NONCOMMUTATIVE BLACKWELL–ROSS MARTINGALE INEQUALITY

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ABSTRACT. We establish a noncommutative Blackwell–Ross inequality for supermartingales under a suitable condition which generalize Khan’s works to the noncommutative setting. We then employ it to deduce an Azuma-type inequality.

1. INTRODUCTION

Blackwell [2] showed that if \( \{X_n, X_0 = 0, n \geq 0\} \) is a martingale such that \( |X_n - X_{n-1}| \leq \alpha \) for all \( n \), then for each positive number \( c \),

\[
\text{Prob}(X_n \geq mc \text{ for some } n \geq m) \leq \exp\left(-\frac{mc^2}{2\alpha^2}\right),
\]

which gives a generalization of a result of Hoeffding [5]. Ross [8] extended Blackwell’s result to the case where the bound on the martingale difference is not symmetric. Indeed, Ross employed a supermartingale argument to show that the same is true when \( -\alpha \leq X_n - X_{n-1} \leq \beta \), where \( \alpha, \beta > 0 \). Khan [7] generalized Blackwell–Ross inequality for martingales (supermartingales) under a subnormal structure on the conditional moment generating function \( \varphi_n(\theta) = \mathbb{E}(\exp(\theta X_n)|\mathcal{F}_{n-1}) \) subject to some mild conditions.

In this paper, we adopt the classical ideas in probability theory and the Golden–Thompson inequality to establish a Blackwell–Ross martingale inequality under a non-symmetric bound on the martingales differences in the framework of noncommutative probability spaces.

A von Neumann algebra \( \mathcal{M} \) on a Hilbert space \( \mathcal{H} \) with unit element 1 equipped with a normal faithful tracial state \( \tau : \mathcal{M} \rightarrow \mathbb{C} \) is called a noncommutative probability space. We denote by \( \preceq \) the usual order on the self-adjoint part \( \mathcal{M}^{sa} \) of \( \mathcal{M} \). For each self-adjoint operator \( x \in \mathcal{M} \), there exists a unique spectral measure \( E \) as a \( \sigma \)-additive mapping with respect to the strong operator topology

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from the Borel $\sigma$-algebra $\mathcal{B}(\mathbb{R})$ of $\mathbb{R}$ into the set of all orthogonal projections such that for every Borel function $f : \sigma(x) \rightarrow \mathbb{C}$ the operator $f(x)$ is defined by $f(x) = \int f(\lambda)dE(\lambda)$, in particular, $1_B(x) = \int_B dE(\lambda) = E(B)$.

The celebrated Golden–Thompson inequality \cite{9} states that for any self-adjoint elements $y_1, y_2$ in a noncommutative probability space $\mathcal{M}$, the inequality

$$\tau(e^{y_1+y_2}) \leq \tau(e^{y_1}e^{y_2}) \quad (1.1)$$

holds; see also \cite{12} for some Golden–Thompson type inequalities.

For $p \geq 1$, the noncommutative $L^p$-space $L^p(\mathcal{M})$ is defined as the completion of $\mathcal{M}$ with respect to the $L^p$-norm $\|x\|_p := \left(\tau(|x|^p)\right)^{1/p}$. The commutative cases of discussed spaces are usual $L^p$-spaces. For further information we refer the reader to \cite{3} and references therein.

Let $\mathfrak{N}$ be a von Neumann subalgebra of $\mathcal{M}$. Then there exists a normal positive contractive projection $E_{\mathfrak{N}} : \mathcal{M} \rightarrow \mathfrak{N}$ satisfying the following properties:

(i) $E_{\mathfrak{N}}(axb) = aE_{\mathfrak{N}}(x)b$ for any $x \in \mathcal{M}$ and $a, b \in \mathfrak{N}$;

(ii) $\tau \circ E_{\mathfrak{N}} = \tau$.

Moreover, $E_{\mathfrak{N}}$ is the unique mapping satisfying (i) and (ii). The mapping $E_{\mathfrak{N}}$ is called the conditional expectation of $\mathcal{M}$ with respect to $\mathfrak{N}$.

