A NOTE ON EXISTENCE RESULTS FOR A NONLINEAR FOURTH-ORDER INTEGRAL BOUNDARY VALUE PROBLEM

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ABSTRACT. In this short note, we present some new existence results for a nonlinear fourth-order two-point boundary value problem with integral condition. The existence results are obtained by using the Leray-Schauder fixed point theorem. Our work improves the main results of Benaicha and Haddouchi [3]. In addition, examples are included to show the validity of our results.

1. INTRODUCTION

Fourth-order ordinary differential equations are models for bending or deformation of elastic beams, and therefore have important applications in engineering and physical sciences. Recently, the two-point and multi-point boundary value problems for fourth-order nonlinear differential equations have received much attention from many authors. Many authors have studied the beam equation under various boundary conditions and by different approaches. We refer the readers to the papers [1, 4, 5, 10, 13, 14, 20, 17, 19, 18, 21, 6, 2, 8, 16, 15, 3].

In 2016, Benaicha and Haddouchi [3], by applying the Krasnoselskii’s fixed point theorem in cones, established the existence of positive solutions for the following fourth-order two-point boundary value problem (BVP) with integral boundary condition:

\[ u''''(t) + f(u(t)) = 0, \quad t \in (0, 1), \quad (1.1) \]

\[ u'(0) = u'(1) = u''(0) = 0, \quad u(0) = \int_0^1 a(s)u(s)ds, \quad (1.2) \]

where

(H1) \( f \in C([0, \infty], [0, \infty]) \);
(H2) \( a \in C([0, 1], [0, \infty]) \) and \( 0 < \int_0^1 a(s)ds < 1 \).

To obtain the existence of at least one positive solutions for this problem, they assumed that the nonlinear term \( f \) is either superlinear or sublinear. That is, defining

\[ f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}, \]

then, \( f_0 = 0 \) and \( f_\infty = \infty \) correspond to the superlinear case, and \( f_0 = \infty \) and \( f_\infty = 0 \) correspond to the sublinear case.

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This note applies the Leray-Schauder fixed point theorem to eliminate half of the assumptions to prove the existence of a solution when using Krasnoselskii’s fixed point theorem of norm type with super and sub linear hypotheses.

Motivated by the work mentioned above, the aim of this note is to improve the results in [3] by showing that the BVP (1.1) and (1.2) has at least a positive solution if $\int_0^1 f(s)ds = 0$ (condition $\int_0^\infty f(s)ds = \infty$ being unnecessary), as well as, for $\int_0^\infty f(s)ds = 0$ (condition $\int_0^1 f(s)ds = \infty$ being also unnecessary).

For our analysis we use the Leray-Schauder’s fixed point theorem.

**Lemma 1.1** ([7]). (Leray-Schauder) Let $\Omega$ be a convex subset in a Banach space $X$, $0 \in \Omega$ and assume that $A : \Omega \to \Omega$ is a completely continuous operator. Then, either (i) $A$ has at least one fixed point in $\Omega$; or (ii) the set \{ $x \in \Omega \mid x = \lambda Ax$, $0 < \lambda < 1$ \} is unbounded.

**2. Some preliminary results**

In order to prove our main results, we need some preliminary results. Consider the following two-point boundary value problem

$$u^{\prime\prime\prime}(t) + y(t) = 0, \ t \in (0, 1), \quad (2.1)$$

$$u'(0) = u'(1) = u''(0) = 0, \ u(0) = \int_0^1 a(s)u(s)ds. \quad (2.2)$$

For problem (2.1), (2.2), we have the following conclusions which are derived from [3].

**Lemma 2.1.** ([3, Lemma 2.2]) The problem (2.1) - (2.2) has a unique solution

$$u(t) = \int_0^1 \left( G(t, s) + \frac{1}{1 - \alpha} \int_0^1 a(\tau)G(\tau, s)d\tau \right)y(s)ds,$$

where $G(t, s) : [0, 1] \times [0, 1] \to \mathbb{R}$ is the Green’s function defined by

$$G(t, s) = \frac{1}{6} \begin{cases} t^3(1-s)^2 - (t-s)^3, & 0 \leq s \leq t \leq 1; \\ t^3(1-s)^2, & 0 \leq t \leq s \leq 1, \end{cases} \quad (2.3)$$

and

$$\alpha = \int_0^1 a(t)dt.$$

**Lemma 2.2.** ([3, Lemma 2.3]) Let $\theta \in ]0, \frac{1}{2}[$ be fixed. Then

(i) $G(t, s) \geq 0$, for all $t, s \in [0, 1]$;

(ii) $\frac{1}{6}\theta^3s(1-s)^2 \leq G(t, s) \leq \frac{1}{6}s(1-s)^2$, for all $(t, s) \in [\theta, 1-\theta] \times [0, 1]$.

