HYPERGRAPH REGULARITY AND HIGHER ARITY VC-DIMENSION

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ABSTRACT. We generalize the fact that graphs with small VC-dimension can be approximated by rectangles, showing that hypergraphs with small VC$_k$-dimension (equivalently, omitting a fixed finite $(k+1)$-partite $(k+1)$-uniform hypergraph) can be approximated by $k$-ary cylinder sets.

In the language of hypergraph regularity, this shows that when $H$ is a $k'$-uniform hypergraph with small VC$_k$-dimension for some $k < k'$, the decomposition of $H$ given by hypergraph regularity only needs the first $k$ levels—one can approximate $H$ using sets of vertices, sets of pairs, and so on up to sets of $k$-tuples—and that on most of the resulting $k$-ary cylinder sets, the density of $H$ is either close to 0 or close to 1.

We also show a suitable converse: $k'$-uniform hypergraphs with large VC$_k$-dimension cannot have such approximations uniformly under all measures on the vertices.

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Date: October 5, 2020.
1. Introduction

We generalize the fact that graphs with small VC-dimension can be approximated by rectangles [AFN07, LS10], showing that hypergraphs with small VC_{k'}-dimension (equivalently, hypergraphs omitting a fixed finite (k + 1)-partite (k + 1)-uniform hypergraph) can be approximated by k-ary cylinder sets.

Our main result is:

**Theorem 1.1.** For every $k < k'$, every $d$ and every $\varepsilon > 0$, there is an $N$ such that whenever $H \subseteq (V_{k'})$ has VC_{k'}-dimension less than $d$, $H$ differs from a union of at most $N$ k-ary cylinder sets by at most $\varepsilon |V_{k'}|^{k'}$ points.

Stated in a more general way, this is Corollary 6.10. We also prove an appropriate converse: that if $H$ has this approximation property with a bound on $N$ which is uniform over all measures on $V$ then $H$ has small VC_{k'}-dimension; this is Theorem 7.1.

To see why we should expect such a result, first recall the situation for graphs. It is convenient to interpret the Szemerédi regularity lemma as saying that when $G = (V, E)$ is a large finite graph, we can present the characteristic function $\chi_E$ of the edge relation $E$ in the form

$$\chi_E = f^\top + f^\perp$$

\[1\] See Definition 3.1.
\[2\] See Remark 3.2.
\[3\] See Definition 2.5.
where $f^\top$ is the “structured” portion of the form

$$f^\top(x, y) = \sum_{i,j \leq n} \alpha_{i,j} \chi_{V_i}(x) \chi_{V_i}(y)$$

where $V = \bigcup_{i \leq n} V_i$ is a partition and the $\alpha_{i,j}$ are real numbers, and $f^\perp$ is quasirandom. That is, we can view $E$ as a finite partition with weights $\alpha_{i,j}$ indicating the density of edges between $V_i$ and $V_j$, with $f^\perp$ representing the random determination of which edges are actually present.

When $G$ has small VC-dimension\(^4\), the $f^\perp$ part is small [APN07, LS10, CS16]. More precisely, for each $d$ and each $\varepsilon > 0$, there is a bound $N$ so that whenever $G$ is a graph with VC-dimension at most $d$, there is a regularity partition into $N$ pieces so that the quasirandom part satisfies

$$\sum_{x,y \in V} |f^\perp(x,y)|^2 < \varepsilon |V|^2. \quad (\text{Indeed, } N \text{ is polynomial in } \varepsilon, \text{ with the degree of the polynomial depending on } d.)$$

This means that the weights $\alpha_{i,j}$ are each either close to 1 or close to 0, so this is equivalent to saying that $G$ is approximately the union of those rectangles where $\alpha_{i,j}$ is close to 1.

We cannot quite get a reverse implication, that $f^\perp$ being small implies small VC-dimension. It cannot be exactly an equivalence because having small VC-dimension is a combinatorial property, while $f^\perp$ has a measure-theoretic character. (For instance, if we take a very large graph of small VC-dimension, and then graft on it a small graph of large VC-dimension, say with size $o(|V|)$, the small graph cannot meaningfully change $f^\perp$.) Instead, having small VC-dimension is equivalent to having $f^\perp$ be small uniformly for all possible measures on $V$.

Now, consider what happens when we generalize to hypergraphs—that is, $H = (V,E)$ with $E \subseteq \binom{V}{k}$ for some $k \geq 2$. Something similar holds if all slices of $E$ have small VC-dimension—that is, for every fixed $z_1, \ldots, z_{k-2}$ in $V$, the binary relation

$$E_{z_1,\ldots,z_{k-2}} = \{(x,y) \mid (x,y,z_1,\ldots,z_{k-1}) \in E\} \subseteq \binom{V}{2}$$

has small VC-dimension\(^5\). When this holds, we have

$$\chi_E = f^\top + f^\perp$$

where the $f^\top$ portion has the form

$$f^\top(x_1, \ldots, x_k) = \sum_{i_1,\ldots,i_k} \alpha_{i_1,\ldots,i_k} \prod_j \chi_{V_{i_j}}(x_j)$$

\(^4\)That is, the family $\{E_x \mid x \in V\}$ of subsets of $V$ has small VC-dimension, where $E_x$ is the fiber $\{y \in V \mid (x,y) \in E\}$. Equivalently, there is a small bipartite graph which $G$ contains no induced copies of.

\(^5\)It is more common to consider a stronger assumption, that all ways of viewing $E$ as a graph on $V \times V^{k-1}$ have small VC-dimension. However the weaker slice-wise assumption here suffices, and is the notion for which we get a converse. There are examples showing that the slice-wise assumption is strictly weaker.
and $\sum_{x \in V} |f^\perp(\bar{x})|^2 < \varepsilon|V|^k$. That is, $H$ is approximated by boxes (CS16), which corresponds to the case $k = 1$ and $k'$ arbitrary of Theorem [17].

This is a very strong conclusion, suggesting that small VC-dimension is a very restrictive condition for a hypergraph. For a general regular hypergraph $H = (V,E)$, the characterization given by hypergraph regularity [NRS06, RS04, Gow07] involves a more complicated decomposition

$$\chi_E = f_{k-1} + \cdots + f_1 + f^\perp$$

where $f_1$ has the form $\sum_{i_1,\ldots,i_k} \alpha_{i_1,\ldots,i_k} \prod_j \chi_{V_j}(x_j)$ as above, but the $f_j$ in general are sums of $j$-ary cylinder sets. (For instance, $f_2$ is, roughly speaking, the portion of $\chi_H$ which can be described using directed graphs.)

Small VC-dimension collapses not only the random part $f^\perp$, but also all the more complex parts $f_{k-1} + \cdots + f_2$. There ought to be a weaker notion than small VC-dimension which corresponds to just $f^\perp$ being small; more generally, there ought to be notions which correspond to collapsing part of this sequence, so that $f^\perp + f_{k-1} + \cdots + f_{j+1}$ is small.

The natural candidate is the notion of VC$_k$-dimension implicit in Shelah’s work in model theory [She14, She17] and studied further in [CPT19].

The proof of the aforementioned result for graphs of finite VC-dimension—which corresponds to the $k = 1$, $k' = 2$ case of Theorem [17]—is fairly short. The key point is that if a graph $E$ has finite VC-dimension, so does the graph $E^* = \{(x,x'), y \in E_x \triangle E_{x'} \}$ on $V^2 \times V$. ($E_x$ is the fiber $\{y \in V \mid (x,y) \in E\}$.) A graph with finite VC-dimension has small $\varepsilon$-nets [HW87]: that is, there is a list of $y_1,\ldots,y_n \in V$ such that, for all pairs $(x,x')$, either the fiber $E_x^* x'$ has density less than $\varepsilon$, or $E_x^* x' \cap \{y_1,\ldots,y_n\} \neq \emptyset$. That is, for any two points $E_x, E_{x'}$, either $E_x \triangle E_{x'}$ is small, or $E_x \triangle E_{x'}$ includes one of the points $y_1,\ldots,y_n$. We call $\{y_1,\ldots,y_n\}$ an “$\varepsilon$-net for differences”: the points $y_1,\ldots,y_n$ are a universal test for whether two fibers can be far apart. There are only finitely many subsets of $\{y_1,\ldots,y_n\}$, so we can then approximate the graph as a union of rectangles of the form

$$\{x \mid (E_x \triangle E_{x_i}) \cap \{y_1,\ldots,y_n\} = \emptyset\} \times E_{x_i}$$

for a short list of points $x_1,\ldots,x_m$.

A quick glance at this paper suggests that the proof of the generalization to hypergraphs will be slightly more complicated.

We carry out our argument in the setting of a Keisler graded probability space. This is the natural infinitary setting for such arguments; in particular, it is the setting one obtains by considering a hypergraph $H \subseteq \prod_{i \leq k} V_i$ with $N \leq \min_i |V_i|$ and letting $N \to \infty$. Many statements which would be approximate, or “up to $o(N^k)$”, or something similar when considering large $N$ become exact in the infinitary setting. Most importantly, in a probability

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6See Definitions $\S$ and $\S$. VC$_1$ is ordinary VC-dimension. A $(k+1)$-graph has small VC$_k$-dimension if it omits a small $(k+1)$-partite hypergraph.

7In fact, using the bounds given by the VC theorem and Sauer-Shelah, of size polynomial in $\varepsilon$. 
space we can identify the “lower dimensional information” mentioned above with the projection onto a \(\sigma\)-algebra. Additionally, this lets us speak of the distinction between finite and infinite VC\(_k\)-dimension, rather than having to speak precisely of quantitative bounds for what it means to have a “small” VC\(_k\)-dimension.

We further work in a compound multipartite setting, where we consider subsets of \(\prod_{i \in [k]} V_i^{m_i}\)—that is, we not only allow separate sets \(V_i\) for each coordinate, we keep track of the possibility that we may have multiple coordinates coming from the same set. (The graph \(E^*\) above, which is naturally viewed as a subset of \(V_1^2 \times V_2\), suggests why this setting shows up in the course of the proof.) For completeness, since it does not seem to have appeared in the literature, we write down the extension of the Keisler graded probability space to this setting in detail in Section 2.2. We need some results about the Gowers uniformity norms and their relationship to conditional expectation in this setting; these results are standard, but have also not been developed in the multipartite setting. We include them for completeness as well, but postpone this discussion to Section 8.

In Section 3.1 we define VC\(_k\)-dimension and recall some standard examples and facts. However we will want to consider not just hypergraphs—that is, sets—but functions with range \([0,1]\). We may think of these functions as weighted hypergraphs, with ordinary hypergraphs as the case where the functions are \([0,1]\)-valued. Such functions show up at intermediate steps anyway—for instance, in the decompositions above, the components \(f^+, f^-\) are naturally functions, not sets. The extension of VC-dimension to functions has appeared in various places (e.g. [Tal87, Tal96, BY09]), and we give the analogous definition of VC\(_k\)-dimension in Definition 3.11.

We include some results showing that various operations preserve VC\(_k\)-dimension of functions; to avoid interrupting the main thread of the argument, we postpone this to Section 10. The last and most difficult of these is Theorem 10.7 showing that given a family of functions of low VC\(_k\)-dimension, the “average” function still has low VC\(_k\)-dimension (more precisely, the VC\(_k\)-dimension of the function \(f'(x_1, \ldots, x_{k+1}) := \int f(x_1, \ldots, x_{k+1})d\mu(x_{k+2})\) can be bounded in terms of the maximum of the VC\(_k\)-dimensions of the functions \(f(x_1, \ldots, x_{k+2})\) over all \(x_{k+2}\)). Our proof combines structural Ramsey theory with a variant of the Aldous-Hoover-Kallenberg theorem on exchangeable arrays of random variables. It provides a higher arity generalization of the main result of [BY09] for \(k = 1\) using different methods.

Section 5 is devoted to proving the existence of “\(\varepsilon\)-nets for differences” for hypergraphs of low VC\(_k\)-dimension. It is a bit surprising that this is possible, because we do not have any analog of the existence of \(\varepsilon\)-nets; it is not even clear what the higher arity generalization of an \(\varepsilon\)-net would be. Nonetheless, we do have an analog of the \(\varepsilon\)-net for differences, in the following sense.

When \(H \subseteq \prod_{i \leq k+1} V_i\) has small VC\(_k\)-dimension, it is no longer reasonable to expect that there is a short list \(x_1, \ldots, x_n \in V\) so that every \(k\)-ary fiber \(H_x\) with \(x \in V\) is close (i.e. has small symmetric difference) to one of the \(H_{x_i}\).
Rather, we have to expect that each fiber $H_x$ is described by the $H_{x_i}$ together with lower dimensional information. This is the content of Proposition 5.1.

The remainder of Section 5 is devoted to further refinement of this result.

To prove Proposition 5.1, we suppose it fails and work with an infinite sequence of fibers which are all far from each other. We then homogenize this sequence using many applications of Ramsey’s Theorem and construct a counterexample to small VC$_k$-dimension from the resulting subsequence. To manage the homogenization of the sequence, we pass to a sequence of indiscernibles in an ultrapower of the original graded probability space; this requires some model theoretic machinery. We treat this machinery as a black box as much as possible, and isolate the model theoretic arguments to Section 9.

Having shown that there are finitely many $k$-ary fibers of $H$ which, up to lower dimensional information, approximate all the fibers, we are able to write down an approximation of $H$ using these fibers in Proposition 6.1.

We then generalize this to the case where $H \subseteq \prod_{i \leq k'} V_i$ for any $k' > k$, concluding the main result of the paper, in Theorem 6.6 and then prove the quantitative Corollary 6.9 using one more detour through the model theoretic techniques of Section 9.

In Section 7 we prove the converse of the main theorem: if a function on $\prod_{i \in [k]} V_i$ has infinite VC$_k$-dimension, then there is some way to put a probability measure on the $V_i$ so that the function has no simple approximation using $(\leq k)$-ary sets.

Finally, in Section 11 we discuss some questions and directions for future work that naturally arise given the results of the paper, along with some applications of our results in model theory.

1.1. Acknowledgements. Artem Chernikov was partially supported by the NSF CAREER grant DMS-1651321. He is grateful to Kota Takeuchi and Itaï Ben Yaacov for helpful discussions. Henry Towsner was partially supported by NSF Grant DMS-1600263. The authors thank the American Institute of Mathematics and the Institut Henri Poincaré for additional support.

2. Preliminaries

2.1. Notation. We summarize the notation used throughout the article for a reference.

(1) $\mathbb{N} = \{0, 1, \ldots\}$. We write $\mathbb{R}_{>0}$ to denote the set of positive reals, $\mathbb{R}_{\neq 0}$ for the set of non-zero reals, $\mathbb{N}_{>0}$ for the set of positive integers, and $\mathbb{Q}^{[0,1]}$ for the set of rational numbers in the interval $[0, 1]$.

(2) For $i \in \mathbb{N}$, by a dyadic rational number of height $i$ we mean a rational number of the form $\frac{m}{2^i}$ with $m \in \mathbb{Z}$ and $n = 2^i$. We let $\mathbb{Q}_i$ be the set of all dyadic rationals of height $i$, and let $\mathbb{Q}_\infty = \bigcup_{i \in \mathbb{N}} \mathbb{Q}_i$ be the set
of all dyadic rationals. We let \( \mathbb{Q}_i^{[0,1]} := \mathbb{Q} \cap [0,1] \), note that it is a
finite set of cardinality \( 2^i + 1 \) for every \( i \in \mathbb{N} \).

3. For \( k \in \mathbb{N}_{>0} \) we will denote by \( [k] \) the set \( \{1, \ldots, k\} \), and \( [0] := \emptyset \).

4. For a set \( V \) and \( k \in \mathbb{N} \), \( \binom{V}{k} := \{ W \subseteq V : |W| = k \} \) and \( \binom{V}{\leq k} = \{ W \subseteq V : |W| \leq k \} \).

5. For any \( i, j \in \mathbb{N} \), \( \delta_{i,j} \) is equal to 1 if \( i = j \) and 0 otherwise. Given \( k \in \mathbb{N}_{>0} \) and \( i \in [k] \), \( \delta^k_i := (\delta_{i,j} : j \in [k]) \). We let
   \[ \vec{0}^k := (0, \ldots, 0), \vec{1}^k := (1, \ldots, 1). \]
   We might omit \( k \) if it is clear from the context, and simply write \( \vec{0}, \vec{1} \).

6. Given \( \vec{n} = (n_1, \ldots, n_k), \vec{m} = (m_1, \ldots, m_k) \) in \( \mathbb{N}^k \), we write \( \vec{n} \leq \vec{m} \) if \( n_i \leq m_i \) for every \( i \in [k] \), and \( \vec{n} < \vec{m} \) if \( \vec{n} \leq \vec{m} \) and \( n_i < m_i \) for at least one \( i \in [k] \).

7. Algebraic operations on tuples of numbers are always performed coordinate-wise. Given \( \vec{n}, \vec{m} \in \mathbb{N}^k \) and \( d \in \mathbb{N} \), we write \( d \cdot \vec{n} \) to
denote the tuple \( (dn_1, \ldots, dn_k) \), \( \vec{n} + \vec{m} \) to denote the tuple \( (n_1 + m_1, \ldots, n_k + m_k) \), \( \vec{n} \cdot \vec{m} \) for the tuple \( (n_1 \cdot m_1, \ldots, n_k \cdot m_k) \), etc.

8. Given two tuples \( \vec{a} = (a_1, \ldots, a_m), \vec{b} = (b_1, \ldots, b_n) \), we write \( \vec{a} \cdot \vec{b} \) for the
concatenated tuple \( (a_1, \ldots, a_m, b_1, \ldots, b_n) \).

9. Given a set \( V \), \( \mathcal{P}(V) \) denotes the set of its subsets.

10. For sets \( V_1, \ldots, V_k \) and \( I \subseteq [k] \) we denote by \( V_I \) the product \( V_I = \prod_{i \in I} V_i \).

11. Given \( I \subseteq [k] \), a tuple \( \vec{a} = (a_i : i \in I) \in V_I \) and a set \( s \subseteq I \), we denote by \( \vec{a}_s \) the subtuple \( (a_i : i \in s) \in V_s \).

12. Let \( R \subseteq V_1 \times \ldots \times V_k \) be a \( k \)-ary relation and \( I \subseteq [k] \). Viewing \( R \) as
a binary relation on \( V_I \times V_{[k] \setminus I} \), for \( b \in V_{[k] \setminus I} \) we denote by \( R_b \) the
fiber
   \[ R_b = \{ a \in V_I : (a, b) \in R \}. \]

13. If \( \vec{a} \in V_1 \times \ldots \times V_k \) and \( \sigma : [k] \to [k] \) is a permutation, then \( \sigma(\vec{a}) := \bigl( a_{\sigma(1)}, \ldots, a_{\sigma(k)} \bigr) \in V_{\sigma(1)} \times \ldots \times V_{\sigma(k)}. \)

14. Given a tuple \( \vec{a} = (a_i : i \in I), i \in I \) and \( b \in V_i \), we let \( \vec{a}_{b, \vec{a}} \) denote
the tuple obtained from \( \vec{a} \) by replacing \( a_i \) by \( b \).

15. Given a relation \( R \subseteq V_1 \times \ldots \times V_k \), \( I \subseteq [k] \) and \( \vec{a} \in V_1 \times \ldots \times V_k \), we let
   \[ R_{\vec{a}, I} := \{ \vec{x} \in V_1 \times \ldots \times V_k : \vec{x}_{a_i \mapsto i, i \in I} \in R \}. \]

16. If \( R \subseteq V_1 \times \ldots \times V_k \) and \( \sigma : [k] \to [k] \) is a permutation, then \( R^\sigma := \{ \vec{a} : \vec{a} \in R \} \subseteq V_{\sigma(1)} \times \ldots \times V_{\sigma(k)}. \)

17. For \( R \subseteq V \), we write \( \chi_R : V \to \{0,1\} \) to denote the characteristic
function of \( R \).

18. For \( R \subseteq V \), we will use the notation \( R^1 := R \) and \( R^{-1} := \neg R := V \setminus R. \)
(19) If $X,Y$ are sets, then $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$ denotes their symmetric difference, and if $\varphi, \psi$ are first-order formulas, then $\phi \triangle \psi$ denotes the formula $(\varphi \land \neg \psi) \lor (\neg \varphi \land \psi)$.

As usual, given a $\sigma$-algebra $\mathcal{B}$, a $\sigma$-subalgebra $\mathcal{B}_0 \subseteq \mathcal{B}$, and a $\mathcal{B}$-measurable function $f$, $E(f \mid \mathcal{B}_0)$ denotes the conditional expectation. We will use freely that the conditional expectation corresponds to orthogonal projection in the corresponding Hilbert space of measurable functions—that is, for any $\mathcal{B}$-measurable functions $f,g$, $\int E(f \mid \mathcal{B}_0) g \, d\mu = \int f \, E(g \mid \mathcal{B}_0) \, d\mu$. As usual, the equality for functions in $L^2(\mu)$ is understood up to a measure 0 set. Given a set of $\mathcal{B}$-measurable functions $G$, for brevity $E(f \mid \mathcal{B}_0 \cup G)$ will denote $E(f \mid \sigma(\mathcal{B}_0 \cup G))$. If $E \in \mathcal{B}$, we might write $E(E \mid \mathcal{B}_0)$ to denote $E(\chi_E \mid \mathcal{B}_0)$.

2.2. Graded probability spaces and cylinder sets. We review and generalize to the partite setting the notion of graded probability spaces, which were introduced by Keisler in [Kei85] and provide a natural setting for the analytic approach to the study of various hypergraph regularity phenomena.

We fix $k \in \mathbb{N}_{>0}$ and sets $(V_i)_{i \in [k]}$, and we are going to be considering the products $V_{n_1,\ldots,n_k} := \prod_{i \in [k]} V_i^{n_i}$ for arbitrary $n_1, \ldots, n_k \in \mathbb{N}$. An element of $\prod_{i \in [k]} V_i^{n_i}$ is a tuple

$$(v_{1,1}, v_{1,2}, \ldots, v_{1,n_1}, v_{2,1}, \ldots, v_{2,n_2}, \ldots, v_{k,1}, \ldots, v_{k,n_k}),$$

which we will usually abbreviate

$$(\vec{v}_1, \ldots, \vec{v}_k)$$

or just $\vec{v}$. It is going to be convenient to define ordered concatenation: if $\vec{v} = (\vec{v}_1, \ldots, \vec{v}_k) \in \prod_{i \in [k]} V_i^{n_i}$ and $\vec{w} = (\vec{w}_1, \ldots, \vec{w}_k) \in \prod_{i \in [k]} V_i^{m_i}$, we define $\vec{v} \oplus \vec{w} = (\vec{v}_1, \vec{w}_1, \vec{v}_2, \vec{w}_2, \ldots, \vec{v}_k, \vec{w}_k) \in \prod_{i \in [k]} V_i^{n_i+m_i}$.

**Definition 2.1.** A $k$-partite graded probability space $\mathcal{P} = (V_{[k]}, \mathcal{B}_{\vec{n}}, \mu_{\vec{n}})_{\vec{n} \in \mathbb{N}^k}$ consists of sets $(V_i)_{i \in [k]}$ and, for every $n_1, \ldots, n_k \in \mathbb{N}$, a $\sigma$-algebra $\mathcal{B}_{n_1,\ldots,n_k} \subseteq \mathcal{P} \left( \prod_{i \in [k]} V_i^{n_i} \right)$ and a probability measure $\mu_{n_1,\ldots,n_k}$ on $\mathcal{B}_{n_1,\ldots,n_k}$ satisfying the following axioms.

1. **(Symmetry)** For every $n_1, \ldots, n_k \in \mathbb{N}$, $i \leq k$, permutation $\pi : [n_i] \to [n_i]$ and $B \in \mathcal{B}_{n_1,\ldots,n_k}$, we let

$$B^\pi := \{ (\vec{v}_1, \ldots, \pi(\vec{v}_i), \ldots, \vec{v}_k) \mid (\vec{v}_1, \ldots, \vec{v}_i, \ldots, \vec{v}_k) \in B \}.$$ 

Then

(a) $B^\pi \in \mathcal{B}_{n_1,\ldots,n_k}$, and

(b) $\mu_{n_1,\ldots,n_k}(B^\pi) = \mu_{n_1,\ldots,n_k}(B)$.

2. **(Closure under products)** If $B \in \mathcal{B}_{n_1,\ldots,n_k}$ and $C \in \mathcal{B}_{m_1,\ldots,m_k}$, then the reordered product $B \times C := \{ \vec{v} \oplus \vec{w} \mid \vec{v} \in B \text{ and } \vec{w} \in C \}$
Remark 2.3. Assume that the Fubini property as in Definition 2.1(3) holds for any functions argument, it also lifts from measures to general integrals. That is, a partite graded probability space canonically induces a graded probability space

\[ V_n^{k+1} = \{ i \in [k] | V_i \subseteq B_i \} \]

Then the Fubini property holds for the algebras \( (B_n, B_m, B_{n+m}) \):

(a) \( B_\bar{n} \subseteq B_{n_1, \ldots, n_k} \) for all \( \bar{w} \in \prod_{i \in [k]} V_i^{m_i} \);

(b) the function \( \bar{w} \mapsto \mu_{n_1, \ldots, n_k}(B_\bar{w}) \) from \( \prod_{i \in [k]} V_i^{m_i} \) to \([0,1]\) is \( \mu_{m_1, \ldots, m_k} \)-measurable; and

(c) \( \mu_{n_1+m_1, \ldots, n_k+m_k}(B) = \int \mu_{n_1, \ldots, n_k}(B_\bar{w}) d\mu_{m_1, \ldots, m_k}(\bar{w}) \).

A graded probability space \( \mathfrak{P} = (V, B_n, \mu_n)_{n \in \mathbb{N}} \) is just a 1-partite graded probability space (equivalently, for any \( k \in \mathbb{N}_{\geq 1} \), it can be identified with a \( k \)-partite graded probability space \( V_1 = \ldots = V_k = V, B_n, \ldots, n_k = B_n \) and \( \mu_{n_1, \ldots, n_k} = \mu_n \) for all \( n_1, \ldots, n_k \) with \( |n_1 + \ldots + n_k| = n \)).

Remark 2.2. (1) A partite graded probability space canonically induces a \( \sigma \)-algebra and measure on any product \( \prod_{i \in [d]} V_{i, j} \) with \( i_j \in [k] \), by identifying elements of \( \prod_{i \in [d]} V_{i, j} \) with elements of \( \prod_{i \in [k]} V_i^{n_i} \) for any appropriate choice of \( n_i \) and a permutation of the coordinates (by symmetry, the choice of permutation does not matter).

(2) Let \( \mathfrak{P} = (V_k, B_n, \mu_n)_{n \in \mathbb{N}_k} \) be a partite graded probability space. Recall that for any set \( V \), \( V^0 = \{ \emptyset \} \). Then given \( i \in [k] \), the set \( \prod_{i \in [k]} V_i^{\delta_i} \) is naturally identified with the set \( V_i \), the algebra \( B_i^{\delta_i} \) is naturally identified with an algebra \( B_i^{\emptyset} \) of subsets of \( V_i \), and the measure \( \mu_i^{\emptyset} \) with a measure \( \mu_i^\emptyset \) on \( B_i^{\emptyset} \) (recall that \( \delta_i \) is the tuple with 1 in the \( i \)th position and 0 in the other positions, see Section 2.1). Then all of the measures \( \mu_n, \bar{n} \in \mathbb{N}_k \) are determined by the measures \( \mu_i^\emptyset \) (by a straightforward induction on \( |n_1 + \ldots + n_2| \) using symmetry and Fubini).

(3) For any \( n_1, \ldots, n_k, m_1, \ldots, m_k \), we let

\[ B_{n_1, \ldots, n_k} \times B_{m_1, \ldots, m_k} = \sigma(\{ B \times C : B \in B_{n_1, \ldots, n_k}, C \in B_{m_1, \ldots, m_k} \}) \]

be the product \( \sigma \)-algebra. Then

\[ B_{n_1, \ldots, n_k} \times B_{m_1, \ldots, m_k} \subseteq B_{n_1+m_1, \ldots, n_k+m_k} \]

(by the closure under products) and \( \mu_{n_1+m_1, \ldots, n_k+m_k} \) extends the product measure \( \mu_{n_1, \ldots, n_k} \times \mu_{m_1, \ldots, m_k} \) (by Fubini property). Note however that in a typical case of interest for us this inclusion of algebras is strict.

Remark 2.3. Assume that the Fubini property as in Definition 2.1(3) holds for \( (B_{\bar{n}}, B_{\bar{m}}, B_{\bar{n}+\bar{m}}) \). Then, via a straightforward approximation by simple functions argument, it also lifts from measures to general integrals. That is, for any \( B_{\bar{n}+\bar{m}} \)-measurable function \( f : V^{n+m} \to \mathbb{R} \) we have:

(1) the fiber \( f_\bar{w} : V^{\bar{n}} \to [0,1], \bar{v} \mapsto f(\bar{v} \oplus \bar{w}) \) is \( \mu_{\bar{n}} \)-measurable for all \( \bar{w} \in V^{\bar{m}} \).
(2) the function \( \bar{w} \mapsto \int f(\bar{v} \oplus \bar{w}) \, d\mu_{\bar{a}}(\bar{v}) \) is \( \mathcal{B}_{\bar{a}} \)-measurable;

(3) the function \( \bar{v} \mapsto \int f(\bar{v} \oplus \bar{w}) \, d\mu_{\bar{a}}(\bar{w}) \) is \( \mathcal{B}_{\bar{a}} \)-measurable (using symmetry in Definition 2.1(1));

\[
\int f(\bar{v} \oplus \bar{w}) \, d\mu_{\bar{a}+\bar{m}}(\bar{v} \oplus \bar{w}) = \int \left( \int f(\bar{v} \oplus \bar{w}) \, d\mu_{\bar{a}}(\bar{v}) \right) \, d\mu_{\bar{a}}(\bar{w}) = \int \left( \int f(\bar{v} \oplus \bar{w}) \, d\mu_{\bar{a}}(\bar{w}) \right) \, d\mu_{\bar{a}}(\bar{v}).
\]

We have the following natural way to form a new partite graded probability space from a given one.

**Remark 2.4.** ("Gluing coordinates") Assume \( (V_{[k]}, \mathcal{B}_{\bar{n}}, \mu_{\bar{n}})_{\bar{n} \in \mathbb{N}^k} \) is a \( k \)-partite graded probability space. Let \( t \in \mathbb{N} \) and \( \bar{n}_i = (n_{i,1}, \ldots, n_{i,k}) \in \mathbb{N}^k \) for \( i \in [t] \) be arbitrary. We define \( V'_i := V_{\bar{n}_i} \) for \( i \in [t] \), and for \( \bar{m} = (m_1, \ldots, m_t) \in \mathbb{N}^t \) we let \( \bar{m}' := m_1 \bar{n}_1 + \ldots + m_t \bar{n}_t \in \mathbb{N}^k \), \( \mathcal{B}'_{\bar{m}} := \mathcal{B}_{\bar{m}'} \), \( \mu'_{\bar{m}} := \mu_{\bar{m}'} \).

Then \( \mathcal{B}'_{\bar{m}} \) can be viewed as an algebra of subsets of \( \prod_{i \in [t]} (V'_i)^{m_i} \) (identifying the product \( \prod_{i \in [t]} (\prod_{j \in [k]} |V'_{j}|)^{m_{i,j}} \) with \( \prod_{j \in [k]} V_j^{\sum_{i \in [t]} m_{i,j}} \) by Remark 2.2(1)), and it is easy to see that \( (V'_{[t]}, \mathcal{B}'_{\bar{n}}, \mu'_{\bar{n}})_{\bar{n} \in \mathbb{N}^t} \) is a \( t \)-partite graded probability space.

**Definition 2.5.** Let \( (V_{[k]}, \mathcal{B}_{\bar{n}}, \mu_{\bar{n}})_{\bar{n} \in \mathbb{N}^k} \) be a partite graded probability space, and fix \( \bar{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k \). Let \( n := \sum_{i \in [k]} n_i \).

1. For each \( i \in [k] \), let \( I_i \subseteq [n_i] \), and \( \bar{I} := (I_1, \ldots, I_k) \). Then \( \mathcal{B}_{\bar{n}, \bar{I}} \) is the \( \sigma \)-subalgebra of \( \mathcal{B}_{\bar{n}} \) generated by all sets of the form

\[
\left\{ \bar{x} = (\bar{x}_1, \ldots, \bar{x}_k) \in \prod_{i \in [k]} V_i^{n_i} : (\bar{x}_1)_{I_1}, \ldots, (\bar{x}_k)_{I_k} \right\} \subseteq X \}
\]

for \( X \in \mathcal{B}_{|I_1|, \ldots, |I_k|} \).

2. For \( m < n \), we let \( \mathcal{B}_{\bar{n}, m} \) be the \( \sigma \)-subalgebra of \( \mathcal{B}_{\bar{n}} \) generated by

\[
\bigcup \left\{ \mathcal{B}_{\bar{n}, \bar{I}} : \sum_{i \in [k]} |I_i| \leq m \right\}.
\]

3. If \( \bar{n} \in \mathbb{N}^r \) for some \( r < k \), then \( \mathcal{B}_{\bar{n}} := \mathcal{B}_{\bar{n} - \bar{0}_k - r} \) — a \( \sigma \)-algebra of subsets of \( V_{\bar{n}} = \prod_{i \in [r]} V_i^{n_i} \), and \( \mu_{\bar{n}} := \mu_{\bar{n} - \bar{0}_k - r} \) a measure on it.

And if \( m < \sum_{i \in [r]} n_i \), then \( \mathcal{B}_{\bar{n}, m} := \mathcal{B}_{\bar{n} - \bar{0}_k - r, m} \).

We refer to the sets in \( \mathcal{B}_{\bar{n}, m} \) as the \( \bar{m} \)-cylinder sets, and to the sets in \( \mathcal{B}_{\bar{n}, t} \) with \( t < k \) as the \( t \)-ary cylinder sets.

In other words, \( \mathcal{B}_{\bar{n}, m} \) is generated by those sets in \( \mathcal{B}_{\bar{n}} \) that can be defined by measurable conditions each of which can involve at most \( m \) out of \( n \) variables. The inclusion \( \mathcal{B}_{\bar{n}, m} \subseteq \mathcal{B}_{\bar{n}} \) is strict in general.
3. VC$_k$-dimension

3.1. VC$_k$-dimension for relations. We review the notion of VC$_k$-dimension, for $k \in \mathbb{N}$, generalizing the usual Vapnik-Chervonenkis dimension in the case $k = 1$. It is implicit in Shelah’s work on $k$-dependent theories in model theory [She17, She14] and is studied in [CPT19]; and further in [Hem16, CH19a, CH19b] for model theory of groups and fields, and in [Ter18] in connection to hypergraph growth rates.

**Definition 3.1.** For $k \in \mathbb{N}$, let $V_1, \ldots, V_{k+1}$ be sets. We say that a $(k+1)$-ary relation $E \subseteq V_1 \times \cdots \times V_{k+1}$ has VC$_k$-dimension $\leq d$, or VC$_k(E) \geq d$, if there is a $k$-dimensional $d$-box $A = A_1 \times \cdots \times A_k$ with $A_i \subseteq V_i$ and $|A_i| = d$ for $i = 1, \ldots, k$ shattered by $E$. That is, for every $S \subseteq A$, there is some $b_S \in V_{k+1}$ such that $S = A \cap E b_S$. We say that VC$_k(E) = d$ if $d$ is maximal such that there is a $d$-box shattered by $E$, and VC$_k(E) = \infty$ if there are $d$-boxes shattered by $E$ for arbitrarily large $d$.

In the case $k = 1$ and $E \subseteq V_1 \times V_2$, VC$_1(E) = d$ simply means that the family $\mathcal{F} := \{E_a : a \in V_2\}$ of all subsets of $V_1$ given by the fibers of $E$ has VC-dimension $d$.

The following equivalence is straightforward (see [CPT19] Proposition 5.2 for the details).

**Remark 3.2.** For $E \subseteq V_1 \times \cdots \times V_{k+1}$, VC$_k(E) \leq d$ implies that $E$ omits some finite $(k+1)$-partite hypergraph as an induced partite hypergraph, with parts of size at most $d' := 2^d$. And if $E$ omits some finite $(k+1)$-partite hypergraph with all parts of size at most $d'$, then VC$_k(E) \leq d'$.

In particular, VC$_k(E) < \infty$ if and only if $E$ omits some finite $(k+1)$-partite hypergraph as an induced partite hypergraph.

**Fact 3.3.** For every $d \in \mathbb{N}$ there exists some $D = D(d) \in \mathbb{N}$ such that: if $E, F \subseteq V_1 \times \cdots \times V_{k+1}$ are two relations with VC$_k(E), V C_k(F) \leq d$, then:

- [CPT19] Corollary 3.15 VC$(\neg E), V C(E \cap F), V C(E \cup F) \leq D$;
- [CPT19] Corollary 5.3 If $\sigma \in S_{k+1}$ is any permutation of the set $\{1, \ldots, k+1\}$, then VC$_k(E^\sigma) \leq D$.

We also extend the definition of VC$_k$-dimension to relations of arity higher than $k + 1$ as follows:

**Definition 3.4.** Let $k < k' \in \mathbb{N}$ be arbitrary. We say that a $k'$-ary relation $E \subseteq V_1 \times \cdots \times V_{k'}$, has VC$_k$-dimension $\leq d$ if for any $I \subseteq [k']$ with $|I| = k' - (k + 1)$ and any $b \in V_I$, the relation $E_b$ (i.e. the fiber of $E$ with the coordinates in $I$ fixed by the elements of the tuple $b$, viewed as a $(k+1)$-ary relation on $V_{[k'] \setminus I}$) has VC$_k'$-dimension $\leq d$ (in the sense of Definition 3.1).

We write VC$_k(E)$ for the least $d$ such that VC$_k$-dimension of $E$ is $\leq d$, or $\infty$ if there is no such $d$.

That is, when $E$ is a $k'$-ary relation with $k' > k + 1$, the VC$_k$-dimension of $E$ is the supremum of the VC$_k$-dimension over all $(k+1)$-ary fibers $E_b$. 


Remark 3.5. It is easy to see that any fiber of a relation with finite \( \text{VC}_k \)-dimension also has finite \( \text{VC}_k \)-dimension; that finite \( \text{VC}_k \)-dimension is preserved under Boolean combinations and permutations of variables (using Fact 3.3); and that if \( k' > k_2 \geq k_1 \) and \( E \) is a \( k' \)-ary relation with \( \text{VC}_{k_1}(E) < \infty \), then also \( \text{VC}_{k_2}(E) < \infty \).

The natural examples of relations with finite \( \text{VC}_k \)-dimension are those which are “essentially \( k \)-ary”—that is, relations which are built from \( k \)-ary relations.

Example 3.6. Let \( E \subseteq V_1 \times \ldots \times V_{k+1} \) be a relation given by a finite Boolean combination of arbitrary relations \( E_1, \ldots, E_m, m \in \mathbb{N} \), such that each \( E_i \) is of the form \( E'_i \times V_{[k+1] \setminus I_i} \), for some \( I_i \subseteq [k+1] \) with \( |I_i| \leq k \) and some \( E'_i \subseteq V_{I_i} \). Then \( \text{VC}_k(E) < \infty \) by Fact 3.3(1), since every relation of arity \( \leq k \) trivially has finite \( \text{VC}_k \)-dimension.

The main result of the paper essentially shows that, up to an error of arbitrarily small measure, every \( k \)-dependent relation is of this form.

Example 3.7. Assume \( V_1 = V_2 = V_3 = V, F, G, H \subseteq V^2 \) are arbitrary (e.g. quasi-random), and let \( E \subseteq V^3 \) consist of those triples \( (x, y, z) \in V^3 \) for which an odd number of the pairs \( (x, y), (x, z), (y, z) \) belongs to \( F, G, H \), respectively. We claim that \( \text{VC}_2(E) \leq 65 \). Consider any \( \{y_1, \ldots, y_5\} \subseteq V \) and \( \{z_1, \ldots, z_6\} \subseteq V \). By Ramsey’s theorem, possibly reordering the elements, we may assume that either \( \{y_1, y_2, y_3\} \times \{z_1, z_2, z_3\} \subseteq H \) or \( \{y_1, y_2, y_3\} \times \{z_1, z_2, z_3\} \cap H = \emptyset \). But then no \( x \in V \) can satisfy \( E_x \cap \{y_1, y_2, y_3\} \times \{z_1, z_2, z_3\} = \{(y_1, z_1), (y_2, z_2), (y_3, z_3)\} \), as this would imply that no two of the values \( \chi_F(x, y_1), \chi_F(x, y_2), \chi_F(x, y_3) \) can be equal, which is impossible.

Example 3.8. Let \( V \) be a \( K \)-vector space, where \( K \) is one of the following fields: \( \mathbb{F}_p, \mathbb{C}, \mathbb{F}^{\text{alg}}_p \) or \( \mathbb{R} \), where \( p \) is a prime number. Let \( f : V \times V \to K \) be a non-degenerate bilinear form. Then every relation definable in the structure \( (V, K, f) \) (on tuples of any arity), in the sense of first order logic, has finite \( \text{VC}_2 \)-dimension. See [CH19b] for the details.

The following is a generalization of the Sauer-Shelah lemma from \( \text{VC}_1 \) to \( \text{VC}_k \)-dimension.

Fact 3.9. [CPT19] Proposition 3.9] If \( E \subseteq V_1 \times \ldots \times V_{k+1} \) satisfies \( \text{VC}_k(E) < d \), then there is some \( \varepsilon = \varepsilon(d) \in \mathbb{R}_{>0} \) such that: for any \( A = A_1 \times \ldots \times A_k \subseteq V_1 \times \ldots \times V_k \) with \( |A_1| = \ldots = |A_k| = m \), there are at most \( 2^{m^{k-\varepsilon}} \) different sets \( S \subseteq A \) such that \( S = A \cap E_b \) for some \( b \in V_{k+1} \).

Remark 3.10. More precisely, if \( \text{VC}_k \leq d \), then the upper bound above is actually given by \( \sum_{i \leq z} \binom{m^k}{i} \leq \binom{2^m k - \varepsilon}{k} \) for \( m \geq k \), where \( z = z_k(m, d + 1) \) is the Zarankiewicz number, i.e. the minimal natural number \( z \) satisfying: every \( k \)-partite \( k \)-hypergraph with parts of size \( m \) and \( \geq z \) edges contains the complete \( k \)-partite \( k \)-hypergraph with each part of size \( d + 1 \). If \( k = 1 \),
then \( z_1(m, d + 1) = d + 1 \), hence the bound in Fact 3.9 coincides with the Sauer-Shelah bound, and for a general \( k \) the bound in Fact 3.9 appears close to optimal (see \([CPT19]\) Proposition 3.9 for the details).

### 3.2. VC\(_k\)-dimension for real-valued functions

We generalize the notion of VC\(_k\)-dimension and some of its basic properties from relations to functions, generalizing \([Tal87, Tal96]\) in the case \( k = 1 \).

**Definition 3.11.** Let \( f : \prod_{i \in [k+1]} V_i \to [0, 1] \) be a function.

1. Given \( r < s \in [0, 1] \), we say that a box \( A = A_1 \times \ldots \times A_k \) with \( A_i \subseteq V_i \) is \((r, s)\)-shattered by \( f \) if for every \( S \subseteq A \) there exists some \( c_S \in V_{k+1} \) so that \( f(\bar{a}, c_S) \leq r \) for every \( \bar{a} \in S \) and \( f(\bar{a}, c_S) \geq s \) for every \( \bar{a} \in A \setminus S \).
2. Given \( d = (d_{r,s})_{r<s \in [0,1]} \) with each \( d_{r,s} \in \mathbb{N} \), we will write \( \text{VC}_k(f) \leq \tilde{d} \) if for every \( r < s \in [0,1] \), there is no box \( A = \prod_{i \in [k]} A_i \) with \( A_i \subseteq V_i \) and \( |A_i| = d_{r,s} \) for each \( i \in [k] \) which is \((r, s)\)-shattered by \( f \).
3. We say that \( f \) has finite VC\(_k\)-dimension, or \( \text{VC}_k(f) < \infty \), if there exists some sequence \( \tilde{d} \) with \( d_{r,s} \in \mathbb{N} \) so that \( \text{VC}_k(f) \leq \tilde{d} \); and that \( f \) has infinite VC\(_k\)-dimension or \( \text{VC}_k(f) = \infty \) otherwise.
4. Given an arbitrary \( k' \in \mathbb{N} \), we say that a function \( f : \prod_{i \in [k']} V_i \to [0,1] \) satisfies \( \text{VC}_k(f) \leq \tilde{d} \) if either \( k' \leq k \) or \( k' > k \) and for any \( I \subseteq [k'] \) with \( |I| = k' - (k + 1) \) and any \( \bar{b} \in V_I \), the function \( f_{\bar{b}} : V_{[k'] \setminus I} \to [0,1], f_{\bar{b}}(\bar{x}) = f(\bar{x} \oplus \bar{b}) \) has \( \text{VC}_{k'}(\cdot) \leq \tilde{d} \).

