Analysis on the Vibration of Cracked Cantilever Beams with Application on Crack Detection

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Abstract. In this research, the transfer matrix of a cracked beam which includes the vibration modes of both longitudinal and transverse deformation is derived based on Timoshenko beam model. The expression of the natural frequencies and mode shapes of vibration using transfer matrix method are proposed which can be effectively used for real-time crack detection. Manifold types of cracks in the cantilever beam which have been simulated by finite element method, and the solutions are compared with those from the transfer matrix method. The difference between results could be used for detecting both the size and location of the crack. Therefore, the transfer matrix method can be successful used for analysis of vibration in the cracked beam, and the errors are within the tolerant range.

1. Introduction
In engineering practice, many vibration problems of the structure could be simplified to the study of vibration of an approximate cantilever beam, for examples, the blades of a Turbomachine, wings on an aircraft and the extensional beam in civil engineering, etc. Three fundamental forms of vibrations of the cantilever beam are longitudinal extension, transverse bending and cross-section reversing. In most cases, the vibrations of the cantilever beam could be decomposed to these three fundamental forms.

Meanwhile, crack in the beam is one of the most important factors that affect its vibration characteristics, which can significantly affect its natural frequencies and corresponding mode shapes. In general, size and location of the crack can be determined by examining changes of vibration character of the beam. The vibration problems of beam with different types of cracks have been researched intensively and many valuable results have been published. For instance, the irregularity-based damage detection technique have been proposed by Wang and Qiao [1], who use it in determining the location and depth of cracks. Location and size parameters of both single crack and multiple cracks could be estimated by the irregularity curve of the mode with this method. Analytical expressions on shifts of wavenumber and frequency of the beam are derived by Kasper et al. [2], in which the beam has been simplified to a rotational spring. Expressions of natural frequencies of the beam, taking account of the effects of cracks, indicate that the shift of wavenumber and frequency are proportional to potential energy at crack location. Law et al. [3] use the dynamic responses of a Euler-Bernoulli beam model to identified the crack. Calculation of the dynamic response are based on modal superposition and numerical optimization technology which requires the regularization of the solution. A model of beam with damage has been studied by Timothy and Whalen [4], who shows the damage is connected with changes of mode shapes and their higher order derivatives. A dynamic stiffness matrix has been studied by Li et al. [5] based on trigonometric shear deformation theory, with which the natural frequencies and modes of vibration are obtained. The multiple cracks detection method
proposed by Lam et al. [6] is able to handle more complex scenario which takes into consideration of the number of cracks as well as depth and location of individual crack. Related experiments have also been conducted to verify the method. The size parameters including location and depth of the cracks in a stepped cantilever Euler-Bernoulli beam with a rigid disk on one end are identified by Al-Said [7]. The analysis of natural frequencies and mode shapes of cracks in a Timoshenko beam model has been present in the paper of Viola [8]. The influence of crack in a geometrically segmented beam was investigated by Chaudhari and Maiti [9]. The work of Kisa and Brandon [10] relates the change on dynamic response of a cantilever beam with geometric characters of crack. The vibration of beam with multiple cracks was studied by Binici [11], which can be used to determine the frequency changes in an axially loaded beam.

In above studies, the frequencies combined with mode shapes play central role in identification of the crack. In this research, a crack detection approach is proposed based on transfer matrices. In following sections, the transfer matrix analytical calculation result which is followed by Timoshenko theory [12] is developed. And validation by FEM calculation is conducted. The possibility of application on crack detection is also explored.

2. Transfer Matrix

The Transfer matrices of beam are derived in this section, the Timoshenko beam theory is employed while solving the equilibrium equations of vibration, so that shearing effect could be taken into consideration. Different types of boundary conditions are also considered. To this end, the coupled transfer matrix of the longitudinal and transverse deformation is derived based on the transitive relation of vibration.

