One-point functions in integrable coupled minimal models

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Abstract

We propose exact vacuum expectation values of local fields for a quantum group restriction of the $C_2^{(1)}$ affine Toda theory which corresponds to two coupled minimal models. The central charge of the unperturbed models ranges from $c = 1$ to $c = 2$, where the perturbed models correspond to two magnetically coupled Ising models and Heisenberg spin ladders, respectively. As an application, in the massive phase we deduce the leading term of the asymptotics of the two-point correlation functions.

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1 Introduction

The vacuum expectation values (VEV)s of local fields play an important role in quantum field theory (QFT) and in statistical mechanics [1, 2]. In statistical mechanics the VEVs determine the “generalized susceptibilities”, i.e., the linear response of a system to external fields. Furthermore, the VEVs provide all the information about correlation functions in QFT defined as a perturbed conformal field theory (CFT) that is not accessible through a direct calculation in conformal perturbation theory [3]. A few years ago, some important progress was made in the calculation of such quantities in integrable (1+1) QFT. In ref. [4], an explicit expression for the VEVs of the exponential field in the sinh-Gordon and sine-Gordon models was proposed. In ref. [5] it was shown that this result can be obtained using the “reflection amplitude” [6] of the Liouville field theory. This method was also applied in the so-called Bullough-Dodd model with real and imaginary coupling. It is known that $c < 1$ minimal CFT perturbed by the operators $\Phi_{12}$ and $\Phi_{13}$ can be obtained by a quantum group (QG) restriction of the sine-Gordon [7] and imaginary Bullough-Dodd model [8] with special values of the coupling. The VEVs of primary fields were then calculated. The same method was applied later to integrable QFTs involving more than one field. For instance, the VEV for a two-parameter family of
integrable QFTs [9] gave rise to the VEV of local operators in parafermionic sine-Gordon models and in integrable perturbed \( SU(2) \) coset CFT [10]. The VEV of simply-laced affine Toda field theories (ATFT)s is known for a long time [11] and the case of non-simply laced dual pairs was recently studied in [12, 13] for which a general expression for the VEVs was derived.

Such perturbed CFTs have recently attracted much attention in condensed matter physics, such as in the context of point contacts in the fractional quantum Hall effect and impurities in quantum wires [14, 15]. In such cases the property of integrability has provide a non-perturbative answer for experimentally important strongly interacting solid state physics problems [13, 16]. Particularly, on-shell results were obtained using exact relativistic scattering and related form factor techniques [17, 18].

In this paper, we are interested in exact off-shell results for two coupled conformal field theories\(^1\) for which the inter-layer coupling preserves integrability. The on-shell dynamics of these models were studied in [19, 20, 21].

Let us briefly recall the ideas of [19, 20, 21]. We consider two planar systems corresponding to two coupled minimal models \( \mathcal{M}_{p/p'} \) which interact through a relevant operator which leads to an integrable theory. The resulting action can be written:

\[
\mathcal{A} = \mathcal{M}_{p/p'} + \mathcal{M}_{p/p'} + \lambda \int d^2 x \Phi_{12}^{(1)} \Phi_{12}^{(2)}
\]

or

\[
\tilde{\mathcal{A}} = \mathcal{M}_{p/p'} + \mathcal{M}_{p/p'} + \hat{\lambda} \int d^2 x \Phi_{21}^{(1)} \Phi_{21}^{(2)},
\]

where we denote respectively \( \Phi_{12}^{(1)} \Phi_{12}^{(2)} \) and \( \Phi_{21}^{(1)} \Phi_{21}^{(2)} \) as two specific primary operators of each unperturbed minimal models and where the parameters \( \lambda \) and \( \hat{\lambda} \) characterize the strength of the interaction. Here, we will be interested in exact one-point functions in such system.

In section 2 we introduce the notations and those known results which are useful for our purpose. In section 3 using the exact result for the VEV of the \( C_2^{(1)} \) ATFT derived in [13], we deduce the exact VEV \( \langle \Phi_{12}^{(1)} (x) \Phi_{12}^{(2)} (x) \rangle \) for any values of \( (r, s), (r', s') \) in the model with action (1). To do it, we relate the parameter \( \lambda \) in (1) to the masses of the particles and we perform the QG restriction of the \( C_2^{(1)} \) ATFT with imaginary coupling which leads to the model (1). For \( (r, s) = (r', s') = (1, 2) \) this VEV can be calculated exactly as well as the bulk free energy. The specific case \( \mathcal{M}_{3/4} + \mathcal{M}_{3/4} \) coupled by \( \Phi_{12}^{(1)} \Phi_{12}^{(2)} \) is considered in details. It corresponds to two layer Ising models coupled by their magnetization operator \( \sigma^{(1)} \sigma^{(2)} \). The previous approach is also extended to the model described by action (2).

In section 4, we extract some limited information about the asymptotics of two-point correlation functions between any pairs of primary operators which belong to the same

\(^1\)In literature, the first example of such integrable coupled models was studied in [19].
or different unperturbed models. More precisely, we will distinguish four different cases:

\[(a) : \langle \Phi_{rs}^{(1)}(x)\Phi_{r's'}^{(2)}(y) \rangle \text{ for } |x-y| \to 0,\]

\[(b) : \langle \Phi_{rs}^{(1)}(x)\Phi_{r's'}^{(2)}(y) \rangle \text{ for } |x-y| \to \infty,\]

\[(c) : \langle \Phi_{rs}^{(1)}(x)\Phi_{r's'}^{(1)}(y) \rangle \text{ for } |x-y| \to 0 \text{ and } i \in \{1, 2\},\]

\[(d) : \langle \Phi_{rs}^{(i)}(x)\Phi_{r's'}^{(i)}(y) \rangle \text{ for } |x-y| \to \infty \text{ and } i \in \{1, 2\}\]

which are depicted in figure 1. We finally give some numerical results for various examples of coupled minimal models such as two energy-energy coupled tricritical Ising, two coupled $A_5$-RSOS models and two energy-energy coupled 3-state Potts model. Perspective and conclusions follow in this final section.

\[\text{Figure 1 - Two coupled two dimensional models. Short distance results are obtained by taking the limit } \epsilon \to 0.\]

### 2 Coupled minimal models as restricted $C_2^{(1)}$ ATFT

The ATFT with real coupling $b$ associated with the affine Lie algebra $C_2^{(1)}$ is described by the action in the Euclidean space:

\[
\mathcal{A} = \int d^2x \left[ \frac{1}{8\pi} (\partial_\nu \varphi)^2 + \mu' e^{-2b\varphi_1} + \mu' e^{2b\varphi_2} + \mu e^{b(\varphi_1 - \varphi_2)} \right]
\]

where we chose the convention that the length squared of the long roots is four. As the different vertex operators do not renormalize in the same way, we introduced two scale parameters $\mu$ and $\mu'$. The fields in eq. (3) are normalized such that:

\[
\langle \varphi_i(x)\varphi_j(y) \rangle = -\delta_{ij} \ln |x-y|^2.
\]

This model possess two fundamental particles with mass $M_a$ which depends on one parameter $m$: 

where we introduced the “deformed” Coxeter number \( H = h(1 - B) + h'\gamma B \) with \( B = \frac{b_2^2}{1 + b^2} \) and \( h = 4, h' = 6 \) are respectively the Coxeter and dual Coxeter numbers. The exact relation between \( m, \mu \) and \( \mu' \) was found in [12] and is given by:

\[
(−πμγ(1 + b^2))(−πμ'γ(1 + 2b^2)) = 2\left(\frac{\Gamma((1 - B)/H)\Gamma(1 + B/H)}{\Gamma(1/H)}\right)^\frac{\mu}{H}. \tag{6}
\]

