WEAKLY MULTIPLICATIVE DISTRIBUTIONS AND WEIGHTED
DIRICHLET SPACES

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Abstract. We show that if $u$ is a compactly supported distribution on the complex plane such
that, for every pair of entire functions $f, g$,

$$\langle u, f \bar{g} \rangle = \langle u, f \rangle \langle u, \bar{g} \rangle,$$

then $u$ is supported at a single point. As an application, we complete the classification of all
weighted Dirichlet spaces on the unit disk that are de Branges–Rovnyak spaces by showing that,
for such spaces, the weight is necessarily a superharmonic function.

1. Introduction

In the first part of this note, we study a family of distributions that possessing a certain weak
multiplicativity property. For general background on distribution theory we refer to the book of
Friedlander [8].

Let $u$ be a compactly supported distribution on the complex plane $\mathbb{C}$. Thus $u$ is a continuous
linear functional on $C_\infty(\mathbb{C})$, where $C_\infty(\mathbb{C})$ is endowed with the Fréchet-space topology of uniform
convergence of all derivatives on compact sets. In what follows, we write $\langle u, \psi \rangle$ for $u(\psi)$.

We say that $u$ is weakly multiplicative if, for every pair of entire functions $f, g$, we have

$$\langle u, f \bar{g} \rangle = \langle u, f \rangle \langle u, \bar{g} \rangle.$$  

Clearly this implies that, for all integers $j, k \geq 0$,

$$\langle u, z^j \bar{z}^k \rangle = \langle u, z^j \rangle \langle u, \bar{z}^k \rangle.$$  

Conversely, if (2) holds for all $j, k \geq 0$, then (1) holds for for all polynomials $f, g$, and because the
Taylor series of an entire function converges to the function in the topology of $C_\infty(\mathbb{C})$, it follows
that (1) holds for all entire functions $f, g$.

It is not hard to see that, if $u$ is strongly multiplicative in the sense that $\langle u, \phi \psi \rangle = \langle u, \phi \rangle \langle u, \psi \rangle$
for all $\phi, \psi \in C_\infty(\mathbb{C})$, then either $u = 0$ or $u = \delta_a$, the Dirac distribution at some $a \in \mathbb{C}$. This
is no longer true for weakly multiplicative distributions, though it is still the case that they must
be supported at a single point. This is the main thrust of our first theorem. As usual, we write
$\partial := (1/2)(\partial_x - i \partial_y)$ and $\overline{\partial} := (1/2)(\partial_x + i \partial_y)$ for the Cauchy–Riemann operators, and $\delta_a$ for the
Dirac distribution at $a$.

Theorem 1.1. Let $u$ be a compactly supported distribution on $\mathbb{C}$. Then $u$ is weakly multiplicative
if and only if $u = 0$ or $u = p(\partial)q(\overline{\partial})\delta_a$, where $a \in \mathbb{C}$ and $p, q$ are complex polynomials with
$p(0) = q(0) = 1$.  

Date: 27 Sep 2021.

2010 Mathematics Subject Classification. 46F05, 46E22.

Key words and phrases. Distribution, weighted Dirichlet space, de Branges–Rovnyak space, superharmonic
function.

First author supported by an NSERC Discovery Grant. Second author supported by grants from NSERC and
the Canada Research Chairs program.
This theorem is a generalization of a result of Youssfi [16, Proposition 1], who treated the special case of measures \( u \) that satisfy (2) (in which case, the conclusion is simply that \( u \) is zero or a Dirac measure). The extra generality afforded by distributions is important for the application that follows.

Youssfi’s proof relies on a result of Luecking [9] characterizing Toeplitz operators on the Bergman space that have finite rank. Luecking’s result applies only to Toeplitz operators with measure-valued symbols, but his result was subsequently extended by Alexandrov and Rozenblum [2] to cover the case of distribution-valued symbols, and in principle their result could be used to deduce Theorem 1.1. However, the Toeplitz operator that arises in our particular situation actually has rank one, which permits a considerable simplification of these ideas, leading to a direct proof of Theorem 1.1. This proof is presented in §2.

