Finite difference schemes for multi-term time-fractional mixed diffusion-wave equations

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Abstract

The multi-term time-fractional mixed diffusion-wave equations (TFMDWEs) are considered and the numerical method with its error analysis is presented in this paper. First, a $L^2$ approximation is proved with first order accuracy to the Caputo fractional derivative of order $\beta \in (1, 2)$. Then the approximation is applied to solve a one-dimensional TFMDWE and an implicit, compact difference scheme is constructed. Next, a rigorous error analysis of the proposed scheme is carried out by employing the energy method, and it is proved to be convergent with first order accuracy in time and fourth order in space, respectively. In addition, some results for the distributed order and two-dimensional extensions are also reported in this work. Subsequently, a practical fast solver with linearithmic complexity is provided with partial diagonalization technique. Finally, several numerical examples are given to demonstrate the accuracy and efficiency of proposed schemes.

Keywords: $L^2$ approximation, compact difference scheme, distributed order, fast solver, convergence

MSC subject classifications: 26A33, 65M06, 65M12, 65M55, 65T50

1. Introduction

In this work, we are concerned with numerical methods for the multi-term time-fractional mixed diffusion-wave equations (TFMDWEs) in the following form

\[
\sum_{i=1}^{s} K_i \frac{C_0 D_0^{\alpha_i}}{\partial t} u(x, t) = \Delta u(x, t) + f(x, t), \quad x \in \Omega, \ t \in (0, T],
\]

where $\Omega$ is spatial domain, $\Delta$ is the Laplace operator, $\alpha_i < \cdots < \alpha_s$ and $K_i > 0$, $i = 1, \ldots, s$. Here $\frac{C_0 D_0^{\alpha_i}}{\partial t}$ denotes the Caputo fractional derivative of order $\alpha_i$, which will be defined later.

As a natural extension of the single-term time-fractional partial differential equations (TFPDEs), e.g. sub-diffusion or diffusion-wave equations, the multi-term TFPDEs are expected to improve the modeling accuracy in depicting the anomalous diffusion process, successfully capturing power-law frequency dependence, adequately modeling various types of viscoelastic damping. For instance, in [2], a two-term mobile and immobile fractional-order diffusion model was proposed to model the total concentration in solute transport. The kinetic equation with two fractional derivatives of different orders appears quite natural when describing sub-diffusive motion in velocity fields [4]. The two-term time-fractional telegraph equations [2], which can

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also be regarded as special cases of Equation (1.1), govern the iterated Brownian motion and the telegraph processes with Brownian time.

There are a few mathematical theories on multi-term TFPDEs. Based on an appropriate maximum principle, Luchko [6] derived some priori estimates for the solution and established its uniqueness using the Fourier method and the method of separation of variables. Daftar Dar-Gejji and Bhalekar [7] considered a multi-term time-fractional diffusion-wave equation (TFDWE) along with the homogeneous/nonhomogeneous boundary conditions and have solved this equation using the method of separation of variables. Based on the orthogonal polynomials of the Laguerre type, Stojanovic [8] found solutions for the diffusion-wave equation in one dimension with \( n \)-term time-fractional derivatives, whose orders belong to the intervals \((0, 1), (1, 2)\) and \((0, 2)\), respectively. By using Luchko’s Theorem and the equivalent relationship between the Laplacian operator and the Riesz fractional derivative, Jiang et al. [9] derived the analytical solutions for the multi-term time-space fractional advection-diffusion equations. Subsequently, Ding et al. [10] presented the analytical solutions for the multi-term time-space fractional advection-diffusion equations with mixed boundary conditions. Recently, Stojanovic has analyzed regularity, existence and uniqueness of the equation with nonlinear source in [11].

Numerical approximations of multi-term TFPDEs have also been discussed by some authors. For the multi-term TFPDEs whose orders, \( \alpha_i \), belong to \((0, 1)\), very recently, Jin et al. [12] developed a fully discrete scheme based on the standard Galerkin finite element method in space combining with a finite difference discretization of the time-fractional derivatives. For the orders, \( \alpha_i \), lying in \((1, 2)\), Chen. et al. [2] presented a finite difference scheme and gave its analysis, following the idea of the method of order reduction proposed by Sun and Wu [13]. In addition, an extension of multi-term TFPDEs, distributed order TFPDEs have also been considered (see e.g. [14, 15]).

However, to the authors’ best knowledge, published works on numerical solution of the multi-term TFMDWEs with the fractional orders, \( \alpha_i \), lying in \((0, 2)\) are still limited in spite of rich literatures on its single-term version [16, 17, 18, 19, 20, 21, 22] and ordinary fractional differential equations (see e.g. [23, 24, 25]). In [26], the authors proposed a numerical method by the method of order reduction for more general problems, i.e., the fractional orders, \( \alpha_i \), belonging to \((0, n)(n > 2)\). However, the corresponding numerical analysis was not provided. Due to the definition of the fractional order derivative, which is a nonlocal operator, the fractional order derivative requires a longer memory of the solution. The global nature of the non-integer order derivative makes the design of stable and accurate methods more difficult. It is a great challenge for memory and storage requirement when all the past history of the solution has to be saved in order to compute the solution at the current time. Introducing new variables and transforming the original problem into an equivalent system will make the design of accurate and robust methods easier. However, it will introduce extra computational cost and data storage meanwhile. Naturally, developing high-order numerical methods for time-fractional partial differential equations is an effective approach to overcome this challenge. But the constraint on the regularity of the underlying problem makes pursuing high-order methods in temporal direction very difficult, since such problem may lack high regularity due to the existence of singularity of the fractional derivatives near the initial time.

The main objective of this paper is to develop a numerical algorithm for multi-term TFMDWEs and give its rigorous numerical analysis. Numerous works focus on time fractional partial differential equations with the orders either lying in \((0, 1)\) or in \((1,2)\). To date, we are not aware of any published papers investigating the multi-term TFMDWEs or their generalization, i.e., distributed-order equations with the order belonging to \((0, 2)\). We shall give a complete analysis of our scheme. By the energy method, a priori estimate for the solution of the Dirichlet boundary value problem of the time-fractional mixed diffusion-wave equations has
been established. Instead of using the method of order reduction in [26], we will utilize the $L^2$ approximation, proposed by Oldham and Spanier [27], to discretize the Caputo derivative of order $\beta \in (0, 2)$ directly, and modify it to be suitable for solving an initial value problem. Combining a modified $L^2$ approximation with the well-known $L^1$ approximation to discretize time-fractional derivatives, we construct a compact finite difference scheme by applying the second-order central difference quotient to approximate the weighted average of the second-order derivatives in spatial derivation. By the discrete analogue of energy or fractional Sobolev inequalities, we provide error analysis of our method rigorously, which can be seen as one of main contributions of this work. In addition, generalization of the distributed order equations and high-dimensional cases are also reported in this paper. To our knowledge, this is the first paper to investigate the distribution order equations with the orders from 0 to 2. This can be seen another main contributions of this paper. Other contributions also include that we present a fast solver for the fully discretized scheme. By partial diagonalization technique, the resulting matrix equation is reduced to the independent linear systems with toeplitz-like structure, being easy and convenient to design fast algorithms.

The rest of this paper is organized as follows. In Section 2, a first-order $L^2$ approximation to the Caputo fractional derivatives is proved and the compact difference scheme is derived. The error analysis of the proposed scheme is showed in Section 3. Extension to distributed-order version and to the two-dimensional counterpart are also considered in Section 4 and Section 5, respectively. In Section 6, a fast solver is suggested with partial diagonalization technique and divide and conquer strategy. To show the effectiveness of the proposed algorithm, the numerical experiments are performed to verify the theoretical results in Section 7. Finally, some remarks and discussions are offered in the conclusion part to close the paper.

2. Numerical method for the one-dimensional two-term time-fractional mixed diffusion-wave equation

For the sake of simplicity but without loss of generality, we consider the one-dimensional two-term TFMD-WEs as follows:

\begin{align*}
K_1 \frac{C}{0} D_t^\alpha u(x,t) + K_2 \frac{C}{0} D_t^\beta u(x,t) &= \partial_x^2 u(x,t) + f(x,t), \quad x \in \Omega, \quad 0 < t \leq T, \quad (2.1) \\
u(x,0) &= \phi_0(x), \quad u_t(x,0) = \phi_1(x), \quad x \in \bar{\Omega}, \quad (2.2) \\
u(0,t) &= \varphi_0(t), \quad u(L,t) = \varphi_1(t), \quad 0 < t \leq T, \quad (2.3)
\end{align*}

where $\Omega = (0,L)$, $K_1, K_2 > 0$, and $0 < \alpha < 1 < \beta < 2$.

In the following analysis of the proposed numerical method, we assume the problem (2.1)-(2.3) has a unique and sufficiently smooth solution. Here it should be pointed out that Alikhanov showed stability of solution by priori estimate for the sub-diffusion and diffusion-wave equations in [29]. Following his idea, one can prove the stability for the solution of the problem (2.1)-(2.3) by the energy method.

2.1. Notations and lemmas

Take an integer $N$. Let $\Omega_{\tau} \equiv \{ t_n \mid 0 \leq n \leq N \}$ be a uniform mesh of the interval $[0,T]$, where $t_n = n\tau$, $0 \leq n \leq N$ with $\tau = T/N$. Two lemmas below are needed. The first one is well-known $L^1$ approximation.

