Mathematical model of the mechanism of the "alpha" type Stirling engine for operation in the timber industry complex

Evgenii Tihonov 1,*, Valentin Bazykin 2, Olga Gerasimova 3, and Sergey Soloviev 3

1Petrosavodsk state university, 185000, Lenina st. 33, Petrozavodsk, Russia
2Institute of agroengineering and environmental problems of agricultural production, 196625, Filrovskoe highway 3, Tyarlevo settlement, St. Petersburg, Russia
3Ivanovo state agricultural academy named after Dmitry Belyaev, 153000, Sovetskaya st. 45, Ivanovo, Russia

Abstract. This article discusses the multi-criteria parameterization of the crank-slider group of the Stirling engine type "alpha". The dependences of the displacement of the piston and the displacer are obtained, taking into account a number of variable geometric parameters. The resulting equation of movement of the piston from displacement of the propellant is composed of a crankshaft angle, which will allow to obtain a numerical model simulation of the crank-slider group that will significantly reduce the time of numerical calculations. The dependence of the torque on the crankshaft with known pressures on the piston and the displacer is obtained. This study will eventually allow us to determine the optimal parameters of the designed steering engine, depending on the power and operating temperatures. This will allow us to design an engine for use in the timber industry (use of logging and woodworking waste as fuel). Effective use for energy generation is an urgent task of the forest complex. Currently, obtained torque equation will allow us to study the working process of the engine when solving the gas-dynamic problem of cyclic flow of gas from the cavity of the piston cylinder to the cavity of the displacer cylinder and back.

1 Introduction

Currently, interest in external combustion engines is growing again. The reason for this is that hydrocarbon fuel is not an inexhaustible resource. The price of traditional energy is growing. But, at the same time, the question of obtaining electricity is becoming more and more urgent.

This is especially true for remote areas with little developed infrastructure. Now the issue of their power supply is usually solved by diesel generators. This solution is expensive both by itself (0.2 kg of diesel fuel is burned per 1 kW*hour [1]) and if we taking into account the cost of fuel delivery.

* Corresponding author: tihonov@petrsu.karelia.ru
At the same time, there are many other types of energy resources in remote areas. Coal, firewood, peat, etc. These resources are suitable as a source of energy for an engine running on the Stirling principle.

Stirling engines (hereinafter - DE) have been known for a long time [2], their development and improvement were based on the basics of thermodynamics and a huge amount of experimental research [3]. Therefore, these engines could not sustain the pace of development of internal combustion engines that have a well-developed theoretical base.

The study of processes occurring in the DE is limited to the study of individual elements and patterns. For example, the "Shuttle effect" in the "displacement piston-cylinder" system [4], heat flows of the "working body-regenerator" system when changing the direction of gas movement, the effect of "dead volume" on efficiency and specific power [5], etc.

A comprehensive review of the DE workflow and the development of a General theory remains an unsolved task. To solve it, it is necessary to apply numerical methods for solving complex problems [6]. Complex problems are understood as problems of hydro-gas dynamics, heat transfer, and mechanics of a deformable body in a single numerical model of the mechanism [7].

With the right approach, it will be possible to obtain optimal DE parameters and obtain an efficient energy machine that implements the Stirling cycle without solving such problems as, for example, sealing the working fluid [8].

Force parameterization of the crank-slider group of Stirling engine type "alpha" considered in [9], and kinematic parameterization of the crank-slider group of Stirling engine type alpha, with calculation schemes of mutual arrangement of elements of the engine and torque depending on the crank angle is considered in [10].

The purpose of this work is to obtain the dependences of the piston and displacer movements, taking into account a number of variable geometric parameters. Determine the boundary conditions for the ratio of the lengths of the connecting rods of the piston and the displacer, taking into account the displacement of the cylinder axes in a plane perpendicular to the axis of the crankshaft.

