On Quantum Versions of the Yao Principle*

Mart de Graaf       Ronald de Wolf

Abstract

The classical Yao principle states that the complexity \( R_\epsilon(f) \) of an optimal randomized algorithm for a function \( f \) with success probability \( 1 - \epsilon \) equals the complexity \( \max_\mu D^\mu_\epsilon(f) \) of an optimal deterministic algorithm for \( f \) that is correct on a fraction \( 1 - \epsilon \) of the inputs, weighed according to the hardest distribution \( \mu \) over the inputs. In this paper we investigate to what extent such a principle holds for quantum algorithms. We propose two natural candidate quantum Yao principles, a “weak” and a “strong” one. For both principles, we prove that the quantum bounded-error complexity is a lower bound on the quantum analogues of \( \max_\mu D^\mu_\epsilon(f) \). We then prove that equality cannot be obtained for the “strong” version, by exhibiting an exponential gap. On the other hand, as a positive result we prove that the “weak” version holds up to a constant factor for the query complexity of all symmetric Boolean functions.

Keywords: Quantum computing, computational complexity.

1 Introduction

1.1 Motivation

In classical computing, the Yao principle [18] gives an equivalence between two kinds of randomness in algorithms: randomness inside the algorithm itself, and randomness on the inputs. Let us fix some model of computation for computing a Boolean function \( f \), like query complexity, communication complexity, etc. Let \( R_\epsilon(f) \) be the minimal complexity among all randomized algorithms that compute \( f(x) \) with success probability at least \( 1 - \epsilon \), for all inputs \( x \). Let \( D^\mu_\epsilon(f) \) be the minimal complexity among all deterministic algorithms that compute \( f \) correctly on a fraction of at least \( 1 - \epsilon \) of all inputs, weighed according to a distribution \( \mu \) on the inputs. The Yao principle now states that these complexities are equal if we look at the “hardest” input distribution \( \mu \):

\[
R_\epsilon(f) = \max_\mu D^\mu_\epsilon(f).
\]

This is a special case of Von Neumann’s minimax theorem in game theory [12, 15].

Since its introduction, the Yao principle has been an extremely useful tool in computational complexity analysis. In particular, it allows us to derive lower bounds on randomized algorithms from lower bounds on deterministic algorithms: choose some “hard” input distribution \( \mu \), prove a lower bound on deterministic algorithms that compute \( f \) correctly for “most” inputs, weighted according to \( \mu \), and then use \( R_\epsilon(f) \geq D^\mu_\epsilon(f) \) to get a lower bound on \( R_\epsilon(f) \). This method is used very often, because it is usually much easier to analyze deterministic algorithms than to analyze randomized ones.

In recent years quantum computation received a lot of attention. Here quantum mechanical principles are employed to realize more efficient computation than is possible with a classical computer. Famous examples are Shor’s polynomial-time factoring algorithm [16] and Grover’s search algorithm [9]. However, the field is still young and open questions are abundant. In particular, there has been a search for good techniques to provide lower bounds on quantum algorithms. Most of these lower bounds are in the query model, where the complexity of an algorithm is measured by the number of queries it needs in order to compute some function (we will provide formal definitions of this and other concepts in the next section). Two general methods...
in this direction are the polynomial method introduced by Beals, Buhrman, Cleve, Mosca, and de Wolf [2] and the method of quantum adversaries of Ambainis [1]. In this paper we investigate the possibility of a third method, a quantum Yao principle. It is our hope that such a principle will prove itself useful as a link between techniques for lower bounds on exact and bounded-error quantum algorithms.

The first difficulty one runs into when investigating a quantum version of the Yao principle, is the question what the proper quantum counterparts of $R_\epsilon(f)$ and $D^\mu(f)$ are. Let us fix the error probability at $\epsilon = \frac{1}{3}$ here (any other value in $(0, \frac{1}{2})$ would do as well). The quantum analogue of $R_{1/3}(f)$ is straightforward: let $Q_2(f)$ denote the minimal complexity among all quantum algorithms that compute $f(x)$ with probability at least $\frac{2}{3}$, for all inputs $x$. However, the inherently “random” nature of quantum algorithms prohibits a straightforward definition of “deterministic” quantum algorithms in analogy of deterministic classical algorithms. We therefore propose two different definitions, a weak and a strong one. In the following, let $f : D \to \{0, 1\}$ be some function that we want to compute, with $D \subseteq \{0, 1\}^N$. If $D = \{0, 1\}^N$ then $f$ is a total function, otherwise $f$ is a promise function. Let $A$ be a quantum algorithm, $P_A(x)$ the acceptance probability of $A$ on input $x$ (the probability of outputting 1 on input $x$, and $\mu : D \to [0, 1]$ a probability distribution over the inputs.

**Definition 1** $A$ is weakly $\frac{2}{3}$-exact for $f$ with respect to $\mu$ iff $\mu(\{x \mid P_A(x) = f(x)\}) \geq \frac{2}{3}$.

**Definition 2** $A$ is strongly $\frac{2}{3}$-exact for $f$ with respect to $\mu$ iff $A$ is weakly $\frac{2}{3}$-exact for $f$ with respect to $\mu$ and $P_A(x) \in \{0, 1\}$ for all inputs $x \in \{0, 1\}^N$.

Informally, in the second definition we require the algorithm to output the same output on the same input, even on inputs $x \in D$ where the algorithm fails and even on $x \in \{0, 1\}^N \setminus D$ (similar to a classical deterministic algorithm). In the first definition, we only require this “input-determines-output” behavior to occur for a $\mu$-fraction of at least $\frac{2}{3}$ of the inputs where the algorithm gives the correct output $f(x)$. Note that a strongly $\frac{2}{3}$-exact algorithm for $f$ with respect to $\mu$ actually computes some total function $g : \{0, 1\}^N \to \{0, 1\}$ with success probability 1, namely the function $g(x) = P_A(x)$. This $g$ will agree with $f$ on at least $\frac{2}{3}$ of the inputs.

These two definitions lead to a weak and a strong quantum counterpart to the classical distributional complexity $D^{\mu}_{1/3}(f)$: let $Q_{WE}^{\mu}(f)$ and $Q_{SE}^{\mu}(f)$ denote the minimal complexity among all weakly and strongly $\frac{2}{3}$-exact algorithms for $f$ with respect to $\mu$, respectively. We can now state two potential quantum versions of the Yao principle:

- Strong quantum Yao principle: $Q_2(f) \geq \max_\mu Q_{SE}^{\mu}(f)$
- Weak quantum Yao principle: $Q_2(f) \geq \max_\mu Q_{WE}^{\mu}(f)$

In this paper we investigate to what extent these two quantum Yao principles hold.

