THE BOGOMOLOV–PANTEV RESOLUTION, AN
EXPOSITORY ACCOUNT

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STATEMENT OF THE RESULT

Bogomolov and Pantev [3] have recently discovered a rather elegant
geometric proof of the weak Hironaka theorem on resolution of singularities:

Theorem 1. Let \( X \) be a projective variety and \( Z \) a proper Zariski
closed subscheme of \( X \). There is a projective birational map \( \varepsilon : \tilde{X} \to X \)
such that \( \tilde{X} \) is smooth, the scheme theoretic inverse image \( \varepsilon^{-1}(Z) \) is a
Cartier divisor, the exceptional locus of \( \varepsilon \) is a divisor and the union of
these two divisors is a divisor with simple normal crossings.

Before the work cited above, and that of Abramovich and de Jong
[1] (appearing at roughly the same time) the only proof of this theo-
rem was as a corollary of the famous result of Hironaka [5]. These new
proofs were inspired by the recent work of de Jong [6], which Bogomolov
and Pantev combine with a beautiful idea of Belyi [2] “simplifying” the
ramification locus of a covering of \( \mathbb{P}^1 \) by successively folding up the \( \mathbb{P}^1 \)
onto itself, over a fixed base. This latter step unfortunately only works
in characteristic zero, limiting the scope of the argument (Abramovich
and de Jong’s paper gives some results even in characteristic \( p \)). Hence,
we work over the field of complex numbers; the argument also works
(with suitable modifications about rationality) over any field of char-
acteristic zero.

The outline of the argument we follow is the same as that of the
paper of Bogomolov and Pantev; however we offer different (and we
hope simpler) proofs of the corresponding lemmas. To begin with, their
argument using Grassmannians is replaced by an application of Noether
normalisation in Section 1. Belyi’s argument to reduce the degree of
individual components of the ramification locus is presented in purely
algebraic form in Section 1. Lemmas 3 and 4. The presentation of
Bogomolov and Pantev refers to “semi-stable families of pointed curves

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of genus 0”; the precise result required from that theory is proved here by means of blowups in Section 2. Finally we also give a summary (in Section 3) of the desingularisation of toroidal embeddings which is used by both the papers [3, 1]. To summarise, the aim of this account is to give all the details of the argument, so that it should be accessible to anyone with a basic knowledge of algebraic geometry (as contained for example in Mumford’s book [8]).

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Some reductions

For completeness, we define the notion of exceptional locus that we use. For a proper birational morphism \( \varepsilon : Y \to X \) of varieties, the exceptional locus is defined as the complement of the largest \( \varepsilon \)-saturated open subset of \( Y \) on which \( \varepsilon \) is an isomorphism. We note that the exceptional locus of the composite \( f \circ g \) is the union of the inverse image under \( g \) of the exceptional locus for \( f \) and the exceptional locus for \( g \).

Let \( f : X_1 \to X \) be the blowup of \( X \) along the scheme \( Z \); we thus ensure that \( f^{-1}(Z) \) supports a Cartier divisor. Now let \( g : X_2 \to X_1 \) be the blowup along the reduced scheme supported on the exceptional locus \( E(f) \) of \( f \). Finally, let \( h : X_3 \to X_2 \) be the blowup along the reduced scheme supported on the singular locus of \( X_2 \). Let \( Z' \) denote the union of the supports of inverse images of \( Z \), \( E(f) \) and \( (X_2)_{\text{sing}} \) in \( X_3 \).

Lemma 1. With notation as above, let \( \varepsilon_1 : \tilde{X} \to X_3 \) be a projective birational morphism with \( \tilde{X} \) smooth such that \( \varepsilon_1^{-1}(Z') \) is contained in a divisor with simple normal crossings. The composite morphism \( \varepsilon = f \circ g \circ h \circ \varepsilon_1 \) then satisfies the conclusion of Theorem 1.

Proof. Since any sub-divisor of a divisor with simple normal crossings (see Section 2 for the definition) is again a divisor with simple normal
crossings it is enough to show that each of \( \varepsilon^{-1}(Z) \) and the exceptional locus \( E(\varepsilon) \) are divisors.

First of all, \( f^{-1}(Z) \) supports a Cartier divisor, thus so does \( \varepsilon^{-1}(Z) \). It follows that the latter is a divisor.

