ON THE IWASAWA INVARIANTS OF KATO’S ZETA ELEMENTS FOR MODULAR FORMS

CHAN-HO KIM, JAEHOON LEE, AND GAUTIER PONSINET

Abstract. We study the behavior under congruences of the Iwasawa invariants of the Iwasawa modules which appear in Kato’s main conjecture without $p$-adic $L$-functions. It generalizes the work of Greenberg–Vatsal, Emerton–Pollack–Weston, B.D. Kim, Greenberg–Iovita–Pollack, and one of us simultaneously. As a consequence, we establish the propagation of Kato’s main conjecture for modular forms of higher weight at arbitrary good prime under the assumption on the mod $p$ non-vanishing of Kato’s zeta elements. The application to the $\pm$ and $\sharp/\flat$-Iwasawa theory for modular forms is also discussed.

Contents

1. Introduction 1
2. Main results and applications 6
3. “Prime-to-$p$ local” Iwasawa theory 9
4. The zeta element side 12
5. The $H^2$-side 18
6. The invariance of the difference of $\lambda$-invariants 20
Acknowledgement 22
References 22

1. Introduction

1.1. Overview. Fix once and for all a prime number $p \geq 5$. In Iwasawa theory for elliptic curves and modular forms, the techniques of congruences of modular forms have played important roles. Especially, in their ground-breaking work [GV00], Greenberg and Vatsal observed that both algebraic and analytic Iwasawa invariants of elliptic curves with good ordinary reduction over the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$ can be described in terms of the information of the residual representations and the local behavior at bad reduction primes under the $\mu = 0$ assumption. As a consequence, it is proved that the Iwasawa main conjecture for an elliptic curve implies the Iwasawa main conjecture for all congruent elliptic curves when one-sided divisibility is given and $\mu = 0$. This fundamental idea has been generalized to a variety of settings including Hida families, elliptic curves with supersingular reduction with $a_p = 0$, arbitrary $\mathbb{Z}_p$-extensions including anticyclotomic $\mathbb{Z}_p$-extensions of imaginary quadratic fields with and without Heegner hypothesis, modular forms at non-ordinary primes, and their mixtures [Wes05, EPW06, Och06, GIP08, Kim09, Hac11, PW11b, CK17, Kim17, CKL17, DL18, Kid18, Pon20, HL19, Fou].

In this article, we give a rather different and unified approach to realize the idea of Greenberg–Vatsal for modular forms of higher weight at arbitrary good primes over the full cyclotomic extension $\mathbb{Q}(\zeta_{p^\infty})$ of $\mathbb{Q}$. In order to do this, we directly work with Kato’s zeta elements

Date: November 22, 2021.

2010 Mathematics Subject Classification. 11R23 (Primary); 11F33 (Secondary).

Key words and phrases. Iwasawa theory, Kato’s Euler systems, Iwasawa main conjecture, congruences of modular forms.
and Kato’s main conjecture without $p$-adic $L$-functions [Kat04]. As corollaries, we can prove Greenberg–Vatsal type results for both good ordinary and non-ordinary cases simultaneously including $\pm$-Iwasawa theory à la Kobayashi–Pollack [Kob03, Pol03] and Lei [Lei11] and $\sharp/\flat$-Iwasawa theory à la Sprung [Spr12] and Lei–Loeffler–Zerbes [LLZ10]. Note that the construction of integral $p$-adic $L$-functions depends genuinely on the reduction type and we do not have to consider this issue at all.

The key ingredients include an extensive use of the localization exact sequence in étale cohomology as well as the mod $p$ multiplicity one and Ihara’s lemma. Since Fontaine–Laffaille theory is implicitly used to obtain the mod $p$ multiplicity one and Ihara’s lemma, the weight $k$ of modular forms we consider is required to satisfy $2 \leq k \leq p - 1$. The weight assumption can be removed in the semi-stable ordinary case thanks to the work of Vatsal.

We also point out where the classical argument breaks down and how to overcome this obstruction. In the argument of Greenberg–Vatsal and its successors, the relevant Selmer group over the Iwasawa algebra has no proper Iwasawa submodule of finite index and it is essentially used to reveal the following phenomenon.

Let $f$ be a newform, $\overline{\rho}_f$ the residual representation, and $\Sigma_0$ a finite set of primes consisting of all bad reduction primes. In both algebraic and analytic sides, we have the following equality:

“ $\lambda$-invariant of $f = \lambda$-invariant of $\overline{\rho}_f$

\[ + \sum_{\ell \in \Sigma_0} \lambda$-invariant of the local behavior of $f$ at $\ell$”

when $\mu = 0$. Here, the algebraic side means the Selmer group part and the analytic side means the $p$-adic $L$-function part in the Iwasawa main conjecture of Mazur–Greenberg type.

In our setting, we do not expect that the relevant Iwasawa modules have no finite Iwasawa submodule in general. Thus, the above formula would not hold as it stands in our setting. However, we are still able to prove the following type of statement:

Fix a 2-dimensional odd irreducible mod $p$ representation $\overline{\rho}$. Let $S_k(\overline{\rho})$ be the set of newforms of weight $k$ such that the residual representation is isomorphic to $\overline{\rho}$ and $p$ does not divide the levels of the newforms. Suppose that the $\mu$-invariant of one form in $S_k(\overline{\rho})$ in the zeta element side vanishes. Then, for all $f \in S_k(\overline{\rho})$, we have the following statements:

- The $\mu$-invariants of $f$ in both the zeta element and $H^2$-sides vanish;
- $(\lambda$-invariant of $f$ in the $H^2$-side)$−(\lambda$-invariant of $f$ in the zeta element side) is constant.

Here, the zeta element side means the Kato’s Euler system part and the $H^2$-side means the the second Iwasawa cohomology part in the Iwasawa main conjecture of Kato type.

In other words, although we are not able to see how Iwasawa invariants vary in each side under congruences, the stability of the difference of Iwasawa invariants is strong enough to deduce a Greenberg–Vatsal type result for Kato’s main conjecture.

The following non-exhaustive list shows how much the idea of Greenberg–Vatsal is generalized. (See §1.3 for the difference between $\mathbb{Q}(\zeta_{p^\infty})$ and $\mathbb{Q}_\infty$.)

- In [GV00], elliptic curves with good ordinary reduction over $\mathbb{Q}_\infty$ are studied;
- In [EPW06], Hida families over the full cyclotomic extension $\mathbb{Q}(\zeta_{p^\infty})$ are studied;
- In [Kim09], elliptic curves with good supersingular reduction ($a_p = 0$) over $\mathbb{Q}_\infty$ are studied (the algebraic side only);
- In [GIP08], modular forms of weight two at non-ordinary primes over $\mathbb{Q}_\infty$ are studied.
• In [Pon20], modular forms of higher weight in the Fontaine–Laffaille range at non-ordinary primes over $\mathbb{Q}_\infty$ are studied (the algebraic side only).

Indeed, some of the above work deal with more general $\mathbb{Z}_p$-extensions. Our main theorem (Theorem 2.1) covers and strengthens all the above results simultaneously for modular forms of weight $k$ with $2 \leq k \leq p - 1$ over the full cyclotomic extension $\mathbb{Q}(\zeta_{p^\infty})$.

The rest of this article is organized as follows.

(1) In the rest of this section, we recall the convention of modular Galois representations, Iwasawa cohomology, Kato’s zeta elements and the Iwasawa main conjecture without $p$-adic $L$-functions closely following [Kat04].

(2) In §2, we state our main result (Theorem 2.1 and Theorem 2.3) and discuss the applications to other types of Iwasawa theory.

(3) In §3, we recall the notion of Euler factors in various settings and their Iwasawa-theoretic variants. Also, we explicitly prove that the characteristic polynomial of the second local Iwasawa cohomology at $\ell \neq p$ and the Iwasawa-theoretic Euler factor at $\ell$ coincide up to the involution.

(4) In §4, we study the mod $p$ behavior of the zeta elements. Theorem 2.1.(1) on the $\mu$-invariant is proved here.

(5) In §5, we study the mod $p$ behavior of the second Iwasawa cohomology.

(6) In §6, we prove Theorem 2.1.(2) on the $\lambda$-invariant.

Notation 1.1. We expect the reader is rather familiar with [Kat04]. We freely use the results and notations of [Kat04]. Especially, we follow the notation of [Kat04] with only few exceptions. We denote by $\pi$ instead of $\lambda$ for the uniformizer of coefficient fields in order to avoid the conflict with $\lambda$-invariants. We also denote by $V_f$ and $T_f$ instead of $V_{F_\lambda}(f)$ and $V_{O_\lambda}(f)$ for modular Galois representations and their lattices, respectively.

1.2. Galois representations. Let $f = \sum_{n \geq 1} a_n(f)q^n \in S_k(\Gamma_1(N), \psi)$ be a newform with $(N, p) = 1$. We fix embeddings $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let $F := \mathbb{Q}(a_n(f) : n)$ and $F_{\pi} := \mathbb{Q}_p(\iota_p(a_n(f)) : n)$, and $O_\pi \subseteq F_{\pi}$ the ring of integers of $F_{\pi}$, $\pi$ a uniformizer of $F_{\pi}$, and $\mathbb{F}$ the residue field of $F_{\pi}$.

Let $S$ be a finite set of places of $\mathbb{Q}$ containing the places dividing $NP\infty$ and $\mathbb{Q}_S$ be the maximal extension of $\mathbb{Q}$ unramified outside $S$. For a field $K$, denote by $G_K$ the absolute Galois group of $K$. Let

$$\rho_f : \text{Gal}(\mathbb{Q}_S/\mathbb{Q}) \to \text{GL}_2(F_{\pi}) \cong \text{GL}_{F_{\pi}}(V_f)$$

be the (cohomological) $\pi$-adic Galois representation associated to $f$ following the convention of [Kat04, §14.10].

1.2.1. Construction. We follow [Kat04, (4.5.1) and §8.3]. Let $N \geq 4$ and $\varpi : E \to Y_1(N)$ the universal elliptic curve over the modular curve and $H^1_p := R^1\varpi_*\mathbb{Z}_p$ the étale $\mathbb{Z}_p$-sheaf on $Y_1(N)$. We define

$$V_{k,\mathbb{Z}_p}(Y_1(N)) := H^1_{\text{ét}}(Y_1(N)_{\overline{\mathbb{Q}}}, \text{Sym}^{k-2}_\mathbb{Z}_p(H^1_p)),$$

and

$$T_f := V_{k,\mathbb{Z}_p}(Y_1(N)) \otimes_{\mathbb{T}(N)} \mathbb{T}(N)/\mathfrak{p}_f, \quad V_f := T_f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

where $\mathbb{T}(N)$ be the image of the abstract Hecke algebra generated by Hecke operators at all primes in the endomorphism ring of $V_{k,\mathbb{Z}_p}(Y_1(N))$ over $\mathbb{Z}_p$ and $\varphi_f$ is the height one prime ideal of $\mathbb{T}$ generated by the Hecke eigensystem of $f$ following [Kat04, §6.3]. When $N < 4$, we are still able to construct Galois representations using the $N \geq 4$ case and trace maps.
1.2.2. Properties. More explicitly, $\rho_f$ satisfies the following properties:

(1) $\det(\rho_f) = \chi_{\text{cyc}}^{-1} \cdot \psi^{-1}$ where $\chi_{\text{cyc}}$ is the cyclotomic character;

(2) for any prime $\ell$ not dividing $NP$, we have

$$\det \left( 1 - \rho_f(Frob_\ell^{-1}) \cdot u : H^0(I_\ell, V_f) \right) = 1 - a_\ell(f) \cdot u + \psi(\ell) \cdot \ell^{k-1} \cdot u^2$$

where $\text{Frob}_\ell$ is the arithmetic Frobenius at $\ell$ in $G_{\mathbb{Q}_\ell}/I_\ell$ and $I_\ell$ is the inertia subgroup of $G_{\mathbb{Q}_\ell}$;

(3) for the prime number $p$ lying under $\pi$, we have

$$\det(1 - \varphi \cdot u : D_{\text{cris}}(V_f)) = 1 - a_p(f) \cdot u + \psi(p) \cdot p^{k-1} \cdot u^2$$

where $\varphi$ is the Frobenius operator acting on $D_{\text{cris}}(V_f)$, Fontaine’s crystalline Dieudonné module associated to the restriction of $V_f$ to $G_{\mathbb{Q}_p}$ [Fon94].

For any Galois module $M$ over $\mathbb{Z}_p$ and any integer $k \in \mathbb{Z}$, let $M(k) := M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(k)$ be the $k$-th Tate twist of $M$.

Let $f^* = \sum_{n \geq 1} a_n(f)q^n \in S_k(\Gamma_1(N), \overline{\psi})$ be the dual modular form of $f$ where $(-)$ means the complex conjugation. Due to the duality of modular Galois representations [Kat04, (14.10.1)], we also have

$$V_{f^*} \simeq \text{Hom}_{F_\pi}(V_f, F_\pi)(1 - k), \quad V_{f^*}(k - r) \simeq \text{Hom}_{F_\pi}(V_f(r), F_\pi(1)).$$

Then the Euler factor of $V_{f^*}(r)$ at $\ell$ not dividing $p$ is

$$1 - a_\ell(f) \cdot \ell^{-r} + \overline{\psi}(\ell) \cdot \ell^{k-1-2r}.$$ 

One can compare this with (1.1) and §3.1.

Let $R$ be any $p$-adic ring including $F_\pi$, $O_\pi$, and $O_\pi/\pi^i$. Then, for any $R$-module $M$, we set $M^* := \text{Hom}_R(M, R)$. Also, the torsion part of $M$ is denoted by $M_{\text{tors}}$.

Let $\overline{\mathfrak{p}} : G_{\mathbb{Q}} \to \text{GL}_2(\mathbb{F})$ be the residual Galois representation of $V_f$. We assume the following condition throughout this article.

Assumption 1.2. The image of $\overline{\mathfrak{p}}$ contains $\text{SL}_2(\mathbb{F}_p)$.

Due to Assumption 1.2, all the content of this article is independent of the choice of a Galois-stable $O_\pi$-lattice $T_f$ of $V_f$.

