AN EXPANSIVE HOMEOMORPHISM OF A 3-MANIFOLD WITH A LOCAL STABLE SET THAT IS NOT LOCALLY CONNECTED

Abstract. In this article we construct an expansive homeomorphism of a compact three-dimensional manifold with a fixed point whose local stable set is not locally connected. This homeomorphism is obtained as a topological perturbation of a quasi-Anosov diffeomorphism that is not Anosov.

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1. Introduction

A homeomorphism $f$ of a metric space $(M, \text{dist})$ is expansive if there is $\eta > 0$ such that if $x, y \in M$ and $\text{dist}(f^n(x), f^n(y)) \leq \eta$ for all $n \in \mathbb{Z}$ then $x = y$. Expansivity is a well known property of Anosov diffeomorphisms and hyperbolic sets. Also, pseudo-Anosov [3, 4] and quasi-Anosov diffeomorphisms [2] are known to be expansive. In [3, 4], Hiraide and Lewowicz proved that on compact surfaces, expansive homeomorphisms are conjugate to pseudo-Anosov diffeomorphisms. In particular, if $M$ is the two-torus then $f$ is conjugate to an Anosov diffeomorphism and there are no expansive homeomorphisms on the two-sphere.

In [5] Vieitez proved that if $f$ is an expansive diffeomorphism of a compact three-dimensional manifold and $\Omega(f) = M$ then $f$ is conjugate to an Anosov diffeomorphism. We recall that $x \in M$ is a wandering point if there is an open set $U \subset M$ such that $x \in U$ and $f^n(U) \cap U = \emptyset$ for all $n \neq 0$. The set of non-wandering points is denoted as $\Omega(f)$. Given a homeomorphism $f : M \to M$ and $\varepsilon > 0$ small, the $\varepsilon$-stable set (local stable set) of a point $x \in M$ is defined as

$$W^s_\varepsilon(x) = \{ y \in M : \text{dist}(f^n(x), f^n(y)) \leq \varepsilon \text{ for all } n \geq 0 \}.$$ 

On three-dimensional manifolds there are several open problems. For instance, it is not known whether the three-sphere admits expansive homeomorphisms. In that article Vieitez asks: allowing wandering points, can we have points with local stable sets that are not manifolds for $f : M \to M$, an expansive diffeomorphism defined on a three-dimensional manifold $M$? We remark that in the papers by Hiraide and Lewowicz that we mentioned, before proving that stable and unstable sets form pseudo-Anosov singular foliations, they show that local stable sets are locally connected.

The purpose of this paper is to give a positive answer to Vieitez’ question for homeomorphisms. We will construct an expansive homeomorphism on a three-dimensional manifold with a point whose local stable set is not locally connected and $\Omega(f)$ consists of an attractor and a repeller. The construction follows the ideas

1991 Mathematics Subject Classification. Primary: 37B45; Secondary: 37B05.

Key words and phrases. topological dynamics, expansive homeomorphism, continuum theory.
in [1] where it is proved that the compact surface of genus two admits a continuum-wise expansive homeomorphism with a fixed point whose local stable set is not locally connected. The key point for the present construction is that on a three-dimensional manifold there is enough room to place two one-dimensional continua meeting in a singleton, even if one of these continua is not locally connected. In Remark 2.2 we explain why our example is not $C^1$.

2. The example

The example is a $C^0$ perturbation of the quasi-Anosov diffeomorphism of [2]. The perturbation will be obtained by a composition with a homeomorphism that is close to the identity. This homeomorphism will be defined in local charts around a fixed point and will be extended as the identity outside this chart. In §2.1 we construct this perturbation in $\mathbb{R}^3$ obtaining a homeomorphism such that the stable set of the origin $(0,0,0)$ is not locally connected. Then, in §2.2 we perform the perturbation of the quasi-Anosov diffeomorphism to obtain our example.

2.1. In local charts. Fix $r \in (0,1/2)$ and let $B \subset \mathbb{R}^3$ be the closed ball of radius $r$ centered at $(\frac{r}{2},0,0)$. Let $T_2: \mathbb{R}^3 \to \mathbb{R}^3$ be defined as

$$T_2(x,y,z) = \frac{1}{2}(x,y,z).$$

Define $B_n = T^n_2(B)$ for $n \in \mathbb{Z}$. Let $E \subset B$ be the non-locally connected continuum given by the union of the following segments:

- a segment parallel to $(1,0,0)$: $[\frac{1}{2} - \frac{r}{2}, \frac{1}{2} + \frac{r}{2}] \times \{(0,0)\}$,
- a segment parallel to $(0,1,0)$: $[\frac{1}{2} - \frac{r}{2}, \frac{1}{2} + \frac{r}{2}] \times \{(0,\frac{r}{2})\}$,
- a countable family of segments parallel to $(0,1,0)$: $\left\{ \frac{3}{2} - \frac{r}{2k}, \frac{3}{2} + \frac{r}{2k} \right\} \times \{0, \frac{r}{2}\}$.

