NEVANLINNA THEORY AND VALUE DISTRIBUTION IN THE UNICRITICAL POLYNOMIALS FAMILY

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Abstract. In the space $\mathbb{C}$ of the parameters $\lambda$ of the unicritical polynomials family $f(\lambda, z) = f_\lambda(z) = z^d + \lambda$ of degree $d > 1$, we establish a quantitative equidistribution result towards the bifurcation current (indeed measure) $\mathcal{T}_f$ of $f$ as $n \to \infty$ on the averaged distributions of all parameters $\lambda$ such that $f_\lambda$ has a superattracting periodic point of period $n$ in $\mathbb{C}$, with a concrete error estimate for $C^2$-test functions on $\mathbb{P}^1$. In the proof, not only complex dynamics but also a standard argument from the Nevanlinna theory play key roles.

1. Introduction

Let $f : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{P}^1$ be the (monic and centered) unicritical polynomials family

$$(1.1) \quad f(\lambda, z) = f_\lambda(z) := z^d + \lambda$$

of degree $d > 1$. Let $c_0 \equiv 0$ on $\mathbb{C}$, which is a marked critical point of the family $f$ in that for every $\lambda \in \mathbb{C}$, $c_0(\lambda)$ is a critical point of $f_\lambda(z) \in \mathbb{C}[z]$. For every $n \in \mathbb{N} \cup \{0\}$, let us define the monic polynomial

$$F_n(\lambda) := f_\lambda^n(c_0(\lambda)) \equiv f_\lambda^n(0) \in \mathbb{Z}[\lambda]$$

of degree $d^{n-1}$. Any zero of $F_n$ is simple (Douady–Hubbard [10, Exposé XIX]; see also [19, Theorem 10.3] for a simple proof). The study of the asymptotic behavior as $n \to \infty$ of the set of all zeros of $F_n$, which is the set of all parameters $\lambda \in \mathbb{C}$ such that $f_\lambda$ has a superattracting periodic point of (not necessarily exact) period $n$ in $\mathbb{C}$, was initiated by Levin [15], and has been developed by Bassanelli–Berteloot [2, 3] and Buff–Gauthier [7] subsequently.

Our aim is, from both complex dynamics and the Nevanlinna theory, to contribute to the quantitative study of the asymptotic behavior of zeros of $F_n$ as $n \to \infty$, partly sharpening Gauthier–Vigny [14].

Notation 1.1. Let $\mu : \mathbb{N} \to \{-1, 0, 1\}$ be the Möbius function from arithmetic (cf. [1, §2]). Let $\log^+ t := \log \max\{1, t\}$ on $\mathbb{R}$. Let $\omega$ be the Fubini-Study area element on $\mathbb{P}^1$ normalized as $\omega(\mathbb{P}^1) = 1$, let $[z, w]$ be the chordal metric on $\mathbb{P}^1$ normalized as $[1, \infty] = 1/\sqrt{1 + |\cdot|^2}$ on $\mathbb{P}^1$ (following the notation in Nevanlinna’s and Tsuji’s books [23, 29]), and let $\delta_x$ be the Dirac measure on $\mathbb{P}^1$ at each $x \in \mathbb{P}^1$. The Laplacian $dd^c$ on $\mathbb{P}^1$ is normalized as

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dd^c(-\log[1, \infty]) = \omega - \delta_\infty \text{ on } \mathbb{P}^1. \text{ Set } \mathbb{D}(x, r) := \{y \in \mathbb{C} : |x - y| < r\} \text{ for}
\text{every } x \in \mathbb{C} \text{ and every } r > 0, \mathbb{D}(r) := \mathbb{D}(0, r) \text{ for every } r > 0, \text{ and } \mathbb{D} := \mathbb{D}(1).

1.1. Main result. Let \( g_{I_0} \) be the Green function with pole \( \infty \) on the escaping locus \( I_{0} := \{\lambda \in \mathbb{C} : \limsup_{n \to \infty} |F_n(\lambda)| = \infty\} \) of the marked critical point \( c_0 \) of \( f; I_{c_0} \) is a punctured open and connected neighborhood of \( \infty \) in \( \mathbb{P}^1 \), and \( \partial I_{c_0} \) and \( \mathbb{C} \setminus I_{c_0} \) respectively coincide with the \( J \)-unstable or bifurcation locus \( B_f \) and the connectedness locus \( M_f \) of \( f \). The function \( g_{I_{c_0}} \) extends to \( \mathbb{C} \) continuously by setting \( g_{I_{c_0}} = 0 \) on \( M_f \), and \( \mu_B := dd^c g_{I_{c_0}} + \delta_\infty \) on \( \mathbb{P}^1 \) coincides with the harmonic measure on \( B_f \) with pole \( \infty \). The measure \((d - 1)d^{-1}\mu_B, \text{ on } \mathbb{P}^1 \) coincides with the bifurcation current \((\text{indeed measure})\) \( T_f \) of \( f \) on \( \mathbb{P}^1 \) (see Subsection 2.1). By a refinement of Przytycki’s argument on the recurrence of critical orbits [25, Proof of Lemma 2] and Buff’s upper estimate of the moduli of the derivatives of polynomials [6] the proof of Theorem 3], we will establish the following \( L^1(\omega) \) estimate

\[
\int_{\mathbb{P}^1} |\log |F_n| - d^{n-1} \cdot g_{I_{c_0}}| \omega \leq \frac{2 \log d}{d - 1} n + O(1)
\]

as \( n \to \infty \), with the concrete coefficient \((2 \log d)/(d - 1)\) of \( n \) in the right hand side; a question on the best possibility of this estimate [12] seems also interesting. As seen in the proof of (1.2) in Section 3, this may be regarded as a counterpart of H. Selberg’s theorem [26, p. 313] from the Nevanlinna theory.

Our principal result is a deduction from (1.2) of the following quantitative equidistribution of the sequence \((F_n^\ast \delta_0/d^n)\) of the averaged distribution of the superattracting parameters of period \( n \) towards \((d - 1)^{-1}T_f = d^{-1}\mu_B\), as \( n \to \infty \).

Theorem 1. Let \( f : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{P}^1 \) be the unicritical \((\text{monic and centered})\) polynomials family of degree \( d > 1 \) defined as in (1.1). Then for every \( \phi \in C^2(\mathbb{P}^1)\),

\[
\left| \int_{\mathbb{P}^1} \phi d ((d - 1) \cdot F_n^\ast \delta_0 - d^n \cdot T_f) \right| \leq \left( \sup_{\mathbb{P}^1} \left| \frac{dd^c \phi}{\omega} \right| \right) \cdot ((2 \log d)n + O(1))
\]

as \( n \to \infty \), where the implicit constant in \( O(1) \) is independent of \( \phi \) and the Radon-Nikodim derivative \((dd^c \phi)/\omega\) on \( \mathbb{P}^1 \) is bounded on \( \mathbb{P}^1 \).

