The $1/N_c$ expansion in hadron effective field theory

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We study the $N_c$ scalings of pion-nucleon and nucleon-nucleon scatterings in hadron effective field theory. By assuming Witten’s counting rules are applied to matrix elements or scattering amplitudes which use the relativistic normalization for the nucleons, we find that the nucleon axial coupling $g_A$ is of order $N_c^0$, and a consistent large $N_c$ counting can be established for the pion-nucleon and nucleon-nucleon scatterings. We also justify the nonperturbative treatment of the low energy nucleon-nucleon interaction with the large $N_c$ analysis and find that the deuteron binding energy is of order $1/N_c$.

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In the seminal paper Ref. [1], ’t Hooft showed that QCD has a hidden expansion parameter $1/N_c$. With the $1/N_c$ expansion the QCD coupling constant $g$ is of order $1/\sqrt{N_c}$, and gluons are denoted by double lines. By counting the number of interaction vertexes and closed color loops, one can figure out the $N_c$ order of a specific Feynman diagram. It is then found that in the large $N_c$ limit, the leading contribution comes from planar diagrams with minimal quark loops. The idea to expand QCD in $1/N_c$ is very attractive, as it explains hadron phenomenology successfully, for instance the OZI suppression rule. The extension of the $1/N_c$ expansion to baryons was first carried out by Witten [2]. The $1/N_c$ expansion of baryons, which are bound states of $N_c$ valence quarks and have masses of order $N_c$, is more complicated than that of mesons. Using quarks and gluons as degrees of freedom, Witten showed that meson-baryon coupling is of order $\sqrt{N_c}$, meson-baryon scattering amplitude is of order $N_c^0$, and baryon-baryon scattering amplitude is of order $N_c$ (these results are called the large $N_c$ counting rules of Witten in this manuscript). It is interesting to study whether a realistic hadron effective field theory using baryons and mesons as degrees of freedom can reproduce the large $N_c$ counting rules of Witten. We will see in the following that this is not a clearly solved problem.

\[ \text{FIG. 1: Pion-nucleon scattering diagrams in leading chiral expansion.} \]

First of all, we consider the pion-nucleon scattering. The leading pion-nucleon scattering diagrams in chiral expansion are shown in Fig. 1. In chiral theory, the pion-nucleon vertex is proportional to the factor $g_A f_\pi$, where $f_\pi$ is the pion decay constant with order $\sqrt{N_c}$, and $g_A$ is the nucleon axial coupling constant which is assumed to be of order $N_c$ in both the nonrelativistic quark model [3] and the skyrme model [4]. The nucleon propagator is of order $N_c^0$ [5]. Therefore, both amplitudes of Fig. 1(a) and (b) are proportional to the factor $(g_A f_\pi)^2$ and are of order $N_c$, and Fig. 1(c) is proportional to $1/f_\pi^2$ and of order $1/N_c$. Clearly these results contradict the large $N_c$ counting rules of Witten. An solution to this puzzle has been proposed in Ref. [6–8], and we give a short review here for the convenience of further discussions. The axial vector

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current matrix element in the nucleon is of order $N_c$ and can be written as

$$\langle N|\bar{\psi}\gamma^i\gamma_5\tau^a\psi|N\rangle = N_c g\langle N|X^{ia}|N\rangle,$$

where the $N_c$ dependence has been explicitly factored out, so $\langle N|X^{ia}|N\rangle$ and $g$ are of order one. The sum of the scattering amplitudes for the pole diagrams of $\pi^a(q) + N(P) \rightarrow \pi^b(q') + N(P')$ reads

$$-iq^i q'^j N_c^2 g^2 \left[ \frac{1}{q^0} X^{jb} X^{ia} - \frac{1}{q'^0} X^{ia} X^{jb} \right],$$

