Odd-\(J\) Pairing in Nuclei

L. Zamick and A. Escuderos

Department of Physics and Astronomy, Rutgers University, Piscataway, NJ 08854, USA

We point out a simplicity that arises when we use an interaction in which only an energy with odd \(J\) is non-zero. The emphasis is on \(J = J_{\text{max}}\) and in particular \(J = 9^+\) in the \(g_{9/2}\) shell. It is noted that high overlaps can be deceptive. In many cases a single set of unitary 9-\(j\) coefficients gives either an exact or a surprisingly good approximation to the wave function of a non-degenerate state. The many degeneracies that occur in these calculations are discussed and explained. As a counterpoint, we compare the results with an interaction in which both the \(J = 0\) and \(J = J_{\text{max}}\) two-body matrix elements are equal (and attractive). Comparisons with a more realistic interaction are also made.

PACS numbers:

I. INTRODUCTION

The purpose of this work is to study the properties of a very simple interaction in a single-\(j\)-shell model space of neutrons and protons. A single proton–neutron (\(pn\)) pair in this space can have a total angular momentum from \(J = 0\) to \(J = J_{\text{max}} = 2j\). The even-\(J\) states have isospin \(T = 1\), i.e. they are members of an isotriplet—there are analog states of two neutrons and of two protons with these even angular momenta. The odd-\(J\) states have isospin \(T = 0\), i.e. isosinglet—in this model space, they only are present in the \(pn\) system.

There has been much study in journals and in textbooks of the \(J = 0\), \(T = 1\) “pairing interacton”, i.e. where only \(J = 0\) two-body matrix elements are non-zero (and attractive) \(1-5\). Although the pairing interaction is not realistic, the consequences of these studies have yielded results whose importance was well beyond expectation. Examples are seniority classifications, reduced isospin and indeed these studies can be regarded as precursors to the BCS theory in condensed matter physics. The Pauli principle is clear. Once one has antisymmetric wave functions of two protons, which is achieved by limiting the angular momenta to even \(J_p\), and likewise the two neutrons to even \(J_n\), one has a wave function which satisfies the Pauli principle. Then they consider the wave function in terms of \(pn\) pairs, in which one \(pn\) pair couples to \(J_1\) and the other to \(J_2\). For this wave function, \(\langle \{pn(J_1)pn(J_2)\}^I | \{pp(J_p)nn(J_n)\}^I \rangle\) is noted that high overlaps can be deceptive. In many cases a single set of unitary 9-\(j\) coefficients gives either an exact or a surprisingly good approximation to the wave function of a non-degenerate state. The many degeneracies that occur in these calculations are discussed and explained. As a counterpoint, we compare the results with an interaction in which only an energy with \(J = J_{\text{max}}\) is non-zero. The emphasis is on \(J = J_{\text{max}}\) pairing interaction, for which all two-body matrix elements are equal (and attractive). Comparisons with a more realistic interaction are also made.

We will here test the ansatz that these overlaps make up the components of an exact or approximate eigenfunction of an odd-\(J\) pairing interaction. In this work we use the symbol \(I\) for the total angular momentum of the state and \(J\) otherwise.

We here note that work we previously did shows that certain \(U9\)-j coefficients form components of the \(J = 0^+\) ground state wave function of a \(J_{\text{odd}}\) pairing interaction. We here expand on this work. We refer to a previous work by E. Moya de Guerra et al. \(7\) and explicitly to Eqs. (70) and (74). In that work we describe the wave functions in a \(\langle \{pp(J_p)nn(J_n)\}^I \rangle\) basis. The Pauli principle is easily satisfied by constraining \(J_p\) and \(J_n\) to be even. Our previous example was \(^{44}\)Ti, but the same mathematics holds for \(^{96}\)Cd. We previously considered various schematic interactions as well and the more realistic MBZ interaction taken from experiment \(8\), for which detailed wave functions were subsequently published in the archives \(9\) (with some modification of the two-body matrix elements).
II. RESULTS

We now proceed to the calculations. We first define a unitary 9-j symbol $U_{9-j}$ as the following:

$$\langle j_1 j_2 | (j_3 j_4) | j_{34} \rangle = \sum_{J_{34}} \langle j_1 j_2 | J_{12} | J_{34} \rangle \langle J_{12} | J_{34} \rangle = \sum_{J_{34}} \langle j_1 j_2 | J_{12} | J_{34} \rangle \langle J_{12} | J_{34} \rangle \frac{\delta_{J_{12}, J_{34}}}{\sqrt{J_{12} J_{34}}} \times |j_{34}|$$

where $f$ is the wave-function components.

