Generic finite size scaling for discontinuous nonequilibrium phase transitions into absorbing states

M. M. de Oliveira\(^1\,^2\), M. G. E. da Luz\(^3\), and C. E. Fiore\(^4\)

\(^1\) Departamento de Física e Matemática, CAP, Universidade Federal de São João del Rei, Ouro Branco-MG, 36420-000 Brazil,

\(^2\) Theoretical Physics Division, School of Physics and Astronomy, University of Manchester, Manchester, M13 9PL, UK

\(^3\) Departamento de Física, Universidade Federal do Paraná, Curitiba-PR, 81531-980 Brazil

\(^4\) Instituto de Física, Universidade de São Paulo, São Paulo-SP, 05314-970 Brazil

(Dated: October 30, 2015)

Based on quasi-stationary distribution ideas, a general finite size scaling theory is proposed for discontinuous nonequilibrium phase transitions into absorbing states. Analogously to the equilibrium case, we show that quantities such as, response functions, cumulants, and equal area probability distributions, all scale with the volume, thus allowing proper estimates for the thermodynamic limit. To illustrate these results, five very distinct lattice models displaying nonequilibrium transitions – to single and infinitely many absorbing states – are investigated. The innate difficulties in analyzing absorbing phase transitions are circumvented through quasi-stationary simulation methods. Our findings (allied to numerical studies in the literature) strongly point to an unifying discontinuous phase transition scaling behavior for equilibrium and this important class of nonequilibrium systems.

Nonequilibrium phase transition (NeqPT) into absorbing states (AS) is key in a wide range of phenomena as chemical reactions, interface growth, epidemics, and population dynamics. Likewise, it is relevant for the emergence of spatio-temporal chaos in different classes of problems, as experimentally verified in liquid crystal electroconvection \(^4\), driven suspensions \(^7\), and superconducting vortices \(^8\). So, much has been done on continuous NeqPT, specially addressing universality \(^3, 5, 9, 10\). But comparatively less attention has been payed to discontinuous transitions in systems with AS \(^11, 12\), the case, e.g., in catastrophic shifts processes \(^13\) (bearing important questions regarding the influence of diffusion and disorder in creating or destroying AS), heterogeneous catalysis \(^14, 15\), ecological \(^16, 17\), granular \(^18\), and replicator dynamics \(^19\), cooperative coagulation \(^20\), language formation \(^21\), and social patterns \(^22\).

Discontinuous transitions to AS conceivably require mechanisms suppressing the formation of absorbing minority islands induced by fluctuations \(^22, 24\). Also, there are strong evidences they cannot occur in 1D if the interactions are short-range: the absence of boundary fields would prevent the stabilization of compact clusters \(^24\). In spite of these presumably universal facts, a general description of discontinuous NeqPT, including to identify a possible scaling behavior, is still lacking.

Equilibrium first-order transitions are characterized by discontinuities in the order parameter \(\phi\) and by thermodynamic “densities”, whose susceptibilities display delta-like shapes. In finite systems, such quantities become continuous functions of the control parameter \(\lambda\). However, the infinite limit still can be estimated from a finite size scaling theory (FSS) \(^22, 23\), when second derivates scale linearly with the volume \(V = L^d\) (for \(d\) the spatial dimension and \(L\) the lattice size). Also \(|\lambda_V - \lambda_0|\) goes with \(1/V\), with \(\lambda_V\) (\(\lambda_0\)) the coexistence point for a finite \(V\) (in the thermodynamic limit).

For NeqPT to AS, precise methods like spreading simulations – available for continuous transitions – as well as a FSS framework (like the above) are absent in the discontinuous case. Actually, a difficulty in its analysis is that the AS often prevent simulations to properly converge, precluding any scaling inference. Even for large systems, eventually the dynamics will end up in an AS via a statistical fluctuation of small, but nonzero, probability. Also, metastable states can make hard to locate or even classify transition points due to doubts if the observed order parameter jump is genuine.

