A LEFSCHETZ (1, 1) THEOREM FOR SINGULAR VARIETIES

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The usual Lefschetz (1, 1) theorem says that given a smooth complex projective variety $X$, an element of $H^2(X, \mathbb{Z})$ is the class of a divisor if and only if it lies in $H^1(X, \mathbb{C})$ or equivalently in $F^1H^2(X, \mathbb{C})$. When $X$ is singular, $H^2(X, \mathbb{C})$ still carries the Hodge filtration associated to the canonical mixed Hodge structure. One of our main results is that an element of $H^2(X, \mathbb{Z})$ lies in $F^1$ if and only if it comes from motivic cohomology $H^2_M(X, \mathbb{Z}(1))$. Along the way, we give a reasonably concrete description of the last group. As in Deligne’s original construction of mixed Hodge structures, one starts by building a suitable simplicial resolution

$$
\ldots \quad \tilde{X}_2 \xrightarrow{p_0} \tilde{X}_1 \xrightarrow{p_1} \tilde{X}_0 \xrightarrow{\pi} X
$$

Very loosely, $\tilde{X}_*\tilde{X}_*\tilde{X}_*\tilde{X}_*$ is a diagram of smooth varieties with the same cohomology as $X$. An element of $H^2_M(X, \mathbb{Z}(1))$ is represented by a pair $(D, f)$, where $D$ is a divisor on $\tilde{X}_0$ and $f$ a rational function on $\tilde{X}_1$, such that $\partial D := p_0^*D - p_1^*D$ is defined and equal to $(f)$ and $\partial f = 1$. An example of such a pair is $(\pi^*C, 1)$ where $C$ is a Cartier divisor on $X$. So in this sense, the elements of $H^2_M(X, \mathbb{Z}(1))$ can be viewed as generalized Cartier divisors on $X$. It is worth noting that Barbieri-Viale and Srinivas [BS] have constructed a normal projective surface where not every element of $H^2(X, \mathbb{Z}) \cap F^1$ can be represented by a Cartier divisor, so generalized divisors are really needed here.

In addition to the Lefschetz theorem, one of our goals is to give a conjectural description of weight 2p Hodge cycles on $H^{2p}(X, \mathbb{Q})$, or equivalently elements of

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$H^{2p}(X, \mathbb{Q}) \cap F^p$, for all degrees $p$. As a first step, we will try to understand what happens on the maximal pure quotient $\hat{H}^{2p}(X) := H^{2p}(X)/W_{2p-1}$. We define a class in $\hat{H}^{2p}(X)$ to be homologically Cartier if it is represented by an algebraic cycle on some resolution. (Since this notion seems more broadly useful, we modify this definition to work in arbitrary characteristic in the first section.) Basic examples of homological Cartier cycles are provided by Chern classes of vector bundles, Weil divisors on normal surfaces and more generally numerically Cartier $\mathbb{Q}$-divisors in the sense of Boucksom, de Fernex, Favre and Urbanati \cite{BFFU}. The Hodge conjecture would imply that any Hodge cycle on $\hat{H}^{2p}(X)$ is given by a homologically Cartier cycle. This however only gives a partial solution to the original problem, since there is in general a nontrivial obstruction $\varepsilon(\alpha)$ for a homologically Cartier cycle $\alpha$ to lift to a Hodge cycle on $H^{2p}(X)$. The search for a natural source of unobstructed classes led the author first to operational Chow groups and then to motivic cohomology.

For our purposes, the most congenial approach to motivic cohomology is due to Hanamura \cite{Ha}. He defines it as the cohomology of a double complex built from Bloch’s cycle complex and a simplicial resolution. Hanamura shows that, with $\mathbb{Q}$-coefficients, the result is well defined and functorial. However, we really need this with $\mathbb{Z}$-coefficients. We handle this by showing that the group defined using Hanamura’s approach coincides with the more intrinsic definition given by Friedlander, Suslin and Voevodsky in their book (specifically \cite{FV}) as the cohomology of a complex of sheaves on the cdh site. Although these results are probably known to some, we include proofs in sections 4 and 5 for lack of a suitable reference.

Returning to the previous discussion, we have a map from motivic cohomology $H_M^{2p}(X, \mathbb{Q}(p)) \to CH^p_{OP}(X)_\mathbb{Q}$, if $\alpha \in CH^p_{OP}(X)_\mathbb{Q}$ lifts we show that $\varepsilon(\alpha) = 0$, so in particular it determines a weight $2p$ Hodge cycle on $H^{2p}(X, \mathbb{Q})$. The proof uses explicit formulas for higher cycle classes due to Kerr, Lewis and Müller-Stach \cite{KLM}. This result leads naturally to a refined Hodge conjecture (conjecture $\mathbf{S.1}$) that if $X$ is defined over $\overline{\mathbb{Q}}$, then any weight $2p$ Hodge cycle on $H^{2p}(X, \mathbb{Q})$ comes from motivic cohomology. Unlike the usual Hodge conjecture, the statement is easy to falsify in general for varieties not defined over $\overline{\mathbb{Q}}$. This is closely related to the fact that kernels of Abel-Jacobi maps on Chow groups of transcendental varieties can be very large. By contrast, according to a conjecture of Bloch and Beilinson, this sort of phenomenon should not occur for varieties over $\mathbb{Q}$. Regarding evidence for conjecture $\mathbf{S.1}$ we note that it holds for $p = 1$ by the Lefschetz theorem stated above and proved in section 7. As a consequence it also holds for products of degree 2 Hodge cycles. In the last section, we prove that the conjecture holds for the $n$-fold self fibre product of an elliptic modular surface. The result is deduced by showing that the algebra of Hodge cycles on these varieties are generated by degree 2 Hodge cycles following a careful analysis of the Leray spectral sequence.

The word “variety” will mean a reduced scheme of finite type over the ground field, which, with the exception of the first section, is always $\mathbb{C}$. We write $H^*(X)$ (respectively $H^*(X, \mathbb{Z})$) for singular cohomology of the associated analytic space with coefficients in $\mathbb{Q}$ (respectively $\mathbb{Z}$) in all but the first section.
Comments by V. Srinivas, B. Totaro and A. Vistoli at an early stage of this project were very helpful in steering me in the right direction. Parts of this paper were written during a short but productive visit to the Simons Center in Stony Brook.

1. Homologically Cartier cycles

In this section we work over an arbitrary algebraically closed field \( k \), but over \( \mathbb{C} \) in the remaining sections. Let \( H^*(-) \) denote either \( \ell \)-adic cohomology, with \( \mathbb{Q}_\ell \)-coefficients, where \( \ell \neq \text{char } k \), or singular cohomology with \( \mathbb{Q} \)-coefficients when \( k = \mathbb{C} \). Let \( H^*(-) \) denote either or ordinary or \( \ell \)-adic Borel-Moore homology \([F, L]\), again with \( \mathbb{Q} \) or \( \mathbb{Q}_\ell \) coefficients. Every \( p \)-dimensional closed subvariety \( V \subset X \) possesses a fundamental class \([V] \in H_{2p}(X)\). Let \( C^p(X) \subset H_{2p}(X) \) denote the \( \mathbb{Q} \)-span of these classes. This can be identified with the quotient of the Chow group \( \text{CH}^p(X) \) tensored with \( \mathbb{Q} \) by homological equivalence. At this point, we need to bring the weight filtration into play. We start with some elementary definitions and properties.

Lemma 1.1. Let \( \pi: \tilde{X} \rightarrow X \) be a nonsingular alteration of a projective variety \( X \). The subspaces
\[
W_{p-1}H^p(X) = \ker[H^p(X) \rightarrow H^p(\tilde{X})] \\
W_{-p}H^p(X) = \text{im}[H_p(\tilde{X}) \rightarrow H_p(X)]
\]
are independent of the choice of \( \tilde{X} \).

Proof. Given a second alteration \( \pi': \tilde{X}' \rightarrow X \), after replacing it by the component of an alteration of \( \tilde{X} \times_X \tilde{X}' \) dominating \( X \), we can assume that \( \tilde{X}' \) factors through a morphism \( \tilde{X}' \rightarrow \tilde{X} \). By Poincaré duality, \( H^p(\tilde{X}) \rightarrow H^p(\tilde{X}') \) is injective. Therefore \( \ker \pi' = \ker \pi'^* \). The second part is similar. \( \square \)

We can see easily that \( \bigoplus W_{-1}H^*(X) \subset H^*(X) \) is an ideal. Set
\[
\tilde{H}^*(X) = H^*(X)/\bigoplus W_{-1}H^*(X) \cong \text{im}[H^*(X) \rightarrow H^*(\tilde{X})]
\]
to the quotient ring. Also let
\[
\tilde{H}_j(X) = W_{-j}H_j(X)
\]

Lemma 1.2. If \( \alpha \in W_{i-1}H^i(X) \) and \( \beta \in \tilde{H}_j(X) \), then \( \alpha \cap \beta = 0 \).

Proof. Choose \( \tilde{\beta} \in H_j(\tilde{X}) \), with \( \tilde{X} \) as above, so that \( \pi_*\tilde{\beta} = \beta \). Then
\[
\alpha \cap \beta = \pi_*(\pi^*\alpha \cap \tilde{\beta}) = 0
\]
\( \square \)

It follows that the cap product descends to a well defined pairing
\[
\tilde{H}^*(X) \otimes \tilde{H}_j(X) \rightarrow \tilde{H}_{j-i}(X)
\]
that we will also refer to as a cap product.

Lemma 1.3. The image of the cycle map \( CH_p(X) \rightarrow H_{2p}(X) \) lies in \( \tilde{H}_{2p}(X) \).

Proof. Given an algebraic cycle \( \beta \in CH_p(X) \), we can find an algebraic cycle \( \tilde{\beta} \in CH_p(\tilde{X}) \) such that \( \pi_*\tilde{\beta} = \beta \). \( \square \)
We come to the key definition. If \( X \) is a projective (possibly reducible) variety, an element \( \alpha \in \check{H}^{2p}(X) \) can be regarded as an element of \( H^{2p}(\hat{X}) \), for any alteration, under the inclusion \( \check{H}^{2p}(X) \subset H^{2p}(\hat{X}) \). We say that \( \alpha \) is \text{homologically Cartier} if it is represented by an algebraic cycle on some nonsingular alteration \( \hat{X} \). Let \( C^p(X) \) denote the space homologically Cartier cycles on \( X \). When \( X \) is nonsingular and \( \pi^*\alpha \) is algebraic then so is \( \alpha = \pi_\ast \pi^* \alpha \). Thus we see that homologically Cartier cycles are just algebraic cycles in this case. For similar reasons, we can see that if \( X \) is irreducible of dimension \( n \), then
\[
C^p(X) = \text{im } H^{2p}(X) \cap C_{\dim X-n}(\hat{X})
\]
for a fixed nonsingular alteration \( \hat{X} \to X \)

\textbf{Proposition 1.4.}

1. \( C^p(\_\_) \) is functorial in the sense that if \( f : X \to Y \) is morphism, then \( f^\ast C^p(Y) \subset C^p(X) \).
2. \( C^\ast(X) \subset \check{H}^{2\ast}(X) \) is a subring.
3. If \( \alpha \in C^p(X) \) and \( \beta \in C_q(X) \), then \( \alpha \cap \beta \in C_{q-p}(X) \)

\textit{Proof.} The first property is clear, because we can find a commutative diagram

\[
\begin{array}{ccc}
\tilde{X} & \longrightarrow & \tilde{Y} \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
\]

where the vertical maps are nonsingular alterations. For (2), it is enough to observe that \( C^\ast(X) \) is an intersection of two subrings of \( \check{H}^{2\ast}(\hat{X}) \) namely \( \text{im } H^{2\ast}(X) \cap C^\ast(\hat{X}) \).

To prove (3), choose \( \tilde{\alpha} \in C^p(\hat{X}) = C_{\dim X-p}(\check{X}) \) with \([\tilde{\alpha}] = \alpha \) and \( \tilde{\beta} \in C_q(\hat{X}) \) with \( \pi_\ast[\tilde{\beta}] = \beta \) (the existence of \( \tilde{\beta} \) is easy c.f. \cite{K} prop 1.3). Then we have \( \alpha \cap \beta = [\pi_\ast(\tilde{\alpha} \cdot \tilde{\beta})] \). \( \square \)

\textbf{Proposition 1.5.} Given a projective variety \( X \), choose a nonsingular alteration \( \pi : \hat{X} \to X \) and a nonsingular alteration \( \hat{X}_1 \to \hat{X}_0 \times_X \hat{X}_0 \) with projections \( p_i : \hat{X}_1 \to \hat{X}_0 \). Then
\[
H^i(\hat{X}) \to H^i(\hat{\tilde{X}}) \to H^i(\hat{X}_1)
\]
is exact, where \( \partial = p_1^* - p_2^* \).

\textit{Proof.} Suppose that \( \text{char } k = 0 \). Since the étale cohomology of \( X \) is invariant under base extension to a larger algebraically closed field, there is no loss assuming that \( k = \mathbb{C} \). By the comparison theorem, we can also assume that \( H^\ast(\check{X}) \) is singular cohomology. Now the proposition follows from \cite{D} prop 8.2.5.

When \( \text{char } k = r > 0 \), we also use a weight argument, but we will need to work out things from scratch. First of all, we can reduce to the case where \( k \) is the algebraic closure of a field \( k_0 \) which is finitely generated over the finite field \( \mathbb{F}_r \).

