Generalized pointwise Hölder spaces

Damien Kreit∗and Samuel Nicolay†‡

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Abstract
In this paper, we introduce a generalization of the pointwise Hölder spaces. We give alternative definitions of these spaces, look at their relationship with the wavelets and introduce a notion of generalized Hölder exponent.

1 Introduction
In [18, 17], the properties of generalized uniform Hölder spaces have been investigated. The idea underlying the definition is to replace the exponent α of the usual spaces Λα(Rd) (see e.g. [16]) with a sequence σ satisfying some conditions. The so-obtained spaces Λσ(Rd) generalize the spaces Λα(Rd); the spaces Λσ(Rd) are actually the spaces B1/∞,∞(Rd), but they present specific properties (induced by L∞-norms) when compared to the more general spaces B1/σ,p,q(Rd) studied in [9, 11, 10, 19, 22] for example. Indeed it is shown in [18, 17] that most of the usual properties holding for the spaces Λα(Rd) can be transposed to the spaces Λσ(Rd).

Here, we introduce the pointwise version of these spaces: the spaces Λσ,M(x0), with x0 ∈ Rd. Let us recall that a function f ∈ L∞,loc(Rd) belongs to the usual pointwise Hölder space Λα(x0) (α > 0) if and only if there exist C, J > 0 and a polynomial P of degree at most α such that

\[ \sup_{|h| \leq 2^{-j}} |f(x_0 + h) - P(h)| \leq C 2^{-j \alpha}. \]  

∗Université de Liège, Institut de Mathématique, Grande Traverse, 12, Bâtiment B37, B-4000 Liège (Sart-Tilman), Belgium.
†Université de Liège, Institut de Mathématique, Grande Traverse, 12, Bâtiment B37, B-4000 Liège (Sart-Tilman), Belgium.
‡Corresponding author. Email: S.Nicolay@ulg.ac.be. Phone: +32(0)43669433. Fax: +32(0)43669547.
As in [18, 17], the idea is again to replace the sequence \((2^{-j\alpha})_j\) appearing in this inequality with a positive sequence \((\sigma_j)_j\) such that \(\sigma_{j+1}/\sigma_j\) and \(\sigma_j/\sigma_{j+1}\) are bounded (for any \(j\)); the number \(M\) stands for the maximal degree of the polynomial (this degree can not be induced by a sequence \(\sigma\)). By doing so, one tries to get a better characterization of the regularity of the studied function \(f\); a usual choice is to replace \(2^{-j\alpha}\) with \(j2^{-j\alpha}\) (see e.g. [14, 13, 4]). Generalizations of the pointwise H"older spaces have already been proposed (see e.g. [4]), but, to our knowledge, the definition we give here is the most general version and leads to the sharpest results.

This work is organized as follows. We first give the definitions leading to generalized pointwise Hölder spaces \(\Lambda^{\sigma,M}(x_0)\) and prove that, under some general conditions, the polynomials appearing in the definition are independent from the scale, as it is the case with the usual Hölder spaces. Next we give some alternative definitions of the spaces \(\Lambda^{\sigma,M}(x_0)\), mimicking the different possible definitions of \(\Lambda^{\alpha}(x_0)\). One of the nicest properties of the Hölder spaces is their relationship with the wavelet theory given in [13]; we show here that this result still holds in the general case. Finally, we give some conditions under which one gets embedded generalized pointwise Hölder spaces and define a generalized Hölder exponent.

Throughout this paper, \(B\) denotes the open unit ball centered at the origin; moreover we set \(B_j = 2^{-j}B\). The floor function is denoted \([\cdot]\) and \(\mathbb{P}[\alpha]\) designates the set of polynomials of degree at most \([\alpha]\). We use the letter \(C\) for generic positive constant whose value may be different at each occurrence.

2 Pointwise generalized Hölder spaces

To present the generalized pointwise Hölder spaces, we first need to recall some notions concerning the admissible sequences. After having introduced the definitions, we point out a major difference between the usual spaces and the generalized ones: the polynomial arising in the definition depends on the scale. It is then natural to look under which condition this constraint can be dropped.

2.1 Definition

The generalization of the Hölder spaces we propose here is based on the notion of admissible sequence [22].

**Definition 1.** A sequence \(\sigma = (\sigma_j)_{j \in \mathbb{N}}\) of real positive numbers is called admissible if and only if there exists a positive constant \(C\) such that

\[
C^{-1}\sigma_j \leq \sigma_{j+1} \leq C\sigma_j,
\]

for any \(j \in \mathbb{N}\).

If \(\sigma\) is such a sequence, we set

\[
\Theta_j = \inf_{k \in \mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k} \quad \text{and} \quad \overline{\Theta}_j = \sup_{k \in \mathbb{N}} \frac{\sigma_{j+k}}{\sigma_k}
\]
and define the lower and upper Boyd indices as follows,

$$s(\sigma) = \lim_{j} \frac{\log_2 \Theta_j}{j} \quad \text{and} \quad \overline{s}(\sigma) = \lim_{j} \frac{\log_2 \overline{\Theta}_j}{j},$$

Since \((\log \Theta_j)_{j \in \mathbb{N}}\) is a subadditive sequence, such limits always exist \[11\]. In this paper, \(\sigma\) will always stand for an admissible sequence and \(M\) for a natural number, possibly zero.

Starting from the definitions of the pointwise H"older spaces \(\Lambda^\alpha(x_0)\) (with \(\alpha > 0\)) and the generalized uniform H"older spaces \(\Lambda^\sigma(\mathbb{R}^d)\) introduced in \[18\], we are naturally led to the following definition.

**Definition 2.** Let \(x_0 \in \mathbb{R}^d\); a continuous function \(f \in L^\infty_{\text{loc}}(\mathbb{R}^d)\) belongs to \(\Lambda^\sigma,M(x_0)\) if there exist \(C,J > 0\) such that

$$\inf_{P \in \mathbb{P}[M]} \|f - P\|_{L^\infty(x_0 + B_j)} \leq C\sigma_j,$$

for any \(j \geq J\).

