Constrained Synchronization for Commutative Automata and Automata with Simple Idempotents

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Abstract
For general input automata, there exist regular constraint languages such that asking if a given input automaton admits a synchronizing word in the constraint language is PSPACE-complete or NP-complete. Here, we investigate this problem for commutative automata over an arbitrary alphabet and automata with simple idempotents over a binary alphabet as input automata. The latter class contains, for example, the Černý family of automata. We find that for commutative input automata, the problem is always solvable in polynomial time, for every constraint language. For input automata with simple idempotents over a binary alphabet and with a constraint language given by a partial automaton with up to three states, the constrained synchronization problem is also solvable in polynomial time.

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1 Introduction
A deterministic semi-automaton (which is an automaton without a distinguished start state and without a set of final states) is synchronizing if it admits a reset word, i.e., a word which leads to a definite state, regardless of the starting state. This notion has a wide range of applications, from software testing, circuit synthesis, communication engineering and the like, see [13, 47, 53].

The famous Černý conjecture [10] states that a minimal synchronizing word, for an \( n \) state automaton, has length at most \( (n - 1)^2 \). The best general upper bound known so far is cubic [48]. Historically, the following bounds have been published:

\[
\begin{align*}
2^n - n - 1 & \quad (1964, \text{Černý [10]}) \\
\frac{1}{2}n^3 - \frac{1}{2}n^2 + n + 1 & \quad (1966, \text{Starke [12]}) \\
\frac{1}{2}n^3 - n^2 + \frac{n}{2} & \quad (1970, \text{Kohavi [34]}) \\
\frac{3}{8}n^3 - n^2 - \frac{1}{2}n + 6 & \quad (1970, \text{Kfourny [33]}) \\
\frac{1}{2}n^3 - \frac{3}{4}n^2 + \frac{25}{6}n - 4 & \quad (1971, \text{Černý et al. [11]}) \\
\frac{7}{18}n^3 - \frac{17}{18}n^2 + \frac{17}{9}n - 3 & \quad n \equiv 0 \pmod{3} \quad (1977, \text{Pin [38]}) \\
\left(\frac{1}{2} + \frac{3}{10}\right)n^3 + o(n^3) & \quad (1981, \text{Pin [19]}) \\
\frac{1}{2}n^3 - \frac{1}{2}n - 1 & \quad (1983, \text{Pin/Frankl [25, 20]}) \\
\alpha n^3 + o(n^3) & \quad \alpha \approx 0.1664 \quad (2018, \text{Szykula [31]}) \\
\alpha n^3 + o(n^3) & \quad \alpha \leq 0.1654 \quad (2019, \text{Shitov [48]}) 
\end{align*}
\]

The Černý conjecture [10] has been confirmed for a variety of classes of automata, for example: circular automata [15, 16, 39], oriented or (generalized) monotonic automata [3, 4, 17] (even the better bound \( n(n - 1) \)), automata with a sink state [43] (even the better bound \( \frac{n(n - 1)}{2} \)), solvable and commutative automata [22, 43, 44] (even the better bound \( n(n - 1) \)).

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weakly acyclic automata \[46\], Eulerian automata \[32\], automata preserving a chain of partial orders \[54\], automata whose transition monoid contains a \(\mathbb{Q}\)-group \[5, 6\], certain one-cluster automata \[50\], automata that cannot recognize \(ta^*ab(a,b)^*\) \[1\], aperiodic automata \[52\] (even the better bound \(2(n-1)^2\)), certain aperiodically 1-contracting automata \[14\] and automata having letters of a certain rank \[8\].

Additionally, the bound \(2(n-1)^2\) has been obtained for the following two classes: automata with simple idempotents \[45\] and regular automata \[41, 42\].

For further information, we refer to the mentioned survey articles for details \[47, 53\].

Due to its importance, the notion of synchronization has undergone a range of generalizations and variations for other automata models. The paper \[21\] introduced the constrained synchronization problem. In this problem, we search for a synchronizing word coming from a specific subset of allowed input sequences. To sketch a few applications:

**Reset State.** In \[21\] one motivating example was the demand that a system, or automaton thereof, to synchronize has to first enter a “directing” mode, perform a sequence of operations, and then has to leave this operating mode and enter the “normal operating mode” again. In the most simple case, this constraint can be modelled by \(ab^*a\), which, as it turns out \[21\], yields an \(\text{NP}\)-complete constrained synchronization problem. Even more generally, it might be possible that a system – a remotely controlled rover on a distant planet, a satellite in orbit, or a lost autonomous vehicle – is not allowed to execute all commands in every possible order, but certain commands are only allowed in certain order or after other commands have been executed. All of this imposes constraints on the possible reset sequences.

**Part Orienters.** Suppose parts arrive at a manufacturing site and they need to be sorted and oriented before assembly. Practical considerations favor methods which require little or no sensing, employ simple devices, and are as robust as possible. This can be achieved as follows. We put parts to be oriented on a conveyor belt which takes them to the assembly point and let the stream of the parts encounter a series of passive obstacles placed along the belt. Much research on synchronizing automata was motivated by this application \[12, 17, 18, 26, 36, 37, 53\] and I refer to \[53\] for an illustrative example. Now, furthermore, assume the passive components can not be placed at random along the belt, but have to obey some restrictions, or restrictions in what order they are allowed to happen. These can be due to the availability of components, requirements how to lay things out or physical restrictions.

**Supervisory Control.** The constrained synchronization problem can also be viewed of as supervisory control of a discrete event system (DES) that is given by an automaton and whose event sequence is modelled by a formal language \[9, 40, 55\]. In this framework, a DES has a set of controllable and uncontrollable events. Dependent on the event sequence that occurred so far, the supervisor is able to restrict the set of events that are possible in the next step, where, however, he can only limit the use of controllable events. So, if we want to (globally) reset a finite state DES \[2\] under supervisory control, this is equivalent to constrained synchronization problem.

In \[21\] it was shown that we can realize \(\text{PSPACE}\)-complete, \(\text{NP}\)-complete or polynomial time solvable constrained problems by appropriately choosing a constraint language. Investigating the reductions from \[21\], we see that most reductions yield automata with a sink state, which then must be the unique synchronizing state. Hence, we can conclude that we can realize these complexities with this type of input automaton. Contrary, for example, unary automata are synchronizing only if they admit no non-trivial cycle, i.e., only
a single self-loop. In this case, we can easily decide synchronizability for any constraint language in polynomial time. Hence, for these simple types of automata, the complexity drops considerably. So, a natural question is, if we restrict the class of input automata, what complexities are realizable? Or more precisely:

What features in the input automata do we need to realize certain complexities?

In [30] this question was investigated for weakly acyclic, or partially ordered, input automata. These are automata where all cycles are trivial, i.e., the only loops are self-loops. It was shown that in this case, the constrained synchronization problem is always in \(\text{NP}\) and, for suitable constraint languages, \(\text{NP}\)-complete problems are realizable.

**Overview and Contribution** We investigate the constrained synchronization when the input is restricted to the class of commutative input automata and to the class of input automata with simple idempotents. Both classes were investigated previously with respect to the Černý conjecture [22, 43, 45] (see the list of the automata classes above for which this conjecture has been confirmed) and with respect to computational problems of computing a shortest synchronizing word [35]. We show in Section 3 that for commutative input automata over an arbitrary alphabet and arbitrary constraint automata, and in Section 4 that for input automata with simple idempotents over a binary alphabet and small constraint automata, the constrained synchronization problem is always solvable in polynomial time.

Section 3 splits into three subsections. The first, Subsection 3.1, is concerned with the set of synchronizing words for commutative semi-automata. In Subsection 3.2, we show an auxiliary result that is also of independent interest, namely that, given \(m\) weakly acyclic and commutative automata, the set of words accepted by them all is recognizable by a weakly acyclic automaton computable in polynomial time for a fixed alphabet. Then, in Subsection 3.3, we combine all these results to show Theorem 3.10, the statement that for commutative input automata the constrained synchronization problem is tractable.

Section 4 uses, stated in Proposition 4.2, that synchronizing automata with simple idempotent over a binary alphabet must have a very specific form. Then, Theorem 4.3, uses this result to give a polynomial time algorithm for every possible constraint automaton over a binary alphabet with at most three states.