Lemma 2.1. (See [13]) For $\alpha \in (0,1)$, suppose $v(t) \in C^2[0,t_n]$ and denote $v^n = v(t_n)$,

\begin{align*}
a^0_0(\alpha) &= \frac{1}{\Gamma(2-\alpha)}, \quad a^0_n(\alpha) = \frac{1}{\Gamma(2-\alpha)}[(n+1)^{1-\alpha} - n^{1-\alpha}], \quad n \geq 1, \quad (2.4)
\end{align*}
and define the backward difference quotient operator

\[ \delta_t^n v = \frac{1}{\tau^n} \left[ a_0^{(\alpha)} v_n - \sum_{k=1}^{n-1} (a_0^{(\alpha)} - a_{n-k}^{(\alpha)}) v_k - a_{n-1}^{(\alpha)} v_0 \right], \quad n \geq 1. \]  

(2.5)

It holds that

\[ C_0^\beta D_1^n v(t_n) = \delta_t^n v + R_1[v(t_n)], \]

(2.6)

where

\[ |R_1(v(t_n))| \leq c_\alpha \max_{0 \leq t_n \leq t} |v''(t)| \tau^{2-\alpha}. \]

In above Lemma \( c_\alpha \) is a bounded constant, which is dependent on \( \alpha \) but independent of step size \( \tau \). For the approximation of derivative of fractional order \( \beta \in (1, 2) \), we have the following result.

**Lemma 2.2.** For \( \beta \in (1, 2) \), suppose \( v(t) \in C^2[0, t_n] \cap C^3(0, t_n] \), and \( |v'''(t)| \in L_1(0, t_n] \). Denote \( v^n = v(t_n) \),

\[ b_0^{(\beta)} := a_0^{(\beta-1)} = \frac{1}{\Gamma(3-\beta)}, \quad b_n^{(\beta)} := a_n^{(\beta-1)} = \frac{1}{\Gamma(3-\beta)} [(n+1)^{2-\beta} - n^{2-\beta}], \quad n \geq 1 \]

(2.7)

and

\[ \Delta_1^n v = \frac{1}{\tau^n} \left[ \sum_{k=2}^{n} b_{n-k}^{(\beta)} (v_k - 2v_k^{(2)} + v_{k-1}^{(2)}) + 2b_{n-1}^{(\beta)} (v_1 - v_0) \right]. \]

(2.8)

It holds that

\[ C_0^\beta D_1^n v(t_n) = \Delta_1^n v^n - 2 \frac{b_{n-1}^{(\beta)}}{\tau^n} v'(t_0) + R_2[v(t_n)], \quad n \geq 1, \]

(2.9)

where

\[ |R_2[v(t_n)]| \leq 9 \frac{\zeta_{n\beta-2}}{\Gamma(3-\beta)} \max_{0 \leq k \leq n-1} \int_{t_0}^{t_k} |v'''(t_k + \theta \tau)| d\theta \cdot \tau. \]

**Proof** Note that

\[ C_0^\beta D_1^n v(t_n) = \frac{1}{\Gamma(2-\beta)} \int_{t_{k-1}}^{t_k} v''(s) (t_n - s)^{\beta-1} ds + \frac{1}{\Gamma(2-\beta)} \int_{t_0}^{t_k} v''(s) (t_n - s)^{\beta-1} ds \]

\[ = \frac{1}{\Gamma(2-\beta)} \sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} \frac{\delta_t^2 v_{k-1}}{(t_n - s)^{\beta-1}} ds + \frac{1}{\Gamma(2-\beta)} \int_{t_0}^{t_1} \frac{\Delta_1^2 v_0}{(t_n - s)^{\beta-1}} ds + R_2[v(t_n)] \]

\[ = \Delta_1^n v^n - 2 \frac{b_{n-1}^{(\beta)}}{\tau^n} v'(t_0) + R_2[v(t_n)], \]

(2.10)

where

\[ \delta_t^2 v_{k-1} = \frac{v(t_k) - 2v(t_{k-1}) + v(t_{k-2})}{\tau^2}, \quad \Delta_1^2 v_0 = \frac{v(t_1) - v(t_0) - \tau v'(t_0)}{\tau^2}, \]

and

\[ R_2[v(t_n)] = \frac{1}{\Gamma(2-\beta)} \sum_{k=2}^{n} \int_{t_{k-1}}^{t_k} \frac{v''(s) - \delta_t^2 v_{k-1}}{(t_n - s)^{\beta-1}} ds + \frac{1}{\Gamma(2-\beta)} \int_{t_0}^{t_1} \frac{v''(s) - \Delta_1^2 v_0}{(t_n - s)^{\beta-1}} ds. \]

(2.11)
By Taylor expansion with the integral remaining term, we find for \( s \in (t_{k-1}, t_k) \) that

\[
|v''(s) - \Delta^2_t v^0| = \left| \int_{t_{k-1}}^s \left( 1 - \frac{1}{2\tau^2}(t_k - t)^2 + (t_k - s)^2 \right) v'''(t)dt - \int_s^{t_k} \frac{1}{2\tau^2}(t_k - t)^2 v'''(t)dt \right|

+ \left| \int_{t_{k-1}}^{t_k} \frac{1}{\tau^2}(t_k - t)^2 v'''(t)dt \right|
\leq \frac{7}{2} \int_{t_{k-1}}^s |v'''(t)|dt + \frac{1}{2} \int_s^{t_k} |v'''(t)|dt + \int_{t_{k-2}}^{t_{k-1}} |v'''(t)|dt
\leq 9 \max_{1 \leq k \leq n} \int_{t_{k-1}}^{t_k} |v'''(t)|dt = 9 \max_{0 \leq k \leq n-1} \int_0^1 |v'''(t_k + \theta \tau)|d\theta \cdot \tau. \quad (2.12)
\]

Similarly we can obtain

\[
|v''(s) - \Delta^2_t v^0| \leq \max_{0 \leq k \leq n-1} \int_0^1 |v'''(t_k + \theta \tau)|d\theta \cdot \tau. \quad (2.13)
\]

Combining (2.12) and (2.13) with (2.11), we get

\[
|R_2[v(t_n)]| \leq \frac{9t_n^2 - \beta}{\Gamma(3 - \beta)} \max_{0 \leq k \leq n-1} \int_0^1 |v'''(t_k + \theta \tau)|d\theta \cdot \tau
\]

\[
= \frac{9t_n^2 - \beta}{\Gamma(3 - \beta)} \max_{0 \leq k \leq n-1} \int_0^1 |v'''(t_k + \theta \tau)|d\theta \cdot \tau. \quad (2.14)
\]

This completes the proof.

\[\square\]

**Remark 2.3.** When \( \beta = 2 \), (2.13) becomes the standard first-order backward difference approximation for second-order derivative, i.e.

\[
v''(t_n) = \frac{v(t_n) - 2v(t_{n-1}) + v(t_{n-2})}{\tau^2} + O(\tau), \quad n \geq 2,
\]

and

\[
v''(t_1) = \frac{2[v(t_1) - v(t_0) - \tau v'(t_0)]}{\tau^2} + O(\tau).
\]

Take an integer \( M \). Let \( \Omega_h \equiv \{ x_i \mid 0 \leq i \leq M \} \) be a uniform mesh of the interval \( [0, L] \), where \( x_i = ih \), \( 0 \leq i \leq M \) with \( h = L/M \). Suppose \( v = \{ v_i \} \) is a grid function on \( \Omega_h \), define

\[
\delta_x v_{i-\frac{1}{2}} = \frac{1}{h}(v_i - v_{i-1}), \quad \delta_x^2 v_i = \frac{1}{h^2}(\delta_x v_{i+\frac{1}{2}} - \delta_x v_{i-\frac{1}{2}}).
\]

To deal with spatial discretization and construct a compact finite difference scheme for solving the problem (2.11)-(2.23), we still need the lemma below.

**Lemma 2.4.** (See [34]) Denote \( \theta(s) = (1-s)^3[5 - 3(1-s)^2] \). If \( w(x) \in C^6[x_{i-1}, x_{i+1}] \), it holds that

\[
\frac{1}{12} \left[ w'(x_{i-1}) + 10w'(x_i) + w'(x_{i+1}) \right] = \frac{w(x_{i-1}) - 2w(x_i) + w(x_{i+1})}{h^2}
\]

\[
\frac{h^4}{360} \int_0^1 [w^{(6)}(x - sh) + w^{(6)}(x + sh)]\theta(s)ds, \quad 1 \leq i \leq M - 1.
\]
The above Lemma can be proved straightforwardly by Taylor expansion with remaining integral.

Define the average operator $A_x$ as

$$A_xv_i = \begin{cases} \frac{1}{12}(v_{i-1} + 10v_i + v_{i+1}), & 1 \leq i \leq M - 1, \\ v_i, & i = 0 \text{ or } M. \end{cases}$$

It is clear that

$$A_xv_i = (I + \frac{h^2}{12}\delta_x^2)v_i, \quad 1 \leq i \leq M - 1,$$

where $I$ denotes the identity operator.

