Angled Crested Like Water Waves with Surface Tension: Wellposedness of the Problem

Siddhant Agrawal

Department of Mathematics, University of Massachusetts, Amherst, MA 01003, USA.
E-mail: agrawal@math.umass.edu

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Abstract: We consider the capillary–gravity water wave equation in two dimensions. We assume that the fluid is inviscid, incompressible, irrotational and the air density is zero. We construct an energy functional and prove a local wellposedness result without assuming the Taylor sign condition. When the surface tension $\sigma$ is zero, the energy reduces to a lower order version of the energy obtained by Kinsey and Wu (Camb J Math 6(2):93–181, 2018) and allows angled crest interfaces. For positive surface tension, the energy does not allow angled crest interfaces but admits initial data with large curvature of the order of $\sigma^{-\frac{1}{3}+\epsilon}$ for any $\epsilon > 0$.

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1. Introduction

We are concerned with the motion of a fluid in dimension two with a free boundary. In this work we will identify 2D vectors with complex numbers. The fluid region $\Omega(t)$ and the air is separated by an interface $\Sigma(t)$, with the fluid being below the air region and $\Sigma(t)$ being a one dimensional curve. We assume that the interface tends point-wise to the real line at infinity but we do not assume that the interface is a graph. The air and the fluid are assumed to have constant densities of 0 and 1 respectively. The fluid is also assumed to be inviscid, incompressible, irrotational and we assume that the bottom is at infinite depth. The gravitational field is assumed to be a constant vector $-i$ pointing in the downward direction. The motion of the fluid is then governed by the Euler equation

$$v_t + (v, \nabla) v = -i - \nabla P \quad \text{on } \Omega(t)$$

$$\text{div } v = 0, \quad \text{curl } v = 0 \quad \text{on } \Omega(t)$$

Along with the boundary conditions

$$P = -\sigma \partial_s \theta \quad \text{on } \Sigma(t)$$

$(1, v)$ is tangent to the free surface $(t, \Sigma(t))$

$$v \to 0, \quad \nabla P \to -i \quad \text{as } |(x, y)| \to \infty$$

Here $\theta = \text{angle the interface makes with the x-axis}$, $\partial_s = \text{arc length derivative}$, $\sigma = \text{coefficient of surface tension} \geq 0$.

The earliest results on local well-posedness for the Cauchy problem are for small data in 2D and were obtained by Nalimov [31], Yoshihara [46,47] and Craig [17]. In the case of zero surface tension, Wu [41,42] obtained the proof of local well-posedness for arbitrary data in Sobolev spaces. See also the works in [3,4,6,7,13,14,18–20,24,28,48].

In the case of non-zero surface tension, the local well-posedness of the equation in Sobolev spaces was established by Beyer and Gunther in [11]. See also the works in [5,8,12,16,21,32,35,38]. The zero surface tension limit of the water wave equations in Sobolev spaces was proved by Ambrose and Masmoudi [9,10]. See also the works in [12,30,33,36,37].

An important quantity related to the well-posedness of the problem in the zero surface tension case is the Taylor sign condition. This says that there should exist a constant $c > 0$ such that

$$\frac{\partial P}{\partial n} \geq c > 0 \quad \text{on } \Sigma(t)$$
In [41] Wu proved that this condition is satisfied for the infinite bottom case if the interface is $C^{1,\alpha}$ for $\alpha > 0$. This was later shown to be true for flat bottoms and with perturbations to flat bottom by Lannes [24]. In the case of non-zero surface tension, the Taylor sign condition is not needed for establishing the local wellposedness for a fixed $\sigma > 0$, but now the time of existence $T$ depends on the value of $\sigma$ and $T \to 0$ as $\sigma \to 0$. The Taylor sign condition again becomes important if one studies the zero surface tension limit, as in this case one needs uniform time of existence for $0 \leq \sigma \leq \sigma_0$ for some fixed $\sigma_0 > 0$. In particular observe that assuming the Taylor sign condition to prove the zero surface tension limit forces one to assume quantitative bounds on the $C^{1,\alpha}$ norm of the initial interface. In fact in the case of non-zero surface tension, all results mentioned above obtain a time of existence $T \lesssim \|\kappa\|^{-1}_\infty$ where $\kappa$ is the initial curvature of the interface.

In the zero surface tension case, for non $C^1$ curves the Taylor sign condition is only satisfied in a weak sense with $-\frac{\partial P}{\partial n} \geq 0$ [23,41] and this makes the quasilinear equation degenerate. Because of this obtaining a local well-posedness result in this setting becomes highly non trivial as standard energy estimates in Sobolev spaces do not work. Kinsey and Wu [23] managed to overcome this difficulty by using a weighted Sobolev norm with the weight depending nonlinearly on the interface and proved a priori estimates for interfaces which can have angled crests. Building upon this work Wu [45] proved a local well-posedness result that allows for angled crested interfaces as initial data. Later on in [2] we studied the evolution of the singularities of these waves.

In this paper we extend the work of Kinsey and Wu [23] for $\sigma = 0$ to the case of $\sigma \geq 0$. We construct an energy functional $E_\sigma(t)$ and prove an a priori estimate\(^1\) for it in Theorem 5.1 which works for all $\sigma \geq 0$. Using this we prove a local well-posedness result for $\sigma > 0$ in Theorem 3.1. The energy $E_\sigma(t)$ has several interesting properties:

1. For $\sigma = 0$ the energy $E_\sigma(t)$ reduces to a lower order version of the energy of Kinsey and Wu [23]. In particular it allows singular interfaces such as interfaces with angled crests and cusps.

2. For $\sigma > 0$ the energy $E_\sigma(t)$ does not allow any singularities in the interface. In particular it does not allow angled crested interfaces.

3. For $\sigma > 0$ even though the energy $E_\sigma(t)$ does not allow singularities in the interface, it does allow interfaces with large curvature. It allows the $L^\infty$ norm of the curvature of the initial interface to grow like $\sigma^{-\frac{1}{3}+\epsilon}$ for any $\epsilon > 0$. In particular for $\sigma$ small and for interfaces close to being angled crested, we obtain a time of existence much larger than all previous results. See Corollary 3.2 for more details.

4. We do not assume the Taylor sign condition in proving the a priori estimate Theorem 5.1 or the local wellposedness result Theorem 3.1 and the energy $E_\sigma(t)$ is an increasing function of $\sigma$. For initial data in appropriate Sobolev spaces we obtain uniform time of existence of solutions for $0 \leq \sigma \leq \sigma_0$ for arbitrary $\sigma_0 > 0$ thereby recovering the uniform time of existence result of Ambrose and Masmoudi [9] in this case.\(^2\)

The growth rate of $\sigma^{-\frac{1}{3}}$ for the $L^\infty$ norm of the curvature can be explained by the following scaling argument. Let us ignore gravity and consider a solution $Z(\alpha, t)$ to the capillary water wave equation with surface tension $\sigma$.\(^3\) Then for any $\lambda > 0$ and $s \in \mathbb{R}$,

\(^1\) Theorem 5.1 actually uses $E_\sigma(t)$ instead of $E_\sigma(t)$ however both are equivalent to each other by Proposition 6.1.

\(^2\) Ambrose and Masmoudi [9] had the restriction of $\sigma_0$ being small which we do not have. See the discussion after (10) and Sect. 3.2 for more details.

\(^3\) The role of gravity will be clarified in a future work.
the function $Z_\lambda(\alpha, t) = \lambda^{-1} Z(\lambda \alpha, \lambda^2 t)$ is a solution to the capillary water wave equation with surface tension $\sigma_\lambda = \lambda^{2s-3} \sigma$. We are interested in the zero surface tension limit, so we want the solutions $Z_\lambda(\cdot, t)$ to exist in the same time interval $[0, T]$ and hence should have the same time scales. Hence we put $s = 0$ to get $Z_\lambda(\alpha, t) = \lambda^{-1} Z(\lambda \alpha, \lambda^2 t)$ and surface tension $\sigma_\lambda = \lambda^{-3} \sigma$. In this case, the curvature $\kappa_\lambda(\alpha, t) = \lambda \kappa(\lambda \alpha, t)$ which yields $\sigma_\lambda \kappa^3(\alpha, t) = \sigma \kappa^3(\lambda \alpha, t)$. Hence $\|\sigma \frac{\partial}{\partial t} \kappa\|_\infty$ is invariant under this scaling and so the curvature grows like $\sigma^{-\frac{1}{3}}$ as $\sigma \to 0$.

In a forthcoming paper we show that in an appropriate regime (see Remark 3.3) the zero surface tension limit of the solutions obtained in Theorem 3.1 allows for angled crested interfaces (more generally non-$C^1$ interfaces) and satisfies the gravity water wave equation.

We will follow the approach of Kinsey and Wu [23] and Wu [45], and work with free surface equations in conformal coordinates, derive quasilinear equations and use weighted Sobolev norms. The presence of surface tension gives rise to several structural and analytical difficulties which are not present in [23] and [45]. We explain the difficulties, new ideas and the main results in detail in Sect. 3. In addition to the work of Kinsey and Wu [23] and Wu [45], we also use ideas from the work of Ambrose and Masmoudi [9] in choosing appropriate variables to work with.

The paper is organized as follows: in Sect. 2 we introduce the notation and prove some basic formulae including the formula for the Taylor sign condition. In Sect. 3 we state our main results and discuss the main ideas and heuristics. In Sect. 4 we derive our quasilinear equations by taking derivatives to the Euler equation. In Sect. 5 we prove the main a priori estimate for the energy $E_\sigma(t)$. In Sect. 6 we prove the equivalence of the energies $E_\sigma(t)$ and $E_{\sigma}(t)$ and explain their relation to the Sobolev norm. In Sect. 7 we prove an existence result in Sobolev space for $\sigma > 0$ and also a blow up criterion. Finally in Sect. 8 we complete the proof of our main results. The “Appendix” contains all the commonly used identities and estimates used throughout the paper.

### 2. Notation and Preliminaries

We will try to be as consistent as possible with the notation used in [23]. Most of this section is essentially taken directly from [23] except for the new definitions and some modifications of the formulae due to surface tension. The Fourier transform is defined as

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-ix\xi} f(x) \, dx$$

We will denote by $D(\mathbb{R})$ the space of smooth functions with compact support in $\mathbb{R}$ and $D'(\mathbb{R})$ will be the space of distributions. $S(\mathbb{R})$ will denote the Schwartz space of rapidly decreasing functions and $S'(\mathbb{R})$ is the space of tempered distributions. A Fourier multiplier with symbol $a(\xi)$ is the operator $\mathcal{T}_a$ defined formally by the relation $\mathcal{T}_a \hat{f} = a(\xi) \hat{f}(\xi)$. The operators $|\partial_x|^s$ for $s \in \mathbb{R}$ are defined as the Fourier multipliers with symbol $|\xi|^s$. The Sobolev space $H^s(\mathbb{R})$ for $s \geq 0$ is the space of functions with $\|f\|_{H^s} = \left\| (1 + |\xi|^2)^{\frac{s}{2}} \hat{f}(\xi) \right\|_{L^2(d\xi)} < \infty$. The homogenous Sobolev space $\dot{H}^{\frac{1}{2}}(\mathbb{R})$ is the space of functions modulo constants with $\|f\|_{\dot{H}^{\frac{1}{2}}} = \left\| |\xi|^{\frac{1}{2}} \hat{f}(\xi) \right\|_{L^2(d\xi)} < \infty$. The Poisson kernel is given by

$$K_\epsilon(x) = \frac{\epsilon}{\pi (\epsilon^2 + x^2)} \quad \text{for } \epsilon > 0$$

(3)
From now on compositions of functions will always be in the spatial variables. We write \( f = f(\cdot, t), g = g(\cdot, t), f \circ g(\cdot, t) := f(g(\cdot, t), t). \) Define the operator \( U_g \) as given by \( U_g f = f \circ g. \) Observe that \( U_f U_g = U_{g \circ f}. \) Let \([A, B] := AB - BA\) be the commutator of the operators \( A \) and \( B.\) If \( A \) is an operator and \( f \) is a function, then \((A + f)\) will represent the addition of the operators \( A \) and the multiplication operator \( T_f \) where \( T_f(g) = fg. \) We will denote the spacial coordinates in \( \Omega(t) \) with \( z = x + iy, \) whereas \( z' = x' + iy' \) will denote the coordinates in the lower half plane \( P_\omega = \{(x, y) \in \mathbb{R}^2 | y < 0\}. \) As we will frequently work with holomorphic functions, we will use the holomorphic derivatives \( \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \) and \( \partial_{z'} = \frac{1}{2}(\partial_x - i\partial_y). \)

Let the interface \( \Sigma(t) : z = \alpha(t), t) \in \mathbb{C} \) be given by a Lagrangian parametrization \( \alpha \) satisfying \( \alpha(t) \neq 0 \) for all \( \alpha \in \mathbb{R}. \) Hence \( z_t(\alpha, t) = v(z(\alpha, t), t) \) is the velocity of the fluid on the interface and \( z_{tt}(\alpha, t) = (v_t + (v, \nabla)v)(z(\alpha, t), t) \) is the acceleration. As \( \frac{z_{\alpha}}{|\alpha|}(\alpha, t) = e^{i\theta(z, t)} \) and \( \frac{1}{|\alpha|} \partial_{\alpha} \) is the arc length derivative in Lagrangian coordinates, the pressure can be rewritten as

\[
P(z(\alpha, t), t) = i\sigma \frac{1}{|\alpha|} \partial_{\alpha} \frac{z_{\alpha}}{|\alpha|}(\alpha, t)
\]

Note that \( \frac{1}{|\alpha|} \partial_{\alpha} \frac{z_{\alpha}}{|\alpha|} \) is purely imaginary. Hence the Euler equation becomes

\[
z_{tt}(\alpha, t) + i = -\hat{n} \frac{\partial P}{\partial n}(z(\alpha, t)) - i \frac{\partial P}{\partial t}(z(\alpha, t))
\]

where

\[
\hat{t} = \frac{z_{\alpha}}{|z_{\alpha}|} = e^{i\theta} = \text{unit tangent vector}
\]

\[
\hat{n} = i \frac{z_{\alpha}}{|z_{\alpha}|} = ie^{i\theta} = \text{unit outward normal vector}
\]

Define

\[
a(\alpha, t) = -\frac{1}{|z_{\alpha}|} \frac{\partial P}{\partial n}(z(\alpha, t)) \in \mathbb{R}
\]

Hence we get

\[
z_{tt} + i = ia z_{\alpha} - \frac{1}{|z_{\alpha}|} \frac{1}{z_{\alpha}} \partial_{\alpha}(P(z(\alpha, t), t))
\]

Therefore

\[
z_{tt} - i = -ia z_{\alpha} - i\sigma \frac{1}{z_{\alpha}} \partial_{\alpha} \frac{z_{\alpha}}{|z_{\alpha}|}
\]

(4)

Let \( \Psi(t, t) : \Omega(t) \to \Omega(t) \) be conformal maps satisfying \( \lim_{z \to \infty} \Psi(z, t) = 1 \) and \( \lim_{\alpha \to \infty} \Psi(t, z, t) = 0. \) With this, the only ambiguity left in the definition of \( \Psi \) is that of the choice of translation of the conformal map at \( t = 0, \) which does not play any role in the analysis. Let \( \Phi(t, t) : \Omega(t) \to \Omega(t) \) be the inverse of the map \( \Psi(t, t) \) and define \( h(t, t) : \mathbb{R} \to \mathbb{R} \) as

\[
h(\alpha, t) = \Phi(z(\alpha, t), t)
\]

(5)
hence $h(\cdot, t)$ is a homeomorphism. As we use both Lagrangian and conformal parameterizations, we will denote the Lagrangian parameter by $\alpha$ and the conformal parameter by $\alpha'$. Let $h^{-1}(\alpha', t)$ be its inverse i.e.
\[ h(h^{-1}(\alpha', t), t) = \alpha' \]
From now on, we will fix our Lagrangian parametrization at $t = 0$ by imposing
\[ h(\alpha, 0) = \alpha \quad \text{for all } \alpha \in \mathbb{R} \]
Hence the Lagrangian parametrization is the same as conformal parametrization at $t = 0$. Define the variables
\[
Z(\alpha', t) = z \circ h^{-1}(\alpha', t) \quad Z_{,\alpha'}(\alpha', t) = \partial_{\alpha'} Z(\alpha', t) \quad \text{Hence } (z_{h^{-1}(\alpha', t)}) \circ h^{-1} = Z_{,\alpha'}
\]
\[
Z_t(\alpha', t) = z_t \circ h^{-1}(\alpha', t) \quad Z_{t,\alpha'}(\alpha', t) = \partial_{\alpha'} Z_t(\alpha', t) \quad \text{Hence } (z_t h^{-1}(\alpha', t)) \circ h^{-1} = Z_{t,\alpha'}
\]
\[
Z_{tt}(\alpha', t) = z_{tt} \circ h^{-1}(\alpha', t) \quad Z_{tt,\alpha'}(\alpha', t) = \partial_{\alpha'} Z_{tt}(\alpha', t) \quad \text{Hence } (z_{tt} h^{-1}(\alpha', t)) \circ h^{-1} = Z_{tt,\alpha'}
\]
Hence $Z(\alpha', t), Z_t(\alpha', t)$ and $Z_{tt}(\alpha', t)$ are the parameterizations of the boundary, the velocity and the acceleration in conformal coordinates and in particular $Z(\cdot, t)$ is the boundary value of the conformal map $\Psi(\cdot, t)$. Note that as $Z(\alpha', t) = z(h^{-1}(\alpha', t), t)$ we see that $\partial_t Z \neq Z_t$. Similarly $\partial_t Z_t \neq Z_{tt}$. The substitute for the time derivative is the material derivative. Define the operators
\[
D_t = \text{material derivative } = \partial_t + b \partial_{\alpha'} \quad \text{where } b = h_t \circ h^{-1}
\]
\[
D_{\alpha'} = \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \quad \overline{D}_{\alpha'} = \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \quad \text{and } |D_{\alpha'}| = \frac{1}{|Z_{,\alpha'}|} \partial_{\alpha'}
\]
\begin{align*}
\mathbb{H} & = \text{Hilbert transform } = \text{Fourier multiplier with symbol } -\text{sgn}(\xi) \\
\mathbb{P}_H & = \text{Holomorphic projection } = \frac{1}{2} (1 + \mathbb{H}) \\
\mathbb{P}_A & = \text{Antiholomorphic projection } = \frac{1}{2} (1 - \mathbb{H}) \\
|\partial_{\alpha'}| & = i \mathbb{H} \partial_{\alpha'} = \sqrt{-\Delta} = \text{Fourier multiplier with symbol } |\xi| \\
|\partial_{\alpha'}|^{1/2} & = \text{Fourier multiplier with symbol } |\xi|^{1/2}
\end{align*}
Now we have $D_t Z = Z_t$ and $D_t Z_t = Z_{tt}$ and more generally $D_t(f(\cdot, t) \circ h^{-1}) = (\partial_t f(\cdot, t)) \circ h^{-1}$ or equivalently $\partial_t(F(\cdot, t) \circ h) = (D_t F(\cdot, t)) \circ h$. This means that $D_t = U_{h^{-1}}^{-1} \partial_t U_h$ i.e. $D_t$ is the material derivative in conformal coordinates. In this work, the material derivative $D_t$ is more heavily used as compared to the time derivative $\partial_t$.

Define $U : \overline{P}_- \to \mathbb{C}$ as
\[
U = \overline{\nu} \circ \Psi
\]
and observe that $U$ is a holomorphic function on $P_-$. Also note that its boundary value is given by $\overline{Z}_t(\alpha', t) = U(\alpha', t)$ for all $\alpha' \in \mathbb{R}$. The Hilbert transform defined above satisfies the following property.

**Lemma 2.1** ([39]). Let $1 < p < \infty$ and let $F(z)$ be a holomorphic function in the lower half plane with $F(z) \to 0$ as $z \to \infty$. Then the following are equivalent.
\( \sup_{y<0} \| F(\cdot + iy) \|_p < \infty \)

(2) \( F(z) \) has a boundary value \( f \), non-tangentially almost everywhere with \( f \in L^p \) and \( \mathbb{H}(f) = f \).

In particular this says if \( U \) decays appropriately at infinity, then the boundary value of \( U \) namely \( \overline{Z}_t \) will satisfy \( \mathbb{H} \overline{Z}_t = \overline{Z}_t \). We can now define the main variables used in this paper

\[
A = (ah_\alpha) \circ h^{-1}
\]

\[
A_{1,\sigma} = A|Z_{\alpha'}|^2
\]

Hence

\[
A_{1,\sigma} = \frac{1}{|Z_{\alpha'}|} \frac{\partial P}{\partial \hat{n}} \circ h^{-1}
\]

\[
A_1 = 1 - \text{Im}[Z_t, \mathbb{H}]\overline{Z}_{t,\alpha'}
\]

\[
g = \theta \circ h^{-1}
\]

Hence

\[
|Z_{\alpha'}| = e^{iz} \quad \text{and} \quad |D_{\alpha'}|g = (\partial_z \theta) \circ h^{-1} = -i D_{\alpha'} \frac{Z_{\alpha'}}{|Z_{\alpha'}|}
\]

\[
\Theta = (\mathbb{I} + \mathbb{H})|D_{\alpha'}|g = -i (\mathbb{I} + \mathbb{H}) D_{\alpha'} \frac{Z_{\alpha'}}{|Z_{\alpha'}|}
\]

\[
\omega = e^{iz} \frac{Z_{\alpha'}}{|Z_{\alpha'}|}
\]

Hence

\[
|D_{\alpha'}|\omega = i \omega \text{Re} \Theta
\] (7)

Observe that \( \text{Re} \Theta = \kappa \circ h^{-1} \) where \( \kappa \) is the curvature of the interface. With this notation, by precomposing (4) with \( h^{-1} \) we get

\[
\overline{Z}_{tt} - i = -i A_{1,\sigma} \frac{Z_{\alpha'} - i \sigma D_{\alpha'} D_{\alpha'} \frac{Z_{\alpha'}}{|Z_{\alpha'}|}}{|Z_{\alpha'}|}
\] (8)

Let us now derive the formulae of \( A_{1,\sigma} \) and \( b \).

2.1. Formula for \( A_{1,\sigma} \). Let \( F = \mathbf{v} \) and hence \( F \) is holomorphic in \( \Omega(t) \) and \( \overline{z}_t = F(z(\alpha, t), t) \).

Hence

\[
\overline{z}_{tt} = F_t(z(\alpha, t), t) + F_z(z(\alpha, t), t)z_t(\alpha, t) \quad \overline{z}_{t\alpha} = F_z(z(\alpha, t), t)z_\alpha(\alpha, t)
\]

Hence

\[
\overline{z}_{tt} = F_t \circ z + \overline{z}_t \frac{z_{t\alpha}}{z_\alpha}
\]

Precomposing with \( h^{-1} \) we obtain \( \overline{Z}_{tt} = F_t \circ Z + \overline{Z}_t \frac{Z_{t,\alpha'}}{Z_{\alpha'}} \). Now multiply by \( i Z_{\alpha'} \) and use (8) to get

\[
A_{1,\sigma} = i Z_{\alpha'} F_t \circ Z + Z_{\alpha'} + i Z_t \overline{Z}_{t,\alpha'} - \sigma \partial_{\alpha'} D_{\alpha'} \frac{Z_{\alpha'}}{|Z_{\alpha'}|}
\]

Apply \( (\mathbb{I} - \mathbb{H}) \) and use the fact that \( \mathbb{H}(Z_{\alpha'} - 1) = Z_{\alpha'} - 1 \) and \( \mathbb{H} 1 = 0 \) to obtain

\[
(\mathbb{I} - \mathbb{H}) A_{1,\sigma} = 1 + i[Z_t, \mathbb{H}] \overline{Z}_{t,\alpha'} - \sigma (\mathbb{I} - \mathbb{H}) \partial_{\alpha'} D_{\alpha'} \frac{Z_{\alpha'}}{|Z_{\alpha'}|}
\]
Now take the real part

\[ A_{1,\sigma} = 1 - \text{Im}[Z_t, \mathbb{H}]Z_{t,\alpha'} + \sigma \partial_{\alpha'} \mathbb{H} D_{\alpha'} \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \]

Hence

\[ A_{1,\sigma} = A_1 + \sigma \partial_{\alpha'} \mathbb{H} D_{\alpha'} \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \quad \text{and in particular} \quad A_{1,\sigma}\big|_{\sigma=0} = A_1 \quad (9) \]

Note that the only non-holomorphic quantity in the above formula is \( iZ_t Z_{t,\alpha'} \). Also note that as \( A_1 = 1 - \text{Im}[Z_t, \mathbb{H}] Z_{t,\alpha'} \), by the calculation in \([23,44]\) we have that \( A_1 \geq 1 \).

From (9) the Taylor sign condition term can be written as

\[ -\frac{\partial P}{\partial \hat{n}} \circ h^{-1} = \frac{A_{1,\sigma}}{|Z_{\alpha'}|} \left( A_1 + \sigma |\partial_{\alpha'}|(\kappa \circ h^{-1}) \right) \quad (10) \]

where \( \kappa \circ h^{-1} = -i D_{\alpha'} \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \) is the curvature in conformal coordinates. For \( \sigma = 0 \), this formula was first derived by Wu \([41]\) to prove the Taylor sign condition for \( C^{1,\alpha} \) interfaces with \( \alpha > 0 \) and was crucially used in Kinsey-Wu \([23]\) to prove a priori estimates for angled crest interfaces. We will also use this formula in an essential way in this paper.

Observe that if the interface is \( C^{1,\alpha} \) then the Taylor sign condition is true for \( \sigma = 0 \) but will fail generically if \( \sigma \) is large. Ambrose and Masmoudi \([9]\) assumed that the Taylor sign condition holds for all \( 0 \leq \sigma \leq \sigma_0 \) which is only true if \( \sigma_0 \) is small. This issue was resolved by Shatah and Zeng \([37]\) where they only assumed the Taylor sign condition for \( \sigma = 0 \) which removed the smallness assumption of \( \sigma_0 \). However in both cases the Taylor sign condition is assumed for \( \sigma = 0 \) which we do not have as we allow interfaces with angled crests for \( \sigma = 0 \) for which only a weak Taylor sign condition \( -\frac{\partial P}{\partial \hat{n}} \geq 0 \) is satisfied.

2.2. Formula for \( b \). Recall that \( h(\alpha, t) = \Phi(z(\alpha, t), t) \) and so by taking derivatives we get

\[ h_t = \Phi_t \circ z + (\Phi_z \circ z) z_t \quad h_\alpha = (\Phi_z \circ z) z_\alpha \]

Hence

\[ h_t = \Phi_t \circ z + \frac{h_\alpha}{z_\alpha} z_t \]

Precomposing with \( h^{-1} \) we obtain

\[ h_t \circ h^{-1} = \Phi_t \circ Z + \frac{Z_t}{Z_{\alpha'}}. \]

Apply \((\mathbb{I} - \mathbb{H})\) and take real part, to get

\[ b = \text{Re}(\mathbb{I} - \mathbb{H}) \left( \frac{Z_t}{Z_{\alpha'}} \right) \quad (11) \]

This formula is the same as the one in \([23]\) and surface tension does not affect the formula.
2.3. **Fundamental equation.** Substituting the formula for $A_{1,\sigma}$ in equation (8), we get

$$Z_{tt} - i = -i \frac{A_1}{Z_{,\alpha'}} - i\sigma D\alpha' \| D\alpha' \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} - i\sigma D\alpha' D\alpha' \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$$

Now combine the second and third term and use $\Theta = -i(\| + \|) D\alpha' \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|}$ to get the fundamental equation

$$Z_{tt} - i = -i \frac{A_1}{Z_{,\alpha'}} + \sigma D\alpha' \Theta \quad (12)$$

Note that as $A_1$ does not depend on $\sigma$, the effect of surface tension is that it adds a holomorphic quantity to the conjugate of the acceleration. We also see that

$$\frac{A_1}{|Z_{,\alpha'}|} = - \frac{\partial P}{\partial h} \circ h^{-1} \bigg|_{\sigma=0} \geq 0$$

and hence it represents the Taylor sign condition in the absence of surface tension. As the equation is written in terms of $A_1$ and not $A_{1,\sigma}$, our energy $E_\sigma$ will always be positive irrespective of the value of surface tension.

2.4. **System.** To summarize the system is in the variables $(Z_{,\alpha'}, Z_t)$ satisfying

$$b = \text{Re}(\| - \|) \left( \frac{Z_t}{Z_{,\alpha'}} \right)$$

$$A_1 = 1 - \text{Im}[Z_t, \|]Z_{t,\alpha'}$$

$$(\partial_t + b\partial_{\alpha'})Z_{,\alpha'} = Z_{t,\alpha'} - b\alpha' Z_{,\alpha'}$$

$$(\partial_t + b\partial_{\alpha'})Z_t = i - i \frac{A_1}{Z_{,\alpha'}} + \frac{\sigma}{Z_{,\alpha'}} \partial_{\alpha'} (\| + \|) \left\{ \text{Im} \left( \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \right) \right\} \quad (13)$$

along with the condition that their harmonic extensions, namely $\Psi_{z'}(\cdot + iy) = K_{-y} * Z_{,\alpha'}$ and $U(\cdot + iy) = K_{-y} * Z_t$ for all $y < 0$, are holomorphic functions on $P_-$ and satisfy

$$\lim_{c \to \infty} \sup_{|z'| \geq c} \left\{ |\Psi_{z'}(z') - 1| + |U(z')| \right\} = 0 \quad \text{and} \quad \Psi_{z'}(z') \neq 0 \quad \text{for all} \quad z' \in P_-$$

We observe that for such a $\Psi_{z'}$ we can uniquely define $\log(\Psi_{z'}) : P_- \to \mathbb{C}$ such that $\log(\Psi_{z'})$ is a continuous function with $\Psi_{z'} = \exp \left\{ \log(\Psi_{z'}) \right\}$ and $(\log(\Psi_{z'}))(z') \to 0$ as $z' \to \infty$. Also note that one can obtain $Z(\cdot, t)$ by the formula

$$Z(\alpha', t) = Z(\alpha', 0) + \int_0^t \left\{ Z_t(\alpha', s) - b(\alpha', s)Z_{,\alpha'}(\alpha', s) \right\} ds$$

In particular instead of the variables $(Z_{,\alpha'}, Z_t)$, one can view the system being in variables $(Z, Z_t)$.

Another important observation one immediate makes is that in the above system, there is no restriction that the function $Z(\cdot, t)$ be injective. Even if the curve $Z(\cdot, t)$ becomes self-intersecting, the system still makes sense and one can still find a solution. Hence the

---

4 Here $K_{-y}$ is the Poisson kernel (3).
above system allows self-intersecting domains similar to work of [12, 13] where solutions with splash and splat singularities were constructed. Observe that we assumed that the interface is non-self intersecting while deriving this system from the Euler equation (1) and (2). If the solution \((Z, Z_t)(t)\) of (13) leads to a non-self intersecting curve then one can go back and obtain a solution to the Euler equation in a similar way as done in [45] section 2.5. However if the interface becomes self-intersecting, then its relation to the Euler equation is lost and the solution becomes non-physical. From now on we will exclusively focus on the system (13) and hence all results in this paper apply to self-intersecting curves as well.

One can rewrite the function \(h(\alpha, t)\) defined in (5) as the solution to the ODE

\[
\frac{dh}{dt} = b(h, t)
\]

where \(b\) is given by (13). From this we easily see that as long as \(\sup_{[0, T]}\|b_{\alpha'}\|_\infty(t) < \infty\) we can solve this ODE and for any \(t \in [0, T]\) we have that \(h(\cdot, t)\) is a homeomorphism. Hence it makes sense to talk about the functions \(z = Z \circ h, z_t = Z_t \circ h\) which are Lagrangian parametrizations of the interface and the velocity on the boundary.

3. Main Results and Discussion

3.1. Results. For \(\sigma \geq 0\) define the energy

\[
E_{\sigma, 1} = \left\| \frac{1}{Z_{\alpha'}} \frac{\partial_{\alpha'} 1}{Z_{\alpha'}} \right\|_2^2 + \left\| \frac{1}{Z_{\alpha'}} \frac{\partial_{\alpha'} 1}{Z_{\alpha'}} \right\|_{H^1_2}^2 + \left\| \sigma \partial_{\alpha'} \Theta \right\|_{H^1_2}^2 + \left\| \frac{1}{Z_{\alpha'}} \frac{\partial_{\alpha'} 1}{Z_{\alpha'}} \right\|_{2}^6
\]

\[
+ \left\| \frac{1}{Z_{\alpha'}} \frac{\partial_{\alpha'} 1}{Z_{\alpha'}} \right\|_{2}^2 + \left\| \frac{1}{Z_{\alpha'}} \frac{\partial_{\alpha'} 1}{Z_{\alpha'}} \right\|_{H^1_2}^2 + \left\| \frac{\sigma}{Z_{\alpha'}} \frac{\partial_{\alpha'} 1}{Z_{\alpha'}} \right\|_{H^1_2}^2 + \left\| \frac{\sigma}{Z_{\alpha'}} \frac{\partial_{\alpha'} 1}{Z_{\alpha'}} \right\|_{2}^2
\]

\[
E_{\sigma, 2} = \left\| \frac{1}{Z_{\alpha'}} \frac{\partial_{\alpha'} 1}{Z_{\alpha'}} \right\|_2^2 + \left\| \frac{1}{Z_{\alpha'}} \frac{\partial_{\alpha'} 1}{Z_{\alpha'}} \right\|_{H^1_2}^2 + \left\| \frac{\sigma}{Z_{\alpha'}} \frac{\partial_{\alpha'} 1}{Z_{\alpha'}} \right\|_{H^1_2}^2 + \left\| \frac{\sigma}{Z_{\alpha'}} \frac{\partial_{\alpha'} 1}{Z_{\alpha'}} \right\|_{2}^2
\]

\[
E_{\sigma} = E_{\sigma, 1} + E_{\sigma, 2}
\]

Note that \(E_{\sigma, 1}\) controls terms based solely on the interface \(Z(\cdot, t)\) and \(E_{\sigma, 2}\) controls the weighted derivatives of the velocity \(Z_t(\cdot, t)\). Hence the energy can be thought of as a weighted Sobolev norm with the weight given by powers of \(\frac{1}{Z_{\alpha'}}\). Also observe that all the terms of \(E_{\sigma}\) are the boundary values of holomorphic functions.

Observe that the energy \(E_{\sigma}(t)\) is well defined and is finite if \((Z_{\alpha'}, -1, Z_{\alpha'}) - 1, Z_t(t) \in H^3(\mathbb{R}) \times H^3(\mathbb{R}) \times H^3(\mathbb{R})\). If we ignore the weights in the energy, we get back the Sobolev norm. See Lemma 6.2 for the precise relationship between the energy \(E_{\sigma}\) and the Sobolev norm of the solution.

The energy \(E_{\sigma}\) is equivalent to \(E_{\sigma}\) which is what we use to prove the main a priori estimate Theorem 5.1 (\(E_{\sigma}\) is defined in Sect. 5 and the equivalence between \(E_{\sigma}\) and \(E_{\sigma}\) is proved in Proposition 6.1). A more intuitive explanation of the energy \(E_{\sigma}(t)\) can be
obtained by applying the fundamental operators $D_t, \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'}$ and $\sigma^{\frac{1}{2}} |D_{\alpha'}|$ occurring in the quasilinear equations (40) and (43) on the variable $g = \theta \circ h^{-1}$. This is explained in more detail in Sect. 3.2.

For $\sigma = 0$, the energy $E_{\sigma}(t)$ allows angled crest interfaces and outward pointing cusps (i.e. interfaces with angled crests and cusps yield $E_{\sigma} < \infty$ if $\sigma = 0$). To see this, we only need to show that the first two terms of $E_{\sigma,1}$ allows angled crests and cusps. The argument is exactly the same as explained in [2,23] and we briefly explain it here. If the interface has an angled crest of angle $\nu \pi$, then $Z(\alpha') \sim (\alpha')^{\nu}$ near $\alpha' = 0$. Hence near $\alpha' = 0$ we have

$$Z_{\alpha'}(\alpha') \sim (\alpha')^{-1} - \frac{1}{Z_{\alpha'}}(\alpha') \sim (\alpha')^{1-\nu} - \partial_{\alpha'} \frac{1}{Z_{\alpha'}}(\alpha') \sim (\alpha')^{-\nu}$$

From this we can see that the first two terms of $E_{\sigma,1}$ allows interfaces with angled crests of angles $\nu \pi$ with $0 < \nu < \frac{1}{2}$. A similar argument shows that cusps i.e. $\nu = 0$ are also allowed. See [2] for more details.

For $\sigma > 0$, in contrast we see that most of the terms with surface tension in $E_{\sigma,1}$ do not allow angled crest interfaces which can be seen by a similar argument as above.

For example consider the term $\frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}(\alpha') \sim (\alpha')^{1-2\nu}$ in the energy. Suppose we have an interface with an angled crest of angle $\nu \pi$ at $\alpha' = 0$ with $\nu \geq 0$. Then by the argument above we see that near $\alpha' = 0$

$$Z_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}(\alpha') \sim (\alpha')^{-\frac{\nu}{2} - \frac{1}{2}}$$

which is unbounded and so the energy does not allow angled crest interfaces if $\sigma > 0$ (i.e. interfaces with angled crests and cusps yield $E_{\sigma} = \infty$ if $\sigma > 0$). In fact if $E_{\sigma} < \infty$ and $\sigma > 0$, then from Lemma 6.2 and Remark 6.3 we automatically have $\|Z_{t,\alpha'}\| \rightarrow^{H^1} + \|\partial_{\alpha'} Z_{\alpha'}\| \rightarrow^{H^{2.5}} < \infty$ i.e. the data has to be in Sobolev spaces and hence do not allow singularities. However this does not say anything about the size of the Sobolev norm which can be very large. In fact the $L^{\infty}$ norm of the curvature can be of the order of $\sigma^{-\frac{1}{2}}$ (See Corollary 3.2 below for an example).

The fact that the energy does not allow singularities for $\sigma > 0$ is quite natural as one would expect that surface tension should be smoothing and hence stable solutions should not have angled crests. Moreover the natural extension of the energy of Kinsey-Wu [23] does not allow any singularities in the interface. Observe that in Kinsey-Wu [23], the quantity $Z_{t,\alpha'} \in L^2$ and $\partial_{\alpha'} \left( \frac{A_1}{Z_{\alpha'}} \right) \in L^2$. Hence if we assume these quantities remain in $L^2$ when $\sigma > 0$, then by the equation (12) we see that $\sigma \partial_{\alpha'} D_{\alpha'} \Theta \in L^2$. Hence $|D_{\alpha'}| \Theta \in L^{\infty}_{loc}$ and as $\text{Re} \Theta = \kappa \circ h^{-1}$, we see that $\partial_{\alpha'} \kappa \in L^{\infty}_{loc}$. Hence the interface has to be at least $C^{2.1}$ which rules out any type of singularity.

We now state the main result of this paper.

**Theorem 3.1.** Let $\sigma > 0$ and assume the initial data $(Z, Z_t)(0)$ satisfies $(Z_{\alpha'} - 1, \frac{1}{Z_{\alpha'}} - 1, Z_t)(0) \in H^{3.5}(\mathbb{R}) \times H^{3.5}(\mathbb{R}) \times H^3(\mathbb{R})$. Then $E_{\sigma}(0) < \infty$ and there exists $T, C_1 > 0$ depending only on $E_{\sigma}(0)$ such that the initial value problem to (13) has a unique
solution \((Z, Z_t)(t)\) in the time interval \([0, T]\) satisfying \((Z, Z_t) - 1, \frac{1}{Z, Z_t} - 1, Z_t) \in C^l([0, T], H^{3.5 - \frac{l}{2}}(\mathbb{R}) \times H^{3.5 - \frac{l}{2}}(\mathbb{R}) \times H^{3 - \frac{l}{2}}(\mathbb{R}))\) for \(l = 0, 1\) and \(\sup_{t \in [0, T]} E_\sigma(t) \leq C_1 < \infty\).

It is important to note that we do not allow angled crested interfaces as initial data when \(\sigma > 0\). However the initial data does allow interfaces with large \(C^{1,\alpha}\) norm (see below). The most important feature about the above result is that the time of existence depends only on \(E_\sigma(0)\) and not the Sobolev norm of the initial data. We now explain some of the important points about the result and the energy \(E_\sigma\).

**Properties**

1. No assumptions on the Taylor sign condition: Observe that no assumptions are made on the Taylor sign condition in the above result. Recall that the Taylor sign condition is \(-\frac{\partial P}{\partial n} \geq c > 0\) and from (10) that the quantity \(-\frac{\partial P}{\partial n}\) depends on the value of \(\sigma\). All previous results on zero surface tension limit assume the Taylor sign condition. In the case of Ambrose and Masmoudi [9] the condition \(-\frac{\partial P}{\partial n} \geq c > 0\) is assumed for all \(0 \leq \sigma \leq \sigma_0\) which forces \(\sigma_0\) to be small. In Shatah and Zeng [37] the condition \(-\frac{\partial P}{\partial n} \geq c > 0\) is assumed only for \(\sigma = 0\) removing the smallness assumption on \(\sigma\).

However in both cases one still needs control on the \(C^{1,\alpha}\) norm of the interface. We do not make any assumptions which allows us to deal with both large \(\sigma\) and large \(C^{1,\alpha}\) norm of the interface.

2. Time of existence is independent of \(\sigma\): Observe that the energy \(E_\sigma(t)\) is an increasing function of \(\sigma\). Hence for arbitrary \(\sigma_0 > 0\), the time of existence is uniform for all \(0 < \sigma \leq \sigma_0\). In particular we recover the uniform time of existence part of the result of Ambrose and Masmoudi [9] in this case.

3. Energy allows angled crest solutions for \(\sigma = 0\): If we put \(\sigma = 0\) in the energy \(E_\sigma\), then it allows solutions with angled crest with angle less than \(90^\circ\) and also allows cusps. These are exactly the solutions allowed by the energy obtained by Kinsey and Wu in [23] and the local wellposedness for angled crested solutions has been proven by Wu in [45]. Our energy for \(\sigma = 0\) is a lower order version of the energy in [23] by half spatial derivative.

4. Energy does not allow angled crest solution for \(\sigma > 0\) but allows large curvature: In the proof of this theorem we show the estimate \(\|\kappa\|_\infty \leq \sigma^{-\frac{1}{2}} C(E_\sigma)\) holds and hence for \(\sigma > 0\) the curvature is bounded, which automatically excludes angled crest solutions. Note however that for small values of \(\sigma\), the energy allows data with quite large curvature of the order of \(\sigma^{-\frac{1}{2}}\). (See Corollary 3.2 below where for any given \(\epsilon > 0\) arbitrarily small, we construct examples where \(E_\sigma = O(1)\) and the curvature of the initial data grows like \(\sigma^{-\frac{1}{2} + \epsilon}\) as \(\sigma \to 0\)).

In the above theorem we do not prove existence when \(\sigma = 0\) because in this case the energy \(E_\sigma(0)\) allows singular solutions such as angled crested solutions which have already been proved to exist in [45]. The above result is proved via a new a priori estimate Theorem 5.1 which is an extension of the a priori estimate of Kinsey and Wu [23] for \(\sigma \geq 0\), and a local existence result for \(\sigma > 0\) in Sobolev spaces Theorem 7.8.

As noted above the main point of the above theorem is that the time of existence depends only on \(E_\sigma(0)\). The usefulness of the energy \(E_\sigma(t)\) comes from the fact that there are interfaces (such as smooth interfaces close to being angled crest) for which the \(C^{1,\alpha}\) norm (for any \(\alpha > 0\)) of the interface of the initial data is very large but \(E_\sigma(0)\) remains bounded. This translates into a longer time of existence if we use the energy
Corollary 3.2. Consider an initial data \((Z, Z_t)(0)\) with \(M < \infty\). Let \((Z, Z_t)(t)\) be the unique solution of equation (13) for \(\sigma = 0\) with initial data \((Z, Z_t)(0)\) as obtained in [45]. For \(0 < \epsilon \leq 1\) and \(\sigma \geq 0\) denote by \((Z^{\epsilon, \sigma}, Z_t^{\epsilon, \sigma})(t)\) the unique solution to the equation (13) with surface tension \(\sigma\) and with initial data \((Z^{\epsilon, \sigma}, Z_t^{\epsilon, \sigma})(0) = (Z * P_e, Z_t * P_e)(0)\) where \(P_e\) is the Poisson kernel. Then we have the following

1. For any \(c > 0\), there exists \(T, C_1 > 0\) depending only on \(c\) and \(M\) such that for all \(\sigma \geq 0\) and \(0 < \epsilon \leq 1\) satisfying \(\frac{\sigma}{\epsilon} \leq c\), the solutions \((Z^{\epsilon, \sigma}, Z_t^{\epsilon, \sigma})(t)\) exist in the time interval \([0, T]\) with \(\sup_{t \in [0, T]} E_\sigma(Z^{\epsilon, \sigma}, Z_t^{\epsilon, \sigma})(t) \leq C_1 < \infty\).
(2) If the initial interface \( Z(\cdot, 0) \) has only one angled crest of angle \( \nu \pi \) with \( 0 < \nu < \frac{1}{2} \), then the \( L^\infty \) norm of the curvature \( \kappa^{e, \sigma} \) of the initial interface \( Z^{e, \sigma}(\cdot, 0) \) satisfies
\[
\|\kappa^{e, \sigma}\|_\infty \sim \epsilon^{-\nu} \quad \text{as } \epsilon \to 0.
\]
In particular for any \( 0 < \delta < \frac{1}{3} \) arbitrarily small, choosing \( \nu = \frac{1}{2} - \frac{3}{2} \delta \) and \( \sigma = \epsilon^{\frac{2}{3}} \) we obtain \( \|\kappa^{e, \sigma}\|_\infty \sim \sigma^{-\frac{1}{3} + \delta} \) as \( \sigma \to 0 \). Hence Theorem 3.1 allows initial interfaces with large curvature when \( \sigma \) is small.

Remark 3.3. In a companion paper [1] we prove that if in addition \( \max\left\{ \frac{\sigma}{\epsilon^{\frac{3}{2}}}, \epsilon \right\} \to 0 \), then the solutions \((Z^{e, \sigma}, Z_t^{e, \sigma})(t) \to (Z, Z_t)(t)\) in an appropriate norm.

Remark 3.4. Note that in the corollary, we prove uniform time of existence under the condition \( \frac{\sigma}{\epsilon^{\frac{3}{2}}} \lesssim 1 \) and in particular in this case \( \sigma \to 0 \) as \( \epsilon \to 0 \). We do not know what happens when \( \sigma = 1 \) and \( \epsilon \to 0 \). It is our belief that one cannot expect uniform time of existence in this case for general initial data. To see why, observe that as explained in the introduction if we ignore gravity then the relevant scaling for the problem of zero surface tension limit is \( Z_\lambda(\alpha, t) = \lambda^{-1} Z(\lambda \alpha, t) \) with \( \sigma_\lambda = \lambda^{-3} \sigma \) and this scaling leaves the quantity \( \|\sigma^{-\frac{1}{3}} \kappa\|_\infty \) invariant. Because of this we believe that one cannot expect a general local wellposedness result which gives a time of existence of order 1 when \( \|\sigma^{-\frac{1}{3}} \kappa\|_\infty \to \infty \). Note that as explained in the second point in the corollary above, the scaling \( \frac{\sigma}{\epsilon^{\frac{3}{2}}} \lesssim 1 \) allows initial interfaces with the \( L^\infty \) norm of the curvature of the order of \( \sigma^{-\frac{1}{3} + \delta} \) for arbitrary small \( \delta > 0 \), thereby getting us arbitrary close to the rate \( \|\sigma^{-\frac{1}{3}} \kappa\|_\infty \sim 1 \).

The novelty of the above result lies in the fact that all previous results on surface tension for large data obtain a time of existence \( T \lesssim \|\kappa\|^{-1}_\infty \) where \( \kappa \) is the initial curvature, even if \( \sigma \) is very small. Hence if the interface \( Z(\cdot, 0) \) has a single angled crest of angle \( \nu \pi \) then \( Z^{e, \sigma}(\cdot, 0) \) has curvature \( \|\kappa^{e, \sigma}\|_\infty \sim \epsilon^{-\nu} \), which yields \( T \lesssim \epsilon^\nu \to 0 \) as \( \epsilon \to 0 \). The above corollary says that these solutions in fact exist on a much longer time interval and that the time of existence is at least \( O(1) \) even as \( \epsilon \to 0 \), provided there is a balance between surface tension and smoothness \( \sigma \lesssim \epsilon^{\frac{3}{2}} \).

The scaling factor of \( \sigma/\epsilon^{\frac{3}{2}} \) comes about as a compatibility condition of two of the main operators in the quasilinear equations (43) and (40) (see Sect. 3.2 for more details). Observe that
\[
\left( \frac{1}{|Z_{, \alpha'}|^2} \partial_{\alpha'} \right)^{-\frac{3}{2}} \left( \frac{\sigma\frac{\sigma}{\epsilon^{\frac{3}{2}}} \partial_{\alpha'}}{|Z_{, \alpha'}|^3} \right)^3 \sim \sigma \partial_{\alpha'}^{\frac{3}{2}}
\]
which naturally gives us the factor \( \sigma/\epsilon^{\frac{3}{2}} \).

3.2. Discussion. We now give a brief heuristic explanation into the nature of our results.

1. Taylor sign condition: In Ambrose and Masmoudi [9] the Taylor sign condition
\[
-\frac{\partial P}{\partial n} \geq c > 0
\]
is assumed for all \( 0 \leq \sigma \leq \sigma_0 \) which forces \( \sigma_0 \) to be small. To see this why this assumption was made, one can do a similar computation as was done in [9] in conformal coordinates and obtain a quasilinear equation of the form
\[
\left\{ D_t^2 + i \mathbb{H} \left( \frac{A_{1, \sigma}}{|Z_{, \alpha'}|^2} \partial_{\alpha'} \right) - i \sigma \mathbb{H} \left( \frac{1}{|Z_{, \alpha'}|} \partial_{\alpha'} \right)^3 \right\} g = l.o.t \quad (14)
\]
where \( g = \theta \circ h^{-1} \) and \( i \Re \partial_{\alpha'} = |\partial_{\alpha'}| \). Hence heuristically the equation looks like

\[
\left\{ \partial_t^2 + \left( \frac{A_{1, \sigma}}{|Z_{\alpha'}|} \right) |\partial_{\alpha'}| + \sigma |\partial_{\alpha'}|^3 \right\} g = \text{l.o.t}
\]

Note that the coefficient in the second term is Taylor sign condition term namely \( \frac{A_{1, \sigma}}{|Z_{\alpha'}|} = -\frac{\partial P}{\partial n} \circ h^{-1} \) and is given by the formula (10). Hence to obtain a positive energy from this equation, the Taylor sign condition at \( t = 0 \) was assumed i.e. \( \frac{A_{1, \sigma}}{|Z_{\alpha'}|} \geq c > 0 \). This condition is satisfied if \( \sigma \) is small enough depending on the initial data, but will fail generically if \( \sigma \) is large.

In Shatah and Zeng [37] it is observed that one can in fact obtain an energy with the coefficient being \( -\frac{\partial P}{\partial n} \big|_{\sigma=0} \) instead of \( -\frac{\partial P}{\partial n} \) which removes the smallness restriction on \( \sigma \). However this was achieved using a variational formulation and as we are interested in studying angled crests, it is not clear how to use that approach in our problem.

To overcome this issue we make the simple observation that if we take an arc length derivative \( |D_{\alpha'}| \) or a material derivative \( D_t \) to the equation (14), then the structure of the equation improves. In fact by taking these derivatives to the equation (14) one obtains an equation of the form

\[
\left\{ D_t^2 + i \Re \left( \frac{A_{1, \sigma}}{|Z_{\alpha'}|^2} \partial_{\alpha'} \right) - i \sigma \Re \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \right)^3 \right\} f = \text{l.o.t}
\]

where \( f = |D_{\alpha'}| g \) or \( f = D_t g \). Note crucially that the second term now has \( A_1 \) instead of \( A_{1, \sigma} \). As \( A_1 \geq 1 \) irrespective of \( \sigma \), we do not need to make any assumptions on \( \sigma \) being small. Note that \( |D_{\alpha'}| g = \Re \Theta \) and \( D_t g = -\Im (\overline{D_{\alpha'}} \overline{Z_t}) \) from (21) and we derive equations for \( \Theta \) and \( \overline{D_{\alpha'}} \overline{Z_t} \) in (43) and (40) respectively. Of course we still need to deal with the issue that \( \frac{\partial P}{\partial n} \big|_{\sigma=0} = \frac{A_{1}}{|Z_{\alpha'}|} \) does not have a positive lower bound. This is resolved in a similar manner as Kinsey and Wu [23] and is explained below.

(2) Heuristic energy estimate:

The main goal of this paper is to extend the work of Kinsey and Wu [23] to the case of non-zero surface tension. In [23] a priori estimates are given for angled crested interfaces in the case of \( \sigma = 0 \) for angles \( \nu \pi \) with \( 0 < \nu < \frac{1}{2} \). Let us do a heuristic energy estimate to understand the difficulties. If the interface is \( C^{1, \alpha} \) then we have \( 0 < c_1 \leq \frac{1}{|Z_{\alpha'}|} \leq c_2 < \infty \) and hence for smooth enough interfaces the main operator in (15) behaves like \( \partial_t^2 + |\partial_{\alpha'}| + \sigma |\partial_{\alpha'}|^3 \) for which standard energy estimates in Sobolev spaces work. If the interface has an angled crest of angle \( \nu \pi \) at \( \alpha' = 0 \), then \( Z(\alpha') \sim (\alpha')^\nu \) and hence \( \frac{1}{|Z_{\alpha'}|} (\alpha') \sim |\alpha'|^{1-\nu} \) near \( \alpha' = 0 \) and hence the quasilinear equation near \( \alpha' = 0 \) behaves like

\[
\left\{ \partial_t^2 + |\alpha'|^{2-2\nu} |\partial_{\alpha'}| + \sigma |\alpha'|^{3-3\nu} |\partial_{\alpha'}|^3 \right\} f
= |\alpha'|^{1-2\nu} f + \sigma |\alpha'|^{2-3\nu} |\partial_{\alpha'}|^2 f + \sigma |\alpha'|^{1-3\nu} |\partial_{\alpha'}| f + \sigma |\alpha'|^{-3\nu} f + \text{other l.o.t}
\]

(16)
We have included a few simplified versions of the lower order terms to demonstrate the issues in proving an energy estimate. We obtain our energies by multiplying the above equation with either $\partial_t f$ or $|\partial_{\alpha'}| \partial_t f$ and then integrating. If we multiply the equation by $\partial_t f$ and integrate, we obtain the energy

$$E_\sigma(t) = \frac{1}{2} \|\partial_t f\|_2^2 + \frac{1}{2} \|\alpha'|^{1-\nu} f\|_{H^{\frac{1}{2}}}^2 + \frac{1}{2} \|\alpha'|^{2-3\nu} |\partial_{\alpha'}| f\|_{H^{\frac{1}{2}}}^2$$  \hspace{1cm} (17)

For simplicity we can also add the term $\frac{1}{2} \|f\|_2^2$ to the energy which is compatible with $\partial_t f \in L^2$ obtained from the energy. To close the energy estimate, we need to control the $L^2$ norm of the right hand side of (16) (we ignore the commutator terms in this heuristic). Hence to control the first term $|\alpha'|^{1-2\nu} f \in L^2$, we obtain the restriction $\nu \leq \frac{1}{3}$ which is one of the main reasons of the restrictions on the angles in Kinsey and Wu [23]. Note that we cannot control the term $\sigma |\alpha'|^{2-3\nu} |\partial_{\alpha'}|^2 f \in L^2$ as we only have control of $3/2$ derivatives on $f$. For smooth enough interfaces, it was observed by Ambrose-Masmoudi [9] that by carefully choosing $f$ (by using variables derived from $\theta$), this term does not appear in the quasilinear equation and we follow the same approach. We do not use the modified tangential velocity as in [9] but instead use the more natural material derivative $D_t$ along with the variable $\theta$ to obtain our highest order quasilinear equation. Hence the equation looks like

$$\begin{align*}
\left\{ \partial_t^2 + |\alpha'|^{2-2\nu} |\partial_{\alpha'}| + \sigma |\alpha'|^{3-3\nu} |\partial_{\alpha'}|^3 \right\} f \\
= |\alpha'|^{1-2\nu} f + \sigma |\alpha'|^{1-3\nu} |\partial_{\alpha'}| f + \sigma |\alpha'|^{-3\nu} f + \text{other l.o.t} \end{align*}$$  \hspace{1cm} (18)

Now we need to control $\sigma |\alpha'|^{1-3\nu} |\partial_{\alpha'}| f \in L^2$ and $\sigma |\alpha'|^{-3\nu} f \in L^2$. As we only have $f \in L^2$, there is no way we can control the term $\sigma |\alpha'|^{-3\nu} f \in L^2$ and this is the reason why we do not allow angled crest data if $\sigma > 0$. If we work with the smooth interface $Z^\epsilon = Z \ast P_\epsilon$ where $P_\epsilon$ is the Poisson kernel, then this has the effect of changing $|\alpha'| \mapsto |-i\epsilon + \alpha'|$ near $\alpha' = 0$. Hence to close the energy estimate, we obtain the restriction $\sigma \epsilon^{-3\nu} \lesssim 1$. For the interface $Z'$, the curvature $\kappa \sim \epsilon^{-\nu}$ and hence this can be written as $\sigma \kappa^3 \lesssim 1$. A similar argument for $\sigma |\alpha'|^{1-3\nu} |\partial_{\alpha'}| f \in L^2$ also yields the same restriction. Note that this is the exact scaling as was mentioned in the introduction and Remark 3.4. Also note that these restrictions do not depend on the choice of $f$, but is purely a consequence of the structure of the quasilinear equation and attempting to prove an $L^2$ based energy estimate.

The main goal is now to construct and prove an energy estimate which mimics this heuristic energy estimate. A key difficulty in implementing this is to find a suitable $f$ and obtain a corresponding quasilinear equation. We need the following properties for $f$

(a) $f$ needs to have appropriate weights so that the energy such as (17) allows angled crests solutions of angles less than $90^\circ$ when $\sigma = 0$

(b) The quasilinear equation for $f$ should not have terms like $\sigma |\alpha'|^{2-3\nu} |\partial_{\alpha'}|^2 f$ in the errors, i.e. it should look like (18) and not (16)

These are rather severe restrictions on $f$ and it is not clear whether one can find such a function. Observe that if $f$ has such properties, then no weighted derivate
of the form \( w \partial_{\alpha'} f \) will satisfy the same properties (Kinsey-Wu [23] used the weighted derivative \( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \) and Ambrose-Masmoudi [9] used the arc length derivative \( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \) to obtain the higher order energies). Fortunately we observe that the material derivative \( D_t f \) will satisfy both the properties if \( f \) satisfies both properties. In this paper we use \( f = D_{\alpha'} Z_t \) as the variable for the energy of the highest order in \( E_{\sigma} \) and it is related to the material derivative of the angle via \( D_t g = -\text{Im}(D_{\alpha'} Z_t) \).