For a former application of Selberg’s theorem (Theorem 3.2) to obtain a quantitative equidistribution result in complex dynamics, see Drasin and the author [12]. As an order estimate, the estimate (1.3) is due to Gauthier–Vigny [14, Theorem A]. The implicit constant in \( O(1) \) in (1.3) will also be computed in the proof. The coefficient \( 2 \log d \) of \( n \) in (1.3) comes from the full strength of de Branges’s theorem (the solution of the Bieberbach conjecture), on which the proof of Buff’s estimate mentioned above essentially relies.
1.2. Non-repelling parameters having exact periods. For every \( n \in \mathbb{N} \), the \( n \)-th dynameic polynomial

\[
\Phi_{f,n}^*(\lambda, z) := \prod_{m \in \mathbb{N} : m | n} (f^{m}_{\lambda}(z) - z)^{\mu(n/m)}
\]

of the family \( f \) is in fact in \( \mathbb{Z}[\lambda, z] \), and for every \( \lambda \in \mathbb{C} \), \( \Phi_{f,n}^*(\lambda, z) \in \mathbb{C}[z] \) is monic and of degree

\[
(1.4) \quad \nu(n) = \nu_d(n) := \sum_{m \in \mathbb{N} : m | n} \mu\left(\frac{n}{m}\right) d^m.
\]

For every \( \lambda \in \mathbb{C} \) and every \( n \in \mathbb{N} \), let \( \text{Fix}_f(\lambda, n) \) be the set of all fixed points of \( f^{n}_{\lambda} \) in \( \mathbb{C} \) and set \( \text{Fix}_f^*(\lambda, n) := \text{Fix}_f(\lambda, n) \setminus (\bigcup_{m \in \mathbb{N} : m | n} \text{Fix}_f(\lambda, m)) \), each element in which is called a periodic point of \( f^{n}_{\lambda} \) in \( \mathbb{C} \) having the exact period \( n \). For every \( n \in \mathbb{N} \) and every \( \lambda \in \mathbb{C} \), a periodic point \( z \) of \( f^{n}_{\lambda} \) in \( \mathbb{C} \) is said to have the formally exact period \( n \) if either (i) \( z \in \text{Fix}_f^*(\lambda, n) \) or (ii) there is an \( m \in \mathbb{N} \) satisfying \( m | n \) and \( m < n \) such that \( z \in \text{Fix}_f^*(\lambda, m) \) and that \( (f^{m}_{\lambda})'(z) \) is a primitive \( (n/m) \)-th root of unity (so in particular \( (f^{m}_{\lambda})'(z) = 1 \)). For every \( \lambda \in \mathbb{C} \) and every \( n \in \mathbb{N} \), let \( \text{Fix}_{f,n}^*(\lambda, n) \) be the set of all periodic points of \( f^{n}_{\lambda} \) in \( \mathbb{C} \) having the formally exact period \( n \), which in fact coincides with \( (\Phi_{f,n}^*(\lambda, \cdot))^{-1}(0) \). For every \( n \in \mathbb{N} \), the \( n \)-th multiplier polynomial

\[
p_f^*(\lambda, w) := \left( \prod_{z \in \text{Fix}_f^*(\lambda, n)} ((f^{n}_{\lambda})'(z) - w) \right)^{1/n}
\]

of \( f \), where for each \( \lambda \in \mathbb{C} \), the product in the right hand side takes into account the multiplicity of each \( z \in \text{Fix}_f^*(\lambda, n) \) as a zero of \( \Phi_{f,n}^*(\lambda, \cdot) \), is indeed in \( \mathbb{Z}[\lambda, w] \) and unique up to multiplication in \( n \)-th roots of unity. For every \( w \in \mathbb{C} \), by a direct computation,

\[
(1.5) \quad \deg_{\lambda} p_f^*(\lambda, w) = (d - 1) \frac{\nu(n)}{d}
\]

and the coefficient of the leading term of \( p_f^*(\lambda, w) \in \mathbb{C}[\lambda] \) equals \( d^{\nu(n)} \), both of which are independent of \( w \). For every \( n \in \mathbb{N} \) and every \( w \in \mathbb{C} \), let \( \text{Per}_f(n, w) \) be the effective divisor on \( \mathbb{P}^1 \) defined by the zeros of \( p_f^*(\lambda, w) \in \mathbb{C}[\lambda] \); as a Radon measure on \( \mathbb{P}^1 \),

\[
\text{Per}_f(n, w) = dd_{\lambda} \log |p_f^*(\lambda, w)| + (d - 1) \frac{\nu(n)}{d} \delta_{\infty}.
\]

For more details, see e.g. [28 \S 4], [40 \S 2.3], [211 \S 3].

**Notation 1.2.** Let \((\sigma_0(n))\) and \((\sigma_1(n))\) be such sequences in \( \mathbb{N} \) that \( 1 = \sum_{m \in \mathbb{N} : m | n} \mu(n/m) \sigma_0(m) \) and \( n = \sum_{m \in \mathbb{N} : m | n} \mu(n/m) \sigma_1(m) \), or equivalently, \( \sigma_0(n) = \sum_{m \in \mathbb{N} : m | n} 1 \) and \( \sigma_1(n) = \sum_{m \in \mathbb{N} : m | n} m \) by Möbius inversion, for every \( n \in \mathbb{N} \).

By an argument similar to that in the proof of Theorem 1 we will also show the following.
Theorem 2. Let \( f : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{P}^1 \) be the unicritical (monic and centered) polynomials family of degree \( d > 1 \) defined as in (1.1). Then for every \( \phi \in \mathcal{C}^2(\mathbb{P}^1) \),

\[
(1.6) \quad \left| \int_{\mathbb{P}^1} \phi \left( \text{Per}_f^*(n, 0) - \nu(n) \cdot T_f \right) \right| \\
\leq \left( \sup_{\mathbb{P}^1} \left| \frac{d^c \phi}{\omega} \right| \right) \cdot ((2 \log d) \sigma_1(n) + O(\sigma_0(n)))
\]
as \( n \to \infty \), where the term \( O(\sigma_0(n)) \) is independent of \( \phi \), and for every \( \phi \in \mathcal{C}^2(\mathbb{P}^1) \) and every \( r \in (0, 1] \),

\[
(1.7) \quad \left| \int_{\mathbb{P}^1} \phi \left( \int_0^{2\pi} \text{Per}_f^*(n, re^{i\theta}) \frac{d\theta}{2\pi} - \nu(n) \cdot T_f \right) \right| \\
\leq \left( \sup_{\mathbb{P}^1} \left| \frac{d^c \phi}{\omega} \right| \right) \cdot ((2 \log d) \sigma_1(n) + O(\sigma_0(n)))
\]
as \( n \to \infty \), where the term \( O(\sigma_0(n)) \) is independent of both \( \phi \) and \( r \). Here the Radon-Nikodim derivative \( (d^c \phi)/\omega \) on \( \mathbb{P}^1 \) is bounded on \( \mathbb{P}^1 \).