where the amplitude is written in the form of an operator acting on nucleon states, and $q^0 = q'^0$ as both initial and final nucleons are on-shell. One can see explicitly that the above amplitude is of order $N_c$ and violates the unitarity. The solution to this puzzle is the commutation relation in the large $N_c$ limit

$$[X^{ia}, X^{jb}] = 0,$$

which is usually called the large $N_c$ consistency condition. The solution of the large $N_c$ consistency condition requires to consider all the possible baryon states degenerate with the nucleon in the intermediate states of the scattering. For instance, besides the nucleon, $\Delta$ should also be included as an intermediate state in the pion-nucleon scattering, and the coupling $g_{\pi N \Delta}$ can be determined in terms of $g_{\pi NN}$ using the large $N_c$ consistency condition. The large $N_c$ consistency condition can be derived from the spin-flavor algebra.

Consider the two-flavor case, the $SU(4)$ generators $J^i, I^a$ and $G^{ia}$ satisfy the spin-flavor algebra

$$[J^i, J^j] = i\epsilon^{ijk} J^k, \quad [I^a, I^b] = i\epsilon^{abc} I^c, \quad [I^a, G^{ib}] = i\epsilon^{abc} G^{ic},$$

$$[J^i, G^{ia}] = i\epsilon^{ijk} G^{ka}, \quad [J^i, I^a] = 0, \quad [G^{ia}, G^{jb}] = \frac{i}{4} \delta^{ab} \epsilon^{ijk} J^k + \frac{i}{4} \delta^{ij} \epsilon^{abc} I^c.$$ (4)

Rescaling the generator $G^{ia}$ by a factor of $1/N_c$ and taking the large $N_c$ limit,

$$X^{ia} = \lim_{N_c \to \infty} \frac{G^{ia}}{N_c},$$ (5)

we can find that the commutation relation for $[G^{ia}, G^{jb}]$ turns into Eq. (3) in the large $N_c$ limit, as the order of baryon matrix elements of $J$ and $I$ are at most of $N_c$. A careful study find that the commutator $[X^{ia}, X^{jb}]$ is of order $1/N_c^2$ [9], hence Eq. (2) is of order $1/N_c$, which is the same as the amplitude of Fig. 1(c). Since Witten’s counting rules suggest that the leading contribution should come from order $N_c^0$ diagrams in the $1/N_c$ expansion, one may wonder why the pion-nucleon scattering amplitude is dominated by order $1/N_c$ diagrams instead of order $N_c^0$ diagrams.

![Diagram](image)

FIG. 2: A typical diagram of order $N_c$ which contributes to the nucleon-nucleon interaction in large $N_c$ QCD.

Secondly, there is difficulty in matching the $N_c$ counting of the nucleon-nucleon potential calculated at the quark-gluon level to that calculated at the hadronic level. The dominant nucleon-nucleon interaction is generically of order $N_c$ with the analysis at the quark-gluon level [2]. This can be understood from the one-
gluon exchange diagrams in Fig. 2, where each gluon coupling has a factor $1/\sqrt{N_c}$, and there are $N_c^2$ ways to choose a pair of quarks. In studying the $N_c$ counting of the nucleon-nucleon potential, Ref. [10, 11] consider the kinematic region $p \sim \sqrt{N_c}$ for the nucleon and assume that quark-line-connected diagrams such as Fig. 2 are of order $N_c$, and therefore the nucleon-nucleon potential is of order $N_c^0$. It should be noted that the kinematic region is different from that was considered in Witten’s paper, where $p \sim \sqrt{N_c}$. The reason is that, if $p \sim N_c$ which is the same order as the nucleon mass, the nucleon is a relativistic particle, and this kinematic region is beyond the scope of the low energy hadron effective field theory [12]. At the hadronic level, it seems straightforward to see that the one-meson exchange potential is of order $N_c$, for example the one-pion exchange potential is proportional to the factor $(\frac{g_A}{f_\pi})^2$ and of order $N_c$. At the two-meson exchange level, diagrams (for instance, the crossed-box diagram) with four nucleon-meson vertexes are of order $N_c^2$ and will contribute to the potential, but Ref. [12] shows that there is a cancelation of the retardation effect of the box graph against the contribution of the crossed-box diagram, therefore the dominant nucleon-nucleon potential remains to be of order $N_c$. Nevertheless, it is difficult to show that whether such cancelations will happen in all multi-meson exchange levels. Actually, it is found that at the three- and higher-meson exchange level the potential derived from the hadron theory can be larger than that of the quark-gluon theory [13]. This problem is called the large $N_c$ nuclear potential puzzle.