The equilibrium equations [13] of the longitudinal and transverse deformation of the continuous body are

\[
\begin{align*}
\frac{\partial^2 u(x,t)}{\partial x^2} & = \frac{1}{b^2} \frac{\partial^2 u(x,t)}{\partial t^2} \\
E I \frac{\partial^2 \theta(x,t)}{\partial x^2} & - G A k (\theta(x,t) - \frac{\partial w(x,t)}{\partial x} ) - \rho I \frac{\partial^2 \theta(x,t)}{\partial t^2} = 0 \\
G A k \left( \frac{\partial \theta(x,t)}{\partial x} - \frac{\partial^2 w(x,t)}{\partial x^2} \right) + \mu \frac{\partial^2 w(x,t)}{\partial t^2} &= 0
\end{align*}
\]

where \( u(x,t) \) is the longitudinal displacement, \( w(x,t) \) is the lateral deflection, and \( \theta(x,t) \) is the rotation angle. The parameter \( b \) and \( \mu \) can be expressed as \( b = \sqrt{E/\rho} \) and \( \mu = \rho A. E, I, G, A, k \) and \( \rho \) are the Young’s modulus, the inertia moment of cross-section, the shear modulus, the area of cross-section, the shearing factor and the density of the material, respectively.

The boundary conditions of the cantilever beam can be written as,

\[
\begin{align*}
u(0) &= u_0, W(0) = W_0, \theta(0) = \theta_0, N(0) = N_0, Q(0) = Q_0, M(0) = M_0 \\
u(l) &= u_l, W(l) = W_l, \theta(l) = \theta_l, N(l) = N_l, Q(l) = Q_l, M(l) = M_l
\end{align*}
\]

The solutions of Eq. (1), with the separation of variables, can be written in the forms of equation (3)

\[
\begin{align*}
u(x) &= u_0 \cos \beta x + \frac{N_0}{E A \beta} \sin \beta x \\
W(x) &= C_1 \sin \lambda_2 x + C_2 \cos \lambda_2 x + C_3 \sinh \lambda_4 x + C_4 \cosh \lambda_4 x \\
\theta(x) &= B_1 \cos \lambda_2 x + B_2 \sin \lambda_2 x + B_3 \cosh \lambda_4 x + B_4 \sinh \lambda_4 x
\end{align*}
\]

where the coefficients \( B_i, C_i (i = 1, 2, 3, 4) \) are to be determined. Substituting the expressions in equation (3) into equation (1), \( B_i, C_i (i = 1, 2, 3, 4) \) can be written as equations (4) and (5), respectively.
\[
\begin{aligned}
\{ B_4 \} &= \frac{(\lambda_1^2 - \lambda_2^2 + \frac{\mu \omega^2}{kGA} \lambda_1^2 \lambda_2^2)Q_0 - (\mu \omega^2 - kGA \lambda_2^2)\theta_0}{kGA(\lambda_1^2 + \lambda_2^2)} \\
B_2 &= \frac{(\mu \omega^2 - kGA \lambda_2^2)(\lambda_1^2 - \frac{\mu \omega^2}{kGA})W_0 - \frac{1}{EI}(\mu \omega^2 - kGA \lambda_2^2)M_0}{kGA \lambda_2(\lambda_1^2 + \lambda_2^2)} \\
B_3 &= \frac{(\lambda_2^2 - \lambda_1^2 - \frac{\mu \omega^2}{kGA} \lambda_1^2 \lambda_2^2)Q_0 + (\mu \omega^2 + kGA \lambda_2^2)\theta_0}{kGA(\lambda_1^2 + \lambda_2^2)} \\
B_4 &= \frac{(\mu \omega^2 + kGA \lambda_2^2)(\lambda_2^2 - \frac{\mu \omega^2}{kGA} \lambda_1^2 \lambda_2^2)Q_0 - \frac{1}{EI}(\mu \omega^2 + kGA \lambda_2^2)M_0}{kGA \lambda_1(\lambda_1^2 + \lambda_2^2)}
\end{aligned}
\]

