Global well-posedness and scattering for the mass critical nonlinear Schrödinger equation with mass below the mass of the ground state

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April 21, 2015

Abstract: In this paper we prove that the focusing, \(d\)-dimensional mass critical nonlinear Schrödinger initial value problem is globally well-posed and scattering for \(u_0 \in L^2(\mathbb{R}^d), \|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)},\) where \(Q\) is the ground state, and \(d \geq 1\). We first establish an interaction Morawetz estimate that is positive definite when \(\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)},\) and has the appropriate scaling. Next, we will prove a frequency localized interaction Morawetz estimate similar to the estimates made in [13], [14], [15]. See also [12] for the energy critical case. Since we are considering an \(L^2\) - critical initial value problem we will localize to low frequencies.

1 Introduction

The semilinear initial value problem

\[
    iu_t + \Delta u = F(u) = -|u|^\frac{4}{d}u, \quad u(0, x) = u_0 \in L^2(\mathbb{R}^d),
\]

is called the \(d\) - dimensional, focusing, \(L^2\) - critical Schrödinger initial value problem. A solution to the problem

\[
    iu_t + \Delta u = F(u) = \mu|u|^pu, \quad \mu = \pm 1,
\]
on \([0, \mathcal{T}]\) gives rise to a family of solutions, for any \(\lambda > 0,\)

\[
    \lambda^{2/p}u(\lambda^2t, \lambda x)
\]
is also a solution to (1.2) on \([0, \mathcal{T}]\). (1.1) is called \(\dot{H}^{s_c}(\mathbb{R}^d)\) - critical, for

\[
    s_c = \frac{d}{2} - \frac{2}{p},
\]
since

\[
    d \geq 1
\]

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\[ \|u(0, x)\|_{H^s_c(R^d)} = \lambda^{2/p} \|u(0, \lambda x)\|_{H^s_c(R^d)}. \]  

Therefore (1.1) is called \( L^2 \) - critical. (1.2) is called defocusing when \( \mu = 1 \) and focusing when \( \mu = -1 \), so (1.1) is a focusing problem.

In this paper we prove global well-posedness and scattering for initial data in \( L^2 \) lying below the ground state threshold.

**Definition 1.1 (Well-posedness)** Suppose that \( Y \) is a function space. An initial value problem with initial data \( u(0) \in X \subset Y \) is well-posed on an interval \( 0 \in I \) if

1. A solution exists for all \( u(0) \in X \).
2. The solution \( u(t) \) is continuous in time, \( u(t) \in C(I; Y) \).
3. The solution \( u(t) \in C(I; Y) \) depends continuously on \( u(0) \in X \). In other words for all \( t, t_0 \in I \), if \( u \) and \( v \) are solutions to the initial value problem with initial data \( u(0), v(0) \in X \), \( f \) is a continuous function, \( f(0) = 0 \),

\[ \|u(t) - v(t)\|_Y \leq f(\|u(0)\|_Y, \|u(0) - v(0)\|_Y). \]  

4. If \( I = \mathbb{R} \) then we say that the initial value problem is globally well-posed.

In this paper a solution to (1.1) refers to a strong solution.

**Definition 1.2 (Strong solution)** \( u : I \times \mathbb{R}^d \to C, I \subset \mathbb{R} \) is a solution to (1.1) if for any compact \( J \subset I \), \( u \in C^0(I) \cap L^2_{t,x}((J \times \mathbb{R}^d), 2^{(d+2)/d}) \), and for all \( t, t_0 \in I \),

\[ u(t) = e^{i(t-t_0)\Delta} u(t_0) + \int_{t_0}^t e^{i(t-\tau)\Delta} F(u)(\tau) d\tau. \]  

If \( u \in L^2_{t,x} \) locally in time, then (1.7) converges in a weak \( L^2(\mathbb{R}^d) \) sense. The norm \( L^2_{t,x}((J \times \mathbb{R}^d) \)) arises from the Strichartz estimates and is invariant under (1.3).

**Definition 1.3 (Scattering)** A solution \( u(t, x) \) to (1.1) is said to scatter forward in time if there exists \( u_+ \in L^2(\mathbb{R}^d) \) such that

\[ \lim_{t \to \infty} \|u(t, x) - e^{it\Delta} u_+\|_{L^2(\mathbb{R}^d)} = 0. \]  

A solution scatters backward in time if there exists \( u_- \in L^2(\mathbb{R}^d) \) such that
\[
\lim_{t \to -\infty} \| u(t, x) - e^{it\Delta} u \|_{L^2(\mathbb{R}^d)} = 0.
\]  

(1.9)

In general it is an open question whether or not the initial value problem (1.2) is globally well-posed and scattering for any initial data \( u_0 \in \dot{H}^{s_c}(\mathbb{R}^d) \), or perhaps any initial data lying below a ground state. However, global well-posedness and scattering is known to be true for small data.

**Theorem 1.1** For any \( 0 \leq s_c \leq 1, \ d \geq 1 \), there exists \( \epsilon(d, s) > 0 \) such that if \( \| u_0 \|_{\dot{H}^{s_c}(\mathbb{R}^d)} < \epsilon(d, s_c) \), then (1.2) is globally well-posed and scatters both forward and backward in time.

**Proof:** See [6], [7] for the proof for \( s_c < 1 \). [42] proved a perturbative estimate for the energy-critical problem (\( s_c = 1 \)) using the inhomogeneous Strichartz estimates of [17]. The proof does not distinguish between focusing and defocusing problems. \( \square \)

**Remark:** [6], [7] also proved (1.1) is locally well-posed for \( u_0 \in L^2_x(\mathbb{R}^d) \) on some interval \([0, T]\), where \( T(u_0) \) depends on the profile of the initial data, not just its size in \( L^2(\mathbb{R}^d) \). The proofs of [6] and [7] made use of the Strichartz estimates of [18], [38], and [49] and the size of the initial data.

Moreover, theorem 1.1 is sharp for small data.

**Theorem 1.2** (1.2) fails to be locally well-posed for \( u_0 \in H^s(\mathbb{R}^d), \ s < s_c \).

**Proof:** See [8] and [9]. \( \square \)

Therefore it is quite natural to endeavor to extend theorem 1.2 to large data. Large data results for (1.2) have generally centered on the mass-critical (\( s_c = 0 \)) and the energy-critical (\( s_c = 1 \)) problems. This is because a solution to (1.2) has the conserved quantities mass,

\[
M(u(t)) = \int |u(t, x)|^2 dx = M(u(0)),
\]

and energy

\[
E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx + \frac{\mu}{p+2} \int |u(t, x)|^{p+2} dx = E(u(0)).
\]

(1.10)

(1.11)

Thus (1.1) is often called the focusing mass-critical initial value problem.

The perturbative estimates of [44] imply that the mass-critical problem is globally well-posed in an \( L^2 \) - neighborhood of \( u(0) \) and scatters to a free solution if and only if, when \( u \) is the solution to the mass-critical problem with initial data \( u(0) \),
\[ \|u\|_{L^{2(d+2)}_{t,x}(\mathbb{R} \times \mathbb{R}^d)} < \infty. \] (1.12)

Therefore, in contrast to scattering is the blow-up phenomenon.

**Definition 1.4** A solution to (1.1) defined on \( I \subset \mathbb{R} \) blows up forward in time if there exists \( t_0 \in I \) such that
\[ \int_{t_0}^{\sup(I)} \int |u(t,x)|^{2(d+2)/d} \, dx \, dt = \infty. \] (1.13)

\( u \) blows up backward in time if there exists \( t_0 \in I \) such that
\[ \int_{\inf(I)}^{t_0} \int |u(t,x)|^{2(d+2)/d} \, dx \, dt = \infty. \] (1.14)

**Theorem 1.3** Given \( u_0 \in L^2(\mathbb{R}^d) \) and \( t_0 \in \mathbb{R} \), there exists a maximal lifespan solution \( u \) to (1.1) defined on \( I \subset \mathbb{R} \) with \( u(t_0) = u_0 \). Moreover,
1. \( I \) is an open neighborhood of \( t_0 \).
2. If \( \sup(I) \) or \( \inf(I) \) is finite, then \( u \) blows up in the corresponding time direction.
3. The map that takes initial data to the corresponding solution is uniformly continuous on compact time intervals for bounded sets of initial data.
4. If \( \sup(I) = \infty \) and \( u \) does not blow up forward in time, then \( u \) scatters forward to a free solution. If \( \inf(I) = -\infty \) and \( u \) does not blow up backward in time, then \( u \) scatters backward to a free solution.

**Proof:** See [6], [7]. \( \square \)

**Theorem 1.4 (Defocusing problem)** The mass-critical, defocusing, nonlinear Schrödinger initial value problem, (1.2), with \( p = \frac{4}{d}, \mu = 1 \), is globally well-posed and scattering for any \( u_0 \in L^2(\mathbb{R}^d) \), \( d \geq 1 \).

