ASYMPTOTIC EXPANSIONS FOR A CLASS OF SINGULAR INTEGRALS EMERGING IN NONLINEAR WAVE SYSTEMS

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We find asymptotic expansions as \( \nu \to 0 \) for integrals of the form

\[
\int_{\mathbb{R}^d} \frac{F(x)}{\omega^2(x) + \nu^2} \, dx
\]

where sufficiently smooth functions \( F \) and \( \omega \) satisfy natural assumptions on their behavior at infinity and all critical points of \( \omega \) in the set \( \{ \omega(x) = 0 \} \) are nondegenerate. These asymptotic expansions play a crucial role in analyzing stochastic models for nonlinear waves systems. We generalize a result of Kuksin that a similar asymptotic expansion occurs in a particular case where \( \omega \) is a nondegenerate quadratic form of signature \((d/2, d/2)\) with even \( d \).

Keywords: singular integral, asymptotic analysis, wave turbulence, nonlinear waves system

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1. Introduction

1.1. Set up and result. We study the asymptotic behavior of integrals

\[
\int_{\mathbb{R}^d} \frac{F(x)}{\omega^2(x) + \nu^2} \, dx \tag{1.1}
\]

as \( \nu \to 0 \), where \( dx = dx_1 \ldots dx_d \), \( d \geq 2 \), and \( \Gamma, F, \) and \( \omega \) are sufficiently smooth real-valued functions whose behavior at infinity satisfies natural assumptions formulated below and \( \Gamma \) is a strictly positive function. We assume that the function \( \omega \) has only nondegenerate critical points on the set \( \Sigma \cap \text{supp} F \), where

\[
\Sigma = \{ x \in \mathbb{R}^d : \omega(x) = 0 \}, \tag{1.2}
\]

and that the number of these critical points is finite.

Integrals (1.1) arise in physical and mathematical studies on the wave turbulence theory. Their singular limits as \( \nu \to 0 \) describe the behavior of certain physical characteristics studying which is the objective of the theory (see Sec. 1.4 for a more detailed discussion). In physical studies, integrals (1.1) typically appear implicitly and become explicit when rigorously analyzing the heuristic constructions used there.

Dividing the numerator and denominator by \( \Gamma^2 \), we see that it suffices to study the case \( \Gamma(x) \equiv 1 \), i.e., integrals of the form

\[
I_\nu = \int_{\mathbb{R}^d} \frac{F(x)}{\omega^2(x) + \nu^2} \, dx.
\]

Moreover, we assume without loss of generality that \( \omega \) has at most one critical point on the set \( \Sigma \cap \text{supp} F \), where \( \Sigma \) is defined in (1.2), and this point is \( x = 0 \).

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Our main result is as follows. We set \( \Sigma^0 = \Sigma \setminus \{0\} \) if \( x = 0 \) is a critical point of \( \omega \) and \( \Sigma^0 = \Sigma \) otherwise. The set \( \Sigma^0 \) is a differentiable manifold of dimension \( d - 1 \). We let \( d_{\Sigma^0} \) denote the volume element on \( \Sigma^0 \) induced from \( \mathbb{R}^d \) endowed with the standard Euclidean structure and consider the integral

\[
I_0 = \pi \int_{\Sigma^0} \frac{F(x)}{|\nabla \omega(x)|} d_{\Sigma^0} x,
\]

(1.3)

where \( |\nabla \omega| \) denotes the Euclidean norm of the gradient of the function \( \omega \). Corollary 2.2 implies that the integral \( I_0 \) converges under Assumptions A1–A4 formulated below.

For \( r > 0 \) and \( a, b \in \mathbb{R} \), we set

\[
\chi_{a,b}(r) = \begin{cases} 
1, & a \neq b, \\
|\ln r|, & a = b.
\end{cases}
\]

(1.4)

**Theorem 1.1.** The following statements hold.

1. Assume that \( x = 0 \) is a unique critical point of the function \( \omega \) on the set \( \Sigma \cap \text{supp} \ F \), this critical point is nondegenerate, and \( d \geq 4 \). Under Assumptions A1–A5 formulated below, we then have

\[
|I_\nu - \nu^{-1}I_0| \leq C_{\chi,d,A}(\nu),
\]

(1.5)

where the constant \( C \) is independent of \( 0 < \nu \leq 1/2 \).

2. Assume that the function \( \omega \) has no critical points on the set \( \Sigma \cap \text{supp} \ F \) and \( d \geq 2 \). Under Assumptions A1–A5, we then have \( |I_\nu - \nu^{-1}I_0| \leq C \).

We prove only item 1 of the theorem because the proof of item 2 can be obtained by simplifying the proof of item 1. We impose the restriction \( d \geq 4 \) because some integrals involved in the analysis of \( I_\nu \) strongly diverge at the critical point \( x = 0 \) (see, e.g., (3.18)).

In the case where \( d = 2n, n \geq 2 \), and \( \omega \) is a nondegenerate quadratic form with index \( (n,n) \), an analogue of Theorem 1.1 for integral (1.1) was proved in [1]. Our argument follows a scheme suggested in that paper, but we encounter additional difficulties. In [1], the set \( \Sigma \) was a cone, but in our case its geometry can be much more complicated. In particular, this results in the lack of the explicit formulas and constructions employed in [1].

In Sec. 6 in [1], it is shown that the optimal upper bound for \( I_\nu \) and related integrals can be obtained by the stationary phase method, and the proof of this bound is short. But the asymptotic form of \( I_\nu \) is inconsistent with this type of methods, as is explained in Sec. 3 in [2], where a class of rapidly oscillating integrals was studied by the abstract stationary phase method. This can be seen by observing that the leading term (1.3) of the asymptotic expansion depends not only on values of the functions \( \omega \) and \( F \) and their derivatives at the critical point \( x = 0 \) but also on their restrictions to the whole manifold \( \Sigma^0 \).

In the next subsection, we formulate the assumptions imposed on \( F \) and \( \omega \), and in Sec. 1.3 we consider an example where \( \omega \) is a quadratic polynomial, which includes the case studied in [1]. In Sec. 1.4, we explain our motivation for this study by mapping our results into the context of the wave turbulence theory.

The rest of the paper is devoted to the proof of Theorem 1.1. In Sec. 2, we study a certain neighborhood \( U_\Theta(\Sigma^0) \) of \( \Sigma^0 \), introduce appropriate coordinates there, write the volume element in these coordinates, and then study the behavior of various integrals over \( \Sigma^0 \). In Sec. 3, we write the integral \( I_\nu \) as a sum of three integrals, the first of which is taken over a small ball around zero, the second over the complement to the neighborhood \( U_\Theta(\Sigma^0) \), and the third over \( U_\Theta(\Sigma^0) \). We show that the first two integrals are negligible when \( \nu \to 0 \) and, using the results in Sec. 2, prove that the behavior of the third integral is governed by the sought asymptotic form.
1.2. Assumptions. For $x \in \mathbb{R}^n$, we use the notation

$$\langle x \rangle := \max(1, |x|),$$

where $|x|$ denotes the Euclidean norm of $x$.

**Assumption A1.** The function $F$ is $C^2$-smooth and for a constant $M_F \in \mathbb{R}$ satisfies

$$|\partial^\alpha F(x)| \leq C \langle x \rangle^{-M_F - |\alpha|_1}$$

for any $0 \leq |\alpha|_1 \leq 2$ and any $x \in \mathbb{R}^d$, where $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_j \geq 0$, is a multi-index and $|\alpha|_1 := \alpha_1 + \cdots + \alpha_d$.

**Assumption A2.** The function $\omega$ is $C^4$-smooth and for a constant $M_\omega \in \mathbb{R}$ satisfies

$$|\partial^\alpha \omega(x)| \leq C \langle x \rangle M_\omega^{-|\alpha|_1}$$

for any $0 \leq |\alpha|_1 \leq 4$ and any $x \in \text{supp} F$.

