Intrinsic Time in Wheeler–DeWitt Conformal Superspace

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Abstract—Intrinsic time in geometrodynamics is obtained using a scaled Dirac mapping. By addition of a background metric, one can construct a scalar field which is suitable for the role of intrinsic time. The Cauchy problem was successfully solved in conformal variables because they are physical. Intrinsic time as a logarithm of the spatial metric determinant was first applied to a cosmological problem by Misner. Global time exists under the condition of a constant mean curvature slicing of spacetime. A coordinate volume of a hypersurface and the so-called York’s mean time are a canonical conjugated pair. So, the volume is intrinsic global time by its sense. The experimentally observed redshift in cosmology is an evidence of its existence.

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1. INTRODUCTION

Geometrodynamics is a theory of space and time in its inner essence. The spatial metric $γ_{ij}$ carries information about intrinsic time. Intrinsic time for cosmological models is constructed of the inner metric characteristic of space. Time is to be a scalar relative to diffeomorphisms of changing the coordinates of space. For this requirement, we use a bimetric formalism, adding some auxiliary spatial metric. Thus we naturally come to interpretations of the observational data from the concepts of Conformal gravity. The generalized Dirac mapping [1] allows for intrinsic time extraction. The spatial metric is factorized into the inner time factor and the conformal metric.

Dirac’s mapping reflects a transition to physical (conformal) variables. In the spirit of the ideas of Conformal cosmology [2], the conformal metric is a metric of space where we live and make observations. The choice of conformal measurement standards allows us to separate the cosmic evolution of observation devices from the evolution of cosmic objects. Thus we avoid the unpleasant artifact of an expanding Universe and the inevitable Big Bang problem in standard cosmology. After performing a de-parametrization procedure in a model of our isotropic Universe, the coordinate volume of the Universe determines its global intrinsic time. In modern papers [3–9] one can see applications of a local intrinsic time interval in geometrodynamics.

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Werner Heisenberg, in the chapter “Quantum mechanics and a talk with Einstein (1925–1926)” [10], quoted Einstein’s statement: “But in principle, it is quite wrong to try founding a theory on observable magnitudes alone. In reality the very opposite happens. It is the theory which decides what we can observe.” This talk between two great scientists on the philosophy of positivism and a status of observable quantities in physical theory (quantum mechanics or general relativity) remains relevant nowadays.

2. ADM VARIATIONAL FUNCTIONAL OF GENERAL RELATIVITY. NOTATIONS

Einstein’s general relativity (GR) was presented in a Hamiltonian form half a century ago [11]. Dirac claimed that the 4D space-time symmetry is not a fundamental property of the physical world. Instead of space-time transformations, one should consider canonical transformations of the phase space variables.

The ADM formalism, based on the Palatini approach, was developed by Arnowitt, Deser and Misner in 1959 [12]. The formalism supposes that a spacetime with the interval 
$$g = g_{μν}(t,x)dx^μ \otimes dx^ν$$
is foliated into a family of spacelike surfaces $Σ_t$ labeled by the time coordinate $t$, with spatial coordinates $x^i$
on each slice. The space-time metric tensor in the ADM form looks like

\[
(g_{\mu\nu}) = \begin{pmatrix} -N^2 + N_i N^i & N_i \\ N_j & \gamma_{ij} \end{pmatrix}.
\]

(1)

The physical meaning of the metric components are the following: the lapse function \(N(t; x, y, z)\) determines an increment of coordinate time \(t\), and the shift vector \(N_i(t; x, y, z)\) determines changes of hypersurface coordinates under a transition to an infinitesimally close hypersurface.

The first quadratic form

\[
\gamma = \gamma_{ik}(t, x)dx^i \otimes dx^k
\]

(2)
defines the induced metric on every slice \(\Sigma_t\). The components of the spatial metric \(\gamma_{ij}(t; x, y, z)\) contain three gauge functions describing the spatial coordinates. The three remaining components describe two polarizations of gravitational waves and many-fingered time. So, we defined the foliation \((\Sigma_t, \gamma_{ij})\).

The group of general coordinate transformations conserving such a foliation was found by Zel’manov [13], this group involves the reparametrization subgroup of coordinate time. This means that the coordinate time, which is not invariant with respect to gauges, in general case, is not observable. A large number of papers were devoted to the choice of reference frames (see, for example, the monograph [14] and references therein).

The components of the extrinsic curvature tensor \(K_{ij}\) of every slice are constructed from the second quadratic form of the hypersurface and can be defined as

\[
K_{ij} := -\frac{1}{2} \mathcal{L}_n \gamma_{ij},
\]

(3)

where \(\mathcal{L}_n\) denotes a Lie derivative along the direction of \(n^a\), a timelike unit normal to the slice. The components of the extrinsic curvature tensor can be found by the formula

\[
K_{ij} = \frac{1}{2N} (\nabla_i N_j + \nabla_j N_i - \gamma_{ij}),
\]

(4)

where \(\nabla_k\) is a Levi–Civita connection associated with the metric \(\gamma_{ij}\):

\[
\nabla_k \gamma_{ij} = 0.
\]

The Hamiltonian dynamics of GR is built in an infinite-dimensional degenerated phase space of 3-metrics \(\gamma_{ij}(x, t)\) and their momentum densities \(\pi^{ij}(x, t)\). The latter are expressed through the extrinsic curvature tensor

\[
\pi^{ij} := -\sqrt{\gamma}(K^{ij} - K \gamma^{ij}),
\]

(5)

where we have introduced the notations

\[
K^{ij} := \gamma^{ik} \gamma^{jl} K_{kl}, \quad K := \gamma^{ij} K_{ij},
\]

\[
\gamma := \det |\gamma_{ij}|, \quad \gamma_{ij} \gamma^{ij} = \delta_k^i.
\]