We can assume that \( X \) is defined by the base change of a variety defined over \( k_0 \). Using \cite{D}, we build a smooth simplicial scheme \( X_\bullet \to X \) augmented over \( X \) as follows. Let \( \hat{X}_0 = \hat{X} \) and \( \hat{X}_1 \) as above. Choose the higher \( \hat{X}_n \) inductively so that the canonical map
\[
\hat{X}_n \to \cosk(sk_{n-1}\hat{X}_\bullet) \to X
\]
is proper and surjective; see [D1] §6 or [S]. This will ensure that \( \tilde{X}_* \to X \) will satisfy cohomological descent, and in particular that we have a descent spectral sequence

\[
E_{1}^{pq} = H^{q}(\tilde{X}_{p}) \Rightarrow H^{p+q}(X)
\]

We can assume that for any fixed constant \( N \), all \( \tilde{X}_{p} \) for \( p \leq N \), and maps between them, are defined over \( k_{0} \), after possibly enlarging it. This will ensure that \( G = \text{Gal}(k/k_{0}) \) will act on the spectral sequence in the range \( p \leq N \). In particular, that the differentials are equivariant in this range. Choose \( \phi \in G \) which maps to a Frobenius in \( \text{Gal}(\mathbb{F}_r/\mathbb{F}_{r^s}) \). The Weil conjectures [D2] will show that the eigenvalues of \( \phi \) on \( E_{1}^{pq} \) and \( E_{1}^{pq'} \) are different whenever \( q \neq q' \). Since \( N \) can be chosen arbitrarily large, this forces degeneration of the spectral sequence at \( E_{2} \). In particular, \( E_{2}^{pq} = E_{\infty}^{pq} \). This implies exactness of

\[
H^{i}(X) \to H^{i}(\tilde{X}_{0}) \to H^{i}(\tilde{X}_{1})
\]

\[\square\]

**Corollary 1.6.** We have an exact sequence

\[
0 \to C^{p}(X) \to C^{p}(\tilde{X}_{0}) \to C^{p}(\tilde{X}_{1})
\]

**Proof.** By definition the first arrow is injective. The sequence is also clearly a complex by functoriality of \( C^{p}(-) \). We just have to show that if \( \alpha \in C^{p}(\tilde{X}_{0}) \) maps to 0 in the third group, then it must come from the first. We have a commutative diagram

\[
\begin{array}{ccc}
0 & \to & C^{p}(X) \\
\downarrow & & \downarrow \\
0 & \to & \tilde{H}^{2p}(X) \\
\end{array}
\begin{array}{ccc}
& & C^{p}(\tilde{X}_{0}) \\
& & \downarrow \\
& & \tilde{H}^{2p}(\tilde{X}) \\
\end{array}
\begin{array}{ccc}
& & C^{p}(\tilde{X}_{1}) \\
& & \downarrow \\
& & \tilde{H}^{2p}(\tilde{X}_{1}) \\
\end{array}
\]

We see that \( \alpha \) maps to 0 in \( \tilde{H}^{2p}(\tilde{X}_{1}) \). Therefore it lies in \( C^{p}(X) = \tilde{H}^{2p}(X) \cap C^{p}(\tilde{X}_{0}) \) by exactness of the bottom row. \(\square\)

**Remark 1.7.** It will be useful to say a few words about the geometry of \( \tilde{X}_{1} \), when \( \tilde{X} \to X \) is desingularization and \( X \) is irreducible. Then we may also assume that \( \tilde{X} \) is irreducible. Let \( \Sigma \subset X \) the maximal closed set over which \( f \) is not an isomorphism. If \( E = f^{-1}\Sigma \), then \( \tilde{X} \times_{X} \tilde{X} \) is a union of \( \tilde{X} \) embedded diagonally and \( E \times_{\Sigma} E \). Thus \( \tilde{X}_{1} \) can be taken to be a disjoint union of \( \tilde{X} \) and an alteration \( \tilde{X}_{1}' \) of \( E \times_{\Sigma} E \). The proposition and its corollary holds when \( \tilde{X}_{1} \) is replaced by \( \tilde{X}_{1}' \).

Let us discuss some examples. Suppose that \( E \) is a vector bundle on \( X \). The usual cohomological Chern class \( c_{p}(E) \in H^{2p}(X) \) is homologically Cartier because it pulls back to an algebraic cycle on \( \tilde{X} \). Let us say that a cycle is Cartier if it is \( \mathbb{Q} \)-linear combination of Chern classes of vector bundles. We will see later that not every homologically Cartier cycle is Cartier. We lay the groundwork now, by giving a different source of examples. Suppose that \( X \) is a normal projective surface with a desingularization \( \pi : \tilde{X} \to X \) with exceptional divisors \( E_{i} \). Given a Weil divisor \( D \) on \( X \) with strict transform \( D' \), Mumford [M] constructed a unique \( \mathbb{Q} \)-divisor \( \pi^{*}D = D' + \sum a_{i}E_{i} \) on \( \tilde{X} \) for which \( \pi^{*}D \cdot E_{j} = 0 \) for all \( j \). We have a Mayer-Vietoris type sequence

\[
H^{2}(X) \to H^{2}(\tilde{X}) \to \bigoplus H^{2}(E_{j})
\]
which shows that $[\pi^*D] \in H^2(X)$. Thus we have proved:

**Lemma 1.8.** A Weil divisor $D$ on a normal projective surface gives a homologically Cartier cycle, namely $\pi^*D$.

If $X$ is a higher dimensional normal projective variety over a field of characteristic zero, Boucksom, de Fernex, Favre and Urbanati [BFFU] generalize this as follows. They call a Weil divisor $D$ on $X$ numerically $\mathbb{Q}$-Cartier if on some desingularization $\pi : \tilde{X} \to X$, there exists a (necessarily unique) $\mathbb{Q}$-divisor $\pi^* D$ on $\tilde{X}$ which is $\pi$-trivial and for which $\pi_* \pi^* D = D$. The $\pi$-triviality condition means that the intersection number $\pi^* D \cdot C = 0$ for any curve that gets contracted under $\pi$. We claim that $[\pi^* D] \in \text{im} \ H^2(X)$. This will imply that $\pi^* D$ is homologically Cartier; in fact, the conditions of being homologically Cartier is really the same as the condition of being numerically Cartier in this case. The claim follows from the next lemma.

**Lemma 1.9.** A $\mathbb{Q}$-divisor $F$ on $\tilde{X}$ is $\pi$-trivial if and only if $[F] \in \text{im} \ H^2(X)$.

**Proof.** One direction is clear, if $[F] \in \text{im} \ H^2(X)$, then $F \cdot C = \pi_* [F] \cdot 0 = 0$ for any curve $C$ contracted by $\pi$.

The converse hinges on the well known fact that numerical equivalence and homological equivalence for divisors on a smooth projective variety coincide (because the Neron-Severi group tensor $\mathbb{Q}$ injects into $H^2$). Thus using proposition 1.5 we have to show that $p_1^* F - p_2^* F$ is numerically trivial on $\tilde{X}_1$, where $p_i : \tilde{X}_1 \to \tilde{X}$ are the projections. Let $C \subset \tilde{X}_1$ be an irreducible curve. Let $C_i = p_i(C)$ and $C' = \pi \circ p_1(C)$ with reduced structures. The diagram

$$
\begin{array}{ccc}
C & \xrightarrow{d_1} & C_1 \\
\downarrow{d_2} & & \Downarrow{\pi_1} \\
C_2 & \xrightarrow{e_2} & C'
\end{array}
$$

commutes. First, let us suppose that $C'$ is a curve. Let us replace the curves in the diagram by their normalizations. The degrees of the maps $d_i, e_1$ are indicated in the diagram. We have that $d_1 e_1 = d_2 e_2$ is the degree of $C \to C'$. Let $F'$ be the pushforward of the zero cycle $F|_C$ under $C \to C'$. Then

$$(p_1^* F - p_2^* F) \cdot C = d_1 F \cdot C_1 - d_2 F \cdot C_2 = d_1 e_1 \deg(F') - d_2 e_2 \deg(F') = 0$$

The remaining case is when $C'$ is a point. Then $C_i$ is either a point or a curve contracted by $\pi$. In either case $p_i^* F \cdot C = F \cdot p_i_* C = 0$.

$\square$

There are various ways in which this construction extends to higher rank sheaves. We look at a particularly simple case. Suppose that $X$ is a smooth projective variety over a field of characteristic 0 on which a finite group $G$ acts. Then the quotient $Y = X/G$ is well known to exists in the category of normal projective varieties. Let $E$ be reflexive sheaf on $Y$. Then we can homologically Cartier “Chern classes” $c_p(E) \in H^{2p}(Y)$. Here is the construction: $E$ restricts to a locally free sheaf on the smooth locus $U$. This can be pulled back to the preimage of $U$ in $X$ and extended to give a locally free sheaf $F$ on $X$. The Chern classes $c_p(F) \in H^{2p}(X)$ are necessarily $G$-invariant, so they define cohomology classes on $Y$ thanks to the isomorphism $H^{2p}(Y) = H^{2p}(X)^G$. These are homologically Cartier by definition because $X \to Y$ is a nonsingular alteration.
An additional source of examples of homologically Cartier cycles will be discussed in section 3.2.

2. Hodge cycles

In this section, we work exclusively over \( \mathbb{C} \) and take \( H^*(X) \) to be singular cohomology with its canonical mixed Hodge structure [1]. The quotient \( \tilde{H}^i(X) = H^i(X)/W_i \) is a pure Hodge structure of weight \( i \). By a Hodge cycle of weight \( 2p \) on a mixed Hodge structure \( H \), we will mean an element of

\[
\text{Hom}_{MHS}(\mathbb{Q}(-p), H) \cong \text{Hom}_{MHS}(\mathbb{Q}(0), H(p)).
\]

More concretely, this is given an element of \((2\pi i)^p \mathbb{H}_Q \cap W_{2p} \cap F^p H\), or simply \((2\pi i)^p \mathbb{H}_Q \cap F^p H\) when \( W_{2p} = H \). Let us now normalize things so that when \( X \) is smooth and projective, the image of the cycle map on \( CH^p(X) \) lies in \( H^{2p}(X, \mathbb{Q}(p)) = H^{2p}(X, (2\pi i)^p \mathbb{Q}) \) (as a lattice \( H^{2p}(X, \mathbb{C}) \)). This will make certain statements appear more natural. Here is the key observation:

**Proposition 2.1.** If \( X \) is a projective variety, the image of \( C^p(X) \to \tilde{H}^{2p}(X, \mathbb{Q}(p)) \) consists of Hodge cycles of weight \( 2p \). The converse is true if the Hodge conjecture, in degree \( 2p \), holds for a resolution of \( X \). In particular, the result holds unconditionally for \( p = 1 \).

**Proof.** This follows immediately from the diagram given in the proof of corollary [1.6]

From the extension

\[
0 \to W_{2p-1}H^{2p}(X) \to H^{2p}(X) \to \tilde{H}^{2p}(X) \to 0
\]

together with the fact that

\[
H^{\text{Hom}}_{MHS}(\mathbb{Q}(-p), W_{2p-1}H^{2p}(X)) = 0
\]

we obtain an injective map:

**Lemma 2.2.**

\[
H^{\text{Hom}}_{MHS}(\mathbb{Q}(-p), H^{2p}(X)) \hookrightarrow H^{\text{Hom}}_{MHS}(\mathbb{Q}(-p), \tilde{H}^{2p}(X))
\]

A homologically Cartier cycle \( \alpha \) gives an element \( H^{\text{Hom}}_{MHS}(\mathbb{Q}(-p), \tilde{H}^{2p}(X)) \)

Under the connecting map, we obtain a class

\[
\varepsilon(\alpha) \in \text{Ext}^1_{MHS}(\mathbb{Q}(-p), W_{2p-1}H^{2p}(X))
\]

Let

\[
\varepsilon_1(\alpha) \in \text{Ext}^1_{MHS}(\mathbb{Q}(-p), Gr_{2p-1}^W H^{2p}(X))
\]

denote the image of the previous class in the \( \text{Ext} \) group above. These give the obstructions to lifting \( \alpha \) to a Hodge cycle in \( H^{2p}(X) \) and \( H^{2p}(X)/W_{2p-2} \) respectively.

We want to describe \( \varepsilon \) and \( \varepsilon_1 \) in more explicit terms. By work of Carlson [C], the above two \( \text{Ext} \) groups can be identified with the intermediate Jacobians

\[
J \text{W}_{2p-1}H^{2p}(X) = \frac{W_{2p-1}H^{2p}(X)}{F^pW_{2p-1}H^{2p}(X, \mathbb{C}) + W_{2p-1}H^{2p}(X, \mathbb{Q})}
\]

and

\[
J \text{Gr}_{2p-1}^W H^{2p}(X) = \frac{Gr_{2p-1}^W H^{2p}(X)}{F^pGr_{2p-1}^W H^{2p}(X, \mathbb{C}) + Gr_{2p-1}^W H^{2p}(X, \mathbb{Q})}
\]
respectively. Carlson gives a recipe for computing the extension classes. Choose lifts (which exist) \( A \in F^p H^{2p}(X) \) and \( B \in H^{2p}(X, \mathbb{Q}) \) of the Hodge cycle \( [\alpha] \), then the difference \( A - B \) lies in \( W_{2p} H^{2p}(X) \). The obstruction \( \varepsilon(\alpha) \) is the class of \( A - B \) in the quotient in (1). We can also consider a sequence of intermediate obstructions \( \varepsilon_1(\alpha), \ldots \) given by the projection of \( \varepsilon(\alpha) \) to (2) etc. To proceed further, fix a smooth projective augmented simplicial scheme \( \tilde{X}_\bullet \to X \) satisfying cohomological descent. We only require that this be a semi or strict simplicial object, which means that there are face maps \( p_i : \tilde{X}_j \to \tilde{X}_{j-1} \), but not degeneracy maps in the backwards direction. In practice, this makes the constructions more economical (see proposition 5.1). Let \( (\mathcal{E}^\bullet(\tilde{X}_j), d) \) denote the de Rham complex. This forms a double complex \( (\mathcal{E}^\bullet(\tilde{X}_\bullet), d, \pm \partial) \), where \( \partial = \sum (-1)^i p_i^* \) denotes the simplicial boundary (we work up to sign). We can form the total complex, \( E^n = \bigoplus_{a+b=n} \mathcal{E}^a(\tilde{X}_b), \) with differential \( d \pm \partial \). This is filtered by \( F^p E^\bullet = \bigoplus_{a \geq p} \mathcal{E}^{a,b}(\tilde{X}_\bullet) \). Then \( A \) can be represented by an element

\[
A = (A_0, A_1, \ldots) \in F^p E^{2p} = F^p \mathcal{E}^{2p}(\tilde{X}_0) \oplus F^p \mathcal{E}^{2p-1}(\tilde{X}_1) \oplus \ldots
\]

Similarly \( B \) can be represented by an element \( (B_0, \ldots) \in E^{2p}_Q \), where \( E_Q \) denotes the total complex of the \( C^\infty \) singular cochain complex with coefficients in \( \mathbb{Q} \). To make sense of \( A - B \), we can either push \( A \) into \( E^{2p}_Q \otimes \mathbb{C} \) under the quasi-isomorphism \( \mathcal{E}^\bullet \to \mathcal{E}^\bullet_Q \otimes \mathbb{C} \) defined by integration; or we can replace \( (B_0, \ldots) \) by a sequence of differential forms with rational periods. In the second case, we may assume that \( A_0 = B_0 \). We are now in a position to extract an explicit description. The torus \( JGr_{2p-1}^W H^{2p}(X) \) is a subquotient of the Griffiths’ intermediate Jacobian

\[
J^p(\tilde{X}_1)_\mathbb{Q} = \frac{H^{2p-1}(\tilde{X}_1)}{F^p H^{2p-1}(\tilde{X}_1) + H^{2p-1}(\tilde{X}_1, \mathbb{Q})}
\]

Any homologically trivial cycle \( Z \) on \( \tilde{X}_1 \) determines an element \( AJ(Z) \in J^p(\tilde{X}_1)_\mathbb{Q} \). We will recall the construction in the proof below.