In the present contribution we address such class of problems, presenting solid arguments for a common finite size scaling behavior. Based on previous suggestions \(^11, 34, 36\) – and in the fact that equilibrium and nonequilibrium phase transitions share important similarities when the later display stationary (steady) states \(^37\) (see below) – we develop a FSS for transitions into single and infinitely many AS by means of the quasi-stationary (QS) concept. We show that, in full analogy with equilibrium, standard quantities follow a same \(1/V\) scaling. Five models are used to illustrate our results.

The quasi-stationary probability distribution (QSPD) idea, powerful for continuous NeqPT \(^35\), is likewise valuable here. In very general terms, such method has as the main purpose to evade just the absorption process. Formally, assume at time \(t\) the microstates \((\sigma)\) probability distribution \(P(\sigma, t)\) and the survival probability \(P_s(t)\), i.e., the probability that the system is still active. Then, the QSPD \(P_{QS}(\sigma) = \lim_{t \to \infty} P(\sigma, t)/P_s(t)\) describes the asymptotic properties of a finite system conditioned to survival \(^38, 40\). In practice, \(P_{QS}\) is calculated by effectively redistributing the flux from the absorbing state to the system non-absorbing subspace when the dynamics is sufficiently close to the absorbing condition. In this case, although the detailed balance is not satisfied, if the redistribution is made compatible with the QS distribution itself (through a self-consistent procedure, see \(^38\)), then the global balance \(^11\) is verified in the non-absorbing
the superposition of two
results from the much longer procedure of performing av-
the saved configurations. This accurately reproduces the
to AS is imminent, the system is "relocated" to one of
must rely on numerical protocols. An efficient scheme
portray QSPDs for systems with spatial structure, one
in [38], which stores and gradually updates a set
must be either empty or occupied by an oxygen atom O or
oxidation on a catalytic surface (a lattice whose sites can
fraction, \( \rho \)) occupying half of the correspond-
 tính, not differing too much if derived by
\( \lambda \) is the same if estimated via equal areas or
maximum of \( \chi \), not differing too much if derived by
U2 maximum. Thus, distinct measures shows that
\( \lambda - \lambda_0 \sim 1/L \), the usual equilibrium scaling.
This description is illustrated by periodic square lattice
models simulated from the QS approach. For the equal
area criterion, whenever \( P_V(\phi) \) have relevant overlap we
consider each \( P_V(\phi) \) occupying half of the corresponding
\( \lambda_0 \) is the control parameter value at the phase transition
in the thermodynamic limit we can expect this resulting effective to fairly
reproduce the macroscopic transition behavior of the original
Moreover, it represents a discontinuous
between two 'normal' phases ±, bearing two scales, the order parameter
\( \phi = \phi_\pm \) at the transition point. Hence, in general for a finite nonetheless reasonable
large \( V \), the bimodal probability distribution is reason-
As already mentioned, in the thermodynamic limit
we can construct a FSS for discontinuous NeqPTs to AS,
we now observe the following. First, the role of inverse
averages only on samples that have not visited the AS at
entered during the time evolution. Whenever a transition
current system is "relocated" to one of the saved configurations. This accurately reproduces the
From equation (39), we can expect this resulting effective to fairly re-
produce the macroscopic transition behavior of the original
system. Moreover, it represents a discontinuous transition between two ‘normal’ phases ±, bearing two scales, the order parameter \( \phi = \phi_\pm \) at the transition point. Hence, in general for a finite nonetheless reasonable
large \( V \), the bimodal probability distribution is reasonably well described by a sum of two Gaussians (see
\[ P_V(\phi) = \sum_{\omega=\pm} P_V^{(\omega)}(\phi), \text{ with } (\bar{\lambda} = \lambda - \lambda_0) \]
\[ P_V^{(\omega)}(\phi) = \frac{\sqrt{V}}{\sqrt{2\pi}} \exp \left[ g(V) \bar{\lambda} \phi - g(V)(\phi - \phi_\omega)^2/(2\chi_\omega) \right] \]
\[ F_\pm(\bar{\lambda}; V) = \sqrt{\chi_\pm} \exp \left[ V \bar{\lambda} \left( \phi_\pm + \frac{\chi_\pm}{2\bar{\lambda}} \right) \right]. \]
Now, the pseudo-transition point \( \lambda_V \) can be estimated, e.g., from (i) the coexisting phases equal probability condition, i.e., equal areas of \( P_V^{(-)} \) and \( P_V^{(+)} \), or yet from the maximum of (ii) variance \( \chi = V((\phi^2) - (\phi)^2) \), and
(iii) moment ratio (reduced cumulant) \( U_2 = \langle \phi^2 \rangle/\langle \phi \rangle^2 \).
In first order in \( \lambda \) [17], both (i) and (ii) lead to \( \lambda_V = \lambda_0 - V^{-1} \ln [\chi/(\chi + 1)] / (2(\phi_+ + \phi_-)) \). For (iii), we get
\[ \lambda_V = \lambda_0 - V^{-1} \ln [\chi/(\chi + 1)] / (2(\phi_+ - \phi_-)) \].
Note \( \lambda_V - \lambda_0 \) is the same if estimated via equal areas or maximum of \( \chi \), not differing too much if derived by
the \( U_2 \) maximum. Thus, distinct measures shows that
\( \lambda_V - \lambda_0 \sim 1/L \), the usual equilibrium scaling.
This description is illustrated by periodic square lattice
models simulated from the QS approach. For the equal
area criterion, whenever \( P_V(\phi) \) have relevant overlap we
consider each \( P_V(\phi) \) occupying half of the corresponding
\( \phi \) interval.
Consider the Ziff-Gulari-Barshad (ZGB) model [14],
which reproduces relevant features of carbon monoxide oxidation on a catalytic surface (a lattice whose sites can be either empty or occupied by an oxygen atom O or a carbon monoxide molecule CO). CO (O2) reach the surface with probability Y (1-Y). Whenever a CO encounters a vacant site, the site becomes occupied. If a O2 molecule encounters two nearest-neighbor empty sites, it dissociates filling the two sites. If 2 atoms O and 1 atom C reach an elementary 2 x 2 lattice cell, they immediately form CO2 and desorb. The model exhibits two transitions – regulated by the CO molecules fraction, \( \rho_{CO} \) – each between an active steady and an

