A COMBINATORIAL REFINEMENT OF THE KRONECKER-HURWITZ CLASS NUMBER RELATION

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ABSTRACT. We give a refinement of the Kronecker-Hurwitz class number relation, based on a tesselation of the Euclidean plane into semi-infinite triangles labeled by $\text{PSL}_2(\mathbb{Z})$ that may be of independent interest.

1. A REFINEMENT OF A CLASSICAL CLASS NUMBER RELATION

We give a refinement, and a new proof, of the following classical result [1 2 3].

Theorem 1 (Kronecker, Gierster, Hurwitz). For any $n \geq 1$ we have

$$
\sum_{t^2 \leq 4n} H(4n - t^2) = \sum_{n \equiv d \equiv 0} \max(a, d).
$$

Here $H(D)$ ($D \geq 0, D \equiv 0, 3 \mod 4$) is the Kronecker-Hurwitz class number, which has initial values

| $D$  | 0   | 3   | 4   | 7   | 8   | 11  | 12  | 15  | 16  | 19  | 20  | 23  | 24  |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $H(D)$ | $-\frac{1}{12}$ | $\frac{1}{3}$ | $\frac{1}{2}$ | 1   | 1   | 1   | $\frac{4}{3}$ | 2   | 3   | 2   | 1   | 2   | 3   |

and for $D > 0$ equals the number of $\text{PSL}_2(\mathbb{Z})$-equivalence classes of positive definite integral binary quadratic forms of discriminant $-D$, with those classes that contain a multiple of $x^2 + y^2$ or of $x^2 - xy + y^2$ counted with multiplicity $1/2$ or $1/3$, respectively.

Let $\Gamma = \text{PSL}_2(\mathbb{Z})$. By the $\Gamma$-equivariant bijection $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow cx^2 + (d-a)xy - by^2$ between integral matrices of determinant $n$ and trace $t$ and quadratic forms of discriminant $t^2 - 4n$, the class number relation can be written as

$$
(1) \quad \sum_{M \in \mathcal{M}_n} \chi(z_M) = \sum_{n \equiv d \equiv 0} \max(a, d) + \begin{cases} 1/6 & \text{if } n \text{ is a square}, \\ 0 & \text{otherwise}, \end{cases}
$$

where $\mathcal{M}_n$ is the set of integral matrices of determinant $n$ modulo $\pm 1$, $z_M$ is the fixed point of an elliptic $M$ in the upper half-plane $\mathbb{H}$, and $\chi : \mathbb{H} \to \mathbb{Q}$ is the modified characteristic function of the standard fundamental domain

$$
\mathcal{F} = \{ z \in \mathbb{H} : -1/2 \leq \text{Re}(z) \leq 1/2, \ |z| \geq 1 \}.
$$

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of $\Gamma$ acting on $\mathcal{H}$ such that $\chi(z)$ is $1/2\pi$ times the angle subtended by $F$ at $z$ (so $\chi$ is 1 in the interior of $F$, 0 outside of $F$, 1/2 on the boundary points different from the corners $\rho = e^{\pi i/3}$ and $\rho^2$, and 1/6 at the corners).

We will prove a refinement of (1) saying that the subsum of the expression on the left over all $M$ in a given orbit of the right action of $\Gamma$ on $\mathcal{M}_n$ always takes on one of the values 0, 1, 2 (or $7/6$ for the orbit $\sqrt{n}\Gamma$ if $n$ is a square). Specifically, let us define for any right coset $K$ in $\mathcal{M}_n/\Gamma$ (more precisely, $K$ is a right coset in $\text{PGL}_2(\mathbb{Q})/\Gamma$, since $\mathcal{M}_n$ is not a group) two positive integers $\delta_K$ and $\delta'_K$ by

$$\delta_K = \gcd(c, d), \quad \delta'_K = \frac{n}{\delta_K},$$

where $(a \ b \ c \ d)$ is any representative of $K$. Then we have:

**Theorem 2.** For each right coset $K \in \mathcal{M}_n/\Gamma$ we have

$$\sum_{\substack{M \in K \text{ elliptic} \ M \in \mathcal{M}_n}} \chi(z_M) = 1 + \text{sgn}(\delta'_K - \delta_K) + \begin{cases} 1/6 & \text{if } K = \sqrt{n}\Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Equation (1) follows immediately by summing the relations in Theorem 2 over all cosets in the disjoint decomposition $\mathcal{M}_n = \bigsqcup (\delta'_\beta \delta \beta_0 \delta) \Gamma$ with $n = \delta'_\beta \delta$ and $0 \leq \beta < \delta'_\beta$.

