ON SOME DISCRETE RANDOM VARIABLES ARISING FROM RECENT STUDY ON STATISTICAL ANALYSIS OF COMPRESSIVE SENSING

ROMEO MEŠTROVIĆ

Abstract. The recent paper [27] provides a statistical analysis for efficient detection of signal components when missing data samples are present. Here we focus our attention to some complex-valued discrete random variables $X_l(m, N)$ $(0 \leq l \leq N - 1, 1 \leq M \leq N)$, which are closely related to the random variables investigated by LJ. Stanković, S. Stanković and M. Amin in [27]. In particular, by using a combinatorial approach, we prove that for $l \neq 0$ the expected value of $X_l(m, N)$ is equal to zero, and we deduce the expression for the variance of the random variables $X_l(m, N)$. The same results are also deduced for the real part $U_l(m, N)$ and the imaginary part $V_l(m, N)$ of $X_l(m, N)$, as well as the facts that the $k$th moments of $U_l(m, N)$ and $V_l(m, N)$ are equal to zero for every positive integer $k$ which is not divisible by $N/\gcd(N, l)$. Moreover, some additional assertions and examples concerning the random variables $X_l(m, N), U_l(m, N)$ and $V_l(m, N)$ are also presented.

1. Motivation, definitions and related examples

Recently, LJ. Stanković, S. Stanković and M. Amin [27] provided a statistical analysis for efficient detection of signal components when missing data samples are present. As noticed in [27], this analysis is important for both the area of L-statistics and compressive sensing. In both cases, few samples are available due to either noisy sample elimination of random undersampling signal strategies. For more information on the development of compressive sensing (also known as compressed sensing, compressive sampling, or sparse recovery), see [4], [7], [23, Chapter 10] and [24]. For an excellent survey on this topic with applications and related references, see [29] (also see [17]).

In [27, Section 2] (cf. [28, Section II] and [21]) the authors considered a set of $N$ signal values $\Theta$ given by

$$\Theta = \{s(1), s(2), \ldots, s(N)\},$$

where a signal which is sparse in the Discrete Fourier Transform (DFT) domain can be written as

$$s(n) = \sum_{i=1}^{K} A_i e^{j2\pi k_0 n/N}, \quad n \in \{1, 2, \ldots, N\},$$

and the level of sparsity is $K \ll N$, while $A_i$ and $k_0$ denote amplitudes and frequencies of the signal components, respectively. Notice that the relation $K \ll N$ means that most of components of a considered signal are zero. The application of the DFT

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to the above sequence \( \Theta \) leads to the set \( \Phi(l, N) \) of the form (the set \( \Phi \) in the equality (3) of \cite{27}):

\[
\Phi(l, N) = \{ e^{-j2\pi nl/N} : n = 1, 2, \ldots, N \} \quad \text{with some fixed } l \in \{0, 1, \ldots, N - 1\}.
\]

As usually, throughout our considerations we use the term “multiset” (often written as “set”) to mean “a totality having possible multiplicities”; so that two (multi)sets will be counted as equal if and only if they have the same elements with identical multiplicities.

Notice that (2) for \( l = 0 \) implies that

\[
\Phi(0, N) = \{1, 1, \ldots, 1\}.
\]

Moreover, it is obvious that \( \Phi(l, N) \) given by (2) is a set consisting of \( N \) (distinct) elements if and only if \( l \) and \( N \) are relatively prime positive integers.

Let \( \mathcal{M} \) denote the collection of all multisets \( \Phi(l, N) \) of the form (2), i.e.,

\[
\mathcal{M} = \{ \Phi(l, N) : N = 1, 2, \ldots ; l = 0, 1, \ldots, N - 1 \}.
\]

For simplicity and for our computational purposes, for fixed \( N \geq 1 \) and \( l \) such that \( 1 \leq l \leq N - 1 \), in the sequel we shall often write \( w := e^{-j2\pi l/N} \). Accordingly, for each \( l = 1, 2, \ldots, N - 1 \) the multiset \( \Phi(l, N) \) defined by (2) can be written as

\[
\Phi(l, N) = \{w, w^2, \ldots, w^N\}.
\]

Furthermore (see \cite{27} Eq. (3)), we have

\[
w + w^2 + \cdots + w^N = 0,
\]

or if we take \( x(n) = e^{-j2\pi nl/N} (n = 1, 2, \ldots N) \), it is equivalent to

\[
x(1) + x(2) + \cdots + x(N) = 0.
\]

Here, as always in the sequel, we will assume that the signal length \( N \) is an arbitrary fixed positive integer. Accordingly, assuming that \( K = 1 \) and \( A_1 = 1 \), for any fixed \( l \in \{1, 2, \ldots, N - 1\} \), in \cite{27} the authors considered a subset \( \Psi(l, N; m) \) of \( \Phi(l, N) \) consisting of \( m \ll N \) randomly positioned available samples (measurements), i.e.,

\[
\Psi(l, N; m) = \{y(1), y(2), \ldots, y(m)\} \subset \Phi(l, N).
\]

Then the random variable corresponding to the DFT over the available set of samples from \( \Phi(l, N) \) is given by

\[
X_l(m, N) =: X_l(m) = \sum_{n=1}^{m} y(n) = \sum_{n=1}^{N} (x(n) + \varepsilon(n)),
\]

where

\[
\varepsilon(n) = \begin{cases} 
0 & \text{for remaining signal samples} \\
-x(n) = - \exp(-2j\pi l/N) & \text{for removed (unavailable) signal samples.}
\end{cases}
\]

Observe that \( X_l(m, N) \) defined by (6) is a complex-valued discrete random variable formed as a sum of \( m \) randomly positioned samples \( y(1), y(2), \ldots, y(m) \in \Psi(l, N; m) \subset \Phi(l, N) \). Let us notice that the theory currently available on compressive sensing predicts that sampling sets chosen uniformly at random among all possible sets of a
given fixed cardinality work well (see, e.g., [7, Chapter 12]). For some variations of this random variable see [26]. If the number \( m \) of randomly positioned available samples (measurements) is not fixed, but randomly chosen (i.e., if the number of terms in the sum (6) is itself a random variable), then the related random variable \( X_l(m, N) \) can be replaced (generalized) with the corresponding the so-called compound random variable. These random variables were firstly systematically studied by W. Feller in his famous book [6]. A combinatorial approach to the introductory study of the compound random variable followed by several examples was given in [12].

Notice that in the above definition of the random variable \( \tilde{X}_l(m, N) \) given by (6), the number of randomly positioned samples, \( m \), is a fixed positive integer such that \( 1 \leq m \leq N \). We believe that in probabilistic study of sparse signal recovery it can be of interest the complex-valued discrete random variable \( \tilde{X}_l(m, N) \) which may be considered as a random analogue (or “free companion” random variable) of the random variable \( X_l(m, N) \), and it is studied and defined in [14] as follows.

**Definition 1.1.** Let \( N, l \) and \( m \) be nonnegative integers such that \( 0 \leq l \leq N - 1 \) and \( 1 \leq m \leq N \). Let \( B_n (n = 1, \ldots, N) \) be a sequence of independent identically distributed Bernoulli random variables (binomial distributions) taking only the values \( 0 \) and \( 1 \) with probability \( 1 - \frac{m}{N} \) and \( \frac{m}{N} \), respectively, i.e.,

\[
B_n = \begin{cases} 
0 & \text{with probability } 1 - \frac{m}{N} \\
1 & \text{with probability } \frac{m}{N}.
\end{cases}
\]

Then the discrete random variable \( \tilde{X}_l(m, N) \) is defined as a sum

\[
\tilde{X}_l(m, N) = \sum_{n=1}^{N} \exp \left( -\frac{2jnl\pi}{N} \right) B_n.
\]

From Definition 1.1 we see that the range of the random variable \( \tilde{X}_l(m, N) \) consists of all possible \( 2^N - 1 \) sums of the elements of (multi)set \( \{ e^{-j2nl\pi/N} : n = 1, 2, \ldots, N \} \).

Observe that for \( l = 0 \) \( \tilde{X}_l(m, N) \) becomes

\[
\tilde{X}_0(m, N) = \sum_{k=1}^{N} B_k \sim B \left( N, \frac{m}{M} \right),
\]

where \( B (N, m/N) \) is the binomial distribution with parameters \( N \) and \( p = m/N \) and the probability mass function given by

\[
\text{Prob} \left( B \left( N, \frac{m}{N} \right) = k \right) = \binom{N}{k} \left( \frac{m}{N} \right)^k \left( 1 - \frac{m}{N} \right)^{N-k}, \quad k = 0, 1, \ldots, N.
\]

Notice also that a Bernoulli probability model, similar to the distribution \( \tilde{X}_l(m, N) \) defined by (9), was often used in the famous paper [1] by Candès, Romberg and Tao. Moreover, the random variables \( \tilde{X}_l(m, N) \) have some similar probabilistic characteristics to those of \( X_l(m, N) \).

