Revealing Non-circular beam effect in WMAP-7 CMB maps with Bi poSH measures of Statistical Isotropy

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Mild, unavoidable deviations from circular-symmetry of instrumental beams in current Cosmic Microwave Background (CMB) experiments pose a significant challenge to deriving high precision inferences from the high sensitivity and resolution of CMB measurements. It is important to be able to measure and characterize this subtle effect since it has bearing all subsequent inferences, including the cosmological parameters. We present analytic results, confirmed by numerical simulations, that CMB maps of cosmological signal that respect underlying statistical isotropy (SI) symmetry, measured with an instrument that has mildly non-circular (NC) beam would, nevertheless, exhibit SI violation. We show that appropriate observable measures constructed within the Bipolar Spherical Harmonic (Bi poSH) representation of SI violation capture subtle NC-beam effects coupled with the scan strategy of the instrument.

Accompanying their 7-year data release, the WMAP team published very high significance measurements of non-zero Bi poSH spectra, $A_{b1}$ and $A_{b2}$, in the “W” and “V” band of the experiment. The Bi poSH measurements at the two frequency channels show significant differences that point against an origin in the achromatic cosmological CMB anisotropy signal. We present a strong case that the quadrupolar components of the NC-beams, $b_{l2}$, of the respective channels have the most dominant contribution towards Bi poSH measurement and create non-trivial qualitative features of the measured Bi poSH spectrum that may be difficult to mimic through other effects. The fact that subtle levels of non-circularity, e.g., $|b_{l2}|/b_0 \lesssim 0.01$ as seen in WMAP beams, lead to measurable Bi poSH spectra points to the immense promise and potential of the Bi poSH representation.

The scan strategy of CMB experiments typically employ multiple visits with varying orientation at different sky positions. Hence, the effective smoothing associated with the NC part of the beam is not expected to be represented by the instantaneous NC-beam response. In general, the effective beam at any point will also have variations over the sky. Using the Bi poSH measurements it is possible to construct an equivalent single hit, ‘parallel transported’ effective NC-beam that matches the angular power spectrum as well as provides a good estimate of violation of SI in the observed maps. Thus, using the Bi poSH formalism, we successfully characterize an important systematic effect that limits all attempts to probe the SI assumption implied in the cosmological principle at high precision through CMB measurements beyond the angular power spectrum.

I. INTRODUCTION

Cosmic Microwave Background (CMB) experiments with high sensitivity, fine angular resolution covering almost the full sky have reached a point where even mild, unavoidable, deviations from circularity of the instrumental beam around its pointing direction poses a challenging systematic effect that must be overcome within finite computational resources to fully realize potential of these measurements for precision cosmology.

The effect of non-circular (NC) beam on angular power spectrum of CMB has been studied in literature and the non-trivial impact on high precision cosmological inferences has been appreciated but not satisfactorily resolved, particularly, within available computational resources. A significant body of literature attempting to deal with NC beam effect on the angular power spectrum exists, E.g., [1–8]. Further, the beam response function can acquire an inherent time dependence which complicates and hinders the deconvolution enormously. Consequently, beam imperfections, coupled with the scan strategy, lead to very complex modification of the signal demanding high computational resources to assess the final effect on the estimation of angular power spectrum and the cosmological parameters. It is important to be able to capture the NC-beam effect in observed CMB maps. We provide a computationally efficient characterization of the NC-beam effect by focusing on its effect beyond the angular power spectrum. Beyond the angular power spectrum, the improvement in quality of CMB measurements now allow observational confrontation of simplifying, theoretical, assumptions implicit in current cosmology. Given CMB fluctuations are a realization of correlated, Gaussian random field, the two point correlation function uniquely describes the statistics of the fluctuations. Statistical Isotropy (SI) implies rotational invariance of the two point correlation function. Breakdown of SI can be parametrized by the expansion coefficients of the two point correlation function in Bipolar spherical harmonic (Bi poSH) basis known as Bi poSH coefficients. Cosmological CMB temperature fluctuations are gener-
ally assumed to be a realization of statistically isotropic, Gaussian, correlated, random field on the sphere, consequently, the angular power spectrum has been the primary observational target of most CMB experiments. However, current and upcoming CMB experiments also hold the promise to observationally constrain the underlying, often implicit, SI assumption (closely linked to the so called, ‘cosmological principle’) employed in cosmology.

SI assumption has been under intense scrutiny with hints of various ‘anomalies’ persisting in successive years of WMAP data [9–13]. Violation of SI can be generated both, from theoretically motivated possibilities such as, cosmic topology, anisotropic cosmologies etc. as well as from observational artifacts, such as, beam non-circularity, anisotropic noise, foreground residuals, etc. [14–23]. Searching for the most plausible candidate of SI violation is a non-trivial endeavor which strongly depends on observational and theoretical hints to narrow down the possibilities. While NC-beam effects do pose as a serious systematic contaminant for SI measurements, we demonstrate that the BipoSH representation measures of SI violation are extremely effective in capturing, characterizing, and possibly, isolating subtle NC-beam effect in CMB maps.

The recent detection of a quadrupolar power asymmetry ‘anomaly’ in WMAP 7 year data (WMAP-7) by the WMAP team using BipoSH representation measures [24] has proved to be an intriguing observation, that is yet to be satisfactorily explained. The published detections were significant both in the V-Band and W-Band frequency maps. Due to mildly significant differences in the BipoSH signal at the two frequencies and the fact that the signal seems to exhibit azimuthal symmetry in the ecliptic coordinate, it was suspected not to be of cosmological origin. Effects of residual galactic foreground emission would, arguably, be expected to have associated symmetries in the Galactic coordinate.

Observed CMB anisotropy (and polarization) on the sky is a convolution of the underlying cosmological CMB signal with the instrumental beam response function. The instrumental beams in most CMB experiments are nearly circularly (azimuthal) symmetric, but mild NC deviations do arise due to unavoidable limitations in instrumental design, function and fabrication, e.g., the primary lobe of the beam exhibits non-circularity due to the off-axis position of detectors on the focal plane, diffraction around the edges of instrument leads to side lobes of the beam, or due to finite response time of detectors the scan may not correspond to the instrument rotating around its beam axis leading to the effective beam response at any pointing direction being sensitive to the scan strategy, etc.. Regardless of the specific origin of non-circularity in the beam response function, the key common point is the potentially measurable SI violation generated in the observed CMB maps. It has been argued whether the NC beam effect in WMAP provides a plausible explanation of the observed non-zero BipoSH measurements [25, 26]. This paper presents the results from a research program to directly assess the nature and amplitude of SI violation that can arise due to NC-beam response within the BipoSH representation and its implication for the WMAP-7 BipoSH ‘anomaly’.

In the absence of circular symmetry, NC-Beam functions can be most generally expanded in BipoSH basis and coefficients of the expansion, \( B^{L M}_{l_{1}l_{2}} \), are referred to as beam-BipoSH coefficients. An ideal, circular symmetric beam ensures vanishing beam-BipoSH coefficients for all \( L > 0 \). Importantly, the beam-BipoSH coefficients at \( L > 0 \) not only capture NC-beam shape but also include the additional modulation arising from the specific scan strategy that sets the orientation of the NC-beam at any pixel. We show that every non-zero beam-BipoSH coefficient, \( B^{L M}_{l_{1}l_{2}} \), would generate a corresponding non-zero CMB BipoSH coefficient, \( A^{L M}_{l_{1}l_{2}} \), in the observed CMB map.

The analytic formulation presented is valid for any arbitrary beam shape but progress to explicit expressions is possible within an idealized ‘parallel-transport’ (PT) scan strategy approximation [1]. Further, mild deviations from circular symmetry, that permit a perturbative approach, and residual, discrete even-fold azimuthal, symmetry in the NC beam imply that the analytic results at the leading order quadrupolar-non-circularity \( (n = 2) \) captures most of the significant features in the present observations. At the leading order of our approximations, we provide simple explicit analytic expressions only for BipoSH coefficients due to mildly NC-beams that retain discrete even-fold azimuthal and reflection symmetry relevant to understanding the WMAP-7 BipoSH results. Assumption of a reflection symmetric beam function restricts the set to even parity BipoSH coefficients [1]. The published WMAP BipoSH results claim that in ecliptic coordinates BipoSH at \( (L = 2, M = 0) \) are the non-zero BipoSH coefficients measured. Our general analytic formulation then suggests that a PT-scan approximation should be a fairly good simplifying approximation for the WMAP scan for the dominant \( m = 2 \) mode of NC-beams in the ecliptic coordinates. We support this assertion with maps quantifying the departures from this approximation for a realistic WMAP scan. Further, the multiple hits of the beam with varying orientation at any pixel reduces (average out) the level of non-circularity. In particular, it can be estimated in the specific case of WMAP scan, the \( m = 2 \) mode, \( b_{2} \), in the raw NC-beam can be expected to reduce by about a factor of \( \sim 0.45 \), and higher \( m \) mode maps are expected to suffer even progressively greater reduction [see Fig.(4)].

Assuming that the entire BipoSH signal measured by WMAP arises due to NC-beam effect, we determine spectrum of quadrupolar \( (m = 2) \) mode, \( b_{2}^{\text{eff}} \), of an effective beam (incorporating multiple visits by the beam to sky pixels) that fits the measured BipoSH spectra. We fit for a constant rescaling of the raw beam \( b_{2}^{\text{eff}} = \alpha b_{2} \). Although, the best-fit \( \alpha \sim 0.45 \) obtained matches our expectation, the reduced \( \chi^{2} \) of the fit is unsatisfactory,
and would, at best, explain part of the BipoSH signal. We find that a linear $l$ dependent correction to a constant scaling, $b_{l}^{eff} = (\alpha + \beta l) b_{2}$, leads to a good simple ‘phenomenological’ fit to the measured WMAP-7 BipoSH spectra.

Detailed, computationally expensive numerical simulations incorporating the two beam differencing, real scan and map making may justify the linear correction to the constant scaling, or, lead to the conclusion that the entire BipoSH signal cannot be attributed to NC-beam leaving room for other uncorrected systematic effects, or even hint at cosmological SI violation.

We verify all analytic results and estimate error-bars through extensive numerical simulations of SI CMB maps scanned by real space convolution with corresponding NC-beam maps. The numerical simulations presented here are limited to the PT-scan, single side, NC-beam studied analytically. As explained later, without simplifying assumptions, realistic simulations using the WMAP beam differencing and scan are more compute intensive primarily because we find that the results are fairly sensitive to side lobe and other spread-out features (out to $\sim 10$ times the FWHM from the beam center), present in the beam maps released by the WMAP team [27]. Hence, we defer the results from ongoing realistic simulations to a future publication.

The paper is arranged as follows. Sec. II provides a brief primer to the BipoSH formalism to characterize SI violations for keeping the paper self contained. In Sec. III we present a novel expansion of the beam response function in the BipoSH basis, referred to as beam-BipoSH coefficients in the article. In Sec. IV we derive expressions for the CMB BipoSH coefficients arising from the convolution of SI CMB anisotropy signal with general NC-beams. We also provide the simpler explicit analytic expressions for beam-BipoSH coefficients for a mildly NC beam within the PT-scan approximation. As a test case, we study the Elliptical Gaussian (EG) NC-beam model that has minimal parameters with clear interpretation. The beam spherical harmonic transforms (beam-SH) have analytic expressions and permit simple, well controlled perturbative treatment of the NC-effects. We also use this case to test and qualify our numerical simulations against analytic results. In Sec. V, we get closer to addressing the WMAP-7 BipoSH measurements by first computing the BipoSH spectra based on the beam-SH of single-side, raw beam maps (Sec. VA). In Sec. VB we fit for parametrized scaling functions, $f_{l}$ of the leading order raw beam-SH that would reproduce the WMAP-7 BipoSH spectra measurements as arising entirely from uncorrected NC beam effect. Sec. VII has discussions and presents the conclusions of this paper. Detailed steps of all analytical calculations are provided for completeness in Appendix A and B. Appendix C provides an analytic derivation of an effective beam arising from multiple hits at varying orientations in terms of a rescaled beam-SH the raw beam map.

