Weyl equation for temperature fields induced by attosecond laser pulses

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Abstract

In this paper the Weyl equation for temperature field induced by laser beam interaction with matter is proposed and solved. Depending on the scattering mechanism the temperature field oscillate or is damped.

**Key words**: Thermal process, Weyl equation. fields.
1 Derivation of the 1+1 dimensional Dirac and Weyl equations for thermal processes

As pointed in papers \[1, 2\] spin-flip occurs only when there is more than one dimension in space. Repeating the discussion of deriving the Dirac equation \[3\] for the case of one spatial dimension, one easily finds that the Dirac matrices $\alpha$ and $\beta$ are reduced to $2 \times 2$ matrices that can be represented by the Pauli matrices \[3\]. This fact simply implies that if there is only one spatial dimension, there is no spin. It should be instructive to show explicitly how to derive the 1+1 dimensional Dirac equation.

As discussed in textbooks \[2, 3\], a wave equation that satisfies relativistic covariance in space-time as well as the probabilistic interpretation should have the form:

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \left[ c\alpha \left( -i\hbar \frac{\partial}{\partial x} \right) + \beta m_0 c^2 \right] \Psi(x, t). \quad (1)$$

To obtain the relativistic energy-momentum relation $E^2 = (pc)^2 + m_0^2 c^4$ we postulate that (1) coincides with the Klein-Gordon equation

$$\left[ \frac{\partial^2}{\partial (ct)^2} - \frac{\partial^2}{\partial x^2} + \left( \frac{m_0 c}{\hbar} \right)^2 \right] \Psi(x, t) = 0. \quad (2)$$

By comparing (1) and (2) it is easily seen that $\alpha$ and $\beta$ must satisfy

$$\alpha^2 - \beta^2 = 1, \quad \alpha\beta + \beta\alpha = 0. \quad (3)$$

Any two of the Pauli matrices can satisfy these relations. Therefore, we may choose $\alpha = \sigma_x$ and $\beta = \sigma_z$ and we obtain:

$$i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \left[ c\sigma_x \left( -i\hbar \frac{\partial}{\partial x} \right) + \sigma_z m_0 c^2 \right] \Psi(x, t), \quad (4)$$

where $\Psi(x, t)$ is a 2-component spinor.

The Eq. (4) is the Weyl equation. We perform a phase transformation on $\Psi(x, t)$ letting $u(x, t) = \exp \left( \frac{im_0 c^2 t}{\hbar} \right) \Psi(x, t)$. Call $u$’s upper (respectively, lower) component $u_+ (x, t)$, $u_- (x, t)$; it follows from (4) that $u_\pm$ satisfies

$$\frac{\partial u_\pm(x, t)}{\partial t} = \mp c \frac{\partial u_\pm}{\partial x} + \frac{im_0 c^2}{\hbar} (u_- - u_+). \quad (5)$$
Following the physical interpretation of the equation (5) it describes the relativistic particle (mass \( m_0 \)) propagates at the speed of light \( c \) and with a certain chirality (like a two component neutrino) except that at random times it flips both direction of propagation (by 180°) and chirality.

In monograph \[4\] we considered a particle moving on the line with fixed speed \( w \) and supposed that from time to time it suffers a complete reversal of direction, \( u(x, t) \leftrightarrow v(x, t) \), where \( u(x, t) \) denotes the expected density of particles at \( x \) and at time \( t \) moving to the right, and \( v(x, t) \equiv \) expected density of particles at \( x \) and at time \( t \) moving to the left. In the following we perform the change of the abbreviation

\[
\begin{align*}
u(x, t) & \rightarrow u_+, \\
v(x, t) & \rightarrow u_-
\end{align*}
\] (6)

Following the results of the paper \[4\] we obtain for the \( u_\pm (x, t) \) the following equations

\[
\begin{align*}
\frac{\partial u_+}{\partial t} & = -w \frac{\partial u_+}{\partial x} - \frac{w}{\lambda} ((1 - k) u_+ - ku_-), \\
\frac{\partial u_-}{\partial t} & = w \frac{\partial u_-}{\partial x} + \frac{w}{\lambda} (ku_+ + (k - 1) u_-).
\end{align*}
\] (7)

In equation (7) \( k_x \) denotes the number of the particles which are moving in left (right) direction after the scattering at \( x \). The mean free path for scattering is equal \( \lambda, \lambda = w\tau \), where \( \tau \) is the relaxation time for scattering.