(3) Constructing the energy:

Observe that the quasilinear equations we derive in Sect. 4 are essentially of the form

\[
\begin{align*}
D_t^2 f + i A_1 \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \right) f - i \sigma \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \right)^3 f &= 0 \quad \text{(19)}
\end{align*}
\]

for suitable \( f \). To obtain an energy we multiply by \( |\partial_{\alpha'}| D_t f \) and integrate to get the energy

\[
\begin{align*}
\| D_t f \|_{L^2}^2 + \left\| \sqrt{A_1} \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} f \right\|_{L^2}^2 + \sigma \left\| \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} |D_{\alpha'} f| \right\|_{L^2}^2.
\end{align*}
\]

Observe that as \( D_t g = -\text{Im}(D_{\alpha'} Z_t) \) we have from (23) that \( \mathbb{P}_H D_t g \approx \frac{i}{2} D_{\alpha'} Z_t \).

The energy \( E_{\sigma} \) defined in Sect. 5 consists of the following

(a) \( E_{\sigma,0} \): ad-hoc but carefully chosen lower order terms introduced to help in closing the energy estimate but which still allows the scaling \( \sigma/\epsilon^3 \) in Corollary 3.2.5

(b) \( E_{\sigma,1} \): Energy as above for \( f \approx Z_t Z_{\alpha'} \approx \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \right)^{-1} \mathbb{P}_H D_t g \)

(c) \( E_{\sigma,2} \): Energy as above for \( f \approx |\partial_{\alpha'}|^{1/2} Z_t \approx \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \right)^{-1/2} \mathbb{P}_H D_t g \)

(d) \( E_{\sigma,3} \): Energy as above for \( f \approx |\partial_{\alpha'}|^{-1/2} \Theta \approx \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \right)^{1/2} \mathbb{P}_H g \)

(e) \( E_{\sigma,4} \): Energy as above for \( f \approx \frac{1}{|Z_{\alpha'}|} \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \approx -2i \mathbb{P}_H D_t g \)

Observe that the variables used as \( f \) consist of applying the operators \( D_t \) and \( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \) on the variable \( g \). These show up as they are part of the fundamental operators in the quasilinear equation (19) which are namely

\[
\begin{align*}
T_a = D_t = \text{material derivative} \quad T_b = \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \quad \text{and} \quad T_c = \frac{\sigma^3}{|Z_{\alpha'}|^2} \partial_{\alpha'}
\end{align*}
\]

where we have ignored \( A_1 \) as we prove in the energy estimate that \( 1 \leq A_1 \leq 1 + \| Z_{t, \alpha'} \|_2^2 \) and hence we can consider \( A_1 \approx 1 \). All terms in the energy \( E_{\sigma} \) and in fact all the terms showing up in Sect. 5.1 can be written entirely in terms of these operators and \( g \) (and also using the operator \( \mathbb{P}_H \)).

\[5\] For example if we introduce the lower order term \( \| \sigma^{1/2} Z_{\alpha'} \|_{L^1} \sim \| T_b^{-1} T_c \mathbb{P}_H (g) \|_{L^1} \) in \( E_{\sigma,0} \), then it significantly simplifies the proof of the energy estimate and still allows angled crested interfaces for \( \sigma = 0 \). However the introduction of this term does not allow the scaling \( \sigma/\epsilon^3 \) in Corollary 3.2 and one would then get a weaker result.
To understand how the energy looks like in the arc length coordinate system, we define the operators

\[ T_1 = D_t = \text{material derivative} \quad T_2 = \sigma \partial_s \quad \text{and} \quad T_3 = \sigma^{\frac{1}{2}} \partial_s \]

where \( \sigma = -\frac{\partial P}{\partial n} \bigg|_{\sigma=0, n=0} \) and \( \partial_s \) is the arc length derivative. The \( D_t \) above is by abuse of notation the material derivative in arc length coordinates. It is clear from (10) that \( \sigma \) in conformal coordinates is exactly \( \frac{1}{|Z_{\alpha'}|} \) and we also see that the arc length derivative \( \partial_s \) in conformal coordinates is the operator \( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \). To get the analog of \( \mathbb{P}_H \) in arc length coordinates, we let \( \mathbb{P}_{hol} \) denote the linear operator defined by the property that for any smooth real valued function \( f : \Sigma \to \mathbb{R} \) vanishing at infinity, \( \mathbb{P}_{hol}(f) : \Sigma \to \mathbb{C} \) is the boundary value of a holomorphic function on \( \Omega \) vanishing at infinity with \( \text{Re}[\mathbb{P}_{hol}(f)] = f/2 \).

We can now give a heuristic representation of \( \mathcal{E}_\sigma \) in both conformal and arc length coordinates systems using these operators. Let \( g, \theta \) be the angle of the interface with the \( x \)-axis in conformal and arc length coordinates respectively. We have the following representation for \( \mathcal{E}_{\sigma,1} \):

1. \( \frac{\partial_{\alpha'} \frac{1}{Z_{\alpha'}}}{2} \sim \frac{T_b^2 \mathbb{P}_H(g)}{H^{\frac{1}{2}}} \sim \frac{T_2^\frac{1}{2} \mathbb{P}_{hol}(\theta)}{H^{\frac{1}{2}}} \)
2. \( \frac{1}{Z_{\alpha'}} \frac{\partial_{\alpha'} \frac{1}{Z_{\alpha'}}}{H^{\frac{1}{2}}} \sim \|T_b \mathbb{P}_H(g)\| H^{\frac{1}{2}} \sim \|T_2 \mathbb{P}_{hol}(\theta)\| H^{\frac{1}{2}} \)
3. \( \frac{\sigma^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}}{2} \sim \|T_c^2 \mathbb{P}_H(g)\| H^{\frac{1}{2}} \sim \|T_3^\frac{1}{2} \mathbb{P}_{hol}(\theta)\| H^{\frac{1}{2}} \)
4. \( \frac{\sigma^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}}{H^{\frac{1}{2}}} \sim \|T_b^{-\frac{1}{2}} T_c^\frac{3}{2} \mathbb{P}_H(g)\| \sim \|T_2^{-\frac{1}{2}} T_3^\frac{3}{2} \mathbb{P}_{hol}(\theta)\| \)
5. \( \frac{\sigma^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}}{2} \sim \frac{T_c^2 \mathbb{P}_H(g)}{H^{\frac{1}{2}}} \sim \frac{T_3^\frac{1}{2} \mathbb{P}_{hol}(\theta)}{H^{\frac{1}{2}}} \)
6. \( \frac{\sigma^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}}{H^{\frac{1}{2}}} \sim \frac{T_b^2 T_c^\frac{3}{2} \mathbb{P}_H(g)}{H^{\frac{1}{2}}} \sim \frac{T_2^\frac{1}{2} T_3^\frac{3}{2} \mathbb{P}_{hol}(\theta)}{H^{\frac{1}{2}}} \)
7. \( \|\sigma \partial_{\alpha'} \theta\| H^{\frac{1}{2}} \sim \|T_b^{-1} T_c^3 \mathbb{P}_H(g)\| H^{\frac{1}{2}} \sim \|T_2^{-1} T_3^3 \mathbb{P}_{hol}(\theta)\| H^{\frac{1}{2}} \)
8. \( \|\sigma Z_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\| H^{\frac{1}{2}} \sim \|T_b^{-\frac{1}{2}} T_c^3 \mathbb{P}_H(g)\| H^{\frac{1}{2}} \sim \|T_2^{-\frac{1}{2}} T_3^3 \mathbb{P}_{hol}(\theta)\| H^{\frac{1}{2}} \)
9. \( \|\sigma Z_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\| H^{\frac{1}{2}} \sim \|T_c^3 \mathbb{P}_H(g)\| H^{\frac{1}{2}} \sim \|T_3^3 \mathbb{P}_{hol}(\theta)\| H^{\frac{1}{2}} \)

As mentioned before we have \( \mathbb{P}_H D_t g \approx \frac{1}{2} \overline{D_{\alpha'}} Z_t \). Hence \( \mathcal{E}_{\sigma,2} \) has the heuristic representation

1. \( \|Z_{t,\alpha'}\| H^{\frac{1}{2}} \sim \|T_a T_b^{-\frac{1}{2}} \mathbb{P}_H(g)\| H^{\frac{1}{2}} \sim \|T_1 T_2^{-\frac{1}{2}} \mathbb{P}_{hol}(\theta)\| H^{\frac{1}{2}} \)
2. \( \|\frac{1}{Z_{\alpha'}} \partial_{\alpha'} \overline{Z_{t,\alpha'}}\| H^{\frac{1}{2}} \sim \|T_a T_b^\frac{1}{2} \mathbb{P}_H(g)\| H^{\frac{1}{2}} \sim \|T_1 T_2^\frac{1}{2} \mathbb{P}_{hol}(\theta)\| H^{\frac{1}{2}} \)
The operators $\mathbb{P}_m$ and $\mathbb{P}_{hol}$ are not fundamental in the representation and are more of a technical nature. One can ignore them to get the essence of the energy. Also heuristically for the arc length representation, one can write $T_1 \theta$ as the arc length derivative of the velocity on the boundary. For example if the velocity on the boundary in arc length coordinates is $z_t$, then one can heuristically write

$$ \| \frac{\partial}{\partial \alpha} Z_{t, \alpha'} \|_2 \sim \left\| T_1 T_2^{-\frac{1}{2}} \mathbb{P}_{hol}(\theta) \right\|_{H^\frac{1}{2}}.$$  

Similarly for other terms of $E_{\sigma, 2}$. Writing the energy with variables being the free surface elevation $\eta$ and the velocity potential $\phi$ is a little more complicated in general due to the restriction that the interface has to be a graph. However if for example there is a single singularity of the interface at the origin and the data is symmetric, then the energy in these variables can be written easily by converting it from the representation in arc length coordinates. However even then one has to be a little careful as the energy $E_{\sigma, 2}$ for $\sigma = 0$ allows interfaces with cusps (as shown in [2]), which would make the slope of the interface at the cusp infinite and the interface non chord-arc, and this may create some difficulties in proving a local wellposedness result in these coordinates.

(4) Analytical difficulties:

In addition to the structural difficulties due to surface tension explained above, we also face numerous analytical difficulties. Even in the special case of $\sigma = 0$, the energy $E_{\sigma, 2}$ is lower order as compared to the energy in Kinsey-Wu [23] by half weighted spacial derivative and we crucially do not have $D_{\alpha'} Z_{tt} \in L^\infty$. This makes our estimates even in the case of $\sigma = 0$ much more subtle. In addition we now have a lot of nonlinear terms due to surface tension which we need to control. To overcome these issues, we define weighted function spaces adapted to our problem and prove estimates for these spaces (see Lemma 5.3). We use these function spaces along with estimates from harmonic analysis to control the nonlinear terms.

4. The Quasilinear Equations

We will now use the fundamental equation (12) as the starting point to derive our quasilinear equations. Our main equation is for the variable $D_{\alpha'} Z_{t, \alpha'}$ which is obtained by applying the operators $D_{\alpha'} D_t$ to the equation (12). We also obtain equations for $Z_{t}, Z_{t, \alpha'}$, and $\Theta$ which should be thought of lower order and auxiliary equations. Let us first derive some simple but useful formulas:

(a) We have

$$ \frac{Z_{t, \alpha'}}{|Z_{t, \alpha'}|} \frac{1}{Z_{t, \alpha'}} = \omega \frac{\bar{\omega}}{|\bar{\omega}|} = \frac{1}{|Z_{t, \alpha'}|} + \omega |D_{\alpha'} \bar{\omega}|$$
Observe that \( \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \) is real valued and \( \omega|D_{\alpha'}|^2 \) is purely imaginary. From this we obtain the relations
\[
\text{Re} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) = \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|}, \quad \text{Im} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) = i (\bar{\omega}|D_{\alpha'}| \omega) = -\text{Re} \Theta
\] (20)

(b) From (7) we see that
\[
\frac{\partial}{\partial t} \frac{Z_{t, \alpha'}}{Z_{\alpha'}} = \frac{\partial_{\alpha'}}{Z_{t, \alpha'}} \frac{1}{Z_{\alpha'}}
\]

and hence using (23) we obtain
\[
= \frac{\partial_{\alpha'}}{Z_{t, \alpha'}} \frac{1}{Z_{\alpha'}}
\]

Now as \( b_{\alpha'} \) is real valued we have
\[
\frac{\partial}{\partial t} b_{\alpha'} = -\text{Im}(\overline{D_{\alpha'}} Z_t)
\] (21)

(c) Observe that for any complex valued function \( f, \overline{\mathbb{H}}(\text{Re} f) = i \text{Im}(\mathbb{H} f) \) and \( \mathbb{H}(i \text{Im} f) = \text{Re}(\mathbb{H} f) \). Hence we get the following useful identities
\[
\begin{align*}
(\mathbb{I} + \mathbb{H})(\text{Re} f) &= f - i \text{Im}(\mathbb{I} - \mathbb{H}) f \\
(\mathbb{I} + \mathbb{H})(i \text{Im} f) &= f - \text{Re}(\mathbb{I} - \mathbb{H}) f
\end{align*}
\] (22) (23)

Now we observe that
\[
A_1 = 1 - \text{Im}[Z_{t, \alpha'}] Z_{t, \alpha'} = \text{Re}(\mathbb{I} - \mathbb{H}) \left\{ 1 + i Z_t Z_{t, \alpha'} \right\}
\]
and hence using (23) we obtain
\[
A_1 = 1 + i Z_t Z_{t, \alpha'} - i (\mathbb{I} + \mathbb{H}) \left\{ \text{Re}(Z_t Z_{t, \alpha'}) \right\}
\] (24)

Similarly as \( b = \text{Re}(\mathbb{I} - \mathbb{H}) \left( \frac{Z_t}{Z_{\alpha'}} \right) \) we again use (23) to get
\[
b = \frac{Z_t}{Z_{\alpha'}} - i (\mathbb{I} + \mathbb{H}) \left\{ \text{Im} \left( \frac{Z_t}{Z_{\alpha'}} \right) \right\}
\]
and hence
\[
b_{\alpha'} = D_{\alpha'} Z_t + Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}} - i \partial_{\alpha'} (\mathbb{I} + \mathbb{H}) \left\{ \text{Im} \left( \frac{Z_t}{Z_{\alpha'}} \right) \right\}
\] (25)

(d) We now record some frequently used commutator identities. They are easily seen by differentiating
\[
[\partial_{\alpha'}, D_t] = b_{\alpha'} \partial_{\alpha'} \quad \quad [D_{\alpha'}, D_t] = \text{Re}(D_{\alpha'} Z_t) |D_{\alpha'}| = \text{Re}(\overline{D_{\alpha'}} Z_t) |D_{\alpha'}|
\] (26)

\[
[D_{\alpha'}, D_t] = (D_{\alpha'} Z_t) D_{\alpha'} \quad \quad [\overline{D_{\alpha'}} D_t] = (\overline{D_{\alpha'}} Z_t) \overline{D_{\alpha'}}
\] (27)

Using the commutator relation \( [\partial_{\alpha'}, D_t] = b_{\alpha'} \partial_{\alpha'} \) we obtain the following formulae
\[
D_t |Z_{\alpha'}| = D_t e^{\text{Re} \log Z_{\alpha'}} = |Z_{\alpha'}|^2 \{ \text{Re}(D_{\alpha'} Z_t) - b_{\alpha'} \}
\] (28)

\[
D_t \frac{1}{Z_{\alpha'}} = -\frac{1}{Z_{\alpha'}} (D_{\alpha'} Z_t - b_{\alpha'}) = \frac{1}{Z_{\alpha'}} \left( b_{\alpha'} - D_{\alpha'} Z_t - \overline{D_{\alpha'}} \overline{Z_t} + \overline{D_{\alpha'}} \overline{Z_t} \right)
\] (29)

Observe that \( (b_{\alpha'} - D_{\alpha'} Z_t - \overline{D_{\alpha'}} \overline{Z_t}) \) is real valued and this fact will be useful later on.
(e) As we will frequently work with the operator \(|D_{\alpha'}|^3\) we record some commonly used expansions

\[
|D_{\alpha'}|^2 f = \left( \frac{1}{|Z_{\alpha'}|} \right) |D_{\alpha'}| f + \frac{1}{|Z_{\alpha'}|^2} \Delta_{\alpha'} f
\]

\[
|D_{\alpha'}|^3 f = \left( \frac{1}{|Z_{\alpha'}|} \right) |D_{\alpha'}|^2 f + \frac{1}{|Z_{\alpha'}|^2} \Delta_{\alpha'} f
\]

\[
+ 3 \left( \frac{1}{|Z_{\alpha'}|} \right) \frac{1}{|Z_{\alpha'}|^2} \Delta_{\alpha'} f
\]

\[
+ \frac{1}{|Z_{\alpha'}|^3} \Delta_{\alpha'} f
\]

\[
(30)
\]

\[
|D_{\alpha'}|^3 f = \partial_{\alpha'} \left\{ \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} f \right) \right\} - \frac{3}{2} \left( \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|^2} \right) \frac{1}{|Z_{\alpha'}|^2} \Delta_{\alpha'} f
\]

\[
- 2 \left( \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|^2} \right) |D_{\alpha'}| f - \frac{1}{2} \left( \frac{1}{|Z_{\alpha'}|^2} \right) |D_{\alpha'}|^2 |D_{\alpha'}| f
\]

\[
(31)
\]

We will now derive formulas for \(\Theta, D_t\Theta\) and \(D_t^2\Theta\). All three of them are derived similarly.

4.1. Formula for \(\Theta\). We know from (20) that \(\text{Re} \Theta = -\text{Im} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)\). Applying \((I + \mathbb{H})\) to this formula and using the identities (22) and (23) we get

\[
\Theta = i \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} - i \text{Re}(I - \mathbb{H}) \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)
\]

\[
(32)
\]

As \(\partial_{\alpha'} \frac{1}{Z_{\alpha'}}\) is holomorphic, this implies that the second term in the above formula is lower order. Hence \(\Theta \approx i \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\) and therefore \(\Theta\) and \(\partial_{\alpha'} \frac{1}{Z_{\alpha'}}\) have the same regularity.

4.2. Formula for \(D_t\Theta\). Apply \(|D_{\alpha'}|\) on the formula for \(D_t g\) in (21) to obtain

\[
D_t |D_{\alpha'}| g = -\text{Im}(D_{\alpha'} \overline{D_{\alpha'}} \overline{Z_t}) - \text{Re}(D_{\alpha'} \overline{Z_t}) |D_{\alpha'}| g
\]

As \(|D_{\alpha'}| g = \text{Re} \Theta\), hence \(\text{Re}(D_{\alpha'} \overline{Z_t}) |D_{\alpha'}| g = \text{Re} \left\{ (D_{\alpha'} \overline{Z_t}) \text{Re} \Theta \right\} = \text{Im} \left\{ i (D_{\alpha'} \overline{Z_t}) \text{Re} \Theta \right\} \). Also observe that \(D_t |D_{\alpha'}| g = \text{Re}(D_t \Theta)\). Hence we have

\[
\text{Re}(D_t \Theta) = -\text{Im} \left\{ (|D_{\alpha'}| + i \text{Re} \Theta) \overline{D_{\alpha'}} \overline{Z_t} \right\}
\]

\[
(33)
\]

Now apply \((I + \mathbb{H})\) on both sides and use the identities (22) and (23) to get

\[
D_t \Theta = i (|D_{\alpha'}| + i \text{Re} \Theta) \overline{D_{\alpha'}} \overline{Z_t} - i \text{Re}(I - \mathbb{H}) \left\{ (|D_{\alpha'}| + i \text{Re} \Theta) \overline{D_{\alpha'}} \overline{Z_t} \right\} + i \text{Im}(I - \mathbb{H}) D_t \Theta
\]

(34)

Note that \(D_{\alpha'} \overline{Z_t}\) and \(\Theta\) are holomorphic this causes the second and third term in the above formula to be of lower order. For example observe that as \((I - \mathbb{H}) \Theta = 0\), we have

\[
(I - \mathbb{H}) D_t \Theta = [D_t, \mathbb{H}] \Theta = [b, \mathbb{H}] \partial_{\alpha'} \Theta
\]

Hence \(D_t \Theta \approx i (|D_{\alpha'}| + i \text{Re} \Theta) \overline{D_{\alpha'}} \overline{Z_t}\).
4.3. Formula for $D_t^2 \Theta$. Apply $D_t$ on the formula for $\text{Re}(D_t \Theta)$ in (33) to obtain

$$\text{Re}(D_t^2 \Theta) = -\text{Im}\left\{ D_t (|D_{\alpha'}| + i\text{Re}\Theta) D_{\alpha'} Z_t \right\}$$

$$= -\text{Im}\left\{ (|D_{\alpha'}| + i\text{Re}\Theta) D_{\alpha'} Z_t \right\}$$

$$+ \text{Im}\{ \text{Re}(D_{\alpha'} Z_t)|D_{\alpha'}| D_{\alpha'} Z_t - i\text{Re}(D_t \Theta) D_{\alpha'} Z_t \}$$

Now apply $(\mathbb{I} + \mathbb{H})$ on both sides and use the identities (22) and (23) to get

$$D_t^2 \Theta = i(|D_{\alpha'}| + i\text{Re}\Theta) D_t \overline{D_{\alpha'} Z_t} - i\text{Re}(\mathbb{I} - \mathbb{H})\{ (|D_{\alpha'}| + i\text{Re}\Theta) D_{\alpha'} Z_t \}$$

$$+ i\text{Im}(\mathbb{I} - \mathbb{H}) D_t^2 \Theta + (\mathbb{I} + \mathbb{H})\text{Im}\{ \text{Re}(D_{\alpha'} Z_t)|D_{\alpha'}| D_{\alpha'} Z_t - i\text{Re}(D_t \Theta) D_{\alpha'} Z_t \}$$

Again in this formula only the first term is the main term and all other terms are lower order. Hence $D_t^2 \Theta \approx i(|D_{\alpha'}| + i\text{Re}\Theta) D_t \overline{D_{\alpha'} Z_t}$.

4.4. Equation for $\overline{Z_t}$. Apply $D_t$ to the fundamental equation (12)

$$\overline{Z_{ttt}} = -i \frac{D_t A_1}{Z_{,\alpha'}} - i \frac{A_1}{Z_{,\alpha'}} \left( Z_{,\alpha'} D_t \frac{1}{Z_{,\alpha'}} \right) - \sigma (D_{\alpha'} Z_t) D_{\alpha'} \Theta + \sigma D_{\alpha'} D_t \Theta$$

Now use the formula for $D_t \frac{1}{Z_{,\alpha'}}$ from (29) and $D_t \Theta$ from (34) to obtain

$$\overline{Z_{ttt}} = -i \frac{1}{Z_{,\alpha'}} \left( D_t A_1 + A_1 (b_{\alpha'} - D_{\alpha'} Z_t - \overline{D_{\alpha'} Z_t}) \right) - i \frac{A_1}{Z_{,\alpha'}} \overline{D_{\alpha'} Z_t} - \sigma (D_{\alpha'} Z_t) D_{\alpha'} \Theta$$

$$+ i \sigma D_{\alpha'} (|D_{\alpha'}| + i\text{Re}\Theta) \overline{D_{\alpha'} Z_t} - i \sigma D_{\alpha'} \text{Re}(\mathbb{I} - \mathbb{H})\{ (|D_{\alpha'}| + i\text{Re}\Theta) \overline{D_{\alpha'} Z_t} \}$$

$$+ i \sigma D_{\alpha'} \text{Im}(\mathbb{I} - \mathbb{H}) D_t \Theta$$

Let us define the real valued variable $J_1$ as

$$J_1 = D_t A_1 + A_1 (b_{\alpha'} - D_{\alpha'} Z_t - \overline{D_{\alpha'} Z_t}) + \sigma \partial_{\alpha'} \text{Re}(\mathbb{I} - \mathbb{H})\{ (|D_{\alpha'}| + i\text{Re}\Theta) \overline{D_{\alpha'} Z_t} \}$$

$$- \sigma \partial_{\alpha'} \text{Im}(\mathbb{I} - \mathbb{H}) D_t \Theta$$

(36)

Using this we get

$$\overline{Z_{ttt}} + i \frac{A_1}{Z_{,\alpha'}} \overline{D_{\alpha'} Z_t} - i \sigma D_{\alpha'} (|D_{\alpha'}| + i\text{Re}\Theta) \overline{D_{\alpha'} Z_t} = -\sigma (D_{\alpha'} Z_t) D_{\alpha'} \Theta - i \frac{J_1}{Z_{,\alpha'}}$$

(37)

We modify this equation slightly to get an equation appropriate for the computation of the lower order term in the energy. Rewrite the above equation as

$$\overline{Z_{ttt}} + i \frac{A_1}{Z_{,\alpha'}} \overline{D_{\alpha'} Z_t} - i \sigma D_{\alpha'} \left\{ \left( |D_{\alpha'}| \frac{1}{Z_{,\alpha'}} \right) Z_{1,\alpha'} + \frac{1}{Z_{,\alpha'}} |D_{\alpha'}| Z_{1,\alpha'} \right\}$$

$$= -\sigma (D_{\alpha'} Z_t) D_{\alpha'} \Theta - \sigma D_{\alpha'} \left\{ (\text{Re}\Theta) \overline{D_{\alpha'} Z_t} \right\} - i \frac{J_1}{Z_{,\alpha'}}$$
Multiply by \( Z_{t,\alpha'} \) and rearrange to get

\[
\begin{align*}
Z_{ttt} Z_{t,\alpha'} + i A_1 \overline{D_{t,\alpha'}} \overline{Z}_t - i \sigma \partial_{\alpha'} \left( \frac{1}{Z_{t,\alpha'}} |D_{t,\alpha'}| \overline{Z}_{t,\alpha'} \right) \\
= i \sigma \partial_{\alpha'} \left\{ \left( |D_{t,\alpha'}| \frac{1}{Z_{t,\alpha'}} \right) \overline{Z}_{t,\alpha'} \right\} - \sigma (D_{t,\alpha'} Z_t) \partial_{\alpha'} \Theta - \sigma \partial_{\alpha'} \{ (\text{Re} \Theta) \overline{D_{t,\alpha'}} \overline{Z}_t \} - i J_1
\end{align*}
\]

(38)

This equation gives rise to the energy \( E_{\sigma,1} \) in the energy estimate.

4.5. **Equation for \( \overline{D_{t,\alpha'}} \overline{Z}_t \).** Apply \( \overline{D_{t,\alpha'}} \) to the equation (37) and use commutator identities to get

\[
\begin{align*}
D_t^2 \overline{D_{t,\alpha'}} \overline{Z}_t + i \frac{A_1}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} \overline{D_{t,\alpha'}} \overline{Z}_t - i \sigma \overline{D_{t,\alpha'}} D_{t,\alpha'} (|D_{t,\alpha'}| + i \text{Re} \Theta) \overline{D_{t,\alpha'}} \overline{Z}_t \\
= -(\overline{D_{t,\alpha'}} \overline{Z}_t) \overline{D_{t,\alpha'}} \overline{Z}_{ttt} - 2 (\overline{D_{t,\alpha'}} \overline{Z}_t) (D_t \overline{D_{t,\alpha'}} \overline{Z}_t) - i \overline{D_{t,\alpha'}} \left( \frac{A_1}{Z_{t,\alpha'}} \right) (\overline{D_{t,\alpha'}} \overline{Z}_t) \\
- \sigma \overline{D_{t,\alpha'}} ((D_{t,\alpha'} Z_t) D_{t,\alpha'} \Theta) - i \left( \overline{D_{t,\alpha'}} \frac{1}{Z_{t,\alpha'}} \right) J_1 - i \frac{i}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} J_1
\end{align*}
\]

Observe from (12) that \( -(\overline{D_{t,\alpha'}} \overline{Z}_t) \overline{D_{t,\alpha'}} \left( \overline{Z}_{ttt} + i \frac{A_1}{Z_{t,\alpha'}} \right) = -\sigma (\overline{D_{t,\alpha'}} \overline{Z}_t) \overline{D_{t,\alpha'}} D_{t,\alpha'} \Theta \). Hence we get

\[
\begin{align*}
D_t^2 \overline{D_{t,\alpha'}} \overline{Z}_t + i \frac{A_1}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} \overline{D_{t,\alpha'}} \overline{Z}_t - i \sigma \overline{D_{t,\alpha'}} D_{t,\alpha'} (|D_{t,\alpha'}| + i \text{Re} \Theta) \overline{D_{t,\alpha'}} \overline{Z}_t \\
= -2 (\overline{D_{t,\alpha'}} \overline{Z}_t) (D_t \overline{D_{t,\alpha'}} \overline{Z}_t) - 2 \sigma \text{Re} (D_{t,\alpha'} Z_t) D_{t,\alpha'} \Theta - \sigma (\overline{D_{t,\alpha'}} D_{t,\alpha'} Z_t) D_{t,\alpha'} \Theta \\
- i \left( \overline{D_{t,\alpha'}} \frac{1}{Z_{t,\alpha'}} \right) J_1 - i \frac{i}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} J_1
\end{align*}
\]

Now

\[
\begin{align*}
\overline{D_{t,\alpha'}} D_{t,\alpha'} &= \frac{1}{Z_{t,\alpha'}} \partial_{\alpha'} \left( \frac{|Z_{t,\alpha'}|}{|Z_{t,\alpha'}|} |D_{t,\alpha'}| \right) = \left( \frac{1}{Z_{t,\alpha'}} \partial_{\alpha'} \frac{|Z_{t,\alpha'}|}{|Z_{t,\alpha'}|} \right) |D_{t,\alpha'}| + |D_{t,\alpha'}|^2 \\
&= (|D_{t,\alpha'}| - i \text{Re} \Theta) |D_{t,\alpha'}|
\end{align*}
\]

Hence \( \overline{D_{t,\alpha'}} D_{t,\alpha'} (|D_{t,\alpha'}| + i \text{Re} \Theta) \)
\[
\begin{align*}
&= (|D_{t,\alpha'}| - i \text{Re} \Theta) |D_{t,\alpha'}| (|D_{t,\alpha'}| + i \text{Re} \Theta) \\
&= (|D_{t,\alpha'}| - i \text{Re} \Theta) (|D_{t,\alpha'}| + i \text{Re} \Theta) |D_{t,\alpha'}| + (|D_{t,\alpha'}| - i \text{Re} \Theta) (i \text{Re} (|D_{t,\alpha'}| \Theta)) \\
&= (|D_{t,\alpha'}|^2 + i \text{Re} (|D_{t,\alpha'}|^2 \Theta) + (\text{Re} \Theta)^2) |D_{t,\alpha'}| + i \text{Re} (|D_{t,\alpha'}|^2 \Theta) + i \text{Re} (|D_{t,\alpha'}| \Theta) |D_{t,\alpha'}| \\
&\quad + (\text{Re} \Theta) \text{Re} (|D_{t,\alpha'}|^2 \Theta) \\
&= |D_{t,\alpha'}|^3 + (2i \text{Re} (|D_{t,\alpha'}|^2 \Theta) + (\text{Re} \Theta)^2) |D_{t,\alpha'}| + i \text{Re} (|D_{t,\alpha'}|^2 \Theta) + (\text{Re} \Theta) \text{Re} (|D_{t,\alpha'}|^2 \Theta) \quad (39)
\end{align*}
\]

Therefore \( \overline{D_{t,\alpha'}} D_{t,\alpha'} (|D_{t,\alpha'}| + i \text{Re} \Theta) \approx |D_{t,\alpha'}|^3 \). Hence we get the main equation for \( \overline{D_{t,\alpha'}} \overline{Z}_t \):

\[
\begin{align*}
\left( D_t^2 + i \frac{A_1}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} - i \sigma |D_{t,\alpha'}|^3 \right) \overline{D_{t,\alpha'}} \overline{Z}_t = R_1 - i \left( \overline{D_{t,\alpha'}} \frac{1}{Z_{t,\alpha'}} \right) J_1 - i \frac{i}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} J_1 \quad (40)
\end{align*}
\]
where

\[ R_1 = -2(\overline{D}_{\alpha'} Z_t)(D_t \overline{D}_{\alpha'} Z_t) - 2\sigma \text{Re}(D_{\alpha'} Z_t) \overline{D}_{\alpha'} D_{\alpha'} \Theta - \sigma (\overline{D}_{\alpha'} D_{\alpha'} Z_t) D_{\alpha'} \Theta + i \sigma \left( 2i \text{Re}(|D_{\alpha'}|^2) + (\text{Re}\Theta)^2 \right) \left| D_{\alpha'} \overline{D}_{\alpha'} Z_t \right| - \sigma \text{Re}\left( |D_{\alpha'}|^2 \Theta \right) \overline{D}_{\alpha'} Z_t \]

\[ + i \sigma (\text{Re}\Theta)(\text{Re}|D_{\alpha'}|^2) \overline{D}_{\alpha'} Z_t \]

(41)

and \( J_1 \) was defined in (36). This equation gives rise to the energy \( E_{\sigma, t} \) in the energy estimate.

4.6. Equation for \( \overline{Z}_{t, \alpha'} \). Multiply the equation for \( \overline{D}_{\alpha'} \overline{Z}_t \) in (40) by \( \overline{Z}_{t, \alpha'} \) to get the equation

\[ \left( D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^3 \right) \overline{Z}_{t, \alpha'} = R_1 \overline{Z}_{t, \alpha'} - i \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)J_1 - i D_{\alpha'} \overline{Z}_{t, \alpha'} \left[ D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^3, \frac{1}{Z_{\alpha'}} \right] \overline{Z}_{t, \alpha'} \]

(42)

This equation gives rise to the energy \( E_{\sigma, 2} \) in the energy estimate. This equation will also be useful to get estimates for the term \( D_{\alpha'} J_1 \).

4.7. Equation for \( \Theta \). Apply \( \overline{D}_{\alpha'} \) to the fundamental equation (12) and use \( \overline{D}_{\alpha'} D_{\alpha'} = (|D_{\alpha'}| - i \text{Re}\Theta)|D_{\alpha'}| \) from (39) to obtain

\[ D_t \overline{D}_{\alpha'} \overline{Z}_t + i \overline{D}_{\alpha'} \left( \frac{A_1}{Z_{\alpha'}} \right) - \sigma (|D_{\alpha'}| - i \text{Re}\Theta)|D_{\alpha'}| \Theta = - (\overline{D}_{\alpha'} \overline{Z}_t)^2 \]

Now applying the operator \( i (|D_{\alpha'}| + i \text{Re}\Theta) \) we obtain

\[ i(|D_{\alpha'}| + i \text{Re}\Theta) D_t \overline{D}_{\alpha'} \overline{Z}_t - |D_{\alpha'}| \overline{D}_{\alpha'} \left( \frac{A_1}{Z_{\alpha'}} \right) - i \sigma (|D_{\alpha'}| + i \text{Re}\Theta) (|D_{\alpha'}| + i \text{Re}\Theta) |D_{\alpha'}| \Theta \]

\[ = -2i \left( \overline{D}_{\alpha'} \overline{Z}_t \right) (|D_{\alpha'}| \overline{D}_{\alpha'} \overline{Z}_t) + (\text{Re}\Theta) \left\{ \left( \overline{D}_{\alpha'} \overline{Z}_t \right)^2 + i \overline{D}_{\alpha'} \left( \frac{A_1}{Z_{\alpha'}} \right) \right\} \]

Observe that

\[ (|D_{\alpha'}| + i \text{Re}\Theta) (|D_{\alpha'}| - i \text{Re}\Theta) |D_{\alpha'}| |\Theta = |D_{\alpha'}| |^{3} \Theta - i \text{Re}(|D_{\alpha'}| |\Theta) |D_{\alpha'}| | + (\text{Re}\Theta)^{2} |D_{\alpha'}| | \Theta \]

Hence we get

\[ i(|D_{\alpha'}| + i \text{Re}\Theta) D_t \overline{D}_{\alpha'} \overline{Z}_t - |D_{\alpha'}| \overline{D}_{\alpha'} \left( \frac{A_1}{Z_{\alpha'}} \right) - i \sigma |D_{\alpha'}|^{3} \Theta \]

\[ = -2i \left( \overline{D}_{\alpha'} \overline{Z}_t \right) (|D_{\alpha'}| \overline{D}_{\alpha'} \overline{Z}_t) + (\text{Re}\Theta) \left\{ \left( \overline{D}_{\alpha'} \overline{Z}_t \right)^2 + i \overline{D}_{\alpha'} \left( \frac{A_1}{Z_{\alpha'}} \right) + i \sigma (\text{Re}\Theta) |D_{\alpha'}| \Theta \right\} \]

\[ + \sigma \text{Re}(|D_{\alpha'}| \Theta) |D_{\alpha'}| \Theta \]

Now recall from (32) that \( \Theta = i \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} - i \text{Re}(\mathbb{I} - \mathbb{H}) \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \). Therefore

\[ -|D_{\alpha'}| \overline{D}_{\alpha'} \left( \frac{A_1}{Z_{\alpha'}} \right) = -|D_{\alpha'}| \left\{ \frac{A_1}{|Z_{\alpha'}|} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) + \frac{1}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} A_1 \right\} \]

\[ = i \frac{A_1}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} \Theta - \frac{A_1}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} \text{Re}(\mathbb{I} - \mathbb{H}) \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \]
- \left( |D_{\alpha'}| \frac{A_1}{|Z_{\alpha'}|} \right) \left( \frac{Z_{\alpha'} |\partial_{\alpha'} \frac{1}{|Z_{\alpha'}|}}{2} \right) - \left( |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|} \frac{\partial_{\alpha'} A_1}{2} \right) \right)

Hence we have
\[ i(|D_{\alpha'}| + i \Re \Theta) D_t \overline{D_{\alpha'}} \overline{Z_t} + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \Theta - i \sigma |D_{\alpha'}|^3 \Theta \]
\[ = -2i(\overline{D_{\alpha'}} \overline{Z_t})(|D_{\alpha'}| \overline{D_{\alpha'}} \overline{Z_t}) + (\Re \Theta) \left\{ (\overline{D_{\alpha'}} \overline{Z_t})^2 + i \overline{D_{\alpha'}} \left( \frac{A_1}{Z_{\alpha'}} \right) + i \sigma (\Re \Theta) |D_{\alpha'}| \Theta \right\} + \sigma \Re(|D_{\alpha'}|) |D_{\alpha'}| \Theta + \left( |D_{\alpha'}| \frac{A_1}{|Z_{\alpha'}|} \left( \frac{Z_{\alpha'} |\partial_{\alpha'} \frac{1}{|Z_{\alpha'}|}}{2} \right) + |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|} \frac{\partial_{\alpha'} A_1}{2} \right) \right) + \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \Re(\Re - \mathbb{H}) \left( \frac{Z_{\alpha'} |\partial_{\alpha'} \frac{1}{|Z_{\alpha'}|}}{2} \right) \]

Recall from (35) that
\[ D_t^2 \Theta = i(|D_{\alpha'}| + i \Re \Theta) D_t \overline{D_{\alpha'}} \overline{Z_t} - i \Re(\Re - \mathbb{H}) \left\{ (|D_{\alpha'}| + i \Re \Theta) D_t \overline{D_{\alpha'}} \overline{Z_t} \right\} + i \Im(\Re - \mathbb{H}) D_t^2 \Theta + (\Re + \mathbb{H}) \Im \left\{ \Re(\overline{D_{\alpha'}} \overline{Z_t}) \right\} D_t |D_{\alpha'}| \overline{D_{\alpha'}} \overline{Z_t} - i \Re(D_t \Theta) \overline{D_{\alpha'}} \overline{Z_t} \]

Hence replacing the term \( i(|D_{\alpha'}| + i \Re \Theta) D_t \overline{D_{\alpha'}} \overline{Z_t} \) in the equation with \( D_t^2 \Theta \) we get our main equation as
\[ \left( D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^3 \right) \Theta = R_2 + i J_2 \] (43)

where
\[ R_2 = -2i(\overline{D_{\alpha'}} \overline{Z_t})(|D_{\alpha'}| \overline{D_{\alpha'}} \overline{Z_t}) + (\Re \Theta) \left\{ (\overline{D_{\alpha'}} \overline{Z_t})^2 + i \overline{D_{\alpha'}} \left( \frac{A_1}{Z_{\alpha'}} \right) + i \sigma (\Re \Theta) |D_{\alpha'}| \Theta \right\} + \sigma \Re(|D_{\alpha'}|) |D_{\alpha'}| \Theta + \left( |D_{\alpha'}| \frac{A_1}{|Z_{\alpha'}|} \left( \frac{Z_{\alpha'} |\partial_{\alpha'} \frac{1}{|Z_{\alpha'}|}}{2} \right) + |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|} \frac{\partial_{\alpha'} A_1}{2} \right) \right) + \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \Re(\Re - \mathbb{H}) \left( \frac{Z_{\alpha'} |\partial_{\alpha'} \frac{1}{|Z_{\alpha'}|}}{2} \right) \]
\[ J_2 = \Im(\Re - \mathbb{H}) (D_t^2 \Theta) - \Re(\Re - \mathbb{H}) \left\{ (|D_{\alpha'}| + i \Re \Theta) D_t \overline{D_{\alpha'}} \overline{Z_t} \right\} \] (45)

Note that the variable \( J_2 \) is real valued. This equation gives rise to the energy \( E_{\sigma,3} \) in the energy estimate.

### 5. Main a Priori Estimate

We now describe our main a priori estimate. Define
\[
E_{\sigma,0} = \left\| \sigma \frac{1}{2} |Z_{\alpha'}| \frac{1}{2} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right\|_2^2 + \left\| \sigma \frac{1}{6} |Z_{\alpha'}| \frac{1}{2} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right\|_6^6 + \left\| \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right\|_2^2
\]
\[+ \left\| \sigma \frac{1}{2} |Z_{\alpha'}| \frac{1}{2} \partial_{\alpha'}^2 \frac{1}{|Z_{\alpha'}|} \right\|_2^2
\]
Lemma 6.2. The rest of the section is devoted to proving the a priori estimate. Implies that sup

to the case of non-zero surface tension. Note that the hypothesis of the theorem easily
result is an extension of the a priori energy estimate obtained by Kinsey and Wu [23]

Theorem 5.1. Let

Remark 5.2. We mention a minor technical point in the statement of the theorem. The
condition from (10). The energies

We also define two new spaces

higher order than

For this term we

universal non-negative coefficients such that for all t

Observe that the variables used above are all very natural. Z_t and Z_{tt} are the velocity
acceleration on the boundary respectively, Θ is twice the holomorphic projection of
the curvature and \( \overline{D_\alpha Z_t} \) is related to the material derivative of the angle by the relation
\( \text{Im}(\overline{D_\alpha Z_t}) = -(\partial_t \theta) \circ h^{-1} \) from (21). The weight \( \frac{1}{|Z_{\alpha'}|} \) is related to the Taylor sign
condition from (10). The energies \( E_{\sigma,i} \) for \( 1 \leq i \leq 4 \) are obtained from the quasilinear
equations derived in Sect. 4 whereas the energy \( E_{\sigma,0} \) is added as a lower order term.
For \( \sigma = 0 \), the energies \( E_{\sigma,3} \) and \( E_{\sigma,4} \) are equivalent, but for \( \sigma > 0 \) the energy \( E_{\sigma,4} \) is
higher order than \( E_{\sigma,3} \).

Theorem 5.1. Let \( \sigma \geq 0 \) and let \((Z, Z_t)(t)\) be a smooth solution to (13) in \([0, T]\) for
\( T > 0 \), such that for all \( s \geq 3 \) we have \((Z_{\alpha'} - 1, \frac{1}{Z_{\alpha'}} - 1, Z_t) \in L^\infty([0, T], H^{s+\frac{1}{2}}(\mathbb{R}) \times
H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R})). Then sup_{t \in [0, T]} E_{\sigma}(t) < \infty \) and there exists a polynomial \( P \) with
universal non-negative coefficients such that for all \( t \in [0, T) \) we have

\[
\frac{dE_\sigma(t)}{dt} \leq P(E_\sigma(t))
\]

Remark 5.2. We mention a minor technical point in the statement of the theorem. The
energy \( E_{\sigma,0} \) contains a term which is the \( L^\infty \) norm of a function and hence may not
in general be differentiable in time even for smooth solutions. Hence for this term we
replace the time derivative by the upper Dini derivative \( \limsup_{s \to 0^+} \frac{||f(t+s)||_\infty - ||f(t)||_\infty}{s} \).

This theorem along with Theorem 7.8 will allow us to prove Theorem 3.1. The above
result is an extension of the a priori energy estimate obtained by Kinsey and Wu [23]
to the case of non-zero surface tension. Note that the hypothesis of the theorem easily
implies that \( \text{sup}_{t \in [0, T]} E_{\sigma}(t) < \infty \). This can also be seen from Proposition 6.1 and
Lemma 6.2. The rest of the section is devoted to proving the a priori estimate.

In this section whenever we write \( f \in L^2 \), what we mean is that there exists a
universal polynomial \( P \) with nonnegative coefficients such that \( \|f\|_2 \leq P(E_\sigma) \). Similar
definitions for \( f \in H^{\frac{1}{2}} \) or \( f \in L^\infty \). We define the norm \( \|f\|_{L^\infty \cap H^{\frac{1}{2}}} = \|f\|_\infty + \|f\|_{H^{\frac{1}{2}}} \).

We also define two new spaces \( \mathcal{C} \) and \( \mathcal{W} \):

1. If \( w \in L^\infty \) and \( |D_\alpha'|w \in L^2 \), then we say \( w \in \mathcal{W} \). Define

\[
\|w\|_\mathcal{W} = \|w\|_\infty + \|D_\alpha'|w\|_2
\]
(2) If \( f \in H^\frac{1}{2} \) and \( f|Z,\alpha'| \in L^2 \), then we say \( f \in C \). Define
\[
\|f\|_C = \|f\|_{H^\frac{1}{2}} + \left(1 + \left\|\frac{1}{Z,\alpha'}\right\|_2\right)\|f|Z,\alpha'|\|_2
\]
We also define the norm \( \|f\|_{\mathcal{W} \cap C} = \|f\|_{\mathcal{W}} + \|f\|_C \). The reason for the introduction of these spaces is that we will frequently have situations where \( f \in H^\frac{1}{2} \), \( w \in L^\infty \) and we want \( fw \in H^\frac{1}{2} \). We will also have situations where \( f \in H^\frac{1}{2} \), \( g|Z,\alpha'| \in L^2 \) and we want \( fg|Z,\alpha'| \in L^2 \). Clearly these are not true in general but in special cases this can be proved and the following lemma addresses this issue for a majority of the situations we encounter.

**Lemma 5.3.** The following properties hold for the spaces \( \mathcal{W} \) and \( C \)
1. If \( w_1, w_2 \in \mathcal{W} \), then \( w_1 w_2 \in \mathcal{W} \). Moreover \( \|w_1 w_2\|_\mathcal{W} \leq \|w_1\|_\mathcal{W} \|w_2\|_\mathcal{W} \)
2. If \( f \in C \) and \( w \in \mathcal{W} \), then \( fw \in C \). Moreover \( \|fw\|_C \lesssim \|f\|_C \|w\|_\mathcal{W} \)
3. If \( f, g \in C \), then \( fg|Z,\alpha'| \in L^2 \). Moreover \( \|fg|Z,\alpha'|\|_2 \lesssim \|f\|_C \|g\|_C \)

In the lemma and in the definitions of \( C \) and \( \mathcal{W} \), the function \( \frac{1}{|Z,\alpha'|} \) is used as a weight but there is nothing special about this function. We can define similar spaces and prove the lemma for any weight. The only property used of the weight is that \( \left\|\frac{1}{|Z,\alpha'|}\right\|_2 < \infty \). See Proposition 9.12 in the “Appendix” for more details and for the proof of the lemma.

In our case we are able to use the weight \( \frac{1}{|Z,\alpha'|} \) as \( \left\|\frac{1}{|Z,\alpha'|}\right\|_2 \) is controlled by the energy \( E_\alpha \).

In this section we will sometimes use the function \( Z^{1/2}_{,\alpha} \). This is defined as
\[
Z^{1/2}_{,\alpha} = e^{\frac{1}{2} \log(Z,\alpha')} \quad \text{where} \quad \log(Z,\alpha') \to 0 \text{ as } |\alpha'| \to \infty
\]
Note that there is no ambiguity in the definition of \( \log(Z,\alpha') \) as it is continuous and we have fixed its value at infinity. We also use the following notation
\[
[f_1, f_2; f_3](\alpha') = \frac{1}{i\pi} \int \left( \frac{f_1(\alpha') - f_1(\beta')}{\alpha' - \beta'} \right) \left( \frac{f_2(\alpha') - f_2(\beta')}{\alpha' - \beta'} \right) f_3(\beta') \, d\beta'
\]
This notation will be used in some of the computations later on in the section.

### 5.1. Quantities controlled by \( E_\sigma \)

Now we come to main part of the section. Here we control all the important terms controlled by the energy \( E_\sigma \). We will frequently use the estimates proved in the “Appendix” to control the terms such as Proposition 9.7, Proposition 9.8, Proposition 9.9 and Proposition 9.10.

For \( \sigma = 0 \) the energy \( E_\sigma \) is lower order as compared to the energy in Kinsey-Wu [23] by half weighted spacial derivative. In particular we do not have control of \( D_{\alpha'} \overline{Z}_{tt} \in L^\infty \) which was heavily used in Kinsey-Wu [23] but we only have \( D_{\alpha'} \overline{Z}_{tt} \in H^\frac{1}{2} \). Because of this, the energy estimate becomes much more subtle even in the case of \( \sigma = 0 \), and we need to prove stronger control of existing terms. For e.g. in [23] it is shown that \( A_1, D_{\alpha'} \overline{Z}_t, b_{\alpha'} \frac{1}{|Z,\alpha'|^2} \partial_{\alpha'} A_1 \in L^\infty \) and we show that in fact...
$A_1, \overline{D_{\alpha'}} \bar{Z}_t, b_{\alpha'}$, $\frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} A_1 \in L^\infty \cap \dot{H}^{\frac{1}{2}}$. Most of the terms for the $\sigma = 0$ case controlled here in $\dot{H}^{\frac{1}{2}}$ are new. Of course estimates for terms involving surface tension are also new.

(1) $\bar{Z}_{t,\alpha'} \in L^2$, $|D_{\alpha'}| \overline{D_{\alpha'}} \bar{Z}_t \in L^2$

Proof: This is true as $A_1 \geq 1$ and as $E_{\sigma,1}$ and $E_{\sigma,4}$ are part of the energy.

(2) $A_1 \in L^\infty \cap \dot{H}^{\frac{1}{2}}$

Proof: Recall that $A_1 = 1 - \text{Im}[Z_t, \mathbb{H}] \bar{Z}_{t,\alpha'}$ and hence

$$\|A_1\|_\infty \leq 1 + \|\{Z_t, \mathbb{H}\} \bar{Z}_{t,\alpha'}\|_\infty \lesssim 1 + \|\bar{Z}_{t,\alpha'}\|_2^2$$

by Proposition 9.8. Similarly by Proposition 9.8 and Sobolev embedding we have

$$\|A_1\|_{\dot{H}^{\frac{1}{2}}} \lesssim \|\partial_{\alpha'} Z_{t,\alpha'}\|_2 \lesssim \|\partial_{\alpha'} Z_t\|_{\text{BMO}} \|\bar{Z}_{t,\alpha'}\|_2 \lesssim \|\bar{Z}_{t,\alpha'}\|_2^3$$

(2) $\partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in L^2$, $\partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in L^2$, $|D_{\alpha'}| \omega \in L^2$ and hence $\omega \in \mathcal{W}$.

Proof: Observe that $\partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in L^2$ as it is part of the energy $E_{\sigma,0}$. From (20) we easily get that $\partial_{\alpha'} \frac{1}{|Z_{\alpha'}|}$ and $|D_{\alpha'}| \omega$ are in $L^2$. Also as $\|\omega\|_\infty = 1$ and $|D_{\alpha'}| \omega \in L^2$ we get that $w \in \mathcal{W}$. Now that we have shown that $\partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in L^2$, we can use Lemma 5.3 from now on.

(4) $\overline{D_{\alpha'}} \bar{Z}_t \in L^\infty$, $|D_{\alpha'}| \bar{Z}_t \in L^\infty$, $D_{\alpha'} \bar{Z}_t \in L^\infty$

Proof: We only need to prove that $\overline{D_{\alpha'}} \bar{Z}_t \in L^\infty$ and the rest follows. Observe that

$$\partial_{\alpha'} (\overline{D_{\alpha'}} \bar{Z}_t)^2 = 2 (\bar{Z}_{t,\alpha'}) (\overline{D_{\alpha'}} \bar{D_{\alpha'}} \bar{Z}_t)$$

As $\overline{D_{\alpha'}} \bar{Z}_t$ decays at infinity, by integrating we get

$$\left\| \overline{D_{\alpha'}} \bar{Z}_t \right\|_\infty \lesssim \int |\bar{Z}_{t,\alpha'}| |\overline{D_{\alpha'}} \bar{D_{\alpha'}} \bar{Z}_t| \, d\alpha' \lesssim \|\bar{Z}_{t,\alpha'}\|_L^2 \|D_{\alpha'} \overline{D_{\alpha'}} \bar{Z}_t\|_L^2$$

Hence $\|\overline{D_{\alpha'}} \bar{Z}_t\|_\infty \lesssim \sqrt{\|\bar{Z}_{t,\alpha'}\|_L^2 \|D_{\alpha'} \overline{D_{\alpha'}} \bar{Z}_t\|_L^2}$

(5) $\overline{D_{\alpha'}} \bar{Z}_t \in L^2$, $|D_{\alpha'}|^2 \bar{Z}_t \in L^2$, $D_{\alpha'}^2 \bar{Z}_t \in L^2$

Proof: We already know that $|D_{\alpha'}| \overline{D_{\alpha'}} \bar{Z}_t \in L^2$ and hence $\overline{D_{\alpha'}} \bar{Z}_t \in L^2$. Now

$$\overline{D_{\alpha'}} \bar{Z}_t = \overline{D_{\alpha'}} (|D_{\alpha'}| \bar{Z}_t) = (\overline{D_{\alpha'}} |\omega|) \overline{D_{\alpha'}} \bar{Z}_t + |\omega|^2 |D_{\alpha'}|^2 \bar{Z}_t$$

Now observe that $|D_{\alpha'}| \omega \in L^2$ and $|D_{\alpha'}| \bar{Z}_t \in L^\infty$ and hence the first term is in $L^2$. Hence we have $|D_{\alpha'}|^2 \bar{Z}_t \in L^2$. A similar argument works for the rest.