II. PRIMER: BIPOLAR SPHERICAL HARMONIC (BipoSH) REPRESENTATION

The CMB anisotropy signal is a random field on the surface of a sphere and can, hence, be expanded in the spherical harmonic (SH) basis,

$$\Delta T(\hat{n}) = \sum_{l,m} a_{lm} Y_{lm}(\hat{n}),$$

and equivalently represented in the set of random variables $a_{lm}$. Statistical properties of a random field are characterized by its $n$-point correlation function. SI implies expectation values of all $N$-point correlation functions are invariant under the rotations of the sky. The statistical distribution of fluctuations in temperature field are observationally consistent with Gaussian statistics. Hence, the two-point function $C(\hat{n}, \hat{n}') \equiv \langle \Delta T(\hat{n}) \Delta T(\hat{n}') \rangle$ is expected to completely encode all information pertaining to the CMB anisotropy, $\Delta T(\hat{n})$. For CMB sky respecting statistically isotropy (SI), the correlation function only depends on the angle between the two directions $\hat{n}_{1}$ and $\hat{n}_{2}$ and the two point function can then be expanded in Legendre polynomials, as

$$C(\hat{n}_{1}, \hat{n}_{2}) \equiv C(\hat{n}_{1} \cdot \hat{n}_{2}) = \sum_{l=0}^{\infty} \frac{(2l+1)}{4\pi} C_{l} P_{l}(\hat{n}_{1} \cdot \hat{n}_{2}),$$

where the coefficients of expansion $C_{l}$ define the angular power spectrum. The random spherical harmonic moments, $a_{lm}$ are then statistically independent leading to a diagonal covariance matrix in the SH basis,

$$\langle a_{lm} a_{l'm'}^{*} \rangle = C_{l} \delta_{ll'} \delta_{mm'},$$

which also implies that (the $m$-independent), $C_{l}$ encodes all the information in a Gaussian, SI field on the full sky (complete sphere, $S^{2}$). In general, SI violation leads to non-zero off-diagonal elements (for specific subset of SI violations, the SH covariance matrix remains diagonal, but $m$ dependent). The two point correlation function, most generally, then depends on both the directions $\hat{n}_{1}$ and $\hat{n}_{2}$ and not just on the angle between them. The most convenient expansion basis for $C(\hat{n}_{1}, \hat{n}_{2}) \neq C(\hat{n}_{1} \cdot \hat{n}_{2})$ is the Bipolar Spherical Harmonic (BipoSH) basis [28-33],

$$C(\hat{n}_{1}, \hat{n}_{2}) = \sum_{l_{1},l_{2},l,M} A_{l_{1}l_{2}L,M}^{LM} \{ Y_{l_{1}}(\hat{n}_{1}) \otimes Y_{l_{2}}(\hat{n}_{2}) \}_{LM},$$

where $A_{l_{1}l_{2}}^{LM}$ are BipoSH coefficients and the bipolar spherical harmonic (BiPoSH) functions,

$$\{ Y_{l_{1}}(\hat{n}_{1}) \otimes Y_{l_{2}}(\hat{n}_{2}) \}_{LM} = \sum_{m_{1}m_{2}} C_{l_{1}m_{1}l_{2}m_{2}}^{LM} Y_{l_{1}m_{1}}(\hat{n}_{1}) Y_{l_{2}m_{2}}(\hat{n}_{2}),$$

are irreducible tensor product of two spherical harmonics spaces that form an orthonormal basis on $S^{2} \times S^{2}$ and, here, $C_{l_{1}m_{1}l_{2}m_{2}}^{LM}$ are Clebsch-Gordon coefficients. The indexes of these coefficients satisfy the triangularity conditions $|l_{1} - l_{2}| \leq L \leq l_{1} + l_{2}$ and $m_{1} + m_{2} = M$. The
transformation properties of BipoSH under rotations are similar to spherical harmonics [34].

The BipoSH coefficients can be shown to be linear combinations of off-diagonal elements of the harmonic space covariance matrix,

\[ A_{l_l}^{LM} = \sum_{m_1m_2} (a_{l_1m_1} a_{l_2m_2} (-1)^{m_2} C_{l_1l_2m_1l_2m_2}^{LM} ) , \]

where \( a_{lm} \)'s are the spherical harmonic coefficients of the CMB maps [28]. The SI part given in Eq. (3), corresponds to \( (L = 0, M = 0) \), since \( C_{00}^{00} \propto \delta \). Non-zero BipoSH coefficients with \( L > 0 \) that capture SI violation also neatly encode the residual symmetries of the two point correlation function [35]. The correlation function is always real and symmetric under the exchange of \( \hat{n}_1 \) and \( \hat{n}_2 \), reflected in the following symmetry properties for BipoSH coefficients,

\[ A_{l_l}^{LM} = \begin{cases} (-1)^{l_1+l_2-L-M} A_{l_l}^{L^*M} & \text{if } L \text{ is even} \\ (-1)^{l_1+l_2-L} A_{l_l}^{LM} & \text{if } L \text{ is odd} \end{cases} \]

As a consequence, BipoSH coefficients are always real if \( M = 0 \) and \( l_1 + l_2 + L \) is even and always imaginary if \( M = 0 \) and \( l_1 + l_2 + L \) is odd. If the observed CMB sky is statistically isotropic, it can be shown that all the BipoSH coefficients vanish except the coefficients of the form \( A_{00}^{00} \) which can be expressed in terms of the CMB angular power spectrum [28],

\[ A_{l_l}^{LM} = (-1)^{l_1} C_{l_1} \prod_{l_1} \delta_{l_1l_2} \delta_{l0} \delta_{m0} . \]

where, for brevity, the standard notation \( \prod_{abc...c} = \sqrt{(2a+1)(2b+1)...(2c+1)} \) is introduced and employed in the rest of this paper [34].

Recently it has also been realized that BipoSH space representation separates into two distinct classes,

- Even parity BipoSH coefficients \( A_{l_l}^{LM(+)} \) where \( l_1 + l_2 + L \) is even obey \( A_{l_l}^{LM(+)} = (-1)^{M} A_{l_l}^{L-M(+)} \), and

- Odd parity BipoSH coefficients \( A_{l_l}^{LM(-)} \) where \( l_1 + l_2 + L \) odd, obey \( A_{l_l}^{LM(-)} = (-1)^{M+1} A_{l_l}^{L-M(-)} \),

where the ‘parity’ terminology here is based on the analogy with parity transformation properties of ordinary spherical harmonic moments.

It is often easy to anticipate, or determine, the parity property of a possible cause of SI violation. This distinction then provides very valuable clues to the origin of SI violations e.g., weak lensing due to scalar (even parity) and tensor (odd parity) perturbations [36], anisotropic primordial power spectrum (even) [18], temperature modulation (even) [37], primordial homogeneous magnetic fields (even) [17, 38]. Importantly, in the context of origin of WMAP BipoSH measurements from NC-beam effect, the absence of significant odd parity BipoSH would justify the simple parallel-transport (PT) scan approximation adopted in our analytic approach.

III. Beam – BipoSH: NON-CIRCULAR BEAMS IN BipoSH REPRESENTATION

It is very important to characterize the non-circularity of the beam through appropriate measures. We show that NC-beam is well represented in BipoSH space. Beam response function characterizes the angular dependence of the sensitivity of the apparatus around the pointing direction \( \hat{n}_1 \). We expand the beam function around the pointing direction \( \hat{n}_1 \) in Spherical Harmonic (SH) basis as,

\[ B(\hat{n}_1, \hat{n}_2) = \sum_{l_m} b_{lm}(\hat{n}_1) Y_{lm}(\hat{n}_2) . \]

The beam-SH coefficients when the beam is pointed at \( \hat{n}_1 \), are given by

\[ b_{lm}(\hat{n}_1) = \int d\Omega_{\hat{n}_2} B(\hat{n}_1, \hat{n}_2) Y_{lm}(\hat{n}_2) . \]

The SH transform of beam at arbitrary pointing direction, \( \hat{n} \equiv (\theta, \phi) \) is given by \( beam-SH, b_{lm'}(\hat{z}) \) – the SH transform of the beam pointed along \( \hat{z} \), the North pole [1],

\[ b_{lm}(\hat{n}) = \sum_{m'} b_{lm'}(\hat{z}) D_{m'm}^{lm}(\phi, \theta, \rho(\hat{n})) , \]

where Wigner D-functions \( D_{m'm}^{lm}(\alpha, \beta, \gamma) \) are the matrix elements of the rotation operator \( (0 \leq \alpha < 2\pi, 0 \leq \beta < \pi, 0 \leq \gamma < 2\pi) \) and \( \alpha, \beta, \gamma \) are the Euler angles that rotate the \( \hat{z} \)-axis to the pointing direction \( \hat{n} = (\theta, \phi) \) and the angle \( \rho(\hat{n}) \) specifies the orientation of the NC-beam with respect to the local Cartesian (\( \hat{x} \equiv \phi, \hat{y} \equiv \theta \)) aligned with spherical coordinates [1]. Such a rotation can be realized by fixing a coordinate system and performing anti-clockwise rotations, first rotating about the \( \hat{z} \)-axis by an angle \( \alpha = \phi \), then rotating about new \( \hat{y} \)-axis by an angle \( \beta = \theta \), and finally about the new \( \hat{z} \)-axis by \( \gamma = \rho(\hat{n}) \). Rotation is such that the beam pointing direction, \( \hat{n} \) and \( \rho(\hat{n}) \) is the relative orientation of the beam around its axis set by the the scan pattern of the experiment.

In this work we introduce a more general NC-beam characterization in the BipoSH representation. Since, a general NC-Beam function depends on two vector directions, it can be expanded in the BipoSH basis (see Sec. II),

\[ B(\hat{n}_1, \hat{n}_2) = \sum_{l_m} B_{l_l}^{LM} \sum_{m_1m_2} C_{l_1m_1l_2m_2}^{LM} Y_{l_1m_1}(\hat{n}_1) Y_{l_2m_2}(\hat{n}_2) , \]

where the coefficients of expansion \( B_{l_l}^{LM} \) are referred to as beam–BipoSH coefficients . As will be clear in the sections ahead, beam-BipoSH coefficients have the advantage of globally capturing the additional effect of scan strategy together with non-circularity of the beam.

The beam-BipoSH coefficient can be readily related to beam function in spherical harmonic basis ,

\[ B_{l_l}^{LM} = \sum_{m_1m_2} C_{l_1m_1l_2m_2}^{LM} \int d\Omega_{\hat{n}} b_{l_2m_2}(\hat{n}) Y_{l_1m_1}^{*}(\hat{n}) . \]
A circularly symmetric beam function around the pointing direction can be expanded in Legendre polynomials, $B(\hat{n}_1, \hat{n}_2) \equiv B(\hat{n}_1 \cdot \hat{n}_2) = (4\pi)^{-1} \sum_{l_1} (l_1 + 1) B_{l_1} P_l(\hat{n}_1 \cdot \hat{n}_2)$. Inverse transforming Eq. (12) and using orthogonality of BipoSH [34], we obtain beam-BipoSH coefficients for circularly symmetric beam function,

$$B_{l_1 l_2}^{L M} = (-1)^l l_1 \prod_m \delta_{l_1 l_2} \delta_{L0} \delta_{M0}, \quad (14)$$

where $B_l$ is the commonly used Legendre transform of the beam function in the circularized beam approximation.

A. General expression for an arbitrary scan strategy

Using Eq. (11) and (13), it turns out that for any arbitrary scanning strategy, the beam-BipoSH can be expressed in terms of the beam-SH as,

$$B_{l_1 l_2}^{L M} = \sum_{m_1 m_2} C_{l_1 m_1 l_2 m_2} b_{l_2 m_2}(\dot{z}) \times \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} d\phi D_{l m_2}^{l_2 m_2}(\phi, \theta, \rho(\theta, \phi)) Y_{l_1 m_1}^{*}(\theta, \phi) \sin \theta d\theta. \quad (15)$$

To separate the azimuthal ($\phi$) and polar ($\theta$) dependencies, it is convenient to expressed Wigner-$d$ functions in terms of Wigner-$d$ through following relation,

$$D_{m m'}^{l}(\phi, \theta, \rho) = e^{-i m \phi} d_{m m'}^{l}(\theta) e^{-i m' \rho}. \quad (16)$$

Eq. (15) is the most general expression of beam-BipoSH coefficients for any given NC-beam specified through $b_{l m}(\dot{z})$ and scan pattern, defined by $\rho(\theta, \phi)$, in any spherical coordinate system (e.g. ecliptic, galactic, etc.). The beam-BipoSH coefficients can be evaluated numerically from the beam maps of the NC-beam. When the beam is circularly symmetric, $b_{l m}(\dot{z}) = B_l \sqrt{\frac{2l+1}{4\pi}} \delta_{l0} \delta_{m0}$, and it is straightforward to establish that Eq. (15) reduces to Eq. (14). Beam-BipoSH depend not only on NC-beam harmonics but also on the scan-strategy that defines $\rho(\hat{n})$. We motivate a particular idealized scan pattern where $\rho(\hat{n})$ is constant in a some coordinate system, that not only allows a completely analytic treatment but is also reasonably well-justified in the context of the BipoSH signature measured in WMAP-7.

