PAPER

Extinction theorem for a temporal gas-plasma boundary

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Abstract

Temporal discontinuity in a medium’s dielectric properties (temporal boundary) is a useful model for considering electromagnetic phenomena in dynamic materials and metamaterials. Here a counterpart of the Ewald–Oseen extinction theorem of classical optics is derived for light scattering at a temporal boundary. In particular, it is shown that the extinction of the initial electromagnetic wave and its replacement by the frequency shifted waves at a temporal gas-plasma boundary can be understood as a result of a superposition of the elementary waves scattered by the suddenly appeared individual free electrons. In contrast to the classical extinction theorem, the extinction at a temporal boundary is closely related to causality and transient effects; the electromagnetic field at any observation point is formed by the elementary waves arriving from a sphere expanding with the speed of light.

1. Introduction

In classical optics, the fundamental Ewald–Oseen extinction theorem explains the reflection and refraction of an electromagnetic wave at a vacuum-dielectric boundary microscopically [1]. In this theorem, the dielectric half-space is considered as a collection of elementary dipoles that are excited by the incident wave and re-emit secondary elementary waves. The interference of the elementary waves extinguishes the incident wave and forms the reflected and refracted waves.

The temporal boundary, i.e., an abrupt change in time of a medium’s dielectric properties, is a time counterpart of a spatial boundary. In the past several years, the concept of the temporal boundary has drawn substantial attention in the context of active photonics and plasmonics, especially based on using time-variant metamaterials and metasurfaces [2–6]. Time-variant media open new possibilities for light control such as broadband frequency shift [7], non-magnetic non-reciprocity [8], time reversal [9], negative extinction [10], and temporal aiming [11].

An efficient way to create a temporal boundary is to increase rapidly the density of free carriers in a material. This approach was implemented by ionization of gases using strong laser and microwave radiation [12–15] and by photogeneration of electron-hole pairs in semiconductors [16–20]. Frequency up-conversion of microwave and terahertz radiations was demonstrated. Recently, photocarrier generation was used for a sudden merging of two metallic meta-atoms into a single one to create a rapidly time-variant metasurface for frequency-converting terahertz radiation [21, 22]. Rapid free-carrier injection into semiconductor microcavities is considered as a new way to creating efficient wavelength converters and modulators for the infrared range [23–25]. In graphene plasmonics, electrical gating allows for modulating the free carrier density in graphene at the gigahertz rate [22, 26] whereas ultrafast optical excitation can provide free carrier generation at the time scales as short as a few tens of femtoseconds [27–29].

Light scattering at a temporal boundary is commonly treated within the framework of macroscopic electrodynamics using appropriate continuity conditions at the boundary. The continuity conditions depend on the medium model being, for example, different for nondispersive dielectrics [30–32], plasmas [33–37], and Lorentz media [36–38], fact not always recognized in the literature [39]. Some subtle issues related to the temporal boundary models were recently discussed in references [40, 41].
In this paper, a counterpart of the classical Ewald–Oseen extinction theorem is derived for temporally dynamic media. The derivation is specifically done for the temporal boundary created by rapid ionization of a dielectric medium with a close to unity dielectric permittivity (gas), i.e., for a gas-plasma temporal boundary. The transformation of an electromagnetic wave at the temporal boundary is considered microscopically, namely, as a result of a superposition of the initial wave and elementary waves scattered by individual plasma particles (free electrons).

According to the macroscopic approach [33–37], the transformation results in the excitation of two frequency upshifted waves, propagating in the opposite directions, and a zero-frequency non-propagating mode. This poses an intriguing question: how can a superposition of elementary waves from the individual electrons born in the field of the initial wave produce the frequency shifted waves? The consideration below reveals the microscopic mechanism of the replacement of the initial wave by the frequency shifted waves. Generally speaking, it is closely related to causality and transient nature of the elementary waves.

