KNOTS AND LINKS WITHOUT PARALLEL TANGENTS

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Steinhaus conjectured that every closed oriented $C^1$-curve has a pair of anti-parallel tangents. The conjecture is not true. Porter [Po] showed that there exists an unknotted curve which has no anti-parallel tangents. Colin Adams raised the question of whether there exists a nontrivial knot in $\mathbb{R}^3$ which has no parallel or antiparallel tangents. In this paper we will solve this problem, showing that any (smooth or polygonal) link $L$ in $\mathbb{R}^3$ is isotopic to a smooth link $\hat{L}$ which has no parallel or antiparallel tangents. If $S(L)$ is the set of all smooth links isotopic to $L$, then the subset $\hat{L}(L)$ of all $\hat{L}$ which has no parallel or antiparallel tangents is not dense in $S(L)$ if it is endowed with $C^\infty$ topology. However, $\hat{L}(L)$ is dense in $S(L)$ under $C^0$ topology. We will show that any neighborhood of $L$ contains such a link $\hat{L}$. See Theorem 7 below. The result has some impact on studying supercrossing numbers, see the recent work of Pahk [Pa].

We refer the readers to [Ro] for concepts about knots and links. Throughout this paper, we will use $I$ to denote a closed interval on $\mathbb{R}$. Denote by $S^2$ the unit sphere in $\mathbb{R}^3$, and by $S_1$ the circle $S^2 \cap \mathbb{R}_{xy}$ on $S^2$, where $\mathbb{R}_{xy}$ denotes the $xy$-plane in $\mathbb{R}^3$. Denote by $Z[z_1, z_2]$ the set $\{v = (x, y, z) \in \mathbb{R}^3 \mid z_1 \leq z \leq z_2\}$. Similarly for $Y[y_1, \infty)$ etc. A curve $\beta : I \rightarrow \mathbb{R}^3$ is an unknotted curve in $Z[z_1, z_2]$, with endpoints on different components of $\partial Z[z_1, z_2]$, and (ii) $\beta$ is rel $\partial$ isotopic in $Z[z_1, z_2]$ to a straight arc.

Given a curve $\alpha : I = [a, b] \rightarrow S^2$ and a positive function $f : I \rightarrow \mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}$, we use $\beta = \beta(f, \alpha, t_0, v_0)$ to denote the integral curve of $f\alpha$ with $\beta(t_0) = v_0$, where $t \in I$. More explicitly, define

$$\beta(t) = \beta(f, \alpha, t_0, v_0)(t) = v_0 + \int_{t_0}^t f(t)\alpha(t) \, dt.$$ 

When $t_0 = a$ and $v_0 = 0$, simply denote it by $\beta(f, \alpha)$.

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If $\gamma : [a, b] \to \mathbb{R}^3$ is a map and $[c, d]$ is a subinterval of $[a, b]$, denote by $\gamma[a, b]$ the restriction of $\gamma$ on $[c, d]$. If $u, v$ are points in $\mathbb{R}^3$, denote by $e(u, v)$ the line segment with endpoints at $u$ and $v$, oriented from $u$ to $v$. Denote by $d(u, v)$ the distance between $u$ and $v$. Denote by $||e||$ the length of $e$ if $e$ is a line segment or a vector in $\mathbb{R}^3$. Thus $d(u, v) = ||e(u, v)|| = ||u - v||$.

Given $n$ points $v_1, ..., v_n$ in $\mathbb{R}^3$, let $e_i = e(v_i, v_{i+1})$. The subscripts are always mod $n$ numbers. Thus $e_n = e(v_n, v_1)$. In generic case, the union of these edges forms a knot, denoted by $K(v_1, ..., v_n)$. To avoid trivial case, we will always assume $n \geq 4$.

**Lemma 1.** Let $K = K(v_1, ..., v_n)$ be a polygonal knot, and let $N$ be a regular neighborhood of $K$. Then there is a number $r > 0$ such that

(i) $N$ contains the $r$-neighborhood $N(K)$ of $K$;

(ii) $K' = K(v'_1, v_1, ..., v'_n, v_n)$ is isotopic to $K$ in $N$ if $d(v_i, v'_i) < r$; and

(iii) $K'' = K(v'_1, ..., v'_n)$ is isotopic to $K$ in $N$ if $d(v_i, v'_i) < r$.

**Proof.** Choose $r > 0$ to satisfy (i) and $r < d/4$, where $d$ is the minimal distance between non-adjacent edges of $K$. The knot $K'$ is contained in $N(K)$. Let $D_i$ be the meridian disk of the $r$-neighborhood $N(e_i)$ of the $i$-th edge $e_i = e(v_i, v_{i+1})$ of $K$, intersecting $e_i$ perpendicularly at its middle point $m_i$. It is easy to check that the distance from $m_i$ to any edge $e_j$ ($j \neq i$) is at least $d/2 = 2r$, hence $D_i$ is a meridian disk of $N(K)$. The edge $e(v'_i, v_i)$ is contained in an $r$-neighborhood of $v_i$, hence is disjoint from all $D_j$. Thus $D_i$ intersects $K'$ at a single point on the edge $e(v_i, v'_{i+1})$, so the disks $D_1, ..., D_n$ cut $N(K)$ into balls $B_1, ..., B_n$, each intersecting $K'$ in an arc consisting of three edges, hence unknotted. Therefore $K'$ is isotopic to $K$ in $N(K)$. This proves (ii).

