Rescattering Effects for $(\varepsilon'/\varepsilon)$

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Abstract

A calculation of the hadronic matrix elements for $K \to \pi\pi$ including $\pi-\pi$ rescattering effects in a dispersion integral is presented. I study the dependence of the results on the matching scale $\mu$ and use them to calculate the CP–parameter $(\varepsilon t/\varepsilon)$. I find improved stability on the matching scale and good agreement with the experimental results.
During the past few years there have been new developments on the problem of direct CP–violation. On the theoretical side there are predictions [1]–[5] which agree with the new experimental results [6, 7] and some calculations which are still far away from the data. The agreement between theory and experiments must be considered a success of QCD since the dominant contributions to \((\varepsilon'/\varepsilon)\) come from the effective Hamiltonian generated through renormalization of the weak interaction. The renormalization coefficients were computed by two groups [8, 9] and agree with each other.

It is clear now that the original calculations of the matrix elements, done in the Vacuum Saturation Approximation (VSA), must be supplemented by rescattering corrections of the two pions, which increase \(\langle Q_6 \rangle_0\) and decrease \(\langle Q_8 \rangle_2\). The rescattering corrections were described in an earlier calculation [1, 2] which made the ratio \((\varepsilon'/\varepsilon)\) positive. A new and improved calculation in the chiral theory of pseudoscalar mesons [3, 4] and the chiral quark model [5] obtained results consistent with large values for \((\varepsilon'/\varepsilon)\). In this article I introduce a new method for calculating the rescattering corrections, which may also be useful for other low–energy calculations. The results indicate that \(\pi–\pi\) rescattering brings important corrections and allow to compute the imaginary parts of the amplitudes.

The operators occurring in the effective Hamiltonian are local operators which satisfy analyticity and threshold conditions of field theory. Consequently their matrix elements can be written as integrals over their singularities. There are two types of contributions to each matrix element: tree diagrams and loops. The four dimensional integrals for the loops can be rewritten as one–dimensional integrals over the energy flowing through the K–meson. We split the integrals in two regions – from \(4m_\pi^2\) to \(\mu^2\) and from \(\mu^2\) to infinity. The higher energy region is computed in QCD and its discontinuity is related to the anomalous dimension. The low energy part can be represented by an effective theory or by experimental data. The final formula for the amplitude at a low energy point \(\sigma = m_K^2\) can be written as
\[
Re A(\sigma) = a + \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{Im A(s)}{s - \sigma} ds = a + \frac{1}{\pi} \int_{4m^2}^{\mu^2} \frac{Im A(s)}{s - \sigma} ds + a\gamma \frac{\alpha_s}{4\pi} \ln \frac{M^2}{\mu^2} + \ldots
\]

\[
= C(\mu) \left\{ a + \frac{1}{\pi} \int_{4m^2}^{\mu^2} \frac{Im A(t)}{s - \sigma} ds \right\}.
\]

(1)

Here \(a\) is the lowest order contribution, which can be substituted by the vacuum saturation or another improved approximation. In the last equation we factorized two terms and identified one of them with the series generated by QCD. We emphasize that we do not need a subtracted dispersion integral because for the high–energy region we use directly the QCD realization of the theory. This is a useful expression for the low–energy value of the amplitude and may have applications beyond the cases described in this article.

In addition, eq. (1) provides a matching between the two regimes at the cutoff scale \(\mu\), which can be varied from 0.8 GeV all the way up to 1.8 GeV where the validity of QCD is more reliable.

In the \(K \to 2\pi\) decays we have many matrix elements of the operators, generated through QCD, but very few observables. To improve the situation we shall try to calculate the amplitudes and their strong phases. For the \(A_2\) amplitude we use the value

\[
A(K^+ \to \pi^+ \pi^0) = (1.837 \pm 0.002) \cdot 10^{-8} \text{ GeV}
\]

and its phase is given, according to Watson’s theorem, by the phase extracted from \(\pi - \pi\) scattering. In fact we will calculate the phase for each matrix element and compare it with the \(\pi - \pi\) phase shift. The imaginary part for each rescattering is given through unitarity by the cut diagram shown in figure 1; where the square is a weak vertex and the circle a strong vertex. We obtain

\[
Im\langle \pi\pi, I = 0|Q_6|K^0\rangle = \langle \pi\pi, I = 0|Q_6|K^0\rangle_{\text{VSA}} \left( \frac{1}{4} \frac{m_K^2 - \frac{1}{2}m_\pi^2}{4\pi F^2_{\pi}} \left( 1 - \frac{4m^2_{\pi}}{m_K^2} \right) \right)^{1/2}
\]

