Parallelism between locally conformal symplectic manifolds and contact manifolds

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Abstract
We give the parallelism between locally conformal symplectic manifolds and contact manifolds. We also give the generalization of exact contact manifolds.

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1 Introduction
Let \( A \) be a commutative algebra with unit \( 1_A \) over a commutative field \( K \) with characteristic zero, and \( \text{Diff}_K(A) \), the Lie algebra of differential operators of order \( \leq 1 \) on \( A \).

We recall that a Lie-Rinehart algebra is a pair \((G, \rho)\) where \( G \) is simultaneously an \( A \)-module and a \( K \)-Lie algebra, which Lie algebra bracket \([,] \), and

\[
\rho : G \rightarrow \text{Diff}_K(A)
\]

is simultaneously a morphism of \( A \)-modules and \( K \)-Lie algebras satisfying

\[
[x, a \cdot y] = [\rho(x)(a) - a \cdot \rho(x)(1_A)] \cdot y + a \cdot [x, y]
\]

for any \( a \in A \) and \( x, y \in G \).

Let \((G, \rho)\) be a Lie-Rinehart algebra and

\[
\mathfrak{L}_{sk}(G, A) = \bigoplus_{p \in \mathbb{N}} \mathfrak{L}_{sk}^p(G, A)
\]

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where $\mathfrak{L}_{sk}^p(G, A)$ is the module of skew-symmetric $A$-multilinear maps of degree $p$ from $G$ to $A$ and finally

$$d_\rho : \mathfrak{L}_{sk}(G, A) \rightarrow \mathfrak{L}_{sk}(G, A)$$

the cohomology operator associated with the representation $\rho$.

We recall that the pair $(\mathfrak{L}_{sk}(G, A), d_\rho)$ is a differential algebra [5].

For any $x \in G$, the map

$$i_x : \mathfrak{L}_{sk}(G, A) \rightarrow \mathfrak{L}_{sk}(G, A)$$

defined by

$$(i_x f)(x_1, x_2, \ldots, x_{p-1}) = f(x, x_1, x_2, \ldots, x_{p-1}),$$

for $x_1, x_2, \ldots, x_{p-1}$ elements of $G$ and for any $f \in \mathfrak{L}_{sk}^p(G, A)$, is a derivation of degree $-1$ [3]. The map

$$\theta_x = [i_x, d_\rho] = i_x \circ d_\rho + d_\rho \circ i_x : \mathfrak{L}_{sk}(G, A) \rightarrow \mathfrak{L}_{sk}(G, A)$$

is a differential operator of order $\leq 1$ and of degree zero satisfying, for any $y \in G$, $a \in A$,

$$[\theta_x, i_y] = i_{[x,y]};$$

$$\theta_x \circ d_\rho = d_\rho \circ \theta_x;$$

$$[\theta_x, \theta_y] = \theta_{[x,y]};$$

$$\theta_x a = [\rho(x)](a).$$

For any $x \in G$, the bracket that defines $\theta_x$ is the graded commutator.

A Lie-Rinehart-Jacobi algebra structure on a Lie-Rinehart algebra $(G, \rho)$ is defined by a skew-symmetric bilinear form

$$\mu : G \times G \rightarrow A$$

such that

$$d_\rho \mu = 0.$$

The triplet $(G, \rho, \mu)$ is a Lie-Rinehart-Jacobi algebra [4]. A Lie-Rinehart-Jacobi algebra $(G, \rho, \mu)$ is a Lie-Rinehart-Poisson algebra if $\rho(x)(1_A) = 0$ for any $x \in G$ [5].

A Lie-Rinehart-Jacobi algebra (a Lie-Rinehart-Poisson algebra respectively), $(G, \rho, \mu)$, is said to be a symplectic Lie-Rinehart-Jacobi algebra (a symplectic Lie-Rinehart-Poisson algebra respectively) if the skew-symmetric bilinear form $\mu$ is nondegenerate [5] i.e. the induced map

$$G \rightarrow G^*, x \mapsto i_x \mu,$$

is an isomorphism of $A$-modules where $G^*$ is the $A$-module of linear forms on $G$.  

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We recall that if a triplet \((G, \rho, \mu)\) is a symplectic Lie-Rinehart-Jacobi algebra (a symplectic Lie-Rinehart-Poisson algebra respectively), then \(A\) is a Jacobi algebra (\(A\) is a Poisson algebra respectively) \([5]\).

The parallelism between symplectic manifolds and (exact) contact manifolds is given in \([7]\).

The main goal of this paper is to show the parallelism between locally conformal symplectic manifolds and contact manifolds. We also will give the generalization of exact contact manifolds.

In what follows, \(M\) denotes a paracompact and connected smooth manifold, \(C^\infty(M)\) the algebra of numerical functions of class \(C^\infty\) on \(M\), \(\mathfrak{X}(M)\) the \(C^\infty(M)\)-module of vector fields on \(M\), 1 the unit of \(C^\infty(M)\), \(\mathcal{D}(M)\) the \(C^\infty(M)\)-module of differential operators of order \(\leq 1\) on \(C^\infty(M)\) and \(\delta\) the cohomology operator associated with the identically map

\[
id : \mathcal{D}(M) \longrightarrow \mathcal{D}(M).
\]

The term ”differential operator” will mean ”differential operator of order \(\leq 1\”).

