) \). Let \( H^* = H^*(\eta^*; \bar{\Delta}) \) be the event that \( \eta^* \subset \bar{B}_{\bar{\Delta}} \). Let \( S = S(\eta; \sigma, \bar{\sigma}, \Delta, \bar{\Delta}) \) be the event that the following occur.

1. \( \tau_\Delta \leq \sigma < \infty \) and \( \tau_{\bar{\Delta}} \leq \bar{\sigma} < \infty \) (here, \( \tau_\Delta = \tau_\beta \) and \( \tau_{\bar{\Delta}} = \tau_{\bar{\beta}} \) with \( \bar{\beta} = \bar{\Delta} \)).
2. \( \eta^* \) (resp. \( \bar{\eta}^* \)) is contained in the \( e^{-2\Delta} \)-neighborhood of the segment \([x, 0] \) (resp. \([y, 0] \)).
3. The conditional probability of \( H^* \) given \( \eta^* \cup \bar{\eta}^* \) is positive.

Also let \( S^* = S^*(\eta; \sigma, \bar{\sigma}, \Delta, \Delta', \bar{\Delta}) \) be the event that the following occur.

1. \( S(\eta; \sigma, \bar{\sigma}, \Delta, \bar{\Delta}) \) occurs.
2. \( \eta([\tau_{\Delta'}, \sigma]) \) (resp. \( \bar{\eta}([\tau_{\Delta'}, \bar{\sigma}]) \)) is contained in \( \bar{B}_{\bar{\Delta}} \).

**Remark B.2.** If the event \( L_{z,n} \) and the times \( \sigma_{z,n} \) and \( \bar{\sigma}_{z,n} \) are defined as in Section 6.2 then we have

\[
L_{z,n} \subset S^*(\eta^0_{z,n}; \sigma_{z,n}, \bar{\sigma}_{z,n}, \Delta, \Delta/2, \bar{\Delta}).
\]

This is our primary interest in the event \( S^*(\cdot) \).

**Remark B.3.** In the case that \( \eta \) is an SLE\(_\kappa\)(\( \rho^L; \rho^R \)) (which is what we consider in the section) one can show that condition 3 in the definition of \( S \) is in fact implied by the other conditions in the definition of \( S \). The idea to establish this is to realize \( \eta \) as a flow line of a GFF, then condition on two counterflow lines (run up to a certain stopping time) with the property that the interface between them is a.s. equal to \( \bar{\eta}^* \).

See MS12 Section 5.4 for a similar argument. We do not need this fact here though, so for the sake of brevity we include condition 3 as a condition.

The main result of this section is the following.

**Lemma B.4.** Let \( \rho^L, \rho^R \in (-2, 0] \), \( \delta > 0 \), and \( x, y \in \partial D \) with \( |x - y| \geq \delta \). Let \( \eta \) be a chordal SLE\(_\kappa\)(\( \rho^L; \rho^R \)) process from \( x \) to \( y \) in \( D \) with force points located at \( x^- \) and \( x^+ \). Let \( \bar{\eta} \) be its time reversal. Let \( \sigma \) be a stopping time for \( \eta \) and let \( \bar{\sigma} \) be a stopping time for the filtration generated by \( \eta^* \) and \( \bar{\eta}^* \). Let \( S^* = S^*(\eta; \sigma, \bar{\sigma}, \Delta, \Delta', \bar{\Delta}) \) as above. Also let \( \eta^* \) and \( H^* = H^*(\eta^*; \bar{\Delta}) \) be as above. Let \( D \) be the connected component of \( D \setminus (\eta^* \cup \bar{\eta}^*) \) containing 0.

If \( \bar{\Delta} \) (and hence also \( \Delta' \) and \( \Delta \)) is chosen sufficiently large, then the conditional law of \( \eta^* \) given a.e. realization of \( \eta^* \cup \bar{\eta}^* \) for which \( S \) occurs and the event \( H^* \) is s.m.a.c. with respect to the law of a chordal SLE\(_\kappa\) from \( \eta(\sigma) \) to \( \bar{\eta}(\bar{\sigma}) \) in \( D \) conditioned on \( H^* \), with deterministic constants depending only on \( \rho^L, \rho^R, \kappa, \Delta, \Delta', \bar{\Delta}, \) and \( \delta \).

