A partition is said to be \((k, d)\)-noncrossing if it avoids \(12 \cdots k 12 \cdots d\). We find an explicit formula of the ordinary generating function for the number of \((k, d)\)-noncrossing partitions of \([n]\) when \(d = 1, 2\).

Keywords: partitions, forbidden subsequences, kernel method.

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same block of \( \Pi \). A partition \( \pi \) is called \( k \)-noncrossing if it avoids \( 12 \cdots k12 \cdots k \) and \( k \)-nonnesting if it avoids \( 12 \cdots kk \cdots 21 \). For further works on this subject, the reader is refereed to Sagan [7] and the references therein.

In this paper, we generalize the concept of \( k \)-noncrossing partitions to \( (k, d) \)-noncrossing partitions. A partition is said to be \( (k, d) \)-noncrossing if it avoids \( 12 \cdots k12 \cdots d \). Let \( \mathcal{N}_{k,d}(n) \) be the set of all \( (k, d) \)-noncrossing partitions of \([n]\). Note that \( \mathcal{N}_{k,k}(n) \) is the set of \( k \)-noncrossing partitions of \([n]\) (see [3]).

For \( d = 0 \), it is easy to see from the definitions that the number of \( (k, 0) \)-noncrossing partitions of \([n]\) is the same as the number of partitions of \([n]\) with at most \( k - 1 \) blocks. Thus,

\[
\# \mathcal{N}_{k,0}(n) = \sum_{i=0}^{k-1} S(n, i),
\]

where \( S(n, i) \) is the Stirling number of the second kind. In this paper we give a complete answer for two cases of \( (n, d) \)-noncrossing partitions, in which \( d \) is either 1 or 2.

| \( k \backslash n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|-----------------|---|---|---|---|---|---|---|---|---|---|----|----|----|
| 2               | 1 | 1 | 2 | 5 | 14| 42 | 132| 429|1430|4862|16796|58786|208012|
| 3               | 1 | 1 | 2 | 5 | 15| 51 | 188| 731|2950|12235|51822|974427|
| 4               | 1 | 1 | 2 | 5 | 15| 52 | 202| 856|3868|18313|89711|450825|2310453|
| 5               | 1 | 1 | 2 | 5 | 15| 52 | 203| 876|4112|20679|109853|608996|3488806|
| 6               | 1 | 1 | 2 | 5 | 15| 52 | 203| 877|4139|21111|115219|666388|4045991|

Table 1. Number the \( (k, 2) \)-noncrossing partitions of \([n]\) for \( k = 2, 3, 4, 5, 6 \) and \( n = 0, 1, \ldots, 12 \).

Table 1 presents the number of \( (k, 2) \)-noncrossing partitions in \( \mathcal{P}_k(n) = \mathcal{N}_{k,2}(n) \), where \( k = 2, 3, 4, 5, 6 \). We will show that the ordinary generating function

\[
\sum_{n \geq 0} \# \mathcal{P}_k(n) x^n
\]

for the number of partitions in \( \mathcal{P}_k(n) \) is rational in \( x \) and \( \sqrt{(1 - kx)^2 - 4x^2} \). Namely, we prove the following result.

**Theorem 1.1.** Let \( k \geq 2 \) and let

\[
y_k = \frac{1 - (k - 2)x - \sqrt{(1 - kx)^2 - 4x^2}}{2x(1 - (k - 2)x)}.
\]

Then the ordinary generating function for the number of \( (k, 2) \)-noncrossing partitions of \([n]\) is given by

\[
\sum_{n \geq 0} \# \mathcal{P}_k(n) x^n = \frac{x^{k-1} y_k}{1 - xy_k} + \frac{\sum_{j=0}^{k-2} \sum_{i=0}^{j} (-1)^{i+j} x^i \beta_{i,j}}{1 - \sum_{j=0}^{k-2} \sum_{i=0}^{j} (-1)^{i+j} x^i \beta_{i,j}},
\]

where \( \beta_{j,j} = 1 \) and

\[
\beta_{i,j} = jx \prod_{s=i+1}^{j-1} (sx - 1)
\]

for \( i = 0, 1, \ldots, j - 1 \).
Theorem 1.1 gives two particular results, namely $k = 2$ and $k = 3$. For $k = 2$,

