A SURVEY OF THE IMPACT OF THURSTON’S WORK ON KNOT THEORY

MAKOTO SAKUMA

Abstract. This is a survey of the impact of Thurston’s work on knot theory, laying emphasis on the two characteristic features, rigidity and flexibility, of 3-dimensional hyperbolic structures. We also lay emphasis on the role of the classical invariants, the Alexander polynomial and the homology of finite branched/unbranched coverings.

Contents

1. Introduction 3
2. Knot theory before Thurston 6
   2.1. The fundamental problem in knot theory 6
   2.2. Seifert surface 7
   2.3. The unique prime decomposition of a knot 9
   2.4. Knot complements and knot groups 10
   2.5. Fibered knots 11
   2.6. Alexander invariants 12
   2.7. Representations of knot groups onto finite groups 15
3. The geometric decomposition of knot exteriors 16
   3.1. Prime decomposition of 3-manifolds 16
   3.2. Torus decomposition of irreducible 3-manifolds 17
   3.3. The Geometrization Conjecture of Thurston 18
   3.4. Geometric decompositions of knot exteriors 20
4. The orbifold theorem and the Bonahon-Siebenmann decomposition of links 22
   4.1. The Bonahon-Siebenmann decompositions for simple links 23
   4.2. 2-bridge links 25
   4.3. Bonahon-Siebenmann decompositions and \( \pi \)-orbifolds 26
   4.4. The orbifold theorem and the Smith conjecture 28
   4.5. Branched cyclic coverings of knots 29

Date: February 4, 2020.

2010 Mathematics Subject Classification. Primary 57M25; Secondary 57M50

The author was supported by JSPS Grants-in-Aid 15H03620.
12.2. Evaluation of Thurston norms in terms of Twisted Alexander polynomials
12.3. Harmonic norm and Thurston norm
13. Finite index subgroups of knot groups and 3-manifold groups
13.1. Universal knots/links and universal groups
13.2. Virtual fibering conjecture
13.3. Profinite completions of knot groups and 3-manifold groups
13.4. Homology growth
References

1. Introduction

Knot theory is the analysis of pairs \((S^3, K)\), where \(K\) is a knot (i.e., an embedded circle) in the 3-sphere \(S^3\), and classification of knots has been one of the main problems in knot theory. The Alexander polynomial is an excellent invariant of knots, and it had been a dominating tool and theme in knot theory, until knot theory was influenced by Thurston’s work and the Jones polynomial was discovered. In fact, the classical text book by Crowell and Fox \([CrF1963]\) is devoted to the calculation of the knot group and the definition of the Alexander polynomial by using the free differential calculus. The influential text book by Rolfsen \([Ro1976]\) lies emphasis on geometric understanding of the Alexander polynomial through surgery description of the infinite cyclic cover (cf. \([Hir2016]\)). However, the Alexander polynomial is far from complete: there are infinitely many nontrivial knots with trivial Alexander polynomial. The famous Kinoshita-Terasaka knot and the Conway knot are related by mutation, and therefore no skein polynomial, including the Alexander polynomial, can distinguish them. The first proof of their inequivalence was given by Riley \([Ri1971]\) by studying parabolic representations of the knot groups to the finite simple group \(\text{PSL}(2, \mathbb{Z}/7\mathbb{Z})\) and the homology of corresponding finite branched/branched coverings. (This work led him to the discovery of the hyperbolic structure of the figure-eight knot complement, which inspired Thurston.) Riley called this a \textit{universal method} for obtaining algebraic invariants of knots. The method turned out to be, at least experimentally, a very powerful tool in knot theory, due to the development of computer technology. However theoretical background of the universal method has not been given yet.

In 1976, around when Rolfsen’s book was published, William Thurston started a series of lectures on “The geometry and topology of 3-manifolds”. His lecture notes \([Ths1979]\) begin with the following words.

\textit{The theme I intend to develop is that topology and geometry, in dimensions up through 3, are intricately related. Because of this relation, many questions which}
seem utterly hopeless from a purely topological point of view can be fruitfully studied. It is not totally unreasonable to hope that eventually all 3-manifolds will be understood in a systematic way.

This prophecy turned out to be true. Thurston’s work has revolutionized 3-dimensional topology, and it has had tremendous impact on knot theory. The first major impact was the proof of the Smith conjecture [MB1984], a result of the efforts by Thurston, Meeks and Yau, Bass, Shalen, Gordon and Litherland, and Morgan. As Morgan predicted in [MB1984, p.6], this was just the beginning of the saga.

In this paper, we give a survey of the impact of Thurston’s work on knot theory. However, the impact is huge, whereas both of my ability and knowledge are poor. Moreover, there already exist excellent surveys, including Callahan-Reid [CR1998], Adams [Ad2005] and Futer-Kalfagianni-Purcell [FKP2019]. So, I decided to lay emphasis on the two characteristic features, rigidity and flexibility, of hyperbolic 3-manifolds.

As the title of Section 5 of Thurston’s lecture notes [Ths1979] represents, hyperbolic structures on 3-manifolds have two different features, rigidity and flexibility.

The Mostow-Prasad rigidity theorem says that complete hyperbolic structure of finite volume on an n-manifold with \( n \geq 3 \) is rigid: it does not admit local deformation, and moreover, such a structure is unique. Thus any geometric invariants determined by the complete hyperbolic structure of an n-manifold \( M \) with \( n \geq 3 \) is automatically a topological invariant of \( M \). Thurston’s uniformization theorem for Haken manifolds implies that almost every knot \( K \) in \( S^3 \) is hyperbolic, namely the complement \( S^3 - K \) admits a complete hyperbolic structure of finite volume. Thus we obtain a plenty of topological invariants of hyperbolic knots, including the volume, the maximal cusp volume, the Euclidean modulus of the cusp torus, the length spectrum, the lengths of geodesic paths joining the cusp to itself, the invariant trace field, the invariant quaternion algebra, etc. In particular, the canonical decomposition (see Subsection 6.1) gives a complete combinatorial invariant for hyperbolic knots, by virtue of the Gordon-Luecke knot complement theorem. The computer program, SnapPea, developed by J. Weeks enables us to calculate the canonical decomposition of hyperbolic knot complements. For example, we can easily detect the inequivalence of the Kinoshita-Terasaka knot and the Conway knot, by checking with SnapPea that the number of 3-cells in the canonical decompositions of the knot complements are 12 and 14, respectively. The rigidity theorem provides us a number of powerful invariants, and it has enriched knot theory by opening new directions of research, namely study of the behavior of the geometric invariants. Enormous amounts of deep research have been made in these new directions. (See Sections 6, 8 and 9.)

There are two kinds of flexibility of hyperbolic structures on 3-manifolds. One of them is that of cusped hyperbolic manifolds: the complete hyperbolic structure admits nontrivial continuous deformations into incomplete hyperbolic structures. By
considering the metric completions of incomplete hyperbolic structures, Thurston established the hyperbolic Dehn filling (surgery) theorem, which says that “almost all” Dehn fillings of an orientable cusped hyperbolic 3-manifold produce complete hyperbolic manifolds. Since every closed orientable 3-manifold is obtained by Dehn surgery of a hyperbolic link, the theorem implies that “almost all closed orientable 3-manifolds” are hyperbolic. This gave strong evidence for Thurston’s geometrization conjecture, which was eventually proved by Perelman. The natural and important problem to study the exceptional surgeries of hyperbolic knot complements attracted attention of many mathematicians and numerous research was made on this problem. Due to the development of Heegaard-Floer homology, this problem now attracts renewed interest.

The other flexibility of the 3-dimensional hyperbolic structure is that of complete hyperbolic structures of infinite volume, in other words, the flexibility of complete hyperbolic structures on the interior of a compact orientable 3-manifold whose boundary contains a component with negative Euler characteristic. The deformation theory of such structures is the heart of the Kleinian group theory, and it is this flexibility that enabled Thurston to prove the hyperbolization theorem of atoroidal Haken 3-manifolds. In particular, the complete hyperbolic structure of a surface bundle over $S^1$ (with pseudo-Anosov monodromy) was constructed by developing the deformation theory of the complete hyperbolic structures on $\Sigma \times \mathbb{R}$, where $\Sigma$ is the fiber surface. The idea of the Cannon-Thurston map, a $\pi_1(\Sigma)$-equivariant sphere filling curve, naturally arose from this construction. Thurston produced various astonishing pictures of (approximations of) Cannon-Thurston maps. (See [Ths1982, Figures 8 and 10], [Ths1998*, Figure 1] and the beautiful book [MSW2002] by Mumford-Series-Wright.) It was indeed a shocking event for the author of this survey (who was ignorant of deformation theory and had no idea that it has something to do with knot theory) to learn that a simple topological object, such as the figure-eight knot, carries such mysterious mathematics under cover.

In conclusion, the flexibility of 3-dimensional hyperbolic structure has enriched knot theory by bringing the concept of deformation into knot theory. (See Sections 7 and 10.)

In this review, we also consider the role of the classical knot invariants, the Alexander polynomial and the homology of finite branched/unbranched coverings. After the appearance of Thurston’s work and the Jones polynomials, the role of these invariants in knot theory might have decreased. However, they continue to be important themes in knot theory. For the Alexander polynomial, its twisted version was defined by Lin [Lin2001] for classical knots and by Wada [Wd1994] in general setting. For a hyperbolic knot, we can consider the hyperbolic torsion polynomial (see [DFJ2012]) as the most natural twisted Alexander polynomial, and beautiful Thurstonian connection (cf. [AD* Section 1.2]) between the topology and geometry of knots is found (see Subsection 12.2). For the homology of finite
branched/unbranched coverings of a knot, Thang Le [Le2018] has proved a mysterious relation between the asymptotic growth of the order of the torsion part and the Gromov norm of the knot (see Subsection 13.4). This result is particularly surprising to the author of this survey, for whom homology of finite coverings is a favorite invariant, but who had never imagined that the whole family of the familiar invariant could contain such deep geometric information.

**Acknowledgements.** The author would like to thank Ken’ichi Ohshika for giving him the challenging opportunity to survey the tremendous impact of Thurston’s work on knot theory. He is also grateful to François Guéritaud, Luisa Paoluzzi, and Han Yoshida for correcting errors and providing valuable comments for Sections 10, 4 and 9 respectively. The author would also like to thank Yuya Koda, Gaven Martin, and Hitoshi Murakami for reading through an early version and for sending him a large number of valuable suggestions and corrections. The author’s thanks also go to Hirota Akiyoshi, Warren Dicks, Hiroshi Goda, Kazuhiro Ichihara, Yuichi Kabaya, Takuya Katayama, Akio Kawamura, Eiko Kin, Thang Le, Hidetoshi Masai, José María Montesinos, Kimihiko Motegi, Kunio Murasugi, Shunsuke Sakai, Masakazu Teragaito, Ken’ichi Yoshida, and Bruno Zimmermann for their valuable information and suggestions on early versions of this survey. Finally, the author would like to thank the referee for his/her extremely careful reading and valuable suggestions, including Remark 7.5(1).

## 2. Knot theory before Thurston

In this section, we recall basic definitions and the classical results in knot theory, which were mostly obtained before knot theory was influenced by Thurston’s work: (i) genera of knots, (ii) Schubert’s unique prime decomposition theorem, (iii) knot groups, consequences of Waldhausen’s work on Haken manifolds, and Gordon-Lueke knot complement theorem, (iv) fibered knots and open book decompositions, (v) definition of the Alexander polynomial, and its effectiveness and weakness, and (vi) representations of knot groups in finite groups.

The book of Adams [Ad1994] is a wonderful introduction to knot theory. For classical results in knot theory, see the text books, Crowell-Fox [CrF1963], Rolfsen [Ro1976], Kauffman [Kf1987], Burde-Zieschang [BrZs1985], Kawauchi [Kw1996], Murasugi [Mrs1996], Lickorish [Lcr1997], Livingston [Liv1993], Prasolov-Sossinsky [PS1997], Cromwell [Crm2004] and Burde-Zieschang-Heusener [BrZsH2014]. See also the special issue edited by Adams [Ad1998] and the handbook Menasco-Thistlethwaite [MeT2005].

### 2.1. The fundamental problem in knot theory

A **knot** $K$ is a smoothly (or piecewise-linearly) embedded circle in the 3-sphere $S^3 = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1 \}$. Two knots $K$ and $K'$ are said to be...
equivalent, denoted by $K \cong K'$, if there is a self-homeomorphism $f$ of $S^3$ such that $f(K) = K'$, i.e., the pair $(S^3, K)$ is homeomorphic to the pair $(S^3, K')$. If the homeomorphism $f$ preserves the orientation of $S^3$ and hence is isotopic to the identity homeomorphism $1_{S^3}$, then $K$ and $K'$ are said to be isotopic. We do not distinguish between a knot $K$ and the equivalence/isotopy class represented by $K$.

A knot is trivial if it is isotopic to a standard circle $O := \{(z_1, 0) \in S^3 \mid |z_1| = 1\}$.

Every knot is represented by a knot diagram, a 4-valent planar graph whose vertices are endowed with over/under information. A vertex of a knot diagram with over/under information is called a crossing.

For a knot $K$, we denote by $K^*$ the image of $K$ by an orientation-reversing homeomorphism of $S^3$, and call it the mirror image of $K$. $K^*$ is represented by the knot diagram which is obtained from that of $K$ by reversing the over/under information at every crossing. A knot $K$ is achiral (or amphichiral) if $K^*$ is isotopic to $K$, otherwise it is chiral.

An oriented knot is a knot $K$ where the circle $K$ is also endowed with an orientation. (We assume that $S^3$ is endowed with the standard orientation.) Two oriented knots $K$ and $K'$ are said to be isotopic, if there is an orientation-preserving self-homeomorphism $f$ of $S^3$ with $f(K) = K'$, such that $f|_K : K \to K'$ is also orientation-preserving. This is equivalent to the condition that there is an isotopy of $S^3$ which carries the oriented circle $K$ to the oriented circle $K'$. For a given oriented knot $K$, we obtain the following three (possibly isotopic) oriented knots, by reversing one or both of the orientations of $S^3$ and the circle $K$ (see Figure 1):

$-K := (S^3, -K)$, \quad $K^* := (-S^3, K) \cong (S^3, K^*)$, \quad $-K^* := (-S^3, -K) \cong (S^3, -K^*)$

A knot $K$ is invertible, positive-amphichiral, or negative-amphichiral, respectively, if $K$ is isotopic to $-K$, $K^*$, or $-K^*$. If the symmetry can realize by an involution, then we say that $K$ is strongly invertible, strongly positive-amphichiral, or strongly negative-amphichiral, respectively.

It is one of the most fundamental problems in knot theory to detect whether two given knots $K$ and $K'$ are equivalent or not, in particular if a given knot $K$ is trivial or not. The problem to detect whether a given knot is chiral (or invertible) is a special case of a refinement for oriented knots of this fundamental problem.

At the end of this section, we note that the first proof of the existence of non-invertible knots due to Trotter [Tr1963] essentially uses 2-dimensional hyperbolic geometry (see the paragraph after Theorem 4.3).

2.2. Seifert surface

A Seifert surface of a knot $K$ in $S^3$ is a connected compact orientable surface $\Sigma$ in $S^3$ with $\partial \Sigma = K$. The existence of a Seifert surface was first proved by Frankel and Pontryagin [FP1930], through a smooth map $f : S^3 - K \to S^1$ which represents

\[\text{This follows [CrF1963], though the terminology "amphichiral" seems to be more popular.}\]
Figure 1. The pretzel knot $P(-3, 5, 7)$ with four different orientations. These oriented knots are not non-isotopic to each other.

Figure 2.

a generator of $H^1(S^3 - K; \mathbb{Z}) \cong \mathbb{Z}$, as the closure of inverse image $f^{-1}(b)$ of a regular point $b \in S^1$. Later, Seifert gave a simple effective method, called the Seifert algorithm, for constructing a Seifert surface from an oriented knot diagram (see Figure 3). The genus $g(K)$ of a knot $K$ is the minimum of the genera of Seifert surfaces for $K$. This is one of the most fundamental invariants of a knot, generalized by Thurston to the concept of Thurston norm. The trivial knot $O$ is characterized by the property $g(O) = 0$. 

Figure 2.
Figure 3. Seifert algorithm: By smoothing all crossings of a knot diagram, we obtain Seifert circles (mutually disjoint circles in the plane). Construct mutually disjoint disks in $\mathbb{R}^3$ bounded by the Seifert circles, and join them by bands. The resulting surface is a Seifert surface.

2.3. The unique prime decomposition of a knot

We recall Shubert’s unique prime decomposition theorem, which reduces the classification problem of knots to that of prime knots. For a given oriented knots $K_1$ and $K_2$, we can define the composition $K_1 \# K_2$ as the pairwise connected sum $(S^3, K_1) \# (S^3, K_2)$ of oriented manifold pairs, as in Figure 4. With respect to the connected sum, the set of all oriented knots up to isotopy becomes a commutative semi-group having the trivial knot $O$ as the unit.

A knot $K$ is prime if $K \cong K_1 \# K_2$ implies $K_1 \cong O$ or $K_2 \cong O$. It is a classical theorem due to Schubert [Scb1949] that every oriented knot has a unique prime decomposition.

Theorem 2.1 (The unique prime decomposition of knots). Every nontrivial oriented knot $(S^3, K)$ can be decomposed as the sum of finitely many nontrivial prime oriented knots. Moreover if $K \cong K_1 \# K_2 \# \cdots \# K_n$ and $K \cong J_1 \# J_2 \# \cdots \# J_m$ with
each $K_i$ and $J_i$, nontrivial prime knots, then $m = n$, and after reordering, $K_i \cong J_i$ as oriented knots.

The existence of a prime decomposition is guaranteed by the additivity of the genus with respect to the connected sum, i.e.,

$$g(K_1 \# K_2) = g(K_1) + g(K_2)$$

for any oriented knots $K_1$ and $K_2$.

The uniqueness of the prime decomposition is proved by a simple cut and paste argument.

2.4. Knot complements and knot groups

The exterior of a knot $K$ is defined by $E(K) := S^3 - \text{int}N(K)$, where $N(K)$ is a regular neighborhood of $K$. The knot complement $S^3 - K$ is homeomorphic to the interior of $E(K)$, and the fundamental group $\pi_1(S^3 - K) \cong \pi_1(E(K))$ is called the knot group, and denoted by $G(K)$. By using the sphere theorem [Papa1957](#Papa1957), we can see that $E(K)$ is aspherical, and hence the homotopy type of $E(K)$ is completely determined by the knot group $G(K)$. A group presentation, called the Wirtinger presentation, of $G(K)$ can be obtained from a knot diagram of $K$ (see [CrF1963](#CrF1963) [Ro1976](#Ro1976)).

The peripheral subgroup $P(K)$ of the knot group $G(K)$ is defined as (the conjugacy class of) the image of the homomorphism $j_* : \pi_1(\partial E(K)) \rightarrow \pi_1(E(K))$ induced by the inclusion map $j : \partial E(K) \rightarrow E(K)$. For the trivial knot $O$, $E(O)$ is homeomorphic to the solid torus, and so $G(O) = P(O) \cong \mathbb{Z}$. Dehn’s lemma, established by Papakyriakopoulos [Papa1957](#Papa1957), gives the following characterization of the trivial knot.

**Theorem 2.2.** A knot $K$ is trivial if and only if the following mutually equivalent conditions hold.

1. $\text{Ker}[j_* : \pi_1(\partial E(K)) \rightarrow \pi_1(E(K))]$ is nontrivial.
2. The peripheral subgroup $P(K)$ is isomorphic to $\mathbb{Z}$.
3. The knot group $G(K)$ is isomorphic to $\mathbb{Z}$.

If $K$ is a nontrivial knot, then the peripheral subgroup $P(K) \cong \pi_1(\partial E(K)) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by two special elements, a meridian $\mu$ and a longitude $\lambda$, which are represented by the simple loops $\mu := \partial D^2 \times \{\ast\}$ and $\lambda := \{\ast\} \times S^1$, respectively, in $\partial E(K) = \partial N(K) = \partial(D^2 \times S^1)$. Here the framing $N(K) \cong D^2 \times S^1$ is chosen so that the linking number $\text{lk}(K, \lambda) = 0$. When $K$ is oriented, the orientations of $\mu$ and $\lambda$ are chosen so that $\text{lk}(K, \mu) = +1$ and that $K$ and $\lambda$ are homologous in $N(K)$.

The classical work of Waldhausen [Wld1968](#Wld1968) on Haken manifolds implies the following theorem which reduces the equivalence problem for knots to a problem of knot groups.

**Theorem 2.3.** For two knots $K$ and $K'$, the following hold.
(1) $E(K)$ and $E(K')$ are homeomorphic if and only if $(G(K), P(K))$ and $(G(K'), P(K'))$ are isomorphic, i.e., there is an isomorphism $\varphi : G(K) \to G(K')$ such that $\varphi(P(K)) = P(K')$ up to conjugacy.

(2) $K$ and $K'$ are equivalent if and only if $(G(K), P(K))$ and $(G(K'), P(K'))$ are isomorphic, i.e., there is an isomorphism $\varphi : (G(K), P(K)) \to (G(K'), P(K'))$ such that $\varphi(\mu) = \mu'^{\pm 1}$ up to conjugacy.

For nontrivial oriented knots $K_1$ and $K_2$, the knot groups $G(K_1 \# K_2)$ and $G(K_1 \# (-K_2^*))$ are isomorphic. In fact, both $E(K_1 \# K_2)$ and $E(K_1 \# (-K_2^*))$ are obtained from $E(K_1)$ and $E(K_2)$ by gluing annuli in their boundaries, and so homotopy equivalent to the space obtained from $E(K_1)$ and $E(K_2)$ by identifying the meridians $\mu_1$ and $\mu_2$. On the other hand, by the unique prime decomposition Theorem 2.1, the oriented knots $K_1 \# K_2$ and $K_1 \# (-K_2^*)$ are isotopic if and only if $K_2$ is negative amphichiral (i.e., $-K_2^*$ is isotopic to $K_2$). Thus, in general, the knot group alone is not a complete invariant for knots.

Building on the cyclic surgery theorem (Theorem 11.2) by Culler, Gordon, Luecke and Shalen [CGLS1987], Whitten [Whn1987] proved that prime knots with isomorphic knot groups have homeomorphic exteriors. On the other hand, we have the following celebrated theorem of Gordon and Luecke [GL1989].

**Theorem 2.4** (Knot Complement Theorem). Two knots are equivalent if and only if they have homeomorphic complements.

Thus we have the following theorem.

**Theorem 2.5.** Two prime knots are equivalent if and only if they have isomorphic knot groups.

### 2.5. Fibered knots

A knot $K$ is **fibered** if $E(K)$ has the structure of a bundle over the circle, namely, there is a connected compact orientable surface $\Sigma$ and an orientation-preserving homeomorphism $\varphi : \Sigma \to \Sigma$, such that

$$E(K) \cong \Sigma \times [0, 1]/(x, 0) \sim (\varphi(x), 1).$$

The homeomorphism $\varphi$ is called the **monodromy** of the fiber structure. Each fiber $\Sigma$ of the bundle structure is a compact orientable surface in $E(K)$ such that $\Sigma \cap \partial E(K) = \partial \Sigma$ is a longitude of $K$. The union of $\Sigma$ and an annulus in $N(K)$ cobounded by $\partial \Sigma$ and $K$ is a minimal genus Seifert surface for $K$. This is the unique minimal genus Seifert surface for $K$ up to isotopy fixing $K$.

We may choose $\varphi$ so that its restriction to $\partial \Sigma$ is the identity map and thus the image of $y \times [0, 1]$ in $E(K)$ is a meridian of $K$ for every $y \in \partial \Sigma$. Then

$$(S^3, K) \cong (\Sigma, \partial \Sigma) \times [0, 1]/[(x, 0) \sim (\varphi(x), 1); \ y \times [0, 1] \sim y \ (\text{for } y \in \partial \Sigma)].$$
This structure is called an open book decomposition with binding $K$, and the homeomorphism $\varphi$ is called the monodromy of the fibered knot $K$. It was proved by Alexander [Al1920] that every connected closed orientable 3-manifold admits an open book decomposition. Later, Giroux [Gi2002] found a very important correspondence between the open book decompositions (up to positive stabilization) of a given closed oriented 3-manifold $M$ and oriented contact structures on $M$ up to isotopy (see [Et2006] for details). The following characterization of fibered knots in terms of knot groups was proved by Stallings [St1961], and attracted the attention of researchers at the time.

**Theorem 2.6.** A knot $K$ in $S^3$ is a fibered knot if and only if the commutator subgroup $G(K)' = [G(K), G(K)]$ is finitely generated.

The only if part follows from the fact that the infinite cyclic covering $E_\infty(K)$ of $E(K)$, introduced in the subsection below, is identified with $\Sigma \times \mathbb{R}$ and so $G(K)' \cong \pi_1(E_\infty(K)) \cong \pi_1(\Sigma)$ is a free group of rank $2g(K)$. The heart of the theorem is that the converse also holds.

### 2.6. Alexander invariants

Though the knot group is a complete invariant for prime knots, it is, in general, not easy to distinguish two given knot groups. The Alexander polynomial serves a convenient and tractable tool for this problem, even though it is not almighty.

Let $K$ be an oriented knot. Then the first integral homology group $H_1(E(K); \mathbb{Z})$ is the infinite cyclic group generated by the image, $t$, of the meridian $\mu$. Thus there is a unique infinite cyclic covering $p_\infty : E_\infty(K) \to E(K)$, and the covering transformation group is identified with the infinite cyclic group $(t)$ generated by $t$. $H_1(E_\infty(K); \mathbb{Z})$ has the structure of a module over the integral group ring $\mathbb{Z}(t)$. This module is called the knot module. As an abelian group, $H_1(E_\infty(K); \mathbb{Z})$ is identified with $G(K)' \cong G_1(\Sigma)$ where $G(K)'$ and $G(K)''$, respectively, are the first and second commutator subgroups of $G(K)$. Moreover the action of the generator $t$ is given by $t[\alpha] = [\mu\alpha\mu^{-1}]$ for $\alpha \in G(K)'$, where $\mu$ is a meridian. Thus the knot module is determined by $G(K)$. In fact, a presentation matrix is obtained from a presentation of the knot group, via Fox's free differential calculus (see [CrF1963], [Kw1996 Chapter 7]). The Alexander polynomial $\Delta_K(t)$ of $K$ is defined as the generator of the first elementary ideal of the knot module.

A more conceptual definition can be given by using the $\mathbb{Q}(t)$-module $H_1(E_\infty(K); \mathbb{Q})$ as follows. Since the rational group ring $\mathbb{Q}(t)$ is a principal ideal domain and since $H_1(E_\infty(K); \mathbb{Q})$ is a finitely generated torsion module over $\mathbb{Q}(t)$, we have

$$H_1(E_\infty(K); \mathbb{Q}) \cong \mathbb{Q}(t) / \left( f_1(t) \right) \oplus \cdots \oplus \mathbb{Q}(t) / \left( f_r(t) \right),$$

where $f_i(t)$ are elements of $\mathbb{Z}(t)$ whose coefficients are relatively prime. Then $\Delta_K(t) \doteq f_1(t) \cdots f_r(t)$, where $\doteq$ means the equality up to multiplication by a unit.
$t^i$ of the integral Laurent polynomial ring $\mathbb{Z}\langle t \rangle$. The Alexander polynomial $\Delta_K(t)$ is an integral Laurent polynomial in the variable $t$, defined up to multiplication of a unit. For the trivial knot $O$, we have $\Delta_O(t) = 1$. We summarize basic properties of the Alexander polynomial.

**Theorem 2.7.** (1) For any knot $K$, its Alexander polynomial $\Delta_K(t)$ satisfies the following condition.

$$\Delta_K(1) = \pm 1, \quad \Delta_K(t^{-1}) = \Delta_K(t)$$

Conversely, for any Laurent polynomial $\Delta(t)$ satisfying the above condition, there is a knot $K$ whose Alexander polynomial is equal to $\Delta(t)$.

(2) For every knot $K$ in $S^3$, we have the following estimate of the genus:

$$g(K) \geq \deg \Delta_K(t).$$

(3) For any fibered knot $K$, the Alexander polynomial $\Delta_K(t)$ is monic, and the equality hold in the estimate (2).

Proof of Theorem 2.7 relies on analysis of the manifold $M := E(K) \setminus \Sigma$, the manifold obtained form $E(K)$ by cutting along a Seifert surface $\Sigma$, in other words, $M$ is the complement of an open regular neighborhood of $\Sigma$ in $E(K)$. Let $\Sigma_+$ and $\Sigma_-$ be copies of $\Sigma$ on $\partial M$, and consider the annulus $A := M \cap \partial E(K)$. Then $(M, \Sigma_+, \Sigma_-, A)$ is a sutured manifold (see [Gb1983a, Gb1984], [Kw1996, Chapter 5]), and this together with the natural homeomorphism $\Sigma_+ \to \Sigma_-$ recovers $E(K)$.

The infinite cyclic covering $E_\infty(K)$ is obtained from the set of copies $\{M_n\}$ of $M$ indexed with $n \in \mathbb{Z}$, by gluing the copy of $\Sigma_-$ in $M_n$ with the copy of $F_+$ in $M_{n+1}$. The homological glueing information is given by the Seifert matrix $V = (\text{lk}(\alpha_i, \alpha_j^+))_{1 \leq i, j \leq 2g}$, where $\{\alpha_i\}_{1 \leq i, j \leq 2g}$ with $g = 2g(\Sigma)$ is a set of oriented simple loops on $\Sigma$ which forms a basis of $H_1(\Sigma)$, $\alpha_j^+$ is a copy of $\alpha_j$ on the $+$-side of $\Sigma$, and $\text{lk}(\cdot, \cdot)$ denotes the linking number. The matrix $tV - V^T$ gives a presentation matrix of $H_1(E_\infty(K))$ as an $\mathbb{Z}\langle t \rangle$, and hence $\Delta_K(t) = \det(tV - V^T)$. Using this formula we can prove Theorem 2.7.

For knots with small crossing numbers, the Alexander polynomial is quite efficient. For any prime knot $K$ up to 10 crossings, equality holds in the estimate of the genus in Theorem 2.7(2). Moreover, such a knot $K$ is fibered if and only if $\Delta_K(t)$ is monic (see Kanenobu [Kn1979]).

The Alexander polynomial is also very efficient for alternating knots. A knot $K$ is said to be alternating if it is represented by an alternating diagram, namely a diagram in which the crossings alternate under and over as one travels along the diagram. A knot diagram is said to be reduced if there is no circle in the plane which intersects the diagram only at a single crossing. Any alternating diagram can be deformed into a (possible trivial) reduced alternating diagram. When $K$ is an alternating knot and $\Sigma$ is a Seifert surface obtained by Seifert algorithm from a reduced alternating diagram of $K$, the complementary sutured manifold $(M, \Sigma_+, \Sigma_-, A)$ has a nice structure,
which in particular implies \( \det(V) \neq 0 \). This shows that the estimate Theorem 2.7(2) is sharp for alternating knots (see Crowell [Crw1955] and Murasugi [Mrs1958]). Moreover, Murasugi [Mrs1963] proved that the converse to Theorem 2.7(3) also holds for alternating knots.

**Theorem 2.8.** For any alternating knot \( K \), the following hold.

1. \( g(K) = \deg \Delta_K(t) \).
2. \( K \) is fibered if and only if \( \Delta_K(t) \) is monic.

In order to prove the above results, Murasugi introduced the concept of a Murasugi sum of two Seifert surfaces. The simplest case corresponds to the connected sum of knots and the second simplest case corresponds to plumbing introduced by Stallings [St1976]. It was later shown by Gabai [Gb1983b] that the Murasugi sum is a natural geometric operation in the following sense: If \( \Sigma \) is a Murasugi sum of \( \Sigma_1 \) and \( \Sigma_2 \), then the following hold.

1. \( \Sigma \) is of minimal genus if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are of minimal genus.
2. \( \Sigma \) is a fiber surface if and only if \( \Sigma_1 \) and \( \Sigma_2 \) are fibre surfaces.

In addition to Theorems 2.7 and 2.8, various applications of the Alexander polynomials were found. Among them, we explain a theorem by Kinoshita [Kn1958], which gives a condition on the Alexander polynomial that a counter-example to the Smith Conjecture must satisfy. As described in Subsection 1.4, the Smith conjecture (Theorem 4.6) was later proved by using Thurston’s geometrization theorem for Haken manifolds.

**Theorem 2.9.** If \( K \) is a fixed point of an orientation-preserving periodic diffeomorphism of period \( n \), then there is an integral Laurent polynomial \( f(t) \) such that

1. \( \Delta_K(t^n) = \prod_{i=0}^{n-1} f(\xi^i t) \) where \( \xi \) is a primitive \( n \)-th root of unity, and
2. \( f(1) = \pm 1 \), \( f(t^{-1}) = f(t) \).

See [Kw1996, Chapter 10] for other applications of the Alexander polynomial to the study of symmetry of knots, including the first proof of the non-invertibility of the knot 8\( _{17} \) by Kawauchi [Kw1979], answering to a question of Fox (cf. [Fx1962, Problem 10]). (Another proof of the non-invertibility of 8\( _{17} \) was announced almost at the same time by Bonahon and Siebenmann, which is based on their characteristic splitting theory (see Section 4).) It should be noted that though the definition of the Alexander polynomial depends on the orientation of \( K \), the resulting \( \Delta_K(t) \) does not depend on the orientation by Theorem 2.7(1). It is interesting that, despite this fact, the Alexander polynomial can be used for the study of invertibility and chirality of knots. Finally, we point out that the Alexander module \( H_1(E_\infty(K)) \) does depend on the orientation of \( K \), though it is not easy to detect the dependence (see [FxS1964, Hl1981]).

Though we have observed the effectiveness of the Alexander polynomial, there are a lot of knots for which the Alexander polynomial is useless. In fact, H. Seifert
J. H. C. Whitehead and Kinoshita-Terasaka gave systematic construction of nontrivial knots with the trivial Alexander polynomial. For example, the pretzel knot $K(-3, 5, 7)$ in Figure 4, the Whitehead double of any nontrivial knot (cf. Figure 6), the Kinoshita-Terasaka knot and the Conway knot in Figure 5 have trivial Alexander polynomial.

Figure 5. The Kinoshita-Terasaka knot and the Conway knot: The circles in the figure represent the Conway spheres which determine the Bonahon-Siebenmann decompositions, described in Subsection 4.1. From this picture, we can see that the Conway knot is a mutant of the Kinoshita Terasaka knot.

The Conway knot is a mutant of the Kinoshita-Terasaka knot (see Construction 4 in Subsection 4.5 for the precise definition). It is known that various invariants coincide for a knot and its mutant, including the Alexander polynomial, the Jones polynomial, the Homflypt polynomial, the double branched covering, and Gromov norm. So it is not easy to distinguish a knot from its mutant.

2.7. Representations of knot groups onto finite groups

The definition of the Alexander polynomial is based on the fact that the knot group $G(K)$ of an oriented knot $K$ admits a unique preferred epimorphism onto the infinite cyclic group $(t)$. By replacing $\mathbb{Z}$ with an arbitrary group $\Gamma$, we obtain the following family of invariants of knots. Let $R(G(K), \Gamma)$ be the set of homomorphisms from $G(K)$ to $\Gamma$, up to conjugacy (i.e., modulo post composition of inner-automorphisms of $G$), is an invariant of $G(K)$. Then its cardinality $|R(G(K), \Gamma)|$ is an invariant of the knot group $G(K)$. We may also consider the quotient of $R(G(K), \Gamma)$ by the action of the automorphism group of $\Gamma$.

Fix a conjugacy class $\gamma$ of an element of $\Gamma$, and let $R(G(K), \Gamma, \gamma)$ be the subset of $R(G(K), \Gamma)$ consisting of the homomorphisms which map the meridian to an element
in the conjugacy class $\gamma$. Then the cardinality $|R(G(K), \Gamma, \gamma)|$ is again an invariant of the oriented knot $K$. If $\Gamma$ is the dihedral group $D_{2p} = \langle a, t \mid a^p, t^2, tat^{-1} = a^{-1} \rangle$ of order $2p$ and if $\gamma$ is the conjugacy class of the element $t$, the Fox $p$-coloring number \cite{FX1961} is essentially equal to $|R(G(K), D_{2p}, t)|$.

Fix a transitive representation of $\Gamma$ to the symmetric group $S_n$ of degree $n$, where $n$ is possibly infinite. Then for each $\phi \in R(G(K), \Gamma)$, we have a (possibly disconnected) $n$-fold covering $E_\phi(K)$ of $E(K)$. Then the family of homology groups $\{H_1(E_\phi(K); \mathbb{Z})\}_{\phi \in R(G(K), \Gamma)}$ forms an invariant of the knot $K$. Furthermore, if the image of the element $\gamma$ in $S_n$ is of finite order, then we obtain a branched covering $M_\phi(K)$ of $S^3$ branched over $K$. The family of homology groups $\{H_1(M_\phi(K); \mathbb{Z})\}_{\phi \in R(G(K), \Gamma, \gamma)}$ forms another invariant of $K$. We can also consider the torsion linking numbers among the components of the inverse image of $K$.

