Anisotropy Parameters for the Effective Description of Crystalline Color Superconductors

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Abstract

In the high density low temperature limit, Quantum Chromodynamics (QCD) exhibits a transition to a phase characterized by color superconductivity and by energy gaps in the fermion spectra. Under specific circumstances the gap parameter has a crystalline pattern, breaking translational and rotational invariance. The corresponding phase is the the crystalline color superconductive phase (or \textit{LOFF} phase). In the effective theory the fermions couple to the phonon arising from the breaking of rotation and translation invariance. We compute the parameters of the low energy effective lagrangian describing the motion of the free phonon in the high density medium and derive the phonon dispersion law.

1 Introduction

QCD at large densities is expected to exhibit color superconductivity from condensation of quark bilinears. It has recently been pointed out that a distinctive phase, called crystalline color superconducting
phase, may appear, in situations where different quark flavors have different chemical potentials, with potential differences lying inside a certain window. Crystalline color superconductivity is also expected to occur simply because of the quark mass differences [1]. However in the following we shall limit the discussion to the case of two massless quarks with different chemical potentials.

Besides its theoretical interest for the study of the phase structure of QCD theory, such a crystalline phase may result relevant for astrophysical dense systems [2], [7], [12]. A pairing ansatz similar to that of the QCD crystalline phase was already encountered in condensed matter (and referred to as LOFF phase from the name of the authors who studied it in superconductivity [13]). Other recent applications of the LOFF phase in organic superconductors, nuclear matter and cold trapped gases are respectively in Refs. [14], [15] and [16].

In the crystalline phase, rotation and translation invariance are both spontaneously broken. The ensuing goldstone structure and the effective lagrangian were studied in [4]. For plane-wave behavior of the crystalline condensate the conclusion was that a single goldstone appears, the phonon. The appearance and description of a single goldstone in a situation where both translations and rotations are spontaneously broken was found in [4] to originate from a functional relation to be satisfied by fictitious fields associated to broken rotations such as to assure small long wavelength fluctuations of the order parameters.

The main purpose of this paper is to calculate at first order in the derivative expansion the parameters of the effective lagrangian for the crystalline phase and derive the phonon dispersion law. Due to the fact that quark condensation singles out one direction in space, the phonon dispersion law is anisotropic.

The paper is organized as follows. In Section 2 we review the velocity-dependent effective lagrangian approach we shall use in this paper. In Section 2 we review the basic facts concerning the crystalline superconducting phase and the modifications that are needed in the effective lagrangian to deal with the anisotropy induced by the LOFF condensates. In Section 4 we derive the gap equations for the BCS and the LOFF condensates by a Schwinger-Dyson equation derived in the framework of the effective theory. In Section 4 we introduce the phonon field associated with the breaking of rotational and translational invariance and we write down its coupling to the fermion quasi-particles. In Section 4 we derive the parameters of the effective lagrangian
2 Review of the velocity-dependent effective lagrangian

We shall employ in the sequel the effective lagrangian approach based on velocity-dependent fields. It has been developed in [17, 18, 19, 20] for the 2SC and CFL models and in [4] for the LOFF phase. In this Section we give a brief review of it. Let us consider the fermion field

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot x} \psi(p)$$

(1)

and decompose the fermion momentum as

$$p^\mu = \mu v^\mu + \ell^\mu$$

(2)

with $v^\mu = (0, \vec{v})$ and $\vec{v}$ the Fermi velocity (for massless fermions $|\vec{v}| = 1$); finally $\ell^\mu$ is a residual momentum.

By the decomposition (2) only the positive energy component $\psi_+$ of the fermion field survives in the lagrangian in the $\mu \to \infty$ limit while the the negative energy component $\psi_-$ can be integrated out. These effective fields are velocity dependent and are related to the original fields by

$$\psi(x) = \sum_\vec{v} e^{-i\mu v \cdot x} [\psi_+(x) + \psi_-(x)] ,$$

(3)

where

$$\psi_\pm(x) = \frac{1 \pm \vec{\alpha} \cdot \vec{v}}{2} \int \frac{d\vec{\ell}}{(2\pi)^3} \int_{-\infty}^{+\infty} d\ell_\parallel \frac{e^{-i\ell \cdot x}}{2\pi} \psi_\vec{v}(\ell) .$$

(4)

Here $\sum_\vec{v}$ means an average over the Fermi velocities and

$$\psi_\pm(x) \equiv \psi_\pm,\vec{v}(x)$$

(5)

are velocity-dependent fields. The positive energy effective field carries color and flavor indices: $\psi_+ \equiv \psi_{+,i,\alpha}$ where $\alpha = 1, 2, 3$ is a color index and $i$ is a flavor index. In this paper $i = 1, 2$ (two flavors). The integration measure in the isotropic (CFL or two-flavor 2SC models) is as follows

$$\int \frac{d\vec{\ell}}{(2\pi)^3} = \int \frac{d\vec{\ell}_\perp}{(2\pi)^3} \int_{-\delta}^{+\delta} d\ell_\parallel = \frac{1}{2\pi} \left[ \int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos \theta \int_{-\delta}^{+\delta} d\ell_\parallel \right]$$

(6)
where \( \ell \parallel \) is along \( \vec{v} \) and \( d\vec{\ell}_\perp \) is a surface element, which on a sphere of radius \( \mu \) is equal to \( \mu^2 d\Omega = \mu^2 d\phi d \cos \theta \); \( \ell \parallel \) is limited by \( |\ell \parallel| \leq \delta \), where \( \delta \) is such that (\( \Delta \) the BCS gap)

\[ \Delta \ll \delta \leq \mu . \]  

(7)

The condition \( |\ell \parallel| \leq \delta \) is equivalent to the requirement

\[ ||\vec{p}| - \mu| \leq \delta \]  

(8)

once one recognizes that, due to the spherical symmetry, one can always choose \( \vec{\ell} \) in the direction of \( \vec{v} \). On the other hand the condition (7) has a twofold implication: \( \delta \leq \mu \) means that one is considering only degrees of freedom near the Fermi surface, while, by the choice \( \Delta \ll \delta \), the physical quantities become independent of the cut-off procedure.

In the case of anisotropic superconductivity, such as the LOFF phase, the situation is different and will be discussed in the next section.

It is useful to use a different basis for the fermion fields by writing

\[
\begin{align*}
\psi_{+i\alpha} &= \sum_{A=0}^{3} \frac{(\sigma_A)_{i\alpha}}{\sqrt{2}} \varphi^A_+ \\
\psi_{+13} &= \varphi^1_+ \\
\psi_{+23} &= \varphi^5_+ ,
\end{align*}
\]

(9)

where \( \sigma_A \) are the Pauli matrices for \( A = 1, 2, 3 \) and \( \sigma_0 = 1 \). \( \varphi^A_+ \) are positive energy, velocity dependent fields:

\[ \varphi^A_+ \equiv \varphi^A_{+\vec{v}} , \]  

(10)

and represent left-handed Weyl fermions. In view of the introduction of a gap term (see next section) we introduce also the fields

\[ \varphi^A_- \equiv \varphi^A_{+\vec{v}} . \]  

(11)

\( \varphi^A_\pm \) should not be confused with the positive and negative energy fields; they are both positive energy fields, but are relative to opposite velocities.

