The non-relativistic formalism introduced by Berry and Robbins that naturally incorporates the spin-statistics connection is generalized relativistically. It is then extended to an arbitrary Kaluza-Klein space-time by a suitable generalization of the Schwinger treatment of angular momenta. This leads, in this approach, to the inclusion of the ‘internal’ quantum numbers in the spin-statistics connection on an equal footing with spin.
Two particles are said to be identical or indistinguishable if in any state of any physical system containing these particles, interchanging them would not lead to a state that could be experimentally distinguished, even in principle, from the original state. Formally, we may say that both particles belong to the same irreducible representation of the symmetry group of physics. Then the two particles have the same quantum numbers that couple to external fields, such as mass, spin, color, weak isospin and hypercharge. Since the only way to observe the two particles is through their interactions, they are then indistinguishable.

This operational view of indistinguishability suggests, but does not imply, that if two identical particles in any state of a system are interchanged then the same state, that must therefore be represented by the same state vector multiplied by a phase factor, may be obtained. Formally, the states carry one dimensional representations of the permutation group. There are only two such representations, the symmetric and anti-symmetric, in which the above mentioned phase factor is +1 and −1, respectively.

As is well known, the choice between them is made by the spin-statistics theorem that states that for integer spin particles this phase factor is +1 and for half-integer spin particles it is −1. This has been derived from quantum field theory in the following two ways, both of which rely on the existence of anti-particle fields: a) The energy spectrum for the Dirac field would not be bounded below unless anti-commutation relations are used for the creation and annihilation operators in the Hamiltonian. b) The commutators of fields at two points that are space-like separated vanish if the fields have integer spin and do not vanish if they have half-integer spin, whereas the reverse is true for anti-commutators. So, in order for causality to be valid we are forced to adopt commutators for integer spins and anti-commutators for half-integer spins. This immediately gives the spin-statistics theorem [1].

The fact that there cannot be such a compelling theorem in non relativistic quantum mechanics is seen as follows: Indistinguishability implies that the
Hamiltonian is invariant under interchange of any two identical particles. (The converse is not true: we can have a Hamiltonian for two distinguishable particles that is symmetric under exchange. Therefore, indistinguishability is not synonymous with the Hamiltonian having this symmetry.) Suppose at some initial time, in violation of this theorem, the state of a set of identical half-integer (integer) spin particles is assumed to be a symmetric (anti-symmetric) wavefunction. Then, owing to the Hamiltonian being symmetric with respect to interchange of the identical particles, this symmetry (anti-symmetry) of the wavefunction would be preserved in time and there would be no inconsistency. This is also true in relativistic quantum mechanics, but in non-relativistic quantum mechanics there are no additional field theoretic considerations which force the spin-statistics connection.

But against this must be considered several arguments which claim to obtain the spin-statistics connection in non-relativistic quantum mechanics [2]. A particularly popular simple argument due to Feynman [3, 4], considers interchanging the ends of a belt, supposed to represent two identical particles, without rotating them. Then the belt acquires a twist, which may be eliminated by rotating one end by $2\pi$ radians. This suggests that exchanging the particles is in some sense equivalent to rotating one of them by $2\pi$ radians, which results in a sign change for half-integer spins but no sign change for integer spins.

This argument is the intuitive basis of a recent paper by Berry and Robbins [5]. They have constructed an elegant and simple formalism which naturally yields the sign change $(-1)^{2S}$ when the position and spin states of two identical particles, with spin $S$, are exchanged, without any reference to relativity or quantum field theory. The fundamental new idea in their paper is the introduction of an exchange operator that interchanges the spin states without the above mentioned twist in the hypothetical ‘belt’ connecting them. Their work, however, raises the following two problems: 1) The proof of the spin-statistics theorem in relativistic quantum field theory is valid but appears to be indepen-
dent of the Berry-Robbins argument. Can there be two independent proofs of the same result in physics? 2) Their result \(^\text{[5]}\) unextended implies the statement that the spatial-spin part of the wavefunction is always symmetric for integer spins and anti-symmetric for half integer spins. But this need not be true if the wavefunction has other degrees of freedom. (In fact the violation of the above statement, together with the spin-statistics theorem led to the discovery of color.) So, it is necessary to include the other quantum numbers in this argument.