1.3. Iwasawa cohomologies. Let

$$\Lambda = O_\pi[[\text{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})]] \simeq O_\pi[\Delta][\Gamma]$$

be the Iwasawa algebra where $\mathbb{Q}(\zeta_{p^n}) = \bigcup_{n \geq 1} \mathbb{Q}(\zeta_{p^n})$ and $\text{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \simeq \Delta \times \Gamma$ with $\Delta \simeq \mu_{p-1}$ and $\Gamma \simeq 1 + p\mathbb{Z}_p$. Let $\mathbb{Q}_\infty := \mathbb{Q}(\zeta_{p^n})^\Delta$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$ and $\mathbb{Q}_n$ the unique extension of $\mathbb{Q}$ in $\mathbb{Q}_\infty$ of degree $p^{n}$. We decompose

$$\Lambda = \bigoplus_{i=0}^{p-2} \Lambda_i$$

where $\Lambda_i := \Lambda/(\sigma - \omega^i(\sigma) : \sigma \in \Delta)$ and $\omega$ is the Teichmüller character. We fix the notational convention for Galois, étale, and Iwasawa cohomologies.

(1) If $K$ is a field, $L$ is a Galois extension of $K$, and $M$ is a topological $\text{Gal}(L/K)$-module, then we denote by

$$H^i(L/K, M) := H^i_{\text{cont}}(\text{Gal}(L/K), M)$$

the $i$-th Galois cohomology group (i.e. the $i$-th continuous cohomology group) of $\text{Gal}(L/K)$ with coefficients in $M$. If $L$ is an algebraic closure of $K$, we write $H^i(K, M)$ instead of $H^i(L/K, M)$. 

4
(2) If \( R \) is a ring and \( M \) is an étale sheaf on \( \text{Spec}(R) \), then we denote by
\[
H^i_{\text{ét}}(R, M) := H^i_{\text{ét}}(\text{Spec}(R), M)
\]
the \( i \)-th étale cohomology group of \( \text{Spec}(R) \) with coefficients in \( M \).

(3) Let \( T \) be a compact \( \mathbb{Z}_p \)-module with continuous action of \( \text{Gal}(\mathbb{Q}_S/\mathbb{Q}) \). We write
\[
H^i_{\text{ét}}(\mathbb{Q}_S/\mathbb{Q}(\zeta_{p^n}), T) := \lim_{\longrightarrow} H^i(\mathbb{Q}_S/\mathbb{Q}(\zeta_{p^n}), T),
\]
\[
H^i_{\text{ét}}(\mathbb{Q}_S/\mathbb{Q}_\infty, T) := \lim_{\longrightarrow} H^i(\mathbb{Q}_S/\mathbb{Q}_\infty, T),
\]
\[
H^i_{\text{ét}}(\mathbb{Q}(\zeta_{p^n}), T) := \lim_{\longrightarrow} H^i(\mathbb{Q}(\zeta_{p^n}), T)
\]
where \( \eta \) is a prime of \( \mathbb{Q}_\infty \) or \( \mathbb{Q}(\zeta_{p^n}) \) and \( \eta_n \) is the prime of \( \mathbb{Q}_n \) or \( \mathbb{Q}(\zeta_{p^n}) \) lying below \( \eta \), respectively.

Let
\[
j_n : \text{Spec}(\mathbb{Q}(\zeta_{p^n})) \to \text{Spec}(\mathbb{Z}[\zeta_{p^n}, 1/p]), \quad j : \text{Spec}(\mathbb{Q}(\zeta_{p^n})) \to \text{Spec}(\mathcal{O}_{\mathbb{Q}(\zeta_{p^n})}[1/p])
\]
be the natural maps and we use the same notations for \( \mathbb{Q}_n \) and \( \mathbb{Q}_\infty \).

**Definition 1.3.** Let \( a = 1 \) or \( 2 \), and \( i \in \{0, 1, \ldots, p-3, p-2\} \).

1. We define the **\( a \)-th Iwasawa cohomology for \( T_f(k-r) \) over \( \Lambda \)** by
\[
H^a_{\text{ét}}(j_n T_f(k-r)) := \lim_{\longrightarrow} H^a_{\text{ét}}(\text{Spec}(\mathbb{Z}[\zeta_{p^n}, 1/p]), j_n, T_f(k-r))
\]
where \( H^a_{\text{ét}}(\text{Spec}(\mathbb{Z}[\zeta_{p^n}, 1/p]), j_n, T_f(k-r)) \) is the étale cohomology group.

2. We define the **\( a \)-th Iwasawa cohomology for \( T_{f,i}(k-r) \) over \( \Lambda_i \)** by
\[
H^a(j_n T_{f,i}(k-r)) := H^a(j_n T_f(k-r))^{\omega^i}
\]
where \( T_{f,i} = T_f \otimes \omega^i \) and \( H^a(j_n T_f(k-r))^{\omega^i} \) is the \( \omega^i \)-isotypic component of \( H^a(j_n T_f(k-r)) \).

**Remark 1.4.** We use \( H^a \) for the extension \( \mathbb{Q}(\zeta_{p^n})/\mathbb{Q} \) and \( \mathbb{H}^a \) for the extension \( \mathbb{Q}_\infty/\mathbb{Q} \).

**Theorem 1.5 (Kato).** For all \( i = 0, \ldots, p-2 \),

1. \( H^1(j_n T_{f,i}(k-r)) \) is free of rank one over \( \Lambda_i \) under Assumption 1.2;
2. \( H^2(j_n T_{f,i}(k-r)) \) is a finitely generated torsion module over \( \Lambda_i \).

**Proof.** See [Kat04, Theorem 12.4.(1) and (2)], where it is proved for the module \( H^i(j_n T_f) \).

Note that the statement is insensitive to Tate twists. \( \square \)

### 1.4. Kato’s zeta elements and the main conjecture

We keep Assumption 1.2 in this subsection to ensure the integrality of Kato’s zeta elements. Let
\[
Z(f, T_f(k-r)) \subseteq H^1(j_n T_f(k-r))
\]
be the module of Kato’s zeta elements generated by \( z^{(p)} \otimes (\zeta_{p^n})_{k-r} \) where \( \gamma \) runs over \( T_f \). Here, \( z^{(p)} \) appears in [Kat04, Theorem 12.5]. More precisely, there exists a map \( T_f \to H^1(j_n T_f) \) defined by \( \gamma \mapsto z^{(p)} \).

**Proposition 1.6.** Under Assumption 1.2, \( Z(f, T_f(k-r)) \) is generated by one element over \( \Lambda \).

**Proof.** It is a well-known fact. See [KN20, Proposition A.12], for example. \( \square \)
**Definition 1.7.** We define the zeta element \( z_{\text{Kato}}(f, k-r) \) to be a generator of \( Z(f, T_f(k-r)) \), and define \( z_{\text{Kato}}(f, i, k-r) \) by the \( \omega^i \)-isotypic component of \( z_{\text{Kato}}(f, k-r) \). Then \( z_{\text{Kato}}(f, i, k-r) \) generates \( Z(f, T_{f,i}(k-r)) \subseteq H^1(j_*T_f(k-r)) \), the \( \omega^i \)-component of \( Z(f, T_f(k-r)) \).

**Remark 1.8.** Via the dual exponential map, \( z_{\text{Kato}}(f, i, k-r) \) interpolates the \( L \)-values of \( f^* \) at \( s = r \) twisted by \( \omega^{-i} \chi^{-i} \) where \( \chi \) runs over finite order characters of \( \Gamma \). See [Kat04, Theorem 12.5].

**Conjecture 1.9** (Kato’s main conjecture). Let \( T_f \) be an \( \mathcal{O}_E \)-lattice of \( V_f \) and \( 1 \leq r \leq k-1 \). Keep Assumption 1.2. The following equivalent statements hold.

1. Let \( \mathfrak{p} \) be a height one prime ideal of \( \Lambda \). Then
   
   \[
   Z(f, T_f(k-r))_{\mathfrak{p}} \subseteq H^1(j_*T_f(k-r))_{\mathfrak{p}}
   \]
   
   and
   
   \[
   \text{length}_{\Lambda_\mathfrak{p}} H^1(j_*T_f(k-r))_{\mathfrak{p}}/Z(f, T_f(k-r))_{\mathfrak{p}} = \text{length}_{\Lambda_\mathfrak{p}} H^2(j_*T_f(k-r))_{\mathfrak{p}}
   \]

2. Let \( \mathfrak{p}_i \) be a height one prime ideal of \( \Lambda_i \). Then
   
   \[
   Z(f, T_{f,i}(k-r))_{\mathfrak{p}_i} \subseteq H^1(j_*T_{f,i}(k-r))_{\mathfrak{p}_i}
   \]
   
   and
   
   \[
   \text{length}_{\Lambda_{i,\mathfrak{p}_i}} H^1(j_*T_{f,i}(k-r))_{\mathfrak{p}_i}/Z(f, T_{f,i}(k-r))_{\mathfrak{p}_i} = \text{length}_{\Lambda_{i,\mathfrak{p}_i}} H^2(j_*T_{f,i}(k-r))_{\mathfrak{p}_i}
   \]
   
   for all \( i = 0, \ldots, p-2 \).

3. \[
\text{char}_{\Lambda_i} H^1(j_*T_{f,i}(k-r))_{\mathfrak{p}_i}/Z(f, T_{f,i}(k-r))_{\mathfrak{p}_i} = \text{char}_{\Lambda_i} H^2(j_*T_{f,i}(k-r))_{\mathfrak{p}_i}
\]
   
   for all \( i = 0, \ldots, p-2 \).

**Remark 1.10.** See [Kat04, Conjecture 12.10] for the first statement (with “\( r = 0 \)”) and [Kur02, Conjecture 6.1] for the third statement (with \( i = 0 \)). Note that Conjecture 1.9 is independent of \( r \). The decomposition using powers of the Teichmüller character is required to consider Iwasawa invariants as in [EPW06].

The following one-sided divisibility is proved in [Kat04, Theorem 12.5.(4)].

**Theorem 1.11** (Kato). Keep Assumption 1.2. Then the inclusion \( \subseteq \) holds in each statement of Conjecture 1.9.

We say that the \( \omega^i \)-component of Kato’s main conjecture holds if the equality holds for \( i \) in the second or equivalently the third statement in Conjecture 1.9.

For a finitely generated torsion \( \Lambda_i \)-module \( M_i \), we denote by \( \text{char}_{\Lambda_i}(M_i) \) the associated characteristic ideal, and by \( \mu(M_i) = \mu(\text{char}_{\Lambda_i}(M_i)) \) and \( \lambda(M_i) = \lambda(\text{char}_{\Lambda_i}(M_i)) \), the Iwasawa invariants of \( M_i \). If two \( \Lambda_i \)-ideals have the same \( \mu \)-invariants and the same \( \lambda \)-invariants and furthermore one divides the other, they must be equal. We hence find the following.

**Corollary 1.12.** If the Iwasawa invariants of the \( \omega^i \)-components of both sides in Conjecture 1.9 coincide, then the \( \omega^i \)-component of Kato’s main conjecture holds.

2. Main results and applications

2.1. The statement of main theorem. Let \( \overline{\rho} \) be a mod \( p \) residual representation with conductor \( N(\overline{\rho}) \). Denote by \( S_k(\overline{\rho}) \) the set of newforms of fixed weight \( k \) such that the residual representation is isomorphic to \( \overline{\rho} \) and \( p \) does not divide the levels of the newforms.

**Theorem 2.1.** Assume that

- \( 2 \leq k \leq p-1 \);
- the image of \( \overline{\rho} \) contains \( \text{SL}_2(\mathbb{F}_p) \);
\[ \mu \left( \frac{\mathbb{H}^1(j_\ast T_{f_0,i}(k-r))}{z_{\text{Kato}}(f_0,i,k-r)} \right) = 0 \text{ for one } f_0 \in S_k(\overline{\rho}). \]

Then we have the following statements:

1. \[ \mu \left( \frac{\mathbb{H}^1(j_\ast T_{f,i}(k-r))}{z_{\text{Kato}}(f,i,k-r)} \right) = 0 \text{ for all } f \in S_k(\overline{\rho}). \]

2. \[ \lambda \left( \frac{\mathbb{H}^1(j_\ast T_{f,i}(k-r))}{z_{\text{Kato}}(f,i,k-r)} \right) - \lambda \left( \frac{\mathbb{H}^2(j_\ast T_{f,i}(k-r))}{z_{\text{Kato}}(f,i,k-r)} \right) \]
   is constant for \( f \in S_k(\overline{\rho}). \)

3. If the \( \omega^i \)-component of Kato’s main conjecture holds for one form in \( S_k(\overline{\rho}) \), then the \( \omega^i \)-component of Kato’s main conjecture holds for all forms in \( S_k(\overline{\rho}) \).

Proof. (1) It is proved in Corollary 4.11 with Kato’s divisibility statement (Theorem 1.11).

(2) It is proved in §6.

(3) It immediately follows from the above two statements due to Corollary 1.12.

\[ \square \]

Remark 2.2. (1) The assumption \( \mu \left( \frac{\mathbb{H}^1(j_\ast T_{f_0,i}(k-r))}{z_{\text{Kato}}(f_0,i,k-r)} \right) = 0 \) in Theorem 2.1 is weaker than the \( \mu = 0 \) assumptions of various \( p \)-adic \( L \)-functions in other literatures.

(2) In Theorem 2.1.(1), we observe in particular that, if the \( \mu = 0 \) conjecture for the fine Selmer group holds for one form in \( S_k(\overline{\rho}) \), then it holds for all forms in \( S_k(\overline{\rho}) \). The \( \mu = 0 \) conjecture for fine Selmer groups is due to Coates–Sujatha [CS05, Conjecture A]. Indeed, the propagation of the \( \mu = 0 \) conjecture for fine Selmer groups can also be obtained by using Proposition 5.3, Proposition 5.5, and (6.2) only.

(3) Theorem 2.1.(2) is weaker than the \( \lambda \)-invariant formulas in the literature. This is due to the fact that it is unclear whether the Iwasawa modules we deal with have no finite Iwasawa submodule.

(4) One may consider the congruences between modular forms of different weights. It is possible in the ordinary case via Hida theory [EPW06].