\[
\begin{aligned}
\{ C_1 \} &= \frac{-\frac{\lambda_1^2 \lambda_2}{\mu \omega^2} Q_0 - \frac{\lambda_2}{kGA} Q_0 + \lambda_2 \theta_0}{\lambda_1^2 + \lambda_2^2} \\
C_2 &= \frac{-M_0}{EI} + \frac{\lambda_1^2 W_0 + \frac{\mu \omega^2}{kGA} W_0}{\lambda_1^2 + \lambda_2^2} \\
C_3 &= \frac{\frac{\lambda_1 \lambda_2}{\mu \omega^2} Q_0 - \frac{\lambda_1}{kGA} Q_0 + \lambda_1 \theta_0}{\lambda_1^2 + \lambda_2^2} \\
C_4 &= \frac{M_0}{EI} + \frac{\lambda_2^2 W_0 - \frac{\mu \omega^2}{kGA} W_0}{\lambda_1^2 + \lambda_2^2}
\end{aligned}
\]

While the relation of the parameter \( \lambda_i (i = 1,2) \) in equation (3) can be expressed as following:

\[
\begin{aligned}
\lambda_1 &= ((a + \frac{g^2}{4})^{1/2} - \frac{g}{2})^{1/2} \\
\lambda_2 &= ((a + \frac{g^2}{4})^{1/2} + \frac{g}{2})^{1/2}
\end{aligned}
\]

In equation (6), \( a = \frac{\mu \omega^2}{kE \bar{A} \bar{G}} \) and \( g = \frac{\mu \omega^2}{E \bar{A} \bar{G}} + \frac{\mu \omega^2}{kGA} \).

According to the relation between the internal forces and displacements [14], the expressions of the axial force, the shearing force and flexural moment can be obtained as,

\[
\begin{aligned}
N(x) &= E \beta (\sin \beta x + \frac{N_0}{E \alpha})
\end{aligned}
\]

\[
\begin{aligned}
Q(x) &= -kGA(C_1 \lambda_2 \cos \lambda_2 x - C_2 \lambda_1 \lambda_2 \sin \lambda_2 x + C_3 \lambda_1 \cosh \lambda_1 x + C_4 \lambda_2 \sinh \lambda_1 x) \\
&\quad - B_1 \cos \lambda_2 x - B_2 \sin \lambda_2 x - B_3 \cosh \lambda_1 x - B_4 \sinh \lambda_1 x
\end{aligned}
\]

\[
\begin{aligned}
M(x) &= E \beta (B_1 \lambda_2 \sin \lambda_2 x + B_2 \lambda_2 \cos \lambda_2 x + B_3 \lambda_1 \sinh \lambda_1 x + B_4 \lambda_1 \cos \lambda_1 x)
\end{aligned}
\]

In general, the transitive relation of the coupled vibration (longitudinal and transverse deformation) in the cantilever beam can be seen in the equation (8).

\[
\begin{bmatrix}
\mathbf{u}(l) \\
\mathbf{W}(l) \\
\mathbf{\theta}(l) \\
\mathbf{N}(l) \\
\mathbf{Q}(l) \\
\mathbf{M}(l)
\end{bmatrix} = \mathbf{T}(\omega)
\begin{bmatrix}
\mathbf{u}(0) \\
\mathbf{W}(0) \\
\mathbf{\theta}(0) \\
\mathbf{N}(0) \\
\mathbf{Q}(0) \\
\mathbf{M}(0)
\end{bmatrix}
\]
where the transfer matrix of the coupled vibration can be expressed by

\[
T(\omega) = \begin{bmatrix}
\cos \beta l & 0 & 0 & \sin \beta l & E\alpha \beta \\
0 & T_{11} & T_{12} & 0 & T_{13} \\
0 & T_{21} & T_{22} & 0 & T_{23} \\
-E\alpha \beta \sin \beta l & 0 & 0 & \cos \beta l & 0 \\
0 & T_{31} & T_{32} & 0 & T_{33} \\
0 & T_{41} & T_{42} & 0 & T_{43} \\
\end{bmatrix} (9)
\]

