On the Positive Energy Theorem for Stationary Solutions to Fourth-Order Gravity

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Abstract

In this paper we prove a positive energy theorem related to fourth-order gravitational theories, which is a higher-order analogue of the classical ADM positive energy theorem of general relativity. We will also show that, in parallel to the corresponding situation in general relativity, this result intersects several important problems in geometric analysis. For instance, it underlies positive mass theorems associated to the Paneitz operator, playing a similar role in the positive $Q$-curvature conformal prescription problem as the Schoen-Yau positive energy theorem does for the Yamabe problem. Several other links to $Q$-curvature analysis and rigidity phenomena are established.

1 Introduction

In this paper we will analyse the properties of a recently proposed energy associated to higher-order gravitational theories in the stationary limit. Specifically, in parallel work, we have analysed gravitational theories described on globally hyperbolic space-times $(V = M \times \mathbb{R}, \bar{g})$ by an action functional of the form

$$S(\bar{g}) = \int_V (\alpha R^2_{\bar{g}} + \beta \langle \text{Ric}_{\bar{g}}, \text{Ric}_{\bar{g}} \rangle) dV_{\bar{g}},$$

where $\alpha$ and $\beta$ are free parameters in this variational setting. Let us highlight that the study of these kinds of higher-order gravitational action functional is well-motivated within contemporary theoretical physics as they appear in connection with effective field theories of gravity [5, 15, 35], as well as in the context of inflationary cosmology [40] and certain approaches to quantum gravity, such as conformal gravity [29, 30].

In order to make sense of the above functional, we can assume that the class of metrics considered above are such that $R^2_{\bar{g}}$ and $\langle \text{Ric}_{\bar{g}}, \text{Ric}_{\bar{g}} \rangle$ are integrable. Then, the functional $\bar{g} \mapsto S(\bar{g})$ is well-defined and we have an $L^2$-gradient for this functional, given by a divergence-free tensor field $A_{\bar{g}} \in \Gamma(T^2_2V)$, which is explicitly given by

$$A_{\bar{g}} = \beta \Box_{\bar{g}} \text{Ric}_{\bar{g}} + (\frac{1}{2} \beta + 2\alpha) \Box_{\bar{g}} R_{\bar{g}} - (2\alpha + \beta) \nabla^2 R_{\bar{g}} - 2\beta \text{Ric}_{\bar{g}} \text{Riem}_{\bar{g}}$$

$$+ 2\alpha R_{\bar{g}} \text{Ric}_{\bar{g}} - \frac{1}{2} \alpha R^2_{\bar{g}} - \frac{1}{2} \beta \langle \text{Ric}_{\bar{g}}, \text{Ric}_{\bar{g}} \rangle_{\bar{g}},$$

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where above we denoted $\text{Ric}_\bar{g} \cdot \text{Riem}_\bar{g} = \text{Ric}_\bar{g}^{kl} \text{Riem}_\bar{g}^{kijl}$. In a parallel situation to what is well-known in the context of general relativity (GR), we have shown that there is a canonical notion of energy, which we denote by $E_{\alpha,\beta}(\bar{g})$, associated to asymptotically Euclidean (AE) solutions of the space-time field equations $A_{\bar{g}} = 0$ which arise as perturbations of solutions $\bar{g}_0$ which possess a time-like Killing field. Although the analysis of such an energy could be quite involved in general, we are able to identify some particular choices of the parameters $\alpha$ and $\beta$ for which its analysis is tractable. More important, we establish positivity and rigidity results for the energy in those cases. These results are intimately connected with the existence of metrics with positive constant $Q$-curvature.

Our aim here is to analyse the particular case of stationary solutions of the fourth-order field equations parameterized by $2\alpha + \beta = 0$. Recall that globally hyperbolic stationary space-times are manifolds of the form $V = M \times \mathbb{R}$ endowed with a Lorentzian metric that can be written as

$$\bar{g} = -N^2 dt^2 + \bar{g},$$

where $N : M \mapsto \mathbb{R}^+$ is the lapse function and $\bar{g} \in \Gamma(T^0_0 M)$ restricts to a Riemannian metric $g$ on each $t = \text{cte}$ hypersurface. In this setting, the appropriate notion of energy associated to the action $S$ and the corresponding field equations $A_{\bar{g}} = 0$ becomes

$$E_{\alpha}(g) = -\alpha \lim_{r \to \infty} \int_{S^{n-1}_r} (\partial_i \partial_j g_{ua} - \partial_j \partial_i g_{ua}) \nu^j d\omega_r.$$  

In this expression, we are assuming that $(M, g)$ is an AE manifold. This means that for a given compact set $K$ the asymptotic region $M \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus B_1(0)$ and it is foliated by topological $(n-1)$-dimensional spheres $S^{n-1}_r$, whose Euclidean volume element is denoted in the expression above by $d\omega_r$. There, $\nu$ stands for the Euclidean unit normal field to these spheres.

The nature of (4) as a conserved quantity in the context of higher-order gravitational theories makes $E_{\alpha,\beta}(g)$ a very good fourth-order analogue to the Arnowitt, Deser and Misner (ADM) energy in the context of GR. Let us recall that the total energy of an isolated gravitational system in GR, whose initial data is modelled as an AE manifold, is given by (see [1])

$$E_{\text{ADM}}(g) = c(n) \lim_{r \to \infty} \int_{S^{n-1}_r} (\partial_i g_{ji} - \partial_j g_{ii}) \nu^i d\omega_r,$$  

where $c(n)$ stands for a dimensional constant (see [3] for the detailed analytical properties of (5)). This energy $E_{\text{ADM}}$ has had a huge impact both within GR and in geometric analysis. Most notoriously, it was the work of R. Schoen in [36] that elucidated the role that the ADM energy plays in the final resolution of the Yamabe problem (see also [26] for a review on this topic). In particular, in order to solve the Yamabe problem in the positive Yamabe class, in dimensions 3,4 and 5 or in locally conformally flat manifolds, Schoen noticed that it was enough to prove that an appropriate constant appearing in the expansion of the Green function $G_{L_\bar{g}}$ associated to the conformal Laplacian $L_\bar{g}$ was non-negative and that the zero case implied rigidity with the round sphere. Furthermore, it was pointed out that this constant was precisely the ADM energy of the AE manifold obtained via a stereographic projection. Therefore, the proof of the positive energy theorem in GR actually underlies the proof of the Yamabe problem in these cases. This beautiful relation shows what a fundamental result the positive energy theorem actually
is within geometric analysis, being a cornerstone in the resolution of the Yamabe problem. Since then, the ADM energy has influenced many other constructions within geometric analysis and mathematical GR, which are not necessarily concerned with the Yamabe problem. For instance, rigidity phenomena associated to positive scalar curvature [4, 8, 9], isoperimetric problems on AE manifolds [1, 16, 23], geometric foliations and center of mass constructions [16, 22, 23, 33] and even gluing constructions [9] (for nice reviews of many of these topics, see [7, 25]).

In view of the above paragraph, we consider that the analysis of the positive energy theorem associated to (4) stands as a highly well-motivated problem both for the development of a program devoted to the mathematical analysis of fourth order theories of gravity, as well as a tool which can potentially play a fundamental role in several fourth order geometric problems. Thus, the main objective of this paper is the establishing of such a positive energy theorem, and, afterwards, we will show how it underlies some fundamental problems in geometric analysis.

Since we will be concerned with positivity issues related to $E_\alpha (g)$, we must actually fix the sign of $\alpha$ a priori. For reasons that will become apparent through this paper, we will fix $\alpha = -1$ and define $E(g) = E_{-1}(g)$. Let us notice that before embarking on the proof of the positive energy theorem, we should first analyse under what conditions $E(g)$ is actually well-defined and prove that it is actually an intrinsic geometric object within a suitable class of AE-manifolds. This will be done in Proposition 1, Proposition 2 and Theorem 1. Once this is done, we will first explore appropriate geometric conditions that could in principle provide a rigidity statement in the critical cases. Explicitly, such conditions will involve a $Q$-curvature condition of the form $Q_g \geq 0$ (see Proposition 3 and Theorem 4). Let us recall that given a Riemannian manifold $(M^n, g)$, $n \geq 3$, its $Q$-curvature is defined by

$$Q_g = \frac{1}{2(n-1)} \Delta_g R_g - \frac{2}{(n-2)^2} |\text{Ric}_g|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R_g^2.$$

In this context, we will establish the following theorem:

**Theorem A (Positive Energy).** Let $(M^n, g)$ be an $n$-dimensional AE manifold, with $n \geq 3$, which satisfies the decaying conditions: (i) $g_{ij} - \delta_{ij} = O_4 (r^{-\tau})$, with $\tau > \max \{0, \frac{n-1}{2}\}$, in some coordinate system associated to an structure of infinity; (ii) $Q_g \in L^1 (M, dV_g)$, and such that $Y ([g]) > 0$ and $Q_g \geq 0$. Then, the fourth order energy $E(g)$ is non-negative and $E(g) = 0$ if and only if $(M, g)$ is isometric to $(\mathbb{R}^n, \cdot)$.

The proof of this statement will be the content of Theorem 2 (see also theorems 3 and 4). Let us notice that some hypotheses in the above theorem can be relaxed while keeping important results. In particular, the positivity and rigidity statements are somehow decoupled. This implies that under hypotheses (i)-(ii) and $R_g > 0$ at infinity, it follows that $E(g) \geq 0$ (see Theorem 3), while under hypotheses (i)-(ii), $Q_g \geq 0$ and $g$ Yamabe positive, if $E(g) = 0$, then the rigidity statement follows (see Theorem 4). Furthermore, let us notice that the borderline case of $n = 4$ is special, since it follows that any 4-dimensional AE manifold satisfies the decaying conditions (i)-(ii) and, in fact, under these conditions $E(g) = 0$. This imposes restrictions on the positive curvature conditions that these manifolds can admit, providing us with the following corollaries.

**Corollary.** Any $n$-dimensional AE-Riemannian manifold $(M^n, g)$, with $n \in \{3, 4\}$, such that $Q_g \geq 0$ and $Y ([g]) > 0$, then it is isometric to $(\mathbb{R}^n, \delta)$.

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1See Appendix B for details on our conventions on $Q$-curvature.
Let us notice that Yamabe positive AE manifolds in low dimensions (specially three) are very natural objects for instance in GR, since maximal vacuum initial data for isolated systems belong to this class of manifolds. Therefore, the above corollary, in some sense, can be used to separate the trivial solutions via an appeal to their $Q$-curvature.

The following result, which in particular gives a conformally invariant statement, is also a direct consequence of the analysis related to the above positive energy theorem for $\mathcal{E}(g)$ (see Proposition 5 for more details).

**Corollary.** Let $(M, g)$ an asymptotically flat four manifold with $\kappa_g = \int_M Q_g \, dv_g \geq 0$ and $Y([g]) \geq 0$, then $(M, g)$ is isometric to the euclidean $\mathbb{R}^4$.

After presenting the above results concerning the positive energy theorem associated to $\mathcal{E}(g)$, we will make contact with important problems in geometric analysis. In particular, our aim is to show that there is a very clear parallel in the role played by the ADM positive energy theorems of Schoen-Yau [37, 38, 39] in the resolution of the Yamabe problem, to the role Theorem A plays in the resolution of the positive $Q$-curvature Yamabe-type problems which are known up to date. This problem concerns finding a conformal deformation of a closed Riemannian manifold $(M, g)$ so that the resulting metric has constant $Q$-curvature. There has been great progress in this program in recent times and the analysis depends on whether $n \geq 5$ or $n = 3, 4$. In particular, for the most updated resolutions of this problem to our knowledge in dimension four see [14, 17, 10] and references therein, while for $n \geq 5$, with different degrees of generality, let us draw the reader’s attention to [21, 18, 34]. Although we will not be concerned with the three dimensional case, let us point out that the problem has been addressed in this case in [19, 20].

Let us now focus on the results of [21]. There, the authors prove that given a closed manifold $(M, g)$ of dimension $n \geq 5$ which is Yamabe non-negative and satisfies $Q_g \geq 0$ not identically zero, there is a conformal deformation to constant positive $Q$-curvature. In particular, this is done in Theorem 4 wherein where in the cases of dimensions $n = 5, 6, 7$ or $n \geq 5$ and locally conformally flat, there is a parallel to Schoen’s resolution of the Yamabe problem. That is, the problem can be reduced to showing that a certain coefficient in the Green function expansion of the Paneitz operator near a pole is non-negative. In analogy to Schoen’s ideas, this constant has been called the mass of the Paneitz operator (whenever defined) and its positivity has been analysed first by Humbert-Raulot in the locally conformally flat case [24], then by Gursky-Malchiodi who incorporated the cases $5 \leq n \leq 7$ without this last restriction, and finally by Hang-Yang [19], who weakened the hypotheses on the scalar curvature imposed in [18] and achieved the final form of this results which was applied in Theorem 4 of [21].

In Section 2.2 we will show that exactly in dimensions $5 \leq n \leq 7$ or if $(M, g)$ has a point $p$ around which it is conformally flat, then the mass of the Green function $G_{P_g}$ associated to the Paneitz operator $P_g$ is positively proportional to the energy $\mathcal{E}(\hat{g})$ of the asymptotically euclidean manifold $(\hat{M} = M \backslash \{p\}, \hat{g} = G_{P_g}^{-1} g)$ obtained via a stereographic projection. Then, we show that the following $Q$-curvature positive mass theorem follows from Theorem A.