By (ii), both $K$ and $K(v'_1, ..., v'_n)$ are isotopic to $K(v_1, v'_1, ..., v_n, v'_n)$ in $N$. Therefore, they are isotopic to each other, and (iii) follows. \(\square\)

Denote by $C(\theta, u, v)$ the solid cone based at $u$ (the vertex of the cone), open in the direction of $v$, with angle $\theta$. More explicitly, if we set up the coordinate system with $u$ the origin and $v$ in the direction of $(0, 0, 1)$, then

$$C(\theta, u, v) = \{(x, y, z) \in \mathbb{R}^3 \mid z \geq \cot \theta \sqrt{x^2 + y^2}\}.$$

A smooth curve $\beta : [a, b] \to B$ in a ball $B$ is $\theta$-allowable if (i) $\beta$ is properly embedded and unknotted in $B$, (ii) the cones $C_a = C(\theta, \beta(a), -\beta'(a))$ and $C_b = C(\theta, \beta(b), \beta'(b))$ are mutually disjoint, each intersecting $B$ only at its cone point.
A smooth arc $\beta : [a, b] \rightarrow \mathbb{R}^3$ is called an $\epsilon$-suspension if it is an embedding into an equilateral triangle $\Delta$ in $\mathbb{R}^3$ with base the line segment $e = e(\beta(a), \beta(b))$ and height $\epsilon$. It is called a round $\epsilon$-suspension if furthermore it is a subarc of a round circle in $\mathbb{R}^3$, and $||\beta'(t)||$ is a constant function. The line segment $e$ is called the base arc of $\beta$, and the disk bounded by $\beta$ and $e$ is called the suspension disk. Put $\theta = 2\epsilon/||e||$. Then the two angles of $\Delta$ adjacent to $e$ is at most $\arctan(2\epsilon/||e||) < \theta$. Therefore $\Delta$, hence the curve $\beta$, is contained in the cones $C(\theta, \beta(a), \beta'(a))$ and $C(\theta, \beta(b), -\beta'(b))$.

Let $K$ be a polygonal knot, with edges $e_1, \ldots, e_n$. A smooth curve $\beta : S^1 \rightarrow \mathbb{R}^3$ is an allowable $\epsilon$-approximation of $K$ if it is a union of arcs $\beta_1, \ldots, \beta_{2n}$, such that

(i) each $\beta_{2k}$ is an $\epsilon$-allowable arc in some ball $B_k$ of radius at most $\epsilon$;

(ii) each $\beta_{2k-1}$ is an $\epsilon$-suspension, such that its base arc $E_k$ is parallel to $e_k$, and the difference between the lengths of $e_k$ and $E_k$ is at most $\epsilon$.

**Lemma 2.** Given any polygonal knot $K = K(v_1, \ldots, v_n)$ and a regular neighborhood $N$ of $K$, there is an $\epsilon > 0$ such that any allowable $\epsilon$-approximation $\gamma$ of $K$ with the same initial point is a knot, which is isotopic to $K$ in $N$.

**Proof.** Rescaling $\mathbb{R}^3$ if necessary, we may assume that the length of each edge of $K$ is at least 3. Let $e_i = e(v_i, v_{i+1})$. Denote by $m$ the minimum distance between nonadjacent edges, and by $r$ the number given in Lemma 1.

Let $\epsilon$ be a very small positive number (for example, $\epsilon < \min(1, m/10n, r/10n)$). Let $\beta_1, \ldots, \beta_{2n}$ be the arcs of $\gamma$, and $B_i$ the ball containing $\beta_{2i}$, as in the definition of allowable $\epsilon$-approximation. Let $\Delta_i$ be the equilateral triangles containing $\beta_{2i-1}$, as in the definition of $\epsilon$-suspension arcs. Denote by $v_i', v_{i+1}'$ the initial and ending points of $\beta_{2i-1}$, respectively. Consider the union of all $\Delta_i$ and $B_i$.

CLAIM. The triangles $\Delta_i$ are mutually disjoint, the balls $B_i$ are mutually disjoint, and $\Delta_i$ intersects $B_j$ only if $j = i$ or $i - 1 \mod n$, in which case they intersects at a single point.

Since the base arc $E_i = e(v_i', v_{i+1}')$ of $\beta_{2i-1}$ is parallel to $e_i$ with length difference at most $\epsilon$, and since $d(v_i', v_{i+1}')$ is at most $2\epsilon$ (the upper bound of diameters of $B_i$), one can show by induction that $d(v_i, v_i') \leq (3i - 2)\epsilon$, and $d(v_i, v_{i+1}') \leq 3i\epsilon$. Put $\delta = 4n\epsilon < m/2$. Then $\beta_{2i-1}$ is in the $\delta$-neighborhood of $e_i$, and $B_i$ is in the $\delta$-neighborhood of $v_{i+1}$. Since the distance between two vertices or nonadjacent edges of $K$ is bounded below by $m$, it follows that the balls $B_i$ are mutually disjoint, $\Delta_i$ is disjoint from $\Delta_j$ when $i$ and $j$ are not adjacent mod $n$, and disjoint from $B_j$ if $j$ is not equal or adjacent to $i \mod n$. Since $||E_i|| > 3 - \epsilon$ and the height of $\Delta_i$ is at most $\epsilon$, the two angles of
\( \Delta_i \) adjacent to \( E_i \) is at most \( 2\epsilon/(3-\epsilon) > \epsilon \). Thus for each endpoint \( v \) of \( E_i \), \( \Delta_i \) is contained in a cone of angle \( \epsilon \) based at \( v \) in the direction of the tangent or negative tangent of \( \beta \) at \( v \). Since \( \beta_{2i} \) is an \( \epsilon \)-allowable arc, it follows from definition that \( \Delta_i \) is disjoint from \( \Delta_{i+1} \), and they each intersect \( B_i \) only at a single point. This completes the proof of the claim.

