REAL COMPACTNESS VIA REAL MAXIMAL IDEALS OF $B_1(X)$

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Abstract. In this paper, constructing a class of ideals of $B_1(X)$ from proper ideals of $C(X)$ a one-one correspondence between the class of real maximal ideals of $C(X)$ and those of $B_1(X)$ is established. The collection of all real maximal ideals of $B_1(X)$ with hull-kernel topology is proved to be homeomorphic to the space of real maximal ideals of $C(X)$ endowed with a topology finer than the subspace topology induced from its structure space. It is also proved that a Tychonoff space is real compact if and only if every real maximal ideal of $B_1(X)$ is fixed. As a consequence, within the class of real compact $T_4$ spaces whose points are $G_δ$, $B_1(X) = B_1^*(X)$ if and only if $X$ is finite.

1. Introduction

The pointwise limit function of a sequence $\{f_n\}$ of real valued continuous functions defined on a topological space $X$ is well known as a Baire class-one (or, Baire one) function. Several characterizations of Baire one functions defined on metric spaces were obtained by different mathematicians [3], [8]. Inspired by the research on rings of continuous functions, in [1], we introduced two rings $B_1(X)$ and $B_1^*(X)$ consisting respectively of real valued Baire one functions and bounded Baire one functions on a topological space $X$. It has been observed in [1] that $B_1(X)$ is a commutative lattice ordered ring with unity with respect to the usual addition and multiplication of functions and it is an over-ring of the ring $C(X)$ of all real valued continuous functions on $X$. The main purpose of this paper is to establish that the class of all real maximal ideals of $B_1(X)$ is in one-one correspondence with the class of all real maximal ideals of $C(X)$. In fact, defining a sort of ‘extension’ of an ideal $I$ of $C(X)$ (denoted by $I_B$), we show that the contraction $I_B \cap C(X)$ coincides with $I$ if and only if $I$ is a real maximal ideal in $C(X)$. That $I_B$ is a real maximal ideal in $B_1(X)$, for any real maximal ideal $I$ in $C(X)$ determines the said one-one correspondence. Moreover, it is proved that the collection $\mathcal{RM}(B_1(X))$ of all real maximal ideals of $B_1(X)$ with the subspace topology (i.e., the hull-kernel topology) of the structure space of $B_1(X)$ is homeomorphic to the collection $\mathcal{RM}(C(X))$ of all real maximal ideals of $C(X)$ equipped with a topology finer than the subspace topology of the structure space of $C(X)$.

The class of topological spaces on which every Baire one function is bounded, is yet to be determined completely. In Section 3, we prove a necessary and sufficient condition for $B_1(X)$ to coincide with $B_1^*(X)$ within the class of all $T_4$, real compact spaces whose points are $G_δ$.

