INVARIANCE OF NON-VANISHING OF FIRST
$p$-COHOMOLOGY UNDER $L^q$-MEASURE EQUIVALENCE

KAJAL DAS

Abstract: The first $p$-cohomology is an algebro-analytical object attached to a finitely generated discrete group and introduced by M. Gromov. It is well known that it is invariant under quasi-isometry. In this article, we prove that the non-vanishing of the first $p$-cohomology of a non-amenable group is invariant under $L^q$-Measured Equivalence, where $q \geq p > 1$. We prove a weaker version of this result for $q \geq p = 1$. We also discuss many corollaries of this result. Some corollaries are new and we reprove the other corollaries. We prove a new result that for hyperbolic (in the sense of Gromov) groups with boundaries having Combinatorial Loewner Property, conformal dimension (of the canonical conformal gauge) of the Gromov boundary is invariant under $L^q$-Measure Equivalence for some large $q$. We reprove that the finitely generated free groups and surface groups are not $L^1$-Measure Equivalent. Finally, we discuss $L^q$-Measure Equivalence between non-amenable 3-manifold groups corresponding to Thurston’s three geometries $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{SL}_2(\mathbb{R})$.

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1. Introduction

Measure equivalence (ME) is an equivalence relation on countable groups introduced by Gromov in [Gro93], as a measure-theoretic analogue of quasi-isometry (QI). The first detailed study of ME was performed in the work of Furman [Fur99a] in the context of ME-rigidity of lattices in higher rank simple Lie groups. $L^p$-measure equivalence ($L^p$-ME) is defined by imposing an $L^p$-condition on the cocycle maps arising from a measure equivalence relation. Such integrability condition first appeared in Margulis’s proof of the normal subgroup theorem for irreducible lattices [Ma79]. $L^p$-measure equivalence is an emerging area in the overlap geometric group theory and measured group theory. The main question in $L^p$-measured group theory is of the following type:

Which algebraic/geometric properties of groups are invariant under $L^p$-ME?

In recent history, there are major works on $L^p$-measure equivalence, by Y. Shalom in [Sh00], [Sh00'] (for lattices), [Sh04] (in terms of $L^\infty$-Measure Equivalence) and later by Bader-Furman-Sauer [BFS13] (in general $L^p$-Measure Equivalence between discrete groups). Subsequently, it appeared in the work of T. Austin (L. Bowen, appendix) [Aus16], Das-Tessera [DT18], Das [Das18], [Das18'], T. Austin [Aus16'], L. Bowen [Bow17], Delabie-Koivisto-Maitre-Tessera [DKMT22] and Horbez-Huang [HH23]. We briefly survey the recent developments.

- Shalom has used $L^2$-measure equivalence in [Sh00'] to induce cocycle from unitary representations of a lattice to the ambient locally compact second countable group. In [Sh04] he proves that if two amenable groups are quasi-isometric, then they are $L^\infty$-measure equivalence.
- In [BFS13] Bader-Furman-Sauer prove $L^1$-measure equivalence rigidity for lattices in $SO(n, 1)$, $n \geq 3$. 
• T. Austin proves that if two nilpotent groups are \( L^1\)-ME, then their asymptotic cones are bi-Lipschitz isomorphic [Aus16]. In the appendix, L. Bowen proves that if two finitely generated groups are \( L^1\)-ME, then they have same ‘growth types’.

• In [DT18] the authors prove that quasi-isometry and Measure Equivalence together do not imply \( L^1 \) or \( L^\infty\)-measure equivalence.

• L. Bowen proves \( L^1\)-orbit equivalence rigidity (a variant of Measure Equivalence) for free groups inside the class of accessible’ groups [Bow17].

• In [DT18] the authors prove that quasi-isometry and Measure Equivalence together do not imply \( L^1 \) or \( L^\infty\)-measure equivalence.

• L. Bowen proves \( L^1\)-orbit equivalence rigidity (a variant of Measure Equivalence) for free groups inside the class of accessible’ groups [Bow17].

• Delabie-Koivisto-Maˆıtre-Tessera prove in [DKMT22] that \( l^p\)-isoperimetric profile is stable under \( L^p\)-Measure Equivalence for all \( p \geq 1\).

• Marrakchi-de la Salle prove in [MdlS23] that the property of admitting an affine isometric action without fixed points on some \( L^p\)-space or admitting a proper affine isometric action on some \( L^p\)-space (\( 1 \leq p < \infty \)) by a locally compact group is invariant under \( L^q\)-ME , where \( q \geq p\).

• Let \( G \) be a right-angled Artin group with \(|\text{Out}(G)| < \infty\) and let \( H \) be a countable group with bounded torsion. In [HH23] Horbez-Huang proves that if there exists an \((L^1, L^0)\)-measure equivalence coupling from \( H \) to \( G \), then \( H \) is finitely generated and quasi-isometric to \( G \).

However, it has been proved by Gromov in [Gro93] that if two finitely generated groups \( \Gamma \) and \( \Lambda \) are QI, then their \( k\)-th \( p\)-cohomology group \( \ell^p H^k(\Gamma) \) and \( \ell^p H^k(\Lambda) \) are isomorphic as a topological vector space. We prove this result for \( k = 1 \) under \( L^q\)-ME in our Main Theorem 1.1.

1.1. Statement of the main theorem and the corollaries.

**Theorem 1.1. (Main Theorem)** Suppose two non-amenable groups \( \Gamma \) and \( \Lambda \) are \( L^q\)-ME for some \( q \geq 1\). Then \( \ell^p H^1(\Gamma) \neq 0 \) if and only if \( \ell^p H^1(\Lambda) \neq 0 \), when \( 1 < p \leq q \). If \( p = 1 \), then \( \ell^p H^1(\Lambda) = 0 \) implies that all affine actions associated to \( \ell^p H^1(\Gamma) \) have bounded orbits.

