Canonical quantization of the electromagnetic field in the presence of non-dispersive bi-anisotropic inhomogeneous magnetodielectric media

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Abstract

By introducing a suitable Lagrangian, a canonical quantization of the electromagnetic field in the presence of a non-dispersive bi-anisotropic inhomogeneous magnetodielectric medium is investigated. A tensor projection operator is defined and the commutation relation between the vector potential and its canonically conjugate variable is written in terms of the projection operator. The quantization method is generalized in the presence of the atomic systems. The spontaneous emission of a two-level atom located in a non-dispersive anisotropic magnetodielectric medium is studied.

Keywords: Bi-anisotropic non-dispersive magnetodielectric media, Constitutive relation, Onsager’s relation, Canonical quantization, Spontaneous emission

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1 introduction

The quantum properties of the electromagnetic field apparently are influenced by the presence of magnetodielectric media. For examples the spontaneous decay constant of atomic systems[1]-[5], the Casimir effect [6] and the statistical properties of light are modified in the presence of a polarizable or magnetizable medium [7]-[9]. The quantization of electromagnetic field is considered usually in the presence of two types of media, the dispersive lossy magnetodielectric media and non-dispersive one. In dispersive magnetodielectric media there is a temporally nonlocal relationship between the polarization (magnetization ) field and the electric (magnetic) field [10]-[13]. In this type of media, in order to inclusion the lossy effects, the medium is modeled by a collection of harmonic oscillators and both the electromagnetic field and the responsive medium are quantized, [14]-[20]. These quantization approaches cover both isotropic and inhomogeneous anisotropic media. In these schemes the permittivity and permeability of the medium is calculated in terms of the classical parameters applied in the lagrangian or Hamiltonian of the total system.

In the dispersionless dielectric media the relationship between the polarization field and the electric field is temporally local. To quantize the electromagnetic field in the presence of such media, the medium is not quantized directly. The polarization effects of the medium is introduced, in the lagrangian density of the total system, only by the its linear and nonlinear susceptibility tensors [21]-[28]. In this case the Euler-Lagrange equations are the macroscopic Maxwell equations in the presence of the non-dispersive medium. This method has been developed for linear or non-linear isotropic dielectric media. In the dispersionless responsive media there is a class of linear polarizable and magnetizable media that the electric polarization (magnetic polarization) is related linearly to both the electric and magnetic fields. Generally, the electromagnetic properties of these media are described by four tensors of the second rank. This kind of media are known as bi-anisotropic magnetodielectric media. In this paper proposing a Lagrangian density a fully canonical quantization of electromagnetic field is represented in the presence of such media. It is worthy to point out that the form of Maxwell’s equations in the curved space-time , written in the cartesian coordinates, in the absence of any responsive medium, are identical to the form of Maxwell’s equations in flat space-time in the presence of a bi-anisotropic magnetodi-
electric media [29, 30]. Therefore the quantization method demonstrated in
the present work is applicable for the quantized electromagnetic field in the
curved space-time in the absence of any responsive medium. The organiza-
tion of this paper is as follows:

In sec. 2 a Lagrangian for the electromagnetic field in the presence of a bi-
anisotropic magnetodielectric medium is proposed and a classical treatment
about the problem is achieved. In sec. 3 using the lagrangian introduced in
section 2 a canonical quantization of the electromagnetic field in the presence
of a bi-anisotropic inhomogeneous magnetodielectric medium is represented.
In sec. 4 the quantization method is generalized in the presence of the atomic
systems. In sec. 5 the spontaneous emission of an initially excited two-level
atom located in a bi-anisotropic medium is investigated. Finally the paper
is closed by a summary in sec. 6.

2 Classical Lagrange equations

In the non-dispersive linear magnetodielectric media the displacement field \( \mathbf{D} \)
and the magnetic field \( \mathbf{H} \) are related to the electric field \( \mathbf{E} \) and the magnetic
induction field \( \mathbf{B} \) as [31]

\[
\begin{align*}
\mathbf{D}(\mathbf{r}, t) &= \varepsilon^{(1)}(\mathbf{r})\mathbf{E}(\mathbf{r}, t) + \varepsilon^{(2)}(\mathbf{r})\mathbf{B}(\mathbf{r}, t), \\
\mathbf{H}(\mathbf{r}, t) &= \mu^{(1)}(\mathbf{r})\mathbf{E}(\mathbf{r}, t) + \mu^{(2)}(\mathbf{r})\mathbf{B}(\mathbf{r}, t).
\end{align*}
\]  

(1)

The electric (the magnetic) properties of the medium are described by the
permittivities \( \varepsilon^{(1)}(\mathbf{r}), \varepsilon^{(2)}(\mathbf{r}) \) (the permeabilities \( \mu^{(1)}(\mathbf{r}), \mu^{(2)}(\mathbf{r}) \)), respectively.
Generally, for bi-anisotropic media the permittivities and permeabilities may
be appeared as four tensors of the second rank. There are some symmetry
relations between the permittivities and permeabilities of a medium. The
most important symmetry relations between the permittivities and permeabilities
of a non-dispersive magnetodielectric medium are the Onsager’s relations [31]

\[
\begin{align*}
\varepsilon_{ij}^{(1)} &= \varepsilon_{ji}^{(1)}, & \mu_{ij}^{(2)} &= \mu_{ji}^{(2)}, & \varepsilon_{ij}^{(2)} &= -\mu_{ji}^{(1)}.
\end{align*}
\]  

(2)

A realization of the constitutive relations [11] and the symmetry relations
[2] is the electromagnetic field in a curved space-time in the absence of any
responsive medium. It is well known that Maxwell’s equations in a curved
space-time in the absence of external sources, when the cartesian coordinates is used, are as the form \[29, 30\]

\[
\begin{align*}
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \cdot \mathbf{D} &= 0 \\
\n\nabla \times \mathbf{H} &= \frac{\partial \mathbf{D}}{\partial t} \\
\n\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}
\end{align*}
\] (3)

where the cartesian components of the fields \( \mathbf{D} \) and \( \mathbf{H} \) are defined by

\[
D_i = \varepsilon_{ij} E_j + (\mathbf{G} \times \mathbf{H})_i \quad \quad \quad \quad B_i = \mu_{ij} H_j + (\mathbf{G} \times \mathbf{E})_i
\] (4)

with

\[
\varepsilon_{ij} = \mu_{ij} = -\sqrt{-g} \frac{g^{ij}}{g_{00}} \quad i, j = 1, 2, 3
\]

\[
G_i = -\frac{g_{0i}}{g_{00}} \quad i = 1, 2, 3
\] (5)

where \( g_{\mu\nu} \) is the space-time metric and \( g \) is its determinant. The definitions (4) for the fields \( \mathbf{D} \) and \( \mathbf{H} \) can be rewritten in the form of constitutive relations (1) with the tensors \( \varepsilon^{(1)}, \varepsilon^{(2)}, \mu^{(1)}, \mu^{(2)} \) given by

\[
\begin{align*}
\varepsilon^{(1)}_{ij} &= -\sqrt{-g} \frac{g^{ij}}{g_{00}} - \epsilon_{i\alpha\beta} \epsilon_{mnj} \frac{g_{0\alpha} g_{0\beta} g_{mn}}{g_{00} \sqrt{-g}} \\
\varepsilon^{(2)}_{ij} &= -\epsilon_{imn} \frac{g_{nj} g_{0m}}{\sqrt{-g}} \\
\mu^{(1)}_{ij} &= -\epsilon_{mnj} \frac{g_{0n} g_{im}}{\sqrt{-g}} \\
\mu^{(2)}_{ij} &= -\frac{g_{00}}{\sqrt{-g}} g_{ij}
\end{align*}
\] (6)

where \( \epsilon_{ijk} \) is the 3-dimensional Levi-Civita symbol. It is easy to investigate that the tensors defined by (6) obey the Onsager’s relation (2). Therefore, applying the cartesian coordinates, the form of Maxwell’s equations in the curved space-time in the absence of a responsive medium is similar to the form of Maxwell’s equations in the flat space-time in the presence of a bi-anisotropic magnetodielectric medium. The above discussions are sufficient motivation to investigate the quantization of the electromagnetic field in the presence of a bi-anisotropic magnetodielectric medium.
Using the constitutive relations (11) Maxwell’s equations (3) can be rewritten as

\[ \nabla \cdot \mathbf{B} = 0 \] (7)