Now we return to the random variable \( X_l(m) = X_l(m, N) \) defined by (6). Let \( (\xi_1, \xi_2, \ldots, \xi_N) \) be a \( n \)-tuple of integers \( \xi_n \) which are chosen uniformly at random from the set \( \{0, 1\} \) under the condition that

\[
\xi_1 + \xi_2 + \cdots + \xi_N = m.
\]
Then the discrete random variable defined by (6) and considered in [27, p. 402] can be written as a sum

\[ X_l(m) = \sum_{n=1}^{N} \xi_n \exp \left( -\frac{2jnl\pi}{N} \right). \]

In fact, the above representation means that the sparse signal considered in [27] “comes” from the set of values of the random variable \( X_l(m) \). In view of the above considerations, in the form of its distribution law, \( X_l(m) \) may be defined as follows.

**Definition 1.2.** Let \( N, l \) and \( m \) be arbitrary nonnegative integers such that \( 0 \leq l \leq N - 1 \) and \( 1 \leq m \leq N \). Let \( \Phi(l, N) \in \mathcal{M} \) be a multiset defined as

\[ \Phi(l, N) = \{ e^{-j2nlt\pi/N} : n = 1, 2, \ldots, N \}. \]

Define the discrete complex-valued random variable \( X_l(m, N) = X_l(m) \) as

\[ \text{Prob} \left( X_l(m) = \sum_{i=1}^{m} e^{-j2n_l t_i \pi/N} \right) = \frac{1}{\binom{N}{m}} \cdot \left| \{ \{ t_1, t_2, \ldots, t_m \} \subset \{ 1, 2, \ldots, N \} : \sum_{i=1}^{m} e^{-j2t_i l \pi/N} = \sum_{i=1}^{m} e^{-j2n_i t_l \pi/N} \} \right| \]

where \( \{ n_1, n_2, \ldots, n_m \} \) is an arbitrary fixed subset of \( \{ 1, 2, \ldots, N \} \) such that \( 1 \leq n_1 < n_2 < \cdots < n_m \leq N \); moreover, \( q(n_1, n_2, \ldots, n_m) \) is the cardinality of a collection of all subsets \( \{ t_1, t_2, \ldots, t_m \} \) of the set \( \{ 1, 2, \ldots, N \} \) such that \( \sum_{i=1}^{m} e^{-j2t_i l \pi/N} = \sum_{i=1}^{m} e^{-j2n_i t_l \pi/N} \).

**Remark 1.3.** The above definition is correct in view of the fact that there are \( \binom{N}{m} \) index sets \( T \subset \{ 1, 2, \ldots, N \} \) with \( m \) elements. Notice also that this quantity grows (in some sense) exponentially with \( m \) and \( N \). For a sake of understanding this definition, see Examples 1.4 and 1.5 given below. Moreover, a very short, but not strongly exact version of Definition 1.2 is given as follows.

**Definition 1.2’.** Let \( N, l \) and \( m \) be arbitrary nonnegative integers such that \( 0 \leq l \leq N - 1 \) and \( 1 \leq m \leq N \). Let \( \Phi(l, N) \in \mathcal{M} \) be a multiset defined as

\[ \Phi(l, N) = \{ e^{-j2nlt\pi/N} : n = 1, 2, \ldots, N \}. \]

Choose a random subset \( S \) of size \( m \) (the so-called \( m \)-element subset) without replacement from the set \( \{ 1, 2, \ldots, N \} \). Then the complex-valued discrete random variable \( X_l(m, N) = X_l(m) \) is defined as a sum

\[ X_l(m) = \sum_{n \in S} e^{-j2nlt\pi/N}. \]

**Example 1.4.** Consider the multiset
\[ \Phi(2, 6) = \{ e^{-j2n\pi/3} : n = 1, 2, 3, 4, 5, 6 \}. \]
If for brevity we put \( \varepsilon = e^{-2j\pi/3} = (-1 - j\sqrt{3})/2 \), then obviously \( \Phi(2, 6) \) can be written as
\[
\Phi(2, 6) = \{ \varepsilon, \varepsilon^2, 1, 1 \}.
\]

Then accordingly to Definition 1.2, \( X_1(1) \) is the uniform random variable with
\[
\text{Prob} (X_1(1) = \varepsilon) = \text{Prob} (X_1(1) = \varepsilon^2) = \text{Prob} (X_1(1) = 1) = \frac{2}{6} = \frac{1}{3}.
\]

If we put \( X_1(1) = U + jV \), where \( U \) is the real part and \( V \) is the imaginary part of \( X_1(1) \), then since \( \varepsilon^2 = e^{-4j\pi/3} = (-1 + j\sqrt{3})/2 \), a routine calculation gives the following probability laws of \( U \) and \( V \):
\[
\text{Prob}(U = 1) = \frac{1}{3}, \quad \text{Prob}(U = -\frac{1}{2}) = \frac{2}{3};
\]
\[
\text{Prob}(V = 0) = \text{Prob}(V = \frac{\sqrt{3}}{2}) = \text{Prob}(V = -\frac{\sqrt{3}}{2}) = \frac{1}{3}.
\]

From the above two distribution laws, we immediately obtain the following probability laws of \( U^2, V^2 \) and \( UV \):
\[
\text{Prob}(U^2 = 1) = \frac{1}{3}, \quad \text{Prob}(U^2 = \frac{1}{4}) = \frac{2}{3};
\]
\[
\text{Prob}(V^2 = 0) = \frac{1}{3}, \quad \text{Prob}(V^2 = \frac{3}{4}) = \frac{2}{3},
\]
and
\[
\text{Prob}(UV = 0) = \frac{1}{3}, \quad \text{Prob}(UV = \frac{\sqrt{3}}{2}) = \frac{1}{9}, \quad \text{Prob}(UV = -\frac{\sqrt{3}}{2}) = \frac{1}{9},
\]
\[
\text{Prob}(UV = \frac{\sqrt{3}}{4}) = \frac{2}{9}, \quad \text{Prob}(UV = -\frac{\sqrt{3}}{4}) = \frac{2}{9}.
\]

Generally, if \( X = U + jV \) is a complex-valued random variable, then the expected value of its square is defined as
\[
\mathbb{E}[X^2] = \mathbb{E}[U^2] + \mathbb{E}[V^2] - 2j\mathbb{E}[UV].
\]

This expression together with above derived probability law implies that
\[
\mathbb{E}[(X_1(1))^2] = \left( \frac{1}{3} + \frac{1}{6} \right) + \frac{1}{2} - 2j \cdot 0 = 1.
\]

**Example 1.5.** Consider the set
\[
\Phi(1, 4) = \{ e^{-jn\pi/2} : n = 1, 2, 3, 4 \}.
\]

Since \( e^{-j\pi/2} = -j \), we have
\[
\Phi(1, 4) = \{ 1, -1, j, -j \}.
\]
Then accordingly to Definition 1.2, the probability law of $X_1(2)$ is given by

$$
\begin{align*}
\text{Prob}(X_1(2) = 0) &= \frac{1}{3}, \quad \text{Prob}(X_1(2) = 1 + j) = \text{Prob}(X_1(2) = -1 + j) \\
&= \text{Prob}(X_1(2) = -1 - j) = \text{Prob}(X_1(2) = 1 - j) = \frac{1}{6}.
\end{align*}
$$

If we set $X_1(2) = U + jV$, where $U$ is the real part and $V$ is the imaginary part of $X_1(2)$, then a simple calculation implies that both random variables $U$ and $V$ are uniformly distributed, i.e.,

$$
\begin{align*}
\text{Prob}(U = 0) &= \text{Prob}(U = 1) = \text{Prob}(U = -1) = \frac{1}{3}, \\
\text{Prob}(V = 0) &= \text{Prob}(V = 1) = \text{Prob}(V = -1) = \frac{1}{3}.
\end{align*}
$$

The random variable $(X_1(2))^2$ is also uniformly distributed; namely,

$$
\begin{align*}
\text{Prob}((X_1(2))^2 = 0) &= \text{Prob}((X_1(2))^2 = 2j) = \text{Prob}((X_1(2))^2 = -2j) = \frac{1}{3}.
\end{align*}
$$

Moreover, the distribution laws of $(X_1(2))^3$ and $(X_1(2))^4$ are respectively given as follows:

$$
\begin{align*}
\text{Prob}((X_1(2))^3 = 0) &= \frac{1}{3}, \quad \text{Prob}((X_1(2))^3 = 2 + 2j) = \text{Prob}(X_1(2) = -2 + 2j) \\
&= \text{Prob}((X_1(2))^3 = -2 - 2j) = \text{Prob}(X_1(2) = 2 - 2j) = \frac{1}{6}; \\
\text{Prob}((X_1(2))^4 = 0) &= \frac{1}{3}, \quad \text{Prob}((X_1(2))^4 = 4) = \frac{2}{3}.
\end{align*}
$$