Since each \( \beta_i \) is an embedding, it follows from the claim that \( \gamma : S^1 \to \mathbb{R}^3 \) is an embedding, hence is a knot. We can isotope \( \beta_{2i-1} \) via the suspension disk to the edge \( E_i \). Since \( \beta_{2i} \) is unknotted in \( B_i \), it can be rel \( \partial \) isotoped to a straight arc \( E_i' \) in \( B_i \). By the claim these isotopies form an isotopy of \( \gamma \) to the polygonal knot \( K_2 = E_1 \cup E_1' \cup ... \cup E_n \cup E_n' = K(v_1', v_1'', ..., v_n', v_n'') \). Since \( d(v_i', v_i'') \) is very small, by Lemma 1(ii) \( K_2 \) is isotopic to the knot \( K(v_1'', ..., v_n'') \), which is isotopic to \( K \) by Lemma 1(iii). \( \square \)

Let \( A \) be a compact 1-manifold. A smooth map \( \alpha : A \to S^2 \) is \textit{admissible} if (i) \( \alpha \) is an embedding, and (ii) it has no antipodal points, i.e., \( \alpha(t) \neq -\alpha(s) \) for all \( t \neq s \). Denote by \( \eta : S^2 \to S^2 \) the antipodal map, and by \( \rho : S^2 \to \mathbb{P}^2 \) the standard double covering map onto the projective plane \( \mathbb{P}^2 \). Then \( \alpha \) is admissible if and only if \( \rho \circ \alpha : A \to \mathbb{P}^2 \) is a smooth embedding.

**Lemma 3.** Suppose \( Y \) is the disjoint union of finitely many circles, and suppose \( A \) is a compact submanifold of \( Y \). Let \( \alpha : A \to S^2 \) be an admissible map such that each circle component of \( \alpha(A) \) bounds a disk \( \Delta \) with interior disjoint from \( \alpha(A) \) and \( \eta(\Delta) \). Then \( \alpha \) extends to an admissible map \( \tilde{\alpha} : Y \to S^2 \).

**Proof.** Let \( I \) be the closures of components of \( Y - A \). We need to extend \( \alpha \) to an admissible map \( \tilde{\alpha} : A \cup I \to S^2 \) which still satisfies the assumption of the lemma. The result would then follow by induction. If \( I \) is a circle, define \( \tilde{\alpha} : I \to S^2 \) to be a smooth map embedding \( I \) into a small disk \( D \) of \( S^2 \) such that \( D, \rho(D) \) and \( \alpha(A) \) are mutually disjoint. So suppose \( I \) is an interval with endpoints \( u_1, u_2 \) on a component \( Y_0 \) of \( Y \). Denote by \( \tilde{\alpha} = \rho \circ \alpha \).

If \( J = Y_0 - \text{Int} I \) is connected, then by assumption \( \tilde{\alpha} \) is an embedding, so there is a small disk neighborhood \( D \) of \( \tilde{\alpha}(J) \) which is disjoint from \( \tilde{\alpha}(A - J) \). Let \( D_1 \) be the component of \( \rho^{-1}(D) \) containing \( \alpha(J) \), and extend \( \alpha \) to a smooth embedding \( \tilde{\alpha} : A \cup I \to S^2 \) so that \( \tilde{\alpha}(I) \subset D \).

Now suppose \( J \) is disconnected. Let \( J_1, J_2 \) be the components of \( J \) containing \( u_1, u_2 \) respectively. Let \( K_1, ..., K_r \) be the circle components of \( A \), and let \( D_i \) be the disk on \( \mathbb{P}^2 \) bounded by \( K_i \). By assumption \( \tilde{\alpha}(J_i) \) are in \( \mathbb{P}^2 - \cup D_i \), so there are two
non-homotopic arcs $\tilde{\gamma}_1, \tilde{\gamma}_2 : I \to P^2$ such that $\tilde{\gamma}_i \cup \tilde{\alpha} : I \cup A \to P^2$ is a smooth embedding. One of the $\tilde{\gamma}_i$ lifts to a path $\gamma : I \to S^2$ connecting $u_1$ to $u_2$. It follows that $\gamma \cup \alpha : I \cup A \to S^2$ is the required extension. □

**Lemma 4.** Suppose $\alpha = (\alpha_1, \alpha_2, \alpha_3) : I = [a, b] \to S^2$ is an admissible curve intersecting $S_1$ transversely at two points in the interior, and $\alpha_3(a) > 0$. Then there is a function $f : I \to \mathbb{R}_+$ such that (i) $f(t) = 1$ in a neighborhood of $\partial I$, and (ii) the integral curve $\beta$ is unknotted in $Z[z_1, z_2]$, where $z_1 = \beta_3(a)$ and $z_2 = \beta_3(b)$.