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \] (8)

\[ \nabla \cdot \left[ \varepsilon^{(1)}(\mathbf{r}) \mathbf{E}(\mathbf{r}, t) + \varepsilon^{(2)}(\mathbf{r}) \mathbf{B}(\mathbf{r}, t) \right] = 0 \] (9)

\[ \nabla \times \left[ \mu^{(1)}(\mathbf{r}) \mathbf{E}(\mathbf{r}, t) + \mu^{(2)}(\mathbf{r}) \mathbf{B}(\mathbf{r}, t) \right] = \frac{\partial}{\partial t} \left[ \varepsilon^{(1)}(\mathbf{r}) \mathbf{E}(\mathbf{r}, t) + \varepsilon^{(2)}(\mathbf{r}) \mathbf{B}(\mathbf{r}, t) \right] \] (10)

The classical Maxwell’s equations (9) and (10) can be obtained as a consequence of the principle of Hamilton’s least action using the Lagrangian

\[ L(t) = \int d^3r \mathcal{L}(\mathbf{r}, t) = \int d^3r \left\{ \frac{1}{2} \varepsilon^{(1)}_{ij}(\mathbf{r}) E^i(\mathbf{r}, t) E^j(\mathbf{r}, t) + \frac{1}{2} \varepsilon^{(2)}_{ij}(\mathbf{r}) B^i(\mathbf{r}, t) B^j(\mathbf{r}, t) \right\} \] (11)

where the Einstein sum rule has been applied. In (11), \( \mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi \) and \( \mathbf{B} = \nabla \times \mathbf{A} \). This means that the vector potential \( \mathbf{A} \) and the scalar potential \( \phi \) constitute the degrees of freedom of the electromagnetic field. It is easy to show that the Lagrange equation for the variable \( \phi \) leads to the Gauss’ law (9) while, using the Onsager’s relations (2), the Lagrange equation for the vector potential \( \mathbf{A} \) gives the Maxwell’s equation (10).

As it is clear in the present formalism the non-dispersive magnetodielectric medium is not quantized directly and the effect of the medium is appeared as the classical tensors \( \varepsilon^{(1)}, \varepsilon^{(2)}, \mu^{(1)} \) and \( \mu^{(2)} \). However in the quantization of the electromagnetic field in the presence of dispersive lossy media, the medium itself is quantized [14]–[20].

2.1 Gauge fixing

For a consistent canonical quantization of the electromagnetic field, we need the extra degrees of freedom to be eliminated from the above Lagrangian using some appropriate gauge conditions. Here we apply the gauge condition

\[ \nabla \cdot \left( \varepsilon^{(1)}(\mathbf{r}) \mathbf{A}(\mathbf{r}, t) \right) = 0, \] (12)
where previously has been used by Glauber and et al. [25]. Combination of this gauge with the Gauss’ law (9) leads to expression

$$\varphi(r, t) = -\int d^3r' \ G(r, r') \ \nabla' \cdot \left[ \varepsilon^{(2)}(r') \ \nabla' \times A(r', t) \right], \quad (13)$$

for the scalar potential, where \( G(r, r') \) is the Green function satisfying the differential equation

$$\nabla \cdot (\varepsilon^{(1)}(r) \ \nabla G(r, r')) = -\delta(r - r'). \quad (14)$$

Let us investigate the symmetry property

$$G(r, r') = G(r', r) \quad (15)$$

for this Green function. To investigate this symmetry feature we consider a generalized Green’s theorem for two arbitrary scalar functions \( \varphi_1(q) \) and \( \varphi_2(q) \) as

$$\int_V d^3q \left( \varphi_1 \ \nabla \cdot (\varepsilon^{(1)} \nabla \varphi_2) - \varphi_2 \ \nabla \cdot (\varepsilon^{(1)} \nabla \varphi_1) \right)$$

$$= \oint_s \left( \varphi_1 \ \varepsilon^{(1)} \nabla \varphi_2 - \varphi_2 \ \varepsilon^{(1)} \nabla \varphi_1 \right) \cdot \hat{n} \ ds \quad (16)$$

which can be proved using the symmetry property of the tensor \( \varepsilon^{(1)} \), that is \( \varepsilon^{(1)}_{ij} = \varepsilon^{(1)}_{ji} \). Choosing \( \varphi_1(q) = G(q, r) \) and \( \varphi_2(q) = G(q, r') \), and regarding (14) and the boundary condition that the Green function should vanish on a surface located at infinity, the expected symmetry property (15) is deduced.

Let us expediently resolve a squared integrable vector field \( F(r) \) into two components, as \( F = F^\parallel + F^\perp \). The component \( F^\parallel \) is defined by

$$F^\parallel (r) = -\int d^3r' \nabla G(r, r') \ \nabla' \cdot \left( \varepsilon^{(1)}(r') \ F(r') \right) \quad (17)$$

which is a conservative field. The other part \( F^\perp \) is given by

$$F^\perp = F - F^\parallel \quad (18)$$

which satisfies the relation \( \nabla \cdot (\varepsilon^{(1)}(r) \ F^\perp(r)) = 0 \). we call \( F^\parallel \) and \( F^\perp \) as \( \varepsilon^{(1)} \)-longitudinal and \( \varepsilon^{(1)} \)-transverse components of the vector field \( F \), respectively. Conveniently, the tensor projection operators \( P^\parallel_{ij}(r, r') \) and \( P^\perp_{ij}(r, r') \)
are introduced as

\[ P_{ij}^\parallel (r, r') = \sum_{l=1}^{3} \left[ \varepsilon_{ij}^{(1)} (r') \frac{\partial}{\partial x_i} \frac{\partial}{\partial x'_l} G(r, r') \right] \]

\[ P_{ij}^\perp (r, r') = \delta_{ij} \delta(r - r') - P_{ij}^\parallel (r, r'). \] (19)

In terms of these projection operators, one can rewrite the definitions (17) and (18) as

\[ F_i^\parallel (r) = \sum_{j=1}^{3} \int d^3 r' \, P_{ij}^\parallel (r, r') \, F_j(r') \quad i = 1, 2, 3 \] (20)

\[ F_i^\perp (r) = \sum_{j=1}^{3} \int d^3 r' \, P_{ij}^\perp (r, r') \, F_j(r') \quad i = 1, 2, 3. \] (21)

Using the poisson’s equation (14) and the symmetry property (15) it is easy to show that the projection operators \( P^\perp \) and \( P^\parallel \) satisfy the transversality relations

\[ \sum_{i,m=1}^{3} \frac{\partial}{\partial x_i} \left( \varepsilon_{im}(r) \, P_{mj}^\perp (r,r') \right) = 0 \quad j = 1, 2, 3, \] (22)

and

\[ \sum_{j=1}^{3} \frac{\partial}{\partial x'_j} P_{ij}^\perp (r,r') = 0 \quad i = 1, 2, 3. \] (23)

The gauge condition (12) shows that \(-\nabla \varphi\) and \(-\frac{\partial A_i}{\partial t}\) are \(\varepsilon^{(1)}\)- longitudinal and \(\varepsilon^{(1)}\)- transverse components of the the electric field \(E\), respectively. This can be seen by operating \(P^\perp\) and \(P^\parallel\) on the electric field, that is

\[ -\frac{\partial A_i}{\partial t} = \sum_{j=1}^{3} \int d^3 r' \, P_{ij}^\perp (r, r') \, E_j(r') \] (24)

\[ -\frac{\partial \varphi}{\partial x_i} = \sum_{j=1}^{3} \int d^3 r' \, P_{ij}^\parallel (r, r') \, E_j(r') \] (25)
According to the Eq. (23), one can obtain the transverse component of any squared integrable vector field $F(r)$ as

$$F^T_i(r) = \sum_{j=1}^{3} \int d^3r' \: P_{ji}(r,r') \: F_j(r') \quad i = 1, 2, 3. \quad (26)$$

which clearly satisfies $\nabla \cdot F^T = 0$.