(6)
The Poisson bracket is a bilinear operation on two arbitrary functionals \(F[\gamma_{ij}, \pi^{ij}]\) and \(G[\gamma_{ij}, \pi^{ij}]\) [15]:

\[
\{F, G\} = \int d^3x \left( \frac{\partial F}{\partial \gamma_{ij}(t, x)} \frac{\partial G}{\partial \pi^{ij}(t, x)} - \frac{\partial F}{\partial \pi^{ij}(t, x)} \frac{\partial G}{\partial \gamma_{ij}(t, x)} \right).
\]

(7)
The canonical variables satisfy the relation

\[
\{\gamma_{ij}(t, x), \pi^{kl}(t, x')\} = \delta^{kl} \delta(x - x'),
\]

(8)

where

\[
\delta^{kl} := \frac{1}{2} (\delta^k_i \delta^l_j + \delta^l_i \delta^k_j),
\]

and \(\delta(x - x')\) is Dirac’s \(\delta\) function for the volume of \(\Sigma_t\).

The super-Hamiltonian of the gravitational field is the functional

\[
\int_{\Sigma_t} (NH_\perp + N^i H_i) d^3x,
\]

(9)

where \(N\) and \(N^i\) are Lagrange multipliers, \(H_\perp\) and \(H_i\) have the meaning of constraints. Among them,

\[
H_\perp := G_{ijkl} \pi^{ij} \pi^{kl} - \sqrt{\gamma} R(\gamma_{ij})
\]

(10)
is obtained from the Gauss scalar relation of the hypersurface embedding theory and is called the Hamiltonian constraint. Here \(R\) is the Ricci scalar of space,

\[
G_{ijkl} := \frac{1}{2\sqrt{\gamma}} (\gamma_{ik} \gamma_{jl} + \gamma_{il} \gamma_{jk} - \gamma_{ij} \gamma_{kl})
\]

is the supermetric of the 6D hyperbolic Wheeler–DeWitt (WDW) superspace [16]. The momentum constraints

\[
H^i := -2\nabla_j \pi^{ij}
\]

(11)
are obtained from the contracted Codazzi equations of the hypersurface embedding theory. They impose restrictions on the possible data \(\gamma_{ij}(x, t), \pi^{ij}(x, t)\) on a space-like hypersurface \(\Sigma_t\). The divergence law, following from (11), is analogous to the Gauss law in Maxwell’s electrodynamics. The Hamiltonian constraint (10) has no analogue in electrodynamics. It yields the dynamics of the space geometry itself. The Hamiltonian dynamics is built from the ADM variational functional

\[
S = \int_{t_0}^{t_1} dt \int_{\Sigma_t} d^3x \left( \pi^{ij} \frac{d\gamma_{ij}}{dt} - N H_\perp - N^i H_i \right),
\]

(12)
using the ADM units: \( c = 1, 16\pi G = 1 \) [12]. The action (12) is obtained from the Hilbert functional after the procedure of \((3+1)\) foliation and a Legendre transformation.

These constraints are of the first class since they belong to the closed algebra

\[
\{\mathcal{H}_\perp(t, x), \mathcal{H}_\perp(t, x')\} = \{\mathcal{H}^i(t, x) + \mathcal{H}^i(t, x'), \delta_3(x - x')\},
\]

\[
\{\mathcal{H}_i(t, x), \mathcal{H}_\perp(t, x')\} = \mathcal{H}_i(t, x)\delta_3(x - x'),
\]

\[
\{\mathcal{H}_i(t, x), \mathcal{H}_j(t, x')\} = \mathcal{H}_i(t, x')\delta_3(x - x') + \mathcal{H}_j(t, x)\delta_3(x - x').
\]

The Poisson brackets between constraints vanish on the constraints hypersurface. In the presence of matter described by the energy-momentum tensor \( T_{\mu\nu} \), the constraints (10), (11) take the form of Einstein’s equations:

\[
\mathcal{H}_\perp = \sqrt{\gamma}(K_{ij}K^{ij} - K^2) - \sqrt{\gamma}R + \sqrt{\gamma}T_{\perp\perp}, \quad (13)
\]

\[
\mathcal{H}^i = -2\sqrt{\gamma}\nabla_j(K^{ij} - \gamma^{ij}K) + \sqrt{\gamma}(T_{\perp\perp})^i, \quad (14)
\]

where

\[
T_{\perp\perp} := n^\mu n^\nu T_{\mu\nu}
\]

(15)

is the matter density, and

\[
(T_{\perp})_i := n^\mu T_{i\mu}
\]

(16)

is the matter momentum density in a normal observer’s (Euler observer’s) reference frame. The Hamiltonian constraint (13) can be expressed in the momentum variables (5):

\[
\mathcal{H}_\perp = \frac{1}{\sqrt{\gamma}}\left(\pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2\right) - \sqrt{\gamma}R + \sqrt{\gamma}T_{\perp\perp}, \quad (17)
\]

as far as

\[
K^{ij} = -\frac{1}{\sqrt{\gamma}}\left(\pi^{ij} - \frac{1}{2}\pi\gamma^{ij}\right), \quad \pi_{ij} := \gamma_{ik}\gamma_{jl}\pi^{kl},
\]

\[
\pi := \gamma_{ij}\pi^{ij}, \quad K = \frac{\pi}{2\sqrt{\gamma}}.
\]

The momentum constraints (14) in the momentum variables are

\[
\mathcal{H}^i = -2\nabla_j\pi^{ij} + \sqrt{\gamma}(T_{\perp})^i. \quad (18)
\]

The Poisson structure (8) is degenerate due to the existence of the constraints (17) and (18). A reduction of the dynamical system on the level of constraints is, in the general case, an open problem.