Let $\mathfrak{N} \subseteq \mathfrak{A}_j$ ($1 \leq j \leq n$) be von Neumann subalgebras of $\mathcal{M}$. We say that the $\mathfrak{A}_j$ are order independent over $\mathfrak{N}$ if for every $2 \leq j \leq n$, the equality

$$E_{j-1}(x) = E_{\mathfrak{N}}(x)$$

holds for all $x \in \mathfrak{A}_j$, where $E_{j-1}$ is the conditional expectation of $\mathcal{M}$ with respect to the von Neumann subalgebra generated by $\mathfrak{A}_1, \ldots, \mathfrak{A}_{j-1}$; cf. \cite{6}. Note that this notion of independence implies that $\mathfrak{N}$ should be the intersection of all $\mathfrak{A}_j$.

In fact, if $x \in \mathfrak{A}_{j-1} \cap \mathfrak{A}_j$, then

$$x = E_{j-1}(x) = E_{\mathfrak{N}}(x) \in \mathfrak{N}.$$  

A filtration of $\mathcal{M}$ is an increasing sequence $(\mathcal{M}_j, E_j)_{0 \leq j \leq n}$ of von Neumann subalgebras of $\mathcal{M}$ together with the conditional expectations $E_j$ of $\mathcal{M}$ with respect to $\mathcal{M}_j$, such that $\bigcup_j \mathcal{M}_j$ is $w^*$-dense in $\mathcal{M}$. It follows from $\mathcal{M}_j \subseteq \mathcal{M}_{j+1}$ that

$$E_i \circ E_j = E_j \circ E_i = E_{\min\{i,j\}} \quad (1.2)$$

for all $i, j \geq 0$. Generally, a sequence $(x_j)_{j \geq 0}$ in $L^1(\mathcal{M})$ is called a martingale (supermartingale, resp.) with respect to the filtration $(\mathcal{M}_j)_{0 \leq j \leq n}$ if $x_j \in L^1(\mathcal{M}_j)$ and $E_j(x_{j+1}) = x_j$ ($E_j(x_{j+1}) \leq x_j$, resp.) for every $j \geq 0$. It follows from (1.2)
that $\mathcal{E}_j(x_i) = x_j$ for all $i \geq j$. Put $dx_j = x_j - x_{j-1}$ ($j \geq 0$) with the convention that $x_{-1} = 0$. Then $dx = (dx_j)_{j \geq 0}$ is called the martingale difference of $(x_j)$. The reader is referred to [13] for more information.

2. Main Results

In this section, we provide a noncommutative Blackwell–Ross inequality. To this end, we will need the following lemma which was proved by Alon, et al. [1].

Lemma 2.1. For $0 \leq \lambda \leq 1$

$$\lambda e^{(1-\lambda)x} + (1 - \lambda)e^{-\lambda x} \leq e^\frac{x^2}{\lambda}.$$ 

We are inspired by some ideas in the commutative case, e.g. [7], to provide our main result.

Theorem 2.2. Let $\{s_n = \sum_{i=1}^n x_n, n \geq 0\}$ be a self-adjoint supermartingale in $\mathcal{M}$ with respect to a filtration $(\mathcal{M}_n, \mathcal{E}_n)_{n \geq 0}$ such that

$$\mathcal{E}_{n-1}(e^{tx_n}) \leq f(t) \leq e^{-\gamma t + \lambda t^2}, \quad (\lambda > 0, \gamma \geq 0, t > 0) \quad (2.1)$$

where $f(t)$ is a continuous positive function on $[0, \infty)$. Then for positive numbers $a$ and $b$, there exists $i \geq 1$ such that for any positive integer $m$

$$\tau\left(\sqrt[n=m+i]{e^{a(tn+\infty)}}(s_n)\right) \leq A^m e^{-\frac{a(b+\gamma)}{\lambda}}, \quad (2.2)$$

where $A = e^{-bt_0} f(t_0) \leq 1$ and $t_0 = \frac{b+\gamma}{\lambda}$. Moreover,

$$\tau\left(\sqrt[n=m+j]{e^{b(tn+\infty)}}(s_n)\right) \leq A^m e^{-\frac{b(b+\gamma)^2}{4\lambda}} \quad (2.3)$$

for some $j \geq 1$, where $A_0 = e^{-\frac{1}{2}(b-\gamma)t_0} f(t_0)$ and $t_0 = \frac{b+\gamma}{2\lambda}$. 