Again, the terms \( O(\sigma_0(n)) \) in Theorem 2 will also be computed in Section 4. As an order estimate, the estimate (1.6) is a consequence of Gauthier–Vigny [14, Theorem A]. The estimate (1.7) quantifies Bassanelli–Berteloot [3, 2. in Theorem 3.1] for \( r \in (0, 1] \).

1.3. Organization of the article. In Section 2, we recall background from the study of the unicritical polynomials family \( f \). In Section 3, we show Theorem 1. In Section 4, we show Theorem 2.

2. Background from the study of the family \( f \)

Let \( f : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{P}^1 \) be the unicritical (monic and centered) polynomials family of degree \( d > 1 \) defined as in (1.1), and recall that \( c_0(\lambda) = 0 \in \mathbb{Z}[\lambda] \) defines a marked critical point of \( f \).

2.1. Douady–Hubbard’s theory on the parameter space \( \mathbb{C} \) of \( f \). For every \( \lambda \in \mathbb{C} \), let \( J_{f\lambda} \) be the Julia set of \( f\lambda \), which is compact in \( \mathbb{C} \). Let \( B_f \) be the \( J \)-unstability or bifurcation locus of the family \( f \), which is the discontinuity locus of the set function \( \lambda \mapsto J_{f\lambda} \) with respect to the Hausdorff topology from \( (\mathbb{P}^1, [z, w]) \), and is closed and nowhere dense in \( \mathbb{C} \) (by Mañé–Sad–Sullivan [17], Lyubich [16]). The escaping locus

\[
I_{c_0} := \{ \lambda \in \mathbb{C} : \limsup_{n \to \infty} |F_n(\lambda)| = \infty \}
\]
of the marked critical point \( c_0 \) of \( f \) is a punctured open and connected neighborhood of \( \infty \) in \( \mathbb{P}^1 \) and coincides with the unique unbounded component of \( \mathbb{C} \setminus B_f \). We have \( B_f = \partial I_{c_0} \), and the connectedness locus

\[
M_f := \{ \lambda \in \mathbb{C} : J_{f\lambda} \text{ is connected} \}
\]
of \( f \) coincides with \( \mathbb{C} \setminus I_{c_0} \) (and is connected). For every \( \lambda \in \mathbb{C} \), \( f\lambda \) has at most one non-repelling cycle in \( \mathbb{C} \) (see, e.g., [20 §8]). Let \( H_f \) be the hyperbolicity locus of \( f \), which coincides with the union of \( I_{c_0} \) and the set of all \( \lambda \in M_f \) such that \( f\lambda \) has the (super)attracting cycle in \( \mathbb{C} \), and is a closed and open
subset in \( \mathbb{C} \setminus B_f \). For example, for every \( n \in \mathbb{N}, 0 \in F^{-1}_n(0) \subset H_f \setminus I_0 \).

For every component \( U \) of \( H_f \setminus I_0 \), there are an \( n_U \in \mathbb{N} \) and a proper holomorphic mapping \( \phi_U : U \to \mathbb{D} \) of degree \( d - 1 \) such that \#\( \phi_U^{-1}(0) \) = 1 and that for every \( w \in \mathbb{D}, \phi_U^{-1}(w) \) coincides with the set of all \( \lambda \in U \) such that \( f_\lambda \) has the (super)attracting cycle in \( \mathbb{C} \) having the exact period \( n_U \) and the multiplier \( w \). For more details, see Douady–Hubbard [11], and for a modern treatment, see McMullen–Sullivan [19, §10].

2.2. The Green functions on the dynamical and parameter spaces.

For every \( \lambda \in \mathbb{C}, J_{f_\lambda} \) coincides with the boundary of the filled-in Julia set \( K_{f_\lambda} := \{ z \in \mathbb{C} : \limsup_{n \to \infty} |f^n_\lambda(z)| < \infty \} \) of \( f_\lambda \), which is compact in \( \mathbb{C} \). For every \( \lambda \in \mathbb{C} \), the uniform limit

\[
(2.1) \quad g_{f_\lambda}(z) := \lim_{n \to \infty} \frac{-\log[f^n_\lambda(z), \infty]}{d^n}
\]

exists on \( \mathbb{C} \), and setting \( g_{f_\lambda}(\infty) := +\infty \), the probability measure \( \mu_{f_\lambda} := d\phi g_{f_\lambda} + \delta_\infty \) on \( \mathbb{P}^1 \) coincides with the harmonic measure on \( J_{f_\lambda} \) with pole \( \infty \). Moreover, \( \mu_{f_\lambda} \) is mixing so ergodic under \( f_\lambda \) (by Brolin [5]). For completeness, we include a proof of the following.

**Lemma 2.1.** For every \( \lambda \in \mathbb{C} \),

\[
(2.2) \quad \sup_{\mathbb{C}} |g_{f_\lambda} + \log[\cdot, \infty]| \leq \frac{1}{d - 1} \sup_{z \in \mathbb{C}} |\log \frac{[z, \infty]^d}{f^n_\lambda(z), \infty}|,
\]

and the function \( \lambda \mapsto \sup_{z \in \mathbb{C}} |\log([z, \infty]^d/[f^n_\lambda(z), \infty])| \) is locally bounded on \( \mathbb{C} \).

**Proof.** For every \( \lambda \in \mathbb{C} \), by the definition (2.1) of \( g_{f_\lambda} \), we have

\[
\sup_{\mathbb{C}} |g_{f_\lambda} + \log[\cdot, \infty]| \leq \sup_{z \in \mathbb{C}} \left| \sum_{j=1}^{\infty} \frac{-\log[f_\lambda(f_\lambda^{j-1}(z)), \infty] + d \cdot \log[f_\lambda^{j-1}(z), \infty]}{d^j} \right|
\]

\[
\leq \frac{1}{d - 1} \sup_{z \in \mathbb{C}} |\log \frac{[z, \infty]^d}{f^n_\lambda(z), \infty}|.
\]