### 2.2. Derivation of the difference scheme

We are now in a position to derive the compact difference scheme. Define grid functions below

$$U_i^n = u(x_i, t_n), \quad (U_i)_0^0 = u_t(x_i, t_0), \quad F_i^n = f(x_i, t_n), \quad 0 \leq i \leq M, \quad 0 \leq n \leq N.$$ 

Considering Equation (2.1) at grid points $(x_i, t_n)$, we have

$$K_1 \frac{C_0}{D_0} u(x_i, t_n) + K_2 \frac{C_0}{D_1} u(x_i, t_n) = \delta^2 x u(x_i, t_n) + f(x_i, t_n), \quad 0 \leq i \leq M, \quad 0 \leq n \leq N. \quad (2.15)$$

By Lemmas 2.1, 2.2, we have

$$K_1 \delta^2_t U^n_i + K_2 \Delta^2_t U^n_i = \delta^2 x u(x_i, t_n) + F^n_i + \frac{2K_2b^{(b)}_0}{\tau^{b-1}} u_t(x_i, 0) + K_1 R_1[U(x_i, t_n)] + K_2 R_2[U(x_i, t_n)], \quad 0 \leq i \leq M, \quad 1 \leq n \leq N. \quad (2.16)$$

Here $R_1$ and $R_2$ are similarly defined as that in Lemmas 2.1 and 2.2 respectively. For the spatial discretization, acting the average operator $A_x$, it follows from Lemma 2.1 that

$$K_1 A_x \delta^2_t U^n_i + K_2 A_x \Delta^2_t U^n_i = \delta^2 x U^n_i + A_x F^n_i + \frac{2K_2b^{(b)}_0}{\tau^{b-1}} A_x(U_i)_0^n + R^n_i, \quad 1 \leq i \leq M - 1, \quad 1 \leq n \leq N. \quad (2.17)$$

where

$$R^n_i = K_1 A_x R_1^i[u(x_i, t_n)] + K_2 A_x R_2^i[u(x_i, t_n)] + R^3_k[u(x_i, t_n)].$$

and

$$R^3_k[u(x_i, t_n)] = \frac{h^4}{360} \int_0^1 [\delta_x^6 u(x_i - sh, t_n) + \delta_x^6 u(x_i + sh, t_n)\theta(s)] ds.$$

It follows from Lemmas 2.1, 2.2, and 2.4 that there exists a constant $C_u$, which depends on the regularity of the solution $u(x, t)$ and the parameters $\alpha$ and $\beta$ but is independent of the step size $h$ and $\tau$, such that

$$|R^n_i| \leq C_u(\tau + h^4), \quad 1 \leq i \leq M - 1, \quad 1 \leq n \leq N. \quad (2.18)$$

Noticing the initial-boundary conditions,

$$U_i^0 = \phi_0(x_i), \quad (U_i)_0^0 = \phi_1(x_i), \quad 0 \leq i \leq M, \quad (2.19)$$

$$U_0^n = \varphi_0(x_0), \quad U^n_M = \varphi_1(x_M), \quad 1 \leq n \leq N. \quad (2.20)$$
omitting the small terms $R_i^n$ and denoting by $u_i^n$ the numerical approximation of $U_i^n$, we get the compact finite difference scheme,

$$K_1 A_x \phi_i^n + K_2 A_x \phi_i^n = \phi_i^n \phi_i^n + A_x F_i^n + \frac{2K_2 \phi_i^n}{\tau^\beta - 1} A_x \phi_i^n, \quad 1 \leq i \leq M - 1, \quad 1 \leq n \leq N, \quad (2.21)$$

$$u_0^0 = \phi_0(x_i), \quad 0 \leq i \leq M,$n

$$u_n^0 = \phi_0(t_n), \quad u_M^1 = \phi_1(t_n), \quad 1 \leq n \leq N. \quad (2.23)$$

**Remark 2.5.** When $\alpha = 1$, $\beta = 2$, we get the following three time-level backward difference scheme

$$K_1 A_x \left(\frac{u_i^n - u_i^{n-1}}{\tau}\right) + K_2 A_x \left(\frac{u_i^n - 2u_i^{n-1} + u_i^{n-2}}{\tau^2}\right) = \phi_i^n \phi_i^n + A_x F_i^n, \quad 1 \leq i \leq M - 1, \quad 2 \leq n \leq N, \quad (2.24)$$

$$u_i^0 = \phi_0(x_i), \quad 0 \leq i \leq M,$n

$$u_0^n = \phi_0(t_n), \quad u_M^n = \phi_1(t_n), \quad 1 \leq n \leq N. \quad (2.27)$$

### 3. Error analysis of the scheme (2.21)-(2.23)

In this section, we shall give the stability and convergence analysis for the scheme (2.21)-(2.23). Let $V_h = \{v \mid v$ is a grid function on $\Omega_h$ and $v_0 = v_M = 0\}$. For any $u, v \in V_h$, we define the discrete inner products

$$(u, v) = \sum_{i=1}^{M-1} u_i v_i, \quad (\delta_x u, \delta_x v) = \sum_{i=1}^{M} \delta_x u_{i-1/2} \cdot \delta_x v_{i-1/2},$$

and induced norms

$$||u|| = \sqrt{(u, u)}, \quad |u|_1 = \sqrt{(\delta_x u, \delta_x u)}.$$ Denote maximum norm by

$$||u||_\infty = \max_{0 \leq i \leq M} |u_i|. \quad (3.1)$$

It is easy to check that

$$\langle \delta_x^2 u, v \rangle = -\langle \delta_x u, \delta_x v \rangle. \quad (3.2)$$

In addition, one has from (3.1) that

$$||u||_\infty \leq \frac{\sqrt{T}}{2} |u|_1. \quad (3.3)$$

Some additional lemmas are still required in order to prove the stability and convergence of the proposed scheme (2.21)-(2.23).

**Lemma 3.1.** (See [31]) For any $u, v \in V_h$, there exists a positive definite operator denoted by $Q_x$ such that

$$(A_x u, v) = (Q_x u, Q_x v).$$
Lemma 3.2. (See [40]) For any \( v \in V_h \), it holds that

\[
\frac{2}{3} \| v \|^2 \leq (A_x v, v) \leq \| v \|^2.
\]

Then we can define the equivalent weighted norm as

\[
\| v \|_A = \sqrt{(A_x v, v)},
\]

and it follows that

\[
\frac{2}{3} \| v \|^2 \leq \| v \|_A \leq \| v \|^2.
\]

Lemma 3.3. (See [17]) Let \( \{c_0, c_1, \ldots, c_n, \ldots\} \) be a sequence of real numbers with the properties below

\[
c_n \geq 0, \quad c_n - c_{n-1} \leq 0, \quad c_{n+1} - 2c_n + c_{n-1} \geq 0.
\]

Then for any positive integer \( m \) and each vector \((v_1, v_2, \ldots, v_m)\) with \( m \) real entries, it holds that

\[
\sum_{n=1}^{m} \left( \sum_{p=0}^{n-1} c_p v_{n-p} \right) v_n \geq 0.
\]

Denote \( \delta_t v^n = \frac{1}{t} (v^n - v^{n-1}) \). Notice that \( \delta_t v^n \) can be reformulated as

\[
\delta_t^{a} v^n = t^{1-a} \sum_{k=0}^{n-1} a_k^{(a)} \delta_t v^{n-k}.
\]

It is not difficult to verify that the coefficients \( \{a_k^{(a)}\} \) defined by (2.4) satisfy

\[
1 = a_0^{(a)} > a_1^{(a)} > a_2^{(a)} > \cdots > a_k^{(a)} > \cdots \to 0,
\]

\[
\frac{(k+1)^{-a}}{\Gamma(1-a)} < a_k^{(a)} < \frac{k^{-a}}{\Gamma(1-a)}.
\]

\[
a_{k+1}^{(a)} - 2a_k^{(a)} + a_{k-1}^{(a)} \geq 0.
\]

Thus, by Lemma [53], we have

\[
\sum_{n=1}^{m} (\delta_t^{a} v^n) \cdot (\delta_t^{a} v^n) = t^{1-a} \sum_{n=1}^{m} \left( \sum_{k=0}^{n-1} a_k^{(a)} \delta_t v^{n-k} \right) \cdot (\delta_t^{a} v^n) \geq 0.
\]

Lemma 3.4. For any \( v = \{v^0, v^1, v^2, v^3, \ldots\} \) and \( \alpha \in (0,1) \), we have

\[
\tau \sum_{n=1}^{m} v^n \delta_t^{a} v^n \geq \frac{1}{2} \tau^{1-a} \sum_{n=1}^{m} a_{m-n}^{(a)} (v^n)^2 - \frac{t^{1-a}}{2\Gamma(2-a)} (v^0)^2.
\]

Proof. Following the proof technique of [21], we can prove that there holds the discrete analogy of the energy inequality i.e.,

\[
v^n \delta_t^{a} v^n \geq \frac{1}{2} \delta_t^{a} (v^n)^2.
\]

Thus it suffices to prove the following inequality

\[
\frac{1}{2} \tau \sum_{n=1}^{m} \delta_t^{a} (v^n)^2 \geq \frac{1}{2} \tau^{1-a} \sum_{n=1}^{m} a_{m-n}^{(a)} (v^n)^2 - \frac{t^{1-a}}{2\Gamma(2-a)} (v^0)^2.
\]
On the one hand, one has
\[
\frac{1}{2} \tau \sum_{n=1}^{m} \delta_t^n (v^n)^2 = \frac{1}{2} \tau^{2-\alpha} \sum_{n=1}^{m} \sum_{i=1}^{n} a_{n-i}^{(\alpha)} \delta_t (v^k)^2 = \frac{1}{2} \tau^{2-\alpha} \sum_{k=1}^{m} \delta_t (v^k)^2 \sum_{n=k}^{m} a_{n-k}^{(\alpha)}
\]
\[
= \frac{1}{2} \tau^{1-\alpha} \sum_{k=1}^{m} [(v^k)^2 - (v^{k-1})^2] \frac{(m-k+1)^{1-\alpha}}{\Gamma(2-\alpha)}
\]

On the other hand
\[
\frac{1}{2} \tau^{1-\alpha} \sum_{k=1}^{m} [(v^k)^2 - (v^{k-1})^2] \frac{(m-k+1)^{1-\alpha}}{\Gamma(2-\alpha)}
\]
\[
= \frac{1}{2\Gamma(2-\alpha)} \tau^{1-\alpha} \left( \sum_{k=1}^{m} (v^k)^2 [(m-k+1)^{1-\alpha} - (m-k)^{1-\alpha}] - (v^0)^2 m^{1-\alpha} \right)
\]
\[
= \frac{1}{2} \tau^{1-\alpha} \sum_{k=1}^{m} a_{n-k}^{(\alpha)} (v^k)^2 - \frac{\tau^{1-\alpha}}{2\Gamma(2-\alpha)} (v^0)^2.
\]

This concludes the proof of Lemma 3.4.

