Moduli space of logarithmic states in AdS$_3$/LCFT$_2$

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Abstract
In this work, we extend results recently obtained concerning partition functions in the AdS$_3$/LCFT$_2$ correspondence. Originally derived by Grumiller, Gaberdiel and Vassilevich in the theory of topologically (and new) massive gravity at the critical point, these partition functions were recently reformulated in terms of Bell polynomials. The latter were shown to be generated by a function that is identical to the plethystic exponential. We exploit these reformulations to study the moduli space of the logarithmic states, and show using algebro-geometric methods that it describes an $S^n(\mathbb{C}^2)$ orbifold. We also show a connection between this quotient of a complex vector space by the symmetric group action and the untwisted sector of symmetric orbifold CFT models, looking at their generating functions. Furthermore, the action of differential operators on symmetric orbifolds is presented.

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1 Introduction

Gravity in three dimensions has for some time now been an interesting model to test theories of -classical and quantum-gravity, and a consistent non-trivial theory would bring the prospect of clarifying many intricate aspects of gravity. A fundamental breakthrough was made in the study of the asymptotics, revealing the emergence of a Virasoro algebra at the boundary [1]. One can thus expect a dual 2d CFT description, and this discovery can be thought of as a precursor of the AdS/CFT correspondence. However, pure Einstein gravity in three dimensions is locally trivial at the classical level and does not exhibit propagating degrees of freedom. Hence there was a need to modify it.

One way of deforming pure Einstein gravity is by introducing a negative cosmological constant, leading to a theory with black hole solutions [2]. Another possibility of deformation is to add gravitational Chern-Simons term. In that case the theory is called topologically massive gravity (TMG), and contains a massive graviton [3] [4]. When both cosmological and Chern-Simons terms are included in a theory, it yields cosmological topologically massive gravity (CTMG). Such theory features both gravitons and black holes.

Following Witten’s proposal in 2007 to find a CFT dual to Einstein gravity [5], the graviton 1-loop partition function was calculated in [6]. However, discrepancies were found in the results. In particular, the left- and right-contributions did not factorize, therefore clashing with the proposal of [5].

Soon after, a non-trivial slightly modified version of Witten’s construct was proposed by Li, Song and Strominger [7]. Their theory, in which Einstein gravity was replaced by chiral gravity can be viewed as a special case of topologically massive gravity [3] [4], at a specific tuning of the couplings, and is asymptotically defined with AdS3 boundary conditions, according to Feferrman-Graham-Brown-Henneaux [1]. A particular feature of the theory was that one of the two central charges vanishes, whilst the other one can have a non-zero value. This gave an indication that the partition function could factorize.

Shortly after the proposal of [7], Grumiller et al. noticed that relaxing the Brown-Henneaux boundary conditions allowed for the presence of a massive mode that forms a Jordan cell with the massless graviton, leading to a degeneracy at the critical point [8]. In addition, it was observed that the presence of the massive mode spoils the chirality of the theory, as well as its unitarity. Based on these results, the dual CFT of critical cosmological topologically massive gravity (CTMG) was conjectured to be logarithmic, and the massive mode was called the logarithmic partner of the graviton. Indeed, Jordan cell structures are a noticeable feature of logarithmic CFTs, that are non unitary theories (see [9] as well as the very nice introductory notes [10] and [11]). The correspondence distinguishes itself on the conjectured dual LCFT side by a left-moving energy-momentum tensor $T$ that has a logarithmic partner state $t$ with identical conformal weight, forming the following Jordan cell

$$L_0 \begin{pmatrix} |T\rangle \\ |t\rangle \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} |T\rangle \\ |t\rangle \end{pmatrix} .$$

(1)

A major achievement towards the formulation of the correspondence was the calculation of correlation functions in TMG [12, 13], which confirmed the existence of logarithmic correlators of the type $\langle T(x)T(y) \rangle = b_L/(x-y)^4$ that arise in LCFT, with $b_L$ commonly referred as the logarithmic anomaly. Subsequently, the 1-loop graviton partition function of cTMG on the thermal AdS3 background was calculated in [14], resulting in the following expression

$$Z_{cTMG}(q, \bar{q}) = \prod_{n=2}^{\infty} \frac{1}{1 - q^n} \prod_{m=2}^{\infty} \prod_{n=0}^{\infty} \frac{1}{1 - q^m \bar{q}^n},$$

(2)

where the first product can be identified as the three-dimensional gravity partition function $Z_{0,1}$ in [6], and is therefore not modular invariant. The double product is the partition function of the logarithmic single and multi particle logarithmic states, and will be the central object of this work.

The corresponding expression of the partition function on the CFT side was derived in [14] and given the form

$$Z_{LCFT}(q, \bar{q}) = Z_{LCFT}^0(q, \bar{q}) + \sum_{h, \bar{h}} N_{h, \bar{h}} q^h \bar{q}^{\bar{h}} \prod_{n=1}^{\infty} \frac{1}{1 - q^n},$$

(3)
with

\[ Z^0_{\text{LCTF}}(q, \bar{q}) = Z_\Omega + Z_t = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \left( 1 + \frac{q^2}{|1-q|^2} \right), \]

where \( \Omega \) is the vacuum of the holomorphic sector, and \( t \) denotes the logarithmic partner of the energy momentum tensor \( T \).

As pointed out in [15], a better understanding of the partition function from the CFT side is desirable, in particular how to precisely match the combinatorics of multi particle logarithmic states on the gravity side to states on the CFT side. Motivated by that, the partition function was reformulated in [16], and recast in terms of Bell polynomials. Furthermore, it was shown that the partition function could be rewritten using the more usual language of Hilbert series, leading to an identity between the generating function of Bell polynomials and the celebrated plethystic exponential. In addition, the Bell polynomials formalism showed an interesting use in revealing hidden symmetry actions on the \( n \)-particle terms of the partition function (this point will be given an interpretation in this work).

Despite the aforementioned achievements made since the conjecture of the \( \text{AdS}_3/\text{LCFT}_2 \) in 2008, it is fair to say that very little is known about the nature of the logarithmic states. In this paper, we try to deal with this issue by exploiting the results of [16] to study the moduli space of the logarithmic sector. The concept of moduli space originates from algebraic geometry. The behaviour of certain geometric objects such as collections of \( n \) distinct ordered points on a given topological space can be understood by finding a space \( X \) which parametrizes these objects, i.e a space whose points are in bijection with equivalence classes of these objects. A space \( X \) with such a correspondence is called a moduli space, and it parametrizes the types of objects of interest, which in our case will be the logarithmic states. The geometry of moduli spaces can be encoded in their generating functions. We take advantage of that fact to give a symmetric group interpretation of the results obtained in [16], and to show that the moduli space of the logarithmic states is an \( S^n(C^2) \) orbifold.

This paper is organized as follows. In section 2, we give a brief description of partition functions in critical massive gravities. In particular, we recall how some infinite products can usefully be rewritten as generating functions of Bell polynomials. This is an interesting application of Bell polynomials in theoretical physics. In section 3, we discuss the symmetric product structure that appears from the sub-partition function of the logarithmic sector. We start by showing that the latter, expressed in terms of Bell polynomials is in fact the cycle index of the symmetric group, i.e a polynomial in several variables that counts objects that are invariant under the action of the symmetric group. Then, in the spirit of Hilbert schemes of point on surfaces, and from their connection to symmetric products, we show that the moduli space of the logarithmic states is the \( n \)-th symmetric product of \( C^2 \). This is followed by showing that the partition function of the logarithmic states is an index of the untwisted sector of symmetric orbifold CFT models. In section 4, we revisit results from [16] through the construction of differential operators acting on the generating function of \( S^n(C) \) orbifolds. Based on the results obtained, we make a conjecture on the nature of the logarithmic states in section 5. Finally, a conclusion and a discussion on future work are given in section 6.

