Abstract. Let $f$ be an operator monotonic function on $I$ and $A, B \in \mathcal{S}A_I(H)$, the class of all selfadjoint operators with spectra in $I$. Assume that $p : [0, 1] \to \mathbb{R}$ is non-decreasing on $[0, 1]$. In this paper we obtained, among others, that for $A \leq B$ and $f$ an operator monotonic function on $I$,

$$0 \leq \int_0^1 p(t) f((1-t)A + tB)dt - \int_0^1 p(t) dt \int_0^1 f((1-t)A + tB)dt \leq \frac{1}{4} [p(1) - p(0)] [f(B) - f(A)]$$

in the operator order.

Several other similar inequalities for either $p$ or $f$ is differentiable, are also provided. Applications for power function and logarithm are given as well.

1. Introduction

Consider a complex Hilbert space $(H, \langle \cdot , \cdot \rangle)$. An operator $T$ is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x \rangle \geq 0$ for all $x \in H$ and also an operator $T$ is said to be strictly positive (denoted by $T > 0$) if $T$ is positive and invertible. A real valued continuous function $f(t)$ on $(0, \infty)$ is said to be operator monotone if $f(A) \geq f(B)$ holds for any $A \geq B > 0$.

In 1934, K. Löwner [7] had given a definitive characterization of operator monotone functions as follows:

**Theorem 1.** A function $f : (0, \infty) \to \mathbb{R}$ is operator monotone in $(0, \infty)$ if and only if it has the representation

$$f(t) = a + bt + \int_0^\infty \frac{t}{t+s} dm(s)$$

where $a \in \mathbb{R}$ and $b \geq 0$ and a positive measure $m$ on $(0, \infty)$ such that

$$\int_0^\infty \frac{dm(s)}{t+s} < \infty.$$

We recall the important fact proved by Löwner and Heinz that states that the power function $f : (0, \infty) \to \mathbb{R}$,

$$f(t) = t^\alpha$$

is an operator monotone function for any $\alpha \in [0, 1]$.
In [3], T. Furuta observed that for \( \alpha_j \in [0,1], j = 1, ..., n \) the functions
\[
g(t) := \left( \sum_{j=1}^{n} t^{-\alpha_j} \right)^{-1} \quad \text{and} \quad h(t) = \sum_{j=1}^{n} (1 + t^{-1})^{-\alpha_j}
\]
are operator monotone in \((0, \infty)\).

Let \( f(t) \) be a continuous function \((0, \infty) \to (0, \infty)\). It is known that \( f(t) \) is operator monotone if and only if \( g(t) = t/f(t) =: f^*(t) \) is also operator monotone, see for instance [3] or [8].

Consider the family of functions defined on \((0, \infty)\) and \( p \in [-1, 2] \setminus \{0, 1\} \) by
\[
f_p(t) := \frac{p-1}{p} \left( \frac{t^p - 1}{t^{p-1} - 1} \right)
\]
and
\[
f_0(t) := \frac{t}{1-t} \ln t,
\]
\[
f_1(t) := \frac{t - 1}{\ln t} \quad \text{(logarithmic mean)}.
\]

We also have the functions of interest
\[
f_{-1}(t) = \frac{2t}{1+t} \quad \text{(harmonic mean)}, \quad f_{1/2}(t) = \sqrt{t} \quad \text{(geometric mean)}.
\]

In [2] the authors showed that \( f_p \) is operator monotone for \( 1 \leq p \leq 2 \).

In the same category, we observe that the function
\[
g_p(t) := \frac{t - 1}{tp - 1}
\]
is an operator monotone function for \( p \in (0, 1], [3] \).

It is well known that the logarithmic function \( \ln \) is operator monotone and in [3] the author obtained that the functions
\[
f(t) = t(1+t) \ln \left( 1 + \frac{1}{t} \right), \quad g(t) = \frac{1}{(1+t) \ln (1 + \frac{1}{t})}
\]
are also operator monotone functions on \((0, \infty)\).