It should be noted that Sun and Wu also gave the proof of the equality (3.9) in [13]. But here we adopted a different technique in [13].

We now turn to investigate the convergence of the scheme (2.21)-(2.23).

**Theorem 3.5 (Convergence).** Suppose \( u(x, t) \) solves the problem (2.1)-(2.3), \( u(x, t) \in C^{0,2}([0, L] \times [0, T]) \) and \( \partial_x^2 u(x, \cdot) \in L^1[0, L] \partial_t^3 u(\cdot, t) \in L^1[0, T] \). Let \( \{u^n_i\} \) be the solution of difference scheme (2.21)-(2.23). Then for \( n\tau \leq T \), there exists a constant \( C \) such that
\[
\max_{0 \leq t \leq M} |u(x_i, t_n) - u^n_i| \leq C(\tau + h^4).
\]

**Proof** Let
\[
e^n_i = U^n_i - u^n_i, \quad 0 \leq i \leq M, \quad 0 \leq n \leq N.
\]

Subtracting (2.21) and (2.22) from (2.17)-(2.19), we get the error equations,
\[
K_1 A_k e^n_i = K_2 A_k \Delta_t e^n_i = \delta_x^n e_i + R^n_i, \quad 1 \leq i \leq M - 1, \quad 1 \leq n \leq N,
\]
\[
e^n_0 = 0, \quad e^n_M = 0, \quad 1 \leq n \leq N,
\]
\[
e^n_0 = 0, \quad 0 \leq i \leq M.
\]

Recall that
\[
\Delta_t e^n_i = \frac{1}{\tau^\beta} \left[ \sum_{k=2}^{n-2} b^{(\beta)}_{n-k} (e^k - 2e^{k-1} + e^{k-2}) + 2b^{(\beta)}_{n-1} (e^1 - e^0) \right].
\]

Denote \( \delta_t e^n = \frac{1}{\tau}(e^n - e^{n-1}) \). Hereafter we stipulate \( \delta_t e^0 = 0 \). Since \( e^0 = 0 \), we can recast \( \Delta_t e^n \) as
\[
\Delta_t e^n = \frac{1}{\tau^\beta} \left[ b^{(\beta)}_0 \delta_t e^n - \sum_{k=1}^{n-2} (b^{(\beta)}_{n-k-1} - b^{(\beta)}_{n-k}) \delta_t e^k - b^{(\beta)}_{n-1} \delta_t e^0 \right] + \frac{b^{(\beta)}_{n-1}}{\tau^\beta} e^1
\]
\[
= \delta_t^{\beta-1} (\delta_t e^n) + \frac{b^{(\beta)}_{n-1}}{\tau^\beta} e^1,
\]

(3.15)
where we used the notation \(2.5\). Taking the inner product of \(3.12\) with \(\tau \delta_t e^n\) and summing up for \(n = 1\) to \(m\) yield

\[
K_1 \tau \sum_{n=1}^{m} (A_x \delta_t^n e^n, \delta_t e^n) + K_2 \tau \sum_{n=1}^{m} (A_x \Delta_t^n e^n, \delta_t e^n) - \tau \sum_{n=1}^{m} (\delta_t^2 e^n, \delta_t e^n) = \tau \sum_{n=1}^{m} (R^n, \delta_t e^n). \tag{3.16}
\]

For the first term on the left hand side of \(3.16\), using Lemma 3.1 and noting the commutation of the operators in different direction, and the inequality \(3.3\), we have

\[
K_1 \tau \sum_{n=1}^{m} (A_x \delta_t^n e^n, \delta_t e^n) = K_1 \tau \sum_{n=1}^{m} (Q_x \delta_t^n e^n, Q_x \delta_t e^n) = K_1 \tau \sum_{n=1}^{m} (\delta_t^n Q_x e^n, \delta_t Q_x e^n) \geq 0. \tag{3.17}
\]

For the second term on the left hand side of \(3.16\), by Lemmas 3.1.3.4 and Equality \(3.15\), we have

\[
K_2 \tau \sum_{n=1}^{m} (A_x \Delta_t^n e^n, \delta_t e^n) = K_2 \tau \sum_{n=1}^{m} (\Delta_t^n Q_x e^n, Q_x \delta_t e^n)
\]

\[
= K_2 \tau \sum_{n=1}^{m} (\delta_{t-1} \delta_t Q_x e^n, \delta_t Q_x e^n) + K_2 \tau \sum_{n=1}^{m} \frac{\delta_{t-1}^{(\beta)}}{\tau^{\beta-1}} (Q_x e^n, \delta_t Q_x e^n)
\]

\[
\geq \frac{K_2 \tau}{2 \Gamma(2-\beta)} \sum_{n=1}^{m} \|\delta_t Q_x e^n\|^2 - K_2 \tau \sum_{n=1}^{m} \frac{\delta_{t-1}^{(\beta)}}{\tau^{\beta-1}} \left( \frac{\epsilon_2}{4 \Gamma(2-\beta)} \|Q_x e^n\|^2 - K_2 \max_{0 \leq k \leq m-1} \frac{\delta_{t-1}^{(\beta)}}{\tau^{\beta-1}} \right) \sum_{n=1}^{m} \|\delta_t Q_x e^n\|^2
\]

\[
= \left( \frac{K_2 \tau}{2 \Gamma(2-\beta)} - K_2 \frac{\delta_{t-1}^{(\beta)}}{\epsilon_2 \tau^{\beta-1}} \right) \tau \sum_{n=1}^{m} \|\delta_t e^n\|^2_\lambda - \frac{\epsilon_2 K_2^2 \tau^{2-\beta}}{4 \Gamma(3-\beta)} \|e^n\|^2_\lambda. \tag{3.18}
\]

where we have used the \(\epsilon\)-type inequality and the commutativity of the operators, i.e. \(Q_x \delta_t = \delta_t Q_x\).

For the third term on the left hand side of \(3.16\), observing the initial value \(3.14\), we obtain

\[
-\tau \sum_{n=1}^{m} (\delta_t^2 e^n, \delta_t e^n) = \tau \sum_{n=1}^{m} |\delta_t e^n, \delta_t \delta_x e^n| = \sum_{n=1}^{m} \left| \delta_t e^n, \delta_x e^n - \delta_x e^n \right|
\]

\[
\geq \frac{1}{2} \sum_{n=1}^{m} \left| |e^n|^2 - |e^{n-1}|^2 \right| = \frac{1}{2} |e^m|^2_1, \tag{3.19}
\]

where we have used the inequality \(a(a - b) \geq \frac{1}{4}(a^2 - b^2)\).

For the term on the right hand side, we have

\[
\tau \sum_{n=1}^{m} (R^n, \delta_t e^n) \leq \tau \sum_{n=1}^{m} \left( \epsilon_2 \|\delta_t e^n\|^2 + \frac{1}{4 \epsilon_2} \|R^n\|^2 \right). \tag{3.20}
\]

Substituting \(3.17\) into \(3.16\) gives

\[
\left( \frac{K_2 \tau}{2 \Gamma(2-\beta)} - K_2 \frac{\delta_{t-1}^{(\beta)}}{\epsilon_2 \tau^{\beta-1}} \right) \tau \sum_{n=1}^{m} \|\delta_t e^n\|^2_\lambda + \frac{1}{2} \|e^n\|^2_1
\]