For every \( \lambda \in \mathbb{C} \), let us define the non-degenerate homogeneous polynomial endomorphism \( f_\lambda : \mathbb{C}^2 \to \mathbb{C}^2 \) of degree \( d \) by \( f_\lambda(p_0, p_1) := (p_0^d, p_0^d f_\lambda(p_1/p_0)) = (p_0^d, p_0^d + \lambda p_0^d) \). Then the function \( (\lambda, (p_0, p_1)) \mapsto |\log \|f_\lambda(p_0, p_1)\|| \) is continuous on \( \mathbb{C} \times (\mathbb{C}^2 \setminus \{(0, 0)\}) \), and for every compact subset \( K \in \mathbb{C} \), we have

\[
\sup_{(\lambda, z) \in K \times \mathbb{C}} |\log \frac{[z, \infty]^d}{f^n_\lambda(z), \infty}| = \sup_{(\lambda, (p_0, p_1)) \in K \times S(1)} |\log \|f_\lambda(p_0, p_1)\||,
\]

where \( \| \cdot \| \) is the Euclidean norm on \( \mathbb{C}^2 \) and \( S(1) := \{(p_0, p_1) \in \mathbb{C}^2 : \|(p_0, p_1)\| = 1\} \). Now the proof is complete by the compactness of \( K \) in \( \mathbb{C} \) and that of \( S(1) \) in \( \mathbb{C}^2 \setminus \{(0, 0)\} \).

Similarly, the locally uniform limit

\[
\lambda \mapsto g_{f_\lambda}(\lambda) := \lim_{n \to \infty} \frac{-\log|F_n(\lambda)|}{d^{n-1}} = d \cdot g_{f_\lambda}(c_0(\lambda)) = g_{f_\lambda}(f_\lambda(c_0(\lambda)))
\]
exists on $\mathbb{C}$, and setting $g_{I_0} := +\infty$, the probability measure
\[ \mu_f := \text{dd}^c g_{I_0} + \delta_\infty \] on $\mathbb{P}^1$
coincides with the harmonic measure on $B_f = \partial I_c$ with pole $\infty$ (by Douady–Hubbard \[11\], Sibony \[27\]). The activity current (indeed measure) of the marked critical point $c_0$ of $f$ is
\[ T_{c_0} := \lim_{n \to \infty} \frac{F_n^\omega}{d^n} = \frac{\mu_f}{d} \]
as currents on $\mathbb{P}^1$ (DeMarco \[8\], Dujardin–Favre \[13\]). For every $\lambda \in \mathbb{C}$, the Lyapunov exponent of $f_\lambda$ with respect to $\mu_{f_\lambda}$ is
\[ L(f_\lambda) := \int_{\mathbb{P}^1} \log |f'_\lambda(z)| \mu_{f_\lambda}(z) = \log d + (d - 1) \frac{g_{I_0}}{d} (\geq \log d > 0) \]
(DeMarco \[9\]). For more details, see, e.g., Berteloot’s survey \[4, 3.2.3\].

3. PROOF OF THEOREM \[1\]

Let $f : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{P}^1$ be the unicritical polynomials family of degree $d > 1$ defined as \[11\]. For every $\lambda \in \mathbb{C}$ and every $n \in \mathbb{N}$, let us define the chordal derivative
\[ (f_\lambda^n)^\# := 1 \sqrt{(f_\lambda^n)^* \omega} = \mathbb{P}^1 \to \mathbb{R}_{\geq 0} \]
of $f_\lambda^n$ on $\mathbb{P}^1$. For every non-empty subset $S$ in $\mathbb{P}^1$, let $\text{diam}_{\#}(S)$ be the chordal diameter of $S$. The result of $(P(z), Q(z)) \in \mathbb{C}[z] \times \mathbb{C}[z]$ is denoted by $\text{Res}(P, Q)$, as usual. Recall that $\{z \in \mathbb{C} : |z| < |r, 0| = \mathbb{D}(0, r)$ for every $r > 0$ and that $|z, w| \leq |z - w|$ on $\mathbb{C} \times \mathbb{C}$.

**Lemma 3.1.** For every $n \in \mathbb{N}$ and every $\lambda \in \mathbb{C} \setminus (H_f \setminus I_c)$ (so in particular for every $\lambda \in B_f$),
\[ |F_n(\lambda)| \geq (\sqrt{2} - 1) \left(2^{d+1} \sup_{z \in \mathbb{P}^1} ((f_\lambda^{n-1})^\#(z)) \right)^{-1/(d-1)}. \]

**Proof.** Fix $n \in \mathbb{N}$, and define the functions $L_{n-1}$ and $\epsilon_n$ on $\mathbb{C}$ by $L_{n-1}(\lambda) := \sup_{z \in \mathbb{P}^1} ((f_\lambda^{n-1})^\#(z))(> 1$ and $\epsilon_n(\lambda) := (2^{d} \cdot L_{n-1}(\lambda))^{-1/(d-1)}( < 1$). For every $\lambda \in \mathbb{C}$, noting that $f_\lambda(0) = \lambda$ and that $f_\lambda(z) - f_\lambda(0) = z^d$ on $\mathbb{C}$, we have
\[ \text{diam}_{\#}((f_\lambda^n)\{z \in \mathbb{C} : |z| < [\epsilon_n(\lambda), 0]\}) = \text{diam}_{\#}((f_\lambda^n)\{z \in \mathbb{D}(0, \epsilon_n(\lambda))\}) \]
\[ = \text{diam}_{\#}((f_\lambda^{n-1})\{z \in \mathbb{D}(\lambda, \epsilon_n(\lambda)^d)\}) \leq L_{n-1}(\lambda) \cdot \text{diam}_{\#}((f_\lambda^n)\{z \in \mathbb{D}(\lambda, \epsilon_n(\lambda)^d)\}) \leq L_{n-1}(\lambda) \cdot 2\epsilon_n(\lambda)^d = \frac{\epsilon_n(\lambda)}{2}, \]
so that if $[f_\lambda^n(0), 0] < [\epsilon_n(\lambda), 0] - \epsilon_n(\lambda)/2$, then $\sup\{w, 0] = w \in f_\lambda^n\{z \in \mathbb{C} : |z| < [\epsilon_n(\lambda), 0]\} \leq ([\epsilon_n(\lambda), 0] - [-\epsilon_n(\lambda)/2] + \epsilon_n(\lambda)/2 = [\epsilon_n(\lambda), 0]$; i.e., $f_\lambda^n\{z \in \mathbb{C} : |z| < [\epsilon_n(\lambda), 0]\} \subseteq \{z \in \mathbb{C} : |z| < [\epsilon_n(\lambda), 0]\}$; then by
Brouwer’s fixed point theorem, Montel’s theorem, and Fatou’s classification of cyclic Fatou components (see e.g. [20, §16]), the domain \( \{ z \in \mathbb{C} : |z| < |e_n(\lambda)| \} \), which contains both the critical point \( c_0(\lambda)(= 0) \) of \( f_\lambda \) and a fixed point of \( f_\lambda^k \), is contained in the immediate basin of a (super)attracting cycle of \( f_\lambda \) in \( \mathbb{C} \). Hence for every \( \lambda \in \mathbb{C} \), we obtain the desired lower estimate

\[
|F_n(\lambda)| \geq (|F_n(\lambda), 0| = |f_\lambda^n(0), 0| \geq [e_n(\lambda), 0] - \frac{e_n(\lambda)}{2} \geq (\sqrt{2} - 1) \frac{e_n(\lambda)}{2} = (\sqrt{2} - 1)(2^{d+1}L_{n-1}(\lambda))^{-1/(d-1)}
\]

of \( |F_n(\lambda)| \) unless 0 is in the immediate basin of a (super)attracting cycle of \( f_\lambda \) in \( \mathbb{C} \). Now the proof is complete. \( \square \)

The following is substantially shown in Buff [6, the proof of Theorem 4].

