On the free surface motion with a vertical vortex sheet under water

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Abstract. The vortex sheet always represents a thin shear layer across which there is a discontinuity in fluid velocity. We show a vertical one in a 2D water flow and simulate the interaction between its evolution and the free surface. It is quite stimulating that this phenomenon may result from a section of the seafloor lifting up along the fault. Since it also reflects the tsunami generation, the significance for this study is highlighted. We numerically solve the Navier-Stokes equations to tackle this issue. By analyzing the data from computation we attempt to discover the nature of this system and to derive some rule controlling the hydrodynamic process for our problem. The identity of this flow structure and the way the free surface moves are explored as well.

1. Introduction
Helmholtz [1] for the first time noticed that an interface of discontinuity is highly unstable. The most remarkable work is the built of Birkhoff-Rott equation [2, 3] which is an integro-differential equation and indicates the temporally evolutionary profile. Moore [4] contributed much theoretical work in this area. Some key results have been verified by Krasny [5], Shelley [6] via numerical approaches and by Caflish and Semmes [7] via analytical means. However, the equivalence between this mathematical model and the 2D Euler system is still a weak form. Hence more researchers choose Direct Numerical Simulation (DNS) to deal with this problem. Hoepffner et al. [8] reported their numerical experiment that local perturbation accounts for a self-similar growth. Lecoanet et al. [9] computed vortex sheet in the scenario of continuous density distribution as a test for the spectral method code. However, only a few publications discussed the effect of vortex sheet upon the free boundary. Evans [10] introduced the propagation of plane surface wave across a vortex sheet constructed by two opposite flows. McKee and Tesoriero [11] extended the study by moving the background into finite depth water. These studies do not analyze the direct interaction of vortex sheet with the free surface. Tryggvason [12] performed the calculation numerically for a free surface with a horizontal submerged vortex sheet and concluded free surface steeply deforms due to the roll-up of the vortex sheet. Tsai and Yue [13] considered the wake of a vertical surface-piercing plate as the shed vortex sheet which interacts with the free surface.
Although the previous work provides sufficient data for supporting the nature of interactions between free surfaces and vortex sheets, few of them ever talked about the scenario that consists of the top free surface and a vertical vortex sheet excited by the discontinuous tangential components of the flow velocity. The plate motions in the ocean may exert significant influence on the flow. We assume that some cases akin to a sharp upwards movement would for somehow result in a vertical flux right over the active sea bottom and finally forms a steep shear layer across the fault. The motivation for the present manuscript is to discover and predict the deformation of presumed quiet ocean surface with a vertical vortex sheet under the water. In addition, what kind of features this sheet may possess is another angle of sight for observation. When KHI occurs the vortex sheet turns to roll up into a spiral, which may benefit water mixing in ocean meteorology, yet the downside may take a powerful impact on submerged structures.

2. Mathematical Model
The actual nature is rather complex. We simplify our model into a 2D plane and in a dimensionless context. We carefully choose \( d, \rho \) and \( V \) as characteristic length, density and velocity scales to nondimensionalize Navier-Stokes equations:

\[
\nabla \cdot u = 0 \tag{1}
\]

\[
u_t + u \cdot \nabla u = -\frac{1}{\rho} \nabla p - g + \frac{1}{Re} \nabla^2 u + \frac{1}{We} \kappa \delta_s n \tag{2}
\]

where \( Re \) and \( We \) are the Reynolds number and the Weber number. Accordingly the initial vertical velocity field turns into

\[
v(x, y, 0) = \begin{cases} 
\sin\left(\frac{\pi}{2} y\right) e^{-x}, & x \geq 0, \quad 0 < y < 1 \\
-\sin\left(\frac{\pi}{2} y\right) e^x, & x < 0, \quad 0 < y < 1 
\end{cases} \tag{3}
\]

3. Numerics
The Gerris Flow Solver [14, 15] is employed in our research for its accurate interface capturing and reconstructing capability of a modified VOF approach. A good thing is that it is an open source code based on the quad-tree (octree for 3D) grids. Here we directly solve the incompressible Navier-Stokes equations (Eq. 1-2). The computational domain consists of 15 2D square boxes and the length of one side each is 2 thus the domain is a rectangle with the length of 30 and the height of 2 in a nondimensionalized frame. As mentioned before, the dimensionless water depth is 1 which in the real circumstance is \( d \). Given the disturbance which water flow may lead to on the free surface, the domain height is high enough to act as the far field. Therefore, the upper boundary is set to be symmetric and the bottom as well. To avoid wave reflecting and absorbing, periodic boundary conditions are imposed on the left and the right sides.