Two possible resolutions to the large $N_c$ nuclear potential puzzle are suggested in Ref. [13]. One possibility is that necessary cancelations might happen if the hadronic-level calculation is reorganized in some other way. A tentative study on this direction is done in Ref. [14], where the hadronic-level calculation is reorganized in the way that the potential is energy-independent. The other possibility is that the $N_c$ scaling rule of the nucleon-nucleon potential proposed in Ref. [10, 11] may be invalid. This can be plausible by noting that some nuclear phenomena may be difficult to understand if the nucleon potential is of order $N_c$, for example, the deuteron binding energy is a small number in reality, i.e., $B = 2.2$ MeV. Although it is generally believed that such small binding energy results from delicate cancellation which occurs at $N_c = 3$ [13, 10], it will still be interesting to find an alternative solution which does not rely on the fact that $N_c = 3$ in the real world. In addition, since the nucleon kinetic energy is order $N_c^{-1}$, if the potential is order $N_c$ nucleon matter forms a crystal in the large $N_c$ limit. However, nuclear matter appears to be in liquid state rather than in crystal state in reality. Generally, nuclear phenomena seem to prefer a small potential value, and this observation motivates Ref. [17, 18] to proposed a refined quark model which gives $g_A \sim N_c^0$, thus the one-pion exchange potential is of order $N_c^{-1}$.

From the above discussions, we can see that it is not clear how to reproduce the large $N_c$ counting rules of Witten consistently in a realistic hadron effective field theory. In this paper we try to propose a simple way to resolve the above difficulties. We find that we can have a consistent large $N_c$ counting in hadron effective field theory if we assume Witten’s counting rules are applied to matrix elements or scattering amplitudes which use the relativistic normalization for the nucleons. Our work is organized as following: we first discuss the difference of $N_c$ scalings between the relativistic and nonrelativistic normalizations. With the relativistic normalization, we then discuss the $N_c$ scalings of pion-nucleon and nucleon-nucleon scatterings. We finally extend our study to the meson-meson scatterings and find that loosely-bound meson-meson molecular states may not exist in the large $N_c$ limit.

The leading non-relativistic chiral Lagrangian for the pion-nucleon coupling reads

$$\mathcal{L}_{\pi N} = \frac{g_A}{2f_\pi} \bar{\psi} \gamma^i \hat{\sigma} \cdot \hat{\epsilon} \pi \psi,$$

(6)

where $\psi$ is the nucleon doublet and $\pi = \tau^i \pi^i$. If $g_A$ is of order $N_c$, the pion-nucleon coupling is proportional to $\frac{g_A}{f_\pi}$ and of order $\sqrt{N_c}$, which is just the result of Witten. But if $g_A \sim N_c^0$ as suggested in Ref. [17, 18], the pion-nucleon coupling is of order $N_c^{-1/2}$, which seems to contradict the large $N_c$ counting rules of Witten. One should note that in the above $N_c$ counting Feynman rules for the external nucleon lines are assumed to be independent of $N_c$, or explicitly one use the nonrelativistic normalization for the nucleon

$$\langle p(\vec{k}, s_2) | p(\vec{p}, s_1) \rangle = \delta^3(\vec{k} - \vec{p}) \delta_{s_1, s_2}. $$

(7)