Proof. Let $y_n = \exp(ts_n - at - bnt)$, $t > 0$. We show that the sequence $(y_n)_{n \geq 0}$ satisfies the following inequality at $t = t_0 = \frac{b+\gamma}{\lambda}$. To this end, note that

$$
\tau(y_n) = \tau(\exp(ts_n - at - bnt)) \\
= \tau(\exp(ts_{n-1} - at - b(n-1)t - bt + tx_n)) \\
\leq \tau(\exp(ts_{n-1} - at - b(n-1)t) \exp(-bt + tx_n)) \quad \text{(by (1.1))} \\
= \tau(E_{n-1}(\exp(ts_{n-1} - at - b(n-1)t) \exp(-bt + tx_n))) \\
= \tau(\exp(ts_{n-1} - at - b(n-1)t)E_{n-1}(\exp(-bt + tx_n))) \\
= \tau(y_{n-1}E_{n-1}(\exp(-bt + tx_n))) \\
= \tau(y_{n-1}E_{n-1}(e^{-bt}e^{tx_n})) \\
= e^{-bt}\tau(y_{n-1}E_{n-1}(e^{tx_n})) \\
= e^{-bt}\tau(\frac{1}{y_{n-1}}E_{n-1}(e^{tx_n}y_{n-1}^{2})) \\
\leq e^{-bt}f(t)\tau(y_{n-1}) \\
= A\tau(y_{n-1}) \\
\leq \tau(y_{n-1})e^{-(b+\gamma)t+\lambda t^2}
$$

if $A = e^{-bt}f(t)$ and $t = t_0 = \frac{b+\gamma}{\lambda}$, in which the first and second inequalities follows from (2.1).

We have $\bigvee_{n=m+i}^k1_{[a+bn,\infty)}(s_n) \preceq \bigvee_{n=m+i}^k1_{[1,\infty)}(y_n)$, for every $k \geq m+i$, in which $m, i$ are positive integers, since

$$
\bigvee_{n=m+i}^k1_{[a+bn,\infty)}(s_n) \cap (\bigwedge_{n=m+i}^k1_{[0,1)}(y_n)) = 0.
$$

To show this assume that $\xi$ is an unit element in

$$
\bigcup_{n=m+i}^k1_{[a+bn,\infty)}(s_n)(\xi) \bigcap (\bigwedge_{n=m+i}^k1_{[0,1)}(y_n)(\xi)) = 0.
$$

Therefore $\langle s_j, \xi \rangle \geq a + bj$ and $\langle e^{c(s_j-a-bj)}\xi, \xi \rangle < 1$, for some $m+i \leq j \leq k$. By the operator version of the classical Jensen’s inequality for the convex function $t \mapsto e^{c(t-a-bj)}$, we get

$$
e^{c(s_j-a-bj)(\xi,\xi)} \leq (e^{c(s_j-a-bj)}\xi, \xi) < 1.
$$
Consequently, \( \langle c(s_j - a - bj)\xi, \xi \rangle < 0 \) and hence \( \langle sj\xi, \xi \rangle < a + bj \) which gives rise to a contradiction. Choose \( i \in \mathbb{N} \) such that \( \frac{A_i}{1-A} \leq 1 \). Hence

\[
\tau\left(\bigvee_{n=m+i}^{\infty} 1_{[a+bn, \infty]}(s_n)\right) \leq \tau\left(\bigvee_{n=m+i}^{\infty} 1_{[1, \infty)}(y_n)\right) \\
\leq \sum_{n=m+i}^{\infty} \tau(y_n) \\
\leq \sum_{n=m+i}^{\infty} A\tau(y_{n-1}) \\
\vdots \\
\leq \sum_{n=m+i}^{\infty} A^n \tau(y_0) \\
\leq A^m e^{-\alpha t_0}
\]

for any positive integer \( m \), and this ensures \( (2.2) \).

