Constacyclic Codes over Finite Chain Rings of Characteristic $p$

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Abstract: Let $\mathbb{R}$ be a finite commutative chain ring of characteristic $p$ with invariants $p, r, k$. In this paper, we study $\lambda$-constacyclic codes of an arbitrary length $N$ over $\mathbb{R}$, where $\lambda$ is a unit of $\mathbb{R}$. We first reduce this to investigate constacyclic codes of length $p^n (N = n_1 p^r, p \nmid n_1)$ over a certain finite chain ring $CR(u^s, r_b)$ of characteristic $p$, which is an extension of $\mathbb{R}$. Then we use discrete Fourier transform (DFT) to construct an isomorphism $\gamma$ between $R[x]/(x^N - \lambda)$ and a direct sum $\oplus_{b \in \mathcal{I}} S(r_b)$ of certain local rings, where $\mathcal{I}$ is the complete set of representatives of $p$-cyclotomic cosets modulo $n_1$. By this isomorphism, all codes over $\mathbb{R}$ and their dual codes are obtained from the ideals of $S(r_b)$. In addition, we determine explicitly the inverse of $\gamma$ so that the unique polynomial representations of $\lambda$-constacyclic codes may be calculated. Finally, for $k = 2$ the exact number of such codes is provided.

Keywords: finite ring; linear code; polynomials; coding theory

1. Introduction

The class of constacyclic codes plays an important role in coding theory and has been a primary area of study (see [1–9]). In the literature, most research has been focused on the situation where the alphabet of these codes is a field. However, many important non-linear codes over finite fields are actually related via the Gray map to linear codes over finite rings and, particularly, over finite chain rings. Constacyclic codes of arbitrary length $N$ over a finite ring $\mathbb{R}$ are identified with ideals of the polynomials ring $R[x]/(x^N - \lambda)$. Let $p$ be the characteristic of the residue field of a finite commutative chain ring $\mathbb{R}$. When the length $N$ is prime relative to $p$, constacyclic codes are easily determined by the unique factorization of $X^N - \lambda$ using Hensel’s Lemma. On the other hand, case $p \mid N$ yields what is called repeated-root codes, which were studied for the first time by Berman [10] in 1967 (for more details, see [2,7,11,12]).

The class of finite chain rings has been extensively used as the alphabet of constacyclic codes [8,13–24]. This class was introduced in [23] to construct new sequences possessing optimal Hamming correlation properties, and these sequences were found to be useful in frequency hopping multiple-access (FHMA) spreading spectrum communication systems. The chain ring $\mathbb{Z}_4$ has been widely considered as alphabet of cyclic codes (special types of constacyclic codes) [17–19,25]. Doughtry et al. [16] generalized the results to cyclic codes of length $N$ over $\mathbb{Z}_{p^r}$. Moreover, Kiah et al. [8] studied cyclic codes of length $p^k$ over $\mathbb{G}(p^k, r)$, while Zhu et al. [26] examined a special class of constacyclic codes over $\mathbb{Z}_{p^m}$. Now, let $\mathbb{R}$ be a finite commutative chain ring of characteristic $p$ with invariants $p, r, k$. Cyclic codes and their dual codes were initially considered over $\mathbb{R}$ with $p = 2$ and $k = 2$ by Bonnecaze et al. [20]. Qian et al. [21] used discrete Fourier transform (DFT) to study cyclic codes over $\mathbb{R}$. Moreover, Dinh [27] studied constacyclic codes of length $p^k$ over $\mathbb{R}$ when $k = 2$. Ozger et al. [22] discussed constacyclic codes over $\mathbb{R}$ under the condition $p = 2, k = 4$. Recently, in [15], Mu Han et al. classified cyclic codes of length $np^k$ over $\mathbb{R}$ in case of $r = 1$ via DFT. Motivated by the above cited studies, the main objective of this paper is to extend the approach of Han et al. [15] and to obtain unique polynomial representations of constacyclic codes of any finite length $N$ over $\mathbb{R}$ with arbitrary invariants.
2. Preliminaries

All rings considered in this paper are finite commutative and possess an identity. In this section, we mention some definitions and introduce notations that will be used in the subsequent discussions.

2.1. Constacyclic Codes

A code of length $N$ over a ring $R$ is a nonempty subset of $R^N$, and $R$ is referred to as the alphabet of the code. A code $C$ is said to be linear if it is also a $R$-submodule of $R^N$. For a given unit $\lambda$ of $R$, a linear code $C$ is said to be constacyclic or more precisely $\lambda$-constacyclic if $(x_{N-1},x_0,x_1,\ldots,x_{N-2}) \in C$, whenever $(x_0,x_1,\ldots,x_{N-2},x_{N-1}) \in C$, i.e., $C$ is closed under $\lambda$-constacyclic shifts. The cyclic and negacyclic codes are obtained when $\lambda = 1$ and $-1$, respectively.

**Proposition 1** ([28,29]). A linear code $C$ of length $N$ is a $\lambda$-constacyclic code over $R$ if and only if $C$ is an ideal of $R[x]/<x^N-\lambda>$.

2.2. Finite Chain Rings of Characteristic $p$

A ring $R$ is a chain ring if it is local and its Jacobson radical $J(R)$ is principal. Every finite chain ring $R$ is associated with five invariants $p,n,r,k$, and $m$. From now on, $R$ is a finite chain ring of characteristic $p$, i.e., $n=1$ and $m=k$. In this case, $R$ is associated with $p,r$, and $k$. We denote $J(R)=<u>$, $k$ the index of nilpotency of $u$, and $p^r$ is the order of the residue field $R/J(R)$. Such chain rings are uniquely determined by their invariants $p,r$, and $k$ [30].

**Proposition 2** ([31,32]). Let $R$ be a finite chain ring of characteristic $p$ with invariants $p,r,k$. Then, the following is the case:

(i) $R$ has a subfield $F$ of order $p^r$;
(ii) $R = F \oplus uF \oplus \ldots \oplus u^{k-1}F$;
(iii) $R \cong F[u]/<u^k>$;
(iv) If $U(R)$ is the group of units of $R$, then $U(R) \cong F^* \times (1 \oplus uF \oplus u^2F \oplus \ldots \oplus u^{k-1}F)$.

By Proposition 2, every unit $\lambda$ of $R$ can be uniquely written as $\lambda = \alpha + u^l(\beta_1 + \ldots + u^{k-l-1}\beta_{k-l-1}) = \alpha + u^l\beta$, where $\beta = \beta_1 + \ldots + u^{k-l-1}\beta_{k-l-1}$. Thus, every unit $\lambda$ of $R$ is of the form $\lambda = \alpha + u^l\beta$, where $\beta$ is either 0 or a unit of $R$. Let the following be the case:

$$\alpha_0 = \alpha^{-p^{-s-1}},$$

where $s = rq + t$ and $0 \leq t \leq r - 1$. Then, $\alpha_0^{pr} = \alpha^{-p^{r(s+1)}} = \alpha^{-1}$.

