SOMERSAULTS ON UNSTABLE ISLANDS

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Abstract. We give extensions of Katok’s horseshoe constructions, comment on related results, and provide a self-contained proof. We consider either a $C^{1+\alpha}$ diffeomorphism preserving a hyperbolic measure or a $C^1$ diffeomorphism preserving a hyperbolic measure whose support admits a dominated splitting.

1. Introduction

Given a diffeomorphism $f$ of a smooth Riemannian manifold $M$, we call a set $\Gamma \subset M$ basic (with respect to $f$) if it is compact and with respect to $f$ invariant, locally maximal, and hyperbolic. Let $M(f|\Gamma)$ denote the space of $f$-invariant Borel probability measure on $\Gamma$. We call an ergodic $f$-invariant Borel probability measure hyperbolic if it has positive and negative but no zero Lyapunov exponents. In this paper we consider in parallel two assumptions:

(A1) $f$ is $C^{1+\alpha}$ diffeomorphism for some $\alpha > 0$ preserving a hyperbolic ergodic measure $\mu$ with positive entropy, or that

(A2) $f$ is a $C^1$ diffeomorphism preserving a hyperbolic ergodic measure $\mu$ with positive entropy whose support admits a dominated splitting.

Theorem 1. Let $f$ be a $C^1$ diffeomorphism of a smooth Riemannian manifold $M$ and $\mu$ a hyperbolic ergodic $f$-invariant Borel probability measure with positive entropy $h_\mu(f) > 0$. Denote $\chi(\mu) = \min_j |\chi_j(\mu)|$, where $\chi_j(\mu)$ denote the Lyapunov exponents of $\mu$. Assume either hypothesis (A1) or hypothesis (A2).

Then for every $e \in (0, h_\mu(f)]$ and every sufficiently small $r > 0$ there exists a set $\Gamma \subset M$ and a positive integer $m$ so that we have

i) $\Gamma$ is basic and topologically mixing (with respect to $f^m$),

ii) $\Gamma$ is contained in the $r$-neighborhood of the support of $\mu$.

iii) $d(\nu, \mu) < r$ for every $\nu \in M(\Gamma)$, where $d$ is a metric that generates the weak* topology,
iv) $|h(f|\Gamma) - e| < r$.

v) For every continuous function $\varphi: M \to \mathbb{R}$ the topological pressure of $\varphi$ (with respect to $f|\hat{\Gamma}$, where $\hat{\Gamma} = \Gamma \cup f(\Gamma) \cup \ldots \cup f^{m-1}(\Gamma)$) satisfies

$$|P(\varphi, f|\hat{\Gamma}) - (e + \int \varphi \, d\mu)| < r,$$

vi) for every Lyapunov regular point $x \in \Gamma$ we have

$$\left| \frac{1}{m} \log |\text{Jac} \, df_{E^u}^m| - \int \log |\text{Jac} \, df_{E^u}| \, d\mu \right| < r,$$

Results of this type are widely referred to Katok [13] or Katok and Mendoza [15, Supplement S.5] (see also [2, 3]).

Let us mention some extensions to other settings. An earlier related statement for continuous and piecewise monotone maps of the interval goes back to Misiurewicz and Szlenk [23]. Corresponding properties of Theorem 1 in the case of a measure with only positive Lyapunov exponent of holomorphic maps were shown by Przytycki and Urbański [27]. The case of general $C^{1+\alpha}$ maps is discussed by Chung [9] and Yang [36] (see also Buzzi [8] for a sketch) and of maps with singular or critical points by Gelfert [11]. A related result is established in the setting of dyadic diophantine approximations by Persson and Schmeling [24, 26]. Katok [14, Section 4] gave a version for flows.

Mendoza in his PhD thesis [19] showed that in the case of a $C^2$ surface diffeomorphism $f$ leaving invariant an ergodic hyperbolic SRB measure, there exists a sequence of horseshoes with unstable dimension (the Hausdorff dimension of its intersection with local unstable manifolds) converging to 1 (see [21]). See also [20, 22] for related work. However, Mendoza’s general arguments that the Hausdorff measure of the approximating horseshoes is uniformly bounded away from zero are not correct [32]. Sánchez-Salas in his PhD thesis [29] continued studying a $C^2$ diffeomorphism of a compact manifold of dimension $\geq 2$ leaving invariant an ergodic hyperbolic SRB measure and showed that this measure can be approximated by ergodic measures supported on horseshoes with arbitrarily large unstable dimension [30]. His studies are based on Markov towers that can also be described by horseshoes with infinitely many branches and variable return times, this enables uniform bounds on the distortion. In [31] he also establishes a version for interval maps using Markov towers with good fractal properties. In [17] Luzzatto and Sánchez-Salas study “variable-time horseshoes” in order to achieve weak* convergence of all measures on the horseshoes towards the hyperbolic one (compare property iii) in the above theorem).

In the $C^1$ setting Pesin-Katok theory does not work in general (e.g. [4] provide examples of $C^1$ surface diffeomorphisms and hyperbolic measures without stable/unstable manifolds). Gan [10] discusses extensions to $C^1$ surface diffeomorphisms. He restricts to the case that the set of all preperiodic points of hyperbolic (neither sink nor source) periodic points possesses
a dominated splitting. This allows him to apply a result by Pujals and Sambarino [28] stating that corresponding local un-/stable manifolds have uniform size and applying a closing lemma by Liao [16]. Detailed construction of invariant manifolds for dominated hyperbolic measures are due to Abdenur, Bonatti, and Crovisier [1, Section 8]. In the proof of Theorem 1 in the $C^1$ setting we will make use of the method of fake foliations presented in [7].

The following result will be an almost immediate consequence of Theorem 1.

**Theorem 2.** If $h(f|M) > 0$ then for every $e \in [0, h(f|M))$ there exists a compact $f$ invariant and hyperbolic set $\Gamma \subset M$ such that $h(f|\Gamma) = e$.

Results related to Theorem 2 were studied for example by Urbanski [33, 34] studying the Hausdorff dimension of invariant sets of expanding maps on a circle with a hole and showing that the dimension varies continuously with the size of the hole. See also Persson [24] for similar results.

In Section 2 we collect preliminary results on Pesin theory (for the $C^{1+\alpha}$ setting) and on dominated splitting (for the $C^1$ setting). Theorems 1 and 2 are proved in Section 3.

### 2. Preliminaries

#### 2.1. Lyapunov regularity and hyperbolicity.

We say that a point $x \in M$ is *Lyapunov regular* if there exist a number $s(x) \geq 1$, numbers $\chi_1(x) < \ldots < \chi_{s(x)}(x)$, and a decomposition $E_{x}^1 \oplus \cdots \oplus E_{x}^{s(x)} = T_x M$ into subspaces of dimension $k_j(x)$ such that for $j = 1, \ldots, s(x)$ we have

$$\chi_j(x) = \lim_{|k| \to \infty} \frac{1}{k} \log \|df_x^k(v)\|$$

whenever $v \in E_{x}^j \setminus \{0\}$. It is a consequence of the Oseledets multiplicative ergodic theorem that with respect to any $f$-invariant Borel probability measure on $M$ the set of Lyapunov regular points has full measure. Moreover, the function $x \mapsto \chi_j(x)$ and the distribution $x \mapsto E_{x}^j$ are Borel measurable and $f$-invariant. A Lyapunov regular point is said to be *hyperbolic* if $s(x) \geq 2$ and there exists $i(x) \geq 1$ so that $\chi_{i(x)}(x) < 0 < \chi_{i(x) + 1}(x)$. The set of points having this property will be denoted by $\mathcal{R}_H$, that is,

$$\mathcal{R}_H \overset{\text{def}}{=} \{ x \in M \text{ Lyapunov regular and hyperbolic} \}.$$  

In this case we also use the notations $E_{x}^s = E_{x}^1 \oplus \cdots \oplus E_{x}^{i(x)}$, $E_{x}^u = E_{x}^{i(x)+1} \oplus \cdots \oplus E_{x}^{s(x)}$.

