Nonequilibrium corrections to energy spectra of massive particles in expanding universe.

A.D. Dolgov

Teoretisk Astrofysik Center
Juliane Maries Vej 30, DK-2100, Copenhagen, Denmark

Abstract

Deviations from kinetic equilibrium of massive particles caused by the universe expansion are calculated analytically in the Boltzmann approximation. For the case of an energy independent amplitude of elastic scattering, an exact partial differential equation is derived instead of the usual integro-differential one. A simple perturbative solution of the former is found. For the case of an energy-dependent amplitude the problem cannot be reduced to the differential equation but the solution of the original integro-differential equation can be found in terms of the Taylor expansion, which in the case of a constant amplitude shows a perfect agreement with the perturbative solution of the differential equation. Corrections to the spectrum of (possibly) massive tau-neutrinos are calculated. The method may be of more general interest and can be applied to the calculation of spectrum distortion in other (not necessarily cosmological) nonequilibrium processes.

1 Introduction

Abundance of stable relics from the hot epoch of the universe evolution are usually calculated with the help of the integrated in momentum Boltzmann kinetic equation, so that the complicated integro-differential equation is reduced to an ordinary differential equation for the number density of particles in question. To make this reduction one has to assume that these particles are in kinetic equilibrium with the

\footnote{Also: ITEP, Bol. Cheremushkinskaya 25, Moscow 113259, Russia.}
cosmic plasma so that their distribution is given by the expression

\[ f(p, t) = \exp\left[\frac{\mu(t) - E}{T(t)}\right] \]  (1)

where \( E \) is the particle energy, \( \mu(t) \) is an effective chemical potential (in total equilibrium usually \( \mu(t) \equiv 0 \)) and \( T(t) \) is the plasma temperature. Under this assumption the particle number density

\[ n(t) = g_s \int \frac{d^3p}{(2\pi)^3} f(p, t) \]  (2)

was shown to satisfy the equation \([1, 2]\)

\[ \dot{n} + 3Hn = \langle \sigma v \rangle (n_{eq}^2 - n^2) \]  (3)

where \( H = \dot{a}/a \) is the Hubble parameter, \( \langle \sigma v \rangle \) is the thermally averaged product of the annihilation cross-section and the particle velocity, and \( n_{eq}^2 \) is the equilibrium number density obtained from the expression (2) with \( \mu = 0 \). This equation was used in particular for the derivation of the cosmological bound on the mass of a possible heavy neutrino \([3, 4]\) and for the calculation of the mass density of heavy particles, e.g. stable supersymmetric relics, which may constitute invisible matter in the universe. A detailed derivation of eq. (3) valid for any form of the cross-section is given in ref. \([5]\) (see also the book \([6]\)).

Usually kinetic equilibrium is maintained long after breaking of chemical equilibrium, characterized by a nonzero \( \mu \). Though elastic and annihilation cross-sections are typically of comparable magnitude, the rate of annihilation, \( \langle \dot{n}/n \rangle_{ann} \sim \sigma v n \), exponentially goes down with decreasing temperature together with the number density of the nonrelativistic annihilating particles, \( n \sim \exp(-m/T) \), while the probability of elastic scattering, \( \langle \dot{n}/n \rangle_{el} \sim \sigma v n_0 \), dies down much slower, because \( n_0 \sim T^3 \). Here \( n_0 \) is the number density of massless (or light particles) which elastically scatter on the heavy ones and preserve kinetic equilibrium.
However for the particular case of tau-neutrino with the mass in MeV range, deviations from the kinetic equilibrium may be essential and can change the results of the calculations [7, 8] based on equation (3). Correspondingly the nucleosynthesis bounds on a possible mass of nu-tau would be different. Due to a nonzero mass, $\nu_\tau$ would cool faster in the course of expansion, than massless particles. This in turn would result in less efficient annihilation of $\nu_\tau\bar{\nu}_\tau$ and in their larger frozen number density. Thus the nucleosynthesis bound on the mass of $\nu_\tau$ would be stronger. In ref.[8] a simple attempt to take into account nonequilibrium corrections to the $\nu_\tau$-distribution in energy was made. Namely it was assumed that the equilibrium form of nu-tau spectrum is maintained down to a certain temperature $T_{el}$, determined by the strength of the elastic scattering, and below $T_{el}$ the distribution function is given by the expression valid for noninteracting particles, $f \sim \exp[\sqrt{(m^2/T_{el}^2 + p^2/T^2)}]$. The effect was found to be relatively weak.