**Proposition 2.3.** If \( \alpha \) is a homologically Cartier cycle, \( \varepsilon_1(\alpha) \) is (up to sign) the image \( AJ(\partial \alpha) \) under the map \( \ker[J^p(\tilde{X}_1) \to J^p(\tilde{X}_2)]_\mathbb{Q} \to JGr_{2p-1}^W H^{2p}(X) \).

**Proof.** The expressions \( \varepsilon_1(\alpha) \) and \( AJ(\partial \alpha) \) will be summed over the connected components of \( \tilde{X}_1 \). So without loss of generality, we can assume that it is connected of dimension \( n \). We have that \( \partial \alpha \) is homologically trivial. Therefore it is the boundary of a rational \( C^\infty \) \( (2n - 2p + 1) \)-chain \( \Gamma \). Under Poincaré duality,

\[
F^{n-p+1} H^{2n-2p+1}(\tilde{X}_1)^* \cong H^{2p-1}(\tilde{X}_1)/F^p H^{2p-1}(\tilde{X}_1)
\]

Integration along \( \Gamma \) defines a functional on \( H^{2n-2p+1}(\tilde{X}_1) \), and therefore an element of the right side of (3). Its image in \( J^p(\tilde{X}_1)_\mathbb{Q} \) is precisely \( AJ(\partial \alpha) \).

We assume that \( B_i \) is a sequence of differential forms with rational periods, and that \( A_0 = B_0 \). Then \( A_1 - B_1 \) determines a closed form whose image in \( JGr_{2p-1}^W H^{2p}(\tilde{X}_1) \) is \( \varepsilon_1(\alpha) \). Regarding \( B_1 \) as a current, we can choose it cohomologous to the current \( \gamma \) given by

\[
\omega \mapsto \int_{\Gamma} \omega
\]
The form $A_1$ defines the current

$$\omega \mapsto \int_{\tilde{X}_1} A_1 \wedge \omega$$

which acts trivially on the left side of (3). Therefore the action of $\pm (A_1 - B_1)$ on (3) is integration on $\mathbb{R}$.

We now give a simple example, where this obstruction is nontrivial.

**Example 2.4.** Let $C \subset \mathbb{P}^2$ be a nonsingular cubic. Let $Q_0 \subset \mathbb{P}^2$ be a very general quartic. The two curves meet in 12 very general points, $p_1, \ldots, p_{12}$. Blow up these points to get a surface $f : \tilde{X} \to \mathbb{P}^2$ with exceptional divisors $E_1, \ldots, E_{12}$. Let $C \subset \tilde{X}$ be the strict transform of $C$ which is abstractly the same curve. Let $Q = f^* Q_0 - \sum E_i$. We have that $Q^2 = 4$ and $Q \cdot \tilde{C} = 0$. Furthermore, $|Q|$ is base point free, so it contracts $\tilde{C}$ to a point $p$ in a normal surface $X$. We build the augmented simplicial scheme

$$\tilde{X}_1 = \tilde{C} \Rightarrow \tilde{X}_0 = \tilde{X} \coprod p \to X$$

Since $\tilde{X}_2 = \emptyset$ and $H^1(\tilde{X}_0) = 0$, we have $J(C) = JGr^W H^2(X)$. Let $D = f^* L - E_1 - E_2 - E_3$, where $L \subset \mathbb{P}^2$ is a line. Then $D$ has degree 0 on $\tilde{C}$, so it is gives a homologically Cartier cycle on $X$. Note however the class of $D$ in the Jacobian of $\tilde{C}$ is nonzero because, the points $p_1, p_2, p_3$ were very general and therefore noncolinear. So $\epsilon_1(D) \neq 0$.

We want to say more about $\epsilon(\alpha)$ when $p = 1$ and $X$ is eventually a surface. In this case, we will work integrally. We define $W_1 H^2(X, \mathbb{Z})$ to be the intersection $W_1 H^2(X, \mathbb{Q})$ with the torsion free part of $H^2(X, \mathbb{Z})$. Choose a simplicial scheme $\tilde{X}_* \to X$ as above. Let $\text{Div}_{g}(\tilde{X}_1)$ denote the space of divisors in general position with respect to the maps $p_i$; more precisely, no component of $D \in \text{Div}_{g}(\tilde{X}_1)$ should contain the image of a component of $\tilde{X}_2$ under any $p_i$. Let $\text{Div}_{g}(\tilde{X}_1) \subset \text{Div}_{g}(\tilde{X}_1)$ denote the subgroup of divisors which are trivial in $H^2(\tilde{X}_1, \mathbb{Z})$. Let $R(\tilde{X}_1)$ be the product of the fields of rational functions on the connected components of $\tilde{X}_1$, and let $R(\tilde{X}_1)^*$ denote the group of units. Define $R_g(\tilde{X}_1)^* \subset R(\tilde{X}_1)^*$ to be the subgroup of functions whose divisor lies in $\text{Div}_g(\tilde{X}_1)$. Let $\mathbb{C}(\tilde{X}_1)^* \subset R_g(\tilde{X}_1)^*$ denote the subgroup of locally constant functions. By an easy moving argument, we can see that the sequence

$$0 \to \mathbb{C}(\tilde{X}_1)^* \to R_g(\tilde{X}_1)^* \to \text{Div}_g(\tilde{X}_1) \to \text{Pic}^0(\tilde{X}_1) \to 0$$

is exact.

We now assume that $X$ is a surface. Then either using remark [1, 7] or proposition [5, 1], we can see that $\tilde{X}_2$ can be chosen to be zero dimensional. We do so. Let $\partial$ denote the multiplicative simplicial coboundary. Then the quotient

$$R_g(\tilde{X}_1)^*/\partial^{-1}\partial \mathbb{C}(\tilde{X}_1)^* \cong \mathbb{C}(\tilde{X}_2)^*/\partial \mathbb{C}(\tilde{X}_1)^*$$

is a finite dimensional multiplicative torus. Following Carlson [C2], we define the group

$$P(\tilde{X}_*) = \text{Div}_g(\tilde{X}_1)/\partial^{-1}\partial \mathbb{C}(\tilde{X}_1)^*$$

which is an extension of $\text{Pic}^0(\tilde{X}_1)$ by the torus $\mathbb{C}(\tilde{X}_2)^*/\partial \mathbb{C}(\tilde{X}_1)^*$. We remark that if we allow $\tilde{X}_2$ to have positive dimensional components, then $P(\tilde{X}_1)$ is the wrong
object to work with as it could be infinite dimensional (\cite{C2} is not very explicit about this issue). Let $D$ be a divisor class on $\tilde{X}_0$ giving a homologically Cartier cycle on $X$. Then $\partial D \in \text{Div}_h(\tilde{X}_1)$ by definition. So we get an induced map $\partial : \text{Pic}^0(\tilde{X}_0) \to P(\tilde{X}_\bullet)$.

**Proposition 2.5.** With the above assumption that $\dim \tilde{X}_2 = 0$, we have an isomorphism

\begin{equation}
\text{Ext}^1_{\text{MHS}}(\mathbb{Z}(-1), W_1 H^2(X, \mathbb{Z})) \cong P(\tilde{X}_\bullet)/\partial \text{Pic}^0(\tilde{X}_0)
\end{equation}

Let $\alpha$ denote a homologically Cartier element in $\tilde{H}^2(X)$, and let $D$ be a divisor on $\tilde{X}$ representing it (which exists by the Lefschetz $(1, 1)$ theorem). Then $\varepsilon(\alpha) = 0$ if and only if the image of $\partial D$ under the isomorphism \ref{eq:ext} vanishes.

**Remark 2.6.** This statement is sufficient for our purposes, although presumably $\varepsilon(\alpha) = \pm \im \partial D$.

**Proof.** By a theorem of Deligne \cite{D1 §10}, the category of polarizable mixed Hodge structures of type $\{(0, 0), (1, 0), (0, 1), (1, 1)\}$ is equivalent to the category of 1-motives. Let $H \subseteq H^2(X)$ be the maximal submixed Hodge of this type. A theorem of Carlson \cite{C2 thm A} says that $H$ corresponds to the 1-motive $\partial : NS(\tilde{X}_0) \to P(\tilde{X}_\bullet)/\partial \text{Pic}^0(\tilde{X}_0)$

Under this identification, the mixed Hodge structure $E \subset H$ given by the extension class $\varepsilon(\alpha)$:

\[ \begin{array}{ccccccc}
0 & \to & W_1 H^2(X, \mathbb{Z}) & \to & E & \to & \mathbb{Z}(-1) & \to & 0 \\
& & \downarrow & & \downarrow \varepsilon(\alpha) & & \downarrow \partial D \\
0 & \to & W_1 H^2(X, \mathbb{Z}) & \to & H & \to & \tilde{H}^2(X, \mathbb{Z}) & \\
\end{array} \]

corresponds to the sub motive $\mathbb{Z}(-1) \to P(\tilde{X}_\bullet)/\partial \text{Pic}^0(\tilde{X}_0)$

The proposition is an immediate consequence. \hfill $\square$

The next example is a variation on one due to Totaro \cite{T}.

**Example 2.7.** Let $X$ be a normal surface constructed as in example 2.4, but with $C$ a nodal cubic. We build a simplicial scheme $\ast \to \tilde{X}_1 = \mathbb{P}^1 \coprod_{p} \tilde{X}_0 = \tilde{X} \coprod_{p} X$ where the maps are built from inclusions, projections and the normalization of $\tilde{C}$. In this case, $\varepsilon_1(D) = 0$ because $J(\tilde{X}_1) = 0$, but the class of $D$ in $P(\tilde{X}_\bullet) = \mathbb{C}^*$ is non torsion, so that $\varepsilon(D) \neq 0$.

The final example, which is a variation on one due to Barbieri Viale and Srinivas \cite{BS}, gives an example of a homologically Cartier cycle which is not Cartier.

**Example 2.8.** Let $X$ be a normal surface constructed as in example 2.4, but with $C$ a cuspidal cubic. We proceed as above, but now $P(\tilde{X}_\bullet) = 0$, so $\varepsilon(D) = 0$. But $D$ has nontrivial class in $\text{Pic}^0(C) = \mathbb{C}$, so it cannot be Cartier. Note that in this case, Hodge theory is too coarse to detect the Picard group.
3. Two false starts

This section contains two initial attempts by the author to answer the main question about where Hodge cycles come from. Although neither gives the correct answer, we have included this material because we feel that it is nevertheless instructive.

3.1. Fulton’s original Chow ring. Fulton [F1] defined a Chow ring for projective variety as the limit

\[ CH^*_F(X) = \lim_{\to} CH^*_Y \]

where \( X \to Y \) varies over all maps to smooth projective varieties. There is an isomorphism \( K_0(X) \cong CH^*(X) \), where \( K_0(X) \) is the Grothendieck group of vector bundles. Then there is cycle map \( CH^*_F(X) \to H^{2*}(X) \), which can be identified with Chern character. Thus the image of this map lies in the space of Cartier cycles and therefore Hodge cycles. However, as we have seen in example 2.8, Hodge cycles need not be Cartier. Many other examples can be found in [ACK]. Therefore the cycle map on \( CH^*_F(X) \) is not surjective in general.

3.2. The operational Chow ring. Let \( CH^p_{OP}(X) \) denote the operational Chow ring of Fulton-Macpherson [F, chap 17]. An element \( \alpha \in CH^p_{OP}(X) \) is a collection of operators “\( f^* \alpha \cap \)" : \( CH^p(X') \to CH^{p-p}(X') \) varying over morphisms \( f : X' \to X \). These are required to commute with pushforwards, flat pullbacks, and Gysin maps in the sense of [F, def 17.1]. This is an associative graded ring which is contravariant, and has cap products. Furthermore, when \( X \) is smooth and \( n \) dimensional, the operational Chow ring is isomorphic to the usual Chow ring \( \bigoplus_i CH_{n-i}(X) \) with the intersection product.

**Theorem 3.1** (Kimura [K, thm 2.3]). Let \( X \) be projective variety, and let \( \pi : \tilde{X} \to X \) be a resolution of singularities. Then

\[ 0 \to CH^p_{OP}(X) \xrightarrow{\pi^*} CH^p(\tilde{X}) \xrightarrow{p_1^*-p_2^*} CH^p_{OP}(\tilde{X} \times_X \tilde{X}) \]

is exact, where \( p_i : \tilde{X} \times_X \tilde{X} \to \tilde{X} \) denote the projections.

**Corollary 3.2.** Suppose that \( \tilde{X}_1 \to \tilde{X} \times_X \tilde{X} \) is a resolution and \( \tilde{X}_1 \) is as in remark 1.7. Then we have exact sequences

\[ 0 \to CH^p_{OP}(X) \to CH^p(\tilde{X}) \to CH^p(\tilde{X}_1) \]

and

\[ 0 \to CH^p_{OP}(X) \to CH^p(\tilde{X}) \to CH^p(\tilde{X}_1) \]

**Corollary 3.3.** There is a natural ring homomorphism \( CH^p_{OP}(X) \to C^*(X) \), which coincides with the usual cycle map, when \( X \) is smooth.

**Proof.** This follows from the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & CH^p_{OP}(X) & \longrightarrow & CH^p(\tilde{X}) & \longrightarrow & CH^p(\tilde{X}_1) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & C^p(X) & \longrightarrow & C^p(\tilde{X}) & \longrightarrow & C^p(\tilde{X}_1)
\end{array}
\]

\[ \square \]
We can see from the previous results that \( \varepsilon_1(\alpha) = 0 \), when \( \alpha \) comes from \( CH^*_P(X) \). However, \( \varepsilon(\alpha) \) need not be zero. To see this, we can use the class \( \alpha = D \) of example 2.7. Applying corollary 3.2 with \( \tilde{X}_1 = \tilde{X} \coprod P^1 \times P^1 \) (see remark 1.7), shows that \( D \) lies in the image of \( CH^p_P(X) \).

The problem with the operational Chow group is that it is too permissive. We need to restrict the classes so as to kill the higher obstructions. If \( \alpha \in CH^p_P(X) \), then in the above notation, it corresponds to a cycle \( \alpha_0 \) on \( \tilde{X}_0 \) such that the difference of the pullbacks \( \partial \alpha_0 = 0 \) in \( CH^p(\tilde{X}_1) \). This means that \( \partial \alpha_0 \) is the boundary of a higher cycle \( \alpha_1 \) in the sense of Bloch (recalled below). If we insist that \( \alpha_1 \) can be chosen so that \( \partial \alpha_1 \) is a boundary in the Bloch complex of \( \tilde{X}_2 \), we get an additional constraint on \( \alpha \) which is sufficient to prove \( \varepsilon_2(\alpha) = 0 \). Continuing in this way eliminates all the obstructions. The precise statement is theorem 6.2.

But first we need to recall basic facts about motivic cohomology.