$$\sum_{n \geq 0} \#\mathcal{P}_2(n)x^n = 1 + \frac{xy_2}{1 - xy_2} = \frac{1}{1 - xy_2} = y_2,$$

where $y_2 = \frac{1 - \sqrt{1 - 4x}}{2x}$. Thus, the number of partitions in $\mathcal{P}_2(n)$ is given by the $n$-th Catalan number. For $k = 3$,

$$\sum_{n \geq 0} \#\mathcal{P}_3(n)x^n = \frac{1 + x^2y_3}{1 - x} = \frac{3 - 3x - \sqrt{1 - 6x + 5x^2}}{2(1 - x)},$$

where $y_3 = \frac{1 - \sqrt{1 - 6x + 5x^2}}{2(1 - x)}$. It follows that the number of partitions in $\mathcal{P}_3(n)$ is given by the $n$-th binomial transform of the Catalan number $\sum_{i=0}^{n}(1)3^{n-i}(n \atop i)(i \atop \lfloor i/2 \rfloor)$ (see [6, Sequence A007313]). For $k = 2$, there is a combinatorial proof that the number of partitions in $\mathcal{P}_2(n)$ is given by the $n$-th Catalan number. For $k = 3$, the formula above counts the number of Schröder paths with no peaks at even level of length $n$ (see [6, Sequence A007317]). It would be interesting to find a bijective proof of this result.

Another bonus from the proof of Theorem 1.1 is the ordinary generating function for the number of $(k, 1)$-noncrossing partition of $[n]$. Specifically, we prove the following result.

**Theorem 1.3.** Let $k \geq 1$. Then the ordinary generating function for the number of $(k, 1)$-noncrossing partitions of $[n]$ is given by

$$\sum_{n \geq 0} N_{k,1}(n)x^n = \frac{1 - x + (1 - x) \sum_{j=1}^{k-2} \sum_{i=0}^{j}(1)i+jx^i\beta_{i,j} + \sum_{i=0}^{k-1}(1)i+k-1x^i\beta_{i,j}}{1 - x - x(1 - x) \sum_{j=1}^{k-2} \sum_{i=0}^{j}(1)i+ji\beta_{i,j} - x \sum_{i=0}^{k-1}(1)i+k-1i\beta_{i,k-1}},$$

where $\beta_{i,j}$ is defined in Theorem 1.1.
For example, Theorem 1.3, for $k = 2, 3, 4, 5, 6$, gives the following ordinary generating functions for the number of $(k, 1)$-noncrossing partitions of $[n]$:

$$\sum_{n \geq 0} \#N_{2,1}(n)x^n = \frac{1-x}{1-2x}, \quad \sum_{n \geq 0} \#N_{3,1}(n)x^n = \frac{1-3x+x^2}{(1-x)(1-3x)}.$$  

$$\sum_{n \geq 0} \#N_{4,1}(n)x^n = \frac{1-6x+9x^2-3x^3}{(1-x)(1-2x)(1-4x)}, \quad \sum_{n \geq 0} \#N_{5,1}(n)x^n = \frac{1-10x+32x^2-37x^3+11x^4}{(1-x)(1-2x)(1-3x)(1-5x)},$$  

$$\sum_{n \geq 0} \#N_{6,1}(n)x^n = \frac{1-15x+81x^2-192x^3+189x^4-53x^5}{(1-x)(1-2x)(1-3x)(1-4x)(1-6x)}.$$  

The numbers of $(k, 1)$-noncrossing partitions of $[n]$ with $k = 2, 3, 4, 5, 6$ are given by

$$\#N_{2,1}(n) = 2^{n-1},$$  

$$\#N_{3,1}(n) = \frac{1}{6}(3^n + 3),$$  

$$\#N_{4,1}(n) = \frac{1}{24}(4^n + 6 \cdot 2^n + 8),$$  

$$\#N_{5,1}(n) = \frac{1}{120}(5^n + 10 \cdot 3^n + 20 \cdot 2^n + 45),$$  

$$\#N_{6,1}(n) = \frac{1}{720}(6^n + 15 \cdot 4^n + 40 \cdot 3^n + 135 \cdot 2^n + 264).$$