Another reason for examining the Berry-Robbins formalism is that their exchange operator may change the spins of the individual particles while keeping the total spin the same. So, it performs an interesting ‘supersymmetry’ transformation on the individual particles. But in order to relate this to the usual supersymmetry which is intimately related to relativity, it would appear necessary to make their formalism relativistic.

In this paper, I shall introduce the Berry-Robbins formalism into relativistic quantum theory. The new formalism will use the structure of the Lorentz group, and not just its rotation subgroup, in an essential way which makes it close to the quantum field theoretic proof, mentioned above. This suggests a link between the Berry-Robbins formalism and relativistic quantum field theory which may overcome problem (1). There is no velocity of light \(c\) in the spin-statistics theorem. It obviously exists in the non-relativistic limit. So, it may well be that non-relativistic quantum mechanics ‘remembers’ the spin-statistics connection in relativistic quantum field theory as I shall argue later. On the other hand, because this theorem has no \(c\) which would have enabled us to take the usual non-relativistic limit through \(c \to \infty\), it is not possible to regard the non-relativistic argument as an approximation to the relativistic one. The connection between the two, if it exists, must be a different one! I then extend this formalism to fields on an arbitrary Kaluza-Klein space-time. This results in the inclusion of “internal” variables into the argument as required in problem (2) above. As
byproducts, the Schwinger formalism for representations of $SU(2)$ is generalized to the Lorentz group and all $SU(n)$.

As is well known, the generators of the Lorentz group or its covering group $SL(2,C)$, denoted $J_i$ and $K_i$, $i = 1, 2, 3$ generating rotations and boosts respectively, satisfy the Lie algebra relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k, [K_i, K_j] = -i\epsilon_{ijk} J_k, [J_i, K_j] = i\epsilon_{ijk} K_k.$$  

On defining $A = \frac{1}{2}(J + iK)$ and $B = \frac{1}{2}(J - iK)$, equivalently

$$[A_i, A_j] = i\epsilon_{ijk} A_k, [B_i, B_j] = i\epsilon_{ijk} B_k, [A_i, B_j] = 0.$$

Therefore, $A$ and $B$ generate two commuting $SU(2)$ groups, called the left and right handed groups, denoted here by $SU(2)_+$ and $SU(2)_-$. Hence, all the irreducible representations of $SL(2,C)$ are the same as all the pairs of irreducible representations of $SU(2)_+$ and $SU(2)_-$. So, they may be labeled by $(A, B)$, where $A$ and $B$ are the “spin” quantum numbers, that are integers or half-integers, for an arbitrary pair of irreducible representations of $SU(2)_+$ and $SU(2)_-$. But in general, each irreducible representation $(A, B)$ of $SL(2,C)$ is reducible with respect to the physical rotation subgroup generated by

$$J = A + B,$$

which will be denoted by $SU(2)_J$. Applying the usual laws for adding “angular momenta” for the $SU(2)_+$ and $SU(2)_-$ representations, the irreducible representations of $SU(2)_J$ contained in $(A, B)$ have spin

$$j = A + B, A + B - 1, ..., |A - B|.$$  

An invariant for all these spin representations is $(-1)^{2j} = (-1)^{2(A+B)}$ for all $j$ in $(A, B)$.

I shall now generalize the Schwinger formalism for non-relativistic spin to the representations of the Lorentz group. The basic idea is to associate with
each of \( SU(2)_+ \) and \( SU(2)_- \) a pair of independent oscillators whose number
eigenstates determine basis states for all representations of this group. Let
\( p, q, r, s \) and \( p^\dagger , q^\dagger , r^\dagger , s^\dagger \), respectively, be the annihilation and creation operators
of the four commuting oscillators. Then \( A \) and \( B \) may be represented by

\[
A = \frac{1}{2} \left( \begin{array}{c} p \cr q \end{array} \right) \sigma \left( \begin{array}{c} p^\dagger \cr q^\dagger \end{array} \right)
\]

and

\[
B = \frac{1}{2} \left( \begin{array}{c} r \cr s \end{array} \right) \sigma \left( \begin{array}{c} r^\dagger \cr s^\dagger \end{array} \right)
\]

where \( \sigma = (\sigma^1, \sigma^2, \sigma^3) \) are the Pauli spin matrices. Then

\[
A_z = \frac{1}{2} (p^\dagger p - q^\dagger q), B_z = \frac{1}{2} (r^\dagger r - s^\dagger s)
\]

Also, \( A \) and \( B \) satisfy \([1]\).

