Abstract. We introduce higher order regionally proximal relations suitable for an arbitrary acting group. For minimal abelian group actions, these relations coincide with the ones introduced by Host, Kra and Maass. Our main result is that these relations are equivalence relations whenever the action is minimal. This was known for abelian actions by a result of Shao and Ye. We also show that these relations lift through extensions between minimal systems. Answering a question by Tao, given a minimal system, we prove that the regionally proximal equivalence relation of order $d$ corresponds to the maximal dynamical Antolín Camarena–Szegedy nilspace factor of order at most $d$. In particular the regionally proximal equivalence relation of order one corresponds to the maximal abelian group factor. Finally by using a result of Gutman, Manners and Varjú under some restrictions on the acting group, it follows that the regionally proximal equivalence relation of order $d$ corresponds to the maximal pronilfactor of order at most $d$ (a factor which is an inverse limit of nilsystems of order at most $d$).

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1. Introduction

1.1. General background. An old result in the field of topological dynamics is a theorem by Ellis and Gottschalk [EG60], which characterizes the equivalence relation $S_{eq}(X)$, induced from the maximal equicontinuous factor of a system $(G, X)$, as the smallest $G$-invariant closed equivalence relation which contains the regionally proximal relation $RP(X)$. Starting with Veech [Vee68], various authors, including Ellis-Keynes [EK71] and McMahon [McM78], came up with various sufficient conditions for $RP(X)$ to be an equivalence relation, whence for $RP(X) = S_{eq}(X)$. In particular, they proved that for a minimal system with an abelian acting group this is indeed the case.

Host, Kra and Maass [HKM10] introduced the higher order regionally proximal relations $RP[d](X)$ ($d \in \mathbb{N}$, $RP[1](X) = RP(X)$) for abelian actions, while investigating a topological dynamical analog of the celebrated Host-Kra structure theorem [HK05]. One of their results was that $RP[d](X)$ are equivalence relations for minimal distal systems $(\mathbb{Z}, X)$. Shao and Ye [SY12] generalized this theorem and showed that the relations $RP[d](X)$ are equivalence relations for all minimal actions by abelian groups.

In this article we present a new definition, the nilpotent regionally proximal relations of order $d$, $NRP[d](X)$ ($d \in \mathbb{N}$), defined for general group actions $(G, X)$. These are closed and $G$-equivariant relations which coincide with the Host-Kra-Maass definition for minimal abelian group actions. However, for non-abelian group actions it may happen that $RP(X) \subset NRP[1](X)$.

Our main result is that for minimal actions $NRP[d](X)$ is an equivalence relation for all $d \in \mathbb{N}$. This result is surprising as the regionally proximal relation $RP(X)$ is known not to be an equivalence relation for some (non-amenable) group actions ([McM76]).

The proof of Shao and Ye for abelian group actions was based on the general structure theory of minimal actions due to Ellis-Glasner-Shapiro [EGS75], McMahon [McM78] and Veech [Vee77]. In this article we present a direct enveloping semigroup proof of this theorem which is very similar to the short proof by Ellis and Glasner of the celebrated theorem by Van der Waerden on the existence of arbitrary long monochromatic arithmetic progressions in finite colorings of the integers ([Gla03, Gla94]). The proof is shorter and yields the result for general group actions. The possibility of applying the Ellis-Glasner proof as a shortcut to Shao and Ye’s proof in the abelian setting was also discovered by Ethan Akin ([Aki]).

Generalizing a result of Shao and Ye in the abelian setting ([SY12]), we show that given an extension of minimal systems $\pi : (G, X) \to (G, Y)$, the nilpotent regionally proximal relation lifts, i.e. $\pi \times \pi(NRP[d](X)) = NRP[d](Y)$. From this one easily concludes that for any minimal system, $(G, X/ NRP[d](X))$ is the maximal factor of $(G, X)$ for which the nilpotent regionally proximal relation of order $d$ is trivial. Following [HKM10], we call such systems systems of order at most $d$. By the theory developed by
Gutman, Manners and Varjú in [GMV16b], it follows that a system \((G, X)\) of order at most \(d\), where \(G\) has a dense subgroup generated by a compact group, is a pronilsystem of order at most \(d\), that is an inverse limit of nilsystems of order at most \(d\). Nilsystems, pronilsequences and the related nilsequences appear in different guises in several areas of mathematics: topological dynamics ([AHG+63]), ergodic theory ([HK05, Zie07]), additive number theory ([GT10]) and additive combinatorics ([Sze12]).

The paper [GMV16b] forms the third part of a series by the same authors [GMV16a, GMV18] extending the ground-breaking work of Antolín Camarena and Szegedy [ACS12], where the concept of nilspaces was introduced. A nilspace is a compact space \(X\) together with closed collections of cubes \(C^n(X) \subseteq X^{2^n}, n = 1, 2, \ldots\), satisfying some natural axioms. We show \((G, X/\text{NRP}[d](X))\) equipped with a natural collection of cubes is the maximal factor of \((G, X)\) which is a nilspace of order at most \(d\). This answers a question by Tao in [Tao15].

Comparing \(S_{\text{eq}}(X)\), the smallest equivalence relation which contains the regionally proximal relation \(\text{RP}(X)\) with \(\text{NRP}[d](X)\), we show that while the former corresponds to the maximal equicontinuous factor, the latter corresponds to the maximal (compact) abelian group factor. Thus unlike in the case of the maximal equicontinuous factor we have an explicit and unknown hitherto form for the equivalence relation corresponding to the maximal abelian group factor for arbitrary minimal actions. One may wonder whether a similar result can be achieved for the maximal (not necessarily abelian) group factor of a general minimal system.

1.2. Structure of the paper. Section 2 contains basic notation. Section 3 introduces the nilpotent higher order regionally proximal relations. In Section 4 we prove several results that play a key role in Section 5. Section 5 is devoted to proving the main result of the paper, namely that the nilpotent higher order regionally proximal relations are equivalence relations for general minimal group actions. In Section 6 we show that the nilpotent regionally proximal equivalence relations lift through dynamical morphisms between minimal systems. In Section 7 we investigate the structure of systems whose nilpotent regionally proximal equivalence relation of order \(d\) is trivial and answer Tao’s question. In Subsection 8.1 we investigate the relation between the classical regionally proximal relation and the nilpotent regionally proximal equivalence relation of order one. In Subsection 8.2 we present a different higher order generalization of the classical regionally proximal relation for arbitrary group actions, about which we know little. In Section 9 we exhibit an example related to Section 7. Section 10 is dedicated to open questions. Finally the Appendix contains technical results.

1.3. Acknowledgements. The first author was partially supported by a grant of the Israel Science Foundation (ISF 668/13). The second author was partially supported by the Marie Curie grant PCIG12-GA-2012-334564 and
by the NCN (National Science Center, Poland) grant 2016/22/E/ST1/00448. The third author was partially supported by NNSF of China 11371339 and 11431012. The work originated in the trimester program on “Universality and Homogeneity” at the Hausdorff Research Institute for Mathematics in Bonn where the first and second authors took part. We are grateful to the organizers of the program, Alexander Kechris, Katrin Tent and Anatoly Vershik. A significant part of the work was carried out during the visit of the second author to the third author at the University of Technology and Science of China at Hefei in October 2015. The second author is grateful for the hospitality and excellent working conditions during the visit. The work was concluded in the Simons Semester “Dynamical Systems” at the Banach Center in Warsaw (September-December 2015) where the first and second authors took part. We acknowledge the partial support by grant 346300 for IMPAN from the Simons Foundation and the matching 2015-2019 Polish MNiSW fund. We are grateful to Yixiao Qiao for help with Figure 9.1. We thank Joe Auslander for sending us comments on a previous version. Finally, we are grateful to the referee for a careful reading and many useful suggestions.

2. Preliminaries

2.1. The underlying system. Throughout the article, \((G, X)\) denotes a topological dynamical system (t.d.s) where \(G\) is a (Hausdorff) topological group and \(X\) is a compact Hausdorff space. To improve readability we sometimes assume without loss of generality that \(X\) is metrizable and use sequences of elements instead of nets of elements. We stress that this assumption is superfluous unless stated explicitly\(^1\). When a metric is evoked we denote it by \(\text{dist}()\). The action of an element \(g \in G\) on \(x \in X\) is denoted by \(gx\). For \(x \in X\), \(O(x, G) = Gx\) denotes the orbit of \(x\). A continuous \(G\)-equivariant map \((G, X) \rightarrow (G, Y)\) is called a dynamical morphism\(^2\). In exceptional and explicitly stated cases, we allow for dynamical morphisms between two t.d.s \((G, X)\) and \((G', X')\), where possibly \(G \neq G'\). In such a case there exist a continuous group homomorphism \(\phi : G \rightarrow G'\) and a continuous map \(f : X \rightarrow X'\) such that for all \(x \in X\) and \(g \in G\), \(f(gx) = \phi(g)f(x)\).

Discrete cubes and their faces will appear abundantly throughout the article. In the next subsections we summarize some related notation.

2.2. Discrete cubes. For an integer \(d \geq 0\) we denote the set of maps \(\{0, 1\}^d \rightarrow X\) by \(X^{[d]}\) (\(X^{[0]} = X\)) and call its elements \(d\)-configurations. For a configuration \(x \in X^{[d]}\), we call the points \(\{x(\omega)\}_{\omega \in \{0, 1\}^d}\) the vertices of \(x\). Let \(\pi : X^{[d]} \rightarrow X^{(0, 1)^d} \triangleq X^{(0, 1)^d \setminus \{\emptyset\}}\) denote the projection onto the \(\{0, 1\}\)-coordinates.\(^1\)

\(^1\)In fact the only results which require that \(X\) is metrizable are Theorem 4.16 and Theorem 7.18.

\(^2\)We do not insist a dynamical morphism to be surjective, however most (but not all) dynamical morphisms appearing in this article are.
last \((2^d - 1)\)-coordinates; i.e., the map which forgets the \(\tilde{0}\)-coordinate. Let \(X^{[d]}_\epsilon = \pi_\epsilon(X^{[d]}) = \prod \{X_\epsilon : \epsilon \neq \tilde{0}\}\) and for \(x \in X^{[d]}\) let \(x_\epsilon = \pi_\epsilon(x) \in X^{[d]}_\epsilon\) denote its projection. Sometimes it is convenient to write \(x = (x_\tilde{0}, x_+).\) For each \(\epsilon \in \{0, 1\}^d\) we denote by \(\pi_\epsilon\) the projection map from \(X^{[d]}\) onto \(X_\epsilon = X.\) For a point \(x \in X\) we let \(x^{[d]} = x^{[d]}_\epsilon \in X^{[d]}\) and \(x^{[d]}_\epsilon \in X^{[d]}_\epsilon\) be the constant configuration, that is, the configuration all of whose vertices are equal to \(\epsilon.\) Sometimes it is convenient to represent \(X^{[d]}\) \((d \geq 1)\) as a product space \(X^{[d]} = X^{[d - 1]} \times X^{[d - 1]}\). When using this decomposition we write \(x = (x_f, x_c)\) and refer to \(x_f\) and \(x_c\) as the floor and ceiling of \(x.\) More explicitly define the identification \(X^{[d]} \to X^{[d - 1]} \times X^{[d - 1]}\) by \(x \mapsto (x_f, x_c)\) with \((x_f)_\epsilon = x_{\epsilon 0}\) and \((x_c)_\epsilon = x_{\epsilon 1}\) for \(\epsilon \in \{0, 1\}^{d - 1}.\) If \(f', f'' : X^{[d - 1]} \to X^{[d - 1]}\) are functions then we define

\[(2.1) \quad f' \times f''(x_f, x_c) = (f'(x_f), f''(x_c))\]

and define \(\pi_f, \pi_c : X^{[d]} \to X^{[d - 1]}\) by \(\pi_f(x) = x_f\) and \(\pi_c(x) = x_c.\) Further in the case of \(X^{[2]}\) we will employ the following identification:

\[(2.2) \quad X^{[2]} \to X \times X \times X \times X \quad x \mapsto (x_{00}, x_{10}, x_{01}, x_{11})\]

2.3. Faces. Let \(d \geq 0.\) A set of the form \(F = \{\omega \in \{0, 1\}^d : \omega_{i_1} = \alpha_1, \omega_{i_2} = \alpha_2, \ldots, \omega_{i_k} = \alpha_k\}\) for some \(k \geq 0, 1 \leq i_1 < i_2 < \cdots < i_k \leq d\) and \(\alpha_i \in \{0, 1\}\) is called a face of codimension \(k\) of the discrete cube \(\{0, 1\}^d.\) One writes \(\text{codim}(F) = k.\) A face of codimension 1 is called a hyperface. If all \(\alpha_i = 1\) we say that the face is upper. Note all upper faces contain \(I\) and there are exactly \(2^d\) upper faces. Similarly if all \(\alpha_i = 0\) we say that the face is lower.

![Figure 2.1. A 3-configuration with a shaded lower hyperface.](image-url)

\(^3\)The case \(k = 0\) corresponds to \(\{0, 1\}^d.\)
3. Nilpotent regionally proximal relations

3.1. Proximity and its generalizations. Let us recall several classical definitions. Two points $x, y \in X$ are said to be \textbf{proximal}, denoted $(x, y) \in \mathcal{P}(X)$, if there is a sequence of elements $g_i \in G$ such that $\lim_{i \to \infty} \text{dist}(g_ix, g_iy) = 0$. The system $(G, X)$ is said to be \textbf{distal}, if $\mathcal{P}(X) = \Delta = \{(x, x) | x \in X\}$. Two points $x, y \in X$ are said to be \textbf{regionally proximal}, denoted $(x, y) \in \mathcal{RP}(X)$, if there are sequences of points $x_i, y_i \in X$ and a sequence of elements $g_i \in G$ such that $\lim_{i \to \infty} x_i = x$, $\lim_{i \to \infty} y_i = y$ and $\lim_{i \to \infty} \text{dist}(g_ix_i, g_iy_i) = 0$. Let $\mathcal{S}_{\text{eq}}(X)$ be the smallest $G$-invariant closed equivalence relation which contains $\mathcal{RP}(X)$. Clearly $\mathcal{P}(X) \subset \mathcal{RP}(X) \subset \mathcal{S}_{\text{eq}}(X)$. It is a remarkable fact that in many cases the regionally proximal relation happens to be an equivalence relation, i.e it holds $\mathcal{RP}(X) = \mathcal{S}_{\text{eq}}(X)$. These cases include, inter alia, the case when $(G, X)$ is proximal or weakly mixing, or when it is minimal and admits an invariant measure ([McM78], see [Aus88, Theorem 9.8]). In particular if $(G, X)$ is minimal with $G$ amenable, or when it is minimal and satisfies the Bronstein condition\footnote{By [EG60, Theorem 2], $\mathcal{S}_{\text{eq}}(X)$ is induced from the maximal equicontinuous factor of $(G, X)$.}. A particular case of the latter occurs when $(G, X)$ is minimal and point-distal ([EE14, Proposition 16.10], first proven in [EK71]). It is also known that the regionally proximal relation can fail to be an equivalence relation for minimal t.d.s. A well known counterexample is given in [McM76, Example 1.8] (for more details see [dV93, V(1.8)(2)]).

In [HKM10] Host, Kra and Maass introduced the regionally proximal relation of order $d$ ($d \in \mathbb{N}$) for $G$ abelian, where the case $d = 1$ corresponds to the classical regionally proximal relation:

\textbf{Definition 3.1.} [HKM10, Definition 3.2] Let $(G, X)$ be a topological dynamical system with $G$ abelian and $d \in \mathbb{N}$. The points $x, y \in X$ are said to be \textbf{regionally proximal of order} $d$, denoted $(x, y) \in \mathcal{RP}^{[d]}(X)$, if there are sequences of elements $g^1_i, g^2_i, \ldots, g^k_i \in G$, $x_i, y_i \in X$, such that for all $(\epsilon_1, \epsilon_2, \ldots, \epsilon_d) \in \{0, 1\}^d$:

$$\lim_{i \to \infty} x_i = x, \quad \lim_{i \to \infty} y_i = y, \quad \lim_{i \to \infty} \text{dist}\left(\sum_{j=1}^{d} \epsilon_j g^j_i x_i, \sum_{j=1}^{d} \epsilon_j g^j_i y_i\right) = 0$$

In order to generalize this definition to general group actions, we introduce several important concepts in the next two subsections.