3. SHAPE DYNAMICS

A. A. Friedmann, in his book [17] on cosmology, found the following remarkable words about the principle of scale invariance: "... moving from country to country, we have to change the scale, i.e., measured in Russia in arshins, Germany—meters, England—feet. Imagine that such a change of scale we had to do from point to point, and then we got the above operation of changing a scale. Scale changing in the geometric world corresponds, in the physical world, to different ways of measuring the length... Properties of the world are divided into two classes: some are independent of the above-said change of scale, better to say, do not change their shape under any scale changes, while others change their shape. Let us agree that real properties of the world belong to the first class and call them scale-invariant. Weyl expands the invariance postulate, adding to it the requirement that all physical laws were scale-invariant properties of the physical world. Consistent with such an extension of the postulate of invariance, we have to demand that the world equations would be expressed in a form satisfactory to not only coordinate, but scale invariance". Radiative breaking of conformal symmetry in a conformally invariant version of the Standard Model of elementary particles was used in [18]. The fruitful idea of initial conformal symmetry of the theory leads to the right value of the Higgs boson mass without using a phenomenological Higgs potential.

Einstein’s GR is covariant under general coordinate transformations. The group of transformations is an infinite-parameter one. The action of the group can be reduced to alternating actions of its two finite-parameter subgroups: the spatial linear group \( SL(3, 1) \) and the conformal group \( SO(4, 2) \). According to Ogievetsky’s theorem [19], invariance under the infinite-parameter general covariance group is equivalent to simultaneous invariance under the affine and the conformal groups. Using an analogy with phenomenological chiral Lagrangians [20], it is possible to obtain a phenomenological affine Lagrangian as a nonlinear joint realization of affine and conformal symmetry groups. A nonlinear realization of the affine group leads to a symmetric tensor field as a Goldstone field. The requirement that the theory correspond simultaneously to a realization of the conformal group as well leads uniquely to the theory of a tensor field whose equations are Einstein’s [21]. York’s method of decoupling of the momentum and Hamiltonian constraints [22] is derived on the basis of a mathematical discovery. The physical principle underlying York’s method is the initial conformal and affine symmetry of the theory, as has been shown in [21].
To recover the initial conformal symmetry of space, one uses an artificial method. One can possibly change the gauge symmetry of GR (spatial diffeomorphisms and local changes of slicing) to a gauge symmetry of form dynamics (spatial diffeomorphisms and local scaling conserving the global slicing) of spacetime [23]. Following [23], let us define a class of metrics $\gamma$ of some hypersurface $\Sigma$ that conserve its volume

$$V_\gamma = \int_{\Sigma} d^3x \sqrt{\gamma(x)},$$

with the help of a conformal mapping of the metric $\gamma_{ij}(x) \rightarrow \exp(4\hat{\phi}(x))\gamma_{ij}(x).$ (19)

Here we have defined the function

$$\hat{\phi}(x) := \phi(x) - \frac{1}{6} \ln \langle e^{6\phi} \rangle_\gamma,$$ (20)

and the averaging operation over a hypersurface $\Sigma$ for some scalar field $f$:

$$\langle f \rangle_\gamma := \frac{1}{V_\gamma} \int_{\Sigma} d^3x \sqrt{\gamma(x)} f(x).$$ (21)

The St{"u}ckelberg scalar field [24] was introduced in space by analogy with Deser’s [25] introducing a Dirac dilaton field [26] in space-time, as well as averaging of functions [27, 28] over an arbitrary manifold. The theorem that the conformal mapping (19) conserves the volume of any hypersurface was proved in [29]. From the definition (20), the conformal factor is expressed as

$$e^{4\hat{\phi}} = \frac{e^{4\phi}}{\langle e^{6\phi} \rangle_\gamma}.$$ (22)

Then the Jacobian of the transformation is transformed by the formula

$$\sqrt{\gamma} \rightarrow e^{6\phi} \sqrt{\gamma}.$$ (23)

A variation of the Jacobian and, accordingly, of the hypersurface volume are

$$\delta \sqrt{\gamma} = \frac{1}{2} \sqrt{\gamma} \gamma^{ab} \delta \gamma_{ab},$$
$$\delta V_\gamma = \frac{1}{2} \sqrt{\gamma} \gamma^{ij} \delta \gamma_{ij}(y).$$ (24)

The volume of the hypersurface $V_\gamma$ is conserved

$$V_\gamma = \int_{\Sigma_t} d^3x \sqrt{\gamma(x)} \rightarrow \int_{\Sigma_t} d^3x e^{6\phi} \sqrt{\gamma(x)} = V_\gamma \frac{\langle e^{6\phi} \rangle_\gamma}{\langle e^{6\phi} \rangle_\gamma} = V_\gamma.$$ (25)

The phase space (the cotangent bundle over $\text{Riem}(\Sigma)$) can be extended with the scalar field $\phi$ and the canonically conjugate momentum density $\pi_\phi$. Let $\mathcal{C}$ be the group of conformal transformations of the hypersurface $\Sigma$:

$$(\gamma_{ij}, \pi_{ij}; \phi, \pi_\phi) \mapsto (\Gamma_{ij}, \Pi_{ij}; \Phi, \Pi_\phi),$$

parameterized by the scalar field $\phi$, with the generation functional [29]

$$F_\phi[\gamma_{ij}, \Pi_{ij}, \phi, \Pi_\phi] := \int_{\Sigma} d^3x \left[ \gamma_{ij}(x)e^{4\phi(x)}\Pi_{ij}(x) + \phi(x)\Pi_\phi(x) \right].$$ (26)

4. CAUCHY PROBLEM IN CONFORMAL GRAVITY

Let us proceed with the solution of the Cauchy problem following York in the conformal variables (denoted by a bar) in detail. “Note that the configuration space that one is led to by the initial-value equations is not superspace (the space of Riemannian three-geometries), but “conformal superspace” for the space whose each point is a conformal equivalence class of Riemannian three-geometries $\times$ the real line (i.e., the time $T$)” [22].

