Hidden Vector ‘Coordinates’ in Particle Physics

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Abstract

Heretofore unrecognized (i.e., “hidden”) Lorentz-invariant vector observables in the fermion sector (i.e., flavor-defining fermion “coordinates”) are shown to indirectly explain most, if not all, of the so-called “accidental” (internal) symmetries associated with fundamental fermions (quarks and leptons), by explaining quark and lepton flavors, flavor doublets and families. Moreover, these new fermion coordinates lead to quantitative constraints on neutrino mixtures that are found to be in good agreement with current experimental observations. The new fermion coordinates arise in the context of a quantized, internal 2-space possessing a non-Euclidean metric. “Quantization” here means that all scalars, vectors and matrices relevant to a description of fundamental fermions in the 2-space, are both discrete and finite in number. In a certain limited sense, the new 2-space description of fundamental fermions is more fundamental, and more general, than that provided by the standard model of particle physics. In particular, it points to a “layer” of new physics—located somewhere between the unification scale, and the region of applicability of the standard model—that is responsible for the 2-space, its quantization, and its associated “selection rules.”

1.0 Introduction

Most physicists would probably agree that the standard model of particle physics [1–3], based on the gauge group $SU(3)_{\text{color}} \times SU(2)_{\text{L}} \times U(1)_{Y}$, is only a low-energy or “macroscopic” approximation to a deeper, more fundamental, unified theory, possibly involving either Planck level physics at an energy scale of $10^{19}$ GeV [4, 5], or a much lower TeV energy scale should the anticipated extra dimensions turn out to be large [6]. Consider for example, the case of the standard-model Lagrangian.

Given the number of flavors of quarks and leptons, and an appropriate (renormalizable) strong-electroweak Lagrangian, the so-called “accidental symmetries” of the Lagrangian [7] are known to “explain” both the existence, and separate (exact or approximate) conservation, of various (global) “charges” associated with quarks and leptons (e.g., lepton number, baryon number, strangeness, charm, truth, beauty, electron-, muon- and tau-numbers). However, there is nothing in such Lagrangians, or their associated accidental symmetries, that would explain quarks and leptons, or tell us how many flavors of quarks and leptons to include in the Lagrangian. Rather, it seems that the Lagrangian is not an explanation for these things, but instead is a result of physics at some deeper, more fundamental, level, possibly involving unification. However, if unification really does occur, we are immediately faced with the
important unanswered question of exactly how the standard model of particle physics, and its associated Lagrangian, arises or emerges from this deeper, more fundamental, “substratum.”

In this paper these questions are addressed in a new way by identifying what may turn out to be an intermediate “layer” of new physics located somewhere between the unification scale (wherever it happens to be), and the region of applicability of the standard model of particle physics. The strategy pursued here is similar to that of S. Goudsmit and G. Ulenbeck [8], who first realized that a description of the observed fine structure in atomic spectra was not possible unless electrons possessed, in addition to their mass and electric charge, a previously hidden spin “coordinate.” In particular, the principal purpose of this paper is to argue that the proposed intermediate “layer” of new physics is described by heretofore unrecognized (i.e., “hidden”) 2-vector observables in the fermion sector (i.e., flavor-defining fermion “coordinates” U, V and Q), which serve to explain, among many other things, the proliferation of matter fields (flavors) in comparison to the relative paucity of force-mediating particles. And, because we will also argue here that the new flavor-defining fermion coordinates (i.e., the 2-vectors Q, U, V where Q = U + V) are Lorentz-invariant, we will be free to perform a variety of physically meaningful algebraic, geometric, and topological manipulations on any one (or more) of these coordinates—without undue concern for how these manipulations “appear” to observers located in arbitrary (inertial) coordinate systems.

From the foregoing, it is clear that once new coordinates of this kind are introduced in the fermion sector, an entirely new description of fundamental fermions (quarks and leptons) must result. Moreover, the proposed description appears to be both more fundamental, and more general (in certain limited respects), than the standard model description of fundamental fermions. For example, by employing these new fermion coordinates, the question of why fundamental fermion families are replicated, and the question of why there are just three families, can both be answered, whereas this is not possible in the standard model description [1–3]. Thus the new flavor-defining fermion coordinates serve both to complement, and to (modestly) extend, the standard model of particle physics.

This work is based, in part, on previous works by the author wherein different aspects of this approach to flavors, families, and the problem of family replication, were discussed [9–13]. These works provide a different emphasis, and in some cases, a more detailed description of many features of this general idea that are only touched upon in the present work, which essentially reviews, clarifies, and consolidates these earlier works.

2.0 The Electric Charge Coordinates Q

It is well known that in quantum mechanics the wave function or state vector \( \Psi_n \) describing an individual (“isolated”) particle, provides all of the information that is available, in principle, about that particle. Here, the subscript on \( \Psi_n \) refers to various kinds of simultaneously observable quantum numbers associated with the particle. Hence, it is possible to effectively define such a quantum state by specifying the specific values of these simultaneous observables [4]. In a similar way, there are known to exist certain simultaneously observable flavor-defining quantum numbers (e.g., strangeness and baryon number) that
serve to define individual flavor or antiflavor eigenstates of quarks and leptons [15, 16]. It is proposed here that these flavor-defining quantum numbers are to be identified with certain global charge-conjugation-reversing (C-reversing) scalars, which are *derived*, in turn, from the flavor-defining fermion coordinates, \( Q, U \) and \( V \).

If ever there was a signal for new coordinates in particle physics, surely it comes from the known fundamental-fermion sector. Unlike the known “force-mediating” fundamental-boson sector (i.e., the photon, gluons, the graviton, and weak intermediate vector bosons), matter-particles or fundamental fermions (quarks and leptons), exhibit a relative plethora of flavors falling into three families. Counting only color-flavor states, there are 48 known fundamental fermions and antifermions [17], compared with only 13 force-mediating bosons [18]. What could be more natural than to assume that this “embarrassment of riches” in the fermion sector, is due to new flavor-defining fermion coordinates, which are *not* carried by the force-mediating bosons [18, 19]?

In a recent paper [10], it was shown that the fermion-number operator \( F(\text{op}) \) itself may be the key to a preliminary understanding of how new flavor-defining fermion coordinates such as the electric-charge coordinate \( Q \) arise. In particular, let us review the salient points in [10] to see how these coordinates arise from an *analytic continuation* of the fermion-number operator \( F(\text{op}) \).

### 2.1 Analytic continuation of \( F(\text{op}) \) and a new internal non-Euclidean 2-space

We have found that an analytic-continuation of a Hermitian matrix \( F(\text{op}) \) representing the conventional fermion-number operator, from an *external* spacetime and Hilbert-space setting, to a new *internal* (real) non-Euclidean space—\( F(\text{op}) \) is continued to a real, generally non-diagonal matrix \( F(v) \) involving a single real parameter \( v \)—“automatically” leads to a new description of fundamental fermions (quarks and leptons) in which families are *replicated* and there are just *three* families. The fact that this happens, suggests that there is some deep connection between the result of the aforementioned analytic-continuation, namely \( F(v) \), and some more fundamental underlying physics (e.g., Planck-level physics) where flavor degrees-of-freedom, and family-replication, presumably originate.

#### 2.1.1 The fermion-number operator \( F(\text{op}) \)

Consider the situation, presumably at some high energy, where we are dealing with “free” (isolated) leptons or “analytically free” quarks. Suppose we want to describe the scalar fermion-number carried by these particles. And, suppose further, that the energies involved are not so high that quantum field-theory (QFT) breaks down. Under these conditions the fermion-number can be represented by a \( U(1) \)-type scalar “charge” [20, 21], namely, a charge associated with the (continuous) group of unitary matrices \( U \) of order 1 known as \( U(1) \).

The fermion-number operator, which can be represented by a Hermitian matrix \( F(\text{op}) \), is said to generate these so-called “gauge” (or phase) transformations, which in turn act on
fermion and antifermion quantum states in Hilbert space. That is, given that $\alpha$ is a real phase one has

$$U = e^{i\alpha F_{\mathrm{op}}},$$

(1)

and for infinitesimal transformations [i.e., $e^{i\delta \alpha F_{\mathrm{op}}} = 1 + i \delta \alpha F_{\mathrm{op}}$] acting on single-particle (free or “asymptotically free”) fermion and antifermion states $|p\rangle$ and $|\bar{p}\rangle$, respectively, one easily establishes that (the fermion-number “charges” are $f_m = -f_a = 1$ for matter and $f_a$ for antimatter)

$$U|p\rangle = e^{i\delta \alpha f_m}|p\rangle$$
$$U|\bar{p}\rangle = e^{i\delta \alpha f_a}|\bar{p}\rangle,$$

(2)

since, by definition, $F_{\mathrm{op}}$ obeys the eigenvalue equations

$$F_{\mathrm{op}}|p\rangle = f_m|p\rangle$$
$$F_{\mathrm{op}}|\bar{p}\rangle = f_a|\bar{p}\rangle.$$  

(3)

Finally, the assumption that the Hamiltonian $H$ is invariant under $U$, namely

$$H = UHU^\dagger,$$

(4)

ensures that $H$ and $F_{\mathrm{op}}$ commute

$$[F_{\mathrm{op}}, H] = 0,$$

(5)

as can be verified by differentiating $UHU^\dagger$ with respect to $\alpha$. Hence, the total fermion-number (the number of fermions minus the number of antifermions) is a constant of the motion.

### 2.1.2 Matrix representation of the fermion-number operator $F_{\mathrm{op}}$

Because the matrix $F_{\mathrm{op}}$ involves just two kinds of quantum states (3), namely $|p\rangle$ and $|\bar{p}\rangle$, it can be expressed as a $2 \times 2$ diagonal Hermitian matrix (6), where one of the adjustable parameters ($\theta$) is a fixed constant (up to $2\pi$) and the other ($\phi$) is freely adjustable. In particular,

$$F_{\mathrm{op}} = \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{+i\phi} & -\cos \theta \end{pmatrix} = \sigma_z,$$

(6)

where

$$\sigma_z = \begin{pmatrix} f_m & 0 \\ 0 & f_a \end{pmatrix}$$

(7)

is one of the familiar Pauli matrices.
This form for $F(\text{op})$ is consistent with (3) where the normalization and orthogonality conditions, namely $\langle p|p \rangle = \langle p|p \rangle = 1$ and $\langle p|p \rangle = \langle p|p \rangle = 0$, respectively, directly yield

$$F(\text{op}) = \left( \begin{array}{c} \langle p|F(\text{op})|p \rangle \\ \langle \overline{p}|F(\text{op})|p \rangle \end{array} \right) = \sigma_z. \quad (8)$$

Note that owing to (7) and (8), $\cos \theta < 1$ in (6) is excluded. Here it should also be noted that $\text{tr} F(\text{op}) = f_m + f_a = 0$, $\det F(\text{op}) = f_m \cdot f_a = -1$, and $F^2(\text{op}) = I_2$ is the $2 \times 2$ identity matrix.

### 2.1.3 The Continuation From $F(\text{op})$ to a ‘generalized fermion-number’ $F(v)$

Now perform an analytic continuation on $F(\text{op})$, namely $F(\text{op}) \to F(v)$, which maintains $F(v)$ real and $\cos \theta \geq 1$. This can only be accomplished by continuing $\theta$ from a real to an imaginary number, and by maintaining $e^{-i\phi}$ imaginary. In particular, to maintain $F(v)$ real, we must make the replacements $\theta \to iv$ and $e^{-i\phi} \to \mp i$, where $v$ is a real number. Then

$$F(v) = \left( \begin{array}{c} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{+i\phi} & -\cos \theta \end{array} \right) \bigg|_{\theta=iv, e^{-i\phi} = \mp i} \quad (9)$$

or

$$F(v) = \left( \begin{array}{c} \cosh v & \pm \sinh v \\ \mp \sinh v & -\cosh v \end{array} \right), \quad (10)$$

where, just as for $F(\text{op})$, the eigenvalues of $F(v)$ are $f_m$ and $f_a$, and so we have $\text{tr} F(v) = f_m + f_a = 0$, $\det F(v) = f_m \cdot f_a = -1$, and $F^2(v) = I_2$.

In Ref. 9, pp. 50 and 54 it is argued that only the upper signs in (10) have physical significance and $v \geq 0$. And, just what is the physical significance of $F(v)$?

Because the continuation “connects” $F(\text{op})$ and $F(v)$, it is natural to assume that both $F(\text{op})$ and $F(v)$ describe, or represent, aspects of the fermion number (i.e., the matter-antimatter “degree-of-freedom” or matter-antimatter “dichotomy”). However, unlike $F(\text{op})$ or $f$, $F(v)$ is a kind of “generalized fermion-number,” which will be shown to describe additional degrees-of-freedom such as the “up”-“down” and quark-lepton degrees-of-freedom. Moreover, it is abundantly clear from (10) that the generally non-Hermitian (when $v \neq 0$) matrix $F(v)$—unlike the Hermitian matrix $F(\text{op})$ in (8), which acts on Hilbert space—does not act on a Hilbert space in an external spacetime setting [22].

### 2.1.4 A new internal non-Euclidean 2-space

Because of the matter-antimatter dichotomy exhibited by fundamental fermions, the fermion-number operator $F(\text{op})$ is necessarily a 2 by 2 matrix, and so the continuation of $F(\text{op})$, namely $F(v)$, is also a 2 by 2 matrix. Therefore, the matrix $F(v)$ must “act” on some new internal 2-dimensional space. What is the nature of this space? In particular, assuming that it is a metric space, what is the metric?
When the matrix $F(v)$ acts on a real column-vector $\{a, b\}$, it leaves the quadratic form $a^2 - b^2$, invariant. Therefore, the 2-space metric is non-Euclidean or “Lorentzian”, and can be represented by the matrix

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (11)$$

Given this metric, the scalar product of two real vectors (i.e., the “projection” of one vector upon another) assumes the form

$$(a, b)\{e, f\} = ae - bf. \quad (12)$$

Similarly, the square of a real vector (i.e., the “projection” of the vector upon itself) is given by

$$(a, b)\{a, b\} = a^2 - b^2. \quad (13)$$

In (12) and (13) above, $(\cdot, \cdot)$ is a row vector while $\{\cdot, \cdot\}$ is a (conformable) column vector.

