Simulation of continuous variable quantum games without entanglement

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Abstract

A simulation scheme of quantum version of Cournot’s duopoly is proposed, in which there is a new Nash equilibrium that may also be Pareto optimal without any entanglement involved. The unique property of this simulation scheme is decoherence-free against the symmetric photon loss. Furthermore, we analyze the effects of the asymmetric information on this simulation scheme and investigate the case of asymmetric game caused by asymmetric photon loss. A second-order phase transition-like behavior of the average profits of firms 1 and 2 in a Nash equilibrium can be observed with the change of the degree of asymmetry of the information or the degree of ‘virtual cooperation’. It is also found that asymmetric photon loss in this simulation scheme plays a similar role as that with the asymmetric entangled states in the quantum game.

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1. Introduction

Recently, significant interests have been focused on generalizing the classical notion of game theory to an analogous quantum version [1, 2], the so-called quantum game theory, which is a new born branch of quantum information theory. Quantum games and quantum strategies can exploit both quantum superposition [1, 3] and quantum entanglement [2, 4]. Meyer has pointed the way for generalizing the classical game theory to quantum domain by utilizing the quantum superposition [1], though it has been argued that Meyer’s scheme can be classically realized [3]. By making use of the entanglement resource, the prisoner’s dilemma has been quantized through the scheme of Eisert et al and shown that the game ceases to pose a dilemma if restricted quantum strategies are allowed for [2, 5, 6], which has been experimentally demonstrated by Du et al [7]. Since then more work has been done on quantum prisoners’ dilemma [8–11] and a number of other games have been generalized to the quantum realm (for review, see [12, 28] and references therein). Besides those games in which the players have finite number
of strategies, the quantization of classical Cournot’s duopoly in which the players can access to a continuous set of classical strategies has also been investigated \[13\textsuperscript{–}20\]. Classical Cournot’s duopoly exhibits a dilemma-like situation, in which the unique Nash equilibrium is inferior to the Pareto optimum. For the quantum version of Cournot’s duopoly, even though two players both act as ‘selfishly’ in the quantum game, they are found to virtually cooperate due to the quantum entanglement between them \[13\]. However, entanglement is usually very fragile against the decoherence caused by the interaction with the surrounding environment. The advantage of most of the previous quantum games is not robust against the noise \[21\] and the unique properties of various quantum games different from their classical counterpart will disappear in the limit of decoherence \[22, 23\]. However, Chen \textit{et al} have discussed decoherence in quantum prisoners’ dilemma \[24\], and found that some kinds of decoherence have no effects on the Nash equilibria in the quantized prisoners’ dilemma with maximally entangled states. Here, we present the simulation schemes of the quantized symmetric or asymmetric Cournot’s duopoly, in which there is not any entanglement involved. However, the players can also escape the dilemma-like situation. In this scheme, a classical measuring apparatus provides more profits than the quantum measuring apparatus. In the asymmetric game, ‘virtual cooperation’ does not give any advantage to the weaker of the two firms if the degree of the asymmetry exceeds a certain threshold value. The most significant aspect of this scheme is its symmetric decoherence-free aspect, namely a certain kind of symmetric decoherence that does not alter the unique property of this quantized Cournot’s duopoly.

2. Continuous variable quantum game without entanglement

To make this paper self-contained, we briefly outline the classical Cournot’s duopoly and its quantization version in \[13\]. In a simple version of Cournot’s model for the duopoly, two firms simultaneously decide the quantities \(q_1\) and \(q_2\) respectively of a homogeneous product released on the market. Suppose \(Q\) is the total quantity, i.e. \(Q = q_1 + q_2\), and the market-clearing price is given by \(P(Q) = a - Q\) for \(Q \leq a\) and \(P(Q) = 0\) for \(Q > a\). The unit cost of producing the product is assumed to be \(c\) with \(c < a\). The payoffs or profits of the firms can be written as

\[
u_i(q_1, q_2) = q_i[P(Q) - c] = q_i[k - (q_1 + q_2)],\]

where \(k = a - c\) is a constant and \(i = 1, 2\). Solving for the Nash equilibrium (immune to unilateral deviations) yields the Cournot equilibrium:

\[
q_1^* = q_2^* = \frac{k}{3}.
\]

At this equilibrium the payoff for the firm \(i\) \((i = 1, 2)\) is \(k^2/9\), which is not the optimal solution. If two firms can co-operate and restrict their quantities to \(q_1' = q_2' = k/4\), they can both get higher payoff \(k^2/8\), which is the highest profit they can attain in this symmetric game. However, in a competing game, the objective of each firm is to maximize its individual payoff, and avoid the unilateral deviation causing decrease of its profit. This individual rationality confines the strategies of the two firms in the Nash equilibrium point.

