THETA FUNCTIONS AND ADIABATIC CURVATURE
ON AN ABELIAN VARIETY

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Abstract. For an ample line bundle $L$ on an Abelian variety $M$, we study the theta functions associated with the family of line bundles $L \otimes T$ on $M$ indexed by $T \in \text{Pic}^0(M)$. Combined with an appropriate differential geometric setting, this leads to an explicit curvature computation of the direct image bundle $E$ on $\text{Pic}^0(M)$, whose fiber $E_T$ is the vector space spanned by the theta functions for the line bundle $L \otimes T$ on $M$. Some algebro-geometric properties of $E$ are also remarked.

1. Introduction

For an Abelian variety $M$ we write $\hat{M}$ for the Picard variety $\text{Pic}^0(M)$ with natural projections $\pi_1 : M \times \hat{M} \to M$ and $\pi_2 : M \times \hat{M} \to \hat{M}$, and $P$ for the Poincaré line bundle on $M \times \hat{M}$. Let $L$ be an ample line bundle on $M$. By considering $\pi_1^* L \otimes P$ on $M \times \hat{M}$ one defines a vector bundle $E$ to be $E := \pi_2^*(\pi_1^* L \otimes P)$ on $\hat{M}$ (regarded as a Fourier-Mukai transform of $L$, cf. [15]). We also study the pull-back $E' := \varphi_L^* E$ where $\varphi_L : M \to \hat{M}$ is the standard isogeny induced by the line bundle $L$, namely, $\varphi_L(\mu)$ is the line bundle $\tau_\mu^* L \otimes L^* \in \text{Pic}^0(M)$ via the translation $\tau_\mu$ induced by $\mu \in M$.

Our first result concerns an algebro-geometric property of $E'$:

\textbf{Theorem 1.1.} (See [5,10] (cf. [1] Theorem 1.2)). Let $E'$ and $L$ be as above. Then $E' \otimes L$ is a holomorphically trivial vector bundle on $M$.

Our second result concerns an application of Theorem 1.1 to the study of full curvature computation on the aforementioned $E$. For this purpose let us introduce a different geometric framework, cf. [7],[1]. The idea is roughly described as follows. Suppose a line bundle $G$ is given on $M$ with the first Chern class $c_1(G)$ represented by a translationally invariant 2-form $\omega_G$. It is well known that there exists a Hermitian metric $h_G$ for $G$, unique up to a constant scaling, such that $c_1(G; h_G) = \omega_G$ as differential forms. The similar reasoning and notation apply to $G \otimes T$ with $T \in \text{Pic}^0(M)$. In the case where $G$ is the above $L$, it is expected that by suitably normalizing the metrics $h_{G \otimes T}$ with $T$ running over $\text{Pic}^0(M)$, one can obtain a globally well-defined metric on the family of line bundles $\{G \otimes T\}_T$ on $M$. To be more precise, a Hermitian metric $\tilde{h}$ is defined on the bundle $\pi_1^* G \otimes P$ such that it restricts to the normalized $h_{G \otimes T}$. The point here is that $\tilde{h}$ can be explicitly described and computed. Moreover $\tilde{h}$ naturally induces an $L^2$-metric $h$ on $\text{Pic}^0(M, G \otimes T)$ which is identified with

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the fiber of $E$ at $T \in \text{Pic}^0(M)$. We can now state:

**Theorem 1.2.** (= Corollary 5.11 + Theorem 5.12) (cf. [4, Theorem 1.1]) The full curvature $\Theta(E, h)$ associated with the above metric $h$ on the bundle $E \rightarrow \hat{M}$ is given by

\[(1.1) \quad \Theta(E, h) = \left( -\pi \sum_{\alpha, \beta=1}^{n} W_{\alpha\beta} \frac{\delta_\alpha \delta_\beta}{\delta_\alpha \delta_\beta} d\tilde{\mu}_\alpha \wedge d\tilde{\mu}_\beta \right) \cdot I_{\Delta \times \Delta}\]

where $I_{\Delta \times \Delta}$ denotes the $\Delta \times \Delta$ identity matrix. And $-c_1(E, h)$ is the closed, positive, translation-invariant integral 2-form $\omega^\vee = \text{Iso}^* \left( \sum_{\alpha=1}^{n} \frac{\Delta}{\delta_\alpha} d\eta_\alpha \wedge d\eta_{n+\alpha} \right)$ where $\text{Iso} : \hat{M} \rightarrow M^*$ is the isomorphism defined in Lemma 5.2 and $\Delta = \det \Delta_\delta = \prod_{\alpha=1}^{n} \delta_\alpha$. See (2.1) for $\delta_\alpha$ and (2.2) for $W_{\alpha\beta}$.

An immediate corollary is that $E$ admits a Hermitian-Einstein metric (i.e. the aforementioned metric $h$) with respect to the invariant Kähler form $\omega^\vee$ of Theorem 1.2 on $\hat{M}$. See [11] for related discussions aiming at computing the Hermitian-Einstein metric on these vector bundles $E$ (alternatively called Picard bundles there); this work [11] is based on algebraic methods involving Mumford’s theta group and invariant theory, but neither the explicit form of the metric nor the curvature is written down or computed. With the curvature (1.1) it is straightforward to calculate the Chern classes and Chern character. In this regard an analogous Picard bundle $E$ (in the terminology of [10]) on $\text{Pic}^d(C)$ where $C$ is a curve of genus $\geq 1$ can be formed, and the total Chern class $C(E)$ can be explicitly determined via the cycle $W^\vee_d(C)$ in $\text{Pic}^d(C)$ ([11, pp. 317-319]) or via the Grothendick-Riemann-Roch formula ([11, pp. 334-336]). It can be checked that Theorem 1.2 above specialized to $\text{dim}_C M = 1$ gives the same Chern class as that of [1] for $C$ an elliptic curve. It is, however, unclear to the authors whether a purely algebraic analogue of the explicit polarization $\omega^\vee$ (cf. Remark 5.13) on $\hat{M}$ above is available in the literature.

Although the idea of our proofs follows closely that of the elliptic curve case [1], the technicality here is much more laborious and a certain amount of computational details reveal their complexity only in high dimensions. We feel it worthwhile to give a separate treatment that helps to streamline the argument. So whenever the proof obviously duplicates the elliptic curve case, we simply point this out and do not reproduce the proof (e.g. Proposition 5.8); we basically give only those proofs that require computation and reasoning not easily foreseen in the one-dimensional case. The reader is invited to turn to [4]; see also [3], [10], [11], [15] and [18] for related discussions. The setting of [18] uses an $L^2$-metric in greater generality and arrives at the “projective flatness” of the bundle $E$ (for families of polarized Abelian varieties); see also their previous work, in particular the part on the poly-stability of Picard bundles [17, Theorems 3 and 4]. For background materials the work

\[\text{A mistake for } c_1(E) \text{ in the elliptic-curve case [4, Theorem 1.1 and Corollary 8.5] has been corrected in [5].}\]
[13] contains a wealth of information on both the analytic and algebraic aspects of Abelian varieties.

Finally let us make the following remarks. The use of theta functions is instrumental for both proofs of Theorem 1.1 and 1.2. By contrast, a purely algebraic proof of Theorem 1.1 over any algebraically closed field (not necessarily of characteristic zero) can be found in [10] by G. Kempf for different purpose. Those algebraic proofs are basically conceptual ones. It seems natural to ask for a more down-to-earth point of view; such a possibility is hopefully provided in this paper. Another merit of this paper lies perhaps in our ongoing work, which is to study certain “spectral bundles” naturally arising from the above differential geometric setting. These spectral bundles include the above bundle $E$ as the lowest-energy level (cf. [4, Introduction]). By making use of the theta functions here, we can explore in depth the holomorphic structure of those spectral bundles on an Abelian variety and explicitly describe the associated eigen-sections at higher level (cf. [6]). Let us mention in passing that the theta function method here is expected to be also relevant to analogous problems in the $p$-adic setting (see e.g. [16]); we hope to discuss it elsewhere in the future.

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2. Holomorphic Line Bundles over $M = V/\Lambda$

In this section we basically follow the notation and convention of [8]. Let $V$ be a complex vector space of dimension $n$ and $\Lambda \subseteq V$ a discrete lattice of rank $2n$. Assume that $M = V/\Lambda$ is an Abelian variety, and let $\omega$ be an invariant integral form, positive of type $(1,1)$. Choose a basis $\lambda_1, ..., \lambda_{2n}$ for $\Lambda$ over $\mathbb{Z}$ such that in terms of dual coordinates $x_1, ..., x_{2n}$ on $V$

\begin{equation}
\omega = \sum_{i=1}^{n} \delta_i \, dx_i \wedge dx_{n+i}, \quad \delta_i \in \mathbb{N}, \quad \delta_i | \delta_{i+1}, \quad i = 1, \ldots, n - 1.
\end{equation}

With the complex basis $v_\alpha = \delta^{-1}_\alpha \lambda_\alpha$ of $V$, $\alpha = 1, ..., n$, writing $\lambda_\alpha = \delta_\alpha v_\alpha$, $\lambda_{n+\alpha} = \sum_{k=1}^{n} \tau_{\alpha k} v_k$ we take the period matrix $\Omega$ of $\Lambda \subseteq V$ to be the $n \times 2n$ matrix $(\Delta_\delta, Z)$ where

\begin{equation}
(\Delta_\delta) = \begin{pmatrix}
\delta_1 & 0 \\
0 & \ddots \\
0 & \delta_n
\end{pmatrix}, \quad Z = (\tau_{\alpha\beta}); \quad (\text{Im} \, Z)^{-1} =: W = (W_{\alpha\beta}).
\end{equation}

Here $Z$ is symmetric and $\text{Im} \, Z$ is positive definite.
Notation 2.1. A vector $v = \sum_{\alpha=1}^{n} z_{\alpha} v_{\alpha} \in V$ is also expressed by its complex coordinates $(z_1, ..., z_n)$, $z_{\alpha} = z_{\alpha 1} + i z_{\alpha 2}$ on $V$, with $dz_{\alpha}$ dual to $v_{\alpha}$. We use the same notation $(z_1, ..., z_n)$ for complex coordinates on $M$ whenever there is no danger of confusion.

One has

$$dz_{\alpha} = \delta_{\alpha} dx_{\alpha} + \sum_{k=1}^{n} \tau_{ak} dx_{n+k}, \quad d\bar{z}_{\alpha} = \delta_{\alpha} dx_{\alpha} + \sum_{k=1}^{n} \tau_{ak} dx_{n+k}, \quad \alpha = 1, ..., n.$$ (2.3)

Or, equivalently (the following is needed for (5.5))

$$\begin{pmatrix} dx_1 \\ \vdots \\ dx_{2n} \end{pmatrix} = \left( \frac{i}{2} \Delta_g^{-1} Z W \right) \begin{pmatrix} dz_1 \\ \vdots \\ dz_{2n} \end{pmatrix} + \left( -\frac{i}{2} \Delta_g^{-1} Z W \right) \begin{pmatrix} d\bar{z}_1 \\ \vdots \\ d\bar{z}_{2n} \end{pmatrix}. \quad (2.4)$$

A holomorphic line bundle over $M$ can be described by its multipliers $\{ e_{\lambda \alpha} \in O^*(V) \}$ [8, p.308].

