TAME TOPOLOGY AND DESINGULARIZATION IN HENSEL MINIMAL STRUCTURES

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Abstract. We deal with Hensel minimal, non-trivially valued fields $K$ of equicharacteristic zero, whose axiomatic theory was introduced in a recent article by Cluckers–Halupczok–Rideau. We additionally require that every definable subset in the imaginary sort $RV$ be already definable in the pure valued field language. This condition is satisfied by many of the classical tame structures on Henselian fields (including Henselian fields with analytic structure, V-minimal fields and polynomially bounded o-minimal structures with a convex subring), and ensures that the residue field is orthogonal to the value group. The main purpose is to carry over many results of our previous papers to the above general axiomatic settings including, among others, theorem on existence of the limit, curve selection, the closedness theorem, several non-Archimedean versions of the Lojasiewicz inequalities as well as the theorems on extending continuous definable functions and on existence of definable retractions. We establish an embedding theorem for regular definable spaces and the definable ultranormality of definable Hausdorff LC-spaces. Also given is an example that the closedness theorem, a key result for numerous applications, may be no longer true after expanding the language for the leading term structure $RV$. In the case of Henselian fields with analytic structure, a more precise version of the theorem on existence of the limit (a kind of Puiseux’s theorem) is provided. Further, we establish definable versions of resolution of singularities (hypersurface case) and transformation to normal crossings by blowing up, on arbitrary strong analytic manifolds in Hensel minimal expansions of analytic structures. Also introduced are meromorphous functions, i.e. continuous quotients of strong analytic functions on strong analytic manifolds. Finally, we prove a finitary meromorphous version of the Nullstellensatz.

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1. Introduction

We are concerned with geometry of Hensel minimal (more precisely, 1-h-minimal), non-trivially valued fields $K$ of equicharacteristic zero, whose axiomatic theory (in an expansion $\mathcal{L}$ of the language of valued fields) was introduced in the recent papers [10, 11]. We shall additionally require that every definable subset in the imaginary sort $RV$ be already definable in the pure valued field language. Then the language for the imaginary sort $RV$, which combines the residue field $K_v$ and value group $vK$, will be the standard algebraic language $L_{rv}$. This condition is satisfied by many of the classical tame structures on Henselian fields (including Henselian fields with analytic structure, V-minimal fields and polynomially bounded o-minimal structures with a convex subring), and ensures that the residue field is orthogonal to the value group (see Section 2).

The main purpose here is to carry over many results of our previous papers [10, 11, 12, 13, 14] to the general settings of Hensel minimality. This article incorporates the previous version of this preprint and is organized as follows.

In Section 2, we provide basic model-theoretic terminology and facts (including the algebraic language for the leading term structure $RV$) and next, following the paper [10], some results from Hensel minimality needed in our approach. In Section 3, we establish the following theorem on existence of the limit (together with its resplendent version).

**Theorem 1.1.** Let $f : E \rightarrow K$ be a 0-definable function on a subset $E$ of $K$. Suppose that 0 is an accumulation point of $E$. Then there is a subset $\tilde{F}$ of $E$, definable over the algebraic closure of $\emptyset$, with accumulation point 0, and a point $w \in \mathbb{P}^1(K)$ such that

$$\lim_{x \to 0} f|F(x) = w,$$

and the set

$$\{(v(x), v(f(x))) : x \in \tilde{F} \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \{\infty\})$$

is contained either in an affine line with rational slope

$$\{(k, l) \in \Gamma \times \Gamma : q \cdot l = p \cdot k + \beta\}$$

with $p, q \in \mathbb{Z}$, $q > 0$, $\beta \in \Gamma$, or in $\Gamma \times \{\infty\}$.

In Section 4, we prove, using Theorem [14], non-Archimedean versions of curve selection and the closedness theorem, stated below. They have numerous applications in geometry and topology of Henselian fields. In
particular, the closedness theorem enables application of resolution of
singularities in much the same way as over the locally compact fields.

**Theorem 1.2.** (Curve selection) Consider an $L$-definable subset $A$ of
$K^n$ with an accumulation point $a_0 \in K^n$, i.e. $a_0$ lies in the closure of
$A \setminus \{a_0\}$. Then there exists a continuous function $a : E \to K^n$, which is
definable (with parameters) in the language $L$ augmented by an angular
component map, such that $0$ is an accumulation point of $E \subset K$, and

$$a(E \setminus \{0\}) \subset A \setminus \{0\}, \quad \lim_{t \to 0} a(t) = a_0.$$ 

We then say that $a(t)$ is a definable curve in $A$ and write $a(t) \to a_0$.

**Remark 1.3.** Although curve selection is available after adding an
angular component map to the initial language $L$, it is a very useful
tool in geometry and topology of Hensel minimal $L$-structures too. In particular, it will be used in the proofs of the embedding theorem
(Theorem 6.2) and Proposition 6.8 that every $L$-definable Hausdorff
LC-space is regular.

**Theorem 1.4.** Given a definable subset $D$ of $K^n$, the canonical pro-
jection

$$\pi : D \times \mathcal{O}_K^m \rightarrow D$$

is definably closed in the $K$-topology, i.e. if $A \subset D \times \mathcal{O}_K^m$ is a closed
definable subset, so is its image $\pi(A) \subset D$.

Note that Theorems 1.4 may be no longer true after expansion of
the language for the leading term structure $RV$, as demonstrated in
Examples 4.4.

**Remark 1.5.** Since the notions of limit, continuity, closedness etc. are
first order properties, one can prove the above theorems by passage to
elementary extensions. Therefore one can assume that the Henselian
field $K$ under study is $\aleph_1$-saturated and, consequently, that an angular
component map $\overline{ac}$ (also called coefficient map, after van den Dries [19])
exists. We shall sometimes make use of this fact somewhere else in this
paper.

Section 5 is devoted to several applications, including piecewise con-
tinuity, several non-Archimedean versions of the Lojasiewicz inequalities
and Hölder continuity.

In Section 6, we study non-Archimedean definable (Hausdorff) spaces
and definable (Hausdorff) LC-spaces, i.e. spaces obtained by gluing
finitely many definable, locally closed subsets of affine spaces $K^n$. We
provide, among others, an embedding theorem for regular definable
spaces (Theorem 6.2), an analogue of the one from [24, Chapter 10], which is based on two results:

1) A closed definable subset $A$ of $K^n$ is the zero set of a continuous definable function $d$ on $K^n$;

2) A criterion for continuity in terms of arc-continuity.

The proof of the first result is based on a version of the Lojasiewicz inequalities (Theorem 5.9) and on a model-theoretic compactness argument; the proof of the second follows directly via curve selection (Theorem 1.2).

Next, relying on curve selection (Theorem 1.2) and the theorem on existence of the limit (Theorem 1.1), we prove that every definable Hausdorff LC-space $X$ is regular (Proposition 6.8). Hence and by quantifier elimination for ordered abelian groups, we establish the ultranormality of definable Hausdorff LC-spaces (Theorem 5.10).

In Section 7, we establish a non-Archimedean version of the extension theorem and the existence of definable retractions onto arbitrary closed definable subsets of definable Hausdorff LC-spaces.

In Section 8, we establish a more precise version of the theorem on existence of the limit in the case of Henselian fields with analytic structure, which may be regarded as a kind of Puiseux’s theorem. Its proof relies on the term structure of definable functions.

In the last section, we give the strengthened definable versions of desingularization (hypersurface case) and transformation to normal crossings by blowing up (Theorems 9.1 and 9.2); namely those on general strong analytic manifolds $M$ in Hensel minimal expansions of analytic structures, and not only on the definably compact ones over Henselian fields with analytic structure from our paper [11]. Also provided is a finitary version of the Nullstellensatz for meromorphic functions (i.e. continuous definable meromorphic functions) on a strong analytic manifold $M$.

Soon after o-minimality had become a fundamental concept in real algebraic geometry (realizing the postulate of both tame topology and tame model theory), numerous attempts were made to find similar approaches in algebraic geometry of valued fields. This led to axiomatically based notions such as C-minimality [28, 33], P-minimality [29], V-minimality [30], b-minimality [15], tame structures [6, 8], and eventually Hensel minimality [10].
The concept of Hensel minimal theories seems to enjoy most natural and desirable properties, being relatively broad and easily verifiable at the same time. It is tame with respect to the leading term structure $RV$ (binding residue field and value group into one structure). In fact, several variants of Hensel minimality were introduced, abbreviated by $l$-$h$-minimality with $l \in \mathbb{N} \cup \{\omega\}$. The $l$-$h$-minimality condition is the stronger, the larger the number $l$ is. Yet already $1$-$h$-minimality provides, likewise $o$-minimality does, powerful geometric tools as, for instance, cell decomposition, a good dimension theory or the Jacobian property (an analogue of the $o$-minimal monotonicity theorem). Actually, the majority of the results from [10], including those applied in our paper, hold for $1$-$h$-minimal theories.

Note finally that many of the classical examples of Hensel minimal fields, including Henselian fields with analytic structure, are even $\omega$-$h$-minimal (op.cit., Section 7). Below we list four basic examples of Hensel minimal structures:

1) Henselian valued fields in the (algebraic) language of valued fields are $\omega$-$h$-minimal.

2) Henselian valued fields with analytic structure are $\omega$-$h$-minimal. More precisely, let $K$ be a Henselian equicharacteristic zero field with separated analytic structure $\mathcal{A}$; in other words, a model of the $\mathcal{L}_{\text{hen},\mathcal{A}}$-theory $T_{\text{hen},\mathcal{A}}$ (cf. Section 8). Then the $\mathcal{L}_{\text{hen},\mathcal{A}}$-theory of $K$ is $\omega$-$h$-minimal (op.cit., Theorem 6.2.1).

3) V-minimal fields are $1$-$h$-minimal (op.cit., Theorem 6.4.2).

4) $T$-convex valued fields, where $T$ is a power-bounded $o$-minimal theory in an expansion $\mathcal{L}$ of the language of ordered fields and $\mathcal{O}_K$ is a $T$-convex subring of $K$ are $1$-$h$-minimal (op.cit., Theorem 6.3.4). Whether this theory is $\omega$-$h$-minimal is an open question as yet.

2. VALUATION- AND MODEL-THEORETICAL PRELIMINARIES.

We begin with basic notions from valuation theory. By $(K,v)$ we mean a field $K$ endowed with a valuation $v$. Let

$$\Gamma = vK, \mathcal{O}_K, M_K \text{ and } \tilde{K} = Kv$$

denote the value group, valuation ring, its maximal ideal and residue field, respectively. Let $r : \mathcal{O}_K \to Kv$ be the residue map. In this paper, we shall consider the equicharacteristic zero case, i.e. the characteristic of the fields $K$ and $Kv$ are assumed to be zero. For elements $a \in K$, the value is denoted by $va$ and the residue by $av$ or $r(a)$ when $a \in \mathcal{O}_K$. 


Then
\[ \mathcal{O}_K = \{ a \in K : va \geq 0 \}, \quad \mathcal{M}_K = \{ a \in K : va > 0 \}. \]

For a ring \( R \), let \( R^\times \) stand for the multiplicative group of units of \( R \). Obviously, \( 1 + \mathcal{M}_K \) is a subgroup of the multiplicative group \( K^\times \). Let
\[ rv : K^\times \to G(K) := K^\times/(1 + \mathcal{M}_K) \]
be the canonical group epimorphism. Since \( vK \cong K^\times/\mathcal{O}_K^\times \), we get the canonical group epimorphism \( \bar{v} : G(K) \to vK \) and the following exact sequence
\[ (2.1) \quad 1 \to \tilde{\mathcal{K}}^\times \to G(K) \to vK \to 0. \]

We put \( v(0) = \infty \) and \( \bar{v}(0) = \infty \). For simplicity, we shall write
\[ v(a) = (v(a_1), \ldots, v(a_n)) \quad \text{or} \quad rv(a) = (rv(a_1), \ldots, rv(a_n)) \]
for an \( n \)-tuple \( a = (a_1, \ldots, a_n) \in K^n \).

We adopt the following 2-sorted algebraic language \( L_{\text{hen}} \) on Henselian fields \( (K, v) \) of equicharacteristic zero, which goes back to Basarab [4].

**Main sort:** a valued field with the language of rings \( (K, 0, 1, +, -) \) or with the language \( L_{vf} \) of valued fields \( (K, 0, 1, +, -) \).

**Auxiliary sort:** \( RV(K) := G(K) \cup \{0\} \) with the language specified as follows: (multiplicative) language of groups \( (1, \cdot) \) and one unary predicate \( P \) such that \( P_K(\xi) \) iff \( \bar{v}(\xi) \geq 0 \); here we put \( \xi \cdot 0 = 0 \) for all \( \xi \in RV(K) \). The predicate
\[ \mathcal{R}(\xi) :\iff [\xi = 0 \lor (\xi \neq 0 \land P(\xi) \land P(1/\xi))] \]
will be construed as the residue field \( K^v = \tilde{K} \) with the language of rings \( (0, 1, +, \cdot) \); obviously, \( \mathcal{R}_K(\xi) \) iff \( \bar{v}(\xi) = 0 \). The sort \( RV \) binds together the residue field and value group.