Notice that from the above described distributions it follows that

$$
(16) \quad \mathbb{E}[X_1(2)] = \mathbb{E}[(X_1(2))^2] = \mathbb{E}[(X_1(2))^3] = 0 \quad \text{and} \quad \mathbb{E}[(X_1(2))^4] = \frac{8}{3}.
$$

Moreover, since

$$
\text{Prob}(|X_1(2)| = 0) = 1/3, \quad \text{Prob}(|X_1(2)| = \sqrt{2}) = 2/3, \quad \text{Prob}(|X_1(2)|^2 = 0) = 1/3
$$

and

$$
\text{Prob}(|X_1(2)| = 2) = 2/3, \quad \text{by definition, we obtain that the variance of } X_1(2) \text{ is equal to}
$$

$$
\text{Var}[X_1(2)] = \mathbb{E}[|X_1(2)|^2] - |\mathbb{E}[X_1(2)]|^2 = \frac{4}{3} - 0 = \frac{4}{3}.
$$

Furthermore, using (15), a routine calculation shows that $X_1(3)$ is the uniformly distributed random variable, i.e.,

$$
\begin{align*}
\text{Prob}(X_1(3) = 1) &= \text{Prob}(X_1(3) = -1) = \text{Prob}(X_1(3) = j) \\
&= \text{Prob}(X_1(3) = -j) = \frac{1}{4},
\end{align*}
$$

whence we see that $X_1(3)$ and $X_1(1)$ are equally distributed random variables.

Example 1.5 addresses the following curious question.

**Question 1.6.** Do there exist positive integers $N \geq 5$, $l$ and $m$ such that $2 \leq m \leq N - 2$ and $1 \leq l \leq N - 1$ for which at least one of the following two assertions holds:
(i) the real part $U_l(m, N)$ of the random variable $X_l(m, N)$ is uniformly distributed;
(ii) the imaginary part $V_l(m, N)$ of the random variable $X_l(m, N)$ is uniformly distributed?

Let us now briefly describe the organization of the paper. In Section 2 we give our main results followed by some remarks. Some of these results are also proved or attributed in [27] and extended in [13]. Three examples and related two assertions concerning certain classes of the random variables $X_l(m)$ are presented in Section 3. As applications, some combinatorial congruences are proved. In the last section, we give proofs of the results of Section 2.

2. The main results

The following antisymmetric property of the random variables $X_l(m, N)$, $U_l(m, N)$ and $V_l(m, N)$ should be useful for related computational purposes.

**Proposition 2.1.** Let $N \geq 2$, $l$ and $m$ be nonnegative integers such that $0 \leq l \leq N-1$ and $1 \leq m \leq N-1$. Then the random variables $X_l(m, N)$ and $-X_l(N-m, N)$ are equally distributed. The same assertion holds for the random variables $U_l(m, N)$ and $V_l(m, N)$.

Since $X_l(1, N)$ is the uniform random variable with $\text{Prob}(X_l(1, N) = e^{-j2l\pi/N}) = 1/N$ for every $l \in \{0, 1, \ldots, N-1\}$, it follows from Proposition 2.1 that $X_l(N-1, N)$ is also the uniform random variable with $\text{Prob}(X_l(N-1, N) = -e^{-j2l\pi/N}) = 1/N$ for every $l \in \{0, 1, \ldots, N-1\}$.

Let $X_l(m, N) = U_l(m, N) + jV_l(m, N)$ be a random variable from Definition 1.2, where $U_l(m, N)$ and $V_l(m, N)$ be its real and imaginary part, respectively. Since obviously, the set $\Phi(l, N)$ given by (3) can also be expressed in the form

$$\Phi(l, N) = \{w, w^2, \ldots, w^N\},$$

we immediately have the following result.

**Proposition 2.2.** Let $N \geq 2$, $l$ and $m$ be nonnegative integers such that $0 \leq l \leq N-1$ and $1 \leq m \leq N$. Then the imaginary part $V_l(m, N)$ of the random variable $X_l(m, N)$ is symmetrically distributed around zero (i.e., around the mean of $V_l(m, N)$) in the sense that for each value $x$ of $V_l(m, N)$ there holds

$$\text{Prob}(V_l(m, N) = -x) = \text{Prob}(V_l(m, N) = x).$$

As an immediate consequence of Proposition 2.2, we obtain the following result.

**Corollary 2.3.** Let $k$ be any positive odd integer. Then the $k$th moment $\mu_k[V_l(m, N)]$ of the random variable $V_l(m, N)$ defined above is equal to zero, that is,

$$\mu_k[V_l(m, N)] := \mathbb{E}[(V_l(m, N))^k] = 0.$$
Theorem 2.4. Let \( N \geq 2, l \) and \( m \) be positive integers such that \( 1 \leq l \leq N - 1 \) and \( 1 \leq m \leq N \). Then the expected value and the variance of the random variable \( X_l(m, N) \) from Definition 1.2 are respectively given by

\[
\mathbb{E}[X_l(m, N)] = 0, \tag{18}
\]

and

\[
\text{Var}[X_l(m, N)] = \mathbb{E}[|X_l(m)|^2] = \frac{m(N-m)}{N-1}. \tag{19}
\]

If we put \( X_l(m, N) = U_l(m, N) + jV_l(m, N) \), where \( U_l(m, N) \) is the real part and \( V_l(m, N) \) is the imaginary part of \( X_l(m, N) \), then

\[
\mathbb{E}[U_l(m, N)] = \mathbb{E}[V_l(m, N)] = 0. \tag{20}
\]

If in addition, we suppose that \( 1 \leq l \leq N - 1 \) and \( N \neq 2l \), then

\[
\mathbb{E}[(U_l(m, N))^2] = \mathbb{E}[(V_l(m, N))^2] = \frac{m(N-m)}{2(N-1)}. \tag{21}
\]

As consequences of Theorem 2.4, we can easily obtain the following two results.

Corollary 2.5. Let \( N \geq 2, l \) and \( m \) be positive integers such that \( 1 \leq l \leq N - 1 \) and \( 1 \leq m \leq N \). If we take \( X_l(m, N) = U_l(m, N) + jV_l(m, N) \), then

\[
\mathbb{E}[(X_l(m, N))^2] = \mathbb{E}[(U_l(m, N))^2] + \mathbb{E}[(V_l(m, N))^2] = \text{Var}[X_l(m, N)] = \frac{m(N-m)}{N-1}, \tag{22}
\]

where \( \mathbb{E}[(X_l(m, N))^2] \) is defined by (14).

Corollary 2.6. Let \( N \geq 2, l \) and \( m \) be positive integers such that \( 1 \leq l \leq N - 1 \), \( N \neq 2l \) and \( 1 \leq m \leq N \). If we put \( X_l(m, N) = U_l(m, N) + jV_l(m, N) \), then

\[
\text{Var}[U_l(m, N)] = \text{Var}[V_l(m, N)] = \frac{m(N-m)}{2(N-1)}, \tag{23}
\]

where \( \text{Var}[U_l(m, N)] \) and \( \text{Var}[V_l(m, N)] \) are the variances of \( U_l(m, N) \) and \( V_l(m, N) \), respectively.

Remark 2.7. Notice that the cases \( l = 0 \) and \( N = 2l \) which are excluded from the above three assertions correspond to the real-valued cases \( X_0(m, N) \) and \( X_l(m, 2l) \) of the random variable \( X_l(m, N) \) considered by Examples 3.1 and 3.2, respectively.

From the expression (19) we see that the value \( \text{Var}[X_l(m, N)] \) does not depend on \( l \). We believe that this fact would be important and helpful for some further investigations of certain classes of the random variables \( X_l(m, N) \) and related applications.

Here we also extend the expression (18) of Theorem 2.4 as follows.

Theorem 2.8. Let \( N, l, m \) and \( k \) be positive integers such that \( 1 \leq l \leq N - 1 \) and \( 1 \leq m \leq N \). If \( k \) is not divisible by \( N/\gcd(N, l) \) (\( \gcd(N, l) \) denotes the greatest common divisor of \( N \) and \( l \)), then the \( k \)th moment \( \mu_k \) of the random variable \( X_l(m, N) \) from Definition 1.2 is equal to zero, i.e.,

\[
\mu_k := \mathbb{E}[(X_l(m, N))^k] = 0. \tag{24}
\]
Remark 2.9. Notice that the equality (18) from Theorem 2.4 is a particular case of the equality (24) with \( k = 1 \). However, in Section 4, we give a direct proof of (18).

In view of Theorem 2.8, it remains an open problem to calculate \( \mathbb{E}[(X_l(m, N))^k] \) in the case when \( k \) is divisible by \( N/\gcd(N, l) \). From (16) of Example 1.5 we see that generally, in this case \( \mathbb{E}[(X_l(m, N))^k] \neq 0 \). However, we are able to prove the following itself interesting result.

