Finite Quantum Fluctuations About Static Field Configurations

E. Farhi\textsuperscript{a}, N. Graham\textsuperscript{a}, P. Haagensen\textsuperscript{a}, and R. L. Jaffe\textsuperscript{a,b}

\textsuperscript{a}Center for Theoretical Physics, Laboratory for Nuclear Science and Department of Physics
Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

\textsuperscript{b}RIKEN BNL Research Center
Brookhaven National Laboratory, Upton, NY 11973

Abstract

We develop an unambiguous and practical method to calculate one-loop quantum corrections to the energies of classical time-independent field configurations in renormalizable field theories. We show that the standard perturbative renormalization procedure suffices here as well. We apply our method to a simplified model where a charged scalar couples to a neutral "Higgs" field, and compare our results to the derivative expansion.

PACS numbers: 11.10.Gh, 11.15.Kc, 11.27.+d, 11.30.Qc
I. INTRODUCTION

Solitons arise in many model field theories where nontrivial time-independent solutions to the classical field equations exist. The fluctuations of quantum fields about classical configurations are difficult to study and can qualitatively affect, even destabilize, solitons. Topological arguments support the stability of some particularly interesting solitons. However, much less is known about the fate of “non-topological solitons” that appear as minima in the classical action, but have no deeper claim to stability when quantum effects are taken into account.

The quantum corrections to the energies of classical field configurations are typically highly divergent in $3 + 1$ dimensions. In some cases, like the Skyrme model, the underlying theory is non-renormalizable, so the quantum contribution to the soliton’s energy is unavoidably cut-off dependent and ambiguous, like any other radiative correction in a non-renormalizable theory.

In this Letter we describe a systematic and efficient procedure for calculating the quantum fluctuations about time-independent field configurations in renormalizable field theories. We show that all divergences can be removed by the same renormalization procedure that renders the perturbative sector of the theory finite. The only “ambiguities” are the well-known scheme and scale dependences of the renormalization prescription that are resolved completely in the perturbative sector. The result of our program is a renormalized quantum “effective energy”, whose non-trivial minima (if they exist) describe solitons in the quantum theory.

First we show how to regulate and renormalize the divergences in the sum over quantum fluctuations. Then we develop calculational methods that are efficient and practical enough that quantum effects can be included in a search for stable field configurations. Our results take the form of an effective energy, $\mathcal{E}(\phi(\vec{x}), m, \{g\})$, depending on the “profile function” of the renormalized field, $\phi(\vec{x})$, the renormalized mass, $m$, and various renormalized couplings,
\{g\}, defined in the usual perturbative sector of the model. One may then search over the parameter space characterizing \(\phi(\vec{x})\) for minima of \(\mathcal{E}\) while holding \(m\) and \(\{g\}\) fixed. Here we treat the simple case of a charged scalar coupled to the field \(\phi\). Our methods can be generalized straightforwardly to models including fermions, gauge fields and self-coupled scalars. However, our approach is limited in that we work only to order \(\hbar\), and we only consider spherically symmetric profile functions.

The possibility that the top quark in the standard model might be described as a non-topological soliton [3] provided the original motivation for our work. The top quark Yukawa coupling, \(g_t\), leads to its mass, \(g_tv\), where \(v\) is the Higgs vacuum expectation value. For large \(g_t\) it would therefore appear favorable to suppress the Higgs condensate in the vicinity of the \(t\)-quark, and the top quark would be a sort of “bag”. However, gradient and potential energy terms in the Higgs sector of the classical action oppose the creation of a such a bag in the Higgs condensate. In order to study the problem at the quantum level it is necessary to regulate and renormalize the divergences in the \(t\)-quark fluctuations about a deformation of the Higgs condensate. It is crucial to hold the renormalized parameters of the standard model fixed while varying the possible profiles of the Higgs field.

Bagger and Naculich studied this problem by making a derivative expansion [4]. They also worked in a large-\(N\) approximation, in which there are no quantum corrections above order \(\hbar\). We would expect that if the \(t\)-quark is a bag, then the Higgs field will vary significantly — \(\Delta\phi \sim v\) — over distance scales of order the Compton wavelength of the top quark — \(\lambda \sim 1/g_tv\). However, the \(t\)-quark mass, \(g_tv\), also sets the scale for the derivative expansion. Thus all derivatives are of the same size, making the expansion unreliable. Our method is designed for such situations.