However from the Dirac theory, we know that the Feynman rule for the external nucleon line is $u_s(p)$, which reduces to $\sqrt{2m_N \chi_s}(\chi_s$ is the two-component spinor) in the nonrelativistic limit and obviously depends on
$N_c$. Thus the relativistic normalization for the nucleon reads

$$\langle p(\vec{k}, s_2)|p(\vec{p}, s_1)\rangle = 2m_N\delta^3(\vec{k} - \vec{p})\delta_{s_1, s_2}, \tag{8}$$

where we have taken the nonrelativistic limit for the Dirac spinor $u_s(p)$ to simplify the $N_c$ counting in the low energy effective field theory. Scattering amplitudes which use Eq. (8) as the normalization condition can be understood as the nonrelativistic reduction of the relativistic form. In this work, we will use the relativistic normalization for the scattering amplitudes, thus the relation between our defined scattering amplitude $M(p_1, p_2 \rightarrow p_f)$ and the $S$-matrix reads

$$S = 1 + iT, \tag{9}$$

where

$$<p_f|iT|p_1, p_2> = i(2\pi)^4\delta^{(4)}(p_1 + p_2 - \sum p_f)M(p_1, p_2 \rightarrow p_f). \tag{10}$$

To obtain the scattering amplitudes with the nonrelativistic normalization $M_{NR}$, one takes the limit $|\vec{p}| \rightarrow 0$ and drops factors of $\sqrt{2m_N}$ which come from external nucleon spinor function $u_s(p)$. For example, for the nonrelativistic nucleon-nucleon elastic scattering, we can have $M_{NR} = M/(2m_N)^2$. One should note that we only discuss the nonrelativistic scattering in this work as the relativistic normalization reduce to the form in Eq. (8). We will find that Witten’s large $N_c$ counting rules can be consistent, if one assume that these rules are applied to matrix elements or scattering amplitudes which use the relativistic normalization for the nucleon.

We now return back to the discussion of the pion-nucleon coupling. With the normalization in Eq. (8), there is a factor of $\sqrt{2m_N}$ for each external nucleon line, and the pion-nucleon coupling is proportional to $m_N\frac{A_\pi}{f_\pi}$. Our matching scheme is that we assume Witten’s counting rules are applied to matrix elements or scattering amplitudes which use the relativistic normalization for the nucleon, we then find that $g_A \sim N_c^0$ as the pion-nucleon coupling is order $\sqrt{N_c}$ from Witten’s rules. Thus we have shown that if $g_A \sim N_c^0$ as suggested in Ref. [17, 18], the $N_c$ counting of the pion-nucleon coupling can be consistent with Witten’s result. In the following we will count $g_A$ as order $N_c^0$, since it is the consequence of our assumption.

Now we come to discuss the $N_c$ scaling of the pion-nucleon scattering. The scattering amplitude for the pion-nucleon scattering given in Eq. (2) is written with the nonrelativistic normalization. But noting that, while factors of $\sqrt{2m_N}$ which come from external nucleon lines are dropped in Eq. (2), such factor is kept for the baryon propagator. This can be obviously since the baryon propagator should absorb a factor of $2m_N$, which is dimension one and of order $A_\pi^2/m_B^2$, all of which are of order $N_c^0$ and consistent with the large $N_c$ counting rules of Witten. Actually what we have illustrated in the above is that if the assumption that Witten’s rules are applied to matrix elements or scattering amplitudes which use the relativistic normalization for the nucleon is adopted for the pion-nucleon coupling, the $N_c$ scaling of the pion-nucleon scattering will be consistent with this assumption.