### 1.2 Results

Our results are threefold. Firstly, we prove that both of these principles hold in the ‘$\leq$’-direction, for all $f$:

- $Q_2(f) \leq \max_\mu Q_{SE}^{\mu}(f)$
- $Q_2(f) \leq \max_\mu Q_{WE}^{\mu}(f)$

Clearly, the second inequality implies the first, since $Q_{WE}^{\mu}(f) \leq Q_{SE}^{\mu}(f)$ for all $f$ and $\mu$. The proof is similar to the classical game-theoretic proof, with a bit more technical complication. We emphasize that this result is perfectly general, and applies to all computational models to which the classical Yao principle applies.

In order to investigate to what extent the ‘$\geq$’-directions of these two quantum Yao principles hold, we instantiate our complexity measures to the query complexity setting. Our second result is an exponential gap between $Q_2(f)$ and $Q_{SE}^{\mu}(f)$ for the query complexity of Simon’s problem [17]:

- There exist $f$ and $\mu$ such that $Q_2(f)$ is exponentially smaller than $Q_{SE}^{\mu}(f)$.
This shows that the strong quantum Yao principle is false. Thirdly, we prove that the weak quantum Yao principle holds up to a constant factor for the query complexity of all symmetric functions:

\[ Q_2(f) = \Theta \left( \max_{\mu} Q_{WE}^\mu(f) \right) \text{ for all symmetric } f \]

For this result we first construct a quantum algorithm that can determine the \( N \)-bit input \( x \) with certainty in \( O(\sqrt{kN}) \) queries if \( k \) is a known upper bound on the Hamming weight of \( x \). We then use that algorithm to construct, for every symmetric function \( f \) and distribution \( \mu \), a quantum algorithm that computes \( f(x) \) with certainty for “most” inputs \( x \). In addition to this result for symmetric functions, we also show that for a particular monotone non-symmetric function \( f \), the \( \max_\mu Q_{WE}^\mu(f) \) complexity lies in between the best known bounds for \( Q_2(f) \).

## 2 Preliminaries

In this section we formalize the notion of query complexity, define several complexity measures, state Von Neumann’s minimax theorem and derive the classical Yao principle from it.

### 2.1 Query Complexity

We assume familiarity with classical computation theory and briefly sketch the basics of quantum computation; an extensive introduction may be found in the book by Nielsen and Chuang [14]. Quantum algorithms operate on qubits as opposed to bits in classical computers. The state of an \( m \)-qubit quantum system can be written as

\[ |\phi\rangle = \sum_{i\in\{0,1\}^m} \alpha_i |i\rangle, \]

where \( |i\rangle \) denotes the basis state \( i \), which is a classical \( m \)-bit string. The \( \alpha_i \)'s are complex numbers known as the amplitudes of the basis states \( |i\rangle \) and we require \( \sum_{i\in\{0,1\}^m} |\alpha_i|^2 = 1 \). Mathematically, the state of a system is thus described by a \( 2^m \)-dimensional complex unit vector. If we measure the value of \( |\phi\rangle \), then we will see the basis state \( |i\rangle \) with probability \( |\alpha_i|^2 \), after which the system collapses to \( |i\rangle \). Operations which are not measurements on a system of qubits correspond to unitary transformations on the vector of amplitudes.

In the query model of computation, the goal is to compute some function \( f : D \to \{0,1\} \) on an input \( x \in D \subseteq \{0,1\}^N \), using as few accesses (“queries”) to the \( N \) input bits as possible. In quantum algorithms, it is by now standard to formalize a query as an application of a unitary transformation \( O \) that acts as follows:

\[ O|i, b, z\rangle = |i, b \oplus x, z\rangle. \]

Here \( i \in \{1,\ldots,N\} \), \( b \in \{0,1\} \), \( \oplus \) denotes the exclusive-or function, and \( z \) denotes the workspace of the algorithm, which is not affected by \( O \). A \( T \)-query quantum algorithm \( A \) then has the form

\[ A = U_TOU_{T-1}O \cdots U_1OU_0, \]

with each \( U_i \) a fixed unitary transformation independent of the input \( x \). \( A \) is assumed to start in the all-zero state \( |0\ldots0\rangle \), and its output (0 or 1) is obtained by measuring the rightmost bit of its final state \( A|0\ldots0\rangle \).

The acceptance probability \( P_A(x) \) of a quantum algorithm \( A \) is defined as the probability of getting output 1 on input \( x \). Its success probability \( S_A(x) \) is the probability of getting the correct output \( f(x) \) on input \( x \).

A quantum algorithm \( A \) computes a function \( f : D \to \{0,1\} \) exactly if \( S_A(x) = 1 \) for all inputs \( x \in D \). Algorithm \( A \) computes \( f \) with bounded-error if \( S_A(x) \geq \frac{3}{4} \) for all \( x \in D \). We use \( Q_{E}(f) \) and \( Q_{2}(f) \) to denote the minimal number of queries required by exact and bounded-error quantum algorithms for \( f \), respectively. These complexities are the quantum versions of the classical deterministic and bounded-error decision tree complexities \( D(f) \) and \( R_2(f) \), respectively. For completeness, we repeat our two alternative quantum versions of the classical distributional complexity \( D^\mu(f) \) from the introduction. Let \( \mu \) be a probability distribution on the set of all possible inputs. An algorithm \( A \) is weakly \( \frac{3}{4} \)-exact for \( f \) with respect to \( \mu \) if \( \mu\{x \mid P_A(x) = \)}
\[ f(x) \geq \frac{2}{n}, \] and \( A \) is strongly \( \frac{2}{n} \)-exact for \( f \) with respect to \( \mu \) if \( A \) is weakly \( \frac{2}{n} \)-exact for \( f \) with respect to \( \mu \) and \( P_A(x) \in (0, 1) \) for all \( x \in \{0, 1\}^n \). By \( Q_{SE}(f) \) and \( Q_{WE}(f) \) we denote the minimal number of queries needed by strongly and weakly \( \frac{2}{n} \)-exact quantum algorithms for \( f \) with respect to \( \mu \), respectively. Note that \( Q_{WE}(f) = Q_{SE}(f) \) for all \( f \) and \( \mu \), hence in particular \( \max_\mu Q_{WE}(f) \leq \max_\mu Q_{SE}(f) \).