Under the conformal transformation

$$\gamma_{ij} := e^{4\phi}\bar{\gamma}_{ij} \equiv \Psi^4 \bar{\gamma}_{ij}$$ (27)

the matter characteristics are transformed according to their conformal weights. We denote the transformed matter characteristics (15) and (16) as

$$\bar{T}_{\perp\perp} := \Psi^8 T_{\perp\perp}; \quad (\bar{T}_{\perp})^i := \Psi^{10} (T_{\perp})^i.$$ (28)

After the traceless decomposition of $K^{ij}$,

$$K^{ij} = A^{ij} + \frac{1}{3} K \gamma^{ij}, \quad \gamma_{ij} A^{ij} = 0,$$ (29)

we decompose the traceless part of $A^{ij}$ according to

$$A^{ij} = \Psi^{-10} \bar{A}^{ij}.$$ (30)

Then, we obtain the conformal variables

$$\bar{A}^{ij} := \Psi^{10} A^{ij}, \quad \bar{A}_{ij} := \bar{\gamma}_{ik} \bar{\gamma}_{jl} \bar{A}^{kl} = \Psi^{-8} \gamma_{ik} \gamma_{jl} A^{kl} = \Psi^2 A_{ij}.$$ (31)

The Hamiltonian constraint (13) in the new variables

$$\bar{\Delta} \Psi - \frac{1}{8} \bar{R} \Psi + \frac{1}{8} \bar{A}_{ij} \bar{A}^{ij} \Psi^{-7} - \frac{1}{12} K^2 \Psi^5 + \frac{1}{8} \bar{T}_{\perp\perp} \Psi^5 = 0$$ (32)

is called the Lichnerowicz–York equation [30]. Here $\bar{\Delta} := \bar{\nabla}_i \bar{\nabla}^i$ is the conformal Laplacian, $\bar{\nabla}_k$ is the
conformal connection associated with the conformal metric $\tilde{g}_{ij}$,
\[ \nabla_k \tilde{g}_{ij} = 0, \]
$\tilde{R}$ is the conformal Ricci scalar expressed from the Ricci scalar $R$ as
\[ R = \Psi^{-4} \tilde{R} - 8 \Psi^{-5} \Delta \Psi. \tag{29} \]
Lichnerowicz originally considered the differential equation (28) without matter and in the case of a maximal slicing gauge $K = 0$ [31].

The momentum constraints (14) after the decomposition (27) take the form:
\[ \nabla_j \tilde{A}^{ij} - \frac{2}{3} \Psi^6 \nabla^i K + \frac{1}{2} (\tilde{T}_\perp)^i = 0. \tag{30} \]
To solve the Cauchy problem in GR, York elaborated the conformal transverse-traceless method [32]. He made the following decomposition of the traceless part $A_{ij}$:
\[ \tilde{A}^{ij} = (\tilde{L}X)^{ij} + \tilde{A}^{ij}_{TT}, \tag{31} \]
where $\tilde{A}^{ij}_{TT}$ is both traceless and transverse with respect to the metric $\tilde{g}_{ij}$:
\[ \tilde{g}_{ij} \tilde{A}^{ij}_{TT} = 0, \quad \nabla_j \tilde{A}^{ij}_{TT} = 0, \]
$\tilde{L}$ is the conformal Killing operator, acting on the vector field $X$:
\[ (\tilde{L}X)^{ij} := \nabla^i X^j + \nabla^j X^i - \frac{2}{3} \tilde{g}^{ij} \nabla_k X^k. \tag{32} \]
The symmetric tensor $(\tilde{L}X)^{ij}$ is called the \textit{longitudinal part} of $\tilde{A}^{ij}$, whereas $\tilde{A}^{ij}_{TT}$ is called the \textit{transverse part} of $\tilde{A}^{ij}$.

Using York’s longitudinal-transverse decomposition (31), the constraint equation (28) can be rewritten in the following form:
\[ \Delta \Psi - \frac{1}{8} \tilde{R} \Psi \]
\[ + \frac{1}{8} \left[ (\tilde{L}X)_{ij} + \tilde{A}^{ij}_{TT} \right] \left[ (\tilde{L}X)^{ij} + \tilde{A}^{ij}_{TT} \right] \Psi^{-7} \]
\[ - \frac{1}{12} K^2 \Psi^5 + \frac{1}{8} \tilde{T}_\perp \Psi^5 = 0, \tag{33} \]
where the following notations are used:
\[ (\tilde{L}X)_{ij} := \tilde{g}_{ik} \tilde{g}_{jl} (\tilde{L}X)^{kl}, \quad \tilde{A}^{ij}_{TT} := \tilde{g}_{ik} \tilde{g}_{jl} \tilde{A}^{kl}_{TT}, \]
and the momentum equations (30) are
\[ \tilde{\Delta}_L X^i - \frac{2}{3} \Psi^6 \nabla^i K + \frac{1}{2} (\tilde{T}_\perp)^i = 0. \tag{34} \]
The second-order operator $\nabla_j (\tilde{L}X)^{ij}$, acting on the vector $X$, is the \textit{conformal vector Laplacian} $\tilde{\Delta}_L$:
\[ \tilde{\Delta}_L X^i := \nabla_j (\tilde{L}X)^{ij} \]
To obtain Eq. (35), we have used the contracted Ricci identity.