**Proposition 2.10.** Let \( N, l, m \) and \( k \) be nonnegative integers such that \( 0 \leq l \leq N - 1 \), \( 1 \leq m \leq N \) and \( k \geq 1 \). Then the \( k \)th moment \( \mathbb{E}[(X_l(m, N))^k] \) of the random variable \( X_l(m, N) \) is a real number.

Notice that in Section 4 we give a constructive proof of Proposition 2.10 which is based on Newton’s identities (Newton-Girard formula).

**Remark 2.11.** Let \( A \) be a \( m \times n \) matrix over the field \( \mathbb{C} \) (or \( \mathbb{R} \)) and let \( a_1, \ldots, a_n \in \mathbb{C}^m \) (or \( \in \mathbb{R}^m \)) be its columns. Then the coherence of \( A \) is the number \( \mu(A) = \mu \) defined as

\[
\mu = \max_{1 \leq i \neq j \leq n} \frac{|\langle a_i, a_j \rangle|}{\|a_i\|_2 \cdot \|a_j\|_2}.
\]

It was noticed in [25, p. 159] (also see [22]) that the ratio \( \frac{\sigma[X_l(m, N)]}{m} = \sqrt{\frac{N-m}{m(N-1)}} \) (where \( \sigma[X_l(m, N)] = \sqrt{\text{Var}[X_l(m, N)]} \) with \( \text{Var}[X_l(m, N)] \) given by (19)) is a crucial parameter (Welch bound [32] for coherence \( \mu \) of measurement matrix \( A \)) for corrected signal detection. More precisely (for a particularly elegant and very short proof of this bound see [8]; also see [7, Chapter 5, Theorem 5.7]), the coherence \( \mu \) of a matrix \( A \in \mathbb{K}^{m \times N} \), where the field \( \mathbb{K} \) can either be \( \mathbb{R} \) or \( \mathbb{C} \), with \( l_2 \)-normalized columns satisfies the inequality

\[
\mu \geq \sqrt{\frac{N-m}{m(N-1)}},
\]

which under above notation can be written as

\[
\mu \geq \frac{\sigma[X_l(m, N)]}{m}.
\]

Equality in the above two inequalities holds if and only if the columns \( a_1, \ldots, a_N \) of the matrix \( A \) form an equiangular tight frame. Ideally, the coherence \( \mu \) of a measurement matrix \( A \) should be small (see [7, Chapter 5]).

Let us observe that if \( m \ll N \), then this bound reduces to approximately \( \mu(A) \geq 1/\sqrt{m} \). There is a lot of possible ways to construct matrices with small coherence. Not surprisingly, one possible option is to consider random matrices \( A \) with each entry generated independently at random (cf. [18, Chapter 11]).

**Remark 2.12.** Based on some recent results by R. Vershynin on sub-Gaussian random variables ([30] and [31]), some new results concerning the random variable \( X_l(m, N) \) are obtained in [13]. In particular, this our investigation is motivated by the fact that Restricted Isometry Property (RIP) introduced in [2] holds with high probability for any matrix generated by a sub-Gaussian random variable (see [3] and [20]).
Furthermore, in [15] it was generalized the random variable $X_l(m, N)$. It was also derived the expression for related expected value and variance. By using these expressions, some probabilistic aspects of compressive sensing are considered in the mentioned paper. In particular, motivated by the observation given in [25, p. 159], the connection between Welch bound on the coherence of a particular $m \times N$ matrix $A$ over $\mathbb{C}$ and the variance of the associated random variable (defined in a suitable manner) was established in [15].

3. SOME PARTICULAR CASES OF THE RANDOM VARIABLES $X_l(m, N)$

Here we consider some particular cases of the random variable $X_l(m, N) = X_l(m)$ from Definition 1.2 with different related values $N$, $m$ and $l$. We believe that these examples will be of interest in future research related to the topics of this paper. Firstly, we consider the only two cases when $X_l(m)$ is a real-valued discrete random variable, or equivalently, when the multiset $\Phi(l, N)$ defined by (2) consists of real numbers.

Example 3.1. For $l = 0$ and an arbitrary positive integer $N \geq 1$, the equation (2) yields

$$
\Phi(0, N) = \{1, \ldots, 1\}. 
$$

Then in view of Definition 1.2, for any fixed $m$ with $1 \leq m \leq N$, $X_0(m)$ is the constant random variable with

$$
\text{Prob}(X_0(m) = m) = 1.
$$

Example 3.2. Let $l$ and $N$ be positive integers such that $l/N = 1/2$, i.e., $N = 2l$. Then the equation (2) yields

$$
\Phi(l, 2l) = \{1, \ldots, 1, -1, \ldots, -1\}.
$$

Let $m$ be any positive integer such that $1 \leq m \leq 2l$. Notice that for each nonnegative integer $k$ with $0 \leq m - l \leq k \leq \min\{m, l\}$ from the multiset $\Phi(l, 2l)$ we can choose $k$ 1’s by $\binom{l}{k}$ manners and $(m - k)$ −1’s by $\binom{l}{m-k}$ manners. Since the sum of sums of $k$ 1’s and sums of $m - k$ −1’s is equal to $2k - m$, it follows that the distribution of the random variable $X_l(m)$ from Definition 1.2 is given by

$$
\text{Prob}(X_l(m) = 2k - m) = \frac{\binom{l}{k} \binom{m-k}{l}}{2^l m!} \text{ for each } k = \max\{0, m - l\}, \ldots, \min\{m, l\}.
$$

In particular, if $m = l$, then (28) yields

$$
\text{Prob}(X_l(l) = 2k - l) = \left(\frac{l}{2^l l!}\right)^2 \text{ for each } k = 0, 1, \ldots, l.
$$

Notice that the distribution given by (28) implies the following special case of one of the most useful identities among binomial coefficients, well known as the *Chu-Vandermonde identity* in Combinatorics and Combinatorial Number Theory (see, e.g.,
\[ \min_{k=0}^{\min\{m,l\}} \binom{l}{k} \binom{m}{m-k} = \binom{2l}{m}, \]

whose special case for \( m = l \) is given as
\[ \sum_{k=0}^{l} \binom{l}{k}^2 = \binom{2l}{l}. \]

Furthermore, from (29) it follows that the expected value of \( X_l(m) \) is equal to
\[
\mathbb{E}[X_l(m)] = \sum_{k=\max\{0,m-l\}}^{\min\{m,l\}} (2k - m) \binom{l}{k} \binom{m-k}{(2l-m)} \\
= \frac{1}{(2l-m)} \cdot \sum_{k=\max\{0,m-l\}}^{\min\{m,l\}} (2k - m) \binom{l}{k} \binom{m-k}{m} = \text{(substitution } k = m - t) \\
= - \frac{1}{(2l-m)} \cdot (2t - m) \sum_{t=\max\{0,m-l\}}^{\min\{m,l\}} \binom{m-t}{t} \binom{l}{t} \\
= - \mathbb{E}[X_l(m)],
\]

whence it follows that
\[ \mathbb{E}[X_l(m)] = 0. \]

Notice that from (19) of Theorem 2.4 we obtain that the variance of \( X_l(m) \) is equal to
\[ \text{Var}[X_l(m)] = \frac{m(2l - m)}{2l - 1}. \]

On the other hand, by using (29) and (30), we find that
\[
\text{Var}[X_l(m)] = \mathbb{E}[(X_l(m))^2] - (\mathbb{E}[X_l(m)])^2 \\
= \mathbb{E}[(X_l(m))^2] - \sum_{k=\max\{0,m-l\}}^{\min\{m,l\}} (2k - m)^2 \binom{l}{k} \binom{m-k}{(2l-m)} \\
= \frac{1}{(2l-m)} \cdot \sum_{k=\max\{0,m-l\}}^{\min\{m,l\}} (2k - m)^2 \binom{l}{k} \binom{m-k}{m}.
\]

By comparing the equalities (31) and (32), we obtain the following combinatorial identity:
\[ \sum_{k=\max\{0,m-l\}}^{\min\{m,l\}} (2k - m)^2 \binom{l}{k} \binom{m-k}{m} = m(2l - m) \binom{2l}{m} \binom{2l}{m} \text{ with } 1 \leq m \leq 2l. \]

If we take \( l = m \) into (33), then it becomes
\[ \sum_{k=0}^{m} (2k - m)^2 \binom{m}{k}^2 = \frac{m^2}{2m - 1} \binom{2m}{m}, \]
whence by using the identity $\frac{m^2}{2m-1} \binom{2m}{m} = 2m \binom{2m-2}{m-1}$, we get the following curious combinatorial identity.