The quantization version of the classical Cournot’s duopoly showed that quantum entanglement creates virtual cooperation between the two firms and the larger entanglement can guarantee the players (in the Nash equilibrium) a payoff closer to the highest feasible payoff, i.e. the Pareto-optimal payoff. Notwithstanding the domination role of entanglement in various quantum strategies, we attempt to present a simulation scheme of quantized Cournot’s duopoly, in which there is not any intermediate quantum entanglement involved. We utilize two single-mode optical fields which are initially in the vacuum state \(|0\rangle_1 \otimes |0\rangle_2\). Then, two
optical fields are sent to firms 1 and 2, respectively. The strategic moves of firms 1 and 2 are represented by the displacement operators $\hat{D}_1$ and $\hat{D}_2$ locally acted on their individual optical fields. The players are restricted to choose their strategies from the sets

$$S_i = \{ \hat{D}_i(x_i) = \exp \left[ \frac{\sqrt{x_i}}{2} (a_i^\dagger - a_i) \right] \mid x_i \in [0, \infty) \}, \quad i = 1, 2,$$

where $a_i$ and $a_i^\dagger$ are the annihilation and creation operators of the $i$th mode optical field, respectively. In this stage, the state of the game becomes a direct product of the two coherent states $| \frac{\sqrt{x_1}}{2} x_1 \rangle$ and $| \frac{\sqrt{x_2}}{2} x_2 \rangle$:

$$|\psi\rangle = | \frac{\sqrt{x_1}}{2} x_1 \rangle \otimes | \frac{\sqrt{x_2}}{2} x_2 \rangle.$$  \hspace{1cm} (4)

Having executed their moves, firms 1 and 2 forward their optical fields to the final measurement, prior to which a beam splitter operation $\hat{J}(\gamma) = \exp [\gamma (a_1^\dagger a_2 + a_2^\dagger a_1)]$ $(\gamma \in [0, \frac{\pi}{2}])$ is carried out. Therefore, the final state prior to the measurement can be expressed as

$$|\psi\rangle = | \frac{\sqrt{x_1}}{2} x_1 \cos \gamma + \frac{\sqrt{x_2}}{2} x_2 \sin \gamma \rangle$$
$$\otimes | \frac{\sqrt{x_2}}{2} x_2 \cos \gamma + \frac{\sqrt{x_1}}{2} x_1 \sin \gamma \rangle.$$ \hspace{1cm} (5)

Then, a measurement on the photon number of the optical fields is carried out, which is usually done by a photon detector. The measurement is also one of the key issues in quantum games. Different kinds of measurement schemes can alter the characteristics of the games [25, 26]. The measurement schemes mainly depend on the measuring apparatus used. In what follows we analyze two kinds of measuring apparatuses and investigate their influence on this simulation scheme: (1) ‘classical’ measuring apparatus which can only give out the expected values of the photon numbers of the optical fields such as the optical power meter; (2) quantum measuring apparatus, such as the highly sensitive quantum photon-counter which can measure the photon number and its distribution.

2.1. ‘Classical’ measuring apparatus

First, we assume that only the expected values of the photon number of the optical fields are measured by the ‘classical’ measuring apparatus, and the expected values of the photon number are $n_i$ $(i = 1, 2)$, which are regarded as the individual quantities. Hence, the payoffs are given by

$$u^Q_i(\hat{D}_1, \hat{D}_2) = u_i(n_1, n_2),$$ \hspace{1cm} (6)

where the superscript ‘$Q$’ denotes ‘quantum’. The classical Cournot’s duopoly can be faithfully represented when $\hat{J}(\gamma)|_{\gamma=0} = I$ (the identity operator), in which the quantities of firms 1 and 2 are $n_1 = x_1^2 / 2$ and $n_2 = x_2^2 / 2$, respectively. For the final state in equation (5), the measurement gives the respective quantities of the two firms

$$n_1 = \frac{1}{2} (x_1^2 \cos^2 \gamma + x_2^2 \sin^2 \gamma),$$
$$n_2 = \frac{1}{2} (x_2^2 \cos^2 \gamma + x_2^2 \sin^2 \gamma).$$ \hspace{1cm} (7)