In ATFT approach to perturbed CFT, one usually identifies the perturbation with the affine extension of the Lie algebra \( \mathcal{G} \). Instead, the perturbation will be associated here with the standard (length 2) root of \( C_2^{(1)} \). Removing the last term in the action (3) leaves a \( D_2 = SO(4) = SU(2) \otimes SU(2) \) model, i.e. two decoupled Liouville models. To associate the two first terms of the \( C_2^{(1)} \) Toda potential to two decoupled conformal field theories, we first introduce for each one a background charge at infinity. Then, the total stress-energy tensor \( T(z) \) is written

\[
T^{(i)}(z) = -\frac{1}{2}(\partial \varphi_i)^2 + Q_i \partial^2 \varphi_i \quad \text{for} \quad i \in \{1, 2\} \tag{7}
\]

ensures the local conformal invariance of the \( D_2 \) model for the choice \( Q_2 = -Q_1 = b + 1/2b \). With our conventions\(^3\), the exponential fields

\[
e^{a_i \varphi_i(x)} \quad \text{for} \quad i \in \{1, 2\} \tag{8}
\]

are spinless conformal primary fields of each Liouville model with conformal dimensions

\[
\Delta(e^{a_i \varphi_i(x)}) = -\frac{a_i^2}{2} + a_i Q_i \quad \text{and} \quad i \in \{1, 2\}. \tag{9}
\]

In particular, the exponential fields \( e^{-2b\varphi_1} \) and \( e^{2b\varphi_2} \) have conformal dimensions 1. As is well known, the “minimal model” \( \mathcal{M}_{p/p'} \) with central charge \( c = 1 - 6\frac{(\mu - \mu')^2}{pp'} \) can be obtained from the Liouville case. Consequently, the \( D_2 \) CFT can be identified with two decoupled minimal models by the substitution:

\[
b \rightarrow i\beta, \quad \mu \rightarrow -\mu, \quad \mu' \rightarrow -\mu'. \tag{10}
\]

and the choice:

\[
\beta^2 = \beta^2_+ = p/2p' \quad \text{or} \quad \beta^2 = \beta^2_- = p'/2p \quad \text{with} \quad p < p'. \tag{11}
\]

Similarly, the primary operators of each minimal model \( \mathcal{M}_{p/p'} \) are obtained through the substitution \( a_j \rightarrow i\eta_j \) for \( j = 1, 2 \) in (8). With these substitutions, the conformal dimension of the perturbing operator becomes:

\[
\Delta_{pert} = \Delta(e^{i\beta(\varphi_1-\varphi_2)}) = 3\beta^2 - 1. \tag{12}
\]

\(^2\)Differently to the simply laced case for which the mass ratios take the classical values, the mass ratios for non-simply laced case get quantum corrections.

\(^3\)Here, the length of the longest root is chosen to be 4.
As long as we consider a relevant perturbation, we are restricted to choose $\beta^2 < 2/3$. In the following we will consider only the cases for which this condition is satisfied, in particular $\beta^2 = \beta_+^2$.

We define respectively $\{\Phi_{rs}^{(1)}\}$ and $\{\Phi_{r's'}^{(2)}\}$ as the two sets of primary fields with conformal dimensions:

$$\Delta_{rs} = \frac{(p'r - ps)^2 - (p - p')^2}{4pp'} \quad \text{for} \quad 1 \leq r < p, \quad 1 \leq s < p' \quad \text{and} \quad p < p'.$$

They are simply related to the vertex operators of each minimal model through the relation:

$$\Phi_{rs}^{(i)}(x) = N_{rs}^{(i)-1} \exp(i\eta_{rs}^{(i)}\varphi_i(x)) \quad \text{with} \quad \eta_1^{rs} = -\eta_2^{rs} = \frac{(1 - r)}{2\beta} - (1 - s)\beta,$$

where we have introduced the normalization factors $N_{rs}^{(i)}$ for each model. These numerical factors depend on the normalization of the primary fields. Here, they are chosen in such a way that the primary fields satisfy the conformal normalization condition:

$$<\Phi_{rs}^{(i)}(x)\Phi_{rs}^{(i)}(y) >_{\text{CFT}} = \frac{1}{|x - y|^{4\Delta_{rs}}} \quad \text{for} \quad i \in \{1, 2\}.$$

For further convenience, we write these coefficients $N_{rs}^{(i)} = N^{(i)}(\eta_i^{rs})$ where:

$$N^{(1)}(\eta) = \left[ -\pi \mu' \gamma(-2\beta^2) \right]^{1/2} \sqrt{\frac{\Gamma(2\beta^2 + 2\eta\beta)\Gamma(1/2 + \eta\beta)\Gamma(2 - 2\beta^2)\Gamma(2 - 1/2 + \eta\beta)\Gamma(1/2\beta)}{\Gamma(2 - 1/2\beta + \eta/\beta)\Gamma(2\beta)\Gamma(2\beta^2)}}$$

and $N^{(2)}(\eta) = N^{(1)}(-\eta)$.

For imaginary values of the coupling $b = i\beta$, the $C_2^{(1)}$ ATFT possesses complex soliton solutions which interpolate between the degenerate vacua. This QFT possesses the QG symmetry associated to $U_q(D_3^{(2)})$ - as we will recall in the next section. In [23] the $S$-matrix of the $B_2^{(1)}$ ATFT was constructed using the $U_q(A_3^{(2)})$ QG symmetry of this QFT. In particular, using the bootstrap procedure, the authors deduced the breather-breather $S$-matrix. It was also shown that a breather-particle identification holds by comparing the $S$-matrix elements of the lowest-breathers (breathers with lowest mass) with the $S$-matrix elements for the quantum particles in real ATFT. Following the conventions of [12], the particle spectrum in real $B_2^{(1)}$ ATFT is given by $M_1 = 2m \sin(\pi/H)$ and $M_2 = m$. Using the results of [23] we find the identification:

$$m = 2M \sin\left(\frac{\pi \xi}{4 - 2\xi}\right) \quad \text{with} \quad \xi = \frac{\beta^2}{1 - \beta^2}$$

where we denote $M$ as the mass of the lowest kink. Eq. (16) holds for our case due to the identification $B_2^{(1)} \equiv C_2^{(1)}$ and $A_3^{(2)} \equiv D_3^{(2)}$.  