To have a consistent $N_c$ counting with the nonrelativistic normalization, one can redefine the nucleon field by dividing a factor of $\sqrt{2m_N}$, then factors of $\sqrt{2m_N}$ are not needed in the Feynman rules for the external nucleon lines. Meanwhile, the nucleon propagator should be $\frac{i}{2m_N\vec{q}^2}$ instead of $\frac{i}{\vec{q}^2}$ as the new defined nucleon field has the dimension one, and $g_A$ should absorb a factor of $2M_N$ which is dimension one and of order $N_c$. With this convention the scattering amplitudes for Fig. (3a,b) are proportional to $(\frac{g_A}{f_\pi})^2\frac{1}{2m_N\vec{q}^2}$, which is also of order $N_c^0$ as $g_A$ is of the order $N_c$. Therefore, we can see that physical results do not depend on the normalization conventions as long as they are used consistently. The advantage of using the relativistic
normalization is that \( g_A \) is a dimensionless constant as usually defined in the text book.

It is worth to mention that, although with the relativistic normalization \( g_A \) is of order \( N_c^0 \), the commutation relation in Eq. (3) still holds in the large \( N_c \) limit. This can be shown by rewriting the axial current matrix element in Eq. (1) with the normalization in Eq. (8)

\[
\langle N | \bar{\psi} \gamma^i \gamma^5 \tau^a \psi | N \rangle = 2m_N g \langle N | X^{ia} | N \rangle = N_c \frac{2m_N}{N_c} g \langle N | X^{ia} | N \rangle,
\]

where \( \langle N | X^{ia} | N \rangle \) and \( g \) are of order one, and the product \( g \langle N | X^{ia} | N \rangle \) is different from that in Eq. (1) by a \( N_c \) independent factor \( \frac{2m_N}{N_c} \). Since the matrix element of \( X^{ia} \) defined in Eq. (12) is different from that in Eq. (1) by a \( N_c \) independent factor, operators \( X^{ia} \) defined in Eq. (12) still satisfy the commutation relation in Eq. (3), i.e., the large \( N_c \) consistency condition. The commutation relation Eq. (3) arises because the axial vector current matrix element grows with \( N_c \). With our normalization, \( g_A \) is of order \( N_c^0 \), but the axial vector current matrix element is still of order \( N_c \), thus the commutation relation Eq. (3) still holds. Taking this contracted spin-flavor symmetry into account, we should also include \( \Delta \) particle in the intermediate state, then the contributions from pole diagrams Fig. (a) and (b) vanish in the large \( N_c \) limit.

![FIG. 3: Feynman diagrams generated by the contact interaction.](image)

We then come to the nucleon-nucleon scatterings. As in the above, we assume Witten’s counting rules are applied to the nucleon-nucleon scattering amplitudes which use the relativistic normalization. In this way, nucleon-nucleon scattering amplitudes with the relativistic normalization are of order \( N_c \). The order \( N_c \) scattering amplitude leads to a potential of order \( N_c^{-1} \), as factors of \( \sqrt{2m_N} \) which come from the external nucleon lines should be dropped in the scattering amplitude to obtain the non-relativistic potential. Therefore our resolution to the nuclear potential puzzle corresponds to the second possibility suggested in Ref. [13], i.e., the assumption that the dominant nucleon-nucleon potential is of order \( N_c \) is somewhat heuristic and may be invalid. We will take the pion-exchange as an example to show explicitly how the large \( N_c \) nuclear potential puzzle can be resolved. The nucleon-nucleon scattering amplitude for the one-pion exchange diagram is proportional to the factor \((m_N g_A)^2\), which is of order \( N_c \) and again consistent with our assumption. The amplitude of two-pion exchange crossed-box diagram is proportional to \( m_N^2 g_A^2 \), which is of order \( N_c^0 \) and thus is suppressed by \( 1/N_c \). It is obvious that more-pion exchange diagrams which are two-nucleon irreducible, will be suppressed by more powers of \( 1/N_c \). Thus one can see that the nucleon-nucleon potential is at most of order \( N_c^{-1} \), and this large \( N_c \) counting rule will not be violated at the multi-pion exchange level. It is straightforward to see that similar conclusion can be obtained for other meson-exchange diagrams, in particularly the sigma-exchange potential is also of order \( N_c^{-1} \).