If \( D \) is any divisorial component of \((X_2)_{\text{sing}}\), then \( X_2 \) is not normal along \( D \). It follows that \( X_3 \to X_2 \) is not an isomorphism along \( D \) and thus \( h^{-1}(D) \) is contained in the exceptional locus for \( h \). The inverse image of any codimension 2 component of \((X_2)_{\text{sing}}\) is clearly contained in the exceptional locus of \( h \). Thus, the exceptional locus for \( h \) is precisely \( h^{-1}((X_2)_{\text{sing}}) \). The exceptional locus of \( f \circ g : X_2 \to X \) is \( g^{-1}(E(f)) \) which supports a Cartier divisor by construction of \( g \). It follows that the exceptional locus of \( f \circ g \circ h : X_3 \to X \) supports a Cartier divisor. Moreover, the complement \( U \) of this divisor in \( X_3 \) is smooth and is mapped isomorphically to an open subset of \( X \) by \( f \circ g \circ h \). Let \( V = \varepsilon_1^{-1}(U) \). The morphism \( \varepsilon_1 : V \to U \) is a proper birational morphism of smooth varieties and thus its exceptional locus is a divisor \( D \) in \( V \). It thus follows that the exceptional locus of \( \varepsilon \) is a divisor. \( \square \)

As a corollary we see that the main result Theorem 1 is equivalent to the apparently weaker result:

**Theorem 2.** Let \( X \) be a projective variety and \( Z \) a proper Zariski closed subset of \( X \). There is a projective birational map \( \varepsilon : \tilde{X} \to X \) such that \( \tilde{X} \) is smooth. Moreover, the set theoretic inverse image \( \varepsilon^{-1}(Z) \) and the exceptional locus of \( \varepsilon \) are contained in a divisor with simple normal crossings.

We will now prove the two equivalent theorems, by proving Theorem 2. We can make the following trivial reductions:

1. We may assume that \( X \) is normal by replacing \( X \) with its normalisation and replacing \( Z \) by the union of its inverse image with the exceptional locus of the normalisation.
2. Any normal curve is smooth and any proper closed subset of a smooth curve is a simple normal crossing divisor (!). Hence the result is true in dimension 1.
3. We prove the theorem by induction on the dimension of \( X \). Thus we may assume that \( n = \dim X > 1 \) and that Theorem 1 holds for all varieties of smaller dimension than \( X \).
4. In order to prove Theorem 2 for a given \( Z \) it is enough to prove it for a larger \( Z' \supset Z \), as long as \( Z' \) is a proper subset of \( X \). Thus, we can expand \( Z \) at any stage in our construction of
the resolution. In particular, whenever we blow up \( X \) we will expand \( Z \) to include the exceptional locus of the blow up.

1. \( \mathbb{P}^1 \)-bundles

We will prove the following:

**Claim 1.** After replacing \( X \) by \( X' \), where the latter is the blow up of \( X \) at a finite set of smooth points, and replacing \( Z \) by the union of its inverse image in \( X' \) and the exceptional divisor of the blow up, we have:

1. There is a finite surjective map \( f: X \to \mathbb{P}_Y(F) \), where \( Y \) is smooth and \( F \) is a rank 2 vector bundle on \( Y \).
2. The morphism \( f \) is finite and \( \acute{e} \)tale outside a divisor \( B = \bigcup_{i=1}^n s_i(Y) \) which is the union of a finite number of sections \( s_i: Y \to \mathbb{P}_Y(F) \) of the \( \mathbb{P}^1 \)-bundle \( \mathbb{P}_Y(F) \).
3. The image \( f(Z) \) of \( Z \) contained in \( B \).

**Lemma 2.** After replacing \( X \) by its blowup at a suitable finite set of smooth points, we have the following situation. There is a smooth variety \( Y \), a line bundle \( L \) and a finite surjective map \( g: X \to Q = \mathbb{P}_N(O \oplus L) \) such that if \( E \) denotes the section corresponding to the quotient homomorphism \( O \oplus L \to O \), then there is a divisor \( B \subset Q \) disjoint from \( E \) and containing \( g(Z) \) and the branch locus of \( g \). Moreover, \( E \) is the image of the exceptional locus of the blowup of \( X \) at the finite set of smooth points.