**Proof:** See [26], [29], and [43] for the proof with radial data in dimensions \( d \geq 2 \), and [13], [14], [15] for the nonradial case. \( \square \)

For the focusing problem (1.1) there are known counterexamples to global well-posedness and scattering. Let \( Q \) be the unique positive solution to
\[ \Delta Q + Q^{1+4/d} = Q. \] (1.15)
Existence of a positive solution to (1.15) was proved in [3], uniqueness in [30]. $Q$ is called the ground state, and provides a stationary solution

$$u(t, x) = e^{it}Q(x)$$

(1.16)

to (1.1) which blows up at infinity both forward and backward in time. Then applying the pseudoconformal transformation to (1.16) yields a solution

$$u(t, x) = |t|^{-d/2}e^{i|x|^2/4t}Q\left(\frac{x}{t}\right)$$

(1.17)

with the same mass that blows up in finite time.

Now observe, as did [6], that by conservation of energy (1.11) and Sobolev embedding, when $\mu = 1$ and $u_0 \in H^1(\mathbb{R}^d)$, (1.2) cannot blow up in finite time.

In the focusing case the mass $\|Q\|_{L^2(\mathbb{R}^d)}$ provides a stark demarcation line for the positive definiteness of the energy.

**Theorem 1.5 (Gagliardo - Nirenberg inequality)** Let $Q$ be the ground state solution to (1.15). Then

$$\int_{\mathbb{R}^d} |f(x)|^{2(d+2)/d} dx \leq \frac{d+2}{d} \left(\frac{\|f\|_{L^2(\mathbb{R}^d)}}{\|Q\|_{L^2(\mathbb{R}^d)}}\right)^{4/d} \int_{\mathbb{R}^d} |\nabla f(x)|^2 dx.$$  

(1.18)

Equality holds if and only if $f = c\tilde{Q}$, where $c$ is some constant and $\tilde{Q}$ is $Q$ under the action of the group of translations and dilations (1.3).

**Proof:** See [48]. □

Therefore, if $\|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}$ the energy (1.11) is positive definite, so if in addition $u_0 \in H^1(\mathbb{R}^d)$, (1.1) cannot blow up in finite time.

On the other hand, when energy is negative, global well-posedness is known to fail. Computing two time derivatives of the variance,

$$\partial_{tt} \int |x|^2|u(t, x)|^2 dx = 16E(u(t)) = 16E(u(0)).$$

(1.19)

For any $\epsilon > 0$ there exists $\|u_0\|_{L^2(\mathbb{R}^d)} = \|Q\|_{L^2(\mathbb{R}^d)} + \epsilon$, $E(u(0)) < 0$,

$$\int |x|^2|u_0(x)|^2 dx < \infty,$$

(1.20)
and

$$\int 2x \cdot \text{Im}[\bar{u}(t, x) \nabla u(t, x)] \, dx < \infty. \quad (1.21)$$

Then \( \int |x|^2 |u(t, x)|^2 \, dx \) is concave in time, which implies that there exists \( T_0 < \infty \) such that \( \int |x|^2 |u(t, x)|^2 \, dx < 0 \) for \( t > T_0 \) and \( t < -T_0 \), which is impossible. Therefore, (1.1) blows up in both time directions in finite time under such \( u_0 \).

For negative energy in general [34] removed the weight condition (1.23) when \( d = 1 \) and [35] removed weight condition (1.9) in dimensions \( d \geq 2 \) for radial initial data.

**Conjecture 1.6** For \( d \geq 1 \), the focusing, mass critical nonlinear Schrödinger initial value problem (1.1) is globally well-posed for \( u_0 \in L^2(\mathbb{R}^d) \), \( \|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)} \), and all solutions scatter to a free solution as \( t \to \pm \infty \). In fact there exists a function \( A : [0, \|Q\|_{L^2(\mathbb{R}^d)}) \to [0, \infty) \) such that

$$\|u\|_{L_{t,x}^{2(d+2)}(\mathbb{R} \times \mathbb{R}^d)} \leq A(\|u_0\|_{L^2(\mathbb{R}^d)}). \quad (1.22)$$

Conjecture 1.6 has been affirmed in the radial case in dimensions \( d \geq 2 \).

**Theorem 1.7** (1.1) is globally well-posed and scattering for \( u_0 \in L^2(\mathbb{R}^d) \) radial, \( \|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)} \).

**Proof:** See [26] when \( d = 2 \) and [29] when \( d \geq 3 \). \( \square \)

In this paper we remove the radial condition and prove

**Theorem 1.8** (1.1) is globally well-posed and scattering for \( u_0 \in L^2(\mathbb{R}^d) \), \( \|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)} \), \( d \geq 1 \).

To prove this result we utilize the concentration compactness method, a method that has been used since at least the 1980’s to study critical elliptic partial differential equations (see for example [5]).

The first large data nonlinear Schrödinger result ([4]) proved global well - posedness and scattering for the energy - critical, defocusing, nonlinear Schrödinger initial value problem with radial data \( u_0 \in \dot{H}^1(\mathbb{R}^d) \) in dimensions \( d = 3, 4 \). The proof utilized induction on energy, proving that it sufficed to treat solutions to the energy critical problem that were localized in both space and frequency. Around the same time [20] proved global well - posedness for the radial, energy - critical, defocusing three dimensional nonlinear Schrödinger equation with radial data using an argument similar to [19] in the energy - critical wave equation. [40] then extended the result of [4] to dimensions \( d \geq 5 \).
Also at the same time, [1] proved a profile decomposition a uniformly bounded solution to the energy-critical nonlinear wave equation. Then if global well-posedness and scattering fails, the minimal energy blowup solution must lie on a single profile. [23] adapted this argument to the focusing energy-critical wave equation. This same argument was also applied to the energy-critical nonlinear Schrödinger equation by [12] for nonradial data, \(d = 3\), in the defocusing case, and by [22] in the focusing case. [24] also proved global well posedness and scattering for the cubic, defocusing, nonlinear Schrödinger equation in three dimensions, which has critical nonlinearity \(s_c = \frac{1}{2}\), under the a priori assumption that the \(\|u(t)\|_{\dot{H}^{1/2}(\mathbb{R}^3)}\) norm is bounded.

This line of work was extended to the defocusing energy-critical nonlinear Schrödinger equation in dimensions \(d \geq 4\) by [37] and [47], and to the focusing energy critical problem by [28] and [16]. [33] extended the work of [23] on the \(\dot{H}^{1/2}\) critical problem to higher dimensions.

For the mass critical problem [29] and [26] used concentration compactness to prove theorem 1.7, and [13], [14], and [15] used concentration compactness to prove theorem 1.4.

**Definition 1.5** A set is precompact in \(L^2(\mathbb{R}^d)\) if it has compact closure in \(L^2(\mathbb{R}^d)\).

A solution \(u(t, x)\) to (1.1) is said to be almost periodic if there exists a group of symmetries \(G\) of the equation such that \(\{u(t)\}/G\) is a precompact set.

For the mass-critical problem the group \(G\) of symmetries is generated by translation,

\[
    u(t, x) \mapsto u(t, x - x_0), \quad x_0 \in \mathbb{R}^d,
\]

Galilean translation,

\[
    u(t, x) \mapsto e^{-it|\xi_0|^2}e^{ix\cdot\xi_0}u(t, x - 2t\xi_0),
\]

and dilation (1.3).

Then to prove theorem 1.8 it suffices to prove two results:

**Theorem 1.9** If conjecture 1.6 fails then there exists a solution to (1.1) with \(\|u(0)\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)}\) on a maximal open interval of existence \(I\) such that \(u(t, x)\) is almost periodic on \(I\).

**Theorem 1.10** If \(u\) is an almost periodic solution to (1.1) and \(\|u(t)\|_{L^2} < \|Q\|_{L^2}\) then \(u \equiv 0\).

**Proof: of theorem 1.9** This result was proved by [45]. The first progress came in two dimensions, when [31], using a result of [32], proved weak convergence modulo symmetries of a sequence of functions with Strichartz norm uniformly bounded above and mass uniformly bounded below. Then [25] proved a profile decomposition in dimensions \(d = 1, 2\), and showed that some mass must
concentrate on a single profile for blowup to occur. [2] made use of the restriction estimate of [39] to prove that blowup solutions to the mass-critical problem must have some mass concentrate on a single profile. [45] then completed the proof. □

In fact [45] proved that something more is true.