**Proof.** By assumption $\alpha_3$ has exactly two zeroes $u, v \in I$, $(u < v)$, so $a_3(t) < 0$ if and only if $t \in (u, v)$. Since $\alpha$ is admissible, $\alpha(u) \neq \pm \alpha(v)$, so by a rotation along the $z$-axis if necessary we may assume that $\alpha_1(u), \alpha_1(v) > 0$, and $\alpha_2(u), \alpha_2(v)$ have different signs. Without loss of generality we may assume that $\alpha_1(t), \alpha_2(t) > 0$ when $t$ is in an $\epsilon$-neighborhood of $u$, and $\alpha_1(t) > 0, \alpha_2(t) < 0$ when $t$ is in an $\epsilon$-neighborhood of $v$, where $0 < \epsilon < \min(u - a, b - v)$.

We start with the constant function $f(t) = 1$ on $I$, and proceed to modify $f(t)$ so that $f(t)$ and the integral curve $\beta = \beta(f, \alpha, t_0, v_0)$ satisfy the conclusion of the lemma. Put $\beta = (\beta_1(t), \beta_2(t), \beta_3(t))$, and choose the base point $v_0$ so that $\beta(u) = 0$. Thus

$$\beta_1(t) = \int_u^t f(t)\alpha_1(t) \, dt.$$  

Since $\alpha_1(u), \alpha_2(u) > 0$, and $\beta_1(u) = \beta_2(u) = 0$, by enlarging $f(t)$ in a small $\epsilon$-neighborhood of $u$, we may assume that $\beta_1(t), \beta_2(t) > 0$ for all $t \in (u, v)$. Since $\alpha_2(t) < 0$ in a neighborhood of $v$, we may then enlarge $f(t)$ near $v$ so that $\beta_2(v) = \beta_2(u) = 0$. This does not affect the fact that $\beta_1(t) > 0$ for $t \in (u, v)$, and $\beta_2(t) > 0$ for $t \in (u, v)$.

The function $\beta_3$ is descending in $[u, v]$ because $\alpha_3(t)$ is negative in this interval. Thus $\beta_3(v) < \beta_3(u)$. Since $\alpha_3$ is positive in $[a, u]$ and $[v, b]$, $\beta_3$ is increasing in these intervals. We may now enlarge $f(t)$ in $(u - \epsilon, u)$ and $(v, v + \epsilon)$, so that $z_3 = \beta_3(u - \epsilon) < \beta_3(v)$ and $z_4 = \beta_3(v + \epsilon) > b_3(u)$. Thus the curve $\beta$ on $[u - \epsilon, v + \epsilon]$ is a proper arc in $Z[z_3, z_4]$. We want to show that it is unknotted.

By the above, the curve $\beta[u, v]$ lies in $Z[z_3, z_4] \cap Y[0, \infty)$, with endpoints on the $xz$-plane. Since $\beta_3$ is descending on $[u, v]$, $\beta$ is rel $\partial$ isotopic in $Z[z_3, z_4] \cap Y[0, \infty)$ to a straight arc $\hat{\beta}[u, v]$ on the $xz$-plane. Since $\alpha_2(t) > 0$ for $t \in [u - \epsilon, u]$, and $\beta_2(u) = 0$, we have $\beta_2(t) < 0$ for $t \in [u - \epsilon, u]$. Similarly, since $\alpha_2(t) < 0$ near $v$, we have $\beta_2(t) < 0$ for $t \in [v, v + \epsilon]$. Therefore, the above isotopy is disjoint from the arcs $\beta[u - \epsilon, u]$ and
\( \beta[v, v+\epsilon] \), hence extends trivially to an isotopy of \( \beta[u-\epsilon, v+\epsilon] \), deforming \( \beta[u-\epsilon, v+\epsilon] \) to the curve \( \hat{\beta} = \beta[u-\epsilon, u] \cup \hat{\beta}[u, v] \cup \beta[v, v+\epsilon] \).

Since \( \alpha_1(t) \) is positive near \( u, v \), \( \beta_1 \) is increasing in \([u-\epsilon, u]\) and \([v, v+\epsilon]\). Since \( \hat{\beta} \) is a straight arc connecting \( \beta(u) \) and \( \beta(v) \), and \( \beta_1(v) > \beta_1(u) \) by the above, the first coordinate function of \( \hat{\beta} \) is also increasing in \([u, v]\). It follows that the first coordinate of \( \hat{\beta} \) is increasing in \([u-\epsilon, v+\epsilon]\), therefore, \( \hat{\beta} \) is unknotted in \( Z[z_3, z_4] \), hence is rel \( \partial \) isotopic to a straight arc \( \hat{\beta} \) in \( Z[z_3, z_4] \).