So far, we have shown that a consistent large \( N_c \) counting in hadron effective field theory can be established, if we assume Witten’s counting rules are applied to matrix elements or scattering amplitudes which use the relativistic normalization for the nucleons. Nevertheless there are still two points in nucleon-nucleon scatterings need to be further investigated. The first point is that one may wonder whether the order \( N_c \) scattering amplitude violates the unitarity. We now show that this does not happen in the nucleon-nucleon scattering. To be specific, let’s consider the single channel \( S \)-wave nucleon-nucleon scattering. The \( S \)-matrix for the \( S \)-wave nucleon-nucleon scattering can be written as \( S = 1 + \frac{i}{2\pi m_N} \mathcal{M} \), where \( p \) is the nucleon momentum and \( \mathcal{M} \) is the scattering amplitude which use the relativistic normalization. Unitary condition, i.e. \( SS^\dagger = 1 \), requires that

\[
\text{Im} \mathcal{M} = \frac{p}{16\pi m_N} |\mathcal{M}|^2.
\]

(13)
We consider the kinematic region $p \sim N_c^0$ as in Ref. [11] and denote the $N_c$ scaling of the amplitude as $\text{Re} \mathcal{M} \sim N_c^{n_1}, \text{Im} \mathcal{M} \sim N_c^{n_2}$. The unitary condition in the large $N_c$ limit can then be written as

$$n_2 = 2 \max(n_1, n_2) - 1.$$  \hspace{1cm} (14)

The solution to the above condition reads

$$\begin{align*}
\text{If } n_2 &\geq n_1, \text{ then } n_1 \leq n_2 = 1, \quad \mathcal{M} \sim \mathcal{O}(N_c); \\
\text{If } n_2 &< n_1, \text{ then } n_2 < n_1 < 1, \quad \mathcal{M} < \mathcal{O}(N_c).
\end{align*}$$  \hspace{1cm} (15)

Thus one can see that the unitary condition can be satisfied if the scattering amplitude is of order $N_c$.

The second point is that in hadron effective field theory deuteron corresponds to a bound state pole which emerges from the non-perturbative summation of the nucleon-nucleon scattering amplitudes, hence it is interesting to investigate whether such non-perturbative summation is justified in the large $N_c$ limit. Following, we will study this point in detail with a low energy pionless effective field theory.

Let’s now consider the low energy pionless effective field theory in the $^3S_1$ channel [19–21]. The leading order operator in the chiral expansion for nucleon-nucleon interactions reads

$$\mathcal{L}_{NN} = -C_0 (\bar{\psi} \psi)^2.$$  \hspace{1cm} (16)

Feynman diagrams generated by this interaction vertex are shown in Fig. 3. We can identify the order $N_c$ quark-connected diagrams, such as Fig. 2, as the tree diagram in Fig. 3, and identify the quark-disconnected diagrams, such as Fig. 4, as the one-loop diagram in Fig. 3. The amplitude for the tree diagram in Fig. 3 reads

$$i \mathcal{M}_{\text{tree}} = -i 4m_N^2 C_0,$$  \hspace{1cm} (17)

where the factor $4m_N^2$ comes from four external nucleon lines. $\mathcal{M}_{\text{tree}}$ is of order $N_c$, according to our assumption that Witten’s counting rules are applied to the scattering amplitudes which use the relativistic normalization. We can then find that $C_0$ is of order $N_c^{-1}$. Amplitudes for the loop diagrams can be obtained after treating the loop integral. In the pionless effective field theory, the loop integral in can be done in the
nonrelativistic approximation. Using the minimal subtraction scheme, the loop integral reads

\[ I = \int \frac{d^D \ell}{(2\pi)^D} \frac{i}{E_k - \ell^0 - \ell^2/(2m_N) + i\epsilon}, \]

