POSITIVE AND NODAL SOLUTIONS FOR PARAMETRIC NONLINEAR ROBIN PROBLEMS WITH INDEFINITE POTENTIAL

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Abstract. We consider a parametric nonlinear Robin problem driven by the \( p \)-Laplacian plus an indefinite potential and a Carathéodory reaction which is \((p-1)\)-superlinear without satisfying the Ambrosetti - Rabinowitz condition. We prove a bifurcation-type result describing the dependence of the set of positive solutions on the parameter. We also prove the existence of nodal solutions. Our proofs use tools from critical point theory, Morse theory and suitable truncation techniques.

1. Introduction. Let \( \Omega \subset \mathbb{R}^N \) be a bounded domain with a \( C^2 \)-boundary \( \partial \Omega \). In this paper, we study the following parametric nonlinear Robin problem:

\[
(P_\lambda) \begin{cases}
-\Delta_p u(z) + (\xi(z) + \lambda)u(z)^{p-1} = f(z,u(z)) & \text{in } \Omega, \\
\frac{\partial u}{\partial n_\beta} + \beta(z)u(z)^{p-1} = 0 & \text{on } \partial \Omega, \lambda > 0, u > 0.
\end{cases}
\]

Here \( \Delta_p \) denotes the \( p \)-Laplace differential operator defined by

\[
\Delta_p u = \text{div}(|Du|^{p-2}Du) \quad \text{for all } u \in W^{1,p}(\Omega), \ 1 < p < \infty.
\]

The potential function \( \xi \in L^\infty(\Omega) \) is indefinite (that is, sign changing) and \( \lambda > 0 \) is a parameter. The reaction term \( f(z,x) \) is a Carathéodory function (that is, for all \( x \in \mathbb{R}, z \mapsto f(z,x) \) is measurable and for a.a. \( z \in \Omega, x \mapsto f(z,x) \) is continuous) which is \((p-1)\)-superlinear in the \( x \)-variable, but without satisfying the usual (in

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such cases) Ambrosetti - Rabinowitz condition. Indeed, we replace such a condition with a weaker one which lets us consider superlinear nonlinearities with slower growth near $\infty$, and not satisfying the Ambrosetti - Rabinowitz condition. Finally, in the boundary condition, \( \frac{\partial u}{\partial n_p} \) denotes the generalized normal derivative defined by

\[
\frac{\partial u}{\partial n_p} = |Du|^{p-2}(Du, n)_{R^N} \quad \text{for all } u \in W^{1,p}(\Omega),
\]

with \( n(\cdot) \) being the outward unit normal on \( \partial \Omega \). This kind of normal derivative is dictated by the nonlinear Green's identity (see, for example, Gasinski - Papageorgiou [11, p. 211]) and in an even more general form can be found also in the work of Lieberman [15]. The boundary weight function \( \beta(\cdot) \) belongs to \( C^{0,\alpha}(\partial \Omega) \) with \( \alpha \in (0,1) \) and \( \beta(z) \geq 0 \) for all \( z \in \partial \Omega \). When \( \beta \equiv 0 \), we have the Neumann problem and in that case from (1) we see that the boundary condition becomes \( \frac{\partial u}{\partial n} = 0 \) on \( \partial \Omega \) (the usual normal derivative).

First, we look for positive solutions and our aim is to establish the precise dependence of the set of positive solutions on the parameter \( \lambda > 0 \). So, we prove a bifurcation-type result and show that there exists a critical parameter value \( \lambda_* > 0 \) such that

- for all \( \lambda > \lambda_* \), problem \( (P_{\lambda}) \) admits at least two positive solutions,
- for all \( \lambda = \lambda_* \), problem \( (P_{\lambda}) \) admits at least one positive solution,
- for all \( \lambda < \lambda_* \), problem \( (P_{\lambda}) \) has no positive solutions.

We also show that for every \( \lambda \in [\lambda_*, +\infty) \) problem \( (P_{\lambda}) \) has a smallest positive solution \( u^*_\lambda \) and we investigate the monotonicity and continuity properties of the map \( \lambda \mapsto u^*_\lambda \) in the relevant function space. Finally, in Section 4, we impose bilateral asymptotic conditions on \( f(z, \cdot) \) and prove the existence of nodal (sign changing) solutions.

Our work here continues and complements the papers of Motreanu - Motreanu - Papageorgiou [17] and Mugnai - Papageorgiou [20]. In [17] the authors examine problem \( (P_{\lambda}) \) when \( \xi \equiv 0, \beta \equiv 0 \) and prove the existence of constant sign and nodal solutions when \( \lambda > 0 \) is big and the reaction term \( f(z, \cdot) \) is \((p-1)-\)superlinear. However, they do not establish the precise dependence of the set of positive solutions on the parameter \( \lambda > 0 \) (bifurcation-type result). On the other hand Mugnai - Papageorgiou [20], study nonparametric Neumann problems (that is, \( \beta \equiv 0 \)) and prove existence and multiplicity theorems for resonant problems. Their work was extended recently to problems with a \((p-1)-\)superlinear reaction term by Fragnelli - Mugnai - Papageorgiou [10] (see also Mugnai - Papageorgiou [21]). We mention also the relevant works of Brock - Iturriaga - Ubilla [5] (nonlinear parametric Dirichlet problems with \( \xi \equiv 0 \)), Cardinali - Papageorgiou - Rubbioni [6] (nonlinear parametric Neumann problems with \( \xi \equiv 0 \) and a superdiffusive reaction term), Gasinski - Papageorgiou [11] (nonlinear parametric Dirichlet problems with \( \xi \equiv 0 \) and a logistic reaction term), Papageorgiou - Radulescu [24] (nonlinear parametric Robin problems with \( \xi \equiv 0 \), the parameter \( \lambda > 0 \) multiplying the reaction term and the latter satisfying certain monotonicity properties) and Takeuchi [28], [29] (semilinear superdiffusive logistic equations driven by the Dirichlet Laplacian with zero potential). Finally, we recall also the work of Mugnai - Papageorgiou [22] on logistic equations on \( \mathbb{R}^N \) driven by the Dirichlet \( p-\)Laplacian with zero potential.
Our approach is variational, based on the critical point theory. In Section 4, in order to generate a nodal solution, we also use tools from Morse theory (critical groups).

2. Mathematical Background. Let $X$ be a Banach space and $X^*$ its topological dual. By $(\cdot, \cdot)$ we denote the duality brackets for the pair $(X^*, X)$. Given $\phi \in C^1(X, \mathbb{R})$, we say that $\phi$ satisfies the Cerami condition (the “C-condition” for short), if the following property holds:

Every sequence $\{u_n\}_{n \geq 1} \subset X$ s.t. $\{\phi(u_n)\}_{n \geq 1} \subset \mathbb{R}$ is bounded and

$$
(1 + \|u_n\|)\phi'(u_n) \to 0 \text{ in } X^* \text{ as } n \to +\infty,
$$

admits a strongly convergent subsequence.

This is a compactness-type condition on functional $\phi$ which compensates for the fact that the ambient space $X$ need not be locally compact ($X$ is in general infinite dimensional). The C-condition is a basic tool in probing a deformation theorem for the sublevel sets of $\phi$, from which one can deduce the minimax theory for the critical values of $\phi$. Prominent in that theory is the well-known “mountain pass theorem” due to Ambrosetti - Rabinowitz [3]. Here we state it in a slightly more general form (see, for example, Gasinski - Papageorgiou [11, p. 648]).

**Theorem 2.1.** If $X$ is a Banach space, $\phi \in C^1(X, \mathbb{R})$, $\phi$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > \rho > 0$,

$$
\max \left\{ \phi(u_0), \phi(u_1) \right\} < \inf \left\{ \phi(u) : \|u - u_0\| < \rho \right\} = m_\rho
$$

and

$$
c = \inf_{\gamma \in \Gamma} \max_{0 \leq t \leq 1} \phi(\gamma(t)), \text{ with } \Gamma = \{ \gamma \in C([0,1], W^{1,p}(\Omega)) : \gamma(0) = u_0, \gamma(1) = u_1 \},
$$

then $c \geq m_\rho$ and $c$ is a critical value of $\phi$.

In the analysis of problem $(P_3)$, in addition to the Sobolev space $W^{1,p}(\Omega)$, we will also use the Banach space $C^1(\bar{\Omega})$ and the boundary space $L^r(\partial\Omega)$ with $r \in [1, +\infty]$.

In what follows by $| \cdot |$ we denote the norm on $\mathbb{R}^N$, by $(\cdot, \cdot)_{\mathbb{R}^N}$ the inner product of $\mathbb{R}^N$ and by $\| \cdot \|$ the norm of the Sobolev space $W^{1,p}(\Omega)$ defined by

$$
\|u\| = \left(\|u\|_p^p + \|Du\|_p^p\right)^{\frac{1}{p}} \text{ for all } u \in W^{1,p}(\Omega).
$$

The space $C^1(\bar{\Omega})$ is an ordered Banach space with positive cone

$$
C_+ = \left\{ u \in C^1(\bar{\Omega}) : u(z) \geq 0 \text{ for all } z \in \bar{\Omega} \right\}.
$$

This cone has a nonempty interior given by

$$
\text{int } C_+ = \left\{ u \in C_+ : u(z) > 0 \text{ for all } z \in \bar{\Omega} \right\}.
$$

On $\partial\Omega$ we consider the $(N - 1)$-dimensional Hausdorff (surface) measure $\sigma(\cdot)$. With this measure on $\partial\Omega$, we can define the Lebesgue spaces $L^r(\partial\Omega), 1 \leq r \leq \infty$. From the theory of Sobolev spaces, we know that there exists a unique linear continuous map $\gamma_0 : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, known as the “trace map”, s.t. $\gamma_0(u) = u|_{\partial\Omega}$ for all $u \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$. So, we understand the trace map as representing the “boundary values” of a Sobolev function $u \in W^{1,p}(\Omega)$. The trace map is
compact into $L^r(\partial \Omega)$ for all $r \in \left[1, \frac{Np - p}{N - p}\right]$ when $p < N$ and into $L^r(\partial \Omega)$ for all $r \in [1, +\infty)$ when $p \geq N$. We know that
\[ \text{im} \gamma_0 = W^{s,p}_r(\partial \Omega) \left( \frac{1}{p} + \frac{1}{p'} = 1 \right) \quad \text{and} \quad \ker \gamma_0 = W^{1,p}_0(\Omega). \]

In the rest of the work, for the sake of notational simplicity, we drop the use of the trace map $\gamma_0$. It is understood that all restrictions of Sobolev functions $u \in W^{1,p}(\Omega)$ on $\partial \Omega$ are defined in the sense of traces.

Let $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ be the nonlinear map defined by
\[ \langle A(u), h \rangle = \int_{\Omega} |Du|^p - 2(Du, Dh)_{\mathbb{R}^N} \, dz \quad \text{for all } u, h \in W^{1,p}(\Omega). \] (2)

From Motreanu - Motreanu - Papageorgiou [18] (p. 40), we have:

**Proposition 1.** The nonlinear map $A : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by (2), is continuous, monotone, hence maximal monotone and of type $(S)_+$, that is, if $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $\limsup_{n \to +\infty} \langle A(u_n), u_n - u \rangle \leq 0$, then $u_n \to u$ in $W^{1,p}(\Omega)$.

Let $f_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function s.t.
\[ |f_0(z,x)| \leq a_0(z)(1 + |x|^{-\tau}) \quad \text{for a.a. } z \in \Omega \text{ and all } x \in \mathbb{R}, \]
with $a_0 \in L^\infty(\Omega)_+ := \{ a \in L^\infty(\Omega) : a \geq 0 \text{ in } \Omega \}$ and $r \in (1, p^*)$, where
\[ p^* = \begin{cases} \frac{Np}{N - p}, & \text{if } p < N \\ +\infty, & \text{if } p \leq N. \end{cases} \]

Moreover, let $g_0 \in C_0^\eta(\partial \Omega \times \mathbb{R})$, $\eta \in (0, 1)$, be such that
\[ |g_0(z,x)| \leq c_0 |x|^\tau \quad \text{for all } (z,x) \in \partial \Omega \times \mathbb{R}, \]
with $c_0 > 0$ and $\tau \in (1, p]$. Finally, set $F_0(z,x) = \int_0^x f_0(z,s) \, ds$, $G_0(z,x) = \int_0^x g_0(z,s) \, ds$ and consider the $C^1$-functional defined by
\[ \varphi_0(u) = \frac{1}{p} \|Du\|^p_p + \int_{\partial \Omega} G_0(z,u) \, ds - \int_{\Omega} F_0(z,u) \, dz \quad \text{for all } u \in W^{1,p}(\Omega). \]

From Papageorgiou - Radulescu [23], we have

**Proposition 2.** If $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\Omega)$-minimizer of $\varphi_0$, that is there exists $\rho_0 > 0$ s.t.
\[ \varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } h \in C^1(\Omega) \text{ with } \|h\|_{C^1(\Omega)} \leq \rho_0, \]
then $u_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and $u_0$ is also a local $W^{1,p}(\Omega)$-minimizer of $\varphi_0$, that is there exists $\rho_1 > 0$ s.t.
\[ \varphi_0(u_0) \leq \varphi_0(u_0 + h) \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \|h\| \leq \rho_1. \]

**Remark 1.** In fact the result is still true even when $f_0(z, \cdot)$ has critical growth, namely $r = p^*$, see Papageorgiou - Radulescu [25].

We will also need a strong comparison result which is of independent interest and for this reason is formulated in a more general setting.

So, let $\Theta \in C^1(0, \infty)$ be a function such that
\[ 0 < \hat{c} \leq \frac{\Theta'(t)t}{\Theta(t)} \leq c_0 \quad \text{and} \quad \hat{c}_1 t^{p-1} \leq \Theta(t) \leq \hat{c}_2 (1 + t^{p-1}) \]
for all $t > 0$, with $\hat{c}_1, \hat{c}_2 > 0$.

We consider a map $a : \mathbb{R}^N \to \mathbb{R}^N$ satisfying the following hypotheses:

$H(a)$: $a(y) = a_0(|y|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0, \infty)$, $t \mapsto a_0(t) t$ is strictly increasing, $\lim_{t \to 0^+} a_0(t) t = 0$ and

$$\lim_{t \to 0^+} \frac{a_0(t) t}{a_0(t)} > -1;$$

(ii) $|Da(y)| \leq \hat{c}_3 \frac{\Theta(|y|)}{|y|}$ for some $\hat{c}_3 > 0$, all $y \in \mathbb{R}^N \setminus \{0\}$;

(iii) $(Da(y) \Xi, \Xi)_{\mathbb{R}^N} \geq \Theta(|y|) \xi^2$ for all $y \in \mathbb{R}^N \setminus \{0\}$, all $\Xi \in \mathbb{R}^N$.

**Remark 2.** These hypotheses are motivated by the nonlinear regularity of Lieberman [15] and the nonlinear maximum principle of Pucci-Serrin [27]. If $a(y) = |y|^{p-2}y$ with $1 < p < \infty$, then the hypotheses above are satisfied with $\Theta(t) = t^{p-1}$, and in this case, for every $u \in W^{1,p}(\Omega)$,

$$\text{div} a(Du) = \Delta_p u.$$

Given two functions $h_1, h_2 \in L^\infty(\Omega)$, we write $h_1 \prec h_2$, if for every compact $K \subseteq \Omega$, there exists $\epsilon = \epsilon(K) > 0$ such that

$$h_1(z) + \epsilon \leq h_2(z) \quad \text{for a.a. } z \in K.$$

Evidently if $h_1, h_2 \in C(\Omega)$ and $h_1(z) < h_2(z)$ for all $z \in \Omega$, then $h_1 \prec h_2$.

The next strong comparison principle extends analogous results for the Dirichlet $p-$Laplacian, by Guedda-Veron [13] and Arcaya-Ruiz [4]. In what follows by $\frac{\partial u}{\partial n_a}$ we denote the generalized directional derivative $\frac{\partial u}{\partial n_a} = (a(Du), n)_{\mathbb{R}^N}$ for all $u \in W^{1,p}(\Omega)$. If $a(y) = |y|^{p-2}y$ ($1 < p < \infty$), then we recover the generalized directional derivative from (1).

**Proposition 3.** If Hypotheses $H(a)$ hold, $\Upsilon, h_1, h_2 \in L^\infty(\Omega)$, $\Upsilon \geq 0$, $h_1 \prec h_2$, $u \in C^1(\overline{\Omega}) \setminus \{0\}$, $v \in \text{int} C_+$, $u \leq v$ and they satisfy

- $\text{div} a(Du(z)) + \Upsilon(z)|u(z)|^{p-2}u(z) = h_1(z)$ in $\Omega$, 
- $\text{div} a(Dv(z)) + \Upsilon(z)|v(z)|^{p-1} = h_2(z)$ in $\Omega$

with $\frac{\partial u}{\partial n_a}|_{\partial \Omega} < 0$ or $\frac{\partial v}{\partial n_a}|_{\partial \Omega} < 0$, then $(v-u)(z) > 0$ for all $z \in \Omega$ and $\frac{\partial (v-u)}{\partial n}|_{\partial \Omega} < 0$, where $\partial \Omega = \{z \in \partial \Omega : u(z) = v(z)\}$.