(5) Under Assumption 1.2, we have

\[ \frac{\mathbb{H}^1(j_\ast T_{f,i}(k-r))}{z_{\text{Kato}}(f,i,k-r)} \sim \frac{\Lambda_i}{g_i(T)} \]

for some \( g_i(T) \in \Lambda_i \) due to [Kat04, Theorem 12.4.(3)]; thus, the information of Iwasawa invariants is encoded in \( g_i(T) \).

In the semi-stable ordinary case, we can remove the Fontaine–Laffaille assumption as follows. Since we do not use any Hida deformation explicitly, our result is weaker than that of [EPW06] but the argument is much simpler than theirs. For example, we do not need to construct two-variable \( p \)-adic \( L \)-functions.

Denote by \( S_k^{\text{ord}}(\overline{\rho}) \) the set of \( p \)-ordinary \( p \)-stabilized newforms of \textit{fixed} weight \( k \) such that the residual representation is isomorphic to \( \overline{\rho} \) and \( p \) divides the levels of the \( p \)-stabilized newforms exactly once. Note that the mod \( p \) multiplicity one certainly fails when \( p^2 \) divides the level (cf. [PW11a, §1.6 and §7]).

Theorem 2.3. Assume that

- the image of \( \overline{\rho} \) contains \( \text{SL}_2(\mathbb{F}_p) \);
- \( \overline{\rho} \) is ordinary at \( p \);
- \( \overline{\rho} \) has distinct Jordan–Hölder factors on the decomposition group at \( p \);
\[ \mu \left( \frac{\zeta_i(j,T_{f_0,i}(k-r))}{\zeta_{Kato}(f_0,i,k-r)} \right) = 0 \text{ for one } f_0 \in S^\text{ord}_k(\mathfrak{p}). \]

Then the same conclusions of Theorem 2.1 holds for \( S^\text{ord}_k(\mathfrak{p}) \).

**Proof.** The key observation is that the Fontaine–Laffaille condition \((2 \leq k \leq p - 1 \text{ and } p \nmid N)\) is required only when we invoke the mod \( p \) multiplicity one (Theorem 4.1) and Ihara’s lemma (Theorem 4.7). For the mod \( p \) multiplicity one, see [Vat03, Theorem 1.13] which uses [Wil95, Theorem 2.1] when \( k = 2 \) and [Hid86] when \( k > 2 \). The equivalent statement can be found in [EPW06, Propositions 3.1.1 and 3.3.1]. See also [EPW06, §3.8] for how Ihara’s lemma is used in the integral period comparison (cf. [Vat13, Proposition 4.5]). In §4, we explain how the integral period comparison yields the congruence between Kato’s zeta elements. □

**Remark 2.4.** Nakamura has recently constructed Kato’s Euler system over the universal deformation space by using the \( p \)-adic local Langlands correspondence and the local-global compatibility result. Combining his construction with the technique in this paper, the Fontaine–Laffaille condition in Theorem 2.1 could be entirely removed. See [Nak] for details.

### 2.2. Applications to main conjectures with \( L \)-functions.

We describe the relation with other main conjectures focusing more on the non-ordinary case. We do not recall the formulations of the Greenberg-style, \( \pm \), and \( \sharp/\flat \)-main conjectures in this article, but one can find details in [Kat04], [Kob03], and [Spr12] for modular forms of weight two.

When \( p \) divides \( a_p(f) \) and the weight of \( f \) is two, let

\[ L^*_p(f^*, -i) = \begin{cases} L^1_p(f^*, -i) & \text{if } i = 0 \\ \frac{\gamma - 1}{\gamma - 1} \cdot L^*_p(f^*, -i) & \text{if } i \neq 0 \text{ and } \bullet = - \end{cases} \]

where \( L^*_p(f^*, -i) \) is the \( \omega^{-i} \)-component of relevant integral \( p \)-adic \( L \)-functions, \( \bullet \in \{+, -, \sharp, \flat\} \), and \( \gamma \) is a generator of \( \Lambda_i \). It is known that \( L^*_p(f^*, -i) \in \Lambda_i \) under Assumption 1.2. Note that we follow Kobayashi’s convention for \( \pm \). Let \( \text{Im}(\text{Col}^{\sharp/\flat,i}) \subseteq \Lambda_i \) be the image of the \( \sharp/\flat \)-Coleman maps defined by \( \text{Col}^{\sharp,i} = (\text{Col}_L)^{\omega^i} \) and \( \text{Col}^{\flat,i} = (\text{Col}_L)^{\omega^i} \), respectively, following [LLZ10].

**Theorem 2.5** (Kato, Kobayashi, Lei, Sprung, Lei–Loeffler–Zerbes). Keep Assumption 1.2. The \( \omega^i \)-component of Kato’s main conjecture (with \( r = 1 \)) is equivalent to:

1. \( L_p(f^*, -i) = \text{char}_\Lambda \text{Sel}(Q_\infty, A_{f^*, -i}(1))^\vee \) when \( a_p(f) \) is a \( \pi \)-adic unit, or
2. \( L^\pm_p(f^*, -i) = \text{char}_\Lambda \text{Sel}^\pm(Q_\infty, A_{f^*, -i}(1))^\vee \) when \( a_p(f) = 0 \) and the weight is two, or
3. \( \text{char}_\Lambda \left( \text{Im} \left( \text{Col}^{\sharp/\flat,i}/L_p^{\sharp/\flat}(f^*, -i) \right) \right) = \text{char}_\Lambda \text{Sel}^{\sharp/\flat}(Q_\infty, A_{f^*, -i}(1))^\vee \) when \( a_p(f) \) is divisible by \( \pi \).

where \( A_{f^*, -i} := V_{f^*, -i}/T_{f^*, -i} \).

**Proof.** This is the combination of [Kat04, §17.13], [Kob03, Theorem 7.4], [Lei11, Corollary 6.8], [Spr12, Proposition 7.19], and [LLZ10, Corollary 6.6]. □

The following corollary is immediate from Theorem 2.1 and Theorem 2.5.

**Corollary 2.6.** We keep all the assumptions in Theorem 2.1. Let \( f \in S_k(\mathfrak{p}) \).

1. Assume that \( f \) is ordinary at \( p \). If \( (L_p(f^*, -i)) = \text{char}_\Lambda \text{Sel}(Q_\infty, A_{f^*, -i}(1))^\vee \),

then

\( (L_p(g^*, -i)) = \text{char}_\Lambda \text{Sel}(Q_\infty, A_{g^*, -i}(1))^\vee \)

for all \( g \in S_k(\mathfrak{p}) \).
(2) Assume that \( a_p(f) = 0 \) and \( k = 2 \). If
\[
\left( \bar{L}_p^\pm(f^*, -i) \right) = \text{char}_A, \text{Sel}^{\pm}(\mathbb{Q}_\infty, A_{f^*, -i}(1))^\vee,
\]
then
\[
\left( \bar{L}_p^\pm(g^*, -i) \right) = \text{char}_A, \text{Sel}^{\pm}(\mathbb{Q}_\infty, A_{g^*, -i}(1))^\vee
\]
for all \( g \in S_k(\overline{p}) \).

(3) Assume that \( p \mid a_p(f) \). If
\[
\text{char}_A \left( \text{Im} \left( \text{Col}^{\ell/\beta, i} / L_p^{\ell/\beta}(f^*, -i) \right) \right) = \text{char}_A, \text{Sel}^{\ell/\beta}(\mathbb{Q}_\infty, A_{f^*, -i}(1))^\vee,
\]
then
\[
\text{char}_A \left( \text{Im} \left( \text{Col}^{\ell/\beta, i} / L_p^{\ell/\beta}(g^*, -i) \right) \right) = \text{char}_A, \text{Sel}^{\ell/\beta}(\mathbb{Q}_\infty, A_{g^*, -i}(1))^\vee
\]
for all \( g \in S_k(\overline{p}) \).

**Remark 2.7.** Note that mod \( p \) non-vanishing of Kato’s zeta elements is weaker than vanishing of \( \mu \)-invariants of all the above \( p \)-adic \( L \)-functions.

**Remark 2.8.** In the work of Skinner–Urban [SU14], the \( (\omega^0 \text{-component of the}) \) main conjecture for a large class of elliptic curves and modular forms at good ordinary primes is proved; more precisely, the main conjecture is proved under the following conditions:

- \( f \) is good ordinary at \( p \);
- \( k \equiv 2 \pmod{p-1} \), so only \( k = 2 \) is allowed in our setting;
- \( \psi = 1 \), so the Nebentypus is trivial;
- \( r = k - 1 \);
- \( i = 0 \), so \( \mathbb{Q}_\infty / \mathbb{Q} \) is only considered;
- there exists a prime \( \ell \neq p \) such that \( \ell \nmid N(\overline{p}) \).

The second and third conditions come from the anticyclotomic input, the work of Vatsal on vanishing of anticyclotomic \( \mu \)-invariants [Vat03]. The second condition on weight would be removed by the work of Chida–Hsieh [CH18, Remark 1 after Theorem C]. The fourth condition follows from the convention of the Galois modules in the Selmer groups in [SU14, §1.1]. Note that our main theorem (Theorem 2.1) applies without these assumptions Skinner–Urban made.

3. “Prime-to-\( p \)” Iwasawa Theory

We generalize [GV00, Proposition 2.4] to the setting of Conjecture 1.9.

3.1. Euler factors. We recall several Euler factors following [Kat04, Propositions 8.7, 8.10 and 8.12]. Let \( \ell \) be a prime different from \( p \).

**Definition 3.1.**

(1) At the level of \( H^1(\mathbb{Z}[1/p], j_*V_{k, \mathbb{Z}_p}(Y(p^nN))(k - r)) \), we define
\[
E_\ell(Y(p^nN), r) := \begin{cases} 
1 - T'(\ell) \cdot \begin{pmatrix} 1/\ell & 0 \\ 0 & 1/\ell \end{pmatrix}^* \cdot \ell^{-r} & \text{if } \ell \nmid N \\
1 - T'(\ell) \cdot \begin{pmatrix} 1/\ell & 0 \\ 0 & 1/\ell \end{pmatrix}^* \cdot \ell^{-r} & \text{if } \ell \mid N
\end{cases}
\]
where \( T'(\ell) \) is the dual Hecke operator at \( \ell \) as defined in [Kat04, §4.9].

(2) At the level of \( H^1(\mathbb{Z}[\zeta_{p^n}, 1/p], j_*V_{k, \mathbb{Z}_p}(Y_1(N))(k - r)) \), we define
\[
E_\ell(Y_1(N), p^n, r) := 1 - T'(\ell) \cdot \sigma_\ell^{-1} \cdot \ell^{-r} + \Delta'(\ell) \cdot \sigma_\ell^{-2} \cdot \ell^{k-1-2r}
\]
where \( \sigma_\ell \) is the arithmetic Frobenius element in \( \text{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}) \), \( \Delta'(\ell) = \begin{pmatrix} \ell & 0 \\ 0 & 1/\ell \end{pmatrix}^* \) if \( \ell \nmid N \) and \( \Delta'(\ell) = 0 \) otherwise.
(3) At the level of $H^1(\mathbb{Z}[\zeta_{p^n},1/p], j_*T_f(k-r))$, we define
\[ E_{\ell}(f^*, p^n, r) := 1 - a_{\ell}(f) \cdot \sigma_{\ell}^{-1} \cdot \ell^{-r} + \overline{\psi}(\ell) \cdot \sigma_{\ell}^{-2} \cdot \ell^{k-1-2r}, \]
where $\sigma_{\ell}$ is the arithmetic Frobenius element in $\text{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$.

(4) At the level of Iwasawa cohomologies, we define
\[ E_{\ell}(f^*, r) := E_{\ell}(f^*, p^\infty, r) = 1 - a_{\ell}(f) \cdot \sigma_{\ell}^{-1} \cdot \ell^{-r} + \overline{\psi}(\ell) \cdot \sigma_{\ell}^{-2} \cdot \ell^{k-1-2r} \in \Lambda \]
and
\[ E_{\ell}(f^*, -i, r) := 1 - a_{\ell}(f) \cdot \omega^{-i} \left( \sigma_{\ell}^{-1} \right) \cdot \langle \sigma_{\ell}^{-1} \rangle \cdot \ell^{-r} + \overline{\psi}(\ell) \cdot \omega^{-i} \left( \sigma_{\ell}^{-2} \right) \cdot \langle \sigma_{\ell}^{-2} \rangle \cdot \ell^{k-1-2r} \in \Lambda_{-i} \]
where $\sigma_{\ell}$ is the arithmetic Frobenius element in $\text{Gal}(\mathbb{Q}(\zeta_{p^n})/\mathbb{Q})$ and $\langle - \rangle : \mathbb{Z}_p^\times \to 1+p\mathbb{Z}_p$ is the projection to 1-units.

3.2. Local cohomologies. Let
- $\ell$ be a prime different from $p$,
- $\eta$ a prime of $\mathbb{Q}_\infty$ lying above $\ell$,
- $s_\ell$ the number of primes of $\mathbb{Q}_\infty$ lying above $\ell$, and
- $\text{Gal}(\mathbb{Q}_{\infty, \eta}/\mathbb{Q}_\ell) \subseteq \text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ the decomposition subgroup at $\eta$.

Note that $s_\ell = [\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) : \text{Gal}(\mathbb{Q}_{\infty, \eta}/\mathbb{Q}_\ell)]$, which is independent of the choice of $\eta$.

Let $\sigma_\eta$ be the arithmetic Frobenius in $\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ at $\eta$ and then $\text{Gal}(\mathbb{Q}_{\infty, \eta}/\mathbb{Q}_\ell)$ is topologically generated by $\sigma_\eta$. Let $\text{Frob}_\ell \in \text{Gal}(\mathbb{Q}_\infty^\times/\mathbb{Q}_\ell)$ be such that $\sigma_\eta$ is the restriction of $\text{Frob}_\ell$ to $\mathbb{Q}_{\infty, \eta}$. For any Galois representation $V$ over $F_\pi$, we set

- $V^* = \text{Hom}_{F_\pi}(V, F_\pi(1))$,
- $(V)_{\ell}$ the maximal quotient of $V$ on which $I_\ell$ acts trivially, and
- $d_\ell = \dim_{F_\pi} H^0(\mathbb{Q}_{\infty, \eta}, V_{f^*, -i}(r)^*)$.