(3)

and

\[
Im\langle \pi\pi, I = 2|Q_8|K^0\rangle = \langle \pi\pi, I = 2|Q_8|K^0\rangle_{\text{VSA}} \left( -\frac{1}{8} \right) \left( \frac{m_K^2 - 2m^2_\pi}{4\pi F^2_{\pi}} \right) \left( 1 - \frac{4m^2_{\pi}}{m_K^2} \right)^{1/2}
\]

(4)
I shall use these equations for computing the imaginary parts throughout this paper. Using them we computed the numerical values for $\text{Im} \langle Q_8 \rangle_2$ in Table (1) of reference [4]. The imaginary part of $\langle Q_6 \rangle_0$ was not included in [4] because it is higher order in the expansion described in that article. The form of the equations indicates that the imaginary part consists of two multiplicative factors: a weak vertex and the $\pi$-$\pi$ scattering amplitude characterized, in our case, by two isospin states $I = 0, 2$.

We proceed now to calculate the real part of the amplitudes. For this calculation we need the imaginary part for values of the center–of–mass energy squared, $s$, in the range $4m^2_\pi \leq s < \mu^2$. The proof of Watson’s theorem holds for the matrix element of each operator and their imaginary parts are given as

$$\text{Im} A_I = \text{Im} \langle \pi\pi, I|Q_i|K^0\rangle = |\langle \pi\pi, I|Q_i|K^0\rangle| \cdot \sin \delta^I_{I=0}(s)$$

with $\delta^I_0$ the experimental phase shifts for isospin $I = 0, 2$ pion–pion scattering. The imaginary part is given by this formula in the elastic region and I adopt this form beyond the elastic region using the experimental phase shifts. For the magnitude of the matrix element we can take the low energy contribution, mentioned earlier, but we are free to introduce a weak energy dependence; for instance the variation introduced by the real part obtained through the dispersion relation. Let us denote by

$$A_{Q_i,I}(s) = \langle \pi\pi, I|Q_i|K^0\rangle.$$  \hfill (6)

Then

$$\text{Re} A_{Q_i,I}(\sigma) = \left\{ a_{Q_i} + \frac{1}{\pi} \int_{4m^2_\pi}^{\mu^2} |\langle \pi\pi, I|Q_i|K^0\rangle| \frac{\sin \delta^I_0(s)}{s - \sigma} ds \right\} C_i(\mu).$$

We shall assume that the absolute value of the matrix element is a slowly varying function of energy over the region of integration and we use the experimental values for $\sin \delta^I_0$ to
perform the principal value integral. We define the functions
\begin{equation}
    f_I(\mu, \sigma) = \frac{1}{\pi} \int_{4m_i^2}^{\mu^2} \frac{\sin \delta_I^I(s)}{s - \sigma} ds
\end{equation}
and calculated the values presented in table 1. These terms bring in a correction to the tree level contribution. For the integrations we use the experimental phase shifts from references [10]–[13]. Data for \( \delta_0^2 \) exist up to the energy of 1.5 GeV and we performed the integral up to this value. For \( \delta_0^0 \) the data show clearly the \( \rho \)-resonance. They extend to 1.8 GeV and I give the additional values in the table. The functional change of the phase shift given in [11] produces an \( f_0(\mu, m_k) \) which is almost constant for \( \mu > 900 \) MeV. The extended ranges presented in table 2 allow us to study the dependence of \( ReA_{Q_{1,l}} \) on the matching scale \( \mu \). As an additional test, I introduced a 20% energy dependence on the magnitude of the matrix element and carried out the integration. The results differ only by one or two units in the second decimal.

As a first test of the approach we consider the \( A_2 \) amplitude. Its Born contribution is known
\begin{equation}
    A_2^{\text{Born}}(K^+ \to \pi^+ \pi^0) = \sqrt{\frac{3}{2}} \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* (z_1 + z_2) (\langle Q_1 \rangle_2 = 2.74 \times 10^{-8} \text{ GeV}. \end{equation}

| \( \mu \) in GeV | \( f_2(\mu, m_k) \) | \( f_0(\mu, m_k) \) |
|-----------------|-----------------|-----------------|
| 0.7             | -0.09           | 0.34            |
| 0.8             | -0.12           | 0.50            |
| 0.9             | -0.17           | 0.60            |
| 1.1             | -0.23           | 0.71            |
| 1.3             | -0.29           | 0.64            |
| 1.5             | -0.34           | 0.55            |
| 1.7             | —               | 0.55            |
| 1.9             | —               | 0.60            |

