Polymath’s combinatorial proof of the density Hales–Jewett theorem

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Abstract
This is an exposition of the combinatorial proof of the density Hales–Jewett theorem, due to D. H. J. Polymath in 2012. The theorem says that for given \( \delta > 0 \) and \( k \), for every \( n > n_0 \) every set \( A \subset \{1, 2, \ldots, k\}^n \) with \( |A| \geq \delta k^n \) contains a combinatorial line. It implies Szemerédi’s theorem, which claims that for given \( \delta > 0 \) and \( k \), for every \( n > n_0 \) every set \( A \subset \{1, 2, \ldots, n\} \) with \( |A| \geq \delta n \) contains a \( k \)-term arithmetic progression.

1 Introduction
The purpose of this text is to familiarize the author, and possibly the interested reader, with the recent remarkable elementary proof [20, 19] of Polymath (a group of mathematicians, see Nielsen [18] and Gowers [9] for more information) for the density Hales–Jewett theorem, one of the deepest results in extremal combinatorics/Ramsey theory, which has as an easy corollary the famous theorem of Szemerédi, indeed the multidimensional generalization thereof. The author hopes to use it in his future book on number theory; other similar online available fragments are [13, 15, 16]. We begin with recalling the mentioned theorems and introducing some notation. Further notation, concepts and auxiliary results will be introduced along the way.

We denote \( \mathbb{N} = \{1, 2, \ldots\} \), \( \mathbb{N}_0 = \{0, 1, \ldots\} \) and, for \( n \in \mathbb{N} \), \( [n] = \{1, 2, \ldots, n\} \). For finite sets \( B \neq \emptyset \) and \( A \), we call the ratio of cardinalities \( \frac{|A \cap B|}{|B|} \in [0, 1] \) the density of \( A \) in \( B \) and write \( \mu_B(A) \) for it; when \( B \) is understood from the context, we write just \( \mu(A) \) and speak of density of \( A \). Later we consider more general densities. Densities and the quantities bounding them are denoted by the Greek letters \( \mu, \delta, \varepsilon, \gamma, \nu, \eta, \theta, \beta \) and are real numbers from the interval \( [0, 1] \). A partition of a set \( A \) is an expression of \( A \) as a disjoint union of possibly empty sets. Note that if \( B = \bigcup_{i \in I} B_i \) is a partition and \( \mu_B(A) \geq \delta \), then \( \mu_{B_i}(A) \geq \delta \) for some \( i \). For \( a, d, k \in \mathbb{N} \), the \( k \)-element set

\[ \{a, a + d, a + 2d, \ldots, a + (k-1)d\} \]
is the $k$-term arithmetic progression. The following is the famous Szemerédi’s theorem \[22\].

**Theorem 1** For every $\delta > 0$ and $k \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ such that for every $n > n_0$ every set $A \subset [n]$ with $\mu(A) \geq \delta$ contains a $k$-term arithmetic progression.

Precursor of Szemerédi’s theorem was its color version, the van der Waerden theorem \[26\] asserting that for every $r, k \in \mathbb{N}$, for any $n > n_0$ in any partition $[n] = A_1 \cup A_2 \cup \cdots \cup A_r$ a block $A_i$ contains a $k$-term arithmetic progression. Clearly, Szemerédi’s theorem implies van der Waerden theorem.

For $k, n \in \mathbb{N}$, the set $[k]^n$ consists of all $k^n$ $n$-tuples, called words, $x = (x_1, x_2, \ldots, x_n)$ with $x_i \in [k]$. For $x \in [k+1]^n \setminus [k]^n$ and $i \in [k]$, we denote by $x(i)$ the word obtained from $x$ by replacing each occurrence of $k+1$ by $i$. The $k$-element subset of $[k]^n$ of these words, $L(x) = \{x(i) \mid i \in [k]\}$, is the combinatorial line (determined by $x$). In 1963 Hales and Jewett \[10\] proved that for every $r, k \in \mathbb{N}$, for any $n > n_0$ in any partition $[k]^n = A_1 \cup A_2 \cup \cdots \cup A_r$ a block $A_i$ contains a combinatorial line. The stronger density version of this theorem was achieved by Furstenberg and Katznelson in 1991 \[7\] (they proved the special case $k = 3$ earlier in \[6\]) by ergodic methods, developed by Furstenberg \[5\] in his proof of Szemerédi’s theorem. Thus, the density Hales–Jewett theorem asserts the following.

**Theorem 2** For every $\delta > 0$ and $k \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ such that for every $n > n_0$ every set $A \subset [k]^n$ with $\mu(A) \geq \delta$ contains a combinatorial line.

We shall prove Theorem 2 following Polymath’s proof in \[20\]. Theorem 2 implies Theorem 1 with the same $k$, by means of the bijection

$$f : [k]^n \to [k^n], \quad f(x) = f((x_1, x_2, \ldots, x_n)) = 1 + \sum_{i=1}^{n} (x_i - 1)k^{i-1}$$

which sends combinatorial lines to $k$-term arithmetic progressions and, being bijection, preserves densities; for the color versions of the theorems the simpler mapping $x \mapsto x_1 + x_2 + \cdots + x_n$ suffices for the reduction.

**Multidimensional Szemerédi’s theorem** claims that for every $\delta > 0$, $r \in \mathbb{N}$ and finite set $H \subset \mathbb{N}^r$, there exists an $n_0 \in \mathbb{N}$ such that for every $n > n_0$ every set $A \subset [n]^r$ with $\mu(A) \geq \delta$ contains a copy of $H$ of the form $a + dH$, $a \in \mathbb{N}^r$, $d \in \mathbb{N}$. The particular case with $r = 2$ and $H = \{(1, 1), (1, 2), (2, 1)\}$ is the corner theorem which was derived by Ajtai and Szemerédi \[1\] from Szemerédi’s theorem. As explained in \[20\] and \[9\], the proof of Theorem 2 in \[20\] is inspired by and modelled after the increment density argument in \[1\].

2
2 The proof of Theorem 2

The combinatorial subspace $S$ of $[k]^n$ with dimension $d$, $d \leq n$, determined by the word $x \in [k+d]^n$ such that each letter $k+1, k+2, \ldots, k+d$ appears in $x$ at least once, is the $k^d$-element subset of $[k]^n$

$$S = S(x) = \{x(y) \mid y \in [k]^d\}$$

where $x(y)$ is the word obtained from $x$ by replacing each occurrence of $k+i$ by $y_i$, $i = 1, 2, \ldots, d$. In other words, $S \subset [k]^n$ is a $d$-dimensional combinatorial subspace of $[k]^n$ if and only if there exist a word $z \in [k]^n$ and $d$ nonempty and disjoint subsets $X_1 \subset [n]$, $1 \leq l \leq d$, such that

$$x \in S \iff x_i = z_i \text{ if } i \in [n] \setminus (X_1 \cup \cdots \cup X_d) \text{ and } x_i = x_j \text{ if } i, j \in X_l.$$ 

The elements of $[n]$ in the union $X_1 \cup \cdots \cup X_d$ are the free coordinates of $S$. The 1-dimensional combinatorial subspaces are exactly combinatorial lines. From now we omit for brevity ‘combinatorial’ for subspaces and lines. The words $[k+d]^n \setminus \bigcup_{l=k+1}^{k+d}([k+d]\setminus \{l\})^n$ and $d$-dimensional subspaces of $[k]^n$ correspond via the mapping $x \mapsto S(x)$, and this is a $d$-to-one correspondence as $S(x) = S(x')$ iff $x$ and $x'$ can be identified by permuting the letters $k+1, k+2, \ldots, k+d$. The set of words $[k]^d$ and any $d$-dimensional subspace $S(x) \subset [k]^n$ are in bijection via $y \mapsto x(y)$. This bijection sends the $c$-dimensional subspaces of $[k]^d$, $c \leq d$, to the $c$-dimensional subspaces of $[k]^n$ contained in $S(x)$, and this is in fact a bijection.

We capture the density increment argument by the next proposition.

**Proposition 3** There is a function

$$c = c(k, \delta) : \mathbb{N} \times (0, 1) \to (0, 1),$$

nondecreasing in $\delta$ for every $k$, such that for every $k, d \in \mathbb{N}$ and $\delta \in (0, 1)$, there is an $n_0$ such that for every $n > n_0$ and every set $A \subset [k]^n$ with $\mu(A) \geq \delta$ and containing no line, there exists a subspace $S \subset [k]^n$ with dimension $d$ and

$$\mu_S(A) \geq \mu(A) + c \geq \delta + c.$$ 

We fix $k \geq 2$ and derive Theorem 2 from Proposition 3. Suppose $\delta > 0$ is given and let $c = c(k, \delta) > 0$. By Proposition 3 for $d = 1$ there is an $n_0$ such that if $n > n_0$, $A \subset [k]^n$ has $\mu(A) \geq \delta$ and avoids lines, then we get (by the bijection between $S$ and $[k]^d$) a set $A' \subset [k]^d = [k]$ that has $\mu(A') \geq \delta + c$. For $d = n_0 + 1$ we have the conclusion for every $n > n_1$ for some $n_1$ and get $A' \subset [k]^d = [k]^{n_0+1}$ free of lines and with $\mu(A') \geq \delta + c$. We apply to $A' \subset [k]^{n_0+1}$ Proposition 3 again and get $A'' \subset [k]$ with $\mu(A'') \geq (\delta + c) + c = \delta + 2c$. We iterate the process and define inductively in a clear way numbers $n_2, n_3, \ldots, n_t$ where $t = \lfloor 1/c \rfloor$. For $n > n_t$, every set $A \subset [k]^n$ with $\mu(A) \geq \delta$ contains a line, for else repeated applications of Proposition 3 produce at the end a subset of $[k]$ with density at least $\delta + (t+1)c > 1$, which cannot exist.
The density increment $c$ of Proposition 4 arises in two steps, embodied in the next two propositions. For $k \geq 2$ and $i \in [k-1]$, a set $D \subset [k]^n$ is called $ik$-insensitive if $x \in D \Rightarrow x' \in D$ for any word $x'$ obtained from $x$ by changing some occurrences of $k$ to $i$ or vice versa. If $D \subset S \subset [k]^n$, where $S$ is a $d$-dimensional subspace, we say that $D$ is $ik$-insensitive in $S$ if $D' \subset [k]^d$ is $ik$-insensitive where $D'$ is the image of $D$ in the bijection between $S$ and $[k]^d$. A set $D \subset [k]^n$ is a $k$-set if $D = \bigcap_{i=1}^{k-1} D_i$ where each $D_i \subset [k]^n$ is $ik$-insensitive. We define $k$-sets $D \subset S$ in a $d$-dimensional subspace $S \subset [k]^n$ similarly, via the bijection between $S$ and $[k]^d$. In the next two propositions we may assume that $k \geq 3$ since they will be used for such $k$.

