Recently Knill, Laflamme, and Milburn (KLM) have proposed a probabilistic scheme for efficient quantum computation using only linear optical elements, sources of entangled photons and efficient photodetectors. This scheme, based on Gottesman and Chuang's discovery that universal quantum computation requires only teleportation and single qubit operations, has attracted much attention since it provides a very interesting alternative to schemes based on controlled nonlinear qubit-qubit interactions, which are extremely difficult to implement on a larger scale. One of the practical challenges of the KLM scheme, however, is the requirement of photodetectors that can distinguish between zero, one and two photons. State of the art photon counters have rather small quantum efficiencies, and high efficiency avalanche detectors cannot discriminate between one and two photons. Another basic requirement of the KLM proposal, as in any photon-based quantum computation scheme, is the ability to generate single-photon wave packets. In this paper we show that both tasks may be solved by making use of resonant nonlinear optical processes based on electromagnetically induced transparency (EIT).

EIT and similar interference effects have led to a new regime of nonlinear optics on the level of few light quanta as they allow making use of the resonantly enhanced nonlinear interaction without suffering from linear absorption and refraction. The potential for strong interactions between individual photons makes EIT-based nonlinear optics a promising candidate for the implementation of quantum gate operations. However, although it has been shown that resonant nonlinear interactions are strong enough to obtain, for example, a conditional phase shift of one photon by another, no scheme exists that allows a conditional phase shift of one photon by another, so far no scheme exists that allows a controlled qubit-qubit interaction for photon wave packets. The situation is different, however, if only one of the interacting fields is a pulse, as is the case in our scheme.

In this paper we show that resonant four-wave mixing in an atomic double lambda configuration with one strong coherent input can be used to filter out a single-photon wave packet from a given input. The same filtering technique can be used to select components of an incoming wave packet according to their photon number and direct them to different high efficiency avalanche photodetectors. In this way efficient photodetectors can be built that can distinguish between zero, one and two photons.

We consider resonantly enhanced forward four-wave mixing in the modified double-lambda system shown in Fig. 1, where nonlinear phase shifts are eliminated by AC-Stark compensation.

FIG. 1: left: Four-wave mixing in a modified double-Λ system with \( \text{sgn}[d_{22}/d_{11}] = -\text{sgn}[d_{32}/d_{31}] \), with \( d_{ij} \) being the dipole moment of the \( |i \rangle - |j \rangle \) transition. right: Cascade detection system consisting of repeated sections of nonlinear medium and beamsplitter. In each section the \( \Omega_1 \) field is partially converted to the \( E_1 \) and \( E_2 \) fields, which are then diverted by the beamsplitter for filtering or detection. Here two fields with slowly varying amplitudes \( \Omega_1 \) and \( \Omega_2 \) are initially excited and form the pump fields. The other fields with slowly varying amplitudes \( E_1 \) and \( E_2 \) are assumed to be initially zero. \( \Omega_1 \) and \( E_1 \) are taken to be exactly on resonance, while the other two fields are detuned by an amount \( \Delta \). Decay from the two lower levels is considered to be negligible and all fields have the same propagation direction. Because of energy conservation there is overall four-photon resonance. It can also be shown that phase matching will favor pairwise two-photon resonance with the \( |1 \rangle - |2 \rangle \) transition.

Semiclassical solutions to this system show that there is a cycling of energy between the pump and generated fields with unit efficiency. In general quantum effects prevent perfect conversion, but conversion efficiency of...
unity or close to unity is possible in certain cases. Due to the resonant nature of the system, the distance required for a complete cycle becomes shorter as the input power is reduced, which is in sharp contrast to ordinary off-resonant four-wave mixing.

This dependence of conversion length on input power raises the possibility of using the double-lambda system for the creation of single-photon states: One chooses the length of the nonlinear medium such that a single-photon input state performs exactly one cycle to the generated fields and back again, i.e. the output state at the exit of the medium is the same as the input, at least up to a phase. Inputs consisting of higher photon numbers, however, will undergo only a partial cycle, resulting in a lower chance of finding photons in the original high photon number state at the exit of the medium. If correctly filtered after each passage, multiple stages successively reduce higher photon number states, until eventually all that is left is a single photon state and vacuum. As we will show, the process of filtering also allows for the use of such a cascade system as an effective detector discriminating between single and double photons.