Let \( f \) be an operator monotone function on \( I \) and \( A, B \in \mathcal{SA}_I(H) \), the class of all selfadjoint operators with spectra in \( I \). Assume that \( p : [0,1] \to \mathbb{R} \) is non-decreasing on \([0,1]\). In this paper we obtain, among others, that for \( A \leq B \) and \( f \) an operator monotonic function on \( I \),
\[
0 \leq \int_{0}^{1} p(t) f ((1-t) A + tB) dt - \int_{0}^{1} p(t) dt \int_{0}^{1} f (((1-t) A + tB) dt \\
\leq \frac{1}{4} [p(1) - p(0)] [f(B) - f(A)]
\]
in the operator order.

Several other similar inequalities for either \( p \) or \( f \) is differentiable, are also provided. Applications for power function and logarithm are given as well.
2. Main Results

For two Lebesgue integrable functions \( h, g : [a, b] \to \mathbb{R} \), consider the Čebyšev functional:

\[
C(h, g) := \frac{1}{b-a} \int_a^b h(t)g(t)dt - \frac{1}{b-a} \int_a^b h(t)dt \frac{1}{b-a} \int_a^b g(t)dt.
\]

(2.1)

It is well known that, if \( h \) and \( g \) have the same monotonicity on \([a, b] \), then

\[
\frac{1}{b-a} \int_a^b h(t)g(t)dt \geq \frac{1}{b-a} \int_a^b h(t)dt \frac{1}{b-a} \int_a^b g(t)dt,
\]

(2.2)

which is known in the literature as Čebyšev’s inequality.

In 1935, Grüss \([4]\) showed that

\[
|C(h, g)| \leq \frac{1}{4} (M - m) (N - n),
\]

(2.3)

provided that there exists the real numbers \( m, M, n, N \) such that

\[
m \leq h(t) \leq M \quad \text{and} \quad n \leq g(t) \leq N \quad \text{for a.e. } t \in [a, b].
\]

(2.4)

The constant \( \frac{1}{4} \) is best possible in (2.1) in the sense that it cannot be replaced by a smaller quantity.

Let \( f \) be a continuous function on \( I \). If \((A, B) \in \mathcal{SA}_I(H), \) the class of all selfadjoint operators with spectra in \( I \) and \( t \in [0, 1], \) then the convex combination \((1 - t)A + tB\) is a selfadjoint operator with the spectrum in \( I \) showing that \( \mathcal{SA}_I(H) \) is a convex set in the Banach algebra \( \mathcal{B}(H) \) of all bounded linear operators on \( H. \)

By the continuous functional calculus of selfadjoint operator we also conclude that \( f((1 - t)A + tB) \) is a selfadjoint operator in \( \mathcal{B}(H). \)

For \( A, B \in \mathcal{SA}_I(H), \) we consider the auxiliary function \( \varphi_{(A, B)} : [0, 1] \to \mathcal{B}(H) \) defined by

\[
\varphi_{(A, B)}(t) := f((1 - t)A + tB).
\]

(2.5)

For \( x \in H \) we can also consider the auxiliary function \( \varphi_{(A, B), x} : [0, 1] \to \mathbb{R} \) defined by

\[
\varphi_{(A, B), x}(t) := \left\langle \varphi_{(A, B)}(t)x, x \right\rangle = \langle f((1 - t)A + tB)x, x \rangle.
\]

(2.6)

**Theorem 2.** Let \( A, B \in \mathcal{SA}_I(H) \) with \( A \leq B \) and \( f \) an operator monotonic function on \( I. \) If \( p : [0, 1] \to \mathbb{R} \) is monotonic nondecreasing on \([0, 1], \) then

\[
0 \leq \int_0^1 p(t)f((1 - t)A + tB)dt - \int_0^1 p(t)dt\int_0^1 f((1 - t)A + tB)dt
\]

\[
\leq \frac{1}{4}[p(1) - p(0)] [f(B) - f(A)].
\]

If \( p : [0, 1] \to \mathbb{R} \) is monotonic nonincreasing on \([0, 1], \) then

\[
0 \leq \int_0^1 p(t)dt\int_0^1 f((1 - t)A + tB)dt - \int_0^1 p(t)f((1 - t)A + tB)dt
\]

\[
\leq \frac{1}{4}[p(0) - p(1)] [f(B) - f(A)].
\]

