In theories with extra dimensions the Standard Model Higgs field can be identified with the internal components of higher-dimensional gauge fields (Higgs-gauge unification). The higher-dimensional gauge symmetry prevents the Higgs mass from quadratic divergences, but at the fixed points of the orbifold this symmetry is broken and divergences can arise if $U(1)$ subgroups are conserved. We show that another symmetry, remnant of the internal rotation group after orbifold projection, can avoid the generation of such divergences.

1 Introduction: Why Studying Higgs-Gauge Unification Theories?

One of the possible motivations for studying Higgs-gauge unification theories in extra dimensions is the so-called little hierarchy problem. If one considers the Standard Model (SM) as an effective theory valid up to a certain scale $\Lambda_{SM}$ and calculates the radiative corrections to the Higgs mass, he finds that these diverge quadratically and that, in order to avoid a fine-tuning of the parameters, $\Lambda_{SM}$ must be smaller than 1 TeV. This means that new physics must enter into the game at the scale of 1 TeV to regularize the ultraviolet behaviour. On the other hand this new physics can be parametrized by adding to the SM lagrangian non-renormalizable operators suppressed by powers of $\Lambda_{LH}$. From the non-observation of dimension-six four-fermion operators at LEP a lower limit on $\Lambda_{LH}$ of 5-10 TeV has been derived. This one order of magnitude discrepancy between the theoretically required upper limit and the experimental lower limit is called little hierarchy problem.

Up to now the best solution to the little (and grand) hierarchy problem is supersymmetry (SUSY). In the supersymmetric extensions of the SM quadratic divergences are absent so that the supersymmetric model can be extended up to $M_{Pl}$ without the need of other new physics: this solves the hierarchy problem. The $\Lambda_{SM}$ is now identified with the mass of the supersymmetric
particles, which can be $O(1 \text{ TeV})$. Moreover, if R-parity is conserved, it induces a suppression in the loop corrections to four-fermions operators which results in the relation $\Lambda_{LH} \sim 4\pi \Lambda_{SM}$, that precisely solves the little hierarchy problem. However SUSY has not yet been discovered and this reintroduces a small amount of fine-tuning in the theory. For this and other reasons we think it can be worthwhile looking for alternative solutions to the little hierarchy problem.

One alternative solution is given by the so-called Higgs-gauge unifications theories, in the context of theories in extra dimensions compactified on orbifolds. If we consider a gauge field in $D$ dimensions, its components can be split into two parts, according to the transformation properties under the four-dimensional (4D) Lorentz group: $A^A_M = (A^A_\mu, A^A_i)$, where $A^A_\mu$ is a 4D Lorentz vector while $A^A_i$ are 4D Lorentz scalars. The latter can be identified with the Higgs fields and they can acquire a non-vanishing vacuum expectation value through the Hosotani mechanism. The good feature of this kind of constructions is that the Higgs mass in the bulk is protected from quadratic divergences by the higher-dimensional gauge invariance and only finite corrections $\propto (1/R)^2$, where $R$ is the compactification radius, can appear. So the picture is the following: we have the 4D SM valid up to the scale $\Lambda_{SM} \sim 1/R$ that can be $O(1 \text{ TeV})$, then we have a non-renormalizable $D$-dimensional theory valid up to a certain scale $\Lambda_D$ which can be greater or equal to 10 TeV and then we have the ultraviolet completion of the theory. As we see also in this case the little hierarchy problem is solved.

In this talk we will deal with some specific features of these theories and, in particular, we will discuss how the mass protection given by the higher-dimensional gauge invariance can be spoiled at the fixed points of the orbifold and under which conditions it can be restored.

2 Symmetries at the Fixed Points of an Orbifold and Allowed Localized Terms

2.1 Gauge Theories on Orbifolds

We begin by considering a gauge theory coupled to fermions in a $D$-dimensional ($D = d+4 > 4$) space-time parametrized by coordinates $x^M = (x^\mu, y^i)$ where $\mu = 0, 1, 2, 3$ and $i = 1, \ldots, d$. The lagrangian is

$$\mathcal{L}_D = -\frac{1}{4} F^A_{MN} F^{AMN} + i \bar{\Psi} \Gamma^M_D D_M \Psi,$$

with $F^A_{MN} = \partial_M A^A_N - \partial_N A^A_M - g f^{ABC} A^B_M A^C_N$, $D_M = \partial_M - ig A^A_M T^A$ and where $\Gamma^M_D$ are the $\Gamma$-matrices corresponding to a $D$-dimensional space-time. This lagrangian is invariant under the gauge group $\mathcal{G}$ and of course under the $D$-dimensional Lorentz group $SO(1, D - 1)$.

