SOME SUBSPACES OF AN FK SPACE AND DEFERRED CESÁRO CONULLITY

Í. DAĞADUR AND Ş. SEZGEK

Abstract. In this paper, we construct new important the subspaces $D_q^p S$, $D_q^p W$, $D_q^p F$ and $D_q^p B$ for a locally convex FK-space $X$ containing $\phi$, the space of finite sequences. Then we show that there is relation among these subspaces. Also, we study deferred Cesàro conullity of one FK-space with respect to another, and we give some important results. Finally, we examine the deferred Cesàro conullity of the absolute summability domain $l_A$, and show that if $l_A$ is deferred Cesàro conull, then $A$ cannot be $l$-replaceable.

2000 Mathematics Subject Classification: 46A45, 40A05, 40C05, 40D05

Keywords and phrases: deferred Cesàro mean, deferred Cesàro conull, FK-space, AK-space, $\sigma_q^p[K]$-space, $\sigma_q^p[B]$-space

1. INTRODUCTION

Let $w$ denote the space of all complex valued sequences. It can be topologized with the seminorms $r_n(x) = |x_n|$, $n = 1, 2, \ldots$, and any vector subspace $X$ of $w$ is a sequence space. A sequence space $X$ with a vector space topology $\tau$ is a K-space provided that the inclusion map $i : (X, \tau) \rightarrow w$, $i(x) = x$, is continuous. If, in addition, $\tau$ is complete, matrizable, locally convex then $(X, \tau)$ is called FK-space. So an FK-space is a complete, metrizable locally convex topological vector space of sequences for which the coordinate functionals are continuous. An FK-space whose topology is normable is called a BK-space. The basic properties of FK-space may be found in (see [3], [13] and [18]).

By $c$, $c_0$, $l_\infty$ we denote the spaces of convergent sequences, null sequences and bounded sequences, respectively. These are FK-spaces under $\|x\| = \sup_n |x_n|$. By $cs, l$ we denote the spaces of all summable sequences, absolute summable sequences, respectively.

Throughout this paper $e$ denotes the sequences of ones; $\delta^j$ ($j = 1, 2, \ldots$) the sequence with the one in the $j$-th position; $\phi$ the linear span of $\delta^j$’s. The linear span of $\phi$ and $e$ is denoted by $\phi_1$. The topological dual of $X$ is denoted by $X'$. The space $X$ is said to have AD if $\phi$ is dense in $X$. A sequence $x$ in a locally convex sequence space $X$ is said the property AK if $x^{(n)} \rightarrow x$ in $X$ where $x^{(n)} = \sum_{k=1}^{n} x_k \delta^k$.

We recall (see [3] and [13]) that the $f$, $\beta$-duals of a subset $X$ of $w$ are

$$X^f = \{ \{ f(\delta^k) \} : f \in X' \},$$

$$X^\beta = \left\{ x \in w : \sum_{n=1}^{\infty} x_k y_k \text{ is convergent for all } y \in X \right\}.$$
In 1932, Agnew [1] defined the deferred Cesàro mean $D_{p,q}$ of the sequences $x$ by

$$(D_{p,q}x)_n = \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k$$

where \(\{p(n)\}\) and \(\{q(n)\}\) are sequences of nonnegative integers satisfying the conditions $p(n) < q(n)$ and $\lim_{n \to \infty} q(n) = \infty$. We note here that $D_{p,q}$ is clearly regular for any choice of \(\{p(n)\}\) and \(\{q(n)\}\). The deferred Cesàro mean is used throughout this paper. We define some new sequence space by using deferred Cesàro mean.

The sequence spaces

$$[\sigma_0]_p^q := \left\{ x \in w : \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k = 0 \right\},$$

$$[\sigma_c]_p^q := \left\{ x \in w : \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k \text { exists} \right\},$$

$$[\sigma_\infty]_p^q := \left\{ x \in w : \sup_{n} \frac{1}{q(n) - p(n)} \left| \sum_{k=p(n)+1}^{q(n)} x_k \right| < \infty \right\},$$

$$\sigma_p^q[s] := \left\{ x \in w : \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j \text { exists} \right\}$$

and

$$\sigma_p^q[b] := \left\{ x \in w : \sup_{n} \frac{1}{q(n) - p(n)} \left| \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j \right| < \infty \right\}$$

are BK-spaces with the norms

$$\| x \|_{[\sigma_0]_p^q} = \sup_{n} \left| \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k \right|$$

and

$$\| x \|_{[\sigma_p^q[s]]} = \sup_{n} \left| \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j \right|.$$

The proof follows the same lines as in (see [4], [5], [7] and [8]), so we omit the details.