2 Materials and methods

The purpose of the theoretical study of this article is to determine the kinematic relationship between the relative position of the piston and the displacement of the DE type "alpha" at different values of geometric parameters and determine the dependence of the torque on the crankshaft from the angle of rotation and pressure on the piston and the displacement. These equations will allow us to study the working process of the engine when solving the gas dynamic problem of cyclic flow of gas from the piston cavity to the displacer cavity and back. The engine diagram is shown in Fig. 1.
The first problem statement is as follows: the movement of the vertical piston (Y) is
given by a harmonic function. Since the only gas-dynamic problem is being studied at
this stage, there is no need to model the mechanical part. Further, it is necessary to deduce the
dependence of the position of the displacer on the position of the piston X(Y), which will
allow further to investigate the mutual change in the volumes of hot and cold cavities and
the dynamics of gas flow. At the same time, it is necessary to consider the possibility of
changing the angle between the axes of the cylinders (see figure 1). Also, it is necessary to
provide offset for the displacement of the cylinder axes.

The statement of the second part of the problem is as follows: the position of the
 crankshaft will be determined by the angle β. The pressure of the "working fluid" in the
system will be determined by the forces acting on the piston and the displacer. It is
necessary to determine the function of the torque from the angle β depending on the
geometric parameters of the crank-slider group. The mutual restrictions of the geometric
parameters of the crank-slider group are defined in [9].

3 Results

As can be seen from figure 1, the angle between the piston axes is 90°. This parameter is
considered optimal for obtaining the maximum specific power. The system of equations of
coordinates of movements of vertical and horizontal pistons depending on the angle of
rotation of the crankshaft will have the following form:

\[ (1) \]

Next, you need to define the form of the equation X(Y). To do this, define the
coordinate of the horizontal piston X in the form (see figure 1):
Next, consider the parameter \( b \). This parameter can be defined as follows:

\[
\begin{align*}
\frac{a^2}{1} &= k^2 - a^2
\end{align*}
\]  

and the 2nd equation of the system (4) we get the following:

\[
\begin{align*}
b &= \sqrt{l^2 - (k^2 - a^2)} = \sqrt{l^2 + k^2 + a^2}
\end{align*}
\]  

Given 1 equation of the system (4), we obtain:

\[
\begin{align*}
b &= \sqrt{l^2 + k^2 + (Y - b)^2} = \sqrt{l^2 + k^2 + Y^2 - 2Yb + b^2}
\end{align*}
\]  

Let's square the resulting equation and Express \( b \):

\[
\begin{align*}
b &= \frac{l^2 + k^2 + Y^2}{2Y}
\end{align*}
\]  

Substitute the resulting expression in equation (2):

\[
\begin{align*}
\text{(8)}
\end{align*}
\]  

Consider the subtractible of the subcortical difference. Simplify it:

\[
\begin{align*}
\text{(9)}
\end{align*}
\]  

Finally, we have the form of an equation describing the relationship between the coordinates of the horizontal piston and the coordinates of the vertical one:

\[
\begin{align*}
\text{(10)}
\end{align*}
\]  

Now we need to remove the restrictions of this model: the same length of connecting rods, the angle between the cylinder axes is equal to 90°, the absence of offset of the
cylinder axes in the plane perpendicular to the axis of rotation of the crankshaft. We will introduce separate variables to account for the lengths of the connecting rods of the piston and the displacer:

\[
(11)
\]

where \( L_Y \) and \( L_X \) are the lengths of the displacer and piston connecting rods, respectively.

After performing the same transformations as above, we obtain an equation of the position of the displacer (X) from the position of the piston (Y), taking into account the different lengths of the connecting rods:

\[
(12)
\]

Next, consider the possibility of changing the angle between the axes of the cylinders. The diagram is shown in figure 2.

**Fig. 2.** DE scheme with a free angle between the cylinder axes and different lengths of connecting rods

Determine the position of the piston (Y) by rotating the coordinate axes. Then, the rotation angle of the crankshaft to calculate the coordinates of the position of the piston is determined by the sum of the angle position of the crank from the vertical and the rotation axis of the cylinder of the piston from the vertical:

\[
(13)
\]
Then, the angle between the cylinder axes:

\[ \text{(14)} \]

Converting the system (11):

\[ \text{(15)} \]

Next, you need to define the form of the equation \( X(Y) \). To do this, define the coordinate of the horizontal piston \( X \) in the form (see figure 2):

\[ X = \sqrt{k^2 - a'^2} + \sqrt{L_y^2 - a'^2} \]

\[ \text{(16)} \]