### 2.2 Quantization of the Imaginary Angle $\theta$ and the Electric-Charge Coordinates $Q$

It is very important to understand that the imaginary angle $\theta$ in (9) must be quantized. That is, when we do the continuation to an imaginary angle, the only allowed values seem to be

$$\theta = iv = i \ln M_c, \quad (14)$$

or

$$e^{-i\theta} = M_c, \quad (15)$$

where $M_c = 1$ or 3 only (see Refs. 24, 25, and Ref 9, p. 79).

This quantization of $\theta$ and/or $v$ comes about in the following way. Because the eigenvalues of $F(v)$ are C-reversing quantized charges, the components of the diagonal and off-diagonal forms of the matrix $F(v)$ are, necessarily, quantized charges. Since the components of $F(v)$ involve hyperbolic functions of $v$, it follows that $e^v$, $e^{-v}$ and $v$ are, necessarily quantized. In particular, as we show elsewhere $fe^{-v} = f/M_c$ is a quantized charge identified with the baryon or lepton number (Ref. 9, pp. 52, 53), depending on the value of the quantum number $M_c$. And, from the spin-statistics theorem, we know that an odd number of fundamental fermions must bind together (“strongly”) to form a baryon or lepton. Hence, the quantum number $M_c$ must be an odd integer $M_c = 1, 3, 5, \ldots$ (see Refs. 24, 25, and Ref. 9, p. 79).
Figure 1. The four “electric charge” 2-vector fermion and antifermion “coordinates” $Q_{q}$, $Q_{q}^{-}$, $Q_{l}$, $Q_{l}^{-}$.

These identifications are remarkable for at least two reasons. First, we discover that the eigenvectors of $F(v)$, and related functions evaluated at $M_{c} = 1$ and $M_{c} = 3$, yield the quantized electric charges of both leptons and quarks, and their associated quantized lepton or baryon numbers, respectively [26]. In particular, choose the upper signs in (10),

$$F(v) = \begin{pmatrix} \cosh v & \sinh v \\ -\sinh v & -\cosh v \end{pmatrix}. \quad (16)$$

Then identify (see Ref. 9, pp. 52–55) the quark and lepton electric charges with the “up”-“down” components of the eigenvectors of the matrix $F(v)$. That is, the quark charges are given by $(M_{c} = 3)$

$$q_{1}(f) = \frac{(M_{c}^{2} - 1)}{2M_{c}(M_{c} - f)} = +\frac{2}{3} \text{ for } f = +1 \text{ and } +\frac{1}{3} \text{ for } f = -1, \quad (17)$$

$$q_{2}(f) = q_{1}(f) - 1, \quad (18)$$

where the baryon number for quarks is the non-Euclidean quadratic form $Q_{q}^{2} = B = q_{1}^{2}(f) - q_{2}^{2}(f) = \pm\frac{1}{3} \text{ for } f = \pm1$, where the quark electric-charge “coordinate” (see Fig. 1) $Q_{q} = Q_{q}(f, M_{c}) = \{q_{1}(f), q_{2}(f)\}$ is an eigenvector of $F(v)$. Similarly, the lepton electric charges
are given by \((M_c = 1)\)

\[
q'_1(f) = \frac{-(M_c^2 - 1)}{2M_c(M_c - f)} = -1 \text{ for } f = +1 \text{ and } 0 \text{ for } f = -1, \\
q'_2(f) = q'_1(f) + 1,
\]

(19)\(20\)

where the lepton number for leptons is the non-Euclidean quadratic form \(Q^2_l = L = [q'_1(f)]^2 - [q'_2(f)]^2 = \pm 1\) for \(f = \pm 1\), where the lepton electric-charge “coordinate” (see Fig. 1) \(Q_l = Q_l(f, M_c) = \{q'_1(f), q'_2(f)\}\) is an eigenvector of \(F(v)\).

Second, it is abundantly clear that the quantum number \(M_c\) can be identified with the strong-color “multiplicity.” In particular, \textit{leptons} being strong-color \textit{singlets} \((M_c = 1)\) do not bind strongly to form a composite particle, whereas \textit{quarks} being strong-color \textit{triplets} \((M_c = 3)\), do bind strongly to form a composite baryon. That is, \(M_c\) is both a measure of the strong-color multiplicity and the \textit{number} of quarks or leptons that bind (strongly) to form composite particles.

It is remarkable that the 2-space description of fundamental fermions has “automatically” yielded, not only the \textit{electric}, \textit{baric} and \textit{leptic} charges of quarks and leptons, but also an unanticipated connection with \textit{quantum chromodynamics} (QCD) and \(SU(3)_{\text{color}}\). In particular, the 2-space and the associated quantized parameter \(v = \ln M_c\), implies that something very similar to \(SU(3)_\text{color}\) must be a symmetry of nature. \textit{We take this circumstance to be a strong indication that the 2-space and/or whatever underlying physics is responsible for it, is more primitive (i.e., more fundamental) than the standard model of particle physics.}

### 2.2.1 Representing charge conjugation C in the 2-space

Given that there are numerous \(C\)-reversing scalars such as \(q_1, q_2, q'_1, q'_2, B\) and \(L\), in the 2-space, there must exist a \(2 \times 2\) matrix, call it \(X\), that serves to transform these \textit{scalars}, various \(2\)-\textit{vector} fermion coordinates (e.g., \(Q, U\) and \(V\)), and various \(2 \times 2\) \textit{matrices} such as \(F(v)\), to their corresponding \(C\)-reversed (2-space) counterparts. For example, a matrix \(X\) should exist such that

\[
XQ = Q^c
\]

(21)

and

\[
XQ^c = Q.
\]

(22)

From (21) and (22) it follows that \(X\) must equal its multiplicative inverse \((X = X^{-1})\), and thus

\[
X^2 = I_2,
\]

(23)

where \(I_2\) is the \(2 \times 2\) identity matrix.
Write $X$ in the general form ($X$ is real)

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

and consider the situation for leptons (see Eqs. 19 and 20), where one of the two charges $(q'^2_1)$ associated with $Q_l = \{q'_1(1) = -1 = q'_1, q'_2(1) = 0 = q'_2\}$ and $Q^c_l = \{q'_1(-1) = 0 = q'_2, q'_2(-1) = +1 = -q'_1\}$ is zero, and the other charge ($q'_1$) is nonzero.

In this particular case, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} q'_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -q'_1 \end{pmatrix},$$

which means that

$$aq'_1 = 0$$

and

$$cq'_1 = -q'_1.$$  

And therefore, since $q'_1 \neq 0$, it must be true from (26) and (27) that $a = 0$ and $c = -1$. Since $XQ^c = Q$ it must also be true that

$$\begin{pmatrix} 0 & b \\ -1 & d \end{pmatrix}\begin{pmatrix} 0 \\ -q'_1 \end{pmatrix} = \begin{pmatrix} q'_1 \\ 0 \end{pmatrix},$$

which means that

$$-bq'_1 = q'_1$$

and

$$-dq'_1 = 0.$$  

Finally, since $q'_1 \neq 0$ it must be true from (29) and (30) that $b = -1$ and $d = 0$.

Collecting the foregoing matrix elements, we have

$$X = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix},$$

or

$$X = -\sigma_X,$$

where $\sigma_X$ is one of the familiar Pauli matrices. In general, the matrix $X = -\sigma_X$ should apply (in the 2-space) to 2-scalars, 2-vectors, 2×2 matrices, and to both quarks and leptons.
2.2.2 Transformations of charge-like scalars associated with Q, U and V under X

As demonstrated in Section 2.2.1, the charge-like scalar components of 2-vector coordinates such as $Q_q$ or $Q_l$ change signs under $X$. However, owing to the 2-space metric given by (11), scalar products of 2-vectors whose components transform like charges, will also be charge like.

For example, we have the square of the 2-vectors $Q$ and $Q^c$, namely,

$$Q^2 = Q \cdot Q = q_1^2 - q_2^2$$

and

$$(Q^c)^2 = Q^c \cdot Q^c = q_2^2 - q_1^2.$$  

Therefore,

$$Q^2 = -(Q^c)^2,$$  \hspace{1cm} (35)

which means that $Q^2$ and $(Q^c)^2$ each transform like C-reversing 2-scalar charges.

As described previously, these particular charges can be identified with the baryon- or lepton-number carried by quarks or leptons, respectively (see Ref. 9, p. 72). We will see in a later section that when charge vectors such as $Q$ (or $Q^c$) are resolved in the 2-space into pairs of linearly independent vectors (e.g., $Q = U + V$), not only are the components of $Q$, $U$ and $V$, C-reversing charges, but also by squaring $Q = U + V$, namely,

$$Q^2 = U^2 + 2U \cdot V + V^2,$$  \hspace{1cm} (36)

$U^2$, $2U \cdot V$ and $V^2$ are, like $Q^2$, C-reversing charges. The foregoing collection of 2-scalar charges will be used to define and describe flavor eigenstates, flavor doublets, and eventually families of fundamental fermions \[27, 28\].

2.2.3 Transformation of the metric and other matrices under X

Any $2 \times 2$ matrix $M$, appropriate to the 2-space description of fundamental fermions, should transform under $X = -\sigma_X$ to its C-reversed counterpart $M^c$ according to the similarity transformation

$$X M X^{-1} = M^c,$$  \hspace{1cm} (37)

or because $X = X^{-1} = -\sigma_X$, equivalently as

$$(-\sigma_X) M (-\sigma_X) = M^c.$$  \hspace{1cm} (38)

For example, the metric $g$ (see Eq. 11) is found to be C-reversing since

$$(-\sigma_X) g (-\sigma_X) = -g.$$  \hspace{1cm} (39)
Similarly, the matrix \( \mathbf{F}(v) \) in (16) is \( \mathbf{C} \)-reversing since \( (-\mathbf{\sigma}_\mathbf{x}) \mathbf{F}(v)(-\mathbf{\sigma}_\mathbf{x}) = -\mathbf{F}(v) \).

A matrix that is \( \mathbf{C} \)-invariant (e.g., the matrix \( \mathbf{X} \)) would, necessarily, have the form

\[
\mathbf{N} = \begin{pmatrix} a & b \\ b & a \end{pmatrix},
\]

where it is clear that

\[
(-\mathbf{\sigma}_\mathbf{x})\mathbf{N}(-\mathbf{\sigma}_\mathbf{x}) = \mathbf{N}.
\]

### 3.0 What Dynamics Governs the Evolution of Flavor-Defining Fermion Coordinates \( Q, U \) and \( V \)?

As a first approximation, it is safe to say that the dynamics of a system consisting of an “isolated” or individual elementary particle, is governed by the evolution of the system wave function or state vector. The state vector in turn evolves according to quantum mechanical equations of motion that can be said to arise from an appropriate strong-electroweak Lagrangian (when the standard model applies) or a more fundamental Lagrangian in case deeper physics applies [29]. The question that naturally arises in the present situation is how such a dynamical description is altered or constrained, if at all, by the new flavor-defining fermion coordinates \( Q, U \) and \( V \)? Even a partial answer to this question may suggest ways that the new 2-space description of fundamental fermions can be experimentally tested. Let us begin the discussion by describing how quantum mechanical state vectors can be partially represented using the new 2-vector fermion coordinates \( Q, U \) and \( V \).

### 3.1 Representing flavor eigenstates and flavor doublets in the 2-space

The continuation from the 2-D Hilbert space to the 2-D non-Euclidean “charge” space (see Sec. 2.1.3) turns out to mean that individual flavors of fundamental fermions can be partially represented in an unconventional way by geometric objects (in the non-Euclidean 2-space) which differ from a quantum state, but from which the quantum states can be inferred or effectively constructed. In particular, in the non-Euclidean 2-space, an object we call a “vector triad” (i.e., the “triad” of flavor-defining fermion coordinates \( Q, U \) and \( V \)) represents “up”-“down” type flavor doublets of fundamental fermions and antifermions—the “up”-“down” type flavor dichotomy. That is, the components of the vectors associated with a given vector-triad are observable Lorentz-invariant (Lorentz 4-scalar) “charges,” which can be used to define the two flavor-eigenstates in a flavor doublet [13, 27, 28, 10, 30].

#### 3.1.1 Flavor eigenstates

As demonstrated in detail in Ref. 9, pp. 16–18, given the charge-like (\( \mathbf{C} \)-reversing) observables associated with the description involving \( \mathbf{F}(v) \), namely, the real \( \mathbf{C} \)-reversing scalar-
components of various matrices and vectors defined on the internal non-Euclidean 2-space, it is possible to write down flavor eigenstates \[ |C_1, C_2, C_3, \ldots, C_n \rangle. \] (42)

Here \( C_1, C_2, C_3, \ldots, C_n \), are the “good” charge-like flavor-defining quantum numbers (charges) associated with a particular flavor. It happens that these observable real-numbers can be identified with quantum numbers such as electric charge, strangeness, charm, the third-component of (global) isospin, truth and beauty (see Ref. 9, p. 72). To discover what charges describe a particular flavor we must first identify the vector-triad associated with that flavor.

Now, each vector-triad represents a flavor doublet, not just an individual flavor. That is, vector-triads provide information on two quantum states (two simultaneous flavor-eigenstates) associated with flavor doublets. Therefore, vector-triads are associated with both individual flavors and flavor doublets. Here we simply summarize how it is the that non-Euclidean vector-triads can be associated with both individual flavors and “up”-“down” type flavor-doublets.

