1. INTRODUCTION

Lattice QCD provides us with a nonperturbative definition of the Strong Interactions. But so far we do not have a lattice definition for the complete Standard Model, which is a chiral gauge theory. Although the Electro-Weak sector is weakly coupled, there are many important questions that cannot be resolved by continuum techniques, which are limited to perturbation theory or to perturbative expansions around classical field configurations such as instantons or sphalerons. To mention a few examples, we need a better understanding of the Electro-Weak phase transition (see ref. [1] for a recent review) and of the origin of the net baryon number in the observed universe. Another deep question has to do with the fact that, due to triviality, the Higgs sector of the standard model is likely to be only a low energy effective lagrangian that originates from a more fundamental theory.

The goal of constructing chiral lattice gauge theories (χLGTs for short) has not been achieved yet. The basic obstacle is the doubling problem [2–4]. In its simplest form, a naive lattice discretization of the continuum lagrangian of a single Weyl fermion leads to sixteen Weyl fermions in the continuum limit. The latter combine into eight Dirac fermions, thus rendering the continuum limit vector-like. A general discussion of the conditions for species doubling was first given by Karsten and Smit [3], who also investigated the relation between the doublers and the anomaly. A precise mathematical statement of these conditions was given by Nielsen and Ninomiya [4].

Very briefly, every massless fermion is identified with a two by two subhamiltonian

\[ H_{2 \times 2}(\vec{p}) = \pm \vec{\sigma} \cdot (\vec{p} - \vec{p}_c) + O((\vec{p} - \vec{p}_c)^2), \tag{1} \]

where \( \vec{p}_c \) is the location of the zero in the Brillouin zone, and \( \pm \) defines the chirality. Mild locality is needed to guarantee continuous first derivatives for \( H(\vec{p}) \). The relevant mathematical theorems then imply the existence of an equal number of left-handed and right-handed fermions in every complex representation of the conserved charges.

From a physicist’s point of view, it is not clear why the No-Go theorems should be insurmountable. In Lattice QCD, the quarks belong to a complex representation of the non-singlet flavour symmetries. If we use Wilson fermions, the No-Go theorems are evaded because all the axial symmetries are broken by the Wilson term. As a result, exactly conserved axial charges do not exist on the lattice. However, this non-conservation is just enough to reproduce the axial anomaly [3].
while all the non-singlet axial symmetries are expected to be restored in the continuum limit. (I return to this point in Sect. 3.)

In the case of chiral gauge theories too, one could try to circumvent the No-Go theorems by invoking a fermion action where gauge invariance is broken by operators that vanish in the classical continuum limit. This approach looks very natural from a perturbative point of view, and it is the basic idea behind the Smit-Swift model. For example, one can show that the consistent anomaly is correctly reproduced in lattice perturbation theory. However, extensive study of the Smit-Swift model at the nonperturbative level has shown that the quantum continuum limit never leads to a chiral gauge theory. (For a review see ref. [8].) In the symmetric phase(s), the continuum limit is always a vector-like theory. This includes realizations where there are no light fermions at all but only a pure glue theory. In the broken (Higgs) phase the doublers (typically referred to as “mirror fermions” in this context) can acquire larger masses, but they do not decouple and remain in the physical spectrum.

There is a simple and important lesson that should be learned from the Smit-Swift model. (Other models that exhibit a similar behaviour will be discussed later.) When the fermion action is not exactly gauge invariant, the longitudinal component of the lattice gauge field couples to the fermions. This is true even if the perturbative spectrum is anomaly free. The longitudinal component, which can also be thought of as a frozen radius Higgs field, is a strongly fluctuating variable. The strong fluctuations can, and usually do, change the fermion spectrum, thus rendering the perturbative analysis highly unreliable.

This review consists of two parts. In Sect. 2, I present a unifying framework that clarifies the physical reasons for the robustness of the No-Go theorems. That framework allows us to understand why the dynamics of the longitudinal component always gives rise to a vector-like spectrum. I begin with the observation that, even if the fermion action is not gauge invariant, gauge invariance is restored by the integration over the lattice gauge orbit. This kinematical observation, which plays a crucial role, is valid as long as one uses the standard lattice gauge field’s measure. Now, invoking asymptotic freedom, the fermion spectrum of any lattice gauge theory can be determined by setting $g_0 = 0$. Switching off the gauge coupling freezes the transversal degrees of freedom, but not the longitudinal ones. This leads to a fermion-Higgs model with an exact global symmetry that corresponds to the original gauge group.

The quantum continuum limit of the fermion-Higgs model (or ψHM for short) is a free fermion theory. The spectrum in a given complex representation can be read off from the zeros of the inverse two point function $\tilde{\Gamma}(p_\mu)$. At $p_0 = 0$, $\tilde{\Gamma}(p_\vec{p})$ serves as our effective Hamiltonian. Under extremely mild assumptions on the locality of the action, the effective Hamiltonian $\tilde{\Gamma}(\vec{p})$ will satisfy all the conditions of the No-Go theorem, with one important exception. Namely, the No-Go theorem can be evaded by zeros in the propagator, or poles in $\tilde{\Gamma}(\vec{p})$. One can show that poles in the bilinear part of the action are really ghost states that render the theory inconsistent. However, the analysis of ref. [12,14] is not directly applicable to $\tilde{\Gamma}(p_\mu)$, which is an effective action. It is very important to settle the question of whether or not this is a real loophole in the No-Go arguments.

The second part of this review consists of three sections, each of which deals with a specific approach to the problem of defining $\chi$LGTs. In Sect. 3, I discuss Kaplan’s domain wall fermions focusing on two implementations: the waveguide model, and overlap formula of Narayanan and Neuberger. In particular, I address the question of how to determine the spectrum of the overlap model nonperturbatively. I also explain how Kaplan fermions might provide us with a better tool for studying the chiral properties of QCD.

Sect. 4 deals with the “gauge fixing” approach originally introduced by the Roma group. This approach, which was followed by the Zaragoza group is so far limited to the context of lattice perturbation theory. I present various considerations which are relevant for a nonperturbative implementation of the gauge fixing ap-
proach. The main conclusion is that one should use a global algorithm, that selects only relatively smooth gauge field configurations \[24\]. I also discuss the relevance of recent work that reveals a proliferation of Gribov copies on the lattice \[21\].

In Sect. 5, I discuss the attempt to give a non-perturbative definition of chiral gauge theories by putting the gauge fields on the lattice, while keeping the fermions in the continuum. Interest in this approach arose following a recent paper by \textquoteleft \textquoteleft t Hooft \[22\]. The bridge between the lattice and the continuum is provided by a continuum interpolation of the lattice gauge field \[23\]. The lattice interpolation is a multiply connected topological space. This feature leads to a quasi-local topological structure in the interpolating field which, in turn, is the source of certain difficulties. A short conclusions section ends this review.

2. THE NO-GO ARGUMENTS

In the continuum, whether a given gauge theory is chiral or not depends on the fermion fields used to define the lagrangian. Thanks to asymptotic freedom, there has to be a high energy scaling region where the lagrangian fields are the relevant degrees of freedom, and physical processes are well described by weak coupling perturbation theory. This justifies the identification of the “elementary particles” of the theory with the lagrangian fields. If we now switch off the gauge coupling we obtain, in the chiral case, a theory of free fermions with definite handedness in a complex representation of the gauge group. The latter, in turn, is reduced from a local to a global symmetry.

On the lattice, there need not be any simple relation between the lagrangian fields and the elementary fermions of the theory. (This point will be clarified below.) But at energies which are well below the lattice cutoff, the above picture based on continuum physics should hold. We will therefore adopt the same criterion as in the continuum to determine whether a given lattice gauge theory is chiral or not. Namely, we will set \( g_0 = 0 \) and ask whether this operation has reduced the continuum limit to a free theory of chiral fermions.

Before we proceed, let me mention two caveats. First, the above criterion ignores the possibility of composite gauge bosons. There are some indications that this could happen in two dimensions \[24\]. However, the compositeness scenario seems not to work in four dimensions \[6\]. Also, while achieving the desired chiral spectrum at \( g_0 = 0 \) would certainly be an important progress, we should be aware that new problems might arise when the dynamical gauge field is turned back on.

