Graphene is a genuine two dimensional material composed of the carbon atoms that form a honeycomb lattice. Three of the valence electrons of each carbon atom participate in the interatomic interaction, while the fourth one contributes to the conductivity of the crystal. Graphene has been studied theoretically for a long time, see e.g., [1], [2]. However, its experimental observation [3] triggered a real boom of both theoretical and experimental analysis [4], [5], [6].

The material manifests extraordinary electronic properties, which are the consequence of an unusual dynamics of the low-energy charge carriers. It was pointed out in [2] that the tight-binding description of the system is reduced to the massless Dirac equation in the low-energy approximation. This makes graphene an ideal test field for (2 + 1)-dimensional QED [6]; due to the low Fermi velocity $v_F$, $c/v_F \sim 300$, it is possible to simulate relativistic effects in condensed matter systems which would be unreachable experimentally otherwise.

It was predicted [2], [7] that the scattering of the relativistic electrons on the potential barrier is qualitatively different from the nonrelativistic case. The particles can tunnel the barrier without reflection, provided that its height tends to infinity. This is in contrast to the nonrelativistic regime where the tunneling would be exponentially suppressed [8]. This phenomenon, known as Klein tunneling, is not experimentally realizable with elementary particles nowadays due to the extreme electric field needed to observe the predicted difference between relativistic and nonrelativistic scattering [9].

The scattering of the low-energy quasiparticles in graphene on the barrier with translational symmetry in one dimension was analyzed in [9], [10], [11]. The absence of backscattering was noticed for normal incidence. The effect is independent of the height of the barrier and, hence, is testable experimentally [12]. A similar phenomenon was observed earlier [13] and discussed theoretically [14], [15], [16] in the context of electron transport in carbon nanotubes. The perfect transmission of the low-energy charge carriers occurs in metallic nanotubes despite the presence of a scattering potential generated by impurities. The absence of backscattering was understood as a consequence of topological singularity identified with a Dirac point, see [17], [18], or as a result of the pseudospin conservation [8].

We provide here a simple, alternative explanation for the absence of backscattering in the carbon nanostructures within the framework of supersymmetric quantum mechanics. We shall discuss a broad class of potentials in graphene as well as in the metallic nanotubes with the range exceeding the interatomic distance.

The honeycomb lattice is a superposition of two triangular sublattices, $A$ and $B$. The eigenstate $\Psi$ of the Hamiltonian can be then written as $\Psi = c_A \Psi_A + c_B \Psi_B$, where $\Psi_A$ and $\Psi_B$ are atomic wave functions of the sublattices whereas $c_A$ and $c_B$ are slowly varying amplitudes. $\Psi$ is a Bloch function, which acquires a nontrivial phase factor when shifted by a translation vector $\mathbf{R}$ of the Bravais lattice, $\Psi(\mathbf{k}, \mathbf{x} + \mathbf{R}) = e^{i \mathbf{k} \cdot \mathbf{R}} \Psi(\mathbf{k}, \mathbf{x})$. Fermi surface of graphene is formed by discrete points. There are six of them in the first Brillouin zone, situated in its corners, see Fig. [1]. In the analysis of the low-energy behavior of the charge carriers, it is sufficient to consider just two of them, denoted as Dirac points $\mathbf{K}$ and $\mathbf{K}' = -\mathbf{K}$. The remaining four Dirac points do not represent distinct electronic states. They can be obtained either from $\mathbf{K}$ or $\mathbf{K}'$ by translations in the reciprocal lattice.

In the vicinity of Dirac points, the behavior of the system is described by the massless Dirac equation. When the effective Hamiltonian is considered in the valley of the point $\mathbf{K}$ and expanded up to the terms linear in the momenta, the energy eigenvalue equation acquires the form (we put $\hbar = 1$)

$$H \psi = -i v_F (\sigma_1 \partial_x + \sigma_2 \partial_y) \psi = E \psi,$$

where $\sigma_{1,2}$ are Pauli matrices. Spinor $\psi$ reads explicitly $\psi = (c_A, c_B) e^{i \delta \mathbf{k} \cdot \mathbf{x}}$, where $\delta \mathbf{k} = \mathbf{k} - \mathbf{K}$ and $t$ is a transposition. Degree of freedom associated with the components $c_A$ and $c_B$ is called pseudospin, see [4], [5].