**Lemma 2.3.** ([3, Lemma 2.4]) Let $y(t) \in C([0, 1], [0, \infty))$ and $\theta \in ]0, \frac{1}{2}[$. The unique solution of (2.1) - (2.2) is nonnegative and satisfies

$$\min_{t \in [\theta, 1-\theta]} u(t) \geq \theta^3(1 - \alpha + \beta)\|u\|,$$

where $\beta = \int_0^{1-\theta} a(t)dt$, $\alpha = \int_0^1 a(t)dt$. 

3. Existence results

In this section, we will state and prove our main results.

Let \( \theta \in ]0, \frac{1}{2}[, X = C([0, 1], \mathbb{R}), \beta = \int_0^{\frac{1}{2}} a(t)dt, \alpha = \int_0^{1} a(t)dt, \) and define the cone

\[
K = \left\{ u \in X, u \geq 0 : \min_{t \in [\theta, 1-\theta]} u(t) \geq \theta^3 (1 - \alpha + \beta) \| u \| \right\}
\]

From lemmas (2.1), (2.2), and (2.3), the function \( u \) is a positive solution of the boundary value problem (1.1) and (1.2) if and only if \( u(t) \) is a fixed point of the operator

\[
Au(t) := \int_0^1 \left( G(t, s) + \frac{1}{1-\alpha} \int_0^1 a(\tau)G(\tau, s)d\tau \right) f(u(s))ds.
\] (3.1)

**Theorem 3.1.** Assume that \( f_0 = 0 \). Then BVP (1.1) and (1.2) has at least one positive solution.

**Proof.** Since \( f_0 = 0 \), there exists \( \rho_1 > 0 \) such that \( f(u) \leq \epsilon u \), for \( 0 < u \leq \rho_1 \), where \( \epsilon > 0 \) satisfies \( \epsilon \leq 1 - \alpha \).

If we denote

\[
\Omega = \left\{ u \in K, \| u \| \leq \rho_1 \right\},
\]

Then \( \Omega \) is a convex subset of \( X \).

For \( u \in \Omega \), according to the proofs of lemmas 2.2 and 2.3, we have

\[
Au(t) = \int_0^1 \left( G(t, s) + \frac{1}{1-\alpha} \int_0^1 a(\tau)G(\tau, s)d\tau \right) f(u(s))ds \\
\leq \int_0^1 \left( g(s) + \frac{1}{1-\alpha} \int_0^1 g(s)a(\tau)d\tau \right) f(u(s))ds \\
= \frac{1}{1-\alpha} \int_0^1 g(s)f(u(s))ds, \quad t \in [0, 1],
\] (3.2)

where \( g(s) = \frac{1}{8}s(1-s)^2 \).

So,

\[
\| Au \| \leq \frac{1}{1-\alpha} \int_0^1 g(s)f(u(s))ds.
\] (3.3)

In view of lemma 2.3 and 3.3, we have \( Au(t) \geq 0 \) and

\[
Au(t) = \int_0^1 \left( G(t, s) + \frac{1}{1-\alpha} \int_0^1 a(\tau)G(\tau, s)d\tau \right) f(u(s))ds \\
\geq \theta^3 \int_0^1 \left( g(s) + \frac{1}{1-\alpha} \int_0^{1-\theta} g(s)a(\tau)d\tau \right) f(u(s))ds \\
= \theta^3 \frac{1-\alpha + \beta}{1-\alpha} \int_0^1 g(s)f(u(s))ds \\
\geq \theta^3 (1 - \alpha + \beta) \| Au \|, \quad t \in [\theta, 1-\theta],
\] (3.4)

Hence,

\[
\min_{t \in [\theta, 1-\theta]} Au(t) \geq \theta^3 (1 - \alpha + \beta) \| Au \|.
\]
On the other hand,
\[
Au(t) \leq \frac{1}{1-\alpha} \int_0^1 g(s)f(u(s))ds \\
\leq \frac{1}{1-\alpha} L \int_0^1 g(s)ds \\
\leq \frac{L}{6(1-\alpha)}.
\] (3.5)

Thus, \(\|Au\| \leq \rho_1\). Hence \(A\Omega \subset \Omega\). \(A : \Omega \to \Omega\) is completely continuous by an application of Arzela-Ascoli theorem.

For \(u \in \mathcal{V}\) with
\[
\mathcal{V} = \left\{ u \in \Omega : u = \lambda Au, \ 0 < \lambda < 1 \right\},
\]
we have
\[
u(t) = \lambda Au(t) < Au(t) \leq \rho_1,
\]
which implies that \(\|u\| \leq \rho_1\).

So, \(\mathcal{V}\) is bounded. By Lemma 1.1, the operator \(A\) has at least one fixed point in \(\Omega\), which is a positive solution of (1.1) and (1.2).

