Standing waves for the Chern-Simons-Schrödinger equation with critical exponential growth

Chao Ji a,†, Fei Fang b,*

a Department of Mathematics, East China University of Science and Technology
Shanghai, 200237, China

b Department of Mathematics, Beijing Technology and Business University
Beijing, 100048, China

Abstract
In this paper, by combing the variational methods and Trudinger-Moser inequality, we study the existence and multiplicity of the positive standing wave for the following Chern-Simons-Schrödinger equation

$$
-\Delta u + u + \lambda \left( \int_0^\infty \frac{h(s)}{s} u^2(s) ds + \frac{h^2(|x|)}{|x|^2} \right) u = f(x, u) + \epsilon k(x) \quad \text{in } \mathbb{R}^2, \tag{0.1}
$$

where $h(s) = \int_0^s \frac{1}{2} u^2(l) dl$, $\lambda > 0$ and the nonlinearity $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ behaves like $\exp(\alpha |u|^2)$ as $|u| \to \infty$. For the case $\epsilon = 0$, we can get a mountain-pass type solution.

Keywords: Chern-Simons gauge field, Schrödinger equation, Critical exponential growth, Variational methods.

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1. Introduction
In this paper, we are concerned with the following nonlinear Chern-Simons-Schrödinger system

$$
\begin{align*}
\left\{ 
\begin{array}{l}
    iD_0 \phi + (D_1 D_1 + D_2 D_2) \phi = f(x, \phi), \\
    \partial_0 A_1 - \partial_1 A_0 = -\text{Im}(\bar{\phi} D_2 \phi), \\
    \partial_0 A_2 - \partial_2 A_0 = \text{Im}(\bar{\phi} D_1 \phi), \\
    \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2} |\phi|^2,
\end{array}
\right.
\tag{1.1}
\end{align*}
$$

where $i$ denotes the imaginary unit, $\partial_0 = \frac{\partial}{\partial x_1}$, $\partial_1 = \frac{\partial}{\partial x_2}$, $\partial_2 = \frac{\partial}{\partial x_3}$ for $(t, x_1, x_2) \in \mathbb{R}^{1+2}$, $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$ is the complex scalar field, $A_\mu : \mathbb{R}^{1+2} \to \mathbb{R}$ is the gauge field, $D_\mu = \partial_\mu + i A_\mu$ is the covariant derivative for $\mu = 0, 1, 2$. This system was proposed in [11, 12] and consists of the Schrödinger equation augmented by the gauge field $A_\mu$. As usual in Chern-Simons theory, this system is invariant under the following gauge transformation

$$
\phi \to \phi e^{i\chi}, \quad A_\mu \to A_\mu - \partial_\mu \chi,
$$

†E-mail: jichao@ecust.edu.cn

*Corresponding author. E-mail: fangfei68@163.com
where $\chi : \mathbb{R}^{1+2} \to \mathbb{R}$ is an arbitrary $C^\infty$ function.

In recent years, the Chern-Simons-Schrödinger systems have received considerable attention, these models are very important for the study of the high-temperature superconductor, Aharovnov-Bohm scattering, and quantum Hall effect. In [4], the authors investigated the system (1.1) with power type nonlinearity, that is $f(x, u) = \lambda |u|^{p-2} u$ (here $p > 2, \lambda > 0$) and sought the standing waves solutions to the system (1.1) of the form

$$
\phi(t, x) = u(|x|)e^{it}, \quad A_0(x, t) = k(|x|),
$$
$$
A_1(x, t) = \frac{x_2}{|x|} h(|x|), \quad A_2(x, t) = -\frac{x_1}{|x|} h(|x|),
$$

(1.2)

where $\omega > 0$ is a given frequency and $u, k$ and $h$ are real valued functions on $[0, +\infty)$ such that $h(0) = 0$. Note that the ansatz (1.2) satisfies the Coulomb gauge condition $\partial_1 A_1 + \partial_2 A_2 = 0$. Inserting the ansatz (1.2) into the system (1.1), the authors in [4] got the following nonlocal semilinear elliptic equation for $u$

$$
-\Delta u + \omega u + \left(\xi + \int_0^{\infty} \frac{h(s)}{s} u^2(s) ds\right) u + \frac{h^2(|x|)}{|x|^2} u = \lambda |u|^{p-2} u \quad \text{in} \ \mathbb{R}^2,
$$

(1.3)

where $h(s) = \int_0^s \frac{4}{3} u^2(l) dl$ and $\xi$ is a constant. Moreover, they showed (1.3) is actually the Euler-Lagrange equation of the following functional

$$
I(u) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + (\omega + \xi) u^2 + \frac{u^2}{|x|^2} \left(\int_0^{\infty} \frac{s}{2} u^2(s) ds\right)^2 dx - \frac{\lambda}{p} \int_{\mathbb{R}^2} |u|^p dx, \quad u \in H^1_r(\mathbb{R}^2)
$$

here $H^1_r(\mathbb{R}^2)$ denotes the set of radially symmetric functions in $H^1(\mathbb{R}^2)$. They also showed that $I \in C^1(H^1_r(\mathbb{R}^2), \mathbb{R})$ and established some existence results of standing waves by applying variational methods.