**Assumption A3.** There is a constant $m_\omega \in \mathbb{R}$ such that

$$|\nabla \omega(x)| \geq C \langle x \rangle^{m_\omega}$$

for any $x \in \Sigma \cap \text{supp} F$ satisfying $|x| \geq 1$. (1.6)

If $x = 0$ is a nondegenerate critical point of $\omega$, we clearly have

$$|\nabla \omega(x)| \geq C |x|$$

for $|x| \leq 1$. (1.7)

**Assumption A4.** The constants $M_F$, $M_\omega$, and $m_\omega$ in Assumptions A1–A3 satisfy

$$M_F > \max(M_{cr}, 2M_{cr}) + d \quad \text{where} \quad M_{cr} := M_\omega - 2m_\omega - 2.$$

Clearly, by Assumptions A2 and A3, we also have

$$m_\omega \leq M_\omega - 1,$$

if the set $\Sigma \cap \text{supp} F$ is unbounded. Otherwise, the parameter $m_\omega$ can be chosen arbitrarily, and we choose it such that (1.8) is satisfied.

We let $D_\kappa$, with $\kappa > 0$, denote the set of points $|x| \geq 1$ “well separated” from the set $\Sigma_0$:

$$D_\kappa = \{x \in B_1^c: |x - \Sigma_0| \geq \kappa |x|^{m_\omega + 2 - M_\omega}\},$$

where $B_1$ is the closed unit ball centered at zero and $B_1^c$ is its complement, while $|x - \Sigma_0|$ denotes the Euclidean distance from $x$ to $\Sigma_0$.

**Assumption A5.** For any $\kappa > 0$, the integral

$$\int_{D_\kappa} \frac{|F(x)|}{\omega^2(x)} \, dx$$

converges.

Assumption A5 is rather implicit, and we therefore give a sufficient condition for it.
Lemma 1.1. Assume that Assumptions A1–A3 hold with \( m_\omega = M_\omega - 1 \) and \( M_F > d - 2M_\omega \), and that inequality (1.6) holds for any \( x \in B_1^n \) (not only for \( x \in \Sigma \cap \text{supp} F \)). Then Assumption A5 holds.

Proof. We first show that \(|\omega(x)| \geq C(\kappa)|x|^{-M_F}\) for any \( x \in D_\kappa \). We argue by contradiction, supposing that for any \( \varepsilon > 0 \), there is \( x_\varepsilon \in D_\kappa \) such that \(|\omega(x_\varepsilon)| \leq \varepsilon|x_\varepsilon|^{-M_F}\). We claim that in this case there is \(|\gamma'| < C_\varepsilon|x_\varepsilon|^{-m_\omega}\) such that \( y_\gamma = x_\varepsilon + \gamma \nabla \omega(x_\varepsilon)\) satisfies \( y_\gamma \in \Sigma \) if \( \varepsilon \) is sufficiently small. Indeed, setting \( \Omega = \text{Hess} \omega \) and using Taylor’s formula, we find

\[
\omega(y_\gamma) = \omega(x_\varepsilon) + |\nabla \omega(x_\varepsilon)|^2 + \frac{\gamma^2}{2} \left( \Omega(x_\varepsilon + \gamma \nabla \omega(x_\varepsilon)) \nabla \omega(x_\varepsilon) \right),
\]

where \(|\gamma'| \leq |\gamma|\). We assume for definiteness that \( \omega(x_\varepsilon) > 0 \) and take \( \gamma := -c_0 \varepsilon|x_\varepsilon|^{-m_\omega} \) with \( c_0 > 0 \) independent of \( \varepsilon \). Then, using Assumptions A3 and A2, we find

\[
\begin{align*}
\omega(y_\gamma) &\leq \varepsilon|x_\varepsilon|^{-M_F} - C_1 c_0 \varepsilon|x_\varepsilon|^{-m_\omega + 2m_\omega} + C_2 \varepsilon^2 + \varepsilon|x_\varepsilon|^{-2m_\omega + 2(M_\omega - 1)}(x_\varepsilon + \gamma \nabla \omega(x_\varepsilon))^{M_\omega - 2} = \\
&= \varepsilon|x_\varepsilon|^{-M_F} (1 - C_1 c_0 + C_2 \varepsilon^2 - \varepsilon|x_\varepsilon|^{-2m_\omega} (x_\varepsilon + \gamma \nabla \omega(x_\varepsilon))^{M_\omega - 2}).
\end{align*}
\]

Because

\[
(1 - C_1 c_0) |x_\varepsilon| \leq |x_\varepsilon + \gamma \nabla \omega(x_\varepsilon)| \leq (1 + C_1 c_0) |x_\varepsilon|,
\]

for \( c_0 = 2/C_1 \) and \( \varepsilon \leq \varepsilon_0(C, C_1, C_2, M_\omega) \) sufficiently small, we have \( \omega(y_\gamma) < 0 \). Consequently, there is \(|\gamma'| < |\gamma| = c_0 \varepsilon|x_\varepsilon|^{-m_\omega}\) such that \( y_\gamma \in \Sigma \). This contradicts the inclusion \( x_\varepsilon \in D_\kappa \) because

\[
|x_\varepsilon - y_\gamma| < |\gamma \nabla \omega(x_\varepsilon)| < \varepsilon C|x_\varepsilon| < \kappa|x_\varepsilon|
\]

if \( \varepsilon \) is sufficiently small, but in the definition of \( D_\kappa \) we have \( m_\omega + 2 - M_\omega = 1 \).

We have seen that \(|\omega(x)| \geq C(\kappa)|x|^{-M_F}\) for \( x \in D_\kappa \). Then

\[
\int_{D_\kappa} \frac{|F(x)|}{\omega^2(x)} \, dx \leq C(\kappa) \int_{\mathbb{R}^d} (x)^{-M_F - 2M_\omega} \, dx < \infty,
\]

because \( M_F > d - 2M_\omega \).

1.3. Example. In this subsection, we apply Theorem 1.1 to the situation where the set \( \Sigma \) is a quadric, a particular case of which was considered in [1]. Let

\[
q(x) = \frac{1}{2} x \cdot B x + a, \quad x \in \mathbb{R}^d,
\]

where \( a \in \mathbb{R} \) and \( B \) is a nondegenerate \( d \times d \) matrix with \( d \geq 2 \) if \( a \neq 0 \) and \( d \geq 4 \) if \( a = 0 \).\(^1\) We consider the integral

\[
J_\nu = \int_{\mathbb{R}^d} \frac{G(x)}{q^2(x) + \nu^2 \Gamma^2(x)} \, dx,
\]

where the real-valued functions \( G \) and \( \Gamma \) are \( C^2 \)- and \( C^4 \)-smooth, satisfying

\[
|\partial^\alpha G(x)| \leq C(\alpha)^{-M_G - |\alpha|_1} \quad \text{with} \quad 0 \leq |\alpha|_1 \leq 2,
\]

\[
\Gamma(x) \geq C^{-1}(x)^{r^*} \quad \text{and} \quad |\partial^\alpha \Gamma(x)| \leq C(x)^{r^* - |\alpha|_1} \quad \text{with} \quad 0 \leq |\alpha|_1 \leq 4,
\]

\(^1\)The case where the quadratic polynomial \( q \) contains a linear term \( l \cdot x, l \in \mathbb{R}^d \), reduces to the present one with \( l = 0 \) by the transformation \( x \mapsto x + B^{-1}l \).
The real constants $M_G$ and $r_*$ satisfy

$$M_G + r_* > d - 2, \quad M_G > d - 4. \tag{1.11}$$

Let

$$J_0(\nu) = \pi \int_{\Sigma_0} \frac{G(x)}{|F(x)|Bx} d_{\Sigma}x,$$

where $\Sigma, \Sigma_0,$ and $d_{\Sigma}x$ are defined as before with the function $\omega$ replaced by $q$.

**Corollary 1.1.** Under Assumptions (1.10) and (1.11),

$$|J_{\nu} - \nu^{-1}J_0| \leq \begin{cases} C_{d,4}(\nu), & a = 0 \text{ and } d \geq 4, \\ C, & a \neq 0 \text{ and } d \geq 2. \end{cases}$$

In [1], the corollary was proved in the case $d = 2n$, $n \geq 2$, $a = 0$, and

$$B = \begin{pmatrix} 0 & \text{Id}_{n \times n} \\ \text{Id}_{n \times n} & 0 \end{pmatrix}. \tag{1.12}$$

Restrictions (1.11) on the parameters $M_G, r_*$, and $d$ coincide with those imposed in [1].