A sequence $x$ in a locally convex sequence space $X$ is said the property $\sigma_p^q[K]$ if

$$\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \to x \text { in } X.$$
Now we determine a new $d$, $d[b]$-type duality of a sequence space $X$ containing $\phi$.

$$X^d = \left\{ x \in w : \lim_{n \to \infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j u_j \text{ exists for all } y \in X \right\}$$

$$= \left\{ x \in w : x.y \in \sigma_p^q[s] \text{ for all } y \in X \right\},$$

$$X^{d[b]} = \left\{ x \in w : \sup_n \frac{1}{q(n) - p(n)} \left| \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j u_j \right| < \infty, y \in X \right\}$$

$$= \left\{ x \in w : x.y \in \sigma_p^q[b] \text{ for all } y \in X \right\},$$

respectively, where $x.y = (x_n y_n)$.

Let $X$, $Y$ be sets of sequences. Then for $\nu = f, \beta, b, d[b]$

i) $X \subset X^{\nu\nu}$,

ii) $X^{\nu\nu} = X^\nu$,

iii) If $X \subset Y$ then $Y^\nu \subset X^\nu$ holds.

**Theorem 1.1.** Let $X$ be an FK-space containing $\phi$ and $\lim_{n \to \infty} \frac{q(n) - i + 1}{q(n) - p(n)} = 1$ ($i \leq q(n)$). Then

i) $X^\beta \subset X^d \subset X^{d[b]} \subset X^f$;

ii) if $X$ is $\sigma_p^q[K]$-space then $X^f = X^d$;

iii) if $X$ is AD-space then $X^{d[b]} = X^d$.

**Proof.** ii) Let $u \in X^d$ and $f(x) = \lim_{n \to \infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j u_j$ for $x \in X$. Then $f \in X^f$ by Banach-Steinhaus Theorem [13; Theorem 1.0.4]. Now we get

$$f(\delta^i) = \lim_{n \to \infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} u_j \delta^i = \lim_{n \to \infty} \frac{q_n - i + 1}{q(n) - p(n)} u_i = u_i$$

so $u \in X^f$. Thus $X^d \subset X^f$.

Let $u \in X^f$. Since $X$ is $\sigma_p^q[K]$-space

$$f(x) = \lim_{n \to \infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j f(\delta^i) = \lim_{n \to \infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j u_j$$

for $x \in X$, then $u \in X^d$. Hence $X^f = X^d$.

iii) Let $u \in X^{d[b]}$. We define

$$f_n(x) = \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} u_j x_j$$

for $x \in X$. Then $\{f_n\}$ is pointwise bounded and hence equicontinuous by Theorem 7.0.2 of [13]. Since

$$\lim_{n \to \infty} f_n(\delta^i) = u_i, i < q(n),$$

we conclude that $\phi \subset \{x : \lim_n f_n(x) \text{ exists} \}$ is a closed subspace of $X$ by the Convergence Lemma (see [13; 1.0.5 and 7.0.3]). Since $X$ is an AD-space, $X = \{x : \}$
\[ \lim_n f_n(x) \text{ exists } \} = \overline{\phi} \text{ and thus } \lim_n f_n(x) \text{ exists for all } x \in X. \text{ Therefore, } u \in X^d. \]
The opposite inclusion is trivial.

i) \( \overline{\phi} \subset X \) by the hypothesis. Since \( \overline{\phi} \) is an AD-space, we find
\[ X^d[b] \subset (\overline{\phi})^d[b] = (\overline{\phi})^d \subset (\overline{\phi})^f = X^f \]
by (ii), (iii) and Theorem 7.2.4 of [13]. \hfill \Box

2. Main Results

We shall define some new subspaces of a locally convex FK-space \( X \) containing \( \phi \), the space of finite sequences, which are the importance of each one on topological sequence spaces theory.

\textbf{Definition 2.1.} Let \( X \) be an FK-space \( \supset \phi \). Then
\[ D^q_p W := D^q_p W(X) = \left\{ x \in X : \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \to x \text{ (weakly) in } X \right\} \]
\[ = \left\{ x \in X : f(x) = \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_j x_j f(\delta^j) \text{ for all } f \in X \right\}, \]
\[ D^q_p S := D^q_p S(X) = \left\{ x \in X : \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \to x \right\} \]
\[ = \left\{ x \in X : x = \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_j x_j \delta^j \right\}. \]
Thus \( X \) is an \( \sigma^q_p[K] \)-space if and only if \( D^q_p S = X \).

\[ D^q_p F^+ := D^q_p F^+(X) \]
\[ = \left\{ x \in w : \lim_n \left\{ \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \right\} \text{ is weakly Cauchy in } X \right\} \]
\[ = \left\{ x \in w : \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_j x_j f(\delta^j) \text{ exists for all } f \in X' \right\} \]
\[ = \left\{ x \in w : \{x_n f(\delta^n)\} \in \sigma^q_p[s] \text{ for all } f \in X' \} = (X^f)^d. \]
\[ D^q_p B^+ := D^q_p B^+(X) = \left\{ x \in w : \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \right\} \text{ is bounded in } X \}
\[ = \left\{ x \in w : (x_n f(\delta^n)) \in \sigma^q_p[b] \text{ for all } f \in X' \right\} \]
also \( D^q_p F = D^q_p F^+ \cap X \) and \( D^q_p B = D^q_p B^+ \cap X \).
We now study some inclusions which are analogous to those given in ([13]; chapter 10). Also, we prove some theorems related to the \( f^-\), \( d^-\) and \( d[b]^-\)-duality of a sequence space \( X \).