Setting \( g_0 = 0 \) amounts to imposing the constraint

\[
\text{Re } \text{tr} \left( I - U_G \right) = 0
\]

(2)
on every plaquette. Notice that this constraint does not imply \( U_{x,\mu} = I \) for all links. But eq. (2) does imply that \( U_{x,\mu} \) is a gauge transform of the identity. Any such pure gauge configuration can be written as

\[
U_{x,\mu} = V_x V_{x+\bar{\mu}}
\]

(3)
Both \( U_{x,\mu} \) and \( V_x \) take values in some compact Lie group \( G \). \( V_x \) can be identified with the longitudinal component of the gauge field. It can also be viewed a frozen radius Higgs field. Below, I will often use the notation \( I(V) \) to denote pure gauge configurations.

Thus, setting \( g_0 = 0 \) reduces the full gauge field’s dynamics to that of the trivial orbit, which is the collection of all pure gauge configurations. If the lattice action is gauge invariant, the \( V_x \) field decouples and we are allowed to set \( U_{x,\mu} = I \) without loosing anything. But proposals for \( \chi \)LGTs often invoke a fermion action which is not gauge invariant. In this case, setting \( g_0 = 0 \) leaves behind a \( \psi \)HM (fermion-Higgs model) where the \( V_x \) field interacts with the fermions.

A crucial feature is that the group \( G \) always turns out to be an exact global symmetry of the \( \psi \)HM. The reader may wonder how this could happen, since we started with a fermion action which is not gauge invariant. The answer is that because of the local gauge invariance of the lattice gauge field’s measure, we can rewrite a lattice gauge theory with an arbitrary action as a lattice gauge theory with a gauge invariant action, at the price of introducing the \( V_x \) field explicitly. Admittedly, \( V_x \) is a constrained scalar field, but
constrained scalar fields are perfectly legitimate on the lattice.

In order to see the emergence of the global symmetry $G$ let us arrive at the $\psi$HM in a somewhat different way \[11\]. Consider a lattice gauge theory defined by the partition function
\[
Z = \int DU D\psi D\bar{\psi} e^{-S(U,\psi,\bar{\psi})},
\]
\[
S(U,\psi,\bar{\psi}) = S_G(U) + S_F(U,\psi,\bar{\psi}).
\]
In writing the measure I have used the shorthand $\mathcal{D}U = \prod_{x,\mu} dU_{x,\mu}$ etc. For simplicity I will assume that $S_G(U)$ is the standard plaquette action. I make no assumptions on the fermion action $S_F$ which may or may not be gauge invariant. Apart from gauge interactions, $S_F$ may contain additional multi-fermion or Yukawa interactions. (However, additional scalar fields are suppressed since their presence changes nothing in the following arguments.) Consider now the effect of a gauge transformation parametrized by $V_x$. The lattice gauge field transforms according to
\[
U_{x,\mu} \rightarrow U_{x,\mu}^{(V)} = V_x U_{x,\mu} V_x^{\dagger}. \tag{6}
\]
A prototype gauge variant fermion action is
\[
S_F = S_K(U) + S_W, \tag{7}
\]
\[
S_K = \sum_x \bar{\psi} (\partial_P R + \mathcal{D}(U) P_L) \psi,
\]
and $S_W$ is the (free) Wilson term. The gauge variance of $S_F$ means that there is no obvious choice for the gauge transformation of the fermion variables. For example, we may decide to apply the transformation only to the left-handed part of the fermion field, i.e.
\[
\psi_{Lx} \rightarrow \psi_{Lx}^{(V)} = V_x \psi_{Lx}, \tag{8}
\]
\[
\psi_{Rx} \rightarrow \psi_{Rx}^{(V)} = \psi_{Rx},
\]
and similarly for $\bar{\psi}_{Lx}$ and $\bar{\psi}_{Rx}$. This choice leaves $S_K$ (but not $S_W$) invariant.

A change of variables in the partition function now leads to
\[
Z = \int DU D\psi D\bar{\psi} e^{-S(U,\psi,\bar{\psi})}, \tag{9}
\]
\[
= \int DU D\psi D\bar{\psi} e^{-S(U,\psi,\bar{\psi})} \tag{10}
\]
The equality of eq. (9) and eq. (10) follows from the invariance of the lattice measure under gauge transformations, and in going from eq. (10) to eq. (11) one averages over all gauge transformations ($DV = \prod_x dV_x$).

We now observe that the new action, given explicitly by
\[
S_G(U) + S_F(U^{(V)},\psi^{(V)},\bar{\psi}^{(V)}), \tag{12}
\]
is gauge invariant! Under a gauge transformation parametrized by $g_x$, the original gauge and fermion fields transforms as before with $g_x$ replacing $V_x$, e.g. $U_{x,\mu} \rightarrow U_{x,\mu}^{(g)}$, whereas the new field $V_x$ transforms according to $V_x \rightarrow V_x^{(g)} = V_x g_x^\dagger$.

The above trick is so simple that the result might look suspicious. One may worry that the gauge invariance of the new action is “fictitious”, if originally $S_F$ was not gauge invariant. I believe that this is not the case. The transition from eq. (9) to eq. (11) shows that in a lattice gauge theory, a local separation of the longitudinal and transversal degrees of freedom is always possible, and the transversal degrees of freedom always couple to a conserved current.

Since the new action is gauge invariant, switching off the gauge coupling is the same as setting $U_{x,\mu} = I$ for all links in eq. (11). We thus arrive at the $\psi$HM defined by the partition function
\[
Z' = \int DV D\psi D\bar{\psi} e^{-S'(V,\psi,\bar{\psi})}, \tag{13}
\]
\[
S'(V,\psi,\bar{\psi}) = S_F(I^{(V)},\psi^{(V)},\bar{\psi}^{(V)}). \tag{14}
\]
Notice that $S_G$ drops out because it is $V$-independent. For $S_F$ of eq. (5), the resulting $\psi$HM is the Smit-Swift model.

What happens if we choose a different transformation law for the fermion variables? For example, instead of eq. (9), we may decide to leave all the fermion variables inert. This would lead to a different action $S'' = S_F(I^{(V)},\psi,\bar{\psi})$ for the $\psi$HM. However, $S''$ reduces to $S'$ if we make the field redefinition $\psi_L \rightarrow \psi_L^* = V \psi_L$ (and similarly for $\psi_L$). Since the field redefinition is unitary, the two partition functions with actions $S'$ and $S''$ define the same $\psi$HM.

As promised, the gauge invariance of $Z$ when expressed in terms of the additional $V_x$ field,
translates into an *exact global symmetry* $G$ of $Z'$. If $G$ is spontaneously broken, we arrive at a *mirror fermion* model [3]. In this review I will be interested in genuine $\chi$LGTs only, so I will restrict my attention to *symmetric* phases.

Typically, there will be one or two symmetric phases. Let us focus on one of them, and ask whether the fermion spectrum is chiral or vector-like with respect to the global symmetry $G$. Considerable simplification occurs because the quantum continuum limit is a theory of *free massless fermions*. Here I am assuming the absence of exactly massless scalars. While a chiral (massless) fermion spectrum should be stable against infinitesimal modifications of the action, the existence of a massless scalar in a symmetric phase of the $\psi$HM is always *accidental*. Allowing for small modifications of the action, I can therefore assume the absence of exactly massless scalars without loss of generality.

Consider now a specific complex representation $\mathcal{C} = \mathcal{C}(G)$. Let $\chi_i = \chi_i(V, \psi, \bar{\psi})$ be a set of local fermion operators that belong to $\mathcal{C}(G)$, which create all the massless fermions in $\mathcal{C}(G)$ (if there are any). The $\chi_i$-s should be chosen such that every $\chi_i$ creates at least one massless fermionic state when acting on the vacuum. Such an economic set will always contain a finite number of $\chi_i$-s. Because of the freedom in making field redefinitions that involve the $V_\mu$ field, there need not be one-to-one correspondence between the lagrangian fields and the asymptotic states of the $\psi$HM. By taking tensor products of $V$ times an odd number of fermion fields, we can build operators that belong to practically every representation of $G$. The question of which of these operators create massless fermions is clearly a *dynamical* one.