2. Proofs

Let us denote by $F_k(x)$ the generating function for the number of partitions in $P_k(n)$:

$$F_k(x) = \sum_{n \geq 0} \#P_k(n)x^n.$$  

Here, instead of dealing with recurrence relations with two indices $n$ and $\ell$, as we mentioned in the introduction, we deal with recurrence relations in terms of ordinary generating functions with one single index $\ell$. Let us denote by $F_{k,\ell}(x)$ the generating function for the number of partitions in $P_{k,\ell}(n)$:

$$F_{k,\ell}(x) = \sum_{n \geq 0} \#P_{k,\ell}(n)x^n.$$  

Here, for the case $\ell = 0$ we have $F_{k,0}(x) = 1$. Clearly, $F_k(x) = \sum_{i \geq 0} F_{k,i}(x)$. Our main result is based on the construction of linear recurrence relations with one single index for the ordinary generating function $F_{k,\ell}(x)$. As we will see later, since the recurrences contain the expression $\sum_{i \geq \ell} F_{k,i}(x)$, we define for clarity

$$G_{k,\ell}(x) = \sum_{i \geq \ell} F_{k,i}(x).$$  

The expression $G_{k,\ell}(x)$ is the ordinary generating function for the number of $j$-increasing partitions of $[n]$, with $j \geq \ell$. It follows directly from the definitions that the generating function $G_{k,\ell}(x)$ is well-defined, since

$$G_{k,\ell}(x) = F_k(x) - \sum_{i = 0}^{\ell-1} F_{k,i}(x).$$  

In our first lemma, we find the recurrence relation for $F_{k,\ell}(x)$, where $1 \leq \ell \leq k-1$.

**Lemma 2.1.** For all $1 \leq \ell \leq k-1$,

$$F_{k,\ell}(x) = \ell x G_{k,\ell}(x) + x^\ell.$$
Proof. Let $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathcal{P}_{k,\ell}(n)$. If $n > \ell$ then $\pi_{\ell+1} \leq \ell$. Thus $\pi \in \mathcal{P}_k(n)$ if and only if the reduced form of $\pi' = \pi_1 \cdots \pi_{\ell+2} \cdots \pi_n$ is a partition in $\bigcup_{j \geq \ell} \mathcal{P}_{k,j}(n-1)$. Therefore,

$$F_{k,\ell}(x) = \ell x (F_{k,\ell}(x) + F_{k,\ell+1}(x) + \cdots) + x^{\ell} = \ell x G_{k,\ell}(x) + x^{\ell},$$

where $x^{\ell}$ counts the unique $\ell$-increasing partition of $[\ell]$, namely $12 \cdots \ell$, as required by the statement.

The above observation together with the the definition of $G_{k,\ell}(x)$ gives the following system:

(2.1)\[
\begin{cases}
F_{k,0}(x) & = 1 \\
x F_{k,0}(x) + F_{k,1}(x) & = x F_k(x) + x \\
2x F_{k,0}(x) + 2x F_{k,1}(x) + F_{k,2}(x) & = 2x F_k(x) + x^2 \\
& \vdots \\
(k-1)x F_{k,0}(x) + \cdots + (k-1)x F_{k,k-2}(x) + F_{k,k-1}(x) & = (k-1)x F_k(x) + x^{k-1}.
\end{cases}
\]

Next, we find an explicit formula for $F_{k,\ell}(x)$ in terms of $F_k(x)$.

Lemma 2.2. For all $1 \leq \ell \leq k-1$,

$$F_{k,\ell}(x) = \sum_{i=0}^{\ell} (-1)^{i+\ell} (ix F_k(x) + x^i) \beta_{i,\ell},$$

where $\beta_{\ell,\ell} = 1$ and $\beta_{i,\ell} = \ell x \prod_{j=i+1}^{\ell-1} (j x - 1)$ for $i = 0, 1, \ldots, \ell - 1$.