We can trace elements of our approach back to Schwinger’s work on QED in strong fields [5]. Schwinger studied the energy of the electron’s quantum fluctuations — the “Casimir energy” — in the presence of a prescribed, static configuration of electromagnetic fields. He isolated the divergences in low orders of perturbation theory. Our work can be viewed
as an extension of Schwinger’s to situations where the field is determined self-consistently by minimizing the total energy of the system including the Casimir energy. In addition we complete the renormalization program and develop practical computational methods in three dimensions. Dashen, Hasslacher and Neveu renormalized the divergent contributions to the energy of the $\phi^4$ kink and sine-Gordon soliton in 1+1 dimensions using a simple version of the method we propose here [8]. Ambiguities in these models recently pointed out and studied by Rebhan and van Nieuwenhuizen [7] can also be resolved with our methods. Studies of solitons in renormalizable models often note that the divergences in the quantum contribution to the soliton energy can be cancelled by the available counterterms [8]. However, we are not aware of any work in 3+1 dimensions in which renormalization of the field configuration energy is done in a manner consistent with on-shell mass and coupling constant renormalization in the perturbative sector.

II. FORMALISM

We consider a renormalizable field theory with a real scalar field $\phi$ coupled to a charged scalar $\psi$. We take the classical potential $V(\phi) \propto (\phi^2 - v^2)^2$, and $\psi$ acquires a mass through spontaneous symmetry breaking. At the quantum level we put aside the $\phi$ self-couplings and consider only the effects of the $\phi - \psi$ interactions. We further restrict ourselves to $\mathcal{O}(\bar{h})$ effects in the quantum theory, which correspond to one-loop diagrams.

Our model is defined by the classical action

$$S[\phi, \psi] = \int d^4x \left\{ \frac{1}{2} (\partial_\mu \phi)^2 - \frac{\lambda}{4!} (\phi^2 - v^2)^2 + \partial_\mu \psi^* \partial^\mu \psi - g \psi^* \phi^2 \psi + a (\partial_\mu \phi)^2 - b (\phi^2 - v^2) - c (\phi^2 - v^2)^2 \right\},$$

where we have separated out the three counterterms necessary for renormalization and written them in a convenient form. At one-loop order in $\psi$, these are the only counterterms required.

We quantize around the classical vacuum $\phi = v$ and define $h = \phi - v$, so that
\[ S[h, \psi] = \int d^4x \left\{ \frac{1}{2}(\partial_\mu h)^2 - \frac{m^2}{8v^2}(h^2 + 2vh)^2 + \partial_\mu \psi^* \partial^\mu \psi - M^2 \psi^* \psi - g(h^2 + 2vh)\psi^* \psi \right\}
\]
\[ + a(\partial_\mu h)^2 - b(h^2 + 2hv) - c(h^2 + 2hv)^2 \}
\]

where \( M = \sqrt{gv} \) is the \( \psi \) mass and \( m^2 = \lambda v^2 / 3 \) is the \( h \) mass.

The one-loop quantum effective action for \( h \) is obtained by integrating out \( \psi \) to leading order in \( \bar{h} \). We are interested in time-independent field configurations \( h = h(\vec{x}) \), for which the effective action yields an effective energy \( \mathcal{E}[h] \) that has three parts:

\[ \mathcal{E}[h] = \mathcal{E}_{\text{cl}}[h] + \mathcal{E}_{\text{ct}}[h] + \mathcal{E}_\psi[h] , \]

where \( \mathcal{E}_{\text{cl}}[h] \) is the classical energy of \( h \),

\[ \mathcal{E}_{\text{cl}}[h] = \int d^3x \left\{ \frac{1}{2}|\vec{\nabla} h|^2 + \frac{m^2}{8v^2}(h^2 + 2vh)^2 \right\} , \]

\( \mathcal{E}_{\text{ct}}[h] \) is the counterterm contribution,

\[ \mathcal{E}_{\text{ct}}[h] = \int d^3x \left\{ a|\vec{\nabla} h|^2 + b(h^2 + 2hv) + c(h^2 + 2hv)^2 \right\} , \]

and \( \mathcal{E}_\psi[h] \) is the one-loop quantum contribution from \( \psi \). \( \mathcal{E}_{\text{ct}}[h] \) and \( \mathcal{E}_\psi[h] \) are divergent, but we will see explicitly that these divergences cancel for any configuration \( h(\vec{x}) \).

