SUBS YMMETRIC BASES HAVE THE FACTORIZATION PROPERTY

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Abstract. We show that every subsymmetric Schauder basis \( (e_j) \) of a Banach space \( X \) has the factorization property, i.e. \( I_X \) factors through every bounded operator \( T: X \to X \) with a \( \delta \)-large diagonal (that is \( \inf_j |\langle T e_j, e_j^* \rangle| \geq \delta > 0 \), where the \( e_j^* \) are the biorthogonal functionals to \( (e_j) \)). Even if \( X \) is a non-separable dual space with a subsymmetric weak* Schauder basis \( (e_j) \), we prove that if \( (e_j) \) is non-\( \ell^p \)-splicing (there is no disjointly supported \( \ell^p \)-sequence in \( X \)), then \( (e_j) \) has the factorization property. The same is true for \( \ell^p \)-direct sums of such Banach spaces for all \( 1 \leq p \leq \infty \).

Moreover, we find a condition for an unconditional basis \( (e_j)_{j=1}^n \) of a Banach space \( X_n \) in terms of the quantities \( \|e_1 + \ldots + e_n\| \) and \( \|e_1^* + \ldots + e_n^*\| \) under which an operator \( T: X_n \to X_n \) with \( \delta \)-large diagonal can be inverted when restricted to \( X_n = \{e_j : j \in \sigma\} \) for a “large” set \( \sigma \subset \{1, \ldots, n\} \) (restricted invertibility of \( T \); see Bourgain and Tzafriri [Israel J. Math. 1987, London Math. Soc. Lecture Note Ser. 1989]). We then apply this result to subsymmetric bases to obtain that operators \( T \) with a \( \delta \)-large diagonal defined on any space \( X_n \) with a subsymmetric basis \( \{e_j\} \) can be inverted on \( X_n \) for some \( \sigma \) with \( |\sigma| \geq cn^{1/4} \).

1. Introduction

Throughout this paper, we assume that \( X \) and \( Y \) are Banach spaces satisfying the following properties (B1)–(B5). We assume that there exists a bilinear map \( \langle \cdot, \cdot \rangle: X \times Y \to \mathbb{R} \) such that:

(B1) whenever \( x \in X \) and \( \langle x, y \rangle = 0 \) for all \( y \in Y \), then \( x = 0 \);
(B2) whenever \( y \in Y \) and \( \langle x, y \rangle = 0 \) for all \( x \in X \), then \( y = 0 \);
(B3) there exists a constant \( C_4 > 0 \) such that \( |\langle x, y \rangle| \leq C_4\|x\|X\|y\|Y \) for all \( x \in X, y \in Y \).

By \( \sigma(X,Y) \), we denote the locally convex topology on \( X \) generated by the collection of seminorms \( \{x \mapsto |\langle x, y \rangle| : y \in Y\} \). In addition, we assume there exist normalized sequences \( (e_j) \) in \( X \) and \( (f_j) \) in \( Y \) such that

(B4) \( \langle e_j, f_j \rangle = 1 \) and \( \langle e_j, f_k \rangle = 0 \), for all \( j \neq k \);
(B5) every \( x \in X \) has the unique representation \( x = \sum_{j=1}^\infty \langle x, f_j \rangle e_j \), where the series converges in the \( \sigma(X,Y) \)-topology.

If \( (z_j) \) is a normalized Schauder basis for the Banach space \( Z \), \( Z^* \) is the dual space and \( (z_j^*) \) denotes the biorthogonal functionals of \( (z_j) \) \( (z_j^*) \) is called a weak* Schauder basis for \( Z^* \), then (B1)–(B5) are satisfied for \( X = Z, Y = Z^*, e_j = z_j, f_j = z_j^*, j \in \mathbb{N} \) and \( C_4 = 1 \). The same is true for \( X = Z^*, Y = Z e_j = z_j^*, f_j = z_j, j \in \mathbb{N} \) and \( C_4 = 1 \). For more information on topological bases we refer to [23, Section 1.b] and [27, Chapter 1, §13].

For each set \( A, |A| \) denotes the cardinality of that set. Given a sequence of vectors \( (x_j) \) in a Banach space \( X \) and \( A \subset \mathbb{N} \), \( \{x_j : j \in A\} \) denotes the norm-closure of span\( \{x_j : j \in A\} \).

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We say that the sequence \((e_j)\) is \(C_u\)-unconditional, if for all sequences of scalars \((a_j)\), \((\gamma_j)\) holds that

\[
\left\| \sum_{j=1}^{\infty} \gamma_j a_j e_j \right\|_X \leq C_u \sup_k |\gamma_k| \left\| \sum_{j=1}^{\infty} a_j e_j \right\|_X,
\]

where we demand the above series converge in the \(\sigma(X,Y)\)-topology. Moreover, we say that \((e_j)\) is \(C_s\)-spreading, if \((e_j)\) is \(C_s\)-equivalent to each of its increasing subsequences, i.e.

\[
\frac{1}{C_s} \left\| \sum_{j=1}^{\infty} a_j e_{n_j} \right\|_X \leq \left\| \sum_{j=1}^{\infty} a_j e_j \right\|_X \leq C_s \left\| \sum_{j=1}^{\infty} a_j e_{n_j} \right\|_X,
\]

for all increasing \((n_j)\). Again, we demand that the above series converge in the \(\sigma(X,Y)\)-topology.

If (1.1) and (1.2) are both satisfied, we say that \((e_j)\) is \(C_u, C_s\)-subsymmetric. If \((e_j)\) is \(C\)-unconditional or \(C\)-spreading or \((C, C)\)-subsymmetric for some \(C\), we say that \((e_j)\) is unconditional or spreading or subsymmetric. For more information on unconditional or subsymmetric bases we refer to [23, 27].

Moreover, we introduce the following notions. Let \(I_X\) denote the identity operator on the Banach space \(X\) and let \(T : X \to X\) denote a bounded linear operator. We say that

\begin{itemize}
  \item \(I_X\) \(C\)-factors through \(T\), if there exist operators \(A, B : X \to X\) such that \(I_X = ATB\) and \\
  \(\|A\| \cdot \|B\| \leq C\).
  \item \(I_X\) factors through \(T\) if \(I_X\) \(C\)-factors through \(T\) for some \(C\).
  \item \(I_X\) almost \(C\)-factors through \(T\) if \(I_X\) \((C + \eta)\)-factors through \(T\), whenever \(\eta > 0\).
  \item \(T\) has \(\delta\)-large diagonal (with respect to \((e_j)\)) if \(\delta := \inf_j \|Te_j \langle f_j \rangle\| > 0\).
  \item \(T\) has large diagonal (with respect to \((e_j)\)) if \(T\) has \(\delta\)-large diagonal for some \(\delta > 0\).
  \item \((e_j)\) has the \(K(\delta)\)-factorization property if \(I_X\) almost \(K(\delta)\)-factors through every bounded linear operator \(T : X \to X\) which has \(\delta\)-large diagonal with respect to \((e_j)\).
  \item \((e_j)\) has the factorization property if \(I_X\) factors through every bounded linear operator \(T : X \to X\) which has a large diagonal with respect to \((e_j)\).
  \item \(X\) is primary, if for every bounded projection \(Q : X \to X\) either \(Q(X)\) or \((I_X - Q)(X)\) is isomorphic to \(X\) (see [23, Definition 3.b.7]).
\end{itemize}

In this context, the notion of a large diagonal first appeared implicitly in [1] and was then later formally introduced in [11]. The factorization property was conceived in [20] and further investigated in [18].

Finally, we will define non-\(\ell^1\)-splicing bases, which were introduced and investigated in [17] (specifically for weak* Schauder bases). We would like to mention that prior to formally introducing non-\(\ell^1\)-splicing, the underlying properties of such bases were exploited e.g. in [14, 4] and can be traced back all the way to Lindenstrauss’ proof showing that \(\ell^\infty\) is prime [22] (see also [23, Theorem 2.a.7]).