Another interesting result is in [15], the authors studied whether $I$ is bounded from below or not for $p \in (1, 3)$. They proved the existence of a threshold value $\omega_0$ such that $I$ is bounded from below if $\omega \geq \omega_0$, and it is not for $\omega \in (0, \omega_0)$. In fact, they given an explicit expression of $\omega_0$, namely:

$$
\omega_0 = \frac{3 - p}{3 + p} 3 \frac{m^{3(p-1)}}{2^{2p} 2^{p+1}} \left(\frac{m^2 (3 + p)}{p - 1}\right)^{-\frac{p+1}{2(p-1)}},
$$

with

$$
m = \int_{-\infty}^{+\infty} \left(\frac{2}{p + 1} \coth^2 \left(\frac{p - 1}{2} r \right)\right)^{\frac{1}{p+1}} dr.
$$

Moreover, in [6], the authors studied the Chern-Simons-Schrödinger system with the general nonlinearity which is a Berestycki, Gallouët and Kavian type nonlinearity [1] and it is the planar version of the Berestycki-Lions type nonlinearity [2, 3]. In [18], the authors researched the Chern-Simons-Schrödinger system without Ambrosetti-Rabinowitz condition. The other related research for system (1.1), we may refer to [13, 16, 20]. However, to our knowledge, the Chern-Simons-Schrödinger system with critical exponential growth was not considered until now, that is, $f$ behaves like $\exp(\alpha |u|^2)$ as $|u| \to \infty$. More precisely, there exists $\alpha_0 > 0$ such that

$$
\lim_{|s| \to \infty} \frac{f(x, s)}{e^{\alpha |s|^2}} = 0, \quad \forall \alpha > \alpha_0, \quad \text{and} \quad \lim_{|s| \to \infty} \frac{|f(x, s)|}{e^{\alpha |s|^2}} = +\infty, \quad \forall \alpha < \alpha_0.
$$
In order to study this class of problems, the Trudinger-Moser inequalities are very important. If $\Omega$ be a bounded domain in $\mathbb{R}^2$, the authors in [14, 17] asserts that
\[
\exp(\alpha |u|^2) \in L^1(\Omega), \quad \forall u \in H^1_0(\Omega), \quad \alpha > 0,
\]
and there exists a constant $C > 0$ such that
\[
\sup_{\|u\|_{L^2(\Omega)} \leq 1} \int_{\Omega} \exp(\alpha |u|^2) \leq C < \infty, \quad \text{if } \alpha \leq 4\pi.
\]
Afterwards, Cao in [5] proved a version of Trudinger-Moser inequality in whole space in $\mathbb{R}^2$, which was improved by do Ó in [8]
\[
\int_{\mathbb{R}^2} \left( \exp(\alpha |u|^2) - 1 \right) dx < +\infty, \quad \forall u \in H^1(\mathbb{R}^2), \quad \alpha > 0. \quad (1.4)
\]
Moreover, if $\alpha < 4\pi$ and $|u|_{L^2(\mathbb{R}^2)} \leq M$, then there exists a constant $C = C(M, \alpha) > 0$ which depends only on $M$ and $\alpha$, such that
\[
\sup_{\|u\|_{L^2(\mathbb{R}^2)} \leq 1} \int_{\mathbb{R}^2} \left( \exp(\alpha |u|^2) - 1 \right) dx \leq C. \quad (1.5)
\]
In this paper, we will firstly study the existence of positive solution of the equation without the perturbation
\[
-\Delta u + u + \lambda \left( \int_0^\infty \frac{h(s)}{s} u^2(s) ds + \frac{h^2(|x|)}{|x|^2} \right) u = f(x, u) \quad \text{in } \mathbb{R}^2. \quad (1.6)
\]
Combing Trudinger-Moser inequalities (1.4), (1.5) and mountain pass theorem, there exists $\lambda_1 > 0$, such that for any $0 < \lambda < \lambda_1$, we can get a positive and classical mountain-pass type solution.

Our next concern is problem (0.1). When the positive parameter $\lambda$ and $\epsilon$ are small enough, we can find a mountain-pass solution. Moreover, by combing Trudinger-Moser inequality and Ekeland’s variational principle [10], we can find a local minimal solution with negative energy.

Since we are interested in the positive solutions, we may assume $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(x, s) = 0$ for $\mathbb{R}^2 \times (-\infty, 0)$. Moreover, we assume the following growth conditions on the nonlinearity $f(x, s)$:

1. $f(x, s) \leq C e^{4\pi s^2}$, for all $(x, s) \in \mathbb{R}^2 \times [0, \infty)$;
2. $\lim_{s \to 0} \frac{f(x, s)}{s} = 0$ uniformly with respect to $x \in \mathbb{R}^2$;
3. There is $0 \leq \sigma < 2$ such that
\[
s f(x, s) - 6F(x, s) \geq -\sigma s^2, \quad \text{for all } (x, s) \in \mathbb{R}^2 \times [0, \infty),
\]
where $F$ is the primitive of $f$.

