Majority & Stabilization in Population Protocols

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Abstract

Population protocols are a distributed model focused on simplicity and robustness. A system of $n$ identical nodes (finite state machines) must perform a global task like electing a unique leader or determining the majority opinion when each node has one of two opinions. Nodes communicate in pairwise interactions. Communication partners cannot be chosen but are assigned randomly. Quality is measured in two ways: the number of interactions and the number of states per node.

Under strong stability requirements, when the protocol may not fail with even negligible probability, the best protocol for leader election requires $O(n \cdot (\log n)^2)$ interactions and $O(\log \log n)$ states [16, SODA’18]. The best protocol for majority requires $O(n \cdot (\log n)^2)$ interactions and $O(\log n)$ states [4, SODA’18]. Both bounds are known to be space-optimal for protocols with subquadratically many interactions.

We present protocols which allow for a trade-off between space and time. Compared to another trade-off result [2, PODC’15], we improve the number of interactions by almost a linear factor. Compared to the state of the art, we match their bounds and, at a moderate cost in terms of states, improve upon the number of interactions.

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1 Introduction

In this paper we consider majority and leader election in the probabilistic population model. Both problems are fundamental problems in distributed computing. For example, leader election is frequently used as a symmetry breaking strategy, enabling the coordination of more complex protocols. The majority problem is defined as follows: There are $n$ agents, each with one of two opinions, say $A$ and $B$. The goal is to agree on the opinion with the largest support. In the leader election problem, $n$ agents start in an identical state and one seeks to let exactly one agent reach a designated leader state.

The population model was introduced in [6] as a model to explore the computational power of resource-limited mobile agents. Agents are modeled by finite-state machines. In every step a pair of agents is chosen uniformly at random, observe each other’s state, and perform a deterministic state transition. This is called an interaction. States are mapped to outputs by a problem-dependent output function. For example, in the case of leader election a possible output can be “leader” or “non-leader”. A system is called stable if no possible future state transition can change the agents’ output.

Another defining feature of the population model is uniformity, in the sense that agents which are in the same state are indistinguishable and a single algorithm is designed to work for populations of any size. Due to the simplicity of transition-based algorithms and the uniformity, the model is well suited to model real-world systems that consist of many but comparatively simple agents, like a flock of birds or large sensor networks aggregating information (count, sum, average, extrema, median, or histogram). In both scenarios the computational power of the agents is bounded and the algorithms should not depend on the number of agents.

The quality of a population protocol is measured in terms of the number of interactions and states per agent required to “successfully compute” the desired output. The number of interactions is sometimes expressed in parallel time, which divides the number of interactions by $n$ to account for the “inherent parallelism” of the system. In order to avoid confusion, we stick to the actual number of interactions throughout the paper.

There are several definitions of what is conceived as a “successful computation”. A typical requirement is that the system must, eventually, reach a stable state with correct output. However, runtime notions differ in when this strict guarantee must be achieved. A natural but strict definition is to measure the number $t$ of interactions after which, with high probability$^2$, the system is in such a stable, correct state. This notion is used in most recent publications, especially for lower bounds (cf. Section 1.1). Another definition considers the number of interactions $t$ after which, with high probability, the system always gives the correct output except for a negligible probability of temporary divergence. The former runtime notion is typically referred to as stabilization and the latter is typically called convergence (see Section 2).

One may wonder what the advantage in measuring the convergence time instead of the stabilization time may be. In [10] the authors introduce a hybrid protocol that combines a “fast” protocol that might never converge to the correct answer with a “slow” one that stabilizes at the correct answer. The hybrid protocol switches its output from the inaccurate but fast protocol to that of the slow protocol when it is likely that the slow protocol has finished. And therein lies the crux: without further safeguards, it is possible, although with only negligible probability, that a correct output reached by the fast protocol at time $t$ is

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$^2$ The expression with high probability refers to a probability of $1 - n^{-\Omega(1)}$. 
temporarily overwritten by a currently still wrong output of the slow protocol. Hence, while the system has converged at time $t$, it is not yet stable.

**Our Contribution.** In this paper we show results for both majority and leader election. Our protocols provide a parameter $s \in \{2, 3, \ldots, n\}$ that enables a trade-off between the number of states and the runtime. For majority, our results also depend on the absolute bias $\alpha$, which is the initial absolute difference between the number of agents supporting opinion $A$ and $B$, respectively. In the following we state the results for the case $\alpha = 1$; see the corresponding theorems for the full statements.

Our first result is a comparatively simple protocol that, with high probability, correctly determines the majority in $O(n \cdot (\log n)^2 / \log s)$ interactions and uses $\Theta(s + \log \log n)$ states (Theorem 6). While this high probability guarantee is – with respect to the typical requirement of stabilization or at least convergence – comparatively week, this protocol is an important building block for our main results listed below.

a) Based on this, we present two hybrid majority protocols, both having a runtime of $T = O(n \cdot (\log n)^2 / \log s)$. One converges with high probability in $T$ interactions and uses $\Theta(s + \log \log n)$ states (Theorem 8). The other one stabilizes with high probability in $T$ interactions but uses $\Theta(s \cdot \log n / \log s)$ states (Theorem 7).

b) We adapt the stabilizing majority protocol above such that it becomes uniform. The transformed protocol has the same guarantee for the stabilization time and uses, with high probability, $O(s \log n \log \log n / \log s)$ states (Theorem 11).

c) For leader election we present a hybrid protocol that stabilizes with high probability in $O(n \cdot (\log n)^2 / \log s)$ interactions and uses $\Theta(s + \log \log n)$ states (Theorem 9).

For a constant $s$, our majority results underline an important difference between stabilization and convergence: while our upper bound of $O(\log n)$ states is asymptotically tight for any protocol that stabilizes with high probability in a subquadratic number of interactions [4], our upper bound of $O(\log \log n)$ states bypasses this lower bound if one considers convergence instead of stabilization.

For $s = \log \log n$, our majority protocols converge / stabilize with high probability in $O(n \cdot (\log n)^2 / \log \log \log n)$ interactions. Together with [11], these are the first majority protocols with $O(\text{polylog } n)$ states that work in $o(n \cdot (\log n)^2)$ interactions.

For $s = \log \log n$, our leader election protocol improves upon [16]. While both protocols use $O(\log \log n)$ states, our protocol stabilizes with high probability in $O(n \cdot (\log n)^2 / \log \log \log n)$ interactions (instead of $O(n \cdot (\log n)^2)$). As above for majority, this is the first protocol with $O(\text{polylog } n)$ states that requires with high probability strictly less than $O(n \cdot (\log n)^2)$ interactions, answering an open question from [16].

An import ingredient for our result is an improvement to the *phase clock* from [16], a distributed synchronization mechanism for population protocols. While this phase clock itself requires just constant states, it is driven by a junta of $n^\epsilon$ agents (for a constant $\epsilon \in [0, 1)$). Selecting this junta requires $\Theta(\log \log n)$ states. By a careful change to the junta internals and the interplay between the junta and the phase clocks, we not only simplify the protocol but allow agents to *recycle* the states that were used to select the junta. See Section 3.2 for a detailed explanation.

### 1.1 Related Literature

This literature overview concentrates on results in the population model. The original population model was introduced by Angluin et al. [5, 6], assuming that the number of states
per agent is constant. Together with Angluin et al. [7, 8], their results show that semilinear predicates (like parity or majority) are stably computable in this model. Subsequent results focused on quantifying the runtime and state requirements for specific problems, in particular majority and leader election, and on generalizing the model.

All results in the following overview are given in number of interactions (instead of in parallel time, which is the number of interactions divided by \( n \)). Bear in mind that original sources may state bounds in parallel time only.

Majority. In majority, \( n \) agents start in one of two states (opinions) and seek to determine which opinion has the larger support. Angluin et al. [9] present a protocol with three states and show that, with high probability, the agents agree on the majority after \( O(n \log n) \) interactions if the initial difference between both opinions (the absolute bias \( \alpha \)) is \( \omega(\sqrt{n \log n}) \). Mertzios et al. [17] show that, if agents are required to succeed with probability 1, at least four states are necessary. They also provide a four state protocol that stabilizes with high probability in \( O(n^2 \log n) \) interactions. The same four state protocol was independently (and earlier) studied by Draief and Vojnovic [15], who proved similar results. Alistarh et al. [2] show a lower bound of \( \Omega(n^2/\alpha) \) on the expected interactions for any four state protocol. For any number of states, they show a lower bound of \( \Omega(n \log n) \) expected interactions.

Mocquard et al. [18] consider the population model but using a super-constant number of states per agent. They present a protocol that calculates \( \alpha \) with high probability in \( O(n \log n) \) interactions and requires \( O(n^{3/2}) \) states. Alistarh et al. [3] show rather general lower bounds for population protocols with certain, natural monotonicity properties. For majority, their bound states that protocols with less than \( (\log \log n)/2 \) states require \( \Omega(n^2/\text{polylog}(n)) \) interactions in expectation in order to stabilize. Alistarh et al. [4] improve this lower bound for majority: Any protocol that solves majority and stabilizes in \( n^{2-\Omega(1)} \) expected interactions requires \( \Omega(\log n) \) states.

A recent series [2–4, 13] of papers showed upper bounds. The currently best result is due to Alistarh et al. [4], who present a protocol that stabilizes with high probability in \( O(n \cdot (\log n)^2) \) interactions and requires \( O(\log n) \) states. Alistarh et al. [2] present a trade-off result of similar nature to ours. For a parameter \( m \leq n \), their algorithm uses \( s = m + O(\log n \cdot \log m) \) states and stabilizes with high probability in \( O(n^2 \cdot (\log n)/(\alpha \cdot m) + n \cdot (\log n)^2) \) interactions.

In a recent, still unpublished result [11], we present a population protocol for majority that stabilizes in \( O(n \cdot (\log n)^{5/3}) \) interactions, both in expectation and with high probability, and that uses \( \Theta(\log n) \) states.

Leader Election. In leader election, \( n \) identical agents seek a state were exactly one of them is in a designated leader state. Doty and Soloveichik [14] show that any population protocol with a constant number of states that stably elects a leader with probability 1 requires \( \Omega(n^2) \) expected interactions; a bound which is matched by a natural two state protocol.

Upper bounds for protocols with a non-constant number of states per agent were presented in [1, 3, 4, 12, 13, 16]. The currently best result is due to Gasieniec and Stachowiak [16]. They present a leader election protocol that stabilizes with high probability in \( O(n \cdot (\log n)^2) \) interactions and requires \( O(\log \log n) \) states, asymptotically matching a corresponding lower bound on the number of states for protocols stabilizing in \( O(n^2/\text{polylog} n) \) interactions [3]. Their protocol is based on a uniform protocol which, with high probability, correctly identifies a designated leader after \( O(n \cdot (\log n)^2) \) interactions. The protocol first chooses a junta of size \( n^\epsilon \) for constant \( \epsilon \in [0, 1] \), which is then used to implement a phase clock to synchronize the agents. The phase clock generalizes results from [10].
Population protocols are a computational model for a distributed system consisting of $n$ agents, in the following referred to as nodes. Nodes are assumed to be identical, finite-state machines. In each time step, an ordered pair of nodes $(u,v)$ is chosen independently and uniformly at random. Node $u$ is called the initiator and node $v$ is called the responder. Let $s_u$ be the state of $u$ and $s_v$ be the state of $v$ at the beginning of such an interaction. Both nodes observe each other’s state and update themselves according to a fixed, deterministic transition function of the form $(s_u, s_v) \mapsto (s'_u, s'_v)$. At any time, the global state of the system can be fully described by a function $c$ that maps each node to its current state. Such a function $c$ is called configuration.

