Instantaneous third post-Newtonian accurate expressions for the radiated energy and angular momentum during hyperbolic encounters of non-spinning compact objects

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We compute the third post-Newtonian (3PN) accurate instantaneous contributions to the radiated gravitational wave (GW) energy and angular momentum arising from the hyperbolic passages of non-spinning compact objects. The present computations employ 3PN-accurate instantaneous contributions to the far-zone energy and angular momentum fluxes and the 3PN-accurate Keplerian type parametric solution for compact binaries in hyperbolic orbits.

I. INTRODUCTION

The routine detection of transient GW events that arise from merging black hole (BH) binaries in bound orbits has inaugurated the era of GW astronomy [1–3]. Further, observations of a neutron star binary coalescence in GWs, and many electromagnetic frequency windows have provided a peak into the benefits of the multi-messenger GW astronomy [4–6]. In contrast, compact binaries in unbound orbits can provide transient GW burst events in the LIGO,LISA and IPTA GW frequency windows [7–10]. Interestingly, GW burst events due to hyperbolic encounters of neutron stars may even be accompanied by electromagnetic flares [11]. Therefore, there are ongoing PN efforts to characterize both the dynamics and associated GW emission aspects of compact binaries in hyperbolic orbits in general relativity [12–17].

The present effort extends the classic computations of Schäfer and his collaborators who computed the radiated energy(ΔE) and angular momentum (ΔJ) during hyperbolic encounters of non-spinning compact objects to 1PN order [18, 19]. Recall that PN approximation allows us to write, for example, the orbital dynamics of non-spinning compact binaries as corrections to Newtonian equations of motion in powers of \( (v/c)^2 \sim GM/(c^2 r) \), where \( v, M, \) and \( r \) are the velocity, total mass and relative separation of the binary [20]. Note that the expressions, available in Refs. [18, 19], provided the next-to-leading order (1PN) contributions to ΔE and ΔJ, influenced by Refs. [21, 22]. The present computation provides 3PN-accurate ‘instantaneous’ contributions to ΔE and ΔJ with the help of Refs. [12, 23, 24] in the modified Harmonic gauge. It turned out that at PN orders beyond the 1PN, the radiative moments and the resulting far-zone fluxes have two distinct contributions [20]. One part of the radiative moments and their fluxes depends only at the usual retarded time and it is customary to refer these terms as the “instantaneous contributions”. In contrast, the second part depends on the dynamics of compact binary in its entire past and therefore these contributions are usually termed as the “hereditary/tail contributions”. In this paper, we focus our efforts on the instantaneous contributions.

The present computations are not a straightforward extensions to 3PN order of what are done in Refs. [18, 19]. This is mainly because of logarithmic terms that appear at the 3PN corrections to far-zone energy and angular momentum fluxes associated with compact binaries in non-circular orbits [23, 24]. We provide a prescription to compute these logarithmic integrals in terms of Clausen function of order two [25].

The manuscript is structured in the following way. The way of our PN-accurate computations and underlying formalism to obtain ΔE and ΔJ and the results are presented in Sec. II. And also we briefly present parabolic limit and the implications of the bremsstrahlung limit. The detail of our computations are presented in the Appendices A, and our PN-accurate expressions in terms of the Newtonian eccentricity is given in B while Sec. III provides a brief summary and on-going investigations.

II. 3PN ACCURATE INSTANTANEOUS CONTRIBUTIONS TO ΔE AND ΔJ

We begin by detailing our approach to compute ΔE_{inst} to 3PN order, influenced by Ref. [18]. The way we tackled the logarithmic terms that appear at the 3PN order is provided in the Appendix A. These computations are repeated to obtain similar PN contributions to the radiated angular momentum in Sec. II C. Thereafter, we explain why our instantaneous contributions to the radiated energy and angular momentum during hyperbolic encounters are exact up to 3PN order and explore their limiting cases.

A. Newtonian order ΔE and ΔJ Computations

We begin by explaining the procedure of Ref. [18] for computing the radiated energy and angular momentum in GWs during hyperbolic encounters at the Newtonian order. The natural starting point is the familiar Newtonian(leading) order far-zone GW energy flux \( J_N \) for
compact binaries in generic orbits \cite{18}

$$\mathcal{F}_{\text{N}} = \frac{8}{15} \frac{G^3 M^2 \mu^2}{c^5 r^4} \left(12 v^2 - 11 r^2 \right). \quad (1)$$

The approach of Ref. \cite{18} requires us to employ the standard Keplerian parametric solution for compact binaries in Newtonian hyperbolic orbits, available in Refs. 26, 27. This is for expressing the total orbital velocity \(v\), the radial velocity \(\dot{r}\) and the radial separation in terms of various elements of the Keplerian parametric solution for hyperbolic orbits. The underlying parametric solution for hyperbolic orbits reads

\[
\begin{align*}
    r &= a \left( e \cosh u - 1 \right) \quad (2a) \\
    \phi - \phi_0 &= 2 \tan^{-1} \left( \sqrt{\frac{e + 1}{e - 1}} \tanh \frac{u}{2} \right) \quad (2b) \\
    n \left(t - t_0\right) &= e \sinh u - u, \quad (2c)
\end{align*}
\]

where \(r\), \(\phi\) and \(t\) are radial orbital separation, the angular variable of the reduced mass \(\mu\) around the total mass \(M\) and coordinate time, respectively. Further, the eccentric anomaly parameter \(u\) has the range \(-\infty < u < \infty\) while \(\phi_0\) and \(t_0\) denote some initial value of \(\phi\) and \(t\). The familiar Newtonian semi-major axis \(a\), orbital eccentricity \(e\) and the mean motion \(n\) are given in terms of the conserved reduced energy \(E\) and angular momentum \(h\)

\[
\begin{align*}
    a &= \frac{GM}{2E} \\
    e^2 &= 1 + 2Eh^2 \\
    n &= \frac{(2E)^{3/2}}{GM},
\end{align*}
\]

where the reduced energy \(E = \frac{\varepsilon}{\mu}\) and the reduced angular momentum \(h = \frac{N}{\mu GM}\). It is fairly straightforward to express \(\dot{r}^2, v^2\) in terms of \(E, h^2, (e \cosh u - 1)\) with the help of

\[
\begin{align*}
    \dot{r}^2 &= \left( \frac{dr}{du} \right)^2 \quad (3a) \\
    \dot{\phi}^2 &= \left( \frac{d\phi}{du} \right)^2 \quad (3b) \\
    v^2 &= \dot{r}^2 + r^2 \dot{\phi}^2. \quad (3c)
\end{align*}
\]

This leads to

\[
\begin{align*}
    \dot{r}^2 &= \left(1 + \frac{2}{e \cosh u - 1} - \frac{(2Eh^2)^2}{(e \cosh u - 1)^2}\right)(2E), \quad (4a) \\
    v^2 &= \left(1 + \frac{2}{e \cosh u - 1}\right)(2E). \quad (4b)
\end{align*}
\]

Using Eqs. (4), we can express the instantaneous energy flux as a polynomial in \((1 - e \cosh u)^{-1}\) and the final expression reads

\[
\begin{align*}
    \mathcal{F}_{\text{N}} &= \frac{1}{n} \frac{du}{dt} \sum_{N=3}^{5} \frac{\tilde{\alpha}_N(E, h)}{(e \cosh u - 1)^N}, \quad (5) \\
    \tilde{\alpha}_N(e) &= \frac{\eta^2}{G} \left( \frac{E}{c} \right)^5 \alpha_N(E, h). \quad (6)
\end{align*}
\]

In the above equation \(\eta\) is defined as dimensionless mass parameter of the binary, namely \(\eta = m_1 m_2 / (m_1 + m_2)^2\), where \(m_1\) and \(m_2\) are the masses of the binary configuration. The coefficients \(\alpha_N(E, h)\) at the Newtonian order are given by

\[
\begin{align*}
    \alpha_3 &= \frac{256}{15} \\
    \alpha_4 &= \frac{512}{15} \\
    \alpha_5 &= \frac{5632}{15} \frac{Eh^2}{c^2}.
\end{align*}
\]

The fact that we have parametrized the far-zone Newtonian energy flux in terms of Newtonian hyperbolic orbital description allows us to write the total radiated energy in GWs at the Newtonian order as

\[
\Delta \mathcal{E}_{\text{N}} = \int_{-\infty}^{\infty} \mathcal{F}_{\text{N}} \, dt,
\]

\[
= \frac{1}{n} \sum_{N=3}^{5} \tilde{\alpha}_N(E, h) \int_{-\infty}^{\infty} \frac{du}{(e_i \cosh u - 1)^N}. \quad (8)
\]

Clearly, we can easily obtain the desired expression for the quadrupolar order \(\Delta \mathcal{E}\) during hyperbolic encounters if we can compute the three integrals that appear on the right-hand side of Eq. (8). With the help of Refs. 18, 22, we find

\[
\int_{-\infty}^{\infty} \frac{du}{(e \cosh u - x)^N} = \frac{2}{(N-1)!} \left[ \left( \frac{d}{dx} \right)^{N-1} \left( \frac{1}{\sqrt{x^2 - e^2}} \arccos \left( -\frac{x}{e} \right) \right) \right]. \quad (9)
\]

This leads to

\[
\Delta \mathcal{E}_{\text{N}} = \frac{2M}{15e^3 h^2} \left[ \sqrt{e^2 - 1} \left( 602 \frac{673e^2}{3} + \right) + \left( 96 + 292e^2 + 37e^4 \right) \arccos \left( -\frac{1}{e} \right) \right], \quad (10a)
\]

where we have used the following Newtonian accurate relation that connects \(E\) to \(h\) and \(e\), namely \(E = \frac{e^2 - 1}{2h}\), to obtain the above result. Indeed, our expression is fully consistent with Eq. (44) of Ref. 19.