One of the first quantum algorithms operating in the query model is Grover’s search algorithm [9, 4]. If \( t = |x| > 0 \) then the algorithm uses \( \frac{1}{8} \sqrt{N/t} \) queries and with high probability outputs an \( i \) such that \( x_i = 1 \). Here we use \(|x|\) to denote the Hamming weight (number of 1’s) in \( x \), and \( x_i \) to denote the \( i \)th bit of \( x \). If \( |x| = 0 \) then the algorithm outputs ‘no solutions’. Brassard, Hoyer, Mosca, and Tapp [4] give an exact version of Grover’s algorithm that can accomplish the same task with probability \( 1 \) if \( t \) (the number of 1’s in the input) is known.

For total functions \( f : \{0, 1\}^n \rightarrow \{0, 1\} \), Beals, Buhrman, Cleve, Mosca, and de Wolf [2] proved that classical deterministic query complexity \( D(f) \) is polynomially related to the exact and bounded-error quantum complexities: \( D(f) = O(Q_{SE}(f)^4) \) and \( D(f) = O(Q_{SE}(f)^6) \).

A function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) is symmetric if its value \( f(x) \) depends only on \( |x| \). For such \( f \), define \( f_k = f(x) \) where \(|x| = k \). In [2] it is proven that \( Q_2(f) = \Theta(\sqrt{N(N - 1)}(\Gamma(f))) \), where \( \Gamma(f) = \min \{|2k - N - 1| \mid f_k \neq f_{k+1} \text{ and } 0 \leq k \leq N - 1 \} \). Informally, the quantity \( \Gamma(f) \) measures the length of the interval around Hamming weight \( \frac{N}{2} \) where \( f \) is constant. A symmetric function \( f \) is a threshold function if there is a \( 0 < t \leq N \), such that \( f(x) = 1 \) iff \(|x| \geq t \). Note that for \( t \leq N/2 \) we have \( Q_2(f) = \Theta(\sqrt{N}) \) as a direct consequence of the bound for symmetric functions. A function \( f : \{0, 1\}^n \rightarrow \{0, 1\} \) is monotone if \((\forall i \, x_i \leq y_i) \Rightarrow f(x) \leq f(y)\).

### 2.2 The Classical Yao Principle

Consider the following game-theoretic setting: player 1 has a choice between some \( m \) “pure” strategies and player 2 has a choice between \( n \) “pure” strategies. If player 1 plays \( i \) and player 2 plays \( j \), then player 1 receives “payoff” \( P_{ij} \). Player 1 wants to maximize the payoff, player 2 wants to minimize. Viewing \( P \) as an \( m \times n \) matrix, and using \( e_i \) and \( e_j \) to denote the appropriate unit column vectors with a 1 in place \( i \), respectively \( j \), the payoff corresponds to the matrix product \( e_i^T P e_j \). However, the players may also use “mixed” strategies (probability distributions over “pure” strategies) to further their goals. Mixed strategies of players 1 and 2 correspond to \( m \)- and \( n \)-dimensional column vectors \( \rho \) and \( \mu \), respectively, of non-negative reals that sum to 1. Now the expected payoff is \( \rho^T P \mu \). Note that if player 1 can choose his strategy \( \rho \) knowing player 2’s strategy \( \mu \), then he would choose \( \rho \) to maximize the payoff \( \rho^T P \mu \); in this situation player 2 would do best to choose \( \mu \) to minimize \( \max_\rho \rho^T P \mu \), giving expected payoff \( \min_\mu \max_\rho \rho^T P \mu \). Conversely, if player 2 could choose his strategy knowing player 1’s strategy, then the expected payoff would be \( \max_\mu \min_\rho \rho^T P \mu \).

Von Neumann’s famous minimax theorem [12, 15] tells us that these two quantities are in fact equal:

\[ \min_\mu \max_\rho \rho^T P \mu = \max_\rho \min_\mu \rho^T P \mu. \]

It is not hard to see that without loss of generality the “inner” choices can be assumed to be pure strategies, so as an easy consequence we also have

\[ \min_\mu \max_i e_i^T P \mu = \max_\rho \min_j \rho^T P e_j. \]

Yao [18] was the first to interpret this result in computational terms. We will sketch the computational interpretation below. Fix some classical model of computation for which the set of deterministic algorithms of complexity \( \leq c \) is finite, for every \( c \). Examples of such models are query complexity, communication complexity, etc. Player 1 chooses an algorithm to compute \( f : D \rightarrow \{0, 1\} \) and player 2 chooses an input \( x \) that is hard for player 1. The pure strategies for player 1 are all deterministic classical algorithms of complexity \( \leq c \) and hence his mixed strategies are all randomized classical algorithms of complexity \( \leq c \). The pure strategies for player 2 are the inputs in \( D \) and his mixed strategies are all probability distributions \( \mu \) over \( D \). We define the payoff matrix such that \( P_{ix} = 1 \) if algorithm \( i \) computes \( f \) correctly on input \( x \), and \( P_{ix} = 0 \) otherwise. In this setting, the minimax theorem states

\[ \min_\mu \max_i e_i^T P \mu = \max_\rho \min_x \rho^T P e_x. \]
Let us interpret both sides of this equation. On the left, the quantity \( e_i^T P \mu \) is the fraction of inputs on which deterministic algorithm \( i \) is correct, weighed according to \( \mu \), and \( \max_i e_i^T P \mu \) denotes this fraction for the optimal deterministic algorithm of complexity \( \leq c \). Thus the left-hand-side of the equation gives this optimal correct fraction for the hardest distribution \( \mu \) achievable by deterministic complexity-\( c \) algorithms.

On the other hand, \( \rho^T P e_x \) is the success probability on input \( x \) achieved by the randomized algorithm given by probability distribution \( \rho \) over deterministic algorithms, and \( \min_x \rho^T P e_x \) is its success probability on the hardest input. Thus the right-hand-side gives the highest worst-case success probability achievable by randomized complexity-\( c \) algorithms. Since these two quantities are equal for all \( c \), we obtain the classical Yao principle:

\[
R_\epsilon(f) = \max_\mu D_\mu(f).
\]

3 Proof of One Half of the Quantum Yao Principle

As a first result we prove that \( Q_2(f) \leq \max_\mu Q_{WE}(f) \). The proof is similar to the derivation of the classical Yao principle above, but the details are a bit more messy.