where \( l \) is the length of the cantilever beam, and the expression of the items in the matrix can be written as follows:

\[
T_{11} = \frac{(\lambda_1^2 + \mu \omega^2) \cos \lambda_2 l + (\lambda_2^2 - \mu \omega^2) \cosh \lambda_1 l}{\lambda_1^2 + \lambda_2^2} = \frac{\lambda_2 \sin \lambda_2 l + \lambda_1 \sinh \lambda_1 l}{\lambda_1^2 + \lambda_2^2}
\]

\[
T_{13} = \frac{\lambda_1^2 \lambda_2^2}{\mu \omega^2} (\lambda_2 \sin \lambda_1 l - \lambda_1 \sin \lambda_2 l) - \frac{1}{k_G A} (\lambda_2 \sin \lambda_2 l + \lambda_1 \sinh \lambda_1 l)
\]

\[
T_{14} = \frac{\cosh \lambda_1 l - \cos \lambda_2 l}{E I (\lambda_1^2 + \lambda_2^2)}
\]

\[
T_{21} = -\frac{(\mu \omega^2 + \mu \omega^2 \lambda_2^2 - k_G A \lambda_1^2 \lambda_2^2) (\lambda_1 \sin \lambda_2 l - \lambda_2 \sinh \lambda_1 l)}{k_G A \lambda_1 \lambda_2 (\lambda_1^2 + \lambda_2^2)}
\]

\[
T_{22} = \frac{\mu \omega^2}{k_G A} (\cosh \lambda_1 l - \cos \lambda_2 l) + \lambda_1^2 \cosh \lambda_1 l + \lambda_2^2 \cos \lambda_2 l
\]

\[
T_{23} = \frac{(\mu \omega^2)}{k_G A} \frac{\lambda_1^2 \lambda_2^2}{\mu \omega^2} (\cos \lambda_2 l - \cosh \lambda_1 l)
\]

\[
T_{24} = \frac{\mu \omega^2}{E I} (\lambda_2 \sin \lambda_1 l - \lambda_1 \sin \lambda_2 l) + \frac{k_G A}{E I} (\lambda_1^2 \lambda_2 \sinh \lambda_1 l + \lambda_1 \lambda_2^2 \sin \lambda_2 l)
\]

\[
T_{31} = -\frac{(\mu \omega^2)^2}{k_G A} \frac{\lambda_1^2 \lambda_2^2}{\lambda_2 (\lambda_1^2 + \lambda_2^2)} \sin \lambda_2 l + \frac{(\mu \omega^2 \lambda_2^2 - \mu \omega^2)^2}{\lambda_1 (\lambda_1^2 + \lambda_2^2)} \sinh \lambda_1 l
\]

\[
T_{32} = \frac{\mu \omega^2 (\cosh \lambda_1 l - \cos \lambda_2 l)}{\lambda_1^2 + \lambda_2^2}
\]

\[
T_{33} = \frac{(\lambda_1^2 + \mu \omega^2 k_G A)}{\lambda_1^2 + \lambda_2^2} \cos \lambda_2 l + \frac{(\lambda_2^2 - \mu \omega^2)}{\lambda_1^2 + \lambda_2^2} \cosh \lambda_1 l
\]

