CONFIGURATIONS OF BALLS IN EUCLIDEAN SPACE THAT BROWNIAN MOTION CANNOT AVOID

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ABSTRACT. We consider a collection of balls in Euclidean space and the problem of determining if Brownian motion has a positive probability of avoiding all the balls indefinitely.

1. INTRODUCTION

We write \( \overline{B}(c;r) \) for the closed ball in \( \mathbb{R}^d \) with centre \( c \) and radius \( r \), and write \( S(c;r) \) for the sphere of that centre and radius. We consider a region that is formed by removing a countable collection of non-overlapping closed balls from \( \mathbb{R}^d \). Thus

\[
\mathbb{R}^d = \bigcap_{n=1}^{\infty} \overline{B}(c_n; r_n);
\]

and we assume, for convenience, that \( 0 \) lies in \( \mathbb{R}^d \). We say that such a collection of balls is avoidable if there is a positive probability that Brownian motion in \( \mathbb{R}^d \), starting from \( 0 \), never hits any of the balls. Thus the collection of balls is avoidable if the balls do not have full harmonic measure w.r.t. the domain \( \mathbb{R}^d \), or if infinity has positive harmonic measure w.r.t. \( \mathbb{R}^d \). We address the problem of obtaining a geometric characterization of avoidable configurations of balls.

The genesis of this problem is to be found in the paper of Ortega-Cerdà and Seip [2]. Motivated by a question of Akeroyd [1], the analogous problem in the setting of the unit disk was completely solved when the centres of the disks that are removed form a uniformly dense sequence.

In the plane it is possible to hide infinity from \( 0 \) with a single disk. This reflects the fact that Brownian motion in the plane is recurrent and that the sphere \( S(c;r) \) has full harmonic measure with respect to \( \mathbb{R}^2 \cap \overline{B}(c;r) \). For this reason, our results are set in Euclidean space of dimension three or more, in which Brownian motion is transient. It is helpful to bear in mind that, in dimension three or more, \( (\mathbb{E}^d \setminus \{c\})^2 \) is the harmonic measure at \( 0 \) of the sphere \( S(c;r) \) with respect to the domain \( \mathbb{R}^d \cap \overline{B}(c;r) \). In fact, the harmonic measure of this sphere at \( x \) is \( \mu(x) = (\mathbb{E}^d \setminus \{c\})^2 \).

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Proposition 1. We suppose that \( d > 3 \). If
\[
\sum_{n=1}^{\infty} \frac{r_n}{j_n^d} < 1
\]
then the collection of balls \( \bigcup_{n=1}^{\infty} B(c_n, r_n) \) is avoidable.

In order to avoid situations in which a number of small balls packed very close together can contribute significantly to the sum in (1.1) but contribute relatively little to the overall harmonic measure, we now require a separation condition on the balls:
\[\text{(S) there is a positive number } \beta \text{ such that } j_n c_n \geq \beta \text{ for } n \geq m.\]

Theorem 1. We suppose that \( d > 3 \). We assume the separation condition (S) and that there is a number \( M \) such that
\[
r_n^d j_n^2 \leq M \text{ for } n \geq 1.
\]
If the collection of balls \( \bigcup_{n=1}^{\infty} B(c_n, r_n) \) is avoidable then
\[
\sum_{n=1}^{\infty} \frac{r_n}{j_n^d} < 1.
\]

The solid angle subtended by the sphere \( S(c; r) \) at 0 is proportional to \( \left( \frac{r}{j} \right)^d \). The appropriate version of Akeroyd’s question in the present setting is whether there is an avoidable sequence of balls for which the sum \( \sum_{n=1}^{\infty} (r_n = j_n^d)^{d-1} \) is finite. If so, it is possible to hide infinity from the origin from the point of view of harmonic measure even though geometrically there is a clear line of sight to infinity except for a set of directions on the sphere \( S^d \) of arbitrarily small \( (d-1) \)-dimensional measure. Consider \( m^{d-1} \) balls of radius \( m^{-1} \), with \( r_n = j_n m^2 \), arranged evenly on the sphere \( S(0;m) \), this for each integer \( m \) greater than some large \( m_0 \). These balls will be non-intersecting and separated, and (1.2) will hold since \( m^2 m^2 = 1 \). But (1.3) does not hold: in fact
\[
\sum_{n=1}^{\infty} \frac{r_n}{j_n^d} = \frac{X}{m^{d-1} m^1} \frac{d^2}{m^2} = \frac{X}{m^{d-2} m^0} \frac{1}{m}.
\]

By Theorem 11 the collection of balls is unavoidable. Even so, \( \sum_{n=1}^{\infty} (r_n = j_n^d)^{d-1} \) is finite.

