Complete Bundle Moduli Reduction in Heterotic String Compactifications

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Abstract

A major problem in discussing heterotic string models is the stabilisation of the many vector bundle moduli via the superpotential generated by world-sheet instantons. In arXiv:1110.6315 we have discussed the method to make a discrete twist in a large and much discussed class of vector bundles such that the generation number gets new contributions (which can be tuned suitably) and at the same time the space of bundle moduli of the new, twisted bundle is a proper subspace (where the 'new', non-generic twist class exists) of the original bundle moduli space; one thus gets a model, closely related to the original model one started with, but with enhanced flexibility in the generation number and where on the other hand the number of bundle moduli is somewhat reduced. Whereas in the previous paper the emphasis was on examples for the new flexibility in the generation number we here classify and describe explicitly the twists and give the precise reduction formula (for the number of moduli) for SU(5) bundles leading to an SU(5) GUT group in four dimensions. Finally we give various examples where the bundle moduli space is reduced completely: the superpotential for such rigid bundles becomes a function of the complex structure moduli alone (besides the exponential Kahler moduli contribution).

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1 Introduction

We consider supersymmetric heterotic string models in four dimensions (4D) arising by compactification of the ten-dimensional theory on a Calabi-Yau threefold $X$ endowed with a holomorphic vector bundle $V' = (V, V_{hid})$ with $V$ a stable bundle embedded in the visible $E_8$ whose commutant gives the unbroken gauge group in 4D. We restrict our attention to $V$ and assume $c_1(V) = 0$.

Besides the geometric (Kahler and complex structure) moduli of $X$ one gets moduli from the parameters of the bundle construction. As for the other moduli one wants to stabilise these moduli to particular values. This represents usually a formidable problem.

So it would be of interest to have a bundle construction which has no bundle moduli at all. One way to reach such a rather unusual situation is to start with an ordinary bundle construction, which comes with a bundle moduli space $\mathcal{M}_V$ of a large dimension

$$\dim \mathcal{M}_V = h^1(\text{End} V)$$

and to make twists which are available only over a subset $\mathcal{S}$ of the bundle moduli space $\mathcal{M}_V$ (cf. [3] for a systematic introduction which also contains further references to related work): turning on such a twist will restrict the moduli to $\mathcal{S}$ if the twist is discrete. The availability of the twist poses a number $\text{CON}_\gamma$ of conditions

$$\#\{\text{conditions for availability of the discrete twist } \gamma\} =: \text{CON}_\gamma$$

and reduces thereby the number for the new, twisted bundle $V'$ to

$$h^1(\text{End} V') = h^1(\text{End} V) - \text{CON}_\gamma$$

(in our spectral example the number of relevant $C$-deformations (cf. below) is just reduced accordingly; no new, other moduli appear after the restriction to $\mathcal{S}$). If the twist really exists, as we assume, one can not, of course, put more conditions than free moduli available; that is we will assume always that we are in a case $\text{CON}_\gamma \leq h^1(\text{End} V)$.

Clearly the goal in this line of thought is to make the number $h^1(\text{End} V')$ of remaining moduli as small as possible. In the present paper we will give various examples where one can reach $\text{CON}_\gamma = h^1(\text{End} V)$, i.e. rigid bundles, thus leaving only a zerodimensional space, i.e. isolated values for all the bundle moduli (it might be a single point of the original space $\mathcal{M}_V$). To avoid misunderstandings we point out that this is not ordinary moduli stabilisation: we do not stabilise the bundle moduli of the original bundle $V$; rather we switch to a new, closely related bundle $V'$ whose numerical, i.e. cohomological
characteristics are however rather ‘near’ to the original bundle (and which has even more ‘tuning freedom’ from new discrete parameters) and whose bundle moduli space is a zerodimensional subspace of the original space \( M_V \).

It is interesting to note that the issue of this type of moduli reduction converges also with another line of research in the heterotic context: in [7] heterotic constructions are made which exist only for a subset of the complex structure moduli space of an original model, leading to a corresponding reduction of freedom in that moduli space.

Structure of the paper

In sect. 2 we recall the set-up [3] of the non-generic twists in the spectral bundle construction of the heterotic string. In sect. 3 we given an overview over the twist classes and describe them explicitly. In sect. 4 we choose the case of a 4D GUT group \( SU(5) \) which corresponds to an \( SU(5) \) vector bundle and determine \( CON_\gamma \) for the main example (out of two possibilities) for \( \gamma \) in this case (the more technical proof of the part (iii) of proposition 2 is postponed to the appendix); finally we give a list of examples of rigid bundles and conclude with an outlook on some perspectives for further research in sect. 5.

2 A concrete set-up

We consider spectral \( SU(n) \) vector bundles on an elliptic Calabi-Yau space \( \pi: X \to B \) with section\(^2\) \( \sigma \). In this case one has \( V = p_*(\mathcal{P} \otimes p^*_C L) \) where (for this standard construction cf. [1]) one chooses a (ramified) \( n \)-fold cover surface \( C \subset X \) over \( B \), of cohomology class \( n\sigma + \pi^* \eta \) with\(^3\) \( \eta \in H^{1,1}(B) \), and a line bundle \( L \) over \( C \); \( \mathcal{P} \) is the Poincare bundle over \( X(1) \times_B X(2) \) restricted here to \( X \times_B C \) and \( p \) and \( p_C \) the projections to the first and second factor, respectively.

The condition \( c_1(V) = 0 \) will fix \( c_1(L) \) in \( H^{1,1}(C) \cap H^2(C, \mathbb{Z}) \) up to a class \( \gamma \) in \( ker(\pi_{C*}) \): one has \( c_1(L) = \frac{n\sigma + \eta + \pi^* \gamma}{2} + \gamma \) where one has \( \pi_{C*} \gamma = 0 \) (here \( \pi_C : C \to B \) is the restricted projection; we will usually suppress the pull-back notation and write just \( \phi \) for \( \pi^* \phi \) or \( \pi_{C*} \phi \)). If one assumes that \( C \) is ample one has \( H^{1,0}(C) = 0 \) and \( L \) is determined by its first Chern class (no further continuous moduli occur); then also the curve \( A_B := C \cap B \subset B\)

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\(^2\)We will identify notationally \( \sigma \), its image and the divisor and cohomology class of that image; we also use the notation \( c_1 := c_1(B) \), often with the pull-back to \( X \) or \( C \) understood, cf. [1].