**One connecting map:** \( rv : K \to RV(K), \quad rv(0) = 0 \).

The valuation ring can be defined by putting \( \mathcal{O}_K = rv^{-1}(\mathcal{P}_K) \). The residue map \( r : \mathcal{O}_K \to K^v \) will be identified with the map
\[ r(x) = \begin{cases} rv(x) & \text{if } x \in \mathcal{O}_K^\times, \\ 0 & \text{if } x \in \mathcal{M}_K. \end{cases} \]

**Remark 2.1.** Addition in the residue field \( \mathcal{R}_K \cup \{0\} \) is the restriction of the following algebraic operation on \( RV(K) \):
\[ rv(x) + rv(y) = \begin{cases} rv(x + y) & \text{if } v(x + y) = \min\{v(x), v(y)\}, \\ 0 & \text{otherwise} \end{cases} \]
for all \( x, y \in K^\times \); clearly, we put \( \xi + 0 = \xi \) for every \( \xi \in RV(K) \).
Remark 2.2. The standard language for the sort $RV$, whose vocabulary has just been introduced, is of course equivalent to the language of rings $(0, 1, +, \cdot)$ from Remark 2.1. In particular, $\bar{v}(\xi) > 0$ iff $1 + \xi = 1$. This language of rings for $RV$ will be denoted by $\mathcal{L}_{rv}$.

It is well known that exact sequence 2.1 splits whenever the residue field $Kv$ is $\aleph_1$-saturated. In this case, there is a section $\theta : G(K) \to \tilde{K}^\times$ of the monomorphism $\iota : \tilde{K}^\times \to G(K)$ and the map

$$(\theta, \bar{v}) : G(K) \to \tilde{K}^\times \times vK$$

is an isomorphism. Generally, the existence of such a section $\theta$ is equivalent to that of an angular component map $\text{ac} = \theta \circ rv$.

Remark 2.3. It is easy to check that the language $\mathcal{L}_{rv}$ with the section $\theta$ is equivalent to the language which consists of two maps $\theta : RV(K) \to Kv$, $\theta(0) = 0$, and $\bar{v} : RV(K) \to vK \cup \{\infty\}$, $\bar{v}(0) = \infty$, of the language of rings $(0, 1, +, -, \cdot)$ on the residue field $Kv$, and of the language of ordered groups $(0, +, -, <)$ on the value group $vK$.

In view of the above remark, the residue field is orthogonal to the value group, i.e. every definable subset $C \subset (Kv)^p \times (vK)^q$ is a finite union of Cartesian products

$$(2.2) \quad C = \bigcup_{i=1}^{k} X_i \times Y_i$$

for some definable subsets $X_i \subset (Kv)^p$ and $Y_i \subset (vK)^q$.

Remark 2.4. The $\mathcal{L}_{hen}$-theory $\mathcal{F}$ of Henselian, non-trivially valued fields of equicharacteristic zero eliminates valued field quantifiers resplendently. More precisely, consider an expansion $\mathcal{L}'_{rv}$ of the language $\mathcal{L}_{rv}$ for the auxiliary sort $RV$; put

$$\mathcal{L}'_{hen} := \mathcal{L}_{hen} \cup \mathcal{L}'_{rv}.$$ 

Let $\omega(x, \xi)$ be an $\mathcal{L}'_{hen}$-formula with $K$-variables $x = (x_1, \ldots, x_n)$ and $RV$-variables $\xi = (\xi_1, \ldots, \xi_m)$. Then $\omega(x)$ is $\mathcal{F}$-equivalent to a finite disjunction of formulae of the form:

$$\phi(x) \land \psi(rv(p(x)), \xi),$$

where $\phi$ is a quantifier-free formula in the language of rings, $\psi$ is an $\mathcal{L}'_{rv}$-formula, and $p(x)$ is a tuple of polynomials with integer coefficients.

The conclusion of Remark 2.4 can be proven through arguments due to Basarab [4], which rely on an embedding theorem and a relative (with respect to $RV$) version of the Ax–Kochen–Ershov theorem. The
research on this topic has a long history, let us mention some papers as \[\text{[2, 3, 13, 15, 18, 19, 23, 24, 26, 27, 16]}.\] Note that relative quantifier elimination, based on two auxiliary sorts (value group and residue field) with an angular component map, was achieved by Pas \[\text{[16]}.\]

In our geometric approach, most essential is which (not how) sets are definable. We shall usually work with a language \(\mathcal{L}\) which is an expansion of the language of valued fields \(\mathcal{L}_{vf}\), along with the auxiliary imaginary sort \(RV\). The words 0-definable and \(A\)-definable will mean \(\mathcal{L}\)-definable and \(\mathcal{L}_A\)-definable; ”definable” will refer to definable in \(\mathcal{L}\) with arbitrary parameters.

Fix a language \(\mathcal{L}\) which is an expansion of \(\mathcal{L}_{hen}\), and a model \(K\) of a 1-h-minimal (complete) \(\mathcal{L}\)-theory \(T\). For the reader’s convenience, we recall below the following three results of Hensel minimality from the paper \[\text{[10]},\] which are crucial for our approach:

1) Domain and range preparation (\textit{op.cit.}, Proposition 2.8.6), which can be derived from a week form of the Jacobian property, namely the valuative Jacobian property (\textit{loc.cit.}, Lemma. 2.8.5);
2) Reparametrized cell decomposition (\textit{op.cit.}, Theorem 5.7.3 ff.);
3) Cell decomposition (\textit{op.cit.}, Theorem 5.2.4 ff.).

Proposition 2.5. (Valuative Jacobian Property) Let \(f : K \to K\) be a 0-definable function. Then there exists a finite 0-definable set \(C \subset K\) such that for every ball \(B\) 1-next to \(C\), either \(f\) is constant on \(B\), or there exists a \(\mu_B \in vK\) such that

1) for every open ball \(B' \subset B\), \(f(B')\) is an open ball of radius \(\mu_B + \text{rad}(B')\);
2) for every \(x_1, x_2 \in B\), we have \(v(f(x_1) - f(x_2)) = \mu_B + v(x_1 - x_2)\).

Proposition 2.6. (Domain and Range Preparation). Let \(f : K \to K\) be a 0-definable function and let \(C_0 \subset K\) be a finite, 0-definable set. Then there exist finite, 0-definable sets \(C, D \subset K\) with \(C_0 \subset C\) such that \(f(C) \subset D\) and for every ball \(B\) 1-next to \(C\), the image \(f(B)\) is either a singleton in \(D\) or a ball 1-next to \(D\); moreover, the conclusions (1) and (2) of the Valuative Jacobian Property hold.

For \(m \leq n\), denote by \(\pi_{\leq m}\) or \(\pi_{\leq m+1}\) the projection \(K^n \to K^m\) onto the first \(m\) coordinates; put \(x_{\leq m} = \pi_{\leq m}(x)\). Let \(C \subset K^n\) be a non-empty 0-definable set, \(j_i \in \{0, 1\}\) and \(c_i : \pi_{<i}(C) \to K\) be 0-definable functions \(i = 1, \ldots, n\). Then \(C\) is called a 0-definable cell with center tuple \(c = (c_i)_{i=1}^n\) and of cell-type \(j = (j_i)_{i=1}^n\) if it is of
the form:
\[ C = \{ x \in K^n : (rv(x_i - c_i(x_{<i})))_{i=1}^n \in R \}, \]
for a (necessarily 0-definable) set
\[ R \subset \prod_{i=1}^n j_i \cdot G(K), \]
where \( 0 \cdot G(K) = 0 \subset RV(K) \) and \( 1 \cdot G(K) = G(K) \subset RV(K) \). One can similarly define \( A \)-definable cells.

In the absence of the condition that algebraic closure and definable closure coincide in \( T = \text{Th}(K) \) (i.e. the algebraic closure \( \text{acl}(A) \) equals the definable closure \( \text{dcl}(A) \) for any Henselian field \( K' \equiv K \) and every \( A \subset K' \)), a concept of reparameterized cells must come into play. Let us mention that one can ensure the above condition via an expansion of the language for the sort \( RV \).

Consider a 0-definable function \( \sigma : C \to RV(K)^k \). Then \((C, \sigma)\) is called a 0-definable reparameterized (by \( \sigma \)) cell if each set \( \sigma^{-1}(\xi) \), \( \xi \in \sigma(C) \), is a \( \xi \)-definable cell with some center tuple \( c_\xi \) depending definably on \( \xi \) and of cell-type independent of \( \xi \).

**Remark 2.7.** If the language \( L \) has an angular component map, then one can take \( \sigma \) from the above definition to be residue field valued (instead of RV-valued).

**Theorem 2.8.** (Reparameterized Cell Decomposition) For every 0-definable set \( X \subset K^n \), there exists a finite decomposition of \( X \) into 0-definable reparameterized cells \((C_k, \sigma_k)\). Moreover, given finitely many 0-definable functions \( f_j : X \to K \), one can require that the restriction of every function \( f_j \) to each cell \( \sigma_k^{-1}(\xi) \) be continuous. \( \square \)

It is of importance that 0-, 1- and \( \omega \)-h-minimality enjoy the resplendency property, i.e. if an \( L \)-theory is 0-, 1- or \( \omega \)-h-minimal, then so is its \( L' \)-theory for any \( RV \)-expansion by predicates \( L' \) of the language \( L \) (op.cit., Section 4.1). If algebraic closure and definable closure coincide in \( RV(K) \), then so do they in \( K \) (op.cit., Section 4.3).

**Theorem 2.9.** (Cell Decomposition) Suppose that algebraic closure and definable closure coincide in a 1-h-minimal theory \( T = \text{Th}(K) \). For every 0-definable set \( X \subset K^n \), there exists a finite decomposition of \( X \) into 0-definable cells \( C_k \).

Furthermore, there exists a finite decomposition of \( X \) into 0-definable subsets \( C_k \) such that each \( C_k \) is, after some permutation of the variables, a 0-definable cell of type \((1, 1, 1, 0, 0, 0)\) with 1-Lipschitz continuous centers \( c_1, \ldots, c_n \). Such cells shall be called 1-Lipschitz cells. \( \square \)
Thus one has definable, 1-Lipschitz cell decomposition for any 1-h-minimal theory after a suitable expansion of the language of the leading term structure $RV$.

Throughout the paper, we shall additionally require that every definable subset in the imaginary sort $RV$ be already definable in the pure valued field language, and consider the standard algebraic language $L_{rv}$ for $RV$.

3. Existence of the limit

In this section, we prove Theorem 1.1 on existence of the limit for definable functions of one variable. By Remarks 1.5 and 2.3 ff., we can assume that the field $K$ has a coefficient map, exact sequence 2.1 splits and the residue field is orthogonal to the value group. Then we have the isomorphism

$$(\theta, \bar{v}) : G(K) \to \tilde{K}^\times \times vK,$$

and thus we can identify $G(K)$ with $\tilde{K}^\times \times vK$. This isomorphism is of significance because topological properties of the valued field $K$ are described in terms of the value group $vK$.

By Proposition 2.6, there exist finite 0-definable subsets $C \subset K$ with $0 \in C$ and $D \subset K$ such that $f(C) \subset D$ and, for every ball $B$ 1-next to $C$, the image $f(B)$ is either a singleton in $D$ or a ball 1-next to $D$. After partitioning of the domain $E$, we can assume without loss of generality that there is a point $d \in D$, say $d = 0$, such that the image $f(B)$ is either $\{0\}$ or a ball 1-next to 0 for every balls $B \subset E$ which are 1-next to 0. In the first case we are done. So suppose the second case. Obviously, the balls 1-next to 0 are of the form $\{rv(x) = \xi\}$, $\xi \in G(K)$.

Now consider the 0-definable set $X \subset G(K)^2$ defined by the formula

$$\{(\xi, \eta) \in G(K)^2 : \{rv(x) = \xi\} \subset E, f(\{rv(x) = \xi\}) = \{rv(y) = \eta\}\}.$$

By Remark 2.3 (orthogonality property), $X$ is defined by a finite disjunction of conjunctions of the form:

$$\phi(\theta(\xi), \theta(\eta)) \land \psi(\bar{v}(\xi), \bar{v}(\eta)).$$

We can assume, without loss of generality, that $X$ is defined by one from those conjunctions and is of the form:

$$\theta(\eta) = \alpha(\theta(\xi)) \land \bar{v}(\eta) = \beta(\bar{v}(\xi))$$

for some 0-definable functions $\alpha$ and $\beta$, where the domain of $\beta$ is a subset $\Delta$ of $vK$ with accumulation point $\infty$. 
Now we apply the following theorem from [1, Corollary 1.10] to the effect that functions definable in ordered abelian groups are piecewise linear.