Then, the quantum profits for the two firms are given by

$$u^Q_1(\hat{D}_1, \hat{D}_2) = \frac{1}{2} (x_1^2 \cos^2 \gamma + x_2^2 \sin^2 \gamma) \left[ k - \frac{1}{2} (x_1^2 + x_2^2) \right],$$
$$u^Q_2(\hat{D}_1, \hat{D}_2) = \frac{1}{2} (x_2^2 \cos^2 \gamma + x_2^2 \sin^2 \gamma) \left[ k - \frac{1}{2} (x_1^2 + x_2^2) \right].$$ \hspace{1cm} (8)

Solving for the Nash equilibrium gives the unique one

$$x_1^* = x_2^* = \sqrt{\frac{2k \cos^2 \gamma}{1 + 2 \cos^2 \gamma}}.$$ \hspace{1cm} (9)
The profits of the two firms at this equilibrium are given by

$$u_1^Q = u_2^Q = \frac{k^2 \cos^2 \gamma}{(1 + 2 \cos^2 \gamma)^2}. \tag{10}$$

From equation (10), we can see that the profit at the equilibrium increases from the classical payoff \(\frac{k^2}{9}\) to the pareto-optimal payoff \(\frac{k^2}{8}\) when \(\gamma\) increases in the range of \(\gamma \in [0, \frac{\pi}{4})\). Obviously, not any intermediate quantum entanglement has been involved in this scheme. But this fact does not impede the successful escaping from the dilemma when \(\gamma \to \frac{\pi}{4}\). It is true that the complete classical system can be used to implement the proposed scheme. For example, by making use of the source of classical light or electromagnetic wave, two modulators, beam splitter and power meters, one can realize this simulation scheme. Though there exists the intermediate entanglement in the quantization scheme of [13], the final state in equation (11) in [13] is also not entangled. In this sense, the present scheme and the one in [13] can be regarded as the same kind but with different definitions of quantum strategies.

2.2. Quantum measuring apparatus

In this subsection, we assume that a quantum measuring apparatus can be used to measure the photon number distributions of the optical fields. The measured value of the photon number is regarded as the quantity of the respective strategy; then, the average payoff is calculated based on the probability distribution of the photon number. For the final state in equation (5), the average payoffs are given by

$$u_i^Q(\hat{D}_1, \hat{D}_2) = \langle u_i(m_1, m_2) \rangle, \tag{11}$$

where

$$\langle u_i(m_1, m_2) \rangle = \sum_{m_1, m_2=0}^{\infty} u_i(m_1, m_2) P_{m_1, m_2} \tag{12}$$

denotes the average of \(u_i(m_1, m_2)\) taken over all possible values of \(m_1\) and \(m_2\) with the Poisson distribution

$$P_{m_1, m_2} = e^{-\frac{1}{2}(x_1^2 + x_2^2)} \left( \frac{1}{2} \left( x_1^2 \cos^2 \gamma + x_2^2 \sin^2 \gamma \right) \right)^{m_1} \frac{m_1!}{m_2!} \left( \frac{1}{2} \left( x_2^2 \cos^2 \gamma + x_1^2 \sin^2 \gamma \right) \right)^{m_2}. \tag{13}$$

For simplicity, we assume that \(a\) and \(c\) tend to infinity but keeping \(k = a - c \geq 1\) a finite constant. In this case, the average quantum payoffs for the two firms are given by

$$u_1^Q(\hat{D}_1, \hat{D}_2) = \frac{1}{2} \left( x_1^2 \cos^2 \gamma + x_2^2 \sin^2 \gamma \right) \left[ k - 1 - \frac{1}{2} \left( x_1^2 + x_2^2 \right) \right],$$

$$u_2^Q(\hat{D}_1, \hat{D}_2) = \frac{1}{2} \left( x_2^2 \cos^2 \gamma + x_1^2 \sin^2 \gamma \right) \left[ k - 1 - \frac{1}{2} \left( x_1^2 + x_2^2 \right) \right]. \tag{14}$$

In this case, when \(\hat{J}(\gamma) = I\) (the identity operator), the scheme cannot return to the classical Cournot’s duopoly. Comparing the payoffs in equation (14) and equation (8), we can find that quantum fluctuation causes the reduction of the payoffs. Solving for the Nash equilibrium yields the unique one

$$x_1^* = x_2^* = \sqrt{\frac{2(k - 1) \cos^2 \gamma}{1 + 2 \cos^2 \gamma}}. \tag{15}$$

The profits of the two firms at this equilibrium are given by

$$u_1^Q = u_2^Q = \frac{(k - 1)^2 \cos^2 \gamma}{(1 + 2 \cos^2 \gamma)^2}. \tag{16}$$
From equation (16), we can see that the profit at the equilibrium increases from \( \frac{(k-1)^2}{8} \) to \( \frac{(k-1)^2}{8} \) when \( \gamma \) increases from 0 to \( \frac{\pi}{2} \). The above results show that the scheme using the classical measuring apparatus has an advantage over the scheme using the quantum measuring apparatus.

