Experimental observation of fractal modes in unstable optical resonators

J.A. Loaiza, E. R. Eliel, and J. P. Woerdman
Huygens Laboratorium, Universiteit Leiden, P.O. Box 9504, 2300 RA Leiden, The Netherlands

We use a spatially resolved cavity ring-down technique to show that the 2D eigenmode of an unstable optical cavity has a fractal pattern, i.e. it looks the same at different length scales. In agreement with theory, we find that this pattern has the maximum conceivable roughness, i.e., its fractal dimension is $3.01 \pm 0.04$. This insight in the nature of unstable cavity eigenmodes may lead to better understanding of wave dynamics in open systems, for both light and matter waves.

PACS numbers: 5.45.df,42.60.Da,47.53.+n

Conventional lasers are based upon stable optical cavities in which the light rays that bounce between the two mirrors are trapped forever and leakage around the mirror edges is negligible. In wave-optics language this picture of bouncing rays corresponds to the well known Hermite-Gaussian eigenmodes. In contrast, in an unstable cavity (UC), consisting for instance of two convex mirrors that face each other, the combination of the curvatures of the mirrors and their separation is such that the bouncing rays run away from the axis towards the mirror edges and ultimately escape; the escape rate is determined by the magnification $M$ of the system. The associated eigenmodes fill the entire volume of the cavity; this has the advantage of optimum energy extraction if an active medium is placed inside the cavity to realize a laser. Another advantage of an unstable-cavity laser is that discrimination against higher-order transverse modes is much better than for a stable-cavity laser; perfect single-mode oscillation is very easily achieved. These advantages have been known since the early days of laser physics, whereas other surprising properties of the UC laser have emerged more recently. In particular we refer here to the much debated phenomenon of UC excess quantum noise, and the prediction that their transverse eigenmodes are fractals, i.e., they are invariant under magnification. While the former effect has been experimentally demonstrated, the latter is still being pursued. What we report here is the first experimental observation of the fractal nature of UC eigenmodes in the optical regime; these results may stimulate extension to matter waves.

On an intuitive level the origin of the fractality is rooted in two arguments: (i) an unstable cavity has a round-trip magnification $M > 1$. (ii) The cavity eigenmode must, by definition, be invariant upon round-trip propagation. The combination of arguments (i) and (ii) implies that an UC eigenmode must be self-similar (i.e. fractal). This line of reasoning must, however, be handled with care since the concept of magnification has its origin in ray-optics, whereas an eigenmode is a wave-optics concept: moreover, these arguments do not lead to predictions for the fractal dimension.

The starting point to describe the mode structure of an UC is the so-called Virtual Source (VS) method, originally developed by Horwitz and Southwell. In the VS method a two-mirror UC is unfolded to create a corridor of virtual images of the confining aperture (e.g. the smallest of the two mirrors). A plane wave is injected at the far end of the corridor and the eigenmode is formed by superposition of the diffracted patterns produced by the sequence of virtual apertures. This superposition (i.e. the eigenmode) contains thus rich spatial structure that depends on the shape of the confining aperture.

Very recently, Berry and co-workers formulated an analytical theory for fractal eigenmodes and excess quantum noise, by introducing further approximations to the VS method. This is an important development since it carries the notion of a fractal pattern beyond that of a phenomenological description, yielding insight and allowing predictions; fractality then presents a convenient test of the theory as a whole. Specifically, the theory predicts that in the asymptotic limit $N_F \to \infty$, the fractal dimension $D$ of the fundamental eigenmode is $3$ for a wide variety of polygon-shaped confining apertures (e.g. a triangle). Here $N_F \equiv a^2/\lambda L$ is the Fresnel number, where $a$ is the (linear) size of the aperture, $L$ is the length of the cavity and $\lambda$ is the wavelength of light. This prediction is very surprising since it implies that the 2D transverse intensity profile, when viewed as a mountain landscape, has the maximum topologically allowed roughness. Contrary to what might be expected, diffraction does not smooth out this profile: in fact, diffraction is the cause of the phenomenon.

We stress that experimental verification of this prediction is essential, since the validity of the theory may be questioned at various levels. First, all theoretical efforts so far, be it numerical or analytical, are based upon standard diffraction theory; this is an inherently approximate theory that derives its simplicity (as compared to solving Maxwell’s equations) from using inconsistent boundary conditions. Although this is hardly ever a problem in practical cases, questions remain when dealing with an unusual problem such as the UC. On top of this, the VS method is not a straightforward implementation of

\[ D \approx 3 - \frac{\ln N_F}{\ln M} \]
Here we report results for triangular and hexagonal aperatures whose typical Fresnel numbers ($N_F = a^2/\lambda L$) range around 90, with $a$ taken as the radius of the polygon’s inscribed circle, typically around 6 mm. The oscillatory behaviour of the cavity round-trip propagator is characterized by means of the parameter $A \equiv 2\pi MF$ \cite{39}, where $F \equiv N_F/2g_2$, $g_2 = 1 - L/R_2$, and $R_2$ is the radius of curvature of the front mirror \cite{12}. In the present case the parameter $A$ takes on values around 500. The confining aperture is $1-1$ imaged on the photocathode of a single-photon sensitive camera, whose exposure time ($\approx 2$ ns) is shorter than the round-trip time of the cavity ($\approx 3$ ns).