At the same time, consider the following:

\[ a' = k \cdot \cos(\alpha + \beta) \]
\[ a = k \cdot \cos(\beta) \]
\[ k = \frac{a'}{\cos(\alpha + \beta)} = \frac{a}{\cos(\beta)} \]

\[ \text{(17)} \]

Then:

\[ a = \frac{a' \cos(\beta)}{\cos(\alpha + \beta)} \]

\[ \text{(18)} \]

In this case, the parameter \( a' \) will be equal to:

\[ a' = Y - b' \]

\[ \text{(19)} \]

Thus, substituting (18) and (19) in dependence (16) we get:

\[ X = \sqrt{k^2 - \left(\frac{Y - b'}{\cos(\alpha + \beta)}\right)^2} + \sqrt{L_y^2 - \left(\frac{Y - b'}{\cos(\alpha + \beta)}\right)^2} \]

\[ \text{(20)} \]

Next, consider the parameter \( b' \). This parameter can be defined by the length of the piston rod \( L_y \) and the angle of the piston rod to the piston travel axis \( a_y \) (see figure 3):
Fig. 3. Scheme to define $b'$. 

At the same time:

$$\sin(\alpha_y) = \frac{k'}{L_y} = \frac{k \cdot \sin(\alpha + \beta)}{L_y}$$  \hspace{1cm} (22)

From here:

$$\alpha_y = \arcsin\left(\frac{k \cdot \sin(\alpha + \beta)}{L_y}\right)$$  \hspace{1cm} (23)

Substituting the dependence (23) with the formula (21), we get the following:

$$b' = L_y \cdot \cos(\arcsin\left(\frac{k \cdot \sin(\alpha + \beta)}{L_y}\right))$$  \hspace{1cm} (24)

Substituting (24) in (20) we get:
Thus, we obtained the dependence of the position of the displacer on the position of the piston.

Substituting the first equation of system (15) to (25), we finally have the equation that describes the relationship of the coordinates of the piston and displacer in dependence on the crankshaft angle, lengths of connecting rods and crankpin offset:

\[
X = \sqrt{k^2 - \left( \frac{(Y - L_y \cdot \cos(\arcsin\left(\frac{k \cdot \sin(\alpha + \beta)}{L_y}\right)) \cos(\beta)}{\cos(\alpha + \beta)} \right)^2 + L_x^2 - \left( \frac{(Y - L_y \cdot \cos(\arcsin\left(\frac{k \cdot \sin(\alpha + \beta)}{L_y}\right)) \cos(\beta)}{\cos(\alpha + \beta)} \right)^2}
\] (25)

Next, we consider the possibility of displacement of the cylinder axes of the piston and the displacer from the initial position in a plane perpendicular to the axis of the crankshaft. The scheme for the definition is shown in figure 4.

\[
X = \sqrt{k^2 - \left( \frac{(k \cdot \cos(\beta + \alpha) + \sqrt{E_x^2 - (k \cdot \sin(\beta + \alpha))^2} - L_y \cdot \cos(\arcsin\left(\frac{k \cdot \sin(\alpha + \beta)}{L_y}\right)) \cos(\beta)}{\cos(\alpha + \beta)} \right)^2 + L_x^2 - \left( \frac{(k \cdot \cos(\beta + \alpha) + \sqrt{E_x^2 - (k \cdot \sin(\beta + \alpha))^2} - L_y \cdot \cos(\arcsin\left(\frac{k \cdot \sin(\alpha + \beta)}{L_y}\right)) \cos(\beta)}{\cos(\alpha + \beta)} \right)^2}
\] (26)

Fig. 4. The scheme of the DC with offset of the axes of the cylinders

Taking into account \(E_y\) and \(E_x\) converting the system of equations (15):
Thus, we have obtained a calculated model of the DC implying the possibility of changing the following parameters: 

\[ Y_e = k \cdot \cos(\beta + \alpha) + \sqrt{L_x^2 - (k \cdot \sin(\beta + \alpha) + E_y)^2} \]

\[ X_e = k \cdot \sin \beta + \sqrt{L_x^2 - (k \cdot \cos \beta + E_x)^2} \]  

(27)