From \((2)\),

\[
J = \frac{1}{2} \left( \begin{array}{c} p \cr q^\dagger \end{array} \right) \sigma \left( \begin{array}{c} p^\dagger \cr q \end{array} \right) + \frac{1}{2} \left( \begin{array}{c} r \cr s^\dagger \end{array} \right) \sigma \left( \begin{array}{c} r^\dagger \cr s \end{array} \right)
\]

Comparing this with Schwinger’s representation of non-relativistic spin \([3]\) by
means of two oscillators, there has been a doubling of the number of degrees
of freedom in the present relativistic case where there are four oscillators. This
corresponds to the existence of anti-particles. It is interesting that this result
has been obtained here from the structure of the Lorentz group, in particular
from the fact that its dimension is twice that of the rotation group. This is
very different from how anti-particles were historically discovered, namely the
existence of negative energy solutions of relativistic wave equations, which in
turn is due to the quadratic dispersion relation \( E^2 = p^2 + m^2 \).

A basis of states for the \((A, B)\) representation of \( SL(2, C) \) are eigenstates
of \( A^2, A_z, B^2, B_z \). These are states with definite numbers of quanta:

\[
|n_p, n_q, n_r, n_s> = p^\dagger^n_p q^\dagger^n_q r^\dagger^n_r s^\dagger^n_s |0>
\]

Then

\[
A = \frac{1}{2} (n_p + n_q), B = \frac{1}{2} (n_r + n_s).
\]

(4)
Consider now two identical particles $1$ and $2$. Their states are spanned by
\[
|n_1, n_2, n_{1q}, n_{2q}, n_{1r}, n_{2r}, n_{1s}, n_{2s} > = p_{1}^{n_{1p}} p_{2}^{n_{2p}} q_{1}^{n_{1q}} q_{2}^{n_{2q}} r_{1}^{n_{1r}} r_{2}^{n_{2r}} s_{1}^{n_{1s}} s_{2}^{n_{2s}} |0 > .
\]
(5)

Here and later the subscripts 1 and 2 refer to the particles 1 and 2. Analogous to the Berry-Robbins exchange angular momentum define
\[
E_p = \frac{1}{2} \left( \begin{array}{c} p_1^\dagger \\ p_2^\dagger \end{array} \right) \sigma \left( \begin{array}{c} p_1 \\ p_2 \end{array} \right)
\]

Similar definitions are given for $E_q$, $E_r$ and $E_s$. Define
\[
E = E_p + E_q + E_r + E_s.
\]

Then,
\[
[E_i, E_j] = i\epsilon_{ijk} E_k, [E_i, S_j] = 0
\]

where $S = J_1 + J_2 = A_1 + A_2 + B_1 + B_2$ is the total spin.

Analogous to the construction of Berry-Robbins, a new wavefunction of the relative coordinates $r$ of the two particles may be defined as
\[
\tilde{\psi}(r) = U(r)\psi(r),
\]
(6)

where $\psi(r)$ is the usual wavefunction and
\[
U(r) = \exp\{ -i\theta n(r) \cdot E \}
\]

with $\theta$ being the angle between $r$ and the $z$-axis, and $n(r)$ is a unit vector perpendicular to the $z$-axis and varying smoothly with $r$. One such smooth choice, used by Berry-Robbins, is to require $n(r)$ to be perpendicular to $r$, as well. The dependence of $U(r)$ on the choice of $z$-axis and the direction of $n(r)$ gives a gauge freedom in the choice of $U(r)$. Each pair of indistinguishable configurations corresponding to $r$ and $-r$ are identified. Requiring single valuedness of $\tilde{\psi}$ in the quotient configuration space then amounts to the condition
\[
\psi(-r) = \exp\{ -i\pi n(r) \cdot E \}\psi(r)
\]
(7)
in terms of the usual wave function.