2010 Mathematics Subject Classification. 26A21, 54C30, 54C45, 54C50.
Key words and phrases. $B_1(X)$, $B_1^*(X)$, real and hyper-real maximal ideals, real compact spaces.
To make this paper self-sufficient, we now introduce some known terminologies and facts. A zero set of \( f \in B_1(X) \) is defined as usual by a set of the form \( Z(f) = \{ x \in X : f(x) = 0 \} \). As an analogue of \( \mathcal{z} \)-filter (or \( \mathcal{z} \)-ultrafilter) on \( X \), we introduced in \([2]\) the \( \mathcal{Z}_B \)-filter (or respectively, \( \mathcal{Z}_B \)-ultrafilter) on \( X \), thereby investigating the duality between ideals (maximal ideals) in \( B_1(X) \) and \( \mathcal{Z}_B \)-filters (respectively, \( \mathcal{Z}_B \)-ultrafilters) on \( X \). The above mentioned duality existing between ideals in \( B_1(X) \) and \( \mathcal{Z}_B \)-filters on \( X \) is manifested by the fact that if \( I \) is an ideal in \( B_1(X) \) then \( \mathcal{Z}_B[I] = \{ Z(f) : f \in B_1(X) \} \) is a \( \mathcal{Z}_B \)-filter on \( X \) and dually for a \( \mathcal{Z}_B \)-filter \( \mathcal{F} \) on \( X \), \( \mathcal{Z}_B^{-1}[\mathcal{F}] = \{ f \in B_1(X) : Z(f) \in \mathcal{F} \} \) is a proper ideal in \( B_1(X) \). The assignment \( M \mapsto \mathcal{Z}_B[M] \) is a bijection from the set of all maximal ideals in \( B_1(X) \) and to the family of all \( \mathcal{Z}_B \)-ultrafilters on \( X \). In the same paper \([2]\), defining suitably the residue class fields \( B_1(X)/M \), the concept of real and hyper-real maximal ideals of \( B_1(X) \) was introduced. A maximal ideal \( M \) of \( B_1(X) \) is called real \([2]\) if \( B_1(X)/M \cong \mathbb{R} \) and in such case \( B_1(X)/M \) is called real residue class field. If \( M \) is not real then it is called hyper-real \([2]\) and \( B_1(X)/M \) is called hyper-real residue class field. Considering the structure space \( \mathcal{M}(B_1(X)) \) of \( B_1(X) \), i.e., the collection \( \mathcal{M}(B_1(X)) \) of all maximal ideals of \( B_1(X) \) with respect to the hull-kernel topology, we get the subspace topology on the collection \( \mathcal{R}\mathcal{M}(B_1(X)) \) of all real maximal ideals of \( B_1(X) \). We show that the subspace topology via the aforesaid bijection induces a topology on \( \mathcal{R}\mathcal{M}(C(X)) \) of all real maximal ideals of \( C(X) \) which is finer than the hull-kernel topology on \( \mathcal{R}\mathcal{M}(C(X)) \).

Throughout this paper, we consider \( X \) as any Hausdorff topological space unless stated otherwise. A function \( f : X \to \mathbb{R} \) is called a Baire one function if there exists a sequence of continuous functions \( \{ f_n \} \) such that for all \( x \in X \), \( \{ f_n(x) \} \) converges to \( f(x) \). We use the notation \( f_n \xrightarrow{p.w.} f \) to denote \( \{ f_n \} \) pointwise converges to \( f \).

**Theorem 1.1.** \([\text{Theorem 5.14 of } 5]\) For a maximal ideal \( M \) of \( C(X) \) the following statements are equivalent:

1. \( M \) is a real maximal ideal.
2. \( Z[M] \) is closed under countable intersection.
3. \( Z[M] \) has countable intersection property.

An analogue of this theorem in the context of Baire one functions is the following:

**Theorem 1.2.** \([\text{Theorem 4.26 of } 2]\) For a maximal ideal \( M \) of \( B_1(X) \) the following statements are equivalent:

1. \( M \) is a real maximal ideal.
2. \( \mathcal{Z}_B[M] \) is closed under countable intersection.
3. \( \mathcal{Z}_B[M] \) has countable intersection property.

2. **A one-one correspondence between the real maximal ideals of \( C(X) \) and the real maximal ideals of \( B_1(X) \)**

It is easy to observe that for each proper ideal \( I \) of \( C(X) \), \( I_B = \{ f \in B_1(X) : \exists \{ f_n \} \subseteq I \text{ such that } f_n \xrightarrow{p.w.} f \} \) is an ideal of \( B_1(X) \) such that \( I \subseteq I_B \cap C(X) \). As a natural example, we obtain that \((M_p)_B \) is a fixed maximal ideal of \( B_1(X) \) \([2]\), where \( M_p = \{ f \in C(X) : f(p) = 0 \} \) is a fixed maximal ideal of \( C(X) \) \([5]\) and this example prompts us to prove a more general result later in this section.
Example 2.1. For each $p \in X$, $(M_p)_B = \hat{M}_p \equiv \{f \in B_1(X) : f(p) = 0\}$.

For each $f \in (M_p)_B$, there exists $\{f_n\} \subseteq M_p$ such that $f_n \xrightarrow{\text{p.w.}} f$. This implies $f_n(p) = 0$, for all $n \in \mathbb{N}$ and therefore, $f(p) = 0$. i.e., $(M_p)_B \subseteq \hat{M}_p$. On the other hand, $f \in \hat{M}_p$ implies $f(p) = 0$. Since $f \in B_1(X)$, there exists a sequence $\{g_n\} \subseteq C(X)$ such that $g_n \xrightarrow{\text{p.w.}} f$. Define $f_n = g_n - g_n(p)$, for all $n \in \mathbb{N}$. Clearly, $f_n(p) = 0$, for all $n \in \mathbb{N}$. Also, $f_n \xrightarrow{\text{p.w.}} f$. Therefore, $f \in (M_p)_B$ and $\hat{M}_p \subseteq (M_p)_B$. Hence, $(M_p)_B = \hat{M}_p$.

