The presence of a phantom field in a Randall–Sundrum scenario

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Abstract

The presence of phantom dark energy in brane world cosmology generates important new effects, causing a premature big rip singularity when we increase the presence of extra dimensions and considerably competing with the other components of our Universe. This article first considers only a field with the characteristic equation $\omega < -1$ and then the explicit form of the scalar field with a potential with a maximum (with the aim of avoiding a big rip singularity). In both cases we study the dynamics robustly through dynamical analysis theory, considering in detail parameters such as the deceleration $q$ and the vector field associated to the dynamical system. Results are discussed with the purpose of treating the cosmology with a phantom field as dark energy in a Randall–Sundrum scenario.

Keywords: cosmology, phantom field, dynamical analysis

(Some figures may appear in colour only in the online journal)

1. Introduction

Recent observations at high redshift of supernovae of Type Ia [1], along with observations of anisotropies of cosmic microwave background radiation (CMB) [2], among others [3], show evidence of the current accelerated expansion of the Universe, suggesting the existence
of a repulsive energy with the capacity to accelerate the Universe, known as dark energy (DE). These same observations also confirm that DE comprises $\sim 67\%$ of the total composition of the Universe and has only played a role in the recent history of the Universe’s evolution.

In addition, from a theoretical treatment of the Raychaudhuri equation, it is possible to see that in order to obtain an accelerated expansion it is necessary that the dark fluid fulfill the equation of state (EoS) $\omega < -1/3$. In this vein and with the aim of explaining DE, the less expensive candidate is the well-known cosmological constant (CC), originally introduced by Einstein, but with a modern point of view regarding its origin, expressing the EoS as $\omega = -1$ to obtain an accelerated expansion. Despite the excellent agreement of the CC with observations [2, 3], the CC has a fundamental problem, since we assume that it comes from the contributions of quantum vacuum fluctuations [4], so that it has $\sim 120$ orders of magnitude of difference between the theoretical expectation value and the observational value [4]. In this sense, the theoretical community has been exploring many alternatives to control this problem, without a clear resolution so far [4–6]. However, this fundamental problem has encouraged the scientific community to propose alternative candidates for DE, such as quintessence, a phantom field, Chaplygin gas and extra dimension models, among others (for a thorough review of all these alternative models see [6, 7]). However, so far this problem remains open, and there are important ongoing theoretical and observational efforts with the aim of finally understanding the elusive nature of DE.

As we previously mentioned, extra dimension models are some of the most accepted candidates to understand accelerated expansion, being a natural solution due to their straightforward way of confronting the origin of the problem. Extra dimension models like the one proposed by Dvali, Gabadadze and Porrati (DGP) [8] is one of the models with the most promise to solve the DE problem, because it is possible to obtain a natural threshold between 4 and 5D physics, explaining how gravity could leak to the bulk and vice versa, imitating the actual accelerated expansion. Other highly successful models are the Randall–Sundrum models [9], originally created to solve the hierarchy problem between the standard model of particles (SM) and gravity. One of them is Randall–Sundrum I (RSI), which is characterized by the introduction of a 5D AdS compactified extra dimension between two Minkowski branes. The second one is Randall–Sundrum II (RSII) which has a non-compactified extra dimension with the same capacity to solve the hierarchy problem in a more economical way.

Furthermore, in the cosmological context, RSII has been the most successful model, since it permits the modification of the structure of Einstein’s field equations. Additionally, it is important to notice that RSII leads to three new tensors: the first one is associated with the second order corrections to the energy-momentum tensor; the second one is a tensor associated with the existence of matter in the bulk; and finally a tensor that contains non-local effects associated with the Weyl tensor [10]. In this context, we also emphasize that the disadvantage of this model is the need to manually introduce the DE fluid, because the geometrical characteristics do not suffice to obtain a natural accelerated period.

Therefore, the RSII model provides a new paradigm for the study of the Universe’s evolution with different components. Following this idea, we propose a dynamical analysis of the modified Friedmann equations with the addition of a matter fluid (dark and baryonic) and phantom DE in order to study the big rip singularity in this context. As we know from the traditional literature [11], phantom DE produces a big rip singularity at 22 Gyr which can be avoided if the potential has a maximum [12]. Another important characteristic is that a phantom field minimally coupled to gravity has the sign of the kinetic term, in contrast to the ordinary scalar fields (see also [13] as complementary literature on the phantom field). Moreover, the presence of a phantom field itself in a RSII scenario will generate a more abrupt big rip coupled with the brane tension $\lambda$, which is the free parameter of the theory. In
this sense, we establish two limits enunciated as: $\rho \gg \lambda$, which is the high energy limit (early times), and $\rho \ll \lambda$, which is the low energy limit (late times). In this vein, there are several reported attempts to constrain the brane tension parameter through table-top experiments [14], astrophysics observations [15] and cosmological analysis like Big Bang nucleosynthesis [16] and CMB [17]. Indeed, the brane tension lies in $\lambda_{\text{CMB}} > 3.44 \times 10^6$ eV$^4$ in the first one and $\lambda_{\text{TT}} \gtrsim 138.59 \times 10^{48}$ eV$^4$ in the latter one\(^5\), showing an enormous difference between the results, but constraining the region of possible lambda values. Setare \textit{et al} [18] made a brane-world model with a non-minimally coupled phantom field where they observed that this non-minimal coupling provides a mechanism for an indirect bulk-brane gravity interaction. For late-time cosmological evolution they achieved the $-1$-crossing of its EoS parameter.