2 Symplectic Lie-Rinehart-Jacobi algebra structure on \(\mathfrak{X}(M)\)

A locally conformal symplectic structure on \(M\) is a pair \((\omega, \alpha)\) made up by a closed 1-form

\[
\alpha : \mathfrak{X}(M) \longrightarrow C^\infty(M)
\]

and a nondegenerate skew-symmetric 2-form

\[
\omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M)
\]

such that

\[
d\omega = -\alpha \Lambda \omega
\]

where \(d\) is the exterior differentiation operator.

When \(\alpha = 0\), then \(M\) is a symplectic manifold.

**Proposition 1** \([6]\) A smooth manifold \(M\) is a locally conformal symplectic manifold (\(M\) is a symplectic manifold respectively) if and only if \(\mathfrak{X}(M)\) admits a symplectic Lie-Rinehart-Jacobi algebra structure (\(\mathfrak{X}(M)\) admits a symplectic Lie-Rinehart-Poisson algebra structure respectively).
3 Symplectic Lie-Rinehart-Jacobi algebra structure on $\mathcal{D}(M)$

When $\varphi \in \mathcal{D}(M)$, $f, g \in C^\infty(M)$, we recall that

\[
\begin{align*}
[\varphi, f] &= \varphi(f) - f \cdot \varphi(1), \\
[f, g] &= 0.
\end{align*}
\]

3.1 Lie-Rinehart algebra structure on $\mathcal{D}(M)$

For any linear form $\alpha : \mathcal{D}(M) \rightarrow C^\infty(M)$, we verify that the map

\[
\rho_\alpha : \mathcal{D}(M) \rightarrow \mathcal{D}(M), \varphi \mapsto \varphi + \alpha(\varphi),
\]

is $C^\infty(M)$-linear.

**Proposition 2** For any linear form $\alpha : \mathcal{D}(M) \rightarrow C^\infty(M)$, then the map

\[
\rho_\alpha : \mathcal{D}(M) \rightarrow \mathcal{D}(M), \varphi \mapsto \varphi + \alpha(\varphi),
\]

is a morphism of Lie algebras if and only if

\[
\delta \alpha = (\delta 1) \Lambda \alpha.
\]

**Proof.** For any $\varphi, \psi \in \mathcal{D}(M)$, we verify that

\[
[\rho_\alpha(\varphi), \rho_\alpha(\psi)] - \rho_\alpha[\varphi, \psi] = [\delta \alpha - (\delta 1) \Lambda \alpha](\varphi, \psi).
\]

And that ends the proof. $\blacksquare$

**Theorem 3** If $M$ is a smooth manifold, then a Lie-Rinehart algebra structure on $\mathcal{D}(M)$ is always of the form $(\mathcal{D}(M), \rho_\alpha)$ where

\[
\alpha : \mathcal{D}(M) \rightarrow C^\infty(M)
\]

is a linear form such that

\[
\delta \alpha = (\delta 1) \Lambda \alpha.
\]
Proof. The previous proposition implies the sufficient condition. For the necessary condition, let be given a Lie-Rinehart algebra structure \((\mathcal{D}(M), \rho)\) on \(\mathcal{D}(M)\). For any \(\varphi \in \mathcal{D}(M)\), for any \(f \in C^\infty(M)\), we get
\[
[\varphi, f] = [\rho(\varphi)](f) - f \cdot [\rho(\varphi)](1).
\]
On the other hand, we get
\[
[\varphi, f] = \varphi(f) - f \cdot \varphi(1).
\]
We deduce that
\[
[\rho(\varphi)](f) - f \cdot [\rho(\varphi)](1) = \varphi(f) - f \cdot \varphi(1).
\]
Therefore
\[
[\rho(\varphi)](f) = \varphi(f) + f \cdot ([\rho(\varphi)](1) - \varphi(1)).
\]
The map
\[
\alpha : \mathcal{D}(M) \longrightarrow C^\infty(M), \varphi \mapsto [\rho(\varphi) - \varphi](1),
\]
is a \(C^\infty(M)\)-linear. Thus
\[
[\rho(\varphi)](f) = \varphi(f) + f \cdot \alpha(\varphi).
\]
We have
\[
\rho(\varphi) = \varphi + \alpha(\varphi).
\]
We finally conclude that \(\rho = \rho_\alpha\). As \(\rho\) has to be a Lie algebras morphism, we deduce that the linear form \(\alpha\) is such that \(\delta \alpha = (\delta 1) \Lambda \alpha\).

Proposition 4 A linear form
\[
\alpha : \mathcal{D}(M) \longrightarrow C^\infty(M)
\]
satisfies
\[
\delta \alpha = (\delta 1) \Lambda \alpha
\]
if and only if
\[
\alpha(1) \in \mathbb{R} \text{ and } d(\alpha|_{\mathfrak{X}(M)}) = 0.
\]
Proof. For any \(\varphi = \varphi(1) + X, \psi = \psi(1) + Y\) two elements of \(\mathcal{D}(M)\) with \(X, Y \in \mathfrak{X}(M)\), we have
\[
[\delta \alpha - (\delta 1) \Lambda \alpha](\varphi, \psi) = \psi(1) \cdot X [\alpha(1)] - \varphi(1) \cdot Y [\alpha(1)] + [d(\alpha|_{\mathfrak{X}(M)})](X, Y).
\]
/ \Longrightarrow As \(\delta \alpha - (\delta 1) \Lambda \alpha = 0\), we have
\[
0 = [\delta \alpha - (\delta 1) \Lambda \alpha](X, 1) = X [\alpha(1)] = (d[\alpha(1)])(X).
\]
As $X$ is arbitrary, we deduce that $d[\alpha(1)] = 0$. Thus $\alpha(1) \in \mathbb{R}$ since $M$ is connected.