Depending on the considered problem, nodes try to reach and stay in a set of target configurations. In general, it is not required that any node is aware that a target configuration has been reached. We specify the target configuration via an output function of the form $s \mapsto o$ that maps a state $s$ to a (problem specific) output value $o$. We are interested in population protocols for the following problems:

**Leader Election** In leader election, all nodes start in the same initial state. We seek a configuration in which exactly one node is in a designated leader state and all others are in a non-leader state.

The output function maps each state $s$ to an output $o \in \{\text{Follower, Leader}\}$. Target configurations are all configurations in which the state of exactly one node is mapped to LEADER.

**Majority** In majority, nodes start in one of two different states (also called opinions). We seek a configuration in which all nodes correctly agree on the opinion with the initially larger support. The *absolute bias* $\alpha$ is the absolute difference between the initial number of supporters for each opinion.

The output function maps each state $s$ to an output $o \in \{+1, -1\}$, representing the two opinions. Target configurations are all configurations in which node states map all to $+1$ (if $+1$ was the initial majority) or map all to $-1$ (if $-1$ was the initial majority).

Protocol quality is measured in terms of the number of interactions and the number of states per node required to reach and stay in target configurations. There are two common ways to formalize what exactly is meant by “reach and stay”: stabilization time and convergence time.

**Convergence Time** The convergence time $T_C$ of a protocol is the random variable that measures the number of interactions until the protocol has reached and remains in a target configuration.

**Stabilization Time** We say a configuration $c$ is stable if in any configuration $c'$ that is reachable (with any sequence of interactions) from $c$, each node has the same output as in $c$. The stabilization time $T_{ST}$ of a protocol is the random variable that measures the number of interactions until the protocol has reached a stable target configuration.

Clearly, $T_C \leq T_{ST}$, since reaching a stable target configuration implies that, whatever future interactions may be, the system will always remain in a target configuration.

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3 The notions as defined here are the ones used predominantly in population protocols in recent literature. However, note that some previous publications (e.g., [4, 13]) refer to stabilization time as convergence time.
As bounds on the convergence and stabilization time are often given in probabilistic terms, one often additionally emphasizes if a protocol is always guaranteed to, eventually, reach a stable target configuration (i.e., $T_{ST} < \infty$ holds with probability 1). Such protocols are called exact.

Most newer results, in particular [4, 16], consider stabilization (for exact protocols). However, from a practical point of view, convergence provides similarly strong runtime guarantees while enabling more efficient protocols. Indeed, Theorem 8 shows that the lower bound on the number of states required by any majority protocol that stabilizes in $n^{2-\Theta(1)}$ interactions does not apply if one considers convergence instead.

3 Auxiliary Population Protocols

In this section we introduce a few auxiliary population protocols that are used as “subroutines” in our protocols. These protocols, or variants of them, are well known and have been used in other work on population protocols, as indicated below.

We start with two comparatively simple primitives (one-way epidemic and load balancing). We will proceed to describe two more involved protocols ($n^\epsilon$-junta creation and phase clocks) which also require slight adaptions and rephrasing to fit into our setting.

One-way Epidemic. A one-way epidemic for $n$ nodes is a population protocol with state space $\{0, 1\}$ and transition function $(x, y) \mapsto (x, \max\{x, y\})$. Nodes with value 0 are denoted as susceptible and nodes with value 1 as infected. We define the infection time $T_{INF}$ as the number of interactions required by a one-way epidemic starting with a single infected node to infect the whole population. The following upper and lower high-probability bounds on $T_{INF}$ have been shown in [10].

$\textbf{Lemma 1 ([10, Lemma 2]).}$ For any constant $a > 0$ there are constants $c_1, c_2 > 0$ such that $\Pr[c_1 \cdot n \log n \leq T_{INF} \leq c_2 \cdot n \log n] \geq 1 - n^{-a}$.

Load Balancing. We define a simple population protocol for load balancing over $n$ nodes. The state space is $\{-m, -(m - 1), \ldots, m - 1, m\}$, where $m \in \mathbb{N}$ is a positive integer (which may depend on $n$). We say a node in state $x$ has load $x$. The transition function is $(x, y) \mapsto (\lceil \frac{x+y}{2} \rceil, \lfloor \frac{x+y}{2} \rfloor)$. Let $\Delta(t)$ denote the discrepancy after $t$ interactions, which is the difference between the maximum and minimum load among all nodes, and set $\Delta := \Delta(0)$. We define the load balancing time $T_{LB}$ as the number of interactions required to reduce the initial discrepancy to at most 2. The following folklore lemma provides an upper high-probability bound on $T_{LB}$; see also [19, 20].

$\textbf{Lemma 2.}$ For any constant $a > 0$, there exists a constant $c > 0$ such that we have $\Pr[T_{LB} \leq c \cdot n \log(n \cdot \Delta)] \geq 1 - n^{-a}$.

3.1 Junta Creation

The next protocol is a variant of a protocol from [16] that rapidly elects a junta of size $O(n^\epsilon)$ for a constant $\epsilon > 0$.

We first describe how the nodes can compute levels with a specific distribution and then proceed to show how these levels can be used to form a junta. For the level computation, the state of a node is a tuple of the form $(l, a)$, where the level $l \in \mathbb{N}_0$ is a counter and the activity bit $a \in \{0, 1\}$ indicates whether a node is active or not. Initially, all nodes have $l = 0$ and $a = 1$. 
To describe the transition function, we distinguish between a node’s first interaction and any subsequent interaction. During its first interaction, a node \( u \) adopts state \((1, 1)\) if it is the initiator and state \((0, 0)\) if it is the responder (note that the initiator/responder assignment is random; we use this property to simulate coin tosses). During any following interaction, \( u \) changes its state only if it is still active \((a = 1)\) and if it is the initiator of the interaction. In these cases, when interacting with a responder in state \((l', a')\), it updates its state as follows:

\[
((l, 1), (l', a')) \mapsto \begin{cases} 
(l + 1, 1) & \text{if } l' \geq l \\
(l, 0) & \text{otherwise.}
\end{cases}
\]

(1)

For \( l \in \mathbb{N}_0 \) let \( B_l \) denote the number of nodes that reach level at least \( l \) before becoming inactive. Let \( L^* \in \mathbb{N} \cup \{ \infty \} \) be the maximum level reached by any node. We prove the following results:

▶ Lemma 3. For any constant \( a > 0 \), with probability at least \( 1 - n^{-a} \) the following properties hold:

- a) all nodes become inactive within \( O(n \log n) \) interactions,
- b) \( \log \log n - 4 \leq L^* \leq \log \log n + 4 \cdot (a + 1) \), and
- c) \( B_{l^*} = O(\sqrt{n} \cdot \log n) \).

▶ Lemma 4. Let \( l^* := \lceil \log \log n \rceil - 4 \). There is a constant \( \epsilon \in [0, 1) \) such that for any constant \( a > 0 \) we have \( \Pr \{ B_{l^*_n} < n^\epsilon \} \geq 1 - n^{-a} \).

The only difference between the protocol in [16] and ours is how nodes behave in their first interaction. This deviation allows us to provide a lower bound on the maximum level \( L^* \).

See Appendix B for the proofs of Lemmas 3 and 4.

It remains to specify how the level information can be used by other protocols to extract a junta. We describe two different variants, resulting in the junta protocols FormJunta (essentially equivalent to the junta protocol from [16]) and FormJuntaExtended. The latter exploits the lower bound on \( L^* \), which not only allows for a simpler junta protocol but also enables nodes to, eventually, free up the \( \Theta(\log \log n) \) states that were required to store the level.

**FormJunta:** Each node stores, additionally, a marker bit \( b \) and a spoiled bit \( s \), both of which are initially 0. The marker bit indicates whether a node is assumed part of the junta or not. A node that just became inactive at a level \( l \geq 1 \) sets \( b \) to 1. If an inactive node at level \( l \geq 1 \) encounters a node on a higher level, it becomes spoiled: it sets \( s \) to 1, \( b \) to 0, and will from now on simply adopt the largest level during any interaction (not changing any of its other state values). If encountered by another node in the level calculation, a spoiled node is treated as if it were in state \((0, 0)\), independently of its actual level counter \( l \).

**FormJuntaExtended:** Each node stores, additionally, a marker bit \( b \). A node sets its marker bit if it reaches level \( l^* := \lceil \log \log n \rceil - 4 \). Inactive nodes below level \( l^* \) enter state \((0, 0)\) (forgetting their level value) when they encountered two nodes who had their marker bit set.

It is easy to see that changing arbitrary nodes that are not at the currently highest level to state \((0, 0)\), as done in FormJunta, maintains the statements from Lemma 3 except for the lower bound on \( L^* \) but does not maintain Lemma 4 (cf. the “spoiled Forming_junta” protocol in [16]).
3.2 Phase Clock

The basis for the following phase clock protocol is due to [16]. It can be used by other population protocols to perform actions in a (probabilistic) synchronized way. It requires $\Theta(\log \log n)$ states to set up the phase clock, but as explained below, our changes to the phase clock and the underlying junta protocol allows us to recycle these states after the setup.

Let the state of a node be a tuple of the form $(p, b)$, where the phase $p \in \mathbb{N}_0$ is a counter and the marker bit $b \in \{0, 1\}$ indicates whether the node is marked. Initially, all nodes have phase $p = 0$ and $n^\epsilon$ nodes have the marker bit set to 1, where $\epsilon \in [0, 1)$ is an arbitrary constant. We will explain below how to incorporate a junta protocol to generate the marked nodes, such that we can start with all marker bits set to 0. Also, for now we assume that $p$ can represent arbitrarily large integers and explain later how to restrict $p$ to a fixed range. The transition function is

$$[(p, b), (p', b')] \mapsto \begin{cases} \left[\left(\max\{p, p' + 1\}, b\right), (p', b')\right] & \text{if } b = 1 \text{ and } p' \geq n^\epsilon, \\ \left[\left(\max\{p, p'\}, b\right), (p', b')\right] & \text{otherwise}. \end{cases} \quad (2)$$

Let $m \in \mathbb{N}$ be a suitable value (which follows from Lemma 5). Consider a node $u$ with phase counter $p$ after $t$ interactions. We define the round $R_u(t) \in \mathbb{N}_0$ of node $u$ after $t$ interactions as $R_u(t) = [p/m]$. For $i \in \mathbb{N}_0$, we say node $u$ reached round $i$ if $p \geq i \cdot m$. Let $R_{\text{Start}}(i)$ (start of round $i$) denote the interaction during which the last node reaches round $i$. Similarly, let $R_{\text{End}}(i)$ (end of round $i$) denote the interaction during which the first node reaches round $i + 1$.

In general, $R_{\text{Start}}(i)$ might be larger than $R_{\text{End}}(i)$. However, the next lemma states that for polynomial many rounds, the constant parameter $m$ can be chosen such that nodes are well synchronized. It is a reformulation of [16, Theorem 3.1] to fit our setting and proofs. A brief proof sketch based on a technical lemma from [16] is given in Appendix C.

Lemma 5. Fix an $i \in \mathbb{N}_0$ with $i = \text{poly}(n)$. For any constants $a, d_1 > 0$ there exists a constant phase clock parameter $m \in \mathbb{N}$ and a constant $d_2 > 0$ such that, with probability at least $1 - n^{-a}$, $d_1 \cdot n \log n \leq R_{\text{End}}(i) - R_{\text{Start}}(i) \leq d_2 \cdot n \log n$.

For a parameter $r \in \mathbb{N}$, we define the population protocol $\text{PhaseClock}_r$ as above but restrict the phase counter to $p \in \{0, 1, \ldots, r \cdot m - 1\}$. Arithmetic on the phase counter is performed modulo $r \cdot m$.

Phase Clock Interface. Consider the first $\text{poly}(n)$ many rounds. Lemma 5 implies that, with high probability, the rounds of any two nodes differ by at most 1. Thus, with high probability, even with $r = 2$, nodes will be able to distinguish whether they are in the same or in different rounds, but not who is in a later round or what their exact round number is. For $r = 3$, nodes can, with high probability, additionally determine who is in the later round. Note that before experiencing their first overflow, nodes know precisely what round they are in. This is also true for the first $r$ rounds (think of $r = \text{polylog}(n)$).