A part of the initial data on $\Sigma_0$ can be freely chosen, and another part is constrained, i.e., determined from the constraint equations (33), (34). One can suggest a constant mean curvature condition on the Cauchy hypersurface $\Sigma_0$:
\[ K = \frac{\pi}{2 \sqrt{\gamma}} = \text{const}. \tag{36} \]
Then the momentum constraints (14) are separated from the Hamiltonian constraint (13) and reduce to
\[ \tilde{\Delta}_L X^i + \frac{1}{2} (\tilde{T}_\perp)^i = 0. \tag{37} \]
Therefore, we obtain the \textit{conformal vector Poisson equation}. It is solvable for closed manifolds, as was proved in [33]. So, we have
\textbf{Free initial data}: the conformal factor $\Psi^4$, the conformal metric $\tilde{g}_{ij}$, the transverse tensor $\tilde{A}^{ij}_{TT}$, and the conformal matter variables $(\tilde{T}_\perp, (\tilde{T}_\perp)^i)$.

\textbf{Constrained data}: the scalar field $K$, the vector $X$ obeying the linear elliptic equations (34).

Note that after solving the Cauchy problem, we are not going to return to the initial variables, unlike York’s approach. The Cauchy problem was successfully solved not by chance after a mathematically formal transition to conformal variables. The point is that we have found just the physical variables.

\section{5. MANY-FINGER INTRINSIC TIME IN GEOMETRODYNAMICS}

Dirac, in search for dynamical degrees of freedom of the gravitational field, introduced the \textit{conformal field variables} $\tilde{\gamma}_{ij}, \tilde{\pi}^{ij}$ [1]
\[ \tilde{\gamma}_{ij} := \tilde{\gamma}_{ij}, \quad \tilde{\pi}^{ij} := \sqrt{\gamma} \left( \pi^{ij} - \frac{1}{3} \gamma^{ij} \right), \tag{38} \]
i.e., our choice of the conformal factor (24):
\[ \Psi = \gamma^{1/12}. \tag{39} \]
Among them are only five independent pairs $(\tilde{\gamma}_{ij}, \tilde{\pi}^{ij})$ per space point, since the determinant of the conformal metric is equal to unity, and the conformal momentum density matrix is traceless:
\[ \tilde{\gamma} := \det ||\tilde{\gamma}_{ij}|| = 1, \quad \tilde{\pi} := \tilde{\gamma}_{ij} \tilde{\pi}^{ij} = 0. \]
The remaining sixth pair $(D, \pi_D)$
\[ D := -\frac{1}{6} \ln \gamma, \quad \pi_D := 2\pi \tag{40} \]
To use an auxiliary metric for a generic case of spatial manifold with arbitrary topology, let us take a local tangent space $T(\Sigma_t)_x$ as a background space for every local region of our manifold $(\Sigma_t)$. In every local tangent space we define a set of three linearly independent vectors $e^a_i$ (dreibein), numered by the first Latin indices $a, b$. The components of the background metric tensor in the tangent space are

$$e^a_i e_b = f_{ab}.$$ 

Along with the dreibein, we introduce three mutual vectors $e^a_i$ defined by the orthogonality conditions

$$e^a_i e^b = \delta^a_i, \quad e^a_i e^a = \delta^i.$$

Then, we can construct three linearly independent Cartan forms, which are invariant under diffeomorphisms [20],

$$\omega^a(d) = e^a_i dx^i. \quad (42)$$

The background metric is defined by the differential forms (42):

$$f = f_{ab} \omega^a(d) \otimes \omega^b(d) = f_{ij} dx^i \otimes dx^j, \quad (43)$$

where the components of the background metric tensor in a coordinate basis are

$$f_{ij} = f_{ab} e^a_i e_b.$$

The components of the inverse background metric denoted by $f^{ij}$ satisfy the condition

$$f^{ik} f_{kj} = \delta^i_j.$$

Now, we can compare the background metric with the metric of the gravitational field at every point of the manifold $\Sigma_t$ due to bijectivity of the mapping

$$\Sigma_t \leftrightarrow T(\Sigma_t)_x.$$

The Levi–Civita connection $\nabla_k$ is associated with the background metric $f_{ij}$:

$$\nabla_k f_{ij} = 0.$$

Let us define scaled Dirac’s conformal variables $(\bar{\gamma}_{ij}, \bar{\pi}^{ij})$ by the formulas

$$\bar{\gamma}_{ij} := \frac{\gamma_{ij}}{\sqrt{\gamma/f}}, \quad \bar{\pi}^{ij} := \sqrt{\frac{f}{2}} \left( \pi^{ij} - \frac{1}{3} \pi \gamma^{ij} \right), \quad (44)$$

where, in addition to the determinant $\gamma$ defined in (6), the background metric determinant $f$ has appeared,

$$f :=\det(f_{ij}).$$

The conformal metric $\bar{\gamma}_{ij}$ (44) is a tensor field, i.e., it transforms according to the tensor representation of the group of diffeomorphisms. The scaling variable $(\gamma/f)$ is a scalar field, i.e., it is invariant under the diffeomorphisms.
We add to the conformal variables (44) a canonical pair: local intrinsic time \( D \) and the Hamiltonian density \( \pi_D \) in the following way:

\[
D := -\frac{1}{6} \ln \left( \frac{\gamma}{f} \right), \quad \pi_D := 2\pi. \tag{45}
\]