## 2 Partition functions of critical massive gravities

The graviton 1-loop partition function of cosmological topologically massive gravity and new massive gravity both at the critical point were calculated in [14]. In the case of topologically massive gravity, the computation was given the form

\[ Z_{\text{cTMG}}(q, \bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \prod_{m=2}^{\infty} \prod_{n=0}^{\infty} \frac{1}{1-q^n q^m}. \]

and for new massive gravity, the partition function was derived as

\[ Z_{\text{cNMG}}(q, \bar{q}) = \prod_{n=2}^{\infty} \frac{1}{|1-q^n|^2} \prod_{m=2}^{\infty} \prod_{n=0}^{\infty} \frac{1}{|1-q^n q^m|} \prod_{l=2}^{\infty} \prod_{t=0}^{\infty} \frac{1}{1-q^l q^t}. \]

Shortly after these results, topologically massive gravity was generalized to higher spins in [17], and the 1-loop partition function for topologically massive higher spin gravity (cTMHSG) for arbitrary spin was...
calculated in [18]. A special attention was given to spin-3 case for which the partition function was expressed as

\[
Z_{cTMHSC}^{(3)}(q, \bar{q}) = \prod_{n=2}^{\infty} \frac{1}{1 - q^n} \prod_{n=3}^{\infty} \frac{1}{1 - q^n q^m} \times \left[ \prod_{n=3}^{\infty} \frac{1}{1 - q^n} \prod_{n=3}^{\infty} \frac{1}{1 - q^n q^m} \prod_{m=4}^{\infty} \frac{1}{1 - q^m q^m} \right].
\]  

(7)

Recently, motivated by the desire for a better understanding of the combinatorics of the logarithmic excitations in the partition function of these critical massive gravities, and with the eventual goal of having a more concrete grasp of their conjectured holographic (L)CFT duals, the partition functions were shown to be usefully expressed in terms of Bell polynomials [16].

Bell polynomials are very useful in many areas of mathematics and have enjoyed many applications in physics as well. For instance, recently expressions of canonical and grand canonical partition functions of interacting quantum gases of Statistical Mechanics systems were rederived in terms of Bell Polynomials by the authors of [19], in which the Bell polynomials are the Mayer (cluster) expansion. Also, in [20] [21] [22] [23] and references therein, the use of Bell polynomials is discussed for partition functions, suggesting some interaction of particles in the theories concerned.

Bell polynomials have also appeared in the study of partition functions of BPS bosonic operators. Following [24], if we consider such partition functions at finite \( N \), and denote them \( Z_k(\beta; N) \) where \( \beta = (\beta_1, \ldots, \beta_k) \) is a set of \( k \) chemical potentials conjugate to \( n_i \), the quantum numbers of the various conserved charges in the superconformal theory in question, and \( N \) the rank of the gauge group, then these partition functions typically take the following expression

\[
Z_k(\beta, p) = \prod_{n_1, n_2, \ldots, n_k \geq 0} \frac{1}{1 - p \exp\left\{(-\beta \cdot \vec{n})\right\}},
\]

(8)

where the infinite product converges if \(|p| < 1\) and Re\((\beta_i) > 0\). Furthermore, the infinite product is the generating function for \( Z_k(\beta; N) \) expressed as

\[
Z_k(\beta, p) = \sum_{N=0}^{\infty} Z_k(\beta; N)p^N.
\]

(9)

Eq. (9) is the grand canonical partition function for bosons in a \( k \)-dimensional harmonic oscillator potential with \( p \) as the fugacity, defined as the chemical potential that keeps track of particle number \( N \). These partition functions correspond for instance to \( \frac{1}{4} \)-BPS or \( \frac{1}{4} \)-BPS states in \( \mathcal{N} = 4 \) SYM when \( k = 1, 2 \) respectively, or to \( \frac{1}{4} \)-BPS states in the M2-brane world-volume for \( k = 4 \) and to \( \frac{1}{4} \)-BPS states in the M5-brane world-volume for \( k = 2 \) (the \((2,0)\) SCFT in six dimensions) [24].

In [16], it was shown that the partition functions of critical gravities can be expressed in terms of Bell polynomials. In the specific case of critical cosmological TMG, writing Eq. (2) as

\[
Z_{cTMG}(q, \bar{q}) = Z_{\text{gravity}}(q, \bar{q}) \cdot Z_{\log}(q, \bar{q}),
\]

(10)

where

\[
Z_{\text{gravity}}(q, \bar{q}) = \prod_{n=2}^{\infty} \frac{1}{1 - q^n q^m}, \quad \text{and} \quad Z_{\log}(q, \bar{q}) = \prod_{m=2}^{\infty} \prod_{m=0}^{\infty} \frac{1}{1 - q^m q^m},
\]

(11)

it was shown that \( Z_{\log}(q, \bar{q}) \) is the generating function of Bell polynomials.
In Eq. (12), $Y_n$ is the (complete) Bell polynomial with variables $g_1, g_2, \ldots, g_n$ such that

$$Y_n(g_1, g_2, \ldots, g_n) = \sum_{\vec{k} \vdash n} \frac{n!}{k_1! \cdots k_n!} \prod_{j=1}^{n} \left( \frac{g_j^2}{j^2} \right)^{k_j},$$

with

$$\vec{k} \vdash n = \{ \vec{k} = (k_1, k_2, \ldots, k_n) \mid k_1 + 2k_2 + 3k_3 + \cdots + nk_n = n \},$$

and

$$g_n = (n-1)! \sum_{m \geq 0, \bar{m} \geq 0} q^{nm} \bar{q}^{\bar{m}} = (n-1)! \frac{1}{|1 - q^n|^2}.$$

Similarly, Eq. (6) takes the form

$$Z_{cNMG}(q, \bar{q}) = Z_{\text{gravity}}(q, \bar{q}) \cdot Z_{\log}(q, \bar{q}) \cdot \bar{Z}_{\log}(q, \bar{q}),$$

while Eq. (7) can be written as

$$Z^{(3)}_{cTMHSG}(q, \bar{q}) = \chi_0(W_3) \times \bar{\chi}_0(W_3) \left( \sum_{m=0}^{\infty} \frac{Y_m}{m!} (q^2)^m \right) \left( \sum_{l=0}^{\infty} \frac{Y_l}{l!} (\bar{q}^2)^l \right),$$

with $\chi_0(W_3)$ and $\bar{\chi}_0(W_3)$ as the holomorphic and antiholomorphic $W_3$ vacuum characters.

The logarithmic partition function can therefore be given a general form that reads

$$Z_{\log}(\nu; q, \bar{q}) = \prod_{m=0}^{\infty} \prod_{n=0}^{\infty} \frac{1}{1 - \nu q^n \bar{q}^m} = \sum_{n=0}^{\infty} \frac{Y_n}{n!} \nu^n,$$

where the variable $\nu$ represents a monomial in (highest) weight that can be holomorphic denoted by $q^h$ or antiholomorphic denoted by $\bar{q}^\bar{h}$, with $h$ and $\bar{h}$ as the conformal weights of holomorphic and antiholomorphic logarithmic partner states.

Since the conjecture of critical topologically massive gravity as the dual of a logarithmic conformal field theory, it is fair to say that little work has been done in the description of the logarithmic states. In the next section, we would like to make a few steps in that direction by using results recently obtained from the partition function of critical massive gravities to extract information about the moduli space of the logarithmic states.
3 Symmetric product and orbifold in AdS$_3$/LCFT$_2$

Moduli spaces of objects associated with a given space $X$ encode interesting and surprising information about the geometry and topology of $X$. In the present case, the objects under consideration are the logarithmic partner states in critical massive gravities, associated with a quotient of the space $X = \mathbb{C}^2$.