**Proof of Corollary 1.1.** It suffices to verify that Assumptions A1–A5 are satisfied for the functions $F = G/T^2$ and $\omega = q/T$. A simple computation shows that (1.10) implies Assumptions A1–A3 with $M_F = M_G + 2r_*$, $M_\omega = 2 - r_*$, and $m_\omega = 1 - r_*$. Then $M_{cr} = r_* - 2$, and hence (1.11) implies Assumption A4. Assumption A5 follows from Lemma 1.1. ■

1.4. Motivation. The wave turbulence theory was created in 1960s to study small-amplitude solutions of nonlinear Hamiltonian PDEs with a large spatial period. Since then, it has been intensively developing at a heuristic level of rigor [3], [4], and mathematical works devoted to its rigorous justification started appearing only several years ago (see [5]–[9] and the references therein). The central object in wave turbulence is a nonlinear kinetic equation, called the wave kinetic equation, that describes the behavior of certain physical characteristics of solutions of the underlying PDE. If the PDE describes the $N$-wave interaction, the $s$th component ($s \in \mathbb{R}^n$) of the kinetic kernel $K$ is given by an integral of the form

$$K_s = \int_{\mathbb{R}^{(N-1)n}} F_s(\xi, \sigma) \delta_{\sigma_1 \ldots \sigma_{q-1}s} \delta(\omega_s(\xi, \sigma)) d\xi_1 \ldots d\xi_p d\sigma_1 \ldots d\sigma_{q-1}, \tag{1.13}$$

where $\xi_i, \sigma_j \in \mathbb{R}^n$ and $p + q = N$ (see Secs. 6.9 and 6.11 in [4]). Here, $\delta_{\sigma_1 \ldots \sigma_{q-1}s}$ denotes the delta function $\{\xi_1 + \cdots + \xi_p = \sigma_1 + \cdots + \sigma_{q-1} + s\}$, while $\delta(\omega_s)$ is the delta function of

$$\omega_s(\xi, \sigma) = f(\xi_1) + \cdots + f(\xi_p) - f(\sigma_1) - \cdots - f(\sigma_{q-1}) - f(s), \tag{1.14}$$

where $f$ is the dispersion relation for the underlying PDE. The delta function $\delta_{\sigma_1 \ldots \sigma_{q-1}s}$ means that the integration is performed over a hyperspace $\approx \mathbb{R}^{(N-2)n}$, and the subsequent multiplication by the delta function $\delta(\omega_s)$ means that the integration is in fact over the set $\Sigma \subset \mathbb{R}^{(N-2)n}$ with respect to a measure proportional to $|\nu\omega_s|$, exactly as in (1.2) and (1.3) with $\omega = \omega_s$ (see [3]). For a rigorous mathematical treatment of delta functions such as $\delta(\omega_s)$, we refer the reader to [10] and Section III.1.3 in [11] (pp. 36, 37).
A possible approach to rigorous studies of wave turbulence is to add small viscosity and a small random force to the PDE. In this setting, nonlinearities of form (1.13) and, in particular, the kinetic kernel \( K \) above appear as limits when \( \nu \to 0 \) of the integrals (1.1) with \( \omega \) given by (1.14), which brings the study of these limits to the forefront.

In [7], following this stochastic approach, we rigorously derived the wave kinetic equation for a quasi-solution of the cubic NLS, describing the four-wave interaction. In that case, we had \( n \geq 2, p = q = 2, \) and \( f(s) = |s|^2 \) in (1.13). Expressing \( \sigma_1 \) in terms of \( \delta \sigma_1 \xi_2 \) and setting \( x = \xi_1 - s \) and \( y = \xi_2 - s \), we then obtained \( \omega_\sigma(x, y) = -2x \cdot y \), where \( x, y \in \mathbb{R}^n \), showing that \( \omega_\sigma \) is a quadratic form given by a matrix proportional to (1.12) and \( d = 2n \geq 4 \). As explained above, the analysis of the limit behavior as \( \nu \to 0 \) of the corresponding integral (1.1) played a crucial role in our research and was the subject of paper [1] cited in the foregoing.

As another example of the equation to which our result applies, we consider the Petviashvili equation, which describes the three-wave interaction. In book [4], most of the wave turbulence postulates are explained just with this example. In this case, we have \( n = 2, p = 2, q = 1, \) and \( f(s) = s^1|s|^2 \), where \( s = (s^1, s^2) \in \mathbb{R}^2 \). Taking the relation \( \xi_2 = s - \xi_1 \) into account, we obtain

\[
\omega_\sigma(\xi_1) = s^1|\xi_1|^2 - 2(\xi^1_1 - s^1)\xi_1 \cdot s - \xi^1_1|s|^2.
\]

Straightforward computation then shows that if \( 2s^1 \neq \pm |s| \), \( \omega_\sigma \) can be written in form (1.9) with \( a \neq 0 \), where \( d = n = 2 \).

To give an example where the function \( \omega_\sigma \) is not a quadratic polynomial, we suggest, e.g., the Charney–Hassegawa–Mima equation, which is a model for planetary Rossby waves and drift waves in inhomogeneous plasmas. This equation also describes three-wave interaction, but its dispersion relation is \( f(s) = -\beta \rho s^1/(1 + \rho^2|s|^2) \), where \( \beta \) and \( \rho \) are parameters of the system and, again, \( s = (s^1, s^2) \in \mathbb{R}^2 \) (see Secs. 7.6 and 13.2.1 in [4]).

The mathematical theory of wave turbulence has been intensively developing in the last several years, and we believe that more wave systems will be studied rigorously in the nearest future using asymptotic expansions of integrals like (1.1).

2. The manifold \( \Sigma_0 \) and its neighborhood

In this section, we let \( C, C_1, \ldots \) denote various constants that never depend on \( \nu \) and \( \Theta \) (the parameter \( \Theta \) introduced below must be sufficiently small). These constants may change from line to line.

We recall that we give the proof in the case \( d \geq 4 \), assuming that \( x = 0 \) is a unique critical point of the function \( \omega \) on the set \( \Sigma \cap \text{supp} F \).

2.1. Neighborhood of \( \Sigma_0 \). We choose differentiable coordinates \( \xi \in \mathbb{R}^{d-1} \) on the manifold \( \Sigma_0 \) and consider the coordinate functions \( \xi \mapsto x_\xi \in \Sigma_0 \). Abusing notation, we often write \( \xi \in \Sigma_0 \). We set

\[
N_\xi = \nabla \omega(x_\xi).
\]

The vector \( N_\xi \) is orthogonal to \( \Sigma_0 \) at the point \( \xi \). We consider the map \( \pi: \Sigma_0 \times \mathbb{R} \mapsto \mathbb{R}^d \) defined as

\[
\pi(\xi, \theta) = x_\xi + \theta N_\xi.
\]  

Let \( 0 < \Theta \leq 1 \) and

\[
U_\Theta(\Sigma_0) = \{ x = \pi(\xi, \theta): \xi \in \Sigma_0, \ |\theta| < \theta_\xi(\Theta) \},
\]
where

\[ \theta_\xi = \frac{\theta(\Theta)}{\langle x_\xi, M \rangle^{\xi}}. \]  

(2.2)

To recall, \( M_\omega \) is defined in Assumption A2. In what follows, we assume \( \Theta \) to be sufficiently small but fixed and independent of \( \nu \).

We use the notation \( \Omega(x) = \text{Hess} \omega(x) \) for the Hessian of \( \omega \) and let \( \| \Omega \| \) denote the operator norm of \( \Omega \). Assumption A2 implies the following result.

**Lemma 2.1.** The following estimates hold.

1. There is a constant \( C > 0 \) such that for any \( \xi \in \Sigma_0 \),

\[ \theta_\xi |N_\xi| \leq C\Theta |x_\xi|, \]  

(2.3a)

\[ \theta_\xi \| \Omega(x_\xi) \| \leq C\Theta. \]  

(2.3b)

2. There is a constant \( C > 0 \) such that for any \( x \in U_{\Theta}(\Sigma_0) \) given by \( x = \pi(\xi, \theta) = x_\xi + \theta N_\xi \), we have

\[ (1 - C\Theta)|x_\xi| \leq |x| \leq (1 + C\Theta)|x_\xi|. \]

**Proof.** 1. Assumption A2 and Eq. (2.2) immediately imply estimate (2.3b) and the relation \( \theta_\xi |N_\xi| \leq C\Theta |x_\xi| \). The latter implies estimate (2.3a) for \( |x_\xi| \geq 1 \), while for \( |x_\xi| < 1 \), the estimate follows from the relation \( \nabla \omega(0) = 0 \) and \( \theta_\xi = \Theta \).