We also have $d(\alpha|_{X(M)}) = 0$. 

$\Leftarrow \Rightarrow$ If $\alpha(1) \in \mathbb{R}$ and $d(\alpha|_{X(M)}) = 0$, we immediately have $\delta \alpha = (\delta 1) \Lambda \alpha$. 

### 3.2 Symplectic Lie-Rinehart-Jacobi algebra structure on $\mathcal{D}(M)$

Let $(\mathcal{D}(M), \rho_\alpha)$ be a Lie-Rinehart algebra structure on $\mathcal{D}(M)$. The linear form

$$\alpha : \mathcal{D}(M) \rightarrow C^\infty(M)$$

is such that $\delta \alpha = (\delta 1) \Lambda \alpha$. In this case, we denote $\delta_\alpha$ the cohomology operator associated with the representation $\rho_\alpha$.

**Proposition 5** For any $\eta \in \mathfrak{L}_{sk\alpha}(\mathcal{D}(M), C^\infty(M))$, then

$$\delta_\alpha \eta = \delta \eta + \alpha \Lambda \eta.$$ 

**Proof.** For any $\eta \in \mathfrak{L}_{sk\alpha}(\mathcal{D}(M), C^\infty(M))$ and for any $\varphi_1, ..., \varphi_{p+1} \in \mathcal{D}(M)$, we have

$$(\delta_\alpha \eta)(\varphi_1, ..., \varphi_{p+1}) = \sum_{i=1}^{p+1} (-1)^i \rho_\alpha(\varphi_i) \eta(\varphi_1, ..., \varphi_i, ..., \varphi_{p+1})$$

$$+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \eta([\varphi_i, \varphi_j], \varphi_1, ..., \varphi_i, ..., \varphi_j, ..., \varphi_{p+1})$$

$$= \sum_{i=1}^{p+1} (-1)^i \varphi_i \eta(\varphi_1, ..., \varphi_i, ..., \varphi_{p+1})$$

$$+ \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} \eta([\varphi_i, \varphi_j], \varphi_1, ..., \varphi_i, ..., \varphi_j, ..., \varphi_{p+1})$$

$$= (\delta \eta + \alpha \Lambda \eta)(\varphi_1, ..., \varphi_{p+1}).$$

That ends the proof. 

The characterization of symplectic Lie-Rinehart-Jacobi algebra structure on $\mathcal{D}(M)$ is the following one:

**Proposition 6** The $C^\infty(M)$-module $\mathcal{D}(M)$ admits a symplectic Lie-Rinehart-Jacobi algebra structure if and only if there exists a $C^\infty(M)$-linear form

$$\alpha : \mathcal{D}(M) \rightarrow C^\infty(M)$$

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and a nondegenerate skew-symmetric bilinear form
\[ \omega: \mathcal{D}(M) \times \mathcal{D}(M) \rightarrow C^\infty(M) \]
such that
1. \( \delta\alpha = (\delta_1)\Lambda\alpha \)
2. \( \delta\omega = -\alpha\Lambda\omega. \)

**Proof.** It is obvious. ■

### 3.3 Structure of contact manifold on \( M \) when \( \mathcal{D}(M) \) admits a symplectic Lie-Rinehart-Jacobi algebra structure

In this part, we consider a symplectic Lie-Rinehart-Jacobi algebra structure on \( \mathcal{D}(M) \) with a linear form
\[ \alpha: \mathcal{D}(M) \rightarrow C^\infty(M) \]
such that
\[ \delta\alpha = (\delta_1)\Lambda\alpha \]
and a nondegenerate skew-symmetric bilinear form
\[ \omega: \mathcal{D}(M) \times \mathcal{D}(M) \rightarrow C^\infty(M) \]
such that
\[ \delta\omega = -\alpha\Lambda\omega. \]

In this case if \( \mathcal{D}(M)^* \) denotes the dual of the \( C^\infty(M) \)-module \( \mathcal{D}(M) \), the map
\[ \mathcal{D}(M) \rightarrow \mathcal{D}(M)^*, \varphi \mapsto i_\varphi\omega, \]
is an isomorphism of \( C^\infty(M) \)-modules.

**Proposition 7** There exists an unique vector field \( H \) on \( M \) such that
\[ i_H\omega = -\delta_1. \]
Moreover the linear form
\[ i_1\omega: \mathcal{D}(M) \rightarrow C^\infty(M), \varphi \mapsto \omega(1, \varphi), \]
is such that
\[ (i_1\omega)(H) = 1. \]

**Proof.** As
\[ \omega: \mathcal{D}(M) \times \mathcal{D}(M) \rightarrow C^\infty(M) \]
is nondegenerate, let \( H \in \mathcal{D}(M) \) be the unique differential operator such that
\[ i_H\omega = -\delta_1. \]
We have \( H(1) = 0 \). Thus \( H \) is a vector field.