**Proposition 4** Let $k, d \in \mathbb{N}$ and $\varepsilon > 0$. There exists an $n_0$ such that for every $n > n_0$, every $k$-set $D \subset [k]^n$ has a partition

$$D = S_1 \cup \cdots \cup S_t \cup F$$

into $d$-dimensional subspaces $S_i \subset [k]^n$ and a set $F \subset [k]^n$ with $\mu(F) < \varepsilon$.

**Proposition 5** There is a function

$$\gamma = \gamma(k, \delta) : \mathbb{N} \times (0, 1) \to (0, 1),$$

nondecreasing in $\delta$ for every $k$, such that for every $k, r \in \mathbb{N}$ and $\delta \in (0, 1)$, there is an $n_0$ such that for every $n > n_0$ and every set $A \subset [k]^n$ with $\mu(A) \geq \delta$ and containing no line, there exist an $r$-dimensional subspace $W \subset [k]^n$ and a $k$-set $D \subset W$ in $W$ satisfying

$$\mu_W(D) \geq \gamma \quad \text{and} \quad \mu_D(A) \geq \mu(A) + \gamma \geq \delta + \gamma.$$ 

We fix $k \geq 2$ and derive Proposition 4 from Propositions 5 and 3. Let $d$ and $\delta$ be given. We set $\gamma = \gamma(k, \delta) > 0$ and take the $n_0$ of Proposition 4 corresponding to $k, d$ and $\varepsilon = \gamma^2/2$. Then we take $n_1$ such that for $n > n_1$ Proposition 4 holds with $k, r = n_0 + 1$ and $\delta$. Now let $n > n_1$ and suppose a set $A \subset [k]^n$ with $\mu(A) \geq \delta$ and free of lines is given. There exist a subspace $W$ and a $k$-set $D \subset W$ of Proposition 4 such that $\mu_W(D) \geq \gamma$, $\mu_D(A) \geq \mu(A) + \gamma$ and $W$ has dimension $n_0 + 1$. Thus $D$ partitions as in Proposition 4 with $[k]^{n_0+1}$ corresponding to $W$ in place of the $[k]^n$ in Proposition 4 and $\varepsilon = \gamma^2/2$. Let $D = E \cup F$ be a partition where $E$ is a disjoint union of $d$-dimensional subspaces of $W$ and $\mu_W(F) < \varepsilon$. Since $\mu_D(A) \geq \mu(A) + \gamma$, $\mu_D(F) = \mu_W(F)/\mu_W(D) < \varepsilon/\gamma = \gamma/2$ and $\mu_F(A) \leq 1$, we get $\mu_E(D) \geq \mu(A) + \gamma/2$. By averaging, there is a $d$-dimensional subspace $S$ of $W$ (contained in $E$) with $\mu_S(A) \geq \mu(A) + \gamma/2$. Proposition 4 follows, with $c(k, \delta) = \gamma(k, \delta)/2$.

Thus to prove Theorem 2 it suffices to deduce Propositions 5 and 1. We shall proceed by induction on $k$. We start by proving Theorem 2 for $k = 2$ and then for every $k \geq 3$ derive Propositions 5 and 6 from validity of Theorem 2 for $k-1$. The derivations rely on formally stronger but equivalent forms of Theorem 2 Propositions 6 and 7. We get the implications

$$T_2 \Rightarrow P_{43} \& P_{53} \Rightarrow P_{33} \Rightarrow T_{23} \Rightarrow P_{44} \& P_{54} \Rightarrow P_{34} \Rightarrow T_{24} \Rightarrow \ldots.$$
which establish Theorem 2 for every \( k \geq 2 \). We start with the easy case \( k = 2 \) and then prepare some results for the derivation of Propositions 5 and 4.

The words of \([2]^n\) 1-1 correspond to the subsets \( X \subseteq [n] \), and the lines 1-1 correspond to the inclusion pairs: pairs \( X \subseteq Y \subseteq [n] \) with \( Y \neq X \). Thus Theorem 2 for \( k = 2 \) follows from the next classical Sperner’s theorem [21].

**Proposition 6** If \( F \) is a family of subsets of \([n]\) containing no inclusion pair (i.e., \( F \) is an antichain to \( \subseteq \)) then

\[
|F| \leq \binom{n}{\lfloor n/2 \rfloor} \leq \frac{2^n}{\sqrt{n}}.
\]

**Proof.** Let \( F \) be an antichain of subsets of \([n]\). The maximal chains \( \{X_0 \subseteq X_1 \subseteq \ldots \subseteq X_n = [n]\}, |X_i| = i \) 1-1 correspond to the \( n! \) permutations \( \pi \) of \([n]\) via \( \pi \mapsto C_\pi = \{\emptyset, \{\pi(1)\}, \{\pi(1), \pi(2)\}, \ldots, \pi([n]) = [n]\}\). We double count the pairs \((\pi, X)\) such that \( X \in C_\pi \cap F \). Grouping the pairs by \( \pi \) we get that their number is \( \leq n! \) as \(|F \cap C_\pi| \leq 1\) for each \( \pi \). Grouping them by \( X \) we get that their number is exactly \( \sum_{X \in F} |X|!(n - |X|)! \), since the summand equals to the number of \( \pi \) with \( X \in C_\pi \). Hence

\[
\sum_{X \in F} |X|!(n - |X|)! \leq n!.
\]

Since \( \binom{n}{j} \leq \binom{n}{\lfloor n/2 \rfloor} \) for any \( 0 \leq j \leq n, |n/2|!(n - |n/2|)! \leq |X|!(n - |X|)! \) and dividing by \( |n/2|!(n - |n/2|)! \) yields the stated inequality. \( \square \)

One may generalize Theorem 2 to subspaces but this is not really stronger than the original theorem.

**Proposition 7** Let \( k \in \mathbb{N}, k \geq 2 \), be given. Assuming Theorem 2 for \( k \), it follows that for every \( \delta > 0 \) and \( d \in \mathbb{N} \), there exists \( n_0 \in \mathbb{N} \) such that for every \( n > n_0 \) every set \( A \subset [k]^n \) with \( \mu(A) \geq \delta \) contains a \( d \)-dimensional subspace.

**Proof.** We proceed by induction on \( d \) where the case \( d = 1 \) is Theorem 2. Now suppose that \( d \geq 2 \) and the result holds for \( d - 1 \) (and every \( \delta \)). Observe that if \( n = n_1 + n_2, n_i \in \mathbb{N} \), and \( A \subset [k]^n \) with \( \mu(A) \geq \delta \), then

\[
\mu(A_1) \geq \delta/2 \quad \text{for} \quad A_1 = \{x \in [k]^{n_1} \mid \mu(\{y \in [k]^{n_2} \mid (x, y) \in A\}) \geq \delta/2\}
\]

(interpreting \((x, y)\) in the obvious way as an element of \([k]^n\)). Let \( \delta > 0 \) be given. We take an \( n_2 \) such that the result holds (with \( n = n_2 \)) for \( d - 1 \) and density \( \delta/2 \) and then an \( n_1 \) such that the conclusion of Theorem 2 holds for every \( n > n_1 \), with density \( \delta/2(k + d - 1)^{n_2} \). Suppose that \( n > n_1 + n_2 \) and \( A \subset [k]^n \) has \( \mu(A) \geq \delta \). Then, using the observation, inductive assumption and pigeonhole principle, we get a set \( A_1 \subset [k]^{n-n_2} \) with \( \mu(A_1) \geq \delta/2(k + d - 1)^{n_2} \) and a \((d - 1)\)-dimensional subspace \( S \subset [k]^{n_2} \) such that \((x, y) \in A\) for every \( x \in A_1 \).
and \( y \in S \). By Theorem 2, \( A_1 \) contains a line \( L \). Hence \( \{(x, y) \mid x \in L, y \in S\} \) is the desired \( d \)-dimensional subspace contained in \( A \).

We will use the fact that almost all words in \([k]^n\) have almost precisely \( n/k \) occurrences of each of the \( k \) letters.

**Proposition 8** Let \( k, n \in \mathbb{N}, j \in [k] \) and \( A \subset [k]^n \) be the set of words with the number of occurrences of \( j \) outside the interval \([n/k - n^{2/3}, n/k + n^{2/3}]\). Then \( \mu(A) < n^{-1/3} \).

**Proof.** (We say more on the tools used in Subsection 2.2.) For \( i \in [n] \) and \( x \in [k]^n \), let \( f_i(x) = 1 \) if \( x_i = j \) and \( f_i(x) = 0 \) else. Then the function \( f = f_1 + \cdots + f_n \) counts occurrences of \( j \) in \( x \), has mean \( P = n/k \) (sum of the means of the \( f_i \)) and variance \( V = (n/k)(1 - 1/k) < n \) (\( V \) is the mean of \( f^2 \) minus the square of the mean of \( f \), which by linearity of means and independence of the \( f_i \) gives \( n/k + n(n-1)/k^2 - (n/k)^2 \)). By Čebyšev’s inequality, \( \mu(\{x \in [k]^n \mid |f(x) - P| > \lambda \sqrt{V}\}) < \lambda^{-2} \) for any \( \lambda > 0 \). Setting \( \lambda = n^{1/6} \) gives the result. \( \square \)

For \( k, n \in \mathbb{N} \), we have on \([k]^n\) the uniform density \( \mu \), given by \( \mu(\{x\}) = 1/k^n \).

For \( k \geq 2 \) and the parameter \( m \leq n \), we define another, non-uniform, density \( \mu'_m \) on \([k]^n\) by

\[
\mu'_m(\{x\}) = \frac{|\{(J, y, z) \in M \mid (J, y, z) = x\}|}{|M|},
\]

for

\[
M = \{(J, y, z) \mid J \subset [n], |J| = m, y \in [k-1]^J, z \in [k^n] \setminus J\}
\]

(\( AB \) denotes, for sets \( A \) and \( B \), the set of all mappings from \( B \) to \( A \)), where any triple \((J, y, z)\) in \( M \) is interpreted as \( x \in [k]^n \) by setting \( x_i = y_i \) if \( i \in J \) and \( x_i = z_i \) else. (We say more about densities in Subsection 2.2.)