We assume now that \((e_j)\) is unconditional and define for each \(\Lambda \subset \mathbb{N}\) the bounded projection \(P_\Lambda : X \to X\) by

\[
P_\Lambda \left( \sum_{j=1}^{\infty} a_j e_j \right) = \sum_{j \in \Lambda} a_j e_j,
\]

where the above series converge in the \(\sigma(X,Y)\)-topology. We then say that the unconditional sequence \((e_j)\) in \(X\) is non-\(\ell^1\)-splicing if for every infinite set \(I \subset \mathbb{N}\) and every \(\theta > 0\) we can find a sequence \((\Lambda_j)\) of pairwise disjoint and infinite subsets of \(I\), such that for all sequences \((x_j) \subset X\) with \(\|x_j\|_X \leq 1, j \in \mathbb{N}\) there exists a sequence of scalars \((a_j) \in \ell^1\) with \(\|(a_j)\|_{\ell^1} = 1\) such that

\[
\left\| \sum_{j=1}^{\infty} a_j P_{\Lambda_j} x_j \right\|_X \leq \theta.
\]
Otherwise, we say that \( (e_j)_{j=1}^{\infty} \) is \( \ell^1 \)-splicing.

If \( (e_j) \) is subsymmetric, J. L. Ansorena [2, Theorem 4] characterized \( \ell^1 \)-splicing bases in the following way: \( (e_j) \) is \( \ell^1 \)-splicing if and only if there is a disjointly supported sequence \( (x_j) \) in \( X \) which is equivalent to the standard unit vector system in \( \ell^1 \). (Although this characterization is stated for weak* Schauder bases, examining the proof reveals that it only uses (1.1) and (1.2) and therefore easily carries over to our topological basis.) Examples of non-\( \ell^1 \)-splicing weak* Schauder bases are provided in [17] and [2].

2. Results

This section is divided into two parts: a subsection devoted to presenting our qualitative infinite dimensional factorization results Theorem 2.1, Theorem 2.2 and Theorem 2.3, and a subsection for the finite dimensional quantitative factorization results Theorem 2.4 and Corollary 2.6. The qualitative infinite dimensional factorization results are achieved by building on the work done in [17] and bringing in a randomization technique from [19].

For related factorization results in various Banach spaces we refer to e.g. [9, 7, 4, 3, 28, 29, 25, 21, 12, 13, 16, 15, 20, 18] ; see also [24].

2.1. Qualitative Factorization results. First, we discuss the the qualitative infinite dimensional factorization results in Banach spaces with a subsymmetric topological basis Theorem 2.1 and Theorem 2.2, which establish that subsymmetric Schauder bases and subsymmetric topological bases that are non-\( \ell^1 \)-splicing have the factorization property. We then conclude this subsection with Theorem 2.3 asserting that the array \( (e_{n,j})_{n,j} \) corresponding to the direct sum of Banach spaces with a subsymmetric and non-\( \ell^1 \)-splicing topological basis also has the factorization property. We would like to point out that mixing of different Banach spaces in the direct sum is allowed, as long as the constants of the subsymmetric topological basis are uniformly bounded.

Theorem 2.1 below can be viewed as a generalization of the factorization theorem in [10] (see also [23, Proposition 3.b.8]).

**Theorem 2.1.** Let \( (e_j) \) be a \((C_u,C_s)\)-subsymmetric Schauder basis for the Banach space \( X \). Then \( (e_j) \) has the \( \left( \frac{2C_u^2C_s^2}{3} \right) \)-factorization property.

Where Theorem 2.1 demands that \( (e_j) \) is a Schauder basis, Theorem 2.2 allows for more general topological bases e.g. a weak* Schauder bases. To compensate for the lacking norm-convergence of the topological basis, we demand that \( (e_j) \) is non-\( \ell^1 \)-splicing.

We are now ready to state our second main result.

**Theorem 2.2.** Let the dual pair \((X,Y,\langle\cdot,\cdot\rangle)\) together with the sequences \((e_j),(f_j)\) satisfy (B1)–(B5) and assume that \( (e_j) \) is \((C_u,C_s)\)-subsymmetric as well as non-\( \ell^1 \)-splicing. Then \( (e_j) \) has the \( \left( \frac{2C_u^2C_s^2}{3} \right) \)-factorization property.

Recall that (B1)–(B5) are satisfied if \( (e_j) \) is a weak* Schauder basis. The proof of Theorem 2.2, together with the proof of Theorem 2.1, is given in Section 4.

In [9], Casazza, Kottman and Lin showed that the spaces \( \ell^p(X), 1 < p < \infty \) and \( c_0(X) \) are primary, whenever \( X \) has a (sub)symmetric Schauder basis and is not isomorphic to \( \ell^1 \). Samuel [26] proved that the spaces \( \ell^p(\ell^q), 1 \leq p,q \leq \infty \) are primary. Capon [7] showed that \( \ell^1(X) \) and \( \ell^\infty(X) \) is primary, whenever \( X \) has a (sub)symmetric Schauder basis. In [17], the author proved that the Banach spaces \( \ell^p(X), 1 \leq p \leq \infty \) are primary, whenever \( X \) is a Banach space with a non-\( \ell^1 \)-splicing subsymmetric weak* Schauder basis.
Theorem 2.3 below can be viewed as a vector valued version of Theorem 2.2, but can also be regarded as an extension of the factorization result [17, Theorem 1.2] in the sense that [17, Theorem 1.2] is a corollary to Theorem 2.3 (see Corollary 5.1). This highlights the close connection between the factorization property of a basis and the primarity of the space. On the other hand, we would like to point out that although the James space is primary [8], the boundedly complete basis of the James space does not have the factorization property [20, Proposition 2.5].

**Theorem 2.3.** For each \( n \in \mathbb{N} \) let the dual pair \((X_n, Y_n, \langle \cdot, \cdot \rangle)\) of infinite dimensional Banach spaces \(X_n\) and \(Y_n\) and the sequences \((e_{n,j})_j \), \((f_{n,j})_j\) satisfy (B1)-(B5) with constant \( C_d \) (uniformly in \( n \)). Assume that \((e_{n,j})_j \) is \((C_u, C_s)\)-subsymmetric (uniformly in \( n \)) and non-\( \ell^1 \)-splicing and let \( 1 \leq p \leq \infty \). Then \((e_{n,j})_j\) has the factorization property in \( \ell^p((X_n))\), i.e. whenever \( T: \ell^p((X_n)) \to \ell^p((X_n)) \) is a bounded operator with

\[
\inf_{n,j} \| T e_{n,j}, f_{n,j} \| > 0,
\]

there exist bounded operators \( E, P : \ell^p((X_n)) \to \ell^p((X_n)) \) such that \( I_{\ell^p((X_n))} = PTE \).

For the proof of Theorem 2.3 we refer to Section 5.

**2.2. Finite dimensional quantitative factorization results.** In [5, Theorem 6.1], Bourgain and Tzafriri obtain the following restricted invertibility result for operators acting on \( n \)-dimensional Banach spaces with an unconditional basis \((e_j)_{j=1}^n\) which satisfies a lower \( r \)-estimate for some \( 1 < r < \infty \): For any \( 0 < \varepsilon < 1 \), there exists a subset \( \sigma \) with \( |\sigma| \geq n^{1-\varepsilon} \) on which a given operator is invertible when restricted to the subspace \([e_j : j \in \sigma]\). Analyzing their proof, we recognize that we can relax the condition that \((e_j)\) has to satisfy a lower \( r \)-estimate (see Theorem 2.4), albeit at the cost of a weaker estimate for \(|\sigma|\) (see Remark 2.5). We would like to point out the closely related recent works [16, 15], in which the dependence on the dimension for quantitative factorization results in one- and two-parameter Hardy and BMO spaces was improved from super-exponential estimates to polynomial estimates.

In this subsection, \((e_j)\) denotes a normalized basis for the Banach space \(X\) and \((e^*_j)\) denotes the biorthogonal functionals to \((e_j)\). We define the function \( \tau_1 : \mathbb{N} \to [0, \infty) \) by putting

\[
\tau_1(n) = \max \left\{ \min \left\{ \max_{1 \leq i \leq n} \left| \frac{\| e_{ij} e_i \|_{X^*}}{\| e_{ij} e_i \|_{X^*}} \right|, \max_{1 \leq i \leq n} \left| \frac{\| e_{ij} e_j^* \|_{X^*}}{\| e_{ij} e_j^* \|_{X^*}} \right| \right\} : e_{ij} \in \{ \pm 1 \}, 1 \leq i, j \leq n \right\}.
\]

Note that if \((e_j)\) is \(C_u\)-unconditional, we have the estimates

\[
C_u^{-1} \leq \tau_1(n) \leq C_u \min \left( \left\| \sum_{i=1}^n e_i \right\|_{X^*} , \left\| \sum_{j=1}^n e_j^* \right\|_{X^*} \right), \quad n \geq 2.
\]

We are now ready to state our first result on restricted invertibility.

