FACTORIZATION OF COMPLETE MULTIPARTITE HYPERGRAPHS

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Abstract. In a mathematics workshop with $mn$ mathematicians from $n$ different areas, each area consisting of $m$ mathematicians, we want to create a collaboration network. For this purpose, we would like to schedule daily meetings between groups of size three, so that (i) two people of the same area meet one person of another area, (ii) each person has exactly $r$ meeting(s) each day, and (iii) each pair of people of the same area have exactly $\lambda$ meeting(s) with each person of another area by the end of the workshop. Using hypergraph amalgamation-detachment, we prove a more general theorem. In particular we show that above meetings can be scheduled if: $3 \mid rm$, $2 \mid rnm$ and $r \mid 3\lambda(n - 1)^\binom{m}{2}$. This result can be viewed as an analogue of Baranyai’s theorem on factorizations of complete multipartite hypergraphs.

1. Introduction

Throughout this paper, $\mathbb{N}$ is the set of positive integers, $m, n, r, \lambda \in \mathbb{N}$, and $[n] := \{1, \ldots, n\}$. In a mathematics workshop with $mn$ mathematicians from $n$ different areas, each area consisting of $m$ mathematicians, we want to create a collaboration network. For this purpose, we would like to schedule daily meetings between groups of size three, so that (i) two people of the same area meet one person of another area, (ii) each person has exactly $r$ meeting(s) each day, and (iii) each pair of people of the same area have exactly $\lambda$ meeting(s) with each person of another area by the end of the workshop. Using hypergraph amalgamation-detachment, we prove a more general theorem. In particular we show that above meetings can be scheduled if: $3 \mid rm$, $2 \mid rnm$ and $r \mid 3\lambda(n - 1)^\binom{m}{2}$.

A hypergraph $\mathcal{G}$ is a pair $(V, E)$ where $V$ is a finite set called the vertex set, $E$ is the edge multiset, where every edge is itself a multi-subset of $V$. This means that not only can an edge occur multiple times in $E$, but also each vertex can have multiple occurrences within an edge. The total number of occurrences of a vertex $v$ among all edges of $E$ is called the degree, $d_G(v)$ of $v$ in $\mathcal{G}$. For $h \in \mathbb{N}$, $\mathcal{G}$ is said to be $h$-uniform if $|e| = h$ for each $e \in E$. For $r, r_1, \ldots, r_k \in \mathbb{N}$, an $r$-factor in a hypergraph $\mathcal{G}$ is a spanning $r$-regular sub-hypergraph, and

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an \((r_1, \ldots, r_k)\)-factorization is a partition of the edge set of \(G\) into \(F_1, \ldots, F_k\) where \(F_i\) is an \(r_i\)-factor for \(i \in [k]\). We abbreviate \((r, \ldots, r)\)-factorization to \(r\)-factorization.

The hypergraph \(K_n^h := (V, \binom{V}{h})\) with \(|V| = n\) (by \(\binom{V}{h}\) we mean the collection of all \(h\)-subsets of \(V\)) is called a complete \(h\)-uniform hypergraph. In connection with Kirkman’s schoolgirl problem [14], Sylvester conjectured that \(K_n^h\) is 1-factorable if and only if \(h \mid n\). This conjecture was finally settled by Baranyai [8]. Let \(K_{m \times n}^3\) denote the 3-uniform hypergraph with vertex partition \(\{V_i : i \in [n]\}\), so that \(V_i = \{x_{ij} : j \in [m]\}\) for \(i \in [n]\), and with edge set \(E = \{\{x_{ij}, x_{ij'}, x_{kl}\} : i, k \in [n], j, j', l \in [m], j \neq j', i \neq k\}\). One may notice that finding an \(r\)-factorization for \(K_{m \times n}^3\) is equivalent to scheduling the meetings between mathematicians with the above restrictions for the case \(\lambda = 1\).