We trivially have the following alternative definition for \(\Lambda^\sigma,M(x_0)\).

**Definition 3.** A function \(f \in L^\infty_{\text{loc}}(\mathbb{R}^d)\) belongs to \(\Lambda^\sigma,M(x_0)\) if and only if there exist \(C,J > 0\) such that, for any \(j \geq J\), there exists a polynomial \(P_j \in \mathbb{P}[M]\) for which

$$\sup_{h \in B_j} |f(x_0 + h) - P_j(x_0 + h)| \leq C\sigma_j. \quad (2)$$

Sometimes, we will also need to impose a slightly stronger condition than continuity to a function.

**Definition 4.** A function \(f\) is uniformly H"older if and only if there exists \(\varepsilon > 0\) such that \(f \in \Lambda^\varepsilon(\mathbb{R}^d)\) (here a function belonging to \(\Lambda^\varepsilon(\mathbb{R}^d)\) is necessarily continuous).

### 2.2 Independence of the polynomial from the scale

It is important to remark that the polynomial occurring in inequality \((2)\) is a function of the scale \(j\). However, for the classical H"older spaces, such polynomial is independent of \(j\). Here, we look under which conditions the independence still holds in the generalized case, i.e. under which conditions \(P_j = P \in \mathbb{P}[M]\) for any \(j \geq J\).

We will need the following Markov inequality (see e.g. \[8\]): Let \(p \in (0, \infty]\), \(k \in \{1, \ldots, d\}\) and \(S \subset \mathbb{R}^d\) be a bounded convex set with non-empty interior; one has

$$\|D_kP\|_{L^p(S)} \leq Cn^2\|P\|_{L^p(S)},$$

for any \(P \in \mathbb{P}[n-1]\), where \(C\) is a function of \(S\) (but is independent of \(P\) and \(n\)). If \(x_0 \in \mathbb{R}^d\), we thus have

$$\|D_kP\|_{L^\infty(x_0 + rB)} \leq \frac{Cr^2}{p}\|P\|_{L^\infty(x_0 + rB)}, \quad (3)$$
for any $r > 0$ and any $P \in P[n-1]$, where $C$ is a constant (and does not depend on $P$, $n$ or $r$).

**Lemma 1.** If $f \in \Lambda^{\sigma,M}(x_0)$ with $M < s(\sigma^{-1})$, the sequence of polynomials $(P_j)_j$ occurring in (2) satisfies

$$\|D^\beta P_k - D^\beta P_j\|_{L^\infty(x_0+R_k)} \leq C |\beta| \sigma_j,$$

for any multi-index $\beta$ such that $|\beta| \leq M$ and $k \geq j \geq J$.

In particular, $(D^\beta P_j(x_0))_j$ is a Cauchy sequence for any multi-index $\beta$ such that $|\beta| \leq M$.

**Proof.** Using the Markov inequality, we get

$$\|D^\beta P_j - D^\beta P_{j+1}\|_{L^\infty(x_0+R_{j+1})} \leq C |\beta| \|P_j - P_{j+1}\|_{L^\infty(x_0+R_{j+1})} \leq C |\beta| \|P_j - f\|_{L^\infty(x_0+R_{j+1})},$$

for any $\beta$ such that $|\beta| \leq M$. Therefore, if $k$ satisfies $k \geq j \geq J$, one gets

$$\|D^\beta (P_j - P_k)\|_{L^\infty(x_0+R_k)} \leq \sum_{l=j}^{k-1} \|D^\beta (P_l - P_{l+1})\|_{L^\infty(x_0+R_l)} \leq C \sum_{l=j}^{k-1} |\beta| \sigma_l \leq C 2^{|\beta|} \sigma_j,$$

which is the desired result.

**Lemma 2.** If $f \in \Lambda^{\sigma,M}(x_0)$ with $M < s(\sigma^{-1})$ and $(P_j)_j$ is a sequence of polynomials satisfying inequality (2), for any multi-index $\beta$ such that $|\beta| \leq M$, the limit

$$f_\beta(x_0) = \lim_{j \to \infty} D^\beta P_j(x_0)$$

is independent of the chosen sequence $(P_j)_j$.

**Proof.** If $(Q_j)_j$ is another sequence of polynomials satisfying inequality (2), one gets

$$|D^\beta Q_j(x_0) - f_\beta(x_0)| \leq |D^\beta Q_j(x_0) - D^\beta P_j(x_0)| + |D^\beta P_j(x_0) - f_\beta(x_0)|.$$
Since one has, using the Markov inequality,
\[ \|D^\beta(P_j - Q_j)\|_{L^\infty(x_0+B_j)} \leq C 2^{j|\beta|} \|P_j - Q_j\|_{L^\infty(x_0+B_j)} \]
\[ \leq C 2^{j|\beta|}(\|P_j - f\|_{L^\infty(x_0+B_j)} + \|f - Q_j\|_{L^\infty(x_0+B_j)}) \]
\[ \leq C 2^{j|\beta|}\sigma_j \to 0, \]
as \( j \to \infty \), one can conclude.

For such functions, we can introduce the notion of Peano derivative; this definition is similar to the ones given in [7, 4].

**Definition 5.** Under the hypothesis of lemma 1, the \( \beta \)-th Peano derivative of \( f \) at \( x_0 \) is \( f_\beta(x_0) = \lim_{j \to \infty} D^\beta P_j(x_0) \).

We can now obtain the result concerning the independence of the polynomials.

**Theorem 3.** If \( M < 2(\sigma^{-1}) \), then \( f \in \Lambda^\sigma,M(x_0) \) if and only if there exist \( C > 0 \) and a unique polynomial \( P \in \mathbf{P}[M] \) such that
\[ \|f - P\|_{L^\infty(x_0+B_j)} \leq C \sigma_j, \quad (5) \]
for \( j \) sufficiently large.