**Theorem 2.4.** Let \((e_j)\) be a normalized \(C_u\)-unconditional basis for the Banach space \(X\), let \((e^*_j)\) denote the biorthogonal functionals to \((e_j)\) and put \(X_n = [e_j : 1 \leq j \leq n], \ n \in \mathbb{N}\). Let \( n \in \mathbb{N} \) and \( \delta, \Gamma, \eta > 0 \) be such that

\[
\frac{\delta \min(1, \eta)}{4 \Gamma n} \leq \tau_1(n) \leq \frac{\delta \min(1, \eta)}{2^{10} \Gamma n} . \frac{[16 + \min(\eta, (1 + \eta)^{-1})n]^2}{n}.
\]

Let \( T : X_n \to X_n \) denote an operator satisfying

\[
\| T \| \leq \Gamma \quad \text{and} \quad |\langle T e_j, e^*_j \rangle| \geq \delta, \quad 1 \leq j \leq n,
\]

and let \( D : X_n \to X_n \) denote the diagonal operator of \( T \), i.e.

\[
D e_i = \langle T e_i, e^*_i \rangle e_i, \quad 1 \leq i \leq n.
\]
For each \( \sigma \subset \{1, \ldots, n\} \) define the restriction operators \( R_\sigma : X_n \to X_n \) by \( R_\sigma(\sum_{i=1}^n a_i e_i) = \sum_{i \in \sigma} a_i e_i \). Then there exists a subset \( \sigma \subset \{1, \ldots, n\} \) with

\[
|\sigma| \geq \sqrt{\frac{\delta \min(1, \eta)}{16\Gamma}} \cdot \sqrt{\frac{n}{\tau(n)}}
\]

(2.5)
such that the operator \( R_\sigma D^{-1}TR_\sigma \) is invertible and satisfies

\[
\| (R_\sigma D^{-1}TR_\sigma)^{-1} \| \leq 1 + \eta.
\]

Moreover, if we define \( X_\sigma = [e_j : j \in \sigma] \), there exist operators \( E : X_\sigma \to X_n \) and \( P : X_n \to X_\sigma \) with \( \|E\|\|P\| \leq C_u^{1+\eta} \) such that \( I_{X_\sigma} = PTE \).

**Theorem 2.4** will be proved in Section 6.

**Remark 2.5.** We will now relate Theorem 2.4 to [5, Theorem 6.1].

To this end, let \( 1 < r, s < \infty \) with \( \frac{1}{r} + \frac{1}{s} = 1 \). Assume that \( (e_j)_{j=1}^n \) satisfies a lower \( r \)-estimate with constant \( c_r \), i.e. there exists a constant \( c_r > 0 \) such that

\[
\left\| \sum_{j=1}^n a_j e_j \right\| \geq c_r \left( \sum_{j=1}^n |a_j|^r \right)^{1/r}
\]

for all scalars \( (a_i) \). Then one can easily verify that \( (e^*_j)_{j=1}^n \) satisfies an upper \( s \)-estimate with constant \( \frac{1}{c_s} \), i.e.

\[
\left\| \sum_{j=1}^n a_j e^*_j \right\| \leq \frac{1}{c_s} \left( \sum_{j=1}^n |a_j|^s \right)^{1/s}.
\]

In particular, we obtain the estimate \( \tau(n) \leq \frac{1}{c_r}(n-1)^{1/s} \). Thus, if we choose

\[
n \geq \left( \frac{c_r \delta \min(1, \eta) \min(\eta, (1 + \eta)^{-1})}{2^{10}\Gamma} \right)^{1/r},
\]

**Theorem 2.4** yields a subset \( \sigma \subset \{1, \ldots, n\} \) with

\[
|\sigma| \geq \sqrt{\frac{c_r \delta \min(1, \eta)}{16\Gamma}} \cdot n^{1/(2r)}.
\]

In [6, Corollary 4.4] Bourgain and Tzafriri obtained linear lower estimates for the set \( |\sigma| \), whenever the subsymmetric basis \( (e_j) \) satisfies certain conditions in terms of Boyd indices. Using **Theorem 2.4**, we can get rid of this restriction entirely, but the prize we pay is a weaker estimate for \( |\sigma| \) (see **Corollary 2.6**, below).

**Corollary 2.6.** Let \( (e_j) \) be a normalized \( (C_u, C_s) \)-subsymmetric basis for the Banach space \( X \), let \( (e^*_j) \) denote the biorthogonal functionals to \( (e_j) \) and put \( X_n = [e_j : 1 \leq j \leq n] \), \( n \in \mathbb{N} \). Let \( n \in \mathbb{N} \) and \( \delta, \Gamma, \eta > 0 \) be such that

\[
n \geq 1 + \frac{C_u \delta \min(1, \eta)}{4\Gamma} \quad \text{and} \quad n \geq \frac{2^{11}C_s^3}{\delta^2 \min(1, \eta^2) \min(\eta^4, (1 + \eta)^{-4})}.
\]

(2.7)

Let \( T : X_n \to X_n \) denote an operator satisfying

\[
\|T\| \leq \Gamma \quad \text{and} \quad |\langle Te_j, e^*_j \rangle| \geq \delta, \quad 1 \leq j \leq n.
\]

(2.8)

As in **Theorem 2.4**, \( D : X_n \to X_n \) denotes the diagonal operator of \( T \) and \( R_\sigma : X_n \to X_n, \sigma \subset \{1, \ldots, n\} \) the restriction operators. Then there exists a subset \( \sigma \subset \{1, \ldots, n\} \) with

\[
|\sigma| \geq 4 \cdot (2C_u C_s)^{-1/4} \cdot \sqrt{\frac{\delta \min(1, \eta)}{\Gamma}} \cdot n^{1/4}
\]

(2.9)
such that the operator $R_\sigma D^{-1}TR_\sigma$ is invertible and satisfies
\[ \|(R_\sigma D^{-1}TR_\sigma)^{-1}\| \leq 1 + \eta. \] (2.10)

Thus, for $k = |\sigma|$, there exist operators $E: X_k \to X_n$ and $P: X_n \to X_k$ with $\|E\|\|P\| \leq C_\sigma^2 C_\sigma^{1+\eta}$ such that $I_{X_k} = PTE$.

The proof of Corollary 2.6 can be found in Section 6.

3. Tools

Here, we provide the mathematical tools that will be used throughout this article. They comprise of subspace annihilation results (see Lemma 3.1, Lemma 3.4) and techniques by which we preserve the large diagonal of an operator under blocking of the basis (see Lemma 3.2 and Proposition 3.3). Moreover, we provide estimates for basic factorization operators that are obtained by blocking a subsymmetric basis (see Proposition 3.5).

**Lemma 3.1.** Let $I \subset \mathbb{N}$ denote an infinite set, let $n \in \mathbb{N}$, $L \in 2 \cdot \mathbb{N}$ and $\eta > 0$. Then for all bounded sequences $(x_j)$ in $X$ and $(y_j)$ in $Y$ and every $n \in \mathbb{N}$ there exists an infinite set $\Lambda \subset I$ such that
\[
\sup \left\{ \left| \sum_{k \in B} \varepsilon_k x_k, y_j \right| : B \subset \Lambda, |B| = L, \varepsilon \in \mathcal{E}(B), 1 \leq j \leq n \right\} \leq \eta,
\]
and every
\[
\sup \left\{ \left| \sum_{k \in B} \varepsilon_k y_k \right| : B \subset \Lambda, |B| = L, \varepsilon \in \mathcal{E}(B), 1 \leq j \leq n \right\} \leq \eta,
\]
where the set $\mathcal{E}(B)$ is given by
\[
\mathcal{E}(B) = \left\{ (\varepsilon_k) \in \{\pm 1\}^B : \sum_{k \in B} \varepsilon_k = 0 \right\}. \tag{3.1}
\]

**Proof.** The proof is a straightforward adaptation of the argument given for [17, Lemma 3.1]. For sake of completeness, we will give a short proof here.