### 3.1.2 Flavor doublets

Consider the eigenvector (call it \( Q \)) of \( F(v) \) for fundamental fermions (see Footnote 26, and Sec. 2.2). Since the space on which \( F(v) \) “acts” is two-dimensional, the observable vector \( Q \) can be “resolved” into two (no more or less) observable, linearly-independent vectors, call them \( U \) and \( V \), as \( Q = U + V \).\(^{31,32}\) Now, because these three vectors (\( Q, U, \) and \( V \)) are simultaneous observables, it makes sense to speak of this “triad” of vectors as being a well defined geometric object, namely, a “vector triad.”

Recognizing that the components of \( Q, U \) and \( V \) are \( C \)-reversing charge-like observables we can write these observable “charge” vectors as

\[
Q = \{q_1, q_2\} \\
U = \{u_1, u_2\} \\
V = \{v_1, v_2\},
\]

where \( q_1, q_2, u_1, u_2, v_1 \) and \( v_2 \) are the various observable “charges” (e.g., \( q_1 \) and \( q_2 \) were found to be electric charges). Given \( Q = U + V \), the non-Euclidean metric (11), and Eqs. (43) through (45), we find the associated observable quadratic “charges”

\[
Q^2 = U^2 + 2U \cdot V + V^2 \\
2U \cdot V = 2(u_1v_1 - u_2v_2) \\
U^2 = u_1^2 - u_2^2 \\
V^2 = v_1^2 - v_2^2.
\]

12
Finally, using the foregoing flavor-defining charges, we can express the two quantum states (simultaneous flavor-eigenstates) associated with a single vector-triad in the form of “ket” vectors as follows (see Ref. 14, and Ref. 9, pp. 16–18)

\[
|q_1, u_1, v_1, Q^2, U^2, 2U \cdot V, V^2 \rangle, \\
|q_2, u_2, v_2, Q^2, U^2, 2U \cdot V, V^2 \rangle.
\] (50)

Here, the state \(|q_1, u_1, v_1, Q^2, U^2, 2U \cdot V, V^2 \rangle\) represents the “up”-type flavor-eigenstate, and \(|q_2, u_2, v_2, Q^2, U^2, 2U \cdot V, V^2 \rangle\) represents the corresponding “down”-type flavor-eigenstate in a flavor doublet of fundamental fermions [16, 30].

In this section we have demonstrated, among other things, that there are essentially two ways the new flavor-defining fermion coordinates (\(Q, U, V\)) can be combined. In particular, when (\(Q, U, V\)) apply to the same fundamental fermion, these vectors can either be added, as in \(Q = U + V\), or they can be multiplied, as in the non-Euclidean scalar products (i.e., “projections”) expressed by Equations 46–49. However, as described in the next section, the linear superposition principle of quantum mechanics severely limits the ways (\(Q, U, V\)) can be combined when two or more fundamental fermion flavors are involved.

3.2 When the new internal coordinates cannot be superimposed

According to the linear superposition principle of quantum mechanics (QM), wave functions or state vectors for two or more particles, and certain of their associated simultaneously observable charge-like quantum numbers or “charges,” can be superimposed. And, because the overarching linear superposition principle of QM must be retained, it turns out that like flavor-defining fermion coordinates associated with (“carried by”) two or more fundamental fermions cannot be superimposed or added \((Q, U, V)\). For, if they could be superimposed, the linear superposition principle of QM could not be retained, nor could flavors be defined in a self-consistent way using the fermion coordinates \((Q, U, V)\).

In particular, one obvious reason that like coordinates \((Q, U, V)\) for two or more fundamental fermions cannot be superimposed, is that such superpositions inevitably lead to nonsensical charge-like quantum numbers (“charges”), which could not be “carried” by a sensible state vector, or used to define the corresponding wave function or state vector for the collection of particles in question. Let us demonstrate the validity of this assertion by giving a few simple examples, which serve to demonstrate some of the rules governing the superposition (or lack thereof) of the 2-vector coordinates \(Q, U\) and \(V\).

3.2.1 Rules for combining \(Q, U\) and \(V\)

Consider the case of three quarks bound together by strong forces to form a baryon with baryon number \(B = 1\). Since the baryon number of each quark is given by \(Q^2 = 1/3\), the baryon number for three bound quarks is simply \(3Q^2 = B = 1\). Now this result makes it clear that we cannot simply add the three vectors \(Q\) together first, and then “square” them to get the baryon number for such a composite particle. In particular, it is clear that the baryon number of a composite consisting of three quarks is not given by \((3Q)^2 = B = 3\).
Now it might be supposed that this algorithmic failure can be attributed to the absence of an appropriate “normalization” factor when adding such vectors. For example, suppose that we were to make the rule that when $N$ of these $Q$-vectors are added, we must divide the vector sum by $\sqrt{N}$. Then for the case of three quarks we would have a baryon number given by $(3Q/\sqrt{3})^2 = B = 1$, which is certainly correct. However, the maximum, and minimum, electric charges of such a composite would then be given, respectively, by the scalar products $(3Q/\sqrt{3})\{1,0\} = +2\sqrt{3}/3$ and $(3Q/\sqrt{3})\{0,-1\} = -\sqrt{3}/3$, where $Q = (2/3, -1/3)$. Clearly, this is incorrect since three quarks with electric charges $+2/3$ obviously form a composite baryon with an electric charge of $+2$ not $+2\sqrt{3}/3$. Similarly, three quarks with electric charges $-1/3$ form a composite baryon with an electric charge of $-1$ not $-\sqrt{3}/3$. Moreover, if we choose not to employ the proposed normalization factor $\sqrt{N}$, it is true that the projections of the vector $3Q$ upon the 2-space coordinate axes $\{1,0\}$ and $\{0,-1\}$ do yield correct electric charges $+2$ and $-1$ as required [33]. However, in this case the baryon number of the composite is wrong, and there are no corresponding projection axes available for the intermediate electric charges between $+2$ and $-1$ that can be formed by combining three quarks in various well known ways. In particular, there is no way to obtain electric charges $0$ and $+1$ by projecting the vector $3Q$ upon the 2-space axes $\{1,0\}$ or $\{0,-1\}$.

Considerations of the foregoing kind lead to the conclusion that like vector coordinates for two or more fundamental fermions cannot be added or superimposed ($Q$ with $Q$, $U$ with $U$, $V$ with $V$), the way quantum states or state vectors can be superimposed. Only the flavor-defining scalar charges associated with the flavor-defining coordinates $Q$, $U$ and $V$ for two or more fundamental fermions, can be combined or superimposed. That is, because flavor eigenstates are to be described in terms of various simultaneously-observable, global, flavor-defining (scalar) charge-like quantum numbers (associated with $Q$, $U$ or $V$), it is simply not physically meaningful to superimpose like coordinates $Q$, $U$ or $V$ for two or more particles [8].

The foregoing discussion clearly demonstrates that the dynamical significance of the new flavor-defining fermion coordinates $Q$, $U$ and $V$ is quite different than say, the intrinsic angular momentum or spin coordinates for two different fermions. Angular momentum vectors for two different particles can be combined, but the corresponding (like) $Q$-, $U$-, and $V$-vector coordinates cannot. Nevertheless, the triad of 2-vector coordinates $(Q$, $U$ and $V)$ used to represent a flavor eigenstate, is expected to change or evolve during certain kinds of transitions or interactions, simply because the wave function or state vector, which depends, in part, on flavor-defining global scalars associated with $Q$, $U$ and $V$, necessarily, evolves.

Clearly, the expectation that the vector triad evolves could have experimental consequences. As we will demonstrate in a later section, the evolution of vector triads, as constrained by new underlying physics, could lead to an explanation for, among other things, recent observations of solar neutrino mixing.
3.3 Quantization of the coordinates U and V

Because the fermion-number operator $F_{(op)}$ is a 2 by 2 matrix, its continuation (see Sec. 2.1.3), namely $F(v)$, must likewise be a 2 by 2 matrix. Thus we have a natural explanation for why there must be two new linearly-independent flavor-defining fermion coordinates $U$ and $V$ associated with quarks and leptons, and their hypothetical superpartners [19]. Note that we have just two new (2-vector) coordinates $U$ and $V$ because the resultant 2-space can maintain at most two linearly independent 2-vector resolutions of $Q$ at a time [31]. The question now arises as to how it is that these fermion coordinates can be quantized.

Since we do not have a first-principles understanding of the proposed quantization in terms of underlying dynamics, we will appeal to simple heuristic arguments that serve not only to place the question in perspective, but also yield definite rules for quantizing these coordinates.

![Figure 2](image-url)

**Figure 2.** This figure illustrates the six $U$-vector coordinates (solid arrows), and the two “electric charge” or $Q$-vector coordinates (dotted arrows) associated with matter particles (quarks $q$ and leptons $l$), namely, $Q_q$ and $Q_l$. The horizontal “dotted” arrow is offset from the origin for purposes of illustration. It is assumed to pass through the origin.
3.3.1 Quantization of the \( U \)-vector

Write the 2-vectors \( U \) and \( V \) in terms of component charges, which we would like to quantize so as to quantize the coordinates \( U \) and \( V \). That is, write them as row \(( \ ) \) or column \(( \ ) \) vectors thus: \( U = (u_1, u_2) \) and \( V = (v_1, v_2) \) or \( U = \{u_1, u_2\} \) and \( V = \{v_1, v_2\} \), respectively. Now argue that it should take only one charge type (i.e., \( u \) or \( v \)) to distinguish “up” and “down” type flavors within any and all “up”-“down” type flavor doublets. If this is so, we can assume, without loss of generality, that this is the charge \( u \) (i.e., \( u_1 \) or \( u_2 \)). Then, since the charge \( v \) does not distinguish “up” and “down” type flavors, it must be true that \( v_1 = v_2 \) for every flavor doublet of fundamental fermions. And, therefore, given the metric in (11), the quantized charge \( V^2 \) has but one value, namely, \( V^2 = (v_1, v_2) \cdot \{v_1, v_2\} = (v_1^2 - v_2^2) = 0 \) for all fundamental fermions. That is, all fundamental fermions are “neutral,” i.e., they are singlets with respect to the global charge \( V^2 \).

Then, since \( Q = U + V \), we can write \( Q \) in column vector form in terms of \( U \) and \( V \) as \( Q = \{q_1, q_2\} = \{u_1, u_2\} + \{v_1, v_2\} \). Given that the difference between “up” and “down” electric charges for all fundamental fermions defines the fundamental (quantum) unit of electric charge (see Eqs. 18 and 20), namely, \( q_1 - q_2 = \pm 1 \), and given that \( v_1 = v_2 \) for all fundamental fermions, the vector \( U \) must have the column-vector form \( U = \pm\{a, a - 1\} \) for all fundamental fermions.

Now consider the nature of the charges \( u \) (i.e., \( u_1 \) and/or \( u_2 \)), which are the components of \( U = \{u_1, u_2\} \). Because the charge \( u \) distinguishes “up” flavors from “down” flavors, and because it is a charge, it must be possible for this charge, like all known charges, to assume the value zero. Then, given that \( u \)-charge can be zero in certain cases, and given the form \( U = \pm\{a, a - 1\} \), we see immediately that the quantized values of \( a \) must include the values \( a = 0 \) or \( 1 \). This means that there are at least four \( U \)-vectors, namely, the vectors \( U = \pm\{0, -1\} \) and \( U = \pm\{1, 0\} \). And, given the metric (11), this means that the quantized charge \( U^2 \) must include the values \( U^2 = \pm 1 \).

Finally, the quantized charge \( U^2 \), like all known charges, should be able to assume the value zero for which \( a = \frac{1}{2} \). Then the quantized values \( a \) are found to be limited to the finite sequence consisting of the three values \( a = 0, \frac{1}{2}, 1 \). Similarly, we find that the quantized values of the charge \( U^2 \) are found to be limited to the finite sequence of three values \( U^2 = 0, \pm 1 \), while the associated quantized \( U \)-vectors are limited to the finite sequence of six vectors (see Fig. 2), namely, \( U = \pm\{1/2, -1/2\}, \pm\{1, 0\}, \pm\{0, -1\} \).

3.3.2 Family replication and the number of families

Clearly, if we limit the discussion to matter particles—the same arguments apply to antimatter particles and the vectors \( Q_q^c \) and \( Q_l^c \)—we note that because the vector \( Q_q \) applies to quarks and \( Q_l \) applies to leptons, some of the \( U \)- and \( V \)-vectors (and associated global charges) will apply to quarks, but not to leptons, and vice versa. So, it is appropriate to ask which of the six possible \( U \)-vectors go with the vector \( Q_q \), and which of the six possible \( U \)-vectors go with the vector \( Q_l \). Once this question is answered, one can easily determine which \( V \)-vectors go with the vector \( Q_q \), and which \( V \)-vectors go with the vector \( Q_l \). In
other words, we will be able to determine which $U$-vectors go with which $V$-vectors to form a vector triad $(Q, U, V)$ that serves to describe flavor doublets of quarks or leptons. This information will tell us, in turn, how many generations of quarks and leptons there are, and hence how many families of fundamental fermions there are.

By inspection of Figure 2 we see that the requisite form for $V$-vectors, namely, $V = \{v, v\}$ cannot be maintained if $U$-vectors in quadrant II (IV) are associated with the vector $Q_q (Q_l)$ in quadrant IV (II). Therefore, we discover that the single vector $Q_q (Q_l)$ in quadrant IV (II), can only be associated with the three $U$-vectors located in the same quadrant. This simple result immediately tells us that, because there are only six $U$-vectors to choose from—three of which fall in quadrant IV, and three of which fall in quadrant II—there are only three generations of quark flavor doublets and three generations of lepton flavor doublets.

This is a remarkable result, for it means that a most unexpected explanation for why there are three families “automatically” emerges from the 2-space description. Of course, the underlying dynamics responsible for the 2-space, its quantization, and its associated selection rules would, if available, presumably provide a much more detailed explanation.