\[
T_{34} = \frac{\mu \omega^2 (\lambda_2 \sin \lambda_1 l - \lambda_1 \sin \lambda_2 l)}{E I \lambda_1 \lambda_2 (\lambda_1^2 + \lambda_2^2)}
\]
\[
T_{41} = \frac{EI(\mu \omega^2 + \mu \omega^2 \lambda_1^2 - \mu \omega^2 \lambda_2^2 - kG\lambda_2^2\lambda_2^2)(\cos \lambda_2 l - \cosh \lambda_1 l)}{kG(\lambda_2^2 + \lambda_2^2)}
\]
\[
T_{42} = \frac{EI(\mu \omega^2)(\lambda_2 \sin \lambda_2 l + \lambda_1 \sinh \lambda_1 l) - \lambda_2^3 \sin \lambda_2 l + \lambda_1^3 \sinh \lambda_1 l}{\lambda_1^2 + \lambda_2^2}
\]
\[
T_{43} = \frac{EI(\frac{\lambda_2^2}{\mu \omega^2} + \frac{\lambda_2^2}{kG} - \frac{\mu \omega^2}{(kG)^2})(\lambda_2 \sin \lambda_2 l + \lambda_1 \sinh \lambda_1 l)}{\lambda_1^2 + \lambda_2^2}
\]
\[
T_{44} = \frac{\mu \omega^2(\cosh \lambda_1 l - \cos \lambda_2 l) + kG(\lambda_2^2 \cosh \lambda_1 l + \lambda_2^2 \cos \lambda_2 l)}{kG(\lambda_1^2 + \lambda_2^2)}
\]

3. The Transfer Matrix in the Cracked Cantilever Beam

Consider a clamped-end cantilever beam show in figure 1, which is loaded on the other end. And a crack with width \(a\) and \(h\) is present, which divide the beam into three segments including the crack itself. The transfer matrices of the segments are \(T_1(\omega), T_2(\omega), T_3(\omega)\) respectively. The expressions of the \(T_i(\omega)(i = 1,2,3)\) can be obtained by equation (9), where the parameter \(l\) is of the length of each segment.

![Figure 1. The model of the cracked cantilever beam.](image)

To formulate the transition matrix between segments, the moment \(M_N\) acting on the interface need to be taken into account. It depends on depth of crack \(h\) and the axial force \(N\). A diagram depicts the forces and moments on interface is shown in figure 2.

![Figure 2. The generalized force in the connection.](image)

In addition, transition matrices are to be introduced by the impact of crack on forces and moments at the interface. The connection of transition matrix and generalized forces and moments is given in equation (10),

\[
\begin{bmatrix}
u \\
W \\
\theta \\
N \\
Q \\
M 
\end{bmatrix}^+ = C_2 \begin{bmatrix}
u \\
W \\
\theta \\
N \\
Q \\
M 
\end{bmatrix}^-
\] (10)

where \( C_2 = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
-\frac{h}{2} & 0 & 1 
\end{bmatrix} \), and one on the other side can be expressed as \( C_1 = \)

\[
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\frac{h}{2} & 0 & 1 
\end{bmatrix} \].

Therefore, the complete transfer matrix of the mean with single crack can be expressed as

\[
T_c(\omega) = T_3(\omega)C_2T_2(\omega)C_1T_1(\omega)
\] (11)

In the case of no crack, the transition matrix degenerates to identity matrix.

4. The Vibration Analysis Based on Transfer Matrix Method

Take the displacement and rotation degree of freedom with generalized forces and moments as variables, and combine them into a vector as,

\[
y(x) = [u(x), \ W(x), \ \theta(x), \ N(x), \ Q(x), \ M(x)]^T
\] (12)

Then, the boundary conditions on ends of the cantilever beam read

\[
u(0) = 0, \ W(0) = 0, \ \theta(0) = 0, \ N(l) = 0, \ Q(l) = 0, \ M(l) = 0
\] (13)

That is to say, the generalized displacements at the clamped end are zero and the generalized forces at the free end are zero. The expressions could be written in matrix form [15] as

\[
B_1y(0) + B_2y(l) = 0
\] (14)

where \( B_1 = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 
\end{bmatrix} \), \( B_2 = \begin{bmatrix}
0 & 0 \\
0 & 1 \\
1 & 1 
\end{bmatrix} \).