(2.7)
Proof. Let $0 \leq t_1 < t_2 \leq 1$ and $A \leq B$. Then
\[(1 - t_2) A + t_2 B - (1 - t_1) A - t_1 B = (t_2 - t_1) (B - A) \geq 0\]
and by operator monotonicity of $f$ we get
\[f ((1 - t_2) A + t_2 B) \geq f ((1 - t_1) A + t_1 B),\]
which is equivalent to
\[\varphi_{(A,B):x} (t_2) = \langle f ((1 - t_2) A + t_2 B) x, x \rangle \geq \langle f ((1 - t_1) A + t_1 B) x, x \rangle = \varphi_{(A,B):x} (t_1)\]
that shows that the scalar function $\varphi_{(A,B):x} : [0, 1] \to \mathbb{R}$ is monotonic nondecreasing for $A \leq B$ and for any $x \in H$.

If we write the inequality (2.2) for the functions $p$ and $\varphi_{(A,B):x}$ we get
\[
\int_0^1 p(t) \langle f ((1 - t) A + t B) x, x \rangle \, dt \geq \int_0^1 p(t) \, dt \int_0^1 \langle f ((1 - t) A + t B) x, x \rangle \, dt,
\]
which can be written as
\[
\left\langle \left( \int_0^1 p(t) f ((1 - t) A + t B) \, dt \right) x, x \right\rangle \geq \left\langle \left( \int_0^1 p(t) \, dt \int_0^1 f ((1 - t) A + t B) \, dt x, x \right) \right\rangle
\]
for $x \in H$, and the first inequality in (2.7) is obtained.

We also have that
\[
\langle f (A) x, x \rangle = \varphi_{(A,B):x} (0) \leq \varphi_{(A,B):x} (t) = \langle f ((1 - t) A + t B) x, x \rangle \leq \varphi_{(A,B):x} (1) = \langle f (B) x, x \rangle
\]
and
\[p (0) \leq p (t) \leq p (1)
\]
for all $t \in [0, 1]$.

By writing Grüss’ inequality for the functions $\varphi_{(A,B):x}$ and $p$, we get
\[
0 \leq \int_0^1 p(t) \langle f ((1 - t) A + t B) x, x \rangle \, dt
- \int_0^1 p(t) \, dt \int_0^1 \langle f ((1 - t) A + t B) x, x \rangle \, dt
\leq \frac{1}{4} [p (1) - p (0)] \| f (B) x \| - \langle f (A) x, x \rangle
\]
for $x \in H$ and the second inequality in (2.7) is obtained.

A continuous function $g : \mathcal{SA}_f (H) \to \mathcal{B} (H)$ is said to be Gâteaux differentiable in $A \in \mathcal{SA}_f (H)$ along the direction $B \in \mathcal{B} (H)$ if the following limit exists in the strong topology of $\mathcal{B} (H)$
\[
(2.9) \quad \nabla g_A (B) := \lim_{s \to 0} \frac{g (A + s B) - g (A)}{s} \in \mathcal{B} (H).
\]
If the limit (2.9) exists for all $B \in \mathcal{B} (H)$, then we say that $g$ is Gâteaux differentiable in $A$ and we can write $g \in \mathcal{G} (A)$. If this is true for any $A$ in an open set $\mathcal{S}$ from $\mathcal{SA}_f (H)$ we write that $g \in \mathcal{G} (\mathcal{S})$. 

If \( g \) is a continuous function on \( I \), by utilising the continuous functional calculus the corresponding function of operators will be denoted in the same way.

For two distinct operators \( A, B \in \mathcal{S} \mathcal{A}_I (H) \) we consider the segment of selfadjoint operators

\[
[A, B] := \{(1 - t) A + t B \mid t \in [0, 1]\}.
\]

We observe that \( A, B \in [A, B] \) and \( [A, B] \subset \mathcal{S} \mathcal{A}_I (H) \).

