Inverse scattering transform for the integrable nonlocal Lakshmanan-Porsezian-Daniel equation

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Abstract

In this work, a generalized nonlocal Lakshmanan-Porsezian-Daniel (LPD) equation is introduced, and its integrability as an infinite dimensional Hamilton dynamic system is established. Motivated by the ideas of Ablowitz and Musslimani (2016 Nonlinearity 29 915), we successfully derive the inverse scattering transform (IST) of the nonlocal LPD equation. The direct scattering problem of the equation is first constructed, and some important symmetries of the eigenfunctions and the scattering data are discussed. By using a novel Left-Right Riemann-Hilbert (RH) problem, the inverse scattering problem is analyzed, and the potential function is recovered. By introducing the special conditions of reflectionless case, the time-periodic soliton solutions formula of the equation is derived successfully. Take $J = \bar{J} = 1, 2, 3$ and 4 for example, we obtain some interesting phenomenon such as breather-type solitons, arc solitons, three soliton and four soliton. Furthermore, the influence of parameter $\delta$ on these solutions is further considered via the graphical analysis. Finally, the eigenvalues and conserved quantities are investigated under a few special initial conditions.

Keywords: integrable nonlocal Lakshmanan-Porsezian-Daniel equation, inverse scattering method, Left-Right Riemann-Hilbert problem, soliton solutions.

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1. Introduction

Nonlinear integrable evolution equations exist in all aspects of scientific research and play a essential role in modern physical branches. There are numerous nonlinear integrable evolution equations which are applied into fluid mechanics, elasticity, lattice dynamics, electromagnetics, etc. For example, the Korteweg-de Vries (KdV) and modified Korteweg-de Vries (mKdV) equations describe the evolution of weakly dispersive and small amplitude waves in quadratic and cubic nonlinear media, respectively [1]. The KdV equation is more famous for its application in shallow water waves. Besides, the integrable cubic nonlinear Schrödinger (NLS) equation which is well-known for its application to the evolution of weakly nonlinear and quasi-monochromatic wave trains in media with cubic nonlinearities [2, 3]. Besides, the Kadomtsev-Petviashvili (KP) equation which arises in plasma physics and internal waves [1, 4] is applied to describe the evolution of weakly dispersive and small amplitude waves with additional weak transverse variation [5, 6]. Based on the importance of nonlinear integrable evolution equations, these are the focus of scholars’ research from the beginning to the end. In order to solve these equations, many novel and effective methods have been produced, such as Hirota bilinear method [7], Darboux and Bäcklund transformation [8] and inverse scattering transform (IST) [1, 4, 9, 10].

However, there is a special kind of equation called nonlocal equation among many nonlinear integrable equations. As the name suggests, nonlinear integrable nonlocal equation refers to the nonlinear integrable evolution equation with nonlocal nonlinear term, for example, \( q(x, t) \) is replaced by \( q^*(−x, t) \), \( q(−x, −t) \) or \( q^*(−x, −t) \). In Ref. [11], Ablowitz and Musslimani had found a new class of nonlocal integrable NLS hierarchy with the infinite number of conservation laws by introducing a new symmetry reduction \( r(x, t) = q^*(−x, t) \). According to the different inversion relation, integrable nonlocal nonlinear equations roughly include the following categories: real (complex) reverse time nonlocal equation, real (complex) reverse space nonlocal equation ans real (complex) reverse space-time nonlocal equation [12]. Recently there are several new nonlocal system have been analyzed, including multidimensional versions of the NLS equation [13, 14], nonlocal reverse-time NLS equations [15], nonlocal mKdV equation [16-18], nonlocal sine-Gordon equation [19], nonlocal Davey-Stewartson equation [20, 21], nonlocal Alice-Bob systems [22], nonlocal (2+1)-D breaking solitons hierarchy [23] and nonlocal integrable equations [24].

As for the method of solving the nonlocal equation, the above mentioned methods are not all effective to it, and the most classical and effective method is IST. IST associates a compatible pair of linear equations with the integrable nonlinear equation. One of the equations is used to determine suitably analytic eigenfunctions and transform the initial data to appropriate scattering data. The other linear equation is used to complete
the evolution of the scattering data. Based on the linear equations (Lax pair), one can find the exact solutions of origin objective equations successfully \cite{1}.

In this work, we consider IST for the integrable nonlocal Lakshmanan-Porsezian-Daniel (LPD) equation

\[ q_t(x, t) + \frac{1}{2} i q_{xx}(x, t) - i \gamma q^2(x, t) q^*(-x, t) - \delta H[q(x, t)] = 0, \quad (1.1) \]

where

\[
H[q(x, t)] = -i q_{xxxx}(x, t) + 6i \gamma q^*(-x, t) q^2(x, t) + 4i \gamma q(x, t) q_x(x, t) q_x^*(-x, t)
+ 8i \gamma q^*(-x, t) q(x, t) q_{xx}(x, t) + 2i \gamma q^2(x, t) q_x^*(-x, t) - 6i q^2(-x, t) q^3(x, t),
\]

which is an NLS type equation with higher order nonlinear terms, such as fourth-order dispersion, second-order dispersion, cubic and quintic nonlinearities. The LPD equation describes the nonlinear effect more clearly in Refs. \cite{25–27}. The integrable non-local LPD equation whose the potential functions satisfy \( r(x, t) = q^*(-x, t) \) is studied via Darboux transformation in \cite{28}. The authors demonstrated the integrability of the nonlocal LPD equation, provided its Lax pair, and presented its rational soliton solutions and self-potential function. However, in this work, by using a ingenious method, we analyze infinite number of conserved quantities and conservation laws for nonlocal LPD equation whose potential functions satisfy \( r(x, t) = \gamma q^*(-x, t) \), where \( \gamma = \pm 1 \). Furthermore, by using IST method \cite{29}, we obtain the time-periodic pure soliton solutions of the integrable nonlocal LPD equation whose potential functions satisfy \( r(x, t) = -q^*(-x, t) \). Generally speaking, the significance of this work is to improve the previous research on the soliton solutions and some important properties of integrable nonlocal LPD equation reported in \cite{28}, which is very helpful for us to better understand and master this class equations.

This work is organized as follows. In Section 2, the Lax pair and the compatibility condition of nonlocal LPD equation are given. Besides, other properties of nonlocal LPD equation are listed in the end of this section. In Section 3, by introducing a novel method, we derive the global conservation laws and the local conservation laws which establish the integrability of the objective equation. In Section 4, the direct scattering problem of the nonlocal LPD equation is constructed and some other important symmetries of the eigenfunctions and the scattering data are discussed. Afterwards, by using the Left-Right RH method, the inverse scattering problem is established and the potential function is recovered successfully. Next, in Section 8, we discuss the time-periodic pure soliton solutions under the reflectionless case. Moreover, in order to understand the soliton solutions, we select \( J = \mathcal{J} = 1, 2, 3 \) and 4 as example to show the graphics of the soliton solutions vividly. In this process, we find some interesting phenomenon such as breather-type solitons, arc solitons, three soliton and four soliton.
What’s more, the influence of parameter $\delta$ on the soliton solutions is considered. In Section 9, we consider some special cases of initial conditions and derive the eigenvalues and conserved quantities. Finally, the conclusions and the acknowledgement are given in the last two sections.

2. Lax representation and compatibility condition

We begin our discussion by considering the following scattering problem

$$\begin{cases}
\phi_t = L\phi, & L = -\zeta J + U, \\
\phi_t = M\phi, & M = \zeta^2 J - \zeta U + \frac{1}{2} V + \delta V_p,
\end{cases} \tag{2.1}$$

with

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad V = \begin{pmatrix} iq(x,t)r(x,t) & -iq_x(x,t) \\ -ir_{r_s}(x,t) & -iq(x,t)r(x,t) \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & q(x,t) \\ r(x,t) & 0 \end{pmatrix}, \quad V_p(x,t) = \begin{pmatrix} iA_p(x,t) & B_p(x,t) \\ -C_p(x,t) & -iA_p(x,t) \end{pmatrix} \tag{2.2}$$

$$A_p = -8\zeta^4 - 4r(x,t)q(x,t)\zeta^2 - 2ir(x,t)q(x,t)\zeta - 2iq(x,t)r(x,t)\zeta$$

$$- 3q^2(x,t)r^2(x,t) + q_s(x,t)r(x,t) + q(x,t)r(x,t) + r(x,t)q_{xx}(x,t),$$

$$B_p = 8q(x,t)\zeta^3 + 4iq_x(x,t)\zeta^2 - 2q_{xx}(x,t)\zeta + 4r(x,t)q^2(x,t) - iq_{xxx}(x,t)$$

$$+ 6iq(x,t)r(x,t)q_s(x,t),$$

$$C_p = -8r(x,t)\zeta^4 - 4ir_s(x,t)\zeta^2 + 2r_{xx}(x,t)\zeta - 4r^2(x,t)q(x,t)\zeta + ir_{xxx}(x,t)$$

$$- 6ir(x,t)r_s(x,t)q(x,t),$$

where $\Phi(x,t) = (\phi_1(x,t), \phi_2(x,t))^T$ is a two-component vector and the potential functions $q(x,t)$ and $r(x,t)$ are complex functions. Under the symmetry reduction $r(x,t) = \gamma q(-x,t)$ ($\gamma = \pm 1$), the zero curve equation $L_t - M_s + [L, M] = 0$ leads to the objective equation (1.1).

In what follows, we list a few important properties of the above equation:

- Time-reverse symmetry: if $q(x,t)$ is a solution, then $q^*(x, -t)$ is also a solution.
- Space-reverse symmetry: if $q(x,t)$ is a solution, then $q(-x, t)$ is also a solution.
- Gauge invariance: if $q(x,t)$ is a solution, then $e^{ip_0} q(x,t)$ is also a solution with real and constant $p_0$.
- Spatial translation invariance: if $q(x,t)$ is a solution, then $q(x + ix_0, t)$ is also a solution for any real and constant $x_0$.
• PT-symmetry: if \( q(x, t) \) is a solution, then \( q^*(-x, -t) \) is also a solution. It is noted that \( V(x, t) = \gamma q(x, t)q^*(-x, t) \) satisfy the special symmetry \( V(x, t) = V^*(-x, t) \), which is referred to as a self-induced potential in the classical optics.

3. Infinite number of conserved quantities and conservation laws

As we all know, a finite number of local conservation laws and global conservation laws of the nonlinear integrable equation is helpful to establish its integrability as an infinite dimensional Hamilton dynamic system. In this section, we will explain how to obtain the local and global conservation laws of Eq. (1.1).

