APPROXIMATION BY POLYNOMIALS IN TWO DIFFEOMORPHISMS

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We denote by $C$ the complex plane. If $f$ and $g$ are complex-valued functions on a set $S$, then $C[f,g]$ denotes the algebra of polynomials in $f$ and $g$, with complex coefficients, regarded as functions on $S$.

**Theorem.** Let $1 \leq k \in \mathbb{Z}$, and let $f$ and $g$ be $C^k$ diffeomorphisms of $C$ into $C$, having opposite degrees. Then $C[f,g]$ is dense in the Fréchet space $C^k(C)$, i.e., given $h \in C^k(C)$, and $X \subset C$ compact, there is a sequence $h_n \in C[f,g]$ such that $h_n$ and its derivatives up to order $k$ tend to $h$ and its derivatives, uniformly on $X$.

In case $f(z) = z$ and $g(z) = z$, the Theorem reduces to a result of Weierstrass.

Since each diffeomorphism of the closed unit disc $D$ into $C$ extends to a diffeomorphism of $C$ into $C$, we deduce the following.

**Corollary.** Let $f$ and $g$ be $C^1$ diffeomorphisms of $D$ into $C$, having opposite degrees. Then $C[f,g]$ is dense in $C(D)$.

This settles an old chestnut in the field of uniform algebras. It remains open whether the Corollary works for $k = 0$, i.e., for all pairs of homeomorphisms of opposite degrees.

**Proof of Theorem.** Without loss of generality, we may take $g = z$, because the chain rule for $D^j(h \circ g)$ is linear in $h$ and involves only $D^i h$ and $D^j g$ for $0 \leq i \leq j$.

Since $f$ has degree $-1$, we deduce that $|f_\bar{z}| > |f_z|$ on $C$. In particular, $f_\bar{z} \neq 0$, so the graph $G = \{(z, f(z)) : z \in C\}$, which is a $C^k$ submanifold of $C^2$, has no complex tangents. By the Range-Siu theorem [2], $C^k(G)$ is the closure of the space $\mathcal{O}(G)$ of all functions holomorphic in a neighbourhood of $G$. If we can show that $G$ has an exhaustion by polynomially-convex compact sets, then by the functional calculus [4, Chapter 8], it will follow that $C[z,w]$ is dense in $\mathcal{O}(G)$, and hence in $C^k(G)$; since $z \mapsto (z, f)$ is a $C^k$ diffeomorphism of $C \to G$, this will imply that $C[z,f]$ is dense in $C^k(C)$. Thus it suffices to show that $X = \{(z, f(z)) : z \in K\}$ is polynomially-convex whenever $K \subset C$ is a closed disc.

Fix a closed disc $K \subset C$. By modifying $f$ off $K$, if need be, we may assume $f$ maps $C$ onto $C$, that $Df$ and $Df^{-1}$ are bounded and uniformly continuous, and that $|f_\bar{z}|$ and $1 - |f_z/f_\bar{z}|$ are bounded away from zero. We need two lemmas, which are essentially classical results of Wermer.
LEMMA 1. There exists a constant $\lambda_1 > 0$ such that

$$(z - a)(f(z) - f(a)) + \lambda f_2(a)$$

is nonzero whenever $0 < \lambda < \lambda_1$, $a \in \mathbb{C}$, and $z \in \mathbb{C}$.

PROOF. Pick $\delta > 0$ such that the modulus of continuity $\omega(\delta)$ of $Df$ at $\delta$ is less than half \((\inf |f_d|)(1 - \sup |f_2/f_2|)\). Applying the mean value theorem to the real and imaginary parts of $f$ we deduce that for $0 < |z - a| < \delta$, the value $f(z) - f(a)$ differs from $f_2(a)(z - a) + f_2(a)(z - a)$ by less than $2\omega(\delta)|z - a|$. Thus

$$\text{Re}\left(\frac{(z - a)(f(z) - f(a))}{f_2(a)}\right) > 0$$

whenever $|z - a| < \delta$. But for $|z - a| \geq \delta$,

$$\left|\frac{(z - a)(f(z) - f(a))}{f_2(a)}\right| \geq \frac{\delta^2(\sup |Df^{-1}|)^{-1}}{\inf |f_2|}.$$

Denoting the right-hand side by $\lambda_1$, we see that $(z - a)(f(z) - f(a))/f_2(a)$ omits $\{-\lambda: 0 < \lambda < \lambda_1\}$, for all $a$ and $z$, so the lemma is proved.

