Particle Tracking With Space Charge Effects Using Graphics Processing Unit

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Abstract—Particle tracking simulations with space charge effects are very important for high-intensity proton rings. Since they include not only Hamiltonian mechanics of a single particle but also constructing charge densities and solving Poisson equations to obtain the electromagnetic field due to the space charge, they are extremely time-consuming. We have newly developed a particle tracking simulation code that can be used in graphics processing units (GPUs). GPUs have strong capacities of parallel processing so that the calculation of single-particle mechanics can be done very fast by complete parallelization. Our new code also includes the space charge effect. It must construct charge densities, which cannot be completely parallelized. For the charge density construction, we can use “shared memory,” which can be accessed very fast from each thread. The usage of shared memory is another advantage of GPU computing. As a result of our new development, we increase the speed of our particle tracking, including the space charge effect approximately ten times faster than that in the case of our conventional code used in CPU.

Index Terms—Graphics processing unit (GPU), proton accelerator, space charge effect.

I. INTRODUCTION

SPACE charge effects limit the number of particles that can be formed into a bunch in proton accelerators. Electromagnetic fields in a bunch force individual protons to the outside of the bunch. For ring accelerators such as synchrotrons, these defocusing effects shift a betatron tune, which is defined as the number of transverse oscillations per one turn. The betatron tune must be precisely controlled by quadrupole magnets to prevent errors and nonlinear terms of magnetic fields from exciting the transverse oscillations; otherwise, space charge effects can cause emittance growth and consequent beam losses. Electromagnetic potentials due to space charge effects generally add nonlinear terms to Hamiltonian of the transverse motion so that the tune shifts of individual protons depend on their transverse amplitude. These different betatron tunes in a bunch can be hardly measured and controlled. We thus rely on numerical simulation for space charge effects.

Particle-in-cell (PIC) methods [1] are widely used for numerical calculation of space charge effects. In PIC methods, the following processes are repeated after simulated spaces are divided into many cells with their representative points called grids.

1) Charge densities are calculated by assigning each particle to the adjacent grids.
2) Potentials and fields are obtained by solving the Poisson equations.
3) Particles are tracked by solving the equations of motion in the obtained electromagnetic fields.

Particles in ring accelerators usually pass thousands of components for a single turn and circulate over thousands of turns. This involves millions of charge densities and potentials to be calculated. In addition, numerous particles ($10^5$–$10^6$) must be simulated for sufficient accuracy. For these reasons, the PIC methods for ring accelerators require large computational resources.

We developed a new PIC simulation code executable by graphics processing units (GPUs). Their high parallel computing performance makes multiparticle tracking efficient. Even calculations of the charge densities and potentials, which are not completely parallelized, can be accelerated using on-chip shared memory provided by GPUs. Although there are several tracking simulations using GPU that are reported [2]–[4], we concentrate on the application for long particle bunches so that 2-D PIC simulation can be used. In case of a $100 \times 100$ grid, for example, all grid cells can be allocated in the on-chip shared memory. This can drastically shorten the time required for making charge distributions. In this article, the details of our new PIC simulation code are described.

This article is organized as follows. In Section II, general features of GPUs are briefly described. In Section III, the treatments of single-particle dynamics in the code are explained. The single-particle dynamics describes particle motions in external fields of the accelerator components such as electromagnets. Section IV describes how the code involves the calculations about space charge effects. This is the main part of this article. In Section V, we show some results of the simulation for the Japan Proton Accelerator Research Complex (J-PARC) Main Ring [5] using the code. They are compared with the results by another existing simulator. The speed of the new code is also compared with a similar PIC code running on CPUs. Finally, we will summarize this article in Section VI.

II. GRAPHIC PROCESSING UNIT

Fig. 1 shows the structure of a GPU from the viewpoint of software. A thread is a basic unit for parallel processing.
Particle tracking is completely parallelized if each particle is assigned to a thread. A block is a group of threads. Each block has a shared memory to which only threads in the block can access. On the other hand, global memory can be accessed by any thread in the GPU. Although shared memory (48–96 KB) is smaller than global memory (16–32 GB), it has much higher bandwidth and lower latency.