Now $\psi(r)$ may be expanded in terms of (3). Using the commutation relations of the annihilation and creation operators,

$$\exp\{-i\pi \mathbf{n} \cdot \mathbf{E}\} p_1 \exp\{i\pi \mathbf{n} \cdot \mathbf{E}\} = e^{i\phi} p_2 \exp\{-i\pi \mathbf{n} \cdot \mathbf{E}\} \exp\{i\pi \mathbf{n} \cdot \mathbf{E}\} = -e^{-i\phi} p_1$$

where $\mathbf{n}$ is perpendicular to the 3-axis and $\phi$ is an inconsequential angle that $\mathbf{n}$ makes with the 2-axis, with similar relations for the $q, r$ and $s$ operators. Then, using (4) and (3),

$$\exp\{-i\pi \mathbf{n} \cdot \mathbf{E}\}|n_1 p, n_2 p, n_1 q, n_2 q, n_1 r, n_2 r, n_1 s, n_2 s > = (-1)^2j|n_2 p, n_1 p, n_2 q, n_1 q, n_2 r, n_1 r, n_2 s, n_1 s >$$

Hence, the RHS of (4) is $(-1)^2\bar{\psi}(r)$, where the bar denotes the exchange of spin quantum numbers of the two particles. This is the sought after spin-statistics connection.

The above argument suggests that the physical argument of Feynman [3], mentioned above, should be made in the abstract three dimensional spaces on which $SU(2)_+$ and $SU(2)_-$ act, which makes use of the full structure of the Lorentz group, and not in the physical space on which $SU(2)_J$ acts. Indeed, the usual argument which gives the spin-statistics theorem in quantum field theory [1] makes use of the transformation of the field under both $SU(2)_+$ and $SU(2)_-$. As mentioned above, this makes use of the degrees of freedom of the anti-particle as well the particle. Since $SU(2)_J$ is a subgroup of $SU(2)_+ \times SU(2)_-$, the ‘belt’ argument may also be used on the physical space on which $SU(2)_J$ acts. Hence, non-relativistic physics ‘remembers’ an argument in which relativity was essentially involved. This may well be the long sought ‘missing link’ between the ‘belt’ argument, which appeared to have nothing to do with relativity, and the usual argument from relativistic quantum field theory.

The Schwinger formalism will now be extended to an arbitrary Lie group $G$, with the Lie algebra relations $[T^i, T^j] = i \sum_k C^{ij}_{kj} T^k$. Let

$$\hat{T}^i = \sum_{p,q=1}^{\nu} a^{p+1} T^i_{pq} q^q,$$  \hspace{1cm} (8)
where $T_{pq}^i, p, q = 1, 2, \ldots \nu$ are the matrix elements of $T^i$ in the fundamental representation and the annihilation and creation operators satisfy

$$[a^p, a^q] = 0,\ [a^q, a^p] = 0, \ [a^p, a^q\dagger] = 0, \ p, q = 1, 2, \ldots \nu.$$  \hfill (9)

It can then be proved, using (9),

$$[\hat{T}^i, \hat{T}^j] = i \sum_{k} C^{ij}_{k} \hat{T}^k.$$  \hfill (10)

Also, since the commuting generators $T^i_C, i = 1, 2, \ldots r$ of the Cartan subalgebra can be made diagonal, where $r$ is the rank of $G$, write $T^i_C = \lambda^i_p \delta_{pq}$. Therefore,

$$\hat{T}^i_C = \sum_{p=1}^{\nu} \lambda^i_p a^p \dagger a^p, \ i = 1, 2, \ldots r.$$  \hfill (10)

The states,

$$|n_1, n_2, \ldots n_\nu > = (a^1\dagger)^{n_1} (a^2\dagger)^{n_2} \ldots (a^\nu\dagger)^{n_\nu} |0 >,$$  \hfill (11)

are simultaneous eigenstates of the Cartan subalgebra. In particular, the single quantum states are $|\chi_p > \equiv a^p \dagger |0 >, \ p = 1, 2, \ldots \nu$. From (8) and (9), the matrix elements $<\chi_p |\hat{T}^i |\chi_q > = T^i_{pq}$. Hence, the fundamental representation of $G$ acts on the vector space spanned by $\{|\chi_p >\}$. Since the creation operators commute, the tensors (11) may be obtained by taking symmetrized tensor products of the vectors $\{|\chi_p >\}$.