**Remark 1.2.** By Proposition 3.3 and Proposition 3.4, we obtain that for finitely generated non-amenable groups \( \Gamma \), \( \ell^p H^1(\Gamma) \subseteq \ell^{p'} H^1(\Lambda) \), where \( 1 \leq p \leq p' \in \mathbb{R}\). The quantity \( p_*(\Gamma) := \inf\{p : \ell^p H^1(\Gamma) \neq 0\} \) is called the critical exponent of first \( p\)-cohomology of the group \( \Gamma \). Using main theorem 1.1, in particular, we obtain that if two groups \( \Gamma \) and \( \Lambda \) are \( L^q\)-ME, \( q > \max\{p_*(\Gamma), p_*(\Lambda)\} \) and \( p_*(\Gamma), p_*(\Lambda) > 1 \), then \( p_*(\Gamma) = p_*(\Lambda) \).

We prove the Main Theorem in Section 5. It is well-known that the conformal dimension of the boundary of a Gromov hyperbolic group is invariant under quasi-isometry. In this article, we ask the question for \( L^q\)-ME. We obtain an affirmative answer for a special class of hyperbolic Coxeter groups. We prove this result as a corollary of our main theorem in Section 6.
Corollary 1.3. Let $\Gamma$ and $\Lambda$ are two hyperbolic groups with boundaries having Combinatorial Loewner Property. Suppose, $\Gamma$ and $\Lambda$ are $L^q$-ME, for $q > \text{Confdim}(\Gamma)$, $\text{Confdim}(\Lambda)$ and $\text{Confdim}(\Gamma)$, $\text{Confdim}(\Lambda) > 1$. Then, the conformal dimensions (of the canonical conformal gauge) of the groups are equal.

Using our main theorem 1.1, we also prove the following corollaries in Section 6. These corollaries were already known, but we reprove the results here.

Corollary 1.4. The free group $F_n$ with $n \geq 2$ and the surface group $\pi_1(\Sigma_g)$ with genus $g \geq 2$ are not $L^1$-ME.

Corollary 1.5. Any cocompact lattice in $\text{Isom}(\mathbb{H}^3)$ is not $L^{2+\epsilon}$-ME with the central extensions of surface groups for all $\epsilon > 0$.

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2. $L^p$-Measure Equivalence

Two countable discrete groups $\Gamma$ and $\Lambda$ are called Measure Equivalent (ME) if there is a nonzero $\sigma$-finite measure space $(X, \mu)$, which admits commuting free $\mu$-preserving actions of $\Gamma$ and $\Lambda$ such that both have finite-measure fundamental domains $X_\Gamma$ and $X_\Lambda$, respectively, i.e.,

$$X = \bigsqcup_{\gamma \in \Gamma} \gamma X_\Gamma = \bigsqcup_{\lambda \in \Lambda} \lambda X_\Lambda.$$  

$(X, \mu)$ is called the coupling space of $\Gamma$ and $\Lambda$. Let $\alpha : \Gamma \times X_\Lambda \to \Lambda$ (resp. $\beta : \Lambda \times X_\Gamma \to \Gamma$) be the corresponding cocycle defined by the rule: for all $x \in X_\Lambda$ and $\gamma \in \Gamma$, $\alpha(\gamma, x)\gamma x \in X_\Lambda$ (and symmetrically for $\beta$). We define $\gamma \cdot x := \alpha(\gamma, x)\gamma x$ for all $\gamma \in \Gamma$ and $x \in X_\Lambda$ and $\lambda \cdot y := \beta(\lambda, y)\lambda y$ for all $\lambda \in \Lambda$ and $y \in X_\Gamma$. We have the following cocycle relations:

$$\alpha(\gamma_1 \gamma_2, x) = \alpha(\gamma_1, \gamma_2 \cdot x)\alpha(\gamma_2, x) \quad \text{and} \quad \beta(\lambda_1 \lambda_2, x) = \beta(\lambda_1, \lambda_2 \cdot x)\beta(\lambda_2, x)$$

where $\gamma_1, \gamma_2 \in \Gamma$, $x \in X_\Lambda$, $\lambda_1, \lambda_2 \in \Lambda$ and $y \in X_\Lambda$. If, for any $\lambda \in \Lambda$ and $\gamma \in \Gamma$, the integrals

$$\int_{X_\Lambda} |\alpha(\gamma, x)|^p d\mu(x) \quad \text{and} \quad \int_{X_\Gamma} |\beta(\lambda, x')|^p d\mu(x')$$

are finite, then the groups are called $L^p$-measure equivalent. The strongest form is when $p = \infty$, in which case the coupling is called uniform, and the groups
uniformly measure equivalent (UME), as it generalizes the case of two uniform lattices in a same locally compact group. For \( p = 1 \), the coupling is called integrable, and the groups are said to be integrable measure equivalent (IME).

**Remark 2.1.** Measure Equivalence is closely related to the concept of Orbit Equivalence (OE) in ergodic theory. Suppose \( \Gamma \) and \( \Lambda \) are Measure Equivalence with \( X_\Gamma = X_\Lambda \). Then the groups are orbit equivalent. Moreover, two groups are ME if and only if they are ‘weakly OE’ or ‘stably OE’.

3. **The first \( l^p \)-cohomology of groups**

3.1. **Definition of \( l^p \)-cohomology.** We fix \( p \in [1, \infty) \). Let \( \Gamma \) be a finitely generated group and fix a Cayley graph of \( \Gamma \). Suppose \( V_\Gamma \) and \( E_\Gamma \) are the vertex set and the edge set of the Cayley graph, respectively.

“The first \( l^p \)-cohomology group of \( \Gamma \),” denoted by \( l^p H^1(\Gamma) \), will be defined as a quotient of the space of functions \( f : V_\Gamma \to \mathbb{R} \). We follow the definition given in [MT10].

Let \( l^p(V_\Gamma) \) (respectively \( l^p(E_\Gamma) \)) denote the collection of \( p \)-summable functions \( f : V_\Gamma \to \mathbb{R} \) (respectively \( g : E_\Gamma \to \mathbb{R} \)).

