Relational particle models: I. Reconciliation with standard classical and quantum theory

Edward Anderson

Department of Physics, P-412 Avadh Bhatia Physics Laboratory, University of Alberta, Edmonton, T6G 2J1, Canada
Peterhouse, Cambridge, CB2 1RD, UK
and
DAMTP, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 0WA, UK

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Abstract
This paper concerns the absolute versus relative motion debate. The Barbour and Bertotti (1982) work may be viewed as an indirectly set up relational formulation of a portion of Newtonian mechanics. I consider further direct formulations of this and argue that the portion in question—universes with zero total angular momentum that are conservative and with kinetic terms that are (homogeneous) quadratic in their velocities—is capable of accommodating a wide range of classical physics phenomena. Furthermore, as I develop in paper II, this relational particle model is a useful toy model for canonical general relativity. I consider what happens if one quantizes relational rather than absolute mechanics, indeed whether the latter is misleading. By exploiting Jacobi coordinates, I show how to access many examples of quantized relational particle models and then interpret these from a relational perspective. By these means, previous suggestions of bad semiclassicality for such models can be eluded. I show how small (particle number) universe relational particle model examples display eigenspectrum truncation, gaps, energy interlocking and counterbalanced total angular momentum. These features mean that these small universe models make interesting toy models for some aspects of closed-universe quantum cosmology. Meanwhile, these features do not compromise the recovery of reality as regards the practicalities of experimentation in a large universe such as our own.

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1. Introduction

Newton’s formulation and conceptual framework for mechanics is based on his notions of absolute space and absolute time [1]. However, Leibniz [2], Berkeley [3] and Mach [4] envisaged that mechanics should rather be relational, i.e., feature solely relative distances,
angles and times. Despite these occasional criticisms, no concrete realizations of such mechanical theories were achieved and absolutism took hold. So while Lagrange [5] and Jacobi [6] invented notable techniques for the study of \(N\)-body Newtonian mechanics (NM): working in the barycentre frame, using relative particle separation coordinates, using the relative particle cluster separation Jacobi coordinates that diagonalize the relative formulation’s kinetic term, eliminating one angular momentum and using the Jacobi form for the action, they appear not to have assembled these techniques into a relationalist program [7]. The first elements of that appear in Dzioebek [8] and Poincaré [9]. A concrete relational synthesis was attained by Barbour and Bertotti in 1982 (BB82) [10] which I present in section 2. This has also been studied in [11–21], reviewed in [22–24] and philosophized about in [16, 18, 25, 27]. It consists of a classical relational reformulation of a portion of NM. BB82-type formulations may be perceived to have an undesirable indirectness in that while their derivation ends up free of absolute space, it proceeds indirectly via what may be viewed as absolutist notions (even if BB82 select these purely on grounds of convenience among various ways of conceptualizing in terms of configuration spaces). It is worth commenting that while it is not logically necessary\(^1\) for a relational viewpoint to lead\(^2\) to a recovery of NM, it is nevertheless interesting for a resolution of the absolute versus relative motion debate to have such a feature.

In this paper, I examine and improve three key features [26] of BB82 relational particle mechanics (RPM). Firstly, it can also be cast in a directly relational form, which I attain in Jacobi action relative particle separation coordinate form in section 3\(^3\) and furthermore recast in terms of relative Jacobi coordinates in section 4. Jacobi coordinates are particularly well suited to RPM and are mathematically central to this paper.

Secondly, by a ‘portion’ I mean the restriction to closed systems/universes of zero total angular momentum \(L = 0\), that are conservative and have a kinetic term that is homogeneous quadratic in the velocities. It is therefore important to investigate whether these are major restrictions. In section 5, I argue that this portion accommodates much, by tricks which counterbalance the subsystem of interest with other subsystems elsewhere. This portion also admits a nonhomogeneous quadratic extension.

Thirdly, I begin to investigate the quantum implications of shifting from an absolutist to a relationalist viewpoint. This is particularly relevant given how GR and QM incorporate incompatible notions of time. This ‘problem of time’ and its conceptual ramifications are good reasons (see, e.g., [15, 18, 22, 29–32]) why these two areas of theoretical physics have not been successfully combined insofar as we do not have a satisfactory account of quantum GR (nor a fully satisfactory account of any distinct theory of quantum gravity at all). In particular, the traditional framework of QM is based on Newton’s formulation of particle mechanics, absolute space and time and all, while GR, as well as being a successful reconciliation of SR and Newtonian gravity and a successful improvement of the theory of gravitation, is also Einstein’s attempt to free physics of absolute structure. Many theoretical physicists consider this feature of GR to be an important lesson about how the universe works, to the extent that quantum gravity schemes that, contrarily, have extraneous ‘background dependence’ [23, 31, 33, 34]\(^4\) can be regarded as not yet conceptually satisfactory. Indeed one major goal of nonperturbative \(M\)-theory is to attain this background independence.

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\(^1\) Starting on relational premises, one can also arrive at distinct theories of particle mechanics, see section 3.

\(^2\) Such a recovery of NM from different premises parallels Wheeler’s promotion of many routes to general relativity (GR) rather than just Einstein’s. See paper II of this series [28] for references and further discussion.

\(^3\) See also Lynden-Bell [17] and Gergely [19] for earlier formulations.

\(^4\) One should also take care to note that background (in)dependence can take on a different meaning in the string- and \(M\)-theory literature, namely vacuum choice independence of string perturbations.
Now, given that the relational reformulation of \( L = 0 \) NM is itself a successful abolition of absolute structure, the following interesting question arises. Q1: does this give a different sort of QM from that of the traditional absolutist approach? A further reason for this study is that the BB82 RPM approach to \( L = 0 \) NM closely parallels \([10, 14, 18, 21, 24]\) to the geometrodynamical formulation of spatially compact without boundary GR \([21, 35–37]\). Indeed, \([11, 14, 15, 18, 22, 23]\) have considered BB82 RPM as a useful toy model arena for the investigation of quantum gravity and quantum cosmology issues such as the problem of time and difficulties with closed universes. Note, however, that there is a perceived obstruction to this program. While Barbour and others have been advocating BB82 RPM as a resolution of the absolute versus relative motion debate for some years, the Barbour–Smolin (BS) preprint \([11]\) also claims that quantizing BB82 does not give a good semiclassical limit on two counts.

**BS1:** that small masses affect the QM spread of large masses.

**BS2:** Due to constituent subsystems being interlocked by additional restrictions.

This position is followed up by arguing against the applicability of standard quantization procedures, leading them instead to ‘seek a radical alternative’. References \([15, 18]\) may be seen as the start of Barbour’s search for this—a particular consistent records program.

Nevertheless, in this paper I begin by addressing Q1. This subdivides into whether different formal mathematics becomes involved and whether there are interpretational differences. I principally consider simple 1D models, regarding extension to \( d > 1 \) models where possible as a useful bonus. I do so by use of relative Jacobi coordinates, noting that these map many relational quantum problems to problems whose formulation is known from the standard quantization of the absolutist formulation of NM. Thus, the first steps of the study of these simplest RPM models do not involve formal mathematical differences, permitting one to benefit from separation and special function techniques. Thereby, I can cover a wide range of RPM models for the minimally relationally nontrivial case of three particles. These go well beyond BS’s (piecewise) constant potentials in 1D, covering additionally a number of interparticle harmonic oscillator and Newton–Coulomb potential problems including those presented in section 7.

The first salient difference is interpretational (section 8). Spreads are now in terms of relational quantities (relative separations suffice in 1D), from which perspective I show that objection BS1 is misplaced. The second is mathematical, but turns out not to spoil the above standard techniques by being treatable after deploying these techniques. This involves the collective of subsystems having interlocked energy and counterbalancing AM, and the subsystem eigenspectra sometimes exhibit truncations and gaps. These features are a further development of the mathematical observations which led to objection BS2, but I also develop further what the implications of these are. For universes that have a large particle number and a diverse content (such as free particles and both positive- and negative-potential subsystems), gaps need no longer occur, truncations become acceptable, while interlocking and counterbalancing are obscured by the practicalities of experimentation involving only a small subsystem. As our own universe is large and diverse in content, the ‘recovery of reality’ is thereby not compromised by BS2. This opens the way for semiclassical exploration of these models, which I consider further, alongside the aforementioned consistent records formulation, in II.7–II.8. On the other hand, small universe models do exhibit these interesting theoretical effects whereby they are useful toy models for understanding subtle ramifications that arise from considering closed-universe quantum cosmologies.

