A new result on recovery sparse signals using orthogonal matching pursuit

Xueping Chen, Jianzhong Liu and Jiandong Chen

School of Mathematics and Physics, Jiangsu University of Technology, Jiangsu, People’s Republic of China

1. Introduction
Orthogonal matching pursuit (OMP) has received growing attention due to its simplicity and competitive reconstruction performance recently. Consider the following compressed linear model:

\[ y = \Phi x, \]  

(1)

where \( x \in \mathbb{C}^n \) is a \( K \)-sparse signal (i.e., \( \| x \|_0 \leq K \)), \( \Phi = [\phi_1, \phi_2, \ldots, \phi_m] \in \mathbb{C}^{m \times n} \) is a known measurement matrix with \( m \ll n \) and \( y \in \mathbb{C}^m \) is the observation signal. It has been demonstrated that under some appropriate conditions on \( \Phi \), OMP can reliably recover the signal \( x \) based on a set of compressive observations \( y \) by iteratively identifying the support of the sparse signal according to the maximum correlation between columns of measurement matrix and the current residual. See Table 1 for a detailed description of the OMP algorithm (Cai & Wang, 2011; Chang & Wu, 2014; Tropp & Gilbert, 2007; Wang & Shim, 2016; Wen et al., 2020, 2017; Wu et al., 2013).

In Table 1, \( \text{supp}(x) \) is the set of nonzero positions in \( x \). \( r^K \) denotes the residual after the \( K \)th iteration of OMP and \( T^K \) the estimated support set within \( K \)th iteration of OMP.

In compressed sensing, a commonly used framework for analysing the recovery performance is the restricted isometry property (RIP) (Cai et al., 2010; Candès & Tao, 2005; Chang & Wu, 2014). A matrix \( \Phi \) is said to satisfy the RIP of order \( K \) if there exists a constant \( \delta \in [0, 1) \) such that

\[ (1 - \delta)\| x \|_2^2 \leq \| \Phi x \|_2^2 \leq (1 + \delta)\| x \|_2^2, \]  

(2)

for all \( K \)-sparse signal \( x \). In particular, the minimum of all constants \( \delta \) satisfying (2) is called the \( K \)-order Restricted Isometry Constant (RIC) and denoted by \( \delta_K \).

Over the years, many RIP-based conditions have been proposed to guarantee exact recovery of any \( K \)-sparse signals via OMP in \( K \) iterations. It has been shown in Davenport and Wakin (2010) that \( \delta_{K+1} < (3\sqrt{K})^{-1} \) is sufficient for OMP to recover any \( K \)-sparse signals in \( K \) iterations. The sufficient reconstruction condition of OMP is then improved to \( \delta_{K+1} < (1 + 2\sqrt{K})^{-1} \) by Huang and Zhu (2011). Mo (2015) demonstrated that \( \delta_{K+1} < (\sqrt{K} + 1)^{-1} \) is a sharp condition for exact recovery of any \( K \)-sparse signal with OMP in \( K \) iterations. Our recent work Liu et al. (2017) provides some sufficient conditions for recovering restricted classes of \( K \)-sparse signals with a more relaxed bound on RIC.

Obviously, running fewer number of iterations of OMP offers many benefits and many efforts have been made to improve the condition (Cai et al., 2009; Chang & Wu, 2014; Li & Wen, 2019; Wen et al., 2017; Wu et al., 2013). In Livshitz (2012), Livshitz showed that with proper choices of \( \alpha \) and \( \beta \) (\( \alpha \sim 2 \times 10^5, \beta \sim 10^{-6} \)), OMP accurately reconstructs \( K \)-sparse signals in \( \alpha K^{1.2} \) iterations under \( \delta_{K+2} = \beta K^{-0.2} \). It has been shown by Zhang (2011) that OMP recovers any \( K \)-sparse signal in 30 \( K \) iterations under \( \delta_{3K} \leq 3^{-1} \). Livshitz and Temlyakov (2014) considered random sparse signals and showed that with high probability, these signals can be recovered within \( (1 + \epsilon)K \) iterations of OMP for any \( \epsilon > 0 \). Recently, Wang and Shim (2016) showed that if

\[ c \geq -\frac{4(1 + \delta)}{1 - \delta} \ln \left( \frac{1}{2} - \sqrt{\frac{\delta}{2 + 2\delta}} \right), \]  

(3)

and \( \delta_{(c+1)K} \leq \delta \), OMP can recover the \( K \)-sparse signals in \( cK \) iterations. It is the best result as we know in the literature.

CONTACT Xueping Chen  chenxueping@jsut.edu.cn

© 2022 The Author(s). Published by Informa UK Limited, trading as Taylor & Francis Group. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.
Table 1. Orthogonal matching pursuit.

| Input:          | $\Phi, y$, and maximum iteration number $k_{\text{max}}$ |
|-----------------|--------------------------------------------------------|
| Initialize:     | $k = 0, r^0 = y, T^0 = \emptyset$                     |
| while           | $k < k_{\text{max}}$, do                              |
|                 | $k = k + 1$,                                           |
|                 | $t^k = \arg\max_{y \in \mathcal{R}^n} | (r^{k-1}, \phi_1) |$ |
|                 | $T^k = T^{k-1} \cup \{ t^k \}$                        |
|                 | $\tilde{r}^k = \arg\min_{y \in \mathcal{R}^n} \| y - \Phi \tilde{r}^k \|_2$ |
|                 | $T^k = y - \Phi \tilde{r}^k$                          |
| Output:         | $Tk$                                                   |

In this paper, we present a new result on how many iterations of OMP would be enough to guarantee exact recovery of sparse signals:

$$
\epsilon \geq \frac{-4(1 + \delta_1)}{1 - \delta} \ln \left( \frac{1 - \delta}{2} \right),
$$

which improves significantly upon the results proposed in Wang and Shim (2016).