If we replace every edge \(e\) of \(G\) by \(\lambda\) copies of \(e\), then we denote the new hypergraph by \(\lambda G\). In this paper, the main result is the following theorem which is obtained by proving a more general result (see Theorem 3.1) using amalgamation-detachment techniques.

**Theorem 1.1.** \(\lambda K_{m \times n}^3\) is \((r_1, \ldots, r_k)\)-factorable if

\[
\begin{align*}
(S1) & \quad 3 \mid r_i m \text{ for } i \in [k], \\
(S2) & \quad 2 \mid r_i mn \text{ for } i \in [k], \text{ and} \\
(S3) & \quad \sum_{i=1}^{k} r_i = 3\lambda(n - 1)\binom{m}{2}.
\end{align*}
\]

In particular, by letting \(r = r_1 = \cdots = r_k\) in Theorem 1.1, we solve the Mathematicians Collaboration Problem in the following case.

**Corollary 1.2.** \(\lambda K_{m \times n}^3\) is \(r\)-factorable if

\[
\begin{align*}
(i) & \quad 3 \mid rm, \\
(ii) & \quad 2 \mid rnm, \text{ and} \\
(iii) & \quad r \mid 3\lambda(n - 1)\binom{m}{2}.
\end{align*}
\]

The two results above can be seen as analogues of Baranyai’s theorem for complete 3-uniform “multipartite” hypergraphs. We note that in fact, Baranyai [9] solved the problem of factorization of complete uniform multipartite hypergraphs, but here we aim to solve this problem under a different notion of “multipartite”. In Baranyai’s definition, an edge can have at most one vertex from each part, but here we allow an edge to have two vertices from each part (see the definition of \(K_{m \times n}^3\) above). More precise definitions together with preliminaries are given in Section 2, the main result is proved in Section 3, and related open problems are discussed in the last section.

Amalgamation-detachment technique was first introduced by Hilton [10] (who found a new proof for decompositions of complete graphs into Hamiltonian cycles), and was more developed by Hilton and Rodger [11]. Hilton’s method was later generalized to arbitrary
graphs [5], and later to hypergraphs [1, 2, 7, 4] leading to various extensions of Baranyai’s theorem (see for example [1, 3]). The results of the present paper, mainly relies on those from [1] and [15]. For the sake of completeness, here we give a self contained exposition.

2. More Terminology and Preliminaries

Recall that an edge can have multiple copies of the same vertex. For the purpose of this paper, all hypergraphs (except when we use the term graph) are 3-uniform, so an edge is always of one of the forms \{u, u, u\}, \{u, u, v\}, and \{u, v, w\} which we will abbreviate to \{u^3\}, \{u^2, v\}, and \{u, v, w\}, respectively. In a hypergraph \(G\), \(\text{mult}_G(.)\) denotes the multiplicity; for example \(\text{mult}_G(u^3)\) is the multiplicity of an edge of the form \{u^3\}. Similarly, for a graph \(G\), \(\text{mult}(u, v)\) is the multiplicity of the edge \{u, v\}. A \(k\)-edge-coloring of a hypergraph \(G\) is a mapping \(K: E(G) \rightarrow [k]\), and the sub-hypergraph of \(G\) induced by color \(i\) is denoted by \(G(i)\). Whenever it is not ambiguous, we drop the subscripts, and also we abbreviate \(d_G(i)(u)\) to \(d_i(u)\), \(\text{mult}_G(i)(u^3)\) to \(\text{mult}_i(u^3)\), etc..

Factorizations of the complete graph, \(K_n\), is studied in a very general form in [12, 13], however for the purpose of this paper, a \(\lambda\)-fold version is needed:

**Theorem 2.1.** (Bahmanian, Rodger [6, Theorem 2.3]) \(\lambda K_n\) is \((r_1, \ldots, r_k)\)-factorable if and only if \(r_i n\) is even for \(i \in [k]\) and \(\sum_{i=1}^{k} r_i = \lambda(n-1)\).