Given the $\chi$-s (from now on I suppress all indices) we calculate the inverse two point function defined by

$$\Gamma^{-1} = \langle \chi(x) \chi^\dagger(y) \rangle .$$

Let us now consider the Fourier transform $\tilde{\Gamma}(\vec{p}_\mu)$ at $p_0 = 0$. The crucial observation is that $\tilde{\Gamma}(\vec{p})$ can serve as an effective hamiltonian, to which the considerations of the No-Go theorem can be applied [10]. A sufficient condition for hermiticity of $\tilde{\Gamma}(\vec{p})$ is the existence of a spectral representa-

tion. Even more generally one can argue that, in a consistent theory, the massless fermion spectrum must be determined by the zeros of the hermitian part of $\tilde{\Gamma}(\vec{p})$, denoted $H_{\text{eff}}(\vec{p})$ from now on. If $\tilde{\Gamma}(\vec{p})$ had an anti-hermitian part that does not vanish at a zero of $H_{\text{eff}}(\vec{p})$, this would mean that a zero energy fermion has a finite probability to decay, which is clearly a pathological situation.

At this stage we have a hermitian matrix $H_{\text{eff}}(\vec{p})$ which is a function of the lattice momentum, whose zeros are in one-to-one correspondence with the massless fermions that belong to $\mathcal{C}(G)$. The No-Go theorem will apply, and the spectrum will be vector-like, if $H_{\text{eff}}(\vec{p})$ has continuous first derivatives.

In order to determine the analytic structure of $H_{\text{eff}}(\vec{p})$ we have to consider three physically distinct regions [11]. The first region is the vicinity of zeros of $H_{\text{eff}}(\vec{p})$. The second (possible) region is the vicinity of *poles* of $H_{\text{eff}}(\vec{p})$, or zeros of the propagator. The third region covers the rest of the Brillouin zone. For the moment let me assume that there are no poles in $H_{\text{eff}}(\vec{p})$. Under extremely mild assumptions on the locality of the action, one expects that $H_{\text{eff}}(\vec{p})$ will have continuous first derivatives away from its zeros. If the action contains only short range couplings, $H_{\text{eff}}(\vec{p})$ should be an *analytic* function of $\vec{p}$ away from the zeros. Thus, what remains is to establish the continuity of the first derivatives at the zeros.

As explained earlier, one can assume the absence of exactly massless scalars. As a result, the quantum continuum limit in a symmetric phase of the $\psi$HM is a theory of free massless fermions. At energies which are small compared to the mass of the *lightest massive excitation*, the effective low energy lagrangian can contain only non-renormalizable interactions, the first possible non-linear term being a four-fermion interaction. Now, a logarithmic term in $H_{\text{eff}}(\vec{p})$ can arise only from diagrams that contain at least two vertices. Non-renormalizable interactions always have dimensionful coupling constants, and when these coupling constants occur in front of a logarithmic term in $H_{\text{eff}}(\vec{p})$, they will be multiplied by at least two powers of $\vec{p}^2$. This ensures the continuity of the first derivatives of $H_{\text{eff}}(\vec{p})$ at the zeros.
It remains to consider the possibility of poles in \( H_{\text{eff}}(\vec{p}) \). Poles in \( H_{\text{eff}}(\vec{p}) \) are, tentatively, the most important loophole in the No-Go arguments. Their existence cannot be easily ruled out by locality arguments, because these arguments usually apply to the \textit{propagator}, and not to the inverse propagator. A pole in the inverse propagator means a zero in the propagator, and the latter is compatible with analyticity.

The danger is that poles in \( H_{\text{eff}}(\vec{p}) \) actually describe \textit{massless ghost states} (bosonic spinors) that couple to the gauge field, leading to an inconsistent theory. At an intuitive level, this can be understood by considering a bilinear fermion action with a pole at \( \vec{p} = \vec{p}_c \). The contribution to the fermion determinant from momentum eigenstates in the vicinity of \( \vec{p}_c \) can be reproduced by a bosonic field with a first order action.

A non-local two component lattice action, in which the fifteen extra zeros (doublers) were traded with poles, was proposed some ten years ago by Rebbi \cite{25}. A calculation of the vacuum polarization reveals the inconsistency of this proposal \cite{13}. The coefficient of the logarithmic term turns out to be \((-14)\), the right answer. The explanation is that \(-14 = 1 - 15\), i.e. the fifteen poles \textit{do} contribute to the logarithmic term, and with the \textit{wrong} sign. Moreover, Pelissetto \cite{14} showed that this phenomenon is completely general, and it occurs whenever the extra zeros are traded with poles, regardless of the detailed form of the action.

The crucial ingredient of the analysis of ref. \cite{13,14} is the Ward identity relating the gauge field’s vertex to the inverse propagator. In addition, the analysis relies on standard diagrammatic rules for calculating the vacuum polarization in terms of the vertex and the propagator. The calculation proceeds by showing that the poles in the inverse propagator reappear in the vertex, and make an important contribution in the limit of small external momentum.

The difference between ref. \cite{13,14} and our general setup is that the inverse propagator \( \tilde{\Gamma}(p_n) \) is an \textit{effective action}. Consequently, a simple diagrammatic relation between \( \tilde{\Gamma}(p_n) \), the vertex function and the vacuum polarization is lacking. However, as the following argument suggests, the difference might be only a technical one. The \( \psi \)-HM allows us to calculate \( n \)-current correlators like the vacuum polarization. Let us take the continuous time limit and analytically continue to Minkowski space. The full vacuum polarization can now be reconstructed from its discontinuities. The discontinuity is a phase space integral of the product of matrix elements of the source current between physical states of the \( \psi \)-HM. These matrix elements are still related to the inverse propagator via the Ward identity. The pole in the inverse propagator may therefore still reappear in the vacuum polarization via the matrix elements of the source current, leading to an inconsistency as before.

In conclusion, the No-Go arguments as presented in this section are incomplete. Nevertheless, the failure of a remarkably diverse range of proposals for \( \chi \)LGTs can be understood in the present framework. Apart from the Smit-Swift model, I refer to the Eichten-Preskill \cite{26}, the waveguide \cite{10} and the staggered fermion \cite{27} models. In all cases, the analytic structure complies with the above general considerations, and there is no evidence for zeros in the propagator. This is particularly clear when weak or strong coupling expansions are available. See ref. \cite{28} for the Eichten-Preskill model, and ref. \cite{14} for the waveguide model.

In closing this chapter let me mention some recent activity in what is historically the oldest approach to \( \chi \)LGTs. Namely, where the action is sufficiently non-local to prevent the existence of continuous first derivatives.

Slavnov \cite{29} recently considered the effect of additional Pauli-Villars fields in a lattice action based on SLAC fermions. He argues that, at least in perturbation theory, a suitable choice of Pauli-Villars fields allows one to eliminate the diseases pointed out by Karsten and Smit \cite{30}. Another proposal based on SLAC fermions whose details have not been worked out yet can be found in ref. \cite{31}. Finally, an attempt to avoid the doubling precisely by poles in the inverse propagator was recently made by Bietenholz and Wiese \cite{32}. Their method of “integrating out of the continuum” does not have a straightforward diagrammatic interpretation. It is still an open question
whether or not this method leads to a consistent expression for the vacuum polarization.

3. DOMAIN WALL FERMIONS

I now turn to Kaplan’s domain wall fermions \[13\]. I will begin by describing the basic idea in the context of a vector-like model \[13\]. The reason is that the vector-like case poses no conceptual problems. Moreover, the use of Kaplan fermions for lattice QCD minimizes the breaking of axial symmetries by lattice artifacts \[14–37\] (see also ref. \[17\]). Two proposals for \(\chi\)LGTs based on domain wall fermions, the waveguide model \[16\] and the overlap formula of Narayanan and Neuberger \[17\], will be described next. In this section I quote only the central results and give very few technical details. I tried to compensate by giving a rather extensive list of references. For a review on domain wall fermions see ref. \[38\].

The basic problem with Wilson fermions is the hard breaking of axial symmetries. The Wilson term has vanishingly small matrix elements between low energy \textit{free} fermion states. But the matrix elements of the QCD hamiltonian between left-handed and right-handed quark states are \(O(1)\). As a result, fine tuning is required in order to obtain the correct continuum limit, both at the level of the action and in the definition of properly renormalized operators.

Kaplan introduces a five dimensional fermion action that contains a Wilson term, and a mass term which depends explicitly on the fifth coordinate \(s\). The mass function \(M(s)\) describes four dimensional \textit{defects}, and these defects support massless four dimensional chiral fermions. Kaplan originally used domain wall defects. In a more economic setup the massless chiral fermions emerge as surface states on the boundaries of a five dimensional slab \[14–37\].

