Orlicz–Sobolev versus Hölder local minimizer and multiplicity results for quasilinear elliptic equations

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Abstract

We study the following boundary value problem

\[
(P) \begin{cases}
-\text{div}(a(|\nabla u|)\nabla u) = f(x, u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

with nonhomogeneous principal part. By assuming the nonlinearity \( f(x, t) \) being subcritical growth, some abstract results of problem \( (P) \) are obtained: (1) Regularity; (2) Orlicz–Sobolev versus Hölder local minimizer; (3) Strong comparison principle. Applying these abstract results and critical point theory, we prove the existence of multiple solutions of problem \( (P) \) in an Orlicz–Sobolev space.

Keywords. Orlicz–Sobolev spaces, Regularity, Strong comparison theorem, Multiplicity results, Critical groups.

2000 Mathematics Subject Classification. 35J70, 35J20, 35J65

1 Introduction

In this paper, the following boundary value problem is discussed

\[
(P) \begin{cases}
-\text{div}(a(|\nabla u|)\nabla u) = f(x, u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]  

where \( \Omega \in R^N \) ia a bounded domain with smooth boundary \( \partial \Omega \), \( a(t) \in C(R) \) and \( f(x, t) \in C(\Omega \times R) \). Especially, when \( a(t) = |t|^{p-2} \), problem \( (P) \) is the well known \( p \)-Laplacian

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equation. There are a large number of papers on the existence of solutions for \( p \)-Laplacian equation. Readers can be referred to ([3, 7, 12, 13, 14, 21, 22, 27]) and the references therein for some results. Problem (P) is studied in an Orlicz-Sobolev space. The study of problem (P) in the Orlicz-Sobolev spaces has been received considerable attention in recent years. We refer to the overview papers [11, 16, 17, 19] for the advances and references of this area.

Problem (P) possesses more complicated nonlinearities, for example, it is inhomogeneous, so in the discussions, some special techniques will be needed. The inhomogeneous nonlinearities have important physical background, e.g.,

1. nonlinear elasticity: \( P(t) = (1 + t^2)^\gamma - 1, \gamma > \frac{1}{2}, \)
2. plasticity: \( P(t) = t^\alpha (\log(1 + t))^{\beta}, \alpha \geq 1, \beta > 0, \)
3. generalized Newtonian fluids: \( P(t) = \int_0^t s^{1-\alpha} (\sinh^{-1} s)^\beta ds, 0 \leq \alpha \leq 1, \beta > 0. \)

This paper is organized as follows: in Section 2, we present some necessary preliminary knowledge.

In Section 3, a regularity results of problem (P) is proved, that is, any weak solution of problem (P) in Orlicz-Sobolev space belongs to \( C^{1,\alpha}(\Omega) \). Applying the regularity results we extend Brezis and Nirenberg’s results ([9]) to problem (P). In [9], Brezis and Nirenberg have proved a famous theorem which asserts that the local minimizers in the space \( C^1 \) are also local minimizers in the space \( H^1 \) for certain variational functionals. This theorem has been extended to the \( p \)--Laplacian case (see [7, 22]). In this section, we assert that local minimizers in the space \( C^1 \) are also local minimizers in the Orlicz–Sobolev space for the functional with respect to problem (P).

In Section 4, we give a general principle of sub-supersolution method for problem (P) based on the regularity results and the comparison principle, which is similar to the \( p \)--laplacian case. However, it is usually very difficult to find a subsolution \( u_0 \) and a supersolution \( v^0 \) of problem (P) with \( u_0 \leq v^0 \). The main difficulty is that problem (P) possesses more complicated nonlinearities than the \( p \)--Laplacian. So some techniques used in the \( p \)--laplacian case cannot be carried out for problem (P). We give a lemma involving the \( L^\infty \)--estimation of the solution of problem (P) with \( f = M \) (a positive constant), which is useful to find a supersolution of problem. In the end of this section, we give an application of our abstract theorems to the eigenvalue problems with respect to problem (P).

In Section 5, we give several applications of our abstract theorems to problem (P). Problem (P) is considered on a bounded domain, where the nonlinearity \( f(x, t) \) is superlinear but does not satisfy the usual Ambrosetti–Rabinowitz condition near infinity, or its dual version near zero. Nontrivial solutions are obtained by computing the critical groups and the sub-supersolution method.
2 Preliminaries

The function \(a\) is such that \(p : R \rightarrow R\) defined by

\[
p(t) = \begin{cases} 
  a(|t|)t, & t \neq 0, \\
  0, & t = 0,
\end{cases}
\]

is an increasing homeomorphism from \(R\) onto itself and \(f(x, t) \in C(\bar{\Omega} \times R, R)\).

Obviously, problem (P) allows a nonhomogeneous function \(p\) in the differential operator defining problem (P). To deal with this situation we introduce an Orlicz-Sobolev space setting for problem (P) as follows.

Let

\[
P(t) := \int_0^t p(s)ds, \quad \widetilde{P}(t) := \int_0^t p^{-1}(s)ds, \quad t \in R,
\]

then \(P\) and \(\widetilde{P}\) are complementary N-functions (see [1, 33]), which define the Orlicz spaces \(L^P : = L^P(\Omega)\) and \(L^{\widetilde{P}} : = L^{\widetilde{P}}(\Omega)\), respectively.

Throughout this paper, we assume the following conditions on \(P\):

\((p_0)\): \(a(t) \in C^1(0, +\infty), a(t) > 0\) and \(a(t)\) is a monotonic function for \(t > 0\),

\((p_1)\): \(1 < p^- := \inf_{t > 0} \frac{tp(t)}{P(t)} \leq p^+ := \sup_{t > 0} \frac{tp(t)}{P(t)} < +\infty,\)

\((p_2)\): \(0 < a^- := \inf_{t > 0} \frac{tp'(t)}{p(t)} \leq a^+ := \sup_{t > 0} \frac{tp'(t)}{p(t)} < +\infty.\)

Under the conditions \((p_0)\) and \((p_1)\), the Orlicz space \(L^P\) coincides with the set (equivalence classes) of measurable functions \(u : \Omega \rightarrow R\) such that

\[
\int_{\Omega} P(|u|)dx < +\infty.
\]

The space \(L^P(\Omega)\) is a Banach space endowed with the Luxemburg norm

\[
|u|_P := \inf \left\{ k > 0, \int_{\Omega} P\left(\frac{|u|}{k}\right)dx < 1 \right\}.
\]

We shall denote by \(W^{1,P}(\Omega)\) the corresponding Orlicz-Sobolev space with the norm

\[
||u||_{W^{1,P}(\Omega)} := |u|_P + ||\nabla u||_P.
\]

And denote by \(W^{1,P}_0(\Omega)\) the closure of \(C^\infty_0\) in \(W^{1,P}(\Omega)\).

Let us now introduce the Orlicz-Sobolev conjugate \(P_*\) of \(P\), which is given by

\[
P_*^{-1}(t) := \int_0^t \frac{P^{-1}(\tau)}{\tau^{1/n}}d\tau
\]

(see [1, 33]), where we suppose that
\[
\lim_{t \to 0} \int_t^1 \frac{P^{-1}(\tau)}{\tau^{N/N}} d\tau < +\infty \quad \text{and} \quad \lim_{t \to +\infty} \int_t^1 \frac{P^{-1}(\tau)}{\tau^{N/N}} d\tau = +\infty.
\]

(2.5)

In the case \(P(t) = t^p\), (2.5) holds if and only if \(N > p\).

We will make the following assumptions on \(f(x,t)\):

\((f.):\) There exists an odd increasing homeomorphism \(h\) from \(R\) to \(R\), and nonnegative constants \(a_1, a_2\) such that

\[ |f(x,t)| \leq a_1 + a_2 h(|t|), \forall t \in R, \forall x \in \tilde{\Omega}. \]

and

\[ \lim_{t \to +\infty} \frac{H(t)}{P_s(kt)} = 0, \forall k > 0, \]

where

\[ H(t) := \int_0^t h(s)ds. \]

(2.6)

Let

\[ \tilde{H}(t) := \int_0^t h^{-1}(s)ds. \]

(2.7)

So we obtain complementary \(N\)-functions which define corresponding Orlicz spaces \(L^H\) and \(L^{\tilde{H}}\).

Set

\[ p^- := \inf_{t > 0} \frac{tP'_s(t)}{P_s(t)}, \quad p^+ := \sup_{t > 0} \frac{tP'_s(t)}{P_s(t)}. \]

(2.8)

As in [11], by L’Hôpital’s rule we have \(p^- = \frac{Np}{N-p}\).

Similar to condition \((p)\), we also assume the following condition on \(H\):

\((h):\) \(1 < h^- := \inf_{t > 0} \frac{th(t)}{H(t)} \leq h^+ := \sup_{t > 0} \frac{th(t)}{H(t)} < +\infty\).

Moreover, we assume that \(h^+\) and \(h^-\) satisfy the following condition:

\[ p^+ < h^- \leq h^+ < p^- . \]

(2.9)

In this paper, the following equivalent norm on \(W^{1,P}_0(\Omega)\) will be used in below:

\[ \|u\| := \inf \left\{ k > 0 : \int_{\Omega} P\left(\frac{\nabla u}{k}\right) dx < 1 \right\}. \]

The reader is referred to [1, 33] for more details on Orlicz–Sobolev spaces theory. In the proofs of our results we shall use the following results.

**Lemma 2.1** ([1]). *Under the condition \((p)\), the spaces \(L^P(\Omega)\), \(W^{1,P}_0(\Omega)\) and \(W^{1,P}(\Omega)\) are separable and reflexive Banach spaces.*
Lemma 2.2 ([1]). Under the condition \((f_2)\), the imbedding
\[
W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)
\]
is compact.

Lemma 2.3. Under the conditions \((p_0),(p_1)\) and \((p_2)\), we easily have

1. If \(0 < t < 1\), then \(P(t) t^{p^*} \leq P(t) \leq P(1) t^{p^*}\).
2. If \(t > 1\), then \(P(t) t^{p^-} \leq P(t) \leq P(1) t^{p^-}\).

Lemma 2.4 ([16]). Let \(\rho(u) = \int_{\Omega} P(u)dx\), we have

1. If \(|u|_p < 1\), then \(|u|_p^{p^*/(p^- - 1)} \leq \rho(u) \leq |u|_p^{p^*/(p^- - 1)}\).
2. If \(|u|_p > 1\), then \(|u|_p^{p^-/(p^* - 1)} \leq \rho(u) \leq |u|_p^{p^-/(p^* - 1)}\).
3. If \(0 < t < 1\), then \(t^{p^*} P(u) \leq \int_{\Omega} P(u)dx \leq t^{p^-} P(u)\).
4. If \(t > 1\), then \(t^{p^-} P(u) \leq \int_{\Omega} P(u)dx \leq t^{p^-} P(u)\).

Lemma 2.5 ([16]). Let \(\tilde{\rho}(u) = \int_{\Omega} \tilde{P}(u)dx\), we have

1. If \(|u|_p < 1\), then \(|u|_p^{p^*/(p^- - 1)} \leq \tilde{\rho}(u) \leq |u|_p^{p^*/(p^- - 1)}\).
2. If \(|u|_p > 1\), then \(|u|_p^{p^-/(p^* - 1)} \leq \tilde{\rho}(u) \leq |u|_p^{p^-/(p^* - 1)}\).
3. If \(0 < t < 1\), then \(t^{p^*} \tilde{P}(u) \leq \int_{\Omega} \tilde{P}(u)dx \leq t^{p^-} \tilde{P}(u)\).
4. If \(t > 1\), then \(t^{p^-} \tilde{P}(u) \leq \int_{\Omega} \tilde{P}(u)dx \leq t^{p^-} \tilde{P}(u)\).