**Proposition 9** If \( \eta \in (0, 1) \) and \( k, m, n \in \mathbb{N} \) satisfy \( k \geq 2 \), \( n \geq (12k/\eta)^{12} \) and \( m \leq n^{1/4} \), then for every set \( A \subset [k]^n \) we have

\[
|\mu'_m(A) - \mu(A)| < \eta.
\]

**Proof.** This is a particular case of the more general Proposition 15, which we prove later. \( \square \)

To deduce Proposition 4, we need Propositions 7 and 9.
2.1 Derivation of Proposition 4

In this subsection we fix a $k \in \mathbb{N}$, $k \geq 3$, assume that Theorem 2 holds for $k - 1$ (and every $\delta > 0$) and deduce from this Proposition 4 for $k$. The main step is to get the required partition if $D$ is an $i k$-insensitive set for an $i \in [k - 1]$; the full proposition follows inductively by iteration. We may of course set $i = 1$.

**Proposition 10** For every $d \in \mathbb{N}$ and $\varepsilon > 0$, there exists an $n_0$ such that for every $n > n_0$ every $1k$-insensitive set $D \subset [k]^n$ has a partition

$$D = S_1 \cup \cdots \cup S_t \cup F$$

into $d$-dimensional subspaces $S_i \subset [k]^n$ and a set $F \subset [k]^n$ with $\mu(F) < \varepsilon$.

**Proof.** Let $d \in \mathbb{N}$ and $\varepsilon > 0$ be given. Applying Theorem 2 for $k - 1$ and Proposition 4 we take an $m \in \mathbb{N}$, $m \geq d$, such that every set $A \subset [k - 1]^m$ with $\mu(A) \geq \varepsilon / 3$ contains a $d$-dimensional subspace. We set

$$n_0 = \lceil 3\varepsilon^{-1}mk^{m-d}(k + d)^m + m^4 + (36k/\varepsilon)^12 \rceil.$$ 

Let $n > n_0$ and $D \subset [k]^n$ be a $1k$-insensitive set. We may assume that $\mu(D) \geq \varepsilon$ for else we set at once $F = D$. We construct, for $r = 0, 1, \ldots$, sets $D = D_0 \supset D_1 \supset \cdots \supset D_r$ and $\emptyset = J_0 \subset J_1 \subset \cdots \subset J_r \subset [n]$, $|J_j| = jm$, with the properties that (i) for each $x \in [k]^J$, the set

$$(D_r)_x = \{ y \in [k]^{[n]\setminus J} \mid (x, y) \in D_r \}$$

is $1k$-insensitive, (ii) $D \setminus D_r$ partitions into $d$-dimensional subspaces and (iii) $\mu(D_r \setminus D_{r+1}) \geq \varepsilon k^{d-m}(k + d)^{-m} / 3$ for $j = 0, 1, \ldots, r - 1$. Such sets trivially exist if $r = 0$, namely $D_r = D$ and $J_r = \emptyset$. We claim that as long as $\mu(D_r) \geq \varepsilon$, the construction can be continued. This establishes the proposition: since $r \leq 3\varepsilon^{-1}k^{m-d}(k + d)^m$ (by (iii)), the construction has to terminate for some $r$ (by the definition of $n_0$, $n$ is so large that without terminating we hit the contradiction $\mu(D) > 1$), and then $\mu(D_r) < \varepsilon$ and $D \setminus D_r$ partitions into $d$-dimensional subspaces.

To prove the claim we assume that $\mu(D_r) \geq \varepsilon$, which is true if $r = 0$. In the initial step when $r = 0$ and $J_r = \emptyset$, we modify the following construction, which is described for the general step, accordingly by omitting the $x$-coordinate. The (uniform) average of the values $\mu((D_r)_x)$, taken over all $x \in [k]^J_r$, equals $\mu(D_r)$ and so is at least $\varepsilon$. Hence the same average of $\mu'_m((D_r)_x)$, where $\mu'_m$ is the (non-uniform) density on $[k]^{[n]\setminus J_r}$ introduced before Proposition 9, is at least $\varepsilon - \eta = 2\varepsilon / 3$, due to Proposition 9 with $\eta = \varepsilon / 3$. In other words, density of the subset of the quadruples $(x, J, y, z)$, where $x \in [k]^J_r$, $J \subset [n]\setminus J_r$, $|J| = m$, $y \in [k - 1]^J$ and $z \in [k]^{[n]\setminus (J_r \cup J)}$, satisfying $(x, J, y, z) \in D_r$ in the set of all quadruples, is at least $2\varepsilon / 3$. Hence there is a $J$ such that density of the triples $(x, y, z)$ (from the stated domains) with $(x, J, y, z) \in D_r$ is at least $2\varepsilon / 3$. And
for this fixed $J$, density of the pairs $(x, z)$ for which $\mu(\{y \in [k-1]^J \mid (x, J, y, z) \in D_r\}) \geq \varepsilon/3$ is at least $\varepsilon/3$. We set

$$J_{r+1} = J_r \cup J.$$  

By the choice of $m$, each of these sets of words $y$ contains a $d$-dimensional subspace $U_{x, z} \subseteq [k-1]^J$. Since $(D_r)_x$ is 1$k$-insensitive, for this $J$ and for each of these pairs $(x, z)$ there is a $d$-dimensional subspace $U_{x, z} \subseteq [k]^J$ such that $(x, y, z) \in D_r$ for every $y \in U_{x, z}$. By the pigeon-hole principle, there is a single $d$-dimensional subspace $U \subseteq [k]^J$ such that the set

$$T = \{(x, z) \in [k]^J \times [k]^n \setminus J_{r+1} \mid (x, y, z) \in D_r \text{ for every } y \in U\}$$

has density at least $\varepsilon/3(k+d)^m$. Note that for each $x$ the set of $z$ with $(x, z) \in T$ is 1$k$-insensitive. We set

$$D_{r+1} = D_r \setminus (T \times U)$$

where $T \times U$ means the words in $[k]^n$ that restrict, for some $(x, z) \in T$ and $y \in U$, on $J_r$ to $x$, on $n \setminus J_{r+1}$ to $z$ and on $J$ to $y$. Clearly, (ii) holds because $T \times U = D_r \setminus D_{r+1}$ is a disjoint union of $d$-dimensional subspaces. Property (i) holds too because for every $x \in [k]^J$ and $y \in [k]^J$, the set of $z \in [k]^n \setminus J_{r+1}$ with $(x, y, z) \in D_{r+1}$ is 1$k$-insensitive. (Consider $u = (x, y, z) \in D_{r+1}$ and $u' = (x, y, z')$ in which $z'$ arises from $z$ by some exchanges of 1s and 0s. Then $u' \in D_r$ by the 1$k$-insensitivity of $(D_r)_x$. If $u' \not\in T \times U$ then $y \in U$ and $(x, z') \in T$, hence $(x, z) \in T$ as noted above, and $u \in T \times U$, which is not the case. So $u' \not\in T \times U$ and $u' \in D_{r+1}$.) Finally, the density of $T \times U$ in $[k]^n$ equals to the density of $T$ in $[k]^J \times [k]^n \setminus J_{r+1}$ times the density of $U$ in $[k]^J$, which is at least $\varepsilon/(3(k+d)^m(k^m-d))$. Thus $J_{r+1}$ and $D_{r+1}$ have the required properties (i)–(iii).  

Remark. The decrease of density of $T \times U$ compared to $T$, caused by density of $U$, seems to be overlooked by Polymath—they claim [20] bottom of p. 1320 that $T \times U$ has density at least $\eta(k+d)^{-m}$ (i.e., $\varepsilon/3(k+d)^m$ in our notation), which reflects in the statement of [20] Lemma 8.1].

We prove Proposition 4. We proceed by induction on the size of intersection defining $D$. Let $j \in [k-1]$. We assume that for every $d$ and $\varepsilon > 0$ Proposition 4 holds for all sets of the form $D = D_1 \cap D_2 \cap \cdots \cap D_j$ where $D_i \subseteq [k]^n$ is ik-insensitive; for $j = 1$ this is true by Proposition 10. From this we deduce (if $j < k-1$) that Proposition 4 holds for all sets $D$ corresponding to the increased parameter value $j+1$. For $j = k-1$ we get the original Proposition 4.

So let $d$ and $\varepsilon > 0$ be given. We take $n_0$ such that for every $n > n_0$ our inductive assumption (for $j < k-1$) holds for subspaces dimension $d$ and bound on the density of the residual set $\varepsilon/2$. Then we take $n_1$ such that for every $n > n_1$ the conclusion of Proposition 10 holds for subspaces dimension $n_0 + 1$ and bound on the density of the residual set $\varepsilon/2$. Now suppose that $n > n_1$ and $D = D_1 \cap \cdots \cap D_{j+1}$ where $D_i \subseteq [k]^n$ is ik-insensitive. Using
Proposition 10 we obtain a partition $D_{j+1} = T_1 \cup \cdots \cup T_s \cup F$ such that the $T_i$ are $(n_0 + 1)$-dimensional subspaces and $\mu(F) < \varepsilon/2$. We have

$$D = \bigcap_{i=1}^{j+1} D_i = \bigcup_{i=1}^s (T_i \cap D_1 \cap \cdots \cap D_j) \cup (F \cap D_1 \cap \cdots \cap D_j).$$

Clearly, each $T_i \cap D_k$ is $hk$-insensitive in $T_i$ and thus using the inductive assumption for $j$ we can express (working in $[k]^{n_0+1}$ via the bijection with $T_i$) each set $T_i \cap D_1 \cap \cdots \cap D_j = (T_i \cap D_1) \cap \cdots \cap (T_i \cap D_j)$ as a disjoint union of $d$-dimensional subspaces (in $T_i$ and thus in $[k]^n$) and a residual set $F_i$ with $\mu_{T_i}(F_i) < \varepsilon/2$. These subspaces, taken for all $i = 1, 2, \ldots, s$, and the set $E = F_1 \cup \cdots \cup F_s \cup (F \cap D_1 \cap \cdots \cap D_j)$ form the desired partition of $D$ because $\mu(E) \leq \mu(F_1 \cup \cdots \cup F_s) + \mu(F) < \max_i \mu_{T_i}(F_i) + \varepsilon/2 < \varepsilon$. This concludes the derivation of Proposition 10.