**Identity 3.3.** Let $m$ be an arbitrary positive integer. Then
\[
\sum_{k=0}^{m}(2k-m)^2 \binom{m}{k}^2 = 2m \binom{2m-2}{m-1}.
\]

**Remark 3.4.** Similarly as in Example 3.2, we can obtain several combinatorial identities. For example, by determining directly the variance $\text{Var}[U_i(m, 3l)]$ associated to the multiset $\Phi(l, 3l) = \{1, \ldots, \frac{1}{l}, \ldots, -\frac{1}{l}, \ldots, -1/2, \ldots, -1/2\}$ ($1 \leq m \leq 3l$) and using the expression (19) of Theorem 2.4, we arrive at the following identity:
\[
\sum_{\ell=1}^{\min\{m, 2l\}} (2m-3k)^2 \binom{l}{m-k} \binom{2l}{k} = \frac{2m(3l-m)}{3l-1} \binom{3l}{m}.
\]
In particular, for $l = m$ the above congruence becomes
\[
\sum_{k=0}^{m}(2m-3k)^2 \binom{m}{k} \binom{2m}{k} = \frac{4m^2}{3m-1} \binom{3m}{m}.
\]

Similarly, by determining directly the variance $\text{Var}[U_i(m, 6l)]$ associated to the multiset $\Phi(l, 6l) = \{1, \ldots, \frac{1}{l}, \ldots, -\frac{1}{l}, \ldots, -1/2, \ldots, -1/2, \ldots, -1/2\}$ ($1 \leq m \leq 6l$), we obtain the following identity:
\[
\sum_{\sum_{i=1}^{4} m_i = m} (2m_1 - 2m_2 + m_3 - m_4)^2 \binom{l}{m_1} \binom{l}{m_2} \binom{2l}{m_3} \binom{2l}{m_4} = \frac{2m(6l-m)}{(6l-1)} \binom{6l}{m},
\]
where the summation ranges over all nonnegative integers $m_i$ ($i = 1, 2, 3, 4$) such that $\sum_{i=1}^{4} m_i = m$.

**Example 3.5.** Let $N = p$ be any prime number and let $m$ be a positive integer such that $1 \leq m \leq p$. Then for $l = 1$, consider the set consisting of all $p$th roots of the unity. Then if we put $\varepsilon = e^{-2j\pi/p}$, we have
\[
\Phi(1, p) = \{1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{p-1}\}.
\]

Now we will prove that the random variable $X_1(m, p)$ from Definition 1.2 is the uniform random variable whose distribution is given by
\[
\text{Prob} (X_1(m, p) = \varepsilon^{n_1} + \varepsilon^{n_2} + \cdots + \varepsilon^{n_m}) = \frac{1}{\binom{p}{m}},
\]
where $\{n_1, n_2, \ldots, n_m\}$ is any subset of $\{0, 1, 2, \ldots, p-1\}$ such that $0 \leq n_1 < n_2 < \cdots < n_m \leq p-1$. In order to show this fact, for the sake of completeness, we will prove the known fact in Number Theory that for every prime number $p$ the polynomial $P_{p-1}(x)$ defined as
\[
P_{p-1}(x) = 1 + x + \cdots + x^{p-1}, \ x \in \mathbb{R},
\]
is an irreducible polynomial of degree $p - 1$ over the field $\mathbb{Q}$ of rational numbers (or equivalently, in the ring $\mathbb{Z}[x]$ of polynomials with integer coefficients). Namely, since $P_{p-1}(x) = (x^p-1)/(x-1)$ for each $x \neq 1$, then by replacing $x - 1 = y$, i.e., $x = y+1$, and using the binomials expansion, we find that for each $y \neq 0$ there holds

$$P_{p-1}(x) = P_{p-1}(y + 1) = \frac{(y + 1)^p - 1}{y} = \frac{\sum_{k=1}^{p} \binom{p}{k} y^k}{y} = \sum_{k=1}^{p} \binom{p}{k} y^{k-1} = y^{p-1} + \sum_{k=1}^{p-1} \binom{p}{k} y^{k-1}.$$  

Applying the well known classical Eisenstein's irreducibility criterion [5] from Number Theory to the above expression for the polynomial $P_{p-1}(x)$, and using the fact that by Kummer's theorem (see, e.g., [11, Section 2, page 6]), the binomial coefficient $\binom{p}{k}$ is divisible by a prime $p$ for every $k = 1, 2, \ldots, p - 1$, it follows that $P_{p-1}(x)$ is an irreducible polynomial over the field $\mathbb{Q}$ of rational numbers. Hence, the polynomial $P_{p-1}(x)$ given by (35) is the minimal polynomial of its root $\varepsilon = e^{-2j\pi/p}$ over the field $\mathbb{Q}$ of rational numbers.

Now if we suppose that for some two distinct subsets $\{n_1, n_2, \ldots, n_m\}$ and $\{t_1, t_2, \ldots, t_m\}$ of the set $\{0, 1, 2, \ldots, p - 1\}$ there holds

$$\varepsilon^{n_1} + \varepsilon^{n_2} + \cdots + \varepsilon^{n_m} = \varepsilon^{t_1} + \varepsilon^{t_2} + \cdots + \varepsilon^{t_m},$$

then obviously, the above equality can be reduced to the form

$$\sum_{i=0}^{p-1} \alpha_i \varepsilon^i = 0, \quad (36)$$

where the coefficients $\alpha_i \in \{0, -1, 1\}$, at least two $\alpha_i \neq 0$ for some $i \in \{0, 1, 2, \ldots, p - 1\}$ and at least one $\alpha_k = 1$ for some $k \in \{1, 2, \ldots, p - 1\}$. Therefore, in view of the above fact that the polynomial $P_{p-1}(x)$ defined by (35) is the minimal polynomial of $\varepsilon$ over the field $\mathbb{Q}$, we conclude that the expression on the left hand side of (36) is $\neq 0$. A contradiction, and thus for all two distinct subsets $\{n_1, n_2, \ldots, n_m\}$ and $\{t_1, t_2, \ldots, t_m\}$ of the set $\{0, 1, 2, \ldots, p - 1\}$ there holds

$$\varepsilon^{n_1} + \varepsilon^{n_2} + \cdots + \varepsilon^{n_m} \neq \varepsilon^{t_1} + \varepsilon^{t_2} + \cdots + \varepsilon^{t_m}.$$ 

This shows that for every prime number $p$, $X_1(m, p)$ is the uniform random variable with distribution given by (34) and its range consists of $\binom{p}{m}$ elements.

If $l$ is any positive integer such that $l \leq p - 1$, then in view of the fact that $N = p$ is a prime number, we have

$$\Phi(l, p) = \Phi(1, p) = \{1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{p-1}\}.$$ 

This together with the result proved above yields the following assertion.

**Claim 3.6.** Let $N = p$ be any prime number and let $m$ be a positive integer such that $1 \leq m \leq p$. Then $X_1(m), X_2(m), \ldots, X_{p-1}(m)$ are equally distributed uniform random variables whose distribution law is given by (34).
For a given prime number $p$ and a nonnegative integer $k$ consider the measurement row matrix $A$ (the basis function) defined by
\[ A = \left( e^{-2kij\pi/p}, e^{-4kij\pi/p}, \ldots, e^{-2pkij\pi/p} \right). \]

Let $k_0$ and $m$ be nonnegative integers such that $k_0 \neq k$ and $1 \leq m \leq p$ (cf. [27]). Let $\Psi_m$ denote the subset of all vectors (signals) $x \in \mathbb{C}^p$ whose elements are $p$-tuples of the form
\[ (\delta_1 e^{2k_0j\pi/p}, \delta_2 e^{4k_0j\pi/p}, \ldots, \delta_p e^{2pk_0j\pi/p})^T, \]
where $\delta_s \in \{0, 1\}$ for all $s = 1, 2, \ldots, p$ and $\sum_{s=1}^{p} \delta_s = m$. Notice that the condition $\sum_{s=1}^{p} \delta_s = m$ means that every (column) vector $x \in \Psi_m$ has exactly $m$ nonzero coordinates, so that it is $m$-sparse vector. Then from considerations presented in Example 3.5 we immediately get the following assertion.

**Claim 3.7.** Let $y \in \mathbb{C}$ be a complex number such that under above notations and definitions, the equation $Ax = y$ has at least one solution $x_0 \in \Psi_m$. Then this solution is unique in the set $\Psi_m$. In other words, in this case the vector $x_0$ is the unique $m$-sparse solution of $Ax = y$ with $x_0 \in \Psi_m$.

Moreover, the exposition of Example 3.5 obviously yields the following result.

**Claim 3.8.** Under above notations and definitions, consider the equation $Ax = 0$, where $x \in \Psi := \cup_{m=0}^{p-1} \Psi_m$ ($\Psi_0$ denotes the zero vector $(0, 0, \ldots, 0) \in \mathbb{C}^p$). Then the null space $\ker A := \{ x \in \Psi : Ax = 0 \}$ does not contain any vector $x \in \Psi$ other than the zero vector. In other words, the matrix $A$ is injective as a map from $\Psi$ to $\mathbb{C}$.