5 References \([38, 39]\) and II also view a more recent scale-invariant model as useful in this respect.

6 While this paper provides relationalist motivation underlying this technique, note that the technique itself is well established in the molecular physics literature \([40]\) due to its practical usefulness.
2. Barbour–Bertotti 1982 formulation in terms of auxiliary variables

Let \( Q \) be the naive \( Nd \)-dimensional configuration space in \( d \) dimensions for \( N \) particles whose positions are \( q_{\alpha A} \).\(^7\) Jacobi actions are used, which implement temporal relationalism since they are reparametrization invariant in the label-time \( \lambda \). The actions considered in this paper are for particle models whose kinetic term \( T \) is homogeneous quadratic in its constituent velocities. These take the form\(^8\)

\[
S_{\text{Jacobi}} = 2 \int d\lambda \sqrt{(E - V)T},
\]

(1)

where \( V \) is the potential term and \( E \) is the total energy, taken to be a fixed constant ‘energy of the universe’ \( E_{\text{Universe}} \). Note that the more usual particle mechanics action, \( S_{\text{Euler–Lagrange}} = \int dt(T - V) \), can indeed be cast \(^41\) into the incipient Jacobi form (1) by adjoining the Newtonian time \( t \) to \( Q \), then noting that \( \dot{t} = dt/d\lambda \) alone features in the action and subsequently eliminating it by Routhian reduction.

The classical particle mechanics notion of spatial relationalism is implemented by passing to a suitable notion of arbitrary frame\(^9\). This is achieved by the introduction of a translational auxiliary \( d \)-vector \( a_\alpha(\lambda) \) and whichever rotational auxiliary corresponds to \( d \): none for \( d = 1 \), a scalar \( b(\lambda) \) for \( d = 2 \) or a 3-vector \( b_\alpha(\lambda) \) for \( d = 3 \), so that the \( A \)th particle’s position \( q_{\alpha A} \) is replaced by

\[
&_{a,b}q_{\alpha A} \equiv q_{\alpha A} - a_\alpha - \epsilon_{\alpha \beta \gamma}b_\beta q_{\gamma A}. \quad (2)
\]

Whereas the 1D and 2D cases can be written in intrinsically 1D and 2D notation, I present all three cases together by using the extraneous 3D \( \epsilon \) symbol, under the provisos that \( b_\alpha = 0 \) for \( d = 1 \) and \( b_\alpha = (0, 0, b_3) \), \( b \equiv b_3 \), \( L_\alpha = (0, 0, L_3) \), \( L \equiv L_3 \) for \( d = 2 \).

While this may look like doubling the allusions to absolute space, \( Q \rightarrow Q \times \text{Eucl}(d) \) (for \( \text{Eucl}(d) \) the Euclidean group of translations and whichever rotations exist in dimension \( d \)), I reassure the reader that this doubling of redundancy leads promptly below to the removal of the redundancy in the fashion familiar in gauge theory. I proceed by requiring the action to be built as best as possible out of objects that transform well under \( \lambda \)-dependent Eucl(\( d \)). In particular, defining the naive or Lagrange relative coordinates (relative separations between particles) \( r_{aAB} \equiv q_{aA} - q_{aB} \)

\[
V = V(|r_{aAB}| \text{ alone}), \quad (4)
\]

wherein the auxiliary corrections straightforwardly cancel each other out. The situation with the kinetic term is more complicated. \( \frac{\partial}{\partial \lambda} \) is not a tensorial operation under \( \lambda \)-dependent Eucl(\( d \)) transformations. Along the lines in \(^42\), \( \frac{\partial}{\partial \lambda} \) should rather be seen as the Lie derivative \( \mathcal{L} \) in a particular frame, which transforms to the Lie derivative with respect to \( \mathcal{L} \) corrected additively by generators of translations and rotations\(^9\). This gives a kinetic term of the form

\[
T(q_1', q_2', \dot{a}, \dot{b}) = \frac{1}{2} \sum_{A=1}^{N} m_A \delta_{\alpha \beta} (\&_{a,b}q_{\alpha A})(\&_{a,b}q_{\beta A}). \quad (5)
\]

\(^7\) Lower-case Greek letters running from 1 to \( d \) are spatial indices (sometimes I use the underline notation instead of the index notation for these); I shall develop models for \( d = 1–3 \). Upper-case Latin letters running from 1 to \( N \) are particle labels and lower-case Latin letters running from 1 to \( N - 1 \) label relative particle (cluster) separations.

\(^8\) See section 4 for a generalization of this form required for the inclusion of physics with linear kinetic terms.

\(^9\) This account is a further development of the matter rather than following the formulation in the BB82 paper verbatim.
So, finally, the proposed action is $S_{\text{Jacobi}}(\dot{q}_I, \dot{\dot{q}}_J, \dot{\dot{a}}, \dot{\dot{b}})$ of form (1) with (4) and (5) substituted into it.

The momenta conjugate to $q_{\alpha A}$ are

$$p^{\alpha A} = \sqrt{\frac{E - V}{T}} m_A \delta_{\alpha \beta} & \dot{a}_b, \dot{b}_{\beta A}.$$  

(6)

By virtue of the particular reparametrization invariance of the Lagrangian, these obey the primary quadratic constraint

$$Q \equiv \sum_{A=1}^{N} \frac{1}{2m_A} \delta_{\alpha \beta} p^{\alpha A} p^{\beta A} + V = E.$$  

(7)

The secondary linear constraints,

$$P^\alpha \equiv \sum_{A=1}^{N} p^{\alpha A} = 0 \quad \text{(zero total momentum constraint)}$$  

(8)

and whichever portion of

$$L^\alpha \equiv \sum_{A=1}^{N} \epsilon^{\alpha \gamma \beta} q_{\beta A} p^{\gamma A} = 0 \quad \text{(zero total AM constraint)}$$  

(9)

is relevant in the corresponding dimension, follow from variation of the cyclic auxiliary coordinates $a_\alpha$ and $b_\alpha$. These constraints obey the Poisson bracket algebra

$$\{P^\alpha, P^\beta\} = 0, \quad \{P^\alpha, Q\} = 0, \quad \{Q, Q\} = 0, \quad \{P^\alpha, L^\beta\} = \epsilon^{\alpha \gamma \beta} P^\gamma, \quad \{L^\alpha, L^\beta\} = \epsilon^{\alpha \gamma \beta} L^\gamma, \quad \{L^\alpha, Q\} = 0$$  

(10)

for $Q = Q - E$. As this is closed, there are no further constraints. The constraints (8), (9) then signify that the physics is not on the doubly redundant configuration space $Q \times \text{Eucl}(d)$ but on the quotient space $Q / \text{Eucl}(d)$, thus indeed rendering absolute space irrelevant.

It is worth counting to establish which are the minimal nontrivial dynamics examples. There are $dN$ degrees of freedom in the $d$-dimensional NM of $N$ particles. There are $d$ translations and $d(d - 1)/2$ rotations which make up $d(d + 1)/2$ irrelevant motions. So the $d$-dimensional $N$-particle BB82 RPM has $d(2N - d - 1)/2$ degrees of freedom. Furthermore, one wants to express one change in terms of another change rather than in terms of an arbitrarily reparametrizable label-time. Thus, dynamical nontriviality dictates $N$ and $d$ be such that $d(2N - d - 1)/2 > 1$ which gives $N \geq 3$ both in 3D and in 1D. In this paper, I focus principally on the simplest nontrivial dynamics: $N = 3$.