where \( E_k = p^2/m_N \) is the total kinetic energy of the two-nucleon in the center of mass frame. In the chiral expansion the three-momentum of the nucleon is treated as small scale and has the same order as \( m_\pi \), thus we assume \( p \) to be independent of \( N_c \) as in Ref. [11]. It is worth mentioning that the nucleon propagator used here is order \( N_c \) and different from that in Eq. (2), as the term \( \ell^2/2m_N \) should also be included in the propagator to avoid the infrared divergence in the two-nucleon reducible diagrams [19]. With the result for loop integral, one can obtain the one-loop amplitude

\[ iM_{\text{loop}} = \frac{-p}{16\pi m_N} M_{\text{tree}}^2 = \frac{-1}{\pi} \frac{m_N^3 C_0^2}{p}. \] (19)

We can find that the one-loop amplitude is of order \( N_c \), which is the same as that of the tree diagram. This result is somewhat surprising at the first sight, as quark-disconnected diagram Fig. 4 seems to be of order \( N_c^2 \). Actually to count the \( N_c \) order of Fig. 4 one should note that such a diagram contains nucleon propagators and loop, thus its \( N_c \) order can be the same as that obtained in the hadron effective field theory, i.e., \( O(N_c) \). It is straightforward to count the \( N_c \) order of all the other diagrams in Fig. 3. One can find that all the diagrams in Fig. 3 are of order \( N_c \), hence the amplitude at the leading order in the \( 1/N_c \) expansion should come from the non-perturbative summation of all the diagrams in Fig. 3. The re-summed amplitude reads

\[ iM_{\text{sum}} = -i \frac{4m_N^3 C_0}{1 + i \frac{C_0 m_N}{4\pi} p}. \] (20)

which is obviously of order \( N_c \). The partial wave \( S \)-matrix can then be written as

\[ S = 1 + i \frac{p}{8\pi m_N} M_{\text{sum}} = 1 - \frac{i m_N p}{2\pi} \frac{C_0}{1 + i \frac{C_0 m_N}{4\pi} p}. \] (21)

One can find that, although \( M_{\text{sum}} \) is of order \( N_c \), the \( S \)-matrix is independent of \( N_c \) and satisfies the unitary condition, i.e. \( SS^\dagger = 1 \). Actually it has already been shown in Ref. [21] that all the diagrams in Fig. 3 need to be summed, because they all are at the leading chiral order \( O(p^{-1}) \). Here we have shown that this summation is also justified in the large \( N_c \) expansion. The deuteron corresponds to a bound state pole at \( E_k = -B \) in Eq. (20), where \( B \) is the binding energy,

\[ B = \frac{16\pi^2}{C_0^2 m_N^3}, \] (22)

which is of order \( N_c^{-1} \). Similar analysis has been done in Ref. [22], but the conclusion is different. Because \( C_0 \) is taken to be \( O(N_c) \), \( B \) is found to be \( O(N_c^{-5}) \) in Ref. [22]. However, if \( C_0 \) is order \( N_c \) and \( B \) is order \( N_c^{-5} \), it will then be a puzzle why the binding energy is so small, while the nonrelativistic potential is large. We can also study the \( N_c \) scaling of the deuteron binding energy in the pionfull effective field theory with the method used in Ref. [22]. In the chiral limit, the Schrödinger equation in coordinate space can be simply
written as \[ 22 \]

\[
\left( \frac{\nabla^2}{m_N} - \frac{3g_\pi}{r^3} \right) \Psi = -B \Psi,
\] (23)

where

\[
\alpha_s = \frac{g_A^2}{16\pi f_\pi^2}.
\] (24)

Because the three-momentum of the nucleon and \( g_A \) are of order \( N_c^0 \), both kinetic energy and tensor potential energy in the left hand side of Eq. (23) are of order \( N_c^{-1} \), and the binding energy \( B \) is of order \( N_c^{-1} \) which is the same as that in the pionless effective field theory. This conclusion remains unchanged even if an additional term \( C_0\delta^{(3)}(r) \) is included in the potential. In contrast, by treating \( g_A \) as \( O(N_c) \), Ref. [22] assumes that the coordinate scales as \( N_c^2 \), or equivalently the three-momentum carried by the nucleon scales as \( 1/N_c^2 \), then \( B \) scales as \( N_c^{-5} \). However, as mentioned in Ref. [22], if the three-momentum carried by the nucleon scales as \( 1/N_c^2 \), the effective field theory will be constrained to threshold.