Of course, the previously proved fact that $X_1(m, p)$ is the uniform random variable does not imply the fact/facts that its real or/and imaginary part is/are also uniformly distributed (cf. Example 1.4).

**Remark 3.9.** The sufficient condition from Example 3.5 that $N = p$ to be a prime number in order that $X_1(m, p)$ to be an uniform random variable for some (and hence for all) $m$ with $2 \leq m \leq N - 1$ is “probably” also necessary condition for this assertion. This fact is suggested by some heuristic arguments and the following examples of the random variables concerning the small composite integer values of $N$ and $m$:

1) $N = 4, m = 2, \varepsilon = j, X_1(2) = \{1, -1, \varepsilon, -\varepsilon\} \Rightarrow 1 - 1 = \varepsilon - \varepsilon = 0$;
2) $N = 6, m = 2; \varepsilon = (-1 + j\sqrt{3})/2, X_1(2) = \{1, -1, \varepsilon, -\varepsilon, \varepsilon^2, -\varepsilon^2\} \Rightarrow 1 + \varepsilon + \varepsilon^2 = -1 - \varepsilon - \varepsilon^2 = 0$;
3) $N = 8, m = 4; \varepsilon = (1 + j)/\sqrt{2}, X_1(4) = \{1, -1, j, -j, \varepsilon, -\varepsilon, \varepsilon, -\varepsilon\} \Rightarrow 1 + (-1) + j + (-j) = \varepsilon + (-\varepsilon) + \varepsilon + (-\varepsilon) = 0$.

Some computations and heuristic arguments suggest the following conjecture.

**Conjecture 3.10.** Let $N \geq 3, l$ and $m$ be positive integers such that $1 \leq l \leq N$ and $2 \leq m \leq N - 1$ and both integers $l$ and $m$ are relatively prime to $N$. Then the random variable $X_l(m, N)$ from Definition 1.2 is uniformly distributed if and only if $N$ is a prime number.

A Number Theory approach to some probabilistic aspects of compressive sensing problems is given in [16].
4. Proofs of the Results

In order to prove Theorem 2.4, we will need the following known identities.

**Lemma 4.1.** Let \( N \) and \( l \) be positive integers such that \( l \leq N - 1 \). Take \( \xi = e^{2jl\pi/N} \). Then

\[
\sum_{k=1}^{N} \xi^k = \sum_{k=1}^{N} \cos \frac{2kl\pi}{N} = \sum_{k=1}^{N} \sin \frac{2kl\pi}{N} = 0.
\]

If in addition, we suppose that \( N \neq 2l \), then

\[
\sum_{k=1}^{N} \cos \frac{4kl\pi}{N} = \sum_{k=1}^{N} \sin \frac{4kl\pi}{N} = 0.
\]

**Proof.**

Take \( \xi = \cos \frac{2l\pi}{N} + j \sin \frac{2l\pi}{N} = e^{2jl\pi/N} \), \( S_1 = \sum_{k=1}^{N} \cos \frac{2kl\pi}{N} \) and \( S_2 = \sum_{k=1}^{N} \sin \frac{2kl\pi}{N} \).

Then by de Moivre’s formula and the equality \( \sum_{k=1}^{N} \xi^k = 0 \), we immediately obtain

\[
S_1 + jS_2 = \sum_{k=1}^{N} \cos \frac{2kl\pi}{N} + j \sum_{k=1}^{N} \sin \frac{2kl\pi}{N} = \sum_{k=1}^{N} \left( \cos \frac{2kl\pi}{N} + j \sin \frac{2kl\pi}{N} \right)
= \sum_{k=1}^{N} \left( \cos \frac{2n\pi}{N} + j \sin \frac{2n\pi}{N} \right) = \sum_{k=1}^{N} \left( \cos \frac{2l\pi}{N} + j \sin \frac{2l\pi}{N} \right)^k
= (\text{since } \xi \neq 1) \sum_{k=1}^{N} \xi^k = \xi \cdot \frac{\xi^N - 1}{\xi - 1} = 0.
\]

The above equality shows that \( S_1 = S_2 = 0 \), which yields (37).

Proceeding in the same manner as above, with the argument \( 4kl\pi/N \) instead of \( 2kl\pi/N \), and using the fact that \( w := \cos \frac{4l\pi}{N} + j \sin \frac{4l\pi}{N} \neq 1 \) (because of the assumption that \( N \neq 2l \)), we obtain both identities of (38).

**Proof of Proposition 2.1.** Proof of Proposition 2.1 immediately follows from Definition 1.2 and the identities given by (37) of Lemma 4.1.

**Proof of Theorem 2.4.** For brevity, take \( w = e^{-j2l\pi/N} \) and \( w_i = w^i \) for every \( i = 1, 2, \ldots, N \). Firstly, we consider the case when \( N \) and \( l \) are relatively prime integers. Then the set \( \Phi(l, N) \) defined by (3) consists of \( N \) distinct elements; namely,

\[
\Phi(l, N) = \{w_1, w_2, \ldots, w_N\}.
\]

Then by Definition 1.2, we have

\[
\mathbb{E}[X_l(m, N)] = \frac{1}{\binom{N}{m}} \sum_{\{i_1, i_2, \ldots, i_m\} \subset \{1, 2, \ldots, N\}} \left( w_{i_1} + w_{i_2} + \cdots + w_{i_m} \right),
\]

where the summation ranges over all subsets \( \{i_1, i_2, \ldots, i_m\} \) of \( \{1, 2, \ldots, N\} \) with \( 1 \leq i_1 < i_2 < \cdots < i_m \leq N \). Since any fixed \( w_s \) with \( s \in \{1, 2, \ldots, N\} \) occurs
exactly \( \binom{N-1}{m-1} \) times in the sum on the right hand side of (40), and using the fact that
\[
\sum_{i=1}^{N} w_i = 0,
\]
we find that
\[
\mathbb{E}[X_l(m, N)] = \frac{1}{\binom{N}{m}} \left( \binom{N-1}{m-1} w_1 + \binom{N-1}{m-1} w_2 + \cdots + \binom{N-1}{m-1} w_N \right)
\]
(41)
\[
= \frac{\binom{N-1}{m-1}}{\binom{N}{m}} (w_1 + w_2 + \cdots + w_N) = 0,
\]
which implies (18). Both equalities from (20) immediately follow from (18) in view of the fact that
\[
\mathbb{E}[X_l(m, N)] = \mathbb{E}[U_l(m, N)] + j\mathbb{E}[V_l(m, N)].
\]

If \( m = 1 \), then clearly, \( X_l(1, N) \) is the uniform random variable with \( \text{Prob}(X_l(1, N) = w_i) = 1/N \) for each \( i = 1, 2, \ldots, N \), and so \( |X_l(1, N)| \) is the constant random variable with \( \text{Prob}(|X_l(1, N)| = 1) = 1 \). Then since by (41) \( \mathbb{E}[X_l(m, N)] = 0 \), we have
\[
\text{Var}[X_l(m)] = \mathbb{E}[|X_l(m)|^2] - |\mathbb{E}[X_l(m)]|^2 = 1.
\]

The above expression coincides with the expression (19) for \( m = 1 \).

Now suppose that \( m \geq 2 \). Then we have
\[
\mathbb{E}[|X_l(m)|^2] = \frac{1}{\binom{N}{m}} \sum_{\{i_1, i_2, \ldots, i_m\} \subseteq \{1, 2, \ldots, N\}} (w_{i_1} + w_{i_2} + \cdots + w_{i_m})(w_{i_1} + w_{i_2} + \cdots + w_{i_m}),
\]
(42)

where the summation ranges over all subsets \( \{i_1, i_2, \ldots, i_m\} \) of \( \{1, 2, \ldots, N\} \) with \( 1 \leq i_1 < i_2 < \cdots < i_m \leq N \). Notice that after multiplication of terms on the right hand side of (42) we obtain that in the obtained sum every factor of the form \( w_i w_i = |w_i|^2 \) \( (i = 1, 2, \ldots, N) \) occurs exactly \( \binom{N-1}{m-1} \) times, while every factor of the form \( w_i \bar{w}_s \) with \( 1 \leq t < s \leq N \), occurs exactly \( \binom{N-2}{m-2} \) times. Accordingly, the equality (42) becomes
\[
\mathbb{E}[|X_l(m)|^2] = \frac{1}{\binom{N}{m}} \left( \binom{N-1}{m-1} \sum_{i=1}^{N} |w_i|^2 + \binom{N-2}{m-2} \sum_{1 \leq t < s \leq N} w_t \bar{w}_s \right),
\]
(43)
whence by using the Pascal’s formula \( \binom{N-1}{m-1} = \binom{N-2}{m-2} + \binom{N-2}{m-1} \), the facts that \( |w_i| = 1 \) \((i = 1, 2, \ldots, N)\), \( \sum_{i=1}^{N} w_i = 0 \) and the identity \( \binom{N}{m} = \frac{N(N-1)(N-2) \cdots (N-m+1)}{m(m-1) \cdots (m-(m-1))} \), we obtain