Theorem 2 provides a correspondence between right cosets and $\Gamma$-conjugacy classes in $\mathcal{M}_n$, which generically assigns two conjugacy classes to each coset with $\delta'_\beta > \delta$. We will deduce it from a similar statement, Theorem 3, which is sharper in two respects (it counts the number of matrices with a fixed point in a smaller domain, and it allows real coefficients), and which gives a generically one-to-one correspondence between cosets and conjugacy classes. To state it, we consider a half-fundamental domain $F^- = \{z \in \mathcal{H} : -1/2 \leq \Re(z) \leq 0, \ |z| \geq 1\}$, and define a function $\alpha : \text{GL}_2^+(\mathbb{R}) \rightarrow \mathbb{Q}$ by

$$\alpha(M) = \begin{cases} \chi^-(z_M) & \text{if } M \text{ is elliptic with fixed point } z_M \in \mathcal{H}, \\ -\frac{1}{12} & \text{if } M \text{ is scalar,} \\ 0 & \text{if } M \text{ is parabolic or hyperbolic,} \end{cases}$$

where $\chi^-$ is defined in the same way as $\chi$ (and hence equals 1 in the interior of $F^-$, 0 outside $F^-$, 1/2 on the internal boundary points of $F^-$, and 1/4 and 1/6 at the corners $i$ and $\rho^2$, respectively). Note that $\alpha(-M) = \alpha(M)$, so $\alpha$ is well-defined on $\mathcal{M}_n \Gamma$.

**Theorem 3.** For $M = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \text{GL}_2(\mathbb{R})$ with $y > 0$, we have

$$\sum_{\gamma \in \Gamma} \alpha(M\gamma) = \frac{1 + \text{sgn}(y - 1)}{2}. \tag{2}$$

Since each coset $K \in \mathcal{M}_n/\Gamma$ contains a representative $M$ with $M\infty = \infty$, Theorem 2 immediately follows from (2), and the fact that the map $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \pm \begin{pmatrix} -a & b \\ -c & -d \end{pmatrix}$ is a bijection between the sets of elements in $\mathcal{M}_n$ having fixed point in the left half and in the right half of the standard fundamental domain for $\Gamma$.

Theorem 3 is proved in Section 3 as an easy consequence of a triangulation of a Euclidean half-plane by triangles associated to elements of $\Gamma$ (Theorem 4). This triangulation may be of independent interest, and we give a self-contained treatment in the next section.
2. A triangulation of a Euclidean half-plane

Let $\Gamma_\infty = \{ \gamma \in \Gamma \mid g_\infty = \infty \}$. We identify $\Gamma \setminus \Gamma_\infty$ with a subset of $\text{SL}_2(\mathbb{Z})$ by choosing representatives $\gamma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$ with $c > 0$, and for such $\gamma$ we define a semi-infinite triangle

$$\Delta(\gamma) = \{ (x, y) \in \mathbb{R}^2 \mid 0 \leq d - cx - ay \leq c \leq -dx - by \}.$$  

(The motivation for this definition is that $(x, y) \in \Delta(\gamma)$ if and only if $\left( \begin{array}{cc} y & 1 \\ 0 & 1 \end{array} \right) \gamma$ has a fixed point in $\mathcal{F}^\infty$.) Note that $\Delta(\gamma)$ is contained in the half-plane

$$H = \{ (x, y) \in \mathbb{R}^2 \mid y \geq 1 \},$$

since $y = c(-dx - by) + d^2 - d(d - cx - ay) \geq c^2 + d^2 - c|d| \geq 1$.

**Theorem 4.** We have a tesselation

$$H = \bigcup_{\gamma \in \Gamma \setminus \Gamma_\infty} \Delta(\gamma)$$

of the half-plane $H$ into semi-infinite triangles with disjoint interiors.