We fix the counterterms by applying renormalization conditions in the perturbative sector of the theory. Having done so, we have defined the theory for all \( h(\vec{x}) \). We choose the on-shell renormalization conditions

\[ \Sigma_1 = 0, \quad \Sigma_2(m^2) = 0, \quad \text{and} \quad \left. \frac{d\Sigma_2}{dp^2} \right|_{m^2} = 0, \]

where \( \Sigma_1 \) and \( \Sigma_2(p^2) \) are the one- and two-point functions arising only from the loop and counterterms as seen in Fig. [4].
We denote the one-loop diagrams with one insertion by $\Omega$ and with two insertions by $\Pi(p^2)$, and find

$$
\Sigma_1 = 2vg\Omega + 2vb ,
$$

$$
\Sigma_2(p^2) = (2vg)^2\Pi(p^2) + g\Omega + b + (2v)^2c + ap^2 .
$$

(7)

Defining

$$
\Pi'(p^2) \equiv \frac{d\Pi(p^2)}{dp^2} ,
$$

(8)

the renormalization conditions eq.(6) then yield

$$
a = -(2vg)^2\Pi'(m^2) , \ b = -g\Omega , \ c = g^2(m^2\Pi'(m^2) - \Pi(m^2)) ,
$$

(9)

which we then substitute into the counterterm energy, eq. (5).

Now we consider the calculation of $E_\psi[h]$. This energy is the sum over zero point energies, $\frac{1}{2}\hbar\omega$, of the modes of $\psi$ in the presence of $h(\vec{x})$,

$$
E_\psi[h] = \sum_\alpha E_\alpha[h]
$$

(10)

where $E_\alpha$ are the positive square roots of the eigenvalues of a single particle Hamiltonian, $\hat{H}$, given by
\[ \hat{H} = -\nabla^2 + M^2 + g(h^2 + 2\nu h). \] (11)

The fact that \( \psi \) is complex accounts for the absence of \( \frac{1}{2} \) in eq. (11).

\( \mathcal{E}_\psi \) is highly divergent. However our model is renormalizable and therefore the counter-terms fixed in the presence of the trivial \( h \) must cancel all divergences in \( \mathcal{E}_\psi \). Rather than attempt to regulate the divergent sum in eq. (10) directly, we study the density of states that defines the sum. We can isolate the terms that lead to divergences in \( \mathcal{E}_\psi \) and renormalize them using conventional methods.

For fixed \( h(\vec{x}) \) the spectrum of \( \hat{H} \) given in eq. (11) consists of a finite number (possibly zero) of normalizable bound states and a continuum beginning at \( M^2 \), parameterized by \( k \), with \( E(k) = +\sqrt{k^2 + M^2} \). Furthermore, \( \hat{H} \) depends on \( h \) only through the combination
\[ \chi = h^2 + 2hv, \] (12)
so we can consider \( \mathcal{E}_\psi \) to be a functional of \( \chi \). We restrict ourselves to spherically symmetric \( h \). Then
\[ \mathcal{E}_\psi[\chi] = \sum_j E_j + \sum_\ell (2\ell + 1) \int dk \rho_\ell(k) E(k) \] (13)
where \( \rho_\ell(k) \) is the density of states in \( k \) in the \( \ell \)th partial wave and the \( E_j \) are the bound state energies. \( \rho_\ell(k) \) is finite, but the sum over \( \ell \) and the integral over \( k \) are divergent. Furthermore
\[ \rho_\ell(k) = \rho_\ell^{\text{free}}(k) + \frac{1}{\pi} \frac{d\delta_\ell(k)}{dk}, \] (14)
where \( \delta_\ell(k) \) is the usual scattering phase shift for the \( \ell \)th partial wave, and \( \rho_\ell^{\text{free}}(k) \) is the free \( (g = 0) \) density of states. This relationship between the density of states and the derivative of the phase shift is shown for example in [3].