## 2 Preliminaries and Some Known Results

We assume the reader to have some basic knowledge in computational complexity theory and formal language theory, as contained, e.g., in [31]. For instance, we make use of regular expressions to describe languages. By \(\Sigma\) we denote the *alphabet*, a finite set. For a word \(w \in \Sigma^*\) we denote by \(|w|\) its length, and, for a symbol \(x \in \Sigma\), we write \(|w|_x\) to denote the number of occurrences of \(x\) in the word. We denote the empty word, i.e., the word of length zero, by \(\epsilon\). We call \(u \in \Sigma^*\) a *prefix* of a word \(v \in \Sigma^*\) if there exists \(w \in \Sigma^*\) such that \(v = uw\). For \(U, V \subseteq \Sigma^*\), we set \(U \cdot V = UV = \{uv \mid u \in U, v \in V\}\) and \(U^0 = \{\epsilon\}\), \(U^{i+1} = U^i U\), and \(U^* = \bigcup_{i \geq 0} U^i\) and \(U^+ = \bigcup_{i > 0} U^i\). We also make use of complexity classes like \(\text{P}\), \(\text{NP}\), or \(\text{PSPACE}\).

A *partial deterministic finite automaton (PDFA)* is a tuple \(A = (\Sigma, Q, \delta, q_0, F)\), where \(\Sigma\) is a finite set of *input symbols*, \(Q\) is the finite *state set*, \(q_0 \in Q\) the *start state*, \(F \subseteq Q\) the *final state set* and \(\delta : Q \times \Sigma \to Q\) the *partial transition function*. The *partial transition function* extends to words from \(\Sigma^*\) in the usual way. Furthermore, for \(S \subseteq Q\) and \(w \in \Sigma^*\), we set \(\delta(S, w) = \{ \delta(q, w) \mid \delta(q, w) \text{ is defined and } q \in S \}\). We call
A a complete (deterministic finite) automaton if \( \delta \) is defined for every \((q,a) \in Q \times \Sigma\). If \(|\Sigma| = 1\), we call \( A \) a unary automaton and \( L \subseteq \Sigma^* \) is also called a unary language. The set \( L(A) = \{ w \in \Sigma^* \mid \delta(q_0, w) \in F \} \) denotes the language recognized by \( A \).

A deterministic and complete semi-automaton (DCSA) \( A = (\Sigma, Q, \delta) \) is a deterministic and complete finite automaton without a specified start state and with no specified set of final states. When the context is clear, we call both deterministic finite automata and semi-automata simply automata. Here, when talking about semi-automata, we always mean complete and deterministic semi-automata, as we do not consider other models of semi-automata. Concepts and notions that only rely on the transition structure carry over from complete automata to semi-automata and vice versa and we assume, for example, that every notions defined for semi-automata has also the same meaning for complete automata with a start state and a set of final states.

Let \( A = (\Sigma, Q, \delta) \) be a semi-automaton. A maximal subset \( S \subseteq Q \) with the property that for every \( s, t \in S \) there exists \( u \in \Sigma^* \) such that \( \delta(s, u) = t \) is called a strongly connected component of \( A \). We also say that a state \( s \in Q \) is connected to a state \( t \in Q \) (or \( t \) is reachable from \( s \)) if there exists \( u \in \Sigma^* \) such that \( \delta(s, u) = t \). View ing the strongly connected components as vertices of a directed graph with edges being induced by the transitions that connect different components, we get an acyclic directed graph.

Let \( A = (\Sigma, Q, \delta) \) be a semi-automaton. Then, \( A \) is called an automaton with simple idempotents, if every \( a \in \Sigma \) either permutes the states or maps precisely two states to a single state and every other state to itself. More formally, every \( a \in \Sigma \) either (1) permutes the states, i.e., \( \delta(Q, a) = Q \), or (2) it is a simple idempotent, i.e., we have \(|\delta(Q, a)| = |Q| - 1 \) and \( \delta(q, aa) = \delta(q, a) \) for every \( q \in Q \). Letters fulfilling condition (1) are also called permutational letters, and letters fulfilling (2) are called simple idempotent letters.

The semi-automaton \( A \) is called commutative, if for all \( a, b \in \Sigma \) and \( q \in Q \) we have \( \delta(q, ab) = \delta(q, ba) \). The semi-automaton \( A \) is called weakly acyclic, if there exists an ordering \( q_1, q_2, \ldots, q_n \) of its states such that if \( \delta(q_i, a) = q_j \) for some letter \( a \in \Sigma \), then \( i \leq j \) (such an ordering is called a topological sorting). An automaton is weakly acyclic, if \( \delta(q, axv) = q \) for \( u, v \in \Sigma^* \) and \( x \in \Sigma \) implies \( \delta(q, x) = q \), i.e., the only loops in the automaton graph are self-loops. This is also equivalent to the fact that the reachability relation between the states is a partial order.

A complete automaton \( A \) is called synchronizing if there exists a word \( w \in \Sigma^* \) with \(|\delta(Q, w)| = 1\). In this case, we call \( w \) a synchronizing word for \( A \). We call a state \( q \in Q \) with \( \delta(Q, w) = \{q\} \) for some \( w \in \Sigma^* \) a synchronizing state. For a semi-automaton (or PDFA) with state set \( Q \) and transition function \( \delta : Q \times \Sigma \rightarrow Q \), a state \( q \) is called a sink state, if for all \( x \in \Sigma \) we have \( \delta(q, x) = q \). Note that, if a synchronizing automaton has a sink state, then the synchronizing state is unique and must equal the sink state.

In [21] the constrained synchronization problem was defined for a fixed PDFA \( B = (\Sigma, P, \mu, p_0, F) \).

Decision Problem 1: [21] \( L(B)\)-Constr-Sync

Input: \( \text{DCSA } A = (\Sigma, Q, \delta) \).

Question: Is there a synchronizing word \( w \in \Sigma^* \) for \( A \) with \( w \in L(B) \)?  

The automaton \( B \) will be called the constraint automaton. If an automaton \( A \) is a yes-instance of \( L(B)\)-Constr-Sync we call \( A \) synchronizing with respect to \( B \). Occasionally, we do not specify \( B \) and rather talk about \( L\)-Constr-Sync.

Previous results have shown that unconstrained synchronization is solvable in polynomial time, and constrained synchronization in polynomial space.
Theorem 2.2 \([\text{[21]}]\). For any constraint automaton \(B = (\Sigma, P, \mu, p_0, F)\) the problem \(L(B)\)-\textsc{Constr-Sync} is in \(\text{PSPACE}\).

In \([21]\), a complete analysis of the complexity landscape when the constraint language is given by small partial automata was done. It is natural to extend this result to other language classes.

Theorem 2.3 \([\text{[21]}]\). Let \(B = (\Sigma, P, \mu, p_0, F)\) be a PDFA. If \(|P| \leq 1\) or \(|P| = 2\) and \(|\Sigma| \leq 2\), then \(L(B)\)-\textsc{Constr-Sync} is in \(\text{PSPACE}\). For \(|P| = 2\) with a ternary alphabet \(\Sigma = \{a, b, c\}\), up to symmetry by renaming of the letters, \(L(B)\)-\textsc{Constr-Sync} is \(\text{PSPACE}\)-complete precisely in the following cases for \(L(B)\):

\[
\begin{align*}
    a(b + c)^* & \quad (a + b + c)(a + b)^* \quad (a + b)(a + c)^* \quad (a + b)c \\
    (a + b)^*ca^* & \quad (a + b)^*c(a + b)^* \quad (a + b)^*cc^* \quad a^*b(a + c)^* \\
    a^*(b + c)(a + b)^* & \quad a^*b(b + c)^* \quad (a + b)^*c(b + c)^* \quad a^*(b + c)(b + c)^*
\end{align*}
\]

and polynomial time solvable in all other cases.