### 3 Canonical quantization

Before beginning a canonical quantization scheme, the extra degrees of freedom should be eliminated from the Lagrangian of the system applying the constraints (12) and (13). This can be done by substituting the scalar potential $\varphi$ from (13) into the Lagrangian (11) and doing some integration by parts and using the antisymmetry relation $\varepsilon^{(1)}_{ij} = -\mu^{(1)}_{ji}$. Then we have

$$L(t) = \int d^3r \left\{ \varepsilon^{(1)}_{ij}(r) \frac{\partial A_i}{\partial t} \frac{\partial A_j}{\partial t} - \varepsilon^{(1)}_{ij}(r) \left[ (\varepsilon^{(1)}(r))^{-1} \varepsilon^{(2)}(r) \nabla \times A(r,t) \right]_{ij} \right. \left. - \frac{1}{2} \mu^{(2)}_{ij} (\nabla \times A)_i (\nabla \times A)_j \right\}$$

$$- \frac{1}{2} \int d^3r \int d^3r' \: G(r,r') \: \nabla \cdot [\varepsilon^{(2)}(r) \nabla \times A(r,t)] \: \nabla' \cdot [\varepsilon^{(2)}(r') \nabla' \times A(r',t)],$$

(27)

Having the Lagrangian density of the system, denoted by $\mathcal{L}$, as the integrand in (27), the $i$'th cartesian component of the canonical conjugate of the dynamical variable $A$ can easily be calculated in a standard way as

$$\Pi_i(r, t) = \frac{\partial \mathcal{L}}{\partial \dot{A}_i}$$

$$= \varepsilon^{(1)}(r) \frac{\partial A_i(r,t)}{\partial t} - \varepsilon^{(1)}(r) \left[ (\varepsilon^{(1)}(r))^{-1} \varepsilon^{(2)}(r) \nabla \times A(r,t) \right]_{i}.$$  

(28)

As it is seen, the canonical conjugate variable $\Pi$ is a purely transverse vector field, that is $\nabla \cdot \Pi = 0$, while according to the gauge condition (12), the vector potential $A$ is a purely $\varepsilon^{(1)}$-transverse vector field. Now the canonical quantization can be accomplished in a standard fashion by imposing the following equal time commutation relations between the cartesian components of the conjugate variables $\Pi$ and $A$ as

$$[A_i(r,t) \: \Pi_j(r',t)] = P_{ij}^+(r,r').$$

(29)
Because of the transversality relations (22) and (23), it is clear that the commutation relations (29) is compatible to the conditions $\nabla \cdot \Pi = 0$ and $\nabla \cdot (\varepsilon^{(1)}(\mathbf{r}) \mathbf{A}) = 0$. The time evolution of any dynamical operator related to the electromagnetic field, in the Heisenberg picture, can be obtained by using of the Hamiltonian of the system. The Hamiltonian of the electromagnetic field can be written in terms of $\mathbf{A}$ and $\Pi$ as (see the Appendix A)

$$H(t) = \frac{1}{2} \int d^3r \left\{ [\varepsilon^{(1)}(\mathbf{r})]^{-1}_{ij} \Pi_i(\mathbf{r}, t) \Pi_j(\mathbf{r}, t) + \mu^{(2)}_{ij}(\mathbf{r}) (\nabla \times \mathbf{A})_i (\nabla \times \mathbf{A})_j \right\}$$

$$+ \frac{1}{2} \int d^3r \left\{ \Pi_i(\mathbf{r}, t) [\varepsilon^{(2)}(\mathbf{r}) \nabla \times \mathbf{A}]_j + [\varepsilon^{(2)}(\mathbf{r}) \nabla \times \mathbf{A}]_i \Pi_j(\mathbf{r}, t) \right\}$$

$$+ \frac{1}{2} \int d^3r \left\{ [\varepsilon^{(1)}(\mathbf{r})]^{-1}_{ij} [\varepsilon^{(2)}(\mathbf{r}) \nabla \times \mathbf{A}]_i [\varepsilon^{(2)}(\mathbf{r}) \nabla \times \mathbf{A}]_j \right\}$$

(30)

Any consistent quantization of the electromagnetic field should be able to give the Maxwell equations as the equations of motion of the electromagnetic field in the Heisenberg picture. It can be shown that the classical Maxwell’s equation (10) can be reobtained in Heisenberg picture if one uses the Hamiltonian (30) and applies the commutation relations (29). The Heisenberg equation for the vector potential gives (see the Appendix B)

$$\varepsilon^{(1)} \dot{\mathbf{A}} = \Pi + \varepsilon^{(1)} \left[ \left( \varepsilon^{(1)} \right)^{-1} \varepsilon^{(2)} \nabla \times \mathbf{A} \right] \perp$$

(31)

which coincides to the definition of the canonical momentum density in (28). Also the Heisenberg equation for the conjugate variable $\Pi$ leads to (see the Appendix B)

$$\dot{\Pi} = \nabla \times \left( \mu^{(1)}(\mathbf{r}) \dot{\mathbf{A}}(\mathbf{r}, t) \right) - \nabla \times \left( \mu^{(2)}(\mathbf{r}) \nabla \times \mathbf{A}(\mathbf{r}, t) \right) + \nabla \times \left( \mu^{(1)}(\mathbf{r}) \nabla \varphi(\mathbf{r}, t) \right)$$

(32)

where the Onsager’s relations (2) has been used. Combination of the two Heisenberg equations (31) and (32) yields

$$\varepsilon^{(1)} \ddot{\mathbf{A}} - \varepsilon^{(1)} \left[ \left( \varepsilon^{(1)} \right)^{-1} \varepsilon^{(2)} \nabla \times \mathbf{A} \right] \perp$$

$$= \nabla \times \left( \mu^{(1)}(\mathbf{r}) \dot{\mathbf{A}}(\mathbf{r}, t) \right) - \nabla \times \left( \mu^{(2)}(\mathbf{r}) \nabla \times \mathbf{A}(\mathbf{r}, t) \right) + \nabla \times \left( \mu^{(1)}(\mathbf{r}) \nabla \varphi(\mathbf{r}, t) \right)$$

(33)

which, regarding the relation (A4) (in Appendix A), is precisely the Maxwell equation (10).
3.1 diagonalization of the Hamiltonian of the system

To obtain a diagonalized form of the Hamiltonian of the electromagnetic field, here for the sake of simplicity, we restrict ourselves to the special case that the tensors $\varepsilon^{(2)}$ and $\mu^{(1)}$ are identically zero. Then, it is clear from Eq.(13) that $\varphi = 0$ and the Heisenberg equation (33) is reduced to

$$- \nabla \times \mu^{(2)}(r) \nabla \times A(r, t) = \varepsilon^{(1)}(r) \dot{A}(r, t). \quad (34)$$

Separating the time and space coordinates and writing $A(r, t) = A(r)T(t)$, yields a differential equation for $T$ as $\ddot{T} = -\omega^2 T$, where $\omega^2$ is a proper constant of separation, and the following equation for $A(r)$

$$\nabla \times \mu^{(2)}(r) \nabla \times A(r) = \omega^2 \varepsilon^{(1)}(r) A(r), \quad (35)$$

In order to solve this equation, we consider a type of eigenvalue equation as

$$\nabla \times \mu^{(2)}(r) \nabla \times F_k(r) = \omega^2 \varepsilon^{(1)}(r) F_k(r) \quad (36)$$

where $F_k$ is a squared integrable vector field that can be interpreted as an eigenvector field corresponding to an eigenvalue $\omega^2_k$. The eigenvectors $F_k$ and the eigenvalues $\omega^2_k$ are obtained using the boundary conditions at infinity and the boundary conditions on the discontinuity surface of the medium. On the discontinuity surface of the medium, in the absence of external sources, the tangential components of the fields $E$, $H$ and the normal components of the fields $D$, $B$ should be continuous. Using the symmetry property of the tensors $\varepsilon^{(1)}$ and $\mu^{(2)}$, given by (2), it can be shown that in the eigenvalue equation (36), those eigenvectors which correspond to the different eigenvalues, satisfy the orthogonality relation

$$\int d^3r \, \varepsilon^{(1)}_{ij}(r) F^*_k(r) F_{k'}(r) \sim \delta(\omega_k - \omega_{k'}) \quad (37)$$

In order to show this, Let us write the complex conjugate of Eq.(36) for the label $k'$

$$\nabla \times \mu^{(2)}(r) \nabla \times F^*_{k'}(r) = \omega^2_{k'} \varepsilon^{(1)}(r) F^{*}_{k'}(r). \quad (38)$$

Multiplying Eq.(36) by $F^*_{k'}(r)$ and Eq.(38) by $F_k(r)$ on the left, then subtracting and integrating the obtained results on the unbounded space and using the boundary conditions at infinity give us

$$(\omega^2_k - \omega^2_{k'}) \int d^3r \, \varepsilon^{(1)}_{ij}(r) F^*_{k'}(r) F_{k'}(r) = 0. \quad (39)$$
For $k = k'$ the integral in the left hand of (39) has a positive real value. To prove this, let $C(r)$ be a complex tensor of the second rank defined by

$$\varepsilon^{(1)}(r) = C^{\dagger}(r) C(r)$$

(40)

where $C^{\dagger}(r)$ is the Hermitian conjugate of $C(r)$. Substitution $\varepsilon^{(1)}$ from (40) into the integral in (39), one can write

$$\int d^3 r \varepsilon^{(1)}_{ij}(r) F^*_k(r) F_{k'}(r) = \int d^3 r |C(r)F_k(r)|^2 > 0$$

(41)

provided that the tensor $\varepsilon^{(1)}$ and accordingly $C(r)$ are assumed to be almost everywhere invertible. Equations (39) and (41) clearly imply the reality of the eigenvalues $\omega^2_k$ and the orthogonality relation (37).