3.2. Host-Kra cube group. Let $H \subset G$ be a subgroup and $F \subset \{0, 1\}^d$. For $h \in H$ we denote by $[h]_F$ the element of $H^{[d]} = H^{\{0, 1\}^d}$ defined as

\footnote{$(G, X)$ is said to satisfy the Bronstein condition if $X \times X$ has a dense set of minimal points. The mentioned result was proven in [Vee77, Theorem 2.6.2] and was also obtained independently by Ellis (unpublished).}
\[ [h]_F(\omega) = h \text{ if } \omega \in F \text{ and } [h]_F = \text{Id otherwise, where Id denotes the unit element of } G. \]

Define:
\[ [H]_F = \{ [h]_F | h \in H \} \]

We call the subgroup of \( G^{[d]} \) generated by \( [G]_F \), where \( F \) ranges over all hyperfaces of \( \{0,1\}^d \), the **Host-Kra cube group** and denote it by \( \mathcal{HK}^{[d]} \).

We call the subgroup of \( G^{[d]} \) generated by \( [G]_F \) where \( F \) ranges over all upper hyperfaces of \( \{0,1\}^d \) the **face cube group** and denote it by \( \mathcal{F}^{[d]} \). It is easy to see that \( \mathcal{HK}^{[d]} \) is generated by \( \mathcal{F}^{[d]} \) and \( \Delta^{[d]}(G) \).

**Example 3.2.** One sees readily that \( \mathcal{F}^{[2]} \) is generated by
\[ \{(\text{Id}, \text{Id}, h, h), (\text{Id}, h', \text{Id}, h') : h, h' \in G\} \]

and \( \mathcal{HK}^{[2]} \) is generated by
\[ \{(\text{Id}, \text{Id}, h, h), (\text{Id}, h', \text{Id}, h'), (k, k, \text{Id}, \text{Id}), (k', \text{Id}, k', \text{Id}) : h, h', k, k' \in G\} \]

Thus, \( \mathcal{HK}^{[2]} \) is generated by \( \mathcal{F}^{[2]} \) and \( \{(t, t, t, t) : t \in G\} = \Delta^{[2]}(G) \).

The Host-Kra and face cube groups originate in [HK05, Section 5] and coincide with the parallelepiped groups and face groups respectively of [HKM10, Definition 3.1] introduced for abelian actions. Notice \( \mathcal{F}^{[d]} \subset \mathcal{HK}^{[d]} \) and for all \( \gamma \in \mathcal{F}^{[d]} \), \( \gamma(\emptyset) = \text{Id} \). The Host-Kra cube and face groups act (coordinate-wise) on \( X^{[d]} \) by \( \gamma c(\omega) = \gamma(\omega)c(\omega) \) for \( \gamma \in \mathcal{HK}^{[d]} \), \( c \in X^{[d]} \), and \( \omega \in \{0,1\}^d \). Similarly the face group act (coordinate-wise) on \( X^{[d]} \).

**Proposition 3.3.** Let \( G \) be a group then \( \mathcal{HK}^{[d]} = \mathcal{F}^{[d]} \Delta^{[d]}(G) \).

**Proof.** By definition of the groups involved it is easy to see \( \mathcal{F}^{[d]} \Delta^{[d]}(G) \subset \mathcal{HK}^{[d]} \). In order to prove the reverse direction fix \( g \in \mathcal{HK}^{[d]} \) where by definition \( g = \prod_{j=1}^{m} [t_j]_{F_j} \) where \( F_j \) is an upper hyperface or \( F_j = \{0,1\}^d \) and \( t_j \in G \). Note that for \( F_j \) an upper hyperface one has:
\[ [t_1]_{\{0,1\}^d} [t_2]_{F_j} = [t_1 t_2 t_1^{-1}]_{F_j} [t_1]_{\{0,1\}^d}. \]

This implies that one can move all occurrences of the form \( [t_1]_{\{0,1\}^d} \) to the right while leaving on the left only expressions of the form \( [t]_{F_j} \) with \( F_j \) an upper hyperface.

Thus we have shown \( g \in \mathcal{F}^{[d]} \Delta^{[d]}(G) \). \( \square \)

### 3.3. Dynamical cubespaces.

The notion of cubespaces and nilspaces originate from Host and Kra’s *parallelepiped structures* in [HK08]. Antolín Camarena and Szegedy [ACS12] carried out a systematic study of nilspaces and described their structure. We recommend [GMV16b, Subsection 1.3] for a succinct introduction to nilspaces (but see also [GMV16a, GMV18, Can17b, Can17a]). In this subsection we introduce the notion of *dynamical*
cubespaces. For the general theory see Subsection 7.2. We stress that the dynamical cubespaces are a subclass of the class of cubespaces (see Proposition A.1).

Let \((G, X)\) be a topological dynamical system. Following Host, Kra and Maass [HKM10, Definition 1.1] we introduce the dynamical cubes as the orbit closure of constant configurations and denote it by

\[
C_G^{[d]}(X) = \{g x^{[d]} | g \in \mathcal{H}K^{[d]}, x \in X\} \quad (d \geq 0).
\]

The pair \((X, C_G^*) \triangleq (X, \{C_G^{[d]}(X)\}_{d \in \mathbb{Z}_+})\) is called a dynamical cubespace induced by \((G, X)\). We also denote:

\[
C_{\tilde{x}}^{[d]}(X) = C_G^{[d]}(X) \cap (\{x\} \times X^*[d])
\]

\[
C_{\tilde{x}*}^{[d]}(X) = \pi_* (C_{\tilde{x}}^{[d]}(X))
\]

**Proposition 3.4.** Let \((G, X)\) be a minimal t.d.s, \(x_0 \in X\) and \(d \geq 1\) then

\[
C_G^{[d]}(X) = \{gx_0^{[d]} | g \in \mathcal{H}K^{[d]}\} = \{gx^{[d]} | g \in \mathcal{F}^{[d]}, x \in X\}
\]

**Proof.** The first equality is trivial. The second follows from Equation (3.1) and Proposition 3.3. \(\square\)

### 3.4. Nilpotent regionally proximal relations

We are ready to introduce the definition of the nilpotent regionally proximal relations for general group actions.

**Definition 3.5.** Given \(x, y \in X\) we let the **lower corner** \(\downarrow^{[d]}(x, y)\) be the configuration defined by: \(\omega \mapsto x\) for \(\omega \neq \overrightarrow{1}\) and \(\overrightarrow{1} \mapsto y\); and the **upper corner** \(\uparrow^{[d]}(x, y)\) by the configuration: \(\overrightarrow{0} \mapsto x\) and \(\omega \mapsto y\) for \(\omega \neq \overrightarrow{0}\).

**Definition 3.6.** Let \((G, X)\) be a topological dynamical system. Let \(d \geq 1\). We say that a pair of points, \(x, y \in X\) are nilpotent regionally proximal of order \(d\) and write \((x, y) \in \text{NRP}^{[d]}(X)\), if and only if \(\downarrow^{[d+1]}(x, y) \in C_G^{[d+1]}(X)\). That is \((x, y) \in \text{NRP}^{[d]}(X)\) if and only if there are sequences \(g_i \in \mathcal{H}K^{[d+1]}\) and \(x_i \in X\) such that

\[
\lim_{i \to \infty} g_i x_i^{[d+1]} = \downarrow^{[d+1]}(x, y).
\]

Host, Kra and Maass [HKM10, Corollary 4.3] showed that if \((\mathbb{Z}, X)\) is minimal and distal, then \((x, y) \in \text{RP}^{[d]}(X)\) if and only if \(\downarrow^{[d+1]}(x, y) \in C_{\mathbb{Z}}^{[d+1]}(X)\). This was generalized to arbitrary minimal abelian actions by Shao and Ye in [SY12, Theorem 3.4]. Thus for minimal abelian actions \((G, X)\),

\[
\text{NRP}^{[d]}(X) = \text{RP}^{[d]}(X) \quad \text{7}
\]

\(\text{7}\)It is easy to see that this statement is not true in general if we remove the minimality assumption.
When $G$ is abelian there are, canonically defined, surjective, group homomorphisms $G^d = G \times G \times \cdots \times G \to \mathcal{F}^{[d]}$ and $G^{d+1} = G \times G \times \cdots \times G \to \mathcal{H}\mathcal{K}^{[d]}$, namely,

$$\left( g_1, g_2, \ldots, g_d \right) \in G^d \mapsto \left( \sum_{j=1}^{d} \epsilon_j g_j \left| (\epsilon_1, \ldots, \epsilon_d) \in \{0, 1\}^d \right. \right)$$

and

$$\left( g_1, g_2, \ldots, g_d, g_{d+1} \right) \in G^{d+1} \mapsto \left( g_{d+1} + \sum_{j=1}^{d} \epsilon_j g_j \left| (\epsilon_1, \ldots, \epsilon_d) \in \{0, 1\}^d \right. \right),$$

respectively. This fact explains why Host and Kra’s definition of $\text{RP}^{[d]}(X)$ (Definition 3.1) is much simpler than Definition 3.5. However, as we will see, when the action is given by a non-commutative $G$, commutation relations in $G$ or, more precisely, its lower central series, determine, through the cubic structure, the behaviour of the $\text{NRP}^{[d]}(X)$ relations.

The reader may wonder why the word "nilpotent" appears in the name of $\text{NRP}^{[d]}(X)$. The reason is that if $\text{NRP}^{[d]}(X)$ is trivial, i.e. $\text{NRP}^{[d]}(X) = \Delta$, then $(G, X)$ is isomorphic to an action by a nilpotent group of nilpotency class at most $d$ (for an exact statement see Proposition A.7). Another natural question is what is the relation between $\text{NRP}^{[1]}(X)$ and $\text{P}(X)$, $\text{RP}(X)$, $\text{Seq}(X)$ defined in Subsection 3.1. It turns out that $\text{P}(X) \subset \text{RP}(X) \subset \text{Seq}(X) \subset \text{NRP}^{[1]}(X)$ (for a proof see Proposition 8.1). Finally we remark that we could have used the upper corner $\gamma^{d+1}(x, y)$ in the definition of $\text{NRP}^{[d]}(X)$ as by Proposition A.8, $(x, y) \in \text{NRP}^{[d]}(X)$ if and only if $\gamma^{d+1}(x, y) \in C^{[d+1]}(X)$.

**Example 3.7.** Fix $d \in \mathbb{N}$. We give two examples of calculating $\text{NRP}^{[d]}(X)$ of very different in flavor. Let $G_{d+1}$ denotes the $(d+1)$-th element of the lower central series of $G$ (see Subsection A.4), then $(x, hx) \in \text{NRP}^{[d]}(X)$ for any $h \in G_{d+1}$. (For a proof of this fact, see Lemma A.5(4)). Hence if $G$ is perfect, that is $G = \{G, G\}$, and the action is minimal, then $\text{NRP}^{[d]}(X) = X^{[d]}$. For the second example let $(G, L/\Gamma)$ be a (generalized) minimal nilsystem, that is, $L$ is a nilpotent Lie group of nilpotency class at most $d$, $\Gamma \subseteq L$ is a discrete cocompact subgroup and the minimal action of $G$ on $L/\Gamma$ is through a continuous group homomorphism $\phi : G \to L$. In [GMV16a, Proposition 2.5] based on [GT10, Lemma E.10] it is proven that \{\text{gG}^{[d+1]} | \text{g} \in \mathcal{H}\mathcal{K}^{[d+1]}\} is compact. By Proposition 3.4 we conclude $C_{G}^{[d+1]}(X) = \{\text{gG}^{[d+1]} | \text{g} \in \mathcal{H}\mathcal{K}^{[d+1]}\}$. By [GMV16a, Proposition 2.6] if $c, c' \in C_{G}^{[d+1]}(X)$ such that $c(\omega) = c'(\omega)$ for $\omega \neq \bar{1}$ then $c(\bar{1}) = c'(\bar{1})$. It follows that if $\omega^{[d+1]}(x, y) \in C_{G}^{[d+1]}(X)$ then $x = y$ as $\omega^{[d+1]} \in C_{G}^{[d+1]}(X)$. We conclude $\text{NRP}^{[d]}(L/\Gamma)$ is trivial. See also Subsection 7.1.
Shao and Ye [SY12, Theorem 3.5] showed that $RP^d(X)$ is an equivalence relation for minimal actions by abelian groups. Our main result is the following theorem:

**Theorem 3.8.** Let $(G, X)$ be a minimal topological dynamical system, then $NRP^d(X)$ $(d \geq 1)$ is a closed $G$-invariant equivalence relation.

The theorem is surprising as the regionally proximal relation $RP(X)$ is known not to be an equivalence relation for some (non-amenable) group actions ([McM76, Example 1.8]; for more details see [dV93, V(1.8)(2)].

4. Minimal subsystems for the Host-Kra and face cube groups

Let $(G, X)$ be a minimal topological dynamical system. In this section we prove several results that play a key role in the proof that $NRP^d(X)$ are equivalence relations for $d \geq 1$. These results are interesting by their own right. The proofs use the theory of the Ellis semigroup which we now recall.

4.1. Ellis semigroup. We very briefly review some theory related with the Ellis semigroup (also known as the enveloping semigroup). A self-contained reference is [Gla76, Chapter I]. We also recommend [SY12, Appendix A].

**Definition 4.1.** The Ellis semigroup $E = E(G, X)$ of a t.d.s $(G, X)$ is the closure of $G$ in the semigroup (with respect to composition) $X^X$ equipped with the product topology. The Ellis semigroup is compact but in general not metrizable (see [GMU08]). A dynamical morphism $\pi : (G, X) \to (G, Y)$ induces a surjective continuous morphism of semigroups $\pi^* : E(G, X) \to E(G, Y)$. Note that for all $q \in E$, right multiplication in $E$ by $q$, $E \to E$, $p \mapsto pq$ is continuous. An element $u \in E$ with $u^2 = u$ is called an idempotent. A non-empty subset $I \subset E$ is a left ideal if $EI \subset I$. A minimal left ideal is a left ideal that does not contain any proper left ideal of $E$. Clearly any left ideal contains a minimal left ideal. An idempotent contained in a minimal left ideal is called a minimal idempotent.

**Proposition 4.2.** Let $(G, X)$ be a t.d.s and $E$ its Ellis semigroup. Suppose $L \subset E$ is a minimal left ideal and and let $J(L)$ be the set of idempotents in $L$, then:

1. $J(L) \neq \emptyset$.
2. A point $x \in X$ is minimal if and only if there exists $u \in J(L)$ with $ux = x$.
3. Let $u$ be an idempotent in $E$. If $p \in Eu$, then $pu = p$.
4. Let $x \in X$ and $u \in E$ an idempotent, then $(x, ux) \in P(X)$. In particular there is a minimal point which is proximal to $x$.
5. $L = \bigcup_{u \in J(L)} uE$ is a partition and every $uL$ is a group with identity $u$.
6. $(G, E)$ is a t.d.s and $(G, L)$ is a minimal subsystem.
7. Let $x_0 \in X$, then the map $(G, E) \to (G, X)$ given by $p \mapsto px_0$ is a dynamical morphism.
Proof. (1) [Gla76, Proposition I.2.3(1)]. (2) [Gla76, Proposition I.3.1(2)]. (3) If \( p \in E u \) then \( p = qu \) for some \( q \in E \). Thus \( pu = (qu)u = q(uu) = qu = p \).

4.2. Induced projections. Let \( E = E(\mathcal{HK}^d, C_G^{[d]}(X)) \) be the enveloping semigroup of \((\mathcal{HK}^d, C_G^{[d]}(X))\). Let \( \pi_\varepsilon : C_G^{[d]}(X) \to X_\varepsilon = X \) be the projection of \( C_G^{[d]}(X) \) on the \( \epsilon \)-coordinate, where \( \epsilon \in \{0, 1\}^d \). We consider the action of the group \( \mathcal{HK}^d \) on the \( \epsilon \)-coordinate via the projection \( \pi_\varepsilon \), i.e., for \( \epsilon \in \{0, 1\}^d \):

\[
\mathcal{HK}^d \times X_\varepsilon \to X_\varepsilon, \quad gx \mapsto g_\varepsilon x.
\]

With respect to this action of \( \mathcal{HK}^d \) on \( X_\varepsilon = X \) the map \( \pi_\varepsilon : (\mathcal{HK}^d, C_G^{[d]}(X)) \to (\mathcal{HK}^d, X_\varepsilon) \) is a dynamical homomorphism. Let \( \pi_\varepsilon^\ast : E(\mathcal{HK}^d, C_G^{[d]}(X)) \to E(\mathcal{HK}^d, X_\varepsilon) \) be the corresponding homomorphisms of enveloping semigroups. Notice that for the action of \( \mathcal{HK}^d \) on \( X_\varepsilon \), \( E(\mathcal{HK}^d, X_\varepsilon) = E(G, X) \) as subsets of \( X^X \).