This section deals with the description of the moduli space of the logarithmic sector appearing in critical massive gravities. In order to study the moduli space, we use the partition function calculated in [14], and exploit the fact that it was rederived in two different ways in [16]. We start this section by showing that the space associated with logarithmic states is invariant under the action of the symmetric group.

3.1 Symmetry and the cycle index

In combinatorics, only few formulae can be applied systematically in all cases of a given problem. Pólya theory is one such example, as it enables to count the number of items under specific constraints, such as number of colors or more generally symmetry.

From a group theory perspective, counting objects such as states "up to symmetry" means counting the orbits of some group of symmetries on the set of states that are being counted. A standard procedure to solve this problem is to use the orbit-counting (Burnside’s) lemma [25]. Alternatively, the counting can be made systematic by using a multivariate polynomial associated with a permutation group, called the cycle index.

Consider a finite set denoted by $\mathcal{S}$. A permutation of $\mathcal{S}$ is a one-to-one mapping of $\mathcal{S}$ onto itself. Given a permutation $\pi$, $\mathcal{S}$ can be split into cycles, which are subsets of $\mathcal{S}$ subject to cyclic permutation by $\pi$. Denoting the length of the cycle by $l$, and any element of the cycle by $c$, the cycle can then be expressed as

$$c, \pi c, \pi^2 c, \ldots, \pi^{l-1} c,$$

with $\pi^2 = \pi\pi$, $\pi^3 = \pi\pi\pi$, etc...

**Definition 3.1.** Let $\mathcal{S} = \{1, \ldots, r\}$ be a finite set. If $\mathcal{S}$ splits into $c_1$ cycles of length 1, $c_2$ cycles of length 2, etc..., then $\pi$ is of type $[c_1, c_2, c_3, \ldots]$.

**Definition 3.2.** Let $G$ be the group whose elements $g$ are the permutations of $\mathcal{S}$, and let $Z_G(x_1, x_2, \ldots, x_l)$ be the polynomial in $l$ variables $x_1, x_2, \ldots, x_l$ such that for each $g \in G$, the type of $g$ is given by the product $z_g(x_1, x_2, \ldots, x_l) = x_1^{c_1}x_2^{c_2} \cdots x_l^{c_l}$ as the partition of $r$ $[1^{c_1}2^{c_2}\ldots l^{c_l}]$, with $r = c_1 + 2c_2 + \ldots + lc_l$. Then, the polynomial

$$Z_G(x_1, x_2, \ldots, x_l) = \frac{1}{|G|} \sum_{g \in G} z_g(x_1, x_2, \ldots, x_l)$$

is defined as the cycle index of $G$, with $f(c_1, \ldots, c_l)$ as the number of permutations of type $[1^{c_1}2^{c_2}\ldots l^{c_l}]$.

The formula above is reminiscent of Burnside’s lemma, except that here, one distinguishes the cycles of different lengths, and specifies the number of cycles there are.

In the present case, we are interested in the cycle index of the symmetric group of degree $n$ denoted by $S_n$. It is defined by the formula [25]

$$Z(S_n) = \sum_{c_1 + 2c_2 + \ldots + lc_l = n} \frac{1}{l!} \prod_{k=1}^{c_k} x_k^{lc_k}. $$

It is well known that the cycle index of the symmetric group $S_n$ can be expressed in terms of (complete) Bell polynomials as follows

$$Z(S_n) = \frac{Y_n(0!a_1, 1!a_2, \ldots, (n-1)!a_n)}{n!}. $$
3.1 In terms of Bell polynomials can be made clear by noting that $g_n = (n - 1)! \frac{1}{|1-q^n|}$. $Z_{\log}(\nu; q, \bar{q})$ can therefore be rewritten as

$$Z_{\log}(\nu; q, \bar{q}) = \sum_{n=0}^\infty Z(S_n) (\nu)^n.$$  \hspace{1cm} (23)

This means that $Z_{\log}(\nu; q, \bar{q})$ counts geometric objects that are invariant under the action of the symmetric group. This will be made more precise in the next section as we discuss the moduli space of the states that are being counted by $Z_{\log}(\nu; q, \bar{q})$.

3.2 Moduli space of logarithmic states

In section 3.1, we made use of the expression of $Z_{\log}(\nu; q, \bar{q})$ in terms of Bell polynomials to show how it encodes information about spaces invariant under the action of the symmetric group. From a more formal perspective, the Bell polynomial formulation will also be useful (section 4) in the combinatorial description of a Fock space created by the action of an algebra. In this section, we use the fact that $Z_{\log}(\nu; q, \bar{q})$ can also be expressed as the generating function of Hilbert series to show the symmetric product orbifold structure of the moduli space of the logarithmic states.

Also called Molien or Poincaré function, the Hilbert series is a generating function familiar in algebraic geometry for counting the dimension of graded components of the coordinate ring (see Appendix C). Its approach has been developed and extensively used in theoretical physics with for instance the work of [26], and notably with several applications under the so-called Plethystic Program initiated in [27, 28]. In connection with the Plethystic Program, the Hilbert series have been the essential instrument of a systematic method that yields the generating function of multi-trace operators in gauge theory from the generating function of single-trace operators at large $N$. This formalism was shown to hold in the present setting of a function generating multi-particle states from single particle ones. Indeed, we recall from [16] that the multivariate Hilbert series

$$G_1(q, \bar{q}) = \sum_{m \geq 0, n \geq 0} q^n \bar{q}^m = \frac{1}{|1-q|^2},$$ \hspace{1cm} (24)

that counts single particles of the logarithmic states, can be taken by the bosonic plethystic exponential $PE^B$ to generate new partition functions such that

$$Z_{\log}(\nu; q, \bar{q}) = PE^B [G_1(q, \bar{q})] = \exp \left( \sum_{n=1}^\infty \frac{\nu^n}{n} G_1(q^n, \bar{q}^n) \right),$$ \hspace{1cm} (25)

with

$$G_1(q^n, \bar{q}^n) = \frac{1}{|1-q^n|^2} = \frac{1}{(1-q^n)(1-\bar{q}^n)}.$$ \hspace{1cm} (26)

The connection between the plethystic exponential in Eq. (25) and the cycle index discussion of section 3.1 in terms of Bell polynomials can be made clear by noting that $a_1$ in Eq. (22) can be identified with the function $G_1(q, \bar{q})$, and accordingly, $a_n = a_n(q, \bar{q}) = G_1(q^n, \bar{q}^n)$. In analogy with the aforementioned applications, this shows that the Hilbert series $G_1(q, \bar{q})$ counts single particle states, while the plethystic exponential $PE^B [G_1(q, \bar{q})]$ counts multi-particle states.

The formalism of Hilbert series acted upon by the plethystic exponential is well known for its use in describing algebraic and geometric aspects of moduli space. We will draw from that knowledge to study the configuration space of logarithmic states. Essential to this will be a discussion of symmetric products in the spirit of Hilbert schemes of points on surfaces [29].
3.2.1 Hilbert schemes of points on surfaces and symmetric products

The Hilbert scheme $X^{[n]}$ of points on a surface is a simple example of a moduli space. It consists in the description of the configuration space of $n$ points on $X$, i.e. the space of unordered $n$-tuples of points of $X$ [29].

Formally, the Hilbert scheme of points can be defined as

$$X^{[n]} := \{ I \mid I \text{ is an ideal of } X[x_1, \ldots, x_n] \text{ with } \dim(X[x_1, \ldots, x_n]/I)=n \}.$$  \hspace{1cm} (27)

where $X[x_1, \ldots, x_n]$ is the coordinate ring of $X$. In the above definition, $X^{[n]}$ is considered as a set. It can be defined in a more geometric flavor as

$$X^{[n]} := \{ Q_Z \mid Q_Z \text{ is a quotient ring of } X \text{ with } \dim(Q_Z)=n \}.$$  \hspace{1cm} (28)

The algebra-geometry correspondence is expressed as

$$0 \to I \to X \to Q_Z \to 0,$$  \hspace{1cm} (29)

where $Z$ is the 0-dimensional subscheme of $X$, and $Q_Z$ is the coordinate ring of $Z$, and allows for flexibility of terminology between schemes and ideals.