\[
\leq \epsilon_2 \tau \sum_{n=1}^{m} \|\delta_t e^n\|^2_\lambda + \frac{\epsilon_2 K_2^2 \tau^{2-\beta}}{4 \Gamma(3-\beta)} \|e^n\|^2_\lambda + \frac{1}{4 \epsilon_2} \tau \sum_{n=1}^{m} \|R^n\|^2, \quad 1 \leq m \leq N.
\]
Noting the norms equivalence \((3.4)\), we have
\[
\left( \frac{K_2 t_m^{1-\beta}}{3 \Gamma(2-\beta)} - \frac{2 K_2 b_0^{(\beta)}}{3 \tau^{\beta-1}} \right) \sum_{n=1}^{m} \| \delta_t e^n \|^2 + \frac{1}{2} |e^m|^2 \\
\leq \epsilon_2 \tau \sum_{n=1}^{m} \| \delta_t e^n \|^2 + K_2 \frac{\epsilon_1 + 2 - \beta}{4 \Gamma(3-\beta)} \frac{e^n_1}{\tau} \| e^n_1 \|^2 + \frac{1}{4 \epsilon_2} \sum_{n=1}^{m} R^n \|^2, \quad 1 \leq m \leq N.
\]
Recall that \(b_0^{(\beta-1)} = \frac{1}{\Gamma(3-\beta)}\). Taking \(\epsilon_1 = \frac{4 \tau (1-\beta) t_m^{\beta-1}}{2-\beta}, \epsilon_2 = \frac{K_2 t_m^{1-\beta}}{12 \tau (2-\beta)}\) leads to
\[
\frac{K_2 t_m^{1-\beta}}{12 \Gamma(2-\beta)} \sum_{n=1}^{m} \| \delta_t e^n \|^2 + \frac{1}{2} |e^m|^2 \leq \frac{K_2 t_m^{1-\beta}}{2-\beta} \frac{\epsilon_1 + 2 - \beta}{\Gamma(3-\beta)} \frac{e^n_1}{\tau} \| e^n_1 \|^2 + \frac{3 \Gamma(2-\beta) t_m^{\beta-1}}{K_2} \frac{1}{\tau} \sum_{n=1}^{m} R^n \|^2, \quad 1 \leq m \leq N.
\]
Taking the inner product of \((3.12)\) with \(\tau \delta_t e^1\) gives
\[
K_1 \tau (A_2 e_1^e, e_1^e) + K_2 \tau (A_2 \Delta_{n-1} e_1^e, e_1^e) - \tau (\delta_t^2 e_1^e, e_1^e) = \tau (R^1, e_1^e).
\]
That is
\[
\frac{K_1 \epsilon_1^{(a)}}{\tau^a} (A_x e^1, e^1) + \frac{2 K_2 b_0^{(\beta)}}{\tau^\beta} (A_x e^1, e^1) - (\delta_t^2 e^1, e^1) = (R^1, e^1).
\]
Similar to the argument as above, we can obtain
\[
\frac{2 K_2 b_0^{(\beta)}}{\tau^\beta} (A_x e^1, e^1) \leq (R^1, e^1) \leq \| R^1 \| \cdot \| e^1 \| \leq \| R^1 \| \cdot \frac{3}{2} \| e^1 \|_A.
\]
Thus, we have
\[
\| e^1 \|_A \leq \frac{3 \tau^\beta}{4 K_2 b_0^{(\beta)}} \| R^1 \| = \frac{3 \Gamma(3-\beta) \tau^\beta}{4 K_2} \| R^1 \|.
\]
Inserting \((3.22)\) into \((3.21)\) leads to
\[
\frac{K_2 t_m^{1-\beta}}{12 \Gamma(2-\beta)} \sum_{n=1}^{m} \| \delta_t e^n \|^2 + \frac{1}{2} |e^m|^2 \leq \frac{9 \Gamma(2-\beta) t_m^{\beta-1}}{16 K_2} \| R^1 \|^2 + \frac{3 \Gamma(2-\beta) t_m^{\beta-1}}{K_2} \tau \sum_{n=1}^{m} R^n \|^2 \leq \frac{3 \Gamma(2-\beta) t_m^{\beta-1}}{K_2} \left( \frac{3}{4} \tau^{\beta-1} + t_m^{\beta-1} \right) L C_a (\tau + h^4)^2, \quad 1 \leq m \leq N.
\]
It follows from \((3.2)\) that
\[
\| e^m \|_\infty \leq \frac{\sqrt{T}}{2} |e^m|_1 \leq \frac{3 \Gamma(2-\beta) \tau^{\alpha-1} + t_m^{\beta-1}}{K_2} L C_a t_m (\tau + h^4) : = C(\tau + h^4).
\]
This completes the proof. \(\Box\)

With the above convergency and consistency condition \((2.18)\), the stability of the difference scheme can be obtained straightforwardly.

**Corollary 3.6 (Stability).** The difference scheme \((2.21), (2.22)\) is unconditionally stable to the initial time value and the right hand term.
4. Extension to the distributed order equation

The diffusion-wave equation with distributed order in time can be written as:

\[
\int_a^b w(\alpha) \frac{\partial}{\partial \alpha} D_\alpha^\alpha u(x, t) d\alpha = \partial_x^2 u(x, t) + f(x, t), \quad 0 < x < L, \quad t > 0, \tag{4.1}
\]

where \(0 < a < b < 2\), and the function \(w(\alpha)\) acting as weight for the order of differentiation is subject to that \(w(0) > 0\) and \(\int_a^b w(\alpha) = \text{Constant} > 0\). If \(a = 0\) and \(b = 1\) we obtain the sub-diffusion equation with distributed order in time, and if \(a = 1\) and \(b = 2\), we have the diffusion-wave equation with distributed order in time. As an extension of multi-term equations, distributed order equations have gained considerable attention recently. For the non-Markovian process which is non self-similar and exhibits a continuous distribution of time-scales, a continuous distribution of fractional time derivatives is introduced in \([32]\). For example, some complicated processes involving a mixture of power laws often lead to the distributed-order fractional derivative in time \([33, 34]\). As a precise tool to explain and describe some real physical phenomena, numerical work of the time fractional differential equations involved with the distributed order operator has also attract the attention of many scholars. For example, Diethelm and Ford \([35]\) introduced a general framework for distributed-order ordinary differential equations by using the quadrature formula, such as the trapezoidal formula, with some suitable numerical solver for the resulting multi-term fractional equations, while a convergence analysis of the method was discussed in \([36]\) recently. Numerical schemes for partial integro-differential equations with distributed fractional order, including the sub-diffusion equation \((4.1)\), have appeared in the literature very recently, see \([37, 38, 39]\). In this work, we are concerned with the case \(a = 0, b = 2\), which, to the best of our knowledge, has not been investigated to date.

In this section, the implicit difference scheme \((2.21)-(2.28)\) is extended to approximate the distributed-order diffusion-wave equation \((4.1)\) with \(a = 0, b = 2\). The initial boundary conditions \((2.2)-(2.3)\) are considered for equation \((4.1)\). Given a positive integer \(J\), take \(\sigma = \frac{1}{J}\). We partition the interval \([0, 2]\) as follows:

\[
0 = \beta_0 < \beta_1 < \ldots < \beta_J = 1 < \beta_{J+1} < \ldots < \beta_{2J} = 2,
\]

where \(\beta_j = j\sigma\) for \(j = 0, 1, 2, \ldots, 2J\). We can use the mid-point quadrature rule for approximating the integral in \((4.1)\). Let \(\alpha_j = \frac{\beta_{j-1} + \beta_j}{2}\). Then,

\[
\int_0^2 w(\alpha) \frac{\partial}{\partial \alpha} D_\alpha^\alpha u(x, t) d\alpha = \sigma \sum_{j=1}^{2J} K_j \frac{\partial}{\partial \alpha} D_\alpha^\alpha u(x, t) - \frac{\sigma^2}{24} \Phi''(\zeta), \tag{4.2}
\]

where \(\zeta \in (0, 2)\), \(\Phi(\alpha) = w(\alpha) \frac{\partial}{\partial \alpha} D_\alpha^\alpha u(x, t)\) and \(K_j = w(\alpha_j)\). By the formula \((4.2)\), the distributed order fractional diffusion-wave equation are reduced to the following multi-term TFMDWEs as follows:

\[
\sigma \sum_{j=1}^{2J} K_j \frac{\partial}{\partial \alpha} D_\alpha^\alpha u(x, t) = \partial_x^2 u(x, t) + f(x, t) + \frac{\sigma^2}{24} \Phi''(\zeta), \quad t > 0, \quad 0 < x < L. \tag{4.3}
\]

Considering the equation \((4.3)\) at grid points \((x_i, t_n)\), we have

\[
\sigma \sum_{j=1}^{J} K_j \frac{\partial}{\partial \alpha} D_\alpha^\alpha u(x_i, t_n) + \sigma \sum_{j=J+1}^{2J} K_j \frac{\partial}{\partial \alpha} D_\alpha^\alpha u(x_i, t_n) = \partial_x^2 u(x_i, t_n) + f(x_i, t_n) + \frac{\sigma^2}{24} \Phi''(\zeta), \quad 0 \leq i \leq M, \quad 1 \leq n \leq N. \tag{4.4}
\]

Following the similar derivation as that in Subsection 3.2, we can obtain

\[
\sigma \sum_{j=1}^{J} K_j A_x \delta^\alpha_{\alpha_j} U^n_i + \sigma \sum_{j=J+1}^{2J} K_j A_x \Delta^\alpha_{\alpha_j} U^n_i = \partial_x^2 U^n_i + A_x F^n_i + \sigma \sum_{j=J+1}^{2J} K_j \frac{2\beta_{\alpha_j}}{\alpha_{\alpha_j}} A_x(U_i)^0 + \tilde{R}^n_i, \quad 1 \leq i \leq M - 1, \quad 1 \leq n \leq N, \tag{4.5}
\]
where
\[ |\tilde{R}_n^\alpha| \leq \tilde{C}_u(\tau + \sigma^2 + h^4). \] (4.6)

Thus, we get the following implicit compact finite difference scheme
\[ \sigma \sum_{j=1}^{J} K_j A_2 \delta_{l} u_i^n + \sigma \sum_{j=1}^{2J} K_j A_2 \Delta_{l} u_i^n = \delta_{l}^2 u_i^n + \Delta_{l} F_i^n + \sigma \sum_{j=1}^{2J} K_j \frac{\partial^2 u_i^n}{\partial x_j^n} A_x \phi_i^1, \]
\[ 1 \leq i \leq M - 1, \quad 1 \leq n \leq N, \] (4.7)
\[ u_i^0 = \phi_0(x_i), \quad 0 \leq i \leq M, \] (4.8)
\[ u_i^n = \varphi_0(t_n), \quad u_M^n = \varphi_1(t_n), \quad 1 \leq n \leq N. \] (4.9)

Following similar arguments of Section 4, we have the following result.

**Theorem 4.1 (convergence and stability).** The scheme \([1.7] - [1.9]\) is unconditionally stable. Furthermore, let \(w(\alpha) \in C[0, 2]\) and \(\Phi(\alpha) \in C^2[0, 2]\), suppose \(u(x, t)\) solves the problem \([1.1]\) with initial-boundary value \([2.2] - [2.3]\), \(u(x, t) \in C_{\infty, \xi}([0, L] \times [0, T])\) and \(\partial_t^2 u(x, \cdot) \in L^1[0, L], \partial_x^2 u(\cdot, t) \in L^1[0, T]\). And \(\{u^n\}\) be the solution of difference scheme \([4.7] - [4.9]\). Then for \(nT \leq T\), there exists a constant \(C\) such that
\[ \max_{0 \leq i \leq M} |u(x_i, t_n) - u_i^n| \leq C(\tau + \sigma^2 + h^4). \]

It should be noted that Gauss quadrature can also be used to approximate the integration with high accuracy when the integrand \(\Phi(\alpha)\) is smooth enough. The subsequent procedure is as identical as the mid-point case, here we skip it.