We now move onto explain briefly how Ref. 19 computed the quadrupolar order contributions to the radiated angular momentum during hyperbolic encounters of
non-spinning compact objects. We begin by the Newtonian order angular momentum flux $\mathcal{G}_N$ for compact binaries in generic orbits \cite{18}:

\[
\mathcal{G}_N = \frac{8}{5} \mathbf{L}_N \cdot \mathbf{e} \left\{ \frac{2v^2 - 3f^2 + 2 \frac{GM}{r}}{c^5 r^5} \right\}
\]

where $\mathbf{L}_N = \mathbf{r} \times \mathbf{v}$ stands for the scaled Newtonian angular momentum vector. The fact that the orbital angular momentum vector remains a constant as we consider only non-spinning compact binaries allows us to compute an expression for the angular momentum flux $dJ/dt$ from the above equation. Thereafter, we pursue the steps involved in the $\Delta E$ computations and this leads to

\[
\mathcal{G}_N = \frac{1}{n} \frac{du}{dt} \sum_{N=2}^{4} A_N(E, h) \left\{ (e \cosh u - 1)^N \right\},
\]

\[
\tilde{A}_N(e) = \frac{M h E^4 \eta^2}{c^5} A_N(E, h).
\]

The three constant coefficients are given by

\[
A_2 = -\frac{128}{5},
\]

\[
A_3 = 0,
\]

\[
A_4 = \frac{768}{5} E h^2.
\]

The radiated angular momentum $\Delta J_N$ at the Newtonian order becomes

\[
\Delta J_N = \int_{-\infty}^{\infty} \mathcal{G}_N dt,
\]

\[
= \frac{1}{n} \sum_{N=2}^{4} \tilde{A}_N(E, h) \int_{-\infty}^{\infty} \frac{du}{(e \cosh u - 1)^N}.
\]

Clearly, these integrals are similar to those we tackled earlier and this eventually leads to

\[
\Delta J_N = \frac{8}{5} \frac{G M^2 \eta^2}{c^5 h^4} \left\{ (13 + 2 e^2) \sqrt{e^2 - 1} + \frac{1}{e} \right\}.
\]

We have verified that the above expression is fully consistent with Eqs. (42) of Ref. \cite{19}. We now move on to extend these calculations to 3PN order while focusing on the instantaneous contributions.

B. 3PN-accurate instantaneous contributions to the radiated energy

It should be obvious that we require two crucial ingredients for our 3PN-accurate $\Delta E$ computation. The first ingredient is the 3PN accurate ‘instantaneous’ contributions to the far-zone GW energy flux from non-spinning compact binaries in non-circular orbits \cite{20}. These instantaneous contributions depend only on the state of the binary at the usual retarded time and they appear usually at the Newtonian, 1PN, 2PN, 2.5PN and 3PN orders. In contrast, the hereditary contributions, as the name suggests, are sensitive to the binary dynamics at all epochs prior to the usual retarded time and appear at 1.5PN (relative) order for the first time \cite{20}. In this paper, we focus our efforts on the 3PN-accurate ‘instantaneous’ contributions to the far-zone fluxes, as given by Eqs. (5.2) in Ref. \cite{23} in the Modified harmonic (MH) coordinates. A close inspection of these contributions reveal that $\Delta E$ computation at 3PN order will be demanding due to the presence of certain ‘logarithmic’ terms, as evident from Eq. (5.2e) of Ref. \cite{23}. The second ingredient for the present computation is the 3PN-accurate generalized quasi-Keplerian parametric solution for compact binaries in hyperbolic orbits, derived in Ref. \cite{12}.

Note that Ref. \cite{12} provided a parametric way to track 3PN-accurate conservative trajectory of compact binaries in hyperbolic orbits. This effort extended the 1PN-accurate derivation of Keplerian type parametric solution for hyperbolic motion that employed the arguments of analytic continuation \cite{26}. At the 3PN order, the radial motion $r(t)$ is conveniently parametrized as

\[
r(t - t_0) = a_r (e_r \cosh u - 1) + \left( \frac{f_4}{e^4} + \frac{f_6}{e^6} \right) \nu + \left( \frac{g_4}{e^4} + \frac{g_6}{e^6} \right) \sin \nu + \frac{h_6}{e^6} \sin 2\nu + \frac{i_6}{e^6} \sin 3\nu,
\]

where $u$ is the eccentric anomaly while $a_r$, $e_r$, $e_i$, $n$ and $t_0$ are certain PN-accurate semi-major axis, radial eccentricity, time eccentricity, mean motion, and initial epoch, respectively. In addition, we have several orbital functions like $g_4, g_6, f_4, f_6, i_6, h_6$ and $h_6$ that appear at 2PN and 3PN orders. Further, the angular motion is described by

\[
\phi - \phi_0 = (1 + k) \left\{ \nu + \left( \frac{f_4 \phi}{e^4} + \frac{f_6 \phi}{e^6} \right) \sin 2\nu + \left( \frac{g_4 \phi}{e^4} + \frac{g_6 \phi}{e^6} \right) \sin 3\nu + \frac{h_6 \phi}{e^6} \sin 4\nu + \frac{i_6 \phi}{e^6} \sin 5\nu \right\}
\]

where

\[
\nu = 2 \tan^{-1} \left( \sqrt{\frac{e_\phi + 1}{e_\phi - 1} \tan \frac{u}{2}} \right)
\]

In the above expressions, $k$ and $e_\phi$ denote PN-accurate rate of periastron advance, and certain angular eccentricity, respectively. Additionally, we have several orbital functions like $f_4 \phi, f_6 \phi, g_4 \phi, g_6 \phi, h_6 \phi$ and $h_6 \phi$ that appear at 2PN and 3PN orders. We note that Ref. \cite{12}
provided 3PN-accurate expressions for various orbital elements and functions in terms of the scaled conserved orbital energy $E$ and the scaled angular momentum $h$.

It is straightforward but tedious to compute 3PN-accurate expressions for $r^2$, $v^2$ in terms of $E$, $h$, $(e_t \cosh u - 1)$. These dynamical variables that appear in the far-zone energy flux is computed by employing the following relations

$$\frac{dt}{du} = \frac{\partial t}{\partial u} + \frac{\partial t}{\partial v} \frac{dv}{du}, \quad (19a)$$

$$r^2 = \left( \frac{dr}{dt} \right)^2, \quad (19b)$$

In what follows, we display 1PN-accurate parametric expressions for $r^2$, $v^2$ and $GM/r$ for introducing the reader to the structure of these expressions

$$\dot{r}_\text{MH}^2 = \left\{ 1 + \frac{2}{(e_t \cosh u - 1)} - \frac{(2Eh^2)}{(e_t \cosh u - 1)^2} \right\} (2E) + \frac{(2E)^2}{c^2} \left\{ - \frac{3}{4} + \frac{9\eta}{4} + \frac{1}{(e_t \cosh u - 1)^2} \right\} - 2 + (2Eh^2)$$

$$\dot{v}_\text{MH}^2 = \left\{ 1 + \frac{2}{(e_t \cosh u - 1)} \right\} (2E) + \frac{(2E)^2}{c^2} \left\{ - \frac{3}{4} + \frac{9\eta}{4} + \frac{(2Eh^2)\eta}{(e_t \cosh u - 1)^3} + \frac{2(\eta - 1)}{(e_t \cosh u - 1)^2} \right\}$$

$$\frac{GM}{r} = \frac{2E}{(e_t \cosh u - 1)} + \frac{(2E)^2}{c^2} \left\{ \frac{1}{1 - e_t \cosh u} \right\} \left\{ \frac{9}{4} - \frac{5\eta}{4} \right\} + \frac{1}{(1 - e_t \cosh u)^2} \left\{ 4 - \frac{3}{2} \eta \right\}$$

The explicit 3PN-accurate expressions for these dynamical variables are provided at https://github.com/subhajittifr/hyperbolic_flux.

We are now in a position to replace dynamical variables $r$, $\dot{r}$ and $\dot{\phi}$ that appear in the 3PN-accurate instantaneous far-zone energy flux expression, given by Eqs. (5.2) of Ref. [23], with the 3PN-accurate version of the above equations. The associated 3PN-accurate expression for the radiated energy during hyperbolic encounters reads

$$\Delta E = \int_{-\infty}^{+\infty} dt \mathcal{F} = \int_{-\infty}^{+\infty} du \left( \frac{dt}{du} \right) \mathcal{F}. \quad (21)$$

The use of 3PN-accurate expressions for $r$, $\dot{r}$ and $\dot{\phi}$ in terms of $E$, $h$, and $(1 - e_t \cosh u)$, as evident from our Eqs. (20), in the Eqs. (5.2) of Ref. [23] leads to

$$\mathcal{F} = \frac{1}{n^2} \int \left\{ \sum_{N=3}^{11} \frac{\bar{\alpha}_N(E, h)}{(e_t \cosh u - 1)^N} \right. $$

$$+ \sum_{N=5}^{9} \frac{\bar{\beta}_N(E, h)}{(e_t \cosh u - 1)^N} \sinh u $$

$$+ \sum_{N=5}^{9} \frac{\bar{\gamma}_N(E, h)}{(e_t \cosh u - 1)^N} \ln(e_t \cosh u - 1) \right\}, \quad (22)$$

where we may write in a compact manner the above constant coefficients as

$$W_N(E, h) = \frac{\eta^2}{G} \left( \frac{E}{c} \right)^5 W_N(E, h),$$

where $W_N$ stands for $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$. We do not list all these lengthy coefficients in the manuscript and they are provided in an ancillary Mathematica file at (https://github.com/subhajittifr/hyperbolic_flux).