**Lemma 1.** Let \( f \) be a continuous function on \( I \) and \( A, B \in \mathcal{S} \mathcal{A}_I (H) \), with \( A \neq B \).

If \( f \in \mathcal{G} ([A, B]) \), then the auxiliary function \( \varphi_{(A,B)} \) is differentiable on \((0,1)\) and

\[
\varphi'_{(A,B)}(t) = \nabla f_{(1-t)A+tB} (B - A).
\]

In particular,

\[
\varphi'_{(A,B)} (0+) = \nabla f_A (B - A)
\]

and

\[
\varphi'_{(A,B)} (1-) = \nabla f_B (B - A).
\]

**Proof.** Let \( t \in (0,1) \) and \( h \neq 0 \) small enough such that \( t + h \in (0,1) \). Then

\[
\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t) = \frac{h}{f ((1-t-h) A + (t+h) B) - f ((1-t) A + t B)}
\]

\[
= \frac{f ((1-t) A + t B + h (B - A)) - f ((1-t) A + t B)}{h}.
\]

Since \( f \in \mathcal{G} ([A, B]) \), hence by taking the limit over \( h \to 0 \) in (2.13) we get

\[
\varphi'_{(A,B)} (t) = \lim_{h \to 0} \frac{\varphi_{(A,B)}(t+h) - \varphi_{(A,B)}(t)}{h}
\]

\[
= \lim_{h \to 0} \frac{f ((1-t) A + t B + h (B - A)) - f ((1-t) A + t B)}{h}
\]

\[
= \nabla f_{(1-t)A+tB} (B - A),
\]

which proves (2.10).

Also, we have

\[
\varphi'_{(A,B)} (0+) = \lim_{h \to 0+} \frac{\varphi_{(A,B)}(h) - \varphi_{(A,B)}(0)}{h}
\]

\[
= \lim_{h \to 0+} \frac{f ((1-h) A + h B) - f (A)}{h}
\]

\[
= \lim_{h \to 0+} \frac{f (A + h (B - A)) - f (A)}{h}
\]

\[
= \nabla f_A (B - A)
\]

since \( f \) is assumed to be Gâteaux differentiable in \( A \). This proves (2.11).

The equality (2.12) follows in a similar way.

**Lemma 2.** Let \( f \) be an operator monotonic function on \( I \) and \( A, B \in \mathcal{S} \mathcal{A}_I (H) \), with \( A \leq B \), \( A \neq B \). If \( f \in \mathcal{G} ([A, B]) \), then

\[
\nabla f_{(1-t)A+tB} (B - A) \geq 0 \text{ for all } t \in (0,1).
\]
Also
\begin{equation}
\nabla f_A (B - A), \nabla f_B (B - A) \geq 0.
\end{equation}

**Proof.** Let $x \in H$. The auxiliary function $\varphi_{(A,B);x}$ is monotonic nondecreasing in the usual sense on $[0, 1]$ and differentiable on $(0, 1)$, and for $t \in (0, 1)$
\begin{align*}
0 \leq \varphi'_{(A,B);x} (t) &= \lim_{h \to 0} \frac{\varphi_{(A,B),x} (t + h) - \varphi_{(A,B),x} (t)}{h} \\
&= \lim_{h \to 0} \left\langle \frac{\varphi_{(A,B)} (t + h) - \varphi_{(A,B)} (t)}{h}, x \right\rangle x \\
&= \left\langle \lim_{h \to 0} \frac{\varphi_{(A,B)} (t + h) - \varphi_{(A,B)} (t)}{h}, x \right\rangle x \\
&= \langle \nabla f_{(1-t)A+tB} (B - A), x \rangle.
\end{align*}
This shows that
\begin{equation}
\nabla f_{(1-t)A+tB} (B - A) \geq 0
\end{equation}
for all $t \in (0, 1)$.

The inequalities (2.15) follow by (2.11) and (2.12). \qed

The following inequality obtained by Ostrowski in 1970, [9] also holds
\begin{equation}
|C (h, g)| \leq \frac{1}{8} (b - a) (M - m) \|g'\|_\infty,
\end{equation}
provided that $h$ is Lebesgue integrable and satisfies (2.4) while $g$ is absolutely continuous and $g' \in L_\infty [a, b]$. The constant $\frac{1}{8}$ is best possible in (2.16).