**Theorem B (Q-curvature Positive Mass [18, 21, 24]).** Let $(M, g)$ be a closed $n$-dimensional Riemannian manifold, with $5 \leq n \leq 7$ or $n \geq 8$ and locally conformally flat around some point $p \in M$. If $Y([g]) \geq 0$ and $(M, g)$ admits a conformal metric with semi-positive $Q$-curvature.
$Q$-curvature, then the mass of $G_p$ at $p$ is non-negative and vanishes if and only if $(M, g)$ is conformal to the standard sphere.

The above theorem is exactly the positive mass theorem used in by Hang-Yang in [21] to solve the $Q$-curvature prescription problem in dimensions $5 \leq n \leq 7$ or $n \geq 5$ and $M$ locally conformally flat around a point. This highlights the potential parallel of energy $E(g)$ in the analysis of fourth-order problems to the role of the ADM energy in classical second order geometric problems.

Finally, along the lines of the remarks of the previous paragraphs, we will analyse the critical four-dimensional case and provide an independent proof a Theorem B in [17] appealing to the techniques derived in this paper. Concretely, the following theorem follows from our analysis:

**Theorem C** (Gursky). Let $(M^4, g)$ a 4-dimensional manifold with $Y([g]) \geq 0$, then $\kappa_g \leq 16\pi^2$ with equality holding iff $(M^4, g)$ is conformal to the standard sphere.

We would like to highlight that, in this case, our techniques make contact with the positive mass theorem of 2-dimensional manifolds, as presented in [25].

Finally, let us comment that, with the aim of delivering a self-contained presentation but trying to avoid a long introduction to preliminary results concerning analysis on AE-manifolds, $Q$-curvature analysis and constructions concerning conformal normal coordinates, which are things very well-know for experts in each of these fields, we have compiled the main results which will be used in this paper is the Appendices A,B and C, where detailed references can be found.

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## 2 Preliminaries

In this section we will collect some definitions and notational conventions which will be used along the paper.

Some notational conventions:

- $(M^n, g)$ will denote an $n$-dimensional Riemannian manifold.
- Given a Riemannian manifold $(M, g)$, we will denote by $\Delta_g u = g^{ij} \nabla_i \nabla_j u$ the negative Laplacian.
- Given a Riemannian manifold $(M, g)$ we denote by $\nabla$ its Riemannian connection and define the curvature tensor by
  \[
  R(V, W)Z = \nabla_V \nabla_W Z - \nabla_W \nabla_V Z - \nabla_{[V, W]} Z, \quad V, W, Z \in \Gamma(TM),
  \]
  and we label its components in some coordinate system $\{x^i\}_{i=1}^n$ via
  \[
  R^k_{ijkl} = dx^i (R(\partial_k, \partial_l)\partial_j).
  \]
  Then, the Ricci tensor is defined locally via the contraction $R_{jl} = R^k_{jkl}$.
- Given a tensor field $T$, indices of $\nabla^k T$ that result of covariant differentiation will be separated by a comma. That is, if $T_{ij}$ are the components of a 2-tensor, then we denote the components of $\nabla^2 T$ by $T_{ij,kl}$.
• Given $\Sigma \hookrightarrow M$ an embedded compact hypersurface in $(M, g)$, we will denote by $\nu$ the outward pointing $g$-unit normal to $\Sigma$.

• Given $\Sigma \hookrightarrow \mathbb{R}^n$ an embedded compact hypersurface, we will denote by $\nu^e$ Euclidean outward pointing unit normal to $\Sigma$.

• $\omega_{n-1}$ will denote the volume of the unit sphere $\mathbb{S}^{n-1} \hookrightarrow \mathbb{R}^n$ and $d\omega_{n-1}$ its canonical volume measure.

• let $(M, g)$ a compact manifold, the conformal class of $g$ is $\{ u g ; u \in C^\infty(M, \mathbb{R}^*_+) \}$, denoted $[g]$.

• let $(M, g)$ a compact manifold, ist Yamabe invariant is defined as

$$ Y([g]) = \inf_\tilde{g} \frac{\int_M R_{\tilde{g}} \, d\tilde{g}}{\text{vol}(M)} $$

• Given a function $f \in C^k(\mathbb{R}^n)$, we say that $f = O_k(|x|^\tau)$ for some $k \geq 0$ and $\tau \in \mathbb{R}$ if for any multi-index $\alpha$ with $|\alpha| \leq k$, the functions $\partial^\alpha f = O(|x|^{|\tau| - |\alpha|})$ either as $|x| \to \infty$ or $|x| \to 0$, depending on the context.

**Definition 1** (Weighted spaces). Let $\sigma = (1 + r^2)^{\frac{\delta}{2}}$ and $\delta \in \mathbb{R}$, we set

- $L^p_\delta(\mathbb{R}^n) = \{ u \in L^p_{loc} | \int_{\mathbb{R}^n} |u|^p \sigma^{-\delta} - n \, dx < +\infty \}$, we equip this set with the norm $\| u \|_{p,\sigma,\delta} = \int_{\mathbb{R}^n} |u|^p \sigma^{-\delta} - n \, dx$.

- $L^\infty_\delta(\mathbb{R}^n) = \{ u \in L^\infty_{loc} | \sup_{\mathbb{R}^n} |u| \sigma^{-\delta} < +\infty \}$, we equip this set with the norm $\| u \|_{\infty,\sigma,\delta} = \sup_{\mathbb{R}^n} |u| \sigma^{-\delta}$.

- $W^{k,p}_\delta(\mathbb{R}^n) = \{ u \in W^{k,p}_{loc} | \sum_{j=0}^k \| d^j u \|_{p,\sigma^{-\delta} - j} < +\infty \}$, we equip this set with the norm $\| u \|_{k,p,\sigma,\delta} = \sum_{j=0}^k \| d^j u \|_{p,\sigma^{-\delta} - j}$.

Then we introduce the notion of asymptotically flat manifolds.

**Definition 2** (AE manifolds). A $(M, g)$ complete Riemannian manifold, with $g \in W^{k,q}_{loc}(M)$ for some $k \geq 1$ and $q > n$ is said asymptotically euclidean (with one end) if there exists $K \subset M$ compact and $\phi : M \setminus K \to \mathbb{R}^n \setminus B_1(0)$ a diffeomorphism such that

- $\phi_*(g)$ is a uniformly positive defined metric, i.e. there exists $\lambda > 1$ such that

$$ \frac{1}{\lambda} |\xi|^2 \leq g_{ij}(x) \xi^i \xi^j \leq \lambda |\xi|^2 \forall x \in \mathbb{R}^n \setminus B_1 \forall \xi \in \mathbb{R}^n. $$

- $\phi_*(g)_{ij} - \delta_{ij} \in W^{k,q}_{\tau}(\mathbb{R}^n \setminus B(0, 1))$

for some $\tau > 0$ called the decreasing rate.

**Important remark:** In this chart we defined $\sigma$ and we remark that the definition of $L^p_\delta$ is independent of the chart. But $W^{k,q}_\delta$ depends on the chart $\phi$, since the partial derivatives will depend on the choice of coordinates. It will be denoted $W^{k,q}_\delta(\phi)$. In the rest of the section, we will call such a chart a **structure at infinity**. Once the structure
at infinity is chosen we naturally extend the definition of $W^{k,q}_\delta(\mathbb{R}^n \setminus B(0,1))$ to $W^{k,q}_\delta(M)$ since on the compact part all choices of chart defined the same structure.

In the following definition, which will be used in the core of this paper, we will restrict to AE manifold which possess more regularity than that of the general definition given above.

**Definition 3.** We will say that a (smooth) AE manifold $(M, g)$ is of order $\tau > 0$ with respect to some structure at infinity $\Phi : M \setminus K \to \mathbb{R}^n \setminus \overline{B}$, if, in such coordinates, $g_{ij} - \delta_{ij} = O_4(|x|^{-\tau})$.

Of course we can define asymptotically flat structures with multiple ends. But since analysis phenomena are determined by the behaviour at infinity, it is very easy to isolate each end and to consider that there is only one.

## 3 The positive energy theorem

The main result of this section will be a positive energy theorem related to (4) with its corresponding rigidity statement. But, before this, we will begin by analysing geometric conditions under which (4) is well-defined. In the following proposition, we will establish such geometric criteria. Let us first set

$$\tau_n = \begin{cases} 0 & \text{if } n = 3, 4 \\ \frac{n}{2} - 2 & \text{if } n \geq 5 \end{cases}.$$

**Proposition 1.** Let $(M^n, g)$ be an AE manifold of dimension $n \geq 3$ satisfying the following conditions

1. There are end rectangular coordinates, given by a structure of infinity $\Phi$, where $g_{ij} = \delta_{ij} + O_4(r^{-\tau})$, where $\tau > \tau_n$;
2. The $Q$-curvature of $g$ is in $L^1(M, dV_g)$.

Then, given an exhaustion of $M$ by compact sets $\Omega_k$ such that $S_k = \Phi(\partial \Omega_k)$ are smooth connected $(n-1)$-dimensional manifolds without boundary in $\mathbb{R}^n$ satisfying

$$R_k \equiv \inf \{|x| : x \in S_k\} \to \infty,$$

$$R_k^{n-1} \text{area}(S_k) \text{ is bounded as } k \to \infty,$$  

the limit

$$E^{(\Phi)}(g) = \lim_{k \to \infty} \int_{S_k} (\partial_j \partial_i g_{aa} - \partial_i \partial_a \partial_j g_{ai}) \nu^i dS,$$

exists and is independent of the sequence of $\{S_k\}$ used to compute it.

**Proof.** From the decaying conditions and the familiar expression

$$R_g = \partial_i \partial_j g_{ij} - \partial_j \partial_j g_{ii} + O_2((g - \delta)\partial^2 g) + O_2((\partial g)^2),$$

$$= \partial_i \partial_j g_{ij} - \partial_j \partial_j g_{ii} + O_2(r^{-2\tau - 2})$$
we get

\[ \Delta_g R_g = g^{ab} \nabla_a \nabla_b R_g = g^{ab} (\partial_a \partial_b R_g + \Gamma^c_{ab} \partial_c R_g), \]

\[ = g^{ab} \partial_a \partial_b \left( \partial_i \partial_j g_{ij} - \partial_j \partial_j g_{ii} \right) + g^{ab} \partial_a \partial_b \left( O_2 (r^{-2\tau - 2}) \right) + \Gamma^c_{ab} \partial_c \left( \partial_i \partial_j g_{ij} - \partial_j \partial_j g_{ii} \right) \]

\[ + \Gamma^c_{ab} \partial_c \left( O_2 (r^{-2\tau - 2}) \right), \]

\[ = g^{ab} \partial_a \partial_b \left( \partial_i \partial_j g_{ij} - \partial_j \partial_j g_{ii} \right) + \left( g^{ab} - \delta^{ab} \right) O(r^{-2\tau - 4}) + O(r^{-2\tau - 4}) \]

\[ + O_3 ((g - \delta) \partial g) O_1 (r^{-\tau - 3}) + O_3 (\partial g) O_1 (r^{-\tau - 3}) + O_3 ((g - \delta) \partial g) O_1 (r^{-\tau - 3}) \]

\[ + O_3 (\partial g) O_1 (r^{-\tau - 3}), \]

\[ = \partial_a \partial_b \left( \partial_i \partial_j g_{ij} - \partial_j \partial_j g_{ii} \right) + O(r^{-2\tau - 4}). \]

In particular, denoting by \( D_k \) the annular region between \( S_{k+1} \) and \( S_k \), we get that

\[ \int_{D_k} \partial_j \partial_i \left( \partial_a \partial_i g_{ui} - \partial_u \partial_a g_{ui} \right) dV_e = \int_{S_{k+1}} \left( \partial_j \partial_a \partial_i g_{ui} - \partial_j \partial_a \partial_u g_{ui} \right) \nu_j dS^{n-1} \]

\[ - \int_{S_k} \left( \partial_j \partial_a \partial_i g_{ui} - \partial_j \partial_a \partial_u g_{ui} \right) \nu_j dS^{n-1}. \]

Therefore, if the left-hand side is integrable over \( \mathbb{R}^n \setminus K \), then the above boundary integrals form a Cauchy sequence as \( r \to \infty \) and therefore (7) is well defined. But from the above computations, we see that this can be reduced to \( \Delta_g R_g \in L^1 (M) \) and \( 2(\tau + 2) > n \), that is \( \tau > \tau_n \). In particular, notice that under our decaying conditions \( R_g, Ric_g = O(r^{-\tau - 2}) \) near infinity, implying that

\[ Q_g = -\frac{1}{2(n - 1)} \Delta_g R_g + O(r^{-2\tau - 4}), \]

therefore the leading order is carried on the first term and \( \Delta_g R_g \in L^1 (M) \) can be replaced by \( Q_g \in L^1 (M) \).

Now, we intend to show that (4) is a geometric object, independent of the structure of infinity we use. With this in mind, let us start by rewriting (4) in more geometric fashion, which will prove to be more useful for its analysis.