However, we list below one of them to show the typical structure of these coefficients

$$\alpha_3(E, h) = \frac{256}{15} + \frac{E}{c^2} \left( \frac{13184}{105} - \frac{128\eta}{21} \right) + \frac{E^2}{c^4} \left( \frac{51328}{63} \right) + \frac{5504\eta}{315} + \frac{640\eta^2}{63} + \frac{E^3}{c^6} \left( \frac{5451968}{1155} + \frac{1803328\eta}{3465} \right) + \frac{181792\eta^2}{3465} - \frac{8000\eta^3}{693}\right) \quad (23)$$

We note that all $\bar{\beta}_N$ terms appear at the 2.5PN order while $\bar{\gamma}_N$ terms are accompanied by the logarithmic terms.
at the 3PN order. Additionally, we have incorporated the dependence on the constant \( r_0 \) into the coefficients \( \bar{\alpha}_N \). Recall that \( r_0 \) is the gauge dependent length scale appearing in the definition of source multiple moments \([28]\) as discussed in Ref. \([23]\). We have indeed verified that the 1PN-accurate version of these coefficients match with Ref. \([18]\).

We now move to tackle how these coefficients contribute to the instantaneous 3PN-accurate expression for the radiated energy during hyperbolic encounters. It is fairly straightforward to infer that all the 2.5PN terms, namely \( \bar{\beta}_N \) terms, do not contribute to \( \Delta \mathcal{E} \) expression and this is because

\[
\int_{-\infty}^{\infty} \frac{\sinh u}{(e_t \cosh u - 1)^N} \, du = 0. \tag{24}
\]

This is obviously due to the fact that these integrands are an odd function in \( u \). The integration of \( \bar{\alpha}_N \) terms should be straightforward as they are essentially similar to the terms that we confronted at the Newtonian order. Therefore, we employ Eq. (9) to compute \( \Delta \mathcal{E} \) contributions that arise from nine \( \bar{\alpha}_N \) terms in Eq. (22).

Clearly, it is very tricky to tackle the logarithmic terms and we pursued an entirely new line of investigations compared to what was done in Ref. \([23]\) for the eccentric orbits. After some detailed efforts, we were able to obtain analytic expressions for these integrals though the final expressions were too lengthy to list here. However, we were eventually able to obtain the most simplified form of these integrals with the help of Clausen identity as detailed in the Appendix A. This allowed us to tackle the \( \bar{\gamma}_N \) contributions to the 3PN-accurate \( \Delta \mathcal{E} \) expression as

\[
\int_{-\infty}^{\infty} \frac{\ln(e \cosh u - x)}{e \cosh u - x} \, du = \frac{2}{\sqrt{e^2 - x^2}} \int_{-\infty}^{\infty} \left[ \text{Cl}_2 \left( 2 \arccos \left( -\frac{x}{e} \right) \right) + \arccos \left( -\frac{x}{e} \right) \right] \ln \left( \frac{2(e^2 - x^2)}{e^2} \right), \tag{25}
\]

where \( \text{Cl}_2(x) \) is the Clausen function of order 2 given by the integral

\[
\text{Cl}_2(x) = -\int_{0}^{x} dy \ln \left| 2 \sin \frac{y}{2} \right|. \tag{26}
\]

Interestingly, the Clausen function admits the following Fourier series representation:

\[
\text{Cl}_2(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k^2}. \tag{27}
\]

The crucial integrals that are associated with the \( \bar{\gamma}_N \) coefficients in Eq. (22) can now tackled by noting that the integrals

\[
\int_{-\infty}^{\infty} \frac{\ln(e \cosh u - 1)}{(e \cosh u - 1)^N} \, du,
\]

can be computed after taking successive derivatives of Eq. (25) with respect to \( x \) and then finally taking \( x \to 1 \).

With these inputs, we compute Eq. (21) for the 3PN-accurate instantaneous contributions to \( \Delta \mathcal{E} \) and the final expression may be written as

\[
\Delta \mathcal{E} = \frac{2}{15} \frac{M \eta^2}{h^2 c^6} \left( I_{\text{I}}^{\text{MH}} + \frac{1}{c^2 h^2} I_{\text{I}}^{\text{1PN}} + \frac{1}{c^4 h^2} I_{\text{I}}^{\text{2PN}} + \frac{1}{c^6 h^6} I_{\text{I}}^{\text{3PN}} \right), \tag{28}
\]

where contributions to \( \Delta \mathcal{E} \) that appear at the Newtonian, 1PN, 2PN and 3PN orders read

\[
I_{\text{I}}^{\text{MH}} = \sqrt{e_t^2 - 1} \left[ \frac{602}{3} + \frac{673 e_t^2}{3} \right] + \arccos \left( -\frac{1}{e_t} \right) \left[ 96 + 292 e_t^2 + 37 e_t^4 \right], \tag{29a}
\]

\[
I_{\text{I}}^{\text{1PN}} = \sqrt{e_t^2 - 1} \left[ \frac{153263}{70} - \frac{1547 e_t^2}{3} + e_t^2 \left( \frac{271849}{70} - \frac{13799 \eta}{6} \right) + e_t^4 \left( -\frac{288513}{280} - 2 \eta \right) \right] \tag{29b}
\]

\[
+ \arccos \left( -\frac{1}{e_t} \right) \left[ \frac{6578}{7} - 168 \eta + e_t^2 \left( \frac{31013}{7} - 1982 \eta \right) + e_t^4 \left( -\frac{223}{4} - \frac{1483 \eta}{2} \right) + e_t^6 \left( -\frac{15219}{56} + 74 \eta \right) \right],
\]

\[
I_{\text{I}}^{\text{2PN}} = \sqrt{e_t^2 - 1} \left[ \frac{405300022}{19845} - \frac{2947852 \eta}{315} + \frac{1173 \eta^2}{4} + e_t^2 \left( \frac{6673495637}{158760} - \frac{114248429 \eta}{2520} + \frac{66217 \eta^2}{8} \right) \right] \tag{29c}
\]

\[
+ e_t^4 \left( -\frac{3823800817}{158760} + \frac{41499527 \eta}{5040} + \frac{3619 \eta^2}{2} + e_t^6 \left( \frac{39802111}{7840} - \frac{67328 \eta}{35} - 103 \eta^2 \right) \right]
\]
+ \arccos \left( \frac{1}{c_t} \right) \left[ \frac{1636769}{189} - \frac{74435 \eta}{21} + 48 \eta^2 + e_t^2 \left( \frac{863156}{189} - \frac{762901 \eta}{21} + \frac{9851 \eta^2}{2} \right) \right]
+ e_t^4 \left( -\frac{554104}{63} - \frac{350943 \eta}{28} + \frac{48063 \eta^2}{8} \right) + e_t^6 \left( -1324649 + 611613 \eta - 1779 \eta^2 \right)
+ e_t^8 \left( \frac{1224929}{672} - \frac{10070 \eta}{6} + 185 \eta^2 \right) \right],
(29c)

\mathcal{T}_{3\text{PN}}^{\text{MH}} = \sqrt{e_t^2 - 1} \left[ \frac{6713608}{1575} + \frac{17868572 e_t^2}{525} + \frac{19300553 e_t^4}{3150} + \frac{17525209 e_t^6}{3150} \right] \log \left[ \frac{e_t^2 - 1}{E_0 h^2 e_t} \right] - \left[ \frac{54784}{35} + \frac{465664 e_t^2}{21} \right]
+ \frac{442637 e_t^4}{105} + \frac{1498956 e_t^6}{105} + \frac{31779 e_t^8}{70} \right] \left\{ \log \left[ \frac{4 E_0 h^2}{e_t} \right] \arccos \left( \frac{1}{e_t} \right) + \text{Cl}_2 \left[ 2 \arccos \left( \frac{1}{e_t} \right) \right] \right\}
+ \sqrt{e_t^2 - 1} \left[ \frac{1959816183329}{8731800} + \left( -\frac{69906810223}{317520} + \frac{11632643 \pi^2}{3360} \right) \eta + \frac{910459 \eta^2}{560} + \frac{689 \eta^3}{24} \right]
+ e_t^2 \left( \frac{11238026145523}{17463600} - \frac{264708911281}{317520} + \frac{998227 \pi^2}{140} \eta + \frac{518874833 \eta^2}{2880} - \frac{64148 \eta^3}{48} \right)
+ e_t^4 \left( -\frac{10459843311391}{139708800} + \frac{28976695225}{254016} + \frac{99671 \pi^2}{896} \eta + \frac{143029027 \eta^2}{2880} - \frac{16908 \eta^3}{24} \right) + e_t^6 \left( \frac{135148514527}{739200} \right)
+ \left( -\frac{15436846447}{211680} - \frac{12303239 \pi^2}{13440} \right) \eta + \frac{12729151 \eta^2}{5040} + \frac{8501 \eta^3}{3} + e_t^8 \left( \frac{112472361473}{4139520} \right)
+ \frac{244676087 \eta}{8820} - \frac{432849 \eta^2}{140} + \frac{1147 \eta^3}{3} \right] + \arccos \left( \frac{1}{e_t} \right) \left[ \frac{20510192533}{207900} + \left( -\frac{69631105}{756} + \frac{1599 \pi^2}{4} \right) \eta \right]
+ \frac{145195 \eta^2}{28} + 9 \eta^3 + e_t^2 \left( \frac{269134761733}{415800} + \left( -376708039 \right) \eta + \frac{556378 \pi^2}{8} \right) + e_t^4 \left( \frac{21177007 \eta^2}{168} - \frac{2228 \eta^3}{4} \right)
+ \frac{125593677691}{554400} + \left( -2952969469 + \frac{172405 \pi^2}{64} \right) \eta + \frac{112045205 \eta^2}{672} - \frac{38125 \eta^3}{16} \right) + e_t^6 \left( \frac{115347955537}{1108800} \right)
+ \left( \frac{72514474763}{60480} - \frac{174619 \pi^2}{128} \right) \eta - \frac{284351 \eta^2}{64} + 3155 \eta^3 \right) + e_t^8 \left( \frac{2256512667}{492800} + \left( -\frac{37470739}{672} - \frac{12177 \pi^2}{128} \right) \eta \right]
+ \frac{268673 \eta^2}{112} - 2075 \eta^3 \right) + e_t^{10} \left( \frac{484326439}{39424} + \frac{502175 \eta}{28} - \frac{25061 \eta^2}{4} + 518 \eta^3 \right) \right].
(29d)