**Theorem 3.1** (Buff). Let \( f \in \mathbb{C}[z] \) be of degree \( d > 1 \), and let \( z_0 \in \mathbb{C} \). If \( |g_f(z_0)| \geq \max_{c \in \mathbb{C}(f) \cap \mathbb{C}} |g_f(c)| \), where \( g_f \) is the Green function of the filled-in Julia set \( K_f \) of \( f \) with pole \( \infty \) and \( C(f) \) is the set of all critical points of \( f \), then \( |f'(z_0)| \leq d^2 \cdot e^{(d-1)|g_f(z_0)|} \), and the equality never holds if \( C(f) \cap \mathbb{C} \) is not contained in \( K_f \).

**Lemma 3.2.** For every \( n \in \mathbb{N} \) and every \( \lambda \in M_f \),

\[
\log \left( \sup_{z \in \mathbb{C}} [(f_\lambda^n)^\#(z)] \right) \leq (2 \log d)n + \frac{d-1}{d} \sup_{z \in \mathbb{C}} \left| \log \left( \frac{|z, \infty|}{|z_\lambda, \infty|} \right) \right|.
\]

**Proof.** For every \( n \in \mathbb{N} \), every \( \lambda \in M_f \), and every \( z \in \mathbb{C} \), by Theorem 3.1 we have \( |(f_\lambda^n)'(z)| \leq (d^n)^2 e^{(d^n-1)|g_\lambda(z)|} \), and by the definition (2.1) of \( g_\lambda \), we have \( 0 \leq (d^n-1)g_\lambda(z) = g_\lambda(f_\lambda^n(z)) - g_\lambda(z) \), so that

\[
(f_\lambda^n)^\#(z) = [(f_\lambda^n)'(z)] \cdot \frac{|f_\lambda^n(z), \infty|^2}{|z, \infty|^2} \leq d^{2n} e^{g_\lambda(f_\lambda^n(z)) - g_\lambda(z)} \cdot e^{2|\log(f_\lambda^n(z), \infty) - \log(z, \infty)|} \leq d^{2n} \cdot e^{2(g_\lambda(f_\lambda^n(z)) + |\log(f_\lambda^n(z), \infty)|) - 2(g_\lambda(z) + |\log(z, \infty)|)} \leq d^{2n} \cdot e^{4 \sup_{\mathbb{C}} |g_\lambda + \log(z, \infty)|}.
\]

This with (2.2) completes the proof. \( \square \)

Recalling the latter half of Lemma 2.1 we can set

\[
C_{B_f} := \sup_{(\lambda, z) \in B_f \times \mathbb{C}} \left| \log \left( \frac{|z, \infty|}{|f_\lambda(z), \infty|} \right) \right| < \infty.
\]

Then for every \( n \in \mathbb{N} \), by Lemmas 3.1 and 3.2 we have

\[
\inf_{B_f} \log |F_n| \geq - \frac{1}{d-1} \left( (d+1) \log 2 + (2 \log d)(n-1) + \frac{4C_{B_f}}{d-1} \right) + \log(\sqrt{2} - 1).
\]
On the other hand, for every \( n \in \mathbb{N} \) and every \( \lambda \in M_f \), by Buff [8, Theorem 1], we also have \( F_n(\lambda) = f_n^{\ast}(c_0(\lambda)) \in K_{f_n} \subset \mathbb{D}(2) \). Hence for every \( n \in \mathbb{N} \), we have the following uniform estimate

\[
(3.1) \quad \sup_{B_f} \| \log |F_n| \| \leq \frac{1}{d - 1} \left( (d+1) \log 2 + (2 \log d)(n-1) + \frac{4C_{B_f}}{d-1} + (d-1) \log (\sqrt{2}+1) \right) =: t_n.
\]

Now let us recall the following classical theorem from the Nevanlinna theory; for a modern formulation, see [30].

**Theorem 3.2** (Selberg [26, p. 311]). Let \( V \) be a bounded and at most finitely connected domain in \( \mathbb{C} \) whose boundary components are piecewise real analytic Jordan closed curves, so that for every \( y \in V \), the Green function \( G_V(\cdot, y) \) on \( V \) with pole \( y \) exists and extends continuously to \( \mathbb{C} \) by setting \( \equiv 0 \) on \( \mathbb{C} \setminus V \). If \( V \) is in \( \mathbb{C} \setminus \{0\} \), then for every \( y \in V \) and every \( r > 0 \), setting \( \theta_V(r) := \int_{\{\theta \in [0,2\pi] : re^{i\theta} \in V\}} d\theta \in [0,2\pi] \), we have

\[
(3.2) \quad \int_0^{2\pi} G_V(re^{i\theta}, y) \frac{d\theta}{2\pi} \leq \min \left\{ \frac{\pi}{2} \frac{\tan \theta_V(r)}{4}, \log^+ \frac{r}{\inf_{z \in V} |z|} \right\}.
\]

Let \( H_1 \) be the component of \( H_f \) containing 0 and set

\[
C_0 := \pi + \int_0^{\infty} \frac{2r}{(1+r^2)^2} \log^+ \frac{r}{\sup\{t > 0 : D(t) \subset H_1\}} \, dr + \int_{H_1} G_{H_1}(\cdot, 0) \omega < \infty.
\]

Fix \( n \in \mathbb{N} \). Recall that \( \deg F_n = d^{n-1} \).