Here, for avoiding the emergence of the situation \( m_1 + m_2 > a \) with nonzero probability, after equation (13) it has been assumed that \( a \) and \( c \) tend to infinity but keeping \( k = a - c \) a finite constant, which guarantees that the payoff summed over \( m_1, m_2 \) from 0 to infinity in equation (12) has the analytical expression in equation (14). For very large but finite value of \( a \) and \( c \), the probability \( P_{m_1,m_2} \) in equation (13) corresponding to \( m_1 + m_2 > a \) tends to be very small, and the summation of those terms with \( m_1 + m_2 > a \) in equation (12) should be written as the summation of \( -cm_1 P_{m_1,m_2} \) or \( -cm_2 P_{m_1,m_2} \) which also tends to be small enough to guarantee the payoff well approximated by equation (14). However, for other cases with small values of \( a \), the situation becomes very complicate. The payoff in equation (14) is not valid and need to be revised. It is very difficult to obtain an analytical result in this situation. Our numerical results show that the optimal payoff of the two firms is between \( 2 \) and \( 8 \) when \( k = \frac{1}{2}, \gamma = \frac{\pi}{4} \). For example, in the case with \( a = 6, c = 1, k = 5, \gamma = \frac{\pi}{4} \), the optimal payoff is about 2.02487, which is near \( (k-1)^2/8 = 2 \); in the case with \( a = 10, c = 5, k = 5, \gamma = \frac{\pi}{4} \), the optimal payoff is about 2.0006. Thus, it is conjectured that the conclusion that the scheme using the classical measuring apparatus has an advantage over the scheme using the quantum measuring apparatus is valid even in the cases with small values of \( a \) and \( c \).

For any strategies with \( D_1(x_1 \neq 0) \) or \( D_2(x_2 \neq 0) \), one may argue that \( u_i(m_1, m_2) \) \( (i = 1, 2) \) in equations (11) and (12) will become negative if \( m_1 + m_2 > k \), which mean the players will unavoidably probabilistically ‘lose money’ in the scheme with the quantum measuring apparatus. Nevertheless, the average profits in the unique Nash equilibria are always nonnegative if only \( k \geq 1 \).

3. Symmetric decoherence-free aspects of this quantized scheme

The most significant characteristics of our simulation scheme is its symmetric decoherence-free aspect. Previous works have shown that the decoherence could destroy the advantage of the quantum game [21–23]. In our simulation scheme, the advantage of the quantum game is robust against the symmetric photon loss, where the decoherence caused by the photon loss can be described by the following master equation [27]:

\[
\frac{\partial \rho(t)}{\partial t} = \sum_{i=1}^{2} \kappa \hat{a}_i \rho(t) \hat{a}_i^\dagger - \frac{\kappa}{2} \left( \hat{a}_i^\dagger \hat{a}_i \rho(t) + \rho(t) \hat{a}_i \hat{a}_i^\dagger \right),
\]

(17)

where \( \kappa \) is the decay rate. \( \rho(t) \) represents the whole state of firms 1 and 2. The evolving state can be expressed as

\[
| \frac{\sqrt{2}}{2} x_1(t) \rangle \langle \frac{\sqrt{2}}{2} x_2(t) |,
\]

(18)

where \( | \frac{\sqrt{2}}{2} x_i(t) \rangle = | \frac{\sqrt{2}}{2} x_i, e^{-i \gamma} \rangle \) \( (i = 1, 2) \). Then, forward the evolving state into the beam splitter and we have

\[
| \Psi(t) \rangle = | \frac{\sqrt{2}}{2} x_1 e^{-\frac{\gamma}{2}} \cos \gamma + \frac{\sqrt{2}}{2} i x_2 e^{-\frac{\gamma}{2}} \sin \gamma \rangle 
\]

\[
\otimes \langle \frac{\sqrt{2}}{2} x_1 e^{-\frac{\gamma}{2}} \cos \gamma + \frac{\sqrt{2}}{2} i x_2 e^{-\frac{\gamma}{2}} \sin \gamma |
\]

(19)

From the above quantum state, we can immediately obtain the following conclusion. If both firms have the complete information about the photon loss, they can adjust their strategies

\[
5
\]
according to the transformation \( x_i \rightarrow x_i e^{\frac{\gamma}{2}} \), which can guarantee that the final payoffs are invariant under the influence of the photon loss process.