Riley \cite{Ri1971} applied this method by choosing $\Gamma$ to be the simplest finite simple group $\text{PSL}(2, p)$, with $p$ a prime $\geq 5$, and setting $\gamma$ to be the parabolic transformation $\left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$. This enabled him to prove that the Kinoshita-Terasaka knot and the Conway knot are different. His proof also showed that none of them is amphicheiral. He then considered parabolic representations of knot groups to $\text{PSL}(2, \mathbb{C})$, and this led him to the discovery of the complete hyperbolic structure of the Figure-eight knot complement in \cite{Ri1975}.

Hartley \cite{Har1983} realized that one can apply this method to the problem of identifying noninvertible knots, as follows. Suppose no automorphism of $\Gamma$ maps $\gamma$ to $\gamma^{-1}$. Then the set $R(G(K), \Gamma, \gamma)$ is possibly different from the set $R(G(K), \Gamma, \gamma^{-1})$, and there is a chance to show noninvertibility of $K$ by comparing the homology invariants associated with $\phi \in R(G(K), \Gamma, \gamma)$ with those associated with $\phi' \in R(G(K), \Gamma, \gamma^{-1})$. Hartley showed that this method is quite effective: he completely determined the 36 non-invertible knots up to 10 crossings claimed by Conway to be noninvertible.

A variation of the method is to consider the subset $R_t(G(K), S_n, \gamma)$ of $R(G(K), S_n, \gamma)$ consisting of the transitive representations, where $n$ is a finite positive integer. By virtue of the development of computer, this turns out to be an extremely efficient method for distinguishing knots. In fact, Thistlethwaite \cite{Ths1985} succeeded in distinguishing knots up to 11 crossings, and later the same method was applied successfully to knots up to 16 crossings in \cite{HTM1998}.

3. The geometric decomposition of knot exteriors

The purpose of this section is to explain the geometric decompositions of knot exteriors into Seifert pieces and hyperbolic pieces, obtained as a special case of the Thurston’s geometrization theorem of Haken manifolds.

We recall (i) the prime decomposition theorem of general compact orientable 3-manifolds (ii) the torus decomposition theorem of compact irreducible orientable
3-manifolds, (iii) the eight homogeneous 3-dimensional geometries, and (iv) the geometrization conjecture, which was finally established by Perelman. In the final subsection, we give detailed exposition of the geometric decompositions of knot exteriors.

For standard facts in 3-manifold theory, see the short note Hatcher [Ht*] and the textbooks Hempel [Hm1976], Jaco [Jc1980], Jaco-Shalen [JS1979], Johannson [Jh1979] and Schultens [Scl2014]. For an introduction to geometric structures, see the surveys Scott [Sct1983] and Bonahon [Bn2002] and the textbook Martelli [Mrt2016].

3.1. Prime decomposition of 3-manifolds

In this subsection, we recall the canonical decomposition of compact orientable 3-manifolds by 2-spheres. Let $M$ be a compact connected orientable 3-manifold. A 2-sphere in $M$ is \textit{essential} if it does not bound a 3-ball in $M$. $M$ is \textit{irreducible} if it contains no essential 2-sphere. Suppose $M$ is not irreducible, and let $S$ be an essential 2-sphere in $M$. If $S$ is \textit{separating} (i.e., $M - S$ consists of two components), then $M$ is the \textit{connected sum} $M_1 \# M_2$ of the two compact orientable 3-manifolds $M_1$ and $M_2$, which are obtained from the closures of the components by capping off the resulting sphere boundaries by adding 3-balls. If $S$ is \textit{non-separating} (i.e., $M - S$ is connected), then $M$ is expressed as the connected sum $(S^2 \times S^1) \# M'$, of $S^2 \times S^1$ with some compact orientable 3-manifold $M'$ (possibly $S^3$).

$M$ is \textit{prime} if whenever $M = M_1 \# M_2$ we have $M_i \cong S^3$ for $i = 1$ or 2. Then we have the following Kneser-Milnor unique prime decomposition theorem [Kn1929, Mln1962] (cf. [Hm1976]).

\textbf{Theorem 3.1} (Unique prime decomposition of compact orientable 3-manifolds). Any compact orientable 3-manifold admits a decomposition $M = P_1 \# \cdots \# P_n$ into prime manifolds $\{P_i\}$. Moreover, the prime factors $\{P_i\}$ are uniquely determined by $M$, up to change of the indices.

3.2. Torus decomposition of irreducible 3-manifolds

In this subsection, we explain the torus decomposition theorem for compact orientable irreducible 3-manifolds. Torus decomposition is a simple version of more intricate JSJ (Jaco-Shalen-Johannson) decomposition, in which decompositions along annuli are also involved. The JSJ decomposition theory grew out of the study to understand homotopy equivalences among 3-manifolds, and its simplified version, the torus decomposition, turned out to be a complete obstruction for hyperbolization of a 3-manifold.

Throughout this subsection, $\Sigma$ denotes a compact orientable surface in $M$ which is \textit{properly embedded} in $M$, i.e., $\Sigma \cap M = \partial \Sigma$. We also assume that $\Sigma \not\cong S^2$. Then $\Sigma$ is \textit{incompressible} in $M$ if for any disk $D$ in $M$ such that $D \cap \Sigma = \partial \Sigma$, the simple
loop $\partial D$ bounds a disk in $\Sigma$. By the loop theorem (see [Hm1976]), this is equivalent to the algebraic condition that the homomorphism $j_* : \pi_1(\Sigma) \to \pi_1(M)$ induced by the inclusion is injective.

$M$ is Haken if it is irreducible and contains a properly embedded compact orientable surface which is incompressible.

A surface $\Sigma$ in $M$ is essential if it is incompressible and is not $\partial$-parallel, i.e., $\Sigma$ does not cut off a 3-manifold in $M$ homeomorphic to $\Sigma \times I$. $M$ is atoroidal if it does not contain an essential torus.

$M$ is a Seifert fibered space if it is expressed as a union of disjoint circles, in a particular way. The quotient of $M$ obtained by collapsing each fiber into a point has the structure of a 2-dimensional orbifold, and is called the base orbifold. If $M$ admits a smooth $S^1$ action without a fixed point (i.e., the stabilizer of any point is not the whole group $S^1$), then $M$ is a Seifert fibered space whose base orbifold is the orbit space $M/S^1$. Seifert fibered spaces are regarded as $S^1$-bundles over 2-dimensional orbifolds, and are completely described by the Seifert invariants. See [Hm1976] for details.

Now we state the torus decomposition theorem, which is a simplified version of the JSJ decomposition theorem due to Jaco and Shalen [JS1979] and Johannson [Jh1979]. (See [NS1997] and [CS2002] for an alternative proof, and see [Ht*] for a simple proof of the torus decomposition theorem.)

**Theorem 3.2** (Torus decomposition theorem). For a compact orientable irreducible 3-manifold $M$, there is a unique (up to isotopy) family $\mathcal{T}$ of disjoint essential tori, satisfying the following properties.

(a) Each closed up component of $M - \mathcal{T}$ is either a Seifert fibered space or atoroidal.
(b) If any component of $\mathcal{T}$ is deleted, Property (a) fails.

In the above theorem, by a closed up component $M - \mathcal{T}$, we mean the closure of a component of $M - N(\mathcal{T})$, where $N(\mathcal{T})$ is a regular neighborhood of $\mathcal{T}$. The subsurface $\mathcal{T}$ is called the characteristic toric family of $M$, and each closed up component of $M - \mathcal{T}$ is called a JSJ-piece of $M$.

It should be noted that the family $\mathcal{T}$ are not only unique up to homeomorphisms but also unique up to isotopy. This forms a sharp contrast to the fact that in the prime decomposition theorem, the family of the splitting spheres are not unique even up to homeomorphisms. (It only says that the resulting prime manifolds are unique up to homeomorphisms.)

3.3. The Geometrization Conjecture of Thurston

Thurston’s geometrization conjecture says that any compact orientable irreducible 3-manifold has a canonical splitting, by tori, into pieces which admit one of the following eight homogeneous geometries.

- The spaces of constant curvature, $S^3$, $E^3$ and $H^3$;
The product spaces $S^2 \times \mathbb{E}^1$ and $\mathbb{H}^2 \times \mathbb{E}^1$; and
The 3-dimensional Lie groups Nil, Sol, and $\text{SL}_2(\mathbb{R})$.

Here a compact connected orientable 3-manifold $M$ is geometric if either it is a 3-ball or its interior can be presented as the quotient $\text{int} M = X/\Gamma$, of one of the above homogeneous spaces, $X$, by a discrete group $\Gamma$ of isometries, acting freely and discontinuously on $X$. If $X = \text{Sol}$ then $M$ is a bundle over $S^1$ or the 1-dimensional orbifold $S^1/(z \sim \bar{z})$ with torus fiber; if $X$ is neither Sol nor $\mathbb{H}^3$, then $M$ is a Seifert fibered space and it is completely described by Seifert invariants. Conversely any Seifert fibered space admits one of the 6 remaining geometries. See the nice expositions [Bn2002, Sct1983] for details.

Thus we have a complete topological classification of the geometric manifolds with $X$ geometry for $X \neq \mathbb{H}^3$, and the study of hyperbolic manifolds forms the crucial part in 3-manifold theory and knot theory.

Thurston proposed the following geometrization theorem as a conjecture, which says that the torus decomposition gives a complete obstruction for a compact orientable 3-manifold to be hyperbolic. Here $M$ is said to be hyperbolic if its interior can be presented as the quotient $\text{int} M = \mathbb{H}^3/\Gamma$ by a discrete torsion-free group $\Gamma$ of isometries of $\mathbb{H}^3$ (cf. Section 15). Thurston proved the conjecture for various cases, including the case when $M$ is Haken, and the whole conjecture was finally proved by Perelman [Pe2002*, Pe2003a*, Pe2003b*]. (See [Mrg2004, Mc2011, Mc2014] for a survey, and see [CZ2006, MrgT2007, MrgT2007, BBBMP2010] for detailed exposition.)

**Theorem 3.3** (Geometrization theorem). Let $M$ be a connected irreducible atoroidal compact orientable 3-manifold. Then $M$ is either a Seifert fibered space or a hyperbolic manifold.

By combining Theorems 3.2 and 3.3 we obtain the following geometric decomposition theorem.

**Theorem 3.4** (Geometric decomposition theorem). For a compact orientable irreducible 3-manifold $M$, there is a unique (up to isotopy) family $\mathcal{T}$ of disjoint essential tori, satisfying the following properties.

(a) Each closed up component of $M - \mathcal{T}$ is either a Seifert fibered space or a hyperbolic manifold.

(b) If any component of $\mathcal{T}$ is deleted, Property (a) fails.

In the above theorem, a closed up component of $M - \mathcal{T}$ is called a Seifert piece or a hyperbolic piece according to whether it is a Seifert fibered space or a hyperbolic manifold.
3.4. Geometric decompositions of knot exteriors

We describe a consequence for knot exteriors of the torus decomposition theorem and the geometrization theorem described in the previous subsection. Let $K$ be a knot and consider its exterior $E(K)$. Then $E(K)$ is irreducible by the Schönflies theorem. Moreover, $E(K)$ is Haken, because a minimal genus Seifert surface is an incompressible surface in $E(K)$.

**Theorem 3.5** (The geometric decomposition of knot exteriors). *Given a knot $K$ in $S^3$, there is a unique (up to isotopy) compact subsurface $T$ in the interior $E(K)$ satisfying the following properties.*

(a) Each component of $T$ is an essential torus.
(b) Each closed up component of $E(K) - T$ is either a Seifert fibered space or a hyperbolic manifold of finite volume.
(c) If any component of $T$ is deleted, Property (b) fails.

We call the JSJ piece of $E(K)$ containing $\partial E(K)$ the *root JSJ piece*. The JSJ-decomposition is intimately related to Schubert’s satellite operation [Scb1953]. To see this, assume that $T \neq \emptyset$ and pick a component $T$ of $T$. Then $T$ bounds a solid torus $V = S^1 \times D^2$ in $S^3$, which satisfies the following conditions.

1. The core $k := S^1 \times 0$ of $V$ forms a nontrivial knot in $S^3$.
2. $K$ is contained in $V$ geometrically essentially, i.e., there is no 3-ball $B$ such that $K \subset B \subset V$. Moreover, $K$ is not isotopic in $V$ to the core $k$ of $V$.

Thus $K$ is a *satellite knot* of the *companion knot* $k \subset S^3$ with pattern $(V,K)$ (cf. [Ro1976] Section 4.D)). The knot $k \subset S^3$ is called a *companion knot* of $K$.

![Diagram](image)

**Figure 6.** A *Whitehead double* of a trefoil knot is a satellite knot whose companion knot is a trefoil knot and whose pattern knot is represented by the Whitehead link.
It should be noted that composition of knots is a special case of the satellite operation. In fact, the composite knot $K_1 \# K_2$ is a satellite of $K_1$ with pattern $(S^3 - \text{int} \, N(\mu_2), K_2)$ where $\mu_2$ is a meridian of $K_2$. It is also a satellite of $K_2$ with pattern $(S^3 - \text{int} \, N(\mu_1), K_1)$.

It turns out that JSJ pieces of knot exteriors are expressed as link exteriors (Theorem 3.7). A link $L$ is a smoothly (or piecewise-linearly) embedded disjoint circles in $S^3$, namely $L$ is a closed 1-submanifold of $S^3$. Thus a knot is a link of 1 component. A link of $\mu$-components is called a $\mu$-component trivial link if it bounds $\mu$ disjoint disks in $S^3$, and we denote it by $O_\mu$. The exterior of a link $L$ is defined by $E(L) := S^3 - \text{int} \, N(L)$, where $N(L)$ is a regular neighborhood of $L$. The links in the example below play a key role in torus decompositions of knot exteriors.

**Example 3.6.** (1) The $\mu + 1$-component key chain link $H_{\mu+1} = K_0 \cup O_\mu$ is a union of the $\mu$-component trivial link $O_\mu$ and the trivial knot $K_0$ which intersects each member of $\mu$ disjoint links bounded by $O_\mu$. Then $E(H_{\mu+1})$ is homeomorphic to $(\mu \text{ holed disk}) \times S^1$, and is called a composing space. If $E(H_{\mu+1})$ is the root JSJ piece of a knot exterior $E(K)$, then $K$ is a connected sum of $\mu$ prime knots.

(2) For a pair $(p, q)$ of relatively prime integers, the $(p, q)$-torus knot $K_{p,q}$ is defined by

$$K_{p,q} := \{(z_1, z_2) \in S^3 \mid z_1^p + z_2^q = 0\}.$$  

$K_{p,q}$ is a regular orbit of the circle action on $S^3$ given by

$$\omega \cdot (z_1, z_2) = (\omega^q z_1, \omega^p z_2) \quad (\omega \in S^1 \subset \mathbb{C}).$$

Thus $E(K_{p,q})$ is a Seifert fibered space. $K_{p,q}$ is contained in the torus

$$T := \{(z_1, z_2) \in S^3 \mid |z_1|^p = |z_2|^q\},$$

and it wraps $q$ times in the $z_1$ direction and $p$ times in the $z_2$ direction. The annulus $A := T \cap E(K)$ divides $E(K_{p,q})$ into two solid tori. By van-Kampen’s theorem in this setting, we see

$$G(K_{p,q}) = \langle a, b \mid a^p = b^q \rangle$$

The cyclic subgroup generated by $a^p = b^q$ forms the infinite cyclic center of $G(K_{p,q})$. Moreover, a knot $K$ is a torus knot if and only if $G(K)$ has a nontrivial center. $K_{p,q}$ is nontrivial if and only if both $p$ and $q$ have absolute value $\geq 2$. If $E(K_{p,q})$ is a JSJ piece of a knot exterior $E(K)$, then $K$ is a satellite of the torus knot $K_{p,q}$.

(3) For a pair $(p, q)$ of relatively prime integers with $p \geq 2$, the $(p, q)$-Seifert link $C_{p,q}$ is defined by

$$C_{p,q} := K_0 \cup K_{p,q} \quad \text{with} \quad K_0 = \{(z_1, z_2) \in S^3 \mid z_2 = 0\}.$$  

If $C_{p,q}$ is the root JSJ piece of a knot exterior $E(K)$, then $K$ is the $(p, q)$-cable of some nontrivial knot.

We have the following characterization of the torus decompositions of knot exteriors.
Theorem 3.7. (1) A compact orientable 3-manifold $M$ is a JSJ piece of $E(K)$ for some nontrivial knot $K$ in $S^3$, if and only if $M \cong E(L)$ for some link $L$ in $S^3$, which is the union of a knot $K_0$ and a trivial link $O_\mu$ (with $\mu$ possibly 0), such that $E(L)$ is either (i) hyperbolic or (ii) a Seifert fibered space homeomorphic to a composing space, a nontrivial torus knot exterior, or a cable space.

(2) Let $K$ be a nontrivial knot in $S^3$, and let $\mathcal{T}$ be a union of disjoint essential tori in $E(K)$, satisfying the following conditions.

(i) Each closed up component of $E(K) - \mathcal{T}$ is homeomorphic to a link exterior $E(L)$ which satisfies the condition in (1).

(ii) There does not exist a pair of adjacent closed up components of $E(K) - \mathcal{T}$, both of which are composing spaces.

Then $\mathcal{T}$ is the characteristic toric family of $E(K)$.

The way JSJ pieces fit together in $E(K)$ is recorded by the companionship tree, defined as follows: The vertices correspond to the JSJ pieces, and the edges correspond to the components of $\mathcal{T}$, where if an edge corresponds to a component $T$ of $\mathcal{T}$, it joins the vertices corresponding to the two JSJ pieces containing $T$ as a boundary component. Since $H_1(S^3) = 0$, this graph is a tree. For a more detailed description of torus decompositions, see [BnSb*, Chapter 2] and [Bd2006].

4. The orbifold theorem and the Bonahon-Siebenmann decomposition of links

In [Ths1979, Chapter 13], Thurston initiated the systematic study of orbifolds, namely quotients of spaces by properly discontinuous group actions which are not necessarily free. In 1978, he announced the orbifold theorem, the geometrization theorem of 3-orbifolds which have non-empty 1-dimensional singular set. Every link $L = \sqcup_j K_j$ determines an infinite family of orbifolds, by regarding each component $K_j$ as the singular locus of cone angle $2\pi/n_j$ for some $n_j \geq 2$. The case when $n_j = 2$ for every $j$ is particularly important, and the Bonahon-Siebenmann decomposition theory of links is essentially the decomposition theory of such orbifolds. Their theory is intimately related with Conway’s ingenious analysis of link diagrams, and gives us a nice method for understanding links directly from their diagrams. In particular, it gives a complete classification of the “algebraic links”, which implies, for example, that the Kinoshita-Terasaka knot and the Conway knot are different and that they admit no symmetry.

The purpose of this section is to recall the orbifold theorem and its impact on knot theory. To be precise, we will give surveys of (i) the Bonahon-Siebenmann decomposition theory, (ii) the classification of 2-bridge links, (iii) the orbifold theorem, and (iv) application of the orbifold theorem to the study of branched cyclic coverings.
4.1. The Bonahon-Siebenmann decompositions for simple links

By the geometric decomposition Theorems 3.5 and 3.7 of knot exteriors, the classification of knots is reduced to that of the links whose exteriors have trivial torus decompositions. Deriving from Montesinos’ work [Mnt1973, Mnt1975] on double branched coverings of links and Thurston’s work on 3-dimensional orbifolds, Bonahon and Siebenmann established a new decomposition theorem for such links. This is essentially a $\mathbb{Z}/2\mathbb{Z}$-equivariant JSJ decomposition theory, applied to the double branched coverings of links.

To explain their results, we introduce a few definitions. A link $L$ in $S^3$ is *splittable* if there is a 2-sphere $S$ in $S^3$ which separates the components of $L$. $L$ is *unsplittable* if it is not splittable. This is equivalent to the condition that $E(L)$ is irreducible. $L$ is *simple for Schubert* if $E(L)$ is irreducible and atoroidal. If $L$ is simple for Schubert, then the JSJ decomposition of $E(L)$ is trivial. The converse also holds for knots, but not for links. For example, the key-chain link $H_{\mu+1}$ is not simple for Schubert, but the torus decomposition of $E(H_{\mu+1})$ is trivial.

Let $(M,L)$ be a pair consisting of a compact orientable 3-manifold and a proper 1-submanifold $L$ in $M$. A Conway sphere in $(M,L)$ is a 2-sphere $\Sigma$ in int $M$ or $\partial M$ which meets $L$ transversely in 4 points. A Conway sphere $\Sigma$ is said to be *pairwise compressible* if there is a disk $D$ in $M-L$ such that $D\cap\Sigma = \partial D$ does not bound a disk in $\Sigma - L$. Otherwise, $\Sigma$ is said to be *pairwise incompressible*. Two Conway spheres $\Sigma$ and $\Sigma'$ in $M$ are said to be *pairwise parallel*, if there is a closed up component $N$ of $M - (\Sigma \cup \Sigma')$ bounded by $\Sigma$ and $\Sigma'$ such that $(N, N \cap L) \cong (\Sigma, \Sigma \cap L) \times [0,1]$. A Conway sphere is *essential* if it is pairwise incompressible and is not pairwise parallel to a boundary component. $(M,L)$ is *simple for Conway* if it does not contain an essential Conway sphere.

A *trivial tangle* is a pair $(B^3,t)$, where $t$ is a union of two arcs properly embedded arcs in $B^3$ which is parallel to a pair of disjoint arc in $\partial B^3$. A *rational tangle* is a trivial tangle $(B^3,t)$ which is endowed with an identification of $\partial(B^3,t)$ with the Conway sphere standardly embedded in $\mathbb{R}^3 \subset S^3$. A rational tangle, up to the natural equivalence relation, is determined uniquely by its *slope* as illustrated in Figure 7. (See [Cn1970] for the original definition, and see [BnSb* Chapter 8.1] or [Crm2004, Section 8.6] for detailed exposition.)

A *Montesinos pair* is a pair $(M,L)$ which is built from a *hollow Montesinos pair* or a *hollow Montesinos pair with a ring* in Figure 8 by plugging some of the holes with rational tangles of finite slope.

Bonahon and Siebenmann established the following decomposition theorem [BnSb* Theorem 3.4].

**Theorem 4.1.** For a link $L$ in $S^3$ that is simple for Schubert, there is a unique (up to isotopy respecting $L$) compact subsurface $\Sigma \subset S^3$ satisfying the following property.

(a) Each component of $\Sigma$ is a pairwise incompressible Conway sphere.
Figure 7. The pair of arcs forming a trivial tangle is parallel to a pair of arcs on the boundary Conway sphere. If we identify the Conway sphere with the quotient of $\mathbb{R}^2$ by the group generated by $\pi$-rotations around the lattice points, then the inverse image of the pair of arcs in $\mathbb{R}^2$ forms a family of mutually disjoint lines of rational slope passing through the lattice points. This slope is the slope of the rational tangle.

Figure 8. (a) a hollow Montesinos pair, (b) a hollow Montesinos pair with a ring

(b) Each closed up component $N$ of $S^3 - \mathcal{S}$ gives a pair $(N, N \cap L)$ that either is simple for Conway, or else is a Montesinos pair.
(c) If any component of $\mathcal{S}$ is deleted, Property (b) fails.

The above decomposition is called the characteristic decomposition (or the Bonahon-Siebenmann decomposition) of $(S^3, L)$. The union of the Montesinos pairs is called the algebraic part or arborescent part of $(S^3, L)$. The link $(S^3, L)$ is called an arborescent link if its arborescent part is equal to the whole pair $(S^3, L)$. This terminology comes from the fact that arborescent parts can be represented by weight planar trees. The classification the arborescent parts and links is given by [BnSb*, Part V].

**Example 4.2.** The Bonahon-Siebenmann decomposition of the Kinoshita-Tersaka knot and the Conway knot are given by the spheres in Figure 5 (cf. [Mnt1975]).
Sections 4.1 and 4.2). This fact gives an intuitive proof of the inequivalence of these two knots. It also shows that both of the knots are arborescent.

For a link \( L \) in \( S^3 \), let \( p : M_2(L) \to S^3 \) be the double branched covering of \( S^3 \) branched over \( L \), and let \( \tau \) be the covering involution. Then the Bonahon-Siebenmann decomposition of \( L \) can be regarded as a \( \mathbb{Z}_2 \)-equivariant version of the torus decomposition of \( M_2(L) \) for the following reasons:

- The inverse image of an essential Conway sphere of \((S^3, L)\) is an essential torus in \( M_2(L) \).
- Let \((N, N \cap L)\) be a piece of the Bonahon-Siebenmann decomposition of \((S^3, L)\) which is a Montesinos pair. Then the inverse image \( p^{-1}(N) \) is a Seifert fibered space, where the base orbifold is orientable or non-orientable according to whether \((N, N \cap L)\) is obtained from a hollow Montesinos pair or that with a ring (see [Mnt1973, Mnt1987]). Moreover, the covering involution \( \tau \) preserves the Seifert fibration of \( p^{-1}(N) \). The image of its fiber in \( S^3 \) is either a circle disjoint from \( L \) or an interval with endpoints in \( L \).
- The above fact implies that the inverse image in \( M_2(L) \) of the arborescent part of \((S^3, L)\) is a graph manifold (cf. Waldhausen [Wld1967]). In particular, if \((S^3, L)\) is an arborescent link then \( M_2(L) \) is a graph manifold.
- For each piece \((N, N \cap L)\) of the Bonahon-Siebenmann decomposition of \((S^3, L)\) which is not a Montesinos pair, the inverse image \( p^{-1}(N) \) is irreducible and atoroidal. Moreover, by the orbifold theorem (Theorem 4.4) explained later in this section, \( p^{-1}(N) \) admits a complete hyperbolic structure of finite volume, with respect to which \( \tau|_{p^{-1}(N)} \) is an isometry.

We note that the Bonahan-Siebenmann decomposition is intimately related with Conway’s ingenious analysis of knot diagrams, which in turn is based on Kirkmann’s idea from the 19th century (see [HTM1998] for the history). In fact, it reveals that Conway’s notation for a link diagram is not merely a convenient tool for describing diagrams but also contains geometric information of the link represented by a diagram. This is certainly the case for algebraic parts of the link. As shown in [BnSb* Theorems 1.4 and 6.11], Conway’s notation for non-algebraic parts also has geometric information under certain conditions.

4.2. 2-bridge links

In this subsection, we introduce 2-bridge links, which form a very special but important class of links. For a rational number \( r \in \mathbb{Q} \cup \{1/0\} \), the 2-bridge link, \( K(r) \), of slope \( r \) is defined as the “sum” of the rational tangles of slopes \( r \) and 1/0. To be precise, it is obtained from the rational tangles, \((B^3, t(r))\) and \((B^3, t(1/0))\), of slopes \( r \) and 1/0, respectively, by gluing \((B^3, t(r))\) and \((-B^3, t(1/0))\) along the boundaries via the identity map. (Note that the boundaries of rational tangles are identified with the Conway sphere standardly embedded in \( \mathbb{R}^3 \).) If \( r = q/p \) where \( p \geq 0 \) and \( q \)
are relatively prime integers, then \(K(r)\) is a knot or a two-component link according to whether \(p\) is odd or even. The following classification theorem was proved by Schubert [Scb1956], by establishing the uniqueness up to isotopy of 2-bridge spheres (2-spheres which divide \(K(r)\) into two trivial tangles).

- Two 2-bridge links \(K(q/p)\) and \(K(q'/p')\) are isotopic, if and only if \(p = p'\) and either \(q \equiv q' \pmod{p}\) or \(qq' \equiv 1 \pmod{p}\). They are homeomorphic, if and only if \(p = p'\) and either \(q \equiv \pm q' \pmod{p}\) or \(qq' \equiv \pm 1 \pmod{p}\).

The double branched covering, \(M_2(K(q/p))\), of \(S^3\) branched over \(K(q/p)\) is the lens space \(L(p,q)\), and the above classification of 2-bridge links can be also deduced from the classification of lens spaces, which in turn was established by Reidemeister [Rdm1935], using the Reidemeister torsion. Moreover, the following characterization of 2-bridge link was obtained by Hodgson and Rubinstein [HR1985], by classifying involutions on lens spaces with 1-dimensional fixed point sets.

- A link \(L\) in \(S^3\) is a 2-bridge link if and only if the double branched covering \(M_2(L)\) is a lens space.

The result of [HR1985] is a special but important case of the orbifold theorem (Theorem 4.4) explained later in this section. They also proved the uniqueness up to isotopy of genus 1 Heegaard surfaces of lens spaces, which in turn gives a purely topological proof of the classification of lens spaces.

Thurston’s uniformization theorem for Haken manifold (cf. Theorems 3.3), together with an analysis of incompressible surfaces in the exterior of 2-bridge links, imply the following (cf. [Ri1979 p.102], [Kb1984 Lemmas 4.4], [HT1985]).

- The 2-bridge link \(K(q/p)\) is hyperbolic if and only if \(q \not\equiv \pm 1 \pmod{p}\).

### 4.3. Bonahon-Siebenmann decompositions and \(\pi\)-orbifolds

For a link \(L\) in \(S^3\), the pair \((S^3, L)\) is homeomorphic to the quotient \((M_2(L), \text{Fix}(\tau))/\tau\), where \(M_2(L)\) is the double branched covering of \(S^3\) branched over \(L\) and \(\tau\) is the covering involution. This means that the good 3-orbifold \(O(L) := M_2(L)/\tau\) has \(S^3\) as the underlying space and \(L\) as the singular set, and each component of the singular set \(L\) has cone angle \(\pi\). The Bonahon-Siebenmann decomposition is regarded as the torus decomposition of this orbifold.

Recall that an \(n\)-orbifold is a metrizable topological space \(\mathcal{O}\) locally modeled on the quotient of \(\mathbb{R}^n\) by a finite subgroup \(G\) of the orthogonal group \(O(n)\). If a point \(x \in \mathcal{O}\) corresponds to the image of the origin of \(\mathbb{R}^n\), then the finite group \(G\) is called the local group at \(x\), and is denoted by \(G_x\). If \(G_x\) is trivial, \(x\) is regular, otherwise \(x\) is singular. The singular locus is the subset, \(\Sigma_{\mathcal{O}}\), of \(\mathcal{O}\) consisting of the singular points. When \(G_x\) is the cyclic group generated by a \(2\pi/m\)-rotation around the codimension 2 subspace \(\mathbb{R}^{n-2} \times \{0\}\), we say that the point \(x\) (and the stratum of the singular set containing \(x\)) has cone angle \(2\pi/m\) or index \(m\).
A quotient space \( O := X/\Gamma \), where \( X \) is a smooth \( n \)-manifold and \( \Gamma \) is a smooth properly discontinuous action, is an \( n \)-orbifold, and its singular locus is the image of the subspace of \( X \) consisting of points with nontrivial stabilizer. If \( \Gamma \) is a finite group, such an orbifold is called a good orbifold. The orbifold fundamental group \( \pi_1^{\text{orb}}(O) \) of \( O \) is defined as the group consisting of all lifts of \( \Gamma \) to the universal covering space \( \tilde{X} \) of \( X \). Thus we have the following exact sequence.

\[
1 \to \pi_1(X) \to \pi_1^{\text{orb}}(O) \to \Gamma \to 1
\]

For a link \((S^3, L)\), the orbifold \( O(L) := (M_2(L), \text{Fix}(\tau))/\tau \) is called the \( \pi \)-orbifold associated with \( L \). The orbifold fundamental group \( \pi_1^{\text{orb}}(O(L)) \) is called the \( \pi \)-orbifold group of \( L \). It is calculated from the link group \( G(L) = \pi_1(S^3 - L) \) and a set of meridians \( \{\mu_1, \ldots, \mu_m\} \) as follows:

\[
\pi_1^{\text{orb}}(O(L)) = G(L)/\langle \langle \mu_1^2, \ldots, \mu_m^2 \rangle \rangle.
\]

Here \( m \) is the number of components of \( L \), and \( \mu_j \) is a meridian of the \( j \)-th component of \( L \). By using the orbifold theorem explained in the next subsection, Boileau and Zimmermann \[BZm1989\] proved the following theorem which shows that \( \pi_1^{\text{orb}}(O(L)) \) is a very strong invariant for links.

**Theorem 4.3.** Let \( L \) be a prime unsplittable link in \( S^3 \) such that \( \pi_1^{\text{orb}}(O(L)) \) is infinite. Then the following hold.

1. For any link \( L' \) in \( S^3 \), the pairs \((S^3, L)\) and \((S^3, L')\) are homeomorphic if and only if their \( \pi \)-orbifold groups \( \pi_1^{\text{orb}}(O(L)) \) and \( \pi_1^{\text{orb}}(O(L')) \) are isomorphic.

2. The natural homomorphism from the symmetry group \( \text{Sym}(S^3, L) \) to the outer-automorphism group \( \text{Out}(\pi_1^{\text{orb}}(O(L))) \) is an isomorphism.

Here the symmetry group \( \text{Sym}(S^3, L) \) is the group of the diffeomorphisms of the pair \((S^3, L)\) up to isotopy. It should be noted that the problem to determine the symmetry group of a knot is a refinement of the fundamental problem to determine if the knot is chiral/invertible.

Using the above theorem, Boileau and Zimmermann \[BZm1987\] determined the symmetry groups of all non-elliptic Montesinos links, i.e., the Montesinos links with infinite \( \pi \)-orbifold groups. (The symmetry groups of elliptic Montesinos links were determined by \[Skm1990\] by using the orbifold theorem.) This result may be regarded as a broad extension of Trotter’s proof \[Tr1963\] of non-invertibility of the pretzel knot \( P(p, q, r) \) with \(|p|, |q|, |r| \) distinct odd integers \( \geq 3 \). Trotter’s proof is based on the fact that \( \pi_1^{\text{orb}}(O(P(p, q, r))) \) is an extension of the hyperbolic triangular reflection group

\[
[p, q, r] = \langle x, y, z \mid x^2, y^2, z^2, (xy)^p, (yz)^q, (zx)^r \rangle
\]

by the infinite cyclic group, which in turn is a consequence of the fact that the \( \pi \)-orbifold \( O(P(p, q, r)) \) is a Seifert fibered orbifold over the 2-dimensional hyperbolic orbifold \( \mathbb{H}^2/[p, q, r] \).
The symmetry groups of the arborescent links are completely determined by Bonahon and Siebenmann in [BnSb*]. In particular, this implies that the symmetry groups of the Kinoshita-Terasaka knot and the Conway knot are trivial, and so they are chiral and noninvertible. The knot 8_{17} is also arborescent, and its symmetry group is the order 2 cyclic group, generated by an orientation-reversing involution representing the negative-amphicheirality of the knot. This is Bonahon-Siebenmann’s proof of the non-invertibility of 8_{17} (cf. Subsection 2.6).

4.4. The orbifold theorem and the Smith conjecture

Many of the concepts for 3-manifolds, such as irreducibility, atoroidality and Seifert fibrations, have natural generalization for 3-orbifolds, and a characteristic splitting (torus decomposition) theorem was established by Bonahon and Siebenmann [BnSb1987] (cf. [BMP2003, Bn2002]). The characteristic splitting Theorem 4.1 for links is a special case of the general splitting theorem, though the detailed analysis for algebraic parts and application to knot theory in [BnSb*] cannot be found in [BnSb1987].

Their theory forms the first step towards the proof of the following geometrization theorem for orbifolds, which was announced by Thurston [Ths1981], and finally proved by Boileau, Leeb and Porti [BLP2005] (see also Cooper-Hodgson-Kerckhoff [CHK2000] and Boileau-Porti [BP2001] for earlier account, and Dinkelbach-Leeb [DL2009] for the generalization to non-orientable orbifolds using equivariant Ricci flow).

**Theorem 4.4 (Orbifold Theorem).** Every compact orientable good 3-orbifold with nonempty singular set has a canonical splitting by spherical 2-dimensional suborbifolds and toric 2-dimensional suborbifolds into geometric 3-orbifolds.

Here a 3-orbifold $O$ is geometric if either it is the quotient of a ball by an orthogonal action, or its interior has one of the eight Thurston geometries, namely $O = X/\Gamma$, where $X$ is one of the eight Thurston’s geometries, and $\Gamma$ is a discrete subgroup of Isom($X$). (If $X$ is different from the constant curvature spaces $H^3$, $E^3$ and $S^3$, then there is no canonical metric on $X$, however, it admits a family of natural metrics for which Isom($X$) are identical. See the beautiful surveys [Sct1983, Bn2002].)

The orbifold theorem was first announced as the following symmetry theorem concerning finite group actions on 3-manifolds.

