Effects of the free evolution in the Arthurs–Kelly model of simultaneous measurement and in the retrodictive predictions of the Heisenberg uncertainty relations

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Abstract Since its conception, the simultaneous measurement approach of Arthurs and Kelly has been a significant tool for the better understanding of the measurement process in quantum mechanics. This model considers a strong interaction Hamiltonian by discarding the free evolution part. In this work, we study the effect of the full dynamics—taking into account the free Hamiltonian—on the optimal limits of retrodictive and predictive accuracy of the simultaneous measurement process of position and momentum observables. To do that, we consider a minimum uncertainty Gaussian state as the system under inspection, which allows to carry out an optimal simultaneous measurement. We show that the inclusion of the free Hamiltonian induces a spreading on the probability density of the measurement setting, which increases the value of the product of the variances of the so-called retrodictive and predictive error operators, this is equivalent to a reduction in the accuracy of the measurement.

1 Introduction

Measurements in physics constitute the bridge between the theory and its predictions. In this context, the limitations on the measurement precision in quantum mechanics are given by the Heisenberg uncertainty relations and the measurement features of quantum mechanics. In classical mechanics, the act of measuring encompasses the comparison of a property of a physical object with another object acting as a meter; thus, the information obtained from it constitutes an accurate result of the measurement. However, when extending this process to quantum mechanics, the resulting theoretical models, experimentally confirmed, point to different processes with different consequences; for example, in an ideal measurement an entangling system–meter interaction which command the process is required [1].

On the other hand, it is well known that in the quantum world, incompatible observables require their own measurement arrangement which are, generally speaking, incompatible. Some consequences of this situation are that in quantum mechanics the notion of exact

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The first person to challenge the classical deterministic conceptions was Heisenberg \[2\], who heuristically proposes an uncertainty relation \(\delta x \delta p \approx \hbar\). However, the widely celebrated uncertainty relation that appears in almost all textbooks of quantum mechanics, i.e.

\[
\delta_q \delta_p \geq \frac{\hbar}{2},
\]  

(1.1)

was mathematically derived by Kennard \[3\] who gave a precise meaning to \(\delta_q\) and \(\delta_p\) as standard deviations.

Today, the quantum physicists community has reached the consensus that the inequality given by Eq. (1.1) is not related to the simultaneous measurement of complementary observables; instead, it establishes a restriction on quantum state preparation \[4–9\]. On the other hand, the first quantum mechanical description where two measuring devices interact with a pure quantum system to simultaneous measure its position and momentum observables was proposed by Arthurs and Kelly \[10\]. They conceived their model as a generalisation of the Von Neumann measurement process \[11\], where faithful tracking of one single observable is achieved \[12\]. Notably, they demonstrated the fact that a joint measurement process inevitably entails a measurement-induced noise that increases the lower bound of Kennard’s uncertainty relation by a factor of \(\hbar/2\), that is,

\[
\delta_q \delta_p \geq \hbar,
\]  

(1.2)

where now the standard deviations characterize the widths of the probability distributions of the pointer readings of a measuring device. Notably, the Arthurs–Kelly model has led to what is known as pointer-based simultaneous measurements of conjugate observables \[13\]; besides, it has been considered in interesting scopes as open systems \[13\] and decoherence process \[14\], weak measurements \[15\], timing in the measurement process \[6\], entanglement generation \[1,16\], entanglement swapping, and remote tomography and noiseless quantum tracking of conjugate observables \[12\]. Additionally, there are many experimental accomplishments of simultaneous measurement, see for example references \[17–19\].

Furthermore, in the Arthurs–Kelly model \[20–22\] it is assumed that only the interaction Hamiltonian rules the dynamics of the measurement process, neglecting the contribution of the free energies. Hence, this approach focuses on the regime where the measuring instruments and the system under investigation are strongly coupled to each other, which is an ideal situation. Besides, using this approach, previous works have pointed out a necessary distinction between the predictive and retrodictive aspects of a joint measurement \[22–25\]; building on these ideas, in references \[8,26,27\] were proposed two kinds of error observables to capture the full content of the experimental accuracy concept in the Arthurs–Kelly measurement process. Recent advances in the retrodictive prediction of two incompatible observables, with a precision below the Standard Quantum Limit, were given in references \[28,29\] using the past quantum state theory \[30\].

In recent years, the Arthurs–Kelly model (AKM) of simultaneous measurement, i.e. using the strong coupling regime where the free evolution Hamiltonian is disregarded, has been still used to analyse different characteristics of experimental and theoretical arrangements of simultaneous measurement. For example, Hacohen-Gourgy et al. \[31\] have carried out a simultaneous measurement of non-commuting observables applying single quadrature measurements using an effective Hamiltonian that resembles the Arthurs–Kelly model, i.e. without the free evolution components. Additionally, Jian and Watanabe \[32\] also use the AKM by generalising it for arbitrary operators to obtain a master equation for continuous measure-
ment and for state preparation, notably these authors conclude that the generalised AKM is only valid for weak measurement, i.e. for small values of the coupling constant. Also, remote tomographic reconstruction and teleportations have been proposed by Roy et al. [33] using the AKM. The influence of entangled measuring apparatuses on the AKM model is studied in references [34,35], for an alternative approach about how entanglement resources could improve the simultaneous measurement see reference [29].

Additionally, the AKM model of simultaneous measurement has been extended to other scenarios. For example, it was used to investigate a cloning machine optimised to simultaneously measure non-commuting observables [36]; similarly, the use of quantum cloning for achieving simultaneous measurement was studied in [37]. It was also proposed by D’Ariano et al. to simultaneously measure the direction of the spin [38], see also Martens and de Muynick [39]. The AKM has been also used for modelling macroscopic measurements [40]. In the experimental realm of the new scenarios, there has been recent interesting realisations of the simultaneous measurement of incompatible observables, in references [41–43] the joint measurability of non-commuting polarisation observables was theoretically analysed through the error statistics, and the nonclassical correlations between measurement errors were experimentally measured. As a highly important result, it was shown that to enforce restrictions on experimental probabilities (to be positive) define a bound on non-local correlations [43].

On the other hand, the simultaneous measurement of incompatible observables was experimentally implemented in a pair of polarisation components of a single photon avoiding the entanglement with a measurement apparatus [44]; in this reference [44], the authors show a simultaneous measurement technique that avoids coupling the quantum system to an ancillary system, showing an alternative to the usual Arthurs–Kelly model.

In this work, we study the effects of the full dynamic in the simultaneous measurement of position and momentum observables; this is done by using a minimum uncertainty Gaussian state as the state under inspection, which allows measurements with optimal accuracy. The analysis of the dynamics of the system is carried out in the Schrödinger picture by employing the time evolution operator method [45]. We found that some of the effects of the free terms of the Hamiltonian increase the noise in the measurement and this effect is done by increasing the lower bound in the uncertainty relation. We also show that the uncertainty relation that we found reach the uncertainty relation found by Arthurs and Kelly in the limit of strong coupling.

This contribution uses \( \hbar = 1 \) and is organized as follows: in Sect. 2, we define the measurement configuration. Besides, we employ the time evolution operator method to obtain the state describing the system’s dynamics; by using this result, we compute the variances of the probability distributions of the pointers and system just after the time of the measurement. Building on these results, in Sect. 3 we show how the full dynamics affect the retrodictive and predictive aspects of accuracy in an optimal simultaneous measurement. The paper closes with the conclusions in Sect. 4.

## 2 The measurement process

This section describes the scheme of simultaneous measurement for the position and momentum observables following the model used by Arthurs and Kelly [10]. We consider the stage known as pre-measurement [46], where a correlation between the measuring instrument and the system arises; then, we obtain the variances of the probability distributions of the pointers readings at the time just after the simultaneous read-out.
2.1 The measurement configuration and its dynamics

Except for the addition of the free Hamiltonian, the set-up that we will consider is the same that was analysed by Arthurs and Kelly [10]; that is to say, there is a measuring device formed by two measuring systems coupled to a third single system that we are interested in. Particularly, we are interested in measuring two non-commuting observables $\hat{x}_3$ and $\hat{p}_3$ of the latter system [10]. The Hamiltonian for this set-up is given by:

$$\hat{H} = \hat{H}_{\text{int}} + \hat{H}_{\text{free}},$$

(2.1)

where the interaction Hamiltonian $\hat{H}_{\text{int}}$ is the original of Arthurs and Kelly

$$\hat{H}_{\text{int}} = \kappa (\hat{x}_3 \hat{p}_1 + \hat{p}_3 \hat{p}_2),$$

(2.2)

where $\kappa$ is a positive constant that governs the coupling strength between the pointer and its linked variable; the measuring systems are represented by $\hat{p}_1$ and $\hat{p}_2$. Further, we take into account the free dynamics that was not considered by Arthurs and Kelly, which is given by

$$\hat{H}_{\text{free}} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + \frac{\hat{p}_3^2}{2m_3}.$$  

(2.3)

The initial state of each system is given by the following wave function:

$$\langle x_i | \phi_i \rangle = \phi_i(x_i) = \sqrt{S_i} \exp \left[ -\frac{(x_i S_i)^2}{2} \right],$$

(2.4)

where $i$ takes the values of $i = 1, 2, 3$, and it will label the variables of the pointer 1, the pointer 2, and the system, respectively. The variances of the probability distributions associated with the wave function, Eq. (2.4), are given by $\sigma_i^2 = (S_i)^{-2}$, with $S_1 = (2/b)^{1/2}$, $S_2 = (2b)^{3/2}$ and $S_3 = 1/(2^{1/2} \delta_q)$, then the resolution of the pointers can be balanced through the manipulation of the $b$ parameter; hence, it is possible to adjust and tune-up the measurement accuracy of any of the two conjugate observables [6,21]; moreover, it is clear to see both that the position probability distribution of the Gaussian system has a variance of $\delta_q^2$ and the conjugate distribution has a variance of $\delta_p^2 = 1/4 \delta_q^2$.