### 2.2 The equal-slices densities $\nu$ and $\tilde{\nu}$

We move to the second and more complicated half of the proof of Theorem 2 for $k$ from Theorem 4 for $k-1$. Similarly to the role of Proposition 7 in the first half of the proof, we need a stronger version of Proposition 10. Proposition 11 which says that any positively dense set $A \subset [k]^n$ contains, for large $n$, a set of lines with positive density. However, Proposition 8 shows that this cannot hold for the uniform density. Consider the set $A \subset [2]^n$ of words in which the numbers of occurrences of 1 and 2 deviate from $n/2$ by less than $n^{2/3}$. Then $\mu(A) \to 1$ as $n \to \infty$ but at the same time $\mu(M) \to 0$ for the set $M \subset [3]^n$ of lines contained in $A$, because the inclusion $L(x) \subset A$, $x \in [3]^n$, forces $x$ to have at most $2n^{2/3}$ occurrences of 3, and such $x$ have in $[3]^n$ density going to 0. Fortunately, the strengthening holds for a non-uniform density, the equal-slices density $\nu$ that we define in a moment, and one can go from the uniform to equal-slices density and back. Since $\nu$ does not behave well to restrictions to subspaces, we need to work also with a variant density $\tilde{\nu}$ fixing this problem, which for large $n$ differs from $\nu$ only little. We begin with discussing densities in general and then introduce the densities $\nu$ and $\tilde{\nu}$.

A density on a finite set $B \neq \emptyset$ is a mapping $\mu'$ from the set of all subsets of $B$ to the interval $[0, 1]$, such that $\mu'(B) = 1$ and $\mu'(A \cup A') = \mu'(A) + \mu'(A')$ whenever $A, A' \subset B$ and $A \cap A' = \emptyset$. Thus $\mu'(\emptyset) = 0$ and $\mu'$ is uniquely determined by its values on singletons. Any choice of values $\mu'\{\{x\}\} \geq 0$, $x \in B$, with $\sum_{x \in B} \mu'(\{x\}) = 1$ gives a density: $\mu'(A) = \sum_{x \in A} \mu'(\{x\})$ for any $A \subset B$. We have been using the uniform density $\mu$, defined by $\mu(\{\{x\}\}) = 1/|B|$ for any $x \in B$, and before Proposition 9 we met the non-uniform density $\mu'_{\max}$. We reserve the letter $\mu$ for the uniform density and primed $\mu'$ for general, possibly non-uniform, density.

Suppose $B$ is a finite set with a density $\mu'$. If $f : B \to \mathbb{R}$, the average, or mean, of the function $f$ (with respect to $\mu'$) is

$$\sum_{x \in B} f(x) \cdot \mu'(\{x\}).$$
We recall a few useful properties of averages, which we already used in the proof of Proposition 10. Linearity: if \( f_i : B \rightarrow \mathbb{R}, i = 1, 2, \) have means \( \mu_i \) and \( a, b \in \mathbb{R} \), then \( af_1 + bf_2 \) has mean \( a\mu_1 + b\mu_2 \). If \( f_1 \) and \( f_2 \) are independent, which means that \( \mu'(f_1^{-1}(c) \cap f_2^{-1}(d)) = \mu'(f_1^{-1}(c)) \cdot \mu'(f_2^{-1}(d)) \) for any two values \( c, d \in \mathbb{R} \), then the mean of \( f_1f_2 \) equals to \( \mu_1\mu_2 \). If \( f \) has average at least \((at most)\) \( c \) then \( f(x) \geq c \) \((f(x) \leq c)\) for some \( x \in B \). Markov’s inequality: If \( f \geq 0 \) has mean \( \mu \) and \( \lambda > 0 \), then \( \mu'(\{x \in B \mid f(x) > \lambda\mu\}) < \lambda^{-1} \). Applying it to the function \( (f(x) - \mu)^2 \) we get Čebyšev’s inequality. If \( V \), the variance of \( f \), is the mean of \( (f(x) - \mu)^2 \) (where \( \mu \) is the mean of \( f \)), then \( \mu'\{x \in B \mid |f(x) - \mu| > \lambda\sqrt{V}\} < \lambda^{-2} \) for any \( \lambda > 0 \). We do not need any stronger result on concentration of \( f \) around its mean.

If \( f : C \rightarrow B \) is a mapping and \( \mu' \) a density on \( C \), we get a density \( \mu'' \) on \( B \) by setting

\[
\mu''(\{x\}) = \mu'(f^{-1}(x)) = \sum_{c \in C, f(x) = c} \mu'(\{c\}).
\]

We refer to this as projection. Another construction of more complicated densities from simpler ones takes a family of sets \( B_i, i \in I \) (all of them finite), with a density \( \mu'_i \) on \( I \) and densities \( \mu''_i \) on the sets \( B_i \), and defines

\[
\mu'(\{(i, b)\}) = \mu'_i(\{i\}) \cdot \mu''_i(\{b\}), \text{ for } i, b \in B_i.
\]

Then \( \mu' \) is a density on the disjoint union \( \bigcup_{i \in I} B_i \), the set of all pairs \((i, b)\) with \( i \in I \) and \( b \in B_i \). We call this construction, which generalizes to triples etc., higher-dimensional density. Both constructions can be combined: to define a non-uniform density on \( B \), one takes a higher-dimensional density, often patched from uniform densities, and projects it to \( B \).

Let us describe one such situation that we already encountered in the proof of Proposition 10 and will encounter again. Suppose that \( \mu' \) is the higher-dimensional density on \( C = \bigcup_{i \in I} B_i \) coming from the densities \( \mu'_i \) on \( I \) and \( \mu''_i \) on \( B_i \), \( f : C \rightarrow B \) is a mapping that is injective for each fixed \( i \) (each \( B_i \) then can be regarded as a subset of \( B \)) and that \( \mu'' \) is the projection of \( \mu' \) to \( B \) via \( f \). Then for each \( A \subset B \), the value \( \mu''(A) \) in fact equals to the average of the function \( i \mapsto \mu''_i(B_i \cap A) \) with respect to \( \mu'_i \).

Important densities live on the sets of words \([k]^n\). The \( n! \) permutations of \([n]\) act on the coordinates of \([k]^n\) and produce a partition \([k]^n = \bigcup_{r \in \mathbb{N}_0} O_r \) into orbits, or slices, where \( O_r \) consists of the words that have equal numbers of occurrences of each letter \( j \in [k] \) and \( I \) is the \( k-1 \)-element set of \( k \)-tuples \( r = (r_1, \ldots, r_k) \in \mathbb{N}_0^k \), \( \sum r_j = n \), recording these numbers. The equal-slices density \( \nu \) on \([k]^n\) is the unique density satisfying \( \nu(\{x\}) = \nu(\{y\}) \) if \( x, y \in O_r \) and \( \nu(O_r) = \nu(O_s) \) for any \( r, s \in I \). Explicitly,

\[
\nu(\{x\}) = \frac{1}{\binom{n+k-1}{k-1}} \frac{n}{(r_1, r_2, \ldots, r_k)}
\]

for \( x \in [k]^n \) with \( r_j \) occurrences of \( j \). We reserve the letter \( \nu \) for the equal-slices densities and refer to the uniform and equal-slices densities as \( \mu \)-density and \( \nu \)-density, respectively. If \( S \subset [k]^n \) is a \( d \)-dimensional subspace and \( A \subset [k]^n \) then
\(\nu_S(A)\) is defined as \(\nu'(A')\) where \(A' \subseteq [k]^d\) is the image of \(A \cap S\) in the bijection between \(S\) and \([k]^d\) and \(\nu'\) is the equal-slices density on \([k]^d\) (this in general differs from \(\nu(A \cap S)/\nu(S)\), whereas for \(\mu\)-density both ways of relativising density to subspaces give the same result).

A slice \(O_r, r = (r_1, \ldots, r_k)\), is degenerate if \(r_j = 0\) for some \(j\), and is non-degenerate else (then each letter \(j \in [k]\) occurs in the words of \(O_r\)); there are \(\binom{n}{k-1}\) non-degenerate slices. The non-degenerate equal-slices density \(\nu\) on \([k]^n\), for \(n \geq k\) (else all slices are degenerate), is obtained from \(\nu\) by setting \(\tilde{\nu}(O_r) = 0\) for every degenerate slice and rescaling \(\nu\) accordingly (by the factor \((1 - \nu(D))^{-1}\) where \(D\) is the union of degenerate slices) on the union of non-degenerate slices.

So

\[
\tilde{\nu}(\{x\}) = \frac{1}{\binom{n}{k-1}(r_1, r_2, \ldots, r_k)}
\]

if \(x \in [k]^n\) has \(r_j \geq 1\) occurrences of \(j\) for each \(j \in [k]\), and \(\tilde{\nu}(\{x\}) = 0\) if \(r_j = 0\) for some \(j\).

For \(n, d, k \in \mathbb{N}\) and two words \(y \in [d]^n, z \in [k]^d\), we define their composition \(y \ast z\) as the word \(x \in [k]^n\) given by \(x_i = z_{y_i}, i = 1, 2, \ldots, n\). Suppose that \(y \in [d]^n\) is non-degenerate (hence \(n \geq d\)). Clearly, \(\{y \ast z \mid z \in [k]^d\} \subseteq [k]^n\) is the \(d\)-dimensional subspace \(S(y')\), where \(y' \in [k + d]^n\) is obtained from \(y\) by replacing letter \(j\) with \(k + j\) (note that \(S(y')\) has no fixed coordinate), and in the factorization \(x = y \ast z\) the word \(z\) is uniquely determined by \(y\); the equality \(x = y \ast z\) captures the way of determining \(x\) by selecting a subspace \(S(y')\) containing \(x\) and then selecting ‘in’ \(S(y')\) the word \(z\) corresponding to \(x\). Note that if \(L = L(z') \subseteq [k]^d, z' \in [k + 1]^d\), is a line, then \(\{y \ast z \mid z \in L\} = L(y \ast z') \subseteq [k]^n\) is a line too. For \(n \geq d\) and \(M = [d]^n \times [k]^d\), we define a new density \(\nu_d'\) on \([k]^n\) by

\[
\nu_d'(\{x\}) = \sum_{(y, z) \in M, y \ast z = x} \tilde{\nu}_1(\{y\}) \cdot \tilde{\nu}_2(\{z\})
\]

where \(\tilde{\nu}_1\) (resp. \(\tilde{\nu}_2\)) is the non-degenerate equal-slices density on \([d]^n\) (resp. on \([k]^d\)); we may clearly assume that in the sum \(y\) is non-degenerate. Below we show that \(\nu_d' = \tilde{\nu}\). Before that we demonstrate that by replacing the densities \(\tilde{\nu}_i\) in the definition of \(\nu_d'\) with \(\nu_i\) (and keeping \(y\) in the sum non-degenerate), we obtain a density \(\nu_d'\) distinct from \(\nu\). Indeed, for \(n = d = k = 2\) and \(x = 11\), the two factorizations \(11 = 21 \ast 11 = 21 \ast 11\) give \(\nu'_2(\{x\}) = 2(3(2^1)^{2})^{-1}(3(2^2_2)^{2})^{-1} = 1/9\), but \(\nu(\{x\}) = (3(2^2_0)^{2})^{-1} = 1/3\).