All irreducible representations of $G$ may be obtained by constructing the vector space $V$ spanned by tensor products of the symmetric tensors (11) and reducing the representation of $G$ that acts on $V$. Shmuel Elitzur has suggested adding another index, say $\tau$, to the creation and annihilation operators to represent the position of each symmetric tensor in the last mentioned tensor product on which they act. These symmetric tensors then correspond to the rows of a Young tableau (see, for example, [7]). I.e. each value of $\tau$ corresponds to a particular row of the Young tableau. By anti- symmetrizing the columns of the Young tableau, irreducible representations of the permutation
group acting on the vector space spanned by the tensors of this Young tableau are obtained. I shall therefore make the creation (annihilation) operators with different values of \( \tau \) anti-commute. The Young tableau has at most \( \nu \) rows because anti-symmetrizing more than \( \nu \) elements gives zero. Therefore, \( \tau \) takes values 1, 2, ... \( \nu \). Hence, (8) and (10) may be generalized to

\[
\hat{\mathcal{T}}_i = \sum_{\tau=1}^{\nu} \sum_{p,q=1}^{\nu} a^{\tau p \dagger} T_{pq}^{i} a^{\tau q}, \quad \hat{T}^i_C = \sum_{\tau=1}^{\nu} \sum_{p=1}^{\nu} \lambda_{p}^{i} a^{\tau p \dagger} a^{\tau p}.
\]

(12)

The representation corresponding to a given Young tableau acts on a vector space spanned by

\[
| n^{\tau p} \rangle \equiv | n^{1 \nu} ... n^{1 \nu} \rangle, \quad \text{where} \ n^{\tau p} = \sum_{\tau=1}^{\nu} n^{\tau p}, \ n^{\tau p} \ \text{are non negative integers}.
\]

The \( \tau \)-th row of this Young tableau has \( n^{\tau} = \sum_{p=1}^{\nu} n^{\tau p} \) elements, where \( n^{\tau p} \) are non negative integers.

But this representation, corresponding to a given Young tableau, may be reducible under the action of \( G \). The irreducible representations may be extracted by contracting the tensors on which this representation acts with tensors that are invariant under \( G \) to obtain invariant lower dimensional representations, and by taking suitable linear combination of these tensors multiplied by Kronecker deltas where appropriate to obtain irreducible tensors. Anti-symmetrization of a column with \( \nu \) elements corresponds to contracting these indices with the epsilon tensor. If the determinant of the transformation matrix is 1, e.g. for \( G = SU(n) \) or \( SO(d) \), this contraction is invariant and a lower rank tensor is obtained. Hence for special groups, the Young tableaux may have at most \( \nu - 1 \) rows, or equivalently \( \tau = 1, 2, ..., \nu - 1 \) only.

For the special case of \( G = SU(2), \nu = 2 \) and therefore the \( \tau \) index takes only one possible value and may be omitted. This corresponds to the Schwinger formalism for angular momentum, used above. More generally for \( G = SU(n), \nu = n \) and \( \tau \) takes \( n - 1 \) possible values. Then, as is well known, each Young tableau uniquely corresponds to an irreducible representation of \( SU(n) \), be-
cause the only invariant tensor is the epsilon tensor which has already been
used to reduce the rows of the Young tableaux to at most \(n - 1\) rows. The
above treatment, with the sum over \(\tau\) restricted to \(1, 2, \ldots n - 1\), then generalizes
the Schwinger oscillator formalism for \(SU(2)\) to representations of \(SU(n)\). This
has the advantage that we would always use \(n(n - 1)\) oscillators, with creation
operators \(a^{\tau p\dagger}\), irrespective of the dimension of the representation.

Consider now an arbitrary Kaluza-Klein space-time of dimension \(d = 2m\)
or \(2m + 1\), where \(m\) is an integer \(\geq 2\). The spinor and tensor fields on this space-
time transform under representations of the covering group of \(SO(1, d-1)\). They
can be built from the fundamental spinor representation of this group, which
may be constructed as follows: First construct the \(\gamma\) matrices to act on tensor
products of \(m\) spin-\(\frac{1}{2}\) representations. A particular set that is anti-commuting
and appropriately normalized is

\[
\begin{align*}
\gamma^0 &= -I \otimes \sigma^1 \otimes \ldots \otimes I, \\
\gamma^j &= i\sigma^j \otimes \sigma^2 \otimes \sigma^3 \otimes \ldots \otimes I, \quad j = 1, 2, 3 \\
\gamma^5 &= iI \otimes \sigma^3 \otimes \sigma^1 \otimes \ldots \otimes I, \\
\gamma^6 &= iI \otimes \sigma^3 \otimes \sigma^2 \otimes \sigma^1 \otimes \ldots \otimes I, \\
\gamma^7 &= iI \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^1 \otimes \ldots \otimes I, \\
\gamma^8 &= iI \otimes \sigma^3 \otimes \sigma^3 \otimes \sigma^2 \otimes \sigma^1 \otimes \ldots \otimes I
\end{align*}
\]

