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N=4 MULTIPLETS IN N=3 HARMONIC
SUPERSPACE

Abstract

It is shown that the N=3 harmonic-superfield equations of motion are invariant with respect to the 4-th supersymmetry. The SU(3) harmonics are also used to analyze a more flexible form of superfield constraints for the Abelian N=4 vector multiplet and its N=3 decomposition. An alternative unusual representation of the N=4 supersymmetry is realized on infinite multiplets of analytic superfields in the N=3 harmonic superspace. U(1) charges of superfields in these multiplets are parametrized by an integer-valued parameter which plays the role of the discrete coordinate. Each superfield term of the N=3 Yang-Mills action has the infinite-dimensional N=4 generalization. The gauge group of this model contains an infinite number of superfield parameters.

1 Introduction

The Yang-Mills theory in the harmonic D=4, N=3 superspace was constructed in Ref. [1] (see also [2, 3] and book [4]). In this approach, physical component fields of the N=3 vector multiplet are included into geometric superfields which have an infinite number of auxiliary fields off mass shell. The superfield geometry of the N=3 gauge theory is based on the preservation of the Grassmann analyticity, and superfield equations are zero-curvature conditions for basic superfields (harmonic potentials) of this theory. The off-shell structure of N=3 harmonic superfields plays an important role, in particular, for constructing the N=3 analogue of the Born-Infeld action [5].

The N=4 vector multiplet was considered in the framework of the SU(4)-harmonic superspace [5]-[10]. In this quite complicated harmonic formalism, one can construct a harmonic projection of superfield strengths satisfying simple equations of motion; however, nobody knows how to solve problems of constructing the corresponding superfield action. It is not excluded that these problems are connected with a manifest SU(4)-invariance and a high dimension of the Grassmann part of the N=4 superspace. We consider a more flexible N=3 superspace and analyze different realizations of the N=4 supersymmetry on N=3 superfields and equations of motion. In particular, the N=4 generalization of the N=3 gauge action for an infinite multiplet of superfields is proposed.

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It is well known that physical field-components of the $N=3$ and $N=4$ vector multiplets are completely identical; however, transformations of the $N=4$ supersymmetry for these component fields form a closed algebra on the equations of motion. The $N=4$ Yang-Mills theory can also be constructed in the $N=2$ harmonic superspace [1], but the algebra of additional supersymmetry transformations of this formulation is closed on-shell only.

Section 2 is devoted to the $N = 4$ supersymmetry realized on superfield solutions of the $N=3$ equations of motion. We construct additional spinor transformations of the $N=3$ superfield strengths and potentials on mass shell. It is shown that the Abelian and non-Abelian gauge equations in $N=3$ superspace are invariant with respect to the 4-th supersymmetry.

Superfield $N=4$ equations (constraints) in the standard superspace can be decomposed in terms of the 4-th spinor coordinate to study the $N = 4$ transformations of the $N=3$ gauge superfields and solutions of constraints with hidden SU(4) symmetry. In section 3, we analyze the SU(3) covariant representation of the $N=4$ superfield constraints for the Abelian gauge multiplet. One can try to use the SU(3)/U(1)×U(1) harmonics in order to understand the geometric meaning of these equations and their relations with the $N=3$ harmonic superspace. Decomposition of $N=4$ superfields in the 4-th spinor coordinate allows us to study $N=4$ transformations of the gauge $N=3$ superfields. It should be said that we do not find an appropriate action without Lagrange multipliers in this approach.

In Ref. [11], the $N=4$ supersymmetry with central charges was realized on an infinite set of chiral $N=2$ superfields which then could be nonlinearly expressed via the single chiral $N=2$ superfield describing the partial spontaneous breaking of the $N=4$ supersymmetry. In the SU(4) harmonic formalism [10], one can also define geometric $N=4$ superfields whose decompositions contain infinite multiplets of $N=3$ superfields of different dimensions. The $N=3$ harmonic superspace gives us a simpler possibility to realize transformations of the $N=4$ supersymmetry and the corresponding version of superfield geometry. In section 4, we study the $N=4$ transformations on an infinite set of $N=3$ superfields parametrized by an integer-valued parameter $k$ which plays the role of the auxiliary discrete coordinate. The 4-th spinor transformation of this set contains the spinor derivatives which preserve the Grassmann analyticity. The multiplication rule for these infinite $N=4$ multiplets is used to construct an $N=4$ invariant generalization of the $N=3$ Yang-Mills action. We analyze also the gauge invariance of this action. The classical equations for additional $N=3$ superfields may have nontrivial solutions, so the consistency of the infinite-superfield model at the quantum level and its relation with the $N=3$ gauge theory require an additional analysis. A possibility of ‘truncated’ realizations of the $N=4$ supersymmetry on a finite number of $N=3$ superfields is also considered; however, superfield actions for these realizations are not constructed.

In the appendix, we give a review of basic formulae of the $N=3$ gauge theory in the harmonic superspace which are used in other sections of the paper.
2 N=4 transformations of N=3 superfield equations

Supersymmetry transformations of the components of the N=4 vector multiplet and their equations of motion are well known. In this approach, the algebra of all supersymmetry transformations is closed on the equations of motion. Formulations of the N=4 Yang-Mills theory in terms of N=1 or N=2 superfields correspond to a manifest realization of the corresponding supersymmetries.