We use the above observations to define an interface to $\text{PhaseClock}_r$, which simplifies the usage of the phase clock in other population protocols.

For $r \geq 1$, $\text{PhaseClock}_r$ supports the following function calls, all correctly with probability 1:

- $\text{PhaseClock}_r(u, v)$: Update the state of $u$ according to the transition function.
- $\text{PCoverflowed}(u)$: True iff $u$’s phase counter overflowed at least once.
PCnewRound($u$): TRUE iff $u$ entered a new round in its last phase counter update.

PCmarked($u$): TRUE iff $u$’s marker bit is set.

For $r \geq 2$ and any constant $a > 0$, PhaseClock$_r$ supports the following function calls for poly($n$) many rounds with probability $1 - n^{-a}$:

PCsameRound($u,v$): TRUE iff $u$ is currently in the same round as $v$.

PCdifferentRound($u,v$): TRUE iff $u$ is currently in a round different from $v$’s.

For $r \geq 3$ and any constant $a > 0$, PhaseClock$_r$ supports the following function calls for poly($n$) many rounds with probability $1 - n^{-a}$:

PCsmallerRound($u,v$): TRUE iff $u$ is currently in a smaller round than $v$.

PLargerRound($u,v$): TRUE iff $u$ is currently in a larger round than $v$.

4 Simple Majority with Phase Clocks

In this section we present and analyze ClockedMajority$_{s,r}$, a population protocol for majority, parameterized by two values $s$ and $r$. We will prove the following theorem:

\begin{theorem}
Consider the majority problem for $n$ nodes with initial absolute bias $\alpha \in \mathbb{N}$. Let $s \in \mathbb{N} \setminus \{1\}$ and $r \in \mathbb{N} \setminus \{1,2\}$. With high probability, protocol ClockedMajority$_{s,r}$ correctly identified the majority for all interactions $t = \Omega(n \log n \log s(n/\alpha))$. It uses $\Theta(s \cdot r + \log \log n)$ states per node.
\end{theorem}

Theorem 6 does not imply that ClockedMajority$_{s,r}$ converges or even stabilizes. In fact, it might not correctly identify the majority (such that the stabilization time is infinite). However, Sections 5 and 6 extend the idea of ClockedMajority$_{s,r}$ in order to derive results for stabilization and convergence. The proof of Theorem 6 is given in Appendix D.

We will now describe the state space and transition function for ClockedMajority$_{s,r}$. A formal description of the transition function in form of pseudocode is also given in Algorithm 1.
State Space. The state of a node $u$ consists of the states required for the PhaseClock$_r$ protocol (cf. Section 3.2) and a load value $\text{load}_u$. The load value $\text{load}_u$ represents $u$’s current opinion (sign) and its “magnitude” (absolute value). It is initialized with either $+1$ or $-1$, depending on $u$’s initial opinion. The output function maps a node’s state to the sign its load value. Thus, the majority guess of a node $u$ is equal to $\text{sign}(\text{load}_u)$.

To simplify the description of the protocol, we assume that the load values can represent arbitrary integers. In the proof of Theorem 6, we will see that, with high probability, load values will not exceed $O(s)$, allowing us to restrict $\text{load}_u$ to $\pm O(s)$ without changing the protocol outcome.

Transition Function. Consider an interaction between two nodes $u$ (initiator) and $v$ (responder). The nodes’ actions can be divided into three parts: load explosion, load balancing, and synchronization. During the load explosion, $u$ uses the phase clock’s PCnewRound() method to check whether this is its first interaction as an initiator in its current phase clock round. If yes, it multiplies its load by a factor of $s$. During the load balancing, the nodes use the phase clock’s PCsameRound() method to check whether they are in the same phase clock round and, if so, perform a simple load balancing step by balancing their respective loads as evenly as possible. During the synchronization, the PhaseClock$_r$ protocol is triggered with initiator $u$ and responder $v$ to update the states of $u$’s phase clock.

```plaintext
ClockedMajority$_{s,r}(u, v)$
1    if PCnewRound(u) then /* load explosion */
2        load$_u$ ← load$_u$ · $s$
3    if PCsameRound(u, v) then /* load balancing */
4        (load$_u$, load$_v$) ← ([load$_u$ + load$_v$]/2, [load$_u$ + load$_v$]/2)
5    PhaseClock$_r$, (u, v) /* synchronization */
```

Algorithm 1: Formal description of the transition function for ClockedMajority$_{s,r}$ for an initiator $u$ and a responder $v$.

5 Stable Majority with Phase Clocks

In this section, we present and analyse StrongMajority$_s$, a population protocol for majority parameterized by a value $s$. We prove the following theorem:

▶ Theorem 7. Consider the majority problem for $n$ nodes with initial absolute bias $\alpha \in \mathbb{N}$. Let $s \in \{2, 3, \ldots, n\}$. Protocol StrongMajority$_s$ is exact and stabilizes with high probability in $O(n \log n \cdot \log_s(n/\alpha))$ interactions. It uses $\Theta(s \cdot \log_s n)$ states per node.

See Appendix E for the proof of Theorem 7. In the following we explain our protocol StrongMajority$_s$, whose formal description is given in form of pseudo code in Algorithm 2.

Consider an interaction between two nodes $u$ (initiator) and $v$ (responder) and let $r := \lceil \log_s(5n) \rceil$. Nodes perform, among others, the actions of ClockedMajority$_{s,r}$. We say node $x$ finished in interaction $t$ if at least one of the following conditions is true:

- It is initiator and reaches a new round in interaction $t$, and its load value load$_x$ (in protocol ClockedMajority$_{s,r}$) is larger than 3 at the beginning of interaction $t$.
- It is initiator and its phase clock overflowed (the phase counter reached $r$ and thus was reset to 0) at the end of its last interaction as an initiator.
A finished node stops performing actions of \textsc{ClockedMajority},\( s, r \). Instead, it sets an error bit if any of the following conditions is true: a) it is not in the same round as its interaction partner\(^4\); b) \( \text{sign}(\text{load}_u) \neq \text{sign}(\text{load}_v) \); or c) its interaction partner has the error bit set. In addition to this, a backup protocol \textsc{BackupMajority} is run in parallel. We use the 4-state protocol from [17] for this, which stabilizes in \( O(n^2 \log n) \) interactions in expectation (which implies a finite stabilization time). The output function maps a node’s state to a majority guess as follows: a) If the phase counter of the phase clock is zero or if the error bit is set, the majority guess of the backup protocol is used. b) Otherwise, node \( u \) uses \( \text{sign}(\text{load}_u) \) as its majority guess.

\begin{algorithm}
\begin{algorithmic}
\State \textbf{StrongMajority},\( s \)(\( u, v \))
\State \textbf{BackupMajority}(\( u, v \))
\If {\( \text{PCnewRound}(u) \text{ and } \text{load}_u \geq 3 \) or \text{PCoverflowed}(u) \text{ then}}
\State \( \text{finished}_u \leftarrow \text{TRUE} \)
\EndIf
\If {\( \text{finished}_u \text{ or } \text{finished}_v \) \text{ then}}
\State \( (\text{finished}_u, \text{finished}_v) \leftarrow (\text{TRUE, TRUE}) \)
\EndIf
\If {\( \text{PCdifferentRound}(u, v) \) or \( \text{sign}(\text{load}_u) \neq \text{sign}(\text{load}_v) \) or \( \text{error}_u \) or \( \text{error}_v \) \text{ then}}
\State \( (\text{error}_u, \text{error}_v) \leftarrow (\text{TRUE, TRUE}) \)
\EndIf
\State \textbf{exit}
\State \textbf{ClockedMajority},\( s, \lceil \log s(5n) \rceil \)(\( u, v \))
\end{algorithmic}
\end{algorithm}

\textbf{Algorithm 2}: Formal description of the transition function for \textbf{StrongMajority},\( s \) for an initiator \( u \) and a responder \( v \).

\section{Convergent Majority with Phase Clocks}

In this section, we present and analyse \textbf{WeakMajority},\( s \), a population protocol for majority parameterized by a value \( s \). We prove the following theorem:

\textbf{Theorem 8}. Consider the majority problem for \( n \) nodes with initial absolute bias \( \alpha > 0 \). Let \( s \in \{2, 3, \ldots, n\} \). Protocol \textbf{WeakMajority},\( s \) is exact and converges with high probability in \( O(n \log n \cdot \log s(n/\alpha)) \) interactions. It uses \( \Theta(s + \log \log n) \) states per node.

See Appendix F for the proof of Theorem 9 and for the pseudocode of the transition function of \textbf{WeakMajority},\( s \). As before, we start by explaining the protocol \textbf{WeakMajority},\( s \).

Consider an interaction between two nodes \( u \) (initiator) and \( v \) (responder). Nodes perform, among others, the actions of \textbf{BackupMajority} and \textbf{ClockedMajority},\( s, 3 \). In addition to the states required by the backup protocol on \textbf{ClockedMajority},\( s, 3 \), every node \( x \) has a constant size counter \( \text{count}_x \in \{0, \ldots, 600\} \). We say node \( x \) finishes in interaction \( t \) if it hits the maximum counter value in interaction \( t \). Observe that once nodes have reached the maximum counter, they no longer update the counter or participate in \textbf{ClockedMajority},\( s, 3 \). They, however, respond in interactions like regular nodes.

The idea of \textbf{WeakMajority},\( s \) is the following. All nodes always perform the backup protocol in parallel, which is guaranteed to eventually stabilize. In addition, nodes count

\footnotetext[4]{Note that this can be decided correctly with probability 1 as long as all nodes have finished fewer than \( r \) rounds (only then the phase counter overflows and a node might be mistaken for a straggler).}
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the number of subsequent interactions with junta nodes. Only after a polynomial number of interactions (which is larger than the time required for the backup protocol to stabilize), the nodes switch to the backup protocol. This hybrid protocol rapidly returns the correct result with high probability, while the backup protocol guarantees that it will eventually output the correct result with probability 1. Once \( \text{ClockedMajority}_{s,3} \) has stabilized, every node returns the correct result with high probability and no node changes its output in a subsequent interaction with high probability.

The output function maps a node’s state to a majority guess as follows: a) If the phase clock of a node has not yet ticked or if the counter has reached 600, the majority guess of the backup protocol is used. b) Otherwise, node \( u \) uses the current output of \( \text{ClockedMajority}_{s,3} \) as its majority guess.

7 Leader Election

This section describes \( \text{LeaderElection}_{s} \), a population protocol for leader election parameterized by a value \( s \). We will prove the following theorem:

▶ Theorem 9. Consider the leader election problem for \( n \) nodes. Let \( s \in \{2, 3, \ldots, n\} \). Protocol \( \text{LeaderElection}_{s} \) is exact and stabilizes with high probability in \( O(n(\log n)^2/\log s) \) interactions. It uses \( \Theta(\max\{\log \log n, s\}) \) states per node.

With \( s = \log \log n \) we get the following corollary, which gives an affirmative answer to the open question from [16], whether leader election can be done with \( O(\log \log n) \) states in \( o(n(\log n)^2) \) interactions.

▶ Corollary 10. Consider the leader election problem for \( n \) nodes. For \( s = \log \log n \), protocol \( \text{LeaderElection}_{s} \) is exact and stabilizes with high probability in \( o(n(\log n)^2) \) interactions. It uses \( \Theta(\log \log n) \) states per node.