4. There exist constants $p > 6$ and $C_p > 0$ such that
\[
f(x, s) \geq C_p s^{p-1}, \quad \text{for all } (x, t) \in \mathbb{R}^2 \times [0, +\infty),
\]
where
\[
C_p > \left[ \frac{6(p - 2)}{p(2 - \sigma)} \right]^{\frac{2}{2}} S_p^p.
\]
and

\[
S_p = \inf_{u \in H^1_0(\mathbb{R}^2))} \left( \frac{\int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2) dx}{\left( \int_{\mathbb{R}^2} |u|^p dx \right)^{\frac{2}{p}}} \right)^\frac{1}{2}.
\]

The following are main results of this paper.

**Theorem 1.1.** Suppose hypotheses \((f_1)-(f_4)\) hold, then there exists \(\lambda_1 > 0\), such that for any \(0 < \lambda < \lambda_1\), problem (1.6) has a positive and classical solution of mountain-pass type.

**Theorem 1.2.** Suppose hypotheses \((f_1)-(f_4)\) hold, for any \(0 \leq h(x) \in H^{-1}\), then there exist \(\lambda_2 > 0\) and \(\varepsilon_1 > 0\), such that for any \(0 < \lambda < \lambda_2\) and \(0 < \varepsilon < \varepsilon_1\), problem (0.1) has at least two nonnegative solutions and one of them has a negative energy.

The paper is organized as follows. In Section 2 we are concerned with the nonperturbation problem (1.4) and prove Theorem 1.1. In Section 3, the proof of Theorem 1.2 is given.

**Notations.** \(C, C_1, C_2\) etc. will denote positive constants whose essential values are inessential. \(H^{-1}\) is dual space of \(H^1(\mathbb{R}^2)\), \(C_0^\infty(\mathbb{R}^2)\) denotes the space of infinitely differential radial functions with compact support in \(\mathbb{R}^2\). \(o_n(1)\) denotes a quantity which goes to zero. \(B_R\) denotes the open ball centered at the origin and radius \(R > 0\) and \(\bar{B}_R\) is its closure. \(u_n \to u\) and \(u_n \rightharpoonup u\) denote the strong convergence and weak convergence of a sequence \(\{u_n\}\) in a Banach space, respectively.

### 2. Proof of Theorem 1.1

From assumptions \((f_1)\) and \((f_2)\), for given \(\eta > 0\) small there exist positive constants \(C_\eta\) and \(\gamma > 1\) such that

\[
F(x, s) \leq \eta \frac{s^2}{2} + C_\eta \left( e^{\gamma s^2} - 1 \right)
\]

for all \((x, s) \in \mathbb{R}^2 \times \mathbb{R}\).

Thus, by the Trudinger-Moser inequalities (1.4), we have \(F(x, u) \in L^1(\mathbb{R}^2)\) for all \(u \in H^1(\mathbb{R}^2)\). Therefore, the functional

\[
J(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) dx + \frac{\lambda}{2} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \int_0^{2\pi} \left( \frac{s}{2} u^2(s) \right) ds dx - \int_{\mathbb{R}^2} F(x, u) dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) dx + \frac{\lambda}{2} \int_{\mathbb{R}^2} \frac{u^2}{4|\nabla|} \left( \frac{1}{2\pi} \int_{B_0} u^2 dx \right) dx - \int_{\mathbb{R}^2} F(x, u) dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) dx + \frac{\lambda}{2} \int_{\mathbb{R}^2} F(x, u) dx - \int_{\mathbb{R}^2} F(x, u) dx
\]

\[
= J_0(u) + \frac{\lambda}{2} c(u)
\]

\(u \in H^1(\mathbb{R}^2)\)

is well defined. Furthermore, using standard arguments (see [19]) we can show that \(J \in C^1(\mathcal{H}_r^1(\mathbb{R}^2), \mathbb{R})\) with

\[
J'(u)\varphi = \int_{\mathbb{R}^2} \nabla u \nabla \varphi + u \varphi dx + \lambda \int_{\mathbb{R}^2} \left( \int_{|\nabla|}^{s} \frac{h(s)}{s} u^2(s) ds \right) u \varphi + \frac{h^2(|\nabla|)}{|\nabla|^2} u \varphi dx - \int_{\mathbb{R}^2} f(x, u) \varphi dx
\]

\(\varphi \in H^1_{r}(\mathbb{R}^2)\).
Consequently, each critical point of the functional $J$ is a solution of problem (1.6).

For functional $c(u)$, there is the following compactness lemma we use later.

**Lemma 2.1.** (see [4]) Suppose that a sequence $\{u_n\}$ converges weakly to a function $u$ in $H^1_\nu(\mathbb{R}^2)$ as $n \to \infty$. Then for each $\varphi \in H^1_\nu(\mathbb{R}^2)$, $c(u_n)$, $c'(u_n)\varphi$ and $c'(u_n)u_n$ converges up to a subsequence to $c(u)$, $c'(u)\varphi$ and $c'(u)u$, respectively, as $n \to \infty$.

In order to show that the weak limit of a sequence in $H^1_\nu(\mathbb{R}^2)$ is a weak solution of problem (1.6), we need the following convergence result.

**Lemma 2.2.** (see [7]) Assume that $\Omega \subset \mathbb{R}^2$ be a bounded domain and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ a continuous function. Let $(u_n)$ be a sequence of functions in $L^1(\Omega)$ converging to $u$ in $L^1(\Omega)$. Assume that $f(x, u_n)$ and $f(x, u)$ are also $L^1$ functions. If

$$\int_\Omega |f(x, u_n)u_n|dx \leq C_1,$$

then $f(x, u_n)$ converges in $L^1$ to $f(x, u)$.