**Claim 1.**

\[
\int_{F_n^{-1}(D(e^{-t_n})]} |\log |F_n| - d^{n-1} \cdot g_{t_n}|| \omega \leq \omega(F_n^{-1}(D(e^{-t_n})])t_n + C_0.
\]

**Proof.** By (3.1), we have \( \inf_{B_f} |F_n| \geq e^{-t_n} \). Let \( \mathcal{F} \) be the family of all components of \( F_n^{-1}(D(e^{-t_n})) \), so that \( \# \mathcal{F} \leq d^{n-1} \). By the description of \( H_f \) in Subsection 2.1, every \( V \in \mathcal{F} \) is a piecewise real analytic Jordan domain in \( H_f \setminus I_{t_0} \) and, since any zero of \( F_n \) is also simple, for every \( V \in \mathcal{F} \), the restriction \( F_n|_V : V \to D(e^{-t_n}) \) is conformal. For every \( V \in \mathcal{F} \), set \( \lambda_V := (F_n|_V)^{-1}(0) \). Let \( V_0 \) be the element of \( \mathcal{F} \) containing 0. Recall the notation in Theorem 3.2. For every \( V \in \mathcal{F} \), by the conformal invariance of the Green functions, we have

\[
\log \frac{e^{-t_n}}{|F_n|} = G_{D(e^{-t_n})}(F_n, 0) = G_V(\cdot, \lambda_V) \quad \text{on} \; V.
\]

For every \( r > 0 \), fixing such \( V_r \in \mathcal{F} \setminus \{V_0\} \) that for every \( V \in \mathcal{F} \setminus \{V_0\} \), \( \theta_{V_r}(r) \geq \theta_V(r) \) (so in particular that for every \( V \in \mathcal{F} \setminus \{V_0, V_r\}, \theta_V(r) \in [0,\pi] \).
since $2\pi \geq \theta_V(r) + \theta_V(r) \geq 2\theta_V(r) \geq 0$, we have

$$\sum_{V \in \mathcal{F}} \int_0^{2\pi} G_V(re^{i\theta}, \lambda_V) \frac{d\theta}{2\pi} = \sum_{V \in \mathcal{F} \setminus \{V_0\}} \int_0^{2\pi} G_V(re^{i\theta}, \lambda_V) \frac{d\theta}{2\pi} + \int_0^{2\pi} G_{V_0}(re^{i\theta}, 0) \frac{d\theta}{2\pi}$$

$$\leq \left( \sum_{V \in \mathcal{F} \setminus \{V_0, V_r\}} \left( \frac{\pi}{2} \tan \frac{\theta_V(r)}{4} + \log^+ \frac{r}{\inf_{z \in V_r} |z|} \right) \right) + \int_0^{2\pi} G_{H_1}(re^{i\theta}, 0) \frac{d\theta}{2\pi}$$

$$\leq \frac{\pi}{2} \cdot \sum_{V \in \mathcal{F} \setminus \{V_0, V_r\}} \frac{\theta_V(r)}{\pi} + \log^+ \sup\{r \in (0; \sup \{\mathbb{D}(t) \subset H_1\}) \} + \int_0^{2\pi} G_{H_1}(re^{i\theta}, 0) \frac{d\theta}{2\pi},$$

where the first inequality is by (3.2) and the monotonicity of the Green functions, and the second inequality is by $\theta_V(r) \in [0, \pi]$ for every $V \in \mathcal{F} \setminus \{V_0, V_r\}$. Hence, since $t_n \geq 0$, we have

$$\int_{F_n^{-1}(\mathbb{D}(e^{-t_n}))} |\log |F_n|| = \int_{F_n^{-1}(\mathbb{D}(e^{-t_n}))} (-\log |F_n|)\omega$$

$$= \omega(F_n^{-1}(\mathbb{D}(e^{-t_n})))t_n + \int_0^{2\pi} \frac{2rdr}{1+r^2} \sum_{V \in \mathcal{F}} \int_0^{2\pi} G_V(re^{i\theta}, \lambda_V) \frac{d\theta}{2\pi}$$

$$\leq \omega(F_n^{-1}(\mathbb{D}(e^{-t_n})))t_n + C_0,$$

which completes the proof. \hfill \Box

**Claim 2.** $\sup_{\mathbb{C} \setminus F_n^{-1}(\mathbb{D}(e^{-t_n}))} |\log |F_n| - d^{n-1} \cdot g_{t_n} | \leq t_n.$

**Proof.** By the description of $H_f$ in Subsection 2.1, the function $|\log |F_n| - d^{n-1} \cdot g_{t_n}$ is not only harmonic on $I_{t_0}$ but also bounded around $\infty$ so, by the removable singularity theorem for subharmonic functions twice, extends harmonically to $I_{t_0} \cup \{\infty\}$. Applying the maximum principle to this harmonic extension on $I_{t_0} \cup \{\infty\}$ twice, by $g_{t_0} \equiv 0 \in M_f$ and (3.1), we have $\sup_{I_{t_0}}|\log |F_n| - d^{n-1} \cdot g_{t_0} | \leq \sup_{B_f}|\log |F_n|| \leq t_n$ (cf. [14] the proof of Lemma 4.1)). Similarly, applying the maximum principle twice to the restriction of $|\log |F_n|$ on $M_f \setminus F_n^{-1}(\mathbb{D}(e^{-t_n}))$, which is harmonic on the interior of $M_f \setminus F_n^{-1}(\mathbb{D}(e^{-t_n}))$, by $g_{t_0} \equiv 0 \in M_f$ and (3.1), we have $\sup_{M_f \setminus F_n^{-1}(\mathbb{D}(e^{-t_n}))} |\log |F_n| - d^{n-1} \cdot g_{t_0} | \leq \sup_{B_f \cup F_n^{-1}(\partial \mathbb{D}(e^{-t_n}))}|\log |F_n|| \leq t_n$. Now the proof is complete. \hfill \Box

**Remark 3.1.** The proof of Claim 2 is independent of the possibility of the existence of a queer component of the interior of $M_f$.

By Claims 1 and 2 we have the following $L^1(\omega)$ estimate

$$\int_{\mathbb{P}^1} |\log |F_n| - d^{n-1} \cdot g_{t_n}|\omega$$

$$\leq (\omega(F_n^{-1}(\mathbb{D}(e^{-t_n})))t_n + C_0) + \omega((\mathbb{C} \setminus F_n^{-1}(\mathbb{D}(e^{-t_n})))t_n = t_n + C_0,$$
so (1.2) holds.