At the outset, we subtract \( \rho^{\text{free}}(k) \) from the density of states since we wish to compare \( \mathcal{E}_\psi[\chi] \) to \( \mathcal{E}_\psi[0] \). Viewing \( \mathcal{E}_\psi[\chi] \) as the sum of one loop diagrams, we see that only the diagrams with one or two insertions of \( g\chi \) are divergent. A diagram with \( n \) insertions corresponds
to the \(n\)th term in the Born expansion, so all possible divergences can be eliminated by subtracting the first and second Born approximations from the phase shifts that determine the density of states. Standard methods allow us to construct the Born approximation for the phase shifts \([9]\), which is a power series in the “potential” \(g\chi\).

We define the combination

\[
\bar{\delta}_\ell(k) \equiv \delta_\ell(k) - \delta^{(1)}_\ell(k) - \delta^{(2)}_\ell(k) ,
\]

(15)

where \(\delta^{(1)}_\ell(k)\) and \(\delta^{(2)}_\ell(k)\) are the first and second Born approximations to \(\delta_\ell(k)\). We then have

\[
E_\psi[\chi] = \sum_j E_j + \sum_\ell (2\ell + 1) \int_0^\infty dk \frac{1}{\pi} \frac{d\delta_\ell(k)}{dk} E(k) + g\Omega \int \frac{d^3 p}{(2\pi)^3} \tilde{\chi}(\mathbf{p})
\]

\[
+g^2 \int \frac{d^3 p}{(2\pi)^3} \Pi(-\mathbf{p}^2) |\tilde{\chi}(\mathbf{p})|^2
\]

where

\[
\tilde{\chi}(\mathbf{p}) = \int d^3 x \chi(x) e^{-i\mathbf{p} \cdot \mathbf{x}} ,
\]

(17)

and likewise for \(\tilde{h}(\mathbf{p})\). Both \(\tilde{h}\) and \(\tilde{\chi}\) are real and depend only on \(q \equiv |\mathbf{p}|\) for spherically symmetric \(h\). We have subtracted out the order \(g\) and \(g^2\) contributions by using \(\bar{\delta}_\ell(k)\) instead of \(\delta_\ell(k)\), and added them back in by using their explicit diagrammatic representation in terms of the divergent constant \(\Omega\) and the divergent function \(\Pi(p^2)\).

We can now combine \(E_\psi\) and \(E_{ct}\) and obtain a finite result:

\[
E_\psi + E_{ct} = \sum_j E_j + \sum_\ell (2\ell + 1) \int_0^\infty dk \frac{1}{\pi} \frac{d\bar{\delta}_\ell(k)}{dk} E(k) + \Gamma_2[h]
\]

(18)

where

\[
\Gamma_2[h] = g^2 \int \frac{q^2 dq}{2\pi^2} \left\{ \left[ \Pi(-q^2) - \Pi(m^2) + m^2 \Pi'(m^2) \right] \tilde{\chi}(q)^2 + 4v^2 q^2 \Pi'(-q^2) \tilde{h}(q)^2 \right\} .
\]

(19)

\(\Pi\) is log divergent, but both \(\{\Pi(-q^2) - \Pi(m^2)\}\) and \(\Pi'\) are finite, so \(\Gamma_2[h]\) is finite as well.

Each term in the Born approximation to the phase shift goes to zero at \(k = 0\), so by Levinson’s theorem \(\bar{\delta}_\ell(0) = \delta_\ell(0) = \pi n_\ell\) where \(n_\ell\) is the number of bound states with angular
momentum \( \ell \). As \( k \to \infty \), \( \delta_{\ell}(k) \) falls off like \( \frac{1}{k} \); \( \delta_{\ell}^{(1)}(k) \) falls off like \( \frac{1}{k} \), and \( \delta_{\ell}^{(2)}(k) \) falls off like \( \frac{1}{k^2} \). Since the Born approximation becomes exact at large \( k \), \( \bar{\delta}_{\ell}(k) \) falls like \( \frac{1}{k^3} \), \( \delta_{\ell}^{(1)}(k) \) falls off like \( \frac{1}{k} \), and \( \delta_{\ell}^{(2)}(k) \) falls off like \( \frac{1}{k^2} \). Thus we see that the first subtraction renders each integral over \( k \) convergent. The second subtraction makes the \( \ell \)-sum convergent. We are then free to integrate by parts in (18), obtaining

\[
\mathcal{E}[h] = \mathcal{E}_{\text{cl}}[h] + \Gamma_2[h] - \frac{1}{\pi} \sum_{\ell} (2\ell + 1) \int_0^{\infty} dk \frac{\bar{\delta}_{\ell}(k)}{E(k)} + \sum_j (E_j - M). \tag{20}
\]

In this expression we see that each bound state contributes its binding energy, \( E_j - M \), so that the energy varies smoothly as we strengthen \( h \) and bind more states.