To avoid mutually exclusive combinations of the above parameters, we will define ranges of acceptable values. The ratio of connecting rod lengths must meet the condition:

\[
\frac{L_x}{L_y} < 1
\]  

(28)

The connecting rod of the displacer must be longer than the connecting rod neck of the crankshaft:

\[
\frac{k}{L_x} < 1
\]  

(29)

Offset of the displacer and piston axes must meet the condition:

\[
|E_y| < L_y - k \\
|E_x| < L_x - k
\]  

(30)

Next, consider the scheme for determining the torque from the force on the piston, which is shown in figure 5.
The torque relative to the O point will be determined by the formula:

\[ M = F_{cr} \cdot k_{cr} \]  

(31)

Where:

\[ F_{cr} = \frac{F_{n}}{\cos \gamma} \]

\[ k_{cr} = \sqrt{k^2 - (l_{kf} - l)^2} \]

According to figure 5, \( l_{kf} \) it will be equal to:

(32)

Now, let's define a parameter \( a \) from the following system of equations:

(33)

From where:

(34)

Express \( a \):

(35)
Next, substitute the resulting expression in (34):

\[
\frac{l_{kf}}{l} = \frac{Y(Y-a)}{l} = \frac{Y(Y-(\frac{Y}{2}+\frac{k^2-l^2}{2}))}{l} = \frac{Y(Y-\frac{Y}{2}-\frac{k^2-l^2}{2})}{l} = \frac{Y^2-\frac{Y}{2}(k^2-l^2)}{l} = \frac{Y^2-k^2-l^2}{l} = \frac{Y^2-k^2-l^2}{l} = \frac{Y^2+k^2-l^2}{l} = \frac{l(Y^2+k^2-l^2)}{2}.
\]  

(36)

Convert the formula (31) by substituting the formulas in it (34), (35), (36):

(37)

Finally, we have an equation describing the relationship of the torque from the pressure force on the piston - \( F_n \), the position of the piston - \( Y \), the length of the connecting rod - \( l \) and the crankshaft shoulder - \( k \).

The relationship of the torque to the angle of rotation of the crankshaft, through the position of the piston, is reflected by the following relationship (see figure 2):

(38)

Then, the dependence for determining the torque from the force on the piston, at a free angle between the axes of the cylinders, will be as follows:

\[
M_y = \frac{2(k \cdot \cos(\beta + \alpha) + \sqrt{l^2 - (k \cdot \sin(\beta + \alpha))^2}) \cdot F_n \cdot l}{(k \cdot \cos(\beta + \alpha) + \sqrt{l^2 - (k \cdot \sin(\beta + \alpha))^2})^2 - k^2 + l^2}.
\]

(39)

Next, we take into account the displacement of the cylinder axis of the piston \( E_y \).

The scheme for determining the influence of displacement \( E_y \) on the torque is shown in figure 6.

To assess the effect of displacement \( E_y \), we apply a clockwise rotation of the polar coordinate system on angle \( \Delta \beta \). Then the problem is reduced to determining the torque according to the scheme shown in figure 6, with the only difference that the angle of rotation of the crankshaft will be determined by the difference \( \beta - \Delta \beta \), and the angle of deflection of the connecting rod - \( (\gamma + \Delta \beta) \). Therefore, in order to determine the torque, it is necessary to determine \( \Delta \beta \).

From the OAA' triangle (see figure 6):
Then, after converting the dependence (39), the equation for determining the torque from the force on the piston when the axis of the piston cylinder is displaced in a plane perpendicular to the axis of the crankshaft will have the form:

\[
M_y = \frac{2(k \cdot \cos(\beta - \arctg(E_y/Y_e) + \alpha) + \sqrt{l^2 - (k \cdot \sin(\beta - \arctg(E_y/Y_e) + \alpha))^2} \cdot F_n \cdot l}}{(k \cdot \cos(\beta - \arctg(E_y/Y_e) + \alpha) + \sqrt{l^2 - (k \cdot \sin(\beta - \arctg(E_y/Y_e) + \alpha))^2})^2 - k^2 + l^2}.
\]