We define $L_0 \to M$ to be the holomorphic line bundle given by the multipliers

$$e_{\lambda \alpha}(v) \equiv 1, \quad e_{\lambda n_{\alpha}}(v) \equiv e^{-2\pi i z_{\alpha} - \pi i r_{\alpha} \omega}, \quad \alpha = 1, ..., n. \quad (2.5)$$

It is known that $c_1(L_0) = [\omega]$, $\omega = \sum_{i=1}^{n} \delta_i dx_i \wedge dx_{n+i}$ as given in (2.1) ([8, p.310]). The associated theta functions satisfy

$$\begin{cases} \theta(z_1, ..., z_{\alpha} + \delta_{\alpha}, ..., z_n) = \theta(z_1, ..., z_n) \\ \theta(z_1 + \tau_{\alpha 1}, z_2 + \tau_{\alpha 2}, ..., z_n + \tau_{\alpha n}) = e^{-2\pi i z_{\alpha} - \pi i r_{\alpha} \omega} \theta(z_1, ..., z_n), \quad \alpha = 1, ..., n. \end{cases} \quad (2.6)$$

By the same token, a Hermitian metric $h_{L_0}(v) > 0$ on $L_0$ is charaterized by the quasi-periodic property:

$$\begin{cases} h_{L_0}(z_1, ..., z_{\alpha} + \delta_{\alpha}, ..., z_n) = h_{L_0}(z_1, ..., z_n) \\ h_{L_0}(z_1 + \tau_{\alpha 1}, z_2 + \tau_{\alpha 2}, ..., z_n + \tau_{\alpha n}) = |e^{2\pi i z_{\alpha} + \pi i r_{\alpha} \omega}|^2 h_{L_0}(z_1, ..., z_n), \quad \alpha = 1, ..., n. \end{cases} \quad (2.7)$$

Lemma 2.2. For the holomorphic line bundle $L_0 \to M$ above, one has the quasi-periodic entire functions on $V$

$$\theta_m(z_1, ..., z_n) := \sum_{k \in \mathbb{Z}^n} \left( e^{2\pi i \sum_{\alpha, \beta=1}^{n} k_{\alpha} k_{\beta} \tau_{\alpha \beta}} e^{2\pi i \sum_{\alpha=1}^{n} \tau_{\alpha \beta} \frac{m_{\alpha}}{s_{\alpha}}} e^{2\pi i \sum_{\beta=1}^{n} W_{\alpha \beta} m_{\alpha} z_{\beta}} \right) \quad (2.8)$$

where $k = (k_1, ..., k_n) \in \mathbb{Z}^n$ and $m \in \mathfrak{M} = \{(m_1, ..., m_n) \mid 0 \leq m_\alpha < \delta_\alpha, \ m_\alpha \in 0 \cup \mathbb{N}, \ \alpha = 1, ..., n\}$, as a basis of global holomorphic sections of $L_0$ and $h_{L_0}(v) = e^{-2\pi i \sum_{\alpha, \beta=1}^{n} W_{\alpha \beta} z_\alpha z_{\beta}}$ as a metric on $L_0$ where $W = (W_{\alpha \beta})$ is the inverse matrix of $\text{Im} Z$. 


Proof. Using the Riemann \( \theta \)-function \( \vartheta(v) = \sum_{k \in \mathbb{Z}^n} \left( e^{2\pi i \sum_{\alpha=1}^n k_\alpha \tau_{\alpha \beta}} e^{2\pi i \sum_{\alpha=1}^n k_\alpha z_\alpha} \right) \) (cf. [8, p.320]) and comparing with (2.8), we have

\[
\theta_m(z_1, \ldots, z_n) = e^{2\pi i \sum_{\alpha=1}^n m_\alpha z_\alpha} \cdot \vartheta(z_1 + \sum_{\beta=1}^n m_\beta \tau_{1\beta}, \ldots, z_n + \sum_{\beta=1}^n m_\beta \tau_{n\beta}).
\]

By the quasi-periodic property of \( \vartheta(v) \) one finds that \( \theta_m(z_1, \ldots, z_n) \) satisfies the quasi-periodic property (2.6) (cf. [4, Lemma 2.1] for the one-dimensional case whose argument can be easily generalized to the present case). The linear independence of \( \{\theta_m\}_m \) is proved later in Lemma 4.2. With the fact \( \dim H^0(M, L_0) = \prod \delta_\alpha \) [8, p.312] it follows from Lemma 2.1 for the one-dimensional case whose argument can be easily generalized that \( \{\theta_m\}_m \) span \( H^0(M, L_0) \).

For any \( \mu = \sum_{\alpha=1}^n \mu_\alpha v_\alpha \in V \) where \( \mu_\alpha = \mu_{\alpha 1} + i\mu_{\alpha 2} \), we have a map \( \tau_\mu : M \to M \) defined by the translation by \( [\mu] \in M \). Let \( L_\mu := \tau_\mu^* L_0 \to M \), which can be given by multipliers

\[
ed_\lambda \alpha (v) \equiv 1, \quad ed_{\alpha + \alpha} \alpha (v) \equiv e^{-2\pi i(z_\alpha + \mu_\alpha) - \pi i \alpha \alpha} \quad \alpha = 1, \ldots, n.
\]

Any global holomorphic sections \( \tilde{\theta} \) of \( L_\mu \to M \) can be described via quasi-periodic entire functions \( \theta \) on \( V \) satisfying

\[
\begin{cases}
\theta(z_1, \ldots, z_n + \delta_\alpha, \ldots, z_n) = \theta(z_1, \ldots, z_n) \\
\theta(z_1 + \tau_{\alpha 1}, z_2 + \tau_{\alpha 2}, \ldots, z_n + \tau_{\alpha n}) = e^{-2\pi i(z_\alpha + \mu_\alpha) - \pi i \alpha \alpha} \theta(z_1, \ldots, z_n),
\end{cases} \quad \alpha = 1, \ldots, n,
\]

with a metric \( h_{L_\mu} (v) \) on \( L_\mu \to M \):

\[
h_{L_\mu}(z_1, \ldots, z_n + \delta_\alpha, \ldots, z_n) = h_{L_\mu}(z_1, \ldots, z_n) \\
h_{L_\mu}(z_1 + \tau_{\alpha 1}, z_2 + \tau_{\alpha 2}, \ldots, z_n + \tau_{\alpha n}) = e^{2\pi i(z_\alpha + \mu_\alpha) + \pi i \alpha \alpha} h_{L_\mu}(z_1, \ldots, z_n), \quad \alpha = 1, \ldots, n.
\]

Since all the holomorphic line bundles on \( M \) having the same first Chern class as that of \( L_0 \) can be represented as a translate of \( L_0 \) [8, p.312], it follows from Lemma 2.2 (2.11) and (2.12) that

**Lemma 2.3.** Fix \( \mu \in V \). For \( L_\mu \to M \) as above, one has the quasi-periodic entire functions on \( V \) (cf. Lemma 2.2 for relevant notations below):

\[
\theta_m(v; \mu) := \theta_m(v + \mu) = \sum_{k \in \mathbb{Z}^n} \left( e^{2\pi i \sum_{\alpha=1}^n k_\alpha \tau_{\alpha \beta}} e^{2\pi i \sum_{\beta=1}^n m_\beta \tau_{\alpha \beta}} e^{2\pi i \sum_{\alpha=1}^n (k_\alpha m_\beta + m_\alpha k_{\beta}) (z_\alpha + \mu_\alpha)} \right),
\]

where \( k \in \mathbb{Z}^n \) and \( m \in \mathbb{M} \)

as a basis of \( H^0(M, L_\mu) \), and \( h_{L_\mu}(v; \mu) := h_{L_0}(v + \mu) = e^{-2\pi i \sum_{\alpha=1}^n W_{\alpha \beta} \operatorname{Im}(z_\alpha + \mu_\alpha) \operatorname{Im}(z_\beta + \mu_\beta)} \) as a metric on \( L_\mu \).
Let $\text{Pic}^0(M) = \hat{M}$ denote the group of holomorphic line bundles on $M$ with the first Chern class zero. In the notation of [8] p.318 for the natural identification
\begin{equation}
\text{Pic}^0(M) \cong \frac{H^1(M, \mathcal{O})}{H^1(M, \mathbb{Z})} \approx \frac{H^{0,1}(M)}{H^1(M, \mathbb{Z})}
\end{equation}
the image of $H^1(M, \mathbb{Z})$ in $\mathbb{V}^* = H^{0,1}(M)$ is the lattice $\mathbb{X}^* = \mathbb{Z}\{dx_1, ..., dx_{2n}\}$ which consists of conjugate linear functionals on $V$ whose real part is half integral on $\Lambda \subseteq \mathbb{V}$. Recall the period matrix $(\Delta_\delta, Z)$, we write the conjugate linear parts of $dx_1, ..., dx_{2n}$ as $(T$ for transpose below)
\begin{equation}
\begin{cases}
(dx_1^*, ..., dx_n^*)^T = (\frac{1}{2}(\Delta_\delta)^{-1}Z(\text{Im } Z)^{-1})(dz_1, ..., dz_n)^T \\
(dx_{n+1}^*, ..., dx_{2n}^*)^T = (\frac{1}{2}(\text{Im } Z)^{-1})(dz_1, ..., dz_n)^T.
\end{cases}
\end{equation}
Reordering $\{dx_i^*\}$ by setting $dy_{i+1}^* = -dx_{i+1}^*$ and $dy_{i+2}^* = dx_{i+2}^*$, for $i = 1, ..., n$. If we let $v_i^* = \frac{\delta_n}{\delta_i} dy_i^*$, we have $\mathbb{X}^* = \mathbb{Z}\{dy_1^*, ..., dy_{2n}^*\}$ with $(dy_1^*, ..., dy_{2n}^*)^T = (\delta_i(\Delta_\delta)^{-1} | \delta_n(\Delta_\delta)^{-1}Z(\Delta_\delta)^{-1})(v_1^*, ..., v_n^*)^T$. One has $\hat{M} = \mathbb{V}^*/\mathbb{X}^*$.