Although, applying (39) and (41), the orthonormality relation

$$\int d^3 r \varepsilon^{(1)}_{ij}(r) F^*_k(r) F_{k'}(r) = \delta_{kk'}$$

(42)

can not be proved for those eigenvectors which are correspond to the degenerate eigenvalues, in many cases it is possible to construct a complete set of the vector fields satisfying the eigenvalue equation (36) and the orthonormality relation (42). In fact in those cases that the tensor $C(r)$, defined by (40), is a hermitian tensor, that is $C^{\dagger} = C$, it can be shown that the differential operator

$$\hat{K} = [C(r)]^{-1} \nabla \times \mu^{(2)}(r) \nabla \times [C(r)]^{-1}$$

(43)

is a hermitian operator on the Hilbert space

$$\Omega = \{ g : R^3 \to R^3 : \int d^3 r g^*(r) \cdot g(r) < \infty \}$$

(44)

where the inner product in $\Omega$ is given by

$$\forall f, g \in \Omega \quad \langle f | g \rangle = \int d^3 r f^*(r) \cdot g(r)$$

(45)

Therefore there is a complete set of vector fields in $\Omega$ that satisfy the eigenvalue equation

$$[C(r)]^{-1} \nabla \times \mu^{(2)}(r) \nabla \times [C(r)]^{-1} f_k = \omega^2_k f_k$$

(46)
and the orthonormality relation

\[
\int d^3r \, f_k^*(r) \cdot f_{k'}(r) = \delta_{kk'}.
\] (47)

It is clear from (36) and (46) that the vector field \(F_k(r)\) will satisfy Eq. (36) iff \(C(r)F_k(r)\) satisfies the eigenvalue equation (46). Accordingly, at least, in those cases that the tensor \(C(r)\) is a hermitian tensor, one can construct a complete set of the squared integrable vector fields which satisfy the eigenvalue equation (36) and the orthonormality relation (42).

It should be noted that, as (36) shows, those eigenvector fields which are correspond to the nonzero eigenvalues satisfy the gauge condition

\[
\nabla \cdot [\varepsilon^{(1)}(r)F_k(r)] = 0.
\] (48)

Regarding to the completeness of the eigenvector fields one can expand the projection operator \(P_{ij} \perp (r, r')\) given by (19), as

\[
P_{ij} \perp (r, r') = \sum_{k}^J \sum_{l}^J \varepsilon_{jl}^{(1)}(r') F_{ki}^*(r) F_{kl}(r') = \sum_{k}^J \sum_{l}^J \varepsilon_{jl}^{(1)}(r') F_{ki}^*(r) F_{kl}(r').
\] (49)

where \(\sum_{k}^J\) denote the summation over those eigenvector fields which are correspond to the nonzero eigenvalues in (36). The condition (48) shows that this expansion is compatible with the transversality relations (22) and (23). Also one can expand the vector potential \(A\) and the canonical momentum density \(\Pi\) as

\[
A(r, t) = \sum_{k}^J \sqrt{\frac{\hbar}{2\omega_k}} \left[ a_k(0)e^{-i\omega_k t} F_k(r) + a_k^\dagger(0)e^{i\omega_k t} F_k^*(r) \right]
\]

\[
\Pi(r, t) = i \varepsilon^{(1)}(r) \sum_{k}^J \sqrt{\frac{\hbar\omega_k}{2}} \left[ a_k^\dagger(0) e^{i\omega_k t} F_k^*(r) - a_k(0) e^{-i\omega_k t} F_k(r) \right]
\] (50)

where \(a_k\) and \(a_k^\dagger\) are the annihilation and creation operators of the bi-anisotropic non-dispersive magnetodielectric medium. Using the canonical commutation
relations (29) and the expansions (49) and (50), the commutation relations between $a_k$ and $a_k^\dagger$ is easily obtained as

$$[a_k(t), a_k^\dagger(t)] = \delta_{kk'}$$

(51)

Regarding (48), one can see that the gauge condition

$$\nabla \cdot [\varepsilon^{(1)}(r)A(r,t)] = 0$$

and

$$\nabla \cdot \Pi(r,t) = 0$$

are satisfied by the expansions (50). Now inserting the expansions (50) in the Hamiltonian (30) and applying Eqs.(36) and (42), we reach to the diagonalized form for the Hamiltonian of the system (in the case $\varepsilon^{(2)} = \mu^{(1)} = 0$) as the following

$$H_F = \sum_k \hbar \omega_k a_k^\dagger a_k$$

(52)

where the normal ordering has been applied.

### 3.2 Homogeneous bulk material

For a homogeneous bulk material, that is when the tensors $\varepsilon^{(1)}$ and $\mu^{(2)}$ are independent of the position vector $r$, it is easy to show that the eigenvalues, $\omega_2^2$, in Eq.(36) are the roots of the following determinant

$$\det \left[ \Lambda(q, \mu^{(2)}) - \omega_2^2 \varepsilon^{(1)} \right] = 0,$$

(53)

where

$$\Lambda_{ij}(q, \mu^{(2)}) = -\epsilon_{i\alpha\beta} \epsilon_{rsj} \mu_{\beta r}^{(2)} q_\alpha q_s.$$  

(54)

Let us denote the roots of the determinant in (53) by $\omega_\rho(q)$, where $\rho$ labels the different roots of the determinant and $q$ is an arbitrary three dimensional vector. Then, the normalized eigenvector field corresponds to the eigenvalue $\omega_\rho(q)$ can be written as

$$F(\rho, \lambda, q, r) = \frac{X(\rho, \lambda, q)}{\sqrt{X^\dagger(\rho, \lambda, q)\varepsilon^{(1)}X(\rho, \lambda, q)}} e^{iq\cdot r}$$

(55)

where $X(\rho, \lambda, q)$ is a three dimensional vector satisfying the algebraic eigenvalue equation

$$\left[ \Lambda(q, \mu^{(2)}) - \omega_\rho^2(q) \varepsilon^{(1)} \right] X(\rho, \lambda, q) = 0$$

(56)

and $\lambda$ indicates the probable degeneracy of the eigenvalue $\omega_\rho^2(q)$ which is known as the polarization of the photon. Here the label $k$ in Eq.(36) is
represented by a triplet \((\rho, \lambda, \mathbf{q})\) and the orthonormality relation (42) is expressed as

\[
\int d^3 r \ v_{ij}^{(1)} F_i^*(\rho, \lambda, \mathbf{q}, \mathbf{r}) F_j(\rho', \lambda', \mathbf{q}', \mathbf{r}) = \delta_{\rho\rho'} \delta_{\lambda\lambda'} \delta(\mathbf{q} - \mathbf{q}')
\]  

(57)

The vector potential and the canonical momentum density \(\Pi\) can be expanded in terms of the eigenvector fields (55) as

\[
\mathbf{A}(\mathbf{r}, t) = \sum_{\rho} \sum_{\lambda} \int d^3 q \sqrt{\frac{\hbar}{2\omega_\rho(\mathbf{q})}} \left[ a(\rho, \lambda, \mathbf{q}) e^{-i\omega_\rho(\mathbf{q})t} \mathbf{F}(\rho, \lambda, \mathbf{q}, \mathbf{r}) + H.C \right]
\]

\[
\Pi(\mathbf{r}, t) = i \ v_{ij}^{(1)} \sum_{\rho} \sum_{\lambda} \int d^3 q \sqrt{\frac{\hbar}{2\omega_\rho(\mathbf{q})}} \left[ a^\dagger(\rho, \lambda, \mathbf{q}) e^{i\omega_\rho(\mathbf{q})t} \mathbf{F}^*(\rho, \lambda, \mathbf{q}, \mathbf{r}) - H.C \right]
\]

(58)

Where \(\sum_{\rho}\) denotes the summation over the nonzero roots in (53). Finally, the Hamiltonian of the electromagnetic field in the presence of the anisotropic homogeneous magnetodielectric medium can be written as

\[
H = \sum_{\rho, \lambda} \int d^3 q \ h\omega_\rho(\mathbf{q}) \ a^\dagger(\rho, \lambda, \mathbf{q}) a(\rho, \lambda, \mathbf{q})
\]  

(59)