**Proof.** To fix things we assume that $\frac{\partial v}{\partial n_a}|_{\partial \Omega} < 0$.

Let $\mathcal{G} := \{z \in \Omega : u(z) = v(z)\}$ and $\mathcal{E} = \{z \in \Omega : Du(z) = Dv(z) = 0\}$.

Claim: $\mathcal{G} \subseteq \mathcal{E}$.

Let $z_0 \in \mathcal{G}$ and set $y = v - u$. Then $y$ attains its infimum at $z_0$ and so $Dv_0(z) = Du_0(z)$.

If $Du(z_0) \neq 0$, then we can find $\rho > 0$ small such that $\overline{B}_\rho(z_0) \subseteq \Omega$ and $|Du(z)| > 0, |Dv(z)| > 0, (Du(z), Dv(z))_{\mathbb{R}^N} > 0$.


for all \( z \in \overline{B}_\rho(z_0) \). By hypothesis, we have
\[
- \operatorname{div}(a(Dv) - a(Du)) = h_2(z) - h_1(z) - \Upsilon(z)(v^{p-1} - u^{p-1}) \text{ in } \Omega.
\] (3)

If \( a = (a_k)_{k=1}^N \), then using the mean value theorem, we have
\[
a_k(\Xi) - a_k(\Xi') = \sum_{i=1}^N \int_0^1 \frac{\partial a_k}{\partial y_i}(\Xi' + t(\Xi - \Xi'))(\Xi_i - \Xi_i')dt
\]
for all \( \Xi = (\Xi_i)_{i=1}^N, \Xi' = (\Xi_i')_{i=1}^N \) in \( \mathbb{R}^N \), \( k \in \{1, ..., N\} \). On \( B_\rho(z_0) \) we define the continuous coefficients
\[
\theta_{k,i}(z) = \int_0^1 \frac{\partial a_k}{\partial y_i}(Du(z) + t(Dv(z) - Du(z)))dt.
\]
Then, we introduce the following linear differential operator:
\[
L(w) = -\operatorname{div}\left( \sum_{i=1}^N \theta_{k,i}(z) \frac{\partial w}{\partial z_i} \right) \text{ for all } w \in W^{1,p}(B_\rho(z_0)).
\]
From (3) we have
\[
L(y) = h_2(z) - h_1(z) - \Upsilon(z)(v^{p-1} - u^{p-1}) \text{ in } B_\rho(z_0).
\] (4)
Recall that \( \overline{B}_\rho(z_0) \subseteq \Omega \), \( h_1 < h_2 \) and \( \Upsilon \in L^\infty(\Omega) \). So, from (4) and choosing \( \rho \in (0,1) \) even smaller if necessary, we can have that \( L(\cdot) \) is uniformly elliptic (see also [4] and [13]). Using the strong maximum principle (see, for example, Gasinski-Papageorgiou [11, p. 738]), we have
\[
y(z) = (v - u)(z) > 0 \text{ for all } z \in B_\rho(z_0).
\]
But \( u(z_0) = v(z_0) \) since \( z_0 \in \mathcal{G} \), and thus a contradiction arises. This proves the Claim.

Since \( v \in \text{int}C_+ \) satisfies \( \frac{\partial v}{\partial n} \big|_{\partial \Omega} < 0 \), we see that \( \mathcal{G} \subseteq \Omega \) is compact. So, we can find a smooth open set \( \Omega_1 \subseteq \Omega \) such that
\[
\mathcal{G} \subseteq \Omega_1 \subseteq \overline{\Omega}_1 \subseteq \Omega.
\]
Then we can find \( \epsilon > 0 \) such that
\[
u(z) + \epsilon \leq v(z) \text{ for all } z \in \partial \Omega_1,
\] (5)
\[
h_1(z) + \epsilon \leq h_2(z) \text{ for a.a. } z \in \Omega_1.
\] (6)
Let \( \delta \in (0, \min\{\epsilon, 1\}) \) be such that
\[
|\Upsilon(z)||s|^{p-2}s - |t|^{p-2}t| \leq \epsilon
\] (7)
for a.a. \( z \in \Omega \), all \( s, t \in [-\|u\|_{\infty}, \|v\|_{\infty}] \) with \( |s - t| \leq \delta \) (recall that \( \Upsilon \in L^\infty(\Omega) \)).

Then we have
\[
- \operatorname{div} a(D(u + \delta)) + \Upsilon(z)|u + \delta|^{p-2}(u + \delta)
= - \operatorname{div} a(Du) + \Upsilon(z)|u + \delta|^{p-2}(u + \delta)
\leq - \operatorname{div} a(Du) + \Upsilon(z)|u|^{p-2}u + \epsilon \quad \text{(see(7))}
= h_1(z) + \epsilon \leq h_2(z) \quad \text{(see(6))}
= - \operatorname{div} a(Dv) + \Upsilon(z)v^{p-1} \text{ for a.a. } z \in \Omega_1.
\] (8)

From (8) and the weak comparison principle (see (5)) it follows that
\[
u(z) + \delta < v(z) \text{ for all } z \in \Omega_1.
\]
But \( G \subseteq \Omega \). Hence \( G = \emptyset \) and so
\[
(v - u)(z) > 0 \quad \text{for all} \; z \in \Omega.
\]
Moreover, from Hopf’s Lemma we have
\[
\frac{\partial (v - u)}{\partial n} \big|_{\partial \Omega} < 0.
\]

\[ \Box \]

**Remark 3.** Consider the following order cone in \( C^1_+(\Omega) := \{ u \in C^1(\Omega) : u|_{\partial \Omega} = 0 \} \):
\[
\hat{C}_+ := \left\{ y \in C^1_+(\Omega) : y(z) \geq 0 \quad \text{for all} \; z \in \Omega, \; \frac{\partial y}{\partial n}|_{\partial \Omega} \leq 0 \right\}.
\]
This cone has a non empty interior given by
\[
\text{int}\hat{C}_+ = \left\{ y \in C^1_+(\Omega) : y(z) > 0 \quad \text{for all} \; z \in \Omega, \; \frac{\partial y}{\partial n}|_{\partial \Omega} < 0 \right\}.
\]
According to Proposition 3, we have \( v - u \in \text{int}\hat{C}_+ \). Of course if \( \partial \Omega = \emptyset \) then \( \hat{C}_+ = C_+ \). Clearly, Proposition 3 is also valid for \( C^1_+(\Omega) \) and \( W^{1,p} := \overline{C^1_+(\Omega)}^{\| \cdot \|} \).

Let \( \xi \in L^\infty(\Omega) \) and \( \beta \in C^{0,\eta}(\partial \Omega) \) with \( \eta \in (0,1) \), \( \beta(z) \geq 0 \) for all \( z \in \partial \Omega \), and consider the following nonlinear eigenvalue problem:
\[
\begin{cases}
-\Delta u(z) + \xi(z)u(z)^{p-2}u(z) = \lambda u(z)^{p-2}u(z), & \text{in } \Omega, \\
\frac{\partial u}{\partial n} + \beta(z)u|^{p-2}u = 0, & \text{on } \partial \Omega.
\end{cases}
\tag{9}
\]

As in Mugnai - Papageorgiou [20] (there \( \beta \equiv 0 \), Neumann problem), we can show that there exists a smallest eigenvalue \( \hat{\lambda}_1(\beta) \) having the following properties:
\[
\hat{\lambda}_1(\beta) \text{ is isolated in the spectrum of } (9), \quad \hat{\lambda}_1(\beta) \text{ is simple}, \quad \hat{\lambda}_1(\beta) = \inf \left\{ \frac{\vartheta(u) + \int_{\partial \Omega} \beta(z)|u|^p d\sigma}{\| u \|^p_p} : u \in W^{1,p}(\Omega), \; u \neq 0 \right\}, \tag{10}
\]
where
\[
\vartheta(u) = \| Du \|^p_p + \int_{\Omega} \xi(x)|u|^p dx \text{ for all } u \in W^{1,p}(\Omega).
\]

The infimum in (10) is realized on the corresponding one dimensional eigenspace. From (10) it is clear that the elements of this eigenspace do not change sign. In what follows, by \( \hat{u}_1(\beta) \) we denote the positive \( L^p \) - normalized eigenfunction (that is \( \| \hat{u}_1(\beta) \|_p = 1 \)) corresponding to \( \hat{\lambda}_1(\beta) \). From the nonlinear regularity theory of Lieberman [15] and the nonlinear maximum principle (see Pucci - Serrin [27, p. 120]), we have that \( \hat{u}_1(\beta) \in \text{int} C_+ \).

As we already mentioned in the Introduction, in Section 4 in order to produce a nodal solution, we will use tools from Morse theory (critical groups). So, let us recall the definition of critical groups at an isolated critical point.

Let \( X \) be a Banach space, \( \varphi \in C^1(X, \mathbb{R}) \) and \( c \in \mathbb{R} \). We introduce the following sets:
\[
\varphi^c := \{ u \in X : \varphi(u) \leq c \}, \quad K_\varphi = \{ u \in X : \varphi'(u) = 0 \}
\]
and
\[
K^c_\varphi = \{ u \in K_\varphi : \varphi(u) = c \}.
\]
For every topological pair \((Y_1, Y_2)\) with \(Y_1 \subseteq Y_2 \subseteq X\) and every \(k \in \mathbb{N}_0\), by \(H_k(Y_1, Y_2)\) we denote the \(k\)th relative singular homology group with integer coefficients. If \(u \in K^c_u\) is isolated, then the critical groups of \(\varphi\) at \(u\) are defined by
\[
C_k(\varphi, u) = H_k(\varphi^c \cap U, \varphi^c \cap U \setminus \{u\}) \quad \text{for all } k \in \mathbb{N}_0,
\]
where \(U\) is a neighborhood of \(u\) s.t. \(K^c_u \cap \varphi^c \cap U = \{u\}\). The excision property of singular homology theory implies that this definition of critical groups is independent of the particular choice of the neighborhood \(U\) as above.

Finally, we fix our notation. By \(|\cdot|_N\) we denote the Lebesgue measure on \(\mathbb{R}^N\). If \(x \in \mathbb{R}\), then we set \(x^\pm = \max\{\pm x, 0\}\). For \(u \in W^{1, p}(\Omega)\) we define \(u^\pm(\cdot) = u(\cdot)^\pm\). We know that \(u^\pm \in W^{1, p}(\Omega)\), \(u = u^+ - u^-\), \(|u| = u^+ + u^-\). Also, if \(g : \Omega \times \mathbb{R} \to \mathbb{R}\) is measurable, then we define the Nemitzskii operator \(N_g(u)(\cdot) = g(\cdot, u(\cdot))\) for all \(u \in W^{1, p}(\Omega)\).

3. Positive solutions. In order to look for positive solutions, our hypotheses on the data of problem \((P_\lambda)\) are the following:

- \(H(\xi) : \xi \in L^\infty(\Omega)\),
- \(H(\beta) : (a) \, \beta \in C^{0, \alpha}(\partial \Omega)\) with \(\alpha \in (0, 1)\) and \(\beta(z) > 0\) for all \(z \in \partial \Omega\), or
- (b) \(\beta \equiv 0\) (Neumann problem);
- \(H(f) : f : \Omega \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory function such that \(f(z, 0) = 0\) for a.a. \(z \in \Omega\) and

(i) \(|f(z, x)| \leq a(z)(1 + x^{r-1})\) for a.a. \(z \in \Omega\) and all \(x \geq 0\), where \(a \in L^\infty(\Omega)_+\) and
\[
p < r < p^* = \begin{cases} \frac{Np}{N-p}, & \text{if } p < N, \\ +\infty, & \text{if } N \leq p; \end{cases}
\]
(ii) if \(F(z, x) = \int_0^x f(z, s)ds\), then \(\lim_{x \to +\infty} \frac{F(z, x)}{x^{p-1}} = +\infty\) uniformly for a.a. \(z \in \Omega\);
(iii) there exist \(\gamma_0 > 0\) and \(\mu \in \left(\max\left\{(r - p)\frac{N}{p} - 1, 1\right\}, p^*\right)\) such that
\[
0 < \gamma_0 \leq \liminf_{x \to +\infty} \frac{f(z, x)x - pF(z, x)}{x^\mu} \quad \text{uniformly for a.a. } z \in \Omega;
\]
(iv) there exists \(\delta > 0\) and \(q \in (1, p)\) such that
\[
c_0x^{q-1} \leq f(z, x) \quad \text{for a.a. } z \in \Omega \text{ and all } x \in [0, \delta].
\]

**Remark 4.** The alternative in \(H(\beta)\) means that we exclude mixed problems.

**Remark 5.** Since we are looking for positive solutions and the above hypotheses concern the positive semiaxis \(\mathbb{R}_+ = [0, +\infty)\), without any loss of generality, we may assume that \(f(z, x) = 0\) for a.a. \(z \in \Omega\) and all \(x \leq 0\).

Hypotheses \(H(f)(ii), (iii)\) imply that
\[
\lim_{x \to +\infty} \frac{f(z, x)}{x^{p-1}} = +\infty \quad \text{uniformly for a.a. } z \in \Omega,
\]
that is \(f(z, \cdot)\) is \((p - 1)\)–superlinear. We point out that we do not employ the Ambrosetti - Rabinowitz condition, which says that there exist \(\tau > p\) and \(M > 0\) such that
\[
0 < \tau F(z, x) \leq f(z, x)x \quad \text{for a.a. } z \in \Omega, \quad \text{all } x \geq M,
\]
\[
\text{essinf}_\Omega F(\cdot, M) > 0 \quad \text{(see Ambrosetti - Rabinowitz [3] and Mugnai [19]).}
\]
By direct integration of the Ambrosetti - Rabinowitz condition, we have
\[ c_0 x^r \leq F(z, x) \] for a.a. \( z \in \Omega \), all \( x \geq M \) and some \( c_0 > 0 \).
Hence, \( F(z, \cdot) \) has at least \( \tau - \) polynomial growth near \( +\infty \).

**Example.** The following functions satisfy hypotheses \( H(f) \) (for the sake of simplicity, we drop the \( z - \) dependence):
\[
\begin{align*}
  f_1(x) &= x^{r-1} + x^{q-1} \text{ for all } x \geq 0 \text{ with } 1 < q < p < r < p^*, \\
  f_2(x) &= x^{p-1} \left( \log x + \frac{1}{p} \right) \text{ for all } x \geq 0.
\end{align*}
\]
Note that \( f_2 \) does not satisfy the Ambrosetti - Rabinowitz condition.

Let \( \mathcal{L} = \{ \lambda > 0 : \text{problem } (P_\lambda) \text{ has a positive solution} \} \) (this is the set of admissible parameters) and \( S(\lambda) \) the set of positive solutions of problem \( (P_\lambda) \) (for \( \lambda \notin \mathcal{L} \), we have \( S(\lambda) = \emptyset \)).

**Proposition 4.** If Hypotheses \( H(\xi), H(\beta), H(f) \) hold, then \( S(\lambda) \subseteq \text{int } C_+ \) for every \( \lambda > 0 \).

**Proof.** We assume that \( \lambda \in \mathcal{L} \) (otherwise \( S(\lambda) = \emptyset \)). Let \( u \in S(\lambda) \). Then
\[
\langle A(u), h \rangle + \int_\Omega (\xi(z) + \lambda)u^{p-1}hdz + \int_{\partial\Omega} \beta(z)u^{p-1}hds = \int_\Omega f(z, u)hdz
\]
for all \( h \in W^{1,p}(\Omega) \), that is
\[
\begin{cases}
-\Delta_p u(z) + (\xi(z) + \lambda)u(z)^{p-1} = f(z, u(z)) \quad \text{for a.a. } z \in \Omega, \\
\frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0 \quad \text{on } \partial\Omega,
\end{cases}
\]
see Papageorgiou - Radulescu [23].

From Winkert [30] and Papageorgiou - Radulescu [25], we have \( u \in L^\infty(\Omega) \). So, we can apply Theorem 2 of Lieberman [15] and have that \( u \in C_+ \setminus \{0\} \).