**Theorem 1.11** Suppose conjecture 1.6 fails. Then there exists a maximal lifespan solution \( u \) on \( I \subset \mathbb{R} \), \( u \) blows up both forward and backward in time, and \( u \) is almost periodic modulo the group \( G = (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \) which consists of scaling symmetries, translational symmetries, and Galilean symmetries. That is, for any \( t \in I \),

\[
    u(t, x) = N(t)^d/2 e^{ix \cdot \xi(t)} k_t(N(t)(x - x(t))),
\]

where \( k_t(x) \in K \subset L^2(\mathbb{R}^d) \), \( K \) is a precompact subset of \( L^2(\mathbb{R}^d) \). Additionally, \( [0, \infty) \subset I \), \( N(t) \leq 1 \) on \( [0, \infty) \), \( N(0) = 1 \), \( \xi(0) = x(0) = 0 \), and

\[
    \int_0^\infty \int_0^\infty |u(t, x)|^{2(d+2)/d} \, dx \, dt = \infty.
\]

**Proof:** See [45] and section four of [43]. □

**Remark:** From the Arzela-Ascoli theorem, a set \( K \subset L^2(\mathbb{R}^d) \) is precompact if and only if there exists a compactness modulus function, \( C(\eta) < \infty \) for all \( \eta > 0 \) such that

\[
    \int_{|x| \geq C(\eta)} |f(x)|^2 \, dx + \int_{|\xi| \geq C(\eta)} |\hat{f}(\xi)|^2 \, d\xi < \eta.
\]

To prove theorem 1.10 we consider solutions in the form of theorem 1.11 satisfying

\[
    \int_0^\infty N(t)^3 \, dt = \infty,
\]

or

\[
    \int_0^\infty N(t)^3 \, dt < \infty.
\]

[13], [14], and [15] do this in the defocusing case as well. [13], [14], and [15] made use of an estimate on the Strichartz estimate for long time. Such estimates were then utilized to prove that if \( u(t, x) \) is a minimal mass solution to (1.1) and \( \int_0^\infty N(t)^3 \, dt < \infty \), then \( u(t, x) \) possesses additional regularity.

**Remark:** Since we are considering one specific \( u(t, x) \) we may let \( A \lesssim B \) denote \( A \leq C(u)B \).
Theorem 1.12 Suppose \( u(t, x) \) is an almost periodic solution to (1.1), \( \mu = \pm 1 \), \( N(0) = 1 \), \( N(t) \leq 1 \) on \([0, \infty)\), \( \xi(0) = x(0) = 0 \), and \( \int_0^\infty N(t)^3 \, dt = K < \infty \). Then when \( 0 \leq s < 1 + \frac{4}{d} \),

\[
\|u(t, x)\|_{L_t^\infty H_x^s([0, \infty) \times \mathbb{R}^d)} \lesssim d K^s.
\] (1.30)

Proof: See theorem 3.13 of [13] for \( d \geq 3 \), theorem 5.1 of [14] for \( d = 2 \), and theorem 5.1 of [15] for \( d = 1 \). \( \square \)

We can make a conservation of energy argument to preclude this scenario in the focusing case when mass is below the mass of the ground state.

In the defocusing case [13], [14], and [15] utilized previously developed interaction Morawetz estimates (of [11], [44], [10], and [36])

\[
\|\|\nabla|^{-\frac{d-3}{2}}|u(t, x)|^2\|_{L_t^2 I_x^2(I \times \mathbb{R}^d)} \lesssim \|u\|_{L_t^\infty \dot{H}_x^{1/2} I_x^2(I \times \mathbb{R}^d)} \|u\|_{L_t^\infty L_x^2(I \times \mathbb{R}^d)}^2.
\] (1.31)

to preclude the scenario

\[
\int_0^\infty N(t)^3 \, dt = \infty.
\] (1.32)

(1.31) was useful for two reasons

1. It is Galilean invariant, (in fact the left hand side is Galilean invariant while the right hand side can be modified to be Galilean invariant, a fact observed in [36])

2. It scales like \( \int_I N(t)^3 \, dt \) and in fact is bounded below by some constant times \( \int_I N(t)^3 \, dt \).

Since we only know that \( u(t) \in L_x^2 \), [13], [14], and [15] relied on a Morawetz estimate localized to low frequencies. (See [12] for such an estimate in the energy-critical case. [12] dealt with the energy-critical equation, \( u(t) \in \dot{H}^1 \), and thus truncated to high frequencies). The long time Strichartz estimates gave control over the error terms arising from truncating in frequency space, which leads to a contradiction in the case when \( \int_0^\infty N(t)^3 \, dt = \infty \).

Here in this paper we have to deal with the opposite issue. The errors arising from Fourier truncation can be estimated for a variety of potentials.

Theorem 1.13 Suppose \( u \) is an almost periodic solution to (1.1), \( \int_0^T N(t)^3 \, dt = K \), and there exists a constant \( R \) such that

\[
|a_j(t, x)| \leq R,
\] (1.33)
\[ |\nabla_x a_j(t,x)| \leq \frac{R}{|x|}, \quad (1.34) \]

\[ a_j(t,x) = -a_j(t,-x), \quad (1.35) \]

and when \( d = 2 \),

\[ \|\partial_t a_j(t,x)\|_{L^1(\mathbb{R}^d)} \leq R. \quad (1.36) \]

Then the Fourier truncation error arising from \( P_{\leq CK} F(u) - F(P_{\leq CK} u) \) is bounded by \( Ro(K) \). \( o(K) \) is a quantity such that \( \frac{o(K)}{K} \to 0 \) as \( K \to \infty \).

**Proof:** When \( d \geq 3 \) see theorem 4 in [13], when \( d = 2 \) see theorem 6 in [15], and when \( d = 1 \) see theorem 6 in [14]. \( \square \)

**Remark:** (1.35) gives Galilean invariance.

The principal difficulty for the focusing problem is that the interaction Morawetz estimates of [11], [44], [10], and [36] rely heavily on \( \mu = +1 \) and fail to be positive definite when \( \mu = -1 \). Even restricting \( \|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)} \) is not enough to guarantee that the interaction Morawetz estimate of [36] is positive definite. In one dimension [10], [36] proved that

\[ \int_0^T \frac{1}{2} |\partial_x| P_{\leq CK} u(t,x) |^2 \|_{L^2_x(\mathbb{R})} dt + \frac{\mu}{4} \|P_{\leq CK} u(t,x)\|^8_{L^8_x(\mathbb{R})} dt \leq \sup_{t \in [0,T]} \int \frac{(x-y)}{|x-y|} Im[\overline{P_{\leq CK} u(t,x)} \partial_x P_{\leq CK} u(t,x)] |Iu(t,y)||^2 dxdy. \quad (1.37) \]

However, the most (1.37) along with standard Hölder embeddings implies is

\[ \|u(t,x)\|^8_{L^8_x(\mathbb{R})} \leq 3 \|u_0\|^4_{L^2_x(\mathbb{R})} \|\partial_x |u(t,x)|^2\|^2_{L^2_x(\mathbb{R})}, \quad (1.38) \]

which only implies that (1.10) is positive definite for all \( \|u_0\|_{L^2(\mathbb{R})} < (\frac{2}{3})^{1/4} \|Q\|_{L^2(\mathbb{R})} \).

Monica Visan informed the author that there are also counterexamples in higher dimensions which prevent the interaction Morawetz estimates of [36] from being positive definite for all \( \|u_0\|_{L^2(\mathbb{R}^d)} < \|Q\|_{L^2(\mathbb{R}^d)} \). Therefore, it is necessary to construct a new interaction Morawetz estimate adapted to the focusing mass - critical initial value problem.

This Morawetz estimate is the principal new development of the paper and consequently occupies most of the paper. Our Morawetz estimate relies on (1.18) and (1.19), which show that the energy
is positive definite whenever \( \|u_0\|_{L^2} < \|Q\|_{L^2} \). The potential used here is similar to the potentials utilized by [34] and [35].

**Outline of the Paper:** In \S 2, we describe some harmonic analysis and properties of the linear Schrödinger equation that will be needed later in the paper. In particular we discuss the Strichartz estimates and Strichartz estimates. Global well-posedness and scattering for small mass is an easy consequence of these estimates. We discuss the movement of \( \xi(t) \) and \( N(t) \) for a minimal mass blowup solution in this section.

In \S\S 3 – 6 we will turn to the case when \( \int_0^\infty N(t)^3 dt = \infty \) and construct an interaction Morawetz estimate that gives the contradiction

\[
K = \int_0^T N(t)^3 dt \lesssim o(K)
\]

for \( K \) sufficiently large. We will postpone the estimate of the error terms arising from truncation in frequency until \S 7.

In \S 7 we complete the proof of theorem 1.10 using the interaction Morawetz estimates constructed in \S\S 3 – 6 and conservation of energy.

**Acknowledgements:** During some of the research of this paper, the author was supported by the National Science Foundation postdoctoral fellowship DMS - 1103914. The author was located at the University of California - Berkeley when the preprint for this paper was posted.

# 2 Harmonic Analysis

In this section we discuss some of the results from harmonic analysis and the corresponding properties of the linear Schrödinger equation that will be used throughout the paper. None of the results in this section are new, and have been proved in many places. The author refers the reader to [41] and [46] for a general introduction to harmonic analysis and partial differential equations.

**Definition 2.1 (Littlewood - Paley decomposition)** Choose \( \phi \in C_0^\infty(\mathbb{R}^d) \), to be a radial, decreasing function, \( 0 \leq \phi \leq 1 \), and

\[
\phi(x) = \begin{cases} 
1, & |x| \leq 1; \\
0, & |x| > 2.
\end{cases}
\]

\[ \quad (2.1) \]

Define the frequency truncation

\[
\mathcal{F}(P_{\leq N} u) = \phi(\frac{\xi}{N}) \hat{u}(\xi).
\]

\[ \quad (2.2) \]
Let \( P_{>N}u = u - P_{\leq N}u \) and \( P_{N}u = P_{\leq 2N}u - P_{\leq N}u \). For convenience of notation let \( u_N = P_{N}u \), \( u_{\leq N} = P_{\leq N}u \), and \( u_{>N} = P_{>N}u \). If \( N = 2^k \) for some \( k \in \mathbb{Z} \) we abbreviate \( P_{2^k} \) and write \( P_k \).