We assume that the drive field $\Omega_{\text{out}}$ is strong, and treat it classically, while the other three fields are treated fully quantum mechanically. We use the setup shown in Fig. 1 which consists of repeated stages of a length of nonlinear medium followed by a beamsplitter. Three stages are shown, but in principle any number can be used. At the input we assume that the strong classical field, $\Omega_{\text{out}}$, is present, as is the few-photon quantum wave packet described by the operator $\hat{\Omega}_1$. The generated fields $\hat{E}_1$ and $\hat{E}_2$ are assumed to be in the vacuum state at the input. The detectors $D^i$ are standard, high-efficiency avalanche photodetectors that will register the presence of a photon with near certainty, but are unable to discriminate between single and multiple photons. The beamsplitters are considered transmitting for the two fields $\hat{\Omega}_1$ and $\hat{\Omega}_2$ and reflective for the fields $\hat{E}_1$ and $\hat{E}_2$. This could be accomplished for example by choosing orthogonal linear polarization for the $\Omega_j$ and $E_j$ fields and using polarizing beam-splitters.

The effective multi-mode Hamiltonian describing the interaction between the fields in the presence of the nonlinear medium is given by [9]

$$\hat{H}_{\text{int}} = \frac{\hbar g c}{\Delta} \int dz \frac{\hat{\Omega}_1^\dagger \hat{\Omega}_2^\dagger \hat{E}_1 \hat{E}_2 + \hat{E}_1^\dagger \hat{E}_2^\dagger \hat{\Omega}_1 \hat{\Omega}_2}{\hat{\Omega}_1^\dagger \hat{\Omega}_1 + \hat{E}_1^\dagger \hat{E}_1}$$  \hspace{2cm} (1)

Here $\hat{\Omega}_1(z), \hat{\Omega}_1^\dagger(z)$ etc. denote dimensionless, slowly-varying (both in time and space) positive and negative-frequency components of the corresponding electric field operators. The three quantum fields $\hat{\Omega}_1$, $\hat{E}_1$, and $\hat{E}_2$ are taken to be independent and thus commute. The commutator between operators $\hat{O}$ and $\hat{O}^\dagger$ corresponding to the same field is approximately a spatial delta function. The coupling constant $g$ in (1) is given by $g = 3N\lambda^2\gamma/(8\pi)$, where $N$ is the atomic number density, $\lambda$ some typical wavelength of the fields and $\gamma$ the typical radiative decay rate. Since the numerator and denominator in (1) commute there is no ambiguity with respect to ordering. The difference between the resonant four-wave mixing process and ordinary off-resonant systems is in the unusual denominator of (1). It results from the saturation of the two-photon transition $|1\rangle - |2\rangle$ by the resonant fields $\hat{\Omega}_1$ and $\hat{E}_1$. Due to the saturation denominator the atomic system has the largest nonlinear response when $\hat{\Omega}_1$ and $\hat{E}_1$ are small.

It can be shown that the operator equations of motion are given by [12]

$$\partial_t(\hat{\Omega}_1^\dagger \hat{\Omega}_1 + \hat{E}_1^\dagger \hat{E}_1) = 0$$

$$\partial_t(\hat{E}_1^\dagger \hat{E}_1 - \hat{E}_2^\dagger \hat{E}_2) = 0$$

$$\partial_t(\hat{\Omega}_1^\dagger \hat{\Omega}_2 \hat{E}_1 \hat{E}_2 + \hat{\Omega}_1 \hat{\Omega}_2 \hat{E}_1^\dagger \hat{E}_2^\dagger) = 0.$$  \hspace{2cm} (3)

These equations indicate that whenever the $\hat{\Omega}_1$ field loses a photon, both $\hat{E}_1$ and $\hat{E}_2$ must gain one. In addition, the photon number of fields $\hat{E}_1$ and $\hat{E}_2$ are perfectly correlated — they will always lose or gain a photon together.