\(^3\)There are further conditions: note first that the effectiveness of \( C \) entails the effectiveness of \( \eta \); furthermore the irreducibility of \( C \) (which one needs to assume for the stability of \( V \)) is given just for \( \eta - nc_1 \) effective and the linear system \( |\eta| \) being base-point free; the latter condition holds on a Hirzebruch surface \( F_k \) if \( \eta \cdot b \geq 0 \) and on a del Pezzo surface \( dP_k \) with \( 2 \leq k \leq 7 \) if \( \eta \cdot E \geq 0 \) for all curves \( E \) with \( E^2 = -1 \) and \( E \cdot c_1 = 1 \) (such curves generate the effective cone) (for notation cf. the end of sect. 4.2).
of class \( \tilde{\eta} := \eta - nc_1 \) is ample. The equation for \( C \) is given by \( w = a_0 z^2 + a_2 x z + a_3 y z + a_4 x^2 + a_5 x y = 0 \) for the cases (which are also phenomenologically the most important ones) \( n = 4 \) or \( 5 \), resp. (with \( a_5 = 0 \) for \( n = 4 \); here \( x, y, z \) are Weierstrass coordinates of the elliptic fibre and \( a_i \) are global sections of \( \mathcal{O}_B(\eta - i c_1) \). One can consider also the \( \tau \) action \( y \rightarrow -y \) which gives the inverse in the group law on the fibers.

If one wants to describe the possible freedom one has in choosing \( \gamma \), generically one can say only the following: the only obvious classes on \( C \) are, besides the section \( \sigma|_C \), the pull-back classes \( \pi^* \phi \) where the class \( \phi \) comes from the base. One finds \([1]\) that \( \pi_C \sigma|_C = \tilde{\eta} := \eta - nc_1 \) and so the only class in \( \ker(\pi_C^* \gamma) \) available in general is

\[
\gamma = n \sigma|_C - \pi_C^* \tilde{\eta}
\]

(or suitable multiples \( \lambda \gamma \) of it; at this point an integrality issue occurs which we do not make explicit here; important is that \( \lambda \) has only discrete freedom). For the (negative of the) generation number \( N_{gen} \) one gets \( \frac{1}{2} c_3(V) = \lambda \eta \tilde{\eta} \) (cf. \([2]\)).

Now let us assume that, at least for a certain subset \( \mathcal{S} \) of the moduli space \( \mathcal{M}_V \), further divisor classes on \( C \) exist (such that further corresponding cohomology classes, denoted by \( \tilde{\chi} \) below, in the expression for \( \gamma \) can occur). Then we can make a more general ansatz for the cohomology class \( \gamma \) (where \( \rho \) here is still a class coming from the base)\(^5\)

\[
\gamma = n \sigma + \rho + \tilde{\chi}
\]

The condition \( \pi_C \gamma = 0 \) amounts now to \( n(\tilde{\eta} + \rho) + \pi_C \tilde{\chi} = 0 \); to secure the divisibility of \( \pi_C \tilde{\chi} \) by \( n \) we are led to the slightly modified ansatz \( \tilde{\chi} := n \chi \), that is

\[
\gamma = n(\chi + \sigma) + \rho = n(\chi + \sigma) - \pi_C^* (\chi + \sigma)
\]

(2.3)

In the last rewriting we made manifest the condition on \( \rho \) which guarantees\(^6\) \( \gamma \in \ker(\pi_C^*) \) (again one may also consider suitable multiples \( \lambda \gamma \) and an integrality issue occurs \([3]\)).

If one turns on, as specified in (2.3), a non-pull-back class \( \tilde{\chi} = n \chi \) in the twist one gets (using \( \pi_C^* \sigma = \tilde{\eta} = \eta - nc_1 \); the formula is further evaluated in (4.5), cf. also \([3]\))

\[
-N_{gen} = \lambda \left[ \eta \tilde{\eta} + \pi_C^* \chi \cdot \pi_C^* \sigma - n \pi_C^* (\chi \cdot \sigma) \right]
\]

(2.4)

(the first terms in the [...] brackets are the standard terms, the rest the corrections).

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\(^4\)with the common abuse of notation (on our rational base surface \( B \)) to denote divisor classes by symbols for corresponding cohomology classes

\(^5\)the pull-back operation itself is suppressed, so \( \rho \) is actually \( \pi_C^* \rho \); if no confusion arises we will also suppress in the following the restriction and write just \( \sigma \) for the class \( \sigma|_C \)

\(^6\)Note that the final term \( \pi_C^* (\chi + \sigma) \) is, in itself, a class projected down to \( B \); if it occurs, as it is the case here, in a formula for the class \( \gamma \) on \( C \), then this means that it has to be read as being pulled-back to \( C \), i.e. this means actually the class \( Q := \pi_C^* \pi_C^* (\chi + \sigma) \); so both terms in the final expression \( P - Q \) on the right hand side of (2.3) fulfill \( \pi_C^* P = n \pi_C^* (\chi + \sigma) = \pi_C^* Q \), such that \( \gamma \in \ker(\pi_C^*) \).
3 The non-generic twist classes

3.1 The two main example classes

In the following two examples of a non-generic twist class one uses the idea that under special conditions on the (bundle) moduli one of the generically present classes $\pi_C^* \phi$ and $\sigma|_C$ can become reducible; then a component of this reducible class represents typically a 'new' class which can be used for a non-generic twist.

The **first case** is the set-up where under certain conditions the preimage $C := \pi_C^{-1}(c)$ of a curve $c \subset B$ becomes reducible in $C$ (a 'vertical' decomposition of a standard class)

*vertical decomposition* \[ C = \pi_C^{-1}(c) = C_1 + C_2 \] \hspace{1cm} (3.1)

We add a remark: whereas the full preimage $C = \pi_C^{-1}(c)$ is an $n$-fold cover of the base curve $c$, a component $C_i$ (which may of course itself be again reducible) is an $k_i$-fold cover of $c$, such that $n = k_1 + k_2$ (we adopt the convention $k_1 \geq k_2$). If none of the $k_i$ is equal to 1 we say we are in the case of an ordinary vertical decomposition, while if one of the $k_i$ is equal to 1 we speak of a special vertical decomposition:

*ordinary vertical decomposition* \[ k_1 \neq 1 \text{ and } k_2 \neq 1 \] \hspace{1cm} (3.2)

*special vertical decomposition* \[ k_1 = 1 \text{ or } k_2 = 1 \] \hspace{1cm} (3.3)

(Ordinary vertical decompositions exist for $n \geq 4$.) We note that one will always have the special vertical decomposition related to $c = A_B = C \cap B \subset B$ which can also be considered as $A_C := B \cap C = \sigma|_C \subset C$ and to lie in $C$ and thus gives itself a component $C_2$ of $C = \pi_C^{-1}(c)$; in this case, however, the other component $C_1$ in this $(n - 1) + 1$ split will not, of course, represent a new class as one has $C_1 = \pi_C^* c - \sigma|_C$. We call this the trivial special vertical decomposition.

In the **second case** the curve $\sigma|_C = A_C$ becomes reducible in $C$ (or equivalently in $B$) under certain conditions (a 'horizontal' decomposition)

*horizontal decomposition* \[ \sigma|_C = D_1 + D_2 \] \hspace{1cm} (3.4)

One has the following injective association relation between these different new classes

\{horizontal decomposition classes\} $\hookrightarrow$ \{nontrivial special vertical decompositions\} \hspace{1cm} (3.5)

(under $D_i \rightarrow c := D_i$). To see this note that if one takes $D_i$ as\(^7\) $c \subset B$ the preimage $C$ decomposes always as a $(n - 1) + 1$ split because $D_i = c$ lies (as $C_2$) also in the surface $C$.