We are now in a position to organize the paper in the following way. Section 2 is dedicated to constructing the modified Friedmann equation from the modified Einstein equation on the brane, as well as setting the necessary condition to obtain an accelerated Universe in this theory. Following these ideas, we construct section 2.1 in order to generate a numerical analysis taking into account the baryonic and the dark matter (DM) components as a dust fluid and the phantom DE. There we also discuss the possibility of an earlier big rip compared with GR predictions. In section 3, we revisit dynamical systems theory, in order to apply it in the following sections. Section 4, is dedicated to studying phantom DE through dynamical systems theory, focusing our attention on the evolution, the vector field and the deceleration parameter, always just considering $\omega < -1$ as the main characteristic of the phantom field. In order to extend our study, we develop section 5 with the aim of generating a detailed study of the phantom scalar field, using a scalar potential with a maximum; similarly, we focus our attention on the evolution, vector field and deceleration parameter in this case. Finally, in section 6, we discuss our results and we draw important conclusions.

We will henceforth use units in which $c = \hbar = k_B = 1$.

2. From the modified Einstein field equation to brane cosmology

We start this analysis by writing the Einstein field equation projected onto the brane as:

$$G_{\mu\nu} + \xi_{\mu\nu} = \kappa_4^2 T_{\mu\nu} + \kappa_5^2 \Pi_{\mu\nu} + \kappa_5^2 F_{\mu\nu}, \tag{1}$$

where $T_{\mu\nu}$ is the 4D energy-momentum tensor of the matter trapped inside the brane, $G_{\mu\nu}$ is the classical Einstein tensor and the rest of the terms on the right and left sides of the equation are explicitly given by:

$$(2a) \quad \kappa_4^2 = 8\pi G_N = \frac{\kappa_5^4}{6} \lambda,$$

$$(2b) \quad \Pi_{\mu\nu} = -\frac{1}{4} T_{\mu\alpha} T_{\nu}^{\alpha} + \frac{T_{\mu\nu}}{12} + \frac{g_{\mu\nu}}{24} (3T_{\alpha\beta} T^{\alpha\beta} - T^2),$$

$$(2c) \quad F_{\mu\nu} = \frac{2T_{\alpha\beta} h^{\alpha\beta}}{3} + \frac{2g_{\mu\nu}}{3} \left( T_{\alpha\beta} h^{\alpha\beta} - \frac{5}{4} T \right),$$

$$(2d) \quad \xi_{\mu\nu} = \frac{5}{4} C_{A.E} R^{A} h^{B} g^{C}_{\mu\nu} S_{A}^{C}.$$

\(^5\)The first one is related to CMB and the latter one to table top experiments (TT).
where \( \lambda \) is related to the brane tension, \( \kappa_{(4)} \) and \( \kappa_{(5)} \) are the 4 and 5D coupling constants of gravity, \( G_N \) is Newton’s gravitational constant, \( \Pi_{\mu\nu} \) represents the quadratic corrections of the energy-momentum tensor on the brane and \( F_{\mu\nu} \) gives the contributions of the energy-momentum tensor in the bulk projected onto the brane through the unit normal vector \( n_A \), having always in mind that Roman capital letters take the values 0,1,2,3,4. In addition \( \xi_{\mu\nu} \) gives the contributions of the 5D Weyl tensor, also projected onto the brane manifold [10].