Let
\[ \left\{ \alpha_j \cdot \ell^{-\ell} \cdot \omega^{-i}(\text{Frob}_{\ell}^{-1}) \right\}_{j=1, \ldots, e_\ell} \]
be the eigenvalues of $\text{Frob}_{\ell}^{-1}$ acting on $(V_{f^*, -i}(r))_{\ell}$ where $e_\ell = \dim_{F_\pi} (V_{f^*, -i}(r))_{\ell}$. Considering
\[ H^0(\mathbb{Q}_{\infty, \eta}, V_{f^*, -i}(r)^*) \subseteq H^0(\ell, V_{f^*, -i}(r)^*) = ((V_{f^*, -i}(r))_{\ell})^* , \]
we know that $\text{Frob}_{\ell}^{-1}$ acts on the dual $((V_{f^*, -i}(r))_{\ell})^*$ by $(\alpha_j)^{-1} \cdot \ell^{r} \cdot \omega^{i}(\text{Frob}_{\ell}^{-1})$.

Since
\[ ((V_{f^*, -i}(r))_{\ell})^* = \text{Hom}((V_{f^*, -i}(r))_{\ell}, F_\pi(1)) \]
\[ = \text{Hom}(V_{f^*, -i}(r), F_\pi(1))_{\ell} \]
\[ = V_{f, i}(k-r)_{\ell} \]
and $\text{Frob}_{\ell}^{-1}$ acts on $V_{f, i}(k-r)_{\ell}$ by $(\alpha_j) \cdot \ell^{r} \cdot \omega^{i}(\text{Frob}_{\ell}^{-1})$, we have
\[ (\alpha_j)^{-1} \cdot \ell^{r} \cdot \omega^{i}(\text{Frob}_{\ell}^{-1}) = (\alpha_j) \cdot \ell^{r} \cdot \omega^{i}(\text{Frob}_{\ell}^{-1}). \]

Since the action of $\text{Gal}(\mathbb{Q}_\ell^\ur / \mathbb{Q}_{\infty, \eta})$ factors through a finite group of order prime to $p$, $d_\ell$ is the number of $j$'s such that $(\alpha_j)^{-1} \cdot \ell^{r} \cdot \omega^{i}(\text{Frob}_{\ell}^{-1})$ is a principal unit in $F_\pi(\alpha_j)$. These are the eigenvalues of $\sigma_\eta^{-1}$ acting on $H^0(\mathbb{Q}_{\infty, \eta}, V_{f^*, -i}(r)^*)$ counting multiplicities.

We interpret $d_\ell$ in terms of the polynomial
\[ P_\eta(\sigma_\eta^{-1}) = \prod_{j=1}^{e_\ell} (1 - \alpha_j \cdot \ell^{-r} \cdot \omega^{-i}(\text{Frob}_{\ell}^{-1}) \cdot \sigma_\eta^{-1}) \in \mathcal{O}_\pi[\text{Gal}(\mathbb{Q}_{\infty, \eta}/\mathbb{Q}_\ell)] \subseteq \mathcal{O}_\pi[\text{Gal}(\mathbb{Q}_\infty/\mathbb{Q})] = \Lambda_{-i}. \]

Then the degree of $P_\eta$ is $d_\ell$ and $P_\eta := \prod_{\eta \mid \ell} P_\eta$ has degree $s_\ell \cdot d_\ell$. Moreover, we have
\[ E_{\ell}(f^*, -i, r) = \prod_{\eta \mid \ell} P_\eta(\sigma_\eta^{-1}) \in \Lambda_{-i}. \]
Thus, we have the following statement.

**Proposition 3.2.** The ideal \( \chi_{\Lambda_i} \left( \bigoplus_{\eta | \ell} H^0(Q_{\infty, \eta}, \text{Hom}(T_{f^*}, (F_\pi / O_\pi)(1))) \right) \) is generated by \( \mathcal{E}_\ell(f^*, -i, r) \) over \( \Lambda_{-i} \) where \((-)^\vee\) means the Pontryagin dual.

**Remark 3.3.** Notably, \( H^0(Q_{\infty, \eta}, \text{Hom}(T_{f^*}, (F_\pi / O_\pi)(1))) \) is a torsion \( \Lambda_{-i} \)-module with \( \mu = 0 \) and \( \lambda = s_\ell \cdot d_\ell \).

**Definition 3.4.** The involution map \( \iota : \Lambda \to \Lambda \) is defined by \( \gamma \mapsto \gamma^{-1} \) where \( \gamma \in \text{Gal}(Q(\zeta_{p_\infty}) / Q) \).

For a \( \Lambda \)-module \( M \), \( M^\iota \) is defined by the same underlying module with the inverse \( \Lambda \)-action via \( \iota \). Especially, if \( M \) is a \( \Lambda_i \)-module, then \( M^\iota \) is a \( \Lambda_{-i} \)-module. Denote by \( f^i \in \Lambda_i \) the image of \( f \in \Lambda_{-i} \) under \( \iota \).

Furthermore, by (3.1), we have
\[
\prod_{j=1}^{\ell} (1 - \alpha_j \cdot \ell^{-r} \cdot \omega^{-i} \cdot \text{Frob}^{-1} \cdot \sigma_{\eta}^{-1}) = \prod_{j=1}^{\ell} (1 - \beta_j \cdot \ell^{-r} \cdot \omega^{-i} \cdot \text{Frob}^{-1} \cdot \sigma_{\eta}^{-1}) \in \Lambda_{-i}.
\]

Since
\[
\mathcal{E}_\ell(f, i, k - r) = 1 - a_\ell(f) \cdot \omega^{i} \cdot \sigma_\ell^{-1} \cdot \ell^{-r} \cdot \psi(\ell) \cdot \omega^{i} \cdot \sigma_\ell^{-2} \cdot \ell^{-r} \cdot \chi^{-1} \cdot \omega^{-i} \in \Lambda_i
\]
we also have
\[
\mathcal{E}_\ell(f, i, k - r) = \left( \prod_{\eta | \ell} \mathcal{P}_{\eta} (\sigma_{\eta}^{-1}) \right)^{\iota} \in \Lambda_i.
\]

By [Gre89, Proposition 2], it is known that
\[
H^0(Q_{\infty, \eta}, \text{Hom}(T_{f^*}, (F_\pi / O_\pi)(1))) \simeq H^1(Q_{\infty, \eta}, A_{f^*}, (F_\pi / O_\pi)(1)) \simeq H^2(Q_{\infty, \eta}, T_{f^*}, (F_\pi / O_\pi)(1))
\]
where \( \sim \) means a pseudo-isomorphism over \( \Lambda_{-i} \). Here, \( \iota \) plays the role in the interpolation formula for \( \chi^{-1} \cdot \omega^{-i} \), not for \( \chi \cdot \omega^{-i} \).

**Remark 3.5.** Proposition 3.2 and (3.3) recover [GV00, Proposition 2.4] when \( f \) corresponds to an elliptic curve, \( k = 2, r = 1 \), and \( i = 0 \). In this case, we have \( a_\ell(f) = a_\ell(f) = \text{tr}(\rho_1(F_\mu^{-1})) = \text{tr}(\rho_2(F_\mu)) \) and \( \rho_1(1) \) is isomorphic to the Tate module. Thus, the content of this subsection is a very slight extension of [GV00, Proposition 2.4].

The local Tate duality (e.g. [Rub00, Theorem 1.4.1]) implies that
\[
H^0(Q_{\infty, \eta}, \text{Hom}(T_{f^*}, (F_\pi / O_\pi)(1))) \simeq H^2(Q_{\infty, \eta}, T_{f^*}, (F_\pi / O_\pi)(1))
\]
for all \( n \geq 1 \) where \( \eta_n \) is the prime of \( Q_n \) lying below \( \eta \). By taking the inverse limit of (3.4) with respect to \( n \), we have
\[
\chi_{\Lambda_{-i}}^{-1} H^0(Q_{\infty, \eta}, \text{Hom}(T_{f^*}, (F_\pi / O_\pi)(1))) \simeq \chi_{\Lambda_{-i}}^{-1} H^2(Q_{\infty, \eta}, T_{f^*}, (F_\pi / O_\pi)(1))
\]
and (3.5). By (3.3) and (3.5), we have
\[
\chi_{\Lambda_{-i}}^{-1} H^1(Q_{\infty, \eta}, A_{f^*}, (F_\pi / O_\pi)(1)) = \chi_{\Lambda_{-i}}^{-1} H^2(Q_{\infty, \eta}, T_{f^*}, (F_\pi / O_\pi)(1))
\]
Thus, we have the following statement.

**Corollary 3.6.** The ideal \( \chi_{\Lambda_{-i}} \left( \bigoplus_{\ell \in \ell} H^2(Q_{\infty, \eta}, T_{f^*}, (F_\pi / O_\pi)(1)) \right) \) is generated by \( \mathcal{E}_\ell(f^*, -i, r) \) over \( \Lambda_{-i} \) with \( \mu = 0 \) and \( \lambda = s_\ell \cdot d_\ell \).

Applying the same argument to the dual representation with (3.2), we have the following statement.

**Corollary 3.7.** The ideal \( \chi_{\Lambda_i} \left( \bigoplus_{\ell \in \ell} H^1(Q_{\infty, \eta}, T_{f}, (k - r)) \right) \) is generated by \( \mathcal{E}_\ell(f, i, k - r) = \mathcal{E}_\ell(f^*, -i, r) \) over \( \Lambda_i \) with \( \mu = 0 \) and \( \lambda = s_\ell \cdot d_\ell \).
Remark 3.8. Indeed, Corollary 3.7 can be observed in terms of the functional equation. For a finite order character $\chi : \text{Gal}(\mathbb{Q}(\zeta_p^\infty)/\mathbb{Q}) \to \mathbb{C}^\times$, we have the functional equation

$$\Gamma_C(s) \cdot L(f, \chi, s) = \varepsilon(f, \chi, s) \cdot \Gamma_C(k - s) \cdot L(f^*, \chi^{-1}, k - s)$$

where $\Gamma_C(s) := \frac{\Gamma(s)}{(2\pi)^s}$, $\varepsilon(f, \chi, s)$ is the global epsilon factor associated to $\pi_f \otimes (\chi \circ \det)$, and $\pi_f$ is the cuspidal automorphic representation associated to $f$. See [Nak17, (40)] for detail. Considering Euler factors of both $L$-functions in (3.6) at $s = k - r$, one can expect that Corollary 3.7 should hold.

4. The zeta element side

Let $S$ be a finite set of places of $\mathbb{Q}$ containing the primes dividing $pN$ and the infinite place and $\mathbb{Q}_S$ the maximal extension of $\mathbb{Q}$ unramified outside $S$. Let $S' = S \setminus \{p, \infty\}$. We identify $V_{k, \mathbb{Z}_p}(Y_1(N)) := H^1_{\text{et}}(Y_1(N)_{\overline{\mathbb{Q}}}, \text{Sym}^{k-2}(H^1_p(L))) \cong H^1(Y_1(N)(\mathbb{C}), \text{Sym}^{k-2}(\mathbb{Z}_p^2))$.

The complex conjugation acts on the latter and it also induces the action on $T_f$. Denote by $T^\pm_f$ the part on which the complex conjugation acts by $\pm 1$, respectively. The same rule applies to $V_f$.

We denote by $\mathbb{T}(N)$ be the image of the abstract Hecke algebra generated by Hecke operators at all primes in the endomorphism ring of $V_{k, \mathbb{Z}_p}(Y_1(N))$ over $\mathbb{Z}_p$. For a maximal ideal $\mathfrak{m} \subseteq \mathbb{T}(N)$, write $\mathbb{T}(N)_{\mathfrak{m}}$ to be the localization of $\mathbb{T}(N)$ at $\mathfrak{m}$. Denote by $M_{\mathfrak{m}}$ the localization of $\mathbb{T}(N)$-module $M$ at $\mathfrak{m}$.

4.1. $\Sigma_0$-imprimitive zeta elements. We pin down $\Sigma_0 = S'$. Indeed, we have

$$z_{\text{Kato}}(f, i, k - r) = \left(z_{\gamma_f}^{(p)} \otimes (\zeta_p)^{k-r} \right)^{\omega_f^i} \in \mathbb{H}^1(j^*_fT_{f,i}(k - r))$$

where $\gamma_f = \gamma_f^+ + \gamma_f^-$ in $T_f$ satisfies $\gamma_f^\pm \in T_f^\pm$ and $T_f = \mathcal{O}_\pi \gamma_f^+ + \mathcal{O}_\pi \gamma_f^-$. This is an explicit description of $z_{\text{Kato}}(f, i, k - r)$ (cf. Definition 1.7).

We define the $\Sigma_0$-imprimitive zeta element of $z_{\text{Kato}}(f, i, k - r)$ by

$$z_{\text{Kato}}^{\Sigma_0}(f, i, k - r) := \left(\prod_{\ell \in \Sigma_0} \mathcal{E}_\ell(f^*, -i, r)^{\iota}\right) \cdot z_{\text{Kato}}(f, i, k - r)$$

where $\mathcal{E}_\ell(f^*, -i, r)^{\iota}$ is the image of $\mathcal{E}_\ell(f^*, -i, r)$ under $\iota : \Lambda_{-i} \to \Lambda_i$ in Definition 3.4. For notational convenience, we write

$$\mu(z) = \mu(\mathbb{H}^1(j^*_fT_{f,i}(k - r))/\mathfrak{z})$$

for $z \in \mathbb{H}^1(j^*_fT_{f,i}(k - r))$. Then we have

$$\mu(z_{\text{Kato}}^{\Sigma_0}(f, i, k - r)) = \mu(z_{\text{Kato}}(f, i, k - r)),$$

$$\lambda(z_{\text{Kato}}^{\Sigma_0}(f, i, k - r)) = \lambda(z_{\text{Kato}}(f, i, k - r)) + \sum_{\ell \in \Sigma_0} \lambda(\bigoplus_{\eta \ell} \mathbb{H}^2_{\text{Iw}}(\mathbb{Q}_{\infty, \eta}, T_{f,i}(k - r)))$$

by Corollary 3.7.
4.2. Mod \( p \) multiplicity one. We recall the mod \( p \) multiplicity one result following Mazur, Wiles, Ribet, Edixhoven, and Faltings–Jordan. The following form of the mod \( p \) multiplicity one is due to Faltings–Jordan [FJ95, Theorem 2.1].