Since \( \beta_3(t) \) is increasing on \([a, u-\epsilon] \cup [v + \epsilon, b]\), the above isotopy extends trivially to an isotopy deforming \( \beta : I \to \mathbb{R}^3 \) to the curve \( \beta[a, u-\epsilon] \cup \hat{\beta} \cup \beta[v + \epsilon, b] \). Since the third coordinate of this curve is always increasing, it is unknotted in \( Z[z_1, z_2] \), where \( z_1 = \beta_3(a) \) and \( z_2 = \beta_3(b) \). Therefore, \( \beta \) is also unknotted in \( Z[z_1, z_2] \).

\[ \square \]

Given \( a \in \mathbb{R} \) and \( \delta > 0 \), let \( \varphi = \varphi[a, \delta](x) \) be a smooth function on \( \mathbb{R}^1 \) which is symmetric about \( a \), \( \varphi(a) = 1 \), \( \varphi(x) = 0 \) for \( |x - a| \geq \delta \), and \( 0 \leq \varphi(x) \leq 1 \) for all \( x \). Given \( a, b \in \mathbb{R} \) with \( a < b \), let \( \psi(x) = \psi[a, b](x) \) be a smooth monotonic function such that \( \psi(x) = 0 \) for \( x \leq a \), and \( \psi(x) = 1 \) for \( x \geq b \). Such functions exist, see for example [GP, Page 7].

For any point \( p \in S^2 \), denote by \( U(p, \epsilon) \) the \( \epsilon \)-neighborhood of \( p \) on \( S^2 \), measured in spherical distance. Thus for any \( q \in U(p, \epsilon) \), the angle between \( p, q \) (considered as vectors in \( \mathbb{R}^3 \)) is less than \( \epsilon \).

**Lemma 5.** Let \( 0 < \epsilon < \pi/8 \), and let \( \alpha = (\alpha_1, \alpha_2, \alpha_3) : I = [a_1, a_2] \to S^2 \) be an admissible arc transverse to \( S_1 \), such that \( \alpha_3(a_i) > \epsilon \). Let \( \mu > 0 \). Then there is a smooth positive function \( f(t) \) such that (i) \( f(t) = 1 \) near \( a_i \), and (ii) the integral curve \( \beta = \beta(f, \alpha, t_0, v_0) \) is an \( \epsilon \)-allowable arc in a ball of radius \( \mu \) in \( \mathbb{R}^3 \).

**Proof.** Notice that \( U(\alpha(a_i), \epsilon) \) are on the upper half sphere \( S^2_+ \). Choose \( 0.1 > \delta > 0 \) sufficiently small, so that \( \alpha(t) \in U(\alpha(a_i), \epsilon) \) for \( t \) in a \( \delta \)-neighborhood of \( a_i \). Choose \( c_0 = a_1 + \delta, c_1, \ldots, c_p = a_2 - \delta \) so that the curve \( \alpha(I_j) \) intersects \( S_1 \) exactly twice in the interior of \( I_j = [c_{j-1}, c_j], j = 1, \ldots, p \).

By Lemma 4 applied to each \( I_j \), we see that there is a function \( f_1(t) \) on \( I \), such that \( f_1(t) = 1 \) near \( c_i \) and on \([a_1, c_0] \cup [c_p, a_2]\), and the part \( \beta_1[c_0, c_p] \) of the integral curve \( \beta_1 = \beta(f_1, \alpha, t_0, v_0) \) is unknotted in \( Z[z_0, z_p] \), where \( z_i = \beta(c_i) \). Without loss of generality we may choose \( t_0 = c_0 \) and \( v_0 = 0 \). Since the curve is compact, the isotopy is within a ball, so there is a disk \( D \) in \( \mathbb{R}^2 \), such that \( \beta_1[c_0, c_n] \) is unknotted in \( D \times [z_0, z_p] \). Choose \( N \) large enough, so that the ball \( B(N) \) of radius \( N \) centered at the origin contains both \( D \times [z_0, z_p] \) and the curve \( \beta_1 \) in its interior. We want to
modify \( f_1(t) \) on \([a_1, c_0] \cup (c_p, a_2]\) to a function \( f_3(t) \), so that \( \beta_3 = \beta(f_3, \alpha, c_0, 0) \) is an \( \epsilon \)-admissible curve in \( B(10N) \), and \( f_3(t) = 10N/\mu \) near \( \partial I \).

First, consider the function

\[
f_2 = f_1 + \left( \frac{10N}{\mu} - 1 \right)(1 - \psi[a_1, a_1 + \epsilon_1] + \psi[a_2 - \epsilon_1, a_2]),
\]

where \( \epsilon_1 \) is a very small positive number, say \( \epsilon_1 < \min(\delta, \mu/10) \). By the property of the \( \psi \) functions, we have \( f_2(t) = f_1(t) \) for \( t \in [c_0, c_p] \), and \( f_2(t) = 10N/\mu \) near \( a, b \).