3 Expectation values in coupled minimal models

In ref. [12, 13] we derived the exact VEV $G(a) = \langle \exp(a \cdot \varphi)(x) \rangle$ for all non-simply laced ATFTs using the so-called “reflection relations” which relate different fields with the same quantum numbers. We refer the reader to these papers for more details. Although the model (1) is very different from the $C_2^{(1)}$ ATFT (3) in its physical content (the model (1) contains solitons and excited solitons), there are good reasons to believe that the expectation values obtained in the real coupling case provide also the expectation values for imaginary coupling. The calculation of the VEVs in both cases ($b$ real or imaginary) within the standard perturbation theory agree through the identification $b = i \beta$. Following the analysis done for the Bullough-Dodd model [5], it is then straightforward to obtain the VEV of primary operators which belong to different minimal models. With the substitutions (10) one gets $G(\eta) = G(i \eta)$:

$$G(\eta) = \left[ \pi^2 \mu \mu' \gamma(1 + \xi) \gamma(1 - \xi) \frac{1 + \xi}{1 + \xi} \right]^{(1+\xi)/2} \left[ \pi \mu' \gamma(1 - \xi) \gamma(1 + \xi) \right]^{(1+\xi)/2} \left( 1 + \eta^2 \right)^{-\eta^2} \times \exp \left[ \int_0^\infty \frac{dt}{t} \left( \frac{\chi(\eta, t)}{\sinh((1 + \xi)t) \sinh(2t\xi)} \sinh((4 - 2\xi)t) - \eta^2 e^{-2t} \right) \right]$$

with

$$\chi(\eta, t) = 2 \left[ \sinh((\eta_1 - \eta_2)\beta(1 + \xi)t) \sinh(((\eta_1 - \eta_2)\beta(1 + \xi) + 2\xi - 2)t) + \sinh^2((\eta_1 + \eta_2)\beta(1 + \xi)t) \sinh(t) \cosh(t\xi) \right. + \left. \sinh(2\eta_1\beta(1 + \xi)t) \sinh((2\eta_1\beta(1 + \xi) - 2)t) + \sinh(2\eta_2\beta(1 + \xi)t) \sinh((2\eta_2\beta(1 + \xi) + 2)t) \right] \sinh((1 - \xi)t).$$

As usual we denote $\gamma(x) = \Gamma(x)/\Gamma(1 - x)$. The integral in (17) is convergent if:

$$- \frac{2}{(1 + \xi)} < (\eta_1 + \eta_2)\beta < \frac{2}{(1 + \xi)} \quad \text{and} \quad -1 < (\eta_1 - \eta_2)\beta < \frac{3 - \xi}{(1 + \xi)},$$

and is defined via analytic continuation outside this domain. In (17) we defined $\Delta_1$ and $\Delta_2$, the conformal dimensions in the imaginary coupling case by $\Delta_1 = \Delta(e^{i\eta_1 \varphi_1}) = \frac{\eta_1^2}{2} + \frac{\eta_2^2}{2} (\frac{\xi - 1}{2})$ and similarly for $\Delta_2$ with the change $\eta_1 \rightarrow -\eta_2$.

If we want to express the final result for the VEV in terms of the parameter $\lambda$ in the action (1), we need the exact relation between $\lambda$ and the parameters $\mu, \mu'$ in the $C_2^{(1)}$ ATFT with imaginary coupling. We obtain:

$$\lambda = \frac{\pi \mu \mu'}{(4\beta^2 - 1)^2} \gamma(4\beta^2) \gamma^2(1 - 2\beta^2),$$

which corresponds to $\lambda = -\mu N^{(1)}(\beta) N^{(2)}(-\beta)$. 

6
Like any other ATFT, the model (3) for imaginary values of the coupling has non-local conserved charges \( \{Q, \overline{Q}_k\} \) for \( k = 0, 1, 2 \) generated respectively by the purely chiral and anti-chiral components:

\[
J_{e^\gamma_k}(z) = e^{-\frac{i}{\beta}e_k^\gamma \varphi(z)} \quad \text{and} \quad \overline{J}_{-e^\gamma_k}(\overline{z}) = e^{-\frac{i}{\beta}e_k^\gamma \overline{\varphi}(\overline{z})},
\]

where the fundamental vector field \( \varphi(z, \overline{z}) = \varphi(z) + \overline{\varphi}(\overline{z}) \) and \( \{e^\gamma_k\}, k = 0, 1, 2 \) is the set of dual simple roots of the non-simply laced affine Lie algebra \( C_2^{(1)} \). We also define the topological charge:

\[
H_k = -\frac{\beta}{2i\pi} \int d^2x \ e_k^\gamma \partial_x \varphi(x, t).
\]

Using the equal-time braiding relations for all \( x, y \):

\[
J_{e^\gamma_k}(x, t)\overline{J}_{-e^\gamma_i}(y, t) = q_i^{-a_{kl}}\overline{J}_{-e^\gamma_i}(y, t)J_{e^\gamma_k}(x, t) \quad \text{with} \quad q_i = e^{-\frac{2i\pi}{|e_k|^2\beta^2}},
\]

where \( a_{kl} = \frac{2e_k \cdot e_l}{|e_k|^2} \) denotes the extended Cartan matrix of \( C_2^{(1)} \), one can show that the charges \( \{Q_k, \overline{Q}_k, H_k\} \) for \( k = 0, 1, 2 \) satisfy the quantum universal enveloping algebra \( U_q(D_3^{(2)}) \) with deformation parameter \( q \equiv q_0 = e^{-\frac{i\pi}{2\beta^2}} \). If we express these generators in terms of the standard Chevalley basis \( \{E_k^+, E_k^-, H_k\} \) by:

\[
Q_k \equiv E_k^+q^{H_k} \quad \text{and} \quad \overline{Q}_k \equiv E_k^-q^{H_k} \quad \text{for} \quad k = 0, 1, 2,
\]

then we have:

\[
[H_k, H_l] = 0, \quad [H_k, E_i^\pm] = \pm a_{kl}E_i^\pm, \quad [E_k^+, E_l^-] = \delta_{kl} \frac{q^{2H_i} - q^{-2H_i}}{q_l^2 - q_l^{-1}}.
\]

The \( U_q(D_3^{(2)}) \) has two subalgebras \( U_q(D_2) \) as well as \( U_q^2(C_2) \) [23] where the subalgebra \( U_q(D_2) \) is generated by \( \{Q_0, \overline{Q}_0, Q_2, \overline{Q}_2, H_0, H_2\} \). As was found in [23] the \( S \)-matrix in the unrestricted form acts as an intertwiner on the modules of the \( U_q(D_2) \) representations:

\[
S_{a,b}(\theta) : \mathcal{V}_{\rho_a} \otimes \mathcal{V}_{\rho_b} \longrightarrow \mathcal{V}_{\rho_b} \otimes \mathcal{V}_{\rho_a}
\]

where \( \mathcal{V}_\rho \) is the module with highest weight \( \rho \). The scattering of two solitons of species \( a \) and \( b \) is then described by \( S_{a,b}(\theta_{12}) \) with \( \theta_{12} \) being their relative rapidity. There are two such fundamental multiplets denoted \( \{4\} \) and \( \{6\} \) in ref. [21]. In addition, there are also scalar bound states and excited solitons depending on the values of \( (p, p') \) chosen.