**Proposition 3.1.** Consider an ordered abelian group $G$ with the language of ordered abelian groups $L_{\text{oag}} = (0, +, <)$. Let $f : G^n \to G$ be an $A$-definable function for a subset $A \subset G$. Then there exists a partition of $G^n$ into finitely many $A$-definable subsets such that, on each subset $S$ of them, $f$ is linear; more precisely, there exist $r_1, \ldots, r_n, s \in \mathbb{Z}$, $s \neq 0$, and an element $\gamma$ from the definable closure of $A$ such that

$$f(a_1, \ldots, a_n) = \frac{1}{s} \cdot (r_1 a_1 + \ldots + a_n r_n + \gamma)$$

for all $(a_1, \ldots, a_n) \in S$. \hfill $\blacksquare$

For $\bar{v}(\xi) \in \Delta$, we thus get the equivalence

$$\bar{v}(\eta) = \beta(\bar{v}(\xi)) \iff \bar{v}(\eta) = \frac{1}{s} \cdot (r \cdot \beta(\bar{v}(\xi)) + \gamma).$$

Then the set

$$F := \{ a \in K : \alpha(\bar{a}\alpha(a)), v(a) \in \Delta \}$$

is a 0-definable subset of $E$ with accumulation point 0. We encounter three cases:

*Case 1.* If $r/s > 0$, then

$$\lim_{x \to 0} f|F(x) = 0.$$

*Case 2.* If $r/s < 0$, then

$$\lim_{x \to 0} f|F(x) = \infty.$$

*Case 3.* Were $r/s = 0$, then $\beta = \delta(\alpha) = \gamma/s$, we would get

$$f(\{rv(x) = (1, \alpha)\}) = \{rv(y) = \gamma/s\}.$$

Then, for any point $b \in K$ with $rv(b) = \gamma/s$, the set $(f|F)^{-1}(b)$ would be an isolated subset of $K$ with accumulation point 0, which is impossible. This contradiction shows that Case 3 cannot happen, which finishes the proof. \hfill $\blacksquare$

We are now going to strengthen Theorem [1] by taking an arbitrary expansion $\mathcal{L}_{rv}'$ of the algebraic language $\mathcal{L}_{rv}$ for the leading term structure $RV$; put $\mathcal{L}' := \mathcal{L} \cup \mathcal{L}_{rv}'$. Clearly, every $\mathcal{L}'$-formula $\chi(x, \xi)$, with
$K$-variables $x$ and $RV$-variables $\xi$, is $T$-equivalent to a finite disjunction of formulae of the form:

$$\phi(x) \land \psi(rv(p(x)), \xi),$$

where $\phi$ is an $L$-formula in the language of the valued field sort, $\psi$ is an $L'_{rv}$-formula, and $p(x) = (p_1(x), \ldots, p_r(x))$ is a tuple of terms in the valued field sort.

**Remark 3.2.** Note that one can replace the above formulae by ones of the form:

$$\phi(x) \land p_1(x) \neq 0 \land \ldots \land p_r(x) \neq 0 \land \psi(rv(p(x)), \xi),$$

because in the cases $p_i(x) = 0$ one can substitute 0 for $p_i(x)$ in the formula $\psi(rv(p(x)), \xi)$.

In our problem, we consider an $L'$-formula $\chi(x_1, x_2)$ that defines the graph of the function $f : E \to K$, denoted also by $f$ for simplicity. We may of course shrink the set $E$, keeping the assumption that 0 is an accumulation point of $E$. Therefore we can assume that $E$ is defined by one formula of the form $\chi(x_1, x_2)$.

The closure $Z \in K^2$ of the set of those points where the terms $p(x)$ are not continuous is $L$-definable (without parameters) of dimension $< 1$. Then the set

$$U := \{a \in K^2 \setminus Z : p_1(a) \neq 0, \ldots, p_r(a) \neq 0, RV(K) \models \psi(rv(p(a)))\}$$

is an open 0-definable subset of $K^2$. Further, the set of those points $a_1 \in K$, over which the fiber of $Z$ is infinite, is finite. Since we are interested in what happens in the vicinity of $0 \in K$, we may thus assume that the fibres of the projection $\pi_{<2} : Z \to K$ are finite. Hence we get

$$f \subset (\{a \in K^2 : K \models \phi(a)\} \cap U) \cup Z.$$

Therefore the graph of $f$ is contained in the $L$-definable set $Y$ of those points $a \in K^2$ which are isolated in the fibre of the set

$$\{a \in K^2 : K \models \phi(a)\}$$

over $a_1$. Clearly, the projection $\pi_{<2} : Y \to K$ is finite-to-one. By Theorem 2.8, $Y$ is a finite union of $L$-definable cells of type $(1, 0)$, reparametrized by residue field valued functions $\sigma$ (cf. Remark 2.7). We can, as before, assume that $Y$ is one of those cells, $Y = C$. Then

$$f \subset C = \bigcup_{\xi} C_\xi, \quad C_\xi = \sigma^{-1}(\xi),$$

and each cell $C_\xi$ is the graph of the center $c_{\xi,2} : \pi_{<2}(C_\xi) \to K$. 
It follows easily from the orthogonality of the residue field and value group (cf. Remark 2.3 ff.) that $\sigma(C)$ is the disjoint union of $L$-definable sets $\Sigma_1$ and $\Sigma_2$ such that 0 is not an accumulation point of the set

$$\pi_{<2} \left( \bigcup_{\xi \in \Sigma_2} C_\xi \right),$$

but 0 is an accumulation point of $\pi_{<2}(C_\xi)$ for every $\xi \in \Sigma_1$.

Let $B_\xi$ be the set of all points from the closure of any cell $C_\xi$, $\xi \in \Sigma_1$, lying over 0. Then the union

$$B := \bigcup_{\xi \in \Sigma_1} B_\xi$$

is a finite $L$-definable (without parameters) set, say $B = \{b_1, \ldots, b_k\}$. By the orthogonality property and Theorem 1.1, we can partition, in a common vicinity of 0, the domains of the centers $c_{\xi,2}$, $\xi \in \Sigma_1$, into pieces $L$-definable over the algebraic closure of $\emptyset$, in order to obtain new functions $c_{\xi,2,j}$ such that

$$c_{\xi,2,j} \subset c_\xi \text{ and } \lim_{x_1 \to 0} c_{\xi,2,j}(x_1) = b_j$$

for all $j \in B_\xi$. Again by the orthogonality property, the centers $c_{\xi,2,j}$, $\xi \in \Sigma_1$, $j \in B_\xi$, are equally continuous at 0. Therefore, since

$$f \subset \bigcup_{\xi \in \Sigma_1} c_{\xi,2}$$

in a neighbourhood of 0, it is not difficult to check that at least one point from the set $B$ is an accumulation point of the graph of $f$. In this fashion, we have established a resplendent version of Theorem 1.1.

4. Curve selection and the closedness theorem

The proofs of both these results rely on the theorem on existence of the limit (Theorem 1.1). To prove Theorem 1.2, we may of course assume that $A$ is a subset of $O^n_K$ and $a \notin A$, the case $a \in A$ being trivial. We begin be stating the following

**Lemma 4.1.** Consider a definable family $X_\xi$, $\xi \in (Kv)^k$, of subsets of $K^n$ and a point $a \in K^n$. Then $a$ lies in the closure of the union $\bigcup_\xi X_\xi$ iff $a$ lies in the closure of $X_{\xi_0}$ for some $\xi_0$. 

Proof. Apply the orthogonality of the residue field and value group (cf. Remark 2.3 ff.) to the set
\[ \bigcup_{\xi \in (Kv)^k} \{\xi\} \times v(X_\xi - a). \]

Hence and by decomposition into cells with residue field valued reparametrization, we are reduced to the case where \( A \) is a \( \xi \)-definable cell \( C_\xi \) for some \( \xi \in Kv \):
\[ C_\xi = \{x \in K^n : (rv(x_i - c_i(x_{<i})))_{i=1}^n \in R\}, \]
for a \( \xi \)-definable set
\[ R \subset \prod_{i=1}^n j_i \cdot G(K), \quad j_i \in \{0, 1\}. \]

Clearly, the assumptions of Theorem 1.2 are satisfied for each projection \( \pi_{\leq i}(C_\xi) \) and \( \pi_{\leq i}(a), \quad i = 1, \ldots, n \). Therefore it suffices, via induction procedure, to consider the case \( n = 2 \).

Since \( a \not\in A, \quad a_1 = c_1 \) because otherwise \( a \) would not lie in the closure of \( A \). Therefore
\[ C_\xi = \{x \in K^2 : (rv(x_1 - c_1), rv(x_2 - c_2(x_1))) \in R\}, \]
for a \( \xi \)-definable set
\[ R \subset G(K) \times (j_2 \cdot RV(K)). \]

The case \( j_2 = 0 \) is easy, because then the cell \( C_\xi \) is the graph of the center \( c_2(x_1) \), and thus the conclusion follows from Theorem 1.1. So consider the case \( j_2 = 1 \).

Again by Lemma 4.1, we can assume that \( A = C_\xi \) is the set of all points in \( K^2 \) for which
\[ (\overline{ac}(x_1 - c_1), \overline{ac}(x_2 - c_2(x_1))) = \eta \quad \text{and} \quad (v(x_1 - c_1), v(x_2 - c_2(x_1))) \in P \]
for an \( \eta \in (Kv \setminus \{0\})^2 \) and a 0-definable set \( P \subset (vK)^2 \).

We still need the following

**Lemma 4.2.** Let \( G \) be an ordered abelian group, \( P \) a definable subset of \( G^n_+ \) with \( G_+ := \{ \gamma \in G : \gamma \geq 0 \} \), and
\[ \pi : G^n \to G, \quad (x_1, \ldots, x_n) \mapsto x_1 \]
be the projection onto the first factor. Suppose that \( \infty \) is an accumulation point of \( \pi(P) \). Then there is an affine semi-line
\[ L = \{(r_1 \tau + \gamma_1, \ldots, r_n \tau + \gamma_n) : \tau \in G, \quad \tau \geq 0\} \]
with \(r_1, \ldots, r_n \in \mathbb{N}, r_1 > 0\), passing through a point \(\gamma = (\gamma_1, \ldots, \gamma_n) \in P\) and such that \(\infty\) is an accumulation point of \(\pi(P \cap L)\) too.

This lemma can be established, by relative quantifier elimination for ordered abelian groups (cf. [9]), in a similar way as we proved [12, Lemma 6.2], recalled below; see also [10, Section 6].

**Lemma 4.3.** Let \(G\) be an ordered abelian group and \(P\) be a definable subset of \(G^n\). Suppose that \((\infty, \ldots, \infty)\) is an accumulation point of \(P\), i.e. for any \(\delta \in G\) the set

\[
\{ x \in P : x_1 > \delta, \ldots, x_n > \delta \} \neq \emptyset
\]

is non-empty. Then there is an affine semi-line

\[
L = \{(r_1 \tau + \gamma_1, \ldots, r_n \tau + \gamma_n) : \tau \in G, \ \tau \geq 0\}
\]

with \(r_1, \ldots, r_n \in \mathbb{N} \setminus \{0\}\), passing through a point \(\gamma = (\gamma_1, \ldots, \gamma_n) \in P\) and such that \((\infty, \ldots, \infty)\) is an accumulation point of the intersection \(P \cap L\) too.

Note also that Lemma 4.3 will be used in the proof of Theorem 6.10 on the ultranormality of definable Hausdorff LC-spaces.

Apply Lemma 4.2 to \(G = vK\) and \(P \subset (vK)_+ \times (vK)_+\). Then the subset \(P \cap L\) has an accumulation point \((\infty, \rho_2)\), where \(\rho_2 = \infty\) or \(\rho_2 = \gamma_2\), according as \(r_2 > 0\) or \(r_2 = 0\).

Now take an element \(w \in K^2\) such that \(\overline{ac}(w) = \eta\) and \(v(w) = \gamma\).

Put

\[
\Delta := \{ \tau \in vK : \tau \geq 0, (r_1 \tau + \gamma_1, \rho_2 \tau + \gamma_2) \in P \}
\]

and

\[
E := \{ t \in K : \overline{ac}(t) = 1 \text{ and } v(t) \in \Delta \}.
\]

Then

\[
\{(c_1 + w_1 t r_1, c_2(c_1 + w_1 t r_1) + w_2 t r_2) \in K^2 : t \in E\} \subset A = C_\xi.
\]

Therefore it follows from Theorem 1.1 that, after perhaps shrinking the domain \(E\), the function \(a : E \to K^2\) given by the formula

\[
a(t) := (c_1 + w_1 t r_1, c_2(c_1 + w_1 t r_1) + w_2 t r_2)
\]

is the one we are looking for. This completes the proof of Theorem 1.2.

Now we shall prove the closedness theorem. Without loss of generality, we may assume that \(m = 1\) and \(n = 1\). The first reduction is obvious. The latter can be achieved by means of curve selection (Theorem 1.2) in exactly the same way, as it was achieved by means of fiber shrinking in the proof of the algebraic versions of the closedness
theorem in our papers \[10,12\]. Note that fiber shrinking is a relaxed version of curve selection. So consider the case \(m = n = 1\).

We must show that if \(A\) is an \(L'\)-definable subset of \(D \times O\), with \(D \subset K\) and a point \(b = 0 \in K\) lies in the closure of \(B := \pi_{<2}(A)\), then there is a point \(a\) in the closure of \(A\) such that \(\pi_{<2}(a) = 0\). As before, we can assume that \(A\) is an \(L'\)-definable cell of type \((1, j_2)\) with centers \(0, c_2(x_1)\). The case \(j_2 = 0\) is obvious by virtue of Theorem 1.1.