The differential of \( f : V_\Gamma \to \mathbb{R} \), denoted \( df \), is the map from \( E_\Gamma \) to \( \mathbb{R} \) defined by \( df(ab) = f(b) - f(a) \). Since \( V_\Gamma \) has bounded valence, \( df \in l^p(E_\Gamma) \), whenever \( f \in l^p(V_\Gamma) \). Since \( V_\Gamma \) is connected, \( df = 0 \) if and only if \( f \) is a constant.

**Definition 3.1.** The “first \( l^p \) cohomology group” of \( \Gamma \) written \( l^p H^1(\Gamma) \), is defined as follows: \( \{ f : V_\Gamma \to \mathbb{R} : df \in l^p(E_\Gamma) \} / (l^p(V_\Gamma) \bigoplus \mathbb{R} \cdot 1) \), where \( 1 \) denotes the constant function: \( 1(v) = 1 \) for all \( v \in V_\Gamma \).

Similarly, we define “first reduced \( l^p \) cohomology group” \( \Gamma \).

**Definition 3.2.** The “first reduced \( l^p \) cohomology group” of \( \Gamma \) written \( l^p \bar{H}^1(\Gamma) \), is defined as follows: \( \{ f : V_\Gamma \to \mathbb{R} : df \in l^p(E_\Gamma) \} / (l^p(V_\Gamma) \bigoplus \mathbb{R} \cdot 1) \), where \( 1 \) denotes the constant function: \( 1(v) = 1 \) for all \( v \in V_\Gamma \) and the closure is with respect to the topology of point-wise convergence.

We remark that the isomorphism type of \( l^p H^1(\Gamma) \) is a quasi-isometry invariant of the underlying graph \( V_\Gamma \), and thus an invariant of \( \Gamma \). We will write \( l^p H^1(\Gamma) \) for the isomorphism class of first \( l^p \)-cohomology groups associated to \( \Gamma \).

3.2. **List of some known results of first \( l^p \)-cohomology of discrete groups:**

- Suppose \( A \) and \( B \) are two finitely generated groups with \( |A| \geq 2 \) and \( |B| \geq 3 \). Then \( \Gamma = A \ast B \) satisfies \( l^p H^1(\Gamma) \neq 0 \) for all \( p \geq 1 \) (see [Bou10], Proposition 1.1). In particular, for the free group with finite number generators, the first \( l^p \)-cohomology is non-zero for all \( p \geq 1 \).
• Suppose $\Gamma$ is a cocompact lattice in $\text{Isom}(\mathbb{H}^n)$, where $n \geq 2$. Then $l^p H^1(\Gamma) = 0$ if $p \in [1, n - 1]$. (see [Pan89'], Proposition 2.1)
• Let $\Gamma$ be a word-hyperbolic group. Then for $p$ large enough we have $l^p H^1(\Gamma) \neq 0$ (see [Bou10] Proposition 2.3, [Yu05]).

3.3. 1-cohomology with coefficients in a Banach Space. Let $\Gamma$ be a discrete group. Let $\pi$ be an isometric representation of $\Gamma$ on a Banach space $B$.

1. A mapping $b : \Gamma \to B$ s.t. $b(gh) = b(g) + \pi(g)b(h)$, for all $g, h \in \Gamma$ is called a 1-cocycle with respect to $\pi$.
2. A 1-cocycle $b : \Gamma \to B$ for which there exists $\xi \in B$ such that $b(g) = \pi(g)\xi - \xi$, for all $g \in \Gamma$, is called a 1-coboundary with respect to $\pi$.
3. The quotient space $H^1(\Gamma, \pi) = Z^1(\Gamma, \pi) / B^1(\Gamma, \pi)$ is called 1-cohomology associated to the representation $\pi$. The reduced 1-cohomology of $\Gamma$ associated to the representation $\pi$ is the quotient vector space $H^1(\Gamma, \pi) = Z^1(\Gamma, \pi) / B^1(\Gamma, \pi)$, where the closure is taken w.r.t. the pointwise convergent topology. Sometimes, we use the notations $H^1(\Gamma, B)$ and $H^1(\Gamma, B)$ for 1-cohomology and reduced 1-cohomology, respectively.

The following proposition relates the first $l^p$-cohomology of a group $\Gamma$ with the 1-cohomology with coefficients in $l^p(\Gamma)$.

Proposition 3.3. $l^p H^1(\Gamma) = H^1(\Gamma, l^p(\Gamma))$, where $\rho_\Gamma$ is acting on $l^p(\Gamma)$ by right regular action. Similarly, $l^p H^1(\Gamma) = H^1(\Gamma, l^p(\Gamma))$, where $p \geq 1$.

Proof. We briefly sketch the proof. Let $f : \Gamma \to \mathbb{R}$ be a representative of a class in $l^p H^1(\Gamma)$. Suppose $S$ is a generating subset of $\Gamma$. We define $b : \Gamma \to l^p(\Gamma)$ by $b(s)(\gamma) = f(v_{\gamma s}) - f(v_\gamma)$. On the other hand, if we have a cocycle $b : \Gamma \to l^p(\Gamma)$, we define $f(e_{\gamma, \gamma s}) = b(s)(\gamma)$, where $e_{\gamma, \gamma s}$ is the edge between $v_{\gamma s}$ and $v_\gamma$ indexed by $s$. □

We remark that the 1-cohomology of a group $\Gamma$ with coefficients in $l^p(\Gamma)$ is monotone in the following sense.

Proposition 3.4. ([Loh90], [Pul06] Proposition 6.2) Let $\Gamma$ be a finitely generated non-amenable group. If $1 < p \leq p' \in \mathbb{R}$, then $H^1(\Gamma, l^p(\Gamma)) \subset H^1(\Gamma, l^{p'}(\Gamma))$.

