WELL-POSEDNESS FOR STOCHASTIC CONSERVATION LAWS ON RIEMANNIAN MANIFOLDS

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Abstract. We consider the scalar conservation law with stochastic forcing

\[ \partial_t u + \text{div}_g f(x, u) = \Phi(x, u) dW, \quad x \in M, \quad t \geq 0 \]

on a smooth compact Riemannian manifold \((M, g)\) where \(W\) is the Wiener process and \(x \mapsto f(x, \xi)\) is a vector field on \(M\) for each \(\xi \in \mathbb{R}\). We introduce admissibility conditions, derive the kinetic formulation and use it to prove well posedness.

1. Introduction

We consider the Cauchy problem for a stochastic scalar conservation law of the form

\[ \partial_t u + \text{div}_g f(x, u) = \Phi(x, u) dW, \quad x \in M, \quad t \geq 0 \tag{1} \]

\[ u|_{t=0} = u_0(x) \in L^\infty(M) \tag{2} \]

on a smooth, compact, \(d\)-dimensional (Hausdorff) Riemannian manifold \((M, g)\). The object \(W\) is the Wiener process which can be finite or infinite dimensional which does not affect the essence of the proofs. Therefore, we shall assume that we work with one-dimensional Wiener process defined on the stochastic basis \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\).

We will assume that the function \(\Phi\) is continuously differentiable and it decays to zero at infinity i.e. \(\Phi \in C_0^1(M \times \mathbb{R})\).

Nowadays, we are witnessing a rapid development of stochastic conservation laws and related equations. The rising interest to this field of research is stipulated by concrete applications in biology, porous media, finances (see e.g. randomly chosen \[1, 4, 27\] and references therein) and, in general, any realistic situation in which we cannot determine parameters precisely (i.e. the coefficients of the equations governing the process).

Moreover, such equations have rich mathematical structure and therefore, they are very interesting and challenging from the mathematical point of view. We have numerous results in different directions beginning with the stochastic conservation laws \[5, 6, 12, 13, 16, 17, 28\], then velocity averaging results for stochastic transport equations \[7, 21\], stochastic degenerate parabolic equations \[14, 29\]. We remark that latter list of references is far from complete. Here, we aim to expand the theory of stochastic scalar conservation laws to manifolds. As for the stochastic PDEs on manifolds, we mention \[2\] where the wave equation was considered.

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Let us now briefly recall the meaning of the divergence on a manifold. We suppose that the map \((x, \xi) \mapsto f(x, \xi), M \times \mathbb{R} \to TM\) is \(C^1\) and that, for every \(\xi \in \mathbb{R}\), \(x \mapsto f(x, \xi) \in X(M)\) (the space of vector fields on \(M\)).

In local coordinates, we write
\[
f(x, \xi) = (f^1(x, \xi), \ldots, f^d(x, \xi)).
\]
The divergence operator appearing in the equation is to be formed with respect to the metric, so in local coordinates we have (cf. (8) below):
\[
\text{div}_g f(x, u) = \text{div}_g (x \mapsto f(x, u(t, x))) = \frac{\partial}{\partial x^k} (f^k(x, u(t, x)) + \Gamma^j_{kj}(x)f^k(x, u(t, x)))
\]
where the \(\Gamma\)-terms are the Christoffel symbols of \(g\) and the Einstein summation convention is in effect.

As we can see, the divergence operator on manifolds is more involved than the one in Euclidean setting. Thus, in order to simplify computation (in particular, derivation of the admissibility conditions), we shall assume the following condition, which can be called the geometry compatibility condition from a purely mathematical point of view [3], or the incompressibility condition from the fluid dynamics point of view:
\[
\text{div}_g f(x, \xi) = 0 \quad \text{for every } \xi \in \mathbb{R}.
\]
Let us explain why, from a physical point of view, this is an incompressibility condition. Indeed, due to conservation of mass of an incompressible fluid, the density in a control volume changes according to the stochastic forcing
\[
\frac{D\rho}{Dt} = \Phi(x, \rho)dW
\]
where \(\rho\) is density of the control volume and \(\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \frac{dx}{dt} \cdot \nabla \rho\) is the material derivative for the flow velocity \(\frac{dx}{dt} = (\frac{dx_1}{dt}, \ldots, \frac{dx_d}{dt})\). If we assume that the function \(\rho\) is smooth, we can rewrite equation (1) in the form
\[
\frac{\partial \rho}{\partial t} + \partial_\xi (f(x, \xi)) \bigg|_{\xi = \rho} \cdot \nabla \rho + \text{div}_g f(x, \xi) \bigg|_{\xi = \rho} = \Phi(x, \rho)dW.
\]
Then, taking as usual \(\frac{dx}{dt} = \partial_\xi (f(x, \xi)) \bigg|_{\xi = \rho}\) and comparing (2) and (3), we arrive at
\[
\text{div}_g f(x, \xi) \bigg|_{\xi = \rho} = 0,
\]
which immediately gives what is called the geometry compatibility condition.

Since the equation we consider is a nonlinear hyperbolic equation, its solution in general contains discontinuities and we need to pass to the weak solution concept. However, this induces uniqueness issues as one can in general construct several weak solutions satisfying the same initial data. Thus, in order to isolate the physically admissible one, we need to introduce entropy type admissibility conditions [19]. We will first derive them locally and then, using the geometry compatibility conditions, we shall show that the conditions hold globally as well.

Having the admissibility conditions, we can derive the kinetic formulation to (1) (see (28)). We will use it to prove both existence and uniqueness to the considered Cauchy problem. The strategy of proof is adapted from [6]. We have tried to be as precise and self contained and intuitive as possible. We therefore proved a simple corollary of the Itô lemma concerning the derivative of the
product of two stochastic processes and derive the uniqueness proof first informally, and then also formally.

The paper is organized as follows. In Section 2 we introduce notions and notations from differential geometry and stochastic calculus. We then move on to derive the kinetic formulation of (11) and heuristically show how to get uniqueness to the solution. In Section 3 we formally prove the uniqueness result. Finally, in Section 4 we show existence of the kinetic solution which in turn implies existence of the entropy admissible solution.

2. Preliminaries from Riemannian geometry and stochastic calculus

We shall split the section into two parts. In the first one, we will provide details from differential geometry, and in the second one, we recall necessary results from stochastic calculus.

2.1. Riemannian geometry. Our standard references for notions from Riemannian and distributional geometry are [15, 22, 23, 25]. As before, \((M, g)\) will be a \(d\)-dimensional Riemannian manifold. If \(v\) is a distributional vector field on \(M\) then its gradient \(\nabla v\) is the vector field metrically equivalent to the exterior derivative \(dv\) of \(v\): \(\langle \nabla v, X \rangle = dv(X) = X(v)\) for any \(X \in \mathfrak{X}(M)\). In local coordinates,

\[
\nabla v = g^{ij} \frac{\partial v}{\partial x^j} \partial_i,
\]

with \(g^{ij}\) the inverse matrix to \(g_{ij} = \langle \partial_{x^i}, \partial_{x^j}\rangle\). For \(T \in \mathcal{T}_1^k(M)\), a divergence of \(T\) is any contraction of one of its \(k\) contravariant slots with the new covariant slot of its covariant differential \(\nabla T \in \mathcal{T}_1^{k+1}(M)\). In particular, if \(k = 1\) then \(T\) possesses a unique divergence \(\text{div} T \in \mathcal{T}_1^0(M)\). We list here the local coordinate expressions for the cases that will be of interest in this paper.

First, if \(X \in \mathcal{T}_0^1 = \mathfrak{X}(M)\) is a \(C^1\) vector field on \(M\) with local representation \(X = X^i \frac{\partial}{\partial x^i}\), then \(\text{div} X \in C(M)\) is locally given by

\[
\text{div} X = \frac{\partial X^k}{\partial x^k} + \Gamma^k_{ij} X^j.
\]

The same expression holds for a distributional vector field \(X\), and similar for the formulae given below, which we formulate in the smooth case with the understanding that they carry over by continuous extension also to the distributional setting. If a \(C^1\) one-form \(\omega \in \mathcal{T}_0^1(M) = \Omega^1(M)\) is locally given by \(\omega = \omega_i dx^i\), then its divergence is defined as the metric contraction of its covariant differential \(\nabla \omega \in \mathcal{T}_0^2(M)\), so

\[
\text{div} \omega = g^{ij} \partial_i \omega_j - \Gamma^k_{ij} g^{ij} \omega_k.
\]

If \(T \in \mathcal{T}_1^1(M)\), \(T = T^k_i \frac{\partial}{\partial x^i} \otimes dx^k\), then \(\text{div} T = (\text{div} T)_i dx^i\), where

\[
(\text{div} T)_i = \partial_i T^k_j + \Gamma^j_{ji} T^k_i - \Gamma^j_{ik} T^k_j.
\]

Finally, again for \(T \in \mathcal{T}_1^1(M)\), \(\text{div}(\text{div}(T)) \in C(M)\) is given in local coordinates by

\[
\text{div}(\text{div}(T)) = g^{ij} [\partial_i \partial_j T^k_k + \Gamma^k_{kl} \partial_l T^k_j - \Gamma^k_{kj} \partial_l T^k_i - \Gamma^k_{ij} \partial_l T^k_k + (\partial_i \Gamma^k_{lj}) T^k_j - (\partial_j \Gamma^k_{li}) T^k_i + \Gamma^k_{ij} \Gamma^k_{kl} T^l_i].
\]

Let us now recall basic notions from stochastic calculus.
2.2. **Stochastic calculus.** What we essentially need from the stochastic calculus is the Itô lemma and some of its corollaries. To this end, let $X_t$ be a stochastic process satisfying the following stochastic differential equation:

$$
dX_t = \mu_1 dt + \sigma_1 dW. \tag{12}$$

By Itô’s lemma, for each twice differentiable scalar function $f = f(t, z)$ the equation

$$
df(X_t) = \left( \frac{\partial f}{\partial t} + \mu_1 \frac{\partial f}{\partial z} + \frac{\sigma_1^2}{2} \frac{\partial^2 f}{\partial z^2} \right) dt + \sigma_1 \frac{\partial f}{\partial z} dW \tag{13}$$

holds.