**Proof.** Let \((P_j)_j\) be a sequence of polynomials for which inequality (2) is satisfied and set
\[ P(x) = \sum_{|\beta| \leq M} f_\beta(x_0) \frac{(x - x_0)^\beta}{|\beta|!}. \]

One has
\[ \|P - P_j\|_{L^\infty(x_0+B_j)} \]
\[ = \left\| \sum_{|\beta| \leq M} (f_\beta(x_0) - D^\beta P_j(x_0)) \frac{(x - x_0)^\beta}{|\beta|!} \right\|_{L^\infty(x_0+B_j)} \]
\[ \leq \sum_{|\beta| \leq M} |f_\beta(x_0) - D^\beta P_j(x_0)| 2^{-j|\beta|}. \]

Since lemma [1] implies
\[ |f_\beta(x_0) - D^\beta P_j(x_0)| \leq C 2^{j|\beta|}\sigma_j, \]
for \( j \) sufficiently large, we have
\[ \|P - P_j\|_{L^\infty(x_0+B_j)} \leq C \sigma_j. \]

This inequality can be used to obtain
\[ \|f - P\|_{L^\infty(x_0+B_j)} \leq \|f - P_j\|_{L^\infty(x_0+B_j)} + \|P_j - P\|_{L^\infty(x_0+B_j)} \leq C \sigma_j, \]
for \( j \) sufficiently large.
which shows the existence of $P$.

If two polynomials $P, Q \in P[M]$ satisfy inequality (5),
\[ \| P - Q \|_{L^\infty(x_0+B_j)} \leq \| P - f \|_{L^\infty(x_0+B_j)} + \| f - Q \|_{L^\infty(x_0+B_j)} \leq C\sigma_j, \]
but if $P \neq Q$,
\[ \| P - Q \|_{L^\infty(x_0+B_j)} \geq C2^{-jM}, \]
for $j$ sufficiently large, so that $2^{jM}\sigma_j$ does not tend to zero.

The polynomial $P$ in inequality (5) is the Taylor expansion of $f$, where the derivative is replaced by the Peano derivative.

Let $\alpha \in (0, \infty)$; the sequence $\sigma = (2^{-j\alpha}) \in \mathbb{N}$ is an admissible sequence with $s(\sigma) = \sigma$, $s(\sigma^{-1}) = \alpha$ and $\Lambda^\alpha(x_0) = \Lambda^{s(\sigma^{-1})}(x_0) = \Lambda^{s(\sigma^{-1})}\sigma_j(x_0)$. The definition given by (1) is very often slightly changed (we will use such a modified version in the sequel). It is easy to check that both definitions lead to the same spaces.

Remark 1. It is easy to check that the polynomial satisfying equation (1) is unique if and only if $\alpha \notin \mathbb{N}$. If $\alpha \in \mathbb{N}$, one rather imposes $P \in P[\alpha-1]$ in order to obtain the uniqueness of the polynomial, so that $\Lambda^\alpha(x_0) = \Lambda^{s(\sigma^{-1})-1}(x_0)$, with $\sigma_j = 2^{-j\alpha}$.

The following proposition rigorously expresses the idea that the space $\Lambda^{s,M}(x_0)$ associated to a sequence $(\sigma_j)_j$ that decreases faster than $2^{-jM}$ is included in the usual Hölder space $\Lambda^M(x_0)$.

**Corollary 4.** If $s(\sigma^{-1}) > M$, one has $\Lambda^{s,M}(x_0) \subset \Lambda^M(x_0)$.

**Proof.** Let $f \in \Lambda^{s,M}(x_0)$, $P$ be defined as in theorem 3, i.e.
\[ P(x) = \sum_{|\beta| \leq M} f_\beta(x_0) \frac{(x-x_0)^\beta}{|\beta|!}, \]
and let us set
\[ Q(x) = \sum_{|\beta| \leq M-1} f_\beta(x_0) \frac{(x-x_0)^\beta}{|\beta|!}. \]

One gets
\[ \| f - Q \|_{L^\infty(x_0+B_j)} \leq \| f - P \|_{L^\infty(x_0+B_j)} + \| P - Q \|_{L^\infty(x_0+B_j)} \leq C\sigma_j + C2^{-jM} \leq C2^{-jM}, \]
since $2^{jM}\sigma_j$ tends to zero.

**3 Alternative definitions of generalized Hölder spaces**

Since the uniform spaces $\Lambda^\alpha(\mathbb{R}^d)$ can be defined via finite differences or convolutions, one can wonder if such characterizations also hold for the pointwise version of these spaces.
3.1 Characterization in terms of finite differences

As usual, $\Delta^n f$ will stand for the finite difference of order $n$: given a function $f$ defined on $\mathbb{R}^d$ and $x, h \in \mathbb{R}^d$,

$$\Delta^1_h f(x) = f(x + h) - f(x) \quad \text{and} \quad \Delta^{n+1}_h f(x) = \Delta^1_h \Delta^n_h f(x),$$

for any $n \in \mathbb{N}$. We also set

$$B^M_h(x_0,j) = \{x : [x, x + (M + 1)h] \subset x_0 + B_j\}$$

In order to obtain a more general result, we drop the continuity condition of definition 2 in this section.

**Proposition 5.** Let $f \in L_\text{loc}^\infty(\mathbb{R}^d)$; one has $f \in \Lambda^{\sigma,M}(x_0)$ if and only if there exist $C, J > 0$ such that

$$\sup_{h \in B_j} \|\Delta^{M+1}_h f\|_{L^\infty(B^M_h(x_0,j))} \leq C\sigma_j, \quad (6)$$

for any $j \geq J$.

