INTRODUCTION TO NETWORK GAMES
WITH LINEAR BEST RESPONSES

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ABSTRACT. This note, suitable for a lecture in an advanced undergraduate or basic graduate course on the economic theory of networks, exposes basic ideas of linear best-response games and their equilibria.

1. BASIC SETUP AND RESULTS

We study the game of Ballester, Calvó-Armengol, and Zenou (2006), who introduced the idea of the connection between Nash equilibria of a certain kind of game and centrality measures that we will derive is due to them.

Consider a complete-information game where each player (also called agent) \( i \in N = \{1, 2, \ldots, n\} \) simultaneously selects a real-valued action \( a_i \geq 0 \) and receives a real-valued payoff \( u_i(a_1, a_2, \ldots, a_n) \) that depends on everyone's action. Suppose that each agent \( i \)'s best-response function is given by

\[
\text{BR}_i(a_{-i}) = \alpha \sum_j w_{ij} a_j + b_i.
\]

Here \( \alpha > 0, \ W = (w_{ij})_{i,j \in N} \), and \( (b_i)_{i \in N} \) are constants—parameters of the model that do not depend on \( a = (a_i)_{i \in N} \). The matrix \( W \) is irreducible,\(^1\) with \( W_{ii} = 0 \) for every \( i \), and all its entries are nonnegative. All the \( b_i \) are positive.

1.1. Existence result. Recall that \( r(A) \) is the spectral radius of a matrix \( A \), which has two equivalent definitions: (i) the maximum absolute value of any eigenvalue; and (b) the definition you studied in Problem Set 1, Problem 4.

\(^{1}\)We call a matrix irreducible if the corresponding weighted, directed graph is strongly connected. A 1-by-1 nonnegative matrix is said to be irreducible if its sole entry is positive.
Theorem 1. If $r(aW) < 1$ then there is exactly one pure-strategy Nash equilibrium of the game described above, given by

$$a^* = (I - aW)^{-1}b$$

$$= \sum_{\ell=0}^{\infty} a^\ell W^\ell b.$$

The result can be established by manipulating the assumed best responses of each agent to show that as long as $(I - aW)^{-1}$ is well-defined, then

$$a^* = (I - aW)^{-1}b.$$ 

It is left as an exercise to show that $(I - aW)^{-1}$ exists, has the claimed Neumann series expansion, and is nonnegative.\(^2\)

1.2. Interpreting the entries of $W^\ell$. For any walk $s = (i(1), i(2), \ldots, i(\ell + 1))$ define its weight to be the product of the weights of the edges constituting that walk:

$$\lambda(s) = w_{i(1)i(2)} w_{i(2)i(3)} \cdots w_{i(\ell)i(\ell+1)}.$$ 

Let $W^\ell(i, j)$ be the set of all walks of length $\ell$ from $i$ to $j$.

Fact 1. The following identity holds: for every positive integer $\ell$,

$$(W^\ell)_{ij} = \sum_{s \in W^\ell(i, j)} \lambda(s).$$

That is, the $(i, j)$ entry of $W^\ell$ is the sum of the weights of walks of length $\ell$ from $i$ to $j$. Think about the special case of $W = G$. This boils down to counting walks.

What do these walks correspond to game-theoretically? In view of the formula $a^* = \sum_{\ell=0}^{\infty} a^\ell W^\ell b$, the indirect effect of perturbations to $b_i$ on the action $a_j$ via chains of best responses of length $\ell$. For a different interpretation of the walks, in terms of a process of iterated best-responses, see (Golub and Sadler, 2016, Section 3.1.3).