**Proposition 2.** Let \((M, g)\) be an AE manifold which satisfies the decaying conditions i) and ii) of Proposition 1. Then, we can rewrite the energy (4) as

\[ \mathcal{E}(g) = -\lim_{r \to \infty} \int_{S^1} \partial_r R_g r^{n-1} d\omega_{n-1}. \]

**Proof.** Let us denote by \( \nu \) the outward g-unit normal to \( S^1 \), with \( r \) sufficiently large, and by \( \nu^e \) the outward euclidean unit normal to the same sphere. Then, it follows that \(|\nu^e - \nu|_g = O(r^{-\tau})\). Thus, we find that

\[ g(\nabla R_g, \nu) = g(\nabla R_g, \nu^e) + g(\nabla R_g, \nu - \nu^e), \]

\[ = \partial_i R_g \nu^e_i + O_1(r^{-2\tau - 3}) + \partial_i R_g (\nu - \nu^e)_i + O_1(r^{-2\tau - 3}) + \frac{(g - \delta)_{ij} \nabla^i R_g (\nu - \nu^e)^j}{O_1(r^{-3\tau - 3})}. \]
Also, we already know that $R_g = \partial_i \partial_j g_{ij} - \partial_j \partial_i g_{ii} + O_2(r^{-2\tau-2})$, which implies
\[
g(\nabla R_g, \nu) = \partial_i (\partial_k \partial_j g_{kjj} - \partial_j \partial_k g_{kk}) \frac{x^i}{r} + O_1(r^{-2\tau-3})
\]
Thus,
\[
\int_{S^1_n} g(\nabla R_g, \nu) r^{n-1} d\omega_{n-1} = \int_{S^1_n} \partial_i (\partial_k \partial_j g_{kjj} - \partial_j \partial_k g_{kk}) \frac{x^i}{r} r^{n-1} d\omega_{n-1} + O_1(r^{n-2\tau-4}).
\]
Now, from our decaying condition $\tau > \tau_n$, we see that $n - 2\tau - 4 < 0$, which implies that in the limit $r \to \infty$ the last term vanishes. Therefore, under these conditions
\[
\lim_{r \to \infty} \int_{S^1_n} g(\nabla R_g, \nu) r^{n-1} d\omega_{n-1} = \lim_{r \to \infty} \int_{S^1_n} \partial_i (\partial_k \partial_j g_{kjj} - \partial_j \partial_k g_{kk}) \nu_r r^{n-1} d\omega_{n-1}.
\]
That is,
\[
\mathcal{E}(g) = - \lim_{r \to \infty} \int_{S^1_n} g(\nabla R_g, \nu) r^{n-1} d\omega_{n-1},
\]
\[
= - \lim_{r \to \infty} \int_{S^1_n} \nabla R_g \cdot \nu_r r^{n-1} d\omega_{n-1},
\]
\[
= - \lim_{r \to \infty} \int_{S^1_n} \partial_r R_g r^{n-1} d\omega_{n-1}.
\]
\[\square\]

With the aid of the above two results, we can establish the following theorem, which establishes that $\mathcal{E}(g)$ is a geometric object within a suitable class of AE manifolds.

**Theorem 1.** Let $(\phi, x)$ and $(\psi, y)$ be two structures of infinity for the AE manifold $(M, g)$ which satisfies the conditions i) and ii) of the above Proposition with decay rates $\tau_1, \tau_2$ respectively, satisfying $\tau = \min\{\tau_1, \tau_2\} > \tau_n$. Then, the energies $\mathcal{E}^{(\phi)}(g)$ and $\mathcal{E}^{(\psi)}(g)$ are well-defined and equal. Moreover, the following coordinate independent identity follows:
\[
\mathcal{E}(g) = - \int_M \Delta_g R_g dV_g.
\]

**Proof.** Appealing to Proposition 11 let us compute the energy $\mathcal{E}^{(\phi)}(g)$, associated to the structure of infinity given by $(\phi, x)$ using a sequence of spheres near infinity of radii $(r_k)_{k=1}^\infty$. From the above lemma, we have that
\[
\mathcal{E}^{(\phi)}(g) = - \lim_{|x| \to \infty} \int_{S^1_n} \nabla R_g \cdot dV, \\
\text{where } dV_e = r^{n-1} dr \wedge d\omega_{n-1} \text{ stands for the canonical Euclidean volume form. Appealing to the decaying conditions of } g, \text{ we know that } \sqrt{\det(g)} = 1 + O(r^{-\tau}) \text{ and therefore}
\]
\[
\nabla R_g \cdot dV_e = \nabla R_g \cdot dV_e + O(r^{n-\tau-4}).
\]
Since $n - \tau - 4 < 0$ under our hypotheses, the above implies that
\[
\mathcal{E}^{(\phi)}(g) = - \lim_{r \to \infty} \int_{S^1_n} \nabla R_g \cdot dV_g.
\]
Finally, since $\text{div}_g(\nabla R_g)dV_g = d(\nabla R_g\omega dV_g)$, we find that

$$E(\phi)(g) = -\lim_{r \to \infty} \int_{D_r} \Delta_g R_g dV_g = -\int_M \Delta_g R_g dV_g,$$

where $D_r \hookrightarrow M$ is the inner region in $M$ with $\partial D_r = S_{r^n-1}$. In particular, the right-hand side of the above expression is independent of the coordinates used near infinity. Thus we can drop the reference to $(\phi, x)$ and claim that for any structure of infinity of $M$ were $g$ satisfies conditions (i)-(ii) of Proposition 1, it holds that

$$E(g) = -\int_M \Delta_g R_g dV_g.$$

Besides establishing sufficient conditions for the energy to be a well-defined intrinsic geometric object, the above theorem provides us with an easy positive energy corollary.

**Corollary 1.** Let $(M^n, g)$ be an $n$-dimensional AE manifold, with $n \geq 3$, which satisfies the decaying conditions i) and ii) of Proposition 1 and such that $\Delta_g R_g \leq 0$. Then, the fourth order energy $E(g)$ is non-negative and $E(g) = 0$ if and only if $(M, g)$ is scalar flat.

The positivity statement in the above corollary is self-evident. On the other hand, the rigidity statement follows from the injectivity of the Laplacian under our hypotheses (see (A.1)).

Let us notice that the kind of rigidity that we get from the above corollary is quite weak. In fact, although it is well-known that topological obstructions exist, the set of AE manifolds with zero scalar curvature is in general quite reach (see, for instance, the discussion after Theorem 2). Therefore, in order to get a stronger rigidity statement, we will need to impose so other geometric condition. We will explore this in the next proposition and in the main theorem given below. The following proposition, which is an adaptation of a result in [18] will be used to insure that $R_g > 0$ as soon as we are not flat.

**Proposition 3.** Let $(M^n, g)$ be an AE Riemannian manifold with $n \geq 3$ satisfying (1) $Q_g \geq 0$ and (2) $R_g \geq 0$. Then, either $R_g > 0$ or $g$ is flat.

**Proof.** If $Q_g \geq 0$, then we find that

$$\Delta_g R_g - c_1(n) R_g^2 \leq -c_2(n) |\text{Ric}_g|^2 \leq 0,$$

for some constants $c_1(n), c_2(n) > 0$. From the the condition $R_g \geq 0$ we can appeal to Lemma 4 in [31] to conclude that if $R_g$ vanishes at a single point, then it must be identically zero. Thus, we already see that either $R_g > 0$ or $R_g \equiv 0$. In the second case, we find that $0 \leq Q_g = -c_2(n) |\text{Ric}_g|^2 \leq 0$, and thus $\text{Ric}_g \equiv 0$, which implies (from asymptotic flatness) that $g$ is flat. Indeed, since $\text{Ric}_g \geq 0$, then by Bishop-Gromov theorem insures that $r \mapsto \frac{\text{vol}(B_g(p,r))}{r^n}$ is non-increasing, hence it must be constant by asymptotic flatness, finally the equality case of Bishop-Gromov theorem implies that we must be flat. \qed

We will now present the main result of this paper. We should highlight that, in particular, the proof of the rigidity statement in the following theorem follows ideas close to the proofs of the positive mass theorems associated to the Paneitz operator of [24, 18, 19].
**Theorem 2.** Let \((M^n, g)\) be an \(n\)-dimensional AE manifold, with \(n \geq 3\), which satisfies the decaying conditions i) and ii) of Proposition \[\text{7}\] and such that \(Q_g, R_g \geq 0\). Then, the fourth-order energy \(\mathcal{E}(g)\) is non-negative and \(\mathcal{E}(g) = 0\) if and only if \(Q_g \equiv 0\) and \((M, g)\) is conformal to \((\mathbb{R}^n, \cdot)\).

**Proof.** Let us first notice that we need only work with the case \(Q_g \geq 0\) and \(R_g > 0\), since in the remaining case the result follows from Proposition \[\text{3}\]. Thus, in what follows we assume that \(Q_g \geq 0\) and \(R_g > 0\).

**Proof of positivity**

Let us start by noticing that under our hypotheses we have \(R_g = O_2(r^{-\tau - 2})\) with \(\tau > \tau_n\). Then, the energy exists and is independent of the sequence of spheres used to compute it. Furthermore, these decaying conditions imply that \(\text{(8)}\) also holds. We can rewrite

\[
\int_{S^{n-1}_1} \partial_r R_g r^{n-1} d\omega_{n-1} = \frac{\partial}{\partial r} \left( r^{n-1} \int_{S^{n-1}_1} R_g d\omega_{n-1} \right) - (n-1)r^{n-2} \int_{S^{n-1}_1} R_g d\omega_{n-1}.
\]

Define

\[
h(r) \equiv r^{n-2} \int_{S^{n-1}_1} R_g d\omega_{n-1}.
\]

Then,

\[
\int_{S^{n-1}_1} \partial_r R_g r^{n-1} d\omega_{n-1} = \frac{\partial}{\partial r} (rh(r)) - (n-1)h(r).
\]

By assumption we have that \(h > 0\) and from our decreasing assumption that there is some \(r_0\) sufficiently large, so that \(R_g(x) \leq Cr^{-\tau - 2}(x)\) for all \(x \in M\) such that \(r(x) > r_0\). In particular, this last condition implies that \(h(r) \leq C\omega_{n-1}r^{-\tau + n-4}\) for all \(r > r_0\). Suppose that there is some \(r^* > r_0\) such that \(\text{(12)}\) is positive at \(r^*\). Then, by continuity, there is some interval \((r^* - \epsilon, r^* + \epsilon)\) so that

\[rh'(r) > (n-2)h(r),\]

which implies

\[\log \left( \frac{h(r)}{h(r^*)} \right) > (n-2) \log \left( \frac{r}{r^*} \right), \text{ for any } r \in [r^*, r^* + \epsilon].\]

That is \(h(r) > \frac{h(r^*)}{r^{n-2}}r^{n-2}\) for all \(r \in [r^*, r^* + \epsilon]\). But notice that also \(h(r) \leq C\omega_{n-1}r^{-\tau + n-4}\) for all such \(r\), which shows that

\[\frac{h(r^*)}{r^{n-2}}r^{n-2} < h(r) \leq C\omega_{n-1}r^{-\tau + n-4}\text{ for any } r \in [r^*, r^* + \epsilon].\]

Since \(r^*\) is fixed, the above implies that there must be some \(\epsilon_{\text{max}}\) where \(\text{(12)}\) is non-positive around \(r^* + \epsilon_{\text{max}}\). That is, there must be some \(r_1 > r^*\) where

\[
r_1^{n-1} \int_{S^{n-1}_1} \partial_r R_g(r_1, \theta) d\omega_{n-1}(\theta) = \frac{\partial}{\partial r} (rh(r)) - (n-1)h(r) \leq 0. \tag{13}
\]
The idea is now to repeat this procedure so as to select a sequence of spheres \( \{S_{r_i}^{n-1}\}_{i=1}^{\infty} \) along which (12) is manifestly non-positive. The fact that such radii can be chosen under our hypotheses is a consequence of the above argument. Then

\[
\mathcal{E}(g)_j = -r_j^{n-1} \int_{S_{r_j}} \partial_r R_g d\omega_{n-1} \geq 0 \text{ for all } j,
\]

which implies

\[
\mathcal{E}(g) = -\lim_{j \to \infty} \int_{S_{r_j}} \partial_r R_g r^{n-1} d\omega_{n-1} \geq 0. \tag{14}
\]

**Proof of rigidity**

Now, the idea is to analyze whether \( \mathcal{E}(g) \) implies \( g \)-flatness. First we look for a conformal metric \( \tilde{g} = u^{\frac{4}{n-2}} g \) such that \( R_{\tilde{g}} \equiv 0 \), that is,

\[
L_g(u) = \Delta_g u - c_n R_g u = 0, \tag{15}
\]

where \( c_n = \frac{n-2}{4(n-1)} \). Setting \( u = 1 + \phi \), this is equivalent to solve

\[
\Delta_g \phi - c_n R_g \phi = c_n R_g, \tag{16}
\]

We know that \( R_g = O_2(r^{-\tau-2}) \), this implies that \( R_g \in H^{k-2}_{\rho} \) for any \( \delta < \tau \) and any \( k \). Thanks to corollary A.1, we know that \( L_g : H^s_{\rho} \rightarrow H^{s-2}_{\rho-2} \) is an isomorphism for \( s > \frac{n}{2} \) and \( 2 - n < \rho < 0 \). Thus we find a (smooth) solution \( \phi \in H^{s}_{\rho} \) for any \( 0 < \delta < c_{n-2} \min\{\tau, n-2\} \).

\[\text{3. I) } n \neq 4\]

Then, let \( \Phi = u^{-\frac{n-4}{4}} \); rewrite \( g = \Phi^{\frac{4}{n-4}} \tilde{g} \) and notice that transformation rule for the Paneitz operator gives us

\[
\frac{n-4}{2} \Phi^{\frac{n-4}{n-2}} Q_g = P_g \Phi.
\]

Also, since \( R_{\tilde{g}} = 0 \), we have that \( Q_{\tilde{g}} = -\frac{2}{(n-2)^2} |\text{Ric}_{\tilde{g}}|_{\tilde{g}}^2 \), and thus

\[
P_g \Phi = \Delta_g^2 \Phi + \text{div}_g (4S_g(\nabla \Phi, \cdot)) - \frac{(n-4)}{(n-2)^2} |\text{Ric}_{\tilde{g}}|_{\tilde{g}}^2 \Phi, \tag{17}
\]

where, \( S_g = \frac{1}{n-2} \text{Ric}_{\tilde{g}} \) since \( R_{\tilde{g}} = 0 \). Therefore, denoting by \( D_r \) the bounded region in \( M \) whose boundary is given by the sphere \( S_r \) in the end of \( M \), we see that

\[
\int_{D_r} \frac{n-4}{2} \Phi^{\frac{n-4}{n-2}} Q_g + \frac{(n-4)}{(n-2)^2} |\text{Ric}_{\tilde{g}}|_{\tilde{g}}^2 \Phi dv_{\tilde{g}} = \int_{S_r} \tilde{g}(\nabla \Delta_g \Phi, \nu) d\omega_{\tilde{g}} + \frac{4}{n-2} \int_{S_r} \text{Ric}_{\tilde{g}}(\nabla \Phi, \nu) d\omega_{\tilde{g}} \tag{18}
\]

Let us now estimate the terms in the right-hand side.