In the second part of the note, we present an application of Theorem 1.1 to weighted Dirichlet spaces. Let \( D \) be the open unit disk and let \( dA \) be normalized area measure on \( D \). We shall call \( \omega \) a weight on \( D \) if \( \omega \in L^1(D, dA) \) and \( \omega \geq 0 \). The weighted Dirichlet space \( D_\omega \) is the set of holomorphic functions \( f \) on \( D \) such that

\[
D_\omega(f) := \int_D |f'(z)|^2 \omega(z) dA(z) < \infty.
\]

It is known that, for certain weights, \( D_\omega \) is a de Branges–Rovnyak space. What this means and why it is significant will be explained in §3. The superharmonic weights \( \omega \) for which \( D_\omega \) is a de Branges–Rovnyak space were classified in [5]. The following result, which is an application of Theorem 1.1, takes care of the case of non-superharmonic weights, thereby completing the classification.

**Theorem 1.2.** If \( \omega \) is a weight on \( D \) such that \( D_\omega \) is a de Branges–Rovnyak space, then \( \omega \) is (almost-everywhere equal to) a function that is superharmonic on \( D \).

The theory of Dirichlet spaces with harmonic weights was developed by Richter [11] and Richter–Sundberg [12], and extended to the case of superharmonic weights by Aleman [1]. There are now many beautiful results in this area, notably Shimorin’s theorem [15] that \( D_\omega \) has a complete Pick kernel whenever \( \omega \) is a superharmonic weight. Theorem 1.2 may be viewed as a further vindication that the superharmonic weights form a natural class. The theorem is proved in §3.

2. **Weakly multiplicative distributions**

To prove Theorem 1.1 we first establish a density result for \( C^\infty(\mathbb{C}^2) \). As usual, we endow this space with the Fréchet-space topology of uniform convergence of all derivatives on compact subsets.

**Lemma 2.1.** Functions of the form \(|r(z_1, z_2)|^2\), where \( r \) is a polynomial, span a dense subspace of \( C^\infty(\mathbb{C}^2) \).

**Proof.** Let \( v \) be a compactly supported distribution on \( \mathbb{C}^2 \) such that, for all polynomials \( r(z_1, z_2) \),

\[
\langle v, |r|^2 \rangle = 0.
\]

By polarization, it follows that, for all polynomials \( p(z_1, z_2) \) and \( q(z_1, z_2) \), we have

\[
\langle v, pq \rangle = 0.
\]

Since the Taylor series of an entire function on \( \mathbb{C}^2 \) converges to the function in the topology of \( C^\infty(\mathbb{C}^2) \), we deduce that, for all entire functions \( f, g \) on \( \mathbb{C}^2 \),

\[
\langle v, fg \rangle = 0.
\]
In particular, taking \( f(z_1, z_2) := e^{(b-ia)z_1/2+(d-ic)z_2/2} \) and \( g(z) := e^{-(b-ia)z_1/2-(d-ic)z_2/2} \), we see that, for all \( a, b, c, d \in \mathbb{R} \),

\[
\langle v, e^{-i(ax_1+by_1+cz_2+dy_2)} \rangle = 0.
\]

This amounts to saying that the Fourier transform \( \hat{v} \) of \( v \) satisfies \( \hat{v}(a, b, c, d) = 0 \) for all \( a, b, c, d \in \mathbb{R} \), whence also \( v = 0 \).

We have shown that the only continuous linear functional on \( C^\infty(\mathbb{C}^2) \) vanishing on all functions of the form \(|r(z_1, z_2)|^2\) is the zero functional. By the Hahn–Banach theorem, functions of this form span a dense subspace of \( C^\infty(\mathbb{C}^2) \).

**Proof of Theorem 1.1.** If \( u = p(\partial)q(\partial)\delta_a \) with \( p(0) = q(0) = 1 \), then, for every pair of entire functions \( f, g \), we have

\[
\langle u, f \rangle = \langle q(\partial)p(\partial)f \rangle(a) = (p(\partial)f)(a)(q(\partial)\delta)(a) = \langle u, f \rangle \langle u, \delta \rangle,
\]

and thus (1) holds. Obviously (1) also holds if \( u = 0 \).