Let \( \rho = \|u\|_\infty \). Hypotheses \( H(f)(i), (iv) \) imply that we can find \( \xi_\rho > 0 \) such that
\[
f(z, x) + \xi_\rho x^{p-1} \geq 0 \quad \text{for a.a. } z \in \Omega \quad \text{and } x \in [0, \rho].
\]
From (11) and (12), we have
\[
\Delta_p u(z) \leq (\|\xi\|_\infty + \lambda + \xi_\rho)u(z)^{p-1} \quad \text{for a.a. } z \in \Omega \quad \text{(see Hypothesis } H(\xi)),
\]
which implies that \( u \in \text{int } C_+ \) by the nonlinear maximum principle (see Pucci - Serrin [27, p. 120]).

Therefore \( S(\lambda) \subseteq \text{int } C_+ \) for all \( \lambda > 0 \). \( \square \)

**Proposition 5.** If Hypotheses \( H(\xi), H(\beta), H(f) \) hold, then \( \mathcal{L} \neq \emptyset \) and \( \lambda \in \mathcal{L} \) implies that \( [\lambda, +\infty) \subseteq \mathcal{L} \).

**Proof.** Let \( \mu > \|\xi\|_\infty \) (see Hypothesis \( H(\xi) \)), and consider the following nonlinear Robin problem:
\[
\begin{cases}
-\Delta_p u(z) + (\xi(z) + \mu)u(z)^{p-1} = 1, \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n_p} + \beta(z)u^{p-1} = 0, \quad \text{on } \partial\Omega, u > 0.
\end{cases}
\]
Let \( V : L^p(\Omega) \to L^{p'}(\Omega) \left( \frac{1}{p} + \frac{1}{p'} = 1 \right) \) be the nonlinear map defined by
\[
V_p(u)(z) = (\xi(z) + \mu)|u(z)|^{p-2}u(z).
\]
Evidently $V_p(\cdot)$ is continuous, strictly monotone (recall $\mu > \|\xi\|_\infty$), hence maximal monotone, too. Moreover, let $B_p : W^{1,p}(\Omega) \to L^p(\partial \Omega)$ be the boundary map defined by

$$B_p(u)(z) = \beta(z)|u(z)|^{p-2}u(z) \quad \text{for all } z \in \partial \Omega.$$ 

Also this map is continuous, monotone (see Hypothesis $H(\beta)$), hence maximal monotone. If $\gamma_0 \in \mathcal{L}(W^{1,p}(\Omega), L^p(\partial \Omega))$ is the trace map, then $\gamma_0^* \in \mathcal{L}(L^p(\partial \Omega), W^{1,p}(\Omega)^*)$. Finally, we introduce the map $K : W^{1,p}(\Omega) \to W^{1,p}(\Omega)^*$ defined by

$$K(u) = A(u) + V_p(u) + (\gamma_0^* \circ B_p)(u).$$

Then $K$ is continuous, maximal monotone (see Gasinski - Papageorgiou [11, p. 326]), and we have

$$\langle K(u), u \rangle = \|Du\|_p^p + \int_\Omega (\xi(z) + \mu)|u|^p dz + \int_{\partial \Omega} \beta(z)|u|^p d\sigma \geq \|Du\|_p^p + (\mu - \|\xi\|_\infty)|u|_p^p \quad \text{(see Hypothesis } H(\beta)),$$

$$\Rightarrow K(\cdot) \text{ is coercive (recall that } \mu > \|\xi\|_\infty).$$

But a maximal monotone, coercive map is surjective (see Gasinski - Papageorgiou [11, p. 319]). So, we can find $\bar{u} \in W^{1,p}(\Omega), \bar{u} \neq 0$ such that

$$A(\bar{u}) + V_p(\bar{u}) + (\gamma_0^* \circ B_p)(\bar{u}) = 1. \quad (14)$$

On (14) we act with $-\bar{u}^- \in W^{1,p}(\Omega)$. Then

$$\|D\bar{u}^-\|_p^p + (\mu - \|\xi\|_\infty)\|\bar{u}^-\|_p^p \leq 0 \quad \text{(see Hypothesis } H(\beta)),$$

$$\Rightarrow \bar{u} \geq 0, \bar{u} \neq 0.$$

From (14) we have

$$\int_\Omega |D\bar{u}|^{p-2}(D\bar{u}, Dh)_{\mathbb{R}^n} dz + \int_\Omega (\xi(z) + \mu)\bar{u}^{p-1} hdz + \int_{\partial \Omega} \beta(z)\bar{u}^{p-1} h d\sigma = \int_\Omega hdz$$

for all $h \in W^{1,p}(\Omega)$, that is

$$\begin{cases} -\Delta_p \bar{u}(z) + (\xi(z) + \mu)\bar{u}(z)^{p-1} = 1 & \text{for a.a. } z \in \Omega, \\ \frac{\partial \bar{u}}{\partial n_p} + \beta(z)\bar{u}^{p-1} = 0 & \text{on } \partial \Omega \end{cases} \quad (15)$$

(see Papageorgiou - Radulescu [23]). As before (see the proof of Proposition 4) using the nonlinear regularity theory of Lieberman [15, Theorem 2], we have $\bar{u} \in C_+ \setminus \{0\}$. Then from (15), we have

$$\Delta_p \bar{u}(z) \leq (\|\xi\|_\infty + \mu)\bar{u}(z)^{p-1} \quad \text{for a.a. } z \in \Omega,$$

$$\Rightarrow \bar{u} \in \text{int } C_+ \quad \text{(see Pucci - Serrin [27, p.120])}. \quad (16)$$
Let \( \bar{m} = \min_{\Omega} \bar{u} > 0 \) (see (16)) and let \( \lambda_0 = \mu + \frac{\|\mathcal{N}_f(\bar{u})\|_{\infty}}{\bar{m}^{p-1}} \) (see Hypothesis \( H(f)(\bar{u}) \)). For all \( h \in W^{1,p}(\Omega) \) with \( h \geq 0 \), we have

\[
\begin{align*}
\int_{\Omega} |D\bar{u}|^{p-2}(D\bar{u}, Dh)_{\mathbb{R}^N} dz + \int_{\Omega} (\xi(z) + \lambda_0)\bar{u}^{p-1}hdz + \int_{\partial\Omega} \beta(z)\bar{u}^{p-1}hds \\
= \int_{\Omega} |D\bar{u}|^{p-2}(D\bar{u}, Dh)_{\mathbb{R}^N} dz + \int_{\Omega} (\xi(z) + \mu + \frac{\|\mathcal{N}_f(\bar{u})\|_{\infty}}{\bar{m}^{p-1}})\bar{u}^{p-1}hdz \\
+ \int_{\partial\Omega} \beta(z)\bar{u}^{p-1}hds \\
\geq \int_{\Omega} |D\bar{u}|^{p-2}(D\bar{u}, Dh)_{\mathbb{R}^N} dz + \int_{\Omega} (\xi(z) + \mu)\bar{u}^{p-1}hdz + \int_{\Omega} f(z, \bar{u})hdz \\
+ \int_{\partial\Omega} \beta(z)\bar{u}^{p-1}hds \quad \text{(recall that } h \geq 0) \\
\geq \int_{\Omega} (1 + f(z, \bar{u}))hdz \quad \text{(see Hypothesis } H(\beta)\text{and recall } h \geq 0) \\
\geq \int_{\Omega} f(z, \bar{u})hdz.
\end{align*}
\]

We consider the following truncations of the reaction term \( f(z, \cdot) \) and of the boundary term \( x \mapsto \beta(z)|x|^{p-2}x \):

\[
\hat{f}(z, x) = \begin{cases} 
  f(z, x) & \text{if } x \leq \bar{u}(z) \\
  f(z, \bar{u}(z)) & \text{if } \bar{u}(z) < x
\end{cases}
\]

(18) for all \( (z, x) \in \Omega \times \mathbb{R} \),

\[
\begin{align*}
\hat{\beta}(z, x) = \begin{cases} 
  0 & \text{if } x < 0 \\
  \beta(z)x^{p-1} & \text{if } 0 \leq x \leq \bar{u}(z) \\
  \beta(z)\bar{u}(z)^{p-1} & \text{if } \bar{u}(z) < x
\end{cases}
\end{align*}
\]

(19) for all \( (z, x) \in \partial\Omega \times \mathbb{R} \), and both are Carathéodory functions.

We set \( \hat{F}(z, x) = \int_{0}^{x} f(z, s)ds \), \( B(z, s) = \int_{0}^{s} \hat{\beta}(z, s)ds \) and consider the \( C^1 \)-functional \( \hat{\varphi} : W^{1,p}(\Omega) \to \mathbb{R} \) defined by

\[
\hat{\varphi}(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{p} \int_{\Omega} (\xi(z) + \lambda_0)\|u\|^p dz + \int_{\partial\Omega} \hat{B}(z, u)ds - \int_{\Omega} \hat{F}(z, u)dz
\]

for all \( u \in W^{1,p}(\Omega) \).

Since \( \lambda_0 > \mu > \|\xi\|_{\infty} \), from (18) and (19) it is clear that \( \hat{\varphi} \) is coercive. Also, using the Sobolev embedding theorem and the trace theorem, we see that \( \hat{\varphi} \) is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find \( u_0 \in W^{1,p}(\Omega) \) such that

\[
\hat{\varphi}(u_0) = \inf \left\{ \hat{\varphi}(u) : u \in W^{1,p}(\Omega) \right\}.
\]

(20)

Let \( t \in (0, 1) \) be small such that

\[
t \bar{u}(\beta) \leq \bar{u} \quad \text{and} \quad t \hat{u}_1(z) \in [0, \delta] \text{ for all } z \in \Omega \]
Then from (18), (19) and (23), equation (22) becomes

\[ \varphi(t\hat{u}_1) = \frac{t^p}{p} \|D\hat{u}_1(\beta)\|_p^p + \frac{t^p}{p} \int_\Omega (\xi(z) + \lambda_0)|\hat{u}_1(\beta)|^p dz + \frac{t^p}{p} \int_{\partial\Omega} \beta(z)|\hat{u}_1(\beta)|^p d\sigma \]

- \int_{\Omega} F(z, \hat{u}_1(\beta)) dz

(see (18) and (19))

\[ \leq \frac{t^p}{p} [\hat{\lambda}_1(\beta) + \lambda_0] - \frac{t^q}{q} c_0 \|\hat{u}_1(\beta)\|^q_p \]

(see Hypothesis H(f)(iv) and recall that \( \|\hat{u}_1(\beta)\|_p = 1 \)).

Since \( q < p \), see Hypothesis H(f)(iv), by choosing \( t \in (0,1) \) even smaller if necessary, from (21) we see that

\[ \varphi(t\hat{u}_1) < 0 = \varphi(0), \]

\[ \Rightarrow \varphi(u_0) < 0 = \varphi(0) \] (see (20)), hence \( u_0 \neq 0 \).

From (20) we have

\[ \varphi'(u_0) = 0 \]

\[ \Rightarrow \langle A(u_0), h \rangle + \int_\Omega (\xi(z) + \lambda_0)|u_0|^{p-2}u_0 h dz + \int_{\partial\Omega} \hat{\beta}(z, u_0) h d\sigma = \int_\Omega \hat{f}(z, u_0) h dz \]

for all \( h \in W^{1,p}(\Omega) \). In (22) we choose \( h = -u_0^- \in W^{1,p}(\Omega) \). Using (18) and (19), we obtain

\[ \|Du_0^-\|_p^p + (\lambda_0 - \|\xi\|_\infty)\|u_0^-\|_p^p \leq 0, \]

\[ \Rightarrow u_0 \geq 0, u_0 \neq 0. \]

Next in (22) we choose \( h = (u_0 - \bar{u})^+ \in W^{1,p}(\Omega) \). Then

\[ \langle A(u_0), (u_0 - \bar{u})^+ \rangle + \int_\Omega (\xi(z) + \lambda_0)|u_0^-|^{p-1}(u_0 - \bar{u})^+ dz + \int_{\partial\Omega} \beta(z)|u_0^-|^{p-1}(u_0 - \bar{u})^+ d\sigma \]

\[ = \int_\Omega f(z, \bar{u})(u_0 - \bar{u})^+ dz \] (see (18) and (19))

\[ \leq \langle A(\bar{u}), (u_0 - \bar{u})^+ \rangle + \int_\Omega (\xi(z) + \lambda_0)|\bar{u}|^{p-1}(u_0 - \bar{u})^+ dz \]

\[ + \int_{\partial\Omega} \beta(z)|\bar{u}|^{p-1}(u_0 - \bar{u})^+ d\sigma \] (see (17)),

so that

\[ \langle A(u_0) - A(\bar{u}), (u_0 - \bar{u})^+ \rangle + \int_\Omega (\lambda_0 - \|\xi\|_\infty)(u_0^{p-1} - \bar{u}^{p-1})(u_0 - \bar{u})^+ dz \leq 0, \]

which implies that \( |\{u_0 > \bar{u}\}|_\Omega = 0 \), and hence \( u_0 \leq \bar{u} \) (recall that \( \lambda_0 > \|\xi\|_\infty \)). So, we have proved that

\[ u_0 \in [0, \bar{u}] = \left\{ u \in W^{1,p}(\Omega) : 0 \leq u(z) \leq \bar{u}(z) \text{ for a.a. } z \in \Omega \right\}, u_0 \neq 0. \] (23)

Then from (18), (19) and (23), equation (22) becomes

\[ \langle A(u_0), h \rangle + \int_\Omega (\xi(z) + \lambda_0)u_0^{p-1} h dz + \int_{\partial\Omega} \beta(z)u_0^{p-1} h d\sigma = \int_\Omega f(z, u_0) h dz \]
for all \( h \in W^{1,p}(\Omega) \). Hence

\[
\begin{cases}
-\Delta_p u_\lambda(z) + (\xi(z) + \lambda)u_\lambda(z)^{p-1} = f(z, u_\lambda(z)) & \text{for a.a. } z \in \Omega, \\
\frac{\partial u_\lambda}{\partial n} + \beta(z)u_\lambda^{p-1} = 0 & \text{on } \partial \Omega
\end{cases}
\]

and \( u_\lambda \in S(\lambda) \subseteq \text{int } C_+ \) (see Proposition 4). Hence \( \lambda_0 \in \mathcal{L} \neq \emptyset \).

Next, let \( \lambda \in \mathcal{L} \) and pick \( \theta > \lambda \). Let \( u_\lambda \in S(\lambda) \subseteq \text{int } C_+ \) (see Proposition 4). We have

\[
-\Delta_p u_\lambda(z) + (\xi(z) + \theta)u_\lambda(z)^{p-1} \\
\geq -\Delta_p u_\lambda(z) + (\xi(z) + \lambda)u_\lambda(z)^{p-1} = f(z, u_\lambda(z)) \text{ for a.a. } z \in \Omega.
\]

(24)

In this case, we truncate the reaction term and the boundary term at \( u_\lambda(z) \) (instead of truncating at \( \hat{u}(z) \), as in (18), (19)). Repeating the argument above with \( \lambda_0 \) replaced by \( \theta \) using this time (24) instead of (17), via the direct method of the calculus of variations, we obtain

\[
\begin{align*}
\text{for a.a. } z \in \Omega, \text{ Hypotheses } H(f)_1 & \text{ hold, and } f(z, \cdot) \text{ differentiable and } f(z, x) = c_0 x^{p-1} + f_0(z, x) \text{ with } f_0(z, \cdot) \text{ differentiable and } f_0(z, x) \geq -\hat{c} x^{d-2} \text{ for } z \in \Omega, \text{ all } x \geq 0, \text{ with } d > q. \text{ For this setting, } \\
\text{Hypothesis } H(f)_1(v) \text{ is satisfied.}
\end{align*}
\]

Remark 6. In fact a careful reading of the above proof, reveals that the solutions of \((P\lambda)\) exhibit the following “monotonicity” property. Given \( \lambda \in \mathcal{L} \) and \( \theta > \lambda \), then \( \theta \in \mathcal{L} \) and there exists \( u_\theta \in S(\theta) \subseteq \text{int } C_+ \) such that \( u_\theta \leq u_\lambda \). This property can be improved provided we strengthen the Hypotheses on \( f(z, \cdot) \).

\[
H(f)_1: f : \Omega \times \mathbb{R} \to \mathbb{R} \text{ is a Carathéodory function such that } f(z, 0) = 0 \text{ for a.a. } z \in \Omega, \text{ Hypotheses } H(f)_1(i), (ii), (iii), (iv) \text{ are the same as the corresponding Hypotheses } \text{H(f)}_1(i), (ii), (iii), (iv) \text{ and }
\]

\[
(v) \text{ for every } \rho > 0, \text{ there exists } \hat{c}_\rho > 0 \text{ such that for a.a. } z \in \Omega,
\]

\[
x \mapsto f(z, x) + \hat{c}_\rho x^{p-1} \text{ is nondecreasing on } [0, \rho].
\]

Remark 7. Clearly this is true if for a.a. \( z \in \Omega, \text{ } f(z, \cdot) \text{ is nondecreasing. More generally, suppose that } f(z, x) = c_0 x^{d-1} + f_0(z, x) \text{ with } f_0(z, \cdot) \text{ differentiable and } f_0(c_0^d, x) \geq -\hat{c} x^{d-2} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0, \text{ with } d > q. \text{ For this setting, } \text{Hypothesis } H(f)_1(v) \text{ is satisfied.}
\]

With this stronger condition on \( f(z, \cdot) \), we can prove the following “strong monotonicity” property.