**Theorem 4.5 (Symmetry Theorem).** Let $M$ be a compact orientable irreducible 3-manifold. Suppose $M$ admits an action by a finite group $G$ of orientation-preserving diffeomorphisms such that some non-trivial element has a fixed point set of dimension one. Then $M$ admits a geometric decomposition preserved by the group action.

This theorem poses a very strong restriction on finite group actions on knots (see [BP1987, Luo1997, Skm1986]). In particular, it includes, as a special case, the following positive answer to the Smith conjecture.
**Theorem 4.6** (The Smith Conjecture). If \( h : S^3 \to S^3 \) is an orientation-preserving periodic diffeomorphism with non-empty fixed point set, then \( h \) is smoothly conjugate to an orthogonal diffeomorphism. In particular, \( \text{Fix}(h) \) is the trivial knot.

The proof of this conjecture recorded in [MB1984] may be regarded as the first major impact of Thurston’s uniformization theorem for Haken manifolds, and it was established using the uniformization theorem, the equivariant loop theorem by Meeks-Yau [MY1981], and a refinement of Bass-Serre theory [Srr1977].

In Theorems 4.5 and 4.6, the smoothness of the action is essential. In fact there is an orientation-preserving periodic homeomorphism \( h \) of \( S^3 \) which has a wild knot as the fixed point set; in particular, the cyclic action generated by \( h \) is not topologically conjugate to an orthogonal action. It is this phenomena that lead Shin’ichi Kinoshita and Hidetaka Terasaka, the founders of knot theory in Japan, into knot theory. It is an amazing coincidence that Terasaka published an introductory book [Te1977] to non-Euclidean geometry for the general public in 1977, around the time Thurston started the series of lectures on the geometry and topology of 3-manifolds.

4.5. Branched cyclic coverings of knots

In Subsection 4.3, we explained the important role of the double branched coverings of knots and links. Not only the double branched covering but also the cyclic branched covering has attracted keen attention of various mathematicians, because it gives a bridge between knot theory and 3-manifold theory and because of its special beauty. In this subsection, we review the impact of Thurston’s work, in particular the orbifold theorem, on the study of branched cyclic coverings of knots.

For a knot \( K \) in \( S^3 \), let \( M_n(K) \) be the \( n \)-fold cyclic branched covering of \( S^3 \) branched over \( K \). We also call \( M_n(K) \) as the \( n \)-fold cyclic branched covering of \( K \). Then we have the following natural question.

**Problem 4.7.** To what extent does the topological type of \( M_n(K) \) determine \( K \)?

It should be noted that \( M_n(K) \) inherits the orientation of the ambient space \( S^3 \), but it is independent of the orientation of the circle \( K \). Namely \( M_n(K) \cong M_n(\overline{K}) \) as oriented manifolds. Thus the precise meaning of the above question is as follows.

To what extent does the topological type of the oriented manifold \( M_n(K) \) determine the isotopy type of the unoriented knot \( K \)?

The positive solution of the Smith conjecture is essentially equivalent to the following partial answer to the above problem (see [MB1984]).

**Theorem 4.8** (Branched covering theorem). A knot \( K \) in \( S^3 \) is trivial if and only if \( M_n(K) \cong S^3 \) for some \( n \geq 2 \).

The orbifold theorem gives a very strong tool for the study of Problem 4.7. Before describing its influence, let us recall two classical constructions of pair of knots sharing the same cyclic branched covering.
Construction 1. Let \( K \) be a non-invertible prime oriented knot. Then, by the unique prime factorization theorem, the knots \( K \# K \) and \( K \# (-K) \) are not isotopic as unoriented knots. However, they share the same \( n \)-fold cyclic branched covering for all \( n \geq 2 \), because:

\[
M_n(K \# K) \cong M_n(K) \# M_n(K) \cong M_n(K) \# M_n(-K) \cong M_n(K \# (-K))
\]

Construction 2. Let \( L = K_1 \cup K_2 \) be a 2-component link consisting of two trivial knots. For integers \( n_1, n_2 \geq 2 \) which are relatively prime to the linking number \( \text{lk}(K_1, K_2) \), the inverse image \( \tilde{K}_1 \) of \( K_1 \) in \( M_{n_2}(K_2) \cong S^3 \) is a knot, and so is the inverse image \( \tilde{K}_2 \) of \( K_2 \) in \( M_{n_2}(K_1) \cong S^3 \). Moreover, both \( M_{n_2}(\tilde{K}_1) \) and \( M_{n_1}(\tilde{K}_2) \) are homeomorphic to the \( (\mathbb{Z}/n_1\mathbb{Z}) \oplus (\mathbb{Z}/n_2\mathbb{Z}) \)-covering of \( S^3 \) branched over \( L \), and hence they are homeomorphic. (There is an analogous construction by using a three component link such that any 2-component sublink is a Hopf link (see [ReZ2001 0.2]).)

Now, we state an important consequence of the orbifold theorem (see [CHK2000]).

Theorem 4.9. Let \( K \) be a hyperbolic knot in \( S^3 \), i.e., \( K \) is a knot which is not a torus knot nor a satellite knot. Then \( M_n(K) \) is hyperbolic for all \( n \geq 3 \), except for the 3-fold covering of the figure eight knot (which is a Euclidean manifold). Moreover, the covering transformation group acts on \( M_n(K) \) as isometries.

Remark 4.10. In the above theorem the assumption \( n \geq 3 \) is essential. In fact, if a hyperbolic knot contains an essential Conway sphere, \( \Sigma \), then the inverse image, \( \tilde{\Sigma} \), of \( \Sigma \) in \( M_2(K) \) is an essential torus and hence \( M_2(K) \) is non-hyperbolic even though \( K \) itself is hyperbolic. Moreover, every arborescent link has a graph manifold as the double branched covering.

The hyperbolic Dehn surgery theorem implies that if \( n \) is sufficiently large, then the branch line forms the unique shortest closed geodesic in \( M_n(K) \) (cf. Subsection 7.3). By using this fact, we can see that \( M_n(K) \) for sufficiently large \( n \) determines the knot \( K \). More generally, Kojima [Kj1986] proved the following theorem, which gives a positive answer to a question of Goldsmith [Kr1978 Problem 1.27].

Theorem 4.11. For each prime knot \( K \) there exists a constant \( n_K \), such that two prime knots \( K \) and \( K' \) are equivalent if their \( n \)-fold cyclic branched covers are homeomorphic for some \( n > \max(n_K, n_{K'}) \).

We can reformulate Problem 4.7 as follows: For a given connected closed orientable 3-manifold \( M \), in how many different ways can \( M \) occur as a cyclic branched covering of a knot in \( S^3 \)? There are two basic cases: the case when \( M \) is a Seifert fibered space and the case when \( M \) is a hyperbolic manifold. (The general case can be treated by using the equivariant sphere theorem and torus decomposition [McS1986] into Seifert fibered space and hyperbolic manifolds.)
When $M$ is a Seifert fibered space, the covering transformation group, $H$, is fiber-preserving by [McS1986] (when $\pi_1(M)$ is infinite) and by the orbifold theorem (when $\pi_1(M)$ is finite). If $H$ reverses the fiber-orientation, then the quotient knot is a Montesinos knot whereas if $H$ preserves the fiber-orientation then the quotient knot is a torus knot.

In the case where $M$ is hyperbolic, we may assume, by the orbifold theorem, that $H$ is a cyclic subgroup of the finite group $\text{Isom}^+(M)$. The group $H$ must be a hyper-elliptic group, namely $H$ is a finite cyclic group such that $\text{Fix} h$ is a circle for every non-trivial element $h \in H$, and $M/H \cong S^3$ (cf. [BFMP2018, Definition 1]). Thus there is a one-to-one correspondence

$$\{\text{knots } K \text{ such that } M_n(K) \cong M \text{ for some } n \geq 2\}/\text{isotopy}$$

$$\leftrightarrow \{\text{hyper-elliptic subgroups of } \text{Isom}^+(M)\}/\text{conjugacy}.$$ 

By Kojima’s theorem [Kj1988], any finite group can be the full isometry group of a closed orientable hyperbolic 3-manifold. However, the geometric condition for a hyper-elliptic group, $H$, implies purely group theoretical conditions on $H$. For example, we can see by using the Smith conjecture (Theorem 4.6) that the normalizer of $H$ in $\text{Isom}^+(M)$ is a finite subgroup of the semi-direct product $(\mathbb{Z}/2\mathbb{Z}) \rtimes (\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z})$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z}$ as multiplication by $-1$ (see [BFMP2018, Remark 3]). Thus we have a chance to apply finite group theory to the study of cyclic branched coverings. For example, if we are interested in the case when the degree $n$ is a prime number $p$, then by Sylow’s theorem, every hyper-elliptic subgroup of order $p$ is conjugate to a cyclic subgroup of a single Sylow $p$-subgroup $S_p$ of $\text{Isom}^+(M)$. This interplay between the study of cyclic branched coverings and finite group theory was initiated by Reni and Zimmermann, and various interesting results were obtained, including the following.

- **Reni-Zimmermann [ReZ2001]**: Let $K$ and $K'$ be two hyperbolic knots such that $M_n(K) \cong M_{n'}(K')$ for some $n, n' \geq 3$. Suppose further that $n$ and $n'$ have a common prime divisor $p > 2$. Then $K$ and $K'$ are related by Construction 2. In particular, if $n = n'$ is not a power of 2, then the same conclusion holds (cf. [Z1998]).

- **Paoluzzi [P2005]**: A hyperbolic knot is determined by any three of its cyclic branched coverings of order $\geq 2$. Indeed, two coverings suffice if their orders are not coprime.

- **Boileau-Franchi-Mecchia-Paoluzzi-Zimmermann [BFMP2018]**: A closed hyperbolic 3-manifold is a cyclic branched covering of at most fifteen inequivalent knots in $S^3$.

A noteworthy aspect of the proof of the last result is the substantial use of finite group theory, in particular of the classification of finite simple groups.

For the double branched coverings, we have the following additional construction.
**Construction 3.** Let $\theta$ be a $\theta$-curve in $S^3$, namely a spatial graph consisting of two vertices and three edges $\alpha_1, \alpha_2$ and $\alpha_3$, each of which connects the two vertices. For $\{i, j, k\} = \{1, 2, 3\}$, suppose $A_k := \alpha_i \cup \alpha_j$ forms a trivial knot. Then the inverse image, $K_k$, of the arc $\alpha_k$ in $M_2(A_k) \cong S^3$ forms a (strongly invertible) knot, and $M_2(K_k)$ is identified with the $(\mathbb{Z}/2\mathbb{Z})^2$-covering of $S^3$ branched over $\theta$. If $A_k$ is a trivial knot for more than one $k \in \{1, 2, 3\}$, we obtain knots in $S^3$ sharing the same double branched coverings. (Similar construction is applied to embeddings of the 1-skeleton of the tetrahedron and the Kuratowski graph in $S^3$, which produce potentially distinct 4 and 9 knots, respectively, sharing the same double branched coverings (see [MR2002]).)

A link $L$ is said to be $\pi$-hyperbolic if $M_2(L)$ is hyperbolic. For double coverings of $\pi$-hyperbolic knots, the following results were obtained.

- **Boileau-Flapan** [BF1995]: If $K$ is a $\pi$-hyperbolic knot, then every knot $K'$ which shares the same double branched covering with $K$ is constructed by repeatedly applying Constructions 2 and 3.
- **Reni** [Ren2000]: There are at most nine different $\pi$-hyperbolic knots with the same double branched coverings. Mecchia-Reni [MR2002] gave a more geometric proof to this estimate and proved that the same estimate holds for $\pi$-hyperbolic links.
- **Kawauchi** [Kw2006]: Reni's estimate is the best possible, i.e., there are nine mutually inequivalent $\pi$-hyperbolic knots $K_i \ (i = 1, \cdots, 9)$, in $S^3$ with the same double branched coverings.

In the proof of the second result, study of the Sylow 2-subgroup of Isom$^+(M)$ of a closed orientable hyperbolic 3-manifold holds a key. The third result was obtained by using Kawauchi's imitation theory, which yields, for a given $(3, 1)$-manifold pair $(M, L)$, a family of $(3, 1)$-manifold pairs $(M^*, L^*)$ which is “topologically similar” to $(M, L)$. A key example in the theory is the Kinoshita-Terasaka knot, which is an imitation of the trivial knot. (This fact was first found by Nakanishi [Na1981] and a beautiful generalization of this fact was given by Kanenobu [Kn1988].)

For the double branched covering of links which are not $\pi$-hyperbolic, we have the following additional construction. (See Paoluzzi [Pl2001] for further construction.)

**Construction 4.** (Mutation) Let $\Sigma$ be an essential Conway sphere of a link $L$ in $S^3$. Cut $(S^3, L)$ along $\Sigma$ and reglue by an orientation-preserving involution of $(\Sigma, \Sigma \cap L)$ whose fixed point set is disjoint from $\Sigma \cap L$. This process, called a mutation, results in a new link $L'$ in $S^3$, called a mutant of $L$. A pair of links are called mutants if they are related by a sequence of mutations. It was proved by Viro [Vr1976, Theorem 1] that if $L$ and $L'$ are mutants then they share the same double branched coverings (cf. [Kw1996, Proposition 3.8.2]).
In [Gre2013], Greene studied the Heegaard Floer homology of the double branched coverings of alternating links, and proved that a reduced alternating link diagram is determined up to mutation by the Heegaard Floer homology of the double branched covering of the link. In particular, the following result follows.

- Two reduced alternating links $L$ and $L'$ share the same double branched covering, if and only if $L$ and $L'$ are mutants.

He also proposes the mysterious conjecture: *if a pair of links have homeomorphic double branched coverings, then either both are alternating or both are non-alternating.*

5. Hyperbolic manifolds and the rigidity theorem

In this section, we recall basic facts concerning hyperbolic manifolds and the Mostow-Prasad rigidity theorem for hyperbolic manifolds of finite volume with dimension $\geq 3$. The rigidity theorem has had tremendous influence on knot theory, because it guarantees that any geometric invariant of the hyperbolic structure of a hyperbolic knot complement is automatically a topological invariant of the knot complement.

For further information on hyperbolic geometry, see the text books Benedetti-Petronio [BnP1992], Ratcliffe [Rat1994], Matsuzaki-Taniguchi [MaT1998], Anderson [An1999] and Marden [Mrd2007b].

5.1. Hyperbolic space

Let $\mathbb{H}^n$ be the hyperbolic $n$-space, i.e., the upper-half space

$$\mathbb{H}^n := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

in $\mathbb{R}^n$ equipped with the Riemannian metric

$$ds^2 = \frac{1}{x_n^2}(dx_1^2 + \cdots + dx_n^2).$$

$\mathbb{H}^n$ is the unique connected, simply connected, complete Riemannian manifold of constant sectional curvature $-1$. The isometry group Isom$(\mathbb{H}^n)$ is a real Lie group and acts transitively on $\mathbb{H}^n$ and the stabilizer of each point is identified with the orthogonal group $O(n)$. If $n \geq 3$, the ideal boundary $\partial \mathbb{H}^n = (\mathbb{R}^{n-1} \times \{0\}) \cup \{\infty\}$ has a natural conformal structure, and the orientation-preserving isometry group $\text{Isom}^+(\mathbb{H}^n)$ is identified with the group of conformal maps of $\partial \mathbb{H}^n$.

Let $\gamma$ be a nontrivial element of $\text{Isom}^+(\mathbb{H}^n)$. Then precisely one of the following holds.

1. $\gamma$ is *elliptic*, i.e., $\gamma$ has a fixed point in $\mathbb{H}^n$.
2. $\gamma$ is *parabolic*, i.e., $\gamma$ has a unique fixed point, $x$, in $\partial \mathbb{H}^n$, called the *parabolic fixed point*. Then $\gamma$ preserves every horoball, $H_x$, centered at $x$. Here, if $x \neq \infty$, then $H_x$ is the intersection of a (closed) Euclidian ball with $\mathbb{H}^n$. 

which touches \( \partial \mathbb{H}^n \) at \( x \), and if \( x = \infty \) then \( H_x \) is the closed upper-half space

\[
H_{\infty, c} := \{ (x_1, \ldots, x_n) \in \mathbb{H}^n \mid x_n \geq c \}
\]

for some \( c > 0 \), which is called the horoball centered at \( \infty \) with height \( c \). The horosphere \( \partial H_x \) inherits a Euclidean metric from the hyperbolic metric, which is invariant by \( \gamma \).

(3) \( \gamma \) is hyperbolic, i.e., \( \gamma \) has precisely two fixed points in \( \partial \mathbb{H}^n \), one of which is repelling and the other is attracting. The geodesic in \( \mathbb{H}^n \) joining the two fixed points is the unique geodesic which is preserved by \( \gamma \), and it is called the axis of \( \gamma \), and denoted by axis \( \gamma \).

For low dimensions \( n = 2 \) and 3, we have:

\[
\text{Isom}^+(\mathbb{H}^2) \cong \text{PSL}(2, \mathbb{R}), \quad \text{Isom}^+(\mathbb{H}^3) \cong \text{PSL}(2, \mathbb{C})
\]

We identify the upper-half space \( \mathbb{H}^3 = \mathbb{R}^2 \times \mathbb{R}_+ \) with \( \mathbb{C} \times \mathbb{R}_+ \) and identify the ideal boundary \( \partial \mathbb{H}^3 \) with the Riemann sphere \( \mathbb{C} \cup \{ \infty \} \). Then the action of \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{C}) \) on \( \partial \mathbb{H}^3 = \mathbb{C} \cup \{ \infty \} \) is given by the linear fractional transformation

\[
A(z) = \frac{az + b}{cz + d}.
\]

Assume that \( A \neq \pm E \), where \( E \) is the identity matrix. Then, as we see in the following, the orientation-preserving isometry of \( \mathbb{H}^3 \) corresponding to \( A \in \text{PSL}(2, \mathbb{C}) \) is elliptic, parabolic, or hyperbolic according as the trace \( \text{tr} \) \( A \) (which is defined up to sign change) belongs to \((-2, 2), \{ \pm 2 \}, \text{or } \mathbb{C} - [-2, 2] \).

Case 1. \( \text{tr} A \neq \pm 2 \). Then \( A \) has precisely two fixed points in \( \partial \mathbb{H}^3 \). After conjugation in \( \text{PSL}(2, \mathbb{C}) \), we may assume that they are 0 and \( \infty \). Thus \( A(z) = az \) for some \( a = re^{\theta \sqrt{-1}} \in \mathbb{C}^* - \{1\} \). The action of the isometry \( A \) on \( \mathbb{H}^3 \) is given by:

\[
A(z, t) = (az, |a|t) = (re^{\theta \sqrt{-1}}z, rt)
\]

This is a screw motion along the geodesic axis \( 0 \times \mathbb{R}_+ \) with (signed) translation length \( \log r \) and rotation angle \( \theta \). The quantity

\[
\mathcal{L}_A := \log r + \theta \sqrt{-1} = \log a \in \mathbb{C}/2\pi \sqrt{-1} \mathbb{Z}
\]

is called the complex translation length of the isometry \( A \). If we interchange 0 and \( \infty \) by conjugation in \( \text{PSL}(2, \mathbb{C}) \), then the complex translation length changes into \(-\log r + \theta \sqrt{-1}\). Thus the complex translation length is defined only modulo \( 2\pi \sqrt{-1} \mathbb{Z} \) and up to multiplication by \( \pm 1 \). In fact, a simple calculation implies \( \mathcal{L}_A \) is characterized by the following identity: (Note that \( \text{tr} \) \( A \) for \( A \in \text{PSL}(2, \mathbb{C}) \) is defined only up to sign.)

\[
\pm \text{tr} A = 2 \cosh \frac{\mathcal{L}_A}{2}
\]
If we fix an orientation of the axis, then the complex translation length is defined as an element in $\mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$. Note that $A$ is elliptic if and only if $\tau = 1$, which equivalent to the condition that $\text{tr} A \in (-2, 2)$ (under the assumption that $A \neq \pm E$). Thus $A$ is hyperbolic or elliptic according to whether $\text{tr} A$ is contained in $(-2, 2)$ or $\mathbb{C} - [-2, 2]$.

Case 2. $\text{tr} A = \pm 2$. Then $A$ has a unique fixed point in $\partial \mathbb{H}^3$ and hence parabolic. After conjugation in $\text{PSL}(2, \mathbb{C})$, we may assume that it is $\infty$. Thus $A(z) = z + \tau$ for some $\tau \in \mathbb{C}^\ast$. The action of the isometry $A$ on $\mathbb{H}^3$ is given by:

$$A(z, t) = (z + \tau, t)$$

In this case, the complex translation length $L_A$ is defined to be 0.

The following lemma can be easily proved.

**Lemma 5.1.** Let $A$ be a nontrivial element in $\text{PSL}(2, \mathbb{C})$ which is not an elliptic element of order 2. Then the centralizer $C(A)$ in $\text{PSL}(2, \mathbb{C})$ is as follows.

1. If $A$ is elliptic or hyperbolic, then $C(A) - \{E\}$ consists of elliptic/hyperbolic elements which share the same axis with $A$. Thus $C(A)$ is isomorphic to the multiplicative group $\mathbb{C}^\ast$.

2. If $A$ is parabolic, then $C(A) - \{E\}$ consists of parabolic elements which share the same parabolic fixed point with $A$. Thus $C(A)$ is isomorphic to the additive group $\mathbb{C}$.

### 5.2. Basic facts for hyperbolic manifolds

By a *hyperbolic structure* on an $n$-manifold $M$, we mean a Riemannian metric on $M$ of constant sectional curvature $-1$: the curvature condition means that every point in $M$ has a neighborhood isometric to an open set of $\mathbb{H}^n$. A hyperbolic structure on $M$ induces a hyperbolic structure on the universal cover $\tilde{M}$ of $M$ which is invariant by the action of the covering transformation group. Thus we obtain a local isometry $D : \tilde{M} \to \mathbb{H}^n$, called the *developing map*, and a homomorphism $\rho : \pi_1(M) \to \text{Isom} \mathbb{H}^n$, called the *holonomy representation*, such that $D$ is $\rho$-equivariant, i.e., $D \circ \gamma = \rho(\gamma) \circ D : \tilde{M} \to \mathbb{H}^n$.

A hyperbolic structure on $M$ is *complete* if the induced metric on $M$ is complete. This condition is equivalent to the condition that the induced metric on $M$ is complete, which in turn is equivalent to the condition that the developing map $D : \tilde{M} \to \mathbb{H}^n$ is an isometry. Then the holonomy representation $\rho : \pi_1(M) \to \text{Isom} \mathbb{H}^n$ is faithful and discrete, nameley $\rho$ gives an isomorphism from $\pi_1(M)$ to a discrete torsion-free subgroup, $\Gamma$, of $\text{Isom} \mathbb{H}^n$. Thus the complete hyperbolic manifold $M$ is identified with $\mathbb{H}^n/\Gamma$.

By a *Kleinian group* we mean a discrete subgroup of $\text{Isom}^+ \mathbb{H}^3$, and by a *Fuchsian group* we mean a discrete subgroup of $\text{Isom}^+ \mathbb{H}^2$. By Lemma [5.1](#), any commutative
torsion-free Kleinian group is conjugate to one of the three groups in the following example.

**Example 5.2 (Commutative torsion-free Kleinian groups).** (1) The infinite cyclic group $J_0 = J_0(r e^{\theta \sqrt{-1}})$ generated by the hyperbolic element $A(z) = r e^{\theta \sqrt{-1}z}$ with $r > 1$. The hyperbolic manifold $\mathbb{H}^3/J_0$ is homeomorphic to the interior of the solid torus, and it has the unique closed geodesic with length $\Re L_A = \log r$. For any $r > 0$, the closed $r$-neighborhood of axis $A$ is invariant by $J_0$, and its quotient by $J_0$ is called a tube around the closed geodesic.

(2) The infinite cyclic group $J_1$ generated by the parabolic transformation $A(z) = z + 1$. The hyperbolic manifold $\mathbb{H}^3/J_1$ is homeomorphic to the product $\text{int} D^* \times \mathbb{R}^+$, where $D^* = D^2 - \{0\}$ is a once-punctured disk. This hyperbolic manifold does not contain a closed geodesic. For any $c > 0$, the horoball $H_{\infty,c}$ is invariant by $J_1$ and its quotient by $J_1$ is called an annulus cusp.

(3) The rank 2 free abelian group $J_2 = J_2(\tau)$ generated by the two parabolic transformations $A(z) = z + 1$ and $B(z) = z + \tau$ with $\tau \in \mathbb{C} - \mathbb{R}$. The hyperbolic manifold $\mathbb{H}^3/J_2$ is homeomorphic to the product $T^2 \times \mathbb{R}^+$, and it does not contain a closed geodesic. For any $c > 0$, the horoball $H_{\infty,c}$ is invariant by $J_2$ and its quotient by $J_2$ is called a torus cusp. The boundary torus $\partial H_{\infty,c}/J_2$ admits a Euclidean structure which is conformally equivalent to the Euclidean torus $\mathbb{C}/(1, \tau)$. Though the cusp neighborhood $H_{\infty,c}/J_2$ is noncompact, its volume $\text{vol}(H_{\infty,c}/J_2) = \frac{1}{2} \text{area}(\partial H_{\infty,c}/J_2)$ is finite. The complex number $\tau$ is called the modulus of the cusp torus with respect to the basis $\{A, B\}$.

For an orientable complete hyperbolic 3-manifold $M = \mathbb{H}^3/\Gamma$ and a point $x \in M$ the injectivity radius $r(x, M)$ of $M$ at $x$ is defined by

$$r(x, M) = \sup\{r > 0 \mid \text{the } r\text{-neighborhood of } x \text{ in } M \text{ is isometric to an } r\text{-ball in } \mathbb{H}^3\}.$$  

For a given constant $\epsilon > 0$, we can decompose $M$ into the $\epsilon$-thick part

$$M_{\geq \epsilon} = \{x \in M \mid r(x, M) \geq \frac{1}{2} \epsilon\}$$

and its complement

$$M_{\leq \epsilon} = \{x \in M \mid r(x, M) < \frac{1}{2} \epsilon\}.$$  

The closure of $M_{\leq \epsilon}$ is denoted by $M_{\leq \epsilon}$ and is called the $\epsilon$-thin part of $M$. (This complicated definition eliminates the trouble which occurs when there is a closed geodesic of length $\epsilon$ [Ths1997, p.254].) The following is a consequence of the Margulis lemma (see [Ths1979, Theorem 5.10.1 and Corollary 5.10.2]).

**Theorem 5.3.** There is a universal constant $\epsilon_0 > 0$, such that for any positive constant $\epsilon < \epsilon_0$ and for any orientable complete hyperbolic manifold $M = \mathbb{H}^3/\Gamma$, the $\epsilon$-thin part $M_{\leq \epsilon}$ is a disjoint union of tubes around (short) simple closed geodesics, annulus cusps, and torus cusps.
The following proposition can be proved by using the above theorem and the concept of convex core introduced in Subsection 10.1. (See [Ths1979, Proposition 5.11.1]).

**Proposition 5.4.** If an orientable complete hyperbolic manifold \( M = \mathbb{H}^3/\Gamma \) has finite volume, then \( M \) is the union of a compact submanifold (bounded by tori) and finitely many torus cusps \( C_1, \ldots, C_m \) for some \( m \geq 0 \). In particular, \( M \) is identified with the interior of a compact 3-manifold \( \tilde{M} \) with (possibly empty) toral boundary.

At the end of this subsection, we recall an important consequence of Thurston’s hyperbolization theorem for Haken manifolds.

**Definition 5.5.** A knot or link \( L \) in \( S^3 \) is hyperbolic if its complement \( S^3 - L \cong \text{int } E(L) \) admits a complete hyperbolic structure of finite volume.

The following theorem is a special case of Theorem 3.5 which in turn is a special case of the geometrization theorem.

**Theorem 5.6.** A prime knot in \( S^3 \) is hyperbolic if and only if it is neither a torus knot nor a satellite knot. More generally, an unsplittable prime link \( L \) is hyperbolic if and only if \( E(L) \) is atoroidal and is not a Seifert fibered space.

5.3. Rigidity theorem for complete hyperbolic manifolds of finite volume

For complete hyperbolic structures of finite volume of dimension \( \geq 3 \), the following strong rigidity theorem is established by Mostow [Mst1968] and Prasad [Pr1973] (cf. [Ths1979, Theorem 5.7.2]).

**Theorem 5.7** (The Mostow-Prasad rigidity theorem). If an orientable \( n \)-manifold with \( n \geq 3 \) admits a complete hyperbolic structure of finite volume, then this structure is unique. To be precise, the following holds. Let \( \Gamma_i \) \( (i = 1, 2) \) be discrete torsion free subgroups of \( \text{Isom}^+ \mathbb{H}^n \) with \( n \geq 3 \) of cofinite volume, i.e., \( \text{vol}(\mathbb{H}^n/\Gamma_i) < \infty \). Then any isomorphism \( \phi : \Gamma_1 \to \Gamma_2 \) is realized by a unique isometry \( f : \mathbb{H}^n/\Gamma_1 \to \mathbb{H}^n/\Gamma_2 \).

This theorem together with Thurston’s hyperbolization theorem had tremendous impact in knot theory. Because Theorem 5.6 says that almost all knots are hyperbolic (moreover, the Geometrization Theorem 3.5 reduces the study of knots to the study of hyperbolic links) and the above theorem imply that geometric invariants, such as volumes, cusp shapes, and lengths of shortest closed geodesics, of the complete hyperbolic structures on knot/link complements are topological invariants of the knots/links.
6. Computation of hyperbolic structures and canonical decompositions of cusped hyperbolic manifolds

Epstein and Penner proved that every cusped hyperbolic manifold of finite volume admits a natural ideal polyhedral decomposition, called the canonical decomposition. This fact (together with the rigidity theorem and the Gordon-Luecke knot complement theorem) has the following striking consequence in knot theory. The combinatorial structure of the canonical decomposition of a hyperbolic knot complement is a complete knot invariant. Moreover the marvelous computer program SnapPea developed by Jeffrey Weeks enabled us to compute the canonical decompositions of knot complements. For example, SnapPea immediately tells us that the Kinoshita-Terasaka knot and the Conway’s knot are different and that they admit no symmetry.

In this section, we recall the Epstein-Penner canonical decomposition and its impact on knot theory. We also recall a method for constructing hyperbolic structures by using ideal triangulation, which was first explained in Thurston’s lecture notes [Ths1979, Chapter 4], and explain a method for finding the canonical decomposition. In the final subsection, we give a list of geometric invariants of hyperbolic knots, which are guaranteed to be knot invariants by the rigidity theorem, and introduce the study of them from the view point of effective geometrization.

6.1. The canonical decompositions of cusped hyperbolic manifolds

Let $M = \mathbb{H}^n/\Gamma$ be an orientable complete hyperbolic $n$-manifold of finite volume with $m \geq 1$ cusps. Pick mutually disjoint cusps $C_1, \ldots, C_m$ of $M$, and set $C = \cup_{i=1}^m C_i$. Then we can canonically construct a spine $\mathcal{F}$ and a canonical ideal polyhedral decomposition $\mathcal{D}$ of $M$ as follows.

Observe that a generic point in $M - C$ has a unique shortest geodesic path to $C$ but that there are exceptional points which have more than one shortest geodesic paths to $C$. Let $\mathcal{F}$ be the subset of $M - C$ consisting of these exceptional points. Namely, $\mathcal{F}$ is the cut locus in $M$ with respect to the cusps $C = \cup_{i=1}^m C_i$. Then $\mathcal{F}$ is a locally finite totally geodesic cell complex in $M$, and there is a deformation retraction of $M$ onto $\mathcal{F}$. We call it the Ford complex or Ford spine of $M$, with respect to the choice of cusps $C_1, C_2, \ldots, C_m$.

By taking the geometric dual to $\mathcal{F}$ as follows, we obtain an ideal polyhedral decomposition $\mathcal{D}$ of $M$. Let $\bar{\mathcal{F}}$ and $\bar{C}$ be the inverse images of $\mathcal{F}$ and $C$ in the universal covering $\mathbb{H}^n$ of $M$. Pick a vertex $x$ of $\bar{\mathcal{F}}$. Then there are finitely many shortest geodesic paths from $x$ to $\bar{C}$, and let $\{v_i\}$ be the ideal points in $\partial \mathbb{H}^n$ forming the centers of the horoball components of $\bar{C}$ which are joined to $x$ by a shortest geodesic path. The convex hull of the ideal points $\{v_i\}$ forms an $n$-dimensional ideal polyhedron of $\mathbb{H}^n$, and the collection of all such ideal polyhedra, where $x$ runs
over the vertices of $\tilde{F}$, determines a $\Gamma$-invariant tessellation of $\mathbb{H}^n$. The tessellation descends to an ideal polyhedral decomposition $D$ of $M = \mathbb{H}^n / \Gamma$.

Epstein and Penner $[EP1988]$ gave a description of the decomposition $D$ by using a convex hull construction in Minkowski space. Their description shows that each cell of $D$ admits a natural (incomplete) Euclidean structure; so, these decompositions are called Euclidean decompositions. In $[AS2003]$, a generalization of the Epstein-Penner construction to cusped hyperbolic manifolds of infinite volume is given, and their relationship to the convex cores are discussed.

The Ford complex $\tilde{F}$ and its geometric dual $D$ depend only on the ratio of the volumes $\text{vol}(C_1) : \text{vol}(C_2) : \cdots : \text{vol}(C_m)$. Moreover, it is proved by Akiyoshi $[Ak2001]$ that the combinatorial structures of $\tilde{F}$, when the ratio varies, are finite.

The ideal polyhedral decomposition $D$, for the case when $C_1, C_2, \ldots, C_m$ have the same volume, is uniquely determined by the hyperbolic manifold $M$, and is called the canonical decomposition of the cusped hyperbolic manifold $M$.

**Example 6.1.** (1) Let $M$ be the hyperbolic thrice punctured sphere, obtained by gluing two ideal hyperbolic triangles through identification of their boundaries via the identity map. Then this decomposition of $M$ into the two copies of ideal triangles is the canonical decomposition of $M$. The corresponding Ford complex of $M$ is a $\theta$-shaped geodesic spine of $M$ consisting of two vertices and three edges.

(2) As shown by Thurston $[Ths1981$ Chapter 4$]$, the complete hyperbolic structure of the figure-eight knot complement $M$ is obtained by gluing two copies of the regular ideal tetrahedron. The decomposition of $M$ into the two copies of the regular ideal tetrahedron is the canonical decomposition of $M$.

Since the complete hyperbolic structure of a given knot complement is unique by the Mostow-Prasad rigidity theorem (Theorem 5.7), and since a knot is determined by its complement by the knot complement theorem (Theorem 2.4), it follows that the combinatorial structure of the canonical decomposition of a hyperbolic knot complement is a complete topological invariant of the knot.

**Theorem 6.2.** (1) Two hyperbolic knots are equivalent, if and only if the canonical decompositions of their complements are combinatorially equivalent.

(2) Let $K$ be a hyperbolic knot and let $D$ the canonical decomposition of $S^3 - K$. Then

$$\text{Sym}(S^3, K) \cong \text{Isom}(S^3 - K) \cong \text{Aut}(D)$$

In the above theorem, $\text{Sym}(S^3, K) := \pi_0 \text{Diff}(S^3, K)$ denotes the symmetry group of the knot $K$, and $\text{Aut}(D)$ denotes the combinatorial automorphism group of $D$.

**Example 6.3.** (1) It is a simple exercise to see that the automorphism group of the canonical decomposition $D$ of the complement figure-eight knot $K$ is isomorphic to the order 8 dihedral group $D_8$. Thus we have $\text{Sym}(S^3, K) \cong \text{Isom}(S^3 - K) \cong D_8$.

(2) The canonical decompositions of the complements of the Kinoshita-Terasaka knot and the Conway knot consist of 12 and 14 ideal tetrahedra, respectively. Hence
they are inequivalent, even though they are mutants of each other and so they share
the same Alexander polynomial, the Jones polynomial, the hyperbolic volume and
the same double branched coverings. Moreover, the automorphism groups of both
of the canonical decompositions are trivial. Thus the symmetry groups of these two
knots are trivial. In particular, both of them are neither amphicheiral nor invertible.
The noninvertibility of $8_{17}$ can be also proved by using the canonical decomposition
of the knot complement.

As explained in the next subsection, the canonical decompositions are amenable
to computer calculation, and wonderful computer programs were developed: Snap-
Pea by Weeks [WK*], Snap by Coulson-Goodman-Hodgson-Neumann [CGHW2000],
SnapPy by Culler-Dunfield-Goerner [CDG*], and a computer verified program HIK-
MOT by Hoffman-Ichihara-Kashiwagi-Masai-Oishi-Takayasu [HIKOMUT2016]. The
results in Example 6.3(2) are, of course, obtained by any of these programs.

