ÉTALE TOPOLOGY OF THE SPACE OF RATIONAL FUNCTIONS
AND THE MODULI STACK OF STABLE ELLIPTIC SURFACES

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ABSTRACT. Motivated by the enumeration of the moduli stack of morphisms Hom$_n$(P$^1$, P(a, b)), where P(a, b) is the 1-dimensional (a, b) weighted projective stack, over char(F$_q$) not dividing a or b in H[P]. We derive its compactly supported ℓ-adic étale cohomology and describe the eigenvalues of the geometric Frobenius map acting on this cohomology. For any a, b € N and n ≥ 1, we find the étale cohomology of the moduli is of Tate type and not étale pure by having the ℓ-adic rational homology type of 3-sphere. As corollaries, we acquire the corresponding ℓ-adic Galois representations of the moduli space Hom$_n$(P$^1$, P$^1$) = Rat$_n$ of non-based rational self maps P$^1$ → P$^1$ and the moduli stack Hom$_n$(P$^1$, M$_{1,1}$) of stable elliptic fibrations over P$^1$, also known as stable elliptic surfaces, with 12n nodal singular ﬁbers and a distinguished section.

1. Introduction

The goal of this paper is to find the étale cohomology and the eigenvalues of Frobenius map for the Hom stack Hom$_n$(P$^1$, P(a, b)), where P(a, b) is the 1-dimensional a, b € N weighted projective stack, with degree n ≥ 1 morphisms. The moduli Hom$_n$(P$^1$, P(a, b)) was formulated in H[P] when the base field K is not dividing a or b. We recall the arithmetic of Hom$_n$(P$^1$, P(a, b)) over F$_q$.

Theorem 1 (Weighted point count of the moduli stack Hom$_n$(P$^1$, P(a, b))). If char(F$_q$) does not divide a or b, then the weighted point count #$_q$(Hom$_n$(P$^1$, P(a, b))) over F$_q$ is

#$_q$(Hom$_n$(P$^1$, P(a, b))) = q$^{(a+b)n+1} - q^{(a+b)n-1}

While having the F$_q$-point count up to weights certainly tells us one side of the arithmetic for Deligne–Mumford stacks of ﬁnite type, one is naturally led to consider running the Weil conjecture backward similar to the innovative work of [FW]. As the absolute Galois group of ﬁnite ﬁelds Gal(F$^\infty$/F$_q$) acting on the étale cohomology is a procyclic group that is topologically generated by the geometric Frobenius morphism, the task of ﬁnding the eigenvalues of Frobenius map can be achieved through the trace formula where the cardinality of the ﬁxed set of Frob$_i : X(F_q) → X(F_q)$ coincides with the weighted point count #$_q$(X) over F$_q$. We recall the Sun–Behrend trace formula [Sun] [Behrend] for Artin stacks of ﬁnite type over ﬁnite ﬁelds.

Theorem 2 (Theorem 1.1 of Sun). Let X be an Artin stack of ﬁnite type over F$_q$. Let Frob$_q$ be the geometric Frobenius on X. Let ℓ be a prime number different from the characteristic of F$_q$, and let v : Q$_\ell$ → C be an isomorphism of ﬁelds. For an integer i, let H$^i_c(X_{\overline{F}_q}; \overline{Q}_\ell)$ be the cohomology with compact support of the constant sheaf Q$_\ell$ on X as in LO. Then the inﬁnite sum regarded as a complex series via v

(1) \[ \sum_{i \in \mathbb{Z}} (-1)^i \text{tr} \left( \text{Frob}_q^* : H^i_c(X_{\overline{F}_q}; \overline{Q}_\ell) \to H^i_c(X_{\overline{F}_q}; \overline{Q}_\ell) \right) \]

