DEL PEZZO SURFACES IN $\mathbb{P}^5$ AND CALABI–YAU THREEFOLDS IN $\mathbb{P}^6$.

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Abstract. We find a simple geometric construction of Tonoli examples of Calabi–Yau threefolds of degree 17 in $\mathbb{P}^6$. We then observe and investigate an analogy between the descriptions in terms of Pfaffians of vector bundles of anti-canonically embedded del Pezzo surfaces in $\mathbb{P}^5$ and of known Calabi–Yau threefolds in $\mathbb{P}^6$.

1. INTRODUCTION

From the famous Serre construction we know that a codimension 2 submanifold $X \subset \mathbb{P}^n$ that is subcanonical (i.e. $\omega_X \cong \mathcal{O}_X(l)$ for some $l \in \mathbb{Z}$) can be seen as the zero locus of a section of a rank two vector bundle $E$. In particular in such case we have an exact sequence

$$0 \to \mathcal{O}_X \to E \to \mathcal{I}_X(c_1(E)) \to 0.$$ 

There is a similar construction in codimension 3. By answering Okonek question (see \[\text{Oko94}\]), Walter showed in \[\text{Wal96}\] (see also \[\text{EPW01}\]) that, if $n - 3$ is not divisible by 4, then each locally Gorenstein subcanonical subscheme of codimension 3 in $\mathbb{P}^n$, admits a Pfaffian resolution

$$0 \to \mathcal{O}_X(-2s - t) \to E^*(-s - t) \to E(-s) \to \mathcal{I}_X \to 0,$$

where the vector bundle $E$ is of rank $2r + 1$ and $s = c_1(E) + rt$. Moreover, in such case $\omega_X = \mathcal{O}_X(t + 2s - n - 1)$. In such situation we shall say that $E$ defines $X$ through the Pfaffian construction.

This Pfaffian construction was applied in \[\text{Cal97}\] to construct canonically embedded surfaces of general type in $\mathbb{P}^5$ and in \[\text{Ton04}\] and \[\text{ST02}\] to construct Calabi–Yau threefolds in $\mathbb{P}^6$. The latter examples will be referred as Tonoli Calabi–Yau threefolds. The resulting examples of both works have degrees $12 \leq d \leq 17$. In particular, in the case of degree 17 the authors of \[\text{Ton04}\] and \[\text{ST02}\] discover three distinct families of Calabi–Yau threefolds.

Tonoli Calabi–Yau threefolds as one of the simplest Calabi–Yau threefolds which are not described as complete intersections in toric varieties are good testing examples for the mirror symmetry conjecture. The simplest of them, i.e. those which are arithmetically Cohen Macaulay, have already been studied with partial success from this point of view (see \[\text{Re00, Bøh08, Kan12}\]). On the other hand there is no single representative in the huge database \[\text{vEXS}\] of Picard-Fuchs operators whose invariants would fit with the invariants of any non ACM Tonoli Calabi–Yau threefold (i.e of degree $d \geq 15$). Motivated by this we tried to understand the geometry of these examples.

The construction of families of degree 17 Calabi–Yau manifolds in $\mathbb{P}^6$ can be summarized as follows. Let $V_3, V_7$, be two vector space of dimension 3 and 7 respectively and let $W = V_3 \otimes V_7$; then $\mathbb{P}(W)$ contains a natural subvariety Seg consisting of classes of simple tensors. We have, Seg is the image of the Segre embedding of $\mathbb{P}(V_3) \times \mathbb{P}(V_7)$ into $\mathbb{P}(W)$. In particular, we have a map $\pi : \text{Seg} \to \mathbb{P}(V_3)$. For any point $p$ in the Grassmannian $G(16, W)$ we shall write $L_p$ for the corresponding linear space of dimension 15 in $\mathbb{P}(W)$. Consider

$$M_k = \{p \in G(16, V_3 \otimes V_7) | \pi|_{L_p \cap \text{Seg}} \text{ has } k - \mathbb{P}^2 \text{ fibers}\}.$$ 

Observe now that each point $p \in G(16, V_3 \otimes V_7)$ defines a unique vector bundle of rank 13 on $\mathbb{P}(V_7)$ in the following way: if $p \in G(16, V_3 \otimes V_7)$ then there is a natural map $L_p \otimes V_7^* \to V_3$ that defines a map

$$L_p \otimes \mathcal{O}_{\mathbb{P}^6} \to V_3 \otimes \mathcal{O}_{\mathbb{P}^6}(1).$$

The kernel of this map is a vector bundle that we shall call $E_p$. Tonoli proves that, for $k = 8, 9, 11$ the family of bundles $E_p$ parametrized by $p \in M_k$ defines a family of Pfaffian Calabi–Yau threefolds of degree 17.

2000 Mathematics Subject Classification. Primary: 14J32.

Key words and phrases. Calabi-Yau threefolds, surfaces of general type, Pfaffian resolutions, geometric syzygies.
The main results of this paper is a simple construction of Tonoli families of Calabi–Yau threefolds of degree $d = 17$. The main ingredient of this construction is the following theorem.

**Theorem 1.1.** For $k \in \{8, 9, 11\}$, under above notation, $p \in \mathcal{M}_k$ if and only if

1. $k = 11$ and $L_p$ contains the graph $\Gamma_{v_1} \subset \text{Seg}$ of a linear embedding $v_1 : \mathbb{P}^2 \to \mathbb{P}^6$;
2. $k = 9$ and $L_p$ contains the graph $\Gamma_{v_2} \subset \text{Seg}$ of a 2-tuple Veronese embedding $v_2 : \mathbb{P}^2 \to \mathbb{P}^6$;
3. $k = 8$ and $L_p$ contains the closure of the graph $\Gamma_{v_3}$ of a birational map $v_3 : \mathbb{P}^2 \to \mathbb{P}^6$ defined by a system of cubics passing through some point.

This puts Tonoli’s construction in a geometrical context which makes it easier to work with Tonoli examples. In particular, the unirationality of Tonoli families becomes clear. Moreover, it is straightforward to write an example in characteristic 0. We also use our construction to recompute the dimensions of the Tonoli families of Calabi–Yau threefolds in $\mathbb{P}^6$, and point out an error in Tonoli’s computation. We prove that a Tonoli Calabi–Yau threefold of degree $d = 17$ corresponding to $k = 11$ has Picard group of rank $\geq 2$. Finally, we conjecture that the considered Tonoli family is locally complete and thus the Hodge numbers of $E$ bundles these descriptions and descriptions of Tonoli Calabi–Yau threefolds in $\mathcal{M}$, and is naturally described by the Pfaffians of $O_2$, and is naturally described by the Pfaffians of $O_2$.

For a Calabi–Yau threefold $X$ a bundle $E$ from Table 1 defining it and in the same way to any Calabi–Yau threefold $X$ a bundle $F_X$ from Table 2. The analogy can then be formalized by the following theorem.

**Theorem 1.2.** Let $X \subset \mathbb{P}^5$ be a general element of a family of Tonoli Calabi–Yau threefolds and let $F_X$ be as above. Then there exists a map $F_X \to 2O_{\mathbb{P}^6}$ such that its kernel $E$ restricted to any $\mathbb{P}^5$ defines a del Pezzo surface of degree $\deg(X) - 9$. Conversely, for a generic del Pezzo surface $D \subset \mathbb{P}^5$ with associated bundle $E_D$ there exists an extension $E_D'$ of the bundle $E_D$ to $\mathbb{P}^6$, such that a generic bundle $F \in \text{Ext}^1(E_D', 2O)$ defines a Calabi–Yau threefold in $\mathbb{P}^6$ of degree $\deg(D) + 9$.