We deduce that
\[ (i_1\omega)(H) = 1. \]
And that ends the proof. ■
Proposition 8  We get
\[ X(M) = \text{Ker} \left[ i_1 \omega \right|_{X(M)} \] \[ \oplus \mathcal{C}^\infty(M) \cdot H. \]

Proof. For any \( X \in X(M) \), we write
\[ X = [X - (i_1 \omega)(X) \cdot H] + (i_1 \omega)(X) \cdot H. \]
We verify that
\[ [X - (i_1 \omega)(X) \cdot H] \in \text{Ker} \left[ i_1 \omega \right|_{X(M)} \]
and
\[ \text{Ker} \left[ i_1 \omega \right|_{X(M)} \cap \mathcal{C}^\infty(M) \cdot H = \{0\}. \]
Thus
\[ X(M) = \text{Ker} \left[ i_1 \omega \right|_{X(M)} \oplus \mathcal{C}^\infty(M) \cdot H. \]
That ends the proof. □

The sets
\[ D(M) \] and
\[ D(M)_{\mathcal{C}^\infty(M),H} = \left\{ \eta \in D(M) \right\} \]
are modules over \( \mathcal{C}^\infty(M). \)

For any \( X \in X(M) \) ( \( X \in \text{Ker} \left[ i_1 \omega \right|_{X(M)} \) respectively), we verify that
\( i_X \omega \in D(M)_{\mathcal{C}^\infty(M),H} \) ( \( i_X \omega \in D(M)_{\mathcal{C}^\infty(M),H} \) respectively).

Proposition 9  The following maps
\[ X(M) \rightarrow D(M)_{\mathcal{C}^\infty(M),H}^*, X \mapsto i_X \omega, \]
and
\[ \text{Ker} \left[ i_1 \omega \right|_{X(M)} \rightarrow D(M)_{\mathcal{C}^\infty(M),H}^*, X \mapsto i_X \omega, \]
are isomorphisms of \( \mathcal{C}^\infty(M) \)-modules.

Proof. Since the map
\[ D(M) \rightarrow D(M)^*, \varphi \mapsto i_\varphi \omega, \]
is an isomorphism of \( \mathcal{C}^\infty(M) \)-modules, then the maps
\[ X(M) \rightarrow D(M)_{\mathcal{C}^\infty(M),H}^*, X \mapsto i_X \omega, \]
and
\[ \text{Ker} \left[ i_1 \omega \right|_{X(M)} \rightarrow D(M)_{\mathcal{C}^\infty(M),H}^*, X \mapsto i_X \omega, \]
are injective.

Let \( \eta \in D(M)_{\mathcal{C}^\infty(M),H}^* \) be a linear form on \( D(M) \) such that \( \eta(H) = 0 \) and let \( \varphi \) be the unique element of \( D(M) \) such that
\[ i_\varphi \omega = \eta. \]
We get

\[ 0 = \eta(H) \]
\[ = i_{\varphi} \omega(H) \]
\[ = -(i_H \omega)(\varphi) \]
\[ = (\delta 1)(\varphi) \]
\[ = \varphi(1). \]

We deduce that \( \varphi \in \mathfrak{X}(M) \). Thus the map

\[ \mathfrak{X}(M) \to \mathcal{D}(M)^*_H, X \mapsto i_X \omega, \]

is also surjective.

Let \( \sigma \in \mathcal{D}(M)^*_C \mathcal{C}_\infty(M,H) \) be a linear form on \( \mathcal{D}(M) \) such that \( \sigma|_{\mathcal{C}_\infty(M)} = 0 \) and \( \sigma(H) = 0 \), and let \( \varphi \) be the unique element of \( \mathcal{D}(M) \) such that

\[ i_{\varphi} \omega = \sigma. \]

As \( \sigma(H) = 0 \), then \( \varphi \in \mathfrak{X}(M) \).

Since \( \sigma|_{\mathcal{C}_\infty(M)} = 0 \), we obtain

\[ 0 = \sigma(1) \]
\[ = (i_{\varphi} \omega)(1) \]
\[ = -[i_1 \omega](\varphi). \]

We deduce that \( \varphi \in \text{Ker} [i_1 \omega|_{\mathfrak{X}(M)}] \). Thus the map

\[ \text{Ker} [i_1 \omega|_{\mathfrak{X}(M)}] \to \mathcal{D}(M)^*_C \mathcal{C}_\infty(M,H), X \mapsto i_X \omega, \]

is also surjective.

**Corollary 10** The restriction

\[ \omega|_{\text{Ker}[i_1 \omega|_{\mathfrak{X}(M)}] \times \text{Ker}[i_1 \omega|_{\mathfrak{X}(M)}]} : \text{Ker} [i_1 \omega|_{\mathfrak{X}(M)}] \times \text{Ker} [i_1 \omega|_{\mathfrak{X}(M)}] \to C^\infty(M) \]

is a nondegenerate skew-symmetric bilinear form on \( \text{Ker} [i_1 \omega|_{\mathfrak{X}(M)}] \).

**Proof.** It is obvious since the map

\[ \text{Ker} [i_1 \omega|_{\mathfrak{X}(M)}] \to \text{Ker} [i_1 \omega|_{\mathfrak{X}(M)}]^*, X \mapsto i_X \omega|_{\text{Ker}[i_1 \omega|_{\mathfrak{X}(M)}]}, \]

is an isomorphism of \( C^\infty(M) \)-modules.