**Remark.** We can extend the triangulation of Theorem 4 to a triangulation of all of $\mathbb{R}^2$ by triangles labeled by all of $\Gamma$ if we define $\Delta(\gamma)$ also for $\gamma \in \Gamma_\infty$ by

$$\Delta(\left( \begin{array}{cc} 1 & n \\ 0 & 1 \end{array} \right)) = [-n - 1, -n] \times (-\infty, 1],$$

and can then interpret the extended triangulation as giving a piecewise-linear action of $\Gamma$ on $\mathbb{R}^2$, with each triangle being a fundamental domain. However we will not use this in the sequel.

**Proof.** The group $\Gamma$ is a free product of its two subgroups generated by the elements $S = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$ and $U = \left( \begin{array}{cc} 0 & -1 \\ 1 & 1 \end{array} \right)$ of orders 2 and 3, respectively, which fix the two corners of $\mathcal{F}^\infty$. Therefore we can view elements of $\Gamma$ as words in $S, U, U^2$ or as vertices of the tree shown in Figure 1. The proof of both Theorems 3 and 4 will follow from the following decomposition into triangles with disjoint interiors:

$$\mathcal{R} := \{ (x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq y - 1 \} = \bigcup_{\gamma \in \mathcal{R}} \Delta(\gamma),$$

![Figure 1. A tree associated to $\Gamma = \text{PSL}_2(\mathbb{Z})$: the vertices are labeled by the elements of $\Gamma$ and the edges by the generators $S, U$ and $U^2$ as shown.](image-url)
where $\mathcal{T} \subset \Gamma$ is the set of words starting in $U$. The regions $\mathcal{H}$ and $\mathcal{R}$ and a few triangles corresponding to words of small length are pictured in Figure 2.

Figure 2. The region $\mathcal{R}$ (shaded) and a few triangles $\triangle(\gamma)$. The finite side of a triangle $\triangle(\gamma)$ has been labeled by the final letter of $\gamma$ as a word in $S, U, U^2$, with the same convention as in Figure 1.

To prove (4), let $\mathcal{T} = T^+ \cup T^-$, where $T^+$ consists of the elements of $\mathcal{T}$ ending in $U$ or $U^2$, while $T^- := T^+ S$ consists of those elements ending in $S$. The set $T^+$ can be enumerated recursively by starting at $U$ and replacing $\gamma = (a \ b \ c \ d)$ at each step by

\[ \gamma SU = \left( \frac{a}{c} + \frac{b}{d}, c, d \right), \quad \gamma SU^2 = \left( \frac{a + b}{c}, \frac{b}{d} \right). \]

From this description we easily obtain the following equivalent characterization:

\[ \gamma \in T^+ \iff 0 \leq \frac{-a}{c} < \frac{-b}{d} \leq 1, \quad \gamma \in T^- \iff 0 \leq \frac{-b}{d} < \frac{-a}{c} \leq 1. \]

Alternatively, $T^+$ consists of those $\gamma \in \Gamma$ having $d > 0$.

For $\gamma \in \Gamma \setminus \Gamma_\infty$, the triangle $\triangle(\gamma)$ has two vertices given by

\[ P_3(-ac - bd + bc, c^2 + d^2 - cd), \quad P_2(-ac - bd, c^2 + d^2), \]

connected by a line segment of slope $-\frac{d}{b}$, and it has two infinite parallel sides of slope $\frac{-a}{c}$. For $\gamma \in \mathcal{T}$ we denote by $\mathcal{C}(\gamma) \subset \mathcal{H}$ the half-cone containing $\Delta(\gamma)$, bounded by half-lines of slopes $-c/a$ and $-b/d$, and having as vertex $P_3$ or $P_2$, depending on whether $\gamma \in T^+$ or $\gamma \in T^-$ respectively (see Figure 3).

Using this information, it is easy to check that for $\gamma \in T^+$ and $\gamma' = \gamma S \in T^-$ we have the following decompositions into sets with disjoint interiors (see the right picture in Figure 3):

\[ \mathcal{C}(\gamma) = \Delta(\gamma) \cup \mathcal{C}(\gamma') \text{,} \quad \mathcal{C}(\gamma') = \Delta(\gamma') \cup \mathcal{C}(\gamma U) \cup \mathcal{C}(\gamma' U^2). \]

By induction we obtain that $\mathcal{R} = \mathcal{C}(U)$ is the union of the triangles indexed by $\mathcal{T}$, proving (4).