**Theorem 2.1 (Littlewood - Paley theorem)** For any \( 1 < p < \infty \),

\[
\| (\sum_k |P_k u|^2)^{1/2} \|_{L^p(\mathbb{R}^d)} \sim_{p,d} \| u \|_{L^p(\mathbb{R}^d)}. \tag{2.3}
\]

**Proof:** See [46]. \( \square \)

**Definition 2.2** A pair \((p, q)\) is admissible if \( \frac{2}{p} = d(\frac{1}{2} - \frac{1}{q}) \), and \( p \geq 2 \) for \( d \geq 3 \), \( p > 2 \) when \( d = 2 \), and \( p \geq 4 \) when \( d = 1 \).

**Theorem 2.2 (Strichartz estimates)** If \( u(t, x) \) solves the initial value problem

\[
iu + \Delta u = F(t),
\]

\[u(0, x) = u_0,\tag{2.4}\]

on an interval \( I \), then

\[
\| u \|_{L_t^p L_x^q(I \times \mathbb{R}^d)} \lesssim_{p,q,\tilde{p},\tilde{q},d} \| u_0 \|_{L_x^2(\mathbb{R}^d)} + \| F \|_{L_t^{\tilde{p}'} L_x^{\tilde{q}'} (I \times \mathbb{R}^d)}, \tag{2.5}
\]

for all admissible pairs \((p, q)\), \((\tilde{p}, \tilde{q})\). \( \tilde{p}' \) denotes the Lebesgue dual of \( \tilde{p} \), \( \frac{1}{p} + \frac{1}{\tilde{p}'} = 1 \).

**Proof:** [38] proved the case when \( p = q, \tilde{p}' = \tilde{q}' \) using Fourier restriction estimates. [18] and [49] extended this result to \( \tilde{p}, p > 2 \). [21] dealt with the case when \( p = 2, \tilde{p} = 2 \), or both.

The Strichartz estimates motivate the definition of the Strichartz space.

**Definition 2.3 (Strichartz space)** Define the norm

\[
\| u \|_{S^0(I \times \mathbb{R}^d)} \equiv \sup_{(p,q) \ \text{admissible}} \| u \|_{L_t^p L_x^q(I \times \mathbb{R}^d)}. \tag{2.6}
\]

\[S^0(I \times \mathbb{R}^d) = \{ u \in C^0_t(I, L^2(\mathbb{R}^d)) : \| u \|_{S^0(I \times \mathbb{R}^d)} < \infty \}. \tag{2.7}\]

We also define the space \( N^0(I \times \mathbb{R}^d) \) to be the space dual to \( S^0(I \times \mathbb{R}^d) \) with appropriate norm. Then

\[
\| u \|_{S^0(I \times \mathbb{R}^d)} \lesssim \| u_0 \|_{L_x^2(\mathbb{R}^d)} + \| F \|_{N^0(I \times \mathbb{R}^d)}. \tag{2.8}
\]
Remark: When $d = 2$, the absence of an endpoint result at $p = 2$ means we need to fix some $\epsilon > 0$ and define

$$\|u\|_{S^0(I \times \mathbb{R}^2)} \equiv \sup_{(p, q) \text{ admissible}, \ p \geq 2 + \epsilon} \|u\|_{L^p_x L^q_t(I \times \mathbb{R}^2)}.$$  \hspace{1cm} (2.9)

When $s_c = 0$ the proof of theorem 1.1 follows immediately from theorem 2.2.

Now recalling (1.25) and (1.27), if $u$ is an almost periodic solution to (1.1) on maximal interval $I$ then there exist $x(t), \xi(t) : I \to \mathbb{R}^d$ and for any $\eta > 0$, there exists $C(\eta) < \infty$ such that

$$\int_{|x-x(t)| \geq \frac{C(\eta)}{N(t)}} |u(t, x)|^2 dx < \eta,$$ \hspace{1cm} (2.10)

and

$$\int_{|\xi-\xi(t)| \geq C(\eta)N(t)} |\hat{u}(t, \xi)|^2 d\xi < \eta.$$ \hspace{1cm} (2.11)

We now recall some information on the movement of $N(t)$ and $\xi(t)$.

**Lemma 2.3** If $u$ is an almost periodic solution to (1.1) and $J$ is an interval with

$$\|u\|_{L^{2d+2}_{t,x} (J \times \mathbb{R}^d)} \leq C,$$  \hspace{1cm} (2.12)

then for $t_1, t_2 \in J$,

$$N(t_1) \sim_C N(t_2).$$ \hspace{1cm} (2.13)

**Proof:** See [26], corollary 3.6. □

**Lemma 2.4** If $u(t, x)$ is an almost periodic solution to (1.1) on an interval $J$, then

$$\int_J N(t)^2 dt \lesssim \|u\|_{L^{2d+2}_{t,x} (J \times \mathbb{R}^d)} \lesssim 1 + \int_J N(t)^2 dt.$$ \hspace{1cm} (2.14)

**Proof:** See [29]. □

**Lemma 2.5** Suppose $u$ is a minimal mass blowup solution with $N(t) \leq 1$. Suppose also that $J$ is some interval partitioned into subintervals $J_k$ with $\|u\|_{L^{2d+2}_{t,x} (J_k \times \mathbb{R}^d)} = 1$ on each $J_k$. Let

$$N(J_k) = \sup_{J_k} N(t).$$ \hspace{1cm} (2.15)

Then,
\[
\sum_{J_k} N(J_k) \sim \int_J N(t)^3 dt. \tag{2.16}
\]

**Proof:** By (2.13), \( N(t_1) \sim N(t_2) \) for \( t_1, t_2 \in J_k \). Therefore it suffices to show that \( |J_k| \sim \frac{1}{N(J_k)^2} \). See lemma 5.13 of [27] for a proof of this fact. □

Then after modifying \( C(\eta) \) by at most a constant,
\[
|N'(t)| \lesssim N(t)^3. \tag{2.17}
\]

Lemma 5.13 of [27] also states that
\[
|\xi(t_1) - \xi(t_2)| \lesssim N(J_k) \tag{2.18}
\]
for all \( t_1, t_2 \in J_k \). Again after adjusting the modulus function \( C(\eta) \) in (2.10), (2.11) by a constant, we can choose \( \xi(t) : I \to \mathbb{R}^d \) such that
\[
\left| \frac{d}{dt} \xi(t) \right| \lesssim N(t)^3. \tag{2.19}
\]

(2.10) and (2.11) also give decay on the \( L^{2(d+2)}_{L_t,x} (J_k \times \mathbb{R}^d) \) norm far away from \( x(t) \) and at frequencies far away from \( \xi(t) \).

**Lemma 2.6** Suppose \( J \) is an interval with
\[
\|u\|_{L^{2(d+2)}_{L_t,x} (J \times \mathbb{R}^d)} = 1, \tag{2.20}
\]
\( N(J) = 1 \). Then
\[
\|P_{|\xi - \xi(t)| \geq R^d}\|_{L^{2(d+2)}_{L_t,x} (J \times \mathbb{R}^d)} + \int_J \int_{|x-x(t)| \geq R} |u(t,x)| \frac{2(d+2)}{d} dx dt \leq o_R(1), \tag{2.21}
\]
o\( R \) \( \to 0 \) as \( R \to \infty \), \( x(t), \xi(t) \) are the same quantities defined in (2.10) and (2.11).

**Proof:** We will prove this only in the case when \( d = 1 \). since a similar argument also works in higher dimensions. By Strichartz estimates
\[
\|u\|_{L^1_x L^\infty_t (J \times \mathbb{R})} \lesssim 1. \tag{2.22}
\]
Interpolating (2.22) with (2.10) and (2.11) proves the lemma. □

By rescaling this implies
\[ \| P_{|\xi - \xi(t)| \geq R_N(t)} \|^{\frac{2(d+2)}{d}}_{L^2_{t,x}(J \times \mathbb{R}^d)} + \int J \int_{|x-x(t)| \geq \frac{R}{N(t)}} |u(t,x)|^{\frac{2(d+2)}{d}} \, dx \, dt \leq o_R(1). \]  

(2.23)

3 \quad d = 1, \; N(t) \equiv 1, \; u \text{ even}

For the defocusing \( L^2 \) - critical initial value problem the case In this section we begin to build the interaction Morawetz estimate that will ultimately preclude the scenario

\[ \int_0^\infty N(t)^3 \, dt = \infty, \]  

(3.1)

\( N(0) = 1, \; N(t) \leq 1 \) on \([0, \infty)\). We start with the simplest case, \( u \) is even, \( N(t) = 1 \) and \( d = 1 \). Because \( u \) is even we make a Morawetz estimate and not an interaction Morawetz estimate. Let \( K = \int_0^T N(t)^3 \, dt \). By (2.19) choose \( C \) very large so that

\[ \int_0^T \left| \frac{d}{dt} \xi(t) \right| \, dt \ll CK. \]  

(3.2)

Then let \( I = P_{\leq CK} \).