The Hilbert space of the whole system is represented by the tensor product of the individual Hilbert spaces $\mathcal{H} = \otimes_{i=1}^3 \mathcal{H}_i$, where every Hilbert space has an infinite dimensional structure; then, the initial quantum state is

$$\psi(x_1, x_2, x_3, t = 0) = \phi_1(x_1) \phi_2(x_2) \phi_3(x_3).$$

(2.5)

This wave function characterises the complete system before the measurement process, it is worth of noting that it is a pure and separable state.

Interestingly, the wave functions given by Eq. (2.4) represent a particular class of minimum uncertainty states known as squeezed vacuum states with squeezing factor $S_i$ [47]. Bearing this in mind, notice that a full Gaussian minimum uncertainty configuration characterises the whole measurement setting. This representation is suitable due to the versatility of Gaussian states and their proliferation in practically any field of quantum mechanics. Moreover, these states are easily prepared and can be controlled through current technology [48–50], allowing their easy implementation in quantum communication protocols [49,51–53], and they allow to reach optimal limits of accuracy in a simultaneous measurement of position and momentum observables.

Hence, the pointers of the measurement devices interact with the system within a given time for extracting the information of the non-commuting observables. Although the laws
of quantum mechanics rule out for performing simultaneous measurement of the position and momentum observables with arbitrary accuracy; however, it is still possible to obtain an estimate of them by inducing a correlation with the observables of the pointers.

The Hamiltonian of Eq. (2.1) describes the dynamics of the measurement process; notice that, unlike previous works [10,20–22], Eq. (2.1) helps us to consider both situations: the strong and the weak coupling. It is worthy of mention that the scenario of strong coupling, which arises when the free Hamiltonian is discarded [10,20–22], cannot be sustained in a physical realistic measurement scenario since a complete infinite coupling is an ideal situation, see Eq. (2.17) below.

We obtain the temporal evolution of the initial state, Eq. (2.5), in the Schrödinger picture through the time evolution operator method [45], that is,

$$\Psi(x_1, x_2, x_3, t) = e^{-i\hat{H}t} \psi(x_1, x_2, x_3, t = 0).$$  \hspace{1cm} (2.6)

Because the Hamiltonian is time-independent, the unitary operator $e^{-i\hat{H}t}$ can be factorized as (see “Appendix A’’)

$$e^{-i\hat{H}t} = e^{\Delta x_1 \hat{P}_1^2} e^{-\frac{i}{2m_2} \hat{P}_2^2} e^{-\frac{i}{2m_3} \hat{P}_3^2} e^{-\frac{it}{\tau} \hat{P}_3 \hat{P}_2} e^{-it \hat{P}_3} e^{-\frac{i}{2\kappa_t} \hat{P}_3^2} e^{-\frac{it}{\tau} \hat{P}_3 \hat{P}_2},$$  \hspace{1cm} (2.7)

The application of the time evolution operator to the initial wave function is shown in “Appendix C”; here, we write down the result in more compact form as:

$$\Psi(\hat{\chi}, t) = \mathcal{N}(t) \exp \left[ - \left\{ \epsilon_1(t)x_1^2 + \epsilon_2(t)x_2^2 + \epsilon_3(t)x_3^2 + \epsilon_4(t)x_1x_2 + \epsilon_5(t)x_1x_3 + \epsilon_6(t)x_2x_3 \right\} \right].$$  \hspace{1cm} (2.8)

where $\mathcal{N}(t)$ and $\epsilon_j(t)$ are complex time-dependent functions defined in “Appendix B’. The wave function given by Eq. (2.8) has associated a three-variable Gaussian probability distribution; besides, it can be verified that it is normalized for all $t$. Furthermore, it describes the dynamics of the pointers plus the system while the measurement is in progress. Since it cannot be expressed as a product of individual functions, it can be considered as an example of three-mode Gaussian entanglement [54] between the pointers and system variables [55]. The temporal dynamics of the mean value for any position and momentum function $Q(\hat{\chi}, \hat{\phi}, t)$ can be obtained through

$$\langle Q(\hat{\chi}, \hat{\phi}, t) \rangle = \int \Psi^*(\chi, t) Q(\hat{\chi}, \hat{\phi}, t) \Psi(\chi, t) \; d\chi,$$  \hspace{1cm} (2.9)

with $\hat{\chi}$ and $\hat{\phi}$ representing the position and momentum dependence of the function, respectively; hence, $d\chi = \prod_{i=1}^3 dx_i$.

The remaining of this paper will address the mean value of the $n$-th moment of some observables at the time in which the measurement is carried out as the reciprocal of the coupling constant $t = \tau = 1/\kappa$ [10]. This choice simplifies the definitions and calculations. The variance is defined as $\sigma_A^2 = \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2$. We emphasise that each marginal probability distribution associated with the wave function given by Eq. (2.8) is Gaussian with zero mean value; hence, their variances coincide with the squared root-mean-square (rms) error [57], which is a reasonable measure of dispersion; it quantifies the degree at which the probability distribution of an observable deviates from another that is pretended to be estimate [58,59].

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1 At this time, the simultaneous read-out takes place; therefore, projective measurements are done in the pointers, leaving the post-measurement Gaussian state as the normalised projection on the eigen-space of the eigenvalues observed, according to the quantum mechanically postulate of a projective measurement [56].
Using the wave function, i.e. Eq. (2.8), and the definition given in Eq. (2.9), we find that the \( \kappa \)-dependent variances of the position probability distributions of the pointers at the time of the measurement are given by

\[
\sigma_{\hat{x}_1}^2 (\kappa) = \delta_q^2 + \frac{b}{2} + \eta_1 (\kappa) + \eta_2 (\kappa),
\]

\[
\sigma_{\hat{x}_2}^2 (\kappa) = \delta_p^2 + \frac{1}{2b} + \eta_3 (\kappa),
\]

while the variances of the conjugate distributions of the Gaussian system at the same time are

\[
\sigma_{\hat{x}_3}^2 (\kappa) = \delta_q^2 + b + \eta_2 (\kappa),
\]

\[
\sigma_{\hat{p}_3}^2 (\kappa) = \delta_p^2 + \frac{1}{b},
\]

where the \( \eta_j (\kappa) \)-functions, \( j = 1, 2, 3 \), represent the contribution—given by the free evolution of the sub-systems—on the whole system, this contribution was not taken into account before in references [10,20–22]. Thus, they represent one of the main results of this paper and are defined as:

\[
\eta_1 (\kappa) = \frac{(m_1 - 6m_3)^2}{36bm_1^2m_3^2k^2},
\]

\[
\eta_2 (\kappa) = \frac{1}{16\delta_q^2m_3^2k^2},
\]

\[
\eta_3 (\kappa) = \frac{b}{m_3^2 k^2}.
\]

It is worth mentioning that they vanish as the coupling \( \kappa \) becomes larger, that is,

\[
\lim_{\kappa \to \infty} \eta_j (\kappa) = 0,
\]

in this limit the strong coupling regime is recovered together with the variances of the probability distributions calculated in [10].

In their seminal work, Arthurs and Kelly focused on adjusting the balance parameter \( b \) in order to minimise the product \( \sigma_{\hat{x}_1}^2 \sigma_{\hat{x}_2}^2 \); thus, they found that it could not take a value less than 1 (in units of \( \hbar^2 \)), with the balance parameter of the pointers adjusted at the rate \( b = \delta_q \delta_p^{-1} \). For the particular measurement setting here considered, this value is given by \( b = 2\delta_q^2 \); hence, substituting it in the product of variances given by Eqs. (2.10) and (2.11) together with the lower bound of Kennard uncertainty relation in Eq. (1.1), the product \( \sigma_{\hat{x}_1}^2 (\kappa) \sigma_{\hat{x}_2}^2 (\kappa) \) in the full dynamical simultaneous measurement process, i.e. when taking into account the free Hamiltonian, is given by

\[
\sigma_{\hat{x}_1}^2 (\kappa) \sigma_{\hat{x}_2}^2 (\kappa) = \sigma_{\hat{x}_1}^2 \sigma_{\hat{x}_2}^2 |_{\text{min}} + \Delta_1 (\kappa) = 1 + \Delta_1 (\kappa),
\]

where \( \sigma_{\hat{x}_1}^2 \sigma_{\hat{x}_2}^2 |_{\text{min}} = 1 \) and the \( \Delta_1 (\kappa) \)-function is given by

\[
\Delta_1 (\kappa) = \frac{11m_1^2 \left( 4\delta_q^4 + \kappa^2 m_3^2 \right) - 24m_1 \left( 4\delta_q^4 + \kappa^2 m_2^2 \right) m_3 + 72 \left( 4\delta_q^4 + \kappa^2 \left[ 16\delta_q^8 m_1^2 + m_2^2 \right] \right) m_3^2}{288 \left( \delta_q^2 \kappa^2 m_1 m_2 m_3 \right)^2}.
\]
This plot shows the behaviour of the product $\sigma^2_{\hat{x}_1}(\kappa) \sigma^2_{\hat{x}_2}(\kappa)$ given by Eq. (2.18) versus the coupling constant $\kappa$, for the values $\delta_q = 1$, $m_1 = 1$. As the coupling strength $\kappa$ decreases, the product $\sigma^2_{\hat{x}_1}(\kappa) \sigma^2_{\hat{x}_2}(\kappa)$ shifts upwards from its minimal value due to the contribution of the $\Delta_1(\kappa)$-function given in Eq. (2.19).