B. ‘Parallel-transport’ scan approximation

The general beam-BipoSH in Eq. (15) can be tackled analytically if when the scan pattern is such that $\rho(\hat{n})$ is a constant. We refer to this case as ‘parallel-transport’ (PT) scan following [1]. The PT-scan approximation implies that the orientation of the beam relative to the local longitude is constant at any point on the sky. A beam response function with reflection symmetry will generate only real, non-vanishing beam-BipoSH coefficients. The leading order $m = 2$ dominates for a mildly NC-beam. For real coefficients only the $\cos(2\rho)$ part is important (see Eq. (15)). Fig. 1 shows the map for $\cos(2\rho)$ for a full year scan. For WMAP scan pattern, the constant $\rho$ approximation for $m = 2$ holds good over the significant (blue) band around the equator in an ecliptic coordinate system. Therefore, for a fair fraction of the sky part of the map our constant $\rho$ approximation for $m = 2$ mode of non-circularity is a fair assumption. The computationally intensive numerical comparison of results from this approximation with the real WMAP scan is underway and will be reported in the near future.

![Fig. 1](image-url)
For constant \( \rho(\hat{n}) \equiv \rho \), the orthogonality relation,

\[
\int_0^{2\pi} d\phi \exp(-i(m_1 + m_2)\phi) = 2\pi \delta_{m_1,\,-m_2},
\]

implies that the integral over \( \phi \) in Eq. (15), will separate from the integral over \( \theta \) and would restrict the non-zero beam-BipoSH to \( M = 0 \),

\[
B_{l_1l_2}^{LM} = 2\pi \delta_{M0} \sqrt{\frac{(2l_1 + 1)}{4\pi}} \sum_{m'} b_{l_2m'}(\hat{z}) \exp(-im'\rho) \times \sum_{m_2} C_{l_1-m_2l_2m_2}^{L0} \int_0^\pi d^2m_{m_2m_2}(\theta)d_{l_1m_20}(\theta) \sin \theta d\theta. \tag{18}
\]

We define

\[
I_{m_2,m'}^{l_1l_2} = (-1)^{m_2} \sqrt{\frac{(2l_1 + 1)}{4\pi}} \int d^2m_{m_2m_2}(\theta)d_{l_1m_20}(\theta) \sin \theta d\theta. \tag{19}
\]

Here we have used the symmetry property of Wigner-\( d \) functions, \( d_{mm'} = (-1)^{m-m'} d_{-m-m'} \). Beam-BipoSH coefficients derived in Eq. (18) thus take the following form in constant scan approximation, \( \rho(\hat{n}) = \rho \),

\[
B_{l_1l_2}^{LM} = 2\pi \delta_{M0} \sum_{m'} b_{l_2m'}(\hat{z}) \exp(-im'\rho) \times \sum_{m_2} C_{l_1-m_2l_2m_2}^{L0} I_{m_2,m'}^{l_1l_2}. \tag{20}
\]

For a mildly NC beam, the summation can be truncated at a low value of \( m' \). Further, NC-beams that retain discrete even-fold azimuthal symmetry will have \( b_{lm}(\hat{z}) = 0 \) for odd values of \( m \). In BipoSH space, the consequence of discrete even-fold azimuthal and reflection symmetric NC-beam translates to restricting non-zero beam-BipoSH to \( M = 0 \) and \( l_1 + l_2 = 0 \). Imposing reflection symmetry, beam-BipoSH are restricted to even values of multipole \( L \). We derive explicit expressions for beam-BipoSH coefficients up to leading order \((m' = 2)\) term.

To make analytical progress, we need to evaluate \( I_{m_2,m'}^{l_1l_2} \) for \( m' = -2, 0, 2 \).

It is important to note that, \( m' = 0 \) corresponds to the usual circular symmetric beam-BipoSH \((L = 0)\) coefficient. The non-circular \( m' = \pm 2 \) will give rise to non-trivial \((quadrupolar, L = 2)\) beam-BipoSH. We derive beam-BipoSH coefficients for circularly symmetric part of beam \((m' = 0)\) using Eq. (20). For \( m' = 0 \), \( I_{m_2,m'}^{l_1l_2} \) can be simplified to,

\[
I_{m_2,0}^{l_1l_2} = (-1)^{m_2} \sqrt{\frac{(2l_1 + 1)}{4\pi}} \frac{2}{2l_2 + 1} \delta_{l_1l_2}. \tag{21}
\]

Therefore, beam-BipoSH coefficients for circular part of the beam are of following form (refer Appendix A),

\[
B_{l_1l_2}^{LM} = (-1)^{l_2} b_{l_10}(\hat{z}) \sqrt{4\pi} \delta_{l_1l_2} \delta_{L0} \delta_{M0} \delta_{m_20}. \tag{22}
\]

For \( m' = \pm 2 \), the integrals are evaluated separately for the \( m_2 = 0 \) and \( m_2 = \mp 2 \) parts of the summation. In the former case when \( m_2 = 0 \) and \( m' = \pm 2 \), the integral

\[
I_{m_2,\pm 2}^{l_1l_2} = \sqrt{\frac{(2l_1 + 1)}{4\pi}} \times \begin{cases} 0 & \text{if } (l_1 + l_2 \equiv \text{odd}) \\ 0 & \text{if } (l_1 > l_2) \\ 4\sqrt{\frac{(l_2-2)}{(l_2+2)!}} & \text{if } (l_1 < l_2) \\ \frac{4l_2}{(l_2+2)!} - \frac{2l_2(l_1+1)}{(l_2+1)!} & \text{if } (l_1 = l_2). \end{cases} \tag{23}
\]

For \( m_2 = 0 \), \( d_{m_2,\pm 2}(\theta) \) is recursively expanded in terms of \( d_{m_20}(\theta) \) to evaluate \( I_{m_2,\pm 2}^{l_1l_2} \) (refer Appendix A).

This work is motivated by the highly significant measurements of BipoSH published by the WMAP team [24]. The publication claimed that for the \( L = 2 \) BipoSH spectra measured are significantly non-zero only for \( M = 0 \) in the ecliptic coordinates. This has been independently confirmed by our BipoSH measurements on WMAP-7, as well. As shown in Sec. IV, this implies that only beam-BipoSH with \( M = 0 \) are relevant for understanding of the WMAP BipoSH results. Consequently, it suggests that if NC-beam is responsible for non-zero BipoSH measurements, the WMAP scan pattern is such that, in ecliptic coordinate, \( \rho(\hat{n}) \approx \text{constant} \) could be a good approximation for the dominant \( m = 2 \) mode of the NC beams of WMAP. This is being quantitatively assessed through more compute intensive, numerical simulations with realistic WMAP scan patterns (i.e., without invoking PT-scan approximation and incorporating map-making from actual two side difference measurement). We take advantage of the simplicity afforded in the PT-scan approximation to proceed to explicit analytic expressions that readily provide broader as well deeper insight than compute expensive numerical simulations.

PT-scan approximation also implies that only even-parity beam-BipoSH would be relevant for our comparison to WMAP BipoSH results where the estimator used is by definition strictly restricted to even-parity BipoSH. NC beam with reflection symmetry have non-vanishing beam-BipoSH with even-parity only,

\[
B_{l_1l_2}^{LM} \equiv B_{l_1l_2}^{LM(+)} = 2\pi \delta_{M0} \left( b_{l_10}(\hat{z}) e^{-i2\rho} + b_{l_12}(\hat{z}) e^{i2\rho} \right) \times \left[ C_{l_10}^{L0} I_{l_20}^{l_1l_2} + \sum_{m_2 \neq 0} C_{l_1-m_2l_2m_2}^{L0} I_{m_2l_2}^{l_1l_2} \right]. \tag{24}
\]

Refer to Appendix A for details. Note that a constant \( \rho \) can be absorbed as phase factor in the redefinition of the complex quantity \( b_{lm}(\hat{z}) \) essentially resetting the orientation of the beam when pointed at North pole. Hence, the beam-BipoSH due to the NC part of the beam in the
PT-scan approximation is of the following form,

\[ B^{LM}_{l_1l_2} = 2\pi \delta_{M0}(b_{l_2}(\hat{z}) + b^*_{l_2}(\hat{z})) \times \left( C^{L0}_{l_102}l_{l_2} + \sum_{m_2 \neq 0} C^{L0}_{l_1-m_2l_2m_2}l_{l_2} \right). \]  

(25)

We emphasize that the above expression for beam-BipoSH coefficient holds for the PT-scan approximation (with constant \(\rho(\hat{n})\)) for a NC-beam that has reflection symmetry. Beam-BipoSH coefficient is a sum of circular part as in Eq. (22) and NC part of the beam function as in Eq. (25) and can be expressed as,

\[ B^{LM}_{l_1l_2} = (-1)^{l_2}b_{l_2}(\hat{z})\sqrt{4\pi} \delta_{l_{12}}\delta_{L0}\delta_{M0}\delta_{m'0} + 2\pi \delta_{M0}(b_{l_2}(\hat{z}) + b^*_{l_2}(\hat{z})) \times \left( C^{L0}_{l_102}l_{l_2} + \sum_{m_2 \neq 0} C^{L0}_{l_1-m_2l_2m_2}l_{l_2} \right). \]  

(26)

Although we have restricted explicit analytic results presented in the text to reflection symmetric beam functions, in general, odd parity beam BipoSH will be non-vanishing in absence of the above mentioned symmetries. Appendix A provides expressions for odd-Parity beam-BipoSH, \(B^{LM}_{l_1l_2} \neq 0\), that can be used as a measure of breakdown of reflection symmetry in NC beams \[\hat{b}\]. Note that the BipoSH estimator \[37\], that differ by a factor from original definition of Hajian & Souradeep \[28, 33\], used by the WMAP team cannot be extended to odd-parity BipoSH. However, it is possible to devise BipoSH estimators that can measure odd-parity BipoSH spectra while matching that employed by WMAP for even-parity BipoSH spectra \[39\].

IV. NON-CIRCULAR BEAM IMPRINT ON BipoSH OF CMB MAPS

The observed CMB map is a convolution of the true CMB anisotropy sky with the instrumental beam. Instrument beam response functions of all experiments have deviations from circular symmetry at some level. However, circular symmetric beam response function around the pointing direction is often assumed to simplify the analysis of the beam effect. This assumption, though, does affect many stages of CMB data analysis and should be measured and characterized.

In this section we show that CMB maps made with NC beam exhibit SI violation in an otherwise SI cosmological CMB signal. BipoSH coefficients have proved to be robust, model independent, measure of SI violation. Further, BipoSH representation provides clear insight into the nature and extent of the residual symmetry and parity \([35, 36]\) and helps pin down the most plausible origin of SI violation.

The measured CMB temperature fluctuations map, \(\tilde{\Delta}T(\hat{n})\) is the convolution of true underlying CMB temperature fluctuations \(\Delta T(\hat{n})\) with the instrument beam,

\[ \tilde{\Delta}T(\hat{n}) = \int d\Omega_{\hat{n}_2} B(\hat{n}_1, \hat{n}_2) \Delta T(\hat{n}_2). \]  

(27)

where the beam response function \(B(\hat{n}_1, \hat{n}_2)\) gives the sensitivity of an experiment around the pointing direction. The observed two point correlation function is,

\[ \tilde{C}(\hat{n}_1, \hat{n}_2) \equiv \langle \tilde{\Delta}T(\hat{n}_1)\tilde{\Delta}T(\hat{n}_2) \rangle = \int d\Omega_n \int d\Omega_{\hat{n}} C(\hat{n}', \hat{n}_1)B(\hat{n}_1, \hat{n}_1)B(\hat{n}_2, \hat{n}_n), \]  

(28)

where \(C(\hat{n}', \hat{n}) \equiv \langle \Delta T(\hat{n}')\Delta T(\hat{n}) \rangle\) is the underlying correlation function determined by cosmology. It is evident from Eq. (28), that SI violation can occur either due to rotational invariance breakdown of the true underlying temperature correlation function, \(C(\hat{n}_1, \hat{n}_2) \neq C(\hat{n}_2, \hat{n}_2)\), or, due to the breakdown of circularity in beam response function \(B(\hat{n}_1, \hat{n}_2) \neq B(\hat{n}_1, \hat{n}_2)\) or, both.