**Theorem 2.2.** Let \( X \) be an FK-space \( \supset \phi \). Then
\[
\phi \subset D^q_p S \subset D^q_p W \subset D^q_p F \subset D^q_p B \subset X \quad \text{and} \quad \phi \subset D^q_p S \subset D^q_p W \subset \overline{\phi}.
\]

**Proof.** The only non-trivial part is \( D^q_p W \subset \overline{\phi} \). Let \( f \in X' \) and \( f = 0 \) on \( \phi \). The definition of \( D^q_p W \) shows that \( f = 0 \) on \( D^q_p W \). Hence, the Hahn-Banach theorem gives the result. \( \Box \)

**Theorem 2.3.** The subspaces \( E = D^q_p S, D^q_p W, D^q_p F, D^q_p F^+, D^q_p B \) and \( D^q_p B^+ \) of \( X \) FK-spaces are monotone i.e., if \( X \subset Y \) then \( E(X) \subset E(Y) \).

**Proof.** The inclusion map \( i : X \to Y \) is continuous by Corollary 4.2.4 of [13], so
\[
\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \to x \quad \text{in} \quad Y \quad \text{implies} \quad \text{the same in} \quad Y.
\]
This proves the assertion for \( D^q_p S \). For \( D^q_p W \), it follows from the fact that \( i \) is weakly continuous by (4.0.11) of [13]. Note \( z \in D^q_p F^+, D^q_p B^+ \) if and only if \( (z_nf(\delta^n)) \in \sigma^q_p[s], \sigma^q_p([b]) \) respectively for all \( f \in X' \), hence for all \( q \in Y' \) since \( g|X \in X' \) by Corollary 4.2.4 of [13]. The result follows for \( D^q_p F^+, D^q_p B^+ \) and so for \( D^q_p F, D^q_p B \). \( \Box \)

Since \( [\sigma_0]^q_p \) is an AK-space, we immediately get the following

**Theorem 2.4.** Let \( X \) be an FK-space \( \supset [\sigma_0]^q_p \). Then \( [\sigma_0]^q_p \subset D^q_p S \subset D^q_p W \).

**Theorem 2.5.** Let \( X \) be an FK-space \( \supset \phi \). Then \( D^q_p B^+ = X^{Id}[b] \).

**Proof.** By Definition 2.1 \( z \in D^q_p B^+ \) if and only if \( z.u \in \sigma^q_p[b] \) for each \( u \in X' \). This is precisely the assertion. \( \Box \)

**Theorem 2.6.** Let \( X \) be an FK-space \( \supset \phi \). Then \( D^q_p B^+ \) is the same for all FK-spaces \( Y \) between \( \overline{\phi} \) and \( X \); i.e., \( \overline{\phi} \subset Y \subset X \implies D^q_p B^+(Y) = D^q_p B^+(X) \). Here the closure of \( \phi \) is calculated in \( X \).

**Proof.** By Theorem 2.3 we have \( D^q_p B^+(\overline{\phi}) \subset D^q_p B^+(Y) \subset D^q_p B^+(X) \). By Theorem 2.5 and by (7.2.4) of [13] the first and the last are equal. \( \Box \)

**Theorem 2.7.** Let \( X \) be an FK-space such that \( D^q_p B \supset \overline{\phi} \). Then \( \overline{\phi} \) has \( \sigma^q_p[K] \) and \( D^q_p S = D^q_p W = \overline{\phi} \).

**Proof.** Suppose first that \( X \) has \( \sigma^q_p[B] \). Define \( f_n : X \to X \) by
\[
f_n(x) = x - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)}.
\]
Then \( \{f_n\} \) is pointwise bounded, hence equicontinuous by (7.0.2) of [13]. Since \( f_n \to 0 \) on \( \phi \) then also \( f_n \to 0 \) on \( \overline{\phi} \) by (7.0.3) of [13]. This is the desired conclusion. \( \Box \)

**Theorem 2.8.** Let \( X \) be an FK-space \( \supset \phi \). Then \( D^q_p F^+ = X^{Id} \).

**Proof.** This may be proved as in Theorem 2.5 with \( d \) instead of \( d[b] \). \( \Box \)

**Theorem 2.9.** Let \( X \) be an FK-space \( \supset \phi \). Then \( D^q_p F^+ \) is the same for all FK-spaces \( Y \) between \( \overline{\phi} \) and \( X \); i.e., \( \overline{\phi} \subset Y \subset X \implies D^q_p F^+(Y) = D^q_p F^+(X) \) (The closure of \( \phi \) is calculated in \( X \)).
The proof is similar to that of Theorem 2.6

**Lemma 2.10.** Let $X$ be an FK-space in which $\overline{\phi}$ has $\sigma_p^q[K]$. Then $D_p^qF^+ = (\overline{\phi})^{dd}$.  