The $\Delta_1(\kappa)$ summarises the contribution of the free evolution to the measurement setting, which will undergo a hyperbolic shift upwards as the coupling strength $\kappa$ becomes smaller; see Fig. 1. However, although the product $\sigma^2_{\hat{x}_1}(\kappa) \sigma^2_{\hat{x}_2}(\kappa)$ gives us a quantitative notion of the accuracy in the simultaneous measurement, it only refers to the retrodictive aspect—defined in the next section—besides the variances $\sigma^2_{\hat{x}_1}(\kappa)$ and $\sigma^2_{\hat{x}_2}(\kappa)$ cannot be directly interpreted as experimental errors [8,26]. In the following section, it will be shown how these free energy contribution, i.e. Eqs. (2.14) to (2.16), affects the accuracy of the simultaneous measurement, whose meaning will be explained in the next section through the so-called retrodictive and predictive error operators.

### 3 Free energy contributions to the accuracy of the measurement process

To interpret the results of the last section, the definitions for the retrodictive and predictive error operators already given in [8,26,27] will be brought back. It is important to note that these definitions can be fully understood when there is a large quantity of measurements on identical prepared systems, in the sense that it forms a probability distribution from which the $n$-th moment of canonical observables is estimated [21]. This is necessary due to the contextuality in quantum mechanics, which asserts that observables do not have a predetermined value before a measurement [60–62].

#### 3.1 Error operators and optimal joint measurements

The formalism of quantum mechanics limits the simultaneous measurement of non-compatible observables; nevertheless, this restriction does not exclude the possibility of drawing inferences about such quantities within certain margin of error. Hence, it is implicit that by reducing such errors the measurement accuracy will be increased. To clarify this conception, assume a measuring apparatus equipped with an initial pointer variable $\hat{x}$ which is coupled through unitary dynamics $\hat{U}$ with the observable $\hat{A}$ which is intended to be measured; it is known that the measurement is carried out with total accuracy if the probability distribution of the pointer variable just after the measurement process coincides with the pre-measurement (or post-measurement) probability distribution of $\hat{A}$; see for example the definition given in [63].

Within the context of the simultaneous measurement introduced by Arthurs and Kelly, it has been pointed out two aspects of accuracy [8,26,27], which quantify the deviation of
the probability distributions of the pointer records from those of the canonical pair before and after the measurement process. In order to study these aspects in the model given by Eq. (2.1) (i.e. taking in to account the free Hamiltonian of the subsystems), in the following their conceptual definitions will be reviewed.

The retrodictive error operators are defined, see the work of Appleby [8,26,27], as:

\[ \hat{\epsilon}_x \equiv \hat{U}^\dagger \hat{x}_1 \hat{U} - \hat{x}_3, \]  
\[ \hat{\epsilon}_p \equiv \hat{U}^\dagger \hat{p}_2 \hat{U} - \hat{p}_3, \]

where in these and all subsequent definitions, the unitary operator \( \hat{U} \) rules the dynamics of the measurement process; hence, \( \hat{U}^\dagger \hat{A} \hat{U} \) describes the dynamics of the observable \( \hat{A} \) in the Heisenberg’s picture. Through a proper measure of dispersion, the retrodictive error operators give information about how the position probability distributions of the pointers deviate from the initial probability distributions of the conjugate pair of observables which are being measured; thus, they have a retrodictive character in the sense that they establish comparison with the information of the observables under inspection before the measurement process.

The predictive error operators are defined as [8,26,27]:

\[ \hat{\epsilon}_x \equiv \hat{U}^\dagger (\hat{x}_1 - \hat{x}_3) \hat{U}, \]
\[ \hat{\epsilon}_p \equiv \hat{U}^\dagger (\hat{p}_2 - \hat{p}_3) \hat{U}, \]

which provide information about how the probability distributions of the pointers deviate from the observables under inspection, just after the time of the measurement process; therefore, they have a predictive character in the sense that they pretend to establish a comparison with the information of the canonical pair under examination after the measurement process.

The simultaneous measurement process of position and momentum is called retrodictively optimal or with maximal retrodictive accuracy if the following equation is satisfied [64]:

\[ \langle \hat{\epsilon}_x \rangle = \langle \hat{\epsilon}_p \rangle = 0; \]  
\[ \{ \sigma_{\hat{\epsilon}_x}^2 \sigma_{\hat{\epsilon}_p}^2 \}_{\text{min}} = \frac{1}{4}. \]

besides, as the product of variances associated with the retrodictive error operators is minimum, we have:

The condition given by Eq. (3.5) means that the mean value of the records of the pointers match the mean value of the initial observables of the system.

The simultaneous measurement process of position and momentum is called predictively optimal or with maximal predictive accuracy if the product of the variances associated with the retrodictive error operators is minimum [64], that is,

\[ \{ \sigma_{\hat{\epsilon}_x}^2 \sigma_{\hat{\epsilon}_p}^2 \}_{\text{min}} = \frac{1}{4}, \]

consequently the following condition is fulfilled:

\[ \langle \hat{\epsilon}_x \rangle = \langle \hat{\epsilon}_p \rangle = 0. \]  

The condition given by Eq. (3.8) means that the pointer’s outputs match on average the mean value of position and momentum observables of the system just after the measurement is carried out.

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If the conditions given by Eqs. (3.5) and (3.8) are fulfilled, it can be proved that, see reference [8]:
\[
\left\{ \hat{\epsilon}_A \hat{B} \right\} = \left\{ \hat{B} \hat{\epsilon}_A \right\} = 0, \quad A \in \{x_i, p_i, x_f, p_f\}, \quad \hat{B} \in \{\hat{x}_3, \hat{p}_3\},
\]
(3.9)
where \(A\) labels the error operators and \(\hat{B}\) represents any of the initial conjugate observables of the Gaussian system to be measured. The last equation will be needed for future calculations.

It is worth mentioning that the measurement instrument considered by Arthurs and Kelly was designed to maximize the retrodictive and predictive aspects of accuracy [8], which become superior within the regime of strong coupling when the system under measurement is a minimum uncertainty Gaussian state. Consequently, it will be shown that the maximal accuracy on the two aspects of the measurement is not maintained when it is considered the full dynamics of the measurement process—i.e. when the free evolution of the sub-systems is taken into account—since it introduces an extra noise on the variances of the error operators that depends on the degree of coupling between the measurement instrument and the system to be measured.

3.2 The accuracy in the simultaneous measurement

The accuracy in the simultaneous measurement will be proved as follows, using the evolution given by Eq. (2.8) we can deduce—by using the definitions given by Eqs. (3.1) and (3.2) and the conditions given in Eqs. (3.5) and (3.9)—that the variances of the retrodictive error operators are
\[
\sigma^2_{\hat{\epsilon}_{x_i}} = \left\{ \hat{\epsilon}^\dagger \hat{x}_1 \hat{\epsilon} \right\}^2 - \left\{ \hat{x}_3^2 \right\} = \sigma^2_{\hat{x}_1} - \delta^2_q,
\]
(3.10)
\[
\sigma^2_{\hat{\epsilon}_{p_i}} = \left\{ \hat{\epsilon}^\dagger \hat{x}_2 \hat{\epsilon} \right\}^2 - \left\{ \hat{p}_3^2 \right\} = \sigma^2_{\hat{x}_2} - \delta^2_p.
\]
(3.11)
Considering the definitions given by Eqs. (3.3) and (3.4), and the condition, given by Eq. (3.8), the variances of the predictive operators are
\[
\sigma^2_{\hat{\epsilon}_{x_f}} = \left\{ \hat{\epsilon}^\dagger (\hat{x}_1 - \hat{x}_3)^2 \hat{\epsilon} \right\} = \sigma^2_{\hat{x}_1} + \sigma^2_{\hat{x}_3} - 2\left\{ \hat{\epsilon}^\dagger \hat{x}_1 \hat{x}_3 \hat{\epsilon} \right\},
\]
(3.12)
\[
\sigma^2_{\hat{\epsilon}_{p_f}} = \left\{ \hat{\epsilon}^\dagger (\hat{x}_2 - \hat{p}_3)^2 \hat{\epsilon} \right\} = \sigma^2_{\hat{x}_2} + \sigma^2_{\hat{p}_3} - 2\left\{ \hat{\epsilon}^\dagger \hat{x}_2 \hat{p}_3 \hat{\epsilon} \right\},
\]
(3.13)
in these equations we have employed the fact that \(\hat{x}_1, \hat{x}_3, \hat{x}_2, \hat{p}_3 = 0\) and \(\hat{\epsilon}^\dagger \hat{U} = \hat{I}\). The last terms in Eqs. (3.12) and (3.13) express the double covariance between the observables that define the predictive error operator; these are obtained in the Schrödinger picture by employing Eq. (2.9), hence
\[
2 \left\{ \hat{\epsilon}^\dagger \hat{x}_1 \hat{x}_3 \hat{\epsilon} \right\} = 2\delta^2_q + b + 2\eta_2(\kappa),
\]
(3.14)
\[
2 \left\{ \hat{\epsilon}^\dagger \hat{x}_2 \hat{p}_3 \hat{\epsilon} \right\} = \frac{1}{b} + 2\delta^2_p.
\]
(3.15)
Substituting Eqs. (2.10) and (2.11) in Eqs. (3.10) and (3.11), the \(\kappa\)-dependent variances of the retrodictive error operators are
\[
\sigma^2_{\hat{\epsilon}_{x_i}}(\kappa) = \frac{b}{2} + \eta_1(\kappa) + \eta_2(\kappa),
\]
(3.16)
\[
\sigma^2_{\hat{\epsilon}_{p_i}}(\kappa) = \frac{1}{2b} + \eta_3(\kappa).
\]
(3.17)
Substituting Eqs. (2.10) to (2.13) and Eqs. (3.14) and (3.15) in Eqs. (3.12) and (3.13), the $\kappa$-dependent variances of the predictive error operators at the same time are

$$\sigma^2_{\hat{\epsilon} x f}(\kappa) = b + \eta_1(\kappa), \quad (3.18)$$

$$\sigma^2_{\hat{\epsilon} p f}(\kappa) = \frac{1}{2b} + \eta_3(\kappa). \quad (3.19)$$

From the variances of the pointer readings, i.e. Eqs. (2.10) and (2.11), it can be inferred that the simultaneous measurement would be accurate if it was given by the initial variances of position and momentum probability distributions of the Gaussian state only, i.e. $\delta^2_q$ and $\delta^2_p$, respectively; then, the extra terms correspond to the noise affecting the retrodictive accuracy of the measurement.