**Theorem 1** For all \( f : D \to \{0, 1\} \), with \( D \) finite, \( Q_2(f) \leq \max_\mu Q_{WE}(f) \).

**Proof.** Consider the (infinite) set of all quantum algorithms of complexity \( \leq \max_\mu Q_{WE}(f) \). Let \( i \) be any algorithm from this set, and \( x \in D \) an input. Consider the quantity \( |S_i(x)| \), which is 1 if algorithm \( i \) computes \( f(x) \) with success probability 1, and which is 0 otherwise. Call algorithms \( i \) and \( j \) similar if \( |S_i(x)| = |S_j(x)| \) for all \( x \in D \). In this way, similarity is an equivalence relation on the set of all quantum algorithms of complexity \( \leq \max_\mu Q_{WE}(f) \). Note that this relation has at most \( 2^{|D|} \) equivalence classes. From each equivalence class, we choose as a representative an algorithm from that class with the least complexity.

Now consider the game in which player 1 wants to compute \( f \), and as pure strategies he has available the (finite) set of representatives of the equivalence classes. Player 2 is an adversary that tries to make life as hard as possible for player 1 by choosing hard inputs \( x \in D \) to \( f \). Let \( S \) be the matrix of success probabilities \( (S_{ix} = S_i(x)) \). Define the payoff matrix as \( P_{ix} = |S_{ix}| \). Now consider the quantity \( \max_i e_i^T P \mu \). This represents the \( \mu \)-fraction of inputs on which the best weakly 2-weakly exact quantum algorithm for \( f \) with respect to that \( \mu \) is correct. By construction, this quantity is at least \( \frac{2}{3} \) for all \( \mu \). Using the minimax theorem, we now obtain:

\[
\frac{2}{3} \leq \min_\mu \max_i e_i^T P \mu = \max_\rho \min_i \rho^T P e_x \leq \max_\rho \min_x \rho^T S e_x.
\]

Here the last term can be interpreted as the success probability of a quantum algorithm formed by a probability distribution \( \rho \) over the set of representatives of the equivalence classes. By the above inequality, this algorithm has success probability \( \geq \frac{2}{3} \) for all inputs \( x \in D \). Since it is a probability distribution over algorithms of complexity \( \leq \max_\mu Q_{WE}(f) \), its complexity is at most \( \max_\mu Q_{WE}(f) \). Hence \( Q_2(f) \leq \max_\mu Q_{WE}(f) \). \( \square \)

**Corollary 1** For all \( f : D \to \{0, 1\} \), with \( D \) finite, \( Q_2(f) \leq \max_\mu Q_{SE}(f) \).

Note that although we restrict our attention to the query model of computation, the proofs of Theorem 1 and Corollary 1 also work for the other models of complexity where the classical Yao principle applies.

4 A Counterexample for the Strong Quantum Yao Principle

In this section we prove that the strong quantum Yao principle does not hold. There exists a problem \( f \) such that for a suitable distribution \( \mu \), \( Q_2(f) \) is exponentially smaller than \( Q_{SE}(f) \). This exponential gap follows from a known result about the classical and quantum complexity of Simon’s problem [17], and the fact that classical deterministic and quantum exact complexity are polynomially related for total problems [2, Theorem 5.4].

**Theorem 2** There exist a problem \( f \) and a distribution \( \mu \) such that \( Q_2(f) = O(n^2) \) and \( Q_{SE}(f) = \Omega(2^n) \).
Proof. Consider Simon’s problem: given a function $\phi : \{0,1\}^n \to \{0,1\}^n$ with the promise that there is an $s \in \{0,1\}^n$ such that $\phi(a) = \phi(b)$ iff $a \oplus b = s$, decide whether $s = 0$ or not. This function $\phi$ is given as an input $x$ of $N = n2^n$ bits, using $n$ 1-bit entries for each function value $\phi(.)$. The input bits can be queried in the usual way. Using Simon’s bounded-error quantum algorithm, this problem can be solved in $O(n^2)$ queries, and hence $Q_2(Simon) = O(n^2)$. Now define a distribution $\mu$ which uniformly places half the total weight on inputs with $s = 0$ and half the total weight on inputs with $s \neq 0$:

$$
\mu(x) = \begin{cases} 
\frac{1}{2(n^2)!} & \text{if } s = 0 \\
\frac{1}{2(n^2-1)(n^2-1)!} & \text{if } s \neq 0 \\
0 & \text{else.}
\end{cases}
$$

Simon proved that under this distribution, any classical algorithm that is correct on a fraction $\geq \frac{2}{3}$ requires $\Omega(\sqrt{2^n})$ queries. Now take any strongly $\frac{2}{3}$-exact quantum algorithm $A$ that solves this problem and makes $T$ queries, then $A$ computes some total function $g$. Since $D(g) = O(Q_E(g)^4)$, this implies that there exists a deterministic classical algorithm that computes $g$ using $O(T^3)$ queries. But this classical algorithm is then exact on a $\mu$-fraction $\frac{2}{3}$ of all Simon inputs. Simon’s lower bound on classical algorithms now implies that $O(T^4) = \Omega(\sqrt{2^n})$, and hence $Q_{SE}^n(Simon) = \Omega(2^{\frac{3}{2}})$. \qed

5 A Positive Result for the Weak Quantum Yao Principle

In this section we show that the weak quantum Yao principle holds for all symmetric functions. This section is divided into three subsections, in the first we prove the result for threshold functions, in the second subsection we extend it to symmetric functions. In the third subsection we investigate the weak quantum Yao principle for the uniform 2-level AND-OR tree, which is monotone and non-symmetric.

5.1 Equality up to a Constant Factor for Threshold Functions

For every distribution $\mu$, we will exhibit a weakly $\frac{2}{3}$-exact quantum algorithm that computes threshold function $f$ with threshold $t$ in time $O(\sqrt{N})$. This, together with Theorem 1 and the (known) fact that $Q_2(f) = \Theta(\sqrt{N})$ for threshold functions $f$ [2], gives the desired result.