First, we put $G^1_k = \{ i \in I : \frac{(k-1)n}{L} < \langle x_i, y_1 \rangle \leq \frac{k-1}{L} \}$ and note that by (B3) $\bigcup_{k \in \mathbb{Z}} G^1_k = I$ and that there are only finitely many non-empty $G^1_k$, $k \in \mathbb{Z}$. Since $I$ is infinite, there exists at least one $k_1 \in \mathbb{Z}$ such that $G^1_{k_1}$ is also infinite. Next, we define $G^2_k = \{ i \in G^1_{k_1} : \frac{(k-1)n}{L} < \langle x_i, y_2 \rangle \leq \frac{k_2}{L} \}$ and repeat the previous step to obtain a $k_2 \in \mathbb{Z}$ such that $G^2_{k_2}$ is infinite. After $n$ steps, we obtain an infinite set $G = G^n_{k_n}$ such that
\[
\sup \left\{ \left| \langle x_{i_0}, y_j \rangle - \langle x_{i_1}, y_j \rangle \right| : i_0, i_1 \in G, 1 \leq j \leq n \right\} \leq \frac{2\eta}{L}. \tag{3.2}
\]

We will now repeat the above process but with the roles of $x$ and $y$ reversed and with $G$ instead of $I$. To illustrate, we now put $H^1_k = \{ i \in G : \frac{(k-1)n}{L} < \langle x_i, y_1 \rangle \leq \frac{k_0}{L} \}$. With the same reasoning as above, we can find $l_1 \in \mathbb{Z}$ such that $H^1_{l_1}$ is infinite. Iterating this procedure and stopping after $n$ steps yields an infinite set $\Lambda = H^n_{l_n} \subset G$ such that
\[
\sup \left\{ \left| \langle x_{i_0}, y_j \rangle - \langle x_{i_1}, y_j \rangle \right| : i_0, i_1 \in \Lambda, 1 \leq j \leq n \right\} \leq \frac{2\eta}{L}. \tag{3.3}
\]
Note that for each $B \subset \Lambda$ with $|B| = L$ and each $(\varepsilon_k) \in \mathcal{E}(B)$, there are exactly $L/2$ $k \in B$ such that $\varepsilon_k = 1$ and $L/2$ $k \in B$ such that $\varepsilon_k = -1$. This observation together with (3.2) and (3.3) proves the assertion.

For each $L \in 2 \cdot \mathbb{N}$ and $A \subset \mathbb{N}$ with $L \leq |A| < \infty$, we define the finite set $\Omega^A_L$ by
\[
\Omega^A_L = \{(B,(\varepsilon_k)) : B \subset A, |B| = L, (\varepsilon_k) \in \mathcal{E}(B)\}. \tag{3.4}
\]
By $\mathbb{E}_L^A$ we denote the average over all elements in $\Omega_L^A$. For each $(B, (\varepsilon_k)) \in \Omega_L^A$, we define
\[ b_{B}^{(\varepsilon_k)} = \sum_{k \in B} \varepsilon_k \epsilon_k \quad \text{and} \quad d_{B}^{(\varepsilon_k)} = \sum_{k \in B} \varepsilon_k f_k. \] (3.5)

Before turning to the next Lemma, we define $\nu: \mathbb{N} \to [0, \infty)$ by
\[ \nu(n) = \sup \left\{ \min \left( \max_{k \in A} \left\| \sum_{k \in B} \epsilon_k \right\| , \max_{k \in A} \left\| \sum_{l \in A \setminus \{k\}} f_l \right\| \right) : A \subset \mathbb{N}, |A| = n \right\}. \] (3.6)

The following lemma uses a randomization technique from [19]. In contrast to the randomization over all possible choices of signs in [19], we average in Lemma 3.2 over all signs in $\mathcal{E}(B)$. This enables us to use Lemma 3.1 which is essential for the non-separable case (see Theorem 2.2), but has the drawback that it introduces a negative bias when averaging $\varepsilon_{k\epsilon_l}$, $k \neq l$ (see (3.8)). This bias can be traced to the term $A_2$ in (3.9) and that in turn, by averaging over possible choices of $\mathcal{B} \subset \mathcal{A}$, leads to the term $B_2$ in (3.12), where it is finally controlled in terms of $\nu$.

**Lemma 3.2.** Let the dual pair $(X, Y, \langle \cdot, \cdot \rangle)$ together with the sequences $(e_j), (f_j)$ satisfy (B1)-(B5). Let $T: X \to X$ denote a bounded linear operator such that $\delta := \inf_{j} \langle T e_j, f_j \rangle > 0$. Let $L \in 2 \cdot \mathbb{N}$, $N \in \mathbb{N}$ with $N \geq L$ and pick any $\mathcal{A} \subset \mathbb{N}$ with $|\mathcal{A}| = N$. Then
\[ \mathbb{E}_L^A (T b_{B}^{(\varepsilon_k)}, d_{B}^{(\varepsilon_k)}) \geq \left[ \delta - C_d \frac{\|T\|_N}{N-1} \nu(N) \right] \cdot L. \]

**Proof.** Let $\mathcal{B} \subset \mathcal{A}$ with $|\mathcal{B}| = L$ be fixed and note that by (3.5), (3.4) and (3.1)
\[ \frac{1}{(L/2)} \sum_{(\varepsilon_k) \in \mathcal{E}(\mathcal{B})} \langle T b_{B}^{(\varepsilon_k)}, d_{B}^{(\varepsilon_k)} \rangle = \sum_{k \in \mathcal{B}} \langle T e_k, f_k \rangle + \sum_{k \notin \mathcal{B}} \frac{1}{(L/2)} \sum_{(\varepsilon_k) \in \mathcal{E}(\mathcal{B})} \varepsilon_k \epsilon_l \langle T e_k, f_l \rangle. \] (3.7)

A straightforward calculation shows that
\[ \frac{1}{(L/2)} \sum_{(\varepsilon_k) \in \mathcal{E}(\mathcal{B})} \varepsilon_k \epsilon_l = \frac{-1}{L-1}, \quad k \neq l. \] (3.8)

Combining (3.7) with (3.8) yields
\[ \frac{1}{(L/2)} \sum_{(\varepsilon_k) \in \mathcal{E}(\mathcal{B})} \langle T b_{B}^{(\varepsilon_k)}, d_{B}^{(\varepsilon_k)} \rangle = \sum_{k \in \mathcal{B}} \langle T e_k, f_k \rangle - \frac{1}{L-1} \sum_{k \notin \mathcal{B}} \langle T e_k, f_l \rangle = A_1 - \frac{1}{L-1} A_2. \] (3.9)

We will now separately average $A_1$ and $A_2$ over all $\binom{N}{L}$ possible selections $\mathcal{B} \subset \mathcal{A}$ with $|\mathcal{B}| = L$. First, averaging $A_1$ yields
\[ \frac{1}{\binom{N}{L}} \sum_{|\mathcal{B}|=L} A_1 = \frac{1}{\binom{N}{L}} \sum_{|\mathcal{B}|=L} \sum_{k \in \mathcal{B}} \langle T e_k, f_k \rangle = \sum_{k \in \mathcal{A}} \langle T e_k, f_k \rangle \sum_{|\mathcal{B}|=L} [\{ \mathcal{B} \subset \mathcal{A} : \mathcal{B} \ni k, |\mathcal{B}| = L \}] \]
\[ = \binom{N-1}{L-1} \binom{N}{L} \sum_{k \in \mathcal{A}} \langle T e_k, f_k \rangle \]
and we record
\[ \frac{1}{\binom{N}{L}} \sum_{|\mathcal{B}|=L} A_1 = \frac{L}{N} \sum_{k \in \mathcal{A}} \langle T e_k, f_k \rangle. \] (3.10)
Secondly, averaging \( A_2 \) gives us
\[
\frac{1}{\binom{N}{L}} \sum_{|B| = L} A_2 = \frac{1}{\binom{N}{L}} \sum_{|B| = L} \sum_{k \neq \ell \in B} \langle T e_k, f_\ell \rangle = \sum_{k \neq \ell \in A} \langle T e_k, f_\ell \rangle \frac{|\{B \subset A : B \ni k, l, |B| = L\}|}{\binom{N}{L}}
\]
\[
= \frac{(N-2)}{\binom{N}{L}} \sum_{k \neq \ell \in A} \langle T e_k, f_\ell \rangle,
\]
from which immediately follows that
\[
\frac{1}{\binom{N}{L}} \sum_{|B| = L} A_2 = \frac{L(L - 1)}{N(N - 1)} \sum_{k \neq \ell \in A} \langle T e_k, f_\ell \rangle. \tag{3.11}
\]
Combining (3.9), (3.10) and (3.11) yields
\[
\mathbb{E}^A_\delta(T b^{(a)}_B, d^{(a)}_B) = \frac{L}{N} \sum_{k \in A} \langle T e_k, f_k \rangle - \frac{L}{N(N - 1)} \sum_{k \neq \ell \in A} \langle T e_k, f_\ell \rangle = \frac{L}{N} B_1 - \frac{L}{N(N - 1)} B_2. \tag{3.12}
\]
By definition of \( \delta \) and \( A \), we obtain
\[
B_1 = \sum_{k \in A} \langle T e_k, f_k \rangle \geq \delta N. \tag{3.13}
\]
Now we will estimate \( B_2 \) in two different ways, each exploiting the linearity of \( \langle \cdot , \cdot \rangle \) in each component. Exploiting the linearity in the second component of the bilinear form and using (B3) yields
\[
B_2 = \sum_{k \neq \ell \in A} \langle T e_k, f_\ell \rangle = \sum_{k \in A} \Bigl( \sum_{\ell \in A\setminus\{k\}} \langle T e_k, f_\ell \rangle \Bigr) \leq C_d \|T\| \sum_{k \in A} \left\| \sum_{\ell \in A\setminus\{k\}} f_\ell \right\| \\
\leq C_d \|T\| N \max_{k \in A\setminus\{k\}} \left\| \sum_{\ell \in A\setminus\{k\}} f_\ell \right\| = C_d \|T\| N \max_{k \in A\setminus\{k\}} \left\| \sum_{\ell \in A\setminus\{k\}} f_\ell \right\|_Y.
\]
Using the exact same steps as above but exploiting the linearity in the first component of the bilinear form yields
\[
B_2 \leq C_d \|T\| N \max_{k \in A\setminus\{k\}} \left\| \sum_{\ell \in A\setminus\{k\}} e_k \right\|_X.
\]
Thus, we have the estimate
\[
B_2 \leq C_d \|T\| N \nu(N). \tag{3.14}
\]
Combining (3.12), (3.13) and (3.14) concludes the proof. \( \square \)