This enabled Hoste, Thistlethwaite and Weeks [HTM1998] to extend (and cor-
rect) Conway’s enumeration of all 11 crossing knots to include all prime knots up
16 crossings. There are $1,701,936$ such knots, and all except for 32 knots are hy-
perbolic! To be precise, Hoste and Weeks used the canonical decomposition, and
Thistlethwaite used the “universal method” described at the end of Subsection
2.7. Thus their table is double checked, and this fact shows the strength of both
methods.

This is something like a magic wand for knot theorists as long as finitely many
knots of reasonable crossing numbers are concerned. However, to understand the
canonical decompositions of infinite families of knots or cusped hyperbolic manifolds
is not easy. For the Farey manifolds, namely punctured torus bundles and 2-bridge
knot complements, the combinatorial structures of the canonical decompositions
are determined by Jorgensen [Jr2003] and Gueritaud [Gu2006a] (cf. [ASWY2007,
Gu2006a, SkWk1995]).

At the end of this subsection, we remark that it is still an open problem if every
orientable cusped hyperbolic 3-manifold of finite volume admits a (geometric) ideal
triangulation. (Since any such manifold $M$ admits an ideal polyhedral decompos-
tion by [EP1988] and since every ideal polyhedron is decomposed into ideal tetrahe-
dra, $M$ admits a partially flat ideal triangulation, namely one in which some of the
tetrahedra degenerate into flat quadrilaterals with distinct vertices (see [PP2000]).
But this does not necessarily lead to a genuine ideal triangulation of $M$.) Wada,
Yamashita and Yoshida [WYY1996] and Yoshida [Ysh1996] have proved the existence
of such triangulations under certain combinatorial conditions on the polyhedral de-
composition, and Luo, Schleimer and Tillman [LST2008] have proved that every
such manifold virtually admits an ideal triangulation, namely some finite cover has
an ideal triangulation. Hodgson, Rubinstein and Segerman [HRS2012] considered
a relaxed version of the problem, and proved, in particular, that every hyperbolic
link complement in $S^3$ admits a topological ideal triangulation with a “strict angled structure”.

6.2. Ideal triangulations and computations of the hyperbolic structures

Let $M = \mathbb{H}^3/\Gamma$ be an orientable complete hyperbolic 3-manifold of finite volume with $m \geq 1$ cusps, and let $p : \pi_1(M) \to \Gamma < \text{PSL}(2, \mathbb{C})$ be the holonomy representation. Then, as we have observed in the previous section, $M$ admits an ideal polyhedral decomposition $\mathcal{D}$. We now assume that $\mathcal{D}$ is an ideal triangulation, namely $\mathcal{D}$ consists of ideal tetrahedra. Here an ideal tetrahedron is a closed convex hull in $\mathbb{H}^3$ of 4 ideal points in $\partial \mathbb{H}^3$, called the ideal vertices. Topologically an ideal tetrahedron is homeomorphic to a 3-simplex minus the vertices. Any ideal tetrahedron (up to isometry) is represented by a complex number $z$ with positive imaginary part, such that the Euclidean triangle cut out of any vertex of $\Delta$ by a horosphere is similar to the triangle in $\mathbb{C}$ with vertices 0, 1, and $z$. In fact, $\Delta$ is isometric to the ideal tetrahedron $\Delta(z)$ spanned by 0, 1, $\infty$ and $z$ in the upper half-space model $\mathbb{C} \times \mathbb{R}_+$ of $\mathbb{H}^3$. We call $z$ the shape parameter of the ideal tetrahedron $\Delta(z)$. (If $z$ has negative imaginary part, then $\Delta(z)$ is regarded as negatively oriented. If $z$ is a real number different from 0 and 1, then $\Delta(z)$ is regarded as a degenerate ideal tetrahedron.)

The complex numbers $z$, $(z - 1)/z$, and $1/(1 - z)$ give isometric ideal tetrahedra, and we give each edge $e$ of $\Delta = \Delta(z)$ one of the three complex numbers by the following rule, and call it the edge parameter of $\Delta$ associated with $e$.

- Edges $[0, \infty]$ and $[1, z]$ have edge parameter $z$.
- Edges $[1, \infty]$ and $[0, z]$ have edge parameter $1/(1 - z)$.
- Edges $[z, \infty]$ and $[0, 1]$ have edge parameter $(z - 1)/z$.

Let $e$ be an edge of the ideal triangulation $\mathcal{D}$ of $M$, and let $z_1, \ldots, z_k$ be the edge parameter of the edges of ideal tetrahedra glued to $e$. Since these ideal tetrahedra close up as one goes around $e$, the parameters satisfies the following equation.

$$
\prod_{i=1}^k z_j = 1 \quad \text{and} \quad \sum_{j=1}^k \arg(z_j) = 2\pi,
$$

This condition is identical to the following equation, which is called the gluing equation around $e$.

$$
\sum_{j=1}^k \log(z_j) = 2\pi \sqrt{-1},
$$

where $\log : \mathbb{C} - \mathbb{R}_{\leq 0} \to \mathbb{C}$ is the branch of the logarithm function whose imaginary part lies in $(-\pi, \pi)$.

Let $T$ be a torus boundary component of the compact 3-manifold $\overline{M}$ whose interior is homeomorphic to the hyperbolic manifold $M = \mathbb{H}^3/\Gamma$, and let $\mu$ be an oriented essential simple loop on $T$. (A simple loop on $T$ is essential if it does not bound
a disk in $T$.) By identifying $T$ with a cusp torus, and considering the intersection with the cusp torus with the ideal triangulation $\mathcal{D}$, we obtain a triangulation of $T$, where the vertices correspond to the edges of $\mathcal{D}$ and the triangles correspond to truncations of ideal tetrahedra around ideal vertices. We may assume $\mu$ intersects the edges of the triangulation transversely and does not intersect the vertices of the triangulation. Each segment of $\mu$ in a triangle cuts off a single vertex of the triangle, and so has an associated edge parameter $z_j$. Define $\epsilon_j = +1$ or $-1$ according to whether the vertex lies to the left of $\mu$ or not. (Here we assume that $\infty$ is a parabolic fixed point of $\Gamma$, $\pi_1(T)$ is identified with the stabilizer $\Gamma_\infty$ of $\infty$, and $T$ is identified with the Euclidean torus $\mathbb{C}/\Gamma_\infty$ via the projection from a horosphere centered at $\infty$ to $\mathbb{C}$. The left/right convention is determined by the standard orientation of $\mathbb{C}$.) Then we can see that the complex translation length of the image $\rho(\mu)$ of $\mu$ by the holonomy representation $\rho$ of the complete hyperbolic manifold $M$ is represented by the complex number

$$\mathcal{L}_\mu := \sum_j \epsilon_j \log(z_j).$$

Since $\rho(\mu)$ is parabolic, we have $\mathcal{L}_\mu = 0$. Thus we have the following completeness equation

$$\sum_i \epsilon_i \log(z_i) = 0.$$

Conversely, let $\bar{M}$ be an orientable compact manifold whose boundary is non-empty and consists of tori, and let $\mathcal{D}$ be a topological ideal triangulation of $M = \text{int } \bar{M}$. Namely $\mathcal{D}$ is a topological triangulation (a cell decomposition whose cells are identified with simplices) of the space $\hat{M} = \bar{M}/\sim$, where $\sim$ is the equivalence relation which identifies all points of each boundary component of $\bar{M}$, such that the vertex set of $\mathcal{D}$ is equal to the finite set consisting of the image of $\partial \bar{M}$. By a simple argument using the Euler characteristic, we see that the number of the edges in $\mathcal{D}$ is equal to the number, $t$, of the tetrahedra in $\mathcal{D}$. Now let $\mathbb{H}_+ = \{z \in \mathbb{C} \mid \Im z > 0\}$ be the upper-half space of the complex plane. Pick a $t$-tuple of complex numbers $z = (z_1, \ldots, z_t) \in (\mathbb{H}_+)^t \subset \mathbb{C}^t$ with positive imaginary parts, and identify the topological ideal tetrahedra $\{\Delta_1, \ldots, \Delta_t\}$ with hyperbolic ideal tetrahedra $\{\Delta(z_1), \ldots, \Delta(z_t)\}$. Since all hyperbolic ideal triangles are isometric, we can realize the topological gluing maps among the faces of the topological ideal tetrahedra by hyperbolic isometries. Thus we obtain a hyperbolic structure of the complement of the 1-skeleton of $\mathcal{D}$. We have the following theorem (see [Ths1979, NeZ1985]).

**Theorem 6.4.** Under the above setting, the following hold for each $z = (z_1, \ldots, z_t) \in (\mathbb{H}_+)^t \subset \mathbb{C}^t$.

1. The hyperbolic structure on the complement of the 1-skeleton of $\mathcal{D}$ extends a hyperbolic structure on the whole $M$ if and only if $z$ satisfies the gluing equation at every edge of $\mathcal{D}$.
When the condition (1) is satisfied, the resulting hyperbolic structure on $M$ is complete if and only if $z$ also satisfies the completeness equation at every boundary component of $M$ (for a single choice of an oriented essential simple loop $\mu$ for each boundary component).

**Remark 6.5.** Let $X$ be the variety of $z = (z_1, \ldots, z_t) \in \mathbb{C}^t$ consisting of the solutions of the gluing equations. Then, by a combinatorial argument, we can see that $X$ has dimension $m$ over $\mathbb{C}$, where $m$ is the number of the boundary components of $\bar{M}$ ([Ths1979, Theorem 5.6], [NeZ1985, Proposition 2.3]). By the rigidity theorem, there is a unique point $z^0 \in X \cap \mathbb{H}^t_+$ which satisfies the completeness equation. It is proved by [NeZ1985, Section 4] that $z^0$ is a smooth point of $X \cap \mathbb{H}^t_+$, namely there is a neighborhood of $z^0$ in $X \cap \mathbb{H}^t_+$ which is biholomorphically equivalent to an open set in $\mathbb{C}^m$. (Moreover, it was proved by Choi [Cho2004] that $X \cap \mathbb{H}^t_+$ is a smooth complex manifold.) This fact plays a crucial role in a proof of the hyperbolic Dehn filling theorem (see Subsection 7.2).

On the other hand, there is a convenient method for obtaining topological ideal triangulations of knot/link complements from the diagrams ([Mns1983, Tk1985, Wk2005]). Thus we have a good chance to construct a complete hyperbolic structure on a given knot/link complement by applying Theorem 6.4. In fact, this works extremely well, though the proof of Thurston’s uniformization theorem is very difficult.

Moreover, if a given ideal triangulation $D$ of $M$ satisfies certain inequality at each codimension 1 face of $D$, then $D$ is the canonical decomposition (see [Wk1993]). If the inequality was not satisfied at some face of $D$, then apply the Pachner 3–2 move to $D$ at the face, if it is geometrically realizable, and check if the conditions for the faces hold. If this does not lead to the canonical decomposition, then retriangulate $D$ randomly, and repeat the above procedure. This is the way SnapPea finds the canonical decompositions. Though there is no theoretical guarantee, this method is extremely efficient (see [Wk1993, Wk2005]). For the treatment of the case when the canonical decomposition is not an ideal triangulation, see the work of Hodgson and Weeks [HW1994].

### 6.3. Other geometric invariants for hyperbolic knots and effective geometrization

In addition to the canonical decomposition, there are various important geometric invariants of hyperbolic knots and links.

- The volumes and the Chern-Simons invariants of the hyperbolic link complements.
- The volumes of the maximal cusps.
- The moduli of the Euclidean cusp tori.
- Length spectrum, i.e., the multi set of the lengths of closed geodesics, in particular the length of the shortest closed geodesic.
The lengths of the vertical geodesic paths, joining maximal cusps to themselves.

- Euclidean length spectrum of the maximal cusp torus.

The volumes of hyperbolic manifolds are treated in Section 8.

In the recent beautiful survey \[FKP2019\], Futer, Kalfagianni, and Purcell discuss these invariants from the viewpoint of effective geometrization or WYSIWYG topology, where WYSIWYG stands for “what you see is what you get”, which aims to determine the geometry of link complements directly from the link diagrams. A typical example in this direction is the following estimate by Lackenby \[Lcb2004\] of the volume of alternating link complements in terms of the twist number.

**Theorem 6.6.** Let $D$ be a reduced alternating diagram of a hyperbolic link $L$ in $S^3$, and let $t(D)$ be the twist number of the diagram $D$. Then

$$\frac{1}{2} V_{tet}(t(D) - 2) \leq \text{vol}(S^3 - L) \leq V_{tet}(16t(D) - 16)$$

where $V_{tet} = 1.0149416...$ is the volume of the regular ideal tetrahedron.

Here the twist number $t(D)$ of a link diagram $D$ is the number of the twists of $D$, where a twist of $D$ is either a connected collection of bigon regions in $D$ arranged in a row, which is maximal in the sense that it is not part of a longer row of bigons, or a single crossing adjacent to no bigon regions.

The article \[FKP2019\] presents a nice survey on the recent great progress towards effective geometrization, including a refinement of the above result.

7. **Flexibility of Incomplete Hyperbolic Structures and the Hyperbolic Dehn Filling Theorem**

By the Mostow-Prasad rigidity theorem, the complete hyperbolic structure on a 3-manifold $M$ of finite volume is rigid. However, when $M$ has a cusp, the complete hyperbolic structure admits nontrivial continuous deformations into incomplete hyperbolic structures (see Remark 6.5). In the generic case, the metric completion yields a pathological topological space which is not even Hausdorff. However, in certain special isolated cases, the metric completion produces a complete hyperbolic manifold. This is a rough idea of Thurston’s hyperbolic Dehn filling Theorem. This theorem has stimulated keen attention of many mathematicians and enormous amount of research grew out of this result. In this section, we give an outline of a proof of this theorem and a brief survey of its influence on knot theory.

7.1. Hyperbolic Dehn filling theorem

We begin by recalling the topological operation, Dehn filling. By an oriented slope on a torus $T$, we mean the isotopy class of an oriented essential simple loop on $T$. Each slope represents a primitive element of $H_1(T;\mathbb{Z})$, and conversely any primitive element of $H_1(T;\mathbb{Z})$ is represented by a unique slope. If we fix a basis $\{\mu, \lambda\}$ of
$H_1(T; \mathbb{Z})$, then a primitive element of $H_1(T; \mathbb{Z})$ is expressed as $p\mu + q\lambda$ where $(p, q)$ is a pair of relatively prime integers. Thus we can identify the set of oriented slopes on $T$ with the set of pairs relatively prime integers $(p, q) \in \mathbb{Z}^2 \subset \mathbb{R}^2 \cup \{\infty\} \cong S^2$.

Let $M$ be a connected compact orientable 3-manifold whose boundary consists of $m$ tori $T_1, \ldots, T_m$. Pick an oriented slope $\nu_j$ on $T_j$ for each $j$, and attach a solid torus $V_j = D^2_j \times S^1$ to $M$ along $T_j$, so that the meridian $\partial D^2_j \times \{\ast\}$ is identified with the slope $\nu_j$. The resulting manifold is denoted $M(\nu) = M(\nu_1, \ldots, \nu_m)$ and called the result of Dehn filling of $M$ along the tuple $\nu = (\nu_1, \ldots, \nu_m)$ of oriented slopes. We extend this operation to the case where some $\nu_j$ is the symbol $\infty$, by the rule that if $\nu_j = \infty$ then we leave the boundary $T_j$ as it is. In particular, $M(\infty, \ldots, \infty) = M$.

The following theorem is proved by Thurston [Ths1979, Chapters 4 and Section 5.8].

**Theorem 7.1** (Hyperbolic Dehn filling Theorem). Let $M$ be a connected compact orientable 3-manifold whose boundary consists of $m$ tori, and suppose that int $M$ admits a complete hyperbolic structure of finite volume. Then, except for finitely many choices of the slopes of $\nu_j$ for each $1 \leq j \leq m$, the manifold $M(\nu_1, \ldots, \nu_m)$ admits a complete hyperbolic structure. To be more precise, there exists a neighborhood $V$ of $(\infty, \ldots, \infty)$ in $(\mathbb{R}^2 \cup \{\infty\})^m$ such that $M(\nu_1, \ldots, \nu_m)$ admits a complete hyperbolic structure for every slope $(\nu_1, \ldots, \nu_m)$ contained in $V$.

**Remark 7.2.** (1) The operation at $T_j$ is actually determined by the slope (the isotopy class of an unoriented essential simple loop on a torus) obtained from $\nu_j$ by forgetting the orientation.

(2) When $M$ is the exterior of an $m$-component link $L = \bigcup_{j=1}^m K_j$ in $S^3$, we fix an orientation of each component $K_j$ of $L$, and choose the meridian-longitude systems $\{\mu_j, \lambda_j\}$ as a preferred basis for $H_1(T_j; \mathbb{Z})$, and represent an oriented slope, $\nu_j$, on $T_j$ by a pair of relatively prime integers $(p_j, q_j)$ with $\nu_j = p_j \mu_j + q_j \lambda_j$. The slope obtained from $\nu_j$ by forgetting the orientation is uniquely determined by the rational number $p_j/q_j \in \mathbb{Q} \cup \{1/0\}$. (It should be noted that slope $1/0$ and the symbol $\infty$ have different meanings.) Moreover, this does not depend on the choice of the the orientation of $K_j$. We denote the manifold $M(\nu_1, \ldots, \nu_m)$ by $M(p_1/q_1, \ldots, p_m/q_m)$, and call it the result of Dehn surgery on $L$ with slope $(p_1/q_1, \ldots, p_m/q_m)$.

In Theorem 7.1 a slope (or a tuple of slopes) which does not produce a hyperbolic manifold is called an **exceptional slope**.

**Example 7.3.** (1) The exceptional slopes of the figure-eight knot $K$ are the slopes $p/q$ with $-4 \leq p \leq 4$ and $-1 \leq q \leq 1$. Thus the set of exceptional slopes is $\{1/0, 0, \pm 1, \pm 2, \pm 3, \pm 4\}$ (see [Ths1979, Section 4.6]).

(2) Let $M$ be the exterior of the Whitehead link $L = K_1 \cup K_2$ in $S^3$. Consider the Dehn filling only along $T_1 = \partial N(K_1)$. Then the exceptional slopes for this Dehn filling are those slopes contained in the parallelogram with vertices $\pm (4, -1)$ and $\pm (0, 1)$ (see [NR1992, Section 6]).
7.2. Outline of a proof and generalized Dehn filling coefficients

We give an outline of the proof of Theorem 7.1 by Neumann-Zagier [NeZ1985] (cf. [BP1992, Section E.6]), when the hyperbolic manifold \( \text{int} M \) admits an ideal triangulation \( \mathcal{D} \). (See Petronio-Porti [PP2000] for a proof without assuming the existence of an ideal triangulation, and using a partially flat ideal triangulation of \( M \).) Let \( \Delta_1, \ldots, \Delta_t \) be the ideal tetrahedra in \( \mathcal{D} \), and let \( z^0 = (z_1^0, \ldots, z_t^0) \) be their shape parameters. By the rigidity theorem and Theorem 6.4, \( z^0 \) is the unique solution of the gluing equations and the completeness equations. Let \( \mathcal{X} \) be the variety of \( z = (z_1, \ldots, z_t) \in \mathbb{C}^t \) consisting of the solutions of the gluing equations. For \( z \in \mathcal{X} \cap (\mathbb{H}_+)^t \), let \( M_z \) be the (almost certainly incomplete) hyperbolic manifold determined by the parameter \( z \), and let \( \rho_z : \pi_1(M) \to \text{PSL}(2, \mathbb{C}) \) be the holonomy representation of \( M_z \). For each boundary component \( T_j \) of \( M (1 \leq j \leq m) \), fix a oriented slope \( \mu_j \). For \( z \in \mathcal{X} \cap (\mathbb{H}_+)^t \), let \( v_j(z) \) be the complex number \( \mathcal{L}_{\mu_j} \), defined as in Subsection 6.2, which represents the complex translation length of \( \rho_z(\mu_j) \). (Though the complex translation length is defined only modulo \( 2\pi \sqrt{-1} \mathbb{Z} \) and up to multiplication by \( \pm 1 \), the construction in Subsection 6.2 gives a well-defined continuous lift to \( \mathbb{C} \).)

For each boundary component \( T_j \), pick an oriented slope \( \lambda_j \) which intersects \( \mu_j \) transversely in a single point and so \( \{\mu_j, \lambda_j\} \) forms a generator system of \( H_1(T_j; \mathbb{Z}) \). Let \( v := (v_1, \ldots, v_m) \) be the map from \( \mathcal{X} \) to \( \mathbb{C}^m \), where \( v_j(z) \) is the complex number \( \mathcal{L}_{\lambda_j} \), defined as in Subsection 6.2, which represents the complex translation length of \( \rho_z(\lambda_j) \).

Recall the key Remark 6.5 that there is a neighborhood of \( z^0 \) in \( \mathcal{X} \cap (\mathbb{H}_+)^t \) which is biholomorphically equivalent to an open set in \( \mathbb{C}^m \). By using this fact, we can see that \( u := (u_1, \ldots, u_m) \) maps a neighborhood of \( z^0 \in \mathcal{X} \cap (\mathbb{H}_+)^t \) biholomorphically onto a neighborhood, \( \mathcal{X}_0 \), of \( 0 \in \mathbb{C}^m \) (cf. [NeZ1985, Section 4]).

We now change notation as follows. For \( u \in \mathcal{X}_0 \), we denote the corresponding hyperbolic manifold and the holonomy representation by \( M_u \) and \( \rho_u \), respectively, and we regard \( v \) as a map from \( \mathcal{X}_0 \) to \( \mathbb{C}^m \).

By replacing \( \mathcal{X}_0 \) with a smaller neighborhood of \( 0 \), if necessary, we can assume \( u \) and \( v \) are independent over \( \mathbb{R} \), for all \( u \in \mathcal{X}_0 - \{0\} \). In fact, there is an analytic function \( \tau = (\tau_1, \ldots, \tau_m) : \mathcal{X}_0 \to \mathbb{C}^m \), satisfying the following conditions [NeZ1985, Lemma 4.1]:

1. \( v_j(u) = \tau_j(u)u_j \) for every \( u = (u_1, \ldots, u_j, \ldots, u_m) \in \mathcal{X}_0 \) and \( j = 1, \ldots, m \).
2. \( \tau_j(0, \ldots, 0) \) is equal to the modulus of the cusp torus \( T_j \) of the complete hyperbolic manifold \( M \) with respect to \( \{\mu_j, \lambda_j\} \).

In particular, we may assume \( \tau_j(u) \) is non-real for every \( u \in \mathcal{X}_0 \), and so \( u \) and \( v \) are independent over \( \mathbb{R} \) for every \( u \in \mathcal{X}_0 - \{0\} \).
Now we define the **generalized Dehn filling coefficients** of the $j$-th boundary torus component $\nu_j \in \mathbb{R}^2 \cup \{\infty\} \cong S^2$ by the formula:

$$
\begin{align*}
\nu_j &= \infty \quad \text{if } u_j = 0 \\
\nu_j &= (p_j, q_j) \quad \text{where } p_j u_j + q_j v_j = 2\pi\sqrt{-1} \quad \text{if } u_j \neq 0
\end{align*}
$$

The hyperbolic Dehn filling Theorem 7.1 is a consequence of the following theorem.

**Theorem 7.4.** Under the above setting, the “generalized Dehn filling coefficients map” $u \mapsto \nu = (\nu_1, \ldots, \nu_m)$ gives a homeomorphism from a neighborhood $U$ of $0$ in $\mathbb{C}^m$ onto a neighborhood $V$ of $(\infty, \ldots, \infty)$ in $(\mathbb{R}^2 \cup \{\infty\})^m$. Moreover, the following hold.

- If $\nu_j = \infty$, the hyperbolic structure at the $j$-th end is complete.
- If $\nu_j = (p_j, q_j)$ where $p_j, q_j \in \mathbb{Z}$ are coprime, then the completion of the $j$-th end is a hyperbolic 3-manifold, which is topologically the Dehn filling such that the simple loop $p_j \mu_j + q_j \lambda_j$ on $T_j$ bounds a disk.
- If $p_j/q_j \in \mathbb{Q} \cup \{\infty\}$, let $m_j, n_j \in \mathbb{Z}$ be coprime integers such that $(p_j, q_j) = d(m_j, n_j)$ for some $d > 0$. The completion is a hyperbolic cone 3-manifold obtained by gluing a solid torus with singular core, such that the simple loop $m_j \mu_j + n_j \lambda_j$ on $T_j$ bounds a disk which has singularity at the center, and that the cone angle of the singular locus is $2\pi/d$.
- If $p_j/q_j \in \mathbb{R} - \mathbb{Q}$, then the metric completion of the $j$-th end is not even topologically a manifold.

In the above, a **hyperbolic cone 3-manifold** is a smooth 3-manifold $C$ equipped with a complete metric (distance function) which is locally isometric to $\mathbb{H}^3$ or to the space, $\mathbb{H}^3(\alpha)$, obtained from a geodesic cheese-cake-shaped polyhedron of angle $\alpha > 0$ by identifying two sides. The singular locus $\Sigma \subset C$ is the set of points modeled on the singular line of some $\mathbb{H}^3(\alpha)$, and $\alpha$ is called the cone angle at a singular point modeled on this singular line (for precise definition, see [HdK1998, Section 1], [CK2000, Chapter 3], [BP2001, Chapter 1], [BLP2005, Section 3]). Hyperbolic 3-cone manifolds play a key role in the proof of the orbifold theorem (Theorem 4.4).

**Remark 7.5.** (1) Assume that a tuple of oriented slopes $\nu = (\nu_1, \ldots, \nu_m)$ is the image of a parameter $u = (u_1, \ldots, u_m) \in U$ in Theorem 7.4, namely the metric completion of the hyperbolic manifold $M_u$ is homeomorphic to the manifold $M(\nu)$ obtained from $M$ by the Dehn filling along $\nu$. Let $\nu'$ be the tuple of oriented slopes obtained from $\nu$ by replacing some component $\nu_j = (p_j, q_j)$ with $-\nu_j = (-p_j, -q_j)$. Then $\nu'$ is the image of the parameter $u'$ obtained from $u$ by replacing the component $u_j$ with $-u_j$. Since $M(\nu')$ is homeomorphic to $M(\nu)$ by a homeomorphism preserving the subspace $M$, the rigidity theorem implies that $M_u$ is isometric to $M_u'$. In fact, such an isometry exists whenever two parameters $u$ and $u'$ are related.
by the involution \((u_1, \ldots, u_j, \ldots, u_m) \mapsto (u_1, \ldots, -u_j, \ldots, u_m)\). Thus deformations of the complete hyperbolic manifold \(M\) is parametrized by the quotient of \(U\) by the \((\mathbb{Z}/2\mathbb{Z})^m\)-action, generated by the above involutions with \(j = 1, \ldots, m\). In another word, the space \(U\) is identified with an \((\mathbb{Z}/2\mathbb{Z})^m\)-branched covering of a deformation space of \(M\). The space \(U\) actually parametrizes the incomplete hyperbolic manifolds \(M_u\) endowed with an ideal triangulation (see [NeZ1985, p.323]).

(2) In Theorem 7.4, the complete hyperbolic manifolds \(\{M(\nu)\}\) are regarded as discrete points in the deformation space \(U/(\mathbb{Z}/2\mathbb{Z})^m \cong V/(\mathbb{Z}/2\mathbb{Z})^m\). Thus the discrete set of the complete hyperbolic manifolds \(\{M(\nu)\}\) are linked together in the connected space \(V/(\mathbb{Z}/2\mathbb{Z})^m\).

7.3. Geometry of the hyperbolic manifolds obtained by Dehn fillings

In the hyperbolic Dehn filling Theorem 7.1, the complete hyperbolic manifolds \(M(\nu) = M(\nu_1, \ldots, \nu_m)\) geometrically converges to the original complete hyperbolic manifold \(\text{int} M\) as \(\nu = (\nu_1, \ldots, \nu_m) \to \infty = (\infty, \ldots, \infty)\) [Ths1979, Section 5.11]. Namely, there are positive numbers \(\epsilon(\nu)\) converging to 0 as \(\nu \to \infty\), and numbers \(k(\nu) > 1\) converging to 1 as \(\nu \to \infty\), such that there is a \(k(\nu)\)-bi-Lipschitz diffeomorphism \(\phi_{\nu} : M(\nu)_{\geq \epsilon(\nu)} \to (\text{int} M)_{\geq \epsilon(\nu)}\) between that \(\epsilon(\nu)\)-thick parts. This in particular implies that the lengths of core loops of the attached solid tori in \(M(\nu)\) converge to 0 as \(\nu \to \infty\). This fact plays an essential role in various researches, including [Kj1986, HW1994, BHW1999, RY2016, Lcb*].

This also implies that the volumes \(\text{vol}(M(\nu))\) of the Dehn filled manifolds converges to the volume \(\text{vol}(\text{int} M)\) of the original hyperbolic manifold as \(\nu \to \infty\). Moreover, Thuston [Ths1979] proved, by using the Gromov norm (cf. Subsection 8.4), that \(\text{vol}(M(\nu))\) is strictly smaller than \(\text{vol}(M)\) if \(\nu \neq \infty\). This is refined to quantitative estimates of \(\text{vol}(M(\nu))\) by Neumann-Zagier [NeZ1985], Hodgson-Kerckhoff [HdK2005] and Futer-Kalfagianni-Purcell [FKP2008].

Gromov and Thurston obtained the following result, by constructing a Riemannian metric of negative curvature on \(M(\nu)\), when each surgery curve is “sufficiently long”, by modifying the complete hyperbolic metric of \(\text{int} M\) (see [BH1996] for a detailed proof).

**Theorem 7.6 (The \(2\pi\)-Theorem).** Let \(M\) be an orientable complete hyperbolic 3-manifold of finite volume, and let \(C_1, \ldots, C_m\) be disjoint torus cusps of \(M\). Suppose \(\nu_i\) is a slope on \(\partial C_i\) represented by a geodesic with length \(> 2\pi\) with respect to the Euclidian metric. Then \(M(\nu_1, \ldots, \nu_m)\) has a Riemannian metric of negative curvature.

The metric on \(M(\nu_1, \ldots, \nu_m)\) outside the filling solid tori is identical to the hyperbolic metric on \(M - \bigcup_{j=1}^m C_j\). The geometrization theorem (Theorem 3.3) established
by Perelman guarantees that the resulting manifold $M(\nu_1, \ldots, \nu_m)$ is actually hyperbolic.

The $2\pi$-theorem was refined to the 6-theorem by Agol [Ag2000] and Lackenby [Lcb2004], and it plays a key role in the study of exceptional surgeries (see the next subsection).

7.4. Exceptional surgeries

For a given hyperbolic knot $K$ in $S^3$, or more generally an orientable complete hyperbolic manifold with one cusp, there are only finitely many exceptional slopes $\nu$ which produce non-hyperbolic manifolds. For example, the figure-eight knot has 10 exceptional slopes (Example 7.3(1)). In the survey [Gor1988], Gordon proposed various interesting conjectures, including one which says that 10 is the largest possible number of exceptional slopes of a hyperbolic knot complement.

The natural and important problem to determine exceptional surgery slopes has attracted attention of many mathematicians, and an enormous amount of research grew out of this problem, including:

- the $2\pi$-theorem of Gromov-Thurston [GromThs1987] and its improvement to the 6-theorem by Agol [Ag2000] and Lackenby [Lcb2004],
- the cyclic surgery theorem by Culler-Gordon-Luecke-Shalen [CGLS1987], obtained by combining two different kinds of arguments, namely (i) arguments using the $\text{SL}(2, \mathbb{C})$-character varieties (cf. Subsection 11.3) and (ii) combinatorial, graph-theoretic analysis of the intersection of two incompressible, planar surfaces in knot exteriors,
- study of finite surgery by Boyer-Zhang [ByZh1987, ByZh2001] and Ni-Zhang [NiZ2018], by mainly using the $\text{SL}(2, \mathbb{C})$-character varieties (Heegaard Floer homology and the Casson-Walker invariant are also used in [NiZ2018]),
- the proof of the Property R conjecture by Gabai [Gb1987], by using taut foliations,
- a universal upper bound of the number of exceptional slopes by Hodgson-Kerckhoff [HdK2005, HdK2008], by developing deformation theory of hyperbolic structures (cf. [HdK1998]),
- the optimal universal upper bound, 10, on the number of exceptional slopes of one cusped hyperbolic manifold by Lackenby-Meyerhoff [LM2013] (see Agol [Ag2010a] for related work),
- the optimal universal upper bound, 8, on the geometric intersection numbers of pairs of exceptional slopes of one cusped hyperbolic manifolds by Lackenby-Meyerhoff [LM2013],
- the complete classification of exceptional surgeries on hyperbolic alternating knots by Ichihara-Masai [IM2016], building on a result of [Lcb2000] and through computer-aided verified computation [HIKMOT2016] using a supercomputer.
The last three results give affirmative answers to some conjectures in [Gor1988]. See the survey articles [By2002, Gor1988, Gor2012] for background and further information.

As for Seifert surgeries of knots, namely surgeries which produce Seifert fibered spaces, Deruelle, Miyazaki and Motegi [DMM2012] embarked on the project to understand the whole shape of relationships among all such surgeries, and various interesting results are obtained in this direction.

Among Seifert surgeries, lens space surgeries are particularly interesting. Berge [Brg*] presented a conjecturally complete list of lens space surgery on knots in $S^3$. Based on Berge’s conjecture, Goda and Teragaito [GT2000] conjectured that if a $p$-surgery on a hyperbolic knot $K$ produces a lens space then $K$ is a fibered knot and its genus $g$ satisfies the inequality $2g + 8 \leq |p| \leq 4g - 1$. (Note that $p$ is an integer by the cyclic surgery theorem.) Rasmussen [Ras2004] attacked this problem by using the Heegaard Floer homology, and obtained the estimate $|p| \leq 4g + 3$. This in fact relies on the fact that lens spaces belong to larger class of spaces, known as $L$-spaces, which are rational homology 3-spheres with the “simplest Heegaard-Floer homology” (see Ozsváth-Szabó [OS2005]). See Greene [Gre2015] and references therein for further information on L-space surgery, and see the reviews [OS2006, Ju2015] for the background.

A nice overall survey (in Japanese) on surgery was recently written by Motegi, and its English translation [Mtg*] will appear soon. This survey is strongly recommended.

8. Volumes of hyperbolic 3-manifolds

The volume is the most basic invariant of hyperbolic manifolds. After quickly recalling a method for calculating hyperbolic volumes, we explain (i) the Jørgensen-Thurston theory concerning the volume spectrum of hyperbolic 3-manifolds, (ii) results concerning small volume hyperbolic manifolds, (iii) relation to Gromov norm, and finally (iv) the volume conjecture, which lies in the two innovations, hyperbolic geometry and quantum topology, in knot theory.

8.1. Calculation of hyperbolic volumes

We explain a method for calculating the volumes of hyperbolic 3-manifolds, which is implemented in SnapPea. The method depends on the fact that every hyperbolic 3-manifold $M$ is obtained by hyperbolic Dehn filling on a cusped hyperbolic manifold, say $M_0$. This follows from the facts that the complement of a simple closed geodesic is a cusped hyperbolic manifold (see [Ski1991]) and that the shortest closed geodesic in $M$ is simple. SnapPea usually succeeds in finding an ideal triangulation of the complete hyperbolic manifold $M_0$, which can be deformed into an ideal triangulation of the incomplete hyperbolic structure on $M_0$ whose completion yields the complete
hyperbolic structure of $M$ (cf. Subsection 7.2). Thus the calculation of $\text{vol}(M)$ is reduced to that of the volumes of ideal tetrahedra.

Recall that the isometry type of an ideal tetrahedron is determined by its shape parameter $z \in \mathbb{H} \subset \mathbb{C}$, which in turn represent the similarity class of the Euclidean triangle with vertex set $\{0, 1, z\}$. Let $\alpha, \beta, \gamma$ be the inner angles of this triangle. Then the volume of the ideal tetrahedron $\Delta(z)$ of shape parameter $z$ is given by the following formula:

$$\text{vol}(\Delta(z)) = \Lambda(\alpha) + \Lambda(\beta) + \Lambda(\gamma),$$

where $\Lambda(\theta)$ is the Lobachevsky function defined by

$$\Lambda(\theta) = -\int_{0}^{\theta} \log |2 \sin t| dt = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin(2n\theta)}{n^2}.$$

The volume function $\text{vol}(\Delta(z))$ takes the maximal value $V_{\text{tet}} = 3\Lambda(\pi/3) = 1.0149416\ldots$ precisely at $z = \exp(\pi\sqrt{-1}/3)$, i.e., exactly when $\Delta(z)$ is a regular ideal tetrahedron. See [Ths1979, Chapters 6 and 7] for details.