Recalling (2.3), we also have $\delta_0 = d^n \cdot T_f = (d - 1) \cdot \delta^n(\log |F_n| - d^n \cdot g_{i_m})$ on $\mathbb{P}^1$, so that by Green’s theorem, for every $\phi \in C^2(\mathbb{P}^1)$, the estimate (3.3) yields

\[
\left| \int_{\mathbb{P}^1} \phi ((d - 1) \cdot F_n^* \delta_0 - d^n \cdot T_f) \right| \leq \left( \sup_{\mathbb{P}^1} \frac{\delta^n(\log |F_n| - d^n \cdot g_{i_m})}{\omega} \right) \cdot (d - 1)(t_n + C_0),
\]

so (1.3) holds. Now the proof of Theorem 1 is complete.

\[\square\]

4. PROOF OF THEOREM 2

Let $f : \mathbb{C} \times \mathbb{P}^1 \to \mathbb{P}^1$ be the unicritical polynomials family of degree $d > 1$ defined as (1.1). Recall the definitions (and properties) of $\Phi_{f,n}(\lambda, z) \in \mathbb{Z}[\lambda, z]$, $p_{f,n}(\lambda, w) \in \mathbb{Z}[\lambda, z]$, and $Fix_f^*(\lambda, n)$ in Subsection 1.2. For every $n \in \mathbb{N}$, it would be convenient to set

\[P_n^*(\lambda, w) = P_{f,n}^*(\lambda, w) := \frac{p_{f,n}^*(\lambda, w)}{d^{\nu(n)}} \in \mathbb{Q}[\lambda, w],\]

so that for every $w \in \mathbb{C}$, $P_n^*(\lambda, w) \in \mathbb{C}[\lambda]$ is monic.

**Lemma 4.1.** For every $n \in \mathbb{N}$ and every $\lambda \in \mathbb{C}$, we have

\[P_n^*(\lambda, 0) = ((-1)^{\nu(n)} \cdot \Phi_{f,n}^*(\lambda, 0))^{d-1} = \left( (-1)^{\nu(n)} \prod_{m \in \mathbb{N} : m \mid n} F_m(\lambda)^{\mu(n/m)} \right)^{d-1}\]

(up to multiplication in $n$-th roots of unity). For every $n > 1$, we have $0 \not\in (P_n^*(\lambda, 0))^{-1}(0)$. For every $n \in \mathbb{N}$ and every $\lambda \in \mathbb{C}$, if $\lambda \in (P_n^*(\cdot, 0))^{-1}(0)$, then $(c_0(\lambda) = 0) \in Fix_f^*(\lambda, n)$ and $\lambda$ is a zero of $P_n^*(\cdot, 0)$ of the order $d - 1$.

**Proof.** For every $n \in \mathbb{N}$ and every $\lambda \in \mathbb{C}$, by the chain rule and the equalities $f_\lambda'(z) = d \cdot z^{d-1}$ and $Fix_f^*(\lambda, n) = (\Phi_{f,n}^*(\lambda, \cdot))^{-1}(0)$, we have

\[\left( p_{f,n}^*(\lambda, 0) \right)^n = \prod_{z \in Fix_f^*(\lambda, n)} \left( f_\lambda^n(z) \right)^n = \frac{d^{\nu(n)}}{(\lambda^d)^n} ((-1)^{\nu(n)} \cdot \Phi_{f,n}^*(\lambda, 0))^{(d-1)n},\]

which (with the definition of $F_m$) yields (1.1). For every $m \in \mathbb{N}$, even by a direct computation, $0$ is a simple zero of $F_m$ in $\mathbb{C}$, so that for every $n > 1$, $0 \not\in (P_n^*(\cdot, 0))^{-1}(0)$ by $\sum_{m \in \mathbb{N} : m \mid n} \mu(n/m) = 0$ and the latter equality in (1.1). For every $n \in \mathbb{N}$ and every $\lambda_0 \in (P_n^*(\cdot, 0))^{-1}(0)$, by the former equality in (1.1), we have $(c_0(\lambda_0) = 0) \in Fix_f^*(\lambda_0, n)$, which with $f_\lambda'(0) = (f_\lambda^n)'(\lambda_0) = 0 \neq 1$ implies even $0 \in Fix_f^*(\lambda_0, n)$. Then by the latter equality in (1.1), $\lambda_0$ is a zero of $P_n^*(\cdot, 0)$ of order $d - 1$ since any zero of $F_n$ is in fact simple.

\[\square\]

Recall the definitions of the sequences $(\sigma_0(n))$ and $(\sigma_1(n))$ in $\mathbb{N}$ (in Notation 1.2).
4.1. **Proof of (1.6).** For every \( n \in \mathbb{N} \), the estimate (3.3) together with (1.4) and (4.1) yields the following \( L^1(\omega) \) estimate

\[
(4.2) \quad \int_{\mathbb{P}^1} \left| \log |P_n^*(\cdot, 0)| - (d - 1)\nu(n)\frac{g_{\lambda_0}}{d} \right| \omega \leq t_n^* + (d - 1)C_0 \cdot \sigma_0(n),
\]

where we set

\[
t_n^* := (d - 1) \sum_{m \in \mathbb{N}: m \mid n} t_m = (2 \log d)\sigma_1(n)
\]

\[
+ \left( (d + 1) \log 2 - 2 \log d + \frac{4C_B l}{d - 1} + (d - 1) \log (\sqrt{2} + 1) \right) \sigma_0(n).
\]

Recall that \( H_1 \) is by definition the component of \( H_f \) containing 0, and set

\[
C_0^* := \pi + \int_0^\infty \frac{2r}{(1 + r^2)^2} \log^+ \frac{r}{\sup\{t > 0 : \mathbb{D}(t) \subset H_1\}} dr.
\]

In the rest of this subsection, for every \( n > 1 \), we also point out a slightly better estimate

\[
(4.3) \quad \int_{\mathbb{P}^1} \left| \log |P_n^*(\cdot, 0)| - (d - 1)\nu(n)\frac{g_{\lambda_0}}{d} \right| \omega \leq t_n^* + (d - 1)C_0^*
\]

than (4.2). In particular, by Green’s theorem, for every \( \phi \in \mathcal{C}^2(\mathbb{P}^1) \) and every \( n > 1 \), we have

\[
(1.6) \quad \left| \int_{\mathbb{P}^1} \phi d(\text{Per}_f(n, 0) - \nu(n) \cdot T_j) \right| \leq \left( \sup_{\mathbb{P}^1} \left| \frac{d\phi}{\omega} \right| \right) \cdot (t_n^* + (d - 1)C_0^*),
\]

which implies (1.6).