2. Macroscopic theory of light scattering at a temporal gas-plasma boundary

First of all, let us summarize the main results of an electromagnetic wave transformation at a temporal gas-plasma boundary in the framework of the macroscopic approach [33–37]. Initially, $t < 0$, a plane wave at frequency $\omega_0$ with the electric $E = (E_x, 0, 0)$ and magnetic $B = (0, B_y, 0)$ fields given by

$$E_x = B_y = B_0 e^{i\omega_0 t - ik_0 z},$$  \hspace{1cm} (1)

where $k_0 = \omega_0/c$ ($c$ is the speed of light), propagates along the $z$ axis in a gas with a close to unity refractive index. (Throughout this paper the usual convention of linear optics is adopted that the true physical quantities are the real components of their complex counterparts.) At $t = 0$, the gas is instantaneously ionized by an external source, i.e., converted to plasma with a uniform electron density $N$. As a result of the flash ionization three new modes are excited. Due to the translational symmetry of the system the modes’ nonzero field components $(E_x, B_y)$ have the same dependence on the spatial variable $z$ as the initial wave and can be written as

$$B_y = B(t) e^{-ik_0 z}, \quad E_x = E(t) e^{-ik_0 z}, \quad t > 0,$$  \hspace{1cm} (2)

with

$$B(t) = B_+ e^{i\omega t} + B_- e^{-i\omega t} + B_0,$$  \hspace{1cm} (3)

$$E(t) = \frac{\omega}{ck_0} B_+ e^{i\omega t} - \frac{\omega}{ck_0} B_- e^{-i\omega t}.$$  \hspace{1cm} (4)

In equation (4), the amplitudes are related to those in equation (3) by using Faraday’s law $\nabla \times E = -\dot{B}/c$ (the overdot denotes time derivative).

The modes at frequencies $\pm \omega$ are the transmitted and reflected electromagnetic waves propagating, respectively, in the $\pm z$ directions. The dispersion relation for electromagnetic waves in plasma $c^2 k^2 = \omega^2 \varepsilon(\omega)$ with $\varepsilon(\omega) = 1 - \Omega_p^2/\omega^2$ and the wave number conservation $k = k_0$ determine the frequencies of the waves as

$$\omega^2 = \omega_0^2 + \Omega_p^2,$$  \hspace{1cm} (5)

where $\Omega_p = (4\pi e^2 N/m)^{1/2}$ is the plasma frequency (with $e$ and $m$ the electron charge and mass). According to equation (5) the waves are frequency upshifted.

The last term on the right-hand side of equation (3) is the so called free-streaming mode. It has no electric field and consists of a static magnetic field and dc current, which is related to the magnetic field by Ampere’s law $\nabla \times B = 4\pi j殊/c$. This mode is non-propagating. The appearance of the free streaming mode is related to the initial conditions at $t = 0$ [33–37]. The electrons created at $t = 0$ with zero quiver velocity are driven by the electric field (equation (4)) and acquire a steady drift velocity along with the quiver velocity [33–37].

The amplitudes in equation (3) can be found by using the continuity of $E_x$, $B_y$, and $j_x$ at $t = 0$ [35–37] or, more conveniently, using continuity of $E_x$ instead of $j_x$ [36]. The amplitudes are given by

$$B_+ = B_0 \frac{\omega_0}{2\omega} \left( \frac{\omega_0}{\omega} \pm 1 \right), \quad B_- = B_0 \frac{\Omega_p^2}{\omega^2}.$$  \hspace{1cm} (6)

The energy density in each mode incorporates both the field energy and the electron mechanical energy. The total energy density remains constant at the temporal gas-plasma boundary: $W_+ + W_- + W_0 = W_0$ [35–37]. The free-streaming mode can consume a substantial part of the initial wave energy: $W_0$ approaches
3. Microscopic theory and extinction theorem

Let us now consider the transformation of the wave (1) at the temporal gas-plasma boundary microscopically. In the microscopic perspective, the reflected, transmitted, and free streaming modes (2) result from the scattering of the initial wave (1) by the free electrons appeared at \( t = 0 \). A remarkable fact is that the superposition of the elementary waves emitted by the electrons, which are excited by the initial wave at frequency \( \omega_0 \), not only extinguishes this wave but also produces the modes at other frequencies, i.e., the frequency upshifted waves (at \( \pm \omega \)) and the free-streaming mode at zero frequency.

According to the microscopic approach, at any point \( \mathbf{r} \) the total magnetic field \( \mathbf{B}(\mathbf{r}, t) \) is the sum of the initial wave (1) and the superposition of elementary fields \( \mathbf{b}(\mathbf{r}, \mathbf{r}', t) \) produced at \( \mathbf{r} \) by the individual electrons at different points \( \mathbf{r}' \) (figure 1):

\[
\mathbf{B}(\mathbf{r}, t) = \hat{\mathbf{y}} B_0 \, e^{i \Theta} - \hat{i} k_0 z + N \oint\oint\oint_{-\infty}^{\infty} dx' \, dy' \, dz' \mathbf{b}(\mathbf{r}, \mathbf{r}', t)
\]