Proof. With the use of Cramer’s Rule on (2.1), we obtain

$$F_{k,\ell}(x) = \sum_{i=0}^{\ell} (-1)^{i+\ell} (ix F_k(x) + x^i) \begin{vmatrix}
(i+1)x & 1 & 0 & \cdots & 0 & 0 \\
(i+2)x & (i+2)x & 1 & \cdots & 0 & 0 \\
& \vdots & & & & \\
(\ell-1)x & (\ell-1)x & (\ell-1)x & \cdots & (\ell-1)x & 1 \\
\ell x & \ell x & \ell x & \cdots & \ell x & \ell x
\end{vmatrix}.$$ 

By making use of the formula

$$\begin{vmatrix}
ax & 1 & 0 & \cdots & 0 & 0 \\
(a+1)x & (a+1)x & 1 & \cdots & 0 & 0 \\
& \vdots & & & & \\
(b-1)x & (b-1)x & (b-1)x & \cdots & (b-1)x & 1 \\
bx & bx & bx & \cdots & bx & bx
\end{vmatrix} = bx \prod_{j=a}^{b-1} (j x - 1),$$

which holds by induction on $b \geq a$, we obtain

$$F_{k,\ell}(x) = \sum_{i=0}^{\ell} (-1)^{i+\ell} (ix F_k(x) + x^i) \beta_{i,\ell},$$

as claimed. \qed
Now, before completing the proof of our main result, Theorem 1.1, let us present two applications of Lemma 2.2. The first one is the ordinary generating function for the number of partitions in \( Q_\ell(n) \); the second one is the ordinary generating function for the number of \((k,1)\)-noncrossing partitions of \([n]\).

### 2.1. Enumerating partitions in \( Q_\ell(n) \).

The formula of the ordinary generating function \( I_\ell(x) \) for the number of partitions in \( Q_\ell(n) \) can be obtained as follows. From the definition of the set \( Q_\ell(n) \) and from the proof of Lemma 2.2 for \( \ell < k \), we obtain that the ordinary generating function for the number of \( m \)-increasing partitions in \( Q_\ell(n) \) is given by

\[
I_{\ell,m}(x) = \sum_{i=0}^{m} (-1)^{i+m}(ixI_{\ell}(x) + x^i)\beta_{i,m}.
\]

On the other hand, \( I_\ell(x) = \sum_{m=0}^{\ell} I_{\ell,m}(x) \). Combining these two equations, we obtain

\[
I_\ell(x) = \sum_{j=0}^{\ell} I_{\ell,j}(x) = 1 + \sum_{j=1}^{\ell} \sum_{i=0}^{j} (-1)^{i+j}(ixI_{\ell}(x) + x^i)\beta_{i,j}.
\]

The solution of this equation gives a formula for \( I_\ell(x) \), as stated in Corollary 1.2.

### 2.2. Enumerating \((k,1)\)-noncrossing partitions of \([n]\).

Let \( J_k(x) \) be the ordinary generating function for the number of \((k,1)\)-noncrossing partitions of \([n]\), that is,

\[
J_k(x) = \sum_{n\geq 0} \#N_{k,1}(n)x^n.
\]

More generally, let \( J_{k,\ell}(x) \) be the ordinary generating function for the number of \((k,1)\)-noncrossing \( \ell \)-increasing partitions of \([n]\). Then, a similar argument as in the proof of Lemma 2.2 gives that

\[
J_{k,\ell}(x) = \sum_{i=0}^{\ell} (-1)^{i+\ell}(ixJ_{k}(x) + x^i)\beta_{i,\ell}, \quad \ell = 1, 2, \ldots, k - 1,
\]

with \( J_{k,0}(x) = 1 \). On the other hand,

\[
J_{k,\ell}(x) = x^{\ell+1-k}J_{k,k-1}(x), \quad \ell = k, k + 1, k + 2, \ldots.
\]

To prove this observation, let \( \pi \) be any \((k,1)\)-noncrossing \( \ell \)-increasing partition of \([n]\) with \( \ell \geq k \). Then \( \pi_1\pi_2\cdots\pi_\ell = 12\cdots\ell \) and \( \pi_{\ell+1} < \ell + 1 \). Since \( \pi \) avoids \( 12\cdots k1 \), then \( \pi_i \notin \{1,2,\ldots,\ell+1-k\} \) for all \( i \geq \ell + 1 \). Thus, the number of \((k,1)\)-noncrossing \( \ell \)-increasing partition of \([n]\) with \( \ell \geq k \) is the same as the number of \((k,1)\)-noncrossing \((k-1)\)-increasing partition of \([n-\ell-1+k]\). This is equivalent to \( J_{k,\ell}(x) = x^{\ell+1-k}J_{k,k-1}(x) \), for all \( \ell \geq k \). Therefore, using the fact that \( J_k(x) = \sum_{\ell \geq 0} J_{k,\ell}(x) \), and the two equations (2.2) and (2.3), we can write

\[
J_k(x) = 1 + \sum_{\ell=0}^{k-1} \sum_{i=0}^{\ell} (-1)^{i+\ell}(ixJ_k(x) + x^i)\beta_{i,\ell} + \sum_{\ell \geq k} x^{\ell+1-k}J_{k,k-1}(x).
\]