We shall use the formalism of the N=3 gauge theory in the harmonic superspace, which is reviewed in the appendix. In the N=3 superspace, three supersymmetries are realized manifestly on the gauge superfield potentials \( V_1^1 \) and \( \bar{W}_3 \) (A.12), so we shall study a possible connection with the 4-th supersymmetry. Superfield solutions of free equations will be marked by symbol \( |_0 \). One can define the following spinor transformations on the free superfield strengths of the Abelian N=3 theory:

\[
\delta_\eta W^1|_0 = -\bar{\eta}^\alpha D^2_\alpha W^1|_0 , \quad \delta_\eta \bar{W}_3|_0 = \eta^\alpha D^2_\alpha W^1|_0 .
\]

The free equations (2.2) are invariant with respect to these spinor transformations

\[
(\hat{D}^1_2 , D^1_3)(W^1, \bar{W}_3) = 0 ,
\]

By definition, transformations (2.7) satisfy the following conditions characterizing the free superfields:

\[
(D^2)^2 \delta_\eta W^1|_0 = 0 , \quad (\hat{D}_2)^2 \delta_\eta \bar{W}_3|_0 = 0 .
\]

Additional spinor transformations together with the N=3 \( \epsilon \)-transformations form the standard N=4 superalgebra without central charges

\[
[\delta_{\eta_2} , \delta_{\eta_1}] W^1|_0 = 2i(\eta_1^\alpha \bar{\eta}_2^\beta - \eta_2^\alpha \bar{\eta}_1^\beta) \partial_{\alpha\beta} W^1|_0 , \quad [\delta_\epsilon , \delta_\eta] W^1|_0 = 0 .
\]

A spinor transformation of the potential corresponding to the transformation \( \delta_\eta \bar{W}_3 \) (2.1) has the following form:

\[
\delta_\eta V^1_2|_0 = 2\eta_\alpha \theta^2_\alpha W^1|_0 .
\]

The spinor transformation of the potential \( V^2_3|_0 \) can be obtained by conjugation

\[
\delta_\eta V^2_3|_0 = -2\eta_\alpha \theta^2_\alpha \bar{W}_3|_0 .
\]

In comparison with (2.6), the Lie bracket of the potential transformations contains gauge-dependent unusual terms. More general transformation of \( V^1_2 \) can contain additional terms which do not contribute to the transformation of \( \bar{W}_3|_0 \) (2.1), but change the algebra of transformations on \( V^1_2|_0 \).
Spinor transformations of the non-Abelian potentials can be chosen in the following form:

\[
\begin{align*}
\hat{\delta}_\eta V_2^1| &= 2\eta_\alpha \theta_2^\alpha W^1| (V_3^2) , \\
\hat{\delta}_\eta V_3^1| &= 2\eta_\alpha \theta_3^\alpha W^1| (V_3^2) - 2\bar{\eta}_i \bar{\theta}^{i\dot{\alpha}} \bar{W}_3| (V_2^1) .
\end{align*}
\] (2.8)

The non-Abelian equations of motion are invariant with respect to these transformations

\[
\hat{\delta}_\eta [\nabla_3^1, \nabla_3^2] = 2\eta_\alpha \theta_2^\alpha \nabla_3^1 W^1| - 2\eta_\alpha \theta_3^\alpha \nabla_2^1 W^1| + 2\bar{\eta}_i \bar{\theta}^{i\dot{\alpha}} \nabla_2^1 \bar{W}_3| = 0 .
\] (2.9)

In this calculation, we have used conditions of the covariant harmonic analyticity of the superfield strengths (A.18).

It is convenient to study nonlinear transformations (2.8) using gauge conditions (A.21) and nonlinear expansions of superfields \(W^1(V_3^2)\) and \(\bar{W}_3(V_2^1)\) (A.23) in terms of non-Abelian potentials. In particular, these formulae allow us to construct spinor transformations of non-Abelian superfield strengths via transformations of the potentials (2.8).

## 3 SU(3) decomposition of N=4 superfield constraints

SU(4) invariant harmonic-superspace formulations of the equations of motion for the \(D=4, N=4\) vector multiplet were analyzed in Refs. [5]-[10]. It is not difficult to embed the physical fields of this multiplet into infinite-dimensional \(N=4\) multiplets in the harmonic superspace; however, it is not clear how to build the superfield action in this formalism.

Let us suppose that the \(N=3\) theory describes some phase of the \(N=4\) theory which has no manifest SU(4) symmetry. In order to study relations of the \(N=4\) and \(N=3\) theories, we consider the SU(3) decomposition of a free \(N=4\) superfield strength \(W^{AB}\)

\[
W^{AB} = (\bar{W}^{ik}, W^{i4}) , \quad W_{AB} = (W_{ik}, \bar{W}_{i4})
\] (3.1)

satisfying the following version of the \(N=4\) reality condition:

\[
\begin{align*}
\bar{W}^{ik} &= \varepsilon^{ikl} \bar{W}_{l4} , \\
W_{ik} &= \varepsilon_{ikl} W^{l4}
\end{align*}
\] (3.2)

where \(A = (i, 4), i, k, l = 1, 2, 3\) are the indices of SU(4) or SU(3) groups, respectively. In the formalism with hidden SU(4) symmetry, one can treat \(W^{i4}\) and \(\bar{W}_{i4}\) as independent \(N=4\) superfields. The superfield equations of the \(N=4\) Abelian vector multiplet have the following form in this setting:

\[
\begin{align*}
D^i_\alpha W^{j4} + D^j_\alpha W^{i4} &= 0 , \\
D^4_\alpha W^{i4} &= 0 ,
\end{align*}
\] (3.3)

\[
\begin{align*}
\bar{D}_{i\dot{a}} W^{j4} - \frac{1}{3} \gamma^j_{\dot{a}\dot{b}} \bar{D}_{k\dot{b}} W^{k4} &= 0 , \\
2 \bar{D}_{i\dot{a}} W^{i4} &= \varepsilon^{ikl} \bar{D}_{k\dot{a}} \bar{W}_{l4} .
\end{align*}
\] (3.4)

\[
2 \bar{D}_{i\dot{a}} W^{i4} = \varepsilon^{ikl} \bar{D}_{k\dot{a}} \bar{W}_{l4} .
\] (3.5)
One can try to study these constraints using the SU(3)/U(1)×U(1) harmonics and the following notation for conjugated harmonic projections of the N=4 superfields:

\[ \mathcal{W}^1 \equiv u_k^1 W^{k4}, \quad \bar{\mathcal{W}}_3 = u_3^5 \bar{W}_{k4}. \]  