We have found the full agreement with Eq.(1.4) of Ref. [29] up to 3PN($v^7$) order in bremsstrahlung limit.

For obtaining the above version of the final results, we have used the following 3PN accurate relation that connects $E$ to $h$ as the integrands of Eqs. (22) involve both $E$ and $h$.

\[ E = \frac{e_t^2 - 1}{2 h^2} + \frac{1}{c_t^4 h^4} \left[ -\frac{9}{8} + e_t^2 \left( \frac{13}{4} - \frac{3 \eta}{4} \right) + e_t^4 \left( -\frac{17}{8} + \frac{7 \eta}{8} \right) - \frac{\eta}{8} \right] + \frac{1}{c_t^4 h^6} \left[ -\frac{81}{16} + \frac{7 \eta}{16} - \frac{\eta^2}{16} + e_t^4 \left( -\frac{339}{16} \right) \right]
+ \frac{213 \eta}{16} - \frac{35 \eta^2}{16} \right) + e_t^4 \left( \frac{243}{16} - \frac{29 \eta}{16} + \frac{3 \eta^2}{16} \right) + e_t^6 \left( \frac{177}{16} - \frac{191 \eta}{16} + \frac{33 \eta^2}{16} \right) + \frac{1}{c_t^6 h^8} \left[ -\frac{3861}{128} + \frac{8833}{384} \right]
- \frac{41 \pi^2}{64} \right) \eta + 5 \eta^2 \right. + 5 \eta^3 \right] + e_t^6 \left( \frac{4641}{32} - \frac{19037 \eta}{96} + \frac{1119 \eta^2}{16} - \frac{231 \eta^3}{32} \right) + e_t^8 \left( \frac{2105}{32} + \frac{157553}{3360} - \frac{41 \pi^2}{32} \right) \eta \]
eccentricity parameter as noted in Ref. [19]. Therefore, we provide 3PN accurate relation that connects
gauge.

\[
\left( e^2 \right)_{\text{3PN}} = 1 + \frac{\eta}{6} + e^4 \left( -4 + \frac{17 \eta}{6} \right) \rho^2 + \frac{17}{4} - \frac{61 \eta}{24} - \frac{\eta^2}{8} + e^4 \left( -\frac{49}{4} + \frac{307 \eta}{24} - \frac{95 \eta^2}{24} \right)
\]

\[
+ e^4 \left( 8 + \frac{23 \eta}{4} - \frac{5 \eta^2}{12} \right) \rho^2 + \frac{23}{4} + \frac{1375}{48} - \frac{41 \eta^2}{64} \eta - \frac{2213 \eta^2}{72} + \frac{\eta^3}{12} + e^4 \left( -\frac{125}{2} + \frac{141199}{1680} \right)
\]

\[
- \frac{123 \eta^2}{64} \eta + \frac{607 \eta^2}{24} + \frac{5 \eta^3}{4} + e^4 \left( \frac{333}{4} - \frac{557 \eta}{16} - \frac{595 \eta^2}{24} + \frac{95 \eta^3}{12} \right) + e^6 \left( -\frac{271}{3} + \frac{733 \eta}{48} - \frac{639 \eta^2}{72} \right)
\]

\[
+ \frac{217 \eta^3}{12} \right) \rho^3 \right),
\]

with the gauge dependent term \( E_0 = \frac{GM}{r_0} \). We have verified that the above expression is fully consistent with Eqs. (C9) of Ref. [14] up-to 2PN order which required us to express \( e_t \) and \( h \) in terms of \( e_r \) and \( a_r \) to 2PN order in the MH
gauge.

We note that it is customary to characterize hyperbolic encounters with the help of an impact parameter \( b \) and an
eccentricity parameter as noted in Ref. [19]. Therefore, we provide 3PN accurate relation that connects \( h \) to \( b \) and \( e_t \)
with the help of Ref. [12].

\[
h^2 = \frac{b}{GM} \sqrt{e_t^2 - 1} \left\{ 1 + \left[ 1 + \frac{\eta}{6} + e^4 \left( -4 + \frac{17 \eta}{6} \right) \rho^2 + \frac{17}{4} - \frac{61 \eta}{24} - \frac{\eta^2}{8} + e^4 \left( -\frac{49}{4} + \frac{307 \eta}{24} - \frac{95 \eta^2}{24} \right) \right]
\]

\[
+ e^4 \left( 8 + \frac{23 \eta}{4} - \frac{5 \eta^2}{12} \right) \rho^2 + \frac{23}{4} + \frac{1375}{48} - \frac{41 \eta^2}{64} \eta - \frac{2213 \eta^2}{72} + \frac{\eta^3}{12} + e^4 \left( -\frac{125}{2} + \frac{141199}{1680} \right)
\]

\[
- \frac{123 \eta^2}{64} \eta + \frac{607 \eta^2}{24} + \frac{5 \eta^3}{4} + e^4 \left( \frac{333}{4} - \frac{557 \eta}{16} - \frac{595 \eta^2}{24} + \frac{95 \eta^3}{12} \right) + e^6 \left( -\frac{271}{3} + \frac{733 \eta}{48} - \frac{639 \eta^2}{72} \right)
\]

\[
+ \frac{217 \eta^3}{12} \right) \rho^3 \right),
\]

where \( \rho = \frac{1}{\sqrt{e_t^2 - 1}} \frac{GM}{b^2} \). We now move on the briefly list our approach to compute PN-accurate the radiated angular
momentum during hyperbolic encounters that extends the 1PN-accurate effort of Ref. [14, 19].

C. 3PN-accurate instantaneous contributions to the radiated angular momentum

The crucial input that is required for our \( \Delta J \) computation is the 3PN-accurate instantaneous contributions to the far-zone GW angular momentum flux from compact binaries in non-circular orbits, given by Eqs. (3.4) in Ref. [24] and therefore in the MH gauge. The
dynamical variables that appear in these PN-contributions that includes \( |\mathbf{L}_N| \) are expressed in terms of \( E, h \) and \( (e_t \cosh u - 1) \) to 3PN order with the help of Ref. [12]. The resulting 3PN extension of Eq. (12) that provides 3PN-accurate instantaneous contributions to the scalar far-zone GW angular momentum flux may be written as

\[
\Delta J = \frac{1}{n} \frac{du}{dt} \left\{ \sum_{N=2}^{10} \frac{\bar{A}_N(E,h)}{(e_t \cosh u - 1)^N} \right\}
\]

\[
\Delta J = \frac{8}{5} \frac{GM^2 \eta^2}{h^4} \left( \mathcal{H}_N^\text{MH} + \frac{1}{c^2 \hbar^2} \mathcal{H}_1^\text{PN} + \frac{1}{c^3 \hbar^4} \mathcal{H}_2^\text{PN} + \frac{1}{c^6 \hbar^6} \mathcal{H}_3^\text{PN} \right),
\]

where the individual contributions that appear at Newtonian, 1PN, 2PN and 3PN orders are given by

\[
\mathcal{H}_N^\text{MH} = \sqrt{e_t^2 - 1} \left( 13 + 2e_t^2 \right) + \arccos \left( -\frac{1}{e_t} \right) \left( 8 + 7e_t^2 \right),
\]

\[
\mathcal{H}_1^\text{PN} = \sqrt{e_t^2 - 1} \left[ \frac{14759}{168} + e_t^4 \left( \frac{11153}{336} - \frac{1975 \eta}{36} \right) - \frac{847 \eta}{18} + e_t^4 \left( -\frac{62}{7} + 4 \eta \right) \right] + \arccos \left( -\frac{1}{e_t} \right) \left[ \frac{1777}{42} \right],
\]

\[
\mathcal{B}_N(E,h) = \frac{\sinh u}{(e_t \cosh u - 1)^N}
\]

\[
+ \sum_{N=4}^{8} \mathcal{B}_N(E,h) \frac{\sinh u}{(e_t \cosh u - 1)^N}
\]

\[
+ \sum_{N=4}^{8} \mathcal{B}_N(E,h) \ln(e_t \cosh u - 1) \right\},
\]

where \( Y_N(E,h) = \frac{\eta^2 E^4 M h}{c^5} Y_N(E,h) \),

where \( Y \) stands for \( A, B, \Gamma \). It is obvious that we can pursue the similar arguments, detailed in Sec. II B, for computing \( \int^t_{-\infty} dt \mathcal{G}(h, e_t, u) \). This leads to the following 3PN-accurate instantaneous contributions to the radiated angular momentum during hyperbolic encounters of non-spinning compact objects
pute 3PN-accurate expressions for the instantaneous con-

why our instantaneous results can be treated to be exact

Eqs. (E6) of Ref. [14] at the 2PN order. We now explain

We have verified that our expressions are consistent with

to the log terms in the far-zone angular momentum flux.