This noise is quantitatively expressed by the variances of the retrodictive error operators, Eqs. (3.16) and (3.17). In the same way, the noise that is deviating the probability distributions of the pointer readings from the canonical pair just after the time of the measurement is represented by the variances of the predictive error operators, Eqs. (3.18) and (3.19). From the conditions given in Eqs. (3.6) and (3.7) and the variances of the error operators, Eqs. (3.16) to (3.19), it is deduced that the joint measurement would be retrodictive and predictively optimal if the $\eta_j(\kappa)$-functions were zero [which only happens in the strong coupling limit, as is shown in Eq. (2.17)], these terms only appear when the free energy operators in the Hamiltonian of the measurement process are taken into account, they arise as a consequence of the spreading of the probability density of the whole Gaussian system.

Nevertheless, it must be noted that even in the optimal situation of accuracy, there exists an unavoidable noise which is proportional to the pre-measurement variances of the probability distributions of the pointers [that is, the $b$-dependent terms that appears from Eqs. (3.16) to (3.19)], this noise is consequence of the unitary dynamics associated with the quantum measurement process [21,65] and it is the cause for the increment of the lower bound of the Kennard uncertainty relation as it was predicted for the first time by Arthurs and Kelly.

The definition of the predictive aspect of accuracy is quite important in the quantum description of a measurement process, unlike classical mechanics where these effects can be depreciated. Concerning the Arthurs–Kelly model, the consideration of the retrodictive and predictive aspects of accuracy allows us to quantitatively define the concept of a disturbance operator [66] like the one defined in [63], which enable to quantify the feedback on the conjugate probability distributions of the system due to the measurement process [8].

Hence, the two aspects of accuracy in the simultaneous measurement will be gradually decreasing as the coupling strength $\kappa$ becomes smaller; this fact will be reflected in the deviation of the statistical distributions of the joint readings from the canonical pair before and after the measurement process.

To graphically illustrate that argument, in Fig. 2 we plot the set of points coming from the joint probability density

$$\rho(x_1, x_2, \tau) = \int_{-\infty}^{+\infty} |\Psi(x_1, x_2, x_3, \tau)|^2 \, dx_3, \quad (3.20)$$

which represents the distribution of the pointer readings at the time of the measurement; then, we compare it versus both, the Wigner quasi-probability distribution of the Gaussian system before (retrodictive aspect) the measurement given by

$$W(x_3, p_3, t = 0) = \pi^{-1} \int_{-\infty}^{+\infty} \phi_{x}^*(x_3 + \xi)\phi_{x}(x_3 - \xi) e^{2i p_3\xi} \, d\xi, \quad (3.21)$$
Fig. 2 Set of 4000 points generated from the joint probability distribution $\rho(x_1, x_2, \tau) = \int_{-\infty}^{+\infty} |\Psi(x_1, x_2, x_3, \tau)|^2 \, dx_3$ with the wave function, Eq. (2.8), at the time of measurement $t = \tau = 1/\kappa$, using the values $\delta q = 1$, $b = 2\delta q^2$, and $m_i = 1$. Each point constitutes a joint record of the pointers, which is registered in the $x_1 x_2$-plane. To graphically inspect the notion of retrodictive and predictive accuracy, the whole set of records is compared versus the contour plots of the Wigner function (with $\delta q = 1$) associated with the Gaussian state under inspection before (a, b) and after (c, d) the measurement, respectively, for the cases of strong coupling ($\kappa = 100$, left Figs.) and weak coupling ($\kappa = 0.5$, right Figs.). The deviation from the Wigner functions becomes larger in the weak coupling regime. Notably, the correlation (that is to say, the rotation angle of the statistical distribution of the joint readings) between the pointer records is greater as the coupling strength is weaker.

and just after the measurement (predictive aspect)

$$W(x_3, p_3, t = \tau) = \pi^{-1} \int_{-\infty}^{+\infty} \Psi^*(x_1, x_2, x_3 + \xi, \tau) \Psi(x_1, x_2, x_3 - \xi, \tau) e^{2ip_3\xi} \, dx_1 dx_2 d\xi;$$

then, the maximal degree of match is determined within the margins stipulated by the variances of the error operators, Eqs. (3.16) to (3.19).

Hence, the minimum product of the pointer variances found by Arthurs and Kelly is recovered at the strong coupling regime, that is

$$\lim_{\kappa \to \infty} \Delta_1(\kappa) = 0.$$
On the other hand, the pioneers in recognising a lower bound similar to the one of the Kennard uncertainty relation for the product of variances of the retrodictive errors operators were Arthurs and Kelly; they show that \[10\]

\[
\sigma_{\xi_i}^2 \sigma_{\xi_i}^2 \geq \frac{1}{4}. \tag{3.24}
\]

The above inequality is known as the Heisenberg uncertainty relation for joint measurements \[63\], which is specifically defined for the position and momentum observables. Its proper interpretation goes according to the statute (B) defined in \[67\], which establishes that although it is impossible to perform a (retrodictive) exact simultaneous measurement of position and momentum it is still possible to make an approximate estimation, where the product of the variances of the (retrodictive) error operators will be constrained by the inequality (3.24).

There is a similar interpretation for the inequality bound of the variances of the predictive error operators of the simultaneous measurement, this bound is \[8,26,27\]

\[
\sigma_{\xi_f}^2 \sigma_{\xi_f}^2 \geq \frac{1}{4}. \tag{3.25}
\]

Hence, from the product of variances, Eqs. (3.6) and (3.7), it is possible to see that an optimal retrodictive and predictive simultaneous measurement saturates the inequalities given by Eqs. (3.24) and (3.25), respectively.

By taking the product of Eqs. (3.16) and (3.17), and using the value of the parameter \(b = 2\delta q^2\) as well as the lower bound of Kennard uncertainty relation, we obtain the value of the minimum product of variances of the retrodictive error operators plus a \(\kappa\)-dependent contribution, that is,

\[
\sigma_{\xi_i}^2(\kappa) \sigma_{\xi_i}^2(\kappa) = \left\{ \sigma_{\xi_i}^2 \sigma_{\xi_i}^2 \right\}_{\text{min}} + \Delta_2(\kappa). \tag{3.26}
\]

In the same way, by taking the product of Eqs. (3.18) and (3.19) and following the same procedure above, we obtain a similar result for the product of variances of the predictive error operators

\[
\sigma_{\xi_f}^2(\kappa) \sigma_{\xi_f}^2(\kappa) = \left\{ \sigma_{\xi_f}^2 \sigma_{\xi_f}^2 \right\}_{\text{min}} + \Delta_3(\kappa), \tag{3.27}
\]

where \(\left\{ \sigma_{\xi_i}^2 \sigma_{\xi_i}^2 \right\}_{\text{min}} = 1/4 = \left\{ \sigma_{\xi_f}^2 \sigma_{\xi_f}^2 \right\}_{\text{min}}\) and the \(\Delta_{1,2}(\kappa)\)-functions are defined as

\[
\Delta_2(\kappa) = \frac{11m_1^2 \left( 8\delta_q^4 + \kappa^2 m_2^2 \right) - 24m_1 \left( 8\delta_q^4 + \kappa^2 m_2^2 \right) m_3 + 72 \left( 8\delta_q^4 + \kappa^2 \left[ 16\delta_q^8 m_1^2 + m_2^2 \right] \right) m_3^2 \left( 24\delta_q^2 \kappa^2 m_1 m_2 m_3 \right)^2}{288 \left( \delta_q^2 \kappa^2 m_1 m_2 m_3 \right)^2}, \tag{3.28}
\]

\[
\Delta_3(\kappa) = \frac{m_1^2 \left( 8\delta_q^4 + \kappa^2 m_2^2 \right) - 12m_1 \left( 8\delta_q^4 + \kappa^2 m_2^2 \right) m_3 + 36 \left( 8\delta_q^4 + \kappa^2 \left[ 16\delta_q^8 m_1^2 + m_2^2 \right] \right) m_3^2 \left( 8\delta_q^4 + \kappa^2 m_1 m_2 m_3 \right)^2}{288 \left( \delta_q^2 \kappa^2 m_1 m_2 m_3 \right)^2}, \tag{3.29}
\]

which are equal to zero in the ideal situation of strong coupling, therefore

\[
\lim_{\kappa \to \infty} \Delta_{2,3}(\kappa) = 0, \tag{3.30}
\]

recovering the lower bounds of the retrodictive and predictive uncertainty relations, Eqs. (3.24) and (3.25). Consequently, the products \(\sigma_{\xi_i}^2(\kappa) \sigma_{\xi_i}^2(\kappa)\) and \(\sigma_{\xi_f}^2(\kappa) \sigma_{\xi_f}^2(\kappa)\) are hyperbolically shifted from their minimum values as the coupling strength becomes smaller;
This plot shows the behaviour of the products $\sigma_{\hat{x}_1}(\kappa)\sigma_{\hat{p}_1}(\kappa)$ (dashed plot) and $\sigma_{\hat{x}_f}(\kappa)\sigma_{\hat{p}_f}(\kappa)$ (continuous plot) given by Eqs. (3.26) and (3.27) versus the coupling constant $\kappa$ for the values $\delta_q = 1$, $m_i = 1$. As the coupling becomes smaller, the free evolution of the measurement setting (quantified by the $\Delta_{1,2}(\kappa)$-functions) causes an hyperbolic shift upwards of the products of variances of the retrodictive and predictive error operator from its minimal values.

see Fig. 3. Therefore, the retrodictive and predictive experimental errors are higher as the coupling strength $\kappa$ becomes smaller.