Comparing formulae (5) and (7) we conclude that the shapes of both equations are the same. In the subsequent we will call the set of the equations (7) the Weyl equation for the particles with velocity \( w \), mean free path \( \lambda \). For thermal processes we define \( T_{\pm} \equiv \) the temperature of the particles with chiralities + and – respectively and with analogy to equation (7) we obtain:

\[
\begin{align*}
\frac{\partial T_+}{\partial t} & = -w \frac{\partial T_+}{\partial x} - \frac{w}{\lambda} ((1 - k) T_+ - kT_-), \\
\frac{\partial T_-}{\partial t} & = w \frac{\partial T_-}{\partial x} + \frac{w}{\lambda} (kT_+ + (k - 1) T_-),
\end{align*}
\] (8)

where \( \frac{w}{\lambda} = \frac{1}{\tau} \).

In one dimensional case we introduce one dimensional cross section for scattering

\[
\sigma(x, t) = \frac{1}{\lambda(x, t)}
\] (9)
2 The solution of the Weyl equation for stationary temperatures in one dimensional wire

In the stationary state thermal transport phenomena \( \frac{\partial T}{\partial t} = 0 \) and Eq. (8) can be written as

\[
\begin{align*}
\frac{dT_+}{dx} &= -\sigma ((1 - k) T_+ + k T_-), \\
\frac{dT_-}{dx} &= \sigma (k - 1) T_- + \sigma k T_+.
\end{align*}
\]

(10)

After the differentiation of the equation (9) we obtain for \( T_+ (x) \)

\[
\frac{d^2 T_+}{dx^2} - \frac{1}{\sigma k} \frac{d}{dx} (\sigma k) \frac{dT_+}{dx} + T_+ \left[ \sigma^2 (2k - 1) + \frac{d\sigma}{dx} (1 - k) + \frac{\sigma (k - 1)}{\sigma k} \frac{d(\sigma k)}{dx} \right] = 0.
\]

Equation (11) can be written in a compact form

\[
\frac{d^2 T_+}{dx^2} + f (x) \frac{dT_+}{dx} + g (x) T_+ = 0,
\]

where

\[
\begin{align*}
f (x) &= -\frac{1}{\sigma} \left( \frac{\sigma dk}{k dx} + \frac{d\sigma}{dx} \right), \\
g (x) &= \sigma^2 (x) (2k - 1) - \frac{\sigma dk}{k dx}.
\end{align*}
\]

(11)

In the case for constant \( \frac{dk}{dx} = 0 \) we obtain

\[
\begin{align*}
f (x) &= \frac{1}{\sigma} \frac{d\sigma}{dx}, \\
g (x) &= \sigma^2 (x) (2k - 1).
\end{align*}
\]

(12)

With functions \( f (x), g (x) \) described by formula (12) the general solution of Eq. (12) has the form:

\[
T_+ (x) = C_1 e^{(1 - 2k) \frac{x}{\sigma}} \int f (x) dx + C_2 e^{-(1 - 2k) \frac{x}{\sigma}} \int f (x) dx
\]

(13)
and

\[ T_-(x) = \frac{\left[ (1 - k) + (1 - 2k)^{\frac{1}{2}} \right]}{k} \times \]

\[ = C_1 e^{(1 - 2k)^{\frac{1}{2}} \int \sigma(x) dx} + \frac{(1 - k) - (1 - 2k)^{\frac{1}{2}}}{(1 - k) + (1 - 2k)^{\frac{1}{2}}} C_2 e^{-(1 - 2k)^{\frac{1}{2}} \int \sigma(x) dx} \].

The formulae (13) and (14) describe three different modes for heat transport. For \( k = \frac{1}{2} \) we obtain \( T_+(x) = T_- (x) \) while for \( k > \frac{1}{2} \), i.e. for heat carrier generation \( T_+(x) \) and \( T_- (x) \) oscillate for \( (1 - 2k)^{\frac{1}{2}} \) is a complex number. For \( k < \frac{1}{2} \) i.e. for absorption \( T_+(x) \) and \( T_- (x) \) decrease as the function of \( x \).

In the subsequent we will consider the solution of Eq. (9) for Cauchy conditions:

\[ T_+(0) = T_0, \quad T_-(a) = 0. \] (15)

Boundary conditions (15) describes the generation of heat carriers by illuminating the left end of one dimensional slab (with length \( a \)) by laser pulse. From formulae (13) and (14) we obtain:

\[ T_+(x) = \frac{2T_0 e^{[f(0) - f(a)]}}{1 + \beta e^{2[f(0) - f(a)]}} \times \left[ (k - 1) + (1 - 2k)^{\frac{1}{2}} \right] \sinh [f(x) - f(a)] \]

\[ \left[ (1 - 2k)^{\frac{1}{2}} - (k - 1) \right] \right], \quad \quad (16) \]

\[ T_-(x) = \frac{2T_0 e^{2[f(0) - f(a)]}}{1 + \beta e^{2[f(a) - f(0)]}} \left[ (k - 1) + (1 - 2k)^{\frac{1}{2}} \right] \sinh [f(x) - f(a)] \]

\[ \left[ (1 + \beta e^{-2[f(0) - f(a)]}) k \right]. \quad \quad (17) \]