So consider the case \(j_2 = 1\). Then

\[
A = \{ x \in K^2 : (rv(x_1), rv(x_2 - c_2(x_1))) \in R \}
\]

for a subset \(R\) of \(G(K) \times G(K)\) such that \(\bar{v}(R) \subset (vK)_+ \times (vK)_+\). By the orthogonality of the residue field and value group (cf. Remark 2.3 ff.), \(R\) is a finite union of Cartesian products

\[
(4.1) \quad C = \bigcup_{i=1}^k X_i \times Y_i
\]

for some non-empty definable subsets

\[
X_i \subset Kv \times Kv \quad \text{and} \quad Y_i \subset (vK)_+ \times (vK)_+.
\]

Let \(\pi : (vK)^2 \to vK\) be the projection onto the first factor. Then \(\infty\) is an accumulation point of the union \(\pi \left( \bigcup_{i=1}^k Y_i \right)\), and thus of \(\pi(Y_{i_0})\) for some \(i_0 \in \{1, \ldots, k\}\). Then we can replace the set \(A\) by the set

\[
\{ x \in K^2 : (rv(x_1), rv(x_2 - c_2(x_1))) \in \{\eta\} \times P \},
\]

where \(\eta \in X_{i_0}\) and \(P = Y_{i_0}\). It follows immediately from Lemma 4.2 that there is an affine semi-line

\[
L = \{(r_1\tau + \gamma_1, r_2\tau + \gamma_2) : \tau \in vK, \ \tau \geq 0\}
\]

with \(r_1, r_2 \in \mathbb{N}, r_1 > 0, r_2 \geq 0\), passing through a point \(\gamma = (\gamma_1, \gamma_2) \in P\) and such that \(\infty\) is an accumulation point of \(\pi(P \cap L)\) too. Again we can replace the set \(A\) by the set

\[
\{ x \in K^2 : (rv(x_1), rv(x_2 - c_2(x_1))) \in \{\eta\} \times (P \cap L) \}.
\]

Now we shall argue likewise we did in the proof of Theorem 1.2.

The subset \(P \cap L\) has an accumulation point \((\infty, \rho_2)\), where \(\rho_2 = \infty\) or \(\rho_2 = \gamma_2\), according as \(r_2 > 0\) or \(r_2 = 0\). Take an element \(w \in K^2\) such that \(\overline{ac}(w) = \eta\) and \(v(w) = \gamma\). Put

\[
\Delta := \{ \tau \in vK : \tau \geq 0, \ (r_1\tau + \gamma_1, r_2\tau + \gamma_2) \in P \}
\]

and

\[
E := \{ t \in K : \overline{ac}(t) = 1 \quad \text{and} \quad v(t) \in \Delta \}.
\]
Then
\[ \{(w_1t^{r_1}, c_2(w_1t^{r_1}) + w_2t^{r_2}) \in K^2 : t \in E\} \]
is contained in \( A \). It follows immediately from Theorem 1.1 that the graph of the center \( c_2(x_1) \) has an accumulation point \((0, c_2(0))\). Hence the closure of the set \( A \) contains the point \((0, c_2(0)) \) or \((0, c_2(0) + w_2)\) according as \( r_2 > 0 \) or \( r_2 = 0 \). This completes the proof of Theorem 1.4.

Finally, we give an example which demonstrates that the closedness theorem may fail after expansion of the language for the leading term structure \( RV \) by predicates. Notice that such expansions are allowed by virtue of resplendency of Hensel minimality (cf. [11, Theorem 4.1.19]).

**Example 4.4.** Suppose that the exact sequence 2.1 splits and the value group \( vK = \mathbb{Z} \). Then we have a (non-canonical) isomorphism \( G(K) \simeq Kv \times vK \) (cf. Remarks 2.2 and 2.3). Then the set
\[ A := \{(x, y) \in K^2 : rv(x, y) = ((1, k), (k, 0)) \in G(K)^2, k \in \mathbb{N}\} \]
is a closed subset of \( K^2 \), but its projection
\[ \pi_1(A) = \{x \in K : rv(x) = (1, k), k \in \mathbb{N}\} \]
is not a closed subset of \( K \), having \( 0 \in K \) as an accumulation point.

5. **Applications of the closedness theorem**

We begin with the following full version of the theorem on existence of the limit.

**Proposition 5.1.** Let \( f : E \to \mathbb{P}^1(K) \) be an 0-definable function on a subset \( E \) of \( K \), and suppose that 0 is an accumulation point of \( E \). Then there exist points \( w_1, \ldots, w_r \in \mathbb{P}^1(K) \) a finite partition of \( E \) into \( \{w_1, \ldots, w_r\} \)-definable sets \( E_1, \ldots, E_r \) and such that
\[ \lim_{x \to 0} f|E_i(x) = w_i \quad \text{for} \quad i = 1, \ldots, r. \]

**Proof.** We may of course assume that \( 0 \notin E \). Put
\[ F := \text{graph}(f) = \{(x, f(x) : x \in E) \subset K \times \mathbb{P}^1(K); \]

obviously, \( F \) is of dimension 1. It follows from the closedness theorem that the frontier \( \partial F \subset K \times \mathbb{P}^1(K) \) is non-empty, and thus of dimension zero; say
\[ \partial F \cap (\{0\} \times \mathbb{P}^1(K)) = \{(0, w_1), \ldots, (0, w_r)\} \]
for some \(w_1, \ldots, w_r \in \mathbb{P}^1(K)\). Take pairwise disjoint neighborhoods \(U_i\) of the points \(w_i, i = 1, \ldots, r\), and set

\[
F_0 := F \cap \left( E \times \left( \mathbb{P}^1(K) \setminus \bigcup_i E_i \right) \right).
\]

Let

\[
\pi : K \times \mathbb{P}^1(K) \longrightarrow K
\]

be the canonical projection. Then

\[
E_0 := \pi(F_0) = f^{-1} \left( \mathbb{P}^1(K) \setminus \bigcup_i E_i \right).
\]

Clearly, the closure \(\overline{F_0}\) of \(F_0\) in \(K \times \mathbb{P}^1(K)\) and \(\{0\} \times \mathbb{P}^1(K)\) are disjoint. Hence and by the closedness theorem, \(0 \not\in E_0\), the closure of \(E_0\) in \(K\). The set \(E_0\) is thus irrelevant with respect to the limit at \(0 \in K\). Therefore it remains to show that

\[
\lim_{x \to 0} f|E_i(x) = w_i \quad \text{for } i = 1, \ldots, r.
\]

Otherwise there is a neighborhood \(V_i \subset U_i\) such that 0 would be an accumulation point of the set

\[
f^{-1}(U_i \setminus V_i) = \pi(F \cap (E \times (U_i \setminus V_i))).
\]

Again, it follows from the closedness theorem that \(\{0\} \times \mathbb{P}^1(K)\) and the closure of \(F \cap (E \times (U_i \setminus V_i))\) in \(K \times \mathbb{P}^1(K)\) would not be disjoint. This contradiction finishes the proof.

Now we prove the theorem on piecewise continuity.

**Theorem 5.2.** Let \(A \subset K^n\) and \(f : A \rightarrow \mathbb{P}^1(K)\) be an 0-definable function. Then \(f\) is piecewise continuous, i.e. there is a finite partition of \(A\) into 0-definable locally closed subsets \(A_1, \ldots, A_s\) of \(K^n\) such that the restriction of \(f\) to each \(A_i\) is continuous.

**Proof.** Consider the graph

\[
E := \{(x, f(x)) : x \in A\} \subset K^n \times \mathbb{P}^1(K).
\]

We proceed with induction with respect to the dimension

\[
d = \dim A = \dim E.
\]

Observe first that every 0-definable subset \(X\) of \(K^n\) is a finite disjoint union of locally closed 0-definable subsets of \(K^n\). This can be easily proven by induction on the dimension of \(X\). Therefore we can assume that the graph \(E\) is a locally closed subset of \(K^n \times \mathbb{P}^1(K)\) of dimension \(d\)
and that the conclusion of the theorem holds for functions with source and graph of dimension $< d$.

Let $F$ be the closure of $E$ in $K^n \times \mathbb{P}^1(K)$ and $\partial E := F \setminus E$ be the frontier of $E$. Since $E$ is locally closed, the frontier $\partial E$ is a closed subset of $K^n \times \mathbb{P}^1(K)$ as well. Let

$$
\pi : K^n \times \mathbb{P}^1(K) \to K^n
$$

be the canonical projection. Then, by virtue of the closedness theorem, the images $\pi(F)$ and $\pi(\partial E)$ are closed subsets of $K^n$. Further,

$$
\dim F = \dim \pi(F) = d
$$

and

$$
\dim \pi(\partial E) \leq \dim \partial E < d.
$$

Putting

$$
B := \pi(F) \setminus \pi(\partial E) \subset \pi(E) = A,
$$

we thus get

$$
\dim B = d \quad \text{and} \quad \dim (A \setminus B) < d.
$$

Clearly, the set

$$
E_0 := E \cap (B \times \mathbb{P}^1(K)) = F \cap (B \times \mathbb{P}^1(K))
$$

is a closed subset of $B \times \mathbb{P}^1(K)$ and is the graph of the restriction

$$
f_0 : B \to \mathbb{P}^1(K)
$$

of $f$ to $B$. Again, it follows immediately from the closedness theorem that the restriction

$$
\pi_0 : E_0 \to B
$$

of the projection $\pi$ to $E_0$ is a definably closed map. Therefore $f_0$ is a continuous function. But, by the induction hypothesis, the restriction of $f$ to $A \setminus B$ satisfies the conclusion of the theorem, whence so does the function $f$. This completes the proof.

We immediately obtain

**Corollary 5.3.** The conclusion of the above theorem holds for any $0$-definable function $f : A \to K$. 

Yet another direct consequence of the closedness theorem is the following

**Proposition 5.4.** Let $f : E \to K^n$ be a continuous definable map on a closed bounded subset $E$ of $K^n$. Then the image $f(E)$ is a closed bounded subset of $K^n$ too.
Proof. Consider $f$ as a continuous map into the projective space $\mathbb{P}^m(K)$ and apply the closedness theorem to the graph $F$ of the map $f$:

$$F := \{(x, y) \in E \times \mathbb{P}^m(K) : y = f(x)\}.$$

Algebraic non-Archimedean versions of the Łojasiewicz inequalities, established in our papers [40, 42], can be carried over to the general settings considered here with proofs repeated almost verbatim. Thus we shall only state the results (Theorems 11.2, 11.5 and 11.6, Proposition 11.3 and Corollary 11.4 from [42]). The main ingredients of the proofs are the closedness theorem, the orthogonality property and relative quantifier elimination for ordered abelian groups. They allow us to reduce the problem under study to that of piecewise linear geometry. We first state the version, which is closest to the classical one.

**Theorem 5.5.** Let $f, g_1, \ldots, g_m : A \to K$ be continuous definable functions on a closed (in the $K$-topology) bounded subset $A$ of $K^m$. If

$$\{x \in A : g_1(x) = \ldots = g_m(x) = 0\} \subset \{x \in A : f(x) = 0\},$$

then there exist a positive integer $s$ and a constant $\beta \in \Gamma$ such that

$$s \cdot v(f(x)) + \beta \geq \min \{v(g_1(x)), \ldots, v(g_m(x))\}$$

for all $x \in A$. Equivalently, there is a $C \in |K|$ such that

$$|f(x)|^s \leq C \cdot |(g_1(x), \ldots, g_m(x))|$$

for all $x \in A$; here

$$|(g_1(x), \ldots, g_m(x))| := \max \{|g_1(x)|, \ldots, |g_m(x)|\}.$$

A direct consequence of Theorem 5.5 is the following result on Hölder continuity of definable functions.

**Proposition 5.6.** Let $f : A \to K$ be a continuous definable function on a closed bounded subset $A \subset K^n$. Then $f$ is Hölder continuous with a positive integer $s$ and a constant $\beta \in \Gamma$, i.e.

$$s \cdot v(f(x) - f(z)) + \beta \geq v(x - z)$$

for all $x, z \in A$. Equivalently, there is a $C \in |K|$ such that

$$|f(x) - f(z)|^s \leq C \cdot |x - z|$$

for all $x, z \in A$. □

We immediately obtain
Corollary 5.7. Every continuous definable function \( f : A \to K \) on a closed bounded subset \( A \subset K^n \) is uniformly continuous.

Now we formulate another, more general version of the Lojasiewicz inequality for continuous definable functions of a locally closed subset of \( K^n \).

Theorem 5.8. Let \( f, g : A \to K \) be two continuous 0-definable functions on a locally closed subset \( A \) of \( K^n \). If
\[
\{ x \in A : g(x) = 0 \} \subset \{ x \in A : f(x) = 0 \},
\]
then there exist a positive integer \( s \) and a continuous 0-definable function \( h \) on \( A \) such that \( f^s(x) = h(x) \cdot g(x) \) for all \( x \in A \).

Finally, put
\[
\mathcal{D}(f) := \{ x \in A : f(x) \neq 0 \} \quad \text{and} \quad \mathcal{Z}(f) := \{ x \in A : f(x) = 0 \}.
\]
The following theorem may be also regarded as a kind of the Lojasiewicz inequality, which is, of course, a strengthening of Theorem 5.8.

Theorem 5.9. Let \( f : A \to K \) be a continuous 0-definable function on a locally closed subset \( A \) of \( K^n \) and \( g : \mathcal{D}(f) \to K \) a continuous 0-definable function. Then \( f^s \cdot g \) extends, for \( s \gg 0 \), by zero through the set \( \mathcal{Z}(f) \) to a (unique) continuous 0-definable function on \( A \).

6. Definable spaces and embedding theorem, definable ultranormality and ultraparacompactness

In this section, we deal with definable spaces \( X \) for Hensel minimal structures on a field \( K \), which are defined by gluing finitely many affine definable sets (i.e. definable subsets of affine spaces \( K^n \)). Their theory, developed by van den Dries (cf. [20]) in the case of \( \alpha \)-minimal structures, carries over to the non-Archimedean settings. Most natural examples of such spaces are projective spaces, their products and definable subspaces. Obviously, the affine spaces \( K^n \) are zero-dimensional with respect to the small inductive dimension; and so are their subspaces since regularity is a hereditary property. Therefore every regular definable space \( X \) is zero-dimensional too.