Again, by (2.2), we have

\[
J_k(x) = 1 + \sum_{\ell=0}^{k-1} \sum_{i=0}^{\ell} (-1)^{i+\ell}(ixJ_k(x) + x^i)\beta_{i,\ell} + \frac{x}{1-x} \sum_{i=0}^{k-1} (-1)^{i+k-1}(ixJ_k(x) + x^i)\beta_{i,k-1}.
\]
The solution of this equation gives a formula for $J_k(x) = \sum_{n \geq 0} \#N_{k,1}(n)x^n$ as stated in Theorem 1.3.

2.3. Proof of Theorem 1.1. We need some extra notation before completing the proof of Theorem 1.1. Let

$$H_k(x, y) = \sum_{\ell=0}^{k-2} F_{k,\ell}(x)y^\ell$$

and

$$F_k(x, y) = \sum_{\ell \geq 0} F_{k,\ell}(x)y^\ell.$$ 

From Lemma 2.2, we can observe that

$$F_k(x, 1) = F_k(x),$$

where $\beta_{i,\ell} = 1$ and $\beta_{i,\ell} = \ell x \prod_{j=i+1}^{\ell-1} (jx - 1)$, for $i = 0, 1, \ldots, \ell - 1$. Now, let us focus on the generating functions $F_{k,\ell}(x)$, where $\ell \geq k - 1$.

Lemma 2.3. For all $j \geq 0$,

$$F_{k,k-1+j}(x) = x^{k-1+j} + \sum_{i=0}^{j-1} x^{j+1-i} G_{k,k-1+i}(x) + (k-1)x G_{k,k-1+j}(x).$$

Proof. The case $j = 0$ holds on the basis of Lemma 2.1. Let us assume that $j \geq 1$. Let $\pi = \pi_1\pi_2 \cdots \pi_n$ be any partition in $\mathcal{P}_{k,k-1+j}(n)$ such that $\pi_{k+j} = i \leq k-1+j$. We want the equation of the generating function for the number partitions in $\mathcal{P}_{k,k-1+j}(n)$, namely $F_{k,k-1+j}(x)$. Let us consider the following two cases:

- If $1 \leq i \leq j$ then $\pi$ is such that $\pi_p \notin \{i+1, i+2, \ldots, j+1\}$, where $p \geq k+1+j$. Thus, the contribution of this case is

  $$x^{j+1-i}(F_{k,k-1+i}(x) + F_{k,k+i}(x) + \cdots) = x^{j+1-i} G_{k,k-1+i}(x).$$

- If $j+1 \leq i \leq k-1+j$ then $\pi$ satisfies the above conditions if and only if the reduced form of $\pi_1 \cdots \pi_{k-1+j} \pi_{k-1+j+1} \cdots \pi_n$ is a partition in $\bigcup_{i \geq 0} \mathcal{P}_{k,k-1+j+i}(n-1)$. Thus, the contribution of this case is

  $$x(F_{k,k-1+j}(x) + F_{k,k+j}(x) + \cdots) = x G_{k,k-1+j}(x).$$

Putting together the above cases, $i = 1, 2, \ldots, k-1+j$, we obtain that

$$F_{k,k-1+j}(x) = x^{k-1+j} + \sum_{i=0}^{j-1} x^{j+1-i} G_{k,k-1+i}(x) + (k-1)x G_{k,k-1+j}(x),$$

where $x^{k-1+j}$ counts the unique $(k-1+j)$-increasing partitions of $[k-1+j]$, namely $12 \cdots (k-1+j)$. \hfill $\square$