Nvidia provides a parallel computing platform called Compute Unified Device Architecture (CUDA) [6]. Using the CUDA platform, software developers can design applications executable by CUDA-enabled GPUs using programming languages, such as C, C++, and Fortran. In fact, our new code is developed using C++.

III. Single-Particle Mechanics

Our developed code separately simulates particle motions by external fields and space charge effects. To describe a single-particle Hamiltonian, we use an independent variable s that is the length along the reference orbit and a 3-D coordinate system \((x, y, \sigma)\), where \(x\) and \(y\) are 2-D coordinates on a plane perpendicular to the beam direction and \(\sigma\) is defined as \(s - c\beta_0d\) using the velocity of the reference particle \(\beta_0c\). Using this coordinate system with their conjugate variables \((p_x, p_y, p_\sigma)\) and a vector potential \(A_\sigma(x, y)\), the Hamiltonian can be written as

\[
H(x, p_x, y, p_y, \sigma, p_\sigma; s) = p_\sigma - (1 + h\chi) \sqrt{(1 + \alpha^2)^2 - p_x^2 - p_y^2} - eA_\sigma(x, y) p_\sigma
\]

\[
\approx A_\sigma(x, y) p_\sigma = H_{\text{approx}}(x, y, \sigma, p_\sigma; s)
\]

where \(\chi, \gamma_0\), and \(p_0\) are the curvature, gamma factor, and momentum of the reference particle, respectively. In addition, the momentum deviation \((p - p_0)/p_0\) is expressed as \(\delta\), which can be approximated as \(p_\delta + (p_\sigma^2/2\gamma_0^2)\). How to solve the equation of motion depends on the types of external fields \(A_\sigma(x, y)\). Three different cases are shown as follows.

A. Uniform Fields

For uniform fields such as dipole magnets and drift spaces, the equation of motion \((dP(t)/dt) = eB(t)c \times \vec{B}\) (where \(t\) is time) can be analytically solved so that we do not need to use the approximate Hamiltonian \(H_{\text{approx}}\) in (1).

First, we consider the 3-D Cartesian coordinate system \((\tilde{x}, \tilde{y}, \tilde{z})\), as shown in Fig. 2. The directions of \(\tilde{x}\) and \(\tilde{y}\) are aligned to the reference orbit at the magnet entrance and the magnetic field \(\vec{B} = (0, B, 0)\), respectively. The \(\tilde{x}\)-axis is chosen so that the system becomes left-handed. The origin is defined as the curvature center of the reference orbit. Since we consider only sector dipoles, we assume \(\tilde{z}(0) = 0\) as an initial condition. The solutions for \((\tilde{x}, \tilde{z})\) as well as the corresponding momenta \((P_x(t), P_z(t))\) are obtained as

\[
\tilde{x}(t) = \frac{P_x(0)}{eB} \cos m\gamma_0 t - \frac{P_z(0)}{eB} \sin m\gamma_0 t + \tilde{x}(0) - \frac{P_z(0)}{eB}
\]

\[
\tilde{z}(t) = \frac{P_x(0)}{eB} \sin m\gamma_0 t + \frac{P_z(0)}{eB} \cos m\gamma_0 t + \frac{P_z(0)}{eB}
\]

where

\[
P_x(0) = \sqrt{2} p_x \left(1 - p_\sigma^2 - p_\sigma^2 \left(1 - p_\sigma^2/2\gamma_0^2\right)\right)
\]

\[
P_z(0) = p_0 p_\sigma(0).
\]

Here, the solution about the direction of the uniform field \(B\) \((\tilde{y})\) is not shown since no force is applied to the direction. Second, the coordinate transformation from \((\tilde{x}, \tilde{y}, \tilde{z}; t)\) to \((x, y, \sigma; s)\) is performed as

\[
x(s = L) = \sqrt{\tilde{x}(t')} + \tilde{z}(t') - \rho
\]

\[
p_x(s = L) = \frac{P_x(t') \cos \frac{L}{\rho} + P_z(t') \sin \frac{L}{\rho}}{\rho}
\]

\[
\sigma(s = L) = \sigma(0) + L - \beta_0d', p_\sigma = \text{Const}
\]

where \(t'\) can be obtained by the equation

\[
\tilde{z}(t') = \tilde{x}(t') \tan \left(\frac{L}{\rho}\right)
\]

as shown in Fig. 2.