**Estimates on \( \text{Ric}_{\tilde{g}}(\nabla \Phi, \nu) \)**

\[\text{In the case } \tau > n-2, \text{ we can achieve } \delta = \sigma.\]
From the above analysis we find that \( u = 1 + O_4(r^{-\delta}) \) for any \( \delta < \sigma = \min\{\tau, n - 2\} \), and therefore
\[
\tilde{g}_{ij} = (1 + O_4(r^{-\delta}))(\delta_{ij} + O_4(r^{-\tau})),
\]
\[
= \delta_{ij} + O_4(r^{-\delta}).
\]
Then, \( \text{Ric}_g = O_2(r^{-\delta-2}) \) and also
\[
\nabla \Phi = -\frac{n - 4}{n - 2} u^{-\frac{n - 4}{n - 2}} \nabla u = O_3(r^{-\delta-1}).
\]
Which implies
\[
\text{Ric}_g(\nabla \Phi, \nu) = \text{Ric}_g(\nabla \Phi, \nu^e) + \text{Ric}_g \nabla^i \Phi (\nu^e - \tilde{\nu})^j = O_2(r^{-2\delta-3}),
\]
\[
\text{Ric}_g(\nabla \Phi, \nu)^{n-1} = O_2(r^{-n-2\delta-4}).
\]
Since \( \sigma > \frac{n}{2} - 2 \), then, we can choose \( \delta \) such that
\[
|Ric_g(\nabla \Phi, \nu)^{n-1} \rightarrow 0 \text{ as } r \rightarrow +\infty.
\]
**Estimates on \( \Delta_g \Phi \)**

Let us rewrite
\[
\tilde{\nabla}_i \Phi = -\frac{n - 4}{n - 2} u^{-\frac{n - 4}{n - 2}} \tilde{\nabla}_i u,
\]
\[
\tilde{\nabla}_j \tilde{\nabla}_i \Phi = -\frac{n - 4}{n - 2} \left( u^{-\frac{n - 4}{n - 2}} \tilde{\nabla}_j \tilde{\nabla}_i u - \frac{n - 3}{n - 2} u^{-\frac{n - 4}{n - 2}} \tilde{\nabla}_j \nabla_i u \right),
\]
from which we get
\[
\Delta_g \Phi = -\frac{n - 4}{n - 2} u^{-\frac{n - 4}{n - 2}} \Delta_g u + 2 \left( \frac{n - 4}{n - 2} \right) \frac{(n - 4)(n - 3)}{(n - 2)^2} u^{-\frac{n - 4}{n - 2}} \\left| \nabla u \right|^2.
\]
Now, we can compute that
\[
\Delta_g u = \tilde{g}^{ij} \tilde{\nabla}_j \nabla_i u = \tilde{g}^{ij} (\nabla_j \nabla_i u - (\Gamma^k_{ij}(\tilde{g}) - \Gamma^k_{ij}(g)) \nabla_k u),
\]
\[
= \Delta_g u + (\tilde{g}^{ij} - g^{ij}) \nabla_j \nabla_i u - \tilde{g}^{ij} (\Gamma^k_{ij}(\tilde{g}) - \Gamma^k_{ij}(g)) \nabla_k u
\]
and we can estimate \( \tilde{g}^{ij} - g^{ij} = O_4(r^{-\delta}) - O_4(r^{-\tau}) = O_4(r^{-\delta}) \) and \( \Gamma^k_{ij}(\tilde{g}) - \Gamma^k_{ij}(g) = O_3(r^{-\delta-1}) - O_3(r^{-\tau-1}) = O_3(r^{-\delta-1}) \), since \( \sigma = \min\{\tau, n - 2\} \), and also \( \nabla u = O_3(r^{-\delta-1}) \). Therefore
\[
\tilde{g}^{ij} (\Gamma^k_{ij}(\tilde{g}) - \Gamma^k_{ij}(g)) \nabla_k u = O_3(r^{-2\delta-2}),
\]
\[
(\tilde{g}^{ij} - g^{ij}) \nabla_j \nabla_i u = O_2(r^{-2\delta-2}). \tag{19}
\]
On the other hand, from \([13]\) we find that
\[
\Delta_g u = c_n R_g + c_n \underbrace{R_g}_{O_2(r^{-\delta-2})} \underbrace{\phi}_{O_4(r^{-\delta})} = c_n R_g + O_2(r^{-2\delta-2}). \tag{20}
\]
Putting together \((19) - (20)\), we finally find that
\[
\Delta_g u = c_n R_g + O_2(r^{-2\delta-2}), \tag{21}
\]
which, in turn, implies
\[
\Delta_g \Phi = -\frac{n-4}{4(n-1)} u^{-\frac{n+4}{n-2}} R_g + O_2(r^{-2\delta-2}),
\]
\[
\nabla(\Delta_g \Phi) = -\frac{n-4}{4(n-1)} u^{-\frac{n+4}{n-2}} \nabla R_g + O_1(r^{-2\delta-3}).
\]

Therefore
\[
\tilde{g}(\tilde{\nabla} \Delta_g \Phi, \nu) = \tilde{g}(\tilde{\nabla} \Delta_g \Phi, \nu^e) + \tilde{g}(\tilde{\nabla} \Delta_g \Phi, \nu - \nu^e),
\]
\[
= \nabla_i \Delta_g \Phi \frac{x^i}{r} + \left( \tilde{g}_{ij} - \delta_{ij} \tilde{g}^i(\Delta_g \Phi) \right) \frac{x^j}{r} + \tilde{g}(\tilde{\nabla} \Delta_g \Phi, \nu - \nu^e),
\]
\[
= -\frac{n-4}{4(n-1)} u^{-\frac{n+4}{n-2}} \nabla_i R_g \frac{x^i}{r} + O_1(r^{-2\delta-3}),
\]

implying
\[
\tilde{g}(\tilde{\nabla} \Delta_g \Phi, \nu)^r r^{n-1} = -\frac{n-4}{4(n-1)} u^{-\frac{n+4}{n-2}} \partial_i R_g r^{n-1} + O_1(r^{n-2\delta-4}).
\]

where we have used, as above, that \( \delta > \frac{n}{2} - 2 \), implying we can choose \( \delta > 0 \) such that \( n - 2\delta - 4 < 0 \). From all this and \( d\omega_{\tilde{g}} = (1 + O(r^{-\delta}))d\omega_r \), we find that
\[
\lim_{r \to \infty} \int_{S_r^{n-1}} \tilde{g}(\tilde{\nabla} \Delta_g \Phi, \nu) d\omega_r = -\frac{n-4}{4(n-1)} \lim_{r \to \infty} \left( \int_{S_r^{n-1}} \partial_i R_g r^{n-1} d\omega + \int_{S_r^{n-1}} \partial_i R_g O(r^{-\delta}) r^{n-1} d\omega \right),
\]
\[
= \frac{n-4}{4(n-1)} E(g),
\]
Putting together the above analysis, from (18), we find that
\[
\int_M \frac{n-4}{2} \Phi^{\frac{n+4}{2}} Q_g + \frac{(n-4)}{(n-2)^2} |\text{Ric}_g|_g^2 \Phi^2 d\nu_g = \frac{n-4}{4(n-1)} E(g).
\]

Thus, if \( E(g) = 0 \), then \( Q_g \equiv 0 \) and \( \text{Ric}_g \equiv 0 \), which implies, through results such as those exposed in the proof of Proposition 8 that \( \tilde{g} \) is flat and \( M \cong \mathbb{R}^n \). Therefore, since \( \tilde{g} = u^{\frac{4}{n-2}}g \), then \( g \) is conformally-flat.

II) \( n = 4 \)

Now, fix \( \Phi = -\ln(u) \) and rewrite \( \tilde{g} = e^{-4\Phi} g \). Then, using the transformation rule (B.7) we find
\[
0 \leq Q_g = e^{-4\Phi} (P_g \Phi + Q_g).
\]

Since \( R_g = 0 \), then \( Q_g = -\frac{1}{2} |\text{Ric}_g|_g^2 \), which implies
\[
0 \leq \int_{D_r} \left( Q_g e^{4\Phi} + \frac{1}{2} |\text{Ric}_g|_g^2 \right) d\nu_g = \int_{S_r} \tilde{g}(\tilde{\nabla} \Delta_g \Phi, \nu) d\omega_g + 2 \int_{S_r} \text{Ric}_g(\nabla \Phi, \nu) d\omega_g
\]

(25)
**Estimates on** $\text{Ric}_{\tilde{g}}(\nabla \Phi, \tilde{\nu})$

We clearly have $u = 1 + O_4(r^{-\delta})$ for some $\delta < \sigma$. Then, from $\nabla \Phi = -\frac{1}{\nu} \nabla u = O_3(r^{-\delta-1})$, we find that

$$\text{Ric}_{\tilde{g}}(\nabla \Phi, \tilde{\nu}) r^3 = \underbrace{\text{Ric}_{\tilde{g}}(\nabla \Phi, \nu_c) r^3}_{O_3(r^{-2\delta-3})} + \underbrace{\text{Ric}_{\tilde{g}}(\nabla \Phi, \tilde{\nu} - \nu_c) r^3}_{O_3(r^{-3\delta-3})} = O_3(r^{-2\delta}).$$

Combining this with $d\omega_{\tilde{g}} = (1 + O(r^{-\delta})) d\omega$, we get

$$\int_{S_r} \text{Ric}_{\tilde{g}}(\nabla \Phi, \tilde{\nu}) d\omega_{\tilde{g}} = o(1).$$

**Estimates on** $\tilde{g}(\nabla \Delta_{\tilde{g}} \Phi, \tilde{\nu})$

Straightforwardly we see that

$$\tilde{\nabla}_j \tilde{\nabla}_i \Phi = -u^{-1} \tilde{\nabla}_j u \tilde{\nabla}_i u + u^{-2} \tilde{\nabla}_j u \tilde{\nabla}_i u = -u^{-1} \tilde{\nabla}_j \tilde{\nabla}_i u + O_3(r^{-2\delta-2}),$$

$$\tilde{\nabla}_j \tilde{\nabla}_i u = \tilde{\nabla}_j \nabla_i u - \Delta_{\tilde{g}}^k \nabla_k u = \tilde{\nabla}_j \nabla_i u + O_3(r^{-2\delta-2}),$$

where $\Delta_{\tilde{g}}^k = \Gamma_{ij}^k(\tilde{g}) - \Gamma_{ij}^k(g)$. Thus, we see that

$$\Delta_{\tilde{g}} \Phi = -u^{-1} \Delta_g u - \underbrace{u^{-1}(\tilde{g}^{ij} - g^{ij}) \tilde{\nabla}_j \nabla_i u + O_3(r^{-2\delta-2})}_{O_3(r^{-2\delta-2})} = -u^{-1} \Delta_g u + O_3(r^{-2\delta-2}),$$

so that,

$$\nabla \Delta_{\tilde{g}} \Phi = -c_n u^{-1} \nabla R_g + O_3(r^{-2\delta-2}),$$

$$\tilde{g}(\nabla \Delta_{\tilde{g}} \Phi, \tilde{\nu}) = -c_n u^{-1} \tilde{g}(\nabla R_g, \nu_c) - c_n u^{-1} \underbrace{\tilde{g}(\nabla R_g, \tilde{\nu} - \nu_c)}_{O_1(r^{-2\delta-3})} + O_1(r^{-2\delta-3}),$$

implying

$$\int_{S_r} \tilde{g}(\nabla \Delta_{\tilde{g}} \Phi, \tilde{\nu}) d\omega_{\tilde{g}} = -c_n \int_{S_r} u^{-1} \tilde{g}(\nabla R_g, \nu_c) d\omega_{\tilde{g}} + O(r^{-2\delta}),$$

$$= -c_n \int_{S_r} u^{-1} \tilde{g}(\nabla R_g, \nu_c) d\omega_r - \int_{S_r} u^{-1} \tilde{g}(\nabla R_g, \nu_c) O(r^{-\delta}) d\omega_r + O(r^{-2\delta}),$$

$$= -c_n \int_{S_r} u^{-1} \tilde{g}(\nabla R_g, \nu_c) d\omega_r + o(1).$$

Therefore, we find that

$$\lim_{r \to \infty} \int_{S_r} \tilde{g}(\nabla \Delta_{\tilde{g}} \Phi, \tilde{\nu}) d\omega_{\tilde{g}} = c_n E(g).$$

Finally, putting together (25), (27) and (29), we find that

$$0 \leq \int_M \left( Q_g \nabla^2 \Phi + \frac{1}{2} |\text{Ric}_{\tilde{g}}|^2_{\tilde{g}} \right) dV_{\tilde{g}} = c_n E(g),$$

which implies that if $E(g) = 0$, then $\text{Ric}_{\tilde{g}} \equiv 0$ and therefore $g$ is conformally-flat with $Q_g \equiv 0$.

