The collinearly-improved Balitsky-Kovchegov equation

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Abstract

The high-energy evolution in perturbative QCD suffers from a severe lack-of-convergence problem, due to higher order corrections enhanced by double and single transverse logarithms. We resum double logarithms to all orders within the non-linear Balitsky-Kovchegov equation, by taking into account successive soft gluon emissions strongly ordered in lifetime. We further resum single logarithms generated by the first non-singular part of the splitting functions and by the one-loop running of the coupling. The resummed BK equation admits stable solutions, which are used to successfully fit the HERA data at small $x$ for physically acceptable initial conditions and reasonable values of the fit parameters.

1. The BK equation: from LO to NLO

The Balitsky-Kovchegov (BK) equation\textsuperscript{[1,2]} describes the pQCD evolution with increasing energy of the forward scattering amplitude for the scattering between a quark-antiquark dipole and a generic hadronic target (another dipole, a proton, or a nucleus), in the limit where the number of colors is large ($N_c \to \infty$). The leading order (LO) version of this equation defines the ‘leading logarithmic approximation’ (LLA): it resums all radiative corrections in which each power of the QCD coupling $\bar{\alpha} \equiv \alpha_s N_c / \pi$, assumed to be fixed and small, is accompanied by the energy logarithm $Y \equiv \ln(s/Q_0^2)$ (the ‘rapidity’), with $s$ the center-of-mass energy squared and $Q_0$ the characteristic transverse scale of the target. The LO BK equation reads

$$\frac{\partial T_{xy}(Y)}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int d^2 z \frac{(x-y)^2}{(x-z)^2(z-y)^2} \left[ -T_{xy}(Y) + T_{xz}(Y) + T_{zx}(Y) - T_{zy}(Y)T_{zy}(Y) \right].$$

where the subscripts $x$, $y$, $z$ denote the transverse coordinates of the original quark-antiquark pair and, respectively, the soft gluon emitted in one step of the evolution. The first term in the r.h.s., which is negative, is a ‘virtual’ correction where the soft gluon has no overlap with the target. The other 3 terms describe ‘real’ corrections where the virtual gluon exists at the time of scattering (see the left diagrams in Fig. 1). In particular, the term quadratic in $T$, which is negative, describes unitarity corrections associated with multiple scattering; these become important when the target looks dense on the resolution scales of the projectile. For what follows though, we shall be mostly interested in the dilute target, or weak scattering, regime, where this quadratic term is negligible and Eq. (1) can be well approximated by its linearized version, the celebrated BFKL equation. We shall moreover focus on the situation where the dipole looks very small on
the transverse scale of the target: \( rQ_0 \ll 1 \), or \( Q^2 \gg Q_0^2 \), with \( r \equiv |x - y| \equiv 1/Q \). Indeed, this regime is characterized by the existence of large radiative corrections, enhanced by the transverse (or ‘collinear’) logarithm \( \rho \equiv \ln(Q^2/Q_0^2) \). These corrections come from gluons emissions which occur far outside the original dipole, such that \( r \ll |x - z| \gg |z - y| \ll 1/Q_0 \). Such gluons look soft compared to their parent dipole but still hard compared to the target, so they scatter only weakly: \( T(z) \ll 1 \). In this regime, \( T(z) \sim z^2 \), hence the (linear) ‘real’ terms in Eq. (1) dominate over the ‘virtual’ one:

\[
\frac{\partial}{\partial Y} T(r, Y) \approx \bar{\alpha} \int_{r^2}^{1/Q_0^2} dz^2 \frac{r^2}{z^2} T(z, Y).
\]

The solution to this equation resums powers of \( \bar{\alpha} Y \rho \) to all orders. This double logarithmic enhancement—an energy logarithm and a collinear one—reflects the soft and collinear singularities of bremsstrahlung. But Eq. (2) is not yet the correct double-logarithmic approximation in QCD at high energy, as we shall see.