We now turn to the converse. Let \( u \) be a compactly supported distribution on \( \mathbb{C} \) such that (1) holds for every pair of entire functions \( f, g \). Consider the tensor product \( u \otimes u \). By definition, this is the unique compactly supported distribution on \( \mathbb{C}^2 \) such that

\[
\langle u \otimes u, \psi_1(z_1)\psi_2(z_2) \rangle = \langle u_1, \psi_1 \rangle \langle u, \psi_2 \rangle \quad (\psi_1, \psi_2 \in C^\infty(\mathbb{C})).
\]

Let \( j, k, m, n \) be non-negative integers, and consider the expression

\[
\langle u \otimes u, (z_1 - z_2)(z_1^j z_2^k + z_1^k z_2^j) \rangle.
\]

Expanding this out using (3) and then (2), we find that it is equal to 0. Any polynomial \( s(z_1, z_2) \) that is symmetric (i.e. \( s(z_1, z_2) = s(z_2, z_1) \)) is a linear combination of polynomials of the form \((z_1^j z_2^k + z_1^k z_2^j)\). Hence, given a symmetric polynomial \( s(z_1, z_2) \) and an arbitrary polynomial \( t(z_1, z_2) \), we have

\[
\langle u \otimes u, (z_1 - z_2)s(z_1, z_2)t(z_1, z_2) \rangle = 0.
\]

In particular, taking \( t(z_1, z_2) := (z_1 - z_2)s(z_1, z_2) \), we deduce that, for all symmetric polynomials \( s(z_1, z_2) \),

\[
\langle u \otimes u, |z_1 - z_2|^2|s(z_1, z_2)|^2 \rangle = 0.
\]

Our goal now is to show that the support of the distribution \( u \otimes u \) is contained in the diagonal set \( \Delta := \{(z_1, z_2) \in \mathbb{C}^2 : z_1 = z_2 \} \). Let \( \psi \in C^\infty(\mathbb{C}^2) \) be a function such that \( \psi = 0 \) on an open neighborhood of \( \Delta \). Define \( \rho : \mathbb{C}^2 \to \mathbb{C} \) by

\[
\rho(w_1, w_2) := \psi \left( \frac{w_1}{2} + \frac{1}{2} \sqrt{w_1^2 - 4w_2} - \frac{w_1}{2} - \frac{1}{2} \sqrt{w_1^2 - 4w_2} \right) + \psi \left( \frac{w_1}{2} - \frac{1}{2} \sqrt{w_1^2 - 4w_2} + \frac{w_1}{2} + \frac{1}{2} \sqrt{w_1^2 - 4w_2} \right).
\]

The symmetry in the definition ensures that \( \rho \) is well defined, and the fact that \( \psi \) vanishes on a neighborhood of \( \Delta \) ensures that \( \rho \in C^\infty(\mathbb{C}^2) \). By Lemma 2.1, \( \rho(w_1, w_2) \) is the limit in \( C^\infty(\mathbb{C}^2) \) of finite linear combinations of functions of the form \(|r(w_1, w_2)|^2\) where \( r \) is a polynomial. Therefore \( \rho(z_1 + z_2, z_1 z_2) \) is the limit in \( C^\infty(\mathbb{C}^2) \) of finite linear combinations of functions of the form \(|r(z_1 + z_2, z_1 z_2)|^2\) where \( r \) is a polynomial. Since \( r(z_1 + z_2, z_1 z_2) \) is a symmetric polynomial, (4) implies that

\[
\langle u \otimes u, |z_1 - z_2|^2|z_1 + z_2, z_1 z_2|^2 \rangle = 0.
\]

By linearity and continuity, it follows that

\[
\langle u \otimes u, |z_1 - z_2|^2 \rho(z_1 + z_2, z_1 z_2) \rangle = 0.
\]
However, a simple computation shows that, for all \( z_1, z_2 \in \mathbb{C} \),
\[
\rho(z_1 + z_2, z_1 z_2) = \psi(z_1, z_2) + \psi(z_2, z_1),
\]
whence, using the symmetry of \( u \otimes u \) and \( |z_1 - z_2|^2 \), we obtain
\[
\left\langle u \otimes u, |z_1 - z_2|^2 \psi(z_1, z_2) \right\rangle = 0.
\]
As this holds for all \( \psi \in C^\infty(\mathbb{C}^2) \) that vanish on an open neighborhood of \( \Delta \), we conclude that \( \text{supp}(u \otimes u) \subset \Delta \), as claimed.