Let $\mathcal{M}$ denote the space of $f$-invariant Borel probability measure on $M$ and let $\mathcal{M}_{\text{erg}}$ be the subspace of ergodic measures. For $\mu \in \mathcal{M}_{\text{erg}}$, $\chi_j$ and $s$ are constant $\mu$-a.e. and we denote these constants by $\chi_j(\mu)$ and $s(\mu)$. An ergodic Borel probability measure $\mu$ is *hyperbolic* if the set of hyperbolic
Lyapunov regular points has full measure. In this case we also use the notation
\[ \chi = \chi(\mu) \overset{\text{def}}{=} \min_j |\chi_j(\mu)|. \]

Henceforth we consider a general diffeomorphism \( f \) preserving a hyperbolic ergodic measure \( \mu \).

2.2. Uniform geometric potential. Consider the function \( \varphi^u : \mathcal{R}_H \to \mathbb{R} \)
\[ \varphi^u(x) \overset{\text{def}}{=} -\log |\text{Jac} df|_{E^u_x}| \]
that is also called geometric potential. In the case that \( E^u_x \) is one-dimensional, then we simply have \( \varphi^u(x) = -\log ||df||_{E^u_x} \). Recall that
\[ \int \varphi^u \, d\mu = -\int \log |\text{Jac} df|_{E^u_x} \, d\mu = -\sum_j \chi_j(\mu)^+, \]
where \( a^+ = \max\{0, a\} \). By the Oseledets multiplicative ergodic theorem, \( \varphi^u \) is an measurable function and for \( \mu \)-almost every \( x \) we have
\[ \sum_j \chi_j(x)^+ = \sum_j \chi_j(\mu)^+ = \lim_{n \to \infty} \psi_n(x), \]
where
\[ (1) \quad \psi_n(x) \overset{\text{def}}{=} \log |\text{Jac} df^n|_{E^u_x}^{1/n} \]
is a sequence of measurable functions \( \psi_n : \mathcal{R}_H \to \mathbb{R} \) which converges \( \mu \)-almost everywhere to \( \sum_j \chi_j(\mu)^+ \) as \( n \to \infty \). Hence, the following lemma is an immediate consequence of the multiplicative ergodic theorem and the Egorov theorem.

**Lemma 1.** Given \( \delta \in (0, 1) \) and \( r > 0 \), there exist a compact set \( \Gamma_J = \Gamma_J(\delta, r) \) and a positive integer \( n_J = n_J(\delta, r) \) such that \( \mu(\Gamma_J) > 1 - \delta \) and for every \( x \in \Gamma_J \) and every \( n \geq n_J \) we have
\[ \left| \frac{1}{n} \log |\text{Jac} df^n|_{E^u_x}^n - \sum_j \chi_j(\mu)^+ \right| \leq r. \]

2.3. Oseledets-Pesin \( \varepsilon \)-reduction. The following results are consequences of the Oseledets-Pesin \( \varepsilon \)-reduction theorem (see [15 S.2 and Theorem S.2.10]). For any sufficiently small \( \gamma > 0 \) one can chose Borel functions \( C_1, C_2 : \mathcal{R}_H \to (0, \infty) \) such that the following holds. Given \( x \in \mathcal{R}_H \), the subspaces \( E^s_x \) and \( E^u_x \) satisfy
\[ (H1_s) \quad ||df^s_x(v)|| \leq C_1(x)e^{-n(\chi-\gamma)}||v|| \quad \text{for} \quad v \in E^s_x \text{ and } n > 0, \]
\[ (H1_u) \quad ||df^u_x(w)|| \leq C_1(x)e^{-n(\chi+\gamma)}||w|| \quad \text{for} \quad w \in E^u_x \text{ and } n > 0. \]
The angle between these subspaces satisfies
\[ (H2) \quad \angle(E^s_x, E^u_x) \geq C_2(x). \]
Furthermore the functions $C_1, C_2$ can be chosen to be $\gamma$-tempered, that is, for any $n \in \mathbb{Z}$ they satisfy

(H3) \[ C_1(f^n(x)) \leq C_1(x) e^{\gamma |n|}, \quad C_2(f^n(x)) \leq C_2(x) e^{\gamma |n|}. \]

Given $\ell \geq 1$, we consider the Lyapunov regular set of level $\ell$ defined by

(2) \[ \mathcal{R}_H^\ell \overset{\text{def}}{=} \{ x \in \mathcal{R}_H : C_1(x) \leq \ell, C_2(x) \geq \ell^{-1} \}. \]

For $\mu \in \mathcal{M}_{\text{erg}}$ being hyperbolic, we have $\mu(\mathcal{R}_H^\ell) \to 1$ as $\ell \to \infty$. Observe that $x \mapsto E_x^s$ is continuous on each $\mathcal{R}_H^\ell$, $a = s, u$, and that $f$ restricted to $\mathcal{R}_H^\ell$ is hyperbolic. However, $\mathcal{R}_H^\ell$ in general fails to be $f$-invariant. The set $\bigcup_{\ell \geq 1} \mathcal{R}_H^\ell$ is $f$-invariant but in general fails to be hyperbolic.

It is convenient to introduce a new scalar product structure on the tangent bundle $T \mathcal{R}_H M$. Given $0 < \gamma < \chi/3$, the so-called Lyapunov inner product is defined by

\[ \langle v, w \rangle_x^\ell = \sum_{n=0}^{\infty} \langle df_x^n(v), df_x^n(w) \rangle e^{-2n\chi(x)} e^{-2n\gamma} \]

if $v, w \in E_x^s$, and

\[ \langle v, w \rangle_x^s = \sum_{n=0}^{\infty} \langle df_x^n(v), df_x^n(w) \rangle e^{2n\chi(x)+1} e^{-2n\gamma} \]

if $v, w \in E_x^u$. Based on the properties (H1)-(H3) one can indeed verify the convergence of the above series. Then $\langle \cdot, \cdot \rangle_x^\ell$ is extended to all vectors by declaring the subspaces $E_x^s$ and $E_x^u$ to be mutually orthogonal with respect to $\langle \cdot, \cdot \rangle_x^\ell$, that is, for vectors $v = v_1 + v_2$ and $w = w_1 + w_2$ with $v_1, v_2 \in E_x^s$ and $w_1, w_2 \in E_x^u$ it is defined by

\[ \langle v, w \rangle_x^\ell = \langle v_1, w_1 \rangle_x^\ell + \langle v_2, w_2 \rangle_x^\ell. \]

This scalar product induces the so-called Lyapunov norm $\| \cdot \|_x^\ell$ on $T_x M$.