Let \( \beta_2 = \beta(f_2, \alpha, z_0, 0) \). Since \( \epsilon_1 \) is very small, one can show that \( ||\beta_2(t)|| < 2N \) for all \( t \in [a, b] \). Let \( b_1, b_2 \) be positive real numbers. Define

\[
f_3(t) = f_2(t) + b_1 \varphi[a_1 + \delta/2, \delta/4](t) + b_2 \varphi[a_2 - \delta/2, \delta/4](t).
\]

Let \( \beta_3 \) be the integral curve \( \beta_3(f_3, \alpha, z_0, 0) \). Since \( L \) is a polygonal knot, \( \sum e_j = 0 \), so we have

\[
\beta_3(a_2) = \beta_2(a_2) + b_2 \int_{z_p}^b \varphi(a_2 - \delta/2, \delta/4)(t) \alpha(t) \, dt = \beta_2(a_2) + b_2 \nu_2.
\]

Since \( \alpha(t) \in U(\alpha(a_2), \epsilon) \) and \( \epsilon < \pi/8 \), the vector \( \nu_2 \) above is nonzero. Since \( ||\beta_2(a_2)|| < 2N \), we may choose \( b_2 > 0 \) so that \( ||\beta_3(a_2)|| = 10N \). Similarly, choose \( b_1 > 0 \) so that \( \beta_3(a_1) = 10N \).

Consider a point \( t \in [c_p, a_2] \) such that \( ||\beta_3(t)|| \geq 10N \). Let \( \theta(t) \) be the angle between \( \beta_3(t) \) and \( \beta'_3(t) \). Put \( u_0 = \beta_3(c_p) \), and notice that \( ||u_0|| < N \). Since \( \alpha(t) \in U(\alpha(a_2), \epsilon) \), the curve \( \beta_3[c_p, b] \) lies in the cone \( C(\epsilon, u_0, \alpha(a_2)) \), so the angle between \( (\beta_3(t) - u_0) \) and \( \alpha(t) \) is at most \( 2\epsilon \). We have

\[
\cos \theta(t) = \frac{\beta_3(t) \cdot \alpha(t)}{||\beta_3(t)||} = \frac{(\beta_3(t) - u_0) \cdot \alpha(t) + u_0 \cdot \alpha(t)}{||\beta_3(t)||} \\
\geq \frac{(10N - N) \cos(2\epsilon) - N}{10N} = 0.9 \cos(2\epsilon) - 0.1 \\
> \frac{1}{2}
\]

Therefore, \( \theta(t) < \pi/3 \). In particular, this implies that the norm of \( \beta_3(t) \) is increasing if it is at least \( 10N \) and \( t \in [c_p, a_2] \); but since \( ||\beta_3(a_2)|| = 10N \), it follows that \( \beta_3(t) \in B(10N) \) for \( t \in [c_p, a_2] \). Similarly, one can show that this is true for \( t \in [a_1, c_0] \).

Therefore, \( \beta_3 \) is a proper arc in \( B(10N) \). It is unknotted because its third coordinate is increasing on \([a_1, c_0] \cup [c_p, a_2]\) and the curve \( \beta_3[c_0, c_p] = \beta_1[c_0, c_p] \) is unknotted in \( D \times [z_0, z_p] \), with \( \beta_3(c_0) \) on \( D \times z_0 \).
We need to show that the cone $C(\epsilon, \beta_3(a_2), \beta'_3(a_2))$ intersects $B(10N)$ only at the cone point, but this is true because $\epsilon + \theta(a_2) < \pi/8 + \pi/3 < \pi/2$. Similarly for $C(\epsilon, \beta_3(a_1), -\beta'_3(a_1))$. Also, notice that the cone $C(\epsilon, \beta_3(a_2), \beta'_3(a_2))$ lies above the $xy$-plane, while $C(\epsilon, \beta_3(a_1), -\beta'_3(a_1))$ lies below the $xy$-plane, so they are disjoint. It follows that $\beta_3$ is an $\epsilon$-allowable curve in $B(10N)$.

Finally, rescale the curve by defining $f(t) = f_3(t)\mu/10N$, and $\beta = \beta(f, \alpha, c_0, 0)$. Then $\beta$ is an $\epsilon$-allowable curve in a ball of radius $\mu$, and $f(t) = 1$ near $\partial I$. \[\Box\]

**Lemma 6.** Suppose the integral curve $\beta = \beta(f, \alpha, a, 0)$ is a round $\epsilon$-suspension. Then for any $k \in \left[\frac{1}{2}, \frac{3}{2}\right]$, there is a positive function $g(t)$ such that (i) $g(t) = f(t)$ near $a, b$, and (ii) the integral curve $\gamma = \beta(g, \alpha, a, 0)$ is a $(k\epsilon)$-suspension with $\gamma(b) - \gamma(a) = k(\beta(b) - \beta(a))$.

**Proof.** Without loss of generality we may assume $[a, b] = [-1, 1]$. Set up the coordinate system so that $\beta$ lies in the triangle with vertices $\beta(a) = (0, 0, 0)$, $\beta(b) = (2u, 0, 0)$ and $(u, \epsilon, 0)$, where $2u = ||\beta(b) - \beta(a)||$. Put $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. Then $\alpha_3(t) = 0$, and $\alpha_2(-t) = -\alpha_2(t)$. Consider the smooth function $\phi = \psi[-1+\delta, -1+2\delta] - \psi[1-2\delta, 1-\delta]$. It is an even function with $\phi(t) = 1$ when $|t| \leq 1 - 2\delta$, and $\phi(t) = 0$ when $|t| \geq 1 - \delta$. Let $g(t) = c + p\phi(t)$, where $p > -c$ is a constant. Since $|\phi(t)| \leq 1$, $g(t)$ is a positive function. We have

$$
\gamma(1) = \int_{-1}^{1} g(t)\alpha(t) \, dt = \beta(1) + p \int_{-1}^{1} \phi(t)\alpha(t) \, dt.
$$