We will now consider a more regular configuration of balls. We say that the balls are regularly located if (i) the separation condition (S) is satisfied, (ii) the balls are uniformly dense, in that there is a positive \( R \) such that any ball \( B(c; R) \) contains at least one centre \( c_n \), (iii) the radius of any ball depends only on the distance from the ball’s centre to the origin, with \( r_n = \langle c_n, j \rangle \) where \( j \) is a decreasing positive function.
Theorem 2. We suppose that \( d \geq 3 \) and that the balls \( B(c_n, r_n) \), \( n \in \mathbb{Z}^+ \), are regularly located. Then the collection of balls is avoidable if and only if
\[
\int_0^1 (r)^d \, dr < 1.
\]

Theorem 2 is a partial converse to Proposition 1 in that if the radii of the balls decrease sufficiently rapidly then the collection of balls is avoidable only if (1.1) holds. Theorem 2 will be proved by showing that condition (1.2) is automatically satisfied if the collection of balls is both regularly located and avoidable. Hence these results do not give rise to a configuration (1.2) is automatically satisfied if the collection of balls is both regularly located and avoidable. Then the combined harmonic measure at \( x \) is unbounded. In fact, the possibility that condition (1.2) is redundant in Theorem 1 has not been ruled out as yet. We address this gap in our final result.

Theorem 3. Suppose that \( f \) is any increasing unbounded function on \( D; 1 \). Then there is a separated and avoidable collection of balls \( B(c_n, r_n) \), \( n \in \mathbb{Z}^+ \), for which
\[
\int_0^1 \frac{r_n^d}{j_n} f(j_n^d) j_n^d \, dr = 1.
\]

We will write \( \langle x; E \rangle \) to denote the harmonic measure at \( x \) of a subset \( E \) on the boundary of a region \( D \) with respect to \( D \).

2. Proof of Proposition 1

We suppose that (1.1) holds and choose \( N \) so large that
\[
\sum_{n=N+1}^{X^d} \frac{r_n^d}{j_n^d} < \frac{1}{2}.
\]

We write \( R = \mathbb{R}^d \) for \( R^d \) \( n \in \mathbb{R}^d \) \( E(c_n, r_n) \). For \( n > N \), the harmonic measure of the sphere \( S(c_n, r_n) \) at 0 with respect to \( E(c_n, r_n) \) is less than its harmonic measure with respect to the larger domain \( R^d \) \( B(c_n, r_n) \), which is \( r_n = j_n^d \). Thus the combined harmonic measure at 0 with respect to \( S(c_n, r_n) \), \( n > N \), is at most 1=2. As a consequence, Brownian motion in \( R^d \) starting from 0 has a positive probability (at least 1=2) of avoiding the set \( E = \bigcup_{n>N} S(c_n, r_n) \) indefinitely.

We write \( u(\kappa) \) for the harmonic measure \( \langle x; E \rangle \) \( \mathbb{N} \), so that \( u(0) < 1=2 \). The set of points \( x \in \mathbb{N} \) at which \( u(\kappa) < 1=2 \) is unbounded. In fact, suppose that it was the case that \( u(1=2) \) on \( (0; R) \) \( \mathbb{N} \). We could then apply the maximum principle to the harmonic function \( u \) in \( B(0; R) \) \( \mathbb{N} \), noting that \( u = 1 \) on the boundary of \( \mathbb{N} \) inside \( B(0; R) \) (that is, on \( E \) \( B(0; R)) \), and deduce that \( u(1=2) \) in \( B(0; R) \) \( \mathbb{N} \).

We now write \( F \) for the bounded set \( \bigcup_{n>N} S(c_n, r_n) \), and choose \( R \) so that \( F \) \( B(0; R) \). Then, for \( \kappa > R \),
\[
\langle x; F \rangle \langle x; F \rangle \langle x; (0; R) \rangle \langle x; (0; R) \rangle = \frac{R^d \langle x; R \rangle}{j_x^d}.
\]
It follows that, as $\|x\|_2 \to 1$, the harmonic measure $\mu(x;F;\gamma)$ tends to 0. Thus we may be sure that there is a point $x_0$ in $E$ for which both $\mu(x_0;F;\gamma) < 1/2$ and $\mu(x_0;B;\gamma) = 0$. The finite boundary of $E$ that is $E = \bigcup_{n=1}^\infty (c_n;\mathbb{R})$, does not have full harmonic measure at $x_0$. By the maximum principle, it does not have full harmonic measure at 0 either, and so the balls $B(c_n;\mathbb{R})$, $n \geq 1$, do not hide infinity from the origin.