**Theorem 3.2.** If \(f_\infty = 0\), then BVP (1.1) and (1.2) has at least one positive solution.

**Proof.** We discuss two possible cases:

Case 1. If \(f\) is bounded. Then, there exists \(L > 0\) such that \(f(u) \leq L\).

For \(u \in K\), we have \(Au \in K\), and \(A\) is completely continuous. Similar to the estimates of (3.2), we obtain
\[
Au(t) \leq \frac{1}{1-\alpha} \int_0^1 g(s)f(u(s))ds \\
\leq \frac{L}{1-\alpha} \int_0^1 g(s)ds \\
\leq \frac{L}{6(1-\alpha)}.
\] (3.6)

Thus, \(\|Au\| \leq \frac{L}{6(1-\alpha)}\). For \(u \in \mathcal{V}\) with
\[
\mathcal{V} = \left\{ u \in K : u = \lambda Au, \ 0 < \lambda < 1 \right\},
\]
we have
\[
u(t) = \lambda Au(t) < Au(t) \leq \frac{L}{6(1-\alpha)},
\]
which implies that \(\|u\| \leq \frac{L}{6(1-\alpha)}\).

So, \(\mathcal{V}\) is bounded. By Lemma 1.1, the operator \(A\) has at least one fixed point in \(K\), which is a positive solution of (1.1) and (1.2).

Case 2. Suppose that \(f\) is unbounded, since \(f_\infty = 0\), there exists \(\rho_2 > 0\) such that \(f(u) \leq \eta u\) for \(u > \rho_2\), where \(\eta > 0\) satisfies
\[
\eta \leq 1 - \alpha.
\]
On the other hand, from condition (H1), there is \( \sigma > 0 \) such that \( f(u) \leq \eta \sigma \), with \( 0 \leq u \leq \rho_2 \).

Now, set
\[
\Omega = \left\{ u \in K, \|u\| \leq \hat{\rho}_2 \right\},
\]
where \( \hat{\rho}_2 = \max\{\sigma, \rho_2\} \).

If \( u \in \Omega \), then we have \( f(u) \leq \eta \hat{\rho}_2 \). Similar to (3.2), we have
\[
Au(t) \leq \frac{1}{1 - \alpha} \int_0^1 g(s)f(u(s))ds
\leq \frac{1}{6} \frac{\eta \hat{\rho}_2}{1 - \alpha}
\leq \hat{\rho}_2.
\]

Thus, \( \|Au\| \leq \hat{\rho}_2 \).

It is easy to check that \( V = \left\{ u \in \Omega / u = \lambda Au, \ 0 < \lambda < 1 \right\} \) is bounded. Therefore, by Lemma [1.1] the boundary value problem (1.1) and (1.2) has at least one positive solution.

\section*{4. Examples}

\textbf{Example 4.1.} Consider the fourth-order boundary value
\[
u''''(t) + u(1 - e^{-u}) = 0, \quad 0 < t < 1,
\]
\[
u'(0) = u'(1) = u''(0) = 0, \quad u(0) = \int_0^1 s^2 u(s)ds,
\]
where \( f(u) = u(1 - e^{-u}) \in C([0, \infty), [0, \infty)) \) and \( a(t) = t^2 \geq 0, \int_0^1 a(s)ds = \int_0^1 s^2 ds = \frac{1}{3} \).

We have
\[
f_0 = \lim_{u \to 0^+} \frac{f(u)}{u} = \lim_{u \to 0^+} (1 - e^{-u}) = 0.
\]

Thus, it follows from Theorem [3.1] that the problem (4.1) and (4.2) has at least one positive solution. Notice that \( f_\infty = 1, f_\infty \neq \infty \), so Theorem 3.1 in [3] cannot be applied to show the existence of positive solutions for the problem (4.1) and (4.2).

\textbf{Example 4.2.} Consider the fourth-order boundary value
\[
u''''(t) + 1 - e^{-u} = 0, \quad 0 < t < 1,
\]
\[
u'(0) = u'(1) = u''(0) = 0, \quad u(0) = \int_0^1 s^2 u(s)ds,
\]
where \( f(u) = 1 - e^{-u} \in C([0, \infty), [0, \infty)) \) and \( a(t) = t^2 \geq 0, \int_0^1 a(s)ds = \int_0^1 s^2 ds = \frac{1}{3} \).

Since
\[
f_\infty = \lim_{u \to +\infty} \frac{f(u)}{u} = \lim_{u \to +\infty} \frac{1 - e^{-u}}{u} = 0,
\]
From Theorem [3.2] the problem (4.3) and (4.4) has at least one positive solution. On the other hand, we have
\[
f_0 = \lim_{u \to +0} \frac{f(u)}{u} = \lim_{u \to +0} \frac{1 - e^{-u}}{u} = 1.
\]
therefore Theorem 3.2 in [3] also cannot be applied to show the existence of positive solutions for the problem (4.3) and (4.4).

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