2. In view of (2.3a),

\[ |x| \leq |x_\xi| + \theta_\xi |N_\xi| \leq (1 + C\Theta)|x_\xi|, \]

\[ |x_\xi| \leq |x| \leq |x_\xi| + \theta_\xi |N_\xi| \leq |x| + C\Theta |x_\xi|, \]

which implies the desired inequality. \( \blacksquare \)

We now show that \((\xi, \theta)\) are coordinates on the set \( U_{\Theta}(\Sigma_0) \) and write the function \( \omega \) in these coordinates.

**Proposition 2.1.** The following statements hold.

1. For any sufficiently small \( \Theta \), the set \( U_{\Theta}(\Sigma_0) \) is uniquely parameterized by coordinates \( \{ (\xi, \theta) : \xi \in \Sigma_0, |\theta| < \theta_\xi \} \) via formula (2.1).

2. Let \( x = \pi(\xi, \theta) \in U_{\Theta}(\Sigma_0) \). Then

\[ \omega(x) = |N_\xi| g_\xi(\theta), \]  

(2.4)

where the function \( \theta \mapsto g_\xi(\theta) \) is \( C^2 \)-smooth and satisfies \( g_\xi(0) = 1 \),

\[ |g_\xi(\theta) - 1| \leq C\Theta, \quad |g_\xi'(\theta)| \leq C\langle x_\xi \rangle^{M_\omega - 2}, \quad |g_\xi''(\theta)| \leq C\langle x_\xi \rangle^{2(M_\omega - 2)}. \]  

(2.5)

**Proof.** 1. We need to show that if \( \Theta \) is sufficiently small, then for any \( \xi_1 \in \Sigma_0 \) and \( |\theta_1| < \theta_\xi_1(\Theta) \), \( i = 1, 2 \), satisfying \( \pi(\xi_1, \theta_1) = \pi(\xi_2, \theta_2) \), we have \( x_{\xi_1} = x_{\xi_2} \) and \( \theta_1 = \theta_2 \). It follows from (2.1) that

\[ x_{\xi_1} - x_{\xi_2} = \theta_2 N_{\xi_2} - \theta_1 N_{\xi_1}. \]  

(2.6)

Taking the scalar product of both sides of this equation with \( x_{\xi_1} - x_{\xi_2} \) yields

\[ |x_{\xi_1} - x_{\xi_2}|^2 = \theta_2 \langle N_{\xi_2}, x_{\xi_1} - x_{\xi_2} \rangle - \theta_1 \langle N_{\xi_1}, x_{\xi_1} - x_{\xi_2} \rangle. \]  

(2.7)
We recall that \( N_\xi_i = \nabla \omega(x_{\xi_i}) \). Because \( \omega(x_{\xi_1}) = \omega(x_{\xi_2}) = 0 \), Taylor’s formula applied to the function \( \omega \) at the point \( x_{\xi_2} \) implies that

\[
\langle N_{\xi_2}, x_{\xi_1} - x_{\xi_2} \rangle \leq \frac{1}{2} \max_{x \in [x_{\xi_1}, x_{\xi_2}]} \|\Omega(x)\| \| x_{\xi_1} - x_{\xi_2} \|^2,
\]

where \([x_{\xi_1}, x_{\xi_2}]\) denotes the segment in \( \mathbb{R}^d \) connecting \( x_{\xi_1} \) and \( x_{\xi_2} \). Clearly, the same estimate holds for the scalar product \( \langle N_{\xi_1}, x_{\xi_1} - x_{\xi_2} \rangle \). Thus, Eq. (2.7) implies

\[
|x_{\xi_1} - x_{\xi_2}|^2 \leq \frac{\theta_{\xi_1} + \theta_{\xi_2}}{2} \max_{x \in [x_{\xi_1}, x_{\xi_2}]} \|\Omega(x)\| \| x_{\xi_1} - x_{\xi_2} \|^2.
\]  

(2.8)

We claim that for any \( x \in [x_{\xi_1}, x_{\xi_2}] \) and \( i = 1, 2 \),

\[
C^{-1}|x_{\xi_i}| \leq |x| \leq C|x_{\xi_i}|
\]  

(2.9)

(with \( i = 1, 2 \)). Together with Assumption A2, this then implies that the right-hand side of (2.8) is bounded by \( C\Theta|x_{\xi_1} - x_{\xi_2}|^2 \), and hence for \( \Theta < C^{-1} \) we have \( x_{\xi_1} = x_{\xi_2} \). Consequently, \( \theta_1 = \theta_2 \), which completes the proof.

It remains to establish (2.9). In view of (2.3a), Eq. (2.6) implies

\[
|x_{\xi_i} - x_{\xi_2}| \leq |\theta_2 N_{\xi_2}| + |\theta_1 N_{\xi_1}| \leq C\Theta(|x_{\xi_1}| + |x_{\xi_2}|),
\]  

(2.10)

whence

\[
C^{-1}|x_{\xi_i}| \leq |x_{\xi_i}| \leq C|x_{\xi_i}|,
\]  

(2.11)

if \( \Theta \) is sufficiently small. We assume for definiteness that \( |x_{\xi_2}| \geq |x_{\xi_1}| \). Any \( x \in [x_{\xi_1}, x_{\xi_2}] \) satisfies

\[
|x_{\xi_2}| - |x_{\xi_1} - x_{\xi_2}| \leq |x| \leq |x_{\xi_1}| + |x_{\xi_2}|.
\]

By (2.10) and (2.11), we then obtain \( C^{-1}|x_{\xi_2}| \leq |x| \leq C|x_{\xi_2}| \), which implies that (2.9) holds.

2. We apply Taylor’s formula at \( \theta = 0 \) to the function \( \theta \mapsto \omega_\xi(\theta) := \omega(\pi(\xi, \theta)) \). For this, we compute the derivatives

\[
\omega'_\xi(0) = \frac{d}{d\theta} \bigg|_{\theta=0} \omega(x_{\xi} + \theta N_{\xi}) = \langle \nabla \omega(x_{\xi}), N_{\xi} \rangle = |N_{\xi}|^2
\]

and \( \omega''_\xi(\theta) = \langle \Omega(x_{\xi} + \theta N_{\xi}) N_{\xi}, N_{\xi} \rangle \). Because \( \omega_\xi(0) = 0 \), we find

\[
\omega_\xi(\theta) = \theta |N_{\xi}|^2 + \int_0^\theta \langle \Omega(x_{\xi} + t N_{\xi}) N_{\xi}, N_{\xi} \rangle (\theta - t) \, dt.
\]

We thus arrive at (2.4) with

\[
g_\xi(\theta) = 1 + \int_0^\theta \langle \Omega(x_{\xi} + t N_{\xi}) n_{\xi}, n_{\xi} \rangle \frac{\theta - t}{\theta} \, dt = 1 + \theta \int_0^1 \langle \Omega(x_{\xi} + s \theta N_{\xi}) n_{\xi}, n_{\xi} \rangle (1 - s) \, ds,
\]

where \( n_{\xi} = N_{\xi}/|N_{\xi}| \). Because \( \omega \) is \( C^4 \)-smooth, the function \( g_\xi \) is \( C^2 \)-smooth.

Let \( B_\xi = [x_{\xi} - \theta_2 N_{\xi}, x_{\xi} + \theta_2 N_{\xi}] \). Using Assumption A2 and then Lemma 2.1(2), we obtain the inequality

\[
|g_\xi(\theta) - 1| \leq \theta_2 \max_{x \in B_\xi} \|\Omega(x)\| \leq \frac{C}{\langle x_{\xi}\rangle^{M_\omega-2}} \max_{x \in B_\xi} \langle x \rangle^{M_\omega-2} \leq C_1 \Theta,
\]

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which is the first inequality in (2.5). Next,

\[ g'(\theta) = \int_0^1 \langle \Omega(x_\xi + s\theta N_\xi)n_\xi, n_\xi \rangle (1 - s) \, ds + \theta \int_0^1 \left( \sum_{i=1}^d \partial_{x_i} \Omega(x_\xi + s\theta N_\xi)n_\xi, n_\xi \right) (N_\xi), s(1 - s) \, ds. \]

Using Assumption A2 and Lemma 2.1(2) again, we find

\[ |g'(\theta)| \leq C \left( \langle x_\xi \rangle^{M_\omega - 2} + \frac{\Theta}{\langle x_\xi \rangle^{M_\omega - 2}} \langle x_\xi \rangle^{M_\omega - 3} \langle x_\xi \rangle^{M_\omega - 1} \right) = C_1 \langle x_\xi \rangle^{M_\omega - 2}, \]

which is the second inequality in (2.5). The third one follows similarly.

