Probit transformation for nonparametric kernel estimation of the copula

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Motivation

Consider some $n$-i.i.d. sample $\{(X_i, Y_i)\}$ with cumulative distribution function $F_{XY}$ and joint density $f_{XY}$. Let $F_X$ and $F_Y$ denote the marginal distributions, and $C$ the copula,

$$F_{XY}(x, y) = C(F_X(x), F_Y(y))$$

so that

$$f_{XY}(x, y) = f_X(x)f_Y(y)c(F_X(x), F_Y(y))$$

We want a nonparametric estimate of $c$ on $[0, 1]^2$. 
Notations

Define uniformized $n$-i.i.d. sample $\{(U_i, V_i)\}$

$$U_i = F_X(X_i) \text{ and } V_i = F_Y(Y_i)$$

or uniformized $n$-i.i.d. pseudo-sample $\{ (\hat{U}_i, \hat{V}_i) \}$

$$\hat{U}_i = \frac{n}{n+1} \hat{F}_{Xn}(X_i) \text{ and } \hat{V}_i = \frac{n}{n+1} \hat{F}_{Yn}(Y_i)$$

where $\hat{F}_{Xn}$ and $\hat{F}_{Yn}$ denote empirical c.d.f.
Standard Kernel Estimate

The standard kernel estimator for $c$, say $\hat{c}^*$, at $(u, v) \in \mathcal{I}$ would be (see Wand & Jones (1995))

$$\hat{c}^*(u, v) = \frac{1}{n|H_{UV}|^{1/2}} \sum_{i=1}^{n} K \left( H_{UV}^{-1/2} \left( \begin{bmatrix} u - U_i \\ v - V_i \end{bmatrix} \right) \right),$$

(1)

where $K : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a kernel function and $H_{UV}$ is a bandwidth matrix.
**Standard Kernel Estimate**

However, this estimator is not consistent along boundaries of $[0,1]^2$

$$
\mathbb{E}(\hat{c}^*(u,v)) = \frac{1}{4} c(u,v) + O(h) \text{ at corners}
$$

$$
\mathbb{E}(\hat{c}^*(u,v)) = \frac{1}{2} c(u,v) + O(h) \text{ on the borders}
$$

if $K$ is symmetric and $H_{UV}$ symmetric.

Corrections have been proposed, e.g. mirror reflection Gijbels (1990) or the usage of boundary kernels Chen (2007), but with mixed results.

**Remark**: the graph on the bottom is $\hat{c}^*$ on the (first) diagonal.
Mirror Kernel Estimate

Use an enlarged sample: instead of only \(((\hat{U}_i, \hat{V}_i))\), add \((-\hat{U}_i, \hat{V}_i)\), \((\hat{U}_i, -\hat{V}_i)\), \((-\hat{U}_i, -\hat{V}_i)\), \((\hat{U}_i, 2 - \hat{V}_i)\), \((2 - \hat{U}_i, \hat{V}_i)\), \((-\hat{U}_i, 2 - \hat{V}_i)\), \((2 - \hat{U}_i, -\hat{V}_i)\) and \((2 - \hat{U}_i, 2 - \hat{V}_i)\).

See Gijbels & Mieleniczuk (1990).

That estimator will be used as a benchmark in the simulation study.
Using Beta Kernels

Use a Kernel which is a product of beta kernels

\[ K_{x_i}(u) \propto \left( \frac{x_{1,i}}{b} [1 - u_1] \frac{x_{1,i}}{b} \right) \cdot \left( \frac{x_{2,i}}{b} [1 - u_2] \frac{x_{2,i}}{b} \right) \]

See Chen (1999).
Probit Transformation

See Devroye & Gyöfi (1985) and Marron & Ruppert (1994).