8.2. Jorgensen-Thurston theory for the volumes of hyperbolic 3-manifolds

Let $V_n \subset \mathbb{R}_+$ be the ordered set consisting of the volumes of complete hyperbolic $n$-manifolds. If $n \neq 3$ then $V_n$ is isomorphic to $\mathbb{N}$, by Gauss-Bonnet theorem for $n = 2$ and by Wang’s theorem [Wn1972] for $n \geq 4$. For the dimension $n = 3$, we have the following surprising theorem due to Jorgensen and Thurston (see [Ths1979]), which forms a sharp contrast to Wang’s theorem.

**Theorem 8.1.** $V_3$ is a well-ordered closed set which is isomorphic to $\omega^\omega$. Moreover, the map

$$\text{vol} : \{\text{complete hyperbolic 3-manifolds of finite volume}\}/(\text{isometry}) \rightarrow V_3$$

is finite to one.

This means that there is a smallest volume $v_1$, a next smallest volume $v_2$, and so on, and these are the volumes of closed hyperbolic 3-manifolds. The increasing sequence $v_1 < v_2 < \cdots < v_k < \cdots$ has a limit $v_\omega$, and this is the volume of a complete hyperbolic 3-manifold with one cusp (cf. Subsection 7.3). There is a smallest volume $v_{\omega+1}$ bigger than $v_\omega$, a second smallest volume $v_{\omega+2}$ bigger than $v_{\omega+1}$, and so on, and these are the volumes of closed hyperbolic 3-manifolds, and their limit $v_{2\omega}$ is the second smallest volume of a complete hyperbolic 3-manifold with one cusp. The increasing sequence $v_\omega < v_{2\omega} < \cdots < v_{k\omega} < \cdots$ has a limit $v_{\omega^2}$, and this is the volume of a complete hyperbolic 3-manifold with two cusps, and so on.

The second statement of Theorem 8.1 says that the volume is “almost” a complete invariant of complete hyperbolic manifolds.
Of course, the volume is not a complete invariant. For example, the comple-
ments of the Kinoshita-Terasaka knot and the Conway knot have the same volume
11.21911773.... In fact, Ruberman [Ru1987] proved that the hyperbolic volume,
more generally the Gromov invariant (cf. Subsection 8.4 below), is unchanged by
mutation. Hodgson and Masai [HM2013] studied the number $N(v)$ of orientable
hyperbolic 3-manifolds with given volume $v \in V_3$: they constructed infinitely many
$v \in V_3$ for which $N(v) = 1$, and proved the exponential growth of $N(v)$ by showing
$N(4nV_{oct}) \geq 2^n/(2n)$. See Chesebro-DeBlois [ChD2014] and Millichap [Mlc2015]
for related results.

8.3. Small volume hyperbolic manifolds

It is a natural and important problem to determine the small volumes, such as $v_1$, $v_\omega$, $v_{\omega^2}$, etc. For the minimal volume $v_{\omega^n}$ of orientable complete hyperbolic 3-manifolds
with $n$-cusps, the following results are established.

- Gabai-Meyerhoff-Milley (2009) [GMM2009]: The Fomenko-Matveev-Weeks
  manifold, which is obtained by $(5, 2)$ and $(5, 1)$ Dehn surgery on the White-
  head link, has the smallest volume $v_1 = 0.94270736...$

- Cao-Meyerhoff (2001) [CM2001]: The figure-eight knot complement and its
  sister, namely $(5, 1)$-filling on one component of the Whitehead link comple-
  ment, have the volume $v_\omega = 2V_{tet} = 2.02988...$, where $V_{tet} = 1.0149416...$
  is the volume of the regular ideal tetrahedron. The figure-eight knot is the
  orientation double cover of the Gieseking manifold, the non-orientable hyper-
  bolic 3-manifold, which has the smallest volume among the all (orientable or
  not) complete non-compact hyperbolic 3-manifolds (see Adams [Ad1987]).

- Agol (2010) [Ag2010b]: The Whitehead link complement and the comple-
  ment of the pretzel link $P(-2, 3, 8)$ have the volume $v_{\omega^2} = V_{oct} = 3.66386...$
  where $V_{oct}$ is the volume of regular ideal octahedron.

- Yoshida (2013) [Ysk2013]: The complement of the minimally twisted hyper-
  bolic 4-chain link has the volume $v_{\omega^4} = 2V_{oct} = 7.32772...$

See the review [GMM2014], for further information. It should be noted that all of the
above small volume hyperbolic manifolds are arithmetic (cf. [Ht1983], [NR1992,
Theorem 5.1] and Subsection 9.3).

As is noted in [GMM2011, Introduction], Thurston had long promoted the idea
that volume is a good measure of the complexity of a hyperbolic 3-manifold. In
fact, in [Ths1979, the end of Chapter 6], he writes as follows: One gets a feeling
that volume is a very good measure of the complexity of a link complement, and
that the ordinal structure is really inherent in three-manifolds. The following conjecture,
due to Thurston, Weeks, Matveev-Fomenko and Mednykh-Vesnin, states the idea
more rigorously, and the results presented above can be regarded as partial answers
to this conjecture.
Conjecture 8.2. The complete low-volume hyperbolic 3-manifolds can be obtained by filling cusped hyperbolic 3-manifolds of small topological complexity.

At the end of this subsection, we explain another approach to Thurston’s idea above, by using the notions of shadows of 3 and 4-manifolds introduced by Turaev [Tur1992, Tur1994]. Costantino and Thurston [CT2008] introduced the shadow complexity $\text{sc}(M)$ of a compact orientable 3-manifold $M$ with (possibly empty) toral boundary, and proved the following estimate of the Gromov norm $||M||$ (cf. Subsection 8.4 below):

$$\frac{V_{\text{tet}}}{2V_{\text{oct}}} ||M|| \leq \text{sc}(M) \leq C||M||^2$$

for some universal constant $C$.

In the same paper, they implicitly introduced the notion of stable map complexity and studied its relation between (branched) shadow complexity as well. Ishikawa and Koda [IK2017] showed the two complexities are actually equal, and moreover, using the result of [FKP2008], they gave an elaborate refinement of the above (left) inequality when $M$ is hyperbolic. They also defined the branched shadow complexity $\text{bsc}(M, L)$ for a link $L$ in a compact orientable 3-manifold $M$ with (possibly empty) toral boundary, and gave a complete characterization of hyperbolic links $L$ in $S^3$ with $\text{bsc}(S^3, L) = 1$.

8.4. Gromov norm

In [Grom1982], Gromov introduced the notion of simplicial volume $||M||$ of a closed manifold $M$ as follows, by using the real singular homology:

$$||M|| := \inf \{||z|| \mid z \text{ is a singular cycle representing the fundamental class } [M] \}$$

Here, for a (real) singular chain $z = \sum_j a_j \sigma_j$, its norm $||z||$ is defined as the sum $\sum_j |a_j|$ of the absolute values of its coefficients. He used it to estimate the "minimal volume" of closed smooth manifold (see [Grom1982]). Building on this work, Thurston [Ths1979, Chapter 6] defined the Gromov norm $||M||$ of a compact orientable 3-manifold $M$ with (possibly empty) toral boundary as follows:

$$||M|| := \lim_{\epsilon \to 0} \inf \{||z|| \mid z \text{ is a singular chain representing } [M, \partial M] \text{ and } ||\partial z|| \leq \epsilon \}$$

He then proved the following.

1. If $M$ is hyperbolic (and hence int $M$ admits a complete hyperbolic structure of finite volume), then

$$||M|| = \frac{1}{V_{\text{tet}}} \text{vol(int } M\text{).}$$

2. If $M$ is a Seifert fibered space, then $||M|| = 0$.

3. Let $T$ be a torus embedded in int $M$ and let $M_T$ be the manifold obtained by cutting $M$ along $T$. Then $||M|| \leq ||M_T||$. 

53
Soma [Som1981] proved that when $T$ is incompressible, equality holds in (3) and that similarly equality holds for an incompressible annulus properly embedded in $M$. He then defined, for a link $L$ in $S^3$, the Gromov invariant $\|L\|$ of $L$ by $\|L\| = \|E(L)\|$, and obtained the following theorem.

**Theorem 8.3** (Soma). For a link $L$ in $S^3$, the following hold.

1. If $L$ is a split sum of two links $L_1$ and $L_2$, then $\|L\| = \|L_1\| + \|L_2\|$.
2. If $L$ is a connected sum of two links $L_1$ and $L_2$, then $\|L\| = \|L_1\| + \|L_2\|$.
3. Suppose $L$ is a non-splittable link, and let $\{M_j\}$ be the hyperbolic pieces of the JSJ decomposition of $E(L)$. Then

$$\|L\| = \sum_j \|M_j\| = \frac{1}{V_{tet}} \sum_j \text{vol}(\text{int} M_j).$$

8.5. The volume conjecture

In addition to the revolution caused by William Thurston, knot theory has experienced yet another revolution through discovery of the Jones polynomial by Vaughan Jones [Jn1985]. The Volume Conjecture, first stated by Rinat Kashaev [Ks1995] and then reformulated and expanded by Hitoshi Murakami and Jun Murakami [MrMr2001], provoked deep interaction between the two innovations, hyperbolic geometry and quantum topology.

The conjecture says that the hyperbolic volume of a hyperbolic knot in $S^3$ (more generally, the Gromov norm of a knot in $S^3$) is determined by the asymptotic behavior of Kashaev’s invariant $\langle K \rangle_N$, which is shown by [MrMr2001] to coincide with the evaluation, $J_N(K)$, of the $N$-colored Jones polynomial (with a certain normalization) at the primitive $N$-th root of unity $\exp(2\pi i/N)$.

**Conjecture 8.4** (Volume Conjecture). For any knot $K$ in $S^3$, the following holds:

$$\|K\| = \frac{2\pi}{V_{tet}} \lim_{N \to \infty} \frac{\log |J_N(K)|}{N}.$$  

In particular, if $K$ is a hyperbolic knot, the following holds:

$$\text{vol}(S^3 - K) = 2\pi \lim_{N \to \infty} \frac{\log |J_N(K)|}{N}.$$  

Moreover, H. Murakami and J. Murakami proved that Kashaev’s invariant also coincides with an evaluation of the generalized Alexander polynomial defined by Y. Akutsu, T. Deguchi and T. Ohtsuki [ADO1992]. They say in [MrMr2001, page 86] that the set of colored Jones polynomials and the set of generalized Alexander polynomials of Akutsu-Deguchi-Ohtsuki intersect at Kashaev’s invariants.

Furthermore, H. Murakami, J. Murakami, M. Okamoto, T. Takata and Y. Yokota [MMOTY2002] proposed the following complexification of Kashaev’s conjecture:
**Conjecture 8.5** (Complexification of the Volume Conjecture). For any hyperbolic knot $K$ in $S^3$, the following holds:

\[
\text{vol}(S^3 - K) + \sqrt{-1} \text{CS}(S^3 - K) = 2\pi \lim_{N \to \infty} \frac{\log J_N(K)}{N}.
\]

In the above conjecture CS$(S^3 - K)$ denotes the Chern-Simons invariant of $S^3 - K$ (see [ChS1974, My1986, Yst1985]). For further information, see the surveys [Mrk2011, Mrk2013] and the recently published book [MY2018].

9. **Commensurability and arithmetic invariants of hyperbolic manifolds**

In [Ths1979, Sections 6.7 and 6.8], Thurston studied the commensurability relation among hyperbolic knot/link complements, and gave various commensurable and incommensurable examples. This work has promoted intimate interaction between knot theory and number theory. In this section, we recall basic arithmetic invariants of commensurability classes of Kleinian groups, and describe application to knot theory. We also describe the dichotomy between arithmetic groups and non-arithmetic groups found by Margulis and Borel. In the final subsection, we recall the solution due to Gehring, Marshal and Martin of the 3-dimensional Siegel problem to determine the minimal volume of hyperbolic orbifolds, lying emphasis on the role of arithmetic groups. For further information on the topic of this section, see the textbook Maclachlan-Reid [MR2003].

9.1. Commensurability classes and invariant trace fields

Two Kleinian groups $\Gamma_1$ and $\Gamma_2$ are said to be commensurable if there is a conjugate, $\Gamma_2' := g^{-1}\Gamma_2 g$ ($g \in \text{PSL}(2, \mathbb{C})$) such that $\Gamma_1 \cap \Gamma_2'$ has finite index both in $\Gamma_1$ and $\Gamma_2'$. This is equivalent to the condition that the two hyperbolic manifolds $M_1 = \mathbb{H}^3/\Gamma_1$ and $M_2 = \mathbb{H}^3/\Gamma_2$ are commensurable, i.e., there is a hyperbolic manifold which is a finite covering of both $M_1$ and $M_2$. As is explained in Subsection 6.1 the canonical decomposition provides us an efficient method for checking if two (cusped) hyperbolic manifolds are isometric. But, the method is not directly applicable for checking commensurability, though there is a nice application of the canonical decomposition for the commensurability problem (see [GHH2008] and Subsection 9.3).

Number theory enables us to define a very useful invariant of the commensurability classes of Kleinian groups of cofinite volume. Let $M = \mathbb{H}^3/\Gamma$ be an orientable complete hyperbolic manifold of finite volume. Consider the set $\text{tr} \Gamma = \{\pm \text{tr}(\gamma) \mid \gamma \in \Gamma\} \subset \mathbb{C}$ and the field $\mathbb{Q}(\text{tr} \Gamma)$ generated by the set. (Note that the trace $\text{tr} \gamma$ for $\gamma \in \text{PSL}(2, \mathbb{C}) \cong \text{Isom}^+(\mathbb{H}^3)$ is well-defined up to sign.) This is called the trace field of the Kleinian group $\Gamma$. It follows from the rigidity theorem that the trace field $\mathbb{Q}(\text{tr} \Gamma)$ has finite degree over $\mathbb{Q}$, i.e., it is a number field. By the rigidity theorem again, this is an invariant of the topological space $M$. 
Though the trace field $\mathbb{Q}(\mathrm{tr} \Gamma)$ itself is not, in general, an invariant of the commensurability class, it provides us a very useful commensurability invariant as follows. Let $\Gamma^{(2)}$ be the subgroup of $\Gamma$ generated by $\{\gamma^2 \mid \gamma \in \Gamma\}$. Then $\Gamma^{(2)}$ is normal in $\Gamma$ and $\Gamma/\Gamma^{(2)}$ is a finite abelian group which is a direct sum of order 2 cyclic groups. The following theorem was proved by Reid [Rei1990].

**Theorem 9.1.** Let $\Gamma$ be a Kleinian group of finite covolume. Then $\mathbb{Q}(\mathrm{tr} \Gamma^{(2)})$ is an invariant of the commensurability class of $\Gamma$. Moreover

$$\mathbb{Q}(\mathrm{tr} \Gamma^{(2)}) = \mathbb{Q}(\{(\mathrm{tr} \gamma)^2 \mid \gamma \in \Gamma\}).$$

The field $\mathbb{Q}(\mathrm{tr} \Gamma^{(2)})$ is denoted by $k(\Gamma)$ and is called the **invariant trace field** of $\Gamma$. By [NR1992, Corollary 2.3], if $M = \mathbb{H}^3/\Gamma$ is a knot complement (or more generally, the complement of a link in $\mathbb{Z}/2\mathbb{Z}$-homology sphere) then $k(\Gamma) = \mathbb{Q}(\mathrm{tr} \Gamma)$: thus the trace field itself is an invariant of commensurability class in this case.

If $M$ is a cusped hyperbolic manifold which admits an ideal triangulation into the hyperbolic ideal tetrahedra $\{\Delta(z_1), \ldots, \Delta(z_t)\}$, then the following holds [NR1992, Theorem 2.4]:

$$k(\Gamma) = \mathbb{Q}(z_1, \ldots, z_t).$$

The **invariant quaternion algebra** of $\Gamma$ is the $k(\Gamma)$-algebra of the $2 \times 2$ matrix algebra $M_2(\mathbb{C})$ generated over $k(\Gamma)$ by the elements of $\Gamma^{(2)}$. It is denoted by $A(\Gamma)$. This algebra is also an invariant of the commensurability class of $\Gamma$. Both $k(\Gamma)$ and $A(\Gamma)$ are preserved by mutation (see [NR1991]).

The computer program “Snap” calculates various arithmetic invariants including the invariant trace field and the invariant quaternion algebra (see [CGHW2000]).

### 9.2. Commensurators and hidden symmetries

For a Kleinian group $\Gamma$ of cofinite volume, the **commensurator of $\Gamma$** is defined by

$$\mathrm{Comm}(\Gamma) = \{g \in \mathrm{Isom} \mathbb{H}^3 \mid [\Gamma; \Gamma \cap G^g] < \infty\},$$

and its orientation-preserving subgroup is denoted by $\mathrm{Comm}^+(\Gamma)$. The commensurator $\mathrm{Comm}(\Gamma)$ is identified with the group of equivalence classes of virtual automorphisms of $\Gamma$. A **virtual automorphism** of $\Gamma$ is an isomorphism $\phi : \Gamma_1 \rightarrow \Gamma_2$ between subgroups of finite index in $\Gamma$, and two virtual automorphisms are defined to be equivalent if they agree on some subgroup of $\Gamma$ of finite index. A virtual automorphism represents an isometry between two finite coverings $\mathbb{H}^3/\Gamma_1$ and $\mathbb{H}^3/\Gamma_2$ of the hyperbolic manifold $M = \mathbb{H}^3/\Gamma$. It is called a **hidden symmetry** of $M$ if it is not a lift of an isometry of $M$. By a **hidden symmetry** of a hyperbolic knot in $S^3$, we mean a hidden symmetry of the knot complement. We can see that the figure-eight knot $K$ has a hidden symmetry as follows. Recall that $S^3 - K = \mathbb{H}^3/\Gamma$ has an ideal triangulation consisting of two copies of the regular ideal tetrahedron $\Delta(\omega)$ with $\omega = \exp(\frac{\pi \sqrt{-1}}{4})$. This implies that the invariant trace field $k(\Gamma)$ is equal to
\[ Q(\omega) = Q(\sqrt{-3}). \] Moreover, we see that \( \Gamma \) is a subgroup \( \text{PSL}(2, \mathbb{O}_3) \) of finite index (actually equal to 24), where \( \mathbb{O}_3 \) is the ring of integers of the number field of \( Q(\sqrt{-3}). \) This implies that \( \text{PGL}(2, Q(\sqrt{-3})) \) belongs to the commensurator of \( \Gamma. \) In fact, we have \( \text{Comm}^+(\Gamma) = \text{PGL}(2, Q(\sqrt{-3})). \) Since \( \text{PGL}(2, Q(\sqrt{-3})) \) is dense in \( \text{PSL}(2, \mathbb{C}) \), the normalizer of \( \Gamma \) must be a proper subgroup of \( \text{Comm}^+(\Gamma) = \text{PGL}(2, Q(\sqrt{-3})). \) Hence \( \Gamma \) (and so the figure-eight knot) has a hidden symmetry.

In addition to the figure-eight knot, the two dodecahedral knots of Aitchison and Rubinstein [AR1992] admit hidden symmetries, and these three are the only known such knots. Neumann and Reid [NR1992, Question 1] conjecture that they are all. For results related to the conjecture, see [RW2008, BBCW2015, BBCW2015, MW2016] and references therein.

9.3. Arithmetic versus non-arithmetic

The above explanation for the existence of hidden symmetries of the figure-eight knot is based on the fact that the figure-eight knot group belongs to the particularly nice family of Kleinian groups, called the arithmetic groups. For the definition of arithmetic groups, see the textbook [ReM2003] or the course notes [Ne1999b, Chapter 3, Section 3]. If we restrict our attention to a cofinite volume Kleinian group \( \Gamma \) such that \( M = \mathbb{H}^3/\Gamma \) has a cusp, then \( \Gamma \) is arithmetic if and only if \( \Gamma \) is conjugate to a subgroup of \( \text{PGL}(2, \mathbb{O}_d) \) for some positive integer \( d. \) Here \( \mathbb{O}_d \) is the ring of integers of the number field of \( Q(\sqrt{-d}). \) In this case, we have \( k(\Gamma) = Q(\sqrt{-d}) \) and \( A(\Gamma) = M_2(Q(\sqrt{-d})), \) and the invariant trace field \( k(\Gamma) \) is the complete commensurability invariant of the arithmetic group \( \Gamma. \) However, most cusped hyperbolic manifolds are non-arithmetic; in particular, the figure-eight knot is the unique hyperbolic knot in \( S^3 \) whose complement is arithmetic (see Reid [Rei1991]).

Margulis [Marg1974] (see also Borel [Brl1981]) established the following striking dichotomy between the arithmetic Kleinian groups and non-arithmetic Kleinian groups.

**Theorem 9.2.** Let \( \Gamma \) be a cofinite volume Kleinian group. Then the following hold.

1. \( \Gamma \) is non-arithmetic if and only if \( \Gamma \) has finite index in \( \text{Comm}^+(\Gamma) \). In this case, \( \text{Comm}^+(\Gamma) \) is the unique maximal element in the commensurability class of \( \Gamma \).

2. \( \Gamma \) is arithmetic if and only if \( \text{Comm}^+(\Gamma) \) is dense in \( \text{PSL}(2, \mathbb{C}) \). In this case, there are infinitely many maximal elements in the commensurability class of \( \Gamma \).

The first assertion of Theorem 9.2 shows that the commensurability class of a non-arithmetic cofinite volume Kleinian group \( \Gamma \) is particularly simple, namely it consists only of conjugates of finite index subgroups of the Kleinian group \( \text{Comm}^+(\Gamma) \). In terms of orbifolds, this means that two non-arithmetic orientable hyperbolic
3-manifolds $M_1$ and $M_2$ are commensurable if and only if they cover a common orbifold. Based on this fact and by using the Epstein-Penner decomposition [EP1988] and finiteness of Epstein-Penner decompositions of a given cusped hyperbolic manifolds (see Akiyoshi [Ak2001]), Goodman-Heard-Hodgson [GHH2008] gave a practical algorithm for determining when two cusped hyperbolic non-arithmetic 3-manifolds are commensurable. Their algorithm is based on the fact that two cusped hyperbolic $n$-manifolds $M$ and $M'$ cover a common orbifold if and only if they admit Epstein-Penner decompositions lifting to isometric tilings of $\mathbb{H}^n$ (see [GHH2008, Theorem 2.4]). Their algorithm is implemented in a computer program, which enabled them to determine the commensurability classes of the complements of all hyperbolic knots and links up to 12 crossings. In particular, they have shown that the complements of the Kinoshita-Terasaka knot and the Conway knot belong to different commensurability classes, even though they share the same invariant trace fields and invariant quaternion algebras. See Chesebro-DeBlois [ChD2014] and Millichap-Worden [MW2016] for related works.

The second assertion of Theorem 9.2 shows that the commensurability class of arithmetic Kleinian groups is very complicated. Walter Neumann describes a geometric way of thinking of this situation as follows, in his course notes [Ne1999b, Chapter 3, Section 6].

A Kleinian (or Fuchsian) group is the symmetry group of some “pattern” in $\mathbb{H}^3$ (respectively $\mathbb{H}^2$). This pattern might just be a tessellation - for instance, a tessellation by fundamental domains, or it might be an Escher-style drawing. If one superposes two copies of this pattern, displaced with respect to each other, one will usually get a pattern which no longer has a Kleinian (or Fuchsian) symmetry group in our sense - the symmetry group has become too small to have finite volume quotient. But in the arithmetic case - and only in this arithmetic case - one can always change the displacement very slightly to make the superposed pattern have a symmetry group that is of finite index in the original group.

In the course notes [Ne1999b], we can also find a beautiful introduction to the idea of Scissor congruence, with a historical background which goes back to Euclid, Dehn and Hilbert. For more details of this important topic, see [Ne1999a].

9.4. Siegel’s problem and arithmetic manifolds

In Subsection 8.3 we surveyed various important results concerning small volume hyperbolic 3-manifolds. It is equally natural and important to study small volume hyperbolic orbifolds. In 1943, Siegel [Si1943, Si1945] posed the problem to identify the infimum

$$\mu(n) = \inf_{\Gamma} \text{vol}(\mathbb{H}^n/\Gamma)$$

where the infimum is taken over the lattices $\Gamma < \text{Isom}^+ \mathbb{H}^n$, i.e., discrete subgroups of cofinite volume. Siegel solved the problem in dimension 2, by showing that the
(2, 3, 7)-triangle group is the unique Fuchsian group of minimal coarea

\[ \mu(2) = 2\pi \left| \frac{1}{2} + \frac{1}{3} + \frac{1}{7} - 1 \right| = \frac{\pi}{21}. \]

In 1986, Kazdan and Margulis [KZ1986] made an important contribution to the Siegel problem, by proving that \( \mu(n) \) is positive and attained for each \( n \).

The arithmetic groups play a crucial role in the study of the Siegel problem. One big reason is that, due to formulas of Borel [Brl1981], various explicit calculations can be made for arithmetic Kleinian groups. According to Gaven Martin [Mrti2015], another reason is that it turns out that nearly all the extremal problems one might formulate are realised by arithmetic groups, perhaps the number theory forcing additional symmetries in a group and therefore making it “smaller” or “tighter”.

After long term collaboration, Gehring, Marshal and Martin [GhMr2009, MrsMrt2012] finally solved the 3-dimensional Siegel problem.

**Theorem 9.3.** The minimum \( \mu(3) \) of the volumes of hyperbolic 3-orbifolds is

\[ \mu(3) = \text{vol}(\mathbb{H}^3/\Gamma_0) = 275^{3/2} 2^{-7} \pi^{-6} \zeta_k(2) \sim 0.03905..., \]

where \( \zeta_k \) is the Dedekind zeta function of the underlying number field \( \mathbb{Q}(\gamma_0) \), with \( \gamma_0 \) a complex root of \( \gamma^4 + 6\gamma^3 + 12\gamma^2 + 9\gamma + 1 \), of discriminant \( -275 \). Here \( \Gamma_0 \) is an arithmetic Kleinian group obtained as a \( \mathbb{Z}/2\mathbb{Z} \)-extension of the index 2 orientation-preserving subgroup of the group generated by reflection in the faces of the 3-5-3-hyperbolic Coxeter tetrahedron. The group \( \Gamma_0 \) is generated by two elliptic elements, one of order 2 and the other of order 5.

**Remark 9.4.** The quotient orbifold \( \mathcal{O}_0 = \mathbb{H}^3/\Gamma_0 \) is as illustrated in Figure 9, where the blue eyeglasses represent the generating pair. This orbifold is obtained from the “Heckoid orbifold \( H(1/4; 5/2) \)” in Figure 10 by an orbifold surgery. Here a Heckoid orbifold is a hyperbolic 3-orbifold whose orbifold fundamental group is a Heckoid group, which is a Kleinian group generated by two parabolic transformations introduced by Riley [Ri1992] as an analogy of Hecke groups and formulated by [LS2013]. Heckoid orbifolds are also intimately related to 2-bridge links. As is noted by Martin [Mrti2015, Mrt2016], most of small volume 3-orbifolds arise from 2-bridge links.

See the survey by Martin [Mrti2015] for backgrounds and details concerning the 3-dimensional Siegel problem, and the surveys by Belolipetsky [Bel2014] and Kellerhals [Ke2014] for higher dimensional Siegel problem.

The above theorem has the following application to finite group actions on hyperbolic 3-manifolds.

**Corollary 9.5.** Let \( M \) be an orientable complete hyperbolic 3-manifold of finite volume, and let \( G \) be a finite group acting on \( M \) effectively and orientation-preservingly.
Then

$$|G| \leq \frac{\text{vol} \, M}{\mu(3)}.$$ 

A refinement of this corollary for hyperbolic knot complements can be found in [FKP2019, Theorem 4.14].

10. **Flexibility of complete hyperbolic manifolds - deformation theory of hyperbolic structures**

Let $M$ be a complete hyperbolic manifold homeomorphic to the interior of a compact orientable 3-manifold $\overline{M}$. If $\partial M$ is a (possibly empty) union of tori, then $\text{vol}(M) < \infty$ and so the complete hyperbolic structure on $M$ is unique by the
Mostow-Prasad rigidity theorem. However, when $\partial M$ contains a component different from a torus, the complete hyperbolic structure of $M$ admits a nontrivial deformation, and there is a rich and deep deformation theory. This deformation theory is one of the central themes in Kleinian group theory and it plays a crucial role in the proof of the geometrization theorem of Haken manifolds. In particular, the existence of complete hyperbolic structures on surface bundles over the circle, e.g. the complements of hyperbolic fibered knots, was established as a consequence the double limit theorem [Ths1998* Theorem 4.1] concerning the deformation space of hyperbolic structures on $\Sigma \times \mathbb{R}$ where $\Sigma$ is a (fiber) surface. The idea of a Cannon-Thurston map, a $\pi_1(\Sigma)$-equivariant sphere filling curve, grew out of this construction.

On the other hand, Agol [Ag2011] proved that a hyperbolic punctured surface bundle over the circle admits a very special topological ideal triangulation, called a veering triangulation, which is canonical in the sense that it is determined by the fiber structure. It was revealed by Guéritaud [Gu2016] that the veering triangulation is intimately related to the Cannon-Thurston map.

The purpose of this section is (i) to give an introduction to deformation theory of Kleinian groups and its relation to the hyperbolic structures of surface bundles over the circle, and (ii) to explain Cannon-Thurston maps and veering triangulations. For further information on deformation theory, see Otal [Ot1996, Ot2001], Matsuzaki-Taniguchi [MaT1998], Kapovich [Kp2000], Ohshika [Oh2002] and Marden [Mrd2007a, Mrd2007b].

10.1. Convex cores and conformal boundaries of hyperbolic manifolds

In this subsection, we recall the basic concepts of convex cores and conformal boundaries of hyperbolic manifolds.

Though the action of a Kleinian group $\Gamma$ on $\mathbb{H}^3$ is properly discontinuous, the action of $\Gamma$ on $\partial \mathbb{H}^3$ does not have this property. To see this, pick a point $x \in \mathbb{H}^3$ and consider its orbit $\Gamma x$. Of course the orbit is discrete in $\mathbb{H}^3$. But, it has nonempty accumulation points in the 3-ball $\mathbb{H}^3 \cup \partial \mathbb{H}^3$ (provided that $\Gamma$ is not a finite group). The set of all accumulation points is independent of the choice of $x$ and forms a $\Gamma$-invariant closed set in $\partial \mathbb{H}^3$. This set is denoted by $\Lambda(\Gamma)$ and is called the limit set of $\Gamma$. The action of $\Gamma$ on $\Lambda(\Gamma)$ is not properly discontinuous and is chaotic. The complement $\Omega(\Gamma) := \partial \mathbb{H}^3 - \Lambda(\Gamma)$ is called the domain of discontinuity of $\Gamma$, and it is a (possibly empty) maximal open domain in $\partial \mathbb{H}^3$ on which $\Gamma$ acts properly discontinuously.

The convex core $C_M$ of a complete hyperbolic manifold $M = \mathbb{H}^3/\Gamma$ is defined as the quotient $C(\Lambda(\Gamma))/\Gamma$, where $C(\Lambda(\Gamma))$ is the convex hull in $\mathbb{H}^3$ of the limit set $\Lambda(\Gamma)$. Note that any closed geodesic in $M$ corresponds to a conjugacy class of a hyperbolic element of $\Gamma$ and that the endpoints of its axis are contained in $\Lambda(\Gamma)$: this implies that the axis is contained in $C(\Lambda(\Gamma))$ and so the closed geodesic is contained in $C_M$. In fact, $C_M$ is the smallest locally convex closed subset of $M$ which contains
all closed geodesics of $M$. The convex core $C_M$ is also characterized as the smallest locally convex submanifold of $M$ whose inclusion is a homotopy equivalence.

On the other hand, since the action of $\Gamma$ on $\partial \mathbb{H}^3$ (and hence on $\Omega(\Gamma)$) is conformal, the quotient space $\partial_{\infty}M := \Omega(\Gamma)/\Gamma$ has a natural conformal structure and forms the boundary of the Klein manifold $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$. The Riemann surface $\partial_{\infty}M = \Omega(\Gamma)/\Gamma$ is called the conformal boundary of $M$.

**Example 10.1** (Infinite cyclic Kleinian group). For the infinite cyclic Kleinian group $\Gamma$ generated by a hyperbolic transformation $A(z) = az$ with $|a| \neq 1$ in Example 5.2(1), the convex core of the quotient hyperbolic manifold $\mathbb{H}^3/\Gamma \cong \text{int}(D^2 \times S^1)$ is equal to the core closed geodesic $(0 \times \mathbb{R}_+)/\Gamma$, and the conformal boundary is the torus $(\mathbb{C} - \{0\})/(z \sim az)$.

In the remainder of this section $\Sigma \cong \text{int} \Sigma_{g,b}$ denotes the closed orientable surface of genus $g$ with $b$ punctures, and with negative Euler characteristic.

**Definition 10.2** (Type-preserving representation). A representation $\rho: \pi_1(\Sigma) \cong \pi_1(\Sigma_{g,b}) \to \text{Isom}^+ \mathbb{H}^3$ is type-preserving if it satisfies the following conditions.

1. $\rho$ maps peripheral elements (elements represented by boundary loops of $\Sigma_{g,b}$) to parabolic elements.
2. $\rho$ is irreducible, i.e., $\rho(\pi_1(\Sigma))$ does not have a common fixed point on $\partial \mathbb{H}^3$.

**Example 10.3** (Fuchsian group). The surface $\Sigma$ admits a complete hyperbolic structure of finite area $\pi_1(\Sigma)$. Pick a complete hyperbolic metric on $\Sigma$ and let $\rho_0: \pi_1(\Sigma) \to \text{Isom}^+ \mathbb{H}^2$ be the holonomy representation. Then it is discrete, faithful and type-preserving, its image $\Gamma_0 = \rho_0(\pi_1(\Sigma))$ is a Fuchsian group. The limit set of the Fuchsian group $\Gamma_0$ is equal to $\partial \mathbb{H}^2$. Regard $\Gamma_0$ as a Kleinian group, i.e., a discrete subgroup of $\text{Isom}^+ \mathbb{H}^3$. Then the limit set $\Lambda(\Gamma_0)$ is the round circle $\partial \mathbb{H}^2$ in $\partial \mathbb{H}^3$, where $\mathbb{H}^2(= \mathbb{R} \times \mathbb{R}_+ \subset \mathbb{C} \times \mathbb{R}_+ = \mathbb{H}^3)$ is the hyperplane of $\mathbb{H}^3$ invariant by $\Gamma$. The Kleinian manifold $(\mathbb{H}^3 \cup \Omega(\Gamma_0))/\Gamma_0$ is homeomorphic to the product of $\Sigma$ and the closed interval $[-\infty, \infty]$, and the convex core is identified with $\Sigma \times 0$.

**Example 10.4** (Quasifuchsian group). The Fuchsian representation $\rho_0: \pi_1(\Sigma) \to \text{PSL}(2, \mathbb{R})$ in the previous example admits a nontrivial deformation into a faithful discrete type-preserving $\text{PSL}(2, \mathbb{C})$-representation $\rho$, such that $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma \cong \Sigma \times [-\infty, \infty]$ where $\Gamma = \rho(\pi_1(\Sigma))$. This condition is equivalent to the condition that the limit set $\Lambda(\Gamma)$ is a topological circle. A Kleinian group isomorphic to $\pi_1(\Sigma)$ satisfying this condition is called a quasifuchsian group and the holonomy representation is called a quasifuchsian representation. Generically, a quasifuchsian group is not conjugate to a Fuchsian group in $\text{PSL}(2, \mathbb{C})$, and in this case, the circle $\Lambda(\Gamma)$ in $\partial \mathbb{H}^3$ is very complicated; in particular its Housdorff dimension is strictly bigger than 1. The convex core $C_M$ of the hyperbolic manifold $M = \mathbb{H}^3/\Gamma$ is identified with $\Sigma \times [-1, 1]$ in $\Sigma \times (-\infty, \infty) \cong M$. Each boundary component $\Sigma \times \{\pm 1\}$ of the convex core has the structure of "hyperbolic surface bent along a geodesic..."
lamination”. (see [Ths1979, Section 8.5], [CEG1987]). The domain of discontinuity $\Omega(\Gamma)$ consists of two components $\Omega_+(\Gamma)$ and $\Omega_-(\Gamma)$, and the Riemann surfaces $S_\pm = \Omega_\pm(\Gamma)$ correspond to the boundary components $\Sigma \times \{\pm\}$ of $\Sigma \times [\infty, +\infty]$.

Example 10.5 (Fiber group). Let $\hat{M} = \mathbb{H}^3/\hat{\Gamma}$ be a complete hyperbolic manifold of finite volume, and assume that $\hat{M}$ has a structure of a $\Sigma$-bundle over $S^1$. Then the fiber group, $\Gamma$, the subgroup of $\hat{\Gamma}$ obtained as the image of the fundamental group of a fiber surface $\Sigma$, is an infinite normal subgroup. This implies that $\Lambda(\Gamma) = \Lambda(\hat{\Gamma}) = \partial \mathbb{H}^3$ (see [Ths1979, Corollary 8.1.3]). Thus the inverse image of a fiber $\Sigma$ in the universal cover $\mathbb{H}^3$ of $\hat{M}$ is a topological plane whose closure contains the whole ideal boundary $\partial \mathbb{H}^3$. It is very difficult to imagine such a plane, and in this sense, the fiber group $\Gamma$ is quite different from a quasifuchsian group, though they are all isomorphic to $\pi_1(\Sigma)$.