\[
\mathbb{E}[|X_i(m)|^2] = \frac{1}{\binom{N}{m}} \left( \binom{N-2}{m-1} \sum_{i=1}^{N} |w_i|^2 \right. \\
+ \left. \binom{N-2}{m-2} \sum_{i=1}^{N} |w_i|^2 + \binom{N-2}{m-2} \sum_{1 \leq t < s \leq N} w_tw_s \right)
\]

\[
= \frac{1}{\binom{N}{m}} \left( \binom{N-2}{m-1} \sum_{i=1}^{N} |w_i|^2 + \binom{N-2}{m-2} \sum_{1 \leq t < s \leq N} w_t w_s \right)
\]

\[
= \frac{1}{\binom{N}{m}} \left( \binom{N-2}{m-1} \sum_{i=1}^{N} |w_i|^2 + \binom{N-2}{m-2} \sum_{i=1}^{N} w_i \sum_{s=1}^{N} w_i \right)
\]

\[
= \frac{N(N-2)}{\binom{N}{m}} \sum_{k=1}^{N} \cos \frac{2kl\pi}{N} = \frac{N(N-2)}{\binom{N}{m}} \left( \sum_{k=1}^{N} \cos \frac{2kl\pi}{N} \right)^2
\]

\[
= \frac{N(N-2)}{\binom{N}{m}} \sum_{k=1}^{N} \cos \frac{2kl\pi}{N} = \frac{N(N-2)}{\binom{N}{m}} \left( \sum_{k=1}^{N} \cos \frac{4kl\pi}{N} \right) \cdot \frac{N}{2} = \frac{N(N-2)}{\binom{N}{m}} \left( \sum_{k=1}^{N} \cos \frac{4kl\pi}{N} \right) \cdot \frac{N}{2}
\]

\[
= \frac{m(N-m)}{2(N-1)}.
\]

From the above expression and (18) we have

\[
\text{Var}[X_i(m)] = \mathbb{E}[|X_i(m)|^2] = \frac{m(N-m)}{N-1}.
\]

This proves the expression (19).

It remains to prove the expressions (20) and (21). Since \( w = e^{-j2\pi/N} \), we have that the real and imaginary part of \( w^k \) are respectively equal to \( \Re(w^k) = \cos \frac{2kl\pi}{N} \) and \( \Im(w^k) = -\sin \frac{2kl\pi}{N} \) for every \( k = 1, 2, \ldots, N \). Then by using the same argument applied in the proof of (19) and the assumptions that \( 1 \leq l \leq N - 1 \) and \( N \neq 2l \), we obtain the following analogous equality to (43):

\[
\mathbb{E}[(U_i(m))^2] = \frac{1}{\binom{N}{m}} \left( \binom{N-2}{m-1} \sum_{k=1}^{N} \cos^2 \frac{2kl\pi}{N} \right.
\]

\[
+ \left. \binom{N-2}{m-2} \sum_{k=1}^{N} \cos \frac{2kl\pi}{N} + \binom{N-2}{m-2} \sum_{1 \leq t < s \leq N} \cos \frac{2tl\pi}{N} \cos \frac{2sl\pi}{N} \right)
\]

\[
= \frac{1}{\binom{N}{m}} \left( \binom{N-2}{m-1} \sum_{k=1}^{N} \cos \frac{2kl\pi}{N} + \binom{N-2}{m-2} \sum_{1 \leq t < s \leq N} \cos \frac{2tl\pi}{N} \cos \frac{2sl\pi}{N} \right)
\]

\[
= \frac{N(N-2)}{\binom{N}{m}} \sum_{k=1}^{N} \cos \frac{2kl\pi}{N} = \frac{N(N-2)}{\binom{N}{m}} \left( \sum_{k=1}^{N} \cos \frac{4kl\pi}{N} \right)
\]

\[
= \frac{N(N-2)}{\binom{N}{m}} \left( \frac{N}{2} + \sum_{k=1}^{N} \cos \frac{4kl\pi}{N} \right) = \frac{N(N-2)}{\binom{N}{m}} \cdot \frac{N}{2} = \frac{N(N-2)}{\binom{N}{m}} \left( \sum_{k=1}^{N} \cos \frac{4kl\pi}{N} \right) \cdot \frac{N}{2}
\]

\[
= \frac{m(N-m)}{2(N-1)}.
\]
This proves the first equality of (20). Using this equality, (44) and the equality
\[ \mathbb{E}[|X_i(m)|^2] = \mathbb{E}[(U_i(m))^2] + \mathbb{E}[(V_i(m))^2], \]
we obtain
\[ \mathbb{E}[(V_i(m))^2] = \frac{m(N - m)}{2(N - 1)}, \]
which together with (45) implies (21). This completes proof of Theorem 2.4.

**Proof of Corollary 2.5.** In order to prove Corollary 2.5, observe that by (21) of Theorem 2.4, we have
\[ \mathbb{E}[(X_i(m))^2] = \mathbb{E}[(U_i(m))^2] + \mathbb{E}[(V_i(m))^2] - 2j\mathbb{E}[U_i(m)V_i(m)] \]
\[ = \frac{m(N - m)}{N - 1} - 2j\mathbb{E}[U_i(m)V_i(m)]. \]
Therefore, the equalities (22) are equivalent to the following one:
\[ (46) \quad \mathbb{E}[U_i(m)V_i(m)] = 0. \]
Observe that
\[ \mathbb{E}[U_i(m)V_i(m)] \]
\[ = \frac{1}{\binom{N}{m}} \sum_{\substack{(k_1, k_2, \ldots, k_m) \subseteq \{1, 2, \ldots, n\} \setminus \{s_1, s_2, \ldots, s_m\} \subseteq \{1, 2, \ldots, n\} \atop (s_1, s_2, \ldots, s_m) \subseteq \{1, 2, \ldots, n\}}} \left( \cos \frac{2k_1 l \pi}{N} + \cos \frac{2k_2 l \pi}{N} + \cdots + \cos \frac{2k_m l \pi}{N} \right) \times \left( \sin \frac{2s_1 l \pi}{N} + \sin \frac{2s_2 l \pi}{N} + \cdots + \sin \frac{2s_m l \pi}{N} \right), \]
where the summation ranges over all subsets \( \{k_1, k_2, \ldots, k_m\} \) and \( \{s_1, s_2, \ldots, s_m\} \) of \( \{1, 2, \ldots, n\} \) with \( 1 \leq k_1 < k_2 < \cdots < k_m \leq N \) and \( 1 \leq s_1 < s_2 < \cdots < s_m \leq N \). After multiplication of terms on the right hand side of (47) we obtain that in the obtained sum every factor of the form \( \cos \frac{2k j l \pi}{N} \sin \frac{2s j l \pi}{N} \) (\( k, s = 1, 2, \ldots, N \)) occurs exactly \( \binom{N-1}{m-1}^2 \) times in related sum. Therefore, by using the trigonometric identity \( \cos \alpha \sin \beta = (\sin(\alpha + \beta) + \sin(\beta - \alpha))/2 \) and the identity (37) of Lemma 4.1, we have
\[ \mathbb{E}[U_i(m)V_i(m)] = \frac{1}{\binom{N}{m}} \left( \binom{N-1}{m-1}^2 \sum_{k=1}^{N} \sum_{s=1}^{N} \cos \frac{2k l \pi}{N} \sin \frac{2s l \pi}{N} \right) \]
\[ = \frac{1}{2 \binom{N}{m}} \left( \binom{N-1}{m-1}^2 \sum_{k=1}^{N} \sum_{s=1}^{N} \left( \sin \frac{2(k+s) l \pi}{N} + \sin \frac{2(s-k) l \pi}{N} \right) \right) \]
\[ = \frac{1}{2 \binom{N}{m}} \left( \sum_{k=1}^{N} \sum_{s=1}^{N} \sin \frac{2(k+s) l \pi}{N} + \sum_{k=1}^{N} \sum_{s=1}^{N} \sin \frac{2(s-k) l \pi}{N} \right) \]
(because of the periodicity of the function \( \sin x \))
\[ = \frac{1}{2 \binom{N}{m}} \left( N \cdot \sum_{s=1}^{N} \sin \frac{2s l \pi}{N} + N \cdot \sum_{s=1}^{N} \sin \frac{2s l \pi}{N} \right) = 0. \]
Hence, the equality (46) holds and the proof of the corollary is completed. \( \square \)
Proof of Corollary 2.6. Both equalities given by (23) immediately follow from the expressions (20) and (21) of Theorem 2.4, taking into account that \( N \) can be written as

\[
\exp(\frac{i kl \pi}{N}) = 1 \quad \text{for} \quad k \text{ not divisible by } \frac{N}{\gcd(N,l)},
\]

whence taking

\[
\exp(\frac{2 i kl \pi}{N}) = -j \sin(\frac{2 kl \pi}{N}) = 1.
\]