We claim that an element of \( E(\mathcal{HK}^d, C_G^{[d]}(X)) \) is determined by its projections. Indeed as every element of \( \mathcal{HK}^d \) acts on \( C_G^{[d]}(X) \) coordinatewise, this is also true for the closure of \( \mathcal{HK}^d \) inside \((C_G^{[d]}(X))_{C_G^{[d]}(X)} \) therefore \( E(\mathcal{HK}^d, C_G^{[d]}(X)) \) may be identified with a subset of \( E(G, X)^{[d]} \) and moreover \( \mathcal{HK}^d \) acts on \( E(\mathcal{HK}^d, C_G^{[d]}(X)) \) coordinatewise.

Let \( x_0 \in X \). Consider the ceiling map \( \pi_\varepsilon : X^{[d]} \to X^{[d-1]} \) from Subsection 2.2. Let us denote its restriction to \( C_{x_0}^{[d]}(X) \) also by \( \pi_\varepsilon \). We thus have a continuous map \( \pi_\varepsilon : C_{x_0}^{[d]}(X) \to X^{[d-1]} \). Similarly we have a ceiling map \( \pi_\varepsilon' : G^{[d]} \to G^{[d-1]} \). Let us denote its restriction to \( \mathcal{HK}^d \) by \( \phi_\varepsilon \). We thus have a continuous group homomorphism \( \phi_\varepsilon : \mathcal{HK}^d \to G^{[d-1]} \).

Lemma 4.3. \( \phi_\varepsilon(F^{[d]}) = \mathcal{HK}^{[d-1]} \).

Proof. Let \( F_i = \{ \omega \in \{0, 1\}^d \mid \omega_i = 1 \} \), \( 1 \leq i \leq d \), be the upper hyperfaces of \( \{0, 1\}^d \). Define the projection \( \sim : \{0, 1\}^d \to \{0, 1\}^{d-1} \) by \( x_1x_2 \cdots x_d \mapsto x_1 \cdots x_{d-1} \). As \( F^{[d]} \) is generated by \( [G]_{F_i} \), \( \phi_\varepsilon(F^{[d]}) \) is generated by \( \phi_\varepsilon([G]_{F_i}) = [G]_{\widetilde{F_i}} \) for \( 1 \leq i \leq d \). Note \( \widetilde{F_1}, \ldots, \widetilde{F_{d-1}} \) are the upper hyperfaces of \( \{0, 1\}^{d-1} \), whereas \( [G]_{\widetilde{F_d}} = \{0, 1\}^{d-1} \) and thus \( [G]_{\widetilde{F_d}} = \Delta^{[d-1]}(G) \). Thus by Proposition 3.3, \( \phi_\varepsilon(F^{[d]}) = \mathcal{HK}^{[d-1]} \).

Lemma 4.4. Let \( (G, X) \) be a minimal t.d.s, then \( \pi_\varepsilon(C_{x_0}^{[d]}(X)) \subset C_{G}^{[d-1]}(X) \).

Proof. It follows from Proposition A.1 but let us give a direct proof. Clearly it is enough to prove \( \pi_\varepsilon(C_{G}^{[d]}(X)) = C_{G}^{[d-1]}(X) \). By an argument similar to the proof of Lemma 4.3, \( \phi_\varepsilon(\mathcal{HK}^{[d]}) = \mathcal{HK}^{[d-1]} \). Using Proposition 3.4 twice we have as desired:
\[\pi_c(C_G^{[d-1]}(X)) = \pi_c(\{gx_0^{[d]}|g \in \mathcal{HK}_G^{[d]}\}) = \{\phi_c(g)x_0^{[d-1]}|g \in \mathcal{HK}_G^{[d]}\} = \{gx_0^{[d-1]}|g \in \mathcal{HK}_G^{[d-1]}\} = C_G^{[d-1]}(X)\]

\[\square\]

Let \((H, X)\) and \((H', X')\) be t.d.s where possibly \(H \neq H'\). Let us say that a pair of maps \((f, \phi)\) is a dynamical morphism between \((H, X)\) and \((H', X')\) if \(f : X \to X'\) is a continuous map, \(\phi : H \to H'\) is a continuous group homomorphism and for all \(x \in X\) and \(g \in H\), \(f(gx) = \phi(g)f(x)\). The next simple lemma will be used in the next subsection.

**Lemma 4.5.** Let \((G, X)\) be a minimal t.d.s, then the pair \((\pi_c, \phi_c)\) is a dynamical morphism between \((\mathcal{F}^{[d]}, C_{G,0}^{[d]}(X))\) and \((\mathcal{HK}_G^{[d-1]}, C_G^{[d-1]}(X))\).

**Proof.** By Lemmas 4.3 and 4.4 respectively \(\pi_c : C_{x_0}^{[d]}(X) \to C_G^{[d-1]}(X)\) is a continuous map and \(\phi_c : \mathcal{F}^{[d]} \to \mathcal{HK}_G^{[d-1]}\) is a continuous group homomorphism. Finally it is easy to see for all \(b \in C_{x_0}^{[d]}(X)\) and \(g \in \mathcal{F}^{[d]}\), \(\pi_c(gb) = \phi_c(g)\pi_c(b)\). \(\square\)

### 4.3. Minimal actions

In [HKM10, Lemma 4.1] it was proven that \((\mathcal{HK}_G^{[d]}, C_G^{[d]}(X))\) is minimal for \((\mathbb{Z}, X)\) minimal and distal t.d.s. It was also mentioned that Glasner had shown (unpublished) that one can remove the distality assumption. Here we show that the same statement holds for a general group action. We note that the essential feature of \(\mathcal{HK}_G^{[d]}\) which is used in the proof is that it contains the diagonal, i.e. \(\Delta^{[d]} \subset \mathcal{HK}_G^{[d]}\).

**Proposition 4.6.** Let \((G, X)\) be a minimal t.d.s, then the t.d.s \((\mathcal{HK}_G^{[d]}, C_G^{[d]}(X))\) is minimal.

**Proof.** Let \(x \in X\) and let \(u\) be a minimal idempotent in \(E(G, X)\) with \(ux = x\) (Proposition 4.2(2)). Then \(\tilde{u} \triangleq u^{[d]} \in E(\mathcal{HK}_G^{[d]}, C_G^{[d]}(X))\) and \(\tilde{u}\) is an idempotent. Our goal is to show that \(\tilde{u}\) is a minimal idempotent of \(E(\mathcal{HK}_G^{[d]}, C_G^{[d]}(X))\). Given that this is true, as \(x^{[d]} = \tilde{u}x^{[d]}\), by Proposition 4.2(2), \(C_G^{[d]}(X)\) which is the orbit closure of \(x^{[d]}\), is \(\mathcal{HK}_G^{[d]}\)-minimal as desired. Choose \(v\) a minimal idempotent in the closed left ideal \(E(\mathcal{HK}_G^{[d]}, C_G^{[d]}(X))\tilde{u}\) (Proposition 4.2(1)). As \(\tilde{u}\) is an idempotent, \(v\tilde{u} = v\) (Proposition 4.2(3)). We will show that \(\tilde{u}v = \tilde{u}\), which implies that the idempotent \(\tilde{u}\) belongs to the minimal left ideal \(E(\mathcal{HK}_G^{[d]}, C_G^{[d]}(X))v\) and thus is minimal. Set, for \(\epsilon \in \{0, 1\}^d\), \(v_\epsilon = \pi_\epsilon^*v\) (\(\pi_\epsilon^*\) is defined in Subsection 4.2). Note that, as an element of \(E(\mathcal{HK}_G^{[d]}, C_G^{[d]}(X))\) is determined by its projections, it suffices to show that for each \(\epsilon\), \(uv_\epsilon = u\). Since for each \(\epsilon\) the map \(\pi_\epsilon^*\) is a semigroup homomorphism, we have that \(v_\epsilon u = v_\epsilon\) as \(v\tilde{u} = v\), and \(v_\epsilon v_\epsilon = v_\epsilon\) as \(vv = v\). In particular we deduce that \(v_\epsilon\) is an idempotent belonging to the minimal left ideal \(E(G, X_\epsilon)u = E(G, X)u\) and thus \(u \in E(G, X_\epsilon)v_\epsilon\) by Proposition
4.2(6). By Proposition 4.2(3), this implies that \( uv = u \) and it follows that indeed \( \tilde{u}v = \tilde{u} \).

Define:

\[
Y_{x}^{[d]}(X) = \overline{\mathcal{F}^{[d]}(x^{[d]})} \subset C_{x}^{[d]}(X)
\]

\[
Y_{x*}^{[d]}(X) = \pi_{*}(Y_{x}^{[d]}) \subset C_{x*}^{[d]}(X)
\]

In [HKM10, Proposition 4.2] it was proven that \( Y_{x}^{[d]}(X) = C_{x}^{[d]}(X) \) and it clearly follows that for \((\mathbb{Z}, X)\) minimal and distal, for each \( x \in X \), the system \((Y_{x}^{[d]}(X), Y_{x}^{[d]}(X))\) is minimal. In [SY12, Theorem 3.1] it was shown, using the structure theory of minimal systems, that for abelian group actions, for each \( x \in X \), the system \((\mathcal{F}^{[d]}, Y_{x}^{[d]}(X))\) is minimal. Here we show that the same statement holds for a general group action using only enveloping semigroup arguments. We start by an auxiliary lemma:

**Lemma 4.7.** Let \( u \in E(G, X) \) be an idempotent. Then \( u_{*}^{[d]} \in E(\mathcal{F}^{[d]}, Y_{x*}^{[d]}(X)) \).

**Proof.** Enumerate the upper hyperfaces of \( \{0,1\}^{d} \) by \( F_{1}, F_{2}, \ldots, F_{d} \). Let \( t_{\alpha} \in G \) be a net in \( G \) such that \( t_{\alpha} \to_{\alpha} u \) in \( E(G, X) \). As \( [t_{i}]F_{1} \in \mathcal{F}^{[d]} \), we have \( [u]F_{1} \in E(\mathcal{F}^{[d]}, Y_{x}^{[d]}(X)) \). As \( [t_{i}]F_{2}[u]F_{1} \in E(\mathcal{F}^{[d]}, Y_{x}^{[d]}(X)) \), \( E \to E \), \( p \mapsto pq \) is continuous and \( u^{2} = u \), we have \( [u]F_{1} \cup F_{2} \in E(\mathcal{F}^{[d]}, Y_{x}^{[d]}(X)) \). We now continue similarly for \( F_{3}, F_{4}, \ldots, F_{d} \).

**Proposition 4.8.** Let \((G, X)\) be a minimal t.d.s, then for each \( x \in X \), the t.d.s \((\mathcal{F}^{[d]}, Y_{x}^{[d]}(X))\), and hence also \((\mathcal{F}^{[d]}, Y_{x*}^{[d]}(X))\), are minimal.

**Proof.** The proof of the minimality of the t.d.s \((\mathcal{F}^{[d]}, Y_{x}^{[d]}(X))\) is almost verbatim the same as in the proof of Proposition 4.6, except that here the claim that for \( u \) a minimal idempotent in \( E(G, X) \), the map \( \tilde{u} = u_{*}^{[d]} \) is in \( E(\mathcal{F}^{[d]}, Y_{x}^{[d]}(X)) \), is not that evident. However, as \( u \) is an idempotent this fact follows from Lemma 4.7.

In [SY12, Theorem 3.1] it was proven that for each \( x \in X \), \((\mathcal{F}^{[d]}, Y_{x}^{[d]}(X))\) is the unique minimal subsystem of \((\mathcal{F}^{[d]}, C_{x}^{[d]}(X))\) for \((G, X)\) minimal t.d.s with \( G \) abelian. Here we show that the same statement holds for a general group action. We start by proving a lemma is a generalization of the “useful lemma” [SY12, Lemma 5.1]. The proof follows closely the original proof with one exception: the use of the pure ceiling-mixed decomposition (Subsection A.5).

**Lemma 4.9.** Let \((G, X)\) be a minimal t.d.s and \( d \geq 1 \). If \((x^{[d-1]}, w) \in C_{x}^{[d]}(X)\) for some \( x \in X \) and \( w \in C_{G}^{[d-1]}(X)\) and \((x^{[d-1]}, w)\) is an \( \mathcal{F}^{[d]}\)-minimal point, then \((x^{[d-1]}, w) \in Y_{x}^{[d]}(X)\).
Proof. We will show that there exists a minimal left ideal \( L \subset E(\mathcal{F}^{[d]}, C^{[d]}_G(X)) \) and an idempotent \( v \in L \) such that \( (\pi^*_f(v)x^{[d-1]}, w) \in Y^{[d]}_x \). Assume this is true. Since, by assumption, \((x^{[d-1]}, w)\) is \( \mathcal{F}^{[d]}_d \)-minimal, there is some minimal idempotent \( u \in J(L) \) such that \( u(x^{[d-1]}, w) = (\pi^*_f(u)x^{[d-1]}, \pi^*_c(u)w) = (x^{[d-1]}, w) \) (Proposition 4.2(2)). Since \( u, v \in L \) are idempotent in the same minimal left ideal \( L \), we have \( u \in Ev \) and this implies \( uv = u \) (Proposition 4.2(3)). Thus \( u(\pi^*_f(v)x^{[d-1]}, w) = (\pi^*_f(u)\pi^*_f(v)x^{[d-1]}, \pi^*_c(u)w) = (\pi^*_f(w)x^{[d-1]}, w) = (\pi^*_f(v)x^{[d-1]}, w) = (x^{[d-1]}, w) \) which implies \((x^{[d-1]}, w)\) in \( Y^{[d]}_x \) as desired.

To construct \( L \) and \( v \) notice that since \((x^{[d-1]}, w)\) is a minimal t.d.s by Proposition 4.6, \((x^{[d-1]}, w)\) is in the \( \mathcal{H}\mathcal{K}^{[d]}_{d} \)-orbit closure of \( x^{[d]} \), i.e. by Proposition A.4 there are sequences \((s_n \times s_n) \subset \mathcal{H}\mathcal{K}^{[d]}_{d} \) and \((\text{id}^{[d-1]} \times h_n) \in \mathcal{F}^{[d]}_d \) where \( s_n, h_n \in G^{[d-1]} \) such that:

\[
(\text{id}^{[d-1]} \times h_n)(s_n \times s_n)(x^{[d-1]}, x^{[d-1]}) = (s_n x^{[d-1]}, h_n s_n x^{[d-1]}) \rightarrow_n (x^{[d-1]}, w).

\]

letting \( a_n \triangleq s_n x^{[d-1]} = \pi_f(s_n x^{[d-1]}, h_n s_n x^{[d-1]}) \in C^{[d-1]}_G(X) \), we have:

\[
\text{(4.1) } (\text{id}^{[d-1]} \times h_n)(a_n, a_n) \rightarrow (x^{[d-1]}, w)
\]

Fix a minimal left ideal \( L \) of \( E(\mathcal{F}^{[d]}, C^{[d]}_G(X)) \). By Proposition 4.2(6) \((\mathcal{F}^{[d]}, L)\) is a minimal subsystem of \( E(\mathcal{F}^{[d]}, C^{[d]}_G(X)) \). Thus by Lemma 4.5 \( \pi_c(L) \subset E(\mathcal{H}\mathcal{K}^{[d]}_{d}, C^{[d]}_G(X)) \) is a minimal \( \mathcal{H}\mathcal{K}^{[d-1]}_{d} \)-subsystem. As \( (\mathcal{H}\mathcal{K}^{[d-1]}_{d}, C^{[d-1]}_G(X)) \) is minimal by Proposition 4.6 it follows from Proposition 4.2(7) that \( \pi_c(L)x^{[d-1]} = C^{[d-1]}_G(X) \). Thus there exist \( p_n \in L \) such that \( a_n = \pi_c(p_n)x^{[d-1]} \). Let \( p \in L \) be an accumulation point of \( \{p_n\} \). As by (4.1), \( \pi_c(p_n)x^{[d-1]} = a_n \rightarrow_n x^{[d-1]} \) we must have

\[
\text{(4.2) } \pi_c(p)x^{[d-1]} = x^{[d-1]}.
\]

If \( \pi_f^*(v)x^{[d-1]} = \pi_f^*(v)x^{[d-1]} \) for some idempotent \( v \in L \) then \( b_n \triangleq (\text{id}^{[d-1]} \times h_n)p_n x^{[d]} \in Y^{[d]}_x(X) \) and \( b_n = (\pi_f^*(p_n)x^{[d-1]}, h_n \pi_c(p_n)x^{[d-1]}) \rightarrow_n (\pi_f^*(v)x^{[d-1]}, w) \) as desired. However as this does not necessarily hold, the idea is to find an element \( q \in E(\mathcal{F}^{[d]}, C^{[d]}_G(X)) \) and \( v \in J(L) \) so that \( v = pq \) and \( \pi_c(pq)x^{[d-1]} = x^{[d-1]} \). Defining \( x_n = p_n q x^{[d]} \in Y^{[d]}_x(X) \), one has \((\text{id}^{[d-1]} \times h_n)x_n \rightarrow_n (\pi_f^*(v)x^{[d-1]}, w) \) as desired (see details below).