An important influence on the primordial nucleosynthesis may produce nonequilibrium electronic neutrinos originating from the interaction with nonequilibrium tau-neutrinos. There are two possible effects. First, there should be an excessive cooling of massless neutrinos due to elastic scattering on colder-than-equilibrium $\nu_\tau$’s. This makes the production of $\nu_\tau$ in the collision of the massless neutrinos (inverse annihilation) less efficient and correspondingly leads to a smaller number density of $\nu_\tau$. It weakens the nucleosynthesis bound. Another effect is the distortion of $\nu_e$ spectrum due to energetic electronic neutrinos coming from the process $\nu_\tau\bar{\nu}_\tau \rightarrow \nu_e\bar{\nu}_e$. Such nonequilibrium $\nu_e$’s give rise to a larger neutron-to-proton ratio. This effect was found to be stronger than the first one and an overall stronger bound on $m_{\nu_\tau}$ was obtained [9].

In ref.[10] it was assumed that massless neutrinos maintain their canonical distribution in momentum but, as the massive ones, acquire a nonzero chemical potentials (see eq.(11)). This assumption permits to reduce the problem to the solution of a
system of ordinary differential equations. It considerably simplifies the calculations. Such overall heating would result in a weaker nucleosynthesis bound but the neglected spectrum distortion have a stronger impact on the nucleosynthesis.

It was claimed in ref.[11], on the basis of numerical solution of the complete system of the exact integro-differential Boltzmann equations, that deviations from kinetic equilibrium generated by massive $\nu_\tau$ are very strong and this gives rise to a much weaker bound on the nu-tau mass than it was previously stated[7, 8]. In view of the arguments presented above the sign of the effect should be just opposite. Recently the authors found an error in their computations (Madsen, private communication) and the result seems to be close to those existing in the literature.

Because of the complexity of numerical calculations it may be interesting to find an approximate analytical way to calculate spectral distortion for both massive and massless neutrinos. In this paper the spectrum of massive particles, which have elastic interaction with the equilibrium bath of massless ones, is found. In the following section the partial differential equation for the spectral density is derived from integro-differential Boltzmann equation for the case of the energy independent amplitude of elastic scattering. This equation permits a simple perturbative solution. For the case of the energy dependent amplitude such simplification was not found but one can solve the integro-differential equation expanding the distortion into the Taylor series. For the case of a constant amplitude of elastic scattering, comparison of this method with the perturbative solution of the differential equation, derived in Sec.2, is made in Sec.3. The agreement is found to be very good. More practically interesting case of weak interactions with the amplitude proportional to particle energies is considered in Sec.4. The results are applied to the calculations of the corrections to the spectrum of massive tau-neutrinos.
2 Differential kinetic equation for a constant amplitude of elastic scattering.

Let us consider elastic scattering of massive particles with mass $m$, momentum $p$, and energy $E$ on massless ones with momentum $k$ and energy $\omega$. The basic Boltzmann equation in the Robertson-Walker expanding universe and in the limit of Boltzmann statistics has the following form:

\[
(\partial_t + H p \partial_p) f_m = \frac{1}{2E} \int \frac{d^3 k}{2\omega(2\pi)^3} \frac{d^3 p'}{E'(2\pi)^3} \frac{d^3 k'}{2\omega'(2\pi)^3} |A_0|^2 (2\pi)^4 \delta^4(p + k - p' - k') \,

[f_m(p)f_0(k) - f_m(p')f_0(k')] \quad (4)
\]

Massless particles are assumed to be in thermal equilibrium so that $f_0(\omega) = \exp(-\omega/T)$.