Note that given a threshold function $f : \{0,1\}^N \to \{0,1\}$ with threshold $t$, in order to be sure that $f(x) = 1$, one will have to find at least $t$ 1’s in the input. The crucial idea behind our algorithm is that if the number of 1’s in the input is large enough, then for each distribution $\mu$ over the inputs, we can pick a substantially smaller part of the input such that there are between $t$ and $100t$ 1’s in this subpart for a large $\mu$-fraction of the inputs. This idea is formally stated in the following technical lemma.\footnote{We need the condition $i \geq 10$ in this lemma in order to be able to approximate the hypergeometric distribution by a binomial distribution with sufficient accuracy.}

Lemma 1 Let $t$ be a threshold, $\mu$ a probability distribution over the $x \in \{0,1\}^N$, and $i$ an integer such that $10 \leq i \leq \log N - \log t - 1$. Denote the event $t2^i \leq |x| \leq t2^{i+1}$ by $I$, and let $x \land y$ denote the bitwise AND of $x$ and $y$. There is a $y \in \{0,1\}^N$ with $|y| = \min\left\{\frac{10N}{2^i}, N\right\}$, such that $Pr_{x} [t \leq |x \land y| \leq 100t \mid I] > 0.7$.

Proof. Fix an $x \in \{0,1\}^N$ with $t2^i \leq |x| \leq t2^{i+1}$ and assume that $\frac{10N}{2^i} \leq N$, for otherwise the lemma trivially holds. We claim that if we pick a $y \in \{0,1\}^N$ with $|y| = \frac{10N}{2^i}$ uniformly at random, then $Pr[t \leq |x \land y| \leq 100t \mid I] > 0.7$. To prove this claim, note that

$$
Pr[|x \land y| = k \mid I] = \frac{\binom{|x|}{k} \binom{N-|x|}{|y|-k}}{\binom{N}{|y|}}.
$$

This means that $|x \land y|$ is hypergeometrically distributed, with expected value $E(|x \land y|) = \frac{|x||y|}{N}$. Note that in this case $10t \leq E(|x \land y|) \leq 20t$. By Markov’s inequality, it then follows directly that $Pr[|x \land y| > 100t \mid I] \leq 0.2$. \hfill $\Box$
We can approximate the above distribution with a binomial distribution since the number of draws is small compared to the size of the sample space, see e.g. [13], and we shall henceforth treat \(|x \land y|\) as if it were binomially distributed, with success probability \(\theta = \frac{|x|}{N}\) and number of draws \(n = |y|\). To bound \(\Pr[|x \land y| < t \mid I]\), we use the Chernoff bound as explained in [11, pp.67-73]:

\[
\Pr[|x \land y| < (1 - \delta)E(|x \land y|) \mid I] < e^{-\frac{(1-\delta)^2 E(|x \land y|)}{2}}.
\]

Choosing \(\delta = \frac{9}{10}\), we obtain \(\Pr[|x \land y| < t \mid I] < e^{-\frac{81}{100}t} < 0.1\). Combining the previous two inequalities, it then follows that \(\Pr[t \leq |x \land y| \leq 100t \mid I] > 0.7\). This proves the above claim.

Now imagine a matrix whose rows are indexed by the \(x\)s satisfying \(t^2i \leq |x| \leq t^2(i + 1)\) and whose columns are indexed by the \(M = \binom{N}{|y|}\) different \(y\) of weight \(|y| = \frac{10N}{2}\). We give the \((x, y)\) entry of this matrix value \(\mu(x \mid I)\) if \(t \leq |x \land y| \leq 100t\) and value 0 otherwise. By the above claim, each \(x\) row will contain at least 70% non zero entries, so the sum of the entries of each \(x\) row is at least \(0.7M \mu(x \mid I)\). Hence, the sum of all entries in the matrix is equal to \(\sum_x 0.7M \mu(x \mid I) = 0.7M\). But then there must be a column with \(\mu\)-weight at least 0.7. The \(y\) corresponding to this column is the \(y\) we are looking for in this lemma. \(\square\)

We will use the fact stated in the previous lemma to successively search for \(t\) 1’s in exponentially smaller parts of the inputs, assuming the presence of increasingly more 1’s in the original input. The following lemma states that this searching can be done efficiently:

**Lemma 2** There exists a quantum algorithm that can find all the 1’s in an input \(x\) of size \(N\) with probability 1, using at most \(\frac{\pi}{2} \sqrt{kn}\) queries, if \(k\) is a known upper bound on the number of 1’s in \(x\).

**Proof.** Consider Algorithm 1. It is easily proven that this algorithm indeed finds all 1’s, as follows. Assume an upper bound \(k \geq |x|\) on the number of 1’s in \(x\). If the exact version of Grover’s algorithm finds an index of a 1 bit, then we set this index to 0 in the search space. Because \(k\) is an upper bound on the number of 1’s in \(x\), we can lower \(k\) each time we find a 1, without \(k\) ever becoming less than the actual number of 1’s in \(x\). If it does not find a 1, then we know that our upper bound was too high and again we can safely lower it by 1. Using these facts, it is easily proven by induction on \(k\) that the algorithm indeed works as claimed.

**Algorithm 1**

```
for i = k down to 1 do
    Apply Grover’s exact search algorithm assuming there are i solutions.
    if A solution has been found then
        mark its index as a zero in the search space
    end if
end for
output the positions of all solutions found
```

The number of queries made by this algorithm is at most:

\[
\sum_{i=1}^{k} \frac{\pi}{4} \sqrt{\frac{N}{i}} \leq \frac{\pi}{4} \sqrt{N} \int_{0}^{k} \frac{di}{\sqrt{i}} = \frac{\pi}{2} \sqrt{kN}.
\]

\(\square\)

We are now ready to prove an upper bound on \(Q_{WE}^{\mu}(f)\) for threshold functions.

**Lemma 3** For threshold function \(f\) with threshold \(t\), and for every distribution \(\mu\), we have \(Q_{WE}^{\mu}(f) = O(\sqrt{N})\).
Proof. Fix a distribution \( \mu \). Invoking Lemmas 1 and 2, our algorithm is as follows. First we count the number of 1’s in the input using Algorithm 1, assuming an upper bound of \( 2^{10}t \) 1’s. If after that we haven’t found at least \( t \) 1’s yet, then we successively assume that there are between \( t2^i \) and \( t2^{i+1} \) 1’s in the input, with \( i \) going up from 10 to \( \log N - \log t - 1 \). For each of these assumptions, we search a smaller part of the input. If we have reached the \( i \) for which \( t2^i \leq |x| \leq t2^{i+1} \), then Lemma 1 guarantees us that for a large \( \mu \)-fraction of the inputs we can find a small subpart containing between \( t \) and 100 \( t \) 1’s. We then count the number of 1’s in this subpart using Algorithm 1. Algorithm 2 is the actual algorithm we will use.