Indeed since \( L \) is a minimal left ideal and \( p \in L \), by Proposition 4.2(5) there exists a minimal idempotent \( v \in J(L) \) such that \( vp = p \). Thus:

\[
\text{(4.3) } \pi_c^*(v)x^{[d-1]} = \pi_c^*(v)\pi_c^*(p)x^{[d-1]} = \pi_c^*(v)p x^{[d-1]} = \pi_c^*(v)p x^{[d-1]} = x^{[d-1]}
\]

By Proposition 4.2(5) \( vL \) is a group. One verifies easily the following is a subgroup:

\[
S = \{a \in vL : \pi_c^*(a)x^{[d-1]} = x^{[d-1]}\}
\]
By (4.2), we have that $vp = p \in S$. Let $S$ so that $pq = v$. Thus $\pi^*_f(pq)x^{[d-1]} = x^{[d-1]}$. Denote $x_n = p_nqx^{[d]} \in Y_x^{[d]}(X)$. Note $\pi_f(x_n) = \pi^*_f(p_nq)x^{[d-1]} \rightarrow_n \pi^*_f(pq)x^{[d-1]} = \pi^*_f(v)x^{[d-1]}$ and $\pi_c(x_n) = \pi^*_c(p_nq)x^{[d-1]} = a_n$. As $id^{[d-1]} \times h_n \in \mathcal{F}^{[d]}$, $(id^{[d-1]} \times h_n)x_n \in Y_x^{[d]}(X)$. By (4.1), $(id^{[d-1]} \times h_n)x_n \rightarrow_n (\pi^*_f(v)x^{[d-1]}, w)$ and thus we conclude as desired $(\pi^*_f(v)x^{[d-1]}, w) \in Y_x^{[d]}(X)$.

With the above preparation we are ready to show:

**Theorem 4.10.** Let $(G, X)$ be a minimal topological dynamical system and $d \geq 1$, then for each $x \in X$, $(\mathcal{F}^{[d]}, Y_x^{[d]}(X))$ is the unique minimal subsystem of $(\mathcal{F}^{[d]}, C_x^{[d]}(X))$. Hence also $(\mathcal{F}^{[d]}, Y_x^{[d]}(X))$ is the unique minimal subsystem of the t.d.s $(\mathcal{F}^{[d]}, C_x^{[d]}(X))$.

**Proof.** For $d = 1$ the claim is obvious as $Y_x^{[1]}(X) = C_x^{[1]}(X)$. We assume by induction that the assertion holds for every $1 \leq j \leq d - 1$ and given $x \in X$, consider a minimal subsystem $Y$ of the t.d.s $(\mathcal{F}^{[d]}, C_x^{[d]}(X))$. Let $\pi_f$ be the floor projection (see Subsection 4.2). We observe that $Y_1 = \pi_f(Y)$ is a minimal subsystem of the t.d.s $(\mathcal{F}^{[d-1]}, C_x^{[d-1]}(X))$ and therefore, by the induction hypothesis $Y_1 = Y_x^{[d-1]}(X) = \mathcal{F}^{[d-1]}x^{[d-1]}$. But then for some $w \in C_x^{[d-1]}(X)$ we have $(x^{[d-1]}, w) \in Y$. Therefore the claim is reduced to the “useful lemma” [SY12, Lemma 5.1] which we reproduce as Lemma 4.9 in the sequel.

**Corollary 4.11.** Let $(G, X)$ be a minimal t.d.s and $d \geq 1$. If $c \in C_x^{[d]}(X)$ then $x^{[d]} \in \mathcal{F}^{[d]}c$.

**Proof.** Assume not, then there is more than one $\mathcal{F}^{[d]}$-minimal subsystem in $C_x^{[d]}(X)$ contradicting Theorem 4.10.

**Corollary 4.12.** Let $(G, X)$ be a minimal t.d.s and $d \geq 1$. Assume $\gamma^{[d]}(x, y) \in C_x^{[d]}(X)$ then $\gamma^{[d]}(x, y) \in Y_x^{[d]}(X)$. In particular $\gamma^{[d]}(x, y)$ is a $\mathcal{F}^{[d]}$-minimal point.

**Proof.** Note $(\mathcal{F}^{[d]}, \mathcal{F}^{[d]}y^{[d]}) \rightarrow (\mathcal{F}^{[d]}, \mathcal{F}^{[d]}\gamma^{[d]}(x, y))$ is an $\mathcal{F}^{[d]}$-isomorphism. Thus $(\mathcal{F}^{[d]}, \mathcal{F}^{[d]}\gamma^{[d]}(x, y)) \subset C_x^{[d]}(X)$ is $\mathcal{F}^{[d]}$-minimal. By Theorem 4.10, $\mathcal{F}^{[d]}\gamma^{[d]}(x, y) = Y_x^{[d]}(X)$ and the result follows.

**Corollary 4.13.** Let $(G, X)$ be a minimal t.d.s and $d \geq 2$. Assume $\gamma^{[d]}(x, y) \in C_x^{[d]}(X)$ then $(x^{[d-1]}, \gamma^{[d-1]}(y, x)) \in Y_x^{[d]}(X)$.

**Proof.** By Corollary 4.11 there is a sequence $f_k \in \mathcal{F}^{[d-1]}$ such that $f_k^{[d-1]}(x, y) \rightarrow x^{[d-1]}$. Conclude

$$(f_k \times f_k)^{[d]}(x, y) = (f_k \times f_k)^{[d-1]}(x, y, y^{[d-1]}) \rightarrow (x^{[d-1]}, \gamma^{[d-1]}(y, x)).$$

By Corollary 4.12, $\gamma^{[d]}(x, y) \in Y_x^{[d]}(X)$. As $f_k \times f_k \in \mathcal{F}^{[d]}$ (see Subsection A.2) the result follows.
In [HKM10, Proposition 4.2] it was proven that $Y_x[d](X) = C_x[d](X)$ for $(Z, X)$ minimal and distal t.d.s for each $x \in X$. Here we show that the same statement holds for a general group action.

**Theorem 4.14.** Let $(G, X)$ be a minimal distal topological dynamical system and $d \geq 1$, then for each $x \in X$, $Y_x[d](X) = C_x[d](X)$ and hence $Y_{2x}[d](X) = C_{2x}[d](X)$.

**Proof.** By [Aus88, Theorem 5.6] $(G[d], X[d])$ is a distal t.d.s. This immediately implies that $(\mathcal{F}[d], C_x[d](X))$ is a distal t.d.s. By [Aus88, Corollary 5.4(iii)], a distal system is semisimple, i.e decomposes into a disjoint union of minimal subsystems. By Theorem 4.10, $(\mathcal{F}[d], C_x[d](X))$ has a unique minimal subsystem $(\mathcal{F}[d], Y_{2x}[d](X))$. We thus conclude $Y_x[d](X) = C_x[d](X)$. 

**Remark 4.15.** There are non-distal minimal t.d.s for which $Y_x[d](X) \neq C_x[d](X)$. See [TY13, Example 3.6].

Let $Z$ be a compact metric space and let $2^Z$ denote the hyperspace consisting of the closed non-empty subsets of $Z$ equipped with the (compact metric) Vietoris topology ([Aki10, p. 124]). A function $X \to 2^Z$ is called lower-semi-continuous at $x \in X$ if for every open set $O \subset Z$ such that $f(x) \cap O \neq \emptyset$, we have that $\{ y \in X \mid f(y) \cap O \neq \emptyset \}$ is a neighborhood of $x$. A function $X \to 2^Z$ is called upper-semi-continuous at $x \in X$ if for every open set $O \subset Z$ such that $f(x) \subset O$, we have that $\{ y \in X \mid f(y) \subset O \}$ is a neighborhood of $x$ ([Aki10, Proposition 7.11]). A function $X \to 2^Z$ is continuous at $x \in X$ with respect to the Vietoris topology if it is both upper and lower semi-continuous at $x \in X$ ([Aki10, Lemma 7.5]).

The following theorem is new even for $G = Z$.

**Theorem 4.16.** Let $(G, X)$ be a minimal topological dynamical system where $X$ is metrizable, then for a dense $G_\delta$ subset $X_0 \subset X$ one has $Y_{2x}[d](X) = C_{2x}[d](X)$.

**Proof.** Consider $\Phi : X \to 2^{X[d]}$ given by $x \mapsto Y_x[d](X)$. It is easy to check that this map is lower-semi-continuous. By [Aki10, Theorem 7.19] the set of continuity points of $\Phi$ is a dense $G_\delta$ subset $X_0 \subset X$. Since by Proposition 3.4 the set $\mathcal{F}[d] \Delta[d](X)$ is dense in $C_G[d](X)$, it follows that at each point of $X_0$ we must have $Y_x[d](X) = C_x[d](X)$. Indeed let $x_0 \in X_0$ and assume $Y_{x_0}[d](X) \neq C_{x_0}[d](X)$. Let $U$ be an open set in $C_G[d](X)$ so that $Y_{x_0}[d](X) \subset C_{x_0}[d](X) \cap U \neq C_{x_0}[d](X)$. As $\Phi$ is upper-semi-continuous at $x_0$ the set $\{ x \in X \mid Y_x[d](X) \subset U \}$ is a neighborhood of $x_0$ and it follows $\mathcal{F}[d] \Delta[d]$ is not dense in $C_{x_0}[d](X)$. 

5. **NRP$^*[d](X)$ is an equivalence relation for minimal actions**

In this section we prove the main theorem of the article, Theorem 3.8:
Proof. Clearly \( \text{NRP}^{[d]}(X) \) (\( d \geq 1 \)) is a closed \( G \)-invariant and reflexive. To prove symmetry assume \((y, x) \in \text{NRP}^{[d]}(X)\), i.e. \( \gamma^{[d]}(y, x) \in C^{[d+1]}_G(X) \). Permuting coordinates (see Proposition A.1) we have \( \gamma^{[d]}(x, y) \in C^{[d+1]}_G(X) \). Projecting we have \( \gamma^{[d]}(x, y) \in C^{[d]}_G(X) \). By Corollary 4.11 there is a sequence \( f_k \in F^{[d]} \) so that \( f_k \gamma^{[d]}(x, y) \to x^{[d]} \). Thus \((f_k \times f_k)\gamma^{[d+1]}(x, y) = (f_k \times f_k)\gamma^{[d]}(y, x) \to (x^{[d]}, \gamma^{[d]}(y, x)) \) (see Subsection A.2). Permuting coordinates again we have \( \gamma^{[d+1]}(x, y) \in C^{[d+1]}_G(X) \) as desired.

To prove transitivity assume \((x, y), (y, z) \in \text{NRP}^{[d]}(X)\). By symmetry \((z, y) \in \text{NRP}^{[d]}(X)\). Permuting coordinates we have \( \gamma^{[d]}(y, z) \in C^{[d+1]}_G(X) \). By Corollary 4.12 \( \gamma^{[d+1]}(y, z) \) induce minimal \( F^{[d+1]} \)-subsystems of \( C^{[d+1]}_G(X) \) and thus by Theorem 4.10 \( \gamma^{[d]}(y, z) \in C^{[d+1]}_G(X) \). As \((F^{[d+1]}, \overline{F^{[d+1]}}(y, z)) \to Y_x^{[d+1]}(X) \) given by \( w \mapsto (x, w_*) \) is an \( F^{[d+1]} \)-isomorphism it follows that \( \gamma^{[d+1]}(x, z) \in Y_x^{[d+1]}(X) \). Permuting coordinates we have \( \gamma^{[d+1]}(z, x) \in C^{[d+1]}_G(X) \). Thus \((z, x) \in \text{NRP}^{[d]}(X)\). By symmetry \((x, z) \in \text{NRP}^{[d]}(X)\) as desired. \( \square \)

6. Lifting \( \text{NRP}^{[d]}(X) \) from factors to extensions

Let \((G, X)\) be a minimal t.d.s. In Lemma A.5 we note that if \( \pi : (G, X) \to (G, Y) \) is a dynamical morphism then \( \pi \times \pi(\text{NRP}^{[d]}(X)) \subset \text{NRP}^{[d]}(Y) \). In [SY12, Theorem 6.4] it is proven that for \( G \) abelian equality holds, i.e., \( \pi \times \pi(\text{NRP}^{[d]}(X)) = \text{NRP}^{[d]}(Y) \). We next show that the same is true for general minimal group actions. Our proof follows the framework of the proof of [SY12, Theorem 6.4].

Theorem 6.1. Let \((G, X)\) be a minimal topological dynamical system. If \( \pi : (G, X) \to (G, Y) \) is a dynamical morphism then \( \pi \times \pi(\text{NRP}^{[d]}(X)) = \text{NRP}^{[d]}(Y) \).

Proof. Let \((y_1, y_2) \in \text{NRP}^{[d]}(Y)\). Our goal is to find \((x_1, x_2) \in \text{NRP}^{[d]}(X)\) such that \( \pi(x_1) = y_1 \) and \( \pi(x_2) = y_2 \). This will be referred to as in the sequel as lifting \((y_1, y_2)\). By Proposition 4.2(4) there is a minimal point \((y'_1, y'_2) \in \overline{O((y_1, y_2), G)}\) such that \((y'_1, y'_2)\) is proximal to \((y_1, y_2)\). Note \((y'_1, y'_2) \in \text{NRP}^{[d]}(Y)\) as \( \text{NRP}^{[d]}(Y) \) is \( G \)-invariant and closed. Since \((y_1, y'_1), (y_2, y'_2) \in P(Y)\), then by [SY12, Lemma 6.3] there are \( x_1, x_2 \in X \) such that \( \pi \times \pi(x_1, x_2) = (y_1, y'_2) \) and \((x'_1, x_1), (x'_2, x_2) \in P(X)\). By Lemma A.5(1) \((x'_1, x_1), (x'_2, x_2) \in \text{NRP}^{[d]}(X)\). Assume we have proven one can lift \((y'_1, y'_2)\), i.e., there is \((x'_1, x'_2) \in \text{NRP}^{[d]}(X)\) with \( \pi \times \pi(x'_1, x'_2) = (y'_1, y'_2) \). By the transitivity of \( \text{NRP}^{[d]}(X) \) (Theorem 3.8), \((x_1, x'_1), (x'_2, x_2) \in \text{NRP}^{[d]}(X)\) imply \((x_1, x_2) \in \text{NRP}^{[d]}(X)\). Hence we can assume without loss of generality that \((y_1, y_2)\) is a minimal point of \((Y \times Y), G)\).

Let \( q_1 \in \pi^{-1}(y_1) \). We will find \( q_2 \in \pi^{-1}(y_2) \) such that \((q_1, q_2) \in \text{NRP}^{[d]}(X)\) and such that some \((x_1, x_2)\) in the orbit closure of \((q_1, q_2)\) lifts \((y_1, y_2)\). As
an intermediary step we construct cubes in $C^{[d+1]}_q(X)$ with an increasing number of vertices whose value is $q_1$.

