Criticality versus $q$ in the $2 + 1$-dimensional $Z_q$ clock model

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Using Monte Carlo simulations we have studied the $d = 3$ $Z_q$ clock model in two different representations, the phase-representation and the loop/dumbbell-gas (LDG) representation. We find that for $q \geq 5$ the critical exponents $\alpha$ and $\nu$ for the specific heat and the correlation length, respectively, take on values corresponding to the case $q \to \infty$, where $\lim_{q \to \infty} Z_q = 3$DXY model. Hence in terms of critical properties the limiting behaviour is reached already at $q = 5$.

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Matter coupled-gauge field theories in $2 + 1$ dimensions have come under renewed scrutiny in the context of condensed matter physics in the past decade, as effective theories of strongly correlated systems\cite{1}. Concepts such as confinement-deconfinement transitions associated with the proliferation and recombination of topological defects of gauge fields, enter for instance in attempts at providing a theoretical foundation for breakdown of Fermi-liquid theory in more than one dimension. A large variety of such gauge-field theories have been proposed, and one model of particular interest is the compact abelian Higgs model\cite{2, 3, 5, 6}. This model consists of a compact gauge field coupled minimally to a bosonic scalar field with the gauge charge $q$. In a particular limit the dual of this model reduces to a loop-gas representation of the global $Z_q$ model\cite{4, 5, 6}. This identification has been the motivation for the current work, for a detailed account of the $q$ dependence of the full theory we refer to Ref. \cite{5, 6}.

The spin $Z_q$ model is a simple planar-spin model, where the direction of the spin is parametrized by a phase. This phase is restricted to the values $2\pi n/q$ with $n \in \mathbb{Z}$, and is defined by the following action

$$S = -\beta \sum_{\langle i,j \rangle} \cos \left( \frac{2\pi}{q} (n_i - n_j) \right). \quad (1)$$

The state is specified by the integer variables $n_i \in \{0,1, \cdots, q-1\}$. Special cases include $q = 2$ which is the Ising model, $q = 3$ which is the three state Potts model, and the limit $q \to \infty$ which corresponds to the XY model. In addition it is easy to see that for $q = 4$ the partition function $Z(2\beta, 4) = Z(\beta, 2) \times Z(\beta, 2)$. The aim of the current paper has been to determine how the critical properties interpolate between the well known Ising ($q = 2$) and XY ($q \to \infty$) limits. We have done this by measuring the exponent combination $(1 + \alpha)/\nu$ as a function of $q$.

In $d = 2$ the model has a quite peculiar phase structure, with an intermediate *incompletely ordered phase* (IOP), where the system shows behaviour similar to the critical Kosterlitz Thouless phase. Upon further cooling, the system will order completely into one of the $q$ completely ordered states\cite{7, 8}. In $d = 3$ the $Z_q$ model does not have an IOP, but there are generalisations of the model which do\cite{7, 8, 11}.

A related case is that of a globally $U(1)$ symmetric theory which is perturbed by a weak crystal field. Using RG and duality arguments it has been shown that for $q \geq 5$ the crystal field is an irrelevant perturbation, whereas for $q \leq 4$ the XY fixed point is rendered unstable\cite{9}.

It is important to emphasise that we have focused on the properties of the $Z_q$ model at the critical point. For $T < T_c$ the discrete nature of the model will always be apparent. A beautiful RG study of the $Z_6$ model shows how the couplings of the model flow towards a fixed point which is ultimately different from the 3DXY fixed point in the $T \to 0$ limit\cite{12, 13}.

Eq. (1) is straightforwardly reformulated as a model of an interacting ensemble of links which either form closed loops or originate and termine at point charges. We start with the partition function

$$Z(\beta, q) = \sum_{\{n_i\}} \exp \left[ \beta \sum_i \left( \sum_{\mu} \cos \left( \frac{2\pi}{q} \Delta_{\mu} n_i \right) \right) \right]. \quad (2)$$

The first step is to replace the cosine with a quadratic potential, this is the Villain approximation\cite{14}. Next, we promote the integers $n_i$ to real-valued phase variables $\theta_i$, at the expense of introducing an auxiliary integer field $Q$, which through the Poisson summation formula\cite{15} restricts the $\theta_i$ variables to the discrete values allowed by original theory. The resulting partition function is then given by
In Eq. 3 \( \{ k \} \) is an integer link field living on the links of the original lattice, and \( \{ Q \} \) is a scalar field living on the sites of the same lattice. The prefactor \( \Xi[\beta] \) and effective coupling \( \beta_V = \beta_V(\beta) \) must be retained to get results which agree with Eq. 2 on a quantitative level, however they do not affect the critical properties and from now on we will assume \( \beta_V = \beta, \Xi[\beta] = 1 \), and omit the \( V \) index on the partition function.