As noted in the Introduction, the representation of the Casimir energy as a regulated sum/integral over phase shifts plus a limited number of Feynman graphs was derived by Schwinger for the case of a prescribed background field. Our aim is to develop it into a practical tool to study the stability of non-trivial field configurations \( h(r) \).

### III. CALCULATIONAL METHODS

In this Section we describe the method that allows us to construct \( \mathcal{E}[h] \) as a functional of \( h \) and search for stationary points. We now consider the calculation of each of the terms in eq. (20) in turn. The classical contribution to the action is evaluated directly by substitution into eq. (1). \( \Gamma_2[h] \) of eq. (13) is obtained from a Feynman diagram calculation,

\[
\Gamma_2[h] = \frac{g^2}{(4\pi)^2} \int \frac{d^2 q}{2\pi^2} \int_0^1 dx \left\{ \log \frac{M^2 + q^2 x(1-x)}{M^2 - m^2 x(1-x)} - \frac{m^2 x(1-x)}{M^2 - m^2 x(1-x)} \right\} \tilde{\chi}(q)^2 \tag{21}
- \frac{x(1-x)}{M^2 - m^2 x(1-x)} 4v^2 q^2 \tilde{h}(q)^2 \right\}.
\]

The partial wave phase shifts and Born approximations are calculated as follows. The radial wave equation is

\[
- u''_{\ell} + \left[ \frac{\ell(\ell + 1)}{r^2} + g\chi(r) \right] u_{\ell} = k^2 u_{\ell}, \tag{22}
\]

where \( k^2 > 0 \), and \( \chi(r) \to 0 \) as \( r \to \infty \). We introduce two linearly independent solutions to eq. (22), \( u_{\ell}^{(1)}(r) \) and \( u_{\ell}^{(2)}(r) \), defined by
\[ u^{(1)}_\ell (r) = e^{i\beta_{\ell,k}(r)} h^{(1)}_\ell (kr) \]
\[ u^{(2)}_\ell (r) = e^{-i\beta_{\ell,k}^*(r)} h^{(2)}_\ell (kr) \]

where \( h^{(1)}_\ell \) is the spherical Hänkel function asymptotic to \( e^{ikr}/r \) as \( r \to \infty \), \( h^{(2)}_\ell (kr) = h^{(1)*}_\ell (kr) \), and \( \beta_{\ell,k}(r) \to 0 \) as \( r \to \infty \), so that \( u^{(1)}_\ell (r) \to e^{ikr} \) and \( u^{(2)}_\ell (r) \to e^{-ikr} \) as \( r \to \infty \).

The scattering solution is then
\[ u_\ell (r) = u^{(2)}_\ell (r) + e^{2i\delta_\ell (k)r} u^{(1)}_\ell (r) \]
and obeys \( u_\ell (0) = 0 \). Thus we obtain
\[ \delta_\ell (k) = 2 \text{ Re } \beta_\ell (k,0). \]

Furthermore, \( \beta_\ell \) obeys a simple, non-linear differential equation obtained by substituting \( u^{(1)}_\ell \) into eq. (22),
\[ -i\beta''_\ell - 2ikp_\ell (kr)\beta'_\ell + 2(\beta'_\ell)^2 + \frac{1}{2}g\chi(r) = 0 \]
where primes denote differentiation with respect to \( r \), and
\[ p_\ell (x) = \frac{d}{dx} \ln \left[ xh^{(1)}_\ell (x) \right] \]
is a simple rational function of \( x \).