The construction of a moduli space such as the Hilbert scheme can be accomplished by taking the quotient of $X$ by a group acting on it. In that endeavor, it is sensible to consider the quotient by the action of the symmetric group, since we do not distinguish between points. This gives the symmetric product

$$S^nX = \frac{X \times X \times \cdots \times X}{S_n}.$$  \hspace{1cm} (30)

However, the symmetric product $S^nX$ (also denoted $X^{(n)}$) is singular. Indeed, if for instance we consider the case $n=2$, the group action is not free along the diagonal $D \subset X \times X$, which yields a singular locus along the diagonal in $S^2X$. More precisely, approaching the diagonal corresponds in $X^2$ to the two points approaching each other, and eventually overlapping. At that stage, the system has lost one degree of freedom. A possible resolution of the problem would be to keep track of the direction the two points approach each other along. That is in fact the difference between $X^{[n]}$ and $S^nX$: The Hilbert scheme $X^{[n]}$ is a resolution of singularities of the symmetric product $S^nX$. When there exist $n$ distinct points $p_1, \ldots, p_n$ in $X$, each point defines both a point in $X^{[n]}$ and a point in $S^nX$ [30], and setting the ideal of Eq. (27) to

$$I := \{ f \in X[x_1, \ldots, x_n] \mid f(p_1) = \cdots = f(p_n) = 0 \},$$  \hspace{1cm} (31)

$I$ is indeed an ideal with $\dim(X[x_1, \ldots, x_n]/I)=n$. This is the case when $\dim X = 1$ (i.e. $n=1$): the Hilbert scheme $X^{[n]}$ is isomorphic to the $n$-th symmetric product $S^nX$ and we have

$$X^{[n]} \simeq S^nX.$$  \hspace{1cm} (32)

A different situation is when some points collide. Looking at the case $n=2$, two types of ideals must be taken into account in $X^{[2]}$. One can either consider an ideal given by two distinct points $p_1$ and $p_2$, or the ideal

$$I = \{ f \mid f(p) = 0, \ df_P(v) = 0 \},$$  \hspace{1cm} (33)

where $p$ is a point of $X$ and $v$ is a vector in the tangent space $T_pX$. The information of the direction in which $p_1$ approaches $p_2$ is remembered in this ideal. In the symmetric product, this information is lost and one just has $2p$. When $n > 2$, more complicated ideals appear.
3.2.2 The $n$-th symmetric product of $\mathbb{C}^2$

In the spirit of the Hilbert scheme of points on surfaces briefly discussed above, we consider the case when $X = \mathbb{C}^2$. More precisely, we consider the family

$$S^n(\mathbb{C}^2) \simeq \mathbb{C} [x_1, y_1; x_2, y_2; \ldots; x_n, y_n] / S^n,$$  \hspace{1cm} (34)

where $(x, y)$ are the coordinates of $\mathbb{C}^2$ and $S_n$ permutes the $n$-tuple of variables $(x_i, y_i)$. The computation of the Hilbert series of the invariant ring $S^n(\mathbb{C}^2)$ then amounts to extending Molien’s Theorem to the bi-graded case. Such extension has already been studied [31, 32]. We now show that in our case, $Z_{log}(\nu; q, \bar{q})$ is a $\nu$-inserted bi-graded Molien series of the symmetric group and can be expressed as

$$Z_{log}(\nu; q, \bar{q}) \equiv Z_{log}(\nu, q, \bar{q}; \mathbb{C}^2) = \sum_{n=0} (\nu)^n Z(S^n(\mathbb{C}^2)) ,$$  \hspace{1cm} (35)

where $Z(S^n(\mathbb{C}^2))$ is defined in terms of Bell polynomials as

$$Z(S^n(\mathbb{C}^2)) = \frac{Y_n(g_1, \ldots, g_n)}{n!} ,$$  \hspace{1cm} (36)

with

$$g_n = (n - 1)! \frac{1}{(1 - q)(1 - \bar{q})} .$$  \hspace{1cm} (37)

For that we will look at the cases when the symmetric group acts on sets of two and three objects.

The symmetric group on two objects can be presented as $S_2 = \langle e, \sigma \rangle$, where $e$ is the identity element and $\sigma$ can be expressed in cycle notation as $\sigma = (12)$. If we consider the action of $S_2$ on $2 \times 2$ matrices by permuting coordinates $q$, we have

$$\begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \xrightarrow{\pi(e)} \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix} \xrightarrow{\pi(\sigma)} \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix} .$$  \hspace{1cm} (38)

Then,

$$\det(I - q \pi(e)) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 - q \end{bmatrix} \begin{bmatrix} q & 0 \\ 0 & q \end{bmatrix}$$  \hspace{1cm} (39a)

$$= \det \begin{bmatrix} 1 - q & 0 \\ 0 & 1 - q \end{bmatrix}$$  \hspace{1cm} (39b)

$$= (1 - q)^2 ,$$  \hspace{1cm} (39c)

and

$$\det(I - q \pi(\sigma)) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 - q \end{bmatrix} \begin{bmatrix} 0 & q \\ q & 0 \end{bmatrix}$$  \hspace{1cm} (40a)

$$= \det \begin{bmatrix} 1 - q & 0 \\ -q & 1 \end{bmatrix}$$  \hspace{1cm} (40b)

$$= (1 - q^2) .$$  \hspace{1cm} (40c)
In the same way, the action of $S_2$ on $2 \times 2$ matrices by permuting coordinates $\bar{q}$ allows us to write $\det (I - \bar{q}\pi(e)) = (1 - q^2)$ and $\det (I - \bar{q}\pi(\sigma)) = (1 - q^2)$. From there, we have

$$Z (S^2(\mathbb{C}^2)) = \frac{Y_n(g_1, g_2)}{2!} = \frac{1}{2!} (g_1^2 + g_2)$$

$$= \frac{1}{2!} \left[ \frac{1}{(1-q)^2(1-q^2)} + \frac{1}{(1-q^2)(1-q^2)} \right]$$

$$= \frac{1}{2!} \left[ \frac{1}{\det (I - q\pi(e))} \det (I - q\pi(\sigma)) + \frac{1}{\det (I - q\pi(\tau))} \right].$$

Next we consider the symmetric group $S_3 := < e, \sigma, \tau >$, where the elements correspond respectively to the identity, the three-element conjugacy class that consists of swapping two coordinates, and the two-element conjugacy class of cyclic permutations. Using cycle notations, $\sigma = \{(12),(13),(23)\}$ and $\tau = \{(13),(12)\}$. The action of $S_3$ on $3 \times 3$ matrices by permuting coordinates $q$ would yield the following. Starting from the identity

$$\det \begin{pmatrix} 1 - q & 0 & 0 \\ 0 & 1 - q & 0 \\ 0 & 0 & 1 - q \end{pmatrix} = (1 - q)^3.$$  

(42)

Then, taking one of the three terms consisting of swaps of two coordinates, say $\sigma = (12)$ identically results into

$$\det \begin{pmatrix} 1 & -q & 0 \\ -q & 1 & 0 \\ 0 & 0 & 1 - q \end{pmatrix} = (1 - q)(1 - q^2).$$

(43)

Finally, taking one of the two cyclic permutation terms, say $\tau = (132)$, identically yields

$$\det \begin{pmatrix} 1 & -q & 0 \\ 0 & 1 & -q \\ -q & 0 & 1 \end{pmatrix} = (1 - q^3).$$

(44)