4 Conclusions

In this paper, we investigate the effects caused by the free dynamics in the model of Arthurs and Kelly of simultaneous measurement and its consequences on the retrodictive and predictive aspects of the accuracy of the measurement process. One of our main results is that the inclusion of the free energy Hamiltonian generates a spreading of the whole probability density, resulting in an extra noise compared with the one that is induced by the measurement process in the optimal regime of accuracy. That is to say, the free evolution increases the statistical distributions of the pointer readings; this causes a $\kappa$-dependent increment of the variances of the retrodictive and predictive error operators.

For clarity, we exemplify these arguments through the analysis of two limit situations. (i) Strong coupling regime: Where, due to the reciprocal relationship between the time of the measurement process and the coupling strength $\kappa$, the measurement can be taken as instantaneous; consequently, the spreading affecting the whole probability density is very low. Hence, the variances of the retrodictive and predictive error operators are minimum, and the optimal accuracy in both aspects can be reached. (ii) Weak coupling regime: Where the pointers are weakly coupled with the Gaussian system to be inspected; hence, the measurement takes a considerable lapse of time and the free energy operators must be considered in the Hamiltonian governing the dynamics; thus, the spreading of the whole probability density becomes substantial, increasing the product of the variances of the retrodictive and predictive error operators far from their minimal values; thus, the two aspects of the accuracy becomes smaller.

In conclusion, taking into account the full dynamics in the measurement process of two non-commuting observables, we found that the (optimal) retrodictive and predictive accuracy decrease as the coupling strength between the measuring apparatuses and the Gaussian system become smaller. However, we establish the fact that the strength of this conclusion should be
maintained for any other state over which the measurement could be carried out, because the free energy operators $e^{a(t)\hat{P}}$ that appears in the time evolution operator, i.e. Eq. (2.7), will cause the free propagation of the wavefunction, i.e. the temporal dispersion of its associated probability distribution [68,69].

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**Author contributions** JAMF worked on the plots and calculations and discussed the results. VMVA discussed the results and provided assistance. LMAA conceived the idea and the conceptualisation of the overarching goals, supervision of the whole project, worked on the calculations, discussed and analysed all the results and data. JAMF and LMAA wrote initial draft preparation. LMAA contributed to review and editing.

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

**Appendices**

**A factorization of the time evolution operator**

In this appendix, we show the process for expressing the time evolution operator as a product of exponential operators, i.e. Eq. (2.7); hence, we follow the methodology exposed in [45]. With the Hamiltonian given by Eq. (2.1), the time evolution operator is

$$
e^{-i\hat{H}t} = e^{-\frac{it}{2m_1} \hat{p}_1^2} e^{-\frac{it}{2m_2} \hat{p}_2^2} e^{-it \left( \frac{\hat{p}_3^2}{2m_3} + \kappa \hat{x}_3 \hat{p}_1 + \kappa \hat{p}_3 \hat{p}_2 \right)}, \tag{A.1}$$

hence, it is necessary to factorise the last exponential of the above equation as $\hat{x}_3$ does not commute with $\hat{p}_3$ and $\hat{p}_3^2$. To perform it, we define the operators

$$\hat{A} = -it \left( \frac{\hat{p}_3^2}{2m_3} + \kappa \hat{x}_3 \hat{p}_1 + \kappa \hat{p}_3 \hat{p}_2 \right), \tag{A.2}$$

$$\hat{B} = -it \kappa \hat{x}_3 \hat{p}_1; \tag{A.3}$$

then, the following commutators are obtained through a test function $f(\hat{x}_3)$

$$[\hat{A}, \hat{B}] = it^2 \left( \frac{\kappa}{m_3} \hat{p}_1 \hat{p}_3 + \kappa^2 \hat{p}_1 \hat{p}_2 \right), \tag{A.4}$$

$$[\hat{A}, [\hat{A}, \hat{B}]] = 0, \tag{A.5}$$

$$[\hat{B}, [\hat{A}, \hat{B}]] = -it^3 \kappa^2 \frac{\hat{p}_1^2}{m_3}. \tag{A.6}$$

---

2 it can be proved through the convolution theorem for inverse Fourier transform, that the application of this operator to an arbitrary state is equivalent to the dynamics with the free particle-propagator in the method of Green’s functions.
We now define an auxiliary function $F(\zeta)$ in terms of an exponential as

$$F(\zeta) = e^{\zeta (\hat{A} + \hat{B})}, \quad (A.7)$$

its derivative is given by

$$F'(\zeta) = (\hat{A} + \hat{B}) F(\zeta), \quad (A.8)$$

where $\zeta$ is an auxiliary parameter which we will define as 1 when we finished all the calculations. Afterwards, we set a generic factorisation that allows us to factorise the last exponential function of Eq. (A.1), that is

$$F(\zeta) = e^{f_0(\zeta)} e^{f_1(\zeta)} \hat{N}_1 e^{f_2(\zeta)} \hat{N}_2 e^{f_3(\zeta)} \hat{N}_3; \quad (A.9)$$

besides, its derivative is

$$F'(\zeta) = f'_0(\zeta) F(\zeta) + f'_1(\zeta) \hat{N}_1 F(\zeta) + f'_2(\zeta) e^{f_0(\zeta)} e^{f_1(\zeta)} \hat{N}_1 \hat{N}_2 e^{f_2(\zeta)} \hat{N}_2 e^{f_3(\zeta)} \hat{N}_3$$

$$+ f'_3(\zeta) e^{f_0(\zeta)} e^{f_1(\zeta)} \hat{N}_1 e^{f_2(\zeta)} \hat{N}_2 e^{f_3(\zeta)} \hat{N}_3.$$ \quad (A.10)

In this work, we consider $\hat{N}_1 = \hat{A}$, $\hat{N}_2 = \hat{B}$, $\hat{N}_3 = \hat{A}$; thus, Eq. (A.10) is given by

$$F'(\zeta) = f'_0(\zeta) F(\zeta) + f'_1(\zeta) \hat{A} F(\zeta) + f'_2(\zeta) e^{f_0(\zeta)} e^{f_1(\zeta)} \hat{A} e^{f_2(\zeta)} \hat{B} e^{f_3(\zeta)} \hat{A}$$

$$+ f'_3(\zeta) e^{f_0(\zeta)} e^{f_1(\zeta)} \hat{A} e^{f_2(\zeta)} \hat{B} e^{f_3(\zeta)} \hat{A}; \quad (A.11)$$

then, according to Eq. (A.8), it is necessary to move the function $F(\zeta)$ at the end of the right in the right-hand side of the above equation, but the third and fourth terms prevent it. The third term of Eq. (A.11) can be arranged as

$$f'_2(\zeta) e^{f_0(\zeta)} e^{f_1(\zeta)} \hat{A} \hat{B} e^{f_2(\zeta)} \hat{B} e^{f_3(\zeta)} \hat{A} = f'_2(\zeta) \left( \hat{B} + f_1(\zeta) [\hat{A}, \hat{B}] \right) F(\zeta), \quad (A.12)$$

where we have employed the fact that for self-adjoint operators $\hat{X}$ and $\hat{Y}$ it is satisfied that $e^{f_1(\zeta) \hat{Y}} e^{-f_1(\zeta) \hat{Y}} = e^{f_1(\zeta) \hat{X}} \hat{Y} e^{-f_1(\zeta) \hat{X}}$. Employing the commutators given by Eqs. (A.4) and (A.5), and the condition [70]:

$$\left( e^{f_1(\zeta) \hat{X}} \hat{Y} e^{-f_1(\zeta) \hat{X}} \right) = \hat{Y} + f_1 \left[ \hat{X}, \hat{Y} \right] + \frac{f_1^2}{2!} \left[ \hat{X}, \left[ \hat{X}, \hat{Y} \right] \right] + \cdots , \quad (A.13)$$

by following the same procedure, and using the commutators given by Eqs. (A.4) and (A.6), the fourth term of Eq. (A.11) is given by

$$f'_3(\zeta) e^{f_0(\zeta)} e^{f_1(\zeta)} \hat{A} e^{f_2(\zeta)} \hat{A} = f'_3(\zeta) \left( \hat{A} - f_2(\zeta) [\hat{A}, \hat{B}] - \frac{(f_2(\zeta))^2}{2} [\hat{B}, [\hat{A}, \hat{B}]] \right) F(\zeta), \quad (A.14)$$

thus, the derivative given in Eq. (A.11) is expressed as

$$F'(\zeta) = \left\{ f'_0(\zeta) + f'_1(\zeta) \hat{A} + f'_2(\zeta) \left( \hat{B} + f_1(\zeta) [\hat{A}, \hat{B}] \right) \right.$$  

$$+ f'_3(\zeta) \left( \hat{A} - f_2(\zeta) [\hat{A}, \hat{B}] - \frac{(f_2(\zeta))^2}{2} [\hat{B}, [\hat{A}, \hat{B}]] \right) \right\} F(\zeta), \quad (A.15)$$

equating Eq. (A.15) with Eq. (A.8), we obtain the following set of differential equations

$$f'_1(\zeta) + f'_3(\zeta) = 1, \quad (A.16)$$

$$f'_2(\zeta) = 1. \quad (A.17)$$
\[ f'_0(\xi) - \frac{f'_2(\xi)}{12}[\hat{B}, \hat{A}, \hat{B}] = 0, \quad (A.18) \]
\[ f'_2(\xi)f_1(\xi) - f'_3(\xi)f_2(\xi) = 0, \quad (A.19) \]

which are subjected to the conditions
\[ f_0(0) = f_1(0) = f_2(0) = f_3(0) = 0, \quad (A.20) \]

then, the solutions are given by
\[ f_0(\xi) = \frac{[\hat{B}, \hat{A}, \hat{B}]}{12} \xi^3, \quad (A.21) \]
\[ f_1(\xi) = \zeta, \quad (A.22) \]
\[ f_2(\xi) = \zeta, \quad (A.23) \]
\[ f_3(\xi) = \zeta. \quad (A.24) \]