Table 1: Numerical results for the principal value integrals

Including next the unitarity corrections, at \( \mu = 0.9 \) GeV we obtain
\begin{align}
A_2^{\text{complete}}(K^+ \to \pi^+\pi^0) &= 2.74(1 - 0.17 - i0.20) \cdot 10^{-8} \\
&= (2.27 - i0.55) \cdot 10^{-8} \text{ GeV} \\
\tan \theta &= -0.24, \quad \theta = -13.7^\circ.
\end{align}

We note that the magnitude of the calculated amplitude is larger than the measured value by 30%. The phase of the amplitude has the correct sign and it is slightly larger than the experimental value. The experimental values reported in the articles [10]–[13] vary among themselves. An approximate value is $-10.5 \pm 1.6^\circ$. A more accurate value was obtained through a dispersion calculation for $\pi-\pi$ scattering [11], but the error quoted is very small and is perhaps an underestimate. Finally, the dependence of the amplitude on the matching scale $\mu$ is small, as discussed for the other two matrix elements below.

The calculation for $A_{Q_{8,2}}$ proceeds along similar lines. We obtain the imaginary part from eq. (4) and the real part from eq. (7) and table 1

\begin{equation}
A_{Q_{8,2}}(m_k) = \langle \pi\pi, I = 2 | Q_8 | K^0 \rangle_{\text{VSA}} (1 - 0.20 - i0.20) y_8(\mu)
\end{equation}

at $\mu = 1.0$ GeV and with [3, 4]

\begin{equation}
\langle \pi\pi, I = 2 | Q_8 | K^0 \rangle_{\text{VSA}} = \frac{\sqrt{3}}{2\sqrt{2}} r^2 F_\pi \left[ 1 + \frac{8m_k^2}{F_\pi^2} (L_5 - 2L_8) - \frac{4m_k^2}{F_\pi^2} (3L_5 - 8L_8) \right]
\end{equation}

and $r = \frac{2m_k^2}{m_s + m}$. The renormalized couplings $L_5$ and $L_8$ are defined in references [3, 4] and have the numerical values

\begin{equation}
L_5 = 2.07 \cdot 10^{-3} \quad \text{and} \quad L_8 = 1.09 \cdot 10^{-3}.
\end{equation}

Substituting the numerical values for $m_s (1 \text{ GeV}) = 150 \text{ MeV}$ we obtain

\begin{equation}
A_{Q_{8,2}}(m_k) = (0.37 - i0.09) y_8(\mu) \text{ GeV}^3.
\end{equation}

The phases in eqs. (10) and (15) are the same, which is an attractive property of the dispersion relation, i.e. all $I = 2$ matrix elements have the same phase. A similar property
holds for the $I = 0$ matrix elements. Finally, we can investigate the dependence on the matching scale $\mu$ which appears on the factor $r^2$, through the running mass of the strange quark, and the Wilson coefficient $y_8(\mu)$. In a specific regularization scheme the product $r^2 y_8(\mu)$ is stable between 1 and 2 GeV, varying by less than 5%. The variation among the three regularization schemes in the same energy region [14] is at most 15%.

The unitarity corrections to the matrix element $Q_6$ are controlled by the phase shift $\delta_0^0$ which is positive, giving a positive correction for the real part of the matrix element. The increase of the matrix element is given by the function $f_0(\mu)$ in table 1, which is stable above 900 MeV. At $\mu = 1$ GeV

$$A_{Q_{6,0}} = \langle \pi\pi|Q_6|K^0\rangle_{VSA} (1 + 0.60 + i0.40) y_6(\mu) = (-0.56 - i0.14) y_6(\mu) \text{ GeV}^3 \quad (16)$$

with [3, 4]

$$\langle \pi\pi|Q_6|K^0\rangle_{VSA} = -4\sqrt{3} r^2 L_5 \frac{m_k^2 - m_\pi^2}{F_\pi}. \quad (17)$$

Again the product $r^2 y_6(\mu)$ is very stable for $1 < \mu < 2$ GeV; the variations mentioned in the previous paragraph for $r^2 y_8(\mu)$ again hold. The phase of this amplitude is positive and equal to $+14^\circ$ which is approximately half the experiment phase shift at $\sqrt{s} = m_k$.