Let us first consider the $I = 0^+$ states in $^{96}$Cd (or $^{44}$Ti). For most interactions, the diagonalization is a fairly complicated procedure. However for certain interactions it is much easier. For example, the interaction used in Ref. [7] was one in which all two-body matrix elements were set equal to zero except for the $I = 0$, $T = 0$ two-body matrix element. In such a case, the matrix element of the secular four-particle Hamiltonian factorizes (the same result holds for any odd-$J$ interaction). This is the key point. The basis states are $|pp(J_n) n(J_n)\rangle$, where $J_n = J_p$, and the Hamiltonian takes the following form:

$$H_{J_p,J_p'} = V(J_{odd}) f(J_p) f(J_p')$$

where $f(J_p)$ is twice the $U_{9-j}$ symbol:

$$f(J_p) = 2 \langle jj | J_p | jj \rangle J_{odd} (jj) J_{odd} (jj) J_{odd} = 2(2J_p + 1)(2J_{odd} + 1) \frac{\delta_{J_p, J_p'}}{\sqrt{J_p J_p'}} \text{ for }$$

If we write the wave function as $\sum X_{J_p J_p'} |pp(J_p) n(J_p)\rangle$, then it was shown in Ref. [7] that $X_{J_p J_p'}$ is proportional to $f(J_p)$. The other $I = 0^+$ eigenstates are degenerate and, if $V(J_{odd})$ is negative, they are at higher energies. In other words, what we have shown in Ref. [7] is that the wave-function components $X_{J_p J_p'}$ of the lowest $I = 0^+$ state are proportional to the overlap factor of Ref. [8]; alternately, they are equal within a normalization to the $U_{9-j}$ coefficients.

The eigenvalue is given by

$$E(I = 0^+) = V(J_{odd}) \sum f(J_p) X_{J_p J_p'}$$

Note that our very simple interactions are charge independent. This means that the lowest (non-degenerate) $I = 0^+$ state has good isospin, presumably $T = 0$. It is amusing that we can assign the isospin quantum number to a wave function with $U_{9-j}$ coefficients.

In Table I we present the wave functions for the following interactions:

**CCGI:** A realistic interaction fit as well as possible to experiment (see Ref. [10]).

$$E(0): V(0) = -2.0000 \text{ MeV; all other matrix elements are zero.}$$

$$E(9): V(9) = -2.0000 \text{ MeV; all other matrix elements are zero.}$$

$$E(0,9): V(9) = V(0) = -2.0000 \text{ MeV; all other matrix elements are zero.}$$

$$E(1): V(1) = -2.0000 \text{ MeV; all other matrix elements are zero.}$$

| $X_{00}$ | $X_{22}$ | $X_{44}$ | $X_{66}$ | $X_{88}$ |
|---------|---------|---------|---------|---------|
| 0.7725  | 0.5280  | 0.2915  | 0.1704  | 0.1020  |
| 0.8563  | 0.1741  | 0.2335  | 0.2807  | 0.3210  |
| 0.6164  | 0.7518  | 0.2385  | 0.2007  | 0.0005  |
| 0.8103  | 0.4814  | 0.2514  | 0.0233  | 0.1831  |
| 0.2903  | 0.5704  | 0.5190  | 0.1718  | -0.5540 |

We consider the second column (CCGI) as the realistic interaction to which the other interactions should be compared. The simplest thing we can do is give the overlaps of the above interactions with CCGI. They are respectively 0.9020, 0.9451, 0.9944, and 0.6484. We find that $E(9)$ gives higher overlap than the much studied $E(0)$ pairing interaction and a much higher overlap than $E(1)$. This might lead one to believe that the idea of $J = 9^+$ pairing is a valid concept. But overlaps can be deceiving. We also present $E(0,9)$, where the only non-vanishing matrix elements are for $J = 9^+$ and $0^+$, both set to $-2.0000 \text{ MeV}$. Now the overlap is even higher—0.9944. This might not be startlingly different than 0.9467, but let us now look at the energies of the lowest even-$I$ states in Table II. They are given respectively for interactions CCGI, $E(9)$, and $E(0,9)$. The results of the second column (CCGI) were previously given [11] and the point was made that the $I = 16^+$ state is isomeric since it lies below the lowest $14^+$ and $15^+$ states. This is in agreement with experiment [12].