Substituting Eqs. (A.21) to (A.24) in Eq. (A.9) with \( \hat{N}_1 = \hat{A}, \hat{N}_2 = \hat{B}, \hat{N}_3 = \hat{A} \) and choosing \( \zeta = 1 \), the time evolution operator, i.e. Eq. (A.1), is expressed as
\[ e^{-iHt} = e^{\Delta x_1 p_1^2} e^{-\frac{it}{2} \hat{p}_1^2} e^{-\frac{it}{2m_2} \hat{p}_2^2} e^{-\frac{it}{2m_3} \hat{p}_3^2} e^{-it\hat{x}_1 \hat{p}_1} e^{-\frac{it}{2m_3} \hat{p}_3^2} e^{-it\hat{p}_3 \hat{p}_2}. \quad (A.25) \]

where we have grouped together the operators with the same of power \( \hat{p}_1^2 \) into one, and we have taken \( \Delta x_1 \) as
\[ \Delta x_1 = -(it/2m_1) + (it^3\kappa^2/12m_3). \quad (A.26) \]

**B Time-dependent coefficients of the wave function describing the dynamics of the simultaneous measurement**

In this appendix, we define the form of the time-dependent functions \( N(t) \) and \( \epsilon_j(t) \) that appears in the wave function given Eq. (2.8), these functions dictates the temporal evolution through the measurement process of the whole configuration.

The \( \epsilon_j(t) \) functions are defined as
\[ \epsilon_j(t) = \frac{\Gamma_j(t)}{\Theta(t)}, \quad j = 1, 2, 3, 4, 5, 6, \quad (B.1) \]

with
\[ \Gamma_1(t) = 3m_1m_3 \left( t \left( -2bt + m_2 + (i + 4\kappa^2m_3t) \right) + 4m_3 \left( m_2 + 2ibt \right) \delta_q^2 \right), \quad (B.2) \]
\[ \Gamma_2(t) = bm_2 \left( 3bm_1m_3 + i \left( -6m_3t + \kappa^2m_1t^3 \right) \right) \left( it + 4m_3\delta_q^2 \right), \quad (B.3) \]
\[ \Gamma_3(t) = m_3 \left( 6ib^2m_1m_3t + 2m_2t \left( 2i\kappa^2m_1t^2 + m_3 \left( -3i + 6\kappa^2m_1t\delta_q^2 \right) \right) \right) + b \left( 12m_3t^2 + m_1 \left( 3m_2 \left( m_3 + \kappa^4m_3t^4 \right) - 8\kappa^2t^3 \left( t - 3im_3\delta_q^2 \right) \right) \right), \quad (B.4) \]
\[ \Gamma_4(t) = 6b\kappa^2m_1m_2m_3 \left( -t + 4im_3\delta_q^2 \right), \quad (B.5) \]
\[ \Gamma_5(t) = -6\kappa m_1m_3 \left( t \left( im_2 + 2b \left( -1 + \kappa^2m_2m_3 \right) t \right) + 4m_3 \left( m_2 + 2ibt \right) \delta_q^2 \right), \quad (B.6) \]
\[ \Gamma_6(t) = 2b \kappa m_2 m_3 t \left(-6i b m_1 m_3 + t \left(5k^2 m_1 t^2 + m_3 \left(-12 - 12i \kappa^2 m_1 t \delta_q^2 \right) \right) \right) ; \] (B.7)

and

\[ \Theta(t) = ibt \left(3m_1 m_2 m_3 + 12m_3 \left(1 - 2 \kappa^2 m_2 m_3 \right) t^2 + \kappa^2 m_1 \left(-2 + 7 \kappa^2 m_2 m_3 \right) t^4 \right) \\
+ 4b m_3 \left(3m_1 m_2 m_3 + 12m_3 t^2 + \kappa^2 m_1 \left(-2 + 3 \kappa^2 m_2 m_3 \right) t^4 \right) \delta_q^2 \\
+ m_2 t \left(6m_3 - \kappa^2 m_1 t^2 \right) \left(t - 4i m_3 \delta_q^2 \right) + 6b^2 m_1 m_3 t \left(-1 + 2 \kappa^2 m_2 m_3 \right) t + 4i m_3 \delta_q^2 \].

(B.8)

While the \( \mathcal{N}(t) \)-function is given by

\[ \mathcal{N}(t) = 2 \left( \frac{2b^2 \delta_q^2}{\pi^3} \right)^{\frac{1}{4}} \left( \frac{3m_1 m_2 m_3^2}{\Theta(t)} \right)^{\frac{1}{4}}. \] (B.9)

\( \mathcal{N}(t) \) plays the role of a normalization constant.

### C Application of the time evolution operator

Because the initial function given by Eq. (2.5) is in the position representation, it is convenient also to express the time evolution operator in the same basis, this is simply done by making the substitution \( \hat{p}_{i'} \rightarrow -i \partial_{x_{i'}} \) for \( i' = 1, 2, 3 \). Thus, the factorisation given by Eq. (A.25) can be expressed as

\[ e^{-i \hat{H} t} = e^{-\Delta x_i \hat{p}_i^2} e^{\hat{\delta}_2(t) \partial_{x_2}^2} e^{\hat{\delta}_3(t) \partial_{x_3}^2} e^{\hat{\delta}_k(t) \partial_{x_k}^2} e^{\hat{\delta}_{i'}(t) \partial_{x_{i'}}^2} e^{\hat{\delta}_1(t) \partial_{x_1}^2} e^{\hat{\delta}_{i'}(t) \partial_{x_{i'}}^2}, \] (C.1)

where we have done

\[ \delta_{i'}(t) = \frac{it}{2m_{i'}}, \quad i' = 1, 2, 3. \] (C.2)

And

\[ \delta_k(t) = it \kappa / 2, \] (C.3)

\[ \delta_k'(t) = -t \kappa. \] (C.4)

We will refer to the exponentials of Eq. (C.1) from right to left, labelling them from first to seventh, respectively; to apply them, we use the one-dimensional and the two-dimensional Fourier transform (FT)

\[ \mathcal{F} \left[ f(q) \right](p) = g(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(q) e^{-iqp} dq, \] (C.5)

\[ \mathcal{F}_{2\mathbb{R}} \left[ m(x, y) \right](u, v) = n(u, v) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} m(x, y) e^{-i(xu+vy)} dx dy, \] (C.6)

and the one-dimensional and two-dimensional inverse Fourier transform (IFT)

\[ \mathcal{F}^{-1} \left[ g(p) \right](q) = f(q) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(p) e^{iqp} dp, \] (C.7)

\[ \mathcal{F}_{2\mathbb{R}}^{-1} \left[ n(u, v) \right](x, y) = m(x, y) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} n(u, v) e^{i(xu+vy)} dudv. \] (C.8)
besides, we use the expansion in McLaurin series of an exponential operator when it is applied to a wave function, that is,

$$e^A\psi = \sum_{n=0}^{\infty} \frac{1}{n!} A^n \psi. \quad (C.9)$$

Applying the first operator given in Eq. (C.1) to the initial function, Eq. (2.5), we have

$$\psi_1 = e^{\delta_k(t)\delta_{x_3}\delta_{x_2}} \phi_1(x_1)\phi_2(x_2)\phi_3(x_3) = \phi_1(x_1) \sum_{n=0}^{\infty} \frac{\left(\delta_k(t)\right)^n}{n!} \frac{d^n\phi_3(x_3)}{dx_3^n} \frac{d^n\phi_2(x_2)}{dx_2^n}, \quad (C.10)$$

Applying two-dimensional FT in $x_2$ and $x_3$ variables to Eq. (C.10), we have

$$\mathcal{F}_{2D}[\psi_1](x_1, p_2, p_3, t) = \phi_1(x_1) \sum_{n=0}^{\infty} \frac{\left(\delta_k(t)\right)^n}{n!} \mathcal{F}\left[\frac{d^n\phi_2(x_2)}{dx_2^n}\right](p_2)\mathcal{F}\left[\frac{d^n\phi_3(x_3)}{dx_3^n}\right](p_3)$$

$$= \phi_1(x_1) \sum_{n=0}^{\infty} \frac{\left(\delta_k(t)\right)^n}{n!} (ip_2)^n (ip_3)^n \mathcal{F}[\phi_2(x_2)](p_2) \mathcal{F}[\phi_3(x_3)](p_3)$$

$$= \phi_1(x_1) e^{-\delta_k(t)p_2p_3} \mathcal{F}[\phi_2(x_2)](p_2) \mathcal{F}[\phi_3(x_3)](p_3), \quad (C.11)$$

where in the second line we have used the case of separable functions (see for example [71] pp. 9-10), and in the third line the derivative theorem of the one-FT

$$\mathcal{F}\left[\frac{d^n f(q)}{dq^n}\right](p) = (ip)^n \mathcal{F}[f(q)](p). \quad (C.12)$$

Using the definition given in Eq. (2.4) for $\phi_3(x_3)$ and $\phi_2(x_2)$ and taking the two-dimensional IFT, the application of the first operator is accomplished, and the result is

$$\psi_1 = \left(\frac{\delta_q}{(2\pi)^{1/2}\sigma_1(t)}\right)_{1/2} \phi_1(x_1)\phi_2(x_2) e^{-\frac{(x_2-2b_k(t)x_3)^2}{4\sigma_1(t)}}. \quad (C.13)$$

with

$$\sigma_1(t) = \left(\frac{\delta_q^2}{b^2} - b^2 \left(\delta_k(t)\right)^2\right). \quad (C.14)$$

As it can be seen, the wave function is now entangled between the position variables $x_2$ and $x_3$; therefore, the initial unentangled structure of the initial wave function has been lost due to the unitary dynamics.