We first give some notation. Let $\mathbb{N} = \{0, 1, 2, \ldots \}$, $\mathbb{N}^+ = \{1, 2, \ldots \}$ and $\Omega_n = \{1, 2, \ldots, n\}$. $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote floor and ceiling function, respectively. For any two sets $\Lambda$ and $\Gamma$, let $\Lambda \setminus \Gamma = \{ i : i \in \Lambda, i \notin \Gamma \}$, and $|\Lambda|$ is the cardinality of $\Lambda$. For $\Lambda \subseteq \Omega_n$ and $\Lambda \neq \emptyset$, $\Phi_\Lambda$ denotes the submatrix of $\Phi$ that contains only the columns indexed by $\Lambda$ and $\chi_\Lambda$ denotes the subvector of $x$ that contains only the entries indexed by $\Lambda$, and $\text{span}(\Phi_\Lambda)$ represents the span of columns in $\Phi_\Lambda$. Let $P_\Lambda = \Phi_\Lambda (\Phi_\Lambda^* \Phi_\Lambda)^{-1} \Phi_\Lambda^*$ stand for an orthogonal projection matrix onto $\text{span}(\Phi_\Lambda)$, where $\Phi^*$ is the conjugate transpose of the matrix $\Phi$, and $(\Phi^*_\Lambda \Phi_\Lambda)^+$ is Moore-Penrose pseudo inverse of $\Phi^*_\Lambda \Phi_\Lambda$. $P_\Lambda^\perp = I_m - P_\Lambda$ is an orthogonal projection matrix onto the orthogonal complement of $\text{span}(\Phi_\Lambda)$, where $I_m$ denotes the identity matrix. In particular, if $\Lambda = \emptyset$, then $\chi_{\emptyset}$ is a 0-by-1 empty vector, $\Phi_{\emptyset}$ is an $m$-by-0 empty matrix, $\Phi_{\emptyset} \chi_{\emptyset}$ is an $m$-by-1 zero matrix and $\text{span}(\Phi_{\emptyset}) = \{0\}$. For further details on empty matrices, see, e.g., Bernstein (2005).

2. Main results

For notational simplicity, we denote $\Gamma^k = T \setminus T^k$.

**Theorem 2.1:** For any $\delta \in (0, 1)$, let

$$
\alpha = \frac{-4(1 + \delta_1)}{1 - \delta} \ln \left( \frac{1 - \delta}{2} \right),
$$

and $\kappa = [\alpha K]$. If the measurement matrix $\Phi$ in (1) satisfies the RIP of order $K + \kappa$ and $\delta_{K+k} \leq \delta$, then $|\Gamma^k| = 0$.

**Remark 2.1 (Performance of Theorem 2.1):** From Theorem 2.1, if

$$
\epsilon \geq \frac{-4(1 + \delta_1)}{1 - \delta} \ln \left( \frac{1 - \delta}{2} \right),
$$

and $\delta_{([c+1])K} \leq \delta$, then OMP perfectly recovers the signal $x$ from the measurements $y = \Phi x$ in $[cK]$ iteration. In the following, we compare the lower bound in (4) with the result of Wang and Shim (2016), which has been showed in (3). We first establish an upper bound for the ratio of (4) to (3) by using monotonicity property of the RIC ($\delta_1 \leq \delta_{K+k}$) as

$$
R(\delta) := \frac{\ln \left( \frac{1 - \delta}{2} \right)}{\ln \left( \frac{1}{2 - \sqrt{\frac{\delta}{2 - 2\delta}} \frac{\delta}{2 - 2\delta}} \right)}.
$$

It is easy to check that $R(\delta) < 1$ for $0 < \delta < 1$, which means the lower bound of $\epsilon$ in this paper is uniformly smaller than the one proposed in Wang and Shim (2016). See Figure 1, for example, we have $R(3^{-1}) = 0.57$ and $R(2^{-1}) = 0.58$. For $0.015 < \delta < 0.993$, we have $0.57 < R(\delta) < 0.8$. That is, nearly 98% of the values of the bounds proposed in this paper is less than 0.8 times of the one in (3).

**Remark 2.2 (Main differences with the previous work):** Our methods have several key distinctions in construction of the more efficient lower and upper bounds of resident in OMP. First, Wang and Shim (2016) obtained the upper bound of residual in OMP algorithm based on the following fundamental set:

$$
\{ \| \chi_{\Gamma} \|_2 : \Gamma \subseteq \Gamma^k \}.
$$

We modified the above set to

$$
\{ \| \Phi_{\Gamma} \chi_{\Gamma} \|_2 : \Gamma \subseteq \Gamma^k \},
$$

which leads to a more efficient upper bound of resident in OMP algorithm (see inequality (17) of Section 3.3). Second, Livshitz and Temlyakov (2014), Wang and Shim (2016), and Zhang (2011) only use RIP to obtain the lower bound of resident. However, in this paper, we not only used RIP, but also utilized the Wielandt inequality (Wang & Ip, 1999) to derive more...
efficient lower bound of resident (see inequality (16) of Section 3.3). The details can be found in the following sections.