### 3.3.3 Quantization of the $V$-vector

Given the four $Q$-vectors (see Fig. 1), the six $U$-vectors from Section 3.3.1 and Figure 2, the associations between vectors such as $Q_q, Q_l$ and the six $U$-vectors (see Sec. 3.3.2), and the relation $Q = U + V$, we have the following sequence of eleven quantized $V$-vectors, consistent with the requisite form $V = \{v, v\}$, namely, $V = \{0,0\}, \pm\{1/6, 1/6\}, \pm\{-1/3, -1/3\}, \pm\{-1/2, -1/2\}, \pm\{2/3, 2/3\}, \pm\{-1, -1\}$ where the upper plus signs refer to matter, and the lower minus signs refer to antimatter. From the foregoing quantized $U$- and $V$-vectors we have the sequence of eleven quantized global charges $U \cdot V = 0, \pm1/6, \pm1/3, \pm1/2, \pm2/3, \pm1$.

Here, the $V$-vectors $\pm\{1/6, 1/6\}, \pm\{-1/3, -1/3\}$ and $\pm\{2/3, 2/3\}$ apply to quarks (upper plus signs) and antiquarks (lower minus signs), while the $V$-vectors $\{0,0\}, \pm\{-1/2, -1/2\}$ and $\pm\{-1, -1\}$ apply to leptons (upper plus signs) and antileptons (lower minus signs). Similarly, the charges $U \cdot V = \pm1/6, \mp1/3, \pm2/3$ apply to quarks (upper signs) and antiquarks (lower signs), while the charges $U \cdot V = 0, \mp1/2, \pm1$ apply to leptons (upper signs) and antileptons (lower signs). Note that the scalar components of $V$-vectors are proportional to the associated charges $U \cdot V$.

### 3.3.4 A natural basis for family membership of flavor doublets

The discussion in Section 3.3.2 makes it clear that there are only three quark-lepton families. But, what determines which quark flavor doublet goes with which lepton flavor doublet to make a particular quark-lepton family? In other words, how does one decide which quark $U$-vector goes with which lepton $U$-vector?

The question of family membership of flavor doublets is not answered in the standard model of particle physics where these family assignments are made, more-or-less, on the basis of the mass hierarchy of the average family member. However, in the 2-space, there are
other, more “natural,” ways of assigning flavor doublets to families (see Ref. 9 for a detailed discussion). We note first the fact that two of the three quark flavor doublets associated with the three \( \mathbf{U} \)-vectors \( \mathbf{U} = \{1/2, -1/2\}, \{1, 0\} \) and \( \{0, -1\} \), are “equidistant” from the “intermediate” \( \mathbf{U} \)-vector \( \{1, 0\} \), with respect to a matrix \( \mathbf{R} \) given by

\[
\mathbf{R} = \begin{pmatrix} 0 & -2 \\ -1 & -1 \end{pmatrix}.
\]

That is, the matrix \( \mathbf{R} \) is a Lorentz-invariant “measure” of the “distance” between certain \( \mathbf{U} \)-vectors, and their associated flavor doublets, thus \[34, 35\]

\[
\{1/2, -1/2\} \xrightarrow{\mathbf{R}} \{1, 0\} \xrightarrow{\mathbf{R}} \{0, -1\}.
\]

Similarly, two of the three lepton flavor doublets associated with the three \( \mathbf{U} \)-vectors \( \{-1/2, 1/2\}, \{-1, 0\} \) and \( \{0, 1\} \), are also “equidistant” from the “intermediate” \( \mathbf{U} \)-vector \( \{-1, 0\} \), with respect to the same matrix \( \mathbf{R} \) since

\[
\{-1/2, 1/2\} \xrightarrow{\mathbf{R}} \{-1, 0\} \xrightarrow{\mathbf{R}} \{0, 1\}.
\]

In other words, the \( \mathbf{U} \)-vector pairs \( \{1/2, -1/2\} \) and \( \{-1/2, 1/2\} \) or \( \{1, 0\} \) and \( \{-1, 0\} \) or \( \{0, -1\} \) and \( \{0, 1\} \) are at the same “distance” or “location” with respect to the single “distance” measure \( \mathbf{R} \). That is, they are associated with, or are “in,” the same family. To state this differently, the \( \mathbf{U} \)-vector describing a quark flavor doublet in a given family is the additive inverse of the \( \mathbf{U} \)-vector describing a lepton flavor doublet in the same family, and vice versa.

Because of the foregoing relations and identifications, it would seem very natural to assign the \( \mathbf{U} \)-vectors \( \pm\{1/2, -1/2\} \) to the first family, \( \pm\{1, 0\} \) to the second, or intermediate family, and \( \pm\{0, -1\} \) to the third family, where the upper + signs refer to quarks and the lower – signs refer to leptons. While the assignment of the \( \mathbf{U} \)-vectors \( \pm\{1, 0\} \) to the second, or intermediate, family seems unambiguous, what prevents us from assigning the \( \mathbf{U} \)-vectors \( \pm\{1/2, -1/2\} \) to the third family, and the \( \mathbf{U} \)-vectors \( \pm\{0, -1\} \) to the first family?

After all, we could just as well have used the multiplicative inverse of the matrix \( \mathbf{R} \), namely, \( \mathbf{R}^{-1} \) where

\[
\mathbf{R}^{-1} = \begin{pmatrix} \frac{1}{2} & -1 \\ -\frac{1}{2} & 0 \end{pmatrix},
\]

to be the Lorentz-invariant measure of “distance,” in which case we would have had the following relations and associations

\[
\pm\{1/2, -1/2\} \xleftarrow{\mathbf{R}^{-1}} \pm\{1, 0\} \xleftarrow{\mathbf{R}^{-1}} \pm\{0, -1\}.
\]

Clearly, from these relations and associations it would seem just as natural to assign the \( \mathbf{U} \)-vectors \( \pm\{1/2, -1/2\} \) to the third family, \( \pm\{1, 0\} \) to the second, or intermediate family,
and ±\{0, -1\} to the first family. How then, are we to decide if the U-vectors ±\{1/2, -1/2\}, are to be assigned to the first or the third family? The 2-space description of quarks and leptons provides a simple “answer” to this question as well.

It turns out (see Ref. 9, pp. 56–58) that the vector triads \((Q, U, V)\) associated with both quarks and leptons have the highest possible degree of symmetry with respect to the matrix transformation \(F(v)\), only when the associated U-vectors are ±\{1/2, -1/2\}. And, because the “ground state” is typically the most probable state, with the highest degree of symmetry into which other less symmetrical states tend to evolve [36], it is natural to assume that the U-vectors ±\{1/2, -1/2\} are associated with the first or “ground state” family. Recall that the members of the first family have the lowest average mass of any family members, and so the first family naturally resembles a kind of “ground state.”

The foregoing associations and identifications, together with the specific family assignments of the experimentally observed quarks and leptons, are elaborated elsewhere (see Ref. 9). However, it should be clear—as we have repeatedly emphasized throughout this paper—that the foregoing heuristic 2-space relations and associations must be a consequence of some “deeper,” more fundamental, “layer” of physics that is responsible for the 2-space, its quantization, and its associated selection rules.

3.4 The nature of transitions of vector triads

Because strong and electromagnetic interactions have no effect on flavor indices, they can have no effect on the flavor-defining fermion coordinates. Hence, vector triads \((Q, U, V)\) will be unaffected by strong and electromagnetic interactions. However, in general, this is not the case for weak interactions.

Because the quantum state of an “isolated” fundamental fermion such as a quark (eventually) evolves via weak interactions, so must the associated vector triad, consisting of the three 2-vectors \((Q, U, V)\) evolve, and vice versa. And, if we knew the dynamics underlying the 2-space, presumably we would be in a position to calculate, from first principles, such things as Cabibbo-Kobayashi-Maskawa (CKM) matrix elements. However, as this dynamics is currently unknown, we will have to satisfy ourselves here with a less ambitious description based on mathematical properties of the 2-space, and certain experimental facts about transitions of fundamental fermions. Consider first the so-called “diagonal” weak transitions of quarks.

Diagonal weak transitions of quarks are “up”-“down” or “down”-“up” type transitions involving the two quark flavors (flavor eigenstates) “located” within the same flavor doublet. From experimental data on CKM matrix elements, these diagonal transitions are known to be far more probable than the so-called “off-diagonal” transitions, which are relatively suppressed. What do these experimental facts teach us about the evolution of vector triads?

In the case of diagonal transitions, it is clear that the vector triad associated with the initial quark state, and the vector triad associated with the final quark state, are identical, i.e., they are congruent. By comparison, the initial and final state triads associated with all “off-diagonal” weak transitions of quarks are different. These observations suggest that
the underlying dynamics tends to “prefer” transitions in which the associated vector triad is unchanged. Clearly, such transitions would be characterized by the “selection rules” (use $Q = U + V$)

$$\Delta Q = \Delta U = \Delta V = 0.$$  \hfill (56)

Moreover, since $Q$, $U$ and $V$ don’t change in such transitions, we also have the selection rules

$$\Delta Q^2 = \Delta U^2 = \Delta U \cdot V = 0,$$  \hfill (57)

which means that the baryon number $B = Q^2$, the charge $U^2$, and the charge $U \cdot V$, are separately maintained or conserved in such transitions [37].

While quark masses play a significant role in determining the magnitudes of CKM matrix elements, there appears to be a trend here, namely, this: “Easy” transitions (i.e., diagonal transitions within a quark generation) are those for which the associated vector triad does not change, and “hard” transitions (i.e., off-diagonal transitions between different quark generations) are those for which it does change.

To be more specific, while there are some 24 flavor and antiflavor eigenstates of quarks and leptons [17], there are just 12 distinct vector triads. That is, the vector triad associated with each flavor doublet is associated with each of two flavor or antiflavor eigenstates. This means that all diagonal transitions involve the same vector triad, and all off-diagonal transitions involve different (distinguishable) vector triads.

What is it that differs from vector triad to vector triad? Of course, charge-like quantum numbers such as

$$Q^2, U^2, \text{ and } U \cdot V$$  \hfill (58)

can sometimes differ. But there are other, more subtle, things that can differ as well. It happens that different vector triads, and certain of their associated 2-vector coordinates, are “isolated” from one another in a topological sense. Take for example the four 2-vectors $Q_q$, $Q_c$, $Q_l$, and $Q_{cl}$ illustrated in Figure 1.

Because there are no continuous transformations (associated with the metric in Eq. 11) available to convert the 2-vectors $Q_q$, $Q_c$, $Q_l$ and $Q_{cl}$ one into the other, these four $Q$-vectors are, necessarily, topologically distinct. It is certainly conceivable that this fact could be an important underlying reason for the separate conservation of baryon number $B$ and lepton number $L$ in most physical processes. In particular, because the $Q$-vectors are topologically distinct, underlying dynamics could conceivably dictate that there are topological energy “barriers” that maintain these topological distinctions, leading to the 2-vector selection rule [39]

$$\Delta Q = 0,$$  \hfill (59)

and consequently to the selection rule (Note that $Q^2 = B$ or $L$)

$$\Delta Q^2 = 0,$$  \hfill (60)
which should then apply to all fundamental fermions \cite{39}.

Of course, the discussion in the previous paragraph involves only one of the three 2-vector coordinates (i.e., $Q$) associated with a vector triad $(Q, U, V)$. There are other topological distinctions that can be drawn, which involve all three 2-vectors associated with a particular vector triad. For example, as shown elsewhere (see Ref. 9, pp. 56, 57, and Appendix A in Ref. 13), and as elaborated in the following section, there are also topological distinctions based on how an entire vector triad $(Q, U, V)$ transforms under the matrix $F(v)$. It happens that some vector triads transform like Möbius strips, and some transform like cylinders under $F(v)$. Clearly, if there were underlying topological energy “barriers” that act to inhibit changes in the topology of entire vector triads, this situation could have experimental consequences \cite{38}. And, if experiments were to eventually confirm that such topological energy “barriers” exist, a new level of dynamics “below” or “beyond” the standard model of particle physics would seem to be required to explain them.

### 3.5 Does the dynamics underlying the 2-space impose topological constraints on 3-flavor neutrino mixtures?

As alluded to in the previous section, it is important to understand that the new 2-space description of fundamental fermions (quarks and leptons) provides a distinction between these particles that goes beyond differences that can be explained by mass differences alone. For example, in the standard model of particle physics the only difference between the $u$, $c$ and $t$ quarks is that they have different masses. Otherwise, these particles experience identical strong and electroweak interactions. Moreover, the separate conservation (in strong and electromagnetic interactions) of quantum numbers such as “charm” and “truth” can be attributed to certain unavoidable “accidental symmetries” associated with the (renormalizable) Lagrangian describing the interactions of these particles \cite{7}.

Taken at face value, these accidental symmetries would seem to imply that there are no internal “wheels and gears” that would distinguish a $u$ quark from a $c$ quark, for example. But, if the string theories are correct, these particles would be associated with different “handles” on the compactified space (see Ref. 4, Vol. 2, p. 408), and so would be different in this additional sense. Likewise, in the present non-Euclidean 2-space description, topological differences in addition to a variety of (global) 2-scalars, which are indirectly related to the accidental symmetries of the Lagrangian, serve to provide further distinctions between matter particles.

A possible experimental signal of such internal differences is to be found in the recent observations at the Super Kamiokande and SNO of (nearly) bi-maximal $\nu_\mu - \nu_\tau$ neutrino mixing \cite{13}. Models which begin by positing a neutrino mass-matrix, and associated mixing-parameters, such as the three-generation model proposed by Georgi and Glashow \cite{40}, do an acceptable job of describing the observations. However, (nearly) bi-maximal $\nu_\mu - \nu_\tau$ mixing may have a deeper explanation in terms of internal topological differences (in the non-Euclidean 2-space) between $\nu_e$, and $\nu_\mu$ or $\nu_\tau$ neutrinos.