4 Generalization of the quantization in the presence of external charges

The quantization method discussed in the previous sections can be generalized in the presence of \(N\) point external charged particles. This is useful, particularly, when we are concerned with the spontaneous emission of the atomic systems within the magnetodielectric media. In the presence of \(N\) charged particles with charges \(q_1, q_2, ..., q_N\) and masses \(m_1, m_2, ..., m_N\) the
generalization of the Hamiltonian (30) is

\[ H(t) = \frac{1}{2} \int d^3r \left\{ \left[ \epsilon^{(1)}(r) \right]_{ij}^- \Pi_i(r, t) \Pi_j(r, t) + \mu_{ij}^{(2)}(r) (\nabla \times A)_i (\nabla \times A)_j \right\} \]

\[ + \frac{1}{2} \int d^3r \left[ \epsilon^{(1)}(r) \right]_{ij}^- \left\{ \Pi_i(r, t) \left[ \epsilon^{(2)}(r) \nabla \times A \right]_j + \left[ \epsilon^{(2)}(r) \nabla \times A \right]_i \Pi_j(r, t) \right\} \]

\[ + \frac{1}{2} \int d^3r \left[ \epsilon^{(1)}(r) \right]_{ij}^- \left\{ \mu_{ij}^{(2)}(r) \left[ \nabla \times A \right]_i \left[ \epsilon^{(2)}(r) \nabla \times A \right]_j \right\} \]

\[ + \sum_{i=1}^N \left| p_i - q_i A(x_i, t) \right|^2 + \sum_{i=1}^N q_i V(x_i, t) \]

(60)

where \( x_i \) and \( p_i \) are the position and momentum operators of the \( i \)'th particle that satisfy the usual commutation relations

\[ [x_i(t), p_j(t)] = i\hbar \delta_{ij} I, \]

(61)

where \( I \) is the unit matrix. In this case the commutation relations between the cartesian components of the conjugate variables \( A \) and \( \Pi \) are the same as (29). In Hamiltonian (60), \( V(x_i, t) \) is the coulomb potential produced by the other particles at the place of the \( i \)'th particle. This coulomb potential is related to the Green function introduced by (14) as

\[ V(x_i, t) = \sum_{j \neq i} q_j G(x_i, x_j) \]

(62)

As the before section, in the special case that \( \epsilon^{(2)} = \mu^{(1)} = 0 \), the conjugate variables \( A \) and \( \Pi \) can be expanded in terms of the eigenvectors \( F_k \) satisfying Eq.(36) as

\[ A(r, t) = \sum_k \sqrt{\frac{\hbar}{2\omega_k}} \left[ a_k(t) F_k(r) + a_k^\dagger(t) F_k^*(r) \right] \]

\[ \Pi(r, t) = i \epsilon^{(1)}(r) \sum_k \sqrt{\frac{\hbar \omega_k}{2}} \left[ a_k^\dagger(t) F_k^*(r) - a_k(t) F_k(r) \right] \]

(63)

From the commutation relations (29) and the expansions (49), it is clear that the annihilation and creation operators \( a_k, a_k^\dagger \) satisfy the same commutation rules as (51). Now by inserting the expansions (63) into (60) and using the
eigenvalue equation (36), the generalized Hamiltonian (60) (in the special case $\varepsilon^{(2)} = \mu^{(1)} = 0$), is reduced to

$$
H = \sum_k \hbar \omega_k a_k^\dagger a_k + \sum_{i=1}^N \frac{[p_i - q_i A(x_i, t)]^2}{2m_i} + \sum_{i=1}^N q_i V(x_i, t) \tag{64}
$$

It should be pointed out that, in the Heisenberg picture, the time dependence of the operators $a_k(t), a_k^\dagger(t)$, appeared in the expansions (63), is no longer sinusoidal as the before section. The time evolution of $a_k(t)$ and any other dynamical variable, in the Heisenberg picture, should be obtained by using the Hamiltonian (64) together with the commutation relations (51) and (61).

5 Spontaneous emission of a two-level atom within an anisotropic magnetodielectric medium

In this section, using the quantization method discussed in the before section, the decay rate of an initially excited two-level atom in the presence of an anisotropic non-dispersive magnetodielectric medium, is investigated. In the Hamiltonian (64), suppose that, except one of the particles, all the remainder have sufficiently large masses, so that they can be taken approximately in fixed positions. Then, in the electric dipole approximation [32, 33], the Hamiltonian (64) can be written as

$$
H = \sum_k \hbar \omega_k a_k^\dagger a_k + \frac{p^2}{2m} + \phi(x) - \frac{e}{m} p \cdot A(r_0) \tag{65}
$$

where $e$ and $m$ are the charge and mass of the particle that can move, respectively. $\phi(x) = e \sum_{q_i \neq e} q_i G(x, x_j)$ is the coulomb potential at the place of the moving particle due to the other fixed particles. In (65) the position vector $r_0$ is the center of the reign over which the moving particle is free to move. Such a collection of particles can constitute a one electron atom such that $r_0$ points to the center of the atom. In the Hamiltonian (65) the term $\frac{e^2}{m} A^2(r_0)$ has been ignored, because in the electric dipole approximation this term dose not affect the decay rate of the atom [33]. To calculate the spontaneous emission of an atom, we restrict ourselves to the ideal model of a two-level atom. In this model the basis of the Hilbert space of the atom
is contained two kets denoted by $|1\rangle$ and $|2\rangle$. The kets $|1\rangle$ and $|2\rangle$ are the eigenstates of the Hamiltonian of the atom corresponding to the eigenvalues $E_1$ and $E_2$, respectively. For the two-level atom the Hamiltonian (65) can now be rewritten as \[ H = H_F + H_{at} + H' \] where \[ H_F = \sum_k \hbar \omega_k a_k^\dagger a_k \] \[ H_{at} = \frac{P^2}{2m} + \phi(x) = \hbar \omega_0 \sigma^\dagger \sigma \] \[ H' = [-i\omega_0 \sigma \cdot d \cdot A(r_0) + \text{H.C}] \] (66)

The simplest way to estimate the decay rate of an initially excited two-level atom is the Weisskopf-Wigner approach\[33\]. In this approach, in the Schrödinger picture, the atom-field state at time $t$ is taken as

\[ |\psi(t)\rangle = c(t)|2\rangle|0\rangle + \sum_k M_k(t)|1\rangle|k\rangle \] (68)

where $|0\rangle$ is the vacuum state of the electromagnetic field and the coefficients $c(t)$ and $M_k(t)$ satisfy the initial conditions $c(0) = 1$, $M_k(0) = 0$. Now applying the Hamiltonian (67) and substituting $|\psi(t)\rangle$ from (68) in the Schrödinger equation

\[ H|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle \] (69)

for sufficiently large times, we have

\[ C(t) = e^{-i\omega_0 t - i\delta\omega t - \gamma t} \] (70)

where $\delta\omega$ denotes the Lamb shift and $\gamma$ is the decay constant of the atom which are given by

\[ \delta\omega = \frac{1}{2\hbar} P \sum_k \frac{\omega_k}{\omega_0 - \omega_k} |d \cdot F_k(r_0)|^2 \] (71)
\[
\gamma = \frac{\pi}{2\hbar} \sum_k \omega_k \delta(\omega_k - \omega_0) \left| \mathbf{d} \cdot \mathbf{F}_k(r_0) \right|^2
\]  

(72)

where \( P \) denotes the Cauchy principle value \[25\].

As an example let us assume that the center of the atom is located at the center of a very small spherical hole of radius \( R \) within the anisotropic magnetodielectric medium. Taking the origin of coordinates at the center of the hole, then the tensors \( \varepsilon^{(1)} \) and \( \mu^{(2)} \) in spherical coordinates have the form

\[
\varepsilon^{(1)}_{ij}(r) = \begin{cases} 
\varepsilon_0 & r < R \\
\varepsilon_{ij} & r > R 
\end{cases}
\]

(73)

\[
\mu^{(2)}_{ij}(r) = \begin{cases} 
\frac{1}{\mu_0} & r < R \\
\mu_{ij} & r > R 
\end{cases}
\]

(74)

where \( \varepsilon_0, \mu_0 \) are the permittivity and permeability of the vacuum and \( \varepsilon_{ij}, \mu_{ij} \) are two constant tensors which describe the electric and magnetic properties of the medium around the atom. Applying these tensors in the eigenvalue equation (36), the following equation is deduced

\[
\nabla \times \mu \nabla \times \mathbf{F}_k - \omega_k^2 \varepsilon \mathbf{F}_k = \nabla \times \left( \mu - \mathbf{I} \right) \mu_0 \nabla \times \mathbf{F}_k - \omega_k^2 \left( \varepsilon - \mathbf{I} \varepsilon_0 \right) \theta(R-r) \mathbf{F}_k
\]