By taking $f(t, X_t) = X_t^2$, we get

$$
dX_t^2 = 2\mu_1 X_t dt + 2\sigma_1 X_t dW. \tag{14}$$

Notice that $2\mu_1 X_t dt + 2\sigma_1 X_t dW = 2X_t(\mu_1 dt + \sigma_1 dW) = 2X_t dX_t$, so (14) becomes

$$
dX_t^2 = 2X_t dX_t + \sigma_1^2 dt. \tag{15}$$

Similarly, if $Y_t$ is a stochastic process satisfying the stochastic differential equation

$$
dY_t = \mu_2 dt + \sigma_2 dW \tag{16}$$

then

$$
dY_t^2 = 2Y_t dY_t + \sigma_2^2 dt, \tag{17}$$

$$
d(X_t + Y_t)^2 = 2(X_t + Y_t) d(X_t + Y_t) + (\sigma_1 + \sigma_2)^2 dt. \tag{18}$$

The left-hand side of (18) is

$$
d(X_t + Y_t)^2 = d(X_t^2 + 2X_tY_t + Y_t^2) = dX_t^2 + 2d(X_t Y_t) + dY_t^2 = 2X_t dX_t + \sigma_1^2 dt + 2d(X_t Y_t) + 2Y_t dY_t + \sigma_2^2 dt, \tag{19}$$

and the right side is

$$2(X_t + Y_t)d(X_t + Y_t) + (\sigma_1 + \sigma_2)^2 dt = 2X_t dX_t + 2X_t dY_t + 2Y_t dX_t + 2Y_t dY_t + \sigma_1^2 dt + 2\sigma_1 \sigma_2 dt + \sigma_2^2 dt. \tag{20}$$

By annuling the same terms on the left and right side respectively, and dividing the equation by 2, we get

$$
d(X_t Y_t) = X_t dY_t + Y_t dX_t + \sigma_1 \sigma_2 dt. \tag{21}$$

### 3. Entropy admissibility and kinetic formulation

In order to derive the admissibility conditions, we shall, as usual, start with the parabolic approximation to (1)

$$
\begin{align*}
  du_{\varepsilon} &+ \text{div}_g(f(x, u_\varepsilon))dt = \Phi(x, u_\varepsilon)dW + \varepsilon \Delta_g u_\varepsilon dt, \quad x \in M, \quad t \in (0, T)
\end{align*} \tag{22}
$$

where, as before, $f = f(x, \lambda) \in C^1(M \times \mathbb{R})$ and $(M, g)$ is a $d$-dimensional Riemannian manifold with the metric $g$. We will assume that $W$ is a Wiener process and $\Phi \in C^1_0(M \times \mathbb{R})$. 

Since we are dealing with the stochastic parabolic equation on a manifold, we cannot say anything about the existence of solution to the appropriate Cauchy problem. However, we shall assume that we can find a smooth solution to (22), (2) and prove later that this indeed holds.

Using the Itô formula, from (22) we get (here and in the sequel, we will set $f'(x, \xi) = \partial_x f(x, \xi)$):

\[ d\theta(u_\varepsilon) = \left( - \theta'(u_\varepsilon) f'(x, u_\varepsilon) \cdot \nabla_g u_\varepsilon + \theta'(u_\varepsilon) \Div_g f(x, \rho) \right)_{\rho = u_\varepsilon} \]

\[ + \varepsilon \Delta_g \theta(u_\varepsilon) - \varepsilon \theta''(u_\varepsilon) |\nabla_g u_\varepsilon|^2 + \frac{\Phi^2(x, u_\varepsilon)}{2} \theta''(u_\varepsilon) \right) \, dt + \Phi(x, u_\varepsilon) \theta'(u_\varepsilon) dW \]

for all twice differentiable scalar functions $\theta$.

Using the standard approximation procedure and taking into account convexity of the function

\[ \theta(u) = |u - \xi|_+ = \begin{cases} u - \xi, & u \geq \xi \\ 0, & \text{else} \end{cases} \]

we know that we can safely plug it into (23). After letting $\varepsilon \to 0$ and assuming that $E(|u_\varepsilon(t, x) - u(t, x)|) \to 0$ as $\varepsilon \to 0$, we get the following distributional inequality:

\[ \frac{d |u - \xi|_+}{dt} \leq - f'(x, u) \nabla_g u \sign_+ (u - \xi) + \theta'(u) \Div_g f(x, \rho)_{\rho = u} \]

\[ + \frac{\Phi^2(x, u)}{2} \delta(u - \xi) + \Phi(x, u) \sign_+ (u - \xi) \frac{dW}{dt}. \]

Taking into account the geometry compatibility condition (4), we have

\[ f'(x, u) \cdot (\nabla_g u) \sign_+ (u - \xi) = \Div_g (\sign_+ (u - \xi) (f(x, u) - f(x, \xi))) \]

\[ + \sign_+ (u - \xi) \Div_g f(x, \xi) = \Div_g (\sign_+ (u - \xi) (f(x, u) - f(x, \xi))) \]

and using the Schwartz lemma on non-negative distributions, we conclude that there exists a non-negative stochastic kinetic measure $m$ (to be precised later) such that the equation (24) can be written as

\[ \partial_t |u - \xi|_+ = - \Div_g (\sign_+ (u - \xi) (f(x, u) - f(x, \xi))) + \frac{\Phi^2(x, u)}{2} \delta(u - \xi) \]

\[ + \Phi(x, u) \sign_+ (u - \xi) W_t - m(t, x, \xi). \]

Next, we find the partial derivative of the expression given in (20) with respect to $\xi$ to get

\[ \partial_t \partial_\xi |u - \xi|_+ = - \Div_g (-f'(x, \xi) \sign_+ (u - \xi)) + \partial_\xi \left( \frac{\Phi^2(x, u)}{2} \delta(u - \xi) \right) \]

\[ + \partial_\xi (\Phi(x, u) \sign_+ (u - \xi) W_t) - \partial_\xi m. \]

Introducing $h(t, x, \xi) = - \partial_\xi |u - \xi|_+ = \sign_+ (u - \xi)$ into (27) gives

\[ h_t + \Div_g (f'(x, \xi) h) = - \partial_\xi \left( \frac{\Phi^2(x, u)}{2} \delta(u - \xi) \right) - \partial_\xi (\Phi(x, u) h W_t) + \partial_\xi m. \]
Notice that
\[
\partial_\xi (\Phi(x,u)hW_t) = \partial_\xi (\Phi(x,u)\text{sign}_+(u-\xi))W_t = -\Phi(x,u)\delta(u-\xi)W_t, \tag{29}
\]

Using \(\frac{\Phi^2(x,u)}{2} \delta(u-\xi) = \frac{\Phi^2(x,\xi)}{2} \delta(u-\xi)\) and \([23]\), and denoting the measure \(-\partial_\xi h = \delta(u-\xi)\) by \(\nu_{t,\xi}(\xi)\), we finally get the weak form of our equation:

\[
h_t + \text{div}_g(f(x,\xi)h) = -\partial_\xi \left( \frac{\Phi^2(x,\xi)}{2} \nu_{t,x}(\xi) \right) + \Phi(x,\xi)\nu_{t,x}(\xi)W_t + \partial_\xi m. \tag{30}
\]

We shall call the latter equation the kinetic formulation of \((1)\).

It is important to notice that the function \(\overline{h} = 1-h\) satisfies

\[
\overline{h}_t + \text{div}_g(f(x,\xi)\overline{h}) = \partial_\xi \left( \frac{\Phi^2(x,\xi)}{2} \nu_{t,x}(\xi) \right) - \Phi(x,\xi)\nu_{t,x}(\xi)W_t - \partial_\xi m. \tag{31}
\]

We can now introduce a definition of an admissible solution. Let us first introduce what we meant under the stochastic measure here.

**Definition 1.** We say that a mapping \(m\) from \(\Omega\) into the space of Radon measures on \([0,T] \times M \times \mathbb{R}\) is a stochastic kinetic measure if:

- for every \(\phi \in C_0([0,T] \times M \times \mathbb{R})\) the action \(\langle m, \phi \rangle\) defines a \(P\)-measurable function \(\langle m, \phi \rangle : \Omega \to \mathbb{R}\);
- for every \(\phi \in C_0(M \times \mathbb{R})\), the process

\[
t \mapsto \int_{[0,t] \times M \times \mathbb{R}} \phi(x,\xi)dm(s,x,\xi)
\]

is predictable.