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Remark B.5. The statement of Lemma B.4 is also true for general values of $\rho^L, \rho^R \in (-2, \infty)$. We give a full proof here only in the case of negative $\rho^L, \rho^R$ because this is the case which we will need in this paper. We give here a brief outline of the adaptations necessary to treat the case of a general choice of $\rho^L, \rho^R \in (-2, \infty)$. The proof in the case when both $\rho^L$ and $\rho^R$ are positive proceeds in the same manner as the proof of Lemma B.4, except that we condition on auxiliary flow lines to turn a $\text{SLE}_\kappa(\rho^L; \rho^R)$ into an ordinary $\text{SLE}_\kappa$, instead of turning an ordinary $\text{SLE}_\kappa$ into an $\text{SLE}_\kappa(\rho^L; \rho^R)$ as in the argument below. The case where $\rho^L$ and $\rho^R$ have different signs follows by starting with an $\text{SLE}_\kappa(\rho^L; \rho^R)$ process with $\rho^L, \rho^R > 0$, conditioning on auxiliary flow lines to turn it into an $\text{SLE}_\kappa(\rho^L; \rho^R)$, and using the case of positive $\rho^L, \rho^R$.

For the proof of Lemma B.4, we will assume neither $\rho^L$ nor $\rho^R$ is equal to 0; the case when one of the force points is equal to 0 is treated similarly but with only a single auxiliary flow line.

Choose $\Delta_0 > \Delta_0 > 0$ satisfying $\Delta_0 < \Delta < \Delta' < \Delta_0 < \Delta$, with $\Delta, \Delta', \Delta$ as in the statement of Lemma B.4. Let $\eta_\rho$ be an ordinary chordal $\text{SLE}_\kappa$ from $x$ to $y$ in $D$. Let $\eta^*_0$ be the time reversal of $\eta_0$. Let $\sigma_0$ (resp. $\overline{\sigma}_0$) be the first time $\sigma(t)$ (resp. $\overline{\sigma}(t)$) hits $B_\Delta$. Let $\eta_{\rho}^*$ be the part of $\eta_\rho$ between $\eta(\sigma_0)$ and $\overline{\eta}(\overline{\sigma}_0)$. Also let

$$
S_0 := S(\eta_\rho; \sigma_0, \Delta, \overline{\Delta_0}), \quad H^*_0 = H^*(\eta^*_0; \overline{\Delta_0}).
$$

(B.1)

We can couple $\eta_\rho$ with a GFF $h$ on $D$ with appropriately chosen boundary data in such a way that $\eta_\rho$ is the zero angle flow line$^8$ (in the sense of Section 2.5) of $h$ started from $x$. Let $\theta^L > 0$ and $\theta^R < 0$ be chosen so that

$$
\frac{\theta^L}{\Lambda} - 2 = \rho^L, \quad -\frac{\theta^R}{\Lambda} - 2 = \rho^R.
$$

(B.2)

Let $\eta_- \eta_+$ be the flow lines of $h$ started from $x$ with angles $\theta^L$ and $\theta^R$, respectively. Since $\rho^L, \rho^R \in (-2, 0)$, the flow lines $\eta_-$ and $\eta_+$ are well defined. Let $D_0$ be the connected component of $D \setminus (\eta_- \cup \eta_+)$ containing the origin. Let $b$ and $\overline{b}$, respectively, be the first and last point on $\partial D_0$ hit by $\eta_\rho$. By the results of [MS12a, Section 7], the conditional law of the part of $\eta_\rho$ which lies in $D_0$ given $\eta_- \cup \eta_+$ is that of a chordal $\text{SLE}_\kappa(\rho^L; \rho^R)$ process with force points located on either side of $b$.

Fix $\alpha \in (0, 1)$. Let $t_-$ and $t_+$ respectively be the first times $\eta_-$ and $\eta_+$ exit $B_{1-\alpha}(0)$.

Throughout the remainder of this subsection, we require all implicit constants, including those in s.m.a.c., to depend only on $\Delta, \Delta, \Delta', \Delta_0, \Delta_0, \alpha, \rho^L, \rho^R, \kappa$, and $\delta$ (in particular, implicit constants are not allowed to depend on the realization of whatever we are conditioning on or on the choice of stopping times $\sigma, \overline{\sigma}$).