**Proof.** This is essentially just Noether normalisation. Let \( X \hookrightarrow \mathbb{P}^N \) be a projective embedding of \( X \) and \( L = \mathbb{P}^{N-n-1} \) a linear subspace in \( \mathbb{P}^N \) that does not meet \( X \), where \( n = \dim X \). The projection from \( L \) gives a finite map \( X \to \mathbb{P}^n \). Let \( B \) be a hypersurface in \( \mathbb{P}^n \) that contains the image of \( Z \) and the branch locus of the map. Let \( p \) be a point of \( \mathbb{P}^n \) not on \( B \). Replace \( X \) by its blowup at the points lying over \( p \) (which are all smooth) and let \( Y = \mathbb{P}^{n-1} \) with \( L = O(1) \). The blowup of \( \mathbb{P}^n \) at \( p \) is naturally isomorphic to \( \mathbb{P}_Y(O \oplus L) \) and the resulting morphism from \( X \) to this \( \mathbb{P}^1 \)-bundle has the required form. \( \square \)

Now \( B \) is a divisor in the \( \mathbb{P}^1 \)-bundle \( Q \to Y \) that does not meet a section \( E \). Thus the projection \( B \to Y \) is finite and flat. In fact we have:

**Lemma 3.** In the above situation, if \( O_Q(1) \) denotes the universal quotient bundle, then \( O_Q(B) = O_Q(b) \) where \( b \) is the degree of the map \( B \to Y \).
Proof. Consider the line bundle $\mathcal{O}_Q(B) \otimes \mathcal{O}_Q(-b)$. Since this line bundle is trivial on the fibres of $Q \to Y$, it is the pullback of a line bundle from $Y$. But it restricts to the trivial bundle on $E$, which is a section of $Q \to Y$. Hence it is the trivial line bundle. \hfill \square

Set $Q = \mathbb{P}_Y(\mathcal{O} \oplus L)$, and let $B$ be a divisor in $Q$ which is finite over $Y$; assume moreover that $B = B_1 \cup \bigcup_{i=1}^r s_i(Y)$, where $s_i$ are sections, and $B_1$ does not meet $E$. Now let $d(B)$ denote the maximum degree over $Y$ of any irreducible component of $B_1$ and $m(B)$ the number of components of this degree. We wish to construct a new map $X \to Q' = \mathbb{P}_Y(\mathcal{O} \oplus L^N)$ for which $B_1$ is empty. Thus we may assume that $d(B) > 1$ and $m(B) > 0$. The following lemma shows that we can arrange for at least one of these numbers to drop, and that completes the required inductive step.

**Lemma 4.** There is a map $h: Q \to Q' = \mathbb{P}_Y(\mathcal{O} \oplus L^d)$ such that if $B'$ is the union of $h(B)$ and the branch locus of $h$ then $(d(B), m(B)) > (d(B'), m(B'))$ in the lexicographic ordering. Moreover, under this map the image of the divisor $E$ in $Q$ is the corresponding divisor $E'$ in $Q'$.

Proof. Let $A$ be an irreducible component of $B_1$ of degree $d = d(B)$ and consider the two maps $\mathcal{O}_Q \to \mathcal{O}_Q(d)$ and $L^d_Q \to \mathcal{O}_Q(d)$ given by $A$ and $d \cdot E$. Since these two divisors do not meet, the direct sum $\mathcal{O}_Q \oplus L^d_Q \to \mathcal{O}_Q(d)$ is surjective. Thus, by the universal property of $Q'$ we obtain a morphism $Q \to Q'$ which on every fibre is a map $\mathbb{P}^1 \to \mathbb{P}^1$ of degree $d$. Its ramification divisor $R$ has the form $R = (d - 1)E + R'$ for some divisor $R'$ that does not meet $E$. Moreover, the Hurwitz formula on the general fibre $\mathbb{P}^1$ (equivalently, computing the canonical divisors) gives $\mathcal{O}_Q(R') = \mathcal{O}_Q(d - 1)$. Thus we have the required result. \hfill \square

We compose the morphism obtained from Lemma 2 with a succession of morphisms obtained from Lemma 4. By the latter lemma the pair $(d(B), m(B))$ can be reduced until eventually $B_1$ becomes empty and thus the composite morphism $f$ is as stated in Claim 1.