**Definition 2.** The measurable function \(u : [0,T] \times M \times \Omega \to \mathbb{R}\) almost surely continuous with respect to time in the sense that \(u(\cdot, \cdot, \omega) \in C(\mathbb{R}^+; \text{calD}'(M))\) for \(P\)-almost every \(\omega \in \Omega\) is an admissible stochastic solution to \((1), (2)\) if

- there exists \(C_2 > 0\) such that \(E(\text{esssup}_{t \in [0,T]} \|u(t)\|_{L^2(M)}) \leq C_2\);
- the kinetic function \(h = \text{sign}_+(u-\xi)\) satisfies \([26]\) with the initial conditions \(h(0,x,\xi) = \text{sign}_+(u_0(x) - \xi)\) in the sense of weak traces and \(\overline{h}\) satisfies \([31]\) with the initial conditions \(\overline{h}(0,x,\xi) = 1 - \text{sign}_+(u_0(x) - \xi)\) in the sense of weak traces.

We shall also need a notion of the kinetic solution.

**Definition 3.** A measurable function \(h = h(t,x,\xi,\omega), (t,x,\xi,\omega) \in \mathbb{R}^+ \times M \times \mathbb{R} \times \Omega\), bounded between zero and one and non-strictly decreasing with respect to \(\xi \in \mathbb{R}\) is the stochastic kinetic solution to \((1), (2)\) if

- There exists a stochastic kinetic measure \(m\) such that \(h\) satisfies \([54]\) and the initial conditions \(h(0,x,\xi) = \text{sign}_+(u_0(x) - \xi)\) in the sense of weak traces.

Clearly, if we have the admissible solution to \((1), (2)\), then we have the kinetic solution as well. Interestingly, vice versa also holds which we will show in the next sections.
4. INFORMAL UNIQUENESS PROOF – DOUBLING OF VARIABLES

In this section, we shall informally show how to get uniqueness (which paves the way for the existence as well). Formal proof does not essentially differ from the procedure given in this section but one needs to introduce several smoothing procedures which significantly complicates following steps of the proof. We also remark that, in order to simplify the notation, we will denote by $dx$ the measure on the manifold instead of usual $d\gamma(x)$.

Let $h^1(t, x, \xi)$ and $h^2(t, y, \zeta)$ be two different kinetic solutions to (11), (12) (see Definition 3). Then

$$\partial_t h^1 + \text{div}_y (f^1(x, \xi) h^1) = -\partial_\xi \left( \frac{\Phi^2(x, \xi)}{2} \nu^1 \right) + \Phi(x, \xi) \nu^1 W_t + \partial_\xi m_1,$$

$$\partial_t h^2 + \text{div}_g (f^1(y, \zeta) h^2) = \partial_\zeta \left( \frac{\Phi^2(y, \zeta)}{2} \nu^2 \right) - \Phi(y, \zeta) \nu^2 W_t - \partial_\zeta m_2.$$  

By (21), the following holds:

$$d(h^1 h^2) = h^1 dh^2 + h^2 dh^1 - \Phi(x, \xi) \Phi(y, \zeta) \nu^1 \otimes \nu^2 dt.$$  

Multiplying (32) by $h^2 = h^2(t, y, \zeta)$, (33) by $h^1 = h^1(t, x, \xi)$, adding them and using the geometry compatibility conditions (1), yields

$$\partial_t h^2 \partial_\xi h^1 + h^1 \partial_t h^2 + h^2 f^1(x, \xi) \cdot \nabla_{\text{g}_x} h^1 + h^1 f^1(y, \zeta) \cdot \nabla_{\text{g}_y} h^2$$

$$= -h^2 \partial_\xi \left( \frac{\Phi^2(x, \xi)}{2} \nu^1 \right) + h^1 \partial_\zeta \left( \frac{\Phi^2(y, \zeta)}{2} \nu^2 \right) + h^2 \Phi(x, \xi) \nu^1 W_t - h^1 \Phi(y, \zeta) \nu^2 W_t$$

$$+ h^2 \partial_\xi m_1(t, x, \xi) - h^1 \partial_\zeta m_2(t, y, \zeta).$$

Inserting (34) into (35), we get

$$\partial_t (h^1 h^2) + \Phi(x, \xi) \Phi(y, \zeta) \nu^1 \otimes \nu^2 + h^2 f^1(x, \xi) \cdot \nabla_{\text{g}_x} h^1 + h^1 f^1(y, \zeta) \cdot \nabla_{\text{g}_y} h^2$$

$$= -h^2 \partial_\xi \left( \frac{\Phi^2(x, \xi)}{2} \nu^1 \right) + h^1 \partial_\zeta \left( \frac{\Phi^2(y, \zeta)}{2} \nu^2 \right) + (h^2 \Phi(x, \xi) \nu^1 - h^1 \Phi(y, \zeta) \nu^2) W_t$$

$$+ h^2 \partial_\xi m_1(t, x, \xi) - h^1 \partial_\zeta m_2(t, y, \zeta).$$

We now choose the non-negative test function $\varphi(t, x, y, \xi, \zeta) = \rho(x - y) \psi(\xi - \zeta)$, where $\rho$ and $\psi$ are smooth non-negative functions defined on appropriate Euclidean spaces. Multiplying (36) with $\varphi$ and integrating over $(0, T) \times M^2 \times \mathbb{R}^2$ we get

$$\int_{M^2} \int_{\mathbb{R}^2} \int_{M^2} \int_{\mathbb{R}^2} h^1(T, x, \xi) h^2(T, y, \zeta) \rho(x - y) \psi(\xi - \zeta) d\xi d\xi dy dx$$

$$- \int_{M^2} \int_{\mathbb{R}^2} h^1_0 h^2_0 \rho(x - y) \psi(\xi - \zeta) d\xi d\xi dy dx$$

$$+ \int_{0}^{T} \int_{M^2} \int_{\mathbb{R}^2} \rho(x - y) \psi(\xi - \zeta) \Phi(x, \xi) \Phi(y, \zeta) d\nu_{(x, y)}^1(\xi) d\nu_{(x, y)}^1(\zeta) d\nu_{(x, y)}^1(\xi) dy dx dt$$
By using integration by parts with respect to \( \zeta \) and \( \xi \) in the first and second and in the last two terms on the right hand side in (37), and using \( \partial_\xi h^1 = -\nu^1 \) and \( \partial_\zeta h^2 = \nu^2 \), we obtain:

\[
\begin{align*}
&+ \int_0^T \int_\mathbb{R}^2 \int_\mathbb{R}^2 \int_\mathbb{R}^2 h^1(x, \xi) \cdot \nabla_{g,x} h^1(t, x, \xi) \overline{h^2} (t, y, \zeta) \rho(x - y) \psi(\xi - \zeta) d\zeta d\xi dy dx dt \\
&+ \int_0^T \int_\mathbb{R}^2 \int_\mathbb{R}^2 \int_\mathbb{R}^2 f'(y, \zeta) \cdot \nabla_{g,y} \overline{h^2} (t, y, \zeta) h^1(t, x, \xi) \rho(x - y) \psi(\xi - \zeta) d\zeta d\xi dy dx dt \\
&= \int_0^T \int_\mathbb{R}^2 \int_\mathbb{R}^2 \frac{\Phi^2(x, \xi)}{2} \overline{h^2} (t, y, \zeta) \rho(x - y) \psi'(\xi - \zeta) d\nu^1_{(t, x)} (\xi) d\zeta dy dx dt \\
&+ \int_0^T \int_\mathbb{R}^2 \int_\mathbb{R}^2 \frac{\Phi^2(y, \zeta)}{2} h^1(t, x, \xi) \rho(x - y) \psi'(\xi - \zeta) d\nu^2_{(t, y)} (\zeta) d\xi dy dx dt \\
&+ \int_0^T \int_\mathbb{R}^2 \int_\mathbb{R}^2 \rho(x - y) \psi(\xi - \zeta) h^1(t, x, \xi) \overline{\Phi(x, \xi) d\nu^1_{(t, x)} (\xi)} d\zeta dy dx W(t) \\
&- \int_0^T \int_\mathbb{R}^2 \int_\mathbb{R}^2 \rho(x - y) \psi'(\xi - \zeta) h^1(t, x, \xi) \overline{\Phi(y, \zeta) d\nu^2_{(t, y)} (\zeta)} d\xi dy dx W(t) \\
&- \int_0^T \int_\mathbb{R}^2 \int_\mathbb{R}^2 \rho(x - y) \psi'(\xi - \zeta) \overline{h^2 (t, y, \zeta)} dm_1(t, x, \xi) d\zeta dy \\
&- \int_0^T \int_\mathbb{R}^2 \int_\mathbb{R}^2 h^1(t, x, \xi) \rho(x - y) \psi'(\xi - \zeta) dm_2(t, y, \zeta) d\xi dx.
\end{align*}
\]
\[
\begin{align*}
\int_{M^2 \times \mathbb{R}^2} \int_0^T \int \frac{\Phi^2(x, \xi)}{2} \rho(x - y) \psi(\xi - \zeta) d\nu_{t, y}^2(\zeta) d\nu_{t, x}^1(\xi) dy dx dt \\
+ \int_{M^2 \times \mathbb{R}^2} \int_0^T \int \frac{\Phi^2(y, \zeta)}{2} \rho(x - y) \psi(\xi - \zeta) d\nu_{t, y}^2(\zeta) d\nu_{t, x}^1(\xi) dy dx dt \\
+ \int_{M^2 \times \mathbb{R}^2} \int_0^T \int \rho(x - y) \psi(\xi - \zeta) \overline{h^2}(t, y, \zeta) \Phi(x, \xi) d\nu_{t, x}^1(\xi) d\zeta dy dx dW(t) \\
- \int_{M^2 \times \mathbb{R}^2} \int_0^T \int \rho(x - y) \psi(\xi - \zeta) h^1(t, x, \xi) \Phi(y, \zeta) d\nu_{t, y}^2(\zeta) d\xi dy dx dW(t) \\
- \int_{M^2 \times \mathbb{R}^2} \int_0^T \int \rho(x - y) \psi(\xi - \zeta) \nu_{t, y}^2(\zeta) d\nu_{t, x}^1(\xi) dt \\
- \int_{M^2 \times \mathbb{R}^2} \int_0^T \int \rho(x - y) \psi(\xi - \zeta) m_1(t, x, \xi) d\zeta dy \\
- \int_{M^2 \times \mathbb{R}^2} \int_0^T \int \rho(x - y) \psi(\xi - \zeta) m_2(t, y, \zeta) d\xi dx.
\end{align*}
\]