\[ F_\pm(\bar{\lambda}; V) = \sqrt{\chi_\pm} \exp \left[ V \bar{\lambda} \left( \phi_\pm + \frac{\chi_\pm}{2\bar{\lambda}} \right) \right]. \]
absorbing (poisoned) state. For large (extreme low) $Y$, the surface becomes saturated by CO (O). The former (latter) transition is discontinuous (continuous, belonging to the DP universality). The discontinuous transition is shown in Fig. 1. The $Y$ region of rapid increase of $\rho_{CO}$ (inset of (a)) corresponds to the maxima of $\chi$ and $U_2$ (which increase with $L^2$, Fig. 1(a) and (b)) and their location scale with $1/L^2$, Fig. 1(d). So we estimate $Y_0 = 0.5253(3)$ (max. of $\chi$) and $Y_0 = 0.5254(3)$ (max. of $U_2$). The $Y_L$ for which the two peaks of $P_{CO}$, Fig. 1(c), have the same area also scales with $1/L^2$. From this we estimate $Y_0 = 0.5253(3)$. These values are in excellent agreement among them and with $Y_0 = 0.5250(6)$, recently obtained by other means [30]. Defining $\chi^* = \chi/L^2$ and $y^* = (Y - Y_0)L^2$, the collapsed data is shown in Fig. 1(c) inset, confirming a $L^2$ scaling.