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\(^2\)Let $A$ be an irreducible $n$-by-$n$ matrix whose entries are all nonnegative. (a) Assume that $\sum_{k=\infty}^{\kappa} A^k$ tends to the all-zeros matrix entrywise as $K \to \infty$. Show that

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k,$$

where $I$ is an appropriately-sized identity matrix. That is, show that the expression $\sum_{k=0}^{\infty} A^k$ is well-defined and that it is the inverse of the matrix $I - A$. (b) Conclude that all the entries of $(I - A)^{-1}$ are nonnegative. (c) Using the Perron-Frobenius Theorem, show that the assumption (first sentence) of (a) holds if the spectral radius of $A$ is strictly less than 1.
2. **Bonacich Centrality**

**Definition 1.** Let $\hat{W}$ be a nonnegative irreducible matrix, $\alpha \in (0, r(\hat{W})^{-1})$ be a real number, and $\hat{b}$ a nonnegative base vector. Then the vector of Bonacich centralities in $\hat{W}$ with parameter $\alpha$ and base vector $\hat{b}$ is defined to be

$$\beta(\hat{W}; \alpha, \hat{b}) = (I - \alpha \hat{W})^{-1}\hat{b}.$$ 

The “default” value of $\hat{b}$ is $1$, the vector of all ones.

We are using hats to distinguish these from the parameters of our game, since the arguments of $\beta$ can be anything (though of course we will apply it to the game shortly). Note that the Bonacich centrality satisfies

$$\beta = \hat{b} + \alpha \hat{W} \beta.$$ 

The idea behind this equation is that $\beta_i$ is a measure of $i$’s network centrality. It is the sum of a “base” level $\hat{b}_i$ and a socially derived part $\alpha \sum_j \hat{W}_{ij} \beta_j$, where $\hat{W}_{ij}$ describes the part of $\beta_j$ that accrues to $i$. The natural interpretation in many cases is that $\hat{W}_{ij}$ is the amount of attention that $i$ gets from $j$, so is naturally interpreted as a link from $j$ to $i$.

The reference for this notion is Bonacich (1987). However, Leontief studied very similar “centrality” measures around 30 years before that, and the Leontief inverse, $(I - \alpha W)^{-1}$, comes up in the current study of production networks (Acemoglu et al., 2012; Baqaee and Farhi, 2017).

Now, we know from the Neumann series that

$$\beta(\hat{W}; \alpha, \hat{b}) = \sum_{\ell=1}^{\infty} \alpha^{\ell} \hat{W}^{\ell} \hat{b}.$$ 

Thus we can interpret an agent’s Bonacich centrality in terms of sums of walk weights. You did a similar thing for unweighted walks in Problem Set 1.

In our applications, we will be interested in the Bonacich centrality vector, $\beta(\hat{W}; \alpha, \hat{b})$, for various matrices $\hat{W}$ and various vectors $\hat{b}$. For example, clearly in our game as set up in the beginning of Section 1.1 we may write the unique equilibrium identified in Theorem 1 as

$$a^* = \beta(W; \alpha, b).$$ 

We can also describe another important aspect of our game via a different application of Bonacich centrality, i.e., with different parameters. Define the total activity level by $A^* = \sum_i a_i^*$ and define the keyness of $i$ to be

$$k_i = \frac{d A^*}{dB_i}.$$
This is the amount by which an exogenous change in $b_i$ affects the aggregate equilibrium action in the game. (Though we have defined it here as a derivative, the aggregate activity level is linear in each $b_i$, and consequently this is the slope in $b_i$ for any magnitude of change.) Players are more “key” if reducing their $b_i$ causes a greater reduction in aggregate activity.\footnote{This is only one way of measuring what players are key, and in order for this to be a guide to interventions, various assumptions about costs and benefits of intervention have to hold. See Galeotti, Golub, and Goyal (2017) for more on this, and other network measures that might show up when we model the intervention differently.}

It is a good exercise to show that the keyness vector can be expressed in terms of Bonacich centrality. That, figure out how to fill in the question marks in:

$$k = \beta(\alpha, \gamma)$$

3. Limits in which long walks matter

3.1. The limit in which actions blow up. Fix all parameters, in particular the vector $b$, and consider taking $\alpha \uparrow r(W)^{-1}$, so that $r(\alpha W) \uparrow 1$. What happens?

Well, let’s stare at the sum

$$a^* = \sum_{\ell=0}^{\infty} \alpha^\ell W^\ell b.$$  

We know that for each $\alpha$ it is well-defined and finite. But what happens to a typical entry in the limit?