With the values derived already we can compute the parameter $(\varepsilon f/\varepsilon)$. The standard derivation leads to the expression

$$\frac{\varepsilon f}{\varepsilon} = \frac{G_F}{2} \frac{\omega}{|\varepsilon| Re A_0} Im \lambda_t \left[ \pi_0 - \frac{1}{\omega} \pi_2 \right] \quad (18)$$

with

$$\pi_0 = |\sum y_i(\mu) \langle Q_i \rangle_0| (1 - \Omega_{\eta\eta'}) \quad (19)$$

$$\pi_2 = |\sum y_i(\mu) \langle Q_i \rangle_2| \quad (20)$$

$$Y = \pi_0 - \frac{1}{\omega} \pi_2 \quad (21)$$

and $\Omega_{\eta\eta'} \sim 0.25 \pm 0.05$ being the isospin breaking in the quark masses ($m_u \neq m_d$). The absolute values originate from the fact that the phases of strong origin were already
extracted in the calculation of $\varepsilon t/\varepsilon$. The overall factor is precisely known $\frac{G}{2} \frac{\omega}{|\varepsilon| Re A_0} = 346$ GeV$^{-3}$ and the Cabbibo–Kobayashi–Maskawa factor was recently estimated $[14]$. 

$$Im \lambda_t = (1.38 \pm 0.33) \times 10^{-4}$$  \hspace{1cm} (22)

This new value is a large improvement over the values reported in the early 90’s and leads to a large reduction of the uncertainties. A second reduction of uncertainties comes from the weak dependence of the amplitudes in eqs. (12), (15) and (16) on the matching scale. We summarize in table 2 the uncertainties for $(\varepsilon t/\varepsilon)$ originating from two sources.

| source        | $\Delta(\varepsilon t/\varepsilon)$ |
|---------------|--------------------------------------|
| $Im \lambda_t$ | $\pm 25\%$                           |
| Matching      | $\pm 20\%$                           |

Table 2: Theoretical Uncertainties

Another uncertainty comes from the strange quark mass which enters the calculation of the matrix elements through the factor $r = \frac{2m_s^2}{m_s+m}$. Values for the running strange quark mass have been computed by various methods and vary considerably. Older estimates $[15]$ and QCD sum rules $[16]$ give higher values $m_s(1 \text{ GeV}) \approx 150 \pm 55 \text{ MeV}$ at the usual renormalization point $\mu = 1 \text{ GeV}$; while recent values from lattice calculations $[17]$ give smaller values $\approx 110 \pm 20 \text{ MeV}$ at $\mu = 2 \text{ GeV}$. It is customary to adopt the range $[4]$ 

$$m_s(1 \text{ GeV}) = 150 \pm 25 \text{ MeV}. \hspace{1cm} (23)$$

For central values of the parameters at 1 GeV and keeping the errors from table 2 in quadrature, I obtain

$$\varepsilon t/\varepsilon = (4.2 \pm 1.3) \cdot (0.0320) \cdot 10^{-2}$$

$$= (15.0 \pm 4.8) \cdot 10^{-4}, \hspace{1cm} (24)$$

where the number 0.0320 comes from $\langle Q_6 \rangle_0$ including $\Omega_{opp}$ minus the contribution from $\langle Q_8 \rangle_2$. I kept only these two operators and found out that the contribution from the
I = 2 amplitude is only 20% of the \( \langle Q_6 \rangle (1 - \Omega_{\eta\eta'}) \) term. This value for the ratio is in good agreement with the average experimental value \( (\varepsilon t/\varepsilon) = (21.2 \pm 4.6) \cdot 10^{-4} \).

I have shown in this article that the VSA for the matrix elements together with rescattering corrections lead to values of \( (\varepsilon t/\varepsilon) \) which are consistent with the experimental measurements. The unitarity corrections improve the stability on the matching scale \( \mu \).

The specific values of the phase–shifts increase \( \langle Q_6 \rangle_0 \) and decrease \( \langle Q_8 \rangle_2 \) making the difference in the function \( Y \) positive definite [1] – a feature which is maintained in many calculations [2, 3, 19]. Two very recent articles [18, 19] use dispersion relations for the K–meson decay amplitudes. They differ in several basic respects from the present article and the interested reader can study them for comparison. Finally, it will be interesting to test if lattice calculations [20, 21], which fullfil unitarity corrections, still give different results.

A more extensive exposition of this work including numerical studies and other applications will be presented in the future. In particular, I wish to study the various contributions to the \( \Delta I = \frac{1}{2} \) rule and compare them with the results of previous calculations [22–24].

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