Further, we shall investigate definable LC-spaces, i.e. those definable spaces which are defined by gluing finitely many definable, locally closed subsets of affine spaces \( K^n \). Such spaces include, in particular, definable topological manifolds obtained by gluing definable open subsets of \( K^n \). We shall show (Theorem 6.10) that every definable Hausdorff LC-space \( X \) is even definably ultranormal or, in other words,
definably zero-dimensional with respect to the large inductive dimension. This means that, for every two disjoint definable closed subsets \( A \) and \( B \) of \( X \), there exists a definable clopen subset \( C \) of \( X \) such that \( A \subset C \) and \( B \subset X \setminus C \). The proofs essentially rely on the closedness theorem (Theorem 1.4) and relative quantifier elimination for ordered abelian groups. A \emph{definable manifold} \( M \) of dimension \( n \) is a definable Hausdorff LC-space \( M \) obtained by gluing definable open subsets of \( K^n \).

We first give an example of a definable Hausdorff space which is not regular.

**Example 6.1.** Construct a definable space \( X \) by gluing the following two definable subsets of \( K^2 \) by means of the identity charts:

\[
U_1 := (K^2 \setminus (K \times \{0\})) \cup \{(0,0)\}, \quad U_2 := (K^2 \setminus (\{0\} \times K)) \, .
\]

It is not difficult to check that \( X \) is a Hausdorff space. Then

\[
A := (K \times \{0\}) \setminus \{(0,0)\} \subset U_2
\]

is a closed definable subset of \( X \), since \( A \cap U_1 = \emptyset \) and

\[
A \cap U_2 = (K \times \{0\}) \cap U_2
\]

is a closed subset of \( U_2 \). But any neighbourhood of \( A \) in \( U_2 \) has \((0,0)\) as an accumulation point. Therefore \( A \) and \((0,0)\) cannot be separated by open neighbourhhods, and thus \( X \) is not a regular definable space. \( \square \)

Clearly curve selection (Theorem 1.2 ff.) remains valid in the case where \( A \) is a definable Hausdorff space. Observe now that the embedding theorem for definable spaces in o-minimal structures from [20, Chapter 10], which originates from [17] in the semialgebraic case, can be carried over to our non-Archimedean settings, as stated below.

**Theorem 6.2.** Every regular definable space \( X \) is affine, i.e. \( X \) can be embedded into an affine space \( K^N \).

Indeed, besides of the following two facts, its proof can be repeated almost verbatim in our non-Archimedean settings. We leave inspection of that proof to the reader. \( \square \)

**Fact 1.** A closed definable subset \( A \) of \( K^n \) is the zero set of a continuous definable function \( d \) on \( K^n \), which can be used in the proof here in place of a distance function applied in the case of o-minimal structures (cf. [20], Claims 1 and 2 in the proof of Theorem 1.8).

**Fact 2.** A criterion for continuity in terms of arc-continuity (\emph{op.cit}, Lemma 1.7).
The latter fact, stated in the following lemma, follows directly via curve selection (Theorem 1.2 ff.).

**Lemma 6.3.** Let \( f : X \to Y \) be a definable map between definable Hausdorff spaces \( X \) and \( Y \). Then \( f \) is continuous at a point and \( a_0 \in X \) iff

\[
(f \circ a)(t) \to f(a_0)
\]

for each continuous curve

\[
a : E \to X, \quad a(t) \to a_0,
\]

which is definable in the initial language \( \mathcal{L} \) augmented by an angular component map; here \( E \) is a subset of \( K \) with 0 as an accumulation point. \( \square \)

The former, in turn, requires a separate proof given below. We shall make use, among others, of a version of the Lojasiewicz inequalities (Theorem 5.9) and of a model-theoretic compactness argument.

**Proposition 6.4.** Every closed 0-definable subset \( A \) of \( K^n \) is the zero set \( \mathcal{Z}(d) \) of a continuous 0-definable function \( d \) on \( K^n \).

**Proof.** We proceed by double induction with respect to the dimension \( n \) of the ambient space and the dimension \( k = \dim A \) of the set under study. So assume that the conclusion holds for the ambient spaces of dimension \( < n \) and the closed definable subsets of \( K^n \) of dimension \( < k \) with \( 1 \leq k \leq n \). First consider the case \( k < n \).

It is easy to check an elementary

**Claim 6.5.** If \( A = \bigcup_{i=1}^r A_i \), and the conclusion of the above proposition holds for every \( A_i \), then it holds for \( A \). \( \square \)

For \( x = (x_1, \ldots, x_n) \) write \( x = (y, z) \) with \( y = (x_1, \ldots, x_k) \) and \( z = (x_{k+1}, \ldots, x_n) \). Let \( \pi : K^n_x \to K^k_y \) be the projection onto the first \( k \) coordinates. For \( y \in K^k \), denote by \( A_y \subset K^{n-k}_y \) the fiber of the set \( A \) over the point \( y \).

By the above claim, we can assume that \( A \) is the closure \( \overline{E} \) of a definable subset \( E \) of dimension \( k \) such that the projection \( F := \pi(E) \) is an open subset of \( K^k \), the restriction of \( \pi \) to \( E \) has finite fibers, and that, for every \( y \in F \), the fiber \( E_y \) has the same cardinality, say \( s \), and the set of \( j \)-th coordinates of points from \( E_y \) has the same cardinality, say \( s_j \), for each \( j = k + 1, \ldots, n \).

Denote by \( \partial E := \overline{E} \setminus E \) the frontier of \( E \); then we have

\[
\dim \partial E < \dim E.
\]
Further, consider the polynomials

\[ P_j(y, Z_j) := \prod_{z \in C_j(y)} (Z_j - z) = \prod_{i=1}^{s_j} (Z_j - c_{ji}(y)), \quad j = k + 1, \ldots, n, \]

where \( C_j(y) = \{ c_{ji}(y), \ i = 1, \ldots, s_j \} \), is the set of the \( j \)-th coordinates of points from the fiber \( E_y \). Obviously, we have

\[ P_j(y, Z_j) = Z_j^{s_j} + b_{j,1}(y)Z_j^{s_{j-1}} + \ldots + b_{j,s_j}(y), \quad j = k + 1, \ldots, n, \]

where \( b_{j,i} : F \to K, \ i = 1, \ldots, s_j \), are 0-definable functions.

We still need the following

**Lemma 6.6.** There exist a finite number of linear functions

\[ \lambda_l : K^{n-k} \to K, \ l = 1, \ldots, p, \]

with integer coefficients such that, for every \( y \in F \), \( \lambda_l \) is injective on the product \( \prod_{j=k+1}^{n} C_j \) for some \( l = 1, \ldots, p \).

**Proof.** The conclusion follows by a routine model-theoretic compactness argument. \( \square \)

It follows from Lemma 6.6 and the claim that we can additionally assume that a linear function

\[ \lambda : K^{n-k} \to K \]

with integer coefficients is injective on every product

\[ \prod_{j=k+1}^{n} C_j(y), \ y \in F. \]

Then consider the polynomial

\[ P(y, Z) := \prod_{z \in E_y} (Z - \lambda(z)) = Z^s + b_1(y)Z^{s-1} + \ldots + b_s(y). \]

The sets of all points at which the functions \( b_{j,i}(y) \) and \( b_i(y) \) are not continuous are definable subsets of \( F \) of dimension \( < k \) (cf. [10, Theorem 5.1.1]). By the claim and the induction hypothesis, we can additionally assume that the functions \( b_{j,i}(y) \) and \( b_i(y) \) are continuous. Under the circumstances, \( E \) is a closed subset of \( F \times K_z^{n-k} \), and thus

\[ \partial A = \partial E \subset \partial F \times K_z^{n-k}. \]

Since \( F \) is supposed to be an open subset of \( K_y^k \), \( \partial F \) is a closed subset of \( K_y^k \). By the induction hypothesis, \( \partial F \) is the zero set of a continuous
0-definable function $f : K^k_y \to K$. It follows from Theorem 5.9 that there is a positive integer $r$ such that the functions

$$f^r(y) \cdot b_j(y)$$

extend by zero through $\partial F$ to continuous functions on $\overline{F}$. And even they extend by zero off $F$ to continuous 0-definable functions on $K^k_y$.

We can thus regard the coefficients of the polynomials

$$f^r(y) \cdot p_j(y, Z_j)$$

and

$$f^r(y) \cdot P(y, Z)$$

as continuous 0-definable functions on $K^k_y$ vanishing off the subset $F$. Put

$$G := \{ x \in K^n : P_{k+1}(x_1, \ldots, x_k, x_{k+1}) = \ldots = P_n(x_1, \ldots, x_k, x_n) = P(x_1, \ldots, x_k, \lambda(x_{k+1}, \ldots, x_n)) = 0 \}.$$ 

Then $G \cap (F \times K^{n-k}_z) = E$ and $G \cap ((K^k_y \setminus F) \times K^{n-k}_z) = (K^k_y \setminus F) \times K^{n-k}_z$.

Put

$$E := \{(b, c, z) \in A \times K^{n-d} : b \in F \land \forall y \in \partial F |z| < |y-b|\}, \quad \tilde{E} := p(E),$$

where

$$p : K^k \times K^{n-k} \times K^{n-k} \supset (y, z, w) \mapsto (y, z + w) \in K^k \times K^{n-k},$$

and let $\tilde{A}$ be the closure of $\tilde{E}$.

By the induction hypothesis, $\partial A = \partial E$ is the zero set of a continuous 0-definable function $e : K^n \to K$. It is easy to check that $\partial \tilde{A} = \partial A$ and that $\tilde{E} \supset E$ is a clopen subset of $F \times K^{n-k}_z$. Hence the function

$$\tilde{e}(x) = \left\{ \begin{array}{ll}
0 & \text{if } x \in \tilde{A}, \\
e(x) & \text{if } x \in K^n \setminus \tilde{A}.
\end{array} \right.$$ 

is continuous. Obviously, we get

$$\tilde{A} = \{ x \in K^n : \tilde{e}(x) = 0 \},$$

and hence

$$(6.1) \quad A = \{ P_{k+1}(x_1, \ldots, x_k, x_{k+1}) = \ldots = P_n(x_1, \ldots, x_k, x_n) = P(x_1, \ldots, x_k, \lambda(x_{k+1}, \ldots, x_n)) = \tilde{e}(x) = 0 \} \subset K^n_x.$$ 

The lemma below is elementary.

**Lemma 6.7.** There exists a continuous definable function $g : K^t_w \to K$ such that

$$g(w) = 0 \iff w = 0.$$
Proof. When \( t = 2 \), put
\[
g(w) = \begin{cases} 
  w_1 & \text{if } |w_2| \leq |w_1|, \\
  w_2 & \text{if } |w_1| \leq |w_2|. 
\end{cases}
\]
\( \square \)

Now the conclusion of the proposition follows immediately from Lemma 6.7 and description 6.1 of the subset \( A \).

Finally, suppose that \( A \) is of dimension \( k = n \). Then \( A = U \cup E \) for an open 0-definable subset \( U \subseteq K^n \) and a closed 0-definable subset \( E \subseteq K^n \) of dimension \( < n \). By the induction hypothesis, \( E \) is the zero set of a continuous 0-definable function \( g : K^n \to K \). Then \( A \) is the zero set of the following continuous definable function
\[
f(x) = \begin{cases} 
  0 & \text{if } x \in A, \\
  g(x) & \text{if } x \in K^n \setminus A. 
\end{cases}
\]
This completes the proof of Proposition 6.4. \( \square \)

We now turn to definable LC-spaces. The proof of the following proposition relies on curve selection and the theorem on existence of the limit.

**Proposition 6.8.** Every definable Hausdorff LC-space \( X \) is regular.

*Proof.* Clearly, it suffices to prove the following

**Lemma 6.9.** Consider a definable chart \((U_1, \phi_1), \phi_1 : U_1 \to U'_1\), where \( U'_1 \) is a locally closed subset of \( K^{n_1} \). Let \( V \) be a definable subset of \( U_1 \) such that \( \phi_1(V) \) is a closed bounded subset of \( K^{n_1} \). Then \( V \) is a closed subset of \( X \).

*Proof.* Observe that the set \( V \) in the above lemma will play a role of an auxiliary neighbourhood of a point to be separated from a closed definable subset. Let \( a_0 \in X \) be an accumulation point of \( V \). Suppose \( a_0 \) lies in a chart \((U_2, \phi_2), \phi_2 : U_2 \to U'_2\) where \( U'_2 \) is a locally closed subset of \( K^{n_2} \). Obviously, \( a_0 \) is an accumulation point of \( V \cap U_2 \) too. Then \( \phi_2(a_0) \) is an accumulation point of \( \phi_2(V) \).

By curve selection (Theorem 1.2 ff.), there exists a continuous curve \( c : E \to K^{n_2} \), which is definable in the initial language \( L \) augmented by an angular component map, such that 0 is an accumulation point of \( E \subseteq K \), and
\[
c(E \setminus \{0\}) \subseteq \phi_2(V \cap U_2) \quad \text{and} \quad \lim_{t \to 0} c(t) = \phi_2(a_0).
\]
Then
\[ \phi_{21} \circ c : E \rightarrow K^n \]
and \( (\phi_{21} \circ c)(E \setminus \{0\}) \subset \phi_1(V) \);
here
\[ \phi_{21} : \phi_2(U_1 \cap U_2) \rightarrow \phi_1(U_1 \cap U_2) \]
is the transition map. It follows directly from Theorem 1.1 that
\[ \lim_{t \to 0} (\phi_{21} \circ c)|F(t) =: b_0 \]
for a definable subset \( F \) of \( E \) with 0 as an accumulation point and a point \( b_0 \in \phi_1(V) \). Since \( X \) is Hausdorff, we get
\[ b_0 = (\phi_{21} \circ \phi_2)(a_0) = \phi_1(a_0) \quad \text{and} \quad a_0 = \phi_1^{-1}(b_0) \in V, \]
which is the desired result. \( \square \)

This completes the proof of the proposition, the details being left to the reader. \( \square \)

We can readily strengthen Proposition 6.8, relying essentially on quantifier elimination for ordered abelian groups.