Finally, moving the third term on the left hand side in (38) to the right hand side and using non-negativity of the measures \( m_1 \) and \( m_2 \) yields

\[
\begin{align*}
\int_{M^2 \times \mathbb{R}^2} \int \overline{h^2}(t, y, \zeta) \rho(x - y) \psi(\xi - \zeta) d\zeta dy dx \\
- \int_{M^2 \times \mathbb{R}^2} \int \overline{h^2}(y, \zeta) \rho(x - y) \psi(\xi - \zeta) d\zeta dx dy \\
+ \int_{M^2 \times \mathbb{R}^2} \int \overline{h^2}(t, y, \zeta) \rho(x - y) \psi(\xi - \zeta) d\xi dy dx dt \\
+ \int_{M^2 \times \mathbb{R}^2} \int \overline{h^2}(t, y, \zeta) \rho(x - y) \psi(\xi - \zeta) d\xi dy dx dt \\
\leq \frac{1}{2} \int_{M^2 \times \mathbb{R}^2} \int (\Phi(x, \xi) - \Phi(y, \zeta))^2 \rho(x - y) \psi(\xi - \zeta) d\nu_{t, y}^2(\zeta) d\nu_{t, x}^1(\xi) dy dx dt \\
+ \int_{M^2 \times \mathbb{R}^2} \int \rho(x - y) \psi(\xi - \zeta) \overline{h^2}(t, y, \zeta) \Phi(x, \xi) d\nu_{t, x}^1(\xi) d\zeta dy dx dW(t) \\
- \int_{M^2 \times \mathbb{R}^2} \int \rho(x - y) \psi(\xi - \zeta) h^1(t, x, \xi) \Phi(y, \zeta) d\nu_{t, y}^2(\zeta) d\xi dy dx dW(t).
\end{align*}
\]
Setting $\psi(\xi) = \delta(\xi)$ and $\rho(x) = \delta(x)$ and rearranging it a bit, we obtain

$$
\int \int_{M \times \mathbb{R}} h^1(T, x, \xi) h^2(T, x, \xi) d\xi dx
\leq \int \int_{M \times \mathbb{R}} h_0^1 h_0^2 d\xi dx - \int \int_{0}^{T} \int \int_{M \times \mathbb{R}} f'(x, \xi) \cdot \nabla g_\xi (h^1(t, x, \xi) h^2(t, x, \xi)) d\xi dx dt
$$

$$
- \int \int_{0}^{T} \int \int_{M \times \mathbb{R}} \Phi(x, \xi) \partial_\xi (h^1(t, x, \xi) h^2(t, x, \xi)) d\xi dx dW(t).
$$

Another integration by parts provides

$$
\int \int_{M \times \mathbb{R}} h^1(T, x, \xi) h^2(T, x, \xi) d\xi dx
\leq \int \int_{M \times \mathbb{R}} h_0^1 h_0^2 d\xi dx + \int \int_{0}^{T} \int \int_{M \times \mathbb{R}} \nabla g_\xi (f'(x, \xi)) (h^1(t, x, \xi) h^2(t, x, \xi)) d\xi dx dt
$$

$$
+ \int \int_{0}^{T} \int \int_{M \times \mathbb{R}} \Phi(x, \xi) (h^1(t, x, \xi) h^2(t, x, \xi)) d\xi dx dW(t).
$$

By using non-negativity of $h^1$ and $h^2$ and the geometry compatibility conditions (41), we have after finding expectation of (41) and taking into account the Itô isometry

$$
E \left( \int \int_{M \times \mathbb{R}} h^1(T, x, \xi) h^2(T, x, \xi) d\xi dx \right)
\leq E \left( \int \int_{M \times \mathbb{R}} h_0^1 h_0^2 d\xi dx \right) + \| \Phi \|_\infty \int_{0}^{T} E \left( \int \int_{M \times \mathbb{R}} (h^1(t, x, \xi) h^2(t, x, \xi)) d\xi dx \right) dt.
$$

From here, using the Gronwall inequality, we get

$$
E \left( \int \int_{M \times \mathbb{R}} h^1(T, x, \xi) h^2(T, x, \xi) d\xi dx \right) \leq C E \left( \int_{M} |u_{10}(x) - u_{20}(x)| dx \right)
$$

From here, if assume that $u_{10} = u_{20}$, we get almost surely for almost every $(t, x, \xi) \in [0, \infty) \times M \times \mathbb{R}$:

$$
h^1(t, x, \xi) (1 - h^2(t, x, \xi)) = 0.
$$

This implies that either $h^1(t, x, \xi) = 0$ or $h^2(t, x, \xi) = 1$. Since we can interchange the roles of $h^1$ and $h^2$, we conclude that 1 and 0 are actually the only values that $h^1$ or $h^2$ can attain and that $h^1 = h^2 = h$. Since $h$ is also non-increasing with respect to $\xi$ on $[0, \infty)$, we conclude (taking into account the initial data $h_0 = \text{sign}_+ (u_0(x) - \xi)$) that there exists a function $u : [0, \infty) \times M \to \mathbb{R}$ such that

$$
h(t, x, \xi) = \text{sign}_+ (u(t, x) - \xi).
$$

We thus have the following corollary which is proven in the final section.
Corollary 4. The stochastic kinetic solution to (11), (2) has the form (44). If the function \( u \) satisfies the second item from Definition 2, then it is an admissible stochastic solution to (11), (2).

5. Uniqueness – rigorous proof

In this section, we shall formalize the arguments from the previous section. To this end, it will be necessary to express (30) in local coordinates. So, assume we are given a stochastic kinetic solution \( h \). To prove uniqueness locally we take a chart \((U, \kappa)\) for \( M \) and assume, without loss of generality, that \( \kappa(U) = \mathbb{R}^d \). Define the local expression of \( h \) as the map (in order to avoid proliferation of symbols, we shall keep the same notations for global and local quantities but we shall write \( \tilde{x} \) to denote the local variable)

\[
h : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}, \quad h(t, \tilde{x}, \xi, \omega) = h(t, \kappa^{-1}(\tilde{x}), \xi, \omega)G(\tilde{x}),
\]

where \( G(\tilde{x}) \) is the Gramian corresponding to the chart \((U, \kappa)\). Similarly, for \( \tilde{x} \in \mathbb{R}^d \) we define

\[
\Phi(\tilde{x}, \xi) = \Phi(\kappa^{-1}(\tilde{x}), \xi),
\]

\[
f(\tilde{x}, \xi) = f(\kappa^{-1}(\tilde{x}), \xi), \quad f'(\tilde{x}, \xi) = f'(\kappa^{-1}(\tilde{x}), \xi) = a(\tilde{x}, \xi)
\]

and \( m(t, \tilde{x}, \xi) \) will be the pushforward measure of \( m \) with respect to the mapping \( \kappa \).