The next-to-leading order (NLO) corrections to Eq. (1) arise from 2-loop diagrams which involve at least one soft gluon (see Fig. 1). The maximal contribution \( a \rho^2 \) expected for such a diagram (after subtracting the respective LLA piece, if any) is of order \((a Y \rho) \times (\bar{\alpha} \rho) = \bar{\alpha}^2 Y \rho^2\); such a contribution would provide a NLO correction \( \bar{\alpha} \rho^2 \) to the BFKL kernel which is enhanced by a collinear log. Yet, the explicit calculation of all such 2-loop graphs in Ref. [3] reveals the existence of even larger corrections, of relative order \( \bar{\alpha} \rho^2 \). These are enhanced by a double collinear logarithm. The complete result at NLO appears to be extremely complicated [3], but it drastically simplifies if one keeps only the terms which are enhanced by at least one transverse logarithm in the regime where \( Q^2 \gg Q_0^2 \). Then it reads (at large \( N_c \))

\[
\frac{\partial T(r, Y)}{\partial Y} \approx \bar{\alpha} \int_{r^2}^{1/Q_0^2} dz^2 \frac{r^2}{z^2} \left(1 - \bar{\alpha} \left(\frac{1}{2} \ln^2 \frac{z^2}{r^2} + \frac{11}{12} \ln \frac{z^2}{r^2} - \tilde{b} \ln r^2 \mu^2 \right)\right) T(z, Y),
\]

which exhibits 3 types of NLO terms: the double-collinear log previously mentioned, a single collinear log which can be recognized as part of the DGLAP evolution (see below), and the one-loop running coupling, \( \bar{\alpha} = (11N_c - 2N_f)/12N_c \) is the first coefficient of the QCD \( \beta \)-function, and \( \mu \) is a renormalization scale at which the coupling is evaluated.) The NLO corrections enhanced by collinear logs are negative and large and lead to numerical instabilities which render the NLO BK equation void of any predictive power [4, 5]. The main source of this difficulty is the double-collinear logarithm (DCL) \( \bar{\alpha} \rho^2 \), whose origin and resummation will be discussed in the next sections.

2. Time ordering and double-collinear logarithms

The NLO correction \( \bar{\alpha} \rho^2 \) to the kernel arises from a particular 2-loop contribution of order \( \bar{\alpha}^2 Y \rho^3 \), which looks anomalously large: it involves a total of 4 (energy or transverse) logarithms, like the respective LLA contribution \( (a Y \rho)^2 \). As a matter of facts, this particular NLO contribution is generated by the same 2-loop diagrams (in terms of topology and kinematics) that are responsible for 2 successive steps
in the LLA evolution described by Eq. (2); namely, Feynman graphs involving 2 gluon emissions which are strongly ordered in both longitudinal momentum and transverse momentum (or transverse size). The physical interpretation of the enhanced contributions becomes most transparent when the 2-loop diagrams are computed within light-cone, or time-ordered, perturbation theory [4].

For example, let us examine the diagram in Fig. 2 (left) where the longitudinal momenta of the emitted gluons obey \( q^+ \gg p^+ \gg k^+ \), whereas their transverse sizes are ordered according to \( r = |x - y| \ll |u - x| \approx |u - y| \approx |z - x| \approx |z - y| \approx |z - u| \ll 1/Q_0 \). (4)