3. Proof of Theorem 1

Let us suppose that (1.2) holds and that $\prod_{n=1}^\infty \frac{r_n}{c_n} = 1$ is divergent. We wish to show that Brownian motion starting from 0 will never escape to infinity.

We set $I_n = \{ n \in \mathbb{N} : 2^{n-1} < \|x_n\|_2 < 2^n \}$ and note that there is a $k_0$ between 1 and $m_0$ inclusive for which

$$\sum_{j=0}^{m_0} \frac{r_j}{2^{m_0-j}} = 1.$$

We ignore all balls whose index does not lie in $I_{k_0} \cap I_{m_0}$ for some $j$, with fewer balls to avoid, it is easier for Brownian motion starting from 0 to escape to infinity in this new domain. The balls that remain lie more or less in annuli whose inner radius is half the outer radius but arranged so that the annuli are far apart, in that the inner radius of each annulus is $2^{m_0-j}$ times that of the previous annulus.

Following the argument of Ortega-Cerdà and Seip [2, p. 909], we write $m_j$ for $k_0 + jm_0$, $R_j$ for $2^{m_j-1}$, $S_j$ for $S(0;R_j)$ and set $P_j$ to be the probability that Brownian motion in $S_j$ starting from 0 hits $S_j \setminus \mathbb{R}$. We need to show that $P_j = 0$ as $j \to \infty$.

We let $Q_j$ be the supremum of the probabilities that Brownian motion with starting point on $S_j \setminus \mathbb{R}$ hits $S_{j+1} \setminus \mathbb{R}$. Then

$$P_{j+1} = Q_j P_j$$

and so

$$P_{n+1} = P_1 \frac{\gamma}{Q_j} \prod_{j=1}^n Q_j.$$

If $0 < a_j < 1$ and $\prod_{j=1}^\infty (1 - a_j)$ is divergent, then the infinite product $\prod_{j=1}^\infty a_j = 0$. Theorem 1 therefore follows from the next lemma.

Lemma 1. We set $C$ to be $1 + \frac{4^{d+3}M}{d}$. Then, for all sufficiently large $j$,

$$1 - Q_j \leq \frac{1}{2^d} \frac{X}{C} \frac{r_j}{2^{m_0-j}}.$$ (3.1)
Proof. We write
\[ j = B \left( 0; R_{j+1} \right) \cap \bigcap_{n \geq m} B \left( c_n; r_n \right) ; \]

Then \( Q_j \supseteq Q_j^* \), where \( Q_j^* \) is the supremum of the probabilities that a Brownian motion in \( j \) with starting point on \( S_j \) hits \( S_{j+1} \). Lemma 1 may be proved, therefore, by showing that
\[ \inf_{x \in S_{j+1}} \mathbb{P} \left[ x \in S_j \right] < 1. \]

Lemma 1 may be proved by showing that
\[ \int_{x \in S_{j+1}} \mathbb{P} \left[ x \in S_j \right] \leq 1. \]

We consider
\[ u \left( x \right) = \sum_{n \geq m} \frac{r_n^{d+2}}{2k^d M^{d+1}} \]

for \( x \in S_{j+1} \). We suppose that \( x \in S \left( c_m; r_m \right) \) for some \( m \geq m \), Then \( r_m = \sum_{n \geq m} r_n \). We now show that the assumption that
\[ r_n^{d+2} \sum_{j \geq m} 4^d M^{d+1} \]

leads to
\[ \sum_{n \geq m} \frac{r_n^{d+2}}{2k^d M^{d+1}} \leq \frac{4^d M^{d+1}}{2k^d} \]

By (1.2), we may assume that \( r_n < \) for \( n \geq m \). Once \( j \) is sufficiently large. The separation condition \( S \) implies that there are at most \( 4^d 2^k d \)
balls whose centres lie at a distance of more than \( 2^k \) but less than \( 2^k + 1 \)
from \( x \), for \( k \geq 1 \). Each putative ball in this annulus contributes at most
\[ \frac{M}{2k^d M^{d+1}} \]

to the sum in (3.3). Since \( m_j + 1 \) annuli centred at \( x \) will cover all balls \( B \left( c_n; r_n \right) \) with \( n \geq m \), we find that
\[ \sum_{n \geq m} \frac{r_n^{d+2}}{2k^d M^{d+1}} \leq \frac{4^d M^{d+1}}{2k^d} \]