Now it is well known that, in general, \( \text{supp}(u \otimes u) = \text{supp} u \times \text{supp} u \) (see e.g. [8, Theorem 4.3.3 (ii)]). Together with the inclusion \( \text{supp}(u \otimes u) \subset \Delta \), this implies that either \( \text{supp} u \) is empty or that it consists of a single point in \( \mathbb{C} \). If \( \text{supp} u \) is empty, then \( u = 0 \). If \( \text{supp} u = \{ a \} \) for some \( a \in \mathbb{C} \), then, by [8, Theorem 3.2.1], \( u \) must be a finite linear combination of derivatives of the Dirac distribution at \( a \), so it can be written as
\[
u = \sum_{j,k \leq N} c_{jk} \partial^j \overline{a}^k \delta_a,
\]
for some choice of coefficients \( c_{jk} \in \mathbb{C} \). In this case, for all integers \( m,n \geq 0 \),
\[
\left\langle u, (z-a)^m \overline{(z-a)^n} \right\rangle = (-1)^{m+n} m! n! c_{mn}.
\]
But also, by (11), we have
\[
\left\langle u, (z-a)^m \overline{(z-a)^n} \right\rangle = \left\langle u, (z-a)^m (u, (z-a)^n) \right\rangle.
\]
It follows that \( c_{mn} = c_{m0} c_{0n} \) for all \( m, n \). In particular, \( c_{00} = c_{00}^2 \), so \( c_{00} = 0 \) or \( 1 \). If \( c_{00} = 0 \), then \( c_{m0} = c_{0n} = 0 \) for all \( m, n \), whence \( c_{mn} = 0 \) for all \( m, n \), and so \( u = 0 \). If \( c_{01} = 1 \), then
\[
u = \left( \sum_{j \leq N} c_{j0} \partial^j \right) \left( \sum_{k \leq N} c_{0k} \overline{a}^k \right) \delta_a = p(\partial)q(\overline{\partial}) \delta_a,
\]
where \( p, q \) are polynomials with \( p(0) = q(0) = 1 \). This completes the proof. \( \square \)

3. Weighted Dirichlet spaces and de Branges–Rovnyak spaces

Given a holomorphic function \( b : \mathbb{D} \to \overline{\mathbb{D}} \), the associated de Branges–Rovnyak space \( \mathcal{H}(b) \) is the unique reproducing kernel Hilbert space with kernel
\[
\frac{1 - b(z)\overline{b(w)}}{1 - z\overline{w}}.
\]
It is always a subspace of the Hardy space \( H^2 \), but not necessarily closed in \( H^2 \). For background on de Branges–Rovnyak spaces, we refer to the books of Sarason [13] and Fricain–Mashreghi [6, 7].

In this section we address the following question: For which weights \( \omega \) is \( \mathcal{D}_\omega \) a de Branges–Rovnyak space? By this we mean that there exists a holomorphic function \( b : \mathbb{D} \to \overline{\mathbb{D}} \) such that \( \mathcal{D}_\omega = \mathcal{H}(b) \) as sets and also such that
\[
\|f\|_{\mathcal{H}(b)}^2 = \|f\|_{H^2}^2 + \mathcal{D}_\omega(f) \quad (f \in \mathcal{D}_\omega).
\]

The first examples of such weights were given by Sarason [14]. He showed that, for each of the weights
\[
\omega_\zeta(z) := \frac{1 - |z|^2}{|z - \zeta|^2} \quad (\zeta \in \partial \mathbb{D}),
\]
the space \( \mathcal{D}_{\omega_\zeta} \) is a de Branges–Rovnyak space. From this fact, he was able to deduce an interesting inequality for the dilations \( f_r(z) := f(rz) \) \((0 < r < 1)\) of a function \( f \in \text{Hol}(\mathbb{D}) \), namely that \( \mathcal{D}_\omega(f_r) \leq \mathcal{D}_\omega(f) \), not just for \( \omega = \omega_\zeta \), but for all harmonic weights \( \omega \) on \( \mathbb{D} \).
In view of the fact that this last inequality holds for all harmonic weights, one might wonder if indeed \( \mathcal{D}_\omega \) is a de Branges–Rovnyak space for every harmonic weight on \( \mathbb{D} \). The authors of [3] showed that this is actually not the case. In fact, among harmonic weights, the only ones for which \( \mathcal{D}_\omega \) is a de Branges–Rovnyak space are positive multiples of those displayed in (6).