With the following three exceptions, our protocol is identical to the protocol from [16].

a) Our protocol runs in \( O(\log n/\log s) \) rounds of \( O(n \log n) \) interactions each.

b) We use the phase clock based on \( \text{FormJuntaExtended} \). This allows a node \( v \) to recycle the junta states.

c) We sample and compare \( \log s - \log \log s \) bits per round. After the sampling we use one-way epidemics to broadcast the maximum of all sampled \( (\log s - \log \log s) \)-bit numbers. If a contender sees a larger number than its own sample, it becomes a Follower.

For further explanation and detail we refer the reader to Appendix G.

8 A Note on Uniformity

Uniformity in population protocols means that a single algorithm is designed to work for populations of any size. In particular, nodes have no information on the population size \( n \). Protocols where nodes are restricted to a constant number of states are always uniform: nodes cannot even store information about \( n \). But most newer protocols allow for a super-constant number of states and use (upper bounds on) \( n \). In particular, protocols that stop their computation once a counter reaches a value of \( \text{polylog}(n) \) fall into this category.

One of the rare examples for a truly uniform, non-constant state protocol is \( \text{FormJunta} \) from [16]. This algorithm “just works” and requires, with high probability, \( O(\log \log n) \)
states\(^5\). In contrast, our junta protocol \textsc{FormJuntaExtended} is not uniform, as nodes need to know \(l^* = \lceil \log \log n \rceil - 4\) in order to mark themselves (cf. Section 3.1).

To the best of our knowledge, there is no uniform majority protocol that stabilizes with high probability in \(n^{2 - \Omega(1)}\) interactions. This is in contrast to leader election, where the protocol from \cite{16} (used with junta created by \textsc{FormJunta} and with phase clock \textsc{PhaseClock}\(_\infty\)) is uniform and stabilizes with high probability in \(O(n \cdot (\log n)^2)\) interactions and uses, with high probability, \(O(\log \log n)\) states. This poses the question whether there exists a uniform majority protocol that stabilizes fast. The following theorem answers this affirmative.

\textbf{Theorem 11.} Consider the majority problem for \(n\) nodes with initial absolute bias \(\alpha \in \mathbb{N}\). Let \(s \in \mathbb{N}\backslash \{1\}\). There is a variant \textsc{UniformMajority}\(_s\) of \textsc{StrongMajority}\(_s\) which is exact, uniform, and stabilizes with high probability in \(O(n \cdot (\log n)^2)\) interactions and uses, with high probability, \(O(\log \log n)\) states per node.

See appendix Appendix H for a proof sketch of Theorem 11.

\textsc{UniformMajority}\(_s\) is identical to \textsc{StrongMajority}\(_s\) when using the phase clock \textsc{PhaseClock}\(_\infty\) (i.e., the phase clock cannot overflow) with \textsc{FormJunta} instead of \textsc{FormJuntaExtended}. Using the original junta instead of ours has the drawback of making it impossible to recycle the \(O(\log \log n)\) junta states, since nodes do not know whey they completed forming the junta (cf. Section 3.2). However, since \textsc{FormJuntaExtended} is inherently non-uniform, this is unavoidable when aiming for a uniform version of \textsc{StrongMajority}\(_s\).

There is one more subtle change: the spoiling phenomenon described for \textsc{FormJunta} in Section 3.1 (see also \cite{16}): Any inactive node with level \(\geq 1\) is marked until it encounters a node on a higher level. Then it becomes unmarked and spoiled (spoiling any active node encountering it). Between becoming active and being spoiled, such a node basically runs a phase clock “on a lower level” (which might run too fast). Thus, whenever a node becomes spoiled it is also reset: the node resets its phase clock phase to 0 and its load to \(\pm 1\), depending on its initial opinion (this requires an extra bit to store the initial opinion). Since the clock on the largest level \(L^*\) spreads via a one-way epidemic (cf. Section 3.2), we can choose the constant phase clock parameter \(m\) such that, with high probability, all nodes are running the correct phase clock before the \(L^*\)-phase clock starts a new round for the first time.

\(^5\) We cannot avoid this probabilistic upper bound on the state number: with non-zero probability, a node can reach an arbitrarily high level.
1 Dan Alistarh and Rati Gelashvili. Polylogarithmic-time leader election in population protocols. In Magnús M. Halldórsson, Kazuo Iwama, Naoki Kobayashi, and Bettina Speckmann, editors, Automata, Languages, and Programming - 42nd International Colloquium, ICALP 2015, Kyoto, Japan, July 6-10, 2015, Proceedings, Part II, volume 9135 of Lecture Notes in Computer Science, pages 479–491. Springer, 2015. 10.1007/978-3-662-47666-6_38. URL https://doi.org/10.1007/978-3-662-47666-6_38.

2 Dan Alistarh, Rati Gelashvili, and Milan Vojnovic. Fast and exact majority in population protocols. In Chryssis Georgiou and Paul G. Spirakis, editors, Proceedings of the 2015 ACM Symposium on Principles of Distributed Computing, PODC 2015, Donostia-San Sebastián, Spain, July 21 - 23, 2015, pages 47–56. ACM, 2015. 10.1145/2767386.2767429. URL http://doi.acm.org/10.1145/2767386.2767429.

3 Dan Alistarh, James Aspnes, David Eisenstat, Rati Gelashvili, and Ronald L. Rivest. Time-space trade-offs in population protocols. In Philip N. Klein, editor, Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2017, Barcelona, Spain, Hotel Porta Fira, January 16-19, pages 2560–2579. SIAM, 2017. 10.1137/1.9781611974782.169. URL https://doi.org/10.1137/1.9781611974782.169.

4 Dan Alistarh, James Aspnes, and Rati Gelashvili. Space-optimal majority in population protocols. In Artur Czumaj, editor, Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018, pages 2221–2239. SIAM, 2018. 10.1137/1.9781611975031.144. URL https://doi.org/10.1137/1.9781611975031.144.

5 Dana Angluin, James Aspnes, Zoë Diamadi, Michael J. Fischer, and René Peralta. Computation in networks of passively mobile finite-state sensors. In Soma Chaudhuri and Shay Kutten, editors, Proceedings of the Twenty-Third Annual ACM Symposium on Principles of Distributed Computing, PODC 2004, St. John's, Newfoundland, Canada, July 25-28, 2004, pages 290–299. ACM, 2004. 10.1145/1011767.1011810. URL http://doi.acm.org/10.1145/1011767.1011810.

6 Dana Angluin, James Aspnes, Zoë Diamadi, Michael J. Fischer, and René Peralta. Computation in networks of passively mobile finite-state sensors. Distributed Computing, 18(4):235–253, 2006. 10.1007/s00446-005-0138-3. URL https://doi.org/10.1007/s00446-005-0138-3.

7 Dana Angluin, James Aspnes, and David Eisenstat. Stably Computable Predicates Are Semilinear. In Proc. PODC, pages 292–299, New York, NY, USA, 2006.

8 Dana Angluin, James Aspnes, David Eisenstat, and Eric Ruppert. The computational power of population protocols. Distributed Computing, 20(4):279–304, 2007.

9 Dana Angluin, James Aspnes, and David Eisenstat. A simple population protocol for fast robust approximate majority. Distributed Computing, 21(2):87–102, 2008. 10.1007/s00446-007-0059-z. URL https://doi.org/10.1007/s00446-008-0059-z.

10 Dana Angluin, James Aspnes, and David Eisenstat. Fast computation by population protocols with a leader. Distributed Computing, 21(3):183–199, 2008. 10.1007/s00446-008-0067-z. URL https://doi.org/10.1007/s00446-008-0067-z.

11 Petra Berenbrink, Robert Elsässer, Tom Friedetzky, Dominik Kaaser, Peter Kling, and Tomasz Radzik. Population protocols for exact majority with o(log^3/3 n) convergence time and θ(log n) states per node. unpublished manuscript, 2 2018.

12 Petra Berenbrink, Dominik Kaaser, Peter Kling, and Lena Otterbach. Simple and efficient leader election. In Raimund Seidel, editor, 1st Symposium on Simplicity in Algorithms, SODA 2018, January 7-10, 2018, New Orleans, LA, USA, volume 61 of
REFERENCES

13 Andreas Bilke, Colin Cooper, Robert Elsässer, and Tomasz Radzik. Brief announcement: Population protocols for leader election and exact majority with $O(\log^2 n)$ states and $O(\log^2 n)$ convergence time. In Elad Michael Schiller and Alexander A. Schwarzmann, editors, Proceedings of the ACM Symposium on Principles of Distributed Computing, PODC 2017, Washington, DC, USA, July 25-27, 2017, pages 451–453. ACM, 2017. 10.1145/3087801.3087858. URL https://doi.org/10.1145/3087801.3087858.

14 David Doty and David Soloveichik. Stable leader election in population protocols requires linear time. In Yoram Moses, editor, Distributed Computing - 29th International Symposium, DISC 2015, Tokyo, Japan, October 7-9, 2015, Proceedings, volume 9363 of Lecture Notes in Computer Science, pages 602–616. Springer, 2015. 10.1007/978-3-662-48653-5_40. URL https://doi.org/10.1007/978-3-662-48653-5_40.

15 Moez Draief and Milan Vojnovic. Convergence speed of binary interval consensus. SIAM J. Control and Optimization, 50(3):1087–1109, 2012. 10.1137/110823018. URL https://doi.org/10.1137/110823018.

16 Leszek Gasieniec and Grzegorz Stachowiak. Fast space optimal leader election in population protocols. In Artur Czumaj, editor, Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2018, New Orleans, LA, USA, January 7-10, 2018, pages 2653–2667. SIAM, 2018. 10.1137/1.9781611975031.169. URL https://doi.org/10.1137/1.9781611975031.169.

17 George B. Mertzios, Sotiris E. Nikoletseas, Christofooros Raptopoulos, and Paul G. Spirakis. Determining majority in networks with local interactions and very small local memory. In Javier Esparza, Pierre Fraigniaud, Thore Husfeldt, and Elias Koutsoupias, editors, Automata, Languages, and Programming - 41st International Colloquium, ICALP 2014, Copenhagen, Denmark, July 7-11, 2014, Proceedings, Part I, volume 8572 of Lecture Notes in Computer Science, pages 871–882. Springer, 2014. 10.1007/978-3-662-43948-7_72. URL https://doi.org/10.1007/978-3-662-43948-7_72.

18 Yves Mocquard, Emmanuelle Anceaume, James Aspnes, Yann Busnel, and Bruno Sericola. Counting with population protocols. In D. R. Avresky and Yann Busnel, editors, 14th IEEE International Symposium on Network Computing and Applications, NCA 2015, Cambridge, MA, USA, September 28-30, 2015, pages 35–42. IEEE Computer Society, 2015. 10.1109/NCA.2015.35. URL https://doi.org/10.1109/NCA.2015.35.

19 Yves Mocquard, Emmanuelle Anceaume, and Bruno Sericola. Optimal proportion computation with population protocols. In Alessandro Pellegrini, Aris Gkoulalas-Divanis, Pierangelo di Sanzo, and Dimitr R. Avresky, editors, 15th IEEE International Symposium on Network Computing and Applications, NCA 2016, Cambridge, Boston, MA, USA, October 31 - November 2, 2016, pages 216–223. IEEE Computer Society, 2016. 10.1109/NCA.2016.7778621. URL https://doi.org/10.1109/NCA.2016.7778621.

20 Thomas Sauerwald and He Sun. Tight bounds for randomized load balancing on arbitrary network topologies. In 53rd Annual IEEE Symposium on Foundations of Computer Science, FOCS 2012, New Brunswick, NJ, USA, October 20-23, 2012, pages 341–350. IEEE Computer Society, 2012. 10.1109/FOCS.2012.86. URL https://doi.org/10.1109/FOCS.2012.86.


