Effect of spiral vortices on the stability of vortex structures in the diffusor

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Abstract:
It is known that the cause of the instability of stationary vortex structures are the convective terms of the Navier-Stokes equations. Here, the formation of vortex structures is fundamentally influenced by the vector product of the velocity curl and the velocity vector. Spiral vortices are contingent on the zero value of this product and are one of the causes of the formation of a non-stationary rope in a diffusor. In the paper, equations of spiral vortices are derived and the condition of the vortex stability in a diffusor is determined depending on the existence of spiral vortices. In this sense, new forms of the Navier-Stokes equations are derived. It is possible to analyze the influence of boundary conditions on the stability of vortex structures. The theoretical knowledge is applied to the analysis of the stability of the vortex rope in a diffusor. In addition, a simple method for detecting the onset of the instability of vortex structures is presented.

1. Introduction
In general, the study investigates the causes of the formation of a vortex structure instability. The main interest focuses on the problem of the origin and elimination of the vortex rope that occurs in the diffusor [1]. Such diffusor might be the draft tube of hydraulic turbine. It is known that in case of hydraulic turbines, especially Francis turbines, operated in regimes far from the best efficiency point the strong swirling flow exits the runner and consequently decelerates in the draft tube [2], [3]. This flow condition is suitable for creation of vortex structure in shape of spiral vortex. The spiral vortex rotation induces the high amplitude pressure pulsations and stress the mechanical parts [4].

Furthermore, the specific shapes of vortex structures originating in a diffusor, both in a stable and unstable region, are presented. In addition to the analysis of static pressure, specific energy and helicity, it is now possible to study the scalar function Λ(x,t). It characterizes the structure of spiral vortices depending on the helicity. The helicity shows the deviation from the quasi-potential flow, expressed as the scalar product of the velocity curl and the velocity vector:

\[ \text{rot } \mathbf{v} \cdot \mathbf{v} = 0 \]  (1)

Based on these characteristics and the newly modified Navier-Stokes equations, it is possible to design the boundary conditions at the outlet of the turbine runner to ensure the stability of the stationary flow [5]. Requirements for the new form of the boundary conditions at the draft tube inlet will also affect the new concept of the turbine runner. Influence of spiral vortices on the stability of stationary flow is analyzed both theoretically and experimentally.
2. Theoretical study

Potential, irrotational liquid flow is described by the equation \( \text{rot} \mathbf{v} = 0 \), \( \mathbf{v} = \text{grad} \Phi \). Rotational, so-called quasi-potential flow is defined by the equation \( \text{rot} \mathbf{v} \cdot \mathbf{v} = 0 \), \( \mathbf{v} = x \text{ grad} \Phi \). In areas where the equation for quasi-potential flow is applicable, spiral vortices cannot occur. This area is characterized by the so-called Helicity equal to 1, since normalized Helicity is defined by:

\[
H_{\text{norm}} = \frac{\mathbf{v} \cdot \text{rot} \mathbf{v}}{|\mathbf{v}| \cdot |\text{rot} \mathbf{v}|}
\]

(2)

Diversion of this value from 1 characterizes secondary flow, of which dominant manifestations are spiral vortices, defined by relation

\[
\boldsymbol{\Omega} \times \mathbf{v} = 0 \quad \boldsymbol{\Omega} = \text{rot} \mathbf{v} = 2\mathbf{\omega}
\]

(3)

\( \mathbf{\omega} \) is the angular velocity vector of the liquid. Secondary flow and stability of steady liquid flow are influenced by convective terms in the Navier-Stokes equations

\[
(\mathbf{v} \cdot \nabla)\mathbf{v} = \text{grad} \frac{|\mathbf{v}|^2}{2} + \boldsymbol{\Omega} \times \mathbf{v}
\]

(4)

The vorticity \( \boldsymbol{\Omega} \) has a significant effect on the magnitude of the dissipation function \( \mathcal{D}_H \) whose principal term depends on

\[
\mathcal{D}_H \sim \boldsymbol{\Omega} \cdot \boldsymbol{\Omega}.
\]