\(^7\)recall that $\sigma|_C = A_C = A_B = B \cap C$ can be considered at the same time to lie in $C$ and in $B$
Note further that on a fibre $F_b$ over $b \in c$ the $n$ points $q_i$ of $C$ satisfy $\sum_{i=1}^{n} q_i = 0$ in the group law of the elliptic fibre. By contrast one does not necessarily have, in the case of a vertical decomposition, $\sum_{i=1}^{k_1} q_i = 0$ (or equivalently $\sum_{i=k_1+1}^{n} q_i = 0$). If this happens in a vertical decomposition we will call it spectral

\[ \text{spectral vertical decomposition} \quad \sum_{i=1}^{k_1} q_i = 0 \quad (3.6) \]

We will also call the class $C_1$ (and equally then $C_2$) spectral. One gets then the following $1:1$ association relation (as being spectral for a 1-cover means just to lie in $B$)

\{ horizontal decomposition classes \} \overset{1:1}{\leftrightarrow} \{ spectral nontrivial special vertical decompositions \} \quad (3.7)

Clearly it is desirable to have a criterion in term of a new class $\chi$ alone which allows to decide whether it is spectral (i.e. whether its fibre points sum to zero). Here one has the following (note that even if $\chi$ is $\tau$-invariant the full preimage $\pi_C^* \pi_{C^*} \chi$ is usually not, not to speak of the total spectral surface $C$; note also that $k_2 = 1$ or $2$ for $3 \leq n \leq 5$)

**Lemma** Let $\chi = C_1$ or $C_2$ be a 'new' class from a vertical decomposition.

(i) Let $3 \leq n$: $\chi$ is $\tau$-invariant $\implies$ $\chi$ is spectral.

(ii) Let $3 \leq n \leq 5$: $C_2$ is $\tau$-invariant $\iff$ $\chi$ is spectral.

### 3.2 The effective twist classes

Clearly in both cases the new (i.e. not generically available) divisor class is given by an effective curve. Let us consider more generally the case where a given new divisor class contains an effective curve; we will call such a class an effective twist class. So one has

\{ vertical decomposition classes \} $\subset$ \{ effective non-generic twist classes \} \quad (3.8)

Let us ask whether the examples provided by the vertical decompositions classes exhaust already the effective non-generic twist classes. So let $h^0(C, \mathcal{O}_C(\chi)) > 0$ and assume we have in the expression for this cohomological number (which effectively depends only on the linear equivalence class of the divisor) a general member of this class already exchanged by an effective member; in other words $\chi$ should denote already an effective divisor on $C$. Consider then $\mathcal{C} := \pi_C^{-1}(\pi_C(\chi))$: this is an ordinary pullback class $\pi_C^{-1}(c)$ (with $c = \pi_C(\chi)$) which furthermore set-theoretically contains $\chi$; as $\chi$ is assumed to be a new class, it must be a proper subvariety of this pullback curve; but the assumed existence of this subvariety means that $\chi$ is a component of the (then reducible) curve $\mathcal{C}$.

\{ vertical decomposition classes \} $=$ \{ effective non-generic twist classes \} \quad (3.9)

We will adopt the terminology 'special/ordinary' also for the effective twist classes.
4 The spectral ordinary effective twist classes for $n=5$

We will focus now on the case $n=5$ and consider certain non-generic twist classes which are effective (non-effective ones are, of course, just differences of the effective ones). We concentrate here on the ordinary classes which are spectral. In sect. 4.1 we characterize this abstractly defined type of new classes by an explicit construction and give a precise condition for the occurrence of this type of classes; in sect. 4.2 we compute the number of conditions relevant here and specify precisely the assumptions one needs to make these computations (the most technical part is postponed to the appendix); in sect. 4.3 we give some cases where the number of necessary conditions is so large that all bundle moduli are already frozen by choosing the corresponding 'new' (i.e. non-generic) twist class.

4.1 Characterization of the new classes

Consider in a first, preliminary step the following factorisation

$$w = (f_1 z + g_1 x + h_1 y) (f_2 z + g_2 x)$$

(4.1)

of the spectral cover equation (here $f_1, g_1, h_1, f_2, g_2$ are sections of suitable line bundles over $B$). If the coefficients $a_i$ can be written in this special way one gets (from the ensuing expressions for the $a_i$ like $a_0 = f_1 f_2, a_2 = f_1 g_2 + f_2 g_1$ and so on) the relation

$$\text{Res} := a_0 a_5^2 - a_2 a_3 a_5 + a_2^2 a_4 = 0$$

(4.2)

(this means here identical vanishing over all of $B$, not yet an equation for a curve in $B$).

Now we do in a second step the proper construction: let $c$ denote a smooth irreducible reduced curve in $B$ and assume, instead of the global factorisability, only the following factorisability of $w$ over the elliptic surface $E_c := \pi^{-1}(c)$ (with $F_1 := f_1|_c$ and so on)$^8$

$$w|_{E_c} = (F_1 z + G_1 x + H_1 y) (F_2 z + G_2 x)|_{E_c}$$

(4.3)

If condition (4.3) is fulfilled one has the ordinary vertical decomposition (3.1) with $C_1$ and $C_2$ corresponding to the first and second factors in (4.3), respectively: the five-fold cover $C$ of $c$ decomposes into a triple cover $C_1$ and a double cover $C_2$; the individual factors have itself again the form of a spectral cover polynomial for $n=3$ and $2$, respectively, so the decomposition is spectral (i.e. the points in the individual $C_i$ sum to zero fibrewise).

Note also that conversely, if one has such a component of $C$ which is spectral and has

$^8$ where $F_1, G_1, H_1, F_2, G_2$ are now sections of suitable line bundles over $c$: for example one has that $F_1 \in H^0\left(c, \mathcal{O}_c\left((\eta - 2c_1)|_c - (G_2)\right)\right)$ as $A_2 = F_1 G_2 + F_2 G_1$ (with $A_i := a_i|_c$) and so on
good projection (i.e. \( c = \pi_C(\chi) \) is smooth irreducible reduced) then there will exist a corresponding spectral cover curve equation which must be a factor in the sense of (4.3).