**Algorithm 2**

Count the number of 1’s in the input using Algorithm 1, assuming an upper bound of \( 2^{10}t \) 1’s

if at least \( t \) 1’s are found then
  output 1
end if

for \( i = 10 \) to \( \log N - \log t - 1 \) do
  Let \( y^{(i)} \in \{0,1\}^N \) be a string of weight \( \min\{N, \frac{10N}{2^i}\} \) satisfying Lemma 1
  Using Algorithm 1, count the number of solutions in the subpart of the input induced by \( y^{(i)} \), assuming an upper bound of 100 \( t \) 1’s.
  if at least \( t \) 1’s are found then
    output 1
  end if
end for

output 0

This algorithm will be correct on all inputs \( x \) with \( |x| < t \) and will produce a correct answer on at least a \( \mu \)-fraction 0.7 of all inputs \( x \) with \( |x| \geq t \) as guaranteed by Lemma 1. Hence it will produce a correct answer on a \( \mu \)-fraction of at least:

\[
\mu(\{x \mid |x| < t\}) + 0.7(1 - \mu(\{x \mid |x| < t\})) \geq 0.7.
\]

Furthermore, its query complexity is equal to:

\[
O(\sqrt{tN}) + \sum_{i=10}^{\log N - \log t - 1} O\left(\sqrt{\frac{tN}{2^i}}\right) = O(\sqrt{tN}),
\]

where the first term corresponds to the cost of searching the entire space once with a small upper bound, and the summation corresponds to searching consecutively smaller subparts \( y^{(i)} \).

Recall that for threshold functions \( f : \{0,1\}^N \rightarrow \{0,1\} \) with threshold \( t \), \( Q_2(f) = \Theta(\sqrt{tN}) \). By Theorem 1 it then follows that \( \max_\mu Q_\mu_{WE}(f) = \Omega(\sqrt{tN}) \). In combination with Lemma 3, this yields:

**Lemma 4** For all threshold functions \( f : \{0,1\}^N \rightarrow \{0,1\} \) with threshold \( t \),

\[
Q_2(f) = \Theta\left(\max_\mu Q_\mu_{WE}(f)\right) = \Theta\left(\sqrt{tN}\right).
\]

5.2 Equality up to a Constant Factor for Symmetric Functions.

With the result about threshold functions in mind, we can easily prove that the quantum Yao principle holds for all symmetric functions as well.

**Theorem 3** For all symmetric functions \( f : \{0,1\}^N \rightarrow \{0,1\} \)

\[
Q_2(f) = \Theta\left(\max_\mu Q_\mu_{WE}(f)\right) = \Theta\left(\sqrt{N(N - \Gamma(f))}\right).
\]
Proof. From [2] we know that \(Q_2(f) = \Theta(\sqrt{N(1 + \Gamma(f))})\). Also, Theorem 1 tells us that \(Q_2(f) \leq \max_\mu Q_{WE}^\mu(f)\). It remains to show that for every distribution \(\mu\), \(Q_{WE}^\mu(f) = O(\sqrt{N(1 + \Gamma(f))})\).

Fix a probability distribution \(\mu\) over the set of all inputs. Note that \(\Gamma(f)\) measures the length of the interval around Hamming weight \(\frac{N}{2}\) where \(f\) is constant, so in order to compute \(f(x)\), it suffices to know \(|x|\) exactly if \(|x| \in [0, \frac{N - \Gamma(f)}{2}] = I_1\) or \(|x| \in (\frac{N + \Gamma(f) - 2}{2}, N] = I_2\), or to know that \(|x| \in [\frac{N - \Gamma(f)}{2}, \frac{N + \Gamma(f) - 2}{2}] = I_3\).

We can use the threshold algorithm of the previous section to determine whether \(x \in I_1\) (with \(\mu\)-error probability reduced to 1/6). We can use another threshold algorithm to determine whether \(x \in I_3\) (with the role of 0’s and 1’s reversed, and also with error \(\leq 1/6\)). Both threshold algorithms take \(O(\sqrt{\frac{N(1 + \Gamma(f))}{6}})\) queries. Now for at least 2/3 of the inputs \(x\), weighted according to \(\mu\), both of these threshold algorithms will give the correct answer. For all such \(x\) we can determine \(f(x)\) with certainty: if we know \(|x| \in I_2\) then we are done, because \(f\) is constant in this interval. If \(|x| \in I_1\) or \(|x| \in I_3\) then we use Algorithm 1 to count \(|x|\), using \(O(\sqrt{\frac{N(1 + \Gamma(f))}{6}})\) queries. Thus we have a weakly \(\frac{2}{3}\)-exact quantum algorithm for \(f\) with respect to \(\mu\), using \(O(\sqrt{\frac{N(1 + \Gamma(f))}{6}})\) queries in total.

\[\Box\]

5.3 A Result for the AND-OR Tree

Above we proved that the weak quantum Yao principle holds (up to a constant factor) for all symmetric functions. A similar result might be provable for all monotone functions. Recall that a Boolean function \(f\) is monotone if the function value cannot change from 1 to 0 if we change some input bits from 0 to 1. In this section we prove a preliminary result in this direction, namely that the known upper and lower bounds on the \(Q_2(f)\)-complexity of the 2-level AND-OR tree carry over to weakly \(\frac{2}{3}\)-exact quantum algorithms. This monotone but non-symmetric function is the AND of \(\sqrt{N}\) independent ORs of \(\sqrt{N}\) variables each. In the sequel, we use AO to denote this \(N\)-bit AND-OR tree.

No tight characterization of \(Q_2(AO)\) is known, but Buhrman, Cleve, and Widgerson [5] proved \(Q_2(AO) = O(\sqrt{N \log N})\) via a recursive application of Grover’s algorithm. Using a result about efficient error-reduction in quantum search from [6], this upper bound can be improved to \(Q_2(AO) = O(\sqrt{N \log N})\). This nearly matches Ambainis’ lower bound of \(\Omega(\sqrt{N})\) [1]. Note that Ambainis’ bound together with our Theorem 1 immediately gives the lower bound \(\max_\mu Q_{WE}^\mu(AO) = \Omega(\sqrt{N})\). Below we show that also the best known upper bound carries over to weakly \(\frac{2}{3}\)-exact algorithms: \(Q_{WE}^\mu(AO) = O(\sqrt{N \log N})\) for all \(\mu\).

To prove this result, we first show that we can efficiently reduce the error in weakly \(\frac{2}{3}\)-exact quantum search algorithms, in analogy with [6]. For every \(\mu\) and \(\epsilon\), we will construct a quantum search algorithm that uses \(O\left(\sqrt{\frac{N \log(1/\epsilon)}{\epsilon}}\right)\) queries and solves the search problem with certainty for \(1 - \epsilon\) of all inputs, weighted by \(\mu\). We first need the following lemma, which states that if an input contains many 1’s, then we can deterministically reduce its size to a smaller search space which will probably still contain at least one 1.