(5)

In an area where \( \boldsymbol{\Omega} \times \mathbf{v} = 0 \) is the ratio of \( \boldsymbol{\Omega} \) to the dissipation function size, as it is valid here

\[
\boldsymbol{\Omega} \times \mathbf{v} = 0 \Rightarrow \boldsymbol{\Omega} = \Lambda(x, t)\mathbf{v},
\]

(6)

therefore,

\[
\mathcal{D}_A \sim \Lambda^2 |\mathbf{v}|^2.
\]

(7)

Condition (6) where the velocity curl is collinear with the velocity vector represents the existence of spiral vortices with special properties that cause the instability of the steady flow and increase the dissipation of part of the mechanical energy. Let us now consider the properties of the vortices, defined by equation (6), assuming an incompressible liquid:

\[
\boldsymbol{\Omega} \times \mathbf{v} = 0, \quad \boldsymbol{\Omega} = \Lambda \mathbf{v} \ , \quad \text{div} \mathbf{v} = 0
\]

(8)

Applying to the \( \boldsymbol{\Omega} \) operation of divergence, it holds

\[
\text{grad} \Lambda \cdot \mathbf{v} = 0.
\]

(9)

Hence, it is obvious that the value \( \Lambda \) increases in a direction perpendicular to the direction of the velocity vector. By applying a rotation operation to \( \boldsymbol{\Omega} \) in (8), we obtain:

\[
\text{rot} \boldsymbol{\Omega} = \text{grad} \Lambda \times \mathbf{v} + \Lambda \text{ rot} \mathbf{v}.
\]

considering (6) it will be

\[
\text{rot} \boldsymbol{\Omega} = \text{grad} \Lambda \times \mathbf{v} + \Lambda^2 \mathbf{v} = 0,
\]

or

\[
\text{rot rot} \mathbf{v} = \text{grad} \Lambda \times \mathbf{v} + \Lambda^2 \mathbf{v} = 0.
\]

(10)

Taking into account even the continuity equation for incompressible liquid, the following applies:

\[
\text{div} \mathbf{v} = 0.
\]

(11)

From equations (10), (11), unknown \( \Lambda \) and \( \mathbf{v} \) can be determined under the given boundary conditions [6]. However, this equation system is not suitable for analysis, therefore we will introduce a simpler variant of these equations. See (24), (25). From equations (8), (9) we can deduce

\[
\boldsymbol{\Omega} \cdot \text{grad} \Lambda = 0.
\]

(12)
If we multiply (10) by $\nabla \Lambda$, we obtain an important relation with respect to validity (9), (12):
\[
\text{rot } \Omega \cdot \nabla \Lambda = 0.
\] (13)

Define the specific energy by expression:
\[
Y = \frac{p}{\rho} + \frac{1}{2}|\mathbf{v}|^2 - \mathbf{g} \cdot \mathbf{x}
\] (14)
and Navier-Stokes equation can be written, considering validity of (6) in the form:
\[
\frac{\partial \mathbf{v}}{\partial t} + \nabla Y - \gamma (\nabla \Lambda \times \mathbf{v} + \Lambda^2 \mathbf{v}) = \mathbf{0}.
\] (15)

Substituting (10) into (15):
\[
\frac{\partial \mathbf{v}}{\partial t} + \nabla Y - \gamma (\nabla \Lambda \times \mathbf{v}) = \mathbf{0}.
\] (16)

After scalar multiplication of (16) by vector $\mathbf{v}$, it can be written:
\[
\frac{1}{2} \frac{\partial}{\partial t} (|\mathbf{v}|^2) + \nabla Y \cdot \mathbf{v} - \gamma \Lambda^2 |\mathbf{v}|^2 = 0.
\] (17)

Multiplying (16) by $\nabla \Lambda$ we obtain:
\[
\frac{\partial \mathbf{v}}{\partial t} \cdot \nabla \Lambda + \nabla \Lambda \cdot \nabla \Lambda = 0.
\] (18)