So assume now conversely we have given an ordinary\(^9\) effective non-generic twist class \( \chi \) which is spectral and has good projection; we had seen in (3.9) that it actually comes from a vertical decomposition over \( c := \pi_C(\chi) \) which in turn implies, as just pointed out, for spectral classes the factorization (4.3). So one gets \textit{in this case} the explicit description

\[
(3.1) \iff (4.3) \quad (4.4)
\]

For example, one gets\(^10\) in this example by taking\(^11\) \( \chi = C_2 \) (as cohomology class) \([3]\)

\[
-N_{gen} = \lambda[\eta\bar{\eta} + (2\eta - 5(g_2))c] \quad (4.5)
\]

Furthermore one gets as necessary condition for the factorizability over \( c \) that the curve given in (4.2) (now read as an equation for a curve in \( B \) and not as an identical vanishing over all of \( B \)) has \( c \) as a component, i.e. that the equation (4.2) is fulfilled along \( c \)

\[
\text{Res}_{|c} = A_0A_5^2 - A_2A_3A_5 + A_3^2A_4 = 0 \quad (4.6)
\]

This gives the following logical implication

\[
(4.3) \implies (4.6) \quad (4.7)
\]

Concerning the reverse arrow let us assume at first that \( c \cong \mathbf{P}^1 \): then the relation (4.6) is not only necessary but also sufficient to have (4.3). For this we show that if the relation (4.6) holds one can introduce various polynomials \( F_1, G_1, H_1, F_2, G_2 \) on \( c \) which turn out to have the necessary relations to the \( A_i \) to get (4.3), i.e.

\[
A_0z^2 + A_2xz + A_3xy + A_4x^2 + A_5xy = (F_1z + G_1x + H_1y)(F_2z + G_2x) \quad (4.8)
\]

So the relations one needs to derive are

\[
A_0 = F_1F_2 \quad (4.9) \\
A_2 = F_1G_2 + G_1F_2 \quad (4.10) \\
A_3 = H_1F_2 \quad (4.11) \\
A_4 = G_1G_2 \quad (4.12) \\
A_5 = H_1G_2 \quad (4.13)
\]

\(^9\) assume we have, as we can in the argument, \( \chi \) chosen already as an effective member of the divisor class

\(^{10}\) the first terms in the \([...]\) brackets on the right hand sides are the standard contributions, the terms proportional to \( c \) are the new contributions

\(^{11}\) taking \( \chi = C_1 \) instead of \( C_2 \) gives, with \((3\eta - 5(h_1))c\) as new term in (4.5), just the negative of the present new term (as \( \eta \cdot c = A_B \cdot c = (a_5) \cdot c = (h_1) \cdot c + (g_2) \cdot c \))
Now note first that because of (4.6) one has \( A_5 | A_3 A_4 \), so that one can write \( A_5 = H_1 G_2 \) with \( H_1 | A_3 \) and \( G_2 | A_4 \) for some polynomials \( H_1, G_2 \); let us write furthermore \( A_3 = H_1 F_2 \) and \( A_4 = G_1 G_2 \) with certain polynomials \( F_2, G_1 \). From (4.6) one gets \( A_5 | A_0 A_5 \) such that also \( A_5 F_2 | A_0 A_5 \) (as \( A_5 F_2 = A_3 G_2 \) ), such that \( F_2 | A_0 \) and one can write \( A_0 = F_1 F_2 \) for some \( F_1 \). From these determinations it follows already, once more with (4.6), that \( A_2 = (A_0 A_5^2 + A_3^2 A_4) / A_3 A_5 = F_1 G_2 + G_1 F_2 \). In other words one gets

\[
(4.3) \iff (4.6)
\]

Alternatively one might, to show this reverse implication, adopt the strategy of an argument of [1]. This relies on a certain interpretation of the relation (4.6) or, equivalently, \( c \subset (Res) \) (cf. (4.2)): the vanishing divisor \( (Res) \subset B \) can be interpreted as the locus of \( b \in B \) where \( \tau \)-conjugated fibre points \( q_i, q_j \) exist \( (i \neq j) \), i.e. points with \( q_i + q_j = 0 \) (note that \( \tau \)-conjugacy is just \( y \rightarrow -y \) in coordinates such that \( Res = \text{Resultant}(P, Q) \) is relevant where \( w = P(x) + Q(x)y \)). So this is the locus in whose preimage in \( C \) a spectral 2-fold cover point set splits off the total 5-fold spectral cover point set, so

\[
c \subset (Res) \iff \mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 \text{ with } \mathcal{C}_2 \text{ spectral}
\]

which shows in view of the Lemma in sect. 3.1 that (here \( c \) is assumed to be smooth irreducible reduced, respectively \( \chi \) is assumed to have good projection)

\[
(4.3) \iff \text{ordinary spectral effective vertical decomposition} \iff (4.6)
\]

Let us summarize some points we have encountered. If one restricts the bundle moduli (the degrees of freedom coming from the \( a_i \)) by posing the condition (4.6) along \( c \), one gets the factorization of the equation (4.3) for \( \mathcal{C} \) and thus the decomposition (3.1) which defines the 'new' class of \( \mathcal{C}_1 \) (or equivalently the 'new' class \( \mathcal{C}_2 \)). Asking conversely which moduli restriction is enforced by demanding the existence of this class (because it is used in a discrete twist) one has to take into account two things: first, what one really uses in the twist construction is a line bundle, thus a divisor class on \( C \); so, for the purpose of our procedure, one has to make sure that an effective representative in this class exists; this is the reason we have assumed from the outset that we treat the case of an effective non-generic twist class. One also has to clarify whether the existence of such a curve (which we hope to play the role of \( \mathcal{C}_1 \)) can arise only in the way (3.1), resp. (4.3), or whether it may exist 'accidentally' already on a larger moduli subspace than the one given by (4.6) (where it exists 'naturally'); this issue is treated and solved in (4.16).

\[\text{here } A_5 \neq 0 \text{ as otherwise } c \subset (a_5) = A_B \text{ and one would be in a special case (of a } 4+1 \text{ split instead of a } 3+2 \text{ split); above we did assume } A_3 \neq 0 \text{: otherwise } A_0 = 0 \text{ and one can take } F_2 = 0, G_2 = 1\]
4.2 The question of moduli reduction

We now want to count the precise number of the conditions which describe the circumstances in which the type of 'new' class considered in (4.16) exists, cf. (1.2). We assume that $c$ is smooth irreducible and reduced (this is an assumption on $c$ if one starts from $c$; it restricts the given twist class $\chi$ on $C$ with $c = \pi_C(\chi)$ to have good projection if one starts from $\chi$; the interesting possibilities of the case of $c$ being reducible will not be studied in the present paper, cf. the discussion in the Outlook in sect. 5).

We remark that $CON_{\gamma}$, where $\gamma = n\chi - \pi_{C*}\chi + n\sigma - \bar{\eta}$, will be the same for $\chi = C_1$ or $C_2$ as their sum, the pullback class $C := \pi_C^{-1}(c)$, exists universally. We will therefore denote $CON_{\gamma}$ henceforth simply by $CON_c$ (we have fixed that we are in the ordinary case, i.e. in the case of the $3 + 2$ split of $n = 5$).