In order to minimize inessential technical details I will stick to domain wall defects below. In a QCD setup, the lagrangian is

\[
S = \sum \{ \bar{\psi} D(U) \psi + \bar{\psi} \Box(U) \psi + \bar{\psi} \gamma_5 \partial_5 \psi + \bar{\psi} \Box_5 \psi + M(s) \bar{\psi} \psi \}. \tag{16}
\]

The sum runs over the four dimensional coordinates \(x_\mu\) and the fifth coordinate \(s\), where \(-2L < s \leq 2L\). Antiperiodic boundary conditions are assumed in the \(s\) direction. The terms involving \(D(U)\) and \(\Box(U)\) are, respectively, covariant four-dimensional kinetic and Wilson terms. The next two terms are (free) kinetic and Wilson terms for the fifth direction. The last term is the \(s\)-dependent mass term. The gauge field that enters eq. (16) is \textit{four dimensional}. Namely, \(U_{x,s,\mu} = U_{x,\mu}\) independently of \(s\), and \(U_{x,s,5} = I\).

The mass function is given by

\[
M(s) = \begin{cases} 
+M, & 0 < s \leq 2L, \\
-M, & -2L < s \leq 0.
\end{cases} \tag{17}
\]

The parameter \(M\) (which has nothing to do with the mass of the four dimensional fermions) is chosen in the range \(0 < M < 1\) \[17\].

The \(s\)-dependent mass term defines two defects: a domain wall between \(s = 0\) and \(s = 1\), and an antidomain wall between \(s = 2L\) and \(s = -2L+1\). In the free field limit, it is easy to check that the five dimensional Dirac equation has a right-handed and a left-handed zero mode, which are localized on the domain wall and the antidomain wall respectively. Since the two zero modes couple to the same gauge field, they actually describe a single massless quark. A generalization which allows for a non-zero current mass is discussed in ref. \[14–37\].

The great advantage of this formulation, is that the left-handed and right-handed parts of the quark field have practically disjoint supports in the fifth dimension. The axially non-conserving part of the interaction hamiltonian has vanishingly small matrix elements between any pair of quark states. As a result, one can rigorously prove \[36\] the restoration of all non-singlet axial symmetries in the limit \(L \to \infty\). In contrast, the arguments for chiral symmetry restoration are only perturbative for ordinary Wilson fermions \[1] \[37\]. At the same time, vectorial flavour symmetries are manifestly preserved (unlike with staggered fermions). The proof holds for all values of the bare coupling \(g_0\) and, hence, also in the continuum limit. It implies that the current mass is only multiplicatively renormalized, and that operator mixings are restricted by axial quantum numbers as in the continuum. (In
the strong coupling limit the massless spectrum changes. It is likely that the axial currents defined in ref. \[16\] become vectorial with respect to the new massless spectrum.

The price paid is the necessity of introducing an extra, unphysical dimension for the fermions. It is not clear yet how large the extra dimension has to be in practice, and this question certainly deserves further study. In order to minimize \( L \) one should use the boundary fermion scheme \[14\]. If the needed value of \( L \) is sufficiently small, the method may turn out be an attractive alternative for numerical simulations.

I now turn to proposals for \( \chi \)LGTs based on domain wall fermions. In order to construct a \( \chi \)LGT we should somehow decouple the extra chiral mode at the antidomain wall. This can be done by introducing the four dimensional gauge field in the vicinity of the domain wall only. We thus substitute

\[
U_{x,s,\mu} = \begin{cases} 
U_{x,\mu}, & -L < s \leq L, \\
I, & \text{otherwise},
\end{cases} \tag{18}
\]

in the action eq. (16). The region \(-L < s \leq L\) is the “waveguide” \[16\]. In the limit of large \( L \), we clearly succeed in decoupling the opposite chirality zero mode on the antidomain wall. However, in going from the previous QCD setup to the waveguide model, we have created two new defects: these are the interfaces at \( s = \pm L \), where charged degrees of freedom inside the waveguide couple directly to neutral ones outside of it. As a result, the fermion action eq. (14) with the gauge field eq. (13) is not gauge invariant. The breaking is very mild from the point of view of the massless modes at the walls, because they are localized far away from both interfaces. The perturbative philosophy is that mild breaking of gauge invariance at the level of the lattice action is welcomed, because the effective action obtained by integrating out the fermions should violate gauge invariance in the case of an anomalous fermion spectrum. Indeed, using lattice perturbation theory we recover the consistent anomaly in the limit of a smooth external gauge field \[10,11\].

As we already know, the perturbative analysis may be misleading. Following the steps of Sect. 2, we can reformulate the waveguide model in a gauge invariant way, by introducing a Higgs field \( V_x \) on the \( s \)-links that make the two interfaces. We now clearly see the danger: If the \( V_x \) field is strongly fluctuating (which is true in a symmetric phase) new massless species (both charged and neutral) may appear at the waveguide boundaries. In fact, this is precisely what happens \[16\]. Let us introduce a Yukawa coupling \( y \) that controls the interaction between the \( V_x \) field and the neutral and charged fermions at the interface. Like the Smit-Swift and Eichten-Preskill models, the phase diagram of the waveguide model contains two symmetric phases, one at small \( y \) and one at large \( y \). The two symmetric phases have different massless spectra. But in both of them, the new species that appear at the interface render the spectrum vector-like. In conclusion, the waveguide model fails to yield a \( \chi \)LGT. Nevertheless, the waveguide model can help us in studying the physical spectrum of the overlap model, to which I now turn.

The overlap formula is an ansatz for the fermionic partition function which was proposed by Narayanan and Neuberger \[13\] for the construction of \( \chi \)LGTs. This formula can be motivated as follows. If we consider domain wall fermions on a lattice with an infinite \( s \) direction, there is no antidomain wall and, hence, no unwanted zero modes with the wrong chirality. Needless to say, one has to construct an explicit realization of the “infinite \( s \)” situation. The question is whether the overlap realization evades being the \( L \to \infty \) limit of the waveguide model, or a variant of it. I will return to this question below.

The basic observation of ref. \[17\] is that a transfer matrix formalism \[12\] is particularly powerful for domain wall fermions. The reason is that the gauge field is \( s \)-independent, and so the transfer matrices that describe the hopping from one four dimensional layer to the next in the \( s \) direction are almost \( s \)-independent. In fact, there are only two different transfer matrices \( T_+ (U) \) that correspond to the two half-spaces of positive and negative \( s \). The difference between \( T_+ (U) \) and \( T_- (U) \) arises because of the changing sign of \( M (s) \). Formal application of the rules of ref. \[12\] suggests that the fermionic partition function, when expressed in a transfer matrix language, could look something
like
\[ Z_F(U) \sim \cdots T_-(U)T_-(U)T_+(U)T_+(U) \cdots \]  \hspace{1cm} (19)

The dots indicate that, formally, the products of \( T_+ \)'s and \( T_- \)'s continue ad infinitum.

In order to arrive at a well defined expression, one observes that in the limit of large \( s \), \( T_+^s(U) \) projects out the ground state of the many body hamiltonian \( H_\pm(U) = -\log T_\pm(U) \). Explicitly
\[ T_\pm^s(U) \to |U\pm\rangle \lambda_\pm \langle U\pm| + \cdots \]  \hspace{1cm} (20)

Here \( \lambda_\pm \) stands for the largest eigenvalue of \( T_\pm(U) \), and \( |U\pm\rangle \) is the corresponding ground state. Focusing on the interface between positive and negative \( s \), one arrives at the following tentative expression
\[ Z_F(U) \sim \langle U-|U+\rangle. \]  \hspace{1cm} (21)

This expression is still not well defined, because states in a Hilbert space are defined only up to a phase. Adopting the Wigner-Brillouin phase choice, Narayanan and Neuberger arrive at the overlap formula
\[ Z_{ov}(U) = \frac{\langle I-|U-\rangle\langle U-|U+\rangle\langle U+|I+\rangle}{\langle I-|U-\rangle \langle I-|U+\rangle |\langle U+|I+\rangle|} \]  \hspace{1cm} (22)

The subscript \( ov \) is a shorthand for overlap. The \( |I\pm\rangle \) are the ground states of the free hamiltonians \( H_\pm^0 = H_\pm(I) \). Notice that the overlap formula picks only the phase of \( \langle I \pm |U\pm\rangle \).

The free overlap \( I - |I+\rangle \) in the denominator is a normalization factor. Eq. (22) is modified if \( \langle I + |U+\rangle = 0 \). (One can show that \( I - |U-\rangle \) never vanishes.) See the last paper of ref. [17] for the explicit expression.