Let \(K^*_n\) denote the 3-uniform hypergraph with \(n\) vertices in which \(\text{mult}(u^2, v) = 1\), and \(\text{mult}(u^3) = \text{mult}(u, v, w) = 0\) for distinct vertices \(u, v, w\). A (3-uniform) hypergraph \(G = (V, E)\) is \(n\)-partite, if there exists a partition \(\{V_1, \ldots, V_n\}\) of \(V\) such that for every \(e \in E\), \(|e \cap V_i| = 1, |e \cap V_j| = 2\) for some \(i, j \in [n]\) with \(i \neq j\). For example, both \(K^*_n\) and \(\mathcal{H}^3_{m\times n}\) are \(n\)-partite. We need another simple but crucial lemma:

**Lemma 2.2.** If \(r_i n\) is even for \(i \in [k]\), and \(\sum_{i=1}^{k} r_i = \lambda(n-1)\), then \(\lambda K^*_n\) is \((3r_1, \ldots, 3r_k)\)-factorable.

**Proof.** Let \(G = \lambda K_n\) with vertex set \(V\). By Theorem 2.1, \(G\) is \((r_1, \ldots, r_k)\)-factorable. Using this factorization, we obtain a \(k\)-edge-coloring for \(G\) such that \(d_G(i)(v) = r_i\) for every \(v \in V\) and every color \(i \in [k]\). Now we form a \(k\)-edge-colored hypergraph \(H\) with vertex set \(V\) such that \(\text{mult}_{H(i)}(u^2, v) = \text{mult}_{G(i)}(u, v)\) for every pair of distinct vertices \(u, v \in V\), and each color \(i \in [k]\). It is easy to see that \(H \cong \lambda K^*_n\) and \(d_H(i)(v) = 3r_i\) for every \(v \in V\) and every color \(i \in [k]\). Thus we obtain a \((3r_1, \ldots, 3r_k)\)-factorization for \(\lambda K^*_n\). \(\square\)

If the multiplicity of a vertex \(\alpha\) in an edge \(e = p\) is \(p\), we say that \(\alpha\) is *incident* with \(p\) distinct *hinges*, say \(h_1(\alpha, e), \ldots, h_p(\alpha, e)\), and we also say that \(e\) is *incident* with \(h_1(\alpha, e), \ldots, h_p(\alpha, e)\).
The set of all hinges in $G$ incident with $\alpha$ is denoted by $H_G(\alpha)$; so $|H_G(\alpha)|$ is in fact the degree of $\alpha$.

Intuitively speaking, an $\alpha$-detachment of a hypergraph $G$ is a hypergraph obtained by splitting a vertex $\alpha$ into one or more vertices and sharing the incident hinges and edges among the subvertices. That is, in an $\alpha$-detachment $G'$ of $G$ in which we split $\alpha$ into $\alpha$ and $\beta$, an edge of the form $\{\alpha^p, u_1, \ldots, u_z\}$ in $G$ will be of the form $\{\alpha^{p-i}, \beta^i, u_1, \ldots, u_z\}$ in $G'$ for some $i$, $0 \leq i \leq p$. Note that a hypergraph and its detachments have the same hinges. Whenever it is not ambiguous, we use $d'$, mult', etc. for degree, multiplicity and other hypergraph parameters in $G'$.

Let us fix a vertex $\alpha$ of a $k$-edge-colored hypergraph $G = (V, E)$. For $i \in [k]$, let $H_i(\alpha)$ be the set of hinges each of which is incident with both $\alpha$ and an edge of color $i$ (so $d_i(\alpha) = |H_i(\alpha)|$). For any edge $e \in E$, let $H^e(\alpha)$ be the collection of hinges incident with both $\alpha$ and $e$. Clearly, if $e$ is of color $i$, then $H^e(\alpha) \subset H_i(\alpha)$.

A family $\mathcal{A}$ of sets is laminar if, for every pair $A, B$ of sets belonging to $\mathcal{A}$, either $A \subset B$, or $B \subset A$, or $A \cap B = \emptyset$. We shall present two lemmas, both of which follow immediately from definitions.