In order to construct the mountain-pass geometry of the functional $J$, we need next two lemmas.

**Lemma 2.3.** (see [9]) Let $\beta > 0$ and $r > 1$. Then for each $\alpha > r$ there exists a positive constant $C = C(\alpha)$ such that for all $s \in \mathbb{R}$,

$$(e^{\beta s^2} - 1)^r \leq C(e^{\alpha \beta s^2} - 1).$$

In particular, if $u \in H^1(\mathbb{R}^2)$ then $(e^{\beta s^2} - 1)^r$ belongs to $L^1(\mathbb{R}^2)$.

**Lemma 2.4.** (see [9]) Suppose $u \in H^1(\mathbb{R}^2)$, $\beta > 0$, $q > 0$ and $\|v\| \leq M$ with $\beta M^2 < 4\pi$, then there exists $C = C(\beta, M, q) > 0$ such that

$$\int_{\mathbb{R}^2} (e^{\beta s^2} - 1)|v|^qdx \leq C\|v\|^q.$$

**Lemma 2.5.** Let $(f_1)$, $(f_2)$ hold. Then functional $J$ satisfy the mountain pass geometry:

1. There exists $\rho > 0$ small enough, such that $\inf_{\|u\| = \rho} J(u) \geq d > 0$;

2. There exists $u_0 \in H^1_\nu(\mathbb{R}^2)$ with $\|u_0\| > \rho$, such that $J(u_0) < 0$.

**Proof.** (1) From $(f_1)$, for any $\eta > 0$, there exists $\delta > 0$ such that $|u| < \delta$

$$F(x, u) \leq \eta|u|^2 \quad \text{for all } x \in \mathbb{R}^2. \quad (2.1)$$

On the other hand, for $q > 2$, by $(f_2)$, there exists $C = C(q, \delta)$ such that $|u| \geq \delta$ implies

$$F(x, u) \leq C|u|^q \left( \exp(4\pi u^2) - 1 \right) \quad \text{for all } x \in \mathbb{R}^2. \quad (2.2)$$

Combing (2.1) and (2.2) yield

$$F(x, u) \leq \eta|u|^2 + C|u|^q \left( \exp(4\pi u^2) - 1 \right) \quad \text{for all } (x, u) \in \mathbb{R}^2 \times \mathbb{R}.$$
where $\alpha > 1$, $\|u\| = \rho$ small enough such that $4\pi\alpha\rho < 4\pi$, $r > 1$ close to 1, $s > 1$ and $\frac{1}{r} + \frac{1}{s} = 1$, $0 < \eta \leq \frac{1}{4}$.

(2) Due to $(f_4)$, $F(x, u) \geq \frac{C_p}{p}|u|^p$. So, for any $u \in H^1_p(\mathbb{R}^2)$, we have

$$J(u) \leq \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + u^2)dx + \frac{\lambda}{2} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left( \int_0^{\|x\|} \frac{s}{2} u^2(s) \right)^2 dx - \int_{\mathbb{R}^2} \frac{C_p}{p}|u|^p dx.$$

Fix $u \in H^1_p(\mathbb{R}^2) \setminus \{0\}$, we have

$$J(tu) \leq \frac{t^2}{2} \int_{\mathbb{R}^2} |\nabla u|^2 + u^2)dx + \frac{t^6}{2} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left( \int_0^{t\|x\|} \frac{s}{2} u^2(s) \right)^2 dx - \frac{C_p }{p} t^p \int_{\mathbb{R}^2} |u|^p dx.$$

Since $p > 6$, there exists $t_0$ sufficiently large such that $\|tu\| > \rho$ and $J(t_0u) < 0$. Set $u_0 = t_0u$, we get the conclusion. \hfill \Box

By the mountain pass theorem (see [19]), there exists a Palais-Smale sequence $\{u_n\} \subset H^1_p(\mathbb{R}^2)$ satisfying

$$J(u_n) \to c \geq b \quad \text{and} \quad J'(u_n) \to 0,$$

where $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) > 0$ and

$$\Gamma = \{ \gamma \in C([0,1], H^1_p(\mathbb{R}^2)) : \gamma(0) = 0, J(\gamma(1)) < 0 \},$$

shortly $\{u_n\}$ is a $(PS)_c$ sequence. Moreover, by the assumptions of $f$, we may assume that the sequence $\{u_n\}$ is nonnegative.

**Lemma 2.6.** If $(u_n) \subset H^1_p(\mathbb{R}^2)$ is a $(PS)_c$ sequence to $J$, then $(u_n)$ is bounded in $H^1_p(\mathbb{R}^2)$.