With respect to the internal transformation $F(v)$, the topology of the non-Euclidean
“vector triad” (see Ref. 9, p. 57, and the qualifying remarks in Appendix A of Ref. 13) associated with the $\nu_e$ ($\nu_\mu$ or $\nu_\tau$), is found to be that of a cylinder (Möbius strip). And, \textit{assuming that a change in topology (of vector triads) during neutrino mixing is suppressed by energy “barriers,” or other topological “barriers” described by physics “beyond” the standard model,} while neutrino mixing without topology-change is (relatively) enhanced \cite{38, 41, 42}, one can readily explain the experimental observation of (nearly) bi-maximal $\nu_\mu - \nu_\tau$ neutrino mixing—at least (nearly) bi-maximal $\nu_\mu - \nu_\tau$ mixing over long distances, where the proposed topological influences are expected to be \textit{cumulative}. If this qualitative explanation (see detailed calculations below) is basically correct, then it follows that the neutrino mass-matrix, and associated mixing-parameters needed to explain (nearly) bi-maximal $\nu_\mu - \nu_\tau$ neutrino mixing, would be the \textit{result} of new physics (i.e., dynamics) associated with these deeper (internal) topological differences between neutrinos, and not their \textit{cause}.

### 3.5.1 Conventional description of 3-flavor neutrino mixing

Except where explicitly prevented by some “absolute” conservation law (e.g., the conservation of electric charge or spin angular momentum), quantum mechanics generally permits transitions between states having \textit{different} topologies \cite{11}. While a change in topology may be energetically (or otherwise) inhibited by underlying dynamics, unavoidable quantum fluctuations are expected to \textit{catalyze} such processes. Hence, there is always the possibility of \textit{mixing} between otherwise similar quantum states (e.g., $\nu_e$, $\nu_\mu$ and $\nu_\tau$) associated with vector triads having \textit{distinct} topologies \cite{12}.

As described previously—with respect to the matrix transformation $\mathbf{F}(v)$—the topology associated with both the $\nu_\mu$ and $\nu_\tau$ is found to be that of a Möbius strip (Ref. 47, p. 143). By contrast, the topology associated with the $\nu_e$ [with respect to $\mathbf{F}(v)$] is that of a cylinder. And because it is reasonable to \textit{assume} that a change in topology during transitions tends to be \textit{suppressed} \cite{38}, the foregoing topological distinctions between neutrinos may help explain recent observations of (nearly) bi-maximal $\nu_\mu-\nu_\tau$ mixing \cite{13}. However, it is important to stress at the outset that there are good reasons for believing that similar topological distinctions (alone) among quarks, while also present, do not play an important role in $d$, $s$, $b$ quark mixing \cite{44}.

At “birth” via weak decays, or upon detection via weak capture interactions, neutrinos have a \textit{definite} flavor and associated topology \cite{15}. However, between birth and detection they are in a mixed state having no definite flavor or associated topology. In this intermediate region the probability of flavor (or topology) maintenance and/or change \textit{oscillates}. Only at great distances from the neutrino source do these oscillations finally “damp out.”

In the conventional description of three-flavor neutrino mixing \textit{flavor eigenstates} are related to neutrino \textit{mass eigenstates} (states of definite mass) via a unitary CKM-like “mixing” matrix $U_{\alpha i}$, as follows (Ref. 3, p. 365 and Ref. 46)

$$\nu_\alpha = \sum_{i=1}^{3} U_{\alpha i} \nu_i. \quad (61)$$
Here, \( \nu_i = \nu_1, \nu_2 \) and \( \nu_3 \) are mass eigenstates with mass eigenvalues \( m_1, m_2 \) and \( m_3 \), respectively, while \( \nu_\alpha = \nu_e, \nu_\mu \) and \( \nu_\tau \) are flavor eigenstates. Using a conventional parameterization for \( U_{\alpha i}, \) (61) becomes

\[
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix} =
\begin{pmatrix}
c_1 & s_1 c_3, & s_1 s_3 \\
-s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta}, & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\
-s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta}, & c_1 s_2 s_3 - c_2 c_3 e^{i\delta}
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix},
\]

(62)

where \( c_i \equiv \cos \theta_i \) and \( s_i \equiv \sin \theta_i \), and \( \delta \) is associated with a Dirac-type CP-noninvariant phase factor \( e^{i\delta} \).

Using (62) it can be shown that the probability of detecting a neutrino of flavor type \( \beta \) at a distance \( X \) from a source of neutrinos of flavor type \( \alpha \) is given by

\[
P_{\nu_\alpha \rightarrow \nu_\beta} = \sum_{i=1}^{3} |U_{\alpha i}|^2 |U_{\beta i}|^2 + \sum_{i \neq j}^{3} U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j} \cos \left( \frac{2\pi X}{l_{ij}} \right).
\]

(63)

Here, the so-called oscillation lengths \( l_{ij} \) are given by \( l_{ij} = 2\pi/(E_i - E_j) \), where the total relativistic energy differences in a beam of neutrinos having fixed momentum \( p \) are \( E_i - E_j = (m_1^2 - m_2^2)/2p \).

Examination of (63) shows that at the neutrino source (\( X = 0 \), and \( t = 0 \)) the probability \( P_{\nu_\alpha \rightarrow \nu_\beta} \) reduces, as expected, to the 3 \( \times \) 3 identity matrix

\[
P_{\nu_\alpha \rightarrow \nu_\beta} \bigg|_{X=0} = I_3,
\]

(64)

while at “intermediate” distances from the neutrino source (\( X \approx l_{ij} \)), the probability \( P_{\nu_\alpha \rightarrow \nu_\beta} \) undergoes oscillations. Finally, at great distances from the neutrino source (\( X \gg l_{ij} \)), all time- or distance-dependent oscillations “damp out,” and we are left with a 3 \( \times \) 3 matrix of time-average probabilities \( \langle P_{\nu_\alpha \rightarrow \nu_\beta} \rangle \), namely,

\[
\langle P_{\nu_\alpha \rightarrow \nu_\beta} \rangle = \sum_{i=1}^{3} |U_{\alpha i}|^2 |U_{\beta i}|^2.
\]

(65)

From (65) it is clear that this matrix (\( \alpha = \) row index, \( \beta = \) column index), which describes long-distance neutrino mixtures, is symmetric. Keeping in mind this symmetry, and calling this matrix \( M \), one has

\[
M = \begin{pmatrix}
\langle P_{\nu_e \rightarrow \nu_e} \rangle, & \langle P_{\nu_e \rightarrow \nu_\mu} \rangle, & \langle P_{\nu_e \rightarrow \nu_\tau} \rangle \\
\langle P_{\nu_\mu \rightarrow \nu_e} \rangle, & \langle P_{\nu_\mu \rightarrow \nu_\mu} \rangle, & \langle P_{\nu_\mu \rightarrow \nu_\tau} \rangle \\
\langle P_{\nu_\tau \rightarrow \nu_e} \rangle, & \langle P_{\nu_\tau \rightarrow \nu_\mu} \rangle, & \langle P_{\nu_\tau \rightarrow \nu_\tau} \rangle
\end{pmatrix}.
\]

(66)

Note that all rows and columns of \( M \) must sum to unity (total probability 1).
Using $M$ we can describe the expected neutrino flavor content at a great distance from a neutrino source (e.g., the sun or a supernova) as follows:

$$\{D_e, D_\mu, D_\tau\} = M\{B_e, B_\mu, B_\tau\},$$

(67)

where $\{\}$ signifies column vectors, and $D_\alpha$ is the number of detected neutrinos of definite flavor $\nu_\alpha$, and $B_\alpha$ their number at “birth” at some distant neutrino source. Note that because neutrinos are assumed to be conserved, the total number of neutrinos at birth equals their number upon “detection,” namely,

$$B_e + B_\mu + B_\tau = D_e + D_\mu + D_\tau.$$

(68)

### 3.5.2 Proposed topological constraints on 3-flavor neutrino mixing

Given that the $\nu_e$ ($\nu_\mu$ or $\nu_\tau$) neutrino flavor has the associated topology of a cylinder (Möbius strip) with respect to the internal transformation $F(v)$, and assuming that topological constraints are the primary determinants of the matrix $M$ describing long-distance neutrino mixtures, the matrix $M$ is easily determined. To accomplish this we need only apply the following very general principle to neutrino-neutrino transitions:

*All other things being equal, any neutrino flavor $\nu_\alpha$ (i.e., $\nu_e$, $\nu_\mu$ or $\nu_\tau$), which under goes neutrino-neutrino transitions that change the associated neutrino topology, will tend to be suppressed, while neutrino-neutrino transitions that maintain the associated neutrino topology will tend to be (relatively) enhanced.*

To this principle we add the following corollary,

*All other things being equal, because the $\nu_\mu$ and $\nu_\tau$ neutrinos have the same associated topology, they will act the same way in all neutrino-neutrino transitions (involving long-distance neutrino mixtures).*

Given these principles, and assuming as stated previously that topological constraints are the primary determinants of the matrix $M$, we immediately have the following topological constraints on long-distance neutrino mixtures:

A. No matter what neutrino flavor ($\nu_\alpha$) and associated topology one starts with at some distant source (say the sun or a supernova), by the time the neutrino mixture reaches its “equilibrium” state (where all time-dependent oscillations have “damped out”), it should contain equal fractions of $\nu_\mu$ and $\nu_\tau$, because these neutrinos have the same associated topology.

B. Because the $\nu_\mu$ and $\nu_\tau$ neutrinos have the same associated topology, if one starts out with either a pure $\nu_\mu$ or a pure $\nu_\tau$ source, one should end up with the same long-distance equilibrium mixture of $\nu_e$, $\nu_\mu$ and $\nu_\tau$.

C. If topology is the controlling factor in describing long-distance neutrino mixtures, then there should be absolutely no (effective) difference between the two functions (of $U$-matrix mixing parameters), which describe $\langle P_{\nu_e \rightarrow \nu_\mu}\rangle$ and $\langle P_{\nu_e \rightarrow \nu_\tau}\rangle$, because the $\nu_\mu$ and $\nu_\tau$
neutrinos are associated with the same topology. Very loosely speaking we are assuming that these mathematical functions are effectively “topological invariants” with respect to the exchange of flavor indices $\mu$ and $\tau$ (see Ref. 47, pp. 20 and 21). That is, not only are the two functions $\langle P_{\nu_e\rightarrow\nu_\mu} \rangle$ and $\langle P_{\nu_\tau\rightarrow\nu_\mu} \rangle$ required to be equal, but they are also required to be equal, term-by-term (i.e., they are required to be term-wise equal). Similarly the three functions $\langle P_{\nu_\mu\rightarrow\nu_\mu} \rangle$, $\langle P_{\nu_\mu\rightarrow\nu_\tau} \rangle$ and $\langle P_{\nu_\tau\rightarrow\nu_\mu} \rangle$ are required to be term-wise equal.

Constraints A), B) and C), together with Eqs. (67) and (68), dictate that the symmetric matrix $M$ describing long-distance neutrino mixtures must have the form expressed by

$$
\begin{pmatrix}
D_e \\
D_\mu \\
D_\tau
\end{pmatrix} =
\begin{pmatrix}
a & b & b \\
b & c & c \\
b & c & c
\end{pmatrix}
\begin{pmatrix}
B_e \\
B_\mu \\
B_\tau
\end{pmatrix}.
$$

(69)

### 3.5.3 Determination of the matrix $M$

When the proposed topological constraints of Section 3.5.2 are applied to the conventional description of neutrino mixing (62), the matrix $M$ in (66), (67) and (69), is uniquely determined. To see how this happens consider the following time-average probabilities (see Appendix B in Ref. 13 for details)

$$
\langle P_{\nu_e\rightarrow\nu_\mu} \rangle = 2s_1^2s_2^2 + 2s_1s_3c_3c_2^2 + 2s_1^2s_2s_3c_1c_2c_3 \cos \delta(s_3^2 - c_3^2),
$$

(70)

and

$$
\langle P_{\nu_e\rightarrow\nu_\tau} \rangle = 2s_1^2s_2^2 + 2s_1s_3c_3c_2^2 - 2s_1^2s_2s_3c_1c_2c_3 \cos \delta(s_3^2 - c_3^2).
$$

(71)

According to the proposed topological constraints of Section 3.5.2, these two time-average probabilities must be term-wise equal, and nonzero. These requirements place three constraints on the $U$-matrix mixing parameters, namely, $s_1^2 > 0$, $s_2^2 = c_2^2 = \frac{1}{2}$, and

$$
s_1^2s_2s_3c_1c_2c_3 \cos \delta(s_3^2 - c_3^2) = 0.
$$

(72)

Note that (70) and (71) are term-wise equal as required, even with the minus sign preceding the last term of (71) because this term is of zero magnitude.

Next, the topological constraint of Section 3.5.2, namely, $\langle P_{\nu_\mu\rightarrow\nu_\mu} \rangle = \langle P_{\nu_\mu\rightarrow\nu_\tau} \rangle$ can be realized provided the $U$-matrix mixing parameters are further constrained by $s_3^2 = c_3^2 = \frac{1}{2}$, which also happens to satisfy (72). That is, given $s_2^2 = c_2^2$ and $s_3^2 = c_3^2$ one has (see Appendix B in Ref. 13 for details)

$$
\langle P_{\nu_\mu\rightarrow\nu_\mu} \rangle = s_1^4c_4^2 + 2c_2^4c_3^4(c_1^2 + 1)^2 + 8c_1^2c_2^4c_3^4 \cos^2 \delta,
$$

(73)

and

$$
\langle P_{\nu_\tau\rightarrow\nu_\mu} \rangle = s_1^4c_4^2 + 2c_2^4c_3^4(c_1^2 + 1)^2 + 8c_1^2c_2^4c_3^4 \cos^2 \delta.
$$

(74)
Now the topological constraints of Section 3.5.2 also require that \( \langle P_{\nu_\mu \rightarrow \nu_\tau} \rangle \) be equal to (73) and (74). This places additional constraints on \( M \), and the \( U \)-matrix mixing parameters.