Define normalized $n$-i.i.d. sample $\{(S_i, T_i)\}$

\[ S_i = \Phi^{-1}(U_i) \quad \text{and} \quad T_i = \Phi^{-1}(V_i) \]

or normalized $n$-i.i.d. pseudo-sample $\{\hat{S}_i, \hat{T}_i\}$

\[ \hat{U}_i = \Phi^{-1}(\hat{U}_i) \quad \text{and} \quad \hat{V}_i = \Phi^{-1}(\hat{V}_i) \]

where $\Phi^{-1}$ is the quantile function of $\mathcal{N}(0, 1)$ (probit transformation).
**Probit Transformation**

\[ F_{ST}(x, y) = C(\Phi(x), \Phi(y)) \]

so that

\[ f_{ST}(x, y) = \phi(x)\phi(y)c(\Phi(x), \Phi(y)) \]

Thus

\[ c(u, v) = \frac{f_{ST}(\Phi^{-1}(u), \Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}. \]

So use

\[ \hat{c}^{(\tau)}(u, v) = \frac{\hat{f}_{ST}(\Phi^{-1}(u), \Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}. \]
The naive estimator
Since we cannot use
\[ \hat{f}^{*}_{ST}(s, t) = \frac{1}{n|H_{ST}|^{1/2}} \sum_{i=1}^{n} K \left( H_{ST}^{-1/2} \left( s - S_i \right), t - T_i \right), \]
where \( K \) is a kernel function and \( H_{ST} \) is a bandwidth matrix, use
\[ \hat{f}_{ST}(s, t) = \frac{1}{n|H_{ST}|^{1/2}} \sum_{i=1}^{n} K \left( H_{ST}^{-1/2} \left( s - \hat{S}_i \right), t - \hat{T}_i \right). \]
and the copula density is
\[ \hat{c}(\tau)(u, v) = \frac{\hat{f}_{ST}(\Phi^{-1}(u), \Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}. \]
The naive estimator

\[ \hat{c}(\tau)(u, v) = \frac{1}{n|H_{ST}|^{1/2}\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))} \sum_{i=1}^{n} K \left( H_{ST}^{-1/2} \left( \Phi^{-1}(u) - \Phi^{-1}(\hat{U}_i) \right) \Phi^{-1}(v) - \Phi^{-1}(\hat{V}_i) \right) \]

as suggested in C., Fermanian & Scaillet (2007) and Lopez-Paz . et al. (2013).

Note that Omelka . et al. (2009) obtained theoretical properties on the convergence of \( \hat{C}(\tau)(u, v) \) (not \( c \)).
Improved probit-transformation copula density estimators

When estimating a density from pseudo-sample, Loader (1996) and Hjort & Jones (1996) define a local likelihood estimator

Around \((s, t) \in \mathbb{R}^2\), use a polynomial approximation of order \(p\) for \(\log f_{ST}\)

\[
\log f_{ST}(\tilde{s}, \tilde{t}) \simeq a_{1,0}(s, t) + a_{1,1}(s, t)(\tilde{s} - s) + a_{1,2}(s, t)(\tilde{t} - t) \doteq P_{a_1}(\tilde{s} - s, \tilde{t} - t)
\]

\[
\log f_{ST}(\tilde{s}, \tilde{t}) \simeq a_{2,0}(s, t) + a_{2,1}(s, t)(\tilde{s} - s) + a_{2,2}(s, t)(\tilde{t} - t)
\]

\[
+ a_{2,3}(s, t)(\tilde{s} - s)^2 + a_{2,4}(s, t)(\tilde{t} - t)^2 + a_{2,5}(s, t)(\tilde{s} - s)(\tilde{t} - t)
\]

\[
\doteq P_{a_2}(\tilde{s} - s, \tilde{t} - t).
\]
Improved probit-transformation copula density estimators

**Remark** Vectors $\mathbf{a}_1(s, t) = (a_{1,0}(s, t), a_{1,1}(s, t), a_{1,2}(s, t))$ and $\mathbf{a}_2(s, t) = (a_{2,0}(s, t), \ldots, a_{2,5}(s, t))$ are then estimated by solving a weighted maximum likelihood problem.

$$\hat{\mathbf{a}}_p(s, t) = \arg \max_{\mathbf{a}_p} \left\{ \sum_{i=1}^{n} K \left( H_{ST}^{-1/2} \begin{pmatrix} s - \hat{S}_i \\ t - \hat{T}_i \end{pmatrix} \right) P_{\mathbf{a}_p}(\hat{S}_i - s, \hat{T}_i - t) \right\}$$

$$-n \iint_{\mathbb{R}^2} K \left( H_{ST}^{-1/2} \begin{pmatrix} s - \tilde{s} \\ t - \tilde{t} \end{pmatrix} \right) \exp \left( P_{\mathbf{a}_p}(\tilde{s} - s, \tilde{t} - t) \right) d\tilde{s} d\tilde{t},$$

The estimate of $f_{ST}$ at $(s, t)$ is then $\tilde{f}_{ST}^{(p)}(s, t) = \exp(\hat{a}_{p,0}(s, t))$, for $p = 1, 2$.