| degree | Vector bundle defining projected del Pezzo surfaces in $\mathbb{P}^6$ |
|--------|-------------------------------------------------------------|
| 3      | $O_{\mathbb{P}^5}(-1) \oplus 2O_{\mathbb{P}^5}(1)$         |
| 4      | $2O_{\mathbb{P}^5} \oplus O_{\mathbb{P}^5}(1)$           |
| 5      | $5O_{\mathbb{P}^5}$                                       |
| 6      | $O_{\mathbb{P}^5}(1) \oplus 2O_{\mathbb{P}^5}$           |
| 7      | $\ker(\psi)$, where $\psi : 11O_{\mathbb{P}^5} \to 2O_{\mathbb{P}^5}(1)$ is a general map |
| 8      | $\ker(\psi)$, where $\psi : 14O_{\mathbb{P}^5} \to 3O_{\mathbb{P}^5}(1)$ is a special map with more syzygies |
| 9      | $\ker(\psi)$, where $\psi : 17O_{\mathbb{P}^5} \to 4O_{\mathbb{P}^5}(1)$ is a special map with special syzygies |

| degree | Vector bundle defining the Tonoli examples of Calabi–Yau threefolds |
|--------|---------------------------------------------------------------------|
| 12     | $O_{\mathbb{P}^5}(-1) \oplus 2O_{\mathbb{P}^5} \oplus 2O_{\mathbb{P}^5}(1)$ |
| 13     | $4O_{\mathbb{P}^5} \oplus O_{\mathbb{P}^5}(1)$                    |
| 14     | $7O_{\mathbb{P}^5}$                                                |
| 15     | $O_{\mathbb{P}^5}(1) \oplus 3O_{\mathbb{P}^5}$                    |
| 16     | $\ker(\psi)$, where $\psi : 13O_{\mathbb{P}^5} \to 2O_{\mathbb{P}^5}(1)$ is a general map |
| 17     | $\ker(\psi)$, where $\psi : 16O_{\mathbb{P}^5} \to 3O_{\mathbb{P}^5}(1)$ is a special map with more syzygies |

Table 1: Vector bundles defining del Pezzo surfaces  
Table 2: Vector bundles defining Tonoli Calabi–Yau threefolds
This observation, being rather straightforward for degree \( d \leq 16 \), is nontrivial for \( d = 17 \). Our proof follows from the description of Theorem 1.1.

Taking one step further we formulate some conjectures for the upper bound on the degree of Calabi–Yau threefolds in \( \mathbb{P}^6 \). More precisely by analogy to the case of del Pezzo surface we expect that there will be no smooth Calabi–Yau threefolds of degree \( d \geq 19 \) in \( \mathbb{P}^6 \). Finally, we make some speculation about the possibility of constructing a degree 18 Calabi–Yau threefolds we description analogous to the one of a del Pezzo surface of degree 9.

The structure of the paper is the following. In Section 2 we recall some basic facts from the theory of Pfaffians and provide some preliminary results needed throughout the paper. In particular, we describe a method to compute dimensions of families of Calabi–Yau threefolds obtained as Pfaffians varieties associated to families of vector bundles.

In Section 3 we quickly go through the constructions of del Pezzo surfaces of degree \( d_P \leq 7 \) and Calabi–Yau threefolds of degree \( d_X \leq 16 \). We observe that they are strictly related. In particular Theorem 1.1 takes a stronger form in these cases. In Section 4 we describe anti-canonically embedded del Pezzo surfaces of degree 8 in \( \mathbb{P}^5 \) in terms of Pfaffians varieties. In Section 5 we prove Theorem 1.1 providing a new description of Tonoli Calabi–Yau threefolds. We then complete the discussion of the analogy between del Pezzo surfaces and Tonoli Calabi–Yau threefolds and finish the proof of Theorem 1.2.

In Section 6 we collect experimental information on the Hartshorne Rao module of a projected del Pezzo surface of degree 9 in \( \mathbb{P}^5 \) in order to put foundations for a future study of surfaces of Calabi–Yau threefolds of degree \( d \geq 18 \) in \( \mathbb{P}^6 \). We finish with a conjecture stating that 18 is the highest degree a canonical surface in \( \mathbb{P}^5 \) can have. This would imply that the same bound holds for the degree of Calabi–Yau threefolds in \( \mathbb{P}^6 \).

Acknowledgments. We would like to thank Ch. Okonek for all his advice and support, and A. Boralevi, S. Cynk, D. Faenzi, L. Gruson, A. Kresch, A. Langer, P. Pragacz, J. Weyman for comments and discussions.

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Lemma 2.3. From a subset of a suitable Grassmannian we obtain a family of subcanonical varieties. For flatness let $E = \text{ker}(\ldots)$.

Proposition 2.4. Starting from an algebraic subset of a suitable Grassmannian we construct a family $M \in M$ of Hartshorne–Rao modules satisfying the assumptions above. To each of these Hartshorne–Rao modules we associate a vector bundle $E$ on $X$.

Corollary 2.5. The dimension of the family of varieties obtained as Pfaffian varieties associated to a bundle $E = \ker(kO_{P^n} \rightarrow lO_{P^n}(1))$ is $h^0(\text{Ext}^1(I_X, J_X) = \text{Ext}^2(I_X)(7)$.

Proof. First arguing as in [KMR98, Prop. 2] we deduce that $N_{X|P^n} = \text{Ext}^1(I_X, J_X) = \text{Ext}^2(I_X)(7)$. Then we obtain from [Wey79] the free resolution of the sheaf $\text{Ext}^2(I_X, J_X)$. □
Proof. We have a map $\phi$ from an open subset of the vector space of global sections of $(\wedge^2 E)(1)$ to a component of the Hilbert scheme of subvarieties of $\mathbb{P}^n$. If $s \in H^0((\wedge^2 E)(1))$ is a fixed general section and $X$ its image under this map, then the map: $H^0((\wedge^2 E)(1)) \to H^0(N_{X|\mathbb{P}^n})$ in Proposition 2.4 is interpreted as the tangent map to $\phi$ in $s$. We want to prove that the dimension of the image of $\phi$ is $h^0(\wedge^2 E)(1) - \dim \text{Hom}(E, E)$. It is hence enough to prove that the rank of this tangent map in a general point is $h^0(\wedge^2 E)(1) - \dim \text{Hom}(E, E)$. From the splitting of the long exact sequence from Proposition 2.4 into short ones we get:

$$0 \to F \to (\wedge^2 E)(1) \to N_{X|\mathbb{P}^n} \to 0,$$

$$0 \to G \to E \otimes E^* \to F \to 0,$$

$$0 \to E^*(-4) \to E(-3) \oplus (S^2 E^*)(-1) \to G \to 0,$$

for some bundles $E, G$ on $\mathbb{P}^n$. We now prove $h^0(G) = h^1(G) = 0$. Indeed, it follows directly from the exact sequences:

$$(2.4) \quad 0 \to E \to k\mathcal{O}_{\mathbb{P}^n} \to l\mathcal{O}_{\mathbb{P}^n}(1)) \to 0,$$

its twist, twisted duals, and the resulting resolution of $(S^2 E^*)(-1)$:

$$0 \to \left(\begin{array}{c} l \\frac{1}{2} \end{array}\right) \mathcal{O}_{\mathbb{P}^n}(-4) \to \left(\begin{array}{c} k \\frac{1}{2} \end{array}\right) \mathcal{O}_{\mathbb{P}^n}(-1) \to kE(-1) \to (S^2 E^*)(-1) \to 0$$

It follows that $h^0(F) = h^0(E \otimes E^*) = \dim \text{Hom}(E, E)$.

To complete our dimension counts we use the following.

Corollary 2.6. The dimension of the image of a family constructed by means of Lemma 2.3 into the deformation space is equal to the dimension of the image of the family $E$ into the deformation space of bundles increased by the dimension of the space $\mathbb{P}(H^0((\wedge^2 E)(1)))$ and decreased by the dimension of the automorphism space of $E_x$ for a general $x \in U$.

Since we are working with Calabi–Yau threefolds and their surface sections it is good to have in mind the following lemma relating the Hartshorne–Rao module of a variety with the Hartshorne–Rao module of its hyperplane section:

Lemma 2.7. Let $X \subset \mathbb{P}^n$ be a variety satisfying $h^2(I_{X|\mathbb{P}^n}(j)) = 0$ for any $j \in \mathbb{Z}$. Let $M$ be the Hartshorne–Rao module of $X$. Assume that $M$ has a presentation

$$l\mathcal{S}_{\mathbb{P}^n}(-1) \xrightarrow{m} k\mathcal{S}_{\mathbb{P}^n} \to M \to 0,$$

given by a matrix $m$ of linear entries in the coordinate ring $\mathcal{S}_{\mathbb{P}^n}$ of $\mathbb{P}^n$. Let $H$ be a hyperplane defined by a linear equation $h = 0$. Then the Hartshorne–Rao module $M'$ of $X \cap H$ has a presentation:

$$l\mathcal{S}_{H}(-1) \xrightarrow{m'} k\mathcal{S}_{H} \to M' \to 0,$$

with $\mathcal{S}_{H} = \mathcal{S}_{\mathbb{P}^n}/_{<h>}$ the coordinate ring of the hyperplane $H$, and $m'$ the image of $m$ via the projection map $\mathcal{S}_{\mathbb{P}^n} \to \mathcal{S}_{\mathbb{P}^n}/_{<h>}$.\n
Proof. For each $j$ we have the exact sequence:

$$0 \to I_{X|\mathbb{P}^n}(j - 1) \xrightarrow{\lambda} I_{X|\mathbb{P}^n}(j) \to I_{(X \cap H)|H}(j) \to 0,$$

where $\lambda$ is given by multiplication by $h$. From the associated cohomology sequence in each degree and the assumed vanishing $h^2(I_{X|\mathbb{P}^n}(j)) = 0$ we obtain $M' = M/(hM)$ and the presentation follows.\n
3. Del Pezzo surfaces of degree $\leq 7$ and Calabi–Yau threefolds of degree $\leq 16$