Now we compactify the extra dimensions on an orbifold. Firstly we build up a $d$-dimensional torus $T^d$ by identifying $(x^\mu, y^i)$ with $(x^\mu, y^i + u^i)$, with $u$ belonging to a $d$-dimensional lattice $\Lambda^d$. Then we act on the torus with the element $k$ of the group $\mathcal{G}$ generated by a discrete subgroup of $SO(d)$ that acts crystallographically on the torus lattice and by discrete shifts that belong to it. The orbifold is finally defined by the identification $(x^\mu, y^i) = (x^\mu, (P_k \ y^i + u^i))$, where $P_k$ is the rotation associated to the element $k \in \mathcal{G}$. This group acts non-freely on the torus, i.e. it leaves some points invariant: these are called fixed points. The construction of the orbifold $S^1/Z_2$ in five dimensions is depicted in Fig. by identifying $y$ with $y + 2n\pi R$ (with $n$ integer) we are left with a segment of length $2\pi R$ and the extrema identified which can be represented by a circle; then by identifying $y$ with $-y$ we end up with a segment of length $\pi R$, with two fixed points in $y_f = 0, \pi R$, which is the orbifold.

Now that we have defined the orbifold we have to specify how it acts on the fields. If $\phi_\mathcal{R}$ is a generic field transforming as an irreducible representation $\mathcal{R}$ of the gauge group $\mathcal{G}$, then the orbifold action is defined by

$$k \cdot \phi_\mathcal{R}(y) = \lambda^k_\mathcal{R} \otimes \mathcal{P}^k_\phi \phi_\mathcal{R}(k^{-1} \cdot y)$$

(2)

where $\lambda_R^k$ is acting on gauge and flavor indices and $P_{\sigma}^k$, where $\sigma$ refers to the field spin, on Lorentz indices. Splitting the action of the orbifold in this way is particularly useful since while $P_{\sigma}^k$ is fixed by requiring the invariance of the lagrangian under this transformation (in particular we obtain $P_0^k = 1$ for scalar fields and $P_1^k = P_k$ for gauge fields), $\lambda_R^k$ is unconstrained and it can be used to break symmetries.\footnote{For a review on symmetry breaking on orbifolds see for instance Ref. [9] and references therein.}

Now we consider the gauge symmetry breaking realized by the orbifold at the generic fixed point $y_f$. First of all we have to understand why we are looking precisely there. The reason is that new lagrangian terms localized at the fixed points can be generated by bulk radiative corrections\footnote{If they are compatible with the existing symmetries.} if they are compatible with the existing symmetries. Since we are interested in the stability of the Higgs mass under radiative corrections, the knowledge of these symmetries can tell us if our theory is stable or not.

In general the orbifold action breaks the gauge group in the bulk $G = \{T^A\}$ to a subgroup $H_f = \{T^{a_f}\}$, at the fixed point $y_f$, defined by the generators of $G$ which commute with $\lambda_R^k$, i.e. $[\lambda_R^k, T^{a_f}] = 0$. This condition must be satisfied by any irreducible representation $R$ of $G$. The symmetry breaking pattern also defines which fields are non-zero at $y_f$. In this case they will be $A^a_{\mu}$, which are the gauge bosons of the unbroken gauge group $H_f$, and $A^i_{\hat{a}}$, for some $i$ and $\hat{a}$, with zero modes, plus some derivatives of non-invariant fields without zero modes. Since also the parameters which define the gauge transformation transform under the orbifold action, the derivatives of some of them which are invariant define a set of local transformations that are called $K$-transformations.\footnote{Eventually at the fixed point $y_f$ there are two symmetries remnant of the original gauge symmetry $G$: $H_f$ and $K$. All this is summarized in Fig.\ref{fig:orbifold}.}

Now that we have discussed the main features of gauge symmetry breaking on orbifold, we are ready to write down the most general $4D$ effective lagrangian. This is given by the integral over the extra coordinates of the $D$-dimensional lagrangian plus the terms localized at the fixed
the appearance of the tadpole (or, equivalently, of the divergent mass term for the Higgs)?