For \(|P| = 3\) and \(|\Sigma| = 2\), the following is known: In \([21]\) it has been shown that \((ab^*a)\)-\textsc{Constr-Sync} is \(\text{NP}\)-complete for general input automata. In \([21]\) Theorem 33 it was shown that \((b(aa + ba)^*)\)-\textsc{Constr-Sync} is \(\text{PSPACE}\)-complete for general input automata. Further, it can be shown that for the following constraint languages the constrained problem is \(\text{PSPACE}\)-complete: \(b^*a(a + ba)^*, \ a(b + ab)^* + b(bb^*a)^*, \) which are all acceptable by a 3-state PDFA over a binary alphabet. So, with Theorem \(2.3\) these are the smallest possible constraint automata over a binary alphabet giving \(\text{PSPACE}\)-complete problems.

For an overview of the results for different classes of input automata, see Table 1.

### Table 1

| Input Aut. Type       | Complexity Class | Hardness          | Reference |
|-----------------------|------------------|-------------------|-----------|
| General Automata      | \(\text{PSPACE}\) | \(\text{PSPACE}\)-hard for \(a(b + c)^*\) | \([21]\) |
| With Sink State       | \(\text{PSPACE}\) | \(\text{PSPACE}\)-hard for \(a(b + c)^*\) | \([21]\) |
| Weakly Acyclic        | \(\text{NP}\)    | \(\text{NP}\)-hard for \(a(b + c)^*\) | \([20]\) |
| Simple Idempotents    | \(\text{P} \text{ for } |\Sigma| = 2, |P| \leq 3\) | -         | Theorem \([13]\) |
| Commutative           | \(\text{P}\)     | -                 | Theorem \([14]\) |

**Synchronizing Commutative Semi-Automata under Arbitrary Regular Constraints**

Here, we show that for commutative input automata and an arbitrary regular constraint language, the constrained synchronization problem is always solvable in polynomial time.

Subsection 3.1 is concerned with the set of synchronizing words for commutative semi-automata. The main result of this subsection is Proposition 3.3 stating that the set of synchronizing words for a commutative automaton can be represented by a weakly acyclic and commutative automaton computable in polynomial time. Then, in Subsection 3.2 we show an auxiliary result that is also of independent interest, namely that, given \(m\) weakly
acyclic and commutative automata, the set of words accepted by them all is recognizable by a weakly acyclic automaton computable in polynomial time for a fixed alphabet. Then, in Subsection 3.3 we combine all these results to show Theorem 3.10 the statement that for commutative input automata the constrained synchronization problem is tractable.

3.1 The Structure of the Set of Synchronizing Words for Commutative Automata

In commutative semi-automata, a synchronizing state must be a sink state, a property not true for general semi-automata.

Lemma 3.1 (22). Let \( A = (\Sigma, Q, \delta) \) be a commutative semi-automaton. Then, the synchronizing state must be a sink state (and hence is unique).

The next lemma is important in the proof of Proposition 3.3.

Lemma 3.2. Let \( A = (\Sigma, Q, \delta) \) be a commutative semi-automaton, \( S \subseteq Q \) a strongly connected component and \( u \in \Sigma^* \). Then the states in \( \delta(S, u) \) are pairwise connected.

For commutative automata, the set of synchronizing words is represented by a weakly acyclic automata that is constructed out of the strongly connected components and computable in polynomial time.

Proposition 3.3. Let \( A = (\Sigma, Q, \delta) \) be a synchronizing commutative semi-automaton with \( n \) states. Then, there exists a weakly acyclic commutative semi-automaton with at most \( n \) states and the same set of synchronizing words computable in polynomial time for a fixed alphabet.

Proof. Define an equivalence relation on \( Q \) by setting two states \( q, q' \in Q \) to be equivalent \( q \sim q' \) iff they are contained in the same strongly connected component. By Lemma 3.2 this is in fact a congruence relation, i.e., if \( q \sim q' \), then \( \delta(q, u) \sim \delta(q', u) \) for every \( u \in \Sigma^* \).

Let \( q_1, \ldots, q_m \in Q \) be a transversal of the equivalence classes, i.e., we pick precisely one element from each class. Define \( C = (\Sigma, S, \mu) \) with \( S = \{ q_1, \ldots, q_m \} \) and \( \mu(q_j, x) = q_j \) iff \( \delta(q_j, x) \sim q_j \). As we have a congruence relation, we get the same automaton for every choice of transversal \( q_1, \ldots, q_m \) (in fact, we only note in passing that \( C \) is a homomorphic image of \( A \) and corresponds to quotiening the automaton by the introduced congruence relation).

Next, we argue that the sets of synchronizing words of both automata coincide.

Let \( s_f \) be the synchronizing state of \( A \). By Lemma 3.1 it is a sink state, and so \( \{ s_f \} \) is a strongly connected component and we can deduce \( s_f \in S \). Suppose \( u \in \Sigma^* \) is a synchronizing word of \( A \). So, for each state \( q_i \) from the transversal, we have \( \delta(q_i, u) = s_f \), which implies, as, inductively, for every prefix \( v \) of \( u \) we have \( \delta(q_i, v) \sim \mu(q_i, v) \), that \( \mu(q_i, u) = s_f \) and \( u \) synchronizes \( C \). Conversely, suppose \( u \in \Sigma^* \) synchronizes \( C \) and let \( q \in Q \). Then \( q \sim q_i \) for some \( q_i \in S \). Again, for every prefix \( v \) of \( u \), we have \( \delta(q, v) \sim \mu(q, v) \), as is easy to see by the definition of \( C \) and as \( \sim \) is a congruence relation. So, \( \delta(q, u) \sim s_f \), which implies, as \( \{ s_f \} \) is a strongly connected component, that \( \delta(q, u) = s_f \).

Remark 4. See Figure 1 for a synchronizing commutative automaton and a weakly acyclic automaton constructed as in the proof of Proposition 3.3 having the same set of synchronizing words. Note that the construction can actually be performed for any commutative automaton, even a non-synchronizing one as shown in Figure 2.

With the method of proof from Proposition 3.3, we can slightly improve the running time stated in Theorem 2.1 for commutative input semi-automata.
Figure 1 A synchronizing commutative automaton and the weakly acyclic automaton from the proof of Proposition 3.3 with the same set of synchronizing words. A shortest synchronizing word is \(baa\). The strongly connected components in the original automaton have been framed by red boxes.

Figure 2 A commutative automaton that is not synchronizing.

> **Corollary 3.5.** Let \(A = (\Sigma, Q, \delta)\) be a commutative semi-automaton. Then it can be decided in time \(O(|Q| + |\Sigma||Q|)\) if \(A\) is synchronizing.

Let us note the following consequence of Lemma 3.9 and the bound \(n - 1\) for a shortest synchronizing word in weakly acyclic automata [46, Proposition 1] and a bound for the shortest synchronizing word with respect to a constraint [29, Proposition 7]. This gives an alternative proof of the tight bound \(n - 1\) for commutative automata from [22, 43].

> **Corollary 3.6.** If \(A = (\Sigma, Q, \delta)\) is a synchronizing commutative semi-automaton with \(n\) states, then there exist a synchronizing word of length at most \(n - 1\) and, for any constraint PDFA \(B\), there exists a synchronizing word in \(L(B)\) of length at most \(|P|\binom{n}{2}\).

**Proof.** By Lemma 3.9 the set of synchronizing words equals the set of synchronizing words of a weakly acyclic automaton, for which we have the known bounds.

In fact, with a little more work and generalizing Lemma 3.9 using [10, Proposition 1], we can show the stronger statement that for commutative \(A\), if there exists a word \(w \in \Sigma^*\) with \(|\delta(Q, w)| = r\), then there exists one of length at most \(n - r\).
3.2 Recognizing the Intersection of Weakly Acyclic Commutative Automata

- Proposition 3.7. Let \( A_i = (\Sigma, Q, \delta, q_i, F_i) \), \( i \in \{1, \ldots, m\} \), be weakly acyclic and commutative automata with at most \( n \) states. Then, \( \bigcap_{i=1}^{m} L(A_i) \) is recognizable by a weakly acyclic commutative automaton of size \( n^{[2]} \) computable in polynomial-time for a fixed alphabet.

Proof. Let \( \Sigma = \{a_1, \ldots, a_k\} \). Define the threshold counting function \( \psi_n : \Sigma^* \to \{0, 1, \ldots, n-1\}^k \) by \( \psi_n(u) = (\min\{n-1, |u|_{a_1}\}, \ldots, \min\{n-1, |u|_{a_k}\}) \).