We start the cosmological analysis by proposing the traditional homogeneous and isotropic line element as:

\[
\text{d}s^2 = -\text{d}t^2 + a(t)^2 \left( \text{d}r^2 + r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\varphi^2) \right),
\]

where \( a(t) \) represents the scale factor and we have assumed a flat geometry, as recent observations indicate [3, 19] i.e. \( k = 0 \). Using equation (1), with matter in the brane in the form of perfect fluids and assuming no matter in the bulk, it is possible to write the modified Friedman equation and the covariant Raychaudhuri equation in the following way [20]:

\[
\dot{H}^2 = \kappa^2 \sum_i \rho_i \left( 1 + \frac{\rho_i}{2\lambda} \right),
\]

\[
\dot{H} = -\frac{3\kappa^2}{2} \sum_i (\rho_i + p_i) \left( 1 + \frac{\rho_i}{\lambda} \right),
\]

where \( H = \dot{a}/a \) is the Hubble parameter, \( \kappa^2 = 8\pi G_N/3 = \kappa_{(4)}^2/3 \) is the renamed gravitational coupling constant and \( \rho_i \) is the energy density of the different components of the Universe. Note that \( \lambda \) is the free parameter of the theory, giving the threshold between low and high energy regimes of the Universe’s evolution. It is important to note how the regime of low energy is recovered when the following ratio is applied:

\[
\rho_i/2\lambda \rightarrow 0,
\]

recovering the traditional cosmological behavior.

In addition, the EoS necessary to accelerate the Universe satisfies the constraint

\[
w_p < -\frac{1}{3} \left[ \frac{1 + 2\rho_p/\lambda}{1 + \rho_p/\lambda} \right],
\]

where in this case, the equation corresponds to a DE fluid. Equation \((4b)\) can be easily calculated, assuming \( \ddot{a}/a > 0 \) to obtain an accelerated Universe [20]. If we are also considering phantom DE we impose the additional condition \( \omega_p < -1 \) [11], which implies that \( \rho_p/\lambda > -2 \) for the phantom field.

2.1. First integrals

First of all, we specify two fundamental quantities that are dominant in the actual stage of the Universe’s evolution: matter (dark and baryonic) and phantom DE, i.e. \( \omega_p < -1 \) [11]. As we previously mentioned, we assume that the other components are negligible for late times and we also consider non-interaction between the different components i.e. non-crossed terms.

Under these assumptions, the Friedmann equation can be written as:

\[
H^2 = \kappa^2 \left[ \frac{\rho_{m0}}{a^3} \left( 1 + \frac{\dot{\rho}_{m0}}{a^2} \right) + \frac{\rho_{p0}}{a^{3(1+w_p)}} \left( 1 + \frac{\dot{\rho}_{p0}}{a^{3(1+w_p)}} \right) \right],
\]

where we define \( \dot{\rho}_{m0} \equiv \rho_{m0}/2\lambda, \dot{\rho}_{p0} \equiv \rho_{p0}/2\lambda \), which depend on the free parameter of the theory. From here, it is possible to write the equations in terms of quadratures with the aim of integrating numerically. Thus, the previous equation can be written as:
\[
\int_{a(\tau_0)}^{a(\tau)} \frac{da}{\sqrt{\Omega_{0m}(a^{-1} + \rho_{0m}a^{-4}) + \Omega_{0p}(a^{7/2} + \rho_{0p}a^5)}} = \Delta \tau,
\]

where it is convenient to define the following dimensionless variables: \(\Omega_{0m} \equiv \kappa^2 \rho_{0m}/H_0^2\), \(\Omega_{0p} \equiv \kappa^2 \rho_{0p}/H_0^2\) and \(\tau \equiv H_0 t\).

As we can see in figure 1, the big rip singularity occurs earlier than predicted by GR, provided that we increase the presence of extra dimensions mediated by the brane tension. In this sense, high energy in early times in the Universe’s evolution could have caused totally different dynamics in contrast to what would be expected under the presence of a phantom field. Moreover, the reader can verify that we reproduce the results obtained by [11] for the big rip singularity at 22 Gyr, using the observational value of the Hubble constant [2]. Thus, we notice that observations can constrain the brane tension parameter to bound the presence of extra dimensions in the case where DE is modeled by a phantom field.

3. Revisiting a dynamical system analysis

Dynamical systems play an important role in cosmology [6], particularly in understanding the evolution of the Universe via the solutions of either numerical or analytical equations. Many of the tools produced by this area have been widely applied in some models, offering the possibility of studying different epochs of the Universe with this process (see for instance [6, 21, 22]).

As we mention above, in the following section we present a model with phantom dark energy in a brane world to be analyzed comprehensively. However, before our analysis we conduct a revision of the theory of dynamical systems.

First, consider the non-linear 3D system

\[
\begin{align*}
    x' &= xf(x, y, z) + \alpha x, \\
    y' &= yf(x, y, z) - \beta y, \\
    z' &= zf(x, y, z) + \gamma z,
\end{align*}
\]

where

\[
f(x, y, z) = -\alpha x^k + \beta y^k - \gamma z^k, \quad k \geq 1,
\]

and \(\alpha, \beta, \gamma \in \mathbb{R}^+ \setminus \{0\}\). For each choice of these parameters we have the associated critical points

\[
s_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}), \quad i = 0, 1, 2, 3,
\]

with \(\delta_0\) the Kronecker delta. The Jacobian matrix of the system is

\[
J = \begin{pmatrix}
-\alpha(x + 1)x^k + \beta y^k - \gamma z^k + \alpha & -\alpha y^k x^{k-1} & -\alpha z^k x^{k-1} \\
-\beta k x y^{k-1} & -\alpha x^k + \beta (x + 1)y^k - \gamma z^k - \beta & \beta k x y^{k-1} \\
-\gamma k x z^{k-1} & -\gamma k y z^{k-1} & -\alpha x^k + \beta y^k - \gamma (x + 1)z^k + \gamma
\end{pmatrix}.
\]