A Probabilistic Tools

Lemma 12 (Chernoff Bounds). Let \( n \in \mathbb{N} \) and consider a sequence \( (X_i)_{i \in [n]} \) of mutually independent binary random variables. Define \( X := \sum_{i \in [n]} X_i \) and let \( \mu_U, \mu_L \geq 0 \) be such that \( \mu_L \leq \mathbb{E}[X] \leq \mu_U \). The following inequalities hold for any \( \delta \geq 0 \) and \( \phi \geq 6\mu_U \):

\[
\Pr[|X - \mu| \geq \delta \mu_U] \leq \delta^2 e^{-\delta^2/4} \tag{4}
\]

\[
\Pr[X \geq (1 + \delta) \cdot \mu_U] \leq e^{-2 \delta \mu_U} \tag{5}
\]

and

\[
\Pr[X \geq \phi] \leq 2^{-\phi} \tag{6}
\]

Let \( \mu := \mathbb{E}[X] \). We often use the following simplified Chernoff bounds:

\[
\Pr[X \leq (1 - \delta) \cdot \mu] \leq n^{-a} \tag{7}
\]

\[
\Pr[X \geq \max \{13a \cdot \log n, (1 + \delta) \cdot \mu\}] \leq n^{-a} \quad \text{and,} \tag{8}
\]

where \( a \geq 0 \) is an arbitrary constant and \( \delta := \sqrt{3a \cdot \log(n)}/\mu \). For convenience, we sometimes combine both bounds into

\[
\Pr[|X - \mu| \geq \max \{13a \cdot \log n, \delta \cdot \mu\}] \leq 2n^{-a} \tag{9}
\]

B Auxiliary Protocols: Junta

In this section we prove Lemmas 3 and 4. As mentioned before, our process is a variant of a process from [16], altered such that we can prove also a lower bound on the maximum level reached by any node (instead of only an upper bound). Note that some of our auxiliary claims – in particular those concerning the upper bound on the maximum level – have been proven in similar form in [16].

Additional Notation. Recall that, for \( l \in \mathbb{N}_0, B_l \) denotes the number of nodes that reach level at least \( l \) before becoming inactive and that \( L^* \in \mathbb{N} \cup \{\infty\} \) denotes the maximum level reached by any node. For a constant \( \delta \in (0, \sqrt{2} - 1) \) define

\[
\xi_l := (1 + \delta)^{2^l - 1} \cdot 2^{-2^l - 1} \quad \text{and} \quad \tilde{\xi}_l := (1 - \delta)^{2^l - 1} \cdot 2^{-3} \cdot 2^{l+2} \tag{10}
\]

for \( l \in \mathbb{N} \). Let \( \xi_0 := \tilde{\xi}_0 := 1 \). Note that, by our constraints on \( \delta \), both the \( \xi_l \) and the \( \tilde{\xi}_l \) are monotonically decreasing in \( l \). Also, for \( l \in \mathbb{N} \) we have \( \xi_{l+1} = (1 + \delta) \cdot \xi_l^2 \) and \( \tilde{\xi}_{l+1} = (1 - \delta) \cdot \tilde{\xi}_l^2/4 \). As usual, all our proofs assume \( n \) to be sufficiently large.

Auxiliary Claims. Before we prove Lemma 3, we give some auxiliary claims. The first claim provides upper and lower high probability bounds on \( B_1 \).

Claim 1. For all constants \( a, \delta > 0 \), \( \Pr[|B_1 - n/2| < \delta \cdot n/2] \geq 1 - n^{-a} \).

Proof. Define a first interaction as an interaction in which at least one of the two involved nodes interacts for the first time. Let the random variable \( K \) denote the number of such interactions in which both involved nodes interact for the first time. There are \( n - 2K \) remaining first interactions, and in each of those exactly one of the involved nodes interacts for the first time. For each of these we define a binary random variable to be 1 if and only if the node which is interacting for the first time is the initiator. These binary random variables are independent and equal to 1 with probability 1/2. Let \( X \) denote the sum of
these $n - 2K$ random variables. Note that $B_l = K + X$. The random variable $K$ takes values from $\{1, 2, \ldots, \lfloor n/2 \rfloor\}$. If $K = n/2$, we have $B_1 = K + 0 = n/2$. Let us now condition on $K = k$ for a $k < n/2$ and define $\mu_k := \mathbb{E}[X \mid K = k] = (n - 2k)/2 = n/2 - k$. For any constant $b > 0$, Chernoff (Equation (9)) gives

$$
\Pr[|X - \mu_k| \geq \max\{13b \cdot \log n, \delta \cdot \mu_k\} \mid K = k] \leq 2n^{-b},
$$

(11)

where $\delta := \sqrt{3b \cdot \log(n)/\mu_k}$. Note that, conditioned on $K = k$, we have $|X - \mu_k| = |k + X - (k + \mu_k)| = |B_1 - n/2|$. We distinguish two cases:

**Case 1:** $\mu_k \leq 13^2b \cdot \log(n)^3/3$

With probability at least $1 - 2n^{-b}$ we have $|B_1 - n/2| \leq \max\{13b \cdot \log n, \delta \cdot \mu_k\} \leq 13b \cdot \log(n)^2 = o(1) \cdot n/2$.

**Case 2:** $\mu_k > 13^2b \cdot \log(n)^3/3$

We have $\delta = o(1)$ and with probability at least $1 - 2n^{-b}$ we get $|B_1 - n/2| \leq \max\{13b \cdot \log n, \delta \cdot \mu_k\} \leq \delta \cdot \mu_k \leq o(1) \cdot n/2$.

(12)

Combining both cases and using the law of total probability to get rid of the conditioning yields $\Pr\left[|B_1 - n/2| \geq o(1) \cdot \left(\frac{b}{2}\right)\right] \leq 2n^{-b}$, which implies the claim’s statement for $b = a + 1$. ▶

The following two claims will be used to prove the upper bound on $L^*$. ▶

**Claim 2.** For all $l \in \mathbb{N}$, $\xi \in [n^{-1/3}, 1)$, and all constants $a, \delta > 0$,

$$
\Pr[B_{l+1} < (1 + \delta) \cdot \xi^2 \cdot n \mid B_l \leq \xi \cdot n] \geq 1 - n^{-a}.
$$

(13)

**Proof.** Fix an $l \in \mathbb{N}$ and consider a node $u$ that just reached level $l$. Node $u$ is still active and will either become inactive or proceed to level $l + 1$ during its next interaction. Let $t$ be $u$’s next interaction. The probability for $u$ to proceed is at most $B_l/n$. This holds for all the $B_l$ nodes that eventually reach level at least $l$. By a straightforward coupling, we get that $B_{l+1}$ is stochastically dominated by $\text{Bin}(B_l, B_l/n)$. Conditioned on $B_l \leq \xi \cdot n$ we can apply Chernoff (Equation (5)) to get

$$
\Pr[B_{l+1} \geq (1 + \delta) \cdot \xi^2 \cdot n \mid B_l \leq \xi \cdot n] \leq e^{-\frac{\xi^2 \cdot 2^a}{4}} \leq e^{-\frac{\xi^2 \cdot n/3}{4}},
$$

(14)

implying the claim’s statement. ▶

**Claim 3.** For all $l \in \mathbb{N}$ and all constants $a > 0$,

$$
\Pr[B_{l+4a} = 0 \mid B_l < 2n^{1/3}] \geq 1 - n^{-a}.
$$

(15)

**Proof.** Note that $B_l < 2n^{1/3}$ implies $B_{l'} \leq B_l < 2n^{1/3}$ forall $l' \geq l$. By Markov’s inequality, $\Pr[B_{l+1} \geq 1 \mid B_{l'} < 2n^{1/3}] \leq \mathbb{E}[B_{l+1} \mid B_{l'} < 2n^{1/3}] \leq 4n^{-1/3}$. We apply Markov’s inequality to the next $4a$ levels and get $\Pr[B_{l+4a} \geq 1 \mid B_l < 2n^{1/3}] \leq (4n^{-1/3})^{4a} \leq n^{-a}$. ▶

Next, we give an auxiliary claim to be used to prove the lower bound on $L^*$. ▶

**Claim 4.** For all $l \in \mathbb{N}$, $\xi \in [n^{-1/2} \log n, 1)$, and all constants $a, \delta > 0$,

$$
\Pr[B_{l+1} \geq (1 - \delta) \cdot \xi^2 \cdot n/4 \mid B_l \geq \xi \cdot n] \geq 1 - n^{-a}.
$$

(16)

6 Run the original process and mark all nodes that reach level $l$. The coupled process uses the same random choices. Proceeding from level $l'$ to $l' + 1$ for $l' \in \mathbb{N} \setminus \{l\}$ works as in the original process. However, for a node to proceed from level $l$ to $l + 1$ its interaction partner must be marked.
Proof. Fix an \( l \in \mathbb{N} \) and consider a node \( u \) that just reached level \( l \). Node \( u \) is still active and will become either inactive or proceed to level \( l + 1 \) during its next interaction. Consider the last \( B_l/2 \) nodes that try to proceed from level \( l \) to level \( l + 1 \). For each of them, the probability to proceed to level \( l + 1 \) is at least \( B_l/(2n) \). Another straightforward coupling\(^7\) shows that \( B_{l+1} \) stochastically dominates Bin\((B_l/2, B_l/(2n))\). Conditioned on \( B_l \geq \xi \cdot n \) we can apply Chernoff (Equation (5)) to get

\[
\Pr \left[ B_{l+1} \leq (1 - \delta) \cdot \xi^2 \cdot n/4 \mid B_l \geq \xi \cdot n \right] \leq e^{-\delta^2 \xi^2 n/4} \leq e^{-\delta^2 \log n},
\]

implying the claim’s statement. \( \blacksquare \)

The last auxiliary claim bounds the time until all nodes become inactive.

\textbf{Claim 5.} For any constant \( a > 0 \), with probability at least \( 1 - n^{-a} \) all nodes become inactive during the first \( (6a + 12) \cdot n \ln n \) interactions.

\textbf{Proof.} The probability that a given node does not interact in a given interaction is \( 1 - 2/n \). Thus, the probability that a given node does not interact at all during the first \( c \cdot n \ln n \) interactions is at most \( (1 - 2/n)^{c \cdot n \ln n} \leq n^{-2c} \) for any \( c > 0 \). By a union bound, we get that all nodes interacted at least once after the first \( c \cdot n \ln n \) interactions with probability at least \( 1 - n^{-2c+1} \). Together with \textbf{Claim 1} and a union bound, we know that, with probability \( 1 - 2n^{-2c+1} \), there are at least \( n/3 \) nodes in state \((0,0)\) after \( c \cdot n \ln n \) interactions. From that point on, the probability for any fixed node to become inactive during a given interaction is at least \( \frac{1}{2n} \) (it is chosen as the initiator of the interaction and its communication partner is one of the \( n/3 \) nodes in state \((0,0)\)). Thus, the probability that any fixed node remains active during the next \( c \cdot n \ln n \) interactions is at most \( (1 - 1/(3n))^{c \cdot n \ln n} \leq n^{-c/3} \). By a union bound, all nodes become inactive during the next \( c \cdot n \ln n \) interactions with probability at least \( 1 - n^{-c/3+1} \). Combining, we get that all nodes become inactive within \( 2c \cdot n \ln n \) interaction with probability at least \( 1 - 2n^{-2c+1} - n^{-c/3+1} \geq 1 - 3n^{-c/3+1} \). We can make this probability to be at least \( 1 - n^{-a} \) by choosing \( c = 3a + 6 \). \( \blacksquare \)

\textbf{Proof of Lemmas 3 and 4.} We are finally ready to restate and prove the main results of this section.

\textbf{Lemma 3.} For any constant \( a > 0 \), with probability at least \( 1 - n^{-a} \) the following properties hold:

\( a) \) all nodes become inactive within \( O(n \log n) \) interactions,

\( b) \) \( \log \log n - 4 \leq L^* \leq \log \log n + 4 \cdot (a + 1) \), and

\( c) \) \( B_{L^*} = O(\sqrt{n} \cdot \log n) \).