3. Passage to direct formulation in terms of relative variables alone

In contrast to the above formulations, there is a distinct, non-Newtonian, failed and often rediscovered theory\(^{10}\) formulated directly in terms of $r_{IJ}$ and without auxiliaries referring to

\(^{10}\) Sometimes called BB 1977 theory [43], as described in [16], this theory is first known to have been discovered by Hoffmann, and was rediscovered by Reissner and then by Schrödinger. While this theory benefits from being able to explain the anomalous perihelion shift of Mercury (while still being a nonrelativistic particle theory!), it additionally predicts a level of mass anisotropy that is unacceptable given the results of the Hughes–Drever experiment [44]. I note furthermore that there are yet other such RPMs [17, 38], but these either disagree with observation or have not yet been studied enough to know whether they disagree.
absolute space. I should point out that this theory has a kinetic term that is exceptionally simple,

\[ T = \frac{\delta a^\beta \delta^I K \delta^J L}{|r_{PQ}|} \dot{r}_{aI} \dot{r}_{\beta IK}, \]  

(11)

which is probably one reason why this theory has been rediscovered so many times. The question then arises whether BB82 RPM can also be cast as a presumably more complicated direct formulation of this kind, whereupon not even indirect or unphysical reference to absolute space is required. I address this below, partly foreshadowed by work of Lynden-Bell [17], Gergely [19] and Jacobi.

I use that the \( d = 3 \) working contains everything under the provisos (3) and then comment on each individual case \( d = 1, 2, 3 \) as these differ significantly. I begin with \( S_{Jacobi}(q_I, \dot{q}_J, \dot{a}, \dot{b}) \). I define \( \dot{m} \) as the total mass \( \sum_{i=1}^N m_i \). I eliminate \( \dot{a}_a \) from its own variational equation

\[ \dot{a}_a = \frac{1}{\dot{m}} \sum_j m_j (\dot{q}_{aI} - \epsilon_{\alpha \beta \gamma} \dot{b}_\beta q_{\gamma I}) \]  

(12)

(the Lagrangian counterpart of the Hamiltonian expression (9)). This results in the (semi-) eliminated action

\[ S^*(\xi_{IJ}, \dot{\xi}_{KL}, \dot{b}) = 2 \int d\lambda \sqrt{(E - V(|r_{GH}|)) T^*(\dot{\xi}_{IJ}, \dot{b} \times \xi_{KL})} \]  

for \( T^*(\dot{\xi}_{KL}, \dot{b} \times \xi_{IJ}) \)

(13)

I next eliminate \( \dot{b}_a \) from its own variational equation\(^{11}\)

\[ \dot{b}_a = (I^{-1}(\xi_{IJ}))^{a \beta} B^\beta (\xi_{IJ}, \dot{\xi}_{KL}) \]  

(14)

where \( I_{a\beta}, L_\alpha \) are the barycentric inertia tensor and AM, respectively, which are easily castable into relative separation coordinate form

\[ I_{a\beta} (\xi_{IJ}) = \sum_{l<J} \sum_j \frac{m_j m_j}{\dot{m}} (|\xi_{IJ}|^2 \delta_{a\beta} - \xi_{aIJ} \xi_{\beta JI}), \]  

\[ L_\alpha (\xi_{IJ}, \dot{\xi}_{KL}) = \epsilon_{\alpha \beta \gamma} \sum_{l<J} \sum_j \frac{m_j m_j}{\dot{m}} \xi_{\betaIJ} \dot{\xi}_{\gamma JI}. \]  

(15)

This results in the eliminated action

\[ S^{**}(\xi_{PQ}, \dot{\xi}_{ST}) = 2 \int d\lambda \sqrt{(E - V(|r_{NO}|)) T^{**}(\dot{\xi}_{PQ}, \dot{\xi}_{ST})} \]  

(16)

for

\[ T^{**}(\dot{\xi}_{PQ}, \dot{\xi}_{ST}) = T^*(\dot{\xi}_{PQ}, 0) - \frac{1}{2} L_\alpha (I^{-1})^{a \beta} B^\beta. \]  

(17)

Equation (16) is the formulation of the stated portion of NM cast entirely in relative terms for \( d = 3 \). For \( d = 1, \dot{b} = 0 \) so the working stops at (13). For \( d = 2 \), (14) simplifies considerably since \( I_{a\beta} \) becomes a mere scalar

\[ B^\beta (\xi_{IJ}) \equiv B^\beta (\xi_{IJ}) = \frac{2-d}{d} \sum_{l<J} \sum_j \frac{m_j m_j}{\dot{m}} |\xi_{IJ}|^2, \]  

(18)

so one has trivial explicit invertibility and the second term in (17) simplifies considerably (see also (29)).

\(^{11}\) This gives the same result as if \( \dot{b} \) is eliminated first or indeed as if \( \dot{a} \) and \( \dot{b} \) are eliminated simultaneously.
4. Coordinate improvements: Jacobi coordinates

As written, my expressions depend on all of $r_{IJ}$ rather than on an $(N-1)$-member basis. I first redress this by noting that

$$r_{IJ}, I = 1 \text{ to } N - 1, \quad J = I + 1 \quad \text{form a basis for the relative separation coordinates},$$

and recasting the previous section’s symmetric but redundant ‘Lagrangian’ expressions asymmetrically yet nonredundantly in terms of these alone (e.g., using $L_{13} = L_{12} + L_{11}$).

Next, I consider mapping the $Nq_i$ position vector coordinates linearly to a maximal number $N - 1$ vector relative particle (cluster) separation coordinates $\omega_i$ and 1 vector absolute coordinate $\omega_N$. The relative coordinates therein can be recognized since then from (19)

$$\omega_I = \sum_{J=1}^{N} A_{IJ} q_J$$

for $A_{IJ}$ constants such that

$$\sum_{J=1}^{N} A_{IJ} = 0.$$

(20)

If one has any such set of coordinates containing a maximal number of relative coordinates $\omega_i, i = 1 \text{ to } N - 1$, the final coordinates $\omega_N$ being no true relative coordinates, the zero total momentum constraint of section 2 gives

$$0 = \sum_{I=1}^{N} p_I = \sum_{I=1}^{N} \sum_{J=1}^{N} \frac{\partial \omega_I}{\partial q_J} m^J = \sum_{I=1}^{N} \sum_{J=1}^{N} A_{IJ} m^J = 0 + \sum_{I=1}^{N} A_{N} \omega^N \Rightarrow \omega^N = 0,$$

(21)

where $m^J$ are the momenta corresponding to the new set of coordinates, and using (20) in the last two steps. Thus, the zero momentum constraint can be absorbed by passing to any such coordinate system, and this causes the absolute coordinates $\omega_N$ therein to drop out of all remaining equations, which thereby involve relative coordinates alone.

Furthermore, I wish $T^*$ not only to be in terms of a basis of relative coordinates but also to be diagonal. This will be particularly useful in this paper when it comes to quantization, and is true of the Jacobi coordinates $R_i$ [40, 45], which exist for all relevant $N$ (i.e., $N \geq 3$). As among the $(N$ arbitrary) Jacobi coordinates, without loss of generality $R_i, i = 1 \text{ to } N - 1$ are relative, I set these to be the particular relative coordinates I use. Because of the existence of the zero momentum constraint, which may be cast in the form (21), it is irrelevant which single absolute coordinate vector I adjoin to these as the constraint will in any case wipe out this choice, so I can choose my last coordinate to be $q_N$ rather than $R_N$ for simplicity of computation. I then find that the relative Jacobi coordinates not only diagonalize $T$ but have a large number of further properties which are particularly well suited to the study of RPM. Namely, the form-preservation under the $q_i$ space to $R_i$ space map of $T$ (both as a function of velocities and of momenta), the moment of inertia $J, I_{i,j}, L_{i},$ and $L_{i}$; see also section II.4 for one more such preserved object and sections 6 and II.6 for major applications of these properties.

As a first application of relative Jacobi coordinates, I recast the indirect formulation of section 2 as the action

$$S^*(R, \dot{R}, b) = \int d\lambda \sqrt{(E - V(|\dot{R}_i|, R_i, R_j))T^*(\dot{R}_i, \dot{b} \times R_i)}$$

for $T^*(\dot{R}_i, \dot{b} \times R_i) = \sum_{i=1}^{N-1} \frac{M_i}{2} |\dot{R}_i - \dot{b} \times R_i|^2.$

(22)
With $P_0 = 0$ reducing to the absence of any additional absolute coordinates, the surviving constraints are
\[ \mathcal{Q} = \sum_{i=1}^{N-1} \frac{P_i^2}{2M_i} + V - E \]
and
\[ L_a(R_i, \dot{R}_j) = \epsilon^{\alpha\beta\gamma} N^{-1} \sum_{i=1}^{N-1} M_i R_{\beta i} P_i^2 = N^{-1} \sum_{i=1}^{N-1} L_{ai} = 0. \]
Note how employing relative Jacobi coordinates retains the separability of the zero AM constraint while preserving the form of the individual AM operators.