Equations (44) and (45) define the scaled Dirac mapping as a mapping of the fiber bundles

\[
(\gamma_{ij}, \pi^{ij}) \mapsto (D, \pi_D; \tilde{\gamma}_{ij}, \tilde{\pi}^{ij}). \tag{46}
\]

Riemannian superspaces of the metrics of \( (3M) \) are defined on a compact Hausdorff manifold \( \Sigma_L \). Denote the set whose each point represents all respective Riemannian metrics as \( \text{Riem}(3M) \). Since the same Riemannian metric can be written in different coordinate systems, we identify all points associated with coordinate transformations of the diffeomorphism group \( \text{Diff}(3M) \). All points received from some point by a coordinate transformations of the group form its orbit. In this way, the WDW superspace is defined as a coset:

\[
(3\mathfrak{G}) := \text{Riem}(3M)/\text{Diff}(3M).
\]

As a saying goes [36, 37], the superspace is the arena of geometrodynamics. Denote by \( (3\mathfrak{G})^* \) the space of corresponding canonically conjugate momentum densities. According to the scaled Dirac mapping (46), we have a functional mapping of the WDW phase superspace of metrics \( \gamma_{ij} \) and the corresponding densities of their momenta \( \pi_{ij} \) to the WDW conformal superspace of metrics \( \tilde{\gamma}_{ij} \) and momentum densities \( \tilde{\pi}^{ij} \), local intrinsic time \( D \), and the Hamiltonian density \( \pi_D \):

\[
(3\mathfrak{G}) \times (3\mathfrak{G})^* \mapsto (3\tilde{\mathfrak{G}}) \times (3\tilde{\mathfrak{G}})^*.
\]

We can conclude that the WDW phase superspace

\[
(3\tilde{\mathfrak{G}}) \times (3\tilde{\mathfrak{G}})^*
\]

is an extended one if we draw an analogy with relativistic mechanics [38].

York constructed the so-called extrinsic time [22] as a trace of the extrinsic curvature tensor. Hence, it is a scalar, and such a definition since then is legalized in the theory of gravitation [39]. We have built the intrinsic time using a ratio of the determinants of spatial metric tensors. So to say, the variables of the canonical pair time—Hamiltonian density in extended phase space, as opposed to York’s pair, are reversed.

Now the canonical variables have acquired a clear physical meaning. The local time \( D \) is constructed in accordance with the conditions imposed on internal time. The spatial metric \( \gamma_{ij} \) carries information on the inner time. According to the scaled Dirac mapping (44), (45), the metric \( \gamma_{ij} \) is factorized into an exponential function of the inner time \( D \) and the conformal metric \( \tilde{\gamma}_{ij} \). So, the extracted intrinsic time has a spatial geometric origin. Unlike homogeneous examples, the intrinsic time \( D \), in the generic case, is a so-called local many-finger time.

The Poisson brackets (7) between the new variables are

\[
\{D(t, x), \pi_D(t, x')\} = -\delta(x - x'),
\]

\[
\{\tilde{\gamma}_{ij}(t, x), \tilde{\pi}^{kl}(t, x')\} = \hat{\delta}^{kl}_{ij}\delta(x - x'),
\]

\[
\{\tilde{\pi}^{ij}(t, x), \tilde{\pi}^{kl}(t, x')\} = \frac{1}{3}(\tilde{\gamma}^{kl}_{ij}\tilde{\pi}^{ij} - \tilde{\gamma}^{ij}_{ij}\tilde{\pi}^{kl})\delta(x - x'),
\]

where

\[
\hat{\delta}^{kl}_{ij} := \delta_i^k\delta_j^l + \delta_i^l\delta_j^k - \frac{1}{3}\tilde{\gamma}^{kl}_{ij} \tilde{\gamma}^{ij}.
\]

is the conformal Kronecker delta function with the properties

\[
\hat{\delta}^{ij}_{ij} = 5, \quad \hat{\delta}^{ij}_{kl} \hat{\delta}^{kl}_{mn} = \hat{\delta}^{ij}_{mn},
\]

\[
\hat{\delta}^{kl}_{ij} \tilde{\gamma}^{ij} = \hat{\delta}^{ijkl} = 0, \quad \hat{\delta}^{ij}_{ij} \tilde{\pi}^{ij} = \tilde{\pi}^{kl}.
\]

The matrix \( \hat{\gamma}^{ij} \) is the inverse conformal metric, i.e.,

\[
\hat{\gamma}^{ij}\hat{\gamma}^{jk} = \delta^i_k, \quad \hat{\gamma}^{ij} = \sqrt{\hat{\gamma}/f} \gamma^{ij}.
\]

There are only five independent pairs \( (\hat{\gamma}_{ij}, \tilde{\pi}^{ij}) \) per space point because of the properties

\[
\hat{\gamma} := \det \|\hat{\gamma}_{ij}\| = f, \quad \tilde{\pi} := \hat{\gamma}_{ij}\tilde{\pi}^{ij} = 0.
\]

The generators \((D(x, t), \pi_D(x, t))\) form a subalgebra of a nonlinear Lie algebra.

Let us notice that the problem of time and conserved dynamical quantities does not exist in asymptotically flat worlds [40]. Time is measured by clocks of observers located at a sufficiently far distance from the gravitating objects. In this case, the super-Hamiltonian (9), constructed from the constraints, is supplemented by surface integrals at infinity. Therefore, we focus our attention in the present paper on the cosmological problems only.