Substituting equation (8) into equation (14) yields

\[
[B_1 + B_2T(\omega)]y(0) = 0
\] (15)

Thus, the natural frequencies of the beam can be obtained by solving following equation,

\[
F(\omega) = \text{det}[B_1 + B_2T(\omega)] = 0
\] (16)

In addition, the mode can be calculated as

\[
y(x) = H(x; \omega)y(0)
\] (17)
where \( y(0) \) is the solution of equation (16), and \( H(x; \omega) \) is the expression of transfer matrix.

5. Validation with Finite Element Method

In this section, a rectangular sectioned cantilever beam with crack in the middle is studied. The beam dimension and material properties are as follows: total length \( l = 0.3 \) m, width \( b = 0.015 \) m, depth \( H = 0.02 \) m, density \( \rho = 7800 \) kg/m\(^2\), modulus of elasticity \( E = 210 \) GPa and shear modulus and \( G = 80 \) GPa. Dimension of the crack are: \( a \) (width), \( h \) (depth), and \( L \) (distance from the fixed end).

Meanwhile, finite element simulations with different modeling approach are performed for comparison with transfer matrix solution. The beam was modeled with beam elements, solid elements and plane stress elements. The results reveal that the 1\(^{st}\) to 3\(^{rd}\) mode and the 5\(^{th}\) mode are flexural vibration, while the 4\(^{th}\) mode is the stretching vibration.

Firstly, a beam without crack is considered. Its natural frequencies of the transfer matrix solution and those of the FEM in the un-cracked cantilever beam is shown in table 1.

| Order | Transfer Matrix (Hz) | FEM Beam (Hz) | FEM Solid (Hz) | FEM Plane Stress (Hz) |
|-------|----------------------|---------------|----------------|-----------------------|
| 1     | 185.485              | 185.61        | 186.29         | 185.75                |
| 2     | 1134.476             | 1139.5        | 1145.54        | 1140.9                |
| 3     | 3063.203             | 3093.2        | 3117.92        | 3099.3                |
| 4     | 4323.954             | 4324.0        | 4333.54        | 4327.3                |
| 5     | 5721.578             | 5815.5        | 5883.4         | 5832.8                |

It is easy to see that the results of the transfer matrix and that of FEM show good consistency, thus validates the transfer matrix approach.

Furthermore, beams with various crack configuration are simulated to investigate how cracks affect its natural frequencies. Only the base mode is considered, and tables 2-4 show the result with varied parameters, \( a/l \), \( h/H \), and \( L/l \).

It is clearly shown that the base frequency is influenced by the location, depth, and width of crack. Among them, the location affects the base frequency most significantly. However, the variation tendency of frequency is the same with each other from the results. That means the base frequency decreases as the size of crack increases. Dependency of base frequency on crack dimension parameters combined with FEM results are shown in figures 3 and 4.

| \( a/l \) | Transfer Matrix (Hz) | FEM Beam (Hz) | FEM Solid (Hz) | FEM Plane Stress (Hz) |
|----------|----------------------|---------------|----------------|-----------------------|
| 1/10000  | 185.473              | 185.60        | 184.61         | 183.67                |
| 1/1000   | 185.372              | 185.50        | 184.6          | 183.54                |
| 1/500    | 185.260              | 185.38        | 184.48         | 183.43                |
| 1/100    | 184.381              | 184.50        | 183.48         | 182.54                |

| \( h/H \) | Transfer Matrix (Hz) | FEM Beam (Hz) | FEM Solid (Hz) | FEM Plane Stress (Hz) |
|-----------|----------------------|---------------|----------------|-----------------------|
| 1/10      | 185.372              | 185.50        | 184.6          | 183.54                |
| 1/8       | 185.336              | 185.46        | 183.72         | 182.41                |
| 1/6       | 185.264              | 185.39        | 181.78         | 180.11                |
| 1/5       | 185.196              | 185.32        | 179.33         | 177.66                |
Table 4. $\frac{a}{l} = 1/1000$, $\frac{h}{H} = 1/10$, and variation in $\frac{L}{l}$.