10.2. Deformation space

We continue to denote by $\Sigma$ a closed orientable surface of genus $g$ with $b$ punctures, which has a negative Euler characteristic. By a marked hyperbolic structure on $\Sigma$, we mean a pair $(S, f)$ of a finite area complete hyperbolic surface $S = \mathbb{H}^2/\Gamma$ and an orientation-preserving homeomorphism $f : \Sigma \to S$. Note that the composition of $f_* : \pi_1(\Sigma) \to \pi_1(S)$ and the holonomy representation $\pi_1(S) \to \Gamma < \text{Isom}^+ \mathbb{H}^2$ determines a type-preserving discrete faithful representation $\rho : \pi_1(\Sigma) \to \text{Isom}^+ \mathbb{H}^2$. Two marked hyperbolic structure $(S_1, f_1)$ and $(S_2, f_2)$ on $\Sigma$ are equivalent if there is an orientation-preserving isometry $h : S_1 \to S_2$ such that $h \circ f_1$ is homotopic to $f_2$. This is equivalent to the condition that the corresponding representations $\rho_1$ and $\rho_2$ are equal up to conjugation by an element of $\text{Isom}^+ \mathbb{H}^2$. Let $H(\Sigma)$ be the set of all marked hyperbolic structure on $\Sigma$ up to equivalence.

In order to introduce a natural topology on $H(\Sigma)$, consider the spaces

$$\text{Hom}_{tp}(\pi_1(\Sigma), \text{Isom}^+ \mathbb{H}^2) := \{\rho : \pi_1(\Sigma) \to \text{Isom}^+ \mathbb{H}^2 \mid \rho \text{ is type-preserving}\}$$

$$\mathcal{R}_{tp}(\Sigma) := \text{Hom}_{tp}(\pi_1(\Sigma), \text{Isom}^+ \mathbb{H}^2)/\text{Isom}^+ \mathbb{H}^2$$

By choosing a finite generating set of $\pi_1(\Sigma)$ of cardinality $k$, $\text{Hom}_{tp}(\pi_1(\Sigma), \text{Isom}^+ \mathbb{H}^2)$ is identified with a subset of the product (topological) space $(\text{Isom}^+ \mathbb{H}^2)^k$, and the subspace topology it inherits is independent of the choice of a finite generating set. The group $\text{Isom}^+ \mathbb{H}^2$ acts by conjugation on the space $\text{Hom}_{tp}(\pi_1(\Sigma), \text{Isom}^+ \mathbb{H}^2)$, and $\mathcal{R}_{tp}(\Sigma)$ is defined to be the quotient space. The set $H(\Sigma)$ is identified with a subset of $\mathcal{R}_{tp}(\Sigma)$, and we denote the resulting topological space by $AH(\Sigma)$.

The space $AH(\Sigma)$ is nothing other than the Teichmüller space $\text{Teich}(\Sigma)$ of $\Sigma$. The Fenchel-Nielsen coordinate gives a homeomorphism from $AH(\Sigma) = \text{Teich}(\Sigma)$ onto the Euclidian space $\mathbb{R}^{6g-6+b}$ (see [IT1992], [Ths1979, Theorem 5.3,5]). It should be noted that $\text{Teich}(\Sigma)$ can be also identified with the space of marked Riemann surface structures on $\Sigma$. 

63
Now we consider hyperbolic structures on the oriented 3-manifold $\Sigma \times \mathbb{R}$. By a marked hyperbolic structure on $\Sigma \times \mathbb{R}$, we mean a pair $(N, f)$ where $N = \mathbb{H}^3/\Gamma$ is an oriented complete hyperbolic 3-manifold and $f : \Sigma \times \mathbb{R} \to N$ is an orientation-preserving homeomorphism which satisfies the following conditions.

- Let $\rho : \pi_1(\Sigma) \to \Gamma < \text{Isom}^+ \mathbb{H}^3$ be the homomorphism obtained as the composition of the homomorphism $(f \circ j)_* : \pi_1(\Sigma) \to \pi_1(N)$, where $j : \Sigma \to \Sigma \times 0 \to \Sigma \times \mathbb{R}$ is the inclusion map, and the holonomy representation $\pi_1(N) \to \text{Isom}^+(\mathbb{H}^3)$ of the hyperbolic manifold $N$. Then we require that $\rho$ is type-preserving. (In other words, we require that the homeomorphism $f$ maps (ends of $\Sigma$) $\times \mathbb{R}$ into the main cusp of $N$ carrying the parabolic elements $\rho$(peripheral elements).)

Thus we restrict our attention to the hyperbolic structures on the pared manifold $(\Sigma_{g,b} \times I, \partial \Sigma_{g,b} \times I, \partial \Sigma_{g,b} \times I) = \Sigma_{g,b} \times \mathbb{R}$ (see [Ths1986b, Section 7]) for the terminology).

Two marked hyperbolic structures $(N_1, f_1)$ and $(N_2, f_2)$ on $\Sigma \times \mathbb{R}$ are regarded as equivalent if there is an orientation-preserving isometry $h : N_1 \to N_2$ such that $h \circ f_1$ is homotopic to $f_2$. This condition is equivalent to the condition that the corresponding representations $\rho_1$ and $\rho_2$ are equal up to conjugation by an element of $\text{Isom}^+ \mathbb{H}^3$. Thus the set $H(\Sigma \times \mathbb{R})$ of all marked hyperbolic structures on $\Sigma \times \mathbb{R}$ up to equivalence is identified with the subset of the space $\mathbb{R}_{tp}(\Sigma \times \mathbb{R}) := \{ \rho : \pi_1(\Sigma) \to \text{Isom}^+ \mathbb{H}^3 \mid \rho \text{ is type-preserving} \}/\text{Isom}^+ \mathbb{H}^3$ consisting of (the images of) discrete faithful representations. The set $H(\Sigma \times \mathbb{R})$ with the subspace topology is denoted by $AH(\Sigma \times \mathbb{R})$. This topology is called the algebraic topology of $H(\Sigma \times \mathbb{R})$. It is well-known that $\mathbb{R}_{tp}(\Sigma \times \mathbb{R})$ is Hausdorff, and $AH(\Sigma \times \mathbb{R})$ is a closed subset of $\mathbb{R}_{tp}(\Sigma \times \mathbb{R})$ (cf. [Mrd2007a, Section 4]).

Let $QF(\Sigma \times \mathbb{R})$ be the subspace of $AH(\Sigma \times \mathbb{R})$ consisting of the quasifuchsian representations. For each quasifuchsian representation $\rho : \pi_1(\Sigma) \to \text{PSL}(2, \mathbb{C})$ with $\Gamma = \rho(\pi_1(\Sigma))$, the Kleinian manifold $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma \cong \Sigma \times [-\infty, \infty]$ is bounded by two marked Riemann surfaces $S_\pm = \Omega_\pm(\Gamma)/\Gamma$, where $S_\pm$ correspond to $\Sigma \times \{ \pm \infty \} \subset \Sigma \times [-\infty, \infty]$. The pair $(S_-, S_+)$ is regarded as a point in the product $\text{Teich}(\Sigma) \times \text{Teich}(\Sigma)$, where $\Sigma$ is the surface $\Sigma$ with reverse orientation. This determines a map $\nu : QF(\Sigma \times \mathbb{R}) \to \text{Teich}(\Sigma) \times \text{Teich}(\Sigma)$.

Bers’ simultaneous uniformization theorem says that $\nu$ is a homeomorphism (see [IT1992]).

The positive solution to Thurston’s Density Conjecture by Brock, Canary and Minsky [BCM2012], obtained as a consequence of deep results by a number of researchers in deformation theory of Kleinian groups, says that $AH(\Sigma \times \mathbb{R})$ is equal to the closure of its open subset $QF(\Sigma \times \mathbb{R})$:

$$AH(\Sigma \times \mathbb{R}) = \overline{QF(\Sigma \times \mathbb{R})}$$
Thus any discrete faithful type-preserving $\text{PSL}(2, \mathbb{C})$-representation of $\pi_1(\Sigma)$ is a limit of quasifuchsian representations. In particular, a fiber Kleinian group of a hyperbolic surface bundle over $S^1$ is obtained as the limit of quasifuchsian groups. Historically, the existence of the fiber Kleinian group (and so the existence of a complete hyperbolic structure on surface bundles) was first proved in the case where $\Sigma$ is a once-punctured torus by Jørgensen [Jr2003]: the simplest case of the figure-eight knot complement was also proved by Riley [Ri1975]. Thurston got inspiration from these works, and proved the hyperbolization theorem for surface bundles in [Ths1998*] (cf. Otal [Ot2001]) via his double limit theorem [Ths1998*, Theorem 4.1].

Cannon and Thurston [CT2007] found the following surprising fact. Let $\rho_0 : \pi_1(\Sigma) \to \text{PSL}(2, \mathbb{C})$ be a Fuchsian representation, and let $\rho : \pi_1(\Sigma) \to \text{PSL}(2, \mathbb{C})$ be the type-preserving discrete faithful representation whose image $\Gamma$ gives the fiber group of a hyperbolic $\Sigma$-bundle over $S^1$. Recall that $\Lambda(\Gamma_0) = \partial \mathbb{H}^2$ and $\Lambda(\Gamma) = \partial \mathbb{H}^3$ (see Example 10.5), and $\pi_1(\Sigma)$ acts on these sets via $\rho_0$ and $\rho$, respectively.

**Theorem 10.6 (Cannon-Thurston map).** There is a $(\rho_0, \rho)$-equivariant surjective continuous map

$$\kappa : \partial \mathbb{H}^2 = \Lambda(\Gamma_0) \to \Lambda(\Gamma) = \partial \mathbb{H}^3.$$  

The map $\kappa$ is called the Cannon-Thurston map. This theorem was first proved by Cannon and Thurston [CT2007] for the closed surface case, and then proved by Bowditch [Bw2007] for the general case. Work of many authors has extended the results in various ways (see the review [Mj*]). For the simplest case where $\Sigma$ is the once-punctured torus, the computer program $\text{OPTi}$ developed by Wada [Wd*] visualizes deformations of the limit sets of quasifuchsian punctured torus groups (see [ASWY2007] for background). We can also see a lot of breathtaking pictures related to the Cannon-Thurston maps (mainly for the once-punctured torus) in the book Indra’s Pearls [MSW2002].

10.3. Nielsen-Thurston classification of surface homeomorphisms and geometrization of surface bundles

We quickly recall the Nielsen-Thurston classification of surface homeomorphisms (see [Ths1988, FLP1991, PM2012]). Let $\text{MCG}(\Sigma)$ be the mapping class group of $\Sigma$ (the closed orientable surface of genus $g$ with $b$ punctures such that $\chi(\Sigma) < 0$), the group of the orientation-preserving homeomorphisms of $\Sigma$ modulo isotopy. We do not distinguish between a homeomorphism of $\Sigma$ and the element (mapping class) of $\text{MCG}(\Sigma)$ represented by it, as long as there is no fear of confusion. Then Nielsen-Thurston theory says that for any $\varphi \in \text{MCG}(\Sigma)$, one of the following holds.

1. $\varphi$ is periodic, namely $\varphi$ has finite order in $\text{MCG}(\Sigma)$. In this case, $\varphi$ is represented by a (periodic) isometry with respect to some finite-volume complete hyperbolic structure on $\Sigma$. 

65
(2) $\varphi$ is reducible, i.e., there is a nonempty family of mutually disjoint essential simple loops whose union is preserved by (a representative of) $\varphi$.

(3) $\varphi$ is pseudo-Anosov. This means that $\Sigma$ has a “half-translation structure” such that the homeomorphism $\varphi$ is “realized by” a diagonal matrix \[ \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \]
with $\alpha > 1$.

The precise meaning of the last condition is as follows. A half-translation structure on $\Sigma$ is a singular Euclidean metric on $\Sigma$, with a finite number of conical singularities of cone angle $k\pi$ ($k \geq 3$), and total cone angle $k'\pi$ ($k' \geq 1$) around each puncture. The surface $\Sigma$ with cone points removed admits an isometric atlas over $\mathbb{R}^2$ whose transition maps are of the form $(x, y) \mapsto (\pm (x, y) + (a, b))$ for some $(a, b) \in \mathbb{R}^2$.

Then $\varphi$ is pseudo-Anosov if there is a half-translation structure on $\Sigma$, such that the homeomorphism $\varphi$ has a local expression $(x, y) \mapsto (\alpha x, \alpha^{-1} y)$ with respect to isometric atlas of the half-translation structure. The constant $\alpha$ is called the expansion factor of the map $\varphi$.

This condition is described as follows in Thurston’s original paper [Ths1988, Theorem 4]: there is a constant real number $\alpha > 1$ and a pair of transverse measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ such that $\varphi(\mathcal{F}^s) = \alpha^{-1}\mathcal{F}^s$ and $\varphi(\mathcal{F}^u) = \alpha\mathcal{F}^u$. Here a measured foliation on $\Sigma$ is a singular foliation endowed with a measure in the transverse direction, where only finitely many singularities of “$k$-pronged saddle” ($k = 1$ or $k \geq 3$) are allowed. The notation $\mathcal{F}_1 = \alpha\mathcal{F}_2$ means that $\mathcal{F}_1$ and $\mathcal{F}_2$ agree as foliations, but the transverse measure of $\mathcal{F}_1$ is $\alpha$ times that of $\mathcal{F}_2$. With respect to the half-translation structure of $\Sigma$ in the above, the measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ are the vertical and horizontal foliations, $\lambda^+$ and $\lambda^-$, equipped with the transverse measures $|dx|$ and $|dy|$ respectively. (Note that every straight line segment in $\Sigma$ belongs to a unique (singular) foliation by parallel straight lines, and so the vertical and horizontal foliations make sense.) Since $\varphi$ is locally expressed by $(x, y) \mapsto (\alpha x, \alpha^{-1} y)$, it preserves the vertical and horizontal measured foliations up to the factors $\alpha^{-1}$ and $\alpha$, respectively.

By considering the “projective classes” of measured foliations, Thurston constructed the projective measured foliation space $\text{PMF}(\Sigma)$ and proved that it forms the boundary of a natural compactification of the Teichmüller space $\text{Teich}(\Sigma)$.

\[ \text{Teich}(\Sigma) = \text{Teich}(\Sigma) \sqcup \text{PMF}(\Sigma) \cong \text{int} B^{6g-6+2b} \sqcup \partial B^{6g-6+2b} \cong B^{6g-6+2b} \]

It is natural in the following sense. The action of $\text{MCG}(\Sigma)$ on $\text{Teich}(\Sigma)$ defined by the rule

$\varphi(S, f) := (S, f \circ \varphi^{-1})$ for $(S, f) \in \text{Teich}(\Sigma)$

extends to the action on the compactification, so that its restriction to the boundary $\text{PML}(\Sigma)$ is the natural action given by

\[ \int_\gamma \varphi_*(\mathcal{F}) = \int_{\varphi^{-1}(\gamma)} \mathcal{F}. \]
Here $\gamma$ is an arc transverse to the foliation $\varphi(F)$, and $\int_{\gamma} \varphi_*(F)$ is the measure of $\gamma$ with respect to the measured foliation $\varphi_*(F)$. It should be noted that the set $S$, of all essential simple loops in $\Sigma$ up to isotopy, is identified with a dense subset of $PMF(\Sigma)$ and that the above action is an extension of the natural action of $\text{MCG}(\Sigma)$ on $S$.

By using this natural compactification of Teichmüller space, Thurston established the classification of surface homeomorphisms, as follows. For a given $\varphi \in \text{MCG}(\Sigma)$, its action on $\text{Teich}(\Sigma) \cong B^{6g-6+2b}$ has a fixed point, by Brower’s fixed point theorem. If there is a fixed point in $\text{Teich}(\Sigma)$, then $\varphi$ is periodic. Suppose there is no fixed points in $\text{Teich}(\Sigma)$ and so all fixed points lie in $PMF(\Sigma)$. If the underlying foliation of some fixed point contains a closed leaf, then $\varphi$ is reducible. Thurston managed to prove that $\varphi$ is pseudo-Anosov in the remaining case.

Now, let $M_{\varphi} := \Sigma \times \mathbb{R}/(x, t) \sim (\varphi(x), t+1)$ be the $\Sigma$-bundle over $S^1$ with monodromy $\varphi$. Then it is easy to observe that if $\varphi$ is periodic then $M_{\varphi}$ is a Seifert fibered space, and that if $\varphi$ is reducible then $M_{\varphi}$ admits a nontrivial torus decomposition. For the remaining case when $\varphi$ is pseudo-Anosov, the following theorem was proved by Thurston, as a special case of the geometrization Theorem 3.4.

**Theorem 10.7.** The surface bundle $M_{\varphi}$ is hyperbolic if and only if $\varphi$ is pseudo-Anosov.

As noted in Subsection 10.2, the corresponding fiber group $\rho \in \text{AH}(\Sigma \times I)$ is a limit of quasi-fuchsian groups. Actually, for any $(S_-, S_+) \in \text{Teich}(\Sigma) \times \text{Teich}(\Sigma)$, $\rho$ is obtained as a limit of a subsequence of the sequence of quasi-fuchsian groups $\{\nu^{-1}(\varphi^{-k}(S_-), \varphi^k(S_+))\}_{k \geq 0}$ (see McMullen [Mc1996, Theorem 3.8]).

10.4. Cannon-Thurston maps and veering triangulations

We now describe the combinatorial structure of the Cannon-Thurston map associated with the $\Sigma$-bundle $M_{\varphi}$ with pseudo-Anosov monodromy $\varphi$. Let $\rho_0 : \pi_1(\Sigma) \rightarrow \text{PSL}(2, \mathbb{C})$ be a Fuchsian representation with image $\Gamma_0$, and let $\rho : \pi_1(\Sigma) \rightarrow \text{PSL}(2, \mathbb{C})$ be the discrete faithful representation whose image $\Gamma$ gives the fiber group of $M_{\varphi}$.

Let $j$ be the inclusion map from $\Sigma = \mathbb{H}^2/\Gamma_0$ to the infinite cyclic cover $M_{\varphi} = \mathbb{H}^3/\Gamma$ of $M_{\varphi}$, and consider its lift $\tilde{j} : \mathbb{H}^2 \rightarrow \mathbb{H}^3$ to the universal cover. Then the Cannon Thurston map $\kappa : \partial \mathbb{H}^2 \rightarrow \partial \mathbb{H}^3$ is the boundary map of the extension of $\tilde{j}$ to a map from $\mathbb{H}^2 \cup \partial \mathbb{H}^2$ to $\mathbb{H}^3 \cup \partial \mathbb{H}^3$.

In order to describe the combinatorics of the Cannon-Thurston map $\kappa$, let $\check{\lambda}^{\pm}$ be the singular foliations of $\mathbb{H}^2$ obtained as the lifts of the vertical/horizontal foliations $\lambda^{\pm}$, invariant by $\varphi$. Then the endpoints of each leaf of $\check{\lambda}^{\pm}$ are mapped by $\kappa$ into the same point, and this turns out to generate the combinatorics of $\kappa$. To be precise, let $\sim^{\pm}$ be the equivalence relation on $\partial \mathbb{H}^2$ which identifies the endpoints of each leaf of $\check{\lambda}^{\pm}$ by allowing for leaves that pass through singularities. Let $\sim$ be the equivalence
relation on $\partial \mathbb{H}^2$ generated by $\sim^+$ and $\sim^-$. Here the relations $\sim^+$ and $\sim^-$ are “almost independent” in the sense that if $x \sim y$ then $x \sim^+ y$ or $x \sim^- y$ or else there is a parabolic fixed point $p$ of $\Gamma_0$ such that either $(x \sim^+ p \sim^- y)$ or $(x \sim^- p \sim^+ y)$. Moreover distinct parabolic fixed points of $\Gamma_0$ cannot be equivalent under $\sim$. It was proved by Bowditch [Bw2007 Section 9] (cf. [CT2007 Section 5]) that

$$\kappa(x) = \kappa(y) \text{ if and only if } x \sim y.$$

In the remainder of this subsection, we assume that the singularities of the invariant foliations $\lambda^\pm$ occur only at punctures of the fiber. (This condition is satisfied if $\Sigma$ is a once-punctured torus.) Then it follows that for a point $q \in \partial \mathbb{H}^3$, the inverse image $\kappa^{-1}(q)$ consists of 1, 2 or countably infinitely many points. The last case happens if and only if $q$ is a parabolic fixed point of $\Gamma$, and in this case $\partial \mathbb{H}^2 - \kappa^{-1}(q)$ consists of countably infinitely many open intervals. Cannon and Dicks [CD2006] studied the way these intervals are mapped onto the complex plane $\mathbb{C} \cong \partial \mathbb{H}^3 - \{q\}$, and constructed a certain fractal tessellation of $\mathbb{C}$ in the case where $\Sigma$ is a once-punctured torus. Dicks and Sakuma [DS2010] then observed that there is an intimate relation between the fractal tessellation and the cusp triangulation (lifted to the universal cover $\tilde{\Sigma}$) induced by the canonical triangulation of the hyperbolic once-punctured torus bundles (see Figure 11).

On the other hand, Agol [Ag2011] introduced veering triangulations, which are (topological) ideal triangulations of cusped hyperbolic 3-manifolds with a very special combinatorial structure. He proved that every hyperbolic surface bundle, for which the singularities of the invariant foliations $\lambda^\pm$ occur only at punctures of the fiber, admits a veering triangulation, which is canonical in the sense that it is uniquely determined by the fiber structure. (More strongly, it is determined by Thurston’s fiber face to which the fibration belongs [MNT2017].)

In the beautiful paper [Gu2016], Guéritaud revealed an intimate relation between the veering triangulation and the fractal tessellation arising from the Cannon-Thurston map for every such hyperbolic surface bundle $M_\varphi$. To this end, he has given a natural construction of the veering triangulation in terms of the invariant foliations. The construction works in the universal cover $\tilde{\Sigma}$, endowed with the half-translation structure associated with the pseudo-Anosov monodromy. He considers maximal rectangles in $\tilde{\Sigma}$ whose sides are vertical and horizontal in $\tilde{\Sigma}$ and whose interiors are disjoint from the singularities. Such maximal rectangles have one singularity in each side; connecting these 4 singularities produces the ideal tetrahedra of the veering triangulation. This construction enabled Guéritaud to describe the relation between the veering triangulation and the fractal tessellation associated with the Cannon-Thurston map.

Roughly speaking, Guéritaud’s construction of the veering triangulation is an analogue of the Delaunay triangulation relative to the singular set, with respect to the $\ell^\infty$-metric arising from the half-translation structure. On the other hand, the canonical decomposition of a cusped hyperbolic manifold is an analogue of the
Delaunay triangulation relative to cusps, with respect to the hyperbolic metric. For hyperbolic once-punctured torus bundles, these two decompositions are equal. However, these two decompositions are quite different in general. In fact, it was shown by Hodgson, Issa, Ahmad and Segerman \cite{HIS2016} that there exist veering triangulations which are not geometric, in the sense that they are not isotopic to hyperbolic ideal triangulations. Moreover, it was recently proved by Futer, Taylor and Worden \cite{FTW*} that generically veering triangulations are not geometric. In spite of this defect from the view point of hyperbolic geometry, nice applications of veering triangulations to the study of curve complexes were given by Minsky and Taylor \cite{MnT2017}.
11. Representations of 3-manifold groups

In Sections 7 and 10, we treated deformations of hyperbolic structures. In Section 7, we considered complete hyperbolic manifolds of finite volume and studied deformations into incomplete hyperbolic structures, whereas in Section 10, we considered complete hyperbolic manifolds of infinite volume and studied deformations keeping the completeness. In both sections, deformations are described in terms of deformations of holonomy representations.

One purpose of this section is to present the definition of $SL(2, \mathbb{C})$ character varieties, which forms a common base ground for both treatments in Sections 7 and 10, and then to give a description of the hyperbolic Dehn filling theorem independent of ideal triangulations, following Boileau-Heusener-Porti [BP2001, Appendix B]. For another treatment, see Hodgson-Kerckhoff [HdK1998, p.49, Remark].

Another purpose of this section is to describe applications of the character varieties to knot theory and 3-manifold theory. We have already observed in Subsection 2.7 that study of representations of knot groups to finite groups gives us a powerful tool in knot theory. The character variety, which is essentially the space of representations of a knot group or a 3-manifold group into the Lie group $SL(2, \mathbb{C})$ up to conjugation by elements of $SL(2, \mathbb{C})$, leads to new versatile tools in knot theory and 3-manifold theory. We give a quick review to the Culler-Shalen theory [ClS1983, ClS1987, CGLS1987] and the A-polynomials due to Cooper, Culler, Gillet, Long and Shalen [CCGLS1994]. For further information, see the survey Shalen [Sh2002].

11.1. Character variety

Let $M$ be a compact connected manifold, and let $R(M) = \text{Hom}(\pi_1(M), SL(2, \mathbb{C}))$ be the space of all representations of $\pi_1(M)$ into $SL(2, \mathbb{C})$. This set has the structure of a complex affine algebraic set, because it is identified with a subspace of $(SL(2, \mathbb{C}))^k \subset \mathbb{C}^{4k}$, where $k$ is the cardinality of a generating set of $\pi_1(M)$, defined by a system of polynomial equations. For a representation $\rho \in R(M)$, the function $\chi_\rho : \pi_1(M) \to \mathbb{C}$ defined by $\chi_\rho(\gamma) = \text{tr}(\rho(\gamma))$ is called the character of $\rho$. (We don’t distinguish between a representation and the element of $R(M)$.) The set $X(M)$ of all characters also has the structure of an affine algebraic set, and it is called the character variety of $M$. This can be seen as follows. For each $\gamma \in \pi_1(M)$, consider the function $I_\gamma : X(M) \to \mathbb{C}$, defined by $I_\gamma(\rho) = \chi(\gamma)$. Then there are finitely many elements $\gamma_1, \ldots, \gamma_d$ for which $I_{\gamma_1} \times \cdots \times I_{\gamma_d} : X(M) \to \mathbb{C}^d$ is an embedding, and its image forms an affine algebraic set [ClS1983, Corollary 1.4.5].

The natural projection from the space $R(M)/SL(2, \mathbb{C})$ of all conjugacy classes of representations onto $X(M)$ fails to be injective only at the conjugacy classes of reducible representations (see Sh2002, Proposition 1.1.1)). In this sense, $X(M)$ is regarded as the quotient $R(M)//SL(2, \mathbb{C})$ in the category of affine algebraic sets.
If $M$ is a hyperbolic 3-manifold, i.e., if int $M$ admits a complete hyperbolic structure, then the holonomy representation $\rho : \pi_1(M) \to \text{PSL}(2, \mathbb{C})$ lifts to an $\text{SL}(2, \mathbb{C})$-representation (see [Cl1986]). In particular, the space, $\mathcal{R}_{\text{tp}}(\Sigma \times \mathbb{R})$, of the conjugacy classes of type-preserving $\text{PSL}(2, \mathbb{C})$-representations of $\pi_1(\Sigma \times \mathbb{R})$ (see Subsection 10.2), is covered by a subspace of $X(\Sigma \times \mathbb{R})$.

11.2. Hyperbolic Dehn filling theorem and character variety

Consider the setting in Subsection 7.1, namely $M$ is a connected compact orientable 3-manifold with $\partial M = \bigsqcup_{j=1}^m T_j$ a non-empty union of tori, such that int $M$ admits a complete hyperbolic structure. Let $\{\mu_j, \lambda_j\}$ is a pair of oriented slopes in the boundary torus $T_j$, which forms a generator system of $H_1(T_j; \mathbb{Z})$. Let $\rho_0$ be a lift of the holonomy representation of the complete hyperbolic structure of int $M$, and let $\chi_0$ be its character. Consider the map $I_\mu = (I_{\mu_1}, \ldots, I_{\mu_m}) : X(M) \to \mathbb{C}^m$. Then the following theorem holds (see [BP2001, Theorem B.1.2]).

**Theorem 11.1.** The map $I_\mu : X(M) \to \mathbb{C}^m$ is locally bianalytic at $\chi_0$.

By using this theorem, we can associate each character in some neighborhood of $\chi_0$ with generalized Dehn filling coefficients. To describe this, recall that the complex translation length $L_A$ of an element $A \in \text{SL}(2, \mathbb{C})$ is defined as an element of $\mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$ up to multiplication by $\pm 1$, by the following formula (see Subsection 5.1).

$$\text{tr } A = 2 \cosh \frac{L_A}{2}$$

In order to have a well-defined complex translation length as an element in $\mathbb{C}$, we consider the $(\mathbb{Z}/2\mathbb{Z})^m$-branched covering map $\psi : \hat{U} \to W$ from a neighborhood $\hat{U} \subset \mathbb{C}^m$ of the origin onto a neighborhood $W \subset X(M)$ of $\chi_0$ such that

$$I_{\mu_j}(\psi(u)) = \epsilon_j \cosh \frac{u_j}{2} \quad \text{for every } u = (u_1, \ldots, u_m) \in \hat{U}$$

where $\epsilon_j \in \{\pm\}$ is chosen so that $I_{\mu_j}(\psi(0)) = \chi_0(\mu_j) = \text{tr}(\rho_0(\mu_j)) = \epsilon_j 2$. Note that Theorem 11.1 guarantees the existence of this covering.

One can define a generalized Dehn surgery coefficients map as in Subsection 7.1 as Theorem 7.4 holds in this setting, where a certain open neighborhood, $U \subset \hat{U}$, of the origin plays the role of the open neighborhood $U$ in the theorem (see [BP2001, Proposition B.1.9]). (In fact, by Remark 7.5, the space $U$ is bi-holomorphic to the space $U$ in Theorem 7.4 when $M$ admits an ideal triangulation.)

In the setting of Subsection 7.2, each parameter $u \in U$ corresponds to a parameter $z$ representing the shapes of ideal tetrahedra, and so it determines an (incomplete) hyperbolic structure on int $M$. In the current setting, we appeal to the fact that a small deformation of a hyperbolic structure is parametrized by deformation of the holonomy representation (see [Ths1981, Proposition 5.1] and [BP2001, Proposition 7.2]).
This is an outline of the proof the hyperbolic Dehn filling Theorem 7.1 without using an ideal triangulation of $M$, given by [BP2001 Appendix B].

Note that the above proof is not effective in the sense that it gives no information about the size or shape of hyperbolic Dehn surgery space $V \subset (\mathbb{R}^2 \cup \{\infty\})^m$. In [HdK2008] (cf. [HdK1998, HdK2005]), Hodgson and Kerckhoff developed a new theory of infinitesimal harmonic deformations for compact hyperbolic 3-manifolds with “tubular neighborhood”, and gave an effective proof of the hyperbolic Dehn filling theorem; they proved that all generalized Dehn filling coefficients outside a disc of “uniform” size yield hyperbolic structures.

11.3. The Culler-Shalen theory and the cyclic surgery theorem

We give a quick survey of the Culler-Shalen theory. See [Sh2002] for a detailed self-contained review. Let $M$ be a compact, connected, orientable, irreducible 3-manifold. Suppose $M$ contains an essential surface $F$. Then by considering the inverse image $\tilde{F}$ of $F$ in the universal covering $\tilde{M}$, we can construct a tree $T$, such that the vertices correspond to the components of $\tilde{M} - \tilde{F}$ and the edges correspond to the connected components of $\tilde{F}$, where the edge corresponding to a component of $\tilde{F}$ joins the two vertices corresponding to the components of $\tilde{M} - \tilde{F}$ abutting the component of $\tilde{F}$. The covering transformation group $\pi_1(M)$ acts on the tree $T$ simplicially, and this action is nontrivial (i.e., no vertex is stabilized by the whole group $\pi_1(M)$) and without inversion (i.e., if an element $\gamma \in \pi_1(M)$ leaves an edge invariant, then $\gamma$ fixes the edge pointwise). Conversely, it is known that if $\pi_1(M)$ acts simplicially on a tree nontrivially and without inversion, then $M$ contains an essential surface.

In [CIS1983], Culler and Shalen established a method for constructing such actions of $\pi_1(M)$ on trees, by using the character variety $X(M)$. The theory says that if $X(M)$ contains an algebraic curve $\mathcal{C}$, then each ideal point of the curve gives rise to such an action of $\pi_1(M)$ and hence an essential surface in $M$. In this theory, Tits-Bass-Serre theory [Srr1977] on the structure of subgroups of $\text{SL}(2,F)$, where $F$ is a field with a discrete valuation, plays a key role. Various applications of this theory are given, including (a) a simpler proof and generalization of the Smith conjecture [CIS1983] and (b) a proof of the Neuwirth conjecture which says that every nontrivial knot group is a free product of two proper subgroups amalgamated along a free product [CIS1987].

In [CGLS1987], they introduced a norm $|| \cdot || : H_1(\partial M; \mathbb{R}) \to \mathbb{R}$ for a compact orientable hyperbolic 3-manifold $M$ with a single torus boundary. A key fact behind this definition is the following: Let $X_0$ be the irreducible component of the character variety $X(M)$ containing the character $\chi_0$ of the (lifted) holonomy representation of the complete hyperbolic structure on $\text{int} M$. Then $X_0$ has complex dimension 1 (see Theorem 11.1). Let $\hat{X}_0$ be the projective completion of the affine algebraic curve...
in which the ideal points are smooth. Then for each \( \gamma \in H_1(\partial M) = \pi_1(\partial M) \), the restriction to \( X_0 \) of the function \( I_\gamma : X(M) \to \mathbb{C} \), defined by \( I_\gamma(\chi) = \chi(\gamma) \), extends to a rational function, \( \hat{I}_\gamma : \hat{X}_0 \to \mathbb{C} \cup \{ \infty \} \), where the ideal points of \( \hat{X}_0 \) (i.e., the points in \( \hat{X}_0 - X_0 \)) are the poles of this rational function. The norm \( \| \cdot \| : H_1(\partial M; \mathbb{R}) \to \mathbb{R} \) is defined to be the norm obtained as the continuous extension of the function \( H_1(\partial M; \mathbb{Z}) \to \mathbb{Z} \) which associates \( \gamma \) with the degree of \( \hat{I}_\gamma \). The norm plays a crucial role in the proof of the cyclic surgery theorem below, established by Culler, Gordon, Luecke, and Shalen \[CGLS1987\]. It was proved by combining (i) arguments using the norm and (ii) graph-theoretic analysis of the intersection of two incompressible, planar surfaces in knot exteriors.

**Theorem 11.2** (Cyclic surgery theorem). Let \( M \) be a compact, connected, orientable, irreducible 3-manifold such that \( \partial M \) is a single torus, and suppose that \( M \) is not a Seifert fibered space. Let \( \alpha \) and \( \beta \) be two non-isotopic essential simple loops on \( \partial M \), such that \( \pi_1(M(\alpha)) \) and \( \pi_1(M(\beta)) \) are cyclic. Then the geometric intersection number of \( \alpha \) and \( \beta \) is equal to 1.

In \[ByZh1987\] and \[ByZh2001\], Boyer and Zhang generalized the above idea and proved an analogue of the above theorem for finite surgeries. See \[By2002\], for further information.

The Culler-Shalen theory was extended by Morgan and Shalen \[MrgS1984, MrgS1988a, MrgS1988b\] to the theory of \( \mathbb{R} \)-trees. Here an \( \mathbb{R} \)-tree is a metric space in which any two points are joined by a unique topological arc. The theory plays a key role in Otal’s proof \[Ot2001\] of the double limit theorem. See the reviews \[Bs2002, Mrg1992\] for further information.