Lemma 2.3. Let $\mathcal{A} = \{H_1(\alpha), \ldots, H_k(\alpha)\} \cup \{H^e(\alpha) : e \in E\}$. Then $\mathcal{A}$ is a laminar family of subsets of $H(\alpha)$.

For each $p \in \{1, 2\}$, and each $U \subset V \setminus \{\alpha\}$, let $H(\alpha^p, U)$ be the set of hinges each of which is incident with both $\alpha$ and an edge of the form $\{\alpha^p\} \cup U$ in $G$ (so $|H(\alpha^p, U)| = p \text{mult}(|\{\alpha^p, U\}|$).

Lemma 2.4. Let $\mathcal{B} = \{H(\alpha^p, U) : p \in \{1, 2\}, U \subset V \setminus \{\alpha\}\}$. Then $\mathcal{B}$ is a laminar family of disjoint subsets of $H(\alpha)$.

If $x, y$ are real numbers, then $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the integers such that $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$, and $x \approx y$ means $\lfloor y \rfloor \leq x \leq \lceil y \rceil$. We need the following powerful lemma:

Lemma 2.5. (Nash-Williams [15, Lemma 2]) If $\mathcal{A}, \mathcal{B}$ are two laminar families of subsets of a finite set $S$, and $n \in \mathbb{N}$, then there exist a subset $A$ of $S$ such that

$$|A \cap P| \approx |P|/n \text{ for every } P \in \mathcal{A} \cup \mathcal{B}.$$

3. Proofs

Notice that $\lambda \mathcal{K}_{m \times n}^{3}$ is a $3\lambda(n-1)\binom{m}{2}$-regular hypergraph with $nm$ vertices and $2\lambda m \binom{n}{2} \binom{m}{2}$ edges. To prove Theorem 1.1, we prove the following seemingly stronger result.

Theorem 3.1. Let $3 \mid r|m$ and $2 \mid r_i mn$ for $i \in [k]$, and $\sum_{i=1}^{k} r_i = 3\lambda(n-1)\binom{m}{2}$. Then for all $\ell = n, n+1, \ldots, mn$ there exists a $k$-edge-colored $\ell$-vertex $n$-partite hypergraph $G = (V, E)$ and a function $g : V \rightarrow \mathbb{N}$ such that the following conditions are satisfied:
Remark 3.2. It is implicitly understood that every other type of edge in \( \mathcal{G} \) is of multiplicity 0.

Before we prove Theorem 3.1, we show how Theorem 1.1 is implied by Theorem 3.1.

Proof of Theorem 1.1. It is enough to take \( \ell = mn \) in Theorem 3.1. Then there exists an \( n \)-partite hypergraph \( \mathcal{G} = (V, E) \) of order \( mn \) and a function \( g : V \to \mathbb{N} \) such that by (C1) \( \sum_{w \in W} g(v) = m \) for each part \( W \) of \( \mathcal{G} \). This implies that \( g(v) = 1 \) for each \( v \in V \) and that each part of \( \mathcal{G} \) has \( m \) vertices. By (C2), \( \text{mult}_W(u^2, v) = \lambda \binom{1}{1}(1) = 0 \) for each pair of vertices \( u, v \) from different parts of \( \mathcal{G} \), and by (C3), \( \text{mult}_G(u, v, w) = \lambda \) for each pair of vertices \( u, v \) from the same part and \( w \) from a different part of \( \mathcal{G} \). This implies that \( \mathcal{G} \cong \lambda X_{m \times n}^3 \). Finally, by (C4), \( \mathcal{G} \) admits a \( k \)-edge-coloring such that \( d_{G(i)}(v) = r_i \) for each color \( i \in [k] \). This completes the proof.

The idea of the proof of Theorem 3.1 is that each vertex \( \alpha \) will be split into \( g(\alpha) \) vertices and that this will be done by “splitting off” single vertices one at a time.