Lemma B.6. Let $\omega_0$ be a realization of $\eta_0^\sigma \cup \overline{\eta}_0^\sigma$ for which $S_0$ occurs. If $\overline{\Delta_0}$ (and hence also $\Delta_0$) is chosen sufficiently large and $\alpha > 0$ is chosen sufficiently small then the following is true for a.e. such $\omega_0$. For a.e. realization of $(\eta_-, \eta_+)$, the conditional law of $\eta_0^\sigma$ given $\omega_0$, $H^*_0$, and $(\eta_-, \eta_+)$ is s.m.a.c. with respect to the conditional law of $\eta_0^\sigma$ given only $\omega_0$ and $H^*_0$.

Proof. Let $P_{\omega_0}$ denote the regular conditional probability given $\omega_0$ and $H^*_0$. Let $A_0^\rho$ be an event with positive $P_{\omega_0}$-probability which is measurable with respect to $\omega_0$ and contained in $H^*_0$. Let $A_0^F$ be the intersection of $H^*_0$ with an event which is measurable with respect to $\eta_0^\sigma \cup \overline{\eta}_0^\sigma$ and $(\eta_-, \eta_+)$ and contained in $S_0$. By Bayes’ rule,

$$
P_{\omega_0}(A_0^F | A_0^\rho) = \frac{P_{\omega_0}(A_0^F | A_0^\rho)P_{\omega_0}(A_0^\rho)}{P_{\omega_0}(A_0^\rho)}.
$$

(B.3)

Hence we are lead to study the conditional law of $(\eta_-, \eta_+)$ given $\omega_0$ and $\eta_0^\sigma$, for varying realizations of $\eta_0^\sigma$ for which $H^*_0$ occurs.

By the results of [MS12a, Section 7.1], the conditional law of $\eta_+$ given $\eta_0$ is that of a chordal $\text{SLE}_\kappa(\rho^L; \rho^R)$ process from $x$ to $y$ in the right connected component of $D \setminus \eta_0$ for certain $\rho^L, \rho^R \in \mathbb{R}$ depending on $\rho^L$ and $\rho^R$. Similar statement holds for $\eta_-$. Furthermore, $\eta_+$ and $\eta_-$ are conditionally independent given $\eta_0$. By [MW14, Lemma 2.8] and condition 2 in the definition of $S_0$, if $\overline{\Delta}_0$ is chosen sufficiently large and $\alpha > 0$ is chosen sufficiently small then the conditional law of the pair $(\eta_-, \eta_+)$ given $\omega_0$ and $\eta_0^\sigma$ for varying realizations of $\eta_0^\sigma$ for which $H^*_0$ occurs are s.m.a.c. By averaging over all such realizations, we get $P_{\omega_0}(A_0^F | A_0^\rho) \asymp P_{\omega_0}(A_0^\rho)$. By (B.3) therefore we have $P_{\omega_0}(A_0^F | A_0^\rho) \asymp P_{\omega_0}(A_0^\rho)$.

$^8$In the case $\kappa = 4$, we replace flow lines by level lines, as defined in [SS13, SS09]. Everything works the same with this replacement.
By [MW14, Lemma 2.5], we have that \( P \) process from \( x \) following are true on \( F \omega \) probability given \( \eta \).

By the discussion just above Lemma B.6, the conditional law of \( \eta \) is minimal amongst all such maps. Let

\[
\eta := \psi(\eta_0 \cap D_0), \quad \bar{\eta}^* := \psi(\eta_0^*).
\]

By the discussion just above Lemma B.6, the conditional law of \( \eta \) given \( \omega_F \) is that of a chordal \( \text{SLE}_{\kappa}(\rho^L; \rho^R) \) process from \( x \) to \( y \) in \( D \).

Fix \( \epsilon > 0 \), to be chosen later, and let \( F \) be the event that the following occur.

1. \( \eta_- \) and \( \eta_+ \) trace all of \( \partial D_0 \) before times \( t_- \) and \( t_+ \).
2. \( |\psi(z) - z| \leq \epsilon \) for each \( z \in D_0 \).

By [MW14] Lemma 2.5], we have that \( P_{\omega_0}(F) > 0 \) for any choice of \( \epsilon > 0 \) and a.e. choice of realization \( \omega_0 \).

By choosing \( \epsilon > 0 \) sufficiently small (depending only on \( \Delta, \Delta', \tilde{\Delta}, \Delta_0, \) and \( \tilde{\Delta}_0 \)), we can arrange that the following are true on \( F \).