As we saw in Problem Set 1, Problem 3, the maximum entry of $W^\ell$ is asymptotically (as $\ell \to \infty$) exactly of order $r(W)^\ell$, i.e. in the class $\Theta(r(W)^\ell)$. From this we can conclude that

**Fact 2.** As $\alpha \uparrow r(W)^{-1}$, the sum \eqref{eq:alpha} tends to $+\infty$.

Give an economic interpretation of this. (Basic idea: feedback effects get out of control because we are weighting longer and longer walks.)

3.2. A coordination game and a related, but nicer, limit. Take an irreducible, nonnegative, row-stochastic\footnote{A matrix is said to be row-stochastic if each of its rows adds up to 1.} matrix $\Gamma$ with $\alpha_{ii} = 0$ for each $i$ and set $W = \Gamma$. Let $b(\alpha) = (1 - \alpha) y$. Here $y$ is fixed. Note that for fixed parameters this is just a special case of the general game we’ve been studying. However, now that $b$ depends on $\alpha$, the asymptotics of this model in $\alpha$ will be different from the case in which $b$ does not depend on $\alpha$ from the previous subsection.

With the parameter values described above, the game we have been studying is a coordination game: every player wants to match a weighted
average of (i) own ideal action $y_i$ and (ii) a weighted average of neighbors’ actions. You should check that in $i$’s best-response function

$$\text{BR}_i(a_{-i}) = \alpha \sum_j y_{ij} a_j + (1 - \alpha) y_i.$$ 

the weights placed on $y_i$ and the various $a_j$’s sum to 1.

Applying Theorem 1 to characterize the equilibrium, we find:

$$a^* = \sum_{\ell=1}^{\infty} \alpha^\ell \Gamma^\ell b = (1 - \alpha) \sum_{\ell=1}^{\infty} \alpha^\ell \Gamma^\ell y.$$ 

You’ll show in Problem Set 2 that every player’s action ends up being an average of ideal points $y_i$ with certain weights.

Since it turns out that $r(\Gamma) = 1$ (the spectral radius of a row-stochastic matrix is equal to 1) the characterization of Theorem 1 holds for all $\alpha < 1$.

Note that at $\alpha = 1$ corresponds to a pure coordination game, and you should convince yourself that the pure-strategy Nash equilibria are exactly the action profiles with everyone taking the same action. In particular, in the $\alpha = 1$ game there is a huge amount of equilibrium multiplicity.

For any $\alpha < 1$, however, the game has a unique equilibrium. Also in Problem Set 2, you’ll verify that the equilibria as $\alpha \uparrow 1$ converge to a well-defined limit in which all players take the same action, no matter what that action is. Thus, we can think of the $\alpha \uparrow 1$ limit as a way to refine the large set of equilibria in the $\alpha = 1$ coordination game.

In the $\alpha \uparrow 1$ limit, long walks (in $\Gamma$) also matter: this follows from (3). But now they matter in a limit that is better-behaved than the explosive limit studied in the previous subsection. Here the long walks will determine the way in which everyone averages the ideal points $y_i$ in setting their actions.

4. THE PERRON-FROBENIUS THEOREM

4.1. Motivation. As just discussed, we are interested in the $\alpha \uparrow r(W)^{-1}$ limit of our game, which will correspond to the type of Bonacich centrality that

- cares as much about network effects as possible subject to being well-defined
- cares a lot about long walks.

The next result, which is a fundamental theorem that will recur repeatedly, will help us think about this limit. Indeed, it will tell us essentially everything about $W^\ell$ for large $\ell$. But getting there takes a little bit of setup.
4.2. **Perron-Frobenius Theorem.** This theorem goes a long way in the economic analysis of networks.\(^5\) A wonderful reference on it is Carl D. Meyer’s *Matrix Analysis and Applied Linear Algebra, Chapter 8*; I recommend this textbook very highly. A shorter self-contained exposition can be found in Debreu and Herstein’s 1953 paper in *Econometrica* (1953).

For any matrix \(A\), we denote by \(\text{spec}(A)\) the set of its eigenvalues. This set is also called its *spectrum*.

**Definition.** The *spectral radius* of \(A\) is defined to be

\[
 r(A) = \max_{\lambda \in \text{spec}(A)} |\lambda|,
\]

which is the maximum absolute value of the eigenvalues of \(A\).