is absolutely convergent to #$_q$(X) over F$_q$. And its limit is the number of F$_q$-points on stacks that are counted with weights, where a point with its stabilizer group G contributes a weight $\frac{1}{|G|}$.
As $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ is a smooth Deligne-Mumford stack of finite type and constant dimension over $\mathbb{F}_q$ (see Proposition 7). The corresponding $\ell$-adic étale cohomology for prime number $\ell$ invertible in $\mathbb{F}_q$ is finite dimensional as a $\mathbb{Q}_\ell$-algebra, making the trace formula holds in $\mathbb{Q}_\ell$-coefficients. For $n \geq 1$ and any $a, b \in \mathbb{N}$, by the $\ell$-adic Leray spectral sequence on the Zariski-locally trivial fibration given by the evaluation morphism $ev_n : \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)) \to \mathcal{P}(a,b)$, we first compute the $\ell$-adic étale Betti numbers $\dim_{\mathbb{Q}_\ell}(H^i_{\text{ét}}(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))/\mathbb{F}_q; \mathbb{Q}_\ell))$ showing the dual $\mathbb{Q}_\ell$-vector spaces are one dimensional for $i = 0, 3$ and vanishes for all other $i$. Consequently, we have Main Theorem of this paper through the above trace formula and weighted point count of the moduli.

**Theorem 3.** The Hom stack $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ over $\text{char}(\mathbb{F}_q)$ not dividing $a$ or $b$ for any $a, b \in \mathbb{N}$ parameterizing the morphisms $f : \mathbb{P}^1 \to \mathcal{P}(a,b)$ with $f^*\mathcal{O}_{\mathcal{P}(a,b)}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$ for $n \geq 1$ has the following Tate type and not étale pure compactly supported étale cohomology by having the $\ell$-adic rational homology type of 3-sphere together with the eigenvalues of Frobenius map $\text{Frob}_q^*$ expressed as isomorphisms of $\ell$-adic Galois representations

$$H^i_{\text{ét}, c}(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))/\mathbb{F}_q; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-(a+b)n+1) & i = 2(a+b)n + 2 \\ \mathbb{Q}_\ell(-(a+b)n-1) & i = 2(a+b)n - 1 \\ 0 & \text{else} \end{cases}$$

We single out two special cases of interest for $\mathcal{P}(a,b)$, One is $\mathcal{P}(1,1) \cong \mathbb{P}^1$ which gives us the space $\text{Hom}_n(\mathbb{P}^1, \mathbb{P}^1) = \text{Rat}_n$ of non-based rational self maps studied by [Segal] in depth.

**Corollary 4.** The space $\text{Hom}_n(\mathbb{P}^1, \mathbb{P}^1)$ of degree $n \geq 1$ non-based rational self maps $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ over $\mathbb{F}_q$ of any prime power has the following Tate type and not étale pure compactly supported étale cohomology by having the $\ell$-adic rational homology type of 3-sphere together with the eigenvalues of Frobenius map $\text{Frob}_q^*$ expressed as isomorphisms of $\ell$-adic Galois representations

$$H^i_{\text{ét}, c}(\text{Hom}_n(\mathbb{P}^1, \mathbb{P}^1)/\mathbb{F}_q; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-2n+1) & i = 4n + 2 \\ \mathbb{Q}_\ell(-2n-1) & i = 4n - 1 \\ 0 & \text{else} \end{cases}$$

The second is $\mathcal{P}(4,6) \cong \overline{\mathcal{M}}_{1,1}$ isomorphic to the proper Deligne-Mumford stack of stable elliptic curves over $\text{char}(K) \neq 2, 3$ with the coarse moduli space $\overline{\mathcal{M}}_{1,1} \cong \mathbb{P}^1$ parameterizing the $j$-invariants. This gives us the Hom stack $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(4,6) = \overline{\mathcal{M}}_{1,1})$ which is isomorphic to the moduli stack of stable elliptic fibrations over $\mathbb{P}^1$ formulated and arithmetically studied in [HP].

**Corollary 5.** The Hom stack $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ over $\text{char}(\mathbb{F}_q) \neq 2, 3$ with $n \geq 1$ isomorphic to the moduli stack of stable elliptic fibrations over $\mathbb{P}^1$, also known as stable elliptic surfaces, with $12n$ nodal singular fibers and a distinguished section has the following Tate type and not étale pure compactly supported étale cohomology by having the $\ell$-adic rational homology type of 3-sphere together with the eigenvalues of Frobenius map $\text{Frob}_q^*$ expressed as isomorphisms of $\ell$-adic Galois representations

$$H^i_{\text{ét}, c}(\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})/\mathbb{F}_q; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-10n + 1) & i = 20n + 2 \\ \mathbb{Q}_\ell(-10n - 1) & i = 20n - 1 \\ 0 & \text{else} \end{cases}$$
2. Étale topology of \( \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)) \)

Let us recall the definition of the 1-dimensional \( a,b \in \mathbb{N} \) weighted projective stack \( \mathcal{P}(a, b) \).