\( (N \text{ was pulled out of the integral as a constant}). \) The Liénard–Weichert field \( \mathbf{b}(\mathbf{r}, \mathbf{r}', t) \) of a nonrelativistically moving electron can be written as [42]

\[
\mathbf{b}(\mathbf{r}, \mathbf{r}', t) = -e \left[ \frac{\mathbf{v} \mathbf{n}}{c^2 R} + \frac{[\mathbf{v} \mathbf{n}]}{c R} \right] \Theta(t'),
\]

where the electron velocity \( \mathbf{v} \) and acceleration \( \mathbf{v}' \) are taken at \( t' = t - R/c, R = | \mathbf{r} - \mathbf{r}' | \) (figure 1), \( \mathbf{n} \) is the unit vector in the direction of \( \mathbf{r} - \mathbf{r}' \), and \( \Theta(t') \) is the Heaviside step function, i.e., \( \Theta(t') = 0 \) for \( t' < 0 \) and \( \Theta(t') = 1 \) for \( t' > 0 \). The electron motion is driven by the total electric field \( \mathbf{E}(\mathbf{r}, t) \), which is related to \( \mathbf{B}(\mathbf{r}, t) \) by the Maxwell equations, according to the equation

\[
\mathbf{m} \ddot{\mathbf{v}} = -e \mathbf{E}(\mathbf{r}, t)
\]

with zero initial velocity \( \mathbf{v}(t=0) = 0 \). The magnetic field contribution to the Lorentz force on the right-hand side of equation (9) is neglected because it is small (of the order of \( |\mathbf{v}|/c \)) in comparison with the electric field contribution for driving fields of non-relativistic strength, which are assumed in linear optics.

It is instructive to emphasize the differences between equations (7)–(9) and the equations, which are typically used for deriving the classical (spatial) Ewald–Oseen theorem. For comparison, we will refer to the derivation given in reference [43] for the normal incidence of a plane monochromatic wave from vacuum onto a dielectric half-space, the case being the closest spatial counterpart of the temporal boundary considered here. The typical equations [43] include (i) an integral, similar to equation (7), that sums the elementary electric fields produced by polarizable molecules (dipoles), (ii) a formula for the electric field of a dipole, instead of equation (8), and (iii) a direct proportion between the dipole moment and the electric field strength through the molecular polarizability, instead of equation (9). Using here equation (8) for the magnetic field of an electron, rather than its analog for the electric field [43], first, simplifies the derivation due to a smaller number of terms in equation (8) than in its electric analog [42, 43] and, second, allows to inherently account for the purely magnetic free-streaming mode (see below). Further, plasma is an inherently dispersive medium, and equation (9) completely accounts for temporal dispersion. The spatial counterpart of temporal dispersion is spatial dispersion, which is, however, neglected in the classical Ewald–Oseen theorem. Finally, the most distinctive difference between the extinction theorem derivations for the temporal and spatial boundaries is in the integration limits in equation (7) and in its spatial counterpart (equation (28) in reference [43]). In the spatial case [43], the integration is performed over the dielectric half-space and, therefore, the integration limits are constants. Here, on the contrary, due to the presence of a time-dependent step function in equation (8) the integration limits in equation (7) become time-dependent and integration goes over an expanding spatial region (see below). Physically, this is related to the causality principle, which has no spatial counterpart.

To proceed, several simplifications can be made by taking into account the symmetry of the system. For a uniform spatial distribution of the electrons (constant electron density \( N \)), the total fields \( \mathbf{E}(\mathbf{r}, t) \) and \( \mathbf{B}(\mathbf{r}, t) \) have the same directions and the same spatial dependence as the initial wave (1), i.e., the fields can be presented in the form of equation (2). (In particular, figure 2 illustrates formation of a \( y \)-directed magnetic field at any point in the \( x, z \) plane by superposition of elementary fields from two electrons placed symmetrically with respect to this plane. A similar consideration can be made for any plane parallel to the \( x, z \) plane.) According to equation (9), the electron acceleration and velocity have only \( x \) components, which
can be written as $\dot{v}_x = \dot{v}(t) \exp(-i k_0 z)$ and $v_x = v(t) \exp(-i k_0 z)$ (the latter follows from the former by integrating over time from 0 to $t$ with the initial condition $v(0) = 0$). The projection of the elementary magnetic field $b$ on the direction of the total field $B$ (y direction) can be found by using the relation $\hat{n} x = (z' - z)/R$. Thus, substituting equation (8) to equation (7) and dividing it by $\exp(-i k_0 z)$ yields

$$B(t) = B_0 e^{i \omega_0 t} e^{i eN \int_0^t dt' \frac{\dot{v}}{\rho} \frac{\dot{v}}{c} \frac{z - z'}{R^2} e^{i k_0 (z - z')} \frac{v}{R} + \frac{\dot{v}}{c} \frac{1}{e^i k_0 \rho \cos \theta} + e^{i k_0 \rho} + \frac{e^{-i k_0 \rho} - e^{i k_0 \rho}}{i k_0 \rho}} \Theta(t') .$$