\textbf{Proof.} Let \( \delta := 1/10 \) and recall the definition of \( \hat{\xi}_l \) and \( \hat{\xi}_l \). For the upper bound on \( L^* \), apply Claims 1 and 2, to get that, for any \( l \in \mathbb{N} \) with \( \hat{\xi}_{l-1} \geq n^{-1/3} \) and for any constant \( a > 0 \),

\[
\Pr \left[ B_l < \hat{\xi}_l \cdot n \mid B_{l-1} \leq \hat{\xi}_{l-1} \cdot n \right] \geq 1 - n^{-a-1}.
\]

Note that, since \( \hat{\xi}_0 = 1 \) and \( B_0 = n \), the conditioning is void for \( l = 1 \). Since \( \hat{\xi}_l < n^{-1/3} \) for \( l \geq \log \log n \), we can apply Equation (18) iteratively to see that there is an \( l \leq \log \log n \) such that

\[ \Pr \left[ B_l < \hat{\xi}_l \cdot n \right] \geq 1 - n^{-a-1}. \]

\( \text{(18)} \)

\textbf{Run the original process and let} \( b \) \text{ denote the number of nodes that reach level} \( l \). Mark the first \( b/2 \) nodes that try to proceed from level \( l \) to level \( l + 1 \). The coupled process uses the same random choices. Proceeding to the next level works as in the original process, except for the last \( b/2 \) nodes that try to proceed from level \( l \) to level \( l + 1 \): such nodes proceed only if their interaction partner is marked.
that $\Pr \left[ B_t < n^{2/3} \right] \geq 1 - l \cdot n^{-a-1}$. Together with another application of Claim 2, we get an $l \leq \log \log n + 3$ such that $\Pr \left[ B_l < (1 + \delta) \cdot n^{1/3} \right] \geq 1 - l \cdot n^{-a-1}$. Combined with Claim 3 we get an $l \leq \log \log n + 4a$ such that $\Pr \left[ B_l = 0 \right] \geq 1 - l \cdot n^{-a-1}$.

For the lower bound on $L^*$, similarly apply Claims 1 and 4 to get that, for any $l \in \mathbb{N}$ with $\hat{\xi}_{l-1} \geq n^{-1/3}$ and for any constant $a > 0$,

$$\Pr \left[ B_t > \hat{\xi}_{l-1} \cdot n \right] \geq 1 - n^{-a-1}$$

As above, since $\xi_0 = 1$ and $B_0 = n$, the conditioning is void for $l = 1$. Since $\hat{\xi}_l \geq n^{-1/3}$ for all $l \leq \log \log n - 3$, we can apply Equation (19) iteratively to see that, for $l = \lfloor \log \log n \rfloor - 3$,

$$\Pr \left[ B_t > n^{2/3} \right] \geq 1 - l \cdot n^{-a-1}.$$  

We have shown that the upper and lower bounds on $L^*$ hold each with probability $1 - O \left( \log \log n \cdot n^{-a-1} \right)$. By Claim 5, all nodes become inactive within $O \left( n \log n \right)$ interactions with probability at least $1 - n^{-a-1}$. Moreover, by applying Claim 4 with $\xi = n^{-1/2} \cdot \log n$, the number of nodes that reach level $L^*$ is, with probability at least $1 - n^{-a-1}$, at most $\sqrt{n} \cdot \log n$. Combining all these via a final union bound yields the lemma’s statement.  

**Lemma 4.** Let $l^* := \lfloor \log \log n \rfloor - 4$. There is a constant $\epsilon \in [0, 1)$ such that for any constant $a > 0$ we have $\Pr \left[ B_t < n^{l^*} \right] \geq 1 - n^{-a}$.

**Proof.** As in the proof of Lemma 3, let $\delta := 1/10$ and remember the definition of $\hat{\xi}_l$. Note that

$$\hat{\xi}_l = (1 + \delta) \cdot 2^{\log n - 1} \cdot 2^{-2^{l^* - 1}} \leq (1 + \delta) \cdot 2^{\log n - 1} \cdot 2^{-2^{l^* - 1}} \leq (1 + \delta) \cdot 2^{\log n - 1} \cdot 2^{-2^{l^* - 1}}$$

$$= \frac{1}{1 + \delta} \cdot (1 + \delta)^{\log(n)/16} \cdot 2^{-2^{l^* - 1}}$$

$$= \frac{1}{1 + \delta} \cdot n^{\log(1 + \delta)/16} \cdot n^{-1/32} = \frac{1}{1 + \delta} \cdot n^{2 \log(1 + \delta)/32} < n^{-0.02}.$$  

Define $\epsilon := -0.02 + 1 = 0.98$. Analogously the proof of Lemma 3, we have for any $l \in \mathbb{N}$ with $\hat{\xi}_{l-1} \geq n^{-l}$ and for any constant $a > 0$

$$\Pr \left[ B_t < \hat{\xi}_{l-1} \cdot n \right] \geq 1 - n^{-a-1}.$$  

Note that, since $\xi_0 = 1$ and $B_0 = n$, the conditioning is void for $l = 1$. Since $\hat{\xi}_l < n^{-l}$ for $l \geq l^*$ (by Equation (20) and the monotonicity of $\hat{\xi}_l$), we can apply Equation (21) iteratively to see that there is an $l \leq l^*$ such that $\Pr \left[ B_l < n^{l^*} \right] \geq 1 - l \cdot n^{-a-1} \geq 1 - l \cdot n^{-a}$. This implies the lemma’s statement.  

**C** Auxiliary Protocols: Phase Clock

The proof of Lemma 5 is a straightforward consequence of the following lemma from [16].

**Lemma 13 ([16, Lemma 3.7]).** For any constant $d > 0$ there is a constant $K > 0$ such that the following holds: Let $p_{\text{max}}$ denote the maximum and $p_{\text{min}}$ the minimum phase counter after an interaction $t \in \mathbb{N}$. Assume $p_{\text{max}} - p_{\text{min}} \leq 2K$. With high probability, there is a $t' > t + d \cdot n \log n$ such that:

a) $t'$ is the first interaction after which the maximum phase counter is $p_{\text{max}} + K$.

b) After interaction $t'$, all nodes have a phase counter value of at least $p_{\text{max}}$.

With this, we are ready to restate and prove Lemma 5.
Lemma 5. Fix an \( i \in \mathbb{N}_0 \) with \( i = \text{poly}(n) \). For any constants \( a, d_1 > 0 \) there exists a constant phase clock parameter \( m \in \mathbb{N} \) and a constant \( d_2 > 0 \) such that, with probability at least \( 1 - n^{-e} \), \( d_1 \cdot n \log n \leq R_{\text{End}}(i) - R_{\text{Start}}(i) \leq d_2 \cdot n \log n \).

Proof. The lower bound on \( R_{\text{End}}(i) - R_{\text{Start}}(i) \) follows by applying Lemma 13 with \( d = d_1 \) and by setting \( m = 2K \). For the upper bound, note that the one-way epidemic (cf. Lemma 1) implies that, with high probability, the maximum phase counter increases within \( O(n \log n) \) rounds (when a leader finally sees the maximum phase counter). Thus, with high probability, it takes at most \( m \cdot O(n \log n) = O(n \log n) \) interactions for a node to leave a given round.

D Simple Majority with Phase Clocks

This appendix restates and proves the following theorem:

Theorem 6. Consider the majority problem for \( n \) nodes with initial absolute bias \( \alpha \in \mathbb{N} \). Let \( s \in \mathbb{N} \setminus \{1\} \) and \( r \in \mathbb{N} \setminus \{1, 2\} \). With high probability, protocol ClockedMajority\((s,r)\) correctly identified the majority for all interactions \( t = \Omega(n \log n \cdot \log_2(n/\alpha)) \). It uses \( \Theta(s \cdot r + \log \log n) \) states per node.

Proof. Recall the definition of a round from phase clocks (Section 3.2), in particular \( R_{\text{Start}}(i) \) (start of round \( i \)) and \( R_{\text{End}}(i) \) (end of round \( i \)). Also recall that Lemma 5 provides us with bounds on the minimal and maximal time all nodes spend together in a round.

Define \( i^* := \lfloor \log_2(3n/\alpha) \rfloor \) and let \( K \) be the first round for which nodes either all have a positive or all have a negative sign when they enter round \( K \). Let \( \text{load}_u(i) \) be the load of node \( u \) at the beginning of its first interaction in round \( i \) and define \( \Phi_i := \lfloor \sum_{u \in [n]} \text{load}_u(i) \rfloor \) (the bias after \( i \) rounds). Note that \( \Phi_0 = 0 \).

By applying Lemma 5 with \( d_1 \) equal to the constant \( c \) from Lemma 2 and using a union bound over the first \( i^* \) rounds, we see that, with high probability: a) whenever two nodes balance during the first \( i^* \) rounds, they are in the same round; b) at the end of any of the first \( i^* \) rounds, the discrepancy between the load values is at most 2; c) at the start of each round \( i \in [\min \{ i^*, K \} - 1] \), each node has a load value in \( \{-2, -1, 0, 1, 2\} \); d) for any round \( i \in [i^*] \), we have \( \Phi_i = \alpha \cdot s^{i^*} \), as no load was lost (it merely canceled) and any node increases its absolute load value from 1 to \( s \) at the beginning of each round. In particular, \( \Phi_{i^*} = \alpha \cdot s^{i^*} \geq \alpha \cdot s^{\log_2(3n/\alpha)} = 3\alpha \). But then, there must be at least one node with load at least \( 3 \) (at most \( s^{-3} \)) and, since the discrepancy is at most \( 2 \), all other nodes have load at least \( 1 \) (at most \( -1 \)). It follows that all nodes have, with high probability, the correct sign after \( i^* \) rounds (and this cannot change subsequently as load values cannot cancel any longer). Thus, with high probability, \( K \leq i^* \).

The runtime bound stated in the theorem follows since, by Lemma 5, with high probability any round takes \( d_2 \cdot n \log n = O(n \log n) \) interactions. The bound on the number of required states follows from the above, since, with high probability, no node increases its load value beyond \( 2s \) in absolute value during any of the interactions \( t \in \min \{ i^*, K \} \). Thus, we can simply cap the load values at \( 2s \) and, with high probability, the protocol outcome will not change. Since the PhaseClock\(r\) requires, in addition to the \( \Theta(\log \log n) \) recyclable states, \( \Theta(r) \) states, the total number of states required by ClockedMajority\((s,r)\) is \( \Theta(s \cdot r + \log \log n) \).

E Stable Majority with Phase Clocks

This appendix restates and proves the following theorem:
Theorem 7. Consider the majority problem for \( n \) nodes with initial absolute bias \( \alpha \in \mathbb{N} \). Let \( s \in \{2, 3, \ldots, n\} \). Protocol \textsc{StrongMajority}_{s} is exact and stabilizes with high probability in \( O(n \log n \cdot \log_{s}(n/\alpha)) \) interactions. It uses \( \Theta(s \cdot \log_{s} n) \) states per node.

Proof. Consider the stabilization time \( T_{ST} \) of \textsc{StrongMajority}_{s}. We have to show that \( T_{ST} < \infty \) with probability 1 and that, with high probability, \( T_{ST} = O(n \log n \cdot \log_{s}(n/\alpha)) \).

We first consider the case that \textsc{FormJuntaExtended} (which is used to generate the marked nodes for the phase clock) marked no node. By Lemma 3, this happens with negligible probability, so it is sufficient to show that \( T_{ST} < \infty \). For this, note that if no node is marked, no node will increase its phase counter beyond zero. Thus, the majority guess of \textsc{StrongMajority}_{s} is the majority guess of the backup protocol. Since the stabilization time of \textsc{BackupMajority} is finite, so is \( T_{ST} \) in this case.