### 4. Motivic Cohomology

We start by recalling Bloch’s complex \([B]\). Let

\[
\Delta^m = \text{Spec } \mathbb{C}[x_0, \ldots, x_m]/(\sum x_i - 1) \cong \mathbb{A}^m
\]

be the algebraic geometer’s simplex. Setting some the variables to 0 defines the faces, which can be labelled in the usual way. Given a variety \( Y \), let \( Z^p_s(Y, -n) \) denote the space of codimension \( p \) cycles on \( Y \times \Delta^n \) meeting the faces properly. The coboundary

\[
\delta \alpha = \sum (-1)^i \alpha \cap \text{ith face}
\]

turns this into a complex, which is Bloch’s complex. The homology of this gives the higher Chow groups

\[
CH^p(Y, n) = H^{-n}Z^p_s(Y, \bullet)
\]

When \( n = 0 \), this coincides with the usual Chow group. We also recall the cubical versions of this, referring to Levine \([Le, \S 4]\) for details. Let \( \square = (\mathbb{P}^1 - \{1\})^n \) with coordinates \( z_i \). Setting \( z_i = 0 \) or \( \infty \) give the faces \( \iota_{i,0}, \iota_{i,1} : \square^{n-1} \to \square^n \). Given a smooth projective variety \( Y \), let \( Z^p_c(Y, -n) \) be the the quotient of the space of codimension \( p \) algebraic cycles on \( X \times \square^n \) meeting intersections of faces properly by the subspace of degenerate cycles. This becomes a complex with differential

\[
\delta = \sum (-1)^{i+j} \iota_{i,j}^* : Z^p_c(Y, -n) \to Z^p_c(Y, -n + 1)
\]

This complex is quasi isomorphic to \( Z^p_c(X, \bullet) \). When working rationally with \( Z^p_c(Y, -n) \otimes \mathbb{Q} \), we may also use the subcomplex of alternating cycles. This eliminates the need to divide by degenerate cycles.

Given a finite collection of subvarieties \( W_i \subset Y \), one can form a subcomplex

\[
Z_s(Y, \bullet \cup \{W_i\}) \subseteq Z_s(Y, \bullet),
\]

of cycles meeting the \( W_i \) properly (and likewise for the cubic complexes). Following Hanamura, we call these distinguished subcomplexes. The following hold.

- The above inclusions are all quasiisomorphisms.
- The intersection of two distinguished subcomplexes is distinguished.
Given a morphism $f : Z \to Y$ between smooth varieties, and a distinguished subcomplex $Z^p(Z, \bullet)'$, there is a distinguished subcomplex $Z^p(Y, \bullet)'$ such that pull back of cycles gives a map $f^* Z^p(Y, \bullet)' \to Z^p(Z, \bullet)'$ of complexes. These properties ensure that $CH^p(-, n)$ is a covariant functor on the category of smooth varieties.

An alternative approach to the higher Chow groups is to identify the $m$ with the cohomology of a complex of sheaves following Friedlander, Suslin and Voevodsky [FV, SV]. Given a scheme $X$, we recall two relatively new Grothendieck topologies. The first is the Nisnevich topology where the covers are étale covers $U_i \to X$ such that for every possibly nonclosed $x \in X$, there is a $u$ in some $U_i$ lying over it with the same residue field $k(u) = k(x)$. For the cdh topology we also allow covers of the form $\tilde{X} \coprod Z \to X$

where these form a blow up square (7) but we allow $\tilde{X}$ to be singular.

Given schemes $U, V$, let $z_{qf}(V)(U)$ be the group of correspondences which are quasifinite over $U$; more precisely, it is the abelian group generated by irreducible subvarieties $V \times U$ which are quasifinite over $U$. The group $z_{qf}(V)(-)$ is contravariant under pull back of correspondences, so it determines a presheaf on $X_{cdh}$. For each integer $n \geq 0$, define the complex of presheaves $Z^X(n) = Z(n)$ on $X_{cdh}$ which assigns to a cdh open $U$,

$$\cdots \to z_{qf}(\mathbb{A}^n \times \Delta^1)(U) \to z_{qf}(\mathbb{A}^n)(U)$$

with coboundary $\delta$ as in (5). This is similar to Bloch’s complex, and in fact we have inclusions

(6) $Z(n)(X) \subset Z^n(X \times \mathbb{A}^n, \bullet)[-2n]$

[MVW, lemma 19.4]; moreover, the image lies in any distinguished subcomplex. Let $Z(n)_{cdh}$, respectively $Z(n)_{zar}$, denote the sheafification of $Z(n)$ in the cdh, respectively Zariski, topologies.

Motivic cohomology is defined as

$$H^i_M(X, \mathbb{Z}(j)) := H^i(X_{cdh}, Z^X(j)_{cdh})$$

and

$$H^i_M(X, \mathbb{Q}(j)) := H^i_M(X, \mathbb{Z}(j)) \otimes \mathbb{Q}$$

We make a few comments about this definition.

(1) Since $Z(j)$ is not bounded below, some care needs to be taken in defining hypercohomology. Given a complex of sheaves $S^\bullet$ on a site with an element $U$, we take

$$H^i(U, S^\bullet) = H^i(\Gamma(U, I^\bullet))$$

where $I^\bullet$ is a $K$-injective resolution of $S^\bullet$ in the sense of [AJS, Sp]. When $S^\bullet$ is bounded below, this coincides with the usual definition using injective resolutions.

(2) The complex $Z(j)$, which is denoted by $Z^{SF}(j)$ in [MVW], is more convenient for our purposes than the definition in lecture 3 [MVW]. The two complexes are quasi-isomorphic [MVW thm 16.7].
(3) The definition of motivic cohomology as above, using the cdh topology is taken from [FV, def 4.3, def 9.2]. When $X$ is smooth, it is possible and more convenient to work in the Zariski topology in the sense that 
\[ H^i_M(X, \mathbb{Z}(j)) \cong H^i(X_{zar}, \mathbb{Z}_X(j)_{zar}) \]
cf [FV thm 5.5]. In the smooth case, motivic cohomology can be identified with higher Chow groups after reindexing, cf [MVW] or theorem 5.2.

(4) There are products 
\[ H^i_M(X, \mathbb{Z}(j)) \otimes H^i_M(X, \mathbb{Z}(j')) \to H^{i+i'}_M(X, \mathbb{Z}(j+j')) \]
which agree with the natural products on higher Chow groups when $X$ is smooth, cf [MVW p 24], [W].

Since we use the cdh topology, we get the following Mayer-Vietoris sequence.

**Proposition 4.1.** Given the blow up square (7),
\[ \ldots \to H^i_M(X, \mathbb{Z}(j)) \to H^i_M(\tilde{X}, \mathbb{Z}(j)) \oplus H^i_M(Z, \mathbb{Z}(j)) \to H^i_M(E, \mathbb{Z}(j)) \to \ldots \]
is exact.

**Proof.** This follows from [SV, prop 4.3.3]. □

### 5. Motivic Cohomology via Simplicial Resolutions

The definition of motivic cohomology given in the previous section is not terribly convenient for our purposes. Instead we will use the approach due to Hanamura [Ha], using simplicial resolutions.

To begin with, we need the existence of finite resolutions.

**Proposition 5.1** ([GNPP chap 1, thm 2.6]). Given an $n$ dimensional quasiprojective variety $X$, we can choose a smooth (semi-) simplicial scheme with a projective augmentation $\tilde{X}_\bullet \to X$ satisfying cohomological descent, such that $\dim \tilde{X}_i \leq n - i$ and in particular $\tilde{X}_i = 0$ for $i > n$.

It will be useful to recall the basic idea of the construction of $\tilde{X}_\bullet$, since it gives slightly more information than what is stated above. We will refer to any simplicial scheme constructed by this method as a **GNPP resolution** of $X$.

**Proof.** We use the simplicial rather than cubical viewpoint of the original source. As a first step, choose a resolution of singularities $\pi : \tilde{X} \to X$ and a proper closed set $Z \subset X$ such that $\pi$ is an isomorphism over $X - Z$. Consider the diagram

\[ E = f^{-1}Z \to \tilde{X} \]
\[ Z \to X \]

which we refer to as a **blow up square**. If $Z$ and $E$ are both nonsingular, then we simply take $\tilde{X}_0 = \tilde{X} \amalg Z$, $\tilde{X}_1 = E$ and $\tilde{X}_2, \ldots = \emptyset$. This has an obvious augmentation to $X$. In general, it is used as the foundation for a more elaborate simplicial object constructed inductively by gluing appropriate GNPP resolutions of $Z$ and $E$. So by construction, a GNPP resolution of $X$ can be decomposed as a disjoint union

\[ \ldots \tilde{X}_1 = \tilde{E}_0 \amalg \tilde{Z}_1 \Rightarrow \tilde{X}_0 = \tilde{X} \amalg \tilde{Z}_0 \to X \]
such that $\tilde{Z}_\bullet$ (with solid arrows on the bottom of (8)) is a GNPP resolution of $Z$ and $\tilde{E}_\bullet$ (with solid arrows on the top) is a GNPP resolution of $E$.

\begin{align*}
\cdots \quad & \begin{array}{c}
\tilde{E}_1 \\
\tilde{E}_0 \\
\vdots \\
\tilde{X}
\end{array} \\
\cdots \quad & \begin{array}{c}
\tilde{Z}_2 \\
\tilde{Z}_1 \\
\vdots \\
X
\end{array}
\end{align*}

Furthermore, the rightmost parallelogram should map to (7).

Given $X$, choose a GNPP resolution $\tilde{X}_\bullet$ as above. Next, choose distinguished sub complexes $Z^p_r(\tilde{X}_\bullet, \cdot)'$ stable under any composition of face maps. We let $\mathcal{X}$ denote both sets of choices. Then we can form a double complex $(Z^p_r(\tilde{X}_\bullet, \cdot)', \delta, \pm \partial)$ and define

$$CH^p_r(\tilde{X}_\bullet, n, \mathcal{X}) = H^{-n}(Tot(Z^p_r(\tilde{X}_\bullet, \cdot)'))$$

Hanamura [Ha] proves that, after tensoring with $\mathbb{Q}$, this is independent of the choice of $\mathcal{X}$. The next theorem will give a different proof of this fact, which works integrally.

**Theorem 5.2.** For any quasiprojective variety, there is an isomorphism

$$H^m_H(X, \mathbb{Z}(n)) \cong CH^m_H(X, 2n - m; \mathcal{X})$$

This is natural in the sense that given $Y \to X$ and a morphism of GNPP resolutions fitting into a commutative diagram

\begin{center}
$\begin{array}{ccc}
\tilde{Y}_\bullet & \rightarrow & \tilde{Y}_\bullet \\
\downarrow & & \downarrow \\
Y & \rightarrow & X
\end{array}$
\end{center}

there exists a commutative diagram

\begin{center}
$\begin{array}{ccc}
H^m_H(X, \mathbb{Z}(n)) & \sim & CH^m_H(X, 2n - m; \mathcal{X}) \\
\downarrow & & \downarrow \\
H^m_H(Y, \mathbb{Z}(n)) & \sim & CH^m_H(Y, 2n - m; \mathcal{Y})
\end{array}$
\end{center}

for some appropriate $\mathcal{Y}$.

The proof of the theorem will be broken into a series of lemmas. We start with the following.

**Lemma 5.3.** For any $i$, there is a natural choice $\mathcal{X}'$ for which we have a canonical isomorphism $CH^p_H(X, 2n - m; \mathcal{X}) \cong CH^p_H(X \times \mathbb{A}^i, 2n - m; \mathcal{X}')$.

**Proof.** The simplicial scheme $\tilde{X}_\bullet \times \mathbb{A}^i \to X \times \mathbb{A}^i$ is a GNPP resolution mapping to $\tilde{X}_\bullet$. We pullback the distinguished complexes to $\tilde{X}_\bullet \times \mathbb{A}^i$. These choices will be denoted by $\mathcal{X}'$. We have a map of double complexes, and hence a morphism of spectral sequences

$$CH^p(\tilde{X}_\bullet, -b) \Rightarrow CH^p_H(X, a - b; \mathcal{X})$$

$$CH^p(\tilde{X}_\bullet \times \mathbb{A}^i, -b) \Rightarrow CH^p_H(X \times \mathbb{A}^i, a - b; \mathcal{X}')$$
By [3] thm 2.1, the vertical maps on the left are isomorphisms, therefore the vertical map on the right is also an isomorphism.

We can form the category, in fact topos, \( Sh(\tilde{X}_{cdh, \bullet}) \) where an object consists of a collection of cdh sheaves \( F_i \) on \( \tilde{X} \) and face maps \( p_i^* F_i \to F_{i+1} \). Morphisms \( F_\bullet \to F'_\bullet \) are collections of morphisms of sheaves compatible with the face maps. The complex \( Z_{\tilde{X}_\bullet}(n)_{cdh} \) can be regarded as a complex in this category. Choose a \( K \)-injective resolution \( I^{\bullet} \) of \( Z_{\tilde{X}_\bullet}(n)_{cdh} \), where the second index on \( I^{\bullet} \) is the simplicial index. Let \( I^{ab} = H^0(\tilde{X}_b, I) \). We have quasiisomorphisms of complexes

\[
Z^n_{\bullet}((\tilde{X}_j \times k^n, \bullet)'[-2n] \to Z(n)(\tilde{X}_j)) \to H^0(\tilde{X}_j, Z_{\tilde{X}_j}(n)_{cdh}) \subset I^{ij}
\]

by [MVW] thm 19.1 (and its proof). These quasiisomorphisms are compatible with the coboundary operator \( \delta \). Thus we have a quasiisomorphism of the total complexes. Consequently, we have

\[
H^m(Tot(I^{\bullet})) \cong H^{m-2n}(Z^m_{\bullet}((\tilde{X}_j \times k^n, \bullet)') \cong CH^m_2(X, 2n - m; \mathcal{X})
\]

So to finish the proof of theorem 5.2 we need.

**Lemma 5.4.** \( H^m(Tot(I^{\bullet})) \cong H^m(X_{cdh}, Z(n)_{cdh}) \).

**Proof.** The first group \( H^m(Tot(I^{\bullet})) \) is nothing but the cohomology of \( Z_{\tilde{X}_\bullet}(n)_{cdh} \) in the topos \( Sh(\tilde{X}_{cdh, \bullet}) \). We have a morphism of topoi \( Sh(\tilde{X}_{cdh, \bullet}) \to Sh(X_{cdh}) \), induced by \( \pi_\bullet \), which induces a morphism of groups

\[
\gamma^m : H^m(X_{cdh}, Z(n)_{cdh}) \to H^m(\tilde{X}_{cdh, \bullet}, Z(n)_{cdh})
\]

(More explicitly, choose a \( K \)-injective resolution \( J^\bullet \) of \( Z_X(n) \), then we can easily construct a map of complexes \( H^0(X, J^\bullet) \to Tot(I^{\bullet}) \) inducing the above map.)