To understand the restricted \( C_2^{(1)} \) (denoted \( RC_2^{(1)} \) below), we use the general framework of superselection sectors (for details see [24, 25, 26]). The model (4) is a perturbation of
the two minimal models by the operator \( \Phi_{12}^{(1)} \Phi_{12}^{(2)} \). Each minimal model contains a finite number of primary fields (13,14). Using the superselection criterion for the present case

\[
\left( \Phi_{2j+1}^{(1)}(z) \Phi_{2j+1}^{(2)}(z) \right) \left( \Phi_{12}^{(1)}(w) \Phi_{12}^{(2)}(w) \right) \sim \frac{1}{(z-w)^l} \Phi_{2j+1}^{(1)}(w) \Phi_{2j+1}^{(2)}(w), \quad l \in \mathbb{Z}
\]

we find \( j + \tilde{j} \in \mathbb{Z} \) where \( j \tilde{j} \) denote the representations of \( U_q(D_2) \) with \( j \) the spin-\( j \) representation of \( SU(2) \) with dimension \( 2j + 1 \) (and similarly for \( \tilde{j} \)). If \( q \) is a root of unity i.e if eq. (11) is satisfied then using the Coulomb gas representation condition (13)

we find

\[
\sum_{j} \quad 0 \leq j \leq p/2 - 1 \quad \text{and} \quad 0 \leq \tilde{j} \leq p/2 - 1.
\]

The superselection sectors \( \mathcal{H}_{j\tilde{j}}^{RC_2^{(1)}} \) of the \( RC_2^{(1)} \) model (4) are thus :

\[
\mathcal{H}_{j\tilde{j}}^{RC_2^{(1)}} = \sum_{(j\tilde{j}) \in \{0,1/2,\ldots,p/2-1\}} \mathcal{H}_{j\tilde{j}}^{RC_2^{(1)}} \quad \text{with} \quad j + \tilde{j} \in \mathbb{Z}.
\]

As shown in [20, 21] for the unitary series \( (p' = p + 1) \), after the quantum group restriction (11) for \( p > 3 \) the fundamental solitons in the \{4\} and \{6\} representation of \( U_q(D_2) \) become the RSOS kink \( K_{j\tilde{j}j_2}^{j\tilde{j}j_1} \). These kinks interpolate between different vacua \( |0_{j\tilde{j}j_1} > \) and \( |0_{j\tilde{j}j_2} > \) which are connected using the \( U_q(sl_2) \) fusion ring at \( q = -e^{-i\pi/2} \):

\[
\min_{j_2 = |j_1 - j|} j_1 \times j_2 = \sum_{j_2 = |j_1 - j|} j_2
\]

and similarly for \( \tilde{j} \). However, for \( p = 3 \), \( (j, \tilde{j}) \in \{0, 1/2\} \) and then the \{6\} is projected out of the spectrum, leaving only the \{4\}. We refer the reader to [20, 21] for more details.

From the previous remarks and the identification \( D_2 = SO(4) = SU(2) \otimes SU(2) \), by analogy with (3) the primary fields \( \Phi_{1s}^{(1)} \) and \( \Phi_{1s'}^{(2)} \) commute with the generators in (25) (for \( k = 0, 2 \)) of the subalgebra \( U_q(D_2) \subset U_q(D_3) \). If one interpret the fields \( \Phi_{2j+1}^{(1)}(z) \Phi_{2j+1}^{(2)}(z) \) as the highest component fields in the multiplet, it can be shown that the primary operators \( \Phi_{r}^{(1)} \) and \( \Phi_{r'}^{(2)} \) are not invariant with respect to \( U_q(D_2) \). Together with some non-local fields they form finite-dimensional representation of this algebra. Consequently, the VEV in (1) should take into account the factor \( d_{rs, rs'}^{j\tilde{j}} \) coming from the QG restriction of the QFT (3). Following the conjecture of (3) it takes the form :

\[
d_{rs, rs'}^{j\tilde{j}} = \frac{\sin \left( \frac{\pi (2j+1)}{p} |p'r - ps| \right) \sin \left( \frac{\pi (2\tilde{j}+1)}{p} |p'r' - ps'| \right)}{\sin \left( \frac{\pi (2j+1)}{p} (p' - p) \right) \sin \left( \frac{\pi (2\tilde{j}+1)}{p} (p' - p) \right)}.
\]

\[\text{We only consider the holomorphic part of the primary operators but keep the same notation.}\]
Using the notations introduced in the previous section, and eqs. (14), (17), the outcome for the VEV between different primary operators is:

\[
<0_{jj}| \Phi_{rs}^{(1)}(x) \Phi_{r's'}^{(2)}(x)|0_{jj}> = \sum_{r,s,r',s'} d^s_{rr,s'} \left[ \frac{-\pi \lambda \gamma \left( \frac{1}{1+\xi} \right) \left( 1+\xi \right) \frac{1+\xi}{\left( 1+\xi \right)}}{\gamma \left( \frac{1}{1+\xi} \right) \gamma \left( \frac{1}{1+\xi} \right)} \right]^{\frac{1+\xi}{2}} (\Delta_r + \Delta_{r',s'})
\]

\[
\times \exp \mathcal{Q}_{12}((1+\xi)r - 2\xi s, (1+\xi)r' - 2\xi s').
\]  

The function \(\mathcal{Q}_{12}(\theta, \theta')\) for \(|\theta \pm \theta'| < 4\xi\) and \(\xi > \frac{1}{3}\) is given by the integral:

\[
\mathcal{Q}_{12}^{\mathcal{I}}(\theta, \theta') = \int_0^{\infty} dt \left( \frac{\Psi_{12}(\theta, \theta', t)}{\sinh((1+\xi)t) \sinh(2t\xi) \sinh((4-2t\xi)t)} - \frac{\theta^2 + \theta'^2 - 2(1-\xi)^2}{4\xi(\xi+1)} e^{-2t} \right)
\]

with

\[
\Psi_{12}(\theta, \theta', t) = \left[ \cosh((\theta + \theta')t) \cosh((\theta - \theta')t) - \cosh((2 - 2\xi)t) \right] \sinh((1 - \xi)t) \cosh((4 - 2\xi)t) \sinh(t) \cosh(t\xi).
\]

and defined by analytic continuation outside this domain. Notice that eq. (31) satisfies:

\[
<0_{jj}| \Phi_{rs}^{(1)}(x) \Phi_{r's'}^{(2)}(x)|0_{jj}> = <0_{jj}| \Phi_{p-r}^{(1)}(x) \Phi_{p-r'}^{(2)}(x)|0_{jj}>
\]  

A particular case of eq. (31) is the expectation value of the perturbing operator which can be calculated explicitly to give the result:

\[
<0_{jj}| \Phi_{12}^{(1)}(x) \Phi_{12}^{(2)}(x)|0_{jj}> = \frac{1}{\lambda} \left[ -\pi \lambda \gamma \left( \frac{1}{1+\xi} \right) \frac{1+\xi}{\left( 1+\xi \right)} \right]^{\frac{1+\xi}{2}} \frac{2\pi^{\frac{3}{2}}}{\pi(\xi-2)} \frac{\gamma(\xi)}{\gamma(\frac{1}{1+\xi})}.
\]  

By using the exact relation between the mass parameter \(\mathfrak{m}\) and the mass of the kink \(M\) (16) and eqs. (3), (19) we immediately obtain the relation between \(M\) and \(\lambda\):

\[
M = \frac{2\pi^{\frac{3}{2}} \Gamma \left( \frac{\xi}{4-2\xi} \right) \Gamma \left( \frac{1}{1+\xi} \right)}{\pi(\xi-2)} \left[ -\pi \lambda \gamma \left( \frac{1}{1+\xi} \right) \frac{1+\xi}{\left( 1+\xi \right)} \right]^{\frac{1+\xi}{2}}.
\]  