O the invariance under rotations of the tangent space \(SO\), following: 4D Lorentz invariance \(SO\), the tangent space) by the smooth compactification and then it is definitively broken down to the residual gauge group \([\mathcal{H}_f]\) and the residual local symmetry \([\mathcal{K}]\).

The \(\mathcal{K}\)-symmetry is very important since it forbids the appearance of direct mass terms like \(\Lambda^2 A_i^a A_j^b\) in the case in which \(A_i^a\) is \(\mathcal{G}_f\)-invariant. Anyway the previously listed symmetries allow localized terms as \((F^a_{\mu\nu})^2\), which corresponds to a localized kinetic term for \(A^a_{\mu}\) and \(F^a_{\mu\nu} F^{a\mu\nu}\) which is a localized anomaly. Moreover if for some \((i, j)\) \(F^a_{ij}\) and \(A_i^a\) are orbifold invariant (this is model-dependent), \((F^a_{ij})^2\) and \((F^a_{ij})^2\) are also allowed, giving rise respectively to localized quartic couplings and kinetic terms for \(A_i^a\). All these operators are dimension-four, that is they renormalize logarithmically. However if \(\mathcal{H}_f\) contains a \(U(1)\) factor

\[
F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i - g f^{abc} A^b_i A^c_j,
\]

where \(\alpha\) is the \(U(1)\) quantum number, is invariant under all the above discussed symmetries and can be generated by bulk radiative corrections at the fixed points. This means that we expect both a tadpole for the derivatives of odd fields and a mass term for the even fields. Since these operators are dimension-two, their respective renormalizations will lead to quadratic divergences, making the theory ultraviolet-sensitive.

Apart from the 5D case where the term \(F_{ij}\) does not exist, for \(D \geq 6\) it does and its generation has been confirmed by direct computation in 6D orbifold field[2,11] and 10D string[13] theories. Of course if these divergent localized mass terms were always present, Higgs-gauge unification theories would not be useful in order to solve the little hierarchy problem. One way out can be that local tadpoles vanish globally, but this requires a strong restriction on the bulk fermion content[10]. A more elegant and efficient solution, based on symmetry arguments, has been presented in Ref. [12] and will be discussed in the following.

### 2.3 The Residual \(O_f\) Symmetry

When we discussed the symmetry breaking induced by the orbifold, we did not consider the \(D\)-dimensional Lorentz group. When compactifying a \(d\)-dimensional space to a smooth Riemannian manifold (with positive signature), at each point a tangent space can be defined and the orthogonal transformations acting on it form the group \(SO(d)\).[13] When the orbifold group acts on the manifold, in the same way as the gauge group \(\mathcal{G}\) is broken down to \(\mathcal{H}_f\), the internal rotation group \(SO(d)\) is broken down to a subgroup \(\mathcal{O}_f\) defined by the generators of \(SO(d)\) which commutes with \(\mathcal{P}_\sigma\), i.e. \([\mathcal{P}_\sigma, \mathcal{O}_f] = 0\). This means that the original Lorentz group \(SO(1, D - 1)\) is firstly broken down to \(SO(1, 3) \otimes SO(d)\) (where \(SO(d)\) must be understood as acting on the tangent space) by the smooth compactification and then it is definitively broken down to \(SO(1, 3) \otimes \mathcal{O}_f\) by the orbifold action. All this is outlined in Fig. [11]

We have then identified an additional symmetry that the lagrangian \(\mathcal{L}_f\) at the fixed point \(y_f\) must conserve. Summarizing, the invariances that we have to take into account are the following: 4D Lorentz invariance \([SO(1, 3)]\), invariance under the action of the orbifold group \([\mathcal{G}_f]\), usual 4D gauge invariance \([\mathcal{H}_f]\), remnant of the bulk gauge invariance \([\mathcal{K}]\) and remnant of the invariance under rotations of the tangent space \([\mathcal{O}_f]\). Now the question is: can this \(\mathcal{O}_f\) forbid the appearance of the tadpole (or, equivalently, of the divergent mass term for the Higgs)?