In the presence of smooth external gauge fields, the overlap formula reproduces all the essential properties of chiral fermions. In particular, the consistent anomaly is recovered in lattice perturbation theory [10][11][17]. More remarkably, the overlap formula vanishes identically in an instanton background due to level crossing in the spectrum of \( H_+(U) \), and when the correct number of fermionic creation (annihilation) operators is inserted, instanton induced transition amplitudes are reproduced as well [17]. Finally, by multiplying the overlap formulae for an equal number of left-handed and right-handed fermions, one arrives at a valid model for lattice QCD \[ 37].

All implementations of Kaplan’s domain wall fermions require subtractions to cancel undesirable effects of the infinitely many four dimensional fields with cutoff mass. These subtractions can be represented as five dimensional Pauli-Villars (PV) fields \[ 13 \]. In perturbation theory, the only difference between the overlap formula and the waveguide model is in the choice of the PV fields. The PV contribution to the effective action (the sum of the one loop diagram) is purely \textit{real}. But the perturbative “signature” of a chiral fermion is the \textit{imaginary} part of the effective action. The overlap formula and the waveguide model have equal imaginary parts for their effective actions to all orders in lattice perturbation theory \[ 14 \]. This is the first sign that the overlap and waveguide models may be closely related, and that the overlap model may be vulnerable to the same dangers that ultimately render the waveguide model vector-like. It is true that the two models differ in the presence of gauge fields whose total topological charge is non-zero \[ 14 \]. However, as I will now explain, the significance of this observation is limited.

Following the reasoning of sect. 2, the spectrum of the overlap model is determined by the \( \psi \)HM defined by the partition function
\[ Z'_{ov} = \int D\Psi \ Z_{ov}(U = i\Psi) \]  \hspace{1cm} (23)

The pure gauge configurations, whose nonperturbative dynamics determines the spectrum of the overlap model, have zero total topological charge. As in Sect. 2, one should first find whether the global symmetry of \( Z'_{ov} \) is spontaneously broken. If it is not, one should proceed to calculate the fermion spectrum. To this end, one should augment eq. (23) by giving its dependence on an appropriate set of external sources.

The partition function \( Z'_{ov} \) does not arise from a local action. This makes its investigation particularly difficult. When specifying the operators that couple to the external sources one has to use some physical intuition, and it is not easy to tell whether extra massless state are hiding somewhere. If no new massless states are generated
and the spectrum is truly chiral, one would expect zeros in the propagator. One would then have to check whether or not these zeros imply some inconsistency. A less dramatic possibility \[10\] is that the spectrum is actually vector-like, and the zeros in the propagator simply reflect an under complete set of external sources (or $\chi$-s in the language of Sect. 2). This could happen if the chosen external sources do not create the new, dynamically generated, massless fermion. The question could be settled in principle by calculating $n$-current correlators within the $\psi$HM (the vacuum polarization and more) and inferring the spectrum from them. But, in practice, this strategy may be hampered by technical complications.

A way to circumvent the above difficulty is to show that, at least in some special cases, the partition function $Z'_{ov}$ does have an interpretation as a local field theory. The main result of ref. \[45\] (which corrects an error in a previous publication \[16\]) is the following. If the target $\chi$LGT contains $4n$ chiral families, then for all $V_x$

$$Z_{ov}(I(V)) = Z_{m.w.g.}(I(V)). \quad (24)$$

$Z_{m.w.g.}(U)$ is the fermionic partition function of a modified waveguide model with a Yukawa coupling $y = 1$. Eq. \(24\) implies that the overlap model and the modified waveguide model lead to the same $\psi$HM. This, in turn, implies that the two models have the same spectrum.

The phase diagram and spectrum of the modified waveguide model can be studied using previously developed techniques \[16, 19\]. The difference between the original and modified waveguide models is in the choice of the PV fields. At least for small $y$, one expects that both variants of the waveguide model (and, hence, the overlap model too) will have the same vector-like fermion spectrum. (The modified waveguide and overlap models will also have massless ghost states.) Another interesting question is whether the equivalence between the overlap formula and some waveguide model can be extended to topologically nontrivial sectors as well. See ref. \[13, 19\] for more details.

Domain wall fermions were also investigated using a five dimensional (5-d) gauge field \[17\]. This approach does not produce a $\chi$LGT. Depending on the gauge coupling in the $s$ direction, the 5-d gauge field is reduced dynamically to a 4-d gauge field (see the first paper of ref. \[18\]), or else the system breaks up into an infinite collection of independent 4-d theories of Wilson fermions, the so-called layered phase.

### 4. GAUGE FIXING

The No-Go arguments of Sect. 2 can be circumvented by imposing constraints on the lattice gauge field’s measure. This procedure is usually called “gauge fixing”, although this term is highly inaccurate in the present context. In lattice QCD the action is gauge invariant, and gauge fixing means picking a single representative out of many equivalent ones. But in trying to construct $\chi$LGTs one uses gauge variant fermion actions. In this case, imposing constraints on the gauge field’s measure is really a part of the definition of the theory.

The gauge fixing approach was originally proposed by the Roma group \[18\] and subsequently followed by the Zaragoza group \[19\]. So far, the results are limited to the context of lattice perturbation theory. In this section I will present considerations which are relevant for a nonperturbative implementation of this approach. The main conclusion is that one needs a global gauge fixing algorithm that selects only relatively smooth lattice gauge field configurations. This diagnosis has already been made by Vink \[20\]. Vink proposed a method that effectively couples the fermions only to maximally smooth gauge field configurations. His method, however, has not been investigated in any detail so far.

Consider the perturbative effective action $S_{eff} = S_{eff}(U)$ defined as the sum of the one loop diagrams on the lattice. If the fermion action is not gauge invariant, the first variation of $S_{eff}$ with respect to a lattice gauge transformation parametrized by $g_x = \exp(i\omega_x)$ will look like

$$\delta S_{eff} = c_0 A_x + \sum_{n \geq 1} c_n a^n \mathcal{O}(n) \quad (25)$$

$A_x$ is some discretized version of the consistent anomaly, and the coefficient $c_0$ will vanish if the perturbative spectrum is anomaly free. In writ-
ing eq. (25) I assume that all other operators of dimension less than or equal to four have been cancelled by counter terms.

As the infinite sum on the r.h.s. of eq. (25) indicates, \( \delta S_{\text{eff}}/\delta \omega_x \) contains more than just the consistent anomaly. In fact, \( A_x \) is the first term in an infinite series in the lattice spacing \( a \). The \( O^{(n)}_x \) are local operators of dimension \( n+4 \). Like \( c_0 \), the coefficients \( c_n, n > 0 \) have a group theoretical origin. All the coefficients will vanish simultaneously if the fermion action is exactly gauge invariant. Unless the fermion action is highly non-local, this implies that already the perturbative spectrum is vector-like.

If the perturbative spectrum is chiral but anomaly free, \( \delta S_{\text{eff}}/\delta \omega_x \) will vanish in the limit of smooth external gauge fields. To see this, we observe that all the \( O^{(n)}_x \) must contain lattice derivatives. Non-derivative terms can be calculated in the limit of zero external momentum, and so they should agree with some continuum regularization. Namely, they should give rise to the non-derivative part of the gauge variation of counter terms, and nothing more. Eq. (25) is therefore effectively an expansion in \( \epsilon p \). (This property becomes manifest in explicit computations based on standard momentum space Feynman rules.)

In the adiabatic limit, the infinite series in eq. (25) tend to zero, because the external field contains only Fourier modes with \( \epsilon p \rightarrow 0 \).

The difficulty arises because the lattice momentum is not conserved under gauge transformations. If we apply a generic lattice gauge transformation to a smooth gauge field, we will find that the typical momentum in the transformed configuration is \( \epsilon p \sim 1 \). Substituting in eq. (25) we see that the the gauge variation of the effective action is now \( O(1) \). Thus, even a formally "mild" breaking of gauge invariance by the fermion action is really very large for a generic lattice gauge field configuration. This large generic breaking is in conflict with the fact that the continuum limit must describe a gauge invariant theory. If a non-trivial continuum limit exists, gauge invariance has to be restored dynamically. Sect. 2 explain why the dynamical restoration of gauge invariance comes at the price of producing a vector-like spectrum.