Let \( i \in \{1, \ldots, m\} \). Suppose \( u \in \Sigma^* \). If \( |u|_{a_j} \geq n \) for some \( j \in \{1, \ldots, k\} \), then \( A_i \) must traverse at least one self-loop labeled by \( a_j \) when reading \( u \), as \( A_i \) is weakly acyclic. So, by not traversing these self-loops an appropriate number of times, we find a word \( u' \in \Sigma^* \) with \( |u'|_{a_j} = \min\{n-1, |u|_{a_j}\} \), which implies \( \psi_n(u) = \psi_n(u') \), and \( \delta_i(q_i, u) = \delta_i(q_i, u') \).

For \( a \in \Sigma \), set \( a^{\leq n-1} = \{\varepsilon, a, \ldots, a^{n-1}\} \). Define \( E_i = \{ \psi_n(u) \mid u \in a_1^{\leq n-1} \cdots a_k^{\leq n-1} \cap L(A_i) \} \). Let \( u \in L(A_i) \), by the above and using that \( A_i \) is commutative, we can deduce \( \psi_n(u) \in E_i \). If \( u \in \psi_n^{-1}(E_i) \), then there exists \( v = a_1^{c_1} \cdots a_k^{c_k} \) with \( c_j \in \{0, 1, \ldots, n-1\} \) for all \( j \in \{1, \ldots, k\} \) and \( \delta_i(q_i, u) = \delta_i(q_i, u') \) and \( \psi_n(u') = \psi_n(u) \).

For \( a \in \Sigma \), set \( a^{\leq n-1} = \{\varepsilon, a, \ldots, a^{n-1}\} \). Define \( E_i = \{ \psi_n(u) \mid u \in a_1^{\leq n-1} \cdots a_k^{\leq n-1} \cap L(A_i) \} \). Let \( u \in L(A_i) \), by the above and using that \( A_i \) is commutative, we can deduce \( \psi_n(u) \in E_i \). If \( u \in \psi_n^{-1}(E_i) \), then there exists \( v = a_1^{c_1} \cdots a_k^{c_k} \) with \( c_j \in \{0, 1, \ldots, n-1\} \) for all \( j \in \{1, \ldots, k\} \) and \( \delta_i(q_i, u) = \delta_i(q_i, u') \) and \( \psi_n(u') = \psi_n(u) \).

Finally, a language of the form \( \psi_n^{-1}(E_i) \) is recognizable by a weakly acyclic commutative automaton: set \( A = (\Sigma, Q, \delta, q_0, E) \) with \( Q = \{0, 1, \ldots, n-1\} \) and \( \delta((s_1, \ldots, s_k), a_j) = (s_1, \ldots, s_{j-1}, \min\{n-1, s_j+1\}, s_{j+1}, \ldots, s_k) \) and \( q_0 = (0, 0, \ldots, 0) \). It is obvious that \( A \) is commutative and \( L(A) = \psi_n^{-1}(E) \) as \( A \) essentially implements the threshold counting expressed by \( \psi_n \).

As \( a_1^{\leq n-1} \cdots a_k^{\leq n-1} \) contains \( n^k \) words, the sets \( E_i \) and their intersection can be computed in polynomial time. Also, the automaton recognizing \( \psi_n^{-1}(\bigcap_{i=1}^{m} E_i) \) is computable in polynomial time.

- Remark 8. With Proposition 3.7, we can deduce that we can decide in time \( O(n^{f(\Sigma)}) \), for some computable function \( f \), if the intersection of the languages recognized by \( m \) weakly acyclic commutative automata is non-empty. Hence, this problem is in \( \text{XP} \) for these types of automata and parameterized by the size of the alphabet (see [24] for an introduction to parameterized complexity theory). It can be shown that the languages recognized by these types of automata recognize star-free languages [7]. Note that in [7], the mentioned conclusion has been improved to an \( \text{XP} \) result for so called totally star-free non-deterministic automata recognizing commutative languages. We refer to the mentioned paper for details. Contrary, for general commutative, even unary, automata this problem is \( \text{NP} \)-complete [26], and, further, for fixed alphabet sizes, \( W[1] \)-complete with the number of input automata as parameter, see [23].

3.3 A Polynomial-Time Algorithm for the Constrained Synchronization Problem with Commutative Input Semi-Automata

- Lemma 3.9. Let \( A = (\Sigma, Q, \delta) \) be a commutative semi-automaton with \( n \) states. Then, the set of synchronizing words is recognizable by an automaton of size \( n^{[2]} \) computable in...
polynomial time for a fixed alphabet.

Proof. First, we can test in polynomial time if \( A \) is synchronizing. If not, an automaton with a single state and empty set of final states recognizes the empty set. Otherwise, by Proposition 3.3 we can compute a weakly acyclic automaton \( \mathcal{C} = (\Sigma, S, \mu) \) having the same set of synchronizing words in polynomial time. By Lemma 3.1 there exists a unique synchronizing state \( s_f \in S \). For \( q \in S \), let \( \mathcal{C}_{q_{s_f}} = (\Sigma, S, \mu, q, \{ s_f \}) \) be the automaton \( \mathcal{C} \) but with start state \( q \) and set of final states \( \{ s_f \} \). Then, the set of synchronizing words of \( \mathcal{C} \) is

\[
\bigcap_{q \in Q} L(\mathcal{C}_{q_{s_f}}).
\]

By Proposition 3.7, we can compute in polynomial time a weakly acyclic commutative automaton of size \( n^{|\Sigma|} \) recognizing the above set.

Now, combining the above results we can prove our main result concerning the constrained synchronization problem.

**Theorem 3.10.** Let \( L \subseteq \Sigma^* \) be regular. Then, for a commutative input semi-automaton \( A \) with \( n \) states, the problem if \( A \) admits a synchronizing word in \( L \) is solvable in polynomial time.

Proof. By Lemma 3.9 we can compute in polynomial time an automaton \( \mathcal{C} \) of size \( n^{|\Sigma|} \) recognizing the set of synchronizing words of \( A \). By the product automaton construction \([31]\), using an automaton for \( L \), we can compute in polynomial time a recognizing automaton for the intersection \( L(\mathcal{C}) \cap L \). Then, checking that the resulting automaton for the intersection recognizes a non-empty language can also be done in polynomial time \([31]\).

The degree of the polynomial measuring the running time depends on the alphabet size. But note that a fixed constraint automaton also fixes the alphabet, hence this parameter is not allowed to vary in the input semi-automata.

4 Synchronizing Automata with Simple Idempotents over a Binary Alphabet

Here, we show that for input automata with simple idempotents over a binary alphabet and a constraint given by a PDFA with at most three states, the constrained synchronization problem is always solvable in polynomial time. Note that, as written at the end of Section 2, the smallest constraint languages giving \( \text{PSPACE} \)-complete problems are given by 3-state automata over a binary alphabet. But, as shown here, if we only allow automata with simple idempotents as input, the problem remains tractable in these cases.

Intuitively, by applying an idempotent letter, we can map at most two states to a single state. Hence, to synchronize an \( n \)-state automaton with simple idempotents, we have to apply at least \( n - 1 \) times an idempotent letter. This is the content of the next lemma.

**Lemma 4.1.** Let \( A = (\Sigma, Q, \delta) \) be a semi-automaton with \( n > 0 \) states with simple idempotent and let \( \Gamma \subseteq \Sigma \) be the set of all idempotent letters that are not permutational letters. Suppose \( w \in \Sigma^* \) is a synchronizing word for \( A \). Then, \( \sum_{\alpha \in \Gamma} |w|_\alpha \geq n - 1 \).

The following was shown in \([35]\, Proposition 6.2]\).

**Proposition 4.2.** Let \( \Sigma = \{a, b\} \) be a binary alphabet and \( A = (\Sigma, Q, \delta) \) be an \( n \)-state automaton with simple idempotents. Suppose \( A \) is synchronizing and \( n > 3 \). Then, up to renaming of the letters, we have only two cases for \( A \):
1. There exists a sink state \( t \in Q \), the letter \( b \) permutes the states in \( Q \setminus \{t\} \) in a single cycle and \( |\delta(Q, a)| = n - 1 \) with \( t \in \delta(Q, \{t\}, a) \).
2. The letter \( b \) permutes the states in \( Q \) in a single cycle and there exists \( 0 < p < n \) coprime to \( n \) and \( s, t \in Q \) such that \( t = \delta(s, a) = \delta(s, b^p) \).