If the underlying CMB signal respects SI symmetry as widely assumed in cosmology, then

\[ C(\hat{n}_1, \hat{n}_2) = \sum_{l} \frac{2l + 1}{4\pi} C_{l} W_{l}(\hat{n}_1, \hat{n}_2), \]  

(29)

where \(W_{l}(\hat{n}_1, \hat{n}_2)\) is the elementary window function \([1]\) that accounts for the effect of finite resolution of beam function, given as

\[ W_{l}(\hat{n}_1, \hat{n}_2) = \int d\Omega_{\hat{n}_1} \int d\Omega_{\hat{n}_2} B(\hat{n}_1, \hat{n}_1)B(\hat{n}_2, \hat{n}_2)P_{l}(\hat{n} \cdot \hat{n}'). \]  

(30)

Inverse transform of Eq. (4), yields the CMB BipoSH coefficients, \(\hat{A}^{LM}_{l_1l_2}\), in terms of beam-BipoSH coefficients as

\[ \hat{A}^{LM}_{l_1l_2} = \int d\Omega_{n_1} \int d\Omega_{n_2} \tilde{C}(\hat{n}_1, \hat{n}_2)\{Y_{l_1}(\hat{n}_1) \otimes Y_{l_2}(\hat{n}_2)\}_{LM}. \]  

Hence, when the underlying CMB signal is statistically isotropic and beams are circular, BipoSH coefficient are expected to vanish for \(L > 0\), refer Eq. (8),

\[ \hat{A}^{LM}_{l_1l_2} = (-1)^{l_1} \prod_{l}{C}_{l_1}B_{l_1l_2}^2 \delta_{l_1l_2} \delta_{L0} \delta_{M0}. \]  

(31)

In Sec. III, we showed that non-circularity of the beam response function is captured by beam-BipoSH coefficients. The BipoSH coefficients of CMB map for a reflection symmetric beam function can then be derived in terms of beam-BipoSH coefficients as (see Appendix A),

\[ \hat{A}^{LM}_{l_1l_2} = \sum_{l} (-1)^{l_1} \prod_{l} \left\{ \frac{l_1}{l_2} \prod_{L} C_{LM} \right\} \delta_{L0} \delta_{M0}. \]  

(32)

In Sec. III, we also argued that it is convenient, as well as observationally motivated, to carry out the analytic analysis in a coordinate system where parallel-transport
(PT) scan approximation holds and, consequently, the non-zero beam-BipoSH are restricted to \( M = 0 \). Eq. (32) then dictates that the corresponding BipoSH coefficients of the CMB maps are also restricted to \( M = 0 \) and are given by

\[
\tilde{A}^L_{i_1 i_2} = \sum_{l} (-1)^l C_l \times \sum_{L_1, L_2} B^L_{i_1 i_2} B^{L_1 L_2} \prod_{L_1, L_2} C^L_{L_1 i_0 L_0} \left\{ \frac{l}{L_1} \frac{i_1}{L_2} \frac{i_2}{L} \frac{l_2}{L} \right\}.
\]  

(33)

It turns out that due to triangularity condition (\(|l_1 - l_2| \leq L \leq l_1 + l_2\), the most dominant terms in the above summation are \( \{l_1 = L, L_2 = 0\} \) and \( \{l_1 = 0, L_2 = L\} \) as they are proportional to \( B^L_{i_1 i_2} B^{L_1 i_0 L_0} \) and \( B^L_{i_1 i_2} B^{0 L_1 L_0} \), which in turn depends on the product of SH coefficients \( b_0 b_i \). In the beam response function \( b_0 \) is significantly larger than \( b_i \), therefore the product \( b_0 b_i \) will be much larger than \( b_2 b_i \) which will contribute as second order terms in Eq. (33).

The BipoSH estimator used by the WMAP team \[24, 37\], differs from the original BipoSH definition in Hajian & Souradeep \[28\] by a factor of \( \prod_{L} / \prod_{L} C^{L_{00}}_{L} \) and are restricted to only even-parity BipoSH. We insert this factor in the BipoSH expression in order to allow comparison to the published WMAP measurements more transparent \[?\].

\[
\tilde{A}^L_{i_1 i_2} \rightarrow \prod_{L} L_i C^L_{i_0 L_0} \tilde{A}^L_{i_1 i_2}.
\]  

(34)

SI violation signals in WMAP-7 were measured in two BipoSH spectra, \( \tilde{A}^{20}_{i_1 i_2} \) and \( \tilde{A}^{20}_{i_1 - i_2} \), we provide leading order expressions for these coefficients arising from the NC-beam as,

\[
\tilde{A}^{20}_{i_1 i_2} = \frac{(-1)^l 2\sqrt{5}}{(l+1)^3 C^{L_{00}}_{L} B^{000}_{i_1 i_2}} C^{L_{00}}_{i_0 i_0} B^{20}_{i_1 i_2},
\]  

(35)

\[
\tilde{A}^{20}_{i_1 - i_2} = \frac{\sqrt{5} (-1)^l}{\prod_{L} L_i} C^{L_{00}}_{i_1 - i_2} \times \left[ \frac{C_{i_1 - i_2} B^{000}_{i_1 - i_2} B^{20}_{i_1 - i_2}}{\prod_{L} L_i} + \frac{C_{i_1 i_2} B^{000}_{i_1 i_2} B^{20}_{i_1 i_2}}{\prod_{L} L_i} \right].
\]  

(36)

Note, that BipoSH expression in Eqs. (35) and (36) are provided in the scaled form that matches the BipoSH estimator employed by the WMAP team.

A. BipoSH from Elliptical Gaussian beam model

Elliptical-Gaussian (EG) functions provide a simple model NC-beam extension to the often used circular-symmetric Gaussian beam function. The non-circularity is clearly parametrized by the eccentricity of the elliptical iso-contour lines. An EG-beam function pointed along \( \hat{z} \) axis can be expressed in spherical polar coordinates, as

\[
B(\hat{z}, \hat{n}) = \frac{1}{2\pi \sigma_1 \sigma_2} \exp \left[ -\frac{\theta^2}{2\sigma_2^2(\phi)} \right],
\]  

(37)

where the azimuth angle dependent beam-width \( \sigma(\phi_1) = (\sigma_1^2 / (1 + \epsilon \sin^2 \phi_1))^{1/2} \) is given by Gaussian widths \( \sigma_1 \) and \( \sigma_2 \) along the semi-major and semi-minor axes. The non-circularity parameter \( \epsilon = (\sigma_1^2 / \sigma_2^2 - 1) \) is related to eccentricity \( e = \sqrt{1 - \sigma_2^2 / \sigma_1^2} \) of the elliptic iso-contours. As expected, the EG beam reduces to circular Gaussian beam for zero eccentricity \( e = 0 \). Higher the value of eccentricity, stronger the deviation from circularity. An analytical expression for the beam-SH of EG-beam is available \[1\]: due to reflection symmetry, for odd \( m \), \( b_{im}(\hat{z}) = 0 \), and for even \( m \),

\[
b_{im}(\hat{z}) = \sqrt{\frac{2l+1}{4\pi(l-m)!}}(l+1)^{m} \times
\]  

(38)

\[
I_{m/2} \left[ (l+1)^{2} \sigma^2 \phi^2 / 4 \right] \exp \left[ -(l+1)^{2} \sigma^2 \phi^2 (1 - \epsilon^2 / 2) \right],
\]

where \( I_m(x) \) is the modified Bessel function. The discrete even-fold azimuthal symmetry and reflection symmetry dictates \( b_{im} = 0 \) for odd \( m \). The reality condition of beam, \( b_{im} = \overline{b_{im}} \) for even \( m \), then implies \( b_{l-m} = b_{lm} \).

![FIG. 2. Harmonic transforms of an EG beam with \( \theta_{FWHM} = 13.579' \) and ellipticity \( e = 0.4 \). The circular symmetric component of the NC EG-beam is given by is \( m = 0 \) and \( m = 2 \) captures the leading order EG-beam effect.](image-url)

For Elliptical-Gaussian beams, the ratio \( b_{lm} / b_{00} \) dies down rapidly with \( |m| \). Hence, for mild eccentricities of the Elliptical beam, it is sufficient to consider only the lowest \( m = 2 \) mode. In Fig. 2, we plot spherical harmonic transform of an Elliptical-Gaussian beam with \( \theta_{FWHM} = 13.579' \), which is close to \( \theta_{FWHM} \) of W band of WMAP with ellipticity \( e = 0.4 \) that best matches WMAP-7 W-band BipoSH spectra amplitude. Similarly, we also consider a EG-beam with \( \theta_{FWHM} = 17.7036' \).
close to the V-band beam. The best fit to V-band BipoSH spectra corresponds ellipticity \( e = 0.46 \). For analysis purpose, it is a good enough approximation to restrict to \( m = 0, \pm 2 \) modes. We estimate beam-BipoSH coefficient in Eq. (25) for the case of “non-rotating” beams, \( \rho = 0 \) by using the closed analytical form of \( b_{lm} \)'s as in Eq. (38). Finally, using Eq. (35) and Eq. (36) we obtain the CMB BipoSH spectra \( A_{ll}^{20} \) and \( A_{l-2l}^{20} \) that were measured to be non-zero in WMAP-7. We verify the analytical results and compute the error bars on the BipoSH coefficients using statistically isotropic Gaussian simulations convolved with EG-beam functions. In Fig. 3, the analytic calculations of CMB-BipoSH for two cases of EG-beam are overlaid on estimates from average measurements with error-bars from 100 numerical simulations of noise free SI maps convolved with corresponding NC-beams. BipoSH coefficients obtained from EG beams serve to crosscheck and validate our analytical expression derived and puts a check on the numerical simulation of CMB maps convolved (in real space) with an NC-beam.

It is clear that NC-beams generate detectable levels of non-zero CMB BipoSH. The CMB-BipoSH spectra \( A_{ll}^{20} \) and \( A_{l-2l}^{20} \) are both proportional to the leading order NC correction \( b_{lb2} \), and the underlying SI CMB power spectrum (see Eqs. (33) and (41)). It is instructive to compare the EG-model beam predictions to the WMAP BipoSH measurements. Using Elliptical-Gaussian beams with FWHM values close to the WMAP W and V band reproduces significant detectable peaks in the BipoSH spectra. The amplitudes can be matched by choosing appropriate values of effective eccentricity for the V (\( e = 0.46 \)) and W bands (\( e = 0.4 \)). Although, the eccentricity of the WMAP beams can be estimated for raw beam maps [2], it is expected that multiple visits by the beam at any pixel with varying orientation in WMAP scan would tend to symmetrize the beam and reduce the effective eccentricity to smaller values. The eccentricities that reproduce the correct amplitudes for the CMB-BipoSH roughly correspond to the reduction also indicated from comparing the corrections to the angular power spectrum [40]). The fact that both the \( A_{ll}^{20} \) and \( A_{l-2l}^{20} \) spectra, are roughly matched in amplitude and qualitative peak structure with the correct relative sign, for identical ellipticity parameters, then suggests that the WMAP-7 BipoSH measurements may have actually captured subtle uncorrected NC-beam effects. This claim is made strong through realistic simulations in the following sections.

V. BIPOSH SIGNATURES OF NON-CIRCULARITY IN WMAP BEAMS

While CMB BipoSH from the test EG beam motivates a more careful estimation of the NC-beam effect in WMAP-7 maps, these fail to reproduce certain features of WMAP BipoSH measurements; viz. the change in sign of the V-band \( A_{ll}^{20} \) at large multipoles. This, however, is not surprising. It is widely-known that the WMAP beams deviate from a Gaussian profile and that an EG beam is not a good approximation [2, 41, 42]. In section V.A, we first use the SH transform of the raw WMAP beam maps to compute the CMB-BipoSH under PT-scan approximation. Interestingly, we recover qualitative features in BipoSH spectra measurements that were not captured by the Elliptical Gaussian beams. As expected the amplitude of the CMB-BipoSH based raw beam are much higher in amplitude relative to the measurements. A rudimentary numerical superposition of raw beam transforms to mimic the effect of multiple hits with varying orientation at a pixel (using a WMAP scan), leads us to expect a constant scaling of the leading order \( b_{l2} \) by a factor \( \alpha \sim 0.45 \), as shown in Fig. 4. Moreover detailed analytical treatment presented in the Appendix C, allows for deviations from PT-scan approximation in averaging over multiple hits, reveals multipole \( l \) dependent correction, \( f_l \) to the constant scaling that depends on the detailed scan strategy of the CMB experiment over all multiple hits.

The analytic expression for \( f_l \) in Appendix C, for simplicity, pertains only to a single beam scan case. In particular, the approach cannot account for any complexities that may be introduced to the effective NC-beam that is used to estimate the BipoSH spectra arising from linear algebraic map-making procedure using two beam differencing. However, it motivates the exploration to obtain a phenomenological effective beam \( b_{lm}^{eff} = \sum_{lm'} F_{lm'} b_{lm} \) expressed in terms of the beam-SH obtained from the raw beam maps. We proceed with the assumption that the WMAP BipoSH measurements arise from NC-beam effect and take the ‘phenomenological’ approach to determine \( b_{l2} \) from BipoSH measurements. Limiting attention to the leading order contribution at \( m = 2 \) we parametrize the function \( f_l \equiv F_{l2} \) and fit to the measured BipoSH spectra. In section V.B, we first fit for a constant value \( \alpha \) that closely reproduces the WMAP BipoSH measurements. While we find that \( \alpha \sim 0.4 \) is the best-fit value, the goodness of fit is not very satisfactory. We find that a linear correction to the constant scaling, \( b_{l2}^{eff} = (\alpha + \beta \cdot l) b_{l2} \), which provides very good fits to the WMAP BipoSH measurements. Numerical simulations incorporating complexities of two beam differencing and map-making procedure and deviations from PT-scan approximation are performed below to decide whether the parameters of the phenomenological scaling obtained are justified.