**Remark 1.** If $F$ is a finite field. The ring $F[x]/ < x^{p^r} - \alpha >$ is a chain ring with maximal ideal $< \alpha_0 x - 1 >$. Thus, $\alpha$-constacyclic codes of length $p^r$ over $F$ are precisely the ideals $< (\alpha_0 x - 1)^i >$, where $0 \leq i \leq p^r$. Each $\alpha$-constacyclic code $< (\alpha_0 x - 1)^i >$ has $p^r(p^r-i)$ codewords.
Definition 1. For any $\lambda$-constacyclic code $C$ of length $p^s$ over $R$ and for $0 \leq i \leq k - 1$, we define the following codes over $F$:

$$\text{Tor}_i(C) = \mu \left( \{ a \mid u^i a \in C \} \right),$$

where $\mu$ is the canonical homomorphism (modulo $u$). Moreover, $\text{Tor}_i(C)$ is called the $i$th torsion code of $C$, $\mu(C) = \text{Tor}_0(C) = \text{Res}(C)$ is the residue code of $C$, and $T_i(C) = T_i$ is called the $i$th-torsional degree of $C$.

Proposition 3 ([16]). Let $C$ be a $\lambda$-constacyclic code over $R$ and $i$ be an integer such that $0 \leq i \leq k - 1$. Then, $T_i(C)$ is an $\alpha$-constacyclic codes of length $p^s$ over $F$ and $T_i(C) = \langle (a_0 x - 1)^{T_i} \rangle >$ for some $0 \leq T_i \leq p^s$. Moreover, we have the following:

(i) $|\text{Tor}_i(C)| = (p^s)^{p^s - 1}$;

(ii) If $u^i((\lambda_0 x - 1)^{T_i} + u g(x))$ in $C$, then $t_i \geq T_i$;

(iii) $p^s \geq T_0 \geq T_1 \geq \ldots \geq T_{k-1} \geq 0$;

(iv) $|C| = (p^s)^{p^s - (t_0 + t_1 + \ldots + t_{k-1})}$.

Remark 2. Obviously, $T_i$ is the smallest degree amongst all the degrees of non-zero polynomials in $\text{Tor}_i(C)$.

All symbols stated above shall retain their meanings throughout the article, in addition, $N = n_1 p^s$, $(n_1, p) = 1$.

3. Constacyclic Codes of Length $p^s$

In this section, we provide a unique representation for any constacyclic code of length $p^s$ over $R$. This representation allows us to compute Hamming distances and dual codes as well as enumerates all constacyclic codes of length $p^s$ over $R$, i.e., ideals of the quotient ring $R_{a, \beta} = R[x]/ < x^{p^s} - (a + u^l \beta) >$. Assume $k_1 = \lceil \frac{k}{\lambda} \rceil$, i.e., $k_1$ is the smallest positive integer greater than $\frac{k}{\lambda}$.

Lemma 1. In $R_{a, \beta}$, $< (a_0 x - 1)^{p^s} >= < u^l >$. In particular, $(a_0 x - 1)$ is nilpotent with nilpotency index $k_1 p^s$.

Proof. Note that the following is the case:

$$(a_0 x - 1)^{p^s} = (a_0 x)^{p^s} + \sum_{i=1}^{p^s-1} \binom{p^s}{i} (a_0 x - 1)^i (-1)^{p^s-i} - 1$$

$$= (a_0 x)^{p^s} - 1 = a_0^{p^s} x^{p^s} - 1 = a_0^{-1} (a + u^l \beta) - 1$$

$$= 1 + u^l a_0^{-1} \beta - 1 = u^l a_0^{-1} \beta.$$

Thus, $< (a_0 x - 1)^{p^s} >= < u^l >$. The last statement follows immediately, since $u^l$ has nilpotency index $k_1$. □

Proposition 4. The ring $R_{a, \beta}$ is a local ring with maximal ideal $< (a_0 x - 1), u >$.

Proof. Due to the fact that $R = F \oplus uF \oplus \ldots \oplus u^{k-1}F$, each element $a$ of $R$ has unique presentation as $a = \sum_{i=0}^{k-1} u^i a_i$, where $a_0, a_1, \ldots, a_{k-1}$ are elements of $F$. This implies that for any polynomial $f(x) \in R_{a, \beta}$, $f(x)$ can be expressed uniquely as follows:

$$f(x) = \sum_{i=0}^{k-1} \sum_{j=0}^{p^s-1} a_{ij} u^j (a_0 x - 1)^j,$$

where $a_{ij}$s are elements of $F$. Due to the fact that $a_0 x - 1$ and $u$ are nilpotent, $f(x)$ is a unit if and only if $a_{00} \neq 0$. Moreover, if $f(x)$ is a zero divisor, i.e., $a_{00} = 0$, then...
\( f(x) \in <(a_0x - 1), u> \) by Lemma 1. Thus, the ideal \(<(a_0x - 1), u>\) consists of all zero divisors of \(R_{a,\beta}\). Therefore, \(R_{a,\beta}\) is a local ring with maximal ideal \(<(a_0x - 1), u>\).

**Remark 3.** If \(k = 1\), \(R_{a,\beta} = R_a\) is a chain ring with maximal ideal \(<a_0x - 1>\).

**Theorem 1.** If \(C\) is a \((a + u')\)-constacyclic code of length \(p^s\) over \(R\), then the following is the case:

\[
C = <g_0(x), g_1(x), \ldots, g_{k-1}(x)>,
\]

where \(g_i(x) = u'((a_0x - 1)^{T_i} + u'^{i+1}h_i(x)), \) if \(T_i < p^s\), \(h_i(x) \in R_{a,\beta}\) such that \(\text{deg } h_i < T_{i+1}\) or \(g_i(x) = 0\) otherwise. Moreover, the \(k\)-tuple \((g_0(x), g_1(x), \ldots, g_{k-1}(x))\) is unique.