**Definition 6.** The 1-dimensional \( a,b \in \mathbb{N} \) weighted projective stack is defined as a quotient stack

\[
\mathcal{P}(a, b) := ([\mathbb{A}^2_{x,y} \setminus 0]/\mathbb{G}_m]
\]

Where \( \lambda \in \mathbb{G}_m \) acts by \( \lambda \cdot (x,y) = (\lambda^a x, \lambda^b y) \). In this case, \( x \) and \( y \) have degrees \( a \) and \( b \) respectively. A line bundle \( \mathcal{O}_{\mathcal{P}(a,b)}(m) \) is defined to be a line bundle associated to the sheaf of homogeneous rational functions without poles on \( \mathbb{A}^2_{x,y} \setminus 0 \). The coarse moduli space of \( \mathcal{P}(a, b) \) is given by the coarse moduli map \( c: \mathcal{P}(a, b) \to \mathbb{P}^1 \) which is a smooth projective line.

When the characteristic of the field \( K \) is not equal to 2 or 3, \cite{Hassett} shows that \((\overline{\mathcal{M}}_{1,1})_K \cong [(\text{Spec } K[a_4, a_6] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(4,6)\) by using the Weierstrass equations, where \( \lambda \cdot a_i = \lambda^i a_i \) for \( \lambda \in \mathbb{G}_m \) and \( i = 4, 6 \). Thus, \( a_i \)’s have degree \( i \)’s respectively. Note that this is no longer true if characteristic of \( K \) is 2 or 3, as the Weierstrass equations are more complicated.

**Proposition 7.** For any \( a,b \in \mathbb{N} \) and \( \text{char}(K) \) not dividing \( a \) or \( b \), the Hom stack \( \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)) \) parameterizing morphisms \( f: \mathbb{P}^1 \to \mathcal{P}(a, b) \) with \( f^* \mathcal{O}_{\mathcal{P}(a,b)}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n) \) for \( n \geq 1 \) is a smooth tame Deligne–Mumford stack of finite type and constant dimension of \((a+b)n + 1\) that is isomorphic to the quotient stack \([T/\mathbb{G}_m]\), admitting \( T \) as a smooth schematic cover. A smooth quasiprojective variety \( T \subset H^0(\mathcal{O}_{\mathbb{P}^1}(an)) \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(bn)) \setminus 0 \) parameterizing pairs \((u,v)\) of polynomials with the degrees equal to either \((\deg(u) = an \text{ and } 0 \leq \deg(v) \leq bn)\) or \((\deg(v) = bn \text{ and } 0 \leq \deg(u) \leq an)\) but not both as they are mutually coprime. The quotient stack \([T/\mathbb{G}_m]\) parameterizes pairs \((u,v)\) and \((u',v')\) that are equivalent when there exists \( \lambda \in \mathbb{G}_m \) so that \( u' = \lambda^a \cdot u \) and \( v' = \lambda^b \cdot v \).

**Proof.** These were established in Proposition 8, Remark 9 and Theorem 1 of \cite{HP} which in turn were worked out by \cite{AOV} Theorem 3.2 and \cite{Olsson}. \( \square \)

Note that \( \mathbb{G}_m \) acts on \( T \) properly but not freely unless \((a,b) = (1,1)\). This leads to both the quotient stacks \( \mathcal{P}(a, b) \) and \( \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)) = [T/\mathbb{G}_m] \) to have the finite stabilizers of cyclic groups which in turn makes them smooth tame Deligne–Mumford stacks over \( \text{char}(K) \) not dividing \( a \) or \( b \).

We define \( \text{Poly}_1^{(k,l)} \) that is the space of monic coprime polynomials with degree \( \deg u = k \) and \( \deg v = l \). Study of this space was done in \cite{HP} which in turn was inspired by \cite{FW}.

**Definition 8.** Fix a field \( K \) with algebraic closure \( \overline{K} \). Fix \( k,l \geq 0 \). Define \( \text{Poly}_1^{(k,l)} \) to be the set of pairs \((u,v)\) of monic polynomials in \( K[z] \) so that:

1. \( \deg u = k \) and \( \deg v = l \).
2. \( u \) and \( v \) have no common root in \( \overline{K} \).