**Proof.** From $(f_3)$, for $n$ large enough, we have

$$6c + 1 + \epsilon_n \|u_n\| \geq 6J(u_n) - J'(u_n)u_n = 2\|u_n\|^2 + \int_{\mathbb{R}^2} \left( u_n f(x, u_n) - 6F(x, u_n) \right) dx$$

$$\geq 2\|u_n\|^2 - \sigma \int_{\mathbb{R}^2} |u_n|^2 dx$$

$$\geq (2 - \sigma)\|u_n\|^2$$

where $\epsilon_n \to 0$, it implies the boundedness of $(u_n)$. \hfill \Box
Lemma 2.7. Assume that \((f_1)\) hold, there exists \(\lambda_1 > 0\), then for any \(0 < \lambda < \lambda_1\), \(d \leq c < \frac{2-\sigma}{6}\).

**Proof.** According to Lemma 2.5, it is clear that \(c \geq b\), so we only need to prove that \(c < \frac{2-\sigma}{6}\). Fix a positive function \(v_p \in H^1_0(\mathbb{R}^2)\) such that
\[
S_p = \left( \frac{\int_{\mathbb{R}^2} |\nabla v_p|^2 + |v_p|^2 \, dx}{\left( \int_{\mathbb{R}^2} |v_p|^p \, dx \right)^{\frac{2}{p}}} \right)^{\frac{1}{2}}.
\]
Note that
\[
\max_{t \geq 0} J_0(tv_p) \leq \max_{t \geq 0} \left\{ \frac{t^2}{2} \int_{\mathbb{R}^2} |\nabla v_p|^2 + v_p^2 \, dx - \frac{C_p t^p}{p} \int_{\mathbb{R}^2} |v_p|^p \, dx \right\} = \frac{(p-2) S_p^{\frac{2p}{p-2}}}{2p C_p^{\frac{p}{p-2}}},
\]
So, there exists \(\lambda_1 > 0\) such that for any \(0 < \lambda < \lambda_1\), we have
\[
\max_{t \geq 0} J(tv_p) \leq \frac{(p-2) S_p^{\frac{2p}{p-2}}}{p C_p^{\frac{p}{p-2}}}.
\]
Moreover, from \((f_4)\)
\[
\frac{(p-2) S_p^{\frac{2p}{p-2}}}{p C_p^{\frac{p}{p-2}}} < 2 - \frac{\sigma}{6}.
\]
So \(c < \frac{2-\sigma}{6}\). \(\square\)

**Proof of Theorem 1.1.** From Lemma 2.5, we obtain a nonnegative \((PS)_c\) sequence \(\{u_n\}\) and from Lemma 2.6, this sequence is bounded, thus for a subsequence still denoted by \(\{u_n\}\) there is a nonnegative function \(u_1 \in H^1_0(\mathbb{R}^2)\) such that \(u_n \rightharpoonup u_1\) in \(H^1_0(\mathbb{R}^2)\) and \(u_n \to u_1\) in \(L^s_{\text{loc}}(\mathbb{R}^2)\) for all \(s \geq 1\) and \(u_n \to u_0\) almost everywhere in \(\mathbb{R}^2\). Now, from \((f_5)\),
\[
c = \lim_{n \to \infty} J(u_n)
\]
\[
= \lim_{n \to \infty} \left[ J(u_n) - \frac{1}{6} J'(u_n) u_n \right]
\]
\[
\geq \lim_{n \to \infty} \left( \frac{1}{3} \|u_n\|^2 + \int_{\mathbb{R}^2} \left( \frac{1}{6} f(x, u_n) u_n - F(x, u_n) \right) \, dx \right)
\]
\[
\geq \frac{2-\sigma}{6} \limsup_{n \to \infty} \|u_n\|^2
\]
which implies
\[
\limsup_{n \to \infty} \|u_n\|^2 = m \leq \frac{6c}{2 - \sigma} < 1.
\]
According to Trudinger-Moser inequality (1.5) and \((f_1)\), for any bounded domain \(\Omega\), \(f(x, u_1)\) and \(f(x, u_n)\) belong to \(L^1(\Omega)\). In virtue of \(J'(u_n) u_n \to 0\) and \(J(u_n) \to c\) as \(n \to \infty\), there exists a
constant $C_2 > 0$ such that
\[ \int_{\mathbb{R}^2} F(x, u_n)dx \leq C_2 \quad \text{and} \quad \int_{\mathbb{R}^2} f(x, u_n)u_n dx \leq C_2. \]

By (f1), we have
\[ \int_{\Omega} f(x, u_n)u_n dx \leq C_2. \]

Therefore, according to Lemma 2.2, one has
\[ \lim_{n \to \infty} \int_{\mathbb{R}^2} f(x, u_n)\nu dx = \int_{\mathbb{R}^2} f(x, u_1)\nu dx \quad \forall \, \varphi \in C_{0, r}^{\infty}(\mathbb{R}^2). \] (2.3)

Combing (2.3) and Lemma 2.1, we have
\[ \lim_{n \to \infty} J'(u_n)v = 0 = J'(u_1)v \quad \forall \, v \in C_{0, r}^{\infty}(\mathbb{R}^2), \]
so $u_0$ is a solution of problem (1.6).

At last, we show that the sequence $(u_n)$ has a convergent subsequence.