Using Proposition 3.4 and Proposition 3.3, we obtain that the first $l^p$-cohomology is monotone (i.e., if $p \leq p'$, $l^p H^1(\Gamma) \subset l^{p'} H^1(\Gamma)$). Now, we mention Theorem 3.5 and Proposition 3.6 which will be useful in the proof of our main theorem 1.1.

Theorem 3.5. Suppose $p \in (1, \infty)$. TFAE:

(a) $\Gamma$ is non-amenable,
(b) $l^p H^1(\Gamma) = l^p H^1(\Gamma)$.

Proof. We refer the readers to [Bou10] (Theorem 1.5, p. 9) for the proof of the proposition. □
Proposition 3.6. If $H^1_1(\Lambda, \pi) = 0$, then $H^1_1(\Lambda, \int X \pi(x) d\mu(x)) = 0$ where $\Lambda$ is a discrete group, $\pi(x) = \pi$ for all $x$ and $\pi$ is the left-regular or right-regular representation of $\Lambda$ on $l^p(\Lambda)$, where $1 \leq p < \infty$.

Proof. We refer the readers to Theorem 8.1 of the Appendix of this article for the proof of this proposition. □

4. Induced Representation and Induced Affine Action

4.1. Induced representation. We assume that $\Gamma$ and $\Lambda$ are Measure Equivalent (not necessarily $L^p$-ME) as in Section 2. Given $1 \leq p \leq \infty$ and some isometric representation $\pi$ of $\Gamma$ on a Banach space $B$, we define the induced representation $\text{Ind}^{p,\Lambda}_{\Gamma} \pi$ to $\Lambda$ on $L^p(X_\Gamma, B) := \{ \psi : X_\Gamma \to B \mid \int_{X_\Gamma} |\psi|^p d\mu < \infty \}$ $\psi$ is Bochner-measurable in the following way:

$$\lambda \psi(x) = \pi(\beta(\lambda^{-1}, x)^{-1})\psi(\lambda^{-1}, x).$$

We refer the reader to the appendix for the definition of ‘Bochner-measurable function’. Observe that this is a “linear induction”, and no integrability assumption on the coupling is required. When $p$ is clear from the context, we denote the induced representation by $\text{Ind}^{p,\Lambda}_{\Gamma} \pi$.

4.2. Induced cocycle in the induced representation. Let $b : \Gamma \to B$ be a 1-cocycle coming from $Z^1(\Gamma, B)$, where $\Gamma$ is acting on $B$ by $\pi$ as in the previous section. We consider the induced representation $\text{Ind}^{p,\Lambda}_{\Gamma} \pi$ of $\Lambda$ with coefficients in $L^p(X_\Gamma, B)$. We assume that $\Gamma$ and $\Lambda$ are $L^q$-ME, where $q \geq p$.

We define 1-cocycle $B \in Z^1(\Lambda, L^p(X_\Gamma, B))$ associated to $\text{Ind}^{p,\Lambda}_{\Gamma} \pi$ by the following way:

$$(4.1) \quad B(\lambda)(y) := b[\beta(\lambda^{-1}, y)^{-1}],$$

where $\lambda \in \Lambda$, $y \in X_\Gamma$.

Proposition 4.1. $B$ is well-defined, i.e., $B$ is $p$-integrable, and $B$ is a cocycle.

Proof. We fix $\lambda \in \Lambda$ and we suppose that $\Gamma$ is generated by a symmetric generating set $S = \{ s_1, \ldots, s_k \}$.

$$\int_{X_\Gamma} \| B(\lambda)(y) \|^p d\nu(y) = \int_Y \| b[\beta(\lambda^{-1}, y)^{-1}] \|^p d\nu(y)$$

$$\leq \max\{ \| b(s_i) \|^p \}_{i=1}^k \int_{X_\Gamma} \| \beta(\lambda^{-1}, y) \|^p d\nu(y) < \infty$$
In the above inequality, we use the properties of 1-cocycle $b$ and $p$-integrability of $\beta$. Now, we check that $B$ is a 1-cocycle. We observe that $\{B(\lambda)\}(y) = b[\{\beta(\lambda^{-1}, y)\}]^{-1} = b[\beta(\lambda, \lambda^{-1} \cdot y)]$ using the cocycle relation for $\beta$. Therefore,

\[
\{B(\lambda_1 \lambda_2)\}(y) = b[\beta(\lambda_1 \lambda_2, \lambda_2^{-1} \lambda_1^{-1} \cdot y)] \\
= b[\beta(\lambda_1, \lambda_2 \lambda_2^{-1} \lambda_1^{-1} \cdot y) \beta(\lambda_2, \lambda_2^{-1} \lambda_1^{-1} \cdot y)] \\
= b[\beta(\lambda_1, \lambda_1^{-1} \cdot y)] + \pi(\beta(\lambda_1, \lambda_1^{-1} \cdot y)) b(\beta(\lambda_2, \lambda_2^{-1} \lambda_1^{-1} \cdot y)) \\
= \{B(\lambda_1)\}(y) + \pi(\beta(\lambda_1, \lambda_1^{-1} \cdot y)) \{B(\lambda_2)\}(\lambda_1^{-1} \cdot y) \\
= \{B(\lambda_1)\}(y) + \{[\text{Ind}^A_\gamma(\lambda_1)] B(\lambda_2)\}(y)
\]

Hence, $B \in Z^1(\Lambda, L^p(X_\Gamma, \mathcal{B}))$.  \hfill \Box

4.3. Induced representation from right-regular action. We define a measure preserving isomorphism $\psi : \Lambda \times X_\Lambda \rightarrow \Gamma \times X_\Gamma$ by

\[
(\lambda', x') \mapsto (\gamma', \beta(\lambda', y')^{-1}, x' \cdot y'),
\]

where $x' = \gamma' y'$, $\gamma' \in \Gamma$ and $y' \in X_\Gamma$. It is easy to see that $\Psi$ induces an isometric isomorphism $\Psi : L^p(X_\Gamma, \mathcal{B}) \rightarrow L^p(X_\Lambda, \mathcal{B})$, for $p \geq 1$.