![Figure 3](image_url) The two cases from Proposition 4.2. In the second case, for \( p = 1 \) with the notation from the statement, we get the automata from the Černy family [10], a family of automata giving the lower bound \((n - 1)^2\) for the length of a shortest synchronizing word.

Note that the set of synchronizing words can be rather complicated in both cases. For example, the language \( \Sigma^*a^n(b((ba^*)^{n-1})*a^n)^{n-2}\Sigma^* \) contains only synchronizing words for automata of the first type in Proposition 4.2 and, similarly, \( \Sigma^*a^n((ba^*)^{n+p}(ba^*)^{n+1})^{n-2}\Sigma^* \) in the second case. However, for example, the language \( (bbba)^*bb(bbba)^*bb(bbba)^* \) contains synchronizing words and non-synchronizing words for automata of the first type.

Finally, for binary automata with simple idempotents and an at most 3-state constraint PDFA, the constrained synchronization problem is always solvable in polynomial time. The proof works by case analysis on the possible sequences of words in the constraint language and if it is possible to synchronize the two automata types listed in Proposition 4.2 with those sequences.

**Theorem 4.3.** Let \( B = (\Sigma, P, \mu, p_0, F) \) be a constraint automaton with \( |P| \leq 3 \) and \( |\Sigma| \leq 2 \). Let \( A = (\Sigma, Q, \delta) \) be an input semi-automaton with simple idempotents. Then, deciding if \( A \) has a synchronizing word in \( L(B) \) can be done in polynomial time.

**Proof sketch.** The cases \( |\Sigma| \leq 1 \) or \( |P| \leq 2 \) and \( |\Sigma| = 2 \) are polynomial time decidable in general as shown in [21] Corollary 9 & Theorem 24. So, we can suppose \( \Sigma = \{a, b\} \) and \( P = \{1, 2, 3\} \) with \( p_0 = 1 \). As in [21], we set \( \Sigma_{i,j} = \{x \in \Sigma : \mu(i, x) = j\} \). If the cases in Proposition 4.2 apply, and if so, which case, can be checked in polynomial time, as we only have to check that one letter is a simple idempotent and the other letter permutes the states in a single cycle or two cycles with the restrictions as written in Proposition 4.2.

So, we can assume \( A \) has one of the two forms as written in Proposition 4.2. Without loss of generality, we assume \( a \) is the idempotent letter and \( b \) the cyclic permutation of the states. Set \( n = |Q| \). We also assume \( n > 4 \), if \( n \leq 4 \), then the cases for \( A \) can be checked in constant time for a given (fixed) constraint language \( L(B) \).

Next, we only handle the first case of Proposition 4.2 by case analysis. The other case can be handled similarly. Further, let \( t \in Q \) be the state as written in the first case of Proposition 4.2.

Further, in this sketch, we only handle the case that the strongly connected components of \( B \) are \{1\} and \{2, 3\} and the subautomaton between the states \{2, 3\} is one of the automata
listed in Table 4.3. These are the most difficult cases, for the remaining cases of B we refer to the full proof in the appendix.

Next, we handle each of these cases separately. We show that for each case, either the input automaton with the assumed form is synchronizing (and hence the problem is only to check that A has the stated form, which can be done in polynomial time as said in the beginning of this proof), or we have another simple to check condition, like if n is even or odd.

| Case | $|\Sigma_{2,2} + \Sigma_{2,3}| = 2$ | Case | $|\Sigma_{3,3} + \Sigma_{3,2}| = 2$ |
|------|---------------------------------|------|---------------------------------|
| 1    | ![Diagram A]                     | 2    | ![Diagram B]                     |
| 3    | ![Diagram C]                     | 4    | ![Diagram D]                     |
| 5    | ![Diagram E]                     | 6    | ![Diagram F]                     |
| 7    | ![Diagram G]                     | 8    | ![Diagram H]                     |
| 9    | ![Diagram I]                     | 10   | ![Diagram J]                     |
| 11   | ![Diagram K]                     | 12   | ![Diagram L]                     |

**Table 2** Cases for a partial subautomaton between the states \{2, 3\} that is not complete. See the proof of Theorem 4.3 for details.

We handle the cases as numbered in Table 4.3:

1. In this case $L(B_{2,(2)}) = (a + bb)^*$. Let $s \in Q \setminus \{t\}$ be the state with $\delta(s, a) = \delta(t, a) = t$. If $n$ is odd, then $|Q \setminus \{t\}|$ is even and the single cycle induced by $b$ on the states in $Q \setminus \{t\}$ splits into two cycles for the word $bb$, i.e., we have precisely two disjoint subsets $A, B \subseteq Q \setminus \{t\}$ of equal size such that the states in one subset can be mapped onto each other by a word in $(bb)^*$ but we cannot map states between those subset by a word from $(bb)^*$. Suppose, without loss of generality, that $s \in A$. Then for each $q \in A$ we have $\delta(q, a) = q$ and $\delta(q, bb) \in A$. Hence, we cannot map a state from $A$ to $s$, and so not to $t$, the unique synchronizing state. As $n \geq 5$ and for every $w \in \Sigma_{1,1}^* \Sigma_{1,2}$ we have $|\delta(Q, w)| \geq n - 1$, we must have $\delta(A, w) \neq \emptyset$ for every $w \in \Sigma_{1,1}^* \Sigma_{1,2}$. So, $A$ cannot be synchronized by a word from $L(B_{1,(2)}) = \Sigma_{1,1}^* \Sigma_{1,2}(a + bb)^*$. If $n$ is even, then $bb$ also permutes the states in $Q \setminus \{t\}$ in a single cycle and the word $(ba)^{n-2}$ synchronizes $A$. So, picking any $u \in \{a, b\}^*$ with $\mu(1, u) = 2$, the word $u(ba)^{n-1} \in L(B)$ synchronizes $A$.

2. In this case $L(B_{2,(2)}) = (aa*bb)^*$. The word $(ab)^{n-1}$ synchronizes $A$. Pick any $u \in L(B_{1,(2)})$, then $u(ab)^{n-1} \in L(B)$ synchronizes $A$.

3. In this case $L(B_{2,(2)}) = (a + ba)^*$. For every $q \neq t$ we have $\delta(q, ba) = \delta(q, b)$ and the word $(ba)^{n-2}$ synchronizes $A$. Let $u \in \{a, b\}^*$ be a word with $\mu(1, u) = 2$, then $\mu(1, ua(ba)^{n-2}) = 2$ synchronizes $A$.

4. In this case $L(B_{2,(2)}) = (ab*ba)^*$. The word $(ab)^{n-2}$ synchronizes $A$ (as $a$ is idempotent, it has the same effect as the synchronizing word $a(ba)^{n-2}$ on the states of $A$). Pick any $u \in L(B_{1,(2)})$, then $u(ba)^{n-2} \in L(B)$ synchronizes $A$.

5. In this case $L(B_{2,(2)}) = (b + ab)^*$. The word $(ab)^{n-1} = a(ba)^{n-2}b$ synchronizes $A$, as $a(ba)^{n-2}$ synchronizes $A$. Pick any $u \in L(B_{1,(2)})$, then $u(ba)^{n-2} \in L(B)$ synchronizes $A$. 

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6. In this case $L(B_{2, (2)} t) = (ba^* b)^*$. As $a$ is idempotent, the words $bab$ and $ba^i b$ for $i > 1$ all have the same effect on the states of $A$. So, only the language $(bb, bab)^* \subseteq L(B_{2, (2)} t)$ is of relevance for the question of synchronizability of $A$ with respect to the constraint language $L(B)$. We have $|\delta(Q, bab)| = n - 1$ and if $n$ is odd, we can argue as in Case [1] that $A$ cannot be synchronized by a word from $L(B_{1, (2)} t)$. If $n$ is even, the word $bb$ induces a single cycle on the states in $Q \setminus \{t\}$ and $(bbab)^{n-1}$ synchronizes $A$. As in previous cases, by appending a suitable prefix, we can construct a synchronizing word in $L(B)$.