Acting in the same way on coordinates $\bar{q}$ allows us to eventually write

$$Z (S^3(\mathbb{C}^2)) = \frac{Y_n(g_1, g_2, g_3)}{3!} = \frac{1}{3!} (g_1^3 + 3g_1g_2 + g_3)$$

$$= \frac{1}{3!} \left[ \frac{1}{(1-q)^3(1-q^3)} + \frac{3}{(1-q)(1-q^2)(1-q^3)(1-q^2)} + \frac{2}{(1-q^2)(1-q^3)} \right]$$

$$= \frac{1}{3!} \left[ \frac{1}{\det (I - q\pi(e)) \det (I - q\pi(\sigma)) \det (I - q\pi(\tau))} \right].$$

(45c)

Repeating the same procedure for all permutations $\pi$ of the elements $g \in S_n$, we can generalize the above analysis to

$$Z_{\ellop}(\nu; q, \bar{q}; \mathbb{C}^2) = \frac{1}{|S_n|} \sum_{g \in S_n} \frac{\nu^n}{\det (I - q\pi(g)) \det (I - \bar{q}\pi(g))}.$$

(46)

$Z_{\ellop}(\nu; q, \bar{q}; \mathbb{C}^2)$ is therefore a ($\nu$-inserted) bi-graded Molien series of $S_n$, and a Hilbert series of the ring of invariants $S^n(\mathbb{C}^2)$.

Closer to our previous discussion on Hilbert schemes of points on surfaces, generating functions taking the exponential form of $Z_{\ellop}(\nu, q, \bar{q}; \mathbb{C}^2)$ were considered in [33] and in [28]. Using our notation, we can then write
\[ Z_{\text{log}}(\nu, q, \bar{q}; C^2) = PE_B \left[ \frac{1}{(1-q)(1-\bar{q})} \right] = \exp \left( \sum_{n=1}^{\infty} \frac{(\nu)^n}{n(1-q^n)(1-\bar{q}^n)} \right). \]  

(47)

In the simplest case of CCTMG for instance, i.e when \( \nu = q^2 \), one can write

\[ Z_{\text{log}}(q^2; q, \bar{q}; C^2) = PE_B \left[ \frac{1}{(1-q)(1-\bar{q})} \right] = \exp \left( \sum_{n=1}^{\infty} \frac{(q^2)^n}{n(1-q^n)(1-\bar{q}^n)} \right). \]  

(48)

\( S^n(C^2) \) is an orbifold locally isomorphic to an open set of the Euclidean space quotiented by the action of the symmetric group \([30, 33]\). The above analysis therefore brings forth the orbifold structure of the moduli space of logarithmic partners states in critical massive gravities.

### 3.3 \( Z_{\text{log}}(\nu, q, \bar{q}; C^2) \) as an index of the untwisted sector of a symmetric orbifold model

In this section we show an interesting connection between the previous results in terms of the \( S^n(C^2) \) orbifold and the symmetric product orbifolds in CFT.

The \( S_n \) orbifold partition function was first derived in \([34]\) using automorphic properties of elliptic genus. It was subsequently rederived in \([35]\) using the group theoretic properties of the symmetric products as permutation orbifolds of the full symmetric group.

From \([16]\), we recall that \( Z_{\text{log}}(\nu, q, \bar{q}; C^2) \) can be obtained starting from writing its logarithmic expression as follows

\[
\log \left[ Z_{\text{log}}(\nu, q, \bar{q}; C^2) \right] = - \sum_{h \geq 0, \bar{h} \geq 0} \log \left( 1 - \nu q^h \bar{q}^{\bar{h}} \right) 
= - \sum_{h \geq 0, \bar{h} \geq 0} \left( - \sum_{n=1}^{\infty} \frac{(\nu)^n}{n} q^{nh} \bar{q}^{\bar{n}\bar{h}} \right) 
= \sum_{h \geq 0, \bar{h} \geq 0} \sum_{n=1}^{\infty} \frac{\nu^n}{n} q^{nh} \bar{q}^{\bar{n}\bar{h}} 
= \sum_{n=1}^{\infty} \frac{\nu^n}{n} \left( \sum_{h \geq 0, \bar{h} \geq 0} q^{nh} \bar{q}^{\bar{n}\bar{h}} \right). 
\]  

(49a)

(49b)

(49c)

(49d)

Then, we simply write

\[
Z_{\text{log}}(\nu, q, \bar{q}; C^2) = \exp \left[ \sum_{n=1}^{\infty} \frac{\nu^n}{n} \left( \sum_{h \geq 0, \bar{h} \geq 0} q^{nh} \bar{q}^{\bar{n}\bar{h}} \right) \right] 
= \exp \left[ \sum_{n=1}^{\infty} \frac{\nu^n}{n} Z(n\tau, n\bar{\tau}) \right]. \]  

(50a)

(50b)

From \([36]\), \( Z_{\text{log}}(\nu, q, \bar{q}; C^2) \) can immediately be recognized as the partition function of the untwisted sector of a totally symmetric orbifold. Specifically, if one considers the "seed" partition function of the untwisted sector of the "totally" symmetric orbifold model to be

\[
Z(\tau, \bar{\tau}) = \sum_{h, \bar{h}} q^h \bar{q}^{\bar{h}}, \]  

(51)

then the generating function of the untwisted partition function can be expressed as
\[ Z_{\text{untwisted}} = \prod_{\hat{h}, \hat{\bar{h}}} \frac{1}{1 - \nu \hat{q} \hat{\bar{q}}}, \] (52)

which is the original double product derived in [14]. Hence, in our specific case

\[ Z(\tau, \bar{\tau}) = Y_1(g_1), \] (53)

and

\[ Z_{\text{log}}(\nu, q, \bar{q}; \mathbb{C}^2) = Z_{\text{untwisted}}. \] (54)

This is in agreement with (Eqs. (4.25) to (4.28) of) [36], with the difference that in our case, \( t \) is specialized to \( \nu \), and \( \rho(\hat{h}, \hat{\bar{h}}) = 1 \). The reason for the setting of \( \rho(\hat{h}, \hat{\bar{h}}) \) is because the partition function counts descendant states constructed by the action on the highest weight state of only one (Virasoro) generator \((L_{-1})\). This agrees with the fact that the \( sl(2) \) algebra is composed of only three generators which are expressed in terms of the Virasoro generators \( \{L_{-1}, L_0, L_1\} \), and for which \( L_{-1} \) creates descendant states. It also agrees with results from [8] where the logarithmic states are created by multiple action of \( L_{-1} \) and \( L_{-1} \).

To make contact with the terminology of [36], we start by writing

\[ \chi(n\tau) = \frac{1}{|1 - q^n|^2}. \] (55)

This gives the following correspondence between the work of [36] and our Bell polynomials \( Y_n \)

\[ P_2(1, 0, \boxed{\cdots} \boxed{\cdots} 1) = \frac{1}{2!} [\chi(\tau) + \chi(2\tau)] = \frac{1}{2!} Y_2, \] (56)

\[ P_3(1, 0, \boxed{\cdots} \boxed{\cdots} 1) = \frac{1}{3!} \left( (\chi(\tau))^3 + 3\chi(\tau)\chi(2\tau) + 2\chi(3\tau) \right) = \frac{1}{3!} Y_3, \] (57)

and similarly for \( n > 0 \). Finally, one gets

\[ Z_{\text{log}}(\nu, q, \bar{q}; \mathbb{C}^2) = \sum_n \left| P_n(1, 0, \boxed{\cdots} \boxed{\cdots} 1) \right|^2 (\nu)^n, \] (58)

which is the application of Eq. (4.26) of [36] to the present case. The specific choice of representation is due to the fact that in a bosonic theory, only the totally symmetric representation of the \( S_n \) characters would be considered.

4 Differential operators on symmetric orbifolds

In this section, using an invariant theoretic language, we revisit some of the work done in [16], and give a more formal interpretation to the hidden structures found in the study of \( Z_{\text{log}}(\nu; q, \bar{q}; \mathbb{C}^2) \).