The effective lagrangian \( \mathcal{L}_0 \) for gapless fermion fields at high density can therefore be written as follows:

\[ \mathcal{L}_0 = \sum_{\vec{v}, A=0}^{5} \sum_{\text{odd}} \varphi^A_+ (iV \cdot \partial) \varphi^A_+ + (L \to R) . \]  

(12)

---

\(^1\)Here \( V^\mu = (1, \vec{v}) \) while \( \tilde{V}^\mu = (1, -\vec{v}) \). The derivation of (12) is in [13]. It can be obtained noting that, in the \( \mu \to \infty \) limit, terms bilinear in the fermion fields must have the same velocity by the Riemann-Lebesgue lemma, while the spin matrices are substituted by the \( V^\mu \) vector by making use of simple algebraic identities [17].
Using the fact that the average over velocities is symmetric we write:

\[ \mathcal{L}_0 = \frac{1}{2} \sum_{\vec{v}} \sum_{A=0}^5 \left( \varphi_+^A (i \vec{V} \cdot \partial) \varphi_+^A + \varphi_-^A (i \vec{V} \cdot \partial) \varphi_-^A \right). \]  

(13)

Introducing now

\[ \chi^A = \frac{1}{\sqrt{2}} \begin{pmatrix} \varphi_+^A \\ C \varphi_-^A \end{pmatrix}, \]

(14)

the lagrangian can be written as follows:

\[ \mathcal{L}_0 = \sum_{\vec{v}} \sum_{A=0}^5 \chi^A \begin{pmatrix} i \vec{V} \cdot \partial & 0 \\ 0 & i \vec{V} \cdot \partial \end{pmatrix} \chi^A. \]

(15)

To this lagrangian involving only the left handed fields, one should add a similar one containing the right handed fermionic fields. Also notice that we have embodied the factor 1/2 appearing in Eq. (13) in the definition of the sum over the Fermi velocities. This extra factor cannot be eliminated by a redefinition of the fields \( \psi_{\pm} \) since it corresponds to a genuine doubling of the degrees of freedom; its appearance is needed because, by introducing the fields with opposite \( \vec{v} \), \( \psi_- = \psi_{+, -\vec{v}} \) we must integrate only over half solid angle.

3 Crystalline colour superconductive phase

As shown in [2] when the two fermions have different chemical potentials \( \mu_1 \neq \mu_2 \), for \( \delta \mu \) of the order of the gap\(^2\) the vacuum state is characterized by a nonvanishing expectation value of a quark bilinear which breaks translational and rotational invariance. The appearance of this condensate is a consequence of the fact that for \( \mu_1 \neq \mu_2 \), and in a given range of \( \delta \mu = |\mu_1 - \mu_2| \), the formation of a Cooper pair with a total momentum

\[ \vec{p}_1 + \vec{p}_2 = 2q \neq \vec{0} \]

(16)

\(^2\)In [19] another factor 1/2 appears in the lagrangian \( \mathcal{L}_D \); here we simplify the notation and get rid of it by a redefinition of the fermion fields.

\(^3\)The actual value of the range compatible with the presence of the LOFF state depends on the calculation by which the crystalline colour state is computed. Assuming a local interaction as in [2] produces a rather small interval, which is enlarged by assuming gluon exchange, as in [3].
is energetically favored in comparison with the normal BCS state. The analysis of \cite{2} shows that two different, but related condensates are possible, one with the two quarks in a spin zero state (scalar condensate) and the other one characterized by total spin 1 (vector condensate). In the BCS state the quarks forming the Cooper pair have necessarily $S = 0$; as a matter of fact, since the quarks have opposite momenta and equal helicities, they must be in an antisymmetric spin state. This is not true if the total momentum is not zero and the two quarks can have both $S = 0$ and $S = 1$.

The possible form of these condensates is discussed in \cite{2}; these authors assume only two flavors and make the ansatz of a plane wave behavior for both condensates: $\propto e^{2i\vec{q} \cdot \vec{x}}$. Though more complicated structures are possible we will make the same hypotheses. The results of \cite{2} are as follows:

$$ - < 0 | \epsilon_{ij} \epsilon_{\alpha\beta\gamma} \psi^i \alpha (\vec{x}) C \psi^j \beta (\vec{x}) | 0 > = 2 \Gamma_{L} e^{2i\vec{q} \cdot \vec{x}} ; \quad (17) $$

besides the scalar condensate (17) one may have a spin 1 vector condensate, symmetric in flavor:

$$ i < 0 | \sigma_{ij} \epsilon_{\alpha\beta\gamma} \psi^i \alpha (\vec{x}) C \sigma^0 \psi^j \beta (\vec{x}) | 0 > = 2 \Gamma_{B} e^{2i\vec{q} \cdot \vec{x}} . \quad (18) $$

The effect of the two non vanishing vacuum expectation values can be taken into account by adding to the lagrangian the term:

$$ L_\Delta = L_\Delta^{(s)} + L_\Delta^{(v)} = $$

$$ = - \frac{e^{2i\vec{q} \cdot \vec{x}}}{2} \epsilon_{\alpha\beta\gamma} \psi^T_i \alpha (x) C \left( \Delta^{(s)} + \vec{\alpha} \cdot \vec{n} \Delta^{(v)} \sigma^1_{ij} \right) \psi_i \beta (x) + \text{h.c. ,} \quad (19) $$

which includes both the scalar and the vector condensate.

Let us consider the lagrangian term relative to the scalar condensate. We shall write it as follows:

$$ L_\Delta^{(s)} = - \frac{\Delta^{(s)}}{2} e^{2i\vec{q} \cdot \vec{x}} \epsilon_{\alpha\beta\gamma} \psi^T_i \alpha (x) C \psi_i \beta (x) - (L \rightarrow R) + \text{h.c. .} \quad (20) $$

Here $\psi(x)$ are positive energy left-handed fermion fields discussed in the previous section. We neglect the negative energy states, consistently with the assumption discussed in the previous section.

In order to introduce velocity dependent positive energy fields $\psi_+, \bar{\psi}_i; \bar{\epsilon}_i$ with flavor $i$ we have to decompose the fermion momenta according to \cite{2}:

$$ \vec{p}_j = \mu_j \vec{v}_j + \vec{\ell}_j \quad (j = 1, 2) \quad . \quad (21) $$
Therefore we have:

\[
\mathcal{L}_\Delta^{(s)} = -\frac{\Delta^{(s)}}{2} \sum_{\vec{v}_1, \vec{v}_2} \exp\{i \vec{x} \cdot \vec{\alpha}(\vec{v}_1, \vec{v}_2, \vec{q})\} \epsilon_{ij} \epsilon^{\alpha\beta} \psi_+(-\vec{v}_1; \alpha(x)C\psi_+,-\vec{v}_j; j\beta(x))
\]
\[
-(L \rightarrow R) + \text{h.c.}
\]

(22)

where

\[
\vec{\alpha}(\vec{v}_1, \vec{v}_2, \vec{q}) = 2\vec{q} - \mu_1 \vec{v}_1 - \mu_2 \vec{v}_2.
\]

(23)

We choose the z-axis along \( \vec{q} \) and the vectors \( \vec{v}_1, \vec{v}_2 \) in the \( x-z \) plane; if \( \alpha_1, \alpha_2 \) are the angles formed by the vectors \( \vec{v}_1, \vec{v}_2 \) with the z-axis we have

\[
\begin{align*}
\alpha_x &= -\mu_1 \sin \alpha_1 \cos \phi_1 - \mu_2 \sin \alpha_2 \cos \phi_2, \\
\alpha_y &= -\mu_1 \sin \alpha_1 \sin \phi_1 - \mu_2 \sin \alpha_2 \sin \phi_2, \\
\alpha_z &= 2q - \mu_1 \cos \alpha_1 - \mu_2 \cos \alpha_2.
\end{align*}
\]

(24)

Because of the already quoted Riemann-Lebesgue lemma, in the \( \mu_1, \mu_2 \rightarrow \infty \) limit the only non vanishing terms in the sum correspond to the condition

\[
\alpha_x = \alpha_y = 0.
\]

(25)

Let us introduce

\[
\begin{align*}
\mu &= \frac{\mu_1 + \mu_2}{2}, \\
\delta \mu &= -\frac{\mu_1 - \mu_2}{2}.
\end{align*}
\]

(26)

Eq. (25) implies

\[
\alpha_1 = \alpha_2 + \pi + \mathcal{O}\left(\frac{1}{\mu}\right),
\]

(27)

i.e. the two velocities are opposite in this limit.