**Proof.** Observe that $D_p^qF^+ = X^{fd}$ by Theorem 2.8. Since $X^f = (\overline{\phi})^f$ by Theorem (7.2.4) of [13], we have $X^{fd} = (\overline{\phi})^{fd}$. Hence, by Theorem 1.9 of [9] the result follows. \qed

An FK-space $X$ is said to have $F\sigma_p^q[K]$ (functional $\sigma_p^q[K]$) if $X \subset D_p^qF$ i.e., $X = D_p^qF$.

**Theorem 2.11.** Let $X$ be an FK-space $\supset \phi$. Then $X$ has $F\sigma_p^q[K]$ if and only if $\overline{\phi}$ has $\sigma_p^q[K]$ and $X \subset (\overline{\phi})^{dd}$.

**Proof.** Necessity. $X$ has $\sigma_p^q[B]$ since $D_p^qF \subset D_p^qB$ so $\overline{\phi}$ has $\sigma_p^q[K]$ by Theorem 2.7.

The remainder of the proof follows from Lemma 2.10. Sufficiency is given by Lemma 2.10. \qed

**Theorem 2.12.** Let $X$ be an FK-space $\supset \phi$. The following are equivalent:

1. $X$ has $F\sigma_p^q[K]$,
2. $X \subset (D_p^qS)^{dd}$,
3. $X \subset (D_p^qW)^{dd}$,
4. $X \subset (D_p^qF)^{dd}$,
5. $X^d = (D_p^qS)^d$,
6. $X^d = (D_p^qF)^d$.

**Proof.** Observe that (ii) implies (iii) and (iii) implies (iv) and that they are trivial since

$$D_p^qS \subset D_p^qW \subset D_p^qF.$$  

If (iv) is true, then $X^f \subset (D_p^qF)^d = (X^f)^{dd} \subset X^d$ so (i) is true by Theorem 1.9 of [9]. If (i) holds, then Theorem 2.11 implies that $\overline{\phi} = D_p^qS$ and that (ii) holds. The equivalence of (v), (vi) with the others is clear. \qed

**Theorem 2.13.** Let $X$ be an FK-space $\supset \phi$. The following are equivalent:

1. $X$ has $S\sigma_p^q[K]$,
2. $X$ has $\sigma_p^q[K]$,
3. $X^d = X'$.

**Proof.** Clearly (ii) implies (i). Conversely if $X$ has $S\sigma_p^q[K]$ it must have AD for $D_p^qW \subset \overline{\phi}$ by Theorem 2.2. It also has $\sigma_p[B]$ since $D_p^qW \subset D_p^qB$. Thus $X$ has $\sigma_p^q[K]$ by Theorem 2.7, this proves that (i) and (ii) are equivalent. Assume that (iii) holds. Let $f \in X'$, then there exists $u \in X^d$ such that

$$f(x) = \lim_{n \to \infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} u_j x_j$$

for $x \in X$. Since $f(\delta^j) = u_j$, it follows that each $x \in D_p^qW$ which shows that (iii) implies (i). That (ii) implies (iii) is known (see [7], page 97). \qed

**Theorem 2.14.** Let $X$ be an FK-space $\supset \phi$. The following are equivalent:

1. $D_p^qW$ is closed in $X$,
2. $\overline{\phi} \subset D_p^qB$,
3. $\overline{\phi} \subset D_p^qF$,
iv) \( \overline{\phi} = D_p^0W \),

v) \( \overline{\phi} = D_p^0S \),

vi) \( D_p^0S \) is closed in \( X \).

**Proof.** (ii) implies (v): By Theorem 2.7, \( \overline{\phi} \) has \( \sigma^p_0[K] \), i.e. \( \overline{\phi} \subset D_p^0S \). The opposite inclusion is Theorem 2.2. Note that (v) implies (iv), (iv) implies (iii) and (iii) implies (ii) because

\[
D_p^0S \subset D_p^0W \subset \overline{\phi}, D_p^0W \subset D_p^0F \subset D_p^0B;
\]

(i) implies (iv) and (vi) implies (v) since \( (\mu) \) implies (v): By Theorem 2.7.