**Theorem 6.10.** Every definable Hausdorff LC-space \( X \) is definably ultranormal.

**Proof.** By Proposition 6.8 and Theorem 6.2, we can assume that \( X \) is a definable locally closed subset of \( K^n \). Let \( A \) and \( B \) be two disjoint closed definable subsets of \( X \). For any \( \beta \in \Gamma = \Gamma_K \), \( \beta > 0 \), put
\[ X_\beta := \{ x \in X : v(x) > -\beta, \forall y \in \partial X \ v(x-y) < \beta \}, \]
\[ A_\beta := A \cap X_\beta, \quad B_\beta := B \cap X_\beta, \]
and
\[ \Lambda := \{(\beta, \gamma) \in \Gamma^2 : \forall x \in A_\beta \forall y \in B_\beta \ \gamma > v(x-y)\}. \]
It is easy to check that \( X_\beta, A_\beta \) and \( B_\beta \) are closed bounded subsets of \( K^n \), and that
\[ \bigcup_{\beta > 0} X_\beta = X. \]
It follows from Proposition 5.4 that every set
\[ A_\beta - B_\beta := \{ a - b \in K^n : a \in A_\beta, \ b \in B_\beta \} \]
is a closed subset of \( K^n \). Therefore, since \( 0 \notin A_\beta - B_\beta \), the fibres \( \{ \gamma \in \Gamma : (\beta, \gamma) \in \Lambda \} \) of \( \Lambda \) over \( \beta \) are non-empty for \( \beta \in \Gamma \) large enough.
Now, it follows from Lemma 4.3 that there is an affine semi-line
\[ L = \{(r\tau + \beta_0, s\tau + \gamma_0) : \tau \in G, \ \tau \geq 0\} \]
with \( r, s \in \mathbb{N} \setminus \{0\} \) and such that \((\infty, \infty)\) is an accumulation point of \( \Lambda \cap L \). Hence
\[
\forall a \in A_\beta \ \forall \beta \ v(a, b) > (s + 1)\beta
\]
for \( \beta \in \Gamma \) large enough; say, for \( \beta > \alpha \). Then the set
\[
U := \bigcup_{\beta > \alpha} (A_\beta + \{x \in K^n : v(x) > (s + 1)\beta\})
\]
is a clopen subset of \( K^n \) such that \( A \subset U \) and \( B \subset K^n \setminus U \), concluding the proof. \( \square \)

**Remark 6.11.** By the additional assumption imposed on the auxiliary sort \( RV \), and thus on the value group \( vK \) too, it is clear that the value \( \alpha \) in the above proof can be taken 0-definable whenever the closed subsets \( A \) and \( B \) are 0-definable. Therefore the subset \( U \) is then 0-definable as well.

A Hausdorff space \( X \) is said to be *definably ultraparacompact* if every finite open definable cover \( \{U_1, \ldots, U_m\} \) can be refined by a partition into a finitely many clopen definable sets; then, of course, there is a clopen definable cover \( \{\Omega_1, \ldots, \Omega_m\} \) such that \( \Omega_i \subset U_i \) for all \( i = 1, \ldots, m \).

It follows easily from Theorem 6.10, by an inductive argument (with respect to the cardinality \( m \) of the open definable cover), the following proposition, which will be applied in Section 9.

**Proposition 6.12.** Every definable Hausdorff LC-space is definably ultraparacompact. \( \square \)

A definable space \( X \) shall be called *definably compact* if the theorem on existence of the limit holds on \( X \); i.e., for every definable curve \( f : E \to X \) on a subset \( E \) of \( K \) with 0 as an accumulation point, there is a definable subset \( F \) of \( E \) with accumulation point 0, and a point \( w \in X \) such that
\[
\lim_{x \to 0} f|F (x) = w.
\]
In view of Theorem 1.11, every definable LC-space \( X \) is locally definably compact. By curve selection (Theorem 1.2 ff.), every definably compact subset \( A \) of \( X \) is a closed subset of \( X \).

**Remark 6.13.** Observe that it is much easier, in comparison to the general case, to prove definable ultranormality and ultraparacompactness for closed bounded subsets of \( K^n \) or, more generally, for definably compact LC-spaces (see e.g. [14], Section 2] or [14], Corollary 2.4]).
7. Extension theorem and definable retractions

Let us recall that the classical Tietze–Urysohn extension theorem says that every continuous (and bounded) real valued map on a closed subset of a normal space $X$ can be extended to a continuous (and bounded) function on $X$. Afterwards the problem of extending maps into metric spaces or locally convex linear spaces was investigated by several mathematicians (e.g. Borsuk, Hausdorff, Dugundji, Arens, Michael).

Ellis [24] established some analogues of their results, concerning the extension of continuous maps defined on closed subsets of zero-dimensional spaces with values in various types of metric spaces. They apply to continuous functions from ultranormal spaces into a complete separable field with non-Archimedean absolute value or from ultraparacompact spaces into an arbitrary complete field with non-Archimedean absolute value; hence, a fortiori, to his analogue of the Tietze–Urysohn theorem from [23] on extending continuous functions from ultranormal spaces into a locally compact field with non-Archimedean absolute value.

In this section we establish a non-Archimedean version of the extension theorem and the existence of definable retractions onto closed definable subsets of definable Hausdorff LC-spaces. Obviously, the latter yields immediately a non-Archimedean definable analogue of the Dugundji theorem on the existence of a linear (and continuous) extender, the problem extensively studied by many specialists (see e.g. [17, 31] for references). We shall state both the results in one theorem because their proofs are similar.

We begin by stating a straightforward generalization of separating closed definable subsets, which follows directly from Theorem 6.10 and Remark 6.11.

**Proposition 7.1.** Consider two closed 0-definable subsets $A$ and $B$ of a definable Hausdorff LC-space $X$. Then there is a closed 0-definable subset $U$ of $X$ such that $U \setminus (A \cap B)$ is a clopen subset of $X \setminus (A \cap B)$, $A \subseteq U$ and $B \setminus A \subseteq X \setminus U$. 

Now we can readily establish the main result.

**Theorem 7.2.** Let $A$ be a closed 0-definable subset of a definable Hausdorff LC-space $X$. Then

1) every continuous 0-definable function $f : A \to K$ can be extended to a continuous 0-definable function $F : X \to K$;

2) there exists a 0-definable retraction $r : X \to A$. 

Proof. As before, we can assume that $X$ is a definable locally closed subset of $K^n$. We proceed by induction with respect to the dimension $k$ of the set $A$. So assume that the conclusion holds for the closed definable subsets of dimension $< k$. We first prove the following

**Claim 7.3.** If $A = \bigcup_{i=1}^r A_i$ and the conclusion of the above theorem holds for every $A_i$, then it holds for $A$.

**Proof.** It is enough to consider the case $r = 2$. Consider the closed 0-definable subset $U$ of $X$ from Proposition 7.1.

For conclusion 1), let $F_1 : X \to K$ be a continuous 0-definable extension of the restriction $f|A_1 : A_1 \to K$ and $g := f - F_1|A : A \to K$; then $g$ vanishes on $A_1$. Let $G : X \to K$ be a continuous 0-definable extension of the restriction $g|A_2 : A_2 \to K$. Then the function

$$F_2(x) = \begin{cases} 0 & \text{if } x \in U, \\ G(x) & \text{if } x \in X \setminus U \end{cases}$$

extends continuously the function $g$ too. Clearly, the function

$$F := F_1 + F_2$$

extends continuously the initial function $f$, as desired.

For conclusion 2), let $r_1 : X \to A_1$ and $r_2 : X \to A_2$ be two 0-definable retractions. Then the map

$$r(x) = \begin{cases} r_1(x) & \text{if } x \in U, \\ r_2(x) & \text{if } x \in X \setminus U. \end{cases}$$

is a 0-definable retraction we are looking for. \[\square\]

Suppose now that $A$ is of dimension $k$. Let $K^n_y = K^k_y \times K^{n-k}_z$ and $\pi : X \to K^k_y$ be the projection onto first $k$ coordinates.

By the above claim, we may impose on the subset $A$ the same conditions as in the proof of Proposition 6.4; in particular, we can assume that $A$ is the closure $\overline{E}$ (in the space $X$) of a 0-definable subset $E$ of dimension $k$ such that the projection $F := \pi(E)$ is an open subset of $K^k$, the restriction of $\pi$ to $E$ has finite fibers, and that, for every $y \in F$, the fiber $E_y$ has the same cardinality, say $s$, and, moreover, that $E$ is a closed subset of $F \times K^{n-k}$.

For conclusion 1), we may assume, by the induction hypothesis, that the function $f : A \to K$ vanishes on the frontier $\partial A = \partial E$ (considered in the space $X$). Then, similarly as in the proofs of Theorem 6.10 and Proposition 7.1, we can find, via a separation argument of Lemma 4.3 (which is based on quantifier elimination for ordered abelian groups) a clopen 0-definable subset $U$ of $F \times K^{n-k}$ such that $U \cup \partial E$ is a closed
subset of \( X \), \( E \subset U \) and that, for every \( y \in F \), the fiber \( U_y \) of \( U \) over \( y \) consists of \( s \) pairwise disjoint balls \( B_i(y) \) in the space \( X \), \( i = 1, \ldots, s \), each of which contains a unique point from the fibre \( E_y \). Then the function
\[
F(y, z) := \begin{cases} 
  f(y, z_i) & \text{if } \exists i \in \{1, \ldots, s\} \ [(y, z_i) \in E \land z, z_i \in B_i(y)], \\
  0 & \text{otherwise for } x = (y, z) \in X.
\end{cases}
\]
is a continuous 0-definable extension of \( f \) we are looking for.

For conclusion 2), we may assume, by the induction hypothesis, that there exists a 0-definable retraction \( p : X \to \partial A \). Then the map
\[
F(y, z) := \begin{cases} 
  (y, z_i) & \text{if } \exists i \in \{1, \ldots, s\} \ [(y, z_i) \in E \land z, z_i \in B_i(y)], \\
  p(x) & \text{otherwise for } x = (y, z) \in X.
\end{cases}
\]
is a 0-definable retraction onto \( A \). This finishes the proof. \( \square \)

8. A non-Archimedean version of Puiseux’s theorem

In this section, we provide a more precise version of Theorem 1.1 on existence of the limit in the case of Henselian fields with analytic structure. We begin by recalling, following the paper \[12\], the concept of a separated analytic structure. The study of analytic structures was initiated by \[18, 22, 21\] and continued thereafter by e.g. \[36, 37, 14, 12, 13\].

Let \( A \) be a commutative ring with unit and with a fixed proper ideal \( I \subsetneq A \); put \( \tilde{A} = A/I \). A separated \((A, I)\)-system is a certain system \( \tilde{A} \) of \( A \)-subalgebras \( A_{m,n} \subset A[\xi, \rho] \), \( m, n \in \mathbb{N} \); here \( A_{0,0} = A \) (op. cit., Section 4.1). Two kinds of variables, \( \xi = (\xi_1, \ldots, \xi_m) \) and \( \rho = (\rho_1, \ldots, \rho_n) \), play different roles. Roughly speaking, the variables \( \xi \) vary over the valuation ring (or the closed unit disc) \( \mathcal{O}_K \) of a valued field \( K \), and the variables \( \rho \) vary over the maximal ideal (or the open unit disc) \( \mathcal{M}_K \) of \( K \).

The \((A, I)\)-system \( \tilde{A} \) is called a separated pre-Weierstrass system if two usual Weierstrass division theorems hold with respect to division by each \( f \in A_{m,n} \) which is \( \xi_m \)-regular or \( \rho_n \)-regular. A pre-Weierstrass system \( \tilde{A} \) is called a separated Weierstrass system if the rings \( C \) of fractions enjoy a certain weak Noetherian property.

Let \( \tilde{A} \) be a separated Weierstrass system and \( K \) a valued field. A separated analytic \( \tilde{A} \)-structure on \( K \) is a collection of homomorphisms \( \sigma_{m,n} \) from \( A_{m,n} \) to the ring of \( \mathcal{O}_K \)-valued functions on \( (\mathcal{O}_K)^m \times (\mathcal{M}_K)^n \), \( m, n \in \mathbb{N} \), such that
1) \( \sigma_{0,0}(I) \subset \mathcal{M}_K \);
2) $\sigma_{m,n}(\xi_i)$ and $\sigma_{m,n}(\rho_j)$ are the $i$-th and $(m+j)$-th coordinate functions on $(\mathcal{O}_K)^m \times (\mathcal{M}_K)^n$, respectively;

3) $\sigma_{m+1,n}$ and $\sigma_{m,n+1}$ extend $\sigma_{m,n}$, where functions on the product $(\mathcal{O}_K)^m \times (\mathcal{M}_K)^n$ are identified with those functions on $(\mathcal{O}_K)^{m+1} \times (\mathcal{M}_K)^n$ or $(\mathcal{O}_K)^m \times (\mathcal{M}_K)^{n+1}$ which do not depend on the coordinate $\xi_{m+1}$ or $\rho_{n+1}$, respectively.