Since $\phi(t)$ is even and $\alpha_2$ is odd, $\gamma_2(1) = \gamma_3(1) = 0$. When $\delta$ approaches $0$, the integral

$$
c \int_{-1}^{1} \phi(t)\alpha_1(t) \, dt
$$

approaches $\beta_1(1) = 2u$. Hence for any $s \in [u, 3u]$, we may choose $\delta$ small and $p \in (-c, c)$ so that $\gamma(1) = (s, 0, 0)$. Note that $\gamma_1'(t) = (c - p)\alpha_1(t) > 0$, so $\gamma$ is an embedding.

Consider $\gamma$ and $\beta$ as curves on the $xy$-plane. Then The tangent of $\gamma$ at $t$ is given by

$$
\frac{\gamma'(t)}{\gamma_1'(t)} = \frac{g(t)\alpha_2(t)}{g(t)\alpha_1(t)} = \frac{\alpha_2(t)}{\alpha_1(t)},
$$

which is the same as the tangent slope of $\beta$ at $t$, and hence is bounded above by $\epsilon$. Thus $\gamma$ is below the line $y = \epsilon x$ on the $xy$-plane. Similarly, it is below the line $y = -\epsilon(x - \gamma_1(1))$. It follows that $\gamma$ is a $k\epsilon$-suspension, where $k = \gamma_1(1)/\beta_1(1) < 2$. \[\Box\]
**Theorem 7.** Given any tame link $L$ in $S^3$ and any neighborhood $\eta(L)$ of $L$, there is a smooth link in $\eta(L)$ which is isotopic to $L$ in $\eta(L)$, and has no parallel or antiparallel tangents.

**Proof.** Without loss of generality we may assume that $L = K_1 \cup ... \cup K_r$ is an oriented polygonal link in general position, with oriented edges $e_1, ..., e_m$, which are also considered as vectors in $\mathbb{R}^3$. Let $d$ be the minimum distance between nonadjacent edges. We may assume each $K_i$ has at least four edges, so $d$ is also an upper bound on the length of $e_i$. For any $\epsilon_1 \in (0, d/3)$ the $\epsilon_1$-neighborhoods of $K_i$, denoted by $N(K_i)$, are mutually disjoint. Choosing $\epsilon_1$ small enough, we may assume that $N(K_i)$ are contained in $\eta(L)$. By Lemma 2 there is an $\epsilon > 0$, such that any allowable $\epsilon$-approximation of $K_i$ is contained in $N(K_i)$ and is isotopic to $K_i$ in $N(K_i)$. Note that $\epsilon \leq d/3$. We will construct such an approximation $\hat{K}_i$ for each $K_i$, with the property that $\hat{L} = \hat{K}_1 \cup ... \cup \hat{K}_r$ has no parallel or antiparallel tangents. Since $N(K_i)$ are mutually disjoint, the union of the isotopies from $\hat{K}_i$ to $K_i$ will be an isotopy from $\hat{L}$ to $L$ in $N(L)$.

Consider the unit tangent vector of $e_i$ as a point $p_i$ on $S^2$, which projects to $\hat{p}_i$ on $P^2$. Since $L$ is in general position, $\hat{p}_1, ..., \hat{p}_n$ are mutually distinct, so by choosing $\epsilon$ smaller if necessary we may assume that they have mutually disjoint $\epsilon$-neighborhoods $\hat{D}_1, ..., \hat{D}_n$, which then lifts to $\epsilon$-neighborhoods $D_1, ..., D_n$ of $p_1, ..., p_n$. Adding some edges near vertices of $L$ if necessary, we may assume that the angle between the unit tangent vectors of two adjacent edges $e_i, e_{i+1}$ of $L$ (i.e. the spherical distance between $p_i$ and $p_{i+1}$), is small (say $\leq \pi/2$).

Bend each edge $e_j$ a little bit to obtain a round $(\epsilon/2)$-suspension $\hat{e}_j : I_j \to \mathbb{R}^3$ with $||\hat{e}_j(t)|| = 1$ (so the length of $I_j$ equals the length of the curve $\hat{e}_j$). Then its derivative $\hat{e}_j'$ is a map $I_j \to S^2$ with image in $D_j$ because $D_j$ has radius $\epsilon$. Let $Y = \cup S^1_i$ be a disjoint union of $r$ copies of $S^1$, and let $A = \cup I_j$ be the disjoint union of $I_j$. Embed $A$ into $Y$ by a map $\eta$ according to the order of $e_i$ in $L$. More precisely, if $e_j$ and $e_k$ are edges of $L$ such that the ending point of $e_j$ equals the initial point of $e_k$ then the ending point of $\eta(I_j)$ and the initial point of $\eta(I_k)$ cobounds a component of $Y - \eta(A)$. The union of the maps $\hat{e}_j' \circ \eta^{-1}$ defines a map $\eta(A) \to S^2$, which is admissible because the disks $\hat{D}_j$ on $P^2$ are mutually disjoint. By Lemma 3, it extends to an admissible map $\hat{\alpha} : Y \to S^2$. It now suffices to show that each $\hat{K}_i$ has an allowable $\epsilon$-approximation $\hat{K}_i : S^1_i \to \mathbb{R}^3$, with $\hat{\alpha}|_{S^1_i}$ as its unit tangent map.