With $v_i^*$ above, we let $(\hat{\mu}_1, ..., \hat{\mu}_n)$ be the complex coordinates of $\hat{\mu} = \sum_{a=1}^n \hat{\mu}_a v_a^*$ on $\mathbb{V}^*$ (and on $\hat{M}$); $d\hat{\mu}_a$ are the forms dual to $v_a^*$. Similarly, we write $\mu = \sum_{a=1}^n \mu_a v_a \in V$ with $v_a$ in the beginning of this section. Denote by $[\hat{\mu}], [\mu]$ the class of $\hat{\mu}, \mu$ in $\hat{M}, M$ respectively.

A well-known map $\varphi_{L_0} : M \to \text{Pic}^0(M) = \hat{M}$ via the translation $\tau_\mu : M \to M$ with $[\mu] \in M$ is given by $\varphi_{L_0}([\mu]) = \tau_\mu^* L_0 \otimes L_0^*$ where $L_0^*$ is the dual line bundle of $L_0$. Denote by $\tilde{\varphi}_{L_0} : V \to \mathbb{V}^*$ the natural lifting map. The following property is well known (see [1] Property 1] and [8] pp. 315-317 for more details):

**Property 2.4.** $\tilde{\varphi}_{L_0} : V \to \mathbb{V}^*$ is a complex linear transformation such that
\begin{equation}
\tilde{\varphi}_{L_0}(v_\alpha) = \frac{\delta_n}{\delta_\alpha} v_\alpha^*, \quad \alpha = 1, ..., n.
\end{equation}
Thus $\hat{\mu}_\alpha = \frac{\delta_n}{\delta_\alpha} \mu_\alpha, \alpha = 1, ..., n$, if $\hat{\mu} = \tilde{\varphi}_{L_0}(\mu)$.

We denote by $P_{[\hat{\mu}]}$ or simply $P_{\hat{\mu}}$ the line bundle corresponding to $[\hat{\mu}] \in \hat{M} = \text{Pic}^0(M)$ in (2.13). From Property 2.4 with $\hat{\mu} = \varphi_{L_0}(\mu)$, one has
\begin{equation}
P_{\hat{\mu}} = P_{\varphi_{L_0}(\mu)} \cong \tau_\mu^* L_0 \otimes L_0^*, \quad \forall \mu \in V.
\end{equation}
Recall the Poincaré line bundle [8] p.328:

**Lemma 2.5.** There is a unique holomorphic line bundle $P \to M \times \hat{M}$ called the Poincaré line bundle satisfying i) $P|_{M \times \{\hat{\mu}\}} \cong P_{\hat{\mu}}$ and ii) $P|_{\{0\} \times \hat{M}}$ is a holomorphically trivial line bundle.
3. A holomorphic line bundle $\tilde{K} \to M \times M$

From the Poincaré line bundle $P \to M \times \tilde{M}$ and the map $\varphi_{L_0} : M \to \tilde{M}$ as introduced above, we are going to define a holomorphic line bundle $\tilde{K} \to M \times M$. Let $M_1 = M_2 = M$ and $\pi : V \times V \to M_1 \times M_2 = V/\Lambda \times V/\Lambda$ with projections $\pi_i : M_1 \times M_2 \to M_i$, $i = 1, 2$. We rewrite the lattice vectors $\lambda_0, \lambda_{n+\alpha}$ ($\alpha = 1, ..., n$) of $M_1$ as $\lambda_{0,0}, \lambda_{n+\alpha,0}$ respectively, and analogously $\lambda_0, \lambda_{n+\beta}$ as $\lambda_{0,\beta}, \lambda_{0,n+\beta}$ for the case of $M_2$.

**Definition 3.1.** Define $\tilde{K} := \pi_1^*L_0 \otimes (\text{Id} \times \varphi_{L_0})^*P \otimes \pi_2^*L_0 \to M_1 \times M_2$ where $P \to M \times \tilde{M}$ is the Poincaré line bundle.

**Proposition 3.2.** With the notations above, a system of multipliers for $\tilde{K}$ can be

\[
\begin{aligned}
&\quad \quad e_{\lambda_{0,\alpha}}(v; \mu) \equiv 1, \quad e_{\lambda_{n+\alpha,\alpha}}(v; \mu) = e^{-2\pi i z_\alpha - 2\pi i \mu_\alpha - \pi i \tau_\alpha}, \\
&\quad e_{\lambda_{0,\beta}}(v; \mu) \equiv 1, \quad e_{\lambda_{0,n+\beta}}(v; \mu) = e^{-2\pi i z_\beta - 2\pi i \mu_\beta - \pi i \tau_\beta}.
\end{aligned}
\]

**Proof.** Recall that a holomorphic line bundle on $M \times M = V/\Lambda \times V/\Lambda$ is essentially described by a system of multipliers $\{e_{\lambda_{0,\alpha}}, e_{\lambda_{n+\alpha,\beta}}, e_{\lambda_{0,n+\beta}} \in \mathcal{O}^*(V \times V)\}$ satisfying the compatibility relations: For $i, j = \{1, ..., 2n\}$ (in (3.2) below, we write $e_{\lambda_{0,\alpha}}(v + \lambda_i; \mu)$ to mean $e_{\lambda_{0,\alpha}}(z_1, ..., z_n + \delta_i, ..., z_n; \mu_1, ..., \mu_n)$, $e_{\lambda_{0,\alpha}}(v; \mu + \lambda_i + \beta)$ to mean $e_{\lambda_{0,\alpha}}(z; \mu_1 + \tau_1, \mu_2 + \tau_2, ..., \mu_n + \tau_n)$ etc. for notational convenience)

\[
\begin{aligned}
&\quad e_{\lambda_{0,\alpha}}(v + \lambda_j; \mu)e_{\lambda_{0,\alpha}}(v; \mu) = e_{\lambda_{0,\alpha}}(v + \lambda_i; \mu)e_{\lambda_{0,\alpha}}(v; \mu), \\
&e_{\lambda_{0,\alpha}}(v; \mu + \lambda_j)e_{\lambda_{0,\alpha}}(v; \mu) = e_{\lambda_{0,\alpha}}(v; \mu + \lambda_i)e_{\lambda_{0,\alpha}}(v; \mu), \\
&\quad e_{\lambda_{0,\alpha}}(v; \mu + \lambda_j)e_{\lambda_{0,\alpha}}(v; \mu) = e_{\lambda_{0,\alpha}}(v; \mu + \lambda_i)e_{\lambda_{0,\alpha}}(v; \mu).
\end{aligned}
\]

Using the multipliers for $\pi_1^*L_0 : \{e_{\lambda_{0,\alpha}}(v; \mu) \equiv 1, \quad e_{\lambda_{n+\alpha,\alpha}}(v; \mu) = e^{-2\pi i z_\alpha - \pi i \tau_\alpha}, \quad e_{\lambda_{0,\beta}}(v; \mu) = e_{\lambda_{0,n+\beta}}(v; \mu) \equiv 1, \quad \alpha, \beta = 1, ..., n\}$ (cf. (2.5)) and the similar multipliers for $\pi_2^*L_0$, we claim that a system of multipliers for $\varphi_{L_0}^*P$ can be chosen to be

\[
\begin{aligned}
&\quad e_{\lambda_{0,\alpha}}(v; \mu) \equiv 1, \quad e_{\lambda_{n+\alpha,\alpha}}(v; \mu) = e^{-2\pi i \mu_\alpha}, \\
&\quad e_{\lambda_{0,\beta}}(v; \mu) \equiv 1, \quad e_{\lambda_{0,n+\beta}}(v; \mu) = e^{-2\pi i \mu_\beta} \quad \alpha, \beta = 1, ..., n.
\end{aligned}
\]

Since (3.3) satisfies (3.2), it defines a holomorphic line bundle $J$ on $M_1 \times M_2$. To prove that $J \cong (\text{Id} \times \varphi_{L_0})^*P$ the idea is to compare $i)$ the system of multipliers for $(\text{Id} \times \varphi_{L_0})^*P|_{M \times \{\mu\}} \cong \tau_\mu^*L_0 \otimes L_0 \cong P_{\varphi_{L_0}^*(\mu)} \to M_1 \times \{\mu\}$ given by $e_{\lambda_{0,\alpha}}(v) \equiv 1, \quad e_{\lambda_{n+\alpha,\alpha}}(v) = e^{-2\pi i \mu_\alpha}$ ($\alpha = 1, ..., n$) with the system of multipliers for $J|_{M \times \{\mu\}}$, and $ii)$ the multipliers for the trivial line bundle $(\text{Id} \times \varphi_{L_0})^*P|_{\{0\} \times M}$ by $e_{\lambda_0}(v) = e_{\lambda_{n+\alpha,0}}(v) \equiv 1$ ($\alpha = 1, ..., n$) with the ones for $J|_{\{0\} \times M}$. The argument for verifying $i)$, $ii)$ here is similar to [4 Prop. 4.2].
In the notation of Lemma 2.3, \( \theta_m(v; \mu) = \theta_m(v + \mu) \) (with \( m \in \mathcal{M} \)) and the metric \( h(v; \mu) = h_{L_0}(v + \mu) \) are considered as functions in the joint variables \((v, \mu) \in V \times V\). From Proposition 3.2 and the quasi-periodic properties of \( \theta_m(\xi) \) and \( h_{L_0}(\xi) \) it holds the following global property of the family of functions \( \{\theta_m(v; \mu)\}_{m \in \mathcal{M}} \) (cf. [4, Prop. 4.3]):

**Proposition 3.3.** For the above holomorphic line bundle \( \tilde{K} \to M_1 \times M_2 \), the functions \( \theta_m(v; \mu) \), \( m \in \mathcal{M} \) just defined serves as a basis of holomorphic sections of \( \tilde{K} \), and \( h(v; \mu) \) as a metric on \( \tilde{K} \), induces the metric \( h_{L_\mu} \) in Lemma 2.3 on the restriction \( \tilde{K}|_{M \times \{\mu\}} \).