In Eq. 3 the \( Q \)-field explicitly accounts for the discrete nature of the \( Z_q \) model. Setting \( Q \equiv 0 \), we recover the Villain representation of the XY model. Due to this similarity, the remaining analysis follows well known steps, which we briefly include for completeness. A Hubbard-Stratonovich decoupling of the quadratic expression in Eq. 3 is performed by introducing an auxiliary field \( v \), thus bringing the partition function onto the form

\[
Z(\beta, q) = \int Dv D\theta \sum_{\{ k, Q \}} \exp \left[ -\sum_i \left( \frac{1}{2\beta} v^2 + iv \cdot (\Delta \theta_i - 2\pi k) + iq\theta Q \right) \right].
\]

In Eq. 4 the \( \{ k \} \) summation can be performed, thereby restricting the velocity field \( v \) to integer values denoted by \( l \). In the term coupling \( \Delta \theta \) and \( l \), a partial integration can be performed, such that \( \theta \) only appears in the combination \( i\theta(\Delta l - qQ) \), from this we get the constraint

\[
(\Delta l - qQ) = 0.
\]

At this stage the transformation to a loop gas is complete, and the partition function is given by

\[
Z(\beta, q) = \sum_{\{ l, Q \}} \delta_{\text{Vill}, qQ} \exp \left[ -\frac{1}{2\beta} \sum_i l^2 \right].
\]

In the compact Abelian Higgs model considered in refs. \[15\] the fields \( \{ l \} \) and \( \{ Q \} \) represent vortices and monopoles, i.e. they are the topological excitations of the matter-field and gauge-field respectively. That interpretation does not apply in the current case, but the

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considered is the connected third order moment of the action [6]
\[
\langle (S - \langle S \rangle)^3 \rangle \propto |\beta - \beta_c|^{1+\alpha},
\]
which recently has been demonstrated to yield surprisingly good scaling results compared to second moments [6]. When approaching the critical point, the correlation length \( \xi \) diverges as \( \xi \propto |\beta - \beta_c|^{-\nu} \). Therefore, in a finite system of linear extent \( L \) we find that the third order moment in Eq. 7 scales with \( L^{1+\alpha/\nu} \).

The main advantages of the third order moment in Eq. 7 are that (1) good quality scaling is achieved for practical system sizes even for models with \( \alpha < 0 \), e.g. the 3DXY model, and (2) one set of measurements gives both the combination \( (1+\alpha)/\nu \) and \(-1/\nu\) independently [6], although it is more difficult to achieve high precision on the latter. A schematic figure of \( \langle (S - \langle S \rangle)^3 \rangle \) as a function of coupling constant is shown in Fig. 2 and figures 3 and 4 show finite-size scaling (FSS) of the peak to peak value.

We have considered systems of size \( L \times L \times L \) with \( L = 8, 10, 12, 16, 20, 24, 32, 40, 48 \), and up to 2-10\(^7\) sweeps over the lattice. In addition to the \( q = 4 \) and \( q = 5 \) presented in figures 3 and 4, we have also studied the \( q \) values \( q = 6, 8, 12, 16, 24 \), ref. 6 shows results of \( q = 2 \) simulations of Eq. 4. We find that the combination \( (1+\alpha)/\nu \) changes abruptly from the \( Z_2 \) value of 1.763 [10] to the XY value of 1.467 [16] when increasing \( q \) from \( q = 4 \) to \( q = 5 \). A further increase of \( q \) beyond \( q = 5 \) does not affect the value of \( (1+\alpha)/\nu \), as shown in Fig. 4.