The above considerations suggest that the aim of the “gauge fixing” procedure should be to keep only relatively smooth lattice gauge field configurations, while excluding all the rest. The restriction to relatively smooth configurations clearly reduces the breaking of gauge invariance, as represented by the r.h.s. of eq. (25). If we intend to adopt this strategy, we first need some understanding of the structure of the lattice gauge orbit. Let me begin by considering an arbitrary configuration of the lattice gauge field, and some algorithm that rotates the configuration to one that satisfies the Landau gauge condition. The question is how will the rotated configuration look like. To answer that question, we observe that the Landau gauge condition is satisfied by extrema of the functional \( F(U) = \text{Re} \sum_{x, \mu} \text{tr} U_{x, \mu} \) along the orbit. The structure of the rotated configuration will therefore depend strongly on whether we have arrived at a local or a global maximum of \( F \).

Recent work reveals a proliferation of solutions to the Landau gauge condition (Gribov copies) on the lattice. The basic reason is that the lattice is a multiply connected topological space. Consider for definiteness a \( U(1) \) theory in two dimensions, and let \( \exp(i \theta_x) \) be the gauge transformation that takes us from one Gribov copy to another. If \( \theta_x \) was an ordinary function, it would have to satisfy the lattice Laplace equation. The scalar laplacian has no non-trivial solutions, which is why an abelian theory is free of Gribov copies in the continuum.

However, \( \theta_x \) is a periodic variable. In other words, a lattice gauge transformation is a mapping from the lattice sites to the unit circle. If we extend it to a mapping \( \Theta(\square) \) from the perimeter of the plaquette to the unit circle, we will find that \( \Theta(\square) \) can be classified according to its homotopy class. Notice that some choice has to be made in the definition of \( \Theta(\square) \). When interpolating between \( x \) and \( x + \hat{\mu} \), we will choose to go along the unit circle in the direction that minimizes the arc length between \( \exp(i \theta_x) \) and \( \exp(i \theta_{x + \hat{\mu}}) \). This procedure allows us to assign a local winding number to the gauge transformation. A winding number \( n = \pm 1 \) means that the gauge transformation creates a singular (anti)vortex inside
the plaquette. (Periodic boundary conditions will force equality of the total number of vortices and antivortices.)

The main result of ref. [21] is that lattice Gribov copies are in one to one correspondence with singular vortices. Fig. 6 of ref. [21] describes a gauge fixed configuration obtained as follows. In the first step, a random lattice gauge transformation was applied to a smooth configuration. In the second step, a standard local algorithm was applied to enforce the Landau gauge condition. A glance at Fig. 6 reveals islands of large $A_\mu$ with smooth $A_\mu$ in between, supporting the picture that the Gribov copies arise due to a localized structure. The conclusion is that local extrema of the Landau gauge functional $\mathcal{F}$ are characterized by a vortex-antivortex gas with finite density in lattice units.

Large values for $\mathcal{O}^{(n)}_x$ in eq. (25) are correlated with large gradients in $U_{x,\mu}$. Such large gradients will exist in the vicinity of every singular vortex. Consequently, in order to minimize the r.h.s. of eq. (25) we have to eliminate all the Gribov copies. This, in turn, requires the use of a global algorithm. (See ref. [20, 21] for specific examples.) It is an open question whether one can construct a global algorithm that reduces the breaking of gauge invariance sufficiently, without at the same time spoiling other important properties of the continuum limit.

5. INTERPOLATING FIELDS

An alternative approach to the definition of chiral gauge theories is to put the gauge field on the lattice, while keeping the fermions in the continuum. Interest in this approach arose following a recent paper by ’t Hooft [22]. (See ref. [12, 49, 50] for earlier relevant works.) A central element of the method is the construction of a continuum interpolation of the lattice gauge field [23]. The second element is a nonperturbative definition of the chiral fermion determinant for any continuum gauge field that can be obtained via the interpolation. The interpolating fields method may be problematic to implement in numerical simulations. Here I will only discuss the conceptual question of whether the method can provide a consistent definition of chiral gauge theories.

For definiteness, I will assume that the continuum chiral Dirac operator is $\hat{D} = \bar{\psi} + i\mathcal{A}_L$. The basic property of $\hat{D}$ is that its eigenvalues $\lambda_i$ are complex and gauge variant, and that a right eigenstate defined by $D\psi_i = \lambda_i \psi_i$ is not the complex conjugate of the corresponding left eigenstate $\chi_i \hat{D} = \chi_i \lambda_i$.

The obvious reason why one may hope that the interpolating fields method will do better than pure lattice approaches, is that there is no fermion doubling in the continuum. As a first step, let us see in what way this new method may change the considerations of the previous section. While the gauge field is still regularized by the lattice cutoff, we need a separate regularization to define the continuum fermion determinant. This regularization introduces a new cutoff scale $M$, which now controls the violations of gauge invariance. Assuming $a M \gg 1$ and using the same operator basis as before, one expects that eq. (23) will be replaced by

$$\frac{\delta S_{\text{eff}}}{\delta \omega_x} = c_0 A_x + \sum_{n \geq 1} c'_n M^{-n} \mathcal{O}^{(n)}_x$$

The coefficient $c_0$ of the (discretized) consistent anomaly is of course the same as in eq. (27). The main change is that eq. (26) represents an expansion in $p_\mu / M$. Remember that the generic momentum of a lattice gauge field is $O(1/a)$. The new expansion parameter is therefore effectively $1/(Ma)$. If we now send $M \to \infty$ at a fixed $a$, we may hope that the r.h.s. of eq. (26) will tend to zero for an anomaly free spectrum. (To avoid confusion, let me stress that the above heuristic considerations are based on perturbative reasoning and, in any event, they are no substitute for a detailed proof.)

Let me begin with a list of features that we expect from the interpolating field. There are three fundamental requirements. First, the interpolating field $A_\mu(x) = A_\mu(x; U)$ should reproduce every lattice link variable $U_{\vec{n},\mu}$ by calculating the parallel transporter along that link from $\vec{n}$ to $\vec{n} + \vec{\mu}$. (In this section I use $\vec{n}$ to denote a lattice site.) The second property is gauge covariance. Consider two lattice gauge fields which are related by a lattice gauge trans-
transformation $U'_{\vec{n},\mu} = g_{\vec{n}}U_{\vec{n},\mu}g^\dagger_{\vec{n}+\hat{t}}$. We demand that the corresponding interpolating fields will be related by a continuum gauge transformation

$$A'_\mu(x) = \Omega(x) (A_\mu(x) - i\partial_\mu) \Omega^\dagger(x),$$

(27)

where $A'_\mu(x) = A_\mu(x;U')$, and $\Omega(x) = \Omega(x;g,U)$ is a continuum interpolation of the lattice gauge transformation, that coincides with $g_{\vec{n}}$ at the lattice points. (Notice that $\Omega(x)$ can depend on the $U_{\vec{n},\mu}$-s too.) The third requirement, which stems from the need to have a well behaved spectrum for $D$, is that the worst singularities in $F_{\mu\nu}(x;U)$ are discontinuities. (This requirement is weaker than “transversal continuity” which is often mentioned in the literature.)

I now turn to specific interpolation techniques. A method based on a linear interpolation kernel is discussed in ref. [28]. The method is very simple, but it is highly specific to non-compact $U(1)$. Here I will focus on the interpolation proposed by Göckeler et al [28]. I will describe the method for compact $U(1)$ in two dimensions. The reason is that the formulae are much simplified in this case, which still contains all the essential (and in particular topological) features of non-abelian theories in four dimensions.

In two dimensions, we have to define the continuum gauge field first on the links, and then inside each plaquette. We define the interpolating field to be constant on all points that make a given link $\{\vec{n},\mu\}$. Explicitly, for $\vec{x} = \vec{n} + t\hat{\mu}$, $0 \leq t < 1$, one defines

$$A_\mu(\vec{x}) = A_\mu(\vec{n}) = i\log U_{\vec{n},\mu}. \quad (28)$$

The logarithm is always taken such that $|A_\mu(\vec{n})| < \pi$. The interpolating field is left undefined on a zero measure subset of lattice gauge fields, where $U_{\vec{n},\mu} = -1$ for some links.