Theorem 2.1. If $I$ is an absolutely convex ideal in $C(X)$ then $I_B$ is an absolutely convex ideal in $B_1(X)$.

Proof. We first prove that $I_B$ is a convex ideal in $B_1(X)$. If so, then $f \in I_B$ implies that there is $\{f_n\} \subseteq I$ such that $f_n \xrightarrow{\text{p.w.}} f$ and hence, $|f_n| \xrightarrow{\text{p.w.}} |f|$. As $I$ is absolutely convex, $\{|f_n\} \subseteq I$ which ensures $|f| \in I_B$. In such case, $I_B$ becomes absolutely convex.

Let $f, g \in B_1(X)$ such that $0 \leq f \leq g$ and $g \in I_B$. Then there is a sequence $\{f_n\}$ in $C(X)$ and $\{g_n\} \subseteq I$ such that $f_n \xrightarrow{\text{p.w.}} f$ and $g_n \xrightarrow{\text{p.w.}} g$. Choosing $h_n = f_n \wedge g_n$, we observe the following:

(i) $h_n \xrightarrow{\text{p.w.}} f \wedge g = f$.

(ii) For each $n \in \mathbb{N}$, $0 \leq h_n \leq g_n$ and $g_n \in I$ implies that $h_n \in I$ (since, $I$ is absolutely convex).

Hence, $f \in I_B$.

For any proper ideal $I$ of $C(X)$, it is clear that $I \subseteq I_B \cap C(X)$. In the following theorem we show that the equality holds precisely for the class of all real maximal ideals of $C(X)$.

Theorem 2.2. $M \in \mathcal{RM}(C(X))$ if and only if $M = M_B \cap C(X)$.

Proof. Let $M$ be a real maximal ideal of $C(X)$. Clearly, $M \subseteq M_B \cap C(X)$. Now let $g \in M_B \cap C(X)$. There exists $\{g_n\} \subseteq M$ such that $g_n \xrightarrow{\text{p.w.}} g$. Since $M$ is real and $g_n \in M$, for all $n \in \mathbb{N}$, $\bigcap_{n=1}^{\infty} Z(g_n) \in Z[M]$. Also, $\bigcap_{n=1}^{\infty} Z(g_n) \subseteq Z(g)$. Hence, $Z(g) \in Z[M]$. By maximality of $M$ it follows that $g \in M$. Therefore, $M_B \cap C(X) \subseteq M$ and it implies that $M = M_B \cap C(X)$.

Conversely, let $M$ be a maximal ideal of $C(X)$ such that $M = M_B \cap C(X)$.

Consider any countable family $\{Z(g_n) : n \in \mathbb{N}\}$ of $Z[M]$. By maximality of $M$, $g_n \in M$, for all $n \in \mathbb{N}$.

We now construct a sequence $\{s_n\}$ as follows: $s_n = \sum_{j=1}^{n} \left(\frac{1}{2^n} \wedge |g_j|\right)$, for each $n \in \mathbb{N}$.

Certainly, for each $j$, $Z(g_j) = Z\left(\frac{1}{2^n} \wedge |g_j|\right)$ implies that $\frac{1}{2^n} \wedge |g_j| \in M$. $M$ being an ideal, finite sum of each such member will also lie within $M$ and hence, $s_n \in M$, for all $n \in \mathbb{N}$.

$s = \sum_{n=1}^{\infty} \left(\frac{1}{2^n} \wedge |g_n|\right)$ is the uniform limit of the sequence $\{s_n\}$ of continuous functions and therefore, $s \in C(X)$. Again $\{s_n\} \subseteq M$ ensures that $s \in M_B \cap C(X) = M$. So, $Z(s) \neq \emptyset$. Following the arguments used in (1.14 (a) of [5]) we obtain $\bigcap_{n=1}^{\infty} Z(g_n) = Z(s) \neq \emptyset$. Hence, by Theorem [1.1] $M$ is real. □
That $M_B$ is not even a proper ideal of $B_1(X)$ when $M$ is hyper-real in $C(X)$ is observed in the next theorem.