Remark: In this section, since \( u \) is even, \( \xi(t) \equiv 0 \). However, by (2.19) it is possible to choose \( C \) such that (3.2) holds, regardless of whether \( u \) has any symmetry at all. Since we will ultimately prove theorem 1.8 without any symmetry assumptions on \( u \), we will go ahead and use (3.2) here.

The error terms arise from the fact that

\[ i\partial_t (Iu) + \Delta (Iu) = IF(u), \]  

(3.3)

and the commutator

\[ F(Iu) - IF(u) \neq 0. \]  

(3.4)

By theorem 1.13 the errors arising from Fourier truncation are bounded by \( Ro(K) \), so we will ignore these error terms and assume

**Theorem 3.1** If \( u \) is an almost periodic solution to (1.1) with \( d = 1 \), \( u \) an even function, and \( N(t) \equiv 1 \), then \( u \equiv 0 \).

**Proof:** \( u \) even implies \( \xi(t) = x(t) \equiv 0 \), so we only need a standard Morawetz estimate here. We use the Morawetz potential of [34], [35]. Let \( \psi \in C^\infty(\mathbb{R}) \), \( \psi(x) \) even,
ψ(x) = \begin{cases} 
1 & \text{if } |x| \leq 1, \\
 \frac{1}{3 |x|} & \text{if } |x| > 2, 
\end{cases} 
(3.5)

and

\partial_x(x\psi(x)) = \phi(x) \geq 0. 
(3.6)

Let

\begin{equation}
M(t) = \int \psi(x) x \text{Im} [\overline{Iu(t,x)} \partial_x Iu(t,x)] dx.
\end{equation}

Integrating by parts,

\begin{equation}
\frac{d}{dt} M(t) = \int \psi(x) x [-4 \partial_x(|\partial_x Iu|^2) + \partial_x^3(|Iu|^2) + \frac{4}{3} \partial_x(|Iu|^6)] dx.
\end{equation}

Now let \( \chi \in C_0^\infty(\mathbb{R}) \), \( \chi \equiv 1 \) for \( |x| \leq \frac{1}{2} \), \( \chi \) supported on \([-1, 1]\).

\begin{equation}
\frac{d}{dt} M(t) = 8 \int \left[ \frac{1}{2} \chi(x) |\partial_x Iu|^2 - \frac{1}{6} \chi(x) |Iu|^6 \right] dx
+ 4 \int \left[ \phi(x) - \chi(x) \right] |\partial_x Iu|^2 dx
+ \frac{4}{3} \int \left[ \phi(x) - \chi(x) \right] |Iu|^6 dx
- \int \partial_x^2(\phi(x)) |Iu|^2 dx.
\end{equation}

Because

\begin{equation}
\chi \cdot \partial_x u = \partial_x (\chi u) - u \partial_x \chi,
\end{equation}

integrating by parts,

\begin{equation}
\frac{d}{dt} M(t) = 8 \int \left[ \frac{1}{2} \chi(x) |\partial_x Iu|^2 - \frac{1}{6} \chi(x) |Iu|^6 \right] dx
+ \frac{1}{R^2} \int \chi''(x) \chi(x) |Iu|^2(t,x) dx
+ 4 \int \left[ \phi(x) - \chi(x) \right] |\partial_x Iu|^2 dx
+ \frac{4}{3} \int \left[ \phi(x) - \chi(x) \right] |Iu|^6 dx
- \int \partial_x^2(\phi(x)) |Iu|^2 dx.
\end{equation}

Theorem 1.5 implies that for \( ||u_0||_{L^2(\mathbb{R})} < ||Q||_{L^2(\mathbb{R})} \), there exists some \( \eta(||u_0||_{L^2}) > 0 \) such that
\begin{equation}
8 \int \frac{1}{2} |\partial_x (\chi(\frac{x}{R}) Iu)|^2 - \frac{1}{6} |\chi(\frac{x}{R}) Iu|^6 dx \geq \eta |\chi(\frac{x}{R}) Iu|^6_{L^6(\mathbb{R})} + \frac{\eta}{3} |\partial_x (\chi(\frac{x}{R}) Iu)|^2_{L^2(\mathbb{R})}
\tag{3.15}
\end{equation}

Because \( \phi(\frac{x}{R}) - \chi(\frac{x}{R})^2 \geq 0 \),
\begin{equation}
\frac{d}{dt} M(t) \geq \eta |\chi(\frac{x}{R}) Iu|_{L^6(\mathbb{R})}^6 + \frac{\eta}{3} |\partial_x (\chi(\frac{x}{R}) Iu)|_{L^2(\mathbb{R})}^2 - \int_{|x| \geq \frac{R}{2}} |Iu(t, x)|^6 dx - \frac{C}{R^2} \|u\|_{L^2(\mathbb{R})}^2.
\tag{3.16}
\end{equation}

By theorem 1.13, lemmas 2.4 and 2.6, (2.10), and (2.11) we can choose \( R(\eta), K(R) \) sufficiently large so that
\begin{equation}
\int_0^K \frac{d}{dt} M(t) dt \geq \int_0^K \eta |\chi(\frac{x}{R}) Iu|_{L^6(\mathbb{R})}^6 dt - \frac{C}{R^2} \|u\|_{L^2(\mathbb{R})}^2 \tag{3.17}
\end{equation}

\begin{align*}
&- \int_0^K \int_{|x| \geq \frac{R}{2}} |Iu(t, x)|^6 dx dt - R(\eta) o(K) \geq \frac{\eta}{2} K \|u(t)\|_{L^6(\mathbb{R}^d)}^6.
\end{align*}

On the other hand, by (2.11),
\begin{equation}
M(t) = \int \text{Im}[\overline{Iu} \partial_x Iu] (t, x) \psi(\frac{x}{R}) dx \lesssim R o(K).
\tag{3.18}
\end{equation}

Since \( K \) is arbitrarily large this implies that \( u \equiv 0 \). \( \Box \)

In the next three sections we will remove the conditions \( d = 1, u \) is even, and \( N(t) \equiv 1 \).

4 \( N(t) \equiv 1, u \) is a radial function

We now generalize theorem 3.1 to higher dimensions.

**Theorem 4.1** If \( u \) is an almost periodic solution to (1.1) with \( N(t) \equiv 1 \) and \( u \) radial, then \( u \equiv 0 \).

**Proof:** Again take \( K = \int_0^T N(t)^3 dt \) and \( I = P_{\leq CK} \) such that (3.2) holds. Let \( \varphi \in C^\infty_0(\mathbb{R}^d) \) be a radial, decreasing function such that \( \varphi = 1 \) for \( |x| \leq 1 \) and \( \varphi \) is supported on \( |x| \leq 2 \). Then let \( \psi(r) \) be the radial function
\begin{equation}
\psi_R(r) = \frac{1}{r} \int_0^r \varphi(\frac{s}{R}).
\tag{4.1}
\end{equation}

Since \( \varphi \leq 1 \) and \( \varphi(\frac{s}{R}) \) is supported on \( |x| \leq R \),
\begin{equation}
\psi_R(r) \leq \frac{2R}{r}.
\tag{4.2}
\end{equation}
Also, by direct computation and the fact that \( \varphi \) is decreasing

\[
r \psi'_{R}(r) = \varphi \left( \frac{r}{R} \right) - \psi_{R}(r) \leq 0.
\] (4.3)

By (4.2),

\[
|\psi_{R}(|x|)|x| \leq 2R
\] (4.4)

and by (4.3), if \( d \geq 2 \),

\[
|\nabla \psi_{R}(|x|) x_j| \lesssim \frac{2R}{|x|}.
\] (4.5)

Then by theorem 1.13, the error term arising from frequency truncation is bounded by \( Ro(K) \) and can be safely ignored.