Following a similar methodology, we apply the second operator of Eq. (C.1) to Eq. (C.13)

$$\psi_2 = e^{\delta_k(t)\delta_{x_3}\delta_{x_2}} \psi_1 = \psi_1 \sum_{n=0}^{\infty} \frac{\left(\delta_k(t)\right)^n}{n!} \frac{d^n\phi_3(x_3)}{dx_3^n} \left\{ \left(\sigma_1(t)\right)^{-1/2} e^{-\frac{(x_3-2b_k(t)x_2)^2}{4\sigma_1(t)}} \right\}, \quad (C.15)$$

where we have left inside the brackets the term $\left(\sigma_1(t)\right)^{-1/2}$ in order to simplify the application of the exponential operator (see for example the methodology exposed in [72]) and we take

$$\psi_1' = \left(\frac{\delta_q}{(2\pi)^{1/2}}\right)_{1/2} \phi_1(x_1)\phi_2(x_2). \quad (C.16)$$

Applying a FT to the $x_3$ variable in Eq. (C.15), we have
\[ \mathcal{F} [\psi_2] (x_1, x_2, p_3, t) \]
\[ = \psi_2' \sum_{n=0}^{\infty} \left( \frac{\delta(t)^n}{n!} \right) \mathcal{F} \left[ \delta_x^n \left\{ (\sigma_1(t))^{-1/2} e^{-\frac{(x_3-2b\delta(t)x_2)^2}{4\sigma_2(t)^2}} \right\} \right] (x_1, x_2, p_3, t) \]
\[ = \psi_2' \sum_{n=0}^{\infty} \left( \frac{\delta(t)^n}{n!} \right) (ip_3)^n \mathcal{F} \left[ \left\{ (\sigma_1(t))^{-1/2} e^{-\frac{(x_3-2b\delta(t)x_2)^2}{4\sigma_2(t)^2}} \right\} \right] (x_1, x_2, p_3, t) \]
\[ = \psi_2' e^{-\frac{(x_1)^2}{4\sigma_2(t)^2}} \mathcal{F} \left[ \left\{ (\sigma_1(t))^{-1/2} e^{-\frac{(x_3-2b\delta(t)x_2)^2}{4\sigma_2(t)^2}} \right\} \right] (x_1, x_2, p_3, t), \quad (C.17) \]

Taking the IFT in \( p_{x_3} \) variable to Eq. (C.17), the application of the second operator is accomplished, the result is given by
\[ \psi_2 = \left( \frac{\delta q}{(2\pi)^{1/2}\sigma_2(t)} \right)^{1/2} \phi_1(x_1) \phi_2(x_2) e^{-\frac{(x_3-2b\delta(t)x_2)^2}{4\sigma_2(t)^2}}, \quad (C.18) \]

with
\[ \sigma_2(t) = \left( \frac{\delta^2_q - b (\delta_\chi(t))^2 + \frac{\delta_\delta(t)}{2} \right) ; \quad (C.19) \]

then, the effect of the second operator is to increase the amplitude of the entangled wave function and to contribute to the dispersion of the Gaussian wave packet associated with the entangled variables \( x_2 \) and \( x_3 \). In general, the operators of the type \( e^{Ci\psi_2} \) have this dispersive effect when they act on Gaussian wave packets; see [72].

The third operator is applied as follows:
\[ \psi_3 = e^{i\delta(t)x_3 \partial_{x_1}} \psi_2 = \sum_{n=0}^{\infty} \left( \frac{\delta(t)^n}{n!} \right) e^{i\delta(t)x_3 \partial_{x_1}} \psi_2 = \psi_2' \sum_{n=0}^{\infty} \left( \frac{\delta(t)^n}{n!} \right) x_3^n \left\{ \partial_{x_1}^n \phi_1(x_1) \right\} , \quad (C.20) \]

where we have done
\[ \psi_2' = \left( \frac{\delta q}{(2\pi)^{1/2}\sigma_2(t)} \right)^{1/2} \phi_2(x_2) e^{-\frac{(x_3-2b\delta(t)x_2)^2}{4\sigma_2(t)^2}}. \quad (C.21) \]

Taking the FT of Eq. (C.20) in the \( x_1 \) variable, we obtain:
\[ \mathcal{F} [\psi_3] (p_1, x_2, x_3, t) \]
\[ = \psi_2' \sum_{n=0}^{\infty} \left( \frac{\delta(t)^n}{n!} \right) \mathcal{F} \left[ \partial_{x_1}^n \phi_1(x_1) \right] (p_1, x_2, x_3, t) \]
\[ = \psi_2' \sum_{n=0}^{\infty} \left( \frac{\delta(t)^n}{n!} \right) \mathcal{F} \left[ \phi_1(x_1) \right] (p_1, x_2, x_3, t) \]
\[ = \psi_2' e^{i\delta(t)x_3 p_1} \mathcal{F} \left[ \phi_1(x_1) \right] (p_1, x_2, x_3, t), \quad (C.23) \]

Taking the IFT of Eq. (C.23) in the \( p_1 \) variable, the application of the third operator is accomplished; thus, we have
\[ \psi_3 = \left( \frac{\delta q}{(2\pi)^{1/2}\sigma_2(t)} \right)^{1/2} \phi_1(x_1 + \delta(t)x_3) \phi_2(x_2) e^{-\frac{(x_3-2b\delta(t)x_2)^2}{4\sigma_2(t)^2}}. \quad (C.24) \]
The effect of the third operator is to displace the $x_1$ variable by an amount proportional to the $x_3$ variable, that is, the effect is to entangle them.

It must be noted that all the operators in Eq. (C.1) that remain to be applied commute; hence, the order of its application does not matter. Following this idea, we apply the seventh operator of Eq. (C.1) to Eq. (C.24)

$$
\psi_7 = e^{-\Delta x_1(t)} \delta_{x_1} \psi_3 = \psi'_3 \sum_{n=0}^{\infty} \frac{(-\Delta x_1(t))^n}{n!} \delta_{x_1}^2 \phi_1 (x_1 + \delta'_k(t) x_3),
$$

where we have taken

$$
\psi'_3 = \left( \frac{\delta_q}{(2\pi)^{1/2} \sigma_2(t)} \right)^{1/2} \phi_2(x_2) e^{-\frac{(x_3 - 2b \delta_{x}(t) x_2)^2}{4 \sigma_2(t)^2}}.
$$

Taking the FT of Eq. (C.25) in the $x_1$ variable, we have

$$
\mathcal{F}[\psi_7](p_1, x_2, x_3, t) = \psi'_3 \sum_{n=0}^{\infty} \frac{(-\Delta x_1(t))^n}{n!} \mathcal{F}[\delta_{x_1}^2 \phi_1 (x_1 + \delta'_k(t) x_3)] (p_1, x_2, x_3, t)
$$

$$
= \psi'_3 \sum_{n=0}^{\infty} \frac{(-\Delta x_1(t))^n}{n!} \mathcal{F}[\phi_1 (x_1 + \delta'_k(t) x_3)] (p_1, x_2, x_3, t)
$$

$$
= \psi'_3 e^{\Delta x_1(t)} p_1^2 \mathcal{F}[\phi_1 (x_1 + \delta'_k(t) x_3)] (p_1, x_2, x_3, t)
$$

Taking the definition for $\phi_1 (x_1 + \delta'_k(t) x_3)$ (that is, Eq. (2.4) proportionally displaced by $x_3$) and the IFT in $p_1$ variable; we finish the application of the seventh operator, the result is

$$
\psi_7 = \left( \frac{2b^2 \delta_2^2}{\pi^3} \right)^{1/4} (\sigma_2(t) \sigma_3(t))^{-1/2} e^{-b \delta_2^2} e^{-\frac{(x_1 + \delta'_k(t) x_3)^2}{\sigma_3(t)^2}} e^{-\frac{(x_3 - 2b \delta_{x}(t) x_2)^2}{4 \sigma_2(t)^2}},
$$

with

$$
\sigma_3(t) = b + 4\Delta x_1(t).
$$

The seventh operator contributes to the amplitude and the dispersion of the wave function associated with the $x_1$ and the $x_3$ variables. Before applying the next operator, we complete the square of the binomial in the argument of the exponentials for the $x_2$ variable; thus, the wave function given by Eq. (C.28) can be written as

$$
\psi_7 = \left( \frac{2b^2 \sigma_2^2}{\pi^3} \right)^{1/4} (\sigma_2(t) \sigma_3(t))^{-1/2} e^{-\frac{(\beta(t) x_2 - \alpha(t) x_3)^2}{4 \sigma_2(t)^2}} e^{-\frac{(x_1 + \delta'_k(t) x_3)^2}{\sigma_3(t)^2}} e^{-\frac{(x_3 - 2b \delta_{x}(t) x_2)^2}{4 \sigma_2(t)^2}},
$$

where we have written

$$
\beta(t) = \left[ (2b \delta_{x}(t))^2 + 4\sigma_2(t) b \right]^{1/2},
$$

$$
\alpha(t) = \frac{2b \delta_{x}(t)}{\beta(t)}.
$$

We now apply the sixth operator of Eq. (C.1) to Eq. (C.30)