As a second application, I recast the direct formulation of section 3 dimension by dimension. For $d = 1$,
\[ S^{**}[R_i] = 2 \int d\lambda \sqrt{(E - V(R_i))T^{**}(\dot{R}_i)\dot{R}_i} \quad \text{for} \quad T^{**}(\dot{R}_i) = \sum_{i=1}^{N-1} \frac{M_i}{2} \dot{R}_i^2. \]
So these coordinates are diagonal and also $d = 1$ ensures that this formulation is not just relative but also fully relational, because $S = S[|R_i|]$ alone.

For $d = 2$,
\[ S^{**}[R_i] = 2 \int d\lambda \sqrt{(E - V(R_i, R_j)T^{**}(\dot{R}_i, \dot{R}_j))} \]
for
\[ T(R_i, \dot{R}_j) = \sum_{i=1}^{N-1} \frac{M_i}{2} |\dot{R}_i|^2 - \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} M_i M_j \frac{\epsilon^{\alpha\beta\gamma} R_{\alpha i} R_{\beta j}}{2I} \dot{R}_i \dot{R}_j \dot{R}_j \gamma \dot{R}_j \beta \]
and
\[ B = \sum_{i=1}^{N-1} M_i |\dot{R}_i|^2. \]
But
\[ \epsilon^{\alpha\beta\gamma} R_{\alpha i} \dot{R}_j \dot{R}_j \dot{R}_j \beta = (R_1 \cdot R_2)(\dot{R}_1 \cdot \dot{R}_2) - (R_1 \cdot \dot{R}_1)(R_2 \cdot \dot{R}_2), \]
which is made out of relative angles and relative separations, and so the Jacobi coordinates serve to give a fully relational formulation. Alternatively (to compare it with (11), with [19] and for future reference), (27) can be written in terms of an (inverse) configuration space metric
\[ T(R_i, \dot{R}_j) = \frac{1}{2} \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} (G^{-1})^{\alpha\beta\gamma\delta} \dot{R}_{\alpha i} R_{\beta j} R_{\gamma j} R_{\delta j}, \]
and
\[ (G^{-1})^{\alpha\beta\gamma\delta} = M_i \delta^{\alpha\beta\gamma\delta} \frac{M_j M_k}{\sum_{k=1}^{N-1} M_k |R_k|^2} \epsilon^{\gamma\delta} R_j^i R_k^j. \]
for $d = 1$ the metric is flat).

For $d = 3$,
\[ T^{**}(R_i, \dot{R}_j) = T^*(\dot{R}_i, 0) - \frac{1}{2} B \frac{\dot{L}_\alpha}{L_\alpha} \frac{B}{L_\beta} \]
(30)

12 Here, the $M_i$ are redefined masses in terms of the original masses, $M_f$. I do not provide $M_i = M_i(m_f)$ relations here as in any case depend on normalization convention, and Jacobi coordinates are nonunique for $N > 3$.

13 $[|\dot{R}_i|^2, \arccos(\frac{\dot{R}_i \cdot \dot{R}_j}{|\dot{R}_i||\dot{R}_j|})]$ and $[|\dot{R}_i|^2, \dot{R}_j \cdot \dot{R}_k]$ clearly span the same set of functions, which are the truly relational quantities.
The inverse configuration space metric is now

\[
(G^{-1})^\alpha\beta_{ij} = M_i \delta^{\alpha\beta} \delta_{ij} - M_i M_j \epsilon^\gamma_\delta \epsilon^\mu_\lambda (I^{-1})^\gamma\mu \epsilon^\lambda_\delta R_i^j R_i^j.
\]

Note the complication through the presence of the nontrivial inverse of the inertia tensor \( I^{-1} \).

I currently can do no better than express this as relational variables plus angles between Jacobi vectors and principal directions of the system’s inertia quadric. This underscores the approach in practice for papers I and II: work on techniques for the 1D problems since these will turn out to already exhibit many of the interesting features of RPM models; direct extensions of these methods to \( d > 1 \) are furthermore worth considering even though they will sometimes be a semi-relational ‘halfway house’ rather than fully relational methods. Moreover, some of the remaining interesting features appear in the scale-invariant 1D set-up (see paper II).

The above are the last formulations of the 1D, 2D and 3D \( L = 0 \) conservative homogeneous quadratic portion of NM in this paper (section II.4 contains yet further formulations and study of the configuration space metrics \((29, 32))\). I next address the question of whether focusing on this portion of NM is a significant restriction.

5. How restrictive is considering only this portion, at a classical level?

BB82 RPM and \( L = 0 \) NM give coincident physical predictions. If \( L \neq 0 \) momentum physics is under study, one can always represent it as a subsystem of a \( L = 0 \) universe [10, 17], so that the restriction to \( L = 0 \) universes is by no means as severe as might be naively suggested.

One concern is that the viability of this rests on the \( N \)-body problem’s theoretical framework being such that initially distant subsystems do not fall together below any desired finite time scale internal to the subsystem under study. Reversely, such falling together requires the unboundedness of some velocities and hence of \( T \) and hence of \( V \). This is termed a singularity. While it used to be thought that this could only occur in situations involving collisions, it has been shown that for five bodies it is possible to attain this by merely coming arbitrarily close to collisions [46]. It is still mere conjecture, however, that singular solutions are of measure zero in the set of all solutions. In any case, sufficiently accurate physical modelling of the universe acts to prevent arbitrarily distant subsystems from falling together in finite time: astrophysics with realistic matter will experience short-range forces interfering with the potential being arbitrarily negative or GR (which respects a positive energy theorem) will take over, while SR will bound infall velocities.

Can one test whether \( L = 0 \) or \( \neq 0 \) in our universe? It is true that \( L = 0 \) and \( \neq 0 \) (sub)systems are capable of evolving qualitatively differently as exemplified by Hill’s work [47]. That BB82 requires \( L = 0 \) has been considered to be a prediction of the BB82 formulation as a separate theoretical entity from NM [27]; moreover, \( L = 0 \) appears to be true for the universe we are in\(^{14}\). Also, the possibility of formulating \( L \neq 0 \) NM relationally should not be dismissed. While this goes against the Poincaré principle that Barbour advocates, that at most positions and velocities should need to be specified in RPM, this principle might just be a simplicity; for sure, \( L \neq 0 \) is known to be much more complicated than \( L = 0 \) by the work of Dziobek [8] and of Poincaré [9].

\(^{14}\) However, detailed cosmological study requires GR considerations.
Finally, there are some robustness issues. Can branches of physics external to particle mechanics be incorporated? Doing so would remove interference by external torques violating AM conservation, and the possibility of absolute space or time being bestowed upon particle mechanics by other branches of physics. Electromagnetism can probably be incorporated (see also p 14). I can accommodate dissipative processes such as linear air resistance and diffusion by a counterbalancing trick [49] that adjoins nonphysical fields (which may be regarded as ‘elsewhere’) into the Lagrangian, but I cannot see any means of extending this to nonlinear dissipative processes such as air resistance quadratic in the velocity. In this paper, geared towards quantization, I stay clear of dissipative processes, which one would hope are in any case recastable in terms of more fundamental processes. What is excluded by considering only conservative dynamics? Is homogeneous quadraticity restrictive? While BS say “pretty well all dynamical systems in both normal mechanics and SR field theory can be cast in such forms”, both Lanczos [41] and I [36] have argued contrarily. In mechanics, the direct study of systems cast with linear ‘gyroscopic’ terms would be precluded; linear terms also arise in the field-theoretic counterpart of this study for moving charges and for spin-1/2 fermions. While in principle nothing goes wrong if the treatment is extended to these, practical difficulties can arise if the ensuing complications thus brought into the actions are extended to their logical conclusion. However, as a compromise, as far as I know, nonhomogeneous quadraticity suffices to describe all established physics while retaining a form that is algebraically manageable\textsuperscript{15}, which gives Jacobi actions of the type

\[ S = \int d\lambda \left( \sqrt{E - V_{\text{Total}}} T_{\text{Quadratic}} + T_{\text{Linear}} \right). \]  

6. Setting up the quantization of relational particle models

I next address what kind of QM follows from such a formulation for the simplest models: \( N = 3 \) conservative, homogeneous quadratic \( \hat{L} = 0 \) models. In \( d = 1 \) the configuration spaces are flat, so one can hope to employ standard mathematics. As discussed in section 4, I choose for the moment to continue to work with now only partly reduced flat configuration spaces when passing to \( d = 2 \) and 3. This is possible because of the good fortune that Jacobi coordinates preserve the separation of \( \hat{L} = 0 \).