**Remark 3.12.** Note that if \( E \subseteq \prod_{i \in [k+1]} V_i \), then \( \text{VC}_k(E) \leq d \) if and only if \( \text{VC}_k(\chi_E) \leq \tilde{d} \) with \( d_{r,s} = d \) for all \( r < s \in [0,1] \).

It is sometimes convenient to speak of the VC\(_k\)-dimension of \( f \) “at \((r, s)\):”

**Definition 3.13.** Let \( f : \prod_{i \in [k+1]} V_i \to [0,1] \) be a function. We will write \( \text{VC}_k^{r,s}(f) \leq d \) if there is no box \( A = \prod_{i \in [k]} A_i \) with \( A_i \subseteq V_i \) and \( |A_i| = d \) for each \( i \in [k] \) which is \((r, s)\)-shattered by \( f \).

That is, \( \text{VC}_k(f) \leq \tilde{d} \) is the same as \( \text{VC}_k^{r,s}(f) \leq d_{r,s} \) for all \( r < s \).

Finally, the following is a straightforward analog of Remark 3.2 for real-valued functions.

**Remark 3.14.** For \( f : \prod_{i \in [k+1]} V_i \to [0,1] \) and \( r < s \), \( \text{VC}_k^{r,s}(f) \leq d \) implies that \( f \) omits some finite \((k + 1)\)-partite \( k\)-uniform hypergraph \( H \) as an “induced” \( k\)-partite hypergraph with parts of size at most \( d' := 2^d \), in the sense that there is no way to identify the \( i \)th part of \( H \) to a subset of \( V_i \) so that, restricting to these sets, \( f \) takes values \( \leq r \) on the edges of \( H \) and \( \geq s \) on the non-edges of \( H \).

And if \( f \) omits some finite \((k + 1)\)-partite hypergraph with all parts of size at most \( d' \) in this sense, then \( \text{VC}_k^{r,s}(E) \leq d' \).
4. Level Sets and Some Lemmas about $L^2$-Norm

Throughout this section, we fix $k \in \mathbb{N}_{\geq 1}$ and let $\Psi = (V_{[k]}, \mathcal{B}_{\bar{n}}, \mu_{\bar{n}})_{\bar{n} \in \mathbb{N}^k}$ be a $k$-partite graded probability space. We fix $\bar{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$, $n = \sum_{i \in [k]} n_i$ and $f : \prod_{i \in [k]} V_i^{n_i} \to [0, 1]$ a $\mathcal{B}_{\bar{n}}$-measurable function.

4.1. Level Sets of functions. We will frequently need to consider the level sets of functions.

Definition 4.1. For $r, q \in \mathbb{R}$, we let

$$f^{<r} := \{ \bar{x} \in V^\bar{n} : f(\bar{x}) < r \},$$

$$f^{\geq r} := V^\bar{n} \setminus f^{<r}, \quad f^{[r,q]} := f^{<q} \cap f^{\geq r}.$$

The next lemma captures the following idea: if $f$ is not $\mathcal{B}$-measurable then there should be points which are “fuzzy” with respect to $\mathcal{B}$, in the sense that there are an $r < s$ so that if we made a random choice of $x$ with respect to $\mathcal{B}$, there should be positive probability that $f(x) < r$ and positive probability that $f(x) > s$. In the language of $\sigma$-subalgebras, this becomes the statement that both $\mathbb{E}(\chi_{f^{<r}} \mid \mathcal{B})(x) \geq \delta$ and $\mathbb{E}(\chi_{f^{\geq s}} \mid \mathcal{B})(x) \geq \delta$ for some $\delta > 0$.

Lemma 4.2. Assume that $f : V^\bar{n} \to [0, 1]$ is a $\mathcal{B}_{\bar{n}}$-measurable function, $\varepsilon \in \mathbb{R}_{>0}$ and $\mathcal{B} \subseteq \mathcal{B}_{\bar{n}}$ is a $\sigma$-algebra such that $\|f - \mathbb{E}(f \mid \mathcal{B})\|_{L^2} \geq \varepsilon$. Then there exist some $t = t(\varepsilon) \in \mathbb{N}$, $r < s \in \mathbb{Q}_{[0,1]}^{\bar{n}}$ and $\delta = \delta(\varepsilon) \in \mathbb{R}_{>0}$ so that

$$\mu_{\bar{n}} \left( \left\{ x \in V^\bar{n} : \mathbb{E}(\chi_{f^{<r}} \mid \mathcal{B})(x) \geq \delta \wedge \mathbb{E}(\chi_{f^{\geq s}} \mid \mathcal{B})(x) \geq \delta \right\} \right) \geq \delta.$$

Proof. Let $\alpha \in \mathbb{R}_{>0}$ be arbitrary, and we fix a sufficiently large $t = t(\alpha) \in \mathbb{N}$ and an even $\ell = \ell(\alpha) \in \mathbb{N}$ and a partition $0 = q_0 < \ldots < q_\ell = 1$ of $[0, 1]$ with $q_i \in \mathbb{Q}_{[0,1]}^{\bar{n}}, q_i - q_{i-1} < \alpha$ for all $i \in [\ell]$. We let $U_{-1} := f^{<q_1}, U_i := f^{\geq q_i} \cap f^{<q_{i+2}}$ for $i \in \{0, \ldots, \ell - 2\}, U_{\ell} := f^{\geq q_\ell}$ for $i \in \{\ell - 1, \ell - 2\}$.

Fix $\gamma \in \mathbb{R}_{>0}$, and let

$$Z := \left\{ x \in V^\bar{n} : \bigwedge_{i \in \{-1, \ldots, \ell\}} \mathbb{E}(\chi_{U_i} \mid \mathcal{B}) < 1 - \gamma \right\}, \quad \text{and for } i \in \{-1, \ldots, \ell\},$$

$$V_i := \left\{ x \in V^\bar{n} \setminus Z : i = \min \left\{ j \in \{-1, \ldots, \ell\} : \mathbb{E}(\chi_{U_j} \mid \mathcal{B})(x) \geq 1 - \gamma \right\} \right\}.$$

Note that $\{Z, V_{-1}, \ldots, V_\ell\}$ is a partition of $V^\bar{n}$, and each of these sets is in $\mathcal{B}$. And for each $i \in \{-1, \ldots, \ell\}$ we have

$$(4.1) \quad \mu_{\bar{n}}(V_i \cap U_i) = \int_{V_i} \chi_{U_i} d\mu_{\bar{n}} = \int_{V_i} \mathbb{E}(\chi_{U_i} \mid \mathcal{B}) d\mu_{\bar{n}} \geq (1 - \gamma)\mu_{\bar{n}}(V_i).$$

Consider the $\mathcal{B}$-measurable function $g := \sum_{i \in \{-1, \ldots, \ell\}} q_i \chi_{V_i}$.

Fix $i \in \{-1, \ldots, \ell\}$ and $x \in V_i \cap U_i$. Then $g(x) = q_i$, and by definition of the $U_i$'s: $f(x) \in [q_i, q_{i+2}]$ if $i \in \{0, \ldots, \ell - 2\}, f(x) \in [q_i, 1]$ if $i \in \{\ell - 1, \ell - 2\}$, and $f(x) \in [0, q_1]$ if $i = -1$. In either case, we get $|f - g|(x) \leq 2\alpha$. 

Then, using the assumption on \( f, g \) and that \( f, g \) are \([0,1]\)-valued, we have
\[
\varepsilon \leq \| f - g \|_{L^2}^2 = \int (f - g)^2 \, d\mu =
\int_Z (f - g)^2 \, d\mu = \sum_{i \in \{-1, \ldots, \ell\}} \int_{V \setminus U_i} (f - g)^2 \, d\mu_i + \sum_{i \in \{-1, \ldots, \ell\}} \int_{V \cap U_i} (f - g)^2 \, d\mu_i
\leq \mu_i(Z) + \sum_{i \in \{-1, \ldots, \ell\}} \mu_i(V_i \setminus U_i) + \sum_{i \in \{-1, \ldots, \ell\}} (2\alpha)^2 \mu_i(V_i \cap U_i)
\leq \mu_i(Z) + \gamma \sum_{i \in \{-1, \ldots, \ell\}} \mu_i(V_i) + (2\alpha)^2 \sum_{i \in \{-1, \ldots, \ell\}} \mu_i(V_i)
\leq \mu_i(Z) + \gamma + (2\alpha)^2.
\]
Assuming \( \gamma + (2\alpha)^2 < \frac{\varepsilon}{2} \), we get \( \mu_i(Z) \geq \frac{\varepsilon}{2} \).

As \( \{U_{2i} : i \in \{0, \ldots, \frac{\ell}{2}\}\} \) and \( \{U_{2i-1} : i \in \{0, \ldots, \frac{\ell}{2}\}\} \) are both partitions of \( V^{\bar{\alpha}} \), we also have
\[
(4.2) \quad \sum_{i \in \{0, \ldots, \frac{\ell}{2}\}} \mathbb{E}(\chi_{U_{2i}} \mid \mathcal{B})(x) = 1 \text{ and } \sum_{i \in \{0, \ldots, \frac{\ell}{2}\}} \mathbb{E}(\chi_{U_{2i-1}} \mid \mathcal{B})(x) = 1.
\]
By definition, \( x \in Z \implies \bigwedge_{i \in \{-1, \ldots, \ell\}} \mathbb{E}(\chi_{U_i} \mid \mathcal{B})(x) \leq 1 - \gamma \). In particular, taking \( \delta_0 := \frac{\gamma}{\ell} > 0 \) and using (4.2), for each \( x \in Z \) there must exist some \( i_0, i_1, j_0, j_1 \in \{-1, \ldots, \ell\} \) such that \( i_0 < i_1 \) are both even, \( j_0 < j_1 \) are both odd, and \( \mathbb{E}(\chi_{U_i} \mid \mathcal{B})(x) \geq \delta \) for each \( i \in \{i_0, i_1, j_0, j_1\} \). As there are at most \( \ell^4 \) possible choices for the quadruple \( (i_0, i_1, j_0, j_1) \), by additivity of \( \mu \) there is a set \( Z' \subseteq Z, Z' \in \mathcal{B} \) with \( \mu_i(Z') \geq \delta_1 := \frac{\mu_i(Z)}{\ell^4} \geq \frac{\varepsilon}{2\ell^4} > 0 \) and so that all \( x \in Z' \) share the same values of \( i_0, i_1, j_0, j_1 \). Then either \( i_0 + 2 < j_1 \) (and so) or \( j_0 + 2 < i_1 \); we let \( r := q_{i_0+2}, s := q_{j_1} \) in the former case, and \( r := q_{j_0+2}, s := q_{i_1} \) in the latter case. Then \( r < s \) and the conclusion of the lemma holds by monotonicity of conditional expectation, with \( \delta := \min\{\delta_0, \delta_1\} > 0 \) (note that the choice of \( \delta \) and \( t \) in the proof only depends on \( \varepsilon \)).

\[
\text{Lemma 4.3. Let } (V, \mathcal{B}, \mu) \text{ be a probability space, and assume that } f_0, f_1 : V \to [0,1] \text{ are } \mathcal{B}-\text{measurable functions so that } \int f_1 \, d\mu > \int f_0 \, d\mu \text{. Then there exist some } r < s \in \mathbb{Q}[0,1] \text{ so that } \mu(f_0^{<r}) > \mu(f_1^{<s}).
\]
\[
\text{Proof. Without loss of generality we may replace } f_b \text{ by } f_b \circ \pi_b^{-1} : [0,1] \to [0,1], \text{ where } \pi_b : V \to [0,1] \text{ is a measure-preserving function (with respect to the Lebesgue measure on } [0,1]) \text{ so that } f_b \circ \pi_b^{-1} \text{ is monotone, for } b \in \{0,1\}. \text{ (We can take } \pi_b(x) := \mu(f_b^{<b}(x)) \text{ and make countably many tweaks for those } r \in [0,1] \text{ for which } \mu(\{x \mid f_b(x) = r\}) \text{ has positive measure.)}
\]
Now we almost have \( \mu(f_b^{<b}(x)) = x \); the exception is if the left-handed derivative of \( f_b \) at \( x \) is equal to \( 0 \) — that is, if the set of \( y \) such that \( f_b(y) = \)
\(f_b(x)\) has positive measure, and \(x\) is in the middle or is the right endpoint of this constant interval. But we at least have \(\mu\left(f_b^{<x}\right) \geq x\), and for all \(r > f_b(x)\), we have \(\mu(f_b^{<r}) \geq x\).

Let \(\varepsilon > 0\) be small enough and define \(f_0'(x) = f_0(x + \varepsilon)\). Then we have \(f_0'(x) - f_1 \leq f_0'(1 - \varepsilon)\). Then, since \(f_0', f_1\) are monotone, there must be an \(x\) with \(f_0'(x) < f_1(x)\). Let \(r \in (f_0'(x), f_1(x))\) and \(s = f_1(x)\). Then \(\mu(f_0^{<r}) \geq x + \varepsilon > x \geq \mu(f_0^{<s})\). \(\square\)

### 4.2. Lemmas about measure and \(L^2\)-norm

In this section we collect some miscellaneous lemmas about measurability and the \(L^2\)-norm that will be needed later in the article.

#### Remark 4.4

Let a \(\sigma\)-algebra \(B \subseteq B_n\), \(\varepsilon \in \mathbb{R}_{>0}\) and a set \(X \in B_n\) be given. If \(|X - \mathbb{E}(X | B)|\|L^2 \leq \frac{\varepsilon^2}{2}\), then there exists some \(Y \in B\) such that \(|\mathbb{E}(X - X)\|L^2 \leq 3\varepsilon\) (and the converse implication obviously holds, with the same \(\varepsilon\)).

**Proof.** As \(\mathbb{E}(X | B)\) is \(B\)-measurable, there must exist a \(B\)-simple function \(h = \sum_{i \in \{m\}}\alpha_i \chi_{C_i}\) for some \(\{m\} \in \mathbb{N}\), \(\alpha_i \in \mathbb{R}\) and pairwise disjoint sets \(C_i \in B\), such that \(|\mathbb{E}(X | B) - h\|L^2 < \frac{\varepsilon^2}{2}\), so \(|\mathbb{E}(X - h)\|L^2 < \varepsilon^2\). But then the measure of the union of those \(C_i\) for which \(\alpha_i \notin \{0, \varepsilon\} \cup (\varepsilon, 1]\) must be at most \(\varepsilon\) (as in Lemma [4.2]). So we may replace \(\sum\alpha_i \chi_{C_i}\) by the union of \(C\) of those \(C_i\) with \(\alpha_i > \varepsilon\). Then the \(L^2\)-distance of \(\chi_{C}\) from \(\sum\alpha_i \chi_{C_i}\) is at most \(2\varepsilon\), so \(|\chi_{X} - \chi_{C}\|L^2 < \varepsilon^2 + 2\varepsilon < 3\varepsilon\). \(\square\)

The following lemma is well known (see e.g. [Ber85, Theorem 1.1]).

#### Fact 4.5

*For any \(\varepsilon \in \mathbb{R}_{>0}\) and \(m \in \mathbb{N}\) there exists some \(N = N(\varepsilon, m) \in \mathbb{N}\) and \(\xi = \xi(\varepsilon, m) \in \mathbb{R}_{>0}\) satisfying the following. Given any probability space \((V, B, \mu)\) and any sequence \((X_i : i \in [N])\) of sets in \(B\) with \(\mu(X_i) \geq \varepsilon\) for all \(i \in [N]\), there exists some subsequence \((X_i : i \in I)\) with \(|I| \geq m\) and such that \(\mu(\cap_{i \in I} X_i) > \xi\).*

#### Lemma 4.6

Let \(R \in B_n\) be such that \(\mu_n(R) \geq \alpha > 0\). For \(\bar{d} = (d_1, \ldots, d_k) \in \mathbb{N}_0^k\), let \(\Sigma\) be the set

\[
\left\{ (\bar{x}_1, \ldots, \bar{x}_k) \in \prod_{i \in [k]} (V_i^{n_i})^{d_i} : (\bar{x}_{1,i_1}, \ldots, \bar{x}_{k,i_k}) \in R \right. \\
\left. \text{for all } i_1 \in [d_1], \ldots, i_k \in [d_k] \right\}
\]

Then \(\Sigma \in B_{\bar{d}n}\) and \(\mu_{\bar{d}n}(\Sigma) \geq \alpha^{d_1 \cdots d_k} > 0\).

**Proof.** Let \(R, \alpha\) and \(d\) as above be fixed. Let \(i \in [k]\) be arbitrary, and let \(R'\) be the set of all tuples \(\bar{x} = (\bar{x}_1, \ldots, \bar{x}_{i-1}, (\bar{x}_{i,1}, \ldots, \bar{x}_{i,d_i}), \bar{x}_{i+1}, \ldots, \bar{x}_k)\) in \(V_1^{n_1} \times \ldots \times V_1^{n_i-1} \times (V_i^{n_i})^{d_i} \times V_{i+1}^{n_{i+1}} \times \ldots \times V_k^{n_k}\) so that

\[(\bar{x}_1, \ldots, \bar{x}_{i-1}, \bar{x}_{i,j}, \bar{x}_{i+1}, \ldots, \bar{x}_k) \in R\]
for every $i \in [d]$. Note that $R' \in \mathcal{B}_{n_{d_i n_i} \to i}$ by closure under products.

Then, by Fubini property and Hölder inequality with $p = d_i, q = \frac{d_i}{d_i - 1}$, we have

$$
\mu_{n_{d_i n_i} 	o i} (R') = \int \mu_{n_{d_i n_i} 	o i} \left( R'_x \right) d\mu_{\bar{n}_0 \to i} (\bar{x}|k) = 
\int \mu_{n_{d_i n_i} 	o i} \left( R'_x \right) d\mu_{\bar{n}_0 \to i} (\bar{x}|k) = 
\int \mu_{n_{d_i n_i} 	o i} \left( R'_x \right) d\mu_{\bar{n}_0 \to i} (\bar{x}|k) \cdot \int \frac{d_i}{d_i - 1} d\mu_{\bar{n}_0 \to i} (\bar{x}|k) 
\geq \left( \int \mu_{n_{d_i n_i} 	o i} \left( R'_x \right) d\mu_{\bar{n}_0 \to i} (\bar{x}|k) \right)^{d_i/d_i} \geq \alpha^{d_i} > 0.
$$

Repeating the same argument for every coordinate $i \in [k]$ (using Remark 2.4(1)), we conclude

$$
\mu_{n_1, n_2, \ldots, n_k} (\Sigma) \geq \alpha^{d_1 \cdots d_k} > 0.
$$

\[\square\]

**Lemma 4.7.** Assume that $\bar{n} = \bar{n}_1 + \bar{n}_2 \in \mathbb{N}^k$ and $f, g : V^{\bar{n}} \to [0, 1]$ are $\mathcal{B}_{\bar{n}}$-measurable functions, and $\varepsilon \in (0, 1)$.

1. The following implications hold:

   $$
   \|f - g\|_{L^2} < \varepsilon \implies \mu_{\bar{n}_2} \left( \{ \bar{x}_2 \in V^{\bar{n}_2} : \|f(\oplus \bar{x}_2) - g(\oplus \bar{x}_2)\|_{L^2(\mu_{\bar{n}_1})} > \varepsilon^{\frac{1}{2}} \} \right) \leq \varepsilon
   $$

   $$
   \implies \|f - g\|_{L^2} < \varepsilon^{\frac{1}{2}}.
   $$

2. More precisely, if $Y \in \mathcal{B}_{\bar{n}_2}$ with $\mu_{\bar{n}_2}(Y) > 0$ and

   $$
   \|f(\oplus \bar{x}_2) - g(\oplus \bar{x}_2)\|_{L^2(\mu_{\bar{n}_1})} < \varepsilon
   $$

   for every $\bar{x}_2 \in Y$, then

   $$
   \left\| \frac{1}{\mu_{\bar{n}_2}(Y)} \int f(\bar{x}_1 \oplus \bar{x}_2) \chi_Y(\bar{x}_2) d\mu_{\bar{n}_2}(\bar{x}_2) \right\|_{L^2(\mu_{\bar{n}_1}(\bar{x}_1))} < \varepsilon
   $$

3. If $f_i, g_i : V^{\bar{n}} \to [0, 1]$ are $\mathcal{B}_{\bar{n}}$-measurable and $\|f_i - g_i\|_{L^2} < \varepsilon$ for $i \in [\ell]$, then

   $$
   \left\| \prod_{i \in [\ell]} f_i - \prod_{i \in [\ell]} g_i \right\|_{L^2} < (2\ell + 1)\varepsilon \quad \text{and} \quad \left\| \sum_{i \in [\ell]} f_i - \sum_{i \in [\ell]} g_i \right\|_{L^2} < \ell\varepsilon.
   $$
Proof. By the Fubini property in graded probability spaces (Remark 2.3) and standard calculations. E.g., for (2), taking \( h := f - g \) we have \( \|h(\oplus \bar{x}_2)\|_{L^2(\mu_{\bar{m}})} < \varepsilon \) for every \( \bar{x}_2 \in \mathcal{Y} \). By Jensen’s inequality, for every fixed \( \bar{x}_1 \),

\[
\left( \frac{1}{\mu_{\bar{n}_2}(Y)} \int_Y h(\bar{x}_1 \oplus \bar{x}_2) d\mu_{\bar{n}_2}(\bar{x}_2) \right)^2 \leq \frac{1}{\mu_{\bar{n}_2}(Y)} \int_Y h(\bar{x}_1 \oplus \bar{x}_2)^2 d\mu_{\bar{n}_2}(\bar{x}_2).
\]

Using this and Fubini, we have

\[
\left\| \frac{1}{\mu_{\bar{n}_2}(Y)} \int_Y h(\bar{x}_1 \oplus \bar{x}_2) \chi_Y(\bar{x}_2) d\mu_{\bar{n}_2}(\bar{x}_2) \right\|^2_{L^2(\mu_{\bar{n}_1}(\bar{x}_1))} = \\
\int \left( \frac{1}{\mu_{\bar{n}_2}(Y)} \int_Y h(\bar{x}_1 \oplus \bar{x}_2) d\mu_{\bar{n}_2}(\bar{x}_2) \right)^2 d\mu_{\bar{n}_1}(\bar{x}_1) \leq \\
\int \left( \frac{1}{\mu_{\bar{n}_2}(Y)} \int_Y h(\bar{x}_1 \oplus \bar{x}_2)^2 d\mu_{\bar{n}_2}(\bar{x}_2) \right) d\mu_{\bar{n}_1}(\bar{x}_1) = \\
\int Y \left( \frac{1}{\mu_{\bar{n}_2}(Y)} \int h(\bar{x}_1 \oplus \bar{x}_2)^2 d\mu_{\bar{n}_1}(\bar{x}_1) \right) d\mu_{\bar{n}_2}(\bar{x}_2) \leq \\
\frac{1}{\mu_{\bar{n}_2}(Y)} \int Y \varepsilon^2 d\mu_{\bar{n}_2}(\bar{x}_2) \leq \varepsilon^2.
\]

\( \square \)

Lemma 4.8. Let \( \mathcal{B} \subseteq \mathcal{B}_{\bar{n}} \) be an arbitrary \( \sigma \)-algebra. Let \( \bar{m} \in \mathbb{N}^k \), and assume that \( g : V^{\bar{n}+\bar{m}} \rightarrow \mathbb{R} \) is a \( \mathcal{B}_{\bar{n}+\bar{m}} \)-measurable function such that the set of \( \bar{y} \in V^{\bar{m}} \) for which the function \( g(-, \bar{y}) : \bar{x} \mapsto g(\bar{x} \oplus \bar{y}) \) is \( \mathcal{B} \)-measurable has \( \mu_{\bar{m}} \)-measure 1. Then the “average fiber” function \( g'(\bar{x}) := \int g(\bar{x} \oplus \bar{y}) d\mu_{\bar{m}}(\bar{y}) \) is also \( \mathcal{B} \)-measurable.

Proof. Let \( h : V^{\bar{n}} \rightarrow \mathbb{R} \) be an arbitrary \( \mathcal{B}_{\bar{n}} \)-measurable function orthogonal to \( L^2(\mathcal{B}) \) (in the space \( L^2(\mathcal{B}_{\bar{n}}) \)). Then, for every fixed \( \bar{y} \in V^{\bar{m}} \) outside of a \( \mu_{\bar{m}} \)-measure 0 set, we have

\[
\langle g(- \oplus \bar{y}), h \rangle_{L^2} = \int g(\bar{x} \oplus \bar{y}) \cdot h(\bar{x}) d\mu_{\bar{n}}(\bar{x}) = 0.
\]

Hence, by Fubini,

\[
\int g'(\bar{x}) \cdot h(\bar{x}) d\mu_{\bar{n}}(\bar{x}) = \int \left( \int g(\bar{x} \oplus \bar{y}) d\mu_{\bar{m}}(\bar{y}) \right) \cdot h(\bar{x}) d\mu_{\bar{n}}(\bar{x}) = \\
\int \left( \int g(\bar{x} \oplus \bar{y}) \cdot h(\bar{x}) d\mu_{\bar{m}}(\bar{x}) \right) d\mu_{\bar{n}}(\bar{y}) = 0
\]

(so \( g' \) has no correlation with any function orthogonal to \( L^2(\mathcal{B}) \)). Now we can write

\[
g' = \mathbb{E}(g' \mid \mathcal{B}) + g^\perp,
\]
Then there exists some $p$ and, for $\varepsilon$ such that: for every $E$ where $\|VC\| = \|E\|$ is finite. Then for every $f < g$, we can choose $\gamma > 0$ small enough so that $\mu(\{f < g\} \setminus [f < (g - \gamma)]) < \varepsilon^2$. Let $p \in \mathbb{N}$ satisfy $p \gamma \geq 1$. Then

$$
\|\chi_{[f<g]} - [f < g]^p\|_{L^2}^2 = \int_{V_{n}\setminus [f<g]} 0d\mu + \int_{[f<(g-\gamma)]} 0d\mu + \int_{[f\leq g]\setminus [f<(g-\gamma)]} (\chi_{[f<g]} - [f < g]^p)^2 d\mu \\
\leq \mu(\{f < g\} \setminus [f < (g - \gamma)]) \leq \varepsilon^2.
$$

For $x, y \in [0, 1], x - y = \max\{0, x - y\} \in [0, 1], x + y = \min\{1, x + y\} \in [0, 1]$, and for $p \in \mathbb{N}$, $p \times x = x + \ldots + x$.

**Lemma 4.9.** Assume that $f, g : V_n \to [0, 1]$ are $B_n$-measurable functions and $\varepsilon \in (0, 1)$. We consider the $B_n$-measurable set

$$[f < g] := \{ \bar{x} \in V_n : f(\bar{x}) < g(\bar{x}) \}$$

and, for $p \in \mathbb{N}$, the $B_n$-measurable function $[f < g]^p : V_n \to [0, 1]$ defined by

$$[f < g]^p := p \times (g - f).$$

Then there exists some $p = p(f, g, \varepsilon) \in \mathbb{N}$ such that $\|\chi_{[f<g]} - [f < g]^p\|_{L^2} < \varepsilon$.

**Proof.** As $[f < g] = \bigcup_{\gamma \in \mathbb{Q} > 0} [f < (g - \gamma)]$, by countable additivity of $\mu_n$ we can choose $\gamma > 0$ small enough so that $\mu_n([f < g] \setminus [f < (g - \gamma)]) < \varepsilon^2$. Let $p \in \mathbb{N}$ satisfy $p \gamma \geq 1$. Then

$$
\|\chi_{[f<g]} - [f < g]^p\|_{L^2}^2 = \int_{V_n\setminus [f<g]} 0d\mu + \int_{[f<(g-\gamma)]} 0d\mu + \int_{[f\leq g]\setminus [f<(g-\gamma)]} (\chi_{[f<g]} - [f < g]^p)^2 d\mu \\
\leq \mu(\{f < g\} \setminus [f < (g - \gamma)]) \leq \varepsilon^2.
$$



5. **Approximation by finitely many fibers for functions of bounded VC$_k$-dimension**

5.1. **Statement and some corollaries of the approximation result.** The aim of this section is to prove the following.

**Proposition 5.1.** Let $\left(V_{[k+1]}, B_n, \mu_n\right)_{n \in \mathbb{N}^{k+1}}$ be a $(k + 1)$-partite graded probability space. Suppose that $f : V_{[k+1]} \to [0, 1]$ is $B_{[k+1]}$-measurable and $VC_k(f)$ is finite. Then for every $\varepsilon > 0$, there exist some $x_1, \ldots, x_N \in V_{k+1}$ such that: for every $x \in V_{k+1}$ we have

$$
\|f_x - \mathbb{E}\left(f_x \mid B_{[k,k-1]} \cup \{f_{x_1}, \ldots, f_{x_N}\}\right)\|_{L^2} < \varepsilon.
$$
Recall that for \( x \in V_{k+1}, f_x : V^k \to [0,1] \) is the function \( \bar{y} \mapsto f(\bar{y}^{-}(x)) \) corresponding to the fiber of \( f \) at \( x \). By Remark 2.3, \( f_x \) is \( \mathcal{B}_{1^k} = \mathcal{B}_{1^k-(0)} \)-measurable (see Definition 2.5(3)) for every \( x \in V_{k+1} \).

For relations (i.e. \((0,1)\)-valued functions) of finite VC\(_k\)-dimension this immediately implies the following (using Remark 4.4).

**Corollary 5.2.** Let \( (V_{[k+1]}, \mathcal{B}_n, \mu_n)_{n \in \mathbb{N}^{k+1}} \) be a \((k+1)\)-partite graded probability space. Suppose that \( E \in \mathcal{B}_{1^k+1} \) and VC\(_k\)(\( E \)) is finite. Then there exist some \( x_1, \ldots, x_N \in V_{k+1} \) such that, for every \( x \in V_{k+1} \), there is a set \( D_x \) which is a Boolean combination of \( E_{x_1}, \ldots, E_{x_N} \) and sets from \( \mathcal{B}_{1^k,k-1} \) such that
\[
\mu_{1^k}(E_x \triangle D_x) < \varepsilon.
\]

**Remark 5.3.** When \( k = 1 \), Corollary 5.2 corresponds to the familiar result for relations of finite VC-dimension discussed in the introduction.

Indeed, in this case the algebra \( \mathcal{B}_{(1),0} = \{\emptyset, V_1\} \) is trivial. Assume \( E \in \mathcal{B}_{1,1} \). Then by Corollary 5.2 there exist finitely many fibers \( E_{x_1}, \ldots, E_{x_N} \) of \( E \) with \( x_i \in V_2 \) so that for every \( x \in V_2 \), \( \mu_{1,0}(E_x \triangle D_x) < \frac{\varepsilon}{2} \) for some \( D_x \) a Boolean combination of \( E_{x_1}, \ldots, E_{x_N} \).

Let \( D_1, \ldots, D_{N'} \) list all Boolean combinations of \( E_{x_1}, \ldots, E_{x_N} \) that appear as \( D_x \) for some \( x \in V_2 \). Then, for each \( D_i \), we may choose some \( x'_i \in V_2 \) with \( \mu_{1,0}(D_i \triangle E_{x'_i}) < \frac{\varepsilon}{2} \).

Now for every \( x \in V_2 \) there exists some \( i \in [N'] \) so that \( \mu_{1,0}(E_x \triangle E_{x'_i}) < \varepsilon \).

That is, up to symmetric difference \( \varepsilon \), \( E \) has at most \( N' \) different fibers.

And using Sauer-Shelah, \( N' \) can be bounded by a polynomial of degree \( d \).

### 5.2. A quantitative statement of the approximation result.

In this section we restate Proposition 5.1 in a more quantitative form. This takes some work to state, because there should be quantitative bounds not only on the length of the sequence of fibers, but also on the complexity of the sets from \( \mathcal{B}_{1^k,k-1} \) used in the approximations.

In fact, most of the extra work is formulating the statement: the quantitative strengthening follows from the qualitative form by a compactness argument. We do not need this stronger form in what follows, so the reader can safely skip this subsection. Nonetheless, we include this stronger version both because the potential for bounds is of independent interest, and because the quantitative form is the form that can be applied directly to large finite hypergraphs.

The main additional definition we need to state the quantitative version will be \( \mathcal{F}^{f,n,b} \), which will be the collection of sets formed by certain fibers of level sets (recall Definition 4.1) of \( f \).

**Definition 5.4.** Let \( k \in \mathbb{N}_{\geq 1}, \mathfrak{Q} = (V_{[k+1]}, \mathcal{B}_n, \mu_n)_{n \in \mathbb{N}^{k+1}} \) be a \((k+1)\)-partite graded probability space, and assume that \( f : V^k \to [0,1] \) is a \( \mathcal{B}_{1^k+1} \)-measurable \((k+1)\)-ary function.
(1) Let \( \bar{b} \) be a tuple from \( V_{k+1} \) (finite or infinite), \( \bar{w} = (\bar{w}_1, \ldots, \bar{w}_k) \in V^m \)
for some \( \bar{m} = (m_1, \ldots, m_k) \in \mathbb{N}^k \) and \( m \in \mathbb{N} \). We let \( \mathcal{F}_w^{f,n,\bar{b}} \) be
the family of all sets in \( \mathcal{B}_{1,k+1}^{i} \) of the form
\[
\left( f_b^q \right)_{a_i \rightarrow i, i \in I} = \left\{ \tilde{x} = (x_1, \ldots, x_k) \in V^k : \tilde{x}_{a_i \rightarrow i, i \in I} (b) \in f^q \right\}
\]
for some \( q \in [0,1]^k \), \( \emptyset \neq I \subseteq \binom{[k]}{\leq k-1} \), \( a_i \in \bar{w}_i \) for \( i \in I \) and \( b \in \bar{b} \).

(2) Given a tuple \( \bar{b} \) from \( V_{k+1} \) (finite or infinite), let
\[
\mathcal{F}_{f,\bar{b}} := \mathcal{F}_{f,n,\bar{b}} : n \in \mathbb{N}, \bar{w} \in V^\bar{m}, \bar{m} \in \mathbb{N}^k \}
\]

(3) We let \( \mathcal{F}_f := \bigcup \{ \mathcal{F}_{f,\bar{b}} : \bar{b} \in V_{k+1} \} \).

(4) We let \( \mathcal{B}_w^{f,n,\bar{b}}, B_{1,k+1}^{f,\bar{b}}, B_{1,k+1}^{f,\bar{b},0} \) be the \( \sigma \)-subalgebras (and \( \mathcal{B}_w^{f,n,\bar{b}}, B_{1,k+1}^{f,\bar{b}}, B_{1,k+1}^{f,\bar{b},0} \)
the Boolean subalgebras) of \( \mathcal{B}_{1,k+1}^{f} \) generated by \( \mathcal{F}_{f,n,\bar{b}}, \mathcal{F}_{f,\bar{b}}, \mathcal{F}_f \)
respectively. Note that when \( \bar{b} \) is finite, we have \( \mathcal{B}_w^{f,n,\bar{b}} = \mathcal{B}_w^{f,n,\bar{b},0} \) are
both finite.

Now we can state a quantitative refinement of Proposition 5.1 (which says,
among other things, that the only sets from \( \mathcal{B}_{1,k+1}^{f} \) needed to approximate
\( f_x \) are the fibers of the level sets of \( f \)).

**Proposition 5.5.** For every \( k \in \mathbb{N} \), \( \bar{d} = (d_{r,s})_{r,s \in \mathbb{Q} \cap [0,1]} \) with \( d_{r,s} \in \mathbb{N} \) and \( \varepsilon \in \mathbb{R}_{>0} \) there exist some \( N, N_0 \in \mathbb{N} \) satisfying the following.

Let \( \left( V_{k+1}, \mathcal{B}_n, \mu_n \right)_{n \in \mathbb{N}^{k+1}} \) be a \( (k+1) \)-partite graded probability space.

Suppose that \( f : V_{k+1}^{1} \rightarrow [0,1] \) is \( \mathcal{B}_{1,k+1}^{f} \)-measurable and \( \text{VC}_k(f) \leq \bar{d} \). Then
there exist some \( x_1, \ldots, x_N \in V_{k+1} \) such that: for every \( x \in V_{k+1} \), there exist
some sets \( D_1, \ldots, D_N \in \mathcal{F}_f, N_0, x_1, x_N, x \) and \( a \left( \{ D_i \}_{i \in [N_0]} \cup \{ f^q \}_{q \in [0,1]} \right) \)

simple\footnote{8} function \( g_x \) with coefficients in \( \mathbb{Q}_{[0,1]} \) such that
\[
\| f_x - g_x \|_{L^2} < \varepsilon.
\]

This version of the proposition is non-trivial if we take the \( V_i \) to be very
large finite sets—\( N, N_0 \) depend only on \( k, \bar{d}, \varepsilon \), so we can choose the \( V_i \) to be
much larger than \( N, N_0 \). In that case the \( \sigma \)-algebras are trivialized—every
set is \( \mathcal{B}_{1,k+1,1}^{f} \)-measurable, since it can be written as a very large finite union
of singleton sets. But the collection \( \mathcal{F}_f, N_0, x_1, x_N, x \) is not all sets, so the
conclusion of the proposition is still useful.

### 5.3. Approximability by conditional expectations.

**Proof of**[5.1] and [5.2]. To prove Proposition 5.1 assume towards a contradiction we are given a \( (k+1) \)-partite graded probability space \( \mathbb{Q}_0 = \left( V_{k+1}, \mathcal{B}_n, \mu_n \right)_{n \in \mathbb{N}^{k+1}} \)
so that the conclusion of Proposition 5.1 fails for some \( \varepsilon \in \mathbb{R}_{>0} \).

8 Recall that this means that \( g_x \) is a finite linear combination of the characteristic functions of these sets.
Then we may select an infinite sequence $x_1, x_2, \ldots$ of elements of $V_{k+1}$ by successively choosing $x_{i+1}$ so that
\[ \left\| f_{x_{i+1}} - E\left( f_{x_{i+1}} \mid \mathcal{B}_{1,k-1} \cup \{ f_{x_1}, \ldots, f_{x_i} \} \right) \right\|_{L^2} \geq \varepsilon.\]

We would like to “homogenize” this sequence. For instance, we would like to ensure that the measures of sets like $f_{x_i}^{<r}$ do not depend on the particular elements $x_i, x_j$ in this sequence (as long as $x_i \neq x_j$). Using Ramsey’s Theorem, we can get part way there: we can find an infinite subsequence so that for any $i < j$, $\mu_{1k}(f_{x_i}^{<r} \cap f_{x_j}^{<r})$ belongs to some interval $(a, a + \delta)$ for some $a \in [0,1]$ and a small $\delta > 0$. (We do this by partitioning $[0,1]$ into finitely many intervals $[0,1] = \bigcup_i I_i$ and coloring pairs $i, j$ by the $r$ such that $\mu_{1k}(f_{x_i}^{<r} \cap f_{x_j}^{<r}) \in I_r$.) However, it will be convenient to pin down $\mu_{1k}(f_{x_i}^{<r} \cap f_{x_j}^{<r})$ exactly so that we do not need to keep track of the extra bounds like $\delta$. Furthermore (for instance, by Fact 1.3), if the measure of this intersection is constant, it must be strictly positive, and similarly for intersections of any number of the sets $f_{x_i}^{<r}$.

We will need to arrange that a sequence of intersections of this kind always has positive measure, not for the sets $f_{x_i}^{<r}$, but with a more complicated set we define below.

Furthermore, we want to take into account an additional property. Each $f_{x_i}$ is measurable with respect to some $\sigma$-algebra $\mathcal{B}_i \subseteq \mathcal{B}_{1k}$, and we can consider the “tail $\sigma$-algebra” $\mathcal{B} = \bigcap_N \bigcup_{i \geq N} \mathcal{B}_i$. It is convenient to take the fibers $f_{x_i}$ to be mutually independent over this tail $\sigma$-algebra, because then we can have $E(f_{x_i} \mid \mathcal{B} \cup \{ f_{x_i} : i < l \}) = E(f_{x_l} \mid \mathcal{B})$ for all $l$. This, too, is essentially a kind of homogenization implied by a de Finetti-style argument.

In order to fully homogenize, we may need to leave the original space $\mathfrak{F}_0$ for a different space $\mathfrak{F}$ (the ultrapower of $\mathfrak{F}$) in which we can find a sequence similar to the one we began with, but which is fully homogeneous. The details of this construction are given in Section 9. For now we treat this as a black box and focus on the combinatorial portion of the proof. We therefore have, by Theorem 0.28.

**Assumption 5.6.** There exists a $(k+1)$-partite graded probability space $\mathfrak{F} = (V_{k+1}, \mathcal{B}, \mu_d, \varepsilon) \in \mathbb{R}_{>0}$, a $\mathcal{B}_{1k}$-measurable function $f : V_{1k+1} \to [0,1]$ and a sequence $(x_l)_{l \in \mathbb{Z}}$ in $V_{k+1}$ satisfying the following:

\[ \text{Assumption 5.6 still follows from Theorem 0.28, so the remainder of the proof is unchanged.} \]
where $B \subseteq \text{this random behavior is consistent enough that we can find a large box to which each other, and with the help of Assumption 5.6(2), (4), and (5), contradicting Assumption 5.6(1).}

Given $\bar{\mu}$ think of $B$ that—as far as $f$ and an $r < r'$ in $[0, 1]$ so that $Y$ is $(r', s')$-shattered by $f$, contradicting Assumption 5.6(1).

By Assumption 5.6(3) and (5), we have

$$
\mu_{\bar{\mu}} \left( \{ \bar{x} \in V^{t_k} \mid E \left( \chi_{f_{x_0}^{<r}} \mid B \right) (x) \geq \delta \wedge \E \left( \chi_{f_{x_0}^{s_+}} \mid B \right) (x) \geq \delta \} \right) > 0,
$$

then for any $l \in \mathbb{N}$,

$$
\mu_{\bar{\mu}} \left( \bigcap_{i \in [l]} \{ \bar{x} \in V^{t_k} \mid E \left( \chi_{f_{x_i}^{<r}} \mid B \right) (x) \geq \delta \wedge \E \left( \chi_{f_{x_i}^{s_+}} \mid B \right) (x) \geq \delta \} \right) > 0;
$$

(3) $\| f_{x_1} - E (f_{x_1} \mid B_{\bar{\mu},k-1} \cup \{ f_{x_i} : i < l \}) \|_{L_2} \geq \varepsilon$ for all $l \in \mathbb{Z}$;

(4) $B_{\bar{\mu},k-1} \subseteq B \subseteq B_{\bar{\mu},k}$;

(5) for all $l \in \mathbb{N}$ we have

$$
E \left( f_{x_1} \mid B_{\bar{\mu},k-1} \cup \{ f_{x_i} : i < l \} \right) = E \left( f_{x_1} \mid B \cup \{ f_{x_i} : i < l \} \right) = E \left( f_{x_1} \mid B \right),
$$

where $B := \sigma \left( \{ f_{x_i} : i < 0 \} \cup B_{\bar{\mu},k-1} \right)$.

We will now show that this leads to a contradiction. The idea is that Assumption 5.6(3) implies that the fibers have some “random behavior” relative to each other, and with the help of Assumption 5.6(2), (4), and (5), this random behavior is consistent enough that we can find a large box $Y \subseteq \prod_{i \in [\varepsilon]} V_i$ and an $r < r'$ in $[0, 1]$ so that $Y$ is $(r', s')$-shattered by $f$, contradicting Assumption 5.6(1).

By Assumption 5.6(3) and (5), we have

$$
\| f_{x_0} - E (f_{x_0} \mid B) \|_{L_2} \geq \varepsilon.
$$

By Lemma 4.2 there exist some $r < s \in \mathbb{Q}^{[0,1]}$ and $\delta \in \mathbb{R}_{>0}$ so that

(5.1) $\mu_{\bar{\mu}} \left( \{ \bar{y} \in V^{t_k} : E \left( \chi_{f_{x_0}^{<r}} \mid B \right) (\bar{y}) \geq \delta \wedge E \left( \chi_{f_{x_0}^{s_+}} \mid B \right) (\bar{y}) \geq \delta \} \right) \geq \delta$.