Consider $\rho: B \to [0,1]$ a smooth function such that $\rho^{-1}(1) = E$ and $\rho^{-1}(0) = \partial B$. Define a vector field $X: \mathbb{R}^3 \to \mathbb{R}^3$ by

$$X(x,y,z) = \begin{cases} 
(0,-\rho(T^{-n}_2(x,y,z))y \log 4,0) & \text{if } \exists n \in \mathbb{Z} \text{ s.t. } (x,y,z) \in B_n, \\
(0,0,0) & \text{otherwise}.
\end{cases}$$

Remark 2.1. The vector field $X$ induces a flow $\phi: \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$. The proof is as follows. Notice that $X$ is smooth on $\mathbb{R}^3 \setminus \{(0,0,0)\}$. Since the regular orbits (i.e., not fixed points) are contained in the compact balls $B_n$ we have that these trajectories are defined for all $t \in \mathbb{R}$. At the origin we have a fixed point, and the continuity of $X$ gives the continuity of the complete flow on $\mathbb{R}^3$.

Let $\phi_t$ be the flow on $\mathbb{R}^3$ induced by $X$. For $t = 1$ we obtain the so called time-one map $\phi_1$. Let

$$E_k = T^{k}_2(E)$$

for all $k \in \mathbb{Z}$.

Remark 2.2. The homeomorphism $\phi_1$ is not $C^1$. This is because the partial derivative $\frac{\partial \phi_1}{\partial y}(x_*,0,0) = (0,1/4,0)$ if $(x_*,0,0)$ is in one of the segments of $E_k$ that is parallel to $(0,1,0)$. Since $\frac{\partial \phi_1}{\partial y}(0,0,0) = (0,1,0)$ and $x_* > 0$ can be taken arbitrarily small, we conclude that $\frac{\partial \phi_1}{\partial y}$ is not continuous at the origin.
Define the homeomorphisms $T_1, f : \mathbb{R}^3 \to \mathbb{R}^3$ as

\begin{equation}
T_1(x, y, z) = (x/2, 2y, 2z)
\end{equation}

and

\[ f = T_1 \circ \phi_1. \]

Consider the set $\tilde{W} = \bigcup_{k \in \mathbb{Z}} T^k_2(E) \cup \{(x, 0, 0) : x \in \mathbb{R}\}$. The \textit{(global) stable set} of a point $a \in \mathbb{R}^3$ associated to the homeomorphism $f$ is the set

\[ W^s_f(a) = \{ b \in \mathbb{R}^3 : \text{dist}(f^n(a), f^n(b)) \to 0 \text{ as } n \to +\infty \}. \]

**Proposition 2.3.** It holds that $W^s_f(0, 0, 0) = \tilde{W}$.

**Proof.** We start proving the inclusion $\tilde{W} \subset W^s_f(0, 0, 0)$. We will show that $f(E_k) = E_{k+1}$. By the definition of the flow we have that

\[ \phi_1(E_k) = \{(x, y/4, 0) : (x, y, 0) \in E_k\}. \]

Then $T_1(\phi_1(E_k)) = \{(x/2, y/2, 0) : (x, y, 0) \in E_k\}$ and this set is $T_2(E_k) = E_{k+1}$. Since $E_k \to (0, 0, 0)$ as $k \to +\infty$ we conclude that $E_k \subset W^s_f(0, 0, 0)$ for all $k \in \mathbb{Z}$. Also, $f(x, 0, 0) = (x/2, 0, 0)$ for all $x \in \mathbb{R}$. This proves that $\{(x, 0, 0) : x \in \mathbb{R}\} \subset W^s_f(0, 0, 0)$.