If \(x = (x, y, z)\), and we consider a small perturbation

\[
x \to s_i + \delta x,
\]

If \(s_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})\), and we consider a small perturbation
we obtain the associated system $\delta x' = J_s \delta x$, where $J_s$ is the Jacobian at the point $s$, and the Hartman–Grobman theorem guarantees the existence of a neighborhood for a critical point on which the flow of the system (8) is topologically equivalent to the linearized one. Thus the eigenvalues are

$$\lambda_i^1 = \alpha - (k + 1)\alpha \delta_{i1} + \beta \delta_{i2} - \gamma \delta_{i3},$$

$$\lambda_i^2 = -\beta - \alpha \delta_{i1} + (k + 1)\beta \delta_{i2} - \gamma \delta_{i3},$$

$$\lambda_i^3 = \gamma - \alpha \delta_{i1} + \beta \delta_{i2} - (k + 1)\gamma \delta_{i3}.$$  

Then, fixing the parameters $\beta = 3/2^{k-1}$ and $\gamma = 6/2^{k-1}$ we define the functions

$$g(i, k, \alpha)(j) = \lambda_i^j,$$

where it is possible to identify the values of $\alpha$ for which $g < 0$, $g > 0$, $g = 0$, and from table 1 we are able to identify the kind of point of the nonlinear system.

Figure 1. Numerical solution of equation (7) for different values of $\bar{\rho}_m$ and $\bar{\rho}_p$, including the GR case with a big rip singularity at 22 Gyr [11]. Notice that the big rip singularity occurs at an earlier time if we increase the brane tension parameter.
Table 1. Possible values for \(\alpha\) and \(g\) for every \(k\).

| \(s_0\) | \(j\) | \(g < 0\) | \(g > 0\) | \(g = 0\) |
|--------|------|-----------|-----------|-----------|
| \(i = 0\) | 1    | \((−\infty, 0)\) | \((0, +\infty)\) | \(\alpha = 0\) |
|         | 2    | \((−\infty, +\infty)\) |           |           |
|         | 3    |           | \((−\infty, +\infty)\) |           |
| \(i = 1\) | 1    | \((0, +\infty)\) | \((−\infty, 0)\) | \(\alpha = 0\) |
|         | 2    | \((−\infty, −\frac{1}{2}\sqrt{2})\) | \((−\frac{1}{\sqrt{14}}, +\infty)\) | \(\alpha = −\frac{3}{2}\sqrt{2}\) |
|         | 3    | \((\frac{6}{\sqrt{7}}, +\infty)\) | \((−\infty, \frac{6}{\sqrt{7}})\) | \(\alpha = \frac{6}{\sqrt{7}}\) |
| \(i = 2\) | 1    | \((−\infty, −\frac{1}{2}\sqrt{2})\) | \((−\frac{1}{\sqrt{14}}, +\infty)\) | \(\alpha = −\frac{3}{2}\sqrt{2}\) |
|         | 2    | \((−\infty, +\infty)\) |           |           |
|         | 3    |           | \((−\infty, +\infty)\) |           |
| \(i = 3\) | 1    | \((−\infty, \frac{6}{\sqrt{7}})\) | \((\frac{6}{\sqrt{7}}, +\infty)\) | \(\alpha = \frac{6}{\sqrt{7}}\) |
|         | 2    | \((−\infty, +\infty)\) |           |           |
|         | 3    | \((−\infty, +\infty)\) |           |           |

From here, we observe that for every \(k\) and \(\alpha \neq 0\) the critical point \(s_0\) is always a saddle point, and if \(\alpha\) decays at the rate \(0 < b < 1/2^{k-1}\), we find stability at the point \(s_3\) since we have an hyperbolic system.