We see that despite the 0.9451 overlap between CCGI and $E(9)$, the even-$I$ spectrum for $E(9)$ in which only the
Table II: Calculated spectra of yrast even-\(I\) states in \(^{96}\)Cd for above mentioned interactions.

| \(I^x\) | CCGI | E(9) | E(0,9) |
|--------|------|------|--------|
| 0\(^+\) | 0.0000 | 1.0587 | 0.0000 |
| 2\(^+\) | 1.0812 | 1.0589 | 1.2740 |
| 4\(^+\) | 2.1096 | 1.0588 | 1.8584 |
| 6\(^+\) | 2.8883 | 1.0588 | 2.3929 |
| 8\(^+\) | 3.2302 | 1.0571 | 2.5125 |
| 10\(^+\) | 4.8815 | 1.0464 | 2.3142 |
| 12\(^+\) | 5.3994 | 0.9670 | 3.1348 |
| 14\(^+\) | 5.4031 | 0.6570 | 2.8247 |
| 16\(^+\) | 5.2247 | 0.0000 | 2.1678 |

\(J = 9^+\) matrix element is non-zero is drastically different than CCGI. First of all, the ground state does not have \(I = 0\), rather it has \(I = J_{\text{max}} = 16\) and indeed the two spectra seem to have nothing to do with each other.

Let us briefly digress and look at the spectrum for E(9) for its own sake. It is quite remarkable. The energies of the \(I = 0, 2, 4, 6, 8\), and \(8\) states are very close to each other, differing at most by 0.002 MeV and the \(I = 10^+\) state is 0.012 MeV lower. All six states are essentially degenerate. Then there is a drop in energy with \(I = 16^+\) becoming the ground state. Such a strange spectrum and this for an interaction that gives a 0.9467 overlap with a realistic interaction for the \(I = 0^+\) state.

In the last column of Table II we improve things by also lowering the \(J = 0^+\) matrix element to the same value as for \(J = 9^+, -2.0000\) MeV. The spectrum is better, with \(I = 0^+\) now the lowest state, but it is far from satisfactory. Even an overlap exceeding 0.99 does not guarantee overall good results. Clearly all two-body matrix elements come into play.

As noted above, the eigenfunction components of the lowest \(I = 0^+\) state for the E(9) interaction are \(N \langle jj \rangle^{I^x} (jj)^{J_n} |(jj)^9(jj)^9\rangle^0\). It can be shown that the normalization factor is \(N = \sqrt{2}\).

For the \(I = 1^+\) states with the E(9) interaction, the secular matrix is also separable. This is not true for other values of \(J_{\text{odd}}\). If we were to replace \(I = 0\) by \(I = 1\) in the expression in the last paragraph, all the \(U9-j\) coefficients would vanish. We must make a different choice. The eigenfunction components of the lowest \(I = 1^+\) state is then exactly given by a single set of \(U9-j\) coefficients: \(2\langle jj \rangle^{I^x} (jj)^{J_n} |(jj)^9(jj)^9\rangle_{\text{spin}}^{I = 1}\).

For states with \(I = 2\) or higher, the secular matrix is no longer separable—rather it is a sum of separable terms. The eigenvalue equation is

\[
4 \sum_{J_x} \langle jj \rangle^{I^x} (jj)^{J_x} |(jj)^9(jj)^9\rangle^{I - 2} \times \sum_{J_p, J_n} \langle jj \rangle^{I^x} (jj)^{J_n} |(jj)^9(jj)^9\rangle^{J_n} X_{J_p, J_n} = \lambda X_{J_p, J_n},
\]

where \(\lambda\) is the eigenvalue and \(X_{J_p, J_n}\) stands for the eigenfunction components. For \(I = 2^+\) there are two terms corresponding to \(J_x = 7\) and 9; for \(I = 3^+\) the values are \(J_x = 6\) and 8, etc.

Despite the complexity of the above equation, there are some surprising results. The eigenfunction components of the lowest \(2^+\) state are numerically extraordinarily close to the single \(U9-j\) symbols \(\sqrt{2} (jj)^{I^x} (jj)^{J_n} |(jj)^9(jj)^9\rangle^{I = 2}\). Furthermore, the next \(2^+\) state has also components exceedingly close to \(2 (jj)^{I^x} (jj)^{J_n} |(jj)^9(jj)^9\rangle^{I = 2}\). This is by no means obvious, because, as mentioned above, the interaction involves a sum of two separable terms corresponding to \(J_x = 7\) and 9.