In the case considered in this section, the amplitude of elastic scattering $A_0$ does not depend on the momenta of the colliding particles. All the integration but one in this expression can be explicitly done (see Appendix). Introducing new variables $x = m/T$ and $y = p/T$ and the new unknown function $C(x, y) = \exp(\sqrt{x^2 + y^2}) f_m(x, y)$ and assuming that the temperature drops with the expansion in accordance with the usual law, $\dot{T} = -HT$, we can rewrite this equation in the form:

\[
H x \partial_x (C e^{-u}) = \frac{(-|A_0|^2)me^{-u/2}}{64\pi^3 uxy} \int_0^\infty \frac{dy'y' e^{-u/2}}{u'}

[C(x, y) - C(x, y')] \left[ e^{-|y-y'|/2} - e^{(y+y')/2} \right] \quad (5)
\]

where $u = \sqrt{x^2 + y^2}$ and $u' = \sqrt{x^2 + y'^2}$.

Successively multiplying eq.(5) by $\exp((u + y)/2)$ and by $\exp(-y)$ and each time differentiating in $y$ we arrive after some straightforward algebra to the following differential equation:

\[
JC'' + 2J'C' = -\frac{64\pi^3 H x^2}{|A_0|^2 m} e^{y/2} \partial_y \left\{ e^{-y} \partial_y \left[ e^{(u+y)/2} uy \partial_x (Ce^{-u}) \right] \right\} \quad (6)
\]

where prime means differentiation with respect to $y$ and $J$ is given by the expression:

\[
J(x, y) = \frac{1}{2} e^{y/2} \int_{u+y}^\infty dze^{-z/2} \left( 1 - \frac{x^2}{z^2} \right) - \frac{1}{2} e^{-y/2} \int_{u-y}^\infty dze^{-z/2} \left( 1 - \frac{x^2}{z^2} \right) \quad (7)
\]
It can be checked that $J$ satisfies the differential equation:

$$J'' - J/4 = -ye^{-u/2}/u \quad (8)$$

We will solve eq.(8) perturbatively, looking for the solution in the form $C(x, y) = C_0(x) + C_1(x, y)$ where $C_0(x)$ is the equilibrium solution; it can be found from the condition of particle number conservation:

$$C_0(x) \int_0^\infty dy y^2 e^{-u(x,y)} \sim 1/x^3 \quad (9)$$

If the rate of the elastic scattering is large in comparison with the rate of the universe expansion, or in other words, the dimensionless coefficient in front of the r.h.s. of eq.(8) is sufficiently small we can neglect $C_1$ in the r.h.s. and explicitly solve this equation in quadratures. Twice integrating by parts we get:

$$C'_1(x, y) = \frac{64\pi^3 C_0(x) H x^2}{|A_0|^2} \left\{ \frac{e^{-u(x,y)/2}}{J} \left[ -\frac{\partial_x C_0}{C_0} \left( u(x, y) + \frac{y^2}{2} - \frac{y^2}{2} \right) \right] + \frac{xy^2}{2u(x, y)} + x + u(x, y) y J' \left( \frac{\partial_x C_0}{C_0} - \frac{x}{u(x, y)} \right) \right\} - \frac{1}{J^2} \int_y^\infty dz \frac{e^{-u(x,z)}}{z^2} \left[ \frac{\partial_x C_0}{C_0} - \frac{x}{u(x, z)} \right] \right\} \quad (10)$$

The function $C_1(x, y)$ is found by the integration of this expression in $y$ and the integration constant (which is a function of $x$) should be chosen from the condition of the particle number conservation:

$$\int dy y^2 e^{-u(x,y)} C_1(x, y) = 0 \quad (11)$$

In a simple case of the universe dominated by relativistic particles the Hubble parameter is given by the expression

$$H = \sqrt{\frac{4\pi^3 g_*}{45} \frac{m^2}{m_{Pl} x^2}} \quad (12)$$
where $m_{Pl}$ is the Planck mass and $g_*$ is the number of relativistic degrees of freedom contributing into the cosmological energy density. Thus $H x^2$ is $x$-independent and it is convenient to consider the relative quantity $r'_0 = (C'_1/C_0)(m/H x^2)/(|A_0|^2/64\pi^3)$. It is presented in fig.1 for $x = 10$ as a function of $y$ and in fig.2 for $y = 3$ as a function of $x$. With rising $x$ the correction $C_1(x, y)$ is getting bigger; asymptotically for large $x$, $C_1 \sim x$. We can trust these results till $C_1/C_0 \ll 1$ at least in the region of $y$ where the suppression due to the factor $\exp[-u(x, y)]$ is not too strong, i.e. for $y^2 \sim 2x$.