(6) $\overline{D_{\alpha'}} \bar{Z}_t \in \mathcal{W} \cap C$, $D_{\alpha'} \bar{Z}_t \in \mathcal{W} \cap C$

Proof: We will first prove $\overline{D_{\alpha'}} \bar{Z}_t \in \mathcal{W} \cap C$. Observe that $D_{\alpha'} \bar{Z}_t \in L^\infty$, $|D_{\alpha'}| \bar{D_{\alpha'}} \bar{Z}_t \in L^2$ and hence we have $D_{\alpha'} \bar{Z}_t \in \mathcal{W}$. Now as $D_{\alpha'} \bar{Z}_t$ is holomorphic i.e. $\mathbb{H} D_{\alpha'} \bar{Z}_t = D_{\alpha'} \bar{Z}_t$, we see that $|\partial_{\alpha'}| D_{\alpha'} \bar{Z}_t = i \partial_{\alpha'} D_{\alpha'} \bar{Z}_t$. Hence we have

$$\|D_{\alpha'} \bar{Z}_t\|_{\dot{H}^{\frac{1}{2}}}^2 = \int (\bar{D}_{\alpha'} Z_t) (|\partial_{\alpha'}| D_{\alpha'} \bar{Z}_t) \, d\alpha' = i \int (\bar{Z}_{t,\alpha'}) (\overline{D_{\alpha'}} D_{\alpha'} \bar{Z}_t) \, d\alpha'$$
Hence $\|D_{\alpha'} \bar{Z}_t\|_{H^\frac{3}{2}} \lesssim \sqrt{\|Z_{t,\alpha'}\|_{L^2}} \|D_{\alpha'} |D_{\alpha'} \bar{Z}_t|_{L^2}}$. Also as $(D_{\alpha'} \bar{Z}_t)|Z_{t,\alpha'}| = \bar{Z}_{t,\alpha'} \in L^2$, we have $D_{\alpha'} \bar{Z}_t \in C$. Now as $|D_{\alpha'} |Z_t| = (D_{\alpha'} \bar{Z}_t)\omega$ we have $\|D_{\alpha'} |Z_t||_{W^2C} \lesssim \|D_{\alpha'} \bar{Z}_t||_{W^2C} \|\omega\|_{W^2}$ by Lemma 5.3. The rest are proved similarly. \[ \square \]

(7) $\partial_{\alpha'} \mathbb{P}_A \left( \frac{Z_t}{Z_{t,\alpha'}} \right) \in L^\infty$

**Proof** We see that $2\partial_{\alpha'} \mathbb{P}_A \left( \frac{Z_t}{Z_{t,\alpha'}} \right) = (\mathbb{I} - \mathbb{H})(D_{\alpha'} Z_t) + (\mathbb{I} - \mathbb{H}) \left( Z_t \partial_{\alpha'} \frac{1}{Z_{t,\alpha'}} \right) = 2D_{\alpha'} Z_t - (\mathbb{I} + \mathbb{H})(D_{\alpha'} Z_t) + (\mathbb{I} - \mathbb{H}) \left( Z_t \partial_{\alpha'} \frac{1}{Z_{t,\alpha'}} \right) = 2D_{\alpha'} Z_t + \left[ \frac{1}{Z_{t,\alpha'}}, \mathbb{H} \right] Z_{t,\alpha'} + [Z_t, \mathbb{H}] \partial_{\alpha'} \frac{1}{Z_{t,\alpha'}}$

Hence $\|\partial_{\alpha'} \mathbb{P}_A \left( \frac{Z_t}{Z_{t,\alpha'}} \right)\|_\infty \lesssim \|D_{\alpha'} Z_t\|_\infty + \|\bar{Z}_{t,\alpha'}\|_2 \|\partial_{\alpha'} \frac{1}{Z_{t,\alpha'}}\|_2$ by Proposition 9.8

\[ \square \]

(8) $|D_{\alpha'}| A_1 \in L^2$ and hence $A_1 \in \mathcal{W}$, $\sqrt{A_1} \in \mathcal{W}$, $\frac{1}{A_1} \in \mathcal{W}$, $\frac{1}{\sqrt{A_1}} \in \mathcal{W}$

**Proof** Observe that $|D_{\alpha'}| A_1 = \text{Re} \left\{ \frac{\omega}{Z_{t,\alpha'}} (\mathbb{I} - \mathbb{H}) \partial_{\alpha'} A_1 \right\} = \text{Re} \{\omega (\mathbb{I} - \mathbb{H}) D_{\alpha'} A_1\} - \text{Re} \{\omega [\mathbb{P}_A \left( \frac{Z_t}{Z_{t,\alpha'}} \right), \mathbb{H}] \partial_{\alpha'} Z_{t,\alpha'} \}$

Using the formula of $A_1$ from (24) we see that $(\mathbb{I} - \mathbb{H}) D_{\alpha'} A_1 = i(\mathbb{I} - \mathbb{H})((D_{\alpha'} Z_t)\bar{Z}_{t,\alpha'}) + i(\mathbb{I} - \mathbb{H}) \left( \frac{Z_t}{Z_{t,\alpha'}} \partial_{\alpha'} \bar{Z}_{t,\alpha'} \right) = i(\mathbb{I} - \mathbb{H})((D_{\alpha'} Z_t)\bar{Z}_{t,\alpha'}) + i \left[ \mathbb{P}_A \left( \frac{Z_t}{Z_{t,\alpha'}} \right), \mathbb{H} \right] \partial_{\alpha'} \bar{Z}_{t,\alpha'}$

Hence using Proposition 9.8 we have

$$\|\|D_{\alpha'}| A_1\|_2 \lesssim \|(\mathbb{I} - \mathbb{H}) D_{\alpha'} A_1\|_2 + \|\partial_{\alpha'} \frac{1}{Z_{t,\alpha'}}\|_2 \|A_1\|_\infty \lesssim \|D_{\alpha'} Z_t\|_\infty \|\bar{Z}_{t,\alpha'}\|_2 + \|\partial_{\alpha'} \mathbb{P}_A \left( \frac{Z_t}{Z_{t,\alpha'}} \right)\|_\infty \|\bar{Z}_{t,\alpha'}\|_2 + \|\partial_{\alpha'} \frac{1}{Z_{t,\alpha'}}\|_2 \|A_1\|_\infty$$

Now as $A_1 \in L^\infty$ and $|D_{\alpha'}| A_1 \in L^2$, we have that $A_1 \in \mathcal{W}$. Similarly using the fact that $A_1 \geq 1$, we easily get that $\sqrt{A_1} \in \mathcal{W}$, $\frac{1}{A_1} \in \mathcal{W}$, $\frac{1}{\sqrt{A_1}} \in \mathcal{W}$

(9) $\Theta \in L^2$, $D_t \Theta \in L^2$

**Proof** Using (32) and the fact that the Hilbert transform is bounded on $L^2$, we easily see that $\|\Theta\|_2 \lesssim \|\partial_{\alpha'} \frac{1}{Z_{t,\alpha'}}\|_2$. We have $D_t \Theta \in L^2$ as it part of the energy $E_{\sigma,3}$ \[ \square \]

(10) $\Theta \frac{1}{|Z_{t,\alpha'}|} \in C$
Proof. We know from $E_{\alpha,3}$ that $\frac{\sqrt{A_1}}{|Z_{,\alpha}|} \Theta \in \dot{H}^{\frac{1}{2}}$. Now as $\|\sqrt{A_1} \Theta\|_2 \leq \|A_1\|_\infty \|\Theta\|_2$ we now have $\frac{\sqrt{A_1}}{|Z_{,\alpha}|} \Theta \in \mathcal{C}$. Hence we get $\|\frac{\Theta}{|Z_{,\alpha}|}\|_{\mathcal{C}} \leq \|\frac{\sqrt{A_1}}{|Z_{,\alpha}|} \Theta\|_{\mathcal{C}} \leq \|\frac{1}{\sqrt{A_1}}\|_{\mathcal{W}}$ from Lemma 5.3 □

(11) $D_{\alpha'} \frac{1}{Z_{,\alpha'}} \in \mathcal{C}$, $|D_{\alpha'}| \frac{1}{Z_{,\alpha'}} \in \mathcal{C}$, $|D_{\alpha'}| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \omega \in \mathcal{C}$

Proof. Observe from (32) that

$$\frac{\Theta}{|Z_{,\alpha'}|} = i \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} - i \text{Re} \left\{ \frac{1}{|Z_{,\alpha'}|} \left( \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\}
$$

$$= i \overline{D_{\alpha'}} \frac{1}{Z_{,\alpha'}} + i \text{Re} \left\{ \left[ \frac{1}{|Z_{,\alpha'}|}, \mathbb{H} \right] \left( \frac{Z_{,\alpha'}}{|Z_{,\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\}
$$

$$- i \text{Re} \left\{ \left[ \frac{1}{Z_{,\alpha'}}, \mathbb{H} \right] (\partial_{\alpha'} \frac{1}{Z_{,\alpha'}}) \right\}
$$

Hence $\| \overline{D_{\alpha'}} \frac{1}{Z_{,\alpha'}} \|_{\dot{H}^{\frac{1}{2}}} \leq \| \frac{\Theta}{|Z_{,\alpha'}|}\|_{\mathcal{C}} + \| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \|_2^2$ from Proposition 9.8 which implies that $\overline{D_{\alpha'}} \frac{1}{Z_{,\alpha'}} \in \mathcal{C}$. As $\overline{\omega} \in \mathcal{W}$, by Lemma 5.3 we get $D_{\alpha'} \frac{1}{Z_{,\alpha'}} \in \mathcal{C}$ and $|D_{\alpha'}| \frac{1}{Z_{,\alpha'}} \in \mathcal{C}$. Observe that

$$\text{Re} \left( \overline{D_{\alpha'}} \frac{1}{Z_{,\alpha'}} \right) = |D_{\alpha'}| \frac{1}{|Z_{,\alpha'}|} \quad \text{Im} \left( \overline{D_{\alpha'}} \frac{1}{Z_{,\alpha'}} \right) = i \left( \frac{\overline{\omega}}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \omega \right)
$$

Hence $|D_{\alpha'}| \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} A_1 \in \mathcal{C}$ and $\frac{\overline{\omega}}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \omega \in \mathcal{C}$. Now again using $\omega \in \mathcal{W}$ and Lemma 5.3 we easily obtain $\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} \omega \in \mathcal{C}$ □

(12) $\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} A_1 \in L^\infty \cap \dot{H}^{\frac{1}{2}}$ and hence $\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} A_1 \in \mathcal{C}$

Proof. Observe that $\frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'} A_1 = \text{Re} \left\{ \frac{\overline{\omega}^2 \overline{\omega}^2}{|Z_{,\alpha'}|} (\mathbb{I} - \mathbb{H}) |D_{\alpha'}| A_1 \right\}$ and hence we first show that $\frac{\overline{\omega}^2}{|Z_{,\alpha'}|} (\mathbb{I} - \mathbb{H}) |D_{\alpha'}| A_1 \in L^\infty \cap \mathcal{C}$. Now

$$\frac{\overline{\omega}^2}{|Z_{,\alpha'}|} (\mathbb{I} - \mathbb{H}) |D_{\alpha'}| A_1 = (\mathbb{I} - \mathbb{H}) \left( \frac{1}{Z_{,\alpha'}^2} \partial_{\alpha'} A_1 \right) - \left[ \frac{\overline{\omega}^2}{|Z_{,\alpha'}|}, \mathbb{H} \right] |D_{\alpha'}| A_1$$

Using the formula of $A_1$ from (24) we see that

$$\left( \mathbb{I} - \mathbb{H} \right) \left( \frac{1}{Z_{,\alpha'}^2} \partial_{\alpha'} A_1 \right) = i \left( \mathbb{I} - \mathbb{H} \right) \left\{ \left( \frac{Z_{,t,\alpha'}}{Z_{,\alpha'}^2} \right) \overline{Z_{,t,\alpha'}} \right\} + i \left( \mathbb{I} - \mathbb{H} \right) \left\{ Z_t \left( \frac{1}{Z_{,\alpha'}^2} \partial_{\alpha'} \overline{Z_{,t,\alpha'}} \right) \right\}
$$

$$= i \left[ \frac{Z_{,t,\alpha'}}{Z_{,\alpha'}^2}, \mathbb{H} \right] \overline{Z_{,t,\alpha'}} + i \left( \mathbb{I} - \mathbb{H} \right) \left( \frac{1}{Z_{,\alpha'}^2} \partial_{\alpha'} \overline{Z_{,t,\alpha'}} \right)$$
Hence from Proposition 9.8 we have
\[
\left\| \frac{\omega^2}{|Z_{t,\omega'}|} (\mathbb{I} - \Xi) \frac{D_{\alpha'}|A_1}{H_{\omega}^{\omega}} \right\|_{L^\infty \cap H_{\omega}^{\omega}} \lesssim \left\| \frac{\partial_{t,\alpha'} Z_{t,\omega'}}{Z_{t,\omega'}^2} \right\|_2 + \left\| \frac{1}{Z_{t,\omega'}} \frac{\partial_{t,\alpha'} Z_{t,\omega'}}{Z_{t,\omega'}} \right\|_2,
\]
and as \(|D_{\alpha'}|A_1 \in L^2\), we have
\[
\frac{\omega^2}{|Z_{t,\omega'}|} (\mathbb{I} - \Xi) \frac{D_{\alpha'}|A_1}{H_{\omega}^{\omega}} \in L^\infty \cap C.
\]
Now using the fact that \(\omega \in \mathcal{W}\) and Lemma 5.3, we can conclude that
\[
\frac{1}{|Z_{t,\omega'}|^2} \frac{\partial_{t,\alpha'} A_1}{Z_{t,\omega'}} \in L^2 \quad \text{and hence} \quad \frac{1}{|Z_{t,\omega'}|^2} \frac{\partial_{t,\alpha'} A_1}{Z_{t,\omega'}} \in \mathcal{W}.
\]

**Proof.** Observe that
\[
|D_{\alpha'}| \left( \frac{1}{|Z_{t,\omega'}|^2} \partial_{t,\alpha'} A_1 \right) = \text{Re} \left\{ \frac{\omega^2 \partial_{t,\omega}}{|Z_{t,\omega'}|} (\mathbb{I} - \Xi) \partial_{t,\alpha'} \left( \frac{1}{|Z_{t,\omega'}|^2} \partial_{t,\alpha'} A_1 \right) \right\}
\]
and hence it is enough to show that
\[
\frac{1}{|Z_{t,\omega'}|^2} \frac{\partial_{t,\alpha'} A_1}{Z_{t,\omega'}} \in L^2.
\]
Now
\[
\frac{\omega^3}{|Z_{t,\omega'}|} (\mathbb{I} - \Xi) \partial_{t,\alpha'} \left( \frac{1}{|Z_{t,\omega'}|^2} \partial_{t,\alpha'} A_1 \right)
\]
\[
= (\mathbb{I} - \Xi) \left\{ \frac{\omega^2}{|Z_{t,\omega'}|^2} \frac{D_{\alpha'} \left( \frac{1}{|Z_{t,\omega'}|^2} \partial_{t,\alpha'} A_1 \right)}{Z_{t,\omega'}} \right\} - \left[ \frac{\omega^3}{|Z_{t,\omega'}|}, \Xi \right] \partial_{t,\alpha'} \left( \frac{1}{|Z_{t,\omega'}|^2} \partial_{t,\alpha'} A_1 \right)
\]
\[
= (\mathbb{I} - \Xi) \left\{ D_{\alpha'} \left( \frac{1}{Z_{t,\omega'}} \partial_{t,\alpha'} A_1 \right) - 2 \left( \frac{\omega}{|Z_{t,\omega'}|^2} \partial_{t,\alpha'} A_1 \right) (D_{\alpha'} \omega) \right\}
\]
\[
= - \left[ \frac{\omega^3}{|Z_{t,\omega'}|}, \Xi \right] \partial_{t,\alpha'} \left( \frac{1}{|Z_{t,\omega'}|^2} \partial_{t,\alpha'} A_1 \right)
\]
Using the formula of \(A_1\) from (24) we see that
\[
(\mathbb{I} - \Xi) \left\{ D_{\alpha'} \left( \frac{1}{Z_{t,\omega'}} \partial_{t,\alpha'} A_1 \right) \right\}
\]
\[
= i (\mathbb{I} - \Xi) \left\{ D_{\alpha'} \left( \left( \frac{Z_{t,\omega'}}{Z_{t,\omega'}} \right) Z_{t,\omega'} + Z_t \left( \frac{1}{Z_{t,\omega'}} \partial_{t,\omega} \bar{Z}_{t,\omega'} \right) \right) \right\}
\]
\[
= i (\mathbb{I} - \Xi) \left\{ \left( \partial_{t,\omega} \frac{Z_{t,\omega'}}{Z_{t,\omega'}} \right) (D_{\alpha'} \bar{Z}_t) + 2 (D_{\alpha'} Z_t) \left( \frac{1}{Z_{t,\omega'}} \partial_{t,\omega} \bar{Z}_{t,\omega'} \right) \right\}
\]
\[
+ i \left[ \mathbb{P}_A \left( \frac{Z_t}{Z_{t,\omega'}} \right), \Xi \right] \partial_{t,\alpha'} \left( \frac{1}{Z_{t,\omega'}} \partial_{t,\omega} \bar{Z}_{t,\omega'} \right)
\]
Hence from Proposition 9.8 we have
\[
\left\| D_{\alpha'} \left( \frac{1}{|Z_{t,\omega'}|^2} \partial_{t,\alpha'} A_1 \right) \right\|_2 \lesssim \left\| \frac{1}{Z_{t,\omega'}} \partial_{t,\omega} \bar{Z}_{t,\omega'} \right\|_2 + \left\| D_{\alpha'} Z_t \right\|_\infty + \left\| \partial_{t,\omega} \mathbb{P}_A \left( \frac{Z_t}{Z_{t,\omega'}} \right) \right\|_\infty.
\]
\[
\left\| \frac{1}{|Z_{\alpha'}|^{3}} \partial_{\alpha'}^{2} A_{1} \right\|_{2} \lesssim \left\| \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} A_{1} \right) \right\|_{2} + \left\| \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} A_{1} \right) \right\|_{\infty}
\]

Now the other term is easily controlled

\[
\left\| \frac{1}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} A_{1} \right\|_{2} \lesssim \left\| \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} A_{1} \right) \right\|_{2} \left\| \frac{1}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} A_{1} \right\|_{\infty} + \left\| \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} A_{1} \right) \right\|_{2}
\]

As \( \frac{1}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} A_{1} \in L^{\infty} \) and \( |D_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} A_{1} \right) \in L^{2} \) we get that \( \frac{1}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} A_{1} \in \mathcal{W} \).

(14) \( b_{\alpha'} \in L^{\infty} \cap \dot{H}^{\frac{1}{2}} \) and \( \mathbb{H}(b_{\alpha'}) \in L^{\infty} \cap \dot{H}^{\frac{1}{2}} \)

**Proof.** Using the formula of \( b_{\alpha'} \) from (25) we see that

\[
(\mathbb{I} - \mathbb{H}) b_{\alpha'} = (\mathbb{I} - \mathbb{H}) \left( \frac{Z_{t,\alpha'}}{Z_{\alpha'}} \right) + [Z_{t}, \mathbb{H}] \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)
\]

Hence \( \| (\mathbb{I} - \mathbb{H}) b_{\alpha'} \|_{L^{\infty} \cap \dot{H}^{\frac{1}{2}}} \lesssim \| Z_{t,\alpha'} \|_{2} \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{2} + \| D_{\alpha'} Z_{t} \|_{L^{\infty} \cap \dot{H}^{\frac{1}{2}}} \) from Proposition 9.8. As \( b_{\alpha'} \) is real valued, this implies \( b_{\alpha'} \in L^{\infty} \cap \dot{H}^{\frac{1}{2}} \) and \( \mathbb{H}(b_{\alpha'}) \in L^{\infty} \cap \dot{H}^{\frac{1}{2}} \)

(15) \( |D_{\alpha'}| b_{\alpha'} \in L^{2} \) and hence \( b_{\alpha'} \in \mathcal{W} \)

**Proof.** Observe that

\[
|D_{\alpha'}| b_{\alpha'} = \text{Re} \left\{ \frac{\omega}{Z_{\alpha'}} (\mathbb{I} - \mathbb{H}) \partial_{\alpha'} b_{\alpha'} \right\}
\]

Using the formula of \( b_{\alpha'} \) from (25) we see that

\[
(\mathbb{I} - \mathbb{H}) D_{\alpha'} b_{\alpha'} = (\mathbb{I} - \mathbb{H}) \left\{ D_{\alpha'}^{2} Z_{t} + (D_{\alpha'} Z_{t}) \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) + \frac{Z_{t}}{Z_{\alpha'}} \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \right) \right\}
\]

Hence \( \| |D_{\alpha'}| b_{\alpha'} \|_{2} \lesssim \| D_{\alpha'}^{2} Z_{t} \|_{2} + \| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_{2} \left\{ \| D_{\alpha'} Z_{t} \|_{\infty} + \| \partial_{\alpha'} \mathbb{P}_{A} \left( \frac{Z_{t}}{Z_{\alpha'}} \right) \|_{\infty} + \| b_{\alpha'} \|_{\infty} \right\} \)

from Proposition 9.8

(16) \( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in L^{2} \), \( D_{t} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in L^{2} \)
Proof. Recall from (29) that \( D_t \frac{1}{Z,\alpha'} = \frac{1}{Z,\alpha'} (b_{\alpha'} - D_{\alpha'} Z_t) \) and hence
\[
\partial_{\alpha'} D_t \frac{1}{Z,\alpha'} = \left( \partial_{\alpha'} \frac{1}{Z,\alpha'} \right) (b_{\alpha'} - D_{\alpha'} Z_t) + D_{\alpha'} b_{\alpha'} - D_{\alpha'}^2 Z_t
\]
Hence \( \| \partial_{\alpha'} D_t \frac{1}{Z,\alpha'} \|_2 \lesssim \| \partial_{\alpha'} \frac{1}{Z,\alpha'} \|_2 \| b_{\alpha'} \|_\infty + \| D_{\alpha'} Z_t \|_\infty \) + \| D_{\alpha'} |b_{\alpha'}| \|_2 + \| D_{\alpha'}^2 Z_t \|_2 \).

Similarly we have \( \| D_t \partial_{\alpha'} \frac{1}{Z,\alpha'} \|_2 \lesssim \| \partial_{\alpha'} D_t \frac{1}{Z,\alpha'} \|_2 + \| b_{\alpha'} \|_\infty \| \partial_{\alpha'} \frac{1}{Z,\alpha'} \|_2 \) \( \square \)

(17) \( \overline{Z}_{it,\alpha'} \in L^2 \)

Proof. From \( E_{\alpha,2} \) we have that \( D_t \overline{Z}_{it,\alpha'} \in L^2 \). Hence \( \| \overline{Z}_{it,\alpha'} \|_2 \lesssim \| D_t \overline{Z}_{it,\alpha'} \|_2 + \| b_{\alpha'} \|_\infty \| \overline{Z}_{it,\alpha'} \|_2 \) \( \square \)

(18) \( \overline{D}_{\alpha'} \overline{Z}_{it} \in C, \| D_{\alpha'} |\overline{Z}_{it} | \in C, D_t \overline{D}_{\alpha'} \overline{Z}_t \in C \) and \( D_t |D_{\alpha'}| \overline{Z}_t \in C \)

Proof. From \( E_{\sigma,4} \) we have that \( D_t \overline{D}_{\alpha'} \overline{Z}_t \in \hat{H}^{\frac{1}{2}} \). Observe that
\[
D_t \overline{D}_{\alpha'} \overline{Z}_t = \overline{D}_{\alpha'} \overline{Z}_t - (\overline{D}_{\alpha'} \overline{Z}_t)^2
\]
and as \( \overline{D}_{\alpha'} \overline{Z}_t \in C \cap W \), by using Lemma 5.3 we get that \( (\overline{D}_{\alpha'} \overline{Z}_t)^2 \in C \). Hence \( \overline{D}_{\alpha'} \overline{Z}_{it} \in \hat{H}^{\frac{1}{2}} \). As \( \overline{Z}_{it,\alpha'} \in L^2 \) we get that \( \overline{D}_{\alpha'} \overline{Z}_{it} \in C \). By again using the equation above, we get that \( D_t \overline{D}_{\alpha'} \overline{Z}_t \in C \). By using \( \overline{\omega} \in W \) and that \( \overline{D}_{\alpha'} \overline{Z}_{it} \in C \) in Lemma 5.3, we obtain \( |D_{\alpha'}| \overline{Z}_t \in C \). Now observe that
\[
D_t |D_{\alpha'}| \overline{Z}_t = |D_{\alpha'}| \overline{Z}_{it} - \text{Re}(\overline{D}_{\alpha'} \overline{Z}_t)|D_{\alpha'}| \overline{Z}_t
\]
As \( \overline{D}_{\alpha'} \overline{Z}_t \in C \) we get that \( \text{Re}(\overline{D}_{\alpha'} \overline{Z}_t) \in C \). Also as \( |D_{\alpha'}| \overline{Z}_t \in V \), using Lemma 5.3 we obtain \( \text{Re}(\overline{D}_{\alpha'} \overline{Z}_t)|D_{\alpha'}| \overline{Z}_t \in C \). Hence \( D_t |D_{\alpha'}| \overline{Z}_t \in C \). \( \square \)

(19) \( D_t A_1 \in L^\infty \cap \hat{H}^{\frac{1}{2}} \)

Proof. Recall that \( A_1 = 1 - \text{Im}[Z_t, \mathbb{H}] \overline{Z}_{it,\alpha'} \). This implies from Proposition 9.1
\[
D_t A_1 = -\text{Im}\{ [Z_{it}, \mathbb{H}] \overline{Z}_{it,\alpha'} + [Z_t, \mathbb{H}] \overline{Z}_{it,\alpha'} - [b, Z_t; \overline{Z}_{it,\alpha'}] \}
\]
Hence \( \| D_t A_1 \|_{L^\infty \cap \hat{H}^{\frac{1}{2}}} \lesssim \| \overline{Z}_{it,\alpha'} \|_2 \| \overline{Z}_{it,\alpha'} \|_2 + \| b_{\alpha'} \|_\infty \| \overline{Z}_{it,\alpha'} \|_2 \) from Proposition 9.8 and Proposition 9.10. \( \square \)

(20) \( D_t (b_{\alpha'} - D_{\alpha'} Z_t - \overline{D}_{\alpha'} \overline{Z}_t) \in L^\infty \cap \hat{H}^{\frac{1}{2}} \) and hence \( D_t b_{\alpha'} \in \hat{H}^{\frac{1}{2}}, \partial_{\alpha'} D_t b \in \hat{H}^{\frac{1}{2}} \)

Proof. Using the formula of \( b_{\alpha'} \) from (25) we see that
\[
b_{\alpha'} - D_{\alpha'} Z_t - \overline{D}_{\alpha'} \overline{Z}_t = Z_t \left( \partial_{\alpha'} \frac{1}{Z,\alpha'} \right) - \overline{D}_{\alpha'} \overline{Z}_t - i \partial_{\alpha'} (\mathbb{I} + \mathbb{H}) \left\{ \text{Im} \left( \frac{Z_t}{Z,\alpha'} \right) \right\}
\]
Observe that \( (b_{\alpha'} - D_{\alpha'} Z_t - \overline{D}_{\alpha'} \overline{Z}_t) \) is real valued and hence by applying \( \text{Re}(\mathbb{I} - \mathbb{H}) \) we get
\[
b_{\alpha'} - D_{\alpha'} Z_t - \overline{D}_{\alpha'} \overline{Z}_t = \text{Re} \left\{ [Z_t, \mathbb{H}] \left( \partial_{\alpha'} \frac{1}{Z,\alpha'} \right) - \left[ \frac{1}{Z,\alpha'}, \mathbb{H} \right] \overline{Z}_{it,\alpha'} \right\}
\]
Applying $D_t$ and using Proposition 9.1 we obtain
\[
D_t(b_{\alpha'} - D_{\alpha'} Z_t - \overline{D_{\alpha'} Z_t}) \\
= \text{Re}\left\{ [Z_{tt}, \mathbb{H}](\partial_{\alpha'} \frac{1}{Z_{\alpha'}}) + [Z_t, \mathbb{H}](\partial_{\alpha'} D_t \frac{1}{Z_{\alpha'}}) - \left[ D_t \frac{1}{Z_{\alpha'}}, \mathbb{H} \right] Z_t - \left[ \frac{1}{Z_{\alpha'}}, \mathbb{H} \right] Z_{tt} + \frac{1}{Z_{\alpha'}}; b, Z_t \right\}
\]

Hence from Proposition 9.8 and Proposition 9.10 we get
\[
\| D_t(b_{\alpha'} - D_{\alpha'} Z_t - \overline{D_{\alpha'} Z_t}) \|_{L^\infty \cap \dot{H}^{1 \frac{1}{2}} \subset \| Z_{tt,\alpha'} \|_2 \| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_2 + \| Z_t,\alpha' \|_2 \| \partial_{\alpha'} D_t \frac{1}{Z_{\alpha'}} \|_2 \\
+ \| b_{\alpha'} \|_\infty \| \overline{Z_t,\alpha'} \|_2 \| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_2.
\]

As $D_t D_{\alpha'} Z_t \in C$ and $D_t \overline{D_{\alpha'} Z_t} \in C$, we get that $D_t b_{\alpha'} \in \dot{H}^{1 \frac{1}{2}}$. Now as $\partial_{\alpha'} D_t b = (b_{\alpha'})^2 + D_t b_{\alpha'}$ we get $\| \partial_{\alpha'} D_t b \| \dot{H}^{1 \frac{1}{2}} \lesssim \| b_{\alpha'} \|_2 \| b_{\alpha'} \|_\infty + \| D_t b_{\alpha'} \| \dot{H}^{1 \frac{1}{2}} \quad \Box$

(21) $\sigma^{\frac{1}{2}} |Z_{\alpha'}|^\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in \dot{L}^\infty$, $\sigma^{\frac{1}{2}} |Z_{\alpha'}|^\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in \dot{L}^\infty$, $\sigma^{\frac{1}{2}} \partial_{\alpha'} \omega \in \dot{L}^\infty$

\[
\sigma^{\frac{1}{2}} |Z_{\alpha'}|^\frac{1}{2} \text{Re}\Theta \in \dot{L}^\infty
\]

Proof. $\sigma^{\frac{1}{2}} |Z_{\alpha'}|^\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in \dot{L}^\infty$ as it part of the energy $E_{\sigma,0}$. Using (20) we easily obtain $\sigma^{\frac{1}{2}} |Z_{\alpha'}|^\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in \dot{L}^\infty$ and $\sigma^{\frac{1}{2}} \partial_{\alpha'} \omega \in \dot{L}^\infty$. Now from (20) we have

\[
\text{Re}\Theta = -i D_{\alpha'} \omega \quad \text{and this implies that} \quad \sigma^{\frac{1}{2}} |Z_{\alpha'}|^\frac{1}{2} \text{Re}\Theta \in \dot{L}^\infty \quad \Box
\]

(22) $\sigma^{\frac{1}{2}} |Z_{\alpha'}|^\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in \dot{L}^2$, $\sigma^{\frac{1}{2}} |Z_{\alpha'}|^\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in \dot{L}^2$, $\sigma^{\frac{1}{2}} \partial_{\alpha'} \omega \in \dot{L}^2$

\[
\sigma^{\frac{1}{2}} |Z_{\alpha'}|^\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in \dot{L}^2 \quad \text{as it part of the energy} \quad E_{\sigma,0}. \text{Again using (20) we can control the other terms.} \quad \Box
\]

(23) $\sigma \partial_{\alpha'} \Theta \in \dot{H}^{1 \frac{1}{2}}$

Proof. We first note that $(Z_{tt} - i)Z_{\alpha'} \in \dot{H}^{1 \frac{1}{2}}$ as it part of the energy $E_{\sigma,1}$. But from the fundamental equation (12) we get

\[
(Z_{tt} - i)Z_{\alpha'} = -i A_1 + \sigma \partial_{\alpha'} \Theta.
\]

We have already proven that $A_1 \in \dot{H}^{1 \frac{1}{2}}$ and hence $\sigma \partial_{\alpha'} \Theta \in \dot{H}^{1 \frac{1}{2}} \quad \Box$

(24) $\sigma^{\frac{1}{2}} \partial_{\alpha'} \Theta \in \dot{L}^2$

Proof. As $\Theta \in \dot{L}^2$ and $\sigma \partial_{\alpha'} \Theta \in \dot{H}^{1 \frac{1}{2}}$ we obtain the estimate from Lemma 9.4. \quad \Box
(25) \( \sigma^\frac{2}{3} \frac{\partial^2}{Z,\alpha'} \in L^2 \), \( \sigma^\frac{2}{3} \frac{\partial^2}{Z,\alpha'} |\frac{1}{Z,\alpha'}| \in L^2 \), \( \sigma^\frac{2}{3} \frac{\partial^2}{Z,\alpha'} |\frac{1}{Z,\alpha'}| \omega \in L^2 \), \( \sigma^\frac{2}{3} \partial_{\alpha'} |D_{\alpha'}| \omega \in L^2 \)

**Proof.** Differentiating the equation (32) we get

\[
\sigma^\frac{2}{3} \partial_{\alpha'} \Theta = i \sigma^\frac{2}{3} \partial_{\alpha'} \left( \frac{Z,\alpha'}{|Z,\alpha'|} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right) - i \sigma^\frac{2}{3} \text{Re} \left\{ \partial_{\alpha'} \left[ \frac{\omega}{Z,\alpha'}^{\frac{1}{2}} \right] \right\} \left( Z,\alpha'^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right)
\]

Hence from Proposition 9.8 we get

\[
\left\| \sigma^\frac{2}{3} \partial_{\alpha'} \left( \frac{Z,\alpha'}{|Z,\alpha'|} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right) \right\|_2 \lesssim \left\| \sigma^\frac{2}{3} \partial_{\alpha'} \Theta \right\|_2 + \left\| \sigma^\frac{1}{2} \partial_{\alpha'} \omega \right\|_\infty \left\| \sigma^\frac{1}{6} Z,\alpha'^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2
\]

From this and (20) we get

\[
\left\| \sigma^\frac{2}{3} \partial_{\alpha'} \left( \frac{1}{Z,\alpha'} \right) \right\|_2 \lesssim \left\| \sigma^\frac{2}{3} \partial_{\alpha'} \left( \frac{Z,\alpha'}{|Z,\alpha'|} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right) \right\|_2 + \left\| \sigma^\frac{1}{2} \partial_{\alpha'} \omega \right\|_\infty \left\| \sigma^\frac{1}{6} Z,\alpha'^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2
\]

\[
\left\| \sigma^\frac{2}{3} \partial_{\alpha'} \left( \frac{1}{|Z,\alpha'|} \omega \right) \right\|_2 \lesssim \left\| \sigma^\frac{2}{3} \partial_{\alpha'} \left( \frac{Z,\alpha'}{|Z,\alpha'|} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right) \right\|_2 + \left\| \sigma^\frac{1}{2} \partial_{\alpha'} \omega \right\|_\infty \left\| \sigma^\frac{1}{6} Z,\alpha'^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2
\]

and we easily obtain \( \sigma^\frac{2}{3} \partial_{\alpha'} |D_{\alpha'}| \omega \in L^2 \) from \( \sigma^\frac{2}{3} \frac{\partial^2}{Z,\alpha'} \omega \in L^2 \) and we have

\[
\left\| \sigma^\frac{2}{3} \partial_{\alpha'} |D_{\alpha'}| \omega \right\|_2 \lesssim \left\| \sigma^\frac{2}{3} \frac{\partial^2}{Z,\alpha'} \omega \right\|_2 + \left\| \sigma^\frac{1}{2} \frac{\partial_{\alpha'} \omega}{Z,\alpha'} \right\|_\infty \left\| \sigma^\frac{1}{6} \frac{Z,\alpha'}{2} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2
\]

\[
(26) \sigma^\frac{1}{2} \Theta \in L^\infty \cap \dot{H}^\frac{1}{2}
\]

**Proof.** As \( \Theta \in L^2 \) and \( \sigma^\frac{2}{3} \partial_{\alpha'} \Theta \in L^2 \) we have \( \sigma^\frac{1}{2} \Theta \in \dot{H}^\frac{1}{2} \) from Lemma 9.4. Now as \( \Theta \) decays at infinity we have

\[
\left\| \sigma^\frac{1}{2} \Theta \right\|_\infty^2 = \left\| \sigma^\frac{2}{3} \Theta^2 \right\|_\infty \lesssim \sigma^\frac{2}{3} \int |\partial_{\alpha'} (\Theta^2)| \, d\alpha' \lesssim \| \Theta \|_2 \| \sigma^\frac{2}{3} \partial_{\alpha'} \Theta \|_2
\]

\[
(27) \sigma^\frac{1}{2} \partial_{\alpha'} \frac{1}{Z,\alpha'} \in L^\infty \cap \dot{H}^\frac{1}{2}, \sigma^\frac{2}{3} \partial_{\alpha'} \frac{1}{|Z,\alpha'|} \in L^\infty \cap \dot{H}^\frac{1}{2}, \sigma^\frac{1}{2} |D_{\alpha'}| \omega \in L^\infty \cap \dot{H}^\frac{1}{2}
\]

**Proof.** This is proved by exactly the same argument used above to show \( \sigma^\frac{1}{2} \Theta \in L^\infty \cap \dot{H}^\frac{1}{2} \)

\[
(28) \sigma \partial_{\alpha'} D_{\alpha'} \Theta \in L^2, \sigma |D_{\alpha'} | \partial_{\alpha'} \Theta \in L^2, \sigma \partial_{\alpha'} |D_{\alpha'}| \Theta \in L^2
\]
Proof. Taking a derivative in the fundamental equation (12) we get
\[
\bar{Z}_{tt, \alpha'} = -i D_{\alpha'} A_1 - i A_1 \partial_{\alpha'} \frac{1}{Z_{\alpha'}} + \sigma \partial_{\alpha'} D_{\alpha'} \Theta
\]
Hence \( \| \sigma \partial_{\alpha'} D_{\alpha'} \Theta \|_2 \lesssim \| \bar{Z}_{tt, \alpha'} \|_2 + \| D_{\alpha'} |A_1| \|_2 + \| A_1 \|_{\infty} \| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_2 \). From this we get that \( \| \sigma |D_{\alpha'}| \partial_{\alpha'} \Theta \|_2 \lesssim \| \sigma \partial_{\alpha'} D_{\alpha'} \Theta \|_2 + \| \sigma \frac{1}{Z_{\alpha'}} \|_{\infty} \| \sigma \frac{1}{Z_{\alpha'}} \|_{\infty} \). We can prove \( \sigma \partial_{\alpha'} |D_{\alpha'}| \Theta \in L^2 \) similarly. \( \square \)

(29) \( \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^3 \frac{1}{|Z_{\alpha'}|} \in L^2, \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^3 \frac{1}{|Z_{\alpha'}|} \in L^2, \frac{\sigma}{|Z_{\alpha'}|} \partial_{1, \alpha'}^3 \omega \in L^2 \)

Proof. We first observe that
\[
\left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^2 \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_2 \lesssim \left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^2 \omega \right\|_2 \left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{\infty} + \left\| \frac{\sigma}{|Z_{\alpha'}|} |D_{\alpha'}| \omega \right\|_{\infty} \left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_2
\]

Hence the difference between them is controlled. This implies that we replace them with each other whenever we want. Now differentiating (32) we get
\[
\frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^2 \Theta = i \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^2 \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) + i \text{Re} \left\{ \left[ \frac{\sigma}{|Z_{\alpha'}|}, \mathbb{H} \right] \partial_{\alpha'}^2 \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\}
\]
\[
- i \text{Re} (\mathbb{I} - \mathbb{H}) \left\{ \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^2 \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\}
\]
Now we can replace \((\mathbb{I} - \mathbb{H}) \left\{ \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^2 \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\}\) above with \((\mathbb{I} - \mathbb{H}) \left\{ \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^3 \frac{1}{Z_{\alpha'}} \right\}\)
and rewrite it as \( \left[ \frac{\sigma}{|Z_{\alpha'}|}, \mathbb{H} \right] \partial_{\alpha'}^3 \frac{1}{Z_{\alpha'}} \). Hence from Proposition 9.8 we have
\[
\left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^2 \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_2 \lesssim \left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^2 \Theta \right\|_2 + \left\| \sigma \frac{1}{Z_{\alpha'}} \right\|_{\infty} \left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_2 + \left\| \sigma \frac{1}{Z_{\alpha'}} \right\|_{\infty} \left\| \sigma \frac{1}{Z_{\alpha'}} \right\|_2 \left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{\infty}
\]
\[
+ \left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^2 \omega \right\|_2 \left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{\infty} + \left\| \sigma \frac{1}{Z_{\alpha'}} \right\|_{\infty} \left\| \sigma \frac{1}{Z_{\alpha'}} \right\|_2 \left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{\infty}
\]
Hence \( \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^3 \frac{1}{Z_{\alpha'}} \in L^2 \). By using (20) we get that \( \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^3 \frac{1}{|Z_{\alpha'}|} \in L^2 \),

\( \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^2 D_{\alpha'} \omega \in L^2 \) and so
\[
\left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^3 \omega \right\|_2 \lesssim \left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^2 D_{\alpha'} \omega \right\|_2 + \left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^2 \omega \right\|_2 \left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{\infty}
\]
We will only show that

\[ \| \sigma^{\frac{1}{2}} |D_{\alpha'}| \omega \|_{\infty} \leq 2 \]

**Proof.** \( \sigma^{\frac{1}{2}} |Z_{\alpha'}|^{-\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in L^2 \) as it part of the energy \( E_{\sigma,0} \). Now using (20) we get

\[ \| \sigma^{\frac{1}{2}} |Z_{\alpha'}|^{-\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_{2} \lesssim \| \sigma^{\frac{1}{2}} |Z_{\alpha'}|^{-\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_{2} + \| \sigma^{\frac{1}{2}} |Z_{\alpha'}|^{-\frac{1}{2}} \partial_{\alpha'} \omega \|_{\infty} \| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_{2} \]

Now differentiating the equation (32) we get using Proposition 9.8

\[ \| \sigma^{\frac{1}{2}} |Z_{\alpha'}|^{-\frac{1}{2}} \partial_{\alpha'} \Theta \|_{2} \lesssim \| \sigma^{\frac{1}{2}} |Z_{\alpha'}|^{-\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_{2} + \left( \| \sigma^{\frac{1}{2}} |Z_{\alpha'}|^{-\frac{1}{2}} \partial_{\alpha'} \omega \|_{\infty} + \| \sigma^{\frac{1}{2}} |Z_{\alpha'}|^{-\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_{2} \right) \| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_{2} \]

**Proof.** We will only show that \( \sigma^{\frac{1}{2}} |Z_{\alpha'}|^{-\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in \mathcal{W} \) and the rest are proved similarly.

As \( \sigma^{\frac{1}{2}} |Z_{\alpha'}|^{-\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in L^\infty \) we only need to show \( |D_{\alpha'}| \left( \sigma^{\frac{1}{2}} |Z_{\alpha'}|^{-\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \in L^2 \). Now

\[ \| D_{\alpha'} \left( \sigma^{\frac{1}{2}} |Z_{\alpha'}|^{-\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \|_{2} \lesssim \| \sigma^{\frac{1}{2}} |Z_{\alpha'}|^{-\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_{2} + \| \sigma^{\frac{1}{2}} |Z_{\alpha'}|^{-\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_{\infty} \| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_{2} \]

**Proof.** \( \sigma^{\frac{5}{2}} |Z_{\alpha'}|^{-\frac{1}{2}} \partial_{\alpha'} \Theta \in L^\infty \cap \dot{H}^{\frac{1}{2}} \)
Proof. As $\frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta$ decays at infinity, we use Proposition 9.11 with $w = \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}}$ to get

$$\left\| \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \right\|_{L^\infty \cap \hat{H}^\frac{1}{2}} \lesssim \left\| \sigma_5^2 \partial_{\alpha'} \Omega \right\|_2 + \left\| \sigma_5^2 \partial_{\alpha'} \Omega \right\|_2 \left\| \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right\|_2$$

(33) \[ \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in L^\infty \cap \hat{H}^\frac{1}{2}, \quad \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in L^\infty \cap \hat{H}^\frac{1}{2}, \quad \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'}^2 \omega \in L^\infty \cap \hat{H}^\frac{1}{2} \]

Proof. This is proved by exactly the same argument used above to show $\frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \in L^\infty \cap \hat{H}^\frac{1}{2}$

(34) \[ \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \in C \]

Proof. It was proved earlier that $\frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \in L^2$. Also $\frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \in \hat{H}^\frac{1}{2}$ as it part of the energy $E_{\sigma,3}$

(35) \[ \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in C, \quad \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in C, \quad \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'}^2 \omega \in C \]

Proof. As $\frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in L^2, \quad \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in L^2, \quad \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'}^2 \omega \in L^2$ we only have to prove the $\hat{H}^\frac{1}{2}$ estimates. Using Lemma 5.3 we get

$$\left\| \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_C \lesssim \left\| \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'}^2 \omega \right\|_{\mathcal{W}} \left\| D_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_C$$

Hence the difference between them is controlled. This implies that we replace them with each other whenever we want. Now by differentiating (32) we get

$$\frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta = i \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) + i \text{Re} \left\{ \left[ \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \mathbb{I}, \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \right] \partial_{\alpha'} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\}$$

$$- i \text{Re} (\mathbb{I} - \mathbb{H}) \left\{ \frac{\sigma_5^2}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\}$$
Now we replace \((\mathbb{I} - \mathbb{H})\) \(\left\{ \frac{\sigma^{\frac{1}{2}}}{Z_{\alpha'}} \partial_{\alpha'} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|^\frac{1}{2}} \frac{1}{Z_{\alpha'}} \right) \right\}\) above with 
\((\mathbb{I} - \mathbb{H})\) \(\left\{ \frac{\omega \sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \frac{\partial_{\alpha'}^2 1}{Z_{\alpha'}} \right\}\) and rewrite it as 
\(\frac{\omega \sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \mathbb{H} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}}\). Hence using Proposition 9.8 we have 
\[
\left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|^\frac{1}{2}} \frac{1}{Z_{\alpha'}} \right) \right\|_{\mathcal{H}^\frac{1}{2}} \\
\geq \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \Theta \right\|_{\mathcal{H}^\frac{1}{2}} + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \frac{\omega}{Z_{\alpha'}} \right\| \left\| \frac{1}{Z_{\alpha'}} \right\|_2 + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \omega \right\| \left\| D_{\alpha'} \right\|_2 \frac{1}{Z_{\alpha'}} \right\|_\mathcal{C}
\]

Note that we can easily show \(\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \frac{\partial_{\alpha'}^2 1}{Z_{\alpha'}} \in L^2, \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \frac{\omega}{Z_{\alpha'}} \in L^2\) by using Leibniz rule and controlling each individual term. Hence \(\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|^\frac{1}{2}} \frac{1}{Z_{\alpha'}} \right) \in \mathcal{C}\), 
\(\frac{\omega \sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \frac{\partial_{\alpha'}^2 1}{Z_{\alpha'}} \in \mathcal{C}\). As \(\bar{\omega} \in \mathcal{W}\) by using Lemma 5.3 we get \(\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \omega \in \mathcal{C}\).

Now using (20) we easily get \(\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \frac{\partial_{\alpha'}^2 1}{Z_{\alpha'}} \in \mathcal{C}, \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'} \omega \in \mathcal{C}\). Hence by Lemma 5.3 we have 
\[
\left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \omega \right\|_\mathcal{C} \lesssim \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'} \omega \right\|_\mathcal{C} \left\| \omega \right\|_\mathcal{W} + \left\| D_{\alpha'} \right\|_\mathcal{C} \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} \left\| \partial_{\alpha'} \omega \right\|_\mathcal{W} \left\| \omega \right\|_\mathcal{W}
\]

(36) \(\sigma \bar{D}_{\alpha'} D_{\alpha'} \Theta \in \mathcal{C}, \sigma D_{\alpha'}^2 \Theta \in \mathcal{C}, \sigma |D_{\alpha'}|^2 \Theta \in \mathcal{C}, \frac{\sigma}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \Theta \in \mathcal{C}\)

Proof. Applying the derivative \(\bar{D}_{\alpha'}\) to the fundamental equation (12) we get 
\[
\bar{D}_{\alpha'} Z_{tt} = -i A_1 \bar{D}_{\alpha'} \frac{1}{Z_{\alpha'}} - \frac{i}{Z_{\alpha'}} \partial_{\alpha'} A_1 + \sigma \bar{D}_{\alpha'} D_{\alpha'} \Theta
\]

Hence using Lemma 5.3 we get 
\[
\left\| \sigma \bar{D}_{\alpha'} D_{\alpha'} \Theta \right\|_\mathcal{C} \lesssim \left\| \bar{D}_{\alpha'} Z_{tt} \right\|_\mathcal{C} + \left\| \bar{D}_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_\mathcal{C} \left\| A_1 \right\|_\mathcal{W} + \left\| \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} A_1 \right\|_\mathcal{C}
\]

Now as \(\bar{\omega} \in \mathcal{W}\), by Lemma 5.3 we get \(\sigma D_{\alpha'}^2 \Theta \in \mathcal{C}\). Now we see that 
\[
\sigma \bar{D}_{\alpha'} D_{\alpha'} \Theta = \sigma \left( \omega |Z_{\alpha'}|^\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \left( \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \Theta \right) + \frac{\sigma}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \Theta
\]
Hence again by Lemma 5.3 we have

$$\left\| \frac{\sigma}{|Z,a'|^2} \partial_{\alpha'}^2 \Theta \right\|_{\mathcal{C}} \lesssim \left\| \sigma \overline{D}_{a'} D_{a'} \Theta \right\|_{\mathcal{C}} + \left\| \frac{\sigma^2}{|Z,a'|^2} \partial_{\alpha'} \Theta \right\|_{\mathcal{C}} \left\| \sigma^2 \left| \frac{1}{Z,a'} \right|^2 \partial_{\alpha'}^2 \left| \frac{1}{Z,a'} \right| \right\|_{\mathcal{L}} \left\| \omega \right\|_{\mathcal{W}}$$

By a similar argument we get $\sigma \left| D_{a'} \right|^2 \Theta \in \mathcal{C}$

(37) $\left| \frac{\sigma}{|Z,a'|^2} \partial_{\alpha'}^3 \left( \frac{1}{Z,a'} \right) \right| \in \mathcal{C}$, $\left| \frac{\sigma}{|Z,a'|^2} \partial_{\alpha'} \left( \frac{1}{Z,a'} \right) \right| \in \mathcal{C}$, $\sigma \partial_{\alpha'} \left( \frac{1}{Z,a'} \right) \in \mathcal{C}$, $\frac{\sigma}{|Z,a'|^2} \partial_{\alpha'} \omega \in \mathcal{C}$

**Proof.** As $\left| \frac{\sigma}{|Z,a'|^2} \partial_{\alpha'} \left( \frac{1}{Z,a'} \right) \right| \in \mathcal{L}^2$, $\left| \frac{\sigma}{|Z,a'|^2} \partial_{\alpha'} \left( \frac{1}{Z,a'} \right) \right| \in \mathcal{L}^2$, $\sigma \partial_{\alpha'} \omega \in \mathcal{L}^2$ we only need to show the $\dot{H}^2$ estimates. Now observe that

$$\left\| \frac{\sigma}{|Z,a'|^2} \partial_{\alpha'} \left( \frac{Z,a'}{|Z,a'|} \partial_{\alpha'} \left( \frac{1}{Z,a'} \right) \right) \right\|_{\mathcal{C}} \lesssim \left\| \frac{\sigma^2}{|Z,a'|^2} \partial_{\alpha'}^2 \left( \frac{Z,a'}{|Z,a'|} \partial_{\alpha'} \left( \frac{1}{Z,a'} \right) \right) \right\|_{\mathcal{C}}$$

Hence the difference between them is controlled. This implies that we replace them with each other whenever we want. Now differentiating the equation (32) we get

$$\frac{\sigma}{|Z,a'|^2} \partial_{\alpha'}^2 \Theta = i \frac{\sigma}{|Z,a'|^2} \partial_{\alpha'} \left( \frac{Z,a'}{|Z,a'|} \partial_{\alpha'} \left( \frac{1}{Z,a'} \right) \right) + i \text{Re} \left\{ \left[ \frac{\sigma}{|Z,a'|^2} \right] \partial_{\alpha'} \left( \frac{Z,a'}{|Z,a'|} \partial_{\alpha'} \left( \frac{1}{Z,a'} \right) \right) \right\}$$

Now we replace $(\mathcal{I} + \mathcal{H}) \left\{ \frac{\sigma}{|Z,a'|^2} \partial_{\alpha'} \left( \frac{Z,a'}{|Z,a'|} \partial_{\alpha'} \left( \frac{1}{Z,a'} \right) \right) \right\}$ and rewrite it as $\left[ \frac{\omega \sigma}{|Z,a'|^2} \right] \partial_{\alpha'} \left( \frac{1}{Z,a'} \right)$. Hence using Proposition 9.8 we have

$$\left\| \frac{\sigma}{|Z,a'|^2} \partial_{\alpha'} \left( \frac{Z,a'}{|Z,a'|} \partial_{\alpha'} \left( \frac{1}{Z,a'} \right) \right) \right\|_{\dot{H}^2}$$

Note that we can easily show $\sigma \partial_{\alpha'} \left( \frac{1}{|Z,a'|^2} \right) \in \mathcal{L}^2$, $\sigma \partial_{\alpha'} \left( \frac{\omega}{|Z,a'|^2} \right) \in \mathcal{L}^2$ by using Leibniz rule and controlling each individual term. Hence $\frac{\sigma}{|Z,a'|^2} \partial_{\alpha'} \left( \frac{Z,a'}{|Z,a'|} \partial_{\alpha'} \left( \frac{1}{Z,a'} \right) \right) \in \mathcal{C}$,
Now using (20) we easily get \( \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{|Z,\alpha'|} \in C, \ \frac{\sigma}{|Z,\alpha'|^2} \frac{\partial_{\alpha'}^2}{Z,\alpha'} D_{\alpha'} \omega \in C. \) Hence using Lemma 5.3 we have

\[
\left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \right\|_C \lesssim \left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{\partial_{\alpha'}^2}{Z,\alpha'} D_{\alpha'} \omega \right\|_C \|\omega\|_W
\]

\[
+ \left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \right\|_C \left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{\partial_{\alpha'}^2}{Z,\alpha'} \omega \right\|_W \|\omega\|_W
\]

\[
\left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \right\|_C \lesssim \left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \right\|_C + \left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \right\|_W \left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \right\|_C + \left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \right\|_W \left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \right\|_C
\]

(38) \( \frac{\sigma}{|Z,\alpha'|^2} \partial_{\alpha'} \bar{Z}_t,\alpha' \in L^2 \), \( \frac{\sigma}{|Z,\alpha'|^2} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \in L^2 \) and \( \frac{\sigma}{|Z,\alpha'|^2} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \in L^2 \)

Proof. We have \( \frac{\sigma}{|Z,\alpha'|^2} \partial_{\alpha'} \bar{Z}_t,\alpha' \in L^2 \) as it part of the energy \( E_{\sigma,2} \). Now observe that

\[
\left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \right\|_2 \lesssim \left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \right\| \|\bar{Z}_t,\alpha'\|_2 + \left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \right\| \|\bar{Z}_t,\alpha'\|_2
\]

We prove \( \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} D_{\alpha'} \bar{Z}_t \in L^2 \) similarly. \( \square \)

(39) \( \frac{\sigma}{|Z,\alpha'|^2} \partial_{\alpha'} \bar{Z}_t,\alpha' \in L^2 \), \( \frac{\sigma}{|Z,\alpha'|^2} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \in L^2 \), \( \frac{\sigma}{|Z,\alpha'|^2} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \in L^2 \) and in the same way \( \frac{\sigma}{|Z,\alpha'|^2} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \in L^2 \)

Proof. Note that \( \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \partial_{\alpha'} D_{\alpha'} \bar{Z}_t \in L^2 \) as it part of the energy \( E_{\sigma,4} \). Now we have

\[
\left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \right\|_2 \lesssim \left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \right\| \|D_{\alpha'} \bar{Z}_t\|_2
\]

\[
+ \left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \right\|_2 \|D_{\alpha'} \bar{Z}_t\|_2
\]

\[
+ \left\| \frac{\sigma}{|Z,\alpha'|^2} \frac{1}{Z,\alpha'} \right\|_2 \|D_{\alpha'} \bar{Z}_t\|_2
\]
Similarly we see that

\[
\left\| \frac{1}{|Z_{\alpha}|^2} \frac{\partial^{2} Z_{t,\alpha}}{Z_{t,\alpha}} \right\|_{2} \lesssim \left\| \frac{1}{|Z_{\alpha}|^2} \frac{\partial^{2} D_{\alpha} Z_{t}}{Z_{t}} \right\|_{2} + \left\| \frac{1}{|Z_{\alpha}|^2} \frac{\partial^{2} \frac{1}{Z_{\alpha}}}{Z_{t}} \right\|_{2} \left\| D_{\alpha} \left| \frac{1}{Z_{\alpha}} \right|_{\infty} \right\|_{\infty} \left\| \frac{1}{|Z_{\alpha}|^2} \partial^{2} Z_{t,\alpha} \right\|_{2}.
\]

We also have

\[
\left\| \frac{1}{|Z_{\alpha}|^2} D_{\alpha}^2 Z_{t,\alpha} \right\|_{2} \lesssim \left\| \frac{1}{|Z_{\alpha}|^2} \frac{\partial^{2} Z_{t,\alpha}}{Z_{t,\alpha}} \right\|_{2} + \left\| \frac{1}{|Z_{\alpha}|^2} \frac{\partial^{2} D_{\alpha} Z_{t}}{Z_{t}} \right\|_{2}.
\]

The estimate for \( \frac{1}{|Z_{\alpha}|^2} \partial^{2} D_{\alpha} Z_{t,\alpha} \in L^2 \) is shown in a similar way. \( \square \)

(40) \( \frac{1}{|Z_{\alpha}|^2} \partial^{2} Z_{t,\alpha} \in C, \frac{1}{|Z_{\alpha}|^2} \partial^{2} D_{\alpha} Z_{t} \in C \)

and also \( \frac{1}{|Z_{\alpha}|^2} \partial^{2} D_{\alpha} Z_{t} \in C \)

Proof. Note that \( \frac{1}{|Z_{\alpha}|^2} \partial^{2} Z_{t,\alpha} \in L^2 \) as it part of the energy \( E_{\sigma,1} \) and \( \frac{1}{|Z_{\alpha}|^2} \partial^{2} Z_{t,\alpha} \in \hat{H}^{\frac{1}{2}} \) as it part of the energy \( E_{\sigma,2} \). Hence \( \frac{1}{|Z_{\alpha}|^2} \partial^{2} Z_{t,\alpha} \in C \). Now observe that

\[
\left\| D_{\alpha} \left( \frac{1}{|Z_{\alpha}|^2} \partial^{2} Z_{t,\alpha} \right) \right\|_{2} \lesssim \left\| \frac{1}{|Z_{\alpha}|^2} \partial^{2} Z_{t,\alpha} \right\|_{2} + \left\| \frac{1}{|Z_{\alpha}|^2} \partial^{2} \frac{1}{Z_{\alpha}} Z_{t,\alpha} \right\|_{2} \left\| \frac{1}{|Z_{\alpha}|^2} \partial^{2} Z_{t,\alpha} \right\|_{2}.
\]

Now as \( \frac{1}{|Z_{\alpha}|^2} \partial^{2} Z_{t,\alpha} \) decays at infinity, we use Proposition 9.11 with \( w = \frac{1}{|Z_{\alpha}|} \) to get

\[
\left\| \frac{1}{|Z_{\alpha}|^2} \partial^{2} Z_{t,\alpha} \right\|_{\infty} \lesssim \left\| \frac{1}{|Z_{\alpha}|^2} \partial^{2} Z_{t,\alpha} \right\|_{2} \left\| D_{\alpha} \left( \frac{1}{|Z_{\alpha}|^2} \partial^{2} Z_{t,\alpha} \right) \right\|_{2}.
\]
Hence we have proved \( \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \in \mathcal{W} \cap C \). Now using Lemma 5.3 we see that
\[
\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \|_{\mathcal{W} \cap C} \lesssim \| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \|_{\mathcal{W} \cap C} \leq \| \omega \|_{\mathcal{W}} \]
\[
+ \| D_{\alpha'} \overline{Z}_t \|_{\mathcal{W} \cap C} \| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \|_{\mathcal{W}}
\]
We prove \( \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \in \mathcal{W} \cap C \) \( \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \in \mathcal{W} \cap C \) similarly. \( \Box \)

(41) \( \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \in L^2 \), \( \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \in L^2 \), \( \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{D}_{\alpha'} \overline{Z}_t \in L^2 \)

**Proof.** We interpolate between \( \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \in L^2 \) and \( \frac{1}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \in L^2 \). We simply decompose
\[
\left| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right|^2 = \left| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right|^2 \left| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right|^2 \|
\]
and use Holder inequality to obtain
\[
\left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2 \lesssim \left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2 \left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2 \left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2
\]
We also see that
\[
\left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{D}_{\alpha'} \overline{Z}_t \right\|_2 \lesssim \left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2 \left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2 \left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2 \left\| \overline{D}_{\alpha'} \overline{Z}_t \right\|_\infty
\]
The proof of \( \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{D}_{\alpha'} \overline{Z}_t \in L^2 \) is similar. \( \Box \)

(42) \( \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \in L^2 \), \( \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \in L^2 \), \( \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{D}_{\alpha'} \overline{Z}_t \in L^2 \)

**Proof.** We observe that
\[
\left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2 \lesssim \left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2 \left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_2
\]
Similarly we have
\[
\left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} D_{\alpha'} \overline{Z}_t \right\|_2 \lesssim \left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\|_2 \left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} D_{\alpha'} \overline{Z}_t \right\|_2
\]
We prove \( \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \overline{D}_{\alpha'} \overline{Z}_t \in L^2 \) in the same way as above. \( \Box \)
(43) \( \sigma^\frac{1}{6} \frac{\overline{Z}_{t,\alpha'}}{|Z_{\alpha'}|^\frac{1}{2}} \in \mathcal{W} \)

Proof. We use Proposition 9.11 with \( w = \frac{\sigma^\frac{1}{6}}{|Z_{\alpha'}|^\frac{1}{2}} \) to get

\[
\left\| \sigma^\frac{1}{6} \frac{\overline{Z}_{t,\alpha'}}{|Z_{\alpha'}|^\frac{1}{2}} \right\|_\infty^2 \lesssim \left\| \overline{Z}_{t,\alpha'} \right\|_2 \left\| \sigma^\frac{1}{3} \partial_{\alpha'} \left| D_{\alpha'} \overline{Z}_t \right| \right\|_2 + \left\| \sigma^\frac{1}{6} \left| Z_{t,\alpha'} \right|^\frac{1}{2} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right\|_2 \left\| D_{\alpha'} Z_t \right\|_\infty
\]