The interaction of few-photon pulses with the double-lambda medium can most easily be described in terms of state vectors in the co-moving frame $(\zeta, \tau)$. The local character of the interaction (1) and the constants of motion (3) allow a reduction of the multi-mode problem to a small number of states depending on the photon number of fields $\hat{\Omega}_1$ and $\hat{E}_1$ and $\hat{E}_2$.

$$i\hbar \partial_\tau |\psi(\tau)\rangle = \hat{H}_{\text{int}} |\psi(\tau)\rangle.$$  \hspace{2cm} (4)

The local character of the interaction (1) and the constants of motion (3) allow a reduction of the multi-mode problem to a small number of states depending on the total number of photons. For example, if we consider an initial single-photon wave packet in $\hat{\Omega}_1$ and vacuum in $\hat{E}_1$ and $\hat{E}_2$, then the initial state is given by

$$|\psi(\tau = 0)\rangle = |\psi^{(1)}_0\rangle \sim \int d\zeta f(\zeta) \hat{\Omega}_1^\dagger(\zeta) |0\rangle \equiv |1, 0, 0\rangle,$$  \hspace{2cm} (5)

where $f(\zeta)$ characterizes the shape of the wave packet and is called the single-photon wave-function. The interaction Hamiltonian couples $|\psi^{(1)}_0\rangle$ to only one other state, namely

$$|\psi^{(1)}_1\rangle \sim \int d\zeta f(\zeta) \hat{E}_1^\dagger(\zeta) \hat{E}_2(\zeta) |0\rangle \equiv |0, 1, 1\rangle.$$  \hspace{2cm} (6)

which represents a two-photon wave packet in the two generated fields and in turn is coupled only back to $|\psi^{(1)}_0\rangle$. Thus for the given input the multi-mode problem can be mapped onto a two-state one. We have labeled the two states as $|1, 0, 0\rangle$ and $|0, 1, 1\rangle$, which denote the total number of photons in $\hat{\Omega}_1$, $\hat{E}_1$, and $\hat{E}_2$. The effect of the effective multi-mode Hamiltonian is described in terms of state vectors in the co-moving frame $(\zeta, \tau)$. The local character of the interaction (1) and the constants of motion (3) allow a reduction of the multi-mode problem to a small number of states depending on the total number of photons. For example, if we consider an initial single-photon wave packet in $\hat{\Omega}_1$ and vacuum in $\hat{E}_1$ and $\hat{E}_2$, then the initial state is given by

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$$|\psi^{(1)}_1\rangle \sim \int d\zeta f(\zeta) \hat{E}_1^\dagger(\zeta) \hat{E}_2(\zeta) |0\rangle \equiv |0, 1, 1\rangle.$$  \hspace{2cm} (6)
Similarly, if the input is a two-photon wave packet in $\Omega_1$

$$|\psi(\tau = 0)\rangle = |\psi_0^{(2)}\rangle \sim \int d\zeta d\zeta' f_2(\zeta, \zeta') \hat{\Omega}_1^\dagger(\zeta) \hat{\Omega}_1^\dagger(\zeta') |0\rangle$$

$$\equiv |2, 0, 0\rangle$$

the interaction Hamiltonian couples it only to the states

$$|\psi_1^{(2)}\rangle \sim \int d\zeta d\zeta' f_2(\zeta, \zeta') \hat{E}_1^\dagger(\zeta) \hat{E}_2^\dagger(\zeta') \hat{\Omega}_1^\dagger(\zeta') |0\rangle$$

$$\equiv |1, 1, 1\rangle$$

and

$$|\psi_2^{(2)}\rangle \sim \int d\zeta d\zeta' f_2(\zeta, \zeta') \hat{E}_1^\dagger(\zeta) \hat{E}_1^\dagger(\zeta') \hat{E}_2^\dagger(\zeta') |0\rangle$$

$$\equiv |0, 2, 2\rangle.$$  

We thus can now introduce a simple notation to label all relevant states of the system. If we assume that there are initially $n$ photons in field $\Omega_1$ and vacuum in $E_1$ and $E_2$ then, due to the constants of motion, we see that we can choose a basis for the radiation field that has the form $|n-j, j, j\rangle$, i.e. a basis indicating how many photons are in each field at any one time.