$$
\psi_6 = e^{(x_2(t) i \delta_2^2)} \psi_7 = \psi'_7 \sum_{n=0}^{\infty} \frac{(\delta_2(t))^n}{n!} \delta_{x_2}^2 \left\{ (\sigma_2(t))^{-1/2} e^{-\frac{(\beta(t) x_2 - \alpha(t) x_3)^2}{4 \sigma_2(t)^2}} \right\},
$$
with
\[
\psi_7 = \left( \frac{2b^2 \delta^2 q}{\pi^3} \right)^{\frac{1}{4}} \left( \sigma_3(t) \right)^{-\frac{1}{2}} e^{-\frac{(x_1 + \zeta(t)x_3)^2}{2\sigma_3(t)^2}} e^{-\frac{x_3^2}{4\sigma_2(t)^2}}, \tag{C.34}
\]
taking the FT on Eq. (C.33) in the \( x_2 \) variable, we have
\[
\mathcal{F}[\psi_6](x_1, p_2, x_3, t) = \sum_{n=0}^{\infty} \frac{\left( \delta_2(t) \right)^n}{n!} \mathcal{F} \left[ \left( \sigma_2(t) \right)^{-\frac{1}{2}} e^{-\frac{(\beta(t) x_1 - \alpha(t)x_3)^2}{4\sigma_2(t)^2}} \right] \bigg|_{x_2 \to p_2}(x_1, p_2, x_3, t) = \psi_7 e^{-\delta_2(t)p_2^2} \mathcal{F}\left[ \left( \sigma_2(t) \right)^{-\frac{1}{2}} e^{-\frac{(\beta(t) x_1 - \alpha(t)x_3)^2}{4\sigma_2(t)^2}} \right](x_1, p_2, x_3, t), \tag{C.35}
\]
Taking the IFT of Eq. (C.35) in the \( p_2 \) variable, we finish the application of the sixth operator; thus, we have
\[
\psi_7 = \left( \frac{2b^2 \delta^2 q}{\pi^3} \right)^{\frac{1}{4}} \left( \sigma_3(t) \sigma_4(t) \right)^{-\frac{1}{2}} e^{-\frac{(\beta(t) x_1 - \alpha(t)x_3)^2}{4\sigma_4(t)^2}} e^{-\frac{(x_1 + \zeta(t)x_3)^2}{2\sigma_3(t)^2}} e^{-\frac{x_3^2}{4\sigma_2(t)^2}}, \tag{C.36}
\]
with \( \sigma_4(t) = \sigma_2(t) + (\beta(t))^2 \delta_2(t) \). \tag{C.37}

The sixth operator also contributes to the amplitude of the whole wave function and the dispersion of the Gaussian wave packet associated with the \( x_2 \) and the \( x_3 \) variables. Before applying the next operator, we complete the square of the binomial in the \( x_3 \) variable in the argument of the exponentials of Eq. (C.36); thus, the wave function given by Eq. (C.36) is written as
\[
\psi_6 = \left( \frac{2b^2 \delta^2 q}{\pi^3} \right)^{\frac{1}{4}} \left( \sigma_3(t) \sigma_4(t) \right)^{-\frac{1}{2}} e^{-\frac{(\lambda(t) x_3 + \frac{\xi_1(x_1,x_2,t)}{2\alpha(t)})^2}{2\sigma_3(t)^2}} e^{\frac{\xi_1(x_1,x_2,t)^2}{4\sigma_2(t)^2}} e^{-\frac{x_3^2}{4\sigma_2(t)^2}}, \tag{C.38}
\]
with
\[
\lambda(t) = \left( \frac{1 - \alpha^2(t)}{4\sigma_2(t)} + \frac{\delta_2'(t)}{\sigma_3(t)} + \frac{(\alpha(t))^2}{4\sigma_4(t)} \right)^{\frac{1}{2}}, \tag{C.39}
\]
\[
\xi_1(x_1,x_2,t) = \frac{2\delta_2'(t)}{\sigma_3(t)} x_1 - \frac{\alpha(t) \beta(t)}{2\sigma_4(t)} x_2 \tag{C.40}
\]
\[
\xi_2(x_1,x_2,t) = \frac{1}{\sigma_3(t)} x_1^2 + \frac{(\beta(t))^2}{4\sigma_4(t)} x_2^2. \tag{C.41}
\]
Hence, we apply the fifth operator of Eq. (C.1) to Eq. (C.38)
\[
\psi_5 = \psi_6 \sum_{n=0}^{\infty} \frac{(\delta_3(t)/2)^n}{n!} \partial_{x_3}^n e^{-\left( \frac{\lambda(t) x_3 + \frac{\xi_1(x_1,x_2,t)}{2\alpha(t)}}{2\sigma_3(t)} \right)^2}, \tag{C.42}
\]
where we have taken
\[ \psi_6 = \left( \frac{2b^2 \delta_q^2}{\pi^3} \right)^{\frac{1}{2}} (\sigma_3(t)\sigma_4(t))^{-\frac{1}{2}} e^{\frac{\xi_5(x_1,x_2,t)}{2\sigma(t)}} e^{-\xi_2(x_1,x_2,t)}, \quad (C.43) \]

taking the FT of Eq. (C.42) in the \( x_3 \) variable, we have
\[
\mathcal{F}[\psi_5](x_1, x_2, p_{x3}, t) = \psi'_5 \sum_{n=0}^{\infty} \frac{(\delta_3(t)/2)^n}{n!} \mathcal{F} \left[ \partial_{x3}^n e^{-\left(\frac{\xi(x_3) + \xi_5(x_1,x_2,t)}{2\sigma(t)}\right)^2} \right](x_1, x_2, p_{x3}, t)
\]
\[
= \psi'_5 \sum_{n=0}^{\infty} \frac{(\delta_3(t)/2)^n (ip_3)^{2n}}{n!} \mathcal{F} \left[ e^{-\left(\frac{\xi(x_3) + \xi_5(x_1,x_2,t)}{2\sigma(t)}\right)^2} \right](x_1, x_2, p_{x3}, t)
\]
\[
= \psi'_5 e^{-\left(\frac{\delta_3(t)}{2}\right)^2} \sum_{n=0}^{\infty} \frac{(\delta_3(t)/2)^n (ip_3)^{2n}}{n!} \mathcal{F} \left[ e^{-\left(\frac{\xi(x_3) + \xi_5(x_1,x_2,t)}{2\sigma(t)}\right)^2} \right](x_1, x_2, p_{x3}, t), \quad (C.44)
\]

Taking the IFT of Eq. (C.44) in the \( p_{x3} \) variable, we finish the application of the fifth operator, the result is
\[ \psi_5 = \left( \frac{2b^2 \delta_q^2}{\pi^3} \right)^{\frac{1}{2}} (\sigma_3(t)\sigma_4(t)\sigma_5(t))^{-\frac{1}{2}} e^{-\left(\frac{\xi(x_3) + \xi_5(x_1,x_2,t)}{2\sigma(t)}\right)^2} e^{-\xi_2(x_1,x_2,t)}, \quad (C.45) \]

with
\[ \sigma_5(t) = 1 + 2(\lambda(t))^2 \delta_3(t); \quad (C.46) \]

then, the fifth operator contributes to the amplitude, and due to the entanglement of all position variables, it also contributes to the temporal dispersion of the whole Gaussian wave function.

Finally, to obtain the final wave function, we apply the fourth operator of Eq. (C.1) to Eq. (C.45)
\[ \Psi(x_1, x_2, x_3, t) = e^{\xi(t)\partial_{x3}\partial_{x2}} \psi_5 = \sum_{n=0}^{\infty} \frac{(\delta_4(t))^n}{n!} \partial_{x3}^n \partial_{x2}^n \psi_5, \quad (C.47) \]

Applying the two-dimensional FT to Eq. (C.47) in the \( x_2 \) and the \( x_3 \) variables, we have
\[
\mathcal{F}_{2D}[e^{\xi(t)\partial_{x3}\partial_{x2}} \psi_5](x_1, p_2, p_{x3}, t) = \sum_{n=0}^{\infty} \frac{(\delta_4(t))^n}{n!} \mathcal{F}_{2D}[\partial_{x3}^n \partial_{x2}^n \psi_5](x_1, p_2, p_{x3}, t)
\]
\[
= \sum_{n=0}^{\infty} \frac{(\delta_4(t))^n}{n!} (ip_2)^n (ip_{x3})^n \mathcal{F}_{2D}[\psi_5](x_1, p_2, p_{x3}, t)
\]
\[
= e^{-\delta_4(t)p_2p_{x3}} \mathcal{F}_{2D}[\psi_5](x_1, p_2, p_{x3}, t), \quad (C.48)
\]

where in the second line we use the derivative theorem of the two-dimensional FT
\[
\mathcal{F}_{2D}[\partial_x^n \partial_y^n f(x, y)](u, v) = (iu)^n (iv)^n \mathcal{F}_{2D}[f(x, y)](u, v). \quad (C.49)
\]

Taking the two-dimensional IFT of Eq. (C.48) in the \( p_2 \) and the \( p_{x3} \) variables, we accomplished the application of the fourth operator, and we got the final result as
\[ \Psi(x_1, x_2, x_3, t) \]
\[ N(t) \exp \left[ - \left( \varepsilon_1(t)x_1^2 + \varepsilon_2(t)x_2^2 + \varepsilon_3(t)x_3^2 + \varepsilon_4(t)x_1x_2 + \varepsilon_5(t)x_1x_3 + \varepsilon_6(t)x_2x_3 \right) \right]. \]

(C.50)

The wave function given by Eq. (C.50) is a Gaussian wave function with temporal dispersion, and it is entangled between the three position variables \( x_1, x_2 \) and \( x_3 \).

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