3. Proof of the main theorem

3.1. Preliminaries

In the following, we introduce the subset $\Gamma^k_\tau$ of $\Gamma^k$ based on the magnitude of elements in the set $\{\|\Phi_{\Gamma} x_{\Gamma}\|_2 : \Gamma \subset \Gamma^k\}$. For $k$, $\tau \in N$ and $\Gamma^k \neq \emptyset$, let

\[ \mathfrak{N}_\tau = \{\Lambda : \Lambda \subset \Gamma^k, |\Lambda| \leq 2^\tau - 1\}, \]

\[ f(\tau) = \min\{\|\Phi_{\Gamma \backslash \Lambda} x_{\Gamma \backslash \Lambda}\| : \Lambda \in \mathfrak{N}_\tau\}. \]

It is clear that $\mathfrak{N}_\tau \subset \mathfrak{N}_{\tau+1}$, and $f(\tau) \geq f(\tau + 1)$ with $\tau \in N$.

**Definition 3.1:** If subset $\Gamma^k_\tau$ in $\mathfrak{N}_\tau$ satisfies: (a) $\|\Phi_{\Gamma \backslash \Gamma^k_\tau} x_{\Gamma \backslash \Gamma^k_\tau}\|_2 = f(\tau)$ and (b) $|\Gamma^k_\tau| = \max\{|\Lambda| : \Lambda \in \mathfrak{N}_\tau, \|\Phi_{\Gamma \backslash \Lambda} x_{\Gamma \backslash \Lambda}\|_2 = f(\tau)\}$, then $\Gamma^k_\tau$ is said to be the $\Phi - k - \tau$ characteristic set of $x$.

It is easy to verify that the characteristic set $\Gamma^k_\tau$ has the following properties:

- (P1) $|\Gamma^k_\tau| \leq 2^\tau - 1$, for $\tau \in N$;
- (P2) If $\Gamma \subset \Gamma^k$ and $\|\Phi_{\Gamma} x_{\Gamma}\|_2 < \|\Phi_{\Gamma^k \backslash \Gamma} x_{\Gamma^k \backslash \Gamma}\|_2$, then $|\Gamma| \leq |\Gamma^k| - 2^\tau$;
- (P3) $0 = |\Gamma^k_0| \leq |\Gamma^k_1| \leq \cdots$, and $\Gamma^k = \Gamma^k_{J_k+1} = \cdots = \Gamma^k$, with $J_k = 1 + [\log_2 |\Gamma^k|]$.

In fact, (P1) is obvious as $\Gamma^k_\tau \subset \mathfrak{N}_\tau$. For (P2), since $\Gamma \subset \Gamma^k$, $\Gamma$ can be rewritten as $\Gamma = \Gamma^k \backslash (\Gamma^k \backslash \Gamma)$. From (6) and $\|\Phi_{\Gamma^k \backslash \Gamma} x_{\Gamma^k \backslash \Gamma}\|_2 = f(\tau)$, we know that $\|\Phi_{\Gamma} x_{\Gamma}\|_2 < \|\Phi_{\Gamma^k \backslash \Gamma} x_{\Gamma^k \backslash \Gamma}\|_2$, then $|\Gamma| \leq |\Gamma^k| - 2^\tau$. Hence $|\Gamma^k \backslash \Gamma| = |\Gamma^k| - |\Gamma^k \backslash \Gamma| \geq 2^\tau$. That is $|\Gamma^k \backslash \Gamma| \leq |\Gamma^k| - 2^\tau$. For (P3), if $f(\tau) = f(\tau + 1)$, it is obvious that $|\Gamma^k_\tau| \leq |\Gamma^k_{\tau+1}|$ based on Definition 3.1. If $f(\tau) > f(\tau + 1)$, then

\[ \|\Phi_{\Gamma \backslash \Gamma^k_\tau} x_{\Gamma \backslash \Gamma^k_\tau}\|_2 < \|\Phi_{\Gamma \backslash \Gamma^k_{\tau+1}} x_{\Gamma \backslash \Gamma^k_{\tau+1}}\|_2. \]

From (P2),

\[ |\Gamma^k| - |\Gamma^k_{\tau+1}| = |\Gamma^k \backslash \Gamma^k_{\tau+1}| \leq |\Gamma^k| - 2^\tau. \]

Thus $|\Gamma^k_{\tau+1}| \geq 2^\tau$. From (P1), $|\Gamma^k_{\tau+1}| \leq 2^\tau - 1 - |\Gamma^k_{\tau+1}|$.

Otherwise, it is easy to verify that $|\Gamma^k_{\tau+1}| \leq 2^\tau - 1$ for $J_k \leq \tau$. Thus we have $\Gamma^k_{\tau+1} = \Gamma^k$ based on Definition 3.1.

From the fact $|\Gamma^k_0| = 0$, $|\Gamma^k_1| = |\Gamma^k|$, and (P3), there exists $0 \leq v_k \leq J_k - 1$ such that $\Gamma^k_0 = \cdots = \Gamma^k_{v_k} = \emptyset$, and $\Gamma^k_{v_k+1} \neq \emptyset$. Then it follows from (P3) that $\Gamma^k_{v_k+i} \neq \emptyset$, $i = 1, 2, \ldots$.

Let $\sigma = (1 - \delta)^{-1}$, thus $\sigma > 1$. It can be verified that there exists the maximum value of $\sigma^i |\Phi_{\Gamma \backslash \Gamma^k_i} x_{\Gamma \backslash \Gamma^k_i}|^2$ over $i \in N$. Since $\|\Phi_{\Gamma \backslash \Gamma^k_i} x_{\Gamma \backslash \Gamma^k_i}\|_2 = 0$ based on the fact that $\Gamma^k_i = \Gamma^k$ from (P3) for $i \geq J_k$. Hence, we give the following definition.