Take $p \notin \tilde{W}$. Note that if $p \notin B_n$ for all $n \in \mathbb{Z}$ then $f^n(p) = T^n_1(p)$ for all $n \geq 0$ and $p \notin W^s_f(0, 0, 0)$. Assume that $f^n(p) \in B_n$ for all $n \in \mathbb{Z}$ and define

\[ (a_n, b_n, c_n) = f^n(p). \]

If $c_0 \neq 0$ then $c_n = 2^n c_0 \to \infty$. Thus, we assume that $c_0 = 0$. Suppose that $b_0 > 0$ (the case $b_0 < 0$ is analogous). Let $y_0 = \sup\{ s \geq 0 : (a_0, s, 0) \in \tilde{W}\}$. Define $l = \{(a_0, y, 0) : y \in \mathbb{R}\}$ and $g : l \to l$ by the equation

\[ T^{-1}_2 \circ f(a_0, y, 0) = (a_0, g(y), 0). \]

In this paragraph we will show that $g(y) > y$ for all $y > y_0$. Let $\alpha : \mathbb{R} \to \mathbb{R}$ be such that $(a_0, \alpha(t), 0) = \phi_t(a_0, y, 0)$ for all $t \in \mathbb{R}$. That is, $\alpha$ satisfies $\alpha(0) = y$ and $\alpha(t) = -\rho(a_0, \alpha(t), 0)\alpha(t) \log 4$. Since $(a_0, y, 0) \notin \tilde{W}$ we have that

\[ \rho(a_0, \alpha(0), 0) < 1. \]

Then

\[ \int_0^1 \frac{\alpha(t)}{\alpha(0)} dt = - \int_0^1 \log 4 \rho(a_0, \alpha(t), 0) dt > - \log 4 \]

and $\log(\alpha(1)) - \log(\alpha(0)) > - \log 4$. Consequently, $\alpha(1) > \frac{1}{4}\alpha(0)$. Notice that

\[ (a_0, g(y), 0) = T^{-1}_2 \circ f(a_0, y, 0) = T^{-1}_2 \circ T_1 \circ \phi_1(a_0, y, 0) = T^{-1}_2 \circ T_1(a_0, \alpha(1), 0) = T^{-1}_2(a_0/2, 2\alpha(1), 0) = (a_0, 4\alpha(1), 0). \]

Then $g(y) = 4\alpha(1)$ and $g(y) > y$.

Recall that $X$ is the vector field that defines $\phi$. Since $X \circ T_2 = T_2 \circ X$, as can be easily checked, we have that $f \circ T_2 = T_2 \circ f$. This implies that

\[ (a_0, g^n(b_0), 0) = (T^{-1}_2 \circ f)^n(a_0, b_0, 0) = T^{-n}_2 \circ f^n(a_0, b_0, 0) = T^{-n}_2(a_n, b_n, 0). \]

and $g^n(b_0) = 2^{-n} b_n$. Since we are assuming that $f^n(p) \in B_n$ for all $n \in \mathbb{Z}$, we have that $2^{-n} b_n$ is bounded. But, as $b_0 > y_0$ and $g(y) > y$ for all $y > y_0$ we have that $g^n(b_0)$ is increasing and bounded. Then, if $b_* > y_0$ is the limit of $g^n(b_0)$ we have that $g(b_*) = b_*$, which contradicts that $g(y) > y$ for all $y > y_0$. This implies that
\( f^n(p) \) cannot be in \( B_n \) for all \( n \geq 0 \), and as we said, this shows that \( p \notin W^s_f(0,0,0) \). This proves the inclusion \( W^s_f(0,0,0) \subset \overline{W} \).

**Remark 2.4.** By the definition of the vector field \( X \) we see that \( \phi_1 \) preserves the horizontal planes (i.e., the planes perpendicular to \((0,0,1)\)). Also, \( \phi_1 \) leaves invariant the cube \([-2,2]^3 \) and is the identity in its boundary.

**2.2. The local perturbation of the quasi-Anosov.** To construct our example we start with a quasi-Anosov diffeomorphism as in [2]. A quasi-Anosov diffeomorphism is an axiom A diffeomorphism of \( M \) such that \( T_x W^s(x) \cap T_x W^u(x) = 0_z \) for all \( x \in M \), where \( T_x W^s(x), \sigma = s, u \), denotes the tangent space of the stable or unstable manifold \( W^s(x) \) at \( x \) and \( 0_z \) is the null vector of \( T_x M \). The quasi-Anosov diffeomorphism of [2], that will be denoted as \( f_{FR} : M \to M \), has the following properties: it is defined on a three-dimensional manifold, it is not Anosov and its non-wandering set is the union of two basic sets. The basic sets are an expanding attractor and a shrinking repeller. Both sets are two-dimensional and locally they are homeomorphic to the product of \( \mathbb{R}^2 \) and a Cantor set.