**Proposition 11** Let \(k, n \in \mathbb{N}\) and \(\nu\) be the equal-slices density and \(\tilde{\nu}\) the non-degenerate equal-slices density on \([k]^n\).

1. If \(m \in \mathbb{N}, j \in [k]\) and \(A \subseteq [k]^n\) are the words with less than \(m\) occurrences of \(j\), then \(\nu(A) < mk/n\).

2. Let \(A \subseteq [k]^n\), \(n \geq k\), and \(D \subseteq [k]^n\) be the union of degenerate orbits. Then (i) \(\nu(D) < k^2/n\), (ii) \(\tilde{\nu}(A) = (1 - \nu(D))^{-1}\nu(A)\) if \(A\) consists of non-degenerate words only, and (iii) \(|\nu(A) - \tilde{\nu}(A)| < k^2/n\) for any \(A\).
3. If $n \geq d \geq k$, the above defined density $\nu'_d$ on $[k]^n$ coincides with $\nu$.

**Proof.** 1. We may set $j = k$. By the definition of $\nu$, $\nu(A)$ equals to the ratio $|M|/\binom{n+k-1}{k-1}$ where $M$ is the set of $k$-tuples $(r_1, \ldots, r_k) \in \mathbb{N}_0^k$, $\sum r_i = n$, with $r_k = l < m$. Thus

$$\nu(A) = \sum_{l=0}^{m-1} \frac{\binom{n+k-2-l}{k-2}}{\binom{n+k-1}{k-1}} = \sum_{l=0}^{m-1} \frac{k-1}{n+k-1} \frac{\binom{n+k-2-l}{k-2}}{\binom{n+k-1}{k-2}} < \frac{mk}{n}.$$  

2. The bound on $\nu(D)$ follows from part 1 with $m = 1$. The second claim is just the rescaling of $\nu$ defining $\nu$. To show the last claim set $A_1 = A \cap ([k]^n \setminus D)$ and $A_2 = A \cap D$. Then (by part 1 and (ii)) $0 \leq \nu(A_1) = \nu(D)\nu(A_1) \leq \nu(D) < \frac{k^2}{n}$ and $0 \leq \nu(A_2) = \nu(A_2) \leq \nu(D) < \frac{k^2}{n}$. Since $A = A_1 \cup A_2$ is a partition, subtraction of the two estimates gives $\nu(A) - \nu(A) < \frac{k^2}{n}$.

3. Let $n \geq d \geq k$, $x \in [k]^n$ be a word, $X_j \subset [n]$ for $j \in [k]$ be the positions of the letter $j$ in $x$ and $r_j = |X_j|$. We assume that all $r_j \geq 1$ because for degenerate $x$ we clearly have $\nu'_d(x) = 0$. The factorizations $x = y \cdot z$, with non-degenerate $y \in [d]^n$ and $z \in [k]^d$, 1-1 correspond to the pairs $(P, l)$ where $P$ is a partition of $[n]$ (a set of nonempty blocks) such that $|P| = d$ and if $B \in P$ then $B \subset X_j$ for some $j$, and $l : P \to [d]$ is a bijection. $P$ and $l$ determine $y$ and $z$ uniquely ($y_i = l \iff i \in B \in P$ with $l(B) = i$ and $z_i = j \iff l(B) = i$ for some $B \in P$ with $B \subset X_j$). We can generate the pairs $(P, l)$ also as follows. We take all $k$-tuples $i = (i_1, \ldots, i_k) \in \mathbb{N}^k$ with $|i| = i_1 + \cdots + i_k = d$, for each $i$ take all $i_j$-tuples $s(j) \in \mathbb{N}^{r_j}$, $j = 1, 2, \ldots, k$, with $|s(j)| = r_j$ (here $s(j)! = s(j) \cdot \ldots \cdot s(j)$), then for each $s(j)$ take all $r_j$-tuples $r_j = (r_{j(1)}, \ldots, r_{j(k)})$ ordered partitions $(Y_{1, 1}, \ldots, Y_{1, i_1})$ of $X_j$, with $|Y_{1, i_1}| = s(j)_i$, and finally we forget the orders of blocks $Y_{1, i_1}$ and label the $d$ blocks in each resulting collection in $d!$ ways with $1, 2, \ldots, d$. This way we produce each pair $(P, l)$ with multiplicity $! = i_1! \cdots i_k!$ ($i_j$ is the number of blocks of $P$ contained in $X_j$).

$$\nu'_d(x) = \sum_{||i||=k, |i|=d, |s(j)|=i_j, |s(j)|=r_j} \frac{d! \cdot \binom{r_{j(1)}}{s(j)_1} \cdots \binom{r_{j(k)}}{s(j)_k}}{(n-1)!(s(j)_1) \cdots (s(j)_k)(d-1)!}$$

where $||i||$ is the arity of a tuple, $s(1)s(2) \cdots s(k)$ means concatenation of the $i_j$-tuples into one $d$-tuple and the denominator gives $\nu_1(\{y\}) \nu_2(\{z\})$. By cancelling the common factors in the summand we simplify the sum to $(r = (r_1, \ldots, r_k))$

$$\frac{(n-d)!(k-1)!(d-k)!}{(n-1)!\binom{n}{r}} \sum_{||i||=k, |i|=d, |s(j)|=i_j, |s(j)|=r_j} 1.$$  

The last sum equals

$$\sum_{||i||=k, |i|=d} \binom{r_1-1}{i_1-1} \binom{r_2-1}{i_2-1} \cdots \binom{r_k-1}{i_k-1} = \binom{n-k}{d-k} = \frac{(n-k)!}{(d-k)!(n-d)!}$$
inequality we deduce that
\[ | - m J, x, y | \]
we are counting \((d-k)\)-element subsets \(Y\) of an \((n-k)\)-element set \(X\) according to the sizes of intersections of \(Y\) with blocks of a fixed partition of \(X\) into blocks with sizes \(r_j - 1\). Hence the sum equals \(1/(\binom{n-1}{k}) = \tilde{v}(\{x\})\).

To go from \(\mu\)-density to \(\nu\)-density, we show that if one weights \([k]^n\) on the minority of \(m\) coordinates uniformly and on the majority of remaining coordinates by equal-slices density, the resulting density is approximately \(\nu\)-density. For \(k, m, n \in \mathbb{N}\) with \(m \leq n\), we define a density \(\mu'_m\) on \([k]^n\) by
\[
\mu'_m(\{z\}) = \sum_{M \supseteq (J, x, y) = z} \mu_1(\{J\}) \cdot \mu_2(\{x\}) \cdot \nu_1(\{y\})
\]
where \(M\) consists of all triples \((J, x, y)\) with \(J \subset [n]\), \(|J| = m\), \(x \in [k]^J\) and \(y \in [k]^n\setminus J\), a triple is projected to \([k]^n\) in the obvious way, \(\mu_1\) (resp. \(\mu_2 = \mu_{2,J}\)) is the uniform density on the set of \(m\)-element subsets of \([n]\) (resp. on \([k]^J\)) and \(\nu_1 = \nu_{1,J}\) is the equal-slices density on \([k]^n\setminus J\).

**Proposition 12** Let \(k, m, n \in \mathbb{N}\), \(m \leq n\) and \(\mu'_m\) be the above defined density on \([k]^n\). Then for every set \(A \subset [k]^n\),
\[
|\mu'_m(A) - \nu(A)| \leq km/n.
\]

**Proof.** We prove the inequality, in fact a stronger one, first for \(m = 1\). Let \(z \in [k]^n\) and \(r_j\) be the number of occurrences of the letter \(j \in [k]\) in \(z\). By the definition of \(\mu'_m\) and \(\nu\),
\[
\frac{\mu'_1(\{z\})}{\nu(\{z\})} = \binom{n+k-1}{k-1} \binom{n}{r_1, r_2, \ldots, r_k} \sum_{j=1, r_j \geq 1}^{k} \frac{r_j/kn}{\binom{n+k-1}{r_1, \ldots, r_j-1, \ldots, r_k}} \leq \frac{k-1}{n} \nu(\{z\}).
\]
So \(|\mu'_1(\{z\}) - \nu(\{z\})| \leq \frac{k-1}{n} \nu(\{z\})\). Summing over \(z \in A\) and using triangle inequality we deduce that
\[
|\mu'_1(A) - \nu(A)| \leq \frac{k-1}{n} \nu(A) < \frac{k}{n}.
\]

We derive from this that \(|\mu'_m(A) - \mu'_{m-1}(A)| \leq k/n\) for every \(A \subset [k]^n\) and \(m \geq 2\). The inequality \(|\mu'_m(A) - \nu(A)| \leq mk/n\) then follows by induction and triangle inequality. Let \(m \geq 2\) and \(A \subset [k]^n\). We partition the set of triples \(M\) defining \(\mu'_{m-1}\) by the equivalence \((J, x, y) \sim (J', x', y')\) iff \(J = J'\) and \(x = x'\). So (projecting \((J, x, y)\) to \([k]^n\) when needed)
\[
\mu'_{m-1}(A) = \sum_{B \in M/\sim} \mu_1(\{J\}) \cdot \mu_2(\{x\}) \sum_{B \supseteq (J, x, y) \in A} \nu_1(\{y\}).
\]
We replace in each inner sum the equal-slices density \( \nu_1 \) on \([k]^{[n]} \setminus J\) (now \(|J| = m - 1\)) with the density \( \mu'_1 \), and this changes the total sum to, say, \( \mu''(A) \). Summing the changes of inner sums, the result for \( m = 1 \) gives that \( |\mu''(A) - \mu'_{m-1}(A)| \leq (k-1)/(n-m+1) \). A moment of reflection reveals that the change of \( \nu_1 \) on each \([k]^{[n]} \setminus J\) to \( \mu'_1 \) gives an equivalent, only a more complicated, way of counting \( \mu'_m(A) \) (it boils down to the identity \( \binom{n}{m} = \frac{n-m+1}{m} \binom{n}{m-1} \)). Thus \( \mu''(A) = \mu'_m(A) \) and \( |\mu'_m(A) - \mu'_{m-1}(A)| \leq (k-1)/(n-m+1) \leq k/n \) as needed, because we may assume that \( n \geq km \) (else the result holds trivially). \( \square \)

Propositions 11 and 12 show that we may replace \( \mu \)-density in Theorem 2 with \( \tilde{\nu} \)-density. Using this we derive Proposition 14, the key strengthening of Theorem 2.