4. Asymmetric information aspects of this simulation scheme

Recently, the quantum game with asymmetric information has been investigated and some novel phenomena caused by asymmetry of information have been revealed [14]. It is very interesting to study how the asymmetric information can alter the aspects of the present simulation scheme of the original quantized game. In the case with asymmetric information, firm 1 does not know what \( c_2 \) (firm 2’s unit cost) is, only knows that \( c_2 \in \{c_H, c_L\} \), with probability \( 1 - \theta \) (\( c_H > c_L \)). Yet firm 2 knows with certainty the unit cost \( c_2 \) of its product as well as that of firm 1’s \((c_1)\). So, equation (8) should be replaced by

\[
\begin{align*}
u_1^Q(\hat{D}_1, \hat{D}_2) &= \frac{1}{2} (x_1^2 \cos^2 \gamma + x_2^2 \sin^2 \gamma) (2 \cos^2 \gamma + 2 \Delta) + (1 - \theta) (2 \cos^2 \gamma + 2 \Delta \
\end{align*}
\]

For convenience we denote the strategy by \( x_i \) when it is \( \hat{D}_i(x_i) \). Let \( \{x_1^*, x_2^{*H}, x_2^{*L}\} \) be the Bayes–Nash equilibrium. Then, \( x_2 = x_2^{*H(L)} \) is chosen to maximize \( \nu_0^Q(\{x_1^*, x_2^{*H}\}) \) and \( x_1 = x_1^* \) is chosen to maximize \( \theta \nu_0^Q(x_1, x_2^{*H}) + (1 - \theta) \nu_0^Q(x_1, x_2^{*L}) \). Solving the three optimization problems yields the Bayes–Nash equilibrium. For simplicity, we assume that \( c_1 = \theta c_H + (1 - \theta) c_L, k = a - c_1, \Delta = c_H - c_L \). After calculation, the unique Bayes–Nash equilibrium can be obtained as

\[
\begin{align*}
x_1^2 &= \frac{2 k \cos^2 \gamma}{1 + 2 \cos^2 \gamma}, \\
x_2^{2H} &= \frac{2 k \cos^2 \gamma - 4 \cos^4 \gamma (a - c_H - (1 - \theta) \Delta)}{1 - 4 \cos^4 \gamma}, \\
x_2^{2L} &= \frac{2 k \cos^2 \gamma - 4 \cos^4 \gamma (a - c_L + \theta \Delta)}{1 - 4 \cos^4 \gamma}.
\end{align*}
\]

In the above derivation, it has been assumed that \( \max[\frac{2(c_H-c_L)}{k-(c_H-c_L)}, 0] < \cos(2\gamma) \). When \( \frac{2(c_H-c_L)}{k-(c_H-c_L)} \geq \cos(2\gamma) > 0 \), the unique Bayes–Nash equilibrium can be obtained as

\[
\begin{align*}
x_1^2 &= \frac{2 \cos^2(\gamma)[\theta k - \theta (1 - \theta) \Delta + k \cos(2\gamma)]}{\theta + \cos(2\gamma)[2 + \cos(2\gamma)]}, \\
x_2^{2H} &= 0, \\
x_2^{2L} &= \frac{2 \cos^2(\gamma)[\theta \Delta + (k + \theta \Delta) \cos(2\gamma)]}{\theta + \cos(2\gamma)[2 + \cos(2\gamma)]}.
\end{align*}
\]