Assuming that $\Lambda$ does not depend on time and considering (9):
\[
\nabla Y \cdot \nabla \Lambda = 0.
\] (19)

When we apply the divergence operation to (15), the following holds in the area of spiral vortices:
\[
\Delta Y = 0.
\] (20)

A very important conclusion follows from equation (17). In case that
\[
\nabla Y \cdot \mathbf{v} \leq 0,
\] (21)
the spiral vortex is time-dependent, therefore the steady flow is unstable in the area where $\Omega \times \mathbf{v} = \mathbf{0}$ is true, and thus unstable spiral vortices are created. If we write equation (6) for the vorticity in the index symbol, the following applies:
\[
(\text{rot } \mathbf{v})_i = \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j} = \Lambda v_i.
\] (22)

Considering (9) in the form
\[
\frac{\partial \Lambda}{\partial x_i} v_i = 0,
\] (23)
equation (22) can be simplified in the following way:
\[
\frac{\partial}{\partial x_j} \left[ \varepsilon_{ijk} v_k - \Lambda x_i v_j \right] = 0.
\] (24)
and supported by the continuity equation:

\[
\frac{\partial v_i}{\partial x_i} = 0,
\]

(25)

Then it is possible to specify \( \Lambda, v_i \), more easily than the (10), (11) at the given boundary conditions. Let us resolve a simple case (in this case, it is actually a problem of eigenvalues.)

\[
\Lambda = \text{konst.}
\]

(26)

Under this assumption, the solution of (24), (25) can be written in the form:

\[
v_i = A_i e^{\Lambda k_i x_i},
\]

\[
\varepsilon_{ijk} k_i A_k - A_i = 0
\]

\[
A_i k_i = 0.
\]

If we express vector \( k \) by a relationship

\[
k^T = (k_1, k_2, k_3),
\]

(27)

We get the equation \(|k|^2 + 1 = 0\), from there it can be deduced that

\[
k = (+)(-)i\hat{k}
\]

(28)

so finally we have:

\[
v_i = A_i \left[ \cos(\Lambda \hat{k}_i x_i) \pm i \sin(\Lambda \hat{k}_i x_i) \right]
\]

(29)

We now solve a more general task when \( \Lambda(x, t) \):

\[
\text{rot } \mathbf{v} = \Lambda(x, t) \mathbf{v}, \quad \text{div } \mathbf{v} = 0
\]

(30)

\[
\varepsilon_{ijk} \frac{\partial v_k}{\partial x_j} - \Lambda v_i = 0
\]

(31)

Under these assumptions, the solution of equations (25) and (31) can be written in the form \( v_i = A_i e^{\alpha \Lambda} \), so we obtain a system of equations for \( \Lambda(x) \):

\[
\alpha \varepsilon_{ijk} A_k \frac{\partial \Lambda}{\partial x_j} - \Lambda A_i = 0
\]

(32)

\[
\alpha A_i \frac{\partial \Lambda}{\partial x_i} = 0.
\]

(33)

Equations (32), (33) are fulfilled if:

\[
\Lambda = \beta e^{k_i x_i}, \quad \alpha = \pm i \phi, \quad \beta = \beta(t).
\]

(34)

The resulting relationship for the liquid velocity components can be written as:

\[
v_i = A_i \left[ \cos(\phi \beta e^{k_i x_i}) \pm i \sin(\phi \beta e^{k_i x_i}) \right]
\]

(35)
From conditions (16), (19), (20) and considering (35) it is possible to deduce important relationships:

\[
\frac{\partial \mathbf{v}}{\partial t} \text{ grad } \Lambda = 0, \quad Y = Y_0(t),
\]

(36)

Considering these results in equation (17), we obtain:

\[
\frac{\partial}{\partial t} |\mathbf{v}|^2 - 2\gamma \Lambda^2 |\mathbf{v}|^2 = 0
\]

(37)

The solution of this equation for \( \Lambda = \Lambda(x) \), corresponds to the function:

\[
|\mathbf{v}|^2 = ce^{2\gamma(A(t))^2t}; \quad c = c(x).
\]

(38)

From there, the instability of the vortex velocity is evident, assuming that \( \text{rot } \mathbf{v} \times \mathbf{v} = 0 \).