7. In this case $L(B_{2, (2)} t) = (b + aa)^*$. Then, choosing any $u \in L(B_{1, (2)} t)$, the word $u(bau)^{n-1}$ synchronizes $A$.

8. In this case $L(B_{2, (2)} t) = (bb^* a)^*$. As $(ba)^{n-1} \in L(B_{2, (2)} t)$ synchronizes $A$ we can pick any $u \in \{a, b\}^*$ such that $\mu(1, u) = 2$ and have the synchronizing word $u(ba)^{n-1} \in L(B)$ for $A$.

9. Here $L(B_{2, (2)} t) = (aa + ba)^*$. Using that $aa(ba)^{n-2} \in L(B_{2, (2)} t)$ synchronizes $A$ as in the previous cases by appending a suitable prefix from $L(B_{1, (2)} t)$.

10. Here $L(B_{2, (2)} t) = (aa + ab)^*$. Using that $aa(ab)^{n-2}aa \in L(B_{2, (2)} t)$ synchronizes $A$ as in the previous cases by appending a suitable prefix from $L(B_{1, (2)} t)$.

11. Here $L(B_{2, (2)} t) = (ab + bb)^*$. Using that $a(ba)^{n-2}b = (ab)^{n-1} \in L(B_{2, (2)} t)$ synchronizes $A$ as in the previous cases by appending a suitable prefix from $L(B_{1, (2)} t)$.

12. Here $L(B_{2, (2)} t) = (ba + bb)^*$. Using that $ba(ba)^{n-2} \in L(B_{2, (2)} t)$ synchronizes $A$ as in the previous cases by appending a suitable prefix from $L(B_{1, (2)} t)$.

5 Conclusion

Here, we have shown that for commutative input automata, the constrained synchronization problem is tractable for every regular constraint language. Hence, this case is the first case of input automata where the question of the computational complexity is settled completely. In [30], it was shown that for weakly acyclic automata, the problem is always in $NP$, and there exists NP-hard instances and tractable instance of the problem. However, it was not shown that these are the only complexities that can arise.

Note that, on the other side, for commutative regular constraint languages, we can realize PSPACE-complete problems (for example for $\{a, b\}^* c(a, b)^*$), NP-complete (for $a^* ba^* ba^*$) or problems in $P$. In fact, a full classification giving a trichotomy that only these three complexities arise has been shown for commutative regular constraints [28].

Furthermore, in the present work, we have started the investigation of the constrained synchronization problem for input automata with simple idempotents. Contrary to the results for commutative automata, which are conclusive, in the case of automata with simple idempotents, the present results are only the starting point. They entail the first non-trivial instances of these automata – for example the automata from the Černý family [11, 53] giving a lower bound for the length of a shortest synchronizing word are binary automata with simple idempotents – but also the first instances of constraint automata that can realize PSPACE-complete or NP-complete problems in the general case [21]. In this respect, it is remarkable that the complexity drops to being polynomial time solvable in this case. However, it is unclear what happens for larger alphabets and different constraint languages when only automata with simple idempotents are considered as input.

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Proofs for Section 3 (Synchronizing Commutative Semi-Automata under Arbitrary Regular Constraints)

**Lemma 3.2** Let $A = (\Sigma, Q, \delta)$ be a commutative semi-automaton, $S \subseteq Q$ a strongly connected component and $u \in \Sigma^*$. Then the states in $\delta(S, u)$ are pairwise connected.

**Proof.** Let $t_1, t_2 \in \delta(S, u)$. Then $t_1 = \delta(s_1, u)$ and $t_2 = \delta(s_2, u)$ for some $s_1, s_2 \in S$. As $S$ is a strongly connected component, we find $v, w \in \Sigma^*$ such that $s_1 = \delta(s_2, v)$ and $s_2 = \delta(s_1, w)$. But then, by commutativity, $t_1 = \delta(s_1, u) = \delta(\delta(s_2, v), u) = \delta(s_2, vu) = \delta(s_2, uv) = \delta(\delta(s_2, u), v)$ and similarly $t_2 = \delta(t_1, v)$. ◀

**Corollary A.1.** Let $A = (\Sigma, Q, \delta)$ be a commutative semi-automaton, $S, T \subseteq Q$ be two (not necessarily distinct) strongly connected components and $u \in \Sigma$. Then either $\delta(S, u) \cap T = \emptyset$ or $\delta(S, u) \subseteq T$.

**Proof.** If $\delta(S, u) \cap T \neq \emptyset$, then, by Lemma 3.2 and the maximality of $T$, $\delta(S, u) \subseteq T$. ◀

**Corollary 3.5.** Let $A = (\Sigma, Q, \delta)$ be a commutative semi-automaton. Then it can be decided in time $O(|Q| + |\Sigma||Q|)$ if $A$ is synchronizing.

**Proof.** For a given directed acyclic graph with $n$ nodes and $m$ edges, a topological sorting can be performed in time $O(n + m)$, see [CLRS09]. Hence, if we consider the directed acyclic graph resulting from the strongly connected components [CLRS09] of the automaton graph of $A$, we can topologically sort it in time $O(|Q| + |\Sigma||Q|)$, as we have at most $|Q|$ strongly connected components and $|\Sigma||Q|$ many edges among them, as this is an upper bound for the edges $A$, as each state contributes $|\Sigma|$ many edges. The strongly connected components themselves can also be computed in time $O(|Q| + |\Sigma||Q|)$ with a similar argument, see [CLRS09].

Another argument computes first the strongly connected components, from which the automaton $C$ from the proof of Proposition 3.3 can be derived, and then uses the fact [Hof21, Corollary 6] that for weakly acyclic automata we can decide synchronizability in $O(m + |\Sigma|m)$, where $m$ denotes the number of strongly connected components of $A$. ◀

**Remark 2.** In the proof of Lemma 3.9, as $C = (\Sigma, S, \mu)$ is weakly acyclic, we can show that

$$\bigcap_{q \in Q} L(C_{q, (s_I)}) = \bigcap_{q \in Q} \bigcap_{r \in Q \setminus \{q\}, \forall u \in \Sigma^*, \mu(p, u) \neq q} L(C_{q, (s_I)}),$$

which improves the running time of the algorithm. In terms of the partial order given on the states by the reachability relation, this is implied as every letter either moves to a larger state or induces a self-loop, which implies that we can take the above intersection over the minimal states with respect to the partial order.

In this section, we take a closer look at the structure of commutative semi-automata which admit a synchronizing word. We show that the connectivity of a strongly connected component is preserved under the action of a letter, that a letter either permutes the states of some component, or maps every state into another component, and lastly that a letter, once it permutes a strongly connected component, it must also permute every component reachable from the one under consideration. Intuitively, this means a letter either moves forward between the components, or it has to stop. Then, we use these properties to give a simple criterion for general synchronizability of a commutative semi-automaton.
Lemma A.3. Let $A = (\Sigma, Q, \delta)$ be a commutative semi-automaton and $S \subseteq Q$ be a strongly connected component. Then, each letter $x \in \Sigma$ either permutes all states in $S$, or maps every state in $S$ to some state outside of $S$, i.e., we either have $\delta(S, x) = S$ or $\delta(S, x) \cap S = \emptyset$.

Proof. Suppose $\delta(S, x) \cap S \neq \emptyset$. Let $t \in S$. As $S$ is a strongly connected component, we find $u \in \Sigma^*$ such that $t \in \delta(t) \cap S, u \subseteq \delta(S, u)$. Hence, $\delta(S, u) \cap S \neq \emptyset$. So, by Corollary A.1 $\delta(S, u) \subseteq S$. By commutativity, $t \in \delta(S, xu) = \delta(S, u) \subseteq \delta(S, x)$. Hence, $S \subseteq \delta(S, x)$. Lastly, we can either note, as $S$ is finite and $|\delta(S, x)| \leq |S|$, that the previous inclusion implies $S = \delta(S, x)$, or we can reason as before, i.e., using Corollary A.1 to deduce $\delta(S, x) \subseteq S$. Hence $S = \delta(S, x)$, i.e., $x$ permutes the states in $S$.