Proposition 6. If Hypotheses \( H(\xi), H(\beta), H(f)_1 \) hold, \( \lambda \in \mathcal{L} \), \( u_\lambda \in S(\lambda) \subseteq \text{int } C_+ \) and \( \theta > \lambda \), then we can find \( u_\theta \in S(\theta) \subseteq \text{int } C_+ \) such that \( u_\lambda - u_\theta \in \text{int } C_+ \) with \( \mathcal{G}_0 := \{ z \in \partial \Omega : u_\lambda(z) = u_\theta(z) \} \), if \( H(\beta)(a) \) holds, and \( u_\lambda - u_\theta \in \text{int } C_+ \), if \( H(\beta)(b) \) is in force.

Proof. As we already mentioned in the previous Remark, we have:

\[
\theta \in \mathcal{L} \text{ and we can find } u_\theta \in S(\theta) \subseteq \text{int } C_+ \text{ st. } u_\theta \leq u_\lambda.
\]

For \( \delta > 0 \), we set \( u_\delta := u_\theta + \delta \in \text{int } C_+ \). Let \( \rho = ||u_\lambda||_{\infty} \) and let \( \hat{c}_\rho > 0 \) be as postulated by Hypothesis \( H(f)_1(v) \). We can always take \( \hat{c}_\rho > 0 \) such that
\[ \hat{\xi}_p + \lambda > \|\xi\|_{\infty}. \] Being \( u_\theta \) bounded, we have
\[ \begin{aligned}
    &- \Delta_\rho u_\theta^\rho + (\xi(z) + \lambda + \hat{\xi}_p)(u_\theta^\rho)^{p-1} \\
    &\leq -\Delta_\rho u_\theta + (\xi(z) + \theta)u_\theta^{p-1} - (\theta - \lambda)u_\theta^{p-1} + \hat{\xi}_p u_\theta^{p-1} + \chi(\delta),
\end{aligned} \]
with \( \chi(\delta) \rightarrow 0^+ \) as \( \delta \rightarrow 0^+ \),
\[ \begin{aligned}
    &\leq -\Delta_\rho u_\lambda + (\xi(z) + \lambda + \hat{\xi}_p)u_\lambda^{p-1} + (\chi(\delta) - (\theta - \lambda)m_\theta^{p-1})
\end{aligned} \]
with \( m_\theta = \min_{\overline{\Omega}} u_\theta > 0 \) (recall \( u_\theta \in \text{int} C_+ \))
\[ = f(z, u_\theta) + \hat{\xi}_p u_\theta^{p-1} + (\chi(\theta) - (\theta - \lambda)m_\theta^{p-1}) \]
(see Hypothesis \( H(f)_1(v) \) and recall \( u_\theta \leq u_\lambda \))
\[ = -\Delta_\rho u_\lambda + (\xi(z) + \lambda + \hat{\xi}_p)u_\lambda^{p-1} + (\chi(\delta) - (\theta - \lambda)m_\theta^{p-1}) \]
\[ < -\Delta_\rho u_\lambda + (\xi(z) + \lambda + \hat{\xi}_p)u_\lambda^{p-1} \text{ for } \delta > 0 \text{ small.} \]

If \( \beta(z) > 0 \) for all \( z \in \partial \Omega \), then from (25) and Proposition 3, we infer that
\[ u_\lambda - u_\theta \in \text{int} \hat{C}_+ \quad \text{with } \mathcal{G}_0 := \{ z \in \partial \Omega : u_\lambda(z) = u_\theta(z) \}. \]

If \( \beta \equiv 0 \) (Neumann problem), multiplying (25) by \( (u_\theta^\delta - u_\lambda)^+ \in W^{1,p}(\Omega) \), integrating over \( \Omega \) and using the nonlinear Green’s identity, we obtain for \( \delta > 0 \) small enough,
\[ \langle A(u_\theta^\delta) - A(u_\lambda), (u_\theta^\delta - u_\lambda)^+ \rangle \]
\[ + (\hat{\xi}_p + \lambda - \|\xi\|_{\infty}) \int_{\Omega} ((u_\theta^\delta)^{p-1} - u_\lambda^{p-1})(u_\theta^\delta - u_\lambda)^+ dz \leq 0, \]
and so \( u_\theta^\delta \leq u_\lambda \) for \( \delta > 0 \) small, which implies that
\[ u_\lambda - u_\theta \in \text{int} C_+. \]

Now, let \( \lambda_* = \inf \mathcal{L} \).

In what follows, for every \( \lambda > 0 \), let \( \varphi_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R} \) be the energy functional for problem \( (P_\lambda) \) defined by
\[ \varphi_\lambda(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{p} \int_{\Omega} (\xi(z) + \lambda) \|u\|^{p_1} + \frac{1}{p} \int_{\partial \Omega} \beta(z) \|u\|^{p_1} dz - \int_{\Omega} F(z, u) dz \]
for all \( u \in W^{1,p}(\Omega) \). Here \( F(z, x) = \int_0^x f(z, s) ds \). Evidently \( \varphi_\lambda \in C^1(W^{1,p}(\Omega)) \).

To be able to fix \( \lambda_* \) more precisely, we need to strengthen the condition on \( f(z, \cdot) \) near zero.

\( H(f)_2 : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory function such that \( f(z, 0) = 0 \) for a.a. \( z \in \Omega \). Hypotheses \( H(f)_2(i), (ii), (iii), (iv) \) are the same as Hypotheses \( H(f)(i), (ii), (iii), (iv) \) and
\[ (v) \] there exists \( D \subseteq \Omega \) with \( |D|_N > 0 \) such that
\[ \lambda_1(\beta)x^{p-1} \leq f(z, x) \quad \text{for a.a. } z \in \Omega \text{ and all } x \geq 0, \]
\[ \hat{\lambda}_1(\beta)x^{p-1} < f(z, x) \quad \text{for a.a. } z \in D \text{ and all } x > 0 \]
(recall \( \hat{\lambda}_1(\beta) \) is the first eigenvalue of (9)).
Example. The following function satisfies Hypotheses $H(f)_2$ above (as before, for the sake of simplicity we drop the $z-$dependence):
\[
f(x) = \eta x^{p-1} \left( \log x + \frac{1}{p} \right) + \xi x^{q-1} \quad \text{for all } x \geq 0
\]
with $1 < q < p$, $\eta > p\hat{\lambda}_1(\beta)$ and $\xi > \hat{\lambda}_1(\beta)$.

**Proposition 7.** If Hypotheses $H(\xi), H(\beta), H(f)_2$ hold, then $\lambda_n > 0$.

**Proof.** We argue indirectly. So, suppose that $\lambda_n = 0$ and let $\lambda_n \Downarrow 0$ as $n \to \infty$. Then $\{\lambda_n\}_{n \geq 1} \subseteq \mathcal{L}$ (see Proposition 5). Moreover, from the last part of the proof of Proposition 5, we know that we can find $\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$ such that $u_n \in S(\lambda_n) \subseteq \text{int} C_+ \ (\text{see Proposition 4})$ and
\[
\varphi_{\lambda_n}(u_n) < 0 \quad \text{for all } n \in \mathbb{N}. \tag{26}
\]

From (26) we have
\[
- \int_{\Omega} pF(z, u_n)dz < -\|Du_n\|_p^p - \int_{\Omega} (\xi(z) + \lambda_n)u_n^p dz - \int_{\partial\Omega} \beta(z)u_n^p d\sigma \quad \text{for all } n \in \mathbb{N}. \tag{27}
\]

In addition, since $u_n \in S(\lambda_n)$ for all $n \in \mathbb{N}$, we have
\[
\langle A(u_n), h \rangle + \int_{\Omega} (\xi(z) + \lambda_n)u_n^{p-1}hdz + \int_{\partial\Omega} \beta(z)u_n^{p-1}h d\sigma = \int_{\Omega} f(s, u_n)hdz \tag{28}
\]
for all $h \in W^{1,p}(\Omega)$ and all $n \in \mathbb{N}$.

In (28) we choose $h = u_n \in W^{1,p}(\Omega)$. Then
\[
\int_{\Omega} f(z, u_n)u_n dz = \|Du_n\|_p^p + \int_{\Omega} (\xi(z) + \lambda_n)u_n^p dz + \int_{\partial\Omega} \beta(z)u_n^p d\sigma \quad \text{for all } n \in \mathbb{N}. \tag{29}
\]

We add (27) and (29). Then
\[
\int_{\Omega} [f(z, u_n)u_n - pF(z, u_n)]dz < 0 \quad \text{for all } n \in \mathbb{N}. \tag{30}
\]

Hypotheses $H(f)_2(i), (iii)$ imply that we can find $\gamma \in (0, \gamma_0)$ and $c_1 > 0$ such that
\[
\gamma x^\mu - c_1 \leq f(z, x)x - pF(z, x) \quad \text{for a.a. } z \in \Omega, \quad \text{all } x \geq 0.
\]

Using this estimate in (30) we obtain that
\[
\{u_n\}_{n \geq 1} \subseteq L^\mu(\Omega) \quad \text{is bounded.} \tag{31}
\]

It is clear from Hypothesis $H(f)_2(iii)$, that we may assume, without loss of generality, that $\mu < r$.

First assume $N \not= p$ and let $t \in (0, 1)$ be such that
\[
\frac{1}{r} = \frac{1 - t}{\mu} + \frac{t}{p^*}. \tag{32}
\]

From the interpolation inequality (see, for example, Gasinski - Papageorgiou [11, p. 905]), we have
\[
\|u_n\|_r \leq \|u_n\|_\mu^{1-t} \|u_n\|_p^t,
\]
so that
\[
\|u_n\|_r^\tau \leq c_2\|u_n\|_p^\tau \quad \text{for some } c_2 > 0, \quad \text{all } n \in \mathbb{N}, \text{ see (31)}. \tag{33}
\]

From Hypothesis $H(f)_1(i)$ we have
\[
f(z, u_n)u_n \leq c_3(1 + u_n^p) \quad \text{for a.a. } z \in \Omega, \quad \text{all } n \in \mathbb{N}, \text{ some } c_3 > 0.
\]
So, if in (28), we choose \( h = u_n \in W^{1,p}(\Omega) \), then
\[
\|Du_n\|_p^p \leq c_4 (1 + \|u_n\|_p^p) \quad \text{for some } c_4 > 0, \quad \forall n \in \mathbb{N}
\]
(see Hypotheses \( H(\xi), H(\beta) \))
\[
\leq c_5 (1 + \|u_n\|_p^p) \quad \text{for some } c_5 > 0, \quad \forall n \in \mathbb{N}, \text{ see (33)}.
\]
Recall that \( u \mapsto \|Du\|_p + \|u\|_\mu \) is an equivalent norm on \( W^{1,p}(\Omega) \) (see, for example, Gasinski - Papageorgiou \([11, \text{p. 227}]\)). So, from (31) and (34) we have
\[
\|u_n\|_p^p \leq c_6 (1 + \|u_n\|_p^p) \quad \text{for some } c_6 > 0, \quad \forall n \in \mathbb{N}.
\]
From (32) we get
\[
tr = \frac{(r - \mu)p^*}{p^* - \mu} < p \quad \text{by Hypothesis } H(f)_2(iii),
\]
so from (35) we infer that
\[
\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega) \quad \text{is bounded.}
\]
If \( N = p \), then \( W^{1,p}(\Omega) \hookrightarrow L^\theta(\Omega) \) for all \( \theta \in [1, \infty) \). So, for the argument above to work, it is enough to replace \( p^* \) by \( s \) for \( s \) large enough. More precisely, from (32) we have
\[
tr = \frac{(r - \mu)s}{s - \mu}
\]
and
\[
\frac{(r - \mu)s}{s - \mu} \to r - \mu < p \quad \text{as } s \to +\infty \quad \text{(see Hypothesis } H(f)_2(iii))
\]
and so for \( s > r \) big, we have \( tr < p \). Then the previous argument works and again we have that
\[
\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega) \quad \text{is bounded.}
\]
So, we may assume that
\[
u_n \to u_* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_* \text{ in } L^p(\Omega) \text{ and in } L^p(\partial\Omega) \text{ as } n \to \infty.
\]
(36)
In (28) we choose \( h = u_n - u_* \in W^{1,p}(\Omega) \), pass to the limit as \( n \to \infty \) and use (36). Then
\[
\lim_{n \to \infty} \langle A(u_n), u_n - u_* \rangle = 0.
\]
Hence
\[
u_n \to u_* \text{ in } W^{1,p}(\Omega) \quad \text{(see Proposition 1). \ (37)}
\]
So, if in (28) we pass to the limit as \( n \to \infty \) and use (37), then
\[
\langle A(u_*), h \rangle + \int_\Omega \xi(z)u_*^{p-1}hdz + \int_{\partial\Omega} \beta(z)u_*^{p-1}hd\sigma = \int_\Omega f(z, u_*)hdz
\]
for all \( h \in W^{1,p}(\Omega) \) (recall that \( \lambda_n \downarrow 0 \)). Hence
\[
\begin{cases}
-\Delta u_* + \xi(z)u_*^{p-1} = f(z, u_*) & \text{for a.a. } z \in \Omega, \\
\frac{\partial u_*}{\partial n} + \beta(z)u_*^{p-1} = 0 & \text{on } \partial\Omega,
\end{cases}
\]
see Papageorgiou - Radulescu \([23]\). It follows that \( u_* \in C_+ \) by Lieberman \([15, \text{Theorem 2}]\).

We will show that \( u_* \neq 0 \). To this end, note that given \( r \in (p, p^*) \) and using Hypotheses \( H(f)_2(i), (ii), (iv) \), we can find \( c_7 > 0 \) such that
\[
f(z, x) \geq c_0 x^{q-1} - c_7 x^{r-1} \quad \text{for a.a. } z \in \Omega, \quad \forall x \geq 0.
\]
(39)
For every $\lambda > \lambda_0 = 0$, we consider the following auxiliary nonlinear Robin problem:

$$
(P)_\lambda \begin{cases}
-\Delta_p u(z) + (\xi(z) + \lambda)u(z)^{p-1} = c_0 u(z)^{q-1} - c_7 u(z)^{r-1} & \text{in } \Omega, \\
\frac{\partial u}{\partial n_p} + \beta(z)u(z)^{p-1} = 0 & \text{on } \partial \Omega,
\end{cases}
$$

As before, let $\mu > \lambda > \lambda_0$. For every $\tilde{\lambda}$ sequentially weakly lower semicontinuous. So, by the Weierstrass Theorem, we can find $\tilde{\lambda}$ such that

$$
\psi: q < p < r
$$

As before (see the proof of Proposition 5), since $q < p < r$ and $\mu > \|\xi\|_{\infty}$, from (40) we infer that $\psi_{\lambda}$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass Theorem, we can find $\tilde{u}_{\lambda} \in W^{1,p}(\Omega)$ such that

$$
\psi_{\lambda}(\tilde{u}_{\lambda}) = \inf \left\{ \psi_{\lambda}(u) : u \in W^{1,p}(\Omega) \right\}. \tag{41}
$$

As before (see the proof of Proposition 5), since $q < p < r$, we see that

$$
\psi_{\lambda}(\tilde{u}_{\lambda}) < 0 = \psi_{\lambda}(0),
$$

and hence $\tilde{u}_{\lambda} \neq 0$. From (41) we have

$$
\psi'_{\lambda}(\tilde{u}_{\lambda}) = 0.
$$

It implies

$$
\langle A(\tilde{u}_{\lambda}), h \rangle + \int_{\Omega} (\xi(z) + \lambda)|\tilde{u}_{\lambda}|^{p-2}\tilde{u}_{\lambda}hdz - \mu \int_{\Omega} (\tilde{u}_{\lambda})^{p-1}hdz + \int_{\partial \Omega} \beta(z)(\tilde{u}_{\lambda})^{p-1}hd\sigma \tag{42}
$$

In (42) we choose $h = -\tilde{u}_{\lambda} \in W^{1,p}(\Omega)$. Then

$$
\|D\tilde{u}_{\lambda}\|_p + (\lambda + \mu - \|\xi\|_{\infty})\|\tilde{u}_{\lambda}\|_p \leq 0,
$$
so that $\tilde{\upsilon}_\lambda \geq 0, \tilde{\upsilon}_\lambda \neq 0$ (recall that $\mu > \|\xi\|_\infty$). Then (42) becomes
\[ \langle A(\tilde{\upsilon}_\lambda), h \rangle + \int_\Omega (\xi(z) + \lambda)\tilde{\upsilon}_\lambda^{p-1}hdz + \int_{\partial\Omega} \beta(z)(\tilde{\upsilon}_\lambda)^{p-1}hds = c_0 \int_\Omega \tilde{\upsilon}_\lambda^{q-1}hdz - c_7 \int_\Omega \tilde{\upsilon}_\lambda^{p-1}hdz \]
for all $h \in W^{1,p}(\Omega)$. Thus $\tilde{\upsilon}_\lambda$ is a positive solution of $(P)_\lambda$ (see Papageorgiou - Radulescu [23]).