If the ground field $K$ is trivially valued, then $\mathcal{M}_K = (0)$ and the analytic structure reduces to the algebraic structure given by polynomials. A separated analytic $\mathcal{A}$-structure on a valued field $K$ can be uniquely extended to any algebraic extension $K'$ of $K$ (op. cit., Theorem 4.5.11). Every valued field with separated analytic structure is Henselian (op. cit., Proposition 4.5.10).

One may assume without loss of generality that $\ker \sigma_{0,0} = (0)$, as replacing $A$ by $A/\ker \sigma_{0,0}$ yields an equivalent analytic structure on $K$ with this property. Then $A = A_{0,0}$ can be regarded as a subring of $\mathcal{O}_K$.

From now on, we shall always assume that the ground field $K$ is non-trivially valued and that $\sigma_{0,0}$ is injective. Under these conditions, for any subfield $F$ of $K$ containing $A$, one can canonically obtain, by extension of parameters, a (unique) separated Weierstrass system $\mathcal{A}(F)$ over $(\mathcal{O}_F, \mathcal{M}_F)$ so that $K$ has separated analytic $\mathcal{A}(F)$-structure (op. cit., Theorem 4.5.7). In particular, $K$ has a (unique) separated analytic $\mathcal{A}(K)$-structure. This technique works for strictly convergent Weierstrass systems as well.

The analytic language $\mathcal{L} = \mathcal{L}_{\text{hen}, \mathcal{A}}$ is the semialgebraic language $\mathcal{L}_{\text{hen}}$ augmented on the valued field sort $K$ by the reciprocal function $1/x$ (with $1/0 := 0$) and the names of all functions of the system $\mathcal{A}$, together with the induced language on the auxiliary sort $RV$ (op. cit., Section 6.2). A power series $f \in A_{m,n}$ is construed via the analytic $\mathcal{A}$-structure on their natural domains and as zero outside them. More precisely, $f$ is interpreted as a function

$$\sigma(f) = f^\sigma : (\mathcal{O}_K)^m \times (\mathcal{M}_K)^n \rightarrow \mathcal{O}_K,$$

extended by zero on $K^{m+n} \setminus (\mathcal{O}_K)^m \times (\mathcal{M}_K)^n$.

In the equicharacteristic case, the induced language on the sort $RV$ coincides with the algebraic language $\mathcal{L}_{\text{rv}}$.

Denote by $\mathcal{L}^*$ the analytic language $\mathcal{L}$ augmented by all Henselian functions

$$h_m : K^{m+1} \times RV(K) \rightarrow K, \quad m \in \mathbb{N},$$
which are defined by means of a version of Hensel’s lemma (cf. [12], Section 6). Let $T_{\text{hen},A}$ be the $L$-theory of all Henselian valued fields of equicharacteristic zero with separated analytic $A$-structure. Two crucial results about analytic structures are Theorems 6.3.7 and 6.3.8 from [12], recalled below.

**Theorem 8.1.** The theory $T_{\text{hen},A}$ eliminates valued field quantifiers, is ball-minimal with centers and preserves all balls. Moreover, $T_{\text{hen},A}$ has the Jacobian property.

**Theorem 8.2.** Let $K$ be a Henselian field with separated analytic $A$-structure. Let $f : X \to K$, $X \subset K^n$, be an $L_B$-definable function for some set of parameters $B$. Then there exist an $L_B$-definable function $g : X \to S$ with $S$ auxiliary and an $L^*_B$-term $t$ such that

$$f(x) = t(x, g(x)) \text{ for all } x \in X.$$  

By Theorem 8.1, the theory $T_{\text{hen},A}$ admits reparametrized cell decompositions with centers (cf. [13]). Now we can readily prove the following more precise, analytic version of the theorem on existence of the limit, which may be regarded as a kind of Puiseux’s theorem.

**Theorem 8.3.** Let $f : E \to K$ be an 0-definable function on a subset $E$ of $K$. Suppose that 0 is an accumulation point of $E$. Then there is a subset $F$ of $E$, definable over algebraic closure of $0$, with accumulation point 0, and a point $w \in \mathbb{P}^1(K)$ such that

$$\lim_{x \to 0} f|_{F}(x) = w.$$  

Moreover, we can require that

$$\{(x, f(x)) : x \in F\} \subset \{(x^r, \phi(x)) : x \in G\},$$

where $r$ is a positive integer and $\phi$ is a definable function on a subset $G$ of $K$, being a composite of some functions induced by series from $A$ and of some algebraic power series over $K$ (coming from the implicit function theorem). Then, in particular, the definable set

$$\{(v(x), v(f(x))) : x \in F \setminus \{0\}\} \subset \Gamma \times (\Gamma \cup \{\infty\})$$

is contained either in an affine line with rational slope

$$\{(k, l) \in \Gamma \times \Gamma : q \cdot l = p \cdot k + \beta\}$$

with $p, q \in \mathbb{Z}$, $q > 0$, $\beta \in \Gamma$, or in $\Gamma \times \{\infty\}$. 

Proof. The proof relies on term structure (Theorem 8.2), which enables induction with respect to the complexity of a term \( t \) corresponding to the function \( f \), on ball-minimality (Theorem 8.1) and on Lemma 4.2.

By Remarks 1.5 and 2.3, we can as before assume that the field \( K \) has a coefficient map, exact sequence 2.1 splits and the residue field is orthogonal to the value group.

Therefore, after shrinking \( E \), we can assume that \( \text{ac}(E) = \{1\} \) and the function \( g \) goes into \( \{\xi\} \times \Gamma^s \) with a \( \xi \in \tilde{K}^s \). Without loss of generality, we may assume that \( \xi = (1, \ldots, 1) \); similar reductions were considered in our papers [40, 42]. For simplicity, we look at \( g \) as a function into \( \Gamma^s \). We shall briefly explain the most difficult case where

\[
 t(x, g(x)) = h_m(a_0(x), \ldots, a_m(x), (1, g_0(x))),
\]

assuming that the theorem holds for the terms \( a_0, \ldots, a_m \); here \( g_0 \) is one of the components of \( g \).

By our assumption, each function \( a_i(x) \) has, after taking a suitable subset \( F \) of \( E \), a limit, say, \( a_i(0) \in K \) when \( x \) tends to zero. Due to Lemma 4.2, we can assume that

\[
 (8.1) \quad pv(x) + qg_0(x) + v(a) = 0
\]

for some \( p, q \in \mathbb{Z}, q > 0, \) and \( a \in K \setminus \{0\} \). By the induction hypothesis, we get

\[
 \{(x, a_i(x)) : x \in F\} \subset \{(x^r, \alpha_i(x)) : x \in G\}, \quad i = 0, 1, \ldots, m,
\]

for some power series \( \alpha_i(x) \) as stated in the theorem. Put

\[
 P(x, T) := \sum_{i=0}^{m} a_i(x)T^i.
\]

By the very definition of \( h_m \) and since we can take a smaller subset \( F \) of \( E \) with accumulation point 0, we may assume that there is an \( i_0 = 0, \ldots, m \) such that

\[
 \forall x \in F \exists u \in K \quad v(u) = g_0(x), \quad \text{ac} u = 1,
\]

and the following formulae hold

\[
 (8.2) \quad v(a_{i_0}(x)u^{i_0}) = \min \{v(a_i(x)u^i), \quad i = 0, \ldots, m\},
\]

\[
 v(P(x, u)) > v(a_{i_0}(x)u^{i_0}), \quad v \left( \frac{\partial P}{\partial T}(x, u) \right) = v(a_{i_0}(x)u^{i_0-1}).
\]

Then \( h_m(a_0(x), \ldots, a_m(x), (1, g_0(x))) \) is a unique \( b(x) \in K \) such that

\[
 P(x, b(x)) = 0, \quad v(b(x)) = g_0(x), \quad \text{ac} b(x) = 1.
\]
Via quantifier elimination for ordered abelian groups, we see in view of [12, Remarks 7.2, 7.3] that the set $F$ contains the set of points of the form $c^r t^{Nqr}$ for some $c \in K$ with $\overline{ac} c = 1$, a positive integer $N$ and all $t \in \mathcal{O}_K$ small enough with $\overline{ac} t = 1$. Hence and by equation 8.1, we get

$$g_0(c^r t^{Nqr}) = g_0(c^r) - v(t^{Npr}).$$

Take $d \in K$ such that $g_0(c^r) = v(d)$ and $\overline{ac} d = 1$. Then

$$g_0(c^r t^{Nqr}) = v(dt^{-Npr}).$$

Thus the homothetic change of variable

$$Z = T/dt^{-Npr} = t^{Npr} T/d$$

transforms the polynomial

$$P(c^r t^{Nqr}, T) = \sum_{i=0}^m \alpha_i (ct^{Nq}) T^i$$

into a polynomial $Q(t, Z)$ to which Hensel’s lemma applies (cf. [10, Lemma 3.5]):

$$P(c^r t^{Nqr}, T) = P(c^r t^{Nqr}, dt^{-Npr} Z) = \sum_{i=0}^m \alpha_i (ct^{Nq}) \cdot (dt^{-Npr} Z)^i = (\alpha_{i_0} (ct^{Nq}) \cdot (dt^{-Npr})^{i_0} \cdot Q(t, Z)).$$

Indeed, formulae 8.2 imply that the coefficients $b_i(t)$, $i = 0, \ldots, m$, of the polynomial $Q$ are power series (of order $\geq 0$) in the variable $t$, and that

$$v(Q(t, 1)) > 0 \quad \text{and} \quad v\left(\frac{\partial Q}{\partial Z}(t, 1)\right) = 0$$

for $t \in K^0$ small enough. Hence

$$v(Q(0, 1)) > 0 \quad \text{and} \quad v\left(\frac{\partial Q}{\partial Z}(0, 1)\right) = 0.$$ 

But, for $x(t) = c^r t^{Nqr}$, the unique zero $T(t) = b(x(t))$ of the polynomial $P(x(t), T)$ such that

$$v(b(x(t))) = v(dt^{-Npr}) \quad \text{and} \quad \overline{ac} b(x(t)) = 1$$

corresponds to a unique zero $Z(t)$ of the polynomial $Q(t, Z)$ such that

$$v(Z(t)) = v(1) \quad \text{and} \quad \overline{ac} Z(t) = 1.$$
Therefore the conclusion of the theorem can be directly obtained via the implicit function theorem (see e.g. [12, Proposition 2.5]) applied to the polynomial

\[ P(A_0, \ldots, A_m, Z) = \sum_{i=0}^{m} A_i Z^i \]

in the variables \( A_i \) substituted for \( a_i(x) \) at the point

\[ A_0 = b_0(0), \ldots, A_m = b_m(0), Z = 1. \]

\[ \square \]

9. **Definable resolution of singularities and transformation to normal crossings in Hensel minimal expansions of analytic structures**

This section is concerned with strong analytic functions in Hensel minimal expansions of analytic structures, introduced in the case of Henselian fields with analytic structure in our earlier article [41]. Strong analyticity, being a model-theoretic strengthening of the weak non-Archimedean concept of analyticity (treated in the classical case e.g., by Serre [49]), works well within definable settings and does not appeal to Grothendieck topologies. Strong analytic category enables application of a model-theoretic compactness argument in the absence of the ordinary topological compactness.

All the results of that earlier article [41], including a definable version of the desingularization algorithm (hypersurface case) of Bierstone–Milman, carry over almost verbatim to Hensel minimal settings. We shall state its strengthened version, dealing with general strong analytic ambient manifolds \( M \) in the Hensel minimal settings, and not only with the definably compact ones. Under the circumstances, the only change in the proof is that one must appeal to definable ultranormality and ultraparacompactness (Theorem 6.10 and Proposition 6.12) achieved for general manifolds \( M \) (cf. Remark 5.13).

Now we fix an \( \mathcal{L} \)-expansion of a Henselian valued field \( K \) with separated analytic structure that is a model of a 1-h-minimal \( \mathcal{L} \)-theory, where \( \mathcal{L} \) is an expansion of the language \( \mathcal{L}_{\text{hen},A} \). Then every definable subset in the imaginary sort \( RV \) is already definable in the standard algebraic language \( \mathcal{L}_{rv} \) for \( RV \).

A *definable analytic manifold \( M \) of dimension \( n \)* is a definable manifold of dimension \( n \) with a finite definable analytic atlas. By strong analytic functions and manifolds, we mean the analytic ones that are
definable in the structure $K$ and remain analytic in each field $L$ elementarily equivalent to $K$ in the language $L_K$. Examples of such functions and manifolds are those obtained by means of the implicit function theorem and the zero loci of strong analytic submersions.

As explained above, we state the following two strengthened versions of the results from our paper [41, Theorems 1 and 2]: on definable resolution of singularities (hypersurface case) and on transformation to normal crossings by blowing up. Note that blowups $\sigma: \tilde{M} \to M$ with smooth (definable) centers are *definably proper* maps, i.e. if $E \subset M$ is a definably compact subset, so is its pre-image $\sigma^{-1}(E)$.