From the metrics \( h(v; \mu) \) on \( \tilde{K} \), \( h_{\pi_1L_0}(v; \mu) = e^{-2\pi \sum_{\alpha, \beta} W_{\alpha \beta} \text{Im}(z_{\alpha}) \text{Im}(z_{\beta})} \) on \( \pi_1^*L_0 \) and \( h_{\pi_2L_0}(v; \mu) = e^{-2\pi \sum_{\alpha, \beta} W_{\alpha \beta} \text{Im}(\mu_{\alpha}) \text{Im}(\mu_{\beta})} \) on \( \pi_2^*L_0 \) (cf. \( h_{L_0}(v) \) in Lemma 2.3), we can equip the line bundle \((\text{Id} \times \varphi_{L_0})^*P = (\pi_1^*L_0)^* \otimes \tilde{K} \otimes (\pi_2^*L_0)^* \) with an induced metric:

\[
(3.4) \quad h((\text{Id} \times \varphi_{L_0})^*P(v; \mu) = (h_{\pi_1L_0}(v; \mu))^{-1} h(v; \mu) (h_{\pi_2L_0}(v; \mu))^{-1} = e^{-4\pi \sum_{\alpha, \beta} W_{\alpha \beta} \text{Im}(z_{\alpha}) \text{Im}(\mu_{\beta})}.
\]

For any metric \( h \), we can calculate its curvature \( \Theta \) by \( \Theta = -\partial \bar{\partial} \log h \):

**Proposition 3.4.** The curvature of the metric in (3.4) is

\[
(3.5) \quad \Theta((\text{Id} \times \varphi_{L_0})^*P(v; \mu) = \pi \sum_{\alpha, \beta=1}^n W_{\alpha \beta}(dz_{\alpha} \wedge d\bar{z}_{\beta} + d\mu_{\alpha} \wedge d\bar{\mu}_{\beta}) \quad \text{on } M_1 \times M_2.
\]

**Proof.** By writing

\[
h((\text{Id} \times \varphi_{L_0})^*P(v; \mu) = e^{-4\pi \sum_{\alpha, \beta=1}^n W_{\alpha \beta} \text{Im}(z_{\alpha}) \text{Im}(\mu_{\beta})} = e^{\pi \sum_{\alpha, \beta=1}^n W_{\alpha \beta}(z_{\alpha} - \bar{z}_{\alpha})(\mu_{\beta} - \bar{\mu}_{\beta})},
\]

a straightforward calculation gives

\[
\Theta((\text{Id} \times \varphi_{L_0})^*P(v; \mu) = -\partial \bar{\partial} \left( \pi \sum_{\alpha, \beta=1}^n W_{\alpha \beta}(z_{\alpha} - \bar{z}_{\alpha})(\mu_{\beta} - \bar{\mu}_{\beta}) \right)
\]

\[
= \pi \sum_{\alpha, \beta=1}^n W_{\alpha \beta}(dz_{\alpha} \wedge d\bar{z}_{\beta} + d\mu_{\beta} \wedge d\bar{\mu}_{\alpha}) = \pi \sum_{\alpha, \beta=1}^n W_{\alpha \beta}(dz_{\alpha} \wedge d\bar{\mu}_{\beta} + d\mu_{\alpha} \wedge d\bar{z}_{\beta})
\]

since \( W = (W_{\alpha \beta}) \) is symmetric. \( \square \)
4. Direct image bundle $K \to M_2$ with an $L^2$-metric

With the projections $\pi_i : M_1 \times M_2 \to M_i, i = 1, 2$, the direct image bundle $K$ for $\tilde{K}$ in Section 3 is given as follows (by projection formula on the third factor of $\tilde{K}$)

\begin{equation}
K := \pi_2^* \tilde{K} = \pi_2^*(\pi_1^* L_0 \otimes (Id \times \varphi_{L_0})^* P \otimes \pi_2^* L_0) \cong \pi_2^*(\pi_1^* L_0 \otimes (Id \times \varphi_{L_0})^* P) \otimes L_0 \to M_2.
\end{equation}

**Definition 4.1.** Define a metric $(\ , \ )$ on $K$ by the inner product using $(\ , \ )_{h_{L_\mu}}$ on $K|_{\mu} = H^0(M, \tilde{K}|_{M \times \{\mu\}})$:

\begin{equation}
(\theta_m(v), \theta_{m'}(v))_{h_{L_\mu}} := \int_M h_{L_\mu}(v) \theta_m(v) \overline{\theta_{m'}(v)} \left(\frac{i}{2}\right)^n dz_1 \wedge d\bar{z}_1 \wedge \ldots \wedge dz_n \wedge d\bar{z}_n.
\end{equation}

The main lemma for our computations is as follows.

**Lemma 4.2.** With the inner product $(\ , \ )_{h_{L_\mu}}$, the basis $\{\theta_m(v; \mu)\}_{m \in \mathcal{M}}$ in Lemma 2.3 constitute an orthogonal basis of $H^0(M, L_\mu)$, with

\[
(\theta_m(v; \mu), \theta_m(v; \mu))_{h_{L_\mu}} = \sqrt{\det \left( \frac{i}{2} \Pi_{\alpha=1} \delta_\alpha \right)} \left( \frac{1}{2} \Pi_{\alpha=1} \delta_\alpha \right)^{2n} e^{2 \sum_{\alpha, \beta, \gamma=1}^{\mu} \tau_{\alpha \beta \gamma}}.
\]

**Proof.** Write $z_\alpha = z_{\alpha_1} + iz_{\alpha_2}$, $\tau_{\alpha \beta} = \tau_{\alpha_1 \beta_1} + i \tau_{\alpha_1 \beta_2}$ ($z_{\alpha_1}, z_{\alpha_2}, \tau_{\alpha_1 \beta_1}, \tau_{\alpha_1 \beta_2} \in \mathbb{R}$) and $d\text{vol} = dz_{11} dz_{12} \ldots dz_{n1} dz_{n2}$. We have, via Definition 4.1 and Lemma 2.2, for $m, m' \in \mathcal{M}$,

\begin{equation}
I_{m, m'} := (\theta_m(v; \mu), \theta_{m'}(v; \mu))_{h_{L_\mu}} = \int_M h_{L_\mu}(v) \theta_m(v) \overline{\theta_{m'}(v)} \ d\text{vol}
\end{equation}

\begin{align*}
&= \int_M e^{-2 \sum_{\alpha, \beta=1}^{\mu} W_{\alpha \beta}(z_{\alpha_1} + m_{\alpha_2} z_{\alpha_2})} \sum_{k, j \in \mathbb{Z}^n} \left( e^{2 \sum_{\alpha, \beta=1}^{\mu} \tau_{\alpha \beta} \frac{m_{\alpha}}{z_{\alpha_1} + m_{\alpha_2}}} e^{2 \sum_{\alpha, \beta, \gamma=1}^{\mu} \tau_{\alpha \beta \gamma} (z_{\alpha_1} + m_{\alpha_2}) (z_{\alpha_2} + m_{\beta_2})} \right) \\
&\quad \times \left( e^{-2 \sum_{\alpha, \beta=1}^{\mu} j_{\alpha_1} j_{\beta_1} \frac{m_{\beta}}{z_{\alpha_1} + m_{\alpha_2}}} e^{-2 \sum_{\alpha, \beta, \gamma=1}^{\mu} \tau_{\alpha \beta \gamma} j_{\beta_2} (z_{\alpha_1} + m_{\beta_2})} \right) d\text{vol} \\
&= \int_M e^{-2 \sum_{\alpha, \beta=1}^{\mu} W_{\alpha \beta}(z_{\alpha_1} + m_{\alpha_2} z_{\alpha_2})} \left( e^{2 \sum_{\alpha, \beta=1}^{\mu} \tau_{\alpha \beta} \frac{m_{\alpha}}{z_{\alpha_1} + m_{\alpha_2}}} e^{2 \sum_{\alpha, \beta, \gamma=1}^{\mu} \tau_{\alpha \beta \gamma} (z_{\alpha_1} + m_{\alpha_2}) (z_{\alpha_2} + m_{\beta_2})} \right) \\
&\quad \times \left( e^{-2 \sum_{\alpha, \beta=1}^{\mu} j_{\alpha_1} j_{\beta_1} \frac{m_{\beta}}{z_{\alpha_1} + m_{\alpha_2}}} e^{-2 \sum_{\alpha, \beta, \gamma=1}^{\mu} \tau_{\alpha \beta \gamma} j_{\beta_2} (z_{\alpha_1} + m_{\beta_2})} \right) \ d\text{vol}
\end{align*}
We first show that \( I_{mm'} = 0 \) if \( m \neq m' \). With the change of variables \( z_{a1} = \delta_{a} t_{a} + \sum_{l=1}^{n} \tau_{al1} t_{n+l} \), 
\( z_{a2} = \sum_{l=1}^{n} \tau_{al2} t_{n+l}, \alpha = 1, ..., n \), we can change the domain of integration from \( M \) to the 2\( n \)-
dimensional unit cube \( I^{2n} \):

\[
(4.4) \quad I_{mm'} = \int_{I^{2n}} e^{-2\pi \sum_{\alpha,\beta=1}^{n} W_{\alpha\beta} \left( \sum_{l=1}^{n} \tau_{al2} t_{a+n+l} + \mu_{\alpha2} \right) + \mu_{\beta2}} \cdot \sum_{j,k \in \mathbb{Z}^{n}} \pi^{\frac{n}{2}} \sum_{\alpha,\beta=1}^{n} \left( k_{a} k_{b} \tau_{\alpha\beta} - j_{a} j_{b} \tau_{\alpha\beta} \right) \cdot e^{2\pi i \sum_{\alpha,\beta=1}^{n} \left( k_{a} \frac{m_{a}}{m_{a}'} \tau_{\alpha\beta} - j_{a} \frac{m_{a}'}{m_{a}} \tau_{\alpha\beta} \right) + 2\pi i \sum_{\alpha=1}^{n} \left[ \left( k_{a} + \frac{m_{a}}{m_{a}'} \right) - j_{a} \frac{m_{a}'}{m_{a}} \right] (\delta_{a} t_{a})} \\
\times \pi^{\left[ \left( k_{a} + \frac{m_{a}}{m_{a}'} \right) - j_{a} \frac{m_{a}'}{m_{a}} \right]} \left( \sum_{l=1}^{n} \tau_{al2} t_{a+n+l} + \tau_{a} \right) \right) J \ dt_{1} ... dt_{2n}
\]

where \( J = \prod_{\alpha=1}^{n} \delta_{\alpha} \ det(\tau_{\alpha2}) \) is the Jacobian.

Checking the terms in the integrand related to \( t_{1}, ..., t_{n} \) we have the third term in the second line above:

\[
(4.5) \quad \sum_{k,j \in \mathbb{Z}^{n}} \int_{I^{n}} e^{\sum_{\alpha=1}^{n} 2\pi i \left( k_{a} \delta_{a} + m_{a} - j_{a} \delta_{a} - m_{a}' \right) t_{a}} \ dt_{1} ... dt_{n}.
\]

Now \( 0 \leq m_{a}, m_{a}' < \delta_{a} \) and \( m_{a}, m_{a}' \in 0 \cup \mathbb{N} \). If \( j \neq k \) then \( \left| (k_{a} - j_{a}) \delta_{a} \right| \geq \delta_{a} \) for some \( \alpha = 1, ..., n \), 
\( k_{a} \delta_{a} + m_{a} - j_{a} \delta_{a} - m_{a}' \in \mathbb{Z} \setminus \{0\} \) since \( |m_{a} - m_{a}'| < \delta_{a} \), and so, for this \( \alpha \)

\[
(4.6) \quad \int_{I} e^{2\pi i \left( k_{a} \delta_{a} + m_{a} - j_{a} \delta_{a} - m_{a}' \right) t_{a}} \ dt_{a} = 0.
\]

It remains to consider the \( j = k \) terms. We have, for all \( k \in \mathbb{Z}^{n} \)

\[
(4.7) \quad \int_{I} e^{2\pi i \left( k_{a} \delta_{a} + m_{a} - j_{a} \delta_{a} - m_{a}' \right) t_{a}} \ dt_{a} = \int_{I} e^{2\pi i \left( m_{a} - m_{a}' \right) t_{a}} \ dt_{a}.
\]

By using (4.7) for (4.4) one sees that the integral \( I_{mm'} \) vanishes if \( m \neq m' \), proving the orthogonal property for \( \{ \theta_{m}(v; \mu) \} \) \( m \in \mathbb{R} \).