It is interesting to extend the above analysis to discuss the existence of \( S \)-wave meson-meson molecular states. We will find that the \( N_c \) analysis of the deuteron cannot be straightforwardly extended to the meson-meson case as the meson-meson scattering amplitudes and meson masses have different \( N_c \) countings. In effective field theory approach, an \( S \)-wave meson-meson molecular state corresponds to a bound state pole in the elastic meson-meson scattering amplitude coming from the summation of all the diagrams in Fig. 3 [23].

Similar to Eq. (22), the binding energy \( \tilde{B} \) for the meson-meson molecular state reads

\[
\tilde{B} = \frac{2\pi^2}{C_0^2 \mu^3},
\] (25)

where \( C_0 \) is the coefficient of the contact four-meson operator, and \( \mu \) is the reduced mass of the two-meson system. \( C_0 \) is of order \( N_c^{-1} \), as the tree level amplitude for the meson-meson scattering scales as \( N_c^{-1} \). \( \mu \) is of order \( N_c^0 \), as the meson mass is independent of \( N_c \). Therefore, we can find that the binding energy of the \( S \)-wave meson-meson molecular state scales as \( N_c^2 \), and the binding momentum is of order \( N_c \). This indicates that if the meson-meson molecular state exists, it should be deeply bounded. However, mesons with the momenta of order \( N_c \) are extremely-relativistic particles, and their scatterings cannot be treated in the nonrelativistic effective field theory. If the momentum of the meson is taken to be independent of \( N_c \) as that of the meson mass, we will find that the diagrams in Fig. 3 have different \( N_c \) scalings. The tree diagram in Fig. 3 scales as \( N_c^{-1} \), and the \( n \)-loop diagram are suppressed which scales as \( N_c^{-(n+1)} \). These results are different from the \( S \)-wave nucleon-nucleon scattering, due to the fact that the meson mass is of order \( N_c^0 \), while the nucleon mass is of order \( N_c \). Thus for meson-meson scatterings, only the tree level diagram contributes to the leading order amplitude in the \( 1/N_c \) expansion, which is just the conclusion in the standard large \( N_c \) analysis for meson-meson interactions [2], and the summation of all the diagrams in Fig. 3 is unnecessary. We then conclude that, there is no loosely-bound meson-meson molecular state in the large \( N_c \) limit, as the meson-meson interaction is weak, and similar conclusion has also been given in Ref. [24]. Finally, we would like to mention that we only discuss the existence of the meson-meson molecular state in the large \( N_c \) limit, for other configurations such as tetraquark and polyquark states one can refer to Ref. [25] and references therein.

In summary, we have tried to propose a possible solution to overcome the difficulties which are encountered in reproducing the large \( N_c \) counting rules of Witten in hadron effective field theory. We find that a consistent large \( N_c \) counting can be established if we assume Witten’s counting rules are applied to matrix elements or scattering amplitudes which use the relativistic normalization for the nucleons. We also find that at the leading order in the \( 1/N_c \) expansion, the \( S \)-wave nucleon-nucleon scattering should be treated nonperturbatively, and the deuteron binding energy is of order \( 1/N_c \) which is consistent with the nuclear phenomena. In contrast, the \( S \)-wave meson-meson interaction is weak, and loosely-bound meson-meson molecular states may not exist in the large \( N_c \) limit.
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