In this section we describe anti-canonically embedded del Pezzo surfaces of degree $d \leq 7$ in $\mathbb{P}^5$ in terms of Pfaffians of vector bundles. Let us first make some general remarks on del Pezzo surfaces embedded in $\mathbb{P}^5$ via a subsystem of the canonical class.
3.1. Del Pezzo surfaces in $\mathbb{P}^5$. Recall that an anti-canonical model of a smooth del Pezzo surface of degree $\geq 3$ is a smooth surface of degree $n$ in $\mathbb{P}^n$ for $3 \leq n \leq 9$. Moreover, for each degree $\neq 8$ there is one family of such surfaces and for $n = 8$ two families.

Consider the anti-canonical embedding of these surfaces in $\mathbb{P}^5$. More precisely for $3 \leq n \leq 7$ we denote $D_n$ the image of the anti-canonical embedding of the del Pezzo surface of degree $n$ composed with a general linear map $\mathbb{P}^n \to \mathbb{P}^5$. For $n = 8$ we have two del Pezzo surfaces $F_1$ and $\mathbb{P}^1 \times \mathbb{P}^1$. Their corresponding embedding will be denoted by $D_8^1$ and $D_8^2$. These embedded surfaces are clearly subcanonical thus by the theorem of Walter they admit Pfaffian resolutions, that we shall study.

It follows from the Kodaira vanishing theorem and the Riemann–Roch theorem that $H^i(\mathcal{I}_D) = 0$ for $i > 1$. This implies that the bundle $\mathcal{E}$ in the Pfaffian resolution of our del Pezzo surface is the sheafification of the module $S_{yz^1}(\bigoplus_{k \in \mathbb{Z}} H^1(\mathcal{I}_D(k)))$ over the coordinate ring of $\mathbb{P}^5$ plus a possible direct sum of line bundles.

Lemma 3.1. The Hilbert function of the Hartshorne–Rao module of a del Pezzo surface $D \subset \mathbb{P}^5$ of degree $d$ is 0 for $d \leq 5$ and for $d \in \{6, 7, 8, 9\}$ take values starting from degree 1:

$(0, 1, 0, \ldots), (0, 2, 1, 0, \ldots), (0, 3, 4, 0, \ldots), (0, 4, 7, 0, \ldots)$ respectively. Moreover these del Pezzo surfaces satisfy the maximal rank assumption.

Proof. We first check the maximal rank assumption by checking a random example and concluding by semi-continuity as in [Kap10, Lemm. 5.1]. The values of the Hilbert function are then computed from the Riemann–Roch theorem as in [Ton04].

3.2. Constructions of degree $\leq 7$ del Pezzo surfaces. We can now get a description of a general del Pezzo surface of degree $d \leq 7$ in $\mathbb{P}^5$.

Corollary 3.2. A general del Pezzo surface of degree $d \leq 7$ in $\mathbb{P}^5$ is described as a Pfaffian variety associated to the bundle:

1. $O_{\mathbb{P}^5}(-1) \oplus 2O_{\mathbb{P}^5}(1)$ for $d = 3$
2. $2O_{\mathbb{P}^5} \oplus O_{\mathbb{P}^5}(1)$ for $d = 4$
3. $5O_{\mathbb{P}^5}$ for $d = 5$
4. $O_{\mathbb{P}^5}(1) \oplus 2O_{\mathbb{P}^5}$ for $d = 6$
5. $\ker(\psi)$ for $d = 7$, where $\psi: 11O_{\mathbb{P}^5} \to 2O_{\mathbb{P}^5}(1)$ is a general map

Proof. From Lemma 3.1 we know the bundles up to a direct sum of line bundles. We next use the results of [KK13, Section 3] and proceed analogously.

3.3. Analogy with Tonoli Calabi–Yau threefolds of degree $\leq 16$. Recall that Tonoli Calabi–Yau threefolds of degree $k \leq 16$ are obtained by the Pfaffian construction applied to the vector bundles in $\mathbb{P}^6$ characterized in Table 2. Comparing the vector bundles appearing in the Pfaffian constructions of del Pezzo surfaces and Tonoli Calabi–Yau threefolds we observe that the description of a general del Pezzo surface of degree $d$ in $\mathbb{P}^5$ is similar to the description of Tonoli Calabi–Yau threefold of degree $d + 9$ in $\mathbb{P}^6$. The relation is partially explained by the following.

Proposition 3.3. Let $E$ and $F$ be vector bundles on $\mathbb{P}^5$ and $\mathbb{P}^6$ respectively, such that they are related by the exact sequence:

$0 \to F \to E|_{\mathbb{P}^5} \to 2O_{\mathbb{P}^5} \to 0.$

Assume moreover that both bundles define smooth codimension 3 varieties $X \subset \mathbb{P}^6$ and $D \subset \mathbb{P}^5$. Then $X$ is a Calabi–Yau threefold of degree $d$ if and only if $D$ is a del Pezzo surface of degree $d - 9$.

Proof. The proof follows from the adjunction formula and from the following formula on the degree of a Pfaffian variety defined by a vector bundle in terms of Chern classes of the vector bundle.

Lemma 3.4 (see [Oko94]). If $E$ is a vector bundle of rank $2r + 1$ on $\mathbb{P}^n$ and $s \in H^0(\Lambda^2 E(1))$ a general section such that it defines via the Pfaffian construction a variety $Y$ of codimension 3. Then

\[ \text{deg}(Y) = r c_1^2(E) H^{n-2} + c_1(E) c_2(E) H^{n-3} + (r^2 + r)c_3(E)H^{n-1} + \]

\[ c_2(E) H^{n-2} - c_3(E) H^{n-3} + \frac{r(2r+1)(2r+2)}{12} H^n \]
Proof. The proof is based on a computation using Hirzebruch–Riemann–Roch theorem, the restriction of the Pfaffian sequence to a general $\mathbb{P}^5$, and the fact that the degree of a set of distinct points is equal to the Euler Characteristic of its structure sheaf.

Remark 3.6. The bundles $F_d$ of degree $d$ in $\mathbb{P}^5$ and $E_D$ be the vector bundle on $\mathbb{P}^5$ defining $D$ through the Pfaffian construction. Consider a Tonoli Calabi–Yau threefold $X$ of degree $d + 9$ and its associated bundle $F_X$.

Observe that for $d \leq 7$ the bundles $E_D$ and $F_X$ are determined by $d$ up to a sum of rank 2 line bundles of the form $\mathcal{O}(\pm i) \oplus \mathcal{O}(\pm i - 1)$. For our purpose we choose the bundles from tables 1 and 2 and denote them $E_d$ and $F_d$ respectively.

**Proposition 3.5.** If $d \leq 7$ the bundle $E_d$ is obtained as the cokernel of a generic surjective map $F_d|_{\mathbb{P}^5} \rightarrow 2O_{\mathbb{P}^5}$. Moreover, the bundle $E_d$ admits an extension $E'_d$ such that a general bundle $F'_d \in \text{Ext}^1(E'_D, 2O_{\mathbb{P}^5})$ is isomorphic to $F_d$.

Proof. For each of the bundles $F_d$ for $d \leq 7$ we compute the restriction to a generic $\mathbb{P}^5$. We get

1. $F_3|_{\mathbb{P}^5} = O_{\mathbb{P}^5}(-1) \oplus 2O_{\mathbb{P}^5} \oplus 2O_{\mathbb{P}^5}(1)$,
2. $F_4|_{\mathbb{P}^5} = 4O_{\mathbb{P}^5} \oplus O_{\mathbb{P}^5}(1)$,
3. $F_5|_{\mathbb{P}^5} = 7O_{\mathbb{P}^5}$,
4. $F_6|_{\mathbb{P}^5} = O_{\mathbb{P}^5}(1) \oplus 4O_{\mathbb{P}^5}$,
5. $F_7|_{\mathbb{P}^5} = 2O_{\mathbb{P}^5}(1) \oplus O_{\mathbb{P}^5} \ker(\psi')$ for $\psi': 13O_{\mathbb{P}^5} \rightarrow 2O_{\mathbb{P}^5}(1)$ is a general map.