The condition on \( \alpha_z \) depends on the behavior of \( q \) in the limit \( \mu \rightarrow \infty \). If we might take the limit \( q \approx \delta \mu \rightarrow \infty \) as well, we would also have \( \alpha_z \approx 0 \); we will make this approximation which is justified by previous analysis [2], which points to a value of \( q \simeq 1.2\delta \mu \) and \( \delta \mu \sim 0.7\Delta_{BCS} \gg \Delta^{(s,v)} \). These conditions have to be justified also in our approach and we will do it in the next section. Notice that this approximation considerably simplifies the loop calculation in presence of the LOFF condensate, which is usually performed by defining a blocking region, where the LOFF condensate is impossible, and an allowed, pairing region, whose definition is mathematically involved. This
approximation may be therefore useful in complex calculations such as the present one, even though it can produce numerical results somehow different by the more precise calculations given in [2]. We have:

\[
\cos \alpha_1 = -\frac{\delta \mu}{q} + \frac{q^2 - \delta \mu^2}{\mu}, \\
\cos \alpha_2 = +\frac{\delta \mu}{q} + \frac{q^2 - \delta \mu^2}{\mu},
\]  

(28)

which, together with (25), has the solution

\[
\alpha_2 \equiv \theta_q = \arccos \frac{\delta \mu}{q} - \frac{\epsilon}{2}, \\
\alpha_1 = \alpha_2 + \pi - \epsilon
\]  

(29)  

(30)

with

\[
\epsilon = 2 \sqrt{\frac{q^2 - \delta \mu^2}{\mu}}.
\]  

(31)

Therefore, as anticipated, \( \epsilon = \mathcal{O} \left( \frac{1}{\mu} \right) \) and in the limit \( \mu \to \infty \) the two velocities are almost antiparallel:

\[
\vec{v}_1 \simeq -\vec{v}_2.
\]  

(32)

Putting

\[
\psi_{+,\pm \vec{v}_i; i\alpha}(x) \equiv \psi_{\pm \vec{v}_i; i\alpha}(x),
\]  

(33)

eq. (22) becomes:

\[
\mathcal{L}_\Delta^{(s)} = -\frac{\Delta^{(s)}}{2} \sum_{\vec{v}} \epsilon_{ij} \epsilon^{\alpha \beta \gamma} \psi_{+, \vec{v}_i; i\alpha}(x) C \psi_{-, \vec{v}_j; j\beta}(x) - (L \to R) + h.c.
\]  

(34)

In a similar way the term corresponding to the vector condensate in the lagrangian can be written as follows:

\[
\mathcal{L}_\Delta^{(v)} = -\frac{\Delta^{(v)}}{2} \sum_{\vec{v}} \sigma_{ij} \epsilon^{\alpha \beta \gamma} \psi_{+, \vec{v}_i; i\alpha}(x) C (\vec{v} \cdot \vec{n}) \psi_{-, \vec{v}_j; j\beta}(x) - (L \to R) + h.c.
\]  

(35)

where \( \vec{n} = \vec{q}/|\vec{q}| \) is the direction corresponding to the total momentum carried by the Cooper pair and we have used \( \psi^T C \vec{\alpha} \cdot \vec{n} \psi_+ = \vec{v} \cdot \vec{n} \psi^T C \psi_+ \).
The procedure of the limit $q \to \infty$ may produce a finite renormalization of the gap parameters. To show this let us consider in more detail how the second of the two equations \(28\) can be obtained. In order to take the limit in the lagrangian for $\mu \to \infty$ or $q \to \infty$ we perform a smearing in the space time integrations; for example, considering the $z$ axis we introduce a smearing $|\Delta z|$ and we write:

$$e^{i2qh \cdot \mathbf{z}} \to \frac{1}{|\Delta z|} \int_{z-|\Delta z|/2}^{z+|\Delta z|/2} e^{i2qh \cdot \mathbf{y}} dy = e^{i2qh \cdot \mathbf{z}} \frac{\sin(qh|\Delta z|)}{qh|\Delta z|} = \frac{\pi}{R} e^{i2qh \cdot \delta_R(h)}$$

(36)

We have put

$$R = q|\Delta z|$$

(37)

and introduced the ”fat delta” $\delta_R(x)$ defined by

$$\delta_R(x) \equiv \frac{\sin(Rx)}{\pi x} ,$$

(38)

which, for large $R$, gives

$$\delta_R(x) \to \delta(x) .$$

(39)

Let us now justify the approximation \(28\). We have ($\theta_2 =$ azimuthal angle of the velocity vector $\mathbf{v}_2$)

$$\int d\cos \theta_2 e^{i2qh(c\cos \theta_2) \cdot \mathbf{z}} \to \frac{2\pi}{2q|\Delta z|} \int \frac{d\cos \theta_2 e^{i2qh(c\cos \theta_2) \cdot \mathbf{z}}}{\pi} \delta_R(h(c\cos \theta_2))$$

$$\approx \frac{\pi}{|\Delta z|\tilde{h}'(c\cos \alpha_2)} \int d\cos \theta_2 \delta(c\cos \theta_2 - \cos \alpha_2)$$

(40)

In the previous equations we have defined\[4] \(\tilde{h}(c\cos \theta_2) = 1 + \frac{1}{2q} \left( -\mu_2 \cos \theta_2 + \cos \theta_2 \sqrt{\mu_2^2 - \frac{4\mu \delta_\mu}{\cos^2 \theta_2}} \right) .$$

(41)

In the sequel we will also use

$$h_{\pm}(c\cos \theta_2) = 1 \pm \frac{1}{2q} \left( -\mu_2 \cos \theta_2 \pm \sqrt{\mu_2^2 \cos^2 \theta_2 - 4\delta_\mu^2} \right)$$

(42)

\[4\] The reader should note that, by our convention, $\delta_\mu < 0$. 

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where one has \( \pm 1 = +\text{sign}(\cos \theta) \). Due to eq. (40), the condensates are modified as
\[
\Delta \to \frac{\pi}{R} \delta_R(h_{\pm}(\cos \theta)) \Delta .
\] (43)

We handle the fat delta according to the Fermi trick in the Golden Rule; in expressions involving the gap parameters we make one substitution
\[
\delta_R(h_{\pm}(\cos \theta)) \to \delta(h_{\pm}(\cos \theta))
\] (44)
in the numerator; correspondingly we will get a factor
\[
k_R = \frac{\pi |\delta \mu|}{qR}
\] (45)
in the numerator (from the region \( \cos \theta < 0 \)); the other fat delta are computed as follows
\[
\frac{\pi \delta_R(h_{-}(\cos \theta))}{R} \to \frac{\pi \delta_R(0)}{R} \to 1.
\] (46)

In the basis introduced in the previous section, the effective lagrangian is
\[
\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_\Delta^{(s)} + \mathcal{L}_\Delta^{(v)} =
\sum \sum_{\vec{v}} \chi^A \left( i \delta_{AB} V \cdot \partial \frac{\Delta^\dagger_{AB}}{\Delta_{AB}} + i \delta_{AB} \vec{V} \cdot \partial \right) \chi^B .
\] (47)