The authors of [5] took this analysis one step further by examining superharmonic weights. They showed that, within this class, there is a new family of weights for which \( \mathcal{D}_\omega \) is a de Branges–Rovnyak space, namely those of the form

\[
\omega(z) := \log \left| \frac{1 - \overline{\zeta} z}{z - \zeta} \right| \quad (\zeta \in \mathbb{D}).
\]

They further showed that, among superharmonic weights, the only ones for which \( \mathcal{D}_\omega \) is a de Branges–Rovnyak space are positive multiples of those displayed in (6) and (7).

The article [5] concluded with the question of what happens in the case of non-superharmonic weights. Theorem 1.2 above answers this question.

To prove this theorem, we need the following lemma, essentially taken from [5].

**Lemma 3.1.** Let \( \omega \) be a weight on \( \mathbb{D} \) such that \( \|\omega\|_{L_1(\mathbb{D})} = 1 \). If \( \mathcal{D}_\omega \) is a de Branges–Rovnyak space, then there exists a holomorphic function \( \phi \) on \( \mathbb{D} \) such that

\[
\frac{|\phi(w)|^2}{1 - |w|^2} = \frac{1}{\pi} \int_{\mathbb{D}} \frac{|w|^2}{|1 - zw|^4} \omega(z) \, dA(z) \quad (w \in \mathbb{D}).
\]

**Proof.** See the proof of [5] Theorem 6.2, and in particular the formula (6.2) in that proof. \( \square \)

**Remarks.** (i) The function \( \phi \) in Lemma 3.1 is closely related to the function \( b \) such that \( \mathcal{D}_\omega = \mathcal{H}(b) \). In fact \( \phi(z) = b(z)/a(z) \), where \( a \) is the unique outer function on \( \mathbb{D} \) such that \( a(0) > 0 \) and \( |a(e^{i\theta})|^2 = 1 - |b(e^{i\theta})|^2 \) a.e. on \( \partial \mathbb{D} \). For more on this, see [5]. However, we do need this fact here.

(ii) The assumption that \( \omega \) be normalized so that \( \|\omega\|_{L_1(\mathbb{D})} = 1 \) is not a serious restriction. Indeed, if \( \mathcal{D}_\omega \) is a de Branges–Rovnyak space, then so is \( \mathcal{D}_{c\omega} \) for each positive constant \( c \).

**Proof of Theorem 1.2.** Let \( \omega \) be a weight on \( \mathbb{D} \) such that \( \mathcal{D}_\omega \) is a de Branges–Rovnyak space. We can suppose that \( \omega \) is normalized so that \( \|\omega\|_{L_1(\mathbb{D})} = 1 \), so, by Lemma 3.1, there exists a holomorphic function \( \phi \) on \( \mathbb{D} \) such that (8) holds. Clearly \( \phi(0) = 0 \), so \( h(w) := \phi(w)/w \) has a removable singularity at \( w = 0 \). Thus

\[
\int_{\mathbb{D}} \frac{1}{|1 - zw|^4} \omega(z) \, dA(z) = |h(w)|^2 \quad (w \in \mathbb{D}).
\]

Extend \( \omega \) to the whole of \( \mathbb{C} \) by defining \( \omega := 0 \) on \( \mathbb{C} \setminus \mathbb{D} \). Then \( \omega \) is a compactly supported distribution on \( \mathbb{C} \) and, in the notation of distributions, (9) may be rewritten as

\[
\frac{1}{\pi} \left< \omega(z), \frac{1 - |w|^2}{|1 - zw|^4} \right> = |h(w)|^2 \quad (w \in \mathbb{D}),
\]

the \( \pi \) appearing because \( dA \) is normalized area measure on \( \mathbb{D} \). Now a computation shows that, for all \( z, w \in \mathbb{D} \),

\[
\Delta_z \left( \frac{1 - |z|^2}{|1 - zw|^2} \right) = 4 \Delta_z \overline{z} \left( \frac{1 - \overline{z} z}{(1 - zw)(1 - \overline{zw})} \right) = -4 \frac{1 - |w|^2}{|1 - zw|^4}.
\]