Let us denote the uniform closure of $C[z, f]$ in $C(K)$ by $A$.

LEMMA 2. Suppose that for each $a \in K$, there exists a sequence $\lambda_n \downarrow 0$ such that $(z - a)(f(z) - f(a)) + \lambda_n f_2(a)$ is invertible in $A$. Then $A = C(K)$.

PROOF. Briefly, let $\mu$ be a measure on $K$, annihilating $A$. It suffices to show that the Cauchy transform $\hat{\mu}(a) = \int d\mu(\zeta)/\zeta - a$ vanishes at every point $a \in K$ at which the Newtonian potential $\int |\mu(\zeta)|/|\zeta - a|$ is finite. But the hypothesis, together with Lemma 1, yields a sequence $f_n \in A$ such that $f_n \to (z - a)^{-1}$, pointwise on $K \sim \{a\}$, and $|f_n(z)| \leq \text{const} |z - a|^{-1}$. Thus the dominated convergence theorem yields the desired result.

We remark that the hypothesis of Lemma 2 can be weakened to “almost all $a \in K$”.

CONCLUSION OF PROOF OF THEOREM. Suppose $X$ is not polynomially-convex. Then $A \neq C(K)$, so by Lemma 2, there exists $a \in K$ and $\lambda_2 > 0$ such that for every $\lambda$ with $0 < \lambda < \lambda_2$, the polynomial $(z - a)(w - f(a)) + \lambda f_2(a)$ has a zero somewhere on the polynomially-convex hull of $X$. Fix $\lambda$, with $0 < \lambda < \min\{\lambda_1, \lambda_2\}$. Then the family of algebraic curves

$$(z - a - t)(w - f(a + t)) + \lambda f_2(a + t) = 0 \quad (0 \leq t < \infty)$$

is a curve of algebraic hypersurfaces which meets the hull of $X$, does not meet $X$ (by Lemma 1), and goes to the hyperplane at infinity (since $f$ maps onto $C$, and $f_2$ is bounded). This contradicts Oka’s characterization of polynomial hulls, as given in [3, (1.2), p. 263]. Thus $X$ is polynomially-convex, and we are done.

We remark that minor modifications to the foregoing proof permit us to strengthen the Corollary, as follows:

Let $f$ be an orientation-reversing homeomorphism of $\mathbb{C}$ into $\mathbb{C}$, which is locally $C^1$ and noncritical off a closed set $E$, having area zero and not separating the plane. Then $C[z, f]$ is dense in $C(C)$.  

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Also, for any compact set \( X \) in \( \mathbb{C} \) and for \( 0 < \alpha < 1 \), suppose \( \text{Lip}(\alpha, X) \) denotes the space of bounded functions \( g \) of \( X \) into \( \mathbb{C} \) such that for some \( K > 0 \), \(|g(z) - g(w)| \leq K|z - w|^{\alpha}\) for all \( z, w \in X \) with \( \text{norm sup}|g| + \text{Least } K \) and suppose \( \text{lip}(\alpha, X) \) denotes those functions \( g \in \text{Lip}(\alpha, X) \) such that, given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \(|g(z) - g(w)| \leq \epsilon|z - w|^{\alpha}\) whenever \( z \) and \( w \) satisfy \(|z - w| < \delta\). In view of the results given in [1, p. 227], the conclusion of the above remark implies \( C[z, f] \) is dense in \( \text{lip}(\alpha, X) \) for any compact set \( X \) in \( \mathbb{C} \).

Finally, we remark that the Theorem of this paper is sharp in the sense that one critical point destroys it.

**References**

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