5. Extension to the two-dimensional case

In this section, we will consider the generalization of the our proposed method to the following two-dimensional equation
\[ K_1 C D_t^\alpha u(x, y, t) + K_2 C D_t^\alpha u(x, y, t) = \partial_x^2 u(x, y, t) + \partial_y^2 u(x, y, t) + f(x, y, t), \quad (x, y) \in \Omega, \quad t \in (0, T], \] (5.1)
which is subject to the initial-boundary conditions
\[ u(x, y, 0) = \phi_0(x, y), \quad u_t(x, y, 0) = \phi_1(x, y), \quad (x, y) \in \Omega, \] (5.2)
\[ u(x, y, t) = 0, \quad (x, y) \in \partial \Omega, \quad t \in (0, T]. \] (5.3)

where \(\Omega = (0, L_1) \times (0, L_2)\). For the spatial approximation, take two integers \(M_1, M_2\) and let \(h_1 = (b-a)/M_1, h_2 = (d-c)/M_2, x_i = ih_1, 0 \leq i \leq M_1, y_j = jh_2, 0 \leq j \leq M_2\). Let \(\tilde{\Omega}_h = \{(x, y)|0 \leq i \leq M_1, 0 \leq j \leq M_2\}\), and \(\Omega_h = \tilde{\Omega}_h \cap \Omega, \partial \Omega_h = \tilde{\Omega}_h \cap \partial \Omega\). For any grid function \(v = \{v_i,j|0 \leq i \leq M_1, 0 \leq j \leq M_2\}\), denote
\[ \delta_x v_{i-\frac{1}{2}, j} = \frac{1}{h_1} (v_{i,j} - v_{i-1,j}), \quad \delta_x^2 v_{i,j} = \frac{1}{h_1} (\delta_x v_{i+\frac{1}{2}, j} - \delta_x v_{i-\frac{1}{2}, j}). \]

Similar notations \(\delta_y v_{i,j-\frac{1}{2}}, \delta_y^2 v_{i,j}\) can be defined. The spatial average operators are defined as
\[ \mathcal{A}_x v_{i,j} = \begin{cases} \frac{1}{12} (v_{i-1,j} + 10v_{i,j} + v_{i+1,j}), & 1 \leq i \leq M_1 - 1, \quad 0 \leq j \leq M_2, \\ v_{i,j}, & i = 0 \text{ or } M_1, \quad 0 \leq j \leq M_2, \end{cases} \]
\[ \mathcal{A}_y v_{i,j} = \begin{cases} \frac{1}{12} (v_{i,j-1} + 10v_{i,j} + v_{i,j+1}), & 1 \leq j \leq M_2 - 1, \quad 0 \leq i \leq M_1, \\ v_{i,j}, & j = 0 \text{ or } M_2, \quad 0 \leq i \leq M_1. \end{cases} \]
Akin to the construction as the one-dimensional case, we present the implicit compact difference scheme for
the problem \((5.1)-(5.3)\) as follows

\[
K_1 A_y A_x \delta^\alpha u_{i,j}^{n+1} + K_2 A_y A_x \Delta^\beta u_{i,j}^n = A_y \delta^2_x u_{i,j}^n + A_x \delta^2_y u_{i,j}^n + A_y A_x F_{i,j}^n + \frac{2K_2 \delta^{(\beta)}_{n-1}}{\tau^\beta} A_y A_x \phi^1_{i,j},
\]

\((x_i, y_j) \in \Omega_h, \quad 1 \leq n \leq N; \quad u_{i,j}^0 = \phi_0(x_i, y_j), \quad (x_i, y_j) \in \Omega_h, \quad (5.4)\]

\[
u_{i,j}^n = 0, \quad (x_i, y_j) \in \Omega_h, \quad 1 \leq n \leq N. \quad (5.5)
\]

The convergence in \(H_1\) norm and stability can be derived by the similar analysis as one-dimensional case.
Here, we will not dwell on this anymore.

6. Implementation of scheme \((2.21)-(2.23)\)

In this section, we propose a fast solver based on the partial diagonalization technique together with dived
and conquer strategy. To fix the idea, we consider the homogeneous boundary condition for the simplicity.
Several lemmas are presented first.

**Lemma 6.1.** A general tri-diagonal Toeplitz matrix of order \(n - 1\) is given as

\[
T = \begin{pmatrix}
    b & c & 0 & \cdots & 0 & 0 \\
    a & b & c & \cdots & 0 & 0 \\
    0 & a & b & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & b & c \\
    0 & 0 & 0 & \cdots & a & b
\end{pmatrix}
\]

the eigenvalues and eigenvectors of the tri-diagonal Toeplitz matrix \(T\) are given by

\[
\lambda_i = b + 2a \sqrt{\frac{c}{a}} \cos\left(\frac{\pi i}{n}\right), \quad i = 1, 2, \ldots, n - 1,
\]

and

\[
\xi_i = \begin{pmatrix}
    \left(\frac{\pi}{n}\right)^{\frac{i}{2}} \sin\left(\frac{\pi i}{n}\right) \\
    \left(\frac{2 \pi}{n}\right)^{\frac{i}{2}} \sin\left(\frac{2 \pi i}{n}\right) \\
    \left(\frac{3 \pi}{n}\right)^{\frac{i}{2}} \sin\left(\frac{3 \pi i}{n}\right) \\
    \vdots \\
    \left(\frac{(n-1) \pi}{n}\right)^{\frac{i}{2}} \sin\left(\frac{(n-1) \pi i}{n}\right)
\end{pmatrix}, \quad i = 1, 2, \ldots, n - 1,
\]

i.e., \(T \xi_i = \lambda_i \xi_i, \quad i = 1, 2, \ldots, n - 1\). The matrix \(T\) is diagonalizable and \(P = (\xi_1, \xi_2, \xi_3, \ldots, \xi_{n-1})\) diagonalizes \(T\), i.e., \(P^{-1} TP = \Lambda\), where \(\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})\). Moreover \(P\) can be orthogonal, that is, \(\sqrt{\frac{n}{2}} P\) is the orthogonal matrix.
Now, we introduce the following matrices:

\[
M_x = \begin{pmatrix}
\frac{10}{12} & \frac{1}{12} & 0 & \cdots & 0 & 0 \\
\frac{1}{12} & \frac{10}{12} & \frac{1}{12} & \cdots & 0 & 0 \\
0 & \frac{1}{12} & \frac{10}{12} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \frac{10}{12} & \frac{10}{12} \\
0 & 0 & 0 & \cdots & \frac{1}{12} & \frac{10}{12}
\end{pmatrix}_{M-1},
\]

\[
S_x = \begin{pmatrix}
2 & 0 & -1 & 0 & \cdots & 0 & 0 \\
-1 & 2 & -1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & -1 & -1 \\
0 & 0 & 0 & \cdots & 1 & 2
\end{pmatrix}_{M-1},
\]

and

\[
M_i^\alpha = \begin{pmatrix}
a_0^{(\alpha)} & 0 & 0 & \cdots & 0 & 0 \\
a_1^{(\alpha)} - a_0^{(\alpha)} & a_0^{(\alpha)} & 0 & \cdots & 0 & 0 \\
a_2^{(\alpha)} - a_1^{(\alpha)} & a_1^{(\alpha)} - a_0^{(\alpha)} & a_0^{(\alpha)} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{N-2}^{(\alpha)} - a_{N-3}^{(\alpha)} & a_{N-3}^{(\alpha)} - a_{N-4}^{(\alpha)} & a_{N-4}^{(\alpha)} - a_{N-5}^{(\alpha)} & \cdots & a_0^{(\alpha)} & 0 \\
a_{N-1}^{(\alpha)} - a_{N-2}^{(\alpha)} & a_{N-2}^{(\alpha)} - a_{N-3}^{(\alpha)} & a_{N-3}^{(\alpha)} - a_{N-4}^{(\alpha)} & \cdots & a_1^{(\alpha)} - a_0^{(\alpha)} & a_0^{(\alpha)}
\end{pmatrix}_N
\]

\[
M_i^\beta = \begin{pmatrix}
2b_0^{(\beta)} & 0 & 0 & \cdots & 0 & 0 \\
2b_1^{(\beta)} - 2b_0^{(\beta)} & b_0^{(\beta)} & 0 & \cdots & 0 & 0 \\
2b_2^{(\beta)} - 2b_1^{(\beta)} + b_0^{(\beta)} & b_1^{(\beta)} - 2b_0^{(\beta)} & b_0^{(\beta)} & \cdots & 0 & 0 \\
2b_3^{(\beta)} - 2b_2^{(\beta)} + b_1^{(\beta)} & b_2^{(\beta)} - 2b_1^{(\beta)} + b_0^{(\beta)} & b_1^{(\beta)} - 2b_0^{(\beta)} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2b_{N-2}^{(\beta)} - 2b_{N-3}^{(\beta)} + b_{N-4}^{(\beta)} & b_{N-3}^{(\beta)} - 2b_{N-4}^{(\beta)} + b_{N-5}^{(\beta)} & b_{N-4}^{(\beta)} - 2b_{N-5}^{(\beta)} + b_{N-6}^{(\beta)} & \cdots & b_0^{(\beta)} & 0 \\
2b_{N-1}^{(\beta)} - 2b_{N-2}^{(\beta)} + b_{N-3}^{(\beta)} & b_{N-2}^{(\beta)} - 2b_{N-3}^{(\beta)} + b_{N-4}^{(\beta)} & b_{N-3}^{(\beta)} - 2b_{N-4}^{(\beta)} + b_{N-5}^{(\beta)} & \cdots & b_1^{(\beta)} - 2b_0^{(\beta)} & b_0^{(\beta)}
\end{pmatrix}_N
\]