Since \( R_j^2 = 2^m \), the estimate (3.3) follows. We have shown that the harmonic function \( u \) satisfies
\[ u \left( x \right) = 1 \sum_{n \geq m} \frac{r_n^{d+2}}{2k^d M^{d+1}} \]
We now need an estimate for the size of $u$ on the sphere $S_{j+1}$. If $j \leq R_j$ and $j \neq R_{j+1}$, then

$$j \leq c_n j_1 R_{j+1} 2R_j = (2^{m_0} 2)R_j 2^{m_0} R_j;$$

Thus, for $x \in S_{j+1}$,

$$u(x) = \frac{1}{2^{2d_1 - 1}} \frac{R_n}{j \in I_m} \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right) \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right) \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right) \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right) \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right)$$

(3.5)

It follows from (3.4), (3.5) and the maximum principle that, for $x \in S_j$,

$$C \leq x \in S_{(c_n, R_n)}; \quad u(x) = \frac{1}{2^{d_1 - 1}} \frac{R_n}{j \in I_m} \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right) \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right) \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right) \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right) \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right) \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right)$$

Finally, we use this inequality with $x \in S_j \setminus j$. For such $x$, we have $j \leq c_n j_1 j \leq j_1 2j_n j$ and so

$$u(x) = \frac{1}{2^{d_1 - 1}} \frac{R_n}{j \in I_m} \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right) \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right) \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right) \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right) \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right) \frac{d}{2} \left( \frac{X}{n2 I_m} \frac{R_n}{j_n} \right)$$

The estimate (3.2) follows immediately. This completes the proof of the lemma, and hence of Theorem 1.

**Remark.** If the centres of the balls lie on a $(d - 1)$-dimensional hyperplane, then the conclusion of Theorem 1 still holds with the assumption (1.2) replaced by the weaker assumption $r_n = \log j_n j$ $M$. Working through the proof of Lemma 1, it is still possible to conclude that $u$ is bounded on the boundary of the balls with index in $I_m$, by a constant that is independent of $j$, as in (3.4). (In fact, there are at most $4^d 2^k 1$ balls ‘whose centres lie at a distance of more than $2^k$ but less than $2^{k+1}$ from $x$, for $k = 1$’. The remainder of the proof of Lemma 1 is unchanged.

If the centres of the balls lie on a $(d - 2)$-dimensional hyperplane then one may replace (1.2) by the weaker assumption $r_n = \log j_n j$ $M$ and still retain the conclusion of Theorem 1. For example, suppose that in $R^3$ we put a ball of radius $r_n$ at the point $(n; 0; 0)$, for $n \geq 2$. Under the assumption that $r_n = M = \log n$, this string of beads in $R^3$ is avoidable if and only if $r_n = n$ is finite.
If the centres of the balls lie on a \((d-3)\)-dimensional hyperplane, then it suffices to assume that the radii of the balls are uniformly bounded in order for Theorem 1 to hold. In this case, however, there can be at most about \(m^{d/4}\) balls whose distance from the origin is about \(m\). Assuming that the radii of the balls are bounded by \(R\), say, it follows that

\[
X_n \sum_{j=1}^{\frac{d}{2}} \frac{r_n}{j} \quad R^d \quad X_m \sum_{j=1}^{\frac{d}{2}} \frac{1}{j} \quad CR^d \quad m^{d/4} \quad \frac{1}{m^{d/2}}
\]

which is finite. A collection of balls of uniformly bounded radius whose centres lie on a \((d-3)\)-dimensional hyperplane will always be avoidable.

4. Proof of Theorem 2

To begin with we note that, in the case of a regularly located configuration of balls, the sum \(\sum_n (r_n = j)\) and the integral \(\int_R (r)^d \, dr\) are comparable. The implication that the balls are avoidable if \(\int_R (r)^d \, dr\) is finite is now an immediate consequence of Proposition 1.

The reverse implication will follow from Theorem 1 once we check that the condition \((1.2)\) is automatically satisfied under the regularity assumption if the balls are avoidable. We establish this in the next lemma, whose proof bears a certain resemblance to that of Lemma 1.