Now let

\[
M(t) = \int \psi_{R}(|x|) x_j \text{Im} \{\overline{Iu}(t, x) \partial_j Iu(t, x)\} dx.
\] (4.6)

**Remark:** We sum over repeated indices.

\[
\frac{d}{dt} M(t) = -4 \int \psi_{R}(|x|) x_j \partial_k \text{Re} \{\overline{Iu}(t, x) \partial_k Iu(t, x)\} dx
\] (4.7)

\[
+ \frac{4}{d+2} \int \psi_{R}(|x|) x_j \partial_j (|Iu(t, x)|^\frac{2(d+2)}{d}) dx + \int \psi_{R}(|x|) x_j \partial_j \partial_k^2 (|Iu(t, x)|^2) dx.
\] (4.8)

Integrating (4.7) and (4.8) by parts,

\[
\frac{d}{dt} M(t) = 4 \int \psi_{R}(|x|) |\nabla Iu(t, x)|^2 dx + 4 \int \psi'_{R}(|x|) \frac{x_j x_k}{|x|} \text{Re} \{\overline{\partial_j Iu(t, x)} \partial_k Iu(t, x)\} dx
\] (4.9)

\[
- \frac{4d}{d+2} \int \psi_{R}(|x|) |Iu(t, x)|^{\frac{2(d+2)}{d}} dx - \frac{4}{d+2} \int \psi'_{R}(|x|) |x||Iu(t, x)|^{\frac{2(d+2)}{d}} dx
\] (4.10)

\[
- \int \Delta ((d-1) \psi_{R}(|x|) + \varphi \left( \frac{|x|}{R} \right)) |Iu(t, x)|^2 dx.
\] (4.11)

The gradient vector can be decomposed into a radial component and an angular component. Let \( \nabla_{r, 0} \) be the radial derivative with origin \( x = 0 \),

\[
\nabla_{r, 0} = \frac{x_j}{|x|} \partial_j,
\] (4.12)
and let $\nabla_0$ be the angular component of $\nabla$. Then

$$(4.9) = 4 \int \varphi(\frac{|x|}{R})|\nabla Iu(t,x)|^2 dx + 4 \int (\psi_R(|x|) - \varphi(\frac{|x|}{R}))|\nabla Iu(t,x)|^2 dx.$$  

Next, by (4.3),

$$(4.10) \geq -\frac{4d}{d+2} \int \varphi(\frac{|x|}{R})|Iu(t,x)|^{2(d+2)} dx - \frac{4d}{d+2} \int [\psi_R(|x|) - \varphi(\frac{|x|}{R})]|Iu(t,x)|^{2(d+2)} dx.$$  

Now again choose $\chi \in C^{\infty}_0(\mathbb{R}^d)$, $\chi$ supported on $|x| \leq \frac{1}{2}$, then by the Gagliardo - Nirenberg inequality and integrating by parts,

$$4 \int \chi(\frac{x}{R})^2 |\nabla Iu(t,x)|^2 dx - \frac{4d}{d+2} \chi(\frac{x}{R}) |Iu(t,x)|^{\frac{2(d+2)}{d}} dx $$

$$= 8 \int \left[ \frac{1}{2} \nabla \chi(\frac{x}{R}) Iu(t,x) \right]^2 - \frac{d}{2(d+2)} |\chi(\frac{x}{R}) Iu(t,x)|^{\frac{2(d+2)}{d}} dx \geq \eta \|Iu(t,x)\|^{\frac{2(d+2)}{d}}_{L^2_x(\mathbb{R}^d)} - \frac{C}{R^2} \|Iu(t)\|^2_{L^2_x(\mathbb{R}^d).}$$  

Then since $\chi \equiv 1$ on $|x| \leq \frac{1}{2}$, theorem 1.13, lemmas 2.4 and 2.6, (2.10), and (2.11) again imply that we can again choose $R(\eta)$, $K(R)$ sufficiently large so that

$$\int_0^K \frac{d}{dt} M(t) dt \geq \eta \int_0^K \|Iu(t)\|^{\frac{2(d+2)}{d}}_{L^2_x(\mathbb{R}^d)} dx - o(1)K - Ro(K) \geq \frac{\eta}{2} K \|u(t)\|_{L^2_x(\mathbb{R}^d)}.$$  

On the other hand, we once again have, by (2.11),

$$M(t) = \int \text{Im} \overline{u_t} \partial_x Iu(t,x) \psi(\frac{x}{R}) dx \lesssim Ro(K).$$  

Since $K$ is arbitrarily large this implies that $u \equiv 0. \square$

5 $N(t) \equiv 1$ in any dimension and so symmetry assumptions

When the radial condition is removed, $x(t)$ and $\xi(t)$ are free to move around. Thus, in this section we will modify the Morawetz centered at the origin $x = 0$ to an interaction Morawetz estimate.

**Theorem 5.1** If $u$ is an almost periodic solution to (1.1) with $N(t) \equiv 1$ then $u \equiv 0$.  

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Proof: We construct an interaction Morawetz estimate. Let $\omega_d$ be the measure of the unit sphere $S^{d-1} \subset \mathbb{R}^d$. For some $M$ to be specified later, let $\varphi$ be a $C^\infty_0(\mathbb{R}^d)$, radial function supported on $|x| \leq R$, $\varphi = 1$ for $|x| \leq R - R^{1/2}$.

$$\phi(|x|) = \frac{1}{\omega_d R^d} \int_{\mathbb{R}^d} \varphi(|x-s|)\varphi(|s|)ds.$$  

(5.1)

Making the change of variables $s \mapsto s - y$,

$$\phi(|x-y|) = \frac{1}{\omega_d R^d} \int_{\mathbb{R}^d} \varphi(|x-s|)\varphi(|y-s|)ds.$$  

(5.2)

Now let

$$\psi_R(r) = \frac{1}{r} \int_0^r \phi\left(\frac{R}{s}\right)ds.$$  

(5.3)

Now, following [11], [44], [10], and [36], define the Morawetz action

$$M(t) = \int \int \psi_R(|x-y|)(x-y)_j \text{Im}[\overline{u}(t,x)\partial_j u(t,x)]|u(t,y)|^2dxdy.$$  

(5.4)

Notice that $\psi_R(|x-y|)(x-y)_j$ satisfies the conditions of theorem 1.13 with $R$ replaced by $R^2$. Therefore, by direct computation,

$$\frac{d}{dt} M(t) = 4 \int \int \psi_R(|x-y|)(x-y)_j \partial_k \text{Re}[(\partial_j \overline{u}(t,x))(\partial_k u(t,x))]|u(t,y)|^2dxdy$$  

(5.5)

$$-4 \int \int \psi_R(|x-y|)(x-y)_j \text{Im}[\overline{u}(t,x)\partial_j u(t,x)] \partial_k \text{Im}[\overline{u}(t,y)\partial_k u(t,y)]dxdy$$  

(5.6)

$$+ \int \int \psi_R(|x-y|)(x-y)_j \partial_j \partial_k^2 (|u(t,x)|^2)|u(t,y)|^2dxdy.$$  

(5.7)

$$+ \frac{4}{d+2} \int \int \psi_R(|x-y|)(x-y)_j \partial_j(|u(t,x)|^{2(d+2)/d})|u(t,y)|^2dxdy + \mathcal{E}(t),$$  

(5.8)

and $\int_0^K \mathcal{E}(t) dt \lesssim R^2 o(K)$.

Integrating by parts,

$$\int \psi_R(|x-y|)|u(t,x)|^{2(d+2)/d}|u(t,y)|^2dxdy$$  

$$+ \frac{4}{d+2} \int \int \psi_R'(|x-y|)|x-y||u(t,x)|^{2(d+2)/d}|u(t,y)|^2dxdy,$$  

(5.9)
and

\[(5.7) = -\int \int \Delta((d-1)\psi_R(|x-y|) + \phi(\frac{|x-y|}{R})|Iu(t,x)|^2|Iu(t,y)|^2dxdy. \tag{5.10}\]

Now let \( \psi_y \) denote the \( \psi \) from the previous section, but with the origin shifted to \( y \). Integrating by parts,

\[(5.5) + (5.6) = 4 \int \int \phi(\frac{|x-y|}{R})|\nabla Iu(t,x)|^2|Iu(t,y)|^2dxdy \]

\[ -4 \int \int \phi(\frac{|x-y|}{R})Im[Iu(t,x)\partial_j Iu(t,x)]Im[\overline{Iu(t,y)\partial_j Iu(t,y)}]dxdy \]

\[ +4 \int \int (\psi_R(|x-y|) - \phi(\frac{|x-y|}{R}))|\nabla_y Iu(t,x)|^2|Iu(t,y)|^2dxdy \]

\[ -4 \int \int (\psi_R(|x-y|) - \phi(\frac{|x-y|}{R}))Im[\overline{Iu(t,x)}\nabla_y Iu(t,x)] \cdot Im[\overline{Iu(t,y)}\nabla_x Iu(t,y)]dxdy. \tag{5.12}\]

Since \( \psi_R \) and \( \phi \) are radial functions, \( (5.12) \geq 0 \).

The quantity \( (5.11) \) is Galilean invariant. We will choose to make a Galilean transform that eliminates the second term in \( (5.11) \).