Fock spaces were designed as an algebraic framework to construct many-particle states in quantum mechanics. They typically represent the state space of an indefinite number of identical particles (an electron gas, photons, etc...). These particles can be classified in two types, bosons and fermions, and their Fock spaces look quite different. The reason why a Fock space are of great interest is that several important algebras can naturally act on it. Fermionic Fock spaces are naturally representations of a Clifford algebra, where the generators correspond to adding or removing a particle in a given energy state. In a similar way, bosonic Fock space is naturally a representation of a Weyl algebra.

Proceeding with our discussion on Hilbert schemes, for the non-compact space \( \mathbb{C}^2 \), a connection between the theory of Hilbert schemes of points on surfaces and the infinite dimensional Heisenberg algebra was
made through the construction of a representation of the Heisenberg algebra on the homology group of the Hilbert scheme, turning the homology group into a Fock space [37]. The construction showed that the Fock space representation on the polynomial ring of infinitely many variables is an important representation of the Heisenberg algebra. In the present case, with an interest on the bosonic Fock space, we construct a combinatorial model of creation and annihilation operators that are generators of a Heisenberg-Weyl algebra and that act on the bosonic Fock space.

In general, a Fock space is considered on a Hilbert space, but in the simplest case and for the purpose of our discussion, the bosonic vector space is obtained by considering a complex vector space that they satisfy the Heisenberg-Weyl algebra and that act on the bosonic Fock space.

Given the \( n \)-dimensional polynomial ring \( \mathbb{C}[x_1, \ldots, x_n] \), a subring of invariant polynomials denoted \( \mathbb{C}[x_1, \ldots, x_n]^{S_n} \) can be constructed from invariants \( g_1, \ldots, g_n \in \mathbb{C}[x_1, \ldots, x_n] \). The polynomials of the invariant ring \( \mathbb{C}[x_1, \ldots, x_n]^{S_n} \) can take the form of Bell polynomials \( Y_n \) with coordinates \( g_1, \ldots, g_n \) such that \( \mathbb{C}[x_1, \ldots, x_n]^{S_n} \simeq \mathbb{C}[g_1, \ldots, g_n] \). From the ring of differential operators

\[
D (\mathbb{C}[g_1, \ldots, g_n]) = \mathbb{C} < g_1, \ldots, g_n, \partial_{g_1}, \ldots, \partial_{g_n} >,
\]

we construct the multiplication operator \( \hat{X}_n = g_1 + \sum_{k=1}^{n} g_{k+1} \frac{\partial}{\partial g_k} \) and the derivative operator \( \hat{D} = \frac{\partial}{\partial g_1} \) such that they satisfy the Heisenberg-Weyl algebra \( [\hat{X}_n, \hat{D}] = 1 \). Then, defining the function

\[
Z (\mathbb{C}[x_1, \ldots, x_n]) = \frac{Y_n(g_1, \ldots, g_n)}{n!},
\]

these operators act as ladder operators on the function \( Z (\mathbb{C}[x_1, \ldots, x_n]) \) at each \( n \) level in the following way.

**Proposition 4.1.** Let \( Z (S^n(\mathbb{C})) \) be defined in terms of Bell polynomials \( Y_n \) as

\[
Z (S^n(\mathbb{C})) = \frac{Y_n(g_1, \ldots, g_n)}{n!}.
\]

The set of operators

\[
\hat{X}_n = g_1 + \sum_{k=1}^{n} g_{k+1} \frac{\partial}{\partial g_k}, \quad \hat{D} = \frac{\partial}{\partial g_1},
\]

generating the Heisenberg-Weyl algebra \( [\hat{D}, \hat{X}_n] = 1, [\hat{D}, 1] = [\hat{X}_n, 1] = 0 \) acts on \( Z (S^n(\mathbb{C})) \) as

\[
\hat{X}_n Z (S^n(\mathbb{C})) = (n + 1) Z (S^{n+1}(\mathbb{C}))
\]

\[
\hat{D} Z (S^n(\mathbb{C})) = Z (S^{n-1}(\mathbb{C}))
\]

\[
\hat{X}_n \hat{D} Z (S^n(\mathbb{C})) = n Z (S^n(\mathbb{C})).
\]

**Proof.** From [16], it is known that

\[
\hat{X}_n Y_n = Y_{n+1},
\]

\[
\hat{D} Y_n = n Y_{n-1},
\]

\[
\hat{X}_n \hat{D} Y_n = n Y_n.
\]

Hence we can write

\[
\hat{X}_n [n! Z (S^n(\mathbb{C}))] = (n + 1)! Z (S^{n+1}(\mathbb{C}))
\]

\[
\Rightarrow \hat{X}_n (Z (S^n(\mathbb{C}))) = (n + 1) Z (S^{n+1}(\mathbb{C})).
\]
Similarly
\[
\hat{D} [n!Z (S^n(\mathbb{C}))] = n \left[ (n-1)!Z (S^{n-1}(\mathbb{C})) \right]
\]
\Rightarrow \hat{D} (Z(S^n(\mathbb{C}))) = Z(S^{n-1}(\mathbb{C})),
\]
and
\[
\check{X}_n \hat{D} [Z (S^n(\mathbb{C}))] = n [Z (S^n(\mathbb{C}))]
\]
\Rightarrow \check{X}_n \hat{D} Z (S^n(\mathbb{C})) = nZ(S^n(\mathbb{C})).
\]

5 Conjectures

In the discussions above, we delved quite intensively into algebro-geometric and invariant theoretic issues that had not yet been addressed in the context of critical massive gravities present in the AdS_3/LCFT_2 correspondence. In this section, we come to the point where we would like to make some conjectures about the nature of the logarithmic states.

Firstly, we would like to mention the following about the "characters" generated by the log-partition \( Z_{\log}(\nu, q, \bar{q}; \mathbb{C}^2) \). From the above analysis, these objects are linear combinations of the variables \( g_1, g_2, \ldots, g_n \) multiplied by a factor \((\nu)^n\). However, \( Z_{\log}(\nu, q, \bar{q}; \mathbb{C}^2) \) also seem to describe the symmetric tensor product of the characters of \( sl(2, \mathbb{R}) \) highest weight representations
\[
\chi_{h}^{sl(2, \mathbb{R})} = \frac{q^h}{1-q}, \quad \frac{\bar{X}_h^{sl(2, \mathbb{R})}}{1-q}
\]
Taking \( h = 2, \bar{h} = 0 \) for instance, it is easy to see that \( \chi_{h=2}^{sl(2, \mathbb{R})} \) and \( \chi_{\bar{h}=0}^{sl(2, \mathbb{R})} \) are respectively the single particle holomorphic and antiholomorphic characters from which \( Z_{\log}(q^2, q, \bar{q}; \mathbb{C}^2) \) yields symmetric tensor product (multiparticle) characters. Yet, if really there were a holomorphic and an antiholomorphic character, the numerator of the antiholomorphic character of CCTMG should not be constant since the antiholomorphic central charge is not null. In other words, there should systematically be a \((\nu)\) appear in the logarithmic partition functions. This argument show that the logarithmic highest weight states have descendants that are both holomorphic and antiholomorphic, just as in the \( c = 0 \) LCFT theory [38].

In the extension of holography to the present non-unitary case, critical massive gravities fitting in the AdS_3/LCFT_2 correspondence are considered as non-unitary AdS_3 holographic duals of two dimensional non-unitary CFTs that are known to exist. We would like to conjecture that in the setting, the logarithmic states of the critical massive gravities are conical singularities. This idea is not new. Indeed, it was already seen from the study of the moduli space of logarithmic states, the latter are singular point particles in an orbifold space. It is therefore quite natural to think of them as conical singularities. Of course it is essential to validate this conjecture with further checks, and that will be the object of future work.