The Improved probit-transformation kernel copula density estimators are

$$\tilde{c}^{(\tau,p)}(u, v) = \frac{\tilde{f}_{ST}^{(p)}(\Phi^{-1}(u), \Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$$
Improved probit-transformation copula density estimators

For the local log-linear \((p = 1)\) approximation

\[
\tilde{c}(\tau,1)(u, v) = \frac{\exp(\tilde{a}_{1,0}(\Phi^{-1}(u), \Phi^{-1}(v)))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}
\]
Improved probit-transformation copula density estimators

For the local log-quadratic \((p = 2)\) approximation

\[
\tilde{c}(\tau,2)(u,v) = \frac{\exp(\tilde{a}_{2,0}(\Phi^{-1}(u), \Phi^{-1}(v)))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}
\]
Asymptotic properties

**A1.** The sample \( \{(X_i, Y_i)\} \) is a \( n \)- i.i.d. sample from the joint distribution \( F_{XY} \), an absolutely continuous distribution with marginals \( F_X \) and \( F_Y \) strictly increasing on their support;

(uniqueness of the copula)
**Asymptotic properties**

A2. The copula $C$ of $F_{XY}$ is such that $(\partial C/\partial u)(u, v)$ and $(\partial^2 C/\partial u^2)(u, v)$ exist and are continuous on $\{(u, v) : u \in (0, 1), v \in [0, 1]\}$, and $(\partial C/\partial v)(u, v)$ and $(\partial^2 C/\partial v^2)(u, v)$ exist and are continuous on $\{(u, v) : u \in [0, 1], v \in (0, 1)\}$. In addition, there are constants $K_1$ and $K_2$ such that

$$\left| \frac{\partial^2 C}{\partial u^2}(u, v) \right| \leq \frac{K_1}{u(1-u)} \quad \text{for } (u, v) \in (0, 1) \times [0, 1];$$

$$\left| \frac{\partial^2 C}{\partial v^2}(u, v) \right| \leq \frac{K_2}{v(1-v)} \quad \text{for } (u, v) \in [0, 1] \times (0, 1);$$

A3. The density $c$ of $C$ exists, is positive and admits continuous second-order partial derivatives on the interior of the unit square $\mathcal{I}$. In addition, there is a constant $K_{00}$ such that

$$c(u, v) \leq K_{00} \min \left( \frac{1}{u(1-u)}, \frac{1}{v(1-v)} \right) \quad \forall (u, v) \in (0, 1)^2.$$

see Segers (2012).
Asymptotic properties

Assume that $K(z_1, z_2) = \phi(z_1)\phi(z_2)$ and $H_{ST} = h^2 I$ with $h \sim n^{-a}$ for some $a \in (0, 1/4)$. Under Assumptions A1-A3, the ‘naive’ probit transformation kernel copula density estimator at any $(u, v) \in (0, 1)^2$ is such that

$$
\sqrt{nh^2} \left( \hat{c}^{(\tau)}(u, v) - c(u, v) - h^2 \frac{b(u, v)}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))} \right) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \sigma^2(u, v)),
$$

where $b(u, v) = \frac{1}{2} \left\{ \frac{\partial^2 c}{\partial u^2}(u, v)\phi^2(\Phi^{-1}(u)) + \frac{\partial^2 c}{\partial v^2}(u, v)\phi^2(\Phi^{-1}(v)) - 3 \left( \frac{\partial c}{\partial u}(u, v)\Phi^{-1}(u)\phi(\Phi^{-1}(u)) + \frac{\partial c}{\partial v}(u, v)\Phi^{-1}(v)\phi(\Phi^{-1}(v)) \right) + c(u, v) \left( \{\Phi^{-1}(u)\}^2 + \{\Phi^{-1}(v)\}^2 - 2 \right) \right\}$ \hspace{1cm} (2)

and $\sigma^2(u, v) = \frac{c(u, v)}{4\pi \phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$. 