Recall the definition of \( \phi \). For any \( s \) the quantity

\[ 4 \int \int \varphi(|\frac{x}{R} - s|)\varphi(|\frac{y}{R} - s|)|\nabla Iu(t,x)|^2|Iu(t,y)|^2dxdy \]

\[ -4 \int \int \varphi(|\frac{x}{R} - s|)\varphi(|\frac{y}{R} - s|)Im[Iu\nabla Iu](t,x) \cdot Im[\overline{Iu\nabla Iu}](t,y)dxdy \tag{5.13}\]

is invariant under the Galilean transformation \( Iu(t,x) \mapsto e^{ix\cdot\xi(s)}Iu(t,x) \). It is convenient to choose \( \xi(s) \) so that

\[ \int \varphi(|\frac{x}{R} - s|)Im[e^{ix\cdot\xi(s)}\overline{Iu\nabla(e^{-ix\cdot\xi(s)}Iu)}](t,x)dx = 0. \tag{5.14}\]

Now take \( \chi \in C^\infty_0 \), \( \chi(x) \) is supported on \(|x| \leq R - R^{1/2} \), \( \chi(x) = 1 \) on \(|x| \leq R - 2R^{1/2} \). By the Gagliardo - Nirenberg inequality and integration by parts, if \( \|u_0\|_{L^2} < \|Q\|_{L^2} \) then there exists \( \eta(\|u_0\|_{L^2}) > 0 \) such that

\[ \int \chi^2(|\frac{x}{R} - s|)|\nabla(e^{ix\cdot\xi(s)}Iu(t,x))|^2dx - \int \chi \frac{2(d+2)}{d}(|\frac{x}{R} - s|)|e^{ix\cdot\xi(s)}Iu(t,x)|^{\frac{2(d+2)}{d}}dx \tag{5.15}\]
Now integrating in \(s\),
\[
\frac{\eta}{\omega_d \omega^d} \int \int \int \chi_{\left( \frac{d+2}{d} \right)} \varphi \left( \left| \frac{x}{R} - s \right| \right) \varphi \left( \left| \frac{y}{R} - s \right| \right) |Iu(t, x)|^{\frac{2(d+2)}{d}} |Iu(t, y)|^{\frac{2(d+2)}{d}} \, dx \, dy \, ds 
\geq c(d) \eta \int \int \chi_{\left| x - y \right| \leq \frac{R^2}{4}} |Iu(t, x)| |Iu(t, y)| \, dx \, dy. 
\]  
(5.17)

Also for \( |x - y| \leq \frac{R^2}{4} \),
\[
\frac{1}{\omega_d R^d} \int \varphi \left( \left| \frac{x}{R} - s \right| \right) - \chi_{\left( \frac{d+2}{d} \right)} \varphi \left( \left| \frac{x}{R} - s \right| \right) \, ds \lesssim \frac{1}{R^{1/2}}. 
\]  
(5.19)

Then by lemma 2.6,
\[
\frac{1}{R^{1/2}} \int_0^K \int_{|x - y| \leq \frac{R^2}{4}} |Iu(t, x)|^{\frac{2(d+2)}{d}} |Iu(t, y)|^{\frac{2(d+2)}{d}} \, dx \, dy \, dt 
+ \int_0^K \int_{|x - y| > \frac{R^2}{4}} |Iu(t, x)|^{\frac{2(d+2)}{d}} |Iu(t, y)|^{\frac{2(d+2)}{d}} \, dx \, dy \, dt \lesssim \frac{1}{R^{1/2} K} + o_R(1) K. 
\]  
(5.20)

Therefore by (5.5) - (5.20), Hölder’s inequality, and (2.10) and (2.11),
\[
\int_0^K \frac{d}{dt} M(t) \, dt \geq c(d) \eta K \|u(t)\|_{L^2}^{\frac{2(d+2)}{d}} - \frac{1}{R^{1/2} K} - o_R(1) K - R^2 o(K). 
\]  
(5.21)

Also by (2.10) and (2.11),
\[
\sup_{t \in [0,K]} |M(t)| \lesssim R^2 o(K). 
\]  
(5.22)

Choosing \( R(\eta) \) sufficiently large, by lemma 2.6, (2.10), (2.11), since \( K \) is arbitrarily large, \( u \equiv 0 \). □

### 6 \( N(t) \) varies

Now we are ready to consider the most general setting, \( N(t) \) varies and \( d \geq 1 \).

**Theorem 6.1** If \( u \) is an almost periodic solution to (1.1) satisfying \( \int_0^\infty N(t)^3 \, dt = \infty \), and \( \|u(0)\|_{L^2} < \|Q\|_{L^2} \), then \( u \equiv 0 \).
Proof: The new complication created when \( N(t) \) is free to move around is that by (2.10), (2.11), \( u \) is concentrated on a ball of radius \( \sim \frac{1}{N(t)} \). Therefore, (5.18) should be replaced by

\[
c(d)\eta \int \int_{|x-y| \leq R^2 N(t)} |Iu(t,x)|^{2(d+2)} |Iu(t,y)|^2 dx dy. \tag{6.1}
\]

The naive idea is to simply replace \( \psi_R \) with \( \frac{R}{N(t)^{1/2}} \). In this case, so that the potential satisfies the conditions of the theorem 1.1, let

\[
M(t) = \int \psi_{\frac{R}{N(t)^{1/2}}}(x-y)(x-y) N(t) |Im| \overline{Iu(t,x)} \partial_j Iu(t,x)| |Iu(t,y)|^2 dx dy. \tag{6.2}
\]

Let

\[
\phi(|x-y|) = \frac{1}{\omega_d R^d} \int \varphi(|\frac{x}{R} - s|) \varphi(|\frac{y}{R} - s|) ds, \tag{6.3}
\]

\[
\psi_{\frac{R}{N(t)^{1/2}}}(r) = \frac{1}{r} \int_0^r \phi(|\frac{s}{R}|) ds. \tag{6.4}
\]

Then by the computations proving theorem 5.1,

\[
\int_0^T \frac{d}{dt} M(t) dt \geq c(d)\eta \int \int_{|x-y| \leq R^2 N(t)} |Iu(t,x)|^{2(d+2)} |Iu(t,y)|^2 dx dy dt \tag{6.5}
\]

\[
- o_R(1) \int_0^T N(t) \int |u(t,x)|^{2(d+2)} dx dt - R^2 o(K) \tag{6.6}
\]

\[
+ \int_0^T \int \int |Im| \overline{Iu(t,x)} \partial_j Iu(t,x)| |Iu(t,y)|^2 \frac{d}{dt} (\psi_{\frac{R}{N(t)^{1/2}}}(x-y)(x-y) N(t)) dx dy dt. \tag{6.7}
\]

Now we compute

\[
\frac{d}{dt} (\psi_{\frac{R}{N(t)^{1/2}}}(x-y)(x-y) N(t)) = \phi(|\frac{x-y}{R}|) \frac{N(t)}{R} (x-y) N'(t). \tag{6.8}
\]

\[
\phi(|\frac{x N(t)}{R} - s|) \varphi(|\frac{y N(t)}{R} - s|) |Im| \overline{Iu(t,x)} \partial_j Iu(t,x)| |Iu(t,y)|^2 (x-y) dx dy \tag{6.10}
\]
is also invariant under the Galilean transformation $I u(t, x) \mapsto e^{ix \cdot \xi(s)} I u(t, x)$. For any $\epsilon > 0$,

$$N'(t) \int \varphi(\frac{xN(t)}{R} - s)\varphi(\frac{yN(t)}{R} - s)|Im[\overline{I u} \partial_j (e^{ix \cdot \xi(s)} I u)](t, x)|I u(t, y)|^2(x - y)_j dxdy \leq \epsilon N(t) \int \varphi(\frac{xN(t)}{R} - s)\varphi(\frac{yN(t)}{R} - s)|\nabla (e^{ix \cdot \xi(s)} I u)(t, x)|^2|I u(t, y)|^2 dxdy$$  \hspace{1cm} (6.11)

$$\leq \epsilon N(t) \int \varphi(\frac{xN(t)}{R} - s)\varphi(\frac{yN(t)}{R} - s)|\nabla (e^{ix \cdot \xi(s)} I u)(t, x)|^2|I u(t, y)|^2 dxdy + C(\epsilon) (\frac{N'(t)}{N(t)^d}) \int \varphi(\frac{xN(t)}{R} - s)\varphi(\frac{yN(t)}{R} - s)|I u(t, x)|^2|I u(t, y)|^2 dxdy. \hspace{1cm} (6.12)$$

For $\epsilon > 0$, (6.12) can be absorbed into (6.5). Combining (2.17) with conservation of mass,

$$\int_0^T (6.13) dt \lesssim C(\epsilon) \int_0^T |N'(t)| dt. \hspace{1cm} (6.14)$$

If $N(t)$ is constant or even monotone, (6.15) $\ll K$ for $K$ sufficiently large.

Therefore, we replace $N(t)$ with $\tilde{N}(t)$ much more slowly varying. Now by (6.1) we need $\tilde{N}(t) \leq N(t)$. Since $\tilde{N}(t)$ is much more slowly varying, we call our algorithm the smoothing algorithm.

### 6.1 Smoothing Algorithm

Partition $[0, \infty)$ into an infinite number of disjoint intervals $[a_n, a_{n+1})$ such that on each interval

$$\|u\|_{L_{t,x}^2([a_n, a_{n+1}] \times \mathbb{R})} = 1. \hspace{1cm} (6.15)$$

Now by lemma 2.3 there exists $J_0 < \infty$ such that for all $t \in [a_n, a_{n+1}]$,

$$\frac{N(a_{n+1})}{J_0} \leq N(t) \leq J_0 N(a_{n+1}). \hspace{1cm} (6.16)$$

Possibly after modifying the $C(\eta)$ in (2.10), (2.11) by a constant, since $N(t) \leq 1$ we can choose $N(t)$ so that for each $n$, $N(a_n) = J_0^{i_n}$ for some $i_n \in \mathbb{Z}_{\leq 0}$. Then by (6.16),

$$\frac{N(a_n)}{N(a_{n+1})} = 1, \quad J_0, \quad \text{or} \quad J_0^{-1}. \hspace{1cm} (6.17)$$

Now for $a_n < t < a_{n+1}$ choose $N(t)$ lying on the line connecting $(a_n, N(a_n))$ and $(a_{n+1}, N(a_{n+1}))$.