Fix arbitrary $r', s' \in \mathbb{Q}^{[0,1]}$ so that $r < r' < s' < s$, and let $E^1 := f^{<r'}, E^{-1} := f^{s'}$, and $E^0 = V^{t_k}$. For $i \in \mathbb{N}$ let

$$
F_{\delta}(x_i) := \{ \bar{y} \in V^{t_k} : E \left( \chi_{E_{x_i}^{t_k}} \mid B \right) (\bar{y}) \geq \delta \wedge E \left( \chi_{E_{x_i}^{t_k}} \mid B \right) (\bar{y}) \geq \delta \} \in B.
$$

This is precisely the set to which Assumption 5.6(2) applies. We should think of $\bar{y} \in F_{\delta}(x_i)$ as the points where $f_{x_i}$ is “ambiguous” to $B$ in the sense that—as far as $B$ can tell—both $f_{x_i}(\bar{y}) < r'$ and $f_{x_i}(\bar{y}) \geq s'$ seem plausible.

**Definition 5.7.** Given $l \in \mathbb{N}$, $u : [l] \to \{-1, 0, 1\}$ and $x_1, \ldots, x_l \in V_{k+1}$, let $S_u \in B_{\varepsilon} \subseteq B_{\varepsilon}$ be the subset of $V^{t_k}$ given by

$$
\bigcap_{i \in [l]} E_{x_i}^{u(i)},
$$
That is, \( u \) specifies a configuration of the \( x_i \)—whether we want our points to be in \( E_{x_i}^1 = f_{x_i}^{< r'} \), in \( E_{x_i}^{-1} = f_{x_i}^{\leq r'} \), or to ignore \( x_i \). \( S_u \) is then all the points which satisfy this configuration.

Now we want to show that for any \( m, l \in \mathbb{N} \) and any sequence of functions \( \left(u_i : i = (i_1, \ldots, i_k) \in [m]^k \right) \) with \( u_l : [l] \to \{-1, 0, 1\} \), we have

\[
\int \prod_{i \in [m]^k} \chi_{S_{u_i}}(y_{1,i_1}, \ldots, y_{k,i_k}) d\mu_{m, \tilde{l}}(\tilde{y}_1, \ldots, \tilde{y}_k) > 0,
\]

where \( \tilde{y}_s = (y_{s,1}, \ldots, y_{s,m}) \) for all \( s \in [k] \). In particular, suppose that we take \( l = 2^m \), let \( \pi : [l] \to \mathcal{P}([m]^k) \) be a bijection, and for each \( i \in [m]^k \) we define

\[
u_i(t) = \begin{cases} 1 & \text{if } i \in \pi(t) \\ -1 & \text{otherwise} \end{cases}
\]

for all \( t \in [l] \). Since the integral is positive, we have \( \mu_{m, \tilde{l}}(Z) > 0 \) for the set \( Z \in \mathcal{B}_{m, \tilde{l}} \) defined by

\[
\chi_Z \left( (y_{s,t})_{s \in [k], t \in [m]} \right) = \prod_{i \in [m]^k} \chi_{S_{u_i}}(y_{1,i_1}, \ldots, y_{k,i_k}).
\]

Then, taking any tuple \( (y_{s,t} : s \in [k], t \in [m]) \in Z \), we have that for any \( A \subseteq Y := \prod_{s \in [k]} \{y_{s,1}, \ldots, y_{s,m}\} \) there is some \( i \in [l] \) so that

\[
A = Y \cap E_{x_i}^1 = Y \cap f_{x_i}^{< r'} \quad \text{and} \quad Y \setminus A = Y \cap E_{x_i}^{-1} = Y \cap f_{x_i}^{\geq r'},
\]

hence the box \( Y \) is \((r', s')\)-shattered by \( f \). This would give a contradiction to Assumption \( 5.6(1) \) starting with some \( m > d', s' \).

We turn to showing that, for any choice of \( m, l \in \mathbb{N} \) and functions \( (u_i : \bar{i} \in [m]^k) \),

\[
\int \prod_{i \in [m]^k} \chi_{S_{u_i}}(y_{1,i_1}, \ldots, y_{k,i_k}) d\mu_{m, \tilde{l}}(\tilde{y}_1, \ldots, \tilde{y}_k) > 0.
\]

Since the inside of this integral is always non-negative, it suffices to find some subset of positive measure on which it is strictly positive.

Let

\[
\tilde{F} := \bigcap_{i \in [l]} F_{\tilde{x}_i}(x_i) \in \mathcal{B}.
\]

By \( 5.1 \) and Assumption \( 5.6(2) \), \( \beta := \mu_{\tilde{l}} \left( \tilde{F} \right) > 0 \). An element of \( \tilde{F} \) is “ambiguous” to \( \mathcal{B} \) for all the \( f_{x_i} \) at once; we should expect (and it follows from the work below) that for any \( u \) and any positive measure \( \mathcal{B} \)-measurable set \( D, \tilde{F} \cap D \cap S_u \) has positive measure.

By Lemma \( 4.6 \) the set

\[
F := \left\{(y_{s,t} : s \in [k], t \in [m]) \in V_{m, \tilde{l}} \mid \forall \bar{i} \in [m]^k \ (y_{1,i_1}, \ldots, y_{k,i_k}) \in \tilde{F} \right\}
\]

is in \( \mathcal{B}_{m, \tilde{l}} \) and \( \mu_{m, \tilde{l}}(F) \geq \beta^m \).
We will show that
\[ \int_F \prod_{i \in [m]^k} \chi_{S_{u_i}}(y_{1,i_1}, \ldots, y_{k,i_k})d\mu_{m-\overline{1}^k}(\overline{y}_1, \ldots, \overline{y}_k) > 0. \]

That is, we will show that we can find a positive measure set of matrices \( \overline{y} = (\overline{y}_1, \ldots, \overline{y}_k) \in F \) so that each column traversal—that is, each sequence \((y_{1,i_1}, \ldots, y_{k,i_k})\) consisting of one element from each column—belongs to \( S_{u_i} \).

If we select \( \overline{y} \) randomly then, for each \( i \in [m]^k \), there is a positive probability that \((y_{1,i_1}, \ldots, y_{k,i_k})\) belongs to \( S_{u_i} \). The claim will then follow by showing that the behavior of each column traversal is sufficiently independent. This is what we now show: that if we focus on one row \( \overline{t}^0 \in [m]^k \), the behavior of all the other column traversals is \( \mathcal{B} \)-measurable.

Pick any \( \overline{t}^0 = (t^0_1, \ldots, t^0_k) \in [m]^k \). Let
\[
W'_{\overline{t}^0} := \left\{ \overline{t} \in [m]^k : i_1 \in [m] \setminus \{t^0_1\}, \ldots, i_k \in [m] \setminus \{t^0_k\} \right\},
\]
\[
W^*_{\overline{t}^0} := \left\{ \overline{t} \in [m]^k : \overline{t} \neq \overline{t}^0 \land \left( \bigvee_{s \in \{k\}} i_s = t^0_s \right) \right\}.
\]

Note that \([m]^k\) is the disjoint union of \( W'_{\overline{t}^0}, W^*_{\overline{t}^0} \) and \( \{\overline{t}^0\} \). Using the Fubini property we have
\[
\int_F \prod_{i \in [m]^k} \chi_{S_{u_i}}(y_{1,i_1}, \ldots, y_{k,i_k})d\mu_{m-\overline{1}^k}(\overline{y}_1, \ldots, \overline{y}_k) =
\]
\[
\int_{F'} \prod_{i \in W'_{\overline{t}^0}} \chi_{S_{u_i}}(y_{1,i_1}, \ldots, y_{k,i_k}) \left( \int_{F^*_{\overline{y}_i} \cap \overline{y}_i^0} \chi_{S_{u_i}}(y_{1,i_1}, \ldots, y_{k,i_k}) \right)
\]
\[
\prod_{i \in W'_{\overline{t}^0}} \chi_{S_{u_i}}(y_{1,i_1}, \ldots, y_{k,i_k})d\mu_{m-\overline{1}^k}(\overline{y}_1', \ldots, \overline{y}_k'),
\]

where \( \overline{y}'_i := (y_{s,t} : t \in [m] \setminus \{t^0_s\}) \), and \( F' \subseteq V^{(m-1)-\overline{1}^k} \) and \( F^*_{\overline{y}_i} \subseteq V^{\overline{1}^k} \) are the analogs to \( F \) on suitable coordinates, i.e.
\[
F' := \left\{ (\overline{y}_1', \ldots, \overline{y}_k') \in V^{(m-1)-\overline{1}^k} \mid \bigwedge_{i \in F'_{\overline{t}^0}} (y_{1,i_1}, \ldots, y_{k,i_k}) \in \overline{F} \right\},
\]
\[
F^*_{\overline{y}_i} := \left\{ (y_{1,i_1}, \ldots, y_{k,i_k}) \in V^{\overline{1}^k} \mid (y_{1,i_1}, \ldots, y_{k,i_k}) \in \overline{F} \land \right\}
\]
\[
\bigwedge_{i \in W^*_{\overline{t}^0}} (y_{1,i_1}, \ldots, y_{k,i_k}) \in \overline{F}.
\]

Obviously \( F' \in \mathcal{B}_{(m-1)-\overline{1}^k} \) since \( \overline{F} \in \mathcal{B} \subseteq \mathcal{B}_{\overline{1}^k} \). Note also that if \( \overline{t} \in W^*_{\overline{t}^0} \), then by definition we must have \( i_s \neq t^0_s \) for at least one \( s \in \{k\} \), so \( \overline{t} \land \overline{t}^0 \).
has length \( \leq k - 1 \) and "\((y_{1,i_1}, \ldots, y_{k,i_k}) \in \bar{F}\)" viewed as a condition on the tuple \((y_{1,i_1}, \ldots, y_{k,i_k})\) can involve at most \(k - 1\) coordinates (with all the other coordinates appearing in \(\bar{y}_1, \ldots, \bar{y}_k\) fixed).

Hence

\[
F^*(\bar{y}_1, \ldots, \bar{y}_k) \in \sigma \left( \{ \bar{F} \} \cup B_{1^k,k-1} \right) \subseteq B
\]

for any \((\bar{y}_1, \ldots, \bar{y}_k) \in V^{(m-1)\bar{1}^k}\). The integral in (5.3) can be rewritten as

\[
\int_{\bar{F}^\pi} \prod_{i \in W} \chi_{S_{u_i}}(y_{1,i_1}, \ldots, y_{k,i_k}) \int_{V(\bar{y}_1, \ldots, \bar{y}_k)} \chi_{S_{u_0^0}} d\mu_{1k}(y_{1,i_1}, \ldots, y_{k,i_k})
\]

\[
d\mu_{(m-1)\bar{1}k}(\bar{y}_1, \ldots, \bar{y}_k),
\]

where the set \(V(\bar{y}_1, \ldots, \bar{y}_k)\) is given by the intersection of \(F^*(\bar{y}_1, \ldots, \bar{y}_k)\) with the set

\[
\left\{(y_{1,i_1}, \ldots, y_{k,i_k}) \in V^{\bar{1}^k} \mid \bigwedge_{i \in W} (y_{1,i_1}, \ldots, y_{k,i_k}) \in S_{u_i} \right\}.
\]

As in the previous paragraph, each condition "\((y_{1,i_1}, \ldots, y_{k,i_k}) \in S_{u_i}\)" here, viewed as a condition on the tuple \((y_{1,i_1}, \ldots, y_{k,i_k})\), can involve at most \(k - 1\) coordinates and is given by some fiber of \(S_{u_i} \in \mathcal{B}_{1^k}\), hence we have

\[
\int_{\bar{F}^\pi} \prod_{i \in W} \chi_{S_{u_0}}(y_{1,i_1}, \ldots, y_{k,i_k}) \int_{V(\bar{y}_1, \ldots, \bar{y}_k)} \chi_{S_{u_0^0}} d\mu_{1k}(y_{1,i_1}, \ldots, y_{k,i_k})
\]

\[
d\mu_{(m-1)\bar{1}k}(\bar{y}_1, \ldots, \bar{y}_k),
\]

(5.4)

\[
\int_{V(\bar{y}_1, \ldots, \bar{y}_k)} \chi_{S_{u_0}}(y_{1,i_1}, \ldots, y_{k,i_k}) \int_{V(\bar{y}_1, \ldots, \bar{y}_k)} \chi_{S_{u_0^0}} d\mu_{1k}(y_{1,i_1}, \ldots, y_{k,i_k})
\]

(5.5)

\[
\int_{(V(\bar{y}_1, \ldots, \bar{y}_k) \cap \bigcap_{r \in [l]} E_{x_r}^{u_0(r)} \chi_{E_{x_l}^{u_0(t)}}(y_{1,i_1}, \ldots, y_{k,i_k})} d\mu_{1k}
\]

(5.6)

\[
\left(\text{as the set over which we integrate is in } \sigma(B \cup \{E_{x_r} : r < l\})\right)
\]

(5.7)

\[
\text{(by Assumption 5.6(5))}
\]

(5.8)

\[
\geq \int_{(V(\bar{y}_1, \ldots, \bar{y}_k) \cap \bigcap_{r \in [l]} E_{x_r}^{u_0(r)})} \delta d\mu_{1k}
\]

(by the definition of \(\delta(x_l)\), as \(V(\bar{y}_1, \ldots, \bar{y}_k) \subseteq F_\delta(x_l)\))
There exists a positive measure set of approximations, in the following sense.

The conclusion of Proposition 5.1 from "there exists an approximation" to 5.4.

A positive measure set of approximations.

and the tuple $\overline{\bar{t}}$ once for each tuple $\bar{t}$.

Hence for the original integral (5.2) we have

$$
\int F \prod_{i \in [m]_k} \chi_{S_{w_i}^k}(y_{1,i_1}, \ldots, y_{k,i_k}) d\mu_{m-1}^k(\overline{\bar{\bar{y}}}_1, \ldots, \overline{\bar{\bar{y}}}k)
$$

This concludes the proof of Propositions 5.1 and 5.5.

5.4. A positive measure set of approximations. Next we will strengthen the conclusion of Proposition 5.1 from “there exists an approximation” to “there exists a positive measure set of approximations”, in the following sense.

Definition 5.8. Fix some $x_1, \ldots, x_l \in V_{k+1}$.

1. Given $t \in \mathbb{N}$, $\bar{w} \in V^m$ for some $\bar{m} \in \mathbb{N}^k$ and $x \in V_{k+1}$, let us denote by $f_{\bar{w},x}^t$ the best $\|\cdot\|_{L^2}$-approximation to $f_x$ using a simple function relative to the Boolean algebra $B^L_{\bar{w}}$ generated by

$$
\mathcal{F}_{\bar{w},(x_1, \ldots, x_l,x)} \cup \{f_{x_i}^q : i \in [l], q \in Q_{x_i}^{0,1}\}
$$

(see Definition 5.4).

2. For $s \in \mathbb{N}$, we also denote by $f_{\bar{w},x}^{t,s}$ the best $\|\cdot\|_{L^2}$-approximation to $f_x$ by a simple function with respect to $B^L_{\bar{w}}$ and with all coefficients in $Q_{x}^{0,1}$.

3. For $\varepsilon \in \mathbb{R}_{\geq 0}$, we say that $f_x$ is $\varepsilon$-nicely approximated (with respect to $x_1, \ldots, x_l$) if there exist some $t, \bar{m} \in \mathbb{N}^k$ such that the set of tuples $\bar{w} \in V^m$ with $\|f_x - f_{\bar{w},x}^t\|_{L^2} \leq \varepsilon$ has positive $\mu_{\bar{m}}$-measure (this set is measurable by Fubini property in graded probability spaces, see the proof of Proposition 6.1 for the details).
Lemma 5.9. Suppose that $\mathcal{P} = (V_{[k+1]}, B_{\mathcal{N}}, \mu_{\mathcal{N}})_{n \in \mathbb{N}^{k+1}}$ is a $(k + 1)$-partite graded probability space, $f : V_{[k+1]}^{k+1} \to [0, 1]$ is $B_{[k+1]}$-measurable, $V_{k+1}(f) = \delta < \infty$ and $\varepsilon \in \mathbb{R}_{>0}$. Then there exist some $l = l(k, d, \varepsilon) \in \mathbb{N}$ and $x_1, \ldots, x_l \in V_{k+1}$ such that: for any $x \in V_{k+1}$, $f_x$ is $\varepsilon$-nicely approximated with respect to $x_1, \ldots, x_l$.

Proof. Fix $\varepsilon > 0$. By Proposition 5.1, there exist some $l \in \mathbb{N}$ (we may assume $l = l(k, d, \varepsilon)$ by Proposition 5.5) and $x_1, \ldots, x_l \in V_{k+1}$ such that, for every $x \in V_{k+1}$,

$$\left\|f_x - \mathbb{E}\left(f_x \mid B_{1\ldots1}^{k+1} \cup \{f_{x_i} : i \in [l]\}\right)\right\|_{L^2} \leq \varepsilon.$$

Fix some $x \in V_{k+1}$. Note that $f_x$ is trivially $\|f_x\|_{L^2(\mu_{\mathcal{N}})}$-nicely approximated. Let

$$\delta := \inf \{\delta' \in \mathbb{R}_{\geq 0} : f_x \text{ is } \delta'-\text{nicely approximated}\}.$$

The function $\mathbb{E}\left(f_x \mid B_{1\ldots1}^{k+1} \cup \{f_{x_i} : i \in [l]\}\right)$ is the best approximation to $f_x$ from all functions measurable with respect to the given $\sigma$-algebra. We need to find an analogous function, which we will call $h$, which is the best approximation to $f_x$ with respect to the same $\sigma$-algebra among those approximations which can be obtained for positive measure of parameters $\bar{w}$. This is not actually a projection on a $\sigma$-algebra, so we cannot use the standard result to show that $h$ exists, but the proof is essentially the same: first we show that any two near optimal approximations of positive measure must be close to each other, and then we use this to construct a Cauchy sequence converging to $h$. This is the content of the two claims that follow.

Claim 5.10. For every $\gamma > 0$, there is a $0 < \theta = \theta(\gamma) < \gamma$ so that: whenever $t_0, t_1 \in \mathbb{N}, \bar{m}_0, \bar{m}_1 \in \mathbb{N}^k$ and $\delta_0, \delta_1 < \delta + \theta$, the set

$$\left\{\bar{w}_0 \oplus \bar{w}_1 \in V^{\bar{m}_0 + \bar{m}_1} : \left\|f_{\bar{w}_0}^{t_0} - f_{\bar{w}_1}^{t_1}\right\|_{L^2} > \gamma \land \bigwedge_{i=0,1} \left\|f_x - f_{\bar{w}_i}^{t_i}\right\|_{L^2} \leq \delta_i\right\}$$

is in $B_{\bar{m}_0 + \bar{m}_1}$ and has $\mu_{\bar{m}_0 + \bar{m}_1}$-measure 0.

Proof. Assume that this set has positive measure. For any $\bar{w}_0 \oplus \bar{w}_1$ in it, by the parallelogram rule for the $L^2$-norm we have

$$\left\|2f_x - \left(f_{\bar{w}_0}^{t_0} + f_{\bar{w}_1}^{t_1}\right)\right\|_{L^2}^2 = \left\|f_{\bar{w}_0}^{t_0} - f_{\bar{w}_1}^{t_1}\right\|_{L^2}^2 + \left\|f_{\bar{w}_1}^{t_1} - f_{\bar{w}_0}^{t_0}\right\|_{L^2}^2,$$

hence

$$\left\|2f_x - \left(f_{\bar{w}_0}^{t_0} + f_{\bar{w}_1}^{t_1}\right)\right\|_{L^2}^2 \leq 2(\delta_0^2 + \delta_1^2) - \gamma^2 < 4(\delta + \theta)^2 - \gamma^2 \leq 4(\delta')^2$$

for some $\delta' < \delta$, assuming that $\theta$ is small enough with respect to $\gamma$ and $\delta$. Hence

$$\left\|f_x - \frac{f_{\bar{w}_0}^{t_0} + f_{\bar{w}_1}^{t_1}}{2}\right\|_{L^2} \leq \delta'.$$
As \( \frac{f_{t_0}^{t_1} + f_{t_1}^{t_2}}{2} \) is a \( B_{\tilde{w}_0 \oplus \tilde{w}_1}^{\max \{t_0,t_1\}} \)-simple function, we have
\[
\left\| f_x - f_{\tilde{w}_0 \oplus \tilde{w}_1}^{\max \{t_0,t_1\}} \right\|_{L^2} \leq \delta' < \delta
\]
for a positive \( \mu_{\tilde{w}_0 + \tilde{w}_1} \)-measure set of \( \tilde{w}_0 \oplus \tilde{w}_1 \), contradicting the choice of \( \delta \).

This allows us to choose the “best positive measure approximation” of \( f_x \), in the following sense.

**Claim 5.11.** There exists a \( \sigma \left( \mathcal{F}f, (x_1, \ldots, x_l, x) \cup \{f_{x_i}^{<q} \}_{i \in [l], q \in \mathbb{Q}_{\infty}^{[0,1]}} \right) \)-measurable function \( h \) such that \( \left\| f_x - h \right\|_{L^2} = \delta \) and for any \( \sigma > 0 \) there is some \( t \in \mathbb{N}, \tilde{m} \in \mathbb{N}^k \) so that the set \( \{ \tilde{w} \in V^{\tilde{m}} : \left\| h - f_{x,\tilde{w}}^t \right\|_{L^2} \leq \sigma \} \in \mathcal{B}_{\tilde{m}} \) has positive \( \mu_{\tilde{m}} \)-measure.

**Proof.** Given \( n \in \mathbb{N} \), let \( \gamma_n := \frac{1}{n} \), and let \( \theta_n > 0 \) be given by Claim 5.10 for \( \gamma_n \). By the choice of \( \delta \), there exists some \( t_n \in \mathbb{N}, \tilde{m}_n \in \mathbb{N}^k \) such that the set
\[
S_n := \{ \tilde{w} \in V^{\tilde{m}_n} : \left\| f_x - f_{x,\tilde{w}}^{t_n} \right\|_{L^2} \leq \delta + \theta_n \} \in \mathcal{B}_{\tilde{m}_n}
\]
has positive \( \mu_{\tilde{m}_n} \)-measure.

By induction on \( r \in \mathbb{N} \) we choose sets \( S_r^n \in \mathcal{B}_{\tilde{m}_n} \), \( n \in \mathbb{N} \) and tuples \( \tilde{w}_r \in S_r^n \) satisfying the following:

\[
\text{(5.11)} \quad S_r^n \subseteq S_r^{n'} \subseteq S_n \quad \text{and} \quad \mu_{\tilde{m}_n}(S_r^n) = \mu_{\tilde{m}_n}(S_n) > 0 \quad \text{for all} \quad r', n \in \mathbb{N};
\]
\[
S_r^n = S_r^{n'} \quad \text{for all} \quad r, r' \leq n' \in \mathbb{N};
\]
\[
\text{(5.12)} \quad \left\| f_{x,\tilde{w}_r}^{t_r} - f_{x,\tilde{w}_r}^{t_r} \right\|_{L^2} \leq \gamma_r \quad \text{for all} \quad \tilde{w} \in S_r^{n+1} \quad \text{with} \quad n \geq r \in \mathbb{N}.
\]

Let \( S_r^n := S_n \), then all the conditions are trivially satisfied. Now assume \( \tilde{w}_1, \ldots, \tilde{w}_{r-1} \) and \( (S_r^n)_{n \in \mathbb{N}} \) satisfying these conditions are given.

For each \( n \in \mathbb{N} \), let
\[
T_r^n := \left\{ \tilde{w} \in S_r^n : \mu_{\tilde{m}_n} \left( \left\{ \tilde{w}' \in S_r^n : \left\| f_{x,\tilde{w}_r}^{t_r} - f_{x,\tilde{w}_r}^{t_r} \right\|_{L^2} > \gamma_r \right\} \right) > 0 \} \in \mathcal{B}_{\tilde{m}_r}.
\]

By Claim 5.10 and Fubini property (using (5.10) and (5.11)), \( \mu_{\tilde{m}_r}(T_r^n) = 0 \) for any \( n \geq r \). Let \( \tilde{S}_r^n := S_r^n \setminus \bigcup_{n \geq r} T_r^n \), then \( \mu_{\tilde{m}_r}(\tilde{S}_r^n) > 0 \). Let \( \tilde{w}_r \) be an arbitrary element in \( \tilde{S}_r^n \). For each \( n > r \) let
\[
S_r^{n+1} := S_r^n \cap \left\{ \tilde{w} \in S_r^n : \left\| f_{x,\tilde{w}_r}^{t_r} - f_{x,\tilde{w}_r}^{t_r} \right\|_{L^2} \leq \gamma_r \right\}.
\]

Then \( \mu_{\tilde{m}_n}(S_r^{n+1}) = \mu_{\tilde{m}_n}(S_r^n) \) for all \( n \geq r \), by the choice of \( t_r, \tilde{w}_r \) and (5.12) is satisfied, concluding the construction.

The sequence \( \left( f_{x,\tilde{w}_r}^{t_r} \right)_{r \in \mathbb{N}} \) is Cauchy in the space
\[
L^2 \left( \sigma \left( \mathcal{F}f, (x_1, \ldots, x_l, x) \cup \{f_{x_i}^{<q} \}_{i \in [l], q \in \mathbb{Q}_{\infty}^{[0,1]}} \right) \right)
\]
since for any \( r_0, r_1 \geq r \) we have \( \| f_{x_0}^{r_0} t_r - f_{x_1}^{r_1} t_r \|_{L^2} \leq \gamma_r \) by \((5.11)\) and \((5.12)\), and \( \gamma_r \to 0 \). By completeness of the \( L^2 \)-space it has a limit which we denote by \( h \).

For an arbitrary \( \sigma > 0 \), let \( r \in \mathbb{N} \) be such that \( \gamma_r \leq \frac{\sigma}{2} \) and \( \| h - f_{x_0}^{r} t_r \|_{L^2} \leq \frac{\sigma}{2} \). By \((5.11)\) and \((5.12)\), the set of \( w \in V^{m_r} \) such that \( \| f_{x_0}^{r} t_r - f_{x_0}^{r} t_r \|_{L^2} \leq \gamma_r \) has positive \( \mu_{m_r} \)-measure, and is contained in the set of \( w \in V^{m_r} \) such that \( \| h - f_{x_0}^{r} t_r \|_{L^2} \leq \sigma \). Similarly we have \( \| f_x - h \|_{L^2} \leq \| f_{x_0}^{r} t_r - f_x \|_{L^2} + \| h - f_{x_0}^{r} t_r \|_{L^2} \leq (\delta + \theta_r) + \sigma \) by \((5.10)\), and as \( \theta_r \to 0 \) and \( \sigma \) can be chosen arbitrarily small, we conclude that \( \| f_x - h \|_{L^2} \leq \delta \).

We will now show that \( \delta \leq \varepsilon \). Towards a contradiction, suppose that \( \delta > \varepsilon \). Then

\[
\mathbb{E} \left( f_x \mid B_{1,k-1} \cup \{ f_{x_i} \}_{i \in [k]} \right) \neq h, \text{ hence }
\|
\mathbb{E} \left( f_x - h \mid B_{1,k-1} \cup \{ f_{x_i} \}_{i \in [k]} \right) \|_{L^2} =
\|
\mathbb{E} \left( f_x \mid B_{1,k-1} \cup \{ f_{x_i} \}_{i \in [k]} \right) - h \|_{L^2} > 0.
\]

So \( f_x - h \) is non-orthogonal to \( B_{1,k-1} \cup \{ f_{x_i} \}_{i \in [k]} \), hence for some \( \sigma \left( \{ f_{x_i} \}_{i \in [k]} \right) \)-measurable function \( g \) we have

\[
\| \mathbb{E} \left( g \cdot (f_x - h) \mid B_{1,k-1} \right) \|_{L^2} > 0.
\]

Naturally, we will use this to show that we can find a positive measure set of parameters which give a strictly better approximation to \( f_x \), contradicting the choice of \( \delta \).

We know that there is some \( B_{1,k-1} \)-measurable function \( u \) so that \( \int u_0 \cdot g \cdot (f_x - h) \, d\mu_{1,k} > 0 \); in standard arguments about projections, we would then choose a \( c \) so that \( h - c \cdot u_0 \cdot g \) would be a better approximation to \( f_x \). However, to contradict the definition of \( \delta \), we cannot take an arbitrary \( B_{1,k-1} \cup \{ f_{x_i} \}_{i \in [k]} \)-measurable function \( u = u_0 \cdot g \) to improve our approximation.

The \textit{Gowers uniformity norms} let us construct \( u \) explicitly from \( g \cdot (f_x - h) \): since \( \| \mathbb{E} \left( g \cdot (f_x - h) \mid B_{1,k-1} \right) \|_{L^2} > 0 \), the \textit{Gowers} \( U^k \)-norm is positive. This fact is by now standard (e.g. \cite{Gow01, Tow17}), but for completeness, we develop it in the partite setting in Section \textsection.

By Proposition \textsection we have

\[
\gamma := \| g \cdot (f_x - h) \|^2_{L^2_{1,k}} > 0
\]

The remainder of the proof consists of writing this integral out explicitly, approximating it with functions of the right kind, and doing the calculations to show that this gives us approximations of \( f_x \) which contradict the definition of \( \delta \).
If we write out \( \|g \cdot (f_x - h)\|_{L^1}^2 \) (Definition 8.1), we get

\[
\int \prod_{\alpha \in \{0,1\}^k} \left( g \left( y_1^{\alpha(1)}, \ldots, y_k^{\alpha(k)} \right) \cdot (f_x - h) \left( y_1^{\alpha(1)}, \ldots, y_k^{\alpha(k)} \right) \right) d\mu_{x,\bar{1}k} \left( y^0 \oplus y^1 \right) = \gamma.
\]

Let \( S \) be the set of all \((y_1^1, \ldots, y_k^1) \in V^k \) such that

\[
(5.13) \quad \int \prod_{\alpha \in \{0,1\}^k} \left( g \left( y_1^{\alpha(1)}, \ldots, y_k^{\alpha(k)} \right) \cdot (f_x - h) \left( y_1^{\alpha(1)}, \ldots, y_k^{\alpha(k)} \right) \right) d\mu_{\bar{1}k} \left( y^0 \right) \geq \gamma.
\]

By the Fubini property \( S \in B_{\bar{1}k} \) and \( \mu_{\bar{1}k}(S) > 0 \).

Let \( \sigma = \sigma(\gamma) > 0 \) be sufficiently small (see below). By Claim 5.11 there exist some \( t \in \mathbb{N}, \bar{m} \in \mathbb{N}^k \) and a set \( T \in B_{\bar{m}} \) with \( \mu_{\bar{m}}(T) > 0 \) so that

\[
\left\| f_{\bar{w},x}^t - h \right\|_{L^2} < \sigma \quad \text{for all} \quad \bar{w} \in T.
\]

We can also choose a sufficiently large \( t_0 \in \mathbb{N} \) so that \( \|f_x - f_x^t\|_{L^2} < \sigma \) and \( \|g - g'\|_{L^2} < \sigma \) for some function \( f_x^t \) that is simple with respect to the Boolean algebra generated by \( \{f_x^q : q \in Q_{t_0}^{[0,1]} \} \) and some function \( g' \) that is simple with respect to the Boolean algebra generated by \( \{f_x^q : i \in [l], q \in Q_{t_0}^{[0,1]} \} \).

Then for any fixed \( y^1 \in S \) and any \( \bar{w} \in T \), replacing \( h \) by \( f_{\bar{w},x}^t \), \( f_x \) by \( f_x^t \) and \( g \) by \( g' \) in the integral (5.13) we get (assuming \( \sigma \) is small enough with respect to \( \gamma \))

\[
\int \prod_{\alpha \in \{0,1\}^k} g' \left( y_1^{\alpha(1)}, \ldots, y_k^{\alpha(k)} \right) \cdot (f_x - f_x^t) \left( y_1^{\alpha(1)}, \ldots, y_k^{\alpha(k)} \right) d\mu_{\bar{1}k} \left( y^0 \right) \geq \frac{\gamma}{2}.
\]

For \((y^1, \bar{w}) \in S \times T\), let \( g'_{y^1,\bar{w}} : V^k \rightarrow \mathbb{R} \) be the function defined by

\[
g'_{y^1,\bar{w}} \left( y_1^0, \ldots, y_k^0 \right) :=
\]

\[
\prod_{\alpha \in \{0,1\}^k} g' \left( y_1^{\alpha(1)}, \ldots, y_k^{\alpha(k)} \right) \cdot \prod_{\alpha \in \{0,1\}^k \setminus \{0, \ldots, 0\}} \left( f_x - f_x^t \right) \left( y_1^{\alpha(1)}, \ldots, y_k^{\alpha(k)} \right).
\]

Then

\[
\int g'_{y^1,\bar{w}} \left( y^0 \right) \cdot (f_x - f_x^t) \left( y^0 \right) d\mu_{\bar{1}k} \left( y^0 \right) \geq \frac{\gamma}{2}, \quad \text{or in other words}
\]

\[
(5.14) \quad \left\langle g'_{y^1,\bar{w}}, f_x - f_x^t \right\rangle_{L^2} \geq \frac{\gamma}{2}.
\]
By the choice of $g'$ and $f''_x$, we have that $g'_{g,\bar{\omega}}$ is a $\mathcal{B}_{g,\bar{\omega}}$-simple function for $t':=\max\{t, t_0\}$, and also $\|g'_{g,\bar{\omega}}\|_{L^2} \leq 1$.

Taking the orthogonal projection of $f''_x - f''_{\bar{w},x}$ in the Hilbert space $L^2(\mathcal{B}_{i,k,k-1})$ onto the closed subspace generated by $g'_{g,\bar{\omega}}$, we can write

$$f''_x - f''_{\bar{w},x} = u + v$$

for some $\mathcal{B}_{g,\bar{\omega}}$-simple function $u$ and some $\mathcal{B}_{i,k,k-1}$-measurable function $v$ orthogonal to this subspace. Note that

$$\mathrm{Proposition\ 6.1.}\ \text{Suppose that}\ \mu = \mu_{\bar{w},x}.$$

Proof of the

By the choice of $\gamma$, there is some $\delta' = \delta(\gamma) < \delta - \sigma$ so that $\|v\|_{L^2} \leq \delta'$. Observe that $f''_x - (f''_{\bar{w},x} + u) = v$ and $f''_{\bar{w},x} + u$ is a $\mathcal{B}_{g,\bar{\omega}}$-simple function. Thus for any $(\bar{g}, \bar{\omega}) \in S \times T$ we have

$$\|f''_x - f''_{\bar{w},x}\|_{L^2} \leq \|f''_x - f''_{\bar{g},\bar{\omega}}\|_{L^2} + \delta' + \sigma < \delta,$$

and $\mu_{\bar{w},x}(S \times T) > 0$. This contradicts the choice of $\delta$. □

6. Main Theorem

6.1. Proof of the ($k+1,k$)-case.

Proposition 6.1. Suppose that $\mathfrak{P} = (V_{k+1}, \mathcal{B}_{\bar{w}}, \mu_{\bar{w}})_{n \in \mathbb{N}}$ is a ($k+1$)-partite graded probability space, $f : V^{k+1} \rightarrow [0,1]$ is a ($k+1$)-ary $\mathcal{B}_{i,k}$-measurable function and $\text{VC}_k(f) < \infty$. Then $f$ is $\mathcal{B}_{i,k+1,k}$-measurable.

More precisely, for every $\varepsilon > 0$ there exist some $N \in \mathbb{N}$, $\gamma_i \in \mathbb{Q}[0,1]$ for $i \in [N]$ and $\mathcal{B}_I(f)$-measurable (see Definition 6.3) $(\leq k)$-ary functions $f^I_i : \prod_{j \in I} V_j \rightarrow [0,1]$ for $i \in [N], I \in \binom{[k+1]}{\leq k}$ so that, defining $g : V^{k+1} \rightarrow [0,1]$ via

$$g(x) := \sum_{i \in [N]} \gamma_i \cdot \prod_{I \in \binom{[k+1]}{\leq k}} f^I_i(x_I),$$

we have $\|f - g\|_{L^2} < \varepsilon$. 

Remark 6.2. Furthermore, $N$ can be bounded depending only on $VC_k(f)$ and $\varepsilon$ (this will be established as part of the more general Corollary 6.9).

The main idea of the proof is not so complicated. By Proposition 5.1 there are $x_1, \ldots, x_N$ so that, for every $x \in V_{k+1}$,

$$\| f_x - \mathbb{E} \left( f_x \mid \mathcal{B}_{1,k-1} \cup \{ f_{x_1}, \ldots, f_{x_N} \} \right) \|_{L^2} < \varepsilon.$$ 

Now $\mathbb{E} \left( f_x \mid \mathcal{B}_{1,k-1} \cup \{ f_{x_1}, \ldots, f_{x_N} \} \right)$ can be approximated by a finite sum of the form

$$\mathbb{E} \left( f_x \mid \mathcal{B}_{1,k-1} \cup \{ f_{x_1}, \ldots, f_{x_N} \} \right)(\bar{x}) \approx \sum_{i \leq N} \gamma_i f_{x_i}(\bar{x}) \prod_{I \in \{1, \ldots, k\}} \chi_{C_{i,I,x}}(\bar{x}_I),$$

where each $C_{i,I,x}$ is some set from $\mathcal{B}_{i,k}$. By countable additivity, outside of a set of measure $< \varepsilon$, $N$ can be bounded uniformly in $x$. (In fact, by Proposition 5.5, $N$ can be bounded uniformly in all $x$.)

We can combine these representations for different $x \in V_{k+1}$ by replacing the sets $C_{i,I,x}$ with the set $C_{i,I} = \{ (\bar{x}, x) \mid \bar{x}_I \in C_{i,I,x} \}$, obtaining (with some rearranging of terms) a single function

$$g = \sum_{i \leq N} \gamma_i f_{x_i}(\bar{x}) \prod_{I \in \{1, \ldots, k\}} \chi_{C_{i,I}}(\bar{x}_I, x).$$

By its form, $g$ is $\mathcal{B}_{1,k+1}$-measurable, and for almost every $x \in V_{k+1}$, $\| f_x - g_x \|_{L^2}$ is small, so $\| f - g \|_{L^2}$ is small as well.

There is one complication: just because each of the sets $C_{i,I,x}$ are measurable, it does not follow that the set $C_{i,I}$ is also measurable. Therefore to carry this argument out correctly, we need to write out this cylinder sets in a way that is sufficiently uniform in $x$ (as, more or less, combinations of level sets of fibers of $f$) to guarantee that $C_{i,I}$ is measurable, and use some averaging arguments relying on Lemma 5.9.

To make this explicit, we define a slight variant of the algebras associated to fibers of a function considered earlier.

Definition 6.3. Let $r \in \mathbb{N}$, $(V_1, \mathcal{B}_1, \mu_1)_{n \in \mathbb{N}^r}$ be an $r$-partite graded probability space, $f : V_1^r \to [0,1]$ a $\mathcal{B}_1$-measurable function and $\bar{w}_1, \ldots, \bar{w}_t \in V_1^r$. Let $t \in \mathbb{N}$. $I \subseteq [r]$, and let $\bar{n}_I \in \mathbb{N}^r$ be defined by $\bar{n}_I := \sum_{i \in I} \bar{i}_i$. We let $\mathcal{B}_{I}^{\bar{n}_I}(\bar{w}_1, \ldots, \bar{w}_t)(f)$ be the finite Boolean subalgebra of $\mathcal{B}_{\bar{n}_I}$ generated by all subsets of $V_{\bar{n}_I} = \prod_{i \in I} V_i$ of the form

$$\left\{ \bar{x} \in V_{\bar{n}_I} : \bar{x} + \langle \bar{w}_1 \rangle |_{I^C} \in f^{-}\right\}$$

for some $i \in [t]$ and $q \in \mathbb{Q}^+[0,1]$. We let $\mathcal{B}_{I}^{\bar{n}_I}(f)$ be the $\sigma$-subalgebra of $\mathcal{B}_{\bar{n}_I}$ generated by $\bigcup_{\bar{w} \in \mathcal{B}_{\bar{n}_I}} \mathcal{B}_{I}^{\bar{n}_I}(\bar{w})$, and $B_{I}(f)$ the $\sigma$-subalgebra of $\mathcal{B}_{\bar{n}_I}$ generated by $\bigcup_{\bar{w} \in \mathcal{B}_{\bar{n}_I}} \mathcal{B}_{I}^{\bar{n}_I}(\bar{w})$.

We are ready to prove Proposition 6.1 in the explicit form stated in Remark 6.2.
Proof of Proposition 6.1. Let $\varepsilon \in \mathbb{R}_{>0}$ be given.

We fix $\delta \in \mathbb{Q}_{>0}, t \in \mathbb{N}, m = (m_1, \ldots, m_k) \in \mathbb{N}^k$, to be determined later.

By Lemma 5.9 there exist some $x_1, \ldots, x_l \in V_{k+1}$ such that for any $x \in V_{k+1}$, $f_x$ is $\delta$-nicely approximated with respect to $x_1, \ldots, x_l$.

Let $0 = r_1 < \ldots < r_d = 1$ list all elements of $\mathbb{Q}_{\epsilon}^{[0,1]}$ in the increasing order.

As usual, for $q, q' \in \mathbb{Q}_{\epsilon}^{[0,1]}$ and $\preceq \in \{<, \geq, =\}$ we let

$$f^{>q} := \{\bar{y} \in V^{1k+1} : f(\bar{y}) \triangleright q\},$$

$$f^q(q') = f^q \cap f^{\geq q}.$$

Let

$$S := \left\{s \mid s : [m_1] \times \ldots \times [m_k] \times [l + 1] \times \left(\frac{[k]}{k - 1}\right) \rightarrow [L]\right\}.$$

Let $m' := 1^{k}(0) + \bar{m}(0) + \bar{m}_{k+1} \in \mathbb{N}^{k+1}$. For $s \in S$ let

$$A^s : V^{m'} \rightarrow \{0, 1\},$$

$$A^s(\bar{y}, \bar{w}, x) := \prod_{(i_1, \ldots, i_k) \in [m_1] \times \ldots \times [m_k] \times [l]} \chi_{f_{x,j}^{\delta_{(i,j,t)}}}\left(\bar{y}_{t,j,i,t} \rightarrow t, t \in I\right) \cdot \prod_{(i_1, \ldots, i_k) \in [m_1] \times \ldots \times [m_k]} \chi_{f_{x,j}^{\delta_{(i,j+t+1,t)}}}\left(\bar{y}_{t,j,i,t} \rightarrow t, t \in I, x\right).$$

By definition (see Definition 5.8), for every $\bar{w} \in V^{\bar{m}}$ and $x \in V_{k+1}$, every atom of the algebra $\mathcal{B}_{\bar{w}}^{t,x}$ has characteristic function of the form $A^s(-, \bar{w}, x)$ for some $s \in S$ (some of the atoms may be repeated in this presentation).

For $\bar{a} = (\alpha_s \in \mathbb{Q}_{\epsilon}^{[0,1]} : s \in S) \in Q := \left(\mathbb{Q}_{\epsilon}^{[0,1]}\right)^S$, we consider the function

$$f^t_{\bar{a}} \rightarrow [0, 1],$$

$$f^t_{\bar{a}}(\bar{y}, \bar{w}, x) := \sum_{s \in S} \alpha_s A^s(\bar{y}, \bar{w}, x).$$

Then $f^t_{\bar{a}}$ is $\mathcal{B}_{\bar{m}'}$-measurable, and every $\mathcal{B}_{\bar{w}}^{t,x}$-simple function with coefficients in $\mathbb{Q}_{\epsilon}^{[0,1]}$ is of the form $f^t_{\bar{a}}(-, \bar{w}, x)$ for some $\bar{a} \in Q$.

Recall (Definition 5.8(2)) that $f^t_{\bar{w}, x}$ denotes the best $L^2$-approximation to $f_x$ using a $\mathcal{B}_{\bar{w}}^{t,x}$-simple function with coefficients in $\mathbb{Q}_{\epsilon}^{[0,1]}$. We can define it explicitly as follows. Let $\prec$ be an arbitrary well order on $Q$. For $\bar{a}, \bar{b} \in Q$,
let
\[ C_{\alpha, \beta} := \left\{ (\bar{w}, x) \in V^{\bar{m}^{-1}} : \| f_x - f_{\beta}^t (-, \bar{w}, x) \|_{L^2} < \| f_x - f_{\alpha}^t (-, \bar{w}, x) \|_{L^2} \right\}, \]
\[ C_{\alpha} := \bigcap_{\beta \in Q} \left( V^{\bar{m}^{-1}} \setminus C_{\alpha, \beta} \right), \]
\[ D_{\alpha} := C_{\alpha} \setminus \bigcup_{\bar{\alpha} \in Q} C_{\bar{\alpha}}. \]

Note that \( C_{\alpha, \beta} \in B_{\bar{m}^{-1}} \) (as \( \| x \|_{L^2} = (\int x^2)^{\frac{1}{2}} \) is a composition of functions preserving measurability using Fubini). So (\( \bar{w}, x \) \in \( C_{\alpha} \) if and only if \( \| f_x - f_{\beta}^t (-, \bar{w}, x) \|_{L^2} \) is minimal among all \( \beta \in Q \). As there can be multiple \( \bar{\alpha} \in Q \) that give equally good approximations, we let \( D_{\alpha} \) consist of those \( (\bar{w}, x) \) for which \( \bar{\alpha} \) is \( \ll \)-minimal giving the best approximation. Then \( \{ D_{\alpha} : \bar{\alpha} \in Q \} \) forms a partition of \( V^{\bar{m}^{-1}} \), and we define a function \( h : V^{\bar{m}} \to [0, 1] \) via
\[ h(\bar{y}, \bar{w}, x) := \sum_{\bar{\alpha} \in Q} \chi_{D_{\alpha}}(\bar{w}, x) \cdot f_{\alpha}^t(\bar{y}, \bar{w}, x). \]

From the definition we see that \( h \) is \( B_{m^{-1}} \)-measurable and for every fixed \( (\bar{w}, x), h(-, \bar{w}, x) = f_{\bar{w}, x}^t \).