With such notations at hand, we now rewrite (30) locally in the chart \((U, \kappa)\) into an equation in terms of \( h_1(t, \tilde{x}, \xi) \) and \( h_2(t, \tilde{x}, \xi) \), which are two kinetic solutions to Cauchy problems corresponding to (11) with the initial data \( u_{10} \) and \( u_{20} \), respectively. Below, we use the Einstein summation convention and we remind that \( a = (a_1, \ldots, a_d) = f' = (f'_1, \ldots, f'_d) \). Also, since the equations are to be understood in the weak sense, we need to add the Gramian in each of the terms below except in \( m_1 \) and \( m_2 \), since the corresponding part in these terms is implied there by the definition of the pushforward measure. This is why we introduce the conventions from (45).

\[
\partial_t h^1(t, \tilde{x}, \xi) + \text{div}_x(a(\tilde{x}, \xi)h^1) + h^1 \Gamma^1_{k,j}(\tilde{x})a_k(t, \tilde{x}, \xi)
\]

\[
= -\partial_\xi \left( \frac{\Phi^2(\tilde{x}, \xi)}{2} \nu^1_{(t, \tilde{x})}(\xi) \right) + \Phi(\tilde{x}, \xi)\nu^1_{(t, \tilde{x})}(\xi)W_t + \partial_\xi m_1,
\]

\[
\partial_t \bar{h}^2(t, \tilde{y}, \zeta) + \text{div}_{\tilde{y}}(a(\tilde{y}, \zeta)\bar{h}^2) + \bar{h}^2 \Gamma^2_{k,j}(\tilde{y})a_k(t, \tilde{y}, \zeta)
\]

\[
= \partial_\zeta \left( \frac{\Phi^2(\tilde{y}, \zeta)}{2} \nu^2_{(t, \tilde{y})}(\zeta) \right) - \Phi(\tilde{y}, \zeta)\nu^2_{(t, \tilde{y})}(\zeta)W_t - \partial_\zeta m_2.
\]

We introduce two mollifying functions \( \omega_1, \omega_2 \in D(\mathbb{R}^d), \omega_2 \in D(\mathbb{R}), \) where \( d \) is the dimension of the manifold \( M \), such that \( \omega_i \geq 0, i = 1, 2 \) and \( \int_{\mathbb{R}^d} \omega_1 = \int_{\mathbb{R}} \omega_2 = 1 \). Taking \( \omega_{\delta,r}(\tilde{x}, \xi) = \frac{1}{\delta \pi^d} \omega_1 \left( \frac{\tilde{x} - \xi}{\delta} \right) \omega_2 \left( \frac{\xi}{r} \right) \), for some \( \delta, r > 0 \), and using convolution, (46) and (47) yield (below and in the sequel, subscripts \( \delta \) and \( r \) denote convolution with respect to the corresponding variables):

\[
\partial_t h^1_{\delta,r} + \text{div}_x(a(\tilde{x}, \xi)h^1_{\delta,r}) + g^1_{\delta,r} + \left( \Gamma^1_{k,j}(\tilde{x})a_k(t, \tilde{x}, \xi)h^1 \right)_{\delta,r}
\]

\[
= -\partial_\xi \left( \frac{\Phi^2(\tilde{x}, \xi)}{2} \nu^1_{(t, \tilde{x})}(\xi) \right)_{\delta,r} + (\Phi(\tilde{x}, \xi)\nu^1_{(t, \tilde{x})}(\xi)W_t)_{\delta,r} + \partial_\delta m_1_{\delta,r},
\]
\[
\frac{\partial}{\partial t} h_{\delta,r}^1 \div \psi(a(\tilde{x}, \xi) h_{\delta,r}^1) + g_{\delta,r}^1 + \left( \Gamma_{kj}^1(\tilde{y}) a_k(t, \tilde{y}, \xi) h_{\delta,r}^2 \right)_{\delta,r} = \partial_\zeta \left( \Phi(\tilde{x}, \xi) \nu_{1(t, \tilde{y})}(\xi) \right)_{\delta,r} - (\Phi(\tilde{y}, \xi) \nu_{1(t, \tilde{y})}(\xi) dW(t))_{\delta,r} - \partial_\zeta m_{2,\delta,r}
\]
where
\[
g_{\delta,r}^1 = \div \psi(a(\tilde{x}, \xi) h_{\delta,r}^1) - \div \psi(a(\xi) h_{\delta,r}^1)
\]
\[
g_{\delta,r}^2 = \div \psi(a(\tilde{y}, \xi) h_{\delta,r}^2) - \div \psi(a(\xi) h_{\delta,r}^2).
\]
This term converges to zero as \( \delta, r \to 0 \) according to the Friedrichs lemma [26].

Now, multiplying (48) and (49) with \( h_{\delta,r}^1 \) and \( h_{\delta,r}^1 \), and integrating (50) over \( (0, T) \times \mathbb{R}^d \times \mathbb{R}^d \), the equation is rewritten in the variational formulation (recall that \( h^1 \) and \( h^2 \) are continuous with respect to \( t \in \mathbb{R}^+ \)):

\[
\int_{\mathbb{R}^{2d} \times \mathbb{R}^2} h_{\delta,r}^1(T, \tilde{x}, \xi) h_{\delta,r}^2(T, \tilde{y}, \xi) \rho_\varepsilon(\tilde{x} - \tilde{y}) \psi_\varepsilon(\xi - \zeta) \varphi \left( \frac{\tilde{x} + \tilde{y}}{2} \right) d\xi d\zeta d\tilde{x} d\tilde{y}
\]

\[
- \int_{\mathbb{R}^{2d} \times \mathbb{R}^2} h_{\delta,r}^1(\tilde{x}, \xi) h_{\delta,r}^2(\tilde{y}, \xi) \rho_\varepsilon(\tilde{x} - \tilde{y}) \psi_\varepsilon(\xi - \zeta) \varphi \left( \frac{\tilde{x} + \tilde{y}}{2} \right) d\xi d\zeta d\tilde{x} d\tilde{y}
\]

\[
+ \int_{\mathbb{R}^{2d} \times \mathbb{R}^2} \left( h_{\delta,r}^2 \div \psi(a(\tilde{x}, \xi) h_{\delta,r}^1) + h_{\delta,r}^1 \div \psi(a(\tilde{y}, \xi) h_{\delta,r}^2) \right) \times
\]

\[
\times \rho_\varepsilon(\tilde{x} - \tilde{y}) \psi_\varepsilon(\xi - \zeta) \varphi \left( \frac{\tilde{x} + \tilde{y}}{2} \right) d\xi d\zeta d\tilde{x} d\tilde{y} dW(t)
\]

\[
+ \int_{\mathbb{R}^{2d} \times \mathbb{R}^2} \left( \Gamma_{kj}^1(\tilde{x}) a_k(t, \tilde{x}, \xi) h^1 \right)_{\delta,r} h_{\delta,r}^2 + \left( \Gamma_{kj}^1(\tilde{y}) a_k(t, \tilde{y}, \xi) h^2 \right)_{\delta,r} h_{\delta,r}^1 \times
\]

Using (21), we obtain

\[
\frac{\partial}{\partial t} h_{\delta,r}^2 + \div \psi(a(\tilde{x}, \xi) h_{\delta,r}^2) + g_{\delta,r}^2 + \left( \Gamma_{kj}^2(\tilde{y}) a_k(t, \tilde{y}, \xi) h_{\delta,r}^2 \right)_{\delta,r} = \partial_\zeta m_{2,\delta,r}
\]

\[
\frac{\partial}{\partial t} h_{\delta,r}^2 = \div \psi(a(\tilde{x}, \xi) h_{\delta,r}^2) - \partial_\zeta m_{2,\delta,r}.
\]
\[ \times \rho(x - y)\psi_{\epsilon}(\xi - \zeta)\varphi \left( \frac{x + y}{2} \right) d\xi d\zeta dxdy \]

\[ = - \int_0^T \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \left( g_{\delta,r}h_{\delta,r}^2 \right) + g_{\delta,r}h_{\delta,r}^1 - h_{\delta,r}^2 (\Phi(\tilde{x}, \xi)\nu_{(t, \tilde{x})}(\xi))_{\delta,r} + h_{\delta,r}^3 (\Phi(\tilde{y}, \zeta)\nu_{(t, \tilde{y})}(\zeta))_{\delta,r} \times \]

\[ \times \rho(x - y)\psi_{\epsilon}(\xi - \zeta)\varphi \left( \frac{x + y}{2} \right) d\xi d\zeta dxdy \]

\[ + \int_0^T \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \left( h_{\delta,r}^1 \partial_{\xi} \left( \frac{\Phi^2(\tilde{y}, \zeta)\nu_{(t, \tilde{y})}(\zeta)}{2} \right) - h_{\delta,r}^2 \partial_{\xi} \left( \frac{\Phi^2(\tilde{x}, \xi)\nu_{(t, \tilde{x})}(\xi)}{2} \right) \right)_{\delta,r} \]

\[ - (\Phi(\tilde{x}, \xi)\nu_{(t, \tilde{x})}(\xi))_{\delta,r} (\Phi(\tilde{y}, \zeta)\nu_{(t, \tilde{y})}(\zeta))_{\delta,r} \rho(x - y)\psi_{\epsilon}(\xi - \zeta)\varphi \left( \frac{x + y}{2} \right) d\xi d\zeta dxdy \]

\[ + \int_0^T \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} \left( h_{\delta,r}^3 (t, \tilde{y}, \zeta)\partial_{\xi} m_{1, \delta,r}(t, \tilde{x}, \xi) - h_{\delta,r}^4 (t, \tilde{y}, \zeta)\partial_{\xi} m_{2, \delta,r}(t, \tilde{y}, \zeta) \right) \times \]

\[ \times \rho(x - y)\psi_{\epsilon}(\xi - \zeta)\varphi \left( \frac{x + y}{2} \right) d\xi d\zeta dxdy. \]