Consequently, according to eqs. (19), (23), (33) and \(\beta^2 < 2/3\), the perturbed CFTs develop a massive spectrum for \((3m(\lambda) = 0)\):

\[
(i) \quad 0 < \xi < 1/3, \quad \lambda > 0 \quad \text{i.e.} \quad 0 < \frac{p}{p'} < \frac{1}{2},
\]

\[
(ii) \quad 1/3 < \xi < 1, \quad \lambda < 0 \quad \text{i.e.} \quad \frac{1}{2} < \frac{p}{p'} < 1.
\]

where \(\xi = \frac{p}{2p' - p}\). In particular, the condition (ii) is always satisfied for the coupled unitary minimal models defined by (11).
Finally, the expectation value (33) can be used to derive the bulk free energy:

\[ f_{12} = - \lim_{V \to \infty} \frac{1}{V} \ln Z, \tag{36} \]

where \( V \) is the volume of the 2D space and \( Z \) is the singular part of the partition function associated with action (1). Using

\[ \partial_\lambda f_{12} = \langle 0 | \Phi_{12}^{(1)} \Phi_{12}^{(2)} | 0 \rangle, \tag{37} \]

and eqs. (33), (34), the result for the bulk free energy follows:

\[ f_{12} = - M^2 \sin\left(\frac{\pi}{4} - 2\xi\right) \frac{\sin\left(\frac{\pi}{4} - \xi\right)}{\sin\left(\frac{\pi(1+\xi)}{4}\right)}. \tag{38} \]

As was suggested in [20, 21], a particular case of the model (1) can be related to the SG theory at the reflectionless point as well as the \( D_8^{(1)} \) ATFT with imaginary coupling. Let us first consider the SG theory with action:

\[ A_{SG} = \int d^2 x \left[ \frac{1}{16\pi} (\partial_\nu \phi)^2 + \Lambda \cos(\hat{\beta} \phi) \right]. \tag{39} \]

This model is integrable and its on-shell solutions, i.e. spectrum of particles and \( S \)-matrix are well-known [27]. This theory contains soliton, antisoliton and soliton-antisoliton bound-state (breathers) \( B_n, n = 1, ..., 1/\xi \) where \( \xi = \frac{\hat{\beta}^2}{1-\hat{\beta}^2} \). The lightest bound-state \( B_1 \) coincides with the field \( \phi \) in the perturbative treatment of the QFT (39). Its mass is given by

\[ m_1 = 2M_{SG} \sin\left(\frac{\pi}{4} \xi \right) \]

where \( M_{SG} \) denotes the soliton mass of the SG model. For \( p = 3, p' = 4 \) the model (1) corresponds to two magnetically coupled Ising models. It is a \( c = 1 \) CFT perturbed by the operator \( \Phi_{12}^{(1)} \Phi_{12}^{(2)} \) with conformal dimension 1/8. Indeed, the VEV of this operator must be simply related to the VEV of \( \langle \cos(\hat{\beta} \phi) \rangle \) in the SG model for \( \hat{\beta}^2 = 1/8 \), which is also a \( c = 1 \) CFT with perturbing operator of conformal dimension 1/8. For this latter value, \( \xi = 1/7 \) and the SG model possesses 6 neutral excitations:

\[ m_a = 2M_{SG} \sin\left(\frac{\pi a}{14}\right), \quad a = 1, ..., 6; \tag{40} \]

\[ m_1 < m_2 < M_{SG} < m_3 < ... < m_6. \]

Similarly, for \( \hat{\beta}^2 = 3/8, \xi = 3/5 \) the spectrum of the model (1) was described in [21] and summarize here. Here, we denote the lowest mass by \( M \). If we compare the spectrum of each model, using the notations of [23] we have:

\[ m_1 \equiv M, \quad m_2 \equiv m_{B_1^{(1)}}, \quad M_{SG} \equiv m_{B_1^{(2)}}, \quad \text{etc...} \tag{41} \]

Furthermore, the bulk free energy of the SG model (39) is well known [28, 4] and given by:

\[ f_{SG} = -\frac{M^2_{SG}}{4} \tan\left(\frac{\pi \xi}{2}\right). \tag{42} \]
By expressing it in terms of the lightest particle \( m_1 \) using (40) and the relation
\[ 8 \sin(\pi/14) \sin(3\pi/14) \cos(\pi/7) = 1 \]
we find the same result as in eq. (38) for \( \beta^2 = 3/8 \), i.e. \( \xi = 3/5 \). Also, using the results of [1] for \( \beta^2 = 1/8 \), one has:

\[
\Lambda = -\frac{2\gamma(1/8)}{\pi} \left[ \frac{M_{SG}\sqrt{\pi}}{2} \frac{\Gamma(4/7)}{\Gamma(1/14)} \right]^\frac{7}{8}. \tag{43}
\]

Similarly, the VEV of the perturbing operator is given by:

\[
< \cos(\hat{\beta}\phi) >_{\beta^2=1/8} = 8 \frac{\Gamma(1/7)\Gamma(6/7)}{\Gamma^2(4/7)\Gamma^2(13/14)} \left[ \frac{\pi}{\gamma(1/8)} \right]^{8/7} (-\frac{\Lambda}{2})^{1/7}. \tag{44}
\]

Comparing eqs. (43), (44) with eqs. (33) and (34) respectively for \( \xi = 3/5 \) and using eq. (41), one find the relations:

\[
\Lambda \equiv 2\frac{1}{2} \lambda \quad \text{and} \quad \cos(\hat{\beta}\phi)|_{\beta^2=1/8} \equiv 2^{-\frac{1}{2}}\Phi_1^{(1)}\Phi_1^{(2)} \tag{45}
\]
in perfect agreement with eq. (B.21) in [1].

Let us now turn to the relation with the \( D_{8}^{(1)} \) ATFT. The calculation of the bulk free energy for the simply-laced \( D_{8}^{(1)} \) ATFT gives [28]:

\[
f_{D_{8}^{(1)}} \big|_{b} = \frac{m^2}{8} \frac{\sin(\frac{\pi}{14})}{\sin(\frac{\pi B}{14}) \sin(\frac{\pi(1-B)}{14})} \tag{46}
\]

where \( B \) is defined in section 2 and the mass of the particles are related with the mass parameter \( \overline{m} \) by: \( m_8^2 = m_7^2 = 2\overline{m}^2 \), \( m_8^2 = 8\overline{m}^2 \sin^2(\frac{\pi a}{14}) \) for \( a = 1, \ldots, 6 \). For imaginary coupling, one expects the particle-breather identification:

\[
8\overline{m}^2 \sin^2(\frac{\pi}{14}) = \left( 2M \sin(\frac{\pi \xi}{14}) \right)^2 \tag{47}
\]

where \( \xi \) is defined as in eq. (16). This yields:

\[
f_{D_{8}^{(1)}} \big|_{b=1/\beta} = -\frac{M^2}{16} \frac{\sin(\frac{\pi \xi}{14})}{\sin(\frac{\pi}{14}) \sin(\frac{\pi(1+\xi)}{14})}, \tag{48}
\]

which for \( \xi = 7 \) is in perfect agreement with eq. (33) evaluated at \( \xi = 3/5 \).