For \( \bar{m} \in \mathbb{N}^k \) and \( t, s \in \mathbb{N} \), let
\[ G_{t, \bar{m}} := \left\{ (\bar{w}, x) \in V^{\bar{m}^{-1}} : \| f_x - f_{\bar{w}, x}^t \|_{L^2} < \delta \right\}, \]
\[ G_{t, s, \bar{m}} := \left\{ (\bar{w}, x) \in V^{\bar{m}^{-1}} : \| f_x - f_{\bar{w}, x}^s \|_{L^2} < 2\delta \right\}. \]

As every \( B_{\bar{w}}^{L^2} \)-simple function can be approximated up to \( L^2 \)-distance \( \delta \) by some \( B_{\bar{w}}^{L^2} \)-simple function with coefficients in \( \mathbb{Q}^{[0,1]} \) assuming \( s \in \mathbb{N} \) is large enough, we have
\[ G_{t, \bar{m}} = \bigcup_{s \in \mathbb{N}} G_{t, s, \bar{m}}. \]

Also, for \( \rho \in \mathbb{Q}_{>0} \), let
\[ G_{t, \bar{m}, \rho} := \{ x \in V_{k+1} : \mu_{\bar{m}}((G_{t, \bar{m}})_x) \geq \rho \}, \]
\[ G_{t, s, \bar{m}, \rho} := \{ x \in V_{k+1} : \mu_{\bar{m}}((G_{t, s, \bar{m}})_x) \geq \rho \}. \]

Then \( G_{t, \bar{m}, \rho}, G_{t, s, \bar{m}, \rho} \in B_{\delta_{k+1}} \) by Fubini. And by the choice of \( x_1, \ldots, x_t \) we have that \( V_{k+1} \) is covered by the sets \( \{ G_{t, \bar{m}, \rho} : t \in \mathbb{N}, \bar{m} \in \mathbb{N}^k, \rho \in \mathbb{Q}_{>0} \} \), hence also covered by the sets \( \{ G_{t, s, \bar{m}, \rho} : t, s \in \mathbb{N}, \bar{m} \in \mathbb{N}^k, \rho \in \mathbb{Q}_{>0} \} \) by (6.1). Hence, by countable additivity of the measure (noting that \( t \leq t' \) and \( s \leq s' \) \( \bar{m} \leq \bar{m}' \) and \( \rho \geq \rho' \) implies \( G_{t, s, \bar{m}, \gamma} \subseteq G_{t', s', \bar{m}', \gamma} \)), we can choose some \( t \in \mathbb{N}, \bar{m} \in \mathbb{N}^k \) and \( \rho \in \mathbb{Q}_{>0} \) so that
\[ \mu_{\delta_{k+1}}(G_{t, t, \bar{m}, \rho}) \geq 1 - \delta. \]
We define
\[ H := \left\{ (\bar{w}, x) \in V^{\bar{m}} : \| f_{x} - h(-, \bar{w}, x) \|_{L^2} < 2\delta \right\}. \]

We also define a \( B_{k+1} \)-measurable (by Fubini) function \( g' : V^{\bar{k}+1} \to [0, 1] \) via
\[ g'(\bar{y}, x) := \frac{1}{\max \{ \rho, \mu_{\bar{m}}(H_{x}) \}} \int h(\bar{y}, \bar{w}, x) \cdot \chi_H(\bar{w}, x) d\mu_{\bar{m}}(\bar{w}). \]

As \( h(-, \bar{w}, x) = f_{\bar{w}, x}^{1/\delta} \) for every fixed \((\bar{w}, x) \in V^{\bar{m}}\), we have \( \mu_{\bar{k}+1}(Z) \geq 1 - \delta \) for
\[ Z := \{ x \in V_{k+1} : \mu_{\bar{m}}(H_x) \geq \rho \}. \]

Note that \( Z \in B_{\bar{k}+1} \) by Fubini. Now, for any \( x \in Z \) and \( \bar{w} \in H_x \),
\[ \| f_{x} - h(-, \bar{w}, x) \|_{L^2} < 2\delta \] by definition of \( H \). And for every fixed \( x \in V_{k+1} \) with \( \mu_{\bar{m}}(H_x) > 0 \) we have
\[ f_{x}(\bar{y}) = \frac{1}{\mu_{\bar{m}}(H_x)} \int f_{x}(\bar{y}) \cdot \chi_H(\bar{w}, x) d\mu_{\bar{m}}(\bar{w}) \]
for all \( \bar{y} \in V^{\bar{k}} \). Then, by Lemma 4.7(2), averaging over \( \bar{w} \in H_x \), we get
\begin{equation}
\| f_{x} - g'(-, x) \|_{L^2(\mu_{\bar{k}})} \leq 2\delta \text{ for every fixed } x \in Z. \tag{6.3}
\end{equation}

But then, as \( \mu(Z) \geq 1 - \delta \) by (6.2), using the second implication in Lemma 4.7(1) we get
\begin{equation}
\| f - g' \|_{L^2} \leq (4\delta^2)^{\frac{1}{2}}. \tag{6.4}
\end{equation}

Next we will approximate \( g' \) by a function of the required form.

**Claim 6.4.** The following functions are \( B_{\{k+1\}}(f) \)-measurable.

1. \( x \in V_{k+1} \mapsto \chi_{D_{\bar{\alpha}}}(\bar{w}, x) \) for every fixed \( \bar{\alpha} \in Q \) and \( \bar{w} \in V^{\bar{m}} \);
2. \( x \in V_{k+1} \mapsto \chi_{H}(\bar{w}, x) \) for every fixed \( \bar{w} \in V^{\bar{m}} \);
3. \( x \in V_{k+1} \mapsto \frac{1}{\max \{ \rho, \mu_{\bar{m}}(H_x) \}} \).

**Proof.** (1) Let \( \bar{\alpha} \) and \( \bar{w} \) be fixed. If we also fix \( \bar{y} \), then the function \( x \mapsto f_{x}(\bar{y}) - f^{\ell}_{\bar{\alpha}}(\bar{y}, \bar{w}, x) \) is clearly \( B_{\{k+1\}}(f) \)-measurable. Then \( x \mapsto (f_{x}(\bar{y}) - f^{\ell}_{\bar{\alpha}}(\bar{y}, \bar{w}, x))^2 \) is also \( B_{\{k+1\}}(f) \)-measurable. Applying Lemma 4.8, the function
\[ x \mapsto \int \left( f_{x}(\bar{y}) - f^{\ell}_{\bar{\alpha}}(\bar{y}, \bar{w}, x) \right)^2 d\mu_{\bar{k}}(\bar{y}) \]
is also \( B_{\{k+1\}}(f) \)-measurable, and using uniform continuity of \( x \mapsto x^{\frac{1}{4}} \) on \([0, 1], x \mapsto \| f_{x}(\bar{y}) - f^{\ell}_{\bar{\alpha}}(\bar{y}, \bar{w}, x) \|_{L^2(\mu_{\bar{k}})} \) is also \( B_{\{k+1\}}(f) \)-measurable. Following the definition of \( D_{\bar{\alpha}} \) and standard arguments, we see that \( x \in V_{k+1} \mapsto \chi_{D_{\bar{\alpha}}}(\bar{w}, x) \) is also \( B_{\{k+1\}}(f) \)-measurable.

(2),(3) similar unwinding the definitions and using Lemma 4.8 every time integration is applied. \( \dashv \)
Let \( J \in \binom{[k+1]}{\leq k} \) be arbitrary, and let \( \bar{m}_J \in \mathbb{N}^{k+1} \) be given by \( \bar{m}_J := \sum_{i \in J} \bar{d}_i \). If \( J \subseteq [k] \) and \( s \in S \), we define \( B^s_J : V^{\bar{m}_J + \bar{m}} \rightarrow [0,1] \) via

\[
B^s_J(\bar{z} \oplus \bar{w}) := \prod_{(i_1, \ldots, i_k) \in [m_1] \times \cdots \times [m_k]} \chi_{f}^{r_{s(i_1),x_i(k) \setminus J},r_{s(i_k),x_k(k) \setminus J}+1}(\bar{z} \oplus \bar{w}^{k+1}_{x_i t_{i,t} \rightarrow t, t \in [k] \setminus J}) .
\]

Otherwise, \( J = I \cup \{ k + 1 \} \) for some \( I \in \binom{[k]}{\leq k-1} \), in particular \( [k] \setminus I \neq \emptyset \), and we define \( B^s_J : V^{\bar{m}_J + \bar{m}} \rightarrow [0,1] \) via

\[
B^s_J(\bar{z} \oplus \bar{w}) := \prod_{(i_1, \ldots, i_k) \in [m_1] \times \cdots \times [m_k]} \chi_{f}^{r_{s(i_1),x_i(k) \setminus I},r_{s(i_k),x_k(k) \setminus I}+1}(\bar{z} \oplus \bar{w}^{k+1}_{x_i t_{i,t} \rightarrow t, t \in [k] \setminus I}) .
\]

Comparing to the definition of \( A^s \), we see that for every \( s \in S \) and \( \bar{y} \in V^k, \bar{w} \in V^{\bar{m}}, x \in V_{k+1} \), taking \( \bar{z} := \bar{y} - (0) + \bar{0} \cdot (x) + \bar{w} \) we have

\[
A^s(\bar{y}, \bar{w}, x) = \prod_{J \in \binom{[k+1]}{\leq k}} B^s_J(\bar{z}_J \oplus \bar{w}) .
\]

And from the definition, for every \( J \in \binom{[k+1]}{\leq k} \) and every fixed \( \bar{w} \in V^{\bar{m}} \),

\[
(6.5) \text{ the function } \bar{z} \in V^{\bar{m}_J} \mapsto B^s_J(\bar{z} \oplus \bar{w}) \in [0,1] \text{ is } B^s_J(f)\text{-measurable (see Definition 6.3).}
\]

Consider the \( \sigma \)-algebra \( C_1 \subseteq B_{1,k+1} \) generated by the collection of sets \( \left\{ h^{(q,r)}_{\bar{w}} \cap \chi_{H_{\bar{w}}}^{=t} \right\} \).

For every fixed \( \bar{w} \in V^{\bar{m}} \), the function \( (\bar{y}, x) \mapsto h(\bar{y}, \bar{w}, x) \cdot \chi_{H(\bar{w}, x)} \) is clearly \( C_1 \)-measurable. Hence, by Lemma 4.8 the function

\[
h_1 : (\bar{y}, x) \mapsto \int h(\bar{y}, \bar{w}, x) \cdot \chi_{H(\bar{w}, x)} d\mu_{\bar{m}}(\bar{w})
\]

is \( C_1 \)-measurable. Then we can approximate it up to \( L^2(\mathbb{R}_{1,k+1}) \)-distance \( \delta \) by a \( C_1 \)-simple function

\[
\sum_i \beta_i \cdot \chi_{h^{(q_i,r_i)}_{\bar{w}_i}}(\bar{y}, x) \cdot \chi_{H_{\bar{w}_i}^{=t_i}}(x)
\]

for some finitely many \( \bar{w}_i \in V^{\bar{m}}, \beta_i, q_i, r_i \in \mathbb{Q}_{[0,1]} \) and \( t_i \in \{0,1\} \).
We consider a single summand, so we fix $\bar{w}$ and $q < r$ and $t$. By definition of $h$ and $f_i^{q,t}$'s,

\[
(6.7) \quad \chi_{\bar{w}}^{(q,r)}(\bar{y}, x) = \sum_{\bar{a} \in Q} \chi_{(D_{\bar{a}})\bar{w}}(x) \cdot \chi_{(f_{\bar{a}}^{q,r})\bar{w}}(\bar{y}, x),
\]

\[
\chi_{(f_{\bar{a}}^{q,r})\bar{w}}(\bar{y}, x) = \sum_{s \in S \cap \alpha_s \in [r, s)} \alpha_s A_s^\bar{w}(\bar{y}, x),
\]

\[
A_s^\bar{w}(\bar{y}, x) = \prod_{J \in \{[k+1]^{\leq k}\}} (B_J^s)((\bar{y}, x), f).
\]

Note that each $(B_J^s)_{\bar{w}}$ is a $B_J(f)$-measurable $k$-ary function by (6.5), each $\chi_{(D_{\bar{a}})\bar{w}}$ is $B_{\{k+1\}}(\bar{f})$-measurable by Claim 6.4(1) and $\chi_{H_{\bar{w}}}^\cdot$ is $B_{\{k+1\}}(f)$-measurable by Claim 6.4(2). Then, replacing each summand in (6.6) by a corresponding expression from (6.7) and regrouping the sum, we conclude that $h_1$ can be approximated up to $L^2(\mu_{1+k+1})$-distance $\delta$ by a finite sum of the form

\[
(6.8) \quad \hat{h}_1 : \bar{x} \in V^{1^{k+1}} \mapsto \sum_i \hat{\beta}_i \cdot \prod_{J \in \{[k+1]^{\leq k}\}} \hat{f}_J^i(\bar{x}, f)
\]

with $\hat{\beta}_i \in \mathbb{Q}^{[0,1]}$ and $\hat{f}_J^i : \prod_{J \in \{I\}} V_i \to [0, 1]$ a $B_J(f)$-measurable $k$-ary function. But then, considering the function $h_2 : (\bar{y}, x) \in V^{1^{k+1}} \mapsto \frac{1}{\max \rho_{\mu_{\bar{w}}(H_{\bar{x}})}}$ and applying Lemma 17.3, $\|g' - h_2 \cdot \hat{h}_1\|_{L^2} = \|h_2 \cdot h_1 - h_2 \cdot \hat{h}_1\|_{L^2} \leq 3\delta$. The map $x \in V_{k+1} \mapsto \frac{1}{\max \rho_{\mu_{\bar{w}}(H_{\bar{x}})}}$ is $B_1(f)-$measurable by Claim 6.4(3), hence multiplying the sum in (6.5) by it and regrouping, the product $h_2 \cdot \hat{h}_1$ is of the form

\[
g : \bar{x} \in V^{1^{k+1}} \mapsto \sum_i \gamma_i \cdot \prod_{J \in \{[k+1]^{\leq k}\}} f_J^i(\bar{x}, f)
\]

for some finitely many $\gamma_i \in \mathbb{Q}^{[0,1]}$ and $B_J(f)$-measurable $k$-ary functions $f_J^i : \prod_{J \in \{I\}} V_i \to [0, 1]$. So $g$ is of the required form, and using (6.4)

\[
\|f - g\|_{L^2} \leq \|f - g'\|_{L^2} + \|g' - g\|_{L^2} \leq (4\delta^2)^{\frac{3}{2}} + 3\delta < \epsilon
\]

assuming we started with $\delta$ sufficiently small with respect to $\epsilon$. \qed

This argument actually gives us an additional uniformity we will need in the next subsection.

**Corollary 6.5.** Suppose that $k' > k \in \mathbb{N}$ and $(V_{[k]}, B_{\bar{w}}, \mu_{\bar{w}})_{\bar{w} \in \mathbb{N}^{k'}}$ is a $k'$-partite graded probability space. Let $I' := [k'] \setminus [k + 1]$ and $\bar{n}' := \sum_{i \in I'} \delta_i \in \mathbb{N}^{k'}$. Suppose $f : V^{1^{k'}} \to [0, 1]$ is $B_\bar{n'}$-measurable and, for every $\tilde{z} \in V^{\bar{n}'}$, $VC_k(f_{\tilde{z}}) < \infty$. Then, for every $\epsilon > 0$, there exist some $N \in \mathbb{N}$, $\gamma_i \in \mathbb{Q}^{[0,1]}$ for $i \in [N]$, and $B_{I \cup I'}(f)$-measurable functions $f_i^1 : (\prod_{J \in \{I\}} V_J) \times V^{\bar{n}'} \to [0, 1]$. 

for $i \in [N], I \in \left(\binom{k+1}{\leq k}\right)$ so that, defining $g : V \rightarrow [0, 1]$ via

$$g : \bar{x} \mapsto \sum_{i \in [N]} \gamma_i \cdot \prod_{I \in \left(\binom{k+1}{\leq k}\right)} f_I^1(\bar{x}_I, \bar{x}_I'),$$

we have $\|f - g\|_{L^2(\mu_{k+1})} < \varepsilon$ for all $\bar{z} \in V$ except for a set of $\mu_{k'}$-measure $\varepsilon$.

Proof. The proof of Proposition 6.5 can be carried out uniformly in all those $\bar{z}$ such that $(m_1, \ldots, m_k)$ are large enough relative to $\text{VC}_k(f_1)$. In particular, by countable additivity, we can choose $(m_1, \ldots, m_k)$ large enough to work except for a set of $\bar{z}$ of measure $< \varepsilon$. \square

6.2. Proof of the general case. We are now ready to prove the main theorem.

**Theorem 6.6.** Suppose that $\left(V_{k'}, \mathcal{B}_{k'}, \mu_{k'}\right)_{\bar{r} \in \mathbb{N}^{k'}}$ is a $k'$-partite graded probability space and $f : V^{k'} \rightarrow [0, 1]$ is a $k'$-ary $\mathcal{B}_{k'}$-measurable function with $\text{VC}_k(f) < \infty$ (see Definition 3.7(4)) for some $k < k'$. Then $f$ is $\mathcal{B}_{k', k}$-measurable.

More precisely, for every $\varepsilon > 0$ there exist some $N \in \mathbb{N}$, $\gamma_i \in \mathbb{Q}[0, 1]$ for $i \in [N]$ and $\mathcal{B}_I(f)$-measurable $(\leq k)$-ary functions $f_I : \prod_{j \in I} V_j \rightarrow [0, 1]$ for $i \in [N], I \in \left(\binom{k'}{\leq k}\right)$ so that, defining $g : V^{k'} \rightarrow [0, 1]$ via

$$g(\bar{x}) := \sum_{i \in [N]} \gamma_i \cdot \prod_{I \in \left(\binom{k'}{\leq k}\right)} f_I^1(\bar{x}_I),$$

we have $\|f - g\|_{L^2(\mu_{k'})} < \varepsilon$.

**Remark 6.7.** Consider the simplest case, where $k' = 3$ and $k = 1$. Corollary 6.5 says

$$f(x_1, x_2, x_3) \approx \sum_{i \in [N]} \gamma_i f_i^1(x_1, x_3) f_i^2(x_2, x_3)$$

for almost all fixed $x_3 \in V_3$, where $f_i^1, f_i^2$ are $\mathcal{B}_{I \cup \{3\}}(f)$-measurable for suitable $I \in \left(\binom{3}{\leq 1}\right)$. The elements of $\mathcal{B}_I(f)$ are built from the levels sets of $f$. While this does not ensure that they are themselves of finite $\text{VC}_1$-dimension, we will show that they are closely approximated by sets of finite $\text{VC}_1$-dimension. This implies that we can apply Proposition 6.1 to the approximations of the functions $f_i^1, f_i^2$ approximating them by unary functions, and putting it together we obtain the desired representation of $f$.

**Proof of Theorem 6.6.** We prove the proposition by induction on $k' - k$. The base case $k' - k = 1$ is given by Proposition 6.1

So let $1 \leq k < k'$ with $k' - k \geq 2$ be fixed, and assume that the claim holds for all pairs $k_0 < k'$ with $k_0' - k_0 < k' - k$. Let $\varepsilon \in \mathbb{R}_{>0}$ be given, and fix $\delta \in \mathbb{R}_{>0}$ sufficiently small with respect to $\varepsilon$, to be determined later.
Assume that \( \left( V_{[k]}, \mathcal{B}_{\overline{N}}, \mu_{\overline{N}} \right)_{n \in \mathbb{N}^{k'}} \) is a \( k' \)-partite graded probability space, and \( f : V^{1^{k'}} \rightarrow [0, 1] \) is a \( k' \)-ary \( \mathcal{B}_{1}^{k'} \)-measurable function with \( \text{VC}_{k}(f) \leq d < \infty \).

As \( k \leq k' - 2 < k' \), the latter implies that also \( \text{VC}_{k'-2}(f) < \infty \), in particular \( \text{VC}_{k'-2}(f_{x_i}) < \infty \) for every \( x_i \in V_{k'} \). Applying Corollary 6.5 with \( k_0 := k' - 2, k_0' := k', \delta \) in place of \( k, k', \varepsilon \), there exist some \( N' \in \mathbb{N} \), \( \gamma_i \in [0, 1] \) for \( i \in [N'] \), and \( \mathcal{B}_{I \cup \{k'\}}(f) \)-measurable (\( \leq k' - 1 \))-ary functions \( f_i^j : \left( \prod_{j \in I} V_j \right) \times V_{k'} \rightarrow [0, 1] \) for \( i \in [N'], I \in \left( \mathbb{N}^{k'-2} \right) \) so that, taking \( g : V^{1^{k'}} \rightarrow [0, 1] \) to be

\[
g : \overline{x} \in V^{1^{k'}} \mapsto \sum_{i \in [N']} \gamma_i \cdot \prod_{j \in \left( \mathbb{N}^{k'-2} \right)} f_i^j(\overline{x}_I, \overline{x}_{k'}) ,
\]

we have

\[
\left\| f_{x_{k'}} - g_{x_{k'}} \right\|_{L^2(\mu_{1^{k'-1}})} < \delta \text{ for all } x_{k'} \in V_{k'} \setminus X_{k'},
\]

for some \( X_{k'} \in \mathcal{B}_{1^{k'}} \) with \( \mu_{1^{k'}}(X_{k'}) < \delta \).

At this point we would like to apply the inductive hypothesis to the (\( \leq k' - 1 \))-ary functions \( f_i^j \), however a priori there is no reason for them to be of finite \( \text{VC}_{k} \)-dimension: if \( \text{VC}_{k}(f) < \infty \), then we might still have \( \text{VC}_{k}(\chi_{f \leq r}) = \infty \) for a fixed \( r \in [0, 1] \). We show that at least these functions can be approximated arbitrarily well in \( L^2 \)-norm by functions of finite \( \text{VC}_{k} \)-dimension.

So fix some \( i \in [N'] \) and \( I \in \left( \mathbb{N}^{k'-2} \right) \), and let \( J := I \cup \{k'\} \in \left( \mathbb{N}^{k'} \right) \). Let \( \delta' > 0 \) be arbitrary. As \( f_i^j \) is \( \mathcal{B}_{J}(f) \)-measurable, by definition of \( \mathcal{B}_{J}(f) \) (see Definition 6.3) we can choose a sufficiently large \( t \in \mathbb{N} \) and some \( \overline{w}_1, \ldots, \overline{w}_t \in V^{1^{k'}} \) so that \( \left\| f_i^j - h \right\|_{L^2(\mu_{1^{k'}})} < \delta' \), where \( m_{J} := \sum_{j \in J} \delta_j \in \mathbb{N}^{k'} \) and \( h \) is a function of the form

\[
h : \overline{x} \in \prod_{j \in J} V_j \mapsto \sum_{u \in [t]} \alpha_u \cdot \chi_{f_{\overline{w}_u}[D_{\{u\}}]}(\overline{x})
\]

for some \( \alpha_u, r_u, s_u \in [0, 1] \). Let \( \delta'' > 0 \) be arbitrary. By Lemma 4.9 we can choose a sufficiently large \( p \in \mathbb{N} \) so that, for every \( u \in [t] \) and \( q \in \{s_u, r_u\} \), taking \( f_{u}^{\leq q} := p \times \left( q - f_{\overline{w}_u}[D_{\{u\}}] \right) \), we have

\[
\left\| \chi_{f_{\overline{w}_u}[D_{\{u\}}]} - f_{u}^{\leq q} \right\|_{L^2(\mu_{m_J})} < \delta''.
\]

Letting \( f_{u}^{\leq q} := f_{u}^{\leq q} \cdot (1 - f_{u}^{\leq q}) \) and using Lemma 4.7(3), we thus have

\[
\left\| \chi_{f_{\overline{w}_u}[D_{\{u\}}]} - f_{u}^{\leq q} \right\|_{L^2(\mu_{m_J})} < 2\delta''.
\]

(6.11)
for every \( u \in [t] \). Let
\[
h' : \bar{x} \in \prod_{j \in I} V_j \mapsto \sum_{u \in [t]} \alpha_i \cdot f_{iu}^{[r_u,s_u]}(\bar{x}),
\]
then, by (6.11) and Lemma 4.7[3] again, we have
\[
(6.12) \quad \left\| f_I' - h' \right\|_{L^2(\mu_{\bar{a},j})} \leq \left\| f_I' - h \right\|_{L^2(\mu_{\bar{a},j})} + \left\| h - h' \right\|_{L^2(\mu_{\bar{a},j})} \\
\leq \delta' + (2t + 1)2\delta'' < 2\delta'
\]
assuming we took \( \delta'' \) small enough with respect to \( \delta' \) and \( t \).

Note that, for every \( u \in [t] \), \( f_{iu}^{[r_u,s_u]} \) is clearly \( \mathcal{B}_J(f) \)-measurable from the
definition, hence also \( h' \) is \( \mathcal{B}_J(f) \)-measurable. Also, since fixing some of the
coordinates or permuting the coordinates preserves finiteness of the
VC-dimension of a function (Proposition 10.6) and \( VC \quad \mathcal{B}_J(f) \), we then have
\[
\forall j \in I: \quad \left\| f_I' - h' \right\|_{L^2(\mu_{\bar{a},j})} < \delta' \quad \text{for every} \quad \bar{a} \in \prod_{j \in I} V_j \mapsto [0,1].
\]
Using Lemma 4.7[3] this implies
\[
(6.13) \quad \left\| g - g' \right\|_{L^2(\mu_{\bar{a},I,\bar{c}})} \leq N' \cdot \left( 2 \left( \frac{|k' - 1|}{k' - 2} \right) + 1 \right) \cdot 3\delta' < \delta,
\]
assuming that we took \( \delta' = \delta'(k', \delta, N') \) sufficiently small.
Regrouping the elements of the expression for $g'$, we see that it is of the form

$$g'(\bar{x}) := \sum_{i \in [N]} \gamma'_i \cdot \prod_{I \in \{k'\}} h_I^i(\bar{x}_I)$$

for some $N \in \mathbb{N}$, $\gamma'_i \in \mathbb{Q}^{[0,1]}$ and $h_I^i$ a $(\leq k)$-ary $B_t(f)$-measurable functions $h_I^i : \prod_{j \in I} V_j \to [0,1]$. Hence $g'$ has the required form, and it remains to show that $g'$ approximates $f$ in $L^2$-norm.

By (6.13) and the first implication in Lemma 4.7(1), there exists some set $X''_{k'} \in B_{\delta_{k'}}$ with $\mu_{\delta_{k'}}(X''_{k'}) \leq \delta$ such that $\|g_{x_{k'}} - g'_{x_{k'}}\|_{L^2(\mu_{\delta_{k'}}^{-1})} \leq \delta^2$ for all $x_{k'} \in V_{k'} \setminus X''_{k'}$. Combining this with (6.10) and taking $X_{k'} := X'_{k'} \cup X''_{k'}$, we thus have $\mu_{\delta_{k'}}(X_{k'}) \leq 2\delta$ and $\|f_{x_{k'}} - g'_{x_{k'}}\|_{L^2(\mu_{\delta_{k'}}^{-1})} \leq \delta + \delta^2$ for every $x_{k'} \in V_{k'} \setminus X_{k'}$. Hence, by the second implication in Lemma 4.7(1), we have $\|f - g'\|_{L^2(\mu_{\delta_{k'}})} \leq \left(\max\{2\delta,(\delta + \delta^2)\}\right)^{\frac{1}{2}} < \varepsilon$ assuming $\delta$ was chosen small enough with respect to $\varepsilon$.

\[\square\]

**Remark 6.8.** We note that there is an alternate approach which avoids the careful analysis of the sets in $B_t(f)$, at the price of using additional machinery about $\sigma$-subalgebras. We illustrate the idea in the simplest case, where $k' = 3$ and $k = 1$. Given $f(x_1, x_2, x_3)$, two applications of Corollary 6.5—once with $x_1$ as the parameter and once with $x_2$ as the parameter—tells us that

$$f(x_1, x_2, x_3) \approx \sum_{i \in [N]} \beta_i \chi_{U_i^1}(x_1, x_2) \chi_{U_i^2}(x_1, x_3)$$

and also

$$f(x_1, x_2, x_3) \approx \sum_{j \in [N]} \gamma_j \chi_{W_j^1}(x_1, x_2) \chi_{W_j^2}(x_2, x_3),$$

for an appropriate choice of the coefficients $\beta_i, \gamma_j$ and sets

$$U_i^1, W_j^1 \in B_{\{1,2\}}(f), U_i^2 \in B_{\{1,3\}}(f), W_j^2 \in B_{\{2,3\}}(f).$$

By rearranging the sums to be over intersections $U_i^1 \cap W_j^2$, we may assume the sums are over the same collection of sets—that is,

$$f(x_1, x_2, x_3) \approx \sum_{i \in [N]} \beta_i \chi_{U_i^1}(x_1, x_2) \chi_{U_i^2}(x_1, x_3) \approx \sum_{i \in [N]} \chi_{U_i^1}(x_1, x_2) \chi_{W_i^2}(x_2, x_3).$$

But then on each of the sets $U_i^1$, we have $\chi_{U_i^2} \approx \chi_{W_i^1}$, which means the sets $U_i^2, W_j^2$ must be close to not depending on $x_1$ or $x_2$, respectively: that is, we could replace $U_i^2$ with $u_i^2(x_3) = \int \chi_{U_i^2}(x_1, x_3) \, d\mu_{\delta_{i}}(x_1)$.

So, after rearranging, we get

$$f(x_1, x_2, x_3) \approx \sum_{i \in [N]} \gamma'_i \chi_{U_i^1}(x_1, x_2) \chi_{U_i^2}(x_3).$$
That is, \( f \) is measurable with respect to the \( \sigma \)-subalgebra of \( B_{I^3} \) generated by sets of the form \( A(x_1, x_2) \times B(x_3) \). (In the notation of [Tow17], this \( \sigma \)-subalgebra is called \( B_{I^3, \{\{0,1\}, \{2\}\}} \).)

This argument is symmetric, so \( f \) also has approximations using sets of the form \( A(x_1, x_3) \times B(x_2) \) and \( A(x_2, x_3) \times B(x_1) \). One can show (for instance, using the generalized Gowers uniformity norms) that a function which has several different representations in terms of restricted kinds of sets also has a simultaneous representation respecting all restrictions at once. In a slightly different setting, this is [Tow18, Lemma 8.23].

Finally, we derive a more quantitative version of Theorem 6.6

**Corollary 6.9.** For every \( k < k' \in \mathbb{N}, \bar{d} < \infty \) and \( \varepsilon \in \mathbb{R}_{>0} \) there exists some \( N = N(k, k', \bar{d}, \varepsilon) \in \mathbb{N} \) satisfying the following.

Suppose that \( (V_{[k]}, B_{\bar{n}}, \mu_{\bar{n}})_{\bar{n} \in \mathbb{N}^{k'}} \) is a \( k' \)-partite graded probability space and \( f : V^{1k} \to [0,1] \) is a \( k' \)-ary \( B_{1k'} \)-measurable function with \( \text{VC}_k(f) < \bar{d} \) (see Definition 6.5).

Then for \( i \in [N] \) there exist some \( \gamma_i \in \mathbb{Q}_N^{[0,1]} \), \( \bar{w}_i \in V^{1k} \) and, for each \( I \in \binom{[k]}{\leq k'} \), a \( (\leq k) \)-ary function \( f_i : \prod_{i \in I} V_i \to [0,1] \) simple with respect to the algebra \( B_{I,\bar{w}_1,\ldots,\bar{w}_N}(f) \) (see Definition 6.3) and with all of its coefficients in \( \mathbb{Q}_N^{[0,1]} \) so that, defining a \( B_{1k',k} \)-measurable function \( g : V^{1k+1} \to [0,1] \) via

\[
g(\bar{x}) := \sum_{i \in [N]} \gamma_i \cdot \prod_{I \in \binom{[k]}{\leq k}} f_i(\bar{x}_I),
\]

we have \( \|f - g\|_{L^2} < \varepsilon \).

**Proof.** This follows from Theorem 6.6 via a compactness argument relying on the techniques of Section 9 as we explain below.

Assume first that \( \mathfrak{P} = (V_{[k]}, B_{\bar{n}}, \mu_{\bar{n}})_{\bar{n} \in \mathbb{N}^{k'}} \) is an arbitrary \( k' \)-partite graded probability space and \( g : V^{1k} \to [0,1] \) is as in the conclusion of Theorem 6.6. Approximating each \( f_i \) by a \( B_{I,\bar{w}_1,\ldots,\bar{w}_N}(f) \)-simple functions for a sufficiently large \( t \) and some \( \bar{w}_1, \ldots, \bar{w}_t \in V^{1k'} \), we may assume that \( g \) is of the form \( g(\bar{x}) := \sum_{i \in [N]} \gamma_i \cdot \prod_{I \in \binom{[k]}{\leq k}} f_i(\bar{x}_I) \) for some \( N \in \mathbb{N} \), \( \gamma_i \in \mathbb{Q}_N^{[0,1]} \) and

\[
f_i := \sum_{j \in [t]} \alpha_j \cdot \chi_{\mathfrak{P},i,\bar{w}_j}(\bar{x}_I)
\]

for some \( t \in \mathbb{N} \), \( \alpha_j \in V^{1k'} \) and \( \gamma_i, \alpha_j, r_{ij} \in \mathbb{Q}_N^{[0,1]} \). Substituting these expressions for \( f_i \)'s into \( g \) and rearranging, we may thus assume that \( g \) is of the form

\[
h_{N,\bar{a},\bar{r},\bar{s},\bar{w}}(\bar{x}) := \sum_{i \in [N]} \alpha_i \cdot \prod_{I \in \binom{[k]}{\leq k}} \chi_{\mathfrak{P},i,\bar{w}_j}(\bar{x}_I)
\]
for some bigger $N \in \mathbb{N}$ and some $\bar{\alpha} = (\alpha_i \in Q_N^{[0,1]} : i \in [N])$, $\bar{r} = (r^i \in Q_N^{[0,1]} : i \in [N], I \in \binom{[k]}{\leq k})$ and $\bar{s} = (s^i \in Q_N^{[0,1]} : i \in [N], I \in \binom{[k]}{\leq k})$ with $r^i < s^i$, and $\bar{w} = (w^i \in V^{1k'} : i \in [N], I \in \binom{[k]}{\leq k})$. Following the proof of Lemma 9.21(3) with straightforward modifications, we see that for every fixed $N, \bar{\alpha}, \bar{r}, \bar{s}$ and $\varepsilon \in \mathbb{R}_{>0}$ there exists a countable collection of $L_\infty$-sentences $\Theta_{i,\bar{\alpha},\bar{r},\bar{s}}$ so that: for any $k'$-partite graded probability space $\mathfrak{P} = (V_{\bar{k}'}, \mathcal{B}_{\bar{n}}, \mu_{\bar{n}})_{\bar{n} \in \mathbb{N}^{k'}}$, a $\mathcal{B}_{\bar{k}'}$-measurable $f$ and any $L_\infty$-structure $\mathcal{M}' \propto \mathcal{M}_{\mathfrak{P},f}$, 

\begin{equation} 
\mathcal{M}' \models \Theta_{i,\bar{\alpha},\bar{r},\bar{s}} \iff 
\end{equation}

for all tuples $\bar{w} = (w^i \in V^{1k'} : i \in [N], I \in \binom{[k]}{\leq k})$, 

$$||f - h_{N,\bar{\alpha},\bar{r},\bar{s},\bar{w}}||_{L^2} \geq \varepsilon.$$ 

Now assume towards a contradiction that the conclusion of the theorem fails for some $k, k', d, \varepsilon$. This means that for every $j \in \mathbb{N}$, there exists some $k'$-partite graded probability space $\mathfrak{P}_j = (V_{\bar{k}'}, \mathcal{B}_{\bar{n}}, \mu_{\bar{n}})_{\bar{n} \in \mathbb{N}^{k'}}$ and some $\mathcal{B}_{\bar{k}'}$-measurable function $f_j : \prod_{i \in [k']} V_i \to [0,1]$ with $\text{VC}_k(f_j) \leq d$ such that, in view of the previous paragraph (7) and that $\mathcal{M}_{\mathfrak{P}_j, f_j} \propto \mathcal{M}_{\mathfrak{P}_j, f_j}$ trivially,

$$\mathcal{M}_{\mathfrak{P}_j, f_j} \models \bigwedge_{\bar{\alpha} \in \left(Q_{\bar{j}}^{[0,1]}\right)^{[j]}} \bigwedge_{\bar{r}, \bar{s} \in \left(Q_{\bar{j}}^{[0,1]}\right)^{[j] \times \binom{[k]}{\leq k}}} \Theta_{i,\bar{\alpha},\bar{r},\bar{s}}.$$

Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Let $\hat{\mathfrak{P}} := (\hat{V}_{\bar{k}'}, \hat{\mathcal{B}}_{\bar{n}}, \hat{\mu}_{\bar{n}})_{\bar{n} \in \mathbb{N}^{k'}}$ be the $k'$-partite graded probability space, the $\mathcal{B}_{\bar{k}'}$-measurable function $\hat{f} : \hat{V}^{1k'} \to [0,1]$ and $\hat{\mathcal{M}}$ the $L_\infty$-structure defined by the corresponding ultraproduct in Section 9.3 (Fact 9.12). By Los’ theorem we then have 

$$\hat{\mathcal{M}} \models \bigwedge_{j \in \mathbb{N}} \bigwedge_{\bar{\alpha} \in \left(Q_{\bar{j}}^{[0,1]}\right)^{[j]}} \bigwedge_{\bar{r}, \bar{s} \in \left(Q_{\bar{j}}^{[0,1]}\right)^{[j] \times \binom{[k]}{\leq k}}} \Theta_{i,\bar{\alpha},\bar{r},\bar{s}}.$$ 

As $\hat{\mathcal{M}} \propto \mathcal{M}_{\hat{\mathfrak{P}}, \hat{f}}$, using (7) this implies that $\hat{f}$ does not satisfy the conclusion of Theorem 6.6 for any $N \in \mathbb{N}$ — a contradiction. \hfill \Box

Specializing to the case of hypergraphs instead of arbitrary functions, we immediately get the following corollary.

**Corollary 6.10.** For every $k < k' \in \mathbb{N}, d \in \mathbb{N}$ and $\varepsilon \in \mathbb{R}_{>0}$ there exists some $N = N(k, k', d, \varepsilon) \in \mathbb{N}$ satisfying the following.

Suppose that $(V_{\bar{k}'}, \mathcal{B}_{\bar{n}}, \mu_{\bar{n}})_{\bar{n} \in \mathbb{N}^{k'}}$ is a $k'$-partite graded probability space and $E \in \mathcal{B}_{\bar{k}'}$ is a $k'$-ary relation with $\text{VC}_k(E) \leq d$. 


Then there exists some \((\leq k)\)-ary fibers \(F_1, \ldots, F_N\) of \(E\) (so each \(F_i\) is obtained from \(E\) by fixing all but at most \(k\) coordinates by some parameters from the corresponding \(V_i\)’s) and \(F\) a Boolean combination of \(F_1, \ldots, F_N\) so that \(\mu_i^{(k)}(E \Delta F) < \varepsilon\).

(Where for \(I \in \binom{[k^i]}{k}\) and an \(|I|\)-ary fiber \(F \subseteq \prod_{i \in I} V_i\), \(F'\) is the \(k'\)-ary relation \(\{\bar{x} \in \prod_{i \in [k^i]} V_i : \bar{x}_I \in F\}\).

**Proof.** Applying Corollary 6.9 to \(\chi_E\), we get that \(\|\chi_E - g\|_{L^2} < \varepsilon\) for some \(g\) of the form

\[
g(\bar{x}) = \sum_{i \in [N]} \alpha_i \cdot \prod_{I \in (\binom{[k^i]}{\leq k})} \chi_{E_{\bar{x}_i}}(\bar{x}_I)
\]

for some \(\alpha_i \in \mathbb{Q}_{[0,1]}\), \(i^I \in \{0,1\}\) and \(\bar{w}^i, I \subseteq V_{k^i}\). As in Remark 4.3, replacing \(N\) by some larger \(N' = N'(N, \varepsilon)\), we may assume that \(\alpha_i \in \{0,1\}\) for all \(i \in N'\) — which gives the required presentation. \(\Box\)

7. High \(VC_{k^i}\)-dimension implies inapproximability

We now consider the converse to the results of the previous section. As pointed out in the introduction, we cannot expect that every \(B_{k^i}\)-measurable \(k'\)-ary function has finite \(VC_{k^i}\)-dimension, because the infinite shattered set could have measure 0. To find the right converse, we should notice that the conclusion of Corollary 6.9 depends only on the \(VC_{k^i}\)-dimension of \(f\); this means that we would have approximations with the same bound on their complexity if we replaced the measures \(\mu_{\bar{n}}\) with different measures. That is, Corollary 6.9 holds uniformly under all measures.\(^{10}\)

So the expected converse is that \(f\) should have finite \(VC_{k^i}\)-dimension if \(f\) has the property that for every \(\varepsilon > 0\) there is an \(N\) so that, for all choices of measures on the \(V_i\), \(f\) can be approximated to within \(\varepsilon\) in \(L^2\)-norm with respect to those measures by a function of the form \(g(\bar{x}) = \sum_{j \in [N]} \gamma_j \cdot \prod_{I \in (\binom{[k]}{\leq k})} f_j^i(\bar{x}_I)\) as in Theorem 6.6.

**Theorem 7.1.** Let \(k' > k\) and \(f : \prod_{i \in \binom{[k^i]}{k}} V_i \to [0,1]\) be given such that, for every \(\varepsilon > 0\) there is an \(N \in \mathbb{N}\) such that: for any \(k'\)-partite graded probability space \((V_{k^i}, B_{\bar{n}}, \mu_{\bar{n}})_{\bar{n} \in [N]^{k'}}\) such that \(f\) is \(B_{k'}\)-measurable, there is a function \(g : \prod_{i \in \binom{[k^i]}{k}} V_i \to [0,1]\) of the form

\[
g(\bar{x}) = \sum_{j \in [N]} \gamma_j \cdot \prod_{I \in (\binom{[k]}{\leq k})} f_j^i(\bar{x}_I),
\]

with some coefficients \(\gamma_j\) and each \(f_j^i\) a \(B_{\sum_{i \in [k]} \delta_i}\)-measurable \((\leq k)\)-ary function, and \(\|f - g\|_{L^2} < \varepsilon\). Then \(VC_{k}(f) < \infty\).

\(^{10}\)Compare the distinction between sets with the Glivenko-Cantelli property, the universal Glivenko-Cantelli property, and the uniform Glivenko-Cantelli property. It is only the last which equivalent to having finite VC dimension [Tal87].
Proof. Let $k' > k$ and $f : \prod_{i \in [k']} V_i \to [0, 1]$ satisfy the assumption of the theorem, and towards a contradiction suppose that $\mathrm{VC}_k(f) = \infty$. By Definition 3.11(4) this means that there exist some $I \subseteq [k']$ with $|I| = |k' - (k + 1)|$ and some $b = (b_i : i \in I) \in V_I$ such that the $k + 1$-ary fiber of $f$ at $b$, $f_b : \prod_{i \in [k'] \setminus I} \to [0, 1]$ has $\mathrm{VC}_k(f) = \infty$. Fix $r,s$ so that $\mathrm{VC}_{k,s}^r(f_b) = \infty$. Then, by Remark 3.14, for every finite $(k + 1)$-partite hypergraph $H$ there is an induced copy of $H$ in $f_b$, in the sense that $f_b$ is $\leq r$ on edges of $H$, and $\geq s$ on non-edges of $H$. Permuting the coordinates if necessary (see Remark 2.2(1)), we may assume that $I = [k + 1]$.