Now, we consider the system

\[
x' = f_1(x, y, z) + xF(x, y, z),
\]

\[
y' = f_2(x, y, z) + yF(x, y, z),
\]

\[
z' = f_3(x, y, z) + zF(x, y, z),
\]

with \(f_1, f_2, f_3, P, Q \in \mathbb{R}[x, y, z]\) (the ring of polynomials in three variables with real coefficients) satisfying \(f_1(0) = f_2(0) = f_3(0) = P(0) = 0\) and \(F = P/Q\) an element of the set of rational function in three variables \(\mathbb{R}(x, y, z)\), with the possibility that it is not defined at the origin. Assume further that \(f_1, f_2, f_3, P\) are not irreducible polynomials. Consider the nonempty set

\[Z(a) = \{x \in \mathbb{R}^3 : f_1(x) = f_2(x) = f_3(x) = P(x) = 0\},\]

with \(a = (f_1, f_2, f_3, P) \in \mathbb{R}[x, y, z]\) being an ideal associated to the system. For the ideal

\[I(Z(a)) = \{p \in \mathbb{R}[x, y, z] : p(x) = 0, \forall x \in Z(a)\},\]

if \(f^n \in I(Z(a))\) for some \(n \in \mathbb{N}\), then \(f \in I(Z(a))\), because \(\mathbb{R}[x, y, z]\) is an integer domain, showing the inclusion \(\sqrt{a} \subseteq I(Z(a))\), where \(\sqrt{a}\) denotes the radical of \(a\). Now, let \(f \in I(Z(a)) \setminus \{0\}\) and consider the ideal

\[b = (f_1, f_2, f_3, P, (wF - 1)) \mathbb{R}[x, y, z, w],\]

in the ring \(\mathbb{R}[x, y, z, w]\). We note that \(Z(b) = \emptyset\), hence \(b = \mathbb{R}[x, y, z, w]\), so there are polynomials \(p_1, p_2, p_3, p_4, p_5\) such that

\[1 = p_1f_1 + p_2f_2 + p_3f_3 + p_4P + p_5(wF - 1).\]

Now, considering the ring \(\mathbb{R}(x, y, z)[w]\) with \(w = 1/f\), we have
\[ 1 = p_1 \left( \frac{1}{f} \right) f_1 + p_2 \left( \frac{1}{f} \right) f_2 + p_3 \left( \frac{1}{f} \right) f_3 + p_4 \left( \frac{1}{f} \right) P, \]  \tag{22}

and for some \( k \in \mathbb{N} \)
\[ f^k = q_1 f_1 + q_2 f_2 + q_3 f_3 + q_4 P, \]  \tag{23}

where \( q_1, q_2, q_3, q_4 \in \mathbb{R}[x, y, z] \), showing the inclusion \( \mathcal{I}(\mathcal{Z}(a)) \subset \sqrt{a} \) and the equality
\[ \mathcal{I}(\mathcal{Z}(a)) = \sqrt{a}. \]  \tag{24}

Using this result, a set of critical points for (17) is \( \mathcal{Z}(a) \setminus \{0\} \), which could possibly correspond to the points in equation (10) in the following cases:

(a) \( s_1 \) if
\[ f_1(x, 0, 0)Q(x, 0, 0) + xP(x, 0, 0) \sim_a f_i(x, 0, 0) \quad i = 2, 3. \]

(b) \( s_2 \) if
\[ f_2(0, y, 0)Q(0, y, 0) + yP(0, y, 0) \sim_b f_i(0, y, 0) \quad i = 1, 3. \]

(c) \( s_3 \) if
\[ f_3(0, 0, z)Q(0, 0, z) + zP(0, 0, z) \sim_c f_i(0, 0, z) \quad i = 1, 2, \]

with \( a, b, c \neq 0 \) and \( \sim \), denoting the equivalence relation for polynomial functions in one variable that vanish at the point \( r \).

4. Phantom dark energy in an RS scenario

We now start rearranging equation (4a) in the form:
\[ H^2 = \frac{8\pi G_N}{3} \sum_i \left( \rho_i + \frac{\rho_i^2}{2\lambda} \right), \]  \tag{25}

where, redefining the expression \( \tilde{\rho}_i \equiv \rho_i^2 / 2\lambda \), it is possible to write:
\[ H^2 = \frac{8\pi G_N}{3} \sum_i (\rho_i + \tilde{\rho}_i). \]  \tag{26}

In order to visualize the dynamic equations we propose two different methods. Both methods will generate the same dynamical information but in some cases we will choose one over the other, due to the different information we will be able to obtain from each of them.

(a) Method 1. When the dimensionless variables are constrained to lie on a 4D sphere, with Friedmann constraint \( 1 = \sum_i \left( x_i^2 + y_i^2 \right) \) and dimensionless variables:
\[ x_i^2 \equiv \frac{8\pi G_N}{3H^2} \rho_i, \quad y_i^2 \equiv \frac{8\pi G_N}{3H^2} \tilde{\rho}_i. \]  \tag{27}

Applying the Friedmann constraint, the dynamical equations reduce to:
\[
\frac{2y_1'}{3y_2} = \frac{3}{2}x_2^2 + y_1^2 - 2y_2^2 + \frac{3}{2}, \quad (28a)
\]
\[
\frac{2y_1'}{3y_1} = -\frac{3}{2}x_2^2 + y_1^2 - 2y_2^2 - 1, \quad (28b)
\]
\[
\frac{2y_2'}{3y_2} = -\frac{3}{2}x_2^2 + y_1^2 - 2y_2^2 + 2. \quad (28c)
\]

We note that this is the dynamical system (8) with \( k = 2 \) and \( \alpha = 9/4 \).