The construction of $\hat{K}_i$ is independent of the other components, so for simplicity we may assume that $L = K(v_1, ..., v_n)$ is a knot, with edges $e_i = e_i(v_i, v_{i+1})$. Since
L is in general position, the three unit vectors \( p_1, p_2, p_3 \) of the edges \( e_1, e_2, e_3 \) are linearly independent, so there is a positive number \( \delta < \epsilon \), such that the ball of radius \( \delta \) centered at the origin is contained in the set \( \{ \sum u_i p_i \mid \epsilon > |u_i| \} \).

We may assume that the intervals \( I_j = [a_j, b_j] \) are sub-intervals of \( I = [0, a_{n+1}] \), with \( a_1 = 0 \) and \( b_j < a_{j+1} \). Put \( \hat{I}_j = [b_j, a_{j+1}] \). Without loss of generality we may assume that the function \( \eta : A \to S^1 \) defined above is the restriction of the function \( \eta : I \to S^1 \) defined by \( \eta(t) = \exp(2\pi it/a_{2n}) \). Put \( \alpha = \hat{\alpha} \circ \eta : I \to S^2 \). Thus \( \alpha(t) = \hat{e}'(t) \) on \( I_j \).

Consider the restriction of \( \alpha \) on \( \hat{I}_j = [b_j, a_{j+1}] \). We have assumed that the spherical distance between \( p_j \) and \( p_{j+1} \) is at most \( \pi/2 \). Since \( \alpha(b_j) \in D_j \) and \( \alpha(b_{j+1}) \in D_{j+1} \), the spherical distance between \( \alpha(b_j) \) and \( \alpha(a_{j+1}) \) is at most \( \pi/2 + 2\epsilon \). As \( \epsilon \) is very small, we may choose a coordinate system for \( \mathbb{R}^3 \) so that the third coordinate \( \alpha_3 \) of \( \alpha \) is greater than \( \epsilon \) at \( b_j \) and \( a_{j+1} \), and by transversality theorem we may further assume that \( \alpha \) is transverse to the circle \( S_1 = S^2 \cap \mathbb{R}_{xy} \) in this coordinate system. Now we can apply Lemma 5 to get a function \( f_j(t) \) on \( \hat{I}_j \) such that \( f_j(t) = 1 \) near \( \partial \hat{I}_j \), and the integral curve \( \gamma_j = \beta(f_j, \alpha|_{\hat{I}_j}) \) is an \( \epsilon \)-allowable curve in a ball of radius \( \delta/2n \). Extend these \( f_j \) to a smooth map on \( I \) by defining \( f(t) = 1 \) on \( I_j \).

Consider the integral curve \( \beta = \beta(f, \alpha) \). It is the union of \( 2n \) curves \( \beta_i, \hat{\beta}_i \) defined on \( I_i \) and \( \hat{I}_i \), where \( \beta_i = \beta(f|_{I_i}, \alpha|_{I_i}, a_i, \beta(a_i)) \) is a translation of \( \hat{e}_j \) because \( f|_{I_i} \equiv 1 \); and \( \hat{\beta}_i = \beta(f|_{\hat{I}_i}, \alpha|_{\hat{I}_i}, b_i, \beta(b_i)) \) is a translation of \( \gamma_i \) because \( f|_{\hat{I}_i} = f_i \). We have

\[
||\beta(a_{n+1}) - \beta(a_0)|| \leq \sum_{1}^{n} ||\beta(a_{j+1}) - \beta(b_j)|| + ||\sum_{1}^{n} (\beta(b_j) - \beta(a_j))||
\leq \sum_{1}^{n} 2(\delta/2n) + ||\sum_{j} e_j|| = \delta.
\]

By the definition of \( \delta \), there are numbers \( u_i \in [-\epsilon, \epsilon] \), such that \( \beta(2n) - \beta(0) = \sum_{i=1}^{3} u_i p_i \). Notice that \( |u_i| < \epsilon < ||e_j||/2 \), so by Lemma 6, we can modify \( f(t) \) on \( [a_j + \epsilon_1, \beta_j - \epsilon_1] \) for \( j = 1, 2, 3 \) and some \( \epsilon_1 > 0 \), to a function \( g(t) \), so that the integral curve \( \gamma = \beta(g, \alpha) \) on \( I_j \) is an \( \epsilon \)-suspension with base arc the vector \( e_j + u_j p_j \). Now we have \( \gamma(a_{n+1}) = \gamma(0) \), so \( \gamma \) is a closed curve. Since \( \gamma'(t) = \alpha(t) \) near 0 and 2n and \( \alpha \) induces a smooth map \( \hat{\alpha} : S^1 \to S^2 \), it follows that \( \gamma \) induces a smooth map \( \hat{\gamma} : S^1 \to \mathbb{R}^3 \).

From the definition we see that \( \hat{\gamma} \) is an allowable \( \epsilon \)-approximation of \( K \). This completes the proof of the theorem. \( \square \)

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