The nonlinear regularity theory implies that $\tilde{\upsilon}_\lambda \in C_+ \setminus \{0\}$. In addition, we have
\[ \Delta_p\tilde{\upsilon}_\lambda(z) \leq (\|\xi\|_\infty + \lambda + c_7\|\tilde{\upsilon}_\lambda\|^{p-1})\tilde{\upsilon}_\lambda(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \]
so that $\tilde{\upsilon}_\lambda \in \text{int} C_+$ (see Pucci - Serrin [27, p.120]).

We will show that $\tilde{\upsilon}_\lambda \in \text{int} C_+$ is the unique positive solution of $(P)_\lambda$. Indeed, suppose that $\tilde{\eta}_\lambda$ is another positive solution of $(P)_\lambda$. As above, we can show that $\tilde{\eta}_\lambda \in \text{int} C_+$.

Let $j : L^1(\Omega) \to \mathbb{R} = \mathbb{R} \cup \{+\infty\}$ be the integral functional defined by
\[ j(u) = \left\{ \begin{array}{ll}
\int_\Omega |Du|^pdz + \int_{\partial\Omega} \beta(z)ud\sigma & \text{if } u \geq 0, u^+ \in W^{1,p}(\Omega), \\
+\infty & \text{otherwise}.
\end{array} \right. \]
From Diaz - Saa [8] we know that $j(\cdot)$ is convex. Set $\text{dom} j = \{ u \in L^1(\Omega) : j(u) < +\infty \}$ (the effective domain of $j$). Since $\tilde{\upsilon}_\lambda, \tilde{\eta}_\lambda \in \text{int} C_+$, for $h \in C^1(\Omega)$ and $|t| < 1$ small, we have
\[ \tilde{\upsilon}_\lambda^p + th, \tilde{\eta}_\lambda^p + th \in \text{dom} j. \]

Then the Gateaux derivative of $j(\cdot)$ at $\tilde{\upsilon}_\lambda^p$ and $\tilde{\eta}_\lambda^p$ in the direction $h \in C^1(\Omega)$ exists and using the chain rule and the nonlinear Green's identity, we obtain
\[ j'(\tilde{\upsilon}_\lambda^p)(h) = \frac{1}{p} \int_\Omega \frac{-\Delta_p \tilde{\upsilon}_\lambda}{\tilde{\upsilon}_\lambda^{p-1}}hdz \]
\[ j'(\tilde{\eta}_\lambda^p)(h) = \frac{1}{p} \int_\Omega \frac{-\Delta_p \tilde{\eta}_\lambda}{\tilde{\eta}_\lambda^{p-1}}hdz. \]

The convexity of $j(\cdot)$ implies the monotonicity of $j'(\cdot)$. Hence
\[ 0 \leq \int_\Omega \left( \frac{-\Delta_p \tilde{\upsilon}_\lambda}{\tilde{\upsilon}_\lambda^{p-1}} - \frac{-\Delta_p \tilde{\eta}_\lambda}{\tilde{\eta}_\lambda^{p-1}} \right) (\tilde{\upsilon}_\lambda^p - \tilde{\eta}_\lambda^p)dz \]
\[ = \int_\Omega \left[ c_0 \left( \frac{1}{\tilde{\upsilon}_\lambda^{q-1}} - \frac{1}{\tilde{\eta}_\lambda^{q-1}} \right) - c_7(\tilde{\upsilon}_\lambda^{r-p} - \tilde{\eta}_\lambda^{r-p}) \right] (\tilde{\upsilon}_\lambda^p - \tilde{\eta}_\lambda^p)dz \leq 0. \]

Thus $\tilde{\upsilon}_\lambda = \tilde{\eta}_\lambda$ and $\tilde{\upsilon}_\lambda \in \text{int} C_+$ is the unique positive solution of $(P)_\lambda$.

Let $u \in S(\lambda)$ and consider the Carathéodory functions $g(z, x)$ for $(z, x) \in \Omega \times \mathbb{R}$ and $\beta_0(z, x)$ for $(z, x) \in \partial\Omega \times \mathbb{R}$ defined by
\[ g(z, x) = \begin{cases} 
0 & \text{if } x < 0, \\
c_0x^{q-1} - c_7x^{r-1} + \mu x^{p-1} & \text{if } 0 \leq x \leq u(z), \\
c_0u(z)^{q-1} - c_7u(z)^{r-1} + \mu u(z)^{p-1} & \text{if } u(z) < x,
\end{cases} \quad (43) \]
\begin{equation}
\beta_0(z, x) = \begin{cases} 
0 & \text{if } x < 0, \\
\beta(z)x^{p-1} & \text{if } 0 \leq x \leq u(z), \\
\beta(z)u(z)^{-1} & \text{if } u(z) < x 
\end{cases}
\tag{44}
\end{equation}

(as before \(\mu > \|\xi\|_{\infty} \)). We set \(G(z, x) = \int_0^x g(z, s)ds\), \(B_0(z, x) = \int_0^x \beta_0(z, s)ds\) and consider the \(C^1\) functional \(\hat{\gamma} : W^{1,p}(\Omega) \to \mathbb{R}\) defined by

\[
\hat{\gamma}(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{p} \int_{\Omega} (\xi(z) + \lambda + \mu) |u|^p dz + \int_{\partial\Omega} B_0(z, u) d\sigma - \int_{\Omega} G(z, u) dz
\]

for all \(u \in W^{1,p}(\Omega)\). Since \(\mu > \|\xi\|_{\infty}\), from (43) and (44) it is clear that \(\hat{\gamma}\) is coercive. Also, it is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find \(\tilde{u}_0^0 \in W^{1,p}(\Omega)\) such that

\[
\hat{\gamma}(\tilde{u}_0^0) = \inf \left\{ \hat{\gamma}(u) : u \in W^{1,p}(\Omega) \right\}.
\tag{45}
\]

Since \(q < p < r\), we have

\[
\hat{\gamma}(\tilde{u}_0^0) < 0 = \hat{\gamma}(0).
\]

Hence \(\tilde{u}_0^0 \neq 0\). From (45) we have

\[
\hat{\gamma}'(\tilde{u}_0^0) = 0,
\]

thus

\[
\langle A(\tilde{u}_0^0), h \rangle + \int_{\Omega} (\xi(z) + \lambda + \mu) |\tilde{u}_0^0|^{p-2}\tilde{u}_0^0 hdz + \int_{\partial\Omega} \beta_0(z, \tilde{u}_0^0) h d\sigma = \int_{\Omega} g(z, \tilde{u}_0^0) hdz
\]

for all \(h \in W^{1,p}(\Omega)\).

In (46), we choose \(h = -(\tilde{u}_0^0)^- \in W^{1,p}(\Omega)\) and using (43), (44) we obtain

\[
\|D(\tilde{u}_0^0)^-\|_p^p + (\lambda + \mu - \|\xi\|_{\infty}) \|(\tilde{u}_0^0)^-\|_p^p \leq 0,
\]

hence \(\tilde{u}_0^0 \geq 0\), \(\tilde{u}_0^0 \neq 0\).

Next in (46) we choose \((\tilde{u}_0^0 - u)^+ \in W^{1,p}(\Omega)\). Using (43) and (44) we have

\[
\langle A(\tilde{u}_0^0), (\tilde{u}_0^0 - u)^+ \rangle + \int_{\Omega} (\xi(z) + \lambda + \mu) |(\tilde{u}_0^0)^{p-1}(\tilde{u}_0^0 - u)^+| dz \\
+ \int_{\partial\Omega} \beta(z)u^{p-1}(\tilde{u}_0^0 - u)^+ d\sigma
\]

\[
= \int_{\Omega} \left[ c_0 u^{q-1} - c_\gamma u^{r-1} + \mu u^{p-1} \right] (\tilde{u}_0^0 - u)^+ dz
\]

\[
\leq \int_{\Omega} |f(z, u) + \mu u^{p-1}|(\tilde{u}_0^0 - u)^+ dz \quad \text{(see (39))}
\]

\[
= \langle A(u), (\tilde{u}_0^0 - u)^+ \rangle + \int_{\Omega} (\xi(z) + \lambda + \mu) u^{p-1}(\tilde{u}_0^0 - u)^+ dz \\
+ \int_{\partial\Omega} \beta(z)u^{p-1}(\tilde{u}_0^0 - u)^+ d\sigma \quad \text{(since } u \in S(\lambda)).
\]

Hence

\[
\langle A(\tilde{u}_0^0) - A(u), (\tilde{u}_0^0 - u)^+ \rangle + (\lambda + \mu - \|\xi\|_{\infty}) \int_{\Omega} ((\tilde{u}_0^0)^{p-1} - u^{p-1})(\tilde{u}_0^0 - u)^+ dz \leq 0
\]

and so

\[
\tilde{u}_0^0 \leq u.
\]
Hence, we have reached a contradiction, and this proves that
\[ \lambda_\lambda \neq 0. \] (47)

In the light of (47), equation (46) becomes
\[
(A(u^0_\lambda), h) + \int_{\Omega} (\xi(z) + \lambda)(u^0_\lambda)^{p-1} + \int_{\partial\Omega} \beta(z)(u^0_\lambda)^{p-1}h\sigma = \int_{\Omega} [c_0(u^0_\lambda)^{q-1} - c_\gamma(u^0_\lambda)^{r-1}]h dz \quad \text{(see (43), (44))},
\]
thus \( u_\lambda^0 \) is a positive solution of \((P)_\lambda\) and
\[ \tilde{u}_\lambda^0 = u_\lambda \in \text{int } C_+ \]
(by the uniqueness of the positive solution of \((P)_\lambda\)).

From (47) we have
\[ \tilde{u}_\lambda \leq u \quad \text{for all } u \in S(\lambda) \] (recall that \( u \in S(\lambda) \) was arbitrary). As in the proof of Proposition 5, we have
\[ \lambda \mapsto \tilde{u}_\lambda \quad \text{is nonincreasing from } (\lambda_*, \infty) \text{ into } C^1(\Omega). \]

So, we have
\[ \tilde{u}_{\lambda_1} \leq \tilde{u}_{\lambda_n} \quad \text{for all } n \in \mathbb{N} \text{ (recall } \lambda_n \downarrow 0). \]
In particular, \( \tilde{u}_{\lambda_1} \leq u_* \) for all \( n \in \mathbb{N} \) (since \( u_n \in S(\lambda_n) \), see (48)). This implies \( \bar{u}_{\lambda_1} \leq u_* \) and so \( u_* \neq 0 \); hence
\[ u_* \in \text{int } C_+ \]
(since \( u_* \in C_+ \) and \( \tilde{u}_{\lambda_1} \in \text{int } C_+ \)).

From the nonlinear Picone’s identity of Allegretto - Huang [2] (see also Motreanu - Motreanu - Papageorgiou [18, p. 255]), we have
\[
0 \leq \|D\tilde{u}_1(\beta)\|_p - \int_{\Omega} |Du_*(\beta)|^{p-1} \left( Du_*, D\left( \frac{\tilde{u}_1(\beta)^p}{u_*^{p-1}} \right) \right)_{\mathbb{R}^N} dz
\]
\[
= \|D\tilde{u}_1(\beta)\|_p - \langle A(u_*), \frac{\tilde{u}_1(\beta)^p}{u_*^{p-1}} \rangle
\]
\[
= \|D\tilde{u}_1(\beta)\|_p + \int_{\Omega} \xi(z)\tilde{u}_1(\beta)^p dz + \int_{\partial\Omega} \beta(z)\tilde{u}_1(\beta)^p d\sigma - \int_{\Omega} f(z, u_*) \frac{\tilde{u}_1(\beta)^p}{u_*^{p-1}} dz
\]
(see (38) with \( \tilde{h} = \frac{\tilde{u}_1^p}{u_*^{p-1}} \in \text{int } C_+ \))
\[
\hat{\lambda}_1(\beta) - \int_{\Omega} f(z, u_*) \frac{\tilde{u}_1(\beta)^p}{u_*^{p-1}} dz \quad \text{(see (9), (10) and recall } \|\tilde{u}_1(\beta)\|_p = 1),
\]
\[
= \hat{\lambda}_1(\beta) - \int_{\Omega} f(z, u_*) \frac{\tilde{u}_1(\beta)^p}{u_*^{p-1}} dz < 0
\]
(see Hypothesis \( H(f)_2(v) \), and recall that \( u_* \in \text{int } C_+ \)).

Hence, we have reached a contradiction, and this proves that \( \lambda_* > 0 \). \( \square \)

Next, we will show that for all \( \lambda \in (\lambda_*, \infty) \) problem \((P)_\lambda\) has at least two positive solutions. To do this, we need to strengthen the conditions on the reaction term \( f(z, x) \) by combining Hypotheses \( H(f)_1 \) and \( H(f)_2 \). So, the new Hypotheses on \( f(z, x) \) are:
we can find 

\begin{align*}
H(f)_3: f : \Omega \times \mathbb{R} \to \mathbb{R}
\end{align*}

is a Carathéodory function such that \( f(z,0) = 0 \) for a.a. \( z \in \Omega \), Hypotheses \( H(f)_3(i), (ii), (iii), (iv), (v) \) are the same as the corresponding Hypotheses \( H(f)_1(i), (ii), (iii), (iv), (v) \) and

\begin{align*}
(vi) \text{for every } \rho > 0, \text{ there exists } \xi_\rho > 0 \text{ such that for a.a. } z \in \Omega
\end{align*}

\[
x \to f(z,x) + \xi_\rho x^{p-1} \text{ is nondecreasing on } [0, \rho].
\]

**Example.** The function \( f(x) = \eta x^{p-1} \left( \log x + \frac{1}{p} \right) + \xi x^{q-1} \) with \( 1 < q < p, \eta > p\lambda_1(\beta), \xi > \lambda_1(\beta) \), given earlier still satisfies \( H(f)_3 \).

**Proposition 8.** If Hypotheses \( H(\xi), H(\beta), H(f)_3 \) hold and \( \lambda > \lambda_* \), then problem \( (P_{\lambda}) \) admits at least two positive solutions \( u_{\lambda}, \tilde{u}_{\lambda} \in \text{int} \, C_+ \).

**Proof.** Let \( \theta_1, \theta_2 \in L \) and suppose that \( \lambda_* < \theta_1 < \lambda < \theta_2 \). Proposition 6 implies that we can find \( u_{\theta_1} \in S(\theta_1) \subseteq \text{int} \, C_+ \) and \( u_{\theta_2} \in S(\theta_2) \subseteq \text{int} \, C_+ \) such that \( u_{\theta_1} - u_{\theta_2} \in \text{int} \, C_+ \).