### 11.4. \( A \)-polynomials

We give a short review of the \( A \)-polynomial of a knot \( K \), which is introduced by Cooper, Culler, Gillet, Long, and Shalen \[CCGLS1994\] by using the character variety \( X(M) \) of the knot exterior \( M \) of \( K \). The idea is to consider the restriction map \( r : X(M) \to X(\partial M) \) induced by the inclusion of \( \pi_1(\partial M) \) into \( \pi_1(M) \). Then even though \( X(M) \) is complicated, its image \( r(X(M)) \) can be very simple. Note that \( \pi_1(\partial M) \) is the free abelian group freely generated by the longitude \( \lambda \) and the meridian \( \mu \). Thus, for any irreducible 1-dimensional component \( \mathcal{C} \) in the image \( r(X(M)) \subset X(\partial M) \), there is a holomorphic map \( f : \mathcal{C} \to \mathbb{C} \times \mathbb{C} \) which assigns the pair of the ‘eigen values’ of the images of \( \lambda \) and \( \mu \) by the corresponding representations. Then the closure of the image \( f(\mathcal{C}) \) becomes an algebraic curve in \( \mathbb{C}^2 \). Such a curve is equal to the zero set of a single defining polynomial, \( F_\mathcal{C}(x, y) \). Now consider the product \( \prod_\mathcal{C} F_\mathcal{C}(x, y) \) of the defining polynomials \( F_\mathcal{C}(x, y) \) where \( \mathcal{C} \) runs over the 1-dimensional irreducible components of \( r(X(M)) \). Then the \textit{A-polynomial} of \( K \) is
defined as

\[ A_K(x, y) = \frac{1}{x - 1} \prod_{c} F_c(x, y) \]

The reason of dividing out by the factor \( x - 1 \) is that \( H_1(M) \) is the free abelian group generated by \( \mu \) and so we always have a component corresponding to abelian representations, which gives rise to the factor \( x - 1 \). By normalizing \( A_K(x, y) \) so that it is an integral polynomial, it is defined up to multiplication by \( \pm x^a y^b \).

It is obvious that \( A_O(x, y) = 1 \) for the trivial knot \( O \), and it is proved that the converse also holds (see Boyer-Zhang \cite{ByZh2005} and Dunfield-Garoufalidis \cite{DG2004}). The most important properties of the \( A \)-polynomials come from the fact that they encode information about the boundary slopes of the knot, via the Newton polygon of \( A_K(x, y) \). Recall that a boundary slope of a knot \( K \) is a slope (isotopy class of an essential simple loop) in the boundary torus of the knot exterior \( M \), such that there is an essential surface in \( M \) whose boundary consists of loops representing the slope. The Newton polygon of the polynomial \( A_K(x, y) \) is the convex hull of the finite set:

\[ \{(i, j) \in \mathbb{Z}^2 \mid \text{the coefficient of } x^i y^j \text{ in } A_K(x, y) \text{ is non-zero}\} . \]

The following striking theorem is proved by \cite[Theorem 3.4]{CCGLS1994}.

**Theorem 11.3.** Slopes of the edges of the Newton polygon of \( A_K(x, y) \) are boundary slopes of the knot \( K \).

### 12. Knot genus and Thurston norm

By generalizing the genus of a knot, Thurston \cite{Ths1986a} defined a (semi-) norm on \( H^1(M; \mathbb{R}) \cong H_2(M, \partial M; \mathbb{R}) \) for a compact orientable 3-manifold \( M \). It is called the **Thurston (semi-) norm** of \( M \). By the work of Gabai \cite{Gb1983a}, the Thurston norm is identical to the Gromov norm on \( H_2(M, \partial M; \mathbb{Z}) \). The Thurston norm can be used to study the set of fibers of \( M \) over the circle, and the work of Fried and McMullen enabled a unified treatment of the fibers of \( M \). After recalling these works, we explain two Thurstonian connections between the topology and geometry of 3-manifolds, related to Thurston norms. Namely, we survey (i) the relation of the Thurston norm with the hyperbolic torsion polynomial due to Dunfield-Fried-Jackson \cite{DFJ2012} and Agol-Dunfield \cite{AD2012}, and (ii) that with harmonic \( L^2 \)-norm with respect to the hyperbolic metric due to Brock-Dunfield \cite{BD2017}.

#### 12.1. Thurston norm

Let \( M \) be a compact oriented 3-manifolds with \( \partial M \) a possibly empty union of tori. For a compact possibly disconnected surface \( \Sigma \), let \( \Sigma_0 \) be the surface consisting of the components of \( \Sigma \) which are neither homeomorphic to \( D^2 \) nor \( S^2 \), and define its
complexity by $\chi_-(\Sigma) := |\chi(\Sigma_0)|$. For an integral homology class $\alpha \in H_2(M, \partial M; \mathbb{Z})$, define its norm $||\alpha||_{\text{Th}}$ by

$$||\alpha||_{\text{Th}} = \min \{ \chi_-(\Sigma) \mid [\Sigma] = \alpha \}$$

**Theorem 12.1.** (1) $||\cdot||_{\text{Th}}$ extends to a continuous map $||\cdot||_{\text{Th}} : H^1(M; \mathbb{R}) \cong H_2(M, \partial M; \mathbb{R}) \to \mathbb{R}_{\geq 0}$, and this gives a semi-norm on $H^1(M; \mathbb{R})$. Moreover, if any compact orientable surface properly embedded in $M$, representing a nontrivial homology class, has a negative Euler characteristic, then $||\cdot||_{\text{Th}}$ is a norm.

(2) Suppose $||\cdot||_{\text{Th}}$ is a norm, then the unit ball

$$B_M = \{ \alpha \in H^1(M; \mathbb{R}) \mid ||\alpha||_{\text{Th}} \leq 1 \}$$

is a finite sided polyhedron whose vertices are rational points.

(3) Suppose $||\cdot||_{\text{Th}}$ is a norm. Then there are codimension one faces $F_1, \ldots, F_k$, of $B_M$ satisfying the following conditions.

(i) Any integral cohomology class in the interior of the cone $\mathbb{R}_+ \cdot F_i$ is a fiber class.

(ii) Conversely, any fiber class is contained in the interior of some cone $\mathbb{R}_+ \cdot F_i$.

Here a class $\phi \in H^1(M; \mathbb{Z})$ is called a fibered class if it is an integral multiple of the cohomology class represented by a bundle projection $p : M \to S^1$. In the above theorem, each $F_i$ is called a fibered face.

The fiber structures contained in the interior of the cone on a fibered face can be given unified treatment, and various interesting results can be obtained. In particular, building on the results of Fried, McMullen [Mc2000] proved that each fibered face $F$ determines a 2-dimensional “lamination” $\mathcal{L}$ of $M$ transverse to every fiber surface $\Sigma$ with (Poincaré dual of $[\Sigma] \in \mathbb{R}_+ \cdot F$, where $\Sigma \cap \mathcal{L}$ is the stable lamination of the monodromy of the fibration. By using this result, he defined the Teichmüller polynomial $\theta_F \in \mathbb{Z}[H_1(M; \mathbb{Z})/\operatorname{Tor} H_1(M; \mathbb{Z})]$ and proved the following results [Mc2000].

- The Teichmüller polynomial is symmetric, i.e., if $\theta_F = \sum_g a_g g$ then $\theta_F = \sum_g a_g g^{-1}$ up to a unit in $\mathbb{Z}[H_1(M)/\operatorname{Tor} H_1(M)]$.
- For any integral cohomology class $\phi \in \mathbb{R}_+ \cdot F$, the expansion factor $k(\varphi)$ of the corresponding monodromy $\varphi$ is equal to the largest root of the one-variable polynomial obtained by evaluating $\theta_F$ by $\phi$.
- The function $\phi \mapsto 1/\log k(\varphi)$ extends to a real-analytic function on $\mathbb{R}_+ \cdot F$ which is strictly concave.
- The cone $\mathbb{R}_+ \cdot F$ is dual to a vertex of the Newton polygon $\subset H_1(M; \mathbb{R})$ of $\theta_F$.
- If the lamination $\mathcal{L}$ is transversely orientable, then the (multivariable) Alexander polynomial of $M$ divides the Teichmüller polynomial $\theta_F$.

At the end of this subsection, we recall an important result of Gabai [Gb1983a], obtained as a corollary of his construction of codimension 1 transversely oriented.
foliations without Reeb components which contain a given Thurston norm minimizing surface as a closed leaf. To explain this, we consider another (semi-) norm \( ||\cdot||_{Th}^{s} \) on \( H^1(M; \mathbb{R}) \cong H_2(M, \partial M; \mathbb{R}) \) for a compact irreducible orientable 3-manifold \( M \), defined by using immersed surfaces instead of embedded surfaces. Namely, for an integral homology class \( \alpha \in H_2(M, \partial M; \mathbb{Z}) \), define \( ||\alpha||_{Th}^{s} \) to be the minimum of \( \chi - (\Sigma) \) of a compact oriented surface \( \Sigma \) for which there is a proper immersion \( f : (\Sigma, \partial \Sigma) \to (M, \partial M) \) such that \( f_*([\Sigma]) = \alpha \), namely, 
\[
||\alpha||_{Th}^{s} = \min \{ \chi - (\Sigma) \mid \exists f : (\Sigma, \partial \Sigma) \to (M, \partial M) \text{ such that } f_*([\Sigma]) = \alpha \}.
\] The new norm \( ||\cdot||_{Th}^{s} \) is defined as a continuous extension of the above norm on the integral homology.

In addition to this, as in Subsection 8.4, the Gromov norm \( ||\cdot||_{Gr} \) is defined by 
\[
||\alpha||_{Gr} := \inf \{ ||z|| \mid z \text{ is a singular cycle representing the homology class } \alpha \},
\] where, for a (real) singular chain \( z = \sum_j a_j \sigma_j \), its norm \( ||z|| \) is defined as the sum \( \sum_j |a_j| \) of the absolute values of its coefficients. The following theorem was proved by Gabai [Gb1983a].

**Theorem 12.2.** Let \( M \) be a connected compact irreducible orientable 3-manifold with possibly empty toral boundary. Then the three norms on \( H^1(M; \mathbb{R}) \cong H_2(M, \partial M; \mathbb{R}) \) coincide, namely, 
\[
||\cdot||_{Th} = ||\cdot||_{Th}^{s} = ||\cdot||_{Gr}.
\]

In particular, for a knot \( K \) in \( S^3 \), its genus \( g(K) \) is equal to the immersed genus of \( K \), which is defined as the minimum of the genus \( g(\Sigma) \) of a compact connected oriented surface \( \Sigma \) such that there is an immersion \( f : \Sigma \to S^3 \), with \( f^{-1}(K) = \partial \Sigma \), whose singular set is disjoint from \( K \). This is a generalization of Dehn’s lemma for higher genus, and gives a partial affirmative answer to a question raised by Papakyriakopoulos [Papa1957], who established Dehn’s lemma.

### 12.2. Evaluation of Thurston norms in terms of Twisted Alexander polynomials

The **twisted Alexander polynomials**, defined by Lin [Lin2001] for classical knots and by Wada [Wd1994] in the general setting, give a powerful tool for studying the Thurston norm. Such a ‘polynomial’ \( \Delta(M, \phi, \rho) \) depends on a class \( \phi \in H^1(M; \mathbb{Z}) \) and a linear representation \( \rho : \pi_1(M) \to \text{GL}(V) \), where \( V \) is a finite dimensional vector space over a field \( F \). Then \( \Delta(M, \phi, \rho) \) is defined as an element of the quotient field \( F[t^{\pm 1}] \) of the group ring \( F[\pi_1(M)] \), and its “degree” \( \text{deg} \Delta(M, \phi, \rho) \) gives a lower bound on the Thurston norm \( ||\phi||_{Th} \) of \( \phi \). Friedl and Viddusi proved a surprising result that given \( M \) and \( \phi \) one can always choose \( \rho \) so that the lower bound is attained (see [FV2011, Kt2013] for surveys).

When \( K \) is a hyperbolic knot in \( S^3 \), it is natural to consider the twisted Alexander polynomial for the representation \( \rho : G(K) \to \text{SL}(2, \mathbb{C}) \) which projects to the
holonomy representation of the complete hyperbolic structure of $S^3 - K$. Though there are precisely two such representations up to conjugacy, there is unique such one for which $\text{tr} \rho(\mu) = +2$, where $\mu$ a meridian of $K$. (For the other lift $\rho'$, we have $\text{tr} \rho'(\mu) = -2$.) Thus we can consider the twisted Alexander polynomial $\Delta(E(K), \phi, \rho)$, where $\phi \in H^1(E(K); \mathbb{Z}) \cong \mathbb{Z}$ is the generator. The invariant is called the hyperbolic torsion polynomial of $K$ and is denoted by $J_K(t)$ (see [DFJ2012]). The artificial choice of the lift $\rho$ is irrelevant, because if $\rho$ is replaced with the other lift $\rho'$, then the corresponding polynomial $J'_K(t)$ is equal to $J_K(-t)$. As a special case of the general results on the twisted Alexander polynomial, the following hold for every hyperbolic knot $K$ in $S^3$.

1. $4g(K) - 2 \geq \deg J_K(t)$.
2. If $K$ is fibered, then $J_K(t)$ is monic.

These may be regarded as analogies of Theorem 2.7(2) and (3) on the classical Alexander polynomial. Dunfield, Friedl and Jackson [DFJ2012] made extensive computer experiments, and confirmed that for all hyperbolic knots with at most 15 crossings, the estimate (1) is sharp and that (2) detect all non-fibered knots. In particular, the hyperbolic torsion polynomial detects that the genera of the Kinoshita-Terasaka knot and the Conway knot are 3 and 5, respectively. (The genera of arborescent links, including these two knots, had been determined by Gabai [Gb1986] through topological study of complementary sutured manifolds.) Thus the hyperbolic torsion polynomials can distinguish knots which are mutants of each other. In [AD*], Agol and Dunfield studied the conjecture posed by [DFJ2012], that the estimate (1) is sharp for every hyperbolic knot, and they verified the conjecture for libroid hyperbolic knots in $S^3$. The libroid knots form a broad class of knots, which is closed under Murasugi sum, and in particular all arborescent are libroid knots.

12.3. Harmonic norm and Thurston norm

Let $M$ be a closed orientable hyperbolic 3-manifold. Then in addition to the topologically defined Thurston norm $\| \cdot \|_{\text{Th}}$, there is yet another canonically defined geometric norm on $H^1(M; \mathbb{R})$. By the rigidity theorem, $M$ admits a unique hyperbolic metric, and by applying Hodge theory to this Riemannian metric, we can identify $H^1(M; \mathbb{R})$ with the space of harmonic 1-forms. Thus the harmonic norm $\| \cdot \|_{L^2}$ determines another norm on $H^1(M; \mathbb{R}) \cong H_2(M; \mathbb{R})$. Here the harmonic norm is the one associated with the usual inner product for 1-forms:

$$ \langle \alpha, \beta \rangle = \int_M \alpha \wedge \star \beta, $$

where $\star$ is the Hodge $\star$-operator. Since it comes from a positive-definite inner product, the unit ball of $\| \cdot \|_{L^2}$ is a smooth ellipsoid. Brock and Dunfield [BD2017] proved the following relation between the topological norm and the geometric norm.
Theorem 12.3. For all closed orientable hyperbolic 3-manifold $M$ one has

$$\frac{\pi}{\sqrt{\text{vol}(M)}} \cdot \|\cdot\|_{\text{Th}} \leq \|\cdot\|_{L^2} \leq \frac{10\pi}{\sqrt{\text{inj}(M)}} \cdot \|\cdot\|_{\text{Th}}.$$ 

In the above theorem, $\text{inj}(M)$ denotes the injectivity radius of $M$, i.e., the half the length of the shortest closed geodesic. Moreover, they also showed that the above estimates are in some sense sharp, by giving families of examples.

These results were obtained as refinements of a result of Bergeron, Sengûn and Venkatesh [BSV2016], which in turn is preceded by the work by Kronheimer and Mrowka [KM1997] that characterize the Thurston norm as the infimum (over all possible Riemannian metrics) of certain scaled harmonic metrics.

13. Finite index subgroups of knot groups and 3-manifold groups

As explained in Subsection 2.7, finite branched/unbranched coverings of knots are a powerful tool for distinguishing knots. This fact reflects the richness of finite index subgroups of knot groups. In this section, we survey the following topics which illustrate this richness: (i) universal groups, which produce all closed orientable 3-manifolds, (ii) positive solution of the virtual fibering conjecture, (iii) Grothendieck rigidity of 3-manifold groups, and (iv) mysterious relation between the Gromov norm and the homology growth of finite coverings.

13.1. Universal knots/links and universal groups

In an unpublished preprint [Ths1982b], W. Thurston presented a very complicated six component link in $S^3$, and proved the surprising fact that every closed orientable 3-manifold can be expressed as a branched cover of the 3-sphere branched over this link. He called links with this property universal links. He asked if a universal knot exists, and if even the figure-eight knot was universal. This question was answered affirmatively by Hilden, Lozano and Montesinos in [HLM1985], where they proved that every hyperbolic 2-bridge knot and link is universal.

Figure 12. The Borromean orbifold $B(p, q, r)$ is a universal orbifold if $p \geq 3$ and both $q$ and $r$ are even integers $\geq 4$ by [HLM2010].
Moreover, it was later proved by Hilden, Lozano, Montesinos and Whitten in [HLMW1987] that every closed orientable 3-manifold $M$ is a covering of $S^3$ branched over the Borromean rings and having branching indices 1, 2 and 4. This implies that the hyperbolic orbifold $\mathcal{U} = \mathbb{H}^3 / \Gamma$ with underlying space $S^3$ and with singular set the Borromean ring where all components have cone angle $\pi/2$, is a universal orbifold in the following sense: for any closed orientable 3-manifold $M$, there is a finite orbifold covering $\mathcal{O} \to \mathcal{U}$ with underlying space $|\mathcal{O}|$ homeomorphic to $M$. In other words, the orbifold fundamental group $\mathcal{U} = \pi_1^{orb}(\mathcal{U})$ is a universal group, i.e., for any closed orientable 3-manifold $M$, there is a finite index subgroup $\Gamma$ of $\mathcal{U}$ such that $|\mathbb{H}^3 / \Gamma| \cong M$. It is surprising that all closed orientable 3-manifolds are constructed from a single group $\mathcal{U}$ and its finite index subgroups. Moreover, universal groups seem to be ubiquitous (cf. [HLM2010]).

13.2. Virtual fibering conjecture

The positive solution of Thurston’s virtual fibering conjecture by Agol [AGM2013] and the geometric solution of Waldhausen’s conjecture for hyperbolic manifolds due to Kahn-Markovich [KM2012], which play a key role in the proof of the virtual fibering conjecture, also reflect the richness of subgroups of Kleinian groups. Please see Bestvina [Bs2014] for a survey of this important topic.

Here, I only recall Walsh’s simple construction [Wls2005] of a nontrivial example of virtual fibering by using knot theory. Let $K$ be a spherical Montesinos knot/link which is not fibered, e.g., the 5_2 knot, the 2-bridge knot of slope 2/7. Then the double branched covering $M_2(K)$ of $S^3$ branched over $K$ is a spherical manifold and so its universal covering $\tilde{M}_2(K)$ is the 3-sphere. The inverse image, $\tilde{K}$, of $K$ in the universal cover is a great circle link in $S^3$, because it is the singular set of the isometric group action of the $\pi$-orbifold group of $K$ (cf. Subsection 4.3). Pick a component $O$ of $\tilde{K}$, and observe that the remaining components form a closed braid around $O$, because $\tilde{K}$ consists of great circles. This shows that the covering $\tilde{M}_2(K) - \tilde{K}$ of $S^3 - K$ is a punctured disk bundle over the circle, though $S^3 - K$ itself does not admit a fiber structure over the circle.

13.3. Profinite completions of knot groups and 3-manifold groups

As explained in Subsection 2.7, representations of knot groups onto finite groups serve a powerful tool for distinguishing knots. Thus it is natural to ask the following question (cf. [IFr¹]).

**Question 13.1.** To what degree does the set of finite quotients of knot groups distinguish knots? More generally, what properties of 3-manifolds are determined by the set of finite quotients of their fundamental groups?

The geometrization theorem and Hempel’s argument [Hm1987] show that every 3-manifold group is residually finite, namely, for any nontrivial element $g \in \pi_1(M)$,
where $M$ is a compact connected orientable 3-manifold, there is a finite quotient of $\pi_1(M)$ in which $g$ remain nontrivial. This implies that the above question can be formulated in terms of the profinite completion of the fundamental group.

Recall that the profinite completion of a group $\Gamma$, is the inverse limit of the inverse system of finite quotients of $\Gamma$: we denote it by $\hat{\Gamma}$. (The profinite completion is actually defined to be a topological group endowed with the profinite topology. But any isomorphism between two profinite groups is a homeomorphism, so we do not care about the topological structure in this subsection.) The natural map $\Gamma \to \hat{\Gamma}$ is injective if and only if $\Gamma$ is residually finite. Let $\mathcal{C}(\Gamma)$ denote the family of finite quotients of $\Gamma$. Then the following holds (see [RjZ2000] p.88-89, [LR2011] Theorem 2.2).

**Theorem 13.2.** For two finitely generated residually finite groups $\Gamma_1$ and $\Gamma_2$, the equality $\mathcal{C}(\Gamma_1) = \mathcal{C}(\Gamma_2)$ holds if and only if $\hat{\Gamma}_1 \cong \hat{\Gamma}_2$, i.e., the profinite completions are isomorphic.

Thus Question [13.1] is reformulated by using the profinite completion; in particular, the following question arises as a special case.

**Question 13.3.** Let $M_1$ and $M_2$ be connected compact orientable 3-manifolds, for which the profinite completions $\pi_1(M_1)$ and $\pi_1(M_2)$ are isomorphic. Are $\pi_1(M_1)$ and $\pi_1(M_2)$ isomorphic?

The answer to the above question is no. In fact, Funar [Fn2013] and Hempel [Hm*] showed that the profinite completion of the fundamental group cannot always distinguish certain pairs of torus bundles nor certain pairs of Seifert fibered spaces. It is still an open question though whether the profinite completion can distinguish any two hyperbolic 3-manifolds. Boileau and Friedl [BFr*] considered a more relaxed Question [13.1] and obtained various interesting results concerning fiberedness and the Thurston norm, and have shown that the figure-eight knot and torus knots are distinguished from other knots by the profinite completions of their knot groups.

On the other hand, the following problem had been posed by Grothendieck [Grot1970].

**Problem 13.4** (Grothendieck). Let $\varphi : \Gamma_1 \to \Gamma_2$ be a homomorphism of finitely presented residually finite groups for which the extension $\hat{\varphi} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ is an isomorphism. Is $\varphi$ an isomorphism?

If $\hat{\varphi} : \hat{\Gamma}_1 \to \hat{\Gamma}_2$ is an isomorphism, then the composition $\Gamma_1 \to \hat{\Gamma}_1 \to \hat{\Gamma}_2$ is an injection and so $\varphi : \Gamma_1 \to \Gamma_2$ must be an injection. Therefore Grothendieck’s problem reduces to the case where $\Gamma_1$ is a subgroup of $\Gamma_2$. Long and Reid [LR2011] introduced the following terminology. For a group $G$ and its subgroup $H < G$, the pair $(G, H)$ is a Grothendieck pair if the inclusion $j : H \to G$ provides a negative answer to Grothendieck’s problem. If for all finitely generated subgroups $H < G$, $(G, H)$ is never a Grothendieck pair then $G$ is Grothendieck rigid.
The following theorem was proved by Long and Reid [LR2011] for the case where $M$ is closed and by Boileau and Friedl [BF] for general case.

**Theorem 13.5.** Let $M$ be a connected, orientable, irreducible, compact 3-manifold. Then $\pi_1(M)$ is Grothendieck rigid.

In the examples of Funar [Fn2013] and Hempel [Hm], the isomorphisms between the profinite completions are not induced by a homomorphism between the 3-manifold groups.

### 13.4. Homology growth

Investigation of the first homology groups of finite (branched or unbranched) coverings has a long history (cf. Subsection 2.7). For the homology of finite abelian coverings of links, it was proved that they are essentially determined by the Alexander invariants of links (see [Fx1954, Myr1982, Skm1995]). In Gor1972, Gordon studied the asymptotic behavior the homology of finite cyclic branched coverings of a knot, and gave a necessary and sufficient condition for $H_1(M_n(K); \mathbb{Z})$ to be periodic with respect to $n$, in terms of the Alexander invariants. This in particular implies that if the Alexander polynomial $\Delta_K(t)$ has a root which is not a primitive root of 1 then $H_1(M_n(K); \mathbb{Z})$ cannot be periodic. In fact, he showed that the order $|H_1(M_n(K); \mathbb{Z})|$ is unbounded under the same assumption, and then asked if the order $|H_1(M_n(K); \mathbb{Z})|$ tends to $\infty$. Riley [Ri1990] and González-Acuña and Short [GS1991], independently, proved that $|H_1(M_n(K); \mathbb{Z})|$ grows exponentially. To be precise, the following was proved:

$$\lim_{n_j \to \infty} \frac{1}{n_j} \log |H_1(M_{n_j}(K); \mathbb{Z})| = \log \mathcal{M}(\Delta_K),$$

where $\{n_j\}$ runs over the natural numbers such that $|H_1(M_{n_j}(K); \mathbb{Z})|$ is finite, and $\mathcal{M}(\Delta_K)$ is the **Mahler measure** of the Alexander polynomial $\Delta_K(t)$. The Mahler measure of a polynomial $f(t)$ is defined by

$$\mathcal{M}(f) = \exp \left( \int_{S^1} \log |f(s)| ds \right)$$

$$= \exp \left( \int_0^1 \log |f(e^{2\pi \sqrt{-1} t})| dt \right)$$

$$= |c| \prod_{f(\omega) = 0} \max(|\omega|, 1) \quad (c \text{ is the leading coefficient of } f(t)).$$

Here, the last equality is a consequence of Jensen’s formula [Ah1966, p. 205].

This result was extended by Silver and Williams [SlW1992] to links in $S^3$, by using the result of Schmidt [Skm1995] on the entropy of a certain dynamical system. Let $L$ be an $m$-component oriented link in $S^3$, with the complement $X = S^3 - L$. 81
For a subgroup \( \Lambda \subset H_1(X; \mathbb{Z}) \cong \mathbb{Z}^m \) of rank \( m \), let \( X^\text{br}_\Lambda \) the corresponding branched covering of \( X \). Set

\[
\langle \Lambda \rangle = \min \{|x| \mid x \in \Lambda - \{0\}\},
\]

where \( |x| = \sqrt{\sum |x_i|^2} \) for \( x = (x_1, \ldots, x_m) \in \mathbb{Z}^m \). Let \( \Delta_L \in \mathbb{Z}[t_1, \ldots, t_m] \) be the (0-th) Alexander polynomial of \( L \).

**Theorem 13.6** (Silver-Williams). Under the above setting, suppose that \( \Delta_L \neq 0 \). Then the following holds:

\[
\limsup_{\langle \Lambda \rangle \to \infty} \ln \frac{|\text{Tor}_Z H_1(X^\text{br}_\Lambda; \mathbb{Z})|}{|\mathbb{Z}^m/\Lambda|} = \log M(\Delta_L).
\]

Here \( M(\Delta_L) \) is the Mahler measure of \( \Delta_L \), defined by

\[
M(\Delta_L) = \exp \left( \int_{T^m} \log |\Delta_L(s)| ds \right),
\]

where \( T^m := (S^1)^m \subset \mathbb{C}^m \) is the multiplicative subgroup in \( \mathbb{C}^m \), and \( ds \) indicates integration with respect to normalized Haar Measure on \( T^m \).

In [Le2014], Thang Le solved a conjecture of Schmidt [Scm1995], and by using the solution, he extended the above result to links in oriented integral homology 3-spheres, and to include the case where the 0-th Alexander polynomial vanishes, by replacing the 0-th Alexander polynomial with the first non-vanishing Alexander polynomial. He also proved that the same formula holds for unbranched abelian coverings.

In [Le2018], Le also studied the asymptotic behavior of the homology of non-abelian coverings, by using the result on \( L^2 \)-torsion by Lück [Lü2002, Theorems 4.3 and 4.9]. Let \( X \) be an irreducible compact orientable 3-manifold with infinite fundamental group with (possibly empty) toral boundary. For a subgroup \( \Gamma \) of \( \pi_1(X) \) of finite index, let \( X_\Gamma \) be the corresponding finite covering of \( X \). A sequence \( \{\Gamma_k\} \) of subgroups of \( \pi_1(X) \) of finite index is said to be nested, if \( \Gamma_{k+1} \subset \Gamma_k \). It is said to be exhaustive if \( \bigcap_k \Gamma_k = \{1\} \).

**Theorem 13.7** (Le). Under the above setting, the following holds for any nested exhaustive sequence \( \{\Gamma_k\} \) of normal subgroups of \( \pi_1(X) \) of finite index:

\[
\limsup_{k \to \infty} \frac{\ln |\text{Tor} H_1(X_{\Gamma_k}; \mathbb{Z})|}{[\pi_1(X) : \Gamma_k]} \leq \frac{V_{\text{tet}} \|X\|}{6\pi},
\]

where \( \|X\| \) is the Gromov norm of \( X \).

For a knot \( K \) in \( S^3 \) with exterior \( X \) and a finite index subgroup \( \Gamma < G(K) \), let \( X^\text{br}_\Gamma \) be the corresponding branched covering of \( S^3 \) branched over \( K \).
Theorem 13.8 (Le). Under the above setting, the following holds for any nested exhaustive sequence \( \{ \Gamma_k \} \) of normal subgroups of \( G(K) = \pi_1(X) \) of finite index.

\[
\limsup_{k \to \infty} \frac{\ln |\text{Tor} H_1(X^\text{br}, \mathbb{Z})|}{[\pi_1(X) : \Gamma_k]} \leq \frac{V_{\text{tet}}||X||}{6\pi},
\]

where \( ||X|| \) is the Gromov norm of \( X \).

For the sake of simplicity, we stated Le’s theorem only for regular coverings. However, the actual statement of his theorem is much more general and it does not restrict to regular coverings. For a precise statement, see the original paper [Le2018]. Moreover, he conjectures that the identity holds in both theorems.

The homology of finite (branched/unbranched) coverings is a common and well-known invariant in knot theory. It is impressive that the asymptotic behavior of this familiar invariant reflects the deep geometric structure of the knot.

References

[Ad1987] C. Adams, The noncompact hyperbolic 3-manifold of minimum volume, Proc. Amer. Math. Soc. 100 (1987), 601–606.
[Ad1994] C. Adams, The Knot Book, W.H. Freeman & Co./Amer. Math. Soc., 1994.
[Ad1998] C. Adams, Knot theory and its applications: expository articles on current research, Chaos, Solitons & Fractals, 9, Issues 4-5, pp. 531–824.
[Ad2005] C. Adams, Hyperbolic knots, Handbook of knot theory, Elsevier B. V., Amsterdam, 2005, pp. 1–18.
[Ag2000] I. Agol, Bounds on exceptional Dehn filling, Geom. Topol. 4 (2000), 431–449.
[Ag2010a] I. Agol, Bounds on exceptional Dehn filling II, Geom. Topol. 14 (2010), 1921–1940.
[Ag2010b] I. Agol, The minimal volume orientable hyperbolic 2-cusped 3-manifolds, Proc. Amer. Math. Soc. 138 (2010), 3723–3732.
[Ag111] I. Agol, Ideal triangulations of pseudo-Anosov mapping tori, Topology and geometry in dimension three, 1–17, Contemp. Math., 560, Amer. Math. Soc., Providence, RI, 2011.
[AD*] I. Agol and N. Dunfield, Certifying the Thurston norm via \( \text{SL}(2, \mathbb{C}) \)-twisted homology, to appear in the Thurston memorial conference proceedings, Princeton Univ. Press, 23 pages. arXiv:1501.02136.
[AGM2013] I. Agol, The virtual Haken conjecture. With an appendix by Agol, Daniel Groves, and Jason Manning, Doc. Math. 18 (2013), 1045–1087.
[Ah1966] L.V. Ahlfors, Complex Analysis, 2nd Edition, McGraw-Hill, New York, 1966.
[AR1992] I. R. Aitchison and J. H. Rubinstein, Combinatorial cubings, cusps, and the dodecahedral knots, from: “Topology ’90 (Columbus, OH, 1990)”, (B Apanasov, W D Neumann, A W Reid, L Siebenmann, editors), Ohio State Univ. Math. Res. Inst. Publ. 1, de Gruyter, Berlin (1992) 17–26
[Ak2001] H. Akiyoshi, Finiteness of polyhedral decompositions of cusped hyperbolic manifolds obtained by the Epstein-Penner’s method, Proc. Amer. Math. Soc. 129 (2001), 2431–2439.
[AS2003] H. Akiyoshi and M. Sakuma, Comparing two convex hull constructions for cusped hyperbolic manifolds, Kleinian groups and hyperbolic 3-manifolds (Warwick, 2001), 209–246, London Math. Soc. Lecture Note Ser., 299. Cambridge Univ. Press, Cambridge, 2003.
[ASWY2007] H. Akiyoshi, M. Sakuma, M. Wada and Y. Yamashita, Punctured torus groups and 2-bride knot groups (I), Lecture Notes in Mathematics 1909, Springer, Berlin, 2007.
[ADO1992] Y. Akutsu, T. Deguchi and T. Ohtsuki, *Invariants of colored links*, J. Knot Theory Ramifications, **1** (1992), 161–184.

[Al1920] J. W. Alexander, *Note on Riemann spaces*, Bull. Amer. Math. Soc., **26** (1920), 370–372.

[An1999] J. Anderson, *Hyperbolic geometry*, Springer Undergraduate Mathematics Series. Springer-Verlag London, Ltd., London, 1999. x+230

[BnP1992] R. Benedetti and C. Petronio, *Lectures on hyperbolic geometry*, Universitext. Springer-Verlag, Berlin, 1992. xiv+330 pp.

[Bel2014] M. Belolipetsky, *Hyperbolic orbifolds of small volume*, Proceedings of the International Congress of MathematiciansSeoul 2014. Vol. II, 837–851, Kyung Moon Sa, Seoul, 2014.

[Bfr*] J. Berge, *Some knots with surgeries yielding lens spaces*, unpublished manuscript, [arXiv:1802.09722](https://arxiv.org/abs/1802.09722) [math.GT].

[BSV2016] N. Bergeron, M. H. Sengun and A. Venkatesh, *Torsion homology growth and cycle complexity of arithmetic manifolds*, Duke Math. J. **165** (2016), 1629–1693.

[BBMP2010] L. Bessières, G. Besson, M. Boileau, S. Maillot, J. Porti, *Geometrisation of 3-manifolds*, EMS Tracts in Mathematics, **13**. European Math. Soc. (EMS), Zürich, 2010. x+237 pp.

[BS2002] M. Bestvina, *R-trees in topology, geometry, and group theory*, Handbook of geometric topology, 55–91, North-Holland, Amsterdam, 2002.

[BS2014] M. Bestvina, *Geometric group theory and 3-manifolds hand in hand: the fulfillment of Thurston’s vision*, Bull. Amer. Math. Soc. **51** (2014), 53–70.

[BH1996] S. Bleiler and C. Hodgson, *Spherical space forms and Dehn filling*, Topology **35** (1996), 809–833

[BHW1999] S. Bleiler, C. Hodgson and J. R. Weeks, *Cosmetic surgery on knots*, Proceedings of the Kirbyfest (Berkeley, CA, 1998), 23–34, Geom. Topol. Monogr., **2**, Geom. Topol. Publ., Coventry, 1999.

[BBCW2015] M. Boileau, S. Boyer, R. Cebanu, G. S. Walsh, *Knot commensurability and the Berge conjecture*, Geom. Topol. **16** (2012), no. 2, 625–664.

[BBCW2015] M. Boileau, S. Boyer, R. Cebanu, G. S. Walsh, *Knot complements, hidden symmetries and reflection orbifolds*, Ann. Fac. Sci. Toulouse Math. (6) **24** (2015), 1179–1201.

[BFl1987] M. Boileau and E. Flapan, *Uniqueness of free actions on $S^3$ respecting a knot*, Canad. J. Math. **39** (1987), 969–982.

[BFl1995] M. Boileau and E. Flapan, *On $\pi$-hyperbolic knots which are determined by their 2-fold and 4-fold cyclic branched coverings*, Topology Appl. **61** (1995), 229–240.

[BFM2018] M. Boileau, C. Franchi, M. Mecchia, L. Paoluzzi and B. Zimmermann, *Finite group actions on 3-manifolds and cyclic branched coverings of knots*, J. Topol. **11** (2018), 283–308.

[BFr*] M. Boileau and S. Friedl, *The profinite completion of 3-manifold groups, fiberedness and the Thurston norm*, [arXiv:1505.07799](https://arxiv.org/abs/1505.07799) [math.GT].

[BFr*] M. Boileau and S. Friedl, *Grothendieck rigidity of 3-manifold groups*, [arXiv:1710.02746](https://arxiv.org/abs/1710.02746) [math.GT].

[BLP2005] M. Boileau, B. Leeb, and J. Porti, *Geometrization of 3-dimensional orbifolds*, Ann. of Math. **162** (2005), 195–290.

[BMP2003] M. Boileau, S. Maillot, and J. Porti, *Three-dimensional orbifolds and their geometric structures*, Panoramas et Synthèses [Panoramas and Syntheses], **15**, Société Mathématique de France, Paris, 2003. viii+167 pp.

[BP2001] M. Boileau and J. Porti, *Geometrization of 3-orbifolds of cyclic type*, Appendix A by Michael Heusener and Porti, Astérisque No. 272 (2001).