For \( n \in \mathbb{N} \) we define the functions \( \lambda, \mu : \mathbb{N} \to [0, \infty) \) by
\[
\lambda(n) = \left\| \sum_{j=1}^{n} e_j \right\|_X \quad \text{and} \quad \mu(n) = \left\| \sum_{j=1}^{n} f_j \right\|_Y. \tag{3.15}
\]
If \( (e_j) \) is \( C_s \)-spreading, then by (3.6) and (1.2), we obtain
\[
\nu(n) \leq C_s \min(\lambda(n - 1), \mu(n - 1)), \quad n \in \mathbb{N}. \tag{3.16}
\]
We note that keeping track of the constants in the proof of Proposition 3.3.6 and Proposition 3.3.4 in [23] yields
\[
\lambda(n) \mu(n) \leq 2 C_u C_s n \tag{3.17}
\]
We are now ready to prove Proposition 3.3 by specializing Lemma 3.2 to the case where \( (e_j) \) is subsymmetric.
Proposition 3.3. Let the dual pair \((X, Y, \langle \cdot, \cdot \rangle)\) together with the sequences \((e_j), (f_j)\) satisfy (B1)-(B5) and assume that \((e_j)\) is \((C_u, C_s)\)-subsymmetric. Let \(T: X \to X\) denote a bounded linear operator such that \(\delta := \inf_j \langle Te_j, f_j \rangle > 0\). For \(L \in 2 \cdot \mathbb{N}, 0 < \kappa < 1\) we define
\[
N = \max\{L, 1 + \frac{2C_u^2C_s^2\|T\|^2}{\kappa^2\delta^2}\}. \tag{3.18}
\]
Then for each \(A \subset \mathbb{N}\) with \(|A| = N\) we have
\[
E^A_L\langle T_b^{(e)}, d^{(e)}_B \rangle \geq (1 - \kappa)\delta L. \tag{3.19}
\]
In particular, there exists a set \(B \subset A\) with \(|B| = L\) and a choice of signs \((\varepsilon_k) \in \mathcal{E}(B)\) such that
\[
\langle T \sum_{k \in B} \varepsilon_k e_k, \sum_{k \in B} \varepsilon_k f_k \rangle \geq (1 - \kappa)\delta L. \tag{3.20}
\]

Proof. Lemma 3.2 yields
\[
E^A_L\langle T_b^{(e)}, d^{(e)}_B \rangle \geq \left[\delta - C_d \frac{\|T\|}{N - 1} \nu(N)\right] \cdot L.
\]
Using (3.16) and (3.17) gives us
\[
\nu(N) \leq C_s \min(\lambda(N - 1), \mu(N - 1)) \leq C_s \sqrt{\lambda(N - 1)\mu(N - 1)} \leq \sqrt{2C_uC_s^2(N - 1)}.
\]
Thus far, we proved
\[
E^A_L\langle T_b^{(e)}, d^{(e)}_B \rangle \geq \left[\delta - C_d \frac{\|T\|}{\sqrt{N - 1}} \cdot \sqrt{2C_uC_s} \right] \cdot L.
\]
The latter inequality together with (3.18) implies (3.19). (3.20) directly follows from (3.19) and definition of \(E^A_L\). \(\Box\)

We will now restate the result \[17, Lemma 3.2\] (which concerns weak* Schauder bases) for our dual system.

Lemma 3.4. Let the dual pair \((X, Y, \langle \cdot, \cdot \rangle)\) together with the sequences \((e_j), (f_j)\) satisfy (B1)-(B5) and assume that \((e_j)_{j=1}^\infty\) is an unconditional non-\(\ell^1\)-splicing sequence. Let \(I \subset \mathbb{N}\) denote an infinite set, \(\eta > 0\) and \(y_1, \ldots, y_n \in Y\) and let \(T: X \to X\) denote a bounded linear operator. Then there exists an infinite set \(\Lambda \subset I\) such that
\[
\sup_{\|x\| \leq 1} |\langle T\Lambda x, y_j \rangle| \leq \eta, \quad 1 \leq j \leq n.
\]

Proof. The same proof given for \[17, Lemma 3.2\] applies to our case for \(n = 1\) and is therefore omitted. Thus we may assume that Lemma 3.4 has already been established for \(n = 1\).

We will now inductively apply the Lemma. In the first step, we apply the Lemma with \(n = 1\) to \(y_1\) and obtain an infinite set \(\Lambda \subset I\) such that \(\sup_{\|x\| \leq 1} |\langle T\Lambda x, y_1 \rangle| \leq \eta/C_u\). Next, we apply the Lemma with \(n = 1\) to \(\Lambda = \Lambda_1\) and \(y_2\) and obtain an infinite set \(\Lambda_2 \subset \Lambda_1\) such that \(\sup_{\|x\| \leq 1} |\langle T\Lambda_2 x, y_2 \rangle| \leq \eta/C_u\). Continuing in this manner and stopping after \(n\) steps yields infinite sets \(\Lambda_1 \supset \Lambda_2 \supset \ldots \supset \Lambda_n\) such that \(\sup_{\|x\| \leq 1} |\langle T\Lambda_n x, y_j \rangle| \leq \eta/C_u, 1 \leq j \leq n\). Defining \(\Lambda = \Lambda_n\) and observing that for \(x \in X\) and \(1 \leq j \leq n\), unconditionality yields that
\[
|\langle T\Lambda x, y_j \rangle| = |\langle T\Lambda P\Lambda x, y_j \rangle| \leq \eta \|P\Lambda x\|/C_u \leq \eta \|x\|
\]
as claimed. \(\Box\)

The following proposition estimates our basic factorization operators \(B, Q\) acting on a subsymmetric sequence \((e_j)\).
Proposition 3.5. Let the dual pair $(X,Y,(\cdot,\cdot))$ together with the sequences $(e_j)$, $(f_j)$ satisfy (B1)-(B5) and assume that $(e_j)_{j=1}^\infty$ is a $(C_u,C_s)$-subsymmetric sequence. Let $L \in \mathbb{N}$ and let $(B_j)$ denote a sequence of sets $B_j \subset \mathbb{N}$ with $|B_j| = L$, $B_j < B_{j+1}$, $j \in \mathbb{N}$, and let $(\varepsilon_k) \in \{\pm 1\}^\mathbb{N}$. Define
\[
b_j = \sum_{k \in B_j} \varepsilon_k e_k \quad \text{and} \quad d_j = \sum_{k \in B_j} \varepsilon_k f_k, \tag{3.21}\]
as well as the operators $B,Q: X \to X$ by
\[
Bx = \sum_{j=1}^\infty \langle x, f_j \rangle b_j \quad \text{and} \quad Qx = \sum_{j=1}^\infty \langle x, d_j \rangle e_j, \quad x \in X. \tag{3.22}\]
The operators are well defined, $QB = L \cdot I_X$ and we have the estimates
\[
\|B\|, \|Q\| \leq C_u C_s \cdot L. \tag{3.23}\]

Proof. The assertions follow by essentially repeating Step 2 in the proof given for [17, Theorem 1.1]. Since the proof is short we include it here.