For each Lyapunov regular point $x \in \mathcal{R}_H$ one can show that there exists a Lyapunov change of coordinates, that is, a linear transformation $C_\gamma(x): \mathbb{R}^{\dim M} \to T_x M$, such that

\[ \langle v, w \rangle = \langle C_\gamma(x)v, C_\gamma(x)w \rangle_x^\ell \]

and that the matrix

\[ A_\gamma(x) = C_\gamma(f(x))^{-1} \circ df_x \circ C_\gamma(x) \]

has Lyapunov block form

(3) \[ A_\gamma(x) = \begin{pmatrix} A_1^s(x) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_1^s(x) \end{pmatrix}, \]

where each block $A_j^s(x)$ is a $k_j(x) \times k_j(x)$ matrix satisfying

\[ e^{\chi_j(x) - \gamma} \leq \| A_j^s(x) \|^{-1} \leq \| A_j^s(x) \| \leq e^{\chi_j(x) + \gamma}. \]
Moreover, the sequence \( (C_\gamma(f^n(x)))_{n \in \mathbb{Z}} \) is a \( \gamma \)-tempered sequence of linear transformations.

For \( x \in \mathcal{A}_\mathcal{H} \), \( \rho > 0 \) let
\[
T_x M(\rho) \overset{\text{def}}{=} \{ v \in T_x M : \|v\|_x \leq \rho \}.
\]
Let us now choose \( \rho_0 > 0 \) so that for each \( x \in M \) the exponential map \( \exp_x : T_x M(2\rho_0) \to M \) is an embedding satisfying \( \| (d\exp_x)_v \| \leq 2 \) and so that \( \exp f(x) \) is injective on \( \exp^{-1}_f \circ f \circ \exp_x(T_x M(2\rho_0)) \). Define the Lyapunov chart
\[
\psi_x \overset{\text{def}}{=} \exp_x \circ C_\gamma(x) : \mathbb{R}^{\dim M} \to M
\]
and define the “lifted map”
\[
F_x \overset{\text{def}}{=} \psi_x^{-1} \circ f \circ \psi_x.
\]

By setting
\[
q(x) \overset{\text{def}}{=} r_0 \min\{\|C_\gamma(x)\|, \|C_\gamma(x)^{-1}\|\}
\]
we obtain a \( \gamma \)-tempered function \( q \) and embeddings \( \psi_x : B(0, q(x)) \to M \) such that \( \psi_x(0) = x \). Moreover, the map \( F_x \) defined above is well-defined on \( B(0, q(x)) \) and its linearization \( (dF_x)_0 \) has Lyapunov block form.

Observe that the above facts so far used only the fact that \( f \) is differentiable and that the cocycle \( A_\gamma : \mathcal{A}_\mathcal{H} \to GL(\dim M, \mathbb{R}) \) satisfies \( \log^+ \|A_\gamma^{\pm 1}\| \in L^1(\mu) \) for every invariant hyperbolic probability measure \( \mu \).

2.4. **Rectangular neighborhoods of regular points.** We will make use of a suitable collection of sets that are “almost rectangles”. Let
\[
L_x(w) \overset{\text{def}}{=} \sqrt{\dim M}^{-1} q(x) \cdot w
\]
be the linear rescaling \( L_x : [-1, 1]^{\dim M} \to [-q(x), q(x)]^{\dim M} \) of the unit cube onto the maximal cube contained in \( B(0, q(x)) \). With the above introduced parametrization \( \psi_x \) in the Lyapunov chart, given \( h \in (0, 1] \) we define
\[
R(x, h) \overset{\text{def}}{=} (\psi_x \circ L_x)([-h, h]^{\dim M}) \subset M
\]
the \( h \)-scaled rectangle centered at \( x \in \mathcal{A}_\mathcal{H} \). For each regular point \( x \in \mathcal{A}_\mathcal{H} \) the set \( R(x, 1) \) is also called the regular neighborhood of \( x \) or Lyapunov chart at \( x \).

The so defined regular neighborhoods of Lyapunov regular points admit local coordinates with respect to which the dynamics behaves uniformly hyperbolic. Recall that vectors tangent to the stable direction \( E^s_x \) (the unstable direction \( E^u_x \)) are contracted forward in time (backward in time). Fix some constant \( L \in (0, 1/2) \). Considering the splitting of the tangent bundle \( T_x M = E^s_x \oplus E^u_x \) and the corresponding splitting of the unit cube \([-1, 1]^{\dim M} = I^s \oplus I^u \), an admissible stable \((s, L, h)\)-manifold is the set
\[
\{(\psi_x \circ L_x)(v, \xi(v)) : v \in I^s\},
\]
where \( \xi: I^s \to I^u \) is a smooth map with Lipschitz constant \( L \). An **admissible unstable** \((u,L,h)\)-manifold is defined analogously. Such stable/unstable admissible manifolds endow \( R(x,h) \) with a product structure as they intersect transversally in a unique point with an angle bounded from zero (whose bound, however, depends on \( x \)). This enables us to define the concept of an **admissible stable rectangle** (admissible unstable rectangle). An admissible \( L^s \)-rectangle is a set in \( R(x,h) \) whose boundaries are smooth sets foliated by segments of admissible stable \((s,L,h)\)-manifolds and unstable admissible unstable \((u,L,h)\)-manifolds such that the stable manifolds stretch fully across \( R(x,h) \). An **admissible \( L^u \)-rectangle** in \( R(x,h) \) is defined analogously.

In this section we will sketch the proof of the following lemma.

**Lemma 2.** Given numbers \( \delta \in (0,1), \gamma, r \in (0, \chi/3), \) and \( \varepsilon > 0 \), there are a compact set \( \Gamma_H = \Gamma_H(\gamma, \varepsilon, \delta) \), a positive number \( h \), a finite collection of closed rectangles \( \{ R(x_i) = R(x_i, h) \}_{i=1}^{\ell} \) in neighborhoods of certain points \( x_i \in \Gamma_H \) and positive numbers \( \lambda \in (0,1), \rho, L \) satisfying the following:

1. \( \mu(\Gamma_H) > 1 - \delta \) and \( e^{-\chi-\gamma} \leq \lambda \leq e^{-\chi+\gamma} \),
2. \( \text{diam}(R(x_i)) < \varepsilon \) for each \( i \),
3. if \( x \in \Gamma_H \cap B(x_i, \rho) \) and \( f^m(x) \in \Gamma_H \cap B(x_j, \rho) \) for some \( m \geq 1 \), then the connected component
   \[ C_x \left( R(x_i) \cap f^{-m}R(x_j) \right) \]
   of \( R(x_i) \cap f^{-m}R(x_j) \) which contains \( x \) is an admissible \( L^s \)-rectangle in \( R(x_i) \) and its image
   \[ f^m \left( C_x \left( R(x_i) \cap f^{-m}R(x_j) \right) \right) \]
   is an admissible \( L^u \)-rectangle in \( R(x_j) \),
4. if \( x, f^m(x) \in \Gamma_H \cap B(x_i, \rho) \) for some \( m \geq 1 \) then for every \( k = 0, \ldots, m \) we have
   \[ \text{diam} f^k \left( C_x \left( R(x_i) \cap f^{-m}R(x_i) \right) \right) \leq 3 \text{diam}(R(x_i)) \max\{\lambda^k, \lambda^{m-k}\} \],
5. if \( x, f^m(x) \in \Gamma_H \cap B(x_i, \rho) \) for some \( m \geq 1 \) then
   \[ \frac{1}{m} \log |\text{Jac} df^m_{x} E_x^m| - \frac{1}{m} \log |\text{Jac} df^m_{x} E_x^m| < r \].