For any \( f \in C^\infty(M) \), the linear form

\[ \delta_\alpha f - [H(f) + f \cdot \alpha(H)] \cdot i_1 \omega - f \cdot [1 + \alpha(1)] \cdot \delta 1 : \mathcal{D}(M) \to C^\infty(M), \]


belongs to $\mathcal{D}(M)_{\mathcal{C}^\infty(M),H}$. We denote $\varphi_f$ the unique element of $\mathcal{D}(M)$ such that 
\[ i_{\varphi_f}\omega = \delta_\alpha f \]
and $X_f$ the unique element of $\text{Ker} \left[i_1\omega|_{\mathcal{X}(M)}\right]$ such that 
\[ i_{X_f}\omega = \delta_\alpha f - [H(f) + f \cdot \alpha(H)] \cdot i_1\omega - f \cdot [1 + \alpha(1)] \cdot \delta 1. \]
For any $f, g \in \mathcal{C}^\infty(M)$, the bracket 
\[ \{ f, g \} = -\omega(\varphi_f, \varphi_g) \]
is a Jacobi bracket on $\mathcal{C}^\infty(M)$. Thus $M$ is a Jacobi manifold \([5]\). We verify that 
\[ \varphi_f = [\rho_\alpha(H)](f) + X_f - f \cdot [1 + \alpha(1)] \cdot H. \]
If we denote $H_\alpha = [1 + \alpha(1)] \cdot \rho_\alpha(H)$, then we have 
\[ \{ f, g \} = -\omega(X_f, X_g) - f \cdot H_\alpha(g) + g \cdot H_\alpha(f). \]

**Remark 11** We recall that as $\omega$ is a nondegenerate skew-symmetric bilinear form on $\mathcal{D}(M)$, then the dimension of $M$ is odd \([7]\).

**Theorem 12** If the dimension of $M$ is $2n + 1$, then the differential form 
\[ [i_1\omega|_{\mathcal{X}(M)}]\Lambda [\omega|_{\mathcal{X}(M)\times\mathcal{X}(M)}]^n \]
is a volume form on $M$.

**Proof.** For any $x \in M$ we have $H(x) \neq 0$ since $\omega \neq 0$. Thus the 1-form 
\[ [i_1\omega|_{\mathcal{X}(M)}]|_x \]
is nonzero everywhere. Let $x \in M$ and let $T_x M$ be the tangent vector space at $x$. As the dimension of $M$ is odd, let $2n + 1$ be the dimension of $M$. The set 
\[ (\text{Ker} \left[i_1\omega|_{\mathcal{X}(M)}\right])_x = \{ X(x) \in T_x M / X \in \text{Ker} \left[i_1\omega|_{\mathcal{X}(M)}\right] \} \]
is a vector space of dimension $2n$. Since \[ \omega|_{\text{Ker} \left[i_1\omega|_{\mathcal{X}(M)}\right] \times \text{Ker} \left[i_1\omega|_{\mathcal{X}(M)}\right]} : \text{Ker} \left[i_1\omega|_{\mathcal{X}(M)}\right] \times \text{Ker} \left[i_1\omega|_{\mathcal{X}(M)}\right] \rightarrow \mathcal{C}^\infty(M) \]
is a nondegenerate skew-symmetric bilinear form on the $\mathcal{C}^\infty(M)$-module 
\[ \text{Ker} \left[i_1\omega|_{\mathcal{X}(M)}\right], \]
then $(\omega|_{\text{Ker} \left[i_1\omega|_{\mathcal{X}(M)}\right] \times \text{Ker} \left[i_1\omega|_{\mathcal{X}(M)}\right]})_x(x)$ is a nondegenerate skew-symmetric bilinear form on the vector space $(\text{Ker} \left[i_1\omega|_{\mathcal{X}(M)}\right])_x$. 

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We deduce that $Ker\left[i_1\omega\mid_{\mathfrak{X}(M)}\right]_x$ is a symplectic vector space and

$$(\omega|_{Ker[i_1\omega\mid_{\mathfrak{X}(M)}] \times Ker[i_1\omega\mid_{\mathfrak{X}(M)}]}\right)^n(x)$$

is a volume form. We also deduce that $(\omega|_{Ker[i_1\omega\mid_{\mathfrak{X}(M)}] \times Ker[i_1\omega\mid_{\mathfrak{X}(M)}]}\right)^n(x) \neq 0$. Let $(v_1, v_2, ..., v_{2n})$ be a basis of $(Ker\left[i_1\omega\mid_{\mathfrak{X}(M)}\right]_x)$. We have

$$(\omega|_{Ker[i_1\omega\mid_{\mathfrak{X}(M)}] \times Ker[i_1\omega\mid_{\mathfrak{X}(M)}]}\right)^n(x)(v_1, v_2, ..., v_{2n}) \neq 0.$$ 

We note that

$$\nu = \left[i_1\omega\mid_{\mathfrak{X}(M)}\right](x)\Lambda(\omega|_{Ker[i_1\omega\mid_{\mathfrak{X}(M)}] \times Ker[i_1\omega\mid_{\mathfrak{X}(M)}]}\right)^n(x)$$

is nonzero since

$$\nu(H(x), v_1, v_2, ..., v_{2n}) = (\omega|_{Ker[i_1\omega\mid_{\mathfrak{X}(M)}] \times Ker[i_1\omega\mid_{\mathfrak{X}(M)}]}\right)^n(x)(v_1, v_2, ..., v_{2n}) \neq 0.$$ 

As $x$ is arbitrary, we conclude that $\left[i_1\omega\mid_{\mathfrak{X}(M)}\right]\Lambda_{\mathfrak{X}(M) \times \mathfrak{X}(M)}^n$ is a volume form on $M$. □

**Corollary 13** If $D(M)$ admits a symplectic Lie-Rinehart-Jacobi algebra structure, then $M$ is a nonexact contact manifold in the sense of André Lichnerowicz.