H_{2PN}^{\text{MH}} = \sqrt{\frac{351}{315} + \frac{210683 e^4}{630} \log \left[ \frac{e^2 - 1}{E_0 h^2 e_t} \right] - \frac{13696}{105} + \frac{98012 e^2}{105} + \frac{23326 e^4}{35} + \frac{2461 e^6}{70} \right]}

× \left\{ \left[ \log \left( \frac{4 E_0 h^2}{e_t} \right) \arccos \left( -\frac{1}{e_t} e_t \right) + \text{Cl}_2 \left[ 2 \arccos \left( -\frac{1}{e_t} e_t \right) \right] \right] + \sqrt{e^2 - 1} \left[ \frac{55475721271}{838528} + \left( -17854035221 \right) \eta - 1905120 \right] \right. 

+ \frac{313336 \pi^2}{1920} \eta + \frac{636197 \eta^3}{18} + \frac{103 \eta^3}{18} + \frac{550589812147}{8385280} \eta - 116779321 \eta^2 - 60480 

+ \frac{12681271 \eta^3}{1890} - \frac{3383 \eta^3}{36} + e^4 \left( \frac{276385167053}{335301120} + \left( \frac{11543781001}{1905120} - 11543781000 \right) \eta - \frac{116779321 \eta^2}{60480} \right) 

- \frac{21775 \eta^3}{144} + e^6 \left( \frac{12794620753}{8279040} - \frac{197812189 \eta}{70560} + \frac{9014755 \eta^2}{8064} - \frac{23497 \eta^3}{288} \right) + e^8 \left( -13784 \eta + 2999 \eta \right) 

- \frac{254 \eta^2 + 28 \eta^3}{144} + \arccos \left( -\frac{1}{e_t} e_t \right) \left[ \frac{4577461991}{1247400} + \left( -\frac{214287779}{4536} + \frac{369 \eta^2}{4} \right) \eta + \frac{7853 \eta^2}{18} \right] 

+ e^6 \left( \frac{13811878057}{1247400} + \left( -\frac{86352541}{5670} + \frac{5781 \pi^2}{32} \right) \eta + \frac{646651 \eta^2}{126} - 488 \eta^4 \right) + e^8 \left( \frac{602403517}{831600} + \left( 150579449 \right) \eta - \frac{60480}{18} \right) 

+ \frac{6273 \pi^2}{256} \eta + \frac{21055 \eta^2}{16} - \frac{4289 \eta^3}{6} + e^6 \left( \frac{2135052803}{1108800} + \left( -\frac{1094353}{672} - \frac{615 \eta^2}{128} \right) \eta + \frac{8513 \eta^2}{12} + \frac{565 \eta^4}{12} \right) 

+ e^8 \left( -\frac{94124017}{2016} + \frac{757831 \eta}{2688} - \frac{306977 \eta^2}{2688} + \frac{129 \eta^3}{32} \right) \right\}

(34c)

Note that the first line contributions in $H_{2PN}^{\text{MH}}$ are due to the log terms in the far-zone angular momentum flux. We have verified that our expressions are consistent with Eqs. (E6) of Ref. [14] at the 2PN order. We now explain why our instantaneous results can be treated to be exact up to 3PN order.

D. On the exact nature of our 3PN results

We note that two crucial inputs are required to compute 3PN-accurate expressions for the instantaneous con-

tributions to $\Delta \mathcal{E}$ and $\Delta \mathcal{J}$. The first input is the 3PN-accurate generalized quasi-Keplerian parametric solution for compact binaries in hyperbolic orbits [12]. This solution, presented in the modified harmonic(MH) gauge, provided analytic expressions for the angular and radial dynamical variables of the 3PN accurate conservative dynamics of compact binaries in hyperbolic orbits. With the help of Ref. [12], we write schematically analytic ex-
pressions for these dynamical variables as
\[ r = R(h, e_t, u), \]
\[ \phi = P(h, e_t, u), \]
\[ \dot{r} = S(h, e_t, u), \]
\[ \dot{\phi} = Q(h, e_t, u), \]
(35)
where \( r \) and \( \dot{r} \) stand for the radial orbital separation and its time derivative. Further, \( \phi \) denotes the angular variable of the reduced mass \( \mu \) around the total mass \( M \) while \( \dot{\phi} \) is the time derivative of the above orbital phase. For the present discussion, we employed certain time eccentricity \( e_t \) and the reduced angular momentum as orbital parameters to characterize the hyperbolic orbit. The temporal evolution arises via the eccentric anomaly \( u \) and it is related to coordinate time \( t \) via the PN-accurate Kepler equation that we symbolically write as
\[ t = T(h, e_t, u). \]
(36)
Let us emphasize that above parametric solution incorporates only the conservative temporal evolution of orbital variables to the 3PN order.

The second crucial ingredient for our present computation is the 3PN-accurate instantaneous contributions to the energy and angular momentum fluxes, given by Eqs. (5.2) of Ref. [23] and Eqs. (3.4) of Ref. [24], in the MH gauge. For the present discussion, we write these fluxes Schematically as
\[ F = F(r, \dot{r}, \dot{\phi}), \]
\[ G = G(r, \dot{r}, \dot{\phi}). \]
(37)
Following Refs. [18, 19], we estimate the radiated energy and angular momentum during hyperbolic encounters by integrating the above fluxes from \( t = -\infty \) to \( t = \infty \). The analytic treatment of these integrals require us to express the dynamical variables that appear in Eqs. (35) and Eqs. (36) with the help of PN-accurate Keplerian parametric solution of Ref. [12]. This leads to
\[ \Delta E := \int_{-\infty}^{+\infty} dt \, F(h, e_t, u), \]
(38a)
\[ \Delta J := \int_{-\infty}^{+\infty} dt \, G(h, e_t, u). \]
(38b)
The above approach is appropriate as it is customary to write these fluxes as \( F = -\partial F / \partial \phi \) and \( G = -\partial G / \partial \phi \). However, the far-zone energy and angular momentum fluxes and the time derivatives of orbital energy and angular momentum are related to each other modulo certain total time derivatives that appear at the 2.5PN order [30–32]. This is why the above equalities hold in an orbital averaged sense in the case of bound elliptical orbits [18]. When we pursue the computations of \( \Delta E \) and \( \Delta J \) to 3PN order, there are certain subtleties that we need to address. This is related to the fact that both \( h \) and \( e_t \) vary with time due to the gravitational radiation reaction effects that appear at the 2.5PN order. This implies that the temporal evolution in the above integrands occur not only through \( u \) but also through \( h \) and \( e_t \). However, the dissipative nature of GW emission allows us to write
\[ e_t(t) = e_{t0} + \frac{\delta e_t(t)}{c^5} + \mathcal{O}(c^{-7}), \]
\[ h(t) = h_0 + \frac{\delta h(t)}{c^5} + \mathcal{O}(c^{-7}), \]
where \( e_{t0} \) and \( h_0 \) are the values of time eccentricity and the scaled angular momentum at periastron (defined by \( u = 0 \)) and hence constants. Therefore, we could ignore temporal evolution in \( e_t \) and \( h \) while computing \( \Delta E \) and \( \Delta J \) expressions upto 2PN order. Further, the resulting expressions involve only the constant scaled orbital angular momentum and time-eccentricity along with the two mass parameters \( m_1 \) and \( m_2 \). However, a close look of Eqs. (39) and its implications for the integrands of Eqs. (38) reveal that the dissipative corrections \( \delta e_t \), \( \delta h \) are required if we plan to obtain 3PN extension of Refs. [18, 19].