Let us count now the number of conditions imposed on the $a_i$ by the factorization (4.3). As described above this implies the vanishing (4.6). Consideration of the association $(a_i) \longrightarrow A_0A_3^2 - A_2A_3A_5 + A_3^2A_4$ gives a map $\mathcal{M}_V \longrightarrow |(3\eta - 10c_1)|_c$ or alternatively, before taking the overall scaling,

$$\bigoplus_{i=0, i\neq 1}^5 H^0(B, \mathcal{O}_B(\eta - ic_1)) \longrightarrow H^0(c, \mathcal{O}_c((3\eta - 10c_1)|_c)) \quad (4.17)$$

We thereby get the following proposition for the number $CON_c$ of conditions posed inside $\mathcal{M}_V$ by demanding (4.6)

**Proposition 1**

(i) One has the estimate

$$CON_c \leq h^0(c, \mathcal{O}_c((3\eta - 10c_1)|_c)) \leq \frac{1}{2}(6\bar{\eta} + 11c_1 - c)c \quad (4.18)$$

(ii) if the map (4.17) is surjective one has an equality in the first inequality in (i)

(iii) if the class $3\bar{\eta} + 6c_1 - c$ is effective one has an equality in the second inequality in (i)

(iv) if the map (4.17) is surjective and the class $3\bar{\eta} + 6c_1 - c$ is effective one has

$$CON_c = \frac{1}{2}(6\bar{\eta} + 11c_1 - c)c \quad (4.19)$$

The assertions (i), (ii) are obvious. To evaluate $h^0(c, \mathcal{O}_c((3\eta - 10c_1)|_c))$ precisely we look for assumptions implying $h^1(c, \mathcal{O}_c((3\eta - 10c_1)|_c))=h^0(c, \mathcal{O}_c((c - c_1 - (3\eta - 10c_1))|_c))=0$. Clearly it will be sufficient to assume that $(3\eta - 9c_1 - c)|_c$ is effective as a divisor on $c$. We will just assume that $3\eta - 9c_1 - c = 3\bar{\eta} + 6c_1 - c$ is effective as a divisor on $B$. Then one gets $h^0(c, \mathcal{O}_c((3\eta - 10c_1)|_c)) = (3\eta - 10c_1)c + \frac{1}{2}e(c) = (3\bar{\eta} + 5c_1)c + \frac{1}{2}(c_1 - c)c$. 

Let us now come back to point (ii): one gets an equality in the first inequality in (4.18) if the map (4.17) is surjective (otherwise one would overcount the real number of conditions on the $a_i$). To make the proposition usefully applicable one has to find a criterion (implying the surjectivity in question) which can be checked easily. Now the map in question can be factorised, in two different ways, as follows

$$
\bigoplus_{i=0,\neq 1}^5 H^0\left(B, \mathcal{O}_B(\eta - ic_1)\right) \stackrel{\alpha}{\longrightarrow} H^0\left(B, \mathcal{O}_B(3\eta - 10c_1)\right)
$$

$$
\downarrow \quad \quad \downarrow \beta
$$

$$
\bigoplus_{i=0,\neq 1}^5 H^0\left(c, \mathcal{O}_c((\eta - ic_1)|c)\right) \longrightarrow H^0\left(c, \mathcal{O}_c((3\eta - 10c_1)|c)\right)
$$

Here elements in the left upper space are mapped as follows

$$
(a_i) \quad \longrightarrow \quad a_0a_5^2 - a_2a_3a_5 + a_4^2a_4
$$

$$
\downarrow \quad \quad \downarrow
$$

$$
(A_i) \quad \longrightarrow \quad A_0A_5^2 - A_2A_3A_5 + A_4^2A_4
$$

Now clearly it will be sufficient for the surjectivity of the total map if both factor maps in a chosen factorisation are surjective. From the two possible paths, the upper way (using as horizontal map the upper map) and the lower way (using as horizontal map the lower map), we will choose the upper way (the lower way is less suitable for our purposes as it leads to stronger conditions). One gets then the following (here assertion (ii) follows as $H^1\left(B, \mathcal{O}_B(3\eta - 10c_1 - c)\right)=0$ holds for $3\eta - 9c_1 - c=3\bar{\eta} + 6c_1 - c$ being ample; part (iii) will be shown in the appendix)

**Proposition 2**

(i) The map (4.17), i.e. the map $\beta \circ \alpha$, is surjective if the maps $\alpha$ and $\beta$ are surjective

(ii) the map $\beta$ is surjective if $3\bar{\eta} + 6c_1 - c$ is ample

(iii) the map $\alpha$ is surjective if $c_1$ and $\bar{\eta}$ are both big and nef.$^{13}$

Note that in the standard examples$^{14}$ for the rational base $B$ the class $c_1$ is always ample except for $F_2$ (where it is however still big and nef) and the Enriques surface (where it is not even effective). The class $\bar{\eta}$ of the curve $A_B = \sigma \cap B \subset B$ is always effective.

---

$^{13}$A divisor $D$ is big if $D^2 > 0$ and nef if $DD' \geq 0$ for all effective divisors $D'$; an ample divisor $D$ is in particular big and nef (the precise condition is being big and $DD' > 0$ for all effective divisors $D' \neq 0$)

$^{14}$Which are given by the Hirzebruch surfaces $F_k$ ($k = 0, 1, 2$), the del Pezzo surfaces $dP_k$ ($k = 0, \ldots, 8$) and the Enriques surface, cf. the explanations given in the Remark below.
Putting everything together one arrives at the following

**Theorem.** Assume that $3\eta + 6c_1 - c$ is ample and that $c_1$ and $\eta$ are both big and nef: then one has

$$CON_c = \frac{1}{2}(6\eta + 11c_1 - c)c \quad (4.20)$$

Note that if $3\eta + 6c_1 - c$ is ample then the condition from proposition 1 that $3\eta + 6c_1 - c$ is effective is redundant as this class will then be such that sufficiently large multiples of it will be (cf. footn. 16) effective as will be then also the class itself (in both cases up to linear equivalence what is sufficient for the use in proposition 1).

**Remark:** For convenience of the reader we recall the essentials about the base surfaces which constitute the standard examples (we will have no need to consider the Enriques surface). The corresponding facts will be used below when we display some examples.

### The Hirzebruch surfaces
The surface $F_k$ is a $\mathbb{P}^1$-fibration over a base $\mathbb{P}^1$ denoted by $b$ (the fibre is denoted by $f$; as no confusion arises $b$ and $f$ will denote also the cohomology classes). One has $c_1(F_k) = 2b + (2 + k)f$ and the curve $b$ of $b^2 = -k$ is a section of the fibration; there is another section ("at infinity") having the cohomology class $b_\infty = b + kf$ and the self-intersection number $+k$; note that $b_\infty \cdot b = 0$. A class

$$(x, y) := xb + yf$$

is ample exactly if\(^{15}\) $(x, y) \cdot f > 0$ and $(x, y) \cdot b > 0$, i.e. if $x > 0, y > kx$. An irreducible non-singular curve of class $xb + yf$ exists exactly if\(^{15}\) the class lies in the ample cone (generated by the ample classes) or is one of the elements $b, f$ or $ab_\infty$ (the last only for $k > 0$; here $a > 0$) on the boundary of the cone; these classes together with their positive linear combinations span the effective cone $(x, y \geq 0)$. $c_1$ is ample for $F_0$ and $F_1$, whereas for $F_2$, where $c_1 = 2b_\infty$ (such that $c_1 \cdot b = 0$), it lies on the boundary of the cone.