We also have

\[
\left\| \partial_{\alpha'} \left( \frac{\sigma^\frac{1}{6} \overline{Z}_{t,\alpha'}}{|Z_{\alpha'}|^\frac{1}{2}} \right) \right\|_2 \lesssim \left\| \frac{\sigma^\frac{1}{6} \partial_{\alpha'} \overline{Z}_{t,\alpha'}}{|Z_{\alpha'}|^\frac{1}{2}} \right\|_2 + \left\| \sigma^\frac{1}{6} \left| Z_{t,\alpha'} \right|^\frac{1}{2} \partial_{\alpha'} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right\|_2 \left\| \partial_{\alpha'} Z_t \right\|_\infty
\]

(44) \( \sigma^\frac{1}{6} \partial_{\alpha'} \overline{Z}_A \left( \frac{Z_t}{Z_{\alpha'}^{\frac{1}{2}}} \right) \in L^\infty \)

Proof. We see that

\[
2\sigma^\frac{1}{6} \partial_{\alpha'} \overline{Z}_A \left( \frac{Z_t}{Z_{\alpha'}^{\frac{1}{2}}} \right) = \sigma^\frac{1}{6} \left( I - H \right) \left( \frac{Z_t}{Z_{\alpha'}^{\frac{1}{2}}} \right) + \sigma^\frac{1}{6} \left( I - H \right) \left( Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}^{\frac{1}{2}}} \right)
\]

\[
= 2\sigma^\frac{1}{6} \frac{Z_{t,\alpha'}}{Z_{\alpha'}^{\frac{1}{2}}} + \sigma^\frac{1}{6} \left[ \frac{1}{Z_{\alpha'}^{\frac{1}{2}}} H \right] Z_{t,\alpha'} + \frac{1}{Z_{\alpha'}^{\frac{1}{2}}} \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}^{\frac{1}{2}}} \right)
\]

Hence using Proposition 9.8 we have

\[
\left\| \sigma^\frac{1}{6} \partial_{\alpha'} \overline{Z}_A \left( \frac{Z_t}{Z_{\alpha'}^{\frac{1}{2}}} \right) \right\|_\infty \lesssim \left\| \frac{\sigma^\frac{1}{6} \overline{Z}_{t,\alpha'}}{|Z_{\alpha'}|^\frac{1}{2}} \right\|_\infty + \left\| \sigma^\frac{1}{6} \left| Z_{t,\alpha'} \right|^\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}^{\frac{1}{2}}} \right\|_2 \left\| Z_{t,\alpha'} \right\|_2
\]

(45) \( \frac{\sigma^\frac{1}{3}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} Z_{t,\alpha'} \in L^\infty \cap \dot{H}^{\frac{1}{2}} \)

Proof. We first observe that

\[
\left\| \partial_{\alpha'} \left( \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} Z_{t,\alpha'} \right) \right\|_2 \lesssim \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 \left\| \sigma^\frac{1}{2} \left| Z_{t,\alpha'} \right|^\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_\infty + \left\| \sigma^\frac{1}{2} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_\infty
\]

We now use Proposition 9.11 with \( w = \frac{\sigma^\frac{1}{3}}{|Z_{\alpha'}|^\frac{1}{2}} \) to get

\[
\left\| \frac{\sigma^\frac{1}{3}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right\|_{L^\infty \cap \dot{H}^{\frac{1}{2}}}^2
\]
\[ \sigma \frac{\partial}{\partial t} b \alpha' \in L^2 \]

**Proof.** Using the formula of \( b \alpha' \) from (25) we see that

\[
(\mathbb{I} - \mathbb{H}) \partial \alpha' b \alpha' = (\mathbb{I} - \mathbb{H}) (\partial \alpha' D \alpha' Z_t) + (\mathbb{I} - \mathbb{H}) \{ Z_{t, \alpha'} \partial \alpha' \frac{1}{Z_{\alpha'}} \} + (\mathbb{I} - \mathbb{H}) \{ Z_t \partial_{\alpha'} \sigma \frac{1}{Z_{\alpha'}} \}
\]

Now we see that

\[
Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}} = Z_t \left( \frac{Z_{1/2} \partial \alpha' \frac{1}{Z_{1/2}}}{Z_{1/2} \partial \alpha' \frac{1}{Z_{1/2}}} \right) + \frac{Z_t}{Z_{1/2} \partial \alpha'} \left( \frac{Z_{1/2} \partial \alpha' \frac{1}{Z_{1/2}}}{Z_{1/2} \partial \alpha' \frac{1}{Z_{1/2}}} \right)
\]

hence

\[
(\mathbb{I} - \mathbb{H}) \{ Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \} = \frac{1}{2} \left[ Z_t, \mathbb{H} \right] \left( \frac{Z_{1/2} \partial \alpha' \frac{1}{Z_{1/2}}}{Z_{1/2} \partial \alpha' \frac{1}{Z_{1/2}}} \right) + \left[ \mathbb{P}_A \left( \frac{Z_t}{Z_{1/2}} \right) \right] \partial \alpha' \left( \frac{Z_{1/2} \partial \alpha' \frac{1}{Z_{1/2}}}{Z_{1/2} \partial \alpha' \frac{1}{Z_{1/2}}} \right)
\]

As \( b \alpha' \) is real valued, by taking real part of \((\mathbb{I} - \mathbb{H}) \partial \alpha' b \alpha'\) and using Proposition 9.8 we get

\[
\sigma \frac{\partial}{\partial t} b \alpha' \in L^2
\]

**Proof.** This is obtained by interpolating between \( \frac{\sigma}{Z_{\alpha'}} \frac{\partial}{\partial t} b \alpha' \in L^2 \) and \( D \alpha' \frac{1}{Z_{\alpha'}} \in L^2 \). We have

\[
\sigma \sigma \frac{\partial}{\partial t} b \alpha' \in L^2
\]
Proof. As \(b_{\alpha'}\) is real valued we have from (46)

\[
\begin{align*}
\frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \partial_{\alpha'} b_{\alpha'} \\
= \text{Re} \left\{ \left[ \frac{1}{|Z_{,\alpha'}|} \frac{\partial_{\alpha'}}{|Z_{,\alpha'}|} \right] Z_{t,\alpha'} + 2D_{\alpha'} Z_{t} + [Z_{t}, \mathbb{H}] \left( \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\}
\end{align*}
\]

Now taking the derivative \(\frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'}\) and using Proposition 9.1 we obtain

\[
\begin{align*}
&\frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} b_{\alpha'} \\
&= \text{Re} \left\{ \left[ \frac{1}{|Z_{,\alpha'}|} \frac{\partial_{\alpha'}}{|Z_{,\alpha'}|} \right] Z_{t,\alpha'} + [Z_{t}, \mathbb{H}] \left( \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) \right\} \\
&\quad - \left[ [Z_{t}, \mathbb{H}] \partial_{\alpha'} \left( \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right) - \left[ \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right] \right]
\end{align*}
\]

Hence using Proposition 9.8 and Proposition 9.10 we have

\[
\begin{align*}
\left\| \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} b_{\alpha'} \right\|_{\infty} &\lesssim \left\| \sigma^\frac{1}{2} |Z_{,\alpha'}|^\frac{1}{2} \partial_{\alpha'} \frac{1}{|Z_{,\alpha'}|} \right\|_{\infty} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{2} \left\| \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} Z_{t,\alpha'} \right\|_{2} \\
&\quad + \left\| \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{2} \left\| \sigma^\frac{1}{2} |Z_{,\alpha'}|^\frac{1}{2} \partial_{\alpha'} Z_{t,\alpha'} \right\|_{2} \\
&\quad + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{2} \left\| \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'} Z_{t,\alpha'} \right\|_{2} + \left\| \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'} b_{\alpha'} \right\|_{\infty}
\end{align*}
\]

(49) \(\frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 b_{\alpha'} \in L^2, \quad \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'} b_{\alpha'} \in L^2\) and \(\frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'} b_{\alpha'} \in L^2\)

\[\frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 b_{\alpha'} \in L^2, \quad \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'} b_{\alpha'} \in L^2\)

\[
\begin{align*}
&
\left\| \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 b_{\alpha'} \right\|_{L^2} \lesssim \left\| \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'} Z_{t,\alpha'} \right\|_{L^2} + \left\| \sigma^\frac{1}{2} |Z_{,\alpha'}|^\frac{1}{2} \partial_{\alpha'} Z_{t,\alpha'} \right\|_{L^2} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2}
\end{align*}
\]

Proof. We will first show that \((\mathbb{I} - \mathbb{H}) \left\{ \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 b_{\alpha'} \right\} \in L^2\). Using the formula of \(b_{\alpha'}\) from (25) we see that

\[
\left\{ \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 b_{\alpha'} \right\} = \left\{ \mathbb{I} - \mathbb{H} \right\} \left\{ \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 b_{\alpha'} \right\} + \left\{ \mathbb{I} - \mathbb{H} \right\} \left\{ \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 b_{\alpha'} \right\}
\]

\[
\begin{align*}
&\left\{ \mathbb{I} - \mathbb{H} \right\} \left\{ \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 b_{\alpha'} \right\} = \left\{ \mathbb{I} - \mathbb{H} \right\} \left\{ \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 b_{\alpha'} \right\} \\
&+ 2(D_{\alpha'} Z_{t}) \left( \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right) + \left( \mathbb{I} - \mathbb{H} \right) \left\{ \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\}
\end{align*}
\]
Now
\[
(\mathbb{I} - \mathbb{H}) \left\{ \left( \frac{Z_t}{Z, \alpha'} \right) \frac{\sigma^{\frac{1}{2}}}{Z^{1/2}} \partial_{\alpha'}^3 \frac{1}{Z, \alpha'} \right\} = -\frac{1}{2} [Z_t, \mathbb{H}] \left\{ \left( \partial_{\alpha'} \frac{1}{Z, \alpha'} \right) \left( \frac{\sigma^{\frac{1}{2}}}{Z^{1/2}} \partial_{\alpha'}^2 \frac{1}{Z, \alpha'} \right) \right\} 
+ \left[ \mathbb{P}_A \left( \frac{Z_t}{Z, \alpha'} \right), \mathbb{H} \right] \partial_{\alpha'} \left( \frac{\sigma^{\frac{1}{2}}}{Z^{1/2}} \partial_{\alpha'}^2 \frac{1}{Z, \alpha'} \right)
\]

Hence using Proposition 9.8 we have
\[
\left\| (\mathbb{I} - \mathbb{H}) \left\{ \frac{\sigma^{\frac{1}{2}}}{Z^{\frac{3}{2}}} \partial_{\alpha'}^2 b_{\alpha'} \right\} \right\|_2 
\lesssim \left\| \frac{\sigma^{\frac{1}{2}}}{|Z, \alpha'|^{\frac{3}{2}}} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_2 + \left\| \frac{1}{|Z, \alpha'|^2} \partial_{\alpha'} Z_t, \alpha' \right\|_2 \left\| \frac{\sigma^{\frac{1}{2}}}{|Z, \alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \frac{1}{Z, \alpha'} \right\|_\infty
+ \left\| \frac{\sigma^{\frac{1}{2}}}{|Z, \alpha'|^{\frac{1}{2}}} \partial_{\alpha'}^2 \frac{1}{Z, \alpha'} \right\|_2 \left\| D_{\alpha'} Z_t \right\|_\infty + \left\| Z_t, \alpha' \right\|_2 \left\| \partial_{\alpha'} \frac{1}{Z, \alpha'} \right\|_2 \}
\]

Now lets come back to prove \( \frac{\sigma^{\frac{1}{2}}}{|Z, \alpha'|^{\frac{3}{2}}} \partial_{\alpha'}^2 b_{\alpha'} \in L^2 \). We see that
\[
\frac{\sigma^{\frac{1}{2}}}{|Z, \alpha'|^{\frac{3}{2}}} \partial_{\alpha'}^2 b_{\alpha'} = \text{Re} \left\{ \frac{\sigma^{\frac{1}{2}}}{Z^{\frac{3}{2}}} (\mathbb{I} - \mathbb{H}) \partial_{\alpha'}^2 b_{\alpha'} \right\}
\]

Hence it is enough to show that \( \frac{\sigma^{\frac{1}{2}}}{Z^{\frac{3}{2}}} (\mathbb{I} - \mathbb{H}) \partial_{\alpha'}^2 b_{\alpha'} \in L^2 \). Now we have
\[
\frac{\sigma^{\frac{1}{2}}}{Z^{\frac{3}{2}}} (\mathbb{I} - \mathbb{H}) \partial_{\alpha'}^2 b_{\alpha'} = -\left[ \frac{\sigma^{\frac{1}{2}}}{Z^{\frac{3}{2}}}, \mathbb{H} \right] \partial_{\alpha'}^2 b_{\alpha'} + (\mathbb{I} - \mathbb{H}) \left\{ \frac{\sigma^{\frac{1}{2}}}{Z^{\frac{3}{2}}} \partial_{\alpha'}^2 b_{\alpha'} \right\}
\]

From this and Proposition 9.8 we finally have the estimate
\[
\left\| \frac{\sigma^{\frac{1}{2}}}{|Z, \alpha'|^{\frac{3}{2}}} \partial_{\alpha'}^2 b_{\alpha'} \right\|_2 \lesssim \left\| b_{\alpha'} \right\|_\infty \left\{ \left\| \frac{\sigma^{\frac{1}{2}}}{|Z, \alpha'|^{\frac{3}{2}}} \partial_{\alpha'}^2 \frac{1}{Z, \alpha'} \right\|_2 + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z, \alpha'|^{\frac{1}{2}}} \partial_{\alpha'} \frac{1}{Z, \alpha'} \right\|_\infty \right\} \left\| \partial_{\alpha'} \frac{1}{Z, \alpha'} \right\|_2 
+ \left\| (\mathbb{I} - \mathbb{H}) \left\{ \frac{\sigma^{\frac{1}{2}}}{Z^{\frac{3}{2}}} \partial_{\alpha'}^2 b_{\alpha'} \right\} \right\|_2
\]

We also see that
\[
\left\| \frac{\sigma^{\frac{1}{2}}}{|Z, \alpha'|^{\frac{3}{2}}} \partial_{\alpha'} |D_{\alpha'}| b_{\alpha'} \right\|_2 \lesssim \left\| \frac{\sigma^{\frac{1}{2}}}{|Z, \alpha'|^{\frac{3}{2}}} \partial_{\alpha'} \frac{1}{|Z, \alpha'|} \right\|_\infty \left\| |D_{\alpha'}| b_{\alpha'} \right\|_2 + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z, \alpha'|^{\frac{3}{2}}} \partial_{\alpha'}^2 b_{\alpha'} \right\|_2
\]

The other term \( \frac{\sigma^{\frac{1}{2}}}{|Z, \alpha'|^{\frac{3}{2}}} \partial_{\alpha'} D_{\alpha'} b_{\alpha'} \in L^2 \) is obtained similarly. □
\[
\frac{\sigma^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'} A_1 \in L^\infty
\]

**Proof.** We know that \( A_1 = 1 - \mathrm{Im}[Z_t, H]Z_{t, \alpha'} \) and hence using Proposition 9.1 we have
\[
\frac{\sigma^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'} A_1 = -\mathrm{Im}\left\{ \left[ \frac{\sigma^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} Z_{t, \alpha'}, H \right] Z_{t, \alpha'} + [Z_t, H] \partial_{\alpha'} \left( \frac{\sigma^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} Z_{t, \alpha'} \right) - \left[ \frac{\sigma^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}}, Z_t; Z_{t, \alpha'} \right] \right\}
\]
Hence using Proposition 9.8 and Proposition 9.10 we have
\[
\frac{\sigma^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'} A_1 \lesssim \frac{\sigma^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{t, \alpha'}|} \| Z_{t, \alpha'} \|_2^2 + \frac{\sigma^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'} \| Z_{t, \alpha'} \|_2 \| Z_{t, \alpha'} \|_2
\]
\[
(50)
\]

\[
\frac{\sigma^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'}^2 A_1 \in L^2
\]

**Proof.** Observe that \( \frac{\sigma^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'}^2 A_1 = \Re \left\{ \frac{\sigma^\frac{1}{2} \omega^\frac{3}{2}}{(Z,\alpha')^\frac{3}{2}} (\mathbb{I} - H) \partial_{\alpha'}^2 A_1 \right\} \) and hence it is enough to show that \( \frac{\sigma^\frac{1}{2}}{(Z,\alpha')^\frac{3}{2}} (\mathbb{I} - H) \partial_{\alpha'}^2 A_1 \in L^2 \). Now
\[
\frac{\sigma^\frac{1}{2}}{(Z,\alpha')^\frac{3}{2}} (\mathbb{I} - H) \partial_{\alpha'}^2 A_1 = (\mathbb{I} - H) \left\{ \frac{\sigma^\frac{1}{2}}{(Z,\alpha')^\frac{3}{2}} \partial_{\alpha'}^2 A_1 - \left[ \frac{\sigma^\frac{1}{2}}{(Z,\alpha')^\frac{3}{2}}, H \right] \partial_{\alpha'}^2 A_1 \right\}
\]
Using the formula of \( A_1 \) from (24) we see that
\[
(\mathbb{I} - H) \left\{ \frac{\sigma^\frac{1}{2}}{(Z,\alpha')^\frac{3}{2}} \partial_{\alpha'}^2 A_1 \right\} = (\mathbb{I} - H) \left\{ \frac{\sigma^\frac{1}{2}}{(Z,\alpha')^\frac{3}{2}} \left( \partial_{\alpha'} Z_{t, \alpha'} \right)(\overline{Z_{t, \alpha'}}) + 2(Z_{t, \alpha'} \partial_{\alpha'} \overline{Z_{t, \alpha'}}) \right\}
\]
\[
+ i[Z_t, H] \left\{ -\frac{\sigma^\frac{1}{2}}{(Z,\alpha')^\frac{3}{2}} \partial_{\alpha'} \overline{Z_{t, \alpha'}} \right\}
\]
With
\[
[Z_t, H] \left\{ \frac{\sigma^\frac{1}{2}}{(Z,\alpha')^\frac{3}{2}} \partial_{\alpha'} \overline{Z_{t, \alpha'}} \right\} = [Z_t, H] \partial_{\alpha'} \left\{ \frac{\sigma^\frac{1}{2}}{(Z,\alpha')^\frac{3}{2}} \partial_{\alpha'} \overline{Z_{t, \alpha'}} \right\}
\]
\[
- \frac{3}{2} [Z_t, H] \left\{ \left( \partial_{\alpha'} \frac{1}{Z,\alpha'} \right) \frac{\sigma^\frac{1}{2}}{(Z,\alpha')^\frac{3}{2}} \partial_{\alpha'} \overline{Z_{t, \alpha'}} \right\}
\]
Hence using Proposition 9.8 we have
\[
\frac{\sigma^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \partial_{\alpha'}^2 A_1 \lesssim \frac{\sigma^\frac{1}{2}}{|Z,\alpha'|^\frac{1}{2}} \frac{1}{|Z,\alpha'|} \| \partial_{\alpha'} \frac{1}{Z,\alpha'} \|_{L^2} \| A_1 \|_{L^\infty}
\]
\[ Hence \text{using Proposition 9.8 and Proposition 9.10 we have} \]
\[ \sigma(H)D_t^2 \Theta \in L^2, (\| - \mathbb{H}) D_t^2 \bar{Z}_{t, \alpha'} \in L^2, (\| - \mathbb{H}) D_t^2 D_{\alpha'} \bar{Z}_t \in H^{\frac{5}{2}} \]

**Proof.** For a function \( f \) satisfying \( \mathbb{P}_A f = 0 \) we use Proposition 9.1 to get
\[ (\| - \mathbb{H}) D_t^2 f = [D_t, \mathbb{H}] D_t f + D_t[D_t, \mathbb{H}] f = [b, \mathbb{H}][\alpha, D_t f] = 2[b, \mathbb{H}][\alpha, \bar{Z}_t f] + [D_t b, \mathbb{H}][\alpha, f] - [b, b; \alpha, f] \]

Hence using Proposition 9.8 and Proposition 9.10 we have
\[ \| (\| - \mathbb{H}) D_t^2 \bar{Z}_{t, \alpha'} \|_2 \leq \| b_{\alpha'} \|_{\mathcal{H}^2} \| D_t \Theta \|_2 + \| \tilde{\alpha}_{\alpha'} \|_{\mathcal{H}^2} \| [D_t b, \mathbb{H}] \|_2 \]
\[ \| (\| - \mathbb{H}) D_t^2 \bar{Z}_{t, \alpha'} \|_2 \leq \| b_{\alpha'} \|_{\mathcal{H}^2} \| D_t \bar{Z}_{t, \alpha'} \|_2 + \| \tilde{\alpha}_{\alpha'} \|_{\mathcal{H}^2} \| \bar{Z}_{t, \alpha'} \|_2 \]
\[ \| (\| - \mathbb{H}) D_t^2 D_{\alpha'} \bar{Z}_t \|_{\mathcal{H}^2} \leq \| b_{\alpha'} \|_{\mathcal{H}^2} \| D_t D_{\alpha'} \bar{Z}_t \|_{\mathcal{H}^2} + \| \tilde{\alpha}_{\alpha'} \|_{\mathcal{H}^2} \| D_{\alpha'} \bar{Z}_t \|_{\mathcal{H}^2} \]

(53) \( \sigma(\| - \mathbb{H}) |D_{\alpha'}|^3 \Theta \in L^2, \sigma(\| - \mathbb{H}) |D_{\alpha'}|^3 \bar{Z}_{t, \alpha'} \in L^2, \sigma(\| - \mathbb{H}) |D_{\alpha'}|^3 D_{\alpha'} \bar{Z}_t \in H^{\frac{5}{2}} \)

**Proof.** We use (30) for a function \( f \) satisfying \( \mathbb{P}_A f = 0 \) to get
\[ \sigma(\| - \mathbb{H}) |D_{\alpha'}|^3 f = \sigma(\| - \mathbb{H}) \left\{ \left( \frac{1}{|Z_{\alpha'}|^2} \tilde{\alpha}_{\alpha'} \right) |D_{\alpha'}|^2 f + \left( \tilde{\alpha}_{\alpha'} \right)^2 |D_{\alpha'}|^2 f \right\} \]
\[ + \sigma \left[ \left( \tilde{\alpha}_{\alpha'} \right)^3 \right] \tilde{\alpha}_{\alpha'} f + \sigma \left[ \left( \tilde{\alpha}_{\alpha'} \right)^3 \right] \tilde{\alpha}_{\alpha'} f \]

Hence using Proposition 9.8 we have
\[ \| \sigma(\| - \mathbb{H}) |D_{\alpha'}|^3 \bar{Z}_{t, \alpha'} \|_2 \]
\[ \leq \| \tilde{\alpha}_{\alpha'} \|_{\mathcal{H}^2} \| \bar{Z}_{t, \alpha'} \|_{\mathcal{H}^2} \| D_{\alpha'} \|_2 \]
\[ + \| \tilde{\alpha}_{\alpha'} \|_{\mathcal{H}^2} \left\{ \left( \frac{1}{|Z_{\alpha'}|^2} \tilde{\alpha}_{\alpha'} \right) \| \bar{Z}_{t, \alpha'} \|_{\mathcal{H}^2} \| \tilde{\alpha}_{\alpha'} \|_{\mathcal{H}^2} \right\} \]
\[ \| \sigma(\| - \mathbb{H}) |D_{\alpha'}|^3 \Theta \|_2 \]
\[ \leq \| \tilde{\alpha}_{\alpha'} \|_{\mathcal{H}^2} \| \Theta \|_{\mathcal{H}^2} \| \tilde{\alpha}_{\alpha'} \|_{\mathcal{H}^2} \| \Theta \|_{\mathcal{H}^2} \]
and also
\[
\| \sigma (\mathbb{I} - \mathbb{H})|D_{a'}|^3 D_{a'} Z_t \|_{\dot{H}_2^1}^1 \\
\lesssim \left\| \sigma \partial_{\alpha}^3 \frac{1}{|Z_{\alpha'}|^3} \right\|_{\dot{H}_2^1} \| D_{a'} Z_t \|_{\dot{H}_2^1} + \left\| \frac{\partial_{\alpha}}{|Z_{\alpha'}|^2} \frac{\partial_{\alpha}}{|Z_{\alpha'}|^2} \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha} D_{a'} Z_t \right\|_{\mathcal{W}} + \left\| \frac{\partial_{\alpha}}{|Z_{\alpha'}|^2} \partial_{\alpha} \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha} D_{a'} Z_t \right\|_{\mathcal{W}}
\]

(54) \[
\left[ D_t^2, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'} \in \mathcal{C}, \left[ D_t^2, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'} \in \mathcal{C}
\]

\textbf{Proof.} We will only show \[
\left[ D_t^2, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'} \in \mathcal{C}
\] and \[
\left[ D_t^2, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'} \in \mathcal{C}
\] is proved similarly. We recall from (29) that \[
D_t \frac{1}{Z_{\alpha'}} = \frac{1}{Z_{\alpha'}} \{(b_{a'} - D_{a'} Z_t - \overline{D}_{a'} Z_t) + \overline{D}_{a'} Z_t\}
\] and hence
\[
\left[ D_t^2, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'} = \overline{Z}_{t, \alpha'} D_t \frac{1}{Z_{\alpha'}} + 2(D_t \overline{Z}_{t, \alpha'}) D_t \frac{1}{Z_{\alpha'}}
\]
\[
= \overline{Z}_{t, \alpha'} D_t \left\{ \frac{1}{Z_{\alpha'}} \{(b_{a'} - D_{a'} Z_t - \overline{D}_{a'} Z_t) + \overline{D}_{a'} Z_t\} \right\}
\]
\[
+ 2(D_t \overline{Z}_{t, \alpha'}) (b_{a'} - D_{a'} Z_t)
\]
\[
= (D_t \overline{Z}_{t, \alpha'}) (b_{a'} - D_{a'} Z_t)^2 + (D_{a'} \overline{Z}_{t, \alpha'}) D_t (b_{a'} - D_{a'} Z_t - \overline{D}_{a'} Z_t)
\]
\[
+ (D_{a'} \overline{Z}_t) D_t (b_{a'} - D_{a'} Z_t - \overline{D}_{a'} Z_t)
\]

Now using Proposition 9.9 we have the estimates
\[
\| (D_t \overline{Z}_{t, \alpha'}) D_t (b_{a'} - D_{a'} Z_t - \overline{D}_{a'} Z_t) \|_2 \lesssim \| Z_{t, \alpha'} \|_2 \| D_t (b_{a'} - D_{a'} Z_t - \overline{D}_{a'} Z_t) \|_\infty
\]
\[
\| (D_{a'} \overline{Z}_{t, \alpha'}) D_t (b_{a'} - D_{a'} Z_t - \overline{D}_{a'} Z_t) \|_{\dot{H}_2^1} \lesssim \| D_{a'} \overline{Z}_{t, \alpha'} \|_{\dot{H}_2^1} \| D_t (b_{a'} - D_{a'} Z_t - \overline{D}_{a'} Z_t) \|_\infty
\]
\[
+ \| D_{a'} \overline{Z}_{t, \alpha'} \|_\infty \| D_t (b_{a'} - D_{a'} Z_t - \overline{D}_{a'} Z_t) \|_{\dot{H}_2^1}
\]

This implies that \[
(D_t \overline{Z}_{t, \alpha'}) D_t (b_{a'} - D_{a'} Z_t - \overline{D}_{a'} Z_t) \in \mathcal{C}
\]. Hence using Lemma 5.3 we have
\[
\left\| \left[ D_t^2, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'} \right\|_C \lesssim \| D_a Z_t \|_C \| b_{a'} - D_{a'} Z_t \|_{\mathcal{W}} + \| (D_{a'} \overline{Z}_t) D_t (b_{a'} - D_{a'} Z_t - \overline{D}_{a'} Z_t) \|_C
\]
\[
+ \left( \| D_{a'} \overline{Z}_{t, \alpha'} \|_C + \| b_{a'} \|_{\mathcal{W}} \| D_{a'} \overline{Z}_{t, \alpha'} \|_C \right) (\| b_{a'} \|_{\mathcal{W}} + \| D_{a'} Z_t \|_{\mathcal{W}})
\]
\[
+ \| D_{a'} \overline{Z}_t \|_{\mathcal{W}} \| D_t \overline{D}_{a'} Z_t \|_C
\]

(55) \[
\left[ i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'}, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'} \in \mathcal{C}, \left[ i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'}, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'} \in \mathcal{C}
\]
Proof. Observe that \[ i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \] \[ Z_{t, \alpha'} = i A_1 (|D_{\alpha'}| Z_t) |D_{\alpha'}| \frac{1}{Z_{\alpha'}} \] and so using Lemma 5.3 we have

\[ \left\| i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} Z_{t, \alpha'} \right\|_C \lesssim \| A_1 \gamma_W \| |D_{\alpha'}| Z_t \gamma_W \| |D_{\alpha'}| \frac{1}{Z_{\alpha'}} \|_C. \]

The other term is proved similarly. \( \square \)

The other term is proved similarly.

(56) \( (\mathbb{I} - \mathbb{H}) \left[ i \sigma |D_{\alpha'}|^3, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'} \in \dot{H}^{\frac{1}{2}}, (\mathbb{I} - \mathbb{H}) \left[ i \sigma |D_{\alpha'}|^3, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'} \in \dot{H}^{\frac{1}{2}} \) and we also have \( |Z_{\alpha'}| \left[ i \sigma |D_{\alpha'}|^3, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'} \in L^2 \), \( |Z_{\alpha'}| \left[ i \sigma |D_{\alpha'}|^3, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'} \in L^2 \) and the other terms are proved similarly. Note that we are not making the stronger claim that \( \left[ i \sigma |D_{\alpha'}|^3, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'} \in C \). This is not true and the use of \( (\mathbb{I} - \mathbb{H}) \) in the \( \dot{H}^{\frac{1}{2}} \) estimate is essential. We have

\[ \left[ i \sigma |D_{\alpha'}|^3, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'} = i \sigma \left( |D_{\alpha'}|^3 \frac{1}{Z_{\alpha'}} \right) Z_{t, \alpha'} + 3 i \sigma \left( |D_{\alpha'}|^2 \frac{1}{Z_{\alpha'}} \right) |D_{\alpha'}| Z_{t, \alpha'} \]

\[ + 3 i \sigma \left( \frac{1}{Z_{\alpha'}} \right) |D_{\alpha'}|^3 Z_{t, \alpha'} \]

\( \square \)

We control each term separately:

(a) We use the expansion in (30) to get

\[ \sigma \left( |D_{\alpha'}|^3 \frac{1}{Z_{\alpha'}} \right) Z_{t, \alpha'} \]

\[ = \sigma \left( \frac{1}{|Z_{\alpha'}|^3} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right) \left( |D_{\alpha'}|^3 \frac{1}{Z_{\alpha'}} \right) Z_{t, \alpha'} + \sigma \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)^2 \left( |D_{\alpha'}|^3 \frac{1}{Z_{\alpha'}} \right) Z_{t, \alpha'} \]

\[ + 3 \sigma \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \left( |D_{\alpha'}|^3 \frac{1}{Z_{\alpha'}} \right) Z_{t, \alpha'} + \sigma \left( \frac{1}{|Z_{\alpha'}|^3} \partial_{\alpha'}^3 \frac{1}{Z_{\alpha'}} \right) \left( |D_{\alpha'}|^3 \frac{1}{Z_{\alpha'}} \right) Z_{t, \alpha'} \]

Hence using Lemma 5.3 we have the estimate

\[ \left\| \sigma \left( |D_{\alpha'}|^3 \frac{1}{Z_{\alpha'}} \right) Z_{t, \alpha'} \right\|_C \]
\[\lesssim \|D_{\alpha'}|Z_t\|_W \left\{ \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \frac{1}{Z_{,\alpha'}} \right\|_C \left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|} \frac{1}{Z_{,\alpha'}} \right\|_W + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z,\alpha'|} \frac{1}{Z_{,\alpha'}} \right\|_C \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \frac{1}{Z_{,\alpha'}} \right\|_W \right\}\]

(b) We observe that
\[\sigma \left( |D_{\alpha'}|^2 \frac{1}{Z_{,\alpha'}} \right) |D_{\alpha'}|Z_{,t,\alpha'} = \sigma \left\{ \left( \frac{\partial_{\alpha'} - \frac{1}{|Z_{,\alpha'}|} \frac{1}{Z_{,\alpha'}} \frac{1}{Z_{,\alpha'}} \right) |D_{\alpha'}|^2 \frac{1}{Z_{,\alpha'}}\right\}
\]

Hence using Lemma 5.3 we have the estimate
\[\| \sigma \left( |D_{\alpha'}|^2 \frac{1}{Z_{,\alpha'}} \right) |D_{\alpha'}|Z_{,t,\alpha'} \|_C \lesssim \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \frac{1}{Z_{,\alpha'}} \right\|_W \left\| |D_{\alpha'}|^2 \frac{1}{Z_{,\alpha'}} \right\|_C \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \frac{1}{Z_{,\alpha'}} \right\|_W + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \frac{1}{Z_{,\alpha'}} \right\|_C \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \frac{1}{Z_{,\alpha'}} \right\|_W\]

(c) We observe that
\[\sigma \left( |D_{\alpha'}|^2 \frac{1}{Z_{,\alpha'}} \right) |D_{\alpha'}|Z_{,t,\alpha'}\]

\[= \sigma \left\{ |D_{\alpha'}|^2 \frac{1}{Z_{,\alpha'}} \right\} \left\{ \left( \frac{\partial_{\alpha'} - \frac{1}{|Z_{,\alpha'}|} \frac{1}{Z_{,\alpha'}} \frac{1}{Z_{,\alpha'}} \right) |D_{\alpha'}|^2 \frac{1}{Z_{,\alpha'}} + \frac{1}{|Z_{,\alpha'}|^2} \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \right\}\]

The first term is easily controlled using Lemma 5.3
\[\| \sigma \left( |D_{\alpha'}|^2 \frac{1}{Z_{,\alpha'}} \right) \left( \partial_{\alpha'} \frac{1}{|Z_{,\alpha'}|} \right) |D_{\alpha'}|Z_{,t,\alpha'} \|_C \lesssim \| |D_{\alpha'}|^2 \frac{1}{Z_{,\alpha'}} \|_C \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \frac{1}{Z_{,\alpha'}} \right\|_W \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \frac{1}{Z_{,\alpha'}} \right\|_W\]

Hence we are only left with \(\sigma \left( |D_{\alpha'}|^2 \frac{1}{Z_{,\alpha'}} \right) \frac{\partial_{\alpha'}^2}{|Z_{,\alpha'}|^2} \frac{1}{Z_{,\alpha'}}\). We see that
\[\| \sigma \left( |D_{\alpha'}|^2 \frac{1}{Z_{,\alpha'}} \right) \frac{\partial_{\alpha'}^2}{|Z_{,\alpha'}|^2} \frac{1}{Z_{,\alpha'}} \|_2 \lesssim \| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \|_\infty \| \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'}^2 \frac{1}{Z_{,\alpha'}} \|_2\]
This conclude the proof of $|Z,\alpha|\left[i\sigma|D,\alpha|^3, \frac{1}{Z,\alpha}\right] Z_{t,\alpha} \in L^2$. To finish the $H^{\frac{1}{2}}$ estimate we rewrite the term \(\frac{1}{|Z,\alpha|^2} \partial^2_{\alpha'} Z_{t,\alpha'}\) as \(\frac{\omega^2}{Z,\alpha'^2} \partial^2_{\alpha'} Z_{t,\alpha'}\) and commute one derivative outside to obtain

\[
(\mathbb{I} - H) \left\{ \sigma \left( \frac{1}{|Z,\alpha'|} \frac{\omega}{|Z,\alpha'|^2} \partial^2_{\alpha'} Z_{t,\alpha'} \right) \right\} = -2(\mathbb{I} - H) \left\{ \sigma \left( \frac{\partial_{\alpha'}}{|Z,\alpha'|^2} \right)^2 \frac{\omega}{|Z,\alpha'|^2} \partial^2_{\alpha'} Z_{t,\alpha'} \right\}
\]

\[+ \sigma \left[ \frac{\omega}{|Z,\alpha'|} \partial_{\alpha'} \frac{1}{Z,\alpha'}, \mathbb{H} \right] \partial_{\alpha'} \left( \frac{1}{Z,\alpha'^2} \partial^2_{\alpha'} Z_{t,\alpha'} \right)\]

We can bound each of the terms using Lemma 5.3 and Proposition 9.8

\[
\left\| \sigma \left( \frac{\omega}{|Z,\alpha'|} \right)^2 \frac{\omega}{|Z,\alpha'|^2} \partial^2_{\alpha'} Z_{t,\alpha'} \right\|_C \leq \left\| D_{\alpha'} \frac{1}{Z,\alpha'} \right\|_C \left\| \sigma^\frac{1}{2} |Z,\alpha'| \frac{1}{Z,\alpha'} \right\|_W \left\| \sigma^\frac{1}{2} \frac{\omega}{|Z,\alpha'|^2} \partial^2_{\alpha'} Z_{t,\alpha'} \right\|_W \left\| \omega \right\|_W
\]

and also

\[
\left\| \sigma \left[ \frac{\omega}{|Z,\alpha'|} \partial_{\alpha'} \frac{1}{Z,\alpha'}, \mathbb{H} \right] \partial_{\alpha'} \left( \frac{1}{Z,\alpha'^2} \partial^2_{\alpha'} Z_{t,\alpha'} \right) \right\|_{\dot{H}^{\frac{1}{2}}} \leq \left\| \frac{1}{Z,\alpha'} \partial_{\alpha'} Z_{t,\alpha'} \right\|_2 \left\{ \left\| \sigma^\frac{3}{2} \frac{\omega}{Z,\alpha'^2} \partial^2_{\alpha'} \omega \right\|_2 \left\| \sigma^\frac{1}{2} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_\infty + \left\| D_{\alpha'} \omega \right\|_2 \left\| \sigma^\frac{1}{2} |Z,\alpha'| \frac{1}{Z,\alpha'} \right\|_\infty^2 + \left\| \sigma^\frac{3}{2} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2 \left\| \sigma^\frac{2}{3} \partial^2_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2 + \left\| \sigma^\frac{3}{2} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_\infty \left\| \sigma^\frac{2}{3} \partial^2_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2 + \left\| \frac{\sigma}{Z,\alpha'} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2 \right\} \]

(57) \(R_1 \in C\)

**Proof.** We recall from (41) the formula of \(R_1\)

\[
R_1 = -2(\overline{D_{\alpha'} Z_t})(D_t \overline{D_{\alpha'} Z_t}) - 2\sigma \text{Re}(D_{\alpha'} Z_t)\overline{D_{\alpha'} D_{\alpha'} \Theta} - \sigma (\overline{D_{\alpha'} D_{\alpha'} Z_t}) D_{\alpha'} \Theta
\]

\[+ i\sigma \left( 2i\text{Re}(|D_{\alpha'}|^2 \Theta) + (\text{Re} \Theta)^2 \right) |D_{\alpha'}| \overline{D_{\alpha'} Z_t} - \sigma \text{Re} \left( |D_{\alpha'}|^2 \Theta \right) \overline{D_{\alpha'} Z_t}
\]

\[+ i\sigma (\text{Re} \Theta)(\text{Re}(|D_{\alpha'}|^2)) \overline{D_{\alpha'} Z_t}
\]
All the terms are easily controlled using Lemma 5.3

\[
\|R_1\|_C \lesssim \|\overline{D_{\alpha'} Z_t}\|_{\mathcal{W}} \|D_t \overline{D_{\alpha'} Z_t}\|_C + \|D_{\alpha'} Z_t\|_{\mathcal{W}} \|\sigma \overline{D_{\alpha'} D_{\alpha'} \Theta}\|_C
\]

\[
+ \left\{ \| \frac{\sigma^{\frac{1}{2}}}{Z_{\alpha'}} \partial_{\alpha'} \Theta \|_C + \left\| \frac{\Theta}{Z_{\alpha'}} \right\|_C \right\} \| \frac{\sigma^{\frac{1}{2}}}{Z_{\alpha'}} \partial_{\alpha'} \overline{D_{\alpha'} Z_t}\|_{\mathcal{W}}
\]

\[
+ \left\| \sigma |D_{\alpha'}|^2 \Theta \right\|_C \overline{D_{\alpha'} Z_t}\|_{\mathcal{W}} + \left\| \sigma^{\frac{1}{2}} |Z_{\alpha'}|^2 \Re \Theta \right\|_{\mathcal{W}} \left\| \frac{\sigma^{\frac{1}{2}}}{Z_{\alpha'}} \partial_{\alpha'} \Theta \right\|_C \| \overline{D_{\alpha'} Z_t}\|_{\mathcal{W}}
\]

(58) \(J_1 \in L^\infty \cap \dot{H}^{\frac{1}{2}}\)

Proof. We recall from (36) the formula for \(J_1\)

\[
J_1 = D_t A_1 + A_1 (b_{\alpha'} - D_{\alpha'} Z_t - \overline{D_{\alpha'} Z_t}) + \sigma \partial_{\alpha'} \Re (\mathbb{I} - \mathbb{H}) \left\{ |D_{\alpha'}| + i \Re \Theta \right\} \overline{D_{\alpha'} Z_t}
\]

\[- \sigma \partial_{\alpha'} \Im (\mathbb{I} - \mathbb{H}) D_t \Theta
\]

We have already shown that \(D_t A_1 \in L^\infty \cap \dot{H}^{\frac{1}{2}}\) and we have using Proposition 9.9

\[
\|A_1 (b_{\alpha'} - D_{\alpha'} Z_t - \overline{D_{\alpha'} Z_t})\|_{L^\infty \cap \dot{H}^{\frac{1}{2}}} \lesssim \|A_1\|_{L^\infty \cap \dot{H}^{\frac{1}{2}}} \|b_{\alpha'} - D_{\alpha'} Z_t - \overline{D_{\alpha'} Z_t}\|_{L^\infty \cap \dot{H}^{\frac{1}{2}}}
\]

Let us now control the other terms.

(a) Observe that

\[
\sigma \partial_{\alpha'} (\mathbb{I} - \mathbb{H}) \left\{ |D_{\alpha'}| \overline{D_{\alpha'} Z_t} \right\} = \sigma \partial_{\alpha'} (\mathbb{I} - \mathbb{H}) \left( \left| D_{\alpha'} \right| \frac{1}{Z_{\alpha'}} \overline{Z_{\alpha'}} + \frac{1}{|Z_{\alpha'}| Z_{\alpha'}} \partial_{\alpha'} \overline{Z_{\alpha'}} \right)
\]

\[
= \sigma \partial_{\alpha'} \left[ |D_{\alpha'}| \frac{1}{Z_{\alpha'}}, \mathbb{H} \right] \overline{Z_{\alpha'}} + \sigma \partial_{\alpha'} \left[ \frac{1}{|Z_{\alpha'}| Z_{\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} \overline{Z_{\alpha'}}
\]

Hence using Proposition 9.8 we have

\[
\|\sigma \partial_{\alpha'} (\mathbb{I} - \mathbb{H}) \left\{ |D_{\alpha'}| \overline{D_{\alpha'} Z_t} \right\}\|_{L^\infty \cap \dot{H}^{\frac{1}{2}}}
\]

\[
\lesssim \left\| \overline{Z_{\alpha'}} \right\|_2 \left\{ \left\| \frac{\sigma}{|Z_{\alpha'}|} \partial_{\alpha'}^3 \frac{1}{Z_{\alpha'}} \right\|_2 + \left\| \sigma \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_2 \right\} \left\| \sigma \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_\infty
\]

\[
+ \left\| \sigma \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_2 \left\| \sigma \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_\infty \right\}
(b) We note that \( i \Re \Theta = D_{\alpha'} \omega \) and hence we have
\[
\sigma \partial_{\alpha'}((I - \mathbb{H}) \{i \Re \Theta)D_{\alpha'}Z_t \} = \sigma \partial_{\alpha'}((I - \mathbb{H}) \{D_{\alpha'} \omega)D_{\alpha'}Z_t \} = \sigma \partial_{\alpha'}\left[ \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \omega, \mathbb{H} \right] Z_{t, \alpha'}
\]
From this and Proposition 9.8 we obtain
\[
\| \sigma \partial_{\alpha'}((I - \mathbb{H}) \{i \Re \Theta)D_{\alpha'}Z_t \} \|_{L^\infty \cap \mathcal{H}^{1/2}} \lesssim \| Z_{t, \alpha'} \|_2 \left\{ \left\| \frac{\sigma}{|Z_{\alpha'}|^2} \partial_{\alpha'}^3 \omega \right\|_2 + \left\| \frac{\sigma^2}{|Z_{\alpha'}|^2} \partial_{\alpha'}^2 \omega \right\|_2 \left\| \sigma^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right\|_\infty \\
+ \left\| \frac{\sigma}{|Z_{\alpha'}|^2} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right\|_2 \left\| \sigma \partial_{\alpha'} |D_{\alpha'}| \omega \right\|_\infty \\
+ \left\| \sigma^{\frac{1}{2}} |Z_{\alpha'}|^2 \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right\|_\infty^2 \|D_{\alpha'}|\omega\|_2 \right\}
\]

(c) We see that as \( \mathbb{P}_A \Theta = 0 \) we have
\[
\sigma \partial_{\alpha'}((I - \mathbb{H}) D_t \Theta) = \sigma \partial_{\alpha'}[D_t, \mathbb{H}] \Theta = \sigma \partial_{\alpha'}[b, \mathbb{H}] \partial_{\alpha'} \Theta
\]
Hence using Proposition 9.8 we have
\[
\| \sigma \partial_{\alpha'}((I - \mathbb{H}) D_t \Theta) \|_{L^\infty \cap \mathcal{H}^{1/2}} \lesssim \| \sigma^{\frac{1}{2}} \partial_{\alpha'} b_{\alpha'} \|_2 \| \sigma^{\frac{3}{2}} \partial_{\alpha'} \Theta \|_2
\]

(59) \( |D_{\alpha'}|J_1 \in L^2 \) and hence \( J_1 \in \mathcal{W} \)

**Proof.** As \( J_1 \) is real valued we have
\[
|D_{\alpha'}|J_1 = \Re \left\{ \frac{\omega}{Z_{\alpha'}}((I - \mathbb{H}) \partial_{\alpha'} J_1 \right\} = \Re \left\{ \omega((I - \mathbb{H})D_{\alpha'}J_1 - \omega \left[ \frac{1}{Z_{\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} J_1 \right\}
\]
We recall the equation of \( Z_{t, \alpha'} \) from (42)
\[
\left( D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^3 \right) Z_{t, \alpha'} = R_1 Z_{\alpha'} - i \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) J_1 - i D_{\alpha'} J_1 - Z_{\alpha'} \left[ D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^3, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'}
\]
By applying \((I - \mathbb{H})\) to the above equation we get
\[
\| (I - \mathbb{H}) D_{\alpha'} J_1 \|_2 \\
\lesssim \| (I - \mathbb{H}) D_t^2 Z_{t, \alpha'} \|_2 + A_1 \| \left[ \frac{1}{Z_{\alpha'}^2} \partial_{\alpha'} Z_{t, \alpha'} \right] \|_2 + \| \sigma (I - \mathbb{H}) |D_{\alpha'}|^3 Z_{t, \alpha'} \|_2 \\
+ \| R_1 |Z_{\alpha'}| \|_2 + \| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \|_2 \| J_1 \|_\infty \\
+ \| Z_{\alpha'} \left[ D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^3, \frac{1}{Z_{\alpha'}} \right] Z_{t, \alpha'} \|_2
\]
Hence using Proposition 9.8 we easily have

$$\|D_{\alpha'}J_1\|_2 \lesssim \|(I - \mathbb{H})D_{\alpha'}J_1\|_2 + \left\|\frac{\partial_{\alpha'}1}{Z_{\alpha'}}\right\|_2 \|J_1\|_{\infty}$$

$$\Box$$

(60) $R_2 \in L^2$

Proof. We recall from (44) the formula for $R_2$

$$R_2 = -2i(\overline{D_{\alpha'}Z_t})(|D_{\alpha'}|\overline{D_{\alpha'}Z_t}) + (\text{Re}\Theta)\left\{\left(\overline{D_{\alpha'}Z_t}\right)^2 + i\overline{D_{\alpha'}}\left(\frac{A_1}{Z_{\alpha'}}\right) + i\sigma(\text{Re}\Theta)|D_{\alpha'}|\Theta\right\}

+ \sigma \text{Re}(|D_{\alpha'}|\Theta)|D_{\alpha'}|\Theta + \left(|D_{\alpha'}|\frac{A_1}{|Z_{\alpha'}|}\right)\left(\frac{Z_{\alpha'}\partial_{\alpha'}1}{|Z_{\alpha'}|}\right) + |D_{\alpha'}|\left(\frac{1}{|Z_{\alpha'}|^2}\partial_{\alpha'}A_1\right)

+ (I + \mathbb{H})\text{Im}\{\text{Re}(\overline{D_{\alpha'}Z_t})|D_{\alpha'}|\overline{D_{\alpha'}Z_t} - i\text{Re}(D_{\bar{t}}\Theta)|D_{\alpha'}|\overline{D_{\alpha'}Z_t}\}

+ \frac{A_1}{|Z_{\alpha'}|^2}\partial_{\alpha'}\text{Re}(I - \mathbb{H})\left(\frac{Z_{\alpha'}\partial_{\alpha'}1}{|Z_{\alpha'}|}\right)

Most of the terms are easily controlled and using Lemma 5.3 we have

$$\left\|R_2 - \frac{A_1}{|Z_{\alpha'}|^2}\partial_{\alpha'}\text{Re}(I - \mathbb{H})\left(\frac{Z_{\alpha'}\partial_{\alpha'}1}{|Z_{\alpha'}|}\right)\right\|_2 \lesssim \|\overline{D_{\alpha'}Z_t}\|_{\infty}\left(\|D_{\alpha'}|\overline{D_{\alpha'}Z_t}\|_2 + \|D_{\bar{t}}\Theta\|_2\right)

+ \|\Theta\|_2\|\overline{D_{\alpha'}Z_t}\|_{\infty}^2 + \left\|\frac{\Theta}{|Z_{\alpha'}|}\right\|_C \left\|D_{\alpha'}\right\|_C \frac{1}{|Z_{\alpha'}|} \|A_1\|_{\infty}

+ \|\Theta\|_2\left\|\frac{1}{|Z_{\alpha'}|^2}\partial_{\alpha'}A_1\right\|_{\infty} + \left\|\frac{\Theta}{|Z_{\alpha'}|}\right\|_C \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}}\partial_{\alpha'}\Theta\right\|_C \left\|\sigma^{\frac{1}{2}}|Z_{\alpha'}|^2\text{Re}\Theta\right\|_{\infty}

+ \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^2}\partial_{\alpha'}\right\|_C^2

+ \left\|\frac{1}{|Z_{\alpha'}|^2}\partial_{\alpha'}A_1\right\|_{\infty} \left\|\partial_{\alpha'}\right\|_2 + \left\|D_{\alpha'}\right\|_C \left\|D_{\alpha'}\right\|_C \frac{1}{|Z_{\alpha'}|} \|A_1\|_{\infty}

+ \left\|\frac{1}{|Z_{\alpha'}|^2}\partial_{\alpha'}A_1\right\|_{\mathcal{W}}$$

We now control the last term. We have

$$\frac{A_1}{|Z_{\alpha'}|^2}\partial_{\alpha'}\text{Re}(I - \mathbb{H})\left(\frac{Z_{\alpha'}\partial_{\alpha'}1}{|Z_{\alpha'}|}\right) = \text{Re}(I - \mathbb{H})\left\{\frac{A_1}{|Z_{\alpha'}|^2}\partial_{\alpha'}\left(\frac{Z_{\alpha'}\partial_{\alpha'}1}{|Z_{\alpha'}|}\right)\right\} - \text{Re}\left\{\frac{A_1}{|Z_{\alpha'}|^2}\mathbb{H}\partial_{\alpha'}\left(\frac{Z_{\alpha'}\partial_{\alpha'}1}{|Z_{\alpha'}|}\right)\right\}$$

The first term can be written as

$$(I - \mathbb{H})\left\{\frac{A_1}{|Z_{\alpha'}|^2}\partial_{\alpha'}\left(\frac{Z_{\alpha'}\partial_{\alpha'}1}{|Z_{\alpha'}|}\right)\right\}$$
Hence using Proposition 9.8 and Lemma 5.3 we have

\[
\left\| \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \text{Re}(\mathbb{I} - \mathbb{H}) \left( \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\|_2 \\
\lesssim \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 \left\{ \| A_1 \|_{\mathcal{W}} \right\} + \left\| \frac{1}{Z_{\alpha'}} A_1 \right\|_C \| \omega \|_{\mathcal{W}} \\
+ \left\| D_{\alpha'} \right\|_{\mathcal{C}} \| A_1 \|_{\mathcal{W}} \| \omega \|_{\mathcal{W}} \right) \right) ||\Theta||_2
\]

(61) \( J_2 \in L^2 \)

**Proof.** Let us recall the equation of \( \Theta \) from (43)

\[
\left( D_i^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^3 \right) \Theta = R_2 + i J_2
\]

Observe that \( J_2 \) is real valued. Hence by applying \( \text{Im}(\mathbb{I} - \mathbb{H}) \) to the above equation and using Proposition 9.8 and Lemma 5.3 we have

\[
\| J_2 \|_2 \lesssim \left\| (\mathbb{I} - \mathbb{H}) D_i^2 \Theta \right\|_2 + \left\{ \| A_1 \|_{\mathcal{W}} \right\} + \left\| D_{\alpha'} \right\|_{\mathcal{C}} \| \omega \|_{\mathcal{W}} \| \Theta \|_2 + \| R_2 \|_2
\]

(62) \( \sigma \left[ \frac{1}{Z_{\alpha'}}, \mathbb{H} \right] \left| D_{\alpha'} \right|^3 Z_{t_{\alpha'}} \in \dot{H}^{\frac{1}{2}} \), \( \sigma \left[ \frac{1}{Z_{\alpha'}}, \mathbb{H} \right] \left| D_{\alpha'} \right|^3 Z_{t_{\alpha'}} \in \dot{H}^{\frac{1}{2}} \)

**Proof.** We will only prove \( \sigma \left[ \frac{1}{Z_{\alpha'}}, \mathbb{H} \right] \left| D_{\alpha'} \right|^3 Z_{t_{\alpha'}} \in \dot{H}^{\frac{1}{2}} \) and the other one is proved exactly in the same way. We first observe that

\[
\frac{\sigma}{Z_{\alpha'}} \partial_{\alpha'}^2 \left( \frac{1}{Z_{\alpha'}} \right) \left| D_{\alpha'} \right| Z_{t_{\alpha'}} = \frac{\sigma}{Z_{\alpha'}} \partial_{\alpha'} \left\{ \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\} \left| D_{\alpha'} \right| Z_{t_{\alpha'}} + \omega \left| D_{\alpha'} \right|^2 Z_{t_{\alpha'}} \}
\]

\[
= \sigma \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \left| D_{\alpha'} \right|^2 Z_{t_{\alpha'}} + \sigma \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right) \left| D_{\alpha'} \right|^2 Z_{t_{\alpha'}} \}
\]

\[
+ \sigma \left( D_{\alpha'} \omega \right) \left| D_{\alpha'} \right|^2 Z_{t_{\alpha'}} + \sigma \left| D_{\alpha'} \right|^3 Z_{t_{\alpha'}}
\]

Hence

\[
\left\| \frac{\sigma}{Z_{\alpha'}} \partial_{\alpha'}^2 \left( \frac{1}{Z_{\alpha'}} \right) \left| D_{\alpha'} \right| Z_{t_{\alpha'}} \right\|_2 \\
\lesssim \left( \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'}^2 \left( \frac{1}{Z_{\alpha'}} \right) \right\|_\infty + \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \omega \right\|_\infty \right) \left\| \frac{1}{Z_{\alpha'}} \right\|_2 \left| D_{\alpha'} \right|^2 Z_{t_{\alpha'}}
\]
Now we have
\[
\sigma \left[ \frac{1}{Z,\alpha'}, \mathbb{H} \right] |D_{\alpha'}|^3 Z_{t,\alpha'} = \left[ \frac{1}{Z,\alpha'}, \mathbb{H} \right] \left\{ \sigma |D_{\alpha'}|^3 Z_{t,\alpha'} - \frac{\sigma}{Z,\alpha'} \partial^2_{\alpha'} \left( \frac{1}{Z,\alpha'} |D_{\alpha'}| Z_{t,\alpha'} \right) \right\}
\]
\[
+ \sigma \left[ \frac{1}{Z,\alpha'}, \mathbb{H} \right] \frac{1}{Z,\alpha'} (\mathbb{P}_H + \mathbb{P}_A) \partial^2_{\alpha'} \left( \frac{1}{Z,\alpha'} |D_{\alpha'}| Z_{t,\alpha'} \right)
\]
We can control each of the terms
(a) The first term is easily controlled using Proposition 9.8
\[
\left\| \left[ \frac{1}{Z,\alpha'}, \mathbb{H} \right] \left\{ \sigma |D_{\alpha'}|^3 Z_{t,\alpha'} - \frac{\sigma}{Z,\alpha'} \partial^2_{\alpha'} \left( \frac{1}{Z,\alpha'} |D_{\alpha'}| Z_{t,\alpha'} \right) \right\} \right\|_{\dot{H}^1}
\]
\[
\lesssim \left\| \partial^2_{\alpha'} \left( \frac{1}{Z,\alpha'} |D_{\alpha'}| Z_{t,\alpha'} \right) \right\|_{\dot{H}^1}
\]
and hence we obtain using Proposition 9.8
\[
\left\| \sigma \left[ \frac{1}{Z,\alpha'}, \mathbb{H} \right] \frac{1}{Z,\alpha'} \mathbb{P}_H \partial^2_{\alpha'} \left( \frac{1}{Z,\alpha'} |D_{\alpha'}| Z_{t,\alpha'} \right) \right\|_{\dot{H}^1}
\]
\[
\lesssim \left\| \frac{1}{|Z,\alpha'|^2} \partial^2_{\alpha'} |Z_{t,\alpha'}| \right\|\left\| \frac{\sigma}{Z,\alpha'} \partial^3_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2 + \left\| \frac{\sigma}{Z,\alpha'} \partial^2_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2\right\|_2
\]
(b) We have
\[
\left[ \frac{1}{Z,\alpha'}, \mathbb{H} \right] \frac{1}{Z,\alpha'} \mathbb{P}_H \partial^2_{\alpha'} \left( \frac{1}{Z,\alpha'} |D_{\alpha'}| Z_{t,\alpha'} \right) = \sigma \left[ \frac{1}{Z,\alpha'}, \mathbb{H} \right] \partial^2_{\alpha'} \mathbb{P}_H \left( \frac{1}{Z,\alpha'} |D_{\alpha'}| Z_{t,\alpha'} \right)
\]
\[
\left\| \sigma \left[ \frac{1}{Z,\alpha'}, \mathbb{H} \right] \mathbb{P}_A \partial^2_{\alpha'} \left( \frac{1}{Z,\alpha'} |D_{\alpha'}| Z_{t,\alpha'} \right) \right\|_{\dot{H}^1}
\]
\[
= -\sigma \left[ \frac{1}{Z,\alpha'}, \mathbb{H} \right] \partial^2_{\alpha'} \left( \frac{1}{Z,\alpha'} |D_{\alpha'}| Z_{t,\alpha'} \right) + \sigma (\mathbb{I} - \mathbb{H}) |D_{\alpha'}|^3 Z_{t,\alpha'}
\]
\[
+ (\mathbb{I} - \mathbb{H}) \left\{ \sigma \frac{1}{Z,\alpha'} \partial^2_{\alpha'} \left( \frac{1}{Z,\alpha'} |D_{\alpha'}| Z_{t,\alpha'} \right) - \sigma |D_{\alpha'}|^3 Z_{t,\alpha'} \right\}
\]
Hence using Proposition 9.8 we have
\[
\left\| \sigma \left[ \frac{1}{Z,\alpha'}, \mathbb{H} \right] \mathbb{P}_A \partial^2_{\alpha'} \left( \frac{1}{Z,\alpha'} |D_{\alpha'}| Z_{t,\alpha'} \right) \right\|_{\dot{H}^1}
\]
\[
\lesssim \left\| \partial_\alpha' \frac{1}{Z_\alpha'} \right\|_2 \left\{ \left\| \sigma^3 \partial_\alpha^2 \frac{1}{Z_\alpha'} \right\|_2 \left\| \frac{\sigma^\frac{1}{2}}{|Z_\alpha'|^2} \partial_\alpha' \frac{\partial_\alpha'}{2} Z_{t,\alpha'} \right\|_\infty + \left\| \sigma (\mathbb{I} - \mathbb{H}) |D_\alpha'|^3 \frac{\partial_\alpha'}{2} Z_{t,\alpha'} \right\|_2 \\
+ \left\| \frac{\sigma}{Z_\alpha'} \partial_\alpha^2 \left( \frac{1}{Z_\alpha'} |D_\alpha'| \frac{\partial_\alpha'}{2} Z_{t,\alpha'} \right) \right\|_2 \right\}
\]