Lemma 2.6. The condition \((p_2)\) implies that
\[
a^- - 1 = \inf_{t > 0} \frac{t a'(t)}{a(t)} \leq a^+ - 1 = \sup_{t > 0} \frac{t a'(t)}{a(t)} < +\infty.
\]
Hence, similar to Lemma 2.4, we easily have

1. If \(0 < t < 1\), then \(t a^{-1} a(u) \leq a(tu) \leq t a^{-1} a(u)\).
2. If \(t > 1\), then \(t a^{-1} a(u) \leq a(tu) \leq t a^{-1} a(u)\).

Lemma 2.7 ([1]). Assume that \(A(t)\) and \(\tilde{A}(t)\) are complementary \(\text{N–functions}\). We have

1. Young inequalities: \(uv \leq A(u) + \tilde{A}(v)\).
2. Hölder inequalities: \(\int_{\Omega} u(x)v(x)dx \leq 2|u|_A|v|_{\tilde{A}}\).
3. \(\tilde{A}(\frac{A(u)}{u}) \leq A(u)\).
4. \(\tilde{A}(\frac{A(u)}{u}) \leq A_s(u)\).
Remark 2.1: Since problem \((P)\) possesses inhomogeneous nonlinearities, Lemma 2.3–2.7 are used to overcome the nonhomogeneous difficulty.

Definition 2.1. \(u \in W_{0}^{1,p}(\Omega)\) is called a weak solution of problem \((P)\) if

\[
\int_{\Omega} a(|\nabla u|) \nabla u \nabla \phi \, dx = \int_{\Omega} f(x,u) \phi \, dx, \quad \forall \phi \in W_{0}^{1,p}(\Omega).
\]  (2.10)

Set

\[
P(u) = \int_{\Omega} P(|\nabla u|) \, dx, \quad \forall u \in W_{0}^{1,p}(\Omega),
\]  (2.11)

\[
F(x,t) = \int_{0}^{t} f(x,s) \, ds, \quad \forall (x,t) \in \bar{\Omega} \times \mathbb{R},
\]  (2.12)

\[
I(u) = \int_{\Omega} P(|\nabla u|) \, dx - \int_{\Omega} F(x,u) \, dx, \quad \forall u \in W_{0}^{1,p}(\Omega).
\]  (2.13)

We know that the critical points of \(I\) are just the weak solutions of problem \((P)\) (see \([11]\)).

Write

\[
\mathcal{F}(u) = \int_{\Omega} F(x,u) \, dx,
\]  (2.14)

then \(I(u) = P(u) - \mathcal{F}(u)\).

Lemma 2.8. (1) \(([17, 19])\) The functional \(P \in C^{1}(W_{0}^{1,p}(\Omega), \mathbb{R})\) is convex and sequentially weakly lower semi-continuous and

\[
P'(u)\phi = \int_{\Omega} p(\nabla u) \nabla \phi \, dx, \quad \forall u, \phi \in W_{0}^{1,p}(\Omega).
\]

Moreover, the mapping \(P' : W_{0}^{1,p}(\Omega) \to W_{0}^{1,p}(\Omega)^{\ast}\) is bounded homeomorphism, and is of type \((S^+)\), namely,

\[
u_{n} \rightharpoonup u \quad \text{and} \quad \limsup_{n \to \infty} P'(u_{n})(u_{n} - u) \leq 0 \quad \text{imply that} \quad u_{n} \to u \quad \text{in} \quad W_{0}^{1,p}(\Omega).
\]  (2.15)

(2) \(([11])\) The functional \(\mathcal{F}(u)\) is sequentially weakly continuous, \(\mathcal{F}(u) \in C^{1}(W_{0}^{1,p}(\Omega), \mathbb{R})\), and for all \(u, \phi \in W_{0}^{1,p}(\Omega)\),

\[
\mathcal{F}'(u)\phi = \int_{\Omega} f(x,u) \phi \, dx.
\]  (2.16)

The mapping \(\mathcal{F}' : W_{0}^{1,p}(\Omega) \to W_{0}^{1,p}(\Omega)^{\ast}\) is weakly–strongly continuous, namely,

\[
u_{n} \rightharpoonup u \quad \text{implies that} \quad \mathcal{F}'(u_{n}) \to \mathcal{F}'(u),
\]  (2.17)

where \(\rightharpoonup\) and \(\to\) denote the weak and strong convergence in \(W_{0}^{1,p}(\Omega)\), respectively.
3 Regularity and $W^{1,p}_0$ versus $C^1$ local minimizers

In this section, we will firstly consider the regularity of weak solutions related to the following problem (3.1).

$$\text{div}(A(x, \nabla u)) = B(x, u), \text{ in } \Omega,$$

where $\Omega$ is an open bounded subset of $\mathbb{R}^N (N \geq 2)$, $A : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$, $B : \Omega \times \mathbb{R} \to \mathbb{R}$. We assume that problem (3.1) satisfies the following growth conditions:

$$A(x, \eta)\eta \geq a_0 P(|\eta|) - c,$$  \hspace{1cm} (3.2)

$$A(x, \eta) \leq a_1 p(|\eta|) + c,$$  \hspace{1cm} (3.3)

$$B(x, u) \leq bh(|\eta|) + c,$$  \hspace{1cm} (3.4)

where $a_0, a_1, b, c$ are positive constant.

**Definition 3.1.** $u \in W^{1,p}(\Omega)$ is said to be a weak solution of problem (3.1) if

$$\int_{\Omega} A(x, Du)D\phi dx - \int_{\Omega} B(x, u)\phi dx = 0,$$ \hspace{1cm} (3.5)

for $\forall \phi \in W^{1,p}_0(\Omega)$.

**Theorem 3.1.** Under the growth conditions (3.2-3.4), if $u \in W^{1,p}_0(\Omega)$ is a weak solution of problem 3.1, then $u \in L^{\infty}(\Omega)$. If in addition $u|_{\partial \Omega}$ is bounded, then $u \in L^{\infty}(\Omega)$.

Before the proof of the Theorem 3.1, we give a lemma which play an important role in the proof of Theorem 3.1.

**Lemma 3.1** (Lemma 2.4, [18]). Let $u \in W^{1,p}_0(\Omega)(p > 1)$. If for any $B_\rho(x_0) \subset \subset \Omega$ with $\rho < R_0$ and any $\sigma \in (0,1)$ and any $k \geq k_0 > 0$

$$\int_{A_{k,\rho\sigma \rho}} |Du|^p dx \leq c \left[ \int_{A_{k,\rho}} \frac{u - k}{\sigma \rho} |D^r| dx + (k^r + 1)|A_{k,\rho}| \right],$$ \hspace{1cm} (3.6)

where $A_{k,\rho} = \{ x \in B_\rho(x_0) : u(x) > k \}$, $0 < r \leq p^*$, $p^*$ is the Sobolev embedding exponent of $p$, $c$ is a positive constant, then $u$ is locally bounded above on $\Omega$.

**Proof of Theorem 3.1.** Let $u$ be a weak solution of problem (3.1). Let $x_0 \in \Omega$. We will prove that $u$ is locally bounded at $x_0$. Take a ball $B_R(x_0) \subset \Omega$.

For arbitrary balls $B_{s}(x) \subset B_{r}(x) \subset B_{R_0}(x_0)$, let $\xi$ be a $C^\infty$ function such that

$$0 \leq \xi \leq 1, \text{ supp}\xi \subset B_{s}, \xi \equiv 1 \text{ on } B_{r}, |D\xi| \leq 2/(s-t).$$ \hspace{1cm} (3.7)

For $k \geq 1$, set $\eta = \xi^{p^*} \max[u - k, 0]$. Then $\eta \in W^{1,p}_0(\Omega)$. By (3.1) we get

$$\int_{A_{k,s}} A(x, Du)Du\xi^{p^*} dx + p^* \int_{A_{k,s}} A(x, Du)D\xi^{p^*-1}(u - k) dx$$
Next let us estimate the terms of the right-hand side of (3.16). As 0 < s − t < 1, we easily have

\[
\int_{A_{k,s}} \xi'^{p^+}(u - k) dx \leq |A_{k,s}|, \tag{3.11}
\]

\[
\int_{A_{k,s}} \xi^{p^+}(u - k) dx \leq \int_{A_{k,s}} |u - k| dx \leq Q + |A_{k,s}|, \tag{3.12}
\]

\[
\int_{A_{k,s}} \xi^{p^+ - 1}|D\xi|(u - k) dx \leq 2 \int_{A_{k,s}} \frac{|u - k|}{s - t} dx \leq 2Q + 2|A_{k,s}|. \tag{3.13}
\]

By 0 < s − t < 1, \( p^+ < h^+ < p^- \) and Young inequalities, we have

\[
\int_{A_{k,s}} h(|u|)\xi^{p^+}(u - k) dx \\
\leq \int_{A_{k,s}} h(|u|)(u - k) dx \\
= \int_{A_{k,s}} \frac{h(|u|)|u|}{|u|} (u - k) dx \\
\leq h^+ \int_{A_{k,s}} H(|u|) (u - k) dx \\
\leq h^+ \int_{A_{k,s}} H(|u|) dx + \int_{A_{k,s}} H(|u - k|) dx \\
\leq h^+ \int_{A_{k,s}} H(|u|) dx + \int_{A_{k,s}} H(|u - k|) dx \\
\leq h^+ \int_{A_{k,s}} H(|u - k| + |k|) dx + \int_{A_{k,s}} H(|u - k|) dx
\]
Similarly, using Young inequalities and taking \( \varepsilon_1 \in (0, 1) \) such that,

\[
a_1(p^+)^2 \varepsilon_1^{p^-} = \frac{a_0}{4},
\]

we get

\[
a_1 p^+ \int_{A_{k,s}} \frac{p(|Du|)|\xi|^{p^-} |D\xi|(u - k)}{|Du|} dx \\
= a_1 p^+ \int_{A_{k,s}} \frac{p(|Du|)|\xi|^{p^-} |D\xi|(u - k)}{|Du|} dx \\
\leq a_1(p^+)^2 \int_{A_{k,s}} \frac{P(|Du|)|\xi|^{p^-} |D\xi|(u - k)}{|Du|} dx \\
\leq a_1(p^+)^2 \left[ \int_{A_{k,s}} P \left( \varepsilon_1 \frac{P(|Du|)|\xi|^{p^-} |D\xi|(u - k)}{|Du|} \right) dx \right. \\
\left. + \int_{A_{k,s}} \left( \frac{1}{\varepsilon_1} |D\xi|(u - k) \right) dx \right] \\
\leq a_1(p^+)^2 \left[ \varepsilon_1^{p^-} \int_{A_{k,s}} |\xi|^{p^+} P \left( \frac{P(|Du|)}{|Du|} \right) dx \right. \\
\left. + \varepsilon_1^{-p^+} \int_{A_{k,s}} \left( \frac{|Du|}{|u - k|} \right) dx \right] \\
\leq \frac{a_0}{4} J + c_3 Q + c_4 |A_{k,s}|. \quad (3.16)
\]

From (3.11)–(3.16), we obtain

\[
J \leq c_5(Q + (k^{h^+} + 1)|A_{k,s}|),
\]

and therefore

\[
\int_{A_{k,s}} |Du|^{p^+} dx \leq c_6 \left[ \int_{A_{k,s}} \frac{|u - k|^{p^-}}{s - t} dx + (k^{h^+} + 1)|A_{k,s}| \right]. \quad (3.18)
\]

By Lemma 3.1, (3.18) implies that \( u \) is bounded above on \( B_{R_0}(x_0) \) and hence \( u \) is locally bounded above on \( \Omega \).