**Lemma 5** For all probability distributions \(\mu\) on \(\{0, 1\}^N\) and integers \(c\), there exists a \(y \in \{0, 1\}^N\) with \(|y| = \min\left\{\frac{cN}{t}, N\right\}\), such that \(\Pr_\mu[|x \wedge y| \geq 1, |x| > t] \geq 1 - e^{-c}\).

**Proof.** If \(\frac{cN}{t} \geq N\) then obviously the lemma holds (pick \(y = 1^N\)), so assume \(\frac{cN}{t} < N\). Fix an \(x \in \{0, 1\}^N\) with \(|x| > t\). If we pick a \(y \in \{0, 1\}^N\) with \(|y| = \frac{cN}{t}\) uniformly at random, then

\[
\Pr[|x \wedge y| = 0 \mid |x| > t] = \frac{\binom{N - |x|}{|y|}}{\binom{N}{|y|}} = \frac{(N - |x|)(N - |x| - 1) \cdots (N - |x| - |y| + 1)}{N(N - 1) \cdots (N - |y| + 1)} \leq \left(1 - \frac{|x|}{N}\right)^{|y|} \leq e^{-|x|/N \cdot t} \leq e^{-c}.
\]

Hence \(\Pr[|x \wedge y| \geq 1 \mid |x| > t] \geq 1 - e^{-c}\). By exactly the same averaging argument as in the proof of Lemma 1, we can show that for every distribution \(\mu\), there exists a \(y\) such that \(\Pr_\mu[|x \wedge y| \geq 1 \mid |x| > t] \geq 1 - e^{-c}\). \(\Box\)

With Lemma 5 at our disposal, we can now prove that we can “cheaply” reduce the error of weakly \(\frac{2}{3}\)-exact quantum search algorithms to small \(\epsilon\).
Lemma 6 For every $\epsilon > 0$ and every probability distribution $\mu$ over $\{0,1\}^N$, there exists a weakly $(1-\epsilon)$-exact quantum search algorithm with respect to $\mu$ that uses $O\left(\sqrt{N \log(1/\epsilon)}\right)$ queries.

Proof. Fix an error bound $\epsilon$ and distribution $\mu$. Our $(1-\epsilon)$-exact search algorithm is inspired by [6]. Let $t_0 = \log(1/\epsilon)$ (assume for simplicity that this is an integer). First we run the exact version of Grover’s algorithm on the input $x$ assuming that $|x| = 1$, then we run it again assuming that $|x| = 2$, and so on until $|x| = t_0$. This takes

$$
\sum_{i=1}^{t_0} \pi \sqrt{\frac{N}{t}} = O(\sqrt{N t_0}) = O\left(\sqrt{N \log(1/\epsilon)}\right)
$$

queries, and finds a 1 with certainty whenever $1 \leq |x| \leq t_0$.

It remains to find a 1 for “most” of the inputs $x$ that have $|x| > t_0$. Let $\mu_1$ be the probability distribution $\mu$ restricted to the $x$ with $|x| > t$. By Lemma 5, we know there exists a $y \in \{0,1\}^N$ with $|y| = O(N/t_0)$ such that $Pr_{\mu_1}[|x \land y| \geq 1] = Pr_{\mu}[|x \land y| \geq 1] | |x| > t_0| \geq \frac{5}{8}$. Now we use a $\frac{\epsilon}{3}$-exact quantum search algorithm with respect to $\mu_1$ to search the subpart of $x$ indicated by $y$. This subpart has size $O(N/t_0)$, and Lemma 3 guarantees us that there is a $\frac{\epsilon}{3}$-exact algorithm with $O(\sqrt{N/t_0})$ queries. Thus we find a 1 with certainty for a $\mu_1$-fraction (and hence also $\mu$-fraction) of at least $\frac{5}{8} - \frac{5}{8} = \frac{1}{2}$ of the inputs with $|x| > t_0$. Now we repeat this idea to “catch” $\frac{1}{2}$ of the remaining inputs. Let $\mu_2$ be $\mu_1$ restricted to the inputs with $|x| > t_0$ where the previous algorithm did not find a 1 with certainty. Using another $\frac{\epsilon}{3}$-exact algorithm (this time with respect to $\mu_2$) for another $y$, we can catch $\frac{1}{8}$ of the remaining inputs. We repeat this $t_0$ times and eventually catch

$$
1 - \left(\frac{1}{2}\right)^{t_0} = 1 - \epsilon
$$

of the inputs (weighed according to $\mu$) in this way. If we still have not found a 1 after all this, we stop and output ‘no solutions’, which ensures that our algorithm is always correct on the all-0 input. Note that the second part of the algorithm uses

$$
t_0 \cdot O(\sqrt{N/t_0}) = O\left(\sqrt{N \log(1/\epsilon)}\right)
$$

queries, so our overall query complexity is $O\left(\sqrt{N \log(1/\epsilon)}\right)$, as promised.

Using Lemma 6 we now show that the best known upper bound for $Q_2(AO)$ also holds for weakly $\frac{2}{3}$-exact quantum algorithms.

Theorem 4 For every distribution $\mu$ on $\{0,1\}^N$ we have $Q_{WE}^\mu(AO) = O(\sqrt{N \log N})$.

Proof. Fix some distribution $\mu$. We will sketch a $\frac{2}{3}$-exact quantum algorithm for AO with respect to $\mu$, along the lines of the recursive-Grover of [5]. For each of the $1 \leq i \leq \sqrt{N}$ OR functions at the “bottom” of the tree, let $\mu_i : \{0,1\}^{\sqrt{N}} \rightarrow [0,1]$ be the distribution over its $\sqrt{N}$ input bits induced by $\mu$, i.e., $\mu_i(y)$ is the sum of $\mu(x)$ over all $x \in \{0,1\}^N$ where the $i$th block of $\sqrt{N}$ variables takes value $y$. Let $A_i$ be a weakly $\left(1 - \frac{1}{\sqrt{\log N}}\right)$-exact quantum algorithm with respect to $\mu_i$ for the $i$th OR. By Lemma 6, each $A_i$ takes

$$
O\left(\sqrt{\sqrt{N \log N}}\right) = O\left(\sqrt{N \log N}\right)
$$

queries. Note that now for $\frac{5}{8}$ of the inputs, weighed according to $\mu$, all $A_i$ deliver the correct answer with certainty. By standard techniques (copying the answer and reversing the computation afterwards [3, 8]) we can “clean up” these computations, setting the workspace back to the initial state and just retaining the answer bit.