| $\frac{L}{l}$ | Transfer Matrix (Hz) | FEM Beam (Hz) | FEM Solid (Hz) | FEM Plane Stress (Hz) |
|---------------|----------------------|---------------|----------------|-----------------------|
| 0             | 185.35               | 185.47        | 184.95         | 184.31                |
| $1/30$        | 185.36               | 185.48        | 184.33         | 183.36                |
| $1/15$        | 185.37               | 185.50        | 184.60         | 183.54                |
| $1/3$         | 185.44               | 185.57        | 185.53         | 184.94                |
| $1/2$         | 185.47               | 185.60        | 185.9          | 185.44                |
| 1             | 185.485              | 185.61        | 186.29         | 185.75                |

Figure 3. $\frac{a}{l} = 1/1000$, $\frac{h}{H} = 1/5$, 1/10, 1/20, and variation in $\frac{L}{l}$.

Figure 4. $\frac{h}{H} = 1/10$, $\frac{a}{l} = 1/1000$, 1/500, 1/100, and variation in $\frac{L}{l}$.

The transfer matrix method is also able to obtain the mode shape by the expressions. For instance, the mode shape of an un-cracked beam could be seen in figure 5.

Figure 5. The former three displacement mode.

Therefore, influence of crack on mode shapes could be examined. The difference between cracked and un-cracked mode shapes could be used to represent that influence. Modes of first 2 orders with variation on $h/H$ are shown in figures 6-8, and those with variation on $a/l$ are shown in figures 9-11. The changes can be obvious seen in the curve shape, especially the localized effects due to the appearance of crack.
Figure 6. Difference between cracked and un-cracked displacement mode curves change in the crack area, $a_l=1/1000$, $L_l=1/15$, and variation in $\frac{h}{H}$. Mode 1 is on the left, and the mode 2 is on the right.

Figure 7. Difference between cracked and un-cracked corner mode curves change in the crack area, $a_l=1/1000$, $L_l=1/15$, and variation in $\frac{h}{H}$. Mode 1 is on the left, and the mode 2 is on the right.

Figure 8. Difference between cracked and un-cracked moment mode curves change in the crack area, $a_l=1/1000$, $L_l=1/15$, and variation in $\frac{h}{H}$. Mode 1 is on the left, and the mode 2 is on the right.
Figure 9. Difference between cracked and un-cracked displacement mode curves change in the crack area, $\frac{h}{H}=1/5$, $\frac{L}{l}=1/15$, and variation in $\frac{a}{l}$. Mode 1 is on the left, and the mode 2 is on the right.

Figure 10. Difference between cracked and un-cracked corner mode curves change in the crack area, $\frac{h}{H}=1/5$, $\frac{L}{l}=1/15$, and variation in $\frac{a}{l}$. Mode 1 is on the left, and the mode 2 is on the right.

Figure 11. Difference between cracked and un-cracked moment mode curves change in the crack area, $\frac{h}{H}=1/5$, $\frac{L}{l}=1/15$, and variation in $\frac{a}{l}$. Mode 1 is on the left, and the mode 2 is on the right.

6. Application on Crack Detection
The high efficiency of transfer matrix solution marks a great advantage in application of crack detection. As the natural frequencies of cracked beam could be rapidly computed in real-time, when a
frequency drop is detected in the beam, the numerical relationship between frequencies and crack parameters could be used to reverse the location, depth and width of crack. For instance, figure 12 shows the frequency drops of the first 4 modes related to the crack.

![Figure 12. Frequencies drop of first 4 modes with related to crack size parameter h/H.](image)

7. Conclusions
In this research, the transfer matrix analytical calculation result which is followed by Timoshenko theory is developed and validated by comparison with FEM calculation. It was shown that natural frequencies of cantilever beam increase as the crack shape becomes larger, and the frequencies is dominated by the depth of the crack. When depth is fixed, the distance between the crack and the clamped-end point of beam also affects frequencies. Due to the advantage of transfer matrix solution procedure on efficiency, the possibility of application on crack detection is also explored.

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