Similarly we can prove that \( -u \) is also locally bounded above on \( \Omega \). So \( u \in L^{\infty}_{\text{loc}}(\Omega) \). If in addition \( \max_{\partial \Omega} |u(x)| = M < +\infty \), then for every \( x_0 \in \partial \Omega \), by the similar argument as above we can prove that (3.18) holds for \( k \geq M \) and therefore \( u \in L^{\infty}(\Omega) \). Theorem 3.1 is proved.

In [29, 30], Lieberman has considered the regularity of weak solutions to the following boundary value problems

\[
\begin{aligned}
-\text{div}(A(x, u, \nabla u)) + B(x, u, \nabla u) &= 0, \quad \text{in } \Omega, \\
u &= \phi, \quad \text{on } \partial \Omega.
\end{aligned}
\]

(3.19)
Applying his results and Theorem 3.1 to problem (P), we have the following corollary.

**Corollary 3.1.** Let \( f(x, t) \) satisfy the condition \((f, )\) and \( u \) be a weak solution in \( W^{1,p}_0(\Omega) \) of problem (P). Then \( u \in C^{1,a}(\Omega) \).

**Remark 3.1:** Corollary 3.1 improves a recent result of Fukagai and Narukawa [17, Lemma 4.1], where the conclusion of corollary 3.1 was obtained under the following assumptions:

(i) \( f(x, 0) = 0, f(x, t) \geq 0 \) for \( x \in \Omega, t > 0 \), and there exists an open set \( \Omega_0 \subset \Omega \) such that \( f(x, t) > 0 \) for \( x \in \Omega_0, t > 0 \).

(ii) \( f(x, t) \leq cP(t) \) for \( x \in \Omega, t \geq 0 \) with some constant \( c > 0 \).

In Corollary 3.1, we only assume that \( f(x, t) \) satisfies the condition \((f, )\), so Corollary 3.1 is natural extension of \( p \)-Laplacian case.

**Lemma 3.2.** There exist constants \( d_1, d_2 \), depending on \( a^- , a^+ \), such that

\[
|a(\eta)\eta - a(\xi)\xi| \leq d_1|\eta - \xi|a(|\eta| + |\xi|).
\] (3.20)

If \( a(t) \) is decreasing for \( t > 0 \), we get

\[
|a(\eta)\eta - a(\xi)\xi| \leq d_2p(|\eta - \xi|),
\] (3.21)

for all \( \eta, \xi \in \mathbb{R}^N \).

**Proof.** Since (3.20) is symmetric in \( \eta, \xi \), we can assume that \( |\xi| \geq |\eta| \). We have

\[
\begin{align*}
|a(\eta)\eta - a(\xi)\xi| &= \left| \int_0^1 \frac{d}{dt} \left[ a(\xi + t(\eta - \xi))((\xi + t(\eta - \xi))(\xi + t(\eta - \xi))) \right] dt \right| \\
&= \left| \int_0^1 a\left(\xi + t(\eta - \xi)\right) \frac{(\xi + t(\eta - \xi)) \cdot (\eta - \xi)}{|\xi + t(\eta - \xi)|} (\xi + t(\eta - \xi)) dt \right| \\
&\quad + \left| \int_0^1 a(\xi + t(\eta - \xi)) (\eta - \xi) dt \right| \\
&\leq c|\eta - \xi| \int_0^1 a(\xi + t(\eta - \xi)) dt.
\end{align*}
\] (3.22)

Since \( |\xi + t(\eta - \xi)| = |(1 - t)\xi + t\eta| \leq |\eta| + |\xi|, \forall t \in (0, 1) \), if \( a(t) \) is increasing, we obtain

\[
|a(\eta)\eta - a(\xi)\xi| \leq d_1|\eta - \xi|a(|\eta + \xi|).
\] (3.23)

To get the (3.20) for \( a(t) \) being decreasing, we have to prove that

\[
\int_0^1 a(\xi + t(\eta - \xi)) dt \leq ca(|\eta| + |\xi|).
\] (3.24)

Note that if \( |\eta - \xi| \leq \frac{|\xi|}{2} \), then

\[
|\xi + t(\eta - \xi)| \geq |\xi| - |\xi - \eta| \geq \frac{|\xi|}{2} \geq \frac{|\eta| + |\xi|}{4},
\] (3.25)
so that (3.24) hold with $c = \left(\frac{4}{7}\right)^{a^{-1}}$.

If $|\eta - \xi| > \frac{|\xi|}{2} > 0$, we set $t_0 = \frac{|\xi|}{|\eta - \xi|} \in (0, 2)$, then
\[
|\xi + t(\eta - \xi)| \geq |\xi| - t|\eta - \xi| = |t - t_0||\eta - \xi|
\geq |t - t_0| \frac{|\xi|}{2} \geq |t - t_0| \frac{|\eta| + |\xi|}{4}. \tag{3.26}
\]

For any $t_0 \in (0, 1)$, we have
\[
\int_0^1 a \left( \frac{|t_0 - t|}{4} (|\eta| + |\xi|) \right) dt \leq ca(|\eta| + |\xi|). \tag{3.27}
\]
So that (3.27) holds with $c = \left(\frac{4}{7}\right)^{a^{-1}} \left(\frac{a^e}{a} + \frac{(1-a)x}{a}\right)$.

Finally, $|\eta - \xi| \leq |\eta| + |\xi|$ implies (3.21). \qed

**Theorem 3.2.** Assume that $f(x, t)$ satisfies the condition (f). If $u \in C^1_0(\overline{\Omega})$ is a local minimizer of $I$ in the $C^1$ topology, then $u_0$ is a local minimizer of $I$ in the $W^{1, P}_0(\Omega)$ topology.

**Proof.** Let $u_0 \in C^1(\overline{\Omega})$ is a local minimizer of $I$ in the $C^1$ topology. Define
\[
G(u) = \int_{\Omega} P(|\nabla u - \nabla u_0|) dx, \forall u \in W^{1, P}_0(\Omega). \tag{3.28}
\]
For $\varepsilon \in (0, 1)$, set $D_\varepsilon = \{u \in W^{1, P}_0(\Omega) : G(u) \leq \varepsilon\}$. Then $D_\varepsilon$ is bounded, closed and convex subset of $W^{1, P}_0(\Omega)$ and is a neighborhood of $u_0$ in $W^{1, P}_0(\Omega)$. Since $f(x, t)$ satisfies the subcritical growth condition, the functional $I : W^{1, P}_0(\Omega) \to R$ is weakly lower semi-continuous and consequently $\inf_{D_\varepsilon} I$ is achieved at some $u_\varepsilon \in D_\varepsilon$. So there is $\mu_\varepsilon \leq 0$ such that $I' = \mu_\varepsilon G'(u_\varepsilon)$, that is
\[
-\operatorname{div} \left(a(|\nabla u_\varepsilon|) \nabla u_\varepsilon \right) + \mu_\varepsilon \operatorname{div} \left(a(|\nabla u_\varepsilon - \nabla u_0|) (\nabla u_\varepsilon - \nabla u_0) \right) = f(x, u_\varepsilon). \tag{3.29}
\]

Arguing by contradiction, assume that $u_0$ is not a local minimizer of $I$ in the $W^{1, P}_0(\Omega)$ topology, then for each $\varepsilon \in (0, 1), u_\varepsilon \neq u_0$ and $I(u_\varepsilon) < I(u_0)$. Note that $u_\varepsilon \to u_0$ in $W^{1, P}_0(\Omega)$ as $\varepsilon \to 0$. Below we shall prove that $u_\varepsilon \to u_0$ in $C^1(\overline{\Omega})$ as $\varepsilon \to 0$, which contradicts with that $u_0$ is a local minimizer of $I$ in the $C^1$ topology.

Dividing both sides of (3.29) by $1 - \mu_\varepsilon$, yields
\[
-\operatorname{div} \left( \frac{1}{1 - \mu_\varepsilon} \left[a(|\nabla u_\varepsilon|) \nabla u_\varepsilon \right] - \mu_\varepsilon a(|\nabla u_\varepsilon - \nabla u_0|) (\nabla u_\varepsilon - \nabla u_0) \right) = f(x, u_\varepsilon). \tag{3.30}
\]
Define $A_\varepsilon : \overline{\Omega} \times R^N \to R^N$ and $B_\varepsilon : \overline{\Omega} \times R \to R$ by
\[
A_\varepsilon(x, \eta) = \frac{1}{1 - \mu_\varepsilon} \left[a(|\eta|)\eta - \mu_\varepsilon a(|\eta - \nabla u_0|) (\alpha - \nabla u_0) \right],
\]
\[ B_\varepsilon(x, t) = \frac{1}{1 - \mu_\varepsilon} f(x, t). \]  

(3.31)

Then \( u_\varepsilon \) is a solution of the following problem:

\[
\begin{cases}
- \text{div}(A_\varepsilon(x, \nabla u)) = B_\varepsilon(x, u), & \text{in } \Omega, \\
 u = 0, & \text{on } \partial \Omega.
\end{cases}
\]  

(3.32)

We can verify that \( A_\varepsilon \) and \( B_\varepsilon \) satisfy the following conditions:

\[ A_\varepsilon(x, \eta) \eta \geq a_0 P(|\eta|) - c, \]  

(3.33)

\[ A_\varepsilon(x, \eta) \leq a_1 p(|\eta|) + c, \]  

(3.34)

\[ B_\varepsilon(x, u) \leq bh(|z|) + c, \]  

(3.35)

where \( a_0, a_1, b, c \) are positive constants independent of \( \varepsilon \in (0, 1) \).

The verification of (3.34) and (3.35) are simple, here we only give the proof of (3.33).

By definition of \( A_\varepsilon(x, \eta) \)

\[
A_\varepsilon(x, \eta) \eta = \frac{1}{1 - \mu_\varepsilon} \left[ (a(|\eta|) \eta - \mu_\varepsilon a(|\eta|) \eta) - \mu_\varepsilon (a(|\eta| - \nabla u_0)|\eta - \nabla u_0| - a(|\eta|) \eta) \right] \eta \\
\geq \frac{1}{1 - \mu_\varepsilon} \left[ (1 - \mu_\varepsilon) P(|\eta|) - \mu_\varepsilon J \right],
\]  

(3.36)

where

\[ J = [a(|\eta| - \nabla u_0)|\eta - \nabla u_0| - a(|\eta|) \eta] \eta. \]  

(3.37)

By lemma 3.2, we get

\[
|J| = |a(|\eta| - \nabla u_0)|\eta - \nabla u_0| - a(|\eta|) \eta| \eta| \\
\leq c |\nabla u_0| a(|\eta - \nabla u_0| + |\eta|) |\eta| \\
\leq cp(|\eta|) + c \leq \frac{1}{2} P(|\eta|) + c, \]  

(3.38)

and when \( a(t) \) is decreasing,

\[
|J| = |a(|\eta| - \nabla u_0)|\eta - \nabla u_0| - a(|\eta|) \eta| \\
\leq ca(|\nabla u_0|) |\nabla u_0| |\eta| \leq \frac{1}{2} P(|\eta|) + c, \]  

(3.39)

where \( c \) is a generic positive constant independent of \( \varepsilon \).