Next we write the volume element \( dx \) in the coordinates \((\xi, \theta)\).

**Proposition 2.2.** On the set \( U_\Theta(\Sigma_0) \subset \mathbb{R}^d \), the volume element \( dx \) written in the coordinates \((\xi, \theta)\) takes the form

\[ dx = |N_\xi| \mu_\xi(\theta) \, d\theta \, m(d\xi), \]  

(2.12)

where \( m(d\xi) \) denotes the volume element on \( \Sigma_0 \) and the density function \( \theta \mapsto \mu_\xi(\theta) \) is a polynomial of degree \( d - 1 \) satisfying \( \mu_\xi(0) = 1 \) and

\[ \mu_\xi(\theta) \geq C^{-1} > 0, \quad \left| \frac{d^p}{d\theta^p} \mu_\xi(\theta) \right| \leq C \langle x_\xi \rangle^{p(M_\omega - 2)} \]

(2.13)

for all \( \xi \in \Sigma_0 \), \( |\theta| < \theta_\xi \), and \( 0 \leq p \leq d - 1 \), if \( \Theta \) is sufficiently small.

**Proof.** We consider the projection \( \Pi : U_\Theta(\Sigma_0) \mapsto \Sigma_0 \) that maps \( x = \pi(\xi, \theta) \) into \( x_\xi = \pi(\xi, 0) \),

\[ \Pi(x) = x_\xi. \]  

(2.14)

The kernel of the differential \( d\Pi(x) \), taken at a point \( x = \pi(\xi, \theta) \), is given by the linear span of the vector \( N_\xi \).

Let \( J_{\Pi}(x) \) be the Jacobian of the linear map \( d\Pi(x) \) restricted to the orthogonal complement \((\text{span}(N_\xi))^\perp\).

According to the coarea formula [12], for any Lebesgue-measurable set, we have \( A \subset U_\Theta(\Sigma_0) \),

\[ \int_A J_{\Pi}(x) \, dx = \int_{\Sigma_0} \text{Leb}(A \cap \Pi^{-1}(x_\xi)) \, dm(\xi), \]

where \( \text{Leb}(\cdot) \) is the Lebesgue measure on the set

\[ \Pi^{-1}(x_\xi) = \{ x = \pi(\xi, \theta) : |\theta| \leq \theta_\xi \}. \]

Using the fact that

\[ \text{Leb}(A \cap \Pi^{-1}(x_\xi)) = |N_\xi| \text{Leb}(\{ \theta : \pi(\xi, \theta) \in A \}), \]

we find \( J_{\Pi}(x) \, dx = |N_\xi| \, d\theta \, dm(\xi) \). Thus, Eq. (2.12) holds with

\[ \mu_\xi(\theta) = \frac{1}{J_{\Pi}(\xi, \theta)}, \quad \text{where} \quad J_{\Pi}(\xi, \theta) := J_{\Pi}(\pi(\xi, \theta)). \]

The analysis of the density function \( \mu \) relies on the following lemma.
Lemma 2.2. We have
\[ J_\Pi(\xi, \theta) = \left( \det(\text{Id}_{d-1} + \theta \Omega_{d-1}(x_\xi)) \right)^{-1}, \]
where \( \text{Id}_{d-1} \) denotes the \((d - 1) \times (d - 1)\) identity matrix and \( \Omega_{d-1}(x_\xi) \) is the matrix obtained from the Hessian \( \Omega(x_\xi) \) by deleting its last row and column, assuming that \( \Omega(x_\xi) \) is written in terms of any orthonormal basis whose last vector coincides with \( N_\xi || N_\xi \).

Proof. We fix \( x = \pi(\xi, \theta) \) and a vector \( v \perp N_\xi \), and take \( t \in \mathbb{R} \) so small that \( x + vt \in U_\Theta(\Sigma_0) \). Then \( x + vt = \pi(\xi(t), \theta(t)) \) for appropriate \( \xi(t) \) and \( \theta(t) \) satisfying \( \xi(0) = \xi \) and \( \theta(0) = \theta \). In other words, \( \pi(\xi, \theta) + vt = \pi(\xi(t), \theta(t)) \), or, in more detail,
\[ \Pi(x) + \theta N_\xi + vt = \Pi(x + vt) + \theta(t)N_{\xi(t)}. \]
Differentiating the last equation with respect to \( t \) at \( t = 0 \), we obtain
\[ v = d\Pi(x)v + \theta'(0)N_\xi + \theta(N_{\xi(t)})'_{t=0}. \tag{2.15} \]
Next,
\[ (N_{\xi(t)})'_{t=0} = \frac{d}{dt} \bigg|_{t=0} \nabla \omega(\Pi(x + vt)) = \Omega(x_\xi)(d\Pi(x)v). \]
Because the vectors \( v \) and \( d\Pi(x)v \) are orthogonal to \( N_\xi \), applying the projection \( \text{Pr} \) to the space \( N_\xi \perp \mathbb{R}^{d-1} \) in (2.15) gives
\[ (\text{Id}_{d-1} + \theta \text{Pr} \ast \Omega(x_\xi))d\Pi(x)v = v, \]
where \( v, d\Pi(x)v \in N_\xi \perp \) are viewed as \((d - 1)\)-vectors. In the basis used in the statement of the lemma, we have \( \text{Pr} \ast \Omega(x_\xi) = \Omega_{d-1}(x_\xi) \). In view of (2.3b), the operator \( \text{Id}_{d-1} + \theta \Omega_{d-1}(x_\xi) \) is invertible if \( \Theta \) is sufficiently small, whence
\[ d\Pi(x) = (\text{Id}_{d-1} + \theta \Omega_{d-1}(x_\xi))^{-1}, \quad J_\Pi(\xi, \theta) = \frac{1}{\det(\text{Id}_{d-1} + \theta \Omega_{d-1}(x_\xi))}. \]
The lemma is proved. \( \blacksquare \)

By Proposition 2.2,
\[ \mu_\xi(\theta) = \det(\text{Id}_{d-1} + \theta \Omega_{d-1}(x_\xi)). \]
Clearly, \( \mu_\xi(0) = 1 \). From relation (2.3b), we obtain the lower bound \( \mu_\xi(\theta) \geq C > 0 \) uniformly in \( \xi \in \Sigma_0 \) and \( |\theta| < \theta_\xi \) if \( \Theta \) is sufficiently small. Thus, it remains to obtain an estimate for the derivatives of \( \mu_\xi \). We write
\[ \mu_\xi(\theta) = \sum_{k=0}^{d-1} \theta^k P_k(x_\xi) \]
where the function \( P_l(x_\xi) \) is a homogeneous polynomial of degree \( l \) in the second derivatives \( \partial_i \partial_j \omega(x_\xi) \) (in particular, \( P_0(x_\xi) = 1 \) and \( P_1(x_\xi) = tr \Omega_{d-1}(x_\xi) \)). Then
\[ \frac{d^p}{d\theta^p} \mu_\xi(\theta) = \sum_{k=0}^{d-p-1} C_{k,p} \theta^k P_{k+p}(x_\xi), \]
and hence, by Assumption A2,
\[ \left| \frac{d^p}{d\theta^p} \mu_\xi(\theta) \right| \leq C \max_{0 \leq k \leq d-p-1} (\theta_\xi)^k (x_\xi)^{(k+p)(M_\omega-2)} \leq C_1 (x_\xi)^{p(M_\omega-2)}, \]
in accordance with (2.2). \( \blacksquare \)
We conclude this section with the following characterization of the set $U_\Theta(\Sigma_0)$.

**Lemma 2.3.** Let $\kappa = \kappa(\Theta)$ be sufficiently small. Then any $x \in \mathbb{R}^d$ satisfying

$$|x - \Sigma_0| \leq \kappa \min(|x|, |x|^{m_\omega+2-M_\omega})$$  \hspace{1cm} (2.16)

belongs to $U_\Theta(\Sigma_0)$.

**Proof.** We first prove that for any $x$ satisfying (2.16), we have

$$|x - \Sigma_0| = |x - x_\xi| \text{ for some } \xi \in \Sigma_0.$$  \hspace{1cm} (2.17)

Indeed, because the set $\Sigma$ is closed, $|x - \Sigma| = |x - y|$ for some $y \in \Sigma$. If $y \in \Sigma_0$, then the claim is true. Otherwise, $y = 0$ and hence $|x - \Sigma_0| = |x|$. But then $x$ does not satisfy (2.16) if $\kappa < 1$.