We introduce the following Carathéodory functions:

\[
e(z,x) = \begin{cases} 
  f(z, u_{\theta_2}(z)) + \mu u_{\theta_2}(z)^{p-1} & \text{if } x < u_{\theta_2}(z), \\
  f(z, x) + \mu x^{p-1} & \text{if } u_{\theta_2}(z) \leq x \leq u_{\theta_1}(z), \\
  f(z, u_{\theta_1}(z)) + \mu u_{\theta_1}(z)^{p-1} & \text{if } u_{\theta_1}(z) < x,
\end{cases}
\]

for all \((z,x) \in \Omega \times \mathbb{R}, \) and

\[
\tau(z,x) = \begin{cases} 
  \beta(z)u_{\theta_2}(z)^{p-1} & \text{if } x < u_{\theta_2}(z), \\
  \beta(z)x^{p-1} & \text{if } u_{\theta_2}(z) \leq x \leq u_{\theta_1}(z), \\
  \beta(z)u_{\theta_1}(z)^{p-1} & \text{if } u_{\theta_1}(z) < x,
\end{cases}
\]

for all \((z,x) \in \partial \Omega \times \mathbb{R}, \) where \( \mu > ||\xi||_\infty. \) We set \( E(z,x) = \int_0^x e(z,s)ds \) and \( T(z,x) = \int_0^x \tau(z,s)ds \) and consider the \( C^1 - \) functional \( \hat{d}_\lambda : W^{1,p}(\Omega) \to \mathbb{R} \) defined by

\[
\hat{d}_\lambda(u) = \frac{1}{p} \|Du\|_p + \frac{1}{p} \int_{\Omega} (\xi(z) + \lambda + \mu)|u|^{p}dz + \int_{\partial \Omega} T(z,u)ds - \int_{\Omega} E(z,u)dz
\]

for all \( u \in W^{1,p}(\Omega). \) Since \( \mu > ||\xi||_\infty, \) from (49) and (50) it is clear that \( \hat{d}_\lambda \) is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find \( u_\lambda \in W^{1,p}(\Omega) \) such that

\[
\hat{d}_\lambda(u_\lambda) = \inf \{ \hat{d}_\lambda(u) : u \in W^{1,p}(\Omega) \}, \text{ so that } \hat{d}'_\lambda(u_\lambda) = 0.
\]

This implies

\[
(A(u_\lambda), h) + \int_{\Omega} (\xi(z) + \lambda + \mu)|u_\lambda|^{p-2}u_\lambda hdz + \int_{\partial \Omega} \tau(z,u_\lambda)hd\sigma = \int_{\Omega} e(z,u_\lambda)hdz.
\]
for all $h \in W^{1,p}(\Omega)$. In (51) first we choose $h = (u_\lambda - u_{\theta_1})^+ \in W^{1,p}(\Omega)$. Using (49) and (50) we obtain

\[
\langle A(u_\lambda), (u_\lambda - u_{\theta_1})^+ \rangle + \int_\Omega (\xi(z) + \lambda + \mu) u_\lambda^{p-1} (u_\lambda - u_{\theta_1})^+ dz \\
+ \int_{\partial \Omega} \beta(z) u_{\theta_1}^{p-1} (u_\lambda - u_{\theta_1})^+ d\sigma \\
= \int_\Omega [f(z, u_{\theta_1}) + \mu u_{\theta_1}^{p-1}](u_\lambda - u_{\theta_1})^+ dz \\
= \langle A(u_{\theta_1}), (u_\lambda - u_{\theta_1})^+ \rangle + \int_\Omega (\xi(z) + \theta_1 + \mu) u_{\theta_1}^{p-1} (u_\lambda - u_{\theta_1})^+ dz \\
+ \int_{\partial \Omega} \beta(z) u_{\theta_1}^{p-1} (u_\lambda - u_{\theta_1})^+ d\sigma \quad \text{(since $u_{\theta_1} \in S(\theta_1)$)}
\]

\[
\leq \langle A(u_{\theta_1}), (u_\lambda - u_{\theta_1})^+ \rangle + \int_\Omega (\xi(z) + \lambda + \mu) u_{\theta_1}^{p-1} (u_\lambda - u_{\theta_1})^+ dz \\
+ \int_{\partial \Omega} \beta(z) u_{\theta_1}^{p-1} (u_\lambda - u_{\theta_1})^+ d\sigma \quad \text{(since $\theta_1 < \lambda$)}
\]

\[
\Rightarrow \langle A(u_\lambda) - A(u_{\theta_1}), (u_\lambda - u_{\theta_1})^+ \rangle + (\lambda + \mu - \|\xi\|_\infty) \int_\Omega (u_\lambda^{p-1} - u_{\theta_1}^{p-1})(u_\lambda - u_{\theta_1})^+ dz \leq 0,
\]

so that $u_\lambda \leq u_{\theta_1}$. Similarly, if in (51) we choose $h = (u_{\theta_2} - u_\lambda)^+ \in W^{1,p}(\Omega)$, then we show that

\[
u_{\theta_2} \leq u_\lambda.
\]

So, we have proved that

\[
u_\lambda \in [u_{\theta_2}, u_{\theta_1}] = \left\{ u \in W^{1,p}(\Omega) : u_{\theta_2}(z) \leq u(z) \leq u_{\theta_1}(z) \text{ for a.a. } z \in \Omega \right\}.
\]

Then, because of (49) and (50), equation (51) becomes

\[
\langle A(u_\lambda), h \rangle + \int_\Omega (\xi(z) + \lambda) u_\lambda^{p-1} h dz + \int_{\partial \Omega} \beta(z) u_\lambda^{p-1} h d\sigma = \int_\Omega f(z, u_\lambda) h dz
\]

for all $h \in W^{1,p}(\Omega)$, and hence $u_\lambda \in S(\lambda) \subseteq \text{int } C_+$ (see Proposition 4).

Moreover, using Hypothesis $H(f)(vi)$, as in the proof of Proposition 6, using the two order cones $C_+, C_+$, we show that

\[
u_\lambda \in \text{int } C_+([\Omega_1, u_{\theta_2}, u_{\theta_1}]].
\]

Using $u_\lambda$ and variational arguments, we will produce a second positive solution for problem $(P_\lambda)$. To this end, we introduce the following Carathéodory functions:

\[
\hat{g}(z, x) = \begin{cases} 
  f(z, u_{\theta_2}(z)) + \mu u_{\theta_2}(z)^{p-1} & \text{if } x \leq u_{\theta_2}(z), \\
  f(z, x) + \mu x^{p-1} & \text{if } u_{\theta_2}(z) < x
\end{cases} \quad \text{(53)}
\]

for all $(z, x) \in \Omega \times \mathbb{R}$ ($\mu > \|\xi\|_\infty$), and

\[
\hat{\beta}(z, x) = \begin{cases} 
  \beta(z) u_{\theta_2}(z)^{p-1} & \text{if } x \leq u_{\theta_2}(z), \\
  \beta(z) x^{p-1} & \text{if } u_{\theta_2}(z) < x
\end{cases} \quad \text{(54)}
\]
for all \((z,x) \in \partial \Omega \times \mathbb{R}\). We set \(\bar{G}(z,x) = \int_0^x \hat{g}(z,s)ds\) and \(\bar{B}(z,x) = \int_0^x \hat{\beta}(z,s)ds\) and consider the \(C^1\)- functional \(d^1_0 : W^{1,p}(\Omega) \to \mathbb{R}\) defined by

\[
d^1_0(u) = \frac{1}{p} \|Du\|_p^p + \frac{1}{p} \int_\Omega (\xi(z) + \lambda + \mu)|u|^{p}dz + \int_{\partial \Omega} \hat{\beta}(z,u)d\sigma - \int_\Omega \bar{G}(z,u)dz
\]

for all \(u \in W^{1,p}(\Omega)\). From (59) and (62), we have

\[
K_{d^1_0} \subset \{u_{\theta_1}\} = \left\{ u \in W^{1,p}(\Omega) : u_{\theta_1}(z) \leq u(z) \text{ for a.a. } z \in \Omega \right\}.
\]

From (49), (50), (53), (54), we see that

\[
\inf_{\|u\|_{W^{1,p}(\Omega)}} \int_{\Omega} (1 + \|u\|_{W^{1,p}(\Omega)}) = 0.
\]

Because of Hypothesis \(H_{\beta,\hat{g}}\) we infer that for every \(u \in \text{int} C_+\), we have

\[
d^1_0(tu) \to -\infty \text{ as } t \to +\infty.
\]

**Claim:** \(d^1_0\) satisfies the \(C^+\) condition.

Indeed, let \(\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)\) be a sequence such that

\[
|d^1_0(u_n)| \leq M_1, \quad \text{for some } M_1 > 0, \text{ all } n \in \mathbb{N},
\]

\[
(1 + \|u_n\|)(d^1_0)'(u_n) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n \to \infty
\]

From (60) we have

\[
|\langle A(u_n), h \rangle + \int_\Omega (\xi(z) + \lambda + \mu)|u_n|^{p-2}u_nhdz + \int_{\partial \Omega} \hat{\beta}(z,u_n)h\sigma - \int_\Omega \hat{g}(z,u_n)hdz| 
\]

\[
\leq \epsilon_n\|h\| + \frac{\epsilon_n\|u_n\|}{1 + \|u_n\|}
\]

for all \(h \in W^{1,p}(\Omega)\) with \(\epsilon_n \downarrow 0\). In (61) we choose \(h = -u_n^- \in W^{1,p}(\Omega)\), so that

\[
\|Du_n^-\|_p^p + (\lambda + \mu - \|\xi\|_\infty)\|u_n^-\|_p^p - \int_{\partial \Omega} \hat{\beta}(z,u_n)u_n^-d\sigma + \int_\Omega \hat{g}(z,u_n)u_n^-dz \leq \epsilon_n
\]

for all \(n \in \mathbb{N}\) (see (53), (54)).

Then, recalling that \(\mu > \|\xi\|_\infty\) and the definition of \(\hat{\beta}\) and of \(\hat{g}\), we get that

\[
\{u_n^-\}_{n \geq 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.}
\]

From (59) and (62), we have

\[
\|Du_n^+\|_p^p + \int_\Omega (\xi(z) + \lambda + \mu)(u_n^+)p^sdz + \int_{\partial \Omega} p\hat{\beta}(z,u_n^+)d\sigma - \int_\Omega p\hat{g}(z,u_n^+)dz \leq M_2
\]

for some \(M_2 > 0, \text{ all } n \in \mathbb{N}\).
Moreover, from (60) and (62), we have
\[
\left| \langle A(u^+_n), h \rangle + \int_\Omega (\xi(z) + \lambda + \mu)(u^+_n)^{p-1}hdz + \int_{\partial\Omega} \hat{\beta}(z, u^+_n)h\sigma - \int_\Omega \hat{g}(z, u^+_n)hdz \right| \\
\leq \epsilon'_n\|h\| + C
\]
\[(64)\]
for all \(h \in W^{1,p}(\Omega)\), with \(\epsilon'_n \downarrow 0\) and \(C > 0\) is a constant.

In (64) we choose \(h = u^+_n \in W^{1,p}(\Omega)\). Then
\[
\int_\Omega \hat{g}(z, u^+_n)u^+_ndz - \|Du^+_n\|_p^p - \int_\Omega (\xi(z) + \lambda + \mu)(u^+_n)^pdz - \int_{\partial\Omega} \hat{\beta}(z, u^+_n)u^+_nd\sigma \\
\leq \epsilon'_n\|u^+_n\| + C
\]
for all \(n \in \mathbb{N}\). Adding (63) and (65) we obtain
\[
\int_\Omega [\hat{g}(z, u^+_n)u^+_n - p\hat{G}(z, u^+_n)]dz + \int_{\partial\Omega} [p\hat{B}(z, u^+_n) - \hat{\beta}(z, u^+_n)u^+_n]d\sigma \leq M_3
\]
for some \(M_3 > 0\) and all \(n \in \mathbb{N}\). Hence, by (53) and (54), we immediately find that
\[
\int_\Omega [f(z, u^+_n)u^+_n - pF(z, u^+_n)]dz \leq M_4
\]
for some \(M_4 > 0\) and all \(n \in \mathbb{N}\).

Using (66) and reasoning as in the proof of Proposition 7 (see the part of the proof from (30) until (37)), we deduce that there exists a subsequence of \(\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)\) such that \(u_n \to u\) in \(W^{1,p}(\Omega)\). This proves the claim.

The claim together with (57) and (58), permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find \(\hat{u}_\lambda \in W^{1,p}(\Omega)\) such that
\[
\hat{u}_\lambda \in K_{d_0^\lambda} \text{ and } m_0^\lambda \leq d_0^\lambda(\hat{u}_\lambda).
\]
\[(67)\]
From (57) and (67), we see that \(\hat{u}_\lambda \neq u_\lambda\). Also, from (55) and (53), (54) we have that \(\hat{u}_\lambda \in S(\lambda) \subseteq \text{int } C_+\) (see Proposition 4).

Next we examine what happens at the critical parameter value \(\lambda_* > 0\) (see Proposition 7).

**Proposition 9.** If Hypotheses \(H(\xi), H(\beta), H(f)\) hold, then \(\lambda_* \in \mathcal{L}\), hence \(\mathcal{L} = [\lambda_*, +\infty)\).

**Proof.** Let \(\{\lambda_n\}_{n \geq 1} \subseteq (\lambda_*, \infty)\) be such that \(\lambda_n \downarrow \lambda_*\). We can find \(u_n \in S(\lambda_n) \subseteq \text{int } C_+\), \(n \in \mathbb{N}\), such that
\[
\varphi_{\lambda_n}(u_n) < 0 \quad \text{for all } n \in \mathbb{N}
\]
(see the proofs of Propositions 5 and 7). Then, reasoning as in the proof of Proposition 7, we show that \(\{u_n\}_{n \geq 1} \subset W^{1,p}(\Omega)\) is bounded and so we may assume that
\[
u_n \to u_* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_* \text{ in } L^r(\Omega) \text{ and in } L^p(\partial\Omega).
\]
\[(68)\]
We have
\[
\langle A(u_n), h \rangle + \int_\Omega (\xi(z) + \lambda_n)u_n^{p-1}hdz + \int_{\partial\Omega} \beta(z)u_n^{p-1}h\sigma = \int_\Omega f(z, u_n)hdz
\]
for all \(h \in W^{1,p}(\Omega)\) and every \(n \in \mathbb{N}\). If in (69) we choose \(h = u_n - u_* \in W^{1,p}(\Omega)\), pass to the limit as \(n \to \infty\) and use (68), then
\[
\lim_{n \to \infty} \langle A(u_n), u_n - u_* \rangle = 0,
\]
so that, by Proposition 1,

\[ u_n \to u_* \text{ in } W^{1,p}(\Omega) \text{ as } n \to \infty. \]

Therefore, if we pass to the limit as \( n \to \infty \) in (69), then we have

\[ \langle A(u_*), h \rangle + \int_\Omega (\xi(z) + \lambda_*) u_*^{p-1} h dz + \int_{\partial \Omega} \beta(z) u_*^{p-1} h ds = \int_\Omega f(z, u_*) hdz \]

for all \( h \in W^{1,p}(\Omega) \). Hence \( u_* \) is a solution of problem \((P_{\lambda_*})\), \( u_* \in C_+ \) by nonlinear regularity. From the proof of Proposition 7 we know that

\[ \tilde{u}_{\lambda_*} \leq u_n \quad \text{for all } n \in \mathbb{N} \quad (\tilde{u}_{\lambda_*} \in \text{int } C_+), \]

and so \( \tilde{u}_{\lambda_*} \leq u_* \). As a consequence,

\[ u_* \in S(\lambda_*) \subset \text{int } C_+ \quad \text{and } \lambda_* \in \mathcal{L}. \]

Therefore from Proposition 5 we conclude that \( \mathcal{L} = [\lambda_*, +\infty) \). \( \square \)

Next we show the existence of a smallest positive solution for problem \((P_{\lambda})\), \( \lambda \in \mathcal{L} \).

**Proposition 10.** If Hypotheses \( H(\xi), H(\beta), H(f)_3 \) hold and \( \lambda \in \mathcal{L} = [\lambda_*, +\infty) \), then problem \((P_{\lambda})\) has a smallest positive solution \( u_{\lambda}^* \in \text{int } C_+ \).

**Proof.** As in Filippakis - Papageorgiou [9], we get that \( S(\lambda) \) is downward directed (that is, if \( u_1, u_2 \in S(\lambda) \), then we can find \( u \in S(\lambda) \) such that \( u \leq u_1, u \leq u_2 \)). Then from Hu - Papageorgiou [14, Lemma 3.10, p. 178], we can find a decreasing sequence \( \{u_n\}_{n \geq 1} \subset S(\lambda) \) such that

\[ \inf S(\lambda) = \lim_{n \to \infty} u_n. \]

Evidently \( \{u_n\}_{n \geq 1} \subset W^{1,p}(\Omega) \) is bounded and so we may assume that

\[ u_n \to u_{\lambda}^* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \to u_{\lambda}^* \text{ in } L^p(\Omega) \text{ and in } L^p(\partial \Omega). \] (70)

We have

\[ \langle A(u_n), h \rangle + \int_\Omega (\xi(z) + \lambda_n) u_n^{p-1} h dz + \int_{\partial \Omega} \beta(z) u_n^{p-1} h ds = \int_\Omega f(z, u_n) h dz \] (71)

for all \( h \in W^{1,p}(\Omega) \). In (71) we choose \( h = u_n - u_{\lambda}^* \in W^{1,p}(\Omega) \), pass to the limit as \( n \to \infty \) and use (70). Then

\[ \lim_{n \to \infty} \langle A(u_n), u_n - u_{\lambda}^* \rangle = 0, \]

which implies that

\[ u_n \to u_{\lambda}^* \text{ in } W^{1,p}(\Omega) \] (72)

(see Proposition 1). From the proof of Proposition 7, we know that

\[ \tilde{u}_{\lambda} \leq u_n \quad \text{for all } n \in \mathbb{N} \quad (\tilde{u}_{\lambda} \in \text{int } C_+), \]

and so, by (72),

\[ \tilde{u}_{\lambda} \leq u_{\lambda}^*, \quad \text{and so } u_{\lambda}^* \neq 0. \]

In (71) we pass to the limit as \( n \to \infty \) and use (72). We obtain

\[ \langle A(u_{\lambda}^*), h \rangle + \int_\Omega (\xi(z) + \lambda) u_{\lambda}^* u_{\lambda}^* h dz + \int_{\partial \Omega} \beta(z) (u_{\lambda}^*)^{p-1} h ds = \int_\Omega f(z, u_{\lambda}^*) h dz \]

for all \( h \in W^{1,p}(\Omega) \). In particular, \( u_{\lambda}^* \in S(\lambda) \subset \text{int } C_+ \) by Proposition 4, and so

\[ \Rightarrow u_{\lambda}^* = \inf S(\lambda). \]
Finally, we examine the properties of the map $\lambda \mapsto u_\lambda^*$ from $\mathcal{L} = [\lambda_*, +\infty)$ into $C^1(\overline{\Omega})$.