We next have to extend the interpolation inside a given plaquette, which we parametrize as $\vec{x} = \vec{n} + t\vec{t}$, where $0 \leq t_1, t_2 \leq 1$. As a first step, let us assume that the link variables satisfy a local axial gauge, where $U'_{\vec{n}+1,2}$ is the only non-trivial link variable. In this special case, the interpolating field is given by

$$A'_1 = 0, \quad A'_2 = it_1 \log U'_{\vec{n}+1,2}. \quad (29)$$

This leads to a constant field strength throughout the plaquette, given by $F = i\log U'_{\vec{n}+1,2}$. The generalization to arbitrary values of the link variables is done as follows. The lattice gauge transformation that enforces the local axial gauge is first extended to a continuum gauge transformation $\Omega(\vec{x})$ throughout the plaquette. The interpolating field $A_\mu(\vec{x})$ is then defined by inverting eq. (28).

Let us compare the field strength $F$ to the directed sum along the perimeter of the $A_\mu$-s defined in eq. (28). In general, we may find

$$F = A_1(\vec{n}) + A_2(\vec{n} + \hat{1}) - A_1(\vec{n} + \hat{2}) - A_2(\vec{n}) + 2\pi k. \quad (30)$$

Here the possible values of $k$ are $k = -2 \ldots 2$, and the value changes whenever a link variable goes through $-1$. Consider now the restriction $\Omega(\vec{x}) = \Omega(\vec{z})$ of the continuum gauge transformation to the perimeter of the plaquette. In fact, $\Omega(\vec{z})$ is completely determined by $A_\mu(\vec{n})$ and $A'_\mu(\vec{n})$ via the gauge covariance condition eq. (27) (up to a constant overall phase). On each link, the phase of $\Omega(\vec{z})$ varies linearly. For the three links where $A'_\mu(\vec{n}) = 0$, this construction coincides with the minimal arc prescription of the previous section. But in general this is not true for the link where $A'_\mu(\vec{n}) \neq 0$. As a mapping into the unit circle, $\Omega(\vec{z})$ is characterized by a winding number. Comparing eq. (28) to the line integral of eq. (27) along the perimeter, we conclude that the winding number of $\Omega(\vec{z})$ is $k$. (A winding number will arise in four dimensional non-abelian theories too, when one extends the gauge transformation to the faces of the hypercube.)

The extension of $\Omega(\vec{x})$ from the perimeter to the entire plaquette depends crucially on the winding number. For $k = 0$, one can extend the phase of $\Omega(\vec{x})$ linearly, resulting in the interpolating field

$$A_1 = (1 - t_2)A_1(\vec{n}) + t_2 A_1(\vec{n} + \hat{1}) \quad (31)$$

$$A_2 = (1 - t_1)A_2(\vec{n}) + t_1 A_2(\vec{n} + \hat{1}).$$

Notice that eq. (31) is rotationally covariant.

If the local winding number $k = k(\vec{n})$ vanishes everywhere, the resulting interpolating field is also transversally continuous. Namely, $A_\mu(x;U)$
is continuous going across a plaquette boundary in the $\nu$ direction for $\nu \neq \mu$, but in general not for $\mu = \nu$. The longitudinal discontinuities lead to an enhanced content of high momentum modes in $\tilde{A}_\mu(q;U)$. This may have undesirable effects on the UV behaviour [23]. However, the UV behaviour can be improved by local smoothing of the interpolating field [22].

If $k(\vec{n}) \neq 0$, there has to be an (anti)vortex singularity inside the plaquette $\vec{n}$. (Similarly, in four dimensions $k(\vec{n}) \neq 0$ implies an (anti)instanton singularity inside the hypercube.) The formulae of ref. [23] lead to a line discontinuity in $\Omega(\vec{x})$. If, say, the line is horizontal, the resulting $A_2(\vec{x})$ will have a $\delta$-function piece localized on that line, while $A_1(\vec{x})$ will not be transversally continuous across it. The two ugly looking singularities cancel of course in $F$.

A different extension of the gauge transformation can be defined by demanding that $\Omega(\vec{x})$ be constant along each ray emanating from the center point $\vec{x}_0$. Explicitly, let $\vec{x} = t\vec{x}_0 + (1-t)\vec{z}$, where $\vec{z} \in \square$ is a boundary point and $0 \leq t < 1$, then we define $\Omega(\vec{x}) = \Omega(\vec{z})$. The resulting singularity in $A_\mu(x)$ is now very similar to that of a continuum vortex.

In summary, the interpolation leads to a well defined quasi-local topological structure, which is characterized by the disorder field $k(\vec{n})$. For $k(\vec{n}) \neq 0$, both the method of ref. [23] and the one described above are not rotationally covariant. This is not a serious problem, however, because rotational covariance can always be restored by averaging over different choices for the direction of the local axial gauge.

The real problem is how to define the continuum fermion determinant nonperturbatively such that, in an anomaly free theory, it will be gauge invariant (up to local counter terms) for all interpolating fields. Using Pauli-Villars regularization, ’t Hooft [22] proves gauge invariance in the limit $Ma \to \infty$ under the assumption that $A_\mu(x;U)$ is globally bounded. (For other proofs see ref. [23].) But $A_\mu(x;U)$ is globally bounded iff the corresponding disorder field $k(\vec{n})$ vanishes everywhere. Notice that interpolating fields with $k(\vec{n}) = 0$ are highly non-representative. In a generic interpolating field, a finite fraction of all plaquettes (hypercubes in four dimensions) will carry a non-zero winding number. Thus, in the thermodynamical limit, the proof of ref. [23] covers only a vanishingly small subset of the interpolating field’s space.

In a similar spirit to the previous section, non-zero values for $k(\vec{n})$ can be suppressed (or even eliminated altogether for non-periodic boundary conditions) if one adopts a global method for constructing the interpolating fields. An inspection of eq. (31) reveals that enforcing $k(\vec{n}) = 0$ has a price. Namely, unlike in eq. (28), one has to allow the interpolating field on the links $A_\mu(\vec{n})$ to lie outside of the interval $(-\pi, \pi)$. In the infinite volume limit, we still cannot establish the existence of a global bound on $|A_\mu(x)|$. The reason now is that the magnitude of the so constructed interpolating field may be infra-red divergent.

6. CONCLUSIONS

The power and beauty of lattice QCD stems from its manifest gauge invariance. In trying to understand why it is so difficult to construct $\chi$LGTs one naturally focuses on the properties of the lattice fermions. In this review I have stressed the complementary role of the lattice gauge field.

The crucial feature is that the transversal degrees of freedom described by the lattice gauge field always couple to a conserved current defined on the lattice links. Only the lattice gauge field’s measure is used in establishing this property. This allows us to apply the No-Go theorems to an effective hamiltonian, constructed in a natural way from the inverse two point function of the massless fermions. The most important open question is whether poles in the inverse two point function can evade the No-Go theorems consistently.

By invoking continuum fermions, the interpolating fields method seems to evade the reasoning of Sect. 2. The real situation, however, is more subtle. Loosely speaking, although our fermions are defined in the continuum, what the lattice gauge field can feel is only an effective fermion field that lives on a lattice too. In more detail, the ultimate role of the interpolation is to define the fermion determinant, or the fermionic partition
function, as a functional of the lattice gauge field. The variation of the fermionic partition function with respect to $U_{x,\mu}$ defines the source current associated with the same link. Since the current lives on lattice links, one expects that its matrix elements could be reproduced by some lattice action. That action could be extremely complicated to write down, but the only thing we need to know is that it exists, and that it is very mildly local. If true, we can apply the reasoning of Sect. 2 to the interpolating fields method too.

Logically, a way out is to constrain the lattice gauge field’s measure. In Sect. 4 I discussed at a heuristic level, how the restriction to relatively smooth gauge fields reduces violations of gauge invariance. As of today, very little is known on this approach. The first question is whether a global algorithm can suppress violations of gauge invariance sufficiently. Ref. [54] finds that non-perturbative counter terms might be needed to completely restore gauge invariance. In my opinion, nonperturbative counter terms are equivalent to an infinite amount of fine tuning. We should therefore hope that we can do without them. Also, the use of any global algorithm introduces some non-locality into the theory. One should check whether this does not spoil some important property of the continuum limit, such as causality or unitarity.

Another issue that I have not addressed in this review, is the fact that proposals for chiral lattice gauge theories tend to have an exactly conserved U(1) symmetry associated with fermion number. This feature leads to an apparent conflict with fermion number non-conservation [55]. On the other hand, there are arguments that the paradox can be resolved [56,54]. The investigation of all these important questions should clearly continue in the future.