Finally (iv) implies (i) and (v) implies (vi). □

3. Combinations of Some Subspaces of an FK-Space

Let \( A = (a_{nk}) \), \( n, k = 1, 2, \ldots \) be an infinite matrix with complex entries and \( c_A = \{x : Ax \in c\} \). Then \( c_A \) is an FK-space with seminorms \( \rho_0(x) = \sup_n |\sum_{k=1}^{\infty} a_{nk} x_k| \) (\( n = 1, 2, \ldots \)), \( \rho_n(x) = |x_n| \) (\( n = 1, 2, \ldots \)); and \( h_n(x) = \sup_m |\sum_{k=1}^{\infty} a_{nk} x_k| \) (\( n = 1, 2, \ldots \)). Also, every \( f \in c'_A \) if and only if

\[
f(x) = \sum_{k=1}^{\infty} \beta_k x_k + \sum_{n=1}^{\infty} t_n \sum_{k=1}^{\infty} a_{nk} x_k + \mu \lim \lambda A x,
\]

where \( t \in \ell, \mu \in \mathbb{C}, (\beta_k) \in c^\beta_A \), the \( \beta \)-dual of \( c_A \) [13]. The representation is not unique; we say that \( A \) is \( \mu \) -unique if all representations for some \( f \) have the same \( \mu \). If \( A \) be \( \mu \) -unique, \( c_A \subset c_D \), \( D \) is conull with respect to \( A \) if and only if \( \mu_A(\lim_D) = 0 \) in [16].

Let \( X \) and \( Y \) be FK-spaces, \( X \) with paranorm \( \rho \) and \( Y \) with paranorm \( s \). It is shown that \( Z = X + Y \) with the unrestricted inductive limit topology is an FK-space as in Theorem 4.5.1 of [13]. The paranorm \( \tau \) of \( Z \) is given by

\[
\tau(z) = \inf_{x+y = z} (\rho(x) + s(y)).
\]

Let \( \{X^n\}_{n=1}^{\infty} \) be a sequence of FK-spaces. \( \rho_n \) the paranorm of \( X^n \) and \( \{s_{nk}\}_{k=1}^{\infty} \) be the seminorms of \( X^n \). Let \( Y = \bigcap_{n=1}^{\infty} X^n \). It is well known that \( Y \) is an FK-space with paranorm \( s = \sum_{n=1}^{\infty} \frac{\rho_n}{2^{n(1+\rho_n)}} \) and seminorms \( \{s_{nk}\}_{n,k=1}^{\infty} \).

We now investigate some important subspaces of a locally convex FK-space \( X \) containing \( \phi \) which are analogous to these given in [8]. To prove the theorems of this section we use the same technique by DeVos in [6].

**Theorem 3.1.** Let \( X, Y \) be FK-spaces and \( Z = X + Y \). Then \( E(X) + E(Y) \subseteq E(Z) \) for \( E = D_p^0S, D_p^0W, D_p^0F \) or \( D_p^0B \).

**Proof.** Let \( E = D_p^0S \). We take \( x \in D_p^0S(X) \) and \( y \in D_p^0S(Y) \). Then

\[
\rho\left(\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} - x\right) \rightarrow 0 \quad \text{and} \quad s\left(\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} y^{(k)} - y\right) \rightarrow 0
\]
as \( n \to \infty \). Hence

\[
\begin{align*}
 r \left( \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} (x+y)^{(k)} - (x+y) \right) \\
\leq \rho \left( \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} - x \right) + s \left( \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} y^{(k)} - y \right)
\end{align*}
\]

which implies that \( x + y \in D^q_p S(Z) \).

Let \( E = D^q_p W \). We take \( x \in D^q_p W(X), y \in D^q_p W(Y) \) and \( f \in Z' \). Then \( f|X \in X' \) and \( f|Y \in Y' \).

\[
f(x + y) = f(x) + f(y) \\
= \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} f(\delta^j) x_j + \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} f(\delta^j) y_j \\
= \lim_{n} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} f(\delta^j) (x_j + y_j) .
\]

The proofs for \( E = D^q_p F \) or \( D^q_p B \) are similar, so the details are omitted. \( \Box \)

**Theorem 3.2.** Let \( \{X^n\}_{n=1}^{\infty} \) be a sequence of FK-spaces and \( Y = \bigcap X^n \). Then \( E(Y) = \bigcap E(X^n) \) for \( E = D^q_p S, D^q_p W, D^q_p F \) or \( D^q_p B \).

**Proof.** By Theorem 2.3, for each \( n \), \( E(Y) \subseteq E(X^n) \), hence \( E(Y) \subseteq \bigcap E(X^n) \) for \( E = D^q_p S, D^q_p W, D^q_p F \) or \( D^q_p B \).