In what follows, we give a generalization of exact and nonexact contact manifolds.

**Proposition 14** We get

$$[1 + \alpha(1)] \cdot \omega = \delta_\alpha(i_1\omega).$$

**Proof.** For any $\varphi, \psi \in D(M)$, we have

$$(\delta\omega)(1, \varphi, \psi) = \omega(\varphi, \psi) - \varphi([i_1\omega](\psi)) + \psi([i_1\omega](\varphi)) - \omega([1, \varphi], \psi) + \omega([1, \psi], \varphi) - \omega([\varphi, \psi], 1).$$

As $[1, \varphi] = [1, \psi] = 0$, we get

$$(\delta\omega)(1, \varphi, \psi) = \omega(\varphi, \psi) - \varphi([i_1\omega](\psi)) + \psi([i_1\omega](\varphi)) + (i_1\omega)([\varphi, \psi]) = [\omega - \delta(i_1\omega)](\varphi, \psi).$$

On the other hand, we get

$$(-\alpha\Lambda\omega)(1, \varphi, \psi) = -\alpha(1) \cdot \omega(\varphi, \psi) + \alpha(\varphi) \cdot (i_1\omega)(\psi) - \alpha(\psi) \cdot (i_1\omega)(\varphi) = [-\alpha(1) \cdot \omega + \alpha\Lambda(i_1\omega)](\varphi, \psi).$$
As $\delta \omega = -\alpha \Lambda \omega$, we conclude that
\[
\omega - \delta(i_1 \omega) = -\alpha(1) \cdot \omega + \alpha \Lambda(i_1 \omega).
\]
Thus
\[
[1 + \alpha(1)] \cdot \omega = \delta(i_1 \omega) + \alpha \Lambda(i_1 \omega) = \delta\alpha(i_1 \omega).
\]
That ends the proof. □

As $\delta\alpha = (\delta 1)\Lambda\alpha$, then $\alpha(1) \in \mathbb{R}$.
If $\alpha(1) \neq -1$, we have
\[
\omega = \delta\alpha \left[ i_1 \left( \frac{1}{1 + \alpha(1)} \cdot \omega \right) \right].
\]
In this case, we will say that $M$ is an exact contact manifold since $\omega$ is $\delta\alpha$-exact.
If $\alpha(1) = -1$, we will say that $M$ is a nonexact contact manifold.

### 3.4 Structure of symplectic Lie-Rinehart-Jacobi algebra on $\mathcal{D}(M)$ when $M$ is a contact manifold

Let $M$ be a contact manifold with dimension $2n + 1$. In this case, there exists an 1-form
\[
\beta : \mathfrak{X}(M) \rightarrow C^\infty(M)
\]
and a skew-symmetric 2-form
\[
\Omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \rightarrow C^\infty(M)
\]
such that
\[
\beta \Lambda \Omega^n
\]
is a volume form on $M$.

Let $E$ be the fundamental vector field of the contact manifold $M$ [4]. We have
\[
\beta(E) = 1
\]
and
\[
i_E \Omega = 0.
\]
We get
\[
\mathfrak{X}(M) = \text{Ker} \beta \oplus C^\infty(M) \cdot E.
\]
The restriction
\[
\Omega|_{\text{Ker} \beta \times \text{Ker} \beta} : \text{Ker} \beta \times \text{Ker} \beta \rightarrow C^\infty(M)
\]
is a nondegenerate skew-symmetric bilinear form on $Ker\beta$.

Thus, we have

$$\mathcal{D}(M) = \mathcal{C}^\infty(M) \oplus Ker\beta \oplus \mathcal{C}^\infty(M) \cdot E.$$  

If

$$\pi : \mathcal{D}(M) \rightarrow \mathcal{X}(M)$$

is the canonical surjection, the linear form

$$\tilde{\beta} = \beta \circ \pi : \mathcal{D}(M) \rightarrow \mathcal{C}^\infty(M)$$

is such that

$$\tilde{\beta}|_{\mathcal{C}^\infty(M)} = 0$$

and

$$\tilde{\beta}|_{\mathcal{X}(M)} = \beta.$$  

For any $\varphi, \psi$ two elements of $\mathcal{D}(M)$, we have

$$\varphi = \varphi(1) + X + \tilde{\beta}(\varphi) \cdot E$$

$$\psi = \psi(1) + Y + \tilde{\beta}(\psi) \cdot E$$

where $X, Y \in Ker\beta$. The map

$$\Omega : \mathcal{D}(M) \times \mathcal{D}(M) \rightarrow \mathcal{C}^\infty(M), (\varphi, \psi) \mapsto \Omega(X, Y),$$

is $\mathcal{C}^\infty(M)$-bilinear and skew-symmetric.