B. Thick Quadrupole Magnets

For thick quadrupole magnets \((eA_\sigma(x, y)/p_0 = -(1/2)k_1(x^2 - y^2))\), analytical solutions of the equation
of motion for $H_{\text{approx}}$

\[
x(s) = x(0) \cos \sqrt{k_1(1 - p_x(0))} s + \sqrt{1 - p_x(0)} \frac{k_1}{k_1} p_x(0) \sin \sqrt{k_1(1 - p_x(0))} s
\]

\[
y(s) = y(0) \cosh \sqrt{k_1(1 - p_x(0))} s + \sqrt{1 - p_x(0)} \frac{k_1}{k_1} p_y(0) \sinh \sqrt{k_1(1 - p_x(0))} s
\]

\[
p_x(s) = p_x(0) \cos \sqrt{k_1(1 - p_x(0))} s - \sqrt{1 - p_x(0)} \frac{k_1}{k_1} x(0) \sin \sqrt{k_1(1 - p_x(0))} s
\]

\[
p_y(s) = p_y(0) \cosh \sqrt{k_1(1 - p_x(0))} s + \sqrt{1 - p_x(0)} \frac{k_1}{k_1} y(0) \sinh \sqrt{k_1(1 - p_x(0))} s
\]

\[
\sigma(s) = \frac{1}{2} \int_0^s ds' \left( p_x(s')^2 + p_y(s')^2 \right) + \frac{p_x(0)}{\gamma_0} s
\]

\[
p_\sigma(s) = p_\sigma(0)
\]

are used since it is difficult to solve the exact equation of motion without any approximations. The solution can be expressed as a symplectic transformation $e^{iH_{\text{approx}}L\vec{q}_0}$, where $L$ and $\vec{q}_0$ are the length of the component along the reference orbit and the initial canonical variables, respectively.

C. Thick Sextupole Magnets

For thick sextupole magnets ($eA_e(x, y)/p_0 = -(1/6) k_2(x^3 - 3xy^2)$), analytical solutions are hardly obtained even for $H_{\text{approx}}$. In this case, $H_{\text{approx}}$ is divided into two parts as

\[
H_{\text{approx}} = H_0 + V
\]

\[
H_0 = \frac{p_x^2 + p_y^2}{2} + \frac{p_\sigma^2}{2\gamma_0} - \frac{p_x^2 + p_x^2}{2} p_\sigma
\]

\[
V = \frac{1}{6} k_2(x^3 - 3xy^2)
\]

where the equations for $H_0$ and $V$ are analytically solvable, and then, the final state is obtained by multiple symplectic transformations described as

\[
e^{iH_0 a_L} e^{iVb_L} e^{iH_0 c_L} e^{iVb_L} e^{iH_0 a_L}
\]

where $a = (1/2)(1 - (1/3)^{1/2})$, $b = (1/2)$, and $c = (1/3)^{1/2}$ [7].

IV. SPACE CHARGE EFFECTS

In this code, we assume that the longitudinal length of a bunch is much larger than the transverse width. The assumption of long bunches is quite reasonable for the J-PARC Rapid Cycle Synchrotron [8] and Main Ring [5]. This corresponds to a 2-D approximation of the Poisson equation

\[
\lambda(z) \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x, y) = -\frac{\lambda(z) f(x, y)}{\epsilon_0}.
\]

Here, $\lambda(z)$ shows the line density of a bunch. For the numerical calculation shown in this section, the line density is created as $\sigma$-distributions with 128 bins where $\sigma(s = c\beta_0 t)$ can be obtained by solving the equation of motion for the Hamiltonian shown in (1). The 2-D potential $u(x, y)$ includes the contribution from not only the charge and current density of a bunch itself but also their images through the beam pipes and magnetic poles. The electric fields cannot penetrate beam pipes (conductors) at any frequencies. On the other hand, the magnetic fields only at low frequencies penetrate the beam pipes so that the image current through the magnetic poles must be considered. Therefore, the effective 3-D potential $\phi(x, y, z)$ can be written as

\[
\phi(x, y, z) = \frac{e}{m\gamma_0 \beta_0^2 c^2} \left( \lambda(z) - \lambda_{\text{dc}}(z) \right) u_{\text{free}}(x, y) + u_{\text{image}, \|}(x, y)
\]

\[
-\beta_0^2 \lambda_{\text{ac}}(z) \left( u_{\text{free}}(x, y) + u_{\text{image}, \perp}(x, y) \right).
\]