SU(3)-harmonic projections of the N=4 constraints are equivalent to the relations

\[ \begin{align*}
(D_{\alpha}^{1}, D_{\alpha}^{4}, \bar{D}_{2\dot{\alpha}}, \bar{D}_{3\dot{\alpha}}) \mathcal{W}^1 & = 0, \quad D_{3}^{3} \mathcal{W}^1 = 0 \\
(D_{\dot{\alpha}}^{1}, D_{4\dot{\alpha}}, \bar{D}_{3\dot{\alpha}}) \bar{\mathcal{W}}_3 & = 0, \quad D_{2}^{2} \bar{\mathcal{W}}_3 = 0
\end{align*} \]  

(3.7)

Equations (3.7) can be formally solved off-shell via some N=4 superfields (potentials) by analogy with the solutions of the N=3 gauge constraints (A.17); however, we do not know a geometric interpretation of these N=4 potentials. Relations (3.8) give directly the equations of motion

\[ \begin{align*}
(\bar{D}_{4})^{2} \mathcal{W}^1 & = 0, \quad (\bar{D}_{2})^{2} \bar{\mathcal{W}}_3 = 0.
\end{align*} \]  

(3.10)

Note that similar equations

\[ \left[(D_{3}^{3})^{2}, (D_{1}^{1})^{2}, (D_{2}^{2})^{2}\right] \bar{\mathcal{W}}_3 = 0 \]  

(3.11)

follow also from the harmonic equations (3.9) and the Grassmann analyticity of \( \bar{\mathcal{W}}_3 \) (3.7).

Let us consider a decomposition of the superfield \( \mathcal{W}^1 \) in terms of the 4-th spinor coordinate

\[ \mathcal{W}^1 = W^1 - \frac{i}{2} \theta^\alpha \bar{\theta}^\dot{\alpha} \partial_{\alpha \dot{\alpha}} W^1 - \frac{1}{32} (\theta)^2 (\bar{\theta})^2 \partial^\alpha \bar{\partial}^{\dot{\alpha}} \partial_{\alpha \dot{\alpha}} W^1 \]

\[ + \bar{\theta}^\dot{\alpha} S^1_{\dot{\alpha}} - \frac{i}{4} \theta^\alpha (\bar{\theta})^2 \partial_{\alpha \dot{\alpha}} S^1_{\dot{\alpha}} + (\bar{\theta})^2 P^1 \]  

(3.12)

where \( \theta \equiv \theta_4, \bar{\theta} \equiv \bar{\theta}^4 \) and \( W^1, S^1_{\dot{\alpha}} \) and \( P^1 \) are the corresponding N = 3 components

\[ (D_{\alpha}^{1}, D_{2\dot{\alpha}}, D_{3\dot{\alpha}}, D_{3\dot{\alpha}})(W^1, S^1_{\dot{\alpha}}, P^1) = 0. \]  

(3.13)

The 4-th supersymmetry transformations of these N = 3 superfields are

\[ \begin{align*}
\delta_\eta W^1 & = -\bar{\eta}^\dot{\alpha} S^1_{\dot{\alpha}}, \quad \delta_\eta S^1_{\dot{\alpha}} = -2i \eta^\alpha \partial_{\alpha \dot{\alpha}} W^1 + 2\bar{\eta}_{\dot{\alpha}} P^1 \\
\delta_\eta P^1 & = i \eta^\alpha \partial_{\alpha \dot{\alpha}} S^1_{\dot{\alpha}}.
\end{align*} \]  

(3.14)

The on-shell constraints (3.8) yield the relations

\[ \begin{align*}
S^1_{\dot{\alpha}} & = \bar{D}_{2\dot{\alpha}} \bar{W}_3, \quad P^1 = 0,
\end{align*} \]  

(3.15)
which give us the equations

\[(\bar{D}_2)^2\bar{W}_3 = 0, \quad D_3^2\bar{D}_{2\alpha}\bar{W}_3 = 0.\] (3.16)

Equations (3.8) can be obtained from a superfield action with Lagrange multipliers; however, these superfield multipliers contain additional degrees of freedom and do not have a geometric interpretation. It is also unclear whether one can construct an \(N=3\) superfield action without Lagrange multipliers for the \(N=4\) supermultiplet \(W^1, S^1_\alpha\) and \(P^1\) (3.13).

4 Infinite-dimensional multiplet of \(N=3\) superfields

4.1 Discrete coordinate and alternative realization of the 4-th supersymmetry

In this section, we define the unusual realization of the \(N=4\) supersymmetry on an infinite set of analytic \(N=3\) superfields. Introduce first a special notation

\[A[k] = (A[-n], A[0], A[n])\] (4.1)

for infinite-dimensional multiplets of \(N=3\) superfields having \(U(1)\times U(1)\) charges corresponding to charges of products of the harmonics \(u_i^2\) or \(u_i^2\)

\[h A[k] = -k, \quad \bar{h} A[k] = k,
\]

\[A[-n] \sim (u_i^{2n} \ldots u_i^{2})^{-1}, \quad A[n] \sim (u_i^2 \ldots u_i^{2n}).\] (4.2)

Note that one can use an alternative notation for these harmonic multiindices

\[A[-2] \equiv A_{22}, \quad A[-1] \equiv A_2, \quad A[0] \equiv A, \quad A[1] \equiv A^2, \quad A[3] \equiv A^{222}.\] (4.3)