3.1. Global conservation laws

The infinite number of conserved quantities of Eq. (1.1) is derived as follows. Before our operation, we suppose that the potential function \( q(x, t) \) decays rapidly at infinity. At the same time, the solutions of the scattering transform can be obtained by defining the four functions which satisfy the following boundary condition

\[
\begin{align*}
\lim_{x \to +\infty} \phi(x, t) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x}, \\
\lim_{x \to -\infty} \psi(x, t) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x}, \\
\lim_{x \to +\infty} \overline{\phi}(x, t) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\xi x}, \\
\lim_{x \to -\infty} \overline{\psi}(x, t) &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\xi x},
\end{align*}
\]

(3.1)

where \( \overline{\phi}(x, t) \) is not the complex conjugate of \( \phi(x, t) \), and \( \overline{\psi}(x, t) \) denotes the complex conjugate of \( \psi(x, t) \). If \( \phi(x, t) = (\phi_1(x, t), \phi_2(x, t))^T \) is the solution to Eq. (2.1) which satisfies the above boundary conditions, we can obtain that \( \phi_1(x, t)e^{i\xi x} \) is analytic for \( \text{Im}(\xi) \geq 0 \) and approaches to 1 as \( x \to \pm \infty \). Substituting \( \phi_1(x, t) = \exp(-i\xi x + \varphi(x, t)) \) into Eq. (2.1), we can find that the function \( \mu(x, t) = \varphi_1(x, t) \) satisfies the Riccati equation

\[
q \frac{\partial}{\partial x} \left( \frac{\mu}{q} \right) + \mu^2 - qr - 2i\xi \mu = 0.
\]

(3.2)

For \( \text{Im}(\xi) > 0 \), we have \( \lim_{|z| \to \infty} \varphi(x, \xi) = 0 \). Substituting the expansion \( \mu(x, \xi) = \sum_{n=0}^{\infty} \frac{\mu_n(x)}{(2i\xi)^n} \) into Eq. (3.2) and equating the powers of \( \xi \), we find

\[
\begin{align*}
\mu_0(x, t) &= -q(x, t)r(x, t) = -\gamma q(x, t)q^*(-x, t), \\
\mu_1(x, t) &= -q(x, t)r_1(x, t) = \gamma q(x, t)q^*_x(-x, t),
\end{align*}
\]

(3.3)

and

\[
\mu_{n+1} = q(x, t) \frac{\partial}{\partial x} \left( \frac{\mu_n}{q} \right) + \sum_{m=0}^{n-1} \mu_m \mu_{n-m-1}, \quad n \geq 1.
\]

(3.4)
3.2. Local conservation laws

From the boundary conditions it follows

$$\lim_{x \to -\infty} \phi_1(x, \zeta)e^{i\zeta x} = 1, \quad \lim_{x \to -\infty} \varphi(x, \zeta) = 0,$$

then we can obtain

$$\ln \alpha(\zeta) = \ln(\phi_1(x, t)e^{i\zeta x}) = \sum_{n=0}^{\infty} \frac{C_n}{(2i\zeta)^{n+1}}, \quad C_n = \int_{-\infty}^{+\infty} \mu_n(x, t)dx.$$  \hspace{1cm} (3.5)

Since $\phi_1(x, t)e^{i\zeta x}$ is time independent for $k$ with $\text{Im}\zeta > 0$, then the above $C_n$ is also time independent for $\zeta$ with $\text{Im}\zeta > 0$. According to Eqs. (3.3), (3.4) and (3.6), we can obtain all conserved quantities. More explicitly, the first few conserved quantities are listed as follows:

$$C_0 = -\gamma \int_{-\infty}^{+\infty} q(x, t)q^*(-x, t)dx, \quad C_1 = \gamma \int_{-\infty}^{+\infty} q(x, t)q^*_x(-x, t)dx,$$

$$C_2 = -\gamma \int_{-\infty}^{+\infty} \left( q(x, t)q^*_x(-x, t) - \gamma q^2(x, t)q^2(-x, t) \right)dx,$$

$$C_3 = \gamma \int_{-\infty}^{+\infty} \left( q(x, t)q^*(-x, t) + \gamma q(x, t)q_x(x, t)q^2(-x, t) - 4q^2(x, t)q^*(-x, t)q^*_x(-x, t) \right)dx,$$

$$C_4 = \gamma \int_{-\infty}^{+\infty} \left( -q(x, t)q^*_xxx(-x, t) + 5\gamma q^2(x, t)q^*_x(-x, t) + 6\gamma q^2(x, t)q^*(-x, t)q^*_xx(-x, t) \right)dx.$$  \hspace{1cm} (3.7)

3.2. Local conservation laws

In order to obtain the local conservation laws, we consider the time-dependent problem

$$\phi_{1t} = A\phi_1 + B\phi_2,$$  \hspace{1cm} (3.8)

where $A$, $B$ denote the $(1, 1)$– and $(1, 2)$–entry of $M$ in Eq. (2.1), respectively. According to the expression of $\mu$ and $\phi$, we find

$$\partial_t \mu(x, t) = \partial_x \left( A_{\text{nonloc}} + \frac{B_{\text{nonloc}}\mu(x, t)}{q(x, t)} \right),$$  \hspace{1cm} (3.9)

where

$$A_{\text{nonloc}} = -8i\delta \zeta^4 + i\zeta^2 (1 - 4\delta q q^*(-x, t)) + 2\delta \zeta (q, q^*(-x, t) - q^*_x(-x, t))$$

$$+ i \left\{ \frac{\gamma}{2} qq^*(-x, t) - 3\delta q^2 q^2(-x, t) - \delta q^*_x(-x, t)q_x(x, t) + \delta \gamma q q^*_x(-x, t) \right\},$$

$$B_{\text{nonloc}} = 8\delta q_x^3 + 4i\delta q^2 \zeta^2 + \left( 4\delta q q^2 q^*(-x, t) - q(x, t) - 2q_{xx} \right)\zeta$$

$$+ i \left\{ 6\gamma qq^*_x(-x, t) + \frac{1}{2} q_x - q_{xxx} \right\}.$$  \hspace{1cm} (3.10)
Substituting Eq. (3.10) and the expansion of \( \mu \) into Eq. (3.9), we have
\[
\partial_t \left( \sum_{n=0}^{\infty} \frac{\mu_n(x,t)}{(2 \lambda \zeta)^{n+1}} \right) = \partial_x \left( A_{\text{nonloc}} + B_{\text{nonloc}} \left[ \sum_{n=0}^{\infty} \frac{\mu_n(x,t)}{(2 \lambda \zeta)^{n+1}} \right] \right),
\]
from which we obtain
\[
\partial_t (\mu_n) = i \partial_x \left( \mu_n S_1 + \frac{1}{2} \mu_{n+1} S_2 - \delta \frac{q_s}{q} \mu_{n+2} + \delta \mu_{n+3} \right), \quad n = 0, 1, 2, 3, \ldots
\]  
where
\[
S_1 = 6 \gamma q_s q^* (-x, t) + \frac{q_s}{2q} - \frac{q_{sxx}}{q}, \quad S_2 = 1 + 2 \frac{q_{xx}}{q} - 4 \delta \gamma q^* (-x, t).
\]

We can write the conservation laws (3.12) as the form
\[
\frac{\partial T}{\partial t} = -i \frac{\partial X}{\partial x},
\]
where \( T = \mu_n \) and \( X = -\mu_n S_1 - \frac{1}{2} \mu_{n+1} S_2 + \delta \frac{q_s}{q} \mu_{n+2} - \delta \mu_{n+3} \) (\( n = 0, 1, 2, 3, \ldots \)) are the so-called densities and fluxes, respectively. The first three local conservation laws are
\[
T = \mu_0, \quad X = -S_1 \mu_0 - \frac{1}{2} S_2 \mu_1 + \delta \frac{q_s}{q} \mu_2 - \delta \mu_3,
\]
\[
T = \mu_1, \quad X = -S_1 \mu_1 - \frac{1}{2} S_2 \mu_2 + \delta \frac{q_s}{q} \mu_3 - \delta \mu_4,
\]
\[
T = \mu_2, \quad X = -S_1 \mu_2 - \frac{1}{2} S_2 \mu_3 + \delta \frac{q_s}{q} \mu_4 - \delta \mu_5,
\]
where
\[
\mu_0 = -\gamma q(x,t)q^*(-x,t), \quad \mu_1 = \gamma q(x,t)q^*_s(-x,t), \quad \mu_2 = -\gamma q(x,t)q^*_{xxs}(-x,t) + q^2(x,t)q^*(-x,t), \quad \mu_3 = \gamma q(x,t)q^*_{xxx}(-x,t) + q(x,t)q_s(x,t)q^*_{xxs}(-x,t) - 4q^2(x,t)q^*(-x,t)q^*_s(-x,t),
\]
\[
\mu_4 = -\gamma q(x,t)q^*_{xxxx}(-x,t) + q(x,t)q^2(-x,t)q^*_s(-x,t) - 6q(x,t)q_s(x,t)q^*_{xx}(-x,t)q^*_s(-x,t) + 5q^2(x,t)q^*_s(-x,t) + 6q^2(x,t)q^*(-x,t)q^*_s(-x,t) - 2\gamma q^3(x,t)q^3(-x,t), \quad \mu_5 = \gamma q^2 q^*_{xxxx}(-x,t) + q^2 q_{xxx} q^2(-x,t) - 6q^3 q^*(-x,t)q^*_s(-x,t) - 2\gamma q^3 q^*_s q^3(-x,t) - 2q^2 q^*(-x,t)q^*_{xxs}(-x,t) - 8q^2 q_{xx} q^*(-x,t)q^*_s(-x,t) + 12q^2 q^* q^*(-x,t)q^*_s(-x,t)
\]
\[
- 2q^2 q^* q^*(x,t)q^*_s(-x,t) - 16q^3 q^*_s(-x,t)q^*_s(-x,t) + 11q^2 q_s q^2(-x,t) - 4\gamma q^3 q q^3(-x,t) + 6\gamma q^4 q^*_s(-x,t)q^2(-x,t) + 10\gamma q^3 q^*_s q^2(-x,t). \tag{3.16}
\]
4. Direct scattering problem

In the following sections, we will consider the scattering problem of the system of Eq. (2.1). For the convenience of the discussion, we define the following Jost functions

\[
M(x, \zeta) = e^{i\zeta x} \phi(x, \zeta), \quad \overline{M}(x, \zeta) = e^{-i\zeta x} \overline{\phi}(x, \zeta),
\]

\[
N(x, \zeta) = e^{-i\zeta x} \psi(x, \zeta), \quad \overline{N}(x, \zeta) = e^{i\zeta x} \overline{\psi}(x, \zeta),
\]

which satisfy the constant boundary condition induced from Eq. (3.1). Furthermore, we can see that the above functions satisfy a linear integral equations and show that $M(x, \zeta), N(x, \zeta)$ are analytic in the upper half complex plane whereas $\overline{M}(x, \zeta), \overline{N}(x, \zeta)$ are analytic in the lower half complex plane $[30]$. Moreover, the large $\zeta$ behavior of the Jost functions are given by $[30]$.

\[
M(x, \zeta) = \left(1 - \frac{1}{2\zeta} \int_{-\infty}^{x} r(z) q(z) dz\right) + O(\zeta^{-2}),
\]

\[
N(x, \zeta) = \left(1 + \frac{1}{2\zeta} \int_{x}^{\infty} r(z) q(z) dz\right) + O(\zeta^{-2}),
\]

\[
\overline{M}(x, \zeta) = \left(1 + \frac{1}{2\zeta} \int_{-\infty}^{x} r(z) q(z) dz\right) + O(\zeta^{-2}),
\]

\[
\overline{N}(x, \zeta) = \left(1 - \frac{1}{2\zeta} \int_{x}^{\infty} r(z) q(z) dz\right) + O(\zeta^{-2}).
\]