Now we have a formula for the generating function $F_k(x, y)$:
Proposition 2.4. We have
\[(1 + \frac{x^2y^2}{(1-y)(1-xy)} + \frac{(k-1)xy}{1-y}) (F_k(x, y) - H_k(x, y))\]
\[= \frac{xy}{1-xy} + \frac{k-1}{1-y} \left(\frac{x^2y}{1-xy} + (k-1)x\right) (F_k(x, 1) - H_k(x, 1)).\]

Proof. Lemma 2.1 together with Lemma 2.3 give
\[F_k(x, y) = \frac{1}{1-xy} + \sum_{j=1}^{k-2} jxG_{k,j}(x)y^j + (k-1)x \sum_{j \geq k-1} G_{k,j}(x)y^j + \frac{x^2y}{1-xy} \sum_{j \geq k-1} G_{k,j}(x)y^j\]
which is equivalent to
\[(1 + \frac{x^2y^2}{(1-y)(1-xy)} + \frac{(k-1)xy}{1-y}) (F_k(x, y) - H_k(x, y))\]
\[= \frac{xy}{1-xy} + \frac{k-1}{1-y} \left(\frac{x^2y}{1-xy} + (k-1)x\right) (F_k(x, 1) - H_k(x, 1)),\]

The functional equation in the statement of Proposition 2.4 can be solved systematically using the kernel method technique (see [1]). Let
\[y = y_k = \frac{1 - (k-2)x - \sqrt{(1-kx)^2 - 4x^2}}{2x(1-(k-2)x)}\]
be one of the roots of the equation \(1 + \frac{x^2y^2}{(1-y)(1-xy)} + \frac{(k-1)xy}{1-y}\). Then Proposition 2.4 gives
\[F_k(x, 1) - H_k(x, 1) = \frac{x^{k-1}y_k}{1-xyk}.\]
Therefore, Lemma 2.2 gives
\[F_k(x, 1) - \sum_{j=0}^{k-2} \sum_{i=0}^{j} (-1)^{i+j}(ixF_k(x, 1) + x^j)\beta_{i,j} = \frac{x^{k-1}y_k}{1-xyk},\]
which implies our main result (see Theorem 1.1).

Theorem 2.5. The ordinary generating function for the number of 12⋯k12-avoiding partitions of \(n\), namely \(F_k(x) = F_k(x, 1)\), is given by
\[\frac{x^{k-1}y_k}{1-xyk} + \frac{\sum_{j=0}^{k-2} \sum_{i=0}^{j} (-1)^{i+j}x^j\beta_{i,j}}{1 - \sum_{j=0}^{k-2} \sum_{i=0}^{j} (-1)^{i+j}ix\beta_{i,j}},\]
where \(\beta_{1,j} = 1\) and \(\beta_{i,j} = jx \prod_{s=i+1}^{j-1}(sx - 1),\) for \(i, j = 0, 1, \ldots, j-1\).

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REFERENCES

[1] C. Banderier, M. Bousquet-Méélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauxchamps, Generating functions for generating trees, Formal Power Series and Algebraic Combinatorics (Barcelona, 1999), *Discr. Math.* **246**:1-3 (2002) 29–55.

[2] William Y. C. Chen, Eva Y.P. Deng, and Rosena R.X. Du, Reduction of m-regular noncrossing Partitions, *Europ. J. Combin.* **26**:2 (2005) 237–243.

[3] William Y. C. Chen, Eva Y.P. Deng, Rosena R.X. Du, R. P. Stanley, and Catherine H. Yan, Crossings and Nestings of Matchings and Partitions, *Trans. Amer. Math. Soc.*, to appear. arXiv: math.CO/0501230.

[4] M. Klazar, On abab-free and abba-free set partitions, *Europ. J. Combin.* **17** (1996) 53–68.

[5] M. Klazar, On trees and noncrossing partitions, *Discr. Appl. Math.* **82** (1998) 263–269.

[6] N.J.A. Sloane, The Online Encyclopedia of Integer Sequences, [www.research.att.com/~njas/sequences/](http://www.research.att.com/~njas/sequences/).

[7] B. E. Sagan, Pattern avoidance in set partitions, preprint.

[8] R. P. Stanley, Enumerative Combinatorics, Vol. 1, Cambridge University Press, Cambridge, UK, 1996.

[9] M. Wachs and D. White, p,q-Stirling numbers and set partition statistics, *J. Combin. Theory, Series A*, **56**:1 (1991) 27–46.