### The del Pezzo surfaces
The surfaces $dP_k$ are the blow-up of $\mathbb{P}^2$ at $k$ points $P_i$ for $k = 0, \ldots, 8$ (lying suitably general, i.e. no three points lie on a line, no six on a conic); the exceptional curves from these blow-ups are denoted by $E_i$, $i = 1, \ldots, k$ (one has $dP_1 \cong F_1$ with $E_1$ corresponding to $b$). The intersection matrix for $H^{1,1}(dP_k)$ in the basis $(l, E_1, \ldots, E_k)$, with $l$ the proper transform of the line $\tilde{l}$ from $\mathbb{P}^2$, is just $Diag(1, -1, \ldots, -1)$; furthermore one has

$$c_1(dP_k) = 3l - \sum_i E_i \quad (4.22)$$

such that $c_1^2(dP_k) = 9 - k$.

\(^{15}\)Cf. Corollary 2.18, Chap. V, *Algebraic Geometry*, R. Hartshorne, Springer Verlag (1977).
4.3 An application: rigid bundles

Finally we recall the number of vector bundle moduli (we assume that $C$, or equivalently $\bar{\eta}$, is ample, so no $\tau$-odd moduli occur). This is computed from the number of different possible shapes of $C$ inside $X$, i.e. by $h^0(X, \mathcal{O}_X(C)) - 1$ which in turn is computed either directly [9] or from the corresponding numbers of degrees of freedom in the coefficients $a_i$ of the spectral cover equation $w = 0$ for $C$ [1]. To get explicit results one assumes in both cases that $\bar{\eta}$ is ample (and $c_1$ also ample or at least big and nef). One then gets

$$h^1(\text{End} V) = n - 1 + \left(\frac{n^3 - n}{6} + n\right)c_1^2 + \frac{n}{2}(\bar{\eta} + nc_1)\bar{\eta} + \bar{\eta}c_1$$  \hspace{1cm} (4.23)

We get for the general equation $h^1(\text{End} V) = CON_c$ in our case

$$8 + 50c_1^2 + 5\bar{\eta}^2 + 27\bar{\eta}c_1 = (6\bar{\eta} + 11c_1 - c)c$$  \hspace{1cm} (4.24)

Here are some solutions to this equation ($3\bar{\eta} + 6c_1 - c$ is always ample; recall $\eta = \bar{\eta} + 5c_1$)

| $B$ | $F_0$ | $F_1$ | $F_2$ | $P^2$ | $dP_5$ | $dP_7$ | $dP_8$ |
|-----|-------|-------|-------|-------|--------|--------|--------|
| $\bar{\eta}$ | (3, 5) | (3, 7) | (3, 8) | 24l   | $2c_1$ | $3c_1$ | $2c_1$ |
| $c$   | (15, 21) | (16, 27) | (15, 36) | 38l   | $9c_1$ | $9c_1$ | $11c_1$ |

The classes $\bar{\eta}$ and $c$ occurring here are actually ample, so on $F_k$ and $P^2$ (where $l$ denotes the class of the general line) there exist smooth irreducible representative curves of the classes in question (for $dP_5$, $c_1$ is known to be very ample so again smooth irreducible representatives will exist by Bertini’s theorem). In all cases $c_1$ is effective so the further conditions, cf. footn. 3, amount just to $\bar{\eta}$ being effective (which obviously is the case) and the linear system $[\eta]$ being base-point free; the latter can be checked via the criteria mentioned in footn. 3 for the $F_k$, for $P^2$ (where it is obvious) and also for $dP_5$ and $dP_7$.

We stop with the checks performed on these special examples at this point although further issues in these particular models could and should be clarified. In the context of the present paper these examples serve, however, merely an illustrative purpose: we wanted to show that one can fix circumstances (cf. the assumptions in the Theorem of the previous subsection) in which one can determine the number of conditions posed on the bundle moduli by the presence of specific non-generic twists in a controlled manner (cf. the propositions; this concerns essentially prop. 1(i) that under specific assumptions one can make sure that the reducibility of the preimage $C = \pi_C^{-1}(c)$ of a base curve $c$ poses actually the maximum number of conditions one expects) such that one gets a precise formula for $CON_c$; and secondly we wanted to show that there is no obstacle in principle that this bundle moduli reduction is so effective that it becomes even complete.
Supersymmetric particle physics models coming from heterotic string theory have, like other string models, to deal with a large number of occurring moduli [9] and their potential stabilisation, here concretely geometric (Kähler and complex structure) moduli and bundle moduli. As the stabilisation of the latter is a difficult and complex task it is interesting to consider cases where only few or no bundle moduli occur. A possibility to achieve this is to make discrete modifications of a given bundle construction which are available only over a subset of the bundle moduli space such that the new, twisted bundle has less parametric freedom (i.e. turning on such discrete ‘twists’ constrains the moduli which thereby are restricted to a subset of their moduli space).

After occasional occurrence of this idea in the literature, usually in slightly different contexts like $F$-theory (cf. for example [4]), this was considered systematically in the heterotic context in [3]. There the question was investigated in the most intensly studied class of heterotic string models, that of spectral cover bundle constructions on elliptic Calabi-Yau spaces $X$ [1]. General formulae were given for the influence of the twist on the cohomological data, most prominently the generation number $N_{gen}$, given by $c_3(V)/2$, and various concrete examples were studied too.

In the present paper the emphasis is on the (bundle) moduli reduction effect of having turned on a discrete twist. The ‘new’ divisor class on the spectral cover surface $C$ (from which the class $\gamma$ is derived which corresponds to the line bundle twist) is here assumed to have an effective representative divisor $\chi$ such that geometrical methods can be brought to bear most directly. Such non-generic effective twists are described as arising from standard classes (like pullback classes $\pi_\ast C_c$, here with $c = \pi_{C_\ast} \chi$) which become reducible. After distinguishing and discussing various types of twist classes we focus on classes we call ‘spectral’ (where also for the component $\chi$, just as for a total pullback class $\pi_\ast C_c$, the points sum fibrewise to zero in the group law on the fibre); furthermore we assume that we are in the ‘ordinary’ case, as we call it, where the corresponding decomposition of $\pi_\ast C_c$ with $c = \pi_{C_\ast} \chi$ does not have a component lying itself already in the base (and thus would have covering degree 1 over $c$; such a case in general, even when the covering curve does not lie in the embedded base, we call ‘special’). These effective non-generic twist classes which are ordinary and spectral (and project smoothly to $B$) can be described quite explicitly by means of a factorization of the corresponding spectral cover equation (when restricted to the elliptic surface $E_c$ over $c$). The condition for such a factorization is provided, in one of the phenomenologically most important cases which is given by $n = 5$ (corresponding to an $SU(5)$ GUT group in four dimensions), by the relation (4.6).
To compute the moduli reduction effect one thus has to count the number of conditions imposed by this relation. This is straightforward in principle by computing the dimension of the space $H^0(c, \mathcal{O}_c((3\eta - 10c_1)|_c))$ in which the expression $A_0A^2_0 - A_2A_3A_5 + A^2_3A_4$ lives. There is, however, the problem to make sure that these expressions fill out that space completely; after all this element of the space $H^0(c, \mathcal{O}_c((3\eta - 10c_1)|_c))$ has by definition built in two restrictions: first being composed in the indicated manner from the expressions $A_i$, and secondly the latter themselves are restrictions from the corresponding global $a_i$ over $B$ to $c$; alternatively this restrictedness may be described in the reversed order: $A_0A^2_0 - A_2A_3A_5 + A^2_3A_4$ is restricted from the corresponding global expression $a_0a^2_0 - a_2a_3a_5 + a^2_3a_4$ which furthermore itself is not a general element of the space $H^0(B, \mathcal{O}_B(3\eta - 10c_1))$ in which it lives but is an element composed of the elements $a_i$ in the indicated manner. What we have described here is the problem pointed out in the diagrams before Proposition 2 in section 4.2. The solution to this problem is given in Proposition 2; the proof of its most nontrivial part is given in the appendix.