**Lemma 2.** Suppose that the balls \(B(c_n, r_n)\) are regularly located and that \(r^2 \, (r)^d \) is an unbounded function of \(r\). Then the collection of balls is not avoidable.

**Proof.** There is a sequence of radii \(R_{j+1} \geq 4R_j\) for which \(R_j^d \, (R_j)^d \) as \(n \to \infty\). We put

\[
C = \frac{A_2}{2A_3}
\]

where the particular numbers \(A_2\) and \(A_3\) that we need depend on the dimension, on the separation number and on the density number \(R\) but on nothing else, and may be worked out in principle from the proof that follows. We assume that \(R_{j+1} > 4R_j\) and that \((R_j)^d \geq C\) for each \(j\). For a technical reason, we change the definition of \(\rho\) in the following way: we set \(\rho(x) = (2R_j)^d \) if \(x \in [R_j, 2R_j]\) for some \(j\) and \(\rho(x) = (x)\) elsewhere. We take new balls \(B(c_n, \rho(x))\). The size of the balls is thereby decreased: thus if the new balls are unavoidable then the original ones are unavoidable too. For the sake of simplicity, we will still denote by \(\rho\) the regularized \(\rho\) and the new smaller balls will still be called \(B(c_n, \rho(x))\). We write \(S_j\) for the sphere \(S(0; R_j)\) and \(\gamma_j = R_j = (2R_j)\).

Arguing as in the proof of Theorem 1 we let \(Q_j\) be the supremum of the probabilities that Brownian motion in \(\gamma_j\) hits...
S_{j+1} \setminus S_j$, and wish to show that $Q_{j+1} - Q_j = 0$, that is that

$$\chi_{S_j}(Q_j) = 1 \quad \text{for} \quad j = 1, \ldots, k.$$  

We write $I_j$ for $n : R_j \setminus S_j \setminus 2R_j$, and write

$$j = B(0;R_{j+1}) \cap \bigcup_{n \in I_j} B(c_n; r_n):$$  

Then $Q_j$ is bounded above by $\hat{Q}_j$, the supremum of the probabilities that Brownian motion with starting point on $S_j \setminus S_{j+1}$ hits $S_{j+1} \setminus S_j$. We will show that, for all sufficiently large $j$,

$$(4.1) \quad \sup_{x \in S_j \setminus S_{j+1}} \hat{Q}_j = \inf_{x \in S_j \setminus S_{j+1}} \sup_{x \in S_j \setminus S_{j+1}} \min_{|x| \leq \tau_n} u(x):$$  

for some positive $A$. Again we consider

$$u(x) = \frac{A}{\tau_n} \frac{1}{d} \frac{x}{c_n} \quad \text{for} \quad x \in S(c_n; r_n)$$  

so that $u$ is harmonic in $S_j$. Since $\tau_n$ is constant on $[R_j; 2R_j]$, we have $\tau_n = (R_j) = j$ for $n \in I_j$ and

$$u(x) = \frac{A}{\tau_n} \frac{1}{d} \frac{x}{c_n} \frac{1}{d} \frac{1}{c_n} \quad \text{for} \quad x \in S(c_n; r_n).$$  

Suppose that $x$ lies on the boundary of a ball $S(c_n; r_n)$ with $m \geq 2I_j$. It is a consequence of the separation condition that there can be at most $A(2^k d 2^k)$ balls with centre at a distance that is between $2^k$ and $2^{k+1}$ from $x$, with $k \geq 1$. Each such ball contributes at most $A(2^k d 2^k)$ to the above sum, making for a combined contribution of at most $A(2^k d 2^k)$. We need only count those $k$ with $2^k \leq 6R_j$, as there are no balls under consideration that are more distant than $6R_j$ from $x$. The ball $B(c_n; r_n)$ itself contributes 1 to $u(x)$, which leads to the estimate

$$u(x) = 1 + \frac{A}{d} \frac{x}{c_n} \frac{1}{d} \frac{1}{c_n} \quad \text{for} \quad x \in S(c_n; r_n).$$  

As $R_j \leq 2$, 1 for sufficiently large $j$,

$$(4.2) \quad u(x) = A_1 R_j \frac{d}{j} \quad \text{for} \quad x \in S(c_n; r_n):$$  