When the "alpha" Stirling engine is running, the pressure of the dynamic gas pressure is also perceived by the displacer. Since the displacer is also connected to the crankshaft (see figure 1), the force on it will produce a torque on the crankshaft. This moment will also be described by equation (41), but without taking into account the angle \(\lambda\) and changing the corresponding variables (see figure 1):
4 Discussion and conclusion

The obtained equations: the first equation of the system (1) and the final equation allow determining the positions of the piston and the displacer at different lengths of connecting rods and the angle between the axes of the cylinders of the piston and the connecting rod differing from 90°. This will allow you to create a numerical model of the Stirling engine workflow without modeling the crank-slide group, and modeling only the working volumes of the piston and displacement chambers. This will allow a comprehensive study of the mutual influence of the design parameters of the crank-rod group on the performance characteristics of the Stirling engine. In this case, it will be possible to change the size of the model to study the effect of a large-scale effect that affects gas-dynamic processes and heat-mass transfer processes in the working body. The geometric constraints set in the model will allow you to determine acceptable ranges of parameter changes for a full-factor numerical study of the Stirling engine workflow.

The obtained equations (41), (42) and the final equation (43) allow determining the torque at different lengths of connecting rods and the angle between the cylinder axes of the piston and the connecting rod differing from 90° and the displacement of the cylinder axes in the plane perpendicular to the axis of the crankshaft.

This study will eventually determine the optimal parameters of the designed Stirling engine depending on power and operating temperatures. This will allow the engine to be designed for use in the timber industry. This implies the use of logging and woodworking waste as fuel for a DC-based power generator. These types of fuel have a certain Gorenje temperature and specific heat of combustion [11-12] (see table 1). Their effective use for energy production is an urgent task of the forest complex at the present time [13-14].

Table 1. Average combustion temperatures of heat sources

| №  | Type of course                     | Temperature, °C | Specific heat of combustion, MJ/kg |
|----|-----------------------------------|----------------|-----------------------------------|
| 1  | Sawdust of normal humidity (pine, birch) | 400            | 8,37                              |
| 2  | Splint                            | 450            | 10,93                             |
| 3  | Bark                              | 450            | 5,69                              |
| 4  | Logging waste                     | 350            | 8,12                              |
The application of the developed mathematical models will allow you to design a DC for a specific type of fuel, which will allow you to get the maximum efficiency. Also, a separate direction of implementing DS in the timber industry will be the development of an engine for heat recovery from secondary sources: exhaust gases of drying chambers, boilers of heating systems, cooling systems of diesel power plants. The temperature of these sources is shown in table 2.

| №  | Type of course                        | Temperature, С° |
|----|---------------------------------------|-----------------|
| 1  | Exhaust gases of drying chambers      | 70              |
| 2  | Boilers heating systems               | 800             |
| 3  | The cooling system of DES             | 130             |

As you can see from the table, for heat utilization of these sources, you will need to develop a Stirling engine with specific parameters. For example, low temperature Stirling engine [15].

In General, Stirling engine, as a generator drive, operated in the conditions of logging enterprises, is promising. The introduction of Stirling energy station working on the logging waste instead of Diesel energy station to solve a few issues: the supply of fuel, noise pollution, pollution by combustion products of diesel fuel, and most importantly, utilization of logging waste.

To successfully solve this task, it is necessary to develop a methodology for the development and design of Stirling engine adapted to the appropriate fuel. Since the processes of heat and mass transfer change dramatically when scaling the Stirling engine, it is necessary to study several power ranges to determine the optimal design parameters. To determine the boundaries of these ranges, you must perform a sequential, multi-parameter optimization of the design with a gradual increase in power. The solution of this problem is associated with a huge complexity of manufacturing a large number of experimental samples. To date, this problem can be solved only by applying methods of numerical modeling of gas dynamics and thermal processes with kinematic and dynamic parameterization of the mechanical part of the studied structures.