\( (\mathbb{I} - \mathbb{H}) D_t^2 \bar{Z}_{t,\alpha'} \in \dot{H}^{\frac{1}{2}} \)

**Proof.** We have already shown that \((\mathbb{I} - \mathbb{H}) D_t^2 |D_\alpha'| \bar{Z}_{t,\alpha'} \in \dot{H}^{\frac{1}{2}} \). Hence

\[
(\mathbb{I} - \mathbb{H}) \left\{ D_t^2 D_\alpha' \bar{Z}_{t,\alpha'} \right\} = (\mathbb{I} - \mathbb{H}) \left\{ \left[ D_t^2, \frac{1}{|Z_\alpha'|^2} |D_\alpha'|^3 \right] \bar{Z}_{t,\alpha'} \right\} + \left[ \frac{1}{Z_\alpha'}, \mathbb{H} \right] D_t^2 \bar{Z}_{t,\alpha'}
\]

Let us recall the equation of \( \bar{Z}_{t,\alpha'} \) from (42)

\[
\left( D_t^2 + i \frac{A_1}{|Z_\alpha'|^2} \partial_\alpha' - i \sigma |D_\alpha'|^3 \right) \bar{Z}_{t,\alpha'} = R_1 \bar{Z}_{t,\alpha'} - i \left( \partial_\alpha' \frac{1}{Z_\alpha'} \right) J_1 - i |D_\alpha'| J_1 - \bar{Z}_{t,\alpha'} \left[ D_t^2 + i \frac{A_1}{|Z_\alpha'|^2} \partial_\alpha' - i \sigma |D_\alpha'|^3, \frac{1}{Z_\alpha'} \right] \bar{Z}_{t,\alpha'}
\]

By replacing \( D_t^2 \bar{Z}_{t,\alpha'} \) with all the other terms from the above equation and using Proposition 9.8 we get

\[
\left\| \left[ \frac{1}{Z_\alpha'}, \mathbb{H} \right] D_t^2 \bar{Z}_{t,\alpha'} \right\|_{\dot{H}^{\frac{1}{2}}}
\]

\[
\lesssim \left\| \sigma \left[ \frac{1}{Z_\alpha'}, \mathbb{H} \right] |D_\alpha'|^3 \bar{Z}_{t,\alpha'} \right\|_{\dot{H}^{\frac{1}{2}}}
\]

\[
+ \left\| \partial_\alpha' \frac{1}{Z_\alpha'} \right\|_2 \left\{ \left\| R_1 |Z_\alpha'| \right\|_2 + \left\| \partial_\alpha' \frac{1}{Z_\alpha'} \right\|_2 \left\| J_1 \right\|_\infty + \left\| |D_\alpha'| J_1 \right\|_2 \right\}
\]

\[
+ \left\| \partial_\alpha' \frac{1}{Z_\alpha'} \right\|_2 \left\| |Z_\alpha'| \right\|_2 \left[ \left\| D_t^2 + i \frac{A_1}{|Z_\alpha'|^2} \partial_\alpha' - i \sigma |D_\alpha'|^3, \frac{1}{Z_\alpha'} \right\|_2 \right]
\]

\[
+ \left\| A_1 \right\|_\infty \left\| \partial_\alpha' \frac{1}{Z_\alpha'} \right\|_2 \left\| \frac{1}{|Z_\alpha'|^2} \partial_\alpha' \bar{Z}_{t,\alpha'} \right\|_2
\]

We can similarly prove that \( \left[ \frac{1}{Z_\alpha'}, \mathbb{H} \right] D_t^2 \bar{Z}_{t,\alpha'} \in \dot{H}^{\frac{1}{2}} \). Using this we have

\[
\left\| \frac{1}{Z_\alpha'} (\mathbb{I} - \mathbb{H}) D_t^2 \bar{Z}_{t,\alpha'} \right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left\| \left[ D_t^2, \frac{1}{Z_\alpha'} \right] \bar{Z}_{t,\alpha'} \right\| + \left\| \left[ \frac{1}{Z_\alpha'}, \mathbb{H} \right] D_t^2 \bar{Z}_{t,\alpha'} \right\|_{\dot{H}^{\frac{1}{2}}}
\]

\[
+ \left\| (\mathbb{I} - \mathbb{H}) D_t^2 D_\alpha' \bar{Z}_{t,\alpha'} \right\|_{\dot{H}^{\frac{1}{2}}}
\]
As \((I - H)D_t^2\overline{Z}_{t,\alpha'} \in L^2\), this implies that \(\frac{1}{Z_{\alpha'}}(I - H)D_t^2\overline{Z}_{t,\alpha'} \in \mathcal{C}\). Now as \(\omega \in \mathcal{W}\), by using Lemma 5.3 we get that \(\frac{1}{Z_{\alpha'}}(I - H)D_t^2\overline{Z}_{t,\alpha'} \in \mathcal{C}\). Now

\[
(I - H)\left\{ D_t^2\overline{D}_{\alpha'}\overline{Z}_t \right\} = (I - H)\left\{ \begin{bmatrix} D_t^2, \frac{1}{Z_{\alpha'}} \end{bmatrix} \overline{Z}_{t,\alpha'} \right\} + \left\{ \frac{1}{Z_{\alpha'}}, H \right\} D_t^2\overline{Z}_{t,\alpha'}
\]

+ \(\frac{1}{Z_{\alpha'}}(I - H)D_t^2\overline{Z}_{t,\alpha'}\)

Hence we obtain

\[
\left\| (I - H)D_t^2\overline{D}_{\alpha'}\overline{Z}_t \right\|_{\dot{H}^\frac{1}{2}} \lesssim \left\| \begin{bmatrix} D_t^2, \frac{1}{Z_{\alpha'}} \end{bmatrix} \overline{Z}_{t,\alpha'} \right\|_{\mathcal{C}} + \left\| \frac{1}{Z_{\alpha'}}, H \right\| D_t^2\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^\frac{1}{2}}
\]

\[
+ \left\| \frac{1}{Z_{\alpha'}}(I - H)D_t^2\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^\frac{1}{2}}
\]

(64) \(\sigma(I - H)|D_{\alpha'}|^3\overline{D}_{\alpha'}\overline{Z}_t \in \dot{H}^\frac{1}{2}\)

**Proof.** We have already shown that \(\sigma(I - H)|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} \in \dot{H}^\frac{1}{2}\). Hence

\[
\left\| \frac{\sigma}{Z_{\alpha'}}(I - H)|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} \right\|_{\dot{H}^\frac{1}{2}} \lesssim \left\| \frac{\sigma}{Z_{\alpha'}}, H \right\| |D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^\frac{1}{2}}
\]

\[
+ \left\| \sigma(I - H)\left\{ |D_{\alpha'}|^3, \frac{1}{Z_{\alpha'}} \right\}\overline{Z}_{t,\alpha'} \right\|_{\dot{H}^\frac{1}{2}}
\]

\[
+ \left\| \sigma(I - H)|D_{\alpha'}|^3\overline{D}_{\alpha'}\overline{Z}_t\right\|_{\dot{H}^\frac{1}{2}}
\]

As \(\sigma(I - H)|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} \in L^2\), this implies that \(\frac{\sigma}{Z_{\alpha'}}(I - H)|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} \in \mathcal{C}\). Now as \(\omega \in \mathcal{W}\), by using Lemma 5.3 we get that \(\frac{\sigma}{Z_{\alpha'}}(I - H)|D_{\alpha'}|^3\overline{Z}_{t,\alpha'} \in \mathcal{C}\). Hence

\[
\left\| \sigma(I - H)|D_{\alpha'}|^3\overline{D}_{\alpha'}\overline{Z}_t \right\|_{\dot{H}^\frac{1}{2}} \lesssim \left\| \sigma(I - H)\left\{ |D_{\alpha'}|^3, \frac{1}{Z_{\alpha'}} \right\}\overline{Z}_{t,\alpha'} \right\|_{\dot{H}^\frac{1}{2}}
\]

\[
+ \left\| \sigma\left[ \frac{1}{Z_{\alpha'}}, H \right] |D_{\alpha'}|^3\overline{Z}_{t,\alpha'} \right\|_{\dot{H}^\frac{1}{2}}
\]

\[
+ \left\| \frac{\sigma}{Z_{\alpha'}}(I - H)|D_{\alpha'}|^3\overline{Z}_{t,\alpha'}\right\|_{\dot{H}^\frac{1}{2}}
\]

(65) \(\frac{1}{|Z_{\alpha'}|^2}\partial_{\alpha'}J_1 \in \dot{H}^\frac{1}{2}\) and hence \(\frac{1}{|Z_{\alpha'}|^2}\partial_{\alpha'}J_1 \in \mathcal{C}\)
Proof. Let us recall the equation of $\overline{D_{\alpha'}}Z_t$ from (40)
\[
\left(D_t^2 + i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^3\right) \overline{D_{\alpha'}Z_t} = R_1 - i \left(\overline{D_{\alpha'}} \frac{1}{Z_{\alpha'}}\right) J_1 - i \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1
\]
We see that
\[
i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \overline{D_{\alpha'}Z_t} = \left(\frac{2i \omega A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \omega\right) \overline{D_{\alpha'}Z_t} + \frac{i \omega^2 A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} D_{\alpha'} \overline{Z_t}
\]
Now observe that $\frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1$ is real valued. Hence by applying $\text{Im}(\mathbb{I} - \mathbb{H})$ to the equation of $\overline{D_{\alpha'}Z_t}$ and using Lemma 5.3 and Proposition 9.8 we get
\[
\left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} J_1 \right\|_{\dot{H}^{1/2}} \lesssim \left\| \overline{D_{\alpha'}} \frac{1}{Z_{\alpha'}} \right\|_C \left\| J_1 \right\|_{\mathcal{V}} + \left\| (\mathbb{I} - \mathbb{H}) D_t^2 \overline{D_{\alpha'}Z_t} \right\|_{\dot{H}^{1/2}} + \left\| \sigma (\mathbb{I} - \mathbb{H}) |D_{\alpha'}|^3 \overline{D_{\alpha'}Z_t} \right\|_{\dot{H}^{1/2}}
\]
\[
+ \left\| D_{\alpha'}^2 \overline{Z_t} \right\|_2 \left( \left\| A_1 \right\|_{\infty} \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 + \left\| A_1 \right\|_{\infty} \left\| |D_{\alpha'}| |\omega| \right\|_2 + \left\| |D_{\alpha'}| A_1 \right\|_2 \right)
\]
\[
+ \left\| D_{\alpha'} \overline{Z_t} \right\|_{\mathcal{V}} \left\| |\omega| \right\|_{\mathcal{V}} \left\| A_1 \right\|_{\mathcal{V}} \left\| \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \omega \right\|_C + \left\| R_1 \right\|_C
\]

5.2. Closing the energy estimate. We are now ready to close the energy $E_\sigma$. To simplify the calculations we will use the following notation: If $a(t), b(t)$ are functions of time we write $a \approx b$ if there exists a universal non-negative polynomial $P$ with $|a(t) - b(t)| \leq P(E_\sigma(t))$. Observe that $\approx$ is an equivalence relation. With this notation proving Theorem 5.1 is equivalent to showing $\frac{dE_\sigma(t)}{dt} \approx 0$.

Lemma 5.4. Let $T > 0$ and let $f, b \in C^2([0, T), H^2(\mathbb{R}))$ with $b$ being real valued. Let $D_t = \partial_t + b \partial_{\alpha'}$. Then
\[
\begin{align*}
(1) & \quad \frac{d}{dt} \int f \, d\alpha' = \int D_t f \, d\alpha' + \int b_{\alpha'} f \, d\alpha' \\
(2) & \quad \left| \frac{d}{dt} \int |f|^2 \, d\alpha' - 2 \text{Re} \int \bar{f}(D_t f) \, d\alpha' \right| \lesssim \| f \|^2_{H^1} \| b_{\alpha'} \|_\infty \\
(3) & \quad \left| \frac{d}{dt} \int (|\partial_{\alpha'}| \bar{f}) f \, d\alpha' - 2 \text{Re} \left\{ \int (|\partial_{\alpha'}| \bar{f}) D_t f \, d\alpha' \right\} \right| \lesssim \| f \|^2_{H^1} \| b_{\alpha'} \|_\infty
\end{align*}
\]

Proof. The first two follow directly from the fact that $D_t = \partial_t + b \partial_{\alpha'}$. For the third estimate we observe that
\[
\frac{d}{dt} \int (|\partial_{\alpha'}| \bar{f}) f \, d\alpha' = 2 \text{Re} \left\{ \int (|\partial_{\alpha'}| \bar{f}) \partial_t f \, d\alpha' \right\}
\]
\[= 2 \text{Re} \left\{ \int (|\partial_{t'} f| D_t f \, d\alpha') \right\} - 2 \text{Re} \left\{ \int (|\partial_{t'} f| (b \partial_{t'} f) \, d\alpha') \right\} \]

\[= 2 \text{Re} \left\{ \int (|\partial_{t'} f| D_t f \, d\alpha') \right\} - 2 \text{Re} \left\{ \int \left( |\partial_{t'} f| \cdot \left( \left[ |\partial_{t'} f| \cdot b \right] \partial_{t'} f \right) \right\} d\alpha' \]

\[+ \int \left| \partial_{t'} f \right|^2 b_{t'} \, d\alpha' \]

The estimate now follows from Proposition 9.8. \qed

This lemma helps us move the time derivative inside the integral as a material derivative. We will now control the time derivative of the energy.

5.2.1. Controlling \( E_{\sigma,0} \)

Recall that

\[ E_{\sigma,0} = \left\| \sigma^{\frac{1}{2}} |Z,\alpha'|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_\infty^2 + \left\| \sigma^{\frac{1}{2}} |Z,\alpha'|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2^2 + \left\| \partial_{t'} \frac{1}{Z,\alpha'} \right\|_2^2 \]

We control the terms individually

(1) As mentioned in Remark 5.2 we will substitute the time derivative with the upper Dini derivative for the \( L^\infty \) term.

\[ \lim_{s \to 0^+} \sup \left\| \sigma^{\frac{1}{2}} |Z,\alpha'|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_\infty (t + s) = \left\| \sigma^{\frac{1}{2}} |Z,\alpha'|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_\infty(t) \]

Now as \( \left\| \sigma^{\frac{1}{2}} |Z,\alpha'|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_\infty (t) \) is part of the energy we only need to concentrate on the second term. As \( \partial_t (f(\cdot, t) \circ h) = (D_t f(\cdot, t)) \circ h \) we use Proposition 9.13 to get

\[ \lim_{s \to 0^+} \sup \left\| \sigma^{\frac{1}{2}} |Z,\alpha'|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_\infty (t + s) \]

\[\leq \left\| D_t \left( \sigma^{\frac{1}{2}} |Z,\alpha'|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right) \right\|_\infty (t) \]
Recall from (28) that $D_t|Z_{\alpha'}| = |Z_{\alpha'}|(\text{Re}(D_{\alpha'}Z_t) - b_{\alpha'})$ and hence we have

$$D_t\left(|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right) = \frac{1}{2} \left(\text{Re}(D_{\alpha'}Z_t) - b_{\alpha'}\right) \left(|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right) - b_{\alpha'}|Z_{\alpha'}|^2 \partial_{\alpha'} \frac{1}{Z_{\alpha'}}$$

$$+ |Z_{\alpha'}|^\frac{1}{2} \partial_{\alpha'} D_t \frac{1}{Z_{\alpha'}}$$

Now as $D_t \frac{1}{Z_{\alpha'}} = \frac{1}{Z_{\alpha'}}(b_{\alpha'} - D_{\alpha'}Z_t)$ we obtain

$$D_t\left(|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right) = \frac{1}{2} \left(\text{Re}(D_{\alpha'}Z_t) - b_{\alpha'} - 2D_{\alpha'}Z_t\right) \left(|Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right)$$

$$+ |Z_{\alpha'}|^\frac{1}{2} D_{\alpha'}(b_{\alpha'} - D_{\alpha'}Z_t)$$

Hence

$$\left\|D_t\left(\sigma^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right)\right\|_\infty \lesssim \left(\|D_{\alpha'}Z_t\|_\infty + \|b_{\alpha'}\|_\infty\right) \left\|\sigma^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_\infty$$

$$+ \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} b_{\alpha'}\right\|_\infty + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'} Z_t\right\|_\infty$$

$$\lesssim P(E_{\sigma})$$

(2) By using the calculation above we first obtain

$$\left\|D_t\left(\sigma^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right)\right\|_2 \lesssim \left(\|D_{\alpha'}Z_t\|_\infty + \|b_{\alpha'}\|_\infty\right) \left\|\sigma^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_2$$

$$+ \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} b_{\alpha'}\right\|_2 + \left\|\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'} Z_t\right\|_2$$

Hence by using Lemma 5.4 we get

$$\frac{d}{dt} \left\|\sigma^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_2^6 \lesssim \|b_{\alpha'}\|_\infty \left\|\sigma^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_2^6 + \left\|\sigma^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_2^5 \left\|D_t\left(\sigma^{\frac{1}{2}} |Z_{\alpha'}|^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right)\right\|_2$$

$$\lesssim P(E_{\sigma})$$

(3) By using Lemma 5.4 we obtain

$$\frac{d}{dt} \left\|\partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_2^2 \lesssim \|b_{\alpha'}\|_\infty \left\|\partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_2^2 + \left\|\partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_2 \left\|D_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_2 \lesssim P(E_{\sigma})$$

(4) We first note from (29) that

$$D_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}} = D_{\alpha'} b_{\alpha'} - D_{\alpha'}^2 Z_t - \left(\partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right) D_{\alpha'} Z_t$$
From this and (28) we see that
\[
\left\| D_t \left( \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}^\prime|} \frac{\partial^2_{\alpha'} 1}{Z_{,\alpha'}} \right) \right\|_2 \\
\lesssim (\|b_{\alpha'}\|_\infty + \|D_{\alpha'} Z_t\|_\infty) \left\| D_t \left( \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}^\prime|} \frac{\partial^2_{\alpha'} 1}{Z_{,\alpha'}} \right) \right\|_2 + \left\| \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}^\prime|} \partial_{\alpha'} D_{\alpha'} b_{\alpha'} \right\|_2 \\
+ \left\| \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}^\prime|} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_2 \left\| \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}^\prime|} \partial_{\alpha'} D_{\alpha'} Z_t \right\|_\infty
\]
Hence by using Lemma 5.4 we get
\[
\frac{d}{dt} \left\| \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}^\prime|} \frac{\partial^2_{\alpha'} 1}{Z_{,\alpha'}} \right\|_2^2 \\
\lesssim \|b_{\alpha'}\|_\infty \left\| \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}^\prime|} \frac{\partial^2_{\alpha'} 1}{Z_{,\alpha'}} \right\|_2^2 + \left\| \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}^\prime|} \partial_{\alpha'} D_{\alpha'} \right\|_2 \left\| D_t \left( \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}^\prime|} \frac{\partial^2_{\alpha'} 1}{Z_{,\alpha'}} \right) \right\|_2 \\
\lesssim P(E_\sigma)
\]
5.2.2. Controlling $E_{\sigma,1}$ Recall that
\[
E_{\sigma,1} = \left\| (\bar{Z}_{tt} - i) Z_{,\alpha'} \right\|_{\bar{H}^\frac{1}{2}}^2 + \left\| \sqrt{A_1 Z_{tt,\alpha'}} \right\|_2^2 + \left\| \frac{\sigma^\frac{1}{2}}{|Z_{,\alpha'}^\prime|} \partial_{\alpha'} Z_{t,\alpha'} \right\|_2^2
\]
We will first simplify the time derivative of each of the individual terms before combining them.

(1) As $b_{\alpha'}, H b_{\alpha'} \in L^\infty$, by using Lemma 5.4 we get
\[
\frac{d}{dt} \int |\partial_{\alpha'}^\frac{1}{2} \{(\bar{Z}_{tt} - i) Z_{,\alpha'}\}|^2 d\alpha' \approx 2\text{Re} \int \{ |\partial_{\alpha'}|((Z_{tt} + i) \bar{Z}_{,\alpha'})\} D_t((\bar{Z}_{tt} - i) Z_{,\alpha'}) d\alpha'
\]
Now from (12) we have
\[
D_t((\bar{Z}_{tt} - i) Z_{,\alpha'}) = \bar{Z}_{tt} Z_{,\alpha'} + (D_{\alpha'} Z_t - b_{\alpha'})(\bar{Z}_{tt} - i) Z_{,\alpha'} \\
= \bar{Z}_{tt} Z_{,\alpha'} + (D_{\alpha'} Z_t - b_{\alpha'})(-i A_1 + \sigma \partial_{\alpha'} \Theta)
\]
and using Proposition 9.9 we observe that
\[
\| (D_{\alpha'} Z_t - b_{\alpha'})(-i A_1 + \sigma \partial_{\alpha'} \Theta) \|_{\bar{H}^\frac{1}{2}} \\
\lesssim \left( \| D_{\alpha'} Z_t \|_{L^\infty \cap \bar{H}^\frac{1}{2}} + \| b_{\alpha'} \|_{L^\infty \cap \bar{H}^\frac{1}{2}} \right) \| A_1 \|_{L^\infty \cap \bar{H}^\frac{1}{2}} + \left( \| \frac{\sigma^\frac{1}{2}}{\bar{H}} \partial_{\alpha'} D_{\alpha'} Z_t \|_2 + \| \frac{\sigma^\frac{1}{2}}{\bar{H}} \partial_{\alpha'} b_{\alpha'} \|_2 \right) \| \sigma^\frac{1}{2} \partial_{\alpha'} \Theta \|_2 + \left( \| D_{\alpha'} Z_t \|_\infty + \| b_{\alpha'} \|_\infty \right) \| \sigma \partial_{\alpha'} \Theta \|_{\bar{H}^\frac{1}{2}}
\]
Hence we have
\[
\frac{d}{dt} \int |\partial_{\alpha'}^\frac{1}{2} \{(\bar{Z}_{tt} - i) Z_{,\alpha'}\}|^2 d\alpha' \approx 2\text{Re} \int (\bar{Z}_{tt} Z_{,\alpha'}) |\partial_{\alpha'}|((Z_{tt} + i) \bar{Z}_{,\alpha'}) d\alpha'
\]
(2) As \( b_{\alpha'} \), \( A_1 \), \( D_t A_1 \) \( \in L^\infty \) and \( Z_{t,\alpha'}, Z_{tt,\alpha'} \) \( \in L^2 \) we get
\[
\frac{d}{dt} \int A_1 |Z_{t,\alpha'}|^2 \, d\alpha' = \int (b_{\alpha'} A_1 + D_t A_1) |Z_{t,\alpha'}|^2 \, d\alpha' + 2\text{Re} \int A_1 \bar{Z}_{t,\alpha'} (-b_{\alpha'} Z_{t,\alpha'} + Z_{tt,\alpha'}) \, d\alpha' \\
\approx 0
\]
Now observe from Proposition 9.9 that
\[
\| A_1 \bar{D}_{\alpha'} \bar{Z}_t \| \bar{H}^{\frac{1}{2}} \lesssim \| A_1 \|_{L^\infty \cap \bar{H}^{\frac{1}{2}}} \| \bar{D}_{\alpha'} \bar{Z}_t \|_{L^\infty \cap \bar{H}^{\frac{1}{2}}}
\]
Hence we have
\[
\frac{d}{dt} \int A_1 |Z_{t,\alpha'}|^2 \, d\alpha' \approx 2\text{Re} \int (i A_1 \bar{D}_{\alpha'} \bar{Z}_t) |\partial_{\alpha'}|( (Z_{tt} + i) \bar{Z}_{\alpha'}) \, d\alpha'
\]
(3) By Lemma 5.4 we get
\[
\sigma \frac{d}{dt} \int \left| \frac{1}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} Z_{t,\alpha'} \right|^2 \, d\alpha' \approx 2\sigma \text{Re} \int \left( \frac{1}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} Z_{t,\alpha'} \right) D_t \left( \frac{1}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} Z_{t,\alpha'} \right) \, d\alpha'
\]
Using (28) we obtain
\[
\sigma \frac{1}{2} D_t \left( \frac{1}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} Z_{t,\alpha'} \right) = \left( -\frac{3}{2} b_{\alpha'} - \frac{1}{2} \text{Re}(D_{\alpha'} Z_t) \right) \left( \frac{\sigma \frac{1}{2}}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} Z_{t,\alpha'} \right) \\
- \left( \frac{\sigma \frac{1}{2}}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} b_{\alpha'} \right) Z_{t,\alpha'} + \frac{\sigma \frac{1}{2}}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} Z_{tt,\alpha'}
\]
As \( b_{\alpha'} \), \( \text{Re}(D_{\alpha'} Z_t) \), \( \frac{\sigma \frac{1}{2}}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} b_{\alpha'} \) \( \in L^\infty \) and \( Z_{t,\alpha'}, \frac{\sigma \frac{1}{2}}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} Z_{t,\alpha'} \) \( \in L^2 \) we have
\[
\sigma \frac{d}{dt} \int \left| \frac{1}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} Z_{t,\alpha'} \right|^2 \, d\alpha' \approx 2\sigma \text{Re} \int \left( \frac{1}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} Z_{t,\alpha'} \right) \left( \frac{1}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} Z_{tt,\alpha'} \right) \, d\alpha'
\]
Now
\[
\frac{\sigma \frac{1}{2}}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} Z_{tt,\alpha'} = \frac{\sigma \frac{1}{2}}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} \left( \frac{1}{Z_{t,\alpha'}} (Z_{tt} + i) \bar{Z}_{t,\alpha'} \right) \\
= \left( \frac{\sigma \frac{1}{2}}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} \frac{1}{Z_{t,\alpha'}} \right) (Z_{tt} + i) \bar{Z}_{t,\alpha'} \\
+ 2 \left( \frac{\sigma \frac{1}{2}}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} \frac{1}{Z_{t,\alpha'}} \right) \partial_{\alpha'} \{ (Z_{tt} + i) \bar{Z}_{t,\alpha'} \} \\
+ \frac{\sigma \frac{1}{2}}{|Z_{t,\alpha'}|^2} \partial_{\alpha'} \{ (Z_{tt} + i) \bar{Z}_{t,\alpha'} \}
\]
Using \((Z_{tt} + i)\overline{Z}_{\alpha'} = iA_1 + \sigma \partial_{\alpha'} \Theta\) from (12) we obtain

\[
\left\| \left( \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right) (Z_{tt} + i)\overline{Z}_{\alpha'} \right\|_2 \\
\lesssim \left\| \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_2 A_1 \|_\infty + \left\| \frac{\sigma^\frac{5}{2}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_\infty \left\| \sigma^\frac{3}{2} \partial_{\alpha'} \Theta \right\|_2
\]

\[
\left\| \left( \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right) \partial_{\alpha'} \left( (Z_{tt} + i)\overline{Z}_{\alpha'} \right) \right\|_2 \\
\lesssim \left\| \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \right\|_\infty \left( \left\| |D_{\alpha'}| A_1 \right\|_2 + \left\| \frac{\sigma}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 \Theta \right\|_2 \right)
\]

Hence

\[
\sigma \frac{d}{dt} \int \left| \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right|^2 d\alpha'
\approx 2\sigma \text{Re} \int \left( \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right) \left( \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 \left( (Z_{tt} + i)\overline{Z}_{\alpha'} \right) \right) d\alpha'
\]

Now using \(\partial_{\alpha'} = i|\alpha'| \pm (\mathbb{I} + \mathbb{H})\partial_{\alpha'}\) and \((Z_{tt} + i)\overline{Z}_{\alpha'} = iA_1 + \sigma \partial_{\alpha'} \overline{\Theta}\) we obtain

\[
\frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 \left( (Z_{tt} + i)\overline{Z}_{\alpha'} \right) = i\frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} |\alpha'| \left( (Z_{tt} + i)\overline{Z}_{\alpha'} \right)
\]

\[
+ \frac{i\sigma^\frac{1}{2}}{|Z_{\alpha'}|^\frac{1}{2}} (\mathbb{I} + \mathbb{H}) \partial_{\alpha'}^2 A_1
\]

By commuting the Hilbert transform outside and using Proposition 9.8 we obtain

\[
\left\| \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|^\frac{1}{2}} (\mathbb{I} + \mathbb{H}) \partial_{\alpha'}^2 A_1 \right\|_2 \\
\lesssim \| A_1 \|_\infty \left( \left\| \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 \frac{1}{|Z_{\alpha'}|} \right\|_2 + \left\| \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right\|_2 \left\| \sigma^\frac{1}{2} |Z_{\alpha'}|^\frac{1}{2} \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right\|_\infty \right)
\]

\[
+ \left\| \frac{\sigma^\frac{1}{2}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'}^2 A_1 \right\|_2
\]

Hence

\[
\sigma \frac{d}{dt} \int \left| \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \overline{Z}_{t,\alpha'} \right|^2 d\alpha'
\approx 2\text{Re} \int \left\{ -i\sigma \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|} |D_{\alpha'}| \overline{Z}_{t,\alpha'} \right) \right\} |\partial_{\alpha'}| \left( (Z_{tt} + i)\overline{Z}_{\alpha'} \right) d\alpha'
\]
Now by combining all the three terms we obtain
\[
\frac{d}{dt} E_{\sigma,1} \approx 2 \text{Re} \int \left\{ \mathcal{Z}_{ttt} Z_{\alpha'} + i A_1 \overline{D_{\alpha'}} \mathcal{Z}_t - i \sigma \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} |D_{\alpha'}| Z_{t,\alpha'} \right) \right\} \left| \partial_{\alpha'} \right| ((Z_{tt} + i) \overline{Z_{\alpha'}}) \, d\alpha'
\]
Recall from (38) that
\[
\mathcal{Z}_{ttt} Z_{\alpha'} + i A_1 \overline{D_{\alpha'}} \mathcal{Z}_t = i \sigma \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \right) \mathcal{Z}_{t,\alpha'} - \sigma (D_{\alpha'} \mathcal{Z}_t) \partial_{\alpha'} \Theta - \sigma \partial_{\alpha'} \left( \Theta \partial_{\alpha'} \overline{D_{\alpha'}} \mathcal{Z}_t \right) - i J_1
\]
Hence it is sufficient to show that each of the terms on the right hand side is in \( \dot{H}^{\frac{1}{2}} \). We have already shown that \( J_1 \in \dot{H}^{\frac{1}{2}} \). We also have from Proposition 9.9 and Lemma 5.3
\[
\| (D_{\alpha'} \mathcal{Z}_t) \partial_{\alpha'} \Theta \|_{\dot{H}^{\frac{1}{2}}} \lesssim \| \sigma \partial_{\alpha'} \|_{\dot{H}^{\frac{1}{2}}} \| D_{\alpha'} \mathcal{Z}_t \|_{\dot{H}^{\frac{1}{2}}} \| \sigma \partial_{\alpha'} \|_{\dot{H}^{\frac{1}{2}}}
\]
Now observe that
\[
i \sigma \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \right) \mathcal{Z}_{t,\alpha'} = i \sigma \left( \partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \right) \mathcal{Z}_{t,\alpha'}
\]
We have the estimates from Lemma 5.3
\[
\| \sigma \partial_{\alpha'} \mathcal{Z}_{t,\alpha'} \|_{\dot{H}^{\frac{1}{2}}} \leq \| \sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \|_{\dot{H}^{\frac{1}{2}}} \| \mathcal{Z}_{t,\alpha'} \|_{\dot{H}^{\frac{1}{2}}}
\]
\[
\| \sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \mathcal{Z}_{t,\alpha'} \|_{\dot{H}^{\frac{1}{2}}} \leq \| \sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \|_{\dot{H}^{\frac{1}{2}}} \| \mathcal{Z}_{t,\alpha'} \|_{\dot{H}^{\frac{1}{2}}}
\]
\[
\| \sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \mathcal{Z}_{t,\alpha'} \|_{\dot{H}^{\frac{1}{2}}} \leq \| \sigma \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \|_{\dot{H}^{\frac{1}{2}}} \| \mathcal{Z}_{t,\alpha'} \|_{\dot{H}^{\frac{1}{2}}}
\]
For the last term we use Proposition 9.12 with 
\[ f = \frac{\sigma^2}{Z_{\alpha'}} \partial^2_{\alpha'} \frac{1}{Z_{\alpha'}} , \quad w = \frac{\sigma^6}{|Z_{\alpha'}|^2} , \quad h = \frac{\sigma^6}{|Z_{\alpha'}|^{\frac{3}{2}}} Z_{t,\alpha'} . \]

Hence
\[
\| \sigma \left( \partial^2_{\alpha'} \frac{1}{Z_{\alpha'}} \right) |D_{\alpha'}| Z_t \|_{\dot{H}^{\frac{1}{2}}} \]
\[
\lesssim \left\| \frac{\sigma^5}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial^2_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{\dot{H}^{\frac{1}{2}}} \left\| \frac{\sigma^5}{|Z_{\alpha'}|^{\frac{3}{2}}} Z_{t,\alpha'} \right\|_{\dot{H}^{\frac{1}{2}}} + \left\| \frac{\sigma^5}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial^2_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{\dot{H}^{\frac{1}{2}}} \left\| \frac{\sigma^5}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \right\|_{\dot{H}^{\frac{1}{2}}} \frac{1}{|Z_{\alpha'}|^{\frac{1}{2}}} Z_t \right\|_{\dot{H}^{\frac{1}{2}}}
\]
\[
+ \left\| \frac{\sigma^5}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial^2_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{\dot{H}^{\frac{1}{2}}} \left\| \frac{\sigma^5}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \right\|_{\dot{H}^{\frac{1}{2}}} \frac{1}{|Z_{\alpha'}|^{\frac{1}{2}}} Z_{t,\alpha'} \right\|_{\dot{H}^{\frac{1}{2}}}
\]

This completes the proof of \( \frac{d}{dt} E_{\sigma,1}(t) \lesssim P(E_{\sigma}(t)) \)

### 5.2.3. Controlling \( E_{\sigma,2} \) and \( E_{\sigma,3} \)

Recall that both \( E_{\sigma,2} \) and \( E_{\sigma,3} \) are of the form

\[
E_{\sigma,i} = \| D_t f \|^2 + \left\| \sqrt{A_{1}} \frac{f}{|Z_{\alpha'}|} \right\|_{\dot{H}^{\frac{1}{2}}}^2 + \left\| \frac{\sigma^5}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} f \right\|_{\dot{H}^{\frac{1}{2}}}^2
\]

Where \( f = \bar{Z}_{t,\alpha'} \) for \( i = 2 \) and \( f = \Theta \) for \( i = 3 \). Also note that \( \mathbb{P}_H f = f \) for these choices of \( f \). We will simplify the time derivative of each of the terms individually before combining them.

1. As \( b_{\alpha'} \in L^\infty \) we have from Lemma 5.4

\[
\frac{d}{dt} \int |D_t f|^2 \, d\alpha' \approx 2 \text{Re} \int (D_t^2 f)(D_t \bar{f}) \, d\alpha'
\]

2. By using Lemma 5.4 we have

\[
\frac{d}{dt} \int \left| \partial_{\alpha'} \right|^{\frac{1}{2}} \left( \sqrt{A_{1}} \right) \frac{f}{|Z_{\alpha'}|} \, d\alpha' \approx 2 \text{Re} \int \left| \partial_{\alpha'} \right| \left( \sqrt{A_{1}} \right) \frac{f}{|Z_{\alpha'}|} \, d\alpha'
\]

Observe from (28) that

\[
D_t \left( \sqrt{A_{1}} \right) \frac{f}{|Z_{\alpha'}|} = \left\{ \frac{D_tA_{1}}{2A_{1}} + b_{\alpha'} - \text{Re}(D_{\alpha'} Z_t) \right\} \sqrt{A_{1}} \frac{f}{|Z_{\alpha'}|} + \frac{\sqrt{A_{1}}}{|Z_{\alpha'}|} D_t \frac{f}{|Z_{\alpha'}|}
\]

We note that for \( f = \bar{Z}_{t,\alpha'} \) or \( f = \Theta \) we have \( \frac{f}{|Z_{\alpha'}|} \in C \). Hence by Lemma 5.3 we have

\[
\left\| \left\{ \frac{D_tA_{1}}{2A_{1}} + b_{\alpha'} - \text{Re}(D_{\alpha'} Z_t) \right\} \right\|_{\dot{H}^{\frac{1}{2}}}
\]
\[
\chi \left\| \frac{f}{|Z_{\alpha'}|} \right\|_C \left( \frac{1}{A_1} \right) \left\| \frac{1}{A_1} \right\|_W \left\{ \left\| D_t A_1 \right\|_W + \| b_{\alpha'} \|_W + \| D_a' Z_t \|_W \right\} \\
+ \| b_{\alpha'} \|_W + \| D_a' Z_t \|_W \}
\]

Hence
\[
\frac{d}{dt} \int \left| \partial_{\alpha'} \right|^2 \left( \frac{\sqrt{A_1}}{|Z_{\alpha'}|} f \right) \, d\alpha' \approx 2 \text{Re} \int \left\{ \frac{\sqrt{A_1}}{|Z_{\alpha'}|} \partial_{\alpha'} \left( \frac{\sqrt{A_1}}{|Z_{\alpha'}|} f \right) \right\} (D_t \tilde{f}) \, d\alpha'
\]

We simplify further using \( |\partial_{\alpha'}| = i \mathbb{H} \partial_{\alpha'} \) and \( \mathbb{H} f = f \)
\[
\frac{\sqrt{A_1}}{|Z_{\alpha'}|} \partial_{\alpha'} \left( \frac{\sqrt{A_1}}{|Z_{\alpha'}|} f \right) = i \left[ \frac{\sqrt{A_1}}{|Z_{\alpha'}|}, \mathbb{H} \right] \partial_{\alpha'} \left( \frac{\sqrt{A_1}}{|Z_{\alpha'}|} f \right) + i \mathbb{H} \left\{ \frac{\sqrt{A_1}}{|Z_{\alpha'}|} \partial_{\alpha'} \left( \frac{\sqrt{A_1}}{|Z_{\alpha'}|} f \right) + \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} f \right\}
\]

Hence by Lemma 5.3, Proposition 9.8 and using \( A_1 \geq 1 \) we have
\[
\left\| \frac{\sqrt{A_1}}{|Z_{\alpha'}|} \partial_{\alpha'} \left( \frac{\sqrt{A_1}}{|Z_{\alpha'}|} f \right) - i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} f \right\|_2 \lesssim \left( \left\| D_{\alpha'} A_1 \right\|_2 + \left\| \frac{\sqrt{A_1}}{|Z_{\alpha'}|} \right\|_2 \left\| \frac{\sqrt{A_1}}{|Z_{\alpha'}|} f \right\|_H \right)^{\frac{1}{2}}
\]

\[
+ \| A_1 \|_W \left\| D_{\alpha'} \right\|_2 \left\| \frac{1}{|Z_{\alpha'}|} \right\|_C \left\| \frac{1}{|Z_{\alpha'}|} f \right\|_C
\]

\[
+ \| f \|_2 \left\| \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} A_1 \right\|_{L^\infty \cap H^\frac{1}{2}}
\]

As \( D_t f \in L^2 \) this shows that
\[
\frac{d}{dt} \int \left| \partial_{\alpha'} \right|^2 \left( \frac{\sqrt{A_1}}{|Z_{\alpha'}|} f \right) \, d\alpha' \approx 2 \text{Re} \int \left( i \frac{A_1}{|Z_{\alpha'}|^2} \partial_{\alpha'} f \right) (D_t \tilde{f}) \, d\alpha'
\]

(3) By Lemma 5.4 we have
\[
\sigma \frac{d}{dt} \int \left| \partial_{\alpha'} \right|^2 \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} f \right) \, d\alpha' \approx 2 \sigma \text{Re} \int \left\{ \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} f \right) \right\} D_t \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \tilde{f} \right) \, d\alpha'
\]

We note that
\[
\sigma \frac{1}{2} D_t \left( \frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} \tilde{f} \right) = \sigma \frac{1}{2} \left( \frac{1}{2} b_{\alpha'} - \frac{3}{2} \text{Re}(D_{\alpha'} Z_t) \right) \left( \frac{1}{|Z_{\alpha'}|^2} \right) \hat{\partial}_{\alpha'} 
\]
\[+ \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} D_t \bar{f}\]

As \(\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} f \in C\) for \(f = \overline{Z_{t,\alpha'}}\) or \(f = \Theta\) we use Lemma 5.3 to obtain

\[
\left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \left( \frac{1}{2} b_{\alpha'} - \frac{3}{2} \text{Re}(D_{\alpha'} Z_t) \right) \left( \frac{1}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \bar{f} \right) \right\|_{\hat{H}^\frac{1}{2}} \\
\lesssim (\|b_{\alpha'}\|_{\mathcal{C}} + \|D_{\alpha'} Z_t\|_{\mathcal{C}}) \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} f \right\|_C
\]

Hence

\[
\frac{\sigma}{d} \int \left| \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} f \right) \right|^2 d\alpha' \\
\approx -2\sigma \text{Re} \int \partial_{\alpha'} \left\{ \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} |\partial_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} f \right) \right\} D_t \bar{f} d\alpha'
\]

Now using \(|\partial_{\alpha'}| = i \mathbb{H} \partial_{\alpha'}\) we see that

\[
\sigma \partial_{\alpha'} \left\{ \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} |\partial_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} f \right) \right\} = i \sigma \partial_{\alpha'} \left\{ \frac{1}{|Z_{\alpha'}|^\frac{1}{2}}, \mathbb{H} \right\} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} f \right) \\
+ i \sigma \mathbb{H} \partial_{\alpha'} \left\{ \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} f \right) \right\}
\]

Using Proposition 9.8 we have the estimate

\[
\left\| \sigma \partial_{\alpha'} \left\{ \frac{1}{|Z_{\alpha'}|^\frac{1}{2}}, \mathbb{H} \right\} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} f \right) \right\|_2 \\
\lesssim \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \right\|_{\hat{H}^\frac{1}{2}} \left\{ \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} f \right) \right\} + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \right\|_2 \left\| \partial_{\alpha'} \right\|_\infty \left\| \partial_{\alpha'} \right\|_2
\]

By using the expansion in (31) for \(f = \overline{Z_{t,\alpha'}}\) we get

\[
\left\| \sigma |D_{\alpha'}|^\frac{3}{2} \overline{Z_{t,\alpha'}} - \sigma \partial_{\alpha'} \left\{ \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} \overline{Z_{t,\alpha'}} \right) \right\} \right\|_2 \\
\lesssim \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \right\|_\infty \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \overline{Z_{t,\alpha'}} \right\|_2 \\
+ \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \right\|_\infty \left\| \frac{1}{|Z_{\alpha'}|^\frac{1}{2}} \partial_{\alpha'} \overline{Z_{t,\alpha'}} \right\|_2
Similarly using the expansion in (31) for $f = \Theta$ and using Lemma 5.3 we obtain

\[
\left\| \sigma |D_{\alpha'}|^{3}\Theta - \sigma \partial_{\alpha'} \left\{ \frac{1}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \right) \right\} \right\|_{2} \leq \left\| \sigma \frac{|Z_{\alpha'}|^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \right\|_{C} + \left\| \sigma \frac{1}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \right) \right\|_{C} + \left\| \frac{1}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} \Theta \right) \right\|_{C} + \left\| \frac{1}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} \Theta \right\|_{C}.
\]

Using these we now have

\[
\sigma \frac{d}{dt} \int |\partial_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} f \right) \, d\alpha' \approx -2 \text{Re} \int \left( i \sigma \mathbb{W} |D_{\alpha'}|^{3} f \right) D_{t} \bar{f} \, d\alpha'
\]

But we have already shown that $\sigma (\mathbb{I} - \mathbb{W}) |D_{\alpha'}|^{3} f \in L^{2}$ for both $f = \bar{Z}_{t,\alpha'}$ and $f = \Theta$. Hence we finally have

\[
\sigma \frac{d}{dt} \int |\partial_{\alpha'}| \left( \frac{1}{|Z_{\alpha'}|^{\frac{3}{2}}} \partial_{\alpha'} f \right) \, d\alpha' \approx 2 \text{Re} \int \left( -i \sigma |D_{\alpha'}|^{3} f \right) D_{t} \bar{f} \, d\alpha'
\]

(4) Combining all three terms we obtain for $i = 2, 3$

\[
\frac{d}{dt} E_{\sigma,i} \approx 2 \text{Re} \int \left( D_{t}^{2} f + i \frac{A_{1}}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} f - i \sigma |D_{\alpha'}|^{3} f \right) (D_{t} f) \, d\alpha'
\]

For $f = \bar{Z}_{t,\alpha'}$ we obtain from (42)

\[
\left( D_{t}^{2} + i \frac{A_{1}}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^{3} \right) \bar{Z}_{t,\alpha'} = R_{1} \bar{Z}_{t,\alpha'} - i \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) J_{1} - i D_{t} \alpha' J_{1} - \bar{Z}_{t,\alpha'}
\]

\[
\left[ D_{t}^{2} + i \frac{A_{1}}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^{3}, \frac{1}{Z_{\alpha'}} \right] \bar{Z}_{t,\alpha'}
\]

We have already shown that $R_{1} \in \mathcal{C}$, $J_{1} \in \mathcal{W}$, $\partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in L^{2}$ and the last term in $L^{2}$. Now for $f = \Theta$ we have from (43)

\[
\left( D_{t}^{2} + i \frac{A_{1}}{|Z_{\alpha'}|^{2}} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^{3} \right) \Theta = R_{2} + i J_{2}
\]

In this case too we have shown that $R_{2}, J_{2} \in L^{2}$. Hence this shows that

\[
\frac{d}{dt} E_{\sigma,i}(t) \lesssim P(E_{\sigma}(t)) \quad \text{for} \quad i = 2, 3
\]
5.2.4. Controlling $E_{\sigma,4}$

Recall that

$$E_{\sigma,4} = \|D_t \overline{D_{\alpha'}} Z_t\|_{H^\frac{1}{2}}^2 + \|\sqrt{A_1} |D_{\alpha'}| \overline{D_{\alpha'} Z_t}\|_2^2 + \left\| \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} |D_{\alpha'}| \overline{D_{\alpha'} Z_t}\right\|_2^2$$

As before we first simplify the terms individually before combining them.

(1) By Lemma 5.4 and the fact that $|\partial_{\alpha'}| = iH\partial_{\alpha'}$ we have

$$\frac{d}{dt} \int |\partial_{\alpha'}| \overline{\frac{1}{2}} (D_t D_{\alpha'} Z_t) \, d\alpha' \approx 2\text{Re} \int (D_t^2 \overline{D_{\alpha'} Z_t}) |\partial_{\alpha'}|(D_t D_{\alpha'} Z_t) \, d\alpha'$$

$$\approx 2\text{Re} \int (\mathbb{I} D_t^2 \overline{D_{\alpha'} Z_t}) (-i \partial_{\alpha'}(D_t D_{\alpha'} Z_t)) \, d\alpha'$$

But we have shown that $(\mathbb{I} - \mathbb{I}) D_t^2 \overline{D_{\alpha'} Z_t} \in H^\frac{1}{2}$. Hence we have

$$\frac{d}{dt} \int |\partial_{\alpha'}| \overline{\frac{1}{2}} (D_t D_{\alpha'} Z_t) \, d\alpha' \approx 2\text{Re} \int (D_t^2 \overline{D_{\alpha'} Z_t}) (-i \partial_{\alpha'}(D_t D_{\alpha'} Z_t)) \, d\alpha'$$

(2) By Lemma 5.4 and as $b_{\alpha'} \in L^\infty$ we have

$$\frac{d}{dt} \int A_1 |D_{\alpha'}| \overline{D_{\alpha'} Z_t} \, d\alpha' \approx \int A_1 \left|D_{\alpha'}| \overline{D_{\alpha'} Z_t}\right|^2 \, d\alpha' + 2\text{Re} \int A_1 \overline{\left|D_{\alpha'}| \overline{D_{\alpha'} Z_t}\right|} D_t \left|D_{\alpha'}| \overline{D_{\alpha'} Z_t}\right| \, d\alpha'$$

As $\frac{D_t A_1}{A_1} \in L^\infty$, the first term is controlled. We now see that

$$D_t |D_{\alpha'}| D_{\alpha'} Z_t = -\text{Re} (D_{\alpha'} Z_t) |D_{\alpha'}| D_{\alpha'} Z_t + |D_{\alpha'}| D_t D_{\alpha'} Z_t$$

Now as $\text{Re}(D_{\alpha'} Z_t) \in L^\infty$ we obtain

$$\frac{d}{dt} \int A_1 |D_{\alpha'}| \overline{D_{\alpha'} Z_t} \, d\alpha' \approx 2\text{Re} \int \left(i A_1 \overline{\frac{1}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'}| \overline{D_{\alpha'} Z_t}} \right) (-i \partial_{\alpha'}(D_t D_{\alpha'} Z_t)) \, d\alpha'$$

(3) By Lemma 5.4 and as $b_{\alpha'} \in L^\infty$ we have

$$\sigma^{\frac{1}{2}} D_t \left\{ \frac{1}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} |D_{\alpha'}| \overline{D_{\alpha'} Z_t} \right\} \, d\alpha'$$

$$\approx 2\sigma \text{Re} \int \left\{ \frac{1}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} |D_{\alpha'}| \overline{D_{\alpha'} Z_t} \right\} D_t \left\{ \frac{1}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} |D_{\alpha'}| D_{\alpha'} Z_t \right\} \, d\alpha'$$

We see that

$$\sigma^{\frac{1}{2}} D_t \left\{ \frac{1}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} |D_{\alpha'}| D_{\alpha'} Z_t \right\}$$

$$= \left(-\frac{3}{2} \text{Re} (D_{\alpha'} Z_t) - \frac{b_{\alpha'}}{2}\right) \left\{ \frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^{\frac{1}{2}}} \partial_{\alpha'} |D_{\alpha'}| D_{\alpha'} Z_t \right\}$$
\[
- \text{Re} \left( \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'} Z_t \right) (|D_{\alpha'}|D_{\alpha'} Z_t) + \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} |D_{\alpha'}| D_t D_{\alpha'} Z_t
\]

As \( D_{\alpha'} Z_t, b_{\alpha'} \in L^\infty \), the first term is controlled in \( L^2 \). The second term is also in \( L^2 \) as we have \( \frac{\sigma^{\frac{1}{2}}}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'} Z_t \in L^\infty \) and \( |D_{\alpha'}|D_{\alpha'} Z_t \in L^2 \). Hence we have

\[
\sigma \frac{d}{dt} \int \left| \frac{1}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'} Z_t \right|^2 d\alpha' \approx 2 \text{Re} \int \left\{ -i \sigma |D_{\alpha'}|^3 \overline{D_{\alpha'} Z_t} \right\} d\alpha'
\]

(4) Combining the three terms we get

\[
\frac{d}{dt} E_{\sigma,4} \approx 2 \text{Re} \int \left( D_t^2 \overline{D_{\alpha'} Z_t} + i \frac{A_1}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} D_{\alpha'} Z_t - i \sigma |D_{\alpha'}|^3 \overline{D_{\alpha'} Z_t} \right) (-i \partial_{\alpha'} (D_t D_{\alpha'} Z_t)) d\alpha'
\]

From equation (40) we see that

\[
\left( D_t^2 + i \frac{A_1}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} - i \sigma |D_{\alpha'}|^3 \right) \overline{D_{\alpha'} Z_t} = R_1 - i \left( \overline{D_{\alpha'} \frac{1}{Z_{,\alpha'}}} \right) J_1 - i \frac{1}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} J_1
\]

But we have already shown that \( R_1, \frac{1}{|Z_{,\alpha'}|^\frac{1}{2}} \partial_{\alpha'} J_1 \in \dot{H}^{\frac{1}{2}} \) and the second term is controlled in \( \dot{H}^{\frac{1}{2}} \) by using Lemma 5.3

\[
\left\| \left( \overline{D_{\alpha'} \frac{1}{Z_{,\alpha'}}} \right) J_1 \right\|_{\dot{H}^{\frac{1}{2}}} \lesssim \left\| \overline{D_{\alpha'} \frac{1}{Z_{,\alpha'}}} \right\|_{\mathcal{C}} \| J_1 \|_{\gamma \mathcal{V}}
\]

Hence we have

\[
\frac{d}{dt} E_{\sigma,4}(t) \lesssim P(E_{\sigma}(t))
\]

This concludes the proof of Theorem 5.1

6. Equivalence of Energy and Relation with Sobolev Norm

In this section we prove the equivalence of the energies \( E_{\sigma}(t) \) and \( E_{\sigma}(t) \). We also explain the relation of the energy \( E_{\sigma}(t) \) to the Sobolev norm of the solution.

**Proposition 6.1.** There exists universal polynomials \( P_1, P_2 \) with non-negative coefficients so that if \((Z, Z_t)(t)\) is a smooth solution to the water wave equation (13) for \( \sigma \geq 0 \) in the time interval \([0, T]\) satisfying \((Z_{,\alpha'} - \frac{1}{Z_{,\alpha'}}, 1, Z_t) \in L^\infty([0, T], H^{s+\frac{1}{2}}(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R}))\) for all \( s \geq 3 \), then for all \( t \in [0, T] \) we have

\[
E_{\sigma}(t) \leq P_1(E_{\sigma}(t)) \quad \text{and} \quad E_{\sigma}(t) \leq P_2(E_{\sigma}(t))
\]
Proof. From now on we will suppress the time variable. Let us first prove that $E_\sigma \leq P_2(E_\sigma)$. Note that from Sect. 5.1 we already have most of the terms of $E_\sigma$ controlled.

The terms which are not controlled namely \[ \frac{\sigma^{\frac{1}{2}}}{Z_{\alpha'}} Z_{\alpha'}^{\frac{1}{2}} \text{ and } \frac{\sigma^{\frac{3}{2}}}{Z_{\alpha'}} Z_{\alpha'}^{\frac{3}{2}} \] can be easily controlled in $H^{\frac{1}{2}}$ as we have $\omega \in \mathcal{W}$ and we have already shown that $$\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} Z_{\alpha'}^{\frac{1}{2}}$$ is in $C$ and using Lemma 5.3

Let us now show that $E_\sigma \leq P_1(E_\sigma)$.

1. As we have $Z_{t,\alpha'} \in L^2$, we see from Sect. 5.1 that $A_1 \in L^\infty \cap H^{\frac{1}{2}}$. Hence we have that $(Z_{tt} - i) Z_{t,\alpha'} \in H^{\frac{1}{2}}$ by using equation (12). We now show that $D_{t,\alpha'} Z_t \in L^\infty$.

Observe that \[
\sigma_{\alpha'}(D_{t,\alpha'} Z_t)^2 = 2(Z_{t,\alpha'})(D_{t,\alpha'}^2 Z_t) = 2(Z_{t,\alpha'})(\sigma_{\alpha'}(Z_{\alpha'}^2) D_{t,\alpha'} Z_t + 2(Z_{t,\alpha'})(\frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_t,\alpha')
\]

Hence we have \[
\|D_{t,\alpha'} Z_t\|_2^2 \leq 2 \|Z_{t,\alpha'}\|_2 \|D_{t,\alpha'} Z_t\|_2 + 2 \|Z_{t,\alpha'}\|_2 \|\frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_t,\alpha'\|_2
\]

Now using the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$ on the first term, we obtain $D_{t,\alpha'} Z_t \in L^\infty$.

2. Following the proof in Sect. 5.1 we now have the terms $|D_{t,\alpha'}| D_{t,\alpha'} Z_t \in L^2$, $\partial_{\alpha'} \frac{1}{|Z_{\alpha'}|} \in L^2$, $\omega \in \mathcal{W}$, $D_{t,\alpha'} Z_t \in \mathcal{W} \cap C$, $\partial_{\alpha'} \mathcal{P} (\frac{Z_t,\alpha'}{Z_{\alpha'}}) \in L^\infty$, $A_1 \in \mathcal{W}$, $b_{\alpha'} \in \mathcal{W}$, $\frac{1}{|Z_{\alpha'}|^2} \partial_{\alpha'} A_1 \in \mathcal{W} \cap C$. We also see that $\Theta \in L^2$, $D_t \Theta \in L^2$ by using the formula (32) and (34). By following the proof of $D_{t,\alpha'} \frac{1}{|Z_{\alpha'}|} \in C$ in Sect. 5.1, we easily obtain $|D_{t,\alpha'}| \frac{1}{|Z_{\alpha'}|} \in C$ and $\frac{\Theta}{|Z_{\alpha'}|} \in C$. Hence we have $\sqrt{A_1} \Theta \in C$ and $\sqrt{A_1} \frac{1}{Z_{t,\alpha'}} \in C$ from Lemma 5.3.