From the boundary condition (3.1), it is seen that the solutions $\phi(x, \zeta)$ and $\overline{\phi}(x, \zeta)$ of the scattering problem (2.1) are linearly dependent. Similarly, the solutions $\psi(x, \zeta)$ and $\overline{\psi}(x, \zeta)$ of the scattering problem (2.1) are linearly dependent. Since the scattering problem (2.1) is a second order linearly ordinary differential equation, $\{\phi, \overline{\phi}\}$ and $\{\psi, \overline{\psi}\}$ are linearly dependent. Moreover, we can express the relation of them as follows

\[
\phi(x, \zeta) = a(\zeta) \overline{\psi}(x, \zeta) + b(\zeta) \psi(x, \zeta),
\]

\[
\overline{\phi}(x, \zeta) = \overline{a}(\zeta) \psi(x, \zeta) + \overline{b}(\zeta) \overline{\psi}(x, \zeta),
\]

where $a(\zeta), \overline{a}(\zeta), b(\zeta)$ and $\overline{b}(\zeta)$ are the scattering data, then we can have

\[
a(\zeta) = W(\phi(x, \zeta), \psi(x, \zeta)), \quad \overline{a}(\zeta) = W(\overline{\psi}(x, \zeta), \overline{\phi}(x, \zeta)),
\]

\[
b(\zeta) = W(\overline{\psi}(x, \zeta), \phi(x, \zeta)), \quad \overline{b}(\zeta) = W(\overline{\phi}(x, \zeta), \psi(x, \zeta)),
\]

where $W(u, v) = u_{1}v_{2} - u_{2}v_{1}$ represents the Wronskian determinant. Furthermore, it can be seen that $a(\zeta)$ and $\overline{a}(\zeta)$ are analytic in the upper half complex plane and the lower half complex plane, respectively, while $b(\zeta)$ and $\overline{b}(\zeta)$ cannot be extend off the real $\zeta$ axis. In addition, the scattering data satisfy the relation $a(\zeta) \overline{a}(\zeta) - b(\zeta) \overline{b}(\zeta) = 1$ for $\text{Im} \zeta = 0$. 
5. Symmetry reduction \( r(x,t) = \gamma q^*(x,t) \): eigenfunctions and the scattering data

5.1. Symmetry of the eigenfunctions

Next, we make efforts to establish the symmetry properties of the eigenfunctions under the symmetry reduction \( r(x,t) = \gamma q^*(x,t) \), \( \gamma = \pm 1 \). We suppose that \( v(x,\zeta) = (v_1(x,t), v_2(x,t))^T \) is the solution of Eq. (2.1), then \( (v_2^*(-x,-k^*), -\gamma v_1^*(-x,-k^*))^T \) is also the solution of Eq. (2.1). Since the solutions of the scattering problem are uniquely determined by the boundary condition (3.1), we have the following important symmetry

\[
\psi(x,\zeta) = \begin{pmatrix} 0 & -\gamma \\ 1 & 0 \end{pmatrix} \phi^*(-x,-k^*), \\
\bar{\psi}(x,\zeta) = \begin{pmatrix} 0 & 1 \\ -\gamma & 0 \end{pmatrix} \bar{\phi}^*(-x,-k^*).
\]

From Eq. (4.1), we obtain the symmetry of the Jost functions

\[
N(x,\zeta) = \begin{pmatrix} 0 & -\gamma \\ 1 & 0 \end{pmatrix} M^*(-x,-k^*),
\]
\[
\bar{N}(x,\zeta) = \begin{pmatrix} 0 & 1 \\ -\gamma & 0 \end{pmatrix} \bar{M}^*(-x,-k^*).
\]

5.2. Symmetry of the scattering data

According to the symmetry of the eigenfunctions (5.1) and the Wronskian representations of the scattering data (4.4), we get

\[
a(\zeta) = a^*(-\zeta^*), \quad \bar{a}(\zeta) = \bar{a}^*(-\zeta^*), \quad \bar{b}(\zeta) = \gamma b^*(-\zeta^*),
\]

which means that if \( \zeta_k = \xi_k + i\eta_k \) is a zero of \( a(\zeta) \) in the upper half complex plane, then \( -\zeta_k^* = -\xi_k + i\eta_k \) is also a zero of \( a(\zeta) \) in the upper half complex plane. Similarly, \( \zeta_k \) is a zero of \( \bar{a}(\zeta) \) in the lower half complex plane, then \( -\zeta_k^* \) is also a zero of \( \bar{a}(\zeta) \) in the lower half complex plane.

6. Inverse scattering problem: Left-Right RH approach

6.1. Left scattering problem

At first, we recall Eq. (4.3)

\[
\phi(x,\zeta) = a(\zeta)\bar{\psi}(x,\zeta) + b(\zeta)\psi(x,\zeta),
\]
\[
\bar{\phi}(x,\zeta) = \bar{a}(\zeta)\psi(x,\zeta) + \bar{b}(\zeta)\bar{\psi}(x,\zeta),
\]

(6.1)
the above system can be rewritten as the following matrix form

\[ \Phi(x, \zeta) = S_L \Psi(x, \zeta), \]  

(6.2)

where \( \Phi(x, \zeta) = (\phi(x, \zeta), \overline{\theta}(x, \zeta))^T \), \( \Psi(x, \zeta) = (\overline{\theta}(x, \zeta), \psi(x, \zeta))^T \) and \( S_L(\zeta) \) is the left scattering matrix

\[ S_L(\zeta) = \begin{pmatrix} a(\zeta) & b(\zeta) \\ \overline{b}(\zeta) & \overline{a}(\zeta) \end{pmatrix}. \]  

(6.3)

Following the results reported in [29], we can formulate the corresponding RH problem on the left and obtain the following linear integral equations which represent the functions \( N(x, \zeta) \) and \( \overline{N}(x, \zeta) \):

\[ N(x, \zeta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \rho(\xi)e^{-2i\xi x} \overline{N}(x, \xi) d\xi, \]

\[ \overline{N}(x, \zeta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \rho(\xi)e^{2i\xi x} N(x, \xi) d\xi, \]  

(6.4)

where \( \rho(\zeta) \) and \( \overline{\rho}(\zeta) \) are the left reflection coefficients defined by

\[ \rho(\zeta) = \frac{b(\zeta)}{a(\zeta)}, \quad \overline{\rho}(\zeta) = \frac{\overline{b}(\zeta)}{\overline{a}(\zeta)}, \]  

(6.5)

and \( C_l \) and \( \overline{C}_l \) are the left norming constants defined by

\[ C_l = \frac{b(\zeta)}{a(\zeta)}, \quad \overline{C}_l = \frac{\overline{b}(\zeta)}{\overline{a}(\zeta)}, \]  

(6.6)

where \( a'(\zeta_l) \) and \( \overline{a}'(\overline{\zeta}_l) \) denote the derivative at \( \zeta_l \) and \( \overline{\zeta}_l \), respectively.

**6.2. Time evolution of the scattering data: Left scattering problem**

According to Eq. (21), we derive the time evolution of the scattering data

\[ a(\zeta, t) = a(\zeta, 0), \quad b(\zeta, t) = e^{i6\delta t^2 - 2\zeta^2/\gamma} b(\zeta, 0), \]

\[ \overline{a}(\zeta, t) = \overline{a}(\zeta, 0), \quad \overline{b}(\zeta, t) = e^{(-i6\delta t^2 + 2\zeta^2/\gamma)} \overline{b}(\zeta, 0). \]  

(6.7)

In what follows, we obtain the time evolution of the left reflection coefficients \( \rho(\zeta) \) and \( \overline{\rho}(\zeta) \) and the left norming constants \( C_l \) and \( \overline{C}_l \) according to Eqs. (6.5) and (6.7)

\[ C_l = C_l(0)e^{(16\delta t^4 - 2\zeta^2)/\gamma}, \quad \rho(\zeta, t) = e^{(16\delta t^4 - 2\zeta^2)/\gamma} b(\zeta, 0)/a(\zeta, 0), \]

\[ \overline{C}_l = \overline{C}_l(0)e^{(-16\delta t^4 + 2\zeta^2)/\gamma}, \quad \overline{\rho}(\zeta, t) = e^{(-16\delta t^4 + 2\zeta^2)/\gamma} \overline{b}(\zeta, 0)/\overline{a}(\zeta, 0). \]  

(6.8)
6.3. Right scattering problem

Next, we consider the following system

\[ \psi(x, \zeta) = \alpha(\zeta) \overline{\phi}(x, \zeta) + \beta(\zeta) \phi(x, \zeta), \]
\[ \overline{\psi}(x, \zeta) = \alpha(\zeta) \phi(x, \zeta) + \overline{\beta}(\zeta) \overline{\phi}(x, \zeta), \]

(6.9)

where \( \alpha(\zeta), \overline{\alpha}(\zeta), \beta(\zeta) \) and \( \overline{\beta}(\zeta) \) are the right reflection data. Similarly, we can rewrite the above system as the matrix form

\[ \Psi(x, \zeta) = S_R \Phi(x, \zeta), \]

(6.10)

where \( \Psi(x, \zeta) = (\psi(x, \zeta), \overline{\psi}(x, \zeta))^T \), \( \Phi(x, \zeta) = (\phi(x, \zeta), \overline{\phi}(x, \zeta))^T \) and \( S_R \) is the right scattering matrix

\[ S_R = \begin{pmatrix} \alpha(\zeta) \\ \beta(\zeta) \end{pmatrix}, \]

(6.11)

We can formulate the corresponding RH problem on the right and obtain the following linear integral equations which govern the functions \( M(x, \zeta) \) and \( \overline{M}(x, \zeta) \):

\[ M(x, \zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{l=1}^{7} \frac{B_l \overline{M}(x, \overline{\zeta}_l)e^{2i\xi x}}{\xi - \overline{\zeta}_l} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{R(\xi)e^{2i\xi x}M(x, \xi)}{\xi - (\zeta + i0)} d\xi, \]
\[ \overline{M}(x, \zeta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{l=1}^{j} \frac{B_l M(x, \zeta_l)e^{-2i\xi x}}{\xi - \zeta_l} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\overline{R}(\xi)e^{-2i\xi x}M(x, \xi)}{\xi - (\zeta - i0)} d\xi, \]

(6.12)

where \( R(\zeta) \) and \( \overline{R}(\zeta) \) are the right reflection coefficients given by

\[ R(\zeta) = \frac{\beta(\zeta)}{\alpha(\zeta)}, \quad \overline{R}(\zeta) = \frac{\overline{\beta}(\zeta)}{\overline{\alpha}(\zeta)}, \]

(6.13)

and \( B_l \) and \( \overline{B}_l \) are the right norming constants defined by

\[ B_l = \frac{\beta(\zeta_l)}{\alpha(\zeta_l)}, \quad \overline{B}_l = \frac{\overline{\beta}(\zeta_l)}{\overline{\alpha}(\zeta_l)}. \]

(6.14)

6.4. Time evolution of the scattering data: Right scattering problem

Similar to the left case, we obtain the time evolution of the right scattering data

\[ \alpha(\zeta, t) = \alpha(\zeta, 0), \quad \beta(\zeta, t) = e^{i6\alpha\zeta^4 - 2i\xi^2 \zeta} \beta(\zeta, 0), \]
\[ \overline{\alpha}(\zeta, t) = \overline{\alpha}(\zeta, 0), \quad \overline{\beta}(\zeta, t) = e^{-i6\overline{\alpha} \zeta^4 + 2i\xi^2 \overline{\zeta}} \overline{\beta}(\zeta, 0). \]