For each $d \in \mathbb{N}$, we choose uniformly at random a finite $(k + 1)$-partite $(k + 1)$-uniform hypergraph $H_d \subseteq [d]^{k + 1}$. With probability $1$, $\lim_{d \to \infty} \frac{|H_d|}{d^{k + 1}} = 1/2$ and $\lim_{d \to \infty} \|H_d - 1/2\|_{U^{1k + 1}} = 0$ (for $U^{1k + 1}$ with respect to the uniform measure; see the proof of [Tow17, Theorem 9.2], for instance, for the second calculation). For each $d$ we define probability measures $\mu^d_{\tilde{b}_i}$ which concentrate on the single element $b_i$ if $i \in I$ and concentrate uniformly on the vertices of the $i$th part in a chosen copy of $H_d$ contained in $V_i$ otherwise. Note that these are atomic measures, so the extension of the $\mu^d_{\tilde{b}_i}$ to a Keisler graded probability space on all subsets of the products of the $V_i$ is immediate: there is a unique extension to all subsets depending on the intersection of a set with the finitely many atoms of the measure (see also Remark 2.2).

This gives us $k'$-partite graded probability spaces $\Psi_d = \left(V_{[k']}, B_{\tilde{b}_i}, \mu^d_{\tilde{b}_i}\right)_{\tilde{b}_i \in [n]}$ (where $B_{\tilde{b}_i}$ is the algebra of all subsets of $V_{[k']}$). Fix an arbitrary $E \in \mathbb{N}_{>0}$. Using the assumption we may choose some $N = N_E \in \mathbb{N}$ and approximations $g^d,E(x) = \sum_{j \in [N]} \gamma^d,E_j \cdot \prod_{I \in ([k']^c)} f^d,E,j(\tilde{x}_I)$ of $f$ to within $\frac{1}{E}$ with respect to $L^2\left(\mu^d_{[1,k']}\right)$.

We fix some non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and consider the ultraproduct $\bar{\Psi} := \left(\bar{V}_{[k']}, \bar{B}_{\tilde{b}_i}, \bar{\mu}_{\tilde{b}_i}\right)_{\tilde{b}_i \in \mathbb{N}_{[k']}}$ of the $\Psi_d$’s, $\bar{f} : \bar{V}^{[1,k']}) \to [0, 1]$ of the functions $f_d = f$, $\bar{f}^d,E,j$ of the functions $f^d,E,j$ and $\bar{g}^E : \bar{V}^{[1,k']}) \to [0, 1]$ of the functions $g^d,E$ as in Section 9.3 (namely, $\bar{\Psi}$ is defined with respect to the ultraproduct of the structures $\mathcal{M}_d := \mathcal{M}_{\bar{\Psi},\bar{f},\bar{g}^E_d,E}(\prod_{I \in ([k']^c)} f^d,E,j(\tilde{x}_I))$ for $d \in \mathbb{N}$ in the notation there). Let $\bar{b} = (\bar{b}_i : i \in I)$ with $\bar{b}_i = (b_{i_1}, b_{i_2}, \ldots) / \mathcal{U} \in \bar{V}_I$.

By the choice of $H_d$ and $\mu^d_{\tilde{b}_i}, i \in [k']$, we have

\begin{align}
\lim_{d \to \infty} \mu^d_{[1,k']}\left(f^\leq r\right) &= \lim_{d \to \infty} \mu^d_{[1,k+1]}\left(f^\leq r\right) = \frac{1}{2}, \quad \text{and} \\
\lim_{d \to \infty} \|\chi f^\leq r - 1/2\|_{U^{1k'}}\left(\mu^d_{[1,k']}\right) &= \lim_{d \to \infty} \|\chi f^\leq r - 1/2\|_{U^{1k+1}}\left(\mu^d_{[1,k+1]}\right) = 0.
\end{align}

This implies that in the ultraproduct we get the exact equalities. Indeed, as in the proof of Lemma 9.21(2), for any $\alpha \in \mathbb{Q}_{>0}$ there exist countable
collections of $L_\infty$-sentences $\Theta_\alpha, \Theta'_\alpha$ so that: for any $k'$-partite graded probability space $\mathfrak{Q} = (V_{[k']}, \mathcal{B}_n,\mu_n)_{n \in \mathbb{N}^k}$, a $\mathcal{B}_{1^{k'}}$-measurable function $f$ and any $L_\infty$-structure $\mathcal{M} \models \mathcal{M}_{\mathfrak{Q}, f}$,

$$\mathcal{M} \models \Theta_\alpha \iff \mu_{1^{k'}}(\chi_f \leq r) \in \left[\frac{1}{2} - \alpha, \frac{1}{2} + \alpha\right],$$

$$\mathcal{M} \models \Theta'_\alpha \iff \|\chi_f - 1/2\|_{L_{1^{k'}}(\mu_{1^{k'}})} \leq \alpha.$$ 

As trivially $\mathcal{M}_d \models \mathcal{M}_d$ for every $d \in \mathbb{N}$, using Los’ theorem and (7.1) we have that

$$\mathcal{M} \models \bigwedge_{\alpha \in \mathbb{Q} > 0} \Theta_\alpha \land \Theta'_\alpha,$$

which together with $\mathcal{M} \models \mathcal{M}_{\mathfrak{Q}, f}$ (Remark 9.13) implies (using that $\tilde{\mu}_i$ is concentrated on the single element $\tilde{b}_i$ for all $i \in [k'] \setminus [k + 1]$) that

$$\tilde{\mu}_{1^{k'}}(\chi_{\tilde{f} \leq r}) = \tilde{\mu}_{1^{k+1}}(\tilde{f}_b \leq r) = \frac{1}{2},$$

and

$$\|\chi_{\tilde{f} \leq r} - 1/2\|_{L_{1^{k'}}(\tilde{\mu}_{1^{k'}})} = \|\chi_{f_b \leq r} - 1/2\|_{L_{1^{k+1}}(\tilde{\mu}_{1^{k+1}})} = 0.$$

By Lemma 8.7 (applied to the $(k + 1)$-partite graded probability space obtained from $\mathfrak{Q}$ by forgetting all but the first $k + 1$ coordinates and the $(k + 1)$-ary function $\tilde{f}_b$ on it, see Remark 2.4), (7.3) implies

$$\|\mathbb{E}(\chi_{\tilde{f} \leq r} - 1/2 \mid \tilde{\mathcal{B}}_{1^{k+1}, k})\|_{L^2(\tilde{\mu}_{1^{k+1}})} = 0.$$

On the other hand, (7.2) implies $\|\chi_{\tilde{f} \leq r} - 1/2\|_{L^2(\tilde{\mu}_{1^{k+1}})} = 1/4$, hence in particular $\tilde{f}_b \leq r$ cannot be $\tilde{\mathcal{B}}_{1^{k+1}, k}$-measurable.

But each of the functions $f^{E,j}_b$ is $\tilde{\mathcal{B}}_{\sum_{i \in I} \delta_i}$-measurable (by definition and Fact 9.12(7)), hence each of the functions $\tilde{g}^E, E \in \mathbb{N}_{> 0}$ is $\tilde{\mathcal{B}}_{1^{k'}, k}$-measurable, and so each of their fibers $\tilde{g}^E_b, E \in \mathbb{N}_{> 0}$ is $\tilde{\mathcal{B}}_{1^{k+1}, k}$-measurable.

Using type-definability of $L^2$-norm and Los’ theorem as above, the assumption that $\|g^{d,E} - f\|_{L^2(\mu_{1^{k'}})} < \frac{1}{8}$ for all $d \in \mathbb{N}$ implies $\|\tilde{g}^E - \tilde{f}\|_{L^2(\tilde{\mu}_{1^{k'}})} < \frac{1}{8}$, which implies $\|\tilde{g}^E - \tilde{f}\|_{L^2(\tilde{\mu}_{1^{k+1}})} < \frac{1}{8}$ by the choice of the measures. As $E \in \mathbb{N}_{> 0}$ was arbitrary, this implies that $\tilde{f}_b$ is $\tilde{\mathcal{B}}_{1^{k+1}, k}$-measurable, a contradiction. \hfill \Box

**Remark 7.2.** As the proof of Theorem 7.1 shows, in order to conclude that $\text{VC}_{k}(\mathfrak{Q}) < \infty$ it is enough that the stated approximation by functions of arity $\leq k$ holds for all $k'$-partite graded probability spaces on $V_{1^{k'}}$ with finitely supported measures $\mu_{\tilde{\delta}_i}, i \in [k']$. 
When the sets \( V_i \) are finite, all functions have finite VC\(_k\)-dimension, so Theorem 7.1 is not directly applicable. To make sense of this result in the finite setting, we have to consider a “modulus of uniform approximability”. Given a function \( \tilde{N} \), we could say \( f : \prod_{i \in [k']} V_i \to [0,1] \) has “\( \tilde{N} \)-uniform approximations” if, for all graded probability spaces on the \( V_i \) and all \( \varepsilon \), \( f \) has an approximation to within \( \varepsilon \) in the form \( g(\bar{x}) = \sum_{j \in [\tilde{N}(\varepsilon)]} \gamma_j \prod_{I \in (\varepsilon)} f_I^j(\bar{x}_I). \)

(To avoid notational issues, it is more convenient to think of \( \tilde{N} \) as a function whose input is the integer \( 1/\varepsilon \), as we do below.)

What we will show is that for any function \( \tilde{N} \), there is a specific \( \tilde{d} \) so that any \( f \) with \( \tilde{N} \)-uniform approximations must satisfy VC\(_k\)(\( f \)) \( \leq \tilde{d} \).

**Corollary 7.3.** Let \( k' > k \) be given. For any function \( \tilde{N} : \mathbb{N} \to \mathbb{N} \) and any \( r < s \) in \([0,1]\) there is a \( \tilde{d} \in \mathbb{N} \) so that whenever \( V_i \) are finite sets and \( f : \prod_{i \in [k']} V_i \to [0,1] \) with VC\(_k\)(\( f \)) \( \geq \tilde{d} \), there is some \( E \in \mathbb{N} \) and some probability measures \( \mu_i \) on the \( V_i \) (uniquely determining a \( k' \)-partite graded probability space on the algebra of all subsets of \( \prod_{i \in [k']} V_i \), see Remark 2.1(2)) such that for every function of the form

\[
g(\bar{x}) = \sum_{j \in [\tilde{N}(E)]]} \gamma_j \cdot \prod_{I \in (\varepsilon)} f_I^j(\bar{x}_I),
\]

we have \( \|f - g\|_{L^2} \geq 1/E \).

**Proof.** Towards a contradiction, suppose this failed, and let \( k' > k, \tilde{N}, r < s \) be a counterexample. That is, for each \( d \in \mathbb{N} \), we have some finite sets \( (V_{id})_{i \in [k']} \) and a function \( f^d : \prod_{i \in [k']} V_{id} \to [0,1] \) satisfying VC\(_k\)(\( f^d \)) \( \geq d \), but such that for any probability measures \( \mu_i \) on \( V_i \), \( f^d \) can be approximated in \( L^2 \)-norm on the corresponding graded probability space up to \( 1/E \) by some function \( g \) of the above form given by a sum of size \( \tilde{N}(E) \).

Taking a non-principal ultraproduct of these examples (see Section 9.3), we obtain a \( k' \)-ary function \( \tilde{f} : \prod_{i \in [k']} \tilde{V}_i \to [0,1] \) with VC\(_k\)(\( \tilde{f} \)) \( = \infty \) (by Lemma 10.1). Then Theorem 7.1 gives us measures \( \mu'_i \) on the \( \tilde{V}_i \) with finite support, \( \varepsilon > 0 \) and a corresponding \( k' \)-partite graded probability space \((\tilde{V}_{[k']}, \mathcal{B}_{[k'], \mathbb{N}}), \mu'_{k'} \) uniquely determined by setting \( \mathcal{B}_{[k'], \mathbb{N}} \) to be the algebra of all internal subsets of \( V_{[k']} \), and \( \mu'_i = \mu'_i \), such that \( \tilde{f} \) is \( \mathcal{B}_{[k'], \mathbb{N}} \)-measurable, but no function \( g : \prod_{i \in [k']} \tilde{V}_i \to [0,1] \) of the form

\[
g(\bar{x}) = \sum_{j \in [\mathbb{N}]} \gamma_j \cdot \prod_{I \in (\varepsilon)} f_I^j(\bar{x}_I),
\]

with some coefficients \( \gamma_j \) and each \( f_I^j \) \( \mathcal{B}_{[k'], \mathbb{N}} \)-measurable \(( \leq k)\)-ary function can satisfy \( \|\tilde{f} - g\|_{L^2(\mu'_{k'})} < \varepsilon \).

Replacing \( \varepsilon \) with \( 1/E \), we may assume \( \varepsilon = 1/E \) for some \( E \). Since the measures \( \mu'_i, i \in [k'] \) have finite support, for each \( i \in [k'] \) and \( d \in \mathbb{N} \)
there exist probability measures $\mu_i^d$ on the $V_i^d$ so that the ultraproduct of $(\mu_i^d : d \in \mathbb{N})$ (in the sense of Section 8.3) is the measure $\mu_i$. But then, by assumption, for each $d$ there also exists an approximation $g^d = \sum_{j \in [N(E)\] \gamma_j^d}$. 

\[ \prod_{I \in \{\leq k\}} f_{I}^{d,j}(\bar{x}_I) \text{ with } ||f^d - g^d||_{L^2(\mu_i^{d,1})} < 1/E \text{ and each } f_{I}^{d,j} \text{ is } B^d_{\sum_i \delta_i^d} \text{-measurable, where } \left(V_i^d, B_i^d, \mu_i^d\right)_{n \in \mathbb{N}} \text{ is the } k'\text{-partite graded probability space with } B_i^d \text{ the algebra of all subsets of } \prod_{i \in [k']} \left(V_i^d\right)^{n_i}\text{ and } \mu_i^d := \mu_i^d \text{ for } i \in [k']. \text{ But then their ultraproduct } \tilde{g} = \sum_{j \in [N(E)\]} \gamma_j \cdot \prod_{I \in \{\leq k\}} f_{I}^{d}(\bar{x}_I) \text{ satisfies } ||\tilde{f} - \tilde{g}||_{L^2(\mu_i^{d,1})} < 1/E \text{, and each } \tilde{f}_{I} \text{ is } B^d_{\sum_i \delta_i^d} \text{-measurable — which is a contradiction.} \]

\[ \square \]

8. Correlation and measurability with respect to subalgebras

In this section, we develop some aspects of the theory of Gowers’ uniformity norms in the context of partite graded probability spaces used throughout the article. Throughout this section, we let \((V_{[k]}, B_{\bar{n}}, \mu_{\bar{n}})_{n \in \mathbb{N}^k}\) be a \(k\)-partite graded probability space. We fix \(\bar{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k, n := \sum_{i \in [k]} n_i\) and a bounded \(B_{\bar{n}}\)-measurable function \(f : \prod_{i \in [k]} V^{n_i} \to \mathbb{R}\).

8.1. Gowers uniformity norms. Gowers’ uniformity norms were introduced in [Gow01]. The crucial property is Proposition 8.7 below, which says that they exactly measure correlation with the \(\sigma\)-algebra \(B_{\bar{n},n-1}\); the useful feature is that it lets us test whether \(f\) has any correlation with \(B_{\bar{n},n-1}\) by evaluating a single integral which only involves \(f\).

The material in this section is standard, and the presentation in this subsection closely follows [GT14 Section 7.4], however we work in the partite setting and include the details for the sake of completeness.

**Definition 8.1.** We define the (partite) **Gowers uniformity seminorm** of \(f\) by

\[
\|f\|_{U^n} = \left[ \int_{V^{2\bar{n}}} \prod_{\alpha_1 \in \{0,1\}^{n_1}} f(x_{1,1}^{\alpha_1,1}, \ldots, x_{1,n_1}^{\alpha_1,n_1}, \ldots, x_{k,1}^{\alpha_k,1}, \ldots, x_{k,n_k}^{\alpha_k,n_k}) d\mu_{2\bar{n}}(x_{1,1}^0, \ldots, x_{1,n_1}^0, x_{1,1}^1, \ldots, x_{1,n_1}^1; \ldots; x_{k,1}^0, \ldots, x_{k,n_k}^0, x_{k,1}^1, \ldots, x_{k,n_k}^1) \right]^{1/2^n}.
\]

The usual Gowers \(U^k\)-norm is the case where \(\bar{n} = (1, \ldots, 1)\). More generally, the integral is taken over two copies of \(V_{\bar{n}}\), and given two elements \(\bar{x}^0, \bar{x}^1 \in V_{\bar{n}}\), the product is taken over all the \(2^n\) possible ways to select an...
element of $V^n$ by choosing, separately for each coordinate, whether to take it from the corresponding component of $\bar{x}^0$ or $\bar{x}^1$.

Given a tuple $\bar{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \prod_{i \in \{0,1\}^{n_i}}$ and $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_k) \in V^n$, we write $\bar{x}^{\bar{\alpha}} = (\bar{x}^{\alpha_1}_1, \ldots, \bar{x}^{\alpha_k}_k)$, with $\bar{x}_i^{\alpha_i} = (x_{i,j}^{\alpha_i} : j \in [n_i])$ for $i \in [k]$.

Lemma 8.2.

$$\left| \int f d\mu_{\bar{n}} \right| \leq \| f \|_{U^n}.$$ 

Proof. Let $i \in [k]$ and $j \in [n_i]$ be arbitrary. By Fubini property,

$$\left| \int f(\bar{x}) d\mu_{\bar{n}}(\bar{x}) \right|^{2^n} = \left| \int \left( \int f(\bar{x}) d\mu_{\bar{n}_i}(x_{i,j}) \right) d\mu_{\bar{n}_{(n_i-1)-i}}(\bar{x}_{i-1}) \right|^{2^n} \cdot \left( \int f(\bar{x}) d\mu_{\bar{n}_{(n_i-1)-i}}(\bar{x}_{i-1}) \right)^{2^{n-1}}$$

(by Cauchy-Schwarz and Fubini again)

$$\leq \left( \int f(\bar{x}) d\mu_{\bar{n}_i}(x_{i,j}) \right) \cdot \left( \int f(\bar{x}) d\mu_{\bar{n}_{(n_i-1)-i}}(\bar{x}_{i-1}) \right)^{2^{n-1}} \cdot \left( \int f(\bar{x}) d\mu_{\bar{n}_{(n_i-1)-i}}(\bar{x}_{i-1}) \right)^{2^{n-1}}$$

Repeating this process for every pair $i \in [k]$ and $j \in [n_i]$, we arrive at

$$\left| \int f(\bar{x}) d\mu_{\bar{n}}(\bar{x}) \right|^{2^n} \leq \left( \int \prod_{\bar{\alpha} \in \prod_{i \in [k]} \{0,1\}^{n_i}} f(\bar{x}^{\bar{\alpha}}) d\mu_{2^n}(\bar{x}^0 + \bar{x}^1) \right) \leq \| f \|_{U^n}.$$ 

$\square$

Lemma 8.3. For each $\bar{I} = (I_1, \ldots, I_k)$ with $I_i \subseteq [n_i]$ and $\sum_{i \in [k]} |I_i| = n - 1$, let $B_{\bar{I}}$ be a set in $\mathcal{B}_{\tilde{n}, \tilde{I}}$. Then $0 \leq \| f \cdot \prod_{i \in [k]} \chi_{B_{I_i}} \|_{U^n} \leq \| f \|_{U^n}$.

Proof. It suffices to show that $0 \leq \| f \cdot \chi_{B_{I_i}} \|_{U^n} \leq \| f \|_{U^n}$ for a single $\bar{I}$ (as each $\chi_{B_{I_i}}$ takes values in $[0,1]$). We consider $\bar{I}$ with $I_i := [n_i]$ for $i \in [k-1]$ and $I_k := [n_k-1]$. As $f = f \cdot \chi_{B_{I_i}} + f \cdot \chi_{B_{I_i}}$, we have

$$\| f \|_{U^n}^{2^n} = \| f \cdot \chi_{B_{I_i}} + f \cdot \chi_{B_{I_i}} \|_{U^n}^{2^n},$$

which in turn expands into a sum of $2^{2^n}$ terms of the form

$$(8.1) \int \prod_{\bar{\alpha} \in \{0,1\}^{n_1}, \ldots, \bar{\alpha} \in \{0,1\}^{n_k}} \left( f \cdot \chi_{S_{\bar{\alpha} \cdot \bar{\alpha}}} \right) (\bar{x}^{\bar{\alpha}_1}, \ldots, \bar{x}^{\bar{\alpha}_k}) d\mu_{2^n}(\bar{x}^0 + \bar{x}^1),$$

where $S_{\bar{\alpha} \cdot \bar{\alpha}}$ denotes the set of tuples $(x_{i,j}^{\alpha_i} : j \in [n_i])$ for $i \in [k]$. This completes the proof.
where each $S_{\bar{a}_1, \ldots, \bar{a}_k}$ is either $B_{\bar{I}}$ or its complement $-B_{\bar{I}}$. Note that $\|f \cdot \chi_{B_{\bar{I}}}\|_{U^n}$ is equal to such a term with each $S_{\bar{a}_1, \ldots, \bar{a}_k} = B_{\bar{I}}$. Thus it suffices to show that all of the $2^n$ terms are non-negative.

Assume that $\bar{a}, \bar{a}' \in \prod_{i \in [k]} \{0,1\}^{n_i}$ are such that $\alpha_i = \alpha'_i$ for all $i \in [k]$ and $j \in I_k$, but $S_{\bar{a}} \neq S_{\bar{a}'}$. As $B_{\bar{I}} \subset B_{\bar{a}, \bar{i}}$ (so whether a tuple $\bar{x} \in V^{\bar{a}}$ belongs to it or not does not depend on the coordinate $x_k$ by the choice of $\bar{I}$), for every tuple $\bar{\bar{v}} = \bar{v}^0 + 2 \bar{v}^1 \in V^{2n}$, we have $\chi_{S_{\bar{a}} \cdot \chi_{S_{\bar{a}'}}} (\bar{\bar{v}}) = \chi_{S_{\bar{a}'} \cdot \chi_{S_{\bar{a}}} (\bar{\bar{v}})} = 0$ — hence the corresponding integral in (8.1) is 0. We thus only need to consider the case where, whenever $\alpha_i = \alpha'_i$ for all $i \in [k], j \in I_k$, then $S_{\bar{a}} = S_{\bar{a}'}$. In this case, using Fubini, we have

$$\int \prod_{\bar{a}_1 \in \{0,1\}^{n_1}, \ldots, \bar{a}_k \in \{0,1\}^{n_k}} (f \cdot \chi_{S_{\bar{a}}}) (x_{\bar{a}_1}^0, \ldots, x_{\bar{a}_k}^0) \, d\mu_{2n} (\bar{x}^0 \oplus \bar{x}^1) =$$

$$\int \prod_{\bar{a}_1 \in \{0,1\}^{n_1}, \ldots, \bar{a}_k \in \{0,1\}^{n_k}} (f \cdot \chi_{S_{\bar{a}}}) (x_{\bar{a}_1}^0, \ldots, x_{\bar{a}_k}^0) \, d\mu_{0,0,2} \left( x^0_{k,n_k} , x^1_{k,n_k} \right)$$

$$d\mu_{2n_{k-2}} (x_{k,n_k}) = \int \prod_{\bar{a}_1 \in \{0,1\}^{n_1}, \bar{a}_{k-1} \in \{0,1\}^{n_{k-1}}, \bar{a}_k \in \{0,1\}^{n_k-1}} (f \cdot \chi_{S_{\bar{a}}}) (x_{\bar{a}_1}^0, \ldots, x_{\bar{a}_{k-1}}^0, x_{\bar{a}_k^1}^{0} , \ldots, x_{\bar{a}_k}^{0} , x_{k,n_k}) \, d\mu_{0,0,1} (x_{k,n_k})^2 \, d\mu_{2n_{k-2}} (x_{k,n_k}).$$

Since the inside of the integral is always non-negative, this term is non-negative. 

**Definition 8.4.** We let the function $D(f) : V^{\bar{a}} \to \mathbb{R}$ be defined by

$$D(f) (\bar{x}^0) := \int \prod_{\bar{a} \in \prod_{i \in [k]} \{0,1\}^{n_i}, \bar{a} \neq (\bar{0}, \ldots, \bar{0})} f (x_{\bar{a}_1}^0, \ldots, x_{\bar{a}_k}^0) \, d\mu_{\bar{a}} (\bar{x}^1).$$

**Remark 8.5.** Observe that, by Fubini, $\|f\|_{U^n}^2 = \int f \cdot D(f) \, d\mu_{\bar{a}} (\bar{x}^0).$

**Lemma 8.6.** The function $D(f)$ is measurable with respect to $\mathcal{B}_{\bar{n},n-1}$.

**Proof.** Note that, for a fixed $\bar{x}^1 \in V^{\bar{a}}$, the function

$$(8.2) \quad \bar{x}^0 \mapsto \prod_{\bar{a} \in \prod_{i \in [k]} \{0,1\}^{n_i}, \bar{a} \neq (\bar{0}, \ldots, \bar{0})} f (\bar{x}^\bar{a})$$

is $\mathcal{B}_{\bar{n},n-1}$-measurable (as for every such $\bar{a}$, at least one of the coordinates in $\bar{x}^\bar{a}$ is then fixed). Then $D(f)$ is also $\mathcal{B}_{\bar{n},n-1}$-measurable by Lemma 4.8. 

Proposition 8.7. \( \|f\|_{U^n} > 0 \) if and only if \( \| \mathbb{E}(f \mid B_{\bar{n},n-1}) \|_{L^2} > 0 \).

Proof. If \( \|f\|_{U^n} > 0 \), then \( \int f \cdot D(f) d\mu_{\bar{n}}(\bar{x}^\beta) > 0 \) (by Remark 8.5). That is, \( f \) is not orthogonal to \( D(f) \) in the space \( L^2(B_{\bar{n}}) \). As \( D(f) \) is \( B_{\bar{n},n-1} \)-measurable by Lemma 8.6, we conclude \( \| \mathbb{E}(f \mid B_{\bar{n},n-1}) \| > 0 \).

For the other direction, assume that \( \| \mathbb{E}(f \mid B_{\bar{n},n-1}) \| > 0 \). The there exist some sets \( B_i \in B_{\bar{n},i} \), for \( \bar{i} = (I_1, \ldots, I_k) \) with \( I_i \subseteq [n_i] \) and \( \sum_{i\in[k]} |I_i| \leq n-1 \), so that \( \int f \cdot \prod_i \chi_{B_i} d\mu_{\bar{n}} \neq 0 \) (as \( f \) and its projection onto the subspace of \( B_{\bar{n},n-1} \)-measurable functions are non-orthogonal). Then, by Lemmas 8.2 and 8.3

\[
0 < \left| \int f \prod_i \chi_{B_i} d\mu_{\bar{n}} \right| \leq \left\| \int f \prod_i \chi_{B_i} \right\|_{U^n} \leq \|f\|_{U^n}.
\]

\[ \square \]

8.2. Subalgebras of fibers. We will later need to know when \( \mathcal{D} \subseteq B_{\bar{n},n-1} \) is large enough that \( \mathbb{E}(f \mid B_{\bar{n},n-1}) = \mathbb{E}(f \mid \mathcal{D}) \) and, slightly more generally, when \( \mathbb{E}(f \mid B_{\bar{n},n-1} \cup G) = \mathbb{E}(f \mid \mathcal{D} \cup G) \) for some set \( G \).

We can determine this by examining the previous subsection more carefully: if \( \| \mathbb{E}(f \mid B_{\bar{n},n-1}) \|_{L^2} > 0 \), we know that it is because \( \int f \cdot D(f) d\mu_{2\bar{n}} > 0 \), so it suffices to investigate exactly which sets are needed to ensure that \( D(f) \) is \( \mathcal{D} \)-measurable. To deal with the more general case, we need to consider not just when \( D(f) \) is measurable, but when functions of the form \( D(f) \cdot g \) are measurable for a certain class of functions \( g \).

Definition 8.8. Let \( \mathcal{D} \) be a \( \sigma \)-subalgebra of \( B_{\bar{n}} \).

1. Let \( \bar{a} \in V_{\bar{n}} \). We say that \( \mathcal{D} \) contains \( \bar{a} \)-fibers of \( f \) if, for each interval \( I \subseteq \mathbb{R} \) and each \( i < k \) and \( j \in [n_i] \),

\[
\{ \bar{x} = (x_{i,j} : i \in [k], j \in [n_i]) \in V_{\bar{n}} : f(\bar{x}_{a_{i,j} \rightarrow (i,j)}) \in I \} \in \mathcal{D}.
\]

Recall that \( \bar{x}_{a_{i,j} \rightarrow (i,j)} \) is the tuple obtained from \( \bar{x} \) by substituting \( a_{i,j} \) into position \((i,j)\) (see Section 2.1).

2. We say that \( \mathcal{D} \) contains \( (n-1) \)-ary fibers of \( f \) if the set of \( \bar{a} \in V_{\bar{n}} \) such that \( \mathcal{D} \) contains \( \bar{a} \)-fibers of \( f \) has \( \mu_{\bar{n}} \)-measure 1.

3. We say that \( \mathcal{D} \) is closed under fibers if for every set \( B \in \mathcal{D} \), \( \mathcal{D} \) contains \( (n-1) \)-ary fibers of \( \chi_B \).

4. Let \( G \) be a set of \( B_{\bar{n}} \)-measurable functions. We say that \( \mathcal{D} \) contains \( (n-1) \)-ary fibers of \( f \) with products from \( G \) if, for every function \( g \) which is a finite product of functions from \( G \), \( \mathcal{D} \) contains \( (n-1) \)-ary fibers of \( g \) and \( f \cdot g \).

Remark 8.9. (1) If \( \mathcal{D} \) is closed under fibers, then for any \( \mathcal{D} \)-measurable function \( f \), \( \mathcal{D} \) contains \( (n-1) \)-ary fibers of \( f \) (by assumption this holds for the indicator functions of sets in \( \mathcal{D} \), and follows for an arbitrary \( \mathcal{D} \)-measurable function approximating it by \( \mathcal{D} \)-simple functions).
(2) The algebra \( \mathcal{B}_{\bar{n}, n-1} \) is both closed under fibers (by Fubini and closure under products) and contains \((n-1)\)-ary fibers of any \( \mathcal{B}_{\bar{n}} \)-measurable function (by Fubini property, see Remark 2.3).

The following is immediate from the definitions (see Definition 8.8).

**Remark 8.10.** (1) Each of the algebras \( \mathcal{B}_{\bar{n}, n-1}^{l, n, \bar{b}}, \mathcal{B}_{\bar{k}, \bar{b}}, \mathcal{B}_{\bar{k}, k-1}^{l} \) is closed under fibers.

(2) For every \( b \in V_{k+1} \) and a tuple \( \bar{b} \) in \( V_{k+1} \), the algebra \( \mathcal{B}_{I, \bar{b}}^{l, (b)\bar{b}} \) contains \((k-1)\)-ary fibers of \( f_b \) with products from \( \{ f_{b'} : b' \in \bar{b} \} \).

(3) For every \( b \) and \( (b_i)_{i \in I} \) in \( V_{k+1} \), where \( I \) is an arbitrary index set, the algebra \( \mathcal{B}_{I, \bar{k}, k-1}^{l} \) contains \((k-1)\)-ary fibers of \( f_b \) with products from \( \{ f_{b_i} : i \in I \} \).

**Lemma 8.11.** If \( \| \mathbb{E}(f \mid \mathcal{B}_{\bar{n}, n-1}) \|_{L^2} > 0 \) and \( \mathcal{D} \) contains \((n-1)\)-ary fibers of \( f \), then \( \| \mathbb{E}(f \mid \mathcal{D}) \|_{L^2} > 0 \).

**Proof.** If \( \| \mathbb{E}(f \mid \mathcal{B}_{\bar{n}, n-1}) \|_{L^2} > 0 \), then \( \| f \|_{L^n} > 0 \) (by Proposition 8.7). If \( \| f \|_{L^n} > 0 \) then, by Remark 8.5, we have

\[
0 < \| f \|_{L^n}^2 = \int f \cdot D(f) d\mu_{2^n}.
\]

As \( \mathcal{D} \) contains \((n-1)\)-ary fibers of \( f \), the function in (8.2) is \( \mathcal{D} \)-measurable for a measure 1 set of \( \bar{x}^1 \in V^{\bar{n}} \). Hence \( D(f) \) is \( \mathcal{D} \)-measurable by Lemma 4.8. Thus \( f \) is not orthogonal to \( L^2(\mathcal{D}) \).

**Lemma 8.12.** If \( \| \mathbb{E}(f \mid \mathcal{B}_{\bar{n}, n-1} \cup G) \|_{L^2} > 0 \) and \( \mathcal{D} \) contains \((n-1)\)-ary fibers of \( f \) with products from \( G \), then \( \| \mathbb{E}(f \mid \mathcal{D} \cup G) \|_{L^2} > 0 \).

**Proof.** Suppose \( \| \mathbb{E}(f \mid \mathcal{B}_{\bar{n}, n-1} \cup G) \|_{L^2} > 0 \). Then there must exist some \( g \), a product of finitely many functions from \( G \), so that \( \| \mathbb{E}(f \cdot g \mid \mathcal{B}_{\bar{n}, n-1}) \|_{L^2} > 0 \), and therefore \( \| \mathbb{E}(f \cdot g \mid \mathcal{D}) \|_{L^2} > 0 \) by Lemma 8.11 hence \( \| \mathbb{E}(f \mid \mathcal{D} \cup G) \|_{L^2} > 0 \).

**Lemma 8.13.** If \( \mathcal{D} \) is closed under fibers and contains \((n-1)\)-ary fibers of \( f \) with products from \( G \), then

\[
\mathbb{E}(f \mid \mathcal{B}_{\bar{n}, n-1} \cup G) = \mathbb{E}(f \mid \mathcal{D} \cup G).
\]

**Proof.** Let \( f^- := f - \mathbb{E}(f \mid \mathcal{D}) \). Consider any \( \bar{a} \in V^{\bar{n}} \) such that \( \mathcal{D} \) contains \( \bar{a} \)-fibers of \( f \). Let \( g \) be a finite product of functions from \( G \). Then, for every \( i \in [k], j \in [n_i] \) and \( \bar{x} \in V^{\bar{n}} \), we have

\[
f^- \cdot g \left( \bar{x}_{a_{i,j} \to (i,j)} \right) = f \cdot g \left( \bar{x}_{a_{i,j} \to (i,j)} \right) - \mathbb{E}(f \mid \mathcal{D}) \cdot g \left( \bar{x}_{a_{i,j} \to (i,j)} \right).
\]

For any interval \( I \), the sets

\[
\{ \bar{x} \mid f \cdot g \left( \bar{x}_{a_{i,j} \to (i,j)} \right) \in I \}, \{ \bar{x} \mid g \left( \bar{x}_{a_{i,j} \to (i,j)} \right) \in I \}
\]

are both in \( \mathcal{D} \), as \( \mathcal{D} \) contains \((n-1)\)-ary fibers of \( f \) with products from \( G \). And \( \{ \bar{x} \mid \mathbb{E}(f \mid \mathcal{D})(\bar{x}_{a_{i,j} \to (i,j)} \in I \} \) also belongs to \( \mathcal{D} \) by Remark 8.9(1), as
\( \mathcal{D} \) is closed under fibers. So, by taking unions and intersections of such sets, 
\( \{ \bar{x} \mid f^-(\bar{x}_{a_{i,j} \rightarrow (i,j)}) \in I \} \) belongs to \( \mathcal{D} \) as well, hence \( \mathcal{D} \) contains \((n - 1)\)-ary fibers of \( f^- \) with products from \( G \).

If \( \mathbb{E}(f \mid \mathcal{D} \cup G) \neq \mathbb{E}(f \mid B_{1,n-1} \cup G) \), then
\[
\|\mathbb{E}(f^- \mid B_{1,n-1} \cup G)\|_{L^2} = \|\mathbb{E}(f^- \mid B_{1,n-1} \cup G)\|_{L^2} - \|\mathbb{E}(f \mid \mathcal{D} \cup G)\| > 0.
\]
Hence \( \|\mathbb{E}(f^- \mid \mathcal{D} \cup G)\|_{L^2} > 0 \) by Lemma \text{S.III} which is a contradiction to the choice of \( f^- \).

\[\square\]

9. Indiscernible sequences of random variables

In this section we gather the model theoretic compactness arguments we need and providing the necessary background on ultraproducts and indiscernible sequences. We also prove a couple of de Finetti-style results that are used in the proof of the main theorem.

9.1. Generic \( k \)-partite \( k \)-uniform hypergraphs. We define some classes of ordered partite hypergraphs and related structures, and discuss their basic model-theoretic properties (see [CPT19] for further discussion).

**Definition 9.1.** For \( k \in \mathbb{N}_{\geq 1} \), let \( G_{k,p} \) denote the countable generic \( k \)-partite \( k \)-uniform ordered hypergraph, viewed as the unique countable first-order structure in the language \( L_{\text{opg}}^k = (R_k, P_1, \ldots, P_k, <) \) with the underlying set \( G \) satisfying the following first-order \( L_{\text{opg}}^k \)-theory \( T_{\text{opg}}^k \):

1. \( P_1, \ldots, P_k \) are unary predicates giving a partition of \( G \);
2. \( R_k \subseteq \prod_{i \in [k]} P_i \);
3. \( < \) is a total linear order on \( G \) and \( P_1 < \ldots < P_k \);
4. \( (P_i, < | P_i) \) is a dense linear ordering for each \( i \in [k] \);
5. for every \( j \in [k] \), any finite disjoint sets \( A_0, A_1 \subseteq \prod_{i \in [k] \setminus \{ j \}} P_i \) and \( b_0 < b_1 \in P_j \), there exists some \( b_0 < b < b_1 \) such that

\[
G_{k,p} \models R_k(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_k) \iff \\
\bar{a} = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k) \in A_0
\]

for all \( \bar{a} \in A_0 \cup A_1 \).

We also let \( G_{k,p} \) be the class of all finite \( k \)-partite \( k \)-uniform ordered hypergraphs (i.e. \( G_{k,p} \) is the class of all finite \( L_{\text{opg}}^k \)-structures satisfying axioms (1)–(3) in Definition 9.1).

**Definition 9.2.** (1) We denote by \( O_{k,p} \) the reduct of \( G_{k,p} \) to the language \( L_{\text{ord}}^k = (P_1, \ldots, P_k, <) \) (i.e. the structure obtained from \( G_{k,p} \) by forgetting the edge relation). We let \( T_{\text{ord}}^k \) be the \( L_{\text{ord}}^k \)-theory consisting of (1), (3) and (4) in Definition 9.1 and \( O_{k,p} \) be the class of all finite \( L_{\text{ord}}^k \)-structures satisfying (1) and (3).

(2) We let \( G_{k,p}^* \) the reduct of \( G_{k,p} \) to the language \( L_{\text{pg}}^k = (R_k, P_1, \ldots, P_k) \) (i.e. the structure obtained from \( G_{k,p} \) by forgetting the ordering). We
let $T_{pg}^k$ be the $L_{pg}^k = (R_k, P_1, \ldots, P_k)$-theory consisting of (1), (2) and the infinite set of sentences expressing the following:

(5)' for every $j \in [k]$ and any finite disjoint sets $A_0, A_1 \subset \prod_{i\in[k]\setminus\{j\}} P_i$ there exists some $b \in P_j$ such that

$$G_{k,p} \models R_k(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_k)$$

$$\iff \exists \bar{a} = (a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k) \in A_0$$

for all $\bar{a} \in A_0 \cup A_1$.

Finally, we let $G_{k,p}'$ be the class of all finite $L_{pg}^k$-structures satisfying (1) and (2) in Definition 9.1.

**Definition 9.3.** Given a structure $\mathcal{M} = (M, (R_i)_{i \in I})$ in a relational language $\mathcal{L} = (R_i : i \in I)$, with $R_i$ a relational symbol of arity $n_i$, and $A \subseteq M$, we let $\mathcal{M}|_A := (A, (R_i \cap A^{n_i})_{i \in I})$ be the substructure induced on $A$.

The following is well-known (we refer to e.g. [Hod93, Chapter 7.1] for the details).

**Fact 9.4.**

1. Each of the theories $T_{opg}^k, T_{pg}^k$ and $T_{rd}^k$ is complete, has quantifier elimination, and is $\aleph_0$-categorical (i.e. there exists a unique, up to isomorphism, countable structure satisfying the corresponding theory).

2. $G_{k,p} (G_{k,p}', O_{k,p})$ is the Fraïssé limit of $G_{k,p} (G_{k,p}', O_{k,p},$ respectively).

3. In particular, $G_{k,p}$ embeds every countable $k$-partite $k$-uniform ordered hypergraphs as an induced substructure; and its finite induced substructures, up to isomorphism, are precisely the structures in $G_{k,p}$.

4. Each of the structures $G_{k,p}, G_{k,p}', O_{k,p}$ is ultrahomogeneous, i.e. every isomorphism between two finite induced substructures extends to an isomorphism of the whole structure.

The following property will be important in Section 10.2.1

**Definition 9.5.** [CT18, Definition 2.17] Let $\mathcal{K}$ be a collection of finite structures in a relational language $\mathcal{L}$. For $n \in \mathbb{N}_{\geq 1}$, we say that $\mathcal{K}$ satisfies the $n$-disjoint amalgamation property ($n$-DAP) if for every collection of $\mathcal{L}$-structures $(\mathcal{M}_i = (M_i, \ldots) : i \in [n])$ so that each $\mathcal{M}_i$ is isomorphic to some structure in $\mathcal{K}$, $M_i = [n] \setminus \{i\}$ and $M_i|_{[n]\setminus\{i,j\}} = M_j|_{[n]\setminus\{i,j\}}$ for all $i \neq j \in [n]$, there exists an $\mathcal{L}$-structure $\mathcal{M} = (M, \ldots)$ isomorphic to some structure in $\mathcal{K}$, and such that $M = [n]$ and $\mathcal{M}|_{[n]\setminus\{i\}} = \mathcal{M}_i$ for every $1 \leq i \leq n$.

We say that an $\mathcal{L}$-structure $\mathcal{M}$ satisfies $n$-DAP if the collection of its finite induced substructures does.

**Proposition 9.6.** $G_{k,p}'$ satisfies $n$-DAP for all $n \in \mathbb{N}_{\geq 1}$.

**Proof.** Fix $k \geq 2$. By Fact 9.4, we need to show that the class of finite structures $G_{k,p}'$ satisfies $n$-DAP. Let $n \in \mathbb{N}$ and $(\mathcal{M}_i : i \in [n])$ with $\mathcal{M}_i \in G_{k,p}'$
as in Definition \[9.5\] be given. In particular, each $\mathcal{M}_i$ satisfies (1) and (2) in Definition \[9.1\]. Then
\[
(9.1) \quad P^{\mathcal{M}_j}_i \cap P^{\mathcal{M}_{j'}}_{i'} = \emptyset \quad \text{for every } i \neq i' \in [k] \text{ and } j, j' \in [n].
\]

Indeed, assume $\ell \in [n]$ is such that $\ell \in P^{\mathcal{M}_j}_i \cap P^{\mathcal{M}_{j'}}_{i'}$. If $j \neq j'$, then necessarily $\ell \in [n] \setminus \{j, j'\}$. By assumption $\mathcal{M}_j|_{[n]\setminus\{j,j'\}} = \mathcal{M}_{j'}|_{[n]\setminus\{j,j'\}}$, hence $\ell \in P^{\mathcal{M}_j}_i \cap P^{\mathcal{M}_{j'}}_{i'}$. But this is impossible as $\mathcal{M}_j$ satisfies (1) of Definition \[9.1\]. Also
\[
(9.2) \quad \ell \in [n] \implies \ell \in P^{\mathcal{M}_j}_i \text{ for some } i \in [k], j \in [n].
\]

Indeed, if $\ell \in [n]$, then $\ell \in M_j$ for any $j \in [n] \setminus \{\ell\}$, hence belongs to $P^{\mathcal{M}_j}_i$ for some $i \in [k]$ as $(P^{\mathcal{M}_j}_i)_{i \in [k]}$ is a partition of $M_j$ by assumption.

For $i \in [k]$, we let $P^M_i := \bigcup_{j \in [n]} P^{\mathcal{M}_j}_i$. Then the sets $P^M_1, \ldots, P^M_k$ give a partition of $M = [n]$ by (9.1) and (9.2).

We let $R^M_k := \bigcup_{j \in [n]} R^M_{k,j}$. As $R^M_{k,j} \subseteq \prod_{i \in [k]} P^{\mathcal{M}_j}_i$ for every $j \in [n]$ by assumption, it follows that $R^M_k \subseteq \prod_{i \in [k]} P^M_i$. Hence the structure $\mathcal{M} := (M, (P^M_i)_{i \in [k]}, R^M_k)$ satisfies (1) and (2) of Definition \[9.1\] hence $\mathcal{M} \in \mathcal{G}_{k,p}^\prime$.