If \(\mathcal{O}_f\) contains among its factors at least one \(SO(2)\), then a corresponding Levi-Civita tensor \(\epsilon^{ij}\) exists, such that the lagrangian term \(\epsilon^{ij} F^{(a)}_{ij}\) is also \(\mathcal{O}_f\)-invariant. In this case tadpoles are
allowed. On the other side, if $O_f$ is given by a product of $SO(p_i)$ with $p_i > 2 \forall i$, then the Levi-Civita tensor has $p_i$ indices and only invariants constructed using $p_i$-forms are allowed. Since $F_{ij}^\alpha$ has two indices this means that in this case tadpoles are not allowed. We have then found a sufficient condition for the absence of localized tadpoles which precisely is that the smallest internal subgroup factor be $SO(p)$ with $p > 2$.

Evidently $O_f$ is orbifold-dependent; in Ref. [12] we analyzed the case of the orbifold $T^d/Z_N$ for $d$ even. In this case the generator of the orbifold group is given by $P_N = \text{diag}(R_1, \ldots, R_{d/2})$, where $R_i$ is the discrete rotation in the $(y_{2i-1}, y_{2i})$-plane. If $Z_{N_f}$ is the orbifold subgroup which leaves invariant the point $y_f$, it can be shown that if $N_f > 2$ then $O_f = \bigotimes_{i=1}^{d/2} SO(2)_i$, where $SO(2)_i$ is the $SO(2) \subseteq SO(d)$ that acts on the $(y_{2i-1}, y_{2i})$-subspace. Then in every subspace $\epsilon^{IJ}$ exists and we expect a tadpoles appearance at the fixed points $y_f$ of the form

$$
\sum_{i=1}^{d/2} C_i \sum_{I,J=2i-1}^{2i} \epsilon^{IJ} F_{IJ}^\alpha \delta^{(d/2)}(y - y_f).
$$

On the contrary if $N_f = 2$ then the generator of the orbifold subgroup is the inversion $P_2 = -1$ that obviously commutes with all the generators of $SO(d)$ so that we have $O_f = SO(d)$. In this case the Levi-Civita tensor is $\epsilon^{1 \ldots d}$ and only a $d$-form can be generated linearly in the localized lagrangian. Therefore tadpoles are only expected in the case of $d = 2 (D = 6)$. This last comment also apply to the case of $Z_2$ orbifolds in arbitrary dimensions (even or odd), since the orbifold generator is always $P = -1$ and then the internal rotation group is always $O_f = SO(d)$.

In Ref. [12] we explicitly checked this result at one- and two-loops for the orbifold $T^d/Z_2$ for any $D$.

3 Conclusions

In orbifold field theories the SM Higgs field can be identified with the internal components of gauge fields. Then the higher-dimensional gauge invariance prevents the Higgs from acquiring a quadratically divergent mass term in the bulk, while at the fixed points a remnant of bulk gauge symmetry after symmetry breaking forbids the appearance of direct mass terms. Still, if the residual gauge symmetry contains a $U(1)$ factor, the corresponding field strength for the 4D scalar fields is invariant under the orbifold action, the 4D Lorentz symmetry, the residual gauge invariance and the residual local symmetry, so that it can be radiatively generated at the fixed points. This is a dimension-two operator and gives rise to a quadratically divergent mass for the Higgs. However we showed that another symmetry must be considered. Indeed, when compactifying on an orbifold, the internal rotation group acting on the tangent space that can be defined at each point of a smooth manifold is broken down at the fixed points, since there a tangent space cannot be defined. How the breaking is realized depends on the particular orbifold but in general a group $O_f$, subgroup of the internal rotation group, will survive and then it shall be taken into account when looking for lagrangian terms that can be radiatively generated. Actually this residual symmetry can forbid the appearance of dangerous divergent terms. Indeed if $O_f$ contains among its factors at least one $SO(2)$, then a Levi-Civita tensor $\epsilon^{ij}$ exists and the previously mentioned invariant field strength will be generated at the considered fixed point in the form $\epsilon^{ij} F_{ij}^{\alpha}$. On the contrary if $O_f$ is given by a product of $SO(p_i)$ with each $p_i > 2$, then only invariants constructed with $p_i$-forms can be generated and our dangerous term will not appear. What we have found is then another sufficient condition for the absence of localized tadpoles. Also we have shown that in the case of the orbifolds $T^d/Z_N$ ($N > 2$, $d$ even) $O_f$ is a product of $SO(2)$ groups (at least at the $Z_N$-fixed points) and then divergent terms will always be allowed. On the other side for the orbifolds $T^d/Z_2$ (any $d$) $O_f$ always coincides with the whole $SO(d)$ and then tadpoles will never appear for $d > 2 (D > 6)$.
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