In the following, we consider the iterative game. When \( \frac{2(c_H-c_L)}{k-(c_H-c_L)} \geq \cos(2\gamma) > 0 \), the average profits in the iterative game are given by

\[
\begin{align*}
\bar{U}_1^Q &= [\cos^2 \gamma (4 \theta (k + \Delta (\theta - 1)) \theta + k \cos 2\gamma) \cdot (-2 \Delta \theta^2 + 2k \theta + 2\Delta \theta + k - 2(k(\theta - 3) + \Delta (\theta - 1)) \cos 2\gamma + k \cos 4\gamma) \cos^2 \gamma + (\theta - 1)(2(\theta - 2) \theta + \Delta \theta) \cos 2\gamma + \theta (-2k + 2\Delta \theta + \Delta \cos 4\gamma) \cdot (2(\Delta \theta - 1) \theta + k(\theta + 2)) \cos 2\gamma + \theta (2k - \Delta + 2\Delta \theta - \Delta \cos 4\gamma))] / [4(2\theta + 4 \cos 2\gamma + \cos 4\gamma + 1)^2],
\end{align*}
\]
\[ \bar{U}_2^Q = [\cos^2 \gamma (\theta (k + \Delta (\theta - 1)) \theta + k \cos 2\gamma) \cdot (2\Delta \theta^2 + 2k\theta + k - 2\Delta - 2k(\theta - 3) + \Delta (\theta^2 - 5\theta + 4)) \cos 2\gamma + (k + 2\Delta (\theta - 1)) \cos 4\gamma) \sin^2 \gamma - (\theta - 1)(\theta(2k + \Delta + 2\Delta \theta + \Delta \cos 4\gamma) - 2(k(\theta - 2) + \Delta (\theta - 3)\theta \cos 2\gamma^2)] / [4(2\theta + 4 \cos 2\gamma + \cos 4\gamma + 1)^2]. \] (23)

When \( \max \left[ \frac{2(\theta - 2)}{\theta^2 - 4}, 0 \right] < \cos(2\gamma) \), the average profits in the iterative game are given by

\[ \bar{U}_2^Q = \frac{4[k^2 - \Delta^2 \theta(1 - \theta)]}{8(2 + \cos 2\gamma)^2} + \frac{[4k^2 + \Delta^2 \theta(1 - \theta) \cos 2\gamma(3 + \cos 2\gamma)] \cos 2\gamma}{8(2 + \cos 2\gamma)^2}, \] (24)

\[ \bar{U}_2 = \bar{U}_1^Q + \frac{1}{4} \Delta^2 \theta(1 - \theta). \]

From the above results, one can find a boundary which separates two parameter regions A and B labeled by the inequality \( \max \left[ \frac{2(\theta - 2)}{\theta^2 - 4}, 0 \right] \leq \cos(2\gamma) \) and \( \max \left[ \frac{2(\theta - 2)}{\theta^2 - 4}, 0 \right] \geq \cos(2\gamma) \), respectively. In the parameter region A, the profits in the Nash equilibrium in equation (24) return to the ones of [14] if replacing \( \cos(2\gamma) \) by \( \exp[-2\gamma] \) in [14]. While in the parameter region B, due to the constraint that the strategy in the Bayes–Nash equilibrium should not exceed the strategy space \([0, \infty)\), we give out the profits of the Bayes–Nash equilibrium in equation (23) which is different from the results in [14].

In figure 1, the rescaled average profits of firms 1 and 2 are plotted as the function of \( \Delta/k \) for different values of \( \gamma \). For a fixed value of \( \theta \), the degree of asymmetry \( \xi \equiv \frac{\Delta(1-\theta)}{\theta} \) is monotonic with \( \Delta/k \). The profit of firm 2 increases with the degree of asymmetry, and the profit of firm 1 decreases with the degree of asymmetry for the case with \( \gamma = 0, \theta = 0.5 \) and \( \Delta/k < \frac{2}{5} \), in which the rescaled profit \( \bar{U}_2^Q/k^2 \) of the firm 1 keeps fixed. When \( \gamma < \frac{\pi}{2} \), a second-order phase transition-like behaviors (i.e. \( \frac{\partial U_2^Q/k^2}{\partial \Delta/k} \) (i = 1, 2) is discontinuous) of the rescaled average profits of firms 1 and 2 in the Bayes–Nash equilibrium may be observed as the degree of asymmetry varies across the boundary of the parameter regions A and B. In figure 2, the rescaled average profits of firms 1 and 2 are plotted as the function of \( \gamma \) for different values of \( \Delta/k \). The asymmetric property of this game can significantly affect the dependence of the average profits of both firms 1 and 2 on the degree of ‘virtual cooperation’ \( \sin(2\gamma) \). For \( \theta = 1/2 \) and \( 0 < \Delta/k < \frac{2\pi}{\sqrt{5}} \), both the profits of firms 1 and 2 first increase with \( \gamma \) and then decrease with \( \gamma \). While for larger asymmetry with \( \theta = 1/2 \) and \( 2 > \Delta/k > \frac{2\pi}{\sqrt{5}} \), both the profits of firms 1 and 2 decrease with \( \gamma \). The above results imply that, in this case, the ‘virtual cooperation’ has an advantage role only in the nearly symmetric or very small asymmetric games. If the degree of asymmetry exceeds a threshold, the ‘virtual cooperation’ in this game can suppress the gain of the firm standing on the advantage side with more information. But for the firm standing on the disadvantage side due to less information, the ‘virtual cooperation’ should be regarded as one disaster after another. Similarly, when \( \theta = 1/2 \) and \( \Delta/k > 1 \), a second-order phase transition-like behavior (i.e. \( \frac{\partial U_2^Q/k^2}{\partial \gamma} \) (i = 1, 2) is discontinuous) of the rescaled average profits of firms 1 and 2 in the Nash equilibrium can be observed as the parameter \( \gamma \) varies across the boundary of the parameter regions A and B. In figure 3, we calculate the total rescaled profit of firms 1 and 2 versus the parameters \( \Delta/k \) or \( \gamma \) for \( \theta = 1/2 \).