**Theorem 2.3.** If $M$ is a hyper-real maximal ideal in $C(X)$ then $M_B = B_1(X)$.

**Proof.** If $M$ is hyper-real then by Theorem 2.2, $M_B \cap C(X) \neq M$. But for any ideal $I, I \subseteq I_B \cap C(X)$ holds. So, $M \subseteq M_B \cap C(X)$. Since $M$ is maximal, $M_B \cap C(X) = C(X)$. Hence, $C(X) \subseteq M_B$, i.e., $1 \in M_B$. Therefore, $M_B = B_1(X)$. □

**Theorem 2.4.** If $M \in \mathcal{RM}(C(X))$ then $M_B \in \mathcal{RM}(B_1(X))$.

**Proof.** Let $f \in B_1(X) \setminus M_B$. Consider the ideal $J$ generated by $M_B \cup \{f\}$. $f \in B_1(X)$ implies there exists $\{f_n\} \subseteq C(X)$ such that $f_n \xrightarrow{p.w.} f$.

Since $M$ is a real maximal ideal in $C(X)$, for each $f_n \in C(X)$, there exists some $r_n \in \mathbb{R}$ such that $M(f_n) = M(r_n)$ and so, $f_n = r_n$ on $Z_n = Z(f_n - r_n) \in Z[M]$. Since $Z = \bigcap_{n=1}^{\infty} Z_n \in Z[M]$, each $f_n = r_n$ on $Z$. As a consequence, $f$ is constant (say, $r$) on $Z \subseteq Z[M] \subseteq Z_B[M_B]$, where $r = \lim_{n \to \infty} r_n$. Since, $Z \subseteq Z(f_n - r_n)$ implies that $Z(f_n - r_n) \in Z[M]$ and $M$ is a z-ideal in $C(X)$, $f_n - r_n \in M$. By definition of $M_B$, $f - r \notin M_B$. Since $f \notin M_B$, $r \neq 0$. But, $r = f - (f - r) \in J$ and $r \neq 0$ implies that $J = B_1(X)$. Hence, $M_B$ is a maximal ideal of $B_1(X)$ such that $f - r \in M_B$. i.e., $M_B(f) = M_B(r)$, for some $r \in \mathbb{R}$. If $f \in M_B$ then $M_B(f) = M_B(0)$ and this proves that $M_B \in \mathcal{RM}(B_1(X))$. □

**Theorem 2.5.** If $\hat{M} \in \mathcal{RM}(B_1(X))$ then $\hat{M} \cap C(X) \in \mathcal{RM}(C(X))$.

**Proof.** Let $\hat{M} \in \mathcal{RM}(B_1(X))$. For each $f \in B_1(X)$, there exists $r_f \in \mathbb{R}$ such that $f - r_f \in \hat{M}$. In particular, for any $f \in C(X)$, there is $r_f \in \mathbb{R}$ such that $f - r_f \in \hat{M}$. So, $f - r_f \in \hat{M} \cap C(X) = M$ (say). Define a function $\phi : C(X)/M \to \mathbb{R}$ by $M(f) \mapsto r_f$, whenever $f - r_f \in M$. We claim that $\phi$ is an isomorphism.

$M(f) = M(g) \Leftrightarrow f - g \in M$. If $\phi(M(f)) = r_f$ and $\phi(M(g)) = r_g$ then $f - r_f, g - r_g \in M$. i.e., $(f - g) - (r_f - r_g) \in M$. Since, $f - g \in M$ and $M$ is an ideal, it follows that $r_f - r_g \in M$ - a contradiction to the fact that $M$ is proper, unless $r_f - r_g = 0$.

Hence $\phi$ is well defined.

$\phi(M(f)) = \phi(M(g))$ implies that $r_f = r_g$ where $f - r_f, g - r_g \in M$. Therefore, $f - g = (f - r_f) - (g - r_g) \in M$ which in turn gives $M(f) = M(g)$, proving $\phi$ to be one-one.