\section{2. Genus 0 fibrations}

We will prove the following:

**Claim 2.** We can replace $X$ by a blow up $X'$ and $Z$ by the union of its inverse image in $X'$ with the exceptional locus of the blow up, so that:

1. There is a finite map $f: X \to W$ with $W$ smooth.
(2) The union of the image \( f(Z) \) of \( Z \), the exceptional locus \( E \) and the branch locus of \( f \) is contained in a divisor \( D \) with simple normal crossings (or strict normal crossings).

For completeness, we recall the definition of a simple normal crossing divisor \( D \) in a smooth projective variety. If \( D = \bigcup_{i=1}^{n} D_i \) is the decomposition of the divisor into irreducible components, then for each \( I \subset \{1, \ldots, n\} \) the scheme theoretic intersection \( D_I = \bigcap_{i \in I} D_i \) is reduced and smooth of codimension equal to the cardinality \( \#I \) of the set \( I \).

Write \( g: X \to P = \mathbb{P}_Y(F) \) for the map constructed in Section \( \[ \] \) and let \( \{s_i\}_{i=1}^{m} \) be sections of \( P \to Y \) such that the union \( \bigcup_{i=1}^{m} s_i(Y) \) contains \( g(Z) \), the branch locus of \( g \) and the image of the exceptional locus of the blowups already performed. Let

\[
Z_1 = \bigcup_{1 \leq i < j \leq m} p_Y(s_i(Y) \cap s_j(Y))
\]

be the divisor in \( Y \) obtained as the image of the pairwise intersections of the sections. We apply the induction hypothesis 0.3 to obtain a map \( \varepsilon_Y: \tilde{Y} \to Y \) such that the inverse image \( \varepsilon_Y^{-1}(Z) \) is a divisor with simple normal crossings in \( \tilde{Y} \). Moreover, note that \( Y \) as constructed in the previous section is \( \mathbb{P}^{n-1} \) and thus normal. Thus, we can also assume (by induction) that the exceptional locus \( E_Y \) of \( \varepsilon_Y \) is a divisor, such that the union \( Z_1 = E_Y \cup \varepsilon_Y^{-1}(Z) \) is also a divisor with simple normal crossings.

Replacing \( Y \) by \( \tilde{Y} \) and \( P \) by its pullback, we obtain a configuration with the following properties (A):

(1) \( p: P \to Y \) is a flat morphism of smooth varieties whose reduced fibres are trees of projective lines \( \mathbb{P}^1 \) (in other words, \( p \) is a genus 0 fibration);
(2) we have a finite collection \( \{s_i\}_{i=1}^{m} \) of sections of \( p \) such that

\[
\bigcup_{1 \leq i < j \leq m} p_Y(s_i(Y) \cap s_j(Y)) \subset Z_1
\]

where \( Z_1 \) is a divisor with simple normal crossings in \( Y \);
(3) the morphism \( p \) is smooth outside \( p^{-1}(Z_1) \) and the latter is a divisor with simple normal crossings.

We wish to perform a succession of blowups resulting in a configuration still having the same properties, but with the sections disjoint.

For each component \( C \) of \( Z_1 \) and each section \( s_i \), the image \( C' = s_i(C) \subset P \) is a codimension 2 subvariety. Write \( n(C') \) for the number of \( j \) such that \( s_j(C) = C' \), and \( n_P \) for the maximum of such \( n(C') \).
Lemma 5. Let $C'$ be so chosen that $n(C') = n_P$. Then for any $j$, either $s_j(C) = C'$ or $s_j(Y) \cap C' = \emptyset$.

Proof. Suppose $j$ is such that $s_j(Y) \cap C'$ is a proper nonempty subset of $C'$, and let $C''$ be an irreducible component of this intersection. Choose $i$ so that $s_i(C) = C'$, let $D'$ be a component of $s_i(Y) \cap s_j(Y)$ containing $C''$, and denote by $D$ the image $p(D')$. On the one hand we have

$$s_j(C) \cap C' \subset s_j(Y) \cap C' \subset C' = s_i(C),$$

while $D' = s_i(D) = s_j(D)$. Thus we see that $D$ and $C$ are different components of $Z_1$ containing $p(C''')$. But $C'' \subset C'$ has codimension 3 in $P$, and $p$ restricts to an isomorphism from $C'$ to $C$. Thus $p(C''')$ is of codimension 2 in $Y$; hence it is an irreducible component of $D \cap C$. Since $Z_1$ is a simple normal crossing divisor we see that $D$ and $C$ are the only components of $Z_1$ that contain $p(C''')$.