Here $A_1$ is some appropriate number that depends only on the dimension and on the separation number.
For \( x \in S_{j+1} \), we have \( x \in B_{n_{j+1}} \) \( 4R_j \). At this point we use the assumption that the balls are uniformly dense to deduce that
\[
    u(x) = \frac{1}{d+2} \sum_{j=2}^{d} \frac{1}{|x - c_{n_j}|^{d+2}}
\]
where the number \( A_2 \) depends only on the dimension and on the number \( R \) that appears in the definition of ‘regularly located’. Thus,
\[
    (4.3) \quad u(x) = \frac{1}{d+2} \frac{A_2}{R_j} R_j^{d+2} \quad \text{for} \quad x \in S_{j+1}.
\]
Finally, for \( x \) on the sphere \( S_{j+1} \) and \( n_{j+1} \), we have \( x \in B_{n_{j+1}} \)
\[
    2R_j \quad 2R_{j+1}.
\]
Hence on \( S_{j+1} \) the function \( u \) satisfies
\[
    u(x) = \frac{1}{d+2} \frac{A_2}{R_{j+1}^2} R_{j+1}^{d+2} \quad \text{for} \quad x \in S_{j+1}.
\]
Since \( (R_j = R_{j+1})^d \leq C \), we obtain that
\[
    (4.4) \quad u(x) = \frac{1}{d+2} A_2 \frac{R_j^2}{R_{j+1}^2} R_{j+1}^{d+2} \quad \text{for} \quad x \in S_{j+1}.
\]
It follows from (4.2), (4.4) and the maximum principle that, for \( x \in S_{j+1} \),
\[
    A_1 R_j^2 \frac{d^{d+2}}{j^{d+2}} x \in S(c_n ; r_n) ; \quad u(x) = \frac{1}{d+2} A_2 \frac{R_j^2}{R_{j+1}^2} R_{j+1}^{d+2}.
\]
Making use of (4.3), we deduce from this last estimate that, for \( x \in S_{j+1} \),
\[
    A_1 \frac{d^{d+2}}{j^{d+2}} x \in S(c_n ; r_n) ; \quad u(x) = \frac{1}{d+2} A_2.
\]
Thus (4.1) has been proven.

**Remark.** With the same proof, one may consider a slightly more general situation where one changes the metric. Assume that a function \( f : \mathbb{R}^d \to \mathbb{R}^+ \) satisfies the smoothness condition (\( y \) ′)(\( x \)) whenever \( y < x < 2y \).

We say that a sequence \( f(c_n, r_n) \) is (regularly located if there is a \( \delta > 0 \) such that the balls \( B(c_n) \) are pairwise disjoint and there is an \( R > 0 \) such that any ball \( B(c_n) \) contains at least a center \( c_n \). Assume finally that we have a sequence of disjoint balls with regularly located centres and the radii of the balls depend on the centre, \( r_n = (\exists c_n) \), where \( x \) is a decreasing positive function. Then the balls are avoidable if and only if
\[
    Z \left( x \right) \frac{1}{d^{d+2}} \frac{d^{d+2}}{j^{d+2}} dx < 1.
\]

The case \( \delta = 1 \) is the case previously considered.
5. Construction of the examples: Proof of Theorem 3

We wish to show by examples that the assumption (1.2) in Theorem 1 is necessary. The examples are of avoidable and separated configurations of balls for which the series \( \sum r_n^d \) is divergent, in which case \( r_n^d j_n^d \) must be unbounded by Theorem 1. In Theorem 3 it is asserted that such configurations of balls are possible even with a growth restriction on \( r_n^d j_n^d \). Leaving the growth restriction to one side for the moment, we first give the details of a plain vanilla example that incorporates the idea behind the general construction.

Proposition 2. There is an avoidable, separated configuration of balls, \( B(c_n; r_n), n \geq 1 \), in \( \mathbb{R}^3 \) for which

\[
\sum_{n=1}^{\infty} \frac{r_n}{j_n^d} = 1
\]

Proof. Consider a string of closed balls \( B_1, B_2, \ldots, B_{2k} \), each of radius \( r = 4 \) and with centres \( c_i \) on the \( x_1 \)-axis at \( m + i, i = 1, 2, \ldots, 2k \). We write \( S(m;k) = \bigcup_{i=1}^{2k} B_i \) and wish to estimate \( \sum_{x \in S(m;k)} j \). We consider, as ever,

\[
u(x) = \sum_{i=1}^{2k} \frac{1}{j_i}
\]