We can explain this result by performing a calculation of the overlap of the two \(U9-j\)’s of the last paragraph. We restrict the sum to even \(J_p\) and even \(J_n\). We first note schematically

\[
4 \sum_{J_p, J_n} = \sum_{J_p, J_n} (1 + (-1)^{J_p})(1 + (-1)^{J_n}) = \sum_{J_p, J_n} + \sum_{J_p, J_n} (-1)^{J_p} + \sum_{J_p, J_n} (-1)^{J_n} + \sum_{J_p, J_n} (-1)^{J_p + J_n}
\]

The first term vanishes because of Eq. (3). In the last term one of the \(U9-j\)’s has two rows that are the same, which means that the only non-vanishing terms in the sum have \((J_p + J_n, J_p + J_n)\) even. Thus, the last term is the same as the first term—zero. The two middle terms are the same, so we get the overlap of the two \(U9-j\)’s to be

\[
\sum_{J_p, J_n} = \frac{1}{2} \sum_{J_p, J_n} (1 + (-1)^{J_p}) \langle jj \rangle^{I^x} (jj)^{J_n} \langle jj \rangle^9 (jj)^9 \rangle^{I = 2} \times \langle jj \rangle^{I^x} (jj)^{J_n} \langle jj \rangle^9 (jj)^9 \rangle^{I = 2} = \frac{1}{2} \langle jj \rangle^9 \langle jj \rangle^9 \rangle^{I = 2} \times \langle jj \rangle^9 \langle jj \rangle^9 \rangle^{I = 2}.
\]

We obtain the above by using the orthogonality relations for \(9-j\)-symbols as shown in Eqs. (3) and (4).

Using similar arguments, one can show that the normalization for the \(|pm(pn)\rangle^{I = 2}\) state is such that its normalization factor is

\[
N(9)^{-2} = \frac{1}{2} \langle jj \rangle^9 \langle jj \rangle^9 \rangle^{I = 2},
\]

For the \(|pm(pn)\rangle^{I = 2}\) state, we obtain

\[
N(7)^{-2} = \frac{1}{4} \langle jj \rangle^9 \langle jj \rangle^9 \rangle^{I = 2} = \frac{1}{4} \langle jj \rangle^9 \langle jj \rangle^9 \rangle^{I = 2}.
\]

To get this latter result, we use the following relationship

\[
\sum_{J_p, J_n} (-1)^{J_p + J_n} |(jj)^9 (jj)^9 \rangle^{I = 2} = 0
\]
From Eqs. (11) and (12), we find that the normalizations are 1.414222 and 1.998497, only slightly different than $\sqrt{2}$ and 2 respectively. Therefore, we obtain that the term in Eq. (10b) is exceedingly small for the $9/2^-$ shell, namely 0.00009113 and, if we include the exact normalization factors, we get 0.00025756.

In lower shells the deviations are larger. To see the trend, we give in Table III the value of the $U9-j$ symbol of Eq. (10b) with 9 replaced by $J_{\text{max}}$ and 7 by $J_{\text{max}}-2$ for various shells. The last column uses approximate normalization factors 2 and $\sqrt{2}$. Clearly the overlap approaches zero in the large-$j$ limit. As we go from one shell to the next, the value of the overlap drops by at least a factor of 10. A study of the large-$j$ behaviour of 9-$j$ symbols has been performed by L. Yu and R.G. Littlejohn [13].

Table III: Value of coupling $U9-j$ symbols for various shells.

| $J$  | $U9-j$  | overlap of Eq. (10b) |
|------|---------|----------------------|
| $p_{3/2}$ | $-0.1800$ | 0.2546 |
| $d_{3/2}$ | $-0.021328$ | 0.03016 |
| $f_{7/2}$ | $-0.002074$ | 0.002933 |
| $g_{9/2}$ | $-0.0001822$ | 0.0002577 |
| $h_{11/2}$ | $-0.00001502$ | 0.00002174 |
| $i_{13/2}$ | $-0.000001185$ | 0.000001676 |

We can see in Table IV that the results for matrix diagonalization for both $I = 2^+$ states yield wave function components which are very close to the normalized $U9-j$ coefficients. In fact, they are so close that one could wonder if they are exactly the same. But they are not. As seen in Eq. (10b), the two $U9-j$ sets corresponding to $J_x = 9$ and $J_x = 7$ are very nearly orthogonal, but not quite.