Comparing (73) and (74) with the following expression from Appendix B in Ref. 13 for \( \langle P_{\nu_\mu \rightarrow \nu_\tau} \rangle \), namely,

\[
\langle P_{\nu_\mu \rightarrow \nu_\tau} \rangle = s_4^1 c_2^3 + 2c_2^2 c_3^4 (c_1^2 + 1)^2 - 8c_2^2 c_3^4 c_3^2 \cos^2 \delta,
\]

we see that the requisite equality \( \langle P_{\nu_\mu \rightarrow \nu_\mu} \rangle = \langle P_{\nu_\tau \rightarrow \nu_\tau} \rangle = \langle P_{\nu_\mu \rightarrow \nu_\tau} \rangle \) leads to the constraint

\[
c_1^2 \cos^2 \delta = 0.
\]

Assuming that \( \sin^2 \delta \neq 1 \) or \( \cos^2 \delta \neq 0 \) (i.e., CP violation is not maximal), we further determine from (76) that

\[
c_1^2 = 0 \quad \text{and} \quad s_1^2 = 1.
\]

Note that (73), (74) and (75) exhibit the requisite term-wise equality [see item C) in Sec. 3.5.2] even with the minus sign preceding the third term of (75), because this term is of zero magnitude.

Gathering together the predicted mixing-parameter constraints, namely, \( c_2^2 = s_2^2, c_3^2 = s_3^2, s_1^2 = 1 \) and \( c_1^2 = 0 \), we can express six of the nine time-average probabilities associated with the (symmetric) matrix \( M \) as follows (see Ref. 13).

\[
\langle P_{\nu_e \rightarrow \nu_e} \rangle = 1 - 2c_3^4,
\]

\[
\langle P_{\nu_e \rightarrow \nu_\mu} \rangle = \langle P_{\nu_e \rightarrow \nu_\tau} \rangle = 2c_3^2 c_3^4,
\]

and

\[
\langle P_{\nu_\mu \rightarrow \nu_\mu} \rangle = \langle P_{\nu_\tau \rightarrow \nu_\tau} \rangle = \langle P_{\nu_\mu \rightarrow \nu_\tau} \rangle = c_2^4 (1 + 2c_3^4).
\]

But, the previous arguments have shown that the \( U \)-matrix mixing parameters satisfy \( c_2^2 = c_3^2 = \frac{1}{2} \), which leads to the following specific numerical predictions

\[
\langle P_{\nu_e \rightarrow \nu_e} \rangle = \frac{1}{2} = a,
\]

\[
\langle P_{\nu_e \rightarrow \nu_\mu} \rangle = \langle P_{\nu_e \rightarrow \nu_\tau} \rangle = \frac{1}{4} = b,
\]

and

\[
\langle P_{\nu_\mu \rightarrow \nu_\mu} \rangle = \langle P_{\nu_\tau \rightarrow \nu_\tau} \rangle = \frac{3}{8} = c.
\]
Given that $M$ is symmetric, note that (81), (82) and (83) are the only matrix elements, which could be consistent with the proposed topological constraints of Section 3.5.2, and the requirement that $\sin^2 \delta \neq 1$ or $\cos^2 \delta \neq 0$.

Employing Eqs. (69) and (81–83), one finally has the prediction \[ M = \frac{1}{8} \begin{pmatrix} 4 & 2 & 2 \\ 2 & 3 & 3 \\ 2 & 3 & 3 \end{pmatrix}. \] (84)

To summarize, the proposed (qualitative) topological constraints on the matrix $M$ (see Sec. 3.5.2) result in quantitative constraints on the $U$-matrix mixing parameters (see Eq. 62), namely, $(s_2^2 = c_2^2 = c_3^2 = s_3^2 = \frac{1}{2}, s_1^2 = 1$ or $c_1^2 = 0$ with $\sin^2 \delta \neq 1$ or $\cos^2 \delta \neq 0$ assumed), which result, in turn, in a unique quantitative determination of the matrix $M$ (see Eq. 84).

### 3.5.4 Topology-maintaining and topology-changing influences in equilibrium?

Not only is the matrix $M$ in (84) a unique solution to the proposed topological constraints of Section 3.5.2 with $\sin^2 \delta \neq 1$ or $\cos^2 \delta \neq 0$, but it also has some very special properties that may eventually help reveal the deeper dynamical significance of this matrix. Note that (66), (69), and (84) have the very special property \[ \langle P_{\nu_e \rightarrow \nu_e} \rangle = \langle P_{\nu_e \rightarrow \nu_\mu} \rangle + \langle P_{\nu_e \rightarrow \nu_\tau} \rangle. \] (85)

This equation says that the time-average probability that the associated $\nu_e$ topology doesn’t change, namely, \[ P_{NC} = \langle P_{\nu_e \rightarrow \nu_e} \rangle = a, \] (86)

and the time-average probability that the associated $\nu_e$ topology does change, namely, \[ P_C = \langle P_{\nu_e \rightarrow \nu_\mu} \rangle + \langle P_{\nu_e \rightarrow \nu_\tau} \rangle = (1 - a), \] (87)

are equal, namely, \[ P_C = P_{NC}. \] (88)

Now this equality looks very much like an “equilibrium” condition between those underlying physical influences that would act to change the associated $\nu_e$ topology (quantum fluctuations), and those underlying physical influences that would act to maintain the associated $\nu_e$ topology (e.g., topological energy “barriers”).

We will now provide further support for this proposal. In particular, consider the “joint” probability \[ P = P_C \cdot P_{NC}, \] (89)
and notice that

\[
\frac{dP}{da} = P_C \frac{dP_{NC}}{da} + P_{NC} \frac{dP_C}{da}.
\]

(90)

From (86) and (87) this last equation reduces to

\[
\frac{dP}{da} = P_C - P_{NC}.
\]

(91)

And, taking the second derivative, we also find

\[
\frac{d^2P}{da^2} = -2 < 0.
\]

(92)

Therefore, when \( P_C = P_{NC} \), (91) and (92) tell us that the joint probability \( P = P_C \cdot P_{NC} \) is a maximum, namely, it characterizes some most probable condition or “state.” And, this of course is the very essence of an “equilibrium” condition [12]. However, it must be understood that this hypothetical (long-distance) “equilibrium” between topology-changing, and topology-maintaining physical influences, is only a (cumulative) result of deeper, and largely unknown (short-distance), dynamical processes in the vacuum, which first begin to act on neutrinos at their source—thereby, eventually establishing the equilibrium condition \( P_C = P_{NC} \)—on time scales very much shorter than the time it takes for the time-dependent oscillations in neutrino mixtures to “damp out.” This is an essential requirement if these hypothetical short-distance processes are to be responsible for “selecting” the (constant) \( U \)-matrix mixing parameters prior to neutrino mixing.

### 3.5.5 The current experimental situation in regard to 3-flavor neutrino mixtures

The most recent analysis [49] of Super Kamiokande and SNO data on solar neutrinos [13] is in good agreement with the prediction given by (84). If this prediction is eventually verified by further detailed observations of neutrinos from distant astronomical sources (e.g., the sun or a supernova), and/or in long-baseline terrestrial experiments (e.g., see a description of proposed Kamland experiments in Ref. 50), this will provide qualitative support for the new description of fundamental fermions, which requires, among other things, that the \( \nu_e \) and (\( \nu_\mu \) or \( \nu_\tau \)) neutrino flavors start, and end, “life” as topologically distinct quantum objects (see Appendix A in Ref. 13, and Refs. 9, 38, 47). However, while it is conceivable that experiments could still falsify (84), together with the assumption [10] that CP violation is not maximal (i.e., \( \sin^2\delta \neq 1 \) or \( \cos^2\delta \neq 0 \)), experimental verification of these things would not confirm the new description or the proposed topological constraints of Section 3.5.2 [51].

A much better theoretical understanding of the dynamical significance of the new 2-space description of quarks and leptons, in terms of the new internal 2-vector coordinates, will be required before one can be confident that, if finally verified, (84) really is a reflection of new physics, e.g., underlying topological constraints [51]. Clearly, additional (crucial) experimental consequences of the 2-space must be found if it is to be considered a serious candidate for a description of nature.
3.6 Does the 2-space imply the Standard Model?

Perhaps the best evidence we have for a “layer” of new physics—located somewhere between the unification scale, and the region of applicability of the standard model of particle physics—is simply that by constraining the possibilities, the 2-space description seems to imply something very similar to the standard model. Consider the following facts in support of this hypothesis.

First, and foremost, we have found that owing to the 2-space parameter \( v = \ln M_c \), where \( M_c \) is a measure of the strong-color “multiplicity,” a symmetry associated with strong interactions such as \( SU(3)_{\text{color}} \) seems to be more-or-less implicit in the 2-space description. Second, because the 2-space determines that there are some 48 fundamental fermions and antifermions, together with their associated global charges—identified with the global charges associated with the accidental symmetries of the strong-electroweak Lagrangian—the form of the Lagrangian is severely restricted. Third, it is clear that the 2-space dictates that we are dealing here with flavor doublets of quarks or leptons. And, this flavor doublet structure imposed by the 2-space, necessarily, severely restricts the possible interactions of fundamental fermions. For example, we know from the 2-space description that the magnitude of the difference in electric charges between the “up” and “down” flavors within a flavor doublet is the fundamental unit of electric charge. And, if there are transitions that change an “up” flavor to a “down” flavor or vice versa, there must exist a weak intermediate boson (with unit electric charge) to accomplish this. Thinking along these lines soon leads to the notion of weak isospin, and \( SU(2)_L \) flavor doublets carrying weak isospin. Fourth and finally, because the electric charge of flavor-doublet members is “explained” by the 2-space description, compatibility with electromagnetism, i.e., there is only one photon associated with flavor-doublet members (see Ref. 3, pp. 342–343), inevitably leads to the concept of weak hypercharge and \( U(1)_Y \). So, while the 2-space certainly does not rigidly determine these things, it does seem to severely limit the possibilities.

In this very general way, it seems that the 2-space, and especially whatever underlying physics is responsible for it, ultimately determines the current form assumed by the standard model of particle physics based on \( SU(3)_{\text{color}} \times SU(2)_L \times U(1)_Y \). However, until the underlying dynamics responsible for the 2-space is fully understood, the foregoing connections will remain speculative in nature.

4.0 Summary and Conclusions

It is widely believed that the explanation for fundamental-fermion (quark and lepton) flavors, flavor doublets, and family replication is to be found in theories of quantum gravity (e.g., superstrings). And, yet, as demonstrated in this paper and elsewhere, an analytic continuation of a Hermitian matrix used to represent the fermion-number operator, leads to a new internal description of quarks and leptons—a “layer” of new physics located somewhere between the unification scale, and the region of applicability of the standard model of particle physics—that also explains such things as family replication. The new description
[9–13] involves an (internal) 2-space, and two new associated 2-vector fermion observables \( U \) and \( V \) (linearly-independent flavor-defining “coordinates”) that are somewhat analogous to the spin coordinate. And, just as the spin coordinate explains certain subtleties of atomic spectra (e.g., the fine structure) these new 2-vector coordinates explain certain subtleties of fundamental-fermion “spectra.” In particular, using the two new flavor-defining fermion coordinates \( U \) and \( V \), we have discovered, among other lesser things, that:

1) The so-called “accidental” symmetries of the Lagrangian describing strong and electroweak interactions (of fundamental fermions) are not accidental, but instead seem to be imposed on these Lagrangians by an underlying non-Euclidean 2-space—or whatever underlying dynamics is responsible for the 2-space, its quantization, and its associated selection rules. In particular, a central conclusion of this work is this:

The global (flavor-defining) charges associated with the “accidental symmetries” of the Lagrangian describing strong and electroweak interactions of fundamental fermions, and the global (flavor-defining) charges associated with the non-Euclidean 2-space, are (essentially) one and the same charges.

2) Family replication is apparently little more than the number of different physically acceptable ways—just three ways are predicted—that the “electric-charge” vector coordinate \( Q \), for both quark and lepton flavor doublets, can be resolved in the 2-space into pairs of linearly independent 2-vector coordinates \( U \) and \( V \) (i.e., \( Q = U + V \)). However, while this description is reasonable, the underlying reasons for this “quantization” remain essentially unknown.

3) Unlike the situation in the standard model, particles such as the \( u \), \( c \) and \( t \) quarks are characterized by certain associated internal topological differences. Similar topological differences between different neutrino flavors may help explain recent observations of (nearly) bi-maximal \( \nu_\mu - \nu_\tau \) mixing. And, this in turn, may be yet another indirect experimental indication of new dynamics (related, in part, to topology) underlying the new 2-space description of quarks and leptons.

Finally, because the (global) accidental symmetries of the Lagrangian are at least partly determined by the 2-space, and because local symmetries such as \( SU(3)_{\text{color}} \), \( SU(2)_L \) and \( U(1)_Y \) are also more-or-less implicit in the 2-space description, this description tends to imply the standard model of particle physics based on \( SU(3)_{\text{color}} \times SU(2)_L \times U(1)_Y \). However, while progress has been made in making these connections, and in understanding the bewildering number of fermion flavors, two of the most important questions remain unanswered. First, what is the detailed nature of the “layer” of new physics, which is supposed to be responsible for the 2-space, its quantization, and its associated selection rules? Second, how does this “layer” of new physics arise or emerge from physics at the unification scale?
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6.0 References and Footnotes

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[8] S. Goudsmit and G. Ulenbeck, *Physica* 5, 266, 1925; *Nature* 117, 264, 1926. While flavor-defining fermion coordinates \((Q, U, V)\) are somewhat analogous to spin angular momentum vectors or spin “coordinates” (i.e., the total angular momentum vector \(J\)), there is one very important difference. All spin \(\frac{1}{2}\) flavors or antiflavors carry the same \(J\) and \(J^2 = 1/2(1/2 + 1)\hbar^2\). That is, neither \(J\) nor \(J^2\) distinguish between flavors or antiflavors whereas, as the term “flavor-defining” implies, every flavor or antiflavor is generally (but not always) associated with a different set of flavor-defining fermion coordinates \((Q, U, V)\). These differences, between \(J\) and \((Q, U, V)\), insure that while \(J\) vectors for any number of flavors and antiflavors can be superimposed (added), one cannot superimpose flavor-defining fermion coordinates for two or more flavors and antiflavors (see Sec. 3.2 in the main text for further discussion). For, if one could do this, these coordinates could not be self-consistently used to define individual “isolated”
flavors. In other words, flavor-defining fermion coordinates define (isolated) flavors or antiflavors, not collections of arbitrary numbers of flavors and antiflavors.