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The Amended version

The last unbounded term in $b$ be easily adjusted.

$$
\hat{c}^{(\tau_{am})}(u,v) = \frac{\hat{f}_{ST}(\Phi^{-1}(u), \Phi^{-1}(v))}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))} \times \frac{1}{1 + \frac{1}{2} h^2 \left(\{\Phi^{-1}(u)\}^2 + \{\Phi^{-1}(v)\}^2 - 2\right)}.
$$

The asymptotic bias becomes proportional to

$$
b^{(am)}(u,v) = \frac{1}{2} \left\{ \frac{\partial^2 c}{\partial u^2}(u,v)\phi^2(\Phi^{-1}(u)) + \frac{\partial^2 c}{\partial v^2}(u,v)\phi^2(\Phi^{-1}(v))
- 3 \left( \frac{\partial c}{\partial u}(u,v)\Phi^{-1}(u)\phi(\Phi^{-1}(u)) + \frac{\partial c}{\partial v}(u,v)\Phi^{-1}(v)\phi(\Phi^{-1}(v)) \right) \right\}.
$$
A local log-linear probit-transformation kernel estimator

\[
\tilde{c}^*(\tau, 1)(u, v) = \tilde{f}_{ST}^{(1)}(\Phi^{-1}(u), \Phi^{-1}(v)) \bigg/ \left( \phi(\Phi^{-1}(u)) \phi(\Phi^{-1}(v)) \right)
\]

Then

\[
\sqrt{nh^2} \left( \tilde{c}^*(\tau, 1)(u, v) - c(u, v) - h^2 \frac{b^{(1)}(u, v)}{\phi(\Phi^{-1}(u)) \phi(\Phi^{-1}(v))} \right) \overset{\mathcal{L}}{\longrightarrow} \mathcal{N} \left( 0, \sigma^{(1)}^2(u, v) \right),
\]

where

\[
b^{(1)}(u, v) = \frac{1}{2} \left\{ \frac{\partial^2 c}{\partial u^2}(u, v) \phi^2(\Phi^{-1}(u)) + \frac{\partial^2 c}{\partial v^2}(u, v) \phi^2(\Phi^{-1}(v)) \right. \\
- \frac{1}{c(u, v)} \left( \left\{ \frac{\partial c}{\partial u}(u, v) \right\}^2 \phi^2(\Phi^{-1}(u)) + \left\{ \frac{\partial c}{\partial v}(u, v) \right\}^2 \phi^2(\Phi^{-1}(v)) \right) \\
- \left( \frac{\partial c}{\partial u}(u, v) \Phi^{-1}(u) \phi(\Phi^{-1}(u)) + \frac{\partial c}{\partial v}(u, v) \Phi^{-1}(v) \phi(\Phi^{-1}(v)) \right) - 2c(u, v) \left\} \right.
\]
Using a higher order polynomial approximation

Locally fitting a polynomial of a higher degree is known to reduce the asymptotic bias of the estimator, here from order $O(h^2)$ to order $O(h^4)$, see Loader (1996) or Hjort (1996), under sufficient smoothness conditions.

If $f_{ST}$ admits continuous fourth-order partial derivatives and is positive at $(s,t)$, then

$$\sqrt{nh^2} \left( \tilde{f}_{ST}^{(2)}(s,t) - f_{ST}(s,t) - h^4 b_{ST}^{(2)}(s,t) \right) \xrightarrow{\mathcal{L}} \mathcal{N} \left( 0, \sigma_{ST}^{(2)}(s,t)^2 \right),$$