There are \(m\) factors in the definition of each \(\gamma^M\) which make the vector space
they act on \(2^m\) dimensional, for either value of \(d\).

The generators of this fundamental spinor representation are then

\[
S^{MN} = \frac{i}{4}[\gamma^M, \gamma^N], \quad M, N = 1, 2, \ldots, d
\]

The generators of Lorentz boosts of ordinary space-time are

\[
S^{0j} = \frac{i}{2}\sigma^1 \otimes \sigma^3 \otimes \ldots \otimes I, \quad j = 1, 2, 3
\]

while the generators of spatial rotations, i.e. \(SU(2)_J\), are

\[
S^{ij} = \frac{1}{2}\epsilon_{ijk}\sigma^k \otimes \ldots \otimes I, \quad i, j = 1, 2, 3.
\]
It is clear that rotation by \(2\pi\) radians that is in \(SU(2)\) gives a sign change to all vectors in this fundamental spinor representation. Other representations may be obtained by making tensor products of the fundamental spinor representations \(n\) times, where \(n\) is any positive integer, and taking the irreducible representations. Clearly, if \(n\) is odd this tensor product would give only half-integer spin fields, because all the elements in this tensor product undergo sign change under the above \(2\pi\) rotation. Similarly, if \(n\) is even then there will be no sign change so that only integer spin fields are obtained. Hence,

\[
(-1)^n = (-1)^{2S},
\]

where \(S\) is the spin of any of the irreducible representations obtained for a given \(n\), although obviously for \(n > 1, n \neq 2S\), in general.

Alternatively, the group \(G\) in the above treatment of its representations by means of oscillators may be taken to be the covering group of \(SO(1, d-1)\). Then \(\nu = 2^m\) and the last mentioned tensor products correspond to states with \(n\) quanta. Consider now two identical particles. Since they are identical, consider their states that belong to the same Young tableau for both particles, which are of the form

\[
|n_1^{\tau_p}, n_2^{\tau_p}\rangle \equiv \left\{ \prod_{\tau=1}^{2m-1} \prod_{p=1}^{2^m} (a_1^{\tau_p\dagger})^{n_1^{\tau_p}} (a_2^{\tau_p\dagger})^{n_2^{\tau_p}} \right\}|0\rangle
\]

where

\[
n = \sum_{\tau=1}^{2m-1} \sum_{p=1}^{2^m} n_1^{\tau_p} = \sum_{\tau=1}^{2m-1} \sum_{p=1}^{2^m} n_2^{\tau_p},
\]

(16)

where \(n_1^{\tau_p}\) and \(n_2^{\tau_p}\) are non negative integers. The generalization of the Berry-Robbins exchange angular momentum is \(E = \sum_{\tau=1}^{2m-1} \sum_{p=1}^{2^m} E^{\tau_p}\), where

\[
E^{\tau_p} = \frac{1}{2} \begin{pmatrix} a_1^{\tau_p\dagger} & a_2^{\tau_p\dagger} \end{pmatrix} \sigma \begin{pmatrix} a_1^{\tau_p} \\ a_2^{\tau_p} \end{pmatrix}
\]

which now generates exchange of Kaluza-Klein spin states. Since each \(E^{\tau_p}\) is quadratic in the creation and annihilation operators, which satisfy (9) for a given
τ and anti-commute for different τs, the E^τp's must commute among themselves. Using these commutation relations,

\[ \exp\{-iπ\bf{n} \cdot \bf{E}\} a_1^{τp\dagger} \exp\{iπ\bf{n} \cdot \bf{E}^τ\} = e^{iφ_1} a_2^{τp\dagger}, \]

\[ \exp\{-iπ\bf{n} \cdot \bf{E}\} a_2^{τp\dagger} \exp\{iπ\bf{n} \cdot \bf{E}^τ\} = e^{-iφ_2} a_1^{τp\dagger}, \]

where n is perpendicular to the 3-axis and φ is the angle that n makes with the 2-axis. It follows from (15), (16) and (14),

\[ \exp\{-iπ\bf{n} \cdot \bf{E}\}|n^{τp}_1, n^{τp}_2\rangle = (-1)^{2S}|n^{τp}_1, n^{τp}_2\rangle. \quad (17) \]