We will split the sketch of the proof of the above lemma in two parts, assuming first (A1) and then (A2).

**Sketch of proof of Lemma 2 assuming (A1).** Let \( f \) be \( C^{1+\alpha} \) for some \( \alpha > 0 \).

Let us write the map \( F_x \) defined in [11] as
\[ F_x(w) = D_\gamma(x)(w) + h_x(w), \]
where \( D_\gamma(x) \overset{\text{def}}{=} C_\gamma(f(x)) \circ df_x \circ C_\gamma(x)^{-1} \).

Note that \( (dF_x)_0 = D_\gamma(x) \) and \( (dh_x)_0 = 0 \). Recall that the linearization \( (dF_x)_0 = A_\gamma(x) \) of the map \( F_x \) in 0 has Lyapunov block form [1]. The map
$F_x$ and its linearization $(dF_x)_0$ are $\gamma$-close in the $C^1$ topology in $B(0, q(x))$. Further, there exists a constant $K > 0$ and a measurable $\gamma$-tempered function $A: \Gamma' \to \mathbb{R}$ on a set $\Gamma \subset \mathcal{R}_H$ of full measure such that for every $y, z \in B(0, q(x))$ we have

$$K \|d(\psi_x(y), \psi_x(z))\| \leq \|y - z\| \leq A(x) \|d(\psi_x(y), \psi_x(z))\|.$$ 

Further, there exists a constant $K > 0$ and a measurable $\gamma$-tempered function $A: \Gamma' \to \mathbb{R}$ on a set $\Gamma \subset \mathbb{R}^H$ of full measure such that for every $y, z \in B(0, q(x))$ we have

$$B(x)^{-1}d(\psi_x(y), \psi_x(z)) \leq d'(x)(\exp_x(y), \exp_x(z)) \leq B(x)d(\psi_x(y), \psi_x(z)),$$

where $d'_x$ denotes the distance on $\exp_x B(0, q(x))$ with respect to the Lyapunov metric $\|\cdot\|'_x$. For details on and proofs of the above sketched properties see [15, Chapters S.3 and S.4 b–d] and [2, 3] and, in particular, [15, Theorems S.3.1 and S.4.16].

Given $\gamma$, there exist a positive integer $\ell$ such that $\mu(\mathcal{R}_H^\ell) \geq 1 - \delta/2$. Hence, the set $\Gamma_H = \Gamma' \cap \mathcal{R}_H^\ell$ has measure at least $1 - \delta$ and on $\Gamma_H$ the above considered functions and the size of the Lyapunov charts are bounded. Moreover, by the Lusin theorem, we can assume that the above considered measurable functions are continuous and hence bounded on $\Gamma_H$ and consequently in each Lyapunov chart the deformation of a neighborhood of a point $x \in \Gamma_H$ under $f$ is approximately given by its linearization $df_x$ which, up to some tempered sequences of coordinate changes, is represented by diagonal matrices.

This sketches the proof of Lemma 2 under the assumption (A1). □

**Sketch of proof of Lemma 2 assuming (A2).** Let us now assume that $f$ is a $C^1$ diffeomorphisms preserving a hyperbolic measure $\mu$ such that the tangent space of $\Lambda = \text{supp} \mu$ decomposes into a dominated splitting $T\Lambda M = E^1 \oplus E^2$, that is, there exists $N \geq 1$ such that for every $x \in \Lambda$, $v \in E^1_x$, $w \in E^2_x$ we have

$$\|df_x^N(v)\| \leq \frac{1}{2} \|df_x^N(w)\|.$$ 

Up to a smooth change of metric on $M$ we can assume that $N = 1$ (see [5] for more details).

Observe that the dominated splitting is continuous and that it coincides with the splitting of the Oseledec decomposition in almost all points. This enables us to obtain the following non-uniform hyperbolicity on a large set without invoking Pesin theory. [11, Lemma 8.4] applied to $f$, $E^1$ and $f^{-1}$, $E^2$ simultaneously gives the following result.
Lemma 3. Given $\gamma \in (0, \chi)$, there exist $N = N(\gamma) \geq 1$ such that for $\mu$-almost every $x$ and every $\ell \geq 1$ we have

$$\lim_{\ell \to \infty} \frac{1}{\ell N} \sum_{k=0}^{\ell-1} \log \|df^{N}_{f_{kN}}(x)\| \leq -\chi + \gamma$$

(6)

$$\lim_{\ell \to \infty} \frac{1}{\ell N} \sum_{k=0}^{\ell-1} \log \|df^{-N}_{f_{-kN}}(x)\| \leq -\chi + \gamma.$$  

A point $x$ satisfying (6) is called $N$-hyperbolic with respect to $E^1 \oplus E^2$. Fixing $\gamma$, up to a smooth change of metric on $M$, we can assume $N = 1$.

We now follow an idea in [7] to construct “fake” foliations to foliate a uniformly sized neighborhood of almost every point (they are called fake because they are close to but, in general, not identical with local stable/unstable manifolds). Observe first that a dominated splitting can be extended to a dominated splitting to some neighborhood and that a dominated splitting persists under $C^1$-perturbations (see [5, Appendix B]). The fake invariant foliations are constructed in two steps: (1) we find foliations tangent to a cone field in each tangent space $T_xM$ and (2) using the exponential map, we project these foliations from some sufficiently small neighborhood of the origin in $T_xM$ to a neighborhood of $x$ in $M$.

As above, we continue to consider $\rho_0 > 0$ so that for each $x \in M$ the exponential map $exp_x : T_xM(2\rho_0) \to M$ is an embedding satisfying $\|(dexp_x)_v\| \leq 2$ and so that $exp_x$ is injective on $exp_x^{-1}(T_xM(2\rho_0))$. Given $\rho \in (0, \rho_0)$, define the map $F_{\rho, x} : TM \to TM$ by

- $F_{\rho, x}(v) = exp^{-1}_{f(x)} \circ f \circ exp_x(v)$ for $\|v\| \leq \rho_0$,
- $F_{\rho, x}(v) = df_x(v)$ for $\|v\| \geq 2\rho$, and
- $\|F_{\rho, x}(\cdot) - df_x(\cdot)\|_{C^1} \to 0$ as $\rho \to 0$, uniformly in $x$.

Regarding the tangent bundle $TM$ as the disjoint union of its fibers, the tangent map $df$ has a dominated splitting. Even though the bundle $TM$ is not compact, all relevant estimates hold uniformly for $F_{\rho, x}$ which is enough to conclude the claimed properties. As this splitting persists after a small $C^1$-perturbation, for sufficiently small $\rho$ the map $F_{\rho, x}$ has also a dominated splitting that we will denote by $\tilde{E}^1 \oplus \tilde{E}^2$. Moreover, if $\rho$ is small enough, each subspace $E^a$ at $v \in T_xM$ lies within the $\epsilon/2$-cone about the corresponding subspace $\tilde{E}^a$ for $a = 1, 2$ (making the usual identification of $T_vT_xM$ with $T_xM$).