(6.15)

According to Eqs. (6.13), (6.14) and (6.15), the time evolution of the right reflection coefficients and norming constants can be obtained by

\[ B_l = B_l(0)e^{i6\alpha\zeta^4 - 2i\xi^2 \zeta}, \quad R(\zeta, t) = e^{i6\alpha\zeta^4 - 2i\xi^2 \zeta} \beta(\zeta, 0)/\alpha(\zeta, 0), \]
\[ \overline{B}_l = \overline{B}_l(0)e^{-i6\overline{\alpha} \zeta^4 + 2i\xi^2 \overline{\zeta}}, \quad \overline{R}(\zeta, t) = e^{-i6\overline{\alpha} \zeta^4 + 2i\xi^2 \overline{\zeta}} \overline{\beta}(\zeta, 0)/\overline{\alpha}(\zeta, 0). \]

(6.16)
6.5. Relationship between the reflection coefficients

According to the matrix forms of the left and the right scattering problem, we have the relationship between the left and the right scattering matrix $S_R = S_L^{-1}$, more explicitly,

\begin{align}
a(\zeta) &= \alpha(\zeta), \quad \overline{a}(\zeta) = \overline{\alpha}(\zeta), \\
\overline{\beta}(\zeta) &= -b(\zeta), \quad \beta(\zeta) = -\overline{b}(\zeta).
\end{align}

Furthermore, we have

\begin{align}
R(\zeta) &= \frac{\beta(\zeta)}{\alpha(\zeta)} = -\frac{\overline{b}(\zeta)}{\overline{a}(\zeta)} = -\gamma \frac{b'(\zeta^*)}{a'(\zeta^*)} = -\gamma p^*(\zeta^*), \\
\overline{R}(\zeta) &= \frac{\overline{b}(\zeta)}{\overline{a}(\zeta)} = -\frac{b(\zeta)}{a(\zeta)} = -\gamma \frac{\overline{b}'(\zeta^*)}{a'(\zeta^*)} = -\gamma \overline{p}^*(\zeta^*).
\end{align}

6.6. Additional symmetry between the eigenfunctions

Suppose that $\zeta$ is the eigenvalue of $a(\zeta)$ in the upper complex plane, i.e., $a(k_i) = 0$, the eigenfunction $\phi(x, \zeta)$ and $\psi(x, \zeta)$ are linear dependent, $\phi(x, \zeta) = b(\zeta)\psi(x, \zeta)$. Moreover,

\begin{align}
M_1(x, \zeta) &= b(\zeta)N_1(x, \zeta)e^{ikx}, \\
M_2(x, \zeta) &= b(\zeta)N_2(x, \zeta)e^{ikx},
\end{align}

then

\begin{align}
M_1(x, \zeta_i)N_2(x, \zeta_i) &= M_2(x, \zeta)N_1(x, \zeta_i).
\end{align}

With the aid of Eq. (5.2), we obtain

\begin{align}
N_2^*(-x, \zeta_i)N_2(x, \zeta_i) &= N_1^*(-x, \zeta_i)N_1(x, \zeta_i).
\end{align}

Similarly, the other important conclusion is given as follows

\begin{align}
\overline{M}_2(-x, \zeta_i)\overline{M}_2(x, \zeta_i) &= \overline{M}_1(-x, \zeta_i)\overline{M}_1(x, \zeta_i).
\end{align}

7. Recovery of the potentials

Based on the above results, we can recover the potential functions $q(x, t)$ and $r(x, t)$ successfully. At first, recall from Eq. (6.4) that

\begin{align}
\overline{N}(x, \zeta) &= \left(1 \frac{1}{2\pi i} \int_{-\infty}^{\infty} \rho(\xi)e^{2i\xi x}N(x, \xi) \, d\xi.
\end{align}

The large $k$ behavior of $\overline{N}_2(x, \zeta)$ is determined by

\begin{align}
\overline{N}_2(x, \zeta) \sim \frac{1}{\zeta} \sum_{i=1}^{J} C_i N_2(x, \zeta_i)e^{2i\xi x} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \rho(\xi)e^{2i\xi x}N_2(x, \xi) \, d\xi.
\end{align}
According to Eq. (4.2), we obtain

\[ N_2(x, \zeta) \sim -\frac{r(x)}{2i\zeta}, \]  

(7.3)

thus, we can recover the potential \( r(x) \) by

\[ r(x) \sim -2i\left( \sum_{l=1}^{J} C_l N_2(x, \zeta_l) e^{2i\zeta_l x} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \rho(\xi) e^{2i\xi x} N_2(x, \xi) d\xi \right). \]  

(7.4)

With the asymptotic relation (4.2)

\[ M_1(x, \zeta) \sim q(x) e^{2i\zeta x}, \]  

(7.5)

and the symmetry relation \( M_1(x, \zeta) = -\gamma N_2(-x, -\zeta^*) \), we obtain the following asymptotic relation of \( q(x) \):

\[ q(x) \sim -2i\gamma \xi N_2^*(-x, -\zeta^*), \quad \gamma = \pm 1. \]  

(7.6)

From Eq. (7.2), we obtain

\[ q(x) = 2i\gamma \sum_{l=1}^{J} C_l^* N_2^*(-x, \zeta_l^*) e^{2i\zeta_l^* x} + \gamma \frac{2}{\pi} \int_{-\infty}^{\infty} \rho^*(\xi) e^{2i\xi x} N_2^*(-x, \xi) d\xi. \]  

(7.7)

According to Eqs. (7.4) and (7.7), it can be seen that the symmetry relation \( r(x) = \gamma q^*(-x) \) still holds.

8. Soliton solutions

In this section, we mainly discuss the pure soliton solutions of nonlocal integrable LPD equation. It is noted that pure soliton solutions arise when the reflection coefficients \( \rho(\xi) \) and \( \tilde{\rho}(\xi) \) vanish. Besides, it can be proved that these types of soliton solutions are only be obtained when \( \gamma = -1 \) (c.f. [29]). According to the special conditions mentioned above, the formula of pure soliton solutions is obtained by

\[ q(x) = -2i \sum_{l=1}^{J} C_l^* N_2^*(-x, \zeta_l^*) e^{2i\zeta_l^* x}. \]  

(8.1)

In order to facilitate the discussion of the properties of soliton solutions, it is necessary to obtain the explicit expression of some critical parameters

\[ C_j = \frac{b_j}{a_j} e^{(16i\delta\xi - 2i\zeta^*)}, \quad \overline{C}_j = \frac{\overline{b}_j}{a_j} e^{-(16i\delta\xi + 2i\zeta^*)}. \]  

(8.2)
where

\[ b_j = e^{\theta_j}, \quad \bar{a}\left(\xi\right) = \prod_{j=1}^{N} \left(\xi - \xi_j\right)^{N} \sum_{i=1}^{N} \frac{\left(\xi_{i} - \xi\right)}{\left(\xi - \xi_{i}\right)}(\zeta_{i} - \xi). \]  

(8.3)

\[ \bar{b}_j = e^{\bar{\theta}_j}, \quad \bar{a}\left(\xi\right) = \prod_{j=1}^{N} \left(\xi - \xi_j\right)^{N} \sum_{i=1}^{N} \frac{\left(\xi_{i} - \xi\right)}{\left(\xi - \xi_{i}\right)}(\zeta_{i} - \xi). \]

with \( \theta_j \) and \( \bar{\theta}_j \) are the amplitude of \( b_j \) and \( \bar{b}_j \), respectively [29].

Next, we will take some special parameters to give the explicit expression of soliton solutions and present them graphically with the aid of mathematic software, which is helpful for studying the properties of soliton solutions.

8.1. One soliton solutions

In this subsection, we discuss the one-soliton solutions of the nonlocal LPD equations by taking \( J = \bar{J} = 1 \) in Eqs. (8.1). Such solution corresponds to soliton eigenvalues

\[ \zeta_1 = \xi_1 + i\eta_1, \quad \eta_1 > 0, \quad \bar{\zeta}_1 = \bar{\xi}_1 + i\bar{\eta}_1, \quad \bar{\eta}_1 < 0. \]  

(8.4)

Taking \( J = 1 \) for Eq. (8.1), we obtain

\[ q(x) = -2iC_1 N_2(-x, \xi_1)e^{2\bar{\zeta}_1 x}, \]  

(8.5)

where \( C_1 \) and \( \bar{C}_1 \) are the norming constants (in \( x \)) whose time evolution is determined by

\[ C_1(t) = C_1(0)e^{i64\bar{\zeta}_1 e^{2\bar{\zeta}_1} e^{2\bar{\zeta}_1}}, \]

\[ \bar{C}_1(t) = \bar{C}_1(0)e^{-i64\bar{\zeta}_1 e^{2\bar{\zeta}_1} e^{2\bar{\zeta}_1}}, \]

(8.6)

and the expression of \( N_2(-x, \xi_1) \) can be obtained by setting \( J = 1 \) into Eq. (8.1)

\[ N_2(-x, \xi_1) = \frac{|\zeta_1 - \bar{\zeta}_1|^2}{|\zeta_1 - \bar{\zeta}_1|^2 - C_1^2 e^{2(i\zeta_1 + \xi_1)x}}, \]  

(8.7)

then we find the one soliton solution

\[ q(x, t) = -2iC_1(t)e^{2\bar{\zeta}_1 x}|\zeta_1 - \bar{\zeta}_1|^2 \]

\[ |\zeta_1 - \bar{\zeta}_1|^2 - C_1^2 e^{2(i\zeta_1 + \xi_1)x}. \]  

(8.8)

The localized structure, the density and the wave propagation of one soliton solution is shown in Fig. 1. From Fig. 1, we can learn that the single soliton wave propagate almost along the axis of \( x = 0 \). Moreover, in the process of wave propagate, the amplitude and the width of the single soliton are not changed.
Figure 1. One-soliton solution with parameters $\delta = 1, \theta_1 = \frac{\pi}{3}, \theta_1 = -\frac{\pi}{3}, \zeta_1 = 0.8i$ and $\zeta_1 = -0.8i$. (a): the structures of the one-soliton solution, (b): the density plot, (c): the wave propagation of the one-soliton solution.