Lemma A.4. Let $A = (\Sigma, Q, \delta)$ be a commutative semi-automaton, $S \subseteq Q$ be a strongly connected component and $u, v \in \Sigma^*$. If $\delta(S, uv) \subseteq \delta(S, u)$, then $\delta(S, uv) = \delta(S, u)$.

Proof. By Lemma A.3, we find a strongly connected component $T \subseteq Q$ containing $\delta(S, u)$. Then, with the assumption, $\delta(T, v) \cap T \neq \emptyset$. But this implies $\delta(T, u) = T$, as by Lemma A.3 if we write $v = x_1 \cdots x_n$ with $x_i \in \Sigma$, $i \in \{1, \ldots, n\}$, the first letter $x_1$ has to permute $T$, for if it does not, it will map inside another component. But then, as the strongly connected components form an acyclic graph, it can never come back to $T$. So, $\delta(T, x_1x_2 \cdots x_n) = \delta(T, x_2 \cdots x_n)$ and with a similar argument, $x_2$ has to permute $T$ and so on. Hence $v$ is a concatenation of permutations on $T$, hence permutes $T$ itself. But then, $v$ acts injective on the subset $\delta(S, u) \subseteq T$, and so $\delta(S, uv) \subseteq \delta(S, u)$ implies $\delta(S, uv) = \delta(S, u)$.

Lemma A.5. Let $A = (\Sigma, Q, \delta)$ be a commutative semi-automaton and $S, T \subseteq Q$ be two strongly connected components. Let $x \in \Sigma$ and suppose we have $u \in (\Sigma\setminus\{x\})^*$ such that $\delta(S, x) = S$ and $\delta(S, u) \subseteq T$. Then $\delta(T, x) = T$.

Proof. We have $\delta(S, u) = \delta(S, xu) = \delta(S, xu) \subseteq \delta(T, x)$. So, $T \cap \delta(T, x) \neq \emptyset$, which implies, by Lemma A.3 that $\delta(T, x) = T$.

By Lemma A.3, Lemma A.3 and Lemma A.5 we already know much about the structure of commutative semi-automata. We know that any letter either maps completely outside of any strongly connected component or permutes the states within this component, and if it maps outside, all target states lie within a common strongly connected component. Also, we know, if we topologically sort the strongly connected components, that along each path, either a letter always maps everything to some strongly connected component strictly greater in the linear ordering given by the topological sorting, or it permutes the states in this components, but then it will also do in every component reachable from the one under consideration. Examples of commutative semi-automata are presented in Figures 1.

Next, we give an easy criterion when a commutative semi-automaton is synchronizing.

Proposition A.6. Let $A = (\Sigma, Q, \delta)$ be a commutative semi-automaton and $S_1, \ldots, S_n \subseteq Q$ be any topological sorting of the strongly connected components of $A$. Then $A$ is synchronizing if and only if $S_n$ is a singleton set and reachable from every other component.

Proof. Assume $A$ is synchronizing with synchronizing state $s_f$. As $s_f$ is reachable from every state and a sink state by Lemma A.3 we must have $S_n = \{s_f\}$ and $S_n$ is reachable from every other component. Conversely, assume $S_n = \{s_f\}$ and we have words $u_1, \ldots, u_{n-1}$ such that $\delta(S_i, u_i) = \{s_f\}$ for $i \in \{1, \ldots, n-1\}$. As $S_n$ is the last component in a topological sorting, $\delta(S_n, x) \subseteq S_n$ for any $x \in \Sigma$. So, $s_f$ is a sink state. Set $u = u_1 \cdots u_{n-1}$. Then, for
any \( S_i, i \in \{1, \ldots, n-1\} \), we have, by commutativity and as \( s_f \) is a sink state,
\[
\delta(S_i, u) = \delta(S_i, u_1 \cdots u_{n-1}) \\
= \delta(S_i, u_i u_1 \cdots u_{i-1} u_{i+1} \cdots u_{n-1}) \\
= \delta(\delta(S_i, u_i), u_1 \cdots u_{i-1} u_{i+1} \cdots u_{n-1}) \\
= \delta(\{s_f\}, u_1 \cdots u_{i-1} u_{i+1} \cdots u_{n-1}) \\
= \{s_f\}.
\]
So \( u \) is a synchronizing word and \( s_f \) a synchronizing state.

- Remark 7. The structure results derived here generalize the structure results for the extremal commutative semi-automata considered in [FH19], i.e., those for which a shortest synchronizing word has maximal length.

B Proofs for Section 4 (Synchronizing Automata with Simple Idempotents over a Binary Alphabet)

- Lemma 4.1. Let \( A = (\Sigma, Q, \delta) \) be a semi-automaton with \( n > 0 \) states with simple idempotent and let \( \Gamma \subseteq \Sigma \) be the set of all idempotent letters that are not permutational letters. Suppose \( w \in \Sigma^* \) is a synchronizing word for \( A \). Then, \( \sum_{a \in \Gamma} |w|_a \geq n - 1 \).

**Proof.** As every letter \( a \in \Gamma \) maps at most two states to a single state in each application, we have \( |S| - 1 \leq |\delta(S, a)| \leq |S| \) for each \( S \subseteq Q \). All other letters permute the states, hence the cardinality of each subset is invariant for them, i.e., \( a \notin \Gamma \) implies \( |S| = |\delta(S, a)| \) for \( S \subseteq Q \). This yields, inductively, \( |S| - \sum_{a \in \Gamma} |w|_a \leq |\delta(S, w)| \) for \( w \in \Sigma^* \). Hence, if \( |\delta(Q, w)| = 1 \) we must have \( \sum_{a \in \Gamma} |w|_a \geq n - 1 \).

The reader might notice, upon comparing [Mar09, Proposition 6.2] with Proposition 4.2, that the parameter \( p \) is actually different. Here, it denotes the number of times we have to apply \( b \) to reach the other state, in [Mar09, Proposition 6.2], by \( p \) a state is denoted for which this relation does not hold true. The change was made because there seems to be a (minor, the result still holds true of course) glitch in [Mar09, Proposition 6.2]. For example, the author writes for “[…] For \( q_2 = n \), the automaton \( A \) is the Černý automaton […]”, but \( \gcd(n, n) = n \), which would imply, according to the following conclusion in [Mar09, Proposition 6.2], that then the automaton is not synchronizing for \( n > 1 \), which is not the case. Or, to give a more concrete example, an automaton with four states \( \{1, 2, 3, 4\} \) and the naming as in [Mar09, Proposition 6.2] and \( q_2 = 3 \) is not synchronizing, but 4 and 3 are coprime.

- Theorem 4.3. Let \( B = (\Sigma, P, \mu, p_0, F) \) be a constraint automaton with \( |P| \leq 3 \) and \( |\Sigma| \leq 2 \). Let \( A = (\Sigma, Q, \delta) \) be an input semi-automaton with simple idempotents. Then, deciding if \( A \) has a synchronizing word in \( L(B) \) can be done in polynomial time.

**Proof.** The cases \( |\Sigma| \leq 1 \) or \(|P| \leq 2 \) and \( |\Sigma| = 2 \) are polynomial time decidable in general as shown in [FGH+19, Corollary 9 & Theorem 24]. So, we can suppose \( \Sigma = \{a, b\} \) and \( P = \{1, 2, 3\} \) with \( p_0 = 1 \). As in [FGH+19], we set \( \Sigma_{i,j} = \{x \in \Sigma : \mu(i, x) = j\} \). If the cases in Proposition 4.2 apply, and if so, which case, can be checked in polynomial time, as we only have to check that one letter is a simple idempotent and the other letter permutes the states in a single cycle or two cycles with the restrictions as written in Proposition 4.2.

So, we can assume \( A \) has one of the two forms as written in Proposition 4.2. Without loss of generality, we assume \( a \) is the idempotent letter and \( b \) the cyclic permutation of the
states. Set \( n = |Q| \). We also assume \( n > 4 \), if \( n \leq 4 \), then the cases for \( \mathcal{A} \) can be checked in constant time for a given (fixed) constraint language \( L(\mathcal{B}) \).

Next, we only handle the first case of Proposition 4.2 by case analysis. The other case can be handled similarly. Further, let \( t \in Q \) be the state as written in the first case of Proposition 4.2.