3 Power series expansion for the case of a constant amplitude.

Here we will present an alternative approach to the solution of the original kinetic equation (3). This equation is not as easy to solve perturbatively (for small deviations from kinetic equilibrium) as the equivalent differential equation (4). However the method considered here can be useful for the solution of the kinetic equation for the case of weak interactions when the amplitude strongly depends on the energies of the colliding particles and the reduction to the more simple form (4) is not found. Below we will find the solution of eq.(3) expanding $C(x, y)$ in powers of $y$ (in fact of $y^2$). Formally the expansion is valid even if the elastic scattering is relatively weak and the deviations from the kinetic equilibrium are large, but the convergence of the expansion is getting worse for larger $x$, when the probability of elastic scattering goes down.

After a simple rearrangement we can rewrite eq.(3) in the form:

\[
\frac{64\pi^3 H x^2 C_0(x) y \exp(-u/2)}{|A_0|^2 m} \left( x - \frac{u \partial_x C_0(x)}{C_0(x)} \right) = C_1(x, y) J(x, y) - 2 \sinh(y/2) \int_0^{\infty} \frac{dy'}{u'} e^{-u'+y'/2} C_1(x, y')\]

\[
+ 2 \int_0^y \frac{dy'}{u'} e^{-u'/2} C_1(x, y') \sinh[(y - y')/2] \]  

(13)
We will look for the solution of this equation in the form of the Taylor expansion

\[ C_1(x, y) = C_{10}(x) + C_{12}(x)y^2/2 + C_{14}(x)y^4/4 + \ldots \]  

(14)

The coefficient \( C_{10} \) is not determined by the equation but should be found from the condition of conservation of the particle number as is discussed above. One can easily check that the odd powers of \( y \) in this expansion are absent. This was evident if fact from the very beginning because the expansion in terms of the three momentum \( |\vec{p}| \) normally goes in terms of \( \vec{p}^2 \) and there is no reason to expect a singularity \( \sqrt{\vec{p}^2} \) in the considered problem.

We will retain in \( C \) only terms up to the fourth power in \( y \), so the last integral in equation (14) which contributes in the order \( y^5 \) can be neglected. After a simple algebra the following system of equations, determining \( C_{12} \) and \( C_{14} \), can be obtained:

\[
\frac{1}{2} C_2 I_3 + \frac{1}{4} C_4 I_5 = x \left( \frac{\partial_x C_0}{C_0} - 1 \right)
\]

(15)

\[
\frac{1}{2} C_2 I_1 - \left( \frac{1}{4x} + \frac{1}{24} \right) \left( \frac{1}{2} C_2 I_3 + \frac{1}{4} C_4 I_5 \right) = -\frac{\partial_x C_0}{2xC_0}
\]

(16)

where

\[
I_n(x) = \int_0^\infty \frac{dy'y^n}{u'} e^{-\left(u'-x+y'\right)/2}
\]

(17)

For example for \( x = 10 \) we find \( C_2 = -1 \) and \( C_4 = 0.007 \). It very well agrees with the results of the previous section up to \( y \approx 10 \). At these and larger \( y \)'s the suppression of the spectrum due to the factor \( \exp(-u) \) is already very strong and this part of the spectrum is not physically significant. It is noteworthy that there is no formal small parameter in the equation (15) and all \( C_n \) would in principle equally contribute. However higher order terms are small numerically, possibly because of nonrelativistic nature of the problem, and thus the neglect of those terms is justified. For example if we neglect the term \( C_4 y^4 \) (which we did not) the resulting value of \( C_2 \) would be only slightly different form the one found together with \( C_4 \).
4 Weak interaction scattering.