Transformations of the 4-th supersymmetry can be realized on three infinite-dimensional multiplets of analytic \(N=3\) superfields:

\[V_2^1[k] \equiv (\ldots V_{222}^1, V_{22}^1, V_2^1, V^1, V^{12}, V^{122}, V^{1222}, \ldots)\] (4.4)

\[V_3^2[k] \equiv (\ldots V_{22222}, V_{2222}, V_{222}, V_{2222}, V_{222}, V_{22222}, \ldots)\] (4.5)

\[V_3^3[k] \equiv (\ldots V_{2222222}, V_{222222}, V_{22222}, V_{22222}, V_{22222}, V_{2222222}, \ldots)\] (4.6)

parametrized by an integer \(k\) or a natural number \(n\). 'External' charge indices correspond to the charges of the \(N=3\) gauge potentials which are included to these multiplets. Let us introduce a common notation for these multiplets \(\mathcal{V}[k] \equiv (V_2^1, V_3^2, V_3^3)[k]\) and treat the parameter \(k\) as the additional discrete (charge) coordinate.
Each superfield in this set has definite values of U(1)-charges which are defined by the parameter \( k \) and the 'external' indices

\[
\begin{align*}
\tilde{h}V_2^1[k] &= k - 1 \\
\tilde{h}V_2^2[k] &= k + 1 \\
\tilde{h}V_3^1[k] &= k + 1
\end{align*}
\]

Define the following additional supersymmetry transformations for each of these multiplets:

\[
\delta \eta \mathcal{V}[k] = \eta^\alpha \bar{D}^2_\alpha \mathcal{V}[k - 1] - \bar{\eta}^{\dot{\alpha}} \bar{D}_{2\dot{\alpha}} \mathcal{V}[k + 1],
\]

where \( \eta^\alpha \) and \( \bar{\eta}^{\dot{\alpha}} \) are the spinor parameters. The given representations contain superfields of the same dimension in distinction with the \( N=3 \) decompositions of \( N=4 \) superfields \( 3.14 \).

The additional transformations commute with the spinor transformations of the \( N=3 \) supersymmetry and preserve all properties of the \( N=3 \) analytic superfields

\[
[\delta(\eta, \bar{\eta}), \delta(\epsilon_i, \bar{\epsilon}^\dot{i})] \mathcal{V}[k] = 0.
\]

The Lie bracket of two \( \eta \)-transformations gives the 4D translations of all superfields

\[
[\delta_{\eta_2}, \delta_{\eta_1}] \mathcal{V}[k] = 2i(\eta_1^\alpha \bar{\eta}_2^{\dot{\alpha}} - \eta_2^\alpha \bar{\eta}_1^{\dot{\alpha}}) \partial_{\alpha\dot{\alpha}} \mathcal{V}[k].
\]

It is easy to understand that multiplication rules for different \( N=4 \) supermultiplets \( \mathcal{V}[k] \) and \( \mathcal{U}[k] \) should include summarizing over the parameter \( k \), e.g.

\[
\mathcal{A}[p] \equiv \sum_k \mathcal{V}[k + p] \mathcal{U}[-k],
\]

where the integer-number parameter \( p \) characterizes the new \( N=4 \) multiplet \( \mathcal{A}[p] \), and \( v \) and \( u \) are some external charges. One can straightforwardly check that the \( \eta \)-variation of this product has the standard \( N=4 \) form \((4.7)\)

\[
\delta_\eta \mathcal{A}[p] = \eta^\alpha \bar{D}^2_\alpha \mathcal{A}[p - 1] - \bar{\eta}^{\dot{\alpha}} \bar{D}_{2\dot{\alpha}} \mathcal{A}[p + 1].
\]

We have used an invariance of infinite sums with respect to translations of the parameter \( k \), for instance,

\[
\sum_k \{ \eta^\alpha \bar{D}^2_\alpha \mathcal{V}[k + p - 1] \mathcal{U}[-k] + \mathcal{V}[k + p] \eta^\alpha \bar{D}^2_\alpha \mathcal{U}[-1 - k] \} = \eta^\alpha \bar{D}^2_\alpha \sum_k \{ \mathcal{V}[k + p - 1] \mathcal{U}[-k] \},
\]

where the change \( k \to k - 1 \) has been made in the 2-nd sum.
4.2 Truncated representations of N=4 supersymmetry

Let us consider the transformation of the $N=3$ potential in the infinite multiplet \( (4.7) \)

$$
\delta \eta V_2^1 = \eta^\alpha D^2_{\alpha} V_2^1 - \bar{\eta}^\dot{\alpha} \bar{D}^2_{\dot{\alpha}} V^1.
$$

We do not know how this representation is related to the realization of the N=4 supersymmetry on mass shell \( (4.6) \), so one discusses a formal possibility of ‘truncation’ of infinite-dimensional representations of the N=4 supersymmetry, when higher terms of supermultiplets \( V[k] \) are expressed via a finite number of independent superfields provided the structure of transformations \( (4.7) \) is preserved. As the simplest example one shall consider the transformations

\[
\begin{align*}
\delta \eta V_2^1 &= c_2(u) \eta^\alpha D^2_{\alpha} V_2^1 + \tilde{c}^2 \eta^\dot{\alpha} \bar{D}^2_{\dot{\alpha}} V^1, \\
\delta \eta V_3^2 &= c_2(u) \eta^\alpha D^2_{\alpha} V_3^2 + \tilde{c}^2 \eta^\dot{\alpha} \bar{D}^2_{\dot{\alpha}} V_2^1,
\end{align*}
\]

where \( c_2(u) \) and \( \tilde{c}^2(u) \) are formal harmonic series satisfying a simple relation

\[
c_2(u)\tilde{c}^2(u) = -c_2(u)\tilde{c}_2(u) = -1.
\]

This relation yields an infinite system of equations for coefficients of the harmonic series. We do not require any regularity for the function \( c_2(u) \), in particular, it cannot be a harmonic polynomial.