Let us consider how the system can be used as a photon filter. To do this we need to know what happens to single- and multi-photon input packets as they pass through each stage of nonlinear medium and beamsplitter. We first consider the case where a single-photon wave packet is injected into the system in the field $\Omega_1$. To determine the evolution of the wave packet in the nonlinear medium we solve the Schrödinger equation and apply the initial condition that $|\psi(0)\rangle = |1, 0, 0\rangle$. We find the solution to be

$$|\psi(\tau)\rangle = \cos[\kappa|\Omega_2|\tau] |1, 0, 0\rangle - i \sin[\kappa|\Omega_2|\tau] |0, 1, 1\rangle$$

where $\kappa = g/\Delta$ showing that the pulse cycles smoothly between the pump and generated fields. If we choose the length of the nonlinear medium to be a multiple of $L_0 \equiv \pi/(\kappa|\Omega_2|)$ we see that on exiting the medium the pulse is once again entirely in the $\Omega_1$ field, with no component of the $E_1$ and $E_2$ fields excited. The shape of the pulse has an identical shape to that of the input, but has undergone a phase shift of multiples of $\pi$.

Now consider the effect of the nonlinear medium on an input pulse including higher photon numbers. Due to the constants of motion, each distinct number state of the initial wave packet will evolve separately in its own subspace. Thus, for example, the $n$-photon component of the initial state can be considered to evolve distinctly from the $n-1$ component. The action of the nonlinear medium on an incoming $n$-photon state and the subsequent elimination of photons in the $E_1$/$E_2$ modes by the beamsplitter is given by

$$|n, 0, 0\rangle \rightarrow \sum_{j=0}^{n} \xi_j^{(n)} |n-j, j, j\rangle \rightarrow \sum_{j=0}^{n} \xi_j^{(n)} |n-j, 0, 0\rangle.$$  

The overbar denotes the fact that the coefficient may have changed phase after the beamsplitter, but its absolute value is unchanged. As the $\xi_j$ all have magnitudes less than one, it is clear that each stage reduces the chance of finding the $\Omega_1$ field in photon occupation modes higher than one. Consequently the system converges on some combination of the $|1, 0, 0\rangle$ state and the vacuum state, both of which are unaffected by the optical elements. The system therefore serves as a method for generating single-photon wave packets. As an example of the efficiency of the scheme, in Figure 2 we present numerical results for a coherent input and choose interaction lengths of $L_0$ and $2L_0$.

As a second possible application, we now demonstrate that the system also serves as a detector capable of discriminating between one and two photons with high efficiency. To do this we need to consider the dynamics of the system more carefully, taking the state of the detectors placed at the exit ports $D^j$ into account and analytically finding values for the $\xi_j$ coefficients.

We indicate the state of the detector subsystem by $|D_1D_2...\rangle$, signifying which detectors have fired. For example $D_0^0$ indicates that no detectors have fired, $D_1^1$ would indicate that detectors $D^1$ and $D^3$ have fired and so on.

First consider a single-photon pulse $|1, 0, 0\rangle$, which is sent into the system with interaction segments of length $L_0$. After each traversal of a section of nonlinear medium it will be in that same state, with a phase shift of $\pi$. The beamsplitter also leaves it in the same state, again albeit with a possible change in phase. This is repeated for each stage in the apparatus, until it reaches the $D^\infty$ detector. Thus, if a single-photon wave packet is injected into the system, none of the detectors put at the exits $D^1$ to $D^n$ will fire, while the $D^\infty$ detector will fire with certainty. This, then, is the signature of a single photon pulse.