We will prove that this map is an isomorphism by induction on the length of \( \tilde{X}_\bullet \). By length, we mean the smallest integer \( d \) such that \( \tilde{X}_{d+i} = \emptyset \) for all \( i > 0 \). As in the proof of proposition 5.1, we can assume that \( \tilde{X}_\bullet \) has the structure given in 3.

Using this we get a commutative diagram

\[
\begin{array}{cccc}
\ldots & H^{m-1}(E) & \to & H^m(X) & \to & H^m(\tilde{X}) \oplus H^m(Z) & \to & \ldots \\
\downarrow{\alpha^{m-1}} & & & & & & \\
H^{m-1}(E_\bullet) & \to & H^m(\tilde{X}_\bullet) & \to & H^m(\tilde{X}) \oplus H^m(Z_\bullet)
\end{array}
\]

where the spaces \( E \) etc. are the same as in the proof of proposition 5.1 and the coefficients are \( Z(n)_{cdh} \). By induction, the arrows \( \alpha^\bullet, \beta^\bullet \) are isomorphisms. Therefore \( \gamma^\bullet \) is an isomorphism by the 5-lemma.

**Remark 5.5.** When \( X = X_1 \cup X_2 \) is a union of smooth varieties meeting transversally, the GNPP algorithm produces

\[
X_1 \cap X_2 \bigcap X_1 \cap X_2 \Rightarrow X_1 \bigcap X_2 \bigcap X_1 \cap X_2 \to X
\]

In this case, it is more efficient to cancel one of the \( X_1 \cap X_2 \) factors and use

\[
X_1 \cap X_2 \Rightarrow X_1 \bigcap X_2 \to X
\]
It is not difficult to see that the resulting double complexes are quasiisomorphic. More generally if \( X = \bigcup X_i \) has global normal crossings, i.e. when the components and their intersections are smooth of expected dimensions, rather than using a GNPP resolution, we can substitute the simpler simplicial resolution
\[
\ldots \coprod X_i \cap X_j \rightrightarrows \coprod X_i \to X
\]
in theorem 5.2.

6. Motivic classes are unobstructed

Let \( \pi : \tilde{X} \to X \) be a desingularization. Then we have an induced map
\[
\pi^* : H^{2p}_{\mathcal{M}}(X, \mathbb{Z}(p)) \to H^{2p}_{\mathcal{M}}(\tilde{X}, \mathbb{Z}(p)) \cong CH^p(\tilde{X})
\]

**Proposition 6.1.** The image \( \pi^*(H^{2p}_{\mathcal{M}}(X, \mathbb{Z}(p))) \subseteq CH^*(\tilde{X}) \) depends only \( X \) and lies in \( CH^*_{\mathcal{D}P}(X) \).

**Proof.** The proof that the image is well defined is similar to the proof of lemma 1.1. So we focus on the last part. We extend \( \tilde{X} \) \( X \) to a GNPP resolution \( \tilde{X}_\bullet \) as in the proof of proposition 5.1 using the same notation as in that proof. So \( \tilde{X}_\bullet \)
takes the form
\[
\ldots \tilde{X}_1 = \tilde{E}_0 \coprod \tilde{Z}_1 \rightrightarrows \tilde{X}_0 = \tilde{X} \coprod \tilde{Z}_0 \to X
\]

Let \( Y \) be a desingularization of \( \tilde{E}_0 \times_{\tilde{Z}_0} \tilde{E}_0 \). We have a commutative diagram
\[
\begin{array}{ccc}
Y & \xrightarrow{p_1, p_2} & \tilde{E}_0 \\
\downarrow g & & \downarrow i \\
\tilde{Z}_0 & \xrightarrow{f} & \tilde{X}
\end{array}
\]

where the arrows are the obvious projections.

From the double complex \( Z^p(X_\bullet, \bullet) \), we get a fourth quadrant spectral sequence (9)
\[
E_1^{ab} = CH^p(\tilde{X}_a, -b) \cong H^{2p-a+b}_{\mathcal{M}}(X_a, \mathbb{Z}(p)) \Rightarrow CH^p_\mathcal{M}(X, a-b) \cong H^{2p-a+b}_{\mathcal{M}}(X, \mathbb{Z}(p))
\]

This certainly does depend on the choice of \( \tilde{X}_\bullet \). However, the image of edge map
\[
H^{2p}_{\mathcal{M}}(X, \mathbb{Z}(0)) \to E_\infty^{00} \subseteq \ldots E_2^{00} \subseteq E_1^{00} = CH^*(X)
\]
does not, because it is just im \( \pi^* \). To complete the proof, we will show that
\[
E_2^{00} = \ker[CH^p(\tilde{X}_0) \to CH^p(\tilde{X}_1)]
\]
lies in \( CH^*_{\mathcal{D}P}(X) \). If \( \alpha \in E_2^{00} \), then \( i^* \alpha = 0 \) in \( CH^*(\tilde{E}_0)/\text{im} CH^*(\tilde{Z}_0) \) or equivalently that \( i^* \alpha = f^* \beta \) for some \( \beta \). Therefore \( (i \circ p_1)^* \alpha = (i \circ p_2)^* \alpha = g^* \beta - g^* \beta = 0 \). Now applying corollary 3.2 implies that \( \alpha \in CH^*_{\mathcal{D}P}(X) \) (\( Y \) plays the role of \( \tilde{X}_1 \) in said corollary).

Thus we get a cycle map given by the composition
\[
H^{2p}_{\mathcal{M}}(X, \mathbb{Q}(p)) \xrightarrow{\varepsilon_\bullet} CH^p_{\mathcal{D}P}(X, \mathbb{Q}) \to \tilde{H}^{2p}(X, \mathbb{Q}(p))
\]
We will prove that if \( \alpha \in H^{2p}_{\mathcal{M}}(X, \mathbb{Q}(p)) \), the obstruction \( \varepsilon(\alpha) \) of the image of this class in \( \tilde{H}^{2p}(X) \) vanishes. Therefore the image of \( \alpha \) lifts to a Hodge cycle in \( \tilde{H}^{2p}(X) \), which by lemma 2.2 is unique. In fact, the statement we prove is a bit more precise.
**Theorem 6.2.** There is a homomorphism

\[ H^{2p}_E(X, \mathbb{Q}(p)) \rightarrow H^{2p}(X, \mathbb{Q}(p)) \]

such that the composite with the projection to \( \tilde{H}^{2p}(X, \mathbb{Q}) \) coincides with \([H]\). Furthermore, the image of this map lands in the space of weight \(2p\) Hodge cycles.

The proof will be given below after the necessary preparation.

Recall that if \( Y \) is an \( n \)-dimensional complex manifold, the space of degree \( p \) currents \( D^p(U) \), over an open set \( U \subseteq Y \), is the topological dual of the space of compactly supported forms \( E^{2n-p}_0(U) \) cf \([GH]\) chap 3 §1]. These form a complex of fine sheaves. We denote the differential by \( d \). This admits a bigrading into \((p, q)\) type, and therefore a Hodge filtration. Any \((p, q)\) differential form \( \alpha \) with locally \( L^1\)-coefficients defines an element \( C(\alpha) \in D^{p,q}(Y) \) given by \( \phi \mapsto \int_Y \alpha \wedge \phi \). When \( \alpha \) is \( C^\infty \), we will usually just conflate \( \alpha \) and \( C(\alpha) \). However, it is a good idea to maintain a distinction when \( \alpha \) is singular because certain operations such as \( d \) do not commute with \( C \) in general. Any piecewise smooth oriented chain \((2n-p)\)-chain \( \Gamma \) in \( Y \), defines an element \( T_\Gamma \in D^p(Y) \) given by \( T_\Gamma(\phi) = \int_\Gamma \phi \). Let \( D^p_\Omega(Y) \) denote the span of such chains with rational coefficients. We give \( D^p(Y) \) the weak topology: \( \eta_i \rightarrow \eta \) if \( \eta_i(\phi) \rightarrow \eta(\phi) \) for every \( \phi \in E^{2n-p}_0(U) \). Smooth forms are dense. Pullbacks of currents under proper \( C^\infty\)-maps are defined as adjoint to pullbacks of compactly supported forms. Pull backs are more delicate. If \( f : Y' \rightarrow Y \) is a \( C^\infty \) map, and \( \eta \in D^p(Y) \), we say that the pullback \( f^* \eta \) exists and is equal to a current \( \xi \in D^p(Y') \) if there is a sequence \( \eta_i \in E^p(Y) \) converging to \( \eta \), such that \( f^* \eta_i \rightarrow \xi \). We note that \( f^* \eta \) need not exist in general. This definition is implicit in a theorem of Hörmander [H] thm 8.2.4], which gives a very general criterion for the existence of pullbacks of distributions. For our purposes, the following criterion is sufficient and easy to check.

**Lemma 6.3.** Suppose that \( \alpha \) is a locally \( L^1 \) differential form on a smooth connected quasiprojective variety \( Y \) such that \( \alpha \) is \( C^\infty \) off of a proper real semialgebraic set \( T \subset Y \). If \( f : Y' \rightarrow Y \) is morphism from another smooth connected quasiprojective variety such that \( f(Y') \nsubseteq T \). Then \( f^* C(\alpha) \) exists and is given by \( C(f^* \alpha) \).

Let us say that \( \alpha \) has mild singularities (along \( T \)) if the assumptions of the last lemma apply.

**Lemma 6.4.** Suppose that we have a commutative diagram

\[
\begin{array}{ccc}
Z' & \xrightarrow{p} & Z \\
\downarrow{p} & & \downarrow{p} \\
Y' & \xrightarrow{f} & Y
\end{array}
\]

of smooth projective varieties and a current \( \omega \) on \( Z \) satisfying the following conditions:

1. \( Z' \) is birational to \( Z \times_Y Y' \).
2. \( \omega \) has mild singularities along \( T \subset Z \).
3. \( F(Z') \nsubseteq T \)
4. There is a Zariski closed set \( \Delta \subset Y \) such that \( f(Y') \nsubseteq p(T) \cup \Delta \) and \( p \) is smooth over \( Y - \Delta \).
Then the currents $f^*p_*\omega$ and $P_*F^*\omega$ both exist, are equal, and have mild singularities.

Proof. As a first case, suppose that $T = \emptyset$ (so that $\omega$ is $C^\infty$), $p$ is smooth, and the diagram is Cartesian. Then $p_*\omega$ and $P_*F^*\omega$ are gotten by integration along the fibres. An easy calculation shows that integration along fibres commutes with pullback, therefore $f^*p_*\omega = P_*F^*\omega$.

In the general case, we can assume that the assumptions of the first case hold for the diagram

$$Z' - P^{-1}(f^{-1}(p(T) \cup \Delta)) \to Z - T \cup p^{-1}\Delta$$

after enlarging $\Delta$ if necessary. Thus we have equality of forms $f^*p_*\omega = P_*F^*\omega$ on the complement on $f^{-1}(p(T) \cup \Delta)$. The only additional thing to observe is that the forms are $L^1$ on $Y$. We can see this by observing that $\int_Y |P_*F^*\omega|d\text{vol} \leq \text{Const.} \int_Z |F^*\omega|d\text{vol}$. □

Given a set of maps $F = \{f_i : Y_i \to Y\}$, let $D^p_F(U) \subseteq D^p(U)$ be the set currents for which the pullback along $f_i \in F$ exists. This is easily seen to give a subsheaf of $C^\infty$-modules. Moreover, $d(D^p_F) \subseteq D^{p+1}_F$. Let $D_{F, \mathbb{Q}}(U) = D_F(U) \cap D_{\mathbb{Q}}(U)$.

The currents of interest to us were constructed by Kerr, Lewis and Müller-Stach [KLM]. Given subvariety $Z \subseteq Y \times \square^n$, we can pull back the coordinates $z_i$ on $\square^n$ to functions on $Z$. We will say that $Z$ is admissible if $Z$ meets all intersections of divisors $(z_i)$ and faces properly, and in particular that $z_i$ does not vanish on $Z$. Choose a desingularization of a compactification $\tilde{Z}$ of $Z$, and pull back $z_i$ to this space. We define the currents and cycles on $\tilde{Z}$ by

$$A'(\tilde{Z}) = C\left(\frac{dz_1}{z_1} \wedge \ldots \wedge \frac{dz_n}{z_n}\right)$$

$$\Gamma_i = z_i^{-1}[-\infty, 0] \quad (\text{as a cycle oriented from } -\infty \text{ to } 0)$$

$$B'(\tilde{Z}) = T_{\Gamma_1 \cap \ldots \cap \Gamma_n}$$

$$C'(\tilde{Z}) = C(\log z_1 \frac{dz_2}{z_2} \wedge \ldots \wedge \frac{dz_n}{z_n} \pm (2\pi i) \log z_1 \frac{dz_2}{z_2} \wedge \ldots \wedge \frac{dz_n}{z_n} T_{\Gamma_1} + \ldots \pm (2\pi i)^{n-1} T_{\Gamma_1 \cap \ldots \cap \Gamma_{n-1}})$$

The logarithm above is the branch with imaginary part in $(-\pi, \pi)$ on $C - [-\infty, 0]$. Now define

$$A(Z) = \pi_*A'(\tilde{Z}), \quad B(Z) = \pi_*B'(\tilde{Z}), \quad C(Z) = \pi_*C'(\tilde{Z})$$

where $\pi : \tilde{Z} \to Y$ is the projection.

Suppose that $Z = \sum n_i Z_i \subseteq Z^p_c(Y, n)$ is a cycle all of whose components are admissible, then we can extend the above definitions by linearity to obtain currents

$$A(Z) \in F^{p}D^{2p-n}(Y), \quad B(Z) \in D^{p-n}_{\mathbb{Q}}(Y), \quad C(Z) \in D^{2p-n-1}(Y)$$

Proposition 6.5 ([KLM (5.5)]). The following relations hold

$$dA(Z) = (2\pi i)A(\delta Z)$$
\[ dB(Z) = B(\delta Z) \]
\[ dC(Z) = A(Z) - (2\pi i)^n B(Z) - 2\pi i C(\delta Z) \]

Finally note that \([\text{KLM}] \, \text{§5.4}\) shows that the subcomplex of admissible cycles \(Z^p(Y, \bullet)_{ad} \subset Z^p(Y, \bullet)\) is quasiisomorphic to the full complex. A similar argument applies to distinguished complexes.

**Proof of theorem 6.2.** We start by proving the weaker statement immediately preceding the theorem that if \(\alpha \in H_M^{2p}(X, \mathbb{Q}(p))\), then \(\varepsilon(\alpha) = 0\). In order to calculate \(\varepsilon(\alpha)\), we use a modification of the set up described in the paragraph before proposition 2.3. We replace the complex of differential forms \((\mathcal{E}^\bullet(\tilde{X}_a), d)\) with the complex of currents \((\mathcal{D}^p(\tilde{X}_a), d)\), where \(P\) is the set of face maps \(p_b : \tilde{X}_{a+1} \to \tilde{X}_a\). This forms a double complex, with the second differential given by the simplicial coboundary \(\partial = \sum (-1)^bp_b^*\). Let \(\mathcal{D}^p\) and \(\mathcal{D}^p_Q\) be the total complexes \(\mathcal{D}^p(\tilde{X}_a)\) and \(\mathcal{D}^p_Q(\tilde{X}_a)\). Then we will choose our representatives \(A = (A_0, A_1, \ldots) \in F^p D^{2p}\) and \(B = (B_0, B_1, \ldots) \in D_Q^{2p}\), and \(\varepsilon(\alpha)\) will be represented by the difference \(A - B\).