8.2. Two soliton solutions

In this subsection, we consider the soliton solutions of the nonlocal LPD equations (1.1) with $J = J = 2$. Suppose the corresponding eigenvalues as follows

$$\zeta_1 = \xi_1 + i\eta_1, \quad \zeta_2 = \xi_2 + i\eta_2, \quad \eta_1, \eta_2 > 0,$$

$$\bar{\zeta}_1 = \bar{\xi}_1 + i\bar{\eta}_1, \quad \bar{\zeta}_2 = \bar{\xi}_2 + i\bar{\eta}_2, \quad \bar{\eta}_1, \bar{\eta}_2 < 0. \quad (8.9)$$

Setting $J = 2$ into Eq. (8.1), we find

$$q(x) = -2iC_1^* N_2^*(-x, \zeta_1)e^{2\zeta_1^* x} - 2iC_2^* N_2^*(-x, \zeta_2)e^{2\zeta_2^* x}, \quad (8.10)$$

where $C_j, \bar{C}_j, j = 1, 2$ are the norming constants whose time evolution is given by

$$C_1(t) = C_1(0)e^{(16i\delta \zeta_1^* - 2i\zeta_1^*) t}, \quad C_2(t) = C_2(0)e^{(16i\delta \zeta_2^* - 2i\zeta_2^*) t},$$

$$\bar{C}_1(t) = \bar{C}_1(0)e^{(-16i\delta \bar{\zeta}_1^* + 2i\bar{\zeta}_1^*) t}, \quad \bar{C}_2(t) = \bar{C}_2(0)e^{(-16i\delta \bar{\zeta}_2^* + 2i\bar{\zeta}_2^*) t}. \quad (8.11)$$

To obtain the functions $N_2^*(-x, \zeta_1)$ and $N_2^*(-x, \zeta_2)$, we need to solve the following system

$$\begin{align*}
\overline{M}_1(x, -\zeta_1) &= \alpha_1 N_2^*(-x, \zeta_1) + \beta_1 N_2^*(-x, \zeta_2), \\
\overline{M}_1(x, -\zeta_2) &= \alpha_2 N_2^*(-x, \zeta_1) + \beta_2 N_2^*(-x, \zeta_2), \\
N_2^*(-x, \zeta_1) &= 1 + \overline{M}_1(x, -\zeta_1) + \overline{\beta}_1 \overline{M}_1(x, -\zeta_2), \\
N_2^*(-x, \zeta_2) &= 1 + \overline{M}_1(x, -\zeta_1) + \overline{\beta}_2 \overline{M}_1(x, -\zeta_2). \quad (8.12)
\end{align*}$$
where
\begin{align*}
\alpha_1 &= C_1(t)e^{2i\xi_1 t}, & \beta_1 &= C_2(t)e^{2i\xi_2 t}, \\
\alpha_2 &= C_1(t)e^{2i\xi_2 t}, & \beta_2 &= C_2(t)e^{2i\xi_1 t}, \\
\bar{\alpha}_1 &= \frac{C_1(t)e^{-2i\xi_1 t}}{\xi_1 - \xi_1^*}, & \bar{\beta}_1 &= \frac{C_2(t)e^{-2i\xi_2 t}}{\xi_1 - \xi_2^*}, \\
\bar{\alpha}_2 &= \frac{C_1(t)e^{-2i\xi_2 t}}{\xi_2 - \xi_1^*}, & \bar{\beta}_2 &= \frac{C_2(t)e^{-2i\xi_1 t}}{\xi_2 - \xi_2^*}. \tag{8.13}
\end{align*}

Solving the above system, we get
\begin{align*}
N_2^*(x, \xi_1) &= \frac{\lambda_4 - \lambda_2}{\lambda_1, \lambda_4 - \lambda_2}, & N_2^*(x, \xi_2) &= \frac{\lambda_1 - \lambda_3}{\lambda_1, \lambda_4 - \lambda_2}, \tag{8.14}
\end{align*}

where
\begin{align*}
\begin{cases}
\lambda_1 &= 1 - \alpha_1 \bar{\alpha}_1 - \alpha_2 \bar{\beta}_1, \\
\lambda_2 &= -\bar{\alpha}_1 \beta_1 - \beta_2 \bar{\beta}_1, \\
\lambda_3 &= -\alpha_1 \bar{\beta}_2 - \alpha_2 \beta_2, \\
\lambda_4 &= 1 - \beta_2 \bar{\beta}_2 - \alpha_2 \beta_1. \tag{8.15}
\end{cases}
\end{align*}

Substituting the above equations into Eq. (8.10), we can obtain the formula of two-soliton solutions.
Two-soliton solutions with parameters $\theta_1 = \frac{2}{3}\pi$, $\theta_2 = \frac{3}{8}\pi$, $\overline{\theta}_1 = \frac{3}{8}\pi$, $\overline{\theta}_2 = \frac{1}{3}\pi$, $\zeta_1 = 0.7 + 0.5i$, $\zeta_2 = -0.7 + 0.5i$, $\overline{\zeta}_1 = 0.7 - 0.5i$ and $\overline{\zeta}_2 = -0.7 - 0.5i$. (a)(b)(c): the structures and the wave propagation of the two-soliton solutions with $\delta = 5$, (d)(e)(f): the structures and the wave propagation of the two-soliton solutions with $\delta = 3$, (g)(h)(i): the structures and the wave propagation of the two-soliton solutions with $\delta = 1$.

The local structure, the density and the wave propagation of two soliton solution is shown in Fig. 2. It is interesting that Fig. 2 shows the whole process of two solitons meet, collide elastically and move away. Furthermore, among these two solitons, one is a ordinary soliton and the other is a breather soliton. Besides, we also find a meaningful phenomenon by select different parameter $\delta$. Observe the three density plots carefully, we find that the angle between two solitons will increase as the parameter $\delta$ increases, which reveals the influence of parameter $\delta$ on the soliton solution graphically.

Breather-type solution with parameters $\delta = 1$, $\theta_1 = \frac{2}{3}\pi$, $\theta_2 = \frac{3}{8}\pi$, $\overline{\theta}_1 = \frac{3}{8}\pi$, $\overline{\theta}_2 = \frac{1}{3}\pi$, $\zeta_1 = 0.1i$, $\zeta_2 = 0.2i$, $\overline{\zeta}_1 = -0.1i$ and $\overline{\zeta}_2 = -0.2i$. (a): the structures of the breather-type solution, (b): the density plot, (c): the wave propagation of the breather-type solution.

By introducing the appropriate parameters, we get the other interesting discovery which is presented in Fig. 3. In Fig. 3, two different breather-type solitons spread alternately forward. Furthermore, the periodicity of the solution is clearly reflected.
8.3. Three soliton solutions

In this section, we consider the three-soliton solutions of the nonlocal LPD equations (1.1). Suppose the corresponding eigenvalues as follows

\[
\begin{align*}
\zeta_1 &= \xi_1 + i\eta_1, & \zeta_2 &= \xi_2 + i\eta_2, & \zeta_3 &= \xi_3 + i\eta_3, & \eta_1, \eta_2, \eta_3 &> 0, \\
\overline{\zeta}_1 &= \overline{\xi}_1 + \overline{\eta}_1, & \overline{\zeta}_2 &= \overline{\xi}_2 + \overline{\eta}_2, & \overline{\zeta}_3 &= \overline{\xi}_3 + \overline{\eta}_3, & \overline{\eta}_1, \overline{\eta}_2, \overline{\eta}_3 &< 0.
\end{align*}
\]

Setting \(J = \overline{J} = 3\) into Eq. (8.1), we find

\[
g(x) = -2iC_1^*N_2^*(-x, \zeta_1)e^{2i\zeta_1x} - 2iC_2^*N_2^*(-x, \zeta_2)e^{2i\zeta_2x} - 2iC_3^*N_2^*(-x, \zeta_3)e^{2i\zeta_3x},
\]

where \(C_j, \overline{C}_j, j = 1, 2, 3\) are the norming constants whose time evolution is given by

\[
\begin{align*}
C_1(t) &= C_1(0)e^{(16i\zeta_1^* - 2i\zeta_1^2)t}, & \overline{C}_1(t) &= \overline{C}_1(0)e^{(-16i\zeta_1^* + 2i\zeta_1^2)t}, \\
C_2(t) &= C_2(0)e^{(16i\zeta_2^* - 2i\zeta_2^2)t}, & \overline{C}_2(t) &= \overline{C}_2(0)e^{(-16i\zeta_2^* + 2i\zeta_2^2)t}, \\
C_3(t) &= C_3(0)e^{(16i\zeta_3^* - 2i\zeta_3^2)t}, & \overline{C}_3(t) &= \overline{C}_3(0)e^{(-16i\zeta_3^* + 2i\zeta_3^2)t}.
\end{align*}
\]

To obtain the functions \(N_1^*(-x, \zeta_1)\), \(N_2^*(-x, \zeta_2)\) and \(N_3^*(-x, \zeta_3)\), we need to solve the following system

\[
\begin{align*}
M_1(x, -\overline{\zeta}_1) &= \alpha_{11}N_2^*(-x, \zeta_1) + \alpha_{12}N_2^*(-x, \zeta_2) + \alpha_{13}N_2^*(-x, \zeta_3), \\
M_1(x, -\overline{\zeta}_2) &= \alpha_{21}N_2^*(-x, \zeta_1) + \alpha_{22}N_2^*(-x, \zeta_2) + \alpha_{23}N_2^*(-x, \zeta_3), \\
M_1(x, -\overline{\zeta}_3) &= \alpha_{31}N_2^*(-x, \zeta_1) + \alpha_{32}N_2^*(-x, \zeta_2) + \alpha_{33}N_2^*(-x, \zeta_3),
\end{align*}
\]

\[
\begin{align*}
N_1^*(-x, \zeta_1) &= 1 + \beta_{11}M_1(x, -\overline{\zeta}_1) + \beta_{12}M_1(x, -\overline{\zeta}_2) + \beta_{13}M_1(x, -\overline{\zeta}_3), \\
N_2^*(-x, \zeta_2) &= 1 + \beta_{21}M_1(x, -\overline{\zeta}_1) + \beta_{22}M_1(x, -\overline{\zeta}_2) + \beta_{23}M_1(x, -\overline{\zeta}_3), \\
N_3^*(-x, \zeta_3) &= 1 + \beta_{31}M_1(x, -\overline{\zeta}_1) + \beta_{32}M_1(x, -\overline{\zeta}_2) + \beta_{33}M_1(x, -\overline{\zeta}_3),
\end{align*}
\]

where

\[
\begin{align*}
\alpha_{11} &= \frac{C_1^*(t)e^{2i\zeta_1^*}}{\xi_1 - \xi_1^*}, & \alpha_{12} &= \frac{C_2^*(t)e^{2i\zeta_2^*}}{\xi_1 - \xi_2^*}, & \alpha_{13} &= \frac{C_3^*(t)e^{2i\zeta_3^*}}{\xi_1 - \xi_3^*}, \\
\alpha_{21} &= \frac{C_1^*(t)e^{2i\zeta_1^*}}{\xi_2 - \xi_1^*}, & \alpha_{22} &= \frac{C_2^*(t)e^{2i\zeta_2^*}}{\xi_2 - \xi_2^*}, & \alpha_{23} &= \frac{C_3^*(t)e^{2i\zeta_3^*}}{\xi_2 - \xi_3^*}, \\
\alpha_{31} &= \frac{C_1^*(t)e^{2i\zeta_1^*}}{\xi_3 - \xi_1^*}, & \alpha_{32} &= \frac{C_2^*(t)e^{2i\zeta_2^*}}{\xi_3 - \xi_2^*}, & \alpha_{33} &= \frac{C_3^*(t)e^{2i\zeta_3^*}}{\xi_3 - \xi_3^*}, \\
\beta_{11} &= \frac{C_1^*(t)e^{-2i\zeta_1^*}}{\overline{\xi}_1 - \overline{\xi}_1^*}, & \beta_{12} &= \frac{C_2^*(t)e^{-2i\zeta_2^*}}{\overline{\xi}_1 - \overline{\xi}_2^*}, & \beta_{13} &= \frac{C_3^*(t)e^{-2i\zeta_3^*}}{\overline{\xi}_1 - \overline{\xi}_3^*}, \\
\beta_{21} &= \frac{C_1^*(t)e^{-2i\zeta_1^*}}{\overline{\xi}_2 - \overline{\xi}_1^*}, & \beta_{22} &= \frac{C_2^*(t)e^{-2i\zeta_2^*}}{\overline{\xi}_2 - \overline{\xi}_2^*}, & \beta_{23} &= \frac{C_3^*(t)e^{-2i\zeta_3^*}}{\overline{\xi}_2 - \overline{\xi}_3^*}, \\
\beta_{31} &= \frac{C_1^*(t)e^{-2i\zeta_1^*}}{\overline{\xi}_3 - \overline{\xi}_1^*}, & \beta_{32} &= \frac{C_2^*(t)e^{-2i\zeta_2^*}}{\overline{\xi}_3 - \overline{\xi}_2^*}, & \beta_{33} &= \frac{C_3^*(t)e^{-2i\zeta_3^*}}{\overline{\xi}_3 - \overline{\xi}_3^*}.
\end{align*}
\]
Solving the above system, we get