A. BipoSH coefficients from single-side WMAP raw beam map

To mimic NC-beams closer to WMAP, we consider the A side raw beam maps of the V2 and W1 differencing Assembly (DA) of WMAP as representative of the V and W band beams, respectively. We compute the beam-SH coefficients for these assemblies numerically for use in semi-
FIG. 3. BipoSH spectra $A_{20}^{30}$ and $A_{20}^{30}$ for an EG beam with: (Top) $\theta_{\text{FWHM}} = 13.579'$ and ellipticity $\epsilon = 0.4$; (Bottom) $\theta_{\text{FWHM}} = 17.7036'$ and ellipticity $\epsilon = 0.46$ smooth (blue) curve are binned CMB-BipoSH computed using the analytic expressions. The red curve with corresponding error-bars are obtained from 100 numerical simulated SI maps convolved with the same EG beam. The underlying SI signal corresponds to the best fit $\Lambda$-CDM model from WMAP7.

analytic estimate of the CMB BipoSH coefficients, using Eqs. (35) and (35). Fig. 5 is a plot of the circular, $b_0$ and leading order $b_2$ beam-SH coefficients of the W1A and V2A raw beam maps. Note that the $b_2$ spectrum changes sign and takes negative values at large $l$ – a key qualitative feature of the WMAP beams that cannot be captured in Elliptical-Gaussian models where $b_2$ does not change sign with $l$. The origin of this curious feature becomes apparent in the raw beam maps of the W1A and V2A channel shown in Fig. 6. The central part of the beam maps show an elliptical peak with marked non-trivial NC ‘shoulder-like’ features. However, more interesting are the right-hand panels, where we highlight a spread out annular distribution of regions with negative response. The integrated power in the negative beam response is $\sim 0.5$ of the total power and has significant impact on the beam-SH. In particular, it leads to negative values of $b_2$ at $l \gtrsim 2\sigma_L^{-1}$. We find that this feature is critical in recovering the qualitative feature in V-band $A_{20}^{30}$ changing sign at large $l$ seen in the WMAP measurements. Since such an unique correspondence between a beam-SH feature and the consequent CMB BipoSH is unlikely to be mimicked by other effects. We claim that this provides a strong hint that WMAP BipoSH measurements are linked to the NC-beam effect.

We use PT-scan approximation in ecliptic coordinates given the fact that the WMAP BipoSH results indicate azimuthal symmetry of SI violation $(M = 0)$ in ecliptic coordinates. The analytical results are verified using numerical simulations of the BipoSH coefficients $A_{l}^{0}$ and $A_{l-2}^{0}$ are shown in Fig. 7. As seen in Fig. 7, the amplitudes of the BipoSH coefficients do not numerically match WMAP7 year’s observed detections. This is expected as the BipoSH coefficients are obtained from the A side beam of a single channel in W and V band without accounting for circularizing of the beam due to multiple hits with varying orientations (this case has been considered later). We collate and list some key similarity of these results to the detections of SI violation in WMAP-7 data:

1. As expected from analytic understanding, the numerical simulations also confirm that the most significant contribution is the $M = 0$ mode in BipoSH spectra $A_{0}^{2M}$ and $A_{0}^{2M}$ when the analysis is done in ecliptic coordinates, giving strength to our PT-scan approximation. In a coordinate system where PT-scan approximation is valid only $M = 0$ mode.
The WMAP beams have a spread-out annular pattern with the main central peak having significant structure. Beam maps present a computational challenge. Implementing numerical simulation with actual WMAP and map-making details should settle the extent to which incorporating the two side WMAP beam differencing, scan in the WMAP-7 maps. Numerically simulated maps in the WMAP-7 maps. Motivation for a deeper analysis of the NC-beam effect is provided by the measured results. We do get very good consistency with semi-analytic results.

should be significant.

2. We notice the NC beam effect is larger in W band than in V band explaining the difference in detected SI violation signal at the two frequencies.

3. We recover the change of sign of the BipoSH coefficients at large l in the V-band.

4. The BipoSH coefficients from NC beam shows a prominent bump roughly around the first acoustic peak (l = 220) for both W and V band. This corresponds mainly to the scale picked by the underlying angular power spectrum C_l. However, the precise peak location also depends on the peak in b_0b_2 for each band and can account for differences in the peak location in the two bands shown in Figure 5.

The exercise of computing BipoSH from single side raw beam maps under PT-scan approximation does provide motivation for a deeper analysis of the NC-beam effect in the WMAP-7 maps. Numerically simulated maps incorporating the two side WMAP beam differencing, scan and map-making details should settle the extent to which approximations used here are valid and clearly settle the question of whether the entire WMAP-7 BipoSH signal arises as an effect of uncorrected NC-beam effects. Implementing numerical simulation with actual WMAP beam maps present a serious computational challenge. Besides the main central peak having significant structure, the WMAP beams have a spread-out annular pattern of negative response that cannot be neglected. This implies that for a careful BipoSH analysis of SI violation, a convolution with much larger regions of the WMAP beam maps of W and V band, extending roughly to, ∼ 10θ_{FWHM} is required, making it computationally expensive. The beam maps of W1A and V2A channel are shown in Fig. 6. In particular, we highlight the spread-out negative part of the beam response that explains some unique qualitative features of the WMAP BipoSH measurements. The power in the negative part of the beam is more than 50 percent of its central positive peak and hence, non-negligible.

B. Fitting BipoSH spectra measurements to an effective NC-beam

In the analytic approach we show that the BipoSH coefficients scale as \( A_{l}^{20} \propto b_0 b_2 \) for mild deviations from circular beams under PT-scan as given in Eq. (33). However, realistically, complex details of relative orientation of A and B side of beams for each DA, the differencing scheme and map-making, the varying orientations for the multiple hits at each pixel in the actual scan strategy, etc., all could have bearing at finer levels on CMB BipoSH generated by the NC-beam.

As shown in Appendix C, the averaging of the NC-beam at any pixel due to multiple-hits at different orientations can be expected to lead to a simple scaling of the form \( b_{eff}^{20} = f_I b_{20} \). This would correspondingly scale BipoSH spectra obtained from A side beam of W1 and V2 channel by \( f_I \).

Retaining PT scan approximation, the superposition of raw beam transforms leads us to expect a constant scaling \( f_I = \alpha \). The value \( \alpha \approx 0.45 \) most closely reproduces the WMAP BipoSH spectra. We confirm this to be consistent with numerical estimates from beam-SH superposition using \( p_i(\hat{n}) \) obtained from WMAP-like multiple hits but with PT-scan approximation (see Appendix C). The numerical value of \( \alpha \) depends on the relative orientations of the A and B side beams (assumed here to be identical) and matches the fit value of \( \alpha \) for relative orientation \( \sim 140^\circ \).

However, the goodness of fit for constant \( f_I \) is not very satisfactory in explaining the entire WMAP-7 BipoSH spectra in terms of NC-beam systematics and suggests that \( l \)-dependence of \( f_I \) may be important for a finer match. While, in principle, it is possible to determine \( f_I \) for the WMAP scan by numerical superposition of beam-SH using Eq. (C13). Practically, it will be plagued by ambiguity of relative orientation of the A and B side NC-beams and many other finer details. Hence, we prefer to take a phenomenological approach and explore the possibility of parametrized \( f_I \) that fit the BipoSH measurements. Minimally, this provides an effective beam that would explain the observed WMAP-7 BipoSH spectra under PT-scan approximation. We believe that this provides a computationally inexpensive approach to quantify...
FIG. 5. Top: Beam spherical harmonic transforms, \( b_{l0} \) and \( b_{l2} \) of beam maps of A side of W1 and V2 DA. It is interesting to note that \( |b_{l2}|/b_{l0} \lesssim 0.01 \) implying the BipoSH representation is sensitive to really really subtle levels of non-circularity in the beams.

Bottom: The plot of the NC-beam leading order NC beam perturbation parameter \( b_{l0}b_{l2} \) vs. \( l \). The solid lines correspond to beam-SH of the raw beam maps, the dashed lines show the \( b_{l0}b_{l2}^{\text{eff}} \) of the phenomenological best-fit effective beam-SH \( b_{l2}^{\text{eff}} = f_l b_{l2} \) obtained by parametrized linear fits to \( f_l \) using the the two measured BipoSH spectra, \( A_{l}^{W0} \) and \( A_{l}^{V0} \). The plots also show that although BipoSH peak structure is largely set by the underlying angular power spectrum \( C_l \) of SI cosmological mode, small differences observed at the two different frequencies can arise because of the difference in the shapes of \( b_{l0}b_{l2} \).

VI. RECOVERING WMAP7 MEASURED BIPOSH WITH ACTUAL SCAN

In order to extract BipoSH coefficients in the previous section, we simulated maps by convolving a statistically isotropic map with the WMAP W-Band and V-Band beam over the sky. For these simulations, we used a very simplistic scan pattern, where beams are parallel transported across the sky, and the angle between the major axis of the elliptical beam and the local meridian does not vary across the sky. These simulations require less computation time to provide the scanned maps. BipoSH coefficients obtained from these simulations match reasonably well with the observed BipoSH of WMAP-7. Evaluating the actual effect coming from the WMAP scan strategy requires convolving the sky with the real WMAP beam followed by the actual WMAP scan procedure, map making procedure and masking as used by WMAP-7. In this section we have presented the BipoSH coefficients obtained from these simulated maps following the WMAP NC-beam systematic effect.

The obvious step beyond constant scaling, would be to add a linear correction \( f_l = \alpha + \beta l \). As shown in Fig. (8) we find very good fits to the WMAP BipoSH measurements with this scaling. Interestingly enough, it seems possible to find values of \( \alpha \) and \( \beta \) that simultaneously provide fairly good fits to both \( A_{l}^{W0} \) and \( A_{l-2l}^{V0} \) BipoSH spectra. Note that the product \( b_{l0}b_{l2} \) still tends to zero for large \( l \) as seen in the bottom panel of Fig. 5.
FIG. 6. Beam response function of A side of W1 differencing assembly (Top) and A side of V2 differencing assembly (Bottom) are shown. The Left-hand panels show 3D zoomed view of the central part of the beam shows a truncated view of the central beam peak to clearly highlight the elliptical contour. Marked NC shoulder features are seen in the central peak. The Right-hand panels cover the entire beam-map images to show the spread-out annular distribution of regions with negative response. (The red circles marks the central beam peak region). The power in negative response is \( \sim 0.5 \) of the positive power in the central peaks. The annular negative power distribution shows quadrupolar feature that modifies the beam-SH \( b_{l2} \) to take negative values at high \( l \) for the V-band (see Fig 7). The corresponding impact of this beam-SH is significant and crucial for understanding the intriguing qualitative zero-crossing feature in the V-band BipoSH spectra. It is apparent then that correctly accounting for WMAP NC-beam effects numerically in our ongoing analysis, requires the convolution of almost the entire beam map region with the SI sky-map leading to enormous increase in computing costs (relative to using only the central peak).

the actual WMAP procedure. Since the entire process is computationally intensive, only 10 simulations using the W1-Band beam maps are generated and the have evaluated BipoSH coefficients from these many simulations [? ]. It can be seen that the results match the WMAP-7 non-zero BipoSH observation to a very good accuracy.

Figure (9) shows a comparison of the computed \( A_{l\ell}^{20} \) and \( A_{l\ell-2\ell}^{20} \), to that measured in WMAP-7 maps. The exercise shows that the WMAP-7 observed SI violation can be completely explained by the effect of NC beams. It can be seen that at low \( \ell \)'s the WMAP-7 detections are lesser then our simulated results. Our re-analysis to compute the BipoSH coefficients for WMAP-7 maps shows that the power at low \( \ell \) is actually more than that reported by the WMAP team. To check this effect a more detailed analysis is being carried out. The error bars plotted on the simulated BipoSH are calculated from the sum of the variances from the full scan simulation error bars and the error bar from the masking. No noise is added to the simulations, as we are only interested in the mean effect. As the error bars are calculated only from 10 realizations, they are not very accurate. To get a more accurate simulations and to know the effect of different beams from all four W-Band and two V-Band maps simulations are ongoing. The full detail of the simulation method and a thorough analysis will be presented in an upcoming publication.
VII. DISCUSSIONS & CONCLUSION

The observed CMB sky is a convolution of the cosmological signal with the instrumental beam response function of the experiment (assuming perfect removal of foreground emissions, etc.). The deconvolution of the beam effect from the signal is relatively straightforward for an ideal circular symmetric beam. Non-Circular (NC) deviations of the beam, however mild, are practically inevitable in all experiments, and affect the results obtained at the limits of the sensitivity and resolution of the recent experiments. The effect of WMAP NC-beam on the angular power spectrum $C_l$ is not very pronounced and is significant only at very large multipoles. It is very therefore very interesting that CMB maps obtained with NC-beams disrupt the rotational invariance of the two point correlation function leading to clearly measurable signatures of SI violation.