The matrix \( \Delta_{AB} \) is as follows:
\[
\Delta_{AB} = 0 \quad (A \text{ or } B = 4 \text{ or } 5)
\] (48)

while for \( A, B = 0, \ldots, 3 \) we have:
\[
\Delta_{AB} = \frac{\pi}{R} \delta_R(h_{\pm}(\cos \theta)) \begin{pmatrix}
\Delta_0 & 0 & 0 & -\Delta_1 \\
0 & -\Delta_0 & -i \Delta_1 & 0 \\
0 & +i \Delta_1 & -\Delta_0 & 0 \\
\Delta_1 & 0 & 0 & -\Delta_0
\end{pmatrix} ,
\] (49)

with
\[
\begin{align*}
\Delta_0 &= \Delta^{(s)} \\
\Delta_1 &= \vec{u} \cdot \vec{n} \Delta^{(v)} .
\end{align*}
\] (50)
Let us now discuss the precise meaning of the average over velocities. As discussed above we embodied a factor of 1/2 in the average over velocities; however in the LOFF case the sum over the velocities is no longer symmetric and, whenever the integrand contains the matrix \( \Delta_{AB} \) now reads:

\[
\sum_{\vec{v}} \equiv \sum_{\vec{v}} \frac{\pi}{R} \delta_{R}(\cos \theta) \rightarrow \frac{k_{R}}{2} \int \frac{d\phi}{2\pi},
\]  

(51)

if

\[
\vec{v} = (\sin \theta q \cos \phi, \sin \theta q \sin \phi, \cos \theta q),
\]  

(52)

and \( \theta_q \) given in (29). As discussed above, this limit corresponds to \( R \) large. For future reference we observe that, in the same limit,

\[
\sum_{\vec{v}} v_i v_j = k_{R} \left( \frac{\sin^2 \theta_{q}}{4} (\delta_{i1} \delta_{j1} + \delta_{i2} \delta_{j2}) + \frac{\cos^2 \theta_{q}}{2} \delta_{i3} \delta_{j3} \right).
\]  

(53)

The effective action for the fermi fields in momentum space reads:

\[
S = \sum_{\vec{v}} \sum_{A,B=0}^{5} \int \frac{d^{4}\ell}{(2\pi)^{4}} \frac{d^{4}\ell'}{(2\pi)^{4}} \chi_{A}^{\dagger}(\ell') D_{AB}^{-1}(\ell', \ell) \chi_{B}(\ell),
\]  

(54)

where \( D_{AB}^{-1}(\ell', \ell) \) is the inverse propagator, given by:

\[
D_{AB}^{-1}(\ell', \ell) = \left( \begin{array}{ccc}
V \cdot \ell \delta_{AB} & \Delta_{AB}^{\dagger} \\
\Delta_{AB} & V \cdot \ell \delta_{AB}
\end{array} \right) \delta^{4}(\ell' - \ell).
\]  

(55)

From these equations one can derive the quark propagator, defined by

\[
\sum_{B} \int \frac{d^{4}\ell'}{(2\pi)^{4}} D_{AB}^{-1}(\ell, \ell') D_{BC}(\ell', \ell'') = \delta_{AC} \delta^{4}(\ell - \ell'').
\]  

(56)

It is given by

\[
D_{AB}(\ell, \ell'') = (2\pi)^{4} \delta^{4}(\ell - \ell'') \times \sum_{C} \left( \begin{array}{ccc}
\tilde{V} \cdot \ell \delta_{AC} & -\Delta_{AC}^{\dagger} \\
\Delta_{AC} & V \cdot \ell \delta_{AC}
\end{array} \right) \frac{\Delta_{BC}(\ell)}{D_{CB}(\ell)}.
\]  

(57)
where
\[ D_{CB}(\ell) = \left( V \cdot \ell \tilde{V} \cdot \ell - \Delta \Delta^\dagger \right)_{CB} \]
\[ \tilde{D}_{CB}(\ell) = \left( V \cdot \ell \tilde{V} \cdot \ell - \Delta^\dagger \Delta \right)_{CB}. \]  

(58)

The propagator for the fields \( \chi^{4,5} \) does not contain gap mass terms and is given by
\[ D(\ell, \ell') = (2\pi)^4 \delta^4(\ell - \ell') \begin{pmatrix} (V \cdot \ell)^{-1} & 0 \\ 0 & (\tilde{V} \cdot \ell)^{-1} \end{pmatrix}. \]

(59)

For the other fields \( \chi^A, A = 0, \ldots, 3 \), it is useful to go to a representation where \( \Delta \Delta^\dagger \) and \( \Delta^\dagger \Delta \) are diagonal. It is accomplished by performing a unitary transformation which transforms the basis \( \chi^A \) into the new basis \( \tilde{\chi}^A \) defined by
\[ \tilde{\chi}^A = R_{AB} \chi^B, \]

(60)

with
\[ R_{AB} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -i & 0 \\ 0 & +i & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}. \]

(61)

In the new basis we have
\[ (\Delta \Delta^\dagger)_{AB} = \lambda_A \delta_{AB} \]
\[ (\Delta^\dagger \Delta)_{AB} = \tilde{\lambda}_A \delta_{AB} \]

(62)

where
\[ \lambda_A = \left( \frac{\pi}{R} \delta_R(h_{\pm}(\cos \theta)) \right)^2 \left( (\Delta_0 + \Delta_1)^2, (\Delta_0 - \Delta_1)^2, (\Delta_0 + \Delta_1)^2, (\Delta_0 - \Delta_1)^2 \right) \]
\[ \tilde{\lambda}_A = \left( \frac{\pi}{R} \delta_R(h_{\pm}(\cos \theta)) \right)^2 \left( (\Delta_0 - \Delta_1)^2, (\Delta_0 - \Delta_1)^2, (\Delta_0 + \Delta_1)^2, (\Delta_0 + \Delta_1)^2 \right) \]

(63)

For further reference we also define
\[ \mu_C = (\Delta_0 + \Delta_1, \Delta_1 - \Delta_0, \Delta_0 + \Delta_1, \Delta_1 - \Delta_0). \]

(64)
Let us finally discuss the integration limits in the residual momentum $\vec{\ell}$ in the LOFF case. We write:

$$\vec{\ell} = \vec{\ell}_\perp + \vec{\ell}_\parallel,$$

(65)

where $\vec{\ell}_\parallel$ is parallel to $\vec{v}_2 \simeq -\vec{v}_1$ and $\vec{\ell}_\perp$ is a 2-dimensional vector orthogonal to $\vec{v}_2$. Clearly $V \cdot \ell = \ell_0 - \ell_\parallel$ and $\tilde{V} \cdot \ell = \ell_0 + \ell_\parallel$, therefore the propagator only depends on $\ell_0$ and $\ell_\parallel$; however, since the integration over velocities is anisotropic, the theory, differently from the CFL and 2SC cases, is not 2-dimensional, but (2+1)-dimensional. One has for the integration measure, whenever the integrand is proportional to the gap

$$\int \frac{d^4\ell}{(2\pi)^4} = \int \frac{d\ell_\perp}{(2\pi)^3} \int_{-\delta}^{+\delta} d\ell_\parallel \int_{-\infty}^{+\infty} d\ell_0 \frac{2}{2\pi} = \int \frac{4\pi\mu^2 k_R}{(2\pi)^3} \int_{-\delta}^{+\delta} d\ell_\parallel \int_{-\infty}^{+\infty} d\ell_0 \frac{2}{2\pi},$$

(66)

which implements the condition (29).

Before closing this Section we note some differences between the present calculation and the one presented in [4]. In [4] we did not work in the simplifying approximation $\alpha_z = 0$ (see eq. (24), which implies that the effective lagrangian and the fermion propagator in [4] contain both $2q$ and $\delta\mu$ and are therefore more involved (there is also a numerical mistake in [4] that renders hermitean the gap matrix $\Delta_{AB}$, which is not true. The correct expression is eq. (49)).