**Proof.** The proof will be carried out by induction. Let \(R_i = F + uF + \ldots + u^F\) and \(R'_i = R_i[x]/<x^{T_i} - (a + u')>, \) where \(0 \leq i \leq k - 1\). First note that if \(k = 1\), \(R = F\), and the case is trivial. Now if \(k = 2\), let \(\mu_1\) be the canonical homomorphism from \(R'_2\) to \(R'_0\). It is clear that \(\text{Ker } \mu_1 = <u>\). Let \(C\) be a constacyclic code of length \(p^s\) over \(R_1\) and \(\mu_C\) be the restriction of \(\mu_1\) on \(C\). Then,

\[
\text{Ker } \mu_C = C \cap <u> = <u(a_0x - 1)^{T_1}>,
\]

where \(T_1 = T_1(C)\). Moreover, \(\text{Im } \mu_C \cong C/\text{Ker } \mu_C\) is a constacyclic code of length \(p^s\) over \(R_0\); thus,

\[
\text{Im } \mu_C = <(a_0x - 1)^{T_0}>,
\]

where \(T_0 = T_0(C)\) and \(0 \leq T_0 \leq p^s\). This implies that \((a_0x - 1)^{T_0} + u\alpha(x) \in C\) for some \(\alpha(x) \in R'_1\), and \(\alpha(x)\) can be expressed as \(\alpha(x) = (a_0x - 1)^{h(x)}\), where \(h(x)\) is either zero or a unit. We can consider \(\text{deg } \alpha \leq T_1\), and, therefore, \(t + \text{deg } h \leq T_1\). Thus, \(C\) is generated by

\[
g_0(x) = (a_0x - 1)^{T_0} + (a_0x - 1)^{h(x)} g_1(x) = u(a_0x - 1)^{T_1}.
\]

In cases when \(T_0 = p^s\), then \(C = \text{Ker } \mu_C\); thus, \(g_0(x) = 0\). Let us assume that the hypothesis is true for \(k - 2\) and we prove it for \(k - 1\). Let \(\mu_{k-1}\) be the natural homomorphism (modulo \(u^{k-1}\)) from \(R'_{k-1}\) to \(R'_{k-2}\). It is obvious that \(\text{Ker } \mu_{k-1} = <u^{k-1}>\). Assume \(C\) is a constacyclic code of length \(p^s\) over \(R_{k-1}\), and \(\mu_C\) is the restriction of \(\mu_{k-1}\) on \(C\). Then,

\[
\text{Ker } \mu_C = <u^{k-1} > \cap C = <u^{k-1} (a_0x - 1)^{T_{k-1}} >.
\]

Now, since \(\text{Im } \mu_C \cong C/\text{Ker } \mu_C\) is a constacyclic code of length \(p^s\) over \(R_{k-2}\), and by the induction step,

\[
\text{Im } \mu_C = <g_0(x), g_1(x), \ldots, g_{k-2}(x)>,
\]

where \(g_0(x), g_1(x), \ldots, g_{k-2}(x)\) satisfy the conditions of the theorem. This implies that there exist \(a_i(x) \in R'_{k-1}\) such that \(g_i(x) = g_i(x) + u^{k-1}a_i(x) \in C\). Moreover, we have \(T_i = T_i(\mu_1)\) for \(i = 0, 1, 2, \ldots, k - 2\). If we write \(a_i(x) = (a_0x - 1)^{h_{k-1-i}(x)}\), then \(h_{k-1-i} + \text{deg } h_{k-1-i} \leq T_{k-1}\). Therefore, we can take \(g_0(x), g_1(x), \ldots, g_{k-1}(x)\) as generators of \(C\), where \(g_{k-1}(x) = u^{k-1}(a_0x - 1)^{T_{k-1}}\). Now suppose

\[
C = <e_0(x), e_1(x), \ldots, e_{k-1}(x)>,
\]

where \(e_0(x), e_1(x), \ldots, e_{k-1}(x)\) is another expression of \(C\) satisfying the conditions of the theorem. Then, the uniqueness follows from the induction step and the fact that \(g_i(x) = e_i(x) \mod \text{Ker } \mu_C\) for \(i = 0, 1, 2, \ldots, k - 2\).

**Corollary 1.** Suppose that \(T_i < p^s\). Then, the smallest degree amongst the polynomials in \(C\) with leading coefficient \(u^i\) is \(T_i\).
Definition 2. Let $C$ be a constacyclic code over $R$. We call the unique $k$-tuple of polynomials described in Theorem 1 to be the representation of $C$.

Next, we construct a one-to-one correspondence between cyclic and $\alpha$-constacyclic codes, where $\alpha$ is a nonzero element of $F$. Consider the map $\Psi : R[x]/ < x^p - 1 > \rightarrow R[x]/ < x^p - \alpha >$ defined by $\Psi(f(x)) = f(a_0 x)$, where $a_0$ as in Equation (2). For polynomials $f(x)$ and $g(x)$ in $R[x]$, $f(x) \equiv g(x) \pmod{x^p - 1}$ if and only if there exists a polynomial $h(x)$ in $R[x]$ such that $f(x) - g(x) = h(x)(x^p - 1)$ if and only if $f(a_0 x) - g(a_0 x) = h(a_0 x)[(a_0 x)^p - 1] = a^{-1}h(a_0 x)[x^p - \alpha]$, if and only if $f(a_0 x) \equiv g(a_0 x) \pmod{x^p - \alpha}$. This means that $\Psi$ is well defined and has one-to-one correspondence. It is easy to show that $\Psi$ is a ring homomorphism. Thus, $\Psi$ is a ring isomorphism.

Proposition 5. The map $\Psi : R[x]/ < x^p - 1 > \rightarrow R[x]/ < x^p - \alpha >$ defined by $\Psi(f(x)) = f(a_0 x)$ is a ring isomorphism. In particular, $C$ is a cyclic code of length $p^e$ over $R$ if and only if $\Psi(C)$ is a $\alpha$-constacyclic code of length $p^e$ over $R$. Moreover, $\Psi$ is Hamming weight preserving.

Hamming Distance and Dual Codes

Definition 3. For a nonzero linear code $C$, the Hamming distances of $C$ and $d(C)$ are defined by the following:

$$d(C) = \min\{\text{wt}(c) \mid c \neq 0, c \in C\}, \quad (5)$$

where $\text{wt}(c)$ is the number of nonzero components of $c = (c_0, c_1, \ldots, c_{N-1})$ in $R^N$. The zero code is conventionally said to have Hamming distance 0.

Theorem 2. Let $C$ be a constacyclic code of length $p^e$ over $R$. Then,

$$d(C) = d(\text{Tor}_{k-1}(C)).$$

Proof. For any nonzero codeword $c(x)$ of $C$, we have $\text{wt}(u^{k-1}c(x)) \leq \text{wt}(c(x))$. Then, it suffices to compute Hamming distance of $u^{k-1}c(x)$, where $c(x) \in C$. As $u^{k-1}c(x)$ and $\tau(x)$ have the same number of nonzero coefficients, then $\text{wt}(u^{k-1}c(x)) = \text{wt}(\tau(x))$. Thus, $d(C) = d(\text{Tor}_{k-1}(C))$. As $\text{Tor}_{k-1}(C)$ is a constacyclic code over $F$, its Hamming distance is completely determined (see [33], Theorem 4.11).