4. Results and discussions
First, we would like to examine the vortex sheet. We deliberately put a scalar so called tracer to mark the two parts on both sides, which is shown in Fig.1. It is hard to measure the evolution of this structure quantitatively, thus We define \( L_{vs} \) as the absolute distance between the horizontal positions of the upper tip and the bottom end of this thin layer (Fig.1b) to uncover the vortex sheet motion.

Fig.2 shows that during the initial stage this vortex sheet experiences a gradually bending process. Since these curves fall into a narrow range and have almost nothing to do with \( V \) and \( d \), they share a similar developing trend. Let us move on to Fig.2a for some details. For this series, \( V \) is the major factor affecting the rise. If we look at the curves, the vortex sheet is expanding
slightly faster when $V$ is bigger, yet after the curve of $V0.36d1$ shows up this tendency is shaken and later on reversed. The tracer map gives us some hints to find the reason. In Fig.3 we may observe the vortex sheet at $t = 0.8$ in different cases. When $V$ is lower than 0.36, it is easy to have the concept that greater the kinetic energy is further the distance expands. However, so far we ignore the KHI. When this singularity turns to be strong enough and the perturbation focuses on the lower part, a dominated spiral will be yielded locally. This structure seriously hinders the expansion, thus as $V$ increases $L_{vs}$ stretches slower. When $V$ approximates 0.6 the free surface breaks. This dissipative procedure may also impede $L_{vs}$ developing. Back to Fig.2b, we may find some twists of the curves but still scattered in a narrow range. A better understanding of Fig.3g, Fig.3b, Fig.3h and Fig.3i may help us know the reason. The change of $d$ in fact shrinks or expands the vertical room, which leads to the front of this layer being confined or fully developed. The first spirals have a quite organized law behind if we zoom in and rearrange them in Fig.4, which is the evidence.

So far we find that vortex sheets move in a similar trend at the beginning. Therefore it would be an important feature for us to judge how much extent this thin layer would have an impact on the free surface.

The most direct way to analyze free surface motion is to follow the water wave evolution. Fig.5a shows directly how the free surface develops. The flat water around the center part indicates that the wave generation procedure has a similar mechanism with the drop waves, which means this wave propagation only depends on the initial disturbance. The two purple lines outline the tail positions of the wave yielded, which corresponds with the wave energy transport. This phenomenon may disappoint us somehow since the underneath singularity seems to have no obvious impact on the free surface evolution. If the continuous influence does not exist, there should be some connection between the vortex sheet and the free surface at the beginning. We examine the maximum value of the free surface elevation during this stage. A strong connection is found in Fig.5b. This linear relation technically reveals, at the beginning,
that the wave rising depends on the initial velocity field under water, and $V$ represents the intensity of the vortex sheet. The potential needed by wave rising procedure is provided by the fluid kinetic power. Fig.6a shows that. All kinetic curves reach the lowest point at the same time and the water wave is on its first crest at that moment as well. This fact tells us at least two things. First, when $d$ is fixed, whatever $V$ is, the real time consuming is equal for kinetic power converting into potential energy. Second, the well agreement of curves implies that the energy magnitude fits the dimensionless scaling rule, which is also verified by Fig.6b. Compared with Fig.2, we find their interaction is quite rapid or instantaneous, since the energy transport follows a dimensional time scale yet as we know $L_{vs}$ goes the other way. If the change of the two items cannot be in the same pace, we hardly see any continuous interaction for both of them although they do have a strong connection illustrated in Fig.5b.

5. Conclusion

We directly solve Navier-Stokes equations by numerical method to study the free surface motion with a vortex sheet under water. By analysis, we find that the vortex sheet bending process does not rely on the choice of characteristic scalars: $V$ and $d$, thus it has a similar trend under dimensionless frame. The expansion procedure of this thin layer has nothing to do with the water wave propagation, but the vortex sheet does have a strong connection with the free surface elevation at the initial stage, and this relation is linear between $V$ and $H_{max}$. In addition, during the initial energy transport, time consuming is equal while kinetic energy is converting into the potential one. We argue that these results may help people understand the interaction between the vertical vortex sheet and the free surface.
Figure 5: (a) Wave profiles with time elapse for case of $V = 0.3$ and $d = 1$. (b) The maximum free surface elevation $H_{\text{max}}$ at the beginning versus $V$ when $d$ is fixed.

Figure 6: Kinetic energy for different cases versus dimensional time. (a) $V$ varies; (b) $d$ varies.

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