Table IV: Comparison for the first two $I = 2^+$ states of the matrix diagonalization with the $E(9)$ interaction and with normalized $U9-j$ components. We give the energy in MeV in the second row.

| $[J_p,J_n]$ | $E(9)$ | $U9-j$ | $E(9)$ | $U9-j$ |
|-------------|--------|--------|--------|--------|
| [0, 2]      | 0.5334 | 0.5338 | 0.1349 | 0.1351 |
| [2, 2]      | $-0.4707$ | $-0.4708$ | 0.5569 | 0.5567 |
| [2, 4]      | 0.3035 | 0.3035 | 0.3188 | 0.3189 |
| [4, 4]      | $-0.1388$ | $-0.1390$ | 0.6300 | 0.6299 |
| [4, 6]      | 0.0531 | 0.0531 | 0.1320 | 0.1320 |
| [6, 6]      | $-0.0137$ | $-0.0138$ | 0.1350 | 0.1350 |
| [6, 8]      | 0.0025 | 0.0025 | 0.0114 | 0.0114 |
| [8, 8]      | $-0.0003$ | $-0.0003$ | 0.0052 | 0.0052 |

It turns out that all the other lowest even-$I$ states have eigenfunctions close although not exactly equal to $\sqrt{2}(|jj\rangle_{J_x=9}\langle jj\rangle_{J_x=7} + |jj\rangle_{J_x=7}\langle jj\rangle_{J_x=9})$. In Table V we compare, as an example, the wave function of the $I = 8^+$ state.

In the second column, we give the single $U9-j$ symbols (normalized) and in the third column we give results of diagonalizing the $E(9)$ interaction. Since the coefficient $[J_p, J_n]$ is the same as $[J_n, J_p]$, we list only one of them. The overlap of the two wave functions is 0.9998.

Table V: Comparing the wave functions of a single $U9-j$ symbol with $J_x = 9$ with a full diagonalization of $E(9)$ for the lowest $I = 8^+$ state in $^{96}$Cd.

| $[J_p,J_n]$ | $U9-j$ | $E(9)$ |
|-------------|--------|--------|
| [0, 8]      | 0.0630 | 0.0644 |
| [2, 6]      | 0.4299 | 0.4271 |
| [2, 8]      | $-0.0522$ | $-0.0513$ |
| [4, 4]      | 0.7444 | 0.7456 |
| [4, 6]      | $-0.1803$ | $-0.1729$ |
| [4, 8]      | 0.0256 | 0.0280 |
| [6, 6]      | 0.0521 | 0.0657 |
| [6, 8]      | $-0.0076$ | $-0.0012$ |
| [8, 8]      | 0.0011 | 0.0047 |

A. Isospin considerations

The $E(9)$ interaction is charge independent and therefore one can assign a definite isospin to a non-degenerate state. A wave function for two protons and two neutrons in a single $j$-shell can be written as

$$\Psi^I = \sum_{J_p, J_n} X_{J_p, J_n} |pp(j_p)nn(j_n)|^I,$$  \hspace{1cm} (14)

where $X_{J_p, J_n}$ is the probability amplitude that the protons are in the state $J_p$ and the neutrons in the state $J_n$. Again, $I$ is the total angular momentum.

If one uses a charge-independent interaction, then for cases where the number of valence protons equals the number of valence neutrons, the states fall into two classes. In the first, $X_{J_p, J_n} = X_{J_n, J_p}$, while in the second class $X_{J_p, J_n} = -X_{J_n, J_p}$. In our case of two protons and two neutrons, for even $I$ the states in the first class must have even isospin, $T = 0$ or 2; in the second class, the states have $T = 1$. For odd $I$ the states in the first class must have odd isospin, $T = 1$; in the second class, $T = 0$ or 2.

For $I = 0$ and $I = 1$, each wave function component has $J_p = J_n$, so all these states belong to class 1. For $I = 0$, the states must have isospins $T = 0$ or $T = 2$; for $I = 1$, all states must have isospin $T = 1$.

We can now make an association of isospin with the quantum number $J_x$ for those states for which this is an exact or reasonably approximate good quantum number. If we interchange two rows (or two columns) of a 9-$j$ symbol, the result is the same 9-$j$ symbol multiplied by a phase factor $(-1)^s$, where $s$ is the sum of all nine angular momenta in the 9-$j$ symbol. If two rows are identical and
$s$ is odd, the $9-j$ will be equal to zero. This leads to the result that for even total angular momentum $I$, we must have $J_x$ odd for states with even isospin, i.e. $T = 0$ or 2, whilst $J_x$ must be even for $T = 1$ states. For odd $I$, $J_x$ must be odd for $T = 0$ and $T = 2$ states and even for $T = 1$ states.