Next, we consider the dual codes. Given $N$-tuples $x = (x_0, x_1, \ldots, x_{N-1})$ and $y = (y_0, y_1, \ldots, y_{N-1})$ in $R^N$, the inner product or dot products is defined as usual, with $x \cdot y = x_0 y_0 + x_1 y_1 + \ldots + x_{N-1} y_{N-1}$, which is evaluated in $R$. Two $N$-tuples $x$ and $y$ are called orthogonal if $x \cdot y = 0$.

Definition 4. For a linear code $C$ over $R$, its dual code $C^\perp$ is the set of $N$-tuples over $R$ that is orthogonal to all codewords of $C$, i.e., $C^\perp = \{x \mid x \cdot y = 0, \forall y \in C\}$.

The following propositions are well known [28,29,34,35].

Proposition 6. Let $\lambda$ be a unit of $R$. Then, the dual of a $\lambda$-constacyclic code over $R$ is a $\lambda^{-1}$-constacyclic code over $R$.

Proposition 7. Let $p$ be a prime and $R$ be a finite chain ring of order $p^e$. The number of codewords in any linear code $C$ of length $N$ over $R$ is $p^e$ for some integer $e \in \{0,1,\ldots,zN\}$. Moreover, the dual code $C^{\perp}$ has $p^e$ codewords, where $e + e' = zN$, i.e., $|C| \cdot |C^{\perp}| = |R|^N$.

Note that in $R_{x,\beta}$, $(\alpha + u^f \beta)^{p^k i} = \alpha^{p^k i}$; thus, the following is the case.

$$(\alpha + u^f \beta)^{p^k i} \alpha^{-p^k i} = 1.$$
Therefore, the following is the case:

\[(\alpha + u\beta)^{-1} = (\alpha + u\beta)^{p^i - 1}\alpha^{-p^i}\]  
\[(\alpha + u\beta)^{-1} = \alpha^{-1} + (u\beta) \sum_{i=1}^{p^i - 1} (p^i - 1) \alpha^{-(i+1)p^i} u^{-p^i}\]  
\[= \alpha^{-1} + (u\beta^i) \sum_{i=1}^{p^i - 1} (p^i - 1) \alpha^{-(i+1)} u^{-p^i}\]  
\[= \alpha^{-1} + (u\beta^i) \alpha^{-1} \sum_{i=1}^{p^i - 1} (p^i - 1) \alpha^{-i} u^{-p^i}\]  
\[= \alpha^{-1} + u\beta^i \alpha^{-1} \zeta_i,\]  
where \(\zeta = \sum_{i=1}^{p^i - 1} (p^i - 1) \alpha^{-i} u^{-p^i}\), which is a unit in \(R_{\alpha, \beta}\).

**Theorem 3.** Let \(C\) be a \((\alpha + u\beta)\)-constacyclic code of length \(p^s\) over \(R\) as in Theorem 1. Then, \(C^\perp\) is a \((\alpha^{-1} + u\beta\alpha^{-1} \zeta)\)-constacyclic code of length \(p^s\) over \(R\), and the following is the case:

\[C^\perp = \langle f_0(x), f_1(x), \ldots, f_{k-1}(x) \rangle,\]  
where \(f_i(x) = u^i(a_0^{-1}x - 1)^{T_i} + u^{i+1}a_i(x)\) for some \(a_i(x) \in R_{\alpha^{-1}, \beta\alpha^{-1} \zeta}\). Moreover, \(T_i = p^s - T_{k-1-i}\) for \(0 \leq i \leq k - 1\).

**Proof.** By Proposition 6, \(C^\perp\) is a \((\alpha^{-1} + u\beta\alpha^{-1} \zeta)\)-constacyclic code of length \(p^s\) over \(R\); thus, by Theorem 1,

\[C^\perp = \langle f_0(x), f_1(x), \ldots, f_{k-1}(x) \rangle,\]  
where \(f_i(x) = u^i(a_0^{-1}x - 1)^{T_i} + u^{i+1}h_i(x)\) for some \(h_i(x) \in R'\) and \(T_i = T_i(C^\perp)\). By the definition of \(C^\perp\), it is easy to deduce that \(u^{k-1-i}(a_0^{-1}x - 1)^{T_i} \in C^\perp\) and then \(T_{k-1-i} \leq p^s - T_i\) for \(0 < i < k - 1\). As \(|C| = (p^s)^{k^p - (T_0 + T_1 + \ldots + T_{k-1})}\) and \(|C^\perp| = (p^s)^{k^p}\) (Proposition 7), then \(|C^\perp| = (p^s)^{T_0 + T_1 + \ldots + T_{k-1}}\). Thus, we must have \(T_{k-1-i} + T_i = p^s\) and so \(T_i = p^s - T_{k-1-i}\). \(\square\)

**Example 1.** Table 1 shows the representation of all proper cyclic codes of length 3 over the chain ring \(R = \mathbb{Z}_3 + u\mathbb{Z}_3\) of characteristic 3.

### Table 1. Proper cyclic codes of length 3 over \(R\).

| \(C \subseteq C < u >\) | \(C \nsubseteq C < u >\) |
|--------------------------|--------------------------|
| \(< 0 >\)               | \(< (x - 1), u(x - 1) >\) |
**Example 2.** Table 2 shows the representation of all proper $(1+u^2)$-constacyclic codes of length 2 over the chain ring $R = \mathbb{Z}_2 + u\mathbb{Z}_2 + u^2\mathbb{Z}_2$ of characteristic 2. We have the following case.

$$l = 2, \alpha = a_0 = \beta = 1, u^2 = (x - 1)^2$$

_and $s = 1$.

**Table 2.** Proper $(1+u^2)$-constacyclic codes of length 2 over $R$.

| $C \subseteq < u >$ | $C \nsubseteq < u >$ |
|-------------------|-------------------|
| $< 0 >$           | $(x - 1) >$       |
| $< u >$           | $(x - 1), u >$    |
| $< u^2 >$         | $(x - 1) + u, u^2 >$ |
| $< u(x - 1) >$    | $(x - 1) + u >$   |
| $< u(x - 1) + u^2 >$ | $(x - 1) + u(u(x - 1) >$ |
| $< u^2(x - 1) >$  | $(x - 1) + u + u^2 >$ |
| $< u(x - 1), u^2 >$ | $(x - 1) + u(u(x - 1) + u >$ |

**Example 3.** Consider the constacyclic codes in Example 2. As $(1+u^2)^{-1} = 1+u^2$ by Equation (10), $C^+ = C$ for any $(1+u^2)$-constacyclic code $C$. This means all $(1+u^2)$-constacyclic codes in Example 2 are self dual codes.