\[ N_2^*(-x, \zeta_j) = \frac{\det(A_j)}{\det(A)}, \quad j = 1, 2, 3, \quad (8.21) \]

where

\[ A = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_4 & \lambda_5 & \lambda_6 \\ \lambda_7 & \lambda_8 & \lambda_9 \end{pmatrix}, \quad (8.22) \]

and

\[ A_1 = \begin{pmatrix} 1 & \lambda_2 & \lambda_3 \\ 1 & \lambda_5 & \lambda_6 \\ 1 & \lambda_8 & \lambda_9 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \lambda_1 & 1 & \lambda_3 \\ \lambda_4 & 1 & \lambda_6 \\ \lambda_7 & 1 & \lambda_9 \end{pmatrix}, \quad A_3 = \begin{pmatrix} \lambda_1 & \lambda_2 & 1 \\ \lambda_4 & \lambda_5 & 1 \\ \lambda_7 & \lambda_8 & 1 \end{pmatrix}, \quad (8.23) \]

with

\[
\begin{align*}
\lambda_1 &= 1 - \alpha_{11}\beta_{11} - \alpha_{21}\beta_{12} - \alpha_{31}\beta_{13}, \\
\lambda_2 &= -\alpha_{12}\beta_{11} - \alpha_{22}\beta_{12} - \alpha_{32}\beta_{13}, \\
\lambda_3 &= -\alpha_{13}\beta_{11} - \alpha_{23}\beta_{12} - \alpha_{33}\beta_{13}, \\
\lambda_4 &= -\alpha_{11}\beta_{21} - \alpha_{21}\beta_{22} - \alpha_{31}\beta_{23}, \\
\lambda_5 &= 1 - \alpha_{12}\beta_{21} - \alpha_{22}\beta_{22} - \alpha_{32}\beta_{23}, \\
\lambda_6 &= -\alpha_{13}\beta_{21} - \alpha_{23}\beta_{22} - \alpha_{33}\beta_{23}, \\
\lambda_7 &= -\alpha_{11}\beta_{31} - \alpha_{21}\beta_{32} - \alpha_{31}\beta_{33}, \\
\lambda_8 &= -\alpha_{12}\beta_{31} - \alpha_{22}\beta_{32} - \alpha_{32}\beta_{33}, \\
\lambda_9 &= 1 - \alpha_{13}\beta_{31} - \alpha_{23}\beta_{32} - \alpha_{33}\beta_{33}. 
\end{align*}
\]

(8.24)

Substituting the above equations into Eq. (8.17), we can obtain the formula of three-soliton solutions.
In Fig. 4, the local structure, the density and the wave propagation of three soliton solutions are shown vividly. Different from the previous three solitons, the three solitons here are composed of two arc solitons on both sides and one breathe-type soliton in the middle. The two arc solitons propagate forward along the left and right half of the circumference respectively, while the breathing solitons propagate forward along the diameter of the circumference, and three solitons meet, collide elastically, and move away at the central diameter of the circumference periodically. There is another obvious point that by change the value of $\delta$. The period of three soliton solutions have changed significantly. Specifically, the period will be shortened as the parameter $\delta$ increases which can be observed clearly form the graphics.

Different from Fig. 4, the following Fig. 5 shows that the local structure and the dynamic behavior of three ordinary solitons. The three solitons propagation along three different to the center($x = 0, t = 0$) and they meet at the center point. Then the three solitons collide elastically and move away along three different directions. During the whole process, the amplitude, energy of three solitons are not changed. It is also worth noting that the change of parameter $\delta$ has an influence on the rebound angle of two solitons on both sides.
Figure 5. Three-soliton solutions with parameters \( \theta_1 = \theta_2 = \theta_3 = \pi/3, \theta_1 = \theta_2 = \theta_3 = \pi/9, \) \( \zeta_1 = \zeta_2 = \zeta_3 = 0, \zeta_1 = -0.3i, \zeta_2 = -0.5i, \zeta_3 = -0.7i. \) \( \text{(a)(b)(c)}: \) the structures and the wave propagation of the three-soliton solutions with \( \delta = 3, \text{(d)(e)(f)}: \) the structures and the wave propagation of the three-soliton solutions with \( \delta = 2, \text{(g)(h)(i)}: \) the structures and the wave propagation of the three-soliton solutions with \( \delta = 1. \)

8.4. Four soliton solutions

In this section, we consider the four-soliton solutions of the nonlocal LPD equations \( \text{(L1).} \) Suppose the corresponding eigenvalues as follows

\[
\begin{align*}
\zeta_1 &= \xi_1 + i\eta_1, \quad \zeta_2 = \xi_2 + i\eta_2, \\
\zeta_3 &= \xi_3 + i\eta_3, \quad \zeta_4 = \xi_4 + i\eta_4, \\
\bar{\zeta}_1 &= \bar{\xi}_1 + i\bar{\eta}_1, \quad \bar{\zeta}_2 = \bar{\xi}_2 + i\bar{\eta}_2, \\
\bar{\zeta}_3 &= \bar{\xi}_3 + i\bar{\eta}_3, \quad \bar{\zeta}_4 = \bar{\xi}_4 + i\bar{\eta}_4, \\
\eta_1, \eta_2, \eta_3, \eta_4 &> 0, \\
\bar{\eta}_1, \bar{\eta}_2, \bar{\eta}_3, \bar{\eta}_4 &< 0.
\end{align*}
\]  

(8.25)
Setting $J = \tilde{J} = 4$ into Eq. (8.27), we find

$$q(x) = -2iC_1^*N_1^*(-x, \xi) e^{2\gamma_i^i\gamma_j} - 2iC_2^*N_2^*(-x, \xi) e^{2\gamma_i^i\gamma_j} - 2iC_3^*N_3^*(-x, \xi) e^{2\gamma_i^i\gamma_j} - 2iC_4^*N_4^*(-x, \xi) e^{2\gamma_i^i\gamma_j},$$  

(8.26)

where $C_j, \bar{C}_j, j = 1, 2, 3$ are the norming constants whose time evolution is given by

$$C_1(t) = C_1(0)e^{16\lambda_j^i\gamma_i^i\gamma_j^j}, \qquad \bar{C}_1(t) = \bar{C}_1(0)e^{16\lambda_j^i\gamma_i^i\gamma_j^j},$$

$$C_2(t) = C_2(0)e^{16\lambda_j^i\gamma_i^i\gamma_j^j}, \qquad \bar{C}_2(t) = \bar{C}_2(0)e^{16\lambda_j^i\gamma_i^i\gamma_j^j},$$

$$C_3(t) = C_3(0)e^{16\lambda_j^i\gamma_i^i\gamma_j^j}, \qquad \bar{C}_3(t) = \bar{C}_3(0)e^{16\lambda_j^i\gamma_i^i\gamma_j^j},$$

$$C_4(t) = C_4(0)e^{16\lambda_j^i\gamma_i^i\gamma_j^j}, \qquad \bar{C}_4(t) = \bar{C}_4(0)e^{16\lambda_j^i\gamma_i^i\gamma_j^j}.$$

(8.27)

To obtain the functions $N_1^*(-x, \xi_1)$, $N_2^*(-x, \xi_2)$ and $N_3^*(-x, \xi_3)$, we need to solve the following system

$$\begin{align*}
M_1(x, \bar{\xi}_1) &= \alpha_{11}N_1^*(-x, \xi_1) + \alpha_{12}N_2^*(-x, \xi_2) + \alpha_{13}N_3^*(-x, \xi_3) + \alpha_{14}N_4^*(-x, \xi_4), \\
M_1(x, \bar{\xi}_2) &= \alpha_{21}N_1^*(-x, \xi_1) + \alpha_{22}N_2^*(-x, \xi_2) + \alpha_{23}N_3^*(-x, \xi_3) + \alpha_{24}N_4^*(-x, \xi_4), \\
M_1(x, \bar{\xi}_3) &= \alpha_{31}N_1^*(-x, \xi_1) + \alpha_{32}N_2^*(-x, \xi_2) + \alpha_{33}N_3^*(-x, \xi_3) + \alpha_{34}N_4^*(-x, \xi_4), \\
M_1(x, \bar{\xi}_4) &= \alpha_{41}N_1^*(-x, \xi_1) + \alpha_{42}N_2^*(-x, \xi_2) + \alpha_{43}N_3^*(-x, \xi_3) + \alpha_{44}N_4^*(-x, \xi_4), \\
N_1^*(-x, \xi_1) &= 1 + \beta_{11}M_1(x, \bar{\xi}_1) + \beta_{12}M_1(x, \bar{\xi}_2) + \beta_{13}M_1(x, \bar{\xi}_3) + \beta_{14}M_1(x, \bar{\xi}_4), \\
N_2^*(-x, \xi_2) &= 1 + \beta_{21}M_1(x, \bar{\xi}_1) + \beta_{22}M_1(x, \bar{\xi}_2) + \beta_{23}M_1(x, \bar{\xi}_3) + \beta_{24}M_1(x, \bar{\xi}_4), \\
N_3^*(-x, \xi_3) &= 1 + \beta_{31}M_1(x, \bar{\xi}_1) + \beta_{32}M_1(x, \bar{\xi}_2) + \beta_{33}M_1(x, \bar{\xi}_3) + \beta_{34}M_1(x, \bar{\xi}_4), \\
N_4^*(-x, \xi_4) &= 1 + \beta_{41}M_1(x, \bar{\xi}_1) + \beta_{42}M_1(x, \bar{\xi}_2) + \beta_{43}M_1(x, \bar{\xi}_3) + \beta_{44}M_1(x, \bar{\xi}_4),
\end{align*}$$

(8.28)

where

$$\alpha_{ij} = \frac{C_j^*(t)e^{2\gamma_i^i\gamma_j^j}}{\xi_i - \xi_j}, \quad \beta_{ij} = \frac{\bar{C}_j(t)e^{-2\gamma_i^i\gamma_j^j}}{\xi_i - \xi_j}, \quad 1 \leq i, j \leq 4.$$  

(8.29)

Solving the above system, we get

$$N_j^*(-x, \xi_j) = \frac{\det(A_j)}{\det(A)}, \quad j = 1, 2, 3, 4,$$

(8.30)

where

$$A = \begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\
\lambda_5 & \lambda_6 & \lambda_7 & \lambda_8 \\
\lambda_9 & \lambda_{10} & \lambda_{11} & \lambda_{12} \\
\lambda_{13} & \lambda_{14} & \lambda_{15} & \lambda_{16}
\end{pmatrix}.$$  