**Proof of (4.3).** For every \( n \in \mathbb{N} \), by (4.1) and (3.1), we have

\[
(5.1) \quad \sup_{B_f} \left| \log |P_n^*(\cdot, 0)| \right| \leq t_n^*,
\]

which is a counterpart to (5.1). Fix \( n > 1 \). By (5.1), \( \inf_{\lambda \in B_f} |P_n^*(\lambda, 0)| \geq e^{-t_n^*} \). As in the proof of Claim 1 in Section 3, let \( \mathcal{F}^* \) be the family of all components of \((P_n^*(\cdot, 0))^{-1}(\mathbb{D}(e^{-t_n^*}))\). By Lemma 4.1 and the description of \( H_f \) in Subsection 2.1, every \( V \in \mathcal{F}^* \) is a piecewise real analytic Jordan domain in \( H_f \setminus (\mathcal{I}_{v_0} \cup H_1) \) now, and for every \( V \in \mathcal{F}^* \), the restriction \( P_n^*(\cdot, 0)|V : V \rightarrow \mathbb{D}(t_n^*) \) is a proper holomorphic mapping of degree \( d - 1 \) now and \( \#((P_n^*(\cdot, 0))^{-1}(0)) \cap V = 1 \). For every \( V \in \mathcal{F}^* \), letting \( \lambda_V \) be the unique point in \(((P_n^*(\cdot, 0))^{-1}(0)) \cap V \), by Myrberg’s theorem [22], we now have

\[
\log \frac{e^{-t_n^*}}{|P_n^*(\cdot, 0)|} = G_{\mathbb{D}(e^{-t_n^*})}(P_n^*(\cdot, 0), 0) = (d - 1) \cdot G_V(\cdot, \lambda_V) \quad \text{on } V.
\]

Recalling \( t_n^* \geq 0 \), by a computation similar to that in the proof of Claim 1 in Section 3, we have

\[
\int_{((P_n^*(\cdot, 0))^{-1}(\mathbb{D}(e^{-t_n^*})))} \left| \log |P_n^*(\cdot, 0)| - \nu(n)(d - 1)\frac{g_{\lambda_0}}{d} \right| \omega
\]

\[
\leq \omega((P_n^*(\cdot, 0))^{-1}(\mathbb{D}(e^{-t_n^*})))t_n^* + (d - 1)C_0^*.
\]
Moreover, by the same argument as that in the proof of Claim 2 in Section 3, we also have \( \sup_{C \setminus (P_\nu(\cdot, 0) - 1)} |\log |P_n^\nu(\cdot, 0)| - \nu(n)(d - 1)\frac{g_{\nu}}{d}| \leq t_n^* \). Hence (4.3) holds.

4.2. Proof of (4.7). As an application of (4.3), we also point out the following \( L^1(\omega) \) estimate

\[
\int_{\mathbb{P}^1} \left| \int_0^{2\pi} \log |P_n^\nu(\lambda, re^{i\theta})| \frac{d\theta}{2\pi} - \nu(n)(d - 1)\frac{g_{\nu}}{d} \left| \omega(\lambda) \leq t_n^* + (d - 1)C_0^* \right.
\]

for every \( n > 1 \) and every \( r \in (0, 1] \) (cf. [3, 2. in Theorem 3.1]). In particular, by Green’s theorem, for every \( \phi \in C^2(\mathbb{P}^1) \), every \( n > 1 \), and every \( r \in (0, 1] \), we will have

\[
\int_{\mathbb{P}^1} \phi \left( \int_0^{2\pi} \text{Per}_n^\nu(\lambda, re^{i\theta}) \frac{d\theta}{2\pi} - \nu(n) \cdot T_f \right) \leq \left( \sup_{\mathbb{P}^1} \left| \frac{dd^c\phi}{\omega} \right| \cdot (t_n^* + (d - 1)C_0^*) \right.
\]

which implies (4.7).

**Proof of (4.3).** For every \( n \in \mathbb{N} \) and every \( \lambda \in \mathbb{C} \setminus (H_f \setminus I_{c_0}) \), we have \( \inf_{z \in \text{Fix}^*_{\nu}(\lambda, n)} |(f_\lambda^n)'(z)| \geq 1 \). Recall the description of components of \( H_f \setminus I_{c_0} \) in Subsection 2.1. For every \( n \in \mathbb{N} \), letting \( H_n^* \) be the union of all components \( U \) of \( H_f \setminus I_{c_0} \) such that \( n_U = n \) (so e.g. \( H_1^* = H_1 \)), there is a holomorphic function \( \lambda \mapsto z_{\lambda} \) on \( H_n^* \) such that for every \( \lambda \in H_n^* \), \( z_{\lambda} \in \text{Fix}^*_{\nu}(\lambda, n) \) and that \((f_\lambda^n)'(z_{\lambda}) \equiv \phi_U(\lambda)\) on each component \( U \) of \( H_n^* \). Fix \( n > 1 \) and \( r \in (0, 1] \), and set \( H_n^*(r) := \{ \lambda \in H_n^* : (f_\lambda^n)'(z_{\lambda}) \in \mathbb{D}(r) \} \). For every \( \lambda \in \mathbb{C} \), by the definitions of \( P_{n, n}^* \) and \( p_{n, n}^* \), we have

\[
\int_0^{2\pi} \log |P_n^\nu(\lambda, re^{i\theta})| \frac{d\theta}{2\pi} = \frac{1}{n} \sum_{z \in \text{Fix}^*_{\nu}(\lambda, n)} \log \max \{ r, |(f_\lambda^n)'(z_{\lambda})| \} - \nu(n) \log d
\]

\[
= \log |P_n^\nu(\lambda, 0)| + \begin{cases} \frac{1}{n} \sum_{j=0}^{n-1} \log |(f_\lambda^j)'(f_\lambda^j(z_{\lambda}))| & \text{if } \lambda \in H_n^*(r), \\ 0 & \text{if } \lambda \in \mathbb{C} \setminus H_n^*(r), \end{cases}
\]

which with (4.3) and the chain rule yields

\[
\int_{\mathbb{P}^1} \left| \int_0^{2\pi} \log |P_n^\nu(\lambda, re^{i\theta})| \frac{d\theta}{2\pi} - \nu(n)(d - 1)\frac{g_{\nu}}{d} \left| \omega(\lambda) \leq (t_n^* + (d - 1)C_0^*) + \int_{H_n^*(r)} \log r \frac{r}{|(f_\lambda^m)'(z_{\lambda})|} \omega(\lambda). \right. \right.
\]

For every component \( V \) of \( H_n^*(r) \), letting \( U \) be the component of \( H_n^*(1) \) containing \( V \), the restriction \( \phi_U|V : V \to \mathbb{D}(r) \) is a proper holomorphic mapping of degree \( d - 1 \), so letting \( \lambda_V \) be the unique point in \( V \cap \phi_U^{-1}(0) \), by Myrberg’s theorem [22], we have

\[
\log r = G_{\mathbb{D}(r)}((\phi_U|V)(\lambda), 0) = (d - 1) \cdot G_V(\lambda, \lambda_V) \quad \text{on } V.
\]
Noting that $H_n^* \subset H_f \setminus (I_{c_0} \cup H_1)$, by a computation similar to that in the proof of Claim 1 in Section 3, we have
\[
\int_{H_n^*(r)} \log \frac{r}{|f_n'(z_\lambda)|} \omega(\lambda) \leq (d - 1) \cdot C_0^*.
\]
Hence (4.3) holds.

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