**Lemma 4.2.** $\Psi \circ [\text{Ind}_\Gamma^A \rho_T(\lambda)] = [\int_{X_\Lambda}^\eta \tau_\Lambda(\lambda)] \circ \Psi$, where $\rho_T$ is the right-regular representation of $\Gamma$ and $\tau_\Lambda$ is the left-regular representation of $\Lambda$.

**Proof.** For $x' \in X_\Lambda$ and $\lambda \in \Lambda$, we have

\[
\{\Psi \circ \text{Ind}_\Gamma^A \rho_T(\lambda) f\}(x')(\lambda') = \{\text{Ind}_\Gamma^A \rho_T(\lambda) f\}(\lambda' \cdot y') (\gamma' \beta(\lambda', y')^{-1}) \\
= \{f(\lambda^{-1} x' \cdot y')\} (\gamma' \beta(\lambda', y')^{-1}) (\gamma \lambda^{-1} x' \cdot y')^{-1}) \\
= \{f(\lambda^{-1} x' \cdot y')\} (\gamma' \beta(\lambda^{-1} x', y')^{-1}) \\
= \{(\int_{X_\Lambda}^\eta \tau_\Lambda(\lambda) \circ \Psi) f\}(x')(\lambda')
\]

Hence, we have our lemma.  \hfill \Box

4.4. Induced affine action. As in Subsection 4.2, we assume that the 1-cocycle $B$ is induced from the 1-cocycle $b \in Z^1(\Gamma, \mathcal{B})$ and associated to the induced representation $\text{Ind}_\Gamma^A \pi$. We denote the affine action of $\Gamma$ associated to $b$ by $\eta$ and the affine action of $\Lambda$ on $L^p(X_\Gamma, \mathcal{B})$ associated to $B \in Z^1(\Lambda, \text{Ind}_\Gamma^A \pi)$ by $\text{Ind}_\Gamma^A \eta$.

We now consider the setting in the proof of Theorem 7.4 in [MdlS23]. Since the coupling space $X$ is equal to $\Gamma X_\Gamma$, we can define $p_1 : X \rightarrow \Gamma$ by $w = \gamma x \mapsto \gamma$ for all $w = \gamma x \in X$, where $\gamma \in \Gamma$ and $x \in X_\Gamma$. Let $L_0(X, \mu, \mathcal{B})^\Gamma$ be the set of all $\Gamma$-equivariant Bochner-measurable maps from $Z$ to $\mathcal{B}$, where $\Gamma$ acts on $\mathcal{B}$ by affine action $\eta$ (we refer the readers to the second paragraph of the Appendix for the definition of Bochner measurable function. We define $f_0 \in L_0(X, \mu, \mathcal{B})^\Gamma$ by
\[ f_0(w) = p_1(w) \cdot 0 = b(\gamma), \] where \( w = \gamma x, \gamma \in \Gamma \) and \( x \in X_\Gamma \). Now, we define an affine subspace

\[ F = \{ f \in L_0(X, \mu, \mathcal{B})^\Gamma : \|f - f_0\|_{p, \Gamma} < \infty \}. \]

We further define \( \Phi : F \to L^p(X_\Gamma, \mathcal{B}) \) by \( f \mapsto f - f_0 \). There is the following natural affine action \( \zeta \) of \( \Lambda \) on \( F \): \( \zeta(\lambda)f)(w) := f(\lambda^{-1}w) \) for all \( \lambda \in \Lambda \) and \( f \in F \). On the other hand, there is the affine action \( \text{Ind}_\Lambda^\Gamma \eta \) of \( \Lambda \) on \( L^p(X_\Gamma, \mathcal{B}) \).

**Lemma 4.3.** \( \Phi \) is equivariant under the affine action \( \zeta \) of \( \Lambda \) on \( F \) and the affine action \( \text{Ind}_\Lambda^\Gamma \eta \) of \( \Lambda \) on \( L^p(X_\Gamma, \mathcal{B}) \).

**Proof.** For all \( x \in X_\Gamma \),

\[
\{\zeta(\lambda)f\}(x) = f(\lambda^{-1}x) = f[\beta(\lambda^{-1}, x)^{-1}(\lambda^{-1} \cdot x)] = \beta(\lambda^{-1}, x)^{-1} \cdot f(\lambda^{-1} \cdot x) = \pi(\beta(\lambda^{-1}, x)^{-1})f(\lambda^{-1} \cdot x) + b(\beta(\lambda^{-1}, x)^{-1}) = [\text{Ind}_\Lambda^\Gamma \eta(\lambda)f](x)
\]

Hence, we have our lemma.

5. **Proof of the main theorem**

In this section, we prove our main theorem 1.1.

**Proof.** Suppose \( \Gamma \) and \( \Lambda \) are \( L^q\)-ME, where \( q \geq 1 \). Let \((X, \mu)\) be the coupling space, and \( X_\Gamma \) and \( X_\Lambda \) be fundamental domains of \( \Gamma \) and \( \Lambda \) respectively. Moreover, we assume that \( \alpha : \Gamma \times X_\Lambda \to \Lambda \) and \( \beta : \Lambda \times X_\Gamma \to \Gamma \) are \( L^q\)-integrable cocycles. Suppose \( \mathcal{L}^pH^1(\Gamma) \neq 0 \), where \( q \geq p \geq 1 \). We prove that \( \mathcal{L}^pH^1(\Lambda) \neq 0 \). From Step 1 to Step 3, we prove our theorem for \( 1 < p < \infty \), and in Step 4 we prove our theorem for \( p = 1 \).