Take any $k$ such that $s_k(C) = C''$; then $s_k(Y) \cap s_j(Y)$ contains a component $E'$ which contains $C''$. Then by the above reasoning we must have $D = p(E')$, and thus

$$D' = s_j(D) = s_j(p(E')) = E' = s_k(p(E')) = s_k(D)$$

But then we get $s_k(D) = D'$ for all $k$ such that $s_k(C) = C''$, and in addition, $s_j(D) = D'$ so that $n(D') = n(C') + 1$. This contradicts the maximality of $n(C')$.

Together with the preceding lemma, the following result shows that blowing up $C'$ with $n(C') = n_P$ again leads to a configuration of type (A).

Lemma 6. Let $p: P \to Y$ be a genus 0 fibration which is smooth outside $Z_1$ so that the inverse image $p^{-1}(Z_1)$ of $Z_1$ is a simple normal crossing divisor in $P$. For $s: Y \to P$ a section and $C$ a component of $Z_1$, consider the blowup $P' \to P$ along $s(C)$.

Then $p': P' \to Y$ is again a genus 0 fibration which is smooth outside $Z_1$ so that the inverse image $p'^{-1}(Z_1)$ of $Z_1$ is a simple normal crossing divisor, and the birational transform (or strict transform) of $s(Y)$ gives a section of $P'$ over $Y$. Moreover, if $t$ is a section of $p$ which is disjoint from $s(C)$ then the birational transform of $t(Y)$ continues to be a section of $p'$.

Proof. The first two statements and the final statement about $P'$ are obvious. The union of $s(Y)$ with $p^{-1}(Z_1)$ is a simple normal crossing divisor since $s$ is a section. Moreover, $s(C)$ is the locus of intersection of $s(Y)$ and one of the components of $p^{-1}(C)$. Thus the blowup preserves the property of being a simple normal crossing divisor. \(\square\)
Suppose that \( n_P > 1 \) and let \( N_P \) be the number of \( C' \) attaining this maximum. For each such \( C' \) and each pair \( i, j \) such that \( s_i(C) = s_j(C') = C' \) let \( m(i, j, C') \) denote the multiplicity of intersection of \( s_i(Y) \) and \( s_j(Y) \) along \( C' \). We are in the situation of the following lemma

**Lemma 7.** Let \( P \) be a smooth variety, \( D_1 \) and \( D_2 \) smooth divisors meeting along a smooth codimension 2 locus \( B \) with multiplicity \( m > 0 \). Let \( P' \) be the blowup of \( P \) along \( B \), and let \( D'_1 \) and \( D'_2 \) be the birational transforms of \( D_1 \) and \( D_2 \) respectively. Then the multiplicity of intersection of \( D'_1 \) and \( D'_2 \) is \( m - 1 \) along a codimension two locus \( B' \) lying over \( B \).

**Proof.** In a neighbourhood of the generic point of \( B \), the given condition can be written as follows

\[
\mathcal{O}_P(D_1) \otimes \mathcal{O}_{D_2} = \mathcal{O}_{D_2}(m \cdot B)
\]

Let \( E \) denote the exceptional divisor of the blowup \( \varepsilon : P' \to P \) and let \( B' \) be the intersection \( D'_2 \cap E \); this is a section of the \( \mathbb{P}^1 \)-bundle \( E \to B \). Now the birational transform of \( D_1 \) represents the divisor class \( \varepsilon^*(D_1) - E \) and thus we see that

\[
\mathcal{O}_{P'}(D'_1) \otimes \mathcal{O}_{D'_2} = (\mathcal{O}_P(D_1) \otimes \mathcal{O}_{D_2}) \otimes \mathcal{O}_{D_2}(-B') = \mathcal{O}_{D_2}((m - 1) \cdot B'),
\]

which proves the result. \( \square \)

From this lemma, we see that the multiplicity of intersection of the birational transforms of \( s_i(Y) \) and \( s_j(Y) \) is \( m(i, j, C') - 1 \). If this becomes zero then \( n(C') \) drops, hence either \( N_P \) decreases or \( n_P \) does so. This completes the argument by induction since all these numbers are positive and we wish to obtain the situation where \( n_P = 1 \).