**Proposition 11.** If Hypotheses $H(\xi), H(\beta), H(f)_3$ hold, then the map $\lambda \mapsto u_\lambda^*$ from $\mathcal{L} = [\lambda_*, +\infty)$ into $C^1(\overline{\Omega})$ is right continuous and strictly decreasing (that is, if $\lambda < \theta$, then $u_\lambda^* - u_\theta^* \in \text{int} \hat{C}_+$ with $\mathcal{H}_0 := \{z \in \partial \Omega : u_\lambda^*(z) = u_\theta^*(z)\}$).

**Proof.** Let $\lambda, \theta \in \mathcal{L}$ with $\lambda < \theta$. From Proposition 6, we know that we can find $u_\theta \in S(\theta) \subseteq \text{int} C_+$ such that

\[
\begin{align*}
&u_\lambda^* - u_\theta \in \text{int} \hat{C}_+ \\
&\text{where } u_\theta \text{ denotes the smallest positive solution of problem } (P_\theta), \text{ see Proposition 10.}
\end{align*}
\]

Hence

\[
\lambda \mapsto u_\lambda^* \text{ is strictly decreasing from } \mathcal{L} = [\lambda_*, +\infty) \text{ into } C^1(\overline{\Omega}). \tag{73}
\]

Now, take $\{\lambda_n\}_{n \geq 1} \subseteq \mathcal{L}$ and suppose that $\lambda_n \to \lambda$. Then $\{u_{\lambda_n}^*\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$ is bounded (see (73)). In addition, using [15, Theorem 2], we can find $\alpha \in (0,1)$ and $M_5 > 0$ such that

\[
u_{\lambda_n}^* \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|u_{\lambda_n}^*\|_{C^{1,\alpha}(\overline{\Omega})} \leq M_5 \quad \text{for all } n \in \mathbb{N}. \tag{74}
\]

Exploiting the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$, from (74) and (73), we have

\[
u_{\lambda_n}^* \to \bar{u}_\lambda \text{ in } C^1(\overline{\Omega}) \text{ as } n \to \infty. \tag{75}
\]

Since $\bar{u}_{\lambda_1} \leq u_{\lambda_n}^*$ for all $n \in \mathbb{N}$ (with $\bar{u}_{\lambda_1} \in \text{int} C_+$, see the proof of Proposition 7), we have $\bar{u}_\lambda \in S(\lambda) \subseteq \text{int} C_+$. We claim that $\bar{u}_\lambda = u_\lambda^*$. Indeed, if this is not true, then we can find $z_0 \in \Omega$ such that

\[
u_\lambda^*(z_0) < \bar{u}_\lambda(z_0),
\]

and so

\[
u_\lambda^*(z_0) < \nu_{\lambda_n}^*(z_0) \quad \text{for all } n \geq n_0, \text{ see (75)}.
\]

But this contradicts (73). Therefore $\bar{u}_\lambda = u_\lambda^*$ and this proves the proposition. \qed

Summarizing the situation for the positive solutions of problem $(P_\lambda), \lambda > 0$, we have the following multiplicity theorem (bifurcation - type result).

**Theorem 3.1.** If Hypotheses $H(\xi), H(\beta), H(f)_3$ hold, then there exists $\lambda_* > 0$ such that

1. for every $\lambda > \lambda_*$ problem $(P_\lambda)$ has at least two positive solutions $u_\lambda, \bar{u}_\lambda \in \text{int} C_+$;
2. for $\lambda = \lambda_*$, problem $(P_\lambda)$ has at least one positive solution $u_* \in \text{int} C_+$;
3. for every $\lambda \in (0, \lambda_*)$ problem $(P_\lambda)$ has no positive solution;
4. for every $\lambda \in [\lambda_*, +\infty)$, problem $(P_\lambda)$ has a smallest positive solution $u_\lambda^* \in \text{int} C_+$ and the map $\lambda \mapsto u_\lambda^*$ from $[\lambda_*, +\infty)$ into $C^1(\overline{\Omega})$ is right continuous and strictly decreasing in the following sense: if $\lambda < \theta$, then $u_\lambda - u_\theta \in \text{int} \hat{C}_+$, if $H(\beta)(a)$ holds, and $u_\lambda^* - u_\theta^* \in \text{int} C_+$, if $H(\beta)(b)$ is in force.
4. Nodal solutions. We consider the following bilateral version of problem $(P_{\lambda})$:

\[
(P_{\lambda})' \begin{cases}
\Delta_p u(z) + (\xi(z) + \lambda)|u(z)|^{p-2}u(z) = f(z, u(z)) & \text{in } \Omega, \\
\frac{\partial u}{\partial n} + \beta(z)|u(z)|^{p-2}u(z) = 0 & \text{on } \partial\Omega, \lambda > 0.
\end{cases}
\]

Imposing bilateral conditions on the reaction term $f(z, x)$ (that is, conditions valid on the whole real axis), we can produce nodal (that is, sign changing) solutions for problem $(P_{\lambda})'$ when $\lambda > 0$ is big.

The conditions on $f(z, x)$ are the following:

$H(f)_4$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

(i) $|f(z, x)| \leq a(z)(1 + |x|^{r-1})$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p < r < p^*$;

(ii) if $f(z, x) = \int_0^x f(z, s)ds$, then $\lim_{x \to \pm\infty} \frac{F(z, x)}{|x|^p} = +\infty$ uniformly for a.a. $z \in \Omega$;

(iii) there exists $\mu \in \left(\max\{(r-p)\frac{N}{p}, 1\}, p^*\right)$ such that

\[0 < \gamma_0 \leq \lim_{x \to \pm\infty} \frac{f(z, x) x - pF(z, x)}{|x|^p}\] uniformly for a.a. $z \in \Omega$;

(iv) there exist $\delta > 0$ and $q \in (1, p)$ such that

\[c_0 |x|^q \leq f(z, x) x \leq qF(z, x)\] for a.a. $z \in \Omega$ and for all $|x| \leq \delta$;

(v) there exists $D \subseteq \Omega$ with $|D|_N > 0$ such that

\[\lambda_1(\beta)|x|^p \leq f(z, x) x \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}\]

\[\lambda_1(\beta)|x|^p < f(z, x) x \text{ for a.a. } z \in D, \text{ all } x \in \mathbb{R} \setminus \{0\};\]

(vi) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $z \in \Omega$ the map $x \mapsto f(z, x) + \xi_\rho |x|^{p-2}x$ is nondecreasing on $[-\rho, \rho]$.

From Theorem 3.1, we know that there exists $\lambda_+^+ > 0$ such that for every $\lambda \in [\lambda_+^+, \infty)$, problem $(P_{\lambda})'$ has a smallest positive solution $u_{\lambda}^+ \in \text{int } C_+$.

Since the conditions in Hypotheses $H(f)_4$ are all bilateral, Theorem 3.1 applies also on the problem restricted on the negative semiaxis $\mathbb{R}_- = (-\infty, 0]$ ans so, we can find $\lambda_-^- > 0$ such that

for every $\lambda \in [\lambda_-^-, \infty)$, problem $(P_{\lambda})'$

has a biggest negative solution $v_{\lambda}^- \in -\text{int } C_+$.

We set

\[\hat{\lambda}_* = \max\{\lambda_+^+, \lambda_-^-\}.
\]

As before for every $\lambda > 0$, $\varphi_{\lambda} : W^{1,p}(\Omega) \to \mathbb{R}$ is the energy functional for problem $(P_{\lambda})'$ defined by

\[\varphi_{\lambda}(u) = \frac{1}{p}||Du||_p^p + \frac{1}{p} \int_{\Omega} (\xi(z) + \lambda)|u|^p dz + \frac{1}{p} \int_{\partial\Omega} \beta(z)|u|^p d\sigma - \int_{\Omega} F(z, u) dz\]

for all $u \in W^{1,p}(\Omega)$, and $\varphi_{\lambda} \in C^1(W^{1,p}(\Omega))$.

**Proposition 12.** If Hypotheses $H(\xi), H(\beta), H(f)_4$ hold, $\lambda > 0$ and $0 \in K_{\varphi_{\lambda}}$ is isolated, then $C_k(\varphi_{\lambda}, 0) = 0$ for all $k \in \mathbb{N}_0$. 
Proof. Let \( \psi_\lambda : W^{1,p}(\Omega) \to \mathbb{R} \) be the \( C^1 \)-functional defined by
\[
\psi_\lambda(u) = \frac{1}{p} \| Du \|_p^p + \frac{1}{p} \int_\Omega (\xi(z) + \lambda + \mu)|u|^p dz + \frac{1}{p} \int_{\partial \Omega} \beta(z)|u|^p d\sigma - \int_\Omega F(z,u)dz
\]
for all \( u \in W^{1,p}(\Omega) \) with \( \mu > \| \xi \|_\infty \).
We have
\[
|\varphi'(u) - \psi'(u)| = \frac{\mu}{p} \| u \|_p^p \quad \text{for all } u \in W^{1,p}(\Omega),
\]
\[
\| \varphi''(u) - \psi''(u) \|_\infty \leq \mu \| u \|_p^{p-1} \quad \text{for all } u \in W^{1,p}(\Omega).
\]
(76)
Note that \( K_\psi \subseteq C^1(\overline{\Omega}) \) and Hypothesis \( H(f)_4(iv) \) implies that \( u = 0 \) is an isolated critical point of \( \psi_\lambda \). From Corvellec - Hantoute [7, Theorem 5.1], we have
\[
C_k(\varphi_k,0) = C_k(\psi_\lambda,0) \quad \text{for all } k \in N_0.
\]
(77)
Hypotheses \( H(f)_4(i),(iv) \) imply that we can find \( c_{10} > 0 \) such that
\[
F(z,x) \geq -\frac{c_0}{q} \langle x \rangle - c_{10} |x|^r \quad \text{and} \quad qF(z,x) - f(z,x)x \geq -c_{10} |x|^r
\]
(78)
for a.a. \( z \in \Omega \), all \( x \in \mathbb{R} \).

Let \( u \in W^{1,p}(\Omega) \) and \( t \in (0,1) \). Using (78) we have
\[
\psi_\lambda(tu) \leq \frac{t^p}{p} \| Du \|_p^p + \frac{t^p}{p} \| (\xi(z) + \lambda + \mu)|u|^p \|_p^p + \frac{t^p}{p} \| \beta \|_L^\infty(\partial \Omega) \| u \|_p^{p(\lambda)}
\]
\[
+ c_{10}t^r \| u \|_r^r - \frac{t^p}{q} c_0 \| u \|_q^q.
\]
Since \( q < p < r \), we see that we can find \( t^* = t^*(u) \in (0,1) \) small enough and such that
\[
\psi_\lambda(tu) < 0 \quad \text{for all } t \in (0,t^*).
\]
(79)
Consider \( u \in W^{1,p}(\Omega) \) with \( 0 < \| u \| \leq 1 \) and \( \psi_\lambda(u) = 0 \). Then
\[
\frac{d}{dt} \psi_\lambda(tu) \big|_{t=1} = \langle \psi_\lambda'(u), u \rangle \quad \text{(by the chain rule)}
\]
\[
= \| Du \|_p^p + \int_\Omega (\xi(z) + \lambda + \mu)|u|^p dz + \int_{\partial \Omega} \beta(z)|u|^p d\sigma - \int_\Omega f(z,u)udz
\]
\[
= \left(1 - \frac{q}{p}\right) \| Du \|_p^p + \left(1 - \frac{q}{p}\right) \int_\Omega (\xi(z) + \lambda + \mu)|u|^p dz
\]
\[
+ \left(1 - \frac{q}{p}\right) \int_{\partial \Omega} \beta(z)|u|^p d\sigma + \int_\Omega [qF(z,u) - f(z,u)u]dz \quad \text{(since } \psi_\lambda(u) = 0)\]
\[
\geq c_{11} \| u \|_p^p - c_{12} \| u \|_r^r \quad \text{for some } c_{11}, c_{12} > 0
\]
(80)
(see (78), Hypothesis \( H(\beta) \) and recall \( q < p, \mu > \| \xi \|_\infty \)).

Since \( r > p \), from (80) we see that for \( \rho \in (0,1) \) small, we have
\[
\frac{d}{dt} \psi_\lambda(tu) \big|_{t=1} > 0 \quad \text{for all } u \in W^{1,p}(\Omega), \text{ with } 0 < \| u \| \leq \rho, \ \psi_\lambda(u) = 0.
\]
(81)
Consider such a \( u \in W^{1,p}(\Omega) \), that is \( 0 < \| u \| \leq \rho \) and \( \psi_\lambda(u) = 0 \). We will show that
\[
\psi_\lambda(tu) \leq 0 \quad \text{for all } t \in [0,1].
\]
(82)
Suppose that (82) is not true. Then we can find \( t_0 \in (0, 1) \) such that \( \psi_\lambda(t_0u) > 0 \). Since by hypothesis \( \psi_\lambda(u) = 0 \), we can find \( t_1 \in (t_0, 1) \) such that \( \psi_\lambda(t_1u) = 0 \). Let
\[
t_* := \min \left\{ t \in (t_0, 1) : \psi_\lambda(tu) = 0 \right\} > t_0 > 0.
\]
We have
\[
\psi_\lambda(tu) > 0 \quad \text{for all } t \in [t_0, t_*]. \tag{83}
\]
Let \( v = t_*u \). We have \( 0 < \|v\| \leq \|u\| \leq \rho \) and \( \psi_\lambda(v) = 0 \). Then from (81) we have
\[
\frac{d}{dt}\psi_\lambda(tu)|_{t=1} > 0. \tag{84}
\]
On the other hand, note that
\[
\frac{d}{dt}\psi_\lambda(tv)|_{t=1} = t_* \frac{d}{dt}\psi_\lambda(tu)|_{t=t_*} = t_* \lim_{t \to t_*} \frac{\psi_\lambda(tu)}{t-t_*} \leq 0 \quad \text{(see (83))}. \tag{85}
\]
Comparing (84), (85), we reach a contradiction. This proves (82).

Now, we can choose \( \rho > 0 \) even smaller if necessary in order to have \( K_{\psi_\lambda} \cap \bar{B}_\rho = \{0\} \) (since \( 0 \in K_{\psi_\lambda} \) is isolated). Then, consider the deformation
\[
h : [0, 1] \times (\psi_\lambda^0 \cap \bar{B}_\rho) \to \psi_\lambda^0 \cap \bar{B}_\rho \quad (\bar{B}_\rho = \{u \in W^{1,p}(\Omega) : \|u\| \leq \rho\})
\]
defined by
\[
h(t, u) = (1-t)u \quad \text{for all } (t, u) \in [0, 1] \times (\psi_\lambda^0 \cap \bar{B}_\rho);
\]
from (82) we see that this deformation is well-defined. So, we see that \( \psi_\lambda^0 \cap \bar{B}_\rho \) is contractible.