1. We have \( B_{\Delta} \subset \psi(B_{\Delta_0}) \subset \psi(B_{\Delta'}) \subset \psi(B_{\tilde{\Delta}_0}) \subset B_{\tilde{\Delta}} \).
2. The image under \( \psi \) of the \( e^{-2\Delta_0} \)-neighborhood of the segment \( [x, 0] \) (resp. \( [y, 0] \)) contains the \( e^{-2\Delta} \)-neighborhood of the segment \( [x, 0] \) (resp. \( [y, 0] \)).

On the event \( F \), let \( \sigma' \) and \( \sigma \) be the stopping times for \( \eta \) and \( \bar{\eta} \) corresponding to \( \sigma_0 \) and \( \bar{\sigma}_0 \), so \( \psi(\eta_0(\sigma_0)) = \eta(\sigma') \), \( \psi(\bar{\eta}_0(\sigma_0)) = \bar{\eta}(\sigma') \), and \( \bar{\eta}^* \) is the part of \( \eta \) between \( \eta(\sigma') \) and \( \bar{\eta}(\sigma') \). Also let \( \eta^* \) be the part of \( \eta \) between \( \sigma \) and \( \bar{\sigma} \), as in the statement of the lemma.

By conditions [1] and [2] above together with condition [2] in the definition of \( S^* \), we have

\[
F \cap S^* \cap H^* \subset F \cap S_0 \cap H_0^*.
\]

(B.4)
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Let \((\omega_0, \omega_F)\) be a realization of \((\eta_0^\sigma \cup \overline{\eta}_0^\sigma, \eta_1^+ \cup \eta_1^-)\) for which \(F \cap S_0\) occurs. We observe the following.

1. By the Markov property, the law of \(\eta_0^\sigma\) given \(\omega_0\) and \(H_0^*\) is that of a chordal \(\text{SLE}_\kappa\) from \(\eta_0(\sigma_0)\) to \(\overline{\eta}_0(\sigma_0)\) in \(D \setminus (\eta_0^\sigma \cup \overline{\eta}_0^\sigma)\), conditioned on \(H_0^*\).

2. It therefore follows from Lemma [B.6] that the law of \(\eta_0^\sigma\) given \(\omega_F\) and \(H_0^*\) is a.s. \(\text{s.m.a.c.}\) with respect to the law of a chordal \(\text{SLE}_\kappa\) from \(\eta_0(\sigma_0)\) to \(\overline{\eta}_0(\sigma_0)\) in \(D \setminus (\eta_0^\sigma \cup \overline{\eta}_0^\sigma)\), conditioned on \(H_0^*\).

3. By [MW14, Lemma 2.8], this latter law is \(\text{s.m.a.c.}\) with respect to the law of a chordal \(\text{SLE}_\kappa\) from \(\eta_0(\sigma_0)\) to \(\overline{\eta}_0(\sigma_0)\) in the connected component of \(D_0 \setminus (\eta_0^\sigma \cup \overline{\eta}_0^\sigma)\) containing \(0\), conditioned on \(H_0^*\).

4. Therefore, the conditional law of \(\tilde{\eta}^*\) given \((\omega_0, \omega_F)\) and \(H_0^*\) is \(\text{s.m.a.c.}\) with respect to the law of a chordal \(\text{SLE}_\kappa\) from \(\eta(\sigma')\) to \(\overline{\eta}(\sigma')\) in the component of \(D \setminus (\eta^\sigma \cup \overline{\eta}^\sigma)\) containing \(0\), conditioned on \(H_0^*\).

5. By [B.4], [B.5], and the Markov property and reversibility of ordinary \(\text{SLE}_\kappa\), assertion 4 implies that the conditional law of \(\eta^*\) given \(\omega_F\); a realization of \(\eta^* \cup \overline{\eta}^*\) for which \(S^*\) occurs; and \(H^*\) is a.s. \(\text{s.m.a.c.}\) with respect to the law of a chordal \(\text{SLE}_\kappa\) from \(\eta(\sigma')\) to \(\overline{\eta}(\sigma')\) in the component of \(D \setminus (\eta^\sigma \cup \overline{\eta}^\sigma)\) containing \(0\), conditioned on \(H^*\).

Since the law of \(\eta\) given a.e. \(\omega_F\) is that of a chordal \(\text{SLE}_\kappa(\rho^L; \rho^R)\) from \(x\) to \(y\) in \(D\) and there is a positive probability event of choices for \(\omega_F\), assertion 5 implies the statement of the lemma.

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