It is now easy to check the first part of the Proposition. For the second part we take $E'_d$ being one of the following

1. $E'_3|_{\mathbb{P}^5} = O_{\mathbb{P}^5}(-1) \oplus 2O_{\mathbb{P}^5}(1)$,
2. $E'_4|_{\mathbb{P}^5} = 2O_{\mathbb{P}^5} \oplus O_{\mathbb{P}^5}(1)$,
3. $E'_5|_{\mathbb{P}^5} = 5O_{\mathbb{P}^5}$,
4. $E'_6|_{\mathbb{P}^5} = O_{\mathbb{P}^5}(1) \oplus O_{\mathbb{P}^5}$,
5. $E'_7|_{\mathbb{P}^5} = \ker(\psi''')$ for $\psi'''$: $11O_{\mathbb{P}^5} \rightarrow 2O_{\mathbb{P}^5}(1)$ is a general map.

It is clear that $E'_d|_{\mathbb{P}^5} = E_d$ for a generic $\mathbb{P}^5 \subset \mathbb{P}^6$ and we conclude by observing that for each $d$ there is an exact sequence

$$0 \rightarrow E'_d \rightarrow F_d \rightarrow 2O_{\mathbb{P}^5} \rightarrow 0,$$

and $F_3$ is always the general element fitting in the exact sequence. Indeed, for $d \leq 6$ we have $\text{Ext}^1(E'_D, 2O_{\mathbb{P}^5}) = 0$ and $F_d = E'_d \oplus 2O_{\mathbb{P}^6}$, whereas for $d = 7$ any bundle $F$ appearing in the exact sequence

$$0 \rightarrow E'_7 \rightarrow F \rightarrow 2O_{\mathbb{P}^5} \rightarrow 0,$$

is the kernel of some map $\theta: 13O_{\mathbb{P}^5} \rightarrow 2O_{\mathbb{P}^5}(1)$, hence $F_d$ is general among them.

□

Remark 3.6. The bundles $F_d|_{\mathbb{P}^5}$ and $E'_d$ for $d \leq 7$ define through the Pfaffian construction general type surfaces in their canonical embedding in $\mathbb{P}^5$ and del Pezzo threefolds in their half-anti-canonical embeddings in $\mathbb{P}^6$ respectively.

**Corollary 3.7.** The Theorem 1.2 holds for del Pezzo surfaces of degree $\leq 7$ and Tonoli Calabi–Yau threefolds of degree $\leq 16$.

Remark 3.8. In view of Propositions 3.3 and 3.5 it is natural to construct the Calabi–Yau threefolds and del Pezzo surfaces in pairs. In fact having $E$ or $F$ one can try to reconstruct the other. The only thing missing and in fact the most important thing in view of the cases with pairs of degrees $(8, 17)$ and $(9, 18)$ is the existence of a section of the new constructed $\wedge^2 F(1)$ and $\wedge^2 E(1)$ defining a smooth Pfaffian variety (in particular of codimension 3).

In the next section we shall study this phenomenon for del Pezzo surfaces of degree 8 and Calabi–Yau threefolds of degree 17.
4. Descriptions of del Pezzo surfaces of degree 8 in $\mathbb{P}^5$

We shall now describe the Pfaffian resolutions of del Pezzo surfaces of degree $d = 8$. We shall look for modules $M$ with Hilbert function $(0, 3, 4, 0, \ldots)$ which admit a minimal resolution $14\mathcal{S}_{\mathbb{P}^5} \to 3\mathcal{S}_{\mathbb{P}^5}(1) \to M \to 0$, where $\mathcal{S}_{\mathbb{P}^5}$ is the homogeneous coordinate ring of $\mathbb{P}^5$. Observe that for such modules $c_1(Sy^{2}(M)) = -3$ and $rk(Sy^{2}(M)) = 11$. It follows by adjunction formula that if $\bigwedge^2(Sy^{2}(M))(1)$ admits a section defining a smooth Pfaffian variety $D$, then $D$ must be a del Pezzo surface in its anti-canonical embedding (composed with a projection). Moreover, by Remark 4.1 if a smooth surface $D$ has a Hartshorne–Rao module of the above form then it is defined by a Pfaffian variety associated to $Sy^{2}(M)$. In this case we shall say that $M$ defines a family of del Pezzo surfaces.

Finding a module with the above minimal presentation is equivalent to find an embedding $\mathbb{P}^{13} \to \mathbb{P}^{17} = (\mathbb{P}^2 \times \mathbb{P}^5)$.

We shall hence also say that $\mathbb{P}^{13} \subset \mathbb{P}^2 \times \mathbb{P}^5$ defines a family of del Pezzo surfaces.

In any case the intersection $\mathbb{P}^{13} \cap (\mathbb{P}^2 \times \mathbb{P}^5) = \mathbb{P}(\mathcal{G})$ can be seen as the projectivization of a sheaf $\mathcal{G}$ on $\mathbb{P}^2$ given by the cokernel of the map

$$4\mathcal{O}_{\mathbb{P}^2}(-1) \to 6\mathcal{O}_{\mathbb{P}^2} \to \mathcal{G}$$

corresponding to the four linear equations defining the $\mathbb{P}^{13}$.

**Proposition 4.1.** If $\mathcal{P}$ is a general $\mathbb{P}^{13}$ such that the bidual sheaf $\mathcal{G}^{**}$ of its corresponding sheaf $\mathcal{G}$ is a rank two vector bundle isomorphic to

(1) $\mathcal{O}_{\mathbb{P}^2}(2) \otimes \mathcal{O}_{\mathbb{P}^2}(2)$

(2) $\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{\mathbb{P}^2}(3)$

then $\mathcal{P}$ defines a family of del Pezzo surface of type $D_8^1$ and $D_8^2$ respectively.

**Proof.** Observe that from the resolution

$$0 \to 4\mathcal{O}_{\mathbb{P}^2}(-1) \to 6\mathcal{O}_{\mathbb{P}^2} \to \mathcal{G}^{**} \to \mathcal{R} \to 0,$$

where $\mathcal{R}$ is a torsion sheaf, we deduce that in the cases stated above $\mathcal{R}$ is supported in 6 and 7 points respectively. We next check on an example that the corresponding bundle has 6 and 7 independent sections respectively and in each case that some section defines a smooth surface which is a del Pezzo surface. We also check that in one case this is $D_8^1$ and in the other $D_8^2$. We conclude by semi-continuity as in [Ton04] Sec 3.1].

**Remark 4.2.** We can adapt the notation of Tonoli to the case of del Pezzo surfaces $M_8^D = \{M \in G(14, 18) : \text{the corresponding } \mathbb{P}(\mathcal{G}) \text{ has } k \text{-special fibers}\}$. From Chern class computation we easily see that if $\mathcal{G}^{**} = \mathcal{O}_{\mathbb{P}^2}(2) \otimes \mathcal{O}_{\mathbb{P}^2}(2)$ then $M \in M_8^D$, whereas if $\mathcal{G}^{**} = \mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{\mathbb{P}^2}(3)$ then $M \in M_8^P$.

**Remark 4.3.** In fact if we consider the space $M_8^D$ and $M_8^P$ then for $M$ on a Zariski open set the corresponding bidual bundle $\mathcal{G}^{**}$ is $\mathcal{O}_{\mathbb{P}^2}(2) \otimes \mathcal{O}_{\mathbb{P}^2}(2)$ and $\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{\mathbb{P}^2}(3)$ respectively. It follows that we can deduce by parameter count as in [Ton04] that our construction gives a generic $D_8^1$ and $D_8^2$. In this way we can prove that we construct generic elements of both components of the Hilbert scheme of del Pezzo surfaces of degree 8 in $\mathbb{P}^5$. We shall however adapt a different method.