Thus we have

\[
A_\varepsilon(x, \eta) \eta \geq \frac{1}{1 - \mu_\varepsilon} \left[ (1 - \mu_\varepsilon) P(|\eta|) - \mu_\varepsilon J \right] \\
\geq \frac{1}{1 - \mu_\varepsilon} \left[ (1 - \mu_\varepsilon) P(|\eta|) - \mu_\varepsilon \left( \frac{1}{2} P(|\eta|) + c \right) \right]
\]
\[ \geq \frac{1}{1 + \mu_e} \left[ (1 + \frac{1}{2} \mu_e) P(|\eta|) - c\mu_e \right] \]
\[ \geq \frac{1}{2} P(|\eta|) - c, \quad (3.40) \]

and (3.33) is proved. It follows from Theorem 3.1 that \( u_e \in L^\infty \) and \(|u_e|_{L^\infty} \) is bounded uniformly for \( \varepsilon \in (0, 1) \), where \( c \) is a positive constant independent of \( \varepsilon \).

Below we shall prove that \( ||u||_{C^{1,\alpha}(\Omega)} \leq c \) for some \( \alpha \in (0, 1) \) by using the results in [29, 30] in the following two cases, respectively.

Case(i): \( \mu_e \in [-1, 0] \).

Note that \( u_0 \) satisfies the equation
\[ -\text{div}(a(|\nabla u_0|)\nabla u_0) = f(x, u_0). \quad (3.41) \]
The formula (3.29) is equivalent to the following:
\[ -\text{div}\left\{ a(|\nabla u_e|)\nabla u_e - \mu_e a(|\nabla u_e - \nabla u_0|)(\nabla u_e - \nabla u_0) - \mu_e a(|\nabla u_0|)\nabla u_0 \right\} \]
\[ = f(x, u_e) - \mu_e f(x, u_0). \quad (3.42) \]

Define \( \tilde{A}_e : \tilde{\Omega} \times R^N \rightarrow R^N \) and \( \tilde{B}_e : \tilde{\Omega} \times R \rightarrow R \) by
\[ \tilde{A}_e(x, \eta) = a(|\eta|)\eta - \mu_e a(|\eta - \nabla u_0|)(\eta - \nabla u_0) - \mu_e a(|\nabla u_0|)\nabla u_0, \]
\[ \tilde{B}_e(x, t) = f(x, t) - \mu_e f(x, u_0). \quad (3.43) \]
Then \( u_e \) is a solution of the following problem:
\[ \begin{cases} 
-\text{div}(\tilde{A}_e(x, \nabla u)) = \tilde{B}_e(x, u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega. 
\end{cases} \quad (3.44) \]

We need to show that, for \( x, y \in \tilde{\Omega}, \eta \in R^N \backslash \{0\}, \xi \in R^N, t \in R \), the following estimations hold:
\[ \tilde{A}_e(x, 0) = 0, \quad (3.45) \]
\[ \sum_{i,j=1}^N \frac{\partial (\tilde{A}_e)}{\partial \eta_i}(x, \eta) \xi_i \xi_j \geq \frac{p(|\eta|)}{|\eta|} |\xi|^2, \quad (3.46) \]
\[ \sum_{i,j=1}^N \left| \frac{\partial (\tilde{A}_e)}{\partial \eta_i}(x, \eta) \right| |\eta| \leq c(1 + p(|\eta|)), \quad (3.47) \]
\[ |\tilde{A}_e(x, \eta) - \tilde{A}_e(y, \eta)| \leq c(1 + p(|\eta|))(|x - y|^\theta), \quad \text{for some } \theta \in (0, 1), \quad (3.48) \]
\[ |\tilde{B}_e(x, t)| \leq c + ch(|t|). \quad (3.49) \]
Inequalities (3.46) and (3.47) follow from Lemma 3.2 and the following derivative

\[ D_\eta(a(\eta)|\eta) = a'(\eta)\frac{\eta \otimes \eta}{|\eta|^2} + a(\eta)|id \]

\[ = a(\eta)\left(id + \frac{a'(\eta)|\eta \otimes \eta}{|\eta|^2}\right). \]  

(3.50)

Inequality (3.48) follows from Lemma 3.2 and the fact that \( \nabla u_0(x) \) is Hölder continuous.

By the regularity results in [29, 30], under the conditions (3.45)–(3.49), \( u_e \in C^{1,\alpha}(\bar{\Omega}) \) and \( \|u_e\|_{C^{1,\alpha}(\bar{\Omega})} \leq c \), where the positive constant \( c \) is independent of \( \mu_e \in [-1, 0] \). From this and \( u_e \to u_0 \) in \( W^{1,p}_0(\Omega) \) it follows that \( u_e \to u_0 \) in \( C^1(\bar{\Omega}) \) as \( \varepsilon \to 0 \).

Case (ii): \( \mu_e < -1 \).

Set \( v_e = u_e - u_0 \). Then from (3.29) we know that \( v_e \) satisfies the equation

\[ -\text{div}\left[a(|\nabla v_e|)\nabla v_e + \frac{1}{|\mu_e|}(a(|\nabla v_e + \nabla u_0|)(\nabla v_e + \nabla u_0) - \frac{1}{|\mu_e|}a(|\nabla u_0|)\nabla u_0\right]

\[ = \frac{1}{|\mu_e|}\left[f(x,v_e + u_0) - f(x,u_0)\right]. \]  

(3.51)

Define

\[ \tilde{A}_e(x, \eta) = a(\eta)\eta + \frac{1}{|\mu_e|}(a(\eta + \nabla u_0)(\eta + \nabla u_0) - \frac{1}{|\mu_e|}a(\nabla u_0)\nabla u_0, \]

\[ \tilde{B}_e(x, t) = \frac{1}{|\mu_e|}\left[f(x,t + u_0) - f(x,u_0)\right]. \]  

(3.52)

Analogously to the case (i), we can prove that \( \tilde{A}_e \) and \( \tilde{B}_e \) satisfy the corresponding conditions (3.45)–(3.49). So by regularity results in [29, 30], \( v_e \in C^{1,\alpha}(\bar{\Omega}) \) and \( \|u_e\|_{C^{1,\alpha}(\bar{\Omega})} \leq c \), furthermore \( v_e \to 0 \) in \( C^1(\bar{\Omega}) \), that is \( v_e \to v_0 \) in \( C^1(\bar{\Omega}) \) as \( \varepsilon \to 0 \). The proof of Theorem 3.2 is complete. \( \square \)

4 Sub-supersolution method and multiplicity results

In this section, firstly, A general principle of sub–supersolution for problem (P) is given based on the regularity results and the comparison principle. Secondly, in order to utilize the results of Section 1-3, an abstract result which provides a method to find a positive minimizer of functional \( f \) in the \( C^1 \)–topology is proved. At last, by several assumptions on \( f(x,t) \), we give an application of the principle of sub–supersolution and the abstract result.

**Definition 4.1.** Let \( u, v \in W^{1,p}(\Omega) \). We say that \( -\Delta_p u \leq -\Delta_p v \) if for all \( \phi \in W^{1,p}_0(\Omega) \) with \( \phi \geq 0 \),

\[ \int_\Omega a(|\nabla u|)\nabla u \nabla \phi dx \leq \int_\Omega a(|\nabla v|)\nabla v \nabla \phi dx, \]  

(4.1)

where \( -\Delta_p u = -\text{div}(a(|\nabla u|)\nabla u) \).
Lemma 4.1 (Comparison principle). (1) Let \( u, v \in W^{1,p}(\Omega) \). If \( -\Delta_P u \leq -\Delta_P v \) and \( u \leq v \) on \( \partial \Omega \) (i.e., \((u - v)^+ \in W^{1,p}_0(\Omega)\)), then \( u \leq v \) in \( \Omega \).

(2) Under the conditions of (1) above, let in addition \( u, v \in C(\bar{\Omega}) \) and denote \( S = \{ x \in \Omega : u(x) = v(x) \} \). If \( S \) is a compact subset of \( \Omega \), then \( S = \emptyset \).

Proof. (1) Taking \( \phi = (u - v)^+ \) as a test function in (4.1), we can obtain \( \nabla (u - v)^+ = 0 \), this implies \((u - v)^+ = 0\) and so \( u \leq v \) in \( \Omega \).

(2) Assume that \( S \) is a compact subset of \( \Omega \) and \( S \neq \emptyset \). Then there is an open subset \( \Omega_1 \) of \( \Omega \) such that \( S \subset \Omega_1 \subset \bar{\Omega_1} \subset \Omega \). Thus \( u < v \) on \( \partial \Omega_1 \) and consequently there is an \( \varepsilon > 0 \) such that \( u < v - \varepsilon \) on \( \partial \Omega_1 \). Noting that \( \nabla(v - \varepsilon) = \nabla v \) and applying the conclusion (1) to \( v - \varepsilon \) on \( \Omega_1 \) we obtain \( u \leq v - \varepsilon \) in \( \Omega_1 \), which contradicts with \( u = v \) on \( S \). \( \square \)

It is well known that, when \( p \neq 2 \), the strong comparison principles for the \( p \)–Laplacian equations are very complicated (see [12, 13, 32]). Here we give a strong comparison principle for problem (P), which is suitable to find a positive \( C^1 \) local minimizer of the energy functional \( I \) in the \( C^1 \) topology. The proof is similar to the \( p \)–Laplacian case (see [20]).

Lemma 4.2. Let \( h_1, h_2 \in L^{\infty}(\Omega), 0 \leq h_1(x) \leq h_2(x) \) and \( u, v \in W^{1,p}_0(\Omega) \). If

\[
-\Delta_P u = h_1 \leq -\Delta_P v = h_2,
\]

and the set

\[
C = \{ x \in \Omega : h_1(x) = h_2(x) \}
\]

has an empty interior, then

\[
\begin{cases}
0 \leq u < v, & \text{in } \Omega, \\
0 \leq \frac{\partial u}{\partial n} < \frac{\partial v}{\partial n}, & \text{on } \partial \Omega,
\end{cases}
\]

and there exists a positive constant \( \varepsilon \) such that

\[
\frac{\partial(v - u)}{\partial n} \geq \varepsilon,
\]

where \( n \) denotes the inward unit normal on \( \partial \Omega \).

Similar to \( p \)–laplacian case, for given \( h \in L^\infty(\Omega) \), the following problem:

\[
\begin{cases}
-\text{div}(a(|\nabla u|)|\nabla u|) = h(x), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

has a unique solution \( u \in W^{1,p}_0(\Omega) \). We denote by \( K(h) := u \) the unique solution. \( K \) is called the solution operator for (4.6). We have the following basic principle of sub–supersolution method for problem (P).
Lemma 4.3. Assume that \( f(x,t) \) satisfies (f) and \( f(x,t) \) is nondecreasing in \( t \in R \). If there exist a subsolution \( u_0 \in W^{1,p}(\Omega) \) and a supersolution \( v^0 \in W^{1,p}(\Omega) \) of problem (P) such that \( u_0 \leq v^0 \), then problem (P) has a minimal solution \( u_* \) and a maximal solution \( v_* \) in the order interval \( [u_0,v^0] \), i.e., \( u_0 \leq u_* \leq v_* \leq v^0 \) and if \( u \) is any solution of problem (P) such that \( u_0 \leq u \leq v^0 \), then \( \leq u_* \leq v_* \).

Proof. Define \( T(u) = K(f(x,u)) \). Under the assumptions of Lemma 4.3, since the imbedding \( W^{1,p}(\Omega) \hookrightarrow L^H(\Omega) \) is compact, we have \( T : L^H(\Omega) \rightarrow L^H(\Omega) \) is completely continuous and increasing, \( u_0 \leq v^0 \), \( u_0, v^0 \in L^H(\Omega) \), \( u_0 \leq T(u_0), v^0 \geq T(v^0) \), and consequently \( T : [u_0,v^0] \rightarrow [u_0,v^0] \). It is clear that the cone of all nonnegative functions in \( L^H(\Omega) \) is normal. So our Theorem 4.1 now follows by applying the well-known fixed point theorem for the increasing operator on the order interval (see [4]). } 

In the practical problems it is often known that the subsolution \( u_0 \) and the supersolution \( v^0 \) are of class \( L^\infty(\Omega) \), so the restriction on the growth condition of \( f \) is needless, hence the following lemma is more suitable.