To prove \( (2.3) \), let \( g(\alpha, n) = m(b - \alpha) + \alpha n \), \( n \geq m \), \( \alpha \leq b \) and note that \( bn \geq g(\alpha, n) \) for every \( n \geq m \). A minimization consideration leads to the choice of \( \alpha = \alpha_0 = \frac{b-n}{2} \). Thus

\[
1_{[bn, \infty)}(s_n) \leq 1_{[\frac{m(b+n)}{2}, \frac{n(b-n)}{2}, \infty)}(s_n)
\]

for any \( n \geq m \). From \( (2.2) \) we infer that

\[
\tau\left(\bigvee_{n=m+j}^{\infty} 1_{[m(b+n)+n(b-n), \infty]}(s_n)\right) \leq A_0^m e^{-\frac{m(b+n)}{4\alpha}};
\]

for some \( j \geq 1 \), where \( A_0 = e^{-\frac{1}{2(b-\gamma)\alpha}} f(t_0) \) and \( t_0 = \frac{b+\gamma}{2\alpha} \). Hence

\[
\tau\left(\bigvee_{n=m+j}^{\infty} 1_{[bn, \infty)}(s_n)\right) \leq \tau\left(\bigvee_{n=m+j}^{\infty} 1_{[m(b+n)+n(b-n), \infty)}\right) \leq A_0^m e^{-\frac{m(b+n)}{4\alpha}},
\]

which implies \( (2.3) \).

Note that, in view of the Jensen inequality for conditional expectations in the above Theorem, we lead to the following inequality:

\[
\mathcal{E}_{n-1}(x_n) \leq -\gamma.
\]

This special case have investigated by Khan. Similar to arguments in [7], we may conclude that if \( \{s_n = \sum_{i=1}^{n} x_n, n \geq 0\} \) is a self-adjoint supermartingale with respect to a filtration \( (\mathfrak{M}_n, \mathcal{E}_n)_{n \geq 0} \) such that \( -\alpha \leq x_n \leq \beta \) and \( \mathcal{E}_{n-1}(x_n) \leq -\gamma \) (\( \alpha > \lambda \geq 0, \beta > 0 \)) for all \( n \), then for positive numbers \( a \) and \( b \), there exists \( i \geq 1 \) such that for any positive integer \( m \)

\[
\tau\left(\bigvee_{n=m+i}^{\infty} 1_{(a+bn, \infty)}(s_n)\right) \leq A^m e^{-\frac{\gamma(b+n)}{2(a+\beta)}}.
\]
in which \( A = e^{-bt_0} f(t_0) \) and \( t_0 = \frac{8(b+\gamma)}{(\alpha+\beta)^2} \), where \( f(t) = e^{-\gamma t} \left(pe^{(\alpha+\beta)tq} + qe^{-(\alpha+\beta)tp}\right) \) and \( p = \frac{\alpha-\gamma}{\alpha+\beta} \) and \( \frac{\beta+\gamma}{\alpha+\beta} \). Similarly,

\[
\tau \left( \bigvee_{n=m+1}^\infty 1_{(bn,\infty)}(s_n) \right) \leq A_m^m e^{-\frac{2m(b+\gamma)^2}{(\alpha+\beta)^2}}
\]

for some \( j \geq 1 \), where \( A_0 = e^{-\frac{4(b+\gamma)}{(\alpha+\beta)^2}} \).

**Corollary 2.3. (Noncommutative Blackwell–Ross inequality)** Let \( x = (x_j)_{0 \leq j \leq n} \) be a self-adjoint martingale in \( \mathcal{M} \) with respect to a filtration \((\mathcal{M}_j, \mathcal{E}_j)_{0 \leq j \leq n}\) with \( x_0 = 0 \) and \( dx_j = x_j - x_{j-1} \) be its associated martingale difference. Assume that \(-\alpha \leq dx_j \leq \beta\) for some positive constants \( \alpha, \beta \) \( 1 \leq j \leq n \). Then for any positive values \( a, b \), there exists \( i \geq 1 \) such that for any positive integer \( m \)

\[
\tau \left( \bigvee_{n=m+i}^\infty 1_{[a+bn,\infty)}(x_n) \right) \leq A^m \exp \left\{ \frac{-8ab}{(\alpha + \beta)^2} \right\},
\]

where

\[
A = \frac{\beta}{\alpha + \beta} \exp \left\{ \frac{-8b(b + \alpha)}{(\alpha + \beta)^2} \right\} + \frac{\alpha}{\alpha + \beta} \exp \left\{ \frac{-8b(b - \beta)}{(\alpha + \beta)^2} \right\} \leq 1.
\]

Moreover,

\[
\tau(\bigwedge_{m=1}^\infty \bigvee_{n=m}^\infty 1_{[1,\infty)}(x_n)) = \lim_{m \to \infty} \tau(\bigvee_{n=m}^\infty 1_{[1,\infty)}(x_n)) = 0
\]