As almost every $x$ is 1-hyperbolic for $E^1 \oplus E^2$ (with respect to $f$ on $\Lambda$) and as the splitting $\tilde{E}^1 \oplus \tilde{E}^2$ is continuous and hence (6) varies continuously in $x$, we can assume that for sufficiently small chosen $\rho$ we obtain that every $v \in T_xM$ is 1-hyperbolic with respect to $\tilde{E}^1 \oplus \tilde{E}^2$ (with respect to $F_{\rho, x}$ on $TM$). One can show (see, for example, [11, Section 6]) that there exist $\rho_0 > 0$ such that in $T_xM$ the ball around the origin of radius $\rho_0$ is foliated by leaves $\mathcal{W}_{\rho, x}(\cdot)$, $a = 1, 2$, having the following properties. For each $v$ the leaf $\mathcal{W}_{\rho, x}(v)$...
is an injectively immersed $C^1$ submanifold tangent to $\tilde{E}_x^1$ containing a ball of uniform size and being invariant in the sense that
def

Having constructed these foliations $\tilde{\mathcal{W}}^a$ of $T_xM$, we apply the exponential map $\exp_x$ to it and obtain a foliation of the ball $B(x,\rho_0)$ that we will denote by $\mathcal{W}^a(x)$, where $a = 1, 2$. If $\rho$ was chosen sufficiently small, in a point $y \in B(x,\rho_0)$ the foliation will be within the $\varepsilon/2$-cone about the parallel translate to $E_a^x$.

As the dominated splitting is transversal in all points, the leafs of the fake foliation intersect transversally and we can now define an almost rectangular box about $x$ that we will denote by $R(x)$. By 1-hyperbolicity of the point $x$ the leafs $\mathcal{W}_1(x)$ and $\mathcal{W}_2(x)$ will be contracted forward and backward in time under the map $f$ and $f^{-1}$, respectively.

Finally, the sequence (1) (for $E^u = E^2$) converges uniformly on any large measure set $\Gamma_H$. Moreover, given $x \in \Gamma_H$, by continuity of the dominated splitting for any point $y$ whose orbit stays close to the orbit of $x$ the derivative in the direction $E^2$ differs only little from the one the one at $x$, proving the claim about the geometric potential.

This sketches the proof of Lemma 2 under the assumption (A2). □

2.5. Uniform Birkhoff averages. Let $\phi: M \to \mathbb{R}$ be a continuous function. The following lemma is an immediate consequence of the Birkhoff ergodic theorem and the Egorov theorem.

Given $n \geq 1$ we define

$$S_n \phi(x) \overset{\text{def}}{=} \phi(x) + \phi(f(x)) + \ldots + \phi(f^{n-1}(x)).$$

**Lemma 4.** Given $\delta \in (0, 1)$ and $r > 0$, there exist a compact set $\Gamma_B = \Gamma_B(\delta, r, \phi)$ and a positive integer $n_B = n_B(\delta, r, \phi)$ such that $\mu(\Gamma_B) > 1 - \delta$ and for every $x \in \Gamma_B$ and every $n \geq n_B$ we have

$$\left| \frac{1}{n} S_n \phi(x) - \int \phi \, d\mu \right| \leq r.$$

2.6. Uniform recurrence. Let $\mathcal{P} = \{P_1, \ldots, P_N\}$ be any finite measurable partition of $M$. Given $x \in M$, denote by $\mathcal{P}(x)$ the partition element which contains $x$.

**Lemma 5.** Given numbers $\delta \in (0, 1), r > 0$, and a measurable set $F$ of positive measure, there exist a compact set $\Gamma_R = \Gamma_R(F, \delta, \mathcal{P}, r)$ with $\mu(\Gamma_R) > 1 - \delta$ and a number $n_R = n_R(F, \delta, \mathcal{P}, r)$ such that for every $x \in \Gamma_R$ and every $n \geq n_R$ we have

$$f^k(x) \in \mathcal{P}(x) \cap F$$

for some number $k \in \{n, \ldots, n + rn\}$.

**Proof.** Let $r \in (0, 1)$ and

$$C \overset{\text{def}}{=} \min \left\{ r, \frac{\mu(P_i)}{4} : i = 1, \ldots, N \right\}.$$
By the Birkhoff ergodic theorem, there is a $\mu$-full measure set $\tilde{A} = \tilde{A}(F, \mathcal{P}) \subset M$ such that for every $i$ for which $\mu(\tilde{A} \cap P_i)$ is positive and for every point in $\tilde{A} \cap P_i$ we have
\[
\lim_{n \to \infty} \frac{1}{n} \operatorname{card}\{k \in \{0, \ldots, n - 1\} : f^k(x) \in F \cap P_i\} = \mu(F \cap P_i).
\]

By the Egorov theorem, we can conclude that for every such $i$ there exists a measurable set $\Gamma_i = \Gamma_i(F, \delta, \mathcal{P}) \subset \tilde{A} \cap P_i$ such that
\[
\mu(\Gamma_i) \geq \mu(\tilde{A} \cap P_i) - \delta \frac{\delta}{N}
\]
and that convergence is uniform on $\Gamma_i$. Since $\mu$ is regular, without loss of generality, we can assume that each $\Gamma_i$ is compact. Hence, we find $n_R = n_R(F, \delta, \mathcal{P}, r) \geq 1$ such that for every such $i$, every $x \in \Gamma_i$, and every $n \geq n_R$ we have
\[
\left| \frac{1}{n} \operatorname{card}\{k \in \{0, \ldots, n - 1\} : f^k(x) \in F \cap P_i\} - \mu(F \cap P_i) \right| \leq C^2.
\]

We can assume that $n_R$ has been chosen large enough such that for every such $i$ we have $n_R \mu(F \cap P_i) - 3C > 1$ and thus for every $x \in \Gamma_i$ and every $n \geq n_R$ we obtain
\[
\operatorname{card}\{k \in \{n, \ldots, n(1 + r) - 1\} : f^k(x) \in F \cap P_i\} \\
\geq n(1 + r)(\mu(F \cap P_i) - C^2) - (n - 1)\mu(F \cap P_i) - (n - 1)C^2 \\
= (nr + 1)\mu(F \cap P_i) - (2n + nr - 1)C^2 \\
\geq nr(\mu(F \cap P_i) - 3C) > 1,
\]
where we also used (7). Let $\Gamma_R \overset{\text{def}}{=} \Gamma_1 \cup \ldots \cup \Gamma_N$. By the above, we have
\[
\mu(\Gamma_R) = \sum_{i=1}^N \mu(\Gamma_i) \geq \sum_{i=1}^N \left( \mu(\tilde{A} \cap P_i) - \frac{\delta}{N} \right) = \sum_{i=1}^N \mu(\tilde{A}) - \delta = 1 - \delta.
\]

This proves the lemma. \(\square\)

2.7. Uniform growth of distinguishable orbits via entropy. To construct a set on which dynamics approximates entropy, the following lemma enable us to produce a sufficiently large number of points which have distinguishable orbits of a certain length.

Given $\varepsilon > 0$, $n \geq 1$, and a compact set $K \subset M$, we call a set $E \subset M$ $(n, \varepsilon)$-spanning for $K$ (with respect to $f$) if
\[
K \subset \bigcup_{x \in E} B_n(x, \varepsilon),
\]
where
\[
B_n(x, \varepsilon) \overset{\text{def}}{=} \{y : d(f^k(x), f^k(y)) \leq \varepsilon \text{ for every } 0 \leq k \leq n - 1\}.
\]

We call a set $E \subset M$ $(n, \varepsilon)$-separated (with respect to $f$) if for every $x \in E$ the set $B_n(x, \varepsilon)$ contains no other point of $E$. 