First we pick any $(n_j^l)$, such that $\{n_j^l : 1 \leq l \leq L\} = B_j$, $j \in \mathbb{N}$ and note that the sequence $(n_j^l)_{j}$ is increasing for each $1 \leq l \leq L$. Now, let $x = \sum_{j=1}^\infty a_j e_j$ be a series that converges in the $\sigma(X,Y)$-topology. Hence, by (1.1) and (1.2) $Bx = \sum_{l=1}^L \sum_{j=1}^\infty \varepsilon_n a_j e_{n_j^l}$ is well defined and satisfies (3.23) as claimed. Similarly, we see that since $Qx = \sum_{l=1}^L \sum_{j=1}^\infty \varepsilon_n a_j e_j$, the operator $Q$ is well defined and satisfies (3.23). The identity $QB = L \cdot I_X$ promptly follows from (B4). \qed

4. Banach spaces with a subsymmetric basis

In this section, we will prove our first two main results Theorem 2.1 and Theorem 2.2. Due to the big overlap of the arguments for Theorem 2.1 and Theorem 2.2, we will show them both simultaneously by bifurcating the proof in three places. This bifurcation is visualized by two color coded columns that will appear side by side. The proofs for Theorem 2.1 and Theorem 2.2 are obtained by reading the text on the white and light teal background, respectively the text on the white and light magenta background.

Proof of Theorem 2.1 and Theorem 2.2. In the case of Theorem 2.1, we put $f_j = e_j^\ast$, $1 \leq j \leq n$, where $(e_j^\ast)$ denotes the sequence of biorthogonal functionals to $(e_j)$.

Let $T: X \to X$ be a bounded linear operator which has $\delta$-large diagonal with respect to $(e_j)$. Given $\eta > 0$, we want to show that $I_X - (\frac{2C_u C_s}{3} \delta + \eta)$-factors through $T$. First, we define the multiplier $M: X \to X$ by $M e_j = \text{sign}((Te_j, f_j)) e_j$ and note that by (1.1) $M$ is a well defined operator satisfying $\|M\| \leq C_u$. Thus, if we define $\tilde{T} = TM$, then
\[
\inf_j \langle \tilde{T} e_j, f_j \rangle = \inf_j |\langle Te_j, f_j \rangle| = \delta > 0. \tag{4.1}\]

Our strategy is to use Lemma 3.1 (and additionally Lemma 3.4 for the proof of Theorem 2.2) to diagonalize the operator with a block basis $(b_j)$ of $(e_j)$, and Proposition 3.3 to transfer the large diagonal that $\tilde{T}$ has with respect to $(e_j)$ over to the block basis $(b_j)$.

Throughout this proof, we will use the following constants:
\[
\kappa = \frac{1}{2 + \frac{4C_u C_s}{9 \delta}}, \quad \eta_i = \frac{2^{-i-1}}{C_d^4}, \quad i \in \mathbb{N}, \quad L = 2, \quad N = \max\left(2, 1 + \left[\frac{2C_u C_s^3 \|T\|^2}{\kappa^4 \delta^2}\right]\right). \tag{4.2}\]

We note that if $f_j = e_j^\ast$, $j \in \mathbb{N}$, then $C_d = 1$ (see (B3)).

Step 1: Construction of the block basis. In each step of the subsequent construction, we will employ Lemma 3.1, Lemma 3.4 and Proposition 3.3 to $\tilde{T}$ and with the constants $\kappa$, $L$ and
$N$ as defined in (4.2). Other parameters will be explicitly specified at the appropriate place in the proof.

For our initial step, we use Proposition 3.3 with $\mathcal{A} = A_1 = \{1, \ldots, N\}$ to obtain a set $B_1 \subset A_1$ with $|B_1| = 2$ and signs $(\varepsilon_k) \in \mathcal{E}(B_1)$ such that
\[
\langle \tilde{T} b_1, d_1 \rangle \geq 2(1 - \kappa)\delta,
\]
where we defined
\[
b_1 = \sum_{k \in B_1} \varepsilon_k e_k \quad \text{and} \quad d_1 = \sum_{k \in B_1} \varepsilon_k f_k.
\]
This completes the initial step of our construction.

Assume that we have already chosen pairwise disjoint sets $B_1 < B_2 < \cdots < B_{i-1}$ with $|B_j| = 2$, $1 \leq j < i - 1$, selected signs $(\varepsilon_k) \in \mathcal{E}(B_j)$, $1 \leq j \leq i - 1$ and that we have defined
\[
b_j = \sum_{k \in B_j} \varepsilon_k e_k \quad \text{and} \quad d_j = \sum_{k \in B_j} \varepsilon_k f_k, \quad 1 \leq j \leq i - 1.
\]

We will now construct $b_i$ and $d_i$.

**Schauder basis (Theorem 2.1).**

In this case, we first use Lemma 3.1 with
\[
\mathcal{I} = \{k \in \mathbb{N} : k > \max B_{i-1}\},
\]
\[
\eta = \eta_i, \quad x_j = \tilde{T} e_j, \quad y_j = d_j
\]
for all $1 \leq j \leq i - 1$ to obtain an infinite set $\Lambda^0_0 \subset \mathcal{I}$ such that
\[
\left| \langle \tilde{T} \sum_{k \in B} \varepsilon_k e_k, d_j \rangle \right| \leq \eta_i
\]
for all $B \subset \Lambda^0_0$ with $|B| = 2$, all $(\varepsilon_k) \in \mathcal{E}(B)$ and $1 \leq j \leq i - 1$.

**Non-$\ell^1$-splicing (Theorem 2.2).**

Here, we will apply Lemma 3.4 with
\[
\mathcal{I} = \{k \in \Lambda^1_{i-1} : k > \max B_{i-1}\},
\]
\[
\eta = \eta_i/C_u, \quad y_j = d_j, \quad 1 \leq j \leq i - 1
\]
to obtain an infinite set $\Lambda^0_i \subset \mathcal{I}$ such that
\[
\sup_{\|x\| \leq 1} |\langle \tilde{T} P_{\Lambda^0_i} x, d_j \rangle| \leq \eta_i/C_u
\]
for all $1 \leq j \leq i - 1$.

In both cases, using Lemma 3.1 with $\mathcal{I} = \Lambda^0_i$, $\eta = \eta_i$, $x_j = T b_j$, $y_j = f_j$, $1 \leq j \leq i - 1$ yields an infinite set $\Lambda^1_i \subset \Lambda^0_i$ such that
\[
\sup \left\{ \left| \langle \tilde{T} b_j, \sum_{k \in B} \varepsilon_k f_k \rangle \right| : B \subset \Lambda^1_i, \ |B| = 2, \ \varepsilon \in \mathcal{E}(B), \ 1 \leq j \leq i - 1 \right\} \leq \eta_i.
\]

Now we select any $A_i \subset A^1_i$ with $|A_i| = N$. Applying Proposition 3.3 to $\mathcal{A} = A_i$ gives us a set $B_i \subset A_i$ with $|B_i| = 2$ and $(\varepsilon_k) \in \mathcal{E}(B_i)$ such that
\[
\langle \tilde{T} b_i, d_i \rangle \geq 2(1 - \kappa)\delta,
\]
where we put
\[
b_i = \sum_{k \in B_i} \varepsilon_k e_k \quad \text{and} \quad d_i = \sum_{k \in B_i} \varepsilon_k f_k.
\]

This concludes the inductive construction of our block basis. To summarize, we proved the following estimates. In both cases, we obtain from (4.8) and (4.9)
\[
|\langle \tilde{T} b_j, d_j \rangle| \leq \eta_i \quad \text{and} \quad |\langle \tilde{T} b_j, d_i \rangle| \geq 2(1 - \kappa)\delta, \quad i, j \in \mathbb{N}, \ i \leq j - 1.
\]

Using different methods, we obtained the following estimates in each case.
Step 2: Factorization. This step of the proof is similar to Step 3 in the proof of [17, Theorem 1.1]. The most notable differences are the following: The operators $B$ and $Q$ have been modified, the additional parameter $\delta$ is now involved, the estimate for $P T z - z$ is different in the proof of Theorem 2.1. For those reasons, we will now give a concise version of this argument.