In the single $j$-shell there are no $I = 0$, $T = 1$ states, so $J_x$ must be odd. For $I = 1$, all states have $T = 1$, so $J_x$ is even. For states with $I = 2$ and 3, and indeed for most other states, one can have all possible isospins ($T = 0, 1, 2$), so one must make separate analyses with even and odd values of $J_x$.

It is not a priori clear if the above mentioned $I = 0$, $T = 1$ state with $J_x = 9$ has isospin $T = 0$ or 2; likewise the above two $I = 2^+$ states. One can show that all these states have isospin $T = 0$. The reason is that a $T = 2$ state will have an absolute energy $E = 0$, since it is a double analog of a state with four identical nucleons. For the latter system, all pairs have $T = 1$, whereas the E(9) interaction acts only for two nucleons in a $T = 0$ state. Four identical nucleons do not see any interaction. The $I = 0$ and both $I = 2$ states above are non-degenerate and have finite absolute energies. For example, as will be shown in a later section on degeneracies, the absolute energy of the lowest $I = 0$ state is $2V(9)$. Indeed this $I = 0$ state and the two $I = 2^+$ states above all have isospin $T = 0$.

The E(9) interaction also yields a $T = 1$, $I = 2$ non-degenerate state with components $2\langle(jj)^9(jj)^8jjjjjjjjjjjj\rangle_i$. This is a pure state—it does not mix with any other $T = 1$ state. This is because there is only one way of forming an $I = 2$, $T = 1$ state from $U9-j$ symbols, i.e only one possible $J_x = 8$. With $J_x = 6$ combined with $J = 9$, we cannot get $I = 2$.

For $I = 3$, $T = 0$ there is also a pure state $2\langle(jj)^9(jj)^7jjjjjjjjjjjj\rangle_i$. This wave function changes sign under the interchange of $J_p$ and $J_n$. It cannot admix with a state with $J_x = 8$ or $J_x = 6$ because those states have $T = 1$.

**B. Comparison with realistic interactions**

At the beginning of Section 11 we compared the overlaps between the $I = 0$ eigenstates of various interactions with a realistic one (CCGI [11]). Now we want to extend the comparison to all the yrast states of the two most relevant interactions in this study: E(9) and E(0,9). Thus, in Table VI we present the two sets of overlaps with the realistic interaction for the lowest even-$I$ states from $I = 0$ to $I = 16$ for the $2p-2n$ system, e.g. $^{96}$Cd. In the second column we have the overlap $\langle\psi_{E(9)}|\psi_{U9-j}\rangle$ between the eigenstates of the E(9) interaction and the normalized $U9-j$ coefficients $\sqrt{2}\langle(jj)^9(jj)^7jjjjjjjjjjjj\rangle_i$. In the third column we have the overlap $\langle\psi_{E(9)}|\psi_{CCGI}\rangle$ between eigenstates of the E(9) interaction and eigenstates of the CCGI interaction. In the fourth column we use the interaction E(0,9). Adding the attractive $J = 0$ pairing interaction only affects states from $I = 0$ to $I = 8$.

**Table VI: Overlaps between the eigenstates of the E(9) and E(0,9) interactions with the more realistic CCGI interaction [10].** We also give the overlap between the eigenstates of E(9) and normalized $U9-j$ coefficients.

| $I$ | $\langle\psi_{E(9)}|\psi_{U9-j}\rangle$ | $\langle\psi_{E(9)}|\psi_{CCGI}\rangle$ | $\langle\psi_{E(0,9)}|\psi_{CCGI}\rangle$ |
|-----|----------------------------------|----------------------------------|----------------------------------|
| 0   | 1                                | 0.9451                           | 0.9944                           |
| 2   | 0.99996                          | 0.9829                           | 0.9904                           |
| 4   | 0.99995                          | 0.9144                           | 0.9773                           |
| 6   | 0.99950                          | 0.6795                           | 0.9361                           |
| 8   | 0.99777                          | 0.2375                           | 0.9858                           |
| 10  | 0.96377                          | 0.6830                           | 0.6830                           |
| 12  | 0.98125                          | 0.9944                           | 0.9944                           |
| 14  | 0.95007                          | 0.9967                           | 0.9967                           |
| 16  | 1                                | 1                                | 1                                |

We see that the overlaps between E(9) and $U9-j$ for $J = 0$ and 16 are 1. The latter is trivial because there is only one configuration for $I = 16$: $J_p = 8$, $J_n = 8$. The $I = 0$ result is explained by the fact that the Hamiltonian is separable in this case. The overlaps for $I = 2$ and 4 are very close to 1. In general, the overlaps are very high, so that the ansatz $|pn(9)pn(9)|^j$ is very good but not 100%. It appears that in the large-$j$ limit it would be exact.