**Proof.** The theorem of Whitney (see e.g. [2]) directly implies that if $f$ satisfies inequality (6), then $f \in \Lambda^{\sigma,M}(x_0)$: One has

$$\inf_{P \in \mathcal{P}[M]} \|f - P\|_{L^\infty(x_0 + B_j)} \leq C \sup_{h \in B_j} \|\Delta^{M+1}_h f\|_{L^\infty(B^M_h(x_0,j))}.$$  

Let us now suppose that $f \in \Lambda^{\sigma,M}(x_0)$ and let $x \in B^M_h(x_0,j)$, $P \in \mathcal{P}[M]$. One has

$$\|\Delta^{M+1}_h f\|_{L^\infty(B^M_h(x_0,j))} = \|\Delta^{M+1}_h (f - P)\|_{L^\infty(B^M_h(x_0,j))} \leq (M + 1)!(M + 2)\|f - P\|_{L^\infty(x_0 + B_j)}.$$  

Now, since there exists a polynomial $P_j \in \mathcal{P}[M]$ such that

$$\|f - P_j\|_{L^\infty(x_0 + B_j)} \leq C\sigma_j,$$

for $j$ sufficiently large, one gets

$$\sup_{h \in B_j} \|\Delta^{M+1}_h f\|_{L^\infty(B^M_h(x_0,j))} \leq C\sigma_j,$$

for $j$ sufficiently large. \hfill $\Box$

3.2 Characterization in terms of convolutions

Let us denote the space of the infinitely differentiable functions with compact support included in $E$ by $C_\infty^\infty(E)$. In this section, $\rho$ will denote a radial function of $C_\infty^\infty(B)$ such that $\rho(x) \in [0, 1]$ for any $x \in \mathbb{R}^d$ and $\|\rho\|_1 = 1$. Moreover, one sets $\rho_j = 2^{-jd}\rho(\cdot/2^j)$, for any $j \in \mathbb{N}$.

In [15], the following result has been obtained:
Lemma 6. Let \( N \in \mathbb{N}_0 \); if \( f \in L^1_{\text{loc}}(\mathbb{R}^d) \) satisfies
\[
\sup_{k \geq j} \|f * \rho_k - f\|_{L^\infty(x_0 + B_j)} \leq C \sigma_j,
\]
for \( j \geq J \), then, for any multi-index \( \beta \) such that \( |\beta| \leq N \), one has
\[
\|D^\beta(f * \rho_j - f * \rho_{j-1})\|_{L^\infty(x_0 + B_j)} \leq C 2^{jN} \sigma_j,
\]
for \( j \geq J \).

Using the same ideas as in [18], one gets the desired characterization.

Theorem 7. If \( f \in \Lambda^{\sigma,M}(x_0) \), then there exists a function \( \Phi \in C^\infty_c(\mathbb{R}^d) \) such that
\[
\sup_{k \geq j} \|f - f * \Phi_k\|_{L^\infty(x_0 + B_j)} \leq C \sigma_j,
\]
for \( j \) sufficiently large.

Conversely, if \( \sigma \to 0 \), \( f \) is uniformly Hölder and if \( f \) satisfies inequality (7) for a function \( \Phi \in C^\infty_c(\mathbb{R}^d) \), then \( f \in \Lambda^{\sigma,M}(x_0) \) for any \( M \in \mathbb{N}_0 \) such that \( M + 1 > \Omega(\sigma^{-1}) \).

Proof. As in [18] (see also [16]), let us set
\[
\Psi(x) = \sum_{j=0}^{m/2-1} (-1)^j \frac{m!}{j!(m-j)!} \frac{1}{2j-m} \rho(\frac{x}{2j-m}),
\]
where \( m \) is large enough (larger than \( M + 1 \)) and \( \Phi = \Psi / \int \Psi dx \). Using the same arguments as in [18], one gets
\[
f * \Phi_k(x) - f(x) = C \int \Delta^{m/2-1}_{2^{-k+1}} f(x) \rho(t) dt,
\]
which leads to inequality (7).

Let us show the converse. Let \( \alpha \in (0,1) \) such that \( f \in \Lambda^{\alpha}(\mathbb{R}^d) \) and set, as in [18],
\[
f_1 = f * \Phi_1 \quad \text{and} \quad f_j = f * (\Phi_j - \Phi_{j-1}),
\]
for \( j > 1 \). Since \( f \) is uniformly Hölder, \( f \) is uniformly equal to \( \sum_{j \geq 1} f_j \) on \( \mathbb{R}^d \) and
\[
\Delta^{M+1}_h f = \sum_{j \geq 1} \Delta^{M+1}_h f_j
\]
uniformly on \( \mathbb{R}^d \), for any \( h \in \mathbb{R}^d \). For \( j \in \mathbb{N} \), let \( n_0 \in \mathbb{N}_0, h \in \mathbb{R}^d \) and \( j_0 \in \mathbb{N}_0 \) be such that \( M + 1 < 2^{n_0}, |h| \leq 2^{-(j+n_0)} \) and \( 2^{-(j+1)\alpha} \leq \sigma_j \leq 2^{-j_0 \alpha} \). One has
\[
\|\Delta^{M+1}_h f\|_{L^\infty(x_0 + B_j)} \leq \sum_{k=1}^{j_0-1} \|\Delta^{M+1}_h f_k\|_{L^\infty(x_0 + B_j)} + \sum_{k=j_0}^j \|\Delta^{M+1}_h f_k\|_{L^\infty(x_0 + B_j)} + \sum_{k \geq j_0 + 1} \|\Delta^{M+1}_h f_k\|_{L^\infty(x_0 + B_j)};
\]
where the second term in the majoration only appears if \( j \leq j_0 \).