Making use of the position representation \( \hat{q}_A = q_A \) and \( \hat{p}_A = -i\hbar \partial/\partial q_A \), and denoting \( \partial^2/\partial q_i^2 \) by \( \triangle q_i \), the constraints of section 2 become the quantum constraints

\[ \hat{\mathcal{L}} \Psi(q_A) \propto \sum_{i=1}^{N} \frac{\partial}{\partial q_i} \Psi(q_A) = 0, \]  

(34)

\[ \hat{\mathcal{Q}} \Psi(q_A) = \left( \sum_{i=1}^{N} -\frac{\hbar^2}{2m_i} \triangle q_i + V(q_A) - E \right) \Psi(q_A) = 0. \]  

(35)

In 1D, that is all, while for \( d = 2 \) or 3 there is also (the portion relevant in dimension \( d \) of)

\[ \hat{\mathcal{L}} \Psi(q_A) \propto \sum_{i=1}^{N} q_i \times \frac{\partial}{\partial q_i} \Psi(q_A) = 0. \]  

(36)

Note that there are no operator ordering ambiguities in any of these constraints (only in the case of the zero AM constraint are there products of \( p_s \) and \( q_s \), and even there the order is

\textsuperscript{15}This nonhomogeneity does however spoil Barbour’s assertion that all energies in the universe contribute to the emergent lapse quantity \( \dot{O} = \sqrt{T_{\text{Total}}/(E - V_{\text{Total}})} \), which is related to his notion of perfect clock.
unambiguous by symmetry–antisymmetry). I am also ‘lucky’ in Dirac’s sense [34]: the full $d$-dimensional set of constraints quantum-closes; moreover, this is in direct parallel with the classical closure (i.e., $\{,\} \rightarrow \frac{1}{i\hbar}[,$]), so I do not present it.

The kinetic term in (35) contains a sum of Laplacians on $\mathbb{R}^d$ in absolute,particle-position Cartesian coordinates $q_I$. Towards addressing relational problems, I recast this in terms of relative coordinates. I wish the Laplacian to remain diagonal so as to exploit separability, so I employ relative Jacobi coordinates $R_I$. The momentum constraint then becomes

$$\frac{\partial \Psi}{\partial N} = 0,$$  \(37\)

which expresses the translation invariance of the wavefunction even more manifestly than (34). Alternatively, one could start in relative Jacobi coordinates, in which case (37) would have been built in automatically. By either route, the remaining constraints are

$$E\Psi = \sum_{i=1}^{N-1} \frac{\hbar^2}{2\mu_{12}} \Delta \Psi + V\Psi$$ \(38\)

and (the portion relevant in dimension $d$ of)

$$\hat{L}\Psi \propto \sum_{i=1}^{N-1} R_I \times \frac{\partial}{\partial R_I} \Psi = 0.$$ \(39\)

Thus, I end up on the quotient space $Q/\{\text{Translations}\} = \mathbb{R}^{N-1} \times \mathbb{R}^d$ in Cartesian coordinates $R_I$.

For $N = 3$, I employ the Jacobi coordinates

$$R_1 = L_{12} = q_1 - q_2,$$ \(40\)

$$R_2 = N((m_1 + m_2)q_3 - m_1 q_1 - m_2 q_2),$$ \(41\)

$$R_3 = q_3;$$ \(42\)

(see figure 1 for their meaning) with an as-yet unfixed normalization $\mathcal{N}$ for convenience.

Then,

$$T\Psi = -\frac{\hbar^2}{2\mu_{12}} (\Delta_1 + \Delta_2) \Psi(R_1, R_2)$$ \(43\)

for $\mu_{12}$ the standard definition of reduced mass for a $m_1, m_2$ pair (I henceforth drop the $12$ indices) and the choice of $\mathcal{N} = \sqrt{m_3/m_1 m_2 M}$. Thus, one has $2d$-dimensional flat space problems, which look like standard time-independent Schrödinger equations (TISEs), except that it is now the relative coordinates $R_1$ and $R_2$, and not the particle positions which play the

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The meaning of (the standard normalization of) the 3-body problem relative Jacobi coordinates $R_1$ and $R_2$. Among other things, Jacobi coordinates are well suited to study the effect of an extra mass $m_3$ on a well-understood subsystem $m_1, m_2$.}
\end{figure}
role of Cartesian coordinates on the reduced configuration space (and, for \( d > 1 \), there is an additional equation restricting the total AM to be zero).

I am principally interested in physically motivated examples such as free, (piecewise) constant, harmonic oscillator (HO) and Newton–Coulomb-type potentials. Furthermore, as each potential involved is expressible as a function of the relative separations \( |r_{ij}| \) alone, choosing to use these potentials is a relational input, as befits the current investigation. This inherent relationalism in many commonly studied potentials is an early indicator of some mathematical similarities\(^{16}\) with the standard absolutist QM. This is worth pointing out, since it means that, at least to start off with, one does not get the wrong sort of formal mathematics by quantizing absolute rather than relational particle mechanics. Interpretational and mathematical differences appear later in the study.

For such potentials, I demonstrate that passing to the relative Jacobi coordinates \( R_1, R_2 \) often permits separation
\[
\Psi(R_1, R_2) = \psi_1(R_1) \otimes \psi_2(R_2)
\]

into recognizable problems. Indeed, the \( q_I \rightarrow R_i \) map I employ is form-preserving for many of these problems. That is, it maps the 3\( d \)-dimensional configuration space quantum problem, for which \( q_I (I = 1–3) \) play the role of Cartesian coordinates and that is separable into three standard \( d \)-dimensional problems for each \( q_I \), to a corresponding 2\( d \)-dimensional configuration space problem, for which \( R_i (i = 1–2) \) play the role of Cartesian configuration space coordinates and that is separable into two of the same kind of standard \( d \)-dimensional problems for each \( R_i \). The normalization of the 1D models’ wavefunctions is then standard and not required for this paper’s applications.

Thus, my strategy is to solve relational quantum problems by setting up \( q_I \rightarrow R_i \) maps to standard separable problems and then exploiting the well-established mathematical machinery for these. Only then do I consider, from a careful relationalist perspective, what the interpretation of the relational QM models is, and explore the new mathematics which comes upon considering these as whole systems/universes.

7. Examples of \( N = 3 \) relational QM

7.1. Example 1: 1D free problem

The free problem straightforwardly separates to two copies of\(^{17}\)
\[
\frac{\hbar^2}{2\mu} \frac{d^2}{dR_i^2} \psi_i + E_i \psi_i = 0,
\]
giving wavefunctions \( \psi_i = e^{\pm i\sqrt{\mu/E_i}R_i/\hbar} \) for a positive continuous spectrum. The solution of the relational problem may then be reassembled as
\[
\Psi_E = \exp \left( \pm i\sqrt{\mu}(\sqrt{E_1} - \sqrt{2E_2N})r_{12}/\hbar \right) \exp \left( \mp i\sqrt{2\muE_2N(m_1 + m_2)r_{23}/\hbar} \right).
\]

Finally, the unusual relational feature is that \( E_1 \) and \( E_2 \) are not independent: \( E_1 + E_2 = E_{\text{Universe}} \).

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\(^{16}\) These similarities arise from the close (but conceptually distinct) parallel between the absolutist ‘intuitively’ working in the barycentre frame since the barycentre motion is ‘mathematically uninteresting’ and the relationalist holding the barycentre motion for a whole system/universe to be \textit{a fortiori} meaningless.