We show that SI violation measures in the Bipolar Spherical Harmonic (BipoSH) representation of the observed CMB maps, are well-suited to capture systematic NC-beam effect. It is important to note that even though the level of non-circularity in WMAP beams is at $\sim 1\%$ percent in $C_l$, the BipoSH spectra generated by this effect are measurable. This points to immense promise and potential of the BipoSH representation also as a diagnostic tool for current and future CMB experiments. A key objective is also to assess whether the recent measurement of non-zero BipoSH spectra, $A_{l0}^{20}$ and $A_{l2}^{20}$ in WMAP-7 in the V & W band maps could arise from uncorrected NC-beam effects in WMAP-7 maps.

In this paper, we introduce the novel and useful concept of expanding the NC-beam response function in the BipoSH basis to define beam-BipoSH coefficients. The beam-BipoSH not only incorporate the effect of the NC-beam (expressed in terms of the beam-SH – the spherical harmonic transform of the beam $b_{lm}(\hat{z})$ pointing at $\hat{z}$), but also the additional effect of the scan strategy that determines the beam orientation angle, $\rho(\hat{n})$ relative to the local longitude at any pixel. Neat and completely analytic results for CMB BipoSH generated by NC-beams can be obtained within the ‘parallel-transport’ (PT) scan approximation where the beam visits the pixels at constant $\rho(\hat{n}) \equiv \rho$. We argue that this approximation is observationally well-motivated due to the azimuthal symmetry ($M=0$) in the WMAP-7 BipoSH measurements in the ecliptic coordinates. All analytical results are verified using corresponding numerical simulations on SI maps convolved with NC-beam in the PT-approximation. The numerical simulations also provide estimates of the error-bars, hence, a measure of the significance of curious features in the predicted BipoSH spectra. As a test case we first work with Elliptical-Gaussian (EG) beam with adjustable eccentricity but FWHM that match the V and W band WMAP beams. EG beams have well interpreted NC parameters and also permits a well-controlled perturbation treatment for NC deviations ($b_{lm}/b_{l0}$ dies rapidly with $|m|$ [1]). This exercise also provides a reliable test case to cross-check analytic results with that obtained from numerical simulations of SI maps convolved with NC-beams. At the leading order ($b_{l0}b_{l0}$), we recover significant CMB BipoSH coefficients $A_{l0}^{20}$ and $A_{l2}^{22}$ with broadly correct qualitative features – relative sign difference in the two spectra and a well understood peak structure defined by the acoustic features in underlying $C_l$ modulated by spectral shape of $b_{l0}b_{l2}$. We choose the eccentricity parameters at fixed FWHM to match the amplitude. We conclude that the multiple visits of the beam reduces the eccentricity by a factor comparable to that indicated by comparing NC beam corrections to WMAP $C_l$ from semi-analytic and numerical estimates [2, 40, 42].

As may be expected that the EG approximation to WMAP beams, completely fails to recover certain qualitative features of the BipoSH measurements, in particular, the change of sign in the V-band $A_{l0}^{20}$ at large $l$. This is addressed next by computing the beam-SH for single (A) side WMAP beam maps for V and W bands. The beam maps show a marked annular region well beyond a few FWHM from the center where the beam response is negative. This leads to the $b_{l2}$ going negative at large multipoles beyond the beam-width and explains the change in sign of V-band BipoSH at larger $l$. Note that EG beams can never reproduce this feature, since $b_{l2} \geq 0$. Again, as expected, the amplitude of the CMB-BipoSH coefficients based on beam-SH, $b_{l2}$, from raw beam maps are much higher. An analytic treatment of superposition of raw beam transforms to mimic the effect of multiple hits with varying orientation at a pixel, leads us to expect a scaling of the leading order term $b_{l2}^{\text{eff}} = f_{l2}b_{l2}$. However, $f_l$ depends on the detailed and accurate scan description of each beam and many other finer details of the instruments. Hence we choose to adopt a ‘phenomenological’ approach to determine the effective beam as a result of multiple hits with different orientations at a pixel and fit a parametrized $f_l$ to the WMAP-7 BipoSH spectra measurements assuming it to be entirely sourced by NC-beam effect. A constant $f_2 = \alpha \approx 0.4$ provides only a fair fit and does not account for the entire signal. A linear correction to the constant scaling, $f_l = \alpha + l\beta$ allows for very good fits to the WMAP BipoSH spectra measurements as shown in Fig. 8.

Complexities of two beam differencing and map-making procedure, deviations from PT-scan approximation, etc., may account for the phenomenologically obtained $f_l$ scaling of raw beam-SH, $b_{l2}$. We simulated 10 maps for W1-band following the actual WMAP procedure and obtained BipoSH spectra from them which match well with the observed BipoSH spectra in WMAP-7 year data. We can firmly conclude that NC-beam effect is dominantly responsible for the WMAP-7 non-zero BipoSH measurements. In any case, our results indicate clearly that a major portion of the BipoSH signal measured in WMAP-7 may be attributed to an experiment specific systematic effect arising from uncorrected NC-Beam effect. However, it is still possible that the BipoSH
measurements are not entirely explained by the NC beam effect leaving room for an achromatic component from intrinsic cosmological SI violation. Since the WMAP-7 BipoSH signal are strong and significant, other full sky CMB experiments with comparable, or better, sensitivity and angular resolution should readily confirm, or disprove this possibility. We will be providing full extensive simulation for W-band and V-band and details of effect of non-circular beams on SI violation in future publication. In near future, good quality full sky CMB polarization maps when available can also be studied in the BipoSH representation [32] and will provide an independent window in to SI violation phenomena.

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Appendix A: Beam BipoSH

Beam-BipoSH are expansion coefficients of the beam response function in BipoSH basis (see Sec III). The most general beam-BipoSH in any coordinate system is given by,

\[ B_{l_1l_2}^{LM} = -\sum_{m_1m_2} C_{l_1m_1l_2m_2}^{LM} b_{l_2m_2} (\hat{z}) \times \] (A1)

\[ \int_0^\pi d(\cos \theta) \int_0^{2\pi} d\phi D_{l_2m_2}^{l_1m_1} (\phi, \theta, \rho(\theta, \phi)) Y_{l_1m_1}^*(\theta, \phi). \]

Wigner-D functions can be expressed in terms of Wigner-d through following relation,

\[ D_{l_1m_1}^l (\phi, \theta, \rho) = e^{-im\phi} d_{l_1m_1}^l (\theta) e^{-im'\rho}, \] (A2)

and reduces to spherical harmonics for \( m' = 0 \),

\[ D_{l_00}^l (\phi, \theta, \rho) = \sqrt{4\pi/(2l+1)} Y_{l0}^* (\theta, \phi). \] (A3)

In the parallel-transport (PT) scan approximation, the beam orientation, with respect to the local Cartesian coordinate aligned with the spherical \((\theta, \phi)\) coordinates, does not vary on sky (i.e., \( \rho(\theta, \phi) = \rho \)), the beam-BipoSH coefficients for such an approximation are,

\[ B_{l_1l_2}^{LM} = -\sqrt{\frac{2l+1}{4\pi}} \sum_{m_1m_2} b_{l_2m_2}(\rho) \times \]

\[ \left( \sum_{m_1m_2} C_{l_1m_1l_2m_2}^{LM} \right)^2 \int_0^{2\pi} d\phi e^{-i(m_1+m_2)\phi} \]

\[ \int_0^\pi d\theta D_{m_2m_1}(\theta)d_{l_1m_1}^l(\theta)d(\cos \theta). \] (A4)

Which simplifies to,

\[ B_{l_1l_2}^{LM} = 2\pi \delta_{M0} \sum_{m_1m_2} b_{l_2m_2}(\rho) e^{-im'\rho} \times \]

\[ \sum_{m_2} (-1)^{m_2} C_{l_1m_1l_2m_2}^{LM} I_{l_1m_1}^{l_2m_2} \] (A5)

and is derived to be non-zero only for \( M = 0 \), using the orthogonality relation,

\[ \int_0^{2\pi} d\phi \exp^{-i(m_1+m_2)\phi} = 2\pi \delta_{m_1,-m_2}. \] (A6)

We define the notation

\[ I_{l_1m_1}^{l_2m_2} = (-1)^{m_2+1} \sqrt{\frac{(2l+1)}{4\pi}} \times \]

\[ \int_0^\pi d\theta D_{m_2m_1}(\theta)d_{l_1m_1}^{l_2}(\theta)d(\cos \theta). \] (A7)

Here we have used \( d_{m_2m_1}(\theta) = (-1)^{m-m'} d_{m-m'}(\theta) \).

To simplify the analytic expressions, we retain only the leading order NC beam spherical harmonic mode \( m' = 2 \), assuming mild NC-beam with discrete even-fold azimuthal symmetry where no odd \( m' \) modes will contribute. Hence, the summation over \( m' \) has three terms corresponding to \( m' = 0, \pm 2 \).

The beam-BipoSH can be then be written as

\[ B_{l_1l_2}^{LM} = B_{l_1l_2}^{L(MC)} + B_{l_1l_2}^{L(NC)} \] (A8)

\[ B_{l_1l_2}^{L(MC)} = 2\pi \delta_{L0} \delta_{M0} b_{l_20}(\rho) \sum_{m_2} C_{l_1m_1l_2m_2}^{L0} r_{l_1m_2}^{l_2} \] (A9)

\[ B_{l_1l_2}^{L(NC)} = 2\pi \delta_{M0} \sum_{m_2 \neq 0} C_{l_1m_1l_2m_2}^{L0} \times \]

\[ \left( b_{l_2-2}(\rho) \exp^{i2\rho} r_{l_1m_2}^{l_2} + b_{l_2-2}(\rho) \exp^{-i2\rho} r_{l_1m_2}^{l_2} \right). \]

First term in Eq. (A8) is the trivial beam-BipoSH, \( B_{l_20}^{(MC)} \), corresponding to the circular symmetric component of the beam response function. NC part of the beam function \( m' = \pm 2 \), gives rise to beam BipoSH having \( L \neq 0 \). First, we evaluate the beam-BipoSH due to circular part of beam function. Orthogonality of Wigner-d functions,

\[ -\int_0^{2\pi} d(\cos \theta) d_{l_1m_1}^{l_2}(\theta)d_{l_2m_1}^{l_1}(\theta) = \frac{2}{2l+1} \delta_{l_1l_2}. \] (A10)

implies

\[ r_{l_1m_2}^{l_2} = (-1)^{m_2} \left( \frac{2}{2l_2+1} \right) \sqrt{\frac{(2l_1+1)}{4\pi}} \delta_{l_1l_2}. \] (A11)
Using the following property of Clebsch-Gordan,
\[ \sum_{m} (-1)^{l-m} C_{lml-m}^{l0} = \sqrt{2l+1} \delta_{l0}. \]  
(A12)

we obtain
\[ B_{01}^{LM} = \sqrt{4\pi} (-1)^{l} b_{00}(\hat{z}) \delta_{11} \delta_{L0} \delta_{M0}. \]  
(A13)

Since,
\[ b_{00}(\hat{z}) = \sqrt{\frac{(2l+1)}{4\pi}} B_l, \]
(A14)

where \( B_l \) is the usual beam transfer function of the circular-symmetrized beam profile,
\[ B_{01}^{LM} = (-1)^{l} \sqrt{2l+1} B_l \delta_{11} \delta_{L0} \delta_{M0}. \]  
(A15)

Next, we analytically calculate the beam-BipoSH \( B_{01}^{LM(\text{NC})} \) due to the NC part of the beam function,
\[ B_{01}^{LM(\text{NC})} = -2\pi \sqrt{\frac{(2l+1)}{4\pi}} \delta_{L0} \sum_{m_2} (-1)^{m_2} C_{l-m_2-m_2}^{l0} \]
\[ \times \left[ b_{l-2}(\hat{z}) \exp \frac{i2\pi}{\theta} \int_{0}^{\pi} d_{m_2-2} d_{m_2} \delta_{l} \left( \hat{z} \right) \delta_{l_0} \delta_{m_0} \theta d(\cos \theta) + b_{l2}(\hat{z}) \exp \frac{-i2\pi}{\theta} \int_{0}^{\pi} d_{m_2+2} d_{m_2} \delta_{l} \left( \hat{z} \right) \delta_{l_0} \delta_{m_0} \theta d(\cos \theta) \right]. \]

In the above expression, the summation is over \( m_2 \). It is convenient to separate the calculation of the \( m_2 = 0 \) and rest of the \( m_2 \neq 0 \) terms.