The map

$$\tilde{\Omega} = \overline{\Omega} + (\delta 1) \Lambda \tilde{\beta} : \mathcal{D}(M) \times \mathcal{D}(M) \rightarrow \mathcal{C}^\infty(M)$$

is a skew-symmetric bilinear form.

**Proposition 15** We get

$$i_1\overline{\Omega} = 0;$$

$$i_2\tilde{\Omega} = \tilde{\beta};$$

$$i_E\tilde{\Omega} = -\delta 1.$$  

**Proof.** It is obvious. ■

**Proposition 16** The skew-symmetric bilinear form

$$\tilde{\Omega} = \overline{\Omega} + (\delta 1) \Lambda \tilde{\beta} : \mathcal{D}(M) \times \mathcal{D}(M) \rightarrow \mathcal{C}^\infty(M)$$

is nondegenerated.
Proof. Let $\varphi \in \mathcal{D}(M)$ such that $\tilde{\Omega}(\varphi, \psi) = 0$ for any $\psi \in \mathcal{D}(M)$. We can write $\varphi = \varphi(1) + X + \tilde{\beta}(\varphi) \cdot E$ with $X \in \text{Ker}\beta$.

For $\psi = 1$, we get

$$
0 = \tilde{\Omega}(\varphi, 1)
= \Omega(\varphi, 1) + \varphi(1) \cdot \tilde{\beta}(1) - 1 \cdot \tilde{\beta}(\varphi)
= -(i_1 \Omega)(\varphi) - \tilde{\beta}(\varphi)
= -\tilde{\beta}(\varphi).
$$

Thus $\tilde{\beta}(\varphi) = 0$.

For $\psi = E$, we get

$$
0 = \tilde{\Omega}(\varphi, E)
= -(i_E \Omega)(\varphi)
= (\delta 1)(\varphi)
= \varphi(1).
$$

Thus $\varphi(1) = 0$.

As $\tilde{\beta}(\varphi) = 0$ and $\varphi(1) = 0$, we have $\varphi = X$. Thus for any $\psi = Y \in \text{Ker}\beta$, we get

$$
0 = \tilde{\Omega}(\varphi, Y)
= \Omega(X, Y).
$$

As

$$
\Omega|_{\text{Ker}\beta \times \text{Ker}\beta} : \text{Ker}\beta \times \text{Ker}\beta \longrightarrow C^\infty(M)
$$

is nondegenerated, we deduce that $X = 0$. We conclude that $\varphi = 0$ and the map

$$
\mathcal{D}(M) \longrightarrow \mathcal{D}(M)^*, \varphi \mapsto i_\varphi \tilde{\Omega},
$$

is injective.

The map

$$
\mathcal{D}(M) \longrightarrow \mathcal{D}(M)^*, \varphi \mapsto i_\varphi \tilde{\Omega},
$$

is also surjective since if

$$
\nu : \mathcal{D}(M) \longrightarrow C^\infty(M)
$$

is a linear form on $\mathcal{D}(M)$ and if $X$ is the unique element of $\text{Ker}\beta$ such that $i_X \Omega = \nu|_{\text{Ker}\beta}$, the differential operator

$$
\varphi = \nu(E) + X - \nu(1) \cdot E
$$

is such that

$$
i_\varphi \tilde{\Omega} = \nu.
$$
Thus
\[ \tilde{\Omega} : \mathcal{D}(M) \times \mathcal{D}(M) \longrightarrow C^\infty(M) \]
is a nondegenerate skew-symmetric bilinear form. ■

In what follows, we give the characterization of a contact manifold in terms of symplectic Lie-Rinehart-Jacobi algebra structure on \( \mathcal{D}(M) \).

We consider the linear form
\[ \alpha = [1 + \alpha(1)] \cdot \delta 1 + i_E \delta \tilde{\beta} + \alpha(E) \cdot \tilde{\beta} : \mathcal{D}(M) \longrightarrow C^\infty(M) \]
on \( \mathcal{D}(M) \) with \( \alpha(1) \in \mathbb{R} \) and \( d[\alpha(E) \cdot \beta + i_E d\beta] = 0 \).

In this case, we have
\[ \delta \alpha = (\delta 1) \Lambda \alpha. \]

We have the following properties:

**Proposition 17** We get
1. \[ [1 + \alpha(1)] \cdot \tilde{\Omega} = \delta \tilde{\beta} + \alpha \Lambda \tilde{\beta}; \]
2. \[ \alpha(E) \cdot \tilde{\Omega} = (\delta 1) \Lambda \alpha - i_E \delta \tilde{\Omega}; \]
3. \[ \beta [X, E] \cdot \tilde{\Omega} = \alpha \Lambda i_X \tilde{\Omega} - i_X \delta \tilde{\Omega}, \text{ for any } X \in \text{Ker}\beta. \]

**Proof.** For any \( x \in M \), as the matrix of \( \tilde{\Omega}(x) \) is regular, then there exists \( f \in C^\infty(M), g \in C^\infty(M) \) and \( h_X \in C^\infty(M) \) for any \( X \in \text{Ker}\beta \) such that
1. \( f \cdot \tilde{\Omega} = \delta \tilde{\beta} + \alpha \Lambda \tilde{\beta}; \)
2. \( g \cdot \tilde{\Omega} = (\delta 1) \Lambda \alpha - i_E \delta \tilde{\Omega}; \)
3. \( h_X : \tilde{\Omega} = \alpha \Lambda i_X \tilde{\Omega} - i_X \delta \tilde{\Omega}. \)