**Theorem 3.** Let $A, B \in \mathcal{S}A_I (H)$ with $A \leq B$, $f$ be an operator monotonic function on $I$ and $p : [0, 1] \to \mathbb{R}$ monotonic nondecreasing on $[0, 1]$.

(i) If $p$ is differentiable on $(0, 1)$, then
\begin{equation}
0 \leq \int_0^1 p(t) f ((1 - t) A + tB) dt - \int_0^1 p(t) dt \int_0^1 f ((1 - t) A + tB) dt \\
\quad \leq \frac{1}{8} \sup_{t \in (0, 1)} p'(t) \left| f (B) - f (A) \right|.
\end{equation}

(ii) If $f \in \mathcal{G} ([A, B])$, then
\begin{equation}
0 \leq \int_0^1 p(t) f ((1 - t) A + tB) dt - \int_0^1 p(t) dt \int_0^1 f ((1 - t) A + tB) dt \\
\quad \leq \frac{1}{8} \left[ p (1) - p (0) \right] \sup_{t \in (0, 1)} \| \nabla f_{(1-t)A+tB} (B - A) \|_H.
\end{equation}

**Proof.** Let $x \in H$. If we use the inequality (2.16) for $g = p$ and $h = \varphi_{(A,B);x}$, then
\begin{align*}
0 &\leq \int_0^1 p(t) \langle f ((1 - t) A + tB), x, x \rangle dt \\
&\quad - \int_0^1 p(t) dt \int_0^1 \langle f ((1 - t) A + tB), x, x \rangle dt \\
&\quad \leq \frac{1}{8} \sup_{t \in (0, 1)} p'(t) \left| \langle f (B), x, x \rangle - \langle f (A), x, x \rangle \right|,
\end{align*}
for any $x \in H$, which is equivalent to (2.17).
If we use the inequality (2.16) for $h = p$ and $g = \varphi_{(A,B)}$ then by Lemmas 1 and 2

\[ 0 \leq \int_0^1 p(t) \langle f((1-t)A + tB)x, x \rangle dt - \int_0^1 p(t) dt \int_0^1 \langle f((1-t)A + tB)x, x \rangle dt \leq \frac{1}{8} \left[ p(1) - p(0) \right] \sup_{t \in (0,1)} \langle \nabla f_{(1-t)A + tB}(B - A)x, x \rangle, \]

for any $x \in H$, which is an inequality of interest in itself.

Observe that for all $t \in (0,1)$,

\[ \langle \nabla f_{(1-t)A + tB}(B - A)x, x \rangle \leq \| \nabla f_{(1-t)A + tB}(B - A) \| \| x \|^2 \]

for any $x \in H$, which implies that

\[ \sup_{t \in (0,1)} \langle \nabla f_{(1-t)A + tB}(B - A)x, x \rangle \leq \sup_{t \in (0,1)} \| \nabla f_{(1-t)A + tB}(B - A) \| \langle 1_H x, x \rangle \]

for any $x \in H$.

By making use of (2.19) and (2.20) we derive

\[ 0 \leq \int_0^1 p(t) \langle f((1-t)A + tB)x, x \rangle dt - \int_0^1 p(t) dt \int_0^1 \langle f((1-t)A + tB)x, x \rangle dt \leq \frac{1}{8} \left[ p(1) - p(0) \right] \sup_{t \in (0,1)} \| \nabla f_{(1-t)A + tB}(B - A) \| \langle 1_H x, x \rangle \]

for any $x \in H$, which is equivalent to (2.18).

Another, however less known result, even though it was obtained by Čebyšev in 1882, [1], states that

\[ |C(h, g)| \leq \frac{1}{12} \| h' \|_\infty \| g' \|_\infty (b - a)^2, \]

provided that $h'$, $g'$ exist and are continuous on $[a, b]$ and $\| h' \|_\infty = \sup_{t \in [a, b]} | h'(t) |$. The constant $\frac{1}{12}$ cannot be improved in the general case.