Armed with the precise formula for the reduction effect (most importantly with the precise conditions under which the formula does hold which is here decisive due to the subtleties described above) one can search for examples of complete moduli reduction. We give in section 4.3 various examples which satisfy the mentioned conditions.

**Outlook**

Several things can now be done. First one should extend the counting method for the moduli reduction effect given here also to other spectral (effective) twist classes of phenomenological interest: i.e. to the special classes for $n = 5$ and to the ordinary and special classes for $n = 4$ (corresponding to an $SO(10)$ GUT group in four dimensions). Having the corresponding formulae for $CON_i$ in these cases one may search for further rigid bundles and also for a comprehensive approach to the computation of the number (1.3) of reduced moduli which produces these results beyond the case-by-case analysis.

Even more important conceptually, in connection with the topic of rigid bundles, seem to be the following two points: first the treatment of the superpotential and secondly the issue of making the described potential 'rigidification' of a bundle an effect as universal as possible. Let us explain what we mean by these two lines of investigations.

First, having a rigid bundle, i.e. a heterotic string model which lacks any bundle moduli, the superpotential becomes now a 'function' of the complex structure moduli alone (besides the well-known explicit exponential dependence on the Kahler moduli; their number for $B$ a Hirzebruch surface or $\mathbb{P}^2$ is 3 or 2, respectively). As all of the difficult questions connected with describing the precise dependence of the world-sheet instanton
generated superpotential on the vector bundle moduli collapse here dramatically (as no continuous bundle moduli are left) one can hope to study now directly all sort of relevant questions, like (Kahler and complex structure) moduli stabilisation (and value of the scalar potential at the stabilised point) and preservation or breaking of supersymmetry, by making explicit the dependence on the remaining complex structure moduli (given in the coefficients \( g_2 \) and \( g_3 \) of the Weierstrass equation of the elliptic fibration). The explicit dependence of the superpotential on these geometric moduli has not been investigated very much (because it is usually intertwined with problems concerning the bundle moduli). Ideally one may think of an explicit expression for the superpotential in terms of (the moduli contained in) \( g_2 \) and \( g_3 \) (together with the known Kahler moduli dependence), something still out of reach at present in a nontrivial situation with vector bundle moduli (despite some progress for results on individual Pfaffian summands of the full superpotential and modest steps towards the an understanding of the latter [8]).

Secondly, in the present paper we have given examples of rigid bundles. To be used most widely, the procedure should be generalised as most as possible. By this we mean that, ideally, one starts with a (spectral cover) bundle \( V \) on \( X \) and can provide a non-generic discrete twist class on the spectral cover surface \( C \) such that the twisted bundle \( V' \) becomes rigid; thereby one would get for each spectral cover bundle a closely related bundle \( V' \) whose phenomenologically relevant cohomological data usually have, from new discrete input parameters, even more flexibility than the original bundle \( V \); thus for all phenomenological relevant questions, as far as expressed in the cohomological data, one could work just as well with \( V' \) instead of \( V \), but one would have solved the many hard problems posed by the presence of the vector bundle moduli, though not in the standard manner of ordinary moduli stabilisation but rather by some sort of rigidification or freezing. To carry out steps in this direction one clearly has to generalize the investigations of the present paper: after all the condition \( h^1(\text{End } V) = \text{CON}_r \) is, in explicit form (cf. (4.24)), a diophantine equation in variables with integral entries (discrete bundle parameteres and similar parameters for \( c = \pi_{C*} \chi \)); so usually one will get only some sort of reduction (corresponding to \( \text{CON}_r < h^1(\text{End } V) \)) instead of a complete reduction. To solve this problem one would like to combine the reduction effects of different twists. But turning on a combined twist will demand only that the combined class exists giving usually a less effective moduli reduction. But only in general. If one is in a set-up where the existence of the sum of the classes implies already the existence of the individual classes one can combine reduction effects (with the goal to reach a complete reduction). This will be described elsewhere.

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A Appendix: Proof of proposition 2 (iii)

In this appendix we give the proof of proposition 2 (iii) concerning the surjectivity of the map \((a_i) \longrightarrow a_0a_5^2 - a_2a_3a_5 + a_2^2a_4\) (the horizontal map in the 'upper way') used above. For convenience of the reader we start to state again the assertion.

**Proposition 2 (iii)** The map

\[
\bigoplus_{i=0, \neq 1}^5 H^0\left(B, \mathcal{O}_B(\eta - ic_1)\right) \longrightarrow H^0\left(B, \mathcal{O}_B(3\eta - 10c_1)\right),
\]

(A.1)

given by \((a_i) \longrightarrow a_0a_5^2 - a_2a_3a_5 + a_2^2a_4\), is surjective if \(c_1\) and \(\bar{\eta}\) are both big and nef.

Note first that the assertion in question is, via a simple rescaling of \(a_2\), trivially equivalent to the same assertion formulated for the expression

\[
a_0a_5^2 - 2a_2a_3a_5 + a_3^2a_4 = \Delta_1a_5 + \Delta_2a_3
\]

(A.2)

where we used the quantities

\[
\Delta_1 = a_0a_5 - a_2a_3, \quad \Delta_2 = a_3a_4 - a_2a_5
\]

(A.3)

These decompositions allow to adopt a two-step strategy to prove the surjectivity in question:

- in a step (1a) and (1b) one proves that the maps from the space
  \(\bigoplus_{i=0, \neq 1}^5 H^0\left(B, \mathcal{O}_B(\eta - ic_1)\right)\) of the \((a_i)\) to the spaces \(H^0\left(B, \mathcal{O}_B(2\bar{\eta} + 5c_1)\right)\) of \(\Delta_1\) and \(H^0\left(B, \mathcal{O}_B(2\bar{\eta} + 3c_1)\right)\) of \(\Delta_2\), respectively, are surjective

- in step (2) one proves that the map from the spaces of the elements \(\Delta_1, a_5, \Delta_2, a_3\) to the space \(H^0\left(B, \mathcal{O}_B(3\eta - 10c_1)\right)\) of the expression (A.2) is surjective.