As $(y_1, y_2) \in \text{NRP}^{[d]}(Y)$, by Corollary 4.13 there is a sequence $f_k \in F^{[d+1]}$, $f_k y_1^{[d+1]} \rightarrow (y_1^{[d-1]}, y^{[d-1]}(y_2, y_1))$. Let $c \in C^{[d+1]}_q(X)$ be an accumulation point of the sequence $f_k q_1^{[d+1]}$. Note $\pi(c) = (y_1^{[d-1]}, y^{[d-1]}(y_2, y_1))$. Let $F_i = \{ \omega \in \{0, 1\}^{d+1} | \omega_i = 0 \}$ be an enumeration of lower hyperfaces of $\{0, 1\}^{d+1}$. Inductively we will construct elements $c_{d+1}, c_d, \ldots, c_1 \in C^{[d+1]}_q(X)$ and $z_{d+1} = y_2, z_d, \ldots, z_1 \in Y$ such that for $i = d + 1, d, \ldots, 1$:

1. $c_i(\omega) = q_1$ for $\omega \in F_{d+1} \cup F_d \cup \cdots \cup F_i$
2. $\pi(c_1)(0 \cdot 0 1 \cdots 1) = z_i$
3. $\pi(c_i)(\omega) = y_1$ for all $\omega \neq (0 \cdots 0 1 \cdots 1)$
4. $(z_i, y_1) \in O((y_1, z_{i+1}), G)$ (only for $i \leq d$)

Assume this has been achieved. Let us consider the element $c_1$. As $F_{d+1} \cup F_d \cup \cdots \cup F_i = \{0, 1\}^{d+1}$, we have $c_1 = \omega_{d+1}(q_1, q_2)$ for some $q_2 \in X$. Thus $(q_1, q_2) \in \text{NRP}^{[d]}(X)$. By (2) $\pi(q_2) = z_1$. By (4)

$$(z_1, y_1) \in \overline{O((y_1, z_2), G)} \subset \overline{O((z_3, y_1), G)} \subset \overline{O((y_1, z_4), G)} \subset \cdots$$

Thus $(z_1, y_1) \in \overline{O((z_{d+1}, y_1), G)}$ or $(z_1, y_1) \in \overline{O((y_1, z_{d+1}), G)}$. Assume without loss of generality the first case. As $z_{d+1} = y_2$ and $(y_1, y_2)$ is a minimal point of $(Y \times Y, G)$, $(y_1, y_2) \in \overline{O((y_1, z_1), G)}$. Let $g_k \in G$ so that $(g_k y_1, g_k z_1) \rightarrow (y_1, y_2)$. Assume without loss of generality $(g_k q_1, g_k q_2) \rightarrow (x_1, x_2)$. As $\text{NRP}^{[d]}(X)$ is $G$-invariant and closed, $(x_1, x_2) \in \text{NRP}^{[d]}(X)$. Moreover we have, as desired:

$$\pi \times \pi(x_1, x_2) = \lim_k \pi(g_k \pi(q_1, g_k \pi(q_2))) = \lim_k (g_k y_1, g_k z_1) = (y_1, y_2)$$

We now return to the inductive construction of $c_{d+1}, c_d, \ldots, c_1 \in C^{[d+1]}_q(X)$.

By Corollary 4.11, there is a sequence $f_k \in F^{[d]}$ such that $f_k c_{d+1} \rightarrow q_1^{[d-1]}$. Let $c_{d+1} \in C^{[d+1]}_q(X)$ be an accumulation point of the sequence $f_k \times f_k c$. Thus $c_{d+1}(\omega) = q_1$ for $\omega \in F_{d+1}$ and property (1) holds for $i = d + 1$.

Combining $f_k \pi(c)_{|F_{d+1}} \rightarrow y_1^{[d-1]}$ with $\pi(c) = (y_1^{[d-1]}, \gamma^{[d-1]}(y_2, y_1))$ we have $\pi(c_{d+1}) = \lim_{k} \pi(f_k \times f_k (y_1^{[d-1]}, \gamma^{[d-1]}(y_2, y_1))) = (y_1^{[d-1]}, \gamma^{[d-1]}(y_2, y_1))$ which implies property (2). Thus denoting $z_{d+1} = y_2$ we have $\pi(c_{d+1})(0 \cdots 0 1) = z_{d+1}$ which is property (2) for $i = d + 1$.

Assume we have already constructed $c_{i+1} \in C^{[d+1]}_q(X)$ and $z_{i+1} \in Y$. By Corollary 4.11, there is a sequence $f_k \in F^{[d]}$ such that $f_k c_{i+1} \rightarrow q_1^{[d-1]}$. Let $c_i \in C^{[d+1]}_q(X)$ be an accumulation point of the sequence $D_i(f_k) c_{i+1}$ (for the notation $D_i(\cdot)$ see Subsection A.2). Clearly $c_{i+1} \rightarrow q_1^{[d-1]}$. In order to establish property (1), we have to show in addition that for $\omega \in (F_{d+1} \cup F_d \cup \cdots \cup F_{i+1}) \setminus F_i$ it holds that $c_i(\omega) = q_1$. Define: $\phi_i : \{0, 1\}^{d+1} \rightarrow F_i$ to
be the projection on $F_i$, i.e., $\phi_i(\omega_1\omega_2\cdots\omega_{d+1}) = (\omega_1\omega_2\cdots\omega_{i}\cdots\omega_{d+1})$.

Fix $\omega \in F_i \setminus F_j$ for $j > i$. As $\omega, \phi_i(\omega) \in F_j$, \(c_{i+1}(\omega) = c_{i+1}(\phi_i(\omega)) = q_1\).

By the definition of doubling, the same is true for $D_i(f_k)$, i.e., $D_i(f_k)(\phi_i(\omega)) = D_i(f_k)(\omega)$ and thus we conclude $c_i(\omega) = c_i(\phi_i(\omega)) = q_1$ as desired, where the last equality follows from $\phi_i(\omega) \in F_i$. Denote $\pi(c_i)(0\cdots 0\,1\cdots 1) = z_i$.

We now establish property (3). If $\phi = \phi_i(\omega)$ for all $\omega \neq (0\cdots 0\,1\cdots 1)$. By the inductive construction $\pi(c_{i+1})(\omega) = y_1$ for all $\omega \neq (0\cdots 0\,1\cdots 1)$.

Note that $\phi_i(\omega) = (0\cdots 0\,i+1\cdots 1)$ implies $\omega = (0\cdots 1\cdots 1)$. Thus as $(0\cdots 0\,1\cdots 1) \in F_i$, we conclude that for $\omega \in F_i^c \setminus \{(0\cdots 0\,1\cdots 1)\}$, $\pi(c_{i+1}(\omega)) = \pi(c_{i+1}(\phi_i(\omega))) = y_1$. By the definition of doubling, $\pi(c_i(\omega)) = \pi(c_i(\phi_i(\omega))) = \pi(q_1) = y_1$ as desired. From property (3) we have $\pi(c_i(0\cdots 0\,i+1\cdots 1)) = y_1$ which implies for $g_k = D_i(f_k)(0\cdots 0\,1\cdots 1) = D_i(f_k)(0\cdots 0\,i+1\cdots 1)$ that

$$y_1 = \lim_{k} g_k \pi(c_{i+1}(0\cdots 0\,i+1\cdots 1)) = \lim_{k} g_k z_{i+1}.$$  

Similarly as $\pi(c_{i+1}(0\cdots 0\,1\cdots 1)) = y_1$, $z_i = \lim_{k} g_k y_1$. We thus have $(z_i, y_1) = \lim_{k} g_k \times g_k(y_1, z_{i+1})$ which is property (4).
7. Systems of order $d$

7.1. Overview. In this section we investigate the structure of minimal systems whose regionally proximal relation of order $d$ is trivial.

Definition 7.1. Let $d \geq 1$. A t.d.s $(G, X)$ is called a system of order $d$ if $d$ is the minimal integer such that $\text{NRP}^{[d]}(X) = \Delta$.

The fundamental example of systems of order $d$ is given by nilsystems:

Definition 7.2. Let $d \geq 1$ be an integer and assume that $L$ is a nilpotent Lie group of nilpotency class $d$ and $\Gamma \subset L$ a discrete, cocompact subgroup of $L$. Denote $X = L/\Gamma$. Notice that $L$ acts naturally on $X$ by left translations: $l \Gamma \to g l \Gamma$ for $g \in L$. Let $G$ be a topological group and let $\phi : G \to L$ be a continuous homomorphism, then the induced action $(G, X)$ is called a nilsystem of order $d$.

Theorem 7.3. Let $(G, X)$ be a minimal nilsystem of order $d$ where $G$ is an arbitrary topological group, then it is a system of order at most $d$.

Proof. See Example 3.7.

A natural question which arises is if one can characterize systems of order $d$ in terms of nilsystems. We will return to this question in Subsection 7.5. In the meantime we will opt for a more abstract treatment. The next corollary provides a canonical way to generate systems of order at most $d$.

Corollary 7.4. Let $(G, X)$ be a minimal t.d.s, then $\text{NRP}^{[d]}(X/\text{NRP}^{[d]}(X)) = \Delta$, i.e. $(G, X/\text{NRP}^{[d]}(X))$ is a system of order at most $d$.

Proof. Let $Y = X/\text{NRP}^{[d]}(X)$ and $\pi : X \to Y$ the associated factor map. By Theorem 6.1, $\text{NRP}^{[d]}(Y) = \pi \times \pi(\text{NRP}^{[d]}(X)) = \Delta$.

The next theorem shows that dividing out by the regionally proximal relation results with the maximal factor which is a system of order at most $d$:

Theorem 7.5. Let $d \geq 1$ and let $(G, X)$ be a minimal topological dynamical system, then $\pi_d : (G, X) \to (G, X/\text{NRP}^{[d]}(X))$ is the maximal factor of order at most $d$ of $(G, X)$. That is, if $\phi : (G, X) \to (G, Y)$ is a factor map where $(G, Y)$ is a system of order at most $d$, then there exists a factor map $\psi : (G, X/\text{NRP}^{[d]}(X)) \to (G, Y)$ such that $\phi = \psi \circ \pi_d$.

Proof. By Corollary 7.4 $(G, X/\text{NRP}^{[d]}(X))$ is a system of order at most $d$. It is enough to show that $(x, y) \in \text{NRP}^{[d]}(X)$, implies $\phi(x) = \phi(y)$. Indeed by Lemma A.5(5) $(x, y) \in \text{NRP}^{[d]}(X)$ implies $(\phi(x), \phi(y)) \in \text{NRP}^{[d]}(Y) = \Delta$ and thus $\phi(x) = \phi(y)$.

---

A Lie group is a second countable topological group $G$ that has a differentiable structure such that the map $G^2 \to G : (g, h) \mapsto gh^{-1}$ is differentiable. Note we do not assume that Lie groups are connected. In particular countable discrete groups are Lie.
Remark 7.6. Systems of finite order are distal.

Proof. By Lemma A.5(1), \( P(X) = \Delta. \)

We now move on to more advanced structure theorems for systems of order \( d. \) The key tool is the theory of nilspaces introduced by Antolín Camarena and Szegedy. We review this theory in Subsection 7.2 and in Subsection 7.3 we prove that minimal systems of finite order are nilspaces. This allows us to adapt the so-called weak structure theorem of Antolín Camarena and Szegedy to the dynamical context in Subsection 7.4. In Subsection 7.5 we quote the stronger Gutman-Manners-Varjú structure theorem for systems of finite order which hold under some restrictions on the acting group.

7.2. Nilspaces. A map \( f = (f_1, \ldots, f_k) : \{0,1\}^d \to \{0,1\}^k \) is called a **morphism of discrete cubes** if each coordinate function \( f_j(\omega_1, \ldots, \omega_d) \) is either identically 0, identically 1, or it equals either \( \omega_i \) or \( \overline{\omega_i} = 1 - \omega_i \) for some \( 1 \leq i = i(j) \leq d. \)

In Subsection 3.3 we introduced dynamical cubespaces. A (general) **cubespaces** is a pair \((X, C^\bullet)\) consisting of a compact metric space \(X\) together with a collection of closed subsets \(C^d(X) \subseteq X^d\), for each integer \(d \geq 0\), called **cubes**, so that for any morphism of discrete cubes \( f : \{0,1\}^d \to \{0,1\}^k \) and any \( c \in C^k(X)\), we have \( c \circ f \in C^d(X)\). We refer to this property as **cub invariance**. When no confusion arises we denote the cubespaces simply by \(X\). It is not hard to verify that dynamical cubespaces are cubespaces (See Proposition A.1). We say that a cubespaces \((X, C^\bullet)\) is **ergodic**, if \(C^1(X) = X^{[1]} = X \times X\), that is to say, if any pair of elements forms a 1-cube.

Let \(X\) be a cubespaces and let \( f : X \to \{0,1\}_d^\bullet \) be a map. We call \( f\) a **d-corner** if \( f|_{\{\omega \in (0,1)^{d+1} : \omega_i = 0\}} \) is a \((d-1)\)-cube for all \(1 \leq i \leq d\). We say that the cubespaces \((X, C^\bullet)\) has **d-completion** if for any d-corner \( f\), there is a cube \( c \in C^d(X)\) such that \( c|_{\{0,1\}_d^\bullet} = f\). We say that \((X, C^\bullet)\) is **fibrant** if it has d-completion for all \(d \geq 1\).

Example 7.7. Recall that \((G, X)\) is called **transitive**, if for every pair of non-empty open subsets \(U\) and \( V\), there is \( g \in G\) such that \( U \cap gV \neq \emptyset\); is called **weakly mixing** if the diagonal action \((\Delta^2(G), X)\) is transitive; and is called **transitive of all orders** if the diagonal action \((\Delta^n(G), X)\) is transitive for all \(n \in \mathbb{N}\). An example of a t.d.s which is fibrant is given by a minimal system which is transitive of all orders\(^9\). See Proposition A.6.

We say that \((X, C^\bullet)\) has **d-uniqueness** if the following holds: whenever \(c, c' \in C^d(X)\) and \(c(\omega) = c'(\omega)\) for all \(\omega \in \{0,1\}_d^\bullet\) then \(c = c'\).  

\(^9\)For \(G\) abelian transitivity of all orders is equivalent to weak mixing ([Gla03, Theorem 1.11]). For \(G\) non-abelian the conditions are not equivalent ([Wei00, p. 277]). If \((G, X)\) is minimal and admits an invariant measure with full support with respect to which it is measurably weakly mixing then it is transitive of all orders ([AAG08, Theorem 6.12])
We say that a cubespace \((X, C^*_G)\) is a nilspace of order \(d\) if it is fibrant and \(d \geq 0\) is the smallest integer such that \(X\) has \((d+1)\)-uniqueness.

Let \(X\) be a cubespace and let \(\sim\) be a closed equivalence relation on \(X\). One endows \(X/\sim\) by a cubespace structure by declaring a configuration \(c \in (X/\sim)^{[d]}\) a cube if and only if there is a cube \(c' \in C^{[d]}(X)\) such that \(\pi(c') = c\). It is clear that \(X/\sim\) is indeed a cubespace.

Let \(X\) be a fibrant cubespace. Define \(x \sim_d y\) if and only if there are two cubes \(c_1, c_2 \in C^{[d+1]}(X)\) such that \(c_1(\omega) = c_2(\omega)\) for \(\omega \neq \tilde{1}\) and \(c_1(\tilde{1}) = x\) and \(c_2(\tilde{1}) = y\). Denote \(\pi_d : X \to X/\sim_d\). By [GMV16a, Proposition 6.3] (following [ACS12, Section 2.4] and [HK08, Section 3.3]) \(\sim_d\) is an equivalence relation and \(\pi_d(X)\) is a nilspace. We call \(X/\sim_d\) the \(d\)-th canonical factor of \(X\). The following remark is trivial:

**Remark 7.8.** Let \(d \geq 0\). A cubespace \(X\) has \((d+1)\)-uniqueness iff \(\sim_d = \Delta\).

The relation between successive canonical factors is elucidated by the so-called weak structure theorem proven by Antolín Camarena and Szegedy in [ACS12, Theorem 1]. A detailed exposition is given in [GMV16a, Chapters 6 & 7]. We quote a partial version of the theorem:

**Theorem 7.9.** Let \(X\) be an ergodic nilspace of order at most \(d\). Then there is an additive compact abelian group \(A_d\) acting continuously and freely on \(X\) such that the orbits of \(A_d\) coincide with the fibres of \(\pi_{d-1} : X \to X/\sim_{d-1}\).

Iterating the theorem we see that a nilspace of finite order can be represented by a finite tower of compact abelian group extensions:

\[
X \to \pi_{d-1}(X) \to \pi_{d-2}(X) \to \ldots \to \pi_0(X) = \bullet
\]

In Subsection 7.4 we will adapt this theorem to the dynamical context.

### 7.3. Minimal distal systems are fibrant.

**Theorem 7.10.** Let \((G, X)\) be a minimal distal topological dynamical system, then the cubespace \((X, C^*_G)\) is ergodic and fibrant.

The fact that \((X, C^*_G)\) is ergodic follows trivially from minimality of \((G, X)\). The proof that \((X, C^*_G)\) is fibrant splits into a number of lemmas, which are based on [HKM10, Section 4.2].