**Proposition 9** (Proposition 16 of \cite{HP}). Fix \( d_1,d_2 \geq 0 \). Then for any prime power \( q \) :

\[
|\text{Poly}_1^{(d_1,d_2)}(\mathbb{F}_q)| = \begin{cases} 
q^{d_1+d_2} - q^{d_1+d_2-1}, & \text{if } d_1, d_2 > 0 \\
q^{d_1+d_2}, & \text{if } d_1 = 0 \text{ or } d_2 = 0
\end{cases}
\]

The following diagram summarizes the relationships of the spaces / stacks we consider.
We are now ready to prove the Theorem 3.

2.1. Proof of Theorem 3.

Proof. We consider the ℓ-adic Leray spectral sequence on the Zariski-locally trivial fibration given by the evaluation morphism $ev_\infty$ to determine the ℓ-adic étale Betti numbers of $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b))$.

**Proposition 10.** The evaluation morphism to the target stack $ev_\infty : \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)) \rightarrow \mathcal{P}(a, b)$ mapping a given non-based morphism $f : \mathbb{P}^1 \rightarrow \mathcal{P}(a, b)$ to its basepoint $f(\infty) \in \mathcal{P}(a, b)$ provides a Zariski-locally trivial fibration over $\mathcal{P}(a, b)$ where the fiber is isomorphic to the stack of based morphisms $\text{Hom}^*_n(\mathbb{P}^1, \mathcal{P}(a, b))$ which can be identified with $\text{Poly}_{1}^{(an, bn)}$ the space of monic coprime polynomials $(u, v)$ with $\deg(u, v) = (an, bn)$.

Proof. It suffices to show that the fiber $\text{Hom}^*_n(\mathbb{P}^1, \mathcal{P}(a, b))$ is isomorphic to the space $\text{Poly}_{1}^{(an, bn)}$. The fiber is the stack of based morphisms $\text{Hom}^*_n(\mathbb{P}^1, \mathcal{P}(a, b))$ consisting of the rational maps $f = u/v$ where $u, v$ are coprime polynomials such that the global sections $u, v$ are not simultaneously vanishing at any points of $\mathbb{P}^1$ and also both $u, v$ are monic polynomials with respective $\deg(u, v) = (an, bn)$ as the rational maps must be based morphisms into $\mathcal{P}(a, b)$. While such pairs $(u, v)$ and $(u', v')$ in $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b))$ are equivalent when there exists $\lambda \in \mathbb{G}_m$ so that $u' = \lambda a u$ and $v' = \lambda b v$, as $u, v$ in $\text{Hom}^*_n(\mathbb{P}^1, \mathcal{P}(a, b))$ are both monic polynomials we have $\lambda \equiv 1$ meaning the $\mathbb{G}_m$ to act freely (i.e., acts with trivial stabilizer and acts properly) allowing us to identify $\text{Hom}^*_n(\mathbb{P}^1, \mathcal{P}(a, b))$ which is priori a stack with a space $\text{Poly}_{1}^{(an, bn)}$. □

Let us recall the (compactly supported) étale cohomology of the base $\mathcal{P}(a, b)$.

**Proposition 11.** There are isomorphisms of ℓ-adic Galois representations

$$H^i_{\text{ét}}(\mathcal{P}(a, b)_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-1) & i = 2 \\ \mathbb{Q}_\ell(0) & i = 0 \\ 0 & \text{else} \end{cases}$$

Next, we need the compactly supported étale cohomology of the fiber $\text{Poly}_{1}^{(an, bn)}$ which follows from [HP, FW].
Proposition 12. There are isomorphisms of $\ell$-adic Galois representations

$$H^i_{\text{et},c}(\text{Poly}_1^{(an, bn)}_{/\mathbb{F}_q}; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-(an + bn)) & i = 2(an + bn) \\ \mathbb{Q}_\ell(-(an + bn - 1)) & i = 2(an + bn) - 1 \\ 0 & \text{else} \end{cases}$$