We can assume every state in \( P \) is reachable from \( p_0 \) and also that at least one state in \( \{2, 3\} \) is reachable from the other state in this set (otherwise, it is a union of two constraint languages over two-state constraint automata, each giving a constraint problem in \( P \), and so, also their union gives a problem in \( P \), as we can check each constraint individually, see also [FGH+19, Lemma 13]). Without loss of generality, we assume \( \Sigma_{2,3} \neq \emptyset \). If the states in \( P \) form a strongly connected component, then in \( \mathcal{B} \) we can map every state back to the starting state and by [FGH+19, Theorem 17] the constrained problem is solvable in polynomial time. If every state forms its own strongly connected component, we can assume \( 3 \in F \) (otherwise, it reduces to the two-state case). As every state is reachable from \( p_0 \), we then have \( \Sigma_{2,1} = \Sigma_{3,1} = \emptyset \). Then

\[
L(\mathcal{B}) = \Sigma_{1,1}^* \Sigma_{1,2} \Sigma_{2,2}^* \Sigma_{2,3} \Sigma_{3,3}^* \cup \Sigma_{1,1}^* \Sigma_{1,3} \Sigma_{3,3}^*.
\]

and either \( \Sigma_{1,2} \cdot \Sigma_{2,3} \neq \emptyset \) or \( \Sigma_{1,3} \neq \emptyset \). If \( \Sigma_{3,3} = \{a, b\} \), then \( \mathcal{A} \) has a synchronizing word in \( L(\mathcal{B}) \) iff it has a synchronizing word at all. For if we take an arbitrary synchronizing word \( u \), we can take any word \( v \) with \( \mu(1, v) = 3 \) and then \( uv \) is synchronizing for \( \mathcal{A} \) (this is actually a special case of [FGH+19, Theorem 15]). As the unconstrained synchronization problem is polynomial time solvable, in the case \( \Sigma_{3,3} = \{a, b\} \) the constrained problem is also polynomial time solvable. As \( \mathcal{B} \) is deterministic, we can furthermore assume \( |\Sigma_{i,1}| \leq 1 \) for all \( i \in \{1, 2, 3\} \) (if, for example \( |\Sigma_{2,2}| = 2 \), this implies \( \Sigma_{2,3} = \emptyset \), which reduces again to the two-state case). However, \( a \) is idempotent, and so, if, for example \( \Sigma_{2,2} = \{a\} \), which implies \( a \not\in \Sigma_{2,3} \), we can suppose the corresponding self-loop in \( \mathcal{B} \) is traversed at most two times.

Considering all such cases, we can deduce that every synchronizing word has the same effect as a synchronizing word where the number of \( a \)'s in it is bounded by three. But as \( n > 4 \), by Lemma 4.1 every synchronizing word \( w \) has to fulfill \( |w| \geq 4 \) and so, for these constraint languages every input semi-automaton \( \mathcal{A} \) with at least five states has no synchronizing word in \( L(\mathcal{B}) \).

So, from now on we can assume that we have one strongly connected component with precisely two states and one with a single state. We handle the case that \( \{2, 3\} \) is a strongly connected component here, the other case that \( \{1, 2\} \) is a strongly connected component can be handled similarly. As every state is reachable from \( p_0 \), under this assumption we must have \( \Sigma_{2,1} = \Sigma_{3,1} = \emptyset \). As either \( \Sigma_{1,2} \neq \emptyset \) or \( \Sigma_{1,3} \neq \emptyset \), we must have \( |\Sigma_{1,1}| \leq 1 \). Note that \( F \cup \{2, 3\} \neq \emptyset \), for otherwise the constrained problem is equivalent to the problem over a one-state automaton, which is in \( P \) (see [FGH+19, Corollary 9]). Further, we can assume \( F = \{2\} \), as if \( \mu(p_0, u) \in F \), by the assumptions there exists a word \( w \) such that \( \mu(p_0, uw) = 2 \) and conversely, if \( \mu(p_0, u) = 2 \), there exists a word \( w \) such that \( \mu(p_0, uw) \in F \) as \( \{2, 3\} \cap F \neq \emptyset \) and both states are reachable from \( p_0 \). As we can append arbitrary words to a synchronizing word and still have a synchronizing word, asking for a synchronizing word in \( L(\mathcal{B}) \) under the stated assumptions is equivalent as asking for a synchronizing word with final state set \( \{2\} \).

We can focus only on the case \( \Sigma_{1,3} = \emptyset \), as showing that the problem is in \( P \) in this case, the case \( \Sigma_{1,3} \neq \emptyset \) can be handled symmetrically under the assumption \( \Sigma_{1,2} \cdot \Sigma_{2,3} = \emptyset \) and the general case can be written as a union of both cases with an appropriate distribution of the final states, which gives that polynomial time solvability by using [FGH+19, Lemma 13].
For $p \in P$ and $E \subseteq P$, we write $B_{p,E} = (\Sigma, P, \mu, p, E)$ for the PDFA that results from $B$ by changing the start state to $p$ and the set of final states to $E$.

So, with the assumption that $\{2,3\}$ is a strongly connected component, and hence $\Sigma_{2,1} = \Sigma_{3,1} = \emptyset$, and $\Sigma_{1,3} = \emptyset$, we can deduce that every word in $\Sigma_{1,1}^* \Sigma_{1,2}$ (note that this language does not equal $\mathcal{L}(B_{1,\{2\}})$) can map at most two states to a single state, and one of these states must be $t$, and every other state is mapped to distinct states, i.e., for every $w \in \Sigma_{1,1}^* \Sigma_{1,2}$, $|\delta(Q, w)| \geq n - 1$ and if $|\delta(Q, w)| = n - 1$, then two states, where one is $t$ itself, are mapped to $t$. If $\Sigma_{1,1} = \emptyset$ this is clear as $w \in \{a, b\}$. If $\Sigma_{1,1} = \{a\}$, we have $w \in a^*b$ and, as $a$ is idempotent, every such word has the same effect on the state set $Q$ of $A$ as the two words $b$ or $ab$. If $\Sigma_{1,1} = \{b\}$, then $w \in b^*a$ and $\delta(Q, b') = \delta(Q, a)$ for all $i \geq 1$, as $b$ permutes all states. The claim about $t$ follows by choice of this state, see Proposition 4.2.

Now, note that if the subautomaton between the states $\{2,3\}$ is complete, then if we have any synchronizing word $u$, by choosing $w, v$ such that $\mu(p_0, w) = 2$ and $\mu(2, uv) = 2$, which can be done by the assumptions, we have a synchronizing word $wwv \in L(B)$, and so the problem to find a synchronizing word in $L(B)$ is equivalent to the unconstrained synchronization problem, which is solvable in polynomial time. Hence, we can assume at least one transition between the states in $\{2,3\}$ is undefined. Further, if strictly more than one transition between these states is not defined, then, as these states form a strongly connected component, we have $\Sigma_{2,2} = \Sigma_{3,3} = \emptyset$ and in this case, if $\Sigma_{2,3} = \Sigma_{3,2}$, we either have $L(B) = \Sigma_{1,1}^* \Sigma_{1,2} (bb)^*$ or $L(B) = \Sigma_{1,1}^* \Sigma_{1,2} (aa)^*$, and it is easy to see that for $n > 4$ we cannot have a synchronizing word for $A$ in these languages. If $\Sigma_{2,3} \neq \Sigma_{3,2}$, then $L(B) = \Sigma_{1,1}^* \Sigma_{1,2} (ab)^*$ or $L(B) = \Sigma_{1,1}^* \Sigma_{1,2} (ba)^*$ and as $a(ba)^n$ synchronizes $A$ if it has the form as stated in case one of Proposition 4.2 by choosing $u \in \Sigma_{1,1}^* \Sigma_{1,2}$, the word $u(a(ba)^{n-2})b$ synchronizes $A$ and is in $\Sigma_{1,1}^* \Sigma_{1,2} (ab)^*$ and $u(a(ba)^{n-2})b$ synchronizing $A$ and is in $\Sigma_{1,1}^* \Sigma_{1,2} (ab)^*$.

So, the only cases left for the subautomaton between the states $\{2,3\}$ are listed in Table 3 which have been handled in the main text.

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