Set $u_n = u_1 + \omega_n$, then $\omega_n \to 0$ in $H_1^1(\mathbb{R}^2)$ and $\omega_n \to 0$ in $L'(\mathbb{R}^2)$ for all $2 < q < \infty$. By the Brézis-Lieb Lemma (see [19]), we get
\[ \|u_n\|^2 = \|u_1\|^2 + \|\omega_n\|^2 + o_n(1). \] (2.4)

We firstly show that
\[ \lim_{n \to \infty} \int_{\mathbb{R}^2} f(x, u_n)u_1 dx = \int_{\mathbb{R}^2} f(x, u_1)u_1 dx. \] (2.5)

In fact, since $C_{0, r}^{\infty}(\mathbb{R}^2)$ is dense in $H_1^1(\mathbb{R}^2)$, for any $\eta > 0$ there exists $\varphi \in C_{0, r}^{\infty}(\mathbb{R}^2)$ such that $\|u_1 - \varphi\| < \eta$. Observe that
\[
\left| \int_{\mathbb{R}^2} f(x, u_n)u_1 dx - \int_{\mathbb{R}^2} f(x, u_1)u_1 dx \right| \leq \int_{\mathbb{R}^2} f(x, u_n)(u_1 - \varphi) dx + \int_{\mathbb{R}^2} f(x, u_1)(u_1 - \varphi) dx + \|\varphi\|_{\infty} \int_{\text{supp} \varphi} \left| f(x, u_n) - f(x, u_1) \right| dx.
\]

For the first integral, using that $|J'(u_n)(u_1 - \varphi)| \leq \eta_n\|u_1 - \varphi\|$ with $\eta_n \to 0$ as $n \to \infty$ and Lemma 2.1, we get
\[
\left| \int_{\mathbb{R}^2} f(x, u_n)(u_1 - \varphi) dx \right| \leq \eta_n\|u_1 - \varphi\| + \left| \int_{\mathbb{R}^2} \nabla u_n \nabla (u_1 - \varphi) + u_n(u_1 - \varphi) dx \right| + \left| c'(u_n)(u_1 - \varphi) \right|
\leq \eta_n\|u_1 - \varphi\| + \|u_n\|\|u_1 - \varphi\| + \left| \left( c'(u_n) - c'(u_1) \right)(u_1 - \varphi) \right| + \left| c'(u_1)(u_1 - \varphi) \right|
\leq C_3\|u_1 - \varphi\| \leq C_3\eta
\]
for $n$ large. Similarly, using that $J'(u_1)(u_1 - \varphi) = 0$, we can estimate the second integral and obtain
\[
\left| \int_{\mathbb{R}^2} f(x, u_1)(u_1 - \varphi) dx \right| \leq C_3\eta.
\]
Combing (2.3) and the previous inequality, we have
\[
\lim_{n \to \infty} \left| \int_{\mathbb{R}^2} f(x, u_n) u_1 dx - \int_{\mathbb{R}^2} f(x, u_1) u_n dx \right| \leq 2C \eta,
\]
this implies (2.5) because \( \eta \) is arbitrary.

From (2.4) and Lemma 2.1, we can write
\[
J'(u_n)u_n = \|u_n\|^2 + c'(u_n)u_n - \int_{\mathbb{R}^2} f(x, u_n)u_n dx
\]
\[
= \|u_1\|^2 + \|\omega_n\|^2 + c'(u_1)u_1 - \int_{\mathbb{R}^2} f(x, u_n)u_1 dx - \int_{\mathbb{R}^2} f(x, u_n)\omega_n dx + o_n(1)
\]
\[
= J'(u_1)u_1 + \|\omega_n\|^2 - \int_{\mathbb{R}^2} f(x, u_n)\omega_n dx + o_n(1),
\]
that is
\[
\|\omega_n\|^2 = \int_{\mathbb{R}^2} f(x, u_n)\omega_n dx + o_n(1).
\]

According to Trudinger-Moser inequality (1.5), for \( \tau > 1, q > 1 \) close to 1 satisfying \( \tau q \frac{\mu - 2\mu}{2\mu} < 1 \), there exists \( C_4 > 0 \) such that the sequence \( h_q(x) = e^{4\pi \tau u_0^2(x)} - 1 \) satisfies
\[
\int_{\mathbb{R}^2} f(x, u_n)\omega_n dx \leq \eta \int_{\mathbb{R}^2} |u_n|\omega_n dx + C_\eta \int_{\mathbb{R}^2} \left( e^{4\pi \tau u_0^2(x)} - 1 \right) |\omega_n| dx
\]
\[
\leq \eta + C_4 \left( \int_{\mathbb{R}^2} |\omega_n|^q dx \right) \left( \int_{\mathbb{R}^2} \left( e^{4\pi \tau u_0^2(x)} - 1 \right) dx \right)^{\frac{1}{q}}
\]
\[
\leq \eta + C_4 \left( \int_{\mathbb{R}^2} |\omega_n|^q dx \right),
\]
where \( \frac{1}{q} + \frac{1}{q'} = 1 \). Since \( q > 1 \) close to 1, \( q' > 2 \), by the compact embedding \( H^1_r(\mathbb{R}^2) \hookrightarrow L^{r'}(\mathbb{R}^2) \) for all \( r > 2 \), we get
\[
\int_{\mathbb{R}^2} f(x, u_n)\omega_n dx \to 0, \quad \text{as } n \to \infty.
\]

So, \( \lim_{n \to \infty} \|\omega_n\|^2 = 0 \). Moreover, by the argument in [4], \( u_1 \in C^2(\mathbb{R}^2) \). Since \( u_1 \) is nonnegative, we have \( u_1 > 0 \) by the strong maximum principle and the proof is completed.

\[\Box\]

3. Proof of Theorem 1.2

In this section, we deal with the problem (0.1) and show that there exist at least two non-negative solutions, one is mountain-pass type solution, another is a local minimal solution with negative energy.