We shall analyze this equality term by term. We start with the terms from (51)-(53). We have:

\[ \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^2} h_{\delta,r}^1 (T, \tilde{x}, \xi)h_{\delta,r}^2 (T, \tilde{y}, \zeta) \rho(x - y)\psi_{\epsilon}(\xi - \zeta)\varphi \left( \frac{x + y}{2} \right) d\xi d\zeta dxdy. \]
\[- \int_0^T \int_{\mathbb{R}^d} \left( a(\bar{x}, \xi) - a(\bar{y}, \zeta) \right) \cdot \nabla \rho_\varepsilon (\bar{x} - \bar{y}) h^1_{\delta, r}(t, \bar{x}, \xi) h^2_{\delta, r}(t, \bar{y}, \zeta) \psi_\varepsilon (\xi - \zeta) \varphi \left( \frac{\bar{x} + \bar{y}}{2} \right) d\xi d\zeta dt \]
\[- \frac{1}{2} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( a(\bar{x}, \xi) + a(\bar{y}, \zeta) \right) \cdot \nabla \varphi \left( \frac{\bar{x} + \bar{y}}{2} \right) h^1_{\delta, r}(t, \bar{x}, \xi) h^2_{\delta, r}(t, \bar{y}, \zeta) \rho_\varepsilon (\bar{x} - \bar{y}) \psi_\varepsilon (\xi - \zeta) d\xi d\zeta d\bar{x} d\bar{y} dt \]

The penultimate term in (58) can be rewritten as:

\[
\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( a(\bar{x}, \xi) - a(\bar{y}, \zeta) \right) \cdot \nabla \rho_\varepsilon (\bar{x} - \bar{y}) h^1_{\delta, r}(t, \bar{x}, \xi) h^2_{\delta, r}(t, \bar{y}, \zeta) \psi_\varepsilon (\xi - \zeta) \varphi \left( \frac{\bar{x} + \bar{y}}{2} \right) d\xi d\zeta d\bar{x} d\bar{y} dt =
\]
(59)

\[
\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left( a(\bar{x}, \xi) - a(\bar{y}, \zeta) \right) \cdot \nabla \left( \frac{1}{\varepsilon^d} \rho(z) \right) \left. \right|_{z = \frac{x - y}{\varepsilon}} h^1_{\delta, r}(t, \bar{x}, \xi) h^2_{\delta, r}(t, \bar{y}, \zeta) \psi_\varepsilon (\xi - \zeta) \varphi \left( \frac{\bar{x} + \bar{y}}{2} \right) d\xi d\zeta d\bar{x} d\bar{y} dt =
\]
(60)

\[
\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{a_k(\varepsilon z + \bar{y}, \xi) - a_k(\bar{y}, \xi)}{\varepsilon \varepsilon_k} z_k \partial_{z_k} \rho(z) h^1_{\delta, r}(t, \bar{y} + \varepsilon z, \xi) h^2_{\delta, r}(t, \bar{y}, \zeta) \psi_\varepsilon (\xi - \zeta) \varphi \left( \bar{y} + \frac{\varepsilon z}{2} \right) d\xi d\zeta d\bar{z} d\bar{y} dt
\]

where \( z = \frac{x - y}{\varepsilon} \). We notice that, as \( r, \delta, \varepsilon \to 0 \) (in any order), this term becomes

\[
\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \partial_{\eta_k} a_k(\bar{y}, \xi) h^1(t, \bar{y}, \xi) h^2(t, \bar{y}, \xi) \varphi(\bar{y}) \int_{\mathbb{R}^d} z_k \partial_{z_k} \rho(z) dz d\xi d\bar{y} dt
\]
(61)

\[
= - \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \text{div}_y a(\bar{y}, \xi) h^1(t, \bar{y}, \xi) h^2(t, \bar{y}, \xi) \varphi(\bar{y}) d\xi d\bar{y} dt
\]

\[
\int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Gamma_{kj}(\bar{y}) a_k(t, \bar{y}, \xi) h^1(t, \bar{y}, \xi) h^2(t, \bar{y}, \xi) d\xi d\bar{y} dt.
\]

due to properties of the mollifier \( \rho \). Thus, from (60) and (61) we conclude that as \( r, \delta, \varepsilon \to 0 \) in any order

\[
(62)
\int_0^T \int_{\mathbb{R}^d} (h^1 \bar{h}^2)(T, \bar{y}, \xi) \varphi(\bar{x}) d\xi d\bar{x} - \int_0^T \int_{\mathbb{R}^d} (h^1 \bar{h}^2)(\bar{x}, \xi) \varphi(\bar{y}) d\xi d\bar{y}
\]
\[- \int \int \int_{0}^{T} \int_{\mathbb{R}^d} \int_{\mathbb{R}} a(\tilde{x}, \xi)(h^1 \overline{h}^2)(t, \tilde{x}, \xi) \nabla \varphi(\tilde{x}) d\xi dt - \int \int \int_{0}^{T} \int_{\mathbb{R}^d} \int_{\mathbb{R}} \Gamma_{k,j}^l(\tilde{x}) a_k(t, \tilde{x}, \xi) h^1(t, \tilde{x}, \xi) \overline{h}^2_{\delta,r}(t, \tilde{x}, \xi) d\xi d\tilde{x} dt.\]

Term (53) is easy to handle. We simply let \( r, \delta, \varepsilon \to 0 \) to conclude

\[\tag{63}\]

In order to prepare handling (54) and (50), we use regularity of the function \( \Phi \) (recall that \( \Phi \in C^0_2(\mathbb{R}^d \times \mathbb{R}) \)). We have

\[\int \int \int_{0}^{T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \left[ \frac{\Phi^2(\tilde{x}, \xi)}{2} \nu^1_{(t, \tilde{x})(\xi)} \nu^2_{(t, \tilde{y}, \delta,r)(\zeta)} - \frac{\Phi^2(\tilde{x}, \xi)}{2} \nu^1_{(t, \tilde{x}, \delta,r)(\xi)} \nu^2_{(t, \tilde{y}, \delta,r)(\zeta)} \right] \times \]

and similarly

\[\tag{64}\]

\[\int \int \int_{0}^{T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \left[ \Phi(\tilde{x}, \xi) \nu^1_{(t, \tilde{x})(\xi)} \Phi(\tilde{y}, \zeta) \nu^2_{(t, \tilde{y}, \delta,r)(\zeta)} - \Phi(\tilde{x}, \xi) \nu^1_{(t, \tilde{x}, \delta,r)(\xi)} \Phi(\tilde{y}, \zeta) \nu^2_{(t, \tilde{y}, \delta,r)(\zeta)} \right] \times \]

\[\tag{65}\]

In a similar fashion, we have

\[\int \int \int_{0}^{T} \int_{\mathbb{R}^d} \int_{\mathbb{R}^2} \left[ \Phi(\tilde{x}, \xi) \nu^1_{(t, \tilde{x})(\xi)} \overline{h}^2_{\delta,r}(t, \tilde{y}, \xi) - \Phi(\tilde{y}, \zeta) \nu^2_{(t, \tilde{y}, \delta,r)(\zeta)} \right] \times \]

\[\tag{66}\]













where we used the procedure leading to (60).

Having in mind (64), (65), and (66), we conclude that (56) has the following asymptotics:

\[ \lim_{r, \delta, \epsilon \to 0} \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi'(\tilde{x}, \xi) h_1(t, \tilde{x}, \xi) h_2(t, \tilde{x}, \xi) \varphi(\tilde{x}) \, d\xi d\tilde{x} dW_t \]

Finally, we want to get rid of the entropy defect measures from (67). We use the fact that \( h^1 \) and \( h^2 \) are decreasing with respect to \( \xi \) (i.e. \( \zeta \)) and that the measures \( m_{11} \) and \( m_{12} \) are non-negative. We have after two integration by parts (keep in mind that \( \partial_\xi \psi(\xi - \zeta) = -\partial_\xi \psi(\xi - \zeta) \))

\[ \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi'(\tilde{x}, \xi) h_1(t, \tilde{x}, \xi) h_2(t, \tilde{x}, \xi) \varphi(\tilde{x}) \, d\xi d\tilde{x} d\tilde{y} dW_t \]

Finally, we want to get rid of the entropy defect measures from (67). We use the fact that \( h^1 \) and \( h^2 \) are decreasing with respect to \( \xi \) (i.e. \( \zeta \)) and that the measures \( m_{11} \) and \( m_{12} \) are non-negative. We have after two integration by parts (keep in mind that \( \partial_\xi \psi(\xi - \zeta) = -\partial_\xi \psi(\xi - \zeta) \))

\[ \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi'(\tilde{x}, \xi) h_1(t, \tilde{x}, \xi) h_2(t, \tilde{x}, \xi) \varphi(\tilde{x}) \, d\xi d\tilde{x} d\tilde{y} dW_t \]
\[ \int_0^T \int_{\mathbb{R}^d} \left( \nu_2^2(t, x, \zeta) m_1(t, x, \zeta) + \nu_1^1(t, x, \zeta) m_2(t, x, \zeta) \right) \times \rho_\varepsilon(\tilde{x} - \tilde{y}) \psi_\varepsilon(\varepsilon - \zeta) \varphi \left( \frac{\tilde{x} + \tilde{y}}{2} \right) d\xi d\tilde{x} d\tilde{y} \leq 0. \]