Here, we divide the charge density $\lambda(z)$ into the dc part $\lambda_{\text{dc}}(z)$ and the ac part $\lambda_{\text{ac}}(z) = \lambda(z) - \lambda_{\text{dc}}$. $u_{\text{free}}$ is the electric potential in free space. $u_{\text{image}, \|}$ is the potential due to the image charge for elimination of the electric field at the beam pipe. The coefficient $-\beta_0^2$ is used for the conversion from electric potentials to magnetic ones. $-\beta_0^2 u_{\text{image}, \perp}(x, y)$ is the potential due to the image current, which eliminates the tangential components of the magnetic field at the magnetic pole. Although our developed code involves the calculation of potentials of (10), there is no experimental or numerical benchmark so far. Therefore, we adopt an additional approximation assuming that the ac part of the bunch is larger than the dc, which means that $\lambda(z) = \lambda_{\text{ac}}(z)$. This approximation is used for other tracking code called “Space-Charge TRacker (SCTR)” [12], [13]. As a result of the approximation, we obtain

\[
\phi(x, y, z) = \frac{e}{m\gamma_0 \beta_0^2 c^2} \left( 1 - \beta_0^2 \right) \lambda(z) u_{\text{free}}(x, y) + u_{\text{image}, \|}(x, y)
\]

\[
= \frac{e}{m\gamma_0 \beta_0^2 c^2} \lambda(z) u_{\text{free}}(x, y) + u_{\text{image}, \|}(x, y).
\]

In this approximation, we just have to solve the 2-D Poisson equation shown in (9) with the boundary condition $u(x, y) = 0$. The transverse kick due to space charge effects is calculated as

\[
-\frac{e}{m\gamma_0 \beta_0^2 c^2} \lambda(z) \left( \frac{\partial}{\partial x} \frac{\partial}{\partial y} \right) u(x, y) \times L
\]

where $L$ is the distance from the previous location at which space charge effects are calculated. The longitudinal kick due to space charge effects is calculated as

\[
-\frac{e}{m\gamma_0 \beta_0^2 c^2} u(x, y) \frac{\partial}{\partial z} \lambda(z) \times L.
\]

In this section, we describe how to make 2-D charge densities $f(x, y)$ and solve 2-D Poisson equations for the potential $u(x, y)$. Although this code enables us to choose the Cartesian $f(x, y)$ or polar $f(r, \theta)$ coordinate system depending on the cross sections of beam pipes, the descriptions in this section are based on the Cartesian coordinate system.
A. Charge Density Calculation

A 2-D charge density \( f(x, y) \) fits the size of shared memory of GPUs when the number of cells is approximately about 10000 (100 \times 100). In fact, a state-of-the-Art GPU can allocate 96-KB shared memory, which corresponds to 12000 double-precision floating-point numbers. Charge densities are calculated using the final states of all particles for each component. When a particle located at \((x, y)\) is in the rectangle whose vertices are four grids labeled as \((x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j+1}), \) and \((x_{i+1}, y_{j+1})\) (see Fig. 3), a 2-D histogram is filled as

\[
Q(x_i, y_j) = \frac{(x_{i+1} - x_i)(y_{j+1} - y_j)}{\delta x \delta y}
\]

\[
Q(x_{i+1}, y_j) = \frac{(x_{i+1} - x_i)(y_{j+1} - y_j)}{\delta x \delta y}
\]

\[
Q(x_i, y_{j+1}) = \frac{(x_{i+1} - x_i)(y_{j+1} - y_j)}{\delta x \delta y}
\]

\[
Q(x_{i+1}, y_{j+1}) = \frac{(x_{i+1} - x_i)(y_{j+1} - y_j)}{\delta x \delta y}
\]

where \(\delta x = x_{i+1} - x_i\) and \(\delta y = y_{j+1} - y_j\).