**Proof.** Note that \( x_n = \sum_{k=1}^n dx_k \) for all \( n \). Let \( t > 0 \). The function \( s \mapsto e^{ts} \) is convex, therefore for any \(-\alpha \leq s \leq \beta\),

\[
e^{st} \leq e^{t\beta} \frac{s + \alpha}{\alpha + \beta} + e^{-t\alpha} \frac{\beta - s}{\alpha + \beta}.
\]

Since \(-\alpha \leq dx_j \leq \beta\), by the functional calculus, we have

\[
e^{tdx_j} \leq e^{t\beta} \frac{dx_j + \alpha}{\alpha + \beta} + e^{-t\alpha} \frac{\beta - dx_j}{\alpha + \beta}.
\]

Since \( \mathcal{E}_{j-1} \) is a positive map and \( \mathcal{E}_{j-1}(dx_j) = 0 \), we reach

\[
\mathcal{E}_{j-1}(e^{tdx_j}) \leq e^{t\beta} \frac{\alpha}{\alpha + \beta} + e^{-t\alpha} \frac{\beta}{\alpha + \beta} \leq e^{\frac{\lambda^2(\alpha + \beta)^2}{8}},
\]

where the second inequality is deduced from Lemma 2.1 with \( \lambda = \frac{\alpha}{\alpha + \beta} \) and \( x = c(\alpha + \beta) \). Hence the desired result can be deduced from Theorem 2.2 with \( f(t) = e^{t\beta} \frac{\alpha}{\alpha + \beta} + e^{-t\alpha} \frac{\beta}{\alpha + \beta} \), \( \gamma = 0 \) and \( \lambda = \frac{(\alpha + \beta)^2}{8} \). \( \square \)

The authors of [10, 11] proved a noncommutative Azuma-type inequality for noncommutative martingales in noncommutative probability spaces, and as applications, the authors obtained a noncommutative Hoeffding inequality. In the
next corollary we give a noncommutative Blackwell inequality from which we deduce an extension of commutative Azuma-type inequality. One may regard the following conclusion as a stronger result than the noncommutative Azuma-type inequality.

**Corollary 2.4.** Let \( x = (x_j)_{0 \leq j \leq n} \) be a self-adjoint martingale in \( \mathcal{M} \) with respect to a filtration \( (\mathcal{M}_j, \mathcal{E}_j)_{0 \leq j \leq n} \) and \( dx_j = x_j - x_{j-1} \) be its associated martingale difference. Assume that \( -\alpha \leq dx_j \leq \beta \) for some nonnegative constants \( \alpha, \beta > 0 \) \((1 \leq j \leq n)\). Then for any positive value \( c \), there exists \( i \geq 1 \) such that for any positive integer \( m \)

\[
\tau \left( \bigvee_{n=m+i}^{\infty} 1_{(cn, \infty)}(x_n) \right) \leq B^m \exp \left\{ -\frac{2mc^2}{(\alpha + \beta)^2} \right\},
\]

where

\[
B = \frac{\beta}{\alpha + \beta} \exp \left\{ -\frac{2c(c + 2\alpha)}{(\alpha + \beta)^2} \right\} + \frac{\alpha}{\alpha + \beta} \exp \left\{ -\frac{2c(c - 2\beta)}{(\alpha + \beta)^2} \right\} \leq 1.
\]

**Proof.** For \( a = \frac{mc}{2} \) and \( b = \frac{c}{2} \), it follows from Corollary 2.3 that

\[
\tau \left( \bigvee_{n=m+i}^{\infty} 1_{\left[\frac{mc}{2}, \frac{mc}{2}+\infty\right]}(x_n) \right) \leq B^m \exp \left\{ -\frac{8mc^2}{4(\alpha + \beta)^2} \right\} \tag{2.4}
\]

for some \( i \geq 1 \). Moreover, we have

\[
1_{[mc, \infty)}(x_n) \leq 1_{\left[\frac{mc}{2}, \frac{mc}{2}+\infty\right]}(x_n) \tag{2.5}
\]

for every \( n \in \mathbb{N} \). Hence the result is deduced from (2.4) and (2.5).