In view of this fact, the above equality yields

\[
\mathbb{E}[(X_l(m))^k] = 0,
\]

as desired. \( \square \)

Proof of Theorem 2.8. Take \( w = e^{-j2l\pi}/N \). Then the multiset \( \Phi(l,N) \) defined by (3) can be written as

\[
\Phi(l,N) = \{1, w, w^2, \ldots, w^{N-1}\}.
\]

Notice that by Definition 1.2, the random variable \( X_l(m) \) is “uniformly” defined on the set \( \Sigma_m \) of all \( m \)-element sums of \( \Phi(l,N) \), i.e., on the set consisting of all sums formed of some \( m \) elements of the set \( \Phi(l,N) \). Therefore, the random variable \( (X_l(m))^k \) is “uniformly” defined on the set

\[
S_k := \{(w^{i_1} + w^{i_2} + \cdots + w^{i_m})^k : 0 \leq i_1 < i_2 < \cdots < i_m \leq N - 1\}.
\]

Notice that the set \( S_k \) is invariant under multiplication by \( w^k \), i.e., there holds

\[
w^k S_k := \{w^k z : z \in S_k \} = S_k.
\]

Accordingly, and taking into account that the random variable \( X_l(m) \) is “uniformly” defined on the set \( \Sigma_m \) in the sense that \( \text{Prob}(X_l(m) = z) = 1/\binom{N}{m} \) for each \( z \in \Sigma_m \), we conclude that the random variables \( (X_l(m))^k \) and \( w^k (X_l(m))^k \) have the same distribution. Therefore, we have

\[
\mathbb{E}[(X_l(m))^k] = \mathbb{E}[w^k (X_l(m))^k],
\]

whence taking \( \mathbb{E}[w^k (X_l(m))^k] = w^k \mathbb{E}[(X_l(m))^k] \), we obtain

\[
(1 - w^k) \mathbb{E}[(X_l(m))^k] = 0.
\]

Since by the assumption of the theorem, \( k \) is not divisible by \( N/\gcd(N,l) \), it follows that \( kl/N \) is not an integer and thus, \( w^k = \cos(\frac{2k\pi}{N}) - j \sin(\frac{2k\pi}{N}) \neq 1 \). In view of this fact, the above equality yields

\[
\mathbb{E}[(X_l(m))^k] = 0,
\]

as desired. \( \square \)

Proof of Proposition 2.10. For brevity, take \( w = e^{-2j\pi}/N \). First notice that the assertion holds for \( l = 0 \) since \( X_0(m) \) is the constant random variable such that

\[
\text{Prob}(X_0(m) = m) = 1.
\]

Now suppose that \( 1 \leq l \leq N - 1 \). By definition of \( \mathbb{E}[(X_l(m))^k] \) and using the additive property for the expectation, we find that

\[
\mathbb{E}[(X_l(m))^k] = \frac{1}{\binom{N}{m}} \sum_{\{i_1, i_2, \ldots, i_m\} \subset \{1,2,\ldots,N\}} (w^{i_1} + w^{i_2} + \cdots + w^{i_m})^k,
\]

where the summation ranges over all subsets \( \{i_1, i_2, \ldots, i_m\} \) of \( \{1,2,\ldots,N\} \) with \( 1 \leq i_1 < i_2 < \cdots < i_m \leq N \). Consider the polynomial \( P_k(x_1, \ldots, x_N) \in \mathbb{R}[x_1, \ldots, x_N] \) of \( N \) real variables \( x_1, \ldots, x_N \) defined as

\[
P_k(x_1, \ldots, x_N) = \frac{1}{\binom{N}{m}} \sum_{\{i_1, i_2, \ldots, i_m\} \subset \{1,2,\ldots,N\}} (x_{i_1} + x_{i_2} + \cdots + x_{i_m})^k,
\]
where the summation ranges over all subsets \( \{i_1, i_2, \ldots, i_m\} \) of \( \{1, 2, \ldots, N\} \) with \( 1 \leq i_1 < i_2 < \cdots < i_m \leq N \). Clearly, \( P_k \) is a homogeneous symmetric polynomial of degree \( k \). Let us recall that a polynomial in \( n \) real (or complex) variables, \( P(x_1, \ldots, x_n) \in \mathbb{R}[x_1, \ldots, x_n] \) (or \( P(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n] \)) is known as a symmetric polynomial if for any permutation \( \sigma \) of the set \( \{1, 2, \ldots, n\} \), \( P(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = P(x_1, \ldots, x_n) \).

Then by fundamental theorem of symmetric functions, the polynomial \( P_k \) defined by (49) can be expressed as a polynomial in the elementary symmetric polynomials on the variables \( x_1, \ldots, x_N \), i.e.,

\[
P_k(x_1, \ldots, x_N) = Q_k(\sigma_1, \ldots, \sigma_N),
\]

where \( Q_k \) is a polynomial in \( \mathbb{R}[x_1, \ldots, x_N] \) and \( \sigma_s \) \( (s = 1, \ldots, N) \) are elementary symmetric polynomials in \( \mathbb{R}[x_1, \ldots, x_N] \) defined as

\[
\sigma_s(x_1, \ldots, x_N) = \sum_{\{i_1, i_2, \ldots, i_s\} \subseteq \{1, 2, \ldots, N\}} x_{i_1} \cdots x_{i_s},
\]

where the summation ranges over all subsets \( \{i_1, \ldots, i_s\} \) of \( \{1, \ldots, N\} \) with \( 1 \leq i_1 < \cdots < i_s \leq N \). The \( N \)th power sum (or the \( N \)th power symmetric function) \( p_n \in \mathbb{R}[x_1, \ldots, x_N] \) is defined as

\[
p_n(x_1, \ldots, x_N) = \sum_{i=1}^{N} x_i^n.
\]

Then by Newton’s identities (also known as the Newton-Girard formula; see, e.g., [9]; cf. [10] Lemma 2.1), for all \( n \geq 1 \) there holds

\[
p_n(x_1, \ldots, x_N) = (-1)^{n-1} n \sigma_n(x_1, \ldots, x_N) + \sum_{i=1}^{n-1} \sigma_{n-i}(x_1, \ldots, x_N)p_i(x_1, \ldots, x_N).
\]

Let us recall that the formulae (49), (50) and (51) are also valid for the complex values \( x_k = w^k \) \((k = 1, \ldots, N)\). Accordingly, for all \( n \in \mathbb{N} \) we have

\[
p_n(\omega, \ldots, \omega^N) = \sum_{k=1}^{N} \omega^{kn} = \begin{cases} 
0 & \text{if } n \text{ is not divisible by } \frac{N}{\gcd(N,l)} \\
N & \text{if } n \text{ is divisible by } \frac{N}{\gcd(N,l)}.
\end{cases}
\]

We will prove by induction on \( n \geq 1 \) that \( \sigma_n(\omega, \ldots, \omega^N) \) is a real number for all \( n \in \mathbb{N} \). For \( n = 1 \) we have

\[
\sigma_1(\omega, \ldots, \omega^N) = \sum_{t=1}^{N} \omega^t = 0,
\]

and thus, the induction base holds. Suppose that \( \sigma_i(\omega, \ldots, \omega^N) \) is a real number for each \( i \geq 1 \) less than \( n \). Then by the identity (51), we have

\[
\sigma_n(x_1, \ldots, x_N)
\]

\[
= (-1)^{n-1} n \left( p_n(x_1, \ldots, x_N) - \sum_{i=1}^{n-1} \sigma_{n-i}(x_1, \ldots, x_N)p_i(x_1, \ldots, x_N) \right).
\]
The above formula with \((w, \ldots, w_N)\) instead of \((x_1, \ldots, x_N)\) together with the equalities (52) and the induction hypothesis implies that \(\sigma_n(w, \ldots, w_N)\) is a real number, which finishes the induction proof. Hence, if we substitute \(x_k = w^k (k = 1, 2, \ldots, N)\) into (49) and (50) and \(a_k = \sigma_k(w, \ldots, w_N)\) into (50) \((k = 1, \ldots, N)\), and comparing then (48) and (49), we immediately obtain

\[
\mathbb{E}[(X_l(m))^k] = Q_k(a_1, \ldots, a_N).
\]

Since \(a_1, \ldots, a_N\) and the all coefficients of the polynomial \(Q_k\) are real numbers, we conclude that \(Q_k(a_1, \ldots, a_N)\) is also a real number. Therefore, from the above equality it follows that \(\mathbb{E}[(X_l(m))^k]\) is a real number. This completes proof of the proposition. \(\square\)

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**Maritime Faculty Kotor, University of Montenegro, 85330 Kotor, Montenegro**

**E-mail address:** romeo@ac.me