4 Gap equation

To derive the gap equation in the CFL case we write a truncated Schwinger-Dyson equation, similarly to the approach followed by [3]. We assume a fictitious gluon propagator given by

$$i D_{ab}^{\mu\nu} = i \frac{g^{\mu\nu}\delta_{ab}}{\Lambda^2},$$

(67)

which corresponds to a local four-fermion coupling. Using the Feynman rules that can be derived by the effective theory one obtains, in the limit of eq.(29) two equations:

$$\Delta_0 = i \frac{\mu^2}{24\Lambda^2 \pi^3} k_R \sum_{A=0}^{3} |\mu_A| I(\mu_A)$$
\[
\Delta_1 = -i \frac{\mu^2}{24 \Lambda^2 \pi^3} k_R (\mu_0 I(\mu_0) + \mu_1 I(\mu_1)) ,
\]

where

\[
I(\Delta) = \int \frac{d^2 \ell}{V \cdot \tilde{V} \cdot \ell - \Delta^2 + i \epsilon} = -i 2\pi \text{arcsinh} \left( \frac{\delta}{|\Delta|} \right).
\]

We fix \( \Lambda \) by using the same equation and the same fictitious gluon propagator at \( \mu = 0 \), where the effect of the order parameter, in this case the chiral condensate, is to produce an effective mass \( M \) for the light quark. This assumption gives the equation

\[
1 = \frac{4}{3 \Lambda^2 \pi^2} \int_0^K dp \frac{p^2}{\sqrt{p^2 + M^2}} .
\]

For \( M = 400 \) MeV and the cutoff \( K = 800 \) MeV (700 MeV) we get \( \Lambda = 181 \) MeV (154 MeV).

On the other hand we can consider the analogous equation for the 2SC model which is given by

\[
1 = \frac{i \mu^2}{3 \Lambda^2 \pi^2} I(\Delta) = \frac{2 \mu^2}{3 \Lambda^2 \pi^2} \text{arcsinh} \left( \frac{\delta}{|\Delta|} \right)
\]

that can be solved explicitly:

\[
|\Delta| = \frac{\delta}{\sinh \left( \frac{3 \Lambda^2 \pi^2}{2 \mu^2} \right)} .
\]

With the same values of \( \Lambda \) (\( \Lambda = 181 \) MeV and 154 MeV) we find the values for the gap parameter of the 2SC model reported in the following table, where we have taken \( \delta \) of the order of \( \mu \).

| \( \delta = K - \mu \) | \( \Delta \) |
|-----------------|--------|
| 400 MeV         | 39 MeV |
| 300 MeV         | 68 MeV |

Table 1. Value of the gap parameter \( \Delta \) in the 2SC model for two values of the cutoff \( K \). The chemical potential has the value \( \mu = 400 \) MeV.
Similar results have been obtained by [21] with a procedure which differs from the present one for two reasons: They include the $O(1/\mu)$ corrections to the gap equation and use a smooth cutoff instead of the sharp cutoff $K$ used here.

The coupled equations for the LOFF case [28] have a non trivial solution:

$$\Delta_0 = \frac{\delta}{\sinh \left( \frac{3\Lambda^2 \pi^2}{\mu^2 k R} \right)}, \quad \Delta_1 = 0 . \quad (73)$$

This is the result for the gap in the LOFF phase in the approximation $\mu \to \infty$, $q \to \infty$. It is known that the crystalline phase holds in a well defined window of values $|\delta\mu|$: $|\delta\mu| \in (\delta\mu_1, \delta\mu_2)$. The determination of $\delta\mu_1$ is obtained by comparing the free energies of the normal and 2SC states and is completely analogous to the one in [2], based on eq. (4.3) of this paper; one gets, in the weak coupling limit $\delta\mu_1 \approx 0.71 \Delta_{2SC}$. On the other hand $\delta\mu_2$ is the limiting value of $|\delta\mu|$ between the LOFF and the normal phase; at $|\delta\mu| \approx \delta\mu_2$ one expects a second order phase transition and $\Delta_0 \to 0$. In our approximation the phase transition occurs because, as it is evident from (28), for sufficiently large $\delta\mu$ and fixed $q$ (in the analysis of [2], in the LOFF window $q$ is constant $\approx 0.90 \Delta_{2SC} \approx 35.3$ MeV) $\cos \theta_q$ is not inside the integration region in $\cos \theta$ and the only solution of the gap equation is $\Delta_0 = 0$. Therefore, within this approximation, $\delta\mu_2 \approx q \approx 0.90 \Delta_{2SC}$, which compares favorably with the result $\approx 0.754 \Delta_{2SC}$ of [2].

Going beyond this approximation implies the use of the eq. (23) alone, instead of (23) and (29); the result (23) is indeed more robust, based as it is only on the limit $\mu \to \infty$ and not also on $q \to \infty$. Now the gap equation would assume the form ($\delta = \mu$):

$$1 = \frac{\mu^2 \pi}{3 \Lambda^2 \pi^2 R} \left( \int_{-1}^{0} dz \delta R(h_-(z)) \arcsinh \frac{\mu}{\pi R \delta R(h_- (\cos \theta) \Delta)} \right) + \int_{0}^{+1} dz \delta R(h_+(z)) \arcsinh \frac{\mu}{\pi R \delta R(h_+(\cos \theta) \Delta)} . \quad (74)$$

Also for this approximation we find non vanishing gaps; for example at $R \approx 1$ and for $\delta\mu = \delta\mu_1$, we get $\Delta_0 \approx 5.5$ MeV; in this case $\delta\mu_2 \approx 1.25 \Delta_{2SC}$; for $R \approx 1.5$ we find smaller values for the gap: $\Delta_0 \approx 0.25$ MeV and $\delta\mu_2 \approx 0.83 \Delta_{2SC}$. We observe explicitly that in the limit of large $R$ one gets, as a solution of (74), the eq. (73) with $\delta = \mu$. 

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We finally notice that we find $\Delta_1 = 0$, whereas in the more precise calculation of ref. [2] it was found a value $\Delta_1 \neq 0$; but very suppressed with respect to $\Delta_0$. Presumably the difference comes from our expansion at the leading order in $\mu$, and this would explain the suppression of the vector condensate.

Notice that, even though our analysis leads to $\Delta_1 = 0$, we will consider in the sequel for completeness the general case where both condensates are present.

5 Phonon-quark interaction

Let us consider again the breaking terms in (19), i.e. $\mathcal{L}_\Delta^{(s)}$ and $\mathcal{L}_\Delta^{(v)}$. This lagrangian explicitly breaks rotations and translations as it induces a lattice structure given by parallel planes perpendicular to $\vec{n}$:

$$\vec{n} \cdot \vec{x} = \frac{\pi k}{q} \quad (k = 0, \pm 1, \pm 2, \ldots).$$

(75)

We can give the following physical picture of the lattice structure of the LOFF phase: Due to the interaction with the medium, the Majorana masses of the red and green quarks oscillate in the direction $\vec{n}$, reaching on subsequent planes maxima and minima. The lattice planes can fluctuate as follows:

$$\vec{n} \cdot \vec{x} \rightarrow \vec{n} \cdot \vec{x} + \frac{\phi}{2q f},$$

(76)

which defines the phonon field, i.e. the Nambu-Goldstone boson that is associated to the breaking of the space symmetries and has zero vacuum expectation value:

$$\langle \phi \rangle_0 = 0.$$  

(77)

Let us observe that, because of (75), we have a fluctuating field $\phi_k$ for any nodal plane, where

$$\phi_k \equiv \phi(t, x, y, z_k = k\pi/q)$$

(78)

Also the vector $\vec{n}$ can fluctuate:

$$\vec{n} \rightarrow \vec{R},$$

(79)

where the field vector $\vec{R}$ satisfies

$$|\vec{R}| = 1,$$

$$\langle \vec{R} \rangle_0 = \vec{n}.$$
Also the vector fields $\vec{R}$ represent, similarly to (78), a collection of fields:

$$\vec{R} \equiv \vec{R}_k .$$

(82)

However we are interested to give an effective description of the fields $\phi_k$ and $\vec{R}_k$ in the low energy limit, i.e. for wavelengths much longer than the lattice spacing $\sim 1/q$; in this limit the fields $\phi_k$ and $\vec{R}_k$ vary almost continuously and can be imagined as continuous functions of three space variables $x$, $y$ and $z$. Therefore we shall use in the sequel the continuous notation $\phi$ and $\vec{R}$ and postpone a discussion on this aspect to the end of the present Section and to the subsequent Section.