For \( k, m, n \in \mathbb{N} \), \( J \subset [n] \) with \(|J| = m \) and \( y \in [k]^{[n]} \setminus J \), we denote by \( S_{J,y} \) the \( m \)-dimensional subspace of \([k]^n\) that has \( J \) as the set of free coordinates and elsewhere is determined by \( y \): \( x \in S_{J,y} \iff x_i = y_i \) for every \( i \in [n] \setminus J \).

**Proposition 13** Let \( k \in \mathbb{N} \), \( k \geq 2 \), be given and assume Theorem 2 for \( k \). It follows that for every \( \delta > 0 \) there is an \( n_0 \in \mathbb{N} \) such that for every \( n > n_0 \) every set \( A \subset [k]^n \) with \( \tilde{\nu}(A) \geq \delta \) contains a line.

**Proof.** Let \( \delta \) be given. We take the \( n_0 \) of Theorem 2 corresponding to uniform density \( \delta/3 \) and set \( m = n_0 + k \). Suppose that \( n > 3km/\delta = 3(n_0 + k)/\delta \) and that \( A \subset [k]^n \) has \( \tilde{\nu}(A) \geq \delta \). By part 2 (ii) of Proposition 11, \( \nu(A) \geq 2\delta/3 \). By Proposition 12 and the definition of density \( \mu'_n \) before it, there exists an \( m \)-dimensional subspace \( S = S_{J,y} \) of \([k]^n\), \( J \subset [n] \) with \(|J| = m \), such that \( \mu_S(A) \geq \nu(A) - km/n \geq \delta/3 \). By the choice of \( m \) and Theorem 2 there is a line in \([k]^n\) contained in \( A \cap S \). \( \square \)

**Proposition 14** Let \( k \in \mathbb{N} \), \( k \geq 2 \), be given and assume Theorem 2 for \( k \). It follows that for every \( \delta > 0 \) there exist an \( n_0 \in \mathbb{N} \) and a \( \theta > 0 \) such that if \( n > n_0 \) and \( A \subset [k]^n \) has \( \nu(A) \geq \delta \), then the set \( M \subset [k+1]^n \) of lines contained in \( A \) has \( \nu(M) \geq \theta \).

**Proof.** Let \( \delta \) be given. We take the \( n_0 \) of Proposition 13 corresponding to the \( \tilde{\nu} \)-density \( \delta/2 \) and set \( d = n_0 + 1 \). Suppose that \( n > d + 4k^2/\delta \) and \( A \subset [k]^n \) has \( \nu(A) \geq \delta \). By part 2 (iii) of Proposition 11, \( \tilde{\nu}(A) \geq 3\delta/4 \). For \( y \in [d]^n \) we define \( C_y = \{ z \in [k]^d \mid y \ast z \in A \} \) (recall the composition of words \( \ast \) introduced before Proposition 11). Let \( B \subset [d]^n \) be the set of (non-degenerate) words \( y \) such that \( \tilde{\nu}_2(C_y) \geq \delta/2 \). By part 3 of Proposition 14 (applied with \( k \)), \( \tilde{\nu}_1(B) \geq 3\delta/4 \) and the definition of \( B \) imply that \( \nu_1(B) \geq \delta/4 \). Deleting degenerate words (they are irrelevant for \( \tilde{\nu}_2 \) anyway), we may assume that all words in every \( C_y \) are non-degenerate. Consider the set

\[ M = \{ x' \in [k+1]^n \mid x' = y \ast z', y \in B, z' \in [k+1]^d, L(z') \subset C_y \} . \]
These are lines contained in $A$: $L(x') \subset A$ for every $x' \in M$. By the choice of $d$, for each $y \in B$ the set $C_y \subset [k]^d$ contains a line $L(z')$ and (due to the purge on $C_y$) $z'$ is non-degenerate. Let $\nu'_2$ (resp. $\nu'$) be the non-degenerate equal-slices density on $[k+1]^d$ (resp. on $[k+1]^n$). Since $\nu_1(B) \geq \delta/4$ and for each $y \in B$ there is at least one non-degenerate $z' \in [k+1]^d$ with $y \neq z' \in M$, giving contribution at least $\nu'_2(\{z'\}) > \tilde{d}^{-k}(k+1)^{-d}$, by part 3 of Proposition [11] (applied with $k+1$) we see that

$$\nu'(M) > \frac{\delta}{4d^k(k+1)^d}.$$  

By part 2 ((i) and (ii)) of Proposition [11] (and since $n > 4k^2$ and $k \geq 2$), the desired lower bound $\nu(M) > (1-(k+1)^2/n)\nu'(M) > (\delta/9)d^{-k}(k+1)^{-d} = \theta > 0$ follows. 

To go from $\nu$-density to $\mu$-density, we show that if one weights $[k]^n$ on the minority of $m$ coordinates by $\nu$-density and on the majority of remaining coordinates uniformly, the resulting density is approximately $\mu$-density. We prove it in greater generality with any density $\mu'$ on the minority of $m$ coordinates. For $k, m, n \in \mathbb{N}$ with $m \leq n$ and a density $\mu'$ on $[k]^m$, we define a density $\mu'_m$ on $[k]^n$ by

$$\mu'_m(\{z\}) = \sum_{M \supseteq \{\sigma, x, y\} = z} \mu_1(\{\sigma\}) \cdot \mu'(\{x\}) \cdot \mu_2(\{y\})$$

where $M$ consists of all triples $(\sigma, x, y)$ with $\sigma : [m] \to [n]$ an injection, $x \in [k]^m$ and $y \in [k]^n \setminus \sigma([m])$, a triple $(\sigma, x, y)$ is projected to $[k]^n$ by setting $z_{\sigma(i)} = x_i$ for $i \in [m]$ and $z_i = y_i$ for $i \in [n] \setminus \sigma([m])$. $\mu_1$ (resp. $\mu_2 = \mu_{2,\sigma}$) is the uniform density on the set of injections from $[m]$ to $[n]$ (resp. on $[k]^n \setminus \sigma([m])$) and $\mu'$ is the given density on $[k]^m$.

**Proposition 15** Let $k, m, n \in \mathbb{N}$ and $\eta > 0$ be such that $m \leq n^{1/4}$ and $n \geq (12k/\eta)^{12}$, $\mu'$ be a density on $[k]^m$ and $\mu'_m$ be the above corresponding density on $[k]^n$. Then for every set $A \subset [k]^n$,

$$|\mu'_m(A) - \mu(A)| < \eta.$$  

**Proof.** It suffices to consider only $\mu' = \mu'_u$ given, for some $u \in [k]^m$, by $\mu' (\{u\}) = 1$ and $\mu' (\{x\}) = 0$ for $x \neq u$, because any density $\mu'$ on $[k]^m$ is a convex combination of these densities, $\mu' = \sum u \lambda_u \mu'_u$ $(\lambda_u \geq 0, \sum \lambda_u = 1)$, and $\mu'_m = \sum u \lambda_u \mu'_{u,m}$: the general result follows by the triangle inequality.

We fix words $u \in [k]^m$ and $z \in [k]^n$ such that $z$ has between $n/k - n^{2/3}$ and $n/k + n^{2/3}$ occurrences of each letter $j \in [k]$ (by Proposition [8] only very few $z$ are not like this). If $p$ (resp. $q$) is the minimum (resp. maximum) number of occurrences of a letter $j$ in $z$ (clearly $p > m$) then

$$\left( \frac{p - m}{n} \right)^m k^{m-n} \leq \mu'_m(\{z\}) \leq \left( \frac{q}{n-m} \right)^m k^{m-n}.$$
because $\mu_1(\{\sigma\}) = 1/n(n-1)\ldots(n-m+1)$ lies between $n^{-m}$ and $(n-m)^{-m}$, the number of $\sigma$ satisfying $u_i = z_{\sigma(i)}$ for $i \in [m]$ is at least $(p-m+1)^m$ and at most $q^m$ ($\sigma$ determines $x$ and $y$) and $\mu_2(\{y\}) = k^{m-n}$. Since $\mu(\{z\}) = k^{-n}$, $n/k - n^{2/3} \leq p \leq n/k + n^{2/3}$ and $m \leq n^{1/4}$, we have

$$(1 - 2kn^{-1/3})^m < \frac{\mu'_m(\{z\})}{\mu(\{z\})} \leq \left(\frac{1 + kn^{-1/3}}{1 - n^{-3/4}}\right)^m < (1 + 2kn^{-1/3})^m.$$ 

Since $1 - \delta < e^{-\delta} < 1 - \delta/2$ and $1 + \delta < e^\delta < 1 + 2\delta$ if $\delta \in (0, \frac{1}{2})$, we deduce that $\frac{\mu'_m(\{z\})}{\mu(\{z\})}$ lies in $(1 - 4kmn^{-1/3}, 1 + 4kmn^{-1/3})$ provided that $4kmn^{-1/3} < \frac{1}{2}$. This is true as $4kmn^{-1/3} \leq 4kn^{-1/12} \leq \eta/3 < \frac{1}{2}$. So

$$|\mu'_m(\{z\})/\mu(\{z\}) - 1| < \eta/3.$$ 

Let $[k]^n = B \cup C$, where $B$ are the words meeting the condition on occurrences of letters and $C$ are the remaining words. By Proposition 8, $\mu(C) < kn^{-1/3} < \eta/3$. Since $\mu'_m(\{z\}) = (1 - \eta/3)\mu(\{z\})$ for $z \in B$, we have $\mu'_m(B) = (1 - \eta/3)\mu(B) > (1 - \eta/3)^2 > 1 - 2\eta/3$ and $\mu'_m(C) < 2\eta/3$. We conclude that

$$|\mu'_m(A) - \mu(A)| \leq \sum_{z \in A \cap B} |\mu'_m(\{z\}) - \mu(\{z\})| + |\mu'_m(A \cap C) - \mu(A \cap C)|$$

$$< \sum_{z \in A \cap B} \mu(\{z\})(\eta/3) + 2\eta/3$$

$$\leq \eta.$$ 

We apply Proposition 15 to three densities $\mu'$ on $[k]^m$, all invariant to permuting the $m$ coordinates. The definition of $\mu'_m$ then simplifies, as one can put the $\sigma$ with the common $m$-element image $J \subset [n]$ together and sum over the triples $(J, x, y)$. The first application with $\mu'(A) = \mu_B(A)$, where $A \subset [k]^m$ and $B = [k-1]^m$, gives Proposition 9. In the other two applications of Proposition 15, $\mu'$ is the equal-slices density on $[k]^m$, respectively the density given by $\mu'(A) = \nu'(A \cap [k-1]^m)$ where $\nu'$ is the equal-slices density on $[k-1]^m$, and we get the next proposition, for which we introduce the following notation. The truncation $S' \subset S \subset [k]^n$ of an $m$-dimensional subspace $S$ is obtained by forbidding $k$ as the value of $x \in S$ on the free coordinates; $S'1$ corresponds with $[k-1]^m$.