The linear independence of \( \{ \theta_{m}(v; \mu) \} \) \( m \in \mathbb{R} \) in \( H^{0}(M, L_{\mu}) \) as needed in the proof of Lemma 2.2 follows from the above orthogonal property.
For $m = m' \in \mathfrak{M}$ we calculate, using (4.3) again and setting $m = m'$ in the third equality below

$$I_m := (\theta_m(v; \mu), \theta_m(v; \mu))_{h_{L_\mu}} = \int_M h_{L_\mu}(v) \theta_m(v; \mu) \theta_m(v; \mu) \, d\text{vol}$$

$$= \int_M \varepsilon_{\alpha,\beta=1}^{n} W_{\alpha\beta}(z_{\alpha2} + \mu_{\alpha2}) (z_{\beta2} + \mu_{\beta2})$$

$$\cdot \sum_{k,j \in \mathbb{Z}^n} \left( e^{\frac{\pi i}{\alpha,\beta=1} \sum_{k=1}^{n} k_{\alpha} k_{\beta} \tau_{\alpha\beta} (z_{\alpha2} + \mu_{\alpha2})} 2\pi i \sum_{\alpha=1}^{n} \tau_{\alpha\beta} \frac{m_{\alpha}}{s_{\alpha}} k_{\beta} e^{\frac{2\pi i}{\alpha=1} \sum_{\alpha=1}^{n} (k_{\alpha} m_{\alpha} + m_{\alpha}) (z_{\alpha} + \mu_{\alpha})} \right)$$

(4.8)

$$\cdot \left( e^{\frac{-\pi i}{\alpha,\beta=1} \sum_{k=1}^{n} k_{\alpha} k_{\beta} \tau_{\alpha\beta} (z_{\alpha2} + \mu_{\alpha2})} - 2\pi i \sum_{\alpha=1}^{n} \tau_{\alpha\beta} \frac{m_{\alpha}}{s_{\alpha}} k_{\beta} e^{\frac{-2\pi i}{\alpha=1} \sum_{\alpha=1}^{n} (k_{\alpha} m_{\alpha} + m_{\alpha}) (z_{\alpha} + \mu_{\alpha2})} \right) \, d\text{vol}.$$

It is slightly tedious but straightforward for (4.8) to complete the square in the following form: (here identities such as $\sum_{\alpha=1}^{n} k_{\alpha} \delta_{\alpha} + m_{\alpha} (z_{\alpha2} + \mu_{\alpha2}) = \sum_{\alpha,\beta,\eta=1}^{n} W_{\alpha\beta} (z_{\alpha2} + \mu_{\alpha2}) \tau_{\beta\eta2} \frac{k_{\alpha} \delta_{\alpha} + m_{\alpha}}{s_{\alpha}}$ since $\sum_{\beta=1}^{n} W_{\alpha\beta} \tau_{\beta\eta} = \delta_{\eta}^{\alpha}$ are useful)

$$I_m = e^{2\pi \sum_{\alpha,\beta=1}^{n} \tau_{\alpha\beta2} \frac{m_{\alpha} m_{\beta}}{s_{\alpha} s_{\beta}}}$$

$$\cdot \int_M \sum_{k,j \in \mathbb{Z}^n} e^{\frac{-\pi i}{\alpha,\beta=1} \sum_{k=1}^{n} W_{\alpha\beta}(z_{\alpha2} + \mu_{\alpha2} + \sum_{\gamma=1}^{n} \tau_{\alpha\gamma2} \frac{m_{\alpha} m_{\gamma}}{s_{\alpha}} (z_{\gamma2} + \mu_{\gamma2} + \sum_{\eta=1}^{n} \frac{k_{\gamma} m_{\eta} + m_{\eta}}{s_{\eta}})} \, d\text{vol}.$$  

(4.9)

With the same change of variables $z_{\alpha1} = \delta_{\alpha} t_{\alpha1} + \sum_{l=1}^{n} \tau_{\alpha l1} t_{n+l}$, $z_{\alpha2} = \sum_{l=1}^{n} \tau_{\alpha l2} t_{n+l}$ as before, we are now integrating over the $2n$-dimensional unit cube $I_{2n}$:

$$I_m = e^{2\pi \sum_{\alpha,\beta=1}^{n} \tau_{\alpha\beta2} \frac{m_{\alpha} m_{\beta}}{s_{\alpha} s_{\beta}}}$$

$$\cdot \int_{I_{2n}} \sum_{k,j \in \mathbb{Z}^n} e^{\frac{-\pi i}{\alpha,\beta=1} \sum_{k=1}^{n} W_{\alpha\beta}(t_{\alpha2} + k_{\alpha} + \sum_{l=1}^{n} W_{\alpha l2}(t_{n+l} + k_{\alpha} + \sum_{\gamma=1}^{n} \tau_{\alpha\gamma2} \frac{m_{\alpha} m_{\gamma}}{s_{\alpha}}) + \sum_{\eta=1}^{n} \frac{k_{\eta} m_{\eta} + m_{\eta}}{s_{\eta}}) \, J \, dt_1 ... dt_{2n}.$$  

(4.10)

It is not absolutely necessary to find this explicit form. If one uses (4.9) and regards it as an implicit function in terms of $t_1, t_2, ..., t_{2n}$, one first observes that the integration involving $dt_1 dt_2 ... dt_n$ is easily obtained as that in (4.11) and then that the complete integration can still be done over $\mathbb{R}^n$ in a way similar to (4.12), with the previous implicit functions. Being on $\mathbb{R}^n$, one returns now to the original variables; in this way one soon obtains the same result via the Gaussian integration (4.13) applied to the quadratic term in the bracket [...] of (4.9).
To simplify the notation, we let \( G(t_{n+1}, \ldots, t_{2n}) = e^{-2\pi \sum_{\alpha,\beta=1}^{n} \tau_{\alpha\beta} t_{\alpha+n\cdot t_{\alpha\beta}}} \) and \( p = (p_1, \ldots, p_n) \) where \( p_{\alpha} = \frac{m_{\alpha}}{\delta_{\alpha}} + \sum_{l=1}^{n} W_{\alpha l \mu} \), and rewrite (4.10) as

\[
I_m = e^{2\pi \sum_{\alpha,\beta=1}^{n} \tau_{\alpha\beta} m_{\alpha} m_{\beta} \delta_{\alpha} \delta_{\beta}} \mathcal{J} \sum_{k \in \mathbb{Z}^n} \left( \int_{I^n} 1 \, dt_1 \ldots dt_n \right) \int_{I^n} G(t_{n+1} + k_1 + p_1, \ldots, t_{2n} + k_n + p_n) \, dt_{n+1} \ldots dt_{2n}.
\]

By Sublemma 4.3 below we have

\[
I_m = e^{2\pi \sum_{\alpha,\beta=1}^{n} \tau_{\alpha\beta} m_{\alpha} m_{\beta} \delta_{\alpha} \delta_{\beta}} \mathcal{J} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \sum_{\alpha,\beta=1}^{n} \tau_{\alpha\beta} m_{\alpha} m_{\beta} \delta_{\alpha} \delta_{\beta}} dt_{n+1} \ldots dt_{2n}.
\]

Applying to (4.12) the Gaussian integral

\[
\int_{\mathbb{R}^n} e^{-X^TAX} dt_{n+1} \ldots dt_{2n} = \left( \frac{\sqrt{\pi}}{\sqrt{\det A}} \right)^n
\]

where \( A \) is a positive definite \( n \times n \) matrix, we get the result :

\[
I_m = \sqrt{\det \left( \frac{\text{Im} \tau_{\alpha\beta}}{2} \right)} \left( \prod_{\alpha=1}^{n} \delta_{\alpha} \right) e^{2\pi \sum_{\alpha,\beta=1}^{n} \text{Im} \tau_{\alpha\beta} m_{\alpha} m_{\beta} \delta_{\alpha} \delta_{\beta}}.
\]

\[
\square
\]

**Sublemma 4.3.** Let \( f(x) : \mathbb{R}^n \to \mathbb{R} \) be an integrable function, \( p \in \mathbb{R}^n \) any fixed point. Then

\[
\sum_{k \in \mathbb{Z}^n} \int_{I^n} f(x + k + p) \, dx = \int_{\mathbb{R}^n} f(x + p) \, dx = \int_{\mathbb{R}^n} f(x) \, dx.
\]

By Lemma 4.2 the value of \((\theta_m(z; \mu), \theta_m(z; \mu))_{h_{\mu}}\) in Definition 4.1 is independent of \( \mu \), giving the first part of the following theorem. The second part of it follows from the first statement, Proposition 3.3, and Lemma 4.2.

**Theorem 4.4.** (1) On \( K \), the curvature tensor of the metric \((, )_h\) defined in Definition 4.1 is identically zero. (2) \( K \) splits holomorphically into a direct sum of holomorphically trivial line bundle \( K = \bigoplus_{m \in \mathbb{M}} K_m \) where each \( K_m \) has the canonical section identified as \( \theta_m \) in Lemma 4.2. Note that for any \( \mu \in M_2(= M) \), \( \theta_m(v; \mu) \in (K_m)_\mu \) is nontrivial although it has zeros along \( v \), so that \( \theta_m(v; \mu) \) is nowhere vanishing as a section of \( K_m \).
5. Canonical Connection on the Poincaré line bundle and Proof of Theorem 1.2

In this section, we view the Abelian variety $M$ as a real $2n$-dimensional manifold and introduce a geometric description of the Poincaré line bundle with a connection on it. Similar to [4], we first define a line bundle $\mathcal{P} \to M \times M^*$ with a connection and then identify it with the Poincaré line bundle $P \to M \times \hat{M}$. We follow the treatment in [3] and [7] and generalize it to the $2n$-dimensional abelian variety $M$.