That the \( Z_q \) model is in the XY universality class for \( q \geq 5 \) must imply that at the critical point the discrete structure is rendered irrelevant for these \( q \) values. To investigate this point further, we have implemented a simple real-space RG procedure, which attempts to probe for what values of \( q \) the discrete nature of \( Z_q \) model is relevant at the critical point. We denote the untransformed phases and fields as \( \theta_0 \). The renormalized phase at level \( n+1 \) is given by the block spin construction

\[
\theta_n(k) = \text{atan} \left( \frac{\sum_k \sin \theta_n(k)}{\sum_k \cos \theta_n(k)} \right),
\]

where the sum over \( k \) in Eq. 9 is over the eight spins in a \( 2 \times 2 \times 2 \) cube. For \( q = 2 \), this transformation is clearly trivial, since adding a number of phases 0 and \( \pi \) will still give 0 or \( \pi \). However, for \( q > 2 \) the effective \( q^* \) will increase with \( n \), and for \( n \to \infty \) the resulting block

FIG. 1: A typical LDG configuration for the \( q = 2 \) (Ising) model. Multiply connected links, like the vertical along the left edge have much lower entropy than loop/dumbbell combinations, and hence give a relatively small contribution to the partition function.

FIG. 2: Schematic figure showing third moment of action, and how data are extracted for FSS analysis. For further details of this method see [6].

FIG. 3: This figure shows the scaling of \( \langle (S - \langle S \rangle)^3 \rangle \) for \( q = 4 \) (■) and \( q = 5 \) (●); the results are obtained using the phase representation Eq. 2. The \( q = 4 \) results show \( Z_2 \) scaling with \( (1+\alpha)/\nu = 1.76 \pm 0.03 \), and the \( q = 5 \) curve shows XY scaling with \( (1+\alpha)/\nu = 1.46 \pm 0.03 \).
spins can take any direction.

We next investigate whether the system flows towards an infinite value of \( q^* \) or not under such a RG transformation. This is tantamount to asking whether the discrete structure is rendered irrelevant or not on long length scales. To this end, at each iteration step \( n \), we have recorded histograms \( h_n(\theta) \) of the phase distributions on the lattice, and monitored the manner in which this histogram flows under rescaling. By purely visual inspection we find that for \( q = 4 \) the discrete nature of the \( Z_q \) model persists, whereas for \( q = 5 \) it is washed away, this is illustrated in Fig. 6.

To study this RG flow at a quantitative level, we have written the phase distribution \( P_n(\theta_n) \) as a sum of harmonic functions

\[
P_n(\theta_n) = a_{n,0} + \sum_k \left( a_{n,k} \cos \left( \frac{k2\pi}{q} \right) + b_{n,k} \sin \left( \frac{k2\pi}{q} \right) \right)
\]

(10)

Here, the coefficient \( a_{n,k} \) in Eq. (10) denotes the \( k \)-th Fourier-cosine component at RG level \( n \). Clearly, the coefficient \( a_{n,q} \) is the interesting component, we have studied how this coefficient flows under repeated rescaling. For \( q = 4 \) this coefficient shows critical fixed point behaviour, whereas for \( q = 5 \) it flows to zero, even for \( T \) well below the critical temperature, this is shown in Fig. 7.

Also the LDG representation Eq. (8) gives a qualitative indication that for \( q \geq 5 \) the discrete nature of the theory is irrelevant. In this representation the discrete nature is represented solely by the \( Q \) excitations, so measurements of \( \langle Q \rangle \) should give a quantitative indication of the importance of the discrete structure. Measurements of \( \langle Q \rangle \) at the critical point give \( \langle Q \rangle \approx 0.07, 5.9 \times 10^{-4} \) and \( 2.75 \times 10^{-6} \) for \( q = 2, 4 \) and 5 respectively, whereas the link density \( \langle l \rangle \approx 0.15 \) for all \( q \). Hence at \( q = 5 \) the discrete \( Q \) excitations have been completely frozen out, and the tangle is essentially identical to the pure-loop tangle of the 3DXY model.

In summary, we have determined the critical exponent combination \((1 + \alpha)/\nu\) in the \( d = 3 \) \( Z_q \) spin model for \( q = 4 \). Using two different representations we have found that for \( q \geq 5 \), the combination \((1 + \alpha)/\nu\) takes a value which is consistent with the value taken in the 3DXY model. Along with other more qualitative indicators this means that at the critical point discrete structure finer than \( q = 5 \) is irrelevant at the critical point, and the long distance properties of the theory are determined by the larger symmetry group \( U(1) \). These results are in accordance with RG studies starting with a \( U(1) \) sym-
FIG. 7: The flow of the coefficient $a_{n,q}$ for $q = 4$ and $q = 5$. For $q = 4$, we see that there is a fixed point at the critical point, whereas for $q = 5$ we see that $a_{n,q}$ flows to zero at the critical point. In the figure $a_{n,5}$ flows to zero also for $T < T_c$; this is a finite size effect. This coefficient will eventually flow to infinity for sufficiently large systems/low $T$.

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