(b) Method 2. When the dimensionless variables are constrained to lie on a 4D plane, with Friedmann constraint \( 1 = \sum_i (x_i + y_i) \) and dimensionless variables:

\[
x_i \equiv \frac{8\pi G_N}{3H^2} \rho_i, \quad y_i \equiv \frac{8\pi G_N}{3H^2} \bar{\rho}_i. \quad (29)
\]

In the same way as in method 1, the Friedmann constraint helps us to reduce the dynamical equations, as in the the case \( k = 1 \) in (8) with \( \alpha = 9/2 \):

\[
\frac{x_2'}{3x_2} = -\frac{3}{2}x_2 + y_1 - 2y_2 + \frac{3}{2}, \quad (30a)
\]
\[
\frac{y_1'}{3y_1} = -\frac{3}{2}x_2 + y_1 - 2y_2 - 1, \quad (30b)
\]
\[
\frac{y_2'}{3y_2} = -\frac{3}{2}x_2 + y_1 - 2y_2 + 2. \quad (30c)
\]

In both cases, the primes denote an \( e \)-folding derivative \( N = \ln(a) \), where we also made use of the fact that \( \omega_{m(DM)} = \omega_1 = 0 \) and \( \omega_p = \omega_2 = -3/2 \), due to our assumption that the only components of the Universe are phantom DE and matter (baryonic and DM). Note that the choice of the phantom EoS is based on Planck satellite observations [3, 9].

For instance, the dynamical system represented by equation (28) can be solved numerically, (graphical solutions are shown in figure 2), establishing the initial conditions for \( \Omega_{0m} \) and \( \Omega_{0p} \) through the Planck satellite constraints [2], while the other initial conditions for \( \Omega_{0m} \) and \( \Omega_{0p} \) can also be constrained with [2] within a permitted region to manipulate the density parameters coupled by the brane tension. Here we separate the different components with the aim of visualizing the behavior. As the reader can observe, the phantom DE coupled with branes dominates in later stages of the Universe’s evolution while in similar conditions matter coupled with branes dominates in earlier stages of the Universe. In this context, it is important to give a more restrictive constraint on the brane tension parameter based on observations, in order to elucidate the effects of extra dimensions.

In addition, the deceleration parameter \( q = -\ddot{a}/aH^2 \) can be written in terms of equation (29) as:

\[
q(N) = \frac{1}{2} \left( 1 - \frac{3}{2}x_2 - \frac{7}{3}y_1 - \frac{10}{3}y_2 \right), \quad (31)
\]

where we have used the Friedmann constraint to eliminate the \( x_1 \) variable. The corresponding plot can be seen in figure 3, assuming the following initial conditions: \( x_2(0) \equiv \Omega_{0m} = 0.6 \) for the four plots and \( y_1(0) \equiv \Omega_{0m} = 0, y_2(0) \equiv \Omega_{0p} = 0 \) (dashed plot), \( \Omega_{0m} = 10^{-4}, \Omega_{0p} = 4 \times 10^{-4} \).
which are permitted small values according to the observations [2]. Note how brane terms, even if they are minimal, can cause an accelerated expansion process. Indeed, phantom dynamics in a non-brane theory has a region where the Universe does not exhibit an acceleration epoch.

Figure 2. Dynamical analysis of equation (28) with appropriate initial conditions for $\Omega_{\text{in}}, \Omega_{\text{in}}, \Omega_{\text{m}}$, and $\Omega_{\text{p}}$. The last two equations are subject to the constraints provided by the Planck satellite [2]. In all the cases there is a noticeable domination of the phantom field coupled with branes at later times while matter coupled with branes dominates the earlier stages of the Universe’s evolution, as can be expected.

(red plot), $\bar{\Omega}_{\text{in}} = 10^{-3}$, $\bar{\Omega}_{\text{p}} = 4 \times 10^{-3}$ (blue plot), $\bar{\Omega}_{\text{m}} = 10^{-2}$, $\bar{\Omega}_{\text{p}} = 4 \times 10^{-2}$ (green plot)
However, we can see that the more the brane effects are present the less pronounced the non-accelerated stages, which is clearly a contradiction to observations.

Another complementary analysis is shown in figure 4 where we present a vectorial dynamical analysis, showing only the region of interest, i.e. the region given by the Friedmann constraint. We start the analysis by finding the equilibrium points and eigenvalues associated with equation (30), defining the critical points as 

\[(x_1, y_1/dN) = 0\]. In this case, as can be seen in table 2, the critical points are associated with matter domination, phantom DE domination, matter coupled with branes domination and phantom coupled with branes domination, respectively.