The element \(\alpha \in H_M^{2p}(X, \mathbb{Q}(p))\) can be represented by a collection of cycles

\[ \alpha_a \in Z^p(\tilde{X}_a, a)_{ad} \]

such that

\[
\begin{align*}
\delta \alpha_0 &= 0 \\
\partial \alpha_0 &= \delta \alpha_1 \\
&\quad \quad \ldots
\end{align*}
\]

(11)

We can associate the currents

\[ A_a = (2\pi i)^{-a} A(\alpha_a) \in F^p D^{2p-a}(\tilde{X}_a) \]
\[ B_a = B(\alpha_a) \in D_Q^{2p-a}(\tilde{X}_a) \]
\[ C_a = (2\pi i)^{-a} C(\alpha_a) \in D^{2p-a-1}(\tilde{X}_a) \]

as above. By lemma \([\text{KLM}] \, \text{§6.4}\) we can see that the pull backs of these currents along the face maps \(p_b : \tilde{X}_{a+1} \to \tilde{X}_a\) exist, and

\[ p_b^* A_a = A(p_b^* \alpha) \]

etc. It follows from this, proposition 6.5 and (11) that

\[
\begin{align*}
dA_a &= (2\pi i)^{1-a} A(\delta \alpha_a) = (2\pi i)^{1-a} A(\partial \alpha_{a-1}) = \partial A_{a-1} \\
 dB_a &= B(\delta \alpha_a) = \partial B_{a-1} \\
 dC_a &= A_a - B_a - \partial C_{a-1}
\end{align*}
\]

This implies that \(A_a \in F^p D^p\) and \(B_a \in D_Q^p\) are cocycles, and that \((A_a - B_a)\) is a coboundary. This proves that \(\varepsilon(\alpha) = 0\). Now to get the full statement, note that \(\alpha \mapsto (2\pi i)^p(B_0, B_1, \ldots)\) determines homomorphism from \(H_M^{2p}(X, \mathbb{Q}(p))\) to the space of weight \(2p\) Hodge cycles in \(H^{2p}(X, \mathbb{Q}(p))\).

\[ \square \]
7. Lefschetz (1, 1) Theorem

We start with some explicit descriptions of degree two motivic cohomology. Throughout this section, \( X \) is a projective variety over \( \mathbb{C} \), with a GNPP resolution \( \tilde{X} \to X \). The face maps are denoted by \( p_i \). We use the notation \( \text{Div}_g(\tilde{X}_i), R_g(\tilde{X}_i)^* \) for divisors or functions in general position with respect to face maps introduced in \( \S 2 \).

**Proposition 7.1.** An element of \( H^2_M(X, \mathbb{Z}(1)) \) is represented by a pair \((D, f)\), where \( D \in \text{Div}_g(\tilde{X}_0) \) and \( f \in R_g(\tilde{X}_1)^* \) such that
\[
\partial D = (f) \quad \partial f = 1 \quad (\partial \text{ on } R_g^* \text{ is multiplicative})
\]

Two pairs \((D_i, f_i)\) represent the same class if there exists \( g \in R_g(\tilde{X}_0)^* \) such that
\[
D_1 - D_2 = (g) \quad f_1/f_2 = \partial g
\]

**Proof.** Consider the diagram

\[
\begin{array}{ccc}
Z^1(\tilde{X}_0, 0)' & \rightarrow & Z^1(\tilde{X}_1, 0)' \\
\downarrow \phi & & \downarrow \phi \\
\text{Div}_g(\tilde{X}_0) & \rightarrow & \text{Div}_g(\tilde{X}_1) \\
\downarrow \phi & & \downarrow \phi \\
Z^1(\tilde{X}_0, 1)' & \rightarrow & Z^1(\tilde{X}_1, 1)' \\
\downarrow \phi & & \downarrow \phi \\
R_g(\tilde{X}_0)^* & \rightarrow & R_g(\tilde{X}_1)^* \\
\downarrow \phi & & \downarrow \phi \\
R_g(\tilde{X}_2)^* & \rightarrow & R_g(\tilde{X}_2)^*
\end{array}
\]

The columns \( R_g(-) \to \text{Div}_g(-) \) of the front face have just the two terms, but the remaining columns \( Z^1(\tilde{X}, \bullet) \) may be longer. The diagonal arrows \( \phi \) are constructed by Nart [N]; they give quasi-isomorphisms between columns. Thus we may use the total complex of the front face to compute \( CH^1_H(X, \bullet) \). In particular, this yields the description of \( H^2_M(X, \mathbb{Z}(1)) = CH^1_H(X, 0) \) stated above. \( \Box \)

We can use this description to construct certain elements of motivic cohomology. If \( E \) is Cartier divisor on \( X \), \((\pi^* E, 1) \in H^2_M(X, \mathbb{Z}(1)) \). Of course, in general, there are additional elements in \( H^2_M(X, \mathbb{Z}(1)) \). Given a simplicial scheme such as \( \tilde{X} \), we can apply the connected components functor \( \pi_0 \) to get a simplicial set called the dual complex \( \Sigma \). Composing this with the free abelian group functor gives a simplicial abelian group whose cohomology we denote by \( H^*(\Sigma, \mathbb{Z}) \). This is the same thing as the singular cohomology Its geometric realization \( |\Sigma| \).

**Corollary 7.2.** We have an exact sequence
\[
1 \to H^1(\Sigma, \mathbb{Z}) \otimes \mathbb{C}^* \to H^2_M(X, \mathbb{Z}(1)) \to \text{Pic}(\tilde{X}_0)
\]

The image of the last map is the set of classes of divisors \( D \) satisfying (12) for some \( f \).
Proof. We have a map $H_2^M(X, \mathbb{Z}(1)) \to Pic(\hat{X}_0)$ which sends $(D, f)$ to the class of $D$. It can be checked that $\{(0, f) \mid f \in \mathbb{C}(\hat{X}_1)^*, \partial f = 1\}$ maps onto the kernel, and that the kernel of this map is precisely $\{(0, \partial g) \mid g \in \mathbb{C}(\hat{X}_0)^*\}$. \hfill $\square$

The proposition also leads to an interpretation of $H_2^M(X, \mathbb{Z}(1))$ as line bundles on $\hat{X}_0$ with descent data.

**Corollary 7.3.** Elements of $H_2^M(X, \mathbb{Z}(1))$ is the group of isomorphism classes of pairs $(L, \sigma : p_0^*L \cong p_1^*L)$, where $L$ is a line bundle on $\hat{X}_0$ and $(p_0^*\sigma)(p_1^*\sigma)^{-1}(p_2^*\sigma) = 1$.

**Proof.** Send $(D, f)$ to $(\mathcal{O}(D), \mathcal{O}(p_0^*D) \xrightarrow{\partial} \mathcal{O}(p_1^*D))$. \hfill $\square$

Let $H^1(\hat{X}_i, \mathcal{O}_{\hat{X}_i}^*)$ denote the cohomology of the Zariski simplicial sheaf $\mathcal{O}_{\hat{X}_i}^*$. From the spectral sequence

$$E_1^{pq} = H^q(\hat{X}_p, \mathcal{O}_{\hat{X}_p}^*) \Rightarrow H^{p+q}(\hat{X}, \mathcal{O}_{\hat{X}}^*)$$

we get an exact sequence

$$0 \to H^1(\Sigma, \mathbb{Z}) \otimes \mathbb{C}^* \to H^1(\hat{X}_i, \mathcal{O}_{\hat{X}_i}^*) \to H^1(\hat{X}_0, \mathcal{O}_{\hat{X}_0}^*)$$

The image of the last map is precisely the $E_2^{01}$.

**Proposition 7.4.** There is a natural isomorphism

$$\eta : H_2^M(X, \mathbb{Z}(1)) \cong H^1(\hat{X}_i, \mathcal{O}_{\hat{X}_i}^*)$$

**Proof.** Let $R^*_g, X$ denote the sheaf $U \mapsto R^*_g(U)$. We can see that $\mathcal{O}_{\hat{X}_i}^* \subset R^*_g, \hat{X}_i$, so there are exact sequences

$$1 \to \mathcal{O}_{\hat{X}_i}^* \to R^*_g, \hat{X}_i \xrightarrow{div} R^*_g, \hat{X}_i / \mathcal{O}_{\hat{X}_i}^* \to 1$$

Thus we have a quasi isomorphism

$$\mathcal{O}_{\hat{X}_i}^* \sim q_i R^*_g, \hat{X}_i \to R^*_g, \hat{X}_i / \mathcal{O}_{\hat{X}_i}^*$$

of complexes of simplicial sheaves. This induces an isomorphism

$$H^1(\hat{X}_i, \mathcal{O}_{\hat{X}_i}^*) \cong H^1(\hat{X}_i, R^*_g, \hat{X}_i \to R^*_g, \hat{X}_i / \mathcal{O}_{\hat{X}_i}^*)$$

We can compute the right side as the cohomology of the total complex

$$T^n = \bigoplus_{i+j+k=n} T_{ijk}$$

where the terms on the right are global sections of injective resolutions fitting into a diagram

$$\begin{array}{cccc}
R^*_g, X_i & \xrightarrow{\partial} & T^{0,i,0} & \xrightarrow{d} & T^{0,i,1} & \xrightarrow{d} \\
\downarrow \text{div} & & \downarrow \text{div} & & \downarrow \text{div} & \\
R^*_g, \hat{X}_i / \mathcal{O}_{\hat{X}_i}^* & \xrightarrow{\partial} & T^{1,i,0} & \xrightarrow{d} & T^{0,i,1} & \xrightarrow{d} \\
\end{array}$$

We need to be a bit careful about the sign. The differential of $T$ on $T^{abc}$ is $\kappa + (-1)^a \partial + (-1)^{a+b} \partial$. One sees that if $(D, f) \in H_2^M(X, \mathbb{Z}(1))$ as in proposition then

$$(D, f, 0) \in T^1 = T^{100} \oplus T^{010} \oplus T^{001}$$
is a cocycle, and hence it defines a class \( \eta(D, f) \in H^1(X_\bullet, \mathcal{O}_X^\bullet) \). We see that this fits into a commutative diagram

\[
\begin{array}{cccc}
1 & H^1(\Sigma, \mathbb{Z}) \otimes \mathbb{C}^* & H^2_M(X, \mathbb{Z}(1)) & Pic(\tilde{X}_0) \\
\downarrow & \downarrow & \downarrow & \\
1 & H^1(\Sigma, \mathbb{Z}) \otimes \mathbb{C}^* & H^1(\tilde{X}_\bullet, \mathcal{O}_{\tilde{X}_\bullet}^\bullet) & H^1(\tilde{X}_0, \mathcal{O}_{\tilde{X}_0}^\bullet) \\
\end{array}
\]

This implies that \( \eta \) is injective. We have to show that it is surjective. Suppose that \((\alpha, \beta, \gamma) \in T^1\) is a cocycle. This implies that

\[d\gamma = 0\]  
\[d\beta + \partial\gamma = 0\]  
\[d\alpha + \kappa\gamma = 0\]  
\[\partial\beta = 0\]

Since \( R^*_p \) is flasque, equation \((13)\) implies that \(\gamma = d\xi\) for some \(\xi \in I^{000}\). After adding the coboundary corresponding to \(-\xi\) to \((\alpha, \beta, \gamma)\), the remaining equations imply that it lies in the image of \(H^2_M(X, \mathbb{Z}(1))\).

**Corollary 7.5.** There is a natural isomorphism

\[\eta : H^2_M(X, \mathbb{Z}(1)) \cong H^1(\tilde{X}_{an, \bullet}, \mathcal{O}_{\tilde{X}_{an, \bullet}}^\bullet)\]

where \(\mathcal{O}_{\tilde{X}_{an, \bullet}}\) is the simplicial sheaf of holomorphic functions.

**Proof.** By standard arguments, we get a map of spectral sequences

\[E_1^{pq} = H^q(\tilde{X}_p, \mathcal{O}_{\tilde{X}_p}^\bullet) \Rightarrow H^{p+q}(\tilde{X}_\bullet, \mathcal{O}_{\tilde{X}_\bullet}^\bullet)\]

By GAGA, we have an isomorphism of \(E_1\)’s and therefore of abutments. The corollary follows from this and the proposition.

By weight 2 Hodge cycles in \(H^2(X, \mathbb{Z}(1))\) we simply mean the preimage of \(F^1\) under the map \(H^2(X, \mathbb{Z}(1)) \to H^2(X, \mathbb{C})\). This includes all the torsion cycles.

**Proposition 7.6.** There exists a natural homomorphism \(c\) which fits into a commutative diagram

\[
\begin{array}{cccc}
H^2_M(X, \mathbb{Z}(1)) & \to & H^2(X, \mathbb{Z}(1)) & \to & Pic(\tilde{X}_0) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\to & H^2(\tilde{X}_0, \mathbb{Z}(1)) & \to & \\
\end{array}
\]

The image of \(c\) lands in the space of weight 2 Hodge cycles.
Proof. By corollary 7.5, we can and will identify motivic cohomology with the first cohomology of \( \mathcal{O}_{\tilde{X}_{an}, \bullet}^* \). We have the exponential sequence

\[
0 \to \mathbb{Z}(1) \to \mathcal{O}_{\tilde{X}_{an}, \bullet} \to \mathcal{O}_{\tilde{X}_{an}, \bullet}^* \to 1
\]

which yields an exact sequence

\[
H^1(\tilde{X}_{an}, \bullet ; \mathcal{O}_{\tilde{X}_{an}, \bullet}^*) \to H^2(X, \mathbb{Z}(1)) \to H^2(\tilde{X}_{an}, \bullet ; \mathcal{O}_{\tilde{X}_{an}, \bullet}^*)
\]

This implies that

\[
\text{im } c \subseteq H^2(X, \mathbb{Z}(1)) \cap \ker[H^2(X, \mathbb{C}) \to (\tilde{X}_{an}, \bullet ; \mathcal{O}_{\tilde{X}_{an}, \bullet})]
\]

By [R1B] thm 4.5, this can be identified with \( H^2(X, \mathbb{Z}(1)) \cap F^1 \), where the intersection is understood as the preimage. □

Corollary 7.7. The map \( c \otimes \mathbb{Q} \) coincides with the map constructed in theorem 6.2.