The functional corresponding to problem (1.1) is
\[
J_{\epsilon}(u) = \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) dx + \frac{\lambda}{2} \int_{\mathbb{R}^2} \frac{u^2}{|x|^2} \left( \int_0^{\frac{|x|}{2}} s u^2(s) \right)^2 dx - \int_{\mathbb{R}^2} F(x, u) dx - \epsilon \int_{\mathbb{R}^2} ku dx.
\]
\[ J'_e(\phi) = \int_{\mathbb{R}^2} \nabla u \nabla \phi + u \phi \, dx + \lambda \int_{\mathbb{R}^2} \left( \int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds \right) u \phi + \frac{h^2(|x|)}{|x|^2} u \phi \, dx - \int_{\mathbb{R}^2} f(x, u) \phi \, dx - \epsilon \int_{\mathbb{R}^2} k \phi \, dx \]

for any \( \phi \in H^1(\mathbb{R}^2) \). So, for searching the solutions of problem (0.1), we may seek the critical points of the functional \( J_e \).

In the next two lemmas we check that the functional \( J_e \) satisfies the geometric conditions of the mountain-pass theorem.

**Lemma 3.1.** Let \((f_1), (f_2)\) hold. Then there exists \(\epsilon_1 > 0\) such that for \(0 < \epsilon < \epsilon_1\), there exists \(\rho_\epsilon > 0\) such that \(J_e(u) > 0\) if \(||u|| = \rho\). Furthermore, \(\rho_\epsilon\) can be chosen such that \(\rho_\epsilon \to 0\) as \(\epsilon \to 0\).

**Proof.** As the same proof of Lemma 2.5, from \((f_1)\) and \((f_2)\), for \(\forall 0 < \eta < \frac{1}{4}\), there exist \(C > 0\) such that for \(q > 2\)

\[ F(x, u) \leq \eta |u|^q + C |u|^q \left( \exp(4\pi u^2) - 1 \right) \quad \text{for all } x \in \mathbb{R}^2 \times \mathbb{R}. \]

So, by Lemma 2.4, we have

\[
J_e(u) \geq \frac{1}{2} ||u||^2 - \int_{\mathbb{R}^2} F(x, u) \, dx - \epsilon \int_{\mathbb{R}^2} k(x) u^2 \, dx \\
\geq \frac{1}{2} ||u||^2 - \eta \int_{\mathbb{R}^2} ||u||^2 \, dx - C \int_{\mathbb{R}^2} ||u||^q \left( \exp(4\pi u^2) - 1 \right) \, dx - \epsilon ||k|| ||u|| \\
\geq \frac{1}{4} ||u||^2 - C \left( \int_{\mathbb{R}^2} ||u||^{q_\eta} \, dx \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^2} \left( \exp(4\pi u^2) - 1 \right) \, dx \right)^{\frac{1}{q}} - \epsilon ||k|| ||u|| \\
\geq \frac{1}{4} ||u||^2 - C ||u||^{q_\eta} - \epsilon ||k|| ||u||, \quad \text{for } ||u|| = \rho \text{ small enough}
\]

where \(r > 1\) close to 1, \(s > 1\) and \(\frac{1}{r} + \frac{1}{s} = 1\). Thus

\[
J_e(u) \geq ||u|| \left( \frac{1}{4} ||u|| - C ||u||^{q_\eta - 1} - \epsilon ||k|| \right). \quad (3.2)
\]

Since \(q > 2\), we may choose \(\rho > 0\) small enough such that \(\frac{1}{4} \rho - C \rho^{q_\eta - 1} > 0\). Thus, if \(\epsilon > 0\) is sufficiently small then we can find some \(\rho_\epsilon > 0\) such that \(J_e(u) > 0\) if \(||u|| = \rho_\epsilon\) and \(\rho_\epsilon \to 0\) as \(\epsilon \to 0\). \(\square\)

**Lemma 3.2.** There exists \(u_0 \in H^1(\mathbb{R}^2)\) with \(||u_0|| > \rho_\epsilon\), such that \(J_e(u_0) < \inf_{||u|| = \rho_\epsilon} J_e(u)\).

**Proof.** Due to \((f_1)\), as the same proof of Lemma 2.1, we may show \(J_e(tu) \to -\infty\) as \(t \to \infty\). Setting \(u_0 = tu\) with \(t\) sufficiently large, we get the result. \(\square\)

From Lemma 3.1 and Lemma 3.2, we can get a \((PS)_\epsilon\) sequence \(\{u_n\} \subset H^1(\mathbb{R}^2)\) satisfying

\[
J_e(u_n) \to c_\epsilon > 0 \quad \text{and} \quad J'_e(u_n) \to 0,
\]
where \( c_\epsilon = \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} J_\epsilon(\gamma(t)) > 0 \) and
\[
\Gamma = \{ \gamma \in C([0, 1], H^1_0(\mathbb{R}^2)) : \gamma(0) = 0, J_\epsilon(\gamma(1)) < 0 \}.
\]

**Lemma 3.3.** Assume that (\( f_1 \)) hold, and let \( \lambda_1 > 0 \) be as in Lemma 2.7, then there exists \( 0 < \epsilon_2 < \epsilon_1 \) such that for any \( 0 < \lambda < \lambda_1 \) and \( 0 < \epsilon < \epsilon_1 \), \( c_\epsilon < \frac{2 \epsilon^\alpha}{\alpha} \).