Proof of Theorem 3.1. We prove the theorem by induction on \( \ell \).

First we prove the basis of induction, case \( \ell = n \). Let \( \mathcal{G} = (V, E) \) be \( \lambda m \binom{m}{2} K^n \) and let \( g(v) = m \) for all \( v \in V \). Since \( \mathcal{G} \) has \( n \) vertices, it is \( n \)-partite (each vertex being a partite set). Obviously, \( \sum_{w \in W} g(v) = g(v) = m \) for each part \( W \) of \( \mathcal{G} \). Also, \( \text{mult}(u^2, v) = \lambda m \binom{m}{2} = \lambda \binom{g(u)}{2} g(v) \) for each pair of vertices \( u, v \) from distinct parts of \( \mathcal{G} \), so (C2) is satisfied. Since there is only one vertex in each part, (C3) is trivially satisfied.

Since for \( i \in [k] \), \( 2 \mid \frac{r_{mn}}{3} \) and \( \sum_{i=1}^{k} \frac{r_{m}}{3} = \lambda m(n-1) \binom{m}{2} \), by Lemma 2.2, \( \mathcal{G} \) is \((mr_1, \ldots, mr_k)\)-factorable. Thus, we can find a \( k \)-edge-coloring for \( \mathcal{G} \) such that \( d_{G(i)}(v) = mr_i = r_i g(v) \) for \( i \in [k] \), and therefore (C4) is satisfied.

Suppose now that for some \( \ell \in \{n, n+1, \ldots, mn-1\} \), there exists a \( k \)-edge-colored \( n \)-partite hypergraph \( \mathcal{G} = (V, E) \) of order \( \ell \) and a function \( g : V \to \mathbb{N} \) satisfying properties (C1)–(C4) from the statement of the theorem. We shall now construct an \( n \)-partite hypergraph \( \mathcal{G}' \) of order \( \ell + 1 \) and a function \( g' : V(\mathcal{G}') \to \mathbb{N} \) satisfying (C1)–(C4).

Since \( \ell < mn \), \( \mathcal{G} \) is \( n \)-partite and (C1) holds for \( \mathcal{G} \), there exists a vertex \( \alpha \) of \( \mathcal{G} \) with \( g(\alpha) > 1 \). The graph \( \mathcal{G}' \) will be constructed as an \( \alpha \)-detachment of \( \mathcal{G} \) with the help of
laminar families

\[ \mathcal{A} := \{H_1(\alpha), \ldots, H_k(\alpha)\} \cup \{H^e(\alpha) : e \in E\} \]

and

\[ \mathcal{B} := \{H(\alpha^p, U) : p \in \{1, 2\}, U \subset V \setminus \{\alpha\}\}. \]

By Lemma 2.5, there exists a subset \( Z \) of \( H(\alpha) \) such that

\[ |Z \cap P| \approx |P|/g(\alpha), \text{ for every } P \in \mathcal{A} \cup \mathcal{B}. \tag{1} \]

Let \( \mathcal{G}' = (V', E') \) with \( V' = V \cup \{\beta\} \) be the hypergraph obtained from \( \mathcal{G} \) by splitting \( \alpha \) into two vertices \( \alpha \) and \( \beta \) in such a way that hinges which were incident with \( \alpha \) in \( \mathcal{G} \) become incident in \( \mathcal{G}' \) with \( \alpha \) or \( \beta \) according to whether they do not or do belong to \( Z \), respectively. More precisely,

\[ H'(\beta) = Z, \quad H'(\alpha) = H(\alpha) \setminus Z. \tag{2} \]