The Lie bracket of transformations \( (4.13) \) has a standard form analogous to relation \( (4.9) \). Note that the introduction of the functions \( c_2(u) \) and \( \tilde{c}^2(u) \) breaks the SU(3) covariance. These harmonic functions allow us to define the truncated solution for the infinite-dimensional N=4 multiplets

\[
(V_2^1, V_3^2)[k] = (c_2)^k (V_2^1, V_3^2), \quad (c_2)^{-n} \equiv -(-1)^n (c^2)^n.
\]

We plan to analyze this and other possible truncated realizations of the N=4 supersymmetry in further investigations and now shall continue to study the infinite-dimensional N=4 multiplets.

4.3 N=4 generalization of free N=3 superfield action

A simple geometric interpretation of infinite-dimensional N=4 multiplets \( V^I_K[k] \) as N=3 potentials depending on the additional discrete coordinate allows us to construct a classical superfield action of this model. The superfield action of the \( N = 3 \) gauge theory is considered in the appendix \( (A.14) \). The free action of the N=3 Abelian superfield is

\[
S_f(N=3) = -\frac{1}{4} \int du d\zeta 13 \{ V_2^1 D_3^2 V_3^1 + V_3^1 D_2^3 V_3^2 + V_3^2 D_3^1 V_2^1 - \frac{1}{2} (V_3^1)^2 \}.
\]
Each term of this action admits the infinite-dimensional $N=4$ invariant generalization constructed in accordance with the multiplication rule for $N=4$ multiplets (4.10). Using the relations
\[(D_2^1, D_2^2, D_3^1, (D_2^2, \bar{D}_{2\alpha}))\mathcal{V}[k] = 0\]
one can readily show that the harmonic derivatives $(D_2^1, D_2^2, D_3^1)\mathcal{V}[k]$ are transformed as $N=4$ multiplets with modified values of external charges (4.7). Let us consider the following $N=4$ multiplets corresponding to all superfield terms of the free $N=3$ action:
\[
\begin{align*}
A_{33}^{11} &\equiv \frac{1}{2} \sum_k V_2^1[k] D_3^2 V_3^1[-k] = V_2^1 D_3^2 V_3^1 + V_2^2 D_3^2 V_3^2 + V_2^3 D_3^2 V_3^3 + \ldots \\
\tilde{A}_{33}^{11} &\equiv \frac{1}{2} \sum_k V_3^1[k] D_2^2 V_3^2[-k] = V_2^1 D_2^2 V_3^1 + V_2^2 D_2^2 V_3^2 + V_2^3 D_2^2 V_3^3 + \ldots \\
B_{33}^{11} &\equiv \frac{1}{2} \sum_k V_3^2[k] D_3^2 V_2^1[-k] = V_2^1 D_3^2 V_2^1 + V_2^2 D_3^2 V_2^2 + V_2^3 D_3^2 V_2^3 + \ldots \\
C_{33}^{11} &\equiv -\frac{1}{2} \sum_k V_3^3[k] V_3^1[-k] = -\frac{1}{2} (V_3^1)^2 - V_3^1 V_3^2 - V_3^1 V_3^3 - \ldots
\end{align*}
\]
All these superfields are constructed by analogy with the component $\mathcal{A}[0]$ in the product of $N=4$ multiplets (4.10), so their $\eta$-variations are combinations of total spinor derivatives, for instance,
\[
\delta_\eta A_{33}^{11} = \eta^\alpha D_\alpha A_{233}^{11} - \bar{\eta}^\dot{\alpha} \bar{D}_{2\dot{\alpha}} A_{33}^{112} ,
\]
The $N=3$ superspace integrals of these superfields are invariant with respect to the $N=4$ supersymmetry.

The action of the $N=4$ Abelian multiplet $\mathcal{V}[k]$ can be constructed as an infinite sum of the Chern-Simons-type superfield terms
\[
S_f(N=4) = -\frac{1}{4} \int du d\zeta_{(33)} \{ A_{33}^{11} + \tilde{A}_{33}^{11} + B_{33}^{11} + C_{33}^{11} \} = \sum_k S_k ,
\]
where $S_0 = S_f(N=3)$ (4.16). This action is invariant with respect to the following Abelian gauge transformations:
\[
\delta_\lambda V_f^J[k] = -D_f^J \lambda[k] , \quad J < J ,
\]
where the analytic gauge parameters $\lambda[k]$ form an infinite supermultiplet which includes the $N = 3$ gauge parameter $\lambda[0] \equiv \lambda$ (4.12). One can readily check this invariance using the relations
\[
\begin{align*}
\delta_\lambda A_{33}^{11} &= \sum_k \lambda[k] \{ D_2^1 D_3^2 V_3^1[-k] - D_3^1 D_3^2 V_3^1[-k] \} + \text{div} \\
\delta_\lambda \tilde{A}_{33}^{11} &= \sum_k \lambda[k] \{ D_3^1 D_2^2 V_3^2[-k] - D_3^2 D_2^1 V_3^2[-k] \} + \text{div}
\end{align*}
\]
\[ \delta_{\lambda}B_{33}^{11} = \sum_{k} \lambda[k] \{ D_3^2 D_3^1 V_2^1[-k] - D_2^1 D_3^1 V_3^2[-k] \} + \text{div} \]
\[ \delta_{\lambda}C_{33}^{11} = -\sum_{k} \lambda[k] D_3^1 V_3^1[-k] + \text{div} \]

where the terms with the total harmonic derivatives are not written down explicitly.