It is also interesting to study the behaviour of the model [1] for \( p' = p + 1 \) in the limit \( p \to \infty \). If we consider the sine-Gordon model with action (39) there is an equivalent description in terms of the massive Thirring model [30]:

\[
\mathcal{A}_{MT} = \int d^2x \left[ \overline{\psi} \gamma^\mu \partial_\mu \psi - M_{SG} \overline{\psi} \psi - \frac{g}{2} (\overline{\psi} \gamma^\mu \psi)^2 \right] \tag{49}
\]

\footnote{At \( \beta^2 = 1/8 \), the operator can be expressed [1, 29] in terms of two independent Ising spin fields, \( \sigma^{(i)} = \Phi_{12}^{(i)} \) for \( i = 1, 2 \).}
where $\psi, \bar{\psi}$ is a Dirac field. The four fermion coupling constant $g$ relates to $\hat{\beta}$ in (33) by $\frac{2}{\pi} = \frac{1}{2\beta^2} - 1$. Also the free fermion point is reached for $\beta^2 \rightarrow \frac{1}{2}$. Using the results of ref. [4] concerning the one-point function $<e^{i\alpha \phi}>$ in the SG model one gets:

\[
<\cos(\hat{\beta}\phi) > \rightarrow \lim_{\epsilon \rightarrow 0} \frac{M_{\text{SG}} \Gamma(\epsilon)}{\pi} \quad \text{and} \quad \Lambda \rightarrow -\frac{M_{\text{SG}}}{\pi} \quad \text{for} \quad \beta^2 \rightarrow \frac{1}{2}. \quad (50)
\]

Using the boson-fermion correspondence, one has the identification $\psi \psi \equiv \cos(\hat{\beta}\phi)/\pi$ and $<\psi\psi> \rightarrow \lim_{\epsilon \rightarrow 0} \frac{M_{\text{SG}} \Gamma(\epsilon)}{\pi}$ for $\hat{\beta}^2 \rightarrow \frac{1}{2}$. Using the results of ref. [4] concerning the one-point function $<e^{i\alpha \phi}>$ in the SG model one gets:

\[
<\psi\psi> \rightarrow \lim_{\epsilon \rightarrow 0} \frac{M_{\text{SG}} \Gamma(\epsilon)}{\pi} \quad \text{and} \quad \Lambda \rightarrow -\frac{M_{\text{SG}}}{\pi} \quad \text{for} \quad \beta^2 \rightarrow \frac{1}{2}. \quad (50)
\]

In fact, this situation is not really surprising: when $p \rightarrow \infty$ the coset algebra of each $M_{p/p+1}$ reduces to a level-1 $SU(2)$ current algebra. The two models are coupled via their primary fields in the spin $\frac{1}{2}$ representation of dimension $\frac{1}{4}$ [21]. As expected, in this limit the model becomes the free field theory of two complex massive fermions with undeformed $SO(4)$ symmetry.

Finally, let us consider the case $p' = p + 1$ and $p = 2$ in the action (1). Each minimal model is then identified to $M_{2/3}$ with central charge $c = 0$. For each model, the unitary representation is the vacuum one with conformal weight $\Delta_{11} = \Delta_{12} = 0$, for which cases we have the identifications $\Phi^{(i)}_{12} = \Phi^{(i)}$ for $i \in \{1, 2\}$. This corresponds to the choice $\beta^2 = \frac{1}{3}$, i.e. $\xi = \frac{1}{2}$ and, using eqs. (33), (34), (38), one can check that

\[
<0|\Phi^{(1)}_{12}\Phi^{(2)}_{12}|0> = 1 \quad \text{and} \quad f_{12} = \lambda, \quad (52)
\]

as expected.

Let us now turn to the model associated with action (2). Due to eq. (11) it corresponds to the choice $\beta^2 = \beta_2^2$. The condition $4p > 3p'$ guarantees that the perturbing operator is relevant. Then, the vacuum structure is expected to be similar to that of (1). From eqs. (30) and (31) and the substitutions:

\[
p \leftrightarrow p', \quad (r, r') \leftrightarrow (s, s'), \quad \xi \rightarrow \frac{1 + \xi}{3\xi - 1} \quad (53)
\]

the result for the VEV immediately follows:

\[
<0_{j j}|\Phi^{(1)}_{r s}(x)\Phi^{(2)}_{r' s'}(x)|0_{j j}> = \delta_{r s, r' s'} \left[ -\pi \hat{\lambda} \gamma(\frac{3\xi - 1}{4\xi})(2\xi)^{\frac{5\xi - 3}{2\xi}} \gamma(\frac{1}{2})(\frac{\xi - 1}{2\xi}) \right] \frac{4\xi - 3}{2\xi - 3}(\Delta_{rs} + \Delta_{r's'}) \times \exp Q_{21}((1 + \xi)r - 2\xi s, (1 + \xi)r' - 2\xi s'). \quad (54)
\]
The function \( Q_{21}(\theta, \theta') \) is given by the integral:
\[
Q_{21}(\theta, \theta') = \int_{0}^{\infty} \frac{dt}{t} \left( \frac{\Psi_{21}(\theta, \theta', t)}{\sinh((1 - \xi)t) \sinh(2t\xi) \sinh((3 - 5\xi)t)} - \frac{\theta^2 + \theta'^2 - 2(1 - \xi)^2 e^{-2t}}{4\xi(\xi + 1)} \right)
\]
with
\[
\Psi_{21}(\theta, \theta', t) = \left[ \cosh((\theta + \theta')t) \cosh((\theta - \theta')t) - \cosh((2 - 2\xi)t) \right] \times \sinh((1 - \xi)t) \cosh((3 - 5\xi)t)
- \left[ \cosh((\theta + \theta')t) + \cosh((\theta - \theta')t) - \cosh((2 - 2\xi)t) - 1 \right] \times \sinh((3\xi - 1)t/2) \cosh((1 + \xi)t/2)
\]
and defined by analytic continuation outside this domain. The prefactor associated with the QG restriction \( d_{rs,r's'}^{ij} = d_{s r,s'r'}^{ij} |_{p=p'} \). For \((r, s) = (r', s') = (2, 1)\) \((54)\) becomes:
\[
< 0_{jj} | \Phi_{21}^{(1)}(x) \Phi_{21}^{(2)}(x) | 0_{jj} > = \frac{1}{\lambda} \left[ -\frac{\pi \lambda \gamma(\frac{3\xi - 1}{4\xi})}{\gamma(\frac{1}{2}) \gamma(\frac{3\xi - 1}{2\xi})} \right] \frac{2^{\frac{3\xi - 1}{2\xi}}}{\pi \Gamma(\frac{2\xi - 3}{5\xi - 3})} \frac{\gamma(\frac{1 + \xi}{10\xi - 6}) \gamma(\frac{3\xi - 1}{10\xi - 6})}{\gamma(\frac{2\xi - 3}{5\xi - 3})} (55)
\]
with the relation between the mass of the lightest kink \( M \) and \( \hat{\lambda} \):
\[
M = \frac{2^{\frac{3\xi - 1}{2\xi}} \Gamma(\frac{1 + \xi}{10\xi - 6}) \Gamma\left(\frac{3\xi - 1}{10\xi - 6}\right)}{\pi \Gamma\left(\frac{2\xi - 3}{5\xi - 3}\right)} \left[ -\frac{\pi \lambda \gamma(\frac{3\xi - 1}{4\xi})}{\gamma(\frac{1}{2}) \gamma(\frac{3\xi - 1}{2\xi})} \right] \frac{2^{\frac{3\xi - 1}{2\xi}}}{\pi \Gamma(\frac{2\xi - 3}{5\xi - 3})} \frac{\gamma(\frac{1 + \xi}{10\xi - 6}) \gamma(\frac{3\xi - 1}{10\xi - 6})}{\gamma(\frac{2\xi - 3}{5\xi - 3})} (56)
\]
For the coupled minimal models defined by \((2)\), the massive phase corresponds to the domain:
\[
(iii) \quad \frac{3}{5} < \xi < 1, \quad \lambda < 0 \quad \text{i.e.} \quad \frac{3}{4} < \frac{p}{p'} < 1. (57)
\]
One also obtains the bulk free energy associated with action \((2)\):
\[
f_{21} = \frac{M^2 \sin(\frac{\pi(3\xi - 1)}{10\xi - 6}) \sin(\frac{\pi(\xi + 1)}{10\xi - 6})}{2 \sin(\frac{\pi(2\xi)}{5\xi - 3})} (58)
\]