**Theorem 4.1** (Faltings–Jordan). Suppose that \( \mathfrak{m} \) is a non-Eisenstein maximal ideal of \( \mathbb{T}(N) \). If \( (N,p) = 1 \) and \( 2 \leq k \leq p - 1 \), then

1. \( V_{k,\mathbb{F}_p}(Y_1(N))|\mathfrak{m} \) is 2-dimensional over \( \mathbb{F} \).
2. \( \mathbb{T}(N)_{\mathfrak{m}} \) is Gorenstein.

**Corollary 4.2.** Under the same assumptions in Theorem 4.1, we have

\[
V_{k,\mathbb{Z}_p}(Y_1(N)) \otimes_{\mathbb{T}(N)} \mathbb{T}(N)/\mathfrak{m} \simeq \mathcal{P},
\]

where \( \mathcal{P} \) is the residual Galois representation corresponding to \( \mathfrak{m} \).

**Remark 4.3.** Note that the conventions of the Galois representations in §1.2.1 and in [FJ95, §2] are dual to each other.

**Proof.** Due to the argument given in [FJ95, §2.1] and the existence of integral perfect paring on \( V_{k,\mathbb{Z}_p}(Y_1(N))|\mathfrak{m} \) as in [DFG04, Corollary 1.6], the subquotients of a Galois-stable filtration of \( V_{k,\mathbb{Z}_p}(Y_1(N))|\mathfrak{m} \) \( \otimes_{\mathbb{T}(N)} \mathbb{T}(N)/\mathfrak{m} \) and the subquotients of a Galois-stable filtration of \( \text{Hom}(V_{k,\mathbb{F}_p}(Y_1(N))|\mathfrak{m}, \mathbb{F}_p(1-k)) \) coincide up to order. Because \( \mathcal{P} \) is irreducible, we have

\[
V_{k,\mathbb{Z}_p}(Y_1(N))|\mathfrak{m} \otimes_{\mathbb{T}(N)} \mathbb{T}(N)/\mathfrak{m} \simeq \text{Hom}(V_{k,\mathbb{F}_p}(Y_1(N))|\mathfrak{m}, \mathbb{F}_p(1-k)).
\]

Thus, we know

\[
\dim_{\mathbb{F}} \left( V_{k,\mathbb{F}_p}(Y_1(N))|\mathfrak{m} \right) = \dim_{\mathbb{F}} \left( V_{k,\mathbb{Z}_p}(Y_1(N)) \otimes_{\mathbb{T}(N)} \mathbb{T}(N)/\mathfrak{m} \right) = 2.
\]

By the construction in §1.2.1, it is isomorphic to \( \mathcal{P} \). \( \square \)

4.3. Congruences between zeta elements: the same level. Let \( f = \sum_{n \geq 1} a_n(f)q^n \in S_k(\Gamma_1(N_f)) \) and \( g = \sum_{n \geq 1} a_n(g)q^n \in S_k(\Gamma_1(N_g)) \) be newforms of weight \( k \geq 2 \). Assume that \( f \) and \( g \) are congruent modulo \( \pi \) and their isomorphic residual representation is irreducible. Let \( \Sigma_0 \) be the set of finite places dividing \( N_f \) and \( N_g \). We put

\[
N^{\Sigma_0} = N(\mathcal{P}) \cdot \prod_{\ell} \ell \cdot \prod_q q^2
\]

where \( \ell \) runs over the primes in \( \Sigma_0 \) exactly dividing \( N(\mathcal{P}) \) once and \( q \) runs over the primes in \( \Sigma_0 \) not dividing \( N(\mathcal{P}) \).

Then

\[
f^{\Sigma_0} \equiv \sum_{n \geq 1} a_n(f^{\Sigma_0})q^n : \sum_{(n,N^{\Sigma_0})=1} a_n(f)q^n, \quad g^{\Sigma_0} \equiv \sum_{n \geq 1} a_n(g^{\Sigma_0})q^n : \sum_{(n,N^{\Sigma_0})=1} a_n(g)q^n
\]

are \( \Sigma_0 \)-imprimitive eigenforms in \( S_k(\Gamma_1(N^{\Sigma_0})) \).

Let \( \mathfrak{m}^{\Sigma_0} \subseteq \mathbb{T}(N^{\Sigma_0}) \) be the non-Eisenstein maximal ideal containing \( \varphi f^{\Sigma_0} \) and \( \varphi g^{\Sigma_0} \). Following [Vat99, §1.3], we associate the cocycles \( \delta^{\Sigma_0}_{f^{\Sigma_0}} \) and \( \delta^{\Sigma_0}_{g^{\Sigma_0}} \) to \( f^{\Sigma_0} \) and \( g^{\Sigma_0} \), respectively, in group cohomology \( H^1(\Gamma_1(N^{\Sigma_0})), \text{Sym}^{k-2}(\mathcal{O}_\pi^2))_{\mathfrak{m}^{\Sigma_0}} \) and these cocycles yield the canonical periods.

**Theorem 4.4.** If \( \mathfrak{m}^{\Sigma_0} \) is non-Eisenstein, \( (N^{\Sigma_0},p) = 1 \), and \( 2 \leq k \leq p - 1 \), then

\[
H^1(\Gamma_1(N^{\Sigma_0})), \text{Sym}^{k-2}(\mathcal{O}_\pi^2))_{\mathfrak{m}^{\Sigma_0}} \simeq H^1(Y_1(N^{\Sigma_0}))(\mathbb{C}), \text{Sym}^{k-2}(\mathcal{O}_\pi^2))_{\mathfrak{m}^{\Sigma_0}} \\
\simeq \text{Hom}_{\mathcal{O}_\pi} \left( \mathbb{T}(N^{\Sigma_0})_{\mathfrak{m}^{\Sigma_0}}, \mathcal{O}_\pi \right) \\
\simeq \mathbb{T}(N^{\Sigma_0})_{\mathfrak{m}^{\Sigma_0}}.
\]

**Proof.** It follows from Theorem 4.1. See [Vat99, Theorem 1.13]. \( \square \)
Considering Theorem 4.4, we have the following identifications and the quotient map
\[ H^1(\Gamma_1(N^{\Sigma_0}), \text{Sym}^{k-2}(\mathcal{O}_\pi^2))^\pm_{m^{\Sigma_0}_0} \simeq H^1(Y_1(N^{\Sigma_0})(\mathbb{C}), \text{Sym}^{k-2}(\mathcal{O}_\pi^2))^\pm_{m^{\Sigma_0}_0} \]
\[ \simeq V_{k,z}(Y_1(N^{\Sigma_0}))^\pm_{m^{\Sigma_0}_0} \to V_{k,z}(Y_1(N^{\Sigma_0}))^\pm_{m^{\Sigma_0}_0} \otimes \mathcal{T}(N^{\Sigma_0})_{m^{\Sigma_0}_0}/\mathcal{O}_{j^{\Sigma_0}} \]
\[ = T^\pm_{f^{\Sigma_0}}. \]
By Chebotarev density theorem, \( T_f \simeq T_{f^{\Sigma_0}} \) as Galois representations. We denote by the same notation the image of \( \delta^\pm_{f^{\Sigma_0}} \) in \( T^\pm_{f^{\Sigma_0}} \). The same rule applies to \( g^{\Sigma_0} \).

Since \( f^{\Sigma_0} \equiv g^{\Sigma_0} \pmod{\pi} \), we have congruences between \( \delta^\pm_{f^{\Sigma_0}} \in T^\pm_{f^{\Sigma_0}} \) and \( \delta^\pm_{g^{\Sigma_0}} \in T^\pm_{g^{\Sigma_0}} \) in the residual representation as in [Vat99, (4), Page 402], i.e.
\[ (4.3) \quad \delta^\pm_{f^{\Sigma_0}} \equiv u^\pm \cdot \delta^\pm_{g^{\Sigma_0}} \pmod{\pi} \]
where \( u^\pm \in \mathcal{O}_\pi^\times \) in the residual representation following the isomorphisms
\[ \mathcal{P}^\pm \simeq V_{k,z}(Y_1(N^{\Sigma_0}))^\pm_{m^{\Sigma_0}_0} \otimes \mathcal{T}(N^{\Sigma_0})_{m^{\Sigma_0}_0}/\mathcal{O}_{\pi}/\pi \]
\[ \simeq V_{k,z}(Y_1(N^{\Sigma_0}))^\pm_{m^{\Sigma_0}_0} \otimes \mathcal{T}(N^{\Sigma_0})_{m^{\Sigma_0}_0}/\mathcal{O}_{g^{\Sigma_0}} \otimes \mathcal{O}_{\pi}/\pi \]
which follow from Corollary 4.2.

**Remark 4.5.** Under the assumptions in Theorem 4.4, \( \delta^\pm_{f^{\Sigma_0}} \) generates \( T^\pm_{f^{\Sigma_0}} \), respectively. In other words, \( \delta^\pm_{f^{\Sigma_0}} \) can play the role of \( \gamma^\pm_{f^{\Sigma_0}} \) in (4.1). It is also equivalent to that \( \delta^\pm_{f^{\Sigma_0}} \) does not vanish modulo \( \pi \). See Remark [Vat99, Remark 1.12] for the mod \( \pi \) non-vanishing of \( \delta^\pm_{f^{\Sigma_0}} \) when \( f^{\Sigma_0} \) is a weight two modular form or \( p \)-ordinary and \( p \)-distinguished. The non-ordinary Fontaine–Laffaille weight case easily reduces to the weight two case by using the weight two specialization argument in [PW11a, §4.4].

We briefly review how to construct zeta elements associated to \( \delta_{f^{\Sigma_0}} = \delta^+_{f^{\Sigma_0}} + \delta^-_{f^{\Sigma_0}} \in T_{f^{\Sigma_0}} \).

Following the explicit construction in [Kat04, §5.5], we have the cocycle
\[ \delta_{1,N^{\Sigma_0}}(k,j,\alpha)^\pm \in V_{k,Q}(Y_1(N^{\Sigma_0}))^\pm := H^1(Y_1(N^{\Sigma_0})(\mathbb{C}), \text{Sym}^{k-2}(\mathbb{Q}^2))^\pm. \]
Write \( \delta(f^{\Sigma_0},j,\alpha)^\pm \in V_{f^{\Sigma_0}}^\pm \) to be the image of \( \delta_{1,N^{\Sigma_0}}(k,j,\alpha)^\pm \) in \( V_{f^{\Sigma_0}}^\pm \) as in [Kat04, §6.3]. Following [Kat04, §13.9], we are able to write
\[ \delta_{f^{\Sigma_0}} = \delta^+_{f^{\Sigma_0}} + \delta^-_{f^{\Sigma_0}} \]
\[ = b_1(f^{\Sigma_0}) \cdot \delta(f^{\Sigma_0},j_1,\alpha_1)^+ + b_2(f^{\Sigma_0}) \cdot \delta(f^{\Sigma_0},j_2,\alpha_2)^-, \]
\[ \delta_{g^{\Sigma_0}} = \delta^+_{g^{\Sigma_0}} + \delta^-_{g^{\Sigma_0}} \]
\[ = b_1(g^{\Sigma_0}) \cdot \delta(g^{\Sigma_0},j_1,\alpha_1)^+ + b_2(g^{\Sigma_0}) \cdot \delta(g^{\Sigma_0},j_2,\alpha_2)^- \]
in \( T_{f^{\Sigma_0}} \) and \( T_{g^{\Sigma_0}} \), respectively, where \( b_1(f^{\Sigma_0}), b_2(f^{\Sigma_0}), b_1(g^{\Sigma_0}), \) and \( b_2(g^{\Sigma_0}) \) lie in \( F_\pi \). Here, \( \alpha_i \in SL_2(\mathbb{Z}) \) and \( j_i \) is an integer such that \( 1 \leq j_i \leq k - 1 \) (\( i = 1, 2 \)). We use the same \( j_i \) and \( \alpha_i \) (\( i = 1, 2 \)) for both \( f^{\Sigma_0} \) and \( g^{\Sigma_0} \) in (4.4). See also [Kat04, Theorem 13.6] for the integrality of \( \delta(f^{\Sigma_0},j_i,\alpha_i)^\pm \) and \( \delta(g^{\Sigma_0},j_i,\alpha_i)^\pm \) for \( i = 1, 2 \).

As in [Kat04, §13.9], the zeta element \( z_{\text{Kato}}(f^{\Sigma_0}, k - r) \) associated to \( \delta_{f^{\Sigma_0}} = b_1(f^{\Sigma_0}) \cdot \delta(f^{\Sigma_0},j_1,\alpha_1)^+ + b_2(f^{\Sigma_0}) \cdot \delta(f^{\Sigma_0},j_2,\alpha_2)^- \) is defined by
\[ z_{\text{Kato}}(f^{\Sigma_0}, k - r) = \left( \mu(c,d,j_1)^{-1} \cdot b_1(f^{\Sigma_0}) \cdot \left( c, d, z_p^{(p)}(f^{\Sigma_0}, j_1, \alpha_1, pN^{\Sigma_0}) \right)_{n \geq 1} \right)^+ \]
\[ + \left( \mu(c,d,j_2)^{-1} \cdot b_2(f^{\Sigma_0}) \cdot \left( c, d, z_p^{(p)}(f^{\Sigma_0}, j_2, \alpha_2, pN^{\Sigma_0}) \right)_{n \geq 1} \right)^+ \otimes (\zeta^{(p)})_{n \geq 1}^{k-r} \]
where $\mu(c, d, j) = \left( c^2 - c^{k+1-j} \cdot \sigma_c \right) \cdot (d^2 - d^{j+1} \cdot \sigma_d) \in \Lambda$ and \( (c, d, z^{(p)}_{\Sigma_0}, k, j, \alpha, p N^{\Sigma_0}) \) is the integral zeta element appeared in [Kat04, (8.1.3)] with $c, d, k, j \in \mathbb{Z}$ and $\alpha \in \text{SL}_2(\mathbb{Z})$ satisfying certain conditions in [Kat04, §5]. Of course, $z_{K\text{ato}}(g_{\Sigma_0}, k - r)$ is defined in the exactly same way. Since $\mu(c, d, j_1)^{-1}$ and $\mu(c, d, j_2)^{-1}$ are independent of $f^{\Sigma_0}$ and $g^{\Sigma_0}$, we choose the same for both $f^{\Sigma_0}$ and $g^{\Sigma_0}$. Then $z_{K\text{ato}}(f^{\Sigma_0}, k - r) \in H^1(j_s T_{f^{\Sigma_0}}(k - r)) \otimes Q(\Lambda)$ \textit{a priori} where $Q(\Lambda)$ is the total quotient ring of $\Lambda$.