A relative coordinate wave function analogous to (6) may now be constructed on Kaluza-Klein space-time, and its single valuedness requires (7) for the usual wave function, with the last defined E, and r is now the d−1 dimensional relative coordinate vector in the Kaluza-Klein space. By expanding this wave function in terms of (13) and using (17), the RHS of (13) is \((-1)^{2S}\tilde{ψ}(r)\). I.e. interchanging Kaluza-Klein spins and positions gives the factor \((-1)^{2S}\) to the state of two identical particles each of which has spin S. This gives the spin-statistics connection in Kaluza-Klein space-time.

When the Kaluza-Klein space-time is compactified by curling up the ‘internal’ dimensions \(B\), this gives the spin-statistics connection in the usual 4 dimensional space-time, as will be shown now. The wave function in the fundamental representation on which (13) act may be written

\[ \Psi^{α_1α_2...α_m}(X) = \sum_{Rf} \psi^{α_1α_2}(x)\phi^{α_3...α_m}(y). \quad (18) \]

Here the Kaluza-Klein coordinates X = (x, y) are split into the usual four dimensional space-time coordinates x and the coordinates y of the internal space B. The isometry group H on B is the gauge group in the usual space-time. R labels each irreducible representation of H which acts on the basis vectors φ^{α_i...α_m}(y), and f labels a vector in this representation. Each α^i index takes two possible values. The Lorentz group acts on the pair of indices α_1, α_2 only,
while the spinor group on $B$ that is induced by $H$ acts on the indices $\alpha^3, \ldots, \alpha^m$.

The usual fields on space-time are $\psi^{\alpha_1^{m_1} \alpha_2^{m_2}}(x)$, where $(\alpha^1, \alpha^2)$ is the 4 dimensional Dirac spinor index, while $(R, f)$ is interpreted as the ‘internal’ index representing weak isospin, hypercharge and color states, etc.

Consider now a state of two identical particles in this fundamental spinor representation. It follows, from the above result in italics, that their wave function must satisfy

$$
\Psi^{\alpha_1^{m_1}, \alpha_2^{m_2}}(X_1, X_2) = -\Psi^{\alpha_2^{m_2}, \alpha_1^{m_1}}(X_2, X_1),
$$

(19)

This wave function may be expanded in terms of the basis states $\phi_{R^f}$ in (18) as

$$
\Psi^{\alpha_1^{m_1}, \alpha_2^{m_2}}(X_1, X_2) = \sum_{R_1 f_1, R_2 f_2} \psi^{\alpha_1^{m_1}, \alpha_2^{m_2}}(x_1, x_2) \phi_{R_1 f_1}^{\alpha_1^{m_1}}(y_1) \phi_{R_2 f_2}^{\alpha_2^{m_2}}(y_2).
$$

(20)

From (19) and (20), the usual space-time wave function satisfies

$$
\psi^{\alpha_1^{m_1}, \alpha_2^{m_2}}(x_1, x_2) = -\psi^{\alpha_2^{m_2}, \alpha_1^{m_1}}(x_2, x_1).
$$

(21)

The proof for the higher dimensional representations is similar to the above except that in (19)-(21) there are now $2mn$ $\alpha$-indices, instead of the $2m$ $\alpha$-indices as in the above special case of $n = 1$, and the $-$ sign in the RHS of (19) and (21) is replaced by the factor $(-1)^n = (-1)^{2S}$. It is emphasized that in the above treatment no a priori preference is given to spin over other variables such as weak isospin, hypercharge and color, unlike in ref. [5].

The extension of this result to more than two identical particles is straightforward: If the state is anti-symmetric (symmetric) with respect to interchange of any pair of identical particles then it must be totally anti-symmetric (symmetric).

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