Let \( z \in \bigcap D^q_p S(X^n) \). Then \( s_{nk} \left( \frac{1}{q(r) - p(r)} \sum_{n=p(r)+1}^{q(r)} z^{(n)} - z \right) \to 0, r \to \infty \), for each fixed \( n \) and \( k \), but these are the seminorms for \( Y \). Hence

\[
\lim_{r \to \infty} \frac{1}{q(r) - p(r)} \sum_{n=p(r)+1}^{q(r)} z^{(n)} = z \text{ in } Y
\]

which implies that \( z \in D^q_p S(Y) \).

Let \( z \in \bigcap D^q_p W(X^n) \) and \( f \in Y' \). Then we have \( f = \sum_{j=1}^{h} f_j \), where \( f_j \in (X^j)'^{'} \) (see [16]: Sections 4.4 and 11.3). Since \( f_j \left( \frac{1}{q(r) - p(r)} \sum_{n=p(r)+1}^{q(r)} z^{(n)} \right) \to f_j(z) \) for \( j = 1, 2, \ldots, h \). Therefore,

\[
f \left( \frac{1}{q(r) - p(r)} \sum_{n=p(r)+1}^{q(r)} z^{(n)} \right) \to f(z) .
\]

Hence \( z \in D^q_p W(Y) \).

The proof for \( E = D^q_p F \) is similar to previous paragraph, so we omit the details.

Let \( z \in \bigcap D^q_p B(X^n) \). Then for any fixed \( l \) and \( k \), \( s_{lk} \left( \frac{1}{q(r) - p(r)} \sum_{n=p(r)+1}^{q(r)} z^{(n)} \right) \leq H_{lk} \) for all \( r \). Hence \( z \in D^q_p B(Y) \). \( \Box \)
In [12], let $X$ be an FK-space containing $\phi_1$ and

$$
(3.1) \quad \zeta^n := e - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} e^{(k)}
$$

$$
= \left(0, 0, \ldots, 0, \frac{1}{q(n) - p(n)}, \frac{2}{q(n) - p(n)}, \frac{3}{q(n) - p(n)}, \ldots, \frac{q(n) - p(n) - 1}{q(n) - p(n)}, 1, 1, \ldots \right).
$$

If $\zeta^n \to 0$ in $X$, then $X$ is called strongly deferred Cesàro conull, where $e^{(k)} := \sum_{j=1}^{k} \delta^j$. If the convergence holds in the weak topology in (3.1) then $X$ is called deferred Cesàro conull. Hence $X$ is deferred Cesàro conull iff

$$
f(e) = \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} f(\delta^j), \quad \forall f \in X'.
$$

Now, we define deferred Cesàro conullity of one FK-space with respect to another.

**Definition 3.3.** Let $X$ be an FK-space with $D^n_p W(X) \neq D^n_p B(X)$ and $Y$ be an FK-space, $X \subseteq Y$. $Y$ is deferred Cesàro conull with respect to $X$ iff $D^n_p B(X) \subseteq D^n_p W(Y)$.

**Theorem 3.4.** Let $X, Y, Z$ be FK-spaces with $X \subseteq Y \subseteq Z$. Then

i) If $Y$ is deferred Cesàro conull with respect to $X$ then $Z$ is deferred Cesàro conull with respect to $X$,

ii) If $Z$ is deferred Cesàro conull with respect to $X$ and $Y$ is closed in $Z$ then $Y$ is deferred Cesàro conull with respect to $X$.

The proof of Theorem is clear by Definition 3.3 and Theorem 2.3.

**Theorem 3.5.** Let $\{Y^m\}_{m=1}^{\infty}$ be FK-spaces such that each $Y^m$ is deferred Cesàro conull with respect to $X$. Then $\bigcap_n Y^n$ is deferred Cesàro conull with respect to $X$.

The proof of Theorem is obtained by Definition 3.3 and Theorem 3.2.

Let $E(c_A) = E(A)$ for $A$ a matrix and $E = D^n_p W$ or $D^n_p B$ and $\mu_A(\lim_D) = \mu_A(D)$. For many cases the following theorem gives an equivalence between Wilansky’s and our extensions of deferred Cesàro conullity.

**Theorem 3.6.** Let $A$ and $D$ be matrices with $D^n_p W(A) \neq D^n_p B(A)$ and $c_A \subset c_D$, $\mu_A(D) = 0$ if and only if $c_D$ is deferred Cesàro conull with respect to $c_A$.

Proof. Let $c_D$ is deferred Cesàro conull with respect to $c_A$. For $x \in D^n_p B(A)$, $\lim_D x = \mu_A(D) \lim_A x + \beta x$.