4 Application and concluding remarks

Accepting the conjectures \((31)\) and \((54)\), one can easily deduce interesting predictions\(^6\) for the short and long distance asymptotic of two-point correlation functions in the model \((4)\) or \((2)\). For each case depicted in figure 1, we can express the result in terms of:
\[
< 0_{jj} | \Phi_{rs}^{(1)}(x) \Phi_{rs'}^{(2)}(x) | 0_{jj} > = G_{rs,r's'}^{ij} (59)
\]
\(^6\)The same method was applied in \((3)\) to obtain a prediction about the long distance asymptotic of two-point correlation function \( < \sigma_n \sigma_n >_{n \to \infty} \) in the XXZ spin chain.
given by eq. (31) or (54), respectively. First, using the short distance approximation (operator product expansion) and eq. (31) we get $$ e^{ia_1(x)}e^{ib_2(y)} \sim e^{i(a_1(x)+b_2(y))}.$$ Then, using the Coulomb gas representation of each primary operator (14) one has

$$ < 0_{jj} | \Phi^{(1)}_{rs}(x) \Phi^{(2)}_{r's'}(y) | 0_{jj} > \underset{|x-y| \to 0}{\longrightarrow} < 0_{jj} | \Phi^{(1)}_{rs}(x) \Phi^{(2)}_{r's'}(x) | 0_{jj} > .$$

(60)

In this limit - case (a) - , the two-point functions then become :

$$ < 0_{jj} | \Phi^{(1)}_{rs}(x) \Phi^{(2)}_{r's'}(y) | 0_{jj} > \underset{|x-y| \to 0}{\longrightarrow} G^{jj}_{rs,r's'}.$$  

(61)

Secondly, in the long distance approximation the asymptotic two-point function simply reduces to the product of two one-point functions as $$ < e^{ia_1(x)}e^{ib_2(y)} > \underset{|x-y| \to \infty}{\longrightarrow} < e^{ia_1(x)} > < e^{ib_2(y)} >$$ when $$|x-y| \to \infty.$$ Then we obtain - case (b) and (d) -

$$ < 0_{jj} | \Phi^{(1)}_{rs}(x) \Phi^{(2)}_{r's'}(y) | 0_{jj} > \underset{|x-y| \to \infty}{\longrightarrow} < 0_{jj} | \Phi^{(1)}_{rs}(x) \Phi^{(2)}_{r's'}(x) | 0_{jj} > ;$$

$$ < 0_{jj} | \Phi^{(1)}_{rs}(x) \Phi^{(1)}_{r's'}(y) | 0_{jj} > \underset{|x-y| \to \infty}{\longrightarrow} < 0_{jj} | \Phi^{(1)}_{rs}(x) \Phi^{(2)}_{r's'}(x) | 0_{jj} > .$$

Indeed, it gives :

$$ < 0_{jj} | \Phi^{(1)}_{rs}(x) \Phi^{(2)}_{r's'}(y) | 0_{jj} > \underset{|x-y| \to \infty}{\longrightarrow} G^{jj}_{rs,11}[\delta_{11}G^{jj}_{r's',11} + \delta_{22}G^{jj}_{r's',r's'}]$$  

for $$i \in \{1,2\}$$ (62)

with $$\delta_{ij}$$ the Kronecker symbol. Obviously similar results are obtained for the two-point function $$< 0_{jj} | \Phi^{(i)}_{rs}(x) \Phi^{(2)}_{r's'}(y) | 0_{jj} >$$ using the $$\mathbb{Z}_2$$ symmetry ($$1 \leftrightarrow 2$$).

Finally, using the fusion rules in the short distance approximation (the two primary fields belong to the same space of states) $$\Phi^{(i)}_{rs} \times \Phi^{(i)}_{r's'} \rightarrow \Phi^{(i)}_{r''s''},$$ the two-point function is expanded in terms of the one-point functions $$< 0_{jj} | \Phi^{(i)}_{r''s''}(x) | 0_{jj} > ;$$

$$< 0_{jj} | \Phi^{(2)}_{r''s''}(x) | 0_{jj} >$$ for $$i \in \{1,2\}$$ respectively. Then, we have for $$i = 1$$ - case (c)

$$ < 0_{jj} | \Phi^{(1)}_{rs}(x) \Phi^{(1)}_{r's'}(y) | 0_{jj} > \underset{|x-y| \to 0}{\longrightarrow} \sum_{r''s''} C^{r''s''}_{rs,r's'} |x-y|^{2(\Delta_{r''s''}-\Delta_{rs}-\Delta_{r's'})} G^{jj}_{r''s'',11}$$

(63)

and similarly for $$i = 2$$ where $$C^{r''s''}_{rs,r's'}$$ are the structure constants of the minimal model operator algebra $[33].$

**Example 1 : Two magnetically coupled Ising models**

It is now straightforward to compute different two-point correlation functions in one of the simplest (non-trivial) cases : two-magnetically coupled Ising models in the massive phase. It corresponds to $$\beta^2 = 3/8$$ i.e. $$\xi = 3/5$$ in $[31].$ In this case we have the identification $$\Phi^{(i)}_{12} = \sigma^{(i)}$$ with conformal dimension $$\Delta_{\sigma} = 1/16$$ - the spin operator - and $$\Phi^{(i)}_{13} = \epsilon^{(i)}$$ with conformal dimension $$\Delta_{\epsilon} = 1/2$$ - the energy operator. There are two degenerate ground
states $|20, 21\rangle$ denoted $|00\rangle \equiv |-\rangle$ and $|1\rangle/2 \equiv |+\rangle$. For simplicity, we write $<\pm|\pm\rangle \equiv |\pm\rangle$. Sometimes, the reflection relation:

$$G(\eta_1, \eta_2) = S_L(\eta_2)G(\eta_1, -\eta_2 + 2\beta - 1/\beta)$$ (64)

with

$$S_L(\eta_2) = \left[ -\pi\mu'\gamma(-2\beta^2) \right]^{-\eta_2\beta + (2\beta^2-1)/2} \times \frac{\Gamma(-2\eta_2\beta + 2\beta^2)\Gamma(\eta_2\beta + 1/2\beta^2)}{\Gamma(2 + 2\eta_2\beta - 2\beta^2)\Gamma(2 - \eta_2\beta - 1/2\beta^2)}$$

is useful for analytic continuation. Using eq. (31) and $d_{11,1s}^\pm = d_{1s,11}^\pm$ for all $s$ we obtain for instance:

$$<\sigma^{(i)} > \pm = G_{11,12}^\pm = \pm1.297197220...(-\lambda)^{1/14}$$
$$<\sigma^{(1)}(0)\sigma^{(2)}(0) > \pm = G_{12,12}^\pm = -2.278284275...(-\lambda)^{4/7}$$
$$<\sigma^{(1)}(0)\sigma^{(2)}(\infty) > \pm = G_{13,12}^\pm = 1.682720628...(-\lambda)^{1/7}$$
$$<\sigma^{(1)}(0)e^{(2)}(0) > \pm = G_{13,13}^\pm = 3.311880669...(-\lambda)^{9/14}$$
$$<\sigma^{(1)}(0)e^{(2)}(\infty) > \pm = G_{13,13}^\pm = 2.955384028...(-\lambda)^{9/14}$$
$$<\epsilon^{(1)}(0)e^{(2)}(\infty) > \pm = G_{13,13}^\pm = 5.160349412...(-\lambda)^{8/7}$$

where the parameter $\lambda$ is related to the mass of the lowest kink by:

$$\lambda = -0.2379062104...M^{7/4}.$$ (65)

Notice that $<\sigma^{(1)}(0)\sigma^{(2)}(0) > \pm$ and $<\sigma^{(1)}(0)\sigma^{(2)}(\infty) > \pm$ differ by less than 0.7\% as expected.