\(^{17}\) In sections 7 and 8, \( i \) denotes a particular component rather than an index to be summed over.
7.2. Example 2: 1D harmonic oscillators

For the single HO relational problem, I exploit the following more widely applicable Move 1: as \( x_{12} = R_1 \), any potential depending at most on \( x_{12} \) is separable in \( R_1, R_2 \) coordinates, with an identity map between the potential coefficients in \( q_r \)-space and \( R \)-space. Thus,

\[
-\frac{\hbar^2}{2\mu} (\Delta_1 + \Delta_2) \psi + V(|R_1|) \psi = E \psi \tag{47}
\]

\[
\Rightarrow -\frac{\hbar^2}{2\mu} \Delta_1 \psi_1 + V(|R_1|) \psi_1 = E_1 \psi_1, \quad -\frac{\hbar^2}{2\mu} \Delta_2 \psi_2 + = E_2 \psi_2 \quad \text{for } E_1 + E_2 = E_{\text{Universe}}. \tag{48}
\]

Thus, let the HO be without loss of generality between particles 1 and 2 so that \( R_1 \) follows the separated-out 1D problem

\[
\frac{\hbar^2}{2\mu} \frac{d^2 \psi_1}{dR_1^2} - \frac{H \cdot R_1^2}{2} \psi_1 + E_1 \psi_1 = 0. \tag{49}
\]

This is solved by wavefunctions \( \psi_i(n_i) = H_n(Y_i) e^{-y_i^2/2} \) for \( Y_i = (\mu H_i/\hbar^2)^{1/4} R_i \) and \( H_n \) the \( n_i \)th Hermite polynomial and corresponding spectrum \( E_i = \sqrt{H_i/\hbar} (\frac{1}{2} + n_i), \ n_i \in \mathbb{N}_0 \).

Meanwhile, \( R_2 \) follows the separated-out free problem of subsection 7.1. Thus,

\[
\Psi_n = (\exp (\pm \sqrt{2\mu E_2 N(m_1 + m_2)r_{23}/\hbar})) \\
\times (\exp(\pm i\sqrt{2\mu E_2 Nm_1r_{12}/\hbar} - \sqrt{\mu H_{1i}/\hbar^2}/2h)H_n((\mu H_i/\hbar^2)^{1/4}r_{12})). \tag{50}
\]

Finally, \( n \equiv n_1 \) and \( E_2 \) are not independent: \( \sqrt{H_i/\hbar} (n + \frac{1}{2}) + E_2 = E_{\text{Universe}} \).

For two or three HO potentials, map according to the following Move 2. Denoting the \( IJ \) pair’s potential by \( h_{1j} \) and the \( i \)th relative coordinate’s potential coefficient by \( H_i \), for two HOs, without loss of generality the potential is

\[
h_{23r_{12}^2} + h_{13r_{13}^2} = \frac{1}{(m_1 + m_2)^2} \left( h_{23}m_1^2 + h_{13}m_2^2 \right) R_1^2 + \frac{h_{23} + h_{13}}{2} \frac{R_1^2}{N^2} + \frac{m_1h_{23} - m_2h_{13}}{N} - 2R_1 \cdot R_2 \tag{51},
\]

so there is the separability restriction \( h_{13}/h_{23} = m_1/m_2 \), and the relations between the \( q_r \) and \( R_0 \) HO’s Hooke coefficients are \( H_1 = h_{23}/m_2, H_2 = h_{23}/m_2m_3 \). For three HOs, the same separability restriction holds again and the relations are \( H_1 = h_{13} + h_{23}/m_2, H_2 = h_{23}/m_2m_3 \). Thus, one obtains two separated-out 1D HO problems, with coefficients in each case as given above. The desired solution in relative separation coordinates is then

\[
\Psi_n = H_{n_1}((\mu H_{1}/\hbar^2)^{1/4}r_{12})H_{n_2}((\mu H_{2}/\hbar^2)^{1/4}N(m_1r_{13} + m_2r_{23})) \\
\times \exp \left(-\sqrt{\mu H_{1}/\hbar^2}r_{12}^2/\hbar + \sqrt{\mu H_{2}/\hbar^2}(N^2(m_1 + m_2)r_{23}^2 + m_1r_{12})^2/\hbar \right). \tag{52}
\]

Finally, \( n_1 \) and \( n_2 \) are not independent: \( \sqrt{H_i/\hbar} (n_1 + \frac{1}{2}) + \sqrt{H_j/\hbar} (n_2 + \frac{1}{2}) = E_{\text{Universe}} \).

Note how the two and three HO solutions are twisted rather than trivial when expressed in terms of \( r_{1j} \) coordinates.

\footnote{These large families serve to provide simple examples in this paper. Completely general two and three HO set-ups can be accommodated by rotating the Jacobi coordinates.}
7.3. Example 3: a simple atomic model in 3D

This is motivated (1) for more accurate real-world modelling purposes. (2) To tackle rotation/AM, which is the interesting and complicated part of relationalism, and the inclusion of which gives toy models that resemble geometrodynamics more closely (see, e.g., II), such as the \( N = 3, d = 3 \) ‘triangle land’ that Barbour’s speculations [18] are based on.

As noted on p 7 AM does not spoil separability. It is furthermore significant that the AM in \( q_I \) space is mapped to its mathematical analogue in \( R_i \) space, so that one can treat AM there as one usually does in \( q_I \) space. Full polar coordinates (i.e., polars in 2D and spherical polars in 3D) for each \( R_i \) are useful in this respect.

For a single Coulomb potential in 3D (representing a hydrogen atom together with a free neutral particle), one immediately has separability by Move 1, into the 3D Helmholtz equation and the simple atomic model. Additionally, the AM are tied between the two problems, so spherical polars \((\rho_i, \theta_i, \phi_i)\) are well suited. Then both of these building blocks have the angular part \( Y_{l_{1}m_{1}}(\theta_1, \phi_1) \) (spherical harmonics).

The radial part of the separated-out free problem is solved by the spherical Bessel functions \( \psi_{\rho_i} = j_l(k_i \rho_i) \propto \sqrt{2\mu E_i/\hbar} \). The radial part of the separated-out simple atomic problem is solved in terms of associated Laguerre polynomials \( \psi_{\rho_i} = \rho_i^{l_{1}}L_{2n_{1}+1}^{2n_{1}+1}(\mu k \rho_i/\hbar^2) e^{-\mu k \rho_i/\hbar} \). The corresponding energy eigenspectrum of bound states is \( E_i = -\mu k/2\hbar^2 n^2 \) for an attractive potential \( V = -k/\rho_1 \). There is also a continuum of positive energies representing ionized states. Thus, overall this simple relational atom model is solved by

\[
\Psi_{l_{1}m_{1}} = Y_{l_{1}m_{1}}(\theta_1, \phi_1) \rho_i^{l_{1}} L_{2n_{1}+1}^{2n_{1}+1}(\mu k \rho_i/\hbar^2) e^{-\mu k \rho_i/\hbar} Y_{l_{1}-m_{1}}(\theta_2, \phi_2) j_{l_{1}}(\sqrt{2\mu E_i \rho_2}/\hbar) \tag{53}
\]

where \( E_1 \) and \( E_2 \) are not independent: \( E_1 + E_2 = E_{\text{Universe}} \) and the angular momentum counterbalancing explained on p 14 is in use. I do not present this here re-expressed in terms of \( R_i \) and then \( r_{IJ} \) as that becomes messy.

As regards further examples, Move 1 is widely applicable, while Move 2 also works for two or three isotropic HOs in 2D or 3D (see [48] for these).

8. Interpretation of \( N = 3 \) relational QM

8.1. Decent semiclassicality

While one common requirement is for the wavelength \( \lambda_Q \) to be smaller than some characteristic scale \( l_C \) of the problem, there is no universal rigorous notion of semiclassical limit for quantum theories. Here are various procedures that one might consider in connection with such an investigation, all of which play some role in paper I or II.

1. Consider the spread (i.e., width) of the wavefunctions (this is essentially what BS do).
2. Furthermore investigate how localized wavepackets are as a whole (as opposed to the spread of each wavefunction in their summand/integrand).
3. Consider what happens to the system for large quantum numbers whereupon the wavefunction becomes wiggly on scales much shorter than \( l_C \).
4. Consider a WKB ansatz for the wavefunction and expand in powers of \( \lambda_Q/l_C \).