Proof. We note that we can express $\text{Poly}_1^{(an, bn)}$ as the quotient of a finite group $S_{an} \times S_{bn}$ acting on the complement of the hyperplane arrangement in $\mathbb{A}^{an+bn}$. It is well known by Kim and Behrend that the Frobenius acts by $q^{-i}$ on $H^i$ of the complement of the hyperplane arrangement. By transfer, we see that Frobenius acts by $q^{-i}$ on $H^i_{\text{et},c}(\text{Poly}_1^{(an, bn)}_{/\mathbb{F}_q}; \mathbb{Q}_\ell)$ as well. In the end, we use the exact point count of Proposition 16 of [HP] together with the Poincaré Duality and the Grothendieck-Lefschetz trace formula for smooth but not necessarily projective variety

$$\left| \text{Poly}_1^{(an, bn)}(\mathbb{F}_q) \right| = \sum_{i=0}^{2(an+bn)} (-1)^i \cdot q^{(an+bn)-i} \cdot \dim_{\mathbb{Q}_\ell}(H^i_{\text{et}}(\text{Poly}_1^{(an, bn)}_{/\mathbb{F}_q}; \mathbb{Q}_\ell)^{\vee})$$

to conclude the above. \hfill \Box

We now consider the $\ell$-adic Leray spectral sequence for Deligne–Mumford algebraic stacks of finite type using the construction of Section 1.3 in Behrend. As some of the spaces / stacks in consideration are smooth but not proper, we will first work with $\ell$-adic étale cohomology in terms of its dual $\mathbb{Q}_\ell$-vector spaces through the Poincaré Duality applied to compactly supported $\ell$-adic étale cohomology. As the $E_2$ page in dual $\mathbb{Q}_\ell$-spaces is the same as $S^1 \to \text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)) \to S^2$, we only need to decide whether we have either $S^1 \times S^2$ case when the transgression differential $d_2^{(2,0)} = 0$ is trivial or $S^3$ case when the transgression differential $d_2^{(2,0)} = \pm m \neq 0$ is an isomorphism.

Proposition 13. Dimensions of $\mathbb{Q}_\ell$-vector spaces $H^i_{\text{et},c}(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b))_{/\mathbb{F}_q}; \mathbb{Q}_\ell)$ are one-dimensional for $i = 2(a+b)n + 2$, $i = 2(a+b)n - 1$ and vanishes for all other $i$.

Proof. If the transgression differential $d_2^{(2,0)} = 0$ was trivial then by the above consideration we would have the first étale Betti number $\dim_{\mathbb{Q}_\ell}(H^1_{\text{et}}(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b))_{/\mathbb{F}_q}; \mathbb{Q}_\ell)^{\vee}) = 1$ which makes the orbifold fundamental group, or the fundamental group of the Deligne-Mumford stack $\pi_1^{\text{orb}}(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)))$ to have the $\mathbb{Z}$ summand and thus is an infinite group. This is not the case, however, as the base $\mathcal{P}(a, b)$ of the fibration is simply-connected, we have surjective map $i_* : \pi_1(\text{Poly}_1^{(an, bn)}) \to \pi_1^{\text{orb}}(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)))$ where $\pi_1(\text{Poly}_1^{(an, bn)}) \cong \mathbb{Z}$ as shown in Segal and ker$(i_*)$ is nonzero by the consideration of a simple root of $u$ moves once around a simple root of $v$ for a given rational function $u/v$ as shown in LEPI which implies $\pi_1^{\text{orb}}(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)))$ is isomorphic to a finite cyclic group. Hence the transgression differential $d_2^{(2,0)} = \pm m \neq 0$ is an isomorphism. \hfill \Box

Remark 14. It leads to conjecture that $\pi_1^{\text{orb}}(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)))$ is isomorphic to $\mathbb{Z}_{(a+b)n}$. This is not obvious as the homotopy groups of algebraic stacks are subtle in general (see Noohi).
As $H^i_{et,f}(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b)); \mathbb{Q}_f)$ are all one dimensional, the trace of $\text{Frob}_q$ on each of these $\mathbb{Q}_f$-vector spaces is just the corresponding eigenvalue $\lambda_i$ of $\text{Frob}_q$.

When $i = 2(an + bn) + 2$, the connectedness of $\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b))$ together with the Poincaré Duality implies that $\lambda_{2(an+bn)+2} = q^{(a+b)n+1}$.

Plugging all of the above into the Sun-Behrend trace formula (1):

$$\#_q(\text{Hom}_n(\mathbb{P}^1, \mathcal{P}(a, b))) = q^{(a+b)n+1} - q^{(a+b)n-1} = \lambda_{2(an+bn)+2} - \lambda_{2(an+bn)-1}$$

which implies that $\lambda_{2(an+bn)-1} = q^{(a+b)n-1}$ as claimed.

This finishes the proof of Theorem 3.

\[ \square \]

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