We solve eq. (23) numerically, integrating from \( r = \infty \) to \( r = 0 \) with \( \beta'_\ell (k,\infty) = \beta'_\ell (k,0) = 0 \), to get the exact phase shifts. To get the Born approximation to \( \beta_\ell \), we solve the equation iteratively, writing \( \beta_\ell = g\beta_{\ell,1} + g^2\beta_{\ell,2} + \ldots \), where \( \beta_{\ell,1} \) satisfies
\[ -i\beta''_{\ell,1} - 2ikp_{\ell,1}(kr)\beta'_{\ell,1} + \frac{1}{2}g\chi(r) = 0 \]
and \( \beta_{\ell,2} \) satisfies
\[ -i\beta''_{\ell,2} - 2ikp_{\ell,2}(kr)\beta'_{\ell,2} + 2(\beta'_{\ell,1})^2 = 0. \]

We can solve efficiently for \( \beta_{\ell,1} \) and \( \beta_{\ell,2} \) simultaneously by combining these two equations into a coupled differential equation for the vector \( (\beta_{\ell,1}, \beta_{\ell,2}) \). This method is much faster.
than calculating the Born terms directly as iterated integrals in \( r \) and will generalize easily to a theory requiring higher-order counterterms.

Having found the phase shifts, we then use Levinson’s theorem to count bound states. We then find the energies of these bound states by using a shooting method to solve the corresponding eigenvalue equation. We use the effective range approximation \(^1\) to calculate the phase shift and bound state energy near the threshold for forming an s-wave bound state.

**IV. RESULTS**

For the model at hand, we calculated the energy \( \mathcal{E}[h] \) for a two-parameter \((d \text{ and } w)\) family of gaussian backgrounds

\[
h(r) = -d v e^{-r^2v^2/2w^2}.
\] (30)

In Fig. 2, we show results which are representative of our findings in general. We plot the energy of configurations with fixed \( d = 1 \) as a function of \( w \), for \( g = 1, 2, 4, 8 \) (from top to bottom). We note that to this order, for \( g = 8 \) the vacuum is unstable to the formation of large \( \phi = 0 \) regions.
To explore whether the charged scalar forms a non-topological soliton in a given \( \phi \) background, we add to \( \mathcal{E}[h] \) the energy of a “valence” \( \psi \) particle in the lowest bound state. We then compare this total energy to \( M \), the energy of the \( \psi \) particle in a flat background, to see if the soliton is favored. This is the scalar model analogue of \( t \)-quark bag formation.

For fixed \( g \) and \( m \), we varied \( h \) looking for bound states with energy \( E \) such that \( E + \mathcal{E}[h] < M \). However, for those values of \( g \) and \( m \) where we did find such solutions, we always found that by increasing \( w \), we could make \( \mathcal{E}[h] < 0 \), so that the vacuum is unstable, as we pointed out above in the case of \( g = 8 \) in Fig. 2. Thus we find that if we stay in the \( g,m \) parameter region where the vacuum is stable, the minimum is at \( h = 0 \), so there are no nontopological solitons.

Although we did not find a non-trivial solution at one-loop order in this simple model, our calculation demonstrates the practicality of our method. We can effectively characterize and search the space of field configurations, \( h(r) \), while holding the renormalized parameters of the theory fixed. The same methods can be used to study solitons in theories with richer
V. DERIVATIVE EXPANSION

Our results are exact to one-loop order. The derivative expansion, which is often applied to problems of this sort, should be accurate for slowly varying \( h(r) \). We found it useful to compare our results with the derivative expansion for two reasons: first, we can determine the range of validity (in \( d \) and \( w \)) of the derivative expansion; and second, where the derivative expansion is expected to be valid, it provides a check on the accuracy of our numerical work and C++ programming. Where expected, the two calculations did agree to the precision we specified (1\%).

In our model, the first two terms in the derivative expansion of the one-loop effective Lagrangian can be calculated to be

\[
L_1 = L_{ct} + \alpha z + \beta z^2 + \frac{g^2 v^4}{32\pi^2} \left[ (1 + z)^2 \ln(1 + z) - z - \frac{3}{2} z^2 \right] + \frac{g}{48\pi^2 v^2} \frac{1}{1 + z} (\partial_\mu z)^2 ,
\]

(31)

where \( z = g\chi/M^2 = (h^2 + 2hv)/v^2 \), \( \alpha \) and \( \beta \) are cutoff-dependent constants, and \( L_{ct} \) is the same counterterm Lagrangian as we used in Sec. 2. For \( \phi^4 \) scalar field theory a similar result was first derived in [10]. The last term above is proportional to \( (\partial h)^2 \), and is completely cancelled by a finite counterterm that implements the renormalization prescription of Sec. 2.