In addition, we define the vector \(x = (x_2, y_1, y_2)\) and consider a linear perturbation of the form (12) and the Jacobian matrix \(J_{\mathcal{E}}\) associated with the linearized system. Table 2 shows the eigenvalues and eigenvectors associated with the critical points of \(J_{\mathcal{E}}\). Then, as seen in the previous section, the critical points can be classified according to the eigenvalues of the Jacobian of the linearized vector field at a specific point. Thus \((0, 0, 1)\) is an attractor and \((0, 1, 0)\) is a source since the eigenvalues associated with these points are all of negative or positive, respectively, while the origin \((0,0,0)\) and \((1,0,0)\) are saddle points of the non-linear system since their eigenvalues have opposite signs.

The region of interest is formed by families of solutions or dynamical fluxes, providing a qualitative description of the evolution of the system as a whole. The dynamics of a particular solution are governed by the initial conditions, \(x_2(0) \equiv \Omega_{\mathcal{E}}\), \(y_1(0) \equiv \Omega_{\text{m}}\), \(y_2(0) \equiv \Omega_{\mathcal{E}}\) which are called solution curves. In figure 4 the vector field and some numerical solutions (solid lines) are shown for different initial conditions, all of them satisfying the Friedmann constraint.

5. Phantom dark energy: a refined analysis

In the previous sections we only considered the phantom field under the constriction \(\omega_\mathcal{E} < -1\). However, we now propose a deeper analysis through the explicit form of the phantom field.

The coupling with gravity is given by the action [6]
with an opposite sign in the kinetic term, where $\phi$ is the phantom scalar field. Therefore, density and pressure are written as

$$
\rho = -\frac{\dot{\phi}^2}{2} + V(\phi),
$$

and

$$
p = -\frac{\dot{\phi}^2}{2} - V(\phi).
$$

Furthermore, as we previously mentioned, it is possible to avoid the big rip singularity if the potential:

$$
V(\phi) = V_0 [\cosh(\sqrt{G_N/\beta} \phi)]^{-1},
$$

has a maximum, where $\beta$ is a constant \[12\].

Thus, the Friedmann equation can be written as:

$$
H^2 = \frac{8\pi G_N}{3} \left\{ \rho_m \left( 1 + \frac{\rho_m}{2\lambda} \right) - \frac{1}{2} \dot{\phi}^2 \left( 1 - \frac{\phi^2}{4\lambda} \right) + \frac{V_0}{\cosh(\sqrt{G_N/\beta} \phi)} \left[ 1 + \frac{1}{2\lambda} \left( \frac{V_0}{\cosh(\sqrt{G_N/\beta} \phi)} - \dot{\phi}^2 \right) \right] \right\},
$$

along with the following equations

$$
\ddot{\phi} + 3H\dot{\phi} + \partial_\phi V = 0, \quad \dot{\rho}_m + 3H\rho_m = 0.
$$
Now, defining the appropriate dimensionless equations:

\[ x^2 \equiv \frac{8\pi G_N \rho_m}{3H^2}, \quad y^2 \equiv \frac{4\pi G_N \dot{\phi}^2}{3H^2}, \quad k^2 \equiv \frac{3H^2}{16\pi G_N \lambda}, \]  

\[ u^2 \equiv \frac{8\pi G_N V_0}{3H^2 \cosh(\sqrt{G_N \beta \phi})}, \]  

\[ \dot{\rho} \equiv \sqrt{\frac{3}{4\pi}} \beta \tanh(\sqrt{G_N \beta \phi}), \]

it is possible to reduce the modified Friedmann equation (34) to:

\[ 1 = x^2 + (u^2 - y^2) + k^2 [x^4 + (u^2 - y^2)^2], \]

recovering the traditional Friedmann equation when \( k \to 0 \). Then, the dynamical system can be written as:

\[ x' = -\left(\frac{3}{2} + \frac{H'}{H}\right)x, \]  

\[ y' = -\left(3 + \frac{H'}{H}\right)y + \frac{1}{2}u^2 \dot{\rho}, \]  

\[ u' = -\left(\frac{1}{2} \dot{\rho} y + \frac{H'}{H}\right)u, \]

together with

\[ l = \Omega_l \left[ \sigma \tanh \left(2\sigma \int y^2 dN\right) \right]^{1/2}, \]

where \( \sigma \equiv \sqrt{3/4\pi} \beta \) is another free parameter, which must be assigned in order to solve the previous equations. Note that we also made use of the Friedmann constraint. Additionally we have:

\[ H' = -\frac{3}{2} x^2 - \frac{1}{2} u^2 \dot{\rho} y - \frac{1}{x^4 + (u^2 - y^2)^2} (6y^4 - 2u^2 \dot{\rho} y^3 - 6u^2 y^2 + \frac{3}{2} u^2 \ddot{\rho} y). \]