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1Recall our convention that $c > 0$. 

Finally we show that the decomposition in (4) implies the decomposition of $\mathcal{H}$ given in Theorem 4. From the parenthetical remark following (3) it is clear that

$$\Delta(T\gamma) = T\Delta(\gamma),$$

where $T = SU = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\Gamma_\infty$ acts on $\mathcal{H}$ by $T^n(x, y) = (x - ny, y)$. The region

$$(5) \quad \mathcal{R}' = \mathcal{R} \cup \Delta(U^2) = \{(x, y) \in \mathcal{H} : 0 \leq x < y\}$$

(see Figure 2) is a fundamental domain for this action of $\Gamma_\infty$ on $\mathcal{H}$, and we obtain the following decompositions into triangles with disjoint interiors

$$\{(x, y) \in \mathcal{H} \mid y - 1 \leq x\} = \bigcup_{\gamma \in \mathcal{T}'} \Delta(\gamma), \quad \{(x, y) \in \mathcal{H} \mid x \leq 0\} = \bigcup_{\gamma \in \mathcal{T}''} \Delta(\gamma),$$

where $\mathcal{T}'$ consists of words starting with $U^2$, but different from $(U^2S)^n = T^{-n}$ with $n > 0$, while $\mathcal{T}''$ consists of words starting with $S$, but different from $(SU)^n = T^n$ with $n > 0$. Theorem 4 follows since $\Gamma \setminus \Gamma_\infty = \mathcal{T} \cup \mathcal{T}' \cup \mathcal{T}''$.  

3. Proof of Theorem 3

Since (2) is invariant under multiplying $M = \begin{pmatrix} \gamma & x \\ 0 & 1 \end{pmatrix}$ on the right by elements in $\Gamma_\infty$, we assume without loss of generality that $0 \leq x < y$. If $M\gamma$ is scalar for $\gamma \in \Gamma$, the only possibility is easily seen to be $M = 1$. In this case, $\alpha(\gamma) \neq 0$ for $\gamma \in \{1, S, U, U^2\}$, and (2) holds since $-\frac{1}{12} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{1}{2}$.

Assuming that $M \neq 1$, it follows that $\alpha(M\gamma) \neq 0$ if and only if $M\gamma$ has a fixed point in $\mathcal{F}^-$, that is $(x, y) \in \Delta(\gamma)$. We conclude from Section 2 that $y \geq 1$, so the point $(x, y)$ belongs to the region $\mathcal{R}'$ in (3), and $\gamma = U^2$ or $\gamma \in \mathcal{T}$ by (2). Therefore the elements $\gamma$ such that $\alpha(M\gamma) \neq 0$ depend on the position of the point $(x, y)$ with respect to the triangulation of $\mathcal{R}'$ as follows (see Figure 3):

- $y = 1$ and $0 < x < 1$: $\alpha(MU^2) = 1/2$;
- $(x, y)$ is in the interior of a triangle $\Delta(\gamma)$: $\alpha(M\gamma) = 1$;
• \((x, y)\) is on a common side between \(\triangle(\gamma)\) and \(\triangle(\gamma')\), but it is not a vertex:

\[
\alpha(M\gamma) + \alpha(M\gamma') = \frac{1}{2} + \frac{1}{2} = 1;
\]

• \((x, y) \in \mathcal{R}\) is the \(P_2\) vertex of the triangle \(\Delta(\gamma)\) for \(\gamma \in T^+\):

\[
\alpha(M\gamma) + \alpha(M\gamma S) + \alpha(M\gamma U) = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1;
\]

• \((x, y) \in \mathcal{R}\) is the \(P_3\) vertex of \(\Delta(\gamma')\) with \(\gamma' \in T^-\):

\[
\alpha(M\gamma') + \alpha(M\gamma' U) + \alpha(M\gamma' U^2) + \alpha(M\gamma' S) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{2} = 1.
\]

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