[9] Gerald L. Fitzpatrick, *The Family Problem-New Internal Algebraic and Geometric Regularities*, Nova Scientific Press, Issaquah, Washington, 1997. Additional information: [http://physicsweb.org/TIPTOP/](http://physicsweb.org/TIPTOP/) or [http://www.amazon.com/exec/obidos/ISBN=0965569500](http://www.amazon.com/exec/obidos/ISBN=0965569500)

[10] G. L. Fitzpatrick, “Continuation of the Fermion-Number Operator and the Puzzle of Families,” [arXiv:physics/0007038](http://arxiv.org/abs/physics/0007038) v1, 13 Jul 2000.

[11] G. L. Fitzpatrick, “Electric Charge as a Vector Quantity,” [arXiv:physics/0011073](http://arxiv.org/abs/physics/0011073) v1, 30 Nov 2000.

[12] G. L. Fitzpatrick, “Topological Constraints on Long-Distance Neutrino Mixtures,” [arXiv:physics/0007039](http://arxiv.org/abs/physics/0007039) v1, 13 Jul 2000. The proposal for such an equilibrium condition was first put forward in this paper, which is a relatively crude “precursor” to Ref. 13 in the present list of references.

[13] G. L. Fitzpatrick, “Evidence for Internal Topological Constraints on Neutrino Mixtures,” [arXiv:physics/0108009](http://arxiv.org/abs/physics/0108009) v1, 7 Aug 2001.

[14] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Fourth Edition revised), Oxford University Press, 1976.

[15] We should remind the reader how flavors are distinguished from one another in the standard model of particle physics. Except for mass differences, the strong and electroweak interactions of quarks are identical. Hence, from the point of view of their interactions alone, all quarks (leptons) are effectively identical. Of course, these particles are not identical, first because the different flavors of quarks and leptons (in principle including neutrinos) can, in fact, be distinguished by their masses. They can also be distinguished by certain Lorentz 4-scalar “charges,” i.e., certain global charge-conjugation-reversing (C-reversing) quantum numbers. These charges, being global in nature, do not in general couple to fields of force like color charges do. They include charges such as baryon and lepton number, strangeness, the third component of global isospin, muon and electron numbers, and other similar global flavor-defining charges. While it is certainly true that these flavor-defining charges can be associated with the so-called accidental symmetries of the Lagrangian describing strong and electroweak interactions, [e.g., SU(3)\text{\text{\text{\text{\text{flavor}}}}]}], we argue in this paper that these charges can also be thought of as arising at a more primitive (more fundamental) level, “below” or “beyond” the conventional standard model structure.

[16] When weak interactions are “turned off,” flavor eigenstates and mass eigenstates are one and the same. For the most part, when we speak here of flavor eigenstates, we are referring to the situation where flavor eigenstates and mass eigenstates are the same.
Not counting spin states, we have 3 quark flavor doublets in three colors each, which yields $3 \times 2 \times 3 = 18$ states, while 3 lepton flavor doublets in one “color” each (i.e., the strong-color “multiplicity” is one), yields $3 \times 2 \times 1 = 6$ states for a total of 24 matter particles. Combining these 24 matter particles with their 24 (distinguishable) antimatter counterparts, yields a total of 48 fundamental fermions and antifermions. Not counting spin or color states, we have a total of 12 flavor eigenstates and 12 antiflavor eigenstates, for a total of 24 flavor and antiflavor eigenstates of fundamental fermions.

We assume that, because there are only 13 known force-mediating fundamental bosons (i.e., not counting spin states, we have the photon, the graviton, 8 gluons and 3 weak intermediate vector bosons—none of which carry a flavor index, and some of which are their own antiparticles) in comparison with the 48 known fundamental fermions and antifermions, the force-mediating bosons do not carry the new 2-vector flavor-defining coordinates. For if they did, we would naturally expect to see many more force-mediating bosons than we do. While there are, no doubt, much deeper reasons for this state of affairs, this “experimental” evidence is a reasonable indication that such flavor-defining coordinates are at least a possibility in the fermion, but not in the boson sector.

If there are supersymmetries, it is natural to assume that the flavor-carrying boson superpartners (squarks and sleptons) to the existing flavor-carrying fundamental fermion matter-particles (i.e., quarks and leptons), will also carry the new 2-vector coordinates $Q, U$ and $V$, but the flavorless fermion superpartners (e.g., photinos, gluinos) to the existing flavorless force-mediating bosons (e.g., photons and gluons) will not.

J. Bernstein, *Elementary Particles and Their Currents*, W. H. Freeman and Co., San Francisco, 1968, pp. 23–25.

T. D. Lee, *Particle Physics and Introduction to Field Theory Vol. I*, Harwood Academic Publishers, New York, 1981, pp. 210–211.

Even though $F(v)_{\text{diag}} = F(\text{op}) = \sigma_z$, these two matrices act on entirely different spaces, since $F(\text{op})$ is associated with the constant $\cos \theta = 1$, and the variable phase-factor $e^{-i\phi}$, whereas $F(v)$ is associated with the variable $\cos \theta = \cosh v \geq 1$, and the constant phase-factor $e^{+i\phi} = +i$.

The term “non-Euclidean” is usually reserved for the geometry of curved spaces. However, to avoid confusing the flat “Lorentzian” 2-D geometry (and discrete transformations therein) with the flat Lorentzian 4-D geometry of spacetime (and continuous transformations therein), we prefer to use a term other than “Lorentzian” to describe the 2-space. In particular, because this 2-space is not Euclidean, we choose to break with tradition, and refer to this flat space as being “not-Euclidean,” or more correctly, non-Euclidean.

O. Bär and U. J. Wiese, “Can One See the Number of Colors?”, arXiv:hep-ph/0105258 v1, 24 May 2001. These authors show that in low-energy experiments involving only
pions and photons, it is impossible to “see” the number of quark colors $M_c$. In particular, they show that the decay width of the $\pi^0 \rightarrow \gamma\gamma$ is not, as is commonly claimed, proportional to $M_c^2$. They further show that in such low-energy situations other values of $M_c$ including $M_c = 5, 7, 9, \ldots$, are just as reasonable as $M_c = 3$. However, when all available experimental data is properly taken into account, the number of quark colors is found to be the widely accepted number $M_c = 3$.

[25] While it seems certain that $M_c = 1(3)$ characterizes ordinary leptons (quarks) it is not at all certain that other extraordinary or “exotic” values of $M_c$, namely, $M_c = 5, 7, 9, \ldots$, are physically excluded. While we may object to such values on the grounds that they lead to as yet unseen “quarks” with unusual electric charges (e.g., $\pm 2/5$), and to corresponding strong interactions based on $SU(5)_{\text{color}}, SU(7)_{\text{color}}, SU(9)_{\text{color}}$, and so on, with their associated color-neutral composites (e.g., when $M_c = 5$, 5 “quarks” would bind “strongly” to form a color-neutral “baryon”) we have found no stronger reasons than these for excluding such states. Moreover, even if all of these states “lived” together in the world at the same time, it is still possible, in principle, that triangle anomalies could be canceled using ordinary leptons ($M_c = 1$) and an infinite “tower” of “quarks” with $M_c = 3, 5, 7, 9, \ldots$, in the same way that triangle anomalies associated with $M_c = 1$ and $M_c = 3$ alone are canceled. It is also possible that something similar to the known three-family structure could be retained. This might be accomplished by arranging for ordinary leptons ($M_c = 1$) to be placed in a particular family along with an infinite tower of “quarks,” starting with ordinary quarks ($M_c = 3$), but also including extraordinary or exotic “quarks” with $M_c = 5, 7, 9, \ldots$. Admittedly, this speculative picture leads to a rather “messy” world, but since the absence of evidence for such unusual “quark” states is not the same thing as evidence of their absence, we cannot simply exclude this possibility out of hand. Moreover, there is potential theoretical justification in string theories for such exotic fractionally-charged particles. However, it is currently unknown if the proposed exotic fractionally-charged “quark” states with $M_c = 5, 7, 9, \ldots$, could be consistent with any existing string theory, since string theories tend to predict a relative plethora of particles having arbitrary fractional charges. Accordingly, until much more is known about the 2-space description, and exactly how it relates to more fundamental treatments such as string theory, it currently seems unlikely that exotic “quarks” with $M_c = 5, 7, 9, \ldots$, exist in nature.

[26] Any acceptable $2 \times 2$ matrix $F(v)$ possesses just two, real linearly-independent eigenvectors, call them $Q$ and $Q^c$, corresponding to the two real eigenvalues $f_m$ and $f_a$, respectively. Therefore, the matrix $F(v)$ can be thought of as “producing” the conventional single-particle fermion numbers $f_m$ and $f_a$ via the 2-space eigenvalue equations

$$F(v)Q = f_m Q$$

and

$$F(v)Q^c = f_a Q^c,$$
respectively.

The 2-vector $Q$ (and its scalar components—the electric charges of quarks and leptons) describes matter, while the linearly-independent 2-vector $Q^c$ describes its antimatter counterpart. The superscript $c$ on $Q^c$ is merely a label signifying antimatter. It is not an exponent or a symbol for complex conjugation. As such, it signifies only that the 2-vectors $Q$ and $Q^c$ are real vectors associated with (“carried by,” “representing,” etc.) individual fundamental-fermions or antifermions, respectively, not state vectors in some Hilbert space.

[27] It is important to point out that the various $C$-reversing 2-scalars associated with the internal 2-space description are taken to be Lorentz 4-scalars in an external spacetime setting. That is, because $q_1$ and $q_2$ are Lorentz 4-scalars in 4-D spacetime, the components of the 2-vectors $U$ and $V$, and the scalar products $Q^2, U^2, V^2, U \cdot V$ are also Lorentz 4-scalars. This connection strengthens the idea that these numbers define flavors in a Lorentz-invariant way.

[28] S. Coleman and J. Mandula, “All Possible Symmetries of the $S$-Matrix,” Phys. Rev., Vol. 159, No. 5, 1967; S. Weinberg, The Quantum Theory of Fields, Vol. III Supersymmetry, Cambridge University Press, New York, 2000, pp. 2, 12–21. According to the Coleman-Mandula theorem, the possible symmetries of the $S$-matrix include ordinary internal symmetries whose generators $B_k$ are independent of energy-momentum and spin. In other words, these group generators are Lorentz 4-“scalars.” In the present paper our internal 2-scalars—these are components, or scalar products, of the new flavor-defining fermion coordinates $Q, U$ and $V$—are also assumed to be Lorentz 4-scalars. These 2-scalars are identified with certain flavor-defining “charges” such as baryon number, lepton number, strangeness, charm and so on. In this way, the new description honors the spirit of the Coleman-Mandula theorem, even though the internal algebraic and geometric structure we propose is, in most cases, quite different from the internal symmetries referred to in the Coleman-Mandula theorem. Our description is, if you will, a kind of “evasion” of the Coleman-Mandula theorem without violating this theorem.

[29] The term “underlying dynamics” or “dynamics” as employed in this paper, refers to quantum dynamics supplemented by whatever “layer” of new physics (e.g., topological “energy barriers”) that may lie between the unification scale, and the region of applicability of the standard model of particle physics.

[30] Strictly speaking, besides the specification of global flavor-defining charges, the overall quantum state of a fundamental fermion would, necessarily, involve a specification of the spin state, the energy-momentum state and so on, together with a specification of the particular mix of local color (gauge)-charges $R, W, B, G$ and $Y$ carried by each fundamental fermion. This color-mix would be determined, in turn, by something like a complementary, local $SU(5)$ color-dependent gauge description.
By definition, in any linear vector 2-space, a 2-vector such as $\mathbf{Q}$ can always be resolved into a pair (no more, or less) of linearly independent vectors $\mathbf{U}$ and $\mathbf{V}$ as $\mathbf{Q} = \mathbf{U} + \mathbf{V}$.

When we say that the vectors $\mathbf{Q}$, $\mathbf{U}$ and $\mathbf{V}$ are observables, we mean that their associated component “charges” are mutually-commuting simultaneous observables. Hence, all of these charge-like components can be known in principle, at the same time, meaning that the vectors $\mathbf{Q}$, $\mathbf{U}$ and $\mathbf{V}$ can be known simultaneously. Thus the vector “triad” $(\mathbf{Q}, \mathbf{U}, \mathbf{V})$ is a well defined geometric object. Moreover, since $\mathbf{Q}$, $\mathbf{U}$, $\mathbf{V}$ are each Lorentz-invariant, vector triads “appear” the same to all observers located in arbitrary inertial coordinate systems.

Given the row vector $\mathbf{Q}_q = (q_1, q_2)$, and the column vectors $\{1,0\}$ and $\{0,-1\}$ we see that the component quark charges associated with $\mathbf{Q}_q$ are just the non-Euclidean “projections” of $\mathbf{Q}_q$ upon the 2-space coordinate “axes” represented by the vectors $\{1,0\}$ and $\{0,-1\}$. In particular, using the non-Euclidean scalar product, we have

$$q_1 = (q_1, q_2)\{1,0\}$$

and

$$q_2 = (q_1, q_2)\{0,-1\}.$$ 

In a similar way, and for reasons of internal consistency, a hypothetical vector such as $3\mathbf{Q}_q$ would have to be projected upon the same axes $\{1,0\}$ and $\{0,-1\}$ to obtain the component charges. However, only the charge projections $3q_1$ and $3q_2$ are obtained in this case. Other charges such as $q = (2q_1 + q_2)$ cannot be obtained by means of such projections.