**Theorem** (Perron-Frobenius). *Let \(A\) be an irreducible, square matrix with no negative entries. Then:*

1. The positive real number \(r(A)\) is an eigenvalue (call it \(\lambda_1\)) of \(A\).
2. There is a unique vector\(^6\) \(p \in \mathbb{R}^n_+\) (called a right-hand Perron vector) satisfying \(\sum_i p_i = 1\) and \(Ap = \lambda_1 p\). All entries of this vector are strictly positive.
3. If there is a \(v \in \mathbb{R}^n_+ \setminus \{0\}\) and \(r' \in \mathbb{R}\) such that \(Av\ r'v\) then \(v\) is a positive scalar multiple of \(p\), and \(r' = \lambda_1\).

We can apply the same result to \(A^\top\) to obtain a unique vector \(q \in \mathbb{R}^n_+\) satisfying \(\sum_i q_i = 1\) and \(A^\top q = r(A^\top)q\).

Because a matrix and its transpose have exactly the same eigenvalues, the matrices \(A\) and \(A^\top\) have the same spectral radius.\(^7\) Thus, taking the transpose of both sides of \(A^\top q = r(A^\top)q\), we find that \(q^\top\) is a left-hand eigenvector of \(A\) satisfying \(q^\top A = r(A)q^\top\). We will call \(q^\top\) the *left-hand Perron vector* of \(A\).

To summarize, the special (positive, real) eigenvalue \(\lambda_1 = r(A)\) of the matrix \(A\) is associated with two special eigenvectors \(p\) (on the right) and \(q^\top\) (on the left), each having only positive entries and each having entries summing to 1. The vector \(p\) is (up to normalization) the unique nonnegative right-hand eigenvector of the matrix \(A\), and the analogous statement holds for \(q^\top\) on the left-hand side.

\(^5\)See Elliott and Golub (2018) for a use of it in the context of characterizing efficient, rather than Nash equilibrium, outcomes.

\(^6\)The notation \(\mathbb{R}^n_+\) means the set of all vectors in \(\mathbb{R}^n\) with nonnegative entries.

\(^7\)The fact that \(r(A^\top) = r(A)\) is also easily deduced from the solution to Problem Set 1, Problem 4, as long as we accept that the definitions of \(r(A)\) given there and in the present note are equivalent.
You have already met $r(A)$ from another angle, in Problem 4 of Problem Set 1, though this is the first time we are discussing the Perron eigenvectors.

4.3. Application to long walks.

**Proposition 1.** Let $A$ be an irreducible, square matrix with no negative entries. Let $r(A)$ be the its largest eigenvalue ($A$’s spectral radius). As $\alpha \uparrow r(A)^{-1}$ we have

$$
(1 - \alpha r(A))(I - \alpha A)^{-1} - \frac{pq^\top}{q^\top p}.
$$

The right-hand side is a rank-1 matrix whose $(i, j)$ entry is $c_{pi}q_j$, where the normalizing constant is the dot product of $p$ and $q$. The proof of this is left as an exercise.

**Definition 2.** A nonnegative matrix $A$ is said to be *primitive* if $A^\ell$ has all strictly positive entries for some positive integer $\ell$.

**Proposition 2.** The nonnegative, irreducible matrix $A$ is primitive if and only if

$$
\lim_{\ell \to \infty} \frac{A^\ell}{r(A)^{\ell}} \to \frac{pq^\top}{q^\top p},
$$

where $p$ and $q^\top$ are the right-hand and left-hand Perron vectors of $A$, respectively.

This is proved in (Meyer, 2000).

5. Comments and Related Models

For our purposes, this resolves the question of how influential various individuals are on the group outcome. The results we just derived about the behavior of long walks are important when we study the long-run behavior of naive learning processes. In particular, the weight of long walks determines one’s influence (Golub and Jackson, 2010). The rate at which the approximations above become good is studied in (DeMarzo et al., 2003; Golub and Jackson, 2012).

A very natural question is how to extend the analysis we have done to incomplete information. This is done in de Martí and Zenou (2015) Golub and Morris (2017a), and Lambert, Martini, and Ostrovsky (2018). Golub and Morris (2017b) shows that the linear algebra we have discussed above is closely related to higher-order expectations, an important object in the study of beliefs and priors generally.
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