The vector field $\vec{R}$ behaves as a vector under rotations while being invariant under translations. If we take the $z$-axis pointing along the direction $\vec{n}$, we can write

$$(\vec{R})_i = R_i(\xi_1, \xi_2) = (e^{i(\xi_1 L_1 + \xi_2 L_2)/f_R})_{i3}$$

(83)

with $\vec{L}$ the generators of the rotation group in the spin 1 representation, i.e.

$$(L_i)_{jk} = -i\epsilon_{ijk}$$

(84)

and $\xi_1, \xi_2$ are fields. At the lowest order in these fields we find

$$\vec{R} \approx \vec{n} + \delta\vec{n}$$

(85)

with

$$\delta\vec{n} = \left(-\frac{\xi_2}{f_R}, \frac{\xi_1}{f_R}, 0\right).$$

(86)

Since the fields $R_i$ transforms as a vector under rotations, it follows that the rotational symmetry is restored by the substitution (79).

If the direction $\vec{q}$ appearing in the exponent in (19) were different from the direction $\vec{n}$ of the spin 1 condensate in the same equation, the physical situation would not differ from magnetic materials; we would need a NGB to take into account the fluctuations of the scalar condensate (17), i.e.

$$\Delta e^{2i\vec{q} \cdot \vec{x}}$$

(87)

and other independent fields $\delta\vec{n}$, behaving as spin waves, i.e. making a precession motion around the direction $\vec{n}$. Here, however, the situation is different, because the two directions coincide. Therefore the value of $\vec{R}$ is strictly related to the symmetry breaking induced by the scalar condensate.
We have shown in [4] that in order to describe the spontaneous breaking of space-time symmetries induced by the condensate (87) one NGB is sufficient. As we stressed in [4], rotations and translations at least locally cannot be seen as transformations breaking the symmetries of the theory in an independent way, because the result of a translation plus a rotation, at least locally, can be made equivalent to a pure translation.

As a matter of fact at the lowest order we write

\[
\frac{\phi}{2qf} = (\vec{n} + \delta\vec{n}) \cdot (\vec{x} + \delta\vec{x}) - \vec{n} \cdot \vec{x} \approx \vec{R} \cdot \vec{x} + \frac{T}{2q} - \vec{n} \cdot \vec{x},
\]

where we have introduced, as in [4], the auxiliary function \(T\), given, in the present approximation, by

\[
T = 2q \vec{n} \cdot \delta\vec{x}.
\]

Now the lattice fluctuations \(\frac{\phi}{2qf}\) must be small, as the phonon is a long-wavelength small amplitude fluctuation of the order parameter. From (88) it follows that \(T\) must depend functionally on \(\vec{R}\), i.e. \(T = F[\vec{R}]\), which, using again (88), means that

\[
\frac{\Phi}{2q} \equiv \vec{n} \cdot \vec{x} + \frac{\phi}{2qf} = \vec{R} \cdot \vec{x} + \frac{F[\vec{R}]}{2q} \equiv G[\vec{R}].
\]

The solution of this functional relation has the form

\[
\vec{R} = \vec{h}[\Phi]
\]

where \(\vec{h}\) is a vector built out of the scalar function \(\Phi\). By this function one can only form the vector \(\vec{\nabla}\Phi\); therefore we get

\[
\vec{R} = \frac{\vec{\nabla}\Phi}{|\vec{\nabla}\Phi|}.
\]

\(^5\)Recently the problem of the Goldstone theorem for Lorentz invariant theories with spontaneous breaking of space-time symmetries has been considered in [22].

\(^6\)In principle \(\vec{R}\) could depend linearly on a second vector, \(\vec{x}\), but this possibility is excluded for the assumed transformation properties under translations.
which satisfies (80) and, owing to (77), also (81). In terms of the phonon field $\phi$ the vector field $\vec{R}$ is given (up to the second order terms in $\phi$) by the expression

$$
\vec{R} = \vec{n} + \frac{1}{2f q} \left[ \vec{\nabla} \phi - \vec{n} (\vec{n} \cdot \vec{\nabla} \phi) \right] + \frac{\vec{n}}{8f^2 q^2} \left[ 3(\vec{n} \cdot \vec{\nabla} \phi)^2 - |\vec{\nabla} \phi|^2 \right] - \frac{\vec{\nabla} \phi}{4f^2 q^2} (\vec{n} \cdot \vec{\nabla} \phi). 
$$

(93)

Note that at the lowest order the fields $\xi_1, \xi_2$ introduced in (86) are given by

$$
\begin{align*}
- \frac{\xi_2}{f_R} &= \frac{1}{2f q} \frac{\partial \phi}{\partial x}, \\
+ \frac{\xi_1}{f_R} &= \frac{1}{2f q} \frac{\partial \phi}{\partial y}.
\end{align*}
$$

(94)

We observe that these relations do not involve $\partial \phi / \partial z$. In fact we are defining the phonons as fluctuations of the lattice planes $z = k \pi q$. In the case of a single plane it is clear that the phonon propagates only along the plane (say, $x$ and $y$ directions), and the only motion along $z$ would be a translation corresponding to a zero mode. However in our problem we have infinite Goldstone bosons propagating on the planes $z = z_k = k \pi / q$. That is we have a collection of fields as in eq. (78). The presence of many planes gives rise to a finite energy mode corresponding to the variation of the relative distance among the planes. Phrased in a different way, different phonon fields interact each other since the fermions propagate in the whole three-dimensional space. Translating the fields in momentum space we have to take a Fourier series with respect to the discrete coordinate $z_k$. This introduces in the theory a quasi-momentum $p_z$ through the combination $\exp(ip_z z_k)$. Therefore $p_z$ and $p_z + 2q$ define the same physical momentum and we can restrict $p_z$ to the first Brillouin zone $-q \leq p_z \leq q$. As already stressed, this point will not affect our final discussion since we will be interested to energies and momenta much smaller than the gap.

The interaction term with the NGB field is contained in

$$
L_{\text{int}} = - e^{i\phi / f} \sum_{\vec{v}} \left[ \Delta^{(e)} \epsilon_{ij} + \Delta^{(v)} (\vec{v} \cdot \vec{R}) \sigma_{ij} \right] e^{\alpha \beta} \bar{\psi}_{i,\alpha,\vec{v}} C \psi_{j,\beta,-\vec{v}} - (L \rightarrow R) + \text{h.c.}
$$

(95)

Notice that we have neglected the breaking of the color symmetry, which has been considered elsewhere [23]. At the first order in the fields one gets the
following three-linear coupling:

\[
\mathcal{L}_{\phi\psi\psi} = -\frac{i\phi}{f} \sum_{\vec{v}} \left[ \Delta^{(s)} \epsilon_{ij} + \vec{\nabla} \Delta^{(v)} \sigma_{ij}^{1} \right] \epsilon^{\alpha\beta} \bar{\psi}_{i,\alpha,\vec{v}} \psi_{j,\beta,-\vec{v}} \\
- \frac{1}{2f q} \sum_{\vec{v}} \vec{v} \cdot \left[ \vec{\nabla} \phi - \vec{n} \cdot \vec{\nabla} \phi \right] \Delta^{(v)} \sigma_{ij}^{1} \epsilon^{\alpha\beta} \bar{\psi}_{i,\alpha,\vec{v}} \psi_{j,\beta,-\vec{v}} \\
- (L \to R) + h.c. \tag{96}
\]