8.1.1. Spread of the wavefunctions. Barbour and Smolin’s first objection BS1 was to the sensitivity of the spread of the wavefunctions of large-mass particles to the values of the mass of small-mass particles. This was in connection with piecewise-constant potential models with two small masses and one large one. I clarify why this is not a problem as follows. First,
bear in mind that in the absolutist quantization, one thinks primarily of particle wavepackets, but in a 1D relationalist quantization one should think of relative distance wavepackets. Then, intuitively, once one rephrases one’s standard quantum intuition about small masses being more spread out in relational terms, it is clear that the position uncertainty of the small mass dominates the relational formulation’s relative separation uncertainty between that small mass and a large mass (figure 2).

BS’s specific example is akin to my Example 1, except that they have a constant \( U \) and set \( E_{\text{Universe}} = 0 \) while I have no \( U \) but a constant \( E_{\text{Universe}} \). Converting, the wavefunction may then be rewritten as

\[
\Psi = \exp \left( i \sqrt{|U|} \left( r_{12} \left( \frac{\sin \theta}{\sqrt{\nu_{13}}} - \frac{n_3 \cos \theta}{\sqrt{\chi}} \right) + r_{23} \left( \frac{\sin \theta}{\sqrt{\nu_{13}}} + \frac{n_1 \cos \theta}{\sqrt{\chi}} \right) \right) / \hbar \right)
\]

for \( n_a = 1/m_a, \nu_{ab} = 1/\mu_{ab}, \chi = n_1^2 \mu_{12} + n_2^2 \mu_{23} + 2n_1n_2n_3 \) and where the mass-independent constants present are parametrized by an angle \( \theta \). This has much nicer 1, 3 symmetry than 1, 2 symmetry, so in setting two of the masses to be equal to match BS’s example, I opt for \( m_1, m_3 = m \ll M = m_2 \). Then, the wavefunction goes as

\[
\Psi \sim e^{iJ \sqrt{Mr_{13}}/\hbar} e^{iK \sqrt{Mr_{23}}/\hbar}
\]

(for \( J, K \) mass-independent constants). Thus, indeed as BS claim, the small masses dominate all the uncertainties. But by my above interpretation, these uncertainties are actually in the separations between a big mass and a small mass, so this situation conforms to standard quantum intuitions rather than constituting an impasse. The truly relevant test to establish whether there is a semiclassical limit problem is rather to check what happens if \( m_1, m_3 = M \gg m = m_2 \), for then there is a big mass–big mass relative separation, and it is this which one would not expect to be influenced much by a small mass somewhere else. And indeed, upon performing the new approximation, and isolating the big mass–big mass separation \( r_{13} \) as the variable whose spread is of relevance, I find that

\[
\Psi \sim \exp(iJ \sqrt{Mr_{13}}/\hbar) \exp \left( -iK \frac{m}{\sqrt{M}} r_{12}/\hbar \right)
\]

(for \( J, K \) mass-independent constants) so that indeed only the big masses contribute significantly to the spread in the big mass–big mass separation. This basic conclusion is unaffected by having the two identical masses replaced by merely similar masses and holds widely throughout the models presented in this paper when suitable pairs of quantities are set to be relatively large and small.

8.1.2. Wavepackets. Consider the 1D problems which separate out of the relational problems of this section, piecemeal and as formal pieces of mathematics in which an eigenspectrum
and wavefunctions are obtained from a differential equation and boundary conditions that depend on some abstract set of variables and parameters. Then the wavepackets built up by summing and/or integrating the wavefunctions (generally with some weighting) over the eigenspectrum may also be considered as formal pieces of mathematics. My first point is that, at the level of the separated-out pieces of the relational problems considered in this section, this formal mathematics is the same as in the usual absolutist quantization. Thus, the piecemeal construction of wavepackets for the 1D quantum problems does not care whether these arise from separation in relational problems or in absolutist ones, so both behave equally well. The best-known examples of these wavepackets are the fixed-size one for the free particle and the pulsating one for the harmonic oscillator (see, e.g., [50, 52]).

A first difference between wavepackets in the absolutist and relational QM schemes arises in their interpretation. This parallels the above situation with the spreads. A second difference is that there are limitations on building composite wavepackets in the relational case. Unlike in the absolutist case, the composition of subsystem wavepackets cannot be extended to include the whole system. This is due to the energy of the universe being a fixed quantity.

Also note that the application of polar coordinates for $d > 1$ brings out that the standard QM interpretation is close to being relational. This is most familiar in the study of the hydrogen atom, for which a simple standard approach is to treat the proton as fixed and then consider the spread of the radial separation $\rho$ between the proton (or more accurately the atom’s barycentre) and the electron. All that is missing as regards obtaining a fully relational perspective is to consider the position of the barycentre not only to be uninteresting but also to be meaningless. Then one considers the spread in $|R_1| (=|\rho_1| = \rho_1)$. The ready availability of this familiar picture is one reason why it is unfortunate that BS restricted their study to 1D examples. I should add that interpreting $d > 1$ RPMs furthermore requires an additional notion of spread in relative angle (figure 2(d)).

8.2. Interesting features of the RPM examples as closed-universe systems

8.2.1. Subsystem energy interlocking and truncation, gaps in the energy spectrum

(1) As mentioned above, there is energy interlocking between constituent subsystems. For example, this requires the above naively free problem to have a line segment rather than a quadrant as its overall eigenspectrum, and the single HO and free particle system to have a set of points rather than an infinity of lines as its overall eigenspectrum, and the coupled HOs have a small set of points rather than a regular lattice as their eigenspectrum (figure 3). By energy interlocking the small universe models built up from individual problems, wavepackets additionally contain a delta function

$$
\delta \left( \sum_{\Delta\epsilon/\text{subsystems}} E_\Delta - E_\text{Universe} \right)
$$

(57)
acting inside the sums and integrals required to build it up, which causes it to differ mathematically from, e.g., the direct product of subsystem wavepackets in a fully separable universe.

Moreover, unlike in the usual interpretation of few-particle QM, the energy here is the energy of the universe, which is not only fixed, but is also a separate attribute of the universe so that the fixed value it takes need bear no relation to the eigenspectra of the universe’s contents. This leads to the following effects.

(2) In my multiple HO example, all the energies \( E_1(n_1) \) and \( E_2(n_2) \) are positive. Then \( E_{\text{Universe}} < E_1(0) + E_2 \) so no wavefunctions exist. Universes failing to meet the zero point energy of their content fail to have a wavefunction.

(3) Such universes could rather fail to meet the energy required for just some of the states which lie above a given energy that is greater than the zero point energy. Then one would obtain not the conventional eigenspectrum but rather a truncation of it.

(4) In my two or three HO examples, one is required to solve \( k_1 n_1 + k_2 n_2 = q \) for \( n_i \in \mathbb{N}_0, k_i = \sqrt{\frac{\mu}{\hbar}} h \) and \( q = E_{\text{Universe}} - \frac{\hbar}{2} \sqrt{\mu + \sqrt{\mu}} \). Then if \( k_i \) and \( q \) mismatch through some being rational and some irrational or even if all are integers but the highest common factor of \( k_1 \) and \( k_2 \) does not divide \( q \), no solutions exist. This can also be set up to give gaps in what would otherwise look like a truncation of the conventional eigenspectrum. Similar effects can be achieved by more complicated matches and mismatches in the other examples’ \( E_i \) dependences on \( n_i \).

8.2.2. How energy interlocking does not affect the ‘recovery of reality’ in practice for large universes. Objection BS2 is due to a less developed version of the above material. However, I develop the following conditional counterstatements.

(5) In universes which furthermore contain free particles, because these have continuous spectra, the missing out of some conventional states by (4) cannot occur. This is the case, e.g., for the one HO example above, while ‘tensoring’ free particles with the two or three HO setting above alleviates this ‘missing state’ problem in the setting of a universe with a slightly larger particle number.

(6) If negative energies are possible, then subsystems can attain energies higher than \( E_{\text{Universe}} \). So if one tensors a negative potential example such as hydrogen with an independent HO pair, one can have less truncation of the HO. One will still have many, or all, states missing, depending on how the coefficients of the two independent subsystems’ potentials are related. But if one then tensors in free particles one can have everything up to the truncation. Thus, truncation can be displaced, at least for some universes, by a modest increase in constituent particle number. It should be noted that there is a mismatch between, e.g., the HO which has excited states unbounded from above in conventional QM and hydrogen which has negative energy states bounded from below. I get out of this difficulty by pointing out that in any case very high positive energy states are unlikely to be physically meaningful; at the very least the physical validity of the model would break down due to, e.g., pair production and ultimately the breakdown of spacetime. To ensure one’s model can attain high enough (but finite) positive energies, wells that are deep and/or numerous enough can be brought in.