### 4. Constacyclic Codes of Length $N$

#### 4.1. Extension Rings

Let $r'$ be a positive integer and let $CR(u^k, r') = R[x] / < f(x) >$, where $f(x)$ is a monic basic irreducible of degree $r'$ over $R$. Note that $f(x)$ can be chosen so that $CR(u^k, r')$ contains $(p^r-1)$th root of unity. Moreover, $CR(u^k, r')$ is a chain ring of characteristic $p$ with maximal ideal $< u >$ and residue field $K = F_{p^{r'}}$. By Theorem 2,

$$CR(u^k, r') = K \oplus uK \oplus \ldots \oplus u^{k-1}K. \quad (12)$$

Let $a$ be the order of $p$ modulo $n_1$, then $F_{p^a}$ contains a primitive $n_1$th root $\xi$ of unity. Assume that $K$ is the splitting field of $x^{n_1} - a$ over $F_{p^a}$, where $a$ is a nonzero element of $F_0$ and $r' = a^a$ for some positive integer $a'$ the degree of the extension. If $\theta$ is a root of $x^{n_1} - a$ in $F_{p^a}$, then $\theta^i\xi^j$, for $0 \leq i \leq n_1-1$ are all distinct roots of $x^{n_1} - a$ in $F_{p^a}$; hence, by Hensel’s Lemma [29], Theorem XIII.4), $CR(u^k, r')$ also contains all those roots. Now $x^{n_1} - a$ factors uniquely into monic irreducible polynomials over $F$, and then again by Hensel’s Lemma, $x^{n_1} - a$ factors into monic basic irreducible polynomials over $R$ as follows.

$$x^{n_1} - a = f_1(x)f_2(x)\ldots f_m(x). \quad (13)$$

For each $0 \leq j \leq n_1 - 1$, there exists a unique $i$, $1 \leq i \leq m$ such that $f_i(\theta^{j}\xi^i) = 0$, and $f_i(x)$ is called the minimal polynomial of $\theta^{j}\xi^i$ over $R$.

Next, we introduce another extension:

$$S(r') = CR(u^k, r')[x] / < x^{p^a} - (a + u^i\beta) > \quad (14)$$

of $CR(u^k, r')$. Note that, for a suitable positive number $r'$, $S$ will be the alphabet of codes of length $p^a$ over $R$ that contains $n$th root of unity. The results of Lemma 1 and Proposition 4 hold for the ring $S$. Moreover, we define the following extension of $R$.

$$R_N = R[x] / < x^{N} - (a + u^i\beta) >. \quad (15)$$

Let $r'$ be the order of $p$ modulo $n_1$. Let $\sim$ be a relation on the set $\{1,2,\ldots,n_1\}$ defined as $i \sim j$ if and only if $\theta^{j}\xi^i$ and $\theta^{i}\xi^j$ are roots of the same minimal polynomial, i.e., there is a unique $b$ such that $f_b(\theta^{i}\xi^j) = f_b(\theta^{j}\xi^i) = 0$. It is easy to show this relation is equivalence.
Now, let \( l \) be the set of all classes of \( \sim \), \( l_b \) be a class containing \( b \), and \( r_b \) is the size of this class, i.e., \( | l_b | = \deg f_b (x) = r_b \).

### 4.2. Discrete Fourier Transform (DFT)

DFT has been used to study repeated-root codes over finite chain rings in [3,5,6]. We employ DFT as a tool to establish the structure of \((\alpha + u^l \beta)\)-constacyclic codes over \( R \) for a given length \( N \).

**Remark 4.** In \( S(r_b) \), we have \((\alpha + u^l \beta)^{p^d-1} = \alpha \), where \( d = \lceil \frac{k}{l} \rceil \); hence, \( x^{p^d+d-1} = \alpha \) and then \((\alpha_0 x)^{p^d+d-1} = 1\).

**Definition 5 (DFT).** Let \( c \) be a vector in \( R^N \) with \( c(x) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_1-1} c_{ij} x^{i+j n_1} \) the corresponding polynomial. The DFT of \( c(x) \) is the following vector:

\[
(\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_{n_1-1}) \in S^{(r')}^{n_1},
\]

where \( \hat{c}_b = c((\alpha_0 w)^{n'} \theta^b) = \sum_{i=0}^{n_1-1} \sum_{j=0}^{n_1-1} c_{ij} w^{i+n' j} (\alpha_0^{n'} \theta^b)^j, b \in I \) and \( n_1 n' \equiv 1 \) (mod \( p^d+d-1 \)). Define the Mattson–Solomon polynomial of \( c \) to be the following.

\[
\hat{c}(z) = \sum_{b=0}^{n_1-1} \hat{c}_{n_1-1-b} z^b.
\]

Note that \( \hat{c}_{n_1} = \hat{c}_0 \).

The following lemma shows that if the Mattson–Solomon polynomial of \( c \) is given, then \( c \) can be recovered. Set \( S = R_{\alpha, \beta} = R[w] / \langle w^p - (\alpha + u^l \beta) \rangle \). Let \( \phi \) be the natural \( R \)-module isomorphism \( \phi : S^{n_1} \rightarrow R^N \) defined by the following case.

\[
\phi(\sum_{i=0}^{p^d-1} c_{0,i} w^i, \ldots, \sum_{i=0}^{p^d-1} c_{n_1-1,i} w^i) = (\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_{n_1-1}).
\]

**Lemma 2.** Let \( c \in R^N \) with \( \hat{c}(z) \) its Mattson–Solomon polynomial. Then, the following is the case:

\[
c = \phi[(1, u^{-n'}, u^{-2n'}, \ldots, u^{-(n_1-1)n'}) \ast \frac{1}{n_1} (\hat{c}(1), \hat{c}(u^1), \ldots, \hat{c}(u^{n_1-1}))],
\]

where \( \ast \) denotes component-wise multiplication.