It is found that in the ‘ideal virtual cooperation’ case with \( \gamma = \frac{\pi}{2} \), the total profit in Nash equilibrium is invariant against the asymmetry of this game. In other cases, the total profit always increases with \( \Delta/k \). The smaller the parameter \( \gamma \), the more significant the influence of the asymmetry on the total profit.
5. Decoherence-induced asymmetric quantum game

In [15], by making use of the asymmetrical entangled states, the quantum model shows some kind of ‘encouraging’ and ‘suppressing’ effect in profit functions of different players. Here, we discuss how the asymmetric decoherence can alter the Bayes–Nash equilibrium of the above quantized scheme of the Cournot’s duopoly, where the asymmetric decoherence means that two firms experience two different degrees of decoherence. Let us consider the following specific case in which firm 2 encounters a decoherence caused by the photon loss with the loss rate $\sqrt{\eta} = e^{-\kappa \tau / 2}$ ($\tau \in [0, \infty]$) just before the final photon counting. Obviously, the final measurement gives the respective quantities of the two firms

$$n_1 = \frac{1}{2} \left( x_1^2 \cos^2 \gamma + x_1^2 \sin^2 \gamma \right),$$

$$n_2 = \frac{\eta}{2} \left( x_2^2 \cos^2 \gamma + x_2^2 \sin^2 \gamma \right).$$

Substituting equation (25) into the payoff function, and solving the two optimization problems yields the Bayes–Nash equilibrium. For $0 < \eta \leq 1$ and $0 \leq \gamma \leq \pi / 4$, the unique Bayes–Nash
Figure 2. (a) The scaled payoff \( U_2/k^2 \) of firm 2 and (b) the scaled payoff \( U_1/k^2 \) of firm 1 at the Nash equilibrium are plotted as the function of \( \gamma \) for different values of \( \Delta/k \) with \( \theta = 0.5 \). (Solid line) \( \Delta/k = 0 \); (dashed line) \( \Delta/k = 0.5 \); (dotted line) \( \Delta/k = 1 \); (dash-dotted line) \( \Delta/k = 1.5 \); (dash-dot-dot line) \( \Delta/k = 2 \).

The Nash equilibrium can be obtained as

\[
x_1^* = \frac{8k \eta \cos^2 \gamma}{1 + \eta(6 + \eta) + 4\eta \cos 2\gamma - (1 - \eta)^2 \cos 4\gamma},
\]

\[
x_2^* = \frac{8k \cos^2 \gamma}{1 + \eta(6 + \eta) + 4\eta \cos 2\gamma - (1 - \eta)^2 \cos 4\gamma}.
\]

(26)

Meanwhile, the corresponding profits at the Bayes–Nash equilibrium can be obtained as

\[
U_1^Q = \xi[1 + \eta - (1 - \eta) \cos 2\gamma],
\]

\[
U_2^Q = \xi\eta[1 + \eta + (1 - \eta) \cos 2\gamma],
\]

(27)

where

\[
\xi = \frac{2k^2 \cos^2 \gamma[1 + \eta]^2 - (1 - \eta)^2 \cos^2 2\gamma}{[1 + \eta(6 + \eta) + 4\eta \cos 2\gamma - (1 - \eta)^2 \cos 4\gamma]^2}.
\]

(28)