(75)

where \( \theta(R-r) \) is the unit step function and \( \mathbf{I} \) is the unit tensor. If the radius \( R \) is assumed to be very smaller than the wavelength \( \lambda = \frac{2\pi c}{\omega_0} \) then, the following approximations may be used for the \( i \)the cartesian component of the right hand of (75)

\[
\left[ -\omega_k^2 \left( \varepsilon - \mathbf{I} \varepsilon_0 \right) \theta(R-r) \mathbf{F}_k(r) \right]_i \simeq -\omega_k^2 \left( \varepsilon_{ij} - \delta_{ij} \varepsilon_0 \right) \theta(R-r) F_{kj}(0)
\]

\[
\left[ \nabla \times \left( \mu - \frac{I}{\mu_0} \right) \theta(R-r) \nabla \times \mathbf{F}_k \right]_i
\]

\[
\simeq \varepsilon_{i\alpha\beta} \varepsilon_{mns} \left( \mu_{\beta m} - \frac{\delta_{\beta m}}{\mu_0} \right) \left[ -\frac{x_{\alpha}}{r} \delta(R-r) F_{ks,n}(0) + \theta(R-r) F_{ks,\alpha n}(0) \right]
\]

(76)

where \( F_{ks,n}(0) \) and \( F_{ks,\alpha n}(0) \) denote \( \frac{\partial F_{ks}}{\partial x_n}(0) \) and \( \frac{\partial^2 F_{ks}}{\partial x_\alpha \partial x_n}(0) \), respectively. The solution of the equation (75) can now be written as the sum of two parts. One part is the solution of the homogeneous equation

\[
\nabla \times \mu \nabla \times \mathbf{F}_k - \omega_k^2 \varepsilon \mathbf{F}_k = 0
\]

(77)
that is
\[ F_k(r) \equiv F(\rho, \lambda, q, r) = \frac{X(\rho, \lambda, q)}{\sqrt{X^T(\rho, \lambda, q) \times X(\rho, \lambda, q)}} e^{iqr} \]  
(78)

with \( X(\rho, \lambda, q) \) given by (53) and (56), where the tensors \( \varepsilon^{(1)} \) and \( \mu^{(2)} \) should now be replaced by \( \varepsilon \) and \( \mu \), respectively. The second part of the solution (75) is the response to the inhomogeneity term in the right hand of (75). The cartesian component of this part can be written as
\[ F_{ki}(r) = \int d^3r' \tilde{G}_{ij}(r', r') J_j(r') \quad i = 1, 2, 3 \]  
(79)

where \( J_i \) is the sum of two approximated terms in (76) and \( \tilde{G}_{ij}(r, r') \) is the Green tensor related to Eq.(75) satisfying
\[ \begin{bmatrix} \varepsilon_{\alpha\beta\epsilon_{mns}} \mu_{\beta m} \frac{\partial^2}{\partial x_\alpha \partial x_n} - \omega_k^2 \varepsilon_{is} \end{bmatrix} \tilde{G}_{sj}(r, r') = \delta_{ij} \delta(r - r') \]  
(80)

Using the technique of Fourier transform, the Green tensor \( \tilde{G} \) is easily obtained as
\[ \tilde{G}(r, r') = \lim_{\eta \to 0^+} \frac{1}{(2\pi)^3} \int_{-\infty}^{+\infty} d^3p \frac{1}{[\Lambda(p, \mu) - \omega^2(\rho, \lambda, q) \varepsilon - i\eta I]} e^{ip \cdot (r - r')} \]  
(81)

where \( I \) is the identity tensor and \( \Lambda(p, \mu) \) is defined by (54) with \( \mu^{(2)} = \mu \). Using (76), (78) and (79), the i'th cartesian component of the solution of Eq.(75) can now be written as
\[ \begin{align*}
F_{ki}(r) &= \frac{X_i(\rho, \lambda, q)}{\sqrt{X^T(\rho, \lambda, q) \times X(\rho, \lambda, q)}} e^{iqr} \\
-\omega^2(\rho, \lambda, q) &\int_{-\infty}^{+\infty} d^3r' \tilde{G}_{ij}(r, r') \theta(R - r') (\varepsilon_{jm} - \varepsilon_0 \delta_{jm}) \ F_{km}(0) \\
+ &\int_{-\infty}^{+\infty} d^3r' \tilde{G}_{ij}(r, r') \left( \varepsilon_{ja\beta\epsilon_{mns}}(\mu_{\beta m} - \frac{\delta_{\beta m}}{\mu_0}) \right) \theta(R - r') F_{ks,sn}(0) \\
+ &\int_{-\infty}^{+\infty} d^3r' \tilde{G}_{ij}(r, r') \left( \varepsilon_{ja\beta\epsilon_{mns}}(\mu_{\beta m} - \frac{\delta_{\beta m}}{\mu_0}) \right) \left[ -\frac{x_j'}{r'^3} \delta(R - r') \ F_{ks,n}(0) \right]
\end{align*} \]  
(82)

In (81) and (82) the eigenfrequency \( \omega_k \) has been replaced approximately by \( \omega_\rho(\rho, \lambda, q) \), that is, the roots of the determinant \( [\Lambda(\rho, \lambda, q) - \omega^2(\rho, \lambda, q) \varepsilon] \). Now
substituting (81) into (82), after integrating with respect to the variable \( r' \), it can be shown straightforwardly that the third term in (82) is negligible for sufficiently small value of \( R \) and accordingly for the i'th cartesian component of the eigenfunctions \( F_k(r) \), for \( R \ll 1 \), one can write

\[
F_{ki}(r) = \frac{X_i(\rho, \lambda, q)}{\sqrt{X^\dagger(\rho, \lambda, q) \in X(\rho, \lambda, q)}} e^{iqr} + \frac{1}{2\pi^2} \lim_{\eta \to 0^+} \left\{ \int_{-\infty}^{+\infty} d^3p \left( \frac{\sin |pR|}{|p|^3} \right) \left[ \Lambda(p, \mu) - \omega^2_p(q) \varepsilon - i\eta I \right]_{ij} \right\} \\
\times \left[ -\omega^2_p(q)(\varepsilon_{jm} - \varepsilon_0\delta_{jm}) \right. \\
\left. F_{km}(0) + \left( \varepsilon_{\alpha\beta}\varepsilon_{\mu\nu}(\mu\beta_m - \delta_{\beta_m}) \right) F_{ks,\alpha\mu}(0) \right]
\]

(83)

Differentiation with respect to the cartesian coordinates \( x_\delta, x_\gamma \) from the both sides of (83) yields

\[
F_{ki,\delta\gamma}(r) = \frac{X_i(\rho, \lambda, q)}{\sqrt{X^\dagger(\rho, \lambda, q) \in X(\rho, \lambda, q)}} (-q_\delta q_\gamma) e^{iqr} + \frac{1}{2\pi^2} \lim_{\eta \to 0^+} \left\{ \int_{-\infty}^{+\infty} d^3p \left( -p_\delta p_\gamma e^{ipr} \frac{\sin |pR|}{|p|^3} \right) \left[ \Lambda(p, \mu) - \omega^2_p(q) \varepsilon - i\eta I \right]_{ij} \right\} \\
\times \left[ -\omega^2_p(q)(\varepsilon_{jm} - \varepsilon_0\delta_{jm}) \right. \\
\left. F_{km}(0) + \left( \varepsilon_{\alpha\beta}\varepsilon_{\mu\nu}(\mu\beta_m - \delta_{\beta_m}) \right) F_{ks,\alpha\mu}(0) \right]
\]

(84)

Consistency condition at \( r = 0 \) for the relations (83) and (84) gives

\[
\Gamma^{(1)}_{im} F_{km}(0) = \frac{X_i(\rho, \lambda, q)}{\sqrt{X^\dagger(\rho, \lambda, q) \in X(\rho, \lambda, q)}} + \Delta^{(1)}_{isan} F_{ks,an}(0)
\]

(85)

and

\[
\Delta^{(2)}_{i\gamma,san} F_{ks,an}(0) = -\frac{X_i(\rho, \lambda, q)(q_\delta q_\gamma)}{\sqrt{X^\dagger(\rho, \lambda, q) \in X(\rho, \lambda, q)}} + \Gamma^{(2)}_{i\gamma,m} F_{km}(0),
\]

(86)

respectively, where the summation should be done over the repeated indices and the tensors \( \Gamma^{(1)}, \Gamma^{(2)}, \Delta^{(1)} \) and \( \Delta^{(2)} \) are given by

\[
\Gamma^{(1)}_{im} = \delta_{im} + \frac{\omega^2_p(q)}{2\pi^2} \lim_{\eta \to 0^+} \left\{ \int_{-\infty}^{+\infty} d^3p \left( \frac{\sin |pR|}{|p|^3} \right) \left[ \Lambda(p, \mu) - \omega^2_p(q) \varepsilon - i\eta I \right]_{ij} \right\} \\
\times (\varepsilon_{jm} - \varepsilon_0\delta_{jm})
\]

