Extracting a model quark propagator’s spectral density

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We propose a practical procedure to extrapolate the space-like quark propagator onto the complex plane, which follows the Schlessinger Point Method and the spectral representation of the propagator. As a feasible example, we employ quark propagators for different flavors, obtained from the solutions of the corresponding Dyson-Schwinger equation. Thus, the analytical structure of the quark propagator is studied, capitalizing on the current-quark mass dependence of the observed features.

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I. INTRODUCTION

The strong-interaction part of the Standard Model, Quantum Chromodynamics (QCD), is characterised by two emergent phenomena: Dynamical Chiral Symmetry Breaking (DCSB) and confinement [1, 2]. DCSB is responsible for the vast majority of the mass of the visible universe and has a crucial impact on the observed hadron spectrum and properties; for example, it explains both the large mass of the proton and the unnaturally light mass of the pion. Confinement entails that colored states, such as QCD’s fundamental degrees of freedom (quarks and gluons), cannot appear in the spectrum. It also guarantees that condensates, typical order parameters of DCSB [3], are wholly contained within hadrons [4]. Thus, DCSB and confinement might be, in fact, intimately connected. Both phenomena can be potentially understood from QCD’s 2-point functions [5], namely, propagators. Studying the analytical properties of the propagators could shed some light [6, 7] on their confinement properties and the connection with DCSB and DSEs have been a cornerstone in handling such endeavours [8, 9]. Thus, focusing on the matter sector, we obtain the quark propagator (in the space-like axis) through its corresponding Dyson-Schwinger equation (DSE) [10–12], properly truncated and with interaction model inputs [13]. Subsequently, the Schlessinger Point Method (SPM) [14, 15] is employed to extrapolate the propagator into the complex plane, allowing us to study the corresponding spectral function and its analytic structure. The article is organized as follows: in Section II we write the DSE for the quark propagator, the truncation and model inputs. Section III describes the SPM and the analytic continuation procedure. It is worth mentioning that the algorithm is quite general and can be employed to study different inputs, such as lattice QCD propagators. Section IV shows the numerical results and Section V summarizes the obtained conclusions.

II. GAP EQUATION

The DSE for the quark propagator, gap equation, is the starting point for analyses of DCSB and confinement in the continuum, as well as the fundamental ingredient for hadron physics studies based upon Bethe-Salpeter or Faddeev equations [16, 17]. The gap equation, in Euclidean space, reads

\[ S_f^{-1}(p) = Z_2(i\gamma \cdot p + m_f^{\text{bm}}) + \Sigma_f(p), \]

\[ \Sigma_f(p) = \frac{4}{3} Z_1 \int_{dq}^\Lambda g^2 D_{\mu\nu}(p-q)\gamma_\mu S_f(q)\Gamma_f^\mu(p,q), \]

where \( f \) stands for a Poincaré invariant regularised integration, with \( \Lambda \) regularisation scale. The rest of the pieces are defined as usual: \( S_f \) is the \( f \)-flavor quark propagator, \( D_{\mu\nu} \) is the gluon propagator and \( \Gamma_f \) the fully-dressed quark-gluon vertex: \( Z_{1,2} \) are the quark-gluon vertex and quark wavefunction renormalisation constants, respectively; \( g \) is the Lagrangian coupling constant and \( m_f^{\text{bm}} \) is the bare-quark mass. The latter is related with the renormalisation point \( \Lambda \) dependent current-quark mass, \( m_f^{\Lambda} \), via Slavnov-Taylor identities [18, 19]. Each Green function involved obeys its own DSE, thus forming an infinite tower of coupled equations, which must be systematically truncated to extract the encoded physics [20, 21]. Regardless of the truncation, a general solution for the fully-dressed quark propagator can be expressed as follows

\[ S_f(p) = Z_f(p^2)(i\gamma \cdot p + M_f(p^2))^{-1}, \]

in analogy with its bare counterpart,

\[ S_f^{(0)}(p) = (i\gamma \cdot p + m_f^{\text{bm}})^{-1}. \]

Here \( Z(p^2) \) and \( M(p^2) \) are dressing functions; the latter is known as the mass function. In the Rainbow approximation [13, 22, 23], Eq. (1) is modified according to the replacement:

\[ g^2 Z_1 D_{\mu\nu}(p-q)\Gamma_f^\mu(p,q) \rightarrow Z_2^2 \tilde{D}_{\mu\nu}^f(p-q)\gamma_\nu. \]
By solving the gap equation, it is almost straightforward to obtain the quark propagator in the $p^2 > 0$ axis. The extension to the complex $p^2$-plane can be numerically challenging since, among other issues, one might encounter singularities [25–30]. Thus, we follow an analytic continuation scheme, based upon the spectral representation [31–35] of the quark propagator and the SPM, to extrapolate our numerical solutions onto the complex plane.