First, let $M = M_0$ be an arbitrary strong analytic manifold, $g$ a strong analytic function on $M$ and $X = X_0 = Z(g)$ be the hypersurface of $M$ determined by $g$. Further, consider a sequence of admissible blowups $\sigma_j: M_j \to M_{j-1}$ along admissible (definable) smooth centers $C_{j-1}, j = 1, 2, \ldots$; let $E_j$ denote the set of exceptional hypersurfaces in $M_j$. Let $X_1, X_2, X_3, \ldots$ denote the successive transforms of $X = X_0$; here the strict and weak transforms coincide. *Admissible* means that $C_j$ and $E_j$ simultaneously have only normal crossings and that the local invariant for desingularization $\text{inv}_{X}(\cdot)$ is locally constant on $C_j$ for all $j$.

**Theorem 9.1.** Under the above assumptions, there exists a finite sequence of blowups with smooth admissible centers $C_j$ such that:

1. for each $j$, either $C_j \subset \text{Sing} X_j$ or $X_j$ is smooth and $C_j \subset X_j \cap E_j$;
2. the final transform $X'$ of $X$ is smooth (unless empty), and $X'$ and the final exceptional hypersurface $E'$ simultaneously have only normal crossings.

\[ \square \]

Next, let $\mathcal{I} = \mathcal{I}_0 \subset \mathcal{O}_{M_0}$ be a sheaf of ideals (in the sheaf $\mathcal{O}_{M_0}$ of strong analytic function germs on $M_0$) generated by a finite number of strong analytic functions $f_1, \ldots, f_s$ on $M_0$. Transforming the sheaf $\mathcal{I} = \mathcal{I}_0$ to normal crossings is a process which applies the successive weak transforms $\mathcal{I}_j$ of the ideal $\mathcal{I}$ when blowing up the maximal strata of the desingularization invariant.

**Theorem 9.2.** Under the above assumptions, there exists a finite sequence of blowups with smooth admissible centers $C_j$ such that the final transform of $\mathcal{I}$ is $\mathcal{I}_k = \mathcal{O}_{M_k}$ and the pull-back

\[ \sigma^{-1}(\mathcal{I}) \cdot \mathcal{O}_{M_k} = E_k \]

of $\mathcal{I}$ is a normal crossing divisor; here $\sigma$ is the composite of the $\sigma_j$.

\[ \square \]

Hence it is not difficult to obtain
Corollary 9.3. Let $X = \mathcal{Z}(f_1, \ldots, f_s)$ be a strong analytic subset of $M = M_0$ of dimension $k$ determined by the functions $f_1, \ldots, f_s$. Then there exist a strong analytic manifold $N$ of dimension $k$ and a definably proper, strong analytic map $\phi : N \rightarrow M$ such that $X = \phi(N)$. □

Observe now that the axiom of global Weierstrass division imposed on a Weierstrass system (both separated and strictly convergent) entails ordinary Weierstrass division and preparation for local rings of analytic function germs induced by that Weierstrass system. We shall use this fact in what follows.

We call a continuous definable function $f : M \rightarrow K$ meromorphic if it is a quotient $h_1/h_2$ off $\mathcal{Z}(h_2)$ of two strong analytic functions $h_1, h_2 : M \rightarrow K$ provided that the zero set $\mathcal{Z}(h_2)$ of $h_2$ is nowhere dense in $M$.

When the ground field $K$ is algebraically closed, every meromorphic function $f : M \rightarrow K$ is just strong analytic. Indeed, by virtue of Weierstrass preparation for the local rings of strong (definable) analytic function germs at points $a \in M$, those local rings possess good algebro-geometric properties. In particular, they are factorial (but also regular and excellent) local rings, and fulfil Rückert’s classic descriptive lemma and Nullstellensatz. The proof of the complex case of the last two results (cf. [38, Chapter III, §3 and §4]) can be repeated here in the non-Archimedean settings. Hence the assertion in question, being local, follows immediately.

Finally, we prove a finitary meromorphic version of the Nullstellensatz.

Proposition 9.4. Consider meromorphic functions $f_1, \ldots, f_s, g : M \rightarrow K$ on a strong analytic manifold $M$. If $g$ vanishes on the zero set $Z = \mathcal{Z}(f_1, \ldots, f_s)$, then $g$ belongs to the radical $\text{Rad}(f_1, \ldots, f_s)$ of the ideal $(f_1, \ldots, f_s)$.

Proof. When $K$ is an algebraically closed field, the conclusion follows directly from the local version of the Nullstellensatz via a routine model-theoretic compactness argument.

So assume that $K$ is not an algebraically closed field. Since there is a polynomial $G_s(x_1, \ldots, x_s)$ in $s$ indeterminates whose only zero on $K^s$ is $(0, \ldots, 0)$ (cf. [11, Lemma 10.1] or the proof of [33, Lemma 15]), we get

$$\mathcal{Z}(f_1, \ldots, f_s) = \mathcal{Z}(f)$$
for a meromorphous function $f \in (f_1, \ldots, f_s)$. Therefore we can assume that $s = 1$, and thus $g$ vanishes on the zero set $\mathcal{Z}(f)$. Then $1/f$ is a definable function on the set

$$\mathcal{D}(f) := \{ x \in M : f(x) \neq 0 \} = M \setminus \mathcal{Z}(f).$$

Now the conclusion follows immediately from Theorems 5.9 and 6.2 and from Proposition 6.8.

Finally, let us mention that, in our recent paper [45], we establish a theorem on extension of Lipschitz maps definable in Hensel minimal structures of equicharacteristic zero. It may be regarded as a definable, non-Archimedean, non-locally compact version of Kirszbraun’s theorem.

To our best knowledge, the only definable, non-archimedian version of Kirszbraun’s theorem was achieved by Cluckers–Martin [16] in the $p$-adic, thus locally compact case; more precisely, for Lipschitz extension of functions which are semi-algebraic, subanalytic or definable in an analytic structure on a finite extension of the field $\mathbb{Q}_p$ of $p$-adic numbers. The easier case of Lipschitz extension of definable $p$-adic functions on the line $\mathbb{Q}_p$ was treated in [35].

References

[1] M. Aschenbrenner, A. Chernikov, A. Gehret, M. Ziegler, Distality in valued fields and related structures, arXiv:2008.09889 [math.LO] (2020).
[2] J. Ax, S. Kochen, Diophantin problems over local fields: I, II, Amer. J. Math. 87 (1965), 605–648.
[3] J. Ax, S. Kochen, Diophantin problems over local fields, III, Ann. Math. 83 (1966), 437–456.
[4] S.A. Basarab, Relative elimination of quantifiers for Henselian valued fields, Ann. Pure Appl. Logic 53 (1991), 51–74.
[5] S.A. Basarab, F.-V. Kuhlmann, An isomorphism theorem for Henselian algebraic extensions of valued fields, Manuscripta Math. 77 (1992), 113–126.
[6] E. Bierstone, P.D Milman, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, Inventiones Math. 128 (1997), 207–302.
[7] R. Cluckers, G. Comte, F. Loeser, Non-Archimedean Yomdin–Gromov parametrizations and points of bounded height, Forum Math. Pi 3 (2015), e5.
[8] R. Cluckers, A. Forey, F. Loeser, Uniform Yomdin–Gromov parametrizations and points of bounded height in valued fields, Algebra Number Theory 14 (2020), 1423–1456.
[9] R. Cluckers, I. Halupczok, Quantifier elimination in ordered abelian groups, Confluentes Math. 3 (4) (2011), 587–615.
[10] R. Cluckers, I. Halupczok, S. Rideau, *Hensel minimality I*, arXiv:1909.13792 [math.LO] (2019).
[11] R. Cluckers, I. Halupczok, S. Rideau, *Hensel minimality II: mixed characteristic and a Diophantine application*, arXiv:2104.09475 [math.LO] (2021).
[12] R. Cluckers, L. Lipshitz, *Fields with analytic structure*, J. Eur. Math. Soc. 13 (2011), 1147–1223.
[13] R. Cluckers, L. Lipshitz, *Strictly convergent analytic structures*, J. Eur. Math. Soc. 19 (2017), 107–149.
[14] R. Cluckers, L. Lipshitz, Z. Robinson, *Analytic cell decomposition and analytic motivic integration*, Ann. Sci. École Norm. Sup. (4) 39 (2006), 535–568.
[15] R. Cluckers, F. Loeser, *b-minimality*, J. Math. Logic 7 (2) (2007), 195–227.
[16] R. Cluckers, F. Martin, *A definable p-adic analogue of Kirszbraun’s theorem on extension of Lipschitz maps*, J. Inst. Math. Jussieu 17 (2018), 39–57.
[17] E.K. van Douwen, D.J. Lutzer, T.C. Przymusiński, *Some extensions of the Tietze–Urysohn theorem*, Amer. Math. Mon. 84 (1977), 435–441.
[18] J. Denef, L. van den Dries, *p-adic and real subanalytic sets*, Ann. Math. 128 (1988), 79–138.
[19] L. van den Dries, *Analytic Ax–Kochen–Ershov theorems*. In: Contemp. Math. 131, AMS (1992), 379–392.
[20] L. van den Dries, *Tame Topology and O-minimal Structures*, Cambridge Univ. Press, 1998.
[21] L. van den Dries, D. Haskell, D. Macpherson, *One dimensional p-adic subanalytic sets*, J. London Math. Soc. 56 (1999), 1–20.
[22] L. van den Dries, A. Macintyre, D. Marker, *The elementary theory of restricted analytic fields with exponentiation*, Ann. Math. 140 (1994), 183–205.
[23] R.L. Ellis, *A non-Archimedean analogue of the Tietze-Urysohn extension theorem*, Indag. Math. 29 (1967), 332–333.
[24] R.L. Ellis, *Extending continuous functions on zero-dimensional spaces*, Math. Ann. 186 (1970), 114–122.
[25] Yu. Ershov, *On the elementary theory of maximal normed fields*, Dokl. Akad. Nauk SSSR 165 (1965), 21–23.
[26] J. Flenner, *Relative decidability and definability in Henselian valued fields*, J. Symbolic Logic 76 (2011), 1240–1260.
[27] Y. Halevi, A. Hassan, *Eliminating valued field quantifiers in strongly dependent Henselian fields*, Proc. AMS 147 (2019), 2213–2230.
[28] D. Haskell, D. Macpherson, *Cell decomposition of C-minimal structures*, Ann. Pure Appl. Logic 66 (1994), 113–162.
[29] D. Haskell, D. Macpherson, *A version of o-minimality for the p-adics*, J. Symbolic Logic 62 (1997), 1075–1092.
[30] E. Hrushovski, D. Kazhdan, *Integration in valued fields*. In: Algebraic Geometry and Number Theory, Progr. Math. 253, pp. 261—405. Birkhäuser, Boston, MMA (2006).
[31] J. Kąkol, A. Kubzdela, W. Śliwa, *A non-Archimedean Dugundji extension theorem*, Czechoslovak Math. J. 63 (138) (2013), 157–164.
[32] J. Kollár, K. Nowak, *Continuous rational functions on real and p-adic varieties*, Math. Zeitschrift 279 (2015), 85–97.
[33] F.-V. Kuhlmann, *Quantifier elimination for Henselian fields relative to additive and multiplicative congruences*, Israel J. Math. 85 (1994), 277–306.
[34] F.-V. Kuhlmann, The algebra and model theory of tame valued fields, J. Reine Angew. Math. 719 (2016), 1–43.
[35] T. Kuijpers, Lipschitz extension of definable p-adic functions, Math. Log. Q. 61 (2015), 151–158.
[36] L. Lipshitz, Z. Robinson, Rings of Separated Power Series and Quasi-Affinoid Geometry, Astérisque 264 (2000).
[37] L. Lipshitz, Z. Robinson, Uniform properties of rigid subanalytic sets, Trans. Amer. Math. Soc. 357 (2005), 4349–4377.
[38] S. Lojasiewicz, Introduction to Complex Analytic Geometry, Birkhäuser, Basel, Boston, Berlin, 1991.
[39] D. Macpherson, C. Steinhorn, On variants of o-minimality, Ann. Pure Appl. Logic 79 (1996), 165–209.
[40] K.J. Nowak, Some results of algebraic geometry over Henselian rank one valued fields, Sel. Math. New Ser. 23 (2017), 455–495.
[41] K.J. Nowak, Definable transformation to normal crossings over Henselian fields with separated analytic structure, Symmetry 11 (7) (2019), 934.
[42] K.J. Nowak, A closedness theorem and applications in geometry of rational points over Henselian valued fields, J. Singul. 21 (2020), 212–233.
[43] K.J. Nowak, A closedness theorem over Henselian fields with analytic structure and its applications. In: Algebra, Logic and Number Theory, Banach Center Publ. 121, Polish Acad. Sci. (2020), 141–149.
[44] K.J. Nowak, Definable retractions and a non-Archimedean Tietze–Urysohn theorem over Henselian valued fields, arXiv:1808.09782 [math.AG] (2018).
[45] K.J. Nowak, Extension of Lipschitz maps definable in Hensel minimal structures, arXiv:2204.05900 [math.LO] (2022).
[46] J. Pas, Uniform p-adic cell decomposition and local zeta functions, J. Reine Angew. Math. 399 (1989), 137–172.
[47] R. Robson, Embedding semialgebraic spaces, Math. Zeitschrift 183 (1983), 365–370.
[48] T. Scanlon, Quantifier elimination for the relative Frobenius. In: Valuation Theory and Its Applications, Fields Inst. Commun., vol. II, pp. 323–352. Amer. Math. Soc., Providence, RI, 2003.
[49] J-P. Serre, Lie Algebras and Lie Groups, Lect. Notes in Math., vol. 1500, Springer, 2006.

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