To construct the bundles defining the del Pezzo surfaces of degree 8 in $\mathbb{P}^5$ one can mimic step by step the construction of [Ton04]. This construction however is rather long and do not shed light on the geometry involved. Another way is the following:

4.1. **Explicit constructions.** For $k = 6, 7$ consider the projectivization of the appropriate bundle $\mathcal{O}_{\mathbb{P}^2}(2) \otimes \mathcal{O}_{\mathbb{P}^2}(2)$ or $\mathcal{O}_{\mathbb{P}^2}(1) \otimes \mathcal{O}_{\mathbb{P}^2}(3)$ respectively. The bundle is embedded in $\mathbb{P}^2 \times \mathbb{P}^l \subset \mathbb{P}^{4l+2}$ with $l = 11, 12$ respectively. For each of these spaces consider the projection $\mathbb{P}^2 \times \mathbb{P}^l \to \mathbb{P}^2 \times \mathbb{P}^5$ from the spaces spanned by $\mathbb{P}^2 \times \mathbb{P}^{k-1}$, where $\mathbb{P}^{k-1}$ is a space spanned by $k$ generic points on the image of the projectivization of the bundle on $\mathbb{P}^l$. We obtain in such a way a $\mathbb{P}^{15} \subset \mathbb{P}^{20}$ containing $k$ special fibers, being proper transforms of the points from which we projected. From Proposition 4.1 the $\mathbb{P}^{13}$ associated to a general module $M \in M_8^D$ must appear as the span of the projection of the corresponding bundle the reason why the projection must be performed from points lying on the projectivization is the appearance of as many special fibers.
The latter construction is slightly clearer than the type of constructions presented in [Ton04]. However it is still not very convenient to work with. Below we present a much simpler way to construct any of these surfaces. It will be a characterization in terms of containing special subspaces, and will appear to be very useful for the explication of the direct relation between descriptions of Calabi–Yau threefolds and del Pezzo surfaces.

4.2. Construction for the del Pezzo surface $F_1 = D_8^1$. In degree 8 let us consider the following construction. Start with a generic map

$$\varphi : 6\mathcal{O}_{P^2} \to \mathcal{O}_{P^2}(2).$$

In fact up to change of coordinates we may assume that $\varphi$ is given by the matrix

$$[a^2, b^2, c^2, ab, ac, bc].$$

The syzygy matrix of $\varphi$ is an 8 $\times$ 6 matrix of linear forms defining a map:

$$\theta : 8\mathcal{O}_{P^2}(-1) \to 6\mathcal{O}_{P^2}.$$

Consider now a general embedding

$$\iota : 4\mathcal{O}_{P^2}(-1) \to 8\mathcal{O}_{P^2}(-1).$$

We can look at the map

$$\theta \circ \iota : 4\mathcal{O}_{P^2}(-1) \to 6\mathcal{O}_{P^2}$$

as at a map

$$4\mathcal{O}_{P^5}(-1) \to 3\mathcal{O}_{P^5}$$

which induces a map

$$\Psi : 14\mathcal{O}_{P^5}(-1) \to 3\mathcal{O}_{P^5}.$$

Proposition 4.4. The kernel of the map $\Psi$ is a vector bundle of rank 11 defining a generic del Pezzo surface $D_8^1$ via the Pfaffian construction.

Proof. observe first that any del Pezzo surface appearing in the construction of Proposition 4.1 appears in this construction. To conclude it is enough to prove that the bidual to the cokernel of $\theta \circ \iota : 4\mathcal{O}_{P^2}(-1) \to 6\mathcal{O}_{P^2}$ is $\mathcal{O}_{P^2}(2) \oplus \mathcal{O}_{P^2}(2)$. But by the construction we have a surjection $G^{**} \to \mathcal{O}_{P^2}(2)$. Its kernel is a line bundle with first Chern class of degree 2 hence $\mathcal{O}_{P^2}(2)$. Since $Ext^1(\mathcal{O}_{P^2}(2), \mathcal{O}_{P^2}(2)) = 0$ we end the proof. □

Remark 4.5. Geometrically to construct the Hartshorne–Rao module of a del Pezzo surface $D_8^1$ one considers a generic $P_{13} \subset P_{17}$ containing the graph of the second Veronese embedding in $P^2 \times P^5 \subset P^{17}$.

Remark 4.6. We saw in particular that a generic $P_{13} \subset P^{17}$ containing the graph of the second Veronese embedding in $P^2 \times P^5 \subset P^{17}$ contains a one parameter family of such graphs.

Proposition 4.7. A general del Pezzo surface of type $D_8^1$ in $P^5$ appears in the construction above.

Proof. By Lemma 2.3 we construct a flat family of del Pezzo surfaces. It is clear that del Pezzo surfaces having different Hartshorne–Rao modules are different. Hence, to compute the dimension of the family constructed we need just to count the dimension of the space of isomorphism types of modules with given properties and then use Corollary 2.5. The dimension of the space of $P^5$ containing the graph of a fixed Veronese embedding is 16 as such a graph spans a $P^9$. From this we subtract 8 the dimension of the space of automorphisms of $P^5$ preserving the Veronese surface and 1 from the fact that by the above discussion the $P^5$ contains a one parameter family of graphs of Veronese embedding. We next add 5 sections of $(\bigwedge^2 E)(1)$ and subtract the dimension of endomorphisms of $E$ which is 0. We get a family of dimension 12. □
4.3. **Construction for the del Pezzo surface** $D_8^2$. Similarly consider a generic map

$$\varphi : 6\mathcal{O}_{P^2} \to \mathcal{O}_{P^2}(1).$$

Up to change of coordinates we may assume that $\varphi$ is given by the matrix

$$[a, b, c, 0, 0, 0].$$

The syzygies of the map $\varphi$ define

$$\theta : 3\mathcal{O}_{P^2}(-1) \oplus 3\mathcal{O}_{P^2} \to 6\mathcal{O}_{P^2}.$$

Consider now a general map

$$\iota : 4\mathcal{O}_{P^2}(-1) \to 3\mathcal{O}_{P^2}(-1) \oplus 3\mathcal{O}_{P^2}.$$

We prove as in Proposition 4.4 that the map

$$\theta \circ \iota : 4\mathcal{O}_{P^2}(-1) \to 6\mathcal{O}_{P^2}$$

defines a del Pezzo surface $D_8^2$.

**Remark 4.8.** Geometrically to construct the Hartshorne–Rao module of a del Pezzo surface $D_1^8$ one considers a generic $\mathbb{P}^{13} \subset \mathbb{P}^{17}$ containing the graph of a linear embedding $\mathbb{P}^2 \to \mathbb{P}^5$ in $\mathbb{P}^2 \times \mathbb{P}^5 \subset \mathbb{P}^{17}$.

**Proposition 4.9.** A general del Pezzo surface of type $D_2^8$ in $\mathbb{P}^5$ appears in the constructions above.

**Proof.** The proof follows from a similar parameter count as in Proposition 4.7. The graph of a linear embedding spans a $\mathbb{P}^5$. The space of $\mathbb{P}^{13}$ containing a fixed linear embedding is 32. From this we subtract 26 the number of automorphisms of $\mathbb{P}^5$ preserving the plane, and add 6 the dimension of the space of sections getting a 12 dimensional family. \(\Box\)

5. **Constructions of Tonoli revisited—the degree 17 Calabi-Yau threefolds**

This section will be parallel to Section 3 since the constructions are analogous. Its aim is to provide constructions for Calabi-Yau threefolds of degree 17 in $\mathbb{P}^6$. It was observed in [Ton04] that for $d = 15, 16, 17$ the corresponding Hartshorne-Rao modules should have a Hilbert function starting from degree 0 with values $(0, 1, 0, \ldots)$, $(0, 2, 1, 0, \ldots)$, $(0, 3, 5, 0, \ldots)$, respectively. Moreover if a Calabi–Yau threefold of degree 18 in $\mathbb{P}^6$ exists then its Hilbert function is $(0, 4, 9, 0, \ldots)$.

In [Ton04] as over-viewed in the introduction the author constructs Calabi–Yau threefolds of degree 17 in $\mathbb{P}^6$ by studying linear embeddings

$$\mathbb{P}^{15} \to \mathbb{P}^{20} = (\mathbb{P}^2 \times \mathbb{P}^6).$$

This is equivalent to finding sheaves $\mathcal{G}$ on $\mathbb{P}^2$ such that

$$5\mathcal{O}_{P^2}(-1) \xrightarrow{\lambda} 7\mathcal{O}_{P^2} \to \mathcal{G} \to 0,$$

where $\lambda$ is induced by the five linear equations defining $\mathbb{P}^{15} \subset \mathbb{P}^{20}$. The three families constructed in [Ton04] correspond to maps $\lambda$ such that the projectivization $\mathbb{P}(\mathcal{G})$ has $k = 8, 9$ or 11 special fibers. More precisely

$$M_k = \{ M \in G(16, 21) : \text{the corresponding } \mathbb{P}(\mathcal{G}) \text{ has } k\text{-special fibers} \}.$$

We can now pass to the proof of Theorem 1.1. We shall interpret Tonoli families in terms analogous to the earlier presented constructions of del Pezzo surfaces.