Lemma 6. Given $\delta \in (0, 1)$, for every $e \in (0, h_\mu(f)]$, $r \in (0, e)$, and $\varepsilon > 0$ there exist a number $n_E = n_E(\delta, e, r, \varepsilon)$ and a measurable set $\Gamma_E$ satisfying $\mu(\Gamma_E) > 1 - \delta$ such that for every $n \geq n_E$ and for every measurable set $F \subset \Gamma_E$ with $\mu(F) > 0$ there exists a $(n, \varepsilon)$-separated set $E \subset F$ such that
$$-\frac{1}{n}|\log \mu(F)| - r \leq \frac{1}{n}|\log \text{card } E - e| \leq r.$$ 

Proof. By ergodicity, by the Brin-Katok theorem [6], for $\mu$-almost every $x$
$$\lim_{\varepsilon \to 0} \limsup_{n \to \infty} -\frac{1}{n} \log \mu(B_n(x, \varepsilon)) = h_\mu(f).$$

Given $r$ and $\varepsilon$, by the Egorov theorem there is a set $\Gamma_E$ of measure $\mu(\Gamma_E) > 1 - \delta$ and a number $n_E \geq 1$ such that for every $n \geq n_E$ and every $x \in \Gamma_E$
$$e^{-n(h_\mu(f)+r)} \leq \mu(B_n(x, \varepsilon)) \leq e^{-n(h_\mu(f)-r)}.$$ 

For any positive measure set $F \subset \Gamma_E$, we construct an $(n, \varepsilon)$-separated set as follows. Choose any point $x_1 \in F$. Let $F_1 := F \setminus B_n(x_1, \varepsilon)$. Continuing inductively, for every $k \geq 2$ choose any $x_k \in F_{k-1}$, let $F_k := F_{k-1} \setminus B_n(x_k, \varepsilon)$. Since we have
$$\mu(F_k) \geq \mu(F) - k\mu(B_n(x_k, \varepsilon)) \geq \mu(F) - ke^{-n(h_\mu(f)-r)},$$
we can repeat this iteration for at least $\lceil \mu(F) \cdot e^{n(h_\mu(f)-r)} \rceil$ times. The resulting set of points $E \overset{\text{def}}{=} \{x_1, x_2, \ldots \} \subset F$ is by construction $(n, \varepsilon)$-separated and satisfies
$$\frac{1}{n} \log \text{card } E \geq h_\mu(f) - r - \frac{1}{n}|\log \mu(F)|.$$ 

Possibly after neglecting some elements of $E$, we can guarantee that
$$h_\mu(f) + r \geq e + r \geq \frac{1}{n} \log \text{card } E \geq e - r - \frac{1}{n}|\log \mu(F)|,$$
which proves the lemma. \qed

3. Proofs

We now provide the proofs of the theorems. Applying Sections 2.4, 2.5, and 2.6 we largely follow [15, Supplement S.5]. We also apply ideas from [27, Section 11.6].

Proof of Theorem 1. Let $U \subset M$ be an open and bounded neighborhood of $\text{supp }\mu$. Pick a countable basis $\{\psi_i\}_{i \geq 1}$ of continuous functions in the space $C^0(\overline{U})$ of all continuous functions on $\overline{U}$. Recall that the space of invariant probabilities on $\overline{U}$ the following function $d: \mathcal{M} \times \mathcal{M} \to [0, 1]$
$$d(\mu, \nu) \overset{\text{def}}{=} \sum_{i=1}^{\infty} 2^{-i} \frac{1}{2\|\psi_i\|_\infty} \left| \int \psi_i d\mu - \int \psi_i d\nu \right|,$$
where $\|\psi\|_\infty \overset{\text{def}}{=} \sup_{x \in \overline{U}} |\psi(x)|$, provides a metric which induces the weak* topology on $\mathcal{M}$. 


Fix some \( \delta \in (0, 1/5) \) and let \( r \in (0, \min\{h_{\mu}(f), \chi/3\}) \).

Let \( K \) be a positive integer satisfying
\[
2^{-K+1} < \frac{r}{2}
\]
and choose \( r_0 \in (0, r) \) such that
\[
(10) \quad r_0(1 - 2^{-K}) \frac{1}{2} \max_{i=1,\ldots,K} \|\psi_i\|^{-1}_\infty < \frac{r}{2}.
\]
Moreover, assume that \( K \) was chosen large enough such that \( \{\psi_1, \ldots, \psi_K\} \) is \( r \)-dense, that is, for every \( \varphi \in C^0(\overline{U}) \) there exists \( \psi_i, i \in \{1, \ldots, K\} \), such that
\[
(11) \quad \|\psi_i - \varphi\|_\infty < r.
\]
Choose \( \varepsilon_0 > 0 \) sufficiently small such that the modulus of continuity of each \( \phi \in \{\psi_1, \ldots, \psi_K\} \) satisfies
\[
(12) \quad \sup_{x,y,d(x,y) \leq \varepsilon_0} |\phi(x) - \phi(y)| < r_0.
\]

For the following arguments, for simplicity, we will consider only one potential function \( \phi \) and will assume that they hold for every potential \( \psi_i \).

First, by Lemma \ref{lem:compact_set} there is a compact set \( \Gamma_J = \Gamma_J(\delta, r) \) and a positive integer \( n_J = n_J(\delta, r) \) such that \( \mu(\Gamma_J) > 1 - \delta \) and for every \( x \in \Gamma_J \) and every \( n \geq n_J \) we have
\[
(13) \quad \left| \frac{1}{n} \log |\text{Jac } df_{E_x^n}| - \sum_j \chi_j(\mu)^+ \right| \leq r.
\]

Next, we will choose points in some Lyapunov regular set. Hence they will have hyperbolic behavior and a nice rectangular cover. Therefor fix
\[
\gamma \in (0, \chi/3), \quad \varepsilon_1 \in (0, \varepsilon_0).
\]

By Lemma \ref{lem:gamma_set} there exist a compact set \( \Gamma_H = \Gamma_H(\gamma, \varepsilon_1, \delta) \) of measure \( > 1 - \delta \) and positive numbers \( \lambda \in (0, 1), \rho, L \) and a finite collection of rectangles \( R(x_1), \ldots, R(x_i), \ldots, R(x_{\ell}) \) with \( x_i \in \Gamma_H \) and so that \( B(x_i, \rho) \subset R(x_i) \) and \( \text{diam } R(x_i) < \varepsilon_1 \) for every \( i \), satisfying properties 0.-5. in the lemma.

By Lemma \ref{lem:gamma_set} there is a compact set \( \Gamma_B = \Gamma_B(\delta, r, \phi) \) and a positive integer \( n_B = n_B(\delta, r, \phi) \) such that \( \mu(\Gamma_B) > 1 - \delta \) and for every \( x \in \Gamma_B \) and every \( n \geq n_B \) we have
\[
(14) \quad \left| \frac{1}{n} S_n \phi(x) - \int \phi \, d\mu \right| \leq r.
\]

Besides the rectangles, let us fix a finite partition \( \mathcal{P} \) of \( M \) of diameter \( < \rho/2 \), where \( \rho \) is the inner size of the rectangular cover. Notice that in
this way for each \( x_i \), the partition element \( \mathcal{P}(x_i) \) is completely contained in \( R(x_i) \).