(7) Unlike (2)–(4), energy interlocking does not go away with particle number increase or the accommodation of a variety of potentials within one’s universe model. However, the large particle number and high quantum number aspects of semiclassicality are relevant here. Subsystems remain well behaved and all experimental studies in practice involve subsystems. But subsystems may be taken to have conventional wavepackets insofar as
these are products of their constituent separated-out problems’ wavepackets. It is still true that these may be truncated rather than built out of arbitrarily many eigenfunctions along the lines of (1)–(3) above, but this will be alleviable in practice by ensuring the inclusion of sufficient additional free particles or particles whose mutual potentials are of an opposing sign to the original subsystem’s. In such a framework, the correlation of a quantity within a subsystem with another in the rest of the universe is overwhelmingly likely to evade detection provided that the rest of the universe contains plenty of other particles. And of course the real universe is indeed well populated with particles.

Thus, BS’s second bad semiclassicality objection should be replaced by: (1)–(4) lead to simple small relational particle universes constituting good toy models for gaining a deeper understanding of closed universes and of the problem of time, while (5)–(7) ensure that thinking of the (diverse, large particle number) real universe in terms of relational particles is not in practice compromised by any manifest bad semiclassicality. I should also caution that solving TISEs is not as complete as setting up the full machinery of QM so more detailed contentions may emerge in further work.

Whether the WKB ansatz may additionally be applied to whole universes is an issue relevant to the semiclassical approach to quantum cosmology in the general (rather than just RPM) context. I touch on this issue below, but mostly leave its discussion to II.8 and [53]. For the moment, I show that increasing the dimension away from 1 reveals further such ‘small closed-universe’ effects.

8.2.3. Angular momentum counterbalancing. 2D RPM solutions depend on two and not four quantum numbers via the AM constraint as follows: \( L_i \psi_i = m_i \psi_i, i = 1, 2 \), arise in each separated problem, but these are ‘joined together’ by \( 0 = \hat{L}_2 \psi = \hat{L}_2 \otimes 1_2 \psi + 1_2 \otimes \hat{L}_2 \psi = m_1 \psi + m_2 \psi \), so \( m_1 = -m_2 \equiv m \). 3D RPM solutions such as Example 3 depend on three and not six quantum numbers via a more elaborate angular momentum counterbalancing as follows: \( m_1 = -m_2 \equiv m \) as for the 2D case using \( 0 = \hat{L}_2 \psi \) in place of \( 0 = \hat{L}_q \psi \). But now, also, \( 0 = \hat{L}_1 \psi \) and \( 0 = \hat{L}_x \psi \), so \( \hat{L}_1 \psi = -\hat{L}_2 \psi \) and \( \hat{L}_1 \psi = -\hat{L}_2 \psi \). Thus, using \( L_i^2 \psi_i = l_i(l_i + 1) \psi_i \Rightarrow (L_i^2 + L_j^2) \psi_i = (l_i(l_i + 1) - m_i^2) \psi_i \) for each of \( i = 1, 2 \), so \( l_1(l_1 + 1) - m_1^2 = l_2(l_2 + 1) - m_2^2 \). But \( m_1 = -m_2 \equiv m \) and \( l_1, l_2 \in N_0 \) so there is no choice but \( l_1 = l_2 \equiv l \). Thus, whole universe models have a variety of additional naively unexpected correlations between their ordinary-looking constituent subsystems due to limited energy resources and due to having to balance out each others’ angular momentum. A similar argument to that presented for energy interlocking leads to one expecting angular momentum counterbalancing not to be noticed in the study of subsystems within a large universe.

8.2.4. Further types of question about the ‘recovery of reality’. Conceptualizing in terms of an atom and some other subsystem lends itself to raising further types of questions. It is all very well that these subsystems can be constructed to have the right sort of spectra in the sense of energy levels (mathematical sets of eigenvalues). But are transitions between these, most obviously manifested by spectra (in the distinct, practical sense of emissive/absorptive frequency patterns) or chemical reactions, possible? After all, the universes in question are governed by TISEs which possess solely stationary solutions? One answer, developed in II.7 and [53], is that this set-up is nevertheless capable of allowing subsystems to take on the appearance of dynamics. This is a semiclassical approach which relies on the WKB ansatz and on (perhaps small) terms that cause nonseparability. One situation which has nonseparability is the multi-Coulomb potential (in some cases through multiple charged particles, while all models will have small gravitational interactions). Consistent records schemes such as
Barbour’s concern a different answer (see II.8. for further comparison of these). Note that, while this discussion of apparent dynamics from TISEs may appear unusual at first sight, it is a simplified discussion about the plausible situation that our own universe, which certainly possesses subsystems which appear to be dynamical, may (at some level) be describable overall by a GR TISE: the so-called Wheeler–DeWitt equation (see II for references).

It is subsequently relevant to ask about mechanism. Insisting on interpreting such models as closed (i.e., self-contained) amounts to not assuming a ‘surrounding photon sea’. Does the absence of this preclude all dynamical processes? The answer is no, insofar that further particles of the same species can also mediate energy at some effective level (this holds as far as Fermi theory, overcoming the non-renormalizability of which leads to quantum electrodynamics, in which photons are then necessary for interactions). For atoms, it is well known that a grazing free particle can result in electron excitation, in ionization or in the (four-particle) Auger effect [55]. If one must furthermore deploy a counterbalancing particle/subsystem in order to recover standard results, at least four- or five-body problems are required to model such situations.

A separate point is that, while photon-free transitions as above are known, many more effects require interaction by photons or taking into account the atomic $E$ and $B$ fields. This is one reason why it would be interesting to extend the present work so as to ‘build in’ electromagnetism by considering closed relational Newton–Maxwell universes. This would also serve both as a robustness check for the results of paper II and as a toy model for closed Einstein–Maxwell universes. Inclusion of intrinsic spin would also be useful.

9. Conclusion

As regards the classical absolute versus relative motion debate, I have provided a concrete synthesis of directly formulated relational particle mechanics (RPM) which is equivalent to a reformulation of the portion of Newtonian mechanics that is conservative, has $\mathcal{L} = 0$ and whose kinetic term is homogeneous quadratic in its velocities. I have presented this in relative Lagrange variables and in relative Jacobi variables. I have furthermore presented improved (counterbalancing subsystem) arguments that adopting this portion or the straightforward nonhomogeneous enlargement of it is not for many purposes a major restriction at the classical level.

As regards QM implications of adopting the relational stance, I have been able to go further than in previous relational studies by bringing in Jacobi variables. Thus, I have been able to work with more complicated, more realistic and more relationally motivated quantum RPM models than Barbour and Smolin (BS) or any other paper in the RPM literature to date. This is based on how many simple problems are separable in Jacobi coordinates and share much formal mathematical structure with the conventional absolutist approach. Thus at the very simplest level, the answer to my question about whether absolutism has misled QM is no, in particular if one continues to adhere with conventional QM’s emphasis (see, e.g., [56]) on formal structures and their manipulation. However, differences in the interpretation of that mathematics are readily manifest due to the one employing particle positions and the other employing interparticle (cluster) separations. Indeed, the bona fide relational interpretation that I provide enables me to reject BS’s spread sensitivity objection to RPM models having a good semiclassical limit.

Moreover, the answer is yes as regards whole closed universes. The models of this paper exhibit several interesting features, which suffice to show that insisting on modelling a closed universe as a whole courts difficulties well beyond those encountered in studying its constituent subsystems. (Of course, there is more to modelling quantum cosmology than
These features are energy interlocking, AM counterbalancing, eigenspectrum gaps due to the universe and its contents being mismatched, and eigenspectrum truncation due to finite resources. While these mathematical observations are an extension of BS’s second objection to RPM models having a good semiclassical limit; moreover, these effects do not affect the ‘recovery of reality’ for a large universe with sufficiently varied contents. For, the presence of free particles and of potentials of both signs serves to overcome gaps and noticeable truncation, while the practicalities of experimentation involving only small subsystems means that interlocking and counterbalancing would be likely to go unnoticed.

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