To begin with, we write $V \cong \mathbb{R}^{2n}$ and $M = V/\Lambda$ where $\Lambda = \mathbb{Z}\{\lambda_1, ..., \lambda_{2n}\}$ and in some standard coordinates of $\mathbb{R}^{2n}$ (see Notation [2,1]), $\lambda_1 = (\delta_1, 0, ..., 0, 0)$, $\lambda_2 = (0, \delta_2, 0, ..., 0, 0)$, ..., $\lambda_n = (0, 0, ..., \delta_n, 0)$, and $\lambda_{n+\alpha} = (\tau_{\alpha1}, \tau_{\alpha2}, ..., \tau_{\alpha n}, \tau_{\alpha n+1})$ for $\alpha = 1, ..., n$; these $\delta$’s and $\tau$’s ($\tau_{\alpha1} = \tau_{\alpha n+1} + i\tau_{\alpha2}$) are from the period matrix $\Omega$ in Section 2. Let $\Lambda^* = \mathbb{Z}\{dx_1, ..., dx_{2n}\}$ be the dual lattice of $\Lambda$ (so $\int_{\gamma_i} dx_j = \delta_{ij}$). Let $V^* := Hom_{\mathbb{R}}(V, \mathbb{R}) = \{\xi_1 dx_1 + ... + \xi_{2n} dx_{2n} \mid \xi_i \in \mathbb{R}, i = 1, ..., 2n\}$.

We define $M^* := V^*/2\pi\Lambda^*$ and write $[\xi]$ for the equivalence class of $\xi$ in $M^*$.

Let $\mathbb{C}|_V : V \times \mathbb{C} \to V$ be the trivial complex line bundle over $V$. An element $\xi = \sum \xi_i dx_i \in V^*$ gives rise to a character $\chi_{\xi} : \Lambda \to U(1)$ by $\chi_{\xi}(\lambda) := e^{-i\langle \xi, \lambda \rangle} = e^{-i\xi(\lambda)}$. The set $\Lambda$ acts on $\mathbb{C}|_V$ by $\lambda \cdot (x, \sigma) := (x + \lambda, \chi_{\xi}(\lambda) \sigma)$. The natural horizontal foliation in $\mathbb{C}|_V$ is preserved by such actions and thus descends to a flat connection $d$ on the quotient $\mathbb{C}|_M := \mathbb{C}|_V / \Lambda$ over $M$. For $\xi \in V^*$, one defines a flat $U(1)$ connection on the complex line bundle $\mathbb{C}|_M$ over $M$ by $\nabla^\xi := d + i\xi$.

The gauge equivalence classes of flat line bundles on $M$ are parametrized by $M^*$. We write $\mathcal{L}_{[\xi]} := (\mathbb{C}|_M, \nabla^\xi)$ for the flat line bundle on $M$ corresponding to the connection $\nabla^\xi, \xi \in V^*$. With the connection $\nabla^\xi$, the parallel transport along the loop is given by $\chi_{\xi}$ above. Dually, for any $x \in V$ we define a character $\chi_x : 2\pi\Lambda^* \to U(1)$ by $\chi_x(2\pi \mu) := e^{-2\pi i \langle \mu, x \rangle}$ and get flat line bundles $\mathcal{L}_x$ over $M^*$ with the parallel transport $\chi_x$. Similar to the 2-dimensional case in [4] and 4-dimensional case in [7], we have the following lemma for the $2n$-dimensional $M$:

**Lemma 5.1.** There is a complex line bundle $\mathcal{P} \to M \times M^*$ with a unitary connection such that the restriction of $\mathcal{P}$ to each $M_{[\xi]} = M \times \{[\xi]\}$ is isomorphic (as a line bundle with connection) to $\mathcal{L}_{[\xi]}$ and the restriction to each $M_{[x]}^* = \{[x]\} \times M^*$ is isomorphic to $\mathcal{L}_x$.

In essence, lifting the action of $2\pi\Lambda^*$ on $M \times V^*$ to the trivial line bundle $\mathbb{C}|_{M \times V^*}$ over $M \times V^*$ endowed with the connection one form $\mathbb{A} = i\xi = i\sum_{k=1}^{2n} \xi_k dx_k (\xi \in V^*)$ by $2\pi\nu \circ (x, \xi, \sigma) := (x, \xi + 2\pi\nu, e^{-2\pi i \langle \nu, x \rangle} \sigma)$ for $\nu \in \Lambda^*$, one sees that this action preserves the connection $d + \mathbb{A}$ on $\mathbb{C}|_{M \times V^*}$ and hence induces a connection on the line bundle $\mathcal{P} := \mathbb{C}|_{M \times V^*}/2\pi\Lambda^* \to M \times M^*$, denoted by $\nabla^{\mathcal{P}}$. We sometimes write

$$\nabla^{\mathcal{P}} = d + \mathbb{A}$$

by abuse of notation. The connection $\nabla^{\mathcal{P}}$ has the curvature $\Theta_{\nabla^{\mathcal{P}}} = d\mathbb{A} = i\sum_{k=1}^{2n} d\xi_k \wedge dx_k.$
If we define a metric on the trivial line bundle $\mathbb{C}|_{M \times V^*}$ by $\langle (x, \xi, \sigma_1), (x, \xi, \sigma_2) \rangle := \sigma_1 \cdot \overline{\sigma_2}$, this metric is preserved by the action of $2\pi \Lambda^*$, and hence induces a metric $h_{\mathcal{P}}$ on $\mathcal{P}$. One sees that $\nabla^\mathcal{P}$ is compatible with $h_{\mathcal{P}}$ as required in Lemma 5.1.

Before we identify the line bundle $\mathcal{P}$ of lemma 5.1 with the Poincaré line bundle $P$, we should first have an isomorphism from $\widehat{M}$ to $M^*$, whose explicit form will be needed in a number of discussions later:

**Lemma 5.2.** One has an isomorphism

$$Iso : \widehat{M} \xrightarrow{\sim} M^*.$$  

**Proof.** $\widehat{M} = \{ \sum_{\alpha=1}^n \hat{\mu}_\alpha \nu_\alpha^* | \hat{\mu}_\alpha \in \mathbb{C} \}/\mathcal{L}$ (where $\hat{\mu}_\alpha = \hat{\mu}_{\alpha 1} + i \hat{\mu}_{\alpha 2}$ and $\mathcal{L} = \{ \sum_{i=1}^{2n} n_i \xi_i^* | n_i \in \mathbb{Z} \}$ in the notation of Section 2) and $M^* = \{ \sum_{i=1}^{2n} \xi_i \pi_i | \xi_i \in \mathbb{R} \}/2\pi \Lambda^*$. The group isomorphism $Iso : \widehat{M} \rightarrow M^*$ sends $dx_i^*$ to $2\pi dx_i$, $i = 1, \ldots, 2n$ and satisfies

$$\begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \\ \xi_{n+1} \\ \vdots \\ \xi_{2n} \end{pmatrix} = 2\pi \delta_n^{-1} \Delta_\delta W \Delta_\delta \begin{pmatrix} \hat{\mu}_{12} \\ \vdots \\ \hat{\mu}_{n2} \\ \hat{\mu}_{11} \\ 2\pi \delta_n^{-1} (\text{Re } Z) W \Delta_\delta \hat{\mu}_{n1} \\ \hat{\mu}_{n2} \end{pmatrix}. \tag{5.2}$$

Here $(\Delta_\delta|\mathcal{L})$ is the period matrix $\Omega$ of $\Lambda \subseteq V$ and $W$ is the inverse matrix of $\text{Im } Z$. Equivalently

$$\begin{pmatrix} \hat{\mu}_1 \\ \vdots \\ \hat{\mu}_n \end{pmatrix} = \left( \frac{1}{2\pi} \delta_n \Delta_\delta^{-1} Z \Delta_\delta^{-1}, \frac{-1}{2\pi} \delta_n \Delta_\delta^{-1} \right) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{2n} \end{pmatrix}. \tag{5.3}$$

In particular, for $n = 1$, the isomorphism reduces to ([1, Lemma 7.1])

$$\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \frac{2\pi \delta}{\tau_2} \text{Im}(\hat{\mu}) \\ \frac{2\pi \delta}{\tau_2} (\tau_1 \text{Im}(\hat{\mu}) - \tau_2 \text{Re}(\hat{\mu})) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \text{Re}(\hat{\mu}) = \frac{1}{2\pi} \xi_1 \\ \text{Im}(\hat{\mu}) = \frac{1}{2\pi} \xi_1 \end{pmatrix} \tag{5.4}$$

where $\Delta_\delta = \delta \in \mathbb{N}$ and $Z = \tau = \tau_1 + i\tau_2 \in \mathbb{C}$. \hfill \Box

We endow $M^*$ with the complex structure inherited from $\widehat{M}$. Recall the Poincaré line bundle $P \rightarrow M \times \widehat{M}$ of Lemma 2.5. To compare $P$ and $\mathcal{P}$, we first show that the global connection $\nabla^\mathcal{P}$ on $\mathcal{P}$ [5.1] gives a holomorphic structure on $\mathcal{P}$ (where this $M$ has been restored as an Abelian variety as considered in the beginning of this section).
For the holomorphic structure on $\mathcal{P}$, write $\widetilde{\text{Iso}} := (\text{Id}, \text{Iso}) : M \times \hat{M} \to M \times M^*$ with the isomorphism $\text{Iso} : \hat{M} \to M^*$ in Lemma 5.2 and form the pull-back bundle $\widetilde{\text{Iso}}^* \mathcal{P}$ equipped with the metric $\widetilde{\text{Iso}} h_{\mathcal{P}}$ and the connection $\widetilde{\text{Iso}}^* \nabla^\mathcal{P}$. By (5.1) $\nabla^\mathcal{P} = d + i\xi, \xi \in V^*$, one has by using (2.4) and (5.2) in the proof of Lemma 5.2 that
\[
(5.5) \quad \widetilde{\text{Iso}}^* \nabla^\mathcal{P} = d + \mathbb{A} = d + \pi \sum_{\alpha, \beta = 1}^n W_{\alpha \beta}\left(\frac{\delta_\beta}{\delta_n} \hat{\mu}_\beta dz_\alpha - \frac{\delta_\alpha}{\delta_n} \hat{\mu}_\alpha dz_\beta\right)
\]
and its curvature
\[
(5.6) \quad \Theta_{\text{Iso}}^* \nabla^\mathcal{P} = d + \mathbb{A} = d + \pi \sum_{\alpha, \beta = 1}^n W_{\alpha \beta}\left(\frac{\delta_\beta}{\delta_n} d\hat{\mu}_\beta \wedge dz_\alpha - \frac{\delta_\alpha}{\delta_n} d\hat{\mu}_\alpha \wedge dz_\beta\right),
\]
giving rise to a holomorphic line bundle structure on $\widetilde{\text{Iso}}^* \mathcal{P}$ since (5.6) is of type $(1, 1)$ (cf. [4]). This holomorphic structure induces a holomorphic structure on $\mathcal{P}$ via the isomorphism $\widetilde{\text{Iso}}^{-1}$. We can now identify $P$ and $\mathcal{P}$ by showing that $P \cong \widetilde{\text{Iso}}^* \mathcal{P}$.