The infinite set of the zero-curvature equations for the free \( N = 4 \) action \((4.22)\) contains three independent series

\[
V_3^1[k] = D_3^1 V_2^2[k] - D_2^2 V_2^1[k] ,
F_{32}^{11}[k] \equiv D_3^1 V_2^2[k] - D_2^1 V_3^3[k] = 0 ,
F_{33}^{21}[k] \equiv D_3^2 V_3^3[k] - D_3^1 V_2^3[k] = 0 .
\]

The 1-st equation is pure algebraic and can be solved off-shell. This solution \( \hat{V}_3^1[k] \) can be used to define the 2-nd order free \( N = 4 \) action on the restricted set of independent potentials

\[
\hat{S}_f(N=4) = -\frac{1}{4} \int du d\zeta^{(33)} \{ B_{33}^{11} - \hat{C}_{33}^{11} \} = \sum_{k} \hat{S}_k ,
\]

where \( B_{33}^{11} \) is given by Eq.\((4.19)\) and the second term can be obtained from \((4.20)\) via the substitution

\[
V_3^1[k] \Rightarrow \hat{V}_3^1[k] = D_2^1 V_3^3[k] - D_3^2 V_2^3[k] .
\]

The structure of solutions to the infinite set of equations can be analyzed in non-supersymmetric gauges. Using the component decomposition, it is not difficult to show that some branches of the infinite sets \((4.24)\) and \((4.25)\) are pure gauge superfields

\[
V_2^1[-n] = -D_2^1 \lambda[-n] ,
V_3^2[n] = -D_3^2 \lambda[n] .
\]

In this gauge with broken 4-th supersymmetry, the following independent superfields remain:

\[
V_3^2[-n] ,
V_3^1 ,
V_2^1 ,
V_2^1[n] ,
\]

as well as a 'residual' gauge invariance with the restricted gauge parameters

\[
D_2^1 \hat{\lambda}[-n] = 0 ,
D_3^2 \hat{\lambda}[n] = 0 .
\]

The superfield \( N = 3 \) equation

\[
F_{32}^{11} = D_3^1 V_2^2 - D_2^1 (D_3^1 V_3^3 - D_3^2 V_2^1) = 0
\]

gives in components free equations of the physical fields of the \( N = 3 \) vector multiplet

\[
\phi^k(x), \tilde{\phi}_k(x), A_m(x), \psi_\alpha^0(x), \tilde{\psi}^{k\dot{\alpha}} , \psi_{\alpha}, \tilde{\psi}^{\alpha} .
\]
Auxiliary component fields of the free superfield $V_2^1$ vanish on mass shell.

Let us consider the following pairs of harmonic equations:

$$F^{11}_{32}[n] = (D_3^1 + D_2^1 D_3^2) V_2^1[n] = 0 \quad , \quad F^{12}_{33}[n] = -(D_3^2)^2 V_2^1[n] = 0 . \quad (4.31)$$

These equations contain the common superfield $V_2^1[n] = (V^1, V^{12}, \ldots)$ in gauge \ref{4.28} and have nontrivial solutions for any $n$ describing 'short' superfields with the components of exotic dimensions and SU(3) representations. For instance, the superfield $V^{12}$ has a dimensionless scalar component $u^k_R k$.

4.4 $N=4$ invariant non-Abelian interaction

We shall construct the $N=4$ generalization of the $N=3$ interaction \ref{A.14} which contains the superfield

$$J_{33}^{11} = V_3^1[V_2^1, V_3^2] . \quad (4.32)$$

The evident $N=4$ generalization of this term is the following infinite double sum of superfields

$$I_{33}^{11} = \sum_{k,p} V_3^1[k] [V_2^1[p-k], V_3^2[-p]] \quad (4.33)$$

where $p, k$ are arbitrary integers and all superfield potentials are considered in the adjoint representation of the Lie algebra of the gauge group $G$.

The $N=4$ transformations of this superfield can be analyzed by analogy with \ref{4.11}, \ref{4.21}. Thus, the $N=4$ invariant generalization of the $N=3$ Yang-Mills theory action has the following form:

$$S_{YM}(N=4) = -\frac{1}{4} \int du d\zeta (33)_{(11)} \text{Tr} \{ A_{33}^{11} + \tilde{A}_{33}^{11} + B_{33}^{11} + C_{33}^{11} + I_{33}^{11} \} , \quad (4.34)$$

where the free terms are given by \ref{4.17}-\ref{4.20}. This Chern-Simons-type action is invariant with respect to the following non-Abelian gauge transformations:

$$\delta_\lambda V_2^1[k] = -D_2^1 \lambda[k] - \sum_{p} [V_2^1[k+p], \lambda[-p]] ,$$

$$\delta_\lambda V_3^2[k] = -D_3^2 \lambda[k] - \sum_{p} [V_3^2[k+p], \lambda[-p]] , \quad (4.35)$$

$$\delta_\lambda V_3^1[k] = -D_3^1 \lambda[k] - \sum_{p} [V_3^1[k+p], \lambda[-p]] ,$$

where $\lambda[k]$ is the $N=4$ set of analytic parameters. The commutator terms in these formulae preserve the $N=4$ structure in accordance with \ref{4.10}. The proof of invariance is completely analogous to the corresponding proof in the $N=3$ theory \ref{4} if one uses the simple properties of infinite sums.
The equations of motion for the non-Abelian $N=4$ action are the infinite-dimensional generalization of the $N=3$ zero-curvature conditions (A.10)

\[
V_3^1[k] = D_2^1V_3^2[k] - D_3^2V_2^1[k] + \sum_p \left[ V_2^1[k+p], V_3^2[-p] \right],
\]
\[
F_{32}^{11}[k] \equiv D_3^1V_2^1[k] - D_2^1V_3^1[k] + \sum_p \left[ V_3^1[k+p], V_2^1[-p] \right] = 0. \tag{4.36}
\]

The $N=3$ potentials $V_2^1$, $V_3^2$ and $V_3^1$ interact with all additional superfields in the infinite $N=4$ multiplets. The problem is how to estimate the role of additional superfields in the $N=4$ invariant phase of this model.

5 Conclusions

It is shown that the $N=3$ harmonic-superspace equations of motion are invariant with respect to the additional spinor transformation of the $N=4$ supersymmetry. The corresponding transformations of the $N=3$ potentials on mass shell have an unusual supersymmetry algebra which contains gauge-dependent terms.