Hence

\[
- \frac{1}{4\pi} \left< \omega(z), \Delta_z \frac{1 - |z|^2}{|1 - zw|^2} \right> = |h(w)|^2 \quad (w \in \mathbb{D}),
\]

and so, writing \( u := -(1/4\pi)(1 - |z|^2)\Delta \omega \), we have

\[
\left< u(z), \frac{1}{|1 - zw|^2} \right> = |h(w)|^2 \quad (w \in \mathbb{D}).
\]
Expanding both sides out in powers of \( w \) and \( \overline{w} \), and equating coefficients of \( \overline{w}^j w^k \), we find that
\[
\langle u, z^j \overline{z}^k \rangle = 7_j h_k,
\]
where \( h(w) = \sum_j h_j w^j \) is the Taylor development of \( h \). Note also that, from formula (9), we have \( h_0 = h(0) = \|\omega\|_{L^1(D)} = 1 \). Hence, for all integers \( j, k \geq 0 \), we have
\[
\langle u, z^j \overline{z}^k \rangle = \langle u, z^j \rangle \langle u, \overline{z}^k \rangle,
\]
in other words, \( u \) is a weakly multiplicative distribution. We now invoke Theorem 1.1. By that theorem, \( u \) is supported at a single point \( a \in \mathbb{C} \). Thus \( u = 0 \) on \( D \setminus \{a\} \), whence \( \Delta \omega = 0 \) on \( D \setminus \{a\} \). By Weyl’s lemma, \( \omega \) is (almost everywhere equal to) a function that is harmonic on \( D \setminus \{a\} \). Finally, since \( \omega \geq 0 \) on \( D \setminus \{a\} \), it can be extended so as to be superharmonic on \( D \) (see e.g. [10] Theorem 3.6.1). This completes the proof. \( \square \)

Remarks. (i) From here it is a small step to recover explicitly the weights \( \omega \) for which \( \mathcal{D}_\omega \) is a de Branges–Rovnyak space. Indeed, knowing that \( \omega \) is a positive superharmonic function on \( D \), we may write it as the sum of a Green potential and positive harmonic function on \( D \) (see e.g. [10] Theorem 4.5.4)). Thus there exist finite positive Borel measures \( \mu \) on \( D \) and \( \nu \) on \( \partial D \) such that
\[
\omega(z) = \int_D \log \left| \frac{1 - \overline{\zeta} z}{\zeta - z} \right| \frac{2}{1 - |\zeta|^2} d\mu(\zeta) + \int_{\partial D} \frac{1 - |z|^2}{|\zeta - z|^2} dv(\zeta) \quad (z \in D).
\]
A computation then shows that, as distributions on \( C \),
\[
-(1/4\pi)(1 - |z|^2)\Delta \omega = \mu + \nu.
\]
But also, from the proof of Theorem 1.1 we know that \( -(1/4\pi)(1 - |z|^2)\Delta \omega \) is supported at a single point. This implies that one of \( \mu, \nu \) is a multiple of a Dirac measure while the other one is zero. This in turn proves that \( \omega \) is a multiple of one of the weights displayed in (6) and (7).

(ii) If \( \mathcal{D}_\omega \) is a de Branges–Rovnyak space, then \( \mathcal{D}_\omega = \mathcal{H}(b) \) for some \( b \) unique up to multiplication by a unimodular constant. We can recover \( b \) from \( \omega \) by exploiting the fact that \( b(z) = \phi(z)a(z) \), where \( \phi \) satisfies (8) and \( a \) is the outer function such that \( |a(e^{i\theta})|^2 = 1/(1 + |\phi(e^{i\theta})|^2) \) a.e. on \( \partial D \). This procedure is carried out in [5], and we do not repeat the details here.

(iii) We have classified those weights \( \omega \) on \( D \) for which \( \mathcal{D}_\omega \) is a de Branges–Rovnyak space in the sense that \( \mathcal{D}_\omega = \mathcal{H}(b) \) for some \( b \) and also such that (5) holds. One might also ask which weights \( \omega \) satisfy \( \mathcal{D}_\omega = \mathcal{H}(b) \) for some \( b \), without requiring that (5) holds. Rather less is known about this problem. Some partial results may be found in [4].

Acknowledgements

The authors are grateful to Trieu Le for drawing the article [2] to their attention, and to Omar El-Fallah, Karim Kellay and Vadim Ognov for helpful discussions.

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