So \( \mathcal{G}' \) is an \( \alpha \)-detachment of \( \mathcal{G} \) and the colors of the edges are preserved. Let \( g' : V' \to \mathbb{N} \) so that \( g'(\alpha) = g(\alpha) - 1, g'(\beta) = 1 \), and \( g'(u) = g(u) \) for each \( u \in V' \setminus \{\alpha, \beta\} \). It is obvious that \( \mathcal{G}' \) is of order \( \ell + 1 \), \( n \)-partite, and \( \sum_{v \in W} g'(v) = m \) for each part \( W \) of \( \mathcal{G}' \) (the new vertex \( \beta \) belongs to the same part of \( \mathcal{G}' \) as \( \alpha \) belongs to). Moreover, it is clear that \( \mathcal{G}' \) satisfies (C2)–(C4) if \( \{\alpha, \beta\} \cap \{u, v, w\} = \emptyset \). For the rest of the argument, we will repeatedly use the definitions of \( \mathcal{A}, \mathcal{B}, (1) \), and (2).

For \( i \in [k] \) we have

\[ d'_i(\beta) = |Z \cap H_i(\alpha)| \approx |H_i(\alpha)|/g(\alpha) = d_i(\alpha)/g(\alpha) = r_i = r_i g'(\beta), \]

\[ d'_i(\alpha) = d_i(\alpha) - d'_i(\beta) = r_i g(\alpha) - r_i = r_i (g(\alpha) - 1) = r_i g'(\alpha), \]

so \( \mathcal{G}' \) satisfies (C4).

Let \( u \in V' \) so that \( u \) and \( \alpha \) (or \( \beta \)) belong to different parts of \( \mathcal{G}' \). We have

\[ \text{mult}'(\beta, u^2) = |Z \cap H(\alpha, \{u^2\})| \approx |H(\alpha, \{u^2\})|/g(\alpha) = \text{mult}(\alpha, u^2)/g(\alpha) \]

\[ = \lambda \left( \frac{g(u)}{2} \right) = \lambda \left( \frac{g'(u)}{2} \right) g'(\beta), \]

\[ \text{mult}'(\alpha, u^2) = \text{mult}(\alpha, u^2) - \text{mult}'(\beta, u^2) = \lambda \left( \frac{g(u)}{2} \right) g(\alpha) - \lambda \left( \frac{g(u)}{2} \right) = \lambda \left( \frac{g'(u)}{2} \right) g'(\alpha). \]

Recall that \( g(\alpha) \geq 2 \), and for every \( e \in E \) and \( i \in [k], |H^e(\alpha)| \leq 2 \), and thus \( |Z \cap H^e(\alpha)| \approx |H^e(\alpha)|/g(\alpha) \leq 1 \). This implies that

\[ \text{mult}'(\beta^2, u) = 0 = \lambda \left( \frac{g'(\beta)}{2} \right) g'(u), \]
and so \( \text{mult}(\alpha^2, u) = \text{mult}'(\alpha^2, u) + \text{mult}'(\alpha, \beta, u) \). Now we have

\[
\text{mult}'(\alpha, \beta, u) = |Z \cap H(\alpha^2, \{u\})| - |H(\alpha^2, \{u\})|/g(\alpha) \\
= 2 \text{mult}(\alpha^2, u)/g(\alpha) = \lambda(g(\alpha) - 1)g(u) = \lambda g'(\alpha)g'(\beta)g'(u), \\
\text{mult}'(\alpha^2, u) = \text{mult}(\alpha^2, u) - \text{mult}'(\alpha, \beta, u) = \lambda \left( \frac{g'(\alpha)}{2} \right) g(u) - \lambda(g(\alpha) - 1)g(u) \\
= \lambda \left( \frac{g'(\alpha)}{2} \right) g'(u).
\]

Therefore \( G' \) satisfies \((C2)\).