**Definition 3.2:** Let $\Gamma^k \neq \emptyset$, integer $L(k)$ is said to be the $k - \sigma$ character of $x$ if it satisfies the following condition:

\[ \sigma^{L(k)} \|\Phi_{\Gamma \backslash \Gamma^k_i} x_{\Gamma \backslash \Gamma^k_i}\|_2^2 = \max_{i \in N} \sigma^i \|\Phi_{\Gamma \backslash \Gamma^k_i} x_{\Gamma \backslash \Gamma^k_i}\|_2^2. \]

By Definition 3.2, it is easy to check that

\[ \|\Phi_{\Gamma \backslash \Gamma^k_0} x_{\Gamma \backslash \Gamma^k_0}\|_2^2 \geq \sigma^{-L(k)} \|\Phi_{\Gamma \backslash \Gamma^k} x_{\Gamma \backslash \Gamma^k}\| > 0 \]

and

\[ \|\Phi_{\Gamma \backslash \Gamma^k_0} x_{\Gamma \backslash \Gamma^k_0}\|_2^2 \leq \sigma^{L(k) - i} \|\Phi_{\Gamma \backslash \Gamma^k_0} x_{\Gamma \backslash \Gamma^k_0}\|_2^2, \quad i \in N. \]

Recalling that $\|\Phi_{\Gamma \backslash \Gamma^k_0} x_{\Gamma \backslash \Gamma^k_0}\|_2^2 = 0$ for $i \geq J_k$ and $\sigma > 1$, we have $v_k \leq L(k) \leq J_k - 1$. Thus

\[ 2^{L(k)} \leq 2^{J_k - 1} \leq \|\Gamma^k\|. \]

Based on $k - \sigma$ character $L(k)$, we define the following integer sequence $\{\kappa_i\}$ as

\[ \kappa_i = \begin{cases} 0 & \text{for } i = 0, \\ \kappa_{i-1} & \text{for } |\Gamma^k| = 0, \\ \kappa_{i-1} + [\alpha 2^{L(k)-1}] & \text{for } |\Gamma^k| \neq 0. \end{cases} \]

Obviously, the sequence $\{\kappa_i\}$ is monotonically non-decreasing, i.e., $\kappa_0 \leq \kappa_1 \leq \cdots$.

3.2. Main idea

Inspired by Wang and Shim (2016) and Zhang (2011), we show that after $k$ iterations, OMP can select a substantial amount of indices in $\Gamma^k$ for a specified number of additional iterations, and the rate of the number of additional iterations to the number of chosen indices is upper bounded by the constant $\alpha$. More precisely, we give the following important proposition, which is the basis of Theorem 2.1.

**Proposition 3.1:** Suppose $\Gamma^k \neq \emptyset$, $k + [\alpha |\Gamma^k|] \leq \kappa$, and $\delta_{K_k} \leq \delta$, then we have

\[ |\Gamma^{k+\alpha [2^{L(k)}]}| \leq |\Gamma^k| - 2^{L(k)}, \]

where $L(k)$ is the $k - \sigma$ character of $x$.

Now, we can prove Theorem 2.1 based on Proposition 3.1. The proof can be divided into two steps.

**Step 1:** We first prove that

\[ \kappa_i + [\alpha |\Gamma^{k_i}|] \leq \kappa, \quad i \in N. \]

Obviously, (11) holds when $i = 0$ since $\kappa_0 + [\alpha |\Gamma^0|] = [\alpha K^0] \leq \alpha K = \kappa$. In the following, we assume that
\[ \kappa_{i-1} + [\alpha|\Gamma^{\kappa_{i-1}}|] \leq \kappa. \] If \(|\Gamma^{\kappa_{i-1}}| = 0\), then \(|\Gamma^\kappa| = 0\) from \(\Gamma^\kappa \subset \Gamma^{\kappa_{i-1}}\). It follows that
\[ \kappa_i + [\alpha|\Gamma^\kappa|] = \kappa_i = \kappa_{i-1} + [\alpha|\Gamma^{\kappa_{i-1}}|] \leq \kappa. \]

If \(|\Gamma^{\kappa_{i-1}}| \neq 0\), by substituting \(k = \kappa_{i-1}\) into (10), we have
\[ |\Gamma^\kappa| = |\Gamma^{\kappa_{i-1}} + [\alpha^2(\kappa_{i-1})]| \leq |\Gamma^{\kappa_{i-1}}| - 2^{L(k_{i-1})}. \]

Thus
\[ \kappa_i + [\alpha|\Gamma^\kappa|] = [\kappa_i + \alpha|\Gamma^{\kappa_{i-1}}|] \leq [\kappa_{i-1} + \alpha^2(\kappa_{i-1}) - \alpha(|\Gamma^{\kappa_{i-1}}| - 2^{L(k_{i-1})})] \]
\[ = \kappa_{i-1} + [\alpha|\Gamma^{\kappa_{i-1}}|] \leq \kappa. \]

This completes the proof of (11).