Consider the integral for \( m_2 = 0 \). For this we use,
\[ d_{m_2}^{l_0} = (-1)^{m+1} d_{m_2}^{l_0} \]
and expansion of Wigner-\( d \)'s in terms of associated Legendre polynomials, \( d_{m_0}^{l_0} = (-1)^{m} \sqrt{2l+1} \left( \frac{l}{2} \right)! \left( \frac{l-m}{2} \right)! \left( \frac{l+m}{2} \right)! \]

\[ \times \left[ \frac{1}{l+2} \int_{\theta=0}^{\pi} P_{l}^{2}(\cos \theta) P_{l_0}(\cos \theta) d(\cos \theta) \right]. \]  
(A16)

Using standard recurrence relations of Associated Legendre functions,
\[ P_{l}^{2}(\cos \theta) = \frac{2 \cos \theta}{\sin \theta} P_{l}^{1}(\cos \theta) - l(l+1) P_{l}(\cos \theta), \]  
(A17)

and orthogonality relations,
\[ -\int_{0}^{\pi} P_{l}(\cos \theta) P_{l}(\cos \theta) d(\cos \theta) = \frac{2}{2l+1} \delta_{l1} \delta_{l2}, \]  
(A19)

\[ \int_{0}^{\pi} \cos \theta P_{l}^{2}(\cos \theta) P_{l_{1}}^{0}(\cos \theta) d(\cos \theta) = \begin{cases} 0 & \text{if } (l_1 + l_2 \text{ is odd}) \\ 0 & \text{if } (l_1 > l_2) \\ 0 & \text{if } (l_1 < l_2) \\ \frac{2l_2}{l_2+1} & \text{if } (l_1 = l_2). \end{cases} \]  
(A20)

Therefore, for \( m_2 = 0 \), the integral simplifies to
\[ I_{0, \pm 2} = \begin{cases} (l_1 + l_2)^{m_2} \sqrt{\frac{(2l+1)}{4\pi}} \times \\ 0 & \text{if } (l_1 + l_2 \text{ is odd}) \\ 0 & \text{if } (l_1 > l_2) \\ 4 \left( \frac{1}{\sqrt{l_2+2}} \right) & \text{if } (l_1 < l_2) \end{cases} \]  
(A21)

Next, we consider the \( m_2 \neq 0 \) terms in the summation in Eq. (A16). Here, \( d_{m_2}^{l_2}(\theta) \) can be recursively reduced to \( d_{m_2}^{l_0} \) using the following recurrence relation,
\[ d_{m_2}^{l_2}(\theta) = \frac{\kappa}{\sin^{2} \theta} \left[ \frac{\kappa_{0} d_{m_2}^{l_0} \theta + \kappa_{1} d_{m_2}^{l_2+1} \theta + \kappa_{-1} d_{m_2}^{l_2-1} \theta + \kappa_{-2} d_{m_2}^{l_2-2} \theta}{2m_2} \right]. \]  
(A22)

where,
\[ \kappa_{0} = \frac{m_2^2}{l_2(l_2+1)^2} - \frac{l_2^2 - m_2^2}{l_2(l_2+1)^2} - \frac{(l_2+1)^2 - m_2^2}{(l_2+1)^2(l_2+1)^2} \]  
\[ \kappa_{1} = \frac{2m_2}{l_2(l_2+1)(2l_2+1)} \]  
\[ \kappa_{-1} = -\frac{2m_2}{l_2(l_2+1)^2(2l_2+1)} \]  
\[ \kappa_{2} = \frac{(l_2+1)^2 - m_2^2}{(l_2+1)^2(l_2+1)^2(l_2+1)^2} \]  
\[ \kappa_{-2} = \frac{(l_2+1)^2 - m_2^2}{l_2(l_2+1)^2(l_2+1)^2} \]  
(A23)

Under reflection Wigner-\( d \)'s transform as, \( d_{m_2}^{l_2}(\pi - \theta) = (-1)^{l+m} d_{-m_2}^{l_2}(-\theta) \).

\[ d_{m_2}^{l_2-2}(\theta) = \frac{\kappa}{\sin^{2} \theta} \left[ \frac{\kappa_{0} d_{m_2}^{l_2} \theta - \kappa_{1} d_{m_2}^{l_2+1} \theta}{2m_2} \right] \]  
(A23)

and then as shown in [2], the integrals for \( m_2 \neq 0 \) terms are obtained as
\[ p_{m_2}^{l_1 l_2} = (-1)^{m_2} \sqrt{\frac{(2l_1 + 1)}{4\pi}} \begin{cases} 
\left( \frac{\kappa_{k_0}}{|m_2|} \sqrt{\left( \frac{(l_2 + m_2)!}{(l_2 - m_2)!} \right) l_1} \right) + \left( \frac{\kappa_{k_0}}{|m_2|} \sqrt{\left( \frac{(l_2 + m_2)!}{(l_2 - m_2)!} \right) l_1} \right) & \text{if } (l_1 = l_2) \\
\left( \frac{\kappa_{k_0}}{|m_2|} \sqrt{\left( \frac{(l_2 + m_2)!}{(l_2 - m_2)!} \right) l_1} \right) + \left( \frac{\kappa_{k_0}}{|m_2|} \sqrt{\left( \frac{(l_2 + m_2)!}{(l_2 - m_2)!} \right) l_1} \right) & \text{if } (l_1 > l_2) \\
\left( \frac{\kappa_{k_0}}{|m_2|} \sqrt{\left( \frac{(l_2 + m_2)!}{(l_2 - m_2)!} \right) l_1} \right) & \text{if } (l_1 < l_2) 
\end{cases} \]

In general, NC-beams would generate both even-parity and odd-parity beam-BipoSH coefficients

\[ B_{i_1 l_2}^{L M^{(NC)}} = B_{i_1 l_2}^{L M^{(+)} + B_{i_1 l_2}^{L M^{(-)}}. \] (A24)

The even-parity beam-BipoSH,

\[ B_{i_1 l_2}^{L M^{(+)} = \begin{cases} 
\delta_{M_0} [b_{l_2}^{1/2} (\hat{z}) \exp(-i2\rho) + b_{l_2}^{1/2} (\hat{z}) \exp(i2\rho)] \left( \frac{(-l_1-l_2)}{4\pi (2l_1+1)(2l_2+1)} \right) + \\
2\pi \sqrt{\frac{(2l_1+1)}{4\pi}} \sum_{|m_2|>0} C_{l_1-m_2}^{L 0} \frac{\kappa_{k_0}}{|m_2|} \sqrt{\left( \frac{(l_2 + m_2)!}{(l_2 - m_2)!} \right) l_1} & \text{if } (l_1 = l_2 \text{ and } l_2 \geq 2) \\
\delta_{M_0} [b_{l_2}^{1/2} (\hat{z}) \exp(-i2\rho) + b_{l_2}^{1/2} (\hat{z}) \exp(i2\rho)] \left( \frac{(-l_1-l_2)}{4\pi (2l_1+1)(2l_2+1)} \right) + \\
2\pi \sqrt{\frac{(2l_1+1)}{4\pi}} \sum_{|m_2|>0} C_{l_1-m_2}^{L 0} \frac{\kappa_{k_0}}{|m_2|} \sqrt{\left( \frac{(l_2 + m_2)!}{(l_2 - m_2)!} \right) l_1} & \text{if } (l_1 > l_2 \text{ and } l_2 \geq 2) \\
\delta_{M_0} [b_{l_2}^{1/2} (\hat{z}) \exp(-i2\rho) + b_{l_2}^{1/2} (\hat{z}) \exp(i2\rho)] \left( \frac{(-l_1-l_2)}{4\pi (2l_1+1)(2l_2+1)} \right) & \text{if } (l_1 < l_2 \text{ and } l_2 \geq 2) 
\end{cases} \]

and odd parity-beam-BipoSH,

\[ B_{i_1 l_2}^{L M^{(-)}} = \begin{cases} 
0 & \text{if } (l_1 = l_2) \\
\delta_{M_0} [b_{l_2}^{1/2} (\hat{z}) \exp(-i2\rho) - b_{l_2}^{1/2} (\hat{z}) \exp(i2\rho)] \left( \frac{(-l_1-l_2)}{4\pi (2l_1+1)(2l_2+1)} \right) + \\
2\pi \sqrt{\frac{(2l_1+1)}{4\pi}} \sum_{|m_2|>0} C_{l_1-m_2}^{L 0} \frac{\kappa_{k_0}}{|m_2|} \sqrt{\left( \frac{(l_2 + m_2)!}{(l_2 - m_2)!} \right) l_1} & \text{if } (l_1 > l_2 \text{ and } l_2 \geq 2) \\
\delta_{M_0} [b_{l_2}^{1/2} (\hat{z}) \exp(-i2\rho) - b_{l_2}^{1/2} (\hat{z}) \exp(i2\rho)] \left( \frac{(-l_1-l_2)}{4\pi (2l_1+1)(2l_2+1)} \right) & \text{if } (l_1 < l_2 \text{ and } l_2 \geq 2) 
\end{cases} \]

To avoid any confusion, we reiterate that the above results hold for PT-scan approximation and a NC-beam.
function with discrete even-fold azimuthal symmetry. Other residual symmetries in NC-beam can reduce the set of non-zero beam BipoSH further. In particular, if the experimental beam has reflection symmetry, then odd parity beam BipoSH will vanish and only even parity ones will be present. This implies that odd parity beam BipoSH can be used as a measure of breakdown of reflection symmetry in NC-beams.

Appendix B: CMB BipoSH due to non-circular beams

The cosmological signal in the observed temperature fluctuations is convolved with instrumental beam response function. So even if the underlying cosmological temperature fluctuations are statistically isotropic, non-circularity of the beam can give rise to detections in BipoSH coefficients. In this appendix we provide a detailed derivation of the BipoSH coefficient arising from NC beam presented in Sec. IV. The measured temperature fluctuation map $\tilde{T}(\hat{n})$ is a convolution

$$\tilde{T}(\hat{n}) = \int d\Omega \, B(\hat{n}) \tilde{T}(\hat{n}) \Delta T(\hat{n}) \, d\Omega, \quad (B1)$$

of the cosmological signal $\Delta T(\hat{n})$ with the beam response function $B(\hat{n})$, that encodes the sensitivity of the instrument around the pointing direction, $\hat{n}$, and can be expanded in the spherical harmonic (SH) basis

$$\Delta T(\hat{n}) = \sum_{lm} \tilde{a}_{lm} Y_{lm}(\hat{n}). \quad (B2)$$

Similarly, the cosmological signal decomposed in the SH basis, as

$$\Delta T(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n}). \quad (B3)$$

Beam response function can be expanded in the BipoSH basis

$$B(\hat{n}) = \sum_{l_{1,2} L M} B_{l_{1,2} l_{1,2}}^{LM} C_{l_{1} m_{1}, l_{2} l_{2}}^{LM} \times Y_{l_{1}, l_{2}}(\hat{n}) Y_{l_{1}, l_{2}}(\hat{n}). \quad (B4)$$

Using orthogonality of spherical harmonics,

$$\int d\Omega \, Y_{lm}(\hat{n}) Y_{lm'}(\hat{n}) = (-1)^{m'} \delta_{l l'} \delta_{m m'}, \quad (B5)$$

we obtain

$$\tilde{\Delta T}(\hat{n}) = \sum_{l_{1} m_{1}} \sum_{l_{2} m_{2}} \sum_{LM} (-1)^{m} a_{lm} B_{l_{1} l_{1}}^{LM} C_{l_{1} m_{1}, l_{2} m_{2}}^{LM} Y_{l_{1} m_{1}}(\hat{n}). \quad (B6)$$

Using the above expansion together with Eq. (B2), we obtain

$$\tilde{a}_{l_{1} m_{1}} = \sum_{LM} \sum_{lm} (-1)^{m} a_{lm} B_{l_{1} l_{1}}^{LM} C_{l_{1} m_{1}, l_{2} m_{2}}^{LM}. \quad (B7)$$

and the harmonic space covariance as

$$\langle \tilde{a}_{l_{1} m_{1}} \tilde{a}_{l_{2} m_{2}} \rangle = \sum_{LM} \sum_{l_{1} m_{1}} \sum_{l_{2} m_{2}} (-1)^{m+m'} \langle a_{lm} a_{l'm'} \rangle \times B_{l_{1} l_{1}}^{LM} B_{l_{2} l_{2}}^{LM} C_{l_{1} m_{1}, l_{2} m_{2}}^{LM} + \sum_{L'M'} B_{l_{1} l_{1}}^{LM} B_{l_{2} l_{2}}^{L'M'} C_{l_{1} m_{1}, l_{2} m_{2}}^{LM} C_{l_{1} m_{1}, l_{2} m_{2}}^{L'M' \prime} \quad (B8)$$