**Corollary 2.5 (Azuma-type inequality).** Let \( Z_n, n \geq 0 \) be a martingale sequence of bounded random variables with respect to a filtration \( (\mathcal{F}_n, \mathbb{E}_n)_{n \geq 1} \) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with \( Z_0 = 0 \). If \( -\alpha \leq dZ_n \leq \alpha \) for all \( n \), then for each \( c > 0 \) there exists \( i \geq 1 \) such that for any positive integer \( m \)

\[
\mathbb{P}(Z_n \geq nc \text{ for some } n \geq m+i) \leq \exp \left\{ -\frac{mc^2}{2\alpha^2} \right\}.
\]

**Proof.** It immediately follows from Corollary 2.4.

Now we can state a version of classical Blackwell-Ross supermartingale inequality as follows; cf. [7].

**Corollary 2.6.** Let \( \{S_n = \sum_{i=1}^{n} X_n, n \geq 0\} \) be a supermartingale of bounded random variables with respect to a filtration \( (\mathcal{F}_n, \mathbb{E}_n)^N_{n=1} \) on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that

\[
\mathbb{E}_{n-1}(e^{tX_n}) \leq f(t) \leq e^{-\gamma t + \lambda t^2} \quad (\lambda > 0, \gamma \geq 0, t > 0),
\]
where \( f(t) \) is a positive continuous function. Then for positive numbers \( a \) and \( b \), there exists \( i \geq 1 \) such that for any positive integer \( m \),

\[
\mathbb{P}(S_n \geq a + bn \text{ for some } n \geq m + i) \leq A^m e^{-\frac{a(b+\gamma)}{\lambda}},
\]

where \( A = e^{-bt_0} f(t_0) < 1 \) and \( t_0 = \frac{b+\gamma}{\lambda} \). Moreover,

\[
\mathbb{P}(S_n \geq bn \text{ for some } n \geq m + j) \leq A^m_0 e^{-\frac{m(b+\gamma)^2}{4\lambda}}
\]

for some \( j \geq 1 \), where \( A_0 = e^{-\frac{1}{2}(b-\gamma)t_0} f(t_0) \) and \( t_0 = \frac{b+\gamma}{2\lambda} \).

**Corollary 2.7.** Let \( \mathcal{N} \subseteq \mathfrak{A}_j(\subseteq \mathfrak{M}) \) be order independent over \( \mathcal{N} \). Let \( x_j \in \mathfrak{A}_j \) be self-adjoint such that \( E_{\mathcal{N}}(x_j) \leq 0 \) and

\[
E_{n-1}(e^{tx_n}) \leq f(t) \leq e^{-\gamma t + \lambda t^2}, \quad (\gamma \geq 0, \lambda > 0, t > 0),
\]

where \( f(t) \) is a continuous positive function on \([0, \infty)\) such that \( f(0) = 1 \). Then for positive numbers \( a \) and \( b \), there exists \( i \geq 1 \) such that for any positive integer \( m \)

\[
\tau \left( \bigvee_{n=m+i}^{\infty} I_{\{a+bn, \infty\}}(s_n) \right) \leq A^m e^{-\frac{a(b+\gamma)}{\lambda}}.
\]

*Proof.* Let \( \mathfrak{M}_0 = \mathcal{N} \) and \( \mathcal{E}_0 = \mathcal{E}_{\mathfrak{M}} \). For every \( 1 \leq j \leq n \), let \( \mathfrak{M}_j \) be the von Neumann subalgebra generated by \( \mathfrak{A}_1, \ldots, \mathfrak{A}_{j-1} \) and \( \mathcal{E}_j \) be the corresponding conditional expectation. Put \( s_0 := 0 \) and \( s_j := \sum_{k=1}^{j} x_k \) for \( 1 \leq j \leq n \). Then

\[
\mathcal{E}_{j-1}(s_j) = \sum_{k=1}^{j-1} x_k + \mathcal{E}_{j-1}(x_k) = \sum_{k=1}^{j-1} x_k + \mathcal{E}_{\mathcal{N}}(x_k) \leq s_{j-1}.
\]

It follows that \( (s_j)_{0 \leq j \leq n} \) is a supermartingale with respect to the filtration \( (\mathfrak{M}_j, \mathcal{E}_j)_{0 \leq j \leq n} \). Hence, the result follows via \( \{s_n = \sum_{i=1}^{n} x_n, \ n \geq 0\} \) in Theorem 2.2. \( \square \)

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