Let us now consider the case where a two-photon pulse is injected into the system. The action of the first nonlinear medium and the subsequent beamsplitter on the
input state is given by
\[ |200\rangle \rightarrow \xi_0^{(2)}|200\rangle + \xi_1^{(2)}|111\rangle + \xi_2^{(2)}|022\rangle \]
\[ \rightarrow \bar{\xi}_0^{(2)}|200\rangle|D^0\rangle + \bar{\xi}_1^{(2)}|100\rangle|D^1\rangle + \bar{\xi}_2^{(2)}|000\rangle|D^1\rangle \]
with the overbar indicating that some phase shift may have taken place, i.e. \(|\bar{\xi}_0^{(2)}\rangle = |\xi_0^{(2)}\rangle\), \(|\bar{\xi}_2^{(2)}\rangle = |\xi_2^{(2)}\rangle\). If we once again assume the length of the nonlinear medium is such that a single photon state will undergo a simple sign change, after \(n\) stages of nonlinear medium and beamsplitter we find that the state of the input pulse reads

\[ \xi_0^{(2)n}|200\rangle|D^0\rangle + \sum_{k=0}^{n-1} (-1)^{k+n} \xi_0^{(2)k}\xi_1^{(2)}|D^{k+1}\rangle|100\rangle + \sum_{k=0}^{n-1} \xi_0^{(2)k}\xi_2^{(2)}|D^{k+1}\rangle|000\rangle. \]

This shows that three results are possible: \(D^\infty\) fires and none of the \(D^i\)‘s does; \(D^\infty\) fires and one of the \(D^i\)‘s fires; one of the \(D^i\)‘s fires and \(D^\infty\) does not. The probability of the \(i\)th detector firing is \(P(D^i) = |\xi_0^{(2)}|^{2i-2} - |\xi_0^{(2)}|^2\). Crucially, the chance that only \(D^\infty\) will fire, thus giving a result indistinguishable from the single-photon case, is given by \(|\xi_0^{(2)}|^{2n}\).

In order to obtain quantitative results we need to evaluate the coefficients \(\xi_0^{(2)}, \xi_1^{(2)}\) and \(\xi_2^{(2)}\). The 3 \(\times\) 3 equations of motion for the two-photon case can easily be integrated and yield

\[ \xi_0^{(2)} = \frac{1}{3} \left( 2 + \cos \sqrt{\frac{5}{2}} |\Omega_2| |\tau| \right) \]
\[ \xi_1^{(2)} = -\frac{i}{\sqrt{3}} \sqrt{\frac{3}{5}} |\Omega_2| \sin \sqrt{\frac{5}{2}} |\Omega_2| |\tau| \]
\[ \xi_2^{(2)} = -2 \frac{\sqrt{2}}{3} \frac{|\Omega_2| \sin^2 \sqrt{\frac{5}{2}} |\Omega_2| |\tau|}{\sqrt{3}} \]

As we have assumed the length of the nonlinear medium to be \(\tau = L_0 = \pi/|\Omega_2|\), we obtain the following numerical values for the coefficients: \(|\xi_0^{(2)}| = 0.4130\), \(|\xi_1^{(2)}| = 0.3746\), and \(|\xi_2^{(2)}| = 0.8301\). Inserting these values into (13) we can calculate the state of the wave packet after each stage, and the associated probability of any particular detector firing. It is immediately clear that this scheme can distinguish between one and two photons with great accuracy. For example, if four stages are used, the chance of a two-photon wave packet causing only the \(D^\infty\) detector to fire is \(|0.4130|^4 = 8 \times 10^{-3}\), i.e. one can distinguish between a single-photon wave packet and a two-photon wave packet with greater than 99.9% accuracy. Using similar arguments one can show that the probability of only \(D^\infty\) firing if the input pulse contains more than two photons is even smaller. Thus the cascaded resonant nonlinear system when combined with avalanche detectors can also be used as an efficient photodetector able to discriminate between zero, one and many photons.

It is worth mentioning that a set-up can be built along similar lines in which only the two-photon or any other fixed photon-number component makes a full return after each interaction zone. In this way a photon source or a detector can be built that is tuned for example to a two- or three-photon wavepacket.

In summary we have shown that successive stages of resonant four wave mixing with one strong coherent cw input along with the filtering out of the generated fields can be used to transform any low-photon input wavepacket into a single-photon wavepacket with large probability. A similar set-up combined with avalanche photodiodes can be used to build detectors that can discriminate between zero, one, and two or more photons. Due to the resonant enhancement the required interaction lengths are rather small and can be in the cm range [13].

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