The problem of a manifestly supersymmetric off-shell description of the $N=4$ Yang-Mills theory has been discussed in the formalism with hidden or broken SU(4) symmetry. The superfield formalism based on the SU(3)/U(1) $\times$ U(1) harmonics seems more flexible in the $N=4$ superspace. A decomposition of the superfield constraints of the $N=4$ strength in terms of the 4-th spinor coordinate yields the equations of motion for the $N=3$ superfield strength $W_1^1$ and auxiliary superfield strengths $S_\dot{\alpha}^1$ and $P^1$; however, one does not succeed in constructing an appropriate superfield action in this setting.

As an illustration of flexibility of the $N=3$ superspace we have considered the new realization of the standard $N=4$ supersymmetry algebra on the infinite set of $N=3$ superfields parametrized by the integer-number parameter $k$. This parameter plays a role of an additional discrete coordinate labeling different superfields of $N=4$ supermultiplets, and this degree of freedom is not connected directly with any Grassmann or harmonic coordinates. It is not difficult to consider the infinite-dimensional $N=4$ generalization of the $N=3$ gauge harmonic-superspace action. This model has a simple geometric interpretation; however, the role of additional superfield degrees of freedom is still unclear and the consistency of this model at the quantum level is not verified. We hope to continue the investigation of possible realizations of the $N=4$ supersymmetry on a finite number of $N=3$ superfields by analogy with the simplest example of such realization in subsection 4.2.

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6 Appendix. N=3 harmonic superspace

In this appendix, one reviews basic formulae of the D=4, N=3 harmonic superspace [1, 2, 4] using some results and the notation of Ref. [6]. The basic coordinates of this superspace are the harmonics $u_i^I$ and $u_i^j$ parametrizing the 6-dimensional coset SU(3)/U(1)$\times$U(1)

$$u_i^I u_j^J = \delta_i^j, \quad u_i^I u_j^I = \delta_i^j, \quad \varepsilon^{ijk} u_i^j u_j^2 u_j^3 = 1$$

(A.1)

where $i, j, k = 1, 2, 3$ are the indices of fundamental representations of the automorphism group SU(3), and $I, J = 1, 2, 3$ are some combinations of U(1)$\times$U(1) charges in the condensed notation of Ref. [6]. The SU(3)-invariant harmonic derivatives $\partial_I^J (I \neq J)$ act on these harmonics

$$\partial_I^J u_K^I = \delta_K^I u_i^I, \quad \partial_I^I u_K^J = -\delta_K^J u_i^I.$$

(A.2)

Harmonic U(1) charges $h$ and $\tilde{h}$ arise in the commutators of the charged derivatives

$$[\partial_1^2, \partial_2^2] = h, \quad [\partial_2^3, \partial_3^2] = \tilde{h}, \quad [h, \partial_2^1] = 2\partial_2^1,$n

$$[h, \partial_1^2] = -2\partial_1^2, \quad [\tilde{h}, \partial_3^2] = 2\partial_3^2, \quad [\tilde{h}, \partial_2^1] = -2\partial_2^1,$n

$$[h, \partial_3^2] = -\partial_3^2, \quad [h, \partial_2^1] = -\partial_1^1, \quad [h, \partial_3^1] = \partial_3^1, \quad [\tilde{h}, \partial_3^1] = [\tilde{h}, \partial_1^1].$$

(A.3)

The N=3 harmonics can be used to define the analytic coordinates $\{x^m, \theta_i^I, \tilde{\theta}^{I\dot{a}}, u\}$ in the harmonic superspace

$$x^m = x^m - i(\sigma^m)_{\alpha\dot{\alpha}}(\theta_\alpha^I \tilde{\theta}^{I\dot{a}} - \theta_\dot{\alpha}^I \tilde{\theta}^{I\dot{a}}),$$

$$\theta_i^I = \theta_i^I u_i^I, \quad \tilde{\theta}^{I\dot{a}} = \tilde{\theta}^{I\dot{a}} u_i^I.$$

(A.4)

where $x^m, \theta_\alpha^I, \tilde{\theta}^{I\dot{a}}$ are the standard coordinates of the D=4, N=3 superspace and $\sigma^m$ are the Weyl matrices of the group SL(2,C).

We shall use the harmonic projections of the standard spinor derivatives $D_k^\alpha$ and $\bar{D}_{k\dot{\alpha}}$

$$D_\alpha^I = u_k^I D_\alpha^k, \quad \bar{D}_{I\dot{\alpha}} = u_k^I \bar{D}_{k\dot{\alpha}},$$

$$\{D_\alpha^I, D_\delta^J\} = 0 = \{\bar{D}_{I\dot{\alpha}}, \bar{D}_{K\dot{\beta}}\}$$

$$\{D_k^\alpha, \bar{D}_{J\dot{\beta}}\} = 2i\delta_k^J \partial_{\alpha\dot{\beta}},$$

(A.5)

where $\partial_{\alpha\dot{\beta}} = (\sigma^m)_{\alpha\beta} \partial_m$.

The special conjugation acts on the harmonics and harmonic derivatives

$$\bar{u}_k^1 = u_k^3, \quad \bar{u}_k^3 = u_k^1, \quad \bar{u}_k^2 = -u_k^2,$r

$$\bar{\partial}_2^1 = -\partial_3^1, \quad \bar{\partial}_3^1 = \partial_2^1.$$

(A.6)
The special conjugation of spinor derivatives can be chosen in the following form:

\[ \tilde{D}_1^\alpha f = -\bar{D}_3\bar{\alpha} \tilde{f}, \quad \tilde{D}_2^\alpha f = \bar{D}_2\bar{\alpha} \tilde{f}, \]  
(A.7)

where \( f \) and \( \tilde{f} \) are mutually conjugated even harmonic superfields.