Following [Kat04, Theorem 12.5], under Assumption 1.2, we have

$$z_{K\text{ato}}(f^{\Sigma_0}, k - r) \in H^1(j_s T f^{\Sigma_0}(k - r))$$

and the assignment

$$\delta_{f^{\Sigma_0}} \mapsto z_{K\text{ato}}(f^{\Sigma_0}, k - r) \mapsto z_{K\text{ato}}(f^{\Sigma_0}, i, k - r)$$

forms a homomorphism

$$T f^{\Sigma_0} \rightarrow H^1(j_s T f^{\Sigma_0}(k - r)) \rightarrow H^1(j_s T f^{\Sigma_0,i}(k - r))$$

where the second map is the projection to the $\omega^i$-isotypic component. Considering the mod $\pi$ reduction of the above map, we have the following commutative diagram

$$\begin{array}{ccc}
T f^{\Sigma_0} & \rightarrow & H^1(j_s T f^{\Sigma_0}(k - r)) \\
\downarrow & & \downarrow \\
H^1_{Iw}(Q \Sigma / Q(\zeta_p), T f^{\Sigma_0}(k - r)) & \rightarrow & H^1_{Iw}(Q \Sigma / Q(\zeta_p), T f^{\Sigma_0,i}(k - r)) \\
\overline{\rho} \simeq T f^{\Sigma_0} / \pi & \rightarrow & H^1_{Iw}(Q \Sigma / Q(\zeta_p), T f^{\Sigma_0}(k - r))/\pi \\
\downarrow & & \downarrow \\
H^1_{Iw}(Q \Sigma / Q(\zeta_p), \overline{\rho} f^{\Sigma_0}(k - r)) & \rightarrow & H^1_{Iw}(Q \Sigma / Q(\zeta_p), \overline{\rho} f^{\Sigma_0,i}(k - r))
\end{array}$$

where $\Sigma = \Sigma_0 \cup \{ p, \infty \}$.

**Proposition 4.6.** In the above diagram, the map in the bottom

$$\overline{\rho} \rightarrow H^1_{Iw}(Q \Sigma / Q(\zeta_p), \overline{\rho}(k - r)) \rightarrow H^1_{Iw}(Q \Sigma / Q(\zeta_p), \overline{\rho}_{f,i}(k - r))$$

is independent of the choice of $f^{\Sigma_0}$ in the set of congruent eigenforms of fixed level $N^{\Sigma_0}$.

**Proof.** For notational convenience, we use $f$ and $N$ instead of $f^{\Sigma_0}$ and $N^{\Sigma_0}$ in this proof. We recall the construction of families of Kato’s zeta elements. Let $T_m = H^1(X_1(N), \text{Sym}^{k-2}(\mathcal{O}_\pi))_m$ be the Galois module free of rank two over the Hecke algebra $T(N)_m$. Let $V_m = T_m \otimes Q_p$. Let $\delta_{1,N}(k, r, \xi)_m \in H^1(X_1(N), \text{Sym}^{k-2}(\mathbb{Q}))_m$ be the $m$-component of $\delta_{1,N}(k, r, \xi)$ where $\xi \in \text{SL}_2(\mathbb{Z})$ [Kat04, §3.5]. By using the idea of [Kat04, Lemma 13.10.(2)] (cf. [FK, §3]), we define the assignment

$$\delta_{1,N}(k, r, \xi)_m \mapsto \left( \left( c^2 - c^{k+1-r} \cdot \sigma_c \right) \cdot (d^2 - d^{j+1} \cdot \sigma_d) \cdot \prod_{\ell \mid N} \left( 1 - T'(\ell) \cdot \ell^{-k} \cdot \sigma_\ell^{-1} \right) \right)^{-1} \cdot c, d, z^{(p)}_{1,N,p}(k, k, r, \xi, p N)_m \otimes (\zeta_p^n)^{\otimes (k-r)}_n$$

and its $T(N)_m$-linear extension yields the map parametrizing the families of Kato’s zeta elements

$$z_{K\text{ato}} \otimes (\zeta_p^n)^{\otimes (k-r)} : V_m \rightarrow H^1(j_s V_m(k - r)) \otimes_{\Lambda \otimes T(N)_m} Q(\Lambda \otimes T(N)_m).$$
Here, \( c_{d,z_1,N,p}\) is the \( m \)-component of \( c_{d,z_1,N,p}\) [Kat04, §8.1.2]. This map is compatible with the specializations to eigenforms whose residual representation is isomorphic to \( p \) due to the combination of the Hecke algebra version of the interpolation formula of zeta modular forms [Kat04, Theorem 5.6.2] and the connection between zeta elements and zeta modular forms via the dual exponential map [Kat04, Theorem 9.7]. Since the \( \xi = a(A) \) case can be done similarly, we omit it. As a result, we have the following commutative diagram

\[
\begin{array}{ccc}
T_m & \rightarrow & V_m \\
\downarrow & & \downarrow \\
\otimes T(N)_m/\varphi_j & \rightarrow & H^1(j_*V_m(k-r)) \otimes_{\Lambda} T(N)_m Q(\Lambda) \otimes T(N)_m \\
T_f & \rightarrow & H^1(j_*T_f(k-r)) \\
\downarrow & & \downarrow \\
\varphi & \rightarrow & H^1(j_*\varphi(k-r)) \\
\end{array}
\]

In the diagram, note that \( V_m \) maps to \( H^1(j_*T_f(k-r)) \subseteq H^1(j_*V_f(k-r)) \otimes_{\Lambda} Q(\Lambda) \) since it factors through \( V_f \) and [Kat04, Theorem 12.5.(1)]. Also, under the large image assumption, \( T_m \) maps to \( H^1(j_*T_f(k-r)) \) since it factors through \( T_f \) and [Kat04, Theorem 12.5.(4)]. Thus, the map \( T_m \rightarrow H^1(j_*\varphi(k-r)) \) is well-defined and even independent of the choice of eigenform \( f \) since it factors through \( \varphi = T_m/mT_m \).

Due to the proposition above, the congruence between cocycles

\[
\delta_{f,\varphi} \equiv u^\pm \cdot \delta_{f,\varphi} \pmod{\pi}
\]

in \( \varphi \) yields the congruence between zeta elements

\[
zh_{Kato}(f_{\Sigma_0}, i, k-r) = \varpi^\pm \cdot zh_{Kato}(g_{\Sigma_0}, i, k-r)
\]

in \( H_{1w}(\mathbb{Q}/\mathbb{Q}, \varphi_{f}(k-r)) \) where \( \varpi^\pm \) is the mod \( \pi \) reduction of \( u^\pm \in \mathcal{O}_p^* \) and its sign coincides with that of \((-1)^i\).

4.4. Congruences between zeta elements: the different levels. We discuss the congruence between two eigenforms of different levels. Let \( m \) be the square-free product of the primes in \( \Sigma_0 \). Consider the following maps

\[
\begin{array}{ccc}
T_f & \rightarrow & T_{f,\Sigma_0} \\
\downarrow & & \downarrow \\
H_{1w}(\mathbb{Q}/\mathbb{Q}, T_f(k-r)) & \rightarrow & H_{1w}(\mathbb{Q}/\mathbb{Q}, T_{f,\Sigma_0}(k-r)) \\
\downarrow_{\text{Tr}} & & \downarrow_{\text{Tr}} \\
H_{1w}(\mathbb{Q}/\mathbb{Q}, T_f(k-r)) & \rightarrow & H_{1w}(\mathbb{Q}/\mathbb{Q}, T_{f,\Sigma_0}(k-r)) \\
\downarrow_{\text{mod } \pi} & & \downarrow_{\text{mod } \pi} \\
H_{1w}(\mathbb{Q}/\mathbb{Q}, \varphi_{f}(k-r)) & \rightarrow & H_{1w}(\mathbb{Q}/\mathbb{Q}, \varphi_{f,\Sigma_0}(k-r))
\end{array}
\]

where \( \text{Tr} \) is the trace map induced from the extension \( \mathbb{Q}(\zeta_m)/\mathbb{Q} \). Let

\[
zh_{Kato,\mathbb{Q}(\zeta_m)}(f, k-r) \in H_{1w}(\mathbb{Q}/\mathbb{Q}, T_f(k-r)),
zh_{Kato,\mathbb{Q}(\zeta_m)}(f^{\Sigma_0}, k-r) \in H_{1w}(\mathbb{Q}/\mathbb{Q}, T_{f,\Sigma_0}(k-r))
\]
be Kato’s zeta elements over $\mathbb{Q}(\zeta_{mp^\infty})$ associated to $\delta_f$ and $\delta_{f^{\Sigma_0}}$, respectively. By [Kat04, Proposition 8.12], we have

$$\text{Tr} \left( z_{\text{Kato}, \mathbb{Q}(\zeta_m)}(f, k-r) \right) = \left( \prod_{t \in \Sigma_0} E_t(f, k-r) \right) \cdot z_{\text{Kato}}(f, k-r)$$

$$(4.7)$$

$$\text{Tr} \left( z_{\text{Kato}, \mathbb{Q}(\zeta_m)}(f^{\Sigma_0}, k-r) \right) = z_{\text{Kato}}(f^{\Sigma_0}, k-r),$$

In order to have a non-vacuous congruence between $z_{\text{Kato}}^{\Sigma_0}(f, k-r)$ and $z_{\text{Kato}}(f^{\Sigma_0}, k-r)$, we need Ihara’s lemma. We follow the convention of [DFG04, §1.7.3].

For positive integers $m$ dividing $N^{\Sigma_0}/N$, we define the morphism

$$\epsilon_m : S_k(\Gamma_1(N), \psi) \to S_k(\Gamma_1(N^{\Sigma_0}), \psi)$$

defined by the double coset operator

$$m^{-1} \left[ U_0(N^{\Sigma_0}) \begin{pmatrix} m^{-1} & 0 \\ 0 & 1 \end{pmatrix} U_0(N) \right]$$

where $U_0(N)$ is the subgroup of the adelic points of $GL_2$ corresponding to the $\Gamma_0(N)$-level structure.

Let $\phi_m$ be the endomorphism of $S_k(\Gamma_1(N), \psi)$ defined by $\phi_1 = 1$, $\phi_\ell = -T_\ell$, $\phi_{p^2} = \psi(p)p^{k-1}$, and $\phi_{m_1 \cdot m_2} = \phi_{m_1} \cdot \phi_{m_2}$ if $(m_1, m_2) = 1$ where $\ell$ is a prime dividing $N^{\Sigma_0}/N$.

We define

$$\gamma^{\Sigma_0} = \sum_m \epsilon_m \cdot \phi_m : S_k(\Gamma_1(N), \psi) \to S_k(\Gamma_1(N^{\Sigma_0}), \psi)$$

where $m$ runs over the divisors of $N^{\Sigma_0}/N$. Then we have $\gamma^{\Sigma_0}(f) = f^{\Sigma_0}$ as in [DFG04, Proposition 1.4(a)].

**Theorem 4.7** (Ihara’s lemma). Assume that $2 \leq k \leq p-1$ and $p$ does not divide $N$. If $\mathfrak{p}$ is irreducible, then the map $\gamma^{\Sigma_0}$ induces an isomorphism

$$V_{k, \mathbb{Z}_p}(Y_1(N))_m[\mathfrak{p}f] \xrightarrow{\sim} V_{k, \mathbb{Z}_p}(Y_1(N^{\Sigma_0}))_{m^{\Sigma_0}}[\mathfrak{p}f^{\Sigma_0}]$$

where $m$ and $m^{\Sigma_0}$ are maximal ideals of $\mathcal{T}(N)$ and $\mathcal{T}(N^{\Sigma_0})$ corresponding to $\mathfrak{p}$, respectively, and $\mathfrak{p}f$ and $\mathfrak{p}f^{\Sigma_0}$ are the height one prime ideals of $\mathcal{T}(N)$ and $\mathcal{T}(N^{\Sigma_0})$ corresponding to $f$ and $f^{\Sigma_0}$, respectively.

**Proof.** See [DFG04, Proposition 1.4(c)].

**Corollary 4.8.** Under the same assumptions in Theorem 4.7, we have an isomorphism

$$V_{k, \mathbb{Z}_p}(Y_1(N))_m \otimes \mathcal{T}(N)/\mathfrak{p}f \xrightarrow{\sim} V_{k, \mathbb{Z}_p}(Y_1(N^{\Sigma_0}))_{m^{\Sigma_0}} \otimes \mathcal{T}(N^{\Sigma_0})_{m^{\Sigma_0}}/\mathfrak{p}f^{\Sigma_0}$$

sending $\delta_f^k$ to $v^+ \cdot \delta_f^{\Sigma_0}$ where $v^+ \in \mathcal{O}_\pi^\times$.

**Proof.** It follows from the combination of Theorem 4.7 and the existence of integral perfect parings on $V_{k, \mathbb{Z}_p}(Y_1(N))_m$ and $V_{k, \mathbb{Z}_p}(Y_1(N^{\Sigma_0}))_{m^{\Sigma_0}}$ as in [DFG04, Corollary 1.6]. More specifically,

$$V_{k, \mathbb{Z}_p}(Y_1(N))_m \otimes \mathcal{T}(N)/\mathfrak{p}f \simeq \text{Hom}(V_{k, \mathbb{Z}_p}(Y_1(N))_m[\mathfrak{p}f], \mathcal{O}_\pi(1-k))$$

$$\simeq \text{Hom}(V_{k, \mathbb{Z}_p}(Y_1(N^{\Sigma_0}))_{m^{\Sigma_0}}[\mathfrak{p}f^{\Sigma_0}], \mathcal{O}_\pi(1-k))$$

$$\simeq V_{k, \mathbb{Z}_p}(Y_1(N^{\Sigma_0}))_{m^{\Sigma_0}} \otimes \mathcal{T}(N^{\Sigma_0})_{m^{\Sigma_0}}/\mathfrak{p}f^{\Sigma_0}$$

[DFG04, Corollary 1.6].