Proof. The two maps induce the same map to \( \tilde{H}^2(X, \mathbb{Q}(1)) \), so they must coincide by lemma 2.2 □

We now come to the main result of this section.

Theorem 7.8. Given a projective variety \( X \), the image of the map

\[
c : H^2_{M}(X, \mathbb{Z}(1)) \to H^2(X, \mathbb{Z}(1)),
\]

is precisely the space of weight 2 Hodge cycles.

Although the proof is almost immediate, we give a second proof for surfaces which although longer gives more geometric insight.

First Proof. The proof of proposition 7.6 actually shows that \( \text{im } c = H^2(X, \mathbb{Z}(1)) \cap F^1 \). But this is exactly what we want to prove. □

Second Proof. For this proof, we will assume that \( X \) is a surface. Let \( \alpha \in \tilde{H}^2(X, \mathbb{Z}) \) be the image of Hodge cycle. By proposition 2.5 \( \alpha \) corresponds to a divisor \( D \) on \( \tilde{X}_0 \) whose class in \( \text{Pic}^0(\tilde{X}_0) \) is zero. In more explicit terms, this means that after translating \( D \) by element of \( \text{Pic}^0(\tilde{X}_0) \), we can assume that \( \partial D = (f) \) for some \( f \in R_\partial(\tilde{X}_1)^* \). Recall that the last condition means that \( \partial f = \partial g \), for some locally constant function \( g \). After replacing \( f \) by \( f g^{-1} \), the relations 12 hold for \( (D, f) \). This gives a class in \( H^2_{M}(X, \mathbb{Z}(1)) \) which maps onto \( \alpha \). □

8. Cohomological Hodge conjecture for singular varieties

Extrapolating from theorem 7.8 suggests the following conjecture:

Conjecture 8.1. Let \( X \) be a complex projective variety be defined over \( \overline{\mathbb{Q}} \). Then every weight 2p Hodge cycle in \( H^{2p}(X, \mathbb{Q}(p)) \) lies in the image of the map from \( H^2_{M}(X, \mathbb{Q}(p)) \) constructed in theorem 6.2.
Note that, unlike the usual Hodge conjecture, this is easy to falsify when the condition of being defined over \( \overline{\mathbb{Q}} \) is dropped. Suppose that \( X \) is a union of two smooth components \( X_1 \cup X_2 \) meeting transversally along a smooth variety \( Y \). Then we can compute motivic cohomology using the resolution

\[
Y \Rightarrow \tilde{X} = X_1 \coprod X_2 \to X
\]

by remark 5.5. As soon as we can find an algebraic cycle \( \alpha \) on \( \tilde{X} \) such that \( \partial \alpha \) is nonzero in \( CH^*(Y)_{\mathbb{Q}} \) but zero in the rational Deligne cohomology of \( Y \) (or equivalently both homologically trivial and trivial in the intermediate Jacobian tensor \( \mathbb{Q} \)), then we get a counterexample. An explicit example was found by Bloch.

**Example 8.2** (Bloch, Appendix 1). Let \( S \subset \mathbb{P}^3 \) be a smooth surface of degree \( \geq 4 \) and \( Y = Bl_x S \) the blow up of \( S \) at a very general point \( x \in Y \). Then the union \( X = Bl_x \mathbb{P}^3 \coprod_{Y} Bl_x \mathbb{P}^3 \) carries a codimension 2 cycle \( \alpha \) as above.

Bloch and Beilinson [Be, Lemma 5.6] have conjectured that when \( Y \) is a smooth projective variety over \( \overline{\mathbb{Q}} \), the cycle map from \( CH^*(Y)_{\mathbb{Q}} \) to rational Deligne cohomology is injective. We can easily see that:

**Proposition 8.3.** Assuming the usual Hodge conjecture and the Bloch-Beilinson conjecture, any weight 2p Hodge cycle in \( H^{2p}(X, \mathbb{Q}(p)) \) on a projective variety \( X \) defined over \( \overline{\mathbb{Q}} \) is represented by an algebraic cycle \( \alpha_0 \) on \( \tilde{X}_0 \) such that \( \partial \alpha_0 = 0 \) in \( CH^p(\tilde{X}_1)_{\mathbb{Q}} \), for any GNPP resolution \( \tilde{X}_\bullet \) defined over \( \overline{\mathbb{Q}} \). In particular, conjecture 8.1 holds if in addition \( \tilde{X}_2 = \emptyset \) or more generally if \( \dim \tilde{X}_2 < p - 1 \).

**Proof.** The argument will use the validity of the Hodge conjecture for \( \tilde{X}_0 \) and Bloch-Beilinson for \( \tilde{X}_1 \). A weight 2p Hodge cycle on \( H^{2p}(X) \) pulls back to an algebraic cycle \( \alpha_0 \) on \( \tilde{X}_0 \) such that \( \partial \alpha_0 \) is homologically trivial. By proposition 8.3, we can assume that \( \partial \alpha_0 = 0 \) in \( J(\tilde{X}_1)_{\mathbb{Q}} \) after modifying \( \alpha_0 \) by adding a homologically trivial cycle to it. Thus \( \partial \alpha_0 = 0 \) in the Chow group. So we can find a higher cycle \( \alpha_1 \) on \( \tilde{X}_1 \) such that \( \delta \alpha_1 = \alpha_0 \). If \( \dim \tilde{X}_2 < p - 1 \), then trivially \( \partial \alpha_1 = 0 \) in Bloch’s complex for \( \tilde{X}_2 \). So \( \alpha_\bullet \) determines a motivic class. \( \square \)

While this does not appear to be enough to imply conjecture 8.1 at the very least, this would appear to rule out “easy” counterexamples of the above type over \( \overline{\mathbb{Q}} \).

We state some basic criteria for the conjecture to hold. There are a few easy cases that can be reduced to the usual Hodge conjecture.

**Lemma 8.4.** Given a blow up square \( \mathcal{I} \), we have a commutative diagram with exact rows

\[
\begin{array}{c}
H^{2p}_M(X, \mathbb{Q}(p)) \\
\downarrow \alpha \\
\tilde{H}^{2p}(X, \mathbb{Q}(p)) \\
\downarrow \beta \\
0 \\
\end{array} \quad \begin{array}{c}
\downarrow \gamma \\
\tilde{H}^{2p}(\tilde{X}, \mathbb{Q}(p)) \oplus H^{2p}_M(Z, \mathbb{Q}(p)) \\
\downarrow \\
H^{2p}(E, \mathbb{Q}(p)) \end{array}
\]

If \( \beta \) is surjective and \( \gamma \) is injective then \( \alpha \) is surjective.

**Proof.** The exactness of the top row is proposition 4.4 for the bottom this is standard, and the commutativity follows from functoriality of the cycle map. The last statement is a consequence of the five lemma. \( \square \)
Corollary 8.5. Suppose that $X$ has isolated singularities, and possess a resolution $\tilde{X}$ for which the Hodge conjecture holds and such that the exceptional divisor $E$ is smooth and has a cellular decomposition (in the sense of [1] ex 19.1.11). Then conjecture $\mathbf{[E]}$ holds for $X$. So in particular, the conjecture holds for a cone over a cellular variety.

Lemma 8.6. If $f : X \to Y$ is a map of projective varieties, and $\alpha \in H^{2p}(Y, \mathbb{Q}(p))$ lies in the image of $H^{2p}_M(Y, \mathbb{Q}(p))$, then $f^* \alpha$ lies in the image of $H^{2p}_M(X, \mathbb{Q}(p))$. In particular, the conclusion holds if $Y$ is smooth and $\alpha$ is algebraic.

Proof. This follows from naturality of the cycle map. \qed

Lemma 8.7. The image of $H^2_M(X, \mathbb{Q}(*)$ in $H^2(X, \mathbb{Q}(*))$ forms a subalgebra.

Proof. Let $\alpha = \alpha_1 \cup \ldots \cup \alpha_n$, where $\alpha_i \in H^2(X, \mathbb{Q}(*))$ is the image of $\beta_i \in H^2_M(X, \mathbb{Q}(*))$ under the cycle map. We can form the product $\beta = \beta_1 \cup \ldots \cup \beta_n \in H^2_M(X, \mathbb{Q}(*))$ (section 1). In order to show that $\beta$ maps to $\alpha$, it is enough to check that their images in $\tilde{H}^2(X)$ agree by lemma 2.2. Let $\pi : \tilde{X} \to X$ be a resolution of singularities. Then $\pi^* \beta$ is the usual intersection product $\pi^* \beta_1 \cdot \pi^* \beta_2 \cdots \in CH^*(X)_Q$. This maps to $\pi^* \alpha_1 \cup \ldots \cup \pi^* \alpha_n \in \tilde{H}^2(X)$. \qed

Corollary 8.8. If $\alpha$ is a sum of products of degree 2 Hodge cycles in $H^2(X, \mathbb{Q}(1))$, then it lies in the image of $H^2_M(X, \mathbb{Q}(*))$.

Lemma 8.9. Let $\pi : X \to Y$ be a finite morphism of normal varieties, then $\pi_* : Sh(X_{cdh}) \to Sh(Y_{cdh})$ is exact.

Proof. By [GK], the topos $Sh(Y_{cdh})$ has enough points, and these correspond to maps of spectra of Henselian valuation rings to $Y$. Given a point $y : \text{Spec} A \to Y$ in this sense, let $\{x_i\}$ be the finite set of points of $X$ corresponding to the valuations of $\Gamma(\mathcal{O}_{\text{Spec} A \times Y, X})$ extending the valuation of $A$. It can be checked that the stalk $(\pi_* \mathcal{F})_y = \prod \mathcal{F}_{x_i}$. Therefore given an epimorphism $\mathcal{F} \to G$ of sheaves, $(\pi_* \mathcal{F})_y \to (\pi_* G)_y$ is surjective for all points $y$. This suffices to prove the lemma. \qed

Proposition 8.10. Let $X$ be a projective variety with an action by a finite group $G$. Let $\pi : X \to Y = X/G$ be the quotient (which is well known to exist in the category of projective varieties). Then $\pi^*$ induces an isomorphism

$$H^i_M(Y, \mathbb{Q}(n)) \cong H^i_M(X, \mathbb{Q}(n))^G$$

Proof. Given a cdh open $U \subset Y$, and an irreducible cycle $V \in z_{qf}(A^n \times \Delta^i)(U)_Q$, then the cycle theoretic pullback

$$\pi^* V = \sum_W \text{card}\{g \in G \mid g|_W = id\}[W]$$

where $W$ runs over irreducible components of $\pi^{-1} U$. This determines a cycle in $z_{qf}(A^n \times \Delta^i)(\pi^{-1} U)_Q$. Moreover, it is seen to induce an isomorphism

$$z_{qf}(A^n \times \Delta^i)(U)_Q \cong z_{qf}(A^n \times \Delta^*(\pi^{-1} U)_Q)^G$$

[G ex 1.7.6]. It is also compatible with the differential $\delta$, and therefore it induces an isomorphism

$$H^i(Y_{cdh}, \mathbb{Q}(n)) \cong H^i(Y_{cdh}, (\pi_* \mathbb{Q}(n))^G)$$
The functor of $G$-invariants on $\mathbb{Q}$-modules is well known to be exact, together with Lemma 8.9, this implies that we can write the last group as

$$H^i(Y_{cdh}, \pi_* \mathbb{Q}(n))^G = H^i(X_{cdh}, \mathbb{Q}(n))^G$$

□

**Corollary 8.11.** If conjecture 8.1 holds for $X$, then it holds for $Y$.

**Proof.** By assumption we have a surjection $H^{2p}_M(X, \mathbb{Q}(p)) \to H^{2p}_H(X, \mathbb{Q}(p))$, where the right side denotes the space of weight $2p$ Hodge cycles. Therefore we have surjections

$$H^{2p}_M(X, \mathbb{Q}(p))^G \to H^{2p}_H(X, \mathbb{Q}(p))^G \cong H^{2p}_M(Y, \mathbb{Q}(p)) \to H^{2p}_H(Y, \mathbb{Q}(p))$$

□

9. Fibre products of modular surfaces

As evidence for conjecture 8.1, we will check it for the following class of examples. Let $\Gamma \subseteq SL_2(\mathbb{Z})$ be a subgroup of finite index such that $-I \notin \Gamma$. Let $\mathbb{H}$ be the upper half plane and let $U = \mathbb{H}/\Gamma$ be the associated modular curve with smooth projective compactification $C \supset U$. This can be interpreted as the moduli space of (generalized) elliptic curves with $\Gamma$-level structures. So in particular we get an associated universal family $f : E \to C$, which is called an elliptic modular surface [Sh]. This is defined over $\mathbb{Q}$.

**Theorem 9.1.** Let $f : E \to C$ be an elliptic modular surface. Then for any $n \geq 1$, conjecture 8.1 holds for the $n$-fold fibre product $X = E \times_C \ldots \times_C E$.

Before starting the proof of the theorem, we will need to recall some facts about elliptic modular surfaces. Let us assume that $f : E \to C$ is a semistable elliptic modular surface. Let $S = C - U$. The cohomology $H^2(E, \mathbb{Q})$ carries a filtration induced by the Leray spectral sequence. Since this degenerates [Z, §15], we can write the subquotients of $H^2(E, \mathbb{Q})$ as

- $L^2 = H^2(C, \mathbb{Q})$
- $L^1/L^2 = H^1(C, R^1 f_* \mathbb{Q})$
- $L^0/L^1 = H^0(C, R^2 f_* \mathbb{Q})$

It is immediate that $L^2$ is generated by the class of a fibre of $f$.

The restriction of $R^2 f_* \mathbb{Q}$ to $U$ is the constant sheaf $\mathbb{Q}_U$. So we have an adjunction map $R^2 f_* \mathbb{Q} \to \mathbb{Q}_C$ leading to an exact sequence

$$0 \to K \to R^2 f_* \mathbb{Q} \to \mathbb{Q} \to 0$$

(17)

where $K = \bigoplus K_s$ is a sum of sheaves supported at $s \in S$. We can interpret this more explicitly by restricting to a small disk $D$ centered at $s \in S$. Let $t \in D - \{s\}$. Then the restriction of (17) to corresponds to the sequence

$$0 \to H^0(K_s) \to H^2(X_s, \mathbb{Q}) \to H^2(X_t, \mathbb{Q}) = \mathbb{Q} \to 0$$

(18)

where $K_s$ is a sum of sheaves supported at $s \in S$. We can interpret this more explicitly by restricting to a small disk $D$ centered at $s \in S$. Let $t \in D - \{s\}$. Then the restriction of (18) to corresponds to the sequence

$$0 \to H^0(K_s) \to H^2(X_s, \mathbb{Q}) \to H^2(X_t, \mathbb{Q}) = \mathbb{Q} \to 0$$

(19)
The map $c$ is the collapsing map induced by the homotopy equivalence followed by restriction $X_s \approx f^{-1}X \supset X_t$. This leads to a sequence

$$0 \to \bigoplus_s H^0(K_s) \to L^0/L^1 \to Q(-1) \to 0$$

(18)

The space on the right is generated by a fundamental class of an irreducible curve which is horizontal in the sense having nonzero intersection number with the general fibre. The space on $H^0(K_s)$ is spanned divisors supported on $X_s$ orthogonal to the horizontal divisor. It follows that $L^0/L^1$ is spanned by divisors.