We deduce the following equations:
\[
\begin{align*}
a/f \cdot i_E \left[ i_1 \tilde{\Omega} \right] &= i_E \left[ i_1 (\delta \tilde{\beta} + \alpha \Lambda \tilde{\beta}) \right], \\
b/g \cdot i_E \left[ i_1 \tilde{\Omega} \right] &= i_E \left[ i_1 ((\delta 1) \Lambda \alpha - i_E \delta \tilde{\Omega}) \right], \\
c/h_X \cdot i_E \left[ i_1 \tilde{\Omega} \right] &= i_E \left[ i_1 (\alpha \Lambda i_X \tilde{\Omega} - i_X \delta \tilde{\Omega}) \right].
\end{align*}
\]
As \( i_E \left[ i_1 \tilde{\Omega} \right] = 1 \), we verify that the unique solutions are: \( f = 1 + \alpha(1); \)
\( g = \alpha(E) \) and \( h_X = \alpha(X) = \beta [X, E] \) for any \( X \in \text{Ker}\beta. \) ■

**Theorem 18** We have
\[ \delta \tilde{\Omega} = -\alpha \Lambda \tilde{\Omega}. \]

**Proof.** We recall that
\[
\begin{align*}
i_1 \tilde{\Omega} &= \tilde{\beta}; \\
i_E \tilde{\Omega} &= -\delta 1; \\
i_1 \delta \tilde{\Omega} &= \tilde{\Omega} - \delta \tilde{\beta}.
\end{align*}
\]
For any $\varphi \in \mathcal{D}(M)$ with $\varphi = \varphi(1) + X + \tilde{\beta}(\varphi) \cdot E, (X \in \text{Ker}\beta)$, we have

\[
i_\varphi(\alpha \Lambda \tilde{\Omega} + \delta \tilde{\Omega}) = (\varphi(1) \cdot \alpha(1) + \alpha(X) + \tilde{\beta}(\varphi) \cdot \alpha(E)) \cdot \tilde{\Omega} + \alpha \Lambda [\varphi(1) \cdot \tilde{\beta} + i_X \tilde{\Omega} + \tilde{\beta}(\varphi) \cdot i_E \tilde{\Omega}] + \varphi(1) \cdot i_1 \delta \tilde{\Omega} + i_X \delta \tilde{\Omega} + \tilde{\beta}(\varphi) \cdot i_E \delta \tilde{\Omega}.
\]

We get

\[
i_\varphi(\alpha \Lambda \tilde{\Omega} + \delta \tilde{\Omega}) = \varphi(1) \cdot \left( [1 + \alpha(1)] \cdot \tilde{\Omega} - \delta \tilde{\beta} - \alpha \Lambda \tilde{\beta} \right) + \alpha(X) \cdot \tilde{\Omega} - \alpha \Lambda i_X \tilde{\Omega} + i_X \delta \tilde{\Omega} + \tilde{\beta}(\varphi) \cdot \left[ \alpha(E) \cdot \tilde{\Omega} - (\delta 1) \Lambda \alpha + i_E \delta \tilde{\Omega} \right].
\]

Proposition 17, above, implies that

\[
i_\varphi(\alpha \Lambda \tilde{\Omega} + \delta \tilde{\Omega}) = 0.
\]

As

\[
i_\varphi(\alpha \Lambda \tilde{\Omega} + \delta \tilde{\Omega}) = 0,
\]

then, for any $\varphi, \psi, \eta \in \mathcal{D}(M)$ we have

\[(\alpha \Lambda \tilde{\Omega} + \delta \tilde{\Omega})(\varphi, \psi, \eta) = 0.
\]

We conclude that

\[
\delta \tilde{\Omega} = -\alpha \Lambda \tilde{\Omega}.
\]

That ends the proof. ■

**Corollary 19** If $M$ is a contact manifold, then $\mathcal{D}(M)$ admits a symplectic Lie-Rinehart-Jacobi algebra structure.

In this paper we showed that $\mathcal{D}(M)$ admits a symplectic Lie-Rinehart-Jacobi algebra structure if and only if $M$ is a contact manifold. Thus a contact structure on a manifold $M$ is due to the existence of a $C^\infty(M)$-linear form

\[
\alpha : \mathcal{D}(M) \rightarrow C^\infty(M)
\]

and a nondegenerate skew-symmetric bilinear form

\[
\omega : \mathcal{D}(M) \times \mathcal{D}(M) \rightarrow C^\infty(M)
\]

such that
1. $\delta \alpha = (\delta 1) \Lambda \alpha$;
2. $\delta \omega = -\alpha \Lambda \omega$.

If $\alpha(1) \neq -1$, we will say that $M$ is an exact contact manifold and if $\alpha(1) = -1$, we will say that $M$ is a nonexact contact manifold.

Thus the parallelism between locally conformal symplectic manifolds and contact manifolds is obvious: a locally conformal symplectic structure on a manifold $M$ is due to the existence of a symplectic Lie-Rinehart-Jacobi algebra structure on $\mathfrak{X}(M)$ whereas a contact structure on a manifold $M$ is due to the existence of a symplectic Lie-Rinehart-Jacobi algebra structure on $\mathcal{D}(M)$.

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