Finally, $g = v^{4 - \delta}$ (or $g = e^{2v} \delta$ if $n = 4$), with $\Delta^2 v = 0$ and $\lim_{r \to \infty} v = 1$ (or $\lim_{r \to \infty} v = 0$), hence, by maximum principle, we get that $v \equiv 1$ (or $v \equiv 0$) which achieves the proof of the theorem.
It is worth noticing that, looking more carefully to the above proof, we can weaken the hypotheses of the previous theorem and still get interesting results. Indeed, in the positivity part, we only use the fact that the scalar curvature is positive at infinity, while for the rigidity statement, we only use the fact that we can make the scalar curvature flat via a conformal transformation, which is ensured by assuming that \( Y([g]) > 0 \) by theorem 5.1 of [13], and that \( Q_g \) is non-negative. Furthermore, under this last condition, equations (23) and (23) also provide proofs of positivity. All these observations give us the following results.

**Theorem 3.** Let \((M^n, g)\) be an \(n\)-dimensional AE manifold, with \(n \geq 3\), which satisfies the decaying conditions i) and ii) of Proposition 1 and such that \( R_g > 0 \) on \( M \setminus K \), for some compact set \( K \), then the following results.

- Equation (23) also provides proofs of positivity. All these observations give us the following results.

**Theorem 4.** Let \((M^n, g)\) be an \(n\)-dimensional AE manifold, with \(n \geq 3\), which satisfies the decaying conditions i) and ii) of Proposition 1 and such that \( Q_g \geq 0 \) and \( Y([g]) > 0 \), then \( \mathcal{E}(g) \geq 0 \) with equality holding if and only if \((M, g)\) is isometric to \((\mathbb{R}^n, \cdot)\).

Let us now notice that the \(Q\)-curvature assumption in Theorem 1 cannot be weakened while keeping the rest of the hypotheses. This can be seen as follows: Consider \((\mathbb{R}^n, \delta)\), with \(n \geq 3\), and let \( \{h_k\}_{k=1}^{\infty} \in C^\infty(U) \), be a sequence of smooth compactly supported symmetric second rank tensor fields, \( U \subset \subset M \), with \(|h_k| < \epsilon\), for some fixed small \(\epsilon > 0\), such that \( h_k \xrightarrow{k \to \infty} 0 \). Then, consider the sequence of metrics \( g_k = \delta + h_k \), where the construction of the \(h_k\) can be made so that \(g_k\) are not conformally flat. Furthermore, this construction can be fit so that \( \{g_k\}_{k=1}^{\infty} \) are all Yamabe positive.\(^4\) Therefore, there are conformal factors \( \{u_k = 1 + \varphi_k\} \), with \( \varphi_k \in H^1_{\delta, \delta} \), with \(\delta < n - 2\) arbitrary, such that \( \Delta_{g_k} u_k - R_{g_k} u_k = 0 \).\(^5\) This means that \( R_{u_k^{-2} g_k} = 0 \) and thus \( Q_{u_k^{-2} g_k} = -c_2(n) |\text{Ric}_{u_k^{-2} g_k}|^2 < 0 \). Furthermore, all the metrics \( u_k^{-2} g_k \) satisfy the decaying conditions of Proposition 1 and are, by construction, Yamabe positive. Nevertheless, from their asymptotics \( u_k^{-2} g_{kij} = (1 + \frac{a_k}{r^{n-2}}) \delta_{ij} + o_k(r^{-(n-2)}) \) we see that \( \mathcal{E}(g_k) = 0 \) for all \(k\), although none of the \(g_k\) are conformally flat. Finally, by considering \(k\) sufficiently large, we can make \(-\frac{1}{k} < Q_{u_k^{-2} g_k} < 0\), which shows that there are Yamabe positive AE metrics satisfying conditions (i)-(ii) of Proposition 1 which have negative \(Q\)-curvature which is as small as we want, zero energy and are not conformally flat.

In dimensions \(n = 3, 4\), under the above hypotheses, we fall into a curious situation, since any such \(n\)-dimensional AE-manifold has \( \mathcal{E}(g) = 0 \) as a consequence of the fall-off conditions.

**Corollary 2.** Any \(n\)-dimensional AE-Riemannian manifold \((M^n, g)\), with \(n \in \{3, 4\}\), such that \( Q_g \geq 0 \) and \( Y([g]) \geq 0 \), then it is conformal to \((\mathbb{R}^n, \delta)\) with vanishing \(Q\)-curvature.

---

\(^4\)For an explicit construction of this see Example 1 in [2]

\(^5\)In fact, from Theorem 1.17 in [13], it follows that

\[ u_k = 1 + \frac{a_k}{r^{n-2}} + o(r^{-(n-2)}) \]

where \(a_k\) are constants.
Rigidity in critical cases

The aim of this section is to comment further on the especial case that occurs in dimension four. In particular, the fact that the energy $E(g)$ is always zero has some topological consequences as a corollary. It is worth to contrast this with the second order case associated to the ADM energy. In this last case, the critical case is given in dimension two, where asymptotic flatness is too strong a condition in order to detect any meaningful information concerning the ADM energy (for details, see Chapter 3 in [25]). Let us start by briefly commenting on this case as a warm-up.

Let $(M, g)$ be an asymptotically flat surface with $\tau > 0$ and non-negative integrable Gaussian curvature. Let us consider a chart at infinity, and recall the definition of the ADM energy

$$m(g) = \lim_{R \to +\infty} \int_{\partial B_R} (\partial_i g_{ij} - \partial_j g_{ii}) \nu^i ds,$$

where $\nu$ is the outer normal. Because of the decreasing assumptions, we easily see that the mass must vanish. Let us write the Gauss-Bonnet formula which gives

$$2\chi(M) = \int_{M \setminus B(0, R)^e} K d\sigma + \int_{\partial B(0, R)} k_g ds.$$

Here we write $\chi(M)$ instead of $\chi(M \setminus B(0, R)^e)$ since they are equal for $R$ large. Then in polar coordinate we have,

$$k_g = -\frac{1}{r^2} g(\nabla_\theta \partial_\theta, \partial_r) = -\frac{1}{r^2} \Gamma^r_{\theta \theta}(g) + O\left(\frac{1}{r^{1+\tau}}\right) = -\frac{1}{r^2} \Gamma^r_{\theta \theta}(\delta) + O\left(\frac{1}{r^{1+\tau}}\right)$$

$$= \frac{1}{r} + O\left(\frac{1}{r^{1+\tau}}\right),$$

hence passing to the limit we obtain that

$$2\pi \chi(M) - 2\pi = \int_M K d\sigma.$$

But we know that

$$\chi(M) = 2 - 2g - r,$$

where $g$ is the genus of $M$ and $r$ its number of ends. Then we necessarily get that $g = 0$, $r = 1$ and $K \equiv 0$. Hence we recover the following well-known proposition, see chapter 3 [25],

**Proposition 4.** Let $(M, g)$ an asymptotically flat surface with non-negative Gauss curvature. Then, $(M, g)$ is the Euclidean plane.

Let us turn to the 4-dimensional case. Let $(M, g)$ be a 4-dimensional AE manifold with $\tau > 0$ with only one end such that $|W_g|^2$ is integrable. From Corollary 2 we know that the mass is necessarily vanishing

$$E(g) = -\lim_{R \to \infty} \int_{S_R} \partial_r R_g r^3 d\omega_3 = 0. \quad (31)$$

Along the lines of the 2-dimensional case, let us try to get a similar proof of the rigidity result. First, we apply the Gauss-Bonnet-Chern formula with boundary (see [11] for instance)

$$32\pi^2 \chi(M) = \int_{K_R} |W|^2 + 16\sigma_2(S_g) dv + 8 \int_{\partial B(0, R)} B_g dv$$
where \( K_R = \left( M \setminus B(0,R) \right)^c \), \( S_g = \frac{1}{2} (\text{Ric} - \frac{1}{6} R_g g) \) is the Schouten tensor and \( B_g = \frac{1}{2} R_g H - \text{Ric}(\nu,\nu) H - R_{\alpha\beta\gamma\tau} \Pi^{\alpha\beta} + \frac{1}{3} H^3 - H|\Pi|^2 + \frac{2}{3} tr(\Pi^3) \), where \( \Pi \) and \( H \) are respectively the second form and the mean curvature of a hypersurface \( \Sigma \hookrightarrow M \), \( H = tr_\Sigma \Pi \), taken with respect to the inner unit normal to \( \Sigma \), the Greek indices are indices tangent to \( \Sigma \), and \( \Pi^3 \) is defined in local coordinates by \( \Pi^3_{ij} = \Pi_i^k \Pi_j^\ell \Pi^\ell_j \). Moreover, in dimension 4, we have

\[
Q_g = -\frac{1}{6} \Delta_g R_g + 4\sigma_2(S_g),
\]

and thanks to our decreasing assumption, we get

\[
B_g = \frac{2}{r^3} + O \left( \frac{1}{r^3 + \tau} \right)
\]

which gives, using once more our decreasing assumption, that

\[
32\pi^2(\chi(M) - 1) = \int_{K_R} |W|^2 + 4Q_g + \frac{2}{3} \Delta_g R_g \, dv_g + O \left( \frac{1}{R^r} \right)
\]

(32)

\[
= \int_{K_R} |W|^2 + 4Q_g \, dv_g + O \left( \frac{1}{R^r} \right)
\]

(33)

Hence passing to the limit, we get

\[
32\pi^2(\chi(M) - 1) = \int_M |W|^2 + 4Q_g \, dv_g
\]

Let us set

\[
\kappa_g = \int_M Q_g \, dv_g.
\]

We can easily deduce the following curvature-topology proposition. This proposition is not really new but we just would like to put them in perspective with our rigidity result

**Proposition 5.** Let \((M, g)\) an AE 4-manifold with \( \kappa_g \geq 0 \) then \( \chi(M) \geq 1 \) with equality if \((M, g)\) is locally conformally flat with vanishing \( Q \)-curvature.

In particular if \( \kappa_g \geq 0 \) and \( Y([g]) \geq 0 \) or the second betti number vanishes, then \( Q_g \equiv 0 \) and \((M, g)\) is conformal to the euclidean \( \mathbb{R}^4 \).

### 3.1 The \( Q \)-curvature positive mass theorem

In this section we will make contact with a series of recent results associated with the positive mass theorem for the Paneitz operator, namely \[18\] [19] [24]. In particular, we will show that these results follow from Theorem 4.

Let us consider a closed manifold \((M^n, g)\) with \( n \geq 5 \). If, the Paneitz operator is coercive, that is to say

\[
\inf_{u \in H^2(M) : \|u\|_2 = 1} \int_M P_g(u) u \, dv_g > 0,
\]

then it admits a Green functions \( G_{P_g} \). This is in particular guaranteed if \( R_g \geq 0 \) and \( Q_g \) is semi-positive, see proposition B of \[18\], or more generally by the fact that

\[
Y_4([g]) = \inf_{u \in H^2(M) : \|u\|_2 = 1} \int_M P_g(u) u \, dv_g > 0,
\]

where \( K_R = \left( M \setminus B(0,R) \right)^c \), \( S_g = \frac{1}{2} (\text{Ric} - \frac{1}{6} R_g g) \) is the Schouten tensor and \( B_g = \frac{1}{2} R_g H - \text{Ric}(\nu,\nu) H - R_{\alpha\beta\gamma\tau} \Pi^{\alpha\beta} + \frac{1}{3} H^3 - H|\Pi|^2 + \frac{2}{3} tr(\Pi^3) \), where \( \Pi \) and \( H \) are respectively the second form and the mean curvature of a hypersurface \( \Sigma \hookrightarrow M \), \( H = tr_\Sigma \Pi \), taken with respect to the inner unit normal to \( \Sigma \), the Greek indices are indices tangent to \( \Sigma \), and \( \Pi^3 \) is defined in local coordinates by \( \Pi^3_{ij} = \Pi_i^k \Pi_j^\ell \Pi^\ell_j \). Moreover, in dimension 4, we have

\[
Q_g = -\frac{1}{6} \Delta_g R_g + 4\sigma_2(S_g),
\]

and thanks to our decreasing assumption, we get

\[
B_g = \frac{2}{r^3} + O \left( \frac{1}{r^3 + \tau} \right)
\]

which gives, using once more our decreasing assumption, that

\[
32\pi^2(\chi(M) - 1) = \int_{K_R} |W|^2 + 4Q_g + \frac{2}{3} \Delta_g R_g \, dv_g + O \left( \frac{1}{R^r} \right)
\]

(32)

\[
= \int_{K_R} |W|^2 + 4Q_g \, dv_g + O \left( \frac{1}{R^r} \right)
\]

(33)

Hence passing to the limit, we get

\[
32\pi^2(\chi(M) - 1) = \int_M |W|^2 + 4Q_g \, dv_g
\]

Let us set

\[
\kappa_g = \int_M Q_g \, dv_g.
\]

We can easily deduce the following curvature-topology proposition. This proposition is not really new but we just would like to put them in perspective with our rigidity result

**Proposition 5.** Let \((M, g)\) an AE 4-manifold with \( \kappa_g \geq 0 \) then \( \chi(M) \geq 1 \) with equality if \((M, g)\) is locally conformally flat with vanishing \( Q \)-curvature.