(10)

It is convenient to convert to spherical coordinates using $x' = x = \rho \sin \theta \cos \varphi$, $y' = y = \rho \sin \theta \sin \varphi$, $z' - z = \rho \cos \theta$, obtaining

$$B(t) = B_0 e^{i \omega_0 t} e^{i eN \int_0^t dt' \frac{\dot{v}}{\rho} \frac{\dot{v}}{c} \frac{z - z'}{R^2} e^{i k_0 (z - z')} \frac{v}{R} + \frac{\dot{v}}{c} \frac{1}{e^i k_0 \rho \cos \theta} + e^{i k_0 \rho} + \frac{e^{-i k_0 \rho} - e^{i k_0 \rho}}{i k_0 \rho}} \cos \theta \int_0^{e^\pi} d\varphi ,$$

(11)

where $v$ and $\dot{v}$ are taken at $t - \rho/c$. In equation (11) and all subsequent integrals, the upper limit $ct$ is positive ($t > 0$). For $t < 0$, the integral contribution to $B(t)$ vanishes due to the Heaviside function in equation (10) and only the initial wave remains. After performing the trivial integration over $\varphi$ and integration over $\theta$ by parts, equation (11) takes the form

$$B(t) = B_0 e^{i \omega_0 t} e^{i eN \int_0^t dt' \frac{\dot{v}}{\rho} \frac{\dot{v}}{c} \frac{z - z'}{R^2} e^{i k_0 (z - z')} \frac{v}{R} + \frac{\dot{v}}{c} \frac{1}{e^i k_0 \rho \cos \theta} + e^{i k_0 \rho} + \frac{e^{-i k_0 \rho} - e^{i k_0 \rho}}{i k_0 \rho}} \cos \theta \int_0^{e^\pi} d\varphi ,$$

(12)
Importantly, in equation (12) the integration on ρ is limited by ct. That is, the integral is taken over the expanding causal sphere of radius ct, so that new and new electrons contribute to the field at the observation point. The arrival of elementary waves from the electrons on the expanding sphere impart nonstationarity to the wave summation. This nonstationarity makes the main difference between the temporal and spatial boundaries; in the latter case all dipoles from a dielectric half-space contribute continuously to the field at the observation point [43].

Representing \( \dot{v}(t - \rho/c) \) as \( \dot{v} = -c \partial v / \partial \rho \) and integrating the terms with \( \partial v / \partial \rho \) by parts, equation (12) can be simplified to the form

\[
B(t) = B_0 e^{i\omega t} - 2\pi eN \int_0^t \rho (e^{-i\omega \rho} - e^{i\omega \rho}) v(t - \rho/c) \, d\rho.
\]  

(13)

To close the system of equations, equation (13) is supplemented by equations

\[
\dot{v}(t) = -c E(t)/m, \quad \dot{B}(t) = i\omega B(t)
\]  

(14)

that follow from equation (9) and the Maxwell equation \( \nabla \times E = -\dot{B}/c \). By eliminating \( E \) from equation (14) and integrating the resultant relation of \( \dot{v} \) and \( \dot{B} \) over time, one can obtain

\[
v(t) = i\epsilon [B(t) - B_0]/(m\omega_0).
\]  

(15)

Finally, substitution of equation (15) into equation (13) yields the integral equation

\[
B(t) = B_0 e^{i\omega t} + \frac{\Omega_p^2}{2\omega_0 c} \int_0^t \rho (e^{-i\omega \rho} - e^{i\omega \rho}) \left[ B(t - \rho/c) - B_0 \right],
\]  

(16)

where the second (integral) term on the right-hand side represents the contribution from electrons to the total magnetic field.