(87)
\[ \Delta_{isan}^{(1)} = \frac{1}{2\pi^2} \lim_{\eta \to 0^+} \left\{ \int_{-\infty}^{+\infty} d^3p \left( \frac{\sin |pR|}{|p|^3} \right) [\Lambda(p, \mu) - \omega_p^2(q) \varepsilon - i\eta I]^{-1}_{ij} \right\} \times \left( \epsilon_{\alpha\beta} \epsilon_{\mu\nu} (\mu_{\beta m} - \frac{\delta_{\beta m}}{\mu_0}) \right) \] (88)

\[ \Gamma_{i\delta\gamma,m}^{(2)} = \frac{\omega_p^2(q)}{2\pi^2} \lim_{\eta \to 0^+} \left\{ \int_{-\infty}^{+\infty} d^3p \left( p_\delta p_\gamma \frac{\sin |pR|}{|p|^3} \right) [\Lambda(p, \mu) - \omega_p^2(q) \varepsilon - i\eta I]^{-1}_{ij} \right\} \times (\varepsilon_{jm} - \varepsilon_0 \delta_{jm}) \] (89)

\[ \Delta_{i\delta\gamma,s\alpha n}^{(2)} = \delta_{i\alpha} \delta_{\delta\alpha} \delta_{\gamma n} + \frac{1}{2\pi^2} \lim_{\eta \to 0^+} \left\{ \int_{-\infty}^{+\infty} d^3p \left( p_\delta p_\gamma \frac{\sin |pR|}{|p|^3} \right) [\Lambda(p, \mu) - \omega_p^2(q) \varepsilon - i\eta I]^{-1}_{ij} \right\} \times \left( \epsilon_{\alpha\beta} \epsilon_{\mu\nu} (\mu_{\beta m} - \frac{\delta_{\beta m}}{\mu_0}) \right) \] (90)

From (86) one can obtain \( F_{ks',\alpha'n'}(0) \) in terms of the cartesian component of \( F(0) \) as

\[ F_{ks',\alpha'n'}(0) = -\frac{1}{\sqrt{X^\dagger(\rho, \lambda, q) \varepsilon X(\rho, \lambda, q)}} \left[ (\Delta_{s\alpha'n',j\delta\gamma}^{(2)})^{-1}_{s\alpha'n',j\delta\gamma} X_j(\rho, \lambda, q) (q_\delta q_\gamma) \right] + \left[ (\Delta_{s\alpha'n',j\delta\gamma}^{(2)})^{-1}_{s\alpha'n',j\delta\gamma} \Gamma_{j\delta\gamma,m}^{(2)} F_{km}(0) \right] \] (91)

where the tensor \((\Delta_{s\alpha'n',i\delta\gamma}^{(2)})^{-1}\) is defined by

\[ (\Delta_{s\alpha'n',j\delta\gamma}^{(2)})^{-1}_{s\alpha'n',j\delta\gamma} = \delta_{\alpha\alpha'} \delta_{ss'} \delta_{nn'} \] (92)

Now, combination of (85) and (91) yields

\[ Q_{im} F_{km}(0) = \frac{X_i(\rho, \lambda, q)}{\sqrt{X^\dagger(\rho, \lambda, q) \varepsilon X(\rho, \lambda, q)}} \left[ \Delta_{isan}^{(1)} (\Delta_{asan,j\delta\gamma}^{(2)})^{-1}_{asan,j\delta\gamma} X_j(\rho, \lambda, q) q_\delta q_\gamma \right] \] (93)

where

\[ Q_{im} = \Gamma_{im}^{(1)} - \Delta_{isan}^{(1)} (\Delta_{asan,j\delta\gamma}^{(2)})^{-1}_{asan,j\delta\gamma} \Gamma_{j\delta\gamma,m}^{(2)} \] (94)
Finally, using the relation (93), one can write the m’th cartesian component of $F(\rho, \lambda, q, 0)$ as follows

$$F_{km}(0) \equiv F_m(\rho, \lambda, q, 0) = \frac{1}{\sqrt{X(\rho, \lambda, q) \cdot X(\rho, \lambda, 0)}} \times \left[ Q^{(-1)}_{mi} X_i(\rho, \lambda, q) - Q^{(-1)}_{mi} \Delta^{(1)}_{isan} (\Delta^{(2)})^{-1}_{isan,j\delta\gamma} X_j(\rho, \lambda, q) \cdot q_{\delta} q_{\gamma} \right]$$

Consequently, Applying the final result (95) in (72), the decay rate of an initially excited two-level atom located at the center of a very small spherical cavity within an anisotropic homogeneous magnetodielectric medium is estimated as

$$\gamma = \frac{\pi}{2\hbar} \sum_{\rho, \lambda} \int_{-\infty}^{+\infty} d^3 q \ \omega_\rho(q) \ \delta(\omega_\rho(q) - \omega_0) \ |d \cdot F(\rho, \lambda, q, 0)|^2$$

It can be shown, straightforwardly, in the case of an isotropic dielectric medium the result (96) coincides to the spontaneous emission of the atom computed by Glauber and et al in the limit $R \to 0$.\[25\]

6 summary

Proposing an appropriate Lagrangian density, a canonical quantization of the electromagnetic field in the presence of a non-dispersive bi-anisotropic inhomogeneous magnetodielectric medium was introduced. The quantization was achieved for both in the absence and in the presence of the atomic systems. The spontaneous emission of a two-level atom embedded in a bi-anisotropic homogeneous magnetodielectric medium was investigated. It was argued that, the form of Maxwell’s equations in the curved space-time in the absence of a responsive medium, when the cartesian coordinates is used, are similar to the form of Maxwell’s equations in the flat space-time in the presence of a bi-anisotropic magnetodielectric medium. Therefore the quantization scheme discussed in this paper is applicable in the curved space-time in the absence of responsive media. This is useful particularly to investigate the effect of the curvature of the space-time on the quantum properties of the electromagnetic field.
Appendix A : Derivation of the Hamiltonian (30)

By definition, the Hamiltonian of the electromagnetic field is written as

\[ H(t) = \int d^3r [\Pi(r, t) \cdot \dot{A}(r, t)] - L(t) \]  \hspace{1cm} (A1)

where the Lagrangian \( L(t) \) is given by (27). If, using (28), the time derivative of the vector potential is obtained in terms of the dynamical variables \( \Pi \) and \( A \), then (A1) reads to

\[ H(t) = \frac{1}{2} \int d^3r \left[ (\varepsilon^{(1)})^{-1} \Pi_i \Pi_j + \mu^{(2)}_{ij} (\nabla \times A)_i (\nabla \times A)_j \right] 
+ \frac{1}{2} \int d^3r \left\{ \Pi \cdot \left[ (\varepsilon^{(1)})^{-1} \varepsilon^{(2)} \nabla \times A \right]_i \right. 
+ \left[ (\varepsilon^{(1)})^{-1} \varepsilon^{(2)} \nabla \times A \right]_j \right\} 
+ \frac{1}{2} \int d^3r \varepsilon^{(1)}_{ij} \left[ (\varepsilon^{(1)})^{-1} \varepsilon^{(2)} \nabla \times A \right]_i \left[ (\varepsilon^{(1)})^{-1} \varepsilon^{(2)} \nabla \times A \right]_j 
+ \frac{1}{2} \int d^3r \int d^3r' G(r, r') \nabla \cdot \left[ \varepsilon^{(2)}(r) \nabla \times A(r, t) \right] \left[ \nabla' \cdot \varepsilon^{(2)}(r') \nabla \times A(r', t) \right] \]  \hspace{1cm} (A2)

Because \( \Pi \) is purely transverse, the second term is clearly equivalent to

\[ \frac{1}{2} \int d^3r \left\{ \Pi \cdot \left[ (\varepsilon^{(1)})^{-1} \varepsilon^{(2)} \nabla \times A \right] + \left[ (\varepsilon^{(1)})^{-1} \varepsilon^{(2)} \nabla \times A \right] \cdot \Pi \right\} \]  \hspace{1cm} (A3)