In particular if \( \kappa_g \geq 0 \) and \( Y([g]) \geq 0 \) or the second betti number vanishes, then \( Q_g \equiv 0 \) and \((M, g)\) is conformal to the euclidean \( \mathbb{R}^4 \).
where \( 2^\# = \frac{2}{n-4} \). This last infimum has the advantage to be conformally invariant and it plays a similar role to the Yamabe invariant for the \( Q \)-curvature. Through this section, we will assume that \( Y_4([g]) > 0 \) and therefore \( G_{P_\gamma} \) exists for every \( g \in [g] \).

Nevertheless, contrary to the conformal Laplacian, nothing here guarantees that \( G_{P_\gamma} \) is positive. This will be one of our assumptions, which is in particular satisfied if \( Y([g]) \geq 0 \) and \( Q_\bar{g} \) is semi-positive for some conformal metric \( \bar{g} \), due to lemma 3.2 in [21].

**Remark 1.** In fact Hang and Yang assume \( Y([g]) > 0 \), but if \( Y([g]) = 0 \) and there is some \( \bar{g} \in [g] \) with \( Q_\bar{g} \) semi-positive, then there exists \( \bar{g} = u^{\frac{n-4}{2}} \bar{g} \) such that \( R_\bar{g} = 0 \) which implies that \( Q_\bar{g} \leq 0 \) which contradicts the fact that

\[
\int_M Q_\bar{g} v \, dv_\bar{g} = \int_M u^{\frac{n+4}{n-4}} Q_\bar{g} \, dv_\bar{g},
\]

with \( v = u^{\frac{n-4}{2}} \). Hence the existence of a conformal metric with semi-positive \( Q \)-curvature forces the Yamabe invariant to be positive.

In fact, if \( Y([g]) \geq 0 \), thanks to, theorem 1.1 of [21], the existence of a positive \( G_{P_\gamma} \) is equivalent to the existence of a conformal metric \( \bar{g} \) with semi-positive \( Q \)-curvature.

Let now us focus on the expansion of the Green function around a singularity. If we assume that \( 5 \leq n \leq 7 \) or \( g \) is locally conformally flat, then, in conformal normal coordinates \( \{x^i\} \) for the conformal metric \( \bar{g} \), the Green function \( G_P \) of \( P_\gamma \) admits an expansion of the form (see Proposition 2.5 in [18])

\[
G_P(p, x) = \frac{\gamma_n}{r^{n-4}} + \alpha + O_4(r), \tag{34}
\]

where \( r(x) \equiv d_\bar{g}(p, x) \), \( \gamma_n \equiv \frac{1}{(n-2)(n-4)\omega_n-1} \) and \( \alpha \) is a constant called the mass.

**Remark 2.** As remarked in [21], under the condition \( \text{Ker}(P_\gamma) = 0 \) (which is itself conformally invariant), the sign of the Green function is a conformal invariant since,

\[
G_{P_\gamma^{-1}P_\gamma}(p, q) = u(p)^{-1}u(q)^{-1}G_{P_\gamma}(p, q).
\]

Hence, the sign of the mass is also a conformal invariant.

This terminology arises in analogy to the Yamabe problem, where the Green function \( G_{L_\gamma} \) of the conformal Laplacian is involved and, in important cases, it admits a similar expansion to (34). In the case of the conformal Laplacian, it was an observation of R. Schoen that the constant which appears in the place of \( \alpha \) is precisely the ADM mass of the AE-manifold obtained by via the stereographic projection \( (M \setminus \{p\}, G_{L_\gamma}^{-1}g) \). This showed that the resolution of the positive mass conjecture in general relativity would amount to completing the resolution of the Yamabe problem, which concerned the cases \( Y(g) > 0 \) and dimensions \( n = 3, 4, 5 \) or \( M \) locally conformally flat (see [36] and [26]).

The positive mass theorem presented in [18] as Theorem 2.9 states that under the conditions described above which provide us with the expansion (34), the mass satisfies \( \alpha \geq 0 \) with equality holding if and only if \( (M, g) \) is conformal to the round sphere. This came about as a generalisation of [24], where this result was proven in the conformally flat case, and, in turn, all this was generalised in [19], where the condition \( R_g \geq 0 \) was replaced by \( Y([g]) > 0 \). Furthermore, in Theorem 1.4 of [21], this positive mass theorem is used to solve the \( Y_4([g]) > 0 \) \( Q \)-curvature prescription problem in very much the same
spirit as the usual positive mass theorem was used by Schoen in [36]. We intend to show that all these results are special cases of the positive energy theorem presented above.

Finally, let us notice that, for instance, in the Q-curvature prescription problem it is the sign of the mass that is actually important rather than its precise value. This becomes especially useful when it is combined with the additional observation that the sign of the mass is itself a conformal invariant, which was highlighted in Remark 2. Thus, being concerned with a conformal problem, given \((M^n, [g])\), we will always consider a choice of \(g\) satisfying the conformal normal coordinate properties \((C.1), (C.3), (C.4)\). Notice that from the discussion presented in that appendix, this can always be achieved by first going to a related conformal metric.

In order to prepare for the main statements of this section, let us first establish a couple of preliminary results.

**Proposition 6.** Let \((M^n, g)\) be a closed manifold satisfying \(n \geq 5\) and whose Panietz operator admits a positive Green function \(G_P\) with an expansion as \((34)\) around some point \(p \in M\). Then, the manifold \((\hat{M} = M \setminus \{p\}, \hat{g} = G_P(p, \cdot) \frac{\rho^2}{\rho^4} g)\) is an asymptotically flat manifold of order \(\tau = 1\) if \(n = 5\) and \(\tau = 2\) if \(n > 5\). Furthermore, either if \(5 \leq n \leq 7\) or \(g\) is flat around \(p\), then \(E(\hat{g}) = 8(n - 1)(n - 2)\omega_{n-1} \gamma_n \alpha\).

**Proof.** Let us fix the conformal normal coordinates \(\{x^i\}\) where \((34)\) holds. Let us appeal to the conformal normal coordinate construction of order \(N \geq 4\) described Appendix B so that \((C.3) - (C.4)\) hold, and start by considering the following expansion near \(p\)

\[
g_{ij} = \delta_{ij} + \frac{1}{3} R_{ijkl}(p) x^k x^l + \frac{1}{6} R_{ijkl,a}(p) x^k x^l x^a + O(r^4),\]

(35)

Now, consider the inverted coordinates \(z^j = \gamma_n^2 x^j / \rho^2\) in a neighborhood of \(p\) and define \(\rho^2 = |z|^2\), so that \(\rho^2 = \gamma_n^{-4} r^{-2}\) and \(x^i = \gamma_n^2 z^i / \rho^2\). Then, it holds that \(\gamma_n^{-4} r^{-(n-4)} = \gamma_n^{-4} r^{n-4}\) and

\[
\frac{\partial}{\partial z^i} = \gamma_n^2 \rho^{-2} \left( \delta_{ij} - 2 \rho^{-2} z^i z^j \right) \frac{\partial}{\partial x^j},
\]

(36)

which, appealing to \((34)\), implies

\[
\hat{g}_{ij}(z) = \left( \frac{\gamma_n^{-4} \rho^{n-4} + \alpha + O_4(\rho^{-1})}{\rho^{n-4}} \right) \left( \delta_{ij} - 2 \rho^{-2} z^i z^j \right) g(\hat{z}_i, \hat{z}_j),
\]

\[
= \left( 1 + \frac{\gamma_n^6 \alpha}{\rho^{n-4}} + O_4(\rho^{3-n}) \right) \left( \delta_{ij} - 2 \rho^{-2} z^i z^j \right) \left( \delta_{ij} - 2 \rho^{-2} z^i z^j \right) g_{kl}(\rho^{-2} z).
\]

Invoking \((35)\), this implies

\[
\hat{g}_{ij}(z) = \left( 1 + \frac{\gamma_n^6 \alpha}{\rho^{n-4}} + O_4(\rho^{3-n}) \right) \left( \delta_{ij} - 2 \rho^{-2} z^i z^j \right) h_{ij}(z),
\]

where

\[
h_{ij}(z) = \left( \delta_{kl} \delta_{ij} - 2 \frac{z^i z^j}{\rho^2} \delta_{kl} - 2 \frac{z^j z^i}{\rho^2} \delta_{kl} + 4 \frac{z^i z^j z^k z^l}{\rho^4} \right) \left( \delta_{kl} + \frac{\gamma_n^6 \alpha}{3 \rho^{n-4}} R_{kmnl}(\rho) \frac{z_m z_n}{\rho^4} \right)
\]

\[
+ \frac{\gamma_n^6 \alpha}{6} R_{kmnl}(\rho) \frac{z_m z_n z_p}{\rho^4} + O_4 \left( \frac{1}{\rho^4} \right),
\]

\[
= \delta_{ij} + H_{ij}^2(z) + H_{ij}^3(z) + O_4 \left( \frac{1}{\rho^4} \right)
\]

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where
\[ \gamma_{n-2}^{-\frac{1}{3}} H_{ij}^2 = \frac{1}{3} R_{mnij}(p) z^m z^n \rho^4 - \frac{2}{3} R_{kmnj}(p) z^k z^m \frac{z^n z^i}{\rho^6} - \frac{2}{3} R_{imnj}(p) z^n z^l \frac{z^m z^j}{\rho^6} \]
\[ + \frac{4}{3} R_{kmnl}(p) z^k z^m \frac{z^n l^i z^j}{\rho^8}, \]
\[ = \frac{1}{3} R_{mnij}(p) z^m z^n \rho^4 \]
and
\[ \gamma_{n-6}^{-\frac{6}{3}} H_{ij}^3 = \frac{1}{6} R_{mnj,a}(p) z^m z^n z^a \rho^6 - \frac{1}{3} R_{kmnj,a}(p) z^k z^m \frac{z^n z^i z^a}{\rho^8} - \frac{1}{3} R_{imnl,a}(p) z^n z^l \frac{z^m z^j z^a}{\rho^8} \]
\[ + \frac{2}{3} R_{kmnl,a}(p) z^k z^m \frac{z^n l^i z^j z^a}{\rho^{10}}, \]
\[ = \frac{1}{6} R_{mnj,a}(p) z^m z^n z^a \rho^6. \]

Putting together all the above, we find that
\[ \hat{g}_{ij}(z) = \left( 1 + \frac{4}{n-4} \frac{\gamma_n \alpha}{\rho^{n-4}} \right) \delta_{ij} + \frac{\gamma_{n-2}}{3} R_{ijab}(p) \frac{z^a z^b}{\rho^4} + \frac{\gamma_{n-6}}{6} R_{ijabc}(p) \frac{z^a z^b z^c}{\rho^6} \]
\[ + O_4(\rho^{-(n-3)}) + O_4(\rho^{-4}), \]
which establishes the asymptotic flatness condition claimed in the proposition. Notice that if $5 \leq n \leq 7$, then, from Proposition [1], we know that $\mathcal{E}(\hat{g})$ is well-defined and the above expansion implies
\[ \partial_a \hat{g}_{ij} = -4 \gamma_n \alpha \frac{z^a}{\rho^{n-2}} \delta_{ij} + \partial_a H_{ij}^2 + \partial_a H_{ij}^3 + O_3 \left( \rho^{-(n-2)} \right), \]
\[ \partial_{ab} \hat{g}_{ij} = -4 \gamma_n \alpha \left( \frac{\delta_{ab}}{\rho^{n-2}} - (n-2) \frac{z^b z^a}{\rho^n} \right) \delta_{ij} + \partial_{ab} H_{ij}^2 + \partial_{ab} H_{ij}^3 + O_2 \left( \rho^{-(n-1)} \right). \]

Appealing to (C.3)-(C.4), it is not difficult to explicitly compute that $\partial_{aa} H_{ii}^3$ and $\partial_{aa} H_{ii}^3$ both vanish. This implies
\[ \partial_{aa} \hat{g}_{ii} = -8 n \gamma_n \alpha \rho^{-(n-2)} + O_2(\rho^{-(n-1)}). \]  
(37)

Similarly, appealing to (C.3)-(C.4), it also holds that $\partial_{ij} H_{ij}^2 = 0$ and $\partial_{ij} H_{ij}^3 = 0$, and therefore we find that
\[ \partial_{ij} \hat{g}_{ij} = -8 \gamma_n \alpha \rho^{-(n-2)} + O_2(\rho^{-(n-1)}). \]  
(38)

Thus, putting together (37)-(38), we see that
\[ (\partial_{ca} \hat{g}_{ii} \frac{z^c}{\rho} - \partial_{ci} \hat{g}_{ij}) \frac{z^i}{\rho} = 8 \gamma_n \alpha (n-1)(n-2) \rho^{-(n-1)} + O_1(\rho^{-n}), \]  
(39)
implying that
\[
\mathcal{E}(\hat{g}) = \lim_{\rho \to \infty} \int_{S_\rho} (\partial_{\rho\rho}\hat{g}_{ij}(z) - \partial_{z\overline{z}}\hat{g}_{ij}(z)) \frac{\omega}{\rho} d\omega \rho = \omega_{n-1} 8\gamma_n \alpha (n-1)(n-2).
\] (40)

Finally, in \(g\) is flat near \(p\), without restriction on the dimension, we know that the expansion (34) holds in rectangular coordinates around \(p\), where \(g_{ij}(x) = \delta_{ij}\). Thus, the same computations as above show that
\[
\hat{g}_{ij}(z) = \left(1 + \frac{4}{n-4} \frac{\gamma_n \alpha}{\rho^{n-4}}\right) \delta_{ij} + O_4(\rho^{-(n-3)}).
\]

In this case the order of decay is improved for \(n \geq 7\) and in particular we know that \(\mathcal{E}(\hat{g})\) is well-defined. From the above expression, the same computations that led us to (40) prove that \(\mathcal{E}(\hat{g}) = 8\omega_{n-1} (n-1)(n-2) \gamma_n \alpha\).