Thus, the difference scheme (2.21), (2.22) can be equivalently reformulated as the following matrix equation:

\[
(K_1 \tau^{-\alpha} M_i^\alpha + K_2 M_i^\beta) u M_x + \frac{\tau^\beta}{k^2} u S_x = b M_x
\]

where \((u)_{i,j} = u^i_j, i = 1, \ldots, N, j = 1, \ldots, M - 1\), and the right hand side matrix \(b \in \mathbb{R}^{N \times (M-1)}\) is given by

\[
b = \tau^\beta F + 2K2\tau \begin{pmatrix}
b_0^{(\beta)} \\
b_1^{(\beta)} \\
b_2^{(\beta)} \\
\vdots \\
b_{N-2}^{(\beta)} \\
b_{N-1}^{(\beta)}
\end{pmatrix}_N \begin{pmatrix}
\phi_1^1, \phi_2^1, \ldots, \phi_{M-2}^1, \phi_{M-1}^1
\end{pmatrix}
\]

with \((F)_{i,j} = F^j_i, i = 1, \ldots, N, j = 1, \ldots, M - 1\). From Lemma 6.1 we know that there exists a normalized orthogonal matrix, \(Q_x \in \mathbb{R}^{(M-1) \times (M-1)}\), such that

\[
\Lambda_x^{(1)} = \text{diag}(\lambda_1^{(1)}, \ldots, \lambda_{M-1}^{(1)}), \\
\Lambda_x^{(2)} = \text{diag}(\lambda_1^{(2)}, \ldots, \lambda_{M-1}^{(2)}),
\]

\[
M_x Q_x = Q_x \Lambda_x^{(1)}, \quad S_x Q_x = Q_x \Lambda_x^{(2)}.
\]

More precisely, we have explicit representation

\[
(Q_x)_{i,j} = \sqrt{\frac{2}{M}} \sin\left(\frac{ij \pi}{M}\right), i, j = 1, 2, \ldots, M - 1,
\]
and
\[ \lambda_i^{(1)} = \frac{5}{6} + \frac{1}{6} \cos \left( \frac{i\pi}{M} \right), \quad \lambda_i^{(2)} = 2 - 2 \cos \left( \frac{i\pi}{M} \right), \quad i = 1, 2, \ldots, M - 1. \]

Hence, multiplying Equation (6.1) by \( Q_x \) gives
\[ (K_1 \tau^\beta - \alpha m_i^\alpha + K_2 M_i^\beta) u Q_x A_x^{(1)} + \frac{\tau^\beta}{h^2} u Q_x A_x^{(2)} = b Q_x A_x^{(1)}. \]  
(6.3)

Let \( v \in \mathbb{R}^{N \times M-1} \) such that
\[ u = v Q_x. \]

Then, Equation (6.3) is equivalent to
\[ (K_1 \tau^\beta - \alpha m_i^\alpha + K_2 M_i^\beta) v A_x^{(1)} + \frac{\tau^\beta}{h^2} v A_x^{(2)} = b Q_x A_x^{(1)} = G A_x^{(1)}, \]  
(6.4)

where \( G = b Q_x \).

Let \( I \in \mathbb{R}^{N \times N} \) be an identity matrix, and \( v_i = (v_{i1}, v_{i2}, \ldots, v_{iN})^T \) and \( G_i = (G_{i1}, G_{i2}, \ldots, G_{iN})^T \). Then the \( i \)-th column of the (6.4) can be written as
\[ \left( K_1 \tau^\beta - \alpha m_i^\alpha + K_2 M_i^\beta + \frac{\lambda_i^{(2)} \tau^\beta}{h^2} I \right) v_i = G_i, \quad i = 1, 2, \ldots, M - 1, \]  
(6.5)

which are equivalent to \( M - 1 \) systems of the following form
\[ \begin{pmatrix} c_0 & 0 & 0 & \cdots & 0 & 0 \\ c_1 & d_1 & 0 & \cdots & 0 & 0 \\ c_2 & d_2 & d_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{N-2} & d_{N-2} & d_{N-3} & \cdots & d_1 & 0 \\ c_{N-1} & d_{N-1} & d_{N-2} & \cdots & d_2 & d_1 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ \vdots \\ e_{N-2} \\ e_{N-1} \end{pmatrix} = \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_{N-2} \\ g_{N-1} \end{pmatrix}. \]  
(6.6)

To fully employ the Toeplitz structure, we recast the above equation as
\[ \begin{pmatrix} d_1 & 0 & \cdots & 0 & 0 \\ d_2 & d_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{N-2} & \vdots & \cdots & d_1 & 0 \\ d_{N-1} & d_{N-2} & \cdots & d_2 & d_1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{N-2} \\ e_{N-1} \end{pmatrix} = \begin{pmatrix} g_1 - c_1 e_0 \\ g_2 - c_2 e_0 \\ \vdots \\ g_{N-2} - c_{N-2} e_0 \\ g_{N-1} - c_{N-1} e_0 \end{pmatrix}. \]  
(6.7)

and \( e_0 = f_0/c_0 \). The resulting lower triangular Toeplitz matrix equation (6.7) can be solved by the divide and conquer strategy proposed in [42], where the authors gave the fast inversion of the lower triangular Toeplitz matrices. The idea of the divide and conquer strategy is also employed by the very recent paper [43] for solving block triangular Toeplitz-like with tri-diagonal block systems from time-fractional partial differential equations. For the completeness of this paper, we sketch it below.

Suppose \( Ax = b \), where \( A \in \mathbb{R}^N \) is a triangular Toeplitz matrix. For simplicity, we assume \( N = 2^n \). Obviously, \( A, x \) and \( b \) can be partitioned as follows:
\[ \begin{pmatrix} A^{(1)} \\ C^{(1)} \end{pmatrix} \begin{pmatrix} x^{(1)} \\ x^{(2)} \end{pmatrix} = \begin{pmatrix} b^{(1)} \\ b^{(2)} \end{pmatrix}. \]  
(6.8)
Thus the original linear system can be equivalently transformed into two half size linear systems

\[
\begin{align*}
A^{(1)}x^{(1)} &= b^{(1)} \\
A^{(1)}x^{(2)} &= b^{(2)} - C^{(1)}x^{(1)}
\end{align*}
\] (6.9)

We note that $A^{(1)}$ is a still triangular Toeplitz matrix and $C^{(1)}$ a Toeplitz matrix. The matrix-vector product $C^{(1)}x^{(1)}$ can be computed efficiently by fast Fourier transform (FFT). The same procedure can be applied to solve both linear systems recursively. Suppose that $\Theta_N$ is the number of operations required to solve lower tri-diagonal linear system. The computation cost can be estimated below

\[
\Theta_N = 2\Theta_{N/2} + N \log(N/2).
\]

By this formula, we can derive the total operations in time direction as $\Theta_N = O(N \log^2 N)$, which has great advantage than the forward substitution method with operations $O(N^2)$.

It should be pointed out that FFT can also applied to matrix-matrix or matrix-vector product in spatial direction due to the periodicity and symmetry of the orthogonal matrix $Q_x$, with operations of $O(M \log M)$ in space. From aforementioned analysis, we can derive that the total operations of our proposed scheme in space and time is $O(MN \log M \log^2 N)$, which enjoys linearithmic complexity.

As to two dimensional case, we can also use partial diagonalization technique to reduce the resulting two-dimensional difference scheme into $(M_1 - 1) \times (M_2 - 1)$ linear systems as (6.7), which can be solved efficiently by the fast direct solver via the divide and conquer idea. Indeed, it suffices to partially diagonalize in the $y$ direction or $x$ direction first, then all the subsequent procedure are as the same as the one-dimensional case. We will not dwell on the fast solver anymore and the robustness and accuracy of the scheme (5.4)-(5.6) will be illustrated by numerical example 7.3.

7. Numerical examples

In order to illustrate the behavior of our proposed numerical method and demonstrate the effectiveness of our theoretical analysis, several examples are presented.

Example 7.1. As the first example, consider the problem

\[
\begin{align*}
\frac{\partial}{\partial t}D^{\alpha_1}_t u(x, t) + \frac{\partial}{\partial t}D^{\alpha_2}_t u(x, t) + u(x, t) &= \partial_x^2 u(x, t) + f(x, t), \quad 0 < x < \pi, \quad 0 < t \leq 1, \\
\end{align*}
\] (7.1)

with the source term and initial-boundary value conditions satisfying that the equation admits the solution

\[u(x, t) = \sin(x)(t^3 + t + 1).\]

Convergence orders, errors and CPU time of the compact scheme (2.21)-(2.23) are examined. Let

\[E(h, \tau) = \max_{1 \leq i \leq M-1} |u(x_i, t_N) - u_i^N|,
\]

where $u(x_i, t_N)$ represents the exact solution and $u_i^N$ is the numerical solution with the step sizes $h$ and $\tau$ at $t_N = 1$. In simulation, we take parameter $\alpha_2 = \alpha_1 + 1$, under which Equation (7.1) becomes the well-known time fractional telegraph equation. In this example, we use the proposed scheme (2.21)-(2.23) to solve this problem and implement it by the fast solver presented in Section 6. The convergence order is computed by

\[\text{Order} = \frac{\log(E(\tau, h_1)/E(\tau, h_2))}{\log(h_1/h_2)}\]
From Tables 8.1-8.2, it is clear to see that the convergence order is one in temporal direction and four in spatial direction, respectively, which is in good agreement with the theoretical analysis. In addition, we can also see from the columns of 'CPU(s)' that the proposed fast solver has a linearithmic complexity, which is in agreement with our prediction.