Let us recall now Noether’s \(AF + BG\) Theorem: this concerns homogeneous polynomials \(F\) and \(G\) on \(\mathbb{P}^2\) whose (zero-)divisors \((F)\) and \((G)\), i.e. vanishing loci, do not have a component in common and intersect transversally; if \(H\) is then a third homogeneous polynomial and one has (with further homogeneous polynomials \(A\) and \(B\)) a relation

\[
H = AF + BG
\]

(A.4)

one gets as necessary relation for such a decomposition the condition \((F) \cap (G) \subset (H)\); the Theorem states that the latter condition is not only necessary but also already sufficient.
To use this in a more general context for various base surfaces $S$ let us introduce the following notation

\begin{align*}
H & \in H^0(S, \mathcal{O}(\delta)) \quad \text{(A.5)} \\
A & \in H^0(S, \mathcal{O}(\kappa)) \quad \text{(A.6)} \\
F & \in H^0(S, \mathcal{O}(\mu)) \quad \text{(A.7)} \\
B & \in H^0(S, \mathcal{O}(\lambda)) \quad \text{(A.8)} \\
G & \in H^0(S, \mathcal{O}(\nu)) \quad \text{(A.9)}
\end{align*}

such that one has the relation

\[ \delta = \kappa + \mu = \lambda + \nu \quad \text{(A.10)} \]

Crucial for the generalization of the proof of the Theorem from $\mathbb{P}^2$ to $S$ is the following quantity

\[ \rho : = \kappa - \nu = \lambda - \mu \quad \text{(A.11)} \]

The analogue of the Theorem will hold on $S$ if

\[ H^1(S, \mathcal{O}(\rho)) \neq 0 \quad \text{(A.12)} \]

(cf. the proof [5] of the standard form of the Theorem). This holds if either $-\rho$ or $\rho + c_1$ is ample or at least (cf. Th. 26, Ch. 1 [6]) big and nef. As in our application one has $\rho = 3c_1$ and $2c_1$ in step (1a) and (1b), respectively, and $\rho = \bar{\eta} + 3c_1$ in step (2) the condition (A.12) will always be satisfied.\footnote{Note that $\alpha$ ample and $\beta$ nef implies $\alpha + \beta$ ample, cf. Exerc. 11, Ch. 1 [6]; similarly assuming $\alpha$ just to be big and nef and $\beta$ nef implies $\alpha + \beta$ big and nef as $\beta^2 \geq 0$ (by Lemma 23, Ch. 1 [6]) and also $\alpha \beta \geq 0$; the latter follows from $na \sim \text{eff.}$ for $n \gg 0$ by Lemma 12, Ch. 1 [6] (where $aH > 0$ for $H$ ample as again $nH \sim \text{eff.}$ for $n \gg 0$ by Lemma 12, Ch. 1 [6] such that $\alpha H \geq 0$ where $\alpha H \neq 0$ by $\alpha^2 > 0$ and the Hodge index theorem).} This shows that we are done because one can easily see that the remaining conditions of the Theorem are also met as we explain now.

For example, in the step (1a) (the other cases are handled analogously) one asks whether an arbitrary element

\[ H \in H^0(B, \mathcal{O}_B(2\bar{\eta} + 5c_1)) \quad \text{(A.13)} \]

from the space where $\Delta_1$ lives can be written as combination $H = AF + BG$ of some elements $F$ and $G$ (representing $a_5$ and $a_3$, respectively) with the help of further elements
A and $B$ where one has (the minus signs in (A.3) do not matter, of course)

\[
A = a_0 \in H^0(B, O_B(\bar{\eta} + 5c_1)) \quad \text{(A.14)}
\]

\[
F = a_5 \in H^0(B, O_B(\bar{\eta})) \quad \text{(A.15)}
\]

\[
B = a_2 \in H^0(B, O_B(\bar{\eta} + 3c_1)) \quad \text{(A.16)}
\]

\[
G = a_3 \in H^0(B, O_B(\bar{\eta} + 2c_1)) \quad \text{(A.17)}
\]

Thus one has in this step

\[
\delta = 2\bar{\eta} + 5c_1 \quad \text{(A.18)}
\]

\[
\kappa = \bar{\eta} + 5c_1 \quad \text{(A.19)}
\]

\[
\mu = \bar{\eta} \quad \text{(A.20)}
\]

\[
\lambda = \bar{\eta} + 3c_1 \quad \text{(A.21)}
\]

\[
\nu = \bar{\eta} + 2c_1 \quad \text{(A.22)}
\]

giving indeed\(^{17}\) the mentioned $\rho = 3c_1$. We ask now whether for any such $H$ there exist $a_5$ and $a_3$ with $(a_5) \cap (a_3) \subset (H)$ (where $(a_5)$ and $(a_3)$ should have no component in common and should intersect transversally). Let us treat the problem in general: these are $D \cdot D'$ conditions (where $D = (a_5) = (F)$ and $D' = (a_3) = (G)$ in our example) while the degrees of freedom available for the effective divisors $D$ and $D'$ (note that here $H$ is given) are given, respectively, by\(^{18}\)

\[
h^0(B, O_B(D)) - 1 = \frac{1}{2}D(D) + c_1 \quad \text{(A.23)}
\]

\[
h^0(B, O_B(D')) - 1 = \frac{1}{2}D'(D') + c_1 \quad \text{(A.24)}
\]

The number $m$ of the remaining degrees of freedom is \[\frac{1}{2}(D^2 + (D + D)c_1 + D^2) - DD'\]

or

\[
m = \frac{1}{2}[(D - D')^2 + (D + D)c_1] \quad \text{(A.25)}
\]

Now $D$ and $D'$ are in any case effective, so the second term is nonnegative for $c_1$ being ample or at least nef; the first term is strictly positive in our case of $((F) - (G))^2 = 4c_1^2$

giving in total a strictly positive dimension for the space of the remaining degrees of freedom. Having elements $D = (F), D' = (G)$ with a non-transversal intersection would be a non-generic case (in a subspace of positive codimension), similarly elements with a common component; such degenerate cases can therefore be avoided.

\[^{17}\text{if one would start instead from } \Delta_1 = a_5a_0 - a_3a_2, \text{ say, one would get } \rho = -3c_1 \text{ and similarly in all the other cases; given the sufficient conditions on } \rho \text{ for (A.12) to hold no difference results in the end.}\]

\[^{18}\text{where we assume that } D + c_1 \text{ and } D' + c_1 \text{ are ample or at least (cf. Th. 26, Ch. 1 [6]) big and nef.}\]
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