We implicitly assumed here that the dipole projectile is a right mover with large momentum \( q^+ \), while the hadronic target is a left mover with large momentum \( p^+ \). As visible in Fig. 2 (left), the softer gluon \( k \) is emitted after and absorbed before the harder one \( p \). This particular time-ordering introduces the energy denominator \( 1/(k^- + p^-) \) which in turn implies that the largest logarithmic contributions occur when the lifetimes of the two gluons are also strongly ordered: \( \tau_k \equiv 2k^+ / k^2 \ll \tau_p \equiv 2p^+ / p^2 \). Here \( p \) is the transverse momentum of the gluon \( p \), related to its transverse size via the uncertainty principle, \( p^2 \sim 1/(u - x)^2 \), and similarly for the gluon \( k \). Indeed, when this condition \( \tau_k \ll \tau_p \) is satisfied, then the four integrations over \( p^+ \), \( p^\perp \), \( k^+ \), and \( z \) are all logarithmic [6]. Different hookings of the two gluons lead to 32 diagrams like the one in Fig. 2 (left). Adding all of these contributions, we find in the regime defined in Eq. (4) (with simplified notations \( |u - x| \to u \) and \( |z - u| \to z \))

\[
\Delta T(r) = \tilde{\alpha} \int \frac{dk^+}{k^+} \int \frac{dz^2}{z^2} \int p^+ \frac{du^2}{u^2} \Theta \left( p^+ u^2 - k^+ z^2 \right) \frac{r^2}{z^2} T\left( z^2 \right),
\]

where the step-function implements the lifetime constraint. If there were not for this constraint, Eq. (5) would look identical as two iterations of the LO equation (2). By integrating out the intermediate gluon \( p^+ \),

\[
\tilde{\alpha} \int_{z^2}^{\infty} \frac{du^2}{u^2} \int_{k^+}^{q^+} \frac{dp^+}{p^+} \Theta \left( p^+ u^2 - k^+ z^2 \right) = \tilde{\alpha} \left( \ln \frac{z^2}{u^2} - \ln \frac{q^+_0}{k^+} - \frac{1}{2} \ln \frac{z^2}{u^2} \right) = \tilde{\alpha} Y_\rho - \tilde{\alpha} \rho^2 / 2,
\]

one finds the expected LLA contribution \( \tilde{\alpha} Y_\rho \) plus a term independent of \( Y \), namely \( -\tilde{\alpha} \rho^2 / 2 \), to be interpreted as a NLO correction to the kernel for emitting the softer gluon \( k^+ \). This correction reproduces the DCL part of the NLO correction in Eq. (3), thus clarifying the physical interpretation of the latter: it expresses the reduction in the rapidity interval \( \Delta Y \) available to the intermediate gluon due to the time-ordering constraint. This argument extends to all orders [5]: the perturbative corrections enhanced by DCLs can be resummed to all orders by enforcing time-ordering within the ‘naive’ LLA. However this procedure has the drawback to produce an evolution equation which is non-local in rapidity [5, 7], as already manifest in Eq. (5). This non-locality reflects the fact that the natural evolution variable at high energy is not the longitudinal momentum \( k^+ \) of a gluon from the projectile, or the associated rapidity \( Y = \ln(q^+ / k^+ \tau) \), but rather its lifetime \( \tau_k = 2k^+ / k^2 \), or equivalently \( \eta = Y - \rho \) with \( \rho = \ln(Q^2 / k^2_\rho) \): the evolution is local in \( \eta \), but not in \( Y \).

\[\text{Fig. 2. Left: a typical diagram yielding a NLO correction enhanced by a DCL. Middle and right: the saturation exponent } \lambda_1 \equiv \text{dln}Q^2_2/\text{dY} \text{ as predicted by the resummed equation (7) with fixed coupling } \bar{\sigma} = 0.25 \text{ (middle) and respectively running coupling (right). The 'DLA at NLO' curves show the instability of the strict NLO approximation, whereas the curves denoted as 'DLA resum' and 'DLA+SL resum' demonstrate the effect of successive resummations in stabilizing and slowing down the evolution.}\]
3. The collinearly-improved BK equation

At this stage, something remarkable happens: the non-local equation with time-ordering can be \textit{equivalently} rewritten as an equation \textit{local} in $Y$, where however both the kernel and the initial condition at $Y = 0$ resum corrections to all orders in $\tilde{a} \rho^2$ \cite{5}. This equation can furthermore be extended to resum the single transverse logarithms that appear at NLO, cf. Eq. (3), namely the single collinear logarithms (SCL) which represent the beginning of the DGLAP evolution and the one-loop running coupling corrections \cite{6}.