(8.31)
and

\[
\begin{align*}
A_1 &= \begin{pmatrix}
1 & \lambda_2 & \lambda_3 & \lambda_4 \\
1 & \lambda_6 & \lambda_7 & \lambda_8 \\
1 & \lambda_{10} & \lambda_{11} & \lambda_{12} \\
1 & \lambda_{14} & \lambda_{15} & \lambda_{16}
\end{pmatrix}, & A_2 &= \begin{pmatrix}
\lambda_1 & 1 & \lambda_3 & \lambda_4 \\
\lambda_5 & 1 & \lambda_7 & \lambda_8 \\
\lambda_9 & 1 & \lambda_{11} & \lambda_{12} \\
\lambda_{13} & 1 & \lambda_{15} & \lambda_{16}
\end{pmatrix}, \\
A_3 &= \begin{pmatrix}
\alpha_1 & \lambda_2 & 1 & \lambda_4 \\
\lambda_5 & \lambda_6 & 1 & \lambda_8 \\
\lambda_9 & \lambda_{10} & 1 & \lambda_{12} \\
\lambda_{13} & \lambda_{14} & 1 & \lambda_{16}
\end{pmatrix}, & A_4 &= \begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3 & 1 \\
\lambda_5 & \lambda_6 & \lambda_7 & 1 \\
\lambda_9 & \lambda_{10} & \lambda_{11} & 1 \\
\lambda_{13} & \lambda_{14} & \lambda_{15} & 1
\end{pmatrix},
\end{align*}
\]  

(8.32)  (8.33)

with

\[
\begin{align*}
\lambda_1 &= 1 - \alpha_1 \beta_{11} - \alpha_2 \beta_{12} - \alpha_3 \beta_{13} - \alpha_4 \beta_{14}, & \lambda_2 &= -\alpha_1 \beta_{11} - \alpha_2 \beta_{12} - \alpha_3 \beta_{13} - \alpha_4 \beta_{14}, \\
\lambda_3 &= -\alpha_1 \beta_{11} - \alpha_2 \beta_{12} - \alpha_3 \beta_{13} - \alpha_4 \beta_{14}, & \lambda_4 &= -\alpha_1 \beta_{11} - \alpha_2 \beta_{12} - \alpha_3 \beta_{13} - \alpha_4 \beta_{14}, \\
\lambda_5 &= -\alpha_1 \beta_{21} - \alpha_2 \beta_{22} - \alpha_3 \beta_{23} - \alpha_4 \beta_{24}, & \lambda_6 &= 1 - \alpha_1 \beta_{21} - \alpha_2 \beta_{22} - \alpha_3 \beta_{23} - \alpha_4 \beta_{24}, \\
\lambda_7 &= -\alpha_1 \beta_{31} - \alpha_2 \beta_{32} - \alpha_3 \beta_{33} - \alpha_4 \beta_{34}, & \lambda_8 &= -\alpha_1 \beta_{31} - \alpha_2 \beta_{32} - \alpha_3 \beta_{33} - \alpha_4 \beta_{34}, \\
\lambda_9 &= -\alpha_1 \beta_{41} - \alpha_2 \beta_{42} - \alpha_3 \beta_{43} - \alpha_4 \beta_{44}, & \lambda_{10} &= -\alpha_1 \beta_{41} - \alpha_2 \beta_{42} - \alpha_3 \beta_{43} - \alpha_4 \beta_{44}, \\
\lambda_{11} &= 1 - \alpha_1 \beta_{31} - \alpha_2 \beta_{32} - \alpha_3 \beta_{33} - \alpha_4 \beta_{34}, & \lambda_{12} &= -\alpha_1 \beta_{31} - \alpha_2 \beta_{32} - \alpha_3 \beta_{33} - \alpha_4 \beta_{34}, \\
\lambda_{13} &= -\alpha_1 \beta_{41} - \alpha_2 \beta_{42} - \alpha_3 \beta_{43} - \alpha_4 \beta_{44}, & \lambda_{14} &= -\alpha_1 \beta_{41} - \alpha_2 \beta_{42} - \alpha_3 \beta_{43} - \alpha_4 \beta_{44}, \\
\lambda_{15} &= -\alpha_1 \beta_{41} - \alpha_2 \beta_{42} - \alpha_3 \beta_{43} - \alpha_4 \beta_{44}, & \lambda_{16} &= 1 - \alpha_1 \beta_{41} - \alpha_2 \beta_{42} - \alpha_3 \beta_{43} - \alpha_4 \beta_{44}.
\end{align*}
\]  

(8.34)

Substituting the above equations into Eq. (8.26), we can obtain the formula of four-soliton solutions.

![Figure 6](image)

\( \text{Figure 6.} \) Four-soliton solution with parameters \( \delta = 1, \theta_1 = \theta_2 = \theta_3 = \theta_4 = \frac{\pi}{8}, \overrightarrow{\theta}_1 = \overrightarrow{\theta}_2 = \overrightarrow{\theta}_3 = \overrightarrow{\theta}_4 = \frac{\pi}{8}, \zeta_1 = 0.1i, \zeta_2 = 0.2i, \zeta_3 = 0.3i, \zeta_4 = 0.4i, \overrightarrow{\zeta}_1 = -0.1i, \overrightarrow{\zeta}_2 = -0.2i, \overrightarrow{\zeta}_3 = -0.3i \) and \( \zeta_4 = -0.4i. \) (a): the structures of the four-soliton solution, (b): the density plot, (c): the wave propagation of the four-soliton solution.
Figure 7. Four-soliton solution with parameters $\delta = 1, \theta_1 = \theta_2 = \theta_3 = \theta_4 = \frac{\pi}{4}, \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0.1 + 0.2i, \zeta_1 = -0.1 + 0.2i, \zeta_3 = 0.3i, \zeta_4 = 0.4i, \xi_1 = 0.1 - 0.2i, \xi_2 = -0.1 - 0.2i, \xi_3 = -0.3i$ and $\xi_4 = -0.4i$. (a): the structures of the four-soliton solution, (b): the density plot, (c): the wave propagation of the four-soliton solution.

Figs. 6 and 7 present the local structure and the dynamic behavior of four soliton solutions. The four solitons include two arc-shaped solitons and two ordinary solitons. Four solitons meet, collide and move away at the center point ($x = 0, t = 0$). Moreover, before and after the collision, the properties of four solitons have no changed. In the following Fig. 8, by select special parameters, we obtain another form of four solitons which is center symmetry about the center point. It is worth noting that the energy of four solitons is exchanged according to symmetry relation. For example, the energy exchange between the leftmost soliton and the rightmost soliton and the energy exchange between the two solitons in the middle.

Figure 8. Four-soliton solution with parameters $\delta = 1, \theta_1 = \theta_2 = \theta_3 = \theta_4 = \frac{\pi}{4}, \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = 0.1 + 0.2i, \zeta_1 = -0.1 + 0.2i, \zeta_3 = 0.4 + 0.3i, \zeta_4 = -0.4 + 0.3i, \xi_1 = 0.1 - 0.2i, \xi_2 = 0.4 - 0.3i$ and $\xi_3 = -0.4 - 0.3i$. (a): the structures of the four-soliton solution, (b): the density plot, (c): the wave propagation of the four-soliton solution.
9. Eigenvlaues and conserved quantities under some special initial conditions

Before this section, we consider pure soliton solutions of the objective equation (1.1) under the condition \( \rho(\xi) = \overline{\rho}(\xi) = 0 \). However, as for more general initial condition \( q(x, 0) \) and \( r(x, 0) \), the reflectionless case may not hold, which makes the objective equation unsolvable by using IST method. Next, we will analysis the eigenvalues the conserved quantities under some special initial conditions.

9.1. Rectangular wave

In what follows, we study the following rectangular initial condition

\[
q(x, 0) = \begin{cases} 
0, & x \in (-\infty, 0), \\
h, & x \in (0, L), \\
0, & x \in (L, \infty),
\end{cases}
\]  

(9.1)

where \( h \) and \( L \) are real and positive constants. Under the symmetry relation \( r(x, 0) = -q^*(x, 0) \), we obtain the initial data of \( r(x, t) \)

\[
r(x, 0) = \begin{cases} 
0, & x \in (-\infty, -L), \\
h, & x \in (-L, 0), \\
0, & x \in (0, \infty).
\end{cases}
\]  

(9.2)

According to the \( t \)-independent scattering problem, we have

\[
\begin{align*}
\phi_{1,x} &= -i\zeta \phi_1 + q(x, t)\phi_2, \\
\phi_{2,x} &= i\zeta \phi_2 + r(x, t)\phi_1.
\end{align*}
\]  

(9.3)

Instituting the above initial condition into Eq. (9.3) and solving the ordinary differential equations, we have

\[
\begin{align*}
0 < x < L, & \quad \begin{pmatrix} \phi_1(x, \zeta) \\ \phi_2(x, \zeta) \end{pmatrix} = \begin{pmatrix} \frac{h}{2\zeta} c_1 e^{i\zeta x} + c_2 e^{-i\zeta x} \\ c_1 e^{i\zeta x} \end{pmatrix}, \\
-L < x < 0, & \quad \begin{pmatrix} \phi_1(x, \zeta) \\ \phi_2(x, \zeta) \end{pmatrix} = \begin{pmatrix} \tilde{c}_1 e^{-i\zeta x} \\ \tilde{c}_2 e^{i\zeta x} + \frac{h}{2\zeta} \tilde{c}_1 e^{-i\zeta x} \end{pmatrix}.
\end{align*}
\]  

(9.4)

In order to match the values of eigenfunctions at the critical point \( x = 0, x = -L \), we obtain

\[
\begin{align*}
\tilde{c}_1 &= 1, & c_1 &= \frac{h}{2i\zeta} (1 - e^{2i\zeta L}), \\
\tilde{c}_2 &= -\frac{h}{2i\zeta} e^{2i\zeta L}, & c_2 &= 1 + \left( \frac{h}{2i\zeta} \right)^2 (e^{2i\zeta L} - 1).
\end{align*}
\]  

(9.5)
At the same time, according to Eqs. (3.1) and (4.3), when \( x > L \), we have

\[
\phi(x, t) = \begin{pmatrix} a(\zeta)e^{-i\zeta x} \\ b(\zeta)e^{i\zeta x} \end{pmatrix}.
\] (9.6)

In the process of matching the value of eigenfunction at \( x = L \), we find

\[
a(\zeta) = 1 + \frac{h}{2i\zeta}(e^{2i\zeta L} - 1) - \left( \frac{h}{2i\zeta} \right)^2 (e^{4i\zeta L} - e^{2i\zeta L}),
\]

\[
b(\zeta) = -he^{i\zeta L} \frac{\sin(\zeta L)}{\zeta},
\] (9.7)

then the eigenvalues, i.e. the zeros of \( a(\zeta) \), can be given implicitly by

\[
e^{2i\zeta L} - 1 \pm \frac{2i\zeta}{h} = 0.
\] (9.8)

Besides, the asymptotic behavior of \( a(\zeta) \) for large and small \( \zeta \) can be derived from Eq. (9.7)

\[
a(\zeta) \sim 1 - \frac{h^2}{(2i\zeta)^2}, \quad \zeta \to \infty,
\]

\[
a(\zeta) \sim 1 - \frac{h^2 L^2}{\zeta}, \quad \zeta \to 0.
\] (9.9)