Relation (2.17) implies that for any $x$ satisfying (2.16), there is a $\xi \in \Sigma_0$ such that

$$x = x_\xi + \theta N_\xi \quad \text{where} \quad |\theta N_\xi| \leq \kappa \min(|x|, |x|^{m_\omega+2-M_\omega}).$$  \hspace{1cm} (2.18)

It remains to show that $|\theta| < \theta_\xi$. For this, we note that according to (2.18),

$$C^{-1}|x_\xi| \leq |x| \leq C|x_\xi|, \quad \text{uniformly in } 0 < \kappa \leq 1/2.$$  \hspace{1cm} (2.19)

We first assume that $|x| \geq 1$. Then $\min(|x|, |x|^{m_\omega+2-M_\omega}) = |x|^{m_\omega+2-M_\omega}$ by (1.8). Hence, by (2.18) and (2.19), $|\theta N_\xi| \leq C\kappa|x_\xi|^{m_\omega+2-M_\omega}$. Using Assumption A3, we then find $|\theta| \leq C\kappa|x_\xi|^{2-M_\omega} < \theta_\xi$ if $\kappa < C^{-1}\Theta$. In the case $|x| \leq 1$, we use the inequality $|\theta N_\xi| \leq C\kappa|x_\xi|$, which follows from (2.18) and (2.19). By (1.7), it then follows $|\theta| \leq C\kappa < \theta_\xi$ if $\kappa < C^{-1}\Theta$. \hfill \blacksquare

**Corollary 2.1.** Assumption A5 implies that

$$\int_{B^*_1 \setminus U_\Theta(\Sigma_0)} \frac{|F(x)|}{\omega^d(x)} \, dx < \infty.$$  

**Proof.** Using Lemma 2.3 together with (1.8), we conclude that if $\kappa$ is sufficiently small, then the domain $B^*_1 \setminus U_\Theta(\Sigma_0)$ is contained in the set $D_n$ from Assumption A5.

**2.2. Integrals over the manifold $\Sigma_0$.** We recall that $B_r$ denotes a closed ball in $\mathbb{R}^d$ of radius $r$ centered at zero. We set

$$R^b_a = B_b \setminus B_a, \quad 0 < a < b$$  \hspace{1cm} (2.20)

and also recall that the function $\chi_{a,b}(r)$ is defined in (1.4).

**Lemma 2.4.** The following statements hold.

1. Assume that $\mathbb{R} \ni \alpha n \leq d - 1$. Then for any $0 < \delta \leq 1/2$,

$$\int_{\Sigma_0 \cap R^b_a} |x_\xi|^{-\alpha n} m(d\xi) \leq C\chi_{d,n+1}(\delta).$$  \hspace{1cm} (2.21)

If $n < d - 1$, then

$$\int_{\Sigma_0 \cap B_\delta} |x_\xi|^{-\alpha n} m(d\xi) \leq C\delta^{d-1-n}.$$  \hspace{1cm} (2.22)

In particular, $\int_{\Sigma_0 \cap B_1} |x_\xi|^{-\alpha n} m(d\xi) < \infty$. 

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2. Assume that $\mathbb{R} \ni n > M_\omega - m_\omega - 2 + d$. Then

$$\int_{\Sigma_0 \setminus B_1} |x_\xi|^{-n} m(d\xi) < \infty.$$ 

**Proof.** We write

$$m(d\xi) = \frac{1}{2} \int_{-\theta_\xi}^{\theta_\xi} |N_\xi|^2 |\mu_\xi(\theta) d\theta m(d\xi)|.$$ 

Then, using (2.12), for any $A \subset \Sigma_0$ we find

$$J := \int_A |x_\xi|^{-n} m(d\xi) = \frac{1}{2} \int \frac{|x_\xi|^{-n}}{\theta_\xi |N_\xi| |\mu_\xi(\theta)|} dx,$$

where $(\xi, \theta) = \pi^{-1}(x)$ and the projection $\Pi$ is defined in (2.14).

1. To establish item 1, we set $A = \Sigma_0 \cap B_1$. By Lemma 2.1(2), $\Pi^{-1}(\Sigma_0 \cap R_1^\delta) \subset R_{\omega_0}^{c_1}$ for appropriate constants $c_0, c_1 > 0$. Then, using the inequality $\theta_\xi \geq C$ for $x_\xi \in R_{\omega_0}^{c_1}$, the estimate in (1.7), and the fact that $\mu_\xi(\theta) \geq C$, we obtain

$$J \leq C \Theta^{-1} \int_{R_{\omega_0}^{c_1}} |x|^{-n-1} dx \leq C_1 \Theta^{-1} \int_{R_{\omega_0}^{c_1}} |x|^{-n-1} dx = C_1 \Theta^{-1} \int_{c_0^\delta} r^{-n+d-2} dr,$$

which implies the first assertion in item 1.$^2$

Equality (2.22) follows by taking $A = \Sigma_0 \cap B_1$ and replacing $R_{\omega_0}^{c_1}$ with $B_{c_0^\delta}$ in the formula above.

2. We set $A = \Sigma_0 \setminus B_1$. Because $\Pi^{-1}(\Sigma_0 \setminus B_1) \subset B_r^\infty$ with some $r > 0$, it follows from Assumption A3 and Eq. (2.2) that

$$J \leq C \Theta^{-1} \int_{B_r^\infty} |x|^{-n+M_\omega-2-m_\omega} dx < \infty$$

if $-n + M_\omega - 2 - m_\omega < -d$. ■

**Corollary 2.2.** Let $g: \Sigma_0 \mapsto \mathbb{R}$ be a measurable function. Then the integral

$$\int_{\Sigma_0} \frac{g(x_\xi)}{|N_\xi|^k} m(d\xi), \quad 0 \leq k < 3,$$

converges if $|g(y)| \leq C(y)^{-n}$ with

$$n > M_\omega - (k+1)m_\omega - 2 + d. \quad (2.23)$$

If (2.23) holds with $k = 3$, then

$$\left| \int_{\Sigma_0 \setminus B_\delta} \frac{g(x_\xi)}{|N_\xi|^3} m(d\xi) \right| \leq C_{\chi, d, 4}(\delta) \quad \text{for} \quad 0 < \delta \leq \frac{1}{2}$$

**Proof** follows immediately from Lemma 2.4, Assumption A3, and Eq. (1.7) together with the restriction $d \geq 4$.

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$^2$The constant $C$ in (2.21) is independent of $\Theta$ because the integral in (2.21) is independent of $\Theta$ (and the argument above works with any sufficiently small but fixed $\Theta$).
3. The integral $I_\nu$

In this section, we prove Theorem 1.1. We fix a sufficiently small $\Theta$ and allow the constants $C, C_1, \ldots$ to depend on $\Theta$ (but not on $\nu$). For a domain $W \subset \mathbb{R}^d$, we let $\langle I_\nu, W \rangle$ denote the integral $I_\nu$ taken not over $\mathbb{R}^d$ but over $W$:

$$\langle I_\nu, W \rangle := \int_W \frac{F(x)}{\omega^2(x) + \nu^2} \, dx.$$  \hfill (3.1)

We set $\delta := \alpha \sqrt{\nu}$, where $\alpha = \alpha(\Theta) > 0$ is a $\nu$-independent constant, which we choose later (typically, it is a large constant). We write the integral $I_\nu$ as the sum

$$I_\nu = \langle I_\nu, B_\delta \rangle + \langle I_\nu, B_\delta^c \setminus U_\Theta(\Sigma_0) \rangle + \langle I_\nu, U_\Theta(\Sigma_0) \setminus B_\delta \rangle$$  \hfill (3.2)

and show that the sum of the first two integrals is bounded by $C\chi_{d,4}(\nu)$, and the third one leads to the desired asymptotic expression.

### 3.1. The first and second integrals in (3.2).

For the first integral in (3.2), we use the trivial estimate

$$|\langle I_\nu, B_\delta \rangle| \leq C\delta^d \nu^{-2} = C\alpha^d \nu^{d/2-2} \leq C_1,$$  \hfill (3.3)

which holds because $d \geq 4$.