In this subsection we will identify \(\{0,1\}^d\) with the collection of all subsets of \(\{1, \ldots, d\}\) and write \(\omega' \subseteq \omega\) for \(\omega', \omega \in \{0,1\}^d\) if \(\omega'(i) \leq \omega(i)\) for all \(i\).

Let \(V \subseteq \{0,1\}^d\) be a downwards-closed subset, i.e. if \(\omega \in V\) and \(\omega' \subseteq \omega\) then \(\omega' \in V\). Denote by \(\text{Hom}(V, X)\) the set of maps \(\alpha : V \to X\) such that for all \(\omega \in V\), \(\alpha|_{\{\omega\} \cap \{\omega' : \omega' \subseteq \omega\}}\) is a cube of \(X\).

**Lemma 7.11.** Let \((G, X)\) be a distal t.d.s and \(V \subseteq \{0,1\}^d\) a downwards-closed subset. Then \((\mathcal{H}K^{[d]}, \text{Hom}(V, X))\) equipped with the coordinate-wise action is a distal system.
Proof. By [Aus88, Chapter 5, Theorem 6] \((G^[[d]], X^[[d]])\) is a distal system. As \(\text{Hom}(V, X) \subset X^[[d]]\) this immediately implies that \((\mathcal{H}K^[[d]], \text{Hom}(V, X))\) is a distal system.

In particular, for \(\alpha_1, \alpha_2 \in \text{Hom}(V, X)\), we have \(\alpha_1 \in \overline{O(\alpha_2, \mathcal{H}K^[[d]])}\) if and only if \(\alpha_2 \in O(\alpha_1, \mathcal{H}K^[[d]])\).

Let \(V \subseteq \{0, 1\}^d\) be a downwards-closed subset. We say that \(\text{Hom}(V, X)\) has the extension property if for every \(\alpha \in \text{Hom}(V, X)\), there exists \(c \in C_G^[[d]](X)\) so that \(c|_V = \alpha\). Note that a cubespace \((X, C^*(X))\) has \(d\)-completion if and only if \(\text{Hom}(\{0, 1\}^d, X)\) has the extension property. Therefore Theorem 7.10 follows from the next lemma.

**Lemma 7.12.** Let \((G, X)\) be a minimal distal t.d.s and let \(V \subseteq \{0, 1\}^d\) be a downwards-closed subset, then \(\text{Hom}(V, X)\) has the extension property.

**Proof.** We prove the lemma by a double induction; first we induct on \(d\), then on the cardinality of \(V\). If \(d = 1\), the claim is clear. We assume that the claim holds for downward-closed subsets in \(\{0, 1\}^{d-1}\) and prove it for downward-closed subsets in \(\{0, 1\}^d\).

Let \(V\) be a downward-closed subset in \(\{0, 1\}^d\). If \(V = \{\bar{0}\}\), the result is clear. Assume \(|V| \geq 2\) and \(V \neq \{0, 1\}^d\). Let \(\bar{1} \neq \bar{\omega} \in V\) be a maximal element in \(V\) and denote \(W = V \setminus \{\bar{\omega}\}\). (Note that \(W \neq \emptyset\).)

Let \(\alpha \in \text{Hom}(V, X)\). We first consider the special case that \(\alpha|_W \equiv x\) for some \(x \in X\). We show that \(\alpha\) can be extended to a cube. Let \(1 \leq i \leq d\) be such that \(\bar{\omega}_i = 0\) and define \(F = \{\omega \in \{0, 1\}^d | \omega_i = 0\}\) and \(E = \{\omega \in \{0, 1\}^d | \omega = 1\}\).

By the inductive assumption, \(\alpha|_{V \cap F}\) can be extended to a map \(c_1 : F \to X\) that is a cube. Let \(c_2 = D_1(c_1)\). By Subsection A.2 \(c_2 \in C_G^[[d]](X)\).

We show that \(c_2\) is an extension of \(\alpha\). This is clearly true on \(F \cap V\). Let \(\omega \in E \cap V\). As \(\tau(\omega) \subseteq \omega\) and \(V\) is a downward-closed subset, we must have \(\tau(\omega) \in V\). Moreover, \(\omega \neq \bar{\omega}\) as \(\bar{\omega}\) is maximal in \(V\). Thus \(c_2(\omega) = c_1 \circ \tau(\omega) = \alpha \circ \tau(\omega) = x = \alpha(\omega)\).

![Figure 7.1](image-url)  
**Figure 7.1.** An example of \(\alpha\) and \(c_2\).

We now return to the general case. By the inductive assumption, \(\alpha|_W\) can be extended to a cube \(c_1 \in C_G^[[d]](X)\). By Proposition 4.6, \((\mathcal{H}K^[[d]], C_G^[[d]](X))\) is
minimal. Therefore, we can find a sequence \( h_i \in \mathcal{H}K[d] \) such that \( \lim_i h_i c_1 = x[d] \).

Let \( \alpha' = \lim_i h_i \alpha \) (we can assume without loss of generality that the limit exists). By Lemma 7.11, \( \text{Hom}(V, X) \) is invariant under the action of \( \mathcal{H}K[d] \), and therefore \( \alpha' \in \text{Hom}(V, X) \). As we have \( \alpha'|_W \equiv x \) in addition, we may conclude by the previous case that \( \alpha' \) can be extended to a cube \( c_2 \).

Using Lemma 7.11, we can find a sequence \( g_i \in \mathcal{H}K[d] \) such that \( \alpha = \lim_i g_i \alpha' \). We conclude that \( \lim_i g_i c_2 \) is an extension of \( \alpha \) (again we can assume without loss of generality that the limit exists). □

We are now ready to prove that minimal systems of finite order are nilspaces. The key observation is that the canonical equivalence relation \( \sim_s \) has the following alternative definition:

**Proposition 7.13.** Let \( X \) be a fibrant cubespace and \( d \geq 1 \), then \( x \sim_d y \) if and only if \( \llbracket d+1 \rrbracket(x, y) \) is a cube.

**Proof.** This is proven in [GMV16a, Lemma 6.6] (see also [ACS12, Lemma 2.3] and [HK08, Proposition 3]). □

We now prove:

**Theorem 7.14.** Let \( d \geq 1 \) and let \((G, X)\) be a minimal topological dynamical system, then \((G, X)\) is a system of order at most \( d \) iff the cubespace \((X, C_G)\) is an ergodic nilspace of order at most \( d \).

**Proof.** Assume that \((G, X)\) is a system of order at most \( d \). By Remark 7.6, \((G, X)\) is distal. In view of Theorem 7.10 one has only to establish that \((X, C_G)\) has \((d + 1)\)-uniqueness. By Remark 7.8 this is equivalent to \( \sim_d = \Delta \). By Proposition 7.13, \( \text{NRP}^d(X) = \sim_d \). As \( \text{NRP}^d(X) = \Delta \), the result follows. Conversely if \((X, C_G^*)\) is a nilspace of order at most \( d \) then \( \text{NRP}^d(X) = \sim_d = \Delta \). □

In [Tao15] Tao asks for "an interpretation of the regionally proximal relation in the nilspace language." We believe the following theorem answers his question:

**Theorem 7.15.** Let \( d \geq 1 \) and let \((G, X)\) be a minimal topological dynamical system, then \( \pi_d : (G, X) \to (G, X/\text{NRP}^d(X)) \) is the maximal factor which is an ergodic nilspace of order at most \( d \).

**Proof.** Follows from Theorem 7.14 and Theorem 7.5. □

7.4. **Weak structure theorem for minimal systems of finite order.**

In this subsection we adapt the so-called weak structure theorem of Antolin Camarena and Szegedy (see Theorem 7.9) to the dynamical context. First we introduce the appropriate terminology:

**Definition 7.16.** (See [Gla03, p.15] and [dV93, V(4.1)]) A dynamical morphism \( f : (G, X) \to (G, X) \) is called an **automorphism** if \( f \) is bijective. The
group of automorphisms equipped with the uniform topology is denoted by \( \text{Aut}(G, X) \). A dynamical morphism \( \pi : (G, X) \to (G, Y) \) is called a principal abelian group extension if there exists a compact abelian group \( K \subset \text{Aut}(G, X) \) such that for all \( x, y \in X \), \( \pi(x) = \pi(y) \) iff there exists a unique \( k \in K \) such that \( kx = y \). If \( Y = \bullet \), then \((G, X)\) is called an abelian group t.d.s. It is not hard to see that \((G, X)\) is a minimal abelian group t.d.s if and only if \( X \) is a compact abelian group and there exists a continuous group homomorphism \( \phi : G \to X \) with \( \phi(G) = X \) such that \( G \) acts through \( gx = \phi(g) + x \).

Our main result in this subsection is:

**Theorem 7.17.** Let \( d \geq 1 \) and let \((G, X)\) be a minimal topological dynamical system of order at most \( d \), then the following is a sequence of principal abelian group extensions:

\[
(7.2) \quad (G, X) \to (G, X/\text{NRP}^{[d-1]}(X)) \to \cdots \to (G, X/\text{NRP}^{[1]}(X)) \to \bullet
\]

In particular \((G, X/\text{NRP}^{[1]}(X))\) is an abelian group t.d.s.

**Proof.** As \( \text{NRP}^{[d]}(X) = \sim_d \), the tower structure (7.2) is a direct consequence of (7.1), however one has to show that the successive maps in (7.2) are principal abelian group extensions. As \( \text{NRP}^{[d]}(X) \) is a \( G \)-equivariant closed equivalence relation the maps are dynamical morphisms. By Theorem 7.9 there is an additive compact abelian group \( K_d \) acting continuously and freely on \( X \) such that the orbits of \( K_d \) coincide with the fibres of \( \pi_{d-1} : (G, X) \to (G, X/\text{NRP}^{[d-1]}(X)) \). We will show \( K_d \subset \text{Aut}(G, X) \). From [GMV16a, p.45]:

\[
K_d = \text{NRP}^{[d-1]}(X)/\sim
\]

where \( (x, x') \sim (y, y') \) if and only if \( (\cup_d(x, x'), \cup_d(y, y')) \in C_G^{[d+1]}(X) \). Denote the equivalence classes by \( [x, x'] \). These classes corresponds to the elements of \( K_d \). Fix \( x \in X \), \( a \in K_d \), \( g \in G \). We have to show that the equality \( a(gx) = g(ax) \) holds. Denote \( x' = ax \). By definition ([GMV16a, p.47]), \( (x, x') \in \text{NRP}^{[d-1]}(X) \) and \( a = [x, x'] \). We conclude \( \cup_d(x, x') \in C_G^{[d]}(X) \).

By doubling \( (\cup_d(x, x'), \cup_d(x, x')) \in C_G^{[d+1]}(X) \) (see Subsection A.2) and this implies \( (\cup_d(x, x'), \cup_d(gx, gx')) \in C_G^{[d+1]}(X) \) by Equation (3.1) in Subsection 3.3. Thus \( (x, x') \sim (gx, gx') \) which implies \( a = [gx, gx'] = [gx, g(ax)] \), i.e \( a(gx) = g(az) \) as desired. \( \square \)

7.5. **Strong structure theorem for some systems of finite order.** In Theorem 7.3 we saw that minimal nilsystems are systems of finite order. It is not hard to see that an inverse limit of nilsystems of uniformly bounded order is a system of finite order. It turns out that under some restrictions on the acting group one can prove that these are the only possible examples. We quote [GMV16b, Theorem 1.29]:

\[
\]
Theorem 7.18. Let \( d \geq 1 \) and let \((G, X)\) be a minimal topological dynamical system of order at most \(d\), where \(G\) has a dense subgroup generated by a compact set and where \(X\) is metrizable. Then \((G, X)\) is a pronilsystem of order at most \(d\).

We recall that the system \((G, X)\) is a pronilsystem of order at most \(d\) when:

- There exists a sequence of nilpotent Lie groups \(G^{(n)}\) of nilpotency class at most \(d\);
- for each \(n\), there is a continuous homomorphism \(\alpha_n : G \to G^{(n)}\);
- for each \(n\), there is a discrete co-compact subgroup \(\Gamma^{(n)} \subseteq G^{(n)}\); and
- for each \(n > m\), there is a continuous homomorphism \(\psi_{m,n} : G^{(n)} \to G^{(m)}\),

such that

- \(\psi_{m,n}(\Gamma_n) \subseteq \Gamma_m\),
- \(\alpha_m = \psi_{m,n} \circ \alpha_n\),
- and \((G, X)\) is isomorphic as a topological dynamical system to the inverse limit of the nilsystems \((G, G^{(n)}/\Gamma^{(n)})\) given by the inverse system of maps induced by \(\psi_{m,n}\), where \(G\) acts on \(G^{(n)}/\Gamma^{(n)}\) via \(\alpha_n\).

Remark 7.19. A minimal t.d.s isomorphic to a tower of principal abelian group extensions as in (7.2) is not necessarily of finite order. Consider the famous Furstenberg counterexample ([Fur61, end of Subsection 3.1], see also [Par81, Chapter 5.5]) of a homeomorphism of the torus \(T = (x, y) \mapsto (x + \alpha, y + \phi(x))\) which is minimal distal but not uniquely ergodic. Denote \(\pi : S^2 \to \mathbb{S}\) by \((x, y) \mapsto x\). Then \(\pi\) realizes \((S^2, T)\) as a circle extension of the maximal equicontinuous factor which is also a circle. Note however that \((S^2, T)\) is not a finite order system. Indeed by a classical Theorem of Green ([AHG+63], see also [Par70]) a minimal \(\mathbb{Z}\)-nilsystem is uniquely ergodic. Thus by the above Theorem 7.18 a finite order \(\mathbb{Z}\)-system is uniquely ergodic.

8. A different generalization of \(\text{RP}^{[d]}(X)\)

8.1. The relation between \(\text{NRP}^{[1]}\) and \(\text{RP}(X)\). Recall the definitions of \(\text{RP}(X)\) and \(\text{Seq}(X)\) from Subsection 3.1 and the introduction. In this section we investigate the relation between \(\text{RP}(X)\) and \(\text{NRP}^{[1]}(X)\) and characterize \((G, X/\text{NRP}^{[1]}(X))\). We start with a simple proposition.

Proposition 8.1. Let \((G, X)\) be a minimal t.d.s. If \((x, y) \in \text{RP}(X)\) then \((x, y) \in \text{NRP}^{[1]}(X)\). Thus \(P(X) \subset \text{RP}(X) \subset \text{Seq}(X) \subset \text{NRP}^{[1]}(X)\).

Proof. Let \(x_i, y_i \in X\), \(g_i \in G\) be sequences such that \(x_i \to x\), \(y_i \to y\), \(g_ix_i \to x\) and \(g_iy_i \to x\). As \((G, X)\) is minimal \((x_i^{[1]}, y_i^{[1]}) \in C_G^{[2]}(X)\). Conclude \((g_ix_i, x_i, g_iy_i, y_i) \in C_G^{[2]}(X)\) (using the identification in Equation (2.2) in
Subsection 2.2). As \((g_{x_{i}}, x_{i}, g_{y_{i}}, y_{i}) \rightarrow \ell^{2}(x, y)\) we have \(\ell^{2}(x, y) \in C^{[2]}_{G}(X)\) and thus \((x, y) \in \text{NRP}^{[1]}(X)\).

**Definition 8.2.** We say \((G, X)\) is a homogeneous t.d.s if and only if \(X = K/H\) where \(K\) is a compact group, \(H \subset K\) is a closed subgroup and there exists a continuous group homomorphism \(\phi : G \rightarrow K\) such that \(G\) acts through \(gx = g(kH)\phi(g)kH\), where \(x = kH \in X = K/H\).

**Theorem 8.3.** Let \((G, X)\) be a minimal topological dynamical system, then \((G, X/\text{S}_{eq}(X))\) is the maximal homogeneous factor of \((G, X)\).

**Proof.** See [Gla03, Theorem 1.8] and [dV93, V(1.6)].

**Lemma 8.4.** If \((G, K)\) is a minimal abelian group t.d.s, then \(\text{NRP}^{[1]}(K) = \Delta\).

**Proof.** Consider \(B^{[2]}_{G}(K) \triangleq \{(x, y, z, x + y - z) | x, y, z \in K\}\). Notice \(B^{[2]}_{G}(K)\) is closed \(\mathcal{H}K^{[2]}\)-invariant and \(\{x^{[2]} | x \in X\} \subset B^{[2]}_{G}(K)\). We conclude \(C^{[2]}_{G}(K) \subset B^{[2]}_{G}(K)\). Thus \((x, x, y, y) \in C^{[2]}_{G}(K)\) implies \(x = y\) and \(\text{NRP}^{[1]}(K) = \Delta\).