In the next column we give the overlaps of the simple E(9) interaction with the more realistic CCGI interaction. We get overlaps bigger than 0.9 for $I = 0, 2, 4, 12, 14$, and 16. Again $I = 16$ is a trivial case. But why does the overlap that goes down suddenly rise up for $I = 12$ and 14? It must be because the interaction E(0), i.e. $J = 0$ pairing, does not come into play here. We see that for $I = 6, 8$, and 10 the simple ansatz does not yield a satisfactory wave function.

In the last column we see that the poor overlaps for $I = 6$ and 8 can be improved significantly by adding a $J = 0$ pairing term to the interaction. We now have $V(0) = V(9)$ (attractive). The overlap for $I = 6$ increases from 0.6795 to 0.9361 and for $I = 8$ from 0.2375 to 0.9858. The overlaps for $I = 0, 2$, and 4 are also improved. For $I = 10$ and higher, one cannot have any pair coupled to zero, so the E(0,9) interaction gives the same result as E(9).

**C. Degeneracies**

With the E(9) interaction, we get several degenerate states with an absolute energy zero. In some detail, for $I = 0^+$ there are five states, three with isospin $T = 0$ and two with $T = 2$. There is one non-degenerate state at an energy $2V(9)$ ($V(9)$ is negative). The other four $I = 0^+$ states have zero energy. For $I = 1^+$ all states have isospin $T = 1$. There is a single non-degenerate state at $V(9)$, the other three have zero energy. For $I = 2^+$ there are
twelve states—six have $T = 0$, four have $T = 1$, and two have $T = 2$. There are two non-degenerate $T = 0$ states with approximate energies $2V(9)$ and $V(9)$ respectively, and one non-degenerate $T = 1$ state with energy $V(9)$. The other nine states have zero energy. To understand this, take a wave function

$$|\Psi^\alpha\rangle = \sum_{J_p,J_n} C^\alpha(J_p,J_n)|pp(J_p)nn(J_n)\rangle$$

and the corresponding energies $E^\alpha = \langle\Psi^\alpha|H|\Psi^\alpha\rangle$. Consider the sum $\sum_\alpha E^\alpha$. We have

$$\sum_\alpha C^\alpha(J_p,J_n)C^\alpha(J_p',J_n') = \delta_{J_p,J_p'}\delta_{J_n,J_n'} \quad (15)$$

Thus

$$\sum_\alpha E^\alpha = \sum_{J_p,J_n} \langle pp(J_p)nn(J_n)\rangle^2 |H|pp(J_p)nn(J_n)\rangle^2 = 4V(9) \sum_{J_pJ_nJ_x} \sum_{J=0}^{\text{even}} |(jj)^4(jj)^4(jj)^0(jj)^4(jj)^4|^2 \quad (17)$$

This expression does not depend on the detailed wave functions. Referring to Eqs. (11) and (12) and neglecting the very small correction terms, we see that $N^{-2}$ is equal to 1/2 for $J_x = 9$ and to 1/4 for all other $J_x$. Basically then Eq. (17) becomes $4V(9) \sum N(J_x)^{-2}$. Hence we obtain $\sum E^\alpha = 2V(9)$ for $I = 0$, $V(9)$ for $I = 1$, and $4V(9)$ for $I = 2$. But we can alternately show, using the explicit wave functions, that for $I = 0$ the energy of the lowest state is $2V(9)$. Hence, all the other states must have zero energy. A similar story for $I = 1$. The $I = 2$ state is a bit more complicated because of the coupling between two states, however small it is. Still one can work it through and see that the $4V(9)$ energy is exhausted by the two $T = 0$ and the one $T = 1$ non-degenerate states.

For $I = 0$ we have two $T = 0$ and two $T = 2$ states, all degenerate. One can remove the degeneracies of $T = 0$ and $T = 2$ by adding to the Hamiltonian an interaction $\delta t(i) \cdot t(j)$. This will not affect the wave functions of the non-degenerate states but will shift the $T = 2$ states away from the formerly degenerate $T = 0$ states.