To analyze the second integral, we take $0 < \hat{\delta} < 1$ sufficiently small but independent of $\nu$ (see below), and write

$$\langle I_\nu, B_\delta^c \setminus U_\Theta(\Sigma_0) \rangle = \langle I_\nu, B_1^c \setminus U_\Theta(\Sigma_0) \rangle + \langle I_\nu, R_\delta^1 \setminus U_\Theta(\Sigma_0) \rangle + \langle I_\nu, R_\delta^3 \setminus U_\Theta(\Sigma_0) \rangle,$$

where we recall notation (2.20). According to Corollary 2.1, the first summand above is bounded by a constant. Because the set $R_\delta^1 \setminus U_\Theta(\Sigma_0)$ is separated from $\Sigma_0$, is bounded, and is independent of $\nu$, the same bound is also true for the second summand.

To analyze the third summand, we invoke Lemma 2.3. By (1.8), for $x \in R_\delta^3$, the right-hand side of (2.16) equals $\kappa|x|$. Then

$$|\langle I_\nu, R_\delta^3 \setminus U_\Theta(\Sigma_0) \rangle| \leq C \int_{\{x \in R_\delta^3: |x - \Sigma_0| > \kappa|x|\}} \frac{dx}{\omega^2(x)}.$$  \hfill (3.4)

According to the Morse lemma, for a sufficiently small $r > 0$, there is a $C^2$-diffeomorphism $x \mapsto y$, $B_r \mapsto y(B_r)$, such that $y(0) = 0$ and the function $q(y) = \omega(x(y))$ is a nondegenerate quadratic form. We choose $\delta, \hat{\delta} > 0$ such that

$$y(B_\delta) \subset B_\hat{\delta} \subset y(B_r).$$

Clearly,

$$\{x \in R_\delta^3: |x - \Sigma_0| > \kappa|x|\} \subset \{x \in R_\delta^3: |x - \Sigma \cap B_r| > \kappa|x|\} =: A.$$

Because

$$C^{-1}|x_1 - x_2| \leq |y(x_1) - y(x_2)| \leq C|x_1 - x_2| \quad \text{for any} \quad x_1, x_2 \in B_r,$$

the image of the set $A$ under the map $x \mapsto y$ is contained in the set

$$\{y \in R_{\hat{\delta}}^3: |y - y(\Sigma \cap B_r)| > c\kappa|y|\}$$
for appropriate $c < 1$. In turn, this set is contained in the set
\[ A^\nu := \{ y \in R^r_{\delta} : |y - y(\Sigma) \cap B_r| > c\kappa|y| \}. \]

Because the set $y(\Sigma) \cap B_r$ is a cone $\Sigma^q = \{ y \in R^d : q(y) = 0 \}$ intersected with the ball $B_r$, we have
\[ A^\nu = \{ y \in R^r_{\delta} : |y - \Sigma^q| > c\kappa|y| \}. \]

The quadratic form $q$ is separated from zero on the set $A^\nu \cap \{|y| = \hat{r} \}$, and $\Sigma^q$ is a cone, and we therefore have $|q(y)| \geq C|y|^2$ for any $y \in A^\nu$. Then, the right-hand side of (3.4) is bounded by
\[ C \int_{A^\nu} \frac{dy}{q^2(y)} \leq C_1 \int_{\epsilon^2} \frac{r^{d-1} dr}{r^4} \leq C_2 \chi_{d,4}(\delta). \]

We thus arrive at
\[ |\langle I^\nu, B^\delta_0 \setminus U_{\Theta}(\Sigma_0) \rangle| \leq C \chi_{d,4}(\delta). \] (3.5)

3.2. The third integral in (3.2). First approximations. Step 1. We consider the set
\[ V_0 = \{ x = \pi(\xi, \theta) \in U_{\Theta}(\Sigma_0) : x_\xi \in \Sigma_0 \setminus B_\delta, |\theta| \leq \theta_\xi \}. \]

With Lemma 2.1(2), it is straightforward to see that the set $V_0 \Delta (U_{\Theta}(\Sigma_0) \setminus B_\delta)$ is contained in the ball $B_{c\delta}$ for an appropriate constant $c \geq 1$. Then, arguing as in (3.3), we see that
\[ |\langle I^\nu, V_0 \rangle - \langle I^\nu, U_{\Theta}(\Sigma_0) \setminus B_\delta \rangle| \leq C_1. \] (3.6)

Thus, it remains to study the integral $\langle I^\nu, V_0 \rangle$. According to Proposition 2.2 written in the $(\xi, \theta)$-coordinates, the integral takes the form
\[ \langle I^\nu, V_0 \rangle = \int_{\Sigma_0 \setminus B_\delta} m(d\xi) \int_{-\theta_\xi}^{\theta_\xi} \frac{|N_\xi|\mu_\xi(\theta)F(\xi, \theta)}{\omega^2(\xi, \theta) + \nu^2} d\theta. \]

Using (2.4), we rewrite it as
\[ \langle I^\nu, V_0 \rangle = \int_{\Sigma_0 \setminus B_\delta} \frac{m(d\xi)}{|N_\xi|^{3/2}} \int_{-\theta_\xi}^{\theta_\xi} \Phi(\xi, \theta) \frac{\theta^2 g_\xi(\theta) + \varepsilon_\xi}{\varepsilon_\xi} d\theta, \] (3.7)

where $\Phi(\xi, \theta) := \mu_\xi(\theta)F(\xi, \theta)$ and $\varepsilon_\xi := \nu|N_\xi|^{-2}$.

Step 2. We consider the inner integral in (3.7),
\[ J^\nu_\xi(\xi) := \int_{-\theta_\xi}^{\theta_\xi} \frac{\Phi(\xi, \theta)}{\theta^2 g_\xi(\theta) + \varepsilon_\xi} d\theta \]

and let $J^\nu_{0}(\xi)$ denote the integral $J^\nu_\xi(\xi)$ in which the functions $\Phi(\xi, \cdot)$ and $g_\xi$ are replaced with their values at zero $\Phi(\xi, 0) = F(\xi, 0)$ and $g_\xi(0) = 1$:
\[ J^\nu_{0}(\xi) := \int_{-\theta_\xi}^{\theta_\xi} \frac{F(\xi, 0)}{\theta^2 + \varepsilon_\xi} d\theta = 2 \frac{F(\xi, 0)}{\varepsilon_\xi} \arctan \frac{\theta_\xi}{\varepsilon_\xi}. \] (3.8)
In this step, we show that it suffices to analyze the integral (3.7) in which the inner integral $J_\nu(\xi)$ is replaced with $J_\nu^0(\xi)$. For this, keeping $\xi$ and $\theta$ fixed, we consider the function

$$f_{\xi,\theta}(t) := \frac{\Phi(\xi, t)}{\theta^2 g_\xi(t) + \varepsilon_\xi^2}.$$ 

By Taylor's formula,

$$f_{\xi,\theta}(t) - f_{\xi,\theta}(0) = f'_{\xi,\theta}(0)t + \frac{1}{2} f''_{\xi,\theta}(\hat{t}(t; \xi, \theta)) t^2,$$

where $|\hat{t}| \leq |t|$. Because $f'_{\xi,\theta}(0) = f'_{\xi,\theta}(0)$, we have

$$J_\nu(\xi) - J_\nu^0(\xi) = \int_{-\theta_\xi}^{\theta_\xi} (f_{\xi,\theta}(\theta) - f_{\xi,\theta}(0)) d\theta = \int_{-\theta_\xi}^{\theta_\xi} \frac{\theta^2}{2} f''_{\xi,\theta}(\hat{t}) d\theta. \quad (3.9)$$

We next estimate the derivative $f''_{\xi,\theta}$. For $x = \pi(\xi, t)$, we have $\partial_t = N_\xi \cdot \nabla_x$, and it therefore follow from Assumptions A1 and A2 that

$$|\partial_t F(\xi, t)| \leq C \langle x \rangle^{-M_F - k} \langle x_\xi \rangle^{k(M_\omega - 1)} \leq C_1 \langle x_\xi \rangle^{-M_F - k + k(M_\omega - 1)},$$

where we use Lemma 2.1(2) in the last inequality. In view of (2.13), for $0 \leq k \leq 2$, we then have

$$|\partial_t \Phi(\xi, t)| \leq \max_{0 \leq i \leq k} C_2 \langle x_\xi \rangle^{-M_F - i + (M_\omega - 1) + (k - i)(M_\omega - 2)} = C \langle x_\xi \rangle^{-M_F + k(M_\omega - 2)}. \quad (3.10)$$