Using lemma 6 and the fact that \( M + 1 > \pi(\sigma^{-1}) \), the mean value theorem allows to write

\[
\sum_{k=1}^{j-1} \| \Delta_h^{M+1} f \|_{L^\infty(x_0+B_j)} \leq \sum_{k=1}^{j-1} C|h|^{M+1} \sup_{|\beta| = M+1} \| D^\beta f_k \|_{L^\infty(x_0+B_{j-1})}
\]

\[
\leq C 2^{-(M+1)} \sum_{k=1}^{j-1} 2^k \| f_k \|_{L^\infty(x_0+B_{j-1})} \sigma_k \leq C \sigma_j.
\]

Moreover,

\[
\sum_{k=j_0}^{j} \| \Delta_h^{M+1} f_k \|_{L^\infty(x_0+B_j)} = \| \Delta_h^{M+1} (f \ast \Phi_{j_0} - f \ast \Phi_{j-1}) \|_{L^\infty(x_0+B_j)}
\]

\[
\leq C \| f \ast \Phi_{j_0} - f \ast \Phi_{j-1} \|_{L^\infty(x_0+B_{j-1})}
\]

\[
\leq C \| f \ast \Phi_{j_0} - f \|_{L^\infty(x_0+B_{j-1})} + \| f \| - f \ast \Phi_{j-1} \|_{L^\infty(x_0+B_{j-1})})
\]

\[
\leq C 2^{-j(\alpha+1)} + C \sigma_{j-1} \leq C \sigma_j.
\]

Finally,

\[
\sum_{k \geq j_0+1} \| \Delta_h^{M+1} f_k \|_{L^\infty(x_0+B_j)} \leq C \sum_{k \geq j_0+1} \| f_k \|_{L^\infty(\mathbb{R}^d)}
\]

\[
\leq C \sum_{k \geq j_0+1} 2^{-k\alpha}
\]

\[
\leq 2^{-j_0\alpha} \leq C \sigma_j.
\]

One then has,

\[
\sup_{h \in B_{j+n_0}} \| \Delta_h^{M+1} f \|_{L^\infty(x_0+B_{j+n_0})} \leq \sup_{h \in B_{j+n_0}} \| \Delta_h^{M+1} f \|_{L^\infty(x_0+B_j)} \leq C \sigma_j \leq C \sigma_{j+n_0},
\]

as wanted.

\[\Box\]

4 Generalized pointwise Hölder spaces and wavelets

The usual Hölder spaces can “nearly” be characterized in terms of wavelet [13]: for the sufficiency of the condition, the function has to be uniformly Hölder and a logarithmic correction appears. We show here that such a result still holds in the generalized case.
4.1 Definitions

Let us briefly recall some definitions and notations (for more precisions, see e.g. [6, 21, 20]). Under some general assumptions, there exist a function \( \phi \) and \( 2^d - 1 \) functions \( \{ \psi^{(i)}(2^j \cdot -k) : 1 \leq i < 2^d, k \in \mathbb{Z}^d, j \in \mathbb{N}_0 \} \), called wavelets, such that

\[
\{ \phi(-k) : k \in \mathbb{Z}^d \} \cup \{ \psi^{(i)}(2^j \cdot -k) : 1 \leq i < 2^d, k \in \mathbb{Z}^d, j \in \mathbb{N}_0 \}
\]

form an orthogonal basis of \( L^2(\mathbb{R}^d) \). Any function \( f \in L^2(\mathbb{R}^d) \) can be decomposed as follows,

\[
f(x) = \sum_{k \in \mathbb{Z}^d} C_k \phi(x - k) + \sum_{j=0}^{+\infty} \sum_{k \in \mathbb{Z}^d} \sum_{1 \leq i < 2^d} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),
\]

where

\[
c_{j,k}^{(i)} = 2^{dj} \int_{\mathbb{R}^d} f(x) \psi^{(i)}(2^j x - k) \, dx,
\]

and

\[
C_k = \int_{\mathbb{R}^d} f(x) \phi(x - k) \, dx.
\]

Let us remark that we do not choose the \( L^2(\mathbb{R}^d) \) normalization for the wavelets, but rather a \( L^\infty \) normalization, which is better fitted to the study of the Hölderian regularity. Hereafter, the wavelets are always supposed to belong to \( C^n(\mathbb{R}^d) \) with \( n > M \), and the functions \( (D^s \phi)_{|s| \leq \gamma}, (D^s \psi^{(i)})_{|s| \leq \gamma} \) are assumed to have fast decay.

A dyadic cube of scale \( j \) is a cube of the form

\[
\lambda = \left[ \frac{k_1}{2^j}, \frac{k_1 + 1}{2^j} \right) \times \cdots \times \left[ \frac{k_d}{2^j}, \frac{k_d + 1}{2^j} \right),
\]

where \( k = (k_1, \ldots, k_d) \in \mathbb{Z}^d \). From now on, wavelets and wavelet coefficients will be indexed with dyadic cubes \( \lambda \). Since \( i \) takes \( 2^d - 1 \) values, we can assume that it takes values in \( \{0, 1\}^d - (0, \ldots, 0) \); we will use the following notations:

- \( \lambda = \lambda(i, j, k) = \frac{k}{2^j} + \frac{i}{2^{j+1}} + \left[0, \frac{1}{2^{j+1}}\right)^d \),
- \( c_\lambda = c_{j,k}^{(i)} \),
- \( \psi_\lambda = \psi_{j,k}^{(i)} = \psi^{(i)}(2^j \cdot -k) \).

The pointwise Hölderian regularity of a function is closely related to the decay rate of its wavelet leaders.

**Definition 6.** The wavelet leaders are defined by

\[
d_\lambda = \sup_{\lambda' \subseteq \lambda} |c_{\lambda'}|.
\]
Two dyadic cubes $\lambda$ and $\lambda'$ are adjacent if they are at the same scale and if $\text{dist}(\lambda, \lambda') = 0$. We denote by $3\lambda$ the set of $3^d$ dyadic cubes adjacent to $\lambda$ and by $\lambda_j(x_0)$ the dyadic cube of side of length $2^{-j}$ containing $x_0$; then

$$d_j(x_0) = \sup_{\lambda \subseteq 3\lambda_j(x_0)} d_\lambda.$$ 

### 4.2 Result

From now on, we will suppose that the wavelets are compactly supported; such wavelets are constructed in [5] and $j_0$ will stand for a natural number such that the support of $\psi^{(i)}$ is included in $2^{j_0}B$, for any $i \in \{1, \ldots, 2^d - 1\}$.