Light from a 75 mW laser diode (Sharp LT024MD0) at $\lambda=786.3$ nm is injected into the cavity through one of its mirrors after first being spectrally filtered by means of a monochromator having a 0.3 nm wide transmission window (Fig.1). Due to the low finesse of the UC we do not need to tune our laser to one of its resonances. The experiment consists of rapidly switching-off the diode laser, and after a short delay, making an exposure by triggering the photocathode and intensifier of the single-photon sensitive camera (Princeton Instruments Pi-Max). Many subsequent exposures are superimposed on the camera until the signal-to-noise ratio of the acquired image is sufficiently large, followed by read-out of the camera. For the longest delay of our experiment ($\approx 30$ ns) we have to accumulate typically $10^5$ exposures to arrive at a spatial pattern with sufficient detail.

Experimentally, we find that for our configuration (Fig. 1), the shape of the intra-cavity intensity distribution stabilizes after approximately 7 round-trips. A typical result of a 2D intensity image, together with its 3D representation, is shown in Fig. 2. The 2D picture shows, with fascinating detail, the high-degree of symmetry and structural self-similarity of the intensity pattern, while the corresponding 3D landscape emphasizes its very abrupt topography. It is worth mentioning that Mandelbrot \cite{10} associates fractal dimensions up to 2.4 to earthy looking landscapes; therefore we expect a higher fractal dimension for the weird landscape shown in Fig. 2.

We use the Fourier-transform method to evaluate the fractal dimension of the intensity profile since this allows a direct comparison with the analytical theory \cite{14, 15}. The method is based on the fact that the power spectrum of a fractal function $f(x)$ follows a power law $P(k) \approx k^{-b}$, and that the phase of its Fourier components is random; the fractal dimension $D$ can be obtained from the exponent $b$ by using the relation \cite{20, 21} $2D = 5 - b$. It does not matter whether we take for $f(x)$ the optical intensity or the field amplitude, both choices lead to the same value of $D$ \cite{22}.

We follow this procedure for 1D cuts of a typical asymptotical intensity profile. Fig. 3, shows a 1D cut taken 9 roundtrips time after laser shut-down, while Fig. 4, shows its spatial power spectrum. This spectrum shows many “spikes” due to the discrete nature of the

![Diagram](image-url)
virtual sources; these have been smoothed by averaging over bands of logarithmically constant length \(2^{23}\). Apparently the power spectrum is described by two power-law contributions dominating at low and high spatial frequencies, respectively. Yates and New \(23\) have shown that the higher the order of a UC eigenmode, the steeper the power law of its spatial power spectrum is. This leads us to associate the low spatial-frequency range in Fig. 3b with residual light in higher-order modes. The high-frequency contribution is then associated with the fundamental eigenmode; a fit of the latter contribution yields \(D_{\text{cut}} = 2.01 \pm 0.04\) for the fundamental mode. The fractal dimension \(D\) of the 2D landscape follows directly from this value, since \(D = D_{\text{cut}} + 1\), yielding \(D \approx 3\), in excellent agreement with theory \(14\). This value of the fractal dimension implies that the intensity landscape has the maximum allowed roughness. The inset in Fig. 3b shows that the Fourier components of the 1D cut have indeed random phases, as required for a fractal curve.

To ascertain that our cavity ring-down method indeed selects a single cavity mode, we have investigated whether the observed asymptotical spatial pattern is sensitive to changes in the initial conditions, i.e., to the spatial pattern of the injected light. To do so, we have introduced a piece of Mylar film as diffuser between the monochromator and the cavity, thereby creating a totally different distribution of input phase and amplitude.

Figure 4 shows, for a hexagonal aperture inside the UC, the initial and asymptotic profiles of the intensity distribution using a narrow injection beam, and a beam that has been diffused by the thin Mylar film before entering the cavity. The asymptotical spatial patterns are very similar, implying that they correspond to a single mode. Our experimental validation of the analytical theory of eigenmodes of UC’s \(14, 15, 16\) allows its use in a broader context. This is important since UC lasers play an important role in laser physics; in particular microlasers. The reason is that microlasers (e.g., semiconductor lasers) must be efficient, which implies that the gain must be localized, e.g., to a \(\lambda\)-sized transverse dimension; pumping outside this region is a waste. Because of the gain localization there is gain guiding in addition to the index guiding that is used to realize the \(\lambda\)-sized optical confinement in the first place. The (unavoidable) combination of gain and index guiding leads to a cavity that is effectively slightly unstable, i.e., transversely open to the outside world; and this increases the quantum noise of the...
FIG. 4: Frames (a) and (b) show the initial distribution of light injected into the cavity without and with a thin sheet of Mylar as a diffuser, respectively; frames (c) and (d) show the corresponding pictures taken after 7 round trips.