We now consider the case that \textsc{FormJuntaExtended} marked at least one node. This implies that all nodes finish with probability 1 (no later than when their phase counters overflow). For a node \( u \) let \( L_{\text{Fin}}(u) \) denote its load at the beginning of the interaction during which it finished. If some node sets its error bit, with probability 1 all nodes will eventually set their error bit. But then, as above, \textsc{StrongMajority}_{s} uses the majority guess of \textsc{BackupMajority}, which implies \( T_{ST} < \infty \). Otherwise, if no node sets its error bit, the following two properties hold for any two nodes \( u \) and \( v \): \textsc{PCdifferentRound}(\( u, v \)) = \textit{false} and \( \text{sign}(L_{\text{Fin}}(u)) = \text{sign}(L_{\text{Fin}}(v)) \). Since nodes finish immediately after their phase counters overflows, with probability 1, \textsc{PCdifferentRound}(\( u, v \)) correctly returns whether \( u \) and \( v \) are in different rounds\(^8\). For the same reason, with probability 1 all load balancing steps are performed between nodes in the same round.

So we know that all nodes were in the same round \( i \leq r \), agreed on their sign just before they turned finished, and all balancing operations were between nodes of the same round. For \( \Phi_{i} \), as defined in the proof of Theorem 6 (the bias after \( i \) rounds), this implies \( \Phi_{i} = \alpha \cdot s^{i} \), which in turn means that all nodes agree on the correct majority and, thus, \( T_{ST} < \infty \). It remains to show that, with high probability, \( T_{ST} = O(n \log n \cdot \log_{s}(n/\alpha)) \). This follows completely analogously to the proof of Theorem 6, but with \( i^{*} = \lceil \log_{s}(5n/\alpha) \rceil \). Indeed, the same argumentation yields that, with high probability, in each of the at most \( r = \log_{s}(5n) \) rounds all nodes stay together, balance out to a discrepancy of at most two, and that each round takes at most \( O(n \log n) \) interactions. The choice \( i^{*} = \lceil \log_{s}(5n/\alpha) \rceil \) guarantees that \( \Phi_{i} = \alpha \cdot s^{i^{*}} \geq 5n \), ensuring that all nodes have an absolute load of at least 3 and, thus, finish until round \( i^{*} \).

It remains to bound the number of states required per node. We need 4 states for \textsc{BackupMajority}, 2 states for the bit finished\(_{u} \), 2 states for the bit error\(_{u} \), and \( \Theta(s \cdot r) = \Theta(\log_{s}(n)) \) states for \textsc{ClockedMajority}_{s,r}, yielding the desired bound.

\(\Box\)

F Convergent Majority with Phase Clocks

This appendix gives the pseudocode for the transition function of \textsc{WeakMajority}_{s} (see Algorithm 3) as well as restates and proves the following theorem:

Theorem 8. Consider the majority problem for \( n \) nodes with initial absolute bias \( \alpha > 0 \). Let \( s \in \{2, 3, \ldots, n\} \). Protocol \textsc{WeakMajority}_{s} is exact and converges with high probability in \( O(n \log n \cdot \log_{s}(n/\alpha)) \) interactions. It uses \( \Theta(s + \log \log n) \) states per node.

\(^{8}\) If they would continue after the overflow, it might give the wrong result for nodes that are a multiple of \( r \) rounds apart.
REFERENCES

| WeakMajority_s(u, v) |
|----------------------|
| BackupMajority(u, v) |
| if count_u = 600 then |
| exit |
| if PCmarked(v) then |
| count_u ← count_u +1 |
| else |
| count_u ← 0 |
| ClockedMajority_s(u, v) |

Algorithm 3: Formal description of the transition function for WeakMajority_s for an initiator u and a responder v.

Proof. We consider the following three disjoint events. Let \( \mathcal{E}_0 \) be the event that the junta remains empty, \( \mathcal{E}_1 \) be the event that 1 to \( n^{0.99} \) nodes form a designated junta, and \( \mathcal{E}_2 \) be the event that more than \( n^{0.99} \) nodes belong to the junta. We first show that in all three events the protocol eventually returns the correct result.

Case 1: \( \mathcal{E}_0 \). Since no junta exists, the phase clocks do not tick. Therefore, the protocol will indefinitely output the result from the backup protocol, which stabilizes after \( n^2 \log n \) interactions.

Case 2: \( \mathcal{E}_1 \lor \mathcal{E}_2 \). Let \( t_{junta} \) be the first time step when at least one node belongs to the junta. At any time step \( t \geq t_{junta} + 600 \), there exists a polynomial probability for any node \( v \) that \( v \) interacts with a junta node 600 times in a row. Therefore, in expectation every node will increment its counter to 600 after a (large) polynomial number of interactions. Once all nodes have counter value 600, they will again indefinitely output the result from the backup protocol, which stabilizes after \( n^2 \log n \) interactions.

Together, these two cases yield the claim that WeakMajority_s stabilizes.

We now consider the event \( \mathcal{E}_1 \). By assumption, the probability that a node samples a junta node is at most \( n^{0.99}/n \). Therefore, the probability that a node samples a junta node 600 times in a row is at most \( n^{-6} \). Taking a coarse union bound over the first \( n^2 \log n \) interactions of any node gives that with probability at least \( 1 - n^{-3} \) no node increments its coin to 600 before it has performed \( n^2 \log n \) interactions. Observe that at the time when any node has performed \( n^2 \log n \) interactions the global number of interactions is larger than the time required for the backup protocol to stabilize.

The following now holds. No node reaches counter value 600 before \( n^2 \log n \) interactions, with high probability, giving all nodes enough time to complete both, ClockedMajority_s and BackupMajority. With high probability, ClockedMajority_s,3 gives the correct answer after \( O(n \log(n/\alpha) \cdot \log s \cdot n) \) interactions, while BackupMajority stabilizes in \( O(n^2 \log n) \) interactions. If ClockedMajority_s does not introduce an error and stabilizes in time, and if no node prematurely counts to 600, all nodes

- give the correct output after \( O(n \log(n/\alpha) \cdot \log s \cdot n) \) interactions, with high probability,
- do not switch to the backup protocol before the backup protocol has stabilized, with high probability,
- thus do not change their output \( O(n \log(n/\alpha) \cdot \log s \cdot n) \) interactions, with high probability,
- and eventually stabilize, with probability 1.

It remains to analyze the state space size. We observe that the junta creation process requires \( O(\log \log n) \) states, which, however, can be recycled. ClockedMajority_s,3 requires \( O(s) \).
states. Together, this implies the theorem.

G Leader Election

G.1 Protocol Description

With three exceptions, the protocol is identical to the protocol from [16]. We first sketch the protocol from [16] and then we describe the necessary changes.

In the original protocol from [16], each node starts in a protocol to form a junta. As soon as a node has concluded the junta protocol, it starts the leader election. The leader election runs in $O(\log n)$ rounds of $O(n \log n)$ interactions each, synchronized by the phase clock. At the beginning of each round, junta nodes sample one random bit. This bit is then sent to all other nodes via one-way epidemic. Junta nodes which have sampled 0 but observe a 1 in the broadcast, meaning that another leader contender exists which has sampled 1, set their state to Follower.

Our Modifications.

- Our protocol runs in $O(\log n/\log s)$ rounds of $O(n \log n)$ interactions each.
- We use the phase clock based on FormJuntaExtended. This allows a node $v$ to recycle the states used by FormJuntaExtended, once one of the following two conditions occur: $v$ has entered the junta, or $v$’s phase clock has ticked at least once. Both conditions ensure the presence of a junta of size at least 1 on a global level, allowing each node to safely leave the FormJuntaExtended process without storing its junta level $l$. Only when nodes have left the junta protocol, they start with the main algorithm.
- We sample more than one random bit in each round to determine which nodes remain possible leaders and which ones do not. We do not only sample and compare one bit per round, but $\log s - \log \log s$ bits. After the sampling we use one-way epidemics to broadcast the maximum of all sampled ($\log s - \log \log s$)-bit numbers. If a contender sees a larger number than its own sample, it becomes a Follower.

Our protocol samples $O(\log n/\log s)$ blocks (one per round) of bit strings strings of length $\log s - \log \log s$ each and compares them block-wise. In each round only those nodes who sampled the largest bit string remain leader contenders. In Lemma 14 we will see that this can be be regarded as comparing long bit stings of length $O(\log n)$ with each other. The node that sampled the largest of these long bit strings will be the final leader, since a bit string of logarithmic length suffices to determine a unique leader with high probability. Note that when talking about the value of a bit string or comparing bit strings we always refer to the number encoded by this bit string in binary representation.

Observe that the protocol in [16], as well as our modification, can be attributed to a class of comparison based leader election protocols. In that class, each node has a random virtual coin (the long bit string). The node with the largest virtual coin wins the leader election. In contrast to comparison based protocols, the protocols from [1, 12] could be attributed to a class of tournament based leader election protocols, where leader contenders do not sample random bits but rather successively increment their coin values.

In the following we describe in detail how the sampling is done.

Sampling $\Theta(\log s)$ Random Bits. As soon as a junta is present, we recycle the states required to form the junta and use the total number of $s$ states available at each node for the
sampling of random bits. Observe, however, that in order to sample $\Theta(\log s)$ random bits, nodes need to count to $\log s$ (to determine which bit to sample next). We therefore sample $\log s - \log \log s$ random bits and use the remaining $\log \log s$ bits to count up to $\log s - \log \log s$.

At the beginning of a new round, each leader contender $v$ starts the sampling procedure. Any contender which initiates the first interaction in a round sets the lowest order bit of its sample randomly (how, we will describe in the next paragraph). In the next interaction which $v$ initiates the next bit is set, and so on. After $\log s - \log \log s$ initiated interactions, each leader contender has sampled $\log s - \log \log s$ random bits. Each leader contender must maintain a counter to determine which bit has to be sampled next. This counter requires $\log(\log s - \log \log s) \leq \log \log s$ bits. Together, the random bits and the counter require at most $2^{\log s - \log \log s} \cdot 2^{\log \log s} = s$ states.

It remains to show that each node has access to a source of randomness that allows the node to generate a random bit with probability $1/2 \cdot (1 \pm o(1))$. We resort to so-called synthetic coins, which were introduced in [3]. The idea is that each node toggles a so-called flip bit in every interaction. The analysis and Lemma 3 from [12] can be directly applied to our setting:

**Lemma (Lemma 3 from [12]).** Let $a > 0$ and consider an interaction $t$ with $n \cdot \ln \log \log n / 2 \leq t \leq n^a$. The number of flip bits that equal zero at the beginning of interaction $t$ lies with probability at least $1 - n^{-a}$ in $(1 \pm 1/\log \log n) \cdot n/2$.

This implies that starting with the second round, the probability to sample a node with flip bits set is $1/2 \cdot (1 - o(1))$ with high probability. Now we are ready to state the following lemma showing that our protocol finds a unique leader with high probability.

**Lemma 14.** Consider the leader election problem for $n$ nodes. Let $s \in \{2, 3, \ldots, n\}$. With high probability, LeaderElection, identifies a unique leader after $t = O(n(\log n)^2 / \log s)$ interactions. It uses $\Omega\left(\max\{\log \log n, s\}\right)$ states per node.

**Proof.** First we consider the stabilization time. Assume that all nodes remain properly synchronized for the first $\Theta(n(\log n)^2 / \log s)$ interactions. Recall that this holds with high probability. We obtain that that after $8 \log n / \log s$ rounds random bit strings of length at least $8 \log n / \log s \cdot (\log s - \log \log s) \geq 4 \log n$ bits have been compared. Let $v$ be a node which has sampled the largest bit string and observe that such a node always exists. We now show that with high probability at most one such node exists. The probability that two nodes have sampled the same $4 \log n$ bits is at most $(1/2 \cdot (1 \pm o(1)))^{-4 \log n} \leq n^{-3}$. Here, the minor difference of the probability to $1/2$ results the use of the flip bits. Taking a union bound over all $n-1$ pairs of nodes $(u, v)$ gives us that with high probability at most one node, $v$, remains a contender. That is, we compare $O(\log n)$ bits in total to determine a unique leader with high probability, and therefore our protocol requires $O(\log n / (\log s - \log \log s)) = O(\log n / \log s)$ rounds. Since sampling $O(\log s)$ bits in every round does not, asymptotically, increase the length of a round, each of these rounds consists of $O(n \log n)$ interactions and thus our protocol requires $O(n(\log n)^2 / \log s)$ interactions in total.