3. Again by following the approach in Sect. 5.1 we automatically have $\sigma^{\frac{1}{2}} |Z_{\alpha'}|^\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in \mathcal{W}$, $\sigma^{\frac{3}{2}} \partial_{\alpha'} \Theta \in L^2$, $\sigma^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in L^2$, $\sigma^{\frac{3}{2}} \Theta \in L^\infty \cap H^{\frac{1}{2}}$, $\sigma^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in L^\infty \cap H^{\frac{1}{2}}$ etc. Hence we now have $\frac{\sigma}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \Theta \in L^2$, $\sigma \partial_{\alpha'} D_{t,\alpha'} \Theta \in L^2$

by following the proof of $\frac{\sigma}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'}^2 \frac{1}{Z_{\alpha'}} \in L^2$ in Sect. 5.1. In particular we now have $D_t \frac{1}{Z_{t,\alpha'}} \in L^2$ by using equation (12).

4. By following the proof of $\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in C$ in Sect. 5.1 we see that $\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'} \Theta \in C$.

Similarly by following the proof of $\frac{\sigma^{\frac{1}{2}}}{|Z_{\alpha'}|^\frac{3}{2}} \partial_{\alpha'}^2 Z_{t,\alpha'} \in L^2$ in Sect. 5.1 we
also obtain \( \frac{\sigma \frac{1}{2}}{|Z_{x',t}|^{\frac{1}{2}}} \partial_{x'} \overline{D_{x'} \overline{Z}_t} \in L^2 \). We use Proposition 9.11 with \( f = \frac{\sigma \frac{1}{2}}{|Z_{x',t}|^{\frac{1}{2}}} \partial_{x'} \overline{Z}_{t,x'} \) and \( w = \frac{1}{|Z_{x'}|} \) to obtain \( \frac{\sigma \frac{1}{2}}{|Z_{x',t}|^{\frac{1}{2}}} \partial_{x'} \overline{Z}_{t,x'} \in C \).

(5) As \( \omega \in \mathcal{W} \) we have \( \frac{\sigma}{|Z_{x',t}|^{\frac{1}{2}}} \partial_{x'} \frac{1}{Z_{x'}} \in C \). Hence by following the proof of \( \frac{\sigma}{|Z_{x',t}|^{\frac{1}{2}}} \partial_{x'} \frac{1}{Z_{x'}} \in C \) in Sect. 5.1 we obtain \( \frac{\sigma}{|Z_{x',t}|^{\frac{1}{2}}} \partial_{x'} \frac{1}{Z_{x'}} \in C \), \( \sigma \overline{D_{x'} D_{x'}} \in C \) etc. Hence by using equation (12) we now have \( \overline{D_{x'} Z}_t \in C \) and hence \( D_t \overline{D_{x'} \overline{Z}_t} \in C \). This finishes the proof of Proposition 6.1 \( \square \)

We now explain the relation between the energy \( E_\sigma (t) \) and the Sobolev norm of the data.

**Lemma 6.2.** Let \((Z, Z_t)(t)\) be a smooth solution to the water wave equation (13) for \( \sigma \geq 0 \) in the time interval \([0, T^*]\) for \( T^* > 0 \), satisfying \((Z, Z_t - 1, \frac{1}{Z_{x'}} - 1, Z_t) \in L^\infty ([0, T^*], H^{s+\frac{1}{2}}(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R}))\) for all \( s \geq 3 \). Then we have the following estimates

1. For \( \sigma > 0 \) there exists universal polynomials \( P_1, P_2 \) with non-negative coefficients so that for each \( t \in [0, T^*] \) we have

\[
\| Z_{x',t} \|_{H^2} (t) + \| \partial_{x'} Z_{x'} \|_{H^{2.5}} (t) \leq P_1 \left( E_\sigma (t) + \| Z_{x'} \|_\infty (t) + \frac{1}{\sigma} \right) \quad \text{and}
\]

\[
E_\sigma (t) \leq P_2 \left( \| Z_{x',t} \|_{H^2} (t) + \| \partial_{x'} Z_{x'} \|_{H^{2.5}} (t) + \| Z_{x'} \|_\infty (t) + \frac{1}{\sigma} \right)
\]

2. For \( \sigma \geq 0 \) there exists universal polynomials \( P_3, P_4 \) with non-negative coefficients so that for each \( t \in [0, T^*] \) we have

\[
\| Z_{x',t} \|_{H^1} (t) + \| \partial_{x'} Z_{x'} \|_{H^{1.5}} (t) \leq P_3 \left( E_\sigma (t) \big|_{\sigma = 0} + \| Z_{x'} \|_\infty (t) \right) \quad \text{and}
\]

\[
E_\sigma (t) \big|_{\sigma = 0} \leq P_4 \left( \| Z_{x',t} \|_{H^1} (t) + \| \partial_{x'} Z_{x'} \|_{H^{1.5}} (t) + \frac{1}{\| Z_{x'} \|_\infty (t)} \right)
\]

3. There exists a universal increasing function \( F : [0, \infty) \to [0, \infty) \) so that if \( 0 \leq T \leq T^* \) and we define

\[
A = \sup_{t \in [0, T]} \left\{ \| Z_{x'} - 1 \|_{H^{3.5}} (t) + \| \frac{1}{Z_{x'}} - 1 \|_{H^{3.5}} (t) + \| Z_t \|_{H^3} (t) \right\} < \infty
\]

\[
B = \sup_{t \in [0, T]} \left\{ \| Z_{x'} - 1 \|_{H^{1.5}} (t) + \| \frac{1}{Z_{x'}} - 1 \|_{H^{1.5}} (t) + \| Z_t \|_{H^2} (t) \right\} < \infty
\]

\[
D = \left\| \frac{1}{Z_{x'}} - 1 \right\|_2 (0) + \| Z_t \|_2 (0) + \sup_{t \in [0, T]} E_\sigma (t) < \infty
\]

Then for \( \sigma > 0 \)

\[
A \leq F \left( D + \| Z_{x'} \|_\infty (0) + T + \sigma + \frac{1}{\sigma} \right)
\]
and for $\sigma \geq 0$

$$B \leq F(D + \|Z_{\alpha'}\|_\infty(0) + T + \sigma + 1)$$

**Remark 6.3.** For $\sigma > 0$ if the interface is non-self intersecting and if $E_\sigma$ is well defined with $E_\sigma < \infty$, then we in fact have $Z_{\alpha'} \in L^\infty$ but the norm $\|Z_{\alpha'}\|_\infty$ depends on $\sigma^{-\frac{1}{2}}$ and the rate at which $Z_{\alpha'} \to 1$ as $|\alpha'| \to \infty$. To see this observe that in the proof of Theorem 5.1 we showed that $\|\sigma^{\frac{1}{2}}\kappa\|_\infty \leq C(E_\sigma)$ and hence the curvature $\kappa \in L^\infty$. Therefore by the Kellogg-Warschawski theorem (see chapter 3 of [34]), the derivative of the Riemann mapping extends continuously to the boundary and hence $Z_{\alpha'} \in L^\infty_{loc}$. As $Z_{\alpha'} \to 1$ when $|\alpha'| \to \infty$, we have that $Z_{\alpha'} \in L^\infty$. Hence by part 1 of the above lemma we have $\overline{Z}_{t,\alpha'} \in H^2(\mathbb{R})$ and $\partial_{\alpha'} Z_{\alpha'} \in H^{2.5}(\mathbb{R})$. Hence for $\sigma > 0$ the condition $E_\sigma < \infty$ is essentially equivalent to the condition that the solution is in a suitable Sobolev space.

**Proof.** We prove each part seperately.

(1) Let $\sigma > 0$ and assume that $E_\sigma + \|Z_{\alpha'}\|_\infty < \infty$. Hence we have that $\overline{Z}_{t,\alpha'} \in L^2$ and we have

$$\left\|\partial^2_{\alpha'} Z_{t,\alpha'}\right\|_2 \lesssim \frac{1}{\sigma^2} \left\|Z_{\alpha'}\right\|_\infty^2 \frac{\sigma_{\alpha'}^2}{Z_{\alpha'}^2} \left\|Z_{t,\alpha'}\right\|_2$$

and

$$\|D_{\alpha'} Z_{\alpha'}\|_2 \lesssim \left\|Z_{\alpha'}\right\|_\infty \left\|\partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_2$$

Hence $\overline{Z}_{t,\alpha'} \in H^2$ and as $Z_{\alpha'} \in L^\infty$ we obtain $Z_{\alpha'} \in W$. Hence from Lemma 5.3 we have

$$\left\|\partial^3_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_{H^\frac{1}{2}} \lesssim \frac{1}{\sigma} \left\|Z_{\alpha'}\right\|_W$$

and

$$\left\|\partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_C \lesssim \frac{1}{\sigma} \left\|Z_{\alpha'}\right\|_W$$

Hence $\partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in H^{2.5}$. As $Z_{\alpha'} \in L^\infty$, we clearly have $\partial_{\alpha'} Z_{\alpha'} \in L^2$ as. Now for $s \geq 1$ we see from Proposition 9.9 that

$$\|\partial_{\alpha'}|^s \partial_{\alpha'} Z_{\alpha'}\|_2 = \left\|\partial_{\alpha'}|^s \left(Z_{\alpha'}^2 \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right)\right\|_2 \lesssim \left\|\partial_{\alpha'}|^s Z_{\alpha'}\right\|_2 \left\|Z_{\alpha'}\right\|_\infty \left\|\partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_\infty + \left\|Z_{\alpha'}\right\|_\infty^2 \left\|\partial_{\alpha'}|^s \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_2$$

Using this for $s = 1, 2, 2.5$ sequentially we obtain $\partial_{\alpha'} Z_{\alpha'} \in H^{2.5}$.

(2) Now assume that $\sigma > 0$ and $\|\overline{Z}_{t,\alpha'}\|_{H^2} + \|\partial_{\alpha'} Z_{\alpha'}\|_{H^{2.5}} + \left\|\frac{1}{Z_{\alpha'}}\right\|_\infty < \infty$. We first observe that $E_{\sigma, 2}$ is easily controlled and that $\sigma^s Z_{\alpha'}^2 \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in L^2$, $\sigma^s Z_{\alpha'}^2 \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in L^\infty$. Now we have

$$\left\|\partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_2 \lesssim \frac{1}{\sigma Z_{\alpha'}} \left\|\partial_{\alpha'} Z_{\alpha'}\right\|_2$$

and hence for $s \geq 1$ we have from Proposition 9.9

$$\left\|\partial_{\alpha'}|^s \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right\|_2 = \left\|\partial_{\alpha'}|^s \left(\frac{1}{Z_{\alpha'}}^2 \partial_{\alpha'} Z_{\alpha'}\right)\right\|_2$$
\[
\begin{align*}
\lambda &\lesssim \left\| \partial_{\alpha'} \right\|^5 \left\| \frac{1}{Z,\alpha'} \right\|_2 \left\| \frac{1}{Z,\alpha'} \right\|_{L^\infty} \left\| \partial_{\alpha'} Z,\alpha' \right\|_{L^\infty} + \left\| \frac{1}{Z,\alpha'} \right\|^2 \left\| \partial_{\alpha'} \right\|^5 \left\| \partial_{\alpha'} Z,\alpha' \right\|_2 \\
\text{Using this for } s = 1, 2, 2.5 \text{ sequentially we obtain } \partial_{\alpha'} \frac{1}{Z,\alpha'} &\in H^{2.5}. \text{ Hence we easily see that} \\
\sigma \frac{1}{Z,\alpha'} \partial_{\alpha'} Z,\alpha' &\in L^2 \text{ and } \frac{\sigma}{Z,\alpha'} \partial_{\alpha'} Z,\alpha' \in L^2. \text{ We also have from Proposition 9.9} \\
\left\| \frac{1}{Z,\alpha'} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_{H^{1/2}} \lesssim \left\| \frac{1}{Z,\alpha'} \right\|_{L^\infty} \left\| \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_{H^{1/2}} + \left\| \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2 \\
\left\| \frac{\sigma}{Z,\alpha'} \partial_{\alpha'} Z,\alpha' \right\|_{H^{1/2}} \lesssim \sigma \left\| \frac{1}{Z,\alpha'} \right\|_{L^\infty} \left\| \partial_{\alpha'} Z,\alpha' \right\|_{H^{1/2}} + \frac{\sigma}{Z,\alpha'} \left\| \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2 \left\| \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2 \\
\text{and similarly} \\
\left\| \frac{\sigma}{Z,\alpha'} \partial_{\alpha'} Z,\alpha' \right\|_{H^{1/2}} \lesssim \sigma \left\| \frac{1}{Z,\alpha'} \right\|_{L^\infty} \left\| \partial_{\alpha'} Z,\alpha' \right\|_{H^{1/2}} + \sigma \left\| \frac{1}{Z,\alpha'} \right\|_2 \left\| \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2 \left\| \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2 \\
\text{We are only left with } \sigma \partial_{\alpha'} \Theta. \text{ We first observe that as } Z,\alpha' = e^{f+ig} \text{ we have} \\
\partial_{\alpha'} Z,\alpha' = Z,\alpha' \partial_{\alpha'} (f + ig) \quad \text{and} \quad \partial_{\alpha'}^2 Z,\alpha' = Z,\alpha' (\partial_{\alpha'} (f + ig))^2 + Z,\alpha' \partial_{\alpha'}^2 (f + ig) \\
\text{and hence we have} \\
\left\| \partial_{\alpha'} \frac{Z,\alpha'}{Z,\alpha'} \right\|_{L^2 \cap L^\infty} = \left\| \partial_{\alpha'} g \right\|_{L^2 \cap L^\infty} \lesssim \left\| \frac{1}{Z,\alpha'} \right\|_{L^\infty} \left\| \partial_{\alpha'} Z,\alpha' \right\|_{L^2 \cap L^\infty} \\
\left\| \partial_{\alpha'}^2 \frac{Z,\alpha'}{Z,\alpha'} \right\|_2 = \left\| \partial_{\alpha'} \left( e^{ig} \partial_{\alpha'} g \right) \right\|_2 \\
\lesssim \left\| \partial_{\alpha'} g \right\|_2 \left\| \partial_{\alpha'} g \right\|_{L^\infty} + \left\| \frac{1}{Z,\alpha'} \right\|_{L^\infty} \left\| \partial_{\alpha'} Z,\alpha' \right\|_2 \\
+ \left\| \frac{1}{Z,\alpha'} \right\|^2 \left\| \partial_{\alpha'} Z,\alpha' \right\|_{L^\infty} \left\| \partial_{\alpha'} Z,\alpha' \right\|_2 \\
\text{From this we see using Proposition 9.9} \\
\left\| \partial_{\alpha'} \right\|^3 \left( \frac{Z,\alpha'}{Z,\alpha'} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right) \lesssim \left\| \partial_{\alpha'}^2 \frac{1}{Z,\alpha'} \right\|_{H^{1/2}} + \left\| \partial_{\alpha'} \frac{2}{Z,\alpha'} \right\|_2 \left\| \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_2 \\
\text{Hence } \sigma \partial_{\alpha'} \Theta \in H^{1/2} \text{ by using the formula (32).} \\
(3) \text{ Now let } \sigma \geq 0 \text{ and assume } E_\sigma |_{\sigma=0+} \left\| Z,\alpha' \right\|_{L^\infty} < \infty. \text{ We clearly see that } Z,\alpha' \in H^1 \\
\text{and } \partial_{\alpha'} Z,\alpha' \in L^2. \text{ By the argument shown earlier, we also have } Z,\alpha' \in \mathcal{W}. \text{ Hence from Lemma 5.3 we have} \\
\left\| \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_{H^{1/2}} \lesssim \left\| Z,\alpha' \right\|_{\mathcal{W}} \left\| \frac{1}{Z,\alpha'} \partial_{\alpha'} \frac{1}{Z,\alpha'} \right\|_C}
\end{align*}
\]
Now we see from Proposition 9.9
\[ \| \partial_{\alpha'} Z_{\alpha'} \|_{H^\frac{1}{2}} = \left\| Z_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{H^\frac{1}{2}} \lesssim \left\| Z_{\alpha'} \right\|_2 \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{H^\frac{1}{2}} \]
\[ + \left\| Z_{\alpha'} \right\|_\infty \left\| \partial_{\alpha'} Z_{\alpha'} \right\|_2 \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 \]

(4) Let \( \sigma \geq 0 \) and assume that \( Z_{t,\alpha'} \in H^1, \partial_{\alpha'} Z_{\alpha'} \in H^\frac{1}{2} \) and \( \frac{1}{Z_{\alpha'}} \in L^\infty \). We easily see that \( Z_{t,\alpha'} \in L^2, \frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_{t,\alpha'} \in L^2 \) and \( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \in L^2 \). We also have from Proposition 9.9
\[ \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{H^\frac{1}{2}} \lesssim \left\| \frac{1}{Z_{\alpha'}} \right\|^2_\infty \left\| \partial_{\alpha'} Z_{\alpha'} \right\|_{H^\frac{1}{2}} + \left\| \frac{1}{Z_{\alpha'}} \right\|_\infty \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 \left\| \partial_{\alpha'} Z_{\alpha'} \right\|_2 \]
and hence again using Proposition 9.9 we have
\[ \left\| \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{H^\frac{1}{2}} \lesssim \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|^2_2 + \left\| \frac{1}{Z_{\alpha'}} \right\|_\infty \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_{H^\frac{1}{2}} \]

(5) Now assume that \( \sigma \geq 0 \) and let
\[ D = \left\| \frac{1}{Z_{\alpha'}} - 1 \right\|_2 (0) + \| Z_t \|_2 (0) + \sup_{t \in [0,T]} E_\sigma (t) \]
Define
\[ M = D + \| Z_{\alpha'} \|_\infty (0) + T + \sigma + 1 \]
In the following \( C(M) \) will denote a constant depending only on \( M \). As \( E_\sigma (0) \leq D \) we see that \( \left\| \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right\|_2 (0) \leq C(M) \) and hence \( \left\| \frac{1}{Z_{\alpha'}} \right\|_\infty (0) \leq C(M) \).

Now the evolution equation (13) gives us
\[ (\partial_t + b \partial_{\alpha'}) Z_{\alpha'} = Z_{t,\alpha'} - b_{\alpha'} Z_{\alpha'} = (D_{\alpha'} Z_t - b_{\alpha'}) Z_{\alpha'} \]
Hence for all \( 0 \leq t \leq T \) we have the estimate
\[ \| Z_{\alpha'} \|_\infty (t) \leq \| Z_{\alpha'} \|_\infty (0) \exp \left\{ \int_0^t (\| D_{\alpha'} Z_t \|_\infty (s) + \| b_{\alpha'} \|_\infty (s)) \, ds \right\} \]
As \( \| D_{\alpha'} Z_t \|_\infty \) and \( \| b_{\alpha'} \|_\infty \) are controlled by \( E_\sigma \), we see that \( \sup_{t \in [0,T]} \| Z_{\alpha'} \|_\infty (t) \leq C(M) \). By a similar argument we also obtain \( \sup_{t \in [0,T]} \left\| \frac{1}{Z_{\alpha'}} \right\|_\infty (t) \leq C(M) \)
We now control \( \left\| \frac{1}{Z_{\alpha'}} - 1 \right\|_2 (t) \) and \( \| Z_t \|_2 (t) \). To do this define
\[ f(t) = \left\| \frac{1}{Z_{\alpha'}} - 1 \right\|^2_2 (t) + \| Z_t \|^2_2 (t) + 1 \]
Observe that \( f(0) \leq C(M) \). We first see some of the quantities controlled by \( f(t) \).
(a) Using the formula (11) we see that
\[ \|b\|_2 \lesssim \|Z_t\|_2 \left\| \frac{1}{Z_{t,\alpha'}} \right\|_\infty \lesssim C(M) f^{\frac{1}{2}} \]
hence using the estimate \( \|b_{\alpha'}\|_{H^\frac{1}{2}} \leq C(E_{\sigma}) \) from Sect. 5.1 and Proposition 6.1 we have
\[ \|b\|_\infty + \|b_{\alpha'}\|_2 \lesssim \|b\|_2 + \|b_{\alpha'}\|_{H^\frac{1}{2}} \lesssim C(M) f^{\frac{1}{2}} + C(M) \lesssim C(M) f^{\frac{1}{2}} \]

(b) Using the formula \( A_1 = 1 - \text{Im}[Z_t, \frac{\partial}{\partial t}Z_{t,\alpha'}] \) from (7) we see that
\[ \|A_1 - 1\|_2 \lesssim \|Z_t\|_\infty \|Z_{t,\alpha'}\|_2 \lesssim (\|Z_t\|_2 + \|Z_{t,\alpha'}\|_2) \|Z_{t,\alpha'}\|_2 \]
\[ \lesssim C(M) f^{\frac{1}{2}} + C(M) \lesssim C(M) f^{\frac{1}{2}} \]

(c) Using (29) we see that
\[ \left\| (\partial_t + b\partial_{\alpha'}) \frac{1}{Z_{t,\alpha'}} \right\|_2 \lesssim \left\| \frac{1}{Z_{t,\alpha'}} (b_{\alpha'} - D_{\alpha'}Z_t) \right\|_2 \]
\[ \lesssim \left\| \frac{1}{Z_{t,\alpha'}} \right\|_\infty \|b_{\alpha'}\|_2 + \left\| \frac{1}{Z_{t,\alpha'}} \right\|_2^2 \|Z_{t,\alpha'}\|_2 \]
\[ \lesssim C(M) f^{\frac{1}{2}} + C(M) \]
\[ \lesssim C(M) f^{\frac{1}{2}} \]

(d) From (12) we see that
\[ \| (\partial_t + b\partial_{\alpha'}) Z_t \|_2 \lesssim \left\| i - i \frac{A_1}{Z_{t,\alpha'}} + \sigma D_{\alpha'} \Theta \right\|_2 \]
\[ \lesssim \left\| \frac{1}{Z_{t,\alpha'}} - 1 \right\|_2 + \left\| \frac{1}{Z_{t,\alpha'}} \right\|_\infty \|A_1 - 1\|_2 + \sigma \left\| \frac{1}{Z_{t,\alpha'}} \right\|_\infty \|\sigma \frac{3}{2} \partial_{\alpha'} \Theta \right\|_2 \]
\[ \lesssim f^{\frac{1}{2}} + C(M) f^{\frac{1}{2}} + C(M) \sigma^{\frac{1}{2}} \]
\[ \lesssim C(M) f^{\frac{1}{2}} \]

where we have used the fact that \( E_{\sigma} \) controls \( \sigma \frac{3}{2} \partial_{\alpha'} \Theta \in L^2 \) from Sect. 5.1.

Hence we see that
\[ \partial_t f \lesssim f^{\frac{1}{2}} \left\{ \left\| \partial_t \frac{1}{Z_{t,\alpha'}} \right\|_2 + \| \partial_t Z_t \|_2 \right\} \]
\[ \lesssim f^{\frac{1}{2}} \left\{ \left\| (\partial_t + b\partial_{\alpha'}) \frac{1}{Z_{t,\alpha'}} \right\|_2 + \|b\|_\infty \left\| \partial_{\alpha'} \frac{1}{Z_{t,\alpha'}} \right\|_2 + \left\| (\partial_t + b\partial_{\alpha'}) Z_t \right\|_2 + \|b\|_\infty \|Z_{t,\alpha'}\|_2 \right\} \]
\[ \lesssim C(M) f \]

Hence \( f(t) \) remains bounded on \([0, T]\) and we have
\[ \sup_{t \in [0, T]} \left\{ \left\| \frac{1}{Z_{t,\alpha'}} - 1 \right\|_2 (t) + \| Z_t \|_2 (t) \right\} \leq C(M) \]
Now using \( \sup_{t \in [0, T]} \| Z_{\alpha'} - 1 \|_2 \leq C(M) \) and the fact that \( \sup_{t \in [0, T]} \| Z_{\alpha'} \|_\infty (t) \leq C(M) \) we see that \( \sup_{t \in [0, T]} \| Z_{\alpha'} - 1 \|_2 \leq C(M) \). Now using part 1 and 2 of this lemma we easily obtain

\[
\sup_{t \in [0, T]} \left\{ \| Z_{\alpha'} - 1 \|_{H^{3.5}} (t) + \frac{1}{\| Z_{\alpha'} \|_{H^{3.5}} (t)} + \| Z_t \|_{H^3} (t) \right\} \leq C \left( M + \frac{1}{\sigma} \right)
\]

for \( \sigma > 0 \) and

\[
\sup_{t \in [0, T]} \left\{ \| Z_{\alpha'} - 1 \|_{H^{1.5}} (t) + \frac{1}{\| Z_{\alpha'} \|_{H^{1.5}} (t)} + \| Z_t \|_{H^2} (t) \right\} \leq C(M)
\]

for \( \sigma \geq 0 \) thereby proving the lemma.

7. Existence in Sobolev Spaces

In this section we prove the existence of solutions in Sobolev spaces for \( \sigma > 0 \) with the results being Theorem 7.8 and also Corollary 7.9. This existence result is then used to complete the proof of Theorem 3.1 in Sect. 8. The existence proof is standard and follows the general approach of [8,12]. Even though [12] already has an existence result in conformal mapping coordinates for the water wave equation with surface tension, we require a much stronger result than the one provided there. First we need to have lower regularity on the initial data in Sobolev spaces, we need lesser restrictions on the lower order terms and we also need a blow up criterion not depending on the chord arc constant of the interface. This existence result is of independent interest as we do not use the vorticity formulation using the Birkhoff-Rott integral as was done in [8,12]. We do not assume that the interface is a graph nor that it is non-self intersecting (as was explained in Sect. 2.4). In this section we fix \( \sigma > 0 \) and constants appearing in the computations may depend on \( \sigma \).

7.1. A priori estimates for exact solutions. In order to prove existence for system (13), it is more convenient to work with an equivalent system in the variables \((g, \nu)\)

\[
c = e^{-i\mathbb{H}g}
\]

\[
\omega = e^{ig}
\]

\[
b^* = 2i\mathbb{H}(c\nu) + i\left[ c^2, \mathbb{H} \right] \left( \frac{\nu}{c} \right)
\]

\[
a = ic\mathbb{H} \left( \frac{\nu}{c} \right)
\]

\[
d = -ie^{ig}c(\| - \mathbb{H} \|) \left( \frac{\nu}{c} \right)
\]

\[
A_{1}^* = 1 - \text{Im}[d, \mathbb{H}]\partial_{\alpha'} \bar{d}
\]

\[
e_2 = \text{Re}(\omega) - A_{1}^* c + \sigma \text{Im}[(c, \mathbb{H})\partial_{\alpha'} (\| + \mathbb{H})(c\partial_{\alpha'} g)]
\]

\[
\partial_t g = -(c\partial_{\alpha'}) v + a(c\partial_{\alpha'}) g - b^* \partial_{\alpha'} g
\]

\[
\partial_t \nu = -i\sigma \mathbb{H}(c\partial_{\alpha'})^2 g - a(c\partial_{\alpha'}) \nu - b^* \partial_{\alpha'} \nu + a^2 (c\partial_{\alpha'}) g + e_2 \quad (47)
\]
To get system (47) from system (13) we use the following transformation

\[ g = \text{Im}(\log(Z_{\alpha'})) \]
\[ v = \text{Im} \left( \frac{Z_{\alpha'} Z_t}{|Z_{\alpha'}|} \right) \]  
\[ (48) \]

and the following to get system (13) from system (47)

\[ Z_{\alpha'} = e^{i(\Pi+\Omega)g} \]
\[ Z_t = d \]  
\[ (49) \]

Let us now prove that these two systems are equivalent.

**Lemma 7.1.** Let \( s \geq 3 \) and \( T \geq 0 \). Then \( (Z, Z_t)(t) \) solves (13) with \( (Z_{\alpha'} - 1, \frac{1}{Z_{\alpha'}} - 1, Z_t) \in L^\infty([0, T], H^{s+\frac{1}{2}}(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R})) \) if and only if \( (g, v)(t) \) solves (47) along with \( (g, v) \in L^\infty([0, T], H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R})) \), where the transformations between them are given by (48) and (49).

**Proof. Step 1** We first assume that \( (Z, Z_t)(t) \) solves (13) and show that \( (g, v)(t) \) solves (47).

1. Now if \( (Z_{\alpha'} - 1, \frac{1}{Z_{\alpha'}} - 1, Z_t) \in L^\infty([0, T], H^{s+\frac{1}{2}}(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R})) \) for \( s \geq 3 \) then for any \( t \in [0, T] \) we see that \( \|Z_{\alpha'}\|_\infty(t) + \|\frac{1}{Z_{\alpha'}}\|_\infty(t) + \|Z_{\alpha'} - 1\|_2(t) \leq M < \infty \) for some \( M > 0 \). Now as \( Z_{\alpha'} - 1 \in H^{s+\frac{1}{2}}(\mathbb{R}) \) we observe that \( \Psi_{\alpha'} \) extends continuously to \( \overline{P}_- \) and hence \( \log(\Psi_{\alpha'}) \) also extends continuously to the boundary. Hence it makes sense to talk about the function \( \log(Z_{\alpha'}) \). Observe that if \( C_1 > 0 \) then

\[ c_1|z| \leq |e^z - 1| \leq c_2|z| \quad \text{for all } z \in \mathbb{R}, |z| \leq C_1 \]

for some \( c_1, c_2 > 0 \) depending only on \( C_1 \). Now as \( |Z_{\alpha'}| = e^{\Re(\log(Z_{\alpha'}))} \) we see that \( \Re(\log(Z_{\alpha'})) \in L^2 \). Hence we see that \( \text{Im}(\log(Z_{\alpha'})) \in L^2 \) and hence \( g \in L^2 \) and \( Z_{\alpha'} = e^{i(\Pi+\Omega)g} \). Now using (48) and the formula \( \partial_\alpha g = \text{Im} \left( \frac{1}{Z_{\alpha'}} \partial_\alpha Z_{\alpha'} \right) \) we see that \( (g, v) \in L^\infty([0, T], H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R})) \).

2. Observe from (48) that \( c = e^{-i\Omega g} = \frac{1}{|Z_{\alpha'}|} \) and \( \omega = e^{ig} = \frac{Z_{\alpha'}}{|Z_{\alpha'}|} \). We also have

\[ \alpha = i\overline{\mathbb{H}} \left( \frac{v}{c} \right) = i\overline{\mathbb{H}}(\text{Im}(\overline{Z}_t Z_{\alpha'})) = c\Re(\overline{\mathbb{H}}(\overline{Z}_t Z_{\alpha'})) = \Re(\overline{\omega Z}_t) \]

3. Observe

\[ b^* = 2i\mathbb{H}(cv) + i \left[ c^2, \mathbb{H} \right] \left( \frac{v}{c} \right) = i\mathbb{H}(cv) + i c^2 \mathbb{H} \left( \frac{v}{c} \right) = i\mathbb{H}(cv) + ca \]

Hence we see that

\[ b^* = i\mathbb{H}(cv) + ca = i\mathbb{H} \left( \text{Im} \left( \frac{Z_t}{Z_{\alpha'}} \right) \right) + \text{Re} \left( \frac{Z_t}{Z_{\alpha'}} \right) = \text{Re}(\mathbb{I} + \mathbb{H}) \left( \frac{Z_t}{Z_{\alpha'}} \right) = b \]
(4) Now observe that
\[
\bar{d} = ie^{-ig}c(1 + \mathbb{H})\left(\frac{v}{c}\right) = \frac{i}{Z_{\alpha'}}(1 + \mathbb{H})\text{Im}(\bar{Z}_t Z_{\alpha'}) = \bar{Z}_t
\]

(5) Hence we also have $A_1^* = A_1$

(6) We can now see that
\[
(\partial_t + b^*\partial_{\alpha'})g = \text{Im}\left((\partial_t + b^*\partial_{\alpha'})\log(Z_{\alpha'})\right)
= \text{Im}\left(\frac{1}{Z_{\alpha'}}(\partial_t + b\partial_{\alpha'})Z_{\alpha'}\right)
= \text{Im}\left(\frac{Z_{t,\alpha'}}{Z_{\alpha'}}\right)
= -\text{Im}(c\omega\partial_{\alpha'}\bar{Z}_t)
= \text{Im}(\bar{Z}_t)c\partial_{\alpha'}(\omega) - \text{Im}(c\partial_{\alpha'}(\omega\bar{Z}_t))
= \text{Im}(i(c\partial_{\alpha'}g\omega\bar{Z}_t) - c\partial_{\alpha'}v
= -c\partial_{\alpha'}v + a(c\partial_{\alpha'}g)
\]

(7) We see that
\[
(\partial_t + b^*\partial_{\alpha'})v
= \text{Im}\left\{(\partial_t + b^*\partial_{\alpha'})(\omega\bar{Z}_t)\right\}
= \text{Im}\left\{i\omega\bar{Z}_t(\partial_t + b^*\partial_{\alpha'})g\right\} + \text{Im}\left\{\omega(\partial_t + b^*\partial_{\alpha'})\bar{Z}_t\right\}
= a(-c\partial_{\alpha'}v + a(c\partial_{\alpha'}g) + \text{Im}\left\{\mathbb{H}(\omega(\partial_t + b^*\partial_{\alpha'})\bar{Z}_t)\right\}
+ \text{Im}\left\{(1 - \mathbb{H})(\omega(\partial_t + b^*\partial_{\alpha'})\bar{Z}_t)\right\}
\]

Observe that
\[
\text{Im}\left\{\mathbb{H}(\omega(\partial_t + b^*\partial_{\alpha'})\bar{Z}_t)\right\} = \text{Im}\left\{\mathbb{H}(\omega(\partial_t + b\partial_{\alpha'})\bar{Z}_t - i\omega)\right\} + \text{Im}\left\{\mathbb{H}(i\omega)\right\}
= -i\mathbb{H}\text{Re}\left\{\omega(\partial_t + b\partial_{\alpha'})\bar{Z}_t - i\omega\right\} + \text{Re}(\mathbb{H}\omega)
= -i\sigma\mathbb{H}(c\partial_{\alpha'})^2 g + \text{Re}(\mathbb{H}\omega)
\]

and we also have
\[
\text{Im}\left\{(1 - \mathbb{H})(\omega(\partial_t + b^*\partial_{\alpha'})\bar{Z}_t)\right\}
= \text{Im}(1 - \mathbb{H})\left\{i\omega - \frac{A_1}{|Z_{\alpha'}|} + \frac{\sigma}{|Z_{\alpha'}|}\partial_{\alpha'}(1 + \mathbb{H})\left(\frac{1}{|Z_{\alpha'}|}\partial_{\alpha'}g\right)\right\}
= \text{Re}(1 - \mathbb{H})(\omega) - \frac{A_1}{|Z_{\alpha'}|} + \sigma\text{Im}\left\{\left[\frac{1}{|Z_{\alpha'}|}, \mathbb{H}\right]\partial_{\alpha'}(1 + \mathbb{H})\left(\frac{1}{|Z_{\alpha'}|}\partial_{\alpha'}g\right)\right\}
\]

We can rewrite $e_2$ as
\[
e_2 = \text{Re}(\omega) - A_1^* c + \sigma\text{Im}\left\{[c, \mathbb{H}]\partial_{\alpha'}(1 + \mathbb{H})(c\partial_{\alpha'}g)\right\}
= \text{Re}(\omega) - \frac{A_1}{|Z_{\alpha'}|} + \sigma\text{Im}\left\{\left[\frac{1}{|Z_{\alpha'}|}, \mathbb{H}\right]\partial_{\alpha'}(1 + \mathbb{H})\left(\frac{1}{|Z_{\alpha'}|}\partial_{\alpha'}g\right)\right\}
\]

Hence combining these we get
\[
(\partial_t + b^*\partial_{\alpha'})v = -i\sigma\mathbb{H}(c\partial_{\alpha'})^2 g - a(c\partial_{\alpha'})v + a^2(c\partial_{\alpha'})g + e_2
\]
**Step 2** We now assume that \((g, v)(t)\) solves (47) and show that \((Z, Z_t)(t)\) solves (13).

1. Observe that if \(C_1 > 0\) then
\[
|e^z - 1| \leq c_2|z| \quad \text{for all } z \in \mathbb{C}, |z| \leq C_1
\]
where \(c_2\) depends only on \(C_1\). Hence via a similar calculation from step 1 and using (49) we see that if \((g, v) \in L^\infty([0, T], H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R}))\) then \((Z_{\alpha'}, 1, \frac{1}{Z_{\alpha'}} - 1, Z_t) \in L^\infty([0, T], H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R}))\). We also observe that in this case we have \(\log(\Psi_{z'}, K - y^* (i(I + \mathbb{H}) g)\) and hence \(\log(\Psi_{z'})\) is well defined.

Hence we easily obtain
\[
\lim_{c \to \infty} \sup_{|z'| \geq c} \{|\Psi_{z'}(z') - 1| + |U(z')|\} = 0 \quad \text{and} \quad \Psi_{z'}(z') \neq 0 \quad \text{for all } z' \in P_-
\]

2. We again have \(c = e^{-iHg} = \frac{1}{|Z_{\alpha'}|}\) and \(\omega = e^{ig} = \frac{Z_{\alpha'}}{|Z_{\alpha'}|}\). Also
\[
\overline{Z}_t Z_{\alpha'} = \overline{d} Z_{\alpha'} = \left\{i e^{-ig} c(I + \mathbb{H}) \left(\frac{v}{c}\right)\right\} Z_{\alpha'} = i (I + \mathbb{H})(v|Z_{\alpha'}|)
\]
Hence by taking imaginary parts we get
\[
\text{Im}(\overline{Z}_t Z_{\alpha'}) = v|Z_{\alpha'}|
\]
and hence we have \(v = \text{Im}(\omega \overline{Z}_t)\). Also observe that \(\overline{d} = \frac{i}{Z_{\alpha'}}(I + \mathbb{H})(v|Z_{\alpha'}|)\) and hence \(\overline{Z}_t\) is the boundary value of a holomorphic function.

3. Hence now from step 1 we automatically have \(a = \text{Re}(\omega \overline{Z}_t), b^* = b\) and \(A_1^* = A_1\).

4. Observe that
\[
b^* = b = \text{Re}(\mathbb{I} - \mathbb{H}) \left(\frac{Z_t}{Z_{\alpha'}}\right) = \mathbb{P}_A \left(\frac{Z_t}{Z_{\alpha'}}\right) + \mathbb{P}_H \left(\frac{\overline{Z}_t}{Z_{\alpha'}}\right)
\]
and hence we have
\[
i[b, \mathbb{H}]g_{\alpha'} = -i[b, \mathbb{H}]\text{Im}\left(Z_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right)
\]
\[
= -\text{Im}\left\{i[b, \mathbb{H}] \left(Z_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right)\right\}
\]
\[
= -\text{Im}\left\{i \left[\frac{Z_t}{Z_{\alpha'}}, \mathbb{H}\right] \left(Z_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right)\right\}
\]
\[
= -\text{Im}\left\{i (\mathbb{I} - \mathbb{H}) \left(Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right)\right\}
\]
\[
= -\text{Re}(\mathbb{I} - \mathbb{H}) \left(Z_t \partial_{\alpha'} \frac{1}{Z_{\alpha'}}\right)
\]
Now from (47) we have
\[(\partial_t + b^* \partial_{\alpha'})g = -c \partial_{\alpha'} v + a(c \partial_{\alpha'})g\]

Now by again using the computation in step 1 we see that
\[(\partial_t + b^* \partial_{\alpha'})g = -c \partial_{\alpha'} v + a(c \partial_{\alpha'})g = \text{Im} \left( \frac{Z_{t, \alpha'}}{Z_{\alpha'}} \right)\]
and we have
\[(\partial_t + b \partial_{\alpha'})Z_{t, \alpha'} = (\partial_t + b \partial_{\alpha'})e^{i(\mathbb{I} + \mathbb{H})}g\]
\[= Z_{t, \alpha'} \left\{ -\text{Re}(\mathbb{I} - \mathbb{H}) \left( Z_{t, \alpha'} \frac{1}{Z_{\alpha'}} \right) + i(\mathbb{I} + \mathbb{H})\text{Im} \left( \frac{Z_{t, \alpha'}}{Z_{\alpha'}} \right) \right\} \]
\[= Z_{t, \alpha'} \left\{ -\text{Re}(\mathbb{I} - \mathbb{H}) \left( Z_{t, \alpha'} \frac{1}{Z_{\alpha'}} \right) + \frac{Z_{t, \alpha'}}{Z_{\alpha'}} - \text{Re}(\mathbb{I} - \mathbb{H}) \left( \frac{Z_{t, \alpha'}}{Z_{\alpha'}} \right) \right\} \]
\[= Z_{t, \alpha'} - Z_{\alpha'}b_{\alpha'}\]

(6) Observe that if \(f\) is function satisfying \(\mathbb{P}_A f = 0\), then we have
\[\mathbb{P}_A \{ (\partial_t + b \partial_{\alpha'})f \} = \frac{1}{2} \left[ b, \mathbb{H} \right] \partial_{\alpha'} f = \frac{1}{2} \left[ \frac{Z_t}{Z_{\alpha'}}, \mathbb{H} \right] \partial_{\alpha'} f = \mathbb{P}_A \left( \frac{Z_t}{Z_{\alpha'}} \partial_{\alpha'} f \right)\]

Hence we see that
\[\mathbb{P}_A \left\{ Z_{t, \alpha'}(\partial_t + b \partial_{\alpha'})Z_t \right\} = \mathbb{P}_A \left\{ Z_{t, \alpha'} \mathbb{P}_A \left\{ (\partial_t + b \partial_{\alpha'})Z_t \right\} \right\} \]
\[= \mathbb{P}_A \left\{ Z_{t, \alpha'} \mathbb{P}_A \left\{ \frac{Z_t}{Z_{\alpha'}} \partial_{\alpha'}Z_t \right\} \right\} \]
\[= \mathbb{P}_A \{ Z_t, Z_{t, \alpha'} \}\]

(7) Now as \(\text{Im}(\omega Z_t) = v\), apply \(\partial_t + b \partial_{\alpha'}\) to this equation to get
\[\text{Im}(\omega(\partial_t + b \partial_{\alpha'})Z_t) + \text{Im}(i \{ (\partial_t + b \partial_{\alpha'})g, \omega Z_t \}) = -i\sigma \mathbb{H}(c \partial_{\alpha'})^2 g - a(c \partial_{\alpha'})v + a^2(c \partial_{\alpha'})g + \epsilon_2\]

But we know that \((\partial_t + b \partial_{\alpha'})g = -(c \partial_{\alpha'})v + a(c \partial_{\alpha'})g\) and that \(\text{Re}(\omega Z_t) = a\). Hence
\[\text{Im}(\omega(\partial_t + b \partial_{\alpha'})Z_t) = -i\sigma \mathbb{H}(c \partial_{\alpha'})^2 g + \text{Re}(\omega) - A_1 c + \sigma \text{Im} \left\{ [c, \mathbb{H}] \partial_{\alpha'}(\mathbb{I} + \mathbb{H})(c \partial_{\alpha'})g \right\}\]

Now observe that
\[\mathbb{H}(c \partial_{\alpha'})^2 g + i \text{Im} \left\{ [c, \mathbb{H}] \partial_{\alpha'}(\mathbb{I} + \mathbb{H})(c \partial_{\alpha'})g \right\} \]
\[= \mathbb{H}(c \partial_{\alpha'})^2 g + i \text{Im}(\mathbb{I} - \mathbb{H}) \{ (c \partial_{\alpha'})(\mathbb{I} + \mathbb{H})(c \partial_{\alpha'})g \} \]
\[= \mathbb{H}(c \partial_{\alpha'})^2 g - \mathbb{H}(c \partial_{\alpha'})^2 g + (c \partial_{\alpha'}) \mathbb{H}(c \partial_{\alpha'})g \]
\[= \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \mathbb{H} \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} g \right)\]
Hence by multiplying both sides by $|Z_{\alpha'}| = \frac{1}{c}$ we get

\[
\text{Im}(Z_{\alpha'}(\partial_t + b\partial_{\alpha'})\bar{Z}_t) = \text{Re}(Z_{\alpha'}) - A_1 - i\sigma \partial_{\alpha'} \mathbb{H}\left(\frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} g\right)
\]

Now apply $i(\mathbb{I} + \mathbb{H})$ to both sides

\[
Z_{\alpha'}(\partial_t + b\partial_{\alpha'})\bar{Z}_t - \text{Re}(\mathbb{I} - \mathbb{H})\left\{Z_{\alpha'}(\partial_t + b\partial_{\alpha'})\bar{Z}_t\right\}
= i(\mathbb{I} + \mathbb{H})\left\{\text{Re}(Z_{\alpha'}) - A_1\right\} + \sigma \partial_{\alpha'}(\mathbb{I} + \mathbb{H})\left(\frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} g\right)
\]

Now observe that

\[
i(\mathbb{I} + \mathbb{H})\text{Re}(Z_{\alpha'}) = i(\mathbb{I} + \mathbb{H})\text{Re}(Z_{\alpha'} - 1) + i(\mathbb{I} + \mathbb{H})\mathbb{I} = i(Z_{\alpha'} - 1) + i = iZ_{\alpha'}
\]

and we also have

\[
-i(\mathbb{I} + \mathbb{H})A_1 = -iA_1 - i\mathbb{H}A_1
= -iA_1 - i\mathbb{H}\left\{1 - \text{Im}(\mathbb{I} - \mathbb{H})(Z_t\bar{Z}_{t,\alpha'})\right\}
= -iA_1 - \text{Re}(\mathbb{I} - \mathbb{H})(Z_t\bar{Z}_{t,\alpha'})
\]

Hence we have

\[
Z_{\alpha'}(\partial_t + b\partial_{\alpha'})\bar{Z}_t - \text{Re}(\mathbb{I} - \mathbb{H})\left\{Z_{\alpha'}(\partial_t + b\partial_{\alpha'})\bar{Z}_t\right\}
= iZ_{\alpha'} - iA_1 - \text{Re}(\mathbb{I} - \mathbb{H})(Z_t\bar{Z}_{t,\alpha'}) + \sigma \partial_{\alpha'}(\mathbb{I} + \mathbb{H})\left(\frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} g\right)
\]

But we have already shown that $(\mathbb{I} - \mathbb{H})\left\{Z_{\alpha'}(\partial_t + b\partial_{\alpha'})\bar{Z}_t\right\} = (\mathbb{I} - \mathbb{H})(Z_t\bar{Z}_{t,\alpha'})$. Hence

\[
Z_{\alpha'}(\partial_t + b\partial_{\alpha'})\bar{Z}_t = iZ_{\alpha'} - iA_1 + \sigma \partial_{\alpha'}(\mathbb{I} + \mathbb{H})\left(\frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} g\right)
\]

Now dividing by $Z_{\alpha'}$ we finally get

\[
(\partial_t + b\partial_{\alpha'})\bar{Z}_t = i - i A_1 Z_{\alpha'}^{-1} + \frac{\sigma}{Z_{\alpha'}} \partial_{\alpha'}(\mathbb{I} + \mathbb{H})\left(\frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} g\right)
\]

As the proof establishes that $b^* = b$ and $A_{1}^* = A_{1}$, we will from now on use the variables $b$, $A_{1}$ instead of $b^*$, $A_{1}^*$ in the system (47). We now prove a priori estimates for (47). Let $N \geq 0$ and define the energy

\[
\mathcal{E}_{3.5} = \frac{1}{2} \|g\|_{H^{2.5}}^2 + \frac{1}{2} \|v\|_{H^{2}}^2
\]

\[
\mathcal{E}_{4.5+i} = \frac{1}{2} \left\|\frac{1}{c^{1/2}} \left\{(c\partial_{\alpha'})^{3+i} v - a(c\partial_{\alpha'})^{3+i} g\right\}\right\|_{L^2}^2 + \frac{\sigma}{2} \left\|(c\partial_{\alpha'})^{3+i} g\right\|_{H^{1/2}}^2
\]

\[
\mathcal{E} = \mathcal{E}_{3.5} + \sum_{i=0}^{N} \mathcal{E}_{4.5+i}
\]

(50)

We also define $K(t) = \|g(\cdot, t)\|_{H^{2.5}} + \|v(\cdot, t)\|_{H^{2}}$. Then we have
**Proposition 7.2.** Fix $N \geq 0$ and let $(g, v)(t)$ be a smooth solution to (47) in the time interval $[0, T]$ with $(g, v) \in C([0, T], H^{s+\frac{1}{2}} \times H^s)$ for all $s \geq 0$. Then there exists a polynomial $C = C(t)$ with non-negative coefficients depending only on $\sigma$ so that for any $t \in [0, T)$ we have
\[
\frac{dE(t)}{dt} \leq C(K(t))E(t)
\]

**Proof.** The proof is divided into 3 steps. We will freely use Lemma 9.5 to simplify the computations.

**Step 1** We first find quantities which can be controlled by the energy. We will use the notation $C(K) = C(K(t))$. Now

1. From the definition of $K$ and $c$, we have
   \[
   \|g\|_{\infty} + \|c\|_{\infty} + \left\| \frac{1}{c} \right\|_{\infty} \leq C(K)
   \]
   Hence from the definitions we easily see that
   \[
   \|c_{\alpha}'\|_{H^{1.5}} + \|\partial_t g\|_{H^1} + \|\partial_c c\|_{H^1} + \|b\|_{H^2} + \|a\|_{H^2} + \|\partial_t b\|_{H^{1.5}} + \|\partial_t a\|_{H^{1.5}} \leq C(K)
   \]

2. Observe that $c_{\alpha}' = (-i \mathbb{H} g_{\alpha})c$ and hence from $E_{3.5}$ we see that
   \[
   \|g\|_{H^{2.5}} + \|c_{\alpha}'\|_{H^{1.5}} + \left\| \partial_{\alpha}' \left( \frac{1}{c} \right) \right\|_{H^{1.5}} \leq C(K)E_{\frac{1}{2}}
   \]
   and hence
   \[
   \left\| (c_{\partial_{\alpha}'})^i g \right\|_{H^{\frac{1}{2}}} \leq C(K)E_{\frac{1}{2}} \quad \text{for } i = 1, 2
   \]
   For $i \geq 3$
   \[
   \left\| \frac{1}{c^2} (c_{\partial_{\alpha}'})^i g \right\|_{\frac{1}{2}}^2 = \int \frac{1}{c} \left( (c_{\partial_{\alpha}'})^i g \right) \left( (c_{\partial_{\alpha}'})^i g \right) d\alpha' \lesssim \left\| (c_{\partial_{\alpha}'})^{i-1} g \right\|_{H^{\frac{1}{2}}} \left\| (c_{\partial_{\alpha}'})^i g \right\|_{H^{\frac{1}{2}}}
   \]
   Hence we see that
   \[
   \left\| (c_{\partial_{\alpha}'})^i g \right\|_2 \leq C(K)E_{\frac{1}{2}}
   \]
   for $1 \leq i \leq N + 3$. Now using Corollary 9.6 we get
   \[
   \|e^i g\|_{H^{N+3}} + \|c_{\alpha}'\|_{H^{N+2}} \leq C(K)E_{\frac{1}{2}}
   \]
   and by using the fact that
   \[
   \|c_{\partial_{\alpha}'})^{N+3} g\|_{H^{\frac{1}{2}}} \leq C(K)E_{\frac{1}{2}}
   \]
   Hence we have
   \[
   \|g\|_{H^{N+3.5}} \leq C(K)E_{\frac{1}{2}}
   \]
   Now observe that for $z \in \mathbb{C}$ we have $|e^z - 1| \leq C_2 |z|$ for all $|z| \leq C_1$, where $C_2$ depends only on $C_1$. Hence we have
   \[
   |c - 1| + |\omega - 1| = \left| e^{-i \mathbb{H} g} - 1 \right| + \left| e^{i g} - 1 \right| \leq C(K)(|\mathbb{H}|g| + |g|)
   \]
   Using this and the fact that
   \[
   \omega_{\alpha}' = \partial_\alpha (e^{ig}) = i \omega g_{\alpha}' \quad \text{and} \quad c_{\alpha}' = (-i \mathbb{H} g_{\alpha}')c
   \]
   we have
   \[
   \|c - 1\|_{H^{N+3.5}} + \left\| \frac{1}{c} - 1 \right\|_{H^{N+3.5}} + \|\omega - 1\|_{H^{N+3.5}} \leq C(K)E_{\frac{1}{2}}
   \]
(3) From the definition of $a$ we have $\|a\|_\infty \leq C(K)$. Now by using the fact that 
$\| (c \partial_{\alpha'})^{3+i} g \|_2 \leq C(K)\mathcal{E}^{1/2}$ for all $0 \leq i \leq N$, using the energy $\mathcal{E}_{4.5+i}$ we now have $\| (c \partial_{\alpha'})^{3+i} v \|_2 \leq C(K)\mathcal{E}^{1/2}$ for all $0 \leq i \leq N$. Hence by using Corollary 9.6 repeatedly we get 
$$\| v \|_{H^{N+3}} \leq C(K)\mathcal{E}^{1/2}$$

Note that with these estimates one can also easily get the estimate $\| (c \partial_{\alpha'})^{2+i} v \|_{\dot{H}^{1/2}} \leq C(K)\mathcal{E}^{1/2}$ for all $0 \leq i \leq N$. Now using the definition of $a$, $d$, $b$, $A_1$ and $e_2$ we easily get using Proposition 9.8

$$\| a \|_{H^{N+3}} + \| d \|_{H^{N+3}} + \| b \|_{H^{N+3}} + \| A_1 - 1 \|_{H^{N+3}} + \| e_2 \|_{H^{N+3}} \leq C(K)\mathcal{E}^{1/2}$$

Now by the equations we get 
$$\| \partial_t g \|_{H^{N+2}} + \| \partial_t v \|_{H^{N+1.5}} \leq C(K)\mathcal{E}^{1/2}$$

Also observe that $\partial_t c = \{-i \mathbb{H}(\partial_t g)\} c$. Hence we now get $\| \partial_t c \|_{H^{N+2}} \leq C(K)\mathcal{E}^{1/2}$.

**Step 2** We now establish some identities which will be useful to prove the energy estimate. We define $D_t = \partial_t + b \partial_{\alpha'}$. We define the following notation: If $a, b : \mathbb{R} \times [0, T] \rightarrow \mathbb{C}$ are functions we write $a \approx_{L^2} b$ if there exists a polynomial $C(t)$ with non-negative coefficients depending only on $\sigma$ such that $\|a - b\|_2 \leq C(K)\mathcal{E}(t)^{1/2}$. Observe that $\approx_{L^2}$ is an equivalence relation.