This mainly arises due to the structure of the relative acceleration at 2.5PN order which may be written as \( \mathbf{x} = a_{2,5PN}(r, \dot{r}, \phi, r, \nu) \) and this ensures both \( dE / dt \neq 0 \) and \( dJ / dt \neq 0 \) at the 2.5PN order and hence contain terms of \( \mathcal{O}(1/c^5) \). Therefore, we may try to parametrise the orbital dynamics at 2.5PN in the following manner
\[ r = R_0 + \frac{1}{c^5} \partial R_0 \cdot \delta, \]
\[ t = T_0 + \frac{1}{c^5} \partial T_0 \cdot \delta + \frac{1}{c^5} C_t(t), \]
\[ \phi = P_0 + \frac{1}{c^5} \partial P_0 \cdot \delta + \frac{1}{c^5} C_\phi(t), \]
\[ \dot{r} = S_0 + \frac{1}{c^5} \partial S_0 \cdot \delta, \]
\[ \dot{\phi} = Q_0 + \frac{1}{c^5} \partial Q_0 \cdot \delta, \]
(40)
where we used a few short hand notation such that
\[ \partial R_0 = \{ \frac{\partial}{\partial h} R(h_0, e_{t0}, u), \frac{\partial}{\partial e_t} R(h_0, e_{t0}, u) \}, \]
\[ \delta = \{ \delta h, \delta e_t \}, \]
and similar notational conventions apply for the other dynamical variables. The above expressions are influenced by the improved ‘method of variation of constants’ detailed in Ref. [33] that provided a way to include the effects of quadrupolar GW emission on the 2PN-accurate Keplerian type parametric solution for eccentric compact binaries. Further, the two new variables \( C_t \) and \( C_\phi \) that...
appear at the 2.5PN order are influenced by the $c_t$ and $c_m$ variables of Ref. [33]. It is not difficult to argue that the temporal evolution of these new variables should follow

$$\frac{dC_t}{dt} = \delta \left( \frac{\partial}{\partial e} \mathcal{R}_0 \mathcal{R}_0 - \frac{\partial}{\partial \phi} \mathcal{R}_0 \mathcal{R}_0 \right)$$

$$+ \frac{d\delta}{dt} \left( - \frac{\partial}{\partial \phi} \mathcal{R}_0 \mathcal{R}_0 \right),$$

$$\frac{dC_\phi}{dt} = \delta \left( - \frac{\partial}{\partial e} \mathcal{R}_0 \mathcal{R}_0 \right)$$

$$+ \delta \left( \frac{\partial}{\partial \phi} \mathcal{R}_0 \mathcal{R}_0 \frac{\partial}{\partial \phi} \mathcal{R}_0 \mathcal{R}_0 \right).$$

The fact that $\mathbf{a}_{2.5PN}$ does not explicitly depend on $\phi$ and $t$ ensure that $C_t$ and $C_\phi$ can not contribute to the variations in $\mathcal{E}$ and $\mathcal{J}$ at 2.5PN order. In other words, we may write the total time derivative of the conserved energy at 2.5PN to be

$$\frac{d\mathcal{E}}{dt} = \dot{\mathcal{E}}_0 + \delta \cdot \dot{\mathcal{E}}_0,$$(41)

where $\dot{\mathcal{E}}_0$ stands for 2.5PN accurate energy flux, evaluated at the periasteron $e_t = e_{10}$ and $h = h_0$. It is easily seen that $\delta \cdot \dot{\mathcal{E}}_0$ represents radiation reaction correction to leading order radiation, which is induced by the deflection. Note that $\dot{\mathcal{E}}_0$ is even function in time, because the choice of $z$ axis (perpendicular to the orbital plane) in the opposite way should be formally equivalent to time reverse operation $t \leftrightarrow -t$, which cannot make any difference in the result and in other words, there is no radiation reaction in $\dot{\mathcal{E}}_0$. Further, $\dot{\mathcal{E}}_0$ is also even in that they are just partial derivatives of $\dot{\mathcal{E}}_0$ with respect to the $e_t$, $h$, so they can not affect to its time dependency structure. On the other hand, $\delta(t)$ comes from time integration of $\dot{\mathcal{E}}_0$ and therefore it should be odd in time. We may now conclude that the reaction correction $\delta \cdot \dot{\mathcal{E}}_0$ is also odd function. Hence, when it comes to total radiation, which involves the integration from $t = -\infty$ to $t = +\infty$, $\delta \cdot \dot{\mathcal{E}}_0$ does not contribute to $\Delta \mathcal{E}$ (likewise to $\Delta \mathcal{J}$) at all.

These arguments ensure that we can express the various PN contributions to the $\Delta \mathcal{E}$ and $\Delta \mathcal{J}$ integrands in terms of variables that are associated with the Keplerian type solution to the PN-accurate conservative dynamics of hyperbolic encounters. In other words, we are fully justified to ignore Newtonian order GW emission induced variations in $\Delta \mathcal{E}$ and $\Delta \mathcal{J}$ while characterizing the orbital dynamics during the 3PN order $\Delta \mathcal{E}$ and $\Delta \mathcal{J}$ computations. Therefore, we employ $e_t$ and $h$ to characterize PN-accurate hyperbolic orbits and can treat them as constant parameters during computing 3PN-accurate expressions for the radiated energy and angular momentum. This is why we have not considered the radiation reaction while we computed the 3PN accurate instantaneous radiation, and hence this argument fully completes our computation.

### E. Parabolic limit

Here, we list the exact values of the parabolic limit $e_t = e = 1$, or $E = 0$ of the total radiations.

$$\Delta \mathcal{E}(E = 0) = \frac{M \eta^2}{15} \left[ \frac{8 \pi}{h^4} \right] + \pi \left( \frac{7 \pi^2}{4 \pi^4} \right)$$

$$\Delta \mathcal{J}(E = 0) = \frac{8 \pi}{5} \left[ \frac{15 \pi}{h^4} \right] - \frac{5 \pi}{4 \pi^4}$$

We have verified that these expressions are in full agreement with the parabolic limit of the radiated energy and angular momentum during one radial period of an eccentric binary available in Refs. [23, 24].

### F. Implications of Post-Bremsstrahlung Expansion

We now probe the implications of the post-bremsstrahlung limit of our $\Delta \mathcal{E}$ and $\Delta \mathcal{J}$ expressions. Recall that the bremsstrahlung limit arises by allowing the eccentricity parameter $\eta \rightarrow \infty$ as noted in Ref. [18], and we are exploring the implications of eccentricity corrections to such bremsstrahlung limit of our 3PN order hyperbolic expressions. This effort is also influenced by the fact that it is rather difficult to obtain closed form expressions for $\Delta \mathcal{E}$ and $\Delta \mathcal{J}$ expressions even when the leading order hereditary contributions are included [16]. Therefore, it is reasonable that our ongoing effort to obtain fully 3PN-accurate expressions for the radiated energy and angular momentum in GWs during hyperbolic encounters will not be exact in orbital eccentricity. In what follows, we probe the implications of post-bremsstrahlung expansion of our 3PN order hyperbolic $\Delta \mathcal{E}$ and $\Delta \mathcal{J}$ expressions with respect to their bound orbit counterparts. These counterparts are essentially 3PN-accurate instantaneous $\delta \mathcal{E}$ and $\delta \mathcal{J}$ expressions
that provide the radiated energy and angular momentum during one radial period of an eccentric binary. These expressions can easily be obtained from Refs. [23, 24], and are exact in orbital eccentricity. Further, we find it convenient to employ the Newtonian eccentricity parameter \( e = \sqrt{1 + 2Eh^2} \) to characterize the both the bound and unbound far-zone quantities to ensure that the same eccentricity parameter is used in our comparisons. We display the relevant \( \Delta \xi \) and \( \Delta J \) expressions in Appendix. B. Further, the explicit 3PN-accurate instantaneous post-bremsstrahlung \( \Delta \xi \) and \( \Delta J \) expressions for hyperbolic encounters are available at https://github.com/subhajittifr/hyperbolic_flux, where \( \Delta \xi \) is expanded up-to \( \mathcal{O}(1/e^5) \) while \( \Delta J \) has been expanded up-to \( \mathcal{O}(1/e^3) \).

In Fig. 1, we plot the fractional differences between Refs. [23, 24] based \( \delta \xi \) and \( \delta J \) 3PN order expression that are exact in \( e \) and the post-bremsstrahlung expansion of our \( \Delta \xi \), \( \Delta J \) expressions while allowing \( e \leq 1 \). These plots reveal that the post-bremsstrahlung versions of our \( \Delta \xi \), and \( \Delta J \) expressions provide excellent proxies to compact binaries both in parabolic and high eccentric (bound) orbits. These post-bremsstrahlung approximants are substantially different from their eccentric counterparts near their circular limits. This is most likely due to the presence of \( \frac{1}{e^2} \) terms and its multiples and similar conclusions are drawn from plots where we change values of \( \eta \) and the dimensionless \( hc \) as evident from Fig. 2. We infer that our post-bremsstrahlung \( \Delta \xi \) and \( \Delta J \) expressions, which should be accurate to describe large eccentric hyperbolic orbit, are not only convergent at the parabolic limit \( (e = 1) \) but also highly eccentric cases \( (e \lesssim 1) \). It will be interesting to explore if numerical relativity simulations display a similar behaviour. This natural convergence at parabolic limit is contrary to the attempt to cover parabolic limit adopted in Sec. IX in Ref. [34], where the physical quantities are divergent at parabolic limit, so the parabolic limit is incorporated by numerical methodologies such as fitting and Pade approximation.

III. SUMMARY AND ON-GOING EFFORTS

We have provided explicit expressions for the 3PN-accurate instantaneous contributions to the radiated energy and angular momentum during hyperbolic encounters. These computations, pursued in the time-domain, are not straightforward extensions of the classic 1PN-accurate efforts by Schäfer and his collaborators [18, 19]. This is essentially due to the presence of certain logarithmic terms in the 3PN order contributions to the far-zone flux expressions for the general orbits [23]. Additionally, we explored the implications of post-Bremsstrahlung expansion of our results from the perspective of eccentric orbits.

There are on-going efforts to compute hereditary contributions to the \( \Delta \xi \) and \( \Delta J \) expressions that are accurate to 3PN order, influenced by Ref. [15]. We are also pursuing efforts to compare our PN-accurate results with those arising from Numerical Relativity [13, 35]. This will be helpful to explore the validity of PN approximation while exploring GWs from hyperbolic encounters. Further, these efforts may allow us to develop a prescription for describing GW emission aspects of highly eccentric compact binaries with constructs that arise from our present and on-going PN-accurate hyperbolic computations.