Given the 2 by 2 matrix $\mathbf{R}$ in Eq. (51) in the main text, we see immediately that the effect of $\mathbf{R}$ on the column vectors $\{1/2,-1/2\}$, $\{1,0\}$ and $\{0,-1\}$ is simply $\mathbf{R}\{1/2,-1/2\} = \{1,0\}$ and $\mathbf{R}\{1,0\} = \{0,-1\}$, or $\mathbf{R}^2\{1/2,-1/2\} = \{0,-1\}$.

It is shown elsewhere (see Ref. 9, pp. 63–65 in the current reference list) that the matrices $\mathbf{R}$ and $\mathbf{R}^{-1}$ are charge-conjugation-reversing (C-reversing). In particular, the components of $\mathbf{R}$ and $\mathbf{R}^{-1}$ transform like C-reversing, Lorentz-invariant (4-scalar) charges. And, this in turn means that $\mathbf{R}$ and $\mathbf{R}^{-1}$ are Lorentz-invariant.

J. Rosen, *A Symmetry Primer for Scientists*, John Wiley and Sons, New York, 1983, p. 16. The symmetry-evolution principle, which is closely associated with the second law of thermodynamics, states: For an isolated physical system the degree of symmetry cannot decrease as the system evolves, but either remains constant or increases.

Note that the change described in $\Delta \mathbf{U} = \mathbf{U}_2 - \mathbf{U}_1$ involves the “addition” of the vectors $\mathbf{U}_2$ and $-\mathbf{U}_1$. However, this addition does not take place at the same spacetime
coordinate. That is, $\Delta U = U_2 - U_1$ describes a transition in which the initial $U$-vector describing a particle is $U_1$, and the final $U$-vector describing the particle (possibly a different particle) is $U_2$. That is, this transition involves only one current. So, the “addition” expressed by the change $\Delta U = U_2 - U_1$ does not violate our rule that like 2-vectors (i.e., $Q$, $U$ or $V$) for two or more fundamental fermions cannot be added or superimposed. The same arguments obviously apply to changes described by $\Delta Q$ and $\Delta V$.

[38] D. J. Thouless, *Topological Quantum Numbers in Nonrelativistic Physics*, World Scientific, Singapore, 1998. Numerous examples from physics and mathematics could be cited to establish the following very general principle:

*In physical systems characterized by a well defined topology, topology-change tends to be suppressed, relative to the condition of topology-maintenance.*

The basis for this principle is that topology change involves either an energy “expense” (an energy “barrier” must be overcome) or a violation of some “topological charge” conservation law, or both. In general, we expect this principle to apply to any discontinuous operation such as the *abstract* equivalent of “tearing,” and subsequently “glueing,” surfaces back together to form states with new topologies. For example, one cannot continuously deform a doughnut into a sphere. It must first be “cut” and then “glued” back together in a new way to achieve such a transformation.

Now, given the “facts” regarding the associated topology of neutrinos with respect to the (abstract) internal transformation $F(v)$, and assuming that there are no “energy barriers” associated with transitions between neutrinos having the same associated topology, the foregoing principle suggests that these facts could have the following dynamical significance:

*Associated topology change in neutrino-neutrino transitions ($\nu_\alpha \rightarrow \nu_\beta$) is suppressed by topological energy and/or topological charge “barriers,” while associated topology maintenance in neutrino-neutrino transitions is relatively enhanced.*

It should be emphasized that even though the $\nu_\mu$ and $\nu_\tau$ neutrinos have the same associated topology with respect to $F(v)$, they are quite distinct in other respects. For example, in conventional weak decays or weak capture reactions involving these particles, mu- and tau-numbers are separately conserved.

[39] The baryon (lepton) asymmetry of the universe insures that $B$ ($L$) are not absolutely conserved. However, violation of these conservation laws is very strongly suppressed as evidenced by the long life of the proton.

[40] H. Georgi and S. L. Glashow, “Neutrinos on Earth and in the Heavens,” arXiv:hep-ph/9808293, and hep-ph/9808293 v2, page 5, Eq. (20). It is interesting, and probably significant that Georgi and Glashow independently arrived at Eq. (84) in the main text of the present paper (see Eq. 20 in their paper) starting from six experimental neutrino
“facts,” some of which are completely different than the facts and assumptions employed in the present paper. In particular, these authors began by assuming

1. There are just three chiral neutrino states having Majorana masses.
2. Atmospheric neutrinos rarely oscillate into electron neutrinos.
3. Atmospheric muon neutrinos suffer maximal, or nearly maximal, two flavor oscillations into tau neutrinos.
4. & 5. Two experimentally (and theoretically) motivated assumptions regarding the order of magnitude of individual neutrino masses, and various mass squared differences.

and

6. Neutrinoless double beta decay either does not occur or is highly suppressed.

Given these six “facts” these authors derived both the neutrino mass matrix, and partially determined the associated CKM-like unitary matrix $U$, which describes neutrino mixtures. From $U$ they then derived Eq. (84) in the present paper. In closing, these authors also noted that CP-violation in this situation could be “superweak.” Clearly, this would be consistent with the assumption in the present paper that CP-violation is not maximal, i.e., $\sin^2 \delta \neq 1$ or $\cos^2 \delta \neq 0$.

While the present approach, and that of Georgi and Glashow in deriving Eq. (20) in their paper, are quite different they are, nevertheless, expected to be compatible. In particular, we anticipate that such things as neutrino masses and mixing parameters, ultimately owe their existence to physics at a deeper level, where topological considerations, of the kind proposed in the present paper, are expected to play an important role.

[41] A. P. Balachandran, “Bringing Up a Quantum Baby,” arXiv:quant-ph/9702055.

[42] G. Holzwarth, “Formation of Extended Topological-Defects During Symmetry-Breaking Phase Transitions in $O(2)$ and $O(3)$ Models”, arXiv:hep-ph/9901296. Analogous examples of fluctuation-induced topology-change in macroscopic objects (e.g., destruction of topological “defects” due to thermal fluctuations at phase transitions) abounds. For example, otherwise persistent (“conserved”) topological defects in crystals can be destroyed by raising the temperature sufficiently (i.e., by melting the crystal). Similarly, otherwise persistent (“conserved”) magnetic flux-tubes in Type II superconductors and/or vortices in a superfluid, can both be destroyed by raising the temperature above the critical temperature $T_c$. And, conversely, topological defects are always created when such macroscopic systems first condense (or crystallize) as the temperature is lowered. We imagine that something roughly similar can happen when quantum-fluctuations (vacuum fluctuations) act on otherwise very similar quantum objects (i.e., quantum objects such as neutrinos having the same electric charge, spin, lepton number, and nearly
identical mass) that also happen to start “life” as topologically-distinct quantum objects. That is, we are assuming that, if not prevented by some absolute conservation law, transitions between such states (e.g., $\nu_e$ and $\nu_\tau$ neutrinos) will be catalyzed by quantum fluctuations.

[43] The Super-Kamiokande, Kamiokande Collaboration, [arXiv:hep-ex/9810001], and the SNO Collaboration, [arXiv:nucl-ex/0106015].

[44] M. Gronau, “Patterns of Fermion Masses, Mixing Angles and CP Violation,” in The Fourth Family of Quarks and Leptons, First International Symposium, edited by D. B. Cline and Amarjit Soni, Annals of The New York Academy of Sciences, New York, Volume 518, 1987, p. 190. According to the scheme described in the present paper, the fundamental fermions $d$, $\nu_e(s, b, \nu_\mu, \nu_\tau)$ have the topology associated with a cylinder (Möbius strip) with respect to the internal transformation $F(v)$. Hence, we imagine that if, in some imaginary world, quark couplings to hypothetical Higgs mass-producing fields were nearly “switched off,” the resulting “low” mass $d$, $s$ and $b$ quark mixtures (located inside strongly-bound quark composites) could conceivably look very much like unbound or “free” $\nu_e, \nu_\mu$ and $\nu_\tau$ low mass neutrino mixtures—owing to topological influences of exactly the kind proposed in the present paper. In particular, in this imaginary world, we would expect the $s$ and $b$ quarks (like the $\nu_\mu$ and $\nu_\tau$ neutrinos, respectively) to exhibit (nearly) bi-maximal mixing. However, because of uncertainty-principle considerations, and because the $b$ quark mass is, in the “real” world, very much larger than the $d$ and $s$ quark masses, very little mixing with the $b$ quark occurs in $d$, $s$, $b$ quark mixtures in spite of topological influences that would tend to encourage this. Thus, in the case of strongly-interacting quarks, it seems very likely that topological constraints of the kind considered in the present paper can, at most, play a minor role in determining such things as CKM-type matrix elements. For example, in the case of $d$ and $s$ quarks, Gronau shows that the mixing angle $\theta_c$ depends on quark masses since the matrix element $V_{12}$, and the $d$, $s$ quark masses $m_d$ and $m_s$, respectively, are known to be empirically related via $V_{12} = \sin \theta_c = \sqrt{m_s/m_d}$, where $m_s > m_d \neq 0$. This is nothing like the corresponding matrix element for $\nu_e$ and $\nu_\mu$ neutrinos found in the present paper. Thus, although a detailed understanding of these matters is far from being achieved, it appears that topological influences are allowed to play a major role in neutrino, but not in quark mixing, because of the small value, and near degeneracy of neutrino masses in comparison to the large differences between $d$, $s$ and $b$ quark masses.

[45] Although conventional treatments of neutrino mixing do not involve topological considerations, they do describe mixing among neutrinos that initially have a definite flavor. And, because we argue that a neutrino with a definite flavor is also associated with a definite topology, hereinafter we will often use these terms together as in “... neutrinos have a definite flavor and associated topology.”
[46] Ta-Pei Cheng and Ling-Fong Li, *Gauge Theory of Elementary Particle Physics*, Clarendon Press, Oxford, 1984, pp. 409–414. While this reference provides a good summary of the formalism involved in neutrino mixing, the reader should be cautioned that there are several “typographical” errors that could cause confusion. For example, in Eq. (13.22) the first term in the expression for the matrix element $U_{32}$ should read $c_1 s_2 c_3$ not $c_1 s_2 s_3$. In Eq. (13.31) the last term in the expression for the time-average probability $\langle P_{\nu_e \rightarrow \nu_\tau} \rangle$ should be preceded by a minus sign, not a plus sign. Finally, the left hand side of Eq. (13.33) should read $P_{\nu_\alpha \rightarrow \nu_\beta}$ not $\langle P_{\nu_\alpha \rightarrow \nu_\beta} \rangle$.

[47] C. Nash and S. Sen, *Topology and Geometry for Physicists*, Academic Press, New York, 1983.

[48] Note that for all practical purposes, time-dependent quantum probabilities $P_{\nu_\alpha \rightarrow \nu_\beta}$ become time-average or classical probabilities $\langle P_{\nu_\alpha \rightarrow \nu_\beta} \rangle$ when $t$ is sufficiently large. For this reason, it makes sense to consider classical probability measures such as the “joint” probability $P = P_C P_{NC}$, where $P_C$ and $P_{NC}$ are time-averages of quantum probabilities.

[49] A. Bandypadhyay et. al., “Three Generation Neutrino Oscillation Parameters after SNO,” arXiv:hep-ph/0110307 v2, 19 Dec 2001. Equation 19 in this paper gives a three-generation mixing matrix $U$, which leads to a matrix describing long distance equilibrium neutrino mixtures given by $M = (M_{11} = 0.6, M_{12} = M_{13} = M_{21} = M_{31} = 0.2, M_{22} = M_{23} = M_{32} = M_{33} = 0.4)$; P. F. Harrison et. al., “Tri-Bimaximal Mixing and the Neutrino Oscillation Data,” arXiv:hep-ph/0202074, 7 Feb 2002. Equation 6 in this paper gives a three-generation mixing matrix $U$, which leads to a matrix describing long distance equilibrium neutrino mixtures given by $M = (M_{11} = 0.56, M_{12} = M_{13} = M_{21} = M_{31} = 0.22, M_{22} = M_{23} = M_{32} = M_{33} = 0.39)$. Both of these experimentally determined matrices compare favorably with the theoretical predictions of the present paper (see Eq. 84 in the main text), namely, $M = (M_{11} = 0.5, M_{12} = M_{13} = M_{21} = M_{31} = 0.25, M_{22} = M_{23} = M_{32} = M_{33} = 0.375)$.

[50] A. Suzuki, Eighth International Workshop on Neutrino Telescopes, Venice, 1999.

[51] Even if the matrix $M$ in Eq. (84) in the main text is verified by experiment, we could not be certain that the constraints expressed by items A), B) and C) in Section 3.5.2, are topological constraints related to the 2-space description of quarks and leptons. There are three basic reasons for this assertion.

First, we do not know why the topology of vector triads [with respect to the internal transformation $F(v)$] should be relevant to neutrino mixing. Second, we do not have any detailed understanding of the hypothetical mechanism that supplies the “energy barriers” or “topological charge conservation laws” that serve to inhibit topology change in neutrino mixing. Third, as indicated in Ref. 13, Appendix A (Sec. A.4) in this list, we cannot be absolutely certain that the $\nu_\mu$ neutrino is associated with the requisite second-family Möbius topology with respect to $F(v)$. What we do know with certainty is that, regardless of their physical origins, the constraints of Section 3.5.2, together
with the additional assumption that \( \sin^2 \delta \neq 1 \) or \( \cos^2 \delta \neq 0 \), definitely determine \( M \) uniquely.

[52] However, all of the flavor-defining charges associated with these 2-space flavor doublets are *global* in nature, so this tells us nothing directly about the *interactions* of flavor doublet members. Nevertheless, the “up”–“down” flavor doublet *structure* imposed by the 2-space, necessarily, severely restricts the possible interactions.