### III. CLOSING REMARKS

In closing, we note that the subject of $J_{\text{max}}$ pairing is currently a very active field. Besides the work of Qi et al. [6], there are related works by Zerguine and Van Isacker [14], Cederwall et al. [15] and Xu et al. [16]. The topic of $J$-pairing interactions has also been addressed by Zhao and Arima [17]. In this work we expand on our 2003 work [2] by noting that the Hamiltonian matrix for a $2p-2n$ system for $I = 0^+$ states in a single $j$-shell is separable for a simple interaction which is non-zero only for a single odd angular momentum. This leads to an eigenfunction with components proportional to a single set of unitary $9\cdot j$ symbols. We apply this to the $J = J_{\text{max}}$ interaction. The single set of $U9\cdot j$ components form the eigenfunction not only for the lowest $I = 0^+$ state, but also of the lowest $I = 1^+$ state and, to a surprisingly excellent approximation, for the lowest two $I = 2^+$ states. A single set of $U9\cdot j$ coefficients yields a good approximation for all yrast even-$I$ states. We also note that the non-degenerate states have good isospin and we show how to assign this quantum number. We have found a quantum number $J_z$ (see Eq. (8)) which can, either exactly or approximately, help classify some of the non-degenerate states. We have shown how to determine the number of degenerate states. We have compared our results of the $E(9)$ interaction with those of $E(0,9)$ in which $J = 0$ and $J = 2j$ matrix elements are equally attractive. Also comparison with a realistic interaction have been made. With $E(9)$ alone, only $I = 12, 14$, and (trivially) 16 have strong overlaps with the realistic interaction, but the situation is dramatically improved with $E(0,9)$. Lastly, we feel that just as there has been an intensive study of the $J = 0$ “pairing interaction” with many positive benefits, so should we as a counterpoint make an intensive study of the $E(J_{\text{max}})$ interaction as well as other schematic interactions. We already have the fascinating result of the closeness, shown in Table V, between $\psi_{E(9)}$ and $\psi_{U9j}$. Perhaps other interesting results will be found later. At the same time, we must also keep a reality check on both the $J = 0$ and $J = J_{\text{max}}$ “pairing interactions” and this we have done by displaying the results in the other columns of the same table.

**Acknowledgments**

We are indebted to Ben Bayman for many useful comments and insights.

[1] G. Racah, *L. Farkas Memorial Volume* (Research Council of Israel, Jerusalem, 1952).
[2] B.H. Flowers, Proc. Roy. Soc. (London) A 212, 248 (1952).
[3] G. Racah and I. Talmi, Phys. Rev. 89, 913 (1953).
[4] A. de-Shalit and I. Talmi, *Nuclear Shell Theory* (Academic Press, New York, 1963).
[5] I. Talmi, *Simple Models of Complex Nuclei* (Harwood Academic, Reading, U.K., 1993).
[6] C. Qi, J. Blomqvist, T. Bäck, B. Cederwall, A. Johnson,
[7] E. Moya de Guerra, A.A. Raduta, L. Zamick, and P. Sarriguren, Nuclear Physics A 727, 3 (2003).
[8] B.F. Bayman, J.D. McCullen, and L. Zamick, Phys. Rev. Lett. 11, 215 (1963).
[9] A. Escuderos, L. Zamick, and B. F. Bayman, “Wave functions in the $f_{7/2}$ shell, for educational purposes and ideas”, http://arxiv.org/abs/nucl-th/0506050 (2005).
[10] L. Coraggio, A. Covello, A. Gargano, and N. Itaco, Phys. Rev. C 85, 034335 (2012).
[11] L. Zamick and A. Escuderos, Nucl. Phys. A 889, 8 (2012).
[12] B.S. Nara Singh et al., Phys. Rev. Lett. 107, 172502 (2011).
[13] L. Yu and R.G. Littlejohn, Phys. Rev. A 83, 052114 (2011).
[14] S. Zerguine and P. Van Isacker, Phys. Rev. C 83, 064314 (2011).
[15] B. Cederwall et al., Nature (London) 469, 68 (2011).
[16] Z.X. Xu, C. Qi, J. Blomqvist, R.J. Liotta, and R. Wyss, Nucl. Phys. A 877, 51 (2012).
[17] Y.M. Zhao and A. Arima, Phys. Rev. C 72, 054307 (2005).