Therefore the critical points for the system (38) are \((\pm 1, 0, 0), (0, \frac{3\pm \sqrt{9 - l^4}}{l}, 1), (0, \frac{3\pm \sqrt{9 - l^4}}{l}, -1)\) and it is possible to plot the dynamical system as shown in figure 5 with the initial conditions: \( \Omega_m = 0.33, \Omega_\phi = 0.33, \Omega_V = 0.72 \) and \( \Omega_l = 10^{-4} \), obtaining the expected behavior for a matter domination at earlier times and a posterior domination of the phantom DE, mainly in the potential of the field (see figure 5). We immediately recognize a state where \( V \gg \dot{\phi}^2 \) for large values of \( N \), producing an accelerated state.

Another conclusive study can be performed through the deceleration parameter \( q(N) \) which can be written in terms of the dimensionless variables (36) as:

\[ q(N) = \frac{1}{2} x^2 - 2x^2 - u^2 + \frac{1 - x^2 - (u^2 - y^2)}{x^4 + (u^2 - y^2)^2} \times (2x^4 + 5y^4 - 4y^2 u^2 - u^4). \]
Its behavior is shown in figure 6. From this, it is possible to observe a transition phase between an unaccelerated and accelerated state at $N \sim -0.4$ as would be expected in the traditional Universe behavior. However our results show a sudden phase transition in a short region of $N$, remaining stable for the value of $q \simeq -1$, and a Universe in a continuous state of acceleration.

Additionally, some extra information comes from the $k$ parameter which has a dynamical equation $k' = kH^2/H$, related to the brane tension. The numerical solution can be seen in figure 7, where we see a domination of the brane tension component in the earlier times of the Universe’s evolution, along with an abrupt peak related to the transition between an unaccelerated and accelerated Universe, and finally a subdominant epoch at later times. This evolution is always constrained by the brane tension which is bounded by current observations [16, 17].
Finally, we explore the vector field of the system (38), fixing the variables \( l = k = 10^{-1} \), which is shown in figure 8. Our results show two repulsers (the baryonic matter and the kinetic part of the phantom) and one attractor associated to the phantom DE potential.

6. Conclusions and discussion

The results presented in figure 1 show that branes generates a premature big rip singularity when the brane tension is accentuated. We emphasize that this important result cannot be obtained in the traditional cosmological analysis with phantom dark energy. In this vein, the analysis developed in this paper unmasks the dominant components in the beginning and in the end of the Universe (see figure 2), showing that the density parameters in the function of the brane tension will dominate in the future. These results are corroborated by figure 4.
where the repulsers associated with the components in the beginning of the Universe and the attractors related with the presence of extra dimensions in future epochs are shown (see also table 2).

Moreover, some important extra information can be obtained from the deceleration parameter $q$, showing the differences from the standard cosmological model ($\Lambda$CDM). Here it is possible to observe an ever-accelerating Universe, as the presence of the extra dimension increases. These results from the deceleration parameter agree with those shown in previous figures. In addition, the TT experiments, or others, could restrict the dynamics of the presence of extra dimensions, mimicking to a large extent the standard cosmological model.

As a complement, we develop an analysis when the form of the phantom DE is explicitly written as a scalar field. Indeed, we assume the same potential used in [12] with the aim of avoiding a big rip singularity, but now in the brane-world context. Thus, this scenario generates a matter dominant era and a posterior domination of the phantom field through the variable $\Omega_V$, which depends on the scalar field potential, implying an accelerated Universe for values $\omega \sim -1$. In addition to this, the deceleration parameter (see figure 6) also gives us information about the Universe passing from a non-accelerated to an accelerated state (which is an expected result). There is an abrupt change of phase (decelerated $\rightarrow$ accelerated) between $N = -0.8$ and $N = 0.2$, coinciding with the region where the brane tension presents an anomalous behavior (see figure 7). It is also possible to see that in figure 7 the density parameter related to brane tension shows the expected behavior, with a dominant brane tension in early epochs and subdominant brane tension in late epochs. Finally, the vector field presented in figure 8 corroborates our results, showing the expected attractor related to phantom dark energy potential and a repulser in the early times of the Universe’s evolution.

As a final comment, the phantom field in a brane-world scenario generates a premature big rip through the presence of brane tension, which can be stopped by the presence of a potential with a local maximum. This last analysis shows us a more adequate and congruent behavior as expected by observations, except for the transition region. Further analysis with observations will help us to constrain the brane tension. However, this is work that will be presented elsewhere.

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