For $A \subset [k]^m$ we define $\nu_S(A)$ as $\nu'(A')$ where $A'$ is the image of $A \cap S'$ in the bijection between $S'$ and $[k-1]^m$ and $\nu'$ is the equal-slices density on $[k-1]^m$.

**Proposition 16** Let $\delta, \eta > 0$ and $k, m, n \in \mathbb{N}$ satisfy $m \leq n^{1/4}, n \geq (12k/\eta)^{12}$ and $A \subset [k]^n$ be a set with $\mu(A) = \delta$. Then the (uniform) averages of the functions $S \mapsto \nu_S(A)$ and $S \mapsto \nu_S'(A)$, over all subspaces $S = S_{J, y}$, $J \subset [n]$, $|J| = m$, and words $y \in [k]^n \setminus J$, are both at least $\delta - \eta$.

To deduce Proposition 8, we need Propositions 11, 12, 14, and 16.
2.3 Derivation of Proposition 5

In this subsection we fix a $k \in \mathbb{N}, k \geq 3$, assume that Theorem 2 holds for $k - 1$ (and every $\delta > 0$) and deduce from this Proposition 5 for $k$. We proceed in three steps. First we show that for any positively $\mu$-dense $A \subset [k]^n$ there is a subspace $S \subset [k]^n$ such that either $A$ gets on $S$ $\nu$-denser (which gives the desired density increment at once), or $A$ gets positively $\nu$-dense on the truncation $S'$ of $S$ (recall that $S'$ has forbidden $k$ on the free coordinates) while losing not too much $\mu$-density on the whole $S$. In the crucial second step we obtain, assuming the second alternative and that $A$ is free of lines, a $\nu$-density increment of $A$ on a $k$-set $D$, an increment large enough to make up for the previous loss. In the third step we convert the $\nu$-density increment of $A$ on $D$ to a $\mu$-density increment.

Proposition 17 Let $\delta, \eta > 0$ and $m, n \in \mathbb{N}$ satisfy $\eta \leq \delta/4$, $m \leq n^{1/4}$, $n \geq (12k/\eta)^{12}$. Then for every set $A \subset [k]^n$ with $\mu(A) = \delta$ there exists an $m$-dimensional subspace $S \subset [k]^n$ such that 1 or 2 holds:

1. $\nu_S(A) \geq \delta + \eta = \mu(A) + \eta$;

2. $\nu_S(A) \geq \delta - 4\eta\delta^{-1} = \mu(A) - 4\eta\delta^{-1}$ and $\nu_{S'}(A) \geq \delta/4$, where $S' \subset S$ is the truncation of $S$ with values on the free coordinates lying in $[k - 1]$.

Proof. We take uniformly the subspaces $S = S_{j,y}$, as described in Proposition 16. Let $M$ be the set of $S$ with $\nu_S(A) < \delta - 4\eta/\delta$ and $N$ be the set of $S$ with $\nu_S(A) < \delta/4$. We assume that 1 does not hold, so $\nu_S(A) < \delta + \eta$ for every $S$, and show that then 2 holds. If $\mu(M) \geq \delta/2$ then the average of $\nu_S(A)$ over $S$ is at most

$$(1 - \delta/2)(\delta + \eta) + (\delta/2)(\delta - 4\eta/\delta) = \delta + (1 - \delta/2)\eta - 2\eta < \delta - \eta,$$

contradicting Proposition 16. So $\mu(M) < \delta/2$. Similarly, if $\mu(N) \geq 1 - \delta/2$ then the average of $\nu_{S'}(A)$ over $S$ is at most

$$\delta/2 + (1 - \delta/2)(\delta/4) < 3\delta/4 \leq \delta - \eta,$$

again contradicting Proposition 16. So $\mu(N) < 1 - \delta/2$. Hence there is a subspace $S = S_{j,y}$ not in $M \cup N$ and 2 holds. \[\square\]

For $x \in [k]^m$ (we have replaced $n$ by $m$ to indicate that we move into $S$) and $j \in [k-1]$, we denote, as before, by $x(j)$ the word obtained from $x$ by changing all $k$s to $j$s. For a set $A_1 \subset [k]^m$ and $j \in [k-1]$, we define

$$C_j = \{x \in [k]^m \mid x(j) \in A_1\} \quad \text{and} \quad C = \bigcap_{j=1}^{k-1} C_j \subset [k]^m.$$

Note that each $C_j$ is $jk$-insensitive and that, crucially, if $A_1$ contains no line then $A_1 \cap C \subset [k-1]^m$. Indeed, if $x \in A_1 \cap C$ had an occurrence of $k$, then $\{x\} \cup \{x(j) \mid j \in [k-1]\}$ would be a line in $[k]^m$ contained in $A_1$. 

17
**Proposition 18** For every \( \delta_1 > 0 \) there is an \( m_0 \in \mathbb{N} \) and a \( \theta > 0 \) such that the following holds. If \( m > m_0 \) and \( A_1 \subset [k]^m \) contains no line, \( \nu(A_1) \geq \delta_1 \) and (measured in \([k-1]^m\)) \( \nu(A_1 \cap [k-1]^m) \geq \delta_1/4 \), then there is a \( k \)-set \( D \subset [k]^m \) satisfying

\[
\nu(A_1 \cap D) \geq \nu(A_1) \nu(D) + \delta_1 \theta/2k \geq \delta_1 \nu(D) + \delta_1 \theta/2k.
\]

**Proof.** Let \( \delta_1 \) be given. Applying Theorem 2 for \( k-1 \) and Proposition 14 we take an \( m_0 \) and a \( \theta > 0 \) such that if \( m > m_0 \) then for every set \( B \subset [k-1]^m \) with \( \nu(B) \geq \delta_1/4 \) the set \( M \subset [k]^m \setminus [k-1]^m \) of lines contained in \( B \) has \( \nu(M) \geq \theta \); we also assume \( m_0 \) so big that \( m > m_0 \) implies \( k/\theta m < \delta_1/2 \). Now let \( m > m_0 \) and \( A_1 \subset [k]^m \) be as stated, with the above defined sets \( C_j \) and \( C \); for convenience we denote \( \delta_1 = \nu(A_1) \). By the assumptions we may take as \( B \) the set \( B = A_1 \cap [k-1]^m \). The lines \( M \) in \( B \) then 1-1 correspond to the words in \( C \setminus [k-1]^m \). Hence \( \nu(C \setminus [k-1]^m) \geq \theta \). We observed above that \( C \setminus [k-1]^m \) is disjoint to \( A_1 \). Therefore using part 1 of Proposition 14 we get

\[
\nu(A_1 \cap C) \leq k/m < \theta \delta_1/2 \leq (\delta_1/2) \nu(C).
\]

For \( j \in [k] \) we set \( D^{(j)} = C_1 \cap \cdots \cap C_{j-1} \cap ([k]^m \setminus C_j) \); \( D^{(1)} = [k]^m \setminus C_1 \) and \( D^{(k)} = C \). Thus \([k]^m = \bigcup_{j=1}^k D^{(j)}\) is a partition. By \( \nu(A_1) = \delta_1 \) and \( \nu(A_1 \cap D^{(k)}) \leq (\delta_1/2) \nu(D^{(k)}) \),

\[
\nu(A_1 \cap (D^{(1)} \cup \cdots \cup D^{(k-1)})) \geq \delta_1 - (\delta_1/2) \nu(D^{(k)}) = \delta_1 (1 - \nu(D^{(k)})) + (\delta_1/2) \nu(D^{(k)}) \geq \delta_1 \nu(D^{(1)} \cup \cdots \cup D^{(k-1)}) + \delta_1 \theta/2.
\]

Thus \( \nu(A_1 \cap D^{(j)}) \geq \delta_1 \nu(D^{(j)}) + \delta_1 \theta/2(k-1) \) for some \( j \in [k-1] \). We set, for this \( j \), \( D_i = C_i \) for \( i < j \), \( D_j = [k]^m \setminus C_j \) and \( D_i = [k]^m \) for \( i > j \). Clearly, each \( D_i \) is \( \delta \)-insensitive. The \( k \)-set \( D = \bigcap_{i=1}^{k-1} D_i = D^{(j)} \) satisfies the displayed inequality. \( \square \)

This is the heart of the proof of Theorem 2, transmuting the inductive assumption on the level \( k-1 \) in a density increment on the level \( k \). The quantities \( m_0 = m_0(\delta_1) \) and \( \theta = \theta(\delta_1) \) come from the validity of Theorem 2 for \( k-1 \). In particular, note that \( \theta \) can be assumed nondecreasing in \( \delta_1 \) (it is obvious from the proof but perhaps is not so clear from the statement).