Now, we replace \( X \) by the normalisation of the fibre product \( X \times_{\mathcal{F}_Y(F)} P \) where \( P \to Y \) is the configuration of type (A) with \( n_P = 1 \) obtained above. We then have a finite morphism \( h : X \to P \). The image \( h(Z) \) of \( Z \) and the branch locus of \( h \) are contained in the simple normal crossing divisor consisting of the finitely many disjoint sections of the configuration (A) and \( p^{-1}(Z_1) \). This proves the Claim.

### 3. Toric singularities

As a last step we must prove the result in the following situation. There is a finite map \( f : X \to W \) with \( W \) smooth and \( X \) normal and \( D = \bigcup_{i=1}^n D_i \) a divisor with simple normal crossings so that \( f \) is étale outside \( D \); moreover \( Z \) is a union of (some of) the components of \( f^{-1}(D) \). So we need to construct a birational morphism \( \varepsilon : \tilde{X} \to X \)
so that $\tilde{X}$ is smooth and $\varepsilon^{-1}f^{-1}(D)$ is a divisor with simple normal crossings in $\tilde{X}$.

We will show (see Lemma 8 below) that the inclusion of $X \setminus f^{-1}(D)$ in $X$ is a strict toroidal embedding in the sense of [7], where the desingularisation problem for such embeddings has been studied and solved. For the sake of completeness we also give a brief summary of their method. Many statements below are given without detailed proofs—readers are invited to complete the arguments on their own or look for proofs in one of the books on toric geometry such as Oda [9] or Fulton [4].

3.1. **Affine toric singularities.** Let us first consider the simple situation where $W = \mathbb{A}^n$ and $D$ is a union of coordinate hyperplanes. In this case $W \setminus D$ is isomorphic to $\mathbb{G}_m^r \times \mathbb{A}^{n-r}$ where $r$ is the number of components of $D$. Hence (because we are working over the complex numbers $\mathbb{C}$) its fundamental group is a product of infinite cyclic groups and the finite cover $X \setminus f^{-1}(D)$ is also isomorphic to $\mathbb{G}_m^r \times \mathbb{A}^{n-r}$. In fact, the homomorphism $f^*$ on rings has the form

$$\mathbb{C}[z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}, z_{r+1}, \ldots, z_n] \rightarrow \mathbb{C}[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}, t_{r+1}, \ldots, t_n]$$

$$z_i \mapsto \begin{cases} m_i & \text{for } i \leq r \\ t_i & \text{for } i > r \end{cases}$$

where the $m_i$ are certain monomials (with negative powers allowed) in $t_1, \ldots, t_r$. The natural action of the torus $\mathbb{G}_m^r$ on $X \setminus f^{-1}(D)$ then descends to an action via $f$ making this map equivariant. Moreover, since $X$ is the normalisation of $W$ in $X \setminus f^{-1}(D)$, the action extends to $X$. Since the map $f$ is equivariant there are only finitely many orbits for the torus action on $X$; in other words, $X$ is an (affine) toric variety.

An explicit description of $X$ can be given as follows. Let $M$ be the free Abelian group of all monomials in the variables $t_1, \ldots, t_r$. Let $M^+$ be the saturated submonoid of $M$ generated by the $m_i$. Then $X = \text{Spec } \mathbb{C}[M^+] \times \mathbb{A}^{n-r}$. For future reference, we note that there is a unique closed orbit in $X$ which maps isomorphically to the closed orbit $0 \times \mathbb{A}^{n-r}$ in $W$. Moreover, $M$ is the group of Cartier divisors supported on $f^{-1}(D)$ and $M^+$ is the submonoid of effective Cartier divisors.

Let $N$ be the dual Abelian group to $M$ and

$$N^+ = \{ n \mid n(m) \geq 0 \text{ for all } m \in M^+ \}$$
the “dual” monoid to $M^+$. Then $N^+$ is a finitely generated saturated monoid in $N$ (like $M^+$ in $M$). Let $\sigma$ be any finitely generated sub-monoid of $N^+$ and

$$M^\sigma = \{m \mid n(m) \geq 0 \text{ for all } n \in N^+\}$$

the dual monoid in $M$ (which contains $M^+$). Then $X(\sigma) = \text{Spec } \mathbb{C}[M^\sigma] \times \mathbb{A}^{n-r}$ is an affine toric variety and the natural morphism $X(\sigma) \to X$ is birational and equivariant for the torus action. Moreover, we see that $X(\sigma)$ is nonsingular and the pullback to $X(\sigma)$ of $D$ is a simple normal crossing divisor if and only if the monoid $\sigma$ is generated over nonnegative integers by a (sub-)basis of $N$; such a monoid is called simplicial.