Further, we want points in \( \Gamma_J \cap \Gamma_H \cap \Gamma_B \), or at least most of them, to be closely recurring at the same, or at least at almost the same, time. Note that \( \mu(\Gamma_J \cap \Gamma_H \cap \Gamma_B) > 1 - 3\delta > 0 \). By Lemma 5 there is a compact set \( \Gamma_R = \Gamma_R(\Gamma_J \cap \Gamma_H \cap \Gamma_B, \delta, \mathcal{P}, r) \) of points with almost uniform recurrence times satisfying \( \mu(\Gamma_R) > 1 - \delta \) and a number \( n_R = n_R(\Gamma_J \cap \Gamma_H \cap \Gamma_B, \delta, \mathcal{P}, r) \) such that for every \( n \geq n_R \) and every \( x \in \Gamma_R \) for some \( k \in \{n, \ldots, n + nr\} \) we have

\[
(15) \quad f^k(x) \in \mathcal{P}(x) \cap \Gamma_J \cap \Gamma_H \cap \Gamma_B.
\]

Consider the set

\[
\Gamma' \overset{\text{def}}{=} \Gamma_J \cap \Gamma_H \cap \Gamma_B \cap \Gamma_R,
\]

and note that it satisfies \( \mu(\Gamma') > 1 - 4\delta > 0 \).

Further, by Lemma 6 given \( e \in (0, h_\mu(f)] \), for every \( \varepsilon \in (0, \varepsilon_1) \) there exist a number \( n_E = n_E(\delta, e, r, \varepsilon) \) and a set \( \Gamma_E \) satisfying \( \mu(\Gamma_E) > 1 - \delta \) such that for every

\[
(16) \quad n \geq \max \left\{ n_J, n_H, n_B, n_E, \log \ell, \frac{\log(1 - 5\delta)}{r} \right\}
\]

there is a \((n, \varepsilon)\)-separated set \( E \subset \Gamma' \cap \Gamma_E \) such that

\[
(17) \quad (1 - 5\delta)e^{n(e-r)} \leq \text{card} E \leq e^{n(e+r)},
\]

where we use the fact that \( \mu(\Gamma' \cap \Gamma_E) \geq 1 - 5\delta > 0 \). Let \( \Gamma'' \overset{\text{def}}{=} \Gamma' \cap \Gamma_E \).

For each \( k \) with \( n \leq k < n + rn \) let

\[
F_k \overset{\text{def}}{=} \left\{ x \in E : f^k(x) \in \mathcal{P}(x) \right\}
\]

be the set of points in \( E \) that have the same time \( k \) of return to their partition element. Pick an index \( m \) satisfying

\[
\text{card} F_m = \max_{n \leq k < n + rn} \text{card} F_k.
\]

Since \( \text{card} E = \sum_{n \leq k < n + rn} \text{card} F_k \), we have \( rn \text{ card} F_m \geq \text{card} E \). With \( rn < e^{nr} \) and (17) we obtain

\[
(1 - 5\delta)e^{n(e-2r)} \leq \frac{\text{card} E}{rn} \leq \text{card} F_m \leq \text{card} E \leq e^{n(e+r)}.
\]

Pick now out of the \( \ell \) rectangles from the rectangular cover the element \( R(x_i), x_i \in \Gamma_H \), for which \( \text{card}(F_m \cap \mathcal{P}(x_i)) \) is maximal. Hence we have

\[
(18) \quad \frac{1}{\ell} (1 - 5\delta)e^{n(e-2r)} \leq \text{card}(F_m \cap \mathcal{P}(x_i)) \leq \frac{1}{\ell} \text{card} F_m \leq \frac{1}{\ell} e^{n(e+r)}.
\]

Recall that exactly after \( m \) iterations each point \( x \in F_m \cap \mathcal{P}(x_i) \) returns to \( \mathcal{P}(x_i) \), and hence to the rectangle \( R(x_i) \), and that \( f^m(x) \in \Gamma_J \cap \Gamma_H \cap \Gamma_B \) (recall (15)). We have \( x \in B(x_i, \rho/2) \) and \( f^m(x) \in B(x_i, \rho/2) \). By items 3.–4. in Lemma 2 the connected components

\[
C_x \left( R(x_i) \cap f^{-m}R(x_i) \right) \quad \text{and} \quad f^m \left( C_x \left( R(x_i) \cap f^{-m}R(x_i) \right) \right)
\]
are an admissible \(L^s\)-rectangle and \(L^u\)-rectangle in \(R(x_i)\), respectively, for some number \(L > 0\). Analogously, one can show that the connected components 

\[ C_{f_m(x)}(R(x_i) \cap f^m R(x_i)) \quad \text{and} \quad f^{-m}(C_{f_m(x)}(R(x_i) \cap f^m R(x_i))) \]

are an admissible \(L^u\)-rectangle and \(L^s\)-rectangle in \(R(x_i)\), respectively. Note that item 4. in Lemma 2 implies that any point \(y \in C_x(R(x_i) \cap f^{-m} R(x_i))\) with \(y \neq x\) satisfies

\[ d(f^k(x), f^k(y)) \leq 3 \text{diam } R(x_i) \cdot \max\{e^{-k(\chi - \gamma)}, e^{-(m-k)(\chi - \gamma)}\} < \varepsilon \]

for each \(k = 0, \ldots, m\), which implies that \(y \notin F_m \cap P(R(x_i))\) since \(F_m\) is an \((n, \varepsilon)\)-separated set and \(x \in F_m \cap P(R(x_i))\). This implies that there are \(\text{card}(F_m \cap P(R(x_i)))\) disjoint admissible \(L^s\)-rectangles which are mapped under \(f^m\) onto \(\text{card}(F_m \cap P(R(x_i)))\) admissible \(L^u\)-rectangles. Let

\[ \Gamma := \bigcap_{n \in \mathbb{Z}} f^{nm} \left( \bigcup_{x \in F_m \cap P(R(x_i))} C_x(R(x_i) \cap f^{-m} R(x_i)) \right). \]

Consider the set \(V = R(x_1) \cup \ldots \cup R(x_\ell)\) and the map \(g = f^m\). Observe that by construction we have

\[ \Gamma := \bigcap_{k \in \mathbb{Z}} g^k(V), \]

that is, \(\Gamma\) is locally maximal with respect to \(g\) and to the closed neighborhood \(V\) built by rectangles. By construction, \(f^m|_\Gamma\) is topologically conjugate to a full one-sided shift in the symbolic space with \(\text{card}(F_m \cap P(R(x_i)))\) symbols. This shows claim i) of the theorem.

Since the rectangular cover was built around some sufficiently small neighborhood of points in the set \(\Gamma_H\) which is a subset of the support of the measure \(\mu\), claim ii) of the theorem follows.