6 Conclusion and outlook

In this work, we used the partition function derived in [14] and reformulated in [16] to extract information about the moduli space of the logarithmic sector in critical massive gravities. Using the relation between the cycle index of the symmetric group and Bell polynomials, we first showed that the partition function of the logarithmic states is the generating function of polynomials invariant under the action of the symmetric group \( S_n \). Then, it was shown that the configuration space of the logarithmic states is the quotient space \( S^n(\mathbb{C}^2) \), by looking at the partition function of the logarithmic states \( Z_{\log}(\nu, q, \bar{q}) \) from an algebraic point of view, and showing that it can be written as a bi-graded Molien series. From a geometric point of view, the quotient space \( S^n(\mathbb{C}^2) \) has the structure of an orbifold, and \( Z_{\log}(\nu, q, \bar{q}; \mathbb{C}^2) \) is then the generating function of an \( S^n(\mathbb{C}^2) \) orbifold. Besides, an interesting connection between the \( S^n(\mathbb{C}^2) \) orbifold and the untwisted
sector of a symmetric orbifold CFT model was made, looking at their respective partition functions. Then, the construction of differential operators on symmetric orbifolds was discussed. The operators were shown to generate a Heisenberg-Weyl algebra, and to act as raising and lowering operators on the Bell polynomials. Finally, based on the arguments developed in the text, a conjecture about the nature of these logarithmic states in terms of conical singularities was made.

On its own right, the logarithmic sector of these critical massive gravities looks like an interesting topic of study, and much more seemingly remains to be unraveled from it. For example, it would be interesting to study the modular properties of $Z_{\text{log}}(\nu, q, \bar{q}; \mathbb{C}^2)$, and how the study contributes to the modular properties of $Z_{\text{TMG}}$. It was also observed that the counting in the multi particle sector of $Z_{\text{log}}(\nu, q, \bar{q}; \mathbb{C}^2)$ can only be done correctly on the account of quantum groupoid (Hopf algebroid) coproducts [40]. This will be discussed in an upcoming publication.

Lastly, the author would like to mention that the realization that the partition functions of critical massive gravities could be recast in terms of Bell polynomials must be credited to [41, 42]. Therefore, inspired by ideas from [41, 42, 43, 44] where partition functions expressed in terms of Bell polynomials and recast into infinite products eventually lead to the construction of quantum group, knots and link invariants, it would be interesting to investigate how $Z_{\text{log}}(\nu, q, \bar{q}; \mathbb{C}^2)$ can be useful in the construction of topological invariants.

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Appendix A Derivations of plethystic exponential and Bell polynomial forms of $Z_{\log}(\nu, q, \bar{q}; \mathbb{C}^2)$

In this appendix, we recall the derivation of $Z_{\log}(\nu, q, \bar{q}; \mathbb{C}^2)$ as a plethystic exponential and in a Bell polynomial form \[\text{(A.1)}\].

Starting from

$$Z_{\log}(\nu, q, \bar{q}; \mathbb{C}^2) = \prod_{m=0}^{\infty} \prod_{\bar{m}=0}^{\infty} \frac{1}{1 - \nu q^m \bar{q}^{\bar{m}}}, \quad \text{(A.1)}$$

we first compute the logarithmic function of $Z_{\log}(\nu, q, \bar{q}; \mathbb{C}^2)$ as

$$\log [Z_{\log}(\nu, q, \bar{q}; \mathbb{C}^2)] = - \sum_{m, \bar{m} \geq 0} \log (1 - \nu q^m \bar{q}^{\bar{m}}). \quad \text{(A.2)}$$

Using the well know Maclaurin series

$$\log(1 - x) = - \sum_{x=1}^{\infty} \frac{x^n}{n}, \quad \text{(A.3)}$$

we write

$$\log [Z_{\log}(\nu, q, \bar{q}; \mathbb{C}^2)] = \sum_{m, \bar{m} \geq 0} \sum_{n=1}^{\infty} \frac{\nu^n}{n} q^m \bar{q}^{\bar{m}}. \quad \text{(A.4)}$$

Then, using the Maclaurin series of geometric series

$$\frac{1}{1 - x} = \sum_{x=0}^{\infty} x^n, \quad \text{(A.5)}$$

we write

$$\log [Z_{\log}(\nu, q, \bar{q}; \mathbb{C}^2)] = \sum_{m, \bar{m} \geq 0} \sum_{n=1}^{\infty} \frac{\nu^n}{n} q^m \bar{q}^{\bar{m}}$$

$$= \sum_{n=1}^{\infty} \frac{\nu^n}{n} \sum_{m, \bar{m} \geq 0} q^m \bar{q}^{\bar{m}}$$

$$= \sum_{n=1}^{\infty} \frac{\nu^n}{n} \frac{1}{1 - q^n} \frac{1}{1 - \bar{q}^{\bar{m}}}. \quad \text{(A.6c)}$$

Finally, exponentiating the above equation yields the plethystic exponential form of $Z_{\log}(\nu, q, \bar{q}; \mathbb{C}^2)$

$$Z_{\log}(\nu, q, \bar{q}; \mathbb{C}^2) = \text{PE}^g \left[ \frac{1}{(1 - q)(1 - \bar{q})} \right] = \exp \left( \sum_{n=1}^{\infty} \frac{(\nu)^n}{n(1 - q^n)(1 - \bar{q}^{\bar{m}})} \right). \quad \text{(A.7)}$$

To obtain the Bell polynomial version of $Z_{\log}(\nu, q, \bar{q}; \mathbb{C}^2)$, it suffices to continue from Eq. (A.6c) as follows

$$\log [Z_{\log}(\nu, q, \bar{q}; \mathbb{C}^2)] = \sum_{n=1}^{\infty} \frac{\nu^n}{n} \frac{1}{1 - q^n} \frac{1}{1 - \bar{q}^{\bar{m}}}. \quad \text{(A.8a)}$$

$$= \sum_{n=1}^{\infty} \frac{\nu^n}{n} \frac{1}{|1 - q^n|}. \quad \text{(A.8b)}$$

$$= \sum_{n=1}^{\infty} \frac{\nu^n}{n!} (n - 1)! \frac{1}{|1 - q^n|}. \quad \text{(A.8c)}$$

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Introducing the function $g_n$ such that

$$g_n = (n-1)! \frac{1}{|1-q^n|},$$

we get

$$\log \left[ Z_{log}(\nu, q, \bar{q}; C^2) \right] = \sum_{n=1}^{\infty} \frac{\nu^n}{n!} g_n.$$  \hspace{1cm} (A.10)

Finally, exponentiation the above logarithmic function gives

$$Z_{log}(\nu, q, \bar{q}; C^2) = \exp \left( \sum_{n=1}^{\infty} \frac{\nu^n}{n!} g_n \right)$$  \hspace{1cm} (A.11a)

$$= \sum_{n=0}^{\infty} \frac{Y_n}{n!} \nu^n, \hspace{1cm} (A.11b)$$

where in Eq. (A.11b), $Z_{log}(\nu, q, \bar{q}; C^2)$ is the generating function of the Bell polynomials $Y_n$.\hspace{1cm}
Appendix B  Cycle index of the symmetric group

In this appendix, the notion of cycle index applied to the symmetric group is reviewed.

**Definition B.1.** The symmetric group $S_n$ defined over a finite set $X$ of $n$ objects, is the group of bijective functions from $X$ to $X$ under the operation of composition, which consists of permutations the $n$ objects.

The permutations of $S_n$ can be expressed in terms of cycles. For instance, considering the set $X = 1, 2, 3, 4, 5, 6$, the permutation $\pi = (124)(35)(6)$ tells us that $\pi$ maps 1 to 2, 2 to 4, 3 to 5, and 6 to itself. In this case, $\pi$ consists of 3 disjoint cycles.