Let $\Sigma = \{\sigma_i\}_{i=1}^n$ be a collection of finitely generated saturated sub-monoids of $N^+$ which give a subdivision of $N^+$; i.e., $N^+$ is the union of all the $\sigma_i$ and for any pair $\sigma_i, \sigma_j$ their intersection is a $\sigma_k$ for some $k$. We then obtain a collection of equivariant birational morphisms $X_i = X(\sigma_i) \to X$ so that if $\sigma_j \subset \sigma_i$ then $X_j \subset X_i$ in a natural way. Thus we can patch together the $X_i$ to obtain $X_\Sigma \to X$ which is birational and equivariant (but $X_\Sigma$ need not be affine any more). Moreover, the condition that the $\sigma_i$ cover $N^+$ implies that $X_\Sigma \to X$ is proper.

Thus to obtain a desingularisation of $X$ it is enough to find a subdivision of $N^+$ consisting entirely of simplicial monoids; an easy enough combinatorial problem solved by barycentric subdivision. The intrepid reader is warned that proving that the resulting morphism is projective is a little intricate since an arbitrary simplicial subdivision need not result in a projective morphism; however, the barycentric subdivision does yield a projective morphism.

3.2. Local toric singularities. Now we examine the general case locally. Let $x \in X$ be any point such that $w = f(x)$ lies in $D$. There is an analytic neighbourhood $U$ of $w$ in $W$ and coordinates on $U$ so that $D \cap U$ is given by the vanishing of a product of coordinate functions; by further shrinking $U$ we can assume that $U$ is a polydisk in these coordinates. Let $V$ be the component of $f^{-1}(U)$ which contains $x$. The normality of $X$ implies the normality of the analytic space $V$. Hence, the open subset $V \setminus f^{-1}(D)$ is connected. So it is a topological cover of $U \setminus D$. The coordinate functions give an inclusion $U \hookrightarrow W' = \mathbb{A}^n$ so that $D \cap U$ is the restriction to $U$ of a union of coordinate hyperplanes in $W'$. The resulting inclusion $U \setminus D \hookrightarrow G_m^r \times \mathbb{A}^{n-r}$ induces an isomorphism of fundamental groups. Thus there is a Cartesian square
in which the horizontal maps are inclusions:

\[
\begin{align*}
V \setminus f^{-1}(D) & \rightarrow G^r_m \times \mathbb{A}^{n-r} \\
\downarrow f & \quad \downarrow f' \\
U \setminus D & \rightarrow G^r_m \times \mathbb{A}^{n-r}
\end{align*}
\]

and \( f' \) is a covering of the form (1) above. Let \( X' \) denote the normalisation of \( W' \) in the cover \( f' \). By the normality of \( X \) (and hence the normality of \( V \) as an analytic space) it follows that we obtain a commutative diagram

\[
\begin{array}{ccc}
V & \rightarrow & X' \\
\downarrow f & & \downarrow f' \\
U & \rightarrow & W'
\end{array}
\]

so that \( V \) is isomorphic to an analytic open neighbourhood of a point in the toric variety \( X' \). By means of the two diagrams above we can carry over the desingularisation of 3.1 to the local analytic space \( V \).

3.3. General toroidal embeddings. The local description 3.2 can be repeated in a suitable neighbourhood of any point \( x \in X \). This shows that the inclusion of \( X \setminus f^{-1}(D) \) in \( X \) is a toroidal embedding. The desingularisations obtained locally need to be constructed in a coherent manner so that they “patch up”.

Let us stratify \( W \) by connected components of intersections of the form

\[
(D_{i_1} \cap \cdots \cap D_{i_r}) \setminus \bigcup_{j \neq i_s} D_j.
\]