3. Test case evaluation

To verify the theory, the test case of swirl generator geometry was used. The geometry was proposed by the team of prof. Resiga at the University of Timisoara [7], [8]. The scheme of the swirl generator is described in Figure 1, where the main parts are the stay vanes, runner blades and the outlet diffusor. Runner blades have the function of a freely rotatable runner that is not braked in any way. The purpose of the freely rotatable runner is to redistribute the total pressure by creating an excess of the axial velocity near the shroud (the outer casing of the generator) and the corresponding deficit near the hub. In this way, a velocity profile is generated behind the generator blades similar to the Francis turbine velocity profile operated at 70% of the optimal flow rate, where the spiral vortex structures are formed. Moreover, the test-rig includes the application of axial water jet from the nozzle situated in the central hub (see Figure 5) [9]. The application of water jet through the turbine runner hub is one of the possible technique which could be used in order to suppress the vortex rope creation in the draft tube [10], [11].

The operating parameters are as follows: flow rate \( Q = 30 \text{ l/s} \), rotation of unbraked blade wheel \( n = 920 \text{ min}^{-1} \), water density \( \rho = 998 \text{ kg/m}^3 \). The jet influence on the spiral vortex is studied by application of \( Q_{\text{jet}} = 0.02Q \) (in other words 2% of nominal flow rate). The CFD simulation was carried out using commercial code Ansys Fluent and final post-processing was done in Matlab. Only constrained domain of convergent-divergent part (starts just behind the runner blades and ends with cylindrical cross-section) was considered. The cylindrical velocity components and turbulent quantities from previously done simulation of whole geometry were prescribed as the inlet boundary condition. The relative static pressure of 0 Pa was set for the outlet boundary. The Reynolds stress turbulence model was used to solve URANS equations with time-step size \( \Delta t = 0.001 \text{ s} \). The more details on CFD simulation might be found in [12]. For the post-processing purpose only the diffusor part was considered.

On the basis of the theoretical relationships derived above, it is shown that the origin of the vortex rope is related to the occurrence of spiral vortices. Proof of this are the results of helicity modeling shown in Figure 2. The area of the normalized helicity \( H_{\text{norm}} = 0.9 \), corresponds to the area of the specific energy \( Y \) and static pressure \( p_{\text{stat}} \) shown in Figure 3 and 4 respectively. Their unsteady behavior in this area corresponds to the velocity distribution according to relation (38). To show the results of different boundary conditions the spiral vortex influenced by 2% jet was studied. The iso-surfaces of normalized helicity \( H_{\text{norm}} \), static pressure \( p_{\text{stat}} \) and specific energy \( Y \) are shown in Figures 6 – 8. The flow field and consequently the spiral vortex structure changed, as it is evident form the shape of iso-surfaces of particular variables.
Figure 1. Cross section through the vortex generator.

Figure 2. Normalized helicity $H_{\text{norm}} = 0.9$ (red surface), $H_{\text{norm}} = -0.9$ (blue surface).

Figure 3. Static pressure $p_{\text{stat}}$.

Figure 4. Specific energy $Y$. 
4. Conclusion

The Eigenshapes of the incompressible liquid velocity in spiral vortices with the assumption $\boldsymbol{\Omega} \times \boldsymbol{v} = \mathbf{0}$, $\boldsymbol{\Omega} = \Lambda (x, t) \boldsymbol{v}$ were derived. Their existence has been demonstrated by computational modelling using the test case of swirl generator geometry. The results proved that the spiral vortices are generated by fluid rotation at the inlet of the area and can also be induced by a change in the curvature of the tubes and their torsion.

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