**Proof.** Let \( 0 \leq l' \leq n_1 - 1 \). Then, the following is the case.

\[
\hat{c}(\zeta^l) = \sum_{b=0}^{n_1-1} \hat{c}_b \zeta^{-b l} = \sum_{b=0}^{n_1-1} \sum_{i=0}^{p^d-1} \sum_{j=0}^{n_1-1} c_{ij} w^{i+n' j} (\alpha_0^{n'} \theta^b)^j \zeta^{-b l} = \sum_{i=0}^{n_1-1} \sum_{j=0}^{p^d-1} c_{ij} w^{i} (\alpha_0^{n'} \theta^j)^i \sum_{b=0}^{n_1-1} \zeta^{b (i-l)} = n_1 ((\alpha_0 w)^{n'} \theta^j)^i \sum_{j=0}^{p^d-1} c_{ij} w^j.
\]

Note that \( \sum_{b=0}^{n_1-1} \xi^{bj} = 0 \), when \( j \neq 0 \) (mod \( n_1 \)). Then, by the definition of \( \phi \), we have

\[
c = \phi[(1, w^{-n'}, w^{-2n'}, \ldots, w^{-(n_1-1)n'}) \ast \frac{1}{n_1} (\hat{c}(1), \hat{c}(u^1), \ldots, \hat{c}(u^{n_1-1}))].
\]
Remark 5. Since $\theta \xi^b \in S(r_b)$, it is easy to verify that $\xi_b \in S(r_b)$. Now let the following be the case.

$$A = \{(\xi_0, \xi_2, \ldots, \xi_{n-1}) \in S(r)^{n_1} | \xi_i \in S(r_b), i \in I_b\}. \tag{19}$$

Note that $A$ with component-wise addition and multiplication is a ring. Moreover, it is clear that $A \cong \oplus_{b \in I} S(r_b)$.

Theorem 4. Let $\gamma$ be the map $\gamma : R_N \rightarrow \oplus_{b \in I} S(r_b)$, given by $\gamma(c(x)) = (\xi_b)_{b \in I}$. Then, $\gamma$ is a ring isomorphism. In particular, if $C$ is a constacyclic code of length $N$ over $R$, then the following is the case:

$$C \cong \oplus_{b \in I} C_b \tag{20}$$

where $C_b$ is the constacyclic code $\{c((a_0w)^d \theta \xi^b) | c \in C\}$ of length $p^d$ over $CR(u^t, r_b)$.

Proof. Define the map $\gamma : R_N \rightarrow A$, where $\gamma(c(x)) = (\xi_0, \xi_2, \ldots, \xi_{n-1})$. Let $a(x), b(x)$ be polynomials over $R$ of degree less than $N$. Then, clearly $\gamma(a(x) + b(x)) = \gamma(a(x)) + \gamma(b(x))$ and also $\gamma(a(x)b(x)) = \gamma(a(x)) \ast \gamma(b(x))$, where $\ast$ denotes the componentwise product. Suppose $\gamma(c(x)) = 0$, then by Lemma 2, $\Sigma_{\gamma(t) = 0} c_{\gamma(t)} w^t = 0$ for any $t'$, where $0 \leq t' \leq n_1 - 1$. It follows that $c(x) = 0$, and this implies $\gamma$ is an injection. Moreover, $|A| = \prod_{b \in I} p^{n_1 n_b} = p^{n_1 N}$, which means that $\gamma$ is a bijection. Therefore, $\gamma$ is an isomorphism. The second statement follows directly because $\gamma$ is a ring isomorphism. \[\square\]

Before we obtain the structure of all constacyclic codes of length $n_1 p^s$ over $R$ in terms of their generator polynomials, we provide the following lemma.

Lemma 3. Let $f_b(x)$ be the minimal polynomial of $\theta \xi^b$ over $R$ for each $b \in I$ and $n'$ a positive integer such that $n_1 n' = 1 \text{ mod } p^{s+d-1}$. Then, the following is the case:

(i) $f_b((a_0w)^d \theta \xi^b)$ is a unit if $i$ is not in $I_b$.

(ii) $f_b((a_0w)^d \theta \xi^b) \in \langle a_0w - 1 \rangle$ but $f_b((a_0w)^d \theta \xi^b)$ is not in $\langle a_0w - 1 \rangle^2$.

Proof.

(i) Since $f_b(x) = \prod_{b \in I_b} (x - \theta \xi^b)$, then, the following is the case.

$$f_b((a_0w)^d \theta \xi^b) = \prod_{b \in I_b} ((a_0w)^d \theta \xi^b - \theta \xi^b)$$

$$= \prod_{b \in I_b} [((a_0w)^d - 1) \theta \xi^b + (\theta \xi^b - \theta \xi^b)].$$

Since $i$ is not in $I_b$, then $\theta \xi^b - \theta \xi^b \neq 0$. Therefore, $f_b((a_0w)^d \theta \xi^b)$ is a unit if $i$ is not in $I_b$.

(ii) We know that $x_i^d - a = \prod_{b \in I} f_b(x)$ and then $\prod_{b \in I} f_b((a_0w)^d \theta \xi^b) = ((a_0w)^d \theta \xi^b)^{n_1} - 1 = a_0w - 1$. However, from (i) we have $f_i((a_0w)^d \theta \xi^b)$, which is a unit for $i \neq b$. Hence, $f_b((a_0w)^d \theta \xi^b) = a((a_0w - 1),$ where $a$ is a unit in $S(r_b)$. It follows that $f_b((a_0w)^d \theta \xi^b) \in \langle a_0w - 1 \rangle$. Now, suppose that $f_b((a_0w)^d \theta \xi^b) \in \langle a_0w - 1 \rangle^2$, which implies that $\langle a_0w - 1 \rangle^2 < \langle a_0w - 1 \rangle^2$. However, this is a contradiction, and this completes the proof. \[\square\]

Next, we introduce the polynomial representations of constacyclic codes over $R$. If $C$ is a constacyclic code of length $N$ over $R$. By Theorem 4, $C \cong \oplus_{b \in I} C_b$, where $C_b$ is a constacyclic code of length $p^s$ over $CR(u^t, r_b)$, and by Theorem 1:

$$C_b = \langle e_{0,b}(w), u e_{1,b}(w), \ldots, u^{k-1} e_{k-1,b}(w) \rangle,$$

where $e_{i,b}(w) = (a_0w - 1)^{t_i} + u h_{i,b}(w)$. Now, fix $i$ and for each $0 \leq j \leq p^s$. We define $F_j(x)$ to be the product of all minimal polynomials of $a^j$ such that $Tor_i(C_b) = \langle (a_0w - 1)^j \rangle$. By Lemma 3, the following is the case:
where \( a_b(a_0w) \) is a unit in \( S(r_b) \). Define the following:

\[
g_i(x) = \prod_{j=0}^{p^k-1} [F_j((a_0w)^{u^j} \theta b^j)]^j = a_b(w)(a_0w - 1)^j,
\]  

(21)

where \( a_b(a_0w) \) is a unit in \( S(r_b) \). Define the following:

\[
g_i(x) = \prod_{j=0}^{p^k-1} [F_j(x)]^j + pb_i(x),
\]  

(22)

where \( b_i(x) = \gamma^{-1}((a_b(a_0w)h_{i,b}(a_0w))_{b \in I}) \).