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REFERENCES
1. K. Jansen, these proceedings.
2. K. Wilson, in New Phenomena in Sub-Nuclear Physics (Erice, 1975), ed. A. Zichichi (Plenum, New York, 1977).
3. L.H. Karsten and J. Smit, Nucl. Phys. B183 (1981) 103.
4. H.B. Nielsen and M. Ninomiya, Nucl. Phys. B185 (1981) 20, B193 (1981) 173. Erratum, Nucl. Phys. B195 (1982) 541.
5. P. Swift, Phys. Lett. B145 (1984) 256. J. Smit, Acta. Phys. Pol. B17 (1986) 531.
6. A. Coste, C. Korthals-Altes and O. Napoly, Nucl. Phys. B289 (1987) 645.
7. M.F.L. Golterman, D.N. Petcher and J. Smit, Nucl. Phys. B370 (1992) 51. M.F.L. Golterman and D.N. Petcher, Nucl. Phys. (Proc. Suppl.) B26 (1992) 483. W. Bock, A.K. De and J. Smit, Nucl. Phys. B388 (1992) 243.
8. D.N. Petcher, Nucl. Phys. (Proc. Suppl.) B30 (1993) 50.
9. I. Montvay, Nucl. Phys. (Proc. Suppl.) B26 (1992) 57, and references therein.
10. Y. Shamir, Phys. Rev. Lett. 71 (1993) 2691; Nucl. Phys. (Proc. Suppl.) B34 (1994) 590; [hep-lat/9307002]
11. D. Foerster, H.B. Nielsen and M. Ninomiya, Phys. Lett. B94 (1980) 135. S. Aoki, Phys. Rev. Lett. 60 (1988) 2109. K. Funakubo and T. Kashiwa, Phys. Rev. Lett. 60 (1988) 2113.
12. J. Smit, Nucl. Phys. (Proc. Suppl.) B4 (1988) 451.
13. M. Campostrini, G. Curci and A. Pelissetto, Phys. Lett. B193 (1987) 279. G.T. Bodwin and E.V. Kovacs, Phys. Lett. B193 (1987) 283.
14. A. Pelissetto, Ann. Phys. 182 (1988) 177.
15. D.B. Kaplan, Phys. Lett. B288 (1992) 342; Nucl. Phys. (Proc. Suppl.) B30 (1993) 597.
16. M.F.L. Golterman, K. Jansen, D.N. Petcher and J.C. Vink, Phys. Rev. D49 (1994) 1606. M.F.L. Golterman and Y. Shamir, Phys. Rev. D51 (1995) 3026.
17. R. Narayanan and H. Neuberger, Phys. Lett. B302 (1993) 62, B348 (1995) 549; Phys. Rev. Lett. 71 (1993) 3251; Nucl. Phys. (Proc. Suppl.) B34 (1994) 587; Nucl. Phys. B412 (1994) 574, B443 (1995) 305.
18. A. Borelli, L. Maiani, G.-C. Rossi, R. Sisto and M. Testa, Nucl. Phys. B333 (1990) 335.
19. J.L. Alonso, Ph. Boucaud, J.L. Cortés and E. Rivas, Nucl. Phys. (Proc. Suppl.) B17 (1990)
461; Phys. Rev. D44 (1991) 3258. J.L. Alonso, Ph. Boucaud, F. Lesmes and A.J. van der Sijs, Nucl. Phys. (Proc. Suppl.) B42 (1995) 595.
20. J.C. Vink, Phys. Lett. B321 (1994) 239.
21. Ph. de Forcrand and J. Hetrick, Nucl. Phys. (Proc. Suppl.) B42 (1995) 861.
22. G. 't Hooft, Phys. Lett. B349 (1995) 491.
23. M. Göckeler, A.S. Kronfeld, G. Schierholz and U.-J. Wiese, Nucl. Phys. B404 (1993) 839.
24. W. Bock, A.K. De, E. Focht and J. Smit, Nucl. Phys. B401 (1993) 481.
25. C. Rebbi, Phys. Lett. B186 (1987) 200.
26. E. Eichten and J. Preskill, Nucl. Phys. B268 (1986) 179.
27. W. Bock, J. Smit and J.C. Vink, Nucl. Phys. B414 (1994) 73, B416 (1994) 645.
28. M.F.L. Golterman, D.N. Petcher and E. Rivas, Nucl. Phys. B395 (1993) 596.
29. A.A. Slavnov, Phys. Lett. B319 (1993) 231, B348 (1995) 553.
30. L.H. Karsten and J. Smit, Nucl. Phys. B144 (1978) 536; Phys. Lett. B85 (1979) 100.
31. S. Zenkin, hep-lat/9506015.
32. W. Bietenholz and U.-J. Wiese, hep-lat/9503022.
33. K. Jansen and M. Schmaltz, Phys. Lett. B296 (1992) 374. K. Jansen, Phys. Lett. B288 (1992) 348.
34. Y. Shamir, Nucl. Phys. B406 (1993) 90.
35. M. Creutz and I. Horvath, Nucl. Phys. (Proc. Suppl.) B34 (1994) 799; Phys. Rev. D50 (1994) 2297.
36. V. Furman and Y. Shamir, Nucl. Phys. B439 (1995) 54.
37. R. Narayanan, H. Neuberger and P. Vranas, Phys. Lett. B353 (1995) 507.
38. K. Jansen, hep-lat/9410013.
39. M. Bochicchio, L. Maiani, G. Martinelli, G.C. Rossi and M. Testa, Nucl. Phys. B262 (1985) 331. C. Curci, Phys. Lett. B167 (1986) 425.
40. M.F.L. Golterman, K. Jansen and D.B. Kaplan, Phys. Lett. B301 (1993) 219. Y. Shamir, Nucl. Phys. B417 (1994) 167. S. Chandrasekharan, Phys. Rev. D49 (1994) 1980. S. Aoki and R.B. Levien, Phys. Rev. D51 (1995) 3790.
41. S. Randjbar-Daemi and J. Strathdee, Nucl. Phys. B443 (1995) 386; Phys. Rev. D51 (1995) 6617; Phys. Lett. B348 (1995) 543.
42. M. Lüscher, Comm. Math. Phys. 54 (1977) 283.
43. S.A. Frolov and A.A. Slavnov, Nucl. Phys. B411 (1994) 647.
44. R. Narayanan and H. Neuberger, hep-lat/9507013.
45. M.F.L. Golterman and Y. Shamir, these proceedings.
46. M.F.L. Golterman and Y. Shamir, Phys. Lett. B353 (1995) 84. Erratum, to appear.
47. Z. Yang, Phys. Lett. B296 (1992) 151. C.P. Korthals-Altes, S. Nicolis and J. Prades, Phys. Lett. B316 (1993) 339. S. Aoki and H. Hirose, Phys. Rev. D49 (1994) 2604. T. Kawano and Y. Kikukawa, Phys. Rev. D50 (1994) 5365. H. Aoki, S. Iso, J. Nishimura and M. Oshikawa, Mod. Phys. Lett. A 9 (1994) 1755.
48. T. deGrand and D. Toussaint, Phys. Rev. D22 (1980) 2478. P. Woit, Nucl. Phys. B262 (1985) 284.
49. R. Flume and D. Wyler, Phys. Lett. B108 (1982) 317. A.S. Kronfeld, Nucl. Phys. (Proc. Suppl.) B4 (1988) 329. M. Göckeler and G. Schierholz, Nucl. Phys. (Proc. Suppl.) B30 (1993) 609.
50. M. Lüscher, Comm. Math. Phys. 85 (1982) 29. A. Phillips and D. Stone, Comm. Math. Phys. 103 (1986) 599.
51. A. Niemi and G. Semenoff, Phys. Rev. Lett. 55 (1985) 927. L. Alvarez-Gaume, S. Della Pietra and V. Della Pietra, Phys. Lett. B166 (1986) 177. R.D. Ball, Phys. Rep. 182 (1989) 1, and references therein.
52. I. Montvay, hep-lat/9505015. J. Hetrick, these proceedings.
53. P. Hernandez and R. Sundrum, hep-ph/9506331. S.D.H. Hsu, hep-th/9503058. A.S. Kronfeld, these proceedings.
54. W. Bock, J. Hetrick and J. Smit, Nucl. Phys. B437 (1995) 585.
55. T. Banks, Phys. Lett. B272 (1991) 75. T. Banks and A. Dabholkar, Phys. Rev. D46 (1992) 4016.
56. M.J. Dugan and A.V. Manohar, Phys. Lett. B265 (1991) 137.