The function $\phi$ is clearly onto, as $\phi(M(r)) = r$, for each $r \in \mathbb{R}$. By routine arguments we easily see that $\phi$ is indeed a ring homomorphism. Hence, $\phi$ is a ring isomorphism and therefore, $M \in \mathcal{RM}(C(X))$. □

**Corollary 2.1.** If $\hat{M} \in \mathcal{RM}(B_1(X))$ then $(\hat{M} \cap C(X))_B = \hat{M}$.

**Proof.** As $\hat{M} \in \mathcal{RM}(B_1(X))$, $\hat{M} \cap C(X) \in \mathcal{RM}(C(X))$ (by Theorem 2.4). Using Theorem 2.1, $(\hat{M} \cap C(X))_B \in \mathcal{RM}(B_1(X))$. Since $(\hat{M} \cap C(X))_B$ is a maximal ideal, it is enough to show that $(\hat{M} \cap C(X))_B \subseteq \hat{M}$.

Let $g \in (\hat{M} \cap C(X))_B$. Then there exists $\{g_n\} \subseteq \hat{M} \cap C(X)$ such that $g_n \xrightarrow{p.w.} g$.

So, $Z(g) \supseteq \bigcap_{i=1}^{\infty} Z(g_n)$. As $Z_B[\hat{M}]$ is a $Z_B$-ultrafilter and $\hat{M}$ is real, it follows that $Z(g) \in Z_B[\hat{M}]$. Hence, $g \in \hat{M}$ and therefore $(\hat{M} \cap C(X))_B \subseteq \hat{M}$. □
In view of Corollary 2.1, Theorem 2.4 and Theorem 2.5, we get a one-one correspondence between $\mathcal{RM}(C(X))$ and $\mathcal{RM}(B_1(X))$.

**Theorem 2.6.** If $\psi : \mathcal{RM}(C(X)) \to \mathcal{RM}(B_1(X))$ is defined by $M \mapsto M_B$ then $\psi$ is a bijection.

**Proof.** Let $\hat{M}$ be any member of $\mathcal{RM}(B_1(X))$. Therefore, by Corollary 2.1 we get $(\hat{M} \cap C(X))_B = \hat{M}$, where $\hat{M} \cap C(X) \in \mathcal{RM}(C(X))$ (By Theorem 2.5). Hence, for $\hat{M} \in \mathcal{RM}(B_1(X))$, we get $\hat{M} \cap (C(X)) \in \mathcal{RM}(C(X))$ such that $\psi(\hat{M} \cap C(X)) = \hat{M}$. This proves that $\psi$ is surjective.

To show that, $\psi$ is injective we assume $\psi(\hat{M}) = \psi(\hat{N})$. This implies $(\hat{M})_B = (\hat{N})_B$. Now by applying Theorem 2.2 we get $\hat{M} = (\hat{M})_B \cap C(X) = (\hat{N})_B \cap C(X) = \hat{N}$. Therefore, $\psi$ is surjective and hence, it is a bijection.

**Corollary 2.2.** $|\mathcal{RM}(C(X))| = |\mathcal{RM}(B_1(X))|$.}

It is well known that $\{\hat{\mathcal{M}}_f : f \in B_1(X)\}$, where each $\hat{\mathcal{M}}_f = \{M \in \mathcal{M}(B_1(X)) : f \in M\}$, forms a base for closed sets for the hull-kernel topology on $\mathcal{M}(B_1(X))$ and certainly $\mathcal{RM}(B_1(X))$ is a subspace of $\mathcal{M}(B_1(X))$. In the following theorem we show that the bijection $\psi$ obtained above becomes a homeomorphism if $\mathcal{RM}(C(X))$ is endowed with a finer topology than the subspace topology induced from the hull-kernel topology of $\mathcal{M}(C(X))$.

**Theorem 2.7.** Let $(X, \tau)$ be a Tychonoff space. Then for each $f \in B_1(X)$, the collection $\mathcal{M}_f^* = \{M \in \mathcal{RM}(C(X)) : f \in M_B\}$ forms a base for closed sets for some topology $\sigma$ on $\mathcal{RM}(C(X))$ which is finer than the subspace topology of the structure space of $\mathcal{M}(C(X))$. Moreover, $\psi : (\mathcal{RM}(C(X)), \sigma) \to \mathcal{RM}(B_1(X))$ given by $M \mapsto M_B$ is a homeomorphism.