Finally, from (62), (63), (66), (67), and (69), we conclude after letting \( r, \delta, \varepsilon \to 0 \) (first \( r, \delta \to 0 \) and then \( \varepsilon \to 0 \)) that (51)–(57) becomes:

\[ \int_{\mathbb{R}^d} \left( \Phi'(t, x, \xi) h^2(t, x, \xi) \varphi(x) \right) d\xi d\tilde{x} d\tilde{y} + \int_0^T \int_{\mathbb{R}^d} \Gamma_{h}(x) a_k(t, x, \xi) h^1(t, x, \xi) h^2(t, x, \xi) \varphi(x) d\xi d\tilde{x} dt \]

\[ \leq \int_{\mathbb{R}^d} \left( \Phi'(t, x, \xi) h^1(t, x, \xi) h^2(t, x, \xi) \right) \varphi(x) d\xi d\tilde{x} d\tilde{y} \]

\[ + \int_0^T \int_{\mathbb{R}^d} \left( h^1(t, x, \xi) h^2(t, x, \xi) \right) \varphi(x) d\xi d\tilde{x} dW(t). \]

From here, using the definition of the integral over a manifold and recalling (65), we see that it holds

\[ \int_{\mathbb{M}} \int_{\mathbb{R}} \left( h^1(T, x, \xi) h^2(T, x, \xi) \varphi(x) \right) d\xi dx \]

\[ \leq \int_{\mathbb{M}} \int_{\mathbb{R}} \left( h^1(x, \xi) h^2(x, \xi) \varphi(x) \right) d\xi dx - \int_0^T \int_{\mathbb{M}} \left( h^1 h^2 \right)(t, x, \xi) \varphi(x) d\xi d\tilde{x} dt \]

\[ + \int_0^T \int_{\mathbb{M}} \left( h^1 h^2 \right)(t, x, \xi) \varphi(x) d\xi d\tilde{x} dW(t). \]

Since we are on the compact manifold, we can take \( \varphi \equiv 1 \) which yields:

\[ \int_{\mathbb{M}} \int_{\mathbb{R}} \left( h^1(T, x, \xi) h^2(T, x, \xi) \varphi(\kappa(x)) \right) d\xi dx \]

\[ \leq \int_{\mathbb{M}} \int_{\mathbb{R}} \left( h^1(x, \xi) h^2(x, \xi) \varphi(\kappa(x)) \right) d\xi dx - \int_0^T \int_{\mathbb{M}} \left( h^1 h^2 \right)(t, x, \xi) \varphi(x) \cdot \nabla g \varphi(x) d\xi d\tilde{x} dt \]

\[ + \int_0^T \int_{\mathbb{M}} \left( h^1 h^2 \right)(t, x, \xi) \varphi(x) d\xi d\tilde{x} dW(t). \]
We arrived to (11) plus a term which does not affect using the Gronwall inequality and Itô isometry which give uniqueness as in (12). Remark that the Gramian has no influence on the procedure since it is a positive bounded function.

6. Existence

Our next aim is to prove that given initial data $u_0 \in L^\infty(M)$, there exists a stochastic kinetic solution $\chi$ in the sense of Definition 3, with the corresponding kinetic measure $m$. To this end, consider the vanishing viscosity approximation (22) augmented with the initial conditions (2). We have the following theorem.

**Theorem 5.** For any $\varepsilon > 0$ the initial value problem (22), (2) with $u_0 \in C^3(M)$ has a unique almost surely continuous stochastic solution $u_\varepsilon$ with $u_\varepsilon(t) \in C^2(M)$ almost surely. It satisfies, for any convex $\theta \in C^2(\mathbb{R})$,

$$
\frac{d\theta(u_\varepsilon)}{dt} \leq \left( -\text{div}_g \int_0^{u_\varepsilon} \theta'(v)f(x,v)dv 
+ \varepsilon \Delta_g \theta(u_\varepsilon) + \frac{\Phi^2(x,u_\varepsilon)}{2}\theta''(u_\varepsilon) \right)dt + \Phi(x,u_\varepsilon)\theta'(u_\varepsilon)dW
\tag{72}
$$

**Proof:** We shall use the result from [16] stating that the Cauchy problem (22), (2) given on a torus $\mathbb{T}^d$ and in the Euclidean setting has a unique almost surely continuous solution with values in $C^2(\mathbb{T}^d)$. The result is given for the case of the flux $f$ independent of the space variable explicitly, but the method given in [16] can be easily adapted to the more general situation given in (22).

In order to use the latter result, we take an arbitrary chart of the manifold $M$ and map it on the unit hyper-cube in $\mathbb{R}^d$ (together with all the coefficients of the equation including the initial data). Then, we extend the coefficients and initial data periodically along the coordinate axis directions on entire $\mathbb{R}^d$. Thus, we obtained a problem posed on $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ admitting a unique almost surely continuous solution with values in $C^2(\mathbb{T}^d)$.

We then patch such obtained solutions to get a unique solution globally defined on the entire $M$. Remark that the patching is possible since otherwise we would have two charts say $U_1$ and $U_2$ on whose intersection we would have two different functions $u_1$ and $u_2$ locally satisfying (22), (2). However, if we map the intersection on the unit hypercube and solve (22), (2) locally on the torus generated by the hypercube, we know by [16] that we need to get a unique solution contradicts the existence of the two different functions $u_1$ and $u_2$ locally satisfying (22), (2).

As for the condition (72), it follows from the Itô formula (which provides (23)) and after elementary algebra on manifolds applied in (23) together with the geometry compatibility conditions (derivation is actually prosecuted in Section 3). \square

From (72), it is not difficult to derive the kinetic formulation for (22). Then, letting the approximation parameter $\varepsilon \to 0$, we reach to the kinetic solution (Definition 3) to (11). Indeed, taking $\theta(u_\varepsilon) = |u_\varepsilon - \xi|$ in (72) (as in Section 3) and remembering Schwartz lemma on non-negative distributions, we get for a non-negative measure stochastic $m_\varepsilon$ after finding derivative with respect to $\xi$:

$$
\partial_t\text{sign}_+(u_\varepsilon - \xi) + \text{div}_g(f(x,\xi)\text{sign}_+(u_\varepsilon - \xi))
\tag{73}
$$
\[ = \varepsilon \Delta_g (\text{sign}_+(u_\varepsilon - \xi)) - \partial_t \left( \frac{\Phi^2(x, \xi)}{2} \nu_\varepsilon \right) + \Phi(x, \xi) \nu_\varepsilon W_t + \partial_\xi m_\varepsilon, \]

with \( \nu_\varepsilon = -\partial_t \text{sign}_+(u_\varepsilon - \xi) \). Finally, taking a weak-* limit of \( (\text{sign}_+(u_\varepsilon - \xi)) \) along a subsequence (denoted by \( h \)), we reach \( (\ref{eq:limit}) \). According to the standard procedure \([8]\) presented in the proof of Theorem \([7]\) below, we conclude that there exists \( u \) such that \( h(t, x, \xi) = \text{sign}_+(u - \xi) \) for a unique entropy admissible solution \( u \) of \((\ref{eq:heat}), (\ref{eq:initial})\). Thus, we have the basic steps of the proof to the existence theorem. Before we prove it, we need the following simple lemma.

**Lemma 6.** Assume that \( E(||u_\varepsilon||_{L^2(U)}) \leq C, U \subset \mathbb{R}^+ \times M \). Define

\[ U_n^t = \{(t, x) \in U : E(|u_n(t, x, \cdot)|) > l\}. \]

Then

\[ \lim_{t \to \infty} \sup_{n \in \mathbb{N}} \text{meas}(U_n^t) = 0. \tag{74} \]

**Proof:** Since \((E(|u_n|^2))\) is bounded in and \( U \subset \mathbb{R}^+ \times M \) it also holds \((E(|u_n|))\) is bounded. Thus, we have

\[ \sup_{n \in \mathbb{N}} \int_{\Omega} E(|u_n(t, x, \cdot)|) dt dx \geq \sup_{n \in \mathbb{N}} \int_{\Omega} l dt dx \implies \]

\[ \implies \lim_{t \to \infty} \sup_{n \in \mathbb{N}} \int_{\Omega} E(|u_n(t, x)|) dt dx \geq \sup_{n \in \mathbb{N}} \text{meas}(\Omega_n^t), \]

implying \( (\ref{eq:limit}) \). \( \Box \)

Before we pass to the proof of the theorem, we need a notion of the truncation operator \([9]\)

\[ T_N(z) = \begin{cases} 
\begin{aligned}
    z, & |z| < N \\
    N, & |z| \geq N \\
    -N, & |z| \leq -N
\end{aligned}
\end{cases} \]

It is by now well known that if we can prove that if for the sequence \((u_n)\) bounded in \( L^p \), \( p > 1 \), the sequence of its truncation \((T_N(u_n))\) converges in \( L_{loc}^1 \) for every \( N \in \mathbb{N} \), then the sequence itself converges in \( L_{loc}^1 \) as well \([9]\).

**Theorem 7.** For any \( u_0 \in L^\infty(M) \) there exists a unique admissible stochastic solution to \((\ref{eq:heat}), (\ref{eq:initial})\).