The case of euclidean norms of the derivative was considered by A. Lupaş in [5] in which he proved that

\[ |C(h, g)| \leq \frac{1}{\pi^2} \| h' \|_2 \| g' \|_2 (b - a), \]

provided that $h$, $g$ are absolutely continuous and $h'$, $g' \in L^2 [a, b]$. The constant $\frac{1}{\pi^2}$ is the best possible.

Using the above inequalities (2.21) and (2.22) and a similar procedure to the one employed in the proof of Theorem 3, we can also state the following result:

**Theorem 4.** Let $A, B \in SA_I(H)$ with $A \leq B$, $f$ be an operator monotonic function on $I$ and $p : [0, 1] \to \mathbb{R}$ monotonic nondecreasing on $[0, 1]$. If $p$ is differentiable
and \( f \in \mathcal{G} ([A, B]) \), then

\begin{align*}
(2.23) \quad 0 & \leq \int_0^1 p(t) f ((1 - t) A + tB) \, dt - \int_0^1 p(t) \, dt \int_0^1 f ((1 - t) A + tB) \, dt \\
& \leq \frac{1}{12} \sup_{t \in (0, 1)} p'(t) \sup_{t \in (0, 1)} \| \nabla f_{(1-t)A+tB} (B - A) \| 1_H
\end{align*}

and

\begin{align*}
(2.24) \quad 0 & \leq \int_0^1 p(t) f ((1 - t) A + tB) \, dt - \int_0^1 p(t) \, dt \int_0^1 f ((1 - t) A + tB) \, dt \\
& \leq \frac{1}{12} \left( \int_0^1 [p'(t)]^2 \, dt \right)^{1/2} \left( \int_0^1 \| \nabla f_{(1-t)A+tB} (B - A) \|^2 \, dt \right)^{1/2} 1_H,
\end{align*}

provided the integrals in the second term are finite.

3. Some Examples

We consider the function \( f : (0, \infty) \to \mathbb{R} \), \( f(t) = -t^{-1} \) which is operator monotone on \((0, \infty)\).

If \( 0 < A \leq B \) and \( p : [0, 1] \to \mathbb{R} \) is monotonic nondecreasing on \([0, 1]\), then by (2.7)

\begin{align*}
(3.1) \quad 0 & \leq \int_0^1 p(t) \, dt \int_0^1 ((1 - t) A + tB)^{-1} \, dt - \int_0^1 p(t) ((1 - t) A + tB)^{-1} \, dt \\
& \leq \frac{1}{4} [p(1) - p(0)] (A^{-1} - B^{-1}).
\end{align*}

Moreover, if \( p \) is differentiable on \((0, 1)\), then by (2.17) we obtain

\begin{align*}
(3.2) \quad 0 & \leq \int_0^1 p(t) \, dt \int_0^1 ((1 - t) A + tB)^{-1} \, dt - \int_0^1 p(t) ((1 - t) A + tB)^{-1} \, dt \\
& \leq \frac{1}{8} \sup_{t \in (0, 1)} p'(t) (A^{-1} - B^{-1}).
\end{align*}

The function \( f(t) = -t^{-1} \) is operator monotone on \((0, \infty)\), operator Gâteaux differentiable and

\( \nabla f_T(S) = T^{-1} ST^{-1} \)

for \( T, S > 0 \).

If \( p : [0, 1] \to \mathbb{R} \) is monotonic nondecreasing on \([0, 1]\), then by (2.18) we get

\begin{align*}
(3.3) \quad 0 & \leq \int_0^1 p(t) \, dt \int_0^1 ((1 - t) A + tB)^{-1} \, dt - \int_0^1 p(t) ((1 - t) A + tB)^{-1} \, dt \\
& \leq \frac{1}{8} [p(1) - p(0)] \\
\times \sup_{t \in (0, 1)} \left\| ((1 - t) A + tB)^{-1} (B - A) ((1 - t) A + tB)^{-1} \right\| 1_H
\end{align*}