**Example 2 : Two energy-energy coupled tricritical Ising models**

The case $p = 4, p' = 5$ in [31] describes two tricritical Ising models which interact through their leading energy density operators $\Phi^{(i)}_{12} = \epsilon^{(i)}$ of conformal dimension $\Delta_\epsilon = 1/10$. It corresponds to $\beta^2 = 2/5$ i.e. $\xi = 2/3$ in (31). Beside $\epsilon^{(i)}$ and the identity operator, each minimal model contains the sub-leading energy density operator $\Phi^{(i)}_{13} = \epsilon^{(i)}$ with $\Delta_\epsilon = 3/5$ (“vacancy operator”), two magnetic operators $\Phi^{(i)}_{22} = \sigma^{(i)}$ with $\Delta_\sigma = 3/80$, $\Phi^{(i)}_{21} = \sigma^{(i)}$ with $\Delta_\sigma = 7/16$ and $\Phi^{(i)}_{14}$. Due to the obvious property $d_{rs,rs'}^\pm = d_{1s,1s'}^\pm d_{rs,11}^\pm$.

\footnote{For instance, it is known that form factors are able to reproduce with high accuracy the UV behaviour of the correlation functions [3, 31, 32].}
similarly to the previous case we find for instance for any vacuum $|j j>$ : 

$$<\sigma^{(1)} >_{j j} = G_{22,11}^{j j} = d_{22}^j \times 1.144656674\ldots (-\lambda)^{3/64};$$

$$<\epsilon^{(1)} >_{j j} = G_{12,11}^{j j} = d_{12}^j \times 1.529866659\ldots (-\lambda)^{1/8};$$

$$<\sigma^{(1)}(0)\sigma^{(2)}(0) >_{j j} = G_{22,22}^{j j} = d_{22}^j \times 1.315726811\ldots (-\lambda)^{3/32};$$

$$<\sigma^{(1)}(0)\sigma^{(2)}(\infty) >_{j j} = G_{22,11}^{j j} G_{11,22}^{j j} = d_{22}^{j j} \times 1.310238901\ldots (-\lambda)^{3/32};$$

$$<\epsilon^{(1)}(0)\epsilon^{(2)}(0) >_{j j} = G_{12,12}^{j j} = d_{12}^{j j} \times 2.419476973\ldots (-\lambda)^{1/4};$$

$$<\epsilon^{(1)}(0)\epsilon^{(2)}(\infty) >_{j j} = G_{12,11}^{j j} G_{11,12}^{j j} = d_{12}^{j j} \times 2.340491994\ldots (-\lambda)^{1/4};$$

where the parameter $\lambda$ is related to the mass of the lowest kink by:

$$\lambda = -0.2566343706\ldots M^{8/5}. \quad (66)$$

Notice that $<\sigma^{(1)}(0)\sigma^{(2)}(0) >_{j j}$ and $<\sigma^{(1)}(0)\sigma^{(2)}(\infty) >_{j j}$, $<\epsilon^{(1)}(0)\epsilon^{(2)}(0) >_{j j}$ and $<\epsilon^{(1)}(0)\epsilon^{(2)}(\infty) >_{j j}$ differ by less than 0.5% and 3% respectively.

**Example 3 : Two coupled $A_5$ RSOS models**

The case $p = 5$, $p' = 6$ in (1) describes two $A_5$ RSOS models coupled by their primary operators $\Phi^{(i)}_2$ with conformal dimension $\Delta_{22} = 1/8$. It corresponds to $\beta^2 = 5/12$ i.e. $\xi = 5/7$ in (31). Each minimal model also contains the primary operator $\Phi^{(1)}_2$ with $\Delta_{22} = 1/40$. As before we obtain:

$$<\Phi^{(1)}_2 >_{j j} = G_{22,11}^{j j} = d_{22,11}^j \times 1.090446894\ldots (-\lambda)^{-30};$$

$$<\Phi^{(1)}_2 >_{j j} = G_{12,11}^{j j} = d_{12,11}^j \times 1.726352342\ldots (-\lambda)^{-1/6};$$

$$<\Phi^{(1)}_2 (0)\Phi^{(2)}_2 (0) >_{j j} = G_{22,22}^{j j} = d_{22}^j \times 1.191588988\ldots (-\lambda)^{-1/15};$$

$$<\Phi^{(1)}_2 (0)\Phi^{(2)}_2 (\infty) >_{j j} = G_{22,11}^{j j} G_{11,22}^{j j} = d_{22}^{j j} \times 1.189074429\ldots (-\lambda)^{-1/15};$$

where the parameter $\lambda$ is related to the mass of the lowest kink by:

$$\lambda = -0.2697511940\ldots M^{3/2}. \quad (67)$$

Notice that $<\Phi^{(1)}_2 (0)\Phi^{(2)}_2 (0) >_{j j}$ and $<\Phi^{(1)}_2 (0)\Phi^{(2)}_2 (\infty) >_{j j}$ differ by $\sim 2\%$.

**Example 4 : Two energy-energy coupled 3-state Potts models**

The case $p = 5$, $p' = 6$ in (2) describes two 3-state Potts models coupled by their energy density operator $\Phi^{(i)}_2 = \epsilon^{(i)}$ with conformal dimension $\Delta_{21} = 2/5$. It corresponds to $\xi = 5/7$ in (31). Each minimal model also contains the primary operator $\Phi^{(i)}_{23} = \sigma^{(i)}$ - the spin operator - with $\Delta_{23} = 1/15$. We obtain for instance:

$$<\sigma^{(1)} >_{j j} = G_{23,11}^{j j} = d_{23}^j \times 1.9079\ldots (-\hat{\lambda})^{1/3};$$

$$<\sigma^{(1)}(0)\sigma^{(2)}(0) >_{j j} = G_{23,23}^{j j} = d_{23}^j \times 4.50\ldots (-\hat{\lambda})^{2/3};$$

$$<\sigma^{(1)}(0)\sigma^{(2)}(\infty) >_{j j} = G_{23,11}^{j j} G_{11,23}^{j j} = d_{23}^{j j} \times 3.64\ldots (-\hat{\lambda})^{2/3};$$
where the parameter $\hat{\lambda}$ is related to the mass of the lowest kink by:

$$\hat{\lambda} = -0.2612863655...M^{2/5}. \quad (68)$$

To conclude, we would like to mention that here we studied only a special case of a much more general class of integrable coupled models [19, 20, 21]. There exist many other examples which can be worked out along the same lines. Let us also note that the exact form factor techniques as well as the truncated conformal space approach may be used and similar numerical analyses can be performed for the correlation functions.

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