**Lemma 3.4.** Suppose (\( f_1 \)) - (\( f_4 \)) hold, and let \( \lambda_1 > 0 \) and \( \epsilon_2 > 0 \) be as in Lemma 3.3. Then for any \( 0 < \lambda < \lambda_1 \) and \( 0 < \epsilon < \epsilon_2 \), problem (0.1) has a mountain pass type solution \( u_2 \).

The proof of Lemma 3.3 is similar to one of Lemma 2.7, the proof of Lemma 3.4 is similar to one of Theorem 1.1, so we omit them.

Now we prove the existence of a local minimal solution with the negative energy.

**Lemma 3.5.** There exists \( \eta > 0 \) and \( v \in H^1_0(\mathbb{R}^2) \) with \( \|v\| = 1 \) such that \( J_\epsilon(tv) < 0 \) for all \( 0 < t < \theta \). In particular, \( \inf_{\|u\|=\eta} J_\epsilon(u) < 0 \).

**Proof.** For each \( k \in H^{-1} \), borrowing the Riesz representation theorem in the Hilbert space \( H^1_0(\mathbb{R}^2) \), the problem
\[
-\Delta v + v = k, \quad x \in \mathbb{R}^2,
\]
has a unique weak solution \( v \) in \( H^1_0(\mathbb{R}^2) \). Thus,
\[
\int_{\mathbb{R}^2} kvdx = \|v\|^2 > 0 \quad \text{for each } k \neq 0.
\]
Since \( f(x, 0) = 0 \), by continuity, there exists \( \theta > 0 \) such that
\[
\frac{d}{dt} J_\epsilon(tv) = t\|v\|^2 + 3t^5 c(v) - \int_{\mathbb{R}^2} f(x, tv)vdx - \epsilon \int_{\mathbb{R}^2} kvdx < 0
\]
for all \( 0 < t < \theta \). Since \( J_\epsilon(0) = 0 \), it is clear that \( J_\epsilon(tv) < 0 \) for all \( 0 < t < \theta \). \( \square \)

By inequality (3.2) and Lemma 3.5, one has
\[
-\infty < c_1 = \inf_{\|u\| \leq \rho_\epsilon} J_\epsilon(u) < 0. \quad (3.3)
\]

**Lemma 3.6.** Let \( \epsilon_2 > 0 \) be as in Lemma 3.3 and each \( \epsilon \) with \( 0 < \epsilon < \epsilon_2 \), problem (0.1) has a minimal type solution \( u_3 \) with \( J_\epsilon(u_3) = c_0 < 0 \).

**Proof.** Let \( \rho_\epsilon \) small be as in Lemma 3.1. We can choose that \( \rho_\epsilon < 1 \). Since \( \tilde{B}_{\rho_\epsilon} \) is a complete metric space with the metric given by the norm of \( H^1_0(\mathbb{R}^2) \), convex and the functional \( J_\epsilon \) is a class of \( C^1 \) and bounded below on \( \tilde{B}_{\rho_\epsilon} \), by Ekeland’s variational principle [10] there exists a sequence \( \{u_n\} \) in \( \tilde{B}_{\rho_\epsilon} \) such that
\[
J_\epsilon(u_n) \to c_1 < 0 \quad \text{and} \quad J_\epsilon'(u_n) \to 0.
\]
By the proof of Theorem 1.1 it follows that there exists a subsequence of \( \{u_n\} \) which converges to a function \( u_3 \). Therefore, \( J_\epsilon(u_3) = c_1 < 0 \). \( \square \)
Proof of Theorem 1.2. From Lemma 3.4 and Lemma 3.6, there exist $\lambda_1 > 0$ and $\epsilon_2 > 0$ such that for any $0 < \lambda < \lambda_1$ and $0 < \epsilon < \epsilon_2$, there exist at least two solutions of problem (0.1), one is a mountain pass type solution, another is a local minimum solution with negative energy. Since $k(x) \geq 0$ almost everywhere in $\mathbb{R}^2$. Let $u \in H^1_r(\mathbb{R}^2)$ be a weak solution of (0.1). Setting $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$ and taking $v = u^-$ in (3.1), we obtain

$$\|u^-\|^2 + 3\lambda c(u^-) = \epsilon \int_{\mathbb{R}^2} kv dx \leq 0,$$

because $f(x,u(x))u^- = 0$ in $\mathbb{R}^2$. So, $u = u^+ \geq 0$. We get two solutions are nonnegative, by the argument in [4], these two solutions belong to $\in C^2(\mathbb{R}^2)$. Moreover, by the strong maximum principle, they are positive. We complete the proof. □

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