**Proof.** To prove $\mathcal{B}^* = \{\mathcal{M}_f^* : f \in B_1(X)\}$ forms a base for closed sets for some topology $\sigma$ on $\mathcal{RM}(C(X))$, it is enough to show that $\emptyset \in \mathcal{B}^*$ and $\mathcal{B}^*$ is closed under finite union. It is easy to observe that, $\emptyset \in \mathcal{M}_f^* \in \mathcal{B}^*$. Now let $\mathcal{M}_f^* \cup \mathcal{M}_g^* \subseteq \mathcal{B}^*$, for some $f, g \in B_1(X)$. Take any $M \in \mathcal{M}_f^* \cup \mathcal{M}_g^*$. Therefore, $gf \in M_B$ and $M \in \mathcal{M}_f^* \cup \mathcal{M}_g^*$. This implies $\mathcal{M}_f^* \cup \mathcal{M}_g^* \subseteq \mathcal{M}_{fg}^*$. On the other hand, if we take any member $M \in \mathcal{M}_{fg}^*$ then we get $M_B \subseteq \mathcal{M}_{fg}^*$. Now let $\mathcal{M}_f^* \cup \mathcal{M}_g^* \subseteq \mathcal{B}^*$, for some $f, g \in B_1(X)$. Take any $M \in \mathcal{M}_f^* \cup \mathcal{M}_g^*$. Therefore, $M_B \subseteq \mathcal{M}_f^* \cup \mathcal{M}_g^*$. So $\mathcal{M}_{fg}^* \subseteq \mathcal{M}_f^* \cup \mathcal{M}_g^*$. This proves that $\mathcal{M}_f^* \cup \mathcal{M}_g^* = \mathcal{M}^*_{fg}$ and hence, $\mathcal{B}^*$ is closed under finite union.

Now to prove that $\psi$ is a homeomorphism, we need to show $\psi$ is bijective and exchanges the basic closed sets of $(\mathcal{RM}(C(X)), \sigma)$ and $\mathcal{RM}(B_1(X))$. The map $\psi$ is already proved in Theorem 2.6. Now for any $f \in B_1(X)$, $\psi(\mathcal{M}_f^*) = \{\psi(M) : f \in M_B\} = \{M_B : f \in M_B\} = \{N \in \mathcal{RM}(B_1(X)) : f \in N\} = \mathcal{M}_f \cap \mathcal{RM}(B_1(X))$, which is a basic closed set of $\mathcal{RM}(B_1(X))$ for the subspace topology induced from the hull-kernel topology on $\mathcal{M}(B_1(X))$. As $\psi$ exchanges the basic closed sets, it is a homeomorphism.

Before we conclude this section, we show that an injective map exists from $\mathcal{H}(C(X))$ into $\mathcal{H}(B_1(X))$, where $\mathcal{H}(C(X))$ and $\mathcal{H}(B_1(X))$ represent the collections of all hyper-real maximal ideals in $C(X)$ and $B_1(X)$ respectively. In what follows, we use the notation $I^*$ for the ideal of $B_1(X)$ generated by the subset $I$ of $B_1(X)$ and $m(I^*)$ for its maximal extension. The next theorem ensures that the ideal of $B_1(X)$ generated by a proper ideal of $C(X)$ is indeed proper, so that it has a maximal extension, say $m(I^*)$. 
Theorem 2.8. For any proper ideal $I$ of $C(X)$, $I^*$ is a proper ideal of $B_1(X)$, where $I^*$ denotes the ideal of $B_1(X)$ generated by $I$ as a subset of $B_1(X)$.