According to the definition (17) and the constraint (13) one can write

\[ \nabla \varphi = \left[ (\varepsilon^{(1)})^{-1} \varepsilon^{(2)} \nabla \times A \right] \| \]  \hspace{1cm} (A4)

and therefore from (18) the third term in (A2) is equal to

\[ \frac{1}{2} \int d^3r \varepsilon^{(1)}_{ij} \left[ (\varepsilon^{(1)})^{-1} \varepsilon^{(2)} \nabla \times A - \nabla \varphi \right]_i \left[ (\varepsilon^{(1)})^{-1} \varepsilon^{(2)} \nabla \times A - \nabla \varphi \right]_j \]  \hspace{1cm} (A5)

which, after integrating by parts, is reduced to

\[ \frac{1}{2} \int d^3r \varepsilon^{(1)}_{ij} \left[ (\varepsilon^{(2)} \nabla \times A)_i \left[ (\varepsilon^{(2)} \nabla \times A)_j + 1 \right] 
+ \frac{1}{2} \int d^3r \varphi \nabla \cdot \left[ \varepsilon^{(2)} \nabla \times A \right] \]  \hspace{1cm} (A6)

Now applying Eq. (13) the second term in (A6) cancels the last term in the Hamiltonian (A2) and one can reach to the form given by (30).
Appendix B: Derivation of the Heisenberg equation (31)

According to the Hamiltonian (30) the time evolution of the \(i\)'th cartesian component of the vector potential contains two parts. The first part is due to the commutator of \(A_i\) and the first term of the Hamiltonian (30). Using the symmetry feature of the tensor \(\varepsilon^{(1)}\) and the commutation relations (29), this part can be expressed as

\[
\frac{i}{\hbar} \int d^3r' \left[ \varepsilon^{(1)}(r') \right]_{\alpha\beta}^{-1} \Pi_\alpha(r', t) \left[ \Pi_\beta(r', t) , A_i(r, t) \right] = \int d^3r' \left[ \varepsilon^{(1)}(r') \right]_{\alpha\beta}^{-1} P^\perp_{\alpha\beta}(r, r') \Pi_\beta(r', t) = \left[ (\varepsilon^{(1)}(r))^{-1} \Pi(r, t) \right]_i \tag{B1}
\]

where the fact that the vector field \((\varepsilon^{(1)})^{-1}\Pi\) is purely \(\varepsilon^1\)-transverse field has been used in the last step. The second part of the time evolution of \(A_i\) is caused by the commutator of the third and fourth terms of the Hamiltonian (30) and \(A_i\), which using the symmetry property of the tensor \(\varepsilon^{(1)}\) can be written as the following

\[
\frac{i}{\hbar} \int d^3r' \left[ \varepsilon^{(1)}(r') \right]_{\alpha\beta}^{-1} \left\{ [\Pi_\alpha(r', t) , A_i(r, t)] \left( \varepsilon^{(2)}(r') \nabla \times A(r', t) \right)_\beta \right\} = \int d^3r' \left[ \varepsilon^{(1)}(r') \right]_{\alpha\beta}^{-1} P^\perp_{\alpha\beta}(r, r') \left( \varepsilon^{(2)}(r') \nabla \times A(r', t) \right)_\beta = \left[ (\varepsilon^{(1)})^{-1} \varepsilon^{(2)} \nabla \times A \right]^\perp_i \tag{B2}
\]

where the definition (21) has been applied in the last step. Now by adding (B1) and (B2) the Heisenberg equation (31) is deduced.

Appendix C: Derivation of the Heisenberg equation (32)

The Hamiltonian (30) shows that the time derivative of the \(i\)'th cartesian component of \(\Pi\) contains three parts. Regarding the symmetry property
\[\mu_{\alpha\beta}^{(2)} = \mu_{\beta\alpha}^{(2)}\] and the commutation relations \(29\), the first part is
\[\frac{i}{2\hbar} \int d^3r' \mu_{\alpha\beta}^{(2)}(r') \left[ (\nabla' \times A(r', t))_{\alpha} (\nabla' \times A(r', t))_{\beta} , \Pi_i(r, t) \right] = \frac{i}{\hbar} \int d^3r' \mu_{\alpha\beta}^{(2)}(r') \left[ (\nabla' \times A(r', t))_{\alpha} \left[ (\nabla' \times A(r', t))_{\beta} , \Pi_i(r, t) \right] \right] = - \int d^3r' \mu_{\alpha\beta}^{(2)}(r') \left( \nabla' \times A(r', t) \right)_{\alpha} \epsilon_{\beta p q} \partial' \partial'_{p} P^\perp_{q i}(r', r)
\]

\(\text{(C1)}\)

Integrating by parts and applying \(26\), the final result in \(\text{(C1)}\) is reduced to
\[= - \int d^3r' \left[ \nabla' \times \mu^{(2)}(r') \nabla' \times A(r', t) \right] q = - \left[ \nabla \times \mu^{(2)}(r) \nabla \times A(r', t) \right] i \quad \text{(C2)}\]

Another part of \(\dot{\Pi}_i\) is related to the commutator of the third and forth terms of the Hamiltonian \(30\) and \(\Pi_i\). Using the symmetry relations \(\left[ \varepsilon^{(1)} \right]_{\alpha\beta}^{-1} = \left[ \varepsilon^{(1)} \right]_{\beta\alpha}^{-1}\) and commutation relations \(29\), this part can be written as
\[= \frac{i}{\hbar} \int d^3r' \left[ \varepsilon^{(1)}(r') \right]_{\alpha\beta}^{-1} \Pi_{a}(r', t) \left[ (\varepsilon^{(2)}(r') \nabla' \times A(r', t))_{\beta} , \Pi_i(r, t) \right] = - \int d^3r' \left[ \varepsilon^{(1)}(r') \right]_{\alpha\beta}^{-1} \varepsilon_{\beta s p q} \Pi_{a}(r', t) \epsilon_{s p q} \partial' \partial'_{p} P^\perp_{q i}(r', r) = \int d^3r' \left[ \varepsilon^{(1)}(r') \right]_{\alpha\beta}^{-1} \mu_{\alpha\beta}^{(1)}(r') \Pi_{a}(r', t) \epsilon_{s p q} \partial' \partial'_{p} P^\perp_{q i}(r', r)
\]

\(\text{(C3)}\)

where the Onsager’s relation \(\varepsilon_{\alpha\beta}^{(2)} = -\mu_{\beta\alpha}^{(1)}\) has been used. Integration by parts and using the relations \(26, 31, 18\) and \(A4\) the final result in \(\text{(C3)}\) is equivalent to
\[= \int d^3r' \left[ \nabla' \times \mu^{(1)}(r')(\varepsilon^{(1)}(r'))^{-1} \Pi'(r', t) \right]_{q} = \left[ \nabla \times \mu^{(1)}(r)(\varepsilon^{(1)}(r))^{-1} \Pi(r, t) \right]_{i} = \left[ \nabla \times \mu^{(1)}(r)\dot{A}(r, t) \right]_{i} + [\nabla \times \mu^{(1)}(r)\nabla\varphi(r, t)]_{i} - \left[ \nabla \times \mu^{(1)}(r)(\varepsilon^{(1)}(r))^{-1}\varepsilon^{(2)}(r) \nabla \times A(r, t) \right]_{i}
\]

\(\text{(C4)}\)
Finally, the third part of time derivative of $\Pi_i$ is caused by the commutator of the last term of the Hamiltonian (30) and $\Pi_i$ which can be computed as

$$
\frac{i}{\hbar} \int d^3r' \left[ \varepsilon^{(1)}(r') \right]_{\alpha \beta}^{-1} \left\{ \left[ \left( \varepsilon^{(2)}(r') \nabla' \times A(r', t) \right) \right]_{\alpha}, \Pi_i(r, t) \right\} \left( \varepsilon^{(2)}(r') \nabla' \times A(r', t) \right)_{\beta} 
$$

$$
= - \int d^3r' \left[ \varepsilon^{(1)}(r') \right]_{\alpha \beta}^{-1} \varepsilon^{(2)}(r') \epsilon_{spq} \partial'_p P_{qi}^\perp (r', r) \left( \varepsilon^{(2)}(r') \nabla' \times A(r', t) \right)_{\beta} \tag{C5}
$$

Using the onsager’s relation $\varepsilon^{(2)}_{\alpha \beta} = -\mu^{(1)}_{\beta \alpha}$, integrating by parts and then, applying the relation (26), this is equal to

$$
= \int d^3r' P_{qi}^\perp (r', r) \left[ \nabla' \times \mu^{(1)}(r')\left( \varepsilon^{(1)}(r') \right)^{-1} \varepsilon^{(2)}(r') \nabla' \times A(r', t) \right]_q 
$$

$$
= \left[ \nabla \times \mu^{(1)}(r)\left( \varepsilon^{(1)}(r) \right)^{-1} \varepsilon^{(2)}(r) \nabla' \times A(r, t) \right]_i \tag{C6}
$$

which cancels the last term in (C4). Therefore, adding the parts (C2), (C4) and (C6) gives the Heisenberg equation (32).

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