**Proof:** First, let us prove that the family \((m_\varepsilon)\) is the family of uniformly bounded functionals on \([L^2_{loc}(\Omega; C_0([0, T]\times M \times [-R, R]))\) for any \( R > 0 \). To this end, we simply take a test function \( \varphi \in C^\infty_0([0, T]\times M \times [-R, R]) \) and test it against \( (\ref{eq:heat}) \). We get

\[ \int_{[0, T] \times M \times [-R, R]} (-\text{sign}_+(u_\varepsilon - \xi) \partial_t \varphi - \text{sign}_+(u_\varepsilon - \xi)) \Phi(x, \xi) \cdot \nabla_g \varphi \ dt dx d\xi - \]

\[ \int_{[0, T] \times M \times [-R, R]} \varepsilon (\text{sign}_+(u_\varepsilon - \xi)) \Delta_g \varphi dt dx d\xi - \int_{[0, T] \times M \times [-R, R]} \partial_\xi \varphi(t, x, \xi) \frac{\Phi^2(x, \xi)}{2} d\nu_\varepsilon dt dx \]

\[ - \int_{[0, T] \times M \times [-R, R]} \varphi(t, x, \xi) \Phi(x, \xi) d\nu_\varepsilon dt dx = - \int_{[0, T] \times M \times [-R, R]} \partial_\xi \varphi(t, x, \xi) dm_\varepsilon, \]
Finding square of the latter expression, using the basic Young inequality and the Ito isometry, we get

$$E\left(\left|\int_{[0,T] \times M \times [-R,R]} \partial_\xi \varphi(t, x, \xi) dm_\xi\right|^2\right) \leq 5 E\left(\left|\int_{[0,T] \times M \times [-R,R]} |\partial_\xi \varphi| dt dx d\xi\right|^2\right) + E\left(\left|\int_{[0,T] \times M \times [-R,R]} |f\varphi| dt dx d\xi\right|^2\right)
$$

By using the Kantor diagonalization procedure, we conclude that there exists a bounded functional on the Bochner space $L^2_{\text{loc}}(\Omega; C^2([0, T] \times M \times [-R, R]))$. Since $m_\varepsilon$ is non-negative, according to the Schwartz lemma on non-negative distributions, we know that $m$ is bounded in $M$. If we fix $K \subset M$, we know that $(m_\varepsilon)$ is bounded functional on the Bochner space $L^2(\Omega; C_0([0, T] \times M \times [-R, R]))$. Thus, $(m_\varepsilon)$ is weakly precompact in $L^2_{\text{meas}}(\Omega; C_0([0, T] \times M \times [-R, R]))$ (see [10] p.606). By using the Kantor diagonalization procedure, we conclude that there exists a subsequence $(m_n)_{n \in \mathbb{N}}$ of $(m_\varepsilon)$ weakly converging toward $m \in L^2_{\text{meas}}(\Omega; C_0([0, T] \times M \times [-R, R]))$.

Now, we need to derive estimates for $u_\varepsilon$. The procedure is essentially the same as when deriving the estimates for $(m_\varepsilon)$. Indeed, take in (72) the entropy $\theta(z) = z^2$ and integrate over $[0, T] \times M$. We get

$$\int_M u_\varepsilon^2(T, x) d\gamma(x) - \int_M u_0^2(T, x) d\gamma(x) \leq \int_0^T \int_M \Phi^2(x, u_\varepsilon) d\gamma(x) + \int_0^T \int_M 2\Phi(x, u_\varepsilon) u_\varepsilon d\gamma(x) dW$$

where we used compactness of the manifold $M$ which provides

$$\int_M \left(- \text{div}_g \int_0^{u_\varepsilon} \theta'(v) f(x, v) dv\right) d\gamma(x) = 0 \quad \text{and} \quad \int_M \Delta_g u_\varepsilon d\gamma(x) = 0.$$ 

Now, we pass $\int_M u_0^2(T, x) d\gamma(x)$ to the right hand side, square both sides of such obtained expression and use the Ito isometry in the last step to discover

$$E\left(\left(\int_M u_\varepsilon^2(T, x) d\gamma(x) - \int_M u_0^2(T, x) d\gamma(x)\right)^2\right) \leq \left(2E\left(\int_0^T \int_M \Phi^2(x, u_\varepsilon) d\gamma(x) dt\right) + \int_0^T \int_M 2\Phi(x, u_\varepsilon) u_\varepsilon d\gamma(x) dt\right)^2.$$ 

From here, using assumed bounds on the function $\Phi$, we conclude that the expected value of $L^2(M)$-norm of the sequence $(u_\varepsilon(T, \cdot))$ is bounded i.e., after integrating everything over $T \in [0, t]$, $t > 0$, we conclude that the expected value of $L^2([0, t] \times M)$-norm of the sequence $(u_\varepsilon)$ is bounded as well.
Denote by \( h \) weak-* limit in \( L^\infty([0, T] \times M \times \mathbb{R} \times \Omega) \) along a subsequence of the family \( \{h_\epsilon\} = \text{sign}_+(u_\epsilon - \lambda) \). It satisfies (40) as well as Definition 3. Thus, it is a unique kinetic solution to (1), (2). According to Corollary 4, it has the form \( h = \text{sign}_+(u - \xi) \). Finally, it remains to prove that the second item from the Definition 2 is satisfied. This follows from the fact that

\[
\text{sign}_+(u_\epsilon(t, x) - \lambda) \xrightarrow{L^\infty} \text{sign}_+(u(t, x) - \lambda), \quad \epsilon \to 0,
\]

from where, since \( \text{sign}(z) = \text{sign}_+(z) - (1 - \text{sign}_+(z)) \) it also follows

\[
\text{sign}(u_\epsilon(t, x) - \lambda) \xrightarrow{L^\infty} \text{sign}(u(t, x) - \lambda), \quad \epsilon \to 0.
\]

Fix \( N > 0 \) and multiply this first by the characteristic function of the interval \((-N, N)\) denoted by \( \chi_{(-N, N)}(\lambda) \) and then by \( \lambda \chi_{(-N, N)}(\lambda) \). We get for the truncation operator \( T_N \):

\[
T_N(u_\epsilon) \xrightarrow{L^\infty} T_N(u) \quad \text{as} \quad \epsilon \to 0
\]

\[
(u_\epsilon^2 - N^2)\chi_{|u_\epsilon|<N} = T_N(u_\epsilon)^2 - N^2 \xrightarrow{L^\infty} (u_\epsilon^2 - N^2)\chi_{|u|<N} = T_N(u)^2 - N^2 \quad \text{as} \quad \epsilon \to 0.
\]

(77)

Now, we consider for a fixed non-negative test function \( \phi \in C_c([0, T] \times M) \):

\[
E \left( \int_{[0,T] \times M} \phi(t, x) (T_N(u_\epsilon) - T_N(u))^2 \, dt \, d\gamma(x) \right)
\]

\[
= E \left( \int_{[0,T] \times M} \phi(t, x) [T_N(u_\epsilon)^2 - T_N(u)^2] \right)
\]

\[
+ 2T_N(u)(T_N(u) - T_N(u_\epsilon)) \, dt \, d\gamma(x) \rightarrow 0
\]

according to (77). From here, we conclude that expectations of the truncations \( T_N(u_\epsilon) \) of the sequence \( \{u_\epsilon\} \) strongly converge in \( L^2_{loc}([0, T] \times M) \) toward

\[
E(T_N(u)) = E(u^N).
\]

From here, we can conclude about the convergence of \( (E(|u_\epsilon - u|^2)) \). First, we prove that the obtained sequence \( \{u^N\} \) converges strongly in \( L^1_{loc}([0, T] \times M) \) as \( N \to \infty \).

To this end, let \( U \subset \subset [0, T] \times M \). It holds

\[
\lim_{l \to \infty} \limsup_n E(\|T_l(u_n) - u_n\|_{L^1(U)}) \to 0.
\]

(78)

Denote by

\[
U^l_n = \{ (t, x) \in U : E(u_n(t, x, \cdot)) > l \}.
\]

Since \( (E(\|u^2\|_{L^2(U)}) \) is bounded, we have

\[
E \left( \int_U |u_n - T_l(u_n)| \, dt \, dx \right) \leq E \left( \int_{U^l_n} |u_n| \, dt \, dx \right) \leq \text{meas}(U^l_n)^{1/2} E(\|u_n\|_{L^2([0, T] \times M)}) \to 0
\]

uniformly with respect to \( n \) according to (19). Thus, (78) is proved.

Next, we have

\[
E(\|u^1 - u^2\|_{L^1(U)}) \leq E(\|u^1 - T_l(u_n)|_{L^1(U)}) + \|T_l(u_n) - u_n\|_{L^1(U)} + E(\|T_l(u_n) - u_n\|_{L^1(U)} + \|T_l(u_n) - u^2\|_{L^1(U)}),
\]

(19)
which together with (78) implies that \((u^l)\) is a Cauchy sequence with respect to expectation of the \(L^1\)-norm. Thus, there exists a measurable function \(u\) such that

\[
E(\|u^l - u\|_{L^1(U)}) \to 0 \text{ as } l \to \infty.
\]

(79)

Now it is not difficult to see that entire \((u_{nk}\) converges toward \(u\) in the same norm as well. Namely, it holds

\[
E(\|u_{nk} - u\|_{L^1(U)}) \leq E(\|u_{nk} - T_l(u_{nk})\|_{L^1(U)}) + \\
+ E(\|T_l(u_{nk}) - u^l\|_{L^1(U)}) + E(\|u^l - u\|_{L^1(U)}),
\]

which by the definition of functions \(u^l\), and convergences (78) and (79) imply the statement. Moreover, since \(u_n\) is bounded in expectation of the \(L^2\)-norms, the function \(u\) must be such as well.

\[\square\]

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