where $\sigma_{ST}^{(2)}(s,t)^2 = \frac{5}{2} \frac{f_{ST}(s,t)}{4\pi}$ and

$$b_{ST}^{(2)}(s,t) = -\frac{1}{8} f_{ST}(s,t) \times \left\{ \left( \frac{\partial^4 g}{\partial s^4} + \frac{\partial^4 g}{\partial t^4} \right) + 4 \left( \frac{\partial^3 g}{\partial s^3} \frac{\partial g}{\partial s} + \frac{\partial^3 g}{\partial t^3} \frac{\partial g}{\partial t} + \frac{\partial^3 g}{\partial s^2 \partial t} \frac{\partial g}{\partial t} + \frac{\partial^3 g}{\partial s \partial t^2} \frac{\partial g}{\partial s} \right) + 2 \frac{\partial^4 g}{\partial s^2 \partial t^2} \right\} (s,t),$$

with $g(s,t) = \log f_{ST}(s,t)$. 
Using a higher order polynomial approximation

**A4.** The copula density $c(u, v) = (\partial^2 C/\partial u \partial v)(u, v)$ admits continuous fourth-order partial derivatives on the interior of the unit square $[0, 1]^2$.

Then

$$\sqrt{n}h^2 \left( \tilde{c}^{(2)}(\tau, 2)(u, v) - c(u, v) - h^4 \frac{b^{(2)}(u, v)}{\phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))} \right) \xrightarrow{L} N \left( 0, \sigma^{(2)}^2(u, v) \right)$$

where $\sigma^{(2)}^2(u, v) = \frac{5}{2} \frac{c(u, v)}{4\pi \phi(\Phi^{-1}(u))\phi(\Phi^{-1}(v))}$
Improving Bandwidth choice

Consider the principal components decomposition of the \((n \times 2)\) matrix \([\hat{S}, \hat{T}] = \mathbf{M}\).

Let \(W_1 = (W_{11}, W_{12})^T\) and \(W_2 = (W_{21}, W_{22})^T\) be the eigenvectors of \(\mathbf{M}^T \mathbf{M}\). Set

\[
\begin{pmatrix}
Q \\
R
\end{pmatrix} = \begin{pmatrix}
W_{11} & W_{12} \\
W_{21} & W_{22}
\end{pmatrix}
\begin{pmatrix}
S \\
T
\end{pmatrix} = \mathbf{W}\begin{pmatrix}
S \\
T
\end{pmatrix}
\]

which is only a linear reparametrization of \(\mathbb{R}^2\), so an estimate of \(f_{ST}\) can be readily obtained from an estimate of the density of \((Q, R)\).

Since \(\{\hat{Q}_i\}\) and \(\{\hat{R}_i\}\) are empirically uncorrelated, consider a diagonal bandwidth matrix \(\mathbf{H}_{QR} = \text{diag}(h_Q^2, h_R^2)\).
Improving Bandwidth choice

Use univariate procedures to select $h_Q$ and $h_R$ independently

Denote $\tilde{f}_Q^{(p)}$ and $\tilde{f}_R^{(p)}$ ($p = 1, 2$), the local log-polynomial estimators for the densities

$h_Q$ can be selected via cross-validation (see Section 5.3.3 in Loader (1999))

$$h_Q = \arg \min_{h>0} \left\{ \int_{-\infty}^{\infty} \left\{ \tilde{f}_Q^{(p)}(q) \right\}^2 dq - \frac{2}{n} \sum_{i=1}^{n} \tilde{f}_Q^{(p)}(\hat{Q}_i) \right\},$$

where $\tilde{f}_Q^{(p)}(\hat{Q}_i)$ is the ‘leave-one-out’ version of $\tilde{f}_Q^{(p)}$. 
Graphical Comparison (loss ALAE dataset)
Simulation Study

$M = 1,000$ independent random samples $\{(U_i, V_i)\}_{i=1}^n$ of sizes $n = 200$, $n = 500$ and $n = 1000$ were generated from each of the following copulas:

- the independence copula (i.e., $U_i$’s and $V_i$’s drawn independently);
- the Gaussian copula, with parameters $\rho = 0.31$, $\rho = 0.59$ and $\rho = 0.81$;
- the Student $t$-copula with 4 degrees of freedom, with parameters $\rho = 0.31$, $\rho = 0.59$ and $\rho = 0.81$;
- the Frank copula, with parameter $\theta = 1.86$, $\theta = 4.16$ and $\theta = 7.93$;
- the Gumbel copula, with parameter $\theta = 1.25$, $\theta = 1.67$ and $\theta = 2.5$;
- the Clayton copula, with parameter $\theta = 0.5$, $\theta = 1.67$ and $\theta = 2.5$.