Proof. If possible let, $I^*$ be not proper. Then $I^* = B_1(X)$ and hence $1$ (the constant function with value 1) can be written as $1 = \sum_{i=1}^{n} \alpha_i f_i$, where $\alpha_i \in B_1(X)$ and $f_i \in I$ for all $i = 1, 2, ..., n$. For each $x \in X$, $\exists k \in \{1, 2, ..., n\}$, such that $f_k(x) \neq 0$, otherwise it contradicts that $1 = \sum_{i=1}^{n} \alpha_i f_i$. We consider the map $g(x) = \sum_{i=1}^{n} f_i^2(x)$, $\forall x \in X$. Clearly, $g \in I \subseteq C(x)$ and $g(x) \neq 0$, $\forall x \in X$. So $g$ is a unit in $I$. i.e., $I = C(X)$ - a contradiction. Hence, $I^*$ is a proper ideal of $B_1(X)$.

Theorem 2.9. If $M$ is a hyperreal maximal ideal of $C(X)$ then $m(M^*)$ is a hyperreal maximal ideal of $B_1(X)$.

Proof. If $m(M^*)$ is a real maximal ideal of $B_1(X)$ then by Theorem 2.8, $m(M^*) \cap C(X)$ is a real maximal ideal of $C(X)$. Since $M \subseteq m(M^*) \cap C(X)$ and $M$ is maximal it follows that $M = m(M^*) \cap C(X)$ - a contradiction to the fact that $M$ is hyperreal.

Theorem 2.10. The function $\zeta : \mathcal{H}(C(X)) \to \mathcal{H}(B_1(X))$ given by $\zeta(M) = m(M^*)$ is an injective function.

Proof. Let $M, N \in \mathcal{H}(C(X))$ be such that $m(M^*) = m(N^*)$. Then by maximality of $M$ and $N$ it follows that $M = m(M^*) \cap C(X) = m(N^*) \cap C(X) = N$.

Corollary 2.3. $|M(C(X))| \leq |M(B_1(X))|.

Proof. This is immediate from Theorem 2.9 and Theorem 2.10.

3. Characterization of Real compact spaces

From the discussion of the last section it follows that there is a one-one correspondence between the collections $RM(C(X))$ and $RM(B_1(X))$ given by $M \mapsto M_B$. It is well known [3] that a Tychonoff space $X$ is real compact if and only if every real maximal ideal of $C(X)$ is fixed. Utilizing the one-one correspondence as mentioned above, we get a characterization of real compact spaces via real maximal ideals of $B_1(X)$.

Theorem 3.1. A Tychonoff space $X$ is real compact if and only if every real maximal ideal of $B_1(X)$ is fixed.

Proof. Let $X$ be a real compact space and $\hat{M} \in RM(B_1(X))$. By Theorem 2.8 there exists $M \in RM(C(X))$ such that $\hat{M} = M_B$. Since $X$ is real compact, $M$ is fixed; i.e., $M = M_p$, for some $p \in X$. Hence, $\hat{M} = M_B = (M_p)_B = \hat{M}_p$ (by Example 2.1).

Conversely, let $M$ be any real maximal ideal of $C(X)$. Then $M_B \in RM(B_1(X))$ and so, $M_B$ is fixed. Therefore, $M (\subseteq M_B)$ is a fixed ideal. Hence, $X$ is real compact.

In Theorem 3.9 of our paper [2], we have proved a result for perfectly normal T$_1$-spaces though the same proof holds true for a larger class of spaces. In the following lemma, we state the result for the bigger class of spaces without proof.
Lemma 3.1. If $X$ is a $T_4$-space in which every point is a $G_δ$ point then the following statements are equivalent:

1. $X$ is finite.
2. Every maximal ideal in $B_1(X)$ is fixed.
3. Every ideal in $B_1(X)$ is fixed.

Theorem 3.2. Let $X$ be a $T_4$ real compact space in which every point is a $G_δ$-point. Then $B_1(X) = B_1^*(X)$ if and only if $X$ is finite.

Proof. If $X$ is finite then certainly, $B_1(X) = B_1^*(X)$. Conversely, let $\hat{M}$ be any maximal ideal of $B_1(X) = B_1^*(X)$. By Theorem 4.21 of [2], $\hat{M}$ is a real maximal ideal. Since $X$ is real compact, by Theorem 3.1 $\hat{M}$ is fixed. Finally, using Lemma 3.1 we can conclude that $X$ is finite. □

Corollary 3.1. Let $X$ be a perfectly normal $T_1$ real compact space. $B_1(X) = B_1^*(X)$ if and only if $X$ is finite.

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