Lemma 4.4. Assume that \( u_0, v^0 \in W^{1,p}(\Omega) \cap L^\infty(\Omega) \), \( u_0, v^0 \in W^{1,p}(\Omega) \) are a subsolution and a supersolution of problem (P) respectively, and \( u_0 \leq v^0 \). If \( f(x,t) \in C(\bar{\Omega} \times R, R) \) satisfies the condition:

\[
f(x,t) \text{ is nondecreasing in } t \in [\inf u_0(x), \sup v^0(x)],
\]

then the conclusion of lemma 4.3 is valid.

The proof Lemma 4.4 is similar to the proof of lemma 4.3 and is omitted here. Remark 4.1: In Lemma 4.4, as was done in the \( p \)-laplacian case (see,[4, 9, 15, 20]), the condition (4.7) can be replaced by some weaker conditions, for example, by the following condition:

there exists a positive constant \( c \) such that

\[
f(x,t) + ct \text{ is nondecreasing in } t \in [\inf u_0(x), \sup v^0(x)].
\]

The following theorem provides a method to find a positive \( C^1 \) local minimizer of the integral functional \( I \) in the \( C^1 \) topology.

Theorem 4.1. Assume that \( u_0, v^0 \in W^{1,p}_0(\Omega) \) are a subsolution and a supersolution of problem (P) respectively, \( -\Delta_p u_0 = h_1(x), -\Delta_p v^0 = h_2(x), h_1, h_2 \in L^\infty(\Omega), 0 \leq h_1 \leq h_2 \), \( h_1(x) \neq h_2(x) \) and \( 0 \leq u_0 \leq v^0 \) in \( \Omega \). Assume that \( f(x,t) \in C(\bar{\Omega} \times R) \) satisfies the condition (4.7) or (4.8). If neither \( u_0 \) nor \( v^0 \) is a solution of problem (P), or neither \( u_0 \) nor \( v^0 \) is a minimizer of \( I \) on \( [u_0,v^0] \cap W^{1,p}_0(\Omega) \) in the case of being a solution of problem (P), then there exists \( u_* \in [u_0,v^0] \cap C^{1,\omega}(\bar{\Omega}) \) such that \( I(u_*) = \inf\{I(u) : u \in [u_0,v^0] \cap W^{1,p}_0(\Omega)\} \), \( u_* \) is a solution of problem (P) and \( u_* \) is a local minimizer of \( I \) in the \( C^1 \) topology.
**Proof.** Under the assumptions of theorem, by Corollary 3.1, $u_0, v^0 \in C^{1,0}(\overline{\Omega})$. Define $\widetilde{f} : \overline{\Omega} \times R \to R$ by

$$
\tilde{f}(x, t) = \begin{cases} 
  f(x, u_0(x)), & \text{if } t < u_0(x), \\
  f(x, s), & \text{if } u_0(x) \leq t \leq v^0(x), \\
  f(x, v^0(x)), & \text{if } t > v^0(x).
\end{cases}
$$

(4.9)

Set $\widetilde{F}(x, t) = \int_0^t \tilde{f}(x, s)ds$ and

$$
I(u) = \int_\Omega P(\nabla u)dx - \int_\Omega \tilde{F}(x, u)dx, \quad \forall u \in W^{1,p}_0(\Omega).
$$

(4.10)

It is easy to see that the minimum of $\overline{I}$ on $W^{1,p}_0(\Omega)$ is achieved at some $u_* \in W^{1,p}_0(\Omega)$. Then $u_*$ satisfies the equation

$$
-\text{div}(a(\nabla u_*)\nabla u_*) = \tilde{f}(x, u_*),
$$

(4.11)

and consequently $u_* \in C^{1,0}(\overline{\Omega})$.

Since

$$
-\text{div}(a(\nabla u_0)\nabla u_0) \leq f(x, u_0) = \tilde{f}(x, u_0) \leq \tilde{f}(x, u_*) = -\text{div}(a(\nabla u_*)\nabla u_*),
$$

(4.12)

using Lemma 4.1(1), $u_0 \leq u_*$. Repeating the same reasoning, we can obtain $u_* \leq v^0$. Note that for all $u \in [u_0, v^0] \cap W^{1,p}_0(\Omega)$, $\tilde{f}(x, u) = f(x, u)$, $\overline{F}(x, u) - F(x, u)$ is a function of $x$, and $\overline{I}(x, u) - I(x, u)$ is a constant. Hence $u_*$ is a solution of problem (P) and is a minimizer of $I$ on $u \in [u_0, v^0] \cap W^{1,p}_0(\Omega)$. It follows from Lemma 4.2 that

$$
\left\{\begin{array}{ll}
0 \leq u_0 < u_* < v_0, & \text{in } \Omega, \\
0 \leq \frac{\partial u_*}{\partial n} < \frac{\partial u}{\partial n} < \frac{\partial v^0}{\partial n}, & \text{on } \partial \Omega,
\end{array}\right.
$$

(4.13)

and there exists a positive constant $\varepsilon$ such that

$$
\frac{\partial(u^* - u_0)}{\partial n}, \quad \frac{\partial(v^0 - u_*)}{\partial n} \geq \varepsilon.
$$

(4.14)

Thus there exists $\delta > 0$ such that

$$
W^{1,p}_0(\Omega) \cap B_{C^1}(u_*, \delta) := \{u \in W^{1,p}_0(\Omega) \cap C^1(\overline{\Omega}) : ||u - u_*||_{C^1(\overline{\Omega})} < \delta\} \subset [u_0, v^0].
$$

(4.15)

Noting that $I(u) = \overline{I}(u) + c$ for $u \in W^{1,p}_0(\Omega) \cap B_{C^1}(u_*, \delta)$, we see that $u_*$ is a local minimizer of $I$ in the $C^1$ topology. The proof is complete. □

As an application of the above abstract theorems, let us consider the following eigen-
value problem:
\[
(P_{\lambda}) \quad \begin{cases} 
-\text{div}(a(|\nabla u|)\nabla u) = \lambda f(x,u) + \mu |u|^{p-1}u, & \text{in } \Omega, \\
\quad u > 0, & \text{in } \Omega, \\
\quad u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

where \(\Omega\) is a bounded smooth domain in \(\mathbb{R}^N\), \(q > p^*\), \(f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})\), \(f(x,t) \geq 0\) and \(t \geq 0\), \(f(x,t)\) is nondecreasing in \(t \geq 0, \mu \geq 0\) is fixed. The energy functional associated with problem \((P_{\lambda})\) is

\[
I_{\lambda} = \int_\Omega P(|\nabla u|)dx - \lambda \int_\Omega F(x,u)dx - \frac{\mu}{q} \int_\Omega |u|^qdx, \quad \forall u \in W^{1,p}_0(\Omega),
\]

where \(F(x,t) = \int_0^t f(x,s)ds\). The corresponding problems in \(p\)-laplacian case have been studied by many authors (see \([3, 6, 9, 22, 26]\))

**Theorem 4.2.** Assume that \(f\) satisfies the condition either

(i) \(f(x,0) \neq 0\) in \(\Omega\), or

(ii) \(f(x,0) \equiv 0\) and there are an open set \(U \subset \Omega\), a closed ball \(\overline{B}(x_0, \varepsilon) \subset U\), the positive constant \(r_0 > 1\) and \(c\), such that \(f(x,t) \geq ct^{q_0-1}\) for \(x \in \overline{B}(x_0, \varepsilon)\) and \(t \in [0,1]\), and \(r_0 < p^*\) for \(x \in \partial U\).

Then we have the following assertions:

(1) For sufficiently small \(\lambda > 0\), Problem \((P_{\lambda})\) has a solution \(u_\lambda\) which is a local minimizer of \(I_{\lambda}\) in the \(C^1\) topology. Moreover, \(||u_\lambda||_{C^1(\overline{\Omega})} \to 0\) as \(\lambda \to 0\).

(2) Define \(\Lambda_0 = \{\lambda > 0: (P_{\lambda})\) has a solution \(u_\lambda\) which is a local minimizer of \(I_{\lambda}\) in the \(C^1\) topology\} and \(\Lambda = \{\lambda > 0: (P_{\lambda})\) has a solution \(u_\lambda\) \}. Then \(\Lambda_0\) and \(\Lambda\) are both intervals, \(\inf \Lambda_0 = \inf \Lambda = 0\) and \(\text{int}\Lambda \subset \Lambda_0\).

(3) In addition, assume that \(\mu > 0, q < p^*_\lambda\) and

\[|f(x,t)| \leq c (1 + r(|t|)) \text{ for } x \in \Omega \text{ and } t \in \mathbb{R},\]

where \(r^* < p^*_\lambda\) and \(r^* < q_\lambda\). Then for each \(\lambda \in \text{int}\Lambda, (P_{\lambda})\) has at least two solutions \(u_\lambda\) and \(v_\lambda\) such that \(u_\lambda < v_\lambda\) and \(u_\lambda\) is a local minimizer of \(I_{\lambda}\) in the \(W^{1,p}_0\) topology.

Before the proof Theorem 4.2, we give a lemma which is useful to find a supersolution of problem \((P_{\lambda})\).

**Lemma 4.5.** Let \(M > 0\) and \(u\) is the unique solution of problem

\[
\begin{cases} 
-\text{div}(a(|\nabla u|)\nabla u) = M, & \text{in } \Omega \\
\quad u = 0, & \text{on } \partial \Omega.
\end{cases}
\]
Set \( m = \frac{1}{2M|\Omega|^{1/N}C_0} \). Then, when \( M \geq m \), \( |u|_\infty \leq C^* M^{\frac{1}{p^-}} \), and when \( M < m \), \( |u|_\infty \leq C_\ast M^{\frac{1}{p^-}} \), where \( C^* \) and \( C_\ast \) are positive constants depending on \( p^+, p^-, N, |\Omega| \) and \( C_0 \).

**Proof.** Let \( u \) be the solution of (4.18), then \( u \geq 0 \). For \( k \geq 0 \), set \( A_k = \{ x \in \Omega : u(x) > k \} \). Taking \((u - k)^+\) as a test function of (4.18), using Young inequalities, we have

\[
\int_{A_k} P(|\nabla u|) \, dx = M \int_{A_k} (u - k) \, dx \\
\leq M |A_k|^{\frac{1}{p}} (u - k)^+_{L^{N/(N-1)}(\Omega)} \leq M |A_k|^{\frac{1}{p}} C_0 \int_{A_k} \varepsilon |\nabla u| \varepsilon^{-1} \, dx \\
\leq M |A_k|^{\frac{1}{p}} C_0 \int_{A_k} (P(\varepsilon|\nabla u|) + \overline{P}(\varepsilon^{-1})) \, dx \\
\leq M |A_k|^{\frac{1}{p}} C_0 \varepsilon^{p^-} \int_{A_k} P(|\nabla u|) \, dx + M |A_k|^{\frac{1}{p}} C_0 \int_{A_k} \overline{P}(\varepsilon^{-1}) \, dx,
\]

(4.19)

where the first inequality is due to the continuous embedding \( W_0^{1,1} \subset L^{\frac{N}{N-1}}(\Omega) \) and \( C_0 \) is the best constant of the embedding.