**Theorem 8.** If $f \in \Lambda^{\sigma, M}(x_0)$, there exist $C > 0$ and $J \in \mathbb{N}_0$ such that

$$d_j(x_0) \leq C \sigma_j, \quad (8)$$

for any $j \geq J$.

Conversely, let $f$ be an uniformly Hölder function; if inequality (8) is satisfied for an admissible sequence $\sigma$ that tends to zero, then $f \in \Lambda^{\tau, M}(x_0)$, where $\tau$ is the admissible sequence defined by $\tau_j = \sigma_j |\log_2 \sigma_j|$ and $M \in \mathbb{N}_0$ is any number satisfying $M + 1 > \pi(\sigma^{-1})$.

**Proof.** If $f \in \Lambda^{\sigma, M}(x_0)$, let $k_0 \in \mathbb{N}_0$ be such that $2^{j_0+1} + 4d \leq 2^{k_0}$. For $j \geq k_0 + 1$ and $\lambda = \lambda(i, j', k') \subset 3\lambda_j(x_0)$, one has

$$|c_\lambda| = |2^{dj'} \int f(x) \psi_\lambda(x) \, dx|$$

$$= |2^{dj'} \int (f(x) - P_{j-k_0}(x)) \psi_\lambda(x) \, dx|$$

$$= |2^{dj'} \int_{x_0+B_{j-k_0}} (f(x) - P_{j-k_0}(x)) \psi_\lambda(x) \, dx|$$

$$\leq 2^{dj'} \int_{x_0+B_{j-k_0}} |f(x) - P_{j-k_0}(x)| \, |\psi_\lambda(x)| \, dx$$

$$\leq C 2^{dj'} \sigma_j \int |\psi_\lambda(x)| \, dx \leq C \sigma_j,$n

which is the desired result.

Now, let us suppose that inequality (8) is satisfied for a function $f \in \Lambda^{\varepsilon}(\mathbb{R}^d)$. Let us set

$$f_{-1} = \sum_{k} C_k \phi(\cdot - k) \quad \text{and} \quad f_j = \sum_{i, k} c_{i, k} \psi_i \lambda,$n

for $j \in \mathbb{N}_0$. In [17], it has been shown that these functions have the same regularity as the wavelets and that $f$ is uniformly equal to $\sum_{j \geq -1} f_j$. Let us define

$$P_j(x) = \sum_{|\beta| \leq M} \frac{(x - x_0)^\beta}{|\beta|!} \sum_{j=-1}^J D^\beta f_j(x_0).$$
and let us choose \( n_d \in \mathbb{N} \) such that \( R > 2^{-j} \) and \( k/2^j \in x + RB \) \((x \in \mathbb{R}^d)\) implies
\[
\frac{k}{2^j} + \frac{i}{2^{j+1}} + [0, \frac{1}{2}^j)^d \subset x + 2^{n_d}RB.
\]
Let us also choose \( m_d \in \mathbb{N} \) such that any ball \( x + B_j \) \((x \in \mathbb{R}^d, j \in \mathbb{N}_0)\) is included in a dyadic cube of length \( 2^{m_d-j} \). If \( J' \) is such that \( \sigma_j < 1 \) for any \( j \geq J' \), we finally choose \( J \) such that \( J \geq \{ J', j_0 + n_d + m_d + 1 \} \). One has
\[
\|f - P_J\|_{L^\infty(x_0 + B_J)}
\leq \sum_{j=-1}^{J} \|f_j(x) - \sum_{|\beta| \leq M} \frac{(x - x_0)^\beta}{|\beta|!} D^\beta f_j(x_0)\|_{L^\infty(x_0 + B_J)}
+ \sum_{j \geq J+1} \|f_j\|_{L^\infty(x_0 + B_J)}
\]
Let us look at the first term of the majoration. Let \( j \leq J \); using the Taylor expansion, one gets
\[
\|f_j(x) - \sum_{|\beta| \leq M} \frac{(x - x_0)^\beta}{|\beta|!} D^\beta f_j(x_0)\|_{L^\infty(x_0 + B_J)}
\leq C 2^{-J(M+1)} \sup_{|\beta| = M+1} \|D^\beta f_j\|_{L^\infty(x_0 + B_J)}.
\]
If \( \beta \) satisfies \( |\beta| = M + 1 \), we have, for any \( x \in x_0 + B_J \),
\[
|D^\beta f_j(x)| \leq \sum_{i,k} 2^{(M+1)} |c_\lambda| |D^\beta \psi_\lambda(x)|
= \sum_{i,k} 2^{(M+1)} |c_\lambda| |D^\beta \psi_\lambda(x)|.
\]
Each coefficient \( c_\lambda \) in the last sum is such that \( \lambda \subset x + B_{j_0-n_d} \). Therefore, if \( j \geq j_0 + n_d + m_d + 1 \),
\[
|c_\lambda| \leq C \sigma_{j+j_0+n_d+m_d+1}.
\]
Otherwise, since \( f \) is uniformly Hölder, \( |c_\lambda| \leq C \leq C \sigma_j \). Therefore,
\[
\|D^\beta f_j\|_{L^\infty(x_0 + B_J)} \leq C 2^{j(M+1)} \sigma_j,
\]
which implies
\[
\|f_j(x) - \sum_{|\beta| \leq M} \frac{(x - x_0)^\beta}{|\beta|!} D^\beta f_j(x_0)\|_{L^\infty(x_0 + B_J)}
\leq C 2^{-J(M+1)} \sum_{j=-1}^{J} 2^{j(M+1)} \sigma_j \leq C \sigma_J.
\]
For the second term in the majoration, let us define $J_1 \in \mathbb{N}$ as the number such that $2^{-\epsilon J_1} \leq \sigma_J < 2^{-\epsilon (J_1-1)}$ and decompose the sum as follows:

$$\sum_{j \geq J+1} \|f_j\|_{L^\infty(x_0+B_J)} = \sum_{j \geq J_1+1} \|f_j\|_{L^\infty(x_0+B_J)} + \sum_{j=J+1}^{J_1} \|f_j\|_{L^\infty(x_0+B_J)}$$