A single wall carbon nanotube can be created by rolling...
up and gluing appropriately a single graphene strip \[^{[10]}\]. The circumference (chiral) vector \( \mathbf{C}_h \) is an integer multiple of the primitive translation vectors \( a_1 \) and \( a_2 \) of the Bravais lattice, \( \mathbf{C}_h = n_1 a_1 + n_2 a_2 \). It defines uniquely the nanotube \[^{[20]}\] by the periodic condition imposed on the wave functions, \( \Psi(x + \mathbf{C}_h) = \Psi(x) \). Let us suppose that the coordinate system is chosen such that the \( y \) axis points in the direction of the chiral vector, \( \mathbf{k} \mathbf{C}_h = k_y |\mathbf{C}_h| \). Taking into account Bloch character of the wave functions, one can see that \( \Psi \) has the required periodicity as long as \( k \mathbf{C}_h = k_y |\mathbf{C}_h| = 2\pi q \) for an integer \( q \). Hence, momentum \( k \) is quantized in the \( y \)-direction and the allowed values form equidistant lines in the first Brillouin zone, with a step \( \frac{2\pi}{|\mathbf{C}_h|} \), see Fig.1.

There are two qualitatively different positions of the Dirac point \( \mathbf{K} \) with respect to the allowed lines, in coherence with two main classes of carbon nanotubes. In case of metallic nanotubes, \( \mathbf{K} \) is crossed by one of the lines. For semiconducting nanotubes there is a minimum distance \( \min |\mathbf{k} - \mathbf{K}| = 2\pi/|3\mathbf{C}_h| \) between the Dirac point and the closest line \[^{[21]}\], which leads to the opening of the gap between the valence and conduction bands.

The low-energy behavior of the charge carriers in the nanotube with the chiral vector \( \mathbf{C}_h \) is approximated by

\[
H_\epsilon \psi = v_F (-i\sigma_1 \partial_x + \epsilon \sigma_2) \psi = E\psi, \tag{2}
\]

where the value of \( \epsilon \in \left\{ 0, \pm \frac{2\pi}{3|\mathbf{C}_h|} \right\} \) depends on the type of the nanotube: it is zero for metallic nanotubes and nonvanishing for semiconducting ones. The parameter \( \epsilon \) can be alternatively regarded as a fictitious magnetic flux \[^{[22]}\] or the mass of the quasiparticle. Thus, the \( H_0 \), given by \((2)\) with \( \epsilon = 0 \), coincides with the Dirac Hamiltonian of the free massless particle in one dimension.

The real-world nanostructures are not perfect. The crystal can have impurities; or it can be in presence of an external field. We suppose that these effects are represented by a potential \( V \), which is vanishing at infinity, varies smoothly on the scale of the interatomic distance and is of the range larger than it (and, hence, does not cause the intervalley transition of the electronic states). Then it can be consistently incorporated into \((1)\) and \((2)\) as the unit matrix multiplied by \( V \)^\[^{[12]}\]. The stationary evolution equation for metallic nanotubes acquires a form

\[
H_V \psi = (-iv_F \sigma_1 \partial_x + V(x))\psi = E\psi, \tag{3}
\]

The same Hamiltonian \( H_V \) can be obtained as a reduction of the Hamiltonian of graphene with a potential barrier, \( H + V \), as long as potential is translationally invariant in the \( y \)-direction, i.e. \( V = V(x) \),

\[
H_V = e^{-i\delta k_y y} (H + V)e^{i\delta k_y y} |_{\delta k_y = 0}. \tag{4}
\]

In this case, \( H_V \) represents the energy operator of the particle with normal incidence (\( \delta k_y = 0 \)) on the barrier. The Hamiltonian \( H_V \) describes reflectionless system, independently on the particular form of the potential \( V \). This peculiar fact was explained by the presence of Berry phase in the wave functions, which causes the Born series for backscattering to vanish identically \[^{[17]}\].

Let us look, however, at the problem from a new perspective. In nonrelativistic quantum mechanics, the reflectionless systems play an important role in the theory of solitons and are intimately related to the nonlinear integrable systems \[^{[23]}\]. Their particular properties come hand in hand with the existence of the supersymmetry, that is based on the Darboux-Crum transformations \[^{[22]}\], \[^{[24]}\], \[^{[25]}\]. The supercharges intertwine such systems with the free-particle model and stay behind the absence of backscattering in the conduction band.