**Step 1:**

By Proposition 3.3, we obtain that \( H^1(\Gamma, \mathcal{L}^p(\Gamma)) = \mathcal{L}^pH^1(\Gamma) \neq 0 \), \( \Gamma \) acts on \( \mathcal{L}^p(\Gamma) \) by right-regular action \( \rho_\Gamma \). Now, we will prove that \( H^1(\Lambda, \text{Ind}_\Lambda^\Gamma(\rho_\Gamma)) \neq 0 \). This proof is a particular case of Theorem 7.4 (i) in [MdS23]. For the sake of completeness, we give a proof of this result here. Let \( b \in Z^1(\Gamma, \mathcal{L}^p(\Gamma)) \) is a non-zero cocycle class in \( H^1(\Gamma, \mathcal{L}^p(\Gamma)) \). We use the notations use in the Subsections 4.1, 4.2, 4.3 and 4.4 (the Banach space \( \mathcal{B} \) is replaced by \( \mathcal{L}^p(\Gamma) \) and the representation \( \pi \) is replaced by \( \rho_\Gamma \)). We assume by contradiction that \( H^1(\Lambda, \text{Ind}_\Lambda^\Gamma(\rho_\Gamma)) = 0 \). We
consider the affine action associated to the representation \( Ind_{\Gamma}^A \rho_T \) and cocycle \( B \). By our assumption, this action has a fixed point in \( L^p(X, \mu) \). By Lemma 4.3, the affine action \( \zeta \) of \( \Lambda \) has a fixed point in \( \mathcal{F} \). Let \( f \in \mathcal{F} \) be a fixed point for \( \zeta \). It implies that \( f : X_A \to L^p(\Gamma) \) is a \( \Gamma \)-equivariant measurable map which is also \( \Lambda \)-invariant. Therefore, we can see \( f \) as a \( \Gamma \)-equivariant map from \( X_A \) to \( L^p(\Gamma) \). Since \( X_A \) admits a \( \Gamma \)-invariant probability measure \( \mu_A \), we can push it forward by \( f \) and obtain a \( \Gamma \)-invariant measure on \( L^p(\Gamma) \). By Lemma 2.14 in [BFGM07], when \( 1 < p < \infty \), we obtain that \( \eta \) has a fixed point in \( L^p(\Gamma) \), which is a contradiction.

**Step 2:**

By Lemma 4.2, we obtain that \( \Psi \circ [Ind_{\Gamma}^A \rho_T(\lambda)] = \left[ \int_{X_A}^\mathbb{S}_p \tau_A(\lambda) \right] \circ \Psi \), for all \( \lambda \in \Lambda \). Since \( \Lambda \) is non-amenable, by Lemma 2 of [BMV05] \( \int_{X_A}^\mathbb{S}_p \tau_A \) does not have almost invariant vectors. By Théorème 1 in [Gui72], we have the following result: Let \( G \) be a locally compact group and \( \Gamma \rtimes_B \mathcal{G} \) be an isometric \( G \)-representation. Assume that \( \pi \) does not weakly contain the trivial representation. Then \( \mathcal{T}_{\mathcal{H}}(G, \pi) = H^1_{\mathcal{H}}(G, \pi) \). Therefore, we obtain that

\[
H^1(\Lambda, Ind_{\Gamma}^A \rho_T, L^p(X, \mu)) = \int_{X_A}^\mathbb{S}_p \tau_A, L^p(X_A, \mu(\lambda))
\]

\[
\mathcal{T}^1(\Lambda, \int_{X_A}^\mathbb{S}_p \tau_A, L^p(X_A, \mu(\lambda))).
\]

**Step 3:**

Since \( \mathcal{T}^1(\Lambda, \int_{X_A}^\mathbb{S}_p \tau_A, L^p(X_A, \mu(\lambda))) \neq 0 \), using Proposition 3.6, we obtain that \( \mathcal{T}^1(\Lambda, \tau_A) \neq 0 \). Now, we define \( \chi : L^p(\lambda) \to L^p(\lambda) \) by extending the map \( \lambda \mapsto \lambda^{-1} \) for all \( \lambda \in \Lambda \). Observe that \( \chi \) is \( \Lambda \)-equivariant, where \( \Lambda \) acts on the domain and the range by left-regular representation \( \tau_A \) and right-regular representation \( \rho_A \), respectively. Since \( \chi \) is a \( \Lambda \)-equivariant invertible isometry, we have \( \mathcal{T}^1(\Lambda, \tau_A) = \mathcal{T}^1(\Lambda, \rho_A) \). Finally, using Proposition 3.3 and Theorem 3.5, we have \( L^p H^1(\lambda) \neq 0 \) for \( 1 < p < \infty \).

**Step 4:**

Now, we deal with case \( p = 1 \). Since Step 2 and Step 3 hold for any \( p \geq 1 \), using these two steps we obtain that \( H^1(\Lambda, Ind_{\Gamma}^A(\rho_T)) \neq 0 \) implies \( \mathcal{T}^1(\Lambda, \int_{X_A}^\mathbb{S}_p \tau_A, L^p(X_A, \mu(\lambda))) \neq 0 \), and which further implies that \( L^p H^1(\lambda) \neq 0 \).
Therefore, contra-positively, \( l^p H^1(\Lambda) = 0 \) implies that \( H^1(\Lambda, Ind_\rho^\Gamma) = 0 \). Now, using Step 1, we obtain that there exists a \( \Gamma \)-invariant measure on \( l^p(\Gamma) \).

Since the argument of the proof of Lemma 2.14 (4) \( \Rightarrow (1) \) in [BFGM07] holds for \( p = 1 \), we obtain that \( \eta \) has bounded orbits in \( l^p(\Gamma) \). Hence we have our theorem for \( p = 1 \).

\[ \square \]

6. Applications

6.1. Conformal dimension and first \( l^p \)-cohomology of groups. There is an excellent result due to Bourdon and Pajot [BP03], which relates geometric quantity ‘conformal dimension’ of the boundary of a class of hyperbolic groups and ‘algebro-analytical quantity’ the first \( l^p \)-cohomology of this class of groups.