To solve equation (16), we differentiate it twice with respect to time using \( \dot{B}(t - \rho/c) = -c \partial \dot{B}/\partial \rho \) and integration by parts, and then eliminate the remaining integral by substituting it from equation (16) (appendix A). This results in the following differential equation:

\[
\ddot{B} + \omega^2 B = \Omega_p^2 B_0,
\]  

(17)

where \( \omega^2 = \omega_0^2 + \Omega_p^2 \) coincides with equation (5). The general solution to equation (17) is given by equation (3). The static term in equation (3) is a particular solution of equation (17) and equals \( B_s = B_0 \Omega_p^2 / \omega^2 \), which agrees with \( B_s \) in equation (6). The oscillatory terms in equation (3) are the general solutions of the complementary homogeneous equation. Their amplitudes \( B_{\pm} \) can be found by using the original integral equation (16). Substituting ansatz (3) to equation (16) and performing the integrals (appendix B) gives

\[
\sum_{\pm} B_{\pm} e^{\pm i\omega t} + B_s = B_0 e^{i\omega t} + \frac{\Omega_p^2}{2\omega_0^2} \sum_{\pm} B_{\pm} \left( \frac{2\omega_0^2 e^{\pm i\omega t}}{\omega^2 - \omega_0^2} + \frac{\omega_0 e^{-i\omega t}}{\omega_0^2 - \omega^2} + \frac{\omega_0 e^{i\omega t}}{\omega_0^2 - \omega^2} \right)
\]  

\[
- \frac{\Omega_p^2}{2\omega_0^2} B_0 \left( e^{-i\omega t} + e^{i\omega t} - 2 \right),
\]  

(18)

where summation over \( \pm \) is implied. It should be emphasized that the result of integration in equation (18) contains terms at frequencies \( \pm \omega_0 \) although only the fields at frequencies \( \pm \omega \) and zero frequency were substituted to the integral in equation (16). This is explained by the time-dependent upper limit of the integral in equation (16). Physically, it is related to the nonstationarity of the elementary wave summation from the electrons inside the expanding causal sphere.

In equation (18), the \( e^{\pm i\omega t} \) terms on both sides balance each other, as well as the static terms. The remaining \( e^{\pm i\omega t} \) terms must have zero coefficients. This gives

\[
B_0 + \frac{\Omega_p^2}{2\omega_0^2} \left( \frac{B_+}{\omega_0 - \omega} + \frac{B_-}{\omega_0 + \omega} - \frac{\omega_0 B_0}{\omega^2} \right) = 0,
\]  

(19)

\[
\frac{B_+}{\omega_0 + \omega} + \frac{B_-}{\omega_0 - \omega} - \frac{\omega_0 B_0}{\omega^2} = 0.
\]  

(20)

Solving the system of equations (19) and (20) determines the amplitudes \( B_{\pm} \), which coincide with \( B_{\pm} \) in equation (6). Thus, the solution of the integral equation (16) is given by equations (3), (5), and (6).

Remarkably, it turns out from equation (19) that the initial wave \( B_0 e^{i\omega_0 t} \) is exactly canceled by the \( e^{i\omega_0 t} \) integral contributions, i.e., equation (19) expresses the extinction of the initial wave by part of the field.
Figure 3. (a) Extinction of the incident wave \((k_0, B_0)\) behind the spatial vacuum-dielectric boundary by the wave with the same wavenumber \(k_0\) and negative amplitude \(-B_0\) is accompanied by creating a single (transmitted) wave in the dielectric (there is also a reflected wave in vacuum). (b) Extinction of the initial wave \((\omega_0, B_0)\) after the temporal gas-plasma boundary by the wave \((\omega_0, -B_0)\) is accompanied by creating three waves in the plasma.

generated by the electrons. Thus, equation (19) is a counterpart of the Ewald–Oseen extinction theorem for a temporal gas-plasma boundary and the main result of the paper. We also showed that the remaining part of the electron field comprises two frequency upshifted waves and static magnetic field (equation (3)), which agree with those obtained within the macroscopic approach (section 2).

The extinction of the initial wave after the temporal boundary has a more complicated character than the extinction of the incident wave behind the spatial boundary in the classical Ewald–Oseen theorem (figure 3). In the conventional spatial case (figure 3(a)), elementary waves from the dipoles excited by the incident wave in the dielectric form two new waves in the dielectric half-space, i.e., the wave with the vacuum wavenumber \(k_0\) that extinguishes the incident wave and the transmitted (refracted) wave with a different wavenumber \(nk_0\) \((n\) is the dielectric refractive index) [43]. All waves have the same frequency. In the temporal case (figure 3(b)), elementary waves emitted by the electrons born at \(t = 0\) form four new waves, i.e., the wave with the initial wave frequency \(\omega_0\) that extinguishes the initial wave, transmitted and reflected waves with frequencies \(\pm\omega\) given by equation (5), and zero frequency mode.