In order to apply the positive energy theorem to establish that \(\alpha \geq 0\), we first need to check that \(\hat{g}\) satisfies its hypotheses. With this in mind, consider the following proposition.

**Proposition 7.** Consider a closed Riemannian manifold \((M^n, g)\), with \(n \geq 5\), which admits a conformal metric with positive \(Q\)-curvature such that \(Y([\hat{g}]) \geq 0\). Then, there exists a conformal metric \(\hat{g}\) such that the asymptotically flat manifold \((\hat{M} = M\setminus \{p\}, \hat{g} = G_{\mathcal{P}_h}^{-\frac{n-4}{2(n-2)}} \hat{g})\) satisfies \(Y([\hat{g}]) > 0\) and \(Q_{\hat{g}} \equiv 0\).

**Proof.** Thanks to Remark 1, \(Y([\hat{g}]) > 0\) and \(G_{\mathcal{P}_h}\) exists and is positive for every \(\hat{g} \in [g]\). Then, thanks to proposition 6 we only have to prove that \(Y([\hat{g}]) > 0\). Since \(Y([\hat{g}]) > 0\), let \(G_{\mathcal{L}_{\hat{g}}}\) the Green function of the conformal Laplacian. We trivially get that \(\hat{g} = \frac{G_{\mathcal{L}_{\hat{g}}}}{G_{\mathcal{P}_h}}\) is scalar flat.

Equipped with the above two propositions, we can recover the following theorem, originally proved by Hang-Yang in [21], and which is an extension of results of Gursky-Malchiodi [18] and Humbert-Raulot [24].

**Theorem 5.** Let \((M, g)\) be a closed \(n\)-dimensional Riemannian manifold, with \(5 \leq n \leq 7\) or \(n \geq 8\) and locally conformally flat around some point \(p \in M\). If \(Y([\hat{g}]) \geq 0\) and \((M, g)\) admits a conformal metric with semi-positive \(Q\)-curvature, then the mass of \(G_{\mathcal{P}}\) at \(p\) is non-negative and vanishes if and only if \((M, g)\) is conformal to the standard sphere.

**Proof.** Under these conditions, we know that the Green function \(G_{\mathcal{P}}\) exists and is positive for every element in \([g]\). From Proposition 6 we also know that there is a conformal metric \(\hat{g}\) such that the manifold \((\hat{M} = M\setminus \{p\}, \hat{g} = G_{\mathcal{P}_h}^{-\frac{n-4}{2(n-2)}} \hat{g})\) is AE, satisfies the decay assumptions of Theorem 4 and, furthermore the energy is positively proportional to the mass of \(G_{\mathcal{P}_h}\).

Finally, from Proposition 7 we know that \(Y([\hat{g}]) > 0\) and \(Q_{\hat{g}} \equiv 0\). Therefore, \((\hat{M}, \hat{g})\) satisfies all the hypotheses of Theorem 4 and thus the non-negativity follows directly from \(\mathcal{E}(g)\) being positively proportional to \(\alpha\). Finally, if \(\alpha = 0\), we find that \(\hat{M}\) is conformally equivalent to \(\mathbb{R}^n\). Being \(\hat{M}\) the one point compactification of \(M\), we see that \(M \cong S^n\) and \(g\) is conformal to the round metric.
3.2 The 4-dimensional case

Finally, we would like to end by briefly discussing some peculiarities concerning the analysis of the four dimensional case. In particular, a large part of what we have done above for the $Q$-curvature positive mass theorem can be translated perfectly well to $n = 4$. Nevertheless, in view of Corollary 2 it should be clear that strong restrictions should appear in the treatment of this special case. Although the origins of some of these restriction may be evident to experts in $Q$-curvature analysis, the aim in what follows is to make explicit the subtle differences of this case and where the particularities arise. In this spirit, let us start by presenting the following lemma which combines results of [32]-[27].

Lemma 1. Let $(M^4, g)$ be a closed Riemannian manifold satisfying $\text{Ker}(P_g) = \mathbb{R}$ and $\kappa_g > 0$. Then, given $p \in M$, there exists a Green function $G_p \in C^\infty(M \backslash \{p\})$, unique up to an additive constant, which satisfies

$$P_g G_p + Q_g = \kappa_g \delta_p$$  \hspace{1cm} (41)

as distributions. Furthermore, near $p$, in $g$-normal coordinates $\{x^i\}_{i=1}^4$ the following expansion holds

$$G_p = \frac{k_p}{16\pi^2} \ln(r^2) + \frac{k_p}{16\pi^2} S_0 + \frac{k_p}{16\pi^2} a_i x^i + \frac{k_p}{16\pi^2} b_{ij} x^i x^j + o_4(r^2).$$  \hspace{1cm} (42)

for some constants $S_0, a_i, b_{ij}$.

Proof. From Lemma 2.1 in [32] we know that if $\text{Ker}(P_g) = \mathbb{R}$, then the Paneitz operator $P_g$ admits a Green function $G_P$, which is to say that for every $u \in C^4(M)$ it holds that

$$u(x) - \bar{u} = \int_M G_P(x, y) P_g(u(y)) dV_g(y),$$  \hspace{1cm} (43)

where $\bar{u} = \text{vol}_g(M)^{-1} \int_M u dV_g$. Also, since $\text{Ker}(P_g) = \mathbb{R}$, we know that there is a smooth function $U$ satisfying

$$P_g U = -(Q_g - \bar{Q}_g).$$

Define $G_p \doteq \kappa_g G_P + U$, so that

$$\langle P_g G_p, u \rangle = \kappa_g u(p) - \kappa_g \bar{u} + \langle P_g U, u \rangle,$$

$$= \kappa_g u_p - \kappa_g \bar{u} - \langle Q_g, u \rangle + \left( \frac{\kappa_g}{\text{vol}_g(M)} \right) u = \kappa u_p - \langle Q_g, u \rangle.$$  \hspace{1cm} (44)

Thus, we see that

$$\langle P_g G_p + Q_g, u \rangle = \kappa_g u(p) \quad \forall u \in C^\infty(M),$$

which is to say that $P_g G_p + Q_g = \kappa_g \delta_p$. The uniqueness claim follows since two different such functions $G_p$ and $\tilde{G}_p$ must satisfy $P_g(G_p - \tilde{G}_p) = 0$, implying $G_p = \tilde{G}_p + c$. If we rewrite (11) as

$$P_g \left( \frac{16\pi^2}{k_p} G_p \right) + \frac{16\pi^2}{k_p} Q_g = 16\pi^2 \delta_p$$

we can appeal quite straightforwardly to the computations of the Appendix in [27] to conclude that, in $g$-normal coordinates, the Green function has the following expansion near $p$:

$$\frac{16\pi^2}{k_p} G_p = -2 \ln(r) + S_0 + a_i x^i + b_{ij} x^i x^j + o_4(r^2).$$  \hspace{1cm} (45)

which proves (12).
Let us now consider the same setting as in the above lemma and consider the inverted coordinates \( z^i = \frac{2}{\rho^2} \) and define \( \tilde{M} = M \setminus \{ p \} \), \( \tilde{g} = e^{2G_r g} \). Then,

\[
Q_{\tilde{g}}(q) = e^{-4G_r(q)} (P_g G_p(q) + Q_g(q)) = 0, \text{ for any } q \in \tilde{M},
\]

and

\[
\hat{g}_{ij}(z) = e^{2\rho \left( \frac{m_g}{16\pi^2} + \frac{1}{2} \chi + \frac{\kappa_g}{16\pi^2} \right)} \left( 4\rho^2 \hat{\Delta} + O(\rho^{-3}) \right) g(\partial_x, \partial_y),
\]

\[
= \rho^4 \kappa_0 \left( \frac{m_g}{16\pi^2} + \frac{\kappa_g}{16\pi^2} \right) \left( 4\rho^2 \hat{\Delta} + O(\rho^{-3}) \right) g(\partial_x, \partial_y),
\]

\[
= \rho^4 \kappa_0 \left( \frac{1}{8\pi^2} \frac{S_0}{\rho^2} + O(\rho^{-2}) \right) \left( \delta_{ij} + O(\rho^{-2}) \right),
\]

(46)

where we have defined \( \Delta_\kappa_g = \frac{\kappa_0}{16\pi^2} - 1 \). Let us notice that the above inversion gives an AE metric \( \hat{g} \) iff \( \Delta_\kappa_g = 0 \), which is to say \( \kappa_g = 16\pi^2 \). In particular, in this case, via a coordinate change of the form \( \hat{z}^i = e^{S_0} z^i \), we find that

\[
\hat{g}_{ij}(\hat{z}) = \left( 1 + 2e^{S_0} \frac{\hat{z}^k}{\rho^2} + O(\hat{\rho}^{-2}) \right) \left( \delta_{ij} + O(\hat{\rho}^{-2}) \right),
\]

(47)

where \( \hat{\rho} = |\hat{z}| = e^{S_0} \rho \). That is, \( \rho^{-k} = e^{kS_0} \hat{\rho}^{-k} \). We see that \( (\hat{M}, \hat{g}) \) is an AE-manifold or order \( \tau = 1 \). Therefore, since by construction \( Q_{\hat{g}} \equiv 0 \), if \( Y(g) > 0 \), in view of Corollary 2 we must conclude that \( (M, g) \) is conformal to the round sphere. That is, the above computations together with Corollary 2 imply the following Corollary, which is not new, since it concerns the rigidity statement involved in Theorem B in [17].

**Corollary 3.** Let \( (M, g) \) be a closed 4-dimensional manifold which satisfies \( \text{Ker}(P_g) = \mathbb{R} \) and \( \kappa_g = 16\pi^2 \). If \( Y(g) > 0 \), then \( (M, g) \) is conformal to the round sphere.

Finally, let us highlight that the techniques developed so far allow us to actually get more than the previous corollary. In fact, we can recover the full statement of Theorem B in [17] by an independent and simple proof.

**Theorem 6 (Gursky).** Let \( (M^4, g) \) a 4-dimensional manifold with \( Y([g]) \geq 0 \), then \( \kappa_g \leq 16\pi^2 \) with equality holding iff \( (M^4, g) \) is conformal to the standard sphere.