Now let us consider a more realistic case of elastic scattering of Dirac massive tau-neutrino on massless $\nu_e, \nu_\mu$, and $e^\pm$, which are in thermal equilibrium. The amplitude of elastic scattering summed over all light lepton species is

$$|A_w|^2 = K[(pk)^2 + (pk')^2]/m^4$$

(18)

where (for $\sin \theta_w = 0.23$)

$$K = 32G_F^2m^4(3 - 4\sin^2 \theta_w + 8\sin^4 \theta_w) \approx 1.09 \times 10^{-20}(m/MeV)^4$$

(19)

We have to substitute the amplitude $A_w$ instead of $A_0$ into eq.(4). The final result of the integration is grossly simplified in nonrelativistic limit when one neglects the terms of the order of $p^2/m^2$ in the amplitude. The details of the integration can be found in the Appendix. Performing it we find, instead of eq.(5), the following equation:

$$H_x \partial_x (Ce^{-u}) = -\frac{12K\mu e^{-u/2}}{64\pi^3 y x^5} \int_0^\infty dy' y' e^{-(u' + y')/2}[C(x, y) - C(x, y')]$$

$$\left[2 \sinh(y/2)(1 + y'/3 + (y^2 + y'^2)/24) - 2y \cosh(y/2)(1/3 + y'/12)\right]$$

(20)

As in the previous sections we make the perturbative expansion $C(x, y) = C_0(x) + C_1(x, y)$ and expand $C_1$ in powers of $y$. In eq.(20) we have kept only the terms which are essential for calculations of $C_1$ up to $y^4$. The exact r.h.s. of this equation is presented in the Appendix.

We will slightly change notations here and expand the relative deviation from the equilibrium distribution:

$$r_w(x, y) = C_1(x, y)/C_0(x) = r_2(x)y^2/2 + r_4(x)y^4/4$$

(21)

For the coefficients $r_n$ the following equations are valid:

$$\frac{1}{2}r_2F_3 + \frac{1}{4}r_4F_5 = S \left(\frac{\partial_x C_0}{C_0} - 1\right)$$

(22)
\[ \frac{r_2}{2} \left[ F_1 + L_3 - \left( \frac{1}{4x} + \frac{1}{2x^2} \right) F_3 \right] + \frac{r_4}{4} \left[ L_5 - \left( \frac{1}{4x} + \frac{1}{2x^2} \right) F_5 \right] = \frac{-S}{2x^2} \]  

(23)

Here \( S = 16\pi^3(Hx^2)x^4/Km^4 \) and

\[ F_n = \int_0^\infty dy y^n \frac{x}{u} \left( 1 + \frac{y}{2} + \frac{y^2}{8} \right) e^{(u-x+y)/2} \]  

(24)

\[ L_n = \int_0^\infty dy y^n \frac{x}{u} \left( \frac{y}{48} - \frac{y^2}{192} \right) e^{(u-x+y)/2} \]  

(25)

For example for \( m = 20 \) MeV and \( x = 10 \) we get \( r_2 = -4.5 \times 10^{-2} \) and \( r_4 = 6.66 \times 10^{-4} \). Behavior of \( r_w(x, y) \) for \( x = 10 \) in units \( S/x^4 \approx 18.6 \text{Mev}^3/m^3 \) is presented in fig.3. The coefficients \( r_2(x) \) and \( r_4(x) \) in the same units are presented in figs.4 and fig.5. The corrections quickly rise with \( x \) (remember the factor \( x^4/S \)) but they remain reasonably small till the freezing of the annihilation of tau neutrinos.

Comparing the annihilation rate for \( \nu_\tau \) in kinetic equilibrium with the rate of annihilation of \( \nu_\tau \) with the spectrum calculated here, we conclude that the frozen number density of mu-tau can be bigger by about 10% than that found in the standard case. However, calculating the spectrum, we have neglected the annihilation of \( \bar{\nu}_\tau \nu_\tau \) and this process may have an essential influence on the spectrum and on the final result on the frozen abundance. We will take this into account in the subsequent work.

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A Appendix.