**Remark 9.7.** $O_{k,p}$ (and hence $G_{k,p}$) do not satisfy 3-DAP.

9.2. **Generalized indiscernibles.** Many combinatorial arguments around VC$k$-dimension can be considerably simplified using a combination of structural Ramsey theory and logical compactness, encapsulated in the model-theoretic notion of *generalized indiscernible sequences* (this method does not typically provide strong bounds however).

**Definition 9.8.** Let $\mathcal{M}$ be a first-order structure in a language $\mathcal{L}$.

1. Let $I$ be a structure in a language $\mathcal{L}_0$. We say that a collection $(a_i)_{i \in I}$ of tuples in $\mathcal{M}$ is $I$-indiscernible over a set of parameters $C \subseteq \mathcal{M}$ if for all $n \in \mathbb{N}$ and all $i_0, \ldots, i_n$ and $j_0, \ldots, j_n$ from $I$ we have:
   \[
   \text{qftp}_{\mathcal{L}_0}(i_0, \ldots, i_n) = \text{qftp}_{\mathcal{L}_0}(j_0, \ldots, j_n) \Rightarrow \text{tp}_{\mathcal{L}}(a_{i_0}, \ldots, a_{i_n}, C) = \text{tp}_{\mathcal{L}}(a_{j_0}, \ldots, a_{j_n}, C).
   \]

2. For two $\mathcal{L}_0$-structures $I$ and $J$, we say that a collection of tuples $(b_i)_{i \in I}$ in $\mathcal{M}$ is based on a collection of tuples $(a_i)_{i \in I}$ in $\mathcal{M}$ over a set of parameters $C \subseteq \mathcal{M}$ if for any finite set $\Delta$ of $\mathcal{L}(C)$-formulas, and for any finite tuple $(j_0, \ldots, j_n)$ from $J$ there is a tuple $(i_0, \ldots, i_n)$ from $I$ such that:
   - $\text{qftp}_{\mathcal{L}_0}(j_0, \ldots, j_n) = \text{qftp}_{\mathcal{L}_0}(i_0, \ldots, i_n)$ and
   - $\text{tp}_\Delta(b_{j_0}, \ldots, b_{j_n}) = \text{tp}_\Delta(a_{i_0}, \ldots, a_{i_n})$. 
Definition 9.9. When $(I,<)$ is an arbitrary linear order and $(a_i)_{i \in I}$ is a sequence of finite tuples in $\mathcal{M}$, we say that the sequence $(a_i)_{i \in I}$ is indiscernible (indiscernible over $C$) if $(a_i)_{i \in I}$ is $(I,<)$-indiscernible over $\emptyset$ (over $C$).

The following is standard, relying on the usual Ramsey theorem for (1), and on the fact that finite ordered partitioned hypergraphs form a Ramsey class [NR77, AH78, NR83] for (2).

Fact 9.10. Let $\mathcal{L}$ be a countable language, $\mathcal{M}$ an $\aleph_1$-saturated $\mathcal{L}$-structure and $C \subseteq \mathcal{M}$ a countable subset.

(1) (see e.g. [TZ12] Lemma 5.1.3]) For every countable infinite linear orders $I$ and $J$ and a sequence $(a_i)_{i \in I}$ of finite tuples in $\mathcal{M}$, there exists some sequence $(b_i)_{i \in I}$ of tuples in $\mathcal{M}$ indiscernible over $C$ and based on $(a_i)_{i \in I}$.

(2) [CPT19] Corollary 4.8] For any $k \in \mathbb{N}_{\geq 1}$ and a collection of finite tuples $(a_g)_{g \in G_{k,p}}$ in $\mathcal{M}$, there is some collection of finite tuples $(b_g)_{g \in G_{k,p}}$ in $\mathcal{M}$ which is $G_{k,p}$-indiscernible over $C$ and is based on $(a_g)_{g \in G_{k,p}}$ over $C$. The same holds with $O_{k,p}$ instead of $G_{k,p}$ everywhere.

9.3. Ultraproducts of functions on partite graded probability spaces.

We assume familiarity with ultraproducts of first-order structures and the construction of Loeb’s measure. There are multiple ways to make sense of ultraproducts and compactness of measure spaces and measurable functions (Keisler’s probability logic [Kei85] and its variants, continuous logic [BYBH08], AML logic [GT14], etc.), but here we use the most basic approach relying on the familiar ultraproduct construction for first-order logic (and similar to the one used e.g. by Hrushovski in [Hru12]).

Definition 9.11. Assume that $\mathfrak{F} = (V_{[k]}, B_{\tilde{a}}, \mu_{\tilde{a}})_{\tilde{a} \in \mathbb{N}_k}$ is a $k$-partite graded probability space, $I$ is a countable set and $\tilde{f} = (f_\alpha : \alpha \in I), f_\alpha : \prod_{i \in [k]} V_i \to [0,1]$ is a collection of $B_{\tilde{a}}$-measurable functions. We associate to it a $k$-sorted first-order structure $\mathcal{M}_{\mathfrak{F},\tilde{f}}$ in a language $\mathcal{L}_{\infty,I}$ (or just $\mathcal{L}_\infty$ when $I$ is clear from the context) with sorts $V_1,\ldots,V_k$ in the following way.

For every $q \in \mathbb{Q}^{[0,1]}$ and $\alpha \in I$, $\mathcal{L}_0$ contains a $k$-ary relational symbol

$$F_{\alpha}^{<q}(x_1,\ldots,x_k)$$

with the variable $x_i$ of sort $V_i$, interpreted in $\mathcal{M}_{\mathfrak{F},\tilde{f}}$ via

$$\mathcal{M}_{\mathfrak{F},\tilde{f}} \models F_{\alpha}^{<q}(x_1,\ldots,x_k) : \iff f_\alpha(x_1,\ldots,x_k) < q$$

for any $(x_1,\ldots,x_k) \in V^I$. We write $F_{\alpha}^{\geq q}$ as an abbreviation for $\neg F_{\alpha}^{<q}$. Note that for every $q,\alpha$, the set $\{\tilde{b} \in V_{\tilde{a}} : \mathcal{M}_{\mathfrak{F},\tilde{f}} \models F_{\alpha}^{<q}(\tilde{b})\}$ is in $B_{\tilde{a}}$ by measurability of $f_\alpha$.

By induction on $i \in \mathbb{N}$, we define a countable language $\mathcal{L}_i$ as follows. In addition to all the symbols in $\mathcal{L}_i$, for every quantifier-free $\mathcal{L}_r$-formula $\varphi(\tilde{x},\tilde{y})$
such that the tuple $\bar{x}$ corresponds to $V^\bar{n}, \bar{n} \in \mathbb{N}^k$ and $r \in \mathbb{Q}$, we add to $\mathcal{L}_{i+1}$ a new relational symbol $m_\bar{x} < r. \varphi(\bar{x}, \bar{y})$ with free variables $\bar{y}$, interpreted by:

$$M_{\Psi, \bar{f}} \models m_\bar{x} < r. \varphi(\bar{x}, \bar{b}) : \iff \mu_\bar{n}(\varphi(\bar{x}, \bar{b})) < r,$$

where as usual $\varphi(\bar{x}, \bar{b}) = \{ \bar{a} \in V^\bar{n} \mid M_{\Psi, \bar{f}} \models \varphi(\bar{a}, \bar{b}) \}$ is the set defined by the corresponding instance of $\varphi$ (note that this set is $\mu_\bar{n}$-measurable by Fubini property in $\Psi$ and induction). Let $\mathcal{L}_\infty := \bigcup_{i \in \mathbb{N}} \mathcal{L}_i$. We will write $m_\bar{x} \geq r$ as an abbreviation for $-m_\bar{x} < r$.

We also write $M_{\bar{f}}$ to denote the $\mathcal{L}_0$-reduct of $M_{\Psi, \bar{f}}$.

Now assume that for each $j \in \mathbb{N}$, $\Psi^j = (V^j_{[k]}, B^j_{\bar{n}}, \mu^j_{\bar{n}})_{\bar{n} \in \mathbb{N}^k}$ is a $k$-partite graded probability space and $f^j_{\bar{a}} : \prod_{i \in [k]} V^j_i \rightarrow [0, 1]$ is a $B^j_{\bar{n}}$-measurable function for $\alpha \in I$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$.

For $i \in [k]$, we let $\tilde{V}_i := \prod_{\bar{n} \in \mathbb{N}^k} V^j_i / \mathcal{U}$. Then for any $\bar{n} \in \mathbb{N}^k$, $\tilde{V}^\bar{n}$ is naturally identified with $\prod_{\bar{n} \in \mathbb{N}^k} \left( \prod_{i \in [k]} \left( V^j_i \right)^{n_i} \right) / \mathcal{U}$.

We let $\tilde{M} := \prod_{j \in \mathbb{N}} M_{\Psi, \bar{f}} / \mathcal{U}$ (i.e., the usual ultraproduct of $\mathcal{L}_\infty$-structures).

For $\alpha \in I$, we define a function $\tilde{f}_\alpha : \tilde{V}^k \rightarrow [0, 1]$ via $\tilde{f}_\alpha(\bar{x}) := \inf\{ q \in [0, 1] : \tilde{M} \models F^\alpha < q(\bar{x}) \}$ (and refer to it as the ultraproduct of $f^j_{\bar{a}}$'s with respect to $\mathcal{U}$).

For $\bar{n} \in \mathbb{N}^k$, we let $\tilde{B}^0_{\bar{n}}$ consist of all subsets of $\tilde{V}^\bar{n}$ of the form $X = \prod_{j \in \mathbb{N}} X_j / \mathcal{U}$ for some $X_j \in B^j_{\bar{n}}$.

For such a set $X$, we define $\tilde{\mu}^0_{\bar{n}}(X) := \lim_{j \rightarrow \mathcal{U}} \mu^j_{\bar{n}}(X_j) \in [0, 1]$.

We let $\tilde{B}_{\bar{n}}$ be the $\sigma$-algebra of subsets of $\tilde{V}^\bar{n}$ generated by $\tilde{B}^0_{\bar{n}}$.

As in the standard construction of Loeb’s measure, we have the following fact.

**Fact 9.12.**

1. For every $\bar{n} \in \mathbb{N}^k$, $\tilde{\mu}^0_{\bar{n}}$ is a finitely-additive probability measure on the Boolean algebra $\tilde{B}^0_{\bar{n}}$.
2. $\tilde{M}$ is an $k_1$-saturated $\mathcal{L}_\infty$-structure (in particular, for every finite tuple of variables $\bar{x}$ and a countable collection of $\mathcal{L}_\infty$-formulas $\varphi_i(\bar{x}, \bar{b}_i)$, with $\bar{b}_i$ an arbitrary tuple of parameters from $\tilde{M}$, if every finite subset of $\{ \varphi_i(\bar{x}, \bar{b}_i) : i \in \mathbb{N} \}$ is realized by some tuple in $\tilde{M}$, then the whole set is realized by some tuple in $\tilde{M}$).
3. For every $\bar{n} \in \mathbb{N}^k$, there exists a unique countably-additive probability measure $\tilde{\mu}_{\bar{n}}$ on $\tilde{B}_{\bar{n}}$ extending $\tilde{\mu}^0_{\bar{n}}$.
4. $\tilde{\Psi} := (\tilde{V}_{[k]}, \tilde{B}_{\bar{n}}, \tilde{\mu}_{\bar{n}})_{\bar{n} \in \mathbb{N}^k}$ is a $k$-partite graded probability space.
5. For every $r \in [0, 1], \alpha \in I$ and $\bar{x} \in \tilde{V}^k$, we have

$$\tilde{f}_\alpha(\bar{x}) < r \implies \tilde{M} \models F^{<r}_\alpha(\bar{x}) \implies \tilde{f}_\alpha(\bar{x}) \leq r.$$
(6) For every $r \in \mathbb{Q}^{[0,1]}$, $\bar{n} \in \mathbb{N}^k$, $\varphi(\bar{x}, \bar{y})$ a quantifier-free $\mathcal{L}_\infty$-formula with $\bar{x}$ corresponding to $\vec{V}^{\bar{n}}$ and $\bar{b}$ a tuple from $\mathcal{M}$, we have

$$\mu_{\bar{n}}(\varphi(\bar{x}, \bar{b})) < r \implies \mathcal{M} \models \exists \bar{x}.(r, \varphi(\bar{x}, \bar{b})) \implies \mu_{\bar{n}}(\varphi(\bar{x}, \bar{b})) \leq r.$$  

(7) The functions $\tilde{f}_\alpha$ are $\tilde{B}_{1k}$-measurable.

Here (1), (5) and (6) hold by Łos’ theorem and basic properties of ultralimits; (2) is a standard model-theoretic fact; (3) follows from $\aleph_1$-saturation restricting to any countable sublanguage and Carathéodory’s extension theorem; (4) is a routine verification, e.g. to check that Fubini property holds in the ultraproduct, one approximates the integral by a sum of $\tilde{B}_{1k}^0$-simple functions, and these are arbitrary close to satisfying Fubini by Łos and the assumption that each $P_j$ satisfies Fubini; (7) holds as $\{ \bar{x} \in \tilde{V}^{1k} : \tilde{f}(\bar{x}) < r \} = \bigcup_{\varepsilon \in \mathbb{Q}_{>0}} \{ \bar{x} \in \tilde{V}^{1k} : \tilde{M} \models \varphi(\bar{x}^-) \}$ by (5), and every set on the right is in $\tilde{B}_{1k}^0$.

The following subtle point can be mostly ignored in the conclusions, but we will have to keep track of it in the proofs.

**Remark 9.13.** Note that the interpretation of the $F_{\alpha}^{< r}$ and $m_{\bar{a}} < r$ predicates may differ in $\mathcal{M}_{\bar{f}, j}$ and $\mathcal{M}$, but not by much: due to Fact 9.12(5) and (6), we have $\tilde{M} \propto \mathcal{M}_{\bar{f}, j}$ in the sense of the following definition.

**Definition 9.14.** Let $\mathcal{M}, \mathcal{M}'$ be two $\mathcal{L}_\infty$-structures. We write $\mathcal{M} \propto \mathcal{M}'$ if the structures $\mathcal{M}, \mathcal{M}'$ have the same underlying sorts $V_1, \ldots, V_k$, and for every $\alpha \in I$, $r \in \mathbb{Q}^{[0,1]}$ and $\varepsilon \in \mathbb{Q}_{>0}$ so that $r + \varepsilon \leq 1$ we have

$$\mathcal{M}' \models F_{\alpha}^{< r}(\bar{b}) \Rightarrow \mathcal{M} \models F_{\alpha}^{< r}(\bar{b}) \Rightarrow \mathcal{M}' \models F_{\alpha}^{< (r+\varepsilon)}(\bar{b})$$

and

$$\mathcal{M} \models m_{\bar{a}} < r.\varphi(\bar{x}, \bar{b}) \Rightarrow \mathcal{M} \models m_{\bar{a}} < r.\varphi(\bar{x}, \bar{b}) \Rightarrow \mathcal{M}' \models m_{\bar{a}} < (r + \varepsilon).\varphi(\bar{x}, \bar{b})$$

for every quantifier-free $\mathcal{L}_\infty$-formula $\varphi(\bar{x}, \bar{y})$ and a tuple $\bar{b}$ from $\mathcal{M}$ of appropriate length.

If $\mathcal{M}, \mathcal{M}'$ are just $\mathcal{L}_0$-structures, we write $\mathcal{M} \propto \mathcal{M}'$ when the first of these two conditions is satisfied.

**9.4. Lemmas on indiscernible sequences.** Throughout this section, $k \in \mathbb{N}_{\geq 1}$, $\bar{a} = (V[k], B_{\bar{a}}, \mu_{\bar{a}})$ is a $(k + 1)$-partite graded probability space and $f : V^{1k+1} \to [0, 1]$ is a $\tilde{B}_{1k+1}$-measurable function. We let $\mathcal{M}_{\bar{a}, f}$ be the associated $\mathcal{L}_\infty$-structure and let $\mathcal{M}'$ be some $\mathcal{L}_\infty$-structure satisfying $\mathcal{M} \propto \mathcal{M}_{\bar{a}, f}$ (Definition 9.11). We verify that various probabilistic conditions on the fibers of $f$ are type-definable in $\mathcal{M}'$, via appropriate finitary approximations, and prove some lemmas on indiscernible sequences in the spirit of the classical de Finetti’s theorem on exchangeable sequences of random variables.

**Definition 9.15.** A set $X \subseteq V^{\bar{a}}$ is type-definable in an $\mathcal{L}_\infty$-structure $\mathcal{M}'$ if there exists a countable set $\{ \varphi_i(\bar{x}, \bar{b}_i) : i \in \mathbb{N} \}$ of $\mathcal{L}_\infty$-formulas with the
tuple of variables $\bar{x}$ corresponding to $V^n$ and parameters in $\mathcal{M}'$ so that $X = \{ \bar{a} \in V^n : \mathcal{M}' \models \varphi_i(\bar{a}, \bar{b}_i) \text{ for all } i \in \mathbb{N} \}$.

**Remark 9.16.** The $\sigma$-algebra $B^f_{k-1}$ (recall Definition 5.4(5)) has a generating set that is uniformly definable in $\mathcal{M}'$. Namely, given $q \in \mathbb{Q}^{[0,1]}$, we consider the $\mathcal{M}'$-definable set

$$F^{<q} := \{ \bar{x} \in V^{k+1} : \mathcal{M}' \models F^{<q}(\bar{x}) \}.$$ 

Using $\mathcal{M}' \cong \mathcal{M}_f^q$ we have $F^{<r} = \bigcup_{\varepsilon \in \mathbb{Q}_{>0}} F^{<r-\varepsilon}$ and $F^{<r} = \bigcup_{\varepsilon \in \mathbb{Q}_{>0}} f^{r-\varepsilon}$, hence $\{ F^{<q} : q \in \mathbb{Q}^{[0,1]} \}$ is a generating set for $\sigma(f)$.

Now, for each $\emptyset \neq I \in \left( \binom{k}{k-1} \right)$ and $q \in \mathbb{Q}^{[0,1]}$, we consider the quantifier-free $L_0$-formula

$$\varphi_{I,q}(\bar{x}, \bar{y}) := F^{<q}(\bar{x}_{y_i \rightarrow r_i, i \in I} \neg(y_{k+1})), $$

where $\bar{x}$ is a tuple of variables corresponding to $V^{1k}$ and $\bar{y}$ is a tuple of variables corresponding to $V^{1k+1}$.

Then, for any $a \in V_{k+1}$, every set in $\mathcal{B}^{f,a}$ (see Definition 5.4) is in the $\sigma$-algebra generated by the sets of solutions of $\varphi_{I,q}(\bar{x}, \bar{b} \neg(a))$ in $\mathcal{M}_f^q$ for some $I \in K := \left( \binom{k}{k-1} \right) \setminus \{ \emptyset \}$, $q \in \mathbb{Q}^{[0,1]}$ and $\bar{b} \in V^{1k}$.

This allows us to uniformly define various other algebras and their generating sets.

**Definition 9.17.** Given $n \in \mathbb{N}$, let

$$S_n := \left\{ s \mid s : [n] \times K \times \mathbb{Q}_n^{[0,1]} \rightarrow \{-1, 1\} \right\},$$

$$U_n := \left\{ u \mid u : [n] \times \mathbb{Q}_n^{[0,1]} \rightarrow \{-1, 1\} \right\},$$

$$Q_n := S_n \times S_n \times U_n.$$ 

Given $n$ and $(s, t, u) \in Q_n$, we consider the quantifier-free $L_0$-formula

$$\varphi^{s,t,u}(\bar{x}; \bar{y}_1, \ldots, \bar{y}_n; z_1, \ldots, z_n) :=$$

$$\bigwedge_{(i,I,q) \in [n] \times K \times \mathbb{Q}_n^{[0,1]}, s(i,I,q)=1} \varphi_{I,q}(\bar{x}, \bar{y}_i) \land$$

$$\bigwedge_{(i,I,q) \in [n] \times K \times \mathbb{Q}_n^{[0,1]}, s(i,I,q)=-1} -\varphi_{I,q}(\bar{x}, \bar{y}_i) \land$$

$$\bigwedge_{(i,q) \in [n] \times \mathbb{Q}_n^{[0,1]}, t(i,q)=1} F^{<q}(\bar{x}, z_i) \land \bigwedge_{(i,q) \in [n] \times \mathbb{Q}_n^{[0,1]}, t(i,q)=-1} -F^{<q}(\bar{x}, z_i).$$

**Remark 9.18.**

1. Every subset of $V^{1k}$ defined by an instance of $\varphi^{s,t,u}$ in $\mathcal{M}'$ is in $\mathcal{B}^{1k}$. 

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(2) For any \( \bar{a} \in V_k^n \) and \( \bar{b}_1, \ldots, \bar{b}_n \in V_{k+1}^n \), the sets
\[
\left\{ \phi^\varphi \left( \bar{x}; \bar{b}_1, \ldots, \bar{b}_n; \bar{a} \right) \mid \bar{v} \in Q_n \right\}
\]
are precisely the atoms of the Boolean algebra generated by
\[
\left\{ \phi_{I,q} \left( \bar{x}, \bar{b}_1 \right) : I \in K, i \in [n], q \in Q_n^{[0,1]} \right\} \cup \left\{ F^{\le q}_{\bar{a}_i} : i \in [n], q \in Q_n^{[0,1]} \right\}.
\]

**Lemma 9.19.** For any \( n \in \mathbb{N}_{\ge 1} \), any quantifier-free \( L_\infty \) formulas \( \varphi_i(\bar{x}, \bar{y}_1, \bar{y}_2) \), \( 1 \le i \le n \) with \( \bar{x} \) corresponding to \( V^n \) and \( \bar{y}_i \) to \( V_{m_i} \), \( \varepsilon \in \mathbb{R}_{>0} \) and \( \beta_1, \ldots, \beta_n \in \mathbb{R} \), there exists countable partial \( L_\infty \) types \( \Gamma^\varphi_{\le \varepsilon, \beta}(\bar{y}_1), \Gamma^\varphi_{\ge \varepsilon, \beta}(\bar{y}_1) \) satisfying the following.

For every \( \mathcal{Q}, f, \mathcal{M}' \propto \mathcal{M}_{\mathcal{Q}, f} \) and \( \bar{b} \in V^m \),
\[
\mathcal{M}' \models \Gamma^\varphi_{\le \varepsilon, \beta}(\bar{b}) \iff \text{for every } \bar{c} \in V^m, \sum_{i=1}^n \beta_i \cdot \mu_n \left( \varphi_i(\bar{x}, \bar{b}, \bar{c}) \right) \le \varepsilon.
\]

And similarly for “\( \leq \)” replaced by “\( \geq \).”

**Proof.** Fix some \( \mathcal{Q}, f, \mathcal{M}' \propto \mathcal{M}_{\mathcal{Q}, f} \). Without loss of generality \( \beta_i \neq 0 \) for all \( i \in [n] \). Then for any \( \bar{b} \in V^m_1 \) we have
\[
\forall \bar{c} \in V^m_2, \sum_{i=1}^n \beta_i \cdot \mu_n \left( \varphi_i(\bar{x}, \bar{b}, \bar{c}) \right) \le \varepsilon \iff \\
\bigwedge_{i=1}^n \bigwedge_{r_i \in \mathbb{Q}_{\ge 0}, \sum_{i=1}^n \beta_i r_i > \varepsilon} \forall \bar{c} \in V^m_2, \mu_n \left( \varphi_i(\bar{x}, \bar{b}, \bar{c}) \right) \gg \varphi_i r_i,
\]
where \( \gg \) means “\( \leq \)” if \( \beta_i > 0 \), and “\( \geq \)” if \( \beta_i < 0 \), for every \( i \in [n] \).

As \( \mathcal{M}' \propto \mathcal{M}_\mathcal{Q} \), for every \( r \in \mathbb{Q} \), \( i \in [n] \) and \( \bar{b} \in V^m_1 \), \( \bar{c} \in V^m_2 \) we have
\[
\mu_n \left( \varphi_i(\bar{x}, \bar{b}, \bar{c}) \right) < r \Rightarrow \mathcal{M}' \models \exists \bar{c} \in V^m_2, \mu_n \left( \varphi_i(\bar{x}, \bar{b}, \bar{c}) \right) < r.
\]

Hence, for any \( \bar{b} \in V^m_1 \),
\[
\forall \bar{c} \in V^m_2, \sum_{i=1}^n \beta_i \cdot \mu_n \left( \varphi_i(\bar{x}, \bar{b}, \bar{c}) \right) \le \varepsilon \iff \\
\mathcal{M}' \models \Gamma^\varphi_{\le \varepsilon, \beta}(\bar{b}) := \bigwedge_{i=1}^n \bigwedge_{r_i \in \mathbb{Q}_{\ge 0}, \sum_{i=1}^n \beta_i r_i > \varepsilon} \forall \bar{c} \in V^m_2, \mu_n \left( \varphi_i(\bar{x}, \bar{b}, \bar{c}) \right) \le r.
\]

Note that the definition of \( \Gamma^\varphi_{\le \varepsilon, \beta} \) does not depend on \( \mathcal{Q}, f, \mathcal{M}' \propto \mathcal{M}_\mathcal{Q}, f \). The argument for “\( \geq \)” is symmetric. \( \square \)

**Definition 9.20.** Given some \( n \in \mathbb{N}, r \in \mathbb{Q}, \alpha = (\alpha_{\bar{v}} \in \mathbb{Q} : \bar{v} \in Q_n) \), tuples \( \left( \bar{b}_j \in V^{k+1} : j \in [n] \right) \) and \( \bar{a} = (a_j \in V_{k+1} : j \in [n]) \), we define the function
\[
h_{n, \alpha, \bar{a}, \bar{b}_1, \ldots, \bar{b}_n}(\bar{x}) = \sum_{\bar{v} \in Q_n} \alpha_{\bar{v}} \chi_\varphi(\bar{x}, \bar{b}_1, \ldots, \bar{b}_n, \bar{a}_1, \ldots, \bar{a}_n).
from $V^k$ to $[0, 1]$ (where, as usual, $\varphi^\bar{v}(...)$ represents the set of solutions of this formula evaluated in $M'$).

Lemma 9.21. For any fixed $n \in \mathbb{N}, r \in \mathbb{Q}, \bar{\alpha} = (\alpha_{\bar{v}} \in \mathbb{Q} : \bar{v} \in Q_n)$ there exist quantifier-free $L_\infty$-formulas $\Theta^n_{<r}, \Theta^n_{\geq r}$ and countable partial $L_\infty$-types $\Lambda^n_{<r}, \Lambda^n_{\geq r}$ satisfying the following for any $\mathfrak{F}, f, M' \propto M_{\mathfrak{F}, f}$:

1. for any $(c, \bar{a}, b_1, \ldots, b_n) \in V^k \times V_{k+1}^n$, 
   $$M' = \Theta^n_{<r}(c, \bar{a}, b_1, \ldots, b_n) \iff h_{n,\bar{a},\bar{b}_1,\ldots,\bar{b}_n}(c) < r;$$

2. for any $(a, \bar{a}, b_1, \ldots, b_n) \in V_{k+1} \times V_{k+1}^n$, 
   $$M' = \Lambda^n_{\geq r}(a, \bar{a}, b_1, \ldots, b_n) \iff \|f_a - h_{n,\bar{a},\bar{b}_1,\ldots,\bar{b}_n}\|_{L^2} \leq r;$$

3. for any $(a, \bar{a}) \in V_{k+1} \times V_{k+1}^n$, 
   $$M' = \Lambda^n_{\geq r}(a, \bar{a}) \iff \forall(b_1, \ldots, b_n) \in \left(\bigvee_{w \in W} \bigwedge_{\bar{v} \in \bar{w}} \varphi^\bar{v}(...\right) \lor \bigwedge_{\bar{v} \in \bar{w}} \neg \varphi^\bar{v}(...\right).$$

And the same for “$\geq$”.

Proof. (1) Let $W := \{w \subseteq Q_n \mid \sum_{\bar{v} \in w} \alpha_{\bar{v}} < r\}$. Then clearly

$$h_{n,\bar{a},\bar{b}_1,\ldots,\bar{b}_n}(c) < r \iff M' = \Theta^n_{<r}(c, \bar{a}, b_1, \ldots, b_n) := \bigvee_{w \in W} \left( \bigwedge_{\bar{v} \in w} \varphi^\bar{v}(c, \bar{b}_1, \ldots, \bar{b}_n, a_1, \ldots, a_n) \land \bigwedge_{\bar{v} \in \bar{w}} \neg \varphi^\bar{v}(c, \bar{b}_1, \ldots, \bar{b}_n, a_1, \ldots, a_n) \right).$$

(2) and (3) Note that, using $M' \propto M_{\mathfrak{F}, f}$, for any $r < s \in \mathbb{Q} \cap [0, 1]$ and $a \in V_{k+1}$, if $\bar{x} \in F_{\alpha}^{\geq r} \cap F_{\alpha}^{< s}$, then $|f_a(\bar{x}) - r| \leq r - s$. Then for any $\varepsilon > 0$ we can choose $\varepsilon > 0$ we can choose $m_\varepsilon, \ell_\varepsilon \in \mathbb{N}$ large enough and a partition $(q_i^\varepsilon : i \in [\ell_\varepsilon])$ of $[0, 1]$ with $q_1^\varepsilon \in \mathbb{Q}^{|\ell_\varepsilon|}, q_\ell^\varepsilon < q_1^\varepsilon, q_1^\varepsilon = 0, q_\ell^\varepsilon = 1$ so that for any $a \in V_{k+1}$, any quantifier-free $L_\infty$-formula $\psi(\bar{x}, \bar{y})$ with $\bar{x}$ corresponding to $V^k$ and any tuple $\bar{c}$ corresponding to $\bar{y}$ we have:

$$\int f_a \cdot \chi_{\psi(\bar{x}, \bar{c})} d\mu^k \approx \varepsilon A^\varepsilon_{\psi}(\bar{x}, \bar{c}) := \sum_{i=1}^{\ell_\varepsilon-2} q_i^\varepsilon \cdot \mu^k \left( F_{a}^{\geq q_i^\varepsilon} \cap F_{a}^{< q_{i+1}^\varepsilon} \cup \psi(\bar{x}, \bar{c}) \right) + q_{\ell_\varepsilon-1}^\varepsilon \cdot \mu^k \left( F_{a}^{\geq q_{\ell_\varepsilon-1}^\varepsilon} \cup \psi(\bar{x}, \bar{c}) \right).$$
For any tuple \((a, \bar{a}, \bar{b}_1, \ldots, \bar{b}_n)\), using (9.3) we have

\[
\left\| f_a - h_{n, \bar{a}, \bar{b}_1, \ldots, \bar{b}_n} \right\|_{L^2}^2 = \\
\int \left( f_a - \sum_{\bar{v} \in Q_n} \alpha_{\bar{v}} \chi_{\varphi^\bar{v}}(\bar{x}, \bar{b}_1, \ldots, \bar{b}_n, a_1, \ldots, a_n) \right)^2 \, d\mu_{\bar{v}}
\]

\[
= \int f_a^2 \, d\mu_{\bar{v}} + \sum_{\bar{v} \in Q_n} (-2\alpha_{\bar{v}}) \int f_a \cdot \chi_{\varphi^\bar{v}}(\bar{x}, \bar{b}_1, \ldots, \bar{b}_n, a_1, \ldots, a_n) \, d\mu_{\bar{v}} + \\
\sum_{\bar{v}, \bar{v}' \in Q_n} \alpha_{\bar{v}} \alpha_{\bar{v}'} \int \chi_{\varphi^\bar{v}'}(\bar{x}, \bar{b}_1, \ldots, \bar{b}_n, a_1, \ldots, a_n) \chi_{\varphi^{\bar{v}'}}(\bar{x}, \bar{b}_1, \ldots, \bar{b}_n, a_1, \ldots, a_n) \, d\mu_{\bar{v}}
\]

As \(f_a\) takes values in \([0, 1]\), as in (9.3) for the first integral we have

\[
\int f_a^2 \, d\mu_{\bar{v}} \approx 2\varepsilon B^\varepsilon := \sum_{i=1}^{\ell_v-2} (\alpha_{\bar{v}} \varphi^\bar{v}(\bar{x}, \bar{b}_1, \ldots, \bar{b}_n, a_1, \ldots, a_n))
\]

Using (9.3) for the second integral we have

\[
\sum_{\bar{v} \in Q_n} (-2\alpha_{\bar{v}}) \int f_a \cdot \chi_{\varphi^\bar{v}}(\bar{x}, \bar{b}_1, \ldots, \bar{b}_n, a_1, \ldots, a_n) \, d\mu_{\bar{v}} \approx \varepsilon |Q_n| C^\varepsilon_{\bar{v}}(\bar{x}, \bar{b}_1, \ldots, \bar{b}_n, a_1, \ldots, a_n)
\]

\[
\sum_{\bar{v} \in Q_n} \sum_{i=1}^{\ell_v-2} (-2\alpha_{\bar{v}} \cdot \varphi^\bar{v}(\bar{x}, \bar{b}_1, \ldots, \bar{b}_n, a_1, \ldots, a_n)) \mu_{\bar{v}} \left( F_a^{\geq \varphi^\bar{v}} \cap F_a^{< \varphi^\bar{v}} \right)
\]

And the third integral is equal to

\[
D^\varepsilon_{\bar{v}} := \sum_{\bar{v}, \bar{v}' \in Q_n} \alpha_{\bar{v}} \cdot \alpha_{\bar{v}'} \cdot \mu_{\bar{v}} \left( \left( \varphi^\bar{v} \wedge \varphi^{\bar{v}'} \right)(\bar{x}, \bar{b}_1, \ldots, \bar{b}_n, a_1, \ldots, a_n) \right)
\]

Combining, we get

\[
\left\| f_a - h_{n, \bar{a}, \bar{b}_1, \ldots, \bar{b}_n} \right\|_{L^2}^2 \approx (|Q_n|+2)\varepsilon E^\varepsilon_{\bar{a}, \bar{b}_1, \ldots, \bar{b}_n} := B^\varepsilon_{\bar{a}, \bar{b}_1, \ldots, \bar{b}_n} + C^\varepsilon_{\bar{a}, \bar{b}_1, \ldots, \bar{b}_n} + D^\varepsilon_{\bar{a}, \bar{b}_1, \ldots, \bar{b}_n}
\]

By definition of \(h_{n, \bar{a}, \bar{b}_1, \ldots, \bar{b}_n}\) and assumption on \(f\), \(\left\| f_a - h_{n, \bar{a}, \bar{b}_1, \ldots, \bar{b}_n} \right\|\) takes values in \([-c, c]\) for some \(c = c(\bar{a}) \in \mathbb{R}_{>0}\), hence

\[
\left\| f_a - h_{n, \bar{a}, \bar{b}_1, \ldots, \bar{b}_n} \right\|_{L^2} \approx \varepsilon \sqrt{|Q_n|+2}\varepsilon E^\varepsilon_{\bar{a}, \bar{b}_1, \ldots, \bar{b}_n} := B^\varepsilon_{\bar{a}, \bar{b}_1, \ldots, \bar{b}_n} + C^\varepsilon_{\bar{a}, \bar{b}_1, \ldots, \bar{b}_n} + D^\varepsilon_{\bar{a}, \bar{b}_1, \ldots, \bar{b}_n}.
\]
By Lemma 9.19 and the definition of $E^{n,\bar{a},b_1,...,b_n}_\varepsilon$, for any $r \in \mathbb{Q}$ there exist some countable partial $\mathcal{L}_\infty$-types $\Gamma_{\varepsilon,r}, \tilde{\Gamma}_{\varepsilon,r}$ over $\emptyset$ such that

$$\mathcal{M}' \models \Gamma_{\varepsilon,r}(a, \bar{a}, b_1, \ldots, b_n) \iff E^{n,\bar{a},b_1,...,b_n}_\varepsilon \leq r,$$

$$\mathcal{M}' \models \tilde{\Gamma}_{\varepsilon,r}(a) \iff \forall(b_1, \ldots, b_n), E^{n,\bar{a},b_1,...,b_n}_\varepsilon \leq r.$$

For each $\varepsilon \in \mathbb{Q}_{>0}$, pick some $\varepsilon' \in \mathbb{Q}_{>0}$ such that $c\sqrt{(|Q_n| + 2)} \cdot \varepsilon' < \varepsilon$. Then

$$\left\| f_a - h_{n,\bar{a},\bar{a},b_1,...,b_n} \right\|_{L^2} \leq r \iff \mathcal{M}' \models \Lambda^{n,\bar{a}}_{\varepsilon'}(a, \bar{a}, b_1, \ldots, b_n),$$

$$\forall(b_1, \ldots, b_n), \left\| f_a - h_{n,\bar{a},\bar{a},b_1,...,b_n} \right\|_{L^2} \leq r \iff \mathcal{M}' \models \Lambda^{n,\bar{a}}_{\varepsilon'}(a),$$

Note that the definitions of $\Lambda^{n,\bar{a}}_{\varepsilon'}, \Lambda^{n,\bar{a}}_{\varepsilon'}$ do not depend on $\mathcal{P}, f, \mathcal{M}' \propto \mathcal{M}_{\mathcal{P},f}$. The argument for “$\geq$” is analogous. \qed

Lemma 9.22. Given an arbitrary countable linear order $I$ and $\varepsilon \in \mathbb{Q}_{>0}$, there exists a countable partial $\mathcal{L}_\infty$-type $\pi_{I,\varepsilon}((z_i)_{i \in I})$ such that the following holds.

For any $\mathcal{P}, f, \mathcal{M}' \propto \mathcal{M}_{\mathcal{P},f}$ and sequence $(a_i)_{i \in I}$ in $V_{k+1}$,

$$\mathcal{M}' \models \pi_{I,\varepsilon}((a_i)_{i \in I}) \iff \left\| f_{a_i} - \mathbb{E} \left( f_{a_i} \mid B_{i,k-1} \cup \{f_{a_j} : j \in I, j < i\} \right) \right\|_{L^2} \geq \varepsilon \text{ for every } i \in I.$$

Proof. Fix $\mathcal{P}, f, \mathcal{M}' \propto \mathcal{M}_{\mathcal{P},f}, (a_i)_{i \in I}$ and $i \in I$.

By Lemma 8.13 (which can be applied here in view of Remarks 8.10 and 8.16), we have

$$\mathbb{E} \left( f_{a_i} \mid B_{i,k-1} \cup \{f_{a_j} : j < i\} \right) = \mathbb{E} \left( f_{a_i} \mid B_{f,(a_j;j \leq i)} \cup \{F_{a_j}^{<q} : j < i, q \in \mathbb{Q}\} \right).$$

Approximating by a simple function, for any $\delta \in \mathbb{R}_{>0}$ there exist some $n \in \mathbb{N}, \bar{a} = (\alpha_0 \in Q_n, \bar{b} : \bar{v} \in Q_n)$, some tuples $\bar{b}_i \in V^{k+1}$ and some $i_1 < \ldots < i_n < i$ in $I$ so that the $\sigma \left( B_{f,(a_j;j \leq i)} \cup \{F_{a_j}^{<q} : j < i, q \in \mathbb{Q}\} \right)$-simple function $h_{n,\bar{a},(a_{i_1},\ldots,a_{i_n}),\bar{b}_1,\ldots,\bar{b}_n}$ (all of them are of these form, see Definition 9.20 and Remark 9.18(2)) satisfies

$$\left\| \mathbb{E} \left( f_{a_i} \mid B_{f,(a_j;j \leq i)} \cup \{F_{a_j}^{<q} : j < i, q \in \mathbb{Q}\} \right) - h_{n,\bar{a},(a_{i_1},\ldots,a_{i_n}),\bar{b}_1,\ldots,\bar{b}_n} \right\|_{L^2} \leq \delta.$$
Hence, for a fixed $i \in I$,
\[
\left\| f_{a_i} - \mathbb{E}\left( f_{a_i} \mid \mathcal{B}_{1:k-1} \cup \{f_{a_j} : j < i\} \right) \right\|_{L^2} \geq \varepsilon \iff \\
\bigwedge_{n \in \mathbb{N}} \bigwedge_{i_1 < \ldots < i_n < i \in I} \bigwedge_{\tilde{a} = (a_0 \in \mathbb{Q}^{[0,1]}, \bar{b})} \forall (\bar{b}_1, \ldots, \bar{b}_n) \left\| f_{a_i} - h_n,\tilde{a},(a_1,\ldots,a_n),\bar{b}_1,\ldots,\bar{b}_n \right\|_{L^2} \geq \varepsilon.
\]

By Lemma 9.21 we thus have
\[
\left\| f_{a_i} - \mathbb{E}\left( f_{a_i} \mid \mathcal{B}_{1:k-1} \cup \{f_{a_j} : j < i\} \right) \right\|_{L^2} \geq \varepsilon \text{ for all } i \in I \iff \\
\mathcal{M}' = \pi_{I,\varepsilon}((a_i)_{i \in I}) := \\
\bigwedge_{i \in I} \bigwedge_{n \in \mathbb{N}} \bigwedge_{i_1 < \ldots < i_n < i \in I} \bigwedge_{\tilde{a} = (a_0 \in \mathbb{Q}^{[0,1]}, \bar{b})} \Lambda_n^{\tilde{a}}(a_i, a_1, \ldots, a_{i_n}).
\]

Note that the definition of the partial type $\pi_{I,\varepsilon}$ does not depend on $\mathcal{Q}$, $\mathcal{F}$, $\mathcal{M}' \propto \mathcal{M}_{\mathcal{Q},f}, (a_i)_{i \in I}$, since neither did $\Lambda_n^{\tilde{a}} \geq \varepsilon$.

\[\Box\]

**Remark 9.23.** It is easy to see from the definition that $(a_i)_{i \in I} \models \pi_{I,\varepsilon} \iff (a_i)_{i \in I'} \models \pi_{I',\varepsilon}$ for every finite $I' \subseteq I$.

The following is a version of de Finetti’s theorem suitable for our context (in particular we observe that $\mathcal{L}_\infty$-indiscernibility implies exchangeability in the probabilistic sense).

**Proposition 9.24.** Assume that $\mathcal{M}' \propto \mathcal{M}_{\mathcal{Q},f}$, $\mathcal{M}'$ is an $\aleph_1$-saturated $\mathcal{L}_\infty$-structure, $I = \mathbb{Z}$ and $(a_i : i \in I)$ in $V_{k+1}$ is an $\mathcal{L}_\infty$-indiscernible sequence in the sense of $\mathcal{M}'$. Let $\mathcal{B} := \sigma\left( \{f_{a_j} : j < 0\} \cup \mathcal{B}_{1:k-1} \right)$. Then:

1. $\mathcal{B}_{1:k-1} \subseteq \mathcal{B} \subseteq \mathcal{B}_{1:k}$;
2. for all $i \in \mathbb{N}$ we have
\[
\mathbb{E}\left( f_{a_i} \mid \mathcal{B}_{1:k-1} \cup \{f_{a_j} : j < i\} \right) = \mathbb{E}\left( f_{a_i} \mid \mathcal{B} \cup \{f_{a_j} : j < i\} \right) = \mathbb{E}\left( f_{a_i} \mid \mathcal{B} \right).
\]

**Proof.** It is obvious that (1) holds for $\mathcal{B}$. In (2), it is enough to show the equality of the first and the last expressions. As in the proof of Lemma 9.22 by Lemma 8.13 and Remark 8.10, we have
\[
\mathbb{E}\left( f_{a_i} \mid \mathcal{B}_{1:k-1} \cup \{f_{a_j} : j < i\} \right) = \mathbb{E}\left( \chi_{E_{a_i}} \mid \mathcal{B}_{1:k-1} \cup \{F_{a_j}^{<q} : j < i, q \in \mathbb{Q}\} \right).
\]

Fix $i \geq 0$ arbitrary. Let $\varepsilon \in \mathbb{Q}_{\geq 0}$ be arbitrary, and assume that
\[
\left\| f_{a_i} - \mathbb{E}\left( f_{a_i} \mid \mathcal{B}_{1:k-1} \cup \{F_{a_j}^{<q} : j < i, q \in \mathbb{Q}\} \right) \right\|_{L^2} \leq \varepsilon.
\]

By definition of $\mathbb{E}$, for any $\delta \in \mathbb{R}_{>0}$ there exist some $n \in \mathbb{N}$, $\alpha,s,t,u \in \mathbb{Q}$, tuples $\bar{b}_j \in V^{1:k+1}$ and $i_1 < \ldots < i_{n} < i$ in $I$, such that taking $\tilde{a} = (a_1, \ldots, a_{i_n})$, the $\sigma\left( \mathcal{B}_{1:k-1} \cup \{F_{a_j}^{<q} : j < i, q \in \mathbb{Q}^{[0,1]}\} \right)$-simple function $h_{n,\tilde{a},\bar{b}_1,\ldots,\bar{b}_n}$ satisfies
\[
\left\| \mathbb{E}\left( f_{a_i} \mid \mathcal{B}_{1:k-1} \cup \{F_{a_j}^{<q} : j < i, q \in \mathbb{Q}\} \right) - h_{n,\tilde{a},\bar{b}_1,\ldots,\bar{b}_n} \right\|_{L^2} \leq \delta,
\]
Assume that \( B \) is \( \aleph_1 \)-saturated \( \mathcal{L}_\infty \)-structure, \( I = \mathbb{Z} \) and \( (a_i : i \in I) \) in \( V_{k+1} \) is an \( \mathcal{L} \)-indiscernible sequence in the sense of \( \mathcal{M} \).