**Theorem 8.5.** Let \((G, X)\) be a minimal topological dynamical system, then \(\pi_{1} : (G, X) \rightarrow (G, X/\text{NRP}^{[1]}(X))\) is the maximal abelian group factor of \((G, X)\). That is, if \(\phi : (G, X) \rightarrow (G, K)\) is a factor map where \((G, K)\) is an abelian group t.d.s., then there exists a map \(\psi : (G, X/\text{NRP}^{[1]}(X)) \rightarrow (G, K)\) such that \(\phi = \psi \circ \pi_{1}\).

**Proof.** By Theorem 7.17, \((G, X/\text{NRP}^{[1]}(X))\) is an abelian group factor of \((G, X)\). It is enough to show that \((x, y) \in \text{NRP}^{[1]}(X)\), implies \(\phi(x) = \phi(y)\). Indeed by Lemma A.5(5) \((x, y) \in \text{NRP}^{[1]}(X)\) implies \((\phi(x), \phi(y)) \in \text{NRP}^{[1]}(K)\) and thus by Lemma 8.4 \(\phi(x) = \phi(y)\).

**Remark 8.6.** It is not hard to show that a compact abelian group is the inverse limit of compact abelian Lie groups (see [Sep07, Theorem 5.2(a)]). Thus Theorem 8.5 implies that a minimal t.d.s \((G, X)\) of order 1 is a pronil-system of order 1. This strengthens Theorem 7.18 in the case \(d = 1\).

Note if \(G\) is not abelian it may happen that \(\text{S}_{eq}(X) \neq \text{NRP}^{[1]}(X)\):

**Example 8.7.** Let \(G = X = A_{5}\), the alternating group on 5 symbols, where \(G\) acts on \(X\) by left multiplication. Clearly the minimal t.d.s \((G, X)\) is equicontinuous so, \(\text{RP}(X) = \text{S}_{eq}(X) = \Delta\). As \(A_{5}\) is simple, it is perfect. By Lemma A.5, \(\text{NRP}^{[d]}(X) = X \times X\) for all \(d \geq 1\).

8.2. **A different generalization of \(\text{RP}^{[d]}(X)\).** We now present a different higher order generalization of the classical regionally proximal relation for arbitrary group actions. This definition has the advantage that for \(d = 1\) and arbitrary acting group it coincides with the classical definition of \(\text{RP}(X)\). Moreover for \(d > 1\) and abelian acting group it coincides with \(\text{RP}^{[d]}(X)\) as defined by Host, Kra and Maass. Therefore we will keep using the notation
RP[^d](X) for the new definition where we put no restriction on the acting group.

**Definition 8.8.** Let \((G, X)\) be a t.d.s. Let \(x, y \in X\). A pair \((x, y) \in X \times X\) is said to be **regionally proximal of order** \(d\), denoted \((x, y) \in \text{RP}[^d](X)\), if there are sequences \(f_i \in \mathcal{F}[^d]\), \(x_i, y_i \in X\), and \(a_* \in X[^d]_*\) so that:

\[
(f_i x_i[^d], f_i y_i[^d]) \to ((x, a_*), (y, a_*)).
\]

**Proposition 8.9.** Let \((G, X)\) be a minimal t.d.s. Then \(\text{RP}[^d](X) \subset \text{NRP}[^d](X)\).

**Proof.** Assume that \((x, y) \in \text{RP}[^d](X)\). By definition there are sequences \(f_i \in \mathcal{F}[^d]\), \(x_i, y_i \in X\), and \(a_* \in X[^d]_*\) so that (8.1) holds. Our first goal is to show that \(((x, a_*), (y, a_*))\) is a cube. Indeed if this is true then \((x, a_*), (y, a_*)) \in C[^d]_G(X)\), and hence by Corollary 4.11 there are \(g_i \in \mathcal{F}[^d]\) such that \(g_i(x, a_*) \to x[^d]\). Thus by doubling (see Subsection A.2), it follows that \((g_i(x, a_*), g_i(y, a_*)) \to (x[^d], y, x_*[^d]) \in C[^d+1]_G(X)\), which implies that \((x, y) \in \text{NRP}[^d](X)\) as desired.

To show that \(((x, a_*), (y, a_*)) \in C[^d+1]_G(X)\), we note that as \((G, X)\) is minimal, we have \((x_i[^d], y_i[^d]) \in C[^d+1]_G(X)\). Thus again by doubling \((f_i x_i[^d], f_i y_i[^d]) \in C[^d+1]_G(X)\) and it follows.

**Remark 8.10.** Let us look at Example 8.7 again. We know that \(\text{NRP}[^d](X) = X \times X\) for all \(d \geq 1\). At the same time \(\text{RP}(X) = \Delta\). This implies that \(\text{RP}[^d](X) = \Delta\), as \(\text{RP}[^d](X) \subset \text{RP}[^1](X) = \text{RP}(X)\) by Lemma A.5. Thus, for this system, \(\text{NRP}[^d](X) \neq \text{RP}[^d](X)\) for all \(d \geq 1\).

9. A MINIMAL SYSTEM WHICH DOES NOT INDUCE A FIBRANT CUBESPACE

According to Theorem 7.10 a minimal distal action induces a fibrant cube-space. Here we exhibit an example of a non-distal minimal \(\mathbb{Z}\)-system which is not fibrant. This is proven by showing that a weaker property, the so-called glueing property, fails to hold for this system. We start by a definition and a proposition:

**Definition 9.1.** We say a cubespace \((X, C^*)\) has the **glueing property** if “glueing” two cubes along a common face yields another cube. Formally, let \(d \geq 1\) and suppose \(c, c' \in C[^d]_G(X)\), \(c = (c_1, c_2)\) and \(c' = (c_2, c_3)\), then \((c_1, c_3) \in C^d(X)\).

**Proposition 9.2.** If a cubespace \((X, C^*)\) is fibrant then it has the glueing property.

**Proof.** See [GMV16a, Proposition 6.2].

**Example 9.3.** We now present an example of a non-distal minimal \(\mathbb{Z}\)-system which is not fibrant. This example is closely related to the examples given in [Gla94, p. 254] and [TY13, Example 3.6]. Let \(S^1 \cong \mathbb{R}/\mathbb{Z}\) be the circle group, also identified with the interval \([0, 1]\) with identified endpoints. Let
$T : S^1 \to S^1$ be the rotation by an irrational number $\alpha$, $Tx = x + \alpha \pmod{1}$. This is a minimal and equicontinuous system. Let $H_1 = [0, \frac{1}{2}]$ and $H_0 = [\frac{1}{2}, 1]$ be subsets of $S^1$. Define $f(n) = \chi_{H_0}([n\alpha])$ for $n \in \mathbb{Z}$ where $[n\alpha] = n\alpha \pmod{1}$. We consider $f$ as an element in the full shift on two letters and define $X$ to be its orbit closure, i.e.

$$X = \overline{O(f, \mathbb{Z})} \subset \{0, 1\}^\mathbb{Z}$$

We will denote the shift on $\{0, 1\}^\mathbb{Z}$ by $\sigma$. The system $(X, \sigma)$ is a particular example of a Sturmian-like system (for an introduction to these systems see [Aus88, p.239]). Define the following natural dynamical morphism $\pi : (X, \sigma) \to (S^1, T)$ by $\pi((x)_{n \in \mathbb{Z}}) = \bigcap_{n \in \mathbb{Z}} T^{-n} H_{x_n}$. Note that for all $x \in X$, the intersection consists of one element exactly of the circle so the map is well defined and continuous. Moreover for any element of the circle which does not belong to the orbit of 0 or $\frac{1}{2}$, i.e. for $x \notin E = \bigcup_{n \in \mathbb{Z}} T^n \{0, \frac{1}{2}\}$, we have $|\pi^{-1}(x)| = 1$. For $x \in E$ one has $|\pi^{-1}(x)| = 2$. This immediately implies that $(X, \sigma)$ is minimal. Denote by $0^+, 0^-$ the preimages of 0 under $\pi$, then $0^+, 0^-$ are proximal as they differ only at the zeroth coordinate. To be specific let us decide that $0^+(0) = 1$ and $0^-(0) = 0$. Let us equip the circle $S^1$ with the anti-clockwise orientation. Given two pairs of points $(x_1, y_1), (x_2, y_2)$ with $|x_i - y_i| < \frac{1}{2}$, we may thus compare their orientations. Define $U_+ = \bigcap_{n=1}^{n=1} T^{-n} H_{0^+(n)}$ and $U_- = \bigcap_{n=0}^{n=0} T^{-n} H_{0^-(n)}$. Clearly $0^+ \in U_+$ and $0^- \in U_-$ and $U_+ \cap U_- = \emptyset$. Moreover there exists some $\epsilon > 0$ such that $[0, \epsilon) \subset \pi(U_+)$ and $(1 - \epsilon, 1] \subset \pi(U_-)$. Let $z \in X$. By minimality $(0^+, 0^-, 0^+, 0^-), (0^-, 0^+, 0^-, 0^+) \in C^2_Z(X)$, where we use convention (2.2). Thus by proximality of the pair $(0^+, 0^-)$ it follows that $(0^+, 0^-, z, z), (0^-, 0^+, z, z) \in C^2_Z(X)$. Assume for a contradiction that $(X, C^*_Z)$ is fibrant. By Proposition 9.2, gluing $(0^+, 0^-, z, z)$ and $(0^-, 0^+, z, z)$, we have $(0^+, 0^-, 0^+, 0^-) \in C^2_Z(X)$. By definition of $C^2_Z(X)$, one may find sequences $w_i, y_i \in X$ and $n_i \in \mathbb{Z}$ such that

$$(w_i, y_i, \sigma^{n_i} w_i, \sigma^{n_i} y_i) \to_{i \to \infty} (0^+, 0^-, 0^+, 0^-)$$

Note that for big enough $i$, $(\pi(\sigma^{n_i} w_i), \pi(\sigma^{n_i} y_i))$ is oriented as $(\pi(w_i), \pi(y_i))$. However $\pi(w_i) \in [0, \epsilon)$ and $\pi(y_i) \in (1 - \epsilon, 1]$, whereas $\pi(\sigma^{n_i} w_i) \in (1 - \epsilon, 1]$ and $\pi(\sigma^{n_i} y_i) \in [0, \epsilon)$. Contradiction.

**Figure 9.1.**
10. Open questions

10.1. Questions relating to NRP\(^d\). In Theorem 7.18 one assumes that \(G\) has a dense subgroup generated by a compact set. We thus ask:

**Question 10.1.** For which groups \(G\) does Theorem 7.18 hold?

Note that given Theorem 7.5, this is equivalent to the following question:

**Question 10.2.** Let \(d \geq 2\). For which groups \(G\) is the maximal factor of \((G, X)\) of order at most \(d\) a pronilsystem?

Note that for \(d = 1\), Remark 8.6 gives a complete solution to this question. As an intermediate step one can try to answer the following question:

**Question 10.3.** Let \((G, X)\) be a minimal system of finite order. Is it uniquely ergodic?

By Theorem 8.5, for any minimal topological dynamical system, \((G, X/\text{NRP}^{[1]}(X))\) is the maximal abelian group factor of \((G, X)\). Thus the following problem is natural:

**Problem 10.4.** Find an explicit description, for minimal topological dynamical systems \((G, X)\), of the equivalence relation \(R(X)\) such that \((G, X/R(X))\) is the maximal (compact) group factor of \((G, X)\).

10.2. Questions relating to RP\(^d\). The following questions refer to RP\(^d\) as defined in Section 8.2.

**Question 10.5.** Let \((G, X)\) be a minimal t.d.s where \(G\) is not abelian. Assume the Bronstein condition (see Subsection 3.1) holds or that \(G\) is amenable. Is RP\(^d\)(X) an equivalence relation for \(d \geq 2\)?

Note that for \(d = 1\) the answer is known to be positive for the first question.

**Question 10.6.** Let \((G, X)\) be a minimal t.d.s, \(d \in \mathbb{N}\) and RP\(^d\)(X) is an equivalence relation. What can be said about the structure of \(X/\text{RP}^{[d]}(X)\) when \(G\) is not abelian?

Note that by Lemma A.5, \(X/\text{RP}^{[d]}(X)\) is a distal system when RP\(^d\)(X) is an equivalence relation.

Appendix

A.1. Cube invariance. We verify a claim made in Subsection 7.2:

**Proposition A.1.** Let \((G, X)\) be a topological dynamical system and let \((X, C^*_G)\) be the dynamical cubespace induced by \((G, X)\). Then \((X, C^*_G)\) has cube invariance.
Proof. From the definition of $C_G^{[d]}(X)$ in Equation (3.1) in Subsection 3.3, it is clearly enough to prove that for any $g \in \mathcal{HK}^{[d]}$ and morphism of discrete cubes $f : \{0,1\}^r \to \{0,1\}^d$ we have $g \circ f \in \mathcal{HK}^{[r]}$. We can assume without loss of generality that $g = [h]_F$ for $h \in G$ and $F = \{ \omega \in \{0,1\}^d \mid \omega = a \}$ a hyperface of $\{0,1\}^d$, where $a \in \{0,1\}$ and $t \in \{1,2,\ldots,d\}$. Let us write explicitly $f = (f_1,\ldots,f_d)$ where $f_j(\omega_1,\ldots,\omega_r)$ equals to either 0, 1, $\omega_i$ or $\overline{\omega}_i = 1 - \omega_i$ for some $1 \leq i = i(j) \leq r$. Denote $H = f^{-1}(F)$, then $H$ is the face of of $\{0,1\}^r$. If $f_i \equiv a$, then $H = \{0,1\}^r$, if $f_i \equiv 1 - a$, then $H = \emptyset$, otherwise $H$ is a hyperface of $\{0,1\}^r$. We conclude $g \circ f = [h]_H \in \mathcal{HK}^{[r]}$. □

A.2. Doubling. Consider the morphisms of discrete cubes $\hat{\pi}_i : \{0,1\}^{d+1} \to \{0,1\}^d$, $i = 1,\ldots,d + 1$ defined by

$$\hat{\pi}_i(\epsilon_1,\epsilon_2,\ldots,\epsilon_i,\ldots,\epsilon_{d+1}) = (\epsilon_1,\epsilon_2,\ldots,\epsilon_i,\ldots,\epsilon_{d+1})$$

Let $F^s_i = \{ \omega \in \{0,1\}^r \mid \omega_i = 1 \}$. Note $\hat{\pi}_i^{-1}(F^d_j) = F^{d+1}_j$ if $j < i$, and $\hat{\pi}_i^{-1}(F^{d}_j) = F^{d+1}_{j+1}$ if $j > i$. Define $D_i(f)(\omega) = f(\hat{\pi}_i(\omega))$ for $f \in \mathcal{F}^{[d]}$ and $\omega \in \{0,1\}^{d+1}$.

Lemma A.2. $D_i(\mathcal{F}^{[d]}) \subset \mathcal{F}^{[d+1]}$.

Proof. By the definition of $\mathcal{F}^{[d]}$ in Subsection 3.2, it is enough to note for $h \in G$, $D_i([h]_{F^d_j}) = [h]_{\hat{\pi}_i^{-1}(F^d_j)} \in \mathcal{F}^{[d+1]}$ for $j = 1,\ldots,d$. □

In fact we see that $D_i(f)$ consists of “painting” $f$ on $F^{d+1}_i$ and on the corresponding parallel lower hyperspace $(F^{d+1}_i)^c$. We refer to this operation as doubling along $F_i$. Notice that using our convention in Equation (2.1) of Subsection 2.2 we have $D_{d+1}(f) = f \times f$.

![Figure A.1. Doubling along $F_3$.](image)

A.3. Pure ceiling and mixed upper faces. Let $d \geq 1$ and let $F$ be an upper face (see Subsection 2.3). If $F$ is contained in the ceiling hyperface $F = \{ \omega \in \{0,1\}^d \mid \omega_0 = 1 \}$ we call it pure ceiling. Otherwise we call it mixed. Note there are $2^{d-1}$ pure ceiling faces and $2^{d-1}$ mixed faces. Fix $g \in G$, pure ceiling face $P$ and mixed face $M$. Note:
(A.1) \[ [g]_P = \text{Id}^{[d-1]} \times [g]_{L_1} \]

(A.2) \[ [g]_M = [g]_{L_2} \times [g]_{L_2} \]

where \( L_1, L_2 \) are some upper faces of \( \{0,1\}^{d-1} \).

\[\text{Figure A.2. An example of a pure ceiling face and a mixed face.}\]

A.4. Lower central series induced representation for the Host-Kra cube group. Let \( G \) be a group. Set \( G = G_0 = G_1 \) and define inductively \( G_{i+1} = [G,G_i] \), where for \( A, B \subseteq G \), \([A,B]\) is the group generated by the commutators \([a,b], a \in A, b \in B\). The sequence \( G = G_0 = G_1 \supseteq G_2 \supseteq \ldots \) is called the **lower central series** of \( G \).