Integration in the r.h.s. of eq.(4) can be done in the following direct way. First using \( \delta^3 \)-function we integrate over \( d^3k \) so that \( \vec{k} = \vec{k}' - \vec{p} + \vec{p}' \). Correspondingly
\[ \omega^2 = \vec{k}'^2 = \omega^2 - 2\omega'R \cos \theta_k + R^2 \text{ where } R = |\vec{p} - \vec{p}'| \text{ and } \theta \text{ is the angle between } \vec{R} \text{ and } \vec{k}'. \]

Since \( d\cos \theta \), which comes from \( d^3k' \), can be whiten as

\[ d\cos \theta_k = -d\omega/\omega'R \quad (A1) \]

we can easily integrate over \( d\omega \) using \( \delta(\omega - \omega' + E - E') \). The limits of the integration over \( \omega' \) are determined by the condition \( |\cos \theta| < 1 \) so we get

\[ \int \frac{d^3k}{\omega'} \frac{d^3k'}{\omega'} \delta^4(p + k - p' - k') = \frac{2\pi}{R} \int_{\omega_{\text{min}}}^{\infty} d\omega' \langle \ldots \rangle_{\phi_k} \quad (A2) \]

where \( \omega_{\text{min}} = (R + E - E')/2 \) and the expression in the angle brackets means averaging over the polar angle of the vector \( \vec{k}' \). For a constant amplitude there is no dependence on this angle.

In the remaining integration over \( d^3p' \) we change the integration over the azimuthal angle to the integration over \( R \) in accordance with the relation \( d\cos \theta_p = -RdR/\omega'p' \). Correspondingly we get

\[ \int \frac{d^3p'}{E'} = (2\pi) \int_0^{\infty} \frac{dp'}{pE'} \int_{R_{\text{min}}}^{R_{\text{max}}} dR \langle \ldots \rangle_{\phi_p} \quad (A3) \]

where, as above the averaging over \( p'-\text{polar angle is done. The limits of integration in } R \text{ are } R_{\text{min}} = |p - p'| \text{ and } R_{\text{max}} = p + p'. \]

If the amplitude of the scattering is constant and if the light particle distribution functions are known, \( f_0 = \exp(-E/T) \), we can easily make all the integration but one (in \( dp' \)) and obtain eq.(4). For the case of weak interactions the amplitude \( (18) \) is a function of momenta and the integration is more complicated. We will make it in the nonrelativistic limit, neglecting terms of the order of \( v^2 = p^2/E^2 \) in the amplitude. In this limit \( |A_w|^2 = K(E/m)^2((\omega^2 + \omega'^2)/m^2) \). The absence of the angular dependence in the amplitude simplifies the calculations very much. Performing the integration in the way described above, we obtain for the r.h.s. of eq.(4) (with the substitution
The following expression:

\[
(r.h.s.) = \frac{KE}{128\pi^3 pm^4} \int_0^\infty \frac{dp'p'}{E'} \int_{|p-p'|}^{p+p'} dR \int_{\omega_{\min}}^{\infty} d\omega' e^{(\omega'+E')/T} \frac{1}{\omega^2 + \omega'^2} \left[ C(p') - C(p) \right]
\]

(26)

where \(\omega = \omega' + E - E'\). Making a trivial integration over \(\omega\) and neglecting terms proportional to \((E - E')^2 \sim m^{-4}\) and also neglecting the integral similar to the last one in eq.(13) we arrive to eq.(20).

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Figure captions.

Fig.1. Relative deviation from the thermal equilibrium of the normalized \( y \)-derivative of \( C_1(x, y) \) for the case of a constant amplitude of elastic scattering for fixed \( x = 10 \) and running \( y, r'(10, y) \). See the definition of \( r' \) at the end of Sec. 2.

Fig.2. The same as in Fig.1 but for fixed \( y = 3 \) and running \( x, r'(x, 3) \).

Fig.3. The correction to the spectral density of nu-tau for \( x = 10, r_w(10, y) \) in units \( S/x^4 \) (see the end of Sec.4).

Fig.4. Taylor expansion coefficient \( r_2(x) \) in units \( S/x^2 \).

Fig.5. Taylor expansion coefficient \( r_4(x) \) in units \( S/x^2 \).