The entries and bins correspond to the charges and grids, respectively. In case that a thread fills the histogram for a single particle, collisions between threads occur when filling a common bin of the histogram. This is why not all threads can be executed in parallel. To reduce these colliding threads, as shown in Fig. 4, all threads in a block fill a subhistogram allocated at shared memory of the block so that fewer threads fill a common histogram [10]. Once all subhistograms are filled, they are summed. It must be noted that colliding threads can still occur, but much less frequently. We thus use a special operation called an atomic operation provided by the CUDA platform to fill the subhistograms. When an atomic operation accesses data at some address, other memory accesses to the same address are blocked until the operation is done.

B. Poisson Solver

Using the charge distribution \( f(x, y) \), potential \( u(x, y) \) is obtained by solving the 2-D Poisson equation

\[
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(x, y) = f(x, y) \tag{15}
\]

with the boundary conditions

\[
u(x, 0) = u(L_x, y) = u(0, y) = u(x, L_y) = 0 \tag{16}
\]

where \(L_x\) and \(L_y\) are the horizontal and vertical lengths of a beam pipe, respectively. The boundary conditions suppress the electric fields at the inner surface of the beam pipe. The CUDA platform involves a library of fast Fourier transform (FFT) called “cuFFT” [11]. The functions in cuFFT are designed to provide high performance on CUDA-enabled GPUs. We thus employ the discrete Fourier transform (DFT) for the Poisson solver.

The differential (15) is discretized as

\[
\frac{u_{i,j-1} - 2u_{i,j} + u_{i+1,j}}{\delta x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\delta y^2} = f_{i,j} \quad i, j = 1, 2, \ldots, m \tag{17}
\]

where \(m\) is the number of cells in one direction. The odd extensions of \(u_{i,j}\) and \(f_{i,j}\), which are labeled as \(V\) and \(F\), are constructed as (18), shown at the bottom of the next page, where only \(V\) is shown, but \(F\) can be constructed in the same manner. Equation (17) of \(V\) and \(F\) instead of \(u\) and \(f\)

\[
\frac{V_{l-1} - 2V_{l} + V_{l+1}}{\delta x^2} + \frac{V_{l-1} - 2V_{l} + V_{l+1}}{\delta y^2} = F_{l,l'} \quad l, l' = 0, 1, 2, \ldots, 2m + 1 \tag{19}
\]

is also satisfied. Using \(V\), the boundary conditions (16) become

\[
V_{l,m+1} = V_{l,0} = V_{0,l'} = V_{m+1,l'} = 0 \tag{20}
\]

which are satisfied by definition of (18).

The 1-D DFT is defined as

\[
DFT_l(g_l) = \sum_{l=0}^{N-1} g_l e^{-\frac{2\pi i l m}{N}} \quad p = 1, 2, \ldots, N \tag{21}
\]

where \(l\) is an index to one direction. Applying DFT about one direction labeled as \(l\) to the first term of the left-hand side in (19), we obtain (22), as shown at the bottom of the next page.

Another DFT to (22) about the other direction labeled as \(l'\) gives

\[
DFT_l \left( DFT_{l'} \left( V_{l-1,l'} - 2V_{l,l'} + V_{l+1,l'} \right) \right) = -4 \sin^2 \frac{\pi}{2(m+1)} \sum_{l=0}^{N-1} \sum_{l'=0}^{N-1} e^{-\frac{2\pi i (l+l')p}{N}} V_{l,l'} \tag{23}
\]
By adding the DFT to the second term of the left-hand side in (19), the total 2-D DFT of the left-hand side in (19) is written as

\[ \text{DFT}_q \left( \text{DFT}_p \left( \frac{V_{l-1,F} - 2V_{l,F} + V_{l+1,F}}{\delta x^2} \right)_p \right)_q \]

\[ + \text{DFT}_q \left( \text{DFT}_p \left( \frac{V_{l,F-1} - 2V_{l,F} + V_{l,F+1}}{\delta y^2} \right)_p \right)_q \]

\[ = -4 \left( \frac{1}{\delta x^2} \sin^2 \frac{p \pi}{2(m+1)} + \frac{1}{\delta y^2} \sin^2 \frac{q \pi}{2(m+1)} \right) \times \text{DFT}_q \left( \text{DFT}_p \left( V_{l,F} \right)_p \right)_q. \]