Assuming the cosmological signal to be statistically isotropic,

$$\langle a_{lm} a_{l'm'} \rangle = (-1)^{m} \delta_{l l'} \delta_{m m'}. \quad (B9)$$

and substituting in Eq. (B8), we obtain the SH-space covariance of the observed map as

$$\langle \tilde{a}_{l_{1} m_{1}} \tilde{a}_{l_{2} m_{2}} \rangle = \sum_{LM} \sum_{L'M'} \sum_{l_{1} m_{1}} \sum_{l_{2} m_{2}} (-1)^{m} C_{l_{11} l_{11}}^{LM} C_{l_{22} l_{22}}^{L'M'} \times \sum_{l_{1} m_{1}} \sum_{l_{2} m_{2}} (-1)^{m} C_{l_{11} m_{11}}^{LM} C_{l_{12} m_{12}}^{L'M'} \times C_{l_{22} m_{22}}^{L'M' \prime} \times C_{l_{12} m_{22}}^{LM} \times \times C_{l_{22} m_{22}}^{L'M'.} \quad (B10)$$

As given in Eq. (6), CMB BipoSH coefficients are related to the SH space covariance matrix in Eq. (B8), leading to

$$\Delta^{L_{1} M_{1}}_{l_{1}, l_{2}} = \sum_{L L' M M'} C_{l_{1} l_{1}}^{LM} C_{l_{2} l_{2}}^{L'M'} \times \sum_{l_{1} m_{1}} \sum_{l_{2} m_{2}} (-1)^{m} C_{l_{11} m_{11}}^{LM} C_{l_{12} m_{12}}^{L'M'} \times C_{l_{22} m_{22}}^{L'M' \prime} \times C_{l_{12} m_{22}}^{LM}. \quad (B11)$$

The sum over product of three Clebsch-Gordan coefficients can be written compactly in terms of a 6-j symbol, as

$$\sum_{\alpha \beta \delta} (-1)^{a-c} C_{abc}^{\alpha \beta \delta} C_{d e f}^{\alpha \beta \delta} C_{g h i}^{\alpha \beta \delta} = K_{1} \prod_{\alpha \beta \delta} \left\{ a \ b \ c \ e \ f \ d \right\}, \quad (B12)$$

where $K_{1} = (-1)^{b+c+d+f}$ and $\prod_{\alpha \beta \delta} = \sqrt{(2L+1)(2L' + 1)}$. In a PT-scan approximation, $M = 0, M' = 0, M_{1} = 0$. Hence, we obtain the expression in Eq. (32) for CMB BipoSH coefficient from NC-beam with PT-scan approximation

$$\Delta^{L_{1} M_{1}}_{l_{1}, l_{2}} = \delta_{M_{1}, 0} \sum_{L L'} C_{l_{1} L_{1}}^{L 0} C_{l_{2} L_{2}}^{L' 0} \left\{ 1 \ l_{1} l_{2} L \right\} \times \sqrt{(2L+1)(2L'+1)} C_{L_{1} L_{2} 0}^{L_{1} L_{2} 0} \left\{ l_{1} \ l_{2} \ L \right\}. \quad (B13)$$

The Clebsch-Gordan coefficient $C_{L_{1} L_{2} 0}^{L_{1} L_{2} 0}$ is zero when the sum $L + L' + L_{1}$ is odd valued, hence, ensures the condition that the summation in the above expression is limited to $L + L' + L_{1}$ being even-valued. When the beam function has an even fold azimuthal symmetry and reflection symmetry beam-BipoSH coefficients are restricted to even parity and follows $l_{1} + l_{2} = even$, then $L$ and $L'$ are restricted to even multipole values. Thereafter, due to the presence of $C_{L_{1} L_{2} 0}^{L_{1} L_{2} 0}$, $L_{1}$ takes up even multipole values.

Appendix C: Effective averaging of beam due to multiple hits with varying orientations

Here we have presented an analytic technique to cover NC-beam effect incorporating the multiple hits at any
pixel \( \hat{n} \) by the NC-beam with fixed shape but at varying orientations, \( \rho_j(\hat{n}) \). This information can be obtained from the instrument design description and the scan-strategy of the experiment over the duration of the data acquisition.

The observed temperature anisotropy for a single hit by the beam at a direction \( \hat{n} \equiv (\theta, \phi) \) with orientation, \( \rho_j(\hat{n}) \) is given by

\[
T(\gamma) = \int d\Omega B(\hat{n}, \hat{n}'; \rho(\hat{n})) T(\hat{n}')
\]

\[
= \int d\Omega B(\hat{n}, \hat{n}'; \rho(\hat{n})) \left( \sum_{l, m} b_{lm}(\hat{n}, \rho(\hat{n})) Y_{lm}(\hat{n}') \right) T(\hat{n}')
\]

\[
= \sum_{l, m} b_{lm}(\hat{n}, \rho(\hat{n})) \int T(\hat{n}') Y_{lm}(\hat{n}') d\Omega
\]

\[
= \sum_{l, m} a_{lm} b_{lm}(\hat{n}, \rho(\hat{n}))
\]

(C1)

If the \( i^{th} \) pixel gets scanned \( n_i \) number of times an approximate observed temperature \( T_s(\gamma) \) of that pixel is given by

\[
T_s(\gamma) = \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{l, m} a_{lm} b_{lm}(\hat{n}, \rho_j(\hat{n}))
\]

(C2)

\[
= \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{l, m} a_{lm} \sum_{m'} b_{lm'}(\hat{z}) d_{mm'}^l(\theta) e^{-im\phi} e^{-im'\rho_j}
\]

Of course, if a pixel get hit \( n_i \) times from different orientations then after following the map making procedure for the differential assembly, its temperature may not be average of all the hits. But the average can provide a very good estimate of the scanned temperature when there is no noise involved and also simplifies the calculation. Here \( \hat{z} \) is a fixed direction in the sky.

Not if we follow the parallel transport with \( \rho = 0 \), then the temperature of any pixel will be given by

\[
T_{p\ell}(\gamma) = \sum_{l, m} a_{lm} b_{lm}^{effective}(\hat{n}, 0)
\]

(C3)

\[
= \sum_{l, m} a_{lm} \sum_{m'} b_{lm'}^{effective}(\hat{z}) d_{mm'}^l(\theta) e^{-im\phi}
\]

(C4)

Now we want to choose the effective beam in such a way that the error in the temperature calculation be minimum. In other words we want to minimize the following factor.

\[
\chi^2 = \int d\Omega \left( \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{l, m} b_{lm'}(\hat{z}) d_{mm'}^l(\theta) e^{-im\phi} e^{-im'\rho_j} - \sum_{l, m} a_{lm} \sum_{m'} b_{lm'}^{effective}(\hat{z}) d_{mm'}^l(\theta) e^{-im\phi} \right)^2
\]

(C5)

\[
= \sum_{l, m} a_{lm}^2 \int d\Omega \left( \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{m'} b_{lm'}(\hat{z}) d_{mm'}^l(\theta) e^{-im'\rho_j} - \sum_{m'} b_{lm'}^{effective}(\hat{z}) d_{mm'}^l(\theta) \right)^2 e^{-2im\phi}
\]

(C6)

As we want to minimize this error by choosing a effective \( b_{lm'}^{effective} \), we have to take the derivative of \( \chi^2 \) with respect to effective \( b_{lm'}^{effective} \) and make it 0. This gives

\[
\frac{\partial \chi^2}{\partial b_{lm'}^{effective}} = \sum_{l, m} a_{lm}^2 e^{-2im\phi} \int d\Omega \left( \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{m'} b_{lm'}(\hat{z}) d_{mm'}^l(\theta) e^{-im'\rho_j} - \sum_{m'} b_{lm'}^{effective}(\hat{z}) d_{mm'}^l(\theta) \right) d_{mm'}^l(\theta)
\]

(C7)

and hence

\[
2 \sum_{l, m} a_{lm}^2 e^{-2im\phi} \int d\Omega \left( \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{m'} b_{lm'}(\hat{z}) d_{mm'}^l(\theta) e^{-im'\rho_j} - \sum_{m'} b_{lm'}^{effective}(\hat{z}) d_{mm'}^l(\theta) \right) d_{mm'}^l(\theta) = 0
\]

(C8)

As \( a_{lm} \) are random quantities, to make the sum 0, each and every coefficients which are with \( a_{lm}^2 \) should be 0. This means for all the \( l \) and \( m \)'s we must have

\[
\int d\Omega \left( \frac{1}{n_i} \sum_{j=1}^{n_i} \sum_{m'} b_{lm'}(\hat{z}) d_{mm'}^l(\theta) e^{-im'\rho_j} - \sum_{m'} b_{lm'}^{effective}(\hat{z}) d_{mm'}^l(\theta) \right) d_{mm'}^l(\theta) = 0
\]

(C9)

The integration over \( d\Omega \) can be replaced by the summation over all the pixels, i.e.
\[ \sum_{N_p} \left( \frac{1}{n_i} \sum_i \sum_m b_{lm'}(\hat{z}) d_{m}^{l\mu}(\theta) e^{-im'\rho_i} - \sum_{m'} f_{lm'}^{\mu}(\hat{z}) d_{m}^{l\mu}(\theta) \right) d_{m}^{l\mu}(\theta) = 0 \] (C10)

\[ \Rightarrow \sum_{N_p} \frac{1}{n_i} \sum_i \sum_m b_{lm'}(\hat{z}) d_{m}^{l\mu}(\theta) d_{m'}^{l\mu}(\theta) e^{-im'\rho_i} = \sum_{N_p} \sum_m b_{lm'}^{\mu}(\hat{z}) d_{m}^{l\mu}(\theta) d_{m}^{l\mu}(\theta) \] (C11)

here the summation over \( N_p \) runs over all the pixels. Few straightforward algebraic manipulation will give us

\[ b_{l2}^{\text{eff}} = \frac{2l+1}{N_p} \sum_{N_p} \frac{1}{n_i} \sum_i \sum_m b_{lm'}(\hat{z}) d_{m}^{l2}(\theta) d_{m'}^{l2}(\theta) e^{-im'\rho_i}. \] (C12)

Now if in the real beam if we consider that the \( m' = 2 \) part is only the important part then we can get

\[ b_{l2}^{\text{eff}} = \frac{2l+1}{N_p} b_{l2}(\hat{z}) \sum_{N_p} \frac{1}{n_i} \sum_i \sum_m d_{m}^{l2}(\theta) d_{m'}^{l2}(\theta) e^{-im'\rho_i}. \] (C13)

If we do the calculation using the WMAP scan strategy, it can be seen that this factor with \( b_{l2}(\hat{z}) \) is almost constant over \( l \) with a value \( \sim 0.45 \) (refer to Fig.(4)). From the plots we can also see that the constant multiplication which is coming is \( \sim 0.45 \) (refer to Fig.(8)), which is very close to the factor calculated here. So we can say that doing only a time efficient parallel transport scan also it is possible to calculate an approximate effect the beam induced BipoSH.

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FIG. 7. BipoSH spectra, $A_{ll}^{20}$ (Left) and $A_{l-2l}^{20}$ (Right) obtained for the raw beam maps A side of V2 channel (Top) and A side of W1 channel (Bottom) of the WMAP experiment. Analytically evaluated BipoSH spectra (blue) overlaid on the average BipoSH spectra (red) with error bars obtained from 100 simulations of statistically isotropic CMB sky convolved W1 and V2 channel of WMAP. While the BipoSH spectra from raw beam-SH recover interesting qualitative features of the measure spectra, the amplitudes need to be corrected for the circularizing effect of averaging raw beam maps with multiple hits with varying orientation at any pixel.
FIG. 8. Comparison of BipoSH spectra generated by effective NC-beams at the V and W bands to the published WMAP-7 BipoSH spectra measurements (approximately digitized from the Fig 16 of Ref. [24]). As described in Appendix C, an effective NC-beam can be constructed from the raw beam-SH, $b_{lm}$ given by $b_{lm}^{\text{eff}} = \sum_{l'} F_{ll'} b_{lm}$. The measured BipoSH spectra, $A_{l}^{20}$ and $A_{l-2}^{20}$, are sourced at the leading order by the $m = 2$ term. Hence we estimate the parametrized fits to the function $f_l \equiv F_{l2}$ using the W and V band BipoSH measurements. We present the best fit results for a constant $f_l = \alpha$ and a linear function $f_l = (\alpha + \beta \cdot l)$. The respective values of the reduced $\chi^2$ of the fit and the best-fit parameters are provided in the figure. The constant, $f_l$, provides a fit that partially accounts for the BipoSH spectra detections in terms of WMAP NC-beam effect. However, remarkably an effective $b_{l2}^{\text{eff}}$ from the linear form of $f_l$ can satisfactorily explain the non-zero BipoSH spectra detected in WMAP-7 in terms of the corresponding V and W band effective NC-beams.

FIG. 9. To illustrate that the SI violation detected by WMAP can be satisfactorily explained by beam asymmetry, we do full simulations of map making for WMAP W1 beam with real scan and real beam. $A_{l}^{20}$ (left) and $A_{l-2}^{20}$ (right) of the simulated maps (red) are consistent with WMAP measurement (grey) and our reanalysis of WMAP-7 (blue). The difference between the last two occurs due to different noise weighting schemes. Results with full simulations with all the detectors will appear in a future publication.