Einstein’s theory of gravity is obtained from shape dynamics by fixing the Stäckelberg field (39):

\[
e^{4\phi} = \sqrt{\hat{\gamma}/f}, \tag{47}
\]

so that the factor is equal to

\[
\frac{\sqrt{\hat{\gamma}}}{f} = e^{6\phi}/\langle e^{6\phi}\rangle_{\gamma}. \tag{48}
\]

6. DEPARAMETRIZATION

To perform deparametrization, we use the substitution (44) and obtain

\[
\tilde{\pi}^{ij}\frac{d}{dt}\hat{\gamma}_{ij} = \left(\pi^{ij} - \frac{1}{3}\pi\hat{\gamma}^{ij}\right)\frac{d}{dt}\hat{\gamma}_{ij}
\]
Then the ADM action functional (12) takes the form

\[ S = \int_{t_i}^{t_f} dt \int d^3x \left[ D_{\pi_{ij}} D_{\gamma_{ij}} - \pi_D D_{\gamma_{ij}} \right] , \quad (49) \]

where the conformal momentum densities are decomposed into longitudinal and traceless-transverse parts, and the momentum is expressed through the extrinsic curvature scalar:

\[ \dot{\pi}_{ij} := \dot{\pi}_{ij}^L + \dot{\pi}_{ij}^T , \quad \kappa = \frac{\pi}{2\sqrt{\gamma}}. \]

The momentum \( \pi_D \) is expressed from the Hamiltonian constraint (28):

\[ \pi_D [\pi^j_L, \pi^j_T, \gamma_{ij}, D] = 2\sqrt{6}\gamma \left[ 8\Psi^{-5} \tilde{A} \Psi - \Psi^{-4} \tilde{R} + \Psi^{-12} \tilde{\pi}_{ij} \tilde{\pi}_{ij} + \tilde{T}_{ij} \right]^{1/2} , \quad (50) \]

where

\[ \Psi = (\gamma/f)^{1/12} . \]

Here we have utilized the property

\[ \tilde{A}_{ij} \tilde{A}^{ij} = \tilde{\pi}_{ij} \tilde{\pi}^{ij} . \]

Picking \( D \) as time and setting an equality of the time intervals, \( dD = dt \), we can partly reduce the action (49).

Substituting the expressed \( \pi_D \) (50) into the ADM action functional (49), we get the functional presimplectic 1-form [41, 42] with local time \( D \):

\[ \omega^1 = \int_{\Sigma_D} d^3x \left[ \left( \pi^j_L + \pi^j_T \right) \dot{\gamma}_{ij} - \pi_D \left( \dot{\pi}^j_L + \dot{\pi}^j_T + \gamma_{ij}, D \right) \right] . \quad (51) \]

7. GLOBAL TIME

Global time exists in homogeneous cosmological models (see, e.g., [43–49]). To get global time in the general case, let us set the CMC gauge [22] (see a special consideration for closed manifolds in [50]) on every slice, labeled by the coordinate time \( t \),

\[ K \equiv -3\kappa = K(t) , \quad (52) \]

where

\[ \kappa := \frac{1}{3}(\kappa_1 + \kappa_2 + \kappa_3) \]

is the mean curvature of the hypersurface \( \Sigma_t \), the arithmetic mean of the principal curvatures \( \kappa_1, \kappa_2, \kappa_3 \).

Let us directly carry out the following transformations with the hypersurface integral:

\[ \int_{\Sigma_t} d^3x \pi_D \frac{dD}{dt} = 4 \int_{\Sigma_t} d^3x \sqrt{\gamma} K \frac{dD}{dt} \]

\[ = -\frac{2}{3} \int_{\Sigma_t} d^3x \frac{K}{\sqrt{\gamma} \frac{d\gamma}{dt}} = -\frac{4}{3} \int_{\Sigma_t} d^3x K \frac{d}{dt} \left( \sqrt{\gamma} \right) \]

\[ = -\frac{4}{3} K(t) \frac{d}{dt} \int_{\Sigma_t} d^3x \sqrt{\gamma} = -\frac{4}{3} K(t) \frac{dV_t}{dt} . \]

The extrinsic curvature scalar \( K(t) \) depends on the slice number \( t \), so it was taken away from the integrand. The coordinate time derivative operator commutes with the space integral due to compactness of the manifold. The diffeo-invariant volume of the hypersurface \( \Sigma_t \) is defined as

\[ V_t := \int_{\Sigma_t} d^3x \sqrt{\gamma} . \]

Let us prove that the hypersurface volume plays the role of global time, and the mean value of the momentum density

\[ H := \frac{2}{3}(\pi) = \frac{2}{3} \int_{\Sigma} \frac{d^3x \pi(x)}{\Sigma} \]

is a Hamiltonian. Let us find the Poisson bracket between these nonlocal characteristics:

\[ \left\{ \int_{\Sigma} d^3y \sqrt{\gamma}(y), \frac{2}{3} \int_{\Sigma} \frac{d^3y \pi(y)}{\Sigma} \right\} . \]

We calculate the functional derivatives of the functionals defined above:

\[ \frac{\delta}{\delta \gamma_{ij}(x)} \int_{\Sigma} d^3y \sqrt{\gamma}(y) = \frac{1}{2} \sqrt{\gamma}(x) \gamma_{ij}(x) , \]

\[ \frac{\delta}{\delta \pi_{ij}(x)} \left( \pi \right) = \frac{1}{V} \frac{\delta}{\delta \pi_{ij}(x)} \int_{\Sigma} d^3y \pi(y) \]

\[ = \frac{1}{V} \frac{\delta}{\delta \pi_{ij}(x)} \int_{\Sigma} d^3y \pi_{ij}(y) \gamma_{ij}(y) = \frac{1}{V} \gamma_{ij}(x) . \]

Hence, one obtains

\[ \left\{ \frac{2}{3} \langle \pi \rangle, \frac{1}{2V} \int_{\Sigma} d^3x \sqrt{\gamma}(x) \gamma_{ij}(x) \gamma_{ij}(x) = 1 . \right\} \]
The corresponding Poisson bracket is canonical as in Ashtekar’s approach [51]. York’s condition (52) sets a slicing allowing for obtaining a global time.