If \( U \) and \( V \) are as in 3.2 then there is a unique stratum \( S \) of \( W \) so that \( S \cap U \) is closed. Under the inclusions of \( U \) in \( W' \) and \( V \) in \( X' \) of 3.2 the strata correspond to the orbits of the torus action. As we have noted in 3.1 the unique closed strata in \( U \) and \( V \) then become isomorphic. Thus, if \( T = f^{-1}(S) \) then \( T \cap V \) is closed in \( V \) and the morphism \( T \cap V \rightarrow S \cap U \) is an isomorphism; indeed \( T \cap V \) and \( S \cap U \) are the restrictions of the closed orbits in \( X' \) and \( W' \) respectively. Thus we see that if \( S \) is any stratum in \( W \) and \( T \) a connected component of \( f^{-1}(S) \) then \( T \rightarrow S \) is étale and proper. We thus stratify \( X \) by connected components of the inverse images of strata in \( W \) under \( f \). Let \( \{ T_a \}_{a \in A} \) denote this stratification; by abuse of notation we define \( S_a = f(T_a) \).
Let \( E = \bigcup_{j=1}^{m} E_j \) be the decomposition into irreducible components of the inverse image \( f^{-1}(D) \) of \( D \). Then there is a function \( i: \{1, \ldots, m\} \to \{1, \ldots, n\} \) so that \( f(E_j) = D_{i(j)} \) for all \( j \). For any stratum \( T_a \) let \( X^a \) denote the complement in \( X \) of all those \( E_j \) that do not meet \( T_a \). Similarly, we denote by \( W^a \) the complement in \( W \) of all \( D_i \) that do not meet \( S_a \). The morphism \( f \) clearly maps \( X^a \) to \( W^a \).

**Lemma 8.** Let \( M_a \) be the Abelian group of all Cartier divisors in \( X^a \) with support in \( E \cap X^a \); let \( M^+_a \) be the submonoid consisting of effective Cartier divisors. Then \( M_a \) has rank \( r_a = \text{codim}_X T_a \). The distinct analytic components of \( E \) in a neighbourhood of any point \( x \) of \( X \) are precisely the algebraic components; i.e., \( X \setminus E \hookrightarrow X \) is a strict toroidal embedding.

**Proof.** Let \( M'_a \) be the group of Cartier divisors on \( W^a \) with support on \( D \cap W^a \). Clearly, this is the free Abelian group on those \( D_i \) which contain \( S_a \). On the other hand, consider an analytic neighbourhood \( V = f^{-1}(U) \) of some point \( x \in T_a \) as in 3.2. If \( M''_x \) denotes the group of all Cartier divisors on \( V \) supported on \( E \cap V \); then we have noted in 3.1 that \( M''_x \) is a free Abelian group of rank \( r_a \). Moreover, \( M_a \) is included as a subgroup of \( M''_x \) under the restriction from \( X^a \) to \( V \).

It follows that the homomorphism \( M_a \to M''_x \) has finite cokernel; but then the normality of \( X \) means that it is surjective. In particular, we see that the distinct analytic components of \( E \cap V \) correspond to distinct algebraic components of \( E \). This concludes the proof. \( \square \)

Let \( M^+_a \) denote the monoid of effective divisors in \( M_a \); under the isomorphism \( M_a \to M''_x = M \) this maps isomorphically onto the submonoid \( M^+ \) considered in 3.1. Thus \( X^a \) is smooth (and the divisor \( E \) is a simple normal crossing divisor) if and only if \( M^+_a \) is simplicial. If \( T_b \) lies in the closure of \( T_a \) then \( X^a \) is an open subset of \( X_b \). This induces by restriction a (surjective) homomorphism \( M_b \to M_a \) which further restricts to a surjection \( M^+_b \to M^+_a \). Thus we have a (finite) projective system of monoids.

Let \( N_a \) and \( N^+_a \) be the dual objects as defined in 3.1. These form a finite injective system of monoids. By a compatible family \( S \) of subdivisions we mean a subdivision \( \Sigma_a \) of \( N^+_a \) for each \( a \) so that the subdivision \( \Sigma_b \) restricts to the subdivision \( \Sigma_a \) on the submonoid \( N^+_a \) of \( N^+_b \). We then obtain a proper birational morphism \( X_{\Sigma_a} \to X_b \) for each \( b \) which restricts to \( X_{\Sigma_a} \to X_a \) on the open subset \( X_a \) of \( X_b \). Thus,
we see that any such compatible family leads to a proper birational morphism $X_S \to X$.

Thus, in order to desingularise, we have to find a compatible family of subdivisions so that each of the new monoids is simplicial. This is achieved by the barycentric subdivision. As seen earlier, this ensures that the morphism $X_S \to X$ is locally projective. Since $X$ is projective, this morphism is indeed projective.

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