**Proposition 5.3.** With the notations above, let $P \to M \times \hat{M}$ be the Poincaré line bundle of Lemma 2.3 and $\mathcal{P} \to M \times M^*$ of Lemma 5.1 with the holomorphic structure just given. Then $P \cong \widetilde{\text{Iso}}^* \mathcal{P}$.

**Proof.** To prove that $P \cong \widetilde{\text{Iso}}^* \mathcal{P}$, by Lemma 2.5 it suffices to show that $\mathcal{P} \to M \times M^*$ satisfies the following two properties : 

(i) for any $[\xi] \in M^*$, the line bundle $\mathcal{L}_{[\xi]} \cong \mathcal{P}|_{M \times [\xi]}$ is holomorphically isomorphic to $P_{\text{Iso}^{-1}[\xi]} = P_{\hat{\mu}}$ and

(ii) $\mathcal{P}|_{\{0\} \times M^*}$ is holomorphically trivial on $\{0\} \times M^*$.

To prove (i) of the claim above, from the action $\lambda \circ (x, \sigma) = (x + \lambda, e^{-i<\xi, \lambda>} \sigma)$ given in the beginning of this section, the holonomy transforms the basis $\{\lambda_1, ..., \lambda_{2n}\}$ by $\chi_\xi(\lambda_j) = e^{-i<\xi, \lambda_j>}, j = 1, ..., 2n$.

The multipliers for $\mathcal{L}_{[\xi]}$ are thus\(^3\)
\[
(5.7) \quad e_{\lambda_j}(v) = e^{i\xi_j}, \quad j = 1, ..., 2n.
\]
On the other hand, we derive from (2.6), (2.11) and (2.15) that the multipliers for $P_{\hat{\mu}}$ with $\hat{\mu} = \sum_\alpha \mu_\alpha v_\alpha^*$ can be given by
\[
(5.8) \quad \begin{cases} 
  e_{\lambda_\alpha}(v) = 1 \\
  e_{\lambda_{n+\alpha}}(v) = e^{-2\pi i \mu_\alpha} e^{-2\pi i \frac{\delta_{n+\alpha}}{\delta_n} \mu_\alpha}.
\end{cases} \quad \alpha = 1, ..., n.
\]
To match the two sets of multipliers (5.7) and (5.8), let $L_{\Delta, \xi} = \mathcal{L}_{[\xi]} \otimes P_{\hat{\mu}}^* \to M$, and accordingly $L_{\Delta, \xi}$ has the multipliers
\[
(5.9) \quad \begin{cases} 
  e_{\lambda_\alpha}(v) = e^{i\xi_\alpha} \\
  e_{\lambda_{n+\alpha}}(v) = e^{-i\xi_{n+\alpha} + 2\pi i \frac{\delta_{n+\alpha}}{\delta_n} \mu_\alpha}.
\end{cases} \quad \alpha = 1, ..., n.
\]
\(^3\)We follow the convention of [8] proof of Lemma in page 310 that the multipliers transform the “coefficient-part” rather than the “basis-part”, and so they look inverse to holonomy transforms above.
By (5.3) expressing $\hat{\mu}$ in terms of $\xi$, we compute the RHS of (5.3) and rewrite the result as

\[ e_{\lambda_\alpha}(v) = e^{i\xi_\alpha}, \quad e_{\lambda_{n+\alpha}}(v) = e^{i\sum_{\beta=1}^{n} \tau_{\alpha\beta} \xi_\beta}, \quad \alpha = 1, \ldots, n. \]

By Lemma 5.4 below $L_{\Delta, \xi}$ is holomorphically trivial so that $L_{[\xi]} \cong P_{\hat{\mu}}$, proving $i$.

For $ii$) of the claim above, the action $2\pi v \circ (x, \xi, \sigma) = (x, \xi + 2\pi v, e^{-2\pi i \langle \nu, x \rangle} \sigma)$ gives $2\pi v \circ (0, \xi, \sigma) = (0, \xi + 2\pi v, \sigma)$ at $x = 0$. Since $\sigma$ is unchanged, it follows that $\mathcal{P}|_{\{0\} \times M^*}$ has trivial multipliers and therefore is a holomorphically trivial line bundle on $\{0\} \times M^*$, proving $ii)$.

**Lemma 5.4.** For any given $\xi \in V^*$ the above line bundle $L_{\Delta, \xi}$ with multipliers (5.10) is holomorphically trivial.

**Proof.** We claim that

\[ \Phi_\xi(v) = e^{i \sum_{\alpha=1}^{n} a_\alpha z_\alpha}, \quad a_\alpha = \frac{\xi_\alpha}{\delta_\alpha} \]

is a global section of $L_{\Delta, \xi} \to M$. From the definition of $\Phi_\xi(v)$, one sees that

\[ \begin{align*}
\Phi_\xi(z_1, \ldots, z_n + \delta_1, \ldots, z_n) &= e^{i \xi_\alpha} \Phi_\xi(z_1, \ldots, z_n) \\
\Phi_\xi(z_1 + \tau_1, z_2 + \tau_2, \ldots, z_n + \tau_n) &= e^{i \sum_{\beta=1}^{n} \tau_{\alpha\beta} \xi_\beta} \Phi_\xi(z_1, \ldots, z_n),
\end{align*} \]

\[ \alpha = 1, \ldots, n. \]

By (5.10) this proves the claim, and hence the lemma since $\Phi_\xi(v)$ is nowhere vanishing.

Now we are in a position to compute the curvature $\Theta(E, h_E)$ (see Corollary 5.11 below). By proposition 5.3 that $P \cong \tilde{\text{Iso}}^* \mathcal{P}$, we can pull back the metric $h_\mathcal{P}$ and the connection $\nabla^\mathcal{P} = d + i \xi$ to $P$ and write

\[ h_P := \tilde{\text{Iso}}^* h_\mathcal{P}, \quad \nabla^P := \tilde{\text{Iso}}^* \nabla^\mathcal{P} \quad \text{on } P. \]

Write $\Theta_P$ for the curvature of $\nabla^P$. Combining (5.6) and Proposition 3.4 using (2.15), one knows that $(\text{Id} \times \varphi_{L_0})^* h_P$ and $h_{(\text{Id} \times \varphi_{L_0})^* P}$ (see (3.1)) differ at most by a multiplicative constant $c$ ([4, Prop. 8.1]). Comparing them on $\{0\} \times M$, one sees that $c = 1$ and hence that they are identical on $M \times M$. More precisely we have

**Proposition 5.5.** Recalling $h_{(\text{Id} \times \varphi_{L_0})^* P}$ and $\Theta_{(\text{Id} \times \varphi_{L_0})^* P}$ on $(\text{Id} \times \varphi_{L_0})^* P \to M \times M$ (see (3.4) and (3.7)), one has $(\text{Id} \times \varphi_{L_0})^* \Theta_P = \Theta_{(\text{Id} \times \varphi_{L_0})^* P}$ and $(\text{Id} \times \varphi_{L_0})^* h_P = h_{(\text{Id} \times \varphi_{L_0})^* P}$ on $M \times M$.

To proceed further, we consider the following bundles:

**Definition 5.6.** $i)$ Define the line bundles

\[ \tilde{E} := \pi_1^* L_0 \otimes P \to M \times \hat{M}, \quad \tilde{E}' := \pi_1^* L_0 \otimes (\text{Id} \times \varphi_{L_0})^* P \to M_1 \times M_2 \]

where $M_1 \cong M_2 \cong M$. 

ii) Define the direct images

\[ E := (\pi_2)_* \tilde{E} \rightarrow \tilde{M}; \quad E' := (\pi_2')_* \tilde{E}' \rightarrow M_2 \]

where \( \pi_2 : M \times \tilde{M} \rightarrow \tilde{M}, \pi_2' : M_1 \times M_2 \rightarrow M_2 \) are projections.

Note that \( E \) and \( E' \) are vector bundles by Grauert’s direct image theorem (cf. [1] Cor. 12.9 and [2] Section 3 of Chapter 3]) since the dimension functions \( h^0(M, L_\mu) \) and \( h^0(M, L_\mu) \) are constant in \( \tilde{\mu} \) and \( \mu \) (cf. Lemmas [2.2 and 2.3]) respectively.

Using the identification \((P, h_P, \nabla_P) \cong (\tilde{\text{Iso}}^* \mathcal{D}, \tilde{\text{Iso}}^* h_\mathcal{D}, \tilde{\text{Iso}}^* \nabla_\mathcal{D})\) we equip \( \tilde{E} \) and \( \tilde{E}' \) with the metric

\[ h_{\tilde{E}} = \pi_1^* h_{L_0} \otimes h_P \quad \text{and} \quad h_{\tilde{E}'} = \pi_1^* h_{L_0} \otimes h_{(\text{Id} \times \varphi_{L_0})^* P} \]

respectively (cf. Lemma [2.2] and [3.4]). One has \( h_{\tilde{E}'} = (\text{Id} \times \varphi_{L_0})^* h_{\tilde{E}} \) by Proposition [5.3]. We can now equip the vector bundle \( E \) with the metric given by the \( L^2 \) inner product using \( h_{\tilde{E}} \) on the fibers \( E_{\tilde{\mu}} = H^0(M, L_{\tilde{\mu}}) \), and similarly \( E' \) with the metric given by that using \( h_{\tilde{E}'} \) on \( E'_\mu = H^0(M, L_\mu) \).

**Notation 5.7.** We denote by \( h_E \) and \( h_{E'} \) those metrics just obtained on \( E \) and \( E' \), respectively.

The statement and proof of the next proposition are the same as those of Proposition 8.3 of [4]:

**Proposition 5.8.** With the notation above, we have \((\tilde{E}', h_{\tilde{E}'}) = (\text{Id} \times \varphi_{L_0})^*(\tilde{E}, h_{\tilde{E}})\). As a consequence, \((E', h_{E'}) = \varphi_{L_0}^*(E, h_E)\) with the curvature \( \Theta(E', h_{E'}) = \varphi_{L_0}^* \Theta(E, h_E) \).