□
By combining (4.6), (4.7), and Corollary 4.8, we have congruence between zeta elements

\( (4.8) \)

\[
\tilde{z}_{\text{Kato}}^0(f, i, k - r) = \overline{\nu}^\pm \cdot z_{\text{Kato}}^0(f^*_{\Sigma_0}, i, k - r)
\]

in \( H^1_{\text{Iw}}(\mathbb{Q}_S/\mathbb{Q}(\zeta_{p^\infty}), \overline{\mathcal{F}}_f(k - r)) \) where \( \overline{\nu}^\pm \in \mathbb{F}_p^\times \) and the sign of \( \nu^\pm \) coincides with that of \((-1)^i\).

4.5. **Putting it all together.** We have the following congruences

\( (4.9) \)

\[
\tilde{z}_{\text{Kato}}^0(f, i, k - r) = \overline{\nu}_1^\pm \cdot z_{\text{Kato}}^0(f^*_{\Sigma_0}, i, k - r)
\]

\[
= \overline{\nu}_2^\pm \cdot z_{\text{Kato}}^0(f^*_{\Sigma_0}, i, k - r)
\]

\[
= \overline{\nu}_3^\pm \cdot z_{\text{Kato}}^0(g, i, k - r)
\]

in \( H^1_{\text{Iw}}(\mathbb{Q}_S/\mathbb{Q}(\zeta_{p^\infty}), \overline{\mathcal{F}}_f(k - r)) \) where \( \overline{\nu}_1^\pm, \overline{\nu}_2^\pm, \) and \( \overline{\nu}_3^\pm \) lie in \( \mathbb{F}_p^\times \). The first and the third equalities follow from (4.8), and the second equality follows from (4.5).

Thus, the following consequence is immediate.

**Theorem 4.9.** If \( \Sigma_0 \) contains the primes dividing \( N_f \) and \( N_g \), then \( \tilde{z}_{\text{Kato}}^0(f, i, k - r) \) and \( \tilde{z}_{\text{Kato}}^0(g, i, k - r) \) coincide in \( H^1_{\text{Iw}}(\mathbb{Q}_S/\mathbb{Q}, \overline{\mathcal{F}}_f(k - r)) \). In particular, we have

\( (4.10) \)

\[
\left[ H^1_{\text{Iw}}(\mathbb{Q}_S/\mathbb{Q}, \overline{\mathcal{F}}_f(k - r)) : \Lambda_1 / \pi f \Lambda_1 \cdot \tilde{z}_{\text{Kato}}^0(f, i, k - r) \right] = \left[ H^1_{\text{Iw}}(\mathbb{Q}_S/\mathbb{Q}, \overline{\mathcal{F}}_f(k - r)) : \Lambda_1 / \pi g \Lambda_1 \cdot \tilde{z}_{\text{Kato}}^0(g, i, k - r) \right].
\]

**Remark 4.10.** If one of (4.10) is finite (“\( \mu = 0 \)”), then the other is also finite. In this case, the index (4.10) contains the information of the \( \lambda \)-invariant of \( H^1(f, T_f(k - r))/\tilde{z}_{\text{Kato}}^0(f, i, k - r) \) and the size of \( H^2_{\text{Iw}}(\mathbb{Q}_S/\mathbb{Q}, T_f(k - r)) \) simultaneously; thus, it seems difficult to obtain the formula on \( \lambda \)-invariants under congruences as in the literature without having the non-existence of finite Iwasawa submodule of \( H^2_{\text{Iw}}(\mathbb{Q}_S/\mathbb{Q}, T_f(k - r)) \).

The following corollary proves Theorem 2.1.(1). The statement is independent of the choice of \( \Sigma_0 \).

**Corollary 4.11.** If \( \tilde{z}_{\text{Kato}}(f_0, i, k - r) \) is non-zero for some \( f_0 \in S_k(\mathcal{P}) \), then \( \tilde{z}_{\text{Kato}}(f, i, k - r) \) is non-zero for all \( f \in S_k(\mathcal{P}) \).

**Proof.** It follows from the equality of \( \mu \)-invariants in (4.2) and Theorem 4.9. \( \square \)

5. **The \( H^2 \)-side**

Recall that \( S \) is a finite set of places of \( \mathbb{Q} \) containing the primes dividing \( pN \) and the infinite place and \( S' = S \setminus \{p, \infty\} \).

**Proposition 5.1** (Kurihara). \( \quad \) (1) The canonical mapping

\[
H^1(j_\ast T_f(k - r)) \xrightarrow{\sim} H^1_{\text{Iw}}(\mathbb{Q}_S/\mathbb{Q}(\zeta_{p^\infty}), T_f(k - r))
\]

is an isomorphism.

(2) The sequence

\[
0 \longrightarrow H^2(j_\ast T_f(k - r)) \longrightarrow H^2_{\text{Iw}}(\mathbb{Q}_S/\mathbb{Q}(\zeta_{p^\infty}), T_f(k - r)) \longrightarrow \bigoplus_{\ell \in S'} H^2_{\text{Iw}}(\mathbb{Q}(\zeta_{p^\infty}), T_f(k - r)) \longrightarrow 0
\]

is an exact sequence of \( \Lambda \)-modules.

**Remark 5.2.** \( \quad \) (1) The first statement is given in [Kob03, Proposition 7.1.i] and [Kur02, §6] using the localization sequence of étale cohomology. We give a direct proof of both statements using the same method.
(2) Proposition 5.1.(2) is an analogue of [GV00, Proposition 2.1]. However, the \( \Lambda \)-torsionness of \( H^2(j_*, T_f(k-r)) \) (Theorem 1.5.(2)) is not required in this setting.

Proof. We use \( j \) instead of \( j_e \) here for notational convenience. The low-degree terms of the Leray spectral sequence

\[
E_2^{a,b} = H^a_{\text{ét}}(\mathbb{Z}[\zeta_{p^n}, 1/p], R^b j_* T_f(k-r)) \Rightarrow H^{a+b}(Q_S/Q(\zeta_{p^n}), T_f(k-r))
\]

induces the following localization exact sequence in étale cohomology

\[
0 \to H^1_{\text{ét}}(\mathbb{Z}[\zeta_{p^n}, 1/p], j_* T_f(k-r)) \\
\to H^1(Q_S/Q(\zeta_{p^n}), T_f(k-r)) \\
\to \bigoplus_{\ell \in S'} \bigoplus_{\eta \mid \ell} H^0_{\text{ét}}(\kappa(\eta), H^1_{\text{ét}}(\mathbb{Q}(\zeta_{p^n})_{\eta}^{\text{ur}}, T_f(k-r))) \\
\to H^2_{\text{ét}}(\mathbb{Z}[\zeta_{p^n}, 1/p], j_* T_f(k-r)) \\
\to \ker\left( H^2(Q_S/Q(\zeta_{p^n}), T_f(k-r)) \to \bigoplus_{\ell \in S'} \bigoplus_{\eta \mid \ell} H^0_{\text{ét}}(\kappa(\eta), H^2_{\text{ét}}(\mathbb{Q}(\zeta_{p^n})_{\eta}^{\text{ur}}, T_f(k-r))) \right) \\
\to \bigoplus_{\ell \in S'} H^1_{\text{ét}}(\kappa(\eta), H^1_{\text{ét}}(\mathbb{Q}(\zeta_{p^n})_{\eta}^{\text{ur}}, T_f(k-r))) \\
\to \bigoplus_{\ell \in S'} H^2_{\text{ét}}(\mathbb{Q}(\zeta_{p^n})_{\eta}, T_f(k-r)) \\
\to 0.
\]

where \( \kappa(\eta) \) is the residue field of \( \mathbb{Z}[\zeta_{p^n}] \) at \( \eta \) and \( F^{\text{ur}} \) is the maximal unramified extension of a local field \( F \). Since the cohomological dimension of \( \kappa(\eta) \) is 1, that \( \mathbb{Q}((\zeta_{p^n})_{\eta}^{\text{ur}}) \) is \( \leq 1 \), and the \( p \)-cohomological dimension of the Galois group of the maximal extension of \( \mathbb{Q}(\zeta_{p^n}) \) unramified outside \( p \) and the infinite places is 2, the above exact sequence becomes

\[
0 \to H^1_{\text{ét}}(\mathbb{Z}[\zeta_{p^n}, 1/p], j_* T_f(k-r)) \\
\to H^1(Q_S/Q(\zeta_{p^n}), T_f(k-r)) \\
\to \bigoplus_{\ell \in S'} \bigoplus_{\eta \mid \ell} H^0_{\text{ét}}(\kappa(\eta), H^1_{\text{ét}}(\mathbb{Q}(\zeta_{p^n})_{\eta}^{\text{ur}}, T_f(k-r))) \\
\to H^2_{\text{ét}}(\mathbb{Z}[\zeta_{p^n}, 1/p], j_* T_f(k-r)) \\
\to H^2(Q_S/Q(\zeta_{p^n}), T_f(k-r)) \\
\to \bigoplus_{\ell \in S'} H^2_{\text{ét}}(\mathbb{Q}(\zeta_{p^n})_{\eta}, T_f(k-r)) \\
\to 0.
\]

For \( \ell \in S' \), we have

\[
\lim_{\eta \to \eta_n} H^0_{\text{ét}}(\kappa(\eta_n), H^1_{\text{ ét}}(\mathbb{Q}(\zeta_{p^n})_{\eta_n}^{\text{ur}}, T_f(k-r))) = 0
\]

where \( \eta_n \) is a prime of \( \mathbb{Q}(\zeta_{p^n}) \) dividing \( \ell \). Thus, we obtain the conclusion. \( \square \)

Since \( H^2(j_*, T_{f,i}(k-r)) \) is a finitely generated torsion \( \Lambda_i \)-module (Theorem 1.5.(2)) and the local \( H^2 \)'s are also finitely generated torsion \( \Lambda_i \)-modules as in Corollary 3.6, \( H^2_{\text{tw}}(Q_S/Q_{\infty}, T_{f,i}(k-r)) \) is also a finitely generated torsion \( \Lambda_i \)-module due to Proposition 5.1.(2).

By Proposition 5.1.(2) and the multiplicative property of characteristic ideals (c.f. [CS06, Proposition 1 in Appendix, Page 104–105]), we have

(5.1)

\[
\text{char}_{\Lambda_i}(H^2_{\text{tw}}(Q_S/Q_{\infty}, T_{f,i}(k-r))) = \text{char}_{\Lambda_i}(H^2(j_*, T_{f,i}(k-r))) \cdot \prod_{\ell \in S'} \text{char}_{\Lambda_i}\left( \bigoplus_{\eta \mid \ell} H^2_{\text{tw}}(Q_{\infty}, \eta, T_{f,i}(k-r)) \right).
\]

Proposition 5.3.
Taking the inverse limit, we have

\[ H^2 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, T_{f,i}(k-r) \right) = H^2 \left( j_* T_{f,i}(k-r) \right). \]

Applying the same argument to the non-existence of finite \( \Lambda_i \)-submodules for Proposition 5.3.(2). Thus, Proposition 5.3 partially generalizes [GV00, Proposition 2.8] and [EPW06, Theorem 4.3.4] by removing these conditions in some sense.

For any \( \Sigma_0 \subseteq S' = S \setminus \{p, \infty\} \), we define the \( \Sigma_0 \)-imprimitive version of \( H^2(j_* T_{f,i}(k-r)) \) by the following exact sequence

\[
0 \longrightarrow H^2,\Sigma_0(j_* T_{f,i}(k-r)) \longrightarrow H^2 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, T_{f,i}(k-r) \right) \longrightarrow \bigoplus_{\ell \in S' \setminus \Sigma_0} \bigoplus_{\eta \mid \ell} H^2 \left( \mathbb{Q}_\infty, T_{f,i}(k-r) \right) \longrightarrow 0.
\]

Applying the same argument to \( H^2,\Sigma_0(j_* T_{f,i}(k-r)) \), we obtain the following statements.

Proposition 5.5.

1. \( \mu \left( H^2 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, T_{f,i}(k-r) \right) \right) = \mu \left( H^2,\Sigma_0(j_* T_{f,i}(k-r)) \right) \).

2. \( \lambda \left( H^2 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, T_{f,i}(k-r) \right) \right) = \lambda \left( H^2,\Sigma_0(j_* T_{f,i}(k-r)) \right) + \sum_{\ell \in S' \setminus \Sigma_0} \lambda \left( \bigoplus_{\eta \mid \ell} H^2 \left( \mathbb{Q}_\infty, T_{f,i}(k-r) \right) \right) \).

6. The invariance of the difference of \( \lambda \)-invariants

In this section, we prove Theorem 2.1.(2).