(1) Let us compute $[D_t, c \partial_{\alpha'}]$. We see that

$$[D_t, c \partial_{\alpha'}] = [\partial_t + b \partial_{\alpha'}, c \partial_{\alpha'}] = \left( \frac{c_t + bc_{\alpha'} - cb_{\alpha'}}{c} \right) c \partial_{\alpha'}$$

Now using the above formula for $c_t$ and the definition of $b^*$ we have

$$\left( \frac{c_t + bc_{\alpha'} - cb_{\alpha'}}{c} \right) = -i \mathbb{H}\{- (c \partial_{\alpha'}) v + a (c \partial_{\alpha'}) g - b \partial_{\alpha'} g\} + b \frac{c_{\alpha'}}{c}$$

$$- \left\{ 2i \mathbb{H}(c_{\alpha'} v + c_{\alpha'} v) + i \partial_{\alpha'} \left[ c^2, \mathbb{H} \right] \left( \frac{v}{c} \right) \right\}$$

$$= - i \mathbb{H}(cv_{\alpha'}) + \left\{ - i \mathbb{H}(a (c \partial_{\alpha'}) g - b \partial_{\alpha'} g) + b \frac{c_{\alpha'}}{c} - \left\{ 2i \mathbb{H}(c_{\alpha'} v) + i \partial_{\alpha'} \left[ c^2, \mathbb{H} \right] \left( \frac{v}{c} \right) \right\} \right\}$$

$$= - i \mathbb{H}(cv_{\alpha'}) + error_1$$

where $error_1$ is defined as the term in the bracket and we observe that $\|error_1\|_{H^{N+2.5}} \leq C(K)\mathcal{E}^{1/2}$. Hence we have

$$[D_t, c \partial_{\alpha'}] = \{- i \mathbb{H}(cv_{\alpha'}) + error_1\} c \partial_{\alpha'}$$

(2) Observe from the definition of $a$

$$a = ic \mathbb{H}\left( \frac{v}{c} \right) = i \mathbb{H} v + i [c, \mathbb{H}] \left( \frac{v}{c} \right)$$
Hence we see that
\[ ca' = i \mathbb{H} (cv_a') + i [c, \mathbb{H}] v_{a'} + ic \partial_a' [c, \mathbb{H}] \left( \frac{v}{c} \right) \]

and hence we have \( \|ca' - i \mathbb{H} (cv_a')\|_{H^{N+2.5}} \leq C(K) E^{\frac{1}{2}} \). Now
\[
D_t (c \partial_a') g \\
= [D_t, c \partial_a'] g + c \partial_a' D_t g \\
= \{ -i \mathbb{H} (cv_a') + error \} cg_a' + c \partial_a' \{ -cv_a' + acg_a' \} \\
= -(c \partial_a')^2 v + a(c \partial_a')^2 g + \left\{ \frac{1}{2} [c, \mathbb{H}] v_{a'} + ic \partial_a' [c, \mathbb{H}] \left( \frac{v}{c} \right) \right\} cg_a' + (error_1)(cg_a') \\
= -(c \partial_a')^2 v + a(c \partial_a')^2 g + error_2
\]

where \( error_2 \) is defined as the term in the bracket and we observe that
\( \|error_2\|_{H^{N+2.5}} \leq C(K) E^{\frac{1}{2}} \)

(3) Using the estimates from above and Proposition 9.9 we have
\[
|\partial_a'|^{\frac{1}{2}} D_t (c \partial_a')^{3+N} g \\
\approx L^2 |\partial_a'|^{\frac{1}{2}} (c \partial_a')^{2+N} D_t (c \partial_a') g \\
\approx L^2 |\partial_a'|^{\frac{1}{2}} (c \partial_a')^{2+N} \left\{ -(c \partial_a')^2 v + a(c \partial_a')^2 g \right\} \\
\approx L^2 - |\partial_a'|^{\frac{1}{2}} (c \partial_a') \left\{ (c \partial_a')^{3+N} v - a(c \partial_a')^{3+N} g \right\}
\]

(4) Using the above estimates and Proposition 9.8 we have
\[
D_t \left\{ (c \partial_a')^{3+N} v - a(c \partial_a')^{3+N} g \right\} \\
\approx L^2 (c \partial_a')^{3+N} D_t v - a(c \partial_a')^{3+N} D_t g \\
\approx L^2 (c \partial_a')^{3+N} \left\{ -i \sigma \mathbb{H} (c \partial_a')^2 g - a(c \partial_a')v + a^2 (c \partial_a') g \right\} \\
\quad - a(c \partial_a')^{3+N} \left\{ -(c \partial_a') v + a(c \partial_a') g \right\} \\
= -i \sigma (c \partial_a')^{3+N} \mathbb{H} (c \partial_a')^2 g - \left[ (c \partial_a')^{3+N}, a \right] (c \partial_a') v - a(c \partial_a')^{4+N} v \\
\quad + a^2 (c \partial_a')^{4+N} g + a(c \partial_a')^{4+N} v - a \left[ (c \partial_a')^{3+N}, a \right] (c \partial_a') g - a^2 (c \partial_a')^{4+N} g \\
\approx L^2 - i \sigma (c \partial_a')^{3+N} \mathbb{H} (c \partial_a')^2 g \\
\approx L^2 - i \sigma (c \partial_a')^{3+N} \mathbb{H} (c \partial_a')^2 g
\]

Step 3 We now prove the energy estimate. Observe that controlling the time derivative of \( E_{3.5} \) and \( E_{4.5+N+i} \) for \( 0 \leq i < N \) is immediate. Hence we now control the time derivative of the highest term in the energy namely \( E_{4.5+N} \). To simplify the calculations we will use the following notation: If \( a(t), b(t) \) are functions of time we write \( a \approx b \) if there exists a non-negative polynomial \( C(t) \) with coefficients depending only on \( \sigma \) so that \( |a(t) - b(t)| \leq C(K(t)) E(t) \). Observe that \( \approx \) is an equivalence relation. With this
notation proving Proposition 7.2 is equivalent to showing \( \frac{dE(t)}{dt} \approx 0 \). Hence now by using Lemma 5.4 we have

\[
\frac{dE_{4.5+N}}{dt} \approx \int \frac{1}{c^2} \left\{ (c\partial_\alpha')^{3+N}v - a(c\partial_\alpha')^{3+N}g \right\} D_1 \left\{ (c\partial_\alpha')^{3+N}v - a(c\partial_\alpha')^{3+N}g \right\} \, d\alpha'
+ \sigma \int \left( |\partial_\alpha'|(c\partial_\alpha')^{3+N}g \right) D_1(c\partial_\alpha')^{3+N}g \, d\alpha'
\]

Observe that \( D_1c = (-iD_t\mathbb{H})c \). Hence

\[
\frac{dE_{4.5+N}}{dt} \approx \int \frac{1}{c} \left\{ (c\partial_\alpha')^{3+N}v - a(c\partial_\alpha')^{3+N}g \right\} D_1 \left\{ (c\partial_\alpha')^{3+N}v - a(c\partial_\alpha')^{3+N}g \right\} \, d\alpha'
+ \sigma \int \left( |\partial_\alpha'|(c\partial_\alpha')^{3+N}g \right) D_1(c\partial_\alpha')^{3+N}g \, d\alpha'
\]

Now using the computation from step 2 we get

\[
\frac{dE_{4.5+N}}{dt} \approx \int \frac{1}{c} \left\{ (c\partial_\alpha')^{3+N}v - a(c\partial_\alpha')^{3+N}g \right\} \left\{ -i\sigma (c\partial_\alpha')^{2}\mathbb{H}(c\partial_\alpha')^{3+N}g \right\} \, d\alpha'
- \sigma \int \left( |\partial_\alpha'|(c\partial_\alpha')^{3+N}g \right) \left( (c\partial_\alpha')^{3+N}v - a(c\partial_\alpha')^{3+N}g \right) \, d\alpha'
\]

\[
\approx 0
\]

where at the last step we used that \( |\partial_\alpha'| = i\mathbb{H}\partial_\alpha' \). This proves the a priori estimate. \( \square \)

We now prove a priori estimate for the difference of two solutions for system (47). This will prove uniqueness of the solution. Let \((g_1, v_1)(t)\) and \((g_2, v_2)(t)\) be two solutions of (47). We will use a subscript to denote which solution we are talking about. For example

\[
c_1 = e^{-i\mathbb{H}g_1} \quad c_2 = e^{-i\mathbb{H}g_2}
\]

and similarly for other variables as well. For \(A_1\) and \(e_2\) we will use the notation \((A_1)_1, (A_1)_2\) and \((e_2)_1, (e_2)_2\) respectively. Now define the energy

\[
E_{\Delta,2} = \frac{1}{2} \|g_1 - g_2\|_{H^1}^2 + \|v_1 - v_2\|_{H^\frac{1}{2}}^2
\]

\[
E_{\Delta,2.5} = \frac{1}{2} \left\| \frac{1}{c_1^2} \left\{ (c_1 \partial_\alpha')(v_1 - v_2) \right\} \right\|_{L^2}^2 + \frac{\sigma}{2} \| (c_1 \partial_\alpha')(g_1 - g_2)\|_{H^\frac{1}{2}}^2
\]

\[
E_{\Delta,3} = \frac{1}{2} \left\| |\partial_\alpha'|^\frac{1}{2} \left\{ (c_1 \partial_\alpha')(v_1 - (c_2 \partial_\alpha')v_2) - a_1|\partial_\alpha'|^\frac{1}{2}((c_1 \partial_\alpha')g_1 - (c_2 \partial_\alpha')g_2) \right\} \right\|_{L^2}^2
+ \frac{\sigma}{2} \left\| \frac{1}{c_1^2} \left\{ (c_1 \partial_\alpha')^2g_1 - (c_2 \partial_\alpha')^2g_2 \right\} \right\|_{L^2}^2
\]

\[
E_\Delta = E_{\Delta,2} + E_{\Delta,2.5} + E_{\Delta,3}
\]

(53)
**Proposition 7.3.** Let \((g_1, v_1)(t)\) and \((g_2, v_2)(t)\) be two solutions of \((47)\) in the time interval \([0, T]\) with \((g_1, v_1) \in C^l([0, T], H^{3.5-\frac{2}{l}} \times H^{3-\frac{3}{2}l})\) for \(l = 0, 1,\) for both \(i = 1, 2.\) Let \(M > 0\) be a constant so that for any \(t \in [0, T]\)

\[
\|g_1(\cdot, t)\|_{H^{3.5}} + \|v_1(\cdot, t)\|_{H^3} + \|g_2(\cdot, t)\|_{H^{3.5}} + \|v_2(\cdot, t)\|_{H^3} \leq M
\]

Then there exists a constant \(C(M) > 0\) depending only on \(M\) and \(\sigma\) such that for all \(t \in [0, T]\) we have

\[
\frac{dE_{\Delta}(t)}{dt} \leq C(M)E_{\Delta}(t)
\]

**Proof.** We define \(D_1^i = \partial_t + b_1\partial_{\alpha'}\) and \(D_2^i = \partial_t + b_2\partial_{\alpha'}\). We will freely use Lemma 9.5 to simplify the computations.

**Step 1** We first find quantities which can be controlled by \(M\) and \(E_\Delta\) in the time interval \([0, T]\).

1. We know that \(\|g_1(\cdot, t)\|_{H^{3.5}} + \|v_1(\cdot, t)\|_{H^3} + \|g_2(\cdot, t)\|_{H^{3.5}} + \|v_2(\cdot, t)\|_{H^3} \leq M.\) As both \((g_1, v_1)\) and \((g_2, v_2)\) solve \((47)\) we can use the same estimates from step 1 of the proof of Proposition 7.2 corresponding to \(N = 0.\) Hence for all \(t \in [0, T]\) we have for \(i = 1, 2\)

\[
\|c_i - 1\|_{H^{3.5}} + \left\| \frac{1}{c_i} - 1 \right\|_{H^{3.5}} + \|\alpha_i - 1\|_{H^{3.5}} \leq C(M)
\]

\[
\|a_i\|_{H^3} + \|d_i\|_{H^3} + \|b_i\|_{H^3} + \|(A_1)_i - 1\|_{H^3} + \|(e_2)_i\|_{H^3} \leq C(M)
\]

\[
\|\partial_t g_i\|_{H^2} + \|\partial_t v_i\|_{H^1} + \|\partial_t c_i\|_{H^2} \leq C(M)
\]

2. From \(E_{\Delta,2}\) we have

\[
\|g_1 - g_2\|_{H^1} + \|v_1 - v_2\|_{H^\frac{1}{2}} \leq C(M)E_{\Delta}^{\frac{1}{2}}
\]

Now from \(E_{\Delta,2.5}\) and bounds on \(c_1\) we have

\[
\|v_1 - v_2\|_{H^1} + \|c_1\partial_{\alpha'}(g_1 - g_2)\|_{H^\frac{1}{2}} \leq C(M)E_{\Delta}^{\frac{1}{2}}
\]

Now using Proposition 9.9 we have

\[
\|\partial_{\alpha'}(g_1 - g_2)\|_{H^\frac{1}{2}} \lesssim \|c_1\partial_{\alpha'}(g_1 - g_2)\|_{H^\frac{1}{2}} \left\| \frac{1}{c_1} \right\|_{\infty} + \|c_1\partial_{\alpha'}(g_1 - g_2)\|_2 \left\| \partial_{\alpha'} \frac{1}{c_1} \right\|_2
\]

Hence \(\|g_1 - g_2\|_{H^{1.5}} \leq C(M)E_{\Delta}^{\frac{1}{2}}.\) Hence from a similar argument to step 1 of Proposition 7.2 we also obtain \(\|c_1 - c_2\|_{H^{1.5}} \leq C(M)E_{\Delta}^{\frac{1}{2}}.\)

3. Observe that

\[
(c_1\partial_{\alpha'})^2 g_1 - (c_2\partial_{\alpha'})^2 g_2 = (c_1 - c_2)\partial_{\alpha'} \{ (c_1\partial_{\alpha'}) g_1 \} + c_2\partial_{\alpha'} \{ (c_1 - c_2)\partial_{\alpha'} g_1 \} + \{ (c_2\partial_{\alpha'}) c_2 \} \partial_{\alpha'} (g_1 - g_2) + c_2^2 \partial_{\alpha'} (g_1 - g_2)
\]
Now using the estimate $\|c_1 - c_2\|_{H^{1.5}} \leq C(M)\mathcal{E}_{\Delta}^{\frac{1}{2}}$ and the fact that $\|g_i\|_{H^{3.5}} + \|c_i - 1\|_{H^{3.5}} + \|\frac{1}{c_i} - 1\|_{H^{3.5}} \leq C(M)$ for $i = 1, 2$ we immediately obtain from $\mathcal{E}_{\Delta,3}$

$$\|g_1 - g_2\|_{H^2} \leq C(M)\mathcal{E}_{\Delta}^{\frac{1}{2}}$$

From the above estimate for $c_1 - c_2$ and the fact that $\|g_2\|_{H^{3.5}} \leq C(M)$ we see that

$$\|(c_1 \partial_{\alpha'})g_1 - (c_2 \partial_{\alpha'})g_2\|_{H^{\frac{1}{2}}} \leq C(M)\mathcal{E}_{\Delta}^{\frac{1}{2}}$$

Hence from using $\|a_1\|_{H^3} \leq C(M)$ and $\mathcal{E}_{\Delta,3}$ we obtain

$$\|(c_1 \partial_{\alpha'})v_1 - (c_2 \partial_{\alpha'})v_2\|_{H^{\frac{1}{2}}} \leq C(M)\mathcal{E}_{\Delta}^{\frac{1}{2}}$$

Hence again by using the estimate for $c_1 - c_2$ we get

$$\|v_1 - v_2\|_{H^{1.5}} \leq C(M)\mathcal{E}_{\Delta}^{\frac{1}{2}}$$

(4) Again by a similar computation from step 1 of Proposition 7.2 we get

$$\|c_1 - c_2\|_{H^2} + \|\omega_1 - \omega_2\|_{H^2} \leq C(M)\mathcal{E}_{\Delta}^{\frac{1}{2}}$$

Now using the definitions of $a, b, A_1$ and $e_2$ we easily get using Proposition 9.8

$$\|a_1 - a_2\|_{H^{1.5}} + \|d_1 - d_2\|_{H^{1.5}} + \|b_1 - b_2\|_{H^{1.5}} + \|(A_1)_1 - (A_1)_2\|_{H^{1.5}}$$

$$+ \|(e_2)_1 - (e_2)_2\|_{H^{1.5}} \leq C(M)\mathcal{E}_{\Delta}^{\frac{1}{2}}$$

Similarly from the equations of $\partial_t g$ and $\partial_t v$ we get

$$\|\partial_t (g_1 - g_2)\|_{H^{\frac{1}{2}}} + \|\partial_t (v_1 - v_2)\|_{2} \leq C(M)\mathcal{E}_{\Delta}^{\frac{1}{2}}$$

Also observe that $\partial_t c = \{-i\mathbb{H}(\partial_t g)\}c$. Hence $\|\partial_t (c_1 - c_2)\|_{H^{\frac{1}{2}}} \leq C(M)\mathcal{E}_{\Delta}^{\frac{1}{2}}$

**Step 2** We now prove some estimates required to prove the energy estimate. We define the following notation: If $a, b : \mathbb{R} \times [0, T] \to \mathbb{C}$ are functions we write $a \approx_{L^2} b$ if there exists a constant $C(M)$ depending only on $M$ and $\sigma$ such that $\|a - b\|_2 \leq C(M)\mathcal{E}_{\Delta}(t)^{\frac{1}{2}}$. Observe that $\approx_{L^2}$ is an equivalence relation.

(1) For the sake of convenience we define

$$\xi(\alpha', t) = |\partial_{\alpha'}|^{\frac{1}{2}}\{(c_1 \partial_{\alpha'})v_1 - (c_2 \partial_{\alpha'})v_2\} - a_1|\partial_{\alpha'}|^{\frac{1}{2}}\{(c_1 \partial_{\alpha'})g_1 - (c_2 \partial_{\alpha'})g_2\}$$

(54)

Using Proposition 9.8 and repeated use of the estimates from step 1 we see that

$$D_1^{\frac{1}{2}}\xi$$

$$\approx_{L^2} |\partial_{\alpha'}|^{\frac{1}{2}}D_1^{\frac{1}{2}}\{(c_1 \partial_{\alpha'})v_1 - (c_2 \partial_{\alpha'})v_2\} - a_1|\partial_{\alpha'}|^{\frac{1}{2}}D_1^{\frac{1}{2}}\{(c_1 \partial_{\alpha'})g_1 - (c_2 \partial_{\alpha'})g_2\}$$
\[ \approx L^2 |\partial \alpha| \frac{1}{2} \left\{ D_t^1 \left( c_1 \partial \alpha \right) v_1 - D_t^2 \left( c_2 \partial \alpha \right) v_2 \right\} \]

\[ - a_1 |\partial \alpha| \frac{1}{2} \left\{ D_t^1 \left( c_1 \partial \alpha \right) g_1 - D_t^2 \left( c_2 \partial \alpha \right) g_2 \right\} \]

\[ \approx L^2 |\partial \alpha| \frac{1}{2} \left\{ \left( c_1 \partial \alpha \right) D_t^1 v_1 - \left( c_2 \partial \alpha \right) D_t^2 v_2 \right\} \]

\[ - a_1 |\partial \alpha| \frac{1}{2} \left\{ D_t^1 \left( c_1 \partial \alpha \right) g_1 - D_t^2 \left( c_2 \partial \alpha \right) g_2 \right\} \]

\[ \approx L^2 \left\{ |\partial \alpha| \frac{1}{2} \left( c_1 \partial \alpha \right) D_t^1 v_1 - a_1 |\partial \alpha| \frac{1}{2} D_t^1 \left( c_1 \partial \alpha \right) g_1 \right\} \]

\[ - \left\{ |\partial \alpha| \frac{1}{2} \left( c_2 \partial \alpha \right) D_t^2 v_2 - a_2 |\partial \alpha| \frac{1}{2} D_t^2 \left( c_2 \partial \alpha \right) g_2 \right\} \]

Now we use the equations (47) and the formulae (52) and (51) to obtain

\[ |\partial \alpha| \frac{1}{2} \left( c_1 \partial \alpha \right) D_t^1 v_1 - a_1 |\partial \alpha| \frac{1}{2} D_t^1 \left( c_1 \partial \alpha \right) g_1 \]

\[ = |\partial \alpha| \frac{1}{2} \left( c_1 \partial \alpha \right) \left\{ -i \sigma \left( c_1 \partial \alpha \right) g_1 - a_1 \left( c_1 \partial \alpha \right) v_1 + a_1^2 \left( c_1 \partial \alpha \right) g_1 + (e_2) \right\} \]

\[ - a_1 |\partial \alpha| \frac{1}{2} \left\{ - \left( c_1 \partial \alpha \right) g_1^2 + a_1 \left( c_1 \partial \alpha \right) g_1 + (error_2) \right\} \]

\[ = -i \sigma |\partial \alpha| \frac{1}{2} \left( c_1 \partial \alpha \right) \left( c_1 \partial \alpha \right) g_1 - |\partial \alpha| \frac{1}{2} \left\{ \left( c_1 \partial \alpha \right) a_1 \right\} - \left| \partial \alpha \right| \frac{1}{2} \left( c_1 \partial \alpha \right) g_1^2 \]

\[ + |\alpha| \frac{1}{2} \left\{ \left( 2a_1 c_1 \partial \alpha \right) a_1 \right\} - \left| \partial \alpha \right| \frac{1}{2} \left( c_1 \partial \alpha \right) g_1^2 \]

\[ - a_1 |\partial \alpha| \frac{1}{2} (error_2) \]

A similar equation is true for the 2nd solution as well. Hence using Proposition 9.8 and the estimates from step 1 we see that

\[ D_t^1 \zeta \approx L^2 - i \sigma |\partial \alpha| \frac{1}{2} \left\{ \left( c_1 \partial \alpha \right) \left( c_1 \partial \alpha \right) g_1 - \left( c_2 \partial \alpha \right) \left( c_2 \partial \alpha \right) g_2 \right\} \]

\[ \approx L^2 - i \sigma |\partial \alpha| \frac{1}{2} \left\{ \left( c_1 \partial \alpha \right) g_1^2 - \left( c_2 \partial \alpha \right) g_2^2 \right\} \]

(2) Using the estimates from step 1 we observe that

\[ D_t^1 \left\{ \left( c_1 \partial \alpha \right)^2 g_1 - \left( c_2 \partial \alpha \right)^2 g_2 \right\} \approx L^2 D_t^1 \left( c_1 \partial \alpha \right)^2 g_1 - D_t^2 \left( c_2 \partial \alpha \right)^2 g_2 \]

\[ \approx L^2 \left( c_1 \partial \alpha \right) D_t^1 \left( c_1 \partial \alpha \right) g_1 - \left( c_2 \partial \alpha \right) D_t^2 \left( c_2 \partial \alpha \right) g_2 \]

\[ \approx L^2 \left( c_1 \partial \alpha \right)^2 \left\{ D_t^1 \left( c_1 \partial \alpha \right) g_1 - D_t^2 \left( c_2 \partial \alpha \right) g_2 \right\} \]

Now we use the formulae (52) and (51) to obtain

\[ D_t^1 \left\{ \left( c_1 \partial \alpha \right)^2 g_1 - \left( c_2 \partial \alpha \right)^2 g_2 \right\} \approx L^2 \left( c_1 \partial \alpha \right)^2 \left\{ \left( c_1 \partial \alpha \right) v_1 + \left( c_2 \partial \alpha \right)^2 v_2 - a_2 \left( c_2 \partial \alpha \right)^2 \right\} \]

\[ \approx L^2 \left( c_1 \partial \alpha \right) \left( c_1 \partial \alpha \right) v_1 - \left( c_2 \partial \alpha \right) \left( c_2 \partial \alpha \right) v_2 - a_1 \left( c_1 \partial \alpha \right) \left( c_2 \partial \alpha \right) g_1 - \left( c_2 \partial \alpha \right) g_2 \right\} \]

(3) Using Proposition 9.8 we observe that

\[ - i \sigma \left( c_1 \partial \alpha \right) \frac{1}{2} \left( c_1 \partial \alpha \right) \zeta \]

\[ = -i \sigma \left( c_1 \partial \alpha \right) \frac{1}{2} \left\{ \left( c_1 \partial \alpha \right) v_1 - \left( c_2 \partial \alpha \right) v_2 - a_1 \left( c_1 \partial \alpha \right) \left( c_2 \partial \alpha \right) g_1 - \left( c_2 \partial \alpha \right) g_2 \right\} \]

\[ \approx L^2 - i \sigma \left( c_1 \partial \alpha \right) \left\{ \left( c_1 \partial \alpha \right) v_1 - \left( c_2 \partial \alpha \right) v_2 - a_1 \left( c_1 \partial \alpha \right) \left( c_2 \partial \alpha \right) g_1 - \left( c_2 \partial \alpha \right) g_2 \right\} \]
Now as $|\partial_{\alpha'}| = i\|\partial_{\alpha'}$ by repeated use of Proposition 9.8 and the estimates from step 1 we obtain
\[
-i\sigma\|\partial_{\alpha'}\left\{c_1|\partial_{\alpha'}|^2(\xi)\right\}
\approx L_2\sigma\partial_{\alpha'}\left\{(c_1\partial_{\alpha'})(v_1 - (c_2\partial_{\alpha'})) - a_1(c_1\partial_{\alpha'})(c_1\partial_{\alpha'})(g_1 - (c_2\partial_{\alpha'}))\right\}
\]

Step 3 We now prove the energy estimate. It is easy to see that the time derivative of $E_{\Delta,2}$ is easily controlled. Hence we now control the time derivative of the energies $E_{\Delta,2.5}$ and $E_{\Delta,3}$. To simplify the calculations as usual we will use the following notation: If $a(t), b(t)$ are functions of time we write $a \approx b$ if there exists a constant $C(M)$ depending only on $M$ and $\sigma$ such that $|a(t) - b(t)| \leq C(M)\Delta(t)$. Observe that $\approx$ is an equivalence relation. With this notation proving Proposition 7.3 is equivalent to showing $\frac{dE_{\Delta}(t)}{dt} \approx 0$.

(1) Observe that
\[
\frac{dE_{\Delta,2.5}}{dt} = \int \frac{1}{c_1}\{(c_1\partial_{\alpha'})(v_1 - (v_2))\partial_{\alpha'}\} \partial_{\alpha'}\{(c_1\partial_{\alpha'})(v_1 - (v_2))\} \, d\alpha'
\]
\[
+ \sigma \int \{|\partial_{\alpha'}|(c_1\partial_{\alpha'})(g_1 - (g_2))\partial_{\alpha'}\} \partial_{\alpha'}\{(c_1\partial_{\alpha'})(g_1 - (g_2))\} \, d\alpha'
\]
\[
\approx - \int \partial_{\alpha'}\{(c_1\partial_{\alpha'})(v_1 - (v_2))\partial_{\alpha'}\} \partial_{\alpha'}\{(c_1\partial_{\alpha'})(g_1 - (g_2))\} \, d\alpha'
\]
\[
+ \sigma \int \{|\partial_{\alpha'}|(c_1\partial_{\alpha'})(g_1 - (g_2))\partial_{\alpha'}\} \partial_{\alpha'}\{(c_1\partial_{\alpha'})(v_1 - (v_2))\} \, d\alpha'
\]

Now we use the equations for $\partial_t g$ and $\partial_t v$ to obtain
\[
\frac{dE_{\Delta,2.5}}{dt} \approx i\sigma \int \partial_{\alpha'}\{(c_1\partial_{\alpha'})(v_1 - (v_2))\|\partial_{\alpha'}\{(c_1\partial_{\alpha'})(g_1 - (g_2))\} \, d\alpha'
\]
\[
- \sigma \int \{|\partial_{\alpha'}|(c_1\partial_{\alpha'})(g_1 - (g_2))\partial_{\alpha'}\} \partial_{\alpha'}\{(c_1\partial_{\alpha'})(v_1 - (v_2))\} \, d\alpha'
\]

Now we use the estimates from step 1 to obtain
\[
\frac{dE_{\Delta,2.5}}{dt} \approx i\sigma \int \partial_{\alpha'}\{(c_1\partial_{\alpha'})(v_1 - (v_2))\|\partial_{\alpha'}\{(c_1\partial_{\alpha'})(g_1 - (g_2))\} \, d\alpha'
\]
\[
- \sigma \int \{|\partial_{\alpha'}|(c_1\partial_{\alpha'})(g_1 - (g_2))\partial_{\alpha'}\} \partial_{\alpha'}\{(c_1\partial_{\alpha'})(v_1 - (v_2))\} \, d\alpha'
\]
\[
= -i\sigma \int \partial_{\alpha'}\{(c_1\partial_{\alpha'})(v_1 - (v_2))\|\partial_{\alpha'}\{(c_1\partial_{\alpha'})(g_1 - (g_2))\} \, d\alpha'
\]
\[
\approx 0
\]

where we used Proposition 9.8 in the last step.

(2) Now we control $E_{\Delta,3}$. Using Lemma 5.4 and the estimates from step 2 we obtain
\[
\frac{dE_{\Delta,3}}{dt} \approx \int (\xi)(D_1^1 \xi) \, d\alpha'
\]
\[
+ \sigma \int \frac{1}{c_1}\{(c_1\partial_{\alpha'})(g_1 - (c_2\partial_{\alpha'})) \|\partial_{\alpha'}\{(c_1\partial_{\alpha'})(g_1 - (c_2\partial_{\alpha'}))\} \, d\alpha'
\]
\[
\approx 0
\]
\[ \approx -i\sigma \int (\xi) \left\{ |\partial_{\alpha'}|^{\frac{1}{2}} (c_1 \partial_{\alpha'}) \frac{H}{H} \left( (c_1 \partial_{\alpha'})^2 g_1 - (c_2 \partial_{\alpha'})^2 g_2 \right) \right\} d\alpha' \\
- \sigma \int \left\{ (c_1 \partial_{\alpha'})^2 g_1 - (c_2 \partial_{\alpha'})^2 g_2 \right\} \partial_{\alpha'} \left( (c_1 \partial_{\alpha'}) (c_1 \partial_{\alpha'}) v_1 - (c_2 \partial_{\alpha'}) v_2 \right) \\
- a_1 (c_1 \partial_{\alpha'}) ((c_1 \partial_{\alpha'}) g_1 - (c_2 \partial_{\alpha'}) g_2) \right\} d\alpha' \\
\approx 0 \]

thereby proving the energy estimate. □

7.2. A priori estimates for approximate solutions. We now work with a mollified system. Fix \( \delta, \epsilon \geq 0 \) as parameters and let \( \phi \) be a smooth bump function satisfying \( \phi(\alpha') \geq 0 \) for all \( \alpha' \in \mathbb{R} \) and \( \int \phi(\alpha') \, d\alpha' = 1 \). For \( s > 0 \) let \( \phi_s(\alpha') = \frac{1}{s} \phi\left( \frac{\alpha'}{s} \right) \). Consider the smoothing operator \( J_\delta \) defined via \( J_\delta(f) = f \ast \phi_\delta \) for \( \delta > 0 \) and \( J_\delta(f) = f \) for \( \delta = 0 \). Consider the following system in the variables \((g, v)\)

\[
c = e^{-iHg}
\]
\[
\omega = e^{ig}
\]
\[
b = 2iH(cv) + i \left[ c^2, \frac{H}{c} \right] \left( \frac{v}{c} \right)
\]
\[
a = icH\left( \frac{v}{c} \right)
\]
\[
d = -ie^{ig}(\mathbb{I} - H)\left( \frac{v}{c} \right)
\]
\[
A_1 = 1 - \text{Im}[d, H] \partial_{\alpha'} \bar{d}
\]
\[
e_2 = \text{Re}(\omega) - A_1 c + \sigma \text{Im}[(c, H) \partial_{\alpha'}(\mathbb{I} + H)(c \partial_{\alpha'} g)]
\]
\[
\partial_t g = J_\delta^2 \{- (c \partial_{\alpha'}) v + a (c \partial_{\alpha'}) g - b \partial_{\alpha'} g \}
\]
\[
\partial_t v = J_\delta^2 \left\{ -\sigma H(c \partial_{\alpha'})^2 g - a (c \partial_{\alpha'}) v - b \partial_{\alpha'} v + a^2 (c \partial_{\alpha'}) g + e_2 \right\} - \epsilon |\partial_{\alpha'}| v
\]

The evolution equations of \( g \) and \( v \) have changed so that now we have a smoothing term \( J_\delta^2 \) in both the equations. We also have a dissipative term for the time evolution equation for \( v \). This is very similar to the system used in [12].

To prove existence for this system we will need some estimates.

**Lemma 7.4.** Let \( f, g \in \mathcal{S}(\mathbb{R}) \) and let \( 0 < \delta, \delta_1, \delta_2 \leq 1 \). Then we have

(1) \( \| J_{\delta_1}(f) - J_{\delta_2} f \|_{L^2} \lesssim \max\{\delta_1, \delta_2\} \| f \|_{H^s} \) for \( 0 < s \leq 1 \)

(2) \( \| [J_{\delta}, f] \partial_{\alpha'} g \|_{L^2} \lesssim \delta \| f \|_{H^s} \| g \|_{L^2} \)

(3) \( \| [J_{\delta}, f] \partial_{\alpha'} g \|_{L^2} \lesssim \| f \partial_{\alpha'} \partial_{\alpha'}^2 \|_{H^s} \| g \|_{L^2} \)

(4) \( \| \partial_{\alpha'} \frac{1}{2} [J_{\delta}, f] \partial_{\alpha'} g \|_{L^2} \lesssim \| f \|_{H^s} \| g \|_{H^s} \)

where the constants in the estimates are independent of \( \delta \).

**Proof.** We prove each of them separately.

(1) It is enough to assume that \( \delta_2 = 0 \). Using the Fourier transform we have

\[
\| J_{\delta}(f) - f \|_{L^2} \lesssim \left\| (\hat{\delta}(\delta \xi) - 1) \hat{f}(\xi) \right\|_{L^2(\delta \xi)}
\]
\[
\lesssim \left\| \hat{f}(\xi)1_{|\delta\xi| \geq 1} \right\|_{L^2(d\xi)} + \left\| (\hat{\phi}(\delta\xi) - \hat{\phi}(0))\hat{f}(\xi)1_{|\delta\xi| \leq 1} \right\|_{L^2(d\xi)} \\
\lesssim \delta^\gamma \| f \|_{H^s}
\]

2. Observe that

\[
\| [J_\delta, f] \partial_{\alpha'} g \|_2 \lesssim \left\| \int \hat{f}(\xi - \eta)\eta \hat{g}(\eta) \left[ \hat{\phi}(\delta\xi) - \hat{\phi}(\delta\eta) \right] d\eta \right\|_{L^2(d\xi)} \\
\lesssim \delta \left\| \int (\xi - \eta) \hat{f}(\xi - \eta)\eta \hat{g}(\eta) \left[ \hat{\phi}(\delta\xi) - \hat{\phi}(\delta\eta) \right] \frac{\delta\eta}{(\delta\xi - \delta\eta)} d\eta \right\|_{L^2(d\xi)} \\
\lesssim \delta \|\xi \hat{f}(\xi)\|_{L^1(d\xi)} \|\xi \hat{g}(\xi)\|_{L^2(d\xi)} \\
\lesssim \delta \| f \|_{H^2} \| g_{\alpha'} \|_2
\]

3. We see that

\[
([J_\delta, f] \partial_{\alpha'} g)(\alpha') = \int \phi_\delta(b')(f(\alpha' - b') - f(\alpha')) g_{\alpha'}(\alpha' - b') \, db'
\]

As \( g_{\alpha'}(\alpha' - b') = -\partial_{b'} g(\alpha' - b') \). Hence if we define \( \chi(b') = \phi'(b')b' \) then

\[
([J_\delta, f] \partial_{\alpha'} g)(\alpha') = \int \partial_{b'} \left[ \phi_\delta(b')(f(\alpha' - b') - f(\alpha')) \right] g(\alpha' - b') \, db'
\]

\[
= \int \left\{ \chi_\delta(b') \left( \frac{f(\alpha' - b') - f(\alpha')}{b'} - \phi_\delta(b')f(\alpha') \right) \right\} g(\alpha' - b') \, db'
\]

from which the estimate follows.

4. Observe that

\[
|\partial_{\alpha'}|^\frac{1}{2} [J_\delta, f] \partial_{\alpha'} g = \left[ |\partial_{\alpha'}|^\frac{1}{2} J_\delta, f \right] \partial_{\alpha'} g - \left[ |\partial_{\alpha'}|^\frac{1}{2}, f \right] \partial_{\alpha'} (J_\delta g)
\]

\[
= J_\delta \left[ |\partial_{\alpha'}|^\frac{1}{2}, f \right] \partial_{\alpha'} g + [J_\delta, f] \partial_{\alpha'} \left( |\partial_{\alpha'}|^\frac{1}{2} g \right) - \left[ |\partial_{\alpha'}|^\frac{1}{2}, f \right] \partial_{\alpha'} (J_\delta g)
\]

The estimate now follows from Proposition 9.8. \( \square \)

We will also require estimates similar to Lemma 5.4 but adapted to this system.

**Lemma 7.5.** Let \( T > 0 \) and let \( f, b \in C^1([0, T), H^2(\mathbb{R})) \) with \( b \) being real valued. Define the operator \( D_t^\delta = \partial_t + J_\delta^2 (b \partial_{\alpha'}) \). Then we have the estimate

\[
(1) \quad \left| \frac{d}{dt} \int |f|^2 \, d\alpha' - 2\text{Re} \int \tilde{f}(D_t^\delta f) \, d\alpha' \right| \lesssim \| f \|_{H^2}^2 \| b_{\alpha'} \|_{\infty}
\]

\[
(2) \quad \left| \frac{d}{dt} \int (|\partial_{\alpha'}|^\frac{1}{2} f) \, d\alpha' - 2\text{Re} \left\{ \int (|\partial_{\alpha'}|^\frac{1}{2} \tilde{f}) D_t^\delta f \, d\alpha' \right\} \right| \lesssim \| f \|_{H^2}^2 \| b_{\alpha'} \|_{\infty}
\]

**Proof.** The proof is very similar to the proof of Lemma 5.4 with the only difference being that we now also use Lemma 7.4. \( \square \)

The energy for this system \( \mathcal{E}(t) \) is again defined by (50) where \( N \geq 0 \)

\[
\mathcal{E}_{3.5} = \frac{1}{2} \| g \|_{H^{2.5}}^2 + \frac{1}{2} \| v \|_{H^2}^2
\]
\[ \mathcal{E}_{4.5i} = \frac{1}{2} \left\| \frac{1}{c} \right\| \left\{ (c \partial \alpha')^{3+i} v - a(c \partial \alpha')^{3+i} g \right\|_2^2 + \frac{\sigma}{2} \left\| (c \partial \alpha')^{3+i} g \right\|_{H^\frac{1}{2}}^2 \]

\[ \mathcal{E} = \mathcal{E}_{3.5} + \sum_{i=0}^{N} \mathcal{E}_{4.5+i} \]  

We again define \( K(t) = \| g(\cdot, t) \|_{H^{2.5}} + \| v(\cdot, t) \|_{H^2} \). We now have

**Proposition 7.6.** Fix \( N \geq 0 \) and let \((g, v)(t)\) be a smooth solution to (55) with parameters \((\delta, \epsilon)\) in the time interval \([0, T]\) with \((g, v) \in C([0, T], H^{3.5+\frac{1}{2}} \times H^3)\) for all \( s \geq 0 \).

1. If \( 0 \leq \delta \leq 1 \) and \( 0 < \epsilon \leq 1 \), then there exists a polynomial \( C_\epsilon = C_\epsilon(t) \) with non-negative coefficients depending only on \( \sigma, \epsilon \) and independent of \( \delta \) so that for any \( t \in [0, T) \) we have

\[ \frac{d \mathcal{E}(t)}{dt} \leq C_\epsilon(K(t)) \mathcal{E}(t) \]

2. If \( \delta = 0 \) and \( 0 \leq \epsilon \leq 1 \), then there exists a polynomial \( C = C(t) \) with non-negative coefficients depending only on \( \sigma \) and independent of \( \epsilon \) so that for any \( t \in [0, T) \) we have

\[ \frac{d \mathcal{E}(t)}{dt} \leq C(K(t)) \mathcal{E}(t) \]

**Proof.** The proof of this proposition is similar to the proof of Proposition 7.2 and we will mostly focus on the changes that we need to make. As before we will freely use Lemma 9.5 to simplify the computations.

**Step 1** The quantities controlled by the energy are the same as in Proposition 7.2. We collect the estimates below. This applies for all \( 0 \leq \delta, \epsilon \leq 1 \)

1. We have \( \| g \|_\infty + \| c \|_\infty + \| \frac{1}{c} \|_\infty \leq C(K) \) and hence

\[ \| c\alpha' \|_{H^{1.5}} + \| \partial_t g \|_{H^1} + \| \partial_t c \|_{H^1} + \| b \|_{H^2} + \| a \|_{H^2} + \| \partial_t b \|_{H^\frac{1}{2}} + \| \partial_t a \|_{H^\frac{1}{2}} \leq C(K) \]

2. \( \| g \|_{H^{3.5+3.5}} + \| c - 1 \|_{H^{3.5+3.5}} + \| \frac{1}{c} - 1 \|_{H^{3.5+3.5}} + \| \omega \|_{H^{3.5+3.5}} \leq C(K) \mathcal{E}^\frac{1}{2} \)

3. We have

\[ \| v \|_{H^{N+3}} + \| a \|_{H^{N+3}} + \| d \|_{H^{N+3}} + \| b \|_{H^{N+3}} + \| A_1 \|_{H^{N+3}} + \| e_2 \|_{H^{N+3}} \leq C(K) \mathcal{E}^\frac{1}{2} \]

and hence

\[ \| \partial_t g \|_{H^{N+2}} + \| \partial_t v \|_{H^{N+1.5}} + \| \partial_t c \|_{H^{N+2}} \leq C(K) \mathcal{E}^\frac{1}{2} \]

**Step 2** We now establish some identities involving \( D^\delta_t = \partial_t + J^\delta_{\omega}(b \partial \alpha') \) similar to the ones obtained in the proof of Proposition 7.2 for the case of \( 0 \leq \delta \leq 1 \) and \( 0 < \epsilon \leq 1 \). As a lot of calculations are similar we will skip some details. We define the following notation: If \( a, b : \mathbb{R} \times [0, T] \rightarrow \mathbb{C} \) are functions we write \( a \approx_{L^2} b \) if there exists a polynomial \( C(t) \) with non-negative coefficients depending only on \( \sigma \) such that

\[ \| a - b \|_2 \leq C(K) \left\{ \mathcal{E}(t)^\frac{1}{2} + \| v \|_{H^{3.5+N}} \right\} \]

Observe that \( \approx_{L^2} \) is an equivalence relation.
(1) If $f_1$, $f_2$ are two functions then
\[ D_1^\delta(f_1, f_2) = f_1(\partial_t f_2) + f_2(\partial_t f_1) + J_2^2(f_1(b \partial_{\alpha'}) f_2 + f_2(b \partial_{\alpha'}) f_1) \]
\[ = f_1(D_1^\delta f_2) + f_2(D_1^\delta f_1) + \left[ J_2^2, f_1 \right](b \partial_{\alpha'}) f_2 + \left[ J_2^2, f_2 \right](b \partial_{\alpha'}) f_1 \]
and we observe that
\[ \left[ J_2^2, f_1 \right](b \partial_{\alpha'}) f_2 = \left[ J_2^2, f_1 b \right] \partial_{\alpha'} f_2 - f_1 \left[ J_2^2, b \right] \partial_{\alpha'} f_2 \]
\[ = J_\delta[J_\delta, f_1 b] \partial_{\alpha'} f_2 + [J_\delta, f_1 b] \partial_{\alpha'}(J_\delta f_2) - f_1 J_\delta[J_\delta, b] \partial_{\alpha'} f_2 \]
\[ - f_1 J_\delta[J_\delta, b] \partial_{\alpha'}(J_\delta f_2) \]
We similarly have an expansion for $\left[ J_2^2, f_2 \right](b \partial_{\alpha'}) f_1$ as well.

(2) Observe that
\[ [D_1^\delta, c \partial_{\alpha'}] g = c[D_1^\delta, \partial_{\alpha'}] g + [D_1^\delta, c] g_{\alpha'} \]
\[ = -c J_\delta^2(b \partial_{\alpha'} g_{\alpha'}) + (D_1^\delta c) g_{\alpha'} + \left[ J_\delta^2, g_{\alpha'} \right](b \partial_{\alpha'}) c + \left[ J_\delta^2, c \right](b \partial_{\alpha'}) g_{\alpha'} \]

(3) Using the above identities and Lemma 7.4 we have
\[ |\partial_{\alpha'}|^2 D_1^\delta(c \partial_{\alpha'})^{3+N} g \]
\[ \approx L_2^2 |\partial_{\alpha'}|^2 (c \partial_{\alpha'})^{3+N} D_1^\delta g \]
\[ = |\partial_{\alpha'}|^2 (c \partial_{\alpha'})^{3+N} J_\delta^2 \{-a(c \partial_{\alpha'}) \partial_{\alpha'} g\} \]
\[ \approx L_2^2 |\partial_{\alpha'}|^2 J_\delta^2 (c \partial_{\alpha'})^{3+N} \{-a(c \partial_{\alpha'}) \partial_{\alpha'} g\} \]
\[ \approx L_2^2 -|\partial_{\alpha'}|^2 J_\delta^2 (c \partial_{\alpha'}) \left\{ (c \partial_{\alpha'})^{3+N} v - a(c \partial_{\alpha'})^{3+N} g \right\} \]

(4) Again by using the above identities and Lemma 7.4 we have
\[ D_1^\delta \left\{ (c \partial_{\alpha'})^{3+N} v - a(c \partial_{\alpha'})^{3+N} g \right\} \]
\[ \approx L_2^2 (c \partial_{\alpha'})^{3+N} D_1^\delta v - a(c \partial_{\alpha'})^{3+N} D_1^\delta g \]
\[ \approx L_2^2 (c \partial_{\alpha'})^{3+N} J_\delta^2 \{-i \sigma \Box (c \partial_{\alpha'})^2 g - a(c \partial_{\alpha'}) v + a^2(c \partial_{\alpha'}) g\} - \epsilon (c \partial_{\alpha'})^{3+N} |\partial_{\alpha'}| v \]
\[ - a(c \partial_{\alpha'})^{3+N} J_\delta^2 \{-a(c \partial_{\alpha'}) \partial_{\alpha'} g\} \]
\[ \approx L_2^2 -i \sigma (c \partial_{\alpha'})^{3+N} J_\delta^2 (c \partial_{\alpha'})^{1+N} J_\delta^2 (c \partial_{\alpha'})^{2} g - \epsilon (c \partial_{\alpha'})^{3+N} |\partial_{\alpha'}| v \]

Step 3 We now prove the energy estimate for the case of $0 \leq \delta \leq 1$ and $0 < \epsilon \leq 1$. As before controlling the time derivative of $E_{3,5}$ and $E_{4,5+i}$ for $0 \leq i < N$ is immediate. Hence we now control the time derivative of the highest term in the energy namely $E_{4,5+N}$. To simplify the calculations we will use the following notation: If $a(t), b(t)$ are functions of time we write $a \approx b$ if there exists a non-negative polynomial $C(t)$ with coefficients depending only on $\sigma$ so that $|a(t) - b(t)| \leq C(K(t)) \cdot (1 + \|v\|_{H^{3,5+N}})$. Observe that $\approx$ is an equivalence relation. Hence now by using Lemma 7.5 and doing a similar computation as done in Proposition 7.2 we obtain
\[ \frac{dE_{4,5+N}}{dt} \]
\[
\approx \frac{1}{c} \left( (c\partial_{\alpha'})^{3+N} v - a(c\partial_{\alpha'})^{3+N} g \right) J_0^T \left( (c\partial_{\alpha'})^{3+N} v - a(c\partial_{\alpha'})^{3+N} g \right) d\alpha' \\
+ \sigma \int \left( |\partial_{\alpha'}| (c\partial_{\alpha'})^{3+N} g \right) J_0^T (c\partial_{\alpha'})^{3+N} g d\alpha'
\]

Hence we have the inequality

\[
\frac{\partial \mathcal{E}_{4.5+N}}{\partial t} \leq C(K)\mathcal{E}(t) + C(K)\mathcal{E}_{4.5+N}(t) ||v||_{H^{3.5+N}} - \epsilon \left( \frac{1}{c} \right) ||v||_{H^{3.5+N}} \]

As \( \epsilon > 0 \) the estimate follows.

**Step 4** We now prove the energy estimate for the case of \( \delta = 0 \) and \( 0 \leq \epsilon \leq 1 \). We will use the following notation: If \( a(t), b(t) \) are functions of time we write \( a \approx b \) if there exists a non-negative polynomial \( C(t) \) with coefficients depending only on \( \sigma \) so that \( |a(t) - b(t)| \leq C(K(t))\mathcal{E}(t) \). Observe that \( \approx \) is an equivalence relation. Hence now by using Lemma 7.5 and doing a similar computation as done in Proposition 7.2 we obtain

\[
\frac{\partial \mathcal{E}_{4.5+N}}{\partial t} \approx \frac{1}{c} \left( (c\partial_{\alpha'})^{3+N} v - a(c\partial_{\alpha'})^{3+N} g \right) \left\{ -\epsilon (c\partial_{\alpha'})^{3+N} |\partial_{\alpha'}| v \right\} d\alpha'
\]

Hence we have the inequality

\[
\frac{\partial \mathcal{E}_{4.5+N}}{\partial t} \leq C(K)\mathcal{E}(t) + \epsilon C(K)\mathcal{E}_{4.5+N}(t) ||v||_{H^{3.5+N}} \leq ||v||_{H^{3.5+N}}
\]

as \( 0 \leq \epsilon \leq 1 \) the estimate follows. \( \square \)
We now prove the a priori estimate for the difference of two solutions for system \((55)\). Let \((g_1, v_1)(t)\) and \((g_2, v_2)(t)\) be two solutions of \((55)\) with parameters \((\delta_1, \epsilon_1)\) and \((\delta_2, \epsilon_2)\) respectively. The energy for the difference \(E_\Delta(t)\) is once again defined by \((53)\). We now have

**Proposition 7.7.** Let \((g_1, v_1)(t)\) and \((g_2, v_2)(t)\) be two solutions of \((47)\) with parameters \((\delta_1, \epsilon_1)\) and \((\delta_2, \epsilon_2)\) respectively in the time interval \([0, T]\). Assume that \((g_i, v_i) \in C^1([0, T], H^{3.5-\frac{2l}{3}} \times H^{3-\frac{2l}{3}})\) for \(l = 0, 1\), for both \(i = 1, 2\). Let \(M > 0\) be a constant so that for any \(t \in [0, T]\)

\[
\|g_1(\cdot, t)\|_{H^{3.5}} + \|v_1(\cdot, t)\|_{H^{3}} + \|g_2(\cdot, t)\|_{H^{3.5}} + \|v_2(\cdot, t)\|_{H^{3}} \leq M
\]

1. If \(0 \leq \delta_1, \delta_2 \leq 1\) and \(0 < \epsilon = \epsilon_1 = \epsilon_2 \leq 1\), then for all \(t \in [0, T]\) we have

\[
\frac{dE_\Delta(t)}{dt} \leq C_\epsilon(M) \left( E_\Delta(t) + \max \left\{ \frac{1}{\delta_1}, \frac{1}{\delta_2} \right\} \right)
\]

where \(C_\epsilon(M)\) is a constant depending on \(M, \epsilon\) and \(\sigma\).

2. If \(\delta_1 = \delta_2 = 0\) and \(0 \leq \epsilon_1, \epsilon_2 \leq 1\), then for all \(t \in [0, T]\) we have

\[
\frac{dE_\Delta(t)}{dt} \leq C(M) (E_\Delta(t) + \max\{\epsilon_1, \epsilon_2\})
\]

where \(C(M)\) is a constant depending only on \(M\) and \(\sigma\).

**Proof.** The proof of this proposition is similar to the proof of Proposition 7.3 and we will mostly focus on the changes that we need to make. We will first do the computation for the first case namely \(0 \leq \delta_1, \delta_2 \leq 1\) and \(0 < \epsilon \leq 1\). We will freely use Lemma 9.5 and Lemma 7.4 to simplify the computations.

**Step 1** As before we first control the quantities controlled by the energy. This is essentially the same as done in step 1 of the proof of Proposition 7.3 so we will just summarize the estimates. This applies for all \(0 \leq \delta_1, \delta_2, \epsilon_1, \epsilon_2 \leq 1\). We again note that \(C(M)\) will denote a constant depending only on \(M\) and \(\sigma\).

\[
\|c_i - 1\|_{H^{3.5}} + \left\| \frac{1}{c_i} - 1 \right\|_{H^{3.5}} + \|\omega_i - 1\|_{H^{3.5}} \leq C(M)
\]

\[
\|a_i\|_{H^3} + \|d_i\|_{H^3} + \|b_i\|_{H^3} + \| (A_1) v_i - 1\|_{H^3} + \| (e_2) i\|_{H^3} \leq C(M)
\]

\[
\|\partial_i c_i\|_{H^2} + \|\partial_i v_i\|_{H^{1.5}} + \|\partial_i c_i\|_{H^2} \leq C(M)
\]

and now the difference of quantities

\[
\|g_1 - g_2\|_{H^2} + \|v_1 - v_2\|_{H^{1.5}} + \|c_1 - c_2\|_{H^2} + \|\omega_1 - \omega_2\|_{H^2} \leq C(M) E_\Delta^\frac{1}{1}
\]

\[
\|a_1 - a_2\|_{H^{1.5}} + \|d_1 - d_2\|_{H^{1.5}} + \|b_1 - b_2\|_{H^{1.5}} \leq C(M) E_\Delta^\frac{1}{2}
\]

\[
\|(A_1) v_1 - (A_1) v_2\|_{H^2} + \|(e_2) v_1 - (e_2) v_2\|_{H^2} \leq C(M) E_\Delta^\frac{1}{2}
\]

For the difference of time derivatives, the estimate becomes a little different

\[
\|\partial_t (g_1 - g_2)\|_{H^\frac{1}{2}} + \|\partial_t (v_1 - v_2)\|_{H^\frac{1}{2}} + \|\partial_t (c_1 - c_2)\|_{H^\frac{1}{2}} \leq C(M) \left\{ E_\Delta^\frac{1}{2} + \max \{\delta_1, \delta_2\} + \max \{\epsilon_1, \epsilon_2\} \right\}
\]
Step 2 We now establish some estimates for the case of $0 \leq \delta_1, \delta_2 \leq 1$ and $0 < \epsilon = \epsilon_1 = \epsilon_2 \leq 1$. We define the following notation: If $a, b : \mathbb{R} \times [0, T] \rightarrow \mathbb{C}$ are functions we write $a \approx_{L^2} b$ if there exists a constant $C(M)$ depending only on $M$ and $\sigma$ such that $\|a - b\| \leq C(M) \left\{ E_\triangle(t)^{\frac{1}{2}} + \|v_1 - v_2\|_{H^2} + \max \left\{ \frac{1}{2}, \frac{1}{2} \right\} \right\}$. Observe that $\approx_{L^2}$ is an equivalence relation. We define $D_t^{\delta_1} = \partial_t + J_{\delta_1}^2 (b_1 \partial_{a'})$ and $D_t^{\delta_2} = \partial_t + J_{\delta_2}^2 (b_2 \partial_{a'})$

(1) Recalling the definition of $\zeta(a', t)$ from (54) and by an analogous computation done in Proposition 7.3 and using Lemma 7.4 we see that

$$D_t^{\delta_1} \zeta$$

$$\approx_{L^2} -i \sigma |\partial_{a'}|^2 (c_1 \partial_{a'}) \left\{ J_{\delta_1}^2 (c_1 \partial_{a'})^2 g_1 - J_{\delta_2}^2 (c_2 \partial_{a'})^2 g_2 \right\}$$

$$- \epsilon |\partial_{a'}|^2 \left\{ (c_1 \partial_{a'}) |\partial_{a'}| v_1 - (c_2 \partial_{a'}) |\partial_{a'}| v_2 \right\}$$

$$\approx_{L^2} -i \sigma |\partial_{a'}|^2 (c_1 \partial_{a'}) \left\{ J_{\delta_1}^2 (c_1 \partial_{a'})^2 g_1 - J_{\delta_2}^2 (c_2 \partial_{a'})^2 g_2 \right\}$$

$$- \epsilon c_1 |\partial_{a'}| |\partial_{a'}|^2 (v_1 - v_2)$$

(2) We also have

$$D_t^{\delta_1} \left\{ (c_1 \partial_{a'})^2 g_1 - (c_2 \partial_{a'})^2 g_2 \right\}$$

$$\approx_{L^2} -c_1 \partial_{a'} \left\{ J_{\delta_1}^2 (c_1 \partial_{a'}) v_1 - J_{\delta_2}^2 (c_2 \partial_{a'}) v_2 \right\}$$

(3) Similarly we have

$$- i \sigma \left\{ c_1 \partial_{a'} \right\} \{ (c_1 \partial_{a'}) |\partial_{a'}| v_1 - (c_2 \partial_{a'}) v_2 \}$$

$$\approx_{L^2} \sigma \left\{ (c_1 \partial_{a'}) \left\{ (c_1 \partial_{a'}) v_1 - (c_2 \partial_{a'}) v_2 \right\} - a_1 (c_1 \partial_{a'}) \left\{ (c_1 \partial_{a'}) v_1 - (c_2 \partial_{a'}) v_2 \right\} \}$$

Step 3 We now prove the energy estimate for the case of $0 \leq \delta_1, \delta_2 \leq 1$ and $0 < \epsilon \leq 1$. We only control the highest order energy as the lower order ones are easily controlled. To simplify the calculations we will use the following notation: If $a(t), b(t)$ are functions of time we write $a \approx b$ if there exists a constant $C(M)$ depending only on $M$ and $\sigma$ so that $|a(t) - b(t)| \leq C(M) \left\{ E_\triangle(t)^{\frac{1}{2}} + E_\triangle(t)^{\frac{1}{2}} \|v_1 - v_2\|_{H^2} + \max \left\{ \frac{1}{2}, \frac{1}{2} \right\} \right\}$. Hence now by using Lemma 7.5, Lemma 7.4 and doing a similar computation as done in Proposition 7.2 we obtain

$$\frac{d E_{\triangle, 3}}{dt}$$

$$\approx \int (\zeta) \left\{ |\partial_{a'}|^2 (c_1 \partial_{a'}) \left\{ J_{\delta_1}^2 (c_1 \partial_{a'})^2 g_1 - J_{\delta_2}^2 (c_2 \partial_{a'})^2 g_2 \right\} - \epsilon c_1 |\partial_{a'}| |\partial_{a'}|^2 (v_1 - v_2) \right\} d\alpha'$$

$$+ \sigma \int \left\{ (c_1 \partial_{a'})^2 g_1 - (c_2 \partial_{a'})^2 g_2 \right\} |\partial_{a'}| \left\{ (c_1 \partial_{a'}) v_1 - J_{\delta_2}^2 (c_2 \partial_{a'}) v_2 \right\}$$

$$- a_1 (c_1 \partial_{a'}) \left\{ J_{\delta_1}^2 (c_1 \partial_{a'}) v_1 - J_{\delta_2}^2 (c_2 \partial_{a'}) v_2 \right\} d\alpha'$$

$$\approx - \epsilon \left\{ \int (\zeta) \left\{ |\partial_{a'}|^2 (c_1 \partial_{a'}) J_{\delta_1}^2 (c_1 \partial_{a'})^2 g_1 - (c_2 \partial_{a'})^2 g_2 \right\} d\alpha' \right\}$$

$$- \epsilon \int (\zeta) \left\{ |\partial_{a'}|^2 |\partial_{a'}|^2 (v_1 - v_2) \right\} d\alpha'$$

$$- \sigma \int \left\{ (c_1 \partial_{a'})^2 g_1 - (c_2 \partial_{a'})^2 g_2 \right\} |\partial_{a'}| J_{\delta_1}^2 \left\{ (c_1 \partial_{a'}) \left\{ (c_1 \partial_{a'}) v_1 - (c_2 \partial_{a'}) v_2 \right\} \right\}$$
As we have the estimate
\[
-a_1(c_1 \partial_{\alpha'})\{(c_1 \partial_{\alpha'})g_1 - (c_2 \partial_{\alpha'})g_2\}
\]
\[
\approx -\epsilon \int |\partial_{\alpha'}|^2 \{(c_1 \partial_{\alpha'})v_1 - (c_2 \partial_{\alpha'})v_2\}\{c_1 \partial_{\alpha'}|\partial_{\alpha'}|^2 (v_1 - v_2)\} \, d\alpha'
\]
\[
\approx -\epsilon \int c_1^2 \partial_{\alpha'}^2 (v_1 - v_2) \, d\alpha'
\]
Hence we have the estimate
\[
\frac{d\mathcal{E}_{\Delta,3}}{dt} \leq C(M)\mathcal{E}_{\Delta}(t) + C(M)\mathcal{E}_{\Delta}^{\frac{1}{2}}(t)\|v_1 - v_2\|_{H^2} + C(M)\max\left\{\delta_1^{\frac{1}{2}}, \delta_2^{\frac{1}{2}}\right\}
\]
\[
- \epsilon \left\| \frac{1}{c_1} \right\|^{-2} \|v_1 - v_2\|_{H^2}^2
\]
As \( \epsilon > 0 \) the estimate follows.

**Step 4** The proof of the energy estimate for \( \delta_1 = \delta_2 = 0 \) and \( 0 \leq \epsilon_1, \epsilon_2 \leq 1 \) is exactly the same as proved in Proposition 7.3 as all the terms involving \( \epsilon_1, \epsilon_2 \) can be controlled by \( C(M)\max\{\epsilon_1, \epsilon_2\} \).

We are now in a position to prove the local existence in Sobolev spaces.

**Theorem 7.8.** For \( \sigma > 0 \) we have the following

1. Let \( s \geq 3 \) and let \( (g, v)(0) \in H^{s+\frac{1}{2}}(\mathbb{R}) \times H^s(\mathbb{R}) \). Then there exists a time \( T > 0 \) so that the initial value problem to (47) has a unique solution \((g, v) \in C^l([0, T], H^{s+\frac{1}{2}-\frac{3}{2}l} \times H^{s-\frac{3}{2}l})\) for \( l = 0, 1 \). Moreover if \( T_{\text{max}} \) is the maximum time of existence then either \( T_{\text{max}} = \infty \) or \( T_{\text{max}} < \infty \) with

\[
\sup_{t \in [0, T_{\text{max}}]} \left\{ \|g(\cdot, t)\|_{H^{3.5}} + \|v(\cdot, t)\|_{H^3} \right\} = \infty
\]

2. If \((g_1, v_1)(t)\) and \((g_2, v_2)(t)\) are two solutions of (47) in \([0, T] \) with

\[
\sup_{t \in [0, T]} \left\{ \|g_1(\cdot, t)\|_{H^{3.5}} + \|v_1(\cdot, t)\|_{H^3} + \|g_2(\cdot, t)\|_{H^{3.5}} + \|v_2(\cdot, t)\|_{H^3} \right\} = M < \infty
\]

Then there is constant \( C(M) \) depending only on \( M \) and \( \sigma \) such that

\[
\sup_{t \in [0, T]} \left\{ \|g_1(\cdot, t) - g_2(\cdot, t)\|_{H^2} + \|v_1(\cdot, t) - v_2(\cdot, t)\|_{H^{1.5}} \right\}
\]
\[
\leq e^{C(M)T} \left\{ \|g_1(\cdot, 0) - g_2(\cdot, 0)\|_{H^2} + \|v_1(\cdot, 0) - v_2(\cdot, 0)\|_{H^{1.5}} \right\}
\]

**Proof.** The proof of this result is essentially the same as the proof of Theorem 5.6 in Ambrose [8]. There are a few minor changes that need to be made which we now describe.