When \( M \geq m \), taking

\[
\varepsilon = \left( \frac{1}{2M|\Omega|^{1/N}C_0} \right)^{\frac{1}{p}} = \left( \frac{m}{M} \right)^{\frac{1}{p^-}},
\]

(4.20)

then \( \varepsilon < 1 \) and

\[
M |A_k|^{\frac{1}{p}} C_0 \varepsilon^{p^-} \int_{A_k} P(|\nabla u|) \, dx \leq \frac{1}{2} \int_{A_k} P(|\nabla u|) \, dx.
\]

(4.21)

Consequently, from this and (4.19), we get

\[
\int_{A_k} P(|\nabla u|) \, dx \leq 2M |A_k|^{\frac{1}{p}} C_0 \int_{A_k} \overline{P}(\varepsilon^{-1}) \, dx \leq 2M |A_k|^{\frac{1}{p}} + 1 C_0 \overline{P}(\varepsilon^{-1}).
\]

(4.22)

So we have

\[
\int_{A_k} (u - k) \, dx = \frac{1}{M} \int_{A_k} P(|\nabla u|) \, dx \leq \gamma |A_k|^{\frac{1}{p}} + 1,
\]

(4.23)

where

\[
\gamma = 2C_0 \overline{P}(\varepsilon^{-1}).
\]

(4.24)

Using Lemma 5.1 in [25, Chapter 2], (4.23) implies that

\[
|u|_\infty \leq \gamma (N + 1) |\Omega|^{\frac{1}{p}}.
\]

(4.25)

From (4.20), (4.24) and (4.25) we obtain

\[
|u|_\infty \leq C^* M^{\frac{1}{p^-}},
\]

(4.26)

where

\[
C^* = \overline{P}(1)(N + 1) (2C_0)^{\frac{p^-}{p^+}} |\Omega|^{\frac{p^-}{N(p^- - 1)}}.
\]

(4.27)
When $M < m$, taking
\[
\varepsilon = \left( \frac{1}{2M|\Omega|^{1/N}C_0} \right)^{p^+} = \left( \frac{m}{M} \right)^{p^+}, \tag{4.28}
\]
(noting that in this case $\varepsilon > 1$) and using arguments similar to those above we can obtain
\[
|u|_\infty \leq C_*M^{-\frac{1}{p^+-1}}, \tag{4.29}
\]
where
\[
C_* = \bar{P}(1)(N+1)(2C_0)^{p^+} |\Omega|^{-\frac{p^+}{q-1}}. \tag{4.30}
\]
The proof is complete. □

**Proof of Theorem 4.2.** (1) Take $0 < M < m$, where $m$ is as in Lemma 4.5, and let $v = v_M$ be the unique positive solution of (4.18). Then by Lemma 4.5, $|v|_\infty \leq C_*M^{1/(p^+-1)}$. Since $q > p^+$, we can choose $M$ small enough such that $\mu(C_*M^{1/(p^+-1)})^{-1} < \frac{M}{q}$, which implies that $\mu v^{q-1} < \frac{M}{q^+}$. Let $\lambda > 0$ be sufficiently small such that $\lambda f(x, v) < \frac{M}{q^+}$. Then for such $\lambda$,
\[
-\text{div}(a(\|\nabla v\|)\nabla v) = M > \lambda f(x, v) + \mu |v|^{q-2}v,
\]
which shows that $v$ is a supersolution of $(P_a)$ and is not a solution of $(P_a)$. By Lemma 4.2, $v > 0$ in $\Omega$ and $\frac{\partial v}{\partial \nu} > 0$ on $\partial \Omega$.

In the case when $f$ satisfies the condition (i), $0$ is a subsolution of $(P_a)$ and $0$ does not satisfy the equation in $(P_a)$. Moreover, by Theorem 4.1, $(P_a)$ has a solution $u_\lambda \in [0, v] \cap C^1(\overline{\Omega})$, which is a local minimizer of $I_\lambda$ in the $C^1$ topology.

In the case when $f$ satisfies the condition (ii), $0$ satisfies the equation in $(P_a)$. We claim that $0$ is not a minimizer of $I_\lambda$ on $[0, v] \cap W^{1,p}_0(\Omega)$. To see this, noting $I_\lambda(0) = 0$, it is sufficient to show that $\inf_{[0, v] \cap W^{1,p}_0(\Omega)} I_\lambda(u) < 0$. For $\delta > 0$ denote $U_\delta = \{x \in U : \text{dist}(x, \partial U) < \delta\}$. By the condition (ii), we can find sufficiently small positive constants $\delta$ such that $\overline{B}(x_0, \varepsilon) \subset U \setminus U_\delta$ and $r_0 < p^-$. Define a function $w \in C^0_w(U)$ such that $0 \leq w \leq 1$ and $w = 1$ on $U \setminus U_\delta$. Then for sufficiently small $t > 0$, we have that $tw \in [0, v]$ and
\[
I_\lambda(tw) = \int_\Omega P(|\nabla tw|)dx - \lambda \int_\Omega F(x, tw)dx \\
\leq \int_{U_\delta} P(|\nabla tw|)dx - \lambda \int_{U \setminus U_\delta} F(x, tw)dx \\
\leq tp^+ \int_{U_\delta} P(|\nabla w|)dx - c\lambda t^{r_0} \int_{U \setminus U_\delta} w^{r_0}dx \leq 0, \tag{4.31}
\]
which shows that the claim is true. By Theorem 4.1, there exists $u_\lambda \in [0, v] \cap C^{1,\alpha}(\overline{\Omega})$ such that $I_\lambda(u_\lambda) = \inf_{[0, v] \cap W^{1,p}_0(\Omega)} I_\lambda$, $u_\lambda$ is a solution of $(P_a)$ and a local minimizer of $I_\lambda$ in the $C^1$ topology.

When $\lambda \to 0$, we can take $M \to 0$, consequently $|v_M|_{L^\infty(\Omega)} \to 0$ and $|u_\lambda|_{L^\infty(\Omega)} \to 0$, furthermore, $\|u_\lambda\| \to 0$ and $|u_\lambda|_{C^{1,\alpha}(\Omega)} \to 0$. Assertion (1) is proved.

(2) Obviously $A_0 \subset A$. By assertion (1), $\inf A_0 = \inf \Lambda = 0$. Let $\lambda_1 \in \Lambda$ and $\lambda \in (0, \lambda_1)$
be given arbitrarily. Let $u_{\lambda_1}$ be a solution of $(P_{\lambda_1})$. Then $u_{\lambda_1}$ is a supersolution of $(P_\lambda)$. It follows from assertion (1) that there exists a sufficiently small $\lambda_2 < \lambda$ such that $(P_{\lambda_2})$ has a solution $u_{\lambda_2}$ and $u_{\lambda_2} < u_{\lambda_1}$ in $\Omega$. Obviously $u_{\lambda_2}$ is a subsolution of $P_\lambda$. By Theorem 4.1, $P_\lambda$ has a solution $u_\lambda$ which is a local minimizer of $I_\lambda$ in the $C^1$ topology, which shows $\lambda \in \Lambda_0$. Assertion (2) is proved.

(3) Note that, under the additional assumptions, it is easy to verify that $I_\lambda \in C^1(W^{1,p}_0(\Omega), R)$ and $I_\lambda$ satisfies the (PS) condition for all $\lambda$(see [11]). Now let $\lambda \in \Lambda \subset \Lambda_0$ be given arbitrarily. Take $\lambda_1, \lambda_2 \in \Lambda_0$ with $\lambda_2 < \lambda < \lambda_1$, and let $u_{\lambda_1}, u_\lambda$ and $u_{\lambda_2}$ be a solution of $(P_{\lambda_1})$, $(P_\lambda)$ and $(P_{\lambda_2})$ respectively, $u_{\lambda_2} \leq u_\lambda \leq u_{\lambda_1}$, and let $u_\lambda$ be a local minimizer of $I_\lambda$ in the $C^1$ topology. Then by Theorem 4.1, $u_\lambda$ is also a local minimizer of $I_\lambda$ in the $W^{1,p}$ topology. Define

$$
\begin{align*}
\tilde{f}_\lambda(x, t) &= \begin{cases} f(x, t), & \text{if } t > u_\lambda(x), \\
    f(x, u_\lambda(x)), & \text{if } t \leq u_\lambda(x),
\end{cases} \quad (4.32)
\end{align*}
$$

and denote the associated functional to (4.35) by $\Lambda_\lambda$. It is easy to see that $u_{\lambda_2}$ and $u_{\lambda_1}$ are a subsolution and a supersolution of (4.35), respectively. By Theorem 4.1, there exists $u_\lambda^* \in [u_{\lambda_2}, u_{\lambda_1}] \cap C^1(\bar{\Omega})$ such that $u_\lambda^*$ is a solution of (4.35) and is a local minimizer of $\Lambda_\lambda$ in the $C^1$ topology. By Lemma 4.1, we can see that $u_\lambda^* \geq u_\lambda$ and consequently $u_\lambda^*$ is also a solution of $(P_\lambda)$. If $u_\lambda^* \neq u_\lambda$ then assertion (3) already holds, hence we can assume that $u_\lambda^* = u_\lambda$. Now $u_\lambda$ is a local minimizer of $\Lambda_\lambda$ in the $C^1$ topology, and so also in the $W^{1,p}$ topology. We can assume that $u_\lambda$ is a strictly local minimizer of $\Lambda_\lambda$ in the $W^{1,p}$ topology, otherwise we have obtained assertion (3). It is easy to verify that, under the additional assumptions in statement $\Lambda_\lambda \in C^1(W^{1,p}_0(\Omega), R)$ and $\Lambda_\lambda$ satisfies the (PS) condition. From $q > p$ and $\mu > 0$, it follows that $\inf\{I_\lambda(u) : u \in W^{1,p}_0(\Omega)\} = -\infty$. Using the mountain pass theorem(see [2]), we know that (4.35) has a solution $v_\lambda$ such that $v_\lambda \neq u_\lambda$. $v_\lambda$ as a solution of (4.35), must satisfy $v_\lambda \geq u_\lambda$, and consequently, by Lemma 4.2, $v_\lambda > u_\lambda$. Noting that $v_\lambda$ is also a solution of $(P_\lambda)$, since $v_\lambda \geq u_\lambda$, thus the proof of assertion (3) is complete.
5 Morse theory and multiplicity results

In this section, we use the results of Section 1–4 and Morse theory to obtain existence of multiple solutions. Morse theory is a very useful tool in treating multiple solution problems in the study of nonlinear differential equations. The main concept in this theory is the critical group $C_q(I, u)$ for a $C^1$-functional $I : X \rightarrow \mathbb{R}$ at an isolated critical point $u$, where $X$ is a Banach space. Let $I(u)$ be a $C^1$–functional defined on a Banach space $X$, then the $k$–th critical group of $I$ at an isolated critical point $u$ with $I(u) = c$ is defined by

$$C_q(I, u) := H_q(I^c \cap U, (I^c \cap U)\setminus \{u\}), \quad q \in \mathbb{N} = \{0, 1, 2, \ldots\},$$

where $U$ is any neighborhood of $u$, $H_*$ is the singular relative homology with coefficients in an Abelian group $G$ and $I^c = I^{-1}(-\infty, c]$.

We say that $I$ satisfies the (C) condition, if any sequence $\{u_n\} \subset X$ such that $\{I(u_n)\}$ is bounded and $(1 + \|u_n\|\|I'(u_n)\|) \rightarrow 0$ has a convergent subsequence; such a sequence is then called a (C) sequence. If $I$ satisfies the (C) condition and the critical values of $I$ are bonded from below by some $\alpha > -\infty$, then the critical groups of at infinity were introduced by Bartsch and Li [8] as

$$C_q(I, \infty) := H_q(X, I^\alpha). \quad (5.1)$$

Note that by the deformation lemma, the right-hand side of (5.1) does not depend on the choice of $\alpha$.