In this prescription, counterterms also cancel the \( \alpha z \) and \( \beta z^2 \) terms above. Thus the \( \mathcal{O}(p^2) \) derivative expansion for the effective energy, to be compared with the phase shift expression for \( \mathcal{E}[h] \), is

\[
\mathcal{E}_{DE}[h] = \int d^3x \left\{ \frac{1}{2} (\nabla h)^2 + \frac{m^2}{8v^2} (h^2 + 2hv)^2 + \frac{g^2 v^4}{32\pi^2} \left[ (1 + z)^2 \ln(1 + z) - z - \frac{3}{2} z^2 \right] \right\} .
\]

(32)

The results of the comparison with the phase shift method can be seen in Fig. 3 for \( d = 0.25 \), and \( g = 4 \). A similar pattern holds in general for other values of both \( d \) and \( g \). As the width becomes larger, the two results merge. This is as expected, since it is for large widths, and thus small gradients, that we expect the derivative expansion to yield accurate
results. As the width tends to zero, both results tend to zero, and the fact that the plot tends to 1 simply indicates that the derivative expansion result goes to zero faster than the phase shift result.

![Graph](image)

FIG. 3. \( \frac{\mathcal{E} - \mathcal{E}_{\text{DE}}}{\mathcal{E}} \) for \( d = 0.25, g = 4 \), as a function of \( w \).

VI. CONCLUSIONS

We have presented a numerically tractable method for evaluating the one-loop effective energy of a static background field configuration in a renormalizable quantum field theory. Since we rely on calculating phase shifts, the method is only suitable for rotationally invariant (or generalized rotationally invariant) backgrounds. The model explored in the present work is particularly simple and does not support any solitons. However, our methods could just as well be applied to any renormalizable field theory. They can be used to study the one-loop quantum stability of field configurations in the standard electroweak model as well as various unified models that support monopoles, strings and the like. Topologically non-trivial field
configurations with maximal symmetry, like the “hedgehog solutions” in chiral models can be studied in this fashion. Ultimately we hope to be able to reliably determine whether large Yukawa couplings may yield solitons in the standard electroweak model.

We would like to thank J. Goldstone, R. Jackiw, K. Johnson, K. Kiers, S. Naculich, B. Scarlet, R. Schrock, M. Tytgat, and P. van Nieuwenhuizen for helpful conversations, suggestions and references. This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under cooperative research agreement #DF-FC02-94ER40818, and by the RIKEN BNL Research Center. N. G. is supported in part by an NSF Fellowship.
REFERENCES

[1] S. Coleman, *Aspects of Symmetry* (Cambridge University Press, Cambridge, 1985); R. Rajaraman, *Solitons and Instantons* (North-Holland, Amsterdam, 1982).

[2] T.H.R. Skyrme, Proc. R. Soc. A260 (1961) 127.

[3] R. Johnson and J. Schechter, Phys. Rev. D36 (1987) 1484.

[4] J. Bagger and S. Naculich, Phys. Rev. Lett. 67 (1991) 2252, Phys. Rev. D45 (1992) 1395.

[5] J. Schwinger, Phys. Rev. 94 (1954) 1362. See also W. Greiner, B. Muller and J. Rafelski, *Quantum Electrodynamics of Strong Fields* (Springer, Berlin, 1985).

[6] R. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D10 (1974) 4114, 4130.

[7] A. Rebhan and P. van Nieuwenhuizen, Nucl. Phys. B508 (1997) 449.

[8] G. Ripka and S. Kahana, Phys. Lett. 155B (1985) 327 and Phys. Rev. D36 (1987) 1233; R. J. Perry, Nucl. Phys. A467 (1987) 717; M. Li and R. J. Perry, Phys. Rev. D37 (1988) 1670; B. Moussalam, Phys. Rev. D40 (1989) 3430; G. Anderson, L. Hall and S. Hsu, Phys. Lett. 249B (1990) 505; J. Zuk, Phys. Rev. D43 (1991) 1358; S. Dimopoulos, B. Lynn, S. Selipsky and N. Tetradis, Phys. Lett. 253B (1991) 237; M. Bordag and K. Kirsten, Phys. Rev. D53 (1996) 5753;

[9] L. Schiff, *Quantum Mechanics* (McGraw-Hill, NY, 1968).

[10] S. Coleman and E. Weinberg, Phys. Rev. D7 (1973) 1888; C. Itzykson, J. Iliopoulos and A. Martin, Rev. Mod. Phys. 47 (1975) 165.