**Theorem 5.** Let \( C \) be a constacyclic code of length \( N \) over \( R \). Then, the following is the case.

*Proof.* By Corollary 3, it suffices to compute \( g_i(x) \) for all \( i \). Next, we provide the enumeration of constacyclic codes of length \( N \) over \( R \) is the following:

\[
\prod_{b \in I} N_b,
\]  

(24)

where \( N_b \) is the number of \((\alpha + u^l \beta)\)-constacyclic codes of length \( N \) over \( R \) is the following:

\[
\prod_{b \in I} \left( \frac{p^{r_b}(z_b+1) - 1}{p^{r_b} - 1} \right),
\]  

(25)

where \( z_b = \min\{\left\lceil \frac{d_b}{2} \right\rceil, p^{s-1}\} \) and \( T_0(C_b) + T_1(C_b) = d_b \leq p^s \).

**Proof.** By Corollary 3, it suffices to compute \( N_b \), \( b \in I \). First fix \( T_1 \); thus, \( T_0 = d_b - T_1 \). Let \( d_p < p^s \). By Theorem 1, \( < (a_0w - 1)^{T_0} + uc(w), (a_0w - 1)^{T_1} > \) is a representation. Moreover, we have \( \left(p^{r_b}\right)^{T_1} \) choices for \( e(w) = \sum_{i=0}^{T_1-1} a_i(a_0w - 1)^i \). Theorem 1 implies that \( T_1 < \min\{p^{s-1}, T_0\} \); hence, \( T_1 < \min\{p^{s-1}, \left\lceil \frac{d_p}{2} \right\rceil\} = z_b \) because \( T_1 + T_0 = d_p \). If
we vary $T_1$ from 0 to $z_0$, then there are $\alpha + u'\beta$-constacyclic codes of length $p^s$ over $CR(u^b, r_b)$. In the case when $d_p = p^s$, we have two options. If $T_0 = p^s$, the only $(\alpha + u'\beta)$-constacyclic code is $< 0, u >$ with $T_0 + T_1 = d_p$. If $T_0 < p^s$, we use a similar discussion as before. \hfill \square

**Remark 7.** When $k > 2$, the enumeration of all constacyclic codes of length $N$ over $R$ is a tedious computation.

### 4.3. Torsion Codes and Hamming Distance

In this subsection, we first obtain the torsion codes of a constacyclic code $C$ of length $N$ over $R$ in terms of the generators of $C$ given in Theorem 5. Then, we reduce the Hamming distance of $C$ to that of its $(k-1)$th torsion code.

**Lemma 4.** Let $C = < g(x), u g_1(x), \ldots, u^{k-1} g_{k-1}(x) >$. If $u'(h(x)) \in C$ such that $h(x) \in Tor_i(C)$, then $deg h \geq deg g_i$.

**Proof.** Assume that $u'(h(x)) \in C$, then $u'(h((a_0 \omega)^u \theta b^b)) \in C$, $b \in I$, $n_1 n' \equiv 1 \pmod{p^s+d-1}$. As $h(x) \in Tor_i(C)$, then $h((a_0 \omega)^u \theta b^b) \in Tor_i(C)$. This means, $h((a_0 \omega)^u \theta b^b) = c(w)(a_0 \omega - 1)^r_i$ for some unit $c(w)$ in $S(r_b)$. Now, let $g(x) = p(x) \prod_{j=0}^{p^s-1} [F_j(x)]^l$, where $F_j(x)$ as defined in the proof of Theorem 5 and $p(x) = \gamma^{-1}(c_b(w)a_b^{-1}(w))_{b \in I}$. By (21), for each $b \in I$, $g((a_0 \omega)^u \theta b^b) = c(w)(a_0 \omega - 1)^r_i$. Thus, $\gamma(h(x)) = \gamma(g(x))$; hence, $h(x) = g(x)$, i.e., $deg \gamma = deg g$. Therefore, $deg g \geq deg \prod_{j=0}^{p^s-1} [F_j]^l = deg g_b$ by (22). \hfill \square

**Theorem 7.** If $C = < g_0(x), u g_1(x), \ldots, u^{k-1} g_{k-1}(x) >$, then $Tor_i(C) = < \overline{g_i}(x) >$.

**Proof.** First, note that $u' \overline{g_i}(x) \in C$; thus, $< \overline{g_i}(x) > \subseteq Tor_i(C)$. Conversely, let $h(x) \in Tor_i(C)$, then by the definition of torsion codes, $u'h(x) \in C$. We make use of Lemma 4, $deg h \geq deg g_i$. By the division algorithm, there are $r(x)$ and $q(x)$ in $R_N$ such that $h(x) - g_i(x)q(x) = r(x)$, where $r(x) = 0$ or $deg r < deg g$. As $u'r(x) \in C$, then by the minimality of $deg g_i$, we must have $r(x) = 0$. In other words, $h(x) \in < g_i(x) >$; thus, $\overline{h(x)} \in < \overline{g_i}(x) >$. Therefore, $Tor_i(C) \subseteq < \overline{g_i}(x) >$, and this ends the proof. \hfill \square

Next, we obtain the Hamming distance of any cyclic code of length $N$ over $R$.

**Theorem 8.** Let $C$ be a cyclic code of length $N$ over $R$. Then, $d(C) = d(Tor_{k-1}(C))$.

**Proof.** By the same argument as in Theorem 2, we obtain $d(C) = d(Tor_{k-1}(C))$, where $Tor_{k-1}(C) = < \overline{g_{k-1}}(x) > = < \prod_{j=0}^{p^s-1} F_j(x) >$ from Theorem 7. \hfill \square

### 4.4. Dual Codes

Define $F_j(x)$ as in the proof of Theorem 5. Let $a_j$ be the constant of $F_j(x)$, $0 \leq j \leq p^s$. Since $\prod_{j=0}^{p^s} F_j(x) = x^{d_1} - a$, then $\prod_{j=0}^{p^s} a_j = -a$. Thus, $a_j$s are units in $R$ and $a_j$s are the leading coefficient of $F_j(x) = x^{deg F_j} F_j(x^{-1})$. Let the following be the case.

$$m_j(x) = a_j^{-1} F_j(x).$$

(26)
Note that \( m_j(x) \)s are monic polynomials and \( \prod_{j=0}^{\mu} a_j^{-1} = -\alpha^{-1} \). Hence, the following is the case.