A better insight in the nature of UC eigenmodes may lead to device architectures that minimize this excess noise.

Finally, the issues discussed in this Letter can also be extended to UC’s (or open systems in general) for matter waves. As demonstrated recently, spherical mirrors for atoms can be made so that an atom-optics based UC is within reach. In fact, an UC for electron waves has already been demonstrated, as a mesoscopic device based upon a 2D electron gas; the unstable nature of this UC determines the conductance of a quantum point contact placed inside. It remains to be seen what the consequences of “fractal matter waves” are for the operation of such devices.

This work is part of the research program of the ‘Stichting voor Fundamenteel Onderzoek der Materie’ (FOM).

Electrical address: javier@molphys.leidenuniv.nl

URL: http://www.molphys.leidenuniv.nl/qo/index.html

1 A. E. Siegman, Laser (University Science, Mill Valley, USA, 1986).

2 A. E. Siegman, Appl. Opt. 13, 353 (1974).

3 Y. J. Cheng, C. G. Fanning, and A. E. Siegman, Phys. Rev. Lett. 77, 629 (1996).

4 M. A. van Eijkelenborg, A. M. Lindberg, M. S. Thijssen, and J. P. Woerdman, Phys. Rev. Lett. 77, 4314 (1996).

5 G. H. C. New, J. Mod. Opt. 42, 799 (1995).

6 A. M. van der Lee, N. J. van Druten, M. P. van Exter, J. P. Woerdman, J.-P. Poizat, and P. Grangier, Phys. Rev. Lett. 85, 4711 (2000).

7 K. Petermann, IEEE J. Quantum Electron. QE-15, 566 (1979).

8 G. P. Karman and J. P. Woerdman, Opt. Lett. 23, 1909 (1998).

9 G. P. Karman, G. S. McDonald, G. H. C. New, and J. P. Woerdman, Nature 402, 138 (1999).

10 B. B. Mandelbrot, The fractal geometry of nature (Freeman, New York, 1982).

11 J. Courtial and M. J. Padgett, Phys. Rev. Lett. 85, 5320 (2000).

12 P. Horwitz, J. Opt. Soc. Am. 63, 1528 (1973).

13 W. H. Southwell, Opt. Lett. 6, 487 (1981).

14 M. V. Berry, Opt. Commun. 200, 321 (2001).

15 M. V. Berry, C. Storm, and W. van Saarloos, Opt. Commun. 197, 393 (2001).

16 M. V. Berry, J. Mod. Opt. 50, 63 (2003).

17 J. J. Stannes, Waves in focal regions (Adam Hilger, Bristol, 1986).

18 A. O’Keefe and D. A. G. Deacon, Rev. Sci. Instrum. 59, 2544 (1988).

19 E. Olivier, D. Chauvat, A. L. Floch, and F. Bretenaker, Opt. Lett. 24, 22 (1999).

20 K. Falconer, Fractal geometry: mathematical foundations and applications (John Wiley, Chichester, 1990).

21 M. V. Berry and Z. V. Lewis, Proc. Roy. Soc. Lond. A 370, 459 (1980).

22 M. V. Berry, J. Phys. A: Math. Gen. 29, 6617 (1996).

23 M. A. Yates and G. H. C. New, Opt. Commun. 208, 377 (2002).

24 V. Milner, J. L. Hanssen, W. C. Campbell, and M. G. Raizen, Phys. Rev. Lett. 86, 1514 (2001).

25 N. Friedman, A. Kaplan, D. Carasso, and N. Davidson, Phys. Rev. Lett. 86, 1518 (2001).

26 J. S. Hersch, M. R. Haggerty, and E. J. Heller, Phys. Rev. E 62, 4873 (2000).

27 J. A. Katine, M. A. Eriksson, A. S. Adourian, R. M. Westervelt, J. D. Edwards, A. Lupu-Sax, E. J. Heller, K. L. Campman, and A. C. Gossard, Phys. Rev. Lett. 79, 4806 (1997).

28 Of dominant concern is the light emitted by the diode laser after it has been nominally turned off (‘afterglow’), for instance due to the recombination of remaining charge carriers in the p-n junction. This is important because the intracavity intensity of the UC has been reduced by many orders of magnitude by the time that the pattern stabilizes. Obviously, one requires any light unintentionally injected in the cavity at that time to be even weaker. As a result of careful optimization of the experimental parameters we obtain on/off ratios of the injected light up to $10^8$.

29 For the convex mirror we use a concave mirror (radius of curvature 6) on a glass substrate ($n = 1.5$) in a reversed way.

30 Note that $A = 2\pi N_{\text{coll}}$, where $N_{\text{coll}}$ is the so called collimated Fresnel number $\tilde{A}$; note also that Berry et al. give an expression for $A$ tailored to the case of a confocal UC.