Step 2: We prove that there exists a constant \(s \in \mathbb{N}\) such that \(|\Gamma^s| = 0\). On one hand, from (11) we have
\[ \kappa_i \leq \kappa_i + [\alpha|\Gamma^\kappa|] \leq \kappa, \quad i \in \mathbb{N}. \quad (12) \]

Thus \(\{\kappa_i\}\) is a monotonically non-decreasing and bounded integer sequence. On the other hand, it is clear that \(\alpha \geq 4 \ln 2 \geq 2\). Then from (9), one can easily show that \(\kappa_{i-1} < \kappa_i\) for \(|\Gamma^{\kappa_{i-1}}| \neq 0\). Therefore, there must exist a constant \(s \in \mathbb{N}\) such that \(|\Gamma^s| = 0\). From (12), we have \(\kappa_s \leq \kappa_s\), then \(|\Gamma^s| \leq |\Gamma^{\kappa_s}|\). Hence \(|\Gamma^s| = 0\), which completes the proof.

### 3.3. Sketch of proof of Proposition 3.1

We here give a sketch of the proof of Proposition 3.1, the remaining details of (16) and (17) can be found in the Appendix. For notational simplicity, let \(k + [\alpha^2(\kappa)] = n_1\). From \(\Gamma^k \neq \emptyset, k + [\alpha|\Gamma^k|] \leq \kappa\), and (8), we have
\[ n_1 = k + [\alpha^2(\kappa)] \leq k + [\alpha|\Gamma^k|] \leq \kappa, \quad (13) \]
and (10) can be rewritten as
\[ |\Gamma^{n_1}| \leq |\Gamma^k| - 2^{L(k)} , \quad (14) \]

By (P.2), a sufficient condition of the above inequality is
\[ \| \Phi_{\Gamma^{n_1}} \chi_{\Gamma^{n_1}} \|_2^2 < \| \Phi_{\Gamma \setminus \Gamma^k} \chi_{\Gamma \setminus \Gamma^k} \|_2^2. \quad (15) \]

Now, what remains is the proof of (15), which is based on the analysis of the residual of OMP. First, by exploiting Wielandt inequality and RIP, we construct a lower bound for \(\| \Gamma^{n_1} \|_2^2\), that is
\[ \| \Gamma^{n_1} \|_2^2 \geq (1 - \delta^2) \| \Phi_{\Gamma^{n_1}} \chi_{\Gamma^{n_1}} \|_2^2. \quad (16) \]

Next, by exploiting some properties of orthogonal projection matrix and RIP, we construct an upper bound for \(\| \Gamma^{n_1} \|_2^2\),
\[ \| \Gamma^{n_1} \|_2^2 < (1 - \delta^2) \| \Phi_{\Gamma^{n_1}} \chi_{\Gamma^{n_1}} \|_2^2. \quad (17) \]

From (16) and (17), it is easy to verify (15) (holds. Hence the remains is the proof of (16) and (17) (see Appendices).

### 4. Conclusion

In this paper, we analyse the number of iterations required for OMP to exactly recover sparse signals. Our analysis shows that OMP can recover any \(K\)-sparse signals in \(|cK|\) iterations \((c \geq -4(1 + s(\kappa)) \ln (\frac{1}{2}))\), which is uniformly smaller than the one proposed in Wang and Shim (2016). For example, to accurately recover \(K\)-sparse signals, it has been shown in Wang and Shim (2016) that OMP requires 30 K iterations with \(\delta_{31} \leq 2\) while our result shows that OMP requires only \([16.8K]\) iterations with \(\delta_{[17,8K]} \leq 2^{-1}\). In practical application, a large number of iterations often lead to a high computational complexity. Our result provides a theoretical basis for the reduction of the number of iterations required for OMP.

### Disclosure statement

No potential conflict of interest was reported by the author(s).

### Funding

The corresponding author gratefully acknowledges support from the National Natural Science Foundation of China No. 11971204, Natural Science Foundation of Jiangsu Province of China No. BK20200108, and the Zhongwu Youth Innovative Talent Program of Jiangsu University of Technology.

### ORCID

Xueping Chen @ http://orcid.org/0000-0003-0499-0972

### References

Bernstein, D. S. (2005). Matrix mathematics: Theory, facts, and formulas with application to linear systems theory. Princeton University Press.

Cai, T., & Wang, L. (2011). Orthogonal matching pursuit for sparse signal recovery with noise. IEEE Transactions on Information Theory, 57(7), 4680–4688. https://doi.org/10.1109/TIT.2011.2146090

Cai, T., Wang, L., & Xu, G. (2010). New bounds for restricted isometry constants. IEEE Transactions on Information Theory, 56(9), 4388–4394. https://doi.org/10.1109/TIT.2010.2054730

Cai, T., Xu, G., & Zhang, J. (2009). On recovery of sparse signals via l1 minimization. IEEE Transactions on Information Theory, 55(7), 3388–3397. https://doi.org/10.1109/TIT.2009.2021377

Candes, E. J., & Tao, T. (2005). Decoding by linear programming. IEEE Transactions on Information Theory, 51(12), 4203–4215. https://doi.org/10.1109/TIT.2005.858979

Chang, L. H., & Wu, J. Y. (2014). An improved RIP-based performance guarantee for sparse signal recovery via orthogonal matching pursuit. IEEE Transactions on Information Theory, 60(9), 405–408. https://doi.org/10.1109/TIT.2014.2304435

Davenport, M. A., & Wakin, M. B. (2010). Analysis of orthogonal matching pursuit using the restricted isometry property. IEEE Transactions on Information Theory, 56(9), 4395–4401. https://doi.org/10.1109/TIT.2010.2054653