With the aid of the large \( \zeta \) asymptotic behavior of \( a(\zeta) \) and Eq. (3.6), we find that the conserved quantities satisfy

\[
C_{2n} = 0, \quad C_{2n+1} = -\frac{h^{2n+2}}{n+1}, \quad n = 0, 1, 2, \ldots.
\] (9.10)

9.2. Arcuated wave

In the second example, we consider the following arcuated initial condition

\[
q(x, 0) = \begin{cases} 
0, & x \in (-\infty, 0), \\
-x^2 + Lx, & x \in (0, L), \\
0, & x \in (L, \infty),
\end{cases}
\] (9.11)

where \( L \) is real and positive constant. Under the symmetry relation \( r(x, 0) = -q^*(-x, 0) \), we obtain the initial data of \( r(x, t) \)

\[
r(x, 0) = \begin{cases} 
0, & x \in (-\infty, -L), \\
x^2 + Lx, & x \in (-L, 0), \\
0, & x \in (0, \infty).
\end{cases}
\] (9.12)
Besides, the asymptotic behavior of

\[ \lim_{x \to \pm \infty} \phi(x, \zeta) = \begin{cases} \frac{1}{2i\zeta} \left( -x^2 + Lx + \frac{1}{(2i\zeta)^2} (2x - L^2) \right), & 0 < x < L, \\ \frac{1}{2i\zeta} \left( -x^2 + Lx + \frac{1}{(2i\zeta)^2} (2x + L^2) \right), & -L < x < 0. \end{cases} \]

And solving the ordinary differential equations, we have

\[
\begin{aligned}
0 < x < L, & \quad \phi_1(x, \zeta) = c_1 e^{-i\zeta x} + c_2 e^{i\zeta x} \\
- \: \: L < x < 0, & \quad \phi_2(x, \zeta) = \tilde{c}_1 e^{-i\zeta x} + \tilde{c}_2 e^{i\zeta x}.
\end{aligned}
\]

Matching the value of the eigenfunction at \( x = 0 \) and \(-L\), we find

\[
\begin{aligned}
\tilde{c}_1 &= 1, \\
\tilde{c}_2 &= e^{2i\zeta L} \left( \frac{2}{(2i\zeta)^3} - \frac{L}{(2i\zeta)^2} \right), \\
c_1 &= 1 + e^{2i\zeta L} \left( \frac{4}{(2i\zeta)^3} - \frac{L^2}{(2i\zeta)^4} \right), \\
c_2 &= e^{2i\zeta L} \left( \frac{2}{(2i\zeta)^3} - \frac{L}{(2i\zeta)^2} \right).
\end{aligned}
\]

Similarly to case 1, when \( x > L \), we have

\[
\phi(x, \zeta) = \begin{cases} \frac{a(\zeta)e^{-i\zeta x}}{b(\zeta)e^{i\zeta x}}, & 0 < x < L, \\ \frac{a(\zeta)e^{-i\zeta x}}{b(\zeta)e^{i\zeta x}}, & -L < x < 0. \end{cases}
\]

In the process of matching the value of eigenfunction at \( x = L \), we find

\[
\begin{aligned}
a(\zeta) &= 1 - \left[ e^{2i\zeta L} \left( \frac{2}{(2i\zeta)^3} - \frac{L}{(2i\zeta)^2} \right) - \left( \frac{2}{(2i\zeta)^3} + \frac{L}{(2i\zeta)^2} \right)^2 \right], \\
b(\zeta) &= e^{i\zeta L} \left( \frac{2}{(2i\zeta)^3} - \frac{L}{(2i\zeta)^2} \right) - e^{-i\zeta L} \left( \frac{2}{(2i\zeta)^3} + \frac{L}{(2i\zeta)^2} \right),
\end{aligned}
\]

then the eigenvalues, i.e. the zeros of \( a(\zeta) \), can be given implicitly by

\[
e^{2i\zeta L} \left( \frac{2}{(2i\zeta)^3} - \frac{L}{(2i\zeta)^2} \right) - \left( \frac{2}{(2i\zeta)^3} + \frac{L}{(2i\zeta)^2} \right) \pm 1 = 0.
\]

Besides, the asymptotic behavior of \( a(\zeta) \) for large and small \( \zeta \) can be derived from Eq. \( (9.7) \)

\[
\begin{aligned}
a(\zeta) &\sim 1 - \frac{L^2}{(2i\zeta)^4}, & \zeta \to \infty, \\
a(\zeta) &\sim 1 - \frac{L^2}{36}, & \zeta \to 0.
\end{aligned}
\]

With the aid of the large \( \zeta \) asymptotic behavior of \( a(\zeta) \) and Eq. \( (3.6) \), we find that the conserved quantities satisfy

\[
C_n = \begin{cases} \frac{L^{2m+1}}{m+1}, & n = 4m + 3, m = 0, 1, 2, \ldots, \\
0, & \text{else}.
\end{cases}
\]
9.3. Triangular wave

In the second example, we consider the following triangular initial condition

\[
q(x, 0) = \begin{cases} 
0, & x \in (-\infty, 0), \\
L - |x - L|, & x \in (0, 2L), \\
0, & x \in (2L, \infty),
\end{cases}
\]

(9.20)

where \(L\) is real and positive constant. Under the symmetry relation \(r(x, 0) = -q(-x, 0)\), we obtain the initial data of \(r(x, t)\)

\[
r(x, 0) = \begin{cases} 
0, & x \in (-\infty, -2L), \\
-L + |x + L|, & x \in (-2L, 0), \\
0, & x \in (0, \infty).
\end{cases}
\]

(9.21)

Instituting the above initial condition into the scattering problem (9.23) and solving the ordinary differential equations, we have

\[
\begin{align*}
-2L < x < -L, & \quad \begin{cases} 
\phi_1(x, \xi) = \left( \frac{c_1 e^{-i\xi x}}{2}, \frac{c_2 e^{i\xi x} + c_1 e^{-i\xi x}}{2} \right), \\
\phi_2(x, \xi) = \left( \frac{c_3 e^{-i\xi x}}{2}, \frac{c_4 e^{i\xi x} - c_3 e^{-i\xi x}}{2} \right), 
\end{cases} \\
-\phi_2(x, \xi), & \quad \begin{cases} 
\phi_1(x, \xi) = \left( \frac{c_5 e^{i\xi x}}{2}, \frac{c_6 e^{-i\xi x}}{2} \right), \\
\phi_2(x, \xi) = \left( \frac{c_7 e^{-i\xi x}}{2}, \frac{c_8 e^{i\xi x}}{2} \right), 
\end{cases} \\
0 < x < L, & \quad \begin{cases} 
\phi_1(x, \xi) = \left( \frac{c_9 e^{i\xi x}}{2}, \frac{c_{10} e^{-i\xi x}}{2} \right), \\
\phi_2(x, \xi) = \left( \frac{c_{11} e^{-i\xi x}}{2}, \frac{c_{12} e^{i\xi x}}{2} \right), 
\end{cases} \\
L < x < 2L, & \quad \begin{cases} 
\phi_1(x, \xi) = \left( \frac{c_{13} e^{i\xi x}}{2}, \frac{c_{14} e^{-i\xi x}}{2} \right), \\
\phi_2(x, \xi) = \left( \frac{c_{15} e^{-i\xi x}}{2}, \frac{c_{16} e^{i\xi x}}{2} \right), 
\end{cases}
\end{align*}
\]

(9.22)

Matching the value of the eigenfunction at \(x = -2L, -L, 0, L,\) we find

\[
c_1 = 1, \quad c_2 = -\frac{1}{(2\xi)^2}e^{4i\xi L}, \quad c_3 = 1, \quad c_4 = \frac{1}{(2\xi)^2}(2e^{2i\xi L} - e^{4i\xi L}), \]

\[
c_5 = -\frac{1}{(2\xi)^2}(e^{2i\xi L} - 1)^2, \quad c_6 = 1 - \frac{1}{(2\xi)^2}(e^{2i\xi L} - 1)^2, \]

\[
c_7 = -\frac{1}{(2\xi)^2}(e^{2i\xi L} - 1)^2, \quad c_8 = 1 - \frac{1}{(2\xi)^2}(e^{2i\xi L} - 1)^2 + \frac{2e^{2i\xi L}}{(2\xi)^4}(e^{2i\xi L} - 1)^2.
\]

(9.23)

Similarly, when \(x > 2L,\) we have

\[
\phi(x, t) = \begin{pmatrix} a(\xi) e^{-i\xi x} \\ b(\xi) e^{i\xi x} \end{pmatrix}.
\]

(9.24)

In the process of matching the value of eigenfunction at \(x = 2L,\) we find

\[
a(\xi) = 1 - \frac{1}{(2\xi)^4}(e^{2i\xi L} - 1)^4, \]

\[
b(\xi) = -e^{2i\xi} \frac{(\sin(\xi L))^2}{\xi^2}.
\]

(9.25)
then the eigenvalues, i.e. the zeros of $a(\zeta)$, can be given implicitly by

$$e^{2i\zeta L} - 1 \pm 2i\zeta = 0, \quad \text{or} \quad e^{2i\zeta L} - 1 \pm 2\zeta = 0. \quad (9.26)$$

However, since $a(\zeta) = a^*(-\zeta^*)$, the eigenvalues are determined uniquely by

$$e^{2i\zeta L} - 1 \pm 2i\zeta = 0. \quad (9.27)$$

Besides, the asymptotic behavior of $a(\zeta)$ for large and small $\zeta$ can be derived from Eq. (9.28)

$$a(\zeta) \sim 1 - \frac{1}{(2i\zeta)^4}, \quad \zeta \to \infty,$$

$$a(\zeta) \sim 1 - L^4, \quad \zeta \to 0. \quad (9.28)$$

With the aid of the large $\zeta$ asymptotic behavior of $a(\zeta)$ and Eq. (3.6), we find that the conserved quantities satisfy

$$C_n = \begin{cases} -\frac{1}{m + 1}, & n = 4m + 3, m = 0, 1, 2, \ldots, \\ 0, & \text{else}. \end{cases} \quad (9.29)$$

10. Conclusions

In this work, a detailed study of the inverse scattering transform for a new nonlocal LPD equation is carried out. Firstly, by an ingenious method, the local and global conservation laws of nonlocal LPD equation is obtained, which establish the integrability as an infinite dimensional Hamilton dynamic system. The direct scattering problem is constructed and some critical symmetries are obtained. Afterwards, with the aid of the novel Left-Right RH approach, the inverse scattering problem is established. Furthermore, the potential function is recovered successfully. By introducing the reflectionless case, the soliton solutions of the nonlocal LPD equation are given. In order to understand the dynamic behavior of soliton solutions more intuitively, we take $J = 7 = 1, 2, 3, 4$ and select some special parameters as examples to present some interesting phenomenon, such as breather-type solitons, arc solitons, three solitons, four solitons, etc. Meanwhile, we also discuss the influence of parameter $\delta$ on soliton solutions. Besides, under some special cases of initial condition such as rectangular wave, arc wave and triangular wave, we consider the zeros of the scattering data $a(\zeta)$ and the conserved quantities.

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