(24)

Since this equals the 2-D DFT of \( F_{l,F} \), the 2-D DFT of \( V_{l,F} \) is derived as

\[ \text{DFT}_q \left( \text{DFT}_p \left( V_{l,F} \right)_p \right)_q \]

\[ = -\frac{\text{DFT}_q \left( \text{DFT}_p \left( F_{l,F} \right)_p \right)_q}{4 \left( \frac{1}{\delta x^2} \sin^2 \frac{p \pi}{2(m+1)} + \frac{1}{\delta y^2} \sin^2 \frac{q \pi}{2(m+1)} \right)}. \]

(25)

Using inverse DFT defined as

\[ i\text{DFT}_p(G_p)_q \equiv \frac{1}{N} \sum_{p=0}^{N-1} G_p e^{2\pi i p q / N}. \]

(26)

The potential \( V_{l,F} \) can be obtained as

\[ V_{l,F} = -\frac{\text{iDFT}_q \left( \text{iDFT}_p \left( \text{DFT}_q \left( F_{l,F} \right)_p \right)_q \right)_q}{4 \left( \frac{1}{\delta x^2} \sin^2 \frac{p \pi}{2(m+1)} + \frac{1}{\delta y^2} \sin^2 \frac{q \pi}{2(m+1)} \right)}. \]

(27)

This solution (27) involves two DFT and two inverse DFT operations, which are calculated using the functions provided by cuFFT. Fig. 5 shows a 2-D charge density and the corresponding potential. The area in both 2-D histograms corresponds to the cross section of a beam pipe. This is the reason why the potential becomes constant at the boundary.

C. Interpolation

The potential at \((x, y)\) as shown in Fig. 6 is obtained by the interpolation using a Bezier surface written as

\[ u(x, y) = \sum_{m=0}^{3} \sum_{n=0}^{3} \left( u_{i+m-1,j+n-1} \right) \times \frac{3!}{m!(3-m)!} \left( \frac{x-x_i}{\delta x} \right)^m \left( \frac{x_{i+1}-x}{\delta x} \right)^{3-m} \times \frac{3!}{n!(3-n)!} \left( \frac{y-y_j}{\delta y} \right)^n \left( \frac{y_{j+1}-y}{\delta y} \right)^{3-n}. \]

(28)

The electric field can be obtained as

\[ \tilde{E}(x, y) = -\nabla u(x, y) = -\left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) u(x, y). \]

(29)

V. J-PARC MAIN RING AS A SIMULATION EXAMPLE

To verify the developed code, we simulated several parameters of the J-PARC Main Ring. Table I shows the main parameters of the J-PARC Main Ring [5]. The simulation parameters are shown in Table II. The components involve drift spaces, RF cavities, and short corrector magnets as well as the magnets shown in Table I. The locations of the space charge calculation are chosen so that their intervals are approximately 1 m or less. For example, the calculation in a bending magnet, whose length approximately 6 m, is performed six times. We fixed the intervals to reduce the beam losses due to numerical errors (unphysical beam losses without any imperfections) down to much smaller than actually observed. This is a very important task because the code should be useful for actual beam tuning.
The transverse beam emittance, momentum spread, and bunch factor are based on the measurements. The bunch factor is increased up to 0.3 during the ring circulation due to the second-harmonic RF technique, which is also implemented in the code. For the chromatic correction, the sextupole strength is decreased by 25% compared to the full correction.