(approximated) MISE relative to the MISE of the mirror-reflection estimator (last column), $n = 1000$. Bold values show the minimum MISE for the corresponding copula (non-significantly different values are highlighted as well).
|        | \( \hat{c}(\tau) \) | \( \hat{c}(\tau_{\text{am}}) \) | \( \hat{c}(\tau_1) \) | \( \hat{c}(\tau_2) \) | \( \hat{c}_1^{(\beta)} \) | \( \hat{c}_2^{(\beta)} \) | \( \hat{c}_1^{(B)} \) | \( \hat{c}_2^{(B)} \) | \( \hat{c}_1^{(p)} \) | \( \hat{c}_2^{(p)} \) | \( \hat{c}_3^{(p)} \) |
|--------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| Indep  | 3.57           | 2.80           | 2.89           | 1.40           | 7.96           | 11.65          | 1.69           | 3.43           | 1.62           | 0.50           | 0.14           |
| Gauss2 | 2.03           | 1.52           | 1.60           | 0.76           | 4.63           | 6.06           | 1.10           | 1.82           | 0.98           | 0.66           | 0.89           |
| Gauss4 | 0.63           | 0.49           | 0.44           | 0.21           | 1.72           | 1.60           | 0.75           | 0.58           | 0.62           | 0.99           | 2.93           |
| Gauss6 | 0.21           | 0.20           | 0.11           | 0.05           | 0.74           | 0.33           | 0.77           | 0.37           | 0.72           | 1.21           | 2.83           |
| Std(4)2| 0.61           | 0.56           | 0.50           | 0.40           | 1.57           | 1.80           | 0.78           | 0.67           | 0.75           | 1.01           | 1.88           |
| Std(4)4| 0.21           | 0.27           | 0.17           | 0.15           | 0.88           | 0.51           | 0.75           | 0.42           | 0.75           | 1.12           | 2.07           |
| Std(4)6| 0.09           | 0.17           | 0.08           | 0.09           | 0.70           | 0.19           | 0.82           | 0.47           | 0.90           | 1.17           | 1.90           |
| Frank2 | 3.31           | 2.42           | 2.57           | 1.35           | 7.16           | 9.63           | 1.70           | 2.95           | 1.31           | 0.45           | 0.49           |
| Frank4 | 2.35           | 1.45           | 1.51           | 0.99           | 4.42           | 4.89           | 1.49           | 1.65           | 0.60           | 0.72           | 6.14           |
| Frank6 | 0.96           | 0.52           | 0.45           | 0.44           | 1.51           | 1.19           | 1.35           | 0.76           | 0.65           | 1.58           | 7.25           |
| Gumbel2| 0.65           | 0.62           | 0.56           | 0.43           | 1.77           | 1.97           | 0.82           | 0.75           | 0.83           | 1.03           | 1.52           |
| Gumbel4| 0.18           | 0.28           | 0.16           | 0.19           | 0.89           | 0.41           | 0.78           | 0.47           | 0.81           | 1.10           | 1.78           |
| Gumbel6| 0.09           | 0.21           | 0.10           | 0.15           | 0.78           | 0.29           | 0.85           | 0.58           | 0.94           | 1.12           | 1.63           |
| Clayton2| 0.63           | 0.60           | 0.51           | 0.34           | 1.78           | 1.99           | 0.78           | 0.70           | 0.79           | 1.04           | 1.79           |
| Clayton4| 0.11           | 0.26           | 0.10           | 0.15           | 0.79           | 0.27           | 0.83           | 0.56           | 0.90           | 1.10           | 1.50           |
| Clayton6| 0.11           | 0.28           | 0.08           | 0.15           | 0.82           | 0.35           | 0.88           | 0.67           | 0.96           | 1.09           | 1.36           |