Firstly, it is convenient to re-express the quark propagator, in Eq. (2), in terms of its vector ($\sigma_V$) and scalar ($\sigma_S$) functions:

$$S_f(p) = -i\gamma \cdot p\sigma_V(p^2) + \sigma_S(p^2).$$  \hspace{1cm} (7)

Both $\sigma_{V,S}(p^2)$ are obtained on a large, discrete set of points ($p_i^2 > 0$, $i = 1, \cdots, N$). Following the SPM, we employ a continued fraction representation such that

$$\sigma(p^2) = \frac{\sigma(p_i^2)}{1 + \frac{\sigma(p_1^2) - \sigma(p_i^2)}{1 + \frac{\sigma(p_2^2) - \sigma(p_1^2)}{1 + \frac{\sigma(p_3^2) - \sigma(p_2^2)}{\ddots}}}}.$$  \hspace{1cm} (8)

interpolates (extrapolates) these functions (the labels $V,S$ are implicit). The coefficients $a_i$ are recursively obtained, ensuring that $\forall p_i^2$, the interpolated value of $\sigma(p_i^2)$ exactly reproduces the numerically obtained one, [14]. Therefore, as discussed in Ref. [34], the SPM gives almost exact reconstructions, of the continuous function, if the number of input points is adequate.

Next we change the space-like $p^2$ to a complex value, $p^2 \rightarrow \tilde{p}^2 = x + iy$, ($x,y \in \mathbb{R}$). An important feature of the SPM is its capability to identify singularities and branch points [36, 37]. However, it is seen that if $\tilde{p}^2$ remains real and positive in the training set, the SPM most likely will not introduce any singular structure besides scattered poles. This is due to the simple shapes of $\sigma_{V,S}$ on the space-like axis: finite, continuous and monotonically decreasing functions. Thus, it is found advantageous to deal with a $\sqrt{\tilde{p}^2}$-grid instead [36]. The training set is simply mapped as:

$$\{p_i^2, \sigma(p_i^2)\} \rightarrow \{\sqrt{\tilde{p}_i^2}, \sigma(\tilde{p}_i^2)\}.$$  \hspace{1cm} (9)

Clearly, this mapping introduces a branch point at $p^2 = 0$, and its corresponding branch cut along its negative real axis. Besides, other singular structures, in the form of complex conjugate poles (CCP), could also appear disperse on the complex plane [25, 26]. The closed contour, which leaves all the poles and branch cut outside, is sketched in Fig. 1. At this point, one can appeal to complex analysis theorems to represent $\sigma(p^2)$ in a convenient way. An arbitrary point inside the chosen contour can be written as a Cauchy integral. Moreover,

![Fig. 1](image-url)

**TABLE I.** One-loop evolved current quark masses and gluon interaction strength are given in Table I.

| Flavor | $m_q^2$ | $\omega_f$ | $D_f^2$ |
|--------|---------|------------|---------|
| $u/d$  | 0.095   | 0.500      | 1.060   |
| $s$    | 0.112   | 0.530      | 1.040   |

The term $G^f(s = k^2) = G^f_{IR}(s) + G^f_{UV}(s)$ is the effective coupling, expressed as [13, 24]:

$$G^f(s = k^2) = G^f_{IR}(s) + G^f_{UV}(s),$$  \hspace{1cm} (4)

$$G^f_{IR}(s) = \frac{8\pi^2D_f^2}{\omega_f^2} e^{-s/\omega_f^2},$$  \hspace{1cm} (5)

$$G^f_{UV}(s) = \frac{8\pi^2\gamma_m F(s)}{\ln[\tau + (1 + s/\Lambda_{QCD})^2]}.$$  \hspace{1cm} (6)

The term $G^f_{IR}(s)$ provides an infrared enhancement, which is controlled by the product $\omega_f D_f^2$. Conversely, $G^f_{UV}(s)$ is set to reproduce the one-loop renormalisation-group behavior of QCD in the gap equation. We have defined: $s F(s) = (1 - \exp[-s/(4m_i^4)])$, $\gamma_m = 12/(33 - 2N_f)$, $\tau = e^{10} - 1$, $m_i = 0.5$ GeV, $\Lambda_{QCD} = 0.234$ GeV and $N_f = 5$. Key features of the gluon propagator, such as the infrared saturation and the connection with perturbation theory, are conveniently captured by the model defined in Eqs. (4) - (6). Thus, we solve Eq. (1) for different current-quark masses, whose specific values and interaction strength are given in Table I.
from the residue theorem [38, 39], the quark propagator can be usefully re-expressed as follows:

\[ \sigma(p^2) = \frac{1}{\pi} \int_0^\infty \frac{\rho(\omega)}{p^2 + \omega} d\omega + \sum_i \left( \frac{R_i}{p^2 - q_i} + \frac{R_i^*}{p^2 - q_i^*} \right), \]

\[ \rho(\omega) = \text{Im}[\sigma(-\omega - i\epsilon)], \]

where \( \epsilon \) is a positive infinitesimal real number; \( R_i \) and \( q_i \) take complex values, which are respectively identified as the residues and poles. The integral part is a standard dispersion relation for the propagator, such that \( \rho(\omega) \) corresponds to its spectral function; the fraction part accounts for the possible presence of complex conjugate poles.

In principle, \( \rho(\omega) \) can be computed straightforwardly from the SPM on the numerical data. However, there are many components that can impact the stability of the extrapolations. We have identified key factors which alter the outcome: the number and distributions of points and, potentially, the mapping of Eq. (9). Thus, in order to get a more accurate and stable result, we proceed as follows:

- Redistribute the big set of \( N_p = m \cdot n \) points into \( n \) subsets of \( m \) points.
- Select one point from each subset, randomly, to form a new small set (of \( n \) points).
- Implement SPM on the latter and obtain \( \sigma(p^2) \).

This strategy makes it easy to control the number of points, while also covering most of the domain of the initial, much bigger, set. The SPM is performed several times, keeping the mean value as the final result and ensuring a small error is produced. Besides, the particular values of \( m \) and \( n \) are properly fixed by the requirement that the produced spectral functions are similar in shape and the location of the poles is stable. The latter is discussed below.

Another important issue is the fact that \( \rho(\omega) \) comes from the integral along the branch cut, but the branch cut might seem artificial, in the sense that it could appear only due to the choice of a \( \sqrt{p^2} \)-grid. This is a standard procedure [36], however. The presence of the branch cut serves a crucial purpose: it allows to write the propagator as in Eq. (10), enabling the access to its pole structure. An effective method to calculate the positions and residues of the poles has been already introduced in Refs. [40, 41], but it is not practical enough to reconstruct the propagator. For instance, we have seen that a single pair of CCP is sufficient to accurately determine the quark propagator. This implies that Eq. (10) can be conveniently reduced to

\[ \sigma(p^2) = \frac{1}{\pi} \int_0^\infty \frac{\rho(\omega)}{p^2 + \omega} d\omega \approx \left( \frac{R}{p^2 - q} + \frac{R^*}{p^2 - q^*} \right) \]

such that, with \( \rho(\omega) \) already determined, \( q \) and \( R \) (poles and residues) can be straightforwardly obtained following standard minimization procedures.

### IV. NUMERICAL RESULTS

From the solutions of the gap equation, Eq. (1), for different quark flavors \( (u/d, s, c, b) \), one gets the spacelike quark propagators. Then the corresponding spectral functions, poles and residues, are identified following the algorithm described in the previous section. The SPM is performed 50 times for each quark flavor; the outputs are averaged to produce a final result and error estimates. The particular values of \( m \) and \( n \), for each case, are specified in Table II.

| Flavor | Vector | Scalar |
|--------|--------|--------|
| u/d    | (42, 24) | (16, 60) |
| s      | (10, 24) | (16, 64) |
| c      | (28, 36) | (32, 8)  |
| b      | (6, 38)  | (42, 24) |

The spectral densities are displayed in Figs. 2 and 5; the latter shows the results separately, for each quark flavor, and also includes the quark propagator dressing functions. As it is clear from the figures, the spectral densities are not positive definite. Moreover, for both scalar and vector parts, spectral densities consistently exhibit peaks (a minimum). The position of the minimum moves towards \( \omega \to \infty \) as the current quark mass increases. For the scalar part, the absolute value of this minimum increases with the quark mass, while the opposite pattern is observed for the vector part. A natural, kindred feature is also observed with the quark propagator dressing functions: at infrared momenta, the vector part of the \( u \) quark is much larger than its scalar counterpart; this is completely reversed for masses above \( m_c \simeq m_c \). The position of the poles (with the associated uncertainty) as a function of the quark’s mass is shown in Fig. 3. Their central values and the corresponding residues are captured in Table III. Clearly, the poles move deeper into the complex plain as the current mass grows, i.e. it takes larger absolute values of both real and imaginary parts. This feature is consistent with realistic DSE studies [42, 43].