Suppose that \( x \) lies on the boundary of one of the balls \( B_j \). Then \( j \leq j_i j_i \), and there are at least \( k \) balls to one side or other of any one ball. It follows that

\[
u(x) \geq \frac{1}{2} \sum_{i=1}^{2k} \frac{1}{j_i} \geq \frac{1}{2} \log k; x \in S(m;k)
\]

By the maximum principle,

\[
\sum_{x \in S(m;k)} j \geq \frac{4k}{\log k}
\]

We construct our counterexample as follows. Let \( S_n = S(n^2, b_n = \log n c) \) and

\[
\sum_{n=1}^{\infty} \frac{r_n}{j_n^d} = 1
\]

Then

\[
\sum_{n=1}^{\infty} \frac{r_n}{j_n^d} = 1
\]

With \( b_n = \log n c \) and

\[
\sum_{n=1}^{\infty} \frac{r_n}{j_n^d} = 1
\]
Thus \( n_0 \) may be chosen to be sufficiently large so that the balls are separated and so that \( 0; \theta; e \) \( 1 \), in which case the balls are avoidable.

On the other hand, the contribution of each string of balls \( S_n \) to the series \( \sum_{n=1}^{\infty} \frac{1}{n \log^2 n} \) is comparable to \( \frac{1}{n \log n} \), and this sum is divergent.

In the examples that follow the balls are arranged in clusters rather than in higher dimensional strings, though the reason the examples work is the same: each ball in a cluster of balls contributes significantly less to the harmonic measure than it would do if taken individually.

**Proof of Theorem 3.** We consider a cluster of \( k^d \) balls, each of radius \( r \) less than \( 1/4 \), whose centres have integer coordinates and are evenly distributed in a large ball that has radius approximately \( k \) and is centred at a distance \( m \) from the origin. We assume that \( km \geq 2 \) and refer to this cluster of balls as \( C(m; k; r) \). We again use the function

\[
(5.1) \quad u(x) = \sum_{i=1}^{\lfloor 2k \rfloor} \frac{1}{c_i^d} \frac{1}{2^d - ak^2};
\]

the \( c_i \) being the centres of the balls. If \( x \) is a point on the boundary of one of these balls and \( 1 \leq i \leq k \), there are at least \( a \frac{d}{2} - 1 \) balls whose centres lie at a distance at most \( 2i \) from \( x \). Here \( a \) represents a number that depends only on the dimension. Moreover, no ball needs to be chosen twice, that is for two different values of \( i \). We find that, for a point \( x \) on the boundary of any ball in the cluster,

\[
\sum_{i=1}^{\lfloor 2k \rfloor} \frac{1}{c_i^d} \frac{1}{2^d - ak^2} \leq \sum_{i=1}^{\lfloor 2k \rfloor} \frac{1}{c_i^d} \frac{1}{2^d - ak^2} \leq \sum_{i=1}^{\lfloor 2k \rfloor} \frac{1}{c_i^d} \frac{1}{2^d - ak^2};
\]

By the maximum principle,

\[
\sum_{i=1}^{\lfloor 2k \rfloor} \frac{1}{c_i^d} \frac{1}{2^d - ak^2};
\]

We suppose that an increasing unbounded function \( f \) on \( [0; 1] \) is given. To each positive integer \( n \) there corresponds a choice of variable \( m_n \) for which \( f(m_n) \leq n^{2d} \) and \( m_n > 2m_{n-1} \). We then choose \( k_n \) to be \( m_n = n^2 \) and choose the radius \( r_n \) so that \( r_n^d = m_n^2 = f(m_n) \). [We assume that the function \( f \) satisfies \( f(x) \leq \frac{1}{2^d - dx^2} \), so that \( x_n < 1/4 \).] We set

\[
\sum_{n=n_0}^{\infty} \frac{1}{n \log^2 n} = \frac{1}{n \log n} \]

Thus \( n_0 \) may be chosen to be sufficiently large so that the balls are separated and so that \( 0; \theta; e \) \( 1 \), in which case the balls are avoidable.
and write $\lambda_n(x)$ for the harmonic measure at $x$ of the finite boundary of $C \subset \mathbb{R}^d \cap \mathbb{R}^d \cap \mathbb{R}^d \cap \mathbb{R}^d$. Then
\[
\lambda_n(x) = \frac{1}{\kappa_n} \int_0^\infty \left( \frac{m_n}{r_n} \right)^{d-2} \frac{1}{n^{d-1}} \, dr
\]
which we can arrange to be strictly less than 1 by taking $n_0$ to be sufficiently large.