**Definition 6.1 (Peaks and valleys)** A peak of length $n$ is an interval $[a, b]$ such that
1. $N(t)$ is constant on $[a, b]$, and $\|u\|^{\frac{2(d+2)}{2(d+2)}}_{L^2_{t,x}([a,b]×\mathbb{R}^d)} = n$,

2. If $[a_-, a], [b, b_+]$, are the small intervals adjacent to $[a, b]$, $N(a_-) < N(a)$, $N(b_+) < N(b)$.
   (This means $N(a_-) = N(b_+) = \frac{N(a)}{J_0}$.

A valley of length $n$ is an interval $[a, b)$ such that

1. $N(t)$ is constant on $[a, b]$, and $\|u\|^{6}_{L^6_{t,x}([a,b]×\mathbb{R})} = n$,

2. If $[a_-, a], [b, b_+]$, are the small intervals adjacent to $[a, b]$, $N(a_-) > N(a)$, $N(b_+) > N(b)$.

If $[a_-, a]$ and $[a, a_+]$ are adjacent small intervals, and $N(a) > N(a_-), N(a_+)$, then we call $\{a\}$ a peak of length $0$. Similarly, if $N(a_-), N(a_+) > N(a)$, then we call $\{a\}$ a valley of length zero.

Remark: We label the peaks $p_k$ and the valleys $v_k$. Because $N(0) = 1$ and $N(t) \leq 1$ we start with a peak. We must alternate between peaks and valleys, $p_0, v_0, p_1, v_1, \ldots$.

Lemma 6.2

$$\int_0^T |N'(t)|dt \leq 2 \sum_{0<p_k<T} N(p_k) + 2. \quad (6.18)$$

Proof: This is proved by the fundamental theorem of calculus.

$$\int_{v_k}^{p_{k+1}} |N'(t)|dt = N(p_{k+1}) - N(v_k) \leq N(p_{k+1}). \quad (6.19)$$

$$\int_{p_k}^{v_k} |N'(t)|dt = N(p_k) - N(v_k) \leq N(p_k). \quad (6.20)$$

□

Now we describe an iterative algorithm to construct progressively less oscillatory $N_m(t)$.

1. Let $N_0(t) = N(t)$.

2. If $[a, b]$ is a peak for $N_m(t)$ and $[a_-, a], [b, b_+]$ are the adjacent small intervals, let $N_{m+1}(t) = N(a_-) = \frac{N(a)}{J_0}$ for $t \in [a_-, b_+]$. 

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Lemma 6.3

\[
\liminf_{T \to \infty} \frac{\int_0^T |N_m'(t)| dt}{\int_0^T N_m(t) \| Iu(t, x) \|_{L_x^{2(d+2)/d}} \, dt} \leq \frac{2}{m}. \tag{6.21}
\]

Proof: We say a peak \([a_l, b_l]\) for \(N_m(t)\) is a parent for a peak \([a_r, b_r]\) for \(N_{m+1}(t)\) if \([a_l, b_l] \subset [a_r, b_r]\). Furthermore every peak for \(N_{m+1}(t)\) must have at least one parent.

Suppose \([a_l, b_l]\) is a peak for \(N_{m+1}(t)\). Let \([a^-, a_r]\) and \([b_r, b^+]\) be the small intervals adjacent to \([a_r, b_r]\). By the above algorithm \(N_m(t)\) is not constant on \([a^-, a_r]\) and \([b_r, b^+]\) and therefore \(N_m(t) = N_{m+1}(t)\) on \([a^-, a_r]\) and \([b_r, b^+]\) and \(N_m(a^-), N_m(b^+) < N_m(a_r) = N_m(b_r)\). Then since \(N_m(t)\) is continuous, \(N_m(t)\) must have a peak in \([a_r, b_r]\)

Now suppose that \([a_l, b_l]\) is any peak for \(N_m(t)\). By the above algorithm, \(N_j(t)\) is constant on \([a_l, b_l]\) for all \(j \geq m\). Therefore, if \([a_r, b_r]\) is any peak for \(N_{m+1}(t)\) and \([a_l, b_l]\) is a peak for \(N_m(t)\), \([a_l, b_l]\) is either disjoint from \([a_r, b_r]\) or a subset of \([a_r, b_r]\).

Furthermore, by construction, if \([a_l, b_l]\) is a parent for a peak \([a_r, b_r]\),

\[
\|u\|_{\frac{2(d+2)}{2(d+2)} L_{x,t}^{\frac{d}{d}}([a_r, b_r] \times \mathbb{R}^d)} \geq \|u\|_{\frac{2(d+2)}{2(d+2)} L_{x,t}^{\frac{d}{d}}([a_l, b_l] \times \mathbb{R}^d)} + 2. \tag{6.22}
\]

By induction this implies that every peak for \(N_m(t)\) is \(\geq 2m\) subintervals long.

Remark: The initial peak \([0, T]\) is only of length \(m\).

Let \(p_k^m\) be the peaks for \(N_m(t)\). By lemma 6.2,

\[
\int_0^T |N'(t)| dt \leq 2 \sum_{0 \leq p_k \leq T} N(p_k^m) + 2. \tag{6.23}
\]

Since each peak is of length \(2m\) other than the first one,

\[
\sum_{J_n \subset [0, T]} N(J_n) \geq m \left( \sum_{0 \leq p_k \leq T} N(p_k^m) \right) - m + \frac{K}{2J_0^m}. \tag{6.24}
\]

This proves the lemma. \(\square\)

Finally notice that by construction \(\frac{|N'(t)|}{N_m(t)}\) is uniformly bounded in both \(t\) and \(m\). This is because if \(N_m'(t) \neq 0\), then \(N_m(t) = N_0(t)\).
Choosing \( m(R(\eta)) \), \( T \) sufficiently large,

\[
(6.13) \leq \frac{\eta}{2} \int_0^T N_m(t) \| u(t) \| \frac{2(\delta+2)}{L_x^{\delta+2}} (R^d) dt.
\]

Therefore, (6.5) - (6.7) imply

\[
\int_0^T \frac{d}{dt} M(t) dt \geq \frac{c(d)\eta}{2} \int_0^T N_m(t) \| I u(t) \| \frac{2(\delta+2)}{L_x^{\delta+2}} (R^d) - R^2 o(K) - o_R(1) K.
\]

Moreover,

\[
|M(t)| \lesssim R^2 o(K),
\]

so since \( K \) is arbitrarily large, \( u \equiv 0 \). \( \square \)

7 Rapid cascade

Finally we address the case when \( \int_0^\infty N(t)^3 dt < \infty \).

**Theorem 7.1** If \( u \) is an almost periodic solution to (1.1) with \( \int_0^\infty N(t)^3 dt = K < \infty \) then \( u \equiv 0 \).

By theorem 1.12 for \( 0 \leq s < 1 + \frac{4}{\delta} \),

\[
\| u(t, x) \|_{L^\infty_t \dot{H}^s_x([0, \infty) \times \mathbb{R}^d)} \lesssim m_0, d \ K^s,
\]

Also by (2.19), there exists \( \xi_\infty \in \mathbb{R}^d \), \( |\xi_\infty| \lesssim K \) such that

\[
\lim_{t \to +\infty} \xi(t) = \xi_\infty.
\]

Make the Galilean transformation

\[
v(t, x) = e^{-i|\xi_\infty|^2} e^{-ix \cdot \xi_\infty} u(t, x + 2t \xi_\infty).
\]

Since \( |\xi_\infty| \lesssim K \), by (7.1),

\[
\| v(t, x) \|_{\dot{H}^s_x(\mathbb{R}^d)} \lesssim K^s.
\]

By interpolation and (2.11), for any \( \eta > 0 \)

\[
\lim_{t \to +\infty} \inf \| v(t, x) \|_{\dot{H}^1_x(\mathbb{R}^d)} \lesssim \lim_{t \to +\infty} \inf C(\eta) N(t)^2 + \eta^{\frac{2}{\delta+2}} K \lesssim \eta^{\frac{2}{\delta+2}} K.
\]

Therefore, by conservation of energy
\[ E(v(t)) = 0. \]  \hspace{1cm} (7.6)

By the Gagliardo - Nirenberg theorem,

\[ E(v(t)) \geq \eta(\|u_0\|_{L^2_x(\mathbb{R}^d)})\|v(t,x)\|_{L^2_x(\mathbb{R}^d)}^{2(d+2)} \frac{d}{2(d+2)} (\mathbb{R}^d). \]  \hspace{1cm} (7.7)

Therefore, by (7.6), \( v \equiv 0 \), and therefore \( u \equiv 0 \). \( \square \)

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