In formulae (16) and (17)

\[ \beta = \frac{(1 - 2k)^{\frac{1}{2}} + (k - 1)}{(1 - 2k)^{\frac{1}{2}} - (k - 1)} \] (18)

and

\[ f(x) = (1 - 2k)^{\frac{1}{2}} \int \sigma(x) dx, \] (19)

\[ f(0) = (1 - 2k)^{\frac{1}{2}} \left[ \int \sigma(x) dx \right]_0, \]

\[ f(a) = (1 - 2k)^{\frac{1}{2}} \left[ \int \sigma(x) dx \right]_a. \]
With formulae (16) and (17) for $T_+ (x)$ and $T_- (x)$ we define the asymmetry $A(x)$ of the temperature $T(x)$

$$A(x) = \frac{T_+ (x) - T_- (x)}{T_+ (x) + T_- (x)},$$

(20)

$$A(x) = \frac{(1 - 2k)^{\frac{1}{2}}}{(1 - 2k)^{\frac{1}{2}} - (k - 1)} \cosh [f(x) - f(a)] - \frac{1 - 2k}{(1 - 2k)^{\frac{1}{2}} - (k - 1)} \sinh [f(x) - f(a)]$$

$$- \frac{(1 - 2k)^{\frac{1}{2}}}{(1 - 2k)^{\frac{1}{2}} - (k - 1)} \cosh [f(x) - f(a)] - \frac{1}{(1 - 2k)^{\frac{1}{2}} - (k - 1)} \sinh [f(x) - f(a)]$$

(21)

From formula (21) we conclude that for elastic scattering, i.e. when $k = \frac{1}{2}$, $A(x) = 0$, and for $k \neq \frac{1}{2}$, $A(x) \neq 0$.

In the monograph [4] we introduced the relaxation time $\tau$ for quantum heat transport

$$\tau = \frac{\hbar}{mv^2}.$$  

(22)

In formula (22) $m$ denotes the mass of heat carriers electrons and $v = \alpha c$, where $\alpha$ is the fine structure constant for electromagnetic interactions. As was shown in monograph [4], $\tau$ is also the lifetime for positron-electron pairs in vacuum.

When the duration time of the laser pulse is shorter than $\tau$, then to describe the transport phenomena we must use the hyperbolic transport equation. Recently the structure of water was investigated with the attosecond ($10^{-18}$s) resolution [3]. Considering that $\tau \approx 10^{-17}$ s we argue that to study performed in [3] open the new field for investigation of laser pulse with matter. In order to apply the equations (9) to attosecond laser induced phenomena we must know the cross section $\sigma (x)$. Considering formulae (9) and (22) we obtain

$$\sigma (x) = \frac{mv}{\hbar} = \frac{me^2}{\hbar^2}$$

(23)

and it occurs $\sigma (x)$ is the Thomson cross section for electron-electron scattering.
With formula (23) the solution of Cauchy problem has the form:

\[
T_+ (x) = \frac{2T_0 e^{-(1-2k)^{\frac{1}{2}} \frac{mc^2}{\hbar}}}{1 + \beta e^{-2(1-2k)^{\frac{1}{2}} \frac{mc^2}{\hbar}a}} \times \\
\left[ (1-2k)^{\frac{1}{2}} \cosh \left( (1-2k)^{\frac{1}{2}} \frac{mc^2}{\hbar^2} (x-a) \right) \right] \\
\left[ (1-2k)^{\frac{1}{2}} - (k-1) \right] \\
+ \frac{(k-1) \sinh \left( (1-2k)^{\frac{1}{2}} \frac{mc^2}{\hbar^2} (x-a) \right)}{(1-2k)^{\frac{1}{2}} - (k-1)},
\]

\[
T_- (x) = \frac{2T_0 e^{-(1-2k)^{\frac{1}{2}} \frac{mc^2}{\hbar}}}{1 + \beta e^{-2(1-2k)^{\frac{1}{2}} \frac{mc^2}{\hbar}a}} \times \\
\left[ (k-1) - (1-2k)^{\frac{1}{2}} \right] \\
\left( 1 + \beta e^{-2(1-2k)^{\frac{1}{2}} \frac{mc^2}{\hbar}a} \right) \right) \\
\sinh \left( (1-2k)^{\frac{1}{2}} \frac{mc^2}{\hbar^2} (x-a) \right).
\]

3 Conclusions

In this paper the one dimensional Weyl type thermal equation was developed and solved. It was shown that depending on the dynamics of the heat carriers scattering the damped or oscillated temperature field can be generated. When the laser pulse generates relativistic electrons the cross section for the generation of electron-positron pairs is equal to the Thomson cross section.
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