For a two-species symbiotic contact process (2SCP) [10], any site is either empty or occupied by an element $A$, by a element $B$, or by one of each. Each individual reproduces (autocatalytic), creating a new at one of its first-neighbors sites at rate $\lambda_A = \lambda_B = \lambda$. In a single occupied site, A or B dies at unitary rate. Sole individuals follow the usual CP dynamics [10]. However, in doubly occupied sites, due to symbiosis both A and B die at a reduced $\mu = \text{const} < 1$ rate. Besides the CP usual active (A and B populations fixed) and absorbing phases, there are two extra symmetric active phases, in which just one species exists.

If $A$ and $B$ diffuse with rate $D$, for $\mu \to 0$ the transition changes from continuous to discontinuous. The order parameter is the density of occupied sites $\rho$. Figure 2 exemplifies this 2SCP for $\mu = 0.01$ and $D = 0.1$, with a discontinuous transition between absorbing and active symmetric phases for $\lambda \approx 0.449$ [10]. Like ZGB, in the transition region there are peaks for $\chi$ and $U_2$, Fig. 2(a) and (b), whose maxima positions $\lambda_L$ increase with $1/L^2$, Fig. 2(d). A $L \to \infty$ extrapolation yields $\lambda_0 = 0.4489(1)$ and $0.4490(1)$, respectively. The equal areas condition for $P_\rho$, Fig. 2(c), shows a $1/L^2$ scaling, leading to $\lambda_0 = 0.4488(1)$. The estimates display excellent agreement among them and with Ref. [10]. Finally, a fair data collapse is shown in Fig. 2(c) inset.

We discuss a model of competitive interactions in bipartite ($k = A$ and $B$) sublattices [48], assuming the version in [49], so instead of critical [48], the phase diagram has three coexistence lines. Also, besides an absorbing, we have a spontaneous breaking symmetry transition. Given a site in the sublattice $k$, the number of particles in its first ($j = 1$) and second ($j = 2$) nearby neighborhood is $n_{jk}$. For $n_{jk}^a$, the number of adjacent particles in $j$, the dynamics is as the following [48]. With probability $(1 + \mu(n_{1k})^2)/(\lambda_1 + \lambda_2 + 1 + \mu(n_{1k})^2)$ we attempt to annihilate a randomly selected particle $P$. If $P$ survives, we choose at will $j = 1, 2$. Then, with probability $p_j$ we try to create a new particle in a free site in the $j$ neighborhood of $P$, with $p_j = \lambda_j/(\lambda_1 + \lambda_2 + 1 + \mu(n_{1k})^2)$ for $n_{jk}^a \geq j$ and zero otherwise (in [48], $\mu = \lambda_0 = 0$).

The absorbing (ab)–active symmetric (as) phases line is discontinuous for lower $\lambda_1$. Proper order parameters are $\rho = (\rho_A + \rho_B)/2$ and $\phi = |\rho_A - \rho_B|$, with $\rho_A$ the X-sublattice density. In the ab phase we have $\rho = \phi = 0$, with $\lambda_1 \to \infty$. From this fair data collapse among the $\rho$ and $\phi$ vs. $1/L^2$ variables, we estimate $\lambda_1 = 0.445(3)$, respectively. The equal areas condition for $P_\rho$, Fig. 2(c), shows a $1/L^2$ scaling, leading to $\lambda_1 = 0.4488(1)$. The estimates display excellent agreement among them and with Ref. [10]. Finally, a fair data collapse is shown in Fig. 2(c) inset.
whereas for the as phase $\rho \neq 0$ and $\phi = 0$. So, for the as phase, the sublattices are equally populated. From Fig. 4 we see that the ab–as transition follows our FSS.