**Example 7.2.** Next, consider the following distributed order time-fractional diffusion-wave equation:

\[
\int_0^2 \Gamma(4-\alpha) \, _0^C D_t^\alpha u(x,t) \, d\alpha = \partial_x^2 u(x,t) + \sin(x)((t^3 + \frac{6t^3 - 6t}{\log t}), \quad 0 < x < \pi, \quad 0 < t \leq 1,
\]

with the exact solution \( u(x,t) = \sin(x)t^3 \).

Setting

\[
\tilde{E}(h, \tau, \sigma) = \max_{1 \leq i \leq M-1} |u(x_i, t_N, \sigma) - u_i^N|,
\]

the errors and convergence orders are displayed in Table 8.3. From the table, we can clearly see that the convergence orders are of first-order in time and fourth-order in space, respectively, and second-order with respect to the variable \( \alpha \), which again verifies the correctness of our theoretical results.

**Example 7.3.** Finally, consider the two-dimensional problem (5.1) - (5.3) with exact solution

\[
u(x, y, t) = \sin(x) \sin(y)(t^3 + t + 1)
\]

on the domain \( \Omega = (0, \pi) \times (0, \pi) \).

Take \( T = 0.5 \). We take \( h = 1/32 \) and \( \tau = 1/1024 \). We compute this problem by the scheme (5.4)-(5.6). Similar to one-dimensional case, we first employ partial diagonalization technique to deal with \( y \) direction, then we can compute the solution following the exactly same procedure as one dimensional problem. From Figure 8.1-8.3 we can see that the numerical solution is perfectly matched with the exact solution when fractional orders take different values. This further verifies the robustness of the scheme (5.4)-(5.6) and the correctness of our theoretical results.

8. Conclusion, remarks and discussion

Numerical methods for multi-term time-fractional diffusion-wave equations are not fully developed yet up to now. The main goal of our work is to investigate the numerical solution and provide the complete error analysis to a class of mixed diffusion-wave equations on bounded domain. First of all, a first-order approximation to the Caputo fractional derivative with order belonging to \((1, 2)\) by modifying the \( L^2 \) approximation is strictly derived in this paper. Then the new approximation, combining with the \( L1 \) approximation to discretize the time derivative, has been applied to solve the multi-term time-fractional mixed diffusion-wave equations. For the error analysis, the discrete version of the fractional Poncaré-Friedrichs Sobolev embedding inequality is provided and proved strictly, which plays an essential role in the numerical analysis. Then, unconditional convergence of the compact difference scheme has been proved by the energy method, and stability can be obtained immediately with the consistency of the scheme. Next, the distribution order equations and two-dimensional extensions are considered. Particularly, we investigated the distributed order fractional equations with orders belonging to \((0, 2)\). In addition, a practical fast solver with linearithmic complexity are presented. Finally, several numerical examples have been given to show the effectiveness and correctness of our proposed schemes.
Finite difference method is widely applied to solve time-fractional equations mainly because it is easy to implement, and fast solver can be developed without too much difficulty, owning to the Toeplitz structure of the resulting linear system. Besides the popular Lagrange interpolant method, $L_1$ and $L_2$ approximation, there are other two commonly used ones, Grünwald-Letnikov approximation and Lubich’s fractional linear multi-step methods. Grünwald-Letnikov approximation

$$\sum_{k=0}^{[t/\tau]} g_k^{(\alpha)} y(t - k\tau) = R^\alpha_0 D_t^\alpha y(t) + O(\tau), \quad g_k^{(\alpha)} = (-1)^k \binom{\alpha}{k}, \quad (8.1)$$

as an another way to discretize fractional differential and integral operators, is based on the straightforward generalization of concepts from classical calculus to the fractional ones. Since the coefficients $g_k^{(\alpha)}$ have the same properties as $a_k^{(\alpha)}$ in $L_1$ approximation, which can be validated readily, the interested readers can also apply Grünwald-Letnikov formula to time-fractional mixed diffusion-wave equations and employ a similar analysis following the proof of this work. Lubich’s fractional linear multi-step methods have fast convergence compared to two former methods but require high regularity of the solution and correction of the error, which limit their applications in practice.

There are also other directions deserved future research. In this paper, we only consider the equations with sufficiently smooth solution, $\frac{\partial^2}{\partial t^2} u(\cdot, t) \in C^2[0, T]$. That is, the second order derivatives can be bounded. We have not touched upon the low regularity case. For example, consider the problem (7.1) with low regularity solution, $u(x, t) = \sin(x) t^n$. From Table 8.4, we can clearly see that convergence rate deteriorates with decrease of regularity of the solution and it behaves like $O(\tau^{n-1})$ in time. Actually, this can be derived directly from the approximation formula (2.9). Due to the potential singularity of the solution, nonuniform mesh based finite difference methods have gained considerable attention for the time-fractional sub-diffusion recently. However, the corresponding results about time-fractional mixed diffusion-wave equation is still scarce. And we will consider this problem and present an effective fast solver in our future work.

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Table 8.1: Temporal convergence orders, errors and CPU time of the scheme \((2.21)-(2.23)\) with fixed stepsize \(h = 1/16\) (Ex. 8.1)

| \(N\) | \(\alpha_1 = 0.2, \alpha_2 = 1.2\) | \(\alpha_1 = 0.5, \alpha_2 = 1.5\) | \(\alpha_1 = 0.7, \alpha_2 = 1.7\) |
|-------|---------------------------------|---------------------------------|---------------------------------|
|       | \(E(h, \tau)\) Order CPU(s) | \(E(h, \tau)\) Order CPU(s) | \(E(h, \tau)\) Order CPU(s) |
| 16    | 4.4722e-2 – 0.0411 | 5.7604e-2 – 0.0216 | 7.5149e-2 – 0.0199 |
| 32    | 2.2796e-2 0.0972 | 2.8439e-2 1.0183 | 3.6662e-2 1.0355 |
| 64    | 1.1489e-2 0.0802 | 1.3953e-2 1.0273 | 1.7712e-2 1.0496 |
| 128   | 5.7626e-3 0.0956 | 6.8480e-3 1.0268 | 8.5411e-3 1.0522 |

Table 8.2: Spatial convergence orders, errors and CPU time the scheme \((2.21)-(2.23)\) with fixed stepsize \(\tau = 1/2^{20}\) (Ex. 8.1)

| \(M\) | \(\alpha_1 = 0.2, \alpha_2 = 1.2\) | \(\alpha_1 = 0.5, \alpha_2 = 1.5\) | \(\alpha_1 = 0.7, \alpha_2 = 1.7\) |
|-------|---------------------------------|---------------------------------|---------------------------------|
|       | \(E(h, \tau)\) Order CPU(s) | \(E(h, \tau)\) Order CPU(s) | \(E(h, \tau)\) Order CPU(s) |
| 4     | 1.0743e-3 – 378.44 | 1.0073e-3 – 365.83 | 9.2285e-4 – 362.97 |
| 6     | 2.1008e-4 4.0248 | 1.9709e-4 4.0234 | 1.7724e-4 4.0694 | 528.73 |
| 8     | 6.664e-5 3.9910 | 6.2494e-5 3.9927 | 5.3393e-5 4.1706 | 720.86 |
| 10    | 2.7652e-5 3.9421 | 2.5945e-5 3.9396 | 2.0317e-5 4.3301 | 868.36 |

Table 8.3: Convergence orders and errors of the scheme \((2.21)-(2.23)\) (Ex. 8.2)

| \(h = 1/16, \sigma = 1/16\) | \(\tau = 1/2^{20}, \sigma = 1/128\) | \(\tau = 1/2^{16}, h = 1/16\) |
|-----------------------|-----------------------|-----------------------|
| \(N\) | \(E(h, \tau, \sigma)\) Order | \(M\) | \(E(h, \tau, \sigma)\) Order | \(J\) | \(E(h, \tau, \sigma)\) Order |
| 16    | 4.4722e-2 – | 4 | 8.5302e-5 – | 2 | 2.8526e-3 – |
| 32    | 2.2796e-2 0.9722 | 6 | 1.5113e-5 4.2683 | 4 | 7.3243e-4 1.9615 |
| 64    | 1.1489e-2 0.9885 | 8 | 3.6059e-6 4.9811 | 6 | 3.3198e-4 1.9516 |
| 128   | 5.7626e-3 0.9955 | 10 | 8.4501e-7 6.5024 | 8 | 1.9127e-4 1.9167 |
Figure 8.1: Numerical solution and relative error at $T = 0.5$ with $\alpha_1 = 0.55$ and $\alpha_2 = 1.1$ (Ex. 8.3)

Figure 8.2: Numerical solution and relative error at $T = 0.5$ with $\alpha_1 = 0.75$ and $\alpha_2 = 1.5$ (Ex. 8.3)
Table 8.4: Temporal convergence orders and errors of the scheme \(2.21\)–\(2.23\) with fixed stepsize \(h = 1/16\),

| \(N\) | \(E(h, \tau)\) | \(\text{Order}\) | \(E(h, \tau)\) | \(\text{Order}\) | \(E(h, \tau)\) | \(\text{Order}\) |
|------|----------------|-----------------|----------------|-----------------|----------------|----------------|
| 16   | 1.9716e-1      | –               | 1.0473e-1      | –               | 6.6505e-2      | –               |
| 32   | 1.7182e-1      | 0.1984          | 7.5392e-2      | 0.4742          | 4.3388e-2      | 0.6161          |
| 64   | 1.4973e-1      | 0.1985          | 5.3799e-2      | 0.4868          | 2.7687e-2      | 0.6481          |
| 128  | 1.3045e-1      | 0.1989          | 3.8218e-2      | 0.4933          | 1.7431e-2      | 0.6676          |

Figure 8.3: Numerical solution and relative error at \(T = 0.5\) with \(\alpha_1 = 0.95\) and \(\alpha_2 = 1.9\) (Ex. 8.3)