Observe that every orbit in \(\Gamma\) is modeled over the shift and the map \(f^m\) sends a point in some rectangle \(R(x_i)\) into a rectangle \(R(x_j)\) by an orbit that stays \(\varepsilon\)-close to an orbit satisfying (14) for \(n = m\). Consider the set

\[ \hat{\Gamma} := \Gamma \cup f(\Gamma) \cup \ldots \cup f^{m-1}(\Gamma) \]

and observe that it is \(f\)-invariant. Together with (12) we obtain that for every \(x \in \hat{\Gamma}\) we have

\[ \int \phi \, d\mu - r \leq \liminf_{n \to \infty} \frac{1}{nm} S_{nm} \phi(x) \leq \limsup_{n \to \infty} \frac{1}{nm} S_{nm} \phi(x) \leq \int \phi \, d\mu + r. \]

In particular, for every \(f\)-invariant probability measure \(\nu\) supported on \(\hat{\Gamma}\)

\[ (19) \quad \left| \int \phi \, d\nu - \int \phi \, d\mu \right| < r. \]
Now recall that we concluded the above for all \( \phi \in \{ \psi_1, \ldots, \psi_K \} \). Hence, with (8), (9), and (10) we obtain

\[
d(\nu, \mu) \leq \sum_{i=1}^{K} 2^{-i} \frac{r_0}{2\|\psi_i\|_{\infty}} + \sum_{i=K+1}^{\infty} 2^{-i} \frac{1}{2\|\psi_i\|_{\infty}} \left| \int \psi_i d\mu - \int \psi_i d\nu \right|
\]

\[
\leq (1 - 2^{-K}) \frac{r_0}{2} \max_{i=1, \ldots, K} \|\psi_i\|_{\infty}^{-1} + 2^{-K} 2 < r.
\]

This proves claim iii) of the theorem.

Since \( f^m|_{\Gamma} \) is conjugate to the full shift, its topological entropy is equal to \( \log \text{card}(F_m \cap \mathcal{P}(x_i)) \) and hence the entropy of \( f|_{\Gamma} \) satisfies

\[
h(f|_{\Gamma}) = \frac{1}{m} \log \text{card}(F_m \cap \mathcal{P}(x_i)).
\]

Together with (18) we obtain

\[
\frac{1}{m} \log \frac{1}{\ell} - \frac{1}{m} \log(1 - 5\delta) | + \frac{n}{m}(e - 2r) \leq h(f|_{\Gamma}) \leq \frac{1}{m} \log \frac{1}{\ell} + \frac{n}{m}(e + r).
\]

Recall that \( n \leq m < n + rn \) and hence \( n/m \geq 1 - r/(1 + r) \) and together with (16)

\[
h(f|_{\Gamma}) \geq e - 2r - \frac{r}{1 + r}(e - 2r) - \frac{1}{m} \log(1 - 5\delta) - \frac{1}{m} \log \ell
\]

\[
> e - r(2 + e) - 2r.
\]

On the other hand

\[
h(f|_{\Gamma}) \leq -\frac{1}{m} \log \ell + e + r < e + r.
\]

This proves that \( h(f|_{\Gamma}) \) is arbitrarily close to \( e \), provided that \( r \) has been chosen small enough. This implies claim iv) of the theorem.

Let \( \nu \in \mathcal{M}(f|_{\widehat{\Gamma}}) \) and \( \varphi : M \to \mathbb{R} \) continuous. Since we want to estimate \( \int \varphi d\nu \), it suffices to assume \( \varphi \in C^0(\overline{U}) \). Then, by (11) there exists some \( \psi_i \) such that \( \|\psi_i - \varphi\|_{\infty} < r \). Hence, together with (19)

\[
\int \varphi d\nu \geq \int \psi_i d\nu - r \geq \int \psi_i d\mu - 2r \geq \int \varphi d\mu - 3r.
\]

Recall that the topological pressure of \( \varphi \) with respect to the compact \( f \)-invariant set \( \widehat{\Gamma} \) satisfies the variational principle (see [35])

\[
P(\varphi, f|_{\widehat{\Gamma}}) = \sup_{\nu \in \mathcal{M}(f|_{\widehat{\Gamma}})} (h_{\nu}(f) + \int \varphi d\nu).
\]

Thus, together with (22) and using (20), we obtain

\[
P(\varphi, f|_{\widehat{\Gamma}}) = \sup_{\nu \in \mathcal{M}(f|_{\widehat{\Gamma}})} (h_{\nu}(f) + \int \varphi d\nu) \geq \sup_{\nu \in \mathcal{M}(f|_{\widehat{\Gamma}})} h_{\nu}(f) + \int \varphi d\mu - 3r
\]

\[
= h(f|_{\widehat{\Gamma}}) + \int \varphi d\mu - 3r \geq e + \int \varphi d\mu - r(7 + e).
\]
Analogously, using (21) instead, we obtain
\[
P(\varphi, f|\hat{\Gamma}) = \sup_{\nu \in \mathcal{M}(f|\hat{\Gamma})} (h_{\nu}(f) + \int \varphi \, d\nu) \leq h(f|\hat{\Gamma}) + \int \varphi \, d\mu + 3r
\]
\[
< e + \int \varphi \, d\mu + 4r.
\]
This implies claim v) of the theorem.

Finally, by item 5. in Lemma 2 and (13) we conclude vi). □

3.1. Proof of Theorem 2. Let \( e \in [0, h(f|_M)) \). By the variational principle, there exists an ergodic measure \( \mu \) with \( h_{\mu}(f) \in (e, h(f|_M)] \). Hence, by Theorem 1 there exists \( m_1 \geq 1 \) and a basic set \( \Gamma_1 \subset M \) (with respect to \( f^{m_1} \)) satisfying \( h(f|_{\Gamma_1}) = e + \zeta_1 \) for some positive number \( \zeta_1 \in (0, h(f|_M) - e) \).

We will construct a sequence of compact \( f \)-invariant sets \( \Gamma_i \) such that \( \Gamma_1 \supset \Gamma_2 \supset \cdots \) and such that the entropy of \( f|_{\Gamma_i} \) monotonically decreases to \( e \). Hence, the set \( \Gamma = \bigcap_i \Gamma_i \) is compact and \( f \)-invariant and has entropy \( e \). The theorem is then a consequence of the following claim.

Claim 1. Given a basic set \( \Gamma_i \) (with respect to \( f^{m_i} \)), there is \( m_{i+1} \geq 1 \) and a basic set \( \Gamma_{i+1} \subset \Gamma_i \) (with respect to \( f^{m_{i+1}} \) for some \( m_{i+1} \geq 1 \)) satisfying \( h(f|_{\Gamma_{i+1}}) \in (e, e + 2^{-i}(h(f|_{\Gamma_i}) - e)] \).

Proof. Recall that, in particular, \( \Gamma_i \) is locally maximal in some neighborhood \( U_i \) (with respect to \( f^{m_i} \)). Let \( \zeta_i := h(f|_{\Gamma_i}) - e \). By the variational principle applied to \( f^{m_i}|_{\Gamma_i} \), there is an ergodic (hyperbolic) measure \( \mu_i \) on \( \Gamma_i \) with
\[
h_{\mu_i}(f) \in (e + 2^{-i}\zeta_i, h(f|_{\Gamma_i})].
\]
We now apply Theorem 1 to \( \mu_i \) (ergodic with respect to \( f^{m_i} \), \( e_i = e + 2^{-i}\zeta_i \), and \( r \in (0, 2^{-i}\zeta_i) \) to obtain a basic set \( \Gamma_{i+1} \) (with respect to \( f^{m_{i+1}} \) for some \( m_{i+1} \geq 1 \)) can be obtained arbitrarily close to the support of \( \mu_i \) being contained in \( U_i \). By local maximality of \( \Gamma_i \) we conclude that \( \Gamma_{i+1} \subset \Gamma_i \). This proves the claim. □

This proves the theorem. □

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