It remains to analyze $L^1/L^2$. Set $\mathcal{L} = R^1f_*\mathbb{Q}$. To begin with, we claim that

$$\mathcal{L} \cong j_* j^* \mathcal{L}$$

(19)

To prove this, it suffices to check isomorphisms at the stalks at each $s \in S$. Choose a small disk $D$ centered at $s$. Then we have to show that $H^1(X_s, \mathbb{Q}) \cong H^1(X_t, \mathbb{Q})^{\pi_1(D^*)}$. Semistability implies that fibre $X_s$ is of type $I_N$, i.e. a polygon of $N$ smooth rational curves, for some $N$. We can choose a symplectic basis $e_1, e_2$ of $H_1(X_s)$ such that the $N$ vanishing cycles are all homologous to $e_2$, and the image of $e_1$ generates $H_1(X_s)$. Thus $H^1(X_s) \to H^1(X_t)$ is injective. The image is precisely the dual $e_1^*$, which by the Picard-Lefschetz formula spans the invariant cycles. Therefore (19) holds. Consequently

$$H^1(C, \mathcal{L}) \cong H^1(C, j_* \mathcal{L}|_U)$$

The right side can be identified with intersection cohomology $IH^1(C, \mathcal{L})$.

Zucker [Z, thm 7.12] showed that intersection cohomology $IH^1(C, \mathcal{L})$ carries an intrinsic Hodge structure which is isomorphic $L^1/L^2$. This comes by identifying this with $L^2$ cohomology with coefficients in $\mathcal{L}$. In a bit more detail, the local system $\mathcal{L}_U$ is associated to a polarized variation of Hodge structure on $U$ with unipotent local monodromy. For any such variation, by work of Schmid [Sc] the vector bundle $\mathcal{V}_U = \mathcal{L}_U \otimes \mathcal{O}_U$ with its Hodge filtration extends to a filtered bundle $(\mathcal{V}, F)$ on $C$. The log complex

$$\mathcal{V} \xrightarrow{\nabla} \mathcal{V} \otimes \Omega^1_C(\log S)$$

is filtered by

$$F^p \to F^{p-1} \otimes \Omega^1_C(\log S)$$

The subcomplex $\mathcal{V} \to \text{im } \nabla$ with induced filtration forms part of a cohomological Hodge complex that computes $IH^*(\mathcal{L})$. Returning to our specific case, we have

$$V^1 = F^1 \cong f_*\Omega^1_{X/C}(\log f^{-1}S) = f_*\omega_{X/C}$$

and

$$V^0 = F^0/F^1 \cong R^1f_*\mathcal{O}_X$$

by [Sc]. An easy computation shows that

$$IH^1(\mathcal{L})_{(p,q)}^{(p,q)} = \begin{cases} H^1(C, V^0) & (p,q) = (0,2) \\ H^1(C, V^1) \xrightarrow{\nabla} V^0 \otimes \Omega^1_C(\log S) & (p,q) = (1,1) \\ H^0(C, V^1 \otimes \Omega^1_C(\log S)) & (p,q) = (2,0) \end{cases}$$

where $\kappa$ is the Kodaira-Spencer class. We can repackage this by defining the graded vector bundle $V = V^0 \oplus V^1$ with Higgs field

$$\theta = \begin{pmatrix} 0 & 0 \\ \kappa & 0 \end{pmatrix} : V \to V \otimes \Omega^1_C(\log S)$$
Then $IH^1(\mathcal{L})$ is the first hypercohomology of the last complex, and the $(p, q)$ decomposition can be recovered from the induced grading. This viewpoint is more convenient for analyzing $IH^1(\mathcal{L}^n)$ below. In order to do this, observe that given two locally unipotent polarized variations of Hodge structure with associated graded Higgs bundles $(W, \theta)$ and $(W', \theta')$, the tensor product is associated to $W'' = W \otimes W'$ with Higgs field $\theta \otimes 1 + 1 \otimes \theta'$ and grading

$$
(W'')^i = \bigoplus_{j+k=i} W^i \otimes (W')^j
$$

Shioda [Sh, eq (4.12)] showed that $\dim IH = 2p_g(\mathcal{E})$. Together with the fact that $p_g(\mathcal{E}) = \dim H^1(C, R^1f_*\mathcal{O}_X) = \dim IH^{(0, 2)}$, we can conclude that

$$
(20) \quad \dim IH^{(1, 1)} = 0
$$

Since $\kappa$ is nonzero, we conclude that

$$
V^1 \xrightarrow{\kappa} V^0 \otimes \Omega^1_C(\log S) \cong \text{coker } \kappa[-1]
$$

in the derived category. Combing this with (20) implies that $\kappa$ is isomorphism.

To extend this analysis to nonsemistable surfaces, we observe the following.

**Lemma 9.2.** If $\mathcal{E} \to C$ is an elliptic modular surface, then there exists a Galois cover $p : C' \to C$ such that $\mathcal{E}' = \mathcal{E} \times_C C'$ is birational to a semistable modular surface.

**Proof.** Let $\Gamma \subset SL_2(\mathbb{Z})$ be the group associated to $\mathcal{E}$. Then we may take $C' \to C$ to be the modular curve associated to $\Gamma \cap \Gamma(N)$, where $\Gamma(N)$ is the principal congruence subgroup of level $N \geq 3$. It is known that all singular fibres of the elliptic modular surface corresponding to $\Gamma(N)$ will be of type $I_N$ [Sh, ex 5.4], and consequently semistable. Semistability will persist over $C'$.

With the notation as in the lemma, let $\pi : \tilde{\mathcal{E}} \to \mathcal{E}'$ be the minimal resolution, $f' : \mathcal{E}' \to C$ and $\tilde{f} : \tilde{\mathcal{E}} \to C$ the projections, and let $G$ be the Galois group of $C'/C$. We claim that the above results carry over to $f' : \mathcal{E} \to C'$. More specifically, there are isomorphisms or exact sequences

$$
(21) \quad f'_*\mathcal{Q} = p_*\mathcal{Q}
$$

$$
R^1f'_*\mathcal{Q} = j_*j^*\mathcal{L}, \quad \mathcal{L}' = R^1f'_*\mathcal{Q}
$$

$$
0 \to \bigoplus_{s \in S} K'_s \to R^2f'_*\mathcal{Q} \to p_*\mathcal{Q} \to 0
$$

where $K'_s$ is supported on $S$ and spanned by algebraic cycles supported on $\mathcal{E}'_s$. We also have that

$$
IH^1(C, \mathcal{L}')^{11} = H^1((V')^1 \xrightarrow{\kappa} (V')^0 \otimes \Omega^1_C(\log S)) = 0
$$

where the graded Higgs bundle is defined as above. These statements follow from straightforward modifications of the previous arguments. We also have that:

**Lemma 9.3.** The Leray spectral sequence for $f'$ degenerates at $E_2$.
**Proof.** The only differential that could be nonzero is indicated as $d'_2$ below.

$$H^0(R^1f^*_e\mathbb{Q}) \xrightarrow{d'_2} H^2(f^*_e\mathbb{Q})$$

The vertical maps are easily seen to be isomorphisms by (21) and the analogous facts for $\tilde{f}$. Since $d''_2 = 0$ by [Z, cor 15.15], we can conclude that $d'_2 = 0$. □

**Lemma 9.4.** Given a polarized variation of Hodge structure $\mathcal{L}$ on $U$, cup product with the fundamental class $[\mathcal{C}]$ induces an isomorphism

$$IH^0(C, \mathcal{L}) \cong IH^2(C, \mathcal{L})$$

**Proof.** This is a special case of the hard Lefschetz theorem of Saito [Sa, thm 5.3.1], but this can proved more directly as follows. Both groups are represented by spaces of $L^2\mathcal{L}$-valued harmonic forms [Z, §7]. Using this representation and the Kähler identities [Z, §2], the map, which given by wedging with the Kähler form, is seen to be an isomorphism by the usual argument [GH, pp 118-122]. □

**Proof of theorem 9.1.** By corollary 8.11 and lemma 9.2, it is enough to prove the conjecture for $X' = E' \times_C \ldots \times_C E'$, because $X = X'/G^n$. From this point onwards, there is no need to refer to the original variety $X$. So in the interest of simplifying the notation, we omit the primes and write $X, f, L, \ldots$ instead of $X', f', L', \ldots$. By corollary 8.8, it is enough to prove that the algebra of Hodge cycles in $H^2(X)$ is generated by degree 2 cycles. If we denote the space of Hodge cycles (respectively products of degree 2 Hodge cycles) by $H^2_{\text{Hodge}}(X)$ (respectively $H^2_{\text{Hodge, 2}}(X)$), then we have to show that $\dim H^2_{\text{Hodge}}(X) = \dim H^2_{\text{Hodge, 2}}(X)$. Toward this end, it suffices to prove that $\dim H^2_{\text{Hodge}}(X) \cap H_i = \dim H^2_{\text{Hodge, 2}} \cap H_i$ for any possibly noncanonical decomposition $H^*(X) = \bigoplus H_i$.

The Leray spectral sequence

$$H^p(C, R^qF_*\mathbb{Q}) \Rightarrow H^{p+q}(X, \mathbb{Q})$$

is compatible with mixed Hodge structures [A]. This degenerates by an argument similar to the proof of lemma 9.3. Thus we will have a noncanonical decomposition

$$(22) \quad \hat{H}^{2k}(X, \mathbb{Q}) \cong \bigoplus_{i+j=2k} H^i(C, R^jF_*\mathbb{Q})$$

We will show that Hodge cycles on the spaces on the right are products of degree 2 cycles. We can decompose the direct images

$$R^jF_*\mathbb{Q} = \bigoplus_{a+b+c=n \atop b+c=2k} (f_*\mathbb{Q})^a \otimes (R^1f_*\mathbb{Q})^b \otimes (R^2f_*\mathbb{Q})^c)^{N(a,b,c)}$$

using Künneth’s formula, where $N(a, b, c)$ is some exponent whose precise value is unimportant for us. The sequence of (21) gives an exact sequence

$$0 \to \bigoplus_{s \in S} K_s\mathbb{C} \to (R^2f_*\mathbb{Q})^c \to \mathbb{Q}^c \cong \mathbb{Q} \to 0$$
where $K_s(c)$ is not the Tate twist, it is merely a notation for a certain skyscraper sheaf supported at $s$. It decomposes noncanonically as

$$K_s(c) \cong \bigoplus_{a+b=c} (K_s \otimes \mathbb{Q}^\otimes b)^{N(a,b)} \cong \bigoplus_{a+b=c} (K_s \otimes a)^{N(a,b)} \cong \bigoplus_{a+b=c} (K_s \otimes a)^{N(a,b)} \cong \bigoplus_{a+b=c} (K_s \otimes a)^{N(a,b)}$$

The components fit into exact sequences

$$0 \to \bigoplus_j \mathbb{Q}^\otimes a \otimes (j_* j^* \mathcal{L}^\otimes b)_s \otimes K_s(c) \to R(a,b,c) \to \bigoplus_j \mathbb{Q}^\otimes a \otimes j_* j^* \mathcal{L}^\otimes b \otimes \mathbb{Q}^\otimes c \to 0$$

Thus we have (noncanonical) isomorphisms

$$(23) \quad H^0(C, R(a,b,c)) \cong \left( \bigoplus_s H^0(K_s(c)) \right) \oplus (\mathcal{L}^\otimes b)^{\pi_1(U)}$$

$$(24) \quad H^i(C, R(a,b,c)) \cong IH^i(C, \mathcal{L}^\otimes b), \quad i \geq 1$$

We analyze each of these summands in turn, and show that Hodge cycles in them are spanned by degree 2 Hodge cycles.

1. The Zariski closure of the image of $\pi_1(U)$ under the monodromy representation associated to $\mathcal{L}$ is $SL_2(\mathbb{Q})$. So by classical invariant theory [FH, appendix F], $(\mathcal{L}^\otimes b)^{\pi_1(U)}$ is a sum of products of sections of $(\mathcal{L}^\otimes 2)^{\pi_1(U)}$, and therefore a sum of products of degree 2 Hodge cycles.

2. The spaces $H^0(K_s(c))$ can be further decomposed into sums of tensor powers of $H^0(K_s)$, and each of these spaces is generated by degree 2 classes.

3. Next, we turn to $IH^2(C, (\mathcal{L}^\otimes b))$. By lemma [9,3] there is an isomorphism

$$\mathcal{L}^\otimes 2^{\pi_1(U)} = IH^0(C, (\mathcal{L}^\otimes b)) \xrightarrow{\sim} IH^2(C, (\mathcal{L}^\otimes b))$$

given by cupping with the fundamental class $[C]$. With this isomorphism, we see that these groups are generated by degree 2 Hodge cycles.

4. Finally consider,

$$T := IH^1(C, \mathcal{L}^\otimes (2q-1))^{(q,q)}$$

By previous remarks, $T$ can be computed as the $q$th summand of the first hypercohomology of the graded Higgs bundle $(V, \theta)^\otimes (2q-1)$. In more explicit terms, $T$ is the 1st hypercohomology of the complex

$$(25) \quad \bigoplus_{\sum i_k = q} V^{i_1} \otimes \ldots \otimes V^{i_{2q-1}} \to \bigoplus_{\sum j_k = q-1} V^{j_1} \otimes \ldots \otimes V^{j_{2q-1}} \otimes \Omega_C^1(\log S)$$

The differential is given as a sum of maps $1 \otimes \kappa \otimes 1$. This is acyclic because $\kappa$ is an isomorphism. Therefore $T = 0$.

When $\mathcal{E} \to C$ is a semistable elliptic modular surface, the singularities of $X = \mathcal{E} \times_{\mathcal{E}} \ldots \times_{\mathcal{E}} \mathcal{E}$ are toroidal. Therefore we have a toroidal resolution of singularities $\pi : \tilde{X} \to X$ (cf [G]).

**Corollary 9.5** (Gordon). The Hodge conjecture holds for $\tilde{X}$. 


Proof. In outline, the cohomology of $\tilde{X}$ is generated by the image of $H^*(X)$ and algebraic cycles supported on the exceptional locus of $\pi$. The Hodge cycles in $\pi^*H^*(X)$ lie in the image of $H^*_{M}(X, \mathbb{Q}(*))$, which factors through $CH^*(X)_\mathbb{Q}$. □

Gordon’s proof is somewhat different. As noted earlier, there does not seem to be anyway of going backwards and deducing the theorem from this result.

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