The reader is referred to [10, 31] for more details on Morse theory. In the proofs of our theorems we shall use the following results.

**Proposition 5.1** (Morse inequalities). Let $X$ be a Banach space. $I \in C^1(X, \mathbb{R})$ satisfies (C) condition. Then

$$\sum_{q=0}^{\infty} M_q t^q = \sum_{q=0}^{\infty} \beta_q t^q + (1 + t)Q(t), \quad (5.2)$$

where $M_q = \sum_{I(u)=c} \text{rank}C_q(I, u)$, $\beta_q = \text{rank}C_q(I, \infty)$ and $Q(t) = \sum_{q=0}^{\infty} a_q t^q$ with $a_q \in \mathbb{N}$.

Here (5.3) means

$$M_q - M_{q-1} + \cdots + (-1)^q M_0 \geq \beta_q - \beta_{q-1} + \cdots + (-1)^q \beta_0,$$

and if $a_q = 0$ for almost all $q$, then

$$\sum_{q=0}^{\infty} (-1)^q M_q = \sum_{q=0}^{\infty} (-1)^q \beta_q. \quad (5.3)$$

In applications, we use critical groups to distinguish critical points, and use Morse inequalities to find unknown critical points.
Proposition 5.2 ([8]). Assume that $I \in C^{1}(X, R)$ satisfies (C) condition and $I$ has only finitely many critical points, then

1. If for some $q \in N$ we have $C_q(I, \infty) \neq 0$, then $I$ has a critical point $u$ with $C_q(I, u) \neq 0$.

2. Let $0$ be an isolated critical point of $I(u)$. If for some $q \in N$ we have $C_q(I, 0) \neq C_q(I, \infty)$, then $I(u)$ has a nonzero critical point.

Assume that $f(x, t) = g(x, t) - k(x)$ and $f(x, t), g(x, t)$ satisfy the following conditions:

1. $g(x, t) \geq 0$, $g(x, 0) = 0$ and $k(x) \in C(\bar{\Omega})$ is positive.

2. $f(x, t)t > 0$ and $f(x, t)$ is superlinear at infinity, that is, the following limit holds uniformly on $x \in \Omega$,

$$\lim_{|t| \to \infty} \frac{f(x, t)}{|t|^{p+1}} = +\infty.$$

3. There exists $\theta \geq 1$ such that for any $s \in [0, 1]$ and $p_1, p_2 \in [p^-, p^+]$ there holds

$$\theta F_{p_1}(x, t) \geq F_{p_2}(x, st), \quad \forall x \in \bar{\Omega}, \quad \forall t \in R,$$

where $F_{p}(x, t) = f(x, t)t - pF(x, t)$.

Theorem 5.1. Assume that the conditions $(f_1)$, $(f_2)$ $(f_3)$ and $(f_3)$ are satisfied. Then problem $(P)$ has at least two nontrivial solutions.

Proof. Below we will take four steps to prove Theorem 5.1.

Step 1: The functional satisfies (C) condition.

Let $\{u_n\}$ be a (C) sequence of $I$. By Lemma 2.8, we only need to show that $\{u_n\}$ is bounded.

If $\{u_n\}$ is unbounded, up to a subsequence we may assume that for some $c \in R$,

$$I(u_n) \to c, \quad ||u_n|| \to \infty, \quad ||I'(u_n)||||u_n|| \to 0. \tag{5.4}$$

So we have

$$\lim_{n \to \infty} \int_{\Omega} \left( \frac{1}{\bar{p}_n} f(x, u_n)u_n - F(x, u_n) \right) \leq \lim_{n \to \infty} \left( I(u_n) - \frac{1}{\bar{p}_n} \langle I'(u_n), u_n \rangle \right) = c, \tag{5.5}$$

where $\bar{p}_n = \int_{\Omega} \frac{p(|u_n|^p)dx}{\|u_n\|^p}$. Let $w_n = \frac{u_n}{||u_n||}$, up to a subsequence we may assume that

$$w_n \rightharpoonup w \text{ in } W^{1,p}_0(\Omega), \quad w_n \to w \text{ in } L^p(\Omega), \quad w_n \to w \text{ a.e } x \in \Omega. \tag{5.6}$$

If $w = 0$, similar to $p$–Laplacian case in [23, 35], we can choose a sequence $\{t_n\} \subset R$ such that

$$I(t_nu_n) = \max_{t \in [0,1]} I(tu_n). \tag{5.7}$$
For any \( m > \frac{1}{2} \), let \( v_n = (2m)^{\frac{1}{2}} w_n \). Since \( v_n \to 0 \) in \( L^H(\Omega) \) and

\[
F(x, t) \leq C(1 + H(t)),
\]

(5.8)

by the continuity of the operator \( F(x, \cdot) \) (see [11]), we see that that \( F(\cdot, v_n) \to 0 \) in \( L^1(\Omega) \). Thus

\[
\lim_{n \to \infty} \int_\Omega F(x, v_n) dx = 0.
\]

(5.9)

So for \( n \) large enough, \( \frac{(2m)^{\frac{1}{2}}}{\|u_n\|} \in (0, 1) \), and we deduce

\[
I(t_n u_n) \geq I(v_n) = \int_\Omega P(\|\nabla v_n\|) - \int_\Omega F(x, v_n) dx \\
\geq \int_\Omega P\left(2m \frac{\|\nabla u_n\|}{\|u_n\|}\right) - \int_\Omega F(x, v_n) dx \\
\geq 2m - \int_\Omega F(x, v_n) dx \quad \text{(by Lemma 2.4)}.
\]

(5.10)

That is \( I(t_n u_n) \to \infty \). Now \( I(0) = 0 \), \( I(u_n) \to c \), we see that \( t_n \in (0, 1) \) and

\[
\int_\Omega p(\|\nabla u_n\|) |t_n \nabla u_n| dx - \int_\Omega f(x, t_n u_n) t_n u_n dx = \langle I'(t_n u_n), t_n u_n \rangle = t_n \frac{d}{dt} \big|_{t=n} I(t u_n) = 0.
\]

(5.11)

Therefore, by the condition \( (f_3) \),

\[
\int_\Omega \left( \frac{1}{\tilde{p}_m} f(x, u_n) u_n - F(x, u_n) \right) \geq \frac{1}{\theta} \int_\Omega \left( \frac{1}{\tilde{p}_m} f(x, t_n u_n) t_n u_n - F(x, t_n u_n) \right) \\
= \frac{1}{\theta} \int_\Omega \left( \frac{1}{\tilde{p}_m} \int_\Omega p(\|\nabla u_n\|) |t_n \nabla u_n| dx - F(x, t_n u_n) \right) \\
= \frac{1}{\theta} I(t_n u_n) \to +\infty,
\]

(5.12)

where \( \tilde{p}_m = \frac{\int_\Omega p(\|\nabla u_n\|) |\nabla u_n| dx}{\int_\Omega p(\|\nabla u_n\|) dx} \). This contradicts with (5.5).

If \( w \neq 0 \), from the third limit in (5.4) we obtain

\[
o(1) = \langle I'(u_n), u_n \rangle = \int_\Omega p(\|\nabla u_n\|) |\nabla u_n| dx - \int_\Omega f(x, u_n) u_n dx \\
\leq p^+ \int_\Omega p(\|\nabla u_n\|) dx - \int_\Omega f(x, u_n) u_n dx \\
\leq p^+ \|u_n\|^{p^*} - \int_\Omega f(x, u_n) u_n dx.
\]

(5.13)

Since \( f(x, u) u \geq 0 \), we deduce

\[
p^+ - o(1) \geq \int_\Omega \frac{f(x, u_n) u_n}{\|u_n\|^{p^*}} dx.
\]
\[
= \left( \int_{w=0} + \int_{w \neq 0} \right) \frac{f(x, u_n) u_n}{|u_n|^p} |w_n|^p \, dx \\
\geq \int_{w \neq 0} \frac{f(x, u_n) u_n}{|u_n|^p} |w_n|^p \, dx.
\]

(5.14)

For \( x \in \Theta := \{ x \in \Omega : w(x) \neq 0 \} \), we have \( |u_n(x)| \to +\infty \). By the condition (f2) we have

\[
\frac{f(x, u_n) u_n}{|u_n|^p} |w_n|^p \, dx \to +\infty.
\]

(5.15)

Note that the Lebesgue measure of \( \Theta \) is positive, using the Fatou Lemma we deduce

\[
\int_{w \neq 0} \frac{f(x, u_n) u_n}{|u_n|^p} |w_n|^p \, dx \to +\infty.
\]

(5.16)

This contradicts with (5.14).

Therefore, \( \{u_n\} \) is a bounded sequence in \( W^{1,P}_0(\Omega) \).

**Step2:** \( I \) is unbounded below.

By the condition (f2), there exists \( C > 0 \) such that

\[
F(x, t) \geq Ct^p.
\]

(5.17)

So there exists \( C_1 > 0 \), such that

\[
I(tu) = \int_{\Omega} P(|\nabla u|) \, dx - \int_{\Omega} F(x, tu) \, dx \\
\leq t^p C \int_{\Omega} P(|\nabla u|) \, dx - C t^p \int_{\Omega} |u|^p \, dx + C_1.
\]

(5.18)

For some constant \( C \), it is easy to see that

\[
\Xi := \{ u : P(|\nabla u|) < C \int_{\Omega} |u|^p \, dx \} \neq \emptyset.
\]

(5.19)

We can choose \( \phi \in \Xi \), and let \( t \to +\infty \), we have

\[
I(t\phi) \to -\infty.
\]

(5.20)

**Step3:** There exits a pair of strict sub-supersolutions for problem (P).

Problem (P) has a strict supersolution \( v = 0 \), and a strict subsolution \( u \):

\[
\begin{cases}
-\text{div}(a(|\nabla u|) \nabla u) = -k(x), & \text{in } \Omega \\
u = 0, & \text{on } \partial\Omega.
\end{cases}
\]

(5.21)

**Step4:** \( I \) has at least two nontrivial critical points.

As in the process of the proof of Theorem 4.1, then \( I \) has a minimizer \( u_0 \) in the \( W^{1,P}_0 \) topology. Since \( I \) is unbounded below, there exists \( \phi \in W^{1,P}_0(\Omega) \) such that \( I(\phi) < I(u_0) \).

Using mountain pass theorem we know that \( I \) has another critical point \( u_1 \).
The proof is complete. □

Before the statement of the Theorem 5.2 and Theorem 5.3, we mention the results about the eigenvalues of the operator $-\Delta_P$(see [19]). In [19], M. García-Huidobro and his coauthors have proved that the principal eigenvalue of the operator $-\Delta_P$ is positive. Similar to $p$–laplacian case, one can obtain an unbounded sequence of minimax eigenvalues \{\lambda_{i+1}\}_{i \in \mathbb{N}} of the operator $-\Delta_P$ by the Lusternik-Schnirlaman theory(see[5]) or Yang index theory(see[28]).

**Theorem 5.2.** Assume that the conditions $(f_1)$, $(f_2)$, $(f_3)$ and $(f_4)$: $f(x,0)=0$ and there exist $t_0>0$ and $\lambda \in (0,\lambda_1)$ such that

$$F(x,t) < \lambda_1 P(t), \text{ for } x \in \Omega, |t| < t_0,$$

are satisfied. Then problem (P) has at least one nontrivial solution.