[BZm1987] M. Boileau and B. Zimmermann, *Symmetries of nonelliptic Montesinos links*, Math. Ann. **277**, (1987), 563-584.
[BZm1989] M. Boileau and B. Zimmermann, The $\pi$-orbifold group of a link, Math. Z. 200 (1989), 187–208.

[Bn2002] F. Bonahon, Geometric structures on 3-manifolds, Handbook of geometric topology, 93–164, North-Holland, Amsterdam, 2002.

[BnSb1987] F. Bonahon and L. Siebenmann, The characteristic toric splitting of irreducible compact 3-orbifolds, Math. Ann. 278 (1987), 441–479.

[BnSb*] F. Bonahon and L. Siebenmann, New Geometric Splittings of Classical Knots, and the Classification and Symmetries of Arborescent Knots, http://www-bcf.usc.edu/~fbonahon/Research/Publications.html

[Brl1981] A. Borel, Commensurability classes and volumes of hyperbolic 3-manifolds, Ann. Scuola Norm. Sup. Pisa 8 (1981), 1–33.

[Bw2007] B. H. Bowditch, The Cannon-Thurston map for punctured-surface groups, Math. Z. 255 (2007), 35–76.

[By2002] S. Boyer, Dehn surgery on knots, Handbook of geometric topology, 165–218, North-Holland, Amsterdam, 2002.

[ByZh1998] S. Boyer and X. Zhang, On Culler-Shalen seminorms and Dehn filling, Ann. of Math. 148 (1998), 737–801.

[ByZh2001] S. Boyer and X. Zhang, A proof of the finite filling conjecture, J. Differential Geom. 59 (2001), 87–176.

[ByZh2005] S. Boyer and X. Zhang, Every nontrivial knot in $S^3$ has nontrivial $A$-polynomial, Proc. Amer. Math. Soc. 133 (2005), 2813–2815.

[BCD2012] J. Brock, R. D. Canary, and Y. N. Minsky, The classification of Kleinian surface groups, II: The ending lamination conjecture. Ann. of Math. 176 (2012), 1–149.

[Bd2006] R. Budney, JSJ-decompositions of knot and link complements in $S^3$, Enseign. Math. 52 (2006), 319–359.

[BrZs1985] G. Burde and H. Zieschang, Knots, De Gruyter Studies in Mathematics, 5. Walter de Gruyter & Co., Berlin, 1985. xii+399 pp.

[BrZsH2014] G. Burde, H. Zieschang and M. Heusener, Knots. Third, fully revised and extended edition, De Gruyter Studies in Mathematics, 5. De Gruyter, Berlin, 2014

[BK2017] J. Brock and N. Dunfield, Norms on the cohomology of hyperbolic 3-manifolds, Invent. Math. 210 (2017), 531–558.

[BK1998] J. W. Cannon and E. Kalfagianni, Geometric estimates from spanning surfaces, Bull. Lond. Math. Soc. 49 (2017), 694–708.

[CR1998] J. W. Cannon and W. P. Thurston, Group invariant Peano curves, Geom. Topol. 2 (1998), 429–454.

[CEG1987] R. D. Canary, D. B. A. Epstein and P. Green, Notes on notes of Thurston, Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984), 3–92, London Math. Soc. Lecture Note Ser., 111, Cambridge Univ. Press, Cambridge, 1987.

[CD2006] J. W. Cannon and W. Dicks, On hyperbolic once-punctured-torus bundles II, Geom. Dedicata 126 (2006), 11–63. Errata and addenda: http://mat.uab.cat/~dicks/spiders.html

[CT2007] J. W. Cannon and W. P. Thurston, Group invariant Peano curves, Geom. Topol. 11 (2007), 1315–1355.

[CM2001] J. W. Cannon and W. P. Thurston, Group invariant Peano curves, Geom. Topol. 11 (2007), 1315–1355.

[CM2001] C. Cao and G. R. Meyerhoff, The orientable cusped hyperbolic 3-manifolds of minimum volume, Invent. Math. 146 (2001), 451–478.

[CZ2006] H.-D. Cao and X.-P. Zhu, A complete proof of the Poincaré and geometrization conjectures, application of the Hamilton-Perelman theory of the Ricci flow, Asian J. Math. 10 (2006), 165–492. Erratum, ibid, 663.
[ChS1974] S.-S. Chern and J. Simons, Characteristic forms and geometric invariants, Ann. of Math. (2) 99 (1974) 48–69.

[ChD2014] E. Chesebro and J. DeBlois, Algebraic invariants, mutation, and commensurability of link complements, Pacific J. Math. 267 (2014), 341–398.

[Cho2004] Y. E. Choi, Positively oriented ideal triangulations on hyperbolic three-manifolds, Topology 43 (2004), 1345–1371.

[Cn1970] J.H. Conway, An enumeration of knots and links, and some of their algebraic properties, in: 1970 Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967), 329–358, Pergamon Press, 1970.

[CCGLS1994] D. Cooper, M. Culler, H. Gillet, D. D. Long, and P. B. Shalen, Plane curves associated to character varieties of 3-manifolds, Invent. Math. 118 (1994), 47–84.

[CHK2000] D. Cooper, C. Hodgson and S. Kerckhoff, Three-dimensional orbifolds and cone-manifolds MSJ Memoirs 5, Math. Soc. of Japan, Tokyo, 2000. x+170 pp.

[CL1998] D. Cooper and D. D. Long, Representation theory and the A-polynomial of a knot, Knot theory and its applications. Chaos Solitons Fractals 9 (1998), 749–763.

[Cs2002] F. Costantino, On a proof of the JSJ theorem, Rend. Sem. Mat. Univ. Politec. Torino 60 (2002), 129–146 (2003).

[CT2008] F. Costantino and D. Thurston, 3-manifolds efficiently bound 4-manifolds, J. Topol. 1, 703–745 (2008).

[CGHW2000] D. Coulson, O. Goodman, C. Hodgson, W. Neumann, Computing arithmetic invariants of 3-manifolds, Experiment. Math. 9 (2000), 127–152.

[Crm2004] P. R. Cromwell, Knots and links, Cambridge Univ. Press, Cambridge, 2004. xviii+328 pp.

[Crow1955] R. Crowell, Genus of alternating link types, Ann. of Math. 69 (1959) 258–275.

[CrF1963] R. Crowell and R. Fox, Introduction to knot theory, Reprint of the 1963 original. Graduate Texts in Mathematics, No. 57. Springer-Verlag, New York-Heidelberg, 1977. x+182 pp.

[CDG*] M. Culler, N. Dunfield, and M. Goerner, SnapPy, Computer Software available at https://www.math.uic.edu/t3m/SnapPy/

[Cl1986] M. Culler, Lifting representations to covering groups, Adv. in Math. 59 (1986), 64–70.

[CGLS1987] M. Culler, C. McA. Gordon, J. Luecke, and P. B. Shalen, Dehn surgery on knots, Ann. of Math. 125 (1987), 237–300.

[CIS1983] M. Culler and P. B. Shalen, Varieties of group representations and splittings of 3-manifolds, Ann. of Math. 117 (1983), 109–146.

[CIS1987] M. Culler and P. B. Shalen, Bounded, separating, incompressible surfaces in knot manifolds, Invent. Math. 75 (1984), 537–545.

[DMM2012] A. Deruelle, K. Miyazaki, and K. Motegi, Networking Seifert surgeries on knots, Mem. Amer. Math. Soc. 217 (2012), no. 1021, viii+130 pp.

[DS2010] W. Dicks and M. Sakuma, On hyperbolic once-punctured-torus bundles III: comparing two tessellations of the complex plane, Topology Appl. 157 (2010) 1873–1899.

[DL2009] J. Dinkelbach and B. Leeb, Equivariant Ricci flow with surgery and applications to finite group actions on geometric 3-manifolds, Geom. Topol. 13 (2009), 1129–1173.

[DFJ2012] N. Dunfield, S. Friedl and N. Jackson, Twisted Alexander polynomials of hyperbolic knots, Exp. Math. 21 (2012) 329–352.

[DG2004] N. Dunfield and S. Garoufalidis, Non-triviality of the A-polynomial for knots in $S^3$, Algebr. Geom. Topol. 4 (2004), 1145–1153.

[EP1988] D. Epstein and R. Penner, Euclidean decompositions of noncompact hyperbolic manifolds, J. Differential Geom. 27 (1988), 67–80.
[Et2006] J. B. Etnyre, *Lectures on open book decompositions and contact structures*, Clay Math. Proc. 5 (the proceedings of the “Floer Homology, Gauge Theory, and Low Dimensional Topology Workshop”), 2006, 103–141.

[FM2012] B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Mathematical Series, 49. Princeton University Press, Princeton, NJ, 2012. xiv+472

[FLP1991] A. Fathi, F. Laudenbach and V. Poénaru, *Travaux de Thurston sur les surfaces*, 2nd edn, Astérisque 66–67, Société Mathématique de France, Paris, 1991.

[Fx1954] R. H. Fox, *Free differential calculus III*, Ann. of Math. 59 (1954), 195–210.

[Fx1961] R. H. Fox, *A quick trip through knot theory*, in: M. K. Fort (Ed.), “Topology of 3-Manifolds and Related Topics”, Prentice-Hall, NJ, 1961, pp. 120–167.

[Fx1962] R. H. Fox, *Some problems in knot theory*, 1962 Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961) pp. 168–176 Prentice-Hall, Englewood Cliffs, N.J.

[FxS1964] R. H. Fox and N. Smythe, *An ideal class invariant of knots*, Proc. Amer. Math. Soc. 15 (1964) 707–709.

[FP1930] F. Frankl, and L. Pontrjagin, *Ein Knotensatz mit Anwendung auf die Dimensionstheorie*, Math. Ann. 102 (1930), 785–789.

[FV2011] S. Friedl and S. Viddusi, *A survey of twisted Alexander polynomials*, The mathematics of knots, 45–94, Contrib. Math. Comput. Sci., 1, Springer, Heidelberg, 2011.

[FV2011] S. Friedl and S. Viddusi, *Twisted Alexander polynomials detect fibered 3-manifolds*, Ann. of Math. 173 (2011), 1587–1643.

[FV2009] S. Friedl and S. Viddusi, *The Thurston norm and twisted Alexander polynomials*, J. Reine Angew. Math. 707 (2015), 87–102.

[Fn2013] L. Funar, *Torus bundles not distinguished by TQFT invariants*, Geometry & Topology 17 (2013), 2289–2344.

[FG2009] D. Futer and F. Guéritaud, *Angled decompositions of arborescent link complements*, Proc. Lond. Math. Soc. 98 (2009), 325–364.

[FG2011] D. Futer and F. Guéritaud, *From angled triangulations to hyperbolic structures*, Interactions between hyperbolic geometry, quantum topology and number theory, Contemp. Math., vol. 541, Amer. Math. Soc., Providence, RI, 2011, pp. 159–182.

[FKP2008] D. Futer, E. Kalfagianni and J. S. Purcell, *Dehn filling, volume, and the Jones polynomial*, J. Differential Geom. 78 (2008), 429–464.

[FKP2019] D. Futer, E. Kalfagianni and J. S. Purcell, *A survey of hyperbolic knot theory*, Knots, low-dimensional topology and applications, 1–30, Springer Proc. Math. Stat., 284, Springer, Cham, 2019.

[FTW*] D. Futer, S. J. Taylor and W. Worden, *Random veering triangulations are not geometric*, arXiv:1808.05586 [math.GT].

[Gb1983a] D. Gabai, *Foliations and the topology of 3-manifolds* J. Differential Geom. 18 (1983), 445–503.

[Gb1983b] D. Gabai, *The Murasugi sum is a natural geometric operation*, Low-dimensional topology (San Francisco, Calif., 1981), 131–143, Contemp. Math., 20, Amer. Math. Soc., Providence, RI, 1983.

[Gb1984] D. Gabai, *Foliations and genera of links*, Topology 23 (1984), 381–394.

[Gb1986] D. Gabai, *Genera of the arborescent links*, Mem. Amer. Math. Soc. 59 (1986), no. 339, i–viii and 1–98.

[Gb1987] D. Gabai, *Foliations and the topology of 3-manifolds. III*, J. Differ. Geom. 26 (1987), 479–536.

[GMM2009] D. Gabai, R. Meyerhoff and P. Milley, *Minimum volume cusped hyperbolic three-manifolds*, J. Amer. Math. Soc. 22 (2009), 1157–1215.
[Jr2003] T. Jørgensen, On pairs of punctured tori, unfinished manuscript, available in Proceedings of the workshop “Kleinian groups and hyperbolic 3-manifolds” (edited by Y. Komori, V. Markovic and C. Series), London Math. Soc., Lect. Notes 299 (2003), 183–207.

[JM1990] T. Jørgensen and A. Marden, Algebraic and geometric convergence of Kleinian groups, Math. Scand. 66 (1990), 47–72.

[Je1980] W. Jaco, Lectures on three-manifold topology, CBMS Regional Conference Series in Mathematics, 43. Amer. Math. Soc., Providence, R.I., 1980. xii+251 pp.

[JS1979] W.H. Jaco and P.B. Shalen, Seifert fibered spaces in 3-manifolds, Mem. Amer. Math. Soc. 21 (1979), no. 220, viii+192 pp.

[Ju2015] A. Juhász, A survey of Heegaard Floer homology, New ideas in low dimensional topology, 237–296, Ser. Knots Everything, 56, World Sci. Publ., Hackensack, NJ, 2015.

[Kn1979] T. Kanenobu, The augmentation subgroup of a pretzel link, Math. Sem. Notes Kobe Univ. 7 (1979), 363–384.

[Kn1988] T. Kanenobu, Unions of knots as cross sections of 2-knots, Kobe J. Math. 4 (1988), 147–162.

[Kp2000] M. Kapovich, Hyperbolic manifolds and discrete groups: Notes on Thurston’s Hyperbolization, Progress in Mathematics 183, Birkhäuser, Basel, 2000.

[Ks1995] R. M. Kashaev, A link invariant from quantum dilogarithm Modern Phys. Lett. A 10 (1995), 1409–1418.

[Kf1987] L. H. Kauffman, On knots, Annals of Mathematics Studies, 115. Princeton University Press, Princeton, NJ, 1987. xvi+481

[Kw1979] A. Kawachi, The invertibility problem on amphicheiral excellent knots, Proc. Japan Acad. 55 (1979), 399–402.

[Kw1996] A. Kawachi, A survey of knot theory, Translated and revised from the 1990 Japanese original by the author. Birkhäuser Verlag, Basel, 1996.

[Kw2006] A. Kawachi, Topological imitations and Reni-Mecchia-Zimmermann’s conjecture, Kyungpook Math. J. 46 (2006), 1–9.

[KZ1986] D. A. Kazhdan and G. A. Margulis, A proof of Selberg’s conjecture, Math. USSR-Sbornik 4 (1968), 147–152.

[Ke2014] R. Kellerhals, Hyperbolic orbifolds of minimal volume, Comput. Methods Funct. Theory 14 (2014), 465–481.

[Kn1958] S. Kinoshita, On knots and periodic transformations, Osaka Math. J. 10 (1958), 43–52.

[KT1957] S. Kinoshita and H. Terasaka, On unions of knots, Osaka Math. J. 9 (1957) 131–153.

[Kr1978] R. Kirby, Problems in low dimensional manifold theory, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, pp. 273–312, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978.

[Kit2013] T. Kitano, Twenty years of twisted Alexander polynomials: refinement of the Alexander polynomial and its applications, (Japanese) Sūgaku 65 (2013), 360–384.

[Kn1929] H. Kneser, Geschlossen Flächen in dreidimensionalen Mannigfaltigkeiten, Jahresbericht der Deutschen Mathematiker Vereinigung, 38 (1929), 248–260.

[Kb1984] T. Kobayashi, Structures of the Haken manifolds with Heegaard splittings of genus two, Osaka J. Math. 21 (1984), 437–455.
[Kj1986] S. Kojima, *Determining knots by branched covers*, Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984), 193–207, London Math. Soc. Lecture Note Ser., 112, Cambridge Univ. Press, Cambridge, 1986.

[Kj1988] S. Kojima, *Isometry transformations of hyperbolic 3-manifolds*, Topology Appl. 29 (1988), 297–297.

[KM1997] P. B. Kronheimer and T. S. Mrowka, *Scalar curvature and the Thurston norm*, Math. Res. Lett. 4 (1997), 931–937.

[Lcb2000] M. Lackenby, *Word hyperbolic Dehn surgery*, Invent. Math. 140 (2000), 243–282.

[Lcb2004] M. Lackenby, *The volume of hyperbolic alternating link complements*, with an appendix by Ian Agol and Dylan Thurston, Proc. London Math. Soc. 88 (2004), 204–224.

[Lcb*] M. Lackenby, *Every knot has characterising slopes*, [arXiv:1707.00457 [math.GT]].

[LM2013] M. Lackenby and R. Meyerhoff, *The maximal number of exceptional Dehn surgeries*, Invent. Math. 191 (2013), 341–382.

[Le2014] T. Le, *Homology torsion growth and Mahler measure*, Comment. Math. Helv. 89 (2014), 719–757.

[Le2018] T. Le, *Growth of homology torsion in finite coverings and hyperbolic volume*, Annales de l’Institut Fourier, 68 (2018), 611–645.

[LS2013] D. Lee and M. Sakuma, *Epmorphisms from 2-bridge link groups onto Heckoid groups (1)*, Hiroshima Math. J. 43 (2013), 239–264.

[Lcr1997] W. B. R. Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, 175, Springer-Verlag, New York, 1997. x+201 pp.

[Lin2001] X. S. Lin, *Representations of knot groups and twisted Alexander polynomials*, Acta Math. Sin. (Engl. Ser.) 17 (2001), 361–380.

[Liv1993] C. Livingston, *Knot theory*, Carus Mathematical Monographs, 24, Mathematical Association of America, Washington, DC, 1993. xviii+240 pp.

[LR2011] D. D. Long and A. W. Reid, *Grothendieck’s problem for 3-manifold groups*, Groups Geom. Dyn. 5 (2011), 479–499.

[Lü2002] W. Lück, *L^2-invariants: theory and applications to geometry and KK-theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics, 44, Springer-Verlag, Berlin, 2002.

[Luo1997] F. Luo, *Actions of finite groups on knot complements*, Pacific J. Math. 154 (1992), no. 2, 317–329.

[LST2008] F. Luo, S. Schleimer and S. Tillman, *Geodesic ideal triangulations virtually exists*, Proc. Amer. Math. Soc. 136 (2008), 2625–2630.

[MR2000] C. Maclachlan and A. W. Reid, *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Mathematics, 219, Springer-Verlag, New York, 2003. xiv+463 pp.

[MrsMrt2012] T. H. Marshall and G. J. Martin, *Minimal co-volume hyperbolic lattices, II: Simple torsion in a Kleinian group*, Ann. of Math. (2) 176 (2012), 261–301.

[Marg1974] G.A. Margulis, *Discrete groups of isometries of manifolds of nonpositive curvature*, Proc. Int. Congress Math. 1974, Vancouver, Vol. 2, pp. 21–34.

[Mrd2007a] A. Marden, *Deformations of Kleinian groups*, Handbook of Teichmüller theory. Vol. I, 411–446, IRMA Lect. Math. Theor. Phys., 11, Eur. Math. Soc., Zürich, 2007.

[Mrd2007b] A. Marden, *Outer circles. An introduction to hyperbolic 3-manifolds*, Cambridge University Press, Cambridge, 2007. xviii+427 pp.

[Mrt2016] B. Martelli, *An Introduction to Geometric Topology*, CreateSpace Independent Publishing Platform, 2016.

[Mrt2015] G. Martin, *The geometry and arithmetic of Kleinian groups*, Handbook of group actions. Vol. I, 411–494, Adv. Lect. Math. (ALM), 31, Int. Press, Somerville, MA, 2015.
[Mrt2016] G. Martin, Siegel’s problem in three dimensions, Notices Amer. Math. Soc. 63 (2016), 1244–1247.

[MaT1998] K. Matsuzaki and M. Taniguchi, Hyperbolic manifolds and Kleinian groups, Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1998. x+253 pp.

[MyTrn1982] J. P. Mayberry and K. Murasugi, Torsion groups of abelian coverings of links, Trans. Amer. Math. Soc. 271 (1982), 143–173.

[Mc1996] C. T. McMullen, Renormalization and 3-manifolds which fiber over the circle, Annals of Mathematics Studies, 142. Princeton University Press, Princeton, NJ, 1996. x+253 pp.

[Mc2000] C. T. McMullen, Polynomial invariants for fibered 3-manifolds and Teichmüller geodesics for foliations, Ann. Sci. École Norm. Sup. 33 (2000), 519–560.

[Mc2011] C. T. McMullen, The evolution of geometric structures on 3-manifolds, Bull. Amer. Math. Soc. 48 (2011), 259–274.

[Mc2014] C. T. McMullen, The evolution of geometric structures on 3-manifolds, The Poincaré conjecture, 31–46, Clay Math. Proc., 19, Amer. Math. Soc., Providence, RI, 2014.

[MR2002] M. Mecchia and M. Reni, Hyperbolic 2-fold branched coverings of links and their quotients, Pacific J. Math. 202 (2002), 429–447.

[MeS1986] W. H. Meeks III and P. Scott, Finite group actions on 3-manifolds, Invent. Math. 86 (1986), 287–346.

[MY1981] W. H. Meeks III and S. T. Yau, The equivariant Dehn’s lemma and loop theorem, Comment. Math. Helv. 56 (1981), 225–239.

[Mns1983] W. W. Menasco, Polyhedra representation of link complements, Low-dimensional topology (San Francisco, Calif., 1981), 305–325, Contemp. Math., 20, Amer. Math. Soc., Providence, RI, 1983.

[MeT2005] W. W. Menasco and M. Thistlethwaite, Handbook of knot theory, Elsevier Science, 2005, 492p.

[My1986] R. Meyerhoff, Density of the Chern-Simons invariant for hyperbolic 3-manifolds, Low-dimensional topology and Kleinian groups (Coventry/Durham, 1984), 217–239, London Math. Soc. Lecture Note Ser., 112, Cambridge Univ. Press, Cambridge, 1986.

[Mct2015] C. Millichap, Factorial growth rates for the number of hyperbolic 3-manifolds of a given volume, Proc. Amer. Math. Soc. 143 (2015), 2201–2214.

[MW2016] C. Millichap and W. Worden, Hidden symmetries and commensurability of 2-bridge link complements, Pacific J. Math. 285 (2016), 453–484.

[Mln1962] J. Milnor, A unique decomposition theorem for 3-manifolds, Amer. J. Math. 84 (1962) 1–7.

[Mnt2017] Y. N. Minsky, and S. J. Taylor, Fibered faces, veering triangulations, and the arc complex, Geom. Funct. Anal. 27 (2017),.1450–1496.

[Mj]* M. Mj, Cannon-Thurston maps, [arXiv:1712.00760 [math.GT]].

[Mnt1973] J. M. Montesinos, Variedades de Seifert que son recubridores cíclicos de dos hojas, Bol. Soc. Mat. Mexicana 18 (1973), 1–32.

[Mnt1975] J. M. Montesinos, Surgery on links and double branched covers of S^3, Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox), pp. 227–259. Ann. of Math. Studies, No. 84, Princeton Univ. Press, Princeton, N.J., 1975.

[Mnt1987] J. M. Montesinos, Classical tessellations and three-manifolds, Universitext. Springer-Verlag, Berlin, 1987. xviii+230 pp.

[Mrg1992] J. W. Morgan, A-trees and their applications, Bull. Amer. Math. Soc. 26 (1992), 87–112.

[Mrg2004] J. W. Morgan, Recent progress on the Poincaré conjecture and the classification of 3-manifolds, Bull. Amer. Math. Soc. 42 (2005), 57–78.
[NR1992] W. D. Neumann and A. W. Reid, *Arithmetic of hyperbolic manifolds*, Topology ’90 (Columbus, OH, 1990), 273–310, Ohio State Univ. Math. Res. Inst. Publ., 1, de Gruyter, Berlin, 1992.

[NS1997] W. D. Neumann and G. Swarup, *Canonical decompositions of 3-manifolds*, Geom. Topol. 1 (1997), 21–40.

[NeZ1985] W. D. Neumann and D. Zagier, *Volumes of hyperbolic three-manifolds*, Topology 24 (1985), 307–332.

[Nw1966] L. P. Neuwirth, *Knot groups*, Annals of Mathematics Studies, No. 56, Princeton University Press, Princeton, N.J. 1966 vi+113 pp.

[NiZ2018] Y. Ni and X. Zhang, *Finite Dehn surgeries on knots in $S^3$*, Algebr. Geom. Topol. 18 (2018), 441–492.

[Oh2002] K. Ohshika *Discrete groups*, Translated from the 1998 Japanese original by the author. Translations of Mathematical Monographs, 207. Iwanami Series in Modern Mathematics. Amer. Math. Soc., Providence, RI, 2002. x+193 pp.

[Ot1996] J. P. Otal, *Thurston’s hyperbolization of Haken manifolds*, Surveys in differential geometry, Vol. III (Cambridge, MA, 1996), 77–194, Int. Press, Boston, MA, 1998.

[Ot2001] J. P. Otal, *The hyperbolization theorem for fibered 3-manifolds*, Translated from the 1996 French original by Leslie D. Kay. SMF/AMS Texts and Monographs, 7. American Mathematical Society, Providence, RI; Société Mathématique de France, Paris, 2001. xiv+126 pp.

[OS2005] P. Ozsváth and Z. Szabó, *On knot Floer homology and lens space surgeries*, Topology 44 (2005), 1281–1300.

[OS2006] P. Ozsváth and Z. Szabó, *An introduction to Heegaard Floer homology*, Floer homology, gauge theory, and low-dimensional topology, 3–27, Clay Math. Proc., 5, Amer. Math. Soc., Providence, RI, 2006.

[PP2000] C. Petronio and J. Porti, *Negatively Oriented Ideal Triangulations and a Proof of Thurston’s Hyperbolic Dehn Filling Theorem*, Expo. Math. 18 (2000), 1–35.

[Pe2002]* G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math/0211159 [math.DG].

[Pe2003a]* G. Perelman, Ricci flow with surgery on three-manifolds, arXiv:math/0303109 [math.DG].

[Pe2003b]* G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds, arXiv:math/0307245 [math.DG].

[PS1997] V. V. Prasolov and A. B. Sossinsky, *Knots, links, braids and 3-manifolds. An introduction to the new invariants in low-dimensional topology*, Translated from the Russian manuscript by Sossinsky. Translations of Mathematical Monographs, 154. American Mathematical Society, Providence, RI, 1997. viii+239.

[Ras2004] J. Rasmussen, *Lens space surgeries and a conjecture of Goda and Teragaito*, Geom. Topol. 8 (2004), 1013–1031.

[Rat1994] J. G. Ratcliffe, *Foundations of hyperbolic manifolds*, Graduate Texts in Mathematics, 149. Springer-Verlag, New York, 1994. xii+747 pp.

[Rei1990] A. W. Reid, *A note on trace-fields of Kleinian groups*, Bull. London Math. Soc. 22 (1990), 349–352.
[Rei1991] A. W. Reid, Arithmeticity of knot complements, J. London Math. Soc. 43 (1991) 171–184.
[ReM2003] A. W, Reid and C. Maclachlan, The arithmetic of hyperbolic 3-manifolds, Graduate Texts in Mathematics, 219. Springer-Verlag, New York, 2003. xiv+463 pp.
[RW2008] A. Reid and G. S. Walsh, Commensurability classes of 2-bridge knot complements, Algebr. Geom. Topol. 8 (2008), 1031–1057.
[Rdm1935] K. Reidemeister, Homotopieringe und Linsenräume, Abh. Math. Sem. Univ. Hamburg 11 (1935), 102–109.
[Ren2000] M. Reni, On pi-hyperbolic knots with the same 2-fold branched coverings, Math. Ann. 316 (2000), 681–697.
[ReZ2001] M. Reni and B. Zimmermann, Hyperbolic 3-manifolds as cyclic branched coverings, Comment. Math. Helv. 76 (2001), 300–313.
[RiZ2000] L. Ribes and P. Zalesskii, Profinite groups, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], 40. Springer-Verlag, Berlin, 2000. xiv+435 pp.
[RY2016] Y. Rieck and Y. Yamashita, Cosmetic surgery and the link volume of hyperbolic 3-manifolds, Algebr. Geom. Topol. 16 (2016), 3445–3521.
[Ri1971] R. Riley, Homomorphisms of knot groups on finite groups, Math. Comp. 25 (1971), 603–619.
[Ri1975] R. Riley, A quadratic parabolic group, Math. Proc. Cambridge Philos. Soc. 77 (1975), 281–288.
[Ri1979] R. Riley, An elliptical path from parabolic representations to hyperbolic structures, Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977), pp. 99–133. Lecture Notes in Math., 722. Springer, Berlin, 1979.
[Ri1990] R. Riley, Growth of order of homology of cyclic branched covers of knots, Bull. London Math. Soc. 22 (1990), 287–297.
[Ri1992] R. Riley, Algebra for Heckeoid groups, Trans. Amer. Math. Soc. 334 (1992), 389–409.
[Rol1976] D. Rolfsen, Knots and links, Mathematics Lecture Series, No. 7. Publish or Perish, Inc., Berkeley, Calif., 1976. ix+439 pp.
[Ru1987] D. Ruberman, Mutation and volumes of knots in $S^3$, Invent. Math. 90 (1987), 189–215.
[Sak1991] T. Sakai, Geodesic knots in a hyperbolic 3-manifold, Kobe J. Math. 8 (1991), 81–87.
[Skm1986] M. Sakuma, Uniqueness of symmetries of knots, Math. Z. 192 (1986), 225–242.
[Skm1990] M. Sakuma, The geometries of spherical Montesinos links, Kobe J. Math. 7 (1990), 167–190.
[Skm1995] M. Sakuma, Homology of abelian coverings of links and spatial graphs, Canad. J. Math. 47 (1995), 201–224.
[Sak1995] M. Sakuma and J. Weeks, Examples of canonical decompositions of hyperbolic link complements, Japan. J. Math. (N.S.) 21 (1995), 393–439.
[Sch1949] H. Schubert, Die eindeutige Zerlegbarkeit eines Knotens in Primknoten, S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1949, (1949), 57–104.
[Sch1953] H. Schubert, Knoten und Vollringe, Acta Math. 90 (1953), 131–286.
[Sch1956] H. Schubert Knoten mit zwei Brücken, Math. Z. 65 (1956) 133–170.
[Sc2014] J. Schultens, Introduction to 3-manifolds, Graduate Studies in Mathematics, 151. American Mathematical Society, Providence, RI, 2014. x+286 pp.
[Scm1995] K. Schmidt, Dynamical systems of algebraic origin, Progress in Mathematics, 128. Birkhäuser Verlag, Basel, 1995.
[Sc1983] P. Scott, The geometries of 3-manifolds, Bull. London Math. Soc. 15 (1983), 401–487.
[Sei1934] H. Seifert, Über das Geschlecht von Knoten, Math. Ann. 110 (1934), 571–592.
[Srr1977] J-P. Serre, Arbres, amalgames, SL2. Avec un sommaire anglais. Rédigé avec la collaboration de Hyman Bass, Astérisque No. 46. Société Mathématique de France, Paris, 1977. 189 pp.
[Sh2002] P. B. Shalen, Representations of 3-manifold groups, Handbook of geometric topology, 955–1044, North-Holland, Amsterdam, 2002.
[Si1943] C. L. Siegel, Discontinuous groups, Ann. of Math. 44 (1943), 674–689.
[Si1945] C. L. Siegel, Some remarks on discontinuous groups, Ann. of Math. 46 (1945), 708–718.
[SlWi2002] D. Silver and S. Williams, Mahler measure, links and homology growth, Topology 41 (2002), 979–991.
[So1981] T. Soma, The Gromov invariant of links, Invent. Math. 64 (1981), 445–454.
[St1961] J. Stallings, On fibering certain 3-manifolds, 1962 Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961) pp. 95–100 Prentice-Hall, Englewood Cliffs, N.J.
[St1976] J. Stallings, Constructions of fibred knots and links, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, pp. 55–60, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., 1978.
[Tk1985] M. Takahashi, On the concrete construction of hyperbolic structures of 3-manifolds, Tsukuba J. Math. 9 (1985), 41–83.
[Te1977] Hidetaka Terasaka Kodansha, Bluebacks; B312, May, 1977.
[Ths1985] M. B. Thistlethwaite, Knot tabulations and related topics, Aspects of topology, 1–76, London Math. Soc. Lecture Note Ser., 93, Cambridge Univ. Press, Cambridge, 1985.
[Ths1979] W. P. Thurston, The geometry and topology of three-manifolds, Lecture notes, Princeton University, 1976–80.
[Ths1981] W. P. Thurston, Three manifolds with symmetry, perprint, 1981.
[Ths1982] W. P. Thurston, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. 6 (1982), 357–381.
[Ths1982b] W. P. Thurston, Universal links, preprint 1882.
[Ths1986a] W. P. Thurston, A norm for the homology of 3-manifolds, Mem. Amer. Math. Soc. 59 (1986), no. 339, i–vi and 99–130.
[Ths1986b] W. P. Thurston, Hyperbolic structures on 3-manifolds. I. Deformation of acylindrical manifolds, Ann. of Math. 124 (1986), 203–246.
[Ths1988] W. P. Thurston, On the geometry and dynamics of diffeomorphisms of surfaces, Bull. Amer. Math. Soc. 19 (1988), 417–431.
[Ths1997] W. P. Thurston, Three-dimensional geometry and topology. Vol. 1. Edited by Silvio Levy. Princeton Mathematical Series, 35. Princeton University Press, Princeton, NJ, 1997. x+311 pp.
[Ths1998] W. P. Thurston, Hyperbolic Structures on 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle, arXiv:math/9801045.
[Tr1963] H. F. Trotter, Non-invertible knots exist, Topology 2 (1963) 275–280.
[Tur1992] V. G. Turaev, Shadow links and face models of statistical mechanics, J. Differ. Geom. 36 (1992), 35–74.
[Tur1994] V. G. Turaev, Quantum invariants of knots and 3-manifolds, In: de Gruyter Studies in Mathematics, vol. 18. Walter de Gruyter & Co., Berlin (1994).
[Vr1976] O. Ja. Viro, Nonprojecting isotopies and knots with homeomorphic coverings, Studies in topology, II. Zap. Nauken. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 66 (1976), 133–147, 207–208.
[Wd1994] M. Wada, Twisted Alexander polynomial for finitely presentable groups, Topology 33 (1994), 241–256.
[Wd*] M. Wada, OPTs, Computer Software available at http://delta-mat.ist.osaka-u.ac.jp/OPTs/
[WYY1996] M. Wada, Y. Yamashita and H. Yoshida, An inequality for polyhedra and ideal triangulations of cusped hyperbolic 3-manifolds, Proc. Amer. Math. Soc. 124 (1996), 3905–3911.
[Wld1967] F. Waldhausen, Eine Klasse von 3-dimensionalen Mannigfaltigkeiten. I, II, Invent. Math. 3 (1967), 308–333; Invent. Math. 4 (1967), 87–117.
[Wld1968] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, Ann. of Math. 87 (1968) 56–88.
[Wls2005] G.S. Walsh, Great circle links and virtually fibered knots, Topology 44 (2005), 947–958.
[Wn1972] H. C. Wang, Topics on totally discontinuous groups, Symmetric spaces (Short Courses, Washington Univ., St. Louis, Mo., 1969–1970), pp. 459–487. Pure and Appl. Math., Vol. 8, Dekker, New York, 1972.
[Wk*] J. Weeks, SnapPea, Computer Software available at http://www.geometrygames.org/
[Wk1993] J. Weeks, Convex hulls and isometries of cusped hyperbolic 3-manifolds, Topology Appl. 52 (1993), 127–149.
[Wk2005] J. Weeks, Computation of hyperbolic structures in knot theory, Handbook of knot theory, 461–480, Elsevier B. V., Amsterdam, 2005.
[Whd1937] J. H. C. Whitehead, On doubled knots, J. London Math. Soc. 12, 63–71 (1937).
[Whn1987] W. Whitten, Knot complements and groups, Topology 26 (1987), 41–44.
[Ysh1996] H. Yoshida, Ideal tetrahedral decompositions of hyperbolic 3-manifolds, Osaka J. Math. 33 (1996), 37–46.
[Ysk2013] K. Yoshida, The minimal volume orientable hyperbolic 3-manifold with 4 cusps, Pacific J. Math. 266 (2013), 457–476.
[Yst1985] T. Yoshida, The η-invariant of hyperbolic 3-manifolds, Invent. Math. 81 (1985), 473–514.
[Z1998] B. Zimmermann, On hyperbolic knots with homeomorphic cyclic branched coverings, Math. Ann. 311 (1998), 665–673.

DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, HIROSHIMA UNIVERSITY, HIGASHI-HIROSHIMA, 739-8526, JAPAN
E-mail address: sakuma@hiroshima-u.ac.jp