When \( \eta \to 0 \), \( U_1^Q \to \frac{k^2}{8\pi^2 58_{\eta,0}} \) and \( U_2^Q \to \frac{k^2 \delta_{\gamma,0}}{9} \), where \( \delta_{\gamma,0} \) equals 1 for \( \gamma = 0 \) and is zero elsewhere. When \( \gamma = 0 \), \( U_1^Q = U_2^Q = \frac{k^2}{9} \), which shows the payoff does not depend on
Figure 3. (a) The scaled total payoff \((U_1 + U_2)/k^2\) at the Nash equilibrium of the two firms is plotted as the function of \(\gamma\) for different values of \(\Delta/k\) with \(\theta = 0.5\). (Solid line) \(\Delta/k = 0\); (dashed line) \(\Delta/k = 0.5\); (dotted line) \(\Delta/k = 1\); (dash-dotted line) \(\Delta/k = 1.5\); (dash-dot dot line) \(\Delta/k = 2\). (b) The scaled total payoff \((U_1 + U_2)/k^2\) at the Nash equilibrium of the two firms is plotted as the function of \(\Delta/k\) for different values of \(\gamma\) with \(\theta = 0.5\). (Solid line) \(\gamma = 0\); (dashed line) \(\gamma = \pi/16\); (dotted line) \(\gamma = \pi/8\); (dash-dotted line) \(\gamma = 3\pi/16\); (dash-dot-dot line) \(\gamma = (\frac{\pi}{4})^2\).

\(\eta\) and implies the Nash equilibrium of the classical game is robust against the asymmetric photon-loss.

In figure 4, the rescaled profits of the two firms in the Nash equilibrium are plotted as the functions of \(\gamma\) and \(\eta\). For \(\gamma \neq 0\), the asymmetric decoherence encourages the profit \(U_1^Q\) of firm 1 and suppresses the profit \(U_2^Q\) of firm 2. We can find that \(U_1^Q\) and \(U_2^Q\) exhibit the sharp decline and ascent in the end of asymmetric photon loss for those cases with very small value of \(\gamma \neq 0\). In the initial stage of photon loss with \(\eta > 0.5\), ‘virtual cooperation’ accelerates the encouragement and suppression effects of the asymmetric photon loss. The asymmetric photon loss plays a role in transferring the profit from firm 2 to firm 1. Surprisingly, the asymmetric photon loss can improve the total profits of firms 1 and 2 in the Nash equilibrium in the situations with \(0 < \gamma < \frac{\pi}{4}\). In the ideal ‘virtual cooperation’, i.e. \(\gamma = \frac{\pi}{4}\), the total profits are kept fixed against the asymmetric photon loss.
6. Conclusions

In this paper, we present a simulation scheme of the continuous variable quantized Cournot’s duopoly, in which not any intermediate quantum entanglement has been involved. The
influence of the measuring apparatus, symmetric or asymmetric photon loss and asymmetric information on their Nash equilibria has been investigated. It is shown that the scheme using the classical measuring apparatus has an advantage over the one using the quantum measuring apparatus. Being different from the previous quantized Cournot’s duopoly involving entanglement, this simulation scheme is also symmetric photon loss free, while for asymmetric photon loss, the profits in the Nash equilibrium exhibit a transfer from one firm to the other. Simultaneously, the total profit in the Nash equilibrium increases with the asymmetric photon loss except for two extreme cases, i.e. the complete no ‘virtual cooperation’ case and the ideal ‘virtual cooperation’ case.

In the cases with asymmetric information, a second-order phase transition-like behavior of the average profits of firms 1 and 2 in the Nash equilibrium can be observed as the degree of asymmetry or the degree of ‘virtual cooperation’ vary. The ‘virtual cooperation’ has an advantage role for total profit in the Nash equilibrium only in the nearly symmetric or very small asymmetric games. If the degree of asymmetry exceeds a threshold value, the ‘virtual cooperation’ in this game can suppress the gain of the firm standing on the advantage side with more information. But for the firm standing on the disadvantage side due to less information, the ‘virtual cooperation’ should be regarded as one disaster after another. For the total profit, it is found that, in the ‘ideal virtual cooperation’ case with \( \gamma = \frac{\pi}{4} \), the total profit in the Nash equilibrium is invariant against the asymmetry of this game. In other cases with \( \gamma < \frac{\pi}{4} \), the total profit always increases with \( \Delta / k \). The smaller the parameter \( \gamma \), the more significant the influence of the asymmetry on the total profit.

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