The betatron amplitude functions and dispersion functions obtained by the code are compared to the calculation by Strategic Accelerator Design (SAD) [9]. The SAD calculations employ transfer matrices. On the other hand, the new code calculates them in different ways. Assuming that the transverse particle distribution is Gaussian, the betatron amplitude function at $s$ ($\beta(s)$) can be written as $\sigma^2(s)/\epsilon_{1\sigma}$, where $\sigma(s)$ is the standard deviation of the transverse distribution at $s$ and $\epsilon_{1\sigma}$ is the $1\sigma$ emittance. For the new code, $\beta(s)$ are obtained by calculating the rms of the transverse coordinates ($x$ or $y$) of all tracked particles without space charge effects. For the dispersion functions in the new code, we use the central orbit distortions of off-momentum particles, which are actually tracked. As shown in Figs. 7 and 8, the results from the new code reproduce the SAD calculation. These validate single-particle mechanics in the new code described in Section III.

To validate space charge effects in the new code, we calculate the betatron tune shift of all tracked particles. Fig. 9 shows the betatron tunes of all tracked particles. The tunes are obtained as the transverse phase advances through one turn. The same plot using “SCTR” [12], [13] is also shown in Fig. 10. Our developed code clearly reproduces the result.
Fig. 8. Dispersion functions of the J-PARC main ring.

Fig. 9. 2-D distribution of the betatron tunes of all tracked particles (our developed code).

Fig. 10. 2-D distribution of the betatron tunes of all tracked particles (SCTR).

from “SCTR.” The betatron tunes without space charge effects (only due to the strength of the quadrupole magnets) are set at \((\nu_x, \nu_y) = (21.35, 21.44)\). The estimated tune spread \(\Delta \nu_{x,y}\) can be obtained as

\[
\Delta \nu_{x,y} = - \frac{N_{\text{ppb}} h r_p}{4 \pi \beta_0^2 \gamma_0^2 \epsilon_{\text{le}} B_f} = -0.45. \tag{30}
\]

As shown in Table I, we use \(3.4 \times 10^{13}\), 9, and 4 for \(N_{\text{ppb}}, h,\) and \(\epsilon_{\text{le}},\) respectively. For the bunch factor, we use 0.3 for (30) and Figs. 9 and 10. \(r_p\) is the classical proton radius \((\approx 1.547 \times 10^{-18} [\text{m}])\). The numerical result shown in Fig. 9 is consistent with the rough estimation using the formula.

The number of turns of the J-PARC Main Ring, which can be simulated per minute, is used as a benchmark. For this test, the same numbers of macroparticles (200,000) are used for both codes. The potential calculation is different between two. Our new code sets a potential boundary on the inner surface of a beam pipe and selects a polar or rectangular coordinate depending on the cross section of the beam pipe. Due to the limitation of shared memories, we adopt a \(100 \times 100\) grid for both coordinates. On the other hand, SCTR uses a common rectangle boundary and a \(128 \times 128\) grid for all locations in the ring. Using the new code on a single TESLA-V100 GPU, we obtained approximately 117 turns per minute in contrast to 9 for SCTR [13] on Intel Xeon Gold 6126 (2.6 GHz). This is a significant difference. In fact, the beam losses in the J-PARC main ring are distributed from the injection until approximately \(5 \times 10^5\)th turn.

VI. CONCLUSION

Particle tracking simulations, including space charge effects, are very important for high-intensity proton rings. Since they include not only Hamiltonian mechanics of a single particle but also constructing charge densities and solving Poisson equations to obtain the electromagnetic field due to the space charge, they are extremely time-consuming. We have newly developed a particle tracking simulation code that can be used in GPU.

GPUs have strong capacities of parallel processing so that the calculation of single-particle mechanics can be done very fast by complete parallelization. Our new code also includes the space charge effect. It must construct charge densities, which cannot be completely parallelized. For the charge density construction, we fill subhistograms in shared memory before constructing the total histogram so that each thread can not only avoid their frequent collisions but also access subhistograms very fast. For the Poisson solver, we employ the DFT to take advantage of the usage of the cuFFT library, which is designed to provide high performance on CUDA-enabled GPUs.

To validate single-particle mechanics and space charge effects implemented in the code, the betatron amplitude functions, dispersion functions, and space-charge-induced tune spread are simulated in the case of the J-PARC Main Ring. These results reproduce the calculations by other simulators. In addition, the new code on a single TESLA-V100 GPU can simulate approximately 117 turns per minute in contrast to 9 for SCTR [13] on Intel Xeon Gold 6126 (2.6 GHz).
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