**Proposition 5.1.** If $P \subset \mathbb{P}^{20}$ is a general $\mathbb{P}^{15}$ such that the bidual of the corresponding $\mathcal{G}$ is a rank two vector bundle isomorphic to:

1. $T_{P^2}(1)$
2. $\mathcal{O}_{P^2}(2) \oplus \mathcal{O}_{P^2}(3)$
3. $\mathcal{O}_{P^2}(1) \oplus \mathcal{O}_{P^2}(4)$

then $M \in M_k$, for $k = 8, 9$ or 11, respectively.

**Proof.** The proof follows from simple Chern class computation. \(\Box\)

We shall in fact see that in this way we obtain the general elements of Tonoli families $M_k$.

The above proposition suggests the following constructions of Tonoli examples of Calabi–Yau threefolds.
5.1. Construction of Calabi–Yau threefolds of degree 17 with $k = 9$. Start with a generic map 

$$\varphi : 7\mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(2).$$

In fact up to change of coordinates we may assume that $\varphi$ is given by the matrix 

$$[a^2, b^2, c^2, ab, ac, bc, 0].$$

The syzygy of $\varphi$ defines a map: 

$$\theta : 8\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}.$$

Consider now a general embedding 

$$\iota : 5\mathcal{O}_{\mathbb{P}^2}(-1) \to 8\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}.$$

If we now take the map 

$$\theta \circ \iota : 5\mathcal{O}_{\mathbb{P}^2}(-1) \to 7\mathcal{O}_{\mathbb{P}^2},$$

think of it as a map 

$$5\mathcal{O}_{\mathbb{P}^6}(-1) \to 3\mathcal{O}_{\mathbb{P}^6}$$

which induces a map 

$$\Gamma : 16\mathcal{O}_{\mathbb{P}^6}(-1) \to 3\mathcal{O}_{\mathbb{P}^6},$$

**Remark 5.2.** Geometrically in the notation from the introduction $\Gamma$ is a map corresponding to a point $p \in G(16, W)$ such that $L_p$ contains the graph of a double Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^6$ in $\text{Seg} \subset \mathbb{P}(W)$.

**Proposition 5.3.** The kernel of $\Gamma$ is a vector bundle of rank 14 defining a Calabi–Yau threefold of degree 17 with $k = 9$ via the Pfaffian construction. A general Tonoli Calabi–Yau threefold of degree 17 and $k = 9$ appears in this way.

**Proof.** This proof is analogous to the proof of Proposition 4.4. Similarly as in the case of del Pezzo surfaces we observe that the bidual sheaf of the corresponding sheaf $G$ is $\mathcal{O}_{\mathbb{P}^2}(2) \oplus \mathcal{O}_{\mathbb{P}^2}(3)$. The proof now follows from Lemma 5.1 and the following dimension count. First we count the dimension of the family of $\mathbb{P}^{15}$ containing a fixed graph of a Veronese embedding. It is 30 since the graph spans a $\mathbb{P}^9$. We add the dimension of Veronese embeddings in $\mathbb{P}^6$ which is $6 + 35$. We get 71 which is also the dimension of $M_9$ as computed in [Ton04]. The assertion follows directly. 

**Corollary 5.4.** Theorem 1.1 is true for $k = 9$.

**Remark 5.5.** To get the dimension of the family of Calabi–Yau threefolds associated we subtract the dimension of the automorphism space of $\mathbb{P}^6$ which is 48 and the dimension of the automorphism space of $\mathbb{P}^2$. We get 15 to which we add 8 the dimension of the space of sections and since there are no automorphisms of $E$ by Corollary 2.5 we get 23 as the dimension of our family.

5.2. Construction of Calabi–Yau threefolds of degree 17 with $k = 11$. Similarly consider a generic map 

$$\varphi : 7\mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1).$$

Up to change of coordinates we may assume that $\varphi$ is given by the matrix 

$$[a, b, c, 0, 0, 0, 0, 0].$$

The syzygies of the map $\varphi$ define 

$$\theta : 3\mathcal{O}_{\mathbb{P}^2}(-1) \oplus 4\mathcal{O}_{\mathbb{P}^2} \to 7\mathcal{O}_{\mathbb{P}^2}.$$

Consider now a general map 

$$\iota : 5\mathcal{O}_{\mathbb{P}^2}(-1) \to 3\mathcal{O}_{\mathbb{P}^2}(-1) \oplus 4\mathcal{O}_{\mathbb{P}^2}.$$

We claim that the map 

$$\theta \circ \iota : 5\mathcal{O}_{\mathbb{P}^2}(-1) \to 7\mathcal{O}_{\mathbb{P}^2}$$

defines a Calabi–Yau threefold of degree 17 with $k = 11$ in the same way as above.

**Remark 5.6.** Geometrically in the notation from the introduction $\Gamma$ is a map corresponding to a point $p \in G(16, W)$ such that $L_p$ contains the graph of a linear embedding $\mathbb{P}^2 \to \mathbb{P}^6$ in $\text{Seg} \subset \mathbb{P}(W)$. 

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Proposition 5.7. A general module $M \in \mathcal{M}_{11}$ appears in the way above. Moreover, the general Tonoli Calabi–Yau threefold of degree 17 with $k = 11$ can be obtained in this way.

Proof. We proceed analogously as before. Let us however make a careful dimension count. The graph of a fixed linear embedding spans a $\mathbb{P}^6$. The Grassmannian parametrizing $\mathbb{P}^{15}$ containing a fixed $\mathbb{P}^5$ is of dimension 50. We now count the number of such graphs. We have an 8 dimensional space of maps to a fixed $\mathbb{P}^5 \subset \mathbb{P}^6$ and a 12 dimensional Grassmannian parametrizing $\mathbb{P}^2$ in $\mathbb{P}^6$. Since a general such $\mathbb{P}^{15}$ contains only one graph of a linear map we have the dimension of our family in 70. Which is indeed the dimension of $\mathcal{M}_{11} = \mathcal{M}_{10}$ by [Ton04]. □

Corollary 5.8. Theorem 1.1 is true for $k = 11$.

Corollary 5.9. The rank of the Picard group of the family of Tonoli Calabi–Yau threefolds of degree 17 with $k = 11$ is not smaller than 2.

Proof. Observe that if we compute further the dimension of the family of Calabi–Yau threefolds corresponding to Corollary 5.8 we get 24 in particular $h^{1,2} \geq 24$. Since by the double point formula we get $2(h^{1,1} - h^{1,2}) = 44$, the Calabi–Yau threefolds constructed do not have Picard number one. □

Remark 5.10. We believe that the Picard number is in this case 2 and in the cases with $k = 9, 8$ is 1, however we cannot prove it. It would be interesting to study this example from the point of view of rationality of the rays of the Kähler cone as in [LP12].

Remark 5.11. The Tonoli Calabi–Yau threefold of degree 17 with $k = 11$ constructed above shows that the Barth-Lefschetz theorem cannot be generalized to subcanonical threefolds in $\mathbb{P}^6$. Another example of this phenomenon is the del Pezzo threefold of degree 7 projected to $\mathbb{P}^6$. It is obtained as the projection to $\mathbb{P}^6$ of the second Veronese embedding of $\mathbb{P}^3$ from a $\mathbb{P}^2$ intersecting it in one point.

5.3. Construction of Calabi–Yau threefolds of degree 17 with $k = 8$. Similarly consider a generic map

$$\varphi : 7\mathcal{O}_{\mathbb{P}^2} \to \mathcal{I}_{\mathbb{P}^2}(3),$$

for a chosen point $p \in \mathbb{P}^2$.

The syzygies of the map $\varphi$ define

$$\theta : \mathcal{O}_{\mathbb{P}^2}(-2) \oplus 7\mathcal{O}_{\mathbb{P}^2}(-1) \to 7\mathcal{O}_{\mathbb{P}^2}.$$ 

Consider now a general map

$$\iota : 5\mathcal{O}_{\mathbb{P}^2}(-1) \to \mathcal{O}_{\mathbb{P}^2}(-2) \oplus 7\mathcal{O}_{\mathbb{P}^2}(-1).$$

We claim that the map

$$\theta \circ \iota : 5\mathcal{O}_{\mathbb{P}^2}(-1) \to 7\mathcal{O}_{\mathbb{P}^2}$$

defines a Calabi–Yau threefold of degree 17 with $k = 8$ in the same way as above.