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Figure 2. Plots that are similar to what are displayed in Fig. 1. For these plots, we let $c^2 h^2 = 10$, $E_0 h^2 = 1/5$ and $\eta = 1/20$. Clearly, our conclusions are rather independent of $\eta$ values.

Appendix A: Hyperbolic Log Integrals

We provide the details of evaluating certain 3PN order log integrals that are crucial for our results. We begin with the following expression

$$ I(x, y) := \int_{-\infty}^{+\infty} du \frac{\ln(e \cosh u - x)}{e \cosh u - y}. \quad (A1) $$

The goal is to first take a differentiation followed by an integration of $I(x, y)$ with respect to $x$. Differentiating the above expression with respect to $x$ yields

$$ -\partial_x I(x, y) = \int_{-\infty}^{+\infty} du \frac{1}{e \cosh u - x} \frac{1}{e \cosh u - y}, $$

$$ = J_1(x, y) + J_2(x, y), \quad (A2a) $$

where

$$ J_1(x, y) := \frac{2}{\sqrt{e^2 - x^2}} \arccos \left(-\frac{x}{e}\right), \quad (A2b) $$

$$ J_2(x, y) := \frac{2}{\sqrt{e^2 - y^2}} \arccos \left(-\frac{y}{e}\right). \quad (A2c) $$

Antiderivative of $-J_2$ with respect to $x$ is

$$ I_2 := -\int dx \ J_2 = \frac{2 \ln(y - x)}{\sqrt{e^2 - y^2}} \arccos \left(-\frac{y}{e}\right). \quad (A3) $$

To integrate $J_1$, we make a change of variable $f := \arccos \left(-\frac{x}{e}\right)$, then

$$ I_1 := -\int dx \ J_1 = \int df \ \frac{2f}{e \cos f + y} = \frac{2i}{\sqrt{e^2 - y^2}} \left[ \text{Li}_2(g) - \text{Li}_2(g^*) \right] $$

finacial support and hospitality of the Pauli Center for Theoretical Studies and the University of Zurich.
\[-\frac{2}{\sqrt{e^2 - y^2}} \left[ \arccos \left( -\frac{y}{e} \right) \log \left( -\frac{e^2 - xy + \sqrt{e^2 - x^2} \sqrt{e^2 - y^2}}{4e(x - y)} \right) \right] + \alpha(y; e), \]  

where

\[
g = \frac{y - i \sqrt{e^2 - y^2}}{e} \left( e + y \right) \sqrt{e^2 - x^2} - \left( e + x \right) \sqrt{e^2 - y^2},
\]

\[
g^* = \frac{y + i \sqrt{e^2 - y^2}}{e} \left( e + y \right) \sqrt{e^2 - x^2} - \left( e + x \right) \sqrt{e^2 - y^2}.
\]

Therein, \( \text{Li}_2 \) is the polylogarithm function of order 2. These results are valid up to modulo a constant \( \alpha(y; e) \) that is independent of \( x \). To determine the expression of \( \alpha \), let us consider \( x \to -\infty \) behavior such that

\[ \lim_{x \to -\infty} I(x,y) = \lim_{x \to -\infty} \frac{2 \log(-x)}{\sqrt{e^2 - y^2}} \arccos \left( -\frac{y}{e} \right). \]

An, it can be easily checked that

\[ \lim_{x \to -\infty} I(x,y) = \lim_{x \to -\infty} I_2(x,y), \]

which implies

\[ \lim_{x \to -\infty} I_1(x,y) = 0, \]

to keep \( I = I_1 + I_2 \) hold. Making use of this condition at \( x \to -\infty \) on Eq. (A4), we reach

\[ \alpha(y; e) = \frac{2i}{\sqrt{e^2 - y^2}} \left\{ \frac{\pi^2}{6} - \text{Li}_2 \left[ \frac{-iy + \sqrt{e^2 - y^2}}{e^2} \right] \right\} - i \arccos \left( -\frac{y}{e} \right) \log \left( \frac{4(y - i \sqrt{e^2 - y^2})}{e} \right). \]

In the above we used \( \lim_{x \to -\infty} \text{Li}_2(g^*) = \frac{\pi^2}{6} \). Thus, we finally obtain

\[ I = \frac{2i}{\sqrt{e^2 - y^2}} \left\{ \frac{\pi^2}{6} + \text{Li}_2(g) - \text{Li}_2(g^*) - \text{Li}_2 \left[ \frac{-iy + \sqrt{e^2 - y^2}}{e^2} \right] \right\} - i \arccos \left( -\frac{y}{e} \right) \log \left( \frac{2(e^2 - x^2)(x - i \sqrt{e^2 - x^2})}{e^2} \right). \]

Now, we focus on a special condition on Eq. (A10) where \( x \to y \), the integral we are interested to compute. Note that when \( y \to x \), there are some divergences because of \( \sim \log(y - x) \) term in \( I_1 \), but with proper calculus, these divergences are cancelled while taking \( y \to x \) limit. We get

\[ \int_{-\infty}^{\infty} du \frac{\ln(e \cosh u - x)}{e \cosh u - x} = \frac{2i}{\sqrt{e^2 - x^2}} \left\{ \frac{\pi^2}{6} - \text{Li}_2 \left[ \frac{-ix + \sqrt{e^2 - x^2}}{e^2} \right] \right\} - i \arccos \left( -\frac{x}{e} \right) \log \left( \frac{2(e^2 - x^2)(x - i \sqrt{e^2 - x^2})}{e^2} \right). \]

Eq. (A11) can be further simplified using the usual definition of Clausen function \( \text{Cl}_2(x) \) [25] of order 2 - given by the integral

\[ \text{Cl}_2(x) = - \int_0^x dx \log \left| 2 \sin \frac{x}{2} \right| \]

There exist a nice expression of Dilogaritum function :\( \text{Li}_2(x) \) in terms of Clausen function, given by

\[ \text{Li}_2(e^{i\theta}) = \frac{\pi^2}{6} - \frac{1}{4} |\theta|(2\pi - |\theta|) + i \text{Cl}_2(\theta), \quad |\theta| \leq 2\pi. \]

By virtue of Eq. (A13) we obtain the most simplified form of the integral \( I(x) \). Our final expression is

\[ \int_{-\infty}^{\infty} du \frac{\ln(e \cosh u - x)}{e \cosh u - x} = \frac{2}{\sqrt{e^2 - x^2}} \left\{ \text{Cl}_2 \left( -\frac{x}{e} \right) + \arccos \left( -\frac{x}{e} \right) \log \left( 2e - \frac{2x^2}{e} \right) \right\}. \]
Appendix B: Total Energy and angular momentum radiations in terms of Newtonian eccentricity

Since the time eccentricity $e_t$ is not gauge invariant, we provide additional expressions of $\Delta \mathcal{E}$ and $\Delta \mathcal{J}$ in terms of energy $E$ and angular momentum $h$ to make them gauge invariant except $E_0$ term, which should be removed by the corresponding hereditary contribution. Here, we use the Newtonian eccentricity $e$ as a shorthand symbol of $\sqrt{1 + 2 E h^2}$ without implying any geometric meaning.

\[
\Delta \mathcal{E}_{\text{inst}} = \frac{2}{15} \frac{M \eta^2}{h^2 e^2} \left( \mathcal{I}_N^M + \frac{1}{h^2 e^2} \mathcal{I}_{1PN}^M + \frac{1}{h^4 e^4} \mathcal{I}_{2PN}^M + \frac{1}{h^6 e^6} \mathcal{I}_{3PN}^M \right),
\]

(B1)

where

\[
\mathcal{I}_N^M = \sqrt{e^2 - 1} \left[ \frac{602}{3} + \frac{673 e^2}{3} \right] + \arccos \left( \frac{1}{e} \right) \left[ 96 + 292 e^2 + 37 e^4 \right],
\]

(B2a)

\[
\mathcal{I}_{1PN}^M = \sqrt{e^2 - 1} \left[ \frac{108 + 12 \eta}{e^2} + \frac{90754}{35} - \frac{2185 \eta}{6} + \frac{e^2}{35040} \left( \frac{141439}{70} - \frac{11441 \eta}{6} \right) + \frac{e^4}{89907} \left( \frac{89907}{280} - \frac{1117 \eta}{2} \right) \right] + \arccos \left( \frac{1}{e} \right) \left[ \frac{1117}{7} - 95 \eta + \frac{37785}{14} - \frac{3051 \eta}{2} + \frac{e^4}{2817} \left( \frac{2817}{4} - \frac{2283 \eta}{2} \right) + \frac{e^6}{2393} \left( \frac{2393}{56} - \frac{111 \eta}{2} \right) \right],
\]