The (4 + 4) Grassmann-analytic superfield \( \Lambda(\zeta, u) \) is defined in the analytic (4|4 + 4)-dimensional superspace with the coordinates

\[ \zeta = (x^m_A, \theta_2^\alpha, \theta_3^\alpha, \bar{\theta}_1^{1\alpha}, \bar{\theta}_2^{2\alpha}) . \]  
(A.8)

The superfield \( \Lambda(\zeta, u) \) satisfies the conditions

\[ (D_1^\alpha, \bar{D}_3\bar{\alpha})\Lambda = 0 . \]  
(A.9)

The harmonic derivatives in the analytic coordinates \( D_I^K \) act covariantly on the spinor derivatives

\[ \{D_I^K, D_J^L\} = \delta^L_I D_J^K, \quad \{D_I^K, \bar{D}_J\bar{\alpha}\} = -\delta^L_J \bar{D}_K\bar{\alpha}. \]  
(A.10)

Note that the following harmonic and spinor derivatives preserve the Grassmann analyticity

\[ \tilde{D}_1^\alpha, \bar{D}_3\bar{\alpha}, D_2^\alpha, \bar{D}_2\bar{\alpha} . \]  
(A.11)

The analytic potentials \( V_1^1, V_3^2 \) and \( V_3^1 \) of the \( N=3 \) Yang-Mills theory can be treated as gauge connections for the harmonic covariant derivatives

\[ \nabla_I^K = D_K^I + V_I^K, \quad I < K, \]  
\[ \delta_\lambda V_I^K = -D_K^I \lambda + [\lambda, V_I^K] , \]  
(A.12)

where \( \lambda \) is an infinitesimal analytic gauge parameter. We use the following conditions of conjugation:

\[ \tilde{V}_2^1 = -V_3^2, \quad \tilde{V}_3^1 = V_3^1, \quad \tilde{\lambda} = -\lambda \]  
(A.13)

The superfield action of the \( N = 3 \) theory was defined on these analytic potentials [1]

\[ S(N=3) = \frac{-1}{4} \int du d\zeta^{(33)}_{(11)} \{V_2^1 D_3^2 V_3^1 + V_3^1 D_3^1 V_3^2 + V_3^2 D_3^1 V_2^1 - \frac{1}{2}(V_3^1)^2 + V_3^1[V_2^1, V_3^2]\} . \]  
(A.14)

In this formula, the analytic integral measure is considered

\[ d\zeta^{(33)}_{(11)} = d^4 x_A d\theta^{(33)}_{(11)}, \quad \int d\theta^{(33)}_{(11)}(\theta_2)^2(\theta_3)^2(\bar{\theta}_1)^2(\bar{\theta}_2)^2 = 1 . \]  
(A.15)

The action yields equations of motion which have the form of zero-curvature conditions for the \( N=3 \) potentials

\[ V_3^1 = D_3^1 V_3^2 - D_3^2 V_3^1 + [V_2^1, V_3^2], \]  
\[ F_3^{12} \equiv D_3^1 V_2^1 - D_2^1 V_3^1 + [V_3^1, V_2^1] = 0 . \]  
(A.16)
The $N=3$ harmonic formalism allows us to introduce the off-shell superfield strengths

\[ W^1 = \frac{1}{4}(\bar{D}_3)^2V_2^3(V_3^2), \quad D_3^2V_2^3 - D_2^3V_3^2 + [V_3^2, V_3^3] = 0, \]

\[ \bar{W}_3 = -\frac{1}{4}(D^1)^2V_1^2(V_2^1), \quad D_1^1V_1^2 - D_2^2V_2^1 + [V_2^1, V_1^1] = 0, \quad (A.17) \]

which are defined via the nonanalytic connections $V_2^2(V_1^2)$ and $V_3^2(V_3^2)$. The harmonic zero-curvature equations for the $N=3$ connections $V_K^I$ guarantee the harmonic analyticities of these superfields, for instance,

\[ \nabla_3^2W^1 = D_3^2W^1 + [V_3^2, W^1] = 0, \quad \nabla_2^1W^1 = D_2^1W^1 + [V_2^1, W^1] = 0. \quad (A.18) \]

The first relation is valid off-shell, while the second one is equivalent to the equations of motion.

It is not difficult to obtain pseudodifferential solutions for the Abelian non-analytic connection

\[ V_1^2 = \frac{1}{2}(D_1^2)^2V_2^1 - \frac{1}{12}(D_1^2)^3D_2^1V_1^1 + \ldots \quad (A.19) \]

using the relation $[D_2^1, D_1^2]V_2^1 = 2V_1^1$. The superfield strength can be expressed via the potential

\[ \bar{W}_3 = -\frac{1}{4}(D^1)^2V_1^2 = -\frac{1}{4}(D^2)^2PV_1^1, \]

\[ P \equiv 1 + \sum_{k=1}^{\infty} (-1)^k \frac{1}{k!(k+1)!}(D_1^2)^k(D_1^1)^k. \quad (A.20) \]

An analogous formula for $W^1$ contains the conjugated operator $\bar{P}$.

It is convenient to use the gauge conditions

\[ D_1^1V_1^1 = 0, \quad D_3^2V_3^2 = 0 \quad (A.21) \]

to simplify the pseudodifferential solutions of Eqs.\((A.17)\). In this case, one can construct a simple nonlinear expansion of the nonanalytic connection $V_1^2$ in terms of the non-Abelian potential

\[ V_1^2(V_2^1) = \frac{1}{2}(D_1^2)^2V_2^1 + \frac{1}{4}[(D_1^2)^2V_2^1, D_1^2V_1^2] + \frac{1}{12}[D_1^2V_2^1, [D_1^2V_2^1, (D_1^2)^2V_2^1]] \]

\[ -\frac{1}{24}[(D_1^2)^2V_2^1, [V_2^1, (D_1^2)^2V_2^1]] + \ldots \quad (A.22) \]

\[ \bar{W}_3(V_2^1) = -\frac{1}{4}(D^1)^2V_1^2(V_2^1) = -\frac{1}{4}(D^2)^2V_2^1 + \ldots \quad (A.23) \]

and analogous expansions for $V_2^3(V_3^2)$ and $W_1(V_3^2)$.
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