The SCL too arises from successive emissions in which the second gluon is much softer, both in transverse and longitudinal momenta, than the first one, but now it is the region $\tau_k \sim \tau_p$ which gives the relevant contribution. Its coefficient $A_1 = 11/12$ can be recognized as the first non-singular term in the small-$\omega$ expansion of the DGLAP anomalous dimension \cite{6}. This implies that in order to resum such SCLs, it suffices to include $A_1$ as an ‘anomalous dimension’, i.e. as a power-law suppression in the evolution kernel. The running coupling corrections can be resummed by choosing the renormalization scale $\mu$ as the hardest scale in the problem: $\tilde{a} \rightarrow \tilde{a}(r_{\min})$, where $r_{\min}$ is the size of the smallest dipole, $r_{\min} \equiv \min([x-y],|x-z|,|y-z|)$.

We are thus led to the following, \textit{collinearly-improved}, version of the BK equation, which faithfully includes the NLO effects enhanced by large (double or single) transverse logarithms, but improves over the strict NLO approximation by resumming similar corrections to all orders:

$$\frac{\partial T_{xy}}{\partial Y} = \int \frac{d^2 z}{2\pi} \tilde{a}(r_{\min}) \frac{(x-y)^2}{(x-z)^2(y-z)^2} \left( \frac{r^2}{z^2} \right)^{\alpha_{\text{eff}}-\tilde{a}} K_{\text{DGLAP}}(\tilde{\rho}^2) \left[ -T_{xy} + T_{xz} + T_{zy} - T_{xz} T_{zy} \right].$$

(7)

In this equation, $z^2 \equiv \min((x-z)^2, (y-z)^2)$, $\tilde{\rho}^2 \equiv \ln(([x-z]^2/r^2] \ln((y-z)^2/r^2)$, and (with $J_1$ the Bessel function)

$$K_{\text{NLO}}(\tilde{\rho}^2) \equiv \frac{J_1(2\sqrt{\tilde{a} \rho^2})}{\sqrt{\tilde{a} \rho^2}} = 1 - \frac{\tilde{a} \rho^2}{2} + \frac{(\tilde{a} \rho^2)^2}{12} + \cdots$$

(8)

is the change in the kernel accounting for DCLs to all orders. A similar kernel was found in transverse momentum space \cite{5}, as an approximations to resummations performed in the context of the BFKL equation. The initial condition to Eq. (7) at $Y = 0$ can be found in Refs. \cite{5,6}, to which we refer for more details.

In contrast to the NLO BK equation, the resummed equation (7) admits stable solutions, which are well-suited for phenomenology. The various resummations considerably slow down the evolution, so the ensuing “evolution speed” — as measured by the saturation exponent $\lambda_s \equiv d \ln Q_s^2/dY$, with $Q_s(Y)$ the saturation momentum — is substantially smaller than that predicted by the LO evolution (see Fig. 2). By using Eq. (7) together with appropriate forms for the initial condition, we have been able to obtain good quality fits to the HERA data \cite{9} for the $ep$ reduced cross section at small Bjorken $x \lesssim 10^{-2}$, with only 4 free parameters (for similar fits, without inclusion of the SCLs, see \cite{10}). The evolution speed extracted from the fits is $\lambda_s = 0.20 \div 0.24$. A remarkable feature about these fits is that they are rather discriminatory: they exclude several models for the initial conditions previously used in the literature and also some previous choices for the running coupling. Conversely, they favor the running-coupling version of the McLerran-Venugopalan model for the initial condition and the ‘smallest dipole’ prescription $\tilde{a}(r_{\min})$ for the running of the coupling.

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