(B2b)

\[
\mathcal{I}_{2PN}^M = \sqrt{e^2 - 1} \left[ \frac{1}{e^4} \left( \frac{243}{2} - 27 \eta - \frac{3 \eta^2}{2} \right) + \frac{1}{e^2} \left( \frac{207423}{112} - 2515 \eta - \frac{179 \eta^2}{16} \right) \right] + \frac{8409586747}{317520} - \frac{9858601 \eta}{720} - \frac{1569 \eta^2}{4} + \frac{e^2}{635040} \left( \frac{3850577 \eta}{1440} + 4098 \eta^2 \right) + \frac{e^4}{154205167} \left( 36653233 \eta - \frac{158383 \eta}{24} - \frac{4787 \eta^2}{16} \right) \]

\[
+ \frac{11581 \eta^2}{2} + \frac{e^6}{1960} \left( \frac{509759}{1120} + \frac{680471 \eta}{16} + \frac{12693 \eta^2}{16} \right) + \arccos \left( \frac{1}{e} \right) \left[ \frac{52925837}{3024} - \frac{158383 \eta}{24} - \frac{4787 \eta^2}{16} \right] + \frac{e^2}{1512} \left( \frac{36730439}{168} - \frac{4415855 \eta}{16} + \frac{387 \eta^2}{2} \right) + \frac{e^4}{479005} \left( \frac{288}{224} - \frac{3069361 \eta}{224} + \frac{25641 \eta^2}{4} \right) + \frac{e^6}{5885} \left( - \frac{5885}{84} \right) - \frac{3489 \eta}{2} + \frac{4339 \eta^2}{2} + \frac{e^8}{745} \left( \frac{12}{224} - \frac{11965 \eta}{224} + \frac{925 \eta^2}{16} \right),
\]

(B2c)

\[
\mathcal{I}_{3PN}^M = \sqrt{e^2 - 1} \left[ \frac{6713608}{1575} + \frac{17868572 e^2}{525} + \frac{1930053 e^4}{525} + \frac{17525209 e^6}{3150} \right] \log \left[ \frac{e^2 - 1}{E_0 h^2 e} \right] - \frac{54784}{35} + \frac{465664 e^2}{21} + \frac{4426374 e^4}{105} + \frac{1498856 e^6}{105} + \frac{31779 e^8}{70} \right] \left\{ \log \left[ \frac{4 E_0 h^2 e}{E} \right] \arccos \left( \frac{1}{e} \right) + C_2 \left[ 2 \arccos \left( \frac{1}{e} \right) \right] \right\} + \sqrt{e^2 - 1} \left( \frac{1}{4 e^8} \left( 729 + 243 \eta + 27 \eta^2 + \eta^3 \right) + \frac{1}{e^4} \left( - \frac{1106055}{448} \right) - \frac{63027 \eta}{448} + \frac{10159 \eta^2}{448} + \frac{167 \eta^3}{192} \right) + \frac{1}{e^2} \left( \frac{15311843}{672} + \frac{2862985}{378} + \frac{123 \eta^2}{2} \right) \eta - \frac{426791 \eta^2}{672} - \frac{79 \eta^3}{4} \right)
\]

\[
+ \frac{4006564486859}{139708800} + \left( - \frac{205531241131}{635040} + \frac{13414913 \pi^2}{3360} \right) \eta + \frac{833407139 \eta^2}{40320} + \frac{206305 \eta^3}{192}
\]
Similarly, the 3PN-accurate instantaneous contributions to the radiated angular momentum terms of $e$ and $h$ become

$$
\Delta G_{\text{inst}} = \frac{8}{5} \frac{GM^2 \eta^2}{h^4} \left( \mathcal{H}_N^{\text{MH}} + \frac{1}{h^2} \frac{c}{e} \mathcal{H}_{1\text{PN}}^{\text{MH}} + \frac{1}{h^4} \frac{c^3}{e^4} \mathcal{H}_{2\text{PN}}^{\text{MH}} + \frac{1}{h^6} \frac{c^5}{e^6} \mathcal{H}_{3\text{PN}}^{\text{MH}} \right),
$$

where

$$
\mathcal{H}_N^{\text{MH}} = \sqrt{e^2 - 1} \left[ 13 + 2e^2 + \arccos \left( -\frac{1}{e} \right) \left[ 8 + 7e^2 \right] \right],
$$

$$
\mathcal{H}_{1\text{PN}}^{\text{MH}} = \sqrt{e^2 - 1} \left[ \frac{9 + \eta}{e^2} + \frac{1125\eta}{168} - \frac{1451\eta}{36} + e^2 \left( \frac{10313}{336} - \frac{2065\eta}{36} \right) + e^4 \left( \frac{109}{28} - \frac{5\eta}{4} \right) \right] + \arccos \left( -\frac{1}{e} \right) \left[ \frac{4877}{84} - \frac{65\eta}{4} + e^2 \left( \frac{869}{21} - \frac{419\eta}{6} \right) + e^4 \left( \frac{4283}{336} - \frac{71\eta}{6} \right) \right],
$$

$$
\mathcal{H}_{2\text{PN}}^{\text{MH}} = \sqrt{e^2 - 1} \left[ -\frac{1}{8e^4} \left( 81 + 18\eta + \eta^2 \right) + \frac{1}{e^2} \left( \frac{41169}{448} - \frac{11839\eta}{672} - \frac{109\eta^2}{64} \right) + \frac{1}{272160} \left( 13984943 - 17944223\eta \right) + \frac{1915\eta^2}{144} + e^2 \left( -\frac{1847311}{272160} - \frac{201089\eta}{540} + \frac{25595\eta^2}{96} \right) + e^4 \left( -\frac{13557}{560} - \frac{870131\eta}{10080} + \frac{15701\eta^2}{144} \right) + e^6 \left( \frac{3245}{4032} - \frac{327\eta^2}{224} + \frac{51\eta^2}{64} \right) + \arccos \left( -\frac{1}{e} \right) \left[ \frac{2051491}{4536} - \frac{46297\eta}{126} - \frac{79\eta^2}{12} + e^2 \left( \frac{752767}{6048} \right) - \frac{12167\eta}{224} + \frac{4433\eta^2}{24} \right] + e^4 \left( -\frac{28297}{378} - \frac{2461\eta}{168} + \frac{783\eta^2}{4} \right) + e^6 \left( \frac{1327}{504} - \frac{3653\eta}{224} + \frac{337\eta^2}{24} \right) \right].
$$

$$
\mathcal{H}_{3\text{PN}}^{\text{MH}} = \sqrt{e^2 - 1} \left[ \frac{99724}{315} + \frac{351067e^2}{315} + \frac{210683e^4}{630} \right] + \arccos \left( -\frac{1}{e} \right) \left[ \frac{c^2 - 1}{E_0 h^2 e} \right] - \left[ \frac{13696}{105} + \frac{98012e^2}{105} + \frac{23326e^4}{35} + \frac{2461e^6}{70} \right].
$$
\[
\begin{align*}
&\times \left\{ \log \left[ \frac{4E_0 h^2}{e} \right] \arccos \left( -\frac{1}{e} \right) + C_2 \left[ 2 \arccos \left( -\frac{1}{e} \right) \right] \right\} + \sqrt{e^2 - 1} \left[ \frac{1}{16 e^6} \left( 243 + 81 \eta + 9 \eta^2 + \eta^3 \right) \right. \\
&+ \frac{1}{e^4} \left( -\frac{257553}{1792} + \frac{16035 \eta}{1792} + \frac{22507 \eta^2}{5376} + \frac{121 \eta^3}{768} \right) + \frac{1}{e^2} \left( \frac{27158603}{32256} + \left( -\frac{165077725}{290304} + \frac{41 \eta^2}{8} \right) \eta \\
&- \frac{71878 \eta^2}{32256} - \frac{237 \eta^3}{512} \right) + \frac{387361378703}{67060224} + \left( -\frac{33779906349}{30481920} + \frac{340423 \eta^2}{1920} \right) \eta + \frac{878650477 \eta^2}{483840} + \frac{344695 \eta^3}{4608} \right) \\
&+ e^2 \left( \frac{1005116464409}{167650560} + \left( -\frac{23645568707}{30481920} + \frac{512123 \eta^2}{3840} \right) \eta + \frac{134389529 \eta^2}{34560} - \frac{70835 \eta^3}{2304} \right) + e^4 \left( \frac{433742948027}{335301120} \right) \\
&+ \left( \frac{5045091193}{7620480} - \frac{4456 \eta^2}{1920} \right) \eta + \frac{60548279 \eta^2}{60480} - \frac{886135 \eta^3}{1152} \right) + e^6 \left( \frac{1172569271}{24837120} + \frac{17091961 \eta}{1128960} + \frac{4248733 \eta^2}{32256} \right) \\
&- \frac{687113 \eta^3}{4608} + \left( \frac{6973}{354816} - \frac{325 \eta}{32256} - \frac{3379 \eta^2}{3584} - \frac{249 \eta^3}{512} \right) \right\} + \arccos \left( -\frac{1}{e} \right) \left[ \frac{26951805241}{5702400} \right. \\
&+ \left( -\frac{503311993}{72576} + \frac{3239 \eta^2}{32} \right) \eta + \frac{12801961 \eta^2}{16128} - \frac{2479 \eta^3}{64} + e^2 \left( \frac{8362307931}{9979200} + \left( -\frac{408126835}{36288} + \frac{5355 \eta^2}{32} \right) \eta \right. \\
&+ \left( \frac{15335105 \eta^2}{4032} - \frac{3907 \eta^3}{48} \right) + e^4 \left( \frac{6781095059}{1663200} + \left( -\frac{18560153}{24192} - \frac{615 \eta^2}{256} \right) \eta + \frac{5148791 \eta^3}{2688} - \frac{71389 \eta^3}{96} \right) \right) \\
&+ e^6 \left( \frac{254451209}{3326400} + \frac{1448255}{6048} - \frac{615 \eta^2}{128} \right) \eta + \frac{391981 \eta^2}{1344} - \frac{16867 \eta^3}{192} \right) + e^8 \left( \frac{227005}{709632} - \frac{14411 \eta}{2016} + \frac{29727 \eta^2}{1792} \right. \right) \\
&- \frac{2693 \eta^3}{192} \right) \right).
\end{align*}
\]
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