It remains to analyze the required number of states. We resort to the protocol FormatMJuntaExtended to form a junta. This protocol requires $O(\log \log n)$ states, which then can be recycled as soon as the nodes conclude the junta protocol. Once the junta has been formed, all nodes use all available $s$ states to generate $\log s - \log \log s$ random bits and to count to $\log s$. Additionally, the protocol requires various Boolean flags, e.g., to store the flip bit. However, these are only of constant size and therefore the resulting state space size is asymptotically dominated by $\max\{s, \log \log n\}$.  

\[\square\]
G.2 Stability

The protocol described above always elects one leader with high probability. However, in order to devise a protocol that stabilizes, we need to apply some further modifications which will be described in the following.

Some of these modifications have been previously described in [16]. There, a hybrid protocol is constructed by combining the leader election protocol with a backup protocol. They use a so-called outer phase clock that ticks once after \( O(n(\log n)^2) \) interactions. Crucial to their approach is that their phase clocks, both ordinary and outer, tick if and only if at least one leader contender is present. If all leader contenders were lost before the outer phase clock has ticked, the ordinary phase clock and the outer phase clock cease to tick. Initially, all nodes output the result from the backup protocol. As soon as the outer phase clock ticks, all nodes stabilize override the backup protocol with the result from the fast protocol. (If more than one leader remains, upon a direct interaction of two leaders, one of them is eliminated.)

Similar to [16], all nodes run the two-state backup protocol from [14] in parallel as a backup protocol. This protocol stabilizes with high probability in \( O(n^2 \log n) \) interactions such that only one leader contender is left.

**Decelerated Phase Clock.** We pick up the idea from [16] and run an additional phase clock which we call decelerated phase clock. This decelerated phase clock is slowed down such that it runs through zero when a node has performed \( \Theta(\log^2 n / \log s) \) interactions. This is achieved based on the following idea.

Each node starts the protocol in a marking phase, with the goal that \( n \cdot \log s / \log n \) nodes mark themselves upon conclusion of the marking phase. We use the same idea and analysis as in [12]. By simply adapting the number of marking trials to \( \log \log n - \log \log s \), the resulting number of marked nodes is, with high probability, in \( \Theta(n \cdot \log s / \log n) \). The proof is analogous to the proof of Proposition 2 from [12].

We use the marked nodes in the following way to artificially slow down the decelerated phase clock. The nodes alternate between proper interaction steps and steps in which they perform an operation on the decelerated phase clock. Whenever a node interacts with a marked node, it will, in its next step, update the decelerated phase clock. This has the effect of decoupling marked nodes from the clock itself, and gives independence between initiating an interaction with a marked node and the operations on the clock. This results in an execution of the phase clock protocol that is slowed down by a factor of \( \log n / \log s \) as stated in the following observation.

**Observation 15.** With high probability, the decelerated phase clock ticks for the first time after \( \Theta\left(n(\log n)^2 / \log s\right) \) interactions.

**Proof.** Let \( X_\beta(t) \) be the number of operations performed on the decelerated phase clock in the first \( t \) interactions when a fraction of \( \beta \) nodes is marked. For polynomial \( t \) and \( \beta = \log s / \log n \) we have \( X_\beta(t) = \Theta(t \cdot \log s / \log n) \), with high probability. Observe that \( X_\beta(t) \) has binomial distribution \( \text{Bin}(t, \beta) \) and thus \( \mathbb{E}\left[X_\beta(c \cdot n \cdot (\log n)^2 / \log s)\right] = c \cdot n \log n \) where \( c \) is a constant. The concentration follows from Chernoff bounds, using the independence between sampling a marked node and the operations on the decelerated phase clock. The observation follows by combining this bound on \( X_\beta(t) \) with Lemma 5 for the phase clocks. ▶
Checking Phase / Stabilizing. We use a similar approach to stabilize the system as described in [13]. As soon as the decelerated phase clock has ticked for the first time, all remaining leader nodes enter a final state $q_l$ and no longer participate in the fast protocol. Any follower which interacts either with a leader in $q_l$ of another follower in $q_f$ enters state $q_f$. In the unlikely event that more than one contender enters state $q_l$, the remaining contenders perform pairwise elimination as in the backup protocol.

Existence of Marked Nodes and the Junta. The entire protocol depends on the existence of marked nodes and the junta. We apply a simple rule to determine whether a junta is present. All nodes initially output the result from the backup protocol. Only when nodes enter states $q_l$ of $q_f$, they output LEADER or FOLLOWER, respectively. (Observe that the existence of marked nodes and the junta is necessary for the decelerated phase clock to ever tick.) That way, the system eventually stabilizes; based on the backup protocol if no marked nodes or no junta exists, or based on the fast protocol, as soon as the nodes enter states $q_l$ or $q_f$.

We now put everything together and show the main theorem of this section.

Proof of Theorem 9. We start by showing that LEADERELECTION$_s$ stabilizes. From the description above of the output function we obtain that there are two cases: a) the decelerated phase clock ticks, thus at least one contender exists, and this contender enters state $q_l$; or b) all contenders have been eliminated before the decelerated phase clock ticked, in which case all nodes indefinitely output the backup protocol. Nodes in $q_l$ eliminate each other analogously to the backup protocol. Since the backup protocol stabilizes, LEADERELECTION$_s$ stabilizes in either case as well.

Now we consider the stabilization time. From Observation 15 we obtain that the decelerated phase clock ticks after $O\left(\frac{n \log n}{\log s}\right)$ interactions for the first time, with high probability. From Lemma 14 we obtain that we have one unique leader at that time. This single leader will enter state $q_l$ as soon as it observes the tick from the decelerated phase clock. After further $O(n \log n)$ interactions required for one-way epidemic to conclude, all other nodes have entered $q_f$, with high probability. Once all nodes are in $q$-states, the system stabilizes.

Finally, we consider the required number of states. We first need to count to $\log \log n$, in order to initialize the flip bits and sample the marker bit. However, once the marking has been done, this counter value can be used for the protocol FORMJUNTAEXTENDED. This protocol requires $O(\log \log n)$ states for the junta level $l$, which again can be recycled as soon as the nodes leave the protocol FORMJUNTAEXTENDED. Finally, all nodes use all available $s$ states to generate $\log s$ random bits in every round. In addition, the protocol requires various Boolean flags, two distinct constant size phase counters for the two types of phase clocks, the states for the backup protocol, and two terminal $q$ states. However, all of these are only of constant size. Therefore, the resulting state space size is asymptotically dominated by $\max\{s, \log \log n\}$.

H A Note on Uniformity

This appendix restates and gives a brief proof sketch for the following theorem:

► Theorem 11. Consider the majority problem for $n$ nodes with initial absolute bias $\alpha \in \mathbb{N}$. Let $s \in \mathbb{N}\setminus\{1\}$. There is a variant UNIFORMMAJORITY$_s$ of STRONGMAJORITY$_s$ which is ex-
act, uniform, and stabilizes with high probability in $O\left(n \log n \cdot \log_4(n/\alpha) + \min\{s, n^2 \cdot \log n\}\right)$ interactions. With high probability, it uses $O\left(s \cdot \log_4(n/\alpha) \cdot \log \log n\right)$ states per node.

**Proof.** Replacing the phase clock’s junta algorithm $\text{FormJuntaExtended}$ by $\text{FormJunta}$ maintains the same runtime guarantees (Lemma 5 still holds) but increases the required number of states by a factor of $\Theta\left(\log \log n\right)$ (we can no longer recycle the junta states). Also, since we use the phase clock $\text{PhaseClock}_r$ with parameter $r = \infty$, it never overflows, voiding the corresponding finishing condition in Algorithm 2. However, it also guarantees that the phase clock function call $\text{PCdifferentRound}(u, v)$ gives always the correct result with probability 1 (cf. Section 3.2).

In contrast to our $\text{FormJuntaExtended}$ protocol, $\text{FormJunta}$ always results in at least one marked node [16, Lemma 4.1], ensuring that each node keeps increasing its phase as long as it is not finished.

We now show that, eventually, all nodes finish with probability 1. Since the finish bit finished$_{u,i}$ of a node $u$ spreads via a one-way epidemic, by Lemma 1 it is sufficient to show that one nodes finishes. To see this, assume no node finishes. Then all nodes reach any round $i \in \mathbb{N}$. Since load balancing is guaranteed to be done only between nodes in the same round, for any such $i$ we have $\Phi_i = \alpha \cdot s^i$, where $\Phi_i$ is defined as in the proof of Theorem 6 (the bias after $i$ rounds). Choosing $i^* = \lceil \log_4(3n/\alpha) \rceil$ guarantees that there is at least one node with absolute $\geq 3$ at the beginning of round $i^* + 1$. This node will become finished.

For a node $u$ let $L_{\text{Fin}}(u)$ denote its load at the beginning of the interaction during which it finishes. As in the proof of Theorem 7, we get that if at least one node sets its error bit, we must have $T_{\text{ST}} < \infty$ (the error spreads via a one-way epidemic and nodes switch to the backup protocol). If no node sets its error bit, by the conditions for setting the error bit in Algorithm 2, all nodes are in same round when they finish and have the same sign. The invariant $\Phi_i = \alpha \cdot s^i$ guarantees that this sign is the initial majority. Thus, also here $T_{\text{ST}} < \infty$.

It remains to show $T_{\text{ST}} = O\left(n \log n \cdot \log_4(n/\alpha) + \min\{s, n^2 \cdot \log n\}\right)$ with high probability. Consider first the case $s = O\left(n^2 \cdot \log n\right)$. In this case, the load balancing time when started with discrepancy $O\left(s\right)$ is, with high probability, $O\left(n \log n\right)$ (cf. Lemma 2). Thus, by Lemma 5, we can choose the constant parameter $m$ of the phase clock parameter such that, with high probability, all nodes balance out to a discrepancy of at most two during any of the first $i^* = \lceil \log_4(5n/\alpha) \rceil$. As before (cf. proof of Theorem 7), this implies that all nodes have agreed on the correct majority opinion after $i^* + 1$ rounds and have finished. Thus, $T_{\text{ST}} = O\left(n \log n \cdot \log_4(n/\alpha)\right)$ in this case.

Consider now the case $s = \Omega\left(n^2 \cdot \log n\right)$. Nodes might not be able to balance out quickly enough, such that they will have too much load at the beginning of the next round and become finished. If no error bit is set, the nodes have the correct output, as argued above. In the (not unlikely) case that an error bit is set (this is likely to happen in the first round, when nodes still have different signs), the backup protocol guarantees stabilization with high probability in $O\left(n^2 \cdot \log n\right)$ interactions.

Together, both cases yield the desired bound on the stabilization time. For the high probability bound on the number of states required per node: We need 4 states for $\text{BackupMajority}$, 2 states for the bit finished$_u$ and 2 states for the bit error$_u$. The phase clock needs, with high probability, $\Theta\left(1\right) \cdot O\left(\log \log n\right)$ states (for the junta generation). Since we saw above that, with high probability, all nodes finish after $O\left(\log_4(n/\alpha)\right)$ rounds, no phase clock counted beyond $O\left(\log_4(n/\alpha)\right)$. Together, with the $\Theta\left(s\right)$ states needed for the load values, this yields the desired bound. ▪