for \( 0 < A \leq B \).
If $p$ is monotonic nondecreasing and differentiable on $(0, 1)$, then by (2.23) and (2.24) we get

\[(3.4) \quad 0 \leq \int_0^1 p(t) \, dt \int_0^1 ((1 - t) \, A + tB)^{-1} \, dt - \int_0^1 p(t) \, ((1 - t) \, A + tB)^{-1} \, dt \]

\[\leq \frac{1}{12} \sup_{t \in (0,1)} p'(t) \times \sup_{t \in (0,1)} \left\| ((1 - t) \, A + tB)^{-1} (B - A) ((1 - t) \, A + tB)^{-1} \right\|_{1_H} \]

and

\[(3.5) \quad 0 \leq \int_0^1 p(t) \, dt \int_0^1 ((1 - t) \, A + tB)^{-1} \, dt - \int_0^1 p(t) \, ((1 - t) \, A + tB)^{-1} \, dt \]

\[\leq \frac{1}{\pi^2} \left( \int_0^1 \left| p'(t) \right|^2 \, dt \right)^{1/2} \times \left( \int_0^1 \left\| ((1 - t) \, A + tB)^{-1} (B - A) ((1 - t) \, A + tB)^{-1} \right\|_2^2 \, dt \right)^{1/2} 1_H, \]

for $0 < A \leq B$.

We note that the function $f(t) = \ln t$ is operator monotonic on $(0, \infty)$.

If $0 < A \leq B$ and $p : [0, 1] \to \mathbb{R}$ is monotonic nondecreasing on $[0, 1]$, then by (2.7) we have

\[(3.6) \quad 0 \leq \int_0^1 p(t) \ln ((1 - t) \, A + tB) \, dt - \int_0^1 p(t) \, dt \int_0^1 \ln ((1 - t) \, A + tB) \, dt \]

\[\leq \frac{1}{4} \left[ p(1) - p(0) \right] (\ln B - \ln A). \]

Moreover, if $p$ is differentiable on $(0, 1)$, then by (2.17) we obtain

\[(3.7) \quad 0 \leq \int_0^1 p(t) \ln ((1 - t) \, A + tB) \, dt - \int_0^1 p(t) \, dt \int_0^1 \ln ((1 - t) \, A + tB) \, dt \]

\[\leq \frac{1}{8} \sup_{t \in (0,1)} p'(t) (\ln B - \ln A). \]

The ln function is operator Gâteaux differentiable with the following explicit formula for the derivative (cf. Pedersen [10, p. 155]):

\[(3.8) \quad \nabla \ln_T (S) = \int_0^\infty (s1_H + T)^{-1} S (s1_H + T)^{-1} \, ds \]

for $T, S > 0$.

If $p : [0, 1] \to \mathbb{R}$ is monotonic nondecreasing on $[0, 1]$, then by (2.18) we get

\[(3.9) \quad 0 \leq \int_0^1 p(t) \ln ((1 - t) \, A + tB) \, dt - \int_0^1 p(t) \, dt \int_0^1 \ln ((1 - t) \, A + tB) \, dt \]

\[\leq \frac{1}{8} \left[ p(1) - p(0) \right] \times \sup_{t \in (0,1)} \left\| \int_0^\infty (s1_H + (1 - t) \, A + tB)^{-1} (B - A) (s1_H + (1 - t) \, A + tB)^{-1} \, ds \right\|_{1_H} \]
and if $p$ is differentiable on $(0, 1)$, then

\begin{equation}
0 \leq \int_0^1 p(t) \ln \left( (1 - t) A + tB \right) dt - \int_0^1 p(t) dt \int_0^1 \ln \left( (1 - t) A + tB \right) dt \\
\leq \frac{1}{12} \sup_{t \in (0, 1)} p'(t) \\
\times \sup_{t \in (0, 1)} \left\| \int_0^\infty (s1_H + (1 - t) A + tB)^{-1} (B - A)(s1_H + (1 - t) A + tB)^{-1} ds \right\|_H
\end{equation}

for $0 < A \leq B$.

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