Remark 5.12. Geometrically in the notation from the introduction $\Gamma$ is a map corresponding to a point $p \in G(16, W)$ such that $L_p$ contains the graph in Seg $\subset \mathbb{P}(W)$ of a birational map $\mathbb{P}^2 \to \mathbb{P}^6$ defined by a system of cubics through a fixed point.

Proposition 5.13. A general module $M \in \mathcal{M}_8$ appears in the way above. Moreover, a general Tonoli Calabi–Yau threefold of degree 17 with $k = 8$ can be obtained in such a way.

Proof. By Chern classes computations we get that our family is contained in $\mathcal{M}_8$ and we perform a dimension count. By the fact that $T_{\mathbb{P}^2}(1)$ is a unique stable bundle with given Chern classes we also know that we are in case 3 of Proposition 5.1. For a chosen point $p \in \mathbb{P}^2$, the space of sections $H^0(\mathcal{I}_{\mathbb{P}^2}(3))$ is 9 dimensional. We hence have a 63 dimensional family of maps $7\mathcal{O}_{\mathbb{P}^2} \to \mathcal{I}_{\mathbb{P}^2}(3)$. The family of graphs for a fixed $p$ is thus of dimension 62 and general of them spans a $\mathbb{P}^{13}$. This gives a 10 dimensional family of $\mathbb{P}^{15}$ for each of them. We now have a unique graph corresponding to a fixed $p$ in each such $\mathbb{P}^{15}$ so we end up with a 72 dimensional family, which is the dimension of $\mathcal{M}_8$. □

Corollary 5.14. Theorem 1.1 is true for $k = 8$. 

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Remark 5.15. The dimension of the family of Calabi–Yau threefolds obtained is again 23 as in [Ton04].

In particular we proved that the converse of Proposition 5.1 is also true. More precisely

Corollary 5.16. If $M \in \mathcal{M}_k$, for $k = 8, 9$ or 11, is a general element of the family then the bidual of the corresponding $\mathcal{G}$ is a rank two vector bundle isomorphic to:

1. $\mathcal{T}_{P_2}(1)$
2. $\mathcal{O}_{P_2}(2) \oplus \mathcal{O}_{P_2}(3)$
3. $\mathcal{O}_{P_2}(1) \oplus \mathcal{O}_{P_2}(4)$

respectively.

Remark 5.17. One can perform an analogous construction of $M \in \mathcal{M}_k$ for smaller $k$. For instance, for $3 \leq k \leq 7$ we get the bidual of $\mathcal{G}$ to be obtained as an extension:

$$0 \to \mathcal{O}_{P_2} \to \mathcal{G}^{**} \to I_Z(5) \to 0$$

with $Z$ a general zero-dimensional scheme of length $15 - k$. We can then proceed further and construct the module $M$ in terms of containing a suitable graph of a projected Veronese embedding.

Remark 5.18. One can consider a similar construction by taking a special map

$$\varphi : 7\mathcal{O}_{P_2} \to \mathcal{O}_{P_2}(3),$$

defined by Pfaffians of a skew-symmetric matrix. The syzygies of this map recover the skew symmetric map:

$$\theta : 7\mathcal{O}_{P_2}(-1) \to 7\mathcal{O}_{P_2}.$$ 

Considering the general map

$$\iota : 5\mathcal{O}_{P_2}(-1) \to \mathcal{O}_{P_2}(-2) \oplus 7\mathcal{O}_{P_2}(-1),$$

we have

$$\theta \circ \iota : 5\mathcal{O}_{P_2}(-1) \to 7\mathcal{O}_{P_2}$$

defines a Calabi–Yau threefold of degree 17 with $k = 9$. However one can check that in this way we can get only special Calabi–Yau threefolds with $k = 9$.

Similarly to the case of del Pezzo surfaces we have also an alternative way to describe the modules involved.

5.4. Another construction. For $k = 8, 9, 11$ consider the projectivization of the appropriate bundle $\mathcal{T}_{P_2}(1)$, $\mathcal{O}_{P_2}(2) \oplus \mathcal{O}_{P_2}(3)$ or $\mathcal{O}_{P_2}(1) \oplus \mathcal{O}_{P_2}(4)$ respectively. The bundle is embedded in $\mathbb{P}^{2} \times \mathbb{P}^l \subset \mathbb{P}^{3l+2}$ with $l = 14, 15, 17$ respectively. For each of these spaces consider the projection $\mathbb{P}^{2} \times \mathbb{P}^l \to \mathbb{P}^{2} \times \mathbb{P}^5$ from the spaces spanned by $\mathbb{P}^2 \times \mathbb{P}^{k-1}$, where $\mathbb{P}^{k-1}$ is a space spanned by $k$ generic points on the image of the projectivization of the bundle on $\mathbb{P}^l$. We obtain in such a way a $\mathbb{P}^{15} \subset \mathbb{P}^{20}$ containing $k$ special fibers, being proper transforms of the points from which we projected. From Proposition 5.1 the $\mathbb{P}^{13}$ associated to a general module $M \in \mathcal{M}_k$ must appear as the span of the projection of the corresponding bundle. The reason why the projection must be performed from points lying on the projectivization is the appearance of special fibers.

5.5. The analogy in degrees $(8, 17)$. Let us now finish the proof that the constructions of del Pezzo surfaces and Calabi–Yau threefolds of codimension 3 are related.

Proof of Thm. 5.2. It remains to prove the case of del Pezzo surface of degree $d_D = 8$ and Tonoli Calabi–Yau threefold of degree $d_X = 17$. On one side we have two families on the other three. Let us start with a del Pezzo surface of degree 8. Its Hartshorne–Rao module defining the bundle $E_D$ corresponds to a subspace of dimension 13 contained in $\mathbb{P}^{17} = (\mathbb{P}^2 \times \mathbb{P}^5)$ such that the intersection $\mathbb{P}^{13} \cap (\mathbb{P}^2 \times \mathbb{P}^5)$ contains either the graph of a linear map $\mathbb{P}^2 \to \mathbb{P}^5$ or the the graph of the second Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^5$. Such space is clearly the projection of a space $\mathbb{P}^{13} \subset \mathbb{P}^{20} \to \mathbb{P}^{2} \times \mathbb{P}^{3}$ with analogous property i.e. such that $\mathbb{P}^{13} \cap (\mathbb{P}^2 \times \mathbb{P}^6)$ contains either the graph of a linear map $\mathbb{P}^2 \to \mathbb{P}^6$ or the the graph of a second Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^6$. The generic such choice of extension defines a bundle $E_D$ on $\mathbb{P}^6$. A generic extension between this bundle and $2\mathcal{O}_{P_6}$ corresponds to a space $\mathbb{P}^{15} \subset \mathbb{P}^{20}$ containing the $\mathbb{P}^{13}$ i.e. in particular $\mathbb{P}^{15} \cap (\mathbb{P}^2 \times \mathbb{P}^6)$ contains either the graph of a linear map $\mathbb{P}^2 \to \mathbb{P}^6$ or the the graph of a second Veronese embedding $\mathbb{P}^2 \to \mathbb{P}^6$. In particular it corresponds to an element of $\mathcal{M}_9$ or $\mathcal{M}_{11}$. To prove that the corresponding bundle defines a Calabi–Yau threefold we observe that a general element of $\mathcal{M}_9$ or $\mathcal{M}_{11}$ arises in this way. Indeed, consider a general
\( M \in M_9 \) or \( M_{11} \) then the corresponding sheaf \( \mathcal{G}^{**} \) is either \( \mathcal{O}_{p_6}(2) \oplus \mathcal{O}_{p_6}(3) \) or \( \mathcal{O}_{p_6}(1) \oplus \mathcal{O}_{p_6}(4) \). It follows that our \( \mathbb{P}^{15} \) contains a the projectivization of the sheaf \( \mathcal{O}_{p_6}(2) \) or \( \mathcal{O}_{p_6}(1) \), hence contains a \( \mathbb{P}^{13} \) containing them also.