Huang, S., & Zhu, J. (2011). Recovery of sparse signals using OMP and its variants: convergence analysis based...
on RIP. Inverse Problems, 27(3), 035003. https://doi.org/10.1088/0266-5611/27/3/035003
Li, H. F., & Wen, J. M. (2019). A new analysis for support recovery with block orthogonal matching pursuit. IEEE Signal Processing Letters, 26(2), 247–251. https://doi.org/10.1109/LSP.2018.2885919
Liu, C., Fang, Y., & Liu, J. Z. (2017). Some new results about sufficient conditions for exact support recovery of sparse signals via orthogonal matching pursuit. IEEE Transactions on Signal Processing, 65(17), 4511–4524. https://doi.org/10.1109/TSP.2017.2711543
Livshitz, E. D. (2012). On the efficiency of the orthogonal matching pursuit in compressed sensing. Sbornik Matematik, 203(2), 33–44. https://doi.org/10.4213/sm
Livshitz, E. D., & Temlyakov, V. N. (2014). Sparse approximation and recovery by greedy algorithms. IEEE Transactions on Information Theory, 60(7), 3989–4000. https://doi.org/10.1109/TIT.2014.2320932
Mo, Q. (2015). A sharp restricted isometry constant bound of orthogonal matching pursuit. arXiv: 1501.01708 [Online]. Available: http://arxiv.org/pdf/1501.01708v1.pdf
Tropp, J. A., & Gilbert, A. C. (2007). Signal recovery from random measurements via orthogonal matching pursuit. IEEE Transactions on Information Theory, 53(12), 4655–4666. https://doi.org/10.1109/TIT.2007.909108
Wang, J., & Shim, B. (2016). Exact recovery of sparse signals using orthogonal matching pursuit: How many iterations do we need. IEEE Transactions on Signal Processing, 64(16), 4194–4202. https://doi.org/10.1109/TSP.2016.2568162
Wang, S. G., & Ip, W.-C. (1999). A matrix version of the Wielandt inequality and its application to statistics. Linear Algebra Its Appl, 2, 118–120. https://doi.org/10.1016/S0024-3795(99)00011-2
Wen, J., Zhang, R., & Yu, W. (2020). Signal-dependent performance analysis of orthogonal matching pursuit for exact sparse recovery. IEEE Transactions on Signal Processing, 68, 5031–5046. https://doi.org/10.1109/TSP.2019.2917787
Wen, J., Zhou, Z., Wang, J., Tang, X., & Mo, Q. (2017). A sharp condition for exact support recovery with orthogonal matching pursuit. IEEE Transactions on Signal Processing, 65(6), 1370–1382. https://doi.org/10.1109/TSP.2016.2634550
Wu, R., Huang, W., & Chen, D.-R. (2013). The exact support recovery of sparse signals with noise via orthogonal matching pursuit. IEEE Signal Processing Letters, 20(4), 403–406. https://doi.org/10.1109/LSP.2012.2233734
Zhang, T. (2011). Sparse recovery with orthogonal matching pursuit under RIP. IEEE Transactions on Information Theory, 57(9), 6215–6221. https://doi.org/10.1109/TIT.2011.2162263

Appendices

Appendix 1. Proof of (16)

The proof of (16) is based on the Wielandt inequality (Wang & Ip, 1999), which is presented in the following Lemma.

Lemma A.1 (Wielandt inequality): Let $A = (A_{11} \ A_{12})$ be an n-order positive-definite matrix with $a_{11} \leq A \leq b_{11}, (a > 0)$. Then we have

$$A_{21} A_{12}^{-1} A_{11} \leq \frac{b - a}{b + a}^{2} A_{22}. \quad \text{(A1)}$$

We now proceed to the proof of (16). The conclusion holds naturally if $\Gamma^{n1} = \emptyset$. In the following, we prove that (16) still holds under $\Gamma^{n1} \neq \emptyset$. Without loss of generality, we assume that $T^{n1} = \{1, 2, \ldots, n_1\}$ and $\Gamma^{n1} = \{n_1 + 1, n_1 + 2, \ldots, n_1 + n_2\}$, where $n_2 = |\Gamma^{n1}|$. Notice that

$\Phi^{n1}_{T^{n1} \cup \Gamma^{n1}, T^{n1} \cup \Gamma^{n1}} = \left( \begin{array}{cc} \Phi^{n1}_{T^{n1}, T^{n1}} & \Phi^{n1}_{T^{n1}, \Gamma^{n1}} \\ \Phi^{n1}_{\Gamma^{n1}, T^{n1}} & \Phi^{n1}_{\Gamma^{n1}, \Gamma^{n1}} \end{array} \right) \quad \text{(A2)}$

By (13), we have $n_1 + n_2 \leq K + \kappa$. Furthermore, from the monotonicity of the RIC and $\delta_{K+\kappa} \leq \delta$, we have $\delta_{n_1 + n_2} \leq \delta_{K+\kappa} \leq \delta$. It implies that

$$|1 - \delta_{n_1 + n_2}| \leq \Delta_{n_1 + n_2} = |1 - \Delta_{K+\kappa}| \leq |1 - \delta|. \quad \text{(A3)}$$