Now let $z \in D^n_p B(A) \setminus D^n_p W(A)$. Firstly, by Theorem 4.2 of [12] we have

$$
\lim_A \left( z - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} z^{(k)} \right) \to 0.
$$
Also, since $D^q_pB(A) \subseteq c_A$ and $\gamma \in c^q_A$ we obtained
\[
\gamma \left( z - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} z^{(k)} \right) = \gamma \left( z - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} z_j \delta^j \right) = \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} z_j \delta^j .
\]

By hypothesis, for each $f \in (cD)'$, \[
f \left( z - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} z^{(k)} \right) \to 0 .
\]

In particular, we take $f = \lim_D \in (cD)'$. Thus $\lim_D \left( z - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} z^{(k)} \right) \to 0$.

Conversely let $f \in (cD)'$. By Theorem 5.2 of [16] we have $\mu_A(f) = \mu_D(f), \mu_A(D) = 0$. Hence $f(x) = t(Ax) + \beta x$ for $x \in c_A$.

Now we are able to write
\[
f(x) = \mu_A(f) \lim_A x + \beta x
\]
with $\gamma = tA + \beta$ for $x \in D^q_pB$ by Corollary 12.5.9 of [13]. Therefore, we get $f(x) = \gamma x$ for $x \in D^q_pB(A)$ which implies that $x \in D^q_pW(D)$. So $c_D$ is deferred Cesàro conull with respect to $c_A$.

We establish some relations among the subspaces $D^q_pS$, $D^q_pW$, $D^q_pF$, $D^q_pF^+$.

**Remark 3.7.** Let $X$ be an FK-space such that weakly convergent sequences are convergent in the FK-topology, $A$ be a matrix such that $X_A \supset \phi$. The subspaces $D^q_pS$, $D^q_pW$ and $D^q_pF$ are calculated in $X_A$.

**Lemma 3.8.** If $X$ is as in Remark 3.7, then for $X$ itself, we have $D^q_pS = D^q_pW = D^q_pF = D^q_pF^+$.

**Proof.** The inclusions $D^q_pS \subseteq D^q_pW \subseteq D^q_pF \subseteq D^q_pF^+$ are trivial by definitions. Conversely, if $x \in D^q_pF^+$, then \[
\left\{ \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \right\}
\]
is weakly Cauchy, hence by (12.0.1) of [13] is Cauchy in the FK-topology of $X$, so convergent, say
\[
\left\{ \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \right\} \to y .
\]

Since $x^{(k)} \to x$ in $w$, we have
\[
\left\{ \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \right\} \to x \text{ in } w .
\]

By the continuity of $i : X \to w, y = x$, and $x \in D^q_pS$. \qed

Now we note that if $X$ is an FK-space containing $\phi_1$, then
\[
(3.2) \quad D^q_pF^+ = X^{fd}.
\]
To see this, it is enough to take $\sigma_p^q[s]$ instead of $cs$ in Theorem 10.4.2 of [13]. If $X$ is also $\sigma_p^q[K]$, $X^{dd} = X$ since, by Theorem 1.9 of [9], $X^{dd} = X^{fd}$ and $D_p^qF^+ = X^{fd}$ by (12.0.1) and (12.0.2). We have $X^{dd} = D_p^qF = D_p^qF^+ \subset H$, hence, the result follows.

**Theorem 3.9.** With $X$, $A$ as in Remark 3.7, for the $X_A$, we have $D_p^qS = D_p^qW = D_p^qF = D_p^qF^+$.

**Proof.** If $x \in D_p^qF^+$, then \( \left\{ \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \right\} \) is weakly Cauchy, hence by (12.0.1) of [13] is Cauchy in the FK-topology of $X$, so convergent. Since by Corollary 4.2.4 of [13] the matrix mapping $A : X_A \to X$ is continuous, \( \left\{ \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} Ax^{(k)} \right\} \) is convergent in $X$, say
\[
\left\{ \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} Ax^{(k)} \right\} \to y.
\]

On the other hand, by Theorem 4.3.8 of [13] $(w_A, \rho \cup h)$ is an AK-space. Hence it is also $\sigma_p^q[K]$-space. Hence
\[
\left\{ \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \right\} \to x.
\]

The matrix mapping $A : w_A \to w$ is continuous, and therefore
\[
\left\{ \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} Ax^{(k)} \right\} \to Ax \quad \text{in} \quad w.
\]

Since $X \subset w$ and $X$ is complete, $Ax = y$. We have
\[
\left\{ \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} Ax^{(k)} \right\} \to Ax \quad \text{in} \quad X.
\]

That is,
\[
r \left( Ax - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} Ax^{(k)} \right) = (r \circ A) \left( x - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \right) \to 0
\]
as $n \to \infty$, where $r$ is a typical seminorm of $X$. Hence $x \in D_p^qS$, which proves Theorem 3.9. \qed

### 3.1. Replaceability, Deferred Cesàro Conullity of $l_A$.

Recall that given a matrix $A$ with $l_A \supset \phi$ is called $l$-replaceable if there is a matrix $D = (d_{nk})$ with $l_D = l_A$, and $\sum d_{nk} = 1$ $k \in \mathbb{N}$ [10]. It is easy to see that $A$ is replaceable if and only if there exists $f \in l'_A$ with $f(\delta^k) = 1$ ($k \in \mathbb{N}$), namely, $f = \sum D$.