**Theorem 6.1.** [BP03][Bourdon-Pajot, Corollaire 0.4] Let \( \Gamma \) be a hyperbolic group without torsion and with boundaries having ‘combinatorial Loewner property’ (CLP). Then

\[
\text{Confdim}\Gamma = \inf \{ p \neq 0 : l^p H^1(\Gamma) \neq 0 \}
\]

Now, using the above theorem and Theorem 1.1, we obtain the following corollary:

**Corollary 6.2.** Let \( \Gamma \) and \( \Lambda \) are two hyperbolic groups without torsion and with boundaries having Combinatorial Loewner Property. Suppose, \( \Gamma \) and \( \Lambda \) are \( L^q\)-ME, for \( q > \text{Confdim}(\Gamma), \text{Confdim}(\Lambda) \) and \( \text{Confdim}(\Gamma), \text{Confdim}(\Lambda) > 1 \) Then, the conformal dimensions (of the canonical conformal gauge) of the groups are equal.

6.2. Free groups and surface groups are not \( L^1 \)-Measure Equivalent. Let \( F_n \) be the free group with finitely many generators \( n \geq 2 \) and \( \Sigma_g \) be the surface group with genus \( g \geq 2 \). We briefly show that there exists an affine action of \( F_n \) on \( l^1(F_n) \) (where the linear part is the left-regular representation on \( l^1(F_n) \)) which has unbounded orbits. We consider the Cayley graph \( G \) of \( F_n \) and we denote the identity element in the Cayley graph as ‘\( o \)’. We define a 1-cocycle \( b : F_n \rightarrow l^1(F_n) \) as follows: \( b(g)(h) = 1 \) if the path from \( g \) to \( h \) directs towards \( o \) and \( b(g)(h) = 0 \) otherwise. In other words, \( b(g) = 1_{\overline{og}} \), where \( \overline{og} \) is the set of points in the line segment connecting \( o \) and \( g \). It is easy to see that \( b \) is a 1-cocycle. Since \( ||b(g)|| = d(o, go) \), we obtain that \( ||b(g)|| \rightarrow \infty \) as \( |g| \rightarrow \infty \). Therefore, the affine action associated to the 1-cocycle \( b \) has unbounded orbits.

Now, since \( l^1 H^1(\pi_1(\Sigma_g)) = 0 \) (see Subsection 3.2), using Main Theorem 1.1, we obtain the following corollary:

**Corollary 6.3.** The free group \( F_n \) with \( n \geq 2 \) and the surface group \( \pi_1(\Sigma_g) \) with genus \( g \geq 2 \) are not \( L^1\)-ME.

This corollary has been proved by different method in Lemma 5.4 of [BFS13].
6.3. The fundamental group of 3-manifolds and \( L^q \)-Measure Equivalent. In this subsection, we discuss \( L^q \)-ME between the fundamental groups of 3-manifolds (the extensions of surface groups and integers) with three geometries \( \mathbb{H}^3 \), \( \mathbb{H}^2 \times \mathbb{R} \) and \( SL_2(\mathbb{R}) \) of Thurston’s eight geometries. Let \( \Gamma \) be the fundamental group of a closed 3-manifold \( M \), with constant sectional curvature -1, and which is a fiber bundle over the circle and whose fibers are a closed surface of genus at least 2. Since \( \Gamma \) is a cocompact lattice in \( \text{Isom} (\mathbb{H}^3) \), \( l^p H^1 (\Gamma) = 0 \) iff \( p \in [1, 2] \).

Now, we consider the non-trivial central extension \( \widetilde{\pi_1} (\Sigma_g) \) of surface group \( \pi_1 (\Sigma_g) \) and the trivial central extension \( \pi_1 (\Sigma_g) \times \mathbb{Z} \). We know from \([DT16]\) (Theorem 1.1) that \( \widetilde{\pi_1} (\Sigma_g) \) and \( \pi_1 (\Sigma_g) \times \mathbb{Z} \) are not \( L^1 \)-ME. Since the center of \( \pi_1 (\Sigma_g) \) and \( \pi_1 (\Sigma_g) \times \mathbb{Z} \) are infinite, by using Proposition 1.9 in \([Bou10]\) we have the first \( l^p \)-cohomology of these two groups are zero for all \( p \in (1, \infty) \). Now, using Theorem 1.1, we obtain the following corollary:

**Corollary 6.4.** Any cocompact lattice in \( \text{Isom}(\mathbb{H}^3) \) is not \( L^{2+\epsilon} \)-ME with \( \widetilde{\pi_1} (\Sigma_g) \) (QI to \( SL_2(\mathbb{R}) \)) or \( \Sigma_g \times \mathbb{Z} \) (QI to \( \mathbb{H}^2 \times \mathbb{R} \)) for all \( \epsilon > 0 \). In particular, \( \pi_1 (\Sigma_g) \) (QI to \( SL_2(\mathbb{R}) \)) or \( \pi_1 (\Sigma_g) \times \mathbb{Z} \) (QI to \( \mathbb{H}^2 \times \mathbb{R} \)) are not \( L^{2+\epsilon} \)-ME with \( \Gamma \) for all \( \epsilon > 0 \), where \( \Gamma \) is the cocompact lattice \( \text{Isom}(\mathbb{H}^3) \) as defined above.

This corollary also follows from Theorem A of \([BFS13]\).

7. Questions

We can define higher \( l^p \)-cohomology groups \( l^p H^k (\Gamma) \) of a group \( \Gamma \) for any \( k \in \mathbb{N} \) (see Section 1.3 in \([Bou10]\) for the details). We do not know whether our main theorem 1.1 is true for higher \( l^p \)-cohomology groups.

**Question 1:** Suppose two non-amenable groups \( \Gamma \) and \( \Lambda \) are \( L^q \)-ME for some \( q > 1 \). Is it true that \( l^p H^k (\Gamma) \neq 0 \) if and only if \( l^p H^k (\Lambda) \neq 0 \), where \( 1 < p \leq q \) and \( k > 1 \).

We study Corollary 6.2 in the context of hyperbolic groups with CLP. We do not know whether it is true without the assumption of CLP or more generally for all hyperbolic groups.