This rises the question whether the perfect tunneling in the carbon nanostructures has a similar algebraic background. The answer is affirmative, despite the fact that the relativistic Hamiltonian \( H_V \) is of the first order, contrary to the second order Hamiltonians of the nonrelativistic reflectionless systems \[^{[26]}\].

Let us define the hermitian operators

\[
\mathcal{H} = \begin{pmatrix} H_V & 0 \\ 0 & H_0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}
\]
where \( \mathbf{1} \) is the unit two-by-two matrix. Besides the operator \( \Gamma \), the Hamiltonian \( \mathcal{H} \) has two other symmetries,

\[
\mathcal{U}_1 = \begin{pmatrix} 0 & U^\dagger \\ U & 0 \end{pmatrix}, \quad \mathcal{U}_2 = i \Gamma \mathcal{U}_1,
\]

where \( U = U(x) \) is a unitary operator of a local chiral rotation (chiral gauge transformation),

\[
U = e^{i \alpha \sigma_1} = \cos \alpha \mathbf{1} + i \sin \alpha \sigma_1, \quad U^\dagger = U^{-1},
\]
dependent on the interaction potential,

\[
\alpha(x) = \frac{1}{v_F} \int_x^\pi V(\tau)d\tau.
\]

They satisfy relations

\[
[\mathcal{H}, \mathcal{U}_a] = 0, \quad \{\mathcal{U}_a, \mathcal{U}_b\} = 2 \delta_{ab} \mathbf{1}, \quad a, b = 1, 2.
\]

The anticommutator of \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) is a zero-order polynomial in \( \mathcal{H} \), to be proportional to the central element \( \mathbf{1} \) which is the unit 4 \times 4 matrix. The grading operator \( \Gamma \) classifies the operators \( \mathcal{U}_a \) and \( \mathbf{1} \) as bosonic (they commute with \( \Gamma \)) while operators \( \mathcal{U}_a \) are fermionic (they anticommute with \( \Gamma \)).

Relations \( \mathcal{U}_a \) constitute the \( N = 2 \) zero-order supersymmetry extended by the central charge \( \mathbf{1} \) and graded by \( \Gamma \). Like in the nonrelativistic case of a reflectionless system with the \( \alpha \)-gap, second order Hamiltonian (where the order \( \alpha \) of supersymmetry is fixed by the number \( \alpha \) of bound states \([25]\)), this structure underlies the absence of the backward scattering in the system given by the first order Hamiltonian \( H_V \). The relation \( U H_V = H_0 U \), implied by the commutator in \( \mathcal{U}_a \) and the unitarity of \( U \), reveal the unitary equivalence of \( H_V \) with the free massless Dirac Hamiltonian \( H_0 \). This proves the absence of backscattering; the setting given by \( H_V \) is unitary equivalent to the free massless particle system and, hence, it shares its trivial scattering properties.

Because of this peculiar supersymmetric structure, based on the nontrivial unitary equivalence, all the integrals of the free-particle system represented by \( H_0 \) have their analogs in the system given by \( H_V \). In particular, the pseudospin is conserved in \( H_V \) just because of \( [H_0, \sigma_1] = [U, \sigma_1] = 0 \). The free-particle momentum \( -i \partial_x \) transforms into the integral \( -i \partial_x + \frac{1}{v_F} \sigma_1 V(x) \equiv -i D_x \) for \( H_V \), that has a form of a covariant derivative with a chirality-dependent charge. Then \( H_V \), like \( H_0 \), is presented as a composition of the integrals, \( H_V = -i v_F \sigma_1 D_x \).

The unitary equivalence of \( H_V \) with the free-particle system is broken when the nonvanishing effective mass (the coefficient of \( \sigma_0 \)) is present in the impurity Hamiltonian. It emerges in semiconducting nanotubes \( H_V = H_e + V \) with \( m = \epsilon = \pm 2 \pi/|3C_h| \), or when other than normal incidence of the particles is considered in graphene system \([4]\), \( H_V = e^{-i k_y y} (H + V) e^{i k_x y} | \delta k_x = m \neq 0 = H_m + V \), where \( H_m = H_0 + v_F m \sigma_2 \).