**Definition B.2.** A $k$-cycle, or cycle of length $k$, is a cycle containing $k$ elements.

Looking back at the example considered above, $\pi = (124)(35)(6)$ contains a 3-cycle, a 2-cycle, and a 1-cycle.

In group theory, the elements of any group may be partitioned into conjugacy classes.

**Definition B.3.** In any group $G$, the elements $g$ and $h$ are conjugates if

$$g = khk^{-1}$$

for $k \in G$. The set of all elements conjugate to a given $g$ is called the conjugacy class of $g$.

Hence, when $S_n$ acts on a set $X$, the cycle decomposition of each $\pi \in S_n$ as product of disjoint cycles is associated to the partitions of the objects in the set. For example, if one considers $S_4$, the partitions of 4 and the corresponding conjugacy classes are

$$
(1,1,1,1) \rightarrow \{(e)\} \quad \text{(B.1a)}
$$

$$
(2,1,1) \rightarrow \{(12),(13),(14),(23),(24),(34)\} \quad \text{(B.1b)}
$$

$$
(2,2) \rightarrow \{(12)(34),(13)(24),(14)(23)\} \quad \text{(B.1c)}
$$

$$
(3,1) \rightarrow \{(123),(132),(124),(142),(134),(143),(234),(243)\} \quad \text{(B.1d)}
$$

$$
(4) \rightarrow \{(1234),(1432),(1423),(1324),(1342),(1243)\} \quad \text{(B.1e)}
$$

To keep track of the cycle decomposition of the elements of $S_4$, one can use the cycle index polynomial. Representing each object of the set by a coordinate, i.e., 1 by $g_1$, 2 by $g_2$, 3 by $g_3$ and 4 by $g_4$, the cycle index of $S_4$ reads

$$Z(S_4) = \frac{1}{24!} \left(g_1^4 + 6g_1^2g_2 + 3g_2^2 + 8g_1g_3 + 6g_4\right). \quad \text{ (B.2)}$$

The coefficients before the monomials (the products of coordinates) count the number of elements in a given conjugacy class, the powers on the monomials indicate the number of times the object appears in a given partition, and the denominator $24!$ is the order of $S_4$, i.e., the total number of elements in $S_4$. As such, the cycle index is simply the average of the number of elements in $X$, that are left invariant by the action of $\pi \in S_4$. 

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Appendix C  Invariant theory

Invariant theory appears in the description of moduli spaces whose points parametrize spaces of interest, and is also useful in the construction of Hilbert schemes with associated Hilbert polynomials. In this appendix, we review some basic concepts of invariant theory (a good reference on this topic can be found in [45]).

C.1 Ring of polynomials

Definition C.1. Let \( V \) be a complex vector space, and denote the dual vector space by \( V^* = \{ f : V \to \mathbb{C} \} \). The coordinate ring \( \mathcal{R}(V) \) of \( V \) is the algebra of functions \( F : V \to \mathbb{C} \) generated by the elements of \( V^* \). The elements of \( \mathcal{R}(V) \) are called polynomial functions on \( V \).

For a fixed basis \( e_1, e_2, \ldots, e_n \) of \( V \), a dual basis of \( V^* \) can be expressed by the coordinates \( x_1, x_2, \ldots, x_n \) such that \( x_i(c_1 e_1 + \cdots + c_n e_n) = c_i \). The coordinate ring \( \mathcal{R}(V) \) obtained is \( \mathbb{C}[x_1, x_2, \ldots, x_n] \), the ring of polynomials in \( n \) variables \( f(x_1, x_2, \ldots, x_n) \) with complex coefficients.

C.2 Invariant rings of the symmetric group

The fundamental question at the heart of invariant theory is to ask whether the orbits of a group \( G \) that acts on a space \( V \) can form a space in their own right. In what follows we will consider the case where \( G = S_n \).

Let the symmetry group \( S_n \) act on the \( n \)-dimensional complex vector space \( V \). The action of \( S_n \) on \( V \) translates into an action of \( S_n \) on the polynomial ring \( \mathbb{C}[x_1, \ldots, x_n] := \mathbb{C}[x] \). The objective is then to describe the subring of invariant polynomials, which in the present case is denoted \( \mathbb{C}[x]^{S_n} \). According to a theorem of Hilbert, \( \mathbb{C}[x]^{S_n} \) is finitely generated as an algebra. This means that there exist invariants \( I_1, \ldots, I_n \in \mathbb{C}[x] \) such that \( \mathbb{C}[x]^{S_n} \) consists exactly of polynomials of the invariant ring \( \mathbb{C}[g] = \mathbb{C}[I_1, \ldots, I_n] \).

We can summarize the results of section 3.1 in the following way. The polynomials invariant under the action of \( S_n \) are precisely the Bell polynomials \( Y \) with coordinates \( g = (g_1, \ldots, g_n) \). In particular, \( Y(g) \in \mathbb{C}[x]^{S_n} \) are uniquely written as polynomial in the \( g_1, \ldots, g_n \) such that we have the isomorphism

\[
\mathbb{C}[x]^{S_n} \sim \mathbb{C}[g]. \tag{C.1}
\]

C.3 Counting the number of invariants

In this subappendix, we are interested in counting the polynomials that remain invariant under the action of the symmetric group. The treatment of this enumerative problem can be made systematic by keeping track of the degrees in which these invariants occur.

Let \( \mathbb{C}[x]^{S_n}_d \) be the set of all homogeneous invariants of degree \( d \). The invariant ring \( \mathbb{C}[x]^{S_n} = \bigoplus_{d=0}^{\infty} \mathbb{C}[x]^{S_n}_d \) is the direct sum of the finite dimensional \( \mathbb{C} \)-vector spaces \( \mathbb{C}[x]^{S_n}_d \). The Hilbert series (or Poincaré series) of the graded algebra \( \mathbb{C}[x]^{S_n} \) is the formal power series in \( t \) defined by

\[
H \left( \mathbb{C}[x]^{S_n}, t \right) = \sum_{d=0}^{\infty} \dim \left( \mathbb{C}[x]^{S_n}_d \right) t^d, \tag{C.2}
\]

which encodes in a convenient way the dimensions of the \( \mathbb{C}[x]^{S_n}_d \)-vector space of degree \( d \).

In 1897, Molien proved that for any group finite group \( G \) acting on \( \mathbb{C}[x]^G \), it is possible to compute \( H \left( \mathbb{C}[x]^G, t \right) \) without first computing \( \mathbb{C}[x]^G \). This is captured in the beautiful theorem below

Theorem C.1. (Molien’s theorem). Let \( \rho : G \to \text{GL}(V) \) be a representation of a finite group \( G \) of order \( |G| \). If \( G \) acts on \( \mathbb{C}[V] = \mathbb{C}[x] \), then the Hilbert series of the invariant ring \( \mathbb{C}[x]^G \) can be expressed as

\[
H \left( \mathbb{C}[x]^G, t \right) = \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det (I - \rho(g)t)}, \tag{C.3}
\]
We refer the reader to [45] for a very readable proof. In the case of the symmetric group, one simply writes the Hilbert series (C.2) as

\[
H (\mathbb{C}[x]^{S_n}, t) = \frac{1}{|S_n|} \sum_{g \in S_n} \frac{1}{\det (I - \rho(g)t)}.
\] (C.4)

### C.4 Rings of differential operator

As the algebra of differential operators on affine \( n \)-spaces, the **Weyl algebra** is perhaps the most important ring of differential operator. It is denoted

\[
D (\mathbb{C}[x_1, \ldots, x_n]) = \mathbb{C} < x_1, \ldots, x_n, \partial x_1, \ldots, \partial x_n >,
\] (C.5)

where the variables \( x_i \) commute with each other, the variables \( \partial x_j = \frac{d}{dx_j} \) commute with each other, and the two sets of variables interact via the product rule \( \partial_j x_i = x_i \partial_j + \delta_{ij} \) [32].
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