Let $B, Q$ be defined as in (4.22) and put $Z = B(X)$. Utilizing Proposition 3.5 yields

$$2 \cdot I_X = QB \quad \text{and} \quad \|B\|, \|Q\| \leq 2C_{\alpha}C_s. \quad (4.14)$$

Next, we define $P: X \to Z$ by

$$Px = \sum_{j=1}^{\infty} \frac{\langle x, d_j \rangle}{\langle T b_j, d_j \rangle} b_j, \quad x \in X, \quad (4.15)$$

and observe that by the large diagonal of $\tilde{T}$ (see (4.1)), the unconditionality of $(e_j)$ (see (1.1)) and Proposition 3.5 we obtain

$$\|P x\|_X \leq \frac{C_{\alpha}^2 C_s}{(1 - \alpha)^3} \|x\|_X, \quad x \in X. \quad (4.16)$$

For each of the two cases, let $z = \sum_{j=1}^{\infty} a_j b_j \in Z$.

Note that $2|a_j| = |\langle z, b_j \rangle| \leq 2C_{\alpha}\|z\|_X$. This estimate together with (4.6), (4.11) and (4.2) yields (apply the triangle inequality to (4.17))

$$\|P T z - z\|_X \leq \frac{\kappa}{(1 - \alpha)} \|z\|_X, \quad z \in Z. \quad (4.19)$$
The same result can be obtained by applying the triangle inequality to (4.18) and using (4.7), (4.11) and (4.2).

Define $J : Z \to X$ by $Jz = z$ and note that by (4.19) $P^\perp J : Z \to Z$ is invertible. Hence, if we define $V = (P^\perp J)^{-1}P$, then by (4.16) and (4.19) we obtain

$$I_Z = V \tilde{T}J \quad \text{and} \quad ||V|| \leq \frac{C_2^2 C_4^3}{(1 - 2\kappa)\delta}.$$  

(4.20)

Combining (4.14) with (4.20) and recalling that $\tilde{T} = TM$ yields

$$I_X = QIZB/2 = QV \tilde{T}JB/2 = QVTMB/2 = ETF,$$  

(4.21)

where we put $E = QV$ and $F = MJB/2$. By (4.14), (4.20), (4.2) and since $||M|| \leq C_u$, we obtain

$$||E|| \cdot ||F|| \leq \frac{2C_2^5 C_4^3}{(1 - 2\kappa)\delta} \leq \frac{2C_2^5 C_4^3}{\delta} + \eta,$$

which together with (4.21) proves both theorems. \hfill \Box

**Remark 4.1.** It is conceivable to prove Theorem 2.1 by adapting the gliding-hump techniques in [10, 9]. However, these techniques do not appear to work in the non-separable case: The exact place in the proof where the seemingly insurmountable problem occurs is illustrated by comparing (4.17) with (4.18). In order to get from the formula in (4.18) to that of (4.17), we need the following identity to be true:

$$\langle \tilde{T} \sum_{j=i+1}^{\infty} a_j b_j, d_i \rangle = \sum_{j=i+1}^{\infty} a_j \langle \tilde{T}b_j, d_i \rangle.$$  

(4.22)

In the non-separable case $\sum_{j=i+1}^{\infty} a_j b_j$ does not need to converge in the norm-topology, therefore we cannot use the norm-continuity of the operator $\tilde{T}$ to swap the operator with the sum.

To compensate we work with the lefthand side of (4.22) directly by exploiting that $(e_j)$ is non-$\ell^1$-splicing (see [17] for a prior use of this approach), whereby we preselect a large subspace whose image under $T$ annihilates all previously constructed block basis elements.

5. **DIRECT SUMS OF BANACH SPACES WITH A SUBSYMMETRIC BASIS**

Here, we will provide a proof for Theorem 2.3. The relationship between the proof of [17, Theorem 1.1] and the proof of [17, Theorem 1.2] is very similar to the relationship between the proof of Theorem 2.2 and that of Theorem 2.3: when the array $(e_{n,j})_{n,j}$ is linearized correctly, we can essentially use the same construction as for the one-parameter basis $(e_j)$. Instead of repeating large portions of a lengthy proof, we will describe the necessary adaptations.

**Proof of Theorem 2.3.** First, we need two-parameter versions of Lemma 3.1, Lemma 3.4 and Proposition 3.3. For the former two, we refer to [17, Lemma 5.1, Lemma 5.2]. The two-parameter version of Proposition 3.3 is just the coordinate-wise application of Proposition 3.3 (i.e. replacing $(e_j)$ and $(f_j)$ with $(e_{n,j})$ and $(f_{n,j})$ for fixed $n$).

Heavily exploiting that $(e_j)$ is non-$\ell^1$-splicing, it was possible to construct the block basis $(b_j)$ in the proof of [17, Theorem 1.2] with the same two basic steps as in the proof of [17, Theorem 1.1]. The $i$th step of the construction reads as follows:

1. **(P1)** annihilating the previously constructed vectors $Tb_j$, $1 \leq j \leq i - 1$ by choosing $b_i$ in $\{|e_k : k \in A_i|\}$;
2. **(P2)** annihilating vectors that will be constructed in later steps by preselecting infinitely many suitable coordinates $A_{i+1}$. 

\[ \text{Proposition 3.3.} \]
For the proof of Theorem 2.3, we can use the same scheme as described above, i.e. we replace the two basic construction steps (P1) and (P2) by the following three steps described in the $i^{th}$ inductive step in the proof of Theorem 2.2:

(F1) using Lemma 3.4 to preselect infinitely many coordinates $\Lambda^0_i$ so that vectors in $TP\lambda^0_i(X)$ annihilate the previously constructed $d_j$, $1 \leq j \leq i - 1$;

(F2) preselecting another infinite set $\Lambda^1_i \subset \Lambda^0_i$ by utilizing Lemma 3.1, so that we can choose among many disjointly supported candidates in $[e_k : k \in \Lambda^1_i]$ for the current block basis element $d_i$, all of which annihilate previously constructed vectors $\tilde{T}d_j$, $1 \leq j \leq i - 1$;

(F3) using Proposition 3.3 to find among those candidates block basis elements $b_i$ and $d_i$ so that the diagonal is kept large, i.e. $\langle \tilde{T}b_i, d_i \rangle \geq 2(1 - \kappa)\delta$.

Finally, we note that Step 3 in the proof of [17, Theorem 1.2] reduces to the following case: we have $I = N$, $J_i = K_i = N$ and $H = \tilde{T}$. Also, that reduced case is essentially Step 2 in the proof of Theorem 2.2. Rerunning those arguments with the described modifications concludes the proof. \(\square\)

We will now show how to obtain [17, Theorem 1.2] from Theorem 2.3 (see Corollary 5.1, below).

**Corollary 5.1 ([17, Theorem 1.2]).** Let the dual pair $(X, Y, \langle \cdot, \cdot \rangle)$ of infinite dimensional Banach spaces $X$ and $Y$ and the sequences $(e_j)$, $(f_j)$ satisfy $(B1)$–$(B5)$ with constant $C_q$. Assume that $(e_j)$ is subsymmetric and non-$\ell^1$-splicing. Then the Banach spaces $\ell^p(X)$, $1 \leq p \leq \infty$ are primary.

**Proof.** Let $Q: \ell^p(X) \to \ell^p(X)$ be a bounded projection, and let $(e_{n,j})_j$ and $(f_{n,j})_j$ denote a copies of $(e_j)$ and $(f_j)$, the $n^{th}$ coordinate of the direct sum. Note that at least one of the sets

$$\{ n \in \mathbb{N} : \text{the set } \{ j \in \mathbb{N} : |\langle Qe_{n,j}, f_{n,j} \rangle| \geq 1/2 \} \text{ is infinite} \},$$

$$\{ n \in \mathbb{N} : \text{the set } \{ j \in \mathbb{N} : |\langle (I_{\ell^p(X)} - Q)e_{n,j}, f_{n,j} \rangle| \geq 1/2 \} \text{ is infinite} \},$$

is infinite. If the first set is infinite we put $H = Q$, and we define $H = I_{\ell^p(X)} - Q$ otherwise. In any case, there exist infinite sets $\mathcal{I} \subset \mathbb{N}$ and $\mathcal{J} \subset \mathbb{N}$, $n \in \mathcal{I}$ so that $H$ has large diagonal with respect to $(e_{n,j})_n \in \mathcal{I}$, $j \in \mathcal{J}$). We define $Z_n = \text{span}\{ e_{n,j} : j \in \mathcal{J}_n \}$, $n \in \mathcal{I}$ and put $Z = \ell^p(\{ Z_n : n \in \mathcal{I} \})$. Let $P: \ell^p(X) \to Z$ denote the natural projection onto $Z$, i.e. $P(e_{n,j}) = e_{n,j}$ if $n \in \mathcal{I}$ and $j \in \mathcal{J}_n$ and $P(e_{n,j}) = 0$ otherwise. We record that by the unconditionality of $(e_{n,j})$, the projection $P$ is bounded. We note that by the subsymmetry of $(e_{n,j})_n$, we have that $Z_n$ is isomorphic to $X$; hence, $Z$ is isomorphic to $\ell^p(X)$ and we denote the isomorphism by $R: \ell^p(X) \to Z$. By Theorem 2.3, $(e_{n,j})_n \in \mathcal{I}$, $j \in \mathcal{J}_n$ has the factorization property; in particular, $I_Z$ factors through the operator $PH: Z \to Z$, i.e. there exist bounded operators $A, B: Z \to Z$ such that $I_Z = BP\text{H}A$. Consequently, $I_{\ell^p(X)} = R^{-1}BP\text{H}A$, that is $I_{\ell^p(X)}$ factors through $H$. Using Pelczyński’s decomposition method (see e.g. Step 4 in the proof of [17, Theorem 1.2]) concludes the proof. \(\square\)