We have

$$\sum_{j \geq J_1+1} \|f_j\|_{L^\infty(x_0+B_J)} \leq \sum_{j \geq J_1+1} \|f_j\|_{L^\infty(\mathbb{R}^d)} \leq C \sum_{j \geq J_1+1} 2^{-\varepsilon j} \leq C2^{-\varepsilon J_1} \leq C \sigma_J.$$  

Now, for $j \in \{J+1, \ldots, J_1\}$ and $x \in x_0 + B_J$, one has

$$|f_j(x)| \leq \sum_i \sum_{k2^{-j} \in x+B_{j-j_0}} |c_{\lambda} \psi_{\lambda}(x)|.$$  

If $j \geq J + j_0 + n_d$, the wavelet coefficients $c_{\lambda}$ in the last sum are such that

$$\lambda \subset x + B_{j-j_0-n_d} \subset x_0 + B_{J-1}$$

and therefore

$$|c_{\lambda}| \leq C \sigma_{J-m_d-1} \leq C \sigma_J.$$  

In the other case,

$$\lambda \subset x + B_{j-j_0-n_d} \subset x_0 + B_{j-j_0-n_d-1}$$

and thus

$$|c_{\lambda}| \leq C \sigma_{j-j_0-n_d-m_d-1} \leq C \sigma_J \leq C \sigma_J.$$  

These inequalities lead to

$$\sum_{j=J+1}^{J_1} \|f_j\|_{L^\infty(x_0+B_J)} \leq C J_1 \sigma_J \leq C |\log_2(\sigma_J)| \sigma_J.$$  

Putting all these inequalities together, one gets

$$\|f - P_J\|_{L^\infty(x_0+B_J)} \leq C |\log_2(\sigma_J)| \sigma_J,$$

as desired.  

The converse part of the previous theorem requires a uniform regularity condition. As shown in [12], a stronger condition than continuity is necessary in the usual case (see also [15], where similar results are obtained in the usual case with a Besov regularity assumption). Similarly, the logarithmic correction is best possible in the usual case [12].
5 A generalized definition of the Hölder exponent

The usual Hölder spaces are embedded: \( \alpha < \beta \) implies \( \Lambda^\beta(x_0) \subset \Lambda^\alpha(x_0) \). A notion of regularity for a function \( f \in L^\infty_{\text{loc}}(\mathbb{R}^d) \) at \( x_0 \) can thus be given by the so-called Hölder exponent,

\[
h_f(x_0) = \sup\{\alpha > 0 : f \in \Lambda^\alpha(x_0)\}.
\]

To do so in the generalized case, one needs some conditions under which \( \Lambda^\sigma,M(x_0) \subset \Lambda^{\sigma',M'}(x_0) \).

5.1 Preliminary results

We first need some technical easy results. From now on, if \( f \in \Lambda^\sigma,M(x_0) \), \((P_j)_j\) will stand for the sequence of polynomials of \( \mathbf{P}[M] \) corresponding to the definition. We will write

\[
P_j(x) = \sum_{|\beta| \leq M} a_j^{(\beta)} x^\beta
\]

and

\[
Q_j(x) = \sum_{|\beta| \leq M-1} a_j^{(\beta)} x^\beta
\]

Lemma 9. Let \( f \in \Lambda^\sigma,M(x_0) \); one has

\[
\sup_{|\beta|=M} |a_j^{(\beta)}| \leq C\left(\sum_{k=1}^{j-1} (2^M \Theta_1)^k + 1\right) \quad (9)
\]

and

\[
\sup_{|\beta|=M} |a_j^{(\beta)}| \leq C(\sigma_j \Theta_1^{j-1} \sum_{k=1}^{j-1} (2^M \Theta_1)^k + 1). \quad (10)
\]

Proof. Using the Markov inequality (3), we get

\[
\|D^\beta(P_j - P_{j+1})\|_{L^\infty(x_0+B_{j+1})} \\
\leq C2^M \|P_j - P_{j+1}\|_{L^\infty(x_0+B_{j+1})} \\
\leq C2^M (\|P_j - f\|_{L^\infty(x_0+B_j)} + \|f - P_{j+1}\|_{L^\infty(x_0+B_{j+1})}) \\
\leq C2^M \sigma_j,
\]
for any $\beta$ such that $|\beta| \leq M$ and $j$ sufficiently large. Therefore, we have

$$\|D^\beta(P_1 - P_j)\|_{L^\infty(x_0 + B_j)}$$

$$\leq \sum_{k=1}^{j-1} \|D^\beta(P_k - P_{k+1})\|_{L^\infty(x_0 + B_j)}$$

$$\leq \sum_{k=1}^{j-1} \|D^\beta(P_k - P_{k+1})\|_{L^\infty(x_0 + B_{k+1})}$$

$$\leq C \sum_{k=1}^{j-1} 2^{kM} \sigma_k \leq C \sum_{k=1}^{j-1} (2^M \Theta_1)^k,$$

for any $j$.