1. First observe that even though the result in [8] is for periodic solutions, the existence proof does not use compactness to prove existence. So the fact that we are working on \( \mathbb{R} \) makes no difference.
2. First fix \( \epsilon, \delta > 0 \) and consider the mollified initial data \((g^{\epsilon, \delta}, v^{\epsilon, \delta})(0) = (J_\epsilon(g(0)), J_\epsilon(v(0)))\). For this smooth initial data we first prove a local existence result to the system (55) by a standard Picard iteration scheme by writing the corresponding integral equation and treating the \( J_\delta^2 \) terms in the equations as forcing terms. We hence obtain smooth solutions \((g^{\delta, \epsilon}, v^{\delta, \epsilon})(t)\) to (55) for \( t \in [0, T] \) where \( T \) depends...
on $\epsilon, \delta$. Then by using the first estimate from Proposition 7.6 we see that in fact $T$ is independent of $\delta$. Hence using Proposition 7.7 and by following the approach of Ambrose [8] we obtain unique smooth solutions $(g^\epsilon, v^\epsilon)(t)$ to the system (55) for $\delta = 0$ in the time interval $[0, T]$ where $T$ depends on $\epsilon$.

(3) Now by the second a priori estimate in Proposition 7.6 for $\delta = 0$, we see that $T$ is in fact independent of $\epsilon$ and now by using Proposition 7.7 and by following the approach of Ambrose [8] we obtain a unique solution $(g, v) \in L^\infty([0, T], H^{s+\frac{1}{2}} \times H^{s})$ to the system (47) with $(\partial_t g, \partial_t v) \in L^\infty([0, T], H^{s-1} \times H^{s-\frac{3}{2}})$. Then by following the approach of Ambrose [8] using the time reversible nature of the system (47) we obtain $(g, v) \in C^l([0, T], H^{s+\frac{1}{2}-\frac{3}{2}l} \times H^{s-\frac{3}{2}l})$ for $l = 0, 1$.

(4) The blow up criterion follows from Proposition 7.6 and part two of the theorem follows from Proposition 7.3.

Corollary 7.9. For $\sigma > 0$ we have the following

1. Let $s \geq 3$ and let $(Z, Z_0, -1, \frac{1}{Z}, -1, Z_1)(0) \in H^{s+\frac{1}{2}}(\mathbb{R}) \times H^{s+\frac{1}{2}}(\mathbb{R}) \times H^{s}(\mathbb{R})$. Then there exists a time $T > 0$ so that the initial value problem to (13) has a unique solution $(Z, Z_1)(t)$ satisfying $(Z, Z_0, -1, \frac{1}{Z}, -1, Z_1) \in C^l([0, T], H^{s+\frac{1}{2}-\frac{3}{2}l}(\mathbb{R}) \times H^{s+\frac{1}{2}-\frac{3}{2}l}(\mathbb{R}))$ for $l = 0, 1$. Moreover if $T_{\text{max}}$ is the maximum time of existence then either $T_{\text{max}} = \infty$ or $T_{\text{max}} < \infty$ with

$$\sup_{t \in [0, T_{\text{max}}]} \left\{ \left\| Z, Z_0 - 1 \right\|_{H^{3.5}}(t) + \left\| \frac{1}{Z}, 1 \right\|_{H^{3.5}}(t) + \left\| Z_1 \right\|_{H^{3}}(t) \right\} = \infty$$

2. Let $(Z_1^1, Z_1^i)(t)$ and $(Z_2^1, Z_2^i)(t)$ be two solutions of (13) in $[0, T]$ with

$$\sup_{t \in [0, T_{\text{max}}]} \left\{ \left\| Z_1^1, Z_0^1 - 1 \right\|_{H^{3.5}}(t) + \left\| \frac{1}{Z_1^1}, 1 \right\|_{H^{3.5}}(t) + \left\| Z_1^i \right\|_{H^{3}}(t) \right\} \leq M < \infty$$

for both $i = 1, 2$ for some $M > 0$. Then there is constant $C(M)$ depending only on $M$ and $\sigma$ such that

$$\sup_{t \in [0, T]} \left\{ \left\| Z_1^1, Z_0^1 - Z_2^1 \right\|_{H^2}(t) + \left\| \frac{1}{Z_1^1}, \frac{1}{Z_2^1} \right\|_{H^2}(t) + \left\| Z_1^i - Z_2^i \right\|_{H^{3.5}}(t) \right\} \leq e^{C(M)T} \left\{ \left\| Z_1^1, Z_0^1 - Z_2^1 \right\|_{H^2}(0) + \left\| \frac{1}{Z_1^1}, \frac{1}{Z_2^1} \right\|_{H^2}(0) + \left\| Z_1^i - Z_2^i \right\|_{H^{3.5}}(0) \right\}$$

(Note that the inequality above is for a weaker norm than the one the problem is stated for; however it is still sufficient to prove uniqueness.)

Proof. The first part is a direct consequence of Theorem 7.8 and Lemma 7.1. The second part follows from Theorem 7.8 and an easy modification of the argument of Lemma 7.1. □
8. Proof of Theorem 3.1 and Corollary 3.2

Proof of Theorem 3.1. Let $\epsilon > 0$ and consider the mollified initial data given by $(Z^\epsilon, Z_t^\epsilon)(0) = (P_\epsilon * Z, P_\epsilon * Z_t)(0)$ where $P_\epsilon$ is the Poisson kernel (3). Observe that there exists an $\epsilon_0 > 0$ small enough so that for all $0 < \epsilon \leq \epsilon_0$ we have

$$E_\sigma(Z^\epsilon, Z_t^\epsilon)(0) \leq 2E_\sigma(Z, Z_t)(0)$$

Define

$$M = \left\| \frac{1}{Z,\alpha'} - 1 \right\|_2(0) + \|Z_t\|_2(0) + \|Z,\alpha'\|_\infty(0) < \infty$$

We observe that for all $0 < \epsilon \leq \epsilon_0$ we have

$$\left\| \frac{1}{Z,\alpha'} - 1 \right\|_2(0) + \|Z_t^\epsilon\|_2(0) + \|Z,\alpha'\|_\infty(0) \leq M$$

Now fix some $0 < \epsilon \leq \epsilon_0$. Hence using Corollary 7.9 we see that there exists a time $T_\epsilon > 0$, so that the initial value problem to (13) with initial data $(Z^\epsilon, Z_t^\epsilon)(0)$ has a unique smooth solution $(Z^\epsilon, Z_t^\epsilon)(t)$ in $[0, T_\epsilon]$ so that for all $s \geq 3$

$$\sup_{t \in [0, T_\epsilon]} \left\{ \|Z^\epsilon - 1\|_{H^{s+\frac{1}{2}}(t)} + \left\| \frac{1}{Z,\alpha'} - 1 \right\|_{H^{s+\frac{1}{2}}}(t) + \|Z_t^\epsilon\|_{H^s}(t) \right\} < \infty$$

Therefore by Theorem 5.1 we see that for all $t \in [0, T_\epsilon)$ we have

$$\frac{dE_\sigma(Z^\epsilon, Z_t^\epsilon)(t)}{dt} \leq P(E_\sigma(Z^\epsilon, Z_t^\epsilon)(t))$$

Hence by using Proposition 6.1, Lemma 6.2 and the blow up criterion from Corollary 7.9 we that there exists $T, C_1 > 0$ both depending only on $E_\sigma(0)$ so that $(Z^\epsilon, Z_t^\epsilon)(t)$ in fact exists in $[0, T]$ with $\sup_{t \in [0, T]} E_\sigma(Z^\epsilon, Z_t^\epsilon)(t) \leq C_1$. Also using Lemma 6.2 we see that

$$\sup_{t \in [0, T]} \left\{ \|Z^\epsilon,\alpha' - 1\|_{H^{s+\frac{1}{2}}(t)} + \|Z^{\epsilon,\alpha'} - 1\|_{H^{s+\frac{1}{2}}}(t) + \|Z_t^\epsilon\|_{H^s}(t) \right\} \leq C_2$$

where $C_2$ depends only on $C_1, M, T$ and $\sigma$. Hence by passing to the limit $\epsilon \to 0$ using Corollary 7.9 we have a unique solution $(Z, Z_t)(t)$ in $[0, T]$ to (13) with $\sup_{t \in [0, T]} E_\sigma(Z, Z_t)(t) \leq C_1$ and

$$\sup_{t \in [0, T]} \left\{ \|Z,\alpha' - 1\|_{H^{s+\frac{1}{2}}(t)} + \|Z^{\epsilon,\alpha'} - 1\|_{H^{s+\frac{1}{2}}}(t) + \|Z_t\|_{H^s}(t) \right\} \leq C_2$$

thereby proving the result. \qed
Proof of Corollary 3.2. Without loss of generality we assume that \( c = 1 \) and define \( \tau = \frac{\epsilon^3}{2} \) which implies that \( \tau \leq 1 \). Observe that the result for \( \sigma = 0 \) and \( 0 < \epsilon \leq 1 \) follows directly from Theorem 3.9 of Wu [45]. Hence from now on we assume \( \sigma > 0 \) and can therefore use Theorem 3.1. Using Theorem 3.1 we only need to show that if \( \tau \leq 1 \) then

\[
\mathcal{E}_\sigma(Z^{\epsilon, \sigma}, Z_t^{\epsilon, \sigma})(0) \leq C(M)
\]

where \( C(M) \) is a constant depending only on \( M \). We now prove this estimate.

To simplify the proof we will suppress the dependence of \( M \) in the inequalities i.e. when we write \( a \lesssim b \), we mean that there exists a constant \( C(M) \) depending only on \( M \) such that \( a \leq C(M) b \). As we only need to prove the estimates for \( t = 0 \), we will suppress the time dependence of the solutions e.g. we will write \((Z * P_\epsilon, Z_t * P_\epsilon)|_{t=0}\) by \((Z, Z_t)_\epsilon\) for simplicity.

1. We first observe that for any \( \epsilon > 0 \) we have

\[
\left\| \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)_\epsilon \right\|_2 \lesssim \sup_{y' < 0} \left\| \partial_z \left( \frac{1}{\Psi_\epsilon} \right) \right\|_{L^2(\mathbb{R}, dx')} \approx 1
\]

Similarly we have

\[
\left\| \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)_\epsilon \right\|_{H^\frac{1}{2}} \lesssim \left\| \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)_\epsilon \right\|_2 \left\| \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)_\epsilon \right\|_2 + \left\| \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)_\epsilon \right\|_2 \left\| \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)_\epsilon \right\|_{L^\infty}
\]

\[
\lesssim \sup_{y' < 0} \left\| \partial_z \left( \frac{1}{\Psi_\epsilon} \right) \right\| \sup_{y' < 0} \left\| \frac{1}{\Psi_\epsilon^2} \partial_{\alpha'} \frac{1}{\Psi_\epsilon} \right\| \left\| \frac{1}{\Psi_\epsilon^2} \partial_{\alpha'} \left( \frac{1}{\Psi_\epsilon} \right) \right\|_{L^2(\mathbb{R}, dx')} \approx 1
\]

2. Observe that

\[
\sup_{y' < 0} \left\| \frac{1}{\Psi_\epsilon} \right\|_{L^\infty(\mathbb{R}, dx')} \lesssim \sup_{y' < 0} \left\| \frac{1}{\Psi_\epsilon} \right\|_{L^2(\mathbb{R}, dx')} = \sup_{y' < 0} \left\| \frac{1}{\Psi_\epsilon} \right\|_{L^2(\mathbb{R}, dx')} \approx 1
\]

Hence using \( \sup_{y' < 0} \|U\|_{H^{1,5}(\mathbb{R}, dx')} \leq M \) and that \( 0 < \sigma \leq 1 \) we obtain

\[
\left\| (Z_{t, \alpha'})_\epsilon \right\|_2^2 + \left\| \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} Z_{t, \alpha'} \right)_\epsilon \right\|_2^2 + \left\| \left( \frac{\sigma^2}{Z_{\alpha'}} \partial_{\alpha'} Z_{t, \alpha'} \right)_\epsilon \right\|_2^2 \lesssim 1
\]

3. By Lemma 9.14 we have

\[
\left\| \left( \sigma^\frac{1}{2} Z_{\alpha'}^\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)_\epsilon \right\|_2 + \left\| \left( \sigma^\frac{1}{2} Z_{\alpha'}^\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)_\epsilon \right\|_\infty \lesssim 1
\]
\[
\varepsilon \lesssim \sup_{y' < 0} \left\| \frac{1}{\Psi} \nabla_\tau \left( \frac{1}{\Psi} \right) \right\|_{L^\infty(\mathbb{R}, dx')} \left( \tau^\frac{1}{3} + \tau^\frac{5}{6} \right) \lesssim 1
\]

Now using (20) we obtain
\[
\left\| \left( \frac{1}{Z_{\tau}} \frac{1}{\tau} \partial_{\tau} \omega \right) \right\|_2 + \left\| \left( \frac{1}{Z_{\tau}} \frac{1}{\tau} \partial_{\tau} \omega \right) \right\|_{\infty} \lesssim 1
\]

(4) Using Lemma 9.14 we obtain
\[
\left\| \left( \frac{1}{Z_{\tau}} \frac{1}{\tau} \partial_{\tau} \right) \frac{1}{Z_{\tau}} \right\|_2 + \left\| \left( \frac{1}{Z_{\tau}} \frac{1}{\tau} \partial_{\tau} \right) \frac{1}{Z_{\tau}} \right\|_{\infty} + \left\| \frac{1}{Z_{\tau}} \partial_{\tau} \left( \frac{1}{Z_{\tau}} \right) \right\|_2 \lesssim 1
\]

Now using (20) we obtain
\[
\left\| \left( \frac{1}{Z_{\tau}} \frac{1}{\tau} \partial_{\tau} \omega \right) \right\|_2 + \left\| \left( \frac{1}{Z_{\tau}} \frac{1}{\tau} \partial_{\tau} \omega \right) \right\|_{\infty} \lesssim 1
\]

We observe that
\[
\left\| \left( \frac{1}{Z_{\tau}} \frac{1}{\tau} \partial_{\tau} \right) \frac{1}{Z_{\tau}} \right\|_2 \lesssim \left\| \frac{1}{Z_{\tau}} \partial_{\tau} \left( \frac{1}{Z_{\tau}} \right) \right\|_2 + \left\| \frac{1}{Z_{\tau}} \partial_{\tau} \left( \frac{1}{Z_{\tau}} \right) \right\|_{\infty} \left\| \frac{1}{Z_{\tau}} \partial_{\tau} \left( \frac{1}{Z_{\tau}} \right) \right\|_2 \lesssim 1
\]

and in particular again by using (20) we obtain
\[
\left\| \left( \frac{1}{Z_{\tau}} \frac{1}{\tau} \partial_{\tau} \right) \frac{1}{Z_{\tau}} \right\|_2 \lesssim \left\| \frac{1}{Z_{\tau}} \partial_{\tau} \left( \frac{1}{Z_{\tau}} \right) \right\|_2 + \left\| \frac{1}{Z_{\tau}} \partial_{\tau} \left( \frac{1}{Z_{\tau}} \right) \right\|_{\infty} \left\| \frac{1}{Z_{\tau}} \partial_{\tau} \left( \frac{1}{Z_{\tau}} \right) \right\|_2 \lesssim 1
\]

and hence we have
\[
\left\| \frac{1}{Z_{\tau}} \partial_{\tau} \left( \frac{1}{Z_{\tau}} \right) \right\|_2 \lesssim \left\| \left( \frac{1}{Z_{\tau}} \frac{1}{\tau} \partial_{\tau} \right) \frac{1}{Z_{\tau}} \right\|_2 + \left\| \frac{1}{Z_{\tau}} \partial_{\tau} \left( \frac{1}{Z_{\tau}} \right) \right\|_{\infty} \left\| \frac{1}{Z_{\tau}} \partial_{\tau} \left( \frac{1}{Z_{\tau}} \right) \right\|_2 \lesssim 1
(4) Using Lemma 9.14 we have
\[ \left\| \left( \sigma \frac{1}{Z_{\alpha'}} \right) \right\|_\infty + \left\| \left( \sigma \frac{2}{Z_{\alpha'}} \right) \right\|_2 + \left\| \left( \sigma \frac{3}{Z_{\alpha'}} \right) \right\|_2 \leq \sup_{y' < 0} \left\| \frac{1}{\Psi_z} \right\| L^2(\mathbb{R}, dx') \left( \tau \right) \]
\[ \leq 1 \]
Hence using (20) we also get \( \left\| \left( \sigma \frac{1}{Z_{\alpha'}} \right) \right\|_\infty \leq 1 \)

(5) Using (20) and Proposition 9.12 with \( f = \left( \frac{1}{Z_{\alpha'}} \right), w = \left( \frac{1}{|Z_{\alpha'}|} \right) \)
and \( h = \omega \) we get
\[ \left\| \left( \sigma \frac{1}{Z_{\alpha'}} \right) \right\|_2 \leq \sup_{y' < 0} \left\| \frac{1}{\Psi_z} \right\| L^2(\mathbb{R}, dx') \left( \tau \right) \]
\[ \leq 1 \]
Hence using (32) and Proposition 9.9 we can finally control
\[ \left\| \left( \sigma \frac{1}{Z_{\alpha'}} \right) \right\|_2 \leq 1 \]

(6) Using Lemma 9.14 we get
\[ \left\| \left( \sigma \frac{1}{Z_{\alpha'}} \right) \right\|_2 \leq \sup_{y' < 0} \left\| \frac{1}{\Psi_z} \right\| L^2(\mathbb{R}, dx') \tau \leq 1 \]
Hence using (20) we have
\[ \left\| \left( \sigma \frac{1}{Z_{\alpha'}} \right) \right\|_2 \leq \sup_{y' < 0} \left\| \frac{1}{\Psi_z} \right\| L^2(\mathbb{R}, dx') \tau \leq 1 \]
Hence we also have

\[ \| \left( \frac{\sigma^{\frac{1}{2}} \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)}{\epsilon} \right) \|_2 \lesssim 1 \]

(7) We observe that

\[ \| \left( \frac{\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)}{\epsilon} \|_2 \lesssim \| \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \|_2 \| \sigma \partial_{\alpha'} \left( \frac{1}{|Z_{\alpha'}|} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \|_2 \lesssim 1 \]

Hence we also have

\[ \| \sigma^{\frac{1}{2}} \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \|_2 \lesssim \| \left( \frac{\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)}{\epsilon} \|_2 \]

\[ + \| \left( \sigma^{\frac{1}{2}} Z_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \|_\infty \| \left( \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \|_2 \lesssim 1 \]

(8) We observe that

\[ \| \left( \frac{\frac{1}{2} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right)}{\epsilon} \|_\frac{\epsilon}{2} \lesssim \| \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \|_2 \| \left\{ \sigma Z_{\alpha'} \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\} \|_2 \]

\[ \lesssim \sup_{\psi < 0} \left\| \frac{1}{\psi_z} \partial_z \left( \frac{1}{\psi_z} \right) \right\|_{L^2(\mathbb{R}, ds')} \| \left\{ \sigma Z_{\alpha'} \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \right\} \|_2 \]

\[ \lesssim 1 \]

(9) Using Lemma 9.14 we get

\[ \| \sigma^{\frac{3}{2}} \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \|_2 \lesssim \| \sigma \partial_{\alpha'} \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \|_{\frac{\epsilon}{2}} \]

\[ \lesssim \sup_{\psi < 0} \left\| \frac{1}{\psi_z} \partial_z \left( \frac{1}{\psi_z} \right) \right\|_{L^2(\mathbb{R}, ds')} \left( \tau^{\frac{3}{2}} + \tau \right) \lesssim 1 \]

Now we use Proposition 9.12 with \( f = \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \), \( w = \left( \frac{1}{Z_{\alpha'}} \right) \) and \( h = Z_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \) to get

\[ \| \sigma \left( \frac{1}{Z_{\alpha'}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \|_{\frac{\epsilon}{2}} \lesssim \| \left( \sigma^{\frac{1}{2}} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \|_{\frac{\epsilon}{2}} \| \left( \sigma^{\frac{1}{2}} Z_{\alpha'} \partial_{\alpha'} \frac{1}{Z_{\alpha'}} \right) \|_{\infty} \]
\[ + \left( \sigma \frac{1}{Z'_{.,\alpha'}} \right) \partial_\alpha' \left( \frac{1}{Z'_{.,\alpha'}} \right) \epsilon \left\| \partial_\alpha' \left( \frac{1}{Z'_{.,\alpha'}} \right) \epsilon \right\|_2 \]

\[ + \left( \sigma \frac{1}{Z'_{.,\alpha'}} \right) \partial_\alpha' \left( \frac{1}{Z'_{.,\alpha'}} \right) \epsilon \left\| \partial_\alpha' \left( \frac{1}{Z'_{.,\alpha'}} \right) \epsilon \right\|_2 \left\| \left( \sigma \frac{1}{Z'_{.,\alpha'}} \right) \partial_\alpha' \left( \frac{1}{Z'_{.,\alpha'}} \right) \epsilon \right\|_\infty \]

\[ \lesssim 1 \]

Hence we finally have

\[ \left\| \left( \sigma \frac{1}{Z'_{.,\alpha'}} \right) \partial_\alpha' \left( \frac{1}{Z'_{.,\alpha'}} \right) \epsilon \right\|_{\dot{H}^1_2} \lesssim \left\| \sigma \partial_\alpha' \left( \frac{1}{Z'_{.,\alpha'}} \right) \epsilon \right\|_{\dot{H}^1_2} \]

\[ + \left\| \sigma \left( \frac{1}{Z'_{.,\alpha'}} \right) \partial_\alpha' \left( \frac{1}{Z'_{.,\alpha'}} \right) \epsilon \right\|_{\dot{H}^1_2} \lesssim 1 \]

This completes the proof of the first part of the corollary. To see the rate of growth of curvature, observe that the \( L^\infty \) norm of the curvature of the initial interface of \( Z^{\epsilon,\sigma} (\cdot, 0) \) is

\[ \left\| \kappa^{\epsilon,\sigma} \right\|_\infty = \left\| \frac{1}{(|Z'_{.,\alpha'}|)\epsilon} \partial_\alpha' (g_\epsilon) \right\|_\infty \quad \text{where} \quad \left( \frac{Z_{.,\alpha'}}{|Z_{.,\alpha'}|} \right)_\epsilon = \epsilon^{-1} \]

Now if the interface \( Z(\cdot, 0) \) has an angled crest at \( \alpha' = 0 \), then we see that \( \partial_\alpha' (g_\epsilon)(0, 0) \sim \epsilon^{-1} \) as \( \epsilon \to 0 \), due to \( g(\cdot, 0) \) having a jump for \( \epsilon = 0 \) and \( g_\epsilon = K_\epsilon * g \) (where \( K_\epsilon \) is the Poisson kernel (3)). But we know from the local description of the conformal map that \( (Z_{.,\alpha'})(0, 0) \sim \epsilon^{-\nu} \) as \( \epsilon \to 0 \) (see [40] for a proof). Hence we see that \( \left\| \kappa^{\epsilon,\sigma} \right\|_\infty \sim \epsilon^{-\nu} \) as \( \epsilon \to 0 \), thereby proving the lemma. \( \Box \)

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Appendix

Here we will prove all the identities and estimates used in the paper. We will state most of the statements only for functions in the Schwartz class and it can be extended to more general functions by an approximation argument. Let us first recall some of the notation used. Let \( D_t = \partial_t + b \partial_\alpha' \) where \( b \) is given by (11) and recall that \([f, g; h]\) is defined as

\[ [f_1, f_2; f_3](\alpha') = \frac{1}{i\pi} \int \left( \frac{f_1(\alpha') - f_1(\beta')}{\alpha' - \beta'} \right) \left( \frac{f_2(\alpha') - f_2(\beta')}{\alpha' - \beta'} \right) f_3(\beta') \, d\beta' \]
Proposition 9.1. Let $f, g, h \in \mathcal{S}^{\mathbb{R}}$. Then we have the following identities

1. $h \partial_{\alpha'} [f, \mathbb{H}] \partial_{\alpha'} g = [h \partial_{\alpha'} f, \mathbb{H}] \partial_{\alpha'} g + [f, \mathbb{H}] \partial_{\alpha'} (h \partial_{\alpha'} g) - [h, f; \partial_{\alpha'} g]$
2. $D_t [f, \mathbb{H}] \partial_{\alpha'} g = [D_t f, \mathbb{H}] \partial_{\alpha'} g + [f, \mathbb{H}] \partial_{\alpha'} (D_t g) - [b, f; \partial_{\alpha'} g]$

Proof. The second identity is a direct consequence of the first. Now we see that

\[
\begin{align*}
&h(\alpha') \partial_{\alpha'} [f, \mathbb{H}] \partial_{\alpha'} g \\
&= h(\alpha') \partial_{\alpha'} \left( \frac{1}{i\pi} \int \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \partial_{\beta'} g(\beta') \, d\beta' \right) \\
&= h(\alpha') f'(\alpha') \left( \frac{1}{i\pi} \int \frac{1}{\alpha' - \beta'} \partial_{\beta'} g(\beta') \, d\beta' \right) \\
&- \frac{1}{i\pi} \int \left( \frac{h(\alpha') - h(\beta')}{\alpha' - \beta'} \right) \left( \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right) \partial_{\beta'} g(\beta') \, d\beta' \\
&- \frac{1}{i\pi} \int \frac{f(\alpha') - f(\beta')}{(\alpha' - \beta')^2} h(\beta') \partial_{\beta'} g(\beta') \, d\beta' \\
&= \frac{1}{i\pi} \int \frac{h(\alpha') f'(\alpha') - h(\beta') f'(\beta')}{\alpha' - \beta'} \partial_{\beta'} g(\beta') \, d\beta' \\
&+ \frac{1}{i\pi} \int \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \partial_{\beta'} (h(\beta') \partial_{\beta'} g(\beta')) \, d\beta' \\
&- \frac{1}{i\pi} \int \left( \frac{h(\alpha') - h(\beta')}{\alpha' - \beta'} \right) \left( \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right) \partial_{\beta'} g(\beta') \, d\beta'.
\end{align*}
\]

\[\square\]

Proposition 9.2. Let $H \in C^1(\mathbb{R}), A_i \in C^1(\mathbb{R})$ for $i = 1, \cdots m$ and $F \in C^\infty(\mathbb{R})$. Define

\[
\begin{align*}
C_1(H, A, f) &= p.v. \int F \left( \frac{H(x) - H(y)}{x - y} \right) \frac{\prod_{i=1}^m (A_i(x) - A_i(y))}{(x - y)^{m+1}} f(y) \, dy \\
C_2(H, A, f) &= p.v. \int F \left( \frac{H(x) - H(y)}{x - y} \right) \frac{\prod_{i=1}^m (A_i(x) - A_i(y))}{(x - y)^m} \partial_y f(y) \, dy
\end{align*}
\]

then there exists constants $c_1, c_2, c_3, c_4$ depending only on $F$ and $\|H\|_{\infty}$ so that

1. $\|C_1(H, A, f)\|_2 \leq c_1 \|A'_1\|_{\infty} \cdots \|A'_m\|_{\infty} \|f\|_2$
2. $\|C_2(H, A, f)\|_2 \leq c_2 \|A'_1\|_2 \|A'_2\|_{\infty} \cdots \|A'_m\|_{\infty} \|f\|_{\infty}$
3. $\|C_2(H, A, f)\|_2 \leq c_3 \|A'_1\|_{\infty} \|A'_2\|_{\infty} \cdots \|A'_m\|_{\infty} \|f\|_2$
4. $\|C_2(H, A, f)\|_2 \leq c_4 \|A'_1\|_2 \|A'_2\|_{\infty} \cdots \|A'_m\|_{\infty} \|f\|_{\infty}$

Proof. The first estimate is a theorem by Coifman, McIntosh and Meyer [15]. See also chapter 9 of [29]. Estimate 2 is a consequence of the Tb theorem and a proof can be found in [43]. The third and fourth estimates can be obtained from the first two by integration by parts. \[\square\]

Proposition 9.3. Let $T : \mathcal{D}(\mathbb{R}) \rightarrow \mathcal{D}'(\mathbb{R})$ be a linear operator with kernel $K(x, y)$ such that on the open set $\{(x, y) : x \neq y\} \subset \mathbb{R} \times \mathbb{R}$, $K(x, y)$ is a function satisfying

\[
|K(x, y)| \leq \frac{C_0}{|x - y|} \quad \text{and} \quad |\nabla_x K(x, y)| \leq \frac{C_0}{|x - y|^2}
\]
where \( C_0 \) is a constant. If \( T \) is continuous on \( L^2(\mathbb{R}) \) with \( \|T\|_{L^2 \to L^2} \leq C_0 \) and if \( T(1) = 0 \), then \( T \) is bounded on \( \dot{H}^s \) for \( 0 < s < 1 \) with \( \|T\|_{\dot{H}^s \to \dot{H}^s} \lesssim C_0 \).

**Proof.** This proposition is a direct consequence of the result of Lemarie [25] where only weak boundedness of \( T \) on \( L^2 \) (in the sense of David and Journe) is assumed. As boundedness on \( L^2 \) implies weak boundedness, the proposition follows. See also chapter 10 of [29] for another proof of the result of Lemarie. \( \Box \)

**Lemma 9.4.** Let \( r, s \in \mathbb{R} \), \( k, m \in \mathbb{Z} \). If \( f \in S(\mathbb{R}) \), then we have the following

1. \( \|\partial^{\alpha_1}_{\alpha_1} \cdots \partial^{\alpha_k}_{\alpha_k} f\|_2 \lesssim \|f\|_2^\theta \|\partial^{\alpha_1}_{\alpha_1} \cdots \partial^{\alpha_k}_{\alpha_k} f\|_2^{1-\theta} \) for \( 0 \leq r < s \) with \( 1 - \theta = \frac{r}{s} \).
2. \( \|\partial^{\alpha_1}_{\alpha_1} \cdots \partial^{\alpha_k}_{\alpha_k} f\|_2 \lesssim \|f''\|_2^\theta \|\partial^{\alpha_1}_{\alpha_1} \cdots \partial^{\alpha_k}_{\alpha_k} f\|_2^{1-\theta} \) for \( 2 \leq r < s \) with \( 1 - \theta = \frac{r-2}{s-2} \).
3. \( \|\partial^{\alpha_1}_{\alpha_1} \partial^m f\|_{\infty} \lesssim \|f\|_{2}^\theta \|\partial^{\alpha_1}_{\alpha_1} \partial^m f\|_{2}^{1-\theta} \) for \( 0 \leq k < m \) with \( 1 - \theta = \frac{k+1}{m} \).
4. \( \|\partial^{\alpha_1}_{\alpha_1} \partial^m f\|_{\infty} \lesssim \|f''\|_{2}^\theta \|\partial^{\alpha_1}_{\alpha_1} \partial^m f\|_{2}^{1-\theta} \) for \( 2 \leq k < m \) with \( 1 - \theta = \frac{k-3}{m-2} \).

**Proof.** The first estimate is a standard interpolation estimate which can be easily proved by using the Fourier transform. We skip its proof. The second one follows from the first by applying it on the function \( f'' \) with \( r, s \) replaced by \( r - 2, s - 2 \) respectively.

The third estimate is a consequence of the Gagliardo-Nirenberg interpolation estimate (see Theorem 12.87 in [26]). The last one follows from the third estimate by applying it on the function \( f'' \) with \( k, m \) replaced by \( k - 2, m - 2 \) respectively. \( \Box \)

**Lemma 9.5.** Let \( k, n \in \mathbb{N} \) and \( f_1, f_2, \ldots, f_k \in S(\mathbb{R}) \). Let \( r_1, r_2, \ldots, r_k \in \mathbb{Z} \) with \( r_1 + \cdots + r_k = n \) and \( r_1, r_2, \ldots, r_k \geq 0 \) for all \( 1 \leq i \leq k \) and. Let \( r = \max\{r_1, r_2, \ldots, r_k\} \geq 1 \). Then

1. \( \|f^{(r_1)}_1 \cdots f^{(r_k)}_k\|_2 \leq C(K) \|f^{(r_1)}_1\|_{H^s} + \cdots + \|f^{(r_k)}_k\|_{H^s} \) for \( s = \max\{r - 1, n - 2\} \).
2. \( \|f^{(r_1)}_1 \cdots f^{(r_k)}_k\|_{H^{\frac{1}{2}}} \leq C(K) \|f^{(r_1)}_1\|_{H^s} + \cdots + \|f^{(r_k)}_k\|_{H^s} \) for \( s = \max\{r - \frac{1}{2}, n - 2\} \).

with \( K = (\|f_1\|_{\infty} + \|f_1\|_{H^1}) + \cdots + (\|f_k\|_{\infty} + \|f_k\|_{H^1}) \) and \( C(K) \) is a constant depending only on \( K \).

**Proof.** Let us begin by proving the first estimate. Without loss of generality \( 0 \leq r_1 \leq r_2 \leq \cdots \leq r_k \). Clearly the estimate holds if \( k = 1 \) or \( r = 1 \). Hence we can now assume that \( k \geq 2 \) and \( r \geq 2 \). If \( r_1 \leq \cdots \leq r_j \leq 1 \) for some \( j < k \) with \( r_{j+1} \geq 2 \), then we have

\[
\|f^{(r_1)}_1 \cdots f^{(r_k)}_k\|_2 \leq C(K) \|f^{(r_{j+1})}_{j+1} \cdots f^{(r_k)}_k\|_2
\]

Hence without loss of generality we can assume that \( r_1 \geq 2 \). As \( k \geq 2 \) this implies that \( n \geq 4 \) and we also have \( r \geq 2, r \leq n - 2 \) and \( s = n - 2 \). Hence using Lemma 9.4 we have

\[
\|f^{(r_1)}_1 \cdots f^{(r_k)}_k\|_2 \\
\leq \|f^{(r_1)}_1\|_{\infty} \cdots \|f^{(r_{k-1})}_{k-1}\|_{\infty} \|f^{(r_k)}_k\|_2 \\
\lesssim \left(\|f^{(s+1)}_1\|_2^{1-\theta_1} \cdots \|f^{(s+1)}_{k-1}\|_2^{1-\theta_{k-1}} \right) \|f^{(s+1)}_1\|_2 \|f^{(s+1)}_2\|_2 \cdots \|f^{(s+1)}_k\|_2^{1-\theta_k}
\]

where \( 1 - \theta_j = \frac{r_j - \frac{3}{2}}{s - 1} \) for \( j < k \) and \( 1 - \theta_k = \frac{r_k - 2}{s - 1} \). Now observe that

\[
(1 - \theta_1) + \cdots + (1 - \theta_k) = \frac{r_1 - \frac{3}{2}}{s - 1} + \cdots + \frac{r_{k-1} - \frac{3}{2}}{s - 1} + \frac{r_k - 2}{s - 1} \leq 1
\]
Hence by using $AM - GM$ inequality the estimate follows. The proof of the second estimate is very similar and we skip it.

**Corollary 9.6.** Let $f, g \in \mathcal{S}(\mathbb{R})$ and let $n \in \mathbb{N}$ with $n \geq 2$. Then

1. \[ \| (f \partial_{\alpha'})^n g - f^n \partial_{\alpha'}^n g \|_2 \leq C(K) \| f' \|_{H^{1/n}} \| g' \|_{H^{1/n}} \] for $s = n - 2$
2. \[ \| (f \partial_{\alpha'})^n g - f^n \partial_{\alpha'}^n g \|_{H^{1/n}} \leq C(K) \| f' \|_{H^{1/n}} \| g' \|_{H^{1/n}} \] for $s = n - \frac{3}{2}$

where $K = \| f \|_{\infty} + \| f' \|_{H^{1/n}} + \| g \|_{\infty} + \| g' \|_{H^{1/n}}$ and $C(K)$ is a constant depending only on $K$.

**Proof.** This follows directly from Lemma 9.5.

**Proposition 9.7.** Let $f \in \mathcal{S}(\mathbb{R})$. Then we have

1. \[ \| f \|_\infty \lesssim \| f \|_{H^{1/2}} \] if $s > \frac{1}{2}$ and for $s = \frac{1}{2}$ we have \[ \| f \|_{BMO} \lesssim \| f \|_{H^{1/2}} \]
2. \[ \int \left| \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right|^2 d\beta' \lesssim \| f' \|_2 \]
3. \[ \sup_{\beta'} \left| \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right| \lesssim \| f' \|_2 \]
4. \[ \| f \|_{H^{1/2}}^2 = \frac{1}{2\pi} \int \left| \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right|^2 d\beta' d\alpha' \]
5. \[ \| \partial_{\beta'} \left( \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right) \|_{L^2(\mathbb{R}^2, d\alpha' d\beta')} \lesssim \| f' \|_{H^{1/2}} \]

**Proof:** (1) This is a standard Sobolev embedding result.
(2) This is a consequence of Hardy’s inequality.
(3) We see that

\[ \sup_{\beta'} \left| \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right| \leq \sup_{\beta'} \frac{\int_{\beta'}^{\beta'} |f'(s)| ds}{|\alpha' - \beta'|} \leq M(f')(\alpha') \]

where $M$ is the uncentered Hardy Littlewood maximal operator. As the maximal operator is bounded on $L^2$, the estimate follows.

(4) Observe that as $|\partial_{\alpha'}| = i \mathbb{H} \partial_{\alpha'}$ and $\mathbb{H}(1) = 0$ we have

\[ \| f \|_{H^{1/2}}^2 = -\frac{1}{\pi} \int \bar{f}(\alpha') \partial_{\alpha'} \left( \int \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} d\beta' \right) d\alpha' \]
\[ = \frac{1}{\pi} \int \bar{f}(\alpha') \left( \int \frac{f(\alpha') - f(\beta')}{(\alpha' - \beta')^2} d\beta' \right) d\alpha' \]
\[ = \frac{1}{\pi} \int \int \left| \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right|^2 d\beta' d\alpha' + \frac{1}{\pi} \int \int \frac{f(\alpha') - f(\beta')}{(\alpha' - \beta')^2} \bar{f}(\beta') d\beta' d\alpha' \]

Now observe that

\[ \frac{1}{\pi} \int \bar{f}(\alpha') \left( \int \frac{f(\alpha') - f(\beta')}{(\alpha' - \beta')^2} d\beta' \right) d\alpha' = -\frac{1}{\pi} \int \int \frac{f(\alpha') - f(\beta')}{(\alpha' - \beta')^2} \bar{f}(\beta') d\beta' d\alpha' \]

The identity now follows.
We see that
\[
\partial^s_{\alpha'} \left( \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right) = \frac{f(\alpha') - f(\beta')}{(\alpha' - \beta')^2} - \frac{f'(\beta')}{\alpha' - \beta'}
\]
\[
= \int_0^1 \frac{f'(\beta' + s(\alpha' - \beta')) - f'(\beta')}{s(\alpha' - \beta')} \, ds
\]
\[
= \int_0^1 \left[ \frac{f'(\beta' + s) - f'(\beta')}{s} \right] \, ds \quad \text{using } \alpha' = \beta' + 1
\]

Hence we have
\[
\left\| \partial^s_{\alpha'} \left( \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right) \right\|_{L^2(\mathbb{R}^2, d\alpha' d\beta')} \lesssim \int_0^1 s \left\| \frac{f'(\beta' + s l) - f'(\beta')}{s l} \right\|_{L^2(\mathbb{R}^2, d\beta' dl)} \, ds
\]
\[
\lesssim \int_0^1 \sqrt{s} \left\| f' \right\|_{\dot{H}^{\frac{1}{2}}} \, ds
\]
\[
\lesssim \left\| f' \right\|_{\dot{H}^{\frac{1}{2}}}
\]

**Proposition 9.8.** Let \( f, g \in S(\mathbb{R}) \) with \( s, a \in \mathbb{R} \) and \( m, n \in \mathbb{Z} \). Then we have the following estimates

1. \( \left\| \partial_\alpha^a [f, \mathbb{H}] (|\partial_\alpha^a|^s g) \right\|_2 \lesssim \left\| \partial_\alpha^a [s + a f] \right\|_{BMO} \| g \|_2 \quad \text{for } s, a \geq 0 \)
2. \( \left\| \partial_\alpha^a [f, \mathbb{H}] (|\partial_\alpha^a|^s g) \right\|_2 \lesssim \left\| \partial_\alpha^a [s + a f] \right\|_{BMO} \| g \|_2 \quad \text{for } s \geq 0 \) and \( a > 0 \)
3. \( \left\| [f, \partial_\alpha^a]^s g \right\|_2 \lesssim \left\| \partial_\alpha^a \right\|_{BMO} \| g \|_2 \)
4. \( \left\| [f, \partial_\alpha^a]^s (|\partial_\alpha^a|^2 g) \right\|_2 \lesssim \left\| \partial_\alpha^a [f] \right\|_{BMO} \| g \|_2 \)
5. \( \left\| \partial_\alpha^m [f, \mathbb{H}] \partial_\alpha^n g \right\|_{L^\infty \cap \dot{H}^{\frac{1}{2}}} \lesssim \left\| \partial_\alpha^{m+n} f \right\|_{\infty} \| g \|_2 \quad \text{for } m \geq 0 \) and \( n \geq 0 \)
6. \( \left\| \partial_\alpha^m [f, \mathbb{H}] \partial_\alpha^n g \right\|_2 \lesssim \left\| \partial_\alpha^{m+n} f \right\|_2 \| g \|_2 \quad \text{for } m \geq 0 \) and \( n \geq 1 \)
7. \( \left\| \partial_\alpha^m [f, \mathbb{H}] \partial_\alpha^n g \right\|_2 \lesssim \left\| \partial_\alpha^{m+n} f \right\|_2 \| g \|_2 \quad \text{for } m \geq 0 \) and \( n \geq 1 \)
8. \( \left\| [f, \mathbb{H}] g \right\|_2 \lesssim \left\| f' \right\|_2 \| g \|_1 \)

**Proof.** The first four estimates are all variants of the Kato Ponce commutator estimate and are proved using the paraproduct decomposition. See Lemma 2.1 in [20] for the first two estimates and Theorem 1.2 in [27] for the third and fourth estimates. The fourth estimate is not explicitly stated as part of Theorem 1.2 in [27] however the proof is identical to the proof of estimate 3 with the only change being at the last step where you move half a derivative from \( g \) to \( f \).

The \( \dot{H}^{\frac{1}{2}} \) estimate of the fifth estimate follows from the first estimate. For the \( L^\infty \) estimate note that
\[
\partial_\alpha^m [f, \mathbb{H}] \partial_\alpha^n g = \partial_\alpha^m \int f(\alpha') - f(\beta') \frac{g(\beta')}{\alpha' - \beta'} \, d\beta'
\]
\[
= \partial_\alpha^m \int_0^1 f'((1-s)\beta' + s\alpha') \frac{g(\beta')}{\alpha' - \beta'} \, ds \, d\beta'
\]
\[
= (-1)^n \int_0^1 s^n (1-s)^n \left( \int f^{(m+n+1)}((1-s)\beta' + s\alpha') g(\beta') \, d\beta' \right) \, ds
\]
The estimate now follows from the Cauchy Schwarz inequality. The sixth and seventh estimates follow from the first two estimates. For the last estimate observe that
\[
\| [ f, g ] \|_{\mathbb{H}} (\alpha') \lesssim \int \left| \frac{f'(\alpha') - f(\beta')}{\alpha' - \beta'} \right| g(\beta') \left| \frac{1}{2} \left| g(\beta') \right| \frac{1}{2} \right| d\beta' \\
\lesssim \left( \int \left| \frac{f'(\alpha') - f(\beta')}{\alpha' - \beta'} \right|^2 g(\beta') \left| \frac{1}{2} \right| \right)^{\frac{1}{2}} \| g \|_{1}^{\frac{1}{2}}
\]
The estimate now follows from Hardy’s inequality as stated in Proposition 9.7. □

**Proposition 9.9.** Let \( f, g, h \in \mathcal{S}(\mathbb{R}) \) with \( s, a \in \mathbb{R} \) and \( m, n \in \mathbb{Z} \). Then we have the following estimates

1. \( \| \partial_{\alpha'}^s (fg) \|_2 \lesssim \| \partial_{\alpha'}^s f \|_2 \| g \|_\infty + \| f \|_\infty \| \partial_{\alpha'}^s g \|_2 \) for \( s > 0 \)
2. \( \| fg \|_{\dot{H}^s} \lesssim \| f \|_{\dot{H}^s} \| g \|_\infty \)
3. \( \| f g \|_{\dot{H}^1} \lesssim \| f \|_2 \| g \|_2 + \| f \|_\infty \| g \|_{\dot{H}^1} \)

**Proof.** See [22] for the first estimate. The second one is a special case of the first. For the third one observe that
\[
\| \partial_{\alpha'}^s (fg) \| \lesssim \| \partial_{\alpha'}^s f \| \| g \|_\infty + \| f \|_\infty \| \partial_{\alpha'}^s g \|
\]
and hence from Proposition 9.8
\[
\| fg \|_{\dot{H}^1} \lesssim \| \partial_{\alpha'}^s f \|_{BMO} \| g \|_\infty + \| f \|_\infty \| g \|_{\dot{H}^1}
\]

**Proposition 9.10.** Let \( f, g, h \in \mathcal{S}(\mathbb{R}) \). Then we have the following estimates

1. \( \| [ f, g, h ] \|_2 \lesssim \| f' \|_2 \| g' \|_2 \| h \|_2 \)
2. \( \| \partial_{\alpha'} [ f, [ g, h ] ] \|_2 \lesssim \| f' \|_2 \| g' \|_2 \| h \|_2 \)
3. \( \| [ f, g, h ] \|_2 \lesssim \| f' \|_\infty \| g' \|_\infty \| h \|_2 \)
4. \( \| [ f, g, h ] \|_{\dot{H}^1} \lesssim \| f' \|_\infty \| g' \|_\infty \| h \|_{\dot{H}^1} \)
5. \( \| [ f, g, h ] \|_{L^\infty \cap \dot{H}^1} \lesssim \| f' \|_\infty \| g' \|_2 \| h \|_2 \)

**Proof.** (1) We see that
\[
\lesssim \int \left| \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right| g(\alpha') - g(\beta') \left| \frac{1}{2} \right| h(\beta') \left| \frac{1}{2} \right| \frac{d\beta'}{2} \lesssim \| f' \|_2 \left( \int \left| \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right|^2 h(\beta') \left| \frac{1}{2} \right| \right)^{\frac{1}{2}}
\]
The estimate now follows from Hardy’s inequality.

(2) We see that
\[
\partial_{\alpha'} [ f, [ g, h ] ]h = \partial_{\alpha'} ( f [ g, h ] - [ g, [ h, f ] ] )h
\]
\[
= \frac{1}{i\pi} \int \frac{(g(\alpha') - g(\beta'))(f(\alpha') - f(\beta'))}{\alpha' - \beta'} h(\beta') \frac{1}{2} \frac{d\beta'}{2}
\]
\[
= - \frac{1}{i\pi} \int \frac{g(\alpha') - g(\beta')}{\alpha' - \beta'} \left( \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right) h(\beta') \frac{1}{2} \frac{d\beta'}{2}
\]
+ g'(\alpha') \left( \frac{1}{i\pi} \int \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} h(\beta') \, d\beta' \right)
+ f'(\alpha') \left( \frac{1}{i\pi} \int \frac{g(\alpha') - g(\beta')}{\alpha' - \beta'} h(\beta') \, d\beta' \right)

The estimate now follows by previous estimates.

(3) This is a special case of Proposition 9.2

(4) From the third estimate we observe that the operator $T$ defined by the action $h \mapsto \left[ f, g; h' \right]$ is bounded on $L^2$. Also we clearly see that $T(1) = 0$. It is also easy to see that the kernel of this operator satisfies the conditions for Proposition 9.3. Hence the operator $T$ is bounded on $H^{\frac{1}{2}}$.

(5) The $L^\infty$ estimate is obtained easily by an application of Cauchy Schwarz and Hardy’s inequality. Now we use $\|f\|_{H^{\frac{1}{2}}} \lesssim \left\| \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right\|_{L^2(\mathbb{R} \times \mathbb{R}, \, da'd\beta')}$ and see that

$$\frac{[f, g; h](\alpha') - [f, g; h](\beta')}{\alpha' - \beta'} = \frac{1}{i\pi} \int \frac{h(s)}{\alpha' - \beta'} \left[ \frac{f(\alpha') - f(s)}{\alpha' - s} - \frac{f(\beta') - f(s)}{\beta' - s} \right] \frac{g(\alpha') - g(s)}{\alpha' - s} \, ds$$

$$+ \frac{1}{i\pi} \int \frac{h(s)}{\alpha' - \beta'} \left[ \frac{g(\alpha') - g(s)}{\alpha' - s} - \frac{g(\beta') - g(s)}{\beta' - s} \right] \frac{f(\beta') - f(s)}{\beta' - s} \, ds$$

Now we use the following notation to simplify the calculation

$$F(a, b) = \frac{f(a) - f(b)}{a - b} \quad \text{and} \quad G(a, b) = \frac{g(a) - g(b)}{a - b}$$

Hence we have

$$\frac{[f, g; h](\alpha') - [f, g; h](\beta')}{\alpha' - \beta'} = - \frac{1}{i\pi} \int \frac{h(s)}{\beta' - s} F(\alpha', s)G(\alpha', s) \, ds + \frac{1}{i\pi} \int \frac{h(s)}{\beta' - s} F(\alpha', \beta')G(\alpha', s) \, ds$$

$$+ \frac{1}{i\pi} \int \frac{h(s)}{\alpha' - s} F(\beta', s)G(\alpha', \beta') \, ds - \frac{1}{i\pi} \int \frac{h(s)}{\alpha' - s} F(\beta', s)G(\beta', s) \, ds$$

$$= - \mathbb{H}(F(\alpha', \cdot)G(\alpha', \cdot)h(\cdot))(\beta') + F(\alpha', \beta')\mathbb{H}(G(\alpha', \cdot)h(\cdot))(\beta')$$

$$+ G(\alpha', \beta')\mathbb{H}(F(\beta', \cdot)h(\cdot))(\alpha') - \mathbb{H}(F(\beta', \cdot)G(\beta', \cdot)h(\cdot))(\alpha')$$

and we see that

$$\| \mathbb{H}(F(\alpha', \cdot)G(\alpha', \cdot)h(\cdot))(\beta') \|_{L^2(\mathbb{R} \times \mathbb{R}, \, da'd\beta')}
\lesssim \left\| \frac{f(\alpha') - f(\beta')}{\alpha' - \beta'} \right\|_{L^2(\mathbb{R}, \, d\beta')} \left\| h(s) \right\|_{L^2(\mathbb{R}, \, da')}$$

$$\lesssim \left\| f' \right\|_{\infty} \left\| h \right\|_{L^2} \left\| G(\alpha', \beta') \right\|_{L^\infty(\mathbb{R}, \, d\beta')} \left\| L^2(\mathbb{R}, \, da') \right\|$$

$$\lesssim \left\| f' \right\|_{\infty} \left\| g' \right\|_{2} \left\| h \right\|_{L^2}$$

The other terms are handled similarly. □
Proposition 9.11. Let $f \in S(\mathbb{R})$ and let $w$ be a smooth non-zero weight with $w, \frac{1}{w} \in L^\infty(\mathbb{R})$ and $w' \in L^2(\mathbb{R})$. Then

1. $\|f\|_2^2 \lesssim \left\| \frac{f}{w} \right\|_2 \|wf'\|_2$

2. $\|f\|_{L^\infty \cap H^\frac{1}{2}}^2 \lesssim \left\| \frac{f}{w} \right\|_2 \|wf')\|_2 + \left\| \frac{f}{w} \right\|_2 \|w'\|_2 \|f\|_\infty$

Proof. (1) We see that

$$\partial_a'(f^2) = 2\left( \frac{f}{w} \right)(wf')$$

Now we integrate and use Cauchy Schwarz to get the estimate.

(2) The $L^\infty$ estimate is obtained from the first estimate by observing that

$$\|f\|_2^2 \lesssim \left\| \frac{f}{w} \right\|_2 \|wf'\|_2 \lesssim \left\| \frac{f}{w} \right\|_2 \|wf')\|_2 + \left\| \frac{f}{w} \right\|_2 \|w'\|_2 \|f\|_\infty$$

Now use the inequality $ab \leq \frac{a^2}{2e} + \frac{eb^2}{2}$ on the last term to obtain the estimate. For the $H^\frac{1}{2}$ estimate, using $|\partial_a'| = iH\partial_a'$ we see that

$$\|f\|_{H^\frac{1}{2}}^2 \lesssim \int \left( \frac{\bar{f}}{w} \right) (wHf') da \lesssim \left\| \frac{f}{w} \right\|_2 \|wHf'\|_2$$

Now as $wHf' = [w, H]f' + H(wf')$ we have

$$s\|f\|_{H^\frac{1}{2}}^2 \lesssim \left\| \frac{f}{w} \right\|_2 (\|w'\|_2 \|f\|_\infty + \|wf'\|_2) \lesssim \left\| \frac{f}{w} \right\|_2 \|w'\|_2 \|f\|_\infty + \left\| \frac{f}{w} \right\|_2 \|wf')\|_2$$

Hence using the inequality $ab \leq \frac{a^2}{2e} + \frac{b^2}{2}$, we see that

$$\|f\|_{H^\frac{1}{2}}^2 \lesssim \left\| \frac{f}{w} \right\|_2 \|wf')\|_2 + \left\| \frac{f}{w} \right\|_2 \|w'\|_2^2 + \|f\|_\infty^2 \lesssim \left\| \frac{f}{w} \right\|_2 \|wf')\|_2 + \left\| \frac{f}{w} \right\|_2 \|w'\|_2^2$$

\[\square\]

Proposition 9.12. Let $f, g \in S(\mathbb{R})$ and let $w, h \in L^\infty(\mathbb{R})$ be smooth functions with $w', h' \in L^2(\mathbb{R})$. Then

$$\|fwh\|_{H^\frac{1}{2}} \lesssim \|fw\|_{H^\frac{1}{2}} \|h\|_\infty + \|f\|_2 \|(wh')\|_2 + \|f\|_2 \|w'\|_2 \|h\|_\infty$$

If in addition we assume that $w$ is real valued then

$$\|fgw\|_2 \lesssim \|fw\|_{H^\frac{1}{2}} \|g\|_2 + \|gw\|_{H^\frac{1}{2}} \|f\|_2 + \|f\|_2 \|g\|_2 \|w'\|_2$$

Proof. (1) We see that

$$|\partial_a'|^\frac{1}{2} (fwh) = [|\partial_a'|^\frac{1}{2}, h]fw + h|\partial_a'|^\frac{1}{2} (fw) = [|\partial_a'|^\frac{1}{2}, hw]f + h[|\partial_a'|^\frac{1}{2}, w]f + h|\partial_a'|^\frac{1}{2} (fw)$$

The estimate now follows from the estimate $\|(|\partial_a'|^\frac{1}{2}, g)\|_2 \lesssim \|\partial_a'|^\frac{1}{2} g \|_{BMO} \|f\|_2 \lesssim \|g'\|_2 \|f\|_2$
(2) We observe that

$$fgw = (P_H f)(P_H g)w + (P_H f)(P_A g)w + (P_A f)(P_H g)w + (P_A f)(P_A g)w$$

$$= (P_H f)(\overline{P_A g})w + (P_H f)(P_A g)w + (P_A f)(P_H g)w + (P_A f)(\overline{P_H g})w$$

We will control only the first term and the other terms are controlled similarly. Now observe that as

$$w$$

we see that

$$\text{Proposition 9.13. Let } f \in C^3([0, T), H^3(\mathbb{R})). \text{ Then for any } t \in [0, T) \text{ we have}$$

$$\lim_{s \to 0^+} \sup_s \frac{\|f(\cdot, t+s)\|_\infty - \|f(\cdot, t)\|_\infty}{s} \leq \|\partial_t f(a, t)\|_\infty$$

**Proof.** Fix $$s > 0$$ satisfying $$t + s \in [0, T)$$ and for every $$\epsilon > 0$$ we find $$a_\epsilon \in \mathbb{R}$$ such that

$$\|f(\cdot, t+s)\|_\infty \leq |f(a_\epsilon, t+s) + \epsilon.$$ Observe that $$|f(a_\epsilon, t)| \leq \|f(\cdot, t)\|_\infty$$ and hence we have

$$\|f(\cdot, t+s)\|_\infty - \|f(\cdot, t)\|_\infty \leq |f| a_\epsilon, t + s| + \epsilon$$

$$\leq |f(a_\epsilon, t + s) - f(a_\epsilon, t)| + \epsilon$$

$$\leq \sup_{a' \in \mathbb{R}} \sup_{u \in (0, s)} |\partial_t f(a', t + u)| s + \epsilon$$

Now let $$\epsilon \to 0$$ to get

$$\lim_{s \to 0^+} \sup_s \frac{\|f(\cdot, t+s)\|_\infty - \|f(\cdot, t)\|_\infty}{s} \leq \sup_{u \in (0, s)} \|\partial_t f(\cdot, t + u)\|_\infty$$

As $$\partial_t^2 f \in L^\infty(\mathbb{R} \times [0, T))$$, we take the limit as $$s \to 0$$ to finish the proof. \(\square\)
Lemma 9.14. Let $K_\epsilon$ be the Poisson kernel from (3). If $f \in L^q(\mathbb{R})$, then for $s \geq 0$ an integer we have

$$
\left\| (\partial_x^s f) * P_\epsilon \right\|_p \lesssim \| f \|_q \epsilon^{-s - \left(\frac{1}{q} - \frac{1}{p}\right)} \quad \text{for} \quad 1 \leq q \leq p \leq \infty
$$

Similarly for $s \in \mathbb{R}$, $s \geq 0$ we have

$$
\left\| (\partial_x^s f) * P_\epsilon \right\|_p \lesssim \| f \|_q \epsilon^{-s - \left(\frac{1}{q} - \frac{1}{p}\right)} \quad \text{for} \quad 1 \leq q \leq p \leq \infty
$$

Proof. The proof follows from basic properties of convolution. \(\Box\)

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