We also write down the quadrilinear coupling:

\[
\mathcal{L}_{\phi\phi\psi\psi} = \frac{\phi^{2}}{2f^{2}} \sum_{\vec{v}} \left[ \Delta^{(s)} \epsilon_{ij} + \vec{\nabla} \Delta^{(v)} \sigma_{ij}^{1} \right] \epsilon^{\alpha\beta} \bar{\psi}_{i,\alpha,\vec{v}} \psi_{j,\beta,-\vec{v}} \\
- \frac{1}{2f q} \sum_{\vec{v}} \vec{v} \cdot \left[ \vec{\nabla} \phi - \vec{n} \cdot \vec{\nabla} \phi \right] \Delta^{(v)} \sigma_{ij}^{1} \epsilon^{\alpha\beta} \bar{\psi}_{i,\alpha,\vec{v}} \psi_{j,\beta,-\vec{v}} \\
- \frac{1}{4f^{2} q^{2}} \sum_{\vec{v}} \vec{v} \cdot \left[ \vec{\nabla} \phi - \vec{n} \cdot \vec{\nabla} \phi \right] \Delta^{(v)} \sigma_{ij}^{1} \epsilon^{\alpha\beta} \bar{\psi}_{i,\alpha,\vec{v}} \psi_{j,\beta,-\vec{v}} - (L \to R) + h.c. \tag{97}
\]

We notice that in (95), (96) and (97) the average over the velocities is defined according to (51).

It is now straightforward to re-write these couplings in the basis of the \( \chi \) fields. One gets

\[
\mathcal{L}_{3} + \mathcal{L}_{4} = \sum_{\vec{v}} \sum_{A=0}^{3} \chi_{A}^{\dagger} \left( \begin{array}{c}
0 \\
g_{3} + g_{4}
\end{array} \right) \chi_{B}, \tag{98}
\]

Here

\[
g_{3} = \left[ \frac{i\phi \Delta_{AB}}{f} + \sigma_{AB} \hat{O}[\phi] \right], \\
g_{4} = \left[ -\frac{\phi^{2} \Delta_{AB}}{2f^{2}} + \sigma_{AB} \left( \frac{i\phi}{f} \hat{O}[\phi] + \hat{Q}[\phi] \right) \right], \tag{99}
\]

with

\[
\hat{O}[\phi] = \frac{1}{2f q} \vec{v} \cdot \left[ \vec{\nabla} \phi - \vec{n} \cdot \vec{\nabla} \phi \right] \Delta^{(v)}, \\
\hat{Q}[\phi] = \frac{\Delta^{(v)}}{4f^{2} q^{2}} \left[ \vec{v} \cdot \vec{n} \left( 3(\vec{n} \cdot \vec{\nabla} \phi)^{2} - |\vec{\nabla} \phi|^{2} \right) - (\vec{v} \cdot \vec{\nabla} \phi)(\vec{n} \cdot \vec{\nabla} \phi) \right], \tag{100}
\]

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\begin{align}
\sigma_{AB} &= \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -i & 0 \\
0 & +i & 0 & 0 \\
+1 & 0 & 0 & 0
\end{pmatrix}.
\end{align}

For the following it is important to observe that the term in \( g_3 \) and \( g_4 \) proportional to \( \Delta_{AB} \) arise from the expansion of \( \exp \frac{i \phi}{f} \) alone, whereas the terms proportional to \( \sigma_{AB} \) get also contribution from the expansion of \( \vec{R} \) in the vector condensate.

Before concluding this section, let us come back to the lattice structure given by the parallel planes of Eq. (75). The effective action for the field \( \phi \), \( S[\phi] \), is obtained by the lagrangian as follows

\begin{align}
S &= \int dt \, dx \, dy \, \frac{\pi}{q} \sum_{k=-\infty}^{+\infty} \mathcal{L}(\phi(t, x, y, k\pi/q)) ,
\end{align}

In the action bilinear terms of the type \( \phi_k \phi_{k'} \) with \( k \neq k' \) may arise. In the continuum limit this terms would correspond to derivatives with respect to the \( z \) direction

We can use the same lagrangian \( \mathcal{L}_3 + \mathcal{L}_4 \) given in (98) with the integration measure \( \int d^4x \) provided we multiply the r.h.s. of (98) by the factor

\begin{align}
\tau(z) &= \frac{\pi}{q} \sum_{k=-\infty}^{+\infty} \delta \left( z - \frac{k\pi}{q} \right) .
\end{align}

As already observed the phonon momentum is a quasi-momentum, i.e. \( p_z \) and \( p_z + 2q \) correspond to the same \( z \)-component of the momentum. Due to this remark one can make use in the calculations of the formula

\begin{align}
\sum_{k=-\infty}^{+\infty} e^{ik\pi p/q} &= \delta \left( \frac{p}{2q} \right) .
\end{align}

6 Effective lagrangian for the phonon field

To introduce formally the NGB in the theory one can use the gradient expansion (see e.g. [24]), i.e. a bosonization procedure similar to the one employed in [13] to describe the goldstones associated to the breaking of \( SU(3) \) in the CFL model. First one introduces an external field with the same quantum
Fig. 1. Self-energy (a) and tadpole (b) diagrams.

numbers of the NGB and then performs a derivative expansion of the generating functional. This gives rise to the effective action for the NGB. At the lowest order one has to consider the diagrams in Fig. 1, i.e. the self-energy, Fig. 1a, and the tadpole, Fig. 1b, whose result we name $\Pi(p)_{\text{s.e.}}$ and $\Pi(p)_{\text{tad}}$ respectively.

To perform the calculation one employs the propagator given in Eq. (57) and the interaction vertices in (98). The result of the calculation of the two diagrams at the second order in the momentum expansion is as follows:

$$
\Pi(p)_{\text{s.e.}} = \frac{i \mu^2}{16\pi^2 f^2} \sum_{\vec{v}} \sum_{C=0}^{3} \int d^2\ell \left[ \frac{4 \lambda_C^2}{D_C^2(\ell)} - \frac{4\lambda_C V \cdot \ell \tilde{V} \cdot \ell}{D_C^2(\ell)} \right] \omega^2(\vec{p}) \left( \frac{2\lambda_C}{D_C^2(\ell)} + \frac{1}{D_C(\ell)} \right) ,
$$

$$
\Pi(p)_{\text{tad}} = \frac{i \mu^2}{16\pi^2 f^2} \sum_{\vec{v}} \sum_{C=0}^{3} \int d^2\ell \left[ \frac{4 \lambda_C}{D_C(\ell)} \left( \frac{\Delta^{(v)}}{q^2} \right) \mu_C \times \left( -p_x^2 - p_y^2 + 2p_z^2 - 2\vec{p} \cdot \vec{v} p_z \right) \right] \right) ,
$$

(105)

where

$$
D_C(\ell) = \ell_0^2 - \ell_\parallel^2 - \lambda_C + i\epsilon ,
$$

(106)

$\mu_c$ defined in (64) and

$$
\omega(\vec{p}) = \vec{p} \cdot \vec{v} - (\vec{p} \cdot \vec{n})(\vec{v} \cdot \vec{n}) .
$$

(107)

One can easily control that the Goldstone theorem is satisfied and the phonon is massless because one has

$$
\Pi(0) = \Pi(0)_{\text{s.e.}} + \Pi(0)_{\text{tad}} = 0 .
$$

(108)
In performing the integration we take into account the circumstance that \( \delta \gg |\mu_C| \). At the second order in the momentum expansion one gets

\[
\Pi(p) = -\frac{\mu^2}{2\pi^2 f^2} \sum_{\vec{v}} \left[ V_{\mu} \tilde{V}_{\nu} p_{\mu} p_{\nu} + \Omega^{(v)}(\vec{p}) \right]. \tag{109}
\]