Let us integrate the equality
\[ \pi = 2K(t)\sqrt{\gamma}, \]
over a hypersurface Σt. Here, \( \pi_D = 2\pi \) is expressed from the Lichnerowicz—York equation (50). One obtains that the mean curvature is connected with the mean value of \( \pi \) over the hypersurface,
\[ K = \frac{1}{2}(\pi). \]

After the Hamiltonian reduction and deparametrization procedures, we obtain the action
\[
S = \int_{\Sigma_t} dV \int d^3x \left[ \pi^{ij} \frac{d\xi^i}{dV} \right] - \int_{\Sigma_t} H dV - \int_{\Sigma_t} dV \int d^3x N^i \mathcal{H}_i
\]
(55)

with the Hamiltonian (53).

The Hamiltonian constraint is algebraic of the second order relative to \( K \), which is characteristic of relativistic theories. The coordinate time \( t \) parameterizes the constrained theory. The Hamiltonian constraint is a result of gauge arbitrariness of the spacetime slicing into space and time.

Let us notice that if York’s time is chosen [52, 53], one has to resolve the Hamiltonian constraint with respect to the variable \( D \), which looks unnaturally difficult. As was noted in the textbook [51], if GR could be deparametrized, the notion of total energy in a closed universe could well emerge. As follows from (50), in a generic case, the energy of the Universe is not conserved.

The CMC gauge make condition to the lapse function \( N \), if suppose the shift vector is zero \( N_i = 0 \):
\[ K_{ij} = -\frac{1}{2N} \gamma_{ij}. \]

Then the trace is
\[ K = \gamma^{ij} K_{ij} = -\frac{1}{2N} \gamma^{ij} \gamma_{ij} = -\frac{1}{2N} (\ln \gamma). \]

So, we find that the lapse function is not arbitrary but is determined according to the restriction
\[ N = -\frac{1}{2K} \frac{d}{dt} \ln \gamma. \]

In the special case where \( K = \text{const} \) for every hypersurface, the Hamiltonian does not depend on time, so the energy of the system is conserved. One maximal slice (\( K = 0 \)) exists at the moment of time symmetry. If additionally we restricted our consideration to conformally flat spaces, we would get a set of waveless solutions [53]. Black holes and wormholes belong to this class of solutions.

8. CONCLUSIONS

In the present paper we have demonstrated that in geometrodynamics, a many-finger intrinsic time is a scalar field. For its construction, a background metric was introduced. To obtain reasonable dynamic characteristics, as a background metric one should choose a spatial metric suitable for the corresponding topology. In a generic case it is a tangent space, in asymptotically flat problems flat space [53], and in cosmological problems a corresponding compact or noncompact manifold. The Hamiltonian approach to obtaining physical observables is not explicitly covariant unlike the Lagrangian one. It seems quite natural: The Lagrangian approach is used to obtain an invariant action functional and covariant field equations, suitable for any reference frame, while the Hamiltonian one for getting physical observables for a given observer in his reference frame.

The idea of introducing background fields is used traditionally while considering various theoretical problems. Let us list here some well-known ones. In problems of studying vacuum polarization and quantum particle creation in curved space, background fields are necessary for extraction of physical quantities [56]. The procedures of regularization and renormalization of Casimir vacuum energy are considered in [57]. Renormalization involves comparing some characteristics to obtain physical observables as a result of subtraction. In construction of cosmological perturbation theory, one uses the Friedmann–Robertson–Walker background metric [58, 59]. The presence of Minkowski spacetime, as shown in [60], is necessary for obtaining conserved quantities of a gravitational field. Minkowski background spacetime is hidden in asymptotically flat space problems [12]. The topological Casimir energy of quantum fields in closed hyperbolic universes is calculated in [61, 62]. The problem of obtaining an energy-momentum tensor of the gravitational field in Ricci-flat backgrounds is discussed in [63].

In geometrodynamics, a many-finger intrinsic time is a scalar field. After implementing York’s gauge, deparametrization leads to global time, a value in the hypersurface of the Universe. In application to problems of the Universe, global time is a function of a FRW model scale. It is in agreement with Einstein’s concept of a stationary Universe [64]. Thus we avoid an unpleasant yet unresolved problem of the initial singularity (Big Bang) in standard cosmology. The Friedmann equation has the sense of a formula.
connecting different (intrinsic, coordinate, conformal) time intervals [2, 65, 66]. If we wish to accept York’s extrinsic time, we get the Friedmann equation as an algebraic one. Hence, the connection between time intervals (geometric coordinate time algebraic one. Hence, the connection between time extrinsic time, we get the Friedmann equation as an time intervals [2, 65, 66]. If we wish to accept Y ork’s connecting di

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... are inexorably from birth to death. But what exactly is it? St. Augustine, who died in AD 430, summed up the problem thus: ‘If nobody asks me, I know what time is, but if I am asked then I am at a loss what to say.’ All agree that time is associated with change, growth and decay, but is it more than this? Questions abound. Does time move forward, bringing into being an ever-changing present? Does the past still exist? Where is the past? Is the future already predetermined, sitting here waiting for us though we know not what it is? [75].

Before the 20th century these questions belonged to philosophers. Einstein’s theory of gravity allows for putting these questions in the framework of physics. The changing volume of the Universe in standard cosmology, or changing or masses of elementary particles in conformal cosmology is a measure of time, not time is the measure of change. The so-called expansion of the Universe is directly tied with intrinsic time of the Universe.

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