References

1. G. Popa, M. A. Gheți, *Locomotive Diesel Engine Operation with Optimal Specific Fuel Consumption, Procedia Manufacturing*, 46, 440-444, ISSN 2351-9789 (2020) https://doi.org/10.1016/j.promfg.2020.03.064.
2. J.-H. Cho, C. S. Lin, C. D. Richards, R. F. Richards, J. Ahn, P. D. Ronney, Demonstration of an external combustion micro-heat engine, Proceedings of the Combustion Institute, 32(2), 3099-3105, ISSN 1540-7489 (2009) https://doi.org/10.1016/j.proci.2008.07.017.
3. P. Breeze, Chapter 6 - Stirling Engines and Free Piston Engines, Editor(s): Paul Breeze, Piston Engine-Based Power Plants, Academic Press, 59-70, ISBN 9780128129043 (2018) https://doi.org/10.1016/B978-0-12-812904-3.00006-9.
4. F. Xin, Z. Liu, S. Wang, W. Liu, Study of heat transfer in oscillatory flow for a Stirling engine heating tube inserted with spiral spring, Applied Thermal Engineering, 143,
5. S. Alfarawi, R. AL-Dadah, S. Mahmoud, Influence of phase angle and dead volume on gamma-type Stirling engine power using CFD simulation, Energy Conversion and Management, 124, 130-140, ISSN 0196-8904 (2016) https://doi.org/10.1016/j.enconman.2016.07.016.

6. B. B. Kosenok, V. B. Balyakin, I. N. Giltsov, The Use of Vector Models to Study the Dynamic Characteristics of an Advanced Two-Shaft Internal Combustion Engine, Procedia Engineering, 176, 37-42, ISSN 1877-7058, (2017) https://doi.org/10.1016/j.proeng.2017.02.270.

7. C. Gu, L. Botto, FPI: A fast numerical method for the simulation of particle-laden fluid interfaces, Computer Physics Communications, 256, 107447, ISSN 0010-4655, (2020) https://doi.org/10.1016/j.cpc.2020.107447.

8. D. Smirnov, V. Dvortsov, A. Saichenko, M. Tkachenko, M. Kukolev, A. Bischi, H. Ouédane, Experimental study of a high-tolerance piston-cylinder pair in the alpha Ross-yoke Stirling refrigerator, International Journal of Refrigeration, 100, 235-245, ISSN 0140-7007 (2019) https://doi.org/10.1016/j.ijrefrig.2019.01.018.

9. B. Rutczyk, I. Szczygieł, A. Kabaj, Evaluation of an α type stirling engine regenerator using a new differential model, Energy, 209, 118369, ISSN 0360-5442 (2020) https://doi.org/10.1016/j.energy.2020.118369.

10. Y. Guo, Y. Chao, B. Wang, Y. Wang, Z. Gan, A general model of Stirling refrigerators and its verification, Energy Conversion and Management, 188, 54-65, ISSN 0196-8904 (2019) https://doi.org/10.1016/j.enconman.2019.03.029.

11. T. Palander, H. Haavikko, K. Kärhä, Towards sustainable wood procurement in forest industry – The energy efficiency of larger and heavier vehicles in Finland, Renewable and Sustainable Energy Reviews, 96, 100-118, ISSN 1364-0321 (2018) https://doi.org/10.1016/j.rser.2018.07.043.

12. M. Manzone, Efficiency of a compactor in wood chip volume reduction, Biomass and Bioenergy, 80, 303-306, ISSN 0961-9534 (2015) https://doi.org/10.1016/j.biombioe.2015.06.007.

13. T. Chungcharoen, N. Srisang, Preparation and characterization of fuel briquettes made from dual agricultural waste: Cashew nut shells and areca nuts, Journal of Cleaner Production, 256, 120434, ISSN 0959-6526, (2020) https://doi.org/10.1016/j.jclepro.2020.120434.

14. C. Song, C. Zhang, S. Zhang, H. Lin, Y. Kim, uthusamy Ramakrishnan, Yanqiang Du, Yan Zhang, Huabao Zheng, Damia Barcelo, Thermochemical liquefaction of agricultural and forestry wastes into biofuels and chemicals from circular economy perspectives, Science of The Total Environment, 141972, ISSN 0048-9697 (2020) https://doi.org/10.1016/j.scitotenv.2020.141972.

15. J. Park, H. Kim, J. Ko, S. In, Y. Hong, H. Yeom, S. Park, Free-piston Stirling cryocooler (FPSC) for low-temperature applications down to 110 K, International Journal of Refrigeration, 117, 218-229, ISSN 0140-7007, (2020) https://doi.org/10.1016/j.ijrefrig.2020.04.020.