Let \( \mathcal{B} = \sigma \left( \left\{ f_{a_j} : j < 0 \right\} \cup \mathcal{B}_{1k, k-1} \right) \), \( \delta \in \mathbb{R}_{>0} \) and \( r < s \in \mathbb{Q}^{[0,1]} \). Let

\[
G^{r,s}_{\delta}(a_i) := \left\{ a \in V_{1k} \mid E\left( \chi_{f_{a_i}} \mid \mathcal{B} \right)(x) \geq \delta \wedge E\left( \chi_{f_{a_i}}^{r,s} \mid \mathcal{B} \right)(x) \geq \delta \right\} \in \mathcal{B}.
\]

Assume that \( \mu_{1k}(G^{r,s}_{\delta}(a_0)) > 0 \). Then \( \mu_{1k} \left( \cap_{i \in [l]} G^{r',s'}_{\delta}(a_i) \right) > 0 \) for any \( l \in \mathbb{N} \) and \( r < r' < s' < s \).

**Proof.** Fix some \( l \in \mathbb{N} \) and \( r < r' < s' < s \). Let

\[
F^{r,s}_{\delta}(a_i) := \left\{ a \in V_{1k} \mid E\left( \chi_{f_{a_i}^{r,s}} \mid \mathcal{B} \right)(x) \geq \delta \wedge E\left( \chi_{F_{a_i}^{r,s}} \mid \mathcal{B} \right)(x) \geq \delta \right\} \in \mathcal{B}.
\]
As $\mathcal{M}' \cong \mathcal{M}_{\mathfrak{B}f}$, by monotonicity of conditional expectation we have $q := 
abla_{\mu_1} (\mathbb{E}^{\mathbb{E}|\mathcal{B}}_\delta (a_0)) \geq \nabla_{\mu_1} (\mathbb{E}^{\mathbb{E}|\mathcal{B}}_\delta (a_0)) > 0$. Let $\xi = \xi (q, l) > 0$ be as given by Fact 4.5. Fix some $0 < \varepsilon < \min \left\{ \frac{q}{2}, \frac{l}{l} \right\}$.

Fix $i \in \mathbb{N}$. Note that $F^{r,s}_{\delta} (a_i) = \bigcup_{n \in \mathbb{N}_0} F^{r,s}_{\delta} (a_i)$, and by countable additivity

\begin{equation}
(9.5) \quad \mu_1 \left( F^{r,s}_{\delta} (a_i) \setminus \mathbb{E}^{\mathbb{E}|\mathcal{B}}_\delta (a_i) \right) \rightarrow 0 \text{ as } \delta \rightarrow 0.
\end{equation}

For arbitrary $\mathcal{B}$, measurable functions $h_1, h_2$ and $\gamma \in \mathbb{R}_0$, we define the set

\[ F_{h_1,h_2,\gamma} := \{ \bar{x} \in V^k \mid h_1 (\bar{x}) \geq \delta + \gamma \wedge h_2 (\bar{x}) \geq \delta + \gamma \} . \]

**Claim 9.26.** There exists some $\gamma > 0$ such that for arbitrary $h_1, h_2$ we have:

\[ \| \mathbb{E} (\chi_{F^{<r}_a} | \mathcal{B}) - h_1 \|_{L_2} < \gamma^2 \wedge \| \mathbb{E} (\chi_{F^{>r}_a} | \mathcal{B}) - h_2 \|_{L_2} < \gamma^2 \implies \mu_{\mathfrak{B}_1} (F^{r,s}_{\delta} (a_i) \setminus F_{h_1,h_2,\gamma}) < \varepsilon. \]

**Proof.** Let

\[ D_1 := \{ \bar{x} \in V^k : \| \mathbb{E} (\chi_{F^{<r}_a} | \mathcal{B}) (\bar{x}) - h_1 (\bar{x}) \|_{L_2} < \gamma \} \in \mathcal{B}_{\mathfrak{B}_1} , \]

\[ D_2 := \{ \bar{x} \in V^k : \| \mathbb{E} (\chi_{F^{>r}_a} | \mathcal{B}) (\bar{x}) - h_2 (\bar{x}) \|_{L_2} < \gamma \} \in \mathcal{B}_{\mathfrak{B}_1} . \]

Then $\mu_{\mathfrak{B}_1} (D_1) < \gamma$ by assumption on $h_1$ for $t \in \{ 1, 2 \}$, and $F_{h_1,h_2,\gamma} \setminus (D_1 \cup D_2) \subseteq F^{r,s}_{\delta} (a_i)$. And similarly $F^{r,s}_{\delta} (a_i) \setminus (D_1 \cup D_2) \subseteq F_{h_1,h_2,\gamma}$. Taking $0 < \gamma < \frac{\xi}{2}$ small enough, by (9.5) we have $\mu_{\mathfrak{B}_1} (F^{r,s}_{\delta} (a_i) \setminus F^{r,s}_{\delta+\gamma} (a_i)) < \frac{\xi}{2}$. But

\[ F^{r,s}_{\delta+\gamma} (a_i) \setminus F_{h_1,h_2,\gamma} \subseteq D_1 \cup D_2 \cup (F^{r,s}_{\delta} (a_i) \setminus F^{r,s}_{\delta+\gamma} (a_i)) , \]

hence $\mu_{\mathfrak{B}_1} (F^{r,s}_{\delta} (a_i) \setminus F_{h_1,h_2,\gamma}) < 2\gamma + \frac{\xi}{2} < \varepsilon$.

From now on, fix some $\gamma > 0$ satisfying the conclusion of Claim 1. By definition of $\mathcal{B}$, for every $i \in \mathbb{N}$, the function $\mathbb{E} (\chi_{F^{<r}_a} | \mathcal{B})$ can be approximated arbitrarily well in $L^2$-norm by functions of the form

\[ h_{n, \bar{a}, (a_1, \ldots, a_j), (\bar{b}_1, \ldots, \bar{b}_n)} \]

with $n \in \mathbb{N}$, $\bar{a} = (a_0 \in \mathbb{Q} : \bar{v} \in Q_n)$, $j_1, \ldots, j_n < 0$, $\bar{b}_1, \ldots, \bar{b}_n \in V^{k+1}$ (and all such functions are $\mathcal{B}$-simple). As in the proof of Proposition 9.24 using $\mathcal{B}^{-}\text{indiscernibility}$ of the sequence $(a_i)_{i \in \mathbb{Z}}$ and type-definability of the corresponding condition, for every such function, $\beta \in \mathbb{R}_0$ and $i, i' \in \mathbb{N}$, we
have
\[
\forall \bar{b}_1 \ldots \forall \bar{b}_n \in V^{1k+1} \left\| \chi_{F_{a_i}^{<r}} - h_{n,\tilde{\alpha},(a_{j_1},\ldots,a_{j_n}),\tilde{b}_1,\ldots,\tilde{b}_n} \right\|_{L^2} \leq \beta \iff \\
\forall \bar{b}_1 \ldots \forall \bar{b}_n \in V^{1k+1} \left\| \chi_{F_{a_i}^{<r}} - h_{n,\tilde{\alpha},(a_{j_1},\ldots,a_{j_n}),\tilde{b}_1,\ldots,\tilde{b}_n} \right\|_{L^2} \leq \beta.
\]

It follows that \( \left\| \chi_{F_{a_i}^{<r}} - \mathbb{E} \left( \chi_{F_{a_i}^{<r}} \mid \mathcal{B} \right) \right\|_{L^2} \) does not depend on \( i \in \mathbb{N} \), and we denote its value by \( \beta_1 \in \mathbb{R}_{\geq 0} \). Similarly \( \beta_2 := \left\| \chi_{F_{a_i}^{>r}} - \mathbb{E} \left( \chi_{F_{a_i}^{>r}} \mid \mathcal{B} \right) \right\|_{L^2} \) does not depend on \( i \in \mathbb{N} \).

**Claim 9.27.** For every \( \gamma > 0 \) there exists \( \gamma' = \gamma'(\gamma, \beta_1, \beta_2) > 0 \) such that: for every \( i \in \mathbb{N} \) and a \( \mathcal{B} \)-measurable function \( h \),
\[
\left\| \chi_{F_{a_i}^{<r}} - h \right\|_{L^2} \leq \beta_1 + \gamma' \implies \left\| \mathbb{E} \left( \chi_{F_{a_i}^{<r}} \mid \mathcal{B} \right) - h \right\|_{L^2} \leq \gamma, \quad \text{and} \\
\left\| \chi_{F_{a_i}^{>r}} - h \right\|_{L^2} \leq \beta_2 + \gamma' \implies \left\| \mathbb{E} \left( \chi_{F_{a_i}^{>r}} \mid \mathcal{B} \right) - h \right\|_{L^2} \leq \gamma.
\]

**Proof.** Assume that \( \left\| \chi_{F_{a_i}^{<r}} - h \right\|_{L^2} \leq \beta_1 + \gamma' \). By the parallelogram rule for the \( L^2 \)-norm, as the function \( \frac{1}{2} \left( \mathbb{E} \left( \chi_{F_{a_i}^{<r}} \mid \mathcal{B} \right) + h \right) \) is \( \mathcal{B} \)-measurable, we have
\[
\left\| \mathbb{E} \left( \chi_{F_{a_i}^{<r}} \mid \mathcal{B} \right) - h \right\|_{L^2}^2 = 2 \left\| \chi_{F_{a_i}^{<r}} - \mathbb{E} \left( \chi_{F_{a_i}^{<r}} \mid \mathcal{B} \right) \right\|_{L^2}^2 + 2 \left\| \chi_{F_{a_i}^{<r}} - h \right\|_{L^2}^2 \]
\[
- 4 \left\| \chi_{F_{a_i}^{<r}} - \frac{1}{2} \left( \mathbb{E} \left( \chi_{F_{a_i}^{<r}} \mid \mathcal{B} \right) + h \right) \right\|_{L^2}^2 \\
\leq 2\beta_1^2 + 2(\beta_1 + \gamma')^2 - 4\beta_1^2 = 2\beta_1 \gamma' + (\gamma')^2 \leq \gamma
\]
assuming that \( \gamma' \) is sufficiently small with respect to \( \beta_1 \) and \( \gamma \). The argument for \( \chi_{F_{a_i}^{>r}} \) is similar.

From now on, fix some \( \gamma' > 0 \) satisfying the conclusion of Claim 2 with respect to \( \gamma' \) instead of \( \gamma \). By the choice of \( \beta_1, \beta_2 \), we can choose \( n, \tilde{\alpha}^1, \tilde{\alpha}^2, b_1, \ldots, b_n \) and \( i_1, \ldots, i_n < 0 \) in \( \mathbb{Z} \) so that, writing \( \tilde{a} := (a_{i_1}, \ldots, a_{i_n}) \),
(9.6) \[
\left\| \chi_{F_{a_0}^{<r}} - h_{n,\tilde{\alpha}^1,\tilde{a},\tilde{b}_1,\ldots,\tilde{b}_n} \right\|_{L^2} \leq \beta_1 + \gamma',
\]
\[
\left\| \chi_{F_{a_0}^{>r}} - h_{n,\tilde{\alpha}^2,\tilde{a},\tilde{b}_1,\ldots,\tilde{b}_n} \right\|_{L^2} \leq \beta_2 + \gamma'.
\]

The set \( F_{h,\tilde{\alpha}^1,\tilde{a},\tilde{b}_1,\ldots,\tilde{b}_n} \) is definable in \( \mathcal{M}' \) by a quantifier-free \( L_{\infty} \)-formula by Lemma 9.21(1). Hence the condition
\[
\mu_{\tilde{\xi}} \left( F_{h,\tilde{\alpha}^1,\tilde{a},\tilde{b}_1,\ldots,\tilde{b}_n} \right) \geq r - \varepsilon
\]
on the tuple \((\tilde{b}_1, \ldots, \tilde{b}_n, a_{i_1}, \ldots, a_{i_m})\) is \(L_\infty\)-type-definable in \(\mathcal{M}'\) by Lemma 9.19. Then the following condition on the tuple \((a_{i_1}, \ldots, a_{i_m}, a_i)\) is also \(L_\infty\)-type-definable in \(\mathcal{M}'\):

\[
\exists \tilde{b}_1 \ldots \exists \tilde{b}_n \left( \left\| X_{F_{a_{i}}^r} - h_{n,\bar{a}^1,\bar{a},\tilde{b}_1,\ldots,\tilde{b}_n} \right\|_{L^2} \leq \beta_1 + \gamma' \land \right.
\]

\[
\left. \left\| X_{F_{a_{i}}^{\geq s'}} - h_{n,\bar{a}^2,\bar{a},\tilde{b}_1,\ldots,\tilde{b}_n} \right\|_{L^2} \leq \beta_2 + \gamma' \land \right.
\]

\[
\mu_{\bar{a}_j} \left( F_{h_{n,\bar{a}^1,\bar{a},\tilde{b}_1,\ldots,\tilde{b}_n}^{\geq s'}, h_{n,\bar{a}^2,\bar{a},\tilde{b}_1,\ldots,\tilde{b}_n}} \right) \geq q - \varepsilon \right). \]

It holds for the tuple \((a_{i_1}, \ldots, a_{i_m}, a_0)\) by (9.16), the choice of \(\gamma'\) and Claims 1 and 2. Hence, by \(L_\infty\)-indiscernibility of the sequence \((a_i)_{i \in \mathbb{Z}}\), it holds for any tuple \((a_{i_1}, \ldots, a_{i_m}, a_i)\) with \(i \in \mathbb{N}\); let \(\tilde{b}_i = (\tilde{b}_{i_1}, \ldots, \tilde{b}_{i_m})\) be some tuple witnessing that.

In particular, for every \(i \in \mathbb{N}\), we have \(\mu_{\bar{a}_j} \left( F_{h_{n,\bar{a}^1,\bar{a},\tilde{b}_1,\ldots,\tilde{b}_n}} \right) \geq q - \varepsilon \geq \frac{q}{2}\) by the choice of \(\varepsilon\). Then, by the choice of \(\xi\) and Fact 9.5 there exist some \(\tilde{j}_1 < \ldots < \tilde{j}_l \in \mathbb{N}\) such that

\[
\mu_{\bar{a}_j} \left( \bigcap_{p \in [l]} F_{h_{n,\bar{a}^1,\bar{a},\tilde{j}_p, h_{n,\bar{a}^2,\bar{a},\tilde{j}_p}}} \right) \geq \xi.
\]

Then, by Lemma 9.19 and \(L_\infty\)-indiscernibility of \((a_i)_{i \in \mathbb{Z}}\) again, there exist some \(\tilde{b}_{1}, \ldots, \tilde{b}_l\) so that:

(a) \(\left\| X_{F_{a_{i}}^r} - h_{n,\bar{a}^1,\bar{a},\tilde{b}_i} \right\|_{L^2} \leq \beta_1 + \gamma' \) for every \(i \in [l]\);

(b) \(\left\| X_{F_{a_{i}}^{\geq s'}} - h_{n,\bar{a}^2,\bar{a},\tilde{b}_i} \right\|_{L^2} \leq \beta_2 + \gamma' \) for every \(i \in [l]\);

(c) \(\mu_{\bar{a}_j} \left( \bigcap_{i \in [l]} F_{h_{n,\bar{a}^1,\bar{a},\tilde{j}_p, h_{n,\bar{a}^2,\bar{a},\tilde{j}_p}}} \right) \geq \xi.\)

By (a), (b), Claim 2 and the choice of \(\gamma'\), for every \(i \in [l]\) we have

\[
\left\| \mathbb{E} \left( X_{F_{a_{i}}^r} \mid B \right) - h_{n,\bar{a}^1,\bar{a},\tilde{b}_i} \right\|_{L^2} \leq \gamma^2 + \left\| \mathbb{E} \left( X_{F_{a_{i}}^{\geq s'}} \mid B \right) - h_{n,\bar{a}^2,\bar{a},\tilde{b}_i} \right\|_{L^2} \leq \gamma^2.
\]

By Claim 1 this implies that \(\mu_{\bar{a}_j} \left( F_{h_{n,\bar{a}^1,\bar{a},\tilde{j}_p}}^{r,s'}(a_i) \triangle F_{h_{n,\bar{a}^2,\bar{a},\tilde{j}_p}}(a_i) \right) < \varepsilon\) for every \(i \in [l]\). But then from (c), \(\mathcal{M}' \preccurlyeq \mathcal{M}_{\mathcal{P},f}\) and monotonicity of conditional expectation, we have

\[
\mu_{\bar{a}_j} \left( \bigcap_{i \in [l]} G_{\delta}^{r,s'}(a_i) \right) \geq \mu_{\bar{a}_j} \left( \bigcap_{i \in [l]} F_{\delta}^{r,s'}(a_i) \right) \geq
\]

\[
\mu_{\bar{a}_j} \left( \bigcap_{i \in [l]} F_{h_{n,\bar{a}^1,\bar{a},\tilde{j}_p, h_{n,\bar{a}^2,\bar{a},\tilde{j}_p}}} - l\varepsilon \geq \xi - l\varepsilon > 0
\]
by the choice of $\varepsilon$. \hfill \Box

9.5. **Passing to an indiscernible counterexample.** Finally, we use the results developed in this section to show how to achieve the Assumption 5.6 in the proof of Proposition 5.1.

**Theorem 9.28.** Let $\tilde{d}$ be fixed and suppose that for each $j$ there is a $(k+1)$-partite graded probability space $\mathfrak{Q}_j = (V_j^{[k+1]}, \mathcal{B}_j^{[k+1]}, \mu_j^{[k+1]})_{i_i \in \mathbb{N}^{k+1}}$, a $\mathcal{B}_j^{[k+1]}$-measurable function $f^j : \prod_{i \in [k+1]} V_j^i \to [0, 1]$ with $\text{VC}_k(f^j) \leq \tilde{d}$ and some $x_1^j, \ldots, x_n^j \in V_j^{[k+1]}$ such that for every $t \leq j$ we have: for any sets $D_1, \ldots, D_j \in \mathcal{F}^{f^j, \tilde{d}, (x_1^j, \ldots, x_n^j)}$ and any \((\{D_i\}_{i \in [j]} \cup \{f_{x_t}^{t+1}\}_{i \in [t-1], q \in \mathbb{Q}_j^{[0,1]}})\)-simple function $g$ with coefficients in $\mathbb{Q}_j^{[0,1]}$, \(\inf_{L^2} g \geq \varepsilon\).

Then there exists a $(k+1)$-partite graded probability space

$$\mathfrak{Q} = (V^{[k+1]}, \mathcal{B}_n, \mu_n)_{n \in \mathbb{N}^{k+1}};$$

$\varepsilon \in \mathbb{R}_{>0}$, a $\mathcal{B}_j^{[k]}$-measurable function $f : V^{[k+1]} \to [0, 1]$ and a sequence $(x_l)_{l \in \mathbb{Z}}$ in $V^{[k+1]}$ satisfying the following:

1. $\text{VC}_k(f) \leq \tilde{d}$;
2. whenever $0 \leq r < r' < s < s \leq s$ are in $\mathbb{Q}$, $\delta \in \mathbb{R}_{>0}$, and

$$\mu_{l_k}(\{\tilde{x} \in V^{[k]} : \mathbb{E}(\chi_{f_{x_t}^r} | \mathcal{B})(\tilde{x}) \geq \delta \wedge \mathbb{E}(\chi_{f_{x_t}^r} | \mathcal{B})(\tilde{x}) \geq \delta\}) > 0,$$

then for any $l \in \mathbb{N}$,

$$\mu_{l_k}(\bigcap_{i \in [l]} \{\tilde{x} \in V^{[k]} : \mathbb{E}(\chi_{f_{x_t}^r} | \mathcal{B})(\tilde{x}) \geq \delta \wedge \mathbb{E}(\chi_{f_{x_t}^r} | \mathcal{B})(\tilde{x}) \geq \delta\}) > 0;$$

3. $\inf_{L^2} f_{x_l} - \mathbb{E}(f_{x_l} | \mathcal{B}_{l_k, l-1} \cup \{f_{x_i} : i < l\}) \geq \varepsilon$ for all $l \in \mathbb{Z}$;
4. $\mathcal{B}_{l_k, l-1} \subseteq \mathcal{B} \subseteq \mathcal{B}_{l_k}$;
5. for all $l \in \mathbb{N}$ we have

$$\mathbb{E}(f_{x_l} | \mathcal{B}_{l_k, l-1} \cup \{f_{x_i} : i < l\}) = \mathbb{E}(f_{x_l} | \mathcal{B} \cup \{f_{x_i} : i < l\}) = \mathbb{E}(f_{x_l} | \mathcal{B}),$$

where $\mathcal{B} := \sigma(\{f_{x_i} : i < 0\} \cup \mathcal{B}_{l_k, l-1})$.

**Proof.** Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Let $\mathfrak{Q} := (\mathcal{V}_k, \mathcal{B}_n, \mu_n)_{n \in \mathbb{N}^{k}}$ be the $(k+1)$-partite graded probability space, the $\mathcal{B}_j^{[k+1]}$-measurable function $\tilde{f} : V^{[k+1]} \to [0, 1]$ and $\mathcal{M}$ the $L^\infty$-structure defined by the corresponding ultraproduct in Section 9.3 (Fact 9.13).

**Claim 9.29.** (1) $\text{VC}_k(\tilde{f}) \leq \tilde{d}'$ for some $\tilde{d}' = \tilde{d}'(\tilde{d})$. 


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(2) There exists an infinite sequence \((x_i : i \in \mathbb{Z})\) in \(\tilde{V}_{k+1}\) such that 
\(\tilde{M} \models \pi_{\mathbb{Z},e}((x_i)_{i \in \mathbb{Z}})\) and \((x_i : i \in \mathbb{Z})\) is \(L_\infty\)-indiscernible in \(\tilde{M}\).

**Proof.** (1) By Lemma 10.1

(2) For \(i \in \mathbb{N}\), let \(\tilde{x}_i := (x^j_i : j \in \mathbb{N}) / U \in \tilde{V}_{k+1}\).

Let \(\pi_0\) be an arbitrary finite set of formulas from \(\pi_{\mathbb{N},e}\), all formulas in \(\pi_0\)
only involve the variables \(z_{i_1}, \ldots, z_{i_n}\) for some \(n \in \mathbb{N}\) and \(i_1 \leq \cdots \leq i_n \in \mathbb{N}\).
From the definition of \(\pi_{\mathbb{N},e}\) and Lemma 9.22 (as obviously \(\mathcal{M}_{\mathbb{Q},f^j} \supseteq \mathcal{M}_{\mathbb{Q},f^j}\)
for every \(j \in \mathbb{N}\), it is not hard to see that for all sufficiently large \(j \in \mathbb{N}\) (so that \(j > n\) and all the rational coefficients appearing among the formulas in
\(\pi_0\) are in \(\mathbb{Q}^{[0,1]}\)) we have

\[\mathcal{M}_{\mathbb{Q},f^j} \models \pi_0 \left( x^j_{i_1}, \ldots, x^j_{i_n} \right),\]
hence by Łos’ theorem

\[\tilde{M} \models \pi_{\mathbb{N},e} \left( (\tilde{x}_i)_{i \in \mathbb{N}} \right),\]
and so

\[\tilde{M} \models \pi_{\mathbb{Z},e} \left( (\tilde{x}_i)_{i \in \mathbb{N}} \right) .\]

As \(\tilde{M}\) is an \(\mathbb{N}_1\)-saturated \(L_\infty\)-structure, by Fact 9.10(1) we can find an
infinite \(L_\infty\)-indiscernible sequence \((x_i : i \in \mathbb{Z})\) in \(\tilde{V}_k\) based on \((\tilde{x}_i)_{i \in \mathbb{N}}\). In
particular, using Remark 9.23 \(\tilde{M} \models \pi_{\mathbb{Z},e} \left( (x_i)_{i \in \mathbb{Z}} \right) .\)

As \(\tilde{M} \propto \mathcal{M}_{\mathbb{Q},f^j}\) by Remark 9.13 by Lemma 9.22 for every \(i \in \mathbb{Z}\) we have

\[\left\| \tilde{f}_{a_i} - \mathbb{E} \left( \tilde{f}_{a_i} \mid B_{1_{i,k-1}} \cup \{ \tilde{f}_{a_j} : j \in I \land j < i \} \right) \right\|_{L^2} \geq \varepsilon .\]

Taking \(\mathfrak{P} := \mathfrak{Q}, \mathcal{M}' := \tilde{M}, f := \tilde{f}\), replacing \(\bar{d}\) by \(d'\) and applying Proposition 9.24 we have thus arrived at the desired situation. \(\square\)

10. OPERATIONS ON FUNCTIONS PRESERVING FINITE VC\(_k\)-DIMENSION

10.1. **Basic operations.** In this section we demonstrate that finiteness of
the VC\(_k\)-dimension is preserved under various natural operations on real-valued functions, obtaining a generalization of Fact 3.3 These results are used in the proof of the main Theorem 6.6 in particular.

**Lemma 10.1.** Assume that, in the notation of Section 9.3, for some \(\bar{d}\) we have
\(\text{VC}_k(f^j) \leq \bar{d}\) for all \(j \in \mathbb{N}\). Then \(\text{VC}_k\left( \tilde{f} \right) \leq \bar{d}'\), where we can take
\(d'_{r,s} := d_{r',s'}\) for any \(r < r' < s' < s\) in \(\mathbb{Q}^{[0,1]}\).

**Proof.** Fix arbitrary \(r < r' < s' < s \in \mathbb{Q} \cap [0,1]\). By assumption, for any
\(j \in \mathbb{N}\), no \(d_{r',s'}\)-box can be \((r',s')\)-shattered by \(f^j\), hence

\[\mathcal{M}_{f^j} \models \neg \exists (x^s_t : s \in [k], t \in [d_{r',s'}]) \left( y_u : u \subseteq [d_{r',s'}]^k \right) \wedge \bigwedge_{u \subseteq [d_{r',s'}]^k} F^{<r'}(x^1_{t_1}, \ldots, x^k_{t_k}) \wedge \bigwedge_{(t_1, \ldots, t_k) \in u} F^{s'}(x^1_{t_1}, \ldots, x^k_{t_k}) ,\]
By Łos’ theorem, the same $L_0$-sentence holds in the ultraproduct $\hat{M}$ as well. As $\hat{M} \preceq M_f$, this implies that no $d_{r',s'}$-box is $(r, s)$-shattered by $\hat{f}$. \hfill \square

The following characterization of finiteness of $VC_k$-dimension in terms of generalized indiscernibles was observed in [CPT19] Lemma 6.2 for relations, and we generalize it to real-valued functions.

**Lemma 10.2.** Let $\mathcal{M}$ be an $\aleph_1$-saturated $\mathcal{L}$-structure in a language $\mathcal{L} \supseteq L_0$, $k < r \in \mathbb{N}$, $\mathcal{M} \models L_0 \times \mathcal{M}_f = (V_1, \ldots, V_r, \ldots)$, $f : \prod_{i \in [r]} V_i \to [0, 1]$ and $\bar{b} \in \prod_{i \in [r]\setminus[k+1]} V_i$ (could be an empty tuple when $r = k + 1$). Then the following are equivalent for $f_{\bar{b}} : \prod_{i \in [k+1]} V_i \to [0, 1], (a_1, \ldots, a_{k+1}) \mapsto f(a_1, \ldots, a_{k+1}, \bar{b})$.

1. $VC_k(f_{\bar{b}}) = \infty$.
2. There exist some $r < s \in \mathbb{Q} \cap [0, 1]$ and elements $(a_g)_{g \in G_{k+1,p}}$ in $\mathcal{M}$ such that:
   (a) $g \in P_i \implies a_g \in V_i$;
   (b) $(a_g)_{g \in G_{k+1,p}}$ is $G_{k+1,p}$-indiscernible over $\emptyset$ (in $\mathcal{M}$);
   (c) For all $(g_1, \ldots, g_{k+1}) \in \prod_{i \in [k+1]} P_i$ we have:
      - $G_{k+1,p} \models R_{k+1}(g_1, \ldots, g_{k+1}) \implies f(a_{g_1}, \ldots, a_{g_{k+1}}, \bar{b}) \leq r$;
      - $G_{k+1,p} \models \neg R_{k+1}(g_1, \ldots, g_{k+1}) \implies f(a_{g_1}, \ldots, a_{g_{k+1}}, \bar{b}) \geq s$.

*Proof.* $(2) \Rightarrow (1)$. Assume that (2) holds, and let $Q_i \subseteq P_i, i \in [k]$ be arbitrary finite sets and $Q := \prod_{i \in [k]} Q_i$. By the definition of $G_{k+1,p}$ (Definition 9.1), for every subset $S \subseteq Q$ there exists some $g_S \in P_{i+1}$ so that for every $(g_1, \ldots, g_k) \in Q$ we have $G_{k+1,p} \models R_{k+1}(g_1, \ldots, g_k, g_S) \iff (g_1, \ldots, g_k) \in S$. By (c) this implies that, taking $A_i := \{a_g : g \in Q_i\} \subseteq V_i$, the box $A := \prod_{i \in [k]} A_i$ is $(r, s)$-shattered by $f_{\bar{b}}$.

$(1) \Rightarrow (2)$. Assume that $r < s \in \mathbb{Q} \cap [0, 1]$ are such that for every $d \in \mathbb{N}$ there exists a finite box $A = \prod_{i \in [k]} A_i \subseteq \prod_{i \in [k]} V_i$ with $|A_i| \geq d$ for each $i \in [k]$ which is $(r, s)$-shattered by $f_{\bar{b}}$. In particular, for any finite $(k+1)$-partite hypergraph $(R; D_1, \ldots, D_k)$ with $R \subseteq \prod_{i \in [k+1]} D_i$ we can choose some sets $A_i \subseteq V_i$ and bijections $\alpha_i : D_i \to A_i$ so that for every $(b_1, \ldots, b_{k+1}) \in \prod_{i \in [k+1]} D_i$, (10.1)

\[
(b_1, \ldots, b_{k+1}) \in R \implies f\left(\alpha_1(b_1), \ldots, \alpha_{k+1}(b_{k+1}), \bar{b}\right) \leq r;
\]

\[
(b_1, \ldots, b_{k+1}) \notin R \implies f\left(\alpha_1(b_1), \ldots, \alpha_{k+1}(b_{k+1}), \bar{b}\right) \geq s.
\]

Fix arbitrary $r', s' \in \mathbb{Q}^{[0,1]}$ with $r < r' < s' < s$ and consider the countable partial $L_0$-type $\pi\left((x_g)_{g \in G_{k+1,p}}\right)$ with a finite tuple of parameters $\bar{b}$ given by

\[
\bigwedge_{(g_1, \ldots, g_{k+1}) \in R_{k+1}} F^{<r'}(x_{g_1}, \ldots, x_{g_{k+1}}, \bar{b}) \wedge \bigwedge_{(g_1, \ldots, g_{k+1}) \in \prod_{i \in [k+1]} P_i \setminus R_{k+1}} F^{\geq s'}(x_{g_1}, \ldots, x_{g_{k+1}}, \bar{b}).
\]
By \([10.1]\) and using \(\mathcal{M} \models \mathcal{M}_f\), every finite set of formulas from \(\pi\) is realized in \(\mathcal{M}\). Then, by \(\aleph_1\)-saturation of \(\mathcal{M}\), we can find some tuples \((a_g)_{g \in G_{k+1,p}}\) (with \(a_g \in V_i\) for \(g \in P_i\)) so that \(\mathcal{M} \models \pi ((a_g)_{g \in G_{k+1,p}})\).

By Fact \([9.10](2)\), let \((a'_g)_{g \in G_{k+1,p}}\) be \(G_{k+1,p}\)-indiscernible over \(b\) in \(\mathcal{M}\) based on \((a_g)_{g \in G_{k+1,p}}\). Then we still have \(\mathcal{M} \models \pi ((a'_g)_{g \in G_{k+1,p}})\). In particular, using \(\mathcal{M} \models \mathcal{M}_f\) again, we get that \((a'_g)_{g \in G_{k+1,p}}\) satisfies (c) with respect to \(r', s'\).

Next we show an analog of Fact \([4.3]\) for real valued functions (generalizing \([BY09]\) Proposition 3.7] in the case \(k = 1\)). We will use the following variant of the Stone-Weierstrass theorem.

**Fact 10.3.** \([BYU10]\) Proposition 1.14] Let \(X\) be a compact Hausdorff space. Assume that \(\mathcal{B} \subseteq C(X, [0, 1])\) satisfies the following:

1. if \(f \in \mathcal{B}\), then \(1 - f \in \mathcal{B}\);
2. if \(f, g \in \mathcal{B}\), then \(f - g \in \mathcal{B}\) (where for any \(x, y \in [0, 1]\), \(x - y := \max\{x - y, 0\}\);
3. if \(f \in \mathcal{B}\), then \(\frac{1}{x} \in \mathcal{B}\);
4. if \(x \neq y \in X\), then \(f(x) \neq f(y)\) for some \(f \in \mathcal{B}\).

Then \(\mathcal{B}\) is dense in \(C(X, [0, 1])\) with respect to the uniform convergence topology.

**Lemma 10.4.**

1. Assume that a sequence of functions \(f_i : \prod_{i \in [k+1]} V_i \rightarrow [0, 1], i \in \mathbb{N}\) converges uniformly to \(g : \prod_{i \in [k+1]} V_i \rightarrow [0, 1]\), and \(\text{VC}_k(f_i) < \infty\) for every \(i \in \mathbb{N}\). Then also \(\text{VC}_k(g) < \infty\).

2. For every \(\bar{d} \in \mathcal{B}\) there exists some \(\bar{d}'\) such that if \(f, g : \prod_{i \in [k+1]} V_i \rightarrow [0, 1]\) and \(\text{VC}_k(f), \text{VC}_k(g) \leq \bar{d}\), then \(\text{VC}_k\left(\frac{f}{g}\right), \text{VC}_k(1 - f), \text{VC}_k(f - g) \leq \bar{d}'\).

**Proof.** (1) Let \(r < s \in [0, 1]\) be arbitrary, and assume that some box \(A = \prod_{i \in [k]} A_i\) with each \(A_i\) infinite is \((r, s)\)-shattered by \(g\). Let \(\varepsilon := \frac{s - r}{2} > 0\). By assumption there exists some \(n \in \mathbb{N}\) such that \(|f_n(\bar{x}) - g(\bar{x})| < \varepsilon\) for every \(\bar{x} \in \prod_{i \in [k+1]} V_i\). But then \(A\) is \((r + \varepsilon, s - \varepsilon)\)-shattered by \(f_n\).

(2) It is clear that if a box \(A \subseteq \prod_{i \in [k]} V_i\) is \((r, s)\)-shattered by \(\frac{f}{g}\), then it is \((2r, 2s)\)-shattered by \(f\), and if \(A\) is \((r, s)\)-shattered by \(1 - f\) then it is \((1 - s, 1 - r)\)-shattered by \(f\).

Suppose \(A\) is \((r, s)\)-shattered by \(f - g\) where \(A = \prod_{i \in [k]} A_i\) with \(|A_i|\) sufficiently large (as determined later). Let \(\varepsilon = s - r\). By Ramsey’s Theorem, we may choose a box \(A' = \prod_{i \in [k]} A'_i\) with

Assume towards a contradiction that there exist some \(\bar{d}\) and \(r < s \in Q^{[0, 1]}\) such that for any \(j \in \mathbb{N}\) there exist some functions \(f^j_1, f^j_2 : \prod_{i \in [k+1]} V_i \rightarrow [0, 1]\) such that \(\text{VC}_k(f^j_1), \text{VC}_k(f^j_2) < \bar{d}\) but \(f^j_3 := f^j_1 - f^j_2\) is \((r, s)\)-shatters some box \(\prod_{i \in [k]} A'_i\) with \(|A'_i|\) sufficiently large. Let \(\tilde{\mathcal{M}} := \prod_{j \in \mathbb{N}} \mathcal{M}_{f^j_1, f^j_2, f^j_3} / U\) and \(\tilde{A}_i := \prod_{j \in U} A^j_i\) for \(i \in [k]\). Then we have:
\( \hat{M} \cong M_{f_1, f_2, f_3} \) (by Fact 9.12);

\( f_3 = \tilde{f}_1 - \tilde{f}_2 \) (easy as \( \hat{M} \cong M_{f_1, f_2, f_3} \));

\( \text{VC}_k(\tilde{f}_1), \text{VC}_k(\tilde{f}_2) \leq d' \) for some \( d' < \infty \) (by Lemma 10.1);

for any \( r', s' \in \text{Q}\{0,1\} \) with \( r < r' < s < s', f_3(r', s') \)-shatters the box \( \prod_{i \in [k]} \tilde{A}_i \) and each \( \tilde{A}_i \) is infinite (by Lemma 10.1).

By Lemma 10.2 (1) \( \Rightarrow \) (2) there exist some \( r < r' < s < s' \in \text{Q}\{0,1\} \) and \( (a_g)_{g \in G_{k+1},} \) (with \( g \in P_i \Rightarrow a_g \in \tilde{V}_i \) for \( i \in [k+1] \)) so that \( (a_g)_{g \in G_{k+1},} \) is \( G_{k+1}, p \)-indiscernible (in \( M \)), and for all \( (g_1, \ldots, g_{k+1}) \in \prod_{i \in [k+1]} P_i \) we have:

- if \( G_{k+1, p} = R_{k+1}(g_1, \ldots, g_{k+1}) \) then \( \tilde{f}_1 - \tilde{f}_2(a_{g_1}, \ldots, a_{g_{k+1}}) \leq r' \);
- if \( G_{k+1, p} = -R_{k+1}(g_1, \ldots, g_{k+1}) \) then \( \tilde{f}_1 - \tilde{f}_2(a_{g_1}, \ldots, a_{g_{k+1}}) \geq s' \).

Fix some \( (g_1, \ldots, g_{k+1}) \in R_{k+1} \) and \( (h_1, \ldots, h_k) \in \prod_{i \in [k]} P_i \setminus R_{k+1} \). By definition of \( \tilde{\cdot} \), one of the following two cases must occur:

- \( \tilde{f}_1(a_{h_1}, \ldots, a_{h_{k+1}}) - \tilde{f}_1(a_{g_1}, \ldots, a_{g_{k+1}}) \geq \frac{s' - r'}{2} \);
- \( \tilde{f}_2(a_{g_1}, \ldots, a_{g_{k+1}}) - \tilde{f}_2(a_{h_1}, \ldots, a_{h_{k+1}}) \geq \frac{s' - r'}{2} \).

In the first case, let \( r'' := \tilde{f}_1(a_{g_1}, \ldots, a_{g_{k+1}}) \) and \( s'' := \tilde{f}_1(a_{h_1}, \ldots, a_{h_{k+1}}) \), then \( r'' < s'' \in [0, 1] \) and we have

- if \( G_{k+1, p} = R_{k+1}(g_1, \ldots, g_{k+1}) \) then \( \tilde{f}_1(a_{g_1}, \ldots, a_{g_{k+1}}) = r'' \);
- if \( G_{k+1, p} = -R_{k+1}(g_1, \ldots, g_{k+1}) \) then \( \tilde{f}_1(a_{g_1}, \ldots, a_{g_{k+1}}) = s'' \).

Indeed, for any \( (g_1, \ldots, g_{k+1}) \in R_{k+1} \) we obviously have

\[ \text{qftp}_{\omega_{k+1}}(g_1, \ldots, g_{k+1}) = \text{qftp}_{\omega_{k+1}}(g_1, \ldots, g_{k+1}). \]

Hence by \( G_{k+1, p} \)-indiscernibility of \( (a_g)_{g \in G_{k+1},} \), for any \( q \in \text{Q}\{0,1\} \) we have

\[ M, \models F_{1}^{<q}(a_{g_1}, \ldots, a_{g_{k+1}}) \iff \hat{M}, \models F_{1}^{<q}(a_{g_1}, \ldots, a_{g_{k+1}}), \]

which using \( \hat{M} \cong M_{f_1} \) implies \( \tilde{f}_1(a_{g_1}, \ldots, a_{g_{k+1}}) = \tilde{f}_1(a_{g_1}, \ldots, a_{g_{k+1}}) \) (and the second bullet is similar). By Lemma 10.2 (2) \( \Rightarrow \) (1) this implies \( \text{VC}_k(1 - \tilde{f}_1) = \infty \), hence \( \text{VC}_k(\tilde{f}_1) = \infty \) — a contradiction.

In the second case, a similar argument shows that \( \text{VC}_k(\tilde{f}_2) = \infty \). \( \square \)

**Proposition 10.5.** Assume that \( n \in \mathbb{N} \) and \( g: [0, 1]^n \to [0, 1] \) is an arbitrary continuous function. Then for any \( \tilde{d} < \infty \) there exists some \( \tilde{D} = \tilde{D}(\tilde{d}, g) \leq \infty \) satisfying the following. Let \( f_1, \ldots, f_n: V_{1}^{k+1} \to [0, 1] \) satisfy \( \text{VC}_k(f_i) < \tilde{d} \) for \( i \in [n] \). Then \( h := g(f_1, \ldots, f_n): V_{1}^{k+1} \to [0, 1] \) satisfies \( \text{VC}_k(h) < \tilde{D} \).

**Proof.** By Fact 10.3 \( g \) can be uniformly approximated by finite compositions of the functions \( \frac{1}{2}, 1-x, x-y \). Plugging \( f_1, \ldots, f_n \) into the arguments of such a composition gives a function of finite \( \text{VC}_k \)-dimension by Lemma 10.4 (2), hence \( g \) has finite \( \text{VC}_k \)-dimension by Lemma 10.4 (1).

Permutation of the variables of function also preserves finiteness of the \( \text{VC}_k \)-dimension.
Proposition 10.6. Given $\bar{d} = (d)_{r<s \in \mathbb{Q}^{[0,1]}} < \infty$, let $\bar{D} = (D_{r,s})_{r<s \in \mathbb{Q}^{[0,1]}} < \infty$ be given by $D_{r,s} := 2^{d_{r,s}}$. Assume $f : \prod_{i \in [k+1]} V_i \to [0,1]$ satisfies $\text{VC}_k(f) \leq \bar{d}$ and $\sigma : [k+1] \to [k+1]$ is an arbitrary permutation. Let $f^\sigma : \prod_{i \in [k+1]} V_{\sigma(i)} \to [0,1]$ be given by $f^\sigma(x_1, \ldots, x_{k+1}) := f(x_{\sigma(1)}, \ldots, x_{\sigma(k+1)})$. Then $\text{VC}_k(f^\sigma) \leq \bar{D}$.

Proof. If some box $A_1 \times \ldots \times A_k$ with $A_i \subseteq V_i, |A_i| = D$ is $(r,s)$-shattered by a function $f : \prod_{i \in [k+1]} V_i \to [0,1]$, then for any $(k+1)$-partite hypergraph $(R; [D], \ldots, [D])$ with $R \subseteq [D]^{k+1}$ we can choose some set $A_{k+1} \subseteq V_{k+1}$ and bijections $\alpha_i : [D] \to A_i$ so that for every $(b_1, \ldots, b_{k+1}) \in [D]^{k+1}$,

$$f((b_1, \ldots, b_{k+1}) \in R \implies f(\alpha_1(b_1), \ldots, \alpha_{k+1}(b_{k+1})) \leq r;$$

$$(b_1, \ldots, b_{k+1}) \notin R \implies f(\alpha_1(b_1), \ldots, \alpha_{k+1}(b_{k+1})) \geq s.$$

Assume $D = 2^{d_{k}}$. Given a permutation $\sigma : [k+1] \to [k+1]$ with $\sigma(k+1) = r^*$, consider the $(k+1)$-partite hypergraph $(R; [D], \ldots, [D])$ so that for every $B \subseteq [D]^k$ there exists some $b_B \in [D]$ satisfying: for every $(b_1, \ldots, b_{r^*-1}, b_{r^*+1}, b_{k+1}) \in [D]^k$,

$$(b_1, \ldots, b_{r^*-1}, b_B, b_{r^*+1}, \ldots, b_{k+1}) \in R \iff (b_1, \ldots, b_{r^*-1}, b_{r^*+1}, b_{k+1}) \in B.$$

Combined with (10.2) and taking $A'_i := \{\alpha_{\sigma(i)}(j) : j \in [d]\}$, this implies that the box $A'_1 \times \ldots \times A'_k$ with $|A'_i| = d, A'_i \subseteq V_{\sigma(i)}$ is $(r,s)$-shattered by $f^\sigma$.

\[\square\]

10.2. Integration preserves finite \text{VC}$_k$\text{-dimension}. The aim of this subsection is to prove the following theorem, after developing some tools for it.