\[
\prod_{j=0}^{\mu} m_j(x) = \left( \prod_{j=0}^{\mu} a_j^{-1} \right) \prod_{j=0}^{\mu} F_j(x) = -\alpha^{-1} x^{\Sigma_{j=0}^{\mu} \deg F_j} \prod_{j=0}^{\mu} F_j(x^{-1}) = -\alpha^{-1} x^{n_1} (x^{-m_1} - 1) = x^{n_1} - \alpha^{-1}.
\]

Therefore, \( m_j(x) \)s are monic coprime divisors of \( x^{n_1} - \alpha^{-1} \) in \( R[x] \). Since \( \text{Tor}_i(C_b) = \langle (a_0 w - 1)^i \rangle \), then \( F_j(\theta b^{n_1}) = 0 \), which implies that \( F_j^\ast(\theta b^{n_1} - b) = 0 \); hence, \( m_j(\theta b^{n_1} - b) = 0 \). It follows that \( m_j(x) \) is the product of all minimal polynomials of \( \alpha^m - b \) such that \( \text{Tor}_i(C_b^\perp) = \langle (a_0 w - 1)j^{i-1} \rangle \). By Lemma 3, the following is the case:

\[
\prod_{j=0}^{\mu} [F_j^\ast(a_0 w^{\mu} \theta b^{n_1})]^j r^{i-1} = a_h(w)(a_0 w - 1)^{r^{i-1}},
\]

(27)

where \( a_h(w) \) is a unit in \( S(r_h) \). Define the following case:

\[
G_i(x) = \prod_{j=0}^{\mu} [F_j^\ast(x)]^r^{i-1} + u c_i(x),
\]

(28)

where \( c_i(x) = \gamma^{-1}((a_b(w) h_{i,b}(w))_{b \in I}) \) and \( h_{i,b}(w) \) as in Theorem 3.

**Theorem 9.** Let \( C \) be a constacyclic code of length \( N \) over \( R \). Then, the following is the case.

\[
C^\perp = \langle G_0(x), u G_1(x), \ldots, u^{k-1} G_{k-1}(x) \rangle.
\]

Furthermore, \( |C^\perp| = p^{t'}, \) where \( t' = \Sigma_{j=0}^{\mu} \deg F_j^\ast \).

**Proof.** By Theorem 4, \( C = \oplus_{b \in I} C_b \), where \( C_b \) is a constacyclic code of length \( p^\mu \) over \( CR(p^\mu, r_b) \). Assume that \( D = \oplus_{b \in I} C_b^\perp \). By the definition of dual code, \( D \subseteq C^\perp \). On the other hand, we have \( |C_b| \cdot |C_b^\perp| = p^{t_k \mu} \). Then, \( |C| \cdot |D| = p^{t_k \mu} \); thus,

\[
C^\perp = D = \oplus_{b \in I} C_b^\perp.
\]

Therefore, by a similar argument to that of the proof of Theorem 5, we obtain

\[
C^\perp = \langle G_0(x), u G_1(x), \ldots, u^{k-1} G_{k-1}(x) \rangle,
\]

where \( G_i(x) \) is defined in (28) and \( 0 \leq i \leq k - 1 \). By Corollary 2 and the fact that \( |C^\perp| \cdot |C| = p^{t_k \mu} \), we obtain \( |C^\perp| = p^{t'} \), where \( t' = \Sigma_{j=0}^{\mu} \deg F_j^\ast \). \( \Box\)

To summarize, the results of this section provide an algorithm for constructing the representation of constacyclic codes of length \( N = n_1 p^\mu \) from those of length \( p^\mu \). This algorithm consists of the following steps:

- **Step 1:** Find \( \theta, \xi, I \) and all \( r_b, b \in I \);
- **Step 2:** Compute \( F_j(x) \) for each \( 0 \leq j \leq p^\mu \) when \( i \) is fixed;
- **Step 3:** Find \( a_h(x), b \in I \) from the relation (21);
- **Step 4:** Extract \( b_1(x) \) by using \( b_1(x) = \gamma^{-1}((a_b(w) h_{1,b}(w))_{b \in I}), 0 \leq i \leq k - 1 \);
- **Step 5:** Compute the polynomials \( g_i(x) \) via Equation (22).
Next, we present an example illustrating the algorithm described above.

**Example 4.** Consider $R = \mathbb{Z}_2 + u\mathbb{Z}_2$ and $N = 6$. First, $n_1 = 3$, $I = \{0, 1\}$, $r_0 = 1$, and $r_1 = 2$. Let $C_0 = \langle (w - 1), u(w - 1) \rangle$, and $C_1 = \langle (w - 1), u \rangle$ be cyclic codes of length 2 over $R$ and $CR(u^2, 2)$, respectively. Next, compute $F_j(x)$ for $i = 0$. As $T_0(C_0) = 1 = T_0(C_1)$ and by the definition of $F_j(x)$,

$$
F_0(x) = 1, F_2(x) = 1 \text{ and } F_1(x) = f_0(x)f_1(x) = (x - 1)(x - a)(x - a^2) = x^3 - 1,
$$

where $a$ is a third primitive root of unity satisfying $a^2 + a + 1 = 0$. Since $h_{0,0}(x) = h_{0,1}(x) = 0$, then $b_0(x) = 0$; thus, $g_0(x) = \prod_{j=0}^{2} F_j(x)^j = x^3 - 1$. Now, find $F_j(x)$ when $i = 1$, note that $T_1(C_0) = 1$ and $T_1(C_1) = 0$. It follows that $F_2(x) = 1$, $F_0(x) = (x - a)(x - a^2)$ and $F_1(x) = (f_0(x)) = (x - 1)$. As $b_1(x) = 0$ since $h_{1,0}(x) = 0 = h_{1,1}(x)$, $g_1(x) = \prod_{j=0}^{2} F_j(x)^j = x - 1$. Therefore, by Theorem 5, the following is the case.

$$
C = \langle x^3 - 1, u(x - 1) \rangle.
$$

**Remark 8.** The same algorithm described above can be applied to compute the generators of dual codes. The main key for performing this is to consider $C_b^\perp$ instead of $C_b$, where $b \in 1$.

### 5. Conclusions

In this article, we have determined a unique representation of any constacyclic code of arbitrary length $N$ over a finite chain ring of characteristic $p$ via discrete Fourier transform (DFT). Such representations allowed us to compute Hamming distance and dual codes easily. Moreover, we managed to provide the number of constacyclic codes of length $N$ over $R$ in terms of that of length $p^s$, where $v_p(N) = s$ and $v_p$ is the $p$-adic valuation. In particular, we provided the exact number of such codes when $k = 2$.

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