Since (A2), (A3), and Lemma A.1, we have

$$\Phi^{n1}_{T^{n1}, T^{n1}} - \Phi^{n1}_{T^{n1}, \Gamma^{n1}} - \Phi^{n1}_{\Gamma^{n1}, T^{n1}} - \Phi^{n1}_{\Gamma^{n1}, \Gamma^{n1}}$$

$$\leq |1 - \delta|^2 \Phi^{n1}_{T^{n1}, \Gamma^{n1}}. \quad \text{(A4)}$$

Hence by (A4), it follows that

$$\|r^{n1}\|_2^2 = \|P_{T^{n1}}^{*} \Phi^{n1}_{T^{n1}, \Gamma^{n1}} \|_2^2$$

$$= \|x^{n1}_{T^{n1}} \Phi^{n1}_{T^{n1}, \Gamma^{n1}} P_{T^{n1}} \|_2^2$$

$$= \|x^{n1}_{T^{n1}} \Phi^{n1}_{T^{n1}, \Gamma^{n1}} x^{n1}_{T^{n1}} - x^{n1}_{T^{n1}} \Phi^{n1}_{T^{n1}, \Gamma^{n1}} P_{T^{n1}} \Phi^{n1}_{T^{n1}, \Gamma^{n1}} x^{n1}_{T^{n1}}$$

$$\geq (1 - |1 - \delta|^2) \|P_{T^{n1}} x^{n1}_{T^{n1}}\|_2^2.$$
(A8) can be proved as follows. It is easy to check that $P_{T^i+1} = P_{T^i+1}P_{T^i}$. Thus

$$r^{i+1} = P_{T^i+1}y = P_{T^i+1}P_{T^i}y = P_{T^i+1}r^i.$$  

Furthermore,

$$\|r^{i+1}\|^2 = \|P_{T^i+1}r^{i+1}\|^2 \geq \frac{\|P_{T^i+1}r^{i+1}\|^2}{\phi_{t^i+1}/\phi_{t^i+1}} \geq \frac{1}{\delta_1}, \quad (A9)$$

where (a), (b), (c), (d) above can be obtained as follows: (a) is from that fact that $\|P_{T^i+1}r^{i+1}\|^2 = \|P_{T^i+1}r^{i+1}\|^2$; (b) is due to $t^{i+1} \in T^i$; (c) is because

$$\|P_{T^i+1}r^{i+1}\|^2 = (r^{i+1})^T P_{T^i+1}r^{i+1} = (r^{i+1})^T (\phi_{t^i+1}/\phi_{t^i+1}) + \phi_{t^i+1}r^i$$

and $\phi_{t^i+1} > 0$; (d) is followed from the definition of RIP.

Now we construct a lower bound for $\|r^i, \phi_{t^i+1}\|^2$ below. Notice that $P_{T^i}^T \Phi_A x_A = 0$ for $A \subset T^i$ and $T \setminus \Gamma^k \subset T^i$. Thus we have

$$r^i = P_{T^i}^T y = P_{T^i}^T \Phi_T x_T = P_{T^i}^T (\Phi_{T^i} x_{T^i} + \Phi_{T \cap T^i} x_{T \cap T^i})$$

$$= P_{T^i}^T \Phi_{T^i} x_{T^i}. \quad (A10)$$

From (A10), it follows that

$$\|r^i - P_{T^i}^T \Phi_{T \cap T^i} x_{T^i}\|^2 = \|r^i - P_{T^i}^T \Phi_{T^i} x_{T^i} - P_{T^i}^T \Phi_{T \cap T^i} x_{T \cap T^i}\|^2$$

$$= \|r^i - P_{T^i}^T \Phi_{T^i} x_{T^i} - P_{T^i}^T (\Phi_{T^i} x_{T^i} - \Phi_{T^i} x_{T^i})\|^2$$

$$\leq \|x_{T \cap T^i}\|^2 \|r^i - P_{T^i}^T \Phi_{T^i} x_{T^i}\|^2 \|x_{T \cap T^i}\|^2. \quad (A11)$$

Notice that $P_{T^i}^T r^i = r^i$. From (A11), we have

$$2\text{Re}(r^i, \Phi_{T \cap T^i} x_{T \cap T^i}) = 2\text{Re}(P_{T^i}^T r^i, \Phi_{T \cap T^i} x_{T \cap T^i})$$

$$= \|r^i - P_{T^i}^T \Phi_{T^i} x_{T^i}\|^2 + \|P_{T^i}^T \Phi_{T^i} x_{T^i} - P_{T^i}^T \Phi_{T \cap T^i} x_{T \cap T^i}\|^2 \geq \|r^i - P_{T^i}^T \Phi_{T^i} x_{T^i}\|^2 \geq \|r^i\|^2 \geq \|r^i\|^2 \|x_{T \cap T^i}\|^2. \quad (A12)$$

Hence by (A7) and (A12), we have

$$2\text{Re}(r^i, \Phi_{T \cap T^i} x_{T \cap T^i}) \geq \|r^i\|^2 - \|\Phi_{T^i} x_{T \cap T^i}\|^2 \geq 0. \quad (A13)$$

and $T \setminus T^i \neq \emptyset$.

Now we prove

$$\|P_{T^i}^T \Phi_{T \cap T^i} x_{T \cap T^i}\|^2 \geq (1 - \delta_{T \cap T^i}) \|x_{T \cap T^i}\|^2. \quad (A14)$$

In fact, if $i = 0$, then $T^0 = \emptyset$ and $P_{T^0} = I_m$. From RIP, we have

$$\|P_{T^0}^T \Phi_{T \cap T^i} x_{T \cap T^i}\|^2 = \|\Phi_{T \cap T^i} x_{T \cap T^i}\|^2 \geq (1 - \delta_{T \cap T^i}) \|x_{T \cap T^i}\|^2$$

$$= (1 - \delta_{T \cap T^i}) \|x_{T \cap T^i}\|^2.$$

For $i \neq 0$, we have

$$\|P_{T^i}^T \Phi_{T \cap T^i} x_{T \cap T^i}\|^2 \geq (1 - \delta_{T \cap T^i}) \|x_{T \cap T^i}\|^2$$

$$\geq (1 - \delta_{T \cap T^i}) \|x_{T \cap T^i}\|^2.$$
for \( i = l, l + 1, \ldots, l' - 1 \). This completes the proof of (A8).