**Proposition 19** Let \( \beta, \delta_2 \in (0,1), m, r \in \mathbb{N}, \beta \geq kr/m \) and let \( A_2 \subset D \subset [k]^m \) be sets satisfying \( \nu(A_2) \geq \delta_2 \nu(D) + 3\beta \). Then there exists a subspace \( V \subset [k]^m \) with dimension \( r \) such that

\[
\mu(V) A_2) \geq \delta_2 \mu(V) + \beta.
\]

**Proof.** The average of \( \mu(V) A_2) - \delta_2 \mu(V) \) over all subspaces \( V = S_{I,Y} \), with \( J \subset [m], |J| = r \), taken uniformly and \( y \in [k]^m \setminus J \) taken according to \( \nu \)-density, equals \( \mu'(A_2) - \delta_2 \mu'_r(D) \) where \( \mu'_r \) is the density on \([k]^m \) introduced
before Proposition [12] By Proposition [12] and the assumptions this is at least 
\((\nu(A_2) - \beta) - \delta_2(\nu(D) + \beta) = \nu(A_2) - \delta_2\nu(D) - 2\beta \geq \beta\). Thus a subspace 
\(V = S_{J,\nu}\) exists that satisfies the displayed inequality. \(\square\)

We prove Proposition [5] Let \(r \in \mathbb{N}\) and \(\delta > 0\) be given \((k \geq 3\) is fixed\) and 
suppose that \(A \subset [k]^n\) contains no line, \(\mu(A) \geq \delta\) and \(n > n_0\); we specify a bound 
on \(n_0\) at the end. We set \(\delta_1 = \delta/2\) and take the \(m_0 = m_0(\delta_1)\) and \(\theta = \theta(\delta_1)\) 
of Proposition [18] Let \(\eta = \delta^2 \theta / 32k\) and \(m = [n/4]\). Note that \(\eta < \delta/4\), 
\(\delta - 4\eta\delta^{-1} > \delta_1\), \(m \leq n^{1/4}\) and \(\delta/4 > \delta_1/4\). By Proposition [17] applied 
for \(\delta = \mu(A), \eta\) and \(A\), \(n \geq (12k/\eta)^{12}\) then there is an \(m\)-dimensional subspace 
\(S \subset [k]^n\) satisfying alternative 1 or alternative 2. We denote by \(A_1 \subset [k]^m\) the 
image of \(S \cap \mathcal{S}\) in the bijection between \(S\) and \([k]^m\). We first consider alternative 1. So \(\nu(A_1) \geq \mu(A) + \eta\). For \(n\) large enough so that \(\eta m / 3k \geq r\), Proposition [19] 
applied for \(\beta = \eta/3\), \(\delta_2 = \mu(A)\), \(m, r, A_2 = A_1\) and \(D = [k]^m\), provides an 
r-dimensional subspace \(V \subset [k]^m\) on which \(\mu_V(A_1) \geq \mu(A) + \eta/3\). We achieved 
a \(\mu\)-density increment of \(A\) on the \(k\)-set \(D = W\) in the \(r\)-dimensional subspace 
\(W \subset [k]^n\) that is the image of \(V\) in the bijection between \([k]^m\) and \(S\), with the 
increment \(\gamma = \eta/3\). Clearly, \(\mu_W(D) = 1 > \gamma\).

Let \(S \subset [k]^n\) satisfy alternative 2 of Proposition [17] So \(\nu(A_1) \geq \mu(A) - 
4\eta/\mu(A) > \delta_1\) and \(\nu(A_1 \cap [k-1]^m) > \delta/4 > \delta_1/4\). By Proposition [18] applied 
for \(\delta_1\) and \(A_1\), for large enough \(n\) (so that \(m > m_0\)) there is a \(k\)-set \(D_1 \subset [k]^m\) 
for which 
\[
\nu(A_1 \cap D_1) \geq \nu(A_1) \nu(D_1) + \delta_1 \theta / 2k \geq \mu(A) \nu(D_1) - 4\eta \delta^{-1} + \delta_1 \theta / 2k \\
> \mu(A) \nu(D_1) + \delta \theta / 8k.
\]

We apply Proposition [19] with \(\beta = \delta \theta / 24k\), \(\delta_2 = \mu(A)\), \(m, r\), \(A_2 = A_1 \cap D_1\) 
and \(D_1\). For large enough \(n\) (so that \(\beta \geq kr/m\)) it provides an \(r\)-dimensional subspace 
\(V \subset [k]^m\) with \(\mu_V(A_2) \geq \mu(A) \mu_V(D_1) + \beta\). Note that \(D_1 \cap V\) is a \(k\)-set 
in \(V\). We achieved a \(\mu\)-density increment of \(A\) on the \(k\)-set \(D = c D_1 \cap V\) in the 
r-dimensional subspace \(W = c(V) \subset [k]^m\), where \(c\) is the bijection between 
\([k]^m\) and \(S\), with the increment \(\gamma = \beta = \delta \theta / 24k\). Clearly, \(\mu_W(D) \geq \beta = \gamma\).

To summarize and integrate both cases, we see that for given \(r \in \mathbb{N}\), \(\delta \in (0, 1)\) and 
any \(n > n_0\), for any set \(A \subset [k]^n\) containing no line and with \(\mu(A) \geq \delta\) 
there is an \(r\)-dimensional subspace \(W \subset [k]^m\) and a \(k\)-set \(D \subset W\) in \(W\) such 
that \(\mu_W(D) \geq \gamma\) and \(\mu_W(A \cap D) \geq \mu(A) \mu_W(D) + \gamma\) (hence \(\mu_D(A) \geq \mu(A) + \gamma\), 
with the desired density increment \(\gamma = \min(\eta/3, \beta) = \eta/3 = \delta^2 \theta / 96k\). We 
observed above that \(\theta\) is nondecreasing in \(\delta_1 = \delta/2\) and so \(\gamma\) is nondecreasing in 
\(\delta\). Finally, the argument shows that the sufficient \(n_0\) to take is, for \(\eta = \delta^2 \theta / 32k\), 
\[
n_0 = \lceil (12k/\eta)^{12} + (3kr/\eta)^4 + m_0^4 + (24k^2r/\delta \theta)^4 \rceil
\]
where \(m_0 = m_0(\delta/2)\) and \(\theta = \theta(\delta/2)\) are the quantities of Proposition [18] guan-
tanteed by Theorem [2] for \(k - 1\). This concludes the derivation of Proposition [5]

The proof of Theorem [3] the density Hales-Jewett theorem, and consequently of Theorem [1] Szemerédi’s theorem, is complete.
3 Concluding remarks and thoughts

In writing this text we were motivated also by the last sentence of the abstract in [20]: “Our proof is surprisingly simple: indeed, it gives arguably the simplest known proof of Szemerédi’s theorem.” How simple/long is then Polymath’s proof of Szemerédi’s theorem? The article [20] has 44 pages but the proof of Theorem 2 only starts after 32 pages in Section 7 and takes about 8 pages, during which it draws on various results and concepts obtained in the preceding part. The original article of Szemerédi [22] has 46 pages and Furstenberg’s ergodic paper 52. In the book of Moreno and Wagstaff, Jr. [17, Chapter 7], one of the few (if not the only one) monographs or textbooks presenting Szemerédi’s combinatorial proof of his theorem, the proof takes 38 pages, and in the write-up of Tao [23] about 26. An article of Tao [24] of 49 pages gives a proof of Szemerédi’s theorem based on a combination of ergodic methods and the approach of Gowers [8]. Towsner [25] gives a (not quite self-contained) model-theoretic proof of Szemerédi’s theorem on 10 pages. (This list of proofs of Theorem 1 in the literature or on the Internet is far from exhaustive.) Our present write-up, a reshuffled and pruned form of Polymath’s proof [20], demonstrates that it is possible to write down a self-contained combinatorial proof of Szemerédi’s theorem well under 20 pages, which justifies the quoted sentence. Of course, it is even a proof of a stronger theorem, the density Hales–Jewett theorem.

As for the correctness of the proof in [20], we pointed in the remark after the proof of Proposition 11 a probably overlooked lower bound factor in [20, Lemma 8.1], but this is trivial to repair (which we did) and we did not notice in [20] anything more serious than that. In recent years formal proofs of various popular theorems were worked out, for example, for the Prime Number Theorem (Avigad et al. [3], Harrison [12]), Dirichlet’s theorem on primes in arithmetic progression (Harrison [13]) or Jordan’s curve theorem (Hales [11]). Szemerédi’s theorem is known for logical intricacy of its proof—an interesting project in formal proofs may be to produce a formal version for it or, for this matter, for the proof of the density Hales–Jewett theorem.

Many arguments of the proof in [20] as we present them are simple instances of the probabilistic method reasoning (see Alon and Spencer [2]), but we evade words ‘probability’, ‘random’ or ‘randomly’ in our write-up (in [20] the last two words appear more than 90 times). We prefer the terminology of densities instead, to emphasize that we give in all cases explicit definitions and constructions of the densities (i.e., probability measures) used, which is not quite done in [20]. We consider it important, for the sake of rigoroussness of the whole approach, to give these explicit definitions. For illustration consider the identity in part 3 of Proposition 11 for which we gave a verificational proof. The original proof of Polymath [20, pp. 1297–8], more elegant, is free of calculations and is based (in our terminology) on representing the non-degenerate equal-slices density \( \nu \) on \([k]^n\), \( k \leq n \), as a projection of a higher-dimensional density built from uniform densities. Informally (20, p. 1295): a \( \nu \)-random word \( x \) arises by selecting \( n \) points \( q_1, \ldots, q_n \) around a circle in a random order, putting randomly \( k \) delimters \( r_1, \ldots, r_k \) in some \( k \) distinct gaps out of the \( n \) gaps determined by the \( n \)
points \( q_i \), and then reading the positions of the letter \( j \in [k] \) in \( x \) in the indices \( i \) of the points \( q_i \) lying between \( r_j \) and the delimiter clockwisely preceding \( r_j \). Formally (not an exact translation): for \( x \in [k]^n \), let

\[
\mu'(\{x\}) = \sum_{(\pi, B') = x} \mu_1(\{\pi\}) \cdot \mu_2(\{B'\})
\]

where \( \pi \) run through the \( n! \) permutations of \([n]\), \( B' \) run through the \( k\binom{n}{k} \) pointed \( k \)-element subsets of \([n]\) (the pairs \((B, b)\) with \( b \in B \subset [n] \), \(|B| = k \)), \( \mu_i \) are uniform densities and \((\pi, B')\) projects to \([k]^n\) as follows. If \( \pi = a_1 a_2 \ldots a_n \) and \( B' = (B, b) \) with \( B = \{b_1 < b_2 < \cdots < b_k\} \) and \( b = b_t \), we project \((\pi, B')\) to \( x \in [k]^n \) by setting, for \( j = 1, 2, \ldots, k \), \( x_{a_i} = j \) exactly for the terms \( a_i \) in \( \pi \) with \( i \) in the interval \( b_{i+j-1} \leq i < b_{i+j} \), where the indices are taken modulo \( k \) and the interval \( b_k \leq i < b_{k+1} = b_1 \) is \([b_k, n] \cup [1, b_1]\). It is immediate to show that \( \mu' = \tilde{\nu} \).

In conclusion, we want to remark that the use of non-uniform densities \( \nu \) and \( \tilde{\nu} \) on words and their interplay with the uniform density is a really interesting and combinatorially beautiful feature of Polymath’s proof [20].

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