The sum in (1.1) for this collection of balls is comparable to
\[
\sum_{n=n_0}^{\infty} \left( \frac{m_n}{r_n} \right)^{d-2} \frac{1}{n^{d-1}} \geq \sum_{n=n_0}^{\infty} n^{d-2} \frac{1}{n^{d-1}} = \sum_{n=n_0}^{\infty} n^{d-2} \frac{1}{n^{d-1}} = 1
\]
The sum in (1.1) is therefore divergent.

6. ADDENDUM: THE UNION OF TWO AVOIDABLE SETS IS AVOIDABLE

At a certain point in our research, it seemed that it might be helpful to know if the union of two avoidable collections of balls would again be avoidable. Put another way, is it possible to split an unavoidable collection of balls into two disjoint avoidable collections? Though the solution to this problem is no longer an essential ingredient in the proofs we have presented here, we cannot resist including the elegant solution to this problem found by Professor Rosay. We are grateful to him for granting us permission to include his proof in this article.

A set $A$ is called avoidable from $p$ if Brownian motion in $\mathbb{R}^d$ starting at $p$ has a probability smaller than one of hitting $A$. We assume that $\mathbb{R}^d \cap A$ is connected: then, by the maximum principle, if $A$ is avoidable from one point it is avoidable from any other point. In this case we just say that the set $A$ is avoidable. Equivalently $A$ is avoidable whenever there is a positive harmonic function $u$ in $\mathbb{R}^d \cap A$ such that $u = 1$ on the boundary of $A$ but $\inf u = 0$.

**Proposition 3.** If two avoidable sets $A$ and $B$ satisfy $\mathbb{R}^d \cap A \cup B$ is connected then $A \cup B$ is avoidable.

The basic lemma required to prove this proposition is the following:

**Lemma 3.** If $E$ is avoidable and $u_E$ is the associated positive harmonic function in $\mathbb{R}^d \cap E$, with $u_E = 1$ on the boundary of $E$ and $\inf u_E = 0$, then there is an $R_0$ such that for all $R > R_0$ the set of points
\[
S_E^R = \{ x \in E : u_E(x) \leq 1 \}
\]
satisfies $S_E^R \supset \{ x \in E : u_E(x) \leq 1 \}$ Here the measure indicated by $\mathcal{J}$ is Lebesgue area measure on $\mathbb{R}^d \cap E$. 
Proof. We take a point $q$ where $u_\mathbb{E}(q) < 1=32$. For any $R$ with $R > \frac{1}{32}$ we denote by $\mathbb{H}^E_R q$ the harmonic measure on the boundary of $B(0;R) \cap \mathbb{E}$ with respect to $q$. Then, since $u = 1$ on $\mathbb{E} \setminus B(0;R)$, we have

$$\frac{1}{32} > R \mathbb{H}^E_R q B(0;R) + \frac{1}{4} R \mathbb{H}^E_R q B(0;R)$$

from which it follows that $R \mathbb{H}^E_R q > 7=8$. We denote by $\mathbb{H}^E_R q$ harmonic measure with base point $q$ with respect to the ball $B(0;R)$, so that $\mathbb{H}^E_R q$ on $S(0;R)$. Thus, $R \mathbb{H}^E_R q > 7=8$ for all $R > \frac{1}{32}$. The harmonic measure $\mathbb{H}^E_R q$ can be given explicitly, but the key property is that as $R \to 1$ it is more and more similar to the normalized area measure on $S(0;R)$. Thus $\mathbb{H}^E_R q > 7=8$ for all large $R$.

Proof of Proposition 3. For the sets $A$ and $B$ we take the corresponding functions $u_A$ and $u_B$. We take $R$ so that $\mathbb{H}^E_A q > \frac{3}{4} \mathbb{H}^E_B q (0;R)$ and $\mathbb{H}^E_B q > \mathbb{H}^E_B q (0;R)$ j This means that there is point $p$ that lies in the intersection $S^R_A \setminus S^R_B$. We define $u = u_A + u_B$: it is a positive and bounded harmonic function defined outside $A \setminus B$. On the boundary of $A \setminus B$ it satisfies $u = 1$ and on the other hand $u(p) = 2$. Thus $A \setminus B$ is avoidable from $p$. Since the complement $R d n (A \setminus B)$ is connected, then it is avoidable from any point.

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