We now proceed to the proof of (17). Let \( k_0 = k \) and

\[
k_i = i + \sum_{r = i+1}^{v_k + i} \left[ \frac{\alpha}{4} \right] \Gamma_{i+1}^k, \quad i = 1, 2, \ldots, L(k) + 1 - v_k.
\]

Let \( l' = k, l = k_i - 1 \) and \( \Gamma = \Gamma_{k_i + l}^k \) in (A5). Notice that \( k_i = k_{i-1} + \left[ \frac{\alpha}{4} [\Gamma_{i+1}^k] \right] \). Then it follows that

\[
\| r_k \|^2 - \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \\
\leq C_{r_{k_i+1}^k} \left( \| r_k \|^2 - \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right),
\]

where

\[
C_{r_{k_i+1}^k} = \exp \left( \frac{(2-\delta)(-\delta)}{2\delta \nu} \Gamma_{k_i+1}^k \right).
\]

In the following, we construct an upper bound for \( C_{r_{k_i+1}^k} \). Recall that \( \alpha \geq 4 \ln 2 \geq 2 \). Then from Appendix 2 in Wang and Shim (2016), we have

\[
\sum_{i=1}^{l} \left[ \frac{\alpha}{4} (2^i - 1) \right] \leq [\alpha 2^{l+1}] - 1.
\]

By (P1) and (A21), it follows that

\[
k_{L(k)+1-v_k} = k + \sum_{r = v_k+1}^{L(k)+1} \left[ \frac{\alpha}{4} \right] \Gamma_{r+1}^k \\
\leq k + \sum_{r = 1}^{L(k)+1} \left[ \frac{\alpha}{4} (2^r - 1) \right] \\
\leq k + [\alpha 2^{L(k)} - 1] \\
\leq k + [\alpha 2^{L(k)}] = n_i.
\]

Notice that \( \Gamma_{k_i}^k \subset T \subset T^k \), and \( T^{k_i-1} \subset T^{L(k)+1-n_i} \) for \( i = 1, 2, \ldots, L(k) + 1 - v_k \). From (13) and (A22), we obtain

\[
\| T_{k_i+1} \|_1 \leq \| T \cup T^{k_i-1} \|_1 \leq \| T \|_1 \leq K + k_i.
\]

Furthermore, from monotonicity of the RIC and \( \delta_{K+k_i} \leq \delta \), we have

\[
\frac{1 - \delta(k_i+1-n)}{1 + \delta} \geq \frac{1 - \delta K+k_i}{1 + \delta} \geq \frac{1 - \delta}{1 + \delta}.
\]

Hence,

\[
C_{r_{k_i+1}^k} = \exp \left( \frac{(1 - \frac{\delta}{2}) \Gamma_{k_i+1}^k}{1 + \delta} \right) \\
\leq \exp \left( \frac{(1 - \frac{\delta}{2}) \Gamma_{k_i+1}^k}{1 + \delta} \right) \\
= (1 - \delta)/2.
\]

Together with (A20), it implies that

\[
\| r_k \|^2 \leq \frac{1 - \delta}{2\delta \nu} \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right) \\
+ \frac{1 + \delta}{2\delta \nu} \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right).
\]

For \( i = 1, 2, \ldots, L(k) + 1 - v_k \), note that we only need to consider \( \| r_k \|^2 \geq \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \) since if \( \| r_k \|^2 \)

\[
< \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2, \quad \text{(A25)}
\]

holds from

\[
\| r_k \|^2 \leq \| r_{k_i+1} \|^2 = \frac{1 - \delta}{2\delta \nu} \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right) + \frac{1 + \delta}{2\delta \nu} \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right) \\
\leq \frac{1 - \delta}{2\delta \nu} \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right) \\
+ \frac{1 + \delta}{2\delta \nu} \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right).
\]

By substituting (7) into (A25), we further obtain

\[
\| r_k \|^2 \leq \frac{1 - \delta}{2\delta \nu} \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right) \\
+ \frac{1 + \delta}{2\delta \nu} \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right).
\]

From (A5) \( \nu_k = 0 \) and (7), we have

\[
\| r_k \|^2 = \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \\
\leq \left( \frac{1 - \delta}{2\delta \nu} \right) \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right) \\
+ \left( \frac{1 + \delta}{2\delta \nu} \right) \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right) \\
\leq \left( \frac{1 - \delta}{2\delta \nu} \right) \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right) + \left( \frac{1 + \delta}{2\delta \nu} \right) \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right).
\]

Notice that \( \sigma = \frac{1}{\Gamma_{k_i}} \) and \( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 > 0 \). Then by (A22), (A26), (A27), we obtain

\[
\| r_k \|^2 \leq \left( \frac{1 - \delta}{2\delta \nu} \right) \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right) \\
+ \left( \frac{1 + \delta}{2\delta \nu} \right) \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right) \\
\leq \left( \frac{1 - \delta}{2\delta \nu} \right) \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right) \\
+ \left( \frac{1 + \delta}{2\delta \nu} \right) \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right) \\
= \left( \frac{1 - \delta}{2\delta \nu} \right) \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right) \\
< \left( \frac{1 - \delta}{2\delta \nu} \right) \left( \| \Phi \Gamma_{k_i+1}^k \Gamma_{k_i+1}^k \|^2 \right).
\]

Thus (17) holds. This completes the proof.