**Proof.** In the proof of Theorem 5.1, we have proved the functional satisfies (PS) condition. So we can compute $C_q(I, \infty)$.

Let $S = \{u \in W_{0}^{1,p}(\Omega) : \|u\| = 1\}$. By the condition $(f_2)$ it is easy to see that for any $u \in S$, we have

$$I(tu) = \int_{\Omega} P(|\nabla tu|)dx - \int_{\Omega} F(x, tu)dx$$

$$\leq t^{p^*} \int_{\Omega} P(|\nabla u|)dx - Ct^{p^*} \int_{\Omega} |u|^{p^*} dx + C_1$$

$$\leq t^{p^*} - Ct^{p^*} \int_{\Omega} |u|^{p^*} dx + C_1 \to -\infty \text{ as } t \to +\infty. \quad (5.22)$$

Choose

$$a < \min \\left\{ \inf_{\|u\| \leq 1} I(u), 0 \right\}. \quad (5.23)$$

Then for any $u \in S$, there exists $t_0 > 1$ such that $I(t_0u) \leq a$. By $(f_3)$, we have

$$\mathcal{F}_{p^*}(x,z) \geq 0, \quad \text{for } (x,z) \in \Omega \times \mathbb{R}. \quad (5.24)$$

Therefore, if

$$I(tu) = \int_{\Omega} P(|\nabla tu|)dx - \int_{\Omega} F(x, tu)dx \leq a, \quad (5.25)$$

using (5.24) we get

$$\frac{d}{dt}I(tu) = \int_{\Omega} p(|\nabla tu|)|\nabla u|dx - \int_{\Omega} uf(x, tu)dx$$

$$= \frac{1}{t} \left\{ \int_{\Omega} p(|\nabla tu|)|\nabla u|dx - \int_{\Omega} tu f(x, tu)dx \right\}$$

$$\leq \frac{1}{t} \left\{ p^+ \int_{\Omega} P(|\nabla u|)dx - \int_{\Omega} tu f(x, tu)dx \right\}$$
\[
\leq \frac{1}{t} \left\{ p^+ a + \int_{\Omega} p^+ F(x, tu) dx - \int_{\Omega} tu f(x, tu) dx \right\} \\
\leq \frac{1}{t} \left\{ p^+ a - \int_{\Omega} \mathcal{F}_p(x, tu) dx \right\} \\
\leq \frac{1}{t} p^+ a < 0.
\] (5.26)

Therefore, by the Implicit Function Theorem, there exists a unique \( T \in C(S, R) \) such that \( I(T(u)u) = a \).

Using the function \( T \), we can follow the argument in [34] to construct a strong deformation retract from \( W_0^{1,p}(\Omega) \) to \( I^a \), and deduce

\[ C_q(I, \infty) = H_q(W_0^{1,p}(\Omega), I^a) \equiv H_q(W_0^{1,p}(\Omega), W_0^{1,p}(\Omega) \setminus \{0\}) \equiv 0. \]

By the theorem \((f_4)\), the function \( 0 \) is a local minimizer of \( I \). Hence, \( C_q(I, u_0) = \delta_{q,0} Z \). Then using Proposition5.2 problem (P) has at least one nontrivial solutions in \( W_0^{1,p}(\Omega) \).

In Theorem5.1 and Theorem5.2, the conditions \((f_2)\) and \((f_3)\) mean that the nonlinearity is superlinear at infinity. It is natural to consider the dual case. In our last result we consider the case that the nonlinearity is sublinear at infinity and superlinear at zero. We assume the following conditions on \( f(x, t) \).

\((f'_1)\): \( f(x, 0) \equiv 0 \).

\((f'_2)\): \( f(x, t) \) is superlinear at zero, that is, the following limit holds uniformly on \( x \in \bar{\Omega} \),

\[ \lim_{|t| \to 0} \frac{f(x, t)}{|t|^{p^+ - 2} t} = +\infty. \]

\((f'_3)\): \( p^- F(x, t) - f(x, t)t > 0, \ \forall x \in \bar{\Omega} \) and \( t \neq 0 \).

\((f'_4)\): \( \lim_{|t| \to +\infty} \frac{f(x, t)}{p(t)} < \lambda_1 \) uniformly in \( x \in \bar{\Omega} \).

**Theorem 5.3.** Assume that the conditions \((f'_1)\), \((f'_2)\) \((f'_3)\) and \((f'_4)\) are satisfied. Then problem (P) has at least three nontrivial solutions.

**Proof.** Below we will take three steps to prove Theorem5.3.

**Step1:** The functional satisfies (PS) condition.

By the condition \((f'_1)\), we easily know that the functional \( I \) is coercive. Hence \( I \) satisfies (PS) condition.

**Step2:** \( C_q(I, 0) = 0 \).

For \( u \in W_0^{1,p}(\Omega) \setminus \{0\} \), we have

\[ \lim_{|s| \to 0} \frac{I(su)}{|s|^{p^+}} = -\infty. \] (5.27)
In fact, for any \( s_n \subset \mathbb{R} \) with \( s_n \to 0 \), let \( v_n = s_n u \). Then

\[
v_n \to 0 \quad \text{in} \quad W_0^{1,p}(\Omega), \quad v_n \to 0 \quad \text{a.e.} \quad x \in \Omega.
\]

(5.28)

By the condition \( (f'_2) \) and Fatou Lemmaw have

\[
\liminf_{n \to \infty} \int_{\Omega} \frac{F(x, v_n)}{|v_n|^p} |u|^p \, dx \geq \int_{\Omega} \liminf_{n \to \infty} \frac{F(x, v_n)}{|v_n|^p} |u|^p \, dx = +\infty.
\]

(5.29)

Thus

\[
\frac{I(s_n u)}{|s_n|^p} = \frac{\int_{\Omega} P(|\nabla s_n u|) \, dx - \int_{\Omega} F(x, s_n u) \, dx}{|s_n|^p} \\
\leq \frac{|s_n|^p \int_{\Omega} P(|\nabla u|) \, dx - \int_{\Omega} F(x, s_n u) \, dx}{|s_n|^p} \\
\leq \int_{\Omega} P(|\nabla u|) \, dx - \int_{\Omega} \frac{F(x, v_n)}{|v_n|^p} |u|^p \, dx \to -\infty,
\]

(5.30)

and (5.29) follows.

Using (5.29), we see that for any \( u \in W_0^{1,p}(\Omega) \setminus \{0\} \), there exists \( s_0 \in (0, 1) \) such that

\[
I(su) < 0, \quad \text{for} \quad s \in (0, s_0).
\]

(5.31)

Assume that \( I(u) \geq 0 \), that is

\[
\int_{\Omega} P(|\nabla u|) \, dx \geq \int_{\Omega} F(x, u) \, dx.
\]

(5.32)

Then according \( (f'_2) \), for \( u \in I^{-1}[0, +\infty) \setminus \{0\} \) we obtain

\[
\left. \frac{d}{ds} \right|_{s=1} I(su) = \left. \frac{d}{ds} \right|_{s=1} \left( \int_{\Omega} P(|\nabla su|) \, dx - \int_{\Omega} F(x, su) \, dx \right) \\
= \int_{\Omega} p(|\nabla u|) \nabla u |u| \, dx - \int_{\Omega} uf(x, u) \, dx \\
\geq \int_{\Omega} p^{-} P(|\nabla u|) \, dx - \int_{\Omega} uf(x, u) \, dx \\
\geq \int_{\Omega} (p^{-} F(x, u) - uf(x, u)) \, dx > 0.
\]

(5.33)

Now we adjust the argument in the proof of [21, Theorem 2.1] or [24, Proposition 2.1] slightly. From (5.31) and (5.33), we see that for any \( u \in I^{-1}[0, +\infty) \setminus \{0\} \), there exists a unique \( T = T(u) > 0 \) such that \( I(Tu) = 0 \). Moreover, since \( I(Tu) = 0 \), from (5.33) we deduce

\[
\left. \frac{d}{dt} \right|_{t=T} I(tu) = \frac{1}{T} \left. \frac{d}{ds} \right|_{s=1} I(sTu) > 0.
\]

(5.34)

Thus, by the Implicit Function Theorem we see that \( T \) is continuous on \( I^{-1}[0, +\infty) \setminus \{0\} \). If \( I(u) \leq 0 \), we set \( T(u) = 1 \). Then \( T : W_0^{1,p}(\Omega) \to \mathbb{R} \) is continuous.
We now define \( \eta : [0, 1] \times W^{1,p}_0(\Omega) \to W^{1,p}_0(\Omega) \) by

\[
\eta(s, u) = (1 - s)u + sT(u)u, \quad (s, u) \in [0, 1] \times W^{1,p}_0(\Omega).
\]  

(5.35)

It is easy to see that \( \eta \) is a continuous deformation from \( (W^{1,p}_0(\Omega), W^{1,p}_0(\Omega)\setminus\{0\}) \) to \( (I^0, I^0\setminus\{0\}) \), hence

\[
C_k(I, 0) = H_k(I^0, \{0\}) \cong H_k(W^{1,p}_0(\Omega), W^{1,p}_0(\Omega)\setminus\{0\}) = 0, \quad k \in \mathbb{N}.
\]  

(5.36)

**Step3:** \( I \) has at least three nontrivial critical points.

By the condition \((f'_q)\), there exist constants \( c \in (0, \lambda_1) \) and \( d > 0 \) such that

\[
f(x, s) \leq ca(|s|)s + d \quad \text{uniformly in } x \in \Omega \text{ and } s > 0,
\]  

(5.37)

and

\[
f(x, s) \geq -ca(|s|)s - d \quad \text{uniformly in } x \in \Omega \text{ and } s < 0.
\]  

(5.38)

We consider the following problem

\[
\begin{cases}
-\div(a(|\nabla u|)\nabla u) = ca(|u|)u + d, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
\]  

(5.39)

Similar to \( p \)-laplacian case, by Riesz representation theorem, the above problem has positive solution \( v \). Lemma4.2 implies that \( v > 0 \) on \( \Omega \) and hence \( v \) is a supersolution of problem (P). Obviously the function \( 0 \) is a subsolution of problem (P). As in the proof of Theorem4.1, then \( I \) has a local minimizer \( u^+ > 0 \) in the \( W^{1,p}_0 \) topology and \( I(u^+) < 0 \).

Similarly, we can obtain another local minimizer \( u^- < 0 \) in the \( W^{1,p}_0 \) topology and \( I(u^-) < 0 \).

Hence the critical groups of \( I \) at its local minimizer are

\[
C_q(I, u^+) = C_q(I, u^-) = \delta_{q0}Z, \quad \forall q \in \mathbb{N}.
\]  

(5.40)

Now if we choose \( a, b \) satisfying

\[
a < \inf I(u) \leq \min\{I(u^+), I(u^-)\} \leq \max\{I(u^+), I(u^-)\} \leq b.
\]  

(5.41)

Assume that \( I \) has only two critical points, that is, \( K^b_a = \{u^+, u^-\} \). Then we have

\[
C_q(I, \infty) = H_q(I^b, I^a) \cong \delta_{q0}Z \oplus Z, \quad \forall q \in \mathbb{N},
\]  

(5.42)

and

\[
H_q(I^b, I^a) \cong H_q(I^b, \emptyset) \cong H_q(I^b), \quad \forall q \in \mathbb{N}.
\]  

(5.43)

On the other hand, since \( I \) is bounded below and satisfies the (PS) condition, we have
\(C_q(I, \infty) = H_q(W_0^{1,p}) = \delta_q Z.\) Therefore it follows immediately from the Morse inequalities and (5.40) that there must be one more critical point \(u \neq 0.\)

The proof is completed. \(\square\)

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