Theorem 10.7. For every $k \in \mathbb{N}_{\geq 1}$ and $\bar{d} = (d_{r,s})_{r<s \in \mathbb{Q}^{[0,1]}}$ with $d_{r,s} \in \mathbb{N}$ there exists some $\bar{D} = (D_{r,s})_{r<s \in \mathbb{Q}^{[0,1]}}$ with $D_{r,s} \in \mathbb{N}$ satisfying the following.

Assume that $(V_{[k+2]}, \mathcal{B}_{\bar{n}}, \mu_{\bar{n}})_{n \in \mathbb{N}_{k+2}}$ is a $(k+2)$-partite graded probability space, $f : V_{1}^{k+2} \to [0,1]$ is a $\mathcal{B}_{1^{k+2}}$-measurable function and $\text{VC}_k(f) \leq \bar{d}$. Then the $(k+1)$-ary “average” function $f' : V_{1}^{k+1} \to [0,1]$ defined by

$$f'(x_1, \ldots, x_{k+1}) := \int f(x_1, \ldots, x_{k+1}) d\mu_{\bar{\delta}_{k+2}}(x_{k+2})$$

satisfies $\text{VC}_k(f') \leq \bar{D}$.

Remark 10.8. Theorem 10.7 generalizes [BY09] Corollary 4.2] in the case $k = 1$.

Corollary 10.9. For every $k \in \mathbb{N}_{\geq 1}$ there exists some $\bar{D} = (D_{r,s})_{r<s \in \mathbb{Q}^{[0,1]}} < \infty$ satisfying the following.

Assume that $(V_{[k+2]}, \mathcal{B}_{\bar{n}}, \mu_{\bar{n}})_{n \in \mathbb{N}_{k+2}}$ is a $(k+2)$-partite graded probability space, and for each $I \in (\mathbb{N}_{k+1}^{[k+1]})$ let $\bar{n}_I := \sum_{i \in I} \bar{\delta}_i + \bar{\delta}_{k+2}$ and $E' \in \mathcal{B}_{\bar{n}_I}$ arbitrary.
Then the \((k + 1)\)-ary function \(f' : V^{1:k+1} \to [0, 1]\) defined by
\[
f'(\vec{x}) \mapsto \mu_{\bar{\delta}_{k+2}} \left( \bigcap_{I \in \binom{[k+1]}{\leq k}} E_{I}^I \right)
\]
satisfies \(VC_k(f') \leq \bar{D}\).

**Proof.** Consider the relation \(F \in \mathcal{B}_{[k+2]}\) defined by
\[
(x_1, \ldots, x_{k+2}) \in F : \iff \bigwedge_{I \in \binom{[k+1]}{\leq k}} \left( \bar{x}^I (x_2) \in E_I^I \right).
\]

Then for any fixed \(b \in V_{k+2}\), the \((k + 1)\)-ary relation \(F_b\) is a conjunction of the \(\leq k\)-ary relations \(E_b^I, I \in \binom{[k+1]}{\leq k}\), hence trivially \(VC_k(F_b) \leq \bar{d}\) with \(d_{r,s} := 1\) for all \(r, s \in \mathbb{Q}[0,1]\). Applying Proposition 10.7 to \(F\) and noting that
\[
\mu_{\bar{\delta}_{k+2}} \left( \bigcap_{I \in \binom{[k+1]}{\leq k}} E_{I}^I \right) = \int \chi_F(x_1, \ldots, x_{k+2}) d\mu_{\bar{\delta}_{k+2}}(x_{k+2}),
\]
we can conclude. \(\square\)

The same holds with any fixed Boolean combination instead of a conjunction.

10.2.1. **Intersections of measurable sets indexed by generic hypergraphs and exchangeability.** In this section we let \(\nu_n\) denote the Lebesgue probability measure on \([0,1]^n\).

Given two collections of random variables \((\xi_i : i \in I)\) on a probability space \((V, \mathcal{B}, \mu)\) and \((\xi'_i : i \in I)\) on a probability space \((V', \mathcal{B}', \mu')\) indexed by the same ordered set \(I\) and taking values in \([0,1]\), we write \((\xi_i : i \in I) \equiv_{\text{dist}} (\xi'_i : i \in I)\) to denote that they have the same joint distribution (that is, for every finite set \(J \subseteq I\) and any \(p_i \in [0,1]\) for \(i \in J\),
\[
\mu\left(\{x \in V : \bigwedge_{i \in J} \xi_i(x) < p_i\}\right) = \mu'\left(\{x \in V' : \bigwedge_{i \in J} \xi'_i(x) < p_i\}\right).
\]

We will need a generalization of the Aldous-Hoover-Kallenberg theorem on exchangeable arrays of random variables \([\text{Ald81, Hoo79, Kal06}]\) for a restricted form of exchangeability with respect to \(k\)-partite generic hypergraphs. We will rely on the setting of \([\text{CT18}]\).

**Definition 10.10.**  
(1) Let \(\mathcal{L}' = \{R'_1, \ldots, R'_{k'}\}\) be a finite relational language, with each \(R'_i\) a relation symbol of arity \(r'_i\). By a random \(\mathcal{L}'\text{-structure}\) we mean a collection of random variables
\[
(\xi^n_i : i \in [k'], \bar{n} \in \mathbb{N}^{r'_i})
\]
on some probability space \((V, \mathcal{B}, \mu)\) with \(\xi^n_i : V \to \{0,1\}\). (Equivalently, we can think of this as equipping the space of all countable \(\mathcal{L}'\text{-structures}\) with a measure, and picking a random \(\mathcal{L}'\text{-structure}\) according to it.)
(2) Let now $L = \{R_1, \ldots, R_k\}$ be another relational language, with $R_i$ a relation symbol of arity $r_i$, and let $M = (\mathbb{N}, \ldots)$ be a countable $L$-structure with domain $\mathbb{N}$. We say that a random $L'$-structure $(\xi^n_i : i \in [k'], \bar{n} \in \mathbb{N}^{r_i})$ is $M$-exchangeable if for any two finite subsets $A = \{a_1, \ldots, a_{\ell}\}, A' = \{a'_1, \ldots, a'_{\ell}'\} \subseteq N$

$qftp_L (a_1, \ldots, a_{\ell}) = qftp_L (a'_1, \ldots, a'_{\ell}') \implies
\left( \xi^n_i : i \in [k'], \bar{n} \in A^{r_i} \right) = ^{\text{dist}} \left( \xi^n_i : i \in [k'], \bar{n} \in (A')^{r_i} \right).

Given a tuple $\bar{n} = (n_1, \ldots, n_r)$ we let $\text{rng} \bar{n}$ denote the set of distinct elements in $\bar{n}$, and write $\bar{m} \subseteq \bar{n}$ if $\bar{m} = (m_{p_1}, \ldots, m_{p_r})$ for an increasing sequence $p_1 < \ldots < p_r \in [r]$.

**Fact 10.11.** ([CT18, Theorem 3.2]) Let $L' = \{R'_i : i \in [k']\}, L = \{R_i : i \in [k]\}$ be finite relational languages with all $R'_i$ of arity at most $r'$, and $M = (\mathbb{N}, \ldots)$ a countable ultrahomogeneous $L$-structure that has $n$-DAP for all $n \geq 1$ (see Definition 7.2). Suppose that $(\xi^n_i : i \in [k'], \bar{n} \in \mathbb{N}^{r_i})$ is a random $L'$-structure that is $M$-exchangeable such that the relations $R'_i$ are symmetric with probability 1.

Then there exists a probability space $(V', \mathcal{B}', \mu')$, $\{0, 1\}$-valued Borel functions $f_1, \ldots, f_{r'}$ and a collection of Uniform$[0,1]$ i.i.d. random variables $(\zeta_s : s \subseteq \mathbb{N}, |s| \leq r')$ on $V'$ so that

$\left( \xi^n_i : i \in [k'], \bar{n} \in \mathbb{N}^{r_i} \right) = ^{\text{dist}} \left( f_i \left( M|_{\text{rng} \bar{n}}, (\zeta_s)_{s \subseteq \text{rng} \bar{n}} \right) : i \in [k'], \bar{n} \in \mathbb{N}^{r_i} \right)\).$

**Remark 10.12.** Given $n \in \mathbb{N}$, let $(\zeta_i : i < n)$ be uniformly distributed $[0, 1]$-valued independent random variables on a probability space $(V, \mathcal{B}, \mu)$. Let $A \subseteq [0, 1]^n$ be a Borel set. Then

$\nu_n(A) = \mu \left( \{x \in V : (\zeta_1(x), \ldots, \zeta_n(x)) \in A\} \right)$.

**Proof.** Assume $\nu_n(A) = r$, and let $\varepsilon > 0$ be arbitrary. As $A$ is measurable with respect to $\nu_n$, we can find some Borel sets $A_{1,j}, \ldots, A_{n,j} \subseteq [0, 1]$ for $j \in \mathbb{N}$ such that $A \subseteq A' := \bigcup_{j \in \mathbb{N}} \prod_{1 \leq i \leq n} A_{i,j}$ and $\nu_n(A') \leq r + \varepsilon$. Then we have:

$\mu \left( \{x \in V : (\zeta_1(x), \ldots, \zeta_n(x)) \in A'\} \right) = \sum_{j \in \mathbb{N}} \mu \left( \left\{ x \in V : (\zeta_1(x), \ldots, \zeta_n(x)) \in \prod_{1 \leq i \leq n} A_{i,j} \right\} \right)$

(by countable additivity and disjointness of the boxes)

$= \sum_{j \in \mathbb{N}} \mu \left( \{x \in V : \zeta_1(x) \in A_{1,j}\} \right) \cdot \ldots \cdot \mu \left( \{x \in V : \zeta_n(x) \in A_{n,j}\} \right)$

(as the random variables $\zeta_1, \ldots, \zeta_n$ are independent)
Then for any finite $Q$ (see Definition 9.2). Assume that for each tuple $G$ arbitrary. Let have some sets $E$ are partite, they may be extended to symmetric relations containing only $G$ tuples with exactly one element from each part. Besides, without loss of generality the domain of $9.6$. Moreover, for any tuple $(\zeta_1(x), \ldots, \zeta_n(x)) \in A')$ = $\nu_n(A') \leq r + \varepsilon$.

Applying the same argument to the complement of $A$ we get that also

$$\mu(\{x \in V : (\zeta_1(x), \ldots, \zeta_n(x)) \in A\}) \geq r - \varepsilon,$$

and, since $\varepsilon > 0$ was arbitrary, the claim follows.

The following can be viewed as an analog of Lemmas 9.25 (which in turn is an “indiscernible” version of Fact 4.5), where instead of indexing by a sequence we are indexing by a generic partite hypergraph.

**Lemma 10.13.** Let $(V, B, \mu)$ be a probability space, and $k \in \mathbb{N}$ and $r \in [0, 1]$ arbitrary. Let $G_{k,p} = (P_1, \ldots, P_k, R_k)$ be the generic $k$-partite hypergraph (see Definition 9.2). Assume that for each tuple $\bar{a} = (a_i)_{i \in [k]} \in \prod_{i \in [k]} P_i$ we have some sets $E_0^\bar{a}, E_1^\bar{a} \in B$ satisfying the following:

1. $\mu(E_0^\bar{a}) > \mu(E_1^\bar{a})$ for some $\bar{a} \in R_k, \bar{a} \notin R_k$;
2. for any for any $m \in \mathbb{N}, \bar{a} = (a_1^i, \ldots, a_m^i) \in P_i$ and $\bar{b} = (b_1^{i'}, \ldots, b_m^{i'}) \in P_i$ for $i \in \{1, \ldots, k\}$

$$\text{qftp}_{L^0_{\text{ops}}} (\bar{a}^1, \ldots, \bar{a}^k) = \text{qftp}_{L^1_{\text{ops}}} (\bar{b}^1, \ldots, \bar{b}^k) \implies \left( \chi_{E^\bar{a}}^{(a_1^1, \ldots, a_m^1)} : t \in \{0, 1\}, (l_1, \ldots, l_k) \in [m]^k \right) = \text{dist} \left( \chi_{E^\bar{a}}^{(b_1^{i'}, \ldots, b_m^{i'})} : t \in \{0, 1\}, (l_1, \ldots, l_k) \in [m]^k \right).$$

Then for any finite $Q_i \subseteq P_i, i \in [k]$, taking $Q := \prod_{i \in [k]} Q_i$, we have

$$\mu \left( \bigcap_{\bar{a} \in Q \cap R_k} E_0^\bar{a} \cap \bigcap_{\bar{a} \in Q \setminus R_k} V \setminus E_1^\bar{a} \right) > 0.$$

**Proof.** Without loss of generality the domain of $G'_{k,p}$ is $\mathbb{N}$, i.e. $(\bigcup_{i \in [k]} P_i)^k = \mathbb{N}$. For each $\bar{a} \in \prod_{i \in [k]} P_i$ and $t \in \{0, 1\}$, let $\xi_t^\bar{a} := \chi_{E^\bar{a}}^t$. For any $\bar{a} \in \mathbb{N}^k \setminus \prod_{i \in [k]} P_i$, let $\xi_t^\bar{a}$ be the constant zero map for $t \in \{0, 1\}$. By assumption (2) it follows that $(\xi_t^\bar{a} : t \in \{0, 1\}, \bar{a} \in \mathbb{N}^k)$ is a $G'_{k,p}$-exchangeable random $\mathcal{L}'$-structure for $\mathcal{L}'$ containing two $k$-ary relational symbols. Since the relations are partite, they may be extended to symmetric relations containing only tuples with exactly one element from each part. Besides, $G'_{k,p}$ is ultrahomogeneous by Fact 9.3(4) and satisfies $n$-DAP for all $n \in \mathbb{N}_{\geq 1}$ by Proposition 9.6. Moreover, for any tuple $(g_1, \ldots, g_k) \in \prod_{i \in [k]} P_i$, there only two possible isomorphism types for the induced substructure $G'_{k,p}|_{\text{opg}(g_1, \ldots, g_k)}$ (see Definition 9.3) — one for $(g_1, \ldots, g_k) \in R_k$ and one for $(g_1, \ldots, g_k) \notin R_k$. Hence,
applying Fact [10.11] there exist a probability space \((V', B', \mu')\), a collection of Uniform\([0, 1]\) i.i.d.
random variables \(\zeta_{\bar{a}} : V' \to [0, 1]\) indexed by the tuples
\(\bar{a} \in \bigcup_{I \subseteq [k]} \prod_{i \in I} P_i\), and Borel measurable functions
\(f^t_{\bar{a}} : [0, 1]^{2^k} \to \{0, 1\}\) for \(t \in \{0, 1\}, s \in \{+, -\}\), such that we have
\begin{equation}
\left( \chi_{E^t_{\bar{a}}} : t \in \{0, 1\}, \bar{a} \in \prod_{i \in [k]} P_i \right) = \text{dist} \left( f^t_{\rho(\bar{a})} ((\zeta_{\bar{a}}_I : I \subseteq [k])) : t \in \{0, 1\}, \bar{a} \in \prod_{i \in [k]} P_i \right),
\end{equation}
where \(\rho(\bar{a}) = +\) if \(\bar{a} \in R_k\) and \(\rho(\bar{a}) = -\) if \(\bar{a} \notin R_k\).

Let \(S_+ := (f^1_+)^{-1}(\{1\})\) and \(S_- := (f^1_-)^{-1}(\{1\})\), both are Borel subsets of
\([0, 1]^{2^k}\). Let \(\bar{a} \in R_k, \bar{a}' \in \prod_{i \in [k]} P_i \setminus R_k\) be as given by assumption (1). Then,
using Remark [10.12] we have
\[\mu(E^0_{\bar{a}}) = \mu' \left\{ x \in V' : f^0_{\rho(\bar{a})} ((\zeta_{\bar{a}}_I : I \subseteq [k])) = 1 \right\} = \mu' \left\{ x \in V' : (\zeta_{\bar{a}}_I (x) : I \subseteq [k]) \in S_+ \right\} = \nu_{2^k} (S_+) \]
Similarly, \(\mu(E^1_{\bar{a}}) = \nu_{2^k} (S_-)\). As \(\mu(E^0_{\bar{a}}) > \mu(E^1_{\bar{a}})\) by assumption, it follows
that \(\nu_{2^k} (S_+ \setminus S_-) > 0\).

Fix any \(\varepsilon \in \mathbb{R}_{>0}\). Then, by the basic properties of Lebesgue measure,
we can choose some \((A_I : I \subseteq [k])\) with each \(A_I\) a Borel subset of \([0, 1]\) with
\(\nu_1 (A_I) > 0\), so that, taking \(A := \prod_{I \subseteq [k]} A_I\), we have
\begin{equation}
\nu_{2^k} (A \cap (S_+ \setminus S_-)) \geq (1 - \varepsilon) \cdot \nu_{2^k} (A).
\end{equation}

Let \(Q_i \subseteq P_i\) be arbitrary finite subsets. It is enough to prove the lemma assuming that for some \(n \in \mathbb{N}\), \(|Q_i| = n\) for all \(i \in [k]\). Let \(K := \sum_{l=0}^{k} (\binom{k}{l}) n^l\).

We let
\[W := \left\{ (x_{\bar{a}} : \bar{a} \in \bigcup_{I \subseteq [k]} \prod_{i \in I} Q_i) \in [0, 1]^K : \right.\]
\[\left. \bigwedge_{\bar{a} \in \prod_{i \in [k]} Q_i \cap R_k} (x_{\bar{a}}_I : I \subseteq [k]) \in S_+ \land \bigwedge_{\bar{a} \in \prod_{i \in [k]} Q_i \setminus R_k} (x_{\bar{a}}_I : I \subseteq [k]) \notin S_- \right\}.\]

Let
\[B := \prod_{\bar{a} \in \bigcup_{I \subseteq [k]} \prod_{i \in I} Q_i} A_I,\]
then \(B\) is a box in \([0, 1]^K\) with \(\nu_K (B) > 0\). For every \(\bar{b} \in \prod_{i \in [k]} Q_i\) let
\[B_{\bar{b}} := \left\{ (x_{\bar{a}} : \bar{a} \in \bigcup_{I \subseteq [k]} \prod_{i \in I} Q_i) \in B : (x_{\bar{b}_I} : I \subseteq [k]) \in A \setminus (S_+ \setminus S_-) \right\}.\]
We have
\[ B \setminus W \subseteq \bigcup_{\bar{b} \in \prod_{i \in [k]} Q_i} B_{\bar{b}}, \]
which by (10.4) and definition of \( B_{\bar{b}} \)'s implies
\[ \nu_K (B \setminus W) \leq \sum_{\bar{b} \in \prod_{i \in [k]} Q_i} \nu_K (B_{\bar{b}}) \leq n^k \cdot \epsilon \cdot \nu_K (B). \]

So, if we take \( \epsilon < \frac{1}{n^k} \), we get \( \nu_K (B \cap W) > 0 \), in particular \( \nu_K (W) > 0 \).

Then, using (10.3) and Remark 10.12, we get
\[ 0 < \mu'(\{ x \in V' : \bigwedge_{\bar{a} \in \prod_{i \in [k]} Q_i \cap R_k} (\zeta_{\bar{a}}(x) : I \subseteq [k]) \in S_+ \wedge \bigwedge_{\bar{a} \in \prod_{i \in [k]} Q_i \setminus R_k} (\zeta_{\bar{a}}(x) : I \subseteq [k]) \notin S_- \}) = \mu\left( \bigcap_{\bar{a} \in \prod_{i \in [k]} Q_i \cap R_k} E^0_{\bar{a}} \cap \bigcap_{\bar{a} \in \prod_{i \in [k]} Q_i \setminus R_k} V \setminus E^1_{\bar{a}} \right). \]

The next fact follows from model-theoretic stability of probability algebras in continuous logic [BYBHU08, Section 16], or a more general [Hru12, Proposition 2.25]. See [Tao13] for a short elementary proof.

**Fact 10.14.** For any real numbers \( 0 \leq p < q \leq 1 \) there exists some \( N = N(p, q) \) satisfying the following. If \( (V, B, \mu) \) is a probability space, and \( A_1, \ldots, A_n, B_1, \ldots, B_n \in B \) satisfy \( \mu(A_i \cap B_j) \geq q \) and \( \mu(A_i \cap B_i) \leq p \) for all \( 1 \leq i < j \leq n \), then \( n \leq N \).

Using this we show that the generic \( k \)-partite ordered hypergraph \( G_{k,p} \)-exchangeability of a collection of random variables implies its exchangeability with respect to the reduct \( G'_{k,p} \) without the ordering (this can be viewed as an analog of Ryll-Nardziewski’s classical result that for a sequence of random variables, spreadability implies exchangeability for our more complicated notion of exchangeability, see e.g. [Kal88]).

**Lemma 10.15.** Let \( (V, B, \mu) \) be a probability space, and assume that for each \( \bar{a} = (a_1, \ldots, a_k) \in \prod_{i \in [k]} P_i \) we have some sets \( E^0_{\bar{a}}, E^1_{\bar{a}} \in B \) such that the following holds: for any \( m \in \mathbb{N} \), \( \bar{a}^i = (a^i_1, \ldots, a^i_m) \in P_i \) and \( \bar{b}^i = (b^i_1, \ldots, b^i_m) \in P_i \) for \( i \in \{1, \ldots, k\} \) such that \( \text{qftp}_{\mathcal{L}_{opg}} (\bar{a}^1, \ldots, \bar{a}^k) = \text{qftp}_{\mathcal{L}_{opg}} (\bar{b}^1, \ldots, \bar{b}^k) \),
we have that
\[ \mu \left( \bigwedge_{(i,v) \in [m]^k \times \{0,1\}} E^{v,t_i}_i (a_1^{i,v}, \ldots, a_{k}^{i,v}) \right) = \mu \left( \bigwedge_{(i,v) \in [m]^k \times \{0,1\}} E^{v,t_i}_i (b_1^{i,v}, \ldots, b_{k}^{i,v}) \right) \]
for every tuple \((t_i^v \in \{0,1\} : v \in \{0,1\}, \bar{t} \in [m]^k)\) (where \(E^{i,1}\) denotes \(E^i\) and \(E^{t,0}\) denotes \(\neg E^i\)). Then the same holds for any pair of tuples satisfying the weaker assumption \(\text{qftp}_{\mathcal{L}_{pg}} (\bar{a}^1, \ldots, \bar{a}^k) = \text{qftp}_{\mathcal{L}_{pg}} (\bar{b}^1, \ldots, \bar{b}^k)\), i.e. Assumption (2) in Lemma 10.13 is satisfied.

**Proof.** It suffices to show the following (under the given assumption of \(G_{k,p}\)-exchangeability). Let \(a_1^i < \ldots < a_m^i\) in \(P_i\) be arbitrary, for \(i \in [k]\), let a tuple \((t_i^v \in \{0,1\} : v \in \{0,1\}, \bar{t} \in [m]^k)\) be fixed, and let \(\sigma\) be a permutation of \([m]\) such that \((a_1^i, \ldots, a_k^i) \in \mathcal{R}_k \iff (a_{\sigma(t_i^v)}^1, a_{\sigma(t_i^v)+1}^1, \ldots, a_k^i) \in \mathcal{R}_k\) for all \(\bar{t} = (t_1^1, \ldots, t_k^1) \in [m]^k\) (i.e. \(\sigma\) preserves the quantifier-free \(\mathcal{L}_{pg}\)-type of the tuple); then

\[ \mu \left( \bigwedge_{(i,v) \in [m]^k \times \{0,1\}} E^{v,t_i}_i (a_1^{i,v}, \ldots, a_{k}^{i,v}) \right) = \mu \left( \bigwedge_{(i,v) \in [m]^k \times \{0,1\}} E^{v,t_i}_i (a_{\sigma(t_i^v)}^{1}, a_{\sigma(t_i^v)+1}^{1}, \ldots, a_k^i) \right) \]

(the case of a permutation \(\sigma\) acting on the elements in \(P_i\) for \(i \neq 1\) is symmetric, and they can be performed separately one by one). As every permutation is a composition of transpositions of consecutive elements, it suffices to show this assuming that \(\sigma\) is a transposition of two consecutive elements. That is, towards a contradiction we assume that there is some \(i^* \in [m], 1 \leq i^* < i^* + 1 \leq m\) such that \(\sigma(i^*) = i^* + 1, \sigma(i^* + 1) = i^*\) and \(\sigma\) is constant on all \(i \in [m] \setminus \{i^*, i^* + 1\}\), and

\[ (10.5) \quad p := \mu \left( \bigwedge_{(i,l_2, \ldots, l_k,v) \in [m]^k \times \{0,1\}} E^{v,t_{(i,l_2, \ldots, l_k,v)}} (a_1^{i,v}, a_2^{i,v}, \ldots, a_k^{i,v}) \right) \]
\[ < q := \mu \left( \bigwedge_{(i,l_2, \ldots, l_k,v) \in [m]^k \times \{0,1\}} E^{v,t_{(i,l_2, \ldots, l_k,v)}} (a_1^{i,v}, a_2^{i,v}, \ldots, a_k^{i,v}) \right) \]

(the case with “>” is symmetric). By the genericity of the hypergraph \(G_{k,p}\) (Definition 9.1) we can find a strictly \(<\)-increasing infinite sequence of elements \((a_i^i : i \in \mathbb{N})\) in \(P_1\) such that:

- for \(i = 2j\) we have
  \[ \text{qftp}_{\mathcal{L}_{opg}} (a_1^1, \ldots, a_{i^* -1}^1, a_i^1, a_{i^* +2}^1, \ldots, a_m^1; a_2^2, \ldots, a_k^2) \]
  \[ = \text{qftp}_{\mathcal{L}_{opg}} (a_1^1, \ldots, a_{i^* -1}^1, a_i^1, a_{i^* +2}^1, \ldots, a_m^1; a_2^2, \ldots, a_k^2); \]
• for $i = 2j + 1$ we have
\[
\text{qftp}_E^{k^1} (a_1^1, \ldots, a_i^1, a_i^{1+1}, a_{i+1}^{1}, a_{i+2}^{1}, \ldots, a_k^{1}) = \text{qftp}_E^{k^1} (a_1^1, \ldots, a_i^1, a_i^{1+1}, a_{i+1}^{1}, a_{i+2}^{1}, \ldots, a_k^{1}).
\]

In particular, for any $j < j' \in \mathbb{N}$ we then have
\[
\begin{align*}
(10.6) & \quad \text{qftp}_E^{k^1} (a_1^1, \ldots, a_i^1, a_j^j, a_{j+1}^{j'}, a_{j+2}^{j'}, a_{i+1}^1, a_{i+2}^{1}, \ldots, a_k^{1}) = \text{qftp}_E^{k^1} (a_1^1, \ldots, a_i^1, a_j^j, a_{j+1}^{j'}, a_{j+2}^{j'}, a_{i+1}^1, a_{i+2}^{1}, \ldots, a_k^{1}); \\
(10.7) & \quad = \text{qftp}_E^{k^1} (a_1^1, \ldots, a_i^1, a_j^j, a_{j+1}^{j'}, a_{j+2}^{j'}, a_{i+1}^1, a_{i+2}^{1}, \ldots, a_k^{1}).
\end{align*}
\]

Then by (10.5), (10.6), (10.7) and the assumption of $G_{k,p}$-exchangeability, we have $\mu(A_i \cap B_j) = p$ for all $i < j$, and $\mu(A_i \cap B_j) = q$ for all $i > j$ — contradicting Fact 10.14.

10.2.2. Proof of Theorem 10.7. Assume towards a contradiction that there exist some $k$, $d$ and $r < s$ in $\mathbb{Q}[0,1]$ such that: for every $j \in \mathbb{N}$ we have a $(k + 2)$-partite graded probability space $(V_j^{k+2}, B^j_n, \mu^j_n)_{n \in \mathbb{N}^{k+2}}$ and a $(k + 2)$-ary $B^j_{k+2}$-measurable function $f^j : (V_j^{k+2})^{k+2} \to [0,1]$ such that $\text{VC}_k(f^j) \leq d$, but such that the function $(f^j)' : (V_j^{k+1})^{k+1} \to [0,1], (f^j)'(x_1, \ldots, x_{k+1}) := \int f(x_1, \ldots, x_{k+2})d\mu^j_{0k+1}(x_{k+2})$ shatters some $k$-box
\[
B^j = \{a_1^{j,1}, \ldots, a_j^{j,1}\} \times \ldots \times \{a_1^{j,k}, \ldots, a_j^{j,k}\}.
\]

As in the proof of Lemma 9.22 for any $r \in [0,1]$ there exist countable partial $L_\infty$-types $\rho_{\leq r}(x_1, \ldots, x_{k+1})$ and $\rho_{\geq r}(x_1, \ldots, x_{k+1})$ satisfying the following: for any $(k + 2)$-partite graded probability space $\Phi = (V_{k+2}, B_\Phi, \mu_{\Phi})_{n \in \mathbb{N}^{k+2}}$, $B^j_{k+2}$-measurable function $f$, an $L_\infty$-structure $M' \models M_{\Phi, f}$ and a tuple $(a_1, \ldots, a_{k+1}) \in V_{k+1}$ we have
\[
(10.8) \quad M' \models \rho_{\leq r}(a_1, \ldots, a_{k+1}) \iff \int f(a_1, \ldots, a_{k+1}, x_{k+2})d\mu^j_{0k+1}(x_{k+2}) \leq r, \text{ and similarly for } \geq r.
\]
Consider the countable partial $L_\infty$-type
\[
\tau ((x_g : g \in G_{k+1,p})) := \bigwedge_{(g_1, \ldots, g_{k+1}) \in R_{k+1}} \left( \rho \leq r (x_{g_1}, \ldots, x_{g_{k+1}}) \land \bigwedge_{(g_1, \ldots, g_{k+1}) \in \prod_{i \in [k+1]} P_i \setminus R_{k+1}} \rho \geq s (g_{g_1}, \ldots, x_{g_{k+1}}) \right).
\]

Let $\tau_0$ be a finite set of formulas from $\tau$ only involving $\ell$ variables from $(x_g : g \in G_{k+1,p})$. As in the proof of Lemma 10.2 (1) $\Rightarrow$ (2), using that trivially $M_{\mathcal{P}_j, f_j} \propto M_{\mathcal{P}_j, f_j}$, by assumption and (10.8), for every $j \geq \ell$ we have that $\tau_0$ is realized in $M_{\mathcal{P}_j, f_j}$. By Los’ theorem this implies that $\tau_0$ is also realized in $\hat{M}$. Hence, by $\aleph_1$-saturation of $\hat{M}$, we have $\hat{M} \models \tau ((a_g : g \in G_{k+1,p}))$ for some $(a_g : g \in G_{k+1,p})$ with $g \in P_i \Rightarrow a_g \in V_i$ for $i \in [k+1]$.

By $\aleph_1$-saturation of $\hat{M}$ and Fact 9.10 (2), let $(a'_g)_{g \in G_{k+1,p}}$ be $G_{k+1,p}$-indiscernible over $0$ in $\hat{M}$ based on $(a_g)_{g \in G_{k+1,p}}$. Then we still have $\hat{M} \models \tau ((a'_g : g \in G_{k+1,p}))$.

For $g = (g_1, \ldots, g_{k+1}) \in \prod_{i \in [k+1]} P_i$, we write $\bar{a}_g := (a'_{g_1}, \ldots, a'_{g_{k+1}})$; and let $\bar{\mu} := \bar{\mu}_{\bar{g}_{k+2}}$. Then, as $\hat{M} \propto M_{\mathcal{P}_j, f_j}$, by definition of $\tau$ and (10.8) we have,
\[
(10.9) \quad G_{k+1,p} \models R_{k+1}(g_1, \ldots, g_{k+1}) \Rightarrow \int \hat{f}(\bar{a}_g, x_{k+2})d\bar{\mu}(x_{k+2}) \leq r,
\]
\[
G_{k+1,p} \models \neg R_{k+1}(g_1, \ldots, g_{k+1}) \Rightarrow \int \hat{f}(\bar{a}_g, x_{k+2})d\bar{\mu}(x_{k+2}) \geq s.
\]

Fix arbitrary $\bar{g}^0 = (g_0^1, \ldots, g_0^{k+1}) \in R_{k+1}, \bar{g}^1 = (g_1^1, \ldots, g_{k+1}^1) \in \prod_{i \in [k+1]} P_i \setminus R_{k+1}$. We let $E^{<q}_{\bar{a}_g} = \left\{ \bar{x}_{k+2} \in \bar{V}_{k+2} : \hat{M} \models E^{<q} (\bar{a}_g, x_{k+2}) \right\}$ for $q \in Q^{[0,1]}$, $\forall \in \{<, \geq\}$ and $\bar{g} \in \prod_{i \in [k+1]} P_i$.

By (10.9), Lemma 4.3 and $\hat{M} \propto M_{\mathcal{P}_j, f_j}$, there exist some $r' < s' \in Q^{[0,1]}$ so that
\[
\bar{\mu} \left( E^{<r'}_{\bar{a}_{g^0}} \right) > \bar{\mu} \left( E^{s'}_{\bar{a}_{g^1}} \right).
\]

For $\bar{g} \in \prod_{i \in [k+1]} P_i$, let $E^0_{\bar{a}_g} := E^{<r'}_{\bar{a}_g}, E^1_{\bar{a}_g} := E^{s'}_{\bar{a}_g}$. As $(a'_g)_{g \in G_{k+1,p}}$ is $G_{k+1,p}$-indiscernible, this implies that the assumption of Lemma 10.13 is satisfied (using that the $E^{<q}$ and $m < q$ predicates are in $L_\infty$ for all $q \in Q^{[0,1]}$). Hence the assumption of Lemma 10.13 is also satisfied, and it follows that for any finite $Q_i \subseteq P_i$ and $Q := \prod_{i \in [k+1]} Q_i$, we have
\[
\bar{\mu} \left( \bigcap_{g \in Q \cap R_{k+1}} E^{<r'}_{\bar{a}_g} \cap \bigcap_{g \in Q \setminus R_{k+1}} E^{\geq s'}_{\bar{a}_g} \right) > 0.
\]

In particular, this intersection is non-empty. Hence, by $\aleph_1$-saturation of $\hat{M}$, there exists some $b \in V_{k+2}$ so that for all $(g_1, \ldots, g_{k+1}) \in \prod_{i \in [k+1]} P_i$ we
have
\[(10.10) \quad G'_{k+1,p} \models R'_{k+1}(g_1, \ldots, g_{k+1}) \Rightarrow \mathcal{N} \models F^{<\mathcal{C}}(a_{g_1}', \ldots, a_{g_{k+1}}', b) \text{ and} \]
\[G'_{k+1,p} \models \neg R'_{k+1}(g_1, \ldots, g_{k+1}) \Rightarrow \mathcal{N} \models F^{\geq\mathcal{C}}(a_{g_1}', \ldots, a_{g_{k+1}}', b) .\]

By Lemma \[10.2(2) \Rightarrow (1)\], this implies that the \((k + 1)\)-ary function \(f'_c\) has infinite VC\(_k\)-dimension — a contradiction to the assumption by Lemma \[10.1\].

Theorem \[10.7\] implies the following slightly more general version.

**Corollary 10.16.** For every \(t \in \mathbb{N}, \bar{d} < \infty\) there exists some \(\bar{D} = \bar{D}(t, \bar{d}) < \infty\) satisfying the following.

Assume that \(k \in \mathbb{N}, (V_\mathcal{B}_n, \mu_n)_{n \in \mathbb{N}^k}\) is a \(k\)-partite graded probability space, \(f : V_\mathcal{B} \to [0, 1]\) is \(\mathcal{B}\)-measurable for some \(\bar{m} = \bar{m} + \bar{m}' \in \mathbb{N}^k\) and \(\text{VC}_t(f) \leq \bar{d}\) (in the sense of Definition \[3.10\](4), i.e. with respect to any partition of the variables of \(f\) into \((t + 1)\) groups). Then the function \(g : V_\bar{m} \to [0, 1]\) defined by
\[
g(\bar{x}') := \int f(\bar{x}' + \bar{x}'')d\mu_\bar{m}'(\bar{x}'')
\]
(so \(g\) is \(\mathcal{B}\)-measurable by Fubini) satisfies \(\text{VC}_t(g) \leq \bar{D}\).

**Proof.** Since permuting the variables preserves finiteness of VC\(_t\)-dimension by Proposition \[10.9\], we only have to show that if \(\bar{m}' = \bar{m} + \bar{m}' + \bar{m}'' \in \mathbb{N}^k\) and \((t + 2)\)-ary function
\[
f'(\bar{x}_1, \ldots, \bar{x}_{t+1}, \bar{x}'') \in \left(\prod_{i \in [t+1]} V_{\bar{m}_i}\right) \times V_\bar{m}'' \to f(\bar{x}_1 + \ldots + \bar{x}_{t+1} + \bar{x}'')
\]
satisfies \(\text{VC}_t \leq \bar{d}\), then the \((t + 1)\)-ary function
\[
g'(\bar{x}_1, \ldots, \bar{x}_{t+1}) \in \prod_{i \in [t+1]} V_{\bar{m}_i} \to \int f(\bar{x}_1 + \ldots + \bar{x}_{t+1} + \bar{x}'')d\mu_\bar{m}''(\bar{x}'');
\]
satisfies \(\text{VC}_t \leq \bar{D}\).

We let \(V'_t := V_{\bar{m}''}\) for \(i \in [t + 1]\), \(V'_{t+2} := V_{\bar{m}'}\) and for \(n = (n_1, \ldots, n_{t+2}) \in \mathbb{N}^{t+2}\), we let \(\bar{n}' := n_1 \bar{m}_1 + \ldots + n_{t+1} \bar{m}_{t+1} + n_{t+2} \bar{m}'\) and \(\mathcal{B}'_{\bar{n}'} := \mathcal{B}_{\bar{n}'}\) as \((t + 2)\)-partite graded probability space and the \((t + 2)\)-ary function
\[
f''(\bar{x}_1, \ldots, \bar{x}_{t+1}, \bar{x}'') \in \prod_{i \in [t+2]} V'_i \to f(\bar{x}_1 + \ldots + \bar{x}_{t+1} + \bar{x}'')
\]
is \(\mathcal{B}'_{\bar{n}''}\)-measurable and satisfies \(\text{VC}_t(f'') \leq \bar{d}\). Then, applying Theorem \[10.7\] there exists some \(\bar{D} = \bar{D}(t, \bar{d})\) so that the function
\[
g''(\bar{x}_1, \ldots, \bar{x}_{t+1}) \in \prod_{i \in [t+1]} V'_i \to \int f(\bar{x}_1 + \ldots + \bar{x}_{t+1} + \bar{x}'')d\mu'_{\bar{n}''}(\bar{x}'')
\]
satisfies \(\text{VC}_t(g'') \leq \bar{D}\). Unwinding, this gives \(\text{VC}_t(g) \leq \bar{D}\). \(\square\)
11. Final remarks

11.1. Directions for future work. It would be interesting to obtain explicit bounds and investigate their optimality for the main results of the paper (Proposition 5.5 and Corollary 6.9).

Problem 11.1. It is possible to finitize our proof of Proposition 5.5, replacing the use of ultraproducts and indiscernible sequences by multiple applications of Ramsey’s theorem and complicated \( \varepsilon - \delta \) bookkeeping. We expect that the bound on \( N_0 \) should be as bad as in the regularity lemma for general hypergraphs (i.e. an exponential tower of height depending on \( \frac{1}{\varepsilon} \)), while we expect \( N \) to be bounded by an exponential tower of height bounded in terms of \( d \). We leave the investigation of these bounds for future work.

In Proposition 5.1 we show that every \( k \)-ary fiber of a \((k+1)\)-ary function of finite VC\( k \)-dimension can be approximated in \( L^2 \) in terms of a fixed finite set of its \( k \)-ary fibers along with smaller arity data. And in Lemma 5.9 we strengthen its conclusion from “there exists an approximation” to “there exists a positive measure set of approximations”. We ask if this can further be strengthened to “there exists a measure 1 set of approximations”:

Problem 11.2. Is it possible to strengthen the conclusion of Lemma 5.9 to “the set of tuples \( \bar{w} \in V^m \) with \( \|f_L - f_{\bar{w},x}\|_{L^2} \leq \delta \) has \( \mu_{\bar{m}} \)-measure converging to 1 when \( l, t \to \infty \)”?

This problem has a positive answer in the case of bounded VC-dimension (i.e. the case \( k = 1 \)) using that a sufficiently long tuple almost surely gives an \( \varepsilon \)-net for differences (see the discussion in the introduction), but for \( k > 2 \), we only know that we get a good choice with positive measure.

11.2. Some model-theoretic consequences. We record a couple of model theoretic corollaries of our results.

As we already mentioned, Theorem 10.7 generalizes [BY09, Corollary 4.2] in the case \( k = 1 \). Using it (and recalling that a first-order theory \( T \) is \( k \)-dependent if every \((k+1)\)-ary relation definable on tuples in a model of \( T \) has finite VC\( k \)-dimension), one immediately obtains the following model-theoretic corollary generalizing the main Theorem 5.3 there.

Corollary 11.3. Let \( T \) be a \( k \)-dependent first-order theory (classical or continuous). Then its Keisler randomization \( T^R \) is also \( k \)-dependent.

We also have the following application to Keisler measures, i.e. finitely additive probability measures on the space of types of a first-order theory. We refer to e.g. [Sta16] for a detailed discussion.

Corollary 11.4. Assume that \( T \) is \( k \)-dependent, \( k' \geq k+1 \), \( M \models T \) and let \( \mu_1, \ldots, \mu_{k'} \) be global Keisler measures on the definable subsets of the sorts \( M^{x_1}, \ldots, M^{x_{k'}} \) respectively, such that each \( \mu_i \) is Borel-definable and all these measures commute, i.e. \( \mu_i \otimes \mu_j \) for all \( i, j \in [k'] \). Then for every formula \( \varphi(x_1, \ldots, x_{k'}) \in \mathcal{L}(M) \) and \( \varepsilon \in \mathbb{R}_{>0} \) there exist some formula \( \psi(x_1, \ldots, x_{k'}) \)
which is a Boolean combination of finitely many \((\leq k)\)-ary formulas each given by an instances of \(\varphi\) with some parameters placed in all but at most \(k\) variables, so that taking \(\mu := \mu_1 \otimes \ldots \otimes \mu_{k'}\) we have \(\mu(\varphi \triangle \psi) < \varepsilon\).

Indeed, for \(\bar{n} = (n_1, \ldots, n_{k'}) \in \mathbb{N}^{k'}\) we let \(M_{\bar{n}}\) be the sort corresponding to \(\prod_{i \in [k']} (M_{x_i})^{n_i}\), \(B^0_{\bar{n}}\) the Boolean algebra of all definable subsets of \(M_{\bar{n}}\) and \(\mu_{n_1, \ldots, n_{k'}} := \mu_1^{\otimes n_1} \otimes \ldots \otimes \mu_{k'}^{\otimes n_{k'}}\). Each Boolean algebra \(B^0_{\bar{n}}\) can be viewed as a Boolean algebra of the clopen subsets of the corresponding space of types \(V_{\bar{n}} := S_{\bar{n}}(M)\), and \(\mu_{\bar{n}}\) as a finitely additive probability measure on it. By Carathéodory’s theorem, it extends uniquely to a regular countably additive probability measure \(\mu'_{\bar{n}}\) on the \(\sigma\)-algebra \(B_{\bar{n}}\) of all Borel subsets of this space. Then we have that \((V_{k'}, B_{\bar{n}}, \mu'_{\bar{n}})\) \(\bar{n} \in \mathbb{N}^{k'}\) is a \(k'\)-partite graded probability space. Indeed, the assumption of pairwise commuting on the \(\mu_i\)’s implies

\[
\left( \mu_1^{\otimes (n_1+m_1)} \otimes \ldots \otimes \mu_{k'}^{\otimes (n_{k'}+m_{k'})} \right) = \\
\left( \mu_1^{\otimes n_1} \otimes \ldots \otimes \mu_{k'}^{\otimes n_{k'}} \right) \otimes \left( \mu_1^{\otimes m_1} \otimes \ldots \otimes \mu_{k'}^{\otimes m_{k'}} \right),
\]

which together with Borel definability imply the Fubini property in Definition 2.1 and the other conditions in the definition are clearly satisfied. Now we apply Corollary 6.10 to \(\varphi\) viewed as a clopen subset in \(B^0_{1^{k'}}\), and approximating Borel sets in the resulting decomposition by the clopen ones from the generating set, we obtain the corollary.

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