6. **Finite dimensional quantitative factorization results**

We first prove Theorem 2.4 and then conclude this work with the proof of Corollary 2.6.

In this section, $(e_j)$ denotes a normalized basis for the Banach space $X$ and $(e'_j)$ denotes the biorthogonal functionals to $(e_j)$. Recall that in (2.1), we defined the function $\tau: \mathbb{N} \to [0, \infty)$ by

$$\tau(n) = \max \left\{ \min \left\{ \max_{1 \leq j \leq n} \left( \sum_{i=1}^{n} |\epsilon_{ij}| e_i \right)_X, \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} |\epsilon_{ij}| e'_j \right)_X \right\} : \epsilon_{ij} \in \{ \pm 1 \}, 1 \leq i, j \leq n \right\}.$$
Mainly, the proof of Theorem 2.4 follows the method used by Bourgain and Tzafriri for [5, Theorem 6.1], but instead of exploiting the lower r-estimate which allows them to extract a large submatrix with small entries, we use our upper estimate for \( \tau(n) \).

**Proof of Theorem 2.4.** We begin by defining the constants

$$
\kappa_1 = \min\{1, \eta\}/4, \quad \kappa_2 = \min\{(1 + \eta)^{-1}\}/4, \quad \alpha = \sqrt{\frac{2\kappa_2}{\Gamma(n)}} \cdot \frac{1}{\sqrt{n\tau(n)}},
$$

and note that \( \alpha \leq 1 \) by the lower estimate in (2.3). Now let \( (\xi_i)_{i=1}^n \) denote independent random variables over the probability space \((\Omega, \Sigma, \mathbb{P})\) taking values in \(\{0, 1\}\) with \(\mathbb{E}\xi_i = \alpha\), where \(\mathbb{E}\) denotes the conditional expectation. For each \(\omega \in \Omega\) we define the operators

$$
A(\omega) = R(\omega)D^{-1}TR(\omega) \quad \text{and} \quad R(\omega)e_i = \xi_i(\omega)e_i, \ 1 \leq i \leq n,
$$

and note that

$$
A\left(\sum_{i=1}^n a_ie_i\right) = \sum_{i,j=1}^{n} a_i\xi_j\frac{\langle Te_i, e_j^*\rangle}{\langle Te_j, e_j^*\rangle}e_j.
$$

Let \( x = \sum_{i=1}^n a_ie_i \) and observe that by (6.3)

$$
\| (A - R)x \| \leq \sum_{i \neq j} |a_i|\xi_j\frac{|\langle Te_i, e_j^*\rangle|}{|\langle Te_j, e_j^*\rangle|} \leq \| x \| \delta \sum_{i \neq j} \xi_j|\langle Te_i, e_j^*\rangle|.
$$

Taking the supremum over all \( \| x \| \leq 1 \) and taking the expectation yields

$$
\mathbb{E}\| A - R \| \leq \frac{1}{\delta} \sum_{i \neq j} \mathbb{E}\xi_j|\langle Te_i, e_j^*\rangle| = \frac{\alpha^2}{\delta} \sum_{i \neq j} |\langle Te_i, e_j^*\rangle|.
$$

Defining \( \varepsilon_{ij} = \text{sign}(\langle Te_i, e_j^*\rangle) \) we obtain from (6.4)

$$
\mathbb{E}\| A - R \| \leq \frac{\alpha^2}{\delta} \sum_i \langle Te_i, \sum_{j \neq i} \varepsilon_{ij}e_j^* \rangle \leq \alpha^2 \frac{\Gamma }{\delta} n \max_i \| \sum_{j \neq i} \varepsilon_{ij}e_j^* \|.
$$

Similarly, we obtain

$$
\mathbb{E}\| A - R \| \leq \alpha^2 \frac{\Gamma }{\delta} n \max_j \| \sum_{i \neq j} \varepsilon_{ij}e_i \|.
$$

Combing (6.5), (6.6), (2.1) and (6.1) yields

$$
\mathbb{E}\| A - R \| \leq \alpha^2 \frac{\Gamma }{\delta} n\tau(n) = \kappa_1.
$$

Now define

$$
\Omega' = \{ \omega \in \Omega : |\sum_{i=1}^n \xi_i(\omega) - \alpha n| \leq \alpha n/2 \}
$$

and observe that by (6.1) and (2.3)

$$
\mathbb{P}(\Omega^c) = \mathbb{P}\left( \left\{ \omega \in \Omega : \left| \sum_{i=1}^n \xi_i - \alpha n \right| > \alpha n/2 \right\} \right) \leq \frac{4}{\alpha^2 n^2} \mathbb{E}\left( \sum_{i=1}^n \xi_i - \alpha n \right)^2 = \frac{4(1 - \alpha)}{\alpha n} \leq \kappa_2.
$$

Thus, combining this measure estimate with (6.7), we can find an \( \omega_0 \in \Omega' \) such that

$$
\| A(\omega_0) - R(\omega_0) \| \leq \kappa_1/(1 - \kappa_2).
$$

Since \( \kappa_1/(1 - \kappa_2) < 1 \), \( A(\omega_0) \) is invertible, hence, we obtain by (6.1)

$$
\| A^{-1}(\omega_0) \| \leq 1 - \kappa_2 \leq 1 + \eta.
$$
We put \( \sigma = \{1 \leq i \leq n : \xi_i(\omega_0) = 1\} \) note that since \( \omega_0 \in \Omega' \) we have \( |\sigma| \geq \alpha n/2 \); appealing to (6.1) shows (2.5). Since \( R(\omega_0) = R_\sigma \), (2.6) follows from (6.9) and the definition of \( A \) (see (6.2)).

We conclude the proof by defining \( E : X_\sigma \rightarrow X_n \) by \( Ex = x = R_\sigma x \) and \( P : X_n \rightarrow X_\sigma \) by \( P = (R_\sigma D^{-1} TR_\sigma)^{-1} R_\sigma D^{-1} \) and using that \( \|D^{-1}\| \leq C_n/\delta \) together with (2.6). \( \square \)

To conclude, we show that Theorem 2.4 implies \([17, \text{Theorem 1.2}], \) below.

**Proof of Corollary 2.6.** Recalling (3.6), we observe that if \((e_j)\) is \(C_u\)-unconditional, then
\[
\tau(n) \leq C_u \nu(n), \quad n \in \mathbb{N}. \quad (6.10)
\]
Moreover, if \((e_j)\) is \((C_u, C_\sigma)\)-subsymmetric, then by (6.10) and (3.16) we obtain
\[
\tau(n) \leq C_u C_\sigma \min(\lambda(n-1), \mu(n-1)), \quad n \in \mathbb{N}. \quad (6.11)
\]
Thus, by (3.17), we also have
\[
\tau(n) \leq \sqrt{2C_u^2 C_\sigma^2 (n-1)}, \quad n \in \mathbb{N}. \quad (6.12)
\]
Using (6.12), (2.7) and that \( \tau(n) \geq 1/C_u, \quad n \geq 2 \), we obtain (2.3); thus, Theorem 2.4 yields (2.9) and (2.10). Noting that \( X_k \) is \( C_\sigma\)-isomorphic to \( X_\sigma \), we obtain the estimate \( \|E\|\|P\| \leq C_u^2 C_\sigma^2 (1+\eta_\sigma) \).

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