Finally the image of this \( \mathbb{P}^{13} \) via a general projection \( \mathbb{P}^2 \times \mathbb{P}^6 \subset \mathbb{P}^{20} \to \mathbb{P}^{13} \supset \mathbb{P}^2 \times \mathbb{P}^5 \) induced by a projection \( \mathbb{P}^6 \to \mathbb{P}^5 \) is a \( \mathbb{P}^{13} \in M_6^D \) or \( M_6^P \). The proposition is hence proven for Calabi–Yau threefolds with \( k = 9 \) or \( 11 \). We are only missing an argument for Calabi–Yau threefolds of degree 17 with \( k = 8 \). In this case we consider a \( \mathbb{P}^{13} \) such that \( \mathcal{G}^{**} = I_{p_2}(1) \), the latter admits a 2 dimensional family of surjections onto \( I_p(3) \) parametrized by \( p \in \mathbb{P}^2 \). The composed map defines a \( \mathbb{P}^{13} \) spanned by the graph of a rational map \( \mathbb{P}^2 \dashrightarrow \mathbb{P}^6 \) defined by a system of cubics passing through a point. We claim that the projection of this \( \mathbb{P}^{13} \) onto \( \mathbb{P}^{17} \) is a general element of \( M_6^D \) and hence defines a del Pezzo surface \( D_8^1 \). Indeed, we just observe that the projected \( \mathbb{P}^{13} \) is associated to a map \( 4\mathcal{O}_{p_5}(-1) \to 6\mathcal{O}_{p_5} \) whose cokernel admits a surjection on \( I_p(3) \). We then compute Chern classes of this cokernel and deduce that we are in \( M_6^D \).

**Remark 5.19.** In Theorem 1.2 we relate Calabi–Yau threefolds to del Pezzo surfaces or more precisely their Hartshorne–Rao modules in two steps, passing through a vector bundle \( E_D' \) on \( \mathbb{P}^6 \). One might wonder if there is variety given by Pfaffians of this bundle. By adjunction and degree formulas such variety if smooth would be a Fano threefold of index 2 and degree \( d - 9 \) in \( \mathbb{P}^6 \). And indeed for Calabi–Yau threefolds of degree \( d \leq 16 \) the considered bundle \( E_D' \) defines a family of such smooth Fano threefolds. For \( d = 17 \) the situation is different. The only Fano threefold of index 2 and degree 8 in \( \mathbb{P}^6 \) is the projection of the double Veronese embedding of \( \mathbb{P}^3 \). Now using our methods one can easily check that such a double Veronese embedding of \( \mathbb{P}^3 \) in \( \mathbb{P}^6 \) is associated to a \( \mathbb{P}^{13} \) corresponding to a skew-symmetric map \( \theta \) as in Remark 5.18. The restriction of the associated bundle to a generic \( \mathbb{P}^5 \) defines a del Pezzo surface \( D_8^2 \). Whereas a general extension bundle with 2\( \mathcal{O}_{p_6} \) corresponding to a \( \mathbb{P}^{15} \) containing our \( \mathbb{P}^{13} \) defines a Calabi–Yau threefold from the special family discussed in Remark 5.18. In particular even in this case we do not recover the whole family of Tonoli Calabi–Yau threefolds of degree 17 with \( k = 9 \). In the general case for any \( k = 8, 9, 11 \) the Pfaffians associated to a general section \( s \in H^0(\wedge^2 E_D') \) do not define a variety of expected codimension. Only after restricting to a general \( \mathbb{P}^5 \) the appropriate sections appear.

6. **Degree 9 del Pezzo surface and degree 18 Calabi–Yau threefolds**

The analogy discussed in Section 5.5 suggests that one might try to construct a Calabi–Yau threefold of degree 18 in \( \mathbb{P}^6 \) if we find an appropriate description of a del Pezzo surface of degree 9 in \( \mathbb{P}^5 \). In this section we collect experimental information on such del Pezzo surface and present a construction of canonically embedded surfaces of general type of degree 18 and Calabi–Yau threefolds with one singular point (being equivalent to a cone over a del Pezzo of degree 9).

**6.1. Del Pezzo surface of degree 9.** Recall that a del Pezzo surface of degree 9 is just \( \mathbb{P}^2 \) and its anticanonical embedding is the triple Veronese embedding. The surface \( D_9 \subset \mathbb{P}^5 =: \mathbb{P}(W) \) is the image of the projection of the image of this embedding from a generic 3-dimensional linear subspace \( \Lambda \subset \mathbb{P}^9 \). Our aim is to understand the module \( M := \bigoplus_{n=0}^\infty H^1(\mathcal{I}_D(n)) \). We can thus write \( M = M_0 \oplus M_1 = H^1(\mathcal{I}_D(2)) \oplus H^1(\mathcal{I}_D(3)) \).

By working out a random example in Macaulay2, we found that \( M \) is generated in degree 2. Moreover the Betti table of the minimal resolution of \( M \) is the following:

\[
\begin{array}{c|c|c|c|c}
0 & M & 4R(2) & 17R(1) & 18R & 4R(-1) \\
& & & \oplus & \oplus & \oplus \\
& & 29R(-1) & 80R(-2) & 81R(-3) & 38R(-4) & 7R(-5) & 0
\end{array}
\]

**6.2. Final remarks and conjectures.** The natural problem that comes to mind is to provide a construction of a general canonical surface of degree 18 by following the analogy to degrees (9, 18). Since this surface should have a similar Pfaffian resolution to the resolution of the generic projection \( D_9 \subset \mathbb{P}^5 \) of the del Pezzo surface of degree 9 it is crucial to understand the bundles on \( \mathbb{P}^5 \) that define such del Pezzo surfaces.

We find that the linear strand of the resolution of \( \mathcal{I}_{D_9} \) have interesting special syzygies. We study the geometry of this linear strand with the help of Macaulay 2. By Lemma 3.1 we know that \( h^1(\mathcal{I}_D(k)) \) equals to 4, 7, 0 for \( k = 1, 2 \) and \( \geq 3 \) respectively.

From the minimal resolution of \( M \) we also see that there are four special linear 3-syzygies. By further computation we get a one dimensional space of sections of the bundle \( \wedge^2 E(1) \), where \( E = \text{Syz}^1(M) \) is the
kernel of the map $17\mathcal{O}_\mathbb{P}^5 \to 4\mathcal{O}_\mathbb{P}^5$ given by the presentation matrix of $M$. For a geometric interpretation of them we find that a linear 3-syzygies corresponds to an element $e$ in the kernel of the Koszul map $\wedge^3 W \otimes M_0 \to \wedge^2 W \otimes M_1$ induced by the natural multiplication $W \otimes M_0 \to M_1$. Thus we can see $e$ as a map $M^*_e \to \wedge^3 W$. Denote by $P_e$ the image of $M^*_e$ in $\mathbb{P}(\wedge^3 W)$.

We compute with Macaulay 2 (using the script symExt from [DE02]) that if we choose 4 generic linear 3-syzygies $e_1, e_2, e_3, e_4$ of $M$ then the corresponding linear spaces $P_{e_i}$ spans a $\mathbb{P}^9$ that we shall denote $A$. We compute that $A \cap G(3, 6) \subset \mathbb{P}(\wedge^3 W)$ is a surface of degree 9 being the triple Veronese embedding of $\mathbb{P}^2$. We moreover can choose a basis $(x_i, y_i, z_i, t_i)$ of each $P_{e_i}$ in such a way that the matrix of 3-syzygies

$$
\begin{align*}
x_1 & y_1 & z_1 & t_1 \\
x_2 & y_2 & z_2 & t_2 \\
x_3 & y_3 & z_3 & t_3 \\
x_4 & y_4 & z_4 & t_4
\end{align*}
$$

is symmetric. In particular for any two $e_1$ and $e_2$ linear three-syzygies the intersection $P_{e_1} \cap P_{e_2}$ is a point.

Having the space of 3-syzygies one can recover the Module of the del Pezzo surface. Unfortunately the condition described above for the 3 syzygies are not sufficient to make sure that the space of syzygies corresponds to a module of a del Pezzo surface of degree 9.

**Problem 6.1.** Find a general symmetric matrix of three forms contained in a Lagrangian $\mathbb{P}^9 \subset \wedge^3 V_6$ spanned by the set of points in the Grassmannian $G(3, 6)$ corresponding to tangents of some Veronese surface in $\mathbb{P}(V_6)$, and such that the corresponding module $M$ has linear strand $4 \ 17 \ 18 \ 4$. We believe that such a module $M$ will be the the Hartshorne-Rao module of a projection of the del Pezzo surface $D'$.

Finally, assuming the analogy in each degree we are tempted to formulate the following conjecture.

**Conjecture 6.2.** There are no canonical surfaces of degree $d \geq 19$ in $\mathbb{P}^5$.

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