Let \( u, v \in V' \) so that \( u, v \) belong to different parts of \( G' \), \( u, \alpha \) belong to the same part of \( G' \), and \( u \notin \{\alpha, \beta\} \). We have

\[
\text{mult}'(\beta, u, v) = |Z \cap H(\alpha, \{u, v\})| - |H(\alpha, \{u, v\})|/g(\alpha) = \text{mult}(\alpha, u, v)/g(\alpha) \\
= \lambda g(u)g(v) = \lambda g'(\beta)g'(u)g'(v), \\
\text{mult}'(\alpha, u, v) = \text{mult}(\alpha, u, v) - \text{mult}'(\beta, u, v) = \lambda(g(\alpha) - 1)g(u)g(v) = \lambda g'(\alpha)g'(u)g'(v).
\]

Finally, let \( u, v \in V' \) so that \( u, v \) belong to the same part of \( G' \), and \( u, \alpha \) belong to different parts of \( G' \), and \( u \notin \{\alpha, \beta\} \). By an argument very similar to the one above, we have

\[
\text{mult}'(u, v, \beta) = \lambda g'(u)g'(v)g'(\beta), \\
\text{mult}'(u, v, \alpha) = \lambda g'(u)g'(v)g'(\alpha).
\]

Therefore \( G' \) satisfies \((C3)\), and the proof is complete.

\[\square\]

\section{4. Final Remarks}

We define \( H^3_{m_1, \ldots, m_n} \) similar to \( H^3_{m \times n} \) with the difference that in \( H^3_{m_1, \ldots, m_n} \) we allow different parts to have different sizes. It seems reasonable to conjecture that

\textbf{Conjecture 4.1.} \( \lambda H^3_{m_1, \ldots, m_n} \) is \((r_1, \ldots, r_k)\)-factorable if and only if

\begin{enumerate}
  \item \( m_i = m_j := m \) for \( i, j \in \{n\} \),
  \item \( 3 \mid r_i m \) for \( i \in \{k\} \), and
  \item \( \sum_{i=1}^{k} r_i = 3\lambda(n - 1)\binom{m}{2} \).
\end{enumerate}

We prove the necessity as follows. Since \( \lambda H^3_{m \times n} \) is factorable, it must be regular. Let \( u \) and \( v \) be two vertices from two different parts, say \( p^{th} \) and \( q^{th} \) parts respectively. Then we
have the following sequence of equivalences:

\[
d(u) = d(v) \iff \sum_{1 \leq i \leq n} \binom{m_i}{2} + (m_p - 1) \sum_{i \neq p} \binom{m_i}{2} = \sum_{1 \leq i \leq n} \binom{m_i}{2} + (m_q - 1) \sum_{i \neq q} \binom{m_i}{2}
\]

\[
\left( \frac{m_p}{2} \right) + \sum_{1 \leq i \leq n} \binom{m_i}{2} + (m_p - 1)(m_q + \sum_{i \neq p,q} \binom{m_i}{2}) = \left( \frac{m_q}{2} \right) + \sum_{1 \leq i \leq n} \binom{m_i}{2} + (m_q - 1)(m_p + \sum_{i \neq p,q} \binom{m_i}{2}) \iff
\]

\[
\left( \frac{m_p}{2} \right) - \left( \frac{m_q}{2} \right) + m_pm_q - m_p - m_pm_q + m_q + (m_p - m_q) \sum_{1 \leq i \leq n} \binom{m_i}{2} = 0
\]

\[
m_p^2 - m_q^2 - 3m_p + 3m_q + 2(m_p - m_q) \sum_{1 \leq i \leq n} \binom{m_i}{2} = 0 \iff
\]

\[
(m_p - m_q)(m_p + m_q - 3 + 2 \sum_{1 \leq i \leq n} \binom{m_i}{2}) = 0 \iff
\]

\[
m_p = m_q := m.
\]

This proves (i). The existence of an \( r_i \)-factor implies that \( 3 \mid r_i m \) for \( i \in [k] \). Since each \( r_i \)-factor is an \( r_i \)-regular spanning sub-hypergraph and \( \lambda \mathcal{X}^3_{m \times n} \) is \( 3\lambda(n - 1)(\binom{m}{2}) \)-regular, we must have \( \sum_{i=1}^{k} r_i = 3\lambda(n - 1)(\binom{m}{2}) \).

In Theorem 1.1, we made partial progress toward settling Conjecture 4.1, however at this point, it is not clear to us whether our approach will work for the remaining cases.

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