If \( F \) is a face of codimension \( d \) and \( d_1, d_2 \) are positive integers with \( d_1 + d_2 = d \) then we can find faces \( F_1 \) and \( F_2 \) of codimension \( d_1 \) and \( d_2 \), respectively, such that \( F_1 \cap F_2 = F \). Note the following key equality:

(A.3) \[ [[g_1]_{F_1}, [g_2]_{F_2}] = [[g_1, g_2]]_F \]

We conclude that the Host-Kra cube group \( \mathcal{H}K^{[d]} \) is generated by \([G_{\text{codim}(F)}]_F\) where \( F \) ranges over all faces of \( \{0,1\}^d \).

A.5. The pure ceiling-mixed decomposition.

**Lemma A.3.** Let \( d \geq 1 \), and fix an ordering \( < \) on \( S_1 = \{\vec{1}\}, S_2, \ldots, S_{2d} = \{0,1\}^d \) of the upper faces that respects inclusion, i.e. if \( S_i \subseteq S_j \) then \( S_i < S_j \). Then any element \( g \in \mathcal{F}^{[d]} \) has a representation as an ordered product \([x_1]_{S_1}[x_2]_{S_2} \cdots [x_{2d}]_{S_{2d-1}}\) where for \( 1 \leq i \leq 2^d - 1 \), \( x_i \in G \) and \([x_i]_{S_i} \in \mathcal{F}^{[d]}\).
Proof. This is essentially proven in [GMV16a, Proposition A.5] (see also [GT10, Appendix E]). Let us sketch the proof. Fix \( S_i < S_j \) and \( g, h \in G \) with. By (A.3) we have \([g]_{S_j} [h]_{S_i} = [g, h]_{S_i \cap S_j} [h]_{S_i} [g]_{S_j}\) where \([g, h] = ghg^{-1}h^{-1}\), as \([g, h]_{S_i \cap S_j} = [[g]_{S_j}, [h]_{S_i}]\). In other words

\[
(A.4) \quad [G]_{S_j} [G]_{S_i} \subset [G]_{S_k} [G]_{S_i} [G]_{S_j}
\]

for some \( S_k \) for which \( S_k \leq S_i \) and \( S_k < S_j \). By definition any \( g \in \mathcal{F}[d] \) is of the form \( \prod_{j=1}^{n} t_j F_j \) where \( F_j \) is an upper hyperface and \( t_j \in G \). Thus one can use (A.4) to move all occurrences of elements of the form \([G]_{S_2d-1}\) to the far right, then move all occurrences of elements of the form \([G]_{S_2d-2}\) to be adjacent to \([G]_{S_2d-1}\), and so on so as to establish \( g = [x_1]_{S_1} [x_2]_{S_2} \cdots [x_{2d}]_{S_{2d-1}} \). \( \square \)

Proposition A.4. Let \( G \) be a group and \( d \geq 1 \). If \( g \in \mathcal{HK}[d] \) then there are elements \((s \times s) \in \mathcal{HK}[d] \) and \((\text{id}^{[d-1]} \times h) \in \mathcal{F}[d] \) such that \( g = (\text{id}^{[d-1]} \times h) (s \times s) \) for some \( h, s \in G^{[d-1]} \).

By Proposition 3.3 there are \( f \in \mathcal{F}[d] \) and \( h \in G \) so that \( g = f[h]_{\{0,1\}^d} \).

Fix an ordering \(< \) on \( S_1 = \{1\}, S_2, \ldots, S_{2d} = \{0,1\}^d \) of the upper faces that respects inclusion, i.e. if \( S_i \subseteq S_j \) then \( S_i < S_j \). Moreover assume that if \( P \) is a pure ceiling upper face and \( M \) is a mixed upper face then \( P < M \) (this is possible as a pure ceiling upper face cannot contain a mixed upper face).

By Lemma A.3 we may write:

\[
(A.5) \quad f = [x_1]_{S_1} [x_2]_{S_2} \cdots [x_{2d-1}]_{S_{2d-1}} [x_{2d-1+1}]_{S_{2d-1+1}} [x_{2d-1+2}]_{S_{2d-1+2}} \cdots [x_{2d}]_{S_{2d}}
\]

Note that by Equation (A.1) the product \([x_1]_{S_1} [x_2]_{S_2} \cdots [x_{2d-1}]_{S_{2d-1}} \in \mathcal{F}[d] \) is of the form \((\text{id}^{[d-1]} \times h) \) where \( h \in G^{[d-1]} \), whereas by Equation (A.2) the product \([x_{2d-1+1}]_{S_{2d-1+1}} [x_{2d-1+2}]_{S_{2d-1+2}} \cdots [x_{2d}]_{S_{2d}} \in \mathcal{F}[d] \) is of the form \((s' \times s') \) where \( s' \in G^{[d-1]} \).

A.6. Elementary properties of \( \text{NRP}^{[d]}(X) \).

Lemma A.5. Let \((G, X)\) be a minimal t.d.s then:

1. \( P(X) \subseteq \cdots \subseteq \text{NRP}^{[d+1]}(X) \subseteq \text{NRP}^{[d]}(X) \) for each \( d \in \mathbb{N} \).
2. \( P(X) \subseteq \cdots \subseteq \text{RP}^{[d+1]}(X) \subseteq \text{RP}^{[d]}(X) \) for each \( d \in \mathbb{N} \).
3. If \( \text{NRP}^{[d]}(X) = \Delta \) for some \( d \geq 1 \) then \((G, X)\) is distal.

\[10\] One can actually prove that \( \mathcal{HK}[d] = [G_{\text{codim}(S_1)}]_{S_1} [G_{\text{codim}(S_2)}]_{S_2} \cdots [G_{\text{codim}(S_{2d})}]_{S_{2d}} \), where \( G = G_0 = G_1 \supseteq G_2 \supseteq \ldots \) is the lower central series of \( G \) (see Subsection A.4). In addition the induced representation for elements in \( \mathcal{HK}[d] \) is unique but we will not need these facts.
(4) If $G_{d+1}$ denotes the $(d+1)$-th element of the lower central series of $G$, then $(x, hx) \in \text{NRP}^{[d]}(X)$ for any $h \in G_{d+1}$. Hence if $G$ is perfect, that is $G = \{G, G\}$, then $\text{NRP}^{[d]}(X) = X \times X$ for all $d \geq 1$.

(5) If $\pi : (G, X) \to (G, Y)$ is a dynamical morphism then $\pi \times \pi(\text{NRP}^{[d]}(X)) \subseteq \text{NRP}^{[d]}(Y)$.

Proof. (1) As $\pi_f(C_G^{[d+2]}(X)) = C_G^{[d+1]}(X)$ it follows directly from Definition 3.6 that $\text{NRP}^{[d+1]}(X) \subseteq \text{NRP}^{[d]}(X)$ for $d \geq 1$. By Proposition 8.1, $P(X) \subseteq \text{NRP}^{[1]}(X)$. We now proceed by induction to show that $P(X) \subseteq \text{NRP}^{[d]}(X)$ for each $d \in \mathbb{N}$. Let $(x, y) \in P(X)$. Assume $(x, y) \in \text{NRP}^{[d]}(X)$ which implies $\ell^{[d+1]}(x, y) \in C_G^{[d+1]}(X)$. By cube invariance (see Subsection 7.2), $c \triangleq (\ell^{[d+1]}(x, y), \ell^{[d+1]}(x, y)) \in C_G^{[d+2]}(X)$. Let $F = \{\omega \in \{0, 1\}^{d+2} : \omega_{d+2} = 0\}$. Note $c_F = \ell^{[d+1]}(x, y)$. As $(G, X)$ is minimal and $(x, y) \in P(X)$ one may find a sequence $g_i \in G$ such that $g_i x \to x$ and $g_i y \to x$. Conclude $(\{g_i\}_F) c = (g_{i-1}^{[d+1]}, \ell^{[d+1]}(x, y), \ell^{[d+1]}(x, y)) \to \ell^{[d+2]}(x, y) \in C_G^{[d+2]}(X)$ which implies $(x, y) \in \text{NRP}^{[d+1]}(X)$ as desired.

(2) Consider the floor map $\pi_f : G^{[d+1]} \to G^{[d]}$ from Subsection 2.2. Let us denote its restriction to $\mathcal{F}^{[d+1]}$ by $\phi_f$. Clearly, $\phi_f(\mathcal{F}^{[d+1]}) = \mathcal{F}^{[d]}$. It follows from Definition 8.8 that $\text{RP}^{[d+1]}(X) \subseteq \text{RP}^{[d]}(X)$ for each $d \in \mathbb{N}$. Now we show that $P(X) \subseteq \text{RP}^{[d]}(X)$ for each $d \in \mathbb{N}$. It follows by the definition that $P(X) \subseteq \text{RP}^{[1]}(X)$ as $(id, g, id, g) \in \mathcal{F}^{[2]}$ for each $g \in G$. We now proceed by induction. Let $(x, y) \in P(X)$ and assume $(x, y) \in \text{RP}^{[d]}(X)$ which implies that there are sequences $f_i \in \mathcal{F}^{[d]}$, $x_i, y_i \in X$ with and $a_s \in X_{s}^{[d]}$ with $(f_i x_i^{[d]}, f_i y_i^{[d]} \to ((x, a_s), (y, a_s))$. As part of the induction one may assume $x_i = x, y_i = y$ for all $i$. Since $(x, y) \in P(X)$ and $(X, G)$ is minimal, there are $g_i \in G$ such that $g_i x \to x$ and $g_i y \to x$. There is a subsequence $\{n_i\}$ such that $g_{n_i} f_{n_i} x^{[d]} \to (x, b_s)$ and $g_{n_i} f_{n_i} y^{[d]} \to (x, b_s)$, here $b_s = \lim g_{n_i} a_s$. Thus

$$(id^{[d]}, g_{n_i}^{[d]}) \cdot (f_{n_i}, f_{n_i})(x^{[d+1]}) = (f_{n_i} x^{[d]}, g_{n_i}^{[d]} f_{n_i} x^{[d]}) \to (x, a_s, x, b_s)$$

and

$$(id^{[d]}, g_{n_i}^{[d]}) \cdot (f_{n_i}, f_{n_i})(y^{[d+1]}) = (f_{n_i} y^{[d]}, g_{n_i}^{[d]} f_{n_i} y^{[d]}) \to (y, a_s, x, b_s).$$

It is clear that $(id^{[d]}, g_{n_i}^{[d]}) \cdot (f_{n_i}, f_{n_i}) \in \mathcal{F}^{[d+1]}$, and the result follows.

(3) By (1) $\text{NRP}^{[d]}(X) = \Delta$ implies $P(X) = \Delta$.

(4) Follows as $\mathcal{H}^{[d]}$ is generated by $[G_{\text{codim}(F)}] F$ where $F$ ranges over all faces of $\{0, 1\}^d$ (see Subsection A.4).

(5) Follows directly from Definition 3.6.

For the next proposition recall the discussion in Example 7.7.
**Proposition A.6.** Let \((G, X)\) be a minimal t.d.s which is transitive of all orders, then:

1. For all \(x \in X\) and \(d \in \mathbb{N}\), \(Y^d_x(X) = x \times X^d_x\).
2. For all \(x \in X\) and \(d \in \mathbb{N}\), \(Y^d_x(X) = C^d_x(X)\).
3. For all \(x \in X\) and \(d \in \mathbb{N}\), \(\text{NRP}^d_x(X) = X \times X\).

**Proof.** We start by proving (1) by induction. Fix \(x \in X\). The case \(d = 1\) follows from minimality. Assume the statement for \(d - 1\), \(d \geq 2\). Note this implies (2) for \(d - 1\) and thus \(C^d_{G}(X) = X^{d-1}\). Let \(a \in X^{d-1}\) be a transitive point. By Proposition 4.6, \(C^d_{G}(X) = X^{d-1}\) is \(\mathcal{HK}^{d-1}\)-minimal. We may thus find a sequence \(g_k \in \mathcal{HK}^{d-1}\) such that \(g_k x^{d-1} \rightarrow a\).

By Proposition 3.3, there is a sequence \(f_k \in \mathcal{F}^{d-1}\) and \(h \in G\) so that \(g_k = f_k h^{d-1}\). Note \((f_k \times f_k)(\text{Id}^{d-1} \times h^{d-1}) \in \mathcal{F}^{d}\). By passing to a subsequence there is \(w \in Y^d_x(X)\) so that \((f_k \times f_k)(\text{Id}^{d-1} \times h^{d-1})(x^{d-1}, x^{d-1}) \rightarrow (w, a)\) and we conclude \((w, a) \in Y^d_x(X)\). Note that for any \(h \in G\), \((\text{Id}^{d-1} \times h^{d-1})(w, a) = (w, h^{d-1}a) \in Y^d_x(X)\). Since the element \(a\) is a transitive point, we have

\[
\{w\} \times X^{d-1} \subset Y^d_x(X).
\]

By Proposition 4.8, \(w\) is \(\mathcal{F}^{d-1}\)-minimal and

\[
\overline{\mathcal{F}^{d-1}}(w) = Y^d_x(X) = \{x\} \times X^{d-1}.
\]

By acting the elements of \(\mathcal{F}^{d}\) on (A.6) and doubling (see Subsection A.2), we have

\[
\overline{\mathcal{F}^{d-1}}(w) \times X^{d-1} \subset Y^d_x(X).
\]

By (A.7) and (A.8), we have

\[
\{x\} \times X^{d-1} \times X^{d-1} = \{x\} \times X^{d} \subset Y^d_x(X).
\]

This completes the proof of (1) for \(d\). Finally trivially (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3). \(\square\)

Let us call two t.d.s \((G, X)\) and \((G', X')\), where possibly \(G \neq G'\), **isomorphic** if there exist a continuous surjective (but not necessarily injective) group homomorphism \(\phi : G \rightarrow G'\) and a homeomorphism \(f : X \rightarrow X'\) such that for all \(x \in X\) and \(g \in G\), \(f(gx) = \phi(g)f(x)\). Let \(\text{Fix}(G, X) = \{g \in G | \forall x \in X, gx = x\}\). It is easy to see \(\text{Fix}(G, X)\) is a closed subgroup of \(G\) and \((G, X)\) and \((G/\text{Fix}(G, X), X)\) are isomorphic.

**Proposition A.7.** Let \((G, X)\) be a system of order at most \(d\), i.e., \(\text{NRP}^d(X) = \Delta\), and denote by \(G_{d+1}\) the \((d + 1)\)-th element of the lower central series of \(G\), then \((G, X)\) is isomorphic to \((H, X)\), where \(H = G/G_{d+1}\) is a nilpotent topological group of nilpotency class at most \(d\).

**Proof.** By Lemma A.5(4) for all \(x \in X\) and \(g \in G_{d+1}\), \((x, gx) \in \text{NRP}^d(X)\) which by assumption implies \(gx = x\). By [MKS66, Lemma 5.1] the elements of the lower central series of \(G\) are normal in \(G\). Thus \(G_{d+1}\) is normal in \(G\).
and \( H = G/G_{d+1} \) is a topological group. We conclude \((G, X)\) is isomorphic to \((H, X)\). Given a group homomorphism \( G' \to H' \) the lower central series of \( G' \) is mapped onto the lower central series of \( H' \). Thus for \( H = G/G_{d+1} \), \( H_{d+1} = \{ \text{Id} \} \) and \( H \) is a nilpotent group of nilpotency class at most \( d \). □

Proposition A.8. \((x, y) \in \text{NRP}^{[d]}(X)\) if and only if \( \gamma_{[d+1]}(x, y) \in C^{[d+1]}_G(X) \).

Proof. By Theorem 3.8, \((x, y) \in \text{NRP}^{[d]}(X)\) iff \((y, x) \in \text{NRP}^{[d]}(X)\) iff \( \gamma^{[d+1]}(y, x) \in C^{[d+1]}_G(X) \). By cube-invariance (e.g applying \( \omega_1, \ldots, \omega_r \leftrightarrow \omega_1, \ldots, \omega_r \)) \( \gamma^{[d+1]}(y, x) \in C^{[d+1]}_G(X) \) iff \( \gamma_{[d+1]}(x, y) \in C^{[d+1]}_G(X) \). □

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