Lemma 6.1. Let \( X \) be a finitely generated torsion \( \mathcal{O}_{\pi}[T] \)-module with \( \mu(X) = 0 \). Then

\[
\frac{\#X}{\pi X} = \left( \frac{\#F}{\pi} \right)^{\lambda(X)}.
\]

Proof. Straightforward. \( \square \)

Consider the exact sequence

\[
0 \longrightarrow H^1 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, T_{f,i}(k-r) \right) \bigg/ \pi H^1 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, T_{f,i}(k-r) \right) \longrightarrow H^1 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, \mathfrak{p}_{f,i}(k-r) \right) \bigg/ \pi H^1 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, \mathfrak{p}_{f,i}(k-r) \right) \longrightarrow H^1 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, T_{f,i}(k-r) \right) \bigg/ \pi H^1 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, T_{f,i}(k-r) \right) \bigg/ \pi H^1 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, \mathfrak{p}_{f,i}(k-r) \right) \longrightarrow 0.
\]

Since \( p \) is odd and \( \text{Gal}(\mathbb{Q}_S / \mathbb{Q}_n) \) has \( p \)-cohomological dimension 2, we have an isomorphism

\[
\frac{H^2 \left( \mathbb{Q}_S / \mathbb{Q}_n, T_{f,i}(k-r) \right)}{\pi H^2 \left( \mathbb{Q}_S / \mathbb{Q}_n, T_{f,i}(k-r) \right)} \simeq H^2 \left( \mathbb{Q}_S / \mathbb{Q}_n, \mathfrak{p}_{f,i}(k-r) \right).
\]

Taking the inverse limit, we have

\[
\frac{H^2 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, T_{f,i}(k-r) \right)}{\pi H^2 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, T_{f,i}(k-r) \right)} \simeq H^2 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, \mathfrak{p}_{f,i}(k-r) \right).
\]

Combining Lemma 6.1, (6.1), and (6.2) with \( \mu \left( H^2 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, T_{f,i}(k-r) \right) \right) = 0 \), we have

\[
\left[ H^1 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, \mathfrak{p}_{f,i}(k-r) \right) : \frac{H^1 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, T_{f,i}(k-r) \right)}{\pi H^1 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, T_{f,i}(k-r) \right)} \right] = \frac{\#H^2 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, \mathfrak{p}_{f,i}(k-r) \right)}{\left( \frac{\#F}{\pi} \right)^{\lambda \left( H^2 \left( \mathbb{Q}_S / \mathbb{Q}_\infty, T_{f,i}(k-r) \right) \right)}}.
\]
Proposition 6.2. Let
\[ z \in H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r)) \]
be a non-zero element and
\[ z \in \frac{H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r))}{\pi H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r))} \subseteq H_{Iw}^1(Q_S/Q_{\infty}, \overline{\mathcal{P}}_{f,i}(k-r)) \]
the image of \( z \) in \( H_{Iw}^1(Q_S/Q_{\infty}, \overline{\mathcal{P}}_{f,i}(k-r)) \). Then we have the following statements.

(1) The following statements are equivalent:
(a) \( \mu (H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r))/\Lambda_i z) = 0 \) and \( \mu (H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r))) = 0. \)
(b) \( \left[H_{Iw}^1(Q_S/Q_{\infty}, \overline{\mathcal{P}}_{f,i}(k-r)): \Lambda_i/\pi \Lambda_i \cdot z \right] < \infty. \)

(2) If (1) holds, then
\[
(6.4) 
\left[H_{Iw}^1(Q_S/Q_{\infty}, \overline{\mathcal{P}}_{f,i}(k-r)) : \Lambda_i/\pi \Lambda_i \cdot z \right] \times \left(\#H_{Iw}^2(Q_S/Q_{\infty}, \overline{\mathcal{P}}_{f,i}(k-r))\right)^{-1} \\
= \left(\#\mathbb{F}\right)^{\lambda(H^1_G(T_{f,i}(k-r))/\Lambda_i z) - \lambda(H_{Iw}^2(Q_S/Q_{\infty}, T_{f,i}(k-r)))}.
\]

Proof. (1) Due to Theorem 1.5, (1) and Proposition 5.1, we have
\[
H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r)) \simeq \Lambda_i.
\]
Thus, the following statements are equivalent:

- \( \left[H_{Iw}^1(Q_S/Q_{\infty}, \overline{\mathcal{P}}_{f,i}(k-r)) : \Lambda_i/\pi \Lambda_i \cdot z \right] < \infty, \)
- \( z \neq 0 \) in \( H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r))/\pi \) and \( \#H_{Iw}^2(Q_S/Q_{\infty}, T_{f,i}(k-r))[\pi] < \infty. \)

Note that the following statements are also equivalent:

- \( \#H_{Iw}^2(Q_S/Q_{\infty}, T_{f,i}(k-r))[\pi] < \infty, \)
- \( \mu (H_{Iw}^2(Q_S/Q_{\infty}, T_{f,i}(k-r))) = 0. \)

(2) Due to (6.3), it suffices to compute the index of
\[
\Lambda_i/\pi \Lambda_i \cdot z \subseteq \frac{H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r))}{\pi H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r))}.
\]
By Lemma 6.1, we have
\[
\frac{H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r))/(z, \pi)}{(H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r))/\Lambda_i z)[\pi]} = \#\mathbb{F}^{\lambda(H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r))/\Lambda_i z)} \\
and (H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r))/\Lambda_i z)[\pi] = 0 \text{ since } z \neq 0. \text{ Thus, the index is} \\
\#\mathbb{F}^{\lambda(H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r))/\Lambda_i z)}
\]
and we obtain the conclusion. Note that we have an isomorphism
\[
H^1_G(T_{f,i}(k-r)) \simeq H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r))
\]
by Proposition 5.1.

Remark 6.3. Indeed, if \( z = z_{Kato}(f, i, k-r) \), then \( \mu (H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r))/\Lambda_i z) = 0 \) implies \( \mu (H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r)))/\Lambda_i z) = 0 \) by using Theorem 1.11, Proposition 5.1, and Proposition 5.3. Thus, Proposition 6.2.(1),(a) can be simplified as \( \mu (H_{Iw}^1(Q_S/Q_{\infty}, T_{f,i}(k-r))/\Lambda_i z) = 0 \) when \( z = z_{Kato}(f, i, k-r) \).

Proof of Theorem 2.1.(2). Now we pin down \( \Sigma_0 = S' = \{ \ell : \ell \mid N_f \cdot N_g \} \). If we choose
\[
z = z_{Kato}^0(f, i, k-r) \text{ or } z_{Kato}^0(g, i, k-r),
\]
then the LHS of (6.4) depends only on $\mathcal{F}$ and $\Sigma_0$ due to Theorem 4.9 under the $\mu = 0$ assumption (Corollary 4.11 and Proposition 6.2(1)). Thus, the following equalities are immediate.

$$
\lambda \left( \frac{\mathbb{H}^1(j_*T_{f,i}(k-r))}{\mathcal{K}_{Kato}(f,i,k-r)} \right) - \lambda \left( \frac{\mathbb{H}^2(j_*T_{f,i}(k-r))}{\mathcal{K}_{Kato}(f,i,k-r)} \right) \\
\lambda \left( \frac{\mathbb{H}^1(j_*T_{g,i}(k-r))}{\mathcal{K}_{Kato}(g,i,k-r)} \right) - \lambda \left( \frac{\mathbb{H}^2,\Sigma_0(j_*T_{f,i}(k-r))}{\mathcal{K}_{Kato}(f,i,k-r)} \right) \quad (4.2) \text{ and Proposition 5.5} \\
\lambda \left( \frac{\mathbb{H}^1(j_*T_{g,i}(k-r))}{\mathcal{K}_{Kato}(g,i,k-r)} \right) - \lambda \left( \frac{\mathbb{H}^2,\Sigma_0(j_*T_{g,i}(k-r))}{\mathcal{K}_{Kato}(g,i,k-r)} \right) \quad \text{Proposition 6.2}(2) \\
\lambda \left( \frac{\mathbb{H}^1(j_*T_{g,i}(k-r))}{\mathcal{K}_{Kato}(g,i,k-r)} \right) - \lambda \left( \frac{\mathbb{H}^2,\Sigma_0(j_*T_{g,i}(k-r))}{\mathcal{K}_{Kato}(g,i,k-r)} \right) \quad (4.2) \text{ and Proposition 5.5.}
$$

Thus, Theorem 2.1.(2) follows. The final statement is independent of the choice of $\Sigma_0$. \qed

Remark 6.4. In [GV00] and its various successors, it is essential to identify the $\lambda$-invariants of Selmer groups over the Iwasawa algebra and their $\mathbb{Z}_p$-coranks (under the $\mu = 0$ assumption) to apply the congruence argument in an appropriate setting. In order to do this, one needs to prove that $\text{Sel}(\mathbb{Q}_\infty, A_{f,0}(1))^\vee$ (or its variant) has no non-zero finite $\Lambda_0$-submodule. See [GV00, Propositions 2.5 and 2.8] for detail. In our setting, we do not expect such a statement (for $\mathbb{H}^2(j_*T_{f,i}(k-r))$) holds in general as described in Remark 5.4. A similar flavor can also be found in [Hac11].

Acknowledgement

This project is inspired from Project $A^1$ of Skinner’s project group at Arizona Winter School 2018. We deeply appreciate Chris Skinner and Francesc Castella for wonderful lectures and great helps during the school. We also thank to other members of the project group: Takahiro Kitajima, Rei Otsuki, Sheng-Chi Shih, Florian Sprung, and Yiwen Zhou.

We thank David Loeffler and Sarah Zerbes very much for the helpful suggestions and discussions in the Arizona Winter School.

We thank Olivier Fouquet, Antonio Lei, David Loeffler, and Xin Wan for helpful comments and Takenori Kataoka for pointing out a gap in an earlier version of this paper.

We thank the referee for various helpful and valuable suggestions and comments.

Chan-Ho Kim is partially supported by a KIAS Individual Grant (SP054102) via the Center for Mathematical Challenges at Korea Institute for Advanced Study and by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. 2018R1C1B6007009).

Jaehoon Lee is partially supported by KAIST Advanced Institute for Science-X Post-Doc Fellowship.

Gautier Ponsinet thanks the Max Planck Institut for Mathematics for its hospitality and support.

References

[CH18] Masataka Chida and Ming-Lun Hsieh, Special values of anticyclotomic L-functions for modular forms, J. Reine Angew. Math. 741 (2018), 87–131.

[CK17] Suh Hyun Choi and Byoung Du Kim, Congruences of two-variable p-adic L-functions of congruent modular forms of different weights, Ramanujan J. 43 (2017), no. 1, 163–195.

[CKL17] Francesc Castella, Chan-Ho Kim, and Matteo Longo, Variation of anticyclotomic Iwasawa invariants in Hida families, Algebra Number Theory 11 (2017), no. 10, 2339–2368.

[CS05] John Coates and Ramadori Sujatha, Fine Selmer groups of elliptic curves over p-adic Lie extensions, Math. Ann. 331 (2005), 809–839.

1http://swc.math.arizona.edu/aws/2018/2018SkinnerProjects.pdf
Représentations $\left[F_{on94}\right]$, Jean-Marc Fontaine, Comparing anticyclotomic Selmer groups of positive corank $s$ for $\left[HL_{19}\right]$, Jeffrey Hatley and Antonio Lei, Galois representations into $\left[FK\right]$, Takako Fukaya and Kazuya Kato, On conjectures of Sharifi $\left[Spr_{12}\right]$, Florian Sprung, Iwasawa theory for elliptic curves at supersingular primes: A pair of main conjectures.

The Iwasawa invariants of the plus/minus Selmer groups $\left[Kim_{09}\right]$, Byoung Du Kim, Remarks on Kato’s Euler systems for elliptic curves with $\left[KN_{20}\right]$, Chan-Ho Kim and Kentaro Nakamura, $\left[Kim_{17}\right]$, Chan-Ho Kim, Anticyclotomic Iwasawa invariants and congruences of modular forms.

Crystal line cohomology and $\left[FJ_{95}\right]$, Gerd Faltings and Bruce Jordan, $\left[Kat_{04}\right]$, Kazuya Kato, $p$-adic representations and values of zeta functions of modular forms, Astérisque $\left[EPW_{06}\right]$, Matthew Emerton, Robert Pollack, and Tom Weston, Variation of Iwasawa invariants in Hida families.

The equivariant Tamagawa number conjecture for modular motives with coefficients $\left[Rub_{00}\right]$, Karl Rubin, Euler Systems.

On the Tate Shafarevich groups over cyclotomic fields of an elliptic curve with $\left[Kur_{02}\right]$, Masato Kurihara, Iwasawa theory for elliptic curves at supersingular primes $\left[Kob_{03}\right]$, Shinichi Kobayashi, Iwasawa theory for elliptic curves at supersingular primes $\left[Nak\right]$, Kentaro Nakamura, Zeta morphisms for rank two universal deformations, arXiv:2006.13647.

$\left[Nak17\right]$, Kentaro Nakamura, Local $\varepsilon$-isomorphisms for rank two $p$-adic representations of $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and a functional equation of Kato’s Euler system, Camb. J. Math. $\left[Och_{06}\right]$, Tadashi Ochiai, On the two-variable Iwasawa main conjecture, Compos. Math. $\left[Pol_{03}\right]$, Robert Pollack, On the $p$-adic $L$-function of a modular form at a supersingular prime, Duke Math. J. $\left[Pon_{20}\right]$, Gautier Ponsinet, On the structure of signed Selmer groups, Math. Z. $\left[PW_{11a}\right]$, Robert Pollack and Tom Weston, Mazur–Tate elements of nonordinary modular forms, Duke Math. J. $\left[PW_{11b}\right]$, On anticyclotomic $\mu$-invariants of modular forms, Compos. Math. $\left[Rub_{00}\right]$, Karl Rubin, Euler Systems, Ann. of Math. Stud., vol. 147, Princeton University Press, 2000.

$\left[Spe_{12}\right]$, Florian Sprung, Iwasawa theory for elliptic curves at supersingular primes: A pair of main conjectures, J. Number Theory.
Christopher Skinner and Eric Urban, *The Iwasawa main conjectures for GL$_2$*, Invent. Math. **195** (2014), no. 1, 1–277.

Vinayak Vatsal, *Canonical periods and congruence formulae*, Duke Math. J. **98** (1999), no. 2, 397–419.

Vinayak Vatsal, *Special values of anticyclotomic L-functions*, Duke Math. J. **116** (2003), no. 2, 219–261.

Vinayak Vatsal, *Integral periods for modular forms*, Ann. Math. Québec **37** (2013), 109–128.

Tom Weston, *Iwasawa invariants of Galois deformations*, Manuscripta Math. **118** (2005), 161–180.

Andrew Wiles, *Modular elliptic curves and Fermat’s last theorem*, Ann. of Math. (2) **141** (1995), 443–551.

*(Chan-Ho Kim) Center for Mathematical Challenges, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Republic of Korea*

*Email address: chanho.math@gmail.com*

*(Jaehoon Lee) KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon 34141, Republic of Korea*

*Email address: jaehoon.lee900907@gmail.com*

*(Gautier Ponsinet) Max Planck Institute für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany*

*Email address: gautier.ponsinet@mpim-bonn.mpg.de*