The non-positivity of the spectral functions (in particular \( \rho_c \)) is often related to confinement [6, 8]. In this connection, we can also study the so-called spaced averaged (SA) Schwinger function [3, 44], which, at \( \vec{p} = 0 \):

\[ \Delta(\tau) := \int d^3 x \int \frac{dp_4}{(2\pi)^4} e^{i(tp_4 + \vec{x} \cdot \vec{p})} \]

\[ = \frac{1}{\pi} \int_0^\infty dp_4 \cos(tp_4) \sigma(p_4^2), \]

where \( \sigma \) is any of the scalar functions of the propagator. The SA Schwinger function for a real, \( m \) massive, scalar particle will decay exponentially, \( \Delta(\tau) \sim e^{-m\tau} \), since the propagator is simply \( \sigma(p^2) = 1/(p^2 + m^2) \) and the mass shell can be reached in the real axis. On the
FIG. 2. [Upper panel] Spectral density associated with the scalar part of the propagator, \( \sigma_s(p^2) \). [Lower panel] The analogous for the vector part. The dark-gray and light-gray shaded areas represent the \( \sigma \) and 2\( \sigma \) confidence intervals, respectively. Consistently, the observed peaks are shifted towards \( \omega \to \infty \) as the quark mass increases.

FIG. 3. Position of the poles (\( q \)) in the complex plane, with extrapolation uncertainty. From left to right: \( u \), \( s \), \( c \) and \( b \) quark propagators.

FIG. 4. [Upper panel] Space averaged Schwinger functions of the scalar part of the propagators: \( u \), \( s \) and \( c \) quarks. The presence of the peaks, which moves towards \( \tau \to 0 \) as the quark mass increases, reveals a change of sign in \( \Delta_s(\tau) \). The first peak of the \( b \) quark propagator lies around \( \tau \sim 1 \), but the Schwinger function becomes numerically unstable for larger values of \( \tau \).

V. CONCLUSIONS AND SUMMARY

Starting with space-like quark propagators, we described a viable procedure to get access to their complex structure. The DSE inputs have been employed merely as an illustration and the proposed method is quite general. This is based upon the SPM, which interpolates the space-like quark propagator and extrapolates it into the complex plane. Then, a proper choice of integration contour, allows us to rewrite the propagator in terms of
TABLE III. Poles (q) and residues (R) for different quark flavors. The mass units are expressed in appropriate powers of GeV.

| Flavor | q          | R [σ+] | R [σ−] |
|--------|------------|--------|--------|
| u/d    | −0.302 ± 0.364i | 0.586 ± 0.542i | −0.013 ± 0.480i |
| s      | −0.646 ± 0.660i | 0.702 ± 0.311i | 0.060 ± 0.719i  |
| c      | −2.325 ± 1.145i | 0.577 ± 0.712i | 1.098 ± 0.157i  |
| b      | −32.942 ± 4.260i | 0.674 ± 0.498i | 5.110 ± 3.287i  |

...a Cauchy integral, Eq. (10); thus defining the propagator’s dressing functions in terms of spectral densities and granting us access to its pole structure. Remarkably, it is seen that a single pair of CCPs is sufficient to accurately represent the quark propagator. Among other things, this could expedite the computation of the form factors, which typically require two pairs of CCPs [43, 45–47] to give a precise result. It is observed that the spectral densities are not positive definite and present peaks, which are shifted towards the ultraviolet region as the mass gets larger. Similarly, the position of the poles moves further into the complex plane with increasing current quark mass. The Schwinger functions exhibit the corresponding analogous features. Such consistency is encouraging. An immediate goal is to study whether or not the features observed are still valid for truncations beyond the Rainbow approximation. This work is underway. Together with the practicability of our approach and the reduced error estimate, we believe that the algorithm discussed herein is also suitable to study other Green functions. Moreover, the SPM has been proven useful in connecting Euclidean and Minkowskian quantities [48–50], a key goal in modern hadron physics [51, 52]. The present approach to obtain the analytical representation of the quark propagator would take potential roles in the calculation of the hadron spectrum [26, 53], especially for the meson excited states [28–30].

VI. ACKNOWLEDGEMENTS

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FIG. 5. The spectral and dressing functions. [Left Panel] Scalar part, $\rho_s(\omega)$. [Middle Panel] Vector part, $\rho_v(\omega)$. [Right Panel] Quark propagator dressing functions: $\sigma_s(p^2)$, $\sigma_v(p^2)$. From top to bottom: u, s, c and b quarks. The depicted spectral functions are the mean result after 20 times SPM and analytic continuation. The dark-gray and light-gray bands represent $\sigma$ and 2$\sigma$ confidence intervals, respectively.