Finally, we address two versions of the second Schlögl model [12]: SL1 [50, 51], corresponding to a lattice version of the stochastic differential equation considered in [13], and SL2 [11], a modification of a pair contact process [14] with time disorder in (e) and (f). Right panels, the order parameters variance $\chi$ versus $1/L$ (insets: their collapsed plots). For SL1, we obtain $\alpha_0 = 0.0743(1)$ (maximum of $\chi$), $0.0472(1)$ for $U_2$ and $0.0472(1)$ (equal areas). All estimates agree very well and are close to 0.0747 in [51] (calculated from the threshold separating ongoing active state and an exponential decay of $\rho$, considering a fully occupied initial configuration). For SL2 $\alpha_0 = 0.0473(1)$ (maximum of $\chi$), $0.0472(1)$ for $U_2$ and $0.0472(1)$ (equal areas), all close to 0.0480 in [11] (derived from the onset for the decay of $\phi$ towards the absorbing regime).

Lastly, we incorporate temporal disorder into the SL1 model by assuming that at each instance, the creation probability, $1 - p_0$, is $\text{Min}\{1/(1 + \alpha) + \delta, 1\}$, with $\delta$ randomly chosen within $[-\sigma, \sigma]$. Results for $\sigma = 0.15$ are shown in Fig. 4 (e) and (f). Here also $\alpha_L$’s scales with $1/L^2$, from which we obtain $\alpha_0 = 0.0680(1)$ (maximum of $\chi$), $0.0683(2)$ for $U_2$ and $0.0680(1)$ (equal areas). Similar conclusions are obtained for $\sigma = 0.25$ (not shown), from which $\alpha_0 = 0.0265(1)$ (maximum of $\chi$ and equal areas). So, in contrast to spatial disorder [15], the present is the first evidence that temporal disorder does not hinder discontinuous absorbing phase transitions (but obviously, more studies should be in order, see, e.g., [53]).

In summary, we propose a general FSS theory for discontinuous NeqPTs to AS. From QS ideas, we obtain an effective system – which reproduces the thermodynamic properties of the original problem – undergoing ’normal’ (i.e., not to AS) discontinuous phase transitions. Moreover it is described by a bimodal distribution for the order parameter, so allowing inference of the $V$ scaling behavior. The only eventual difficulty to implement such universal scheme would be if the particular system hinders a QSPD. However, the known examples displaying such feature are very specific [54]. Our study is particularly useful given that this class of NeqPTs have no equilibrium counterparts and there are no universal treatments for discontinuous absorbing phase transitions for $d \geq 2$.

We acknowledge CNPq, Capes, CT-Infra and Fapesp for research grants. We are in debt to Hans J. Herrmann and Mario de Oliveira for insightful discussions.

[1] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press, Cambridge, 1999).
[2] G. Odor, *Universality in Nonequilibrium Lattice Systems: Theoretical Foundations* (World Scientific, Singapore, 2007).
[3] M. Henkel, H. Hinrichsen, and S. Lubeck, *Nonequilibrium Phase Transitions Volume I: Absorbing Phase Transitions* (Springer-Verlag, Dordrecht, 2008).
[4] H. Hinrichsen, Adv. Phys. 49, 815 (2000).
[5] G. Odor, Rev. Mod. Phys. 76, 663 (2004).
[6] K. A. Takeuchi, M. Kuroda, H. Chaté, and M. Sano, Phys. Rev. Lett. 99, 234503 (2007).
[7] L. Corté, P. M. Chaikin, J. P. Gollub, and D. J. Pine, Nature Physics 4, 420 (2008).
[8] S. Okuma, Y. Tsubawa, and A. Motohashi, Phys. Rev. B 83, 012503 (2011).
[9] S. Lubeck, Int. J. Mod. Phys. B 18, 3977 (2004).
[10] M. Henkel and M. Pleimling, *Non-Equilibrium Phase Transitions Volume II: Ageing and Dynamical Scale* (Springer-Verlag, Dordrecht, 2010).
[11] C. E. Fiore, Phys. Rev. E 89, 022104 (2014).