Now, let \( u \in \bar{B}_\rho \) be such that \( \psi_\lambda(u) > 0 \). We show that there exists a unique \( t(u) \in (0, 1) \) such that
\[
\psi_\lambda(t(u)u) = 0. \tag{86}
\]
The function \( t \to \psi_\lambda(tu) \) is continuous and by hypothesis \( \psi_\lambda(u) > 0 \). Combining these facts with (79) and using Bolzano’s Theorem, we see that we can find \( t(u) \in (0, 1) \) such that (86) hold. We need to show the uniqueness of this \( t(u) \in (0, 1) \). Arguing by contradiction, suppose we can find \( 0 < \hat{t}_1 = t(u)1 < \hat{t}_2 = t(u)2 < 1 \) both satisfying (86). From (82) we have
\[
\theta_\lambda(t) = \psi_\lambda(t\hat{t}_2u) \leq 0 \quad \text{for all } t \in [0, 1].
\]
Note that \( \frac{\hat{t}_1}{\hat{t}_2} \in (0, 1) \) is a maximizer of \( \theta_\lambda(\cdot) \). Hence
\[
\frac{d}{dt}\theta_\lambda(t)|_{t=\frac{\hat{t}_1}{\hat{t}_2}} = 0,
\]
that is
\[
\left[ \frac{\hat{t}_1}{\hat{t}_2} \frac{d}{dt}\psi_\lambda(t\hat{t}_2u)|_{t=\frac{\hat{t}_1}{\hat{t}_2}} \right]_{t=\frac{\hat{t}_1}{\hat{t}_2}} = \frac{d}{dt}\psi_\lambda(t\hat{t}_1u)|_{t=1} = 0,
\]
which contradicts (81). This proves the uniqueness of \( t(u) \in (0, 1) \) satisfying (86), and so we have
\[
\psi_\lambda(tu) < 0 \quad \text{if } t \in (0, t(u)), \ \text{see (82)},
\]
\[
\psi_\lambda(tu) > 0 \quad \text{if } t \in (t(u), 1] \ (\text{recall that } \psi_\lambda(u) > 0 \text{ and see (86)}).
\]

At this point, we consider the function \( \gamma : \bar{B}_\rho \setminus \{0\} \to (0, 1] \) defined by
\[
\gamma(u) = \begin{cases} 
1, & \text{if } u \in \bar{B}_\rho \setminus \{0\}, \ \psi_\lambda(u) \leq 0, \\
t(u), & \text{if } u \in \bar{B}_\rho \setminus \{0\}, \ \psi_\lambda(u) > 0.
\end{cases} \tag{87}
\]
It is easy to see that $\gamma(\cdot)$ is continuous. Then we introduce the map $\zeta : \bar{B}_p \setminus \{0\} \to (\psi_0 \cap \bar{B}_p) \setminus \{0\}$ defined by
\[ \zeta(u) = \gamma(u)u. \] (88)
Using the continuity of $\gamma(\cdot)$, we have the continuity of $\zeta(\cdot)$.
Moreover, we see that
\[ \zeta|_{(\psi_0 \cap \bar{B}_p) \setminus \{0\}} = Id|_{(\psi_0 \cap \bar{B}_p) \setminus \{0\}} \quad \text{(see (88))}, \]
so that
\[ (\psi_0 \cap \bar{B}_p) \setminus \{0\} \text{ is a retract of } \bar{B}_p \setminus \{0\}. \] (89)
Note that $\bar{B}_p \setminus \{0\}$ is contractible. Then, by (89), also $(\psi_0 \cap \bar{B}_p) \setminus \{0\}$ is contractible, as well. Hence
\[ H_k((\psi_0 \cap \bar{B}_p), (\psi_0 \cap \bar{B}_p) \setminus \{0\}) = 0 \]
for all $k \in \mathbb{N}_0$ (see Motreanu - Motreanu - Papageorgious [18, Propositions 6.24 and 6.25]). Thus
\[ C_k(\psi_0, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0, \]
and so, by (77),
\[ C_k(\varphi_0, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \]

**Remark 8.** The first computation of critical groups of functionals exhibiting a concave nonlinearity near the origin was conducted by Moroz (20) for functionals defined on $H^1_0(\Omega)$ with $\xi \equiv 0$ and a reaction term satisfying more restrictive condition. We stress that in the “Dirichlet case”, the arguments simplify due to the validity of Poincaré inequality.

Now, fix $\lambda \in [\bar{\lambda}_*, +\infty)$ and let $u_0^* \in \text{int} \ C_+$, $v_0^* \in -\text{int} \ C_+$ be the extremal constant sign solutions of problem $(P_\lambda)'$. We introduce the following Carathéodory functions (as before $\mu > \|\xi\|_\infty$):
\[ \tilde{g}(z, x) = \begin{cases} f(z, v_0^*(z)) + \mu|v_0^*(z)|^{p-2}v_0^*(z) & \text{if } x < v_0^*(z) \\ f(z, x) + \mu|x|^{p-2}x & \text{if } v_0^*(z) \leq x \leq u_0^*(z) \\ f(z, u_0^*(z)) + \mu u_0^*(z)^{p-1} & \text{if } u_0^*(z) < x, \end{cases} \] (90)
for every $(z, x) \in \Omega \times \mathbb{R}$, and
\[ \tilde{\beta}(z, x) = \begin{cases} \beta(z)|v_0^*(z)|^{p-2}v_0^*(z) & \text{if } x < v_0^*(z) \\ \beta(z)|x|^{p-2}x & \text{if } v_0^*(z) \leq x \leq u_0^*(z) \\ \beta(z)u_0^*(z)^{p-1} & \text{if } u_0^*(z) < x, \end{cases} \] (91)
for all $(z, x) \in \partial \Omega \times \mathbb{R}$. We set
\[ \tilde{G}(z, x) = \int_0^x \tilde{g}(z, s)ds \quad \text{and} \quad \tilde{B}(z, x) = \int_0^x \tilde{\beta}(z, s)ds \]
and consider the $C^1-$ functional $\tilde{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by
\[ \tilde{\varphi}_\lambda(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{p} \int_\Omega (\xi(z) + \lambda + \mu)|u|^p dz + \int_{\partial \Omega} \tilde{B}(z, u)d\sigma - \int_\Omega \tilde{G}(z, u)dz \]
for all $u \in W^{1,p}(\Omega)$.

**Proposition 13.** If Hypotheses $H(\xi)$, $H(\beta)$, $H(f)$ hold, $\lambda \in [\bar{\lambda}_*, +\infty)$ and $0 \in K_{\varphi_\lambda}$ is isolated, then $C_k(\tilde{\varphi}_0, 0) = C_k(\varphi_0, 0)$ for all $k \in \mathbb{N}_0$. 
Proof. Consider the homotopy $h_{\lambda}(t, u)$ defined by
\[
h_{\lambda}(t, u) = t\hat{\varphi}_\lambda(u) + (1 - t)\varphi_{\lambda}(u) \quad \text{for all } (t, u) \in [0, 1] \times W^{1,p}(\Omega).
\]
Suppose we can find $\{n\}_{n \geq 1} \subseteq [0, 1]$ and $\{u_n\}_{n \geq 1} \subseteq W^{1,p}(\Omega)$ such that
\[
t_n \to t, \ u_n \to 0 \quad \text{in } W^{1,p}(\Omega) \quad \text{and } \ (h_{\lambda})'_{u}(t_n, u_n) = 0 \quad \text{for all } n \in \mathbb{N}.
\] (92)

From (90), we have
\[
\langle A(u_n), v \rangle + \int_{\Omega}(\xi(z) + \lambda)|u_n|^{p-2}u_nvdz + t_n\mu \int_{\Omega}|u_n|^{p-2}u_nvdz
\]
\[
+ (1 - t_n)\int_{\partial\Omega}\beta(z)|u_n|^{p-2}u_nvd\sigma + t_n\int_{\partial\Omega}\tilde{\beta}(z, u_n)vvd\sigma
\]
\[
= t_n\int_{\Omega}g(z, u_n)vzdz + (1 - t_n)\int_{\Omega}f(z, u_n)vzdz
\]
for all $v \in W^{1,p}(\Omega)$ and all $n \in \mathbb{N}$. This implies
\[
-\Delta_pu_n(z) + (\xi(z) + \lambda + t_n\mu)|u_n(z)|^{p-2}u_n(z) = t_n\tilde{g}(z, u_n(z)) + (1 - t_n)f(z, u_n(z))
\]
for a.a. $z \in \Omega$ and
\[
\frac{\partial u_n}{\partial n} + t_n\tilde{\beta}(z, u_n) + (1 - t_n)\beta(z)|u_n|^{p-2}u_n = 0 \quad \text{on } \partial\Omega.
\]

We know (see [25], [30]) that there exists $M_6 > 0$ such that
\[
\|u_n\|_{\infty} \leq M_6 \quad \text{for all } n \in \mathbb{N}.
\]

Using Lieberman [15, Theorem 2], we can find $\eta \in (0, 1)$ and $M_7 > 0$ such that
\[
u_n \in C^{1,\eta}(\Omega) \quad \text{and } \|u_n\|_{C^{1,\eta}(\Omega)} \leq M_7 \quad \text{for all } n \in \mathbb{N}.
\] (93)

From (92), (93) and the compact embedding of $C^{1,\eta}(\Omega)$ into $C^{1}(\Omega)$, we have that
\[
u_n \to 0 \quad \text{in } C^{1}(\Omega),
\]
and so
\[
u_n \in [v_{\lambda}^\ast, u_{\lambda}^\ast] \quad \text{for all } n \geq n_0,
\]
which means that
\[
\{u_n\}_{n \geq n_0} \quad \text{are all distinct nontrivial solutions of } (P_\lambda)',
\]
see (90) and (91). This contradicts our hypothesis that $0 \in K_{\varphi_{\lambda}}$ is isolated.

Thus, (92) can not happen and we can use the homotopy invariance of critical groups (see Corvellec - Hantoute [7, Theorem 5.2]) and conclude that
\[
C_k(\varphi_{\lambda}, 0) = C_k(\hat{\varphi}_{\lambda}, 0) \quad \text{for all } k \in \mathbb{N}_0.
\]

\[
\square
\]

As a straightforward consequence of Propositions 12 and 13, we have the following

Corollary 1. If Hypotheses $H(\xi)$, $H(\beta)$ and $H(f)$ hold, $\lambda \in [\hat{\lambda}_s, +\infty)$ and $0 \in K_{\varphi_{\lambda}}$ is isolated, then $C_k(\hat{\varphi}_{\lambda}, 0) = 0$ for all $k \in \mathbb{N}_0$.

Now we are ready to produce nodal solutions. For this purpose, we introduce the positive and negative truncations of $\tilde{g}(z, \cdot)$ and $\tilde{\beta}(z, \cdot)$, that is,
\[
\tilde{g}_\pm(z, x) = \tilde{g}(z, \pm x) \quad \text{and } \tilde{\beta}_\pm(z, x) = \tilde{\beta}(z, \pm x^\pm).
\]
These are Carathéodory functions. We set
\[ \tilde{G}_\pm(z,x) = \int_0^x \tilde{g}_\pm(z,s)ds \quad \text{and} \quad \tilde{B}_\pm(z,x) = \int_0^x \tilde{b}_\pm(z,s)ds. \]

We introduce the \( C^1 \) functionals \( \tilde{\phi}_\lambda^\pm : W^{1,p}(\Omega) \to \mathbb{R} \) defined by
\[
\tilde{\phi}_\lambda^\pm(u) = \frac{1}{p} \| Du \|_p^p + \frac{1}{p} \int_I (\xi(z) + \lambda + \mu) |u|^p dz + \int_{\partial\Omega} \tilde{B}_\pm(z,u)d\sigma - \int_\Omega \tilde{G}_\pm(z,u)dz
\]
for all \( u \in W^{1,p}(\Omega) \).

**Proposition 14.** If Hypotheses \( H(\xi) \), \( H(\beta) \), \( H(f)_4 \) hold and \( \lambda \in [\lambda_*, +\infty) \), then problem \( (P_\lambda) \) has a nodal solution \( y_\lambda \in [v_\lambda^*, u_\lambda^*] \cap C^1(\Omega) \).

**Proof.** As before (see the proof of Proposition 8), we can show that
\[ K_{\tilde{\phi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*], \quad K_{\tilde{\phi}_\lambda^+} \subseteq [0, u_\lambda^*], \quad K_{\tilde{\phi}_\lambda^-} \subseteq [v_\lambda^*, 0]. \]
The extremality of \( u_\lambda^* \) and \( v_\lambda^* \), implies that
\[ K_{\tilde{\phi}_\lambda} \subseteq [v_\lambda^*, u_\lambda^*], \quad K_{\tilde{\phi}_\lambda^+} = \{0, u_\lambda^*\}, \quad K_{\tilde{\phi}_\lambda^-} = \{v_\lambda^*, 0\}. \]  
(94)

From (90) and (91) it is clear that \( \tilde{\phi}_\lambda^+ \) is coercive. Also, it is sequentially weakly lower semicontinuous. So we can find \( u_\lambda^* \in W^{1,p}(\Omega) \) such that
\[ \tilde{\phi}_\lambda^+ (\tilde{u}_\lambda^*) = \inf \{ \tilde{\phi}_\lambda^+ (u) : u \in W^{1,p}(\Omega) \}. \]  
(95)

Using Hypothesis \( H(f)_4(iv) \) and the fact that \( q < p \), we have
\[ \tilde{\phi}_\lambda^+ (\tilde{u}_\lambda^*) < 0 = \tilde{\phi}_\lambda^+ (0) \quad \text{(see the proof of Proposition 5)}, \]
and so
\[ \tilde{u}_\lambda^* = u_\lambda^* \in \text{int} \, C_+, \quad \text{see (94), (95)}. \]

Note that \( \tilde{\phi}_\lambda|_{\text{C}_+} = \tilde{\phi}_\lambda^+|_{\text{C}_+} \). So, it follows that \( u_\lambda^* \in \text{int} \, C_+ \) is a local \( C^1(\bar{\Omega}) \)-minimizer of \( \tilde{\phi}_\lambda \), hence by Proposition 2 it is also a local \( W^{1,p}(\Omega) \)-minimizer of \( \tilde{\phi}_\lambda \). Similarly for \( v_\lambda^* \in \text{int} \, C_- \), using the functional \( \tilde{\phi}_\lambda^- \). Without any loss of generality we assume that
\[ \tilde{\phi}_\lambda(v_\lambda^*) \leq \tilde{\phi}_\lambda(u_\lambda^*), \]
the reasoning being the same if the opposite inequality is true.

We assume that \( K_{\tilde{\phi}_\lambda} \) is finite or otherwise we already have a sequence of distinct nodal solutions for problem \( (P_\lambda)' \) (see (94) and recall that \( u_\lambda^* \in \text{int} \, C_+ \) and \( v_\lambda^* \in \text{int} \, C_- \) are extremal constant sign solutions). Since \( u_\lambda^* \in \text{int} \, C_+ \) is an isolated local minimizer of \( \tilde{\phi}_\lambda \), we can find \( \rho \in (0,1) \) small such that
\[ \tilde{\phi}_\lambda(v_\lambda^*) \leq \tilde{\phi}_\lambda(u_\lambda^*) < \inf \left\{ \tilde{\phi}_\lambda(u) : \| u - u_\lambda^* \| = \rho \right\} = m_\lambda, \quad \| v_\lambda^* - u_\lambda^* \| > \rho \]  
(96)

(see Aizicovizi - Papageorgiou - Staicu [1, proof of Proposition 29]). Since \( \tilde{\phi}_\lambda \) is coercive (see (90), (91)) it satisfies the \( C^- \) condition (see, for example, Papageorgiou - Winkert [26]). This fact and (96) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find \( y_\lambda \in W^{1,p}(\Omega) \) such that
\[ y_\lambda \in K_{\tilde{\phi}_\lambda} \quad \text{and} \quad m_\lambda \leq \tilde{\phi}_\lambda(y_\lambda). \]  
(97)

From (94), (96), (97) we have
\[ y_\lambda \in [v_\lambda^*, u_\lambda^*] \setminus \{v_\lambda^*, u_\lambda^*\}. \]

Since \( y_\lambda \) is a critical point of \( \tilde{\phi}_\lambda \) of mountain pass - type, we have
\[ C_1(\tilde{\phi}_\lambda, y_\lambda) \neq 0, \]  
(98)
see Motreanu - Motreanu - Papageorgiou [18, Proposition 6.100].

From Corollary 1, we have
\[ C_k(\tilde{\phi}, 0) = 0 \quad \text{for all } k \in \mathbb{N}_0. \]  \hfill (99)

Comparing (98), (99), we see that \( y_{\lambda} \neq 0 \), an so \( y_{\lambda} \) must be nodal.

The nonlinear regularity theory implies that \( y_{\lambda} \in [v_{\lambda}^+, u_{\lambda}^-] \bigcap C^1(\bar{\Omega}). \)

So, summarizing the situation for problem \((P_{\lambda})'\), we can formulate the following multiplicity theorem:

**Theorem 4.1.** If Hypotheses \( H(\xi), H(\beta), H(f)_4 \) hold, then there exists \( \hat{\lambda}_* > 0 \) such that

1. for all \( \lambda \in (\hat{\lambda}_*, +\infty) \) problem \((P_{\lambda})'\) has at least five nontrivial solutions\n\[ u_{\lambda}, \bar{u}_{\lambda} \in \text{int } C_+, v_{\lambda}, \bar{v}_{\lambda} \in -\text{int } C_+, y_{\lambda} \in C^1(\bar{\Omega}) \text{ nodal}; \]
2. for \( \lambda = \hat{\lambda}_* \) problem has at least three nontrivial solutions\n\[ u_{\lambda} \in \text{int } C_+, v_{\lambda} \in -\text{int } C_+, y_{\lambda} \in [v_{\lambda}^+, u_{\lambda}^-] \bigcap C^1(\bar{\Omega}) \text{ nodal}. \]

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