The zero frequency mode is not excited if a dielectric medium is created at the temporal boundary instead of plasma (see, for example, figure 1 in reference [44] for a time-varying nondispersive dielectric). However, in the case of a temporally modulated dielectric medium other modes appear due to the inherent material dispersion, which can be accounted for by using the models of Debye or Lorentz media [40, 41]. To derive the extinction theorem for temporal boundaries in such media, a separate consideration is required.

4. Conclusion

To conclude, a temporal counterpart of the Ewald–Oseen extinction theorem of classical optics was derived for an electromagnetic wave scattering at a temporal gas-plasma boundary. The temporal extinction theorem is expressed mathematically by equation (19). The derivation of the theorem, in particular equation (12), reveals the nonstationary character of the microscopic mechanism of the wave scattering. In the microscopic perspective, scattering of an electromagnetic wave at a temporal gas-plasma boundary can be viewed as a result of a superposition of the initial wave and elementary waves scattered by individual suddenly appeared electrons. Despite all the electrons appear in the field of the initial wave at frequency \(\omega_0\), the superposition produces frequency upshifted waves and zero frequency free-streaming mode. This results from the inherently transient nature of the system. Namely, at any point, new and new elementary waves emitted by the electrons on an expanding sphere of the \(ct\) radius permanently arrive and contribute to the total electromagnetic field at this point.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Reduction of the integral equation (16) to the differential form (17)

Differentiating equation (16) with respect to time and using $B(0) = B_0$, we obtain

$$
\dot{B}(t) = i\omega_0 B_0 e^{i\omega_0 t} + \frac{\Omega_p^2}{2k\omega_0 c} \int_0^t \frac{d\rho}{\rho} B(t - \frac{\rho}{c}) (e^{-ik_0\rho} - e^{ik_0\rho}).
$$

(A.1)

Substituting $\dot{B}(t - \rho/c) = -c\partial B(t - \rho/c)/\partial \rho$ into equation (A.1) and integrating by parts, equation (A.1) is reduced to

$$
\dot{B}(t) = i\omega_0 B_0 e^{i\omega_0 t} - \frac{\Omega_p^2}{2k\omega_0 c} B_0 (e^{-ik_0\rho} - e^{ik_0\rho}) - \frac{\Omega_p^2}{2c} \int_0^t d\rho \frac{\partial B(t - \rho/c)}{\partial \rho} (e^{-ik_0\rho} + e^{ik_0\rho}).
$$

(A.2)

Differentiating equation (A.2) with respect to time and using again $\dot{B}(t - \rho/c) = -c\partial B(t - \rho/c)/\partial \rho$, we obtain

$$
\dot{B}(t) = -i\omega_0^2 B_0 e^{ik_0\rho} + \frac{\Omega_p^2}{2} \int_0^t d\rho \frac{\partial B(t - \rho/c)}{\partial \rho} (e^{-ik_0\rho} + e^{ik_0\rho}).
$$

(A.3)

Integration by parts yields

$$
\dot{B}(t) = -i\omega_0^2 B_0 e^{ik_0\rho} + \frac{\Omega_p^2}{2} B_0 (e^{-ik_0\rho} + e^{ik_0\rho}) - \Omega_p^2 B(t) + \frac{\Omega_p^2}{2c} \int_0^t d\rho \frac{\partial B(t - \rho/c)}{\partial \rho} (e^{-ik_0\rho} - e^{ik_0\rho}).
$$

(A.4)

Eliminating the remaining integral in equation (A.4) by substituting it from equation (16), we finally arrive at the differential equation (17).

Appendix B. Derivation of equation (18)

Substituting ansatz (3) to equation (16) gives

$$
\sum_\pm B_\pm e^{\pm ik_0 t} + B_c = B_0 e^{i\omega_0 t} + \frac{\Omega_p^2}{2k\omega_0 c} \sum_\pm B_\pm e^{\pm ik_0 t} \int_0^t \frac{d\rho}{\rho} e^{\pm ik_0\rho} (e^{-ik_0\rho} - e^{ik_0\rho})
$$

$$
+ \frac{\Omega_p^2}{2k\omega_0 c} (B_c - B_0) \int_0^t \frac{d\rho}{\rho} (e^{-ik_0\rho} - e^{ik_0\rho}),
$$

(B.1)

where $k = \omega/c$. Performing the integrals, we arrive at equation (18).

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