Now, let $\beta$ be a multi-index such that $|\beta| = M$; inequality (9) follows from

$$\|D^\beta(P_1 - P_j)\|_{L^\infty(x_0 + B_j)} \geq C(|a_j^{(\beta)}| - |a_1^{(\beta)}|),$$

while inequality (10) can be obtained in the same way, using

$$\|D^\beta(P_1 - P_j)\|_{L^\infty(x_0 + B_j)} \leq C \sum_{k=1}^{j-1} 2^{kM} \sigma_k \leq C \sigma_j \Theta_1^{-j} \sum_{k=1}^{j-1} (2^M \Theta_1)^k,$$

valid for any $j$. □

Corollary 10. Let $f \in \Lambda^{\sigma,M}(x_0)$; we have the following inequalities:

- if $2^M \Theta_1 < 1$,
  $$\|f - Q_j\|_{L^\infty(x_0 + B_j)} \leq C(\sigma_j + 2^{-jM}).$$

- if $2^M \Theta_1 > 1$,
  $$\|f - Q_j\|_{L^\infty(x_0 + B_j)} \leq C(\sigma_j + \Theta_1^{-j}).$$

- if $2^M \Theta_1 = 1$,
  $$\|f - Q_j\|_{L^\infty(x_0 + B_j)} \leq C(\sigma_j + 2^{-jM}).$$

Corollary 11. Let $f \in \Lambda^{\sigma,M}(x_0)$; we have the following inequalities:

- if $2^M \Theta_1 < 1$,
  $$\|f - Q_j\|_{L^\infty(x_0 + B_j)} \leq C(\sigma_j (2^M \Theta_1)^{-j} + 2^{-jM}).$$

- if $2^M \Theta_1 > 1$,
  $$\|f - Q_j\|_{L^\infty(x_0 + B_j)} \leq C(\sigma_j + 2^{-jM}).$$

- if $2^M \Theta_1 = 1$,
  $$\|f - Q_j\|_{L^\infty(x_0 + B_j)} \leq C(\sigma_j + 2^{-jM}).$$
5.2 Definitions

Before introducing a definition of generalized Hölder exponent, we must first consider embedded spaces of type $\Lambda^{\sigma,M}(x_0)$. Once the definitions given, we provide sufficient conditions for generalized Hölder spaces to be embedded.

**Definition 7.** If for any $\alpha > 0$, $\sigma^{(\alpha)}$ is an admissible sequence, the application $\sigma^{(\cdot)} : \alpha > 0 \mapsto \sigma^{(\alpha)}$ is called a family of admissible sequences.

**Definition 8.** Let $x_0 \in \mathbb{R}^d$; a family $\sigma^{(\cdot)}$ of admissible sequences is decreasing for $x_0$ if $\alpha < \beta$ implies $\Lambda^{\alpha,[\alpha]}(x_0) \subset \Lambda^{\beta,[\beta]}(x_0)$.

**Definition 9.** Let $\sigma^{(\cdot)}$ be a decreasing family of admissible sequences for $x_0$; if $f \in L^\infty_{\text{loc}}(\mathbb{R}^d)$, the Hölder exponent of $f$ at $x_0$ for the family $\sigma^{(\cdot)}$ is given by

$$h_{\sigma^{(\cdot)}}(x_0) = \sup\{\alpha > 0 : f \in \Lambda^{\alpha,[\alpha]}(x_0)\}.$$

The following proposition is a simple corollary of the results obtained in the previous section; it helps to check if a family of admissible sequences is decreasing. If $\sigma^{(\cdot)}$ is a family of admissible sequences, we set

$$\Theta_{\sigma^{(\alpha)}}(j) = \inf_{k \in \mathbb{N}} \frac{\sigma^{(\alpha)}_{j+k}}{\sigma^{(\alpha)}_k} \text{ and } \overline{\Theta}_{\sigma^{(\alpha)}}(j) = \sup_{k \in \mathbb{N}} \frac{\sigma^{(\alpha)}_{j+k}}{\sigma^{(\alpha)}_k}.$$

**Proposition 12.** Let $\sigma^{(\cdot)}$ be a family of admissible sequences and $x_0 \in \mathbb{R}^d$; $\sigma^{(\cdot)}$ is decreasing for $x_0$ if it satisfies the two following conditions:

1. if $m \leq \alpha < \beta < m + 1$, with $m \in \mathbb{N}_0$, there exist $C,J > 0$ such that

$$\sigma^{(\beta)}_j \leq C\sigma^{(\alpha)}_j,$$

for any $j \geq J$,

2. for any $m \in \mathbb{N}$, at least one of the two following conditions is satisfied:

   (a) there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0,\varepsilon_0)$, there exist $C,J > 0$ for which $\sigma^{(m)}_j \leq C\sigma^{(m-\varepsilon)}_j$ and

   - if $1 < 2^m \Theta_{\sigma^{(m)}}(j)$, then $(\Theta_{\sigma^{(m)}}(j))^j \leq C\sigma^{(m-\varepsilon)}_j$,
   - if $1 > 2^m \Theta_{\sigma^{(m)}}(j)$, then $2^{-jm} \leq C\sigma^{(m-\varepsilon)}_j$,
   - if $1 = 2^m \Theta_{\sigma^{(m)}}(j)$, then $j2^{-jm} \leq C\sigma^{(m-\varepsilon)}_j$,

   for any $j \geq J$,

   (b) there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0,\varepsilon_0)$, there exist $C,J > 0$ for which $2^{-jm} \leq C\sigma^{(m-\varepsilon)}_j$ and
\[
\begin{align*}
&\text{• if } 1 < 2^m \Omega_j^{(m)}, \quad \sigma_j^{(m)} \leq C \sigma_j^{(m-\varepsilon)}, \\
&\text{• if } 1 > 2^m \Omega_j^{(m)}, \quad \sigma_j^{(m)} (2^m \Omega_j^{(m)})^{-j} \leq C \sigma_j^{(m-\varepsilon)}, \\
&\text{• if } 1 = 2^m \Omega_j^{(m)}, \quad j \sigma_j^{(m)} \leq C \sigma_j^{(m-\varepsilon)},
\end{align*}
\]

for any \( j \geq J \).

This result is similar to the one obtained in [18] (under the hypothesis of proposition [12], one gets a decreasing family of admissible sequences for the uniform case), but the proof given for these generalized uniform Hölder spaces cannot be adapted for the pointwise case.

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