We set $\eta_{\xi,\theta}(t) := \theta^2 g_\xi^2(t) + \varepsilon_\xi^2$. For $0 \leq k \leq 2$, in accordance with (2.5), we have

$$\left| \frac{d^k}{dt^k} \Phi(\xi, t) \right| = \frac{d^k}{dt^k} \Phi(\xi, t) \leq C \theta^2 \langle x_\xi \rangle^{k(M_\omega - 2)}. \quad (3.11)$$

Combining the estimates in (3.10) and (3.11) and the estimate $|\eta_{\xi,\theta}(t)| \geq C \theta^2$ that follows from (2.5), we obtain

$$\left| \partial_t \Phi(\xi, t) \right|, \left| \partial_\xi \Phi(\xi, t) \right|, \left| \Phi(\eta_{\xi,\theta}^2) \right|, \left| \Phi(\eta_{\xi,\theta}^2)^2 \right| \leq C \theta^{-2} \langle x_\xi \rangle^{-M_F + 2(M_\omega - 2)},$$

Then $|f''_{\xi,\theta}(t)| \leq C \theta^{-2} \langle x_\xi \rangle^{-M_F + 2(M_\omega - 2)}$, so, by (3.9),

$$|J_\nu(\xi) - J_\nu^0(\xi)| \leq C \theta \langle x_\xi \rangle^{-M_F + 2(M_\omega - 2)} \leq C \langle x_\xi \rangle^{-M_F + M_\omega - 2}. \quad (3.12)$$

Let $I'_\nu$ denote the integral (3.7) in which the inner integral $J_\nu(\xi)$ is replaced with $J_\nu^0(\xi)$,

$$I'_\nu = \int_{\Sigma_{\theta} \setminus B_\delta} \frac{J_\nu^0(\xi) m(\xi)}{|N_\xi|^3}. \quad (3.13)$$

According to (3.12), we use Corollary 2.2 and Assumption A4 to obtain

$$\langle I'_\nu, V_\delta \rangle - I'_\nu \leq C \int_{\Sigma_{\theta} \setminus B_\delta} \frac{\langle x_\xi \rangle^{-M_F + M_\omega - 2} m(\xi)}{|N_\xi|^3} \leq C_1 \chi_{d,4}(\delta).$$

It then follows from representation (3.2) and estimates (3.3), (3.5), and (3.6) that to prove the theorem, it suffices to establish the desired asymptotic form (1.5) for the integral $I_\nu$, replaced with $I'_\nu$. 

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3.3. Study of the integral $I^f_\nu$. We recall that the integral $I^f_\nu$ is defined in (3.13), where the function $J^0_\nu(\xi)$ is given by (3.8). Using the inequality

$$0 < \frac{\pi}{2} - \arctan\frac{1}{\gamma} < \gamma,$$

which holds for $0 < \gamma \leq 1/2$, for $\gamma = \varepsilon_\xi/\theta_\xi$, we obtain

$$\left| \frac{\pi F(\xi,0)}{\varepsilon_\xi} - J^0_\nu(\xi) \right| < \frac{2F(\xi,0)}{\theta_\xi}$$

(3.14)

if

$$\frac{\varepsilon_\xi}{\theta_\xi} = \nu(x_\xi)^{M_\omega-2} \leq \frac{1}{2}.$$  

(3.15)

We first assume that $|x_\xi| \geq 1$. Then, according to Assumption A3,

$$\frac{\varepsilon_\xi}{\theta_\xi} \leq C\Theta^{-1} \nu|x_\xi|^{M_\omega-2-2m_\omega},$$

and hence inequality (3.15) is satisfied for sufficiently small $\nu$ if $M_{cr} := M_\omega - 2m_\omega - 2 \leq 0$ or $M_{cr} > 0$ and

$$|x_\xi| \leq C_{cr}(\Theta \nu^{-1})^{1/M_{cr}}$$

(3.16)

with an appropriate constant $C_{cr} > 0$.

In the case $|x_\xi| < 1$ with $\xi \in \Sigma_0 \setminus B_3$, according to (1.7), we have

$$\frac{\varepsilon_\xi}{\theta_\xi} \leq C \frac{\nu}{|\xi|^2} \leq C \Theta^{-1} \nu \delta^{-2} = C \Theta^{-1} \alpha^{-2},$$

where we recall the definition $\delta := \alpha \sqrt{\nu}$ of $\delta$. Choosing $\alpha = \sqrt{2C\Theta^{-1/2}}$, we see that (3.15) is satisfied.

We introduce a subset $\Sigma^\leq_0 \supseteq \xi$ of $\Sigma_0$ such that inequality (3.15) is satisfied in $\Sigma^\leq_0 \setminus B_3$. More specifically, if $M_{cr} > 0$, we define (cf. (3.16))

$$\Sigma^\leq_0 = \{ \xi \in \Sigma_0 : |x_\xi| \leq C_{cr}(\Theta \nu^{-1})^{1/M_{cr}} \}$$

and if $M_{cr} \leq 0$, we set $\Sigma^\leq_0 = \Sigma_0$. We write $\Sigma^\geq_0 = \Sigma_0 \setminus \Sigma^\leq_0$ and

$$I^f_\nu = \left( \int_{\Sigma^\leq_0 \setminus B_3} + \int_{\Sigma^\geq_0} \right) \frac{m(d\xi)}{|\xi|^3} J^0_\nu(\xi),$$

where we assume $\nu$ to be so small that $\Sigma^\geq_0 \cap B_1 = \emptyset$. In view of (3.8), the integral over $\Sigma^\geq_0$ is bounded by

$$C \nu^{-1} \int_{\Sigma^\geq_0} |F(\xi,0)| \frac{m(d\xi)}{|\xi|^3} \leq C(C_{cr} \Theta)^{-1} \int_{\Sigma^\geq_0} |x_\xi|^{M_{cr}} |F(\xi,0)| \frac{m(d\xi)}{|\xi|^3}.$$  

(3.17)

According to Corollary 2.2 and Assumptions A1 and A4, the last integral is bounded by a $\nu$-independent constant.

It remains to study the integral over $\Sigma^\leq_0 \setminus B_3$. We let $K_\nu$ denote the integral obtained from the last one by approximating $J^0_\nu(\xi)$ via (3.14):

$$K_\nu = \pi \int_{\Sigma^\leq_0 \setminus B_3} \frac{F(\xi,0) m(d\xi)}{\varepsilon_\xi |\xi|^3} = \pi \nu^{-1} \int_{\Sigma^\leq_0 \setminus B_3} \frac{F(\xi,0) m(d\xi)}{|\xi|^3}. $$

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According to Corollary 2.2, this integral converges, as does integral (1.3):

\[ I_0 = \pi \int_{\Sigma_0} \frac{F(\xi, 0) m(\xi)}{|N_\xi|} d\xi = \pi \int_{\Sigma_0} \frac{F(x)}{|\nabla \omega(x)|} |N_\xi| d\xi. \]

The inequality (3.14) is satisfied for \( \xi \in \Sigma_0^- \) whence

\[ \left| \int_{\Sigma_0^- \setminus B_\delta} J^0_\nu(\xi) \frac{m(\xi)}{|N_\xi|^3} - K_\nu \right| \leq 2 \int_{\Sigma_0^- \setminus B_\delta} \frac{|F(\xi, 0)| m(\xi)}{\theta_\xi |N_\xi|^3} \leq C_1 \chi_{d_4}(\delta), \] (3.18)

again by Corollary 2.2.

To conclude the proof of the theorem it remains to note that by (1.7),

\[ |K_\nu - \nu^{-1} I_0| \leq \pi \nu^{-1} \left( \int_{\Sigma_0 \cap B_\delta} + \int_{\Sigma_0^- \setminus B_\delta} \right) \frac{|F(\xi, 0)| m(\xi)}{|N_\xi|}. \]

For \( d \geq 4 \), in accordance with Lemma 2.4(1), the first integral in the right-hand side is bounded by

\[ C \nu^{-1} \int_{\Sigma_0 \cap B_\delta} |x_\xi|^{-1} m(\xi) \leq C_1 \nu^{-1} \delta^2 \leq C_2, \]

while the second integral coincides with the left-hand side of (3.17) and is therefore bounded by a constant. The proof of Theorem 1.1 is finished.

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