Recall that \( K \rightarrow M_2 \) has the fiber \( K_\mu = H^0(M, K|_{M \times \{\mu\}}) \) (cf. Section 4). By [4.1], [5.13] and [5.14] we have \( K = E' \otimes L_0 \), that is \( E' = K \otimes L_0^* \) (where \( L_0^* \) is the dual bundle of \( L_0 \)), and it follows from Theorem [4.4] that

\[ E' = (K \otimes L_0^*) = \bigoplus_{m \in \mathbb{N}} (K_m \otimes L_0^*) \cong \bigoplus_{m \in \mathbb{N}} L_0^*. \]

In view of Theorem [4.4], [5.16] and \( \Theta_{L_0} = -\overline{\partial} \log h_{L_0} = \pi \sum_{\alpha, \beta = 1} W_{\alpha \beta} d\mu_\alpha \wedge d\overline{\mu}_\beta \) (Lemma [2.2]), the curvature of \( E' \) is now immediate:

**Theorem 5.9.** Let us denote by \( I_{\Delta \times \Delta} \) the \( \Delta \times \Delta \) identity matrix where \( \Delta = \det(\Delta_\delta) = \prod_{\alpha = 1}^n \delta_\alpha = \dim H^0(M, L_0) \). Then

\[ \Theta(E', h_{E'}) = -\Theta_{L_0} \cdot I_{\Delta \times \Delta} = \left( -\pi \sum_{\alpha, \beta = 1} W_{\alpha \beta} d\mu_\alpha \wedge d\overline{\mu}_\beta \right) \cdot I_{\Delta \times \Delta} \]

\[ c_1(E', h_{E'}) = -\frac{i}{2\pi} \Delta \Theta_{L_0} = -\frac{i}{2} \Delta \sum_{\alpha, \beta = 1} W_{\alpha \beta} d\mu_\alpha \wedge d\overline{\mu}_\beta, \quad \text{on } M. \]
Remark 5.10. By (2.3), we have

\[ d\mu_\alpha = \delta_\alpha dx_\alpha + \sum_{k=1}^n \tau_{ak} dx_{n+k}, \quad d\tilde{\mu}_\alpha = \delta_\alpha dx_\alpha + \sum_{k=1}^n \tau_{ak} dx_{n+k}, \quad \alpha = 1, \ldots, n \]

so that \( c_1(E', h_{E'}) = -\frac{i}{2} \Delta \sum_{\alpha, \beta=1}^n W_{\alpha\beta} d\mu_\alpha \wedge d\tilde{\mu}_\beta = -\Delta \sum_{i=1}^n \delta_i dx_i \wedge dx_{n+i} = -\Delta \cdot c_1(L_0). \)

Corollary 5.11. On \( \hat{M} \), we have the curvature \( \Theta(E, h_E) = \left( -\pi \sum_{\alpha, \beta=1}^n W_{\alpha\beta} \frac{\delta_\alpha \delta_\beta}{\delta_\mu_n} d\tilde{\mu}_\alpha \wedge d\tilde{\mu}_\beta \right) \cdot I_{\Delta \times \Delta} \)

and the first Chern class \( c_1(E, h_E) = \left( -\frac{i}{2} \Delta \sum_{\alpha, \beta=1}^n W_{\alpha\beta} \frac{\delta_\alpha \delta_\beta}{\delta_\mu_n} d\tilde{\mu}_\alpha \wedge d\tilde{\mu}_\beta \right). \)

Proof. With the notation from Definition 5.6 and Proposition 5.8 gives us the following commutative diagram

\[
\begin{array}{c}
\tilde{E} \xrightarrow{(Id \times \varphi_{L_0})^*} \tilde{E}' \\
\pi_2 \downarrow \quad \quad \quad \downarrow \pi'_2 \\
E \xrightarrow{\varphi_{L_0}} \quad \quad \quad \quad E'
\end{array}
\]

Since \( \varphi_{L_0} : M \to \hat{M} \) is a local diffeomorphism and we have the formula for \( \Theta(E', h_{E'}) \) and \( c_1(E', h_{E'}) \) of \( E' \to M \) (see (5.17), (5.18)), the corollary follows from the equations \( \tilde{\mu}_\alpha = \frac{\delta_\alpha}{\delta_\mu_n} \mu_\alpha \) (see Property 2.4).

On \( M^* = V^*/2\pi \Lambda^* \) (cf. Lemma 5.2), if we write \( V^* := HOM_R(V, \mathbb{R}) = \{ \xi_1 dx_1 + \ldots + \xi_{2n} dx_{2n} \mid \xi_i \in \mathbb{R} \} = \{ \eta_1(2\pi dx_1) + \ldots + \eta_{2n}(2\pi dx_{2n}) \mid \eta_i \in \mathbb{R} \} \) and \( 2\pi \Lambda^* = \mathbb{Z}\{2\pi dx_1, \ldots, 2\pi dx_{2n}\} \), then \( \eta_1, \ldots, \eta_{2n} \) are the dual real coordinates on \( V^* \) with \( d\eta_1, \ldots, d\eta_n \) the corresponding 1-forms on \( M^* \). It is known that \( H^k(M^*, \mathbb{Z}) = \mathbb{Z}\{d\eta_I\}_{#I=k} \) (\textit{R} p.302).

Theorem 5.12. By the map \( Iso : \hat{M} \to M^* \) (Lemma 5.2) we have, with \( \Delta = \prod_{\alpha=1}^n \delta_\alpha \)

\[ c_1((Iso^{-1})^*E, (Iso^{-1})^*(h_E)) = -\sum_{i=1}^n \frac{\Delta}{\delta_i} d\eta_i \wedge d\eta_{n+i} \in H^2(M^*, \mathbb{Z}). \]

Remark 15.3. \( \omega^\vee := \sum_{i=1}^n \frac{\Delta}{\delta_i} d\eta_i \wedge d\eta_{n+i} \) can be regarded as a “dual polarization” on the dual Abelian variety \( \hat{M} \) (via the map \( Iso \)) while \( \omega = \sum_{i=1}^n \delta_i dx_i \wedge dx_{n+i} \) in (2.1) is the polarization on \( M \).

For the elliptic curve \( \omega^\vee = d\eta_1 \wedge d\eta_2 \) represents 1 in \( H^2(\hat{M}, \mathbb{Z}) = \mathbb{Z}. \)
Proof of Theorem 5.12. From (5.2), \( \hat{\mu}_\alpha = \frac{\delta}{\delta_\alpha} \mu_\alpha, \alpha = 1, \ldots, n \) (Property 2.4), and \( 2\pi \eta_i = \xi_i, i = 1, \ldots, 2n \), one sees that

\[
(5.21) \quad \begin{pmatrix} d\eta_1 \\ \vdots \\ d\eta_n \end{pmatrix} = \Delta_\delta W \begin{pmatrix} d\mu_{12} \\ \vdots \\ d\mu_{n2} \end{pmatrix}, \quad \begin{pmatrix} d\eta_{n+1} \\ \vdots \\ d\eta_{2n} \end{pmatrix} = -\begin{pmatrix} d\mu_{11} \\ \vdots \\ d\mu_{n1} \end{pmatrix} + (\text{Re } Z) W \begin{pmatrix} d\mu_{12} \\ \vdots \\ d\mu_{n2} \end{pmatrix}
\]

where \( (\Delta_\delta, Z) \) is the period matrix of \( \Lambda \) (see (2.2)). We want to calculate the 2-form \( -\omega^\vee = -\sum_{\alpha=1}^n \Delta_{\delta_i} d\eta_i \wedge d\eta_{n+i} \) and relate it to \( c_1(E', h_{E'}) = -\frac{i}{2} \Delta \sum_{\alpha,\beta=1}^n W_{\alpha\beta} d\mu_\alpha \wedge d\mu_\beta \). First, writing \( f_\alpha = \sum_{\beta=1}^n W_{\alpha\beta} d\mu_{\beta 2} \) \((\alpha = 1, \ldots, n)\) we see from (5.21) that

\[
(5.22) \quad \begin{pmatrix} (\Delta_\delta^{-1} \Delta) \\ \vdots \\ d\eta_{n} \end{pmatrix} T = (\Delta_\delta^{-1} \Delta) \Delta_\delta W \begin{pmatrix} d\mu_{12} \\ \vdots \\ d\mu_{n2} \end{pmatrix} \wedge (\text{Re } Z) W \begin{pmatrix} d\mu_{12} \\ \vdots \\ d\mu_{n2} \end{pmatrix}
\]

\[= \Delta \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} T \wedge (\text{Re } Z) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \Delta \sum_{\alpha,\beta=1}^n (\text{Re } Z)_{\alpha\beta} f_\alpha \wedge f_\beta = 0 \]

since \( \text{Re } Z \) is symmetric and \( f_\alpha \)'s are 1-forms. Therefore, by (5.21) again and using (5.22)

\[
\varphi^*_{L_0} \text{is}^\vee = \varphi^*_{L_0} \text{is}^\vee \left(-\sum_{\alpha=1}^n \Delta_{\delta_i} d\eta_i \wedge d\eta_{n+i} \right) = \left(\Delta_\delta^{-1} \Delta \right) \Delta_\delta W \begin{pmatrix} d\mu_{12} \\ \vdots \\ d\mu_{n2} \end{pmatrix} T \wedge \begin{pmatrix} -d\mu_{11} \\ \vdots \\ -d\mu_{n1} \end{pmatrix}
\]

\[
(5.23) \quad = -\Delta \sum_{\alpha,\beta=1}^n W_{\alpha\beta} d\mu_{\alpha 1} \wedge d\mu_{\beta 2} = -\frac{i}{2} \Delta \sum_{\alpha,\beta=1}^n W_{\alpha\beta} d\mu_\alpha \wedge d\mu_\beta \overset{(5.18)}{=} c_1(E', h_{E'}). \]

Now (5.20) follows since \( \varphi^*_{L_0}(c_1(E, h_E)) = c_1(E', h_{E'}) \) by Proposition 5.8 and \( \varphi_{L_0} \) is a local diffeomorphism. \( \square \)

Remark 5.14. The motivation for the \textit{a priori} choice \( \omega^\vee = \sum_{i=1}^n \Delta_{\delta_i} d\eta_i \wedge d\eta_{n+i} \) in Theorem 5.12 is that suppose \( c_1(E, h_E) = \sum_{i=1}^n (-a_i) d\eta_i \wedge d\eta_{n+i} \in H^2(\tilde{M}, \mathbb{Z}) \) for some \( a_i \in \mathbb{Z} \) (as yet unknown).
Then ($\hat{M}$ and $M^*$ are identified via $Iso$)

\begin{equation}
\int_{\hat{M}} (\varphi^*_{L_0} c_1(E, h_E))^n = \deg(\varphi_{L_0}) \int_{\hat{M}} c_1^n(E, h_E) = (-1)^{\frac{n(n+1)}{2}} (n!) \Delta^2 \prod_{i=1}^{n} a_i
\end{equation}

where $\deg(\varphi_{L_0}) = \Delta^2$ by [8, pp. 315-317]. Furthermore, by Remark 5.10

\begin{equation}
\int_{M} c_1^n(E', h_{E'}) = \int_{M} (-\Delta) \sum_{i=1}^{n} \delta_i dx_i \wedge dx_{n+i})^n = (-1)^{\frac{n(n+1)}{2}} (n!) \Delta^{n+1}.
\end{equation}

We see from (5.24) and (5.25) that for $\varphi^*_{L_0} c_1(E, h_E) = c_1(E', h_{E'})$ to hold, one may try $a_i = \frac{\Delta}{\alpha}$ because $\Delta = \prod_{\alpha=1}^{n} \delta_{\alpha}$.

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