On the attractor there is a hyperbolic fixed point \( p \). Take closed balls \( U, V \subset M \) and \( C^0 \) local charts \( \varphi : [-2,2]^3 \to U \subset M \) and \( \psi : [-1,1] \times [-4,4]^2 \to V \subset M \) satisfying the following conditions:

- **C0:** \( \varphi([ -1,1] \times [-2,2]^2) = \psi([ -1,1] \times [-2,2]^2) \),
- **C1:** \( f_{FR}|U| = \psi \circ T_1 \circ \varphi^{-1} \) where \( T_1 \) was defined in (1),
- **C2:** in the local charts stable sets of \( f_{FR} \), the leaves are lines parallel to \((1,0,0)\),
- **C3:** there is \( r \in (0,1/2) \) such that \( W^u_{f_{FR}}(q) \) in the local chart is transverse to the horizontal planes if \( \varphi^{-1}(q) \) is in a neighborhood of \( B \) where, as in §2.1, \( B \subset \mathbb{R}^3 \) is the ball of radius \( r \) centered at \((3/2,0,0)\),
- **C4:** if \( \tilde{B}_n = \varphi(B_n) \) for all \( n \geq 0 \), we assume that \( f_{FR}^k(\tilde{B}_0) \cap \tilde{B}_0 = \emptyset \) for all \( k \geq 1 \).

Let \( \tilde{\phi} : M \to M \) be the homeomorphism given by

\[
\tilde{\phi}(x) = \begin{cases} 
\varphi \circ \phi_1 \circ \varphi^{-1}(x) & \text{if } x \in U, \\
\phi_1(x) & \text{if } x \notin U,
\end{cases}
\]

where \( \phi_1 \) is the time-one of the flow induced by the vector field \( X \) of §2.1. Define the homeomorphism \( \tilde{f} : M \to M \) as

\[
\tilde{f} = f_{FR} \circ \tilde{\phi}
\]

Notice that

\[
\tilde{\phi}(x) = x \text{ for all } x \notin \cup_{n \geq 0} B_n.
\]

**Remark 2.5.** This implies that if \( r \) is small then \( \tilde{f} \) is close to \( f_{FR} \) in the \( C^0 \) topology of homeomorphisms of \( M \).

**Theorem 2.6.** The homeomorphism \( \tilde{f} : M \to M \) is expansive and the local stable set of the fixed point \( p \) is connected but not locally connected.

**Proof.** By Proposition 2.3 and the condition C1 we have that the local stable set of \( p \) is \( \varphi(W) \). Since \( W \) is not locally connected and \( \varphi \) is a homeomorphism we conclude that the local stable set of \( p \) is not locally connected.

Let us show that \( \tilde{f} \) is expansive. By (2) we have that \( \tilde{f} \) and \( f_{FR} \) coincide on \( M \setminus \cup_{n \geq 0} \text{int}(B_n) \). Therefore, if \( \tilde{f}^n(x) \notin \cup_{k \geq 0} B_k \) for all \( n \in \mathbb{Z} \) then \( \tilde{f}^n(x) = f_{FR}^n(x) \) for all \( n \in \mathbb{Z} \). Let \( \eta_1 \) be an expansivity constant of \( f_{FR} \). Thus, if \( x, y \in M \) are such that \( \tilde{f}^n(x), \tilde{f}^n(y) \notin \cup_{k \geq 0} B_k \) for all \( n \in \mathbb{Z} \) then \( \sup_{n \in \mathbb{Z}} \text{dist}(\tilde{f}^n(x), \tilde{f}^n(y)) > \eta_1 \).
From our analysis on local charts of §2.1 we have that \( \tilde{B}_{n+1} \subset f_{FR}(\tilde{B}_n) = \tilde{f}(\tilde{B}_n) \) for all \( n \geq 0 \). This and condition C4 implies that if \( \tilde{f}^{n_0}(x) \in \tilde{B}_{k_0} \) for some \( n_0 \in \mathbb{Z} \) and \( k_0 \geq 0 \) then \( \tilde{f}^{n_0-k_0}(x) \in \tilde{B}_0 \). Let \( \eta \in (0, \eta_1) \) be such that if \( x \in \tilde{B}_0 \) and \( \text{dist}(y, x) < \eta \) then, in the local chart, \( W^s_\eta(x) \) is contained in a horizontal plane and \( W^u_\eta(y) \) is transverse to the horizontal planes. We have applied conditions C2 and C3. This implies that \( W^s_\eta(x) \cap W^u_\eta(y) \) contains at most one point. This proves that \( \eta \) is an expansivity constant of \( \tilde{f} \).

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