In these cases, the unitary transformation of \( H_V \) yields \( U H_V U^{-1} = H_0 + n v_F \pi \sigma_2(x), \sigma_2(x) = U \sigma_2 U^{-1} = \cos 2 \alpha \sigma_2 - \sin 2 \alpha \sigma_3 \), and

\[
U H_V = H_m U - 2 v_F m \sin \alpha \sigma_3.
\]

Therefore, the scale of supersymmetry breaking in the massive case is of the order of \( m \), and the contribution of the potential is controlled by the factor \( |\sin \alpha| \leq 1 \). Hence, for the close-to-the-normal incidence \( m = \delta k_y \sim 0 \), the potential barrier remains almost perfectly transparent for any \( V(x) \). This is coherent with \([10]\), where the scattering on \( n-p \) junction was analyzed. For general values of the effective mass, the scattering properties of \( H_V \) are nontrivial, however, and depend on the explicit form of the potential. The quasiparticles can be confined in graphene, the charge carriers get localized in semiconducting nanotubes, see \([10, 27]\).

Zero-order supersymmetry, based on a unitary equivalence of the superpartner Hamiltonians, can be formally constructed for any quantum system. It is sufficient to make a unitary transformation of an initial Hamiltonian to get its superpartner. However, the supersymmetry is nontrivial and manifests its predictive power when the Hamiltonians describe different physics, like in \([5]\). There, it provided a simple explanation for the absence of backscattering in the considered carbon nanotubes. As an example of another application, it may be used to determine the s-wave of the Dirac operator in polar coordinates \([6]\), \( H(r, \theta) = H_0(r) - i v_F \sigma_2 \frac{1}{r} \partial_r + V(r) \), where \( H_0(r) = i v_F \sigma_1 (\partial_r + \frac{1}{r}) \), for any potential \( V(r) \). Indeed, \( H(r, \theta) \) acts on the subspace of s-waves \( \psi = \psi(r) \) as \( H_0(r) + V(r) \). We can use an \( r \)-dependent unitary mapping to get rid of the interaction term and to obtain the operator \( H_0(r) \) with analytically computable eigenstates.

In the context of graphene, the supersymmetry appeared earlier in the analysis of the zero-energy states in presence of external magnetic field \([23]\), and in the study of the quantum Hall effect in particular \([24]\). In that case, the first order supercharges are proportional to the Dirac Hamiltonian, while the supersymmetric Hamiltonian is of the second order. The structure in \([5]\) is completely different: the supersymmetric Hamiltonian \( \mathcal{H} \) is of the first order and the existence of the zero-order supercharges \( \mathcal{U}_a \) provides a complete information on the eigenstates of \( H_V \). Together with \([10]\), it suggests to analyze the system with small effective mass perturbatively.

The algebraic framework presented here can be broadened in different ways. For instance, the apparent similarity with the nonrelativistic reflectionless systems suggests to extend the analysis by employing the Darboux-Crum transformations in the context of the higher-order (nonlinear) supersymmetry \([24, 30]\). We believe that in such a generalized form the supersymmetry can serve as a useful tool in the study of the low-energy excitations.
of the charge carriers in the wrapped graphene, where the dynamics is governed by the Dirac Hamiltonian in a curved space. Further discussion on this problem, and other possible applications indicated above goes beyond the scope of the present article.

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[30] Both massless and massive free Dirac particle systems (given by \(H_0\) and \(H_m\)) possess their own hidden supersymmetries that reflect different spectral properties of the systems. In addition to \(\sigma_1\) and \(-i\partial_b \equiv Q_1/v_F\), \(H_0\) has a nonlocal integral \(\hat{\Gamma} = R\sigma_2\), \(\hat{\Gamma}^2 = 1\), where \(R\) is a reflection operator \((R x = -x R, R^2 = 1)\), \(\hat{\Gamma}\) anticommutes with \(Q_1\) and can be identified as a grading operator for the massless system \(H_0\). The integrals \(Q_1\) and \(Q_2 = i\hat{\Gamma}Q_1\) play then the role of the supercharges, \([\hat{\Gamma}, Q_a] = 0\), which generate a second order \(N = 2\) supersymmetry, \([Q_a, Q_b] = 2\delta_{ab}(H_m + m)(H_m - m)\), in which the mass represents the bosonic central charge, cf. [28]. Zero-energy doublet for \(H_0\) and singlet states of energies \(E = \pm m\) for \(H_m\) are the zero modes of the both supercharges \(Q_a\) in correspondence with the distinctive structure of the anticommutation relations. The system \(H_V = H_0 + V\) is characterized by the second order supersymmetry of the same structure as the unitary equivalent free Hamiltonian \(H_0\). In the massive case \((H_V = H_m + V)\), however, potential barrier of a general form breaks such a hidden supersymmetry.