We now want to run a $\frac{2}{3}$-exact quantum algorithm for AND on top of these $\sqrt{N}$ subtrees to compute the AND-OR tree. Let $\mu' : \{0,1\}^{\sqrt{N}} \rightarrow [0,1]$ be the induced input distribution for the top-AND, i.e., $\mu'(y)$ is the sum of $\mu(x)$ over all $x \in \{0,1\}^N$ where the $i$th OR takes the value $y_i$, for all $1 \leq i \leq \sqrt{N}$. Let $A$ be a weakly $\frac{2}{3}$-exact quantum algorithm for the $\sqrt{N}$-variable AND with respect to $\mu'$. By Lemma 3 such an algorithm makes $O(N^{1/4})$ queries. If we replace, in $A$, a query to the $i$th bit by a call to $A_i$, then we
obtain an $O(\sqrt{N \log N})$-query algorithm that is correct with certainty on a $\mu$-fraction at least $\frac{5}{6} - \frac{1}{6} = \frac{2}{3}$ of all inputs.

6 Summary and Open Problems

In this paper we investigated to what extent quantum versions of the classical Yao principle hold. We formulated a strong and a weak version of the quantum Yao principle, showed that both hold in one direction, falsified the other direction for the strong version, and proved the weak version for the query complexity of all symmetric functions.

The main question left open by this research is the general validity of the weak quantum Yao principle. On the one hand, we may be able to find a counterexample to the weak principle as well, perhaps based on the query complexity of the order-finding problem. Shor showed that the order-finding problem can be solved by a bounded-error quantum algorithm using $O(\log N)$ queries [16]. Using Cleve’s $\Omega(N^{1/3}/\log N)$ lower bound on classical algorithms for order-finding [7], we can exhibit a $\mu$ such that any strongly $\frac{2}{3}$-exact quantum algorithm for $f$ with respect to $\mu$ requires $N^{O(1)}$ queries (in the same way as Theorem 1). This gives another counterexample to the strong quantum Yao principle. The same problem may even provide a counterexample to the weak quantum Yao principle, as it seems hard to construct even weakly $\frac{2}{3}$-exact quantum algorithms for this problem.

On the other hand, we may try to extend the class of functions for which we know the weak quantum Yao principle does hold. A good starting point here might be the class of all monotone functions. We discussed one such function, the 2-level AND-OR tree, in Section 5.3. Unfortunately, at the time of writing no general characterization of the $Q_2(f)$-complexity of all monotone functions is known, in contrast to the case of symmetric functions. Also, in this direction it might be a fruitful idea to further explore the rapidly growing field of quantum game theory (see for example [10]) and the possible connections between that area and our work.

Acknowledgments

We thank Harry Buhrman for initiating this research, for coming up with the counterexample of Theorem 2, and for useful comments on a preliminary version of this paper. We thank him and Peter Høyer for their contributions to an initial proof of the weak quantum Yao principle for the OR function, which forms the basis for the current proof of Theorem 3. We also thank Leen Torenvliet for useful discussions.

References

[1] A. Ambainis. Quantum lower bounds by quantum arguments. In Proceedings of 32nd ACM STOC, pages 636–643, 2000. quant-ph/0002066.

[2] R. Beals, H. Buhrman, R. Cleve, M. Mosca, and R. de Wolf. Quantum lower bounds by polynomials. In Proceedings of 39th IEEE FOCS, pages 352–361, 1998. quant-ph/9802049.

[3] E. Bernstein and U. Vazirani. Quantum complexity theory. SIAM Journal on Computing, 26(5):1411–1473, 1997. Earlier version in STOC’93.

[4] G. Brassard, P. Høyer, M. Mosca, and A. Tapp. Quantum amplitude amplification and estimation. quant-ph/0005055. To appear in Quantum Computation and Quantum Information: A Millennium Volume, AMS Contemporary Mathematics Series, 15 May 2000.

[5] H. Buhrman, R. Cleve, and A. Wigderson. Quantum vs. classical communication and computation. In Proceedings of 30th ACM STOC, pages 63–68, 1998. quant-ph/9802040.

[6] H. Buhrman, R. Cleve, R. de Wolf, and Ch. Zalka. Bounds for small-error and zero-error quantum algorithms. In Proceedings of 40th IEEE FOCS, pages 358–368, 1999. cs.CC/9904019.
[7] R. Cleve. The query complexity of order-finding. In Proceedings of 15th IEEE Conference on Computational Complexity, pages 54–59, 2000. quant-ph/9911124.

[8] R. Cleve, W. van Dam, M. Nielsen, and A. Tapp. Quantum entanglement and the communication complexity of the inner product function. In Proceedings of 1st NASA QCQC conference, volume 1509 of Lecture Notes in Computer Science, pages 61–74. Springer, 1998. quant-ph/9708019.

[9] L. K. Grover. A fast quantum mechanical algorithm for database search. In Proceedings of 28th ACM STOC, pages 212–219, 1996. quant-ph/9605043.

[10] D. Meyer. Quantum games and quantum algorithms. quant-ph/0004092. To appear in Quantum Computation and Quantum Information: A Millenium Volume, AMS Contemporary Mathematics Series, 5 april 2000.

[11] R. Motwani and P. Raghavan. Randomized Algorithms. Cambridge University Press, 1995.

[12] J. von Neumann and O. Morgenstern. Theory of Games and Economic Behavior. Princeton University Press, 1947.

[13] W. L. Nicholson. On the normal approximation to the hypergeometric distribution. Annals of Mathematical Statistics, 27:471–483, 1956.

[14] M. A. Nielsen and I. L. Chuang. Quantum Computation and Quantum Information. Cambridge University Press, 2000.

[15] G. Owen. Game Theory. Academic Press, second edition, 1982.

[16] P. W. Shor. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. SIAM Journal on Computing, 26(5):1484–1509, 1997. Earlier version in FOCS’94. quant-ph/9508027.

[17] D. Simon. On the power of quantum computation. SIAM Journal on Computing, 26(5):1474–1483, 1997. Earlier version in FOCS’94.

[18] A. C-C. Yao. Probabilistic computations: Toward a unified measure of complexity. In Proceedings of 18th IEEE FOCS, pages 222–227, 1977.