Here

\[
\Omega^{(v)}(\vec{p}) = -\left( \frac{\Delta^{(v)}}{q} \right)^2 \omega^2(\vec{p}) \left( 2 - \frac{1}{2} \sum_{C=0}^{3} \arcsinh \frac{\delta}{|\mu_C|} \right) + \frac{\Delta^{(v)}}{2q^2} \Phi(\vec{p}) \sum_{C=0}^{3} \mu_C \times \arcsinh \frac{\delta}{|\mu_c|} \approx \frac{\Delta^{(v)}}{2q^2} \Phi(\vec{p}) \sum_{C=0}^{3} \mu_C \times \arcsinh \frac{\delta}{|\mu_c|} \approx
\]

\[
-2 \left( \frac{\Delta^{(v)}}{q} \right)^2 \left( 1 - \log \frac{2\delta}{\Delta_0} \right) \left( \omega^2(\vec{p}) + \vec{v} \cdot \vec{n} \Phi(\vec{p}) \right) \tag{110}
\]

where we have taken \( \Delta^{(v)} \ll \Delta_0 \) and

\[
\Phi(\vec{p}) = \left( 3p_z^2 - \vec{p}^2 \right) \vec{v} \cdot \vec{n} - 2\vec{p} \cdot \vec{v} p_z \tag{111}
\]

After averaging over the fermi velocities we obtain

\[
\Pi(p) = -\frac{\mu^2k_R}{4\pi^2 f^2} \left[ p_0^2 - v_\perp^2 (p_x^2 + p_y^2) - v_\parallel^2 p_z^2 \right]. \tag{112}
\]

One obtains canonical normalization for the kinetic term provided

\[
f^2 = \frac{\mu^2k_R}{2\pi^2} . \tag{113}
\]

On the other hand

\[
v_\perp^2 = \left[ \frac{1}{2} \sin^2 \theta_q + \left( 1 - 3 \cos^2 \theta_q \right) \left( 1 - \log \frac{2\delta}{\Delta_0} \right) \left( \frac{\Delta^{(v)}}{q} \right)^2 \right] \tag{114}
\]

\[
v_\parallel^2 = \cos^2 \theta_q . \tag{115}
\]

We note that \( f^2 \) depends on \( k_R \). We expect that \( k_R \) is in the range \((1, 2)\), where 1 corresponds to a smearing distance equal to the lattice spacing \( |\Delta z| = \pi/q \), and 2 corresponds to realistic values for the gap. It is remarkable however that the phonon dispersion relation is not affected by this
uncertainty. We also notice that the only source of anisotropy is in the asymmetric average over the Fermi velocity, that is the integration over a circle instead of the integration over a sphere. In fact, if we were going to average the previous result over $\cos \theta_q$ we would get the isotropic result $v_\perp^2 = v_\parallel^2 = 1/3$. In particular the contribution from $\Delta^{(v)} v^2$ averages to zero. In fact, this contribution is just the one arising from the terms in $g_3$ and $g_4$ proportional to $\sigma_{AB}$. For the sake of the argument let us forget for a moment the relation between $\Phi$ and $\vec{R}$. In this case the previous contribution is nothing but the ”zero-momentum” contribution to the two point function for the fields $\xi_i$ (or $\vec{\delta}n$) of eq. In the actual calculation this is a second order term in the momenta simply because the fields $\xi_i$ are derivatives of $\phi$. It is a simple consequence of the symmetries to see that, after averaging the velocity over all the Fermi sphere, this term is necessarily proportional to $\xi_1^2 + \xi_2^2 \approx (\vec{\delta}n)^2$. In fact, the only vectors involved are $\delta \vec{n}$, $\vec{n}$ and $\vec{v}$. Since $\delta \vec{n} \cdot \vec{n} = 0$, after averaging the result is necessarily proportional to $(\vec{\delta} \vec{n})^2$. However, the Goldstone theorem tells us that Goldstone fields should have no mass. This is true also for $\vec{\delta}n$, since being a gradient of $\phi$ it still satisfies the same wave equation of $\phi$. This explains the particular expression for this term, which has to average to zero integrating the velocity over all the Fermi sphere.

In any event the term proportional to the vector condensate $\Delta^{(v)}$ is very small and the result for $v_\perp^2$ can be approximated as follows

$$v_\perp^2 \simeq \frac{1}{2} \sin^2 \theta_q .$$

(116)

In conclusion, the dispersion law for the phonon is

$$E(p) = \sqrt{v_\perp^2 (p_x^2 + p_y^2) + v_\parallel^2 p_z^2}$$

(117)

which is anisotropic. Besides the anisotropy related to $v_\perp \neq v_\parallel$, there is another source of anisotropy, due to the fact that $p_z$, the component of the momentum perpendicular to the planes of $\phi$, differently from $p_x$ and $p_y$ is a quasi momentum and not a real momentum. The difference can be better appreciated in coordinate space, where the effective lagrangian reads

$$\mathcal{L} = \frac{1}{2} \left[ (\dot{\phi}_k)^2 - v_\perp^2 (\partial_x \phi_k)^2 - v_\perp^2 (\partial_y \phi_k)^2 - v_\parallel^2 \left( \frac{q}{\pi} \right)^2 (\phi_k - \phi_{k-1})^2 \right] .$$

(118)

However, in the long distance limit $\ell >> \pi/q$, the set of fields $\phi_k(x, y)$ becomes a function $\phi(x, y, z)$ and the last term can be approximated by $v_\parallel^2 (\partial_z \phi)^2$. 

24
7 Conclusions

The formalism we have used employs velocity dependent fields and it has been adapted to describe anisotropic phases, such as the crystalline phase of interest here. We have considered QCD with two massless flavors. For the two-quark condensates (scalar and vector) we have made the currently used plane-wave ansatz. For different values of the chemical potentials of up- and down-quark the two fermion momenta require independent decompositions. The quark propagators can be derived from the effective action for the fermi fields, as provided by the velocity dependent formalism, neglecting antiparticles (with respect to the Fermi sphere). A careful discussion is required for the integration over velocities, due to the anisotropy of the LOFF phase, implying that the effective theory is (2+1)-dimensional, differently from the situation that applied to CFL or to 2SC where it was (1+1)-dimensional.

We have derived the gap equation for the normal BCS state and the LOFF phase within the same formalism. Our approach is based on a few approximations, most notably the neglecting of the negative energy states and subleading terms in the $\mu \to \infty$ limit. These approximations considerably simplify the formalism and render it suitable for more complicated computations such as those arising from crystalline patterns more involved than the simple plane wave ansatz assumed in this paper. In spite of this approximations we are able to obtain results that qualitatively agree with more complete calculations existing in the literature.

The breaking of rotations and translations induces within the plane-wave ansatz a lattice structure of parallel planes corresponding to maxima and minima of the fermion Majorana masses. Fluctuations of the lattice planes define the phonon field. The phonon momentum has to be regarded as a quasi-momentum, but one can restrict to the first Brillouin zone. The formal introduction of the Nambu-Goldstone boson has been performed through bosonization, leading to the effective boson action.

The main result of the present paper is the application of the formalism to derive the parameters of the phonon field, i.e. the goldstone field associated to the breaking of the rotational and translational invariance in the crystalline color superconductive phase of QCD. Lowest order calculation requires self-energy and tadpole diagrams. They have to be calculated in terms of the fermion propagators and the three-linear and quadrilinear phonon-quark interactions. Besides explicit verification of the zero value of the phonon mass, we have derived the values of the phonon constant $f$ and the expressions
for the transverse and parallel velocities. The resulting phonon dispersion relation is anisotropic, both because of the difference in the velocity components, and of the quasi-momentum aspect of one momentum component. In coordinate space, the effective lagrangian exhibits explicitly such anisotropy in a most apparent form.

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