**Question 2:** Is the theorem true for all hyperbolic groups?

8. Appendix

In the literature, it has not been written much so far on the group cohomology with coefficients in a direct integral of Banach spaces (in particular for \( l^p \)-spaces). We give a short introduction of this topic and we prove Proposition 3.6.
We consider a locally compact second countable group $G$ with Haar measure $\nu$ and a measurable field of strongly continuous representations on a Banach space $(E, \|\cdot\|)$ parametrized by the measure space $(X, \mu)$ and denoted by $\{\tau_G^{(x)} : x \in X\}$. We define the direct integral of the representations $\tau^{(x)}_\Lambda$ over the measure space $(X, \mu)$, denoted by $\int_X \tau^{(x)}_\Lambda d\mu(x)$, using 'Bochner integral'. We first define 'Bochner Measurable' function. The function $s : X \to E$ defined as $s = \sum_{i=1}^k \chi_{X_i} \xi_i$ (where $X_i$'s are disjoint measurable subsets of $X$ and $\xi_i$'s are vectors in $E$) is called a 'simple function'. Now, a function $f : X \to E$ is called a Bochner Measurable function if there exists a sequence of simple functions $\{s_j\}_{j=1}^\infty$ such that $\|f(x) - s_j(x)\| \to 0$ as $j \to \infty$ for almost all $x \in X$. Now, we define $\int_X \tau^{(x)}_\Lambda d\mu(x)$ in the following way:

$$\int_X \tau^{(x)}_\Lambda d\mu(x) := \{f : X \to E : \int_X \|f(x)\|^p d\mu(x) < \infty \text{ and } f \text{ is Bochner measurable}\},$$

where $G$ acts on the above-mentioned space in the canonical way. We prove the following proposition.

**Theorem 8.1.** If $\overline{H}^n(G, \tau^{(x)}_G) = 0$ for a.e. $x$, then $\overline{H}^n(G, \int_X \tau^{(x)}_\Lambda d\mu(x)) = 0$. In particular, $\overline{H}^1(\Lambda, \pi) = 0$ implies that $\overline{H}^1(\Lambda, \int_X \tau^{(x)}_\pi d\mu(x)) = 0$ where $\Lambda$ is a discrete group, $\tau^{(x)}_\pi = \pi$ for all $x$ and $\pi$ is the left-regular or right-regular representation of $\Lambda$ on $L^1(\Gamma)$ where $1 \leq p < \infty$.

We first introduce some notations and definitions. Suppose $p \in [1, \infty)$. We denote by $L^p_{loc}(G^n, E)$ the set of measurable functions $f$ from $G^n$ to $E$ such that $f|_K \in L^p(K, E)$ for all $K \in K(G^n)$, where we consider the product measure $\nu^n := \nu \times \cdots \times \nu$ (n-times) and $K(G^n)$ is the collection of compact subsets of $G^n$. One gives the topology defined by the semi-norms

$$P_K(f) = \left(\int_K \|f(x)\|^p d\nu^n(x)\right)^{1/p}$$

We denote by $L^p_C(G^n, E)$ the space of $p$-integrable functions with compact support. This space is an inductive limit of the spaces $L^p(K, E)$, where $K \in K(G^n)$. We define $C(G^n, E)$ as the collection of continuous functions $f$ from $G$ to $E$ with compact convergence topology.

The reduced cohomology defined by the following chain is denoted by $\overline{H}^{n,p}(G, E)$:

$$0 \longrightarrow L^p_{loc}(G, E) \longrightarrow L^p_{loc}(G^2, E) \longrightarrow \cdots$$

**Lemma 8.2.** The dual of $L^p_{loc}(G^n, E)$ can be identified algebraically with $L^q_C(G^n, E')$ by the following duality:

$$\langle \phi, \psi \rangle := \int_{G^n} \langle \phi(x), \psi(x) \rangle d\nu^n(x),$$
where $1/p + 1/q = 1$, $E'$ is the dual space of $E$ and $<\phi(x), \psi(x)> = \psi(x)(\phi(x))$ for all $x \in G^n$. We use Hahn-Banach Theorem to conclude the above lemma.

By the above duality, we obtain the following complex:

$$\cdots \to L^q_C(G^2, E') \to L^q_C(G, l^q(\Lambda)) \to 0,$$

where $1/p + 1/q = 1$.

**Proof of Theorem 3.6:**

**Step 1:** We claim that: if $H_{n,p}(G, \tau_G(x)G) = 0$ for a.e. $x$, then $H_{n,p}(G, \int_G \tau_G(x) \mu(x)) = 0$. We use the same argument given in Proposition 2.6 (page 190, [Gui80]) to prove the claim, where use the duality between $L^p_{loc}(G^n, E)$ and $L^q_C(G^n, E')$ as given in Lemma 8.2.

**Step 2:** Using property (i) of D.2.2 (p. 340, [Gui80]), we obtain that $C(G^n, E)$ can be canonically identified with a dense subset of $L^p_{loc}(G^n, E)$. Hence the $n$-th reduced cohomology w.r.t. the chain $C(G^n, E)$, i.e., $\overline{H}^n(G, E)$, is equal to the $n$-th reduced cohomology w.r.t. to the chain $L^p_{loc}(E)$, i.e., $\overline{H}^{n,p}(G, E)$.

**Step 3:** Using Step 2, we obtain that $\overline{H}^{n,p}(G, \tau_G(x)) = \overline{H}^n(G, \tau_G(x))$ for all $x$ in $X$ and $\overline{H}^n(G, \int_X \tau_G(x) \mu(x)) = \overline{H}^{n,p}(G, \int_X \tau_G(x) d\mu(x))$. Hence, we have our theorem using Step 1 and Step 2. □

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DEPARTMENT OF MATHEMATICS, SRM UNIVERSITY AP, ANDHRA PRADESH, INDIA 522240
Email address: kdas.math@gmail.com, kajaldas.m@srmup.edu.in