**Theorem 3.10.** Suppose that $D_p^qF = l_A$. Then $A$ is $l$-replaceable if and only if $l_A \subset \sigma_p^q[s]$.
Proof. Assume that $A$ is $l$-replaceable. Then it follows from [10] that $A$ is $l$-replaceable if and only if there is $f \in l'_A$ such that $f(\delta^j) = 1$ for all $j \in \mathbb{N}$. Since $D^q_p F = l_A$,

$$
\lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j f(\delta^j) = \lim_n \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j
$$

exists for all $x \in l_A$, hence $x \in \sigma^q_p[s]$.

Conversely, if $l_A \subset \sigma^q_p[s]$ then

$$
f(x) = \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j
$$

defines an element $f$ of $l'_A$, and we get $f(\delta^\nu) = 1$, $(\nu = 1, 2, \ldots)$. \hfill \Box

We now establish a relation between deferred Cesàro conullity and replaceability.

**Theorem 3.11.** If $l_A$ is deferred Cesàro conull space, then $A$ is not $l$-replaceable.

**Proof.** Suppose that $A$ is $l$-replaceable. Then it follows from [10] that $A$ is $l$-replaceable if and only if there is $f \in l'_A$ such that $f(\delta^j) = 1$ for all $j \in \mathbb{N}$. Hence

$$
f(e) - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} f(\delta^j) = \left( \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} (k) \right) = \left( f(e) - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} (q(n) - p(n))(q(n) + p(n) + 1) \right) = \left( f(e) - \frac{q(n) + p(n) + 1}{2} \right) \rightarrow 0 \quad \text{as } n \rightarrow \infty ,
$$

so, $l_A$ is not deferred Cesàro conull. \hfill \Box

**References**

[1] Agnew R.P., *On deferred Cesàro means*, Annals of Math. 33[3] (1932) 413-421.
[2] Armitage D.H. and Maddox I.J., *A new type of Cesàro mean*, Analysis, 9 (1989) 195-204.
[3] Boos J., *Classical and modern methods in summability*, Oxford University Press Inc., New York 2000.
[4] Buntinas M., *Convergent and bounded Cesàro sections in FK-spaces*, Math. Z., 121 (1971), 191-200.
[5] Dağduran I., *On some subspaces of an FK-space*, Mathematical Communications 7 (2002) 15-20.
[6] DeVos R., *Combinations of distinguished subsets and conullity*, Math. Z. 192 (1986) 447-451.
[7] Goes G. and Goes S., *Sequences of bounded variation and sequences of Fourier coefficients I.*, Math. Z. 118 (1970) 93-102.
[8] Goes G., *Sequences of bounded variation and sequences of Fourier coefficients*, II, J. Math. Anal. Appl., 39 (1972), 477-494.
[9] Goes G., *Summan von FK-Räumen funktionale Abschnittskonvergenz und Umkehrsatz*, Tôhoku. Math. Journ. 26(1974), 487504.
[10] Macphail M. S. and Orhan C. *Some properties of absolute summability domains*, Analysis, 9, 1989, 317-322.
[11] Sember J. J., *Variational FK-spaces and two norm convergence*, Math. Z. 119 (1971), 153-159.
[12] Sezgek Ş. and Dağadur İ., *Deferred Cesàro conull FK spaces*, (Summitted).
[13] Wilansky A., *Summability through functional analysis*, North Holland, 1984.
[14] Wilansky A., *Functional Analysis*, Blaisdell Press, 1964.
[15] Wilansky A., *An application of Banach linear functionals to summability*, Trans. Amer. Math. Soc. 67 (1949). 59-68.
[16] Wilansky A., *The μ property of FK-spaces*, Comment. Math. 21 (1978) 371-380.
[17] Yurimyae E., *Einige Fragen ber verallgemeinerte Matrixverfahren, co-regular und co-null Verfahren*, Eesti Tead. Akad. Toimetised Tehn. Füüs. Math. 8 (1959) 115-121.
[18] Zeller K., *Allgemeine Eigenschaften von limitierungsverfahren*, Math. Z. 53 (1951) 463-487.

MERSIN UNIVERSITY, FACULTY OF SCIENCE AND LITERATURE, DEPARTMENT OF MATHEMATICS, MERSIN - TURKEY.

E-mail address: seydasezgek@gmail.com
E-mail address: ilhandagadur@yahoo.com