**Proof.** Since we need only pay attention to the cases \( \kappa_g > 0 \), let us start by noticing that, due to Theorem A in [17], under our hypotheses \( \text{Ker}(P_g) = \mathbb{R} \). Furthermore, the case \( Y([g]) = 0 \) is also trivial, since in this case \( \kappa_g \leq 0 \), thus we will assume \( Y([g]) > 0 \). Since our hypotheses are conformally invariant, let us start assuming that \( g \) has been picked within \([g]\) so as to be the metric of a conformal normal coordinate system. We can now appeal to Lemma 1 and the expansion (46) to construct the manifold \( \tilde{M} = M \setminus \{ p \}, \tilde{g} \). Although in general this manifold will not be AE, it holds that \( Q_{\tilde{g}} \equiv 0 \). Also, since \( Y([g]) > 0 \), we now that the Green function \( G_L \) of the conformal Laplacian (with a pole at \( p \in M \)) exists, it is positive and therefore we can construct the metric \( \tilde{g} = G_L^2 g \) on \( \tilde{M} \). In inverted normal coordinates around \( p \), from [20], we know that

\[
\tilde{g}(z)(\partial_z, \partial_z) = \delta_{ij} + O(\rho^{-2}),
\]
which shows that \((\hat{M}, \hat{g})\) is AE of order \(\tau = 2\). Furthermore, \(\hat{g}\) and \(\tilde{g}\) are related via
\[
\hat{g} = e^{-\ln(G^2_L)} e^{2G_P} \tilde{g} = e^{2(G_P - \ln(G_L))} \tilde{g}
\]
and therefore, from the conformal covariance associate to the Paneitz operator, we see that
\[
0 = e^{4\Phi} Q_{\hat{g}} = P_{\hat{g}} \Phi + Q_{\hat{g}}, \quad (48)
\]
where we have defined \(\Phi = G_P - \ln(G_L)\). Following the proof theorem 2, we know that \(Q_{\tilde{g}} = -\frac{1}{2} |\text{Ric}_{\tilde{g}}|^2_{\tilde{g}}\), and therefore we find that
\[
\int_{D_\rho} \frac{1}{2} |\text{Ric}_{\tilde{g}}|^2_{\tilde{g}} dV_{\tilde{g}} = \int_{S_\rho} \tilde{g}(\nabla \Delta_{\tilde{g}} \Phi, \tilde{\nu}) d\tilde{\omega} + 2 \int_{S_\rho} \text{Ric}_{\tilde{g}}(\nabla \Phi, \tilde{\nu}) d\tilde{\omega}. \quad (49)
\]
Now, the main difference with respect to theorem 2 is that we cannot estimate the derivatives of \(\Phi\) in the same way, since it does not solve the same equation as in that proof. Nevertheless, in this case, we have an explicit expression for \(\Phi\), at least asymptotically. Thus, let us notice that for sufficiently large \(\rho\) the following holds
\[
\Phi = \ln(\rho^{2\alpha}) + S_0 + O_4(\rho^{-1}) - \ln(\rho^2 + A + O_4(\rho^{-1})),
\]
\[
= \ln(\rho^{2\alpha}) + S_0 - \ln(\rho^2) - \ln(1 + A \rho^{-2} + O_4(\rho^{-3})) + O_4(\rho^{-1}),
\]
\[
= \ln(\rho^{2(\alpha - 1)}) + S_0 - \ln(1 + A \rho^{-2} + O_4(\rho^{-3})) + O_4(\rho^{-1}),
\]
where above we have defined \(\alpha = \frac{\kappa_g}{16\pi^2}\) and both \(A\) and \(S_0\) are constants. From all this we can directly compute that
\[
\nabla_i \Phi = 2(\alpha - 1) \frac{\zeta_i}{\rho^2} + O_3(\rho^{-2}),
\]
\[
\Delta_{\tilde{g}} \Phi = \frac{4(\alpha - 1)}{\rho^2} + O_3(\rho^{-4}),
\]
\[
\nabla_i \Delta_{\tilde{g}} \Phi = -\frac{8(\alpha - 1)}{\rho^3} \frac{\zeta_i}{\rho} + O_2(\rho^{-5}).
\]
From the above expression, it follows that
\[
\text{Ric}_{\tilde{g}}(\nabla \Phi, \tilde{\nu}) = O(\rho^{-5}),
\]
\[
\tilde{g}(\nabla \Delta_{\tilde{g}} \Phi, \tilde{\nu}) = \frac{8(1 - \alpha)}{\rho^3} + O_2(\rho^{-5}),
\]
implying that
\[
\int_{D_\rho} \frac{1}{2} |\text{Ric}_{\tilde{g}}|^2_{\tilde{g}} dV_{\tilde{g}} = 8\omega_3(1 - \alpha) + o(1). \quad (50)
\]
Finally, passing to the limit, we find that
\[
\int_M |\text{Ric}_{\tilde{g}}|^2_{\tilde{g}} dV_{\tilde{g}} = 8\omega_3(1 - \alpha) \geq 0. \quad (51)
\]
This implies that \(\alpha \leq 1\) and the equality case has already been dealt with in the previous corollary, which establishes the theorem. \(\square\)
### 3.3 The 3-dimensional case

In this section, we briefly explain how to recover the following theorem due to Hang and Yang [20].

**Theorem 7** (Proposition 2.4 [20]). Assume the Yamabe invariant $Y([g]) > 0$, ker $P_g = 0$. If there is some $p \in M$ such that $G_{P_g}(p, \cdot) < 0$ on $M \setminus \{p\}$, then $G_{P_g}(p) < 0$ except when $(M, g)$ is conformally equivalent to the standard $S^3$.

In this case the proof runs along the same lines as in the previous theorem. In particular, its hypotheses and conclusions are conformally invariant. Thus, we can assume that $g$ is the metric of a conformal normal coordinate system. Then, let $\hat{M} = M \setminus \{p\}$ and define $\hat{g} = G_{P_g}^{-4}g$ and $\tilde{g} = G_{L_g}^4 g$ on this manifold. Let us recall the following expansions, valid in conformal normal coordinates around $p$ (see, [20] and [26])

$$G_{P} = A + O_4(r),$$

$$G_{L} = \frac{1}{r} + \alpha + O_4(r),$$

where $A$ and $\alpha$ are constants. After going to inverted coordinates $z = \frac{x}{\rho}$, with $\rho = r^{-1}$, we find that

$$\hat{g}_{ij}(z) = \left(1 + \frac{\alpha}{\rho}\right) \delta_{ij} + O(\rho^{-2}),$$

and, clearly, on $\tilde{g} = \Phi^{-4}\hat{g}$, with $\Phi \doteq G_{P}G_{L}$ and $Q_{\tilde{g}} \equiv 0$. Therefore, from the conformal covariance of the Paneitz operator, we find that $P_{\tilde{g}}\Phi \equiv 0$, which translates to

$$0 = \Delta_{\tilde{g}}^2 \Phi - 4\text{div}_{\tilde{g}}(\text{Ric}_{\tilde{g}}(\nabla \Phi, \cdot)) + |\text{Ric}_{\tilde{g}}|_{\tilde{g}}^2 \Phi.$$  \hspace{1cm} (52)

As in the previous theorem, we now can compute explicitly the terms in the right-hand side. That is,

$$\Phi = (A + O_4(\rho^{-1}))(\rho + \alpha + O_4(\rho^{-1})) = A\rho + O_4(\rho^0)$$

$$\nabla_{\tilde{g}}\Phi = \frac{2A}{\rho} + O_2(\rho^{-2}),$$

$$\Delta_{\tilde{g}}\Phi = -\frac{2A}{\rho^3} + O_1(\rho^{-3}),$$

which, together with $\text{Ric}_{\tilde{g}} = O_2(\rho^{-3})$ implies that

$$\int_{D_{\rho}} |\text{Ric}_{\tilde{g}}|_{\tilde{g}}^2 \Phi dV_{\tilde{g}} = 8\pi A + O(\rho^{-1}).$$

Passing to the limit and remembering that $\Phi < 0$, we find that $A \leq 0$ with equality holding iff $\text{Ric}_{\tilde{g}} \equiv 0$. That is, if $M \cong \mathbb{R}^3$ and $\tilde{g} = \delta$, which implies the final result.
A Appendix: Some analytic results concerning AE manifolds

In this appendix we will collect some facts results concerning AE manifolds which are used in the core of the paper. Most of these results are well-known for experts. We include them for the sake of completeness and to deliver a self-contained presentation. For detailed proofs and discussions on these topics, we refer the reader to Bartnik [3], Lee-Parker [26].

Let us start with the following fundamental theorem regarding the properties of the Laplacian on AE manifolds.

**Theorem A.1.** Let \((M, g)\) an asymptotically flat manifold with a structure at infinity \(\phi : M \setminus K \rightarrow \mathbb{R}^n \setminus B_1\) with decay rate \(\tau\). If \(\delta \not\in (\mathbb{Z} \setminus \{−1, \cdots, 3−n\})\) then
\[
\Delta_g : W^{2,q}_\delta(\phi) \rightarrow L^q_{4−2}
\]
is Fredholm. Moreover \(\Delta_g\) is
\[
\begin{cases}
\text{if } \delta > 2−n \text{ then } \Delta_g \text{ is surjective}, \\
\text{if } 2−n < \delta < 0 \text{ then } \Delta_g \text{ is bijective}, \\
\text{if } \delta < 0 \text{ then } \Delta_g \text{ is injective}.
\end{cases}
\quad (A.1)
\]

**Remark A.1.** The decay rates in the set \(\mathbb{Z} \setminus \{−1, \cdots, 3−n\}\) are called exceptional and we say that \(\delta\) is non-exceptional if \(\delta \not\in \mathbb{Z} \setminus \{−1, \cdots, 3−n\}\).

Let us now analyse the relation between different potential structures of infinity. In particular, the following theorem concerns the existence of harmonic coordinates.

**Theorem A.2.** Let \((M, g)\) be an AE manifold, with \(g \in W^{k,q}_{loc}\), \(k > 1, q > n\), and \((\Phi, x) : M \setminus K \rightarrow E_R\) where \(K \subset M, R \geq 1\) is a structure at infinity of order \(\tau > 0\) with \(1−\tau\) non-exceptional, and fix \(1 < \eta < 2\). There are functions \(y^i \in W^{k,q}_{\eta}\), \(i = 1, \cdots, n\), such that \(\Delta_g y^i = 0\) and \((x^i − y^i) \in W^{k,q}_{1−\tau}(E_R)\), which implies
\[
|y^i| = o_k(r^{1−\tau}), \\
|g(\partial x^i, \partial y^i) − g(\partial y^i, \partial y^i)| = o_k(r^{−\tau}).
\quad (A.2)
\]
Furthermore, the set of functions \(\{1, y^1\}\) is a basis for \(H_1 = \{u \in W^{k,q}_\eta : \Delta_g u = 0\}\).

**Proof.** Let us first extend the functions \(x^i\) smoothly to all of \(M\). Then, near infinity, \(\Delta_g x^i = g^{ij} \Gamma^j_{kl} \Gamma^l x^i \in W^{k−1,q}_{1−\tau}\). Let us denote by \(T^g_{x^i} \doteq \Delta_g : W^{k,q}_{1−\tau} \rightarrow W^{k−2,q}_{1−\tau}\), and recall that if \(1−\tau\) is non-exceptional, then \(\text{Im}(T^g_{x^i})\) is closed. Thus, in particular \(\text{Im}(T^g_{x^i}) = \text{Ker}((T^g_{x^i})^\perp)\) and clearly we must have \(\Gamma^i \in \text{Ker}((T^g_{x^i})^\perp)\). Therefore, there is some \(v^i \in W^{k+1,q}_1\) satisfying \(\Delta_g v^i = \Delta_g x^i\). That is, \(\Delta_g (x^i − v^i) = 0\). Let us then define \(y^i = x^i − v^i\), which implies the first estimate in (A.2), and furthermore, since \(\frac{\partial y^i}{\partial x^j} = \delta^i_j + o(r^{−\tau})\), we see that near infinity \(\{y^i\}\) are coordinates which are asymptotically Cartesian. This, in turn, implies the second estimate in (A.2). The final claim concerning \(H_1\) follows since \(\{1, y^i\}\) is an \((n + 1)\)-dimensional subspace of \(H_1\), but from Proposition 2.2 in [3] \(\dim(H_1) = n + 1\).

The following theorem, which can be found in [3] as Corollary 3.2, presents the relation between two different structures of infinity.
Theorem A.3. Let \((M,g)\) an asymptotically flat manifold, \(g \in W_{\text{loc}}^{k,q}\), \(k \geq 1\) and \(q > n\), with two structures at infinity \(\phi, \psi : M \setminus K \to \mathbb{R}^n \setminus \overline{B}(0,1)\) with decay rates \(\tau_\phi\) and \(\tau_\psi\), where of each of these weights satisfies that \(1 - \tau\) is non-exceptional. There exists \((O,a) \in O(n) \times \mathbb{R}^n\) such that

\[
x^i - (O^i_j z^j + a^i) \in W^{k,q}_{1-\tau}(\mathbb{R}^n)
\]

which implies

\[
|x^i - (O^i_j z^j + a^i)| = o_k(r^{1-\tau}), \quad (A.3)
\]

where \(\tau = \min\{\tau_\phi, \tau_\psi\}\), \(x = \phi^{-1}\) and \(z = \psi^{-1}\).

Proof. Let \(y^i\) and \(w^i\) be the harmonic coordinates constructed in the previous theorem associated to \(\phi\) and \(\psi\) respectively. Then, since \(H_1\) is intrinsic to \(M\) and \(\{1, y^i\}\) and \(\{1, w^i\}\) are bases for this space, we get that

\[
w^i = A^i_j y^j + a^i,
\]

where \(A \in \text{GL}(n, \mathbb{R})\) a priori, but actually, from the construction of the previous theorem, we know that \(z^i = A^i_j x^j + a^i + A^i_j y^j - \bar{v}^i\), with \(v^i \in W^{k,q}_{1-\tau_\phi}\) and \(\bar{v}^i \in W^{k,q}_{1-\tau_\psi}\). Since these last two systems are Cartesian, then \(A \in O(n)\) and we explicitly see that \(z^i = A^i_j x^j - a^i \in W^{k,q}_{1-\tau}\).

The above estimate is the best possible and it is determined by the Ricci curvature as shown by the next theorem which corresponds to Proposition 3.3 in [3], see also [12].

Theorem A.4. Let \((M,g)\) an asymptotically flat manifold with a structure at infinity \(\phi : M \setminus K \to \mathbb{R}^n \setminus B(0,1)\) with decay rate \(\tau\) such that \((\phi,g - \delta) \in W^{2,q}_{-\eta}(\mathbb{R}^n \setminus B(0,1))\) for \(q > n\) and such that

\[
\text{Ric}_g \in L^{q}_{2-\eta}(M) \text{ for some } \eta > \tau \text{ and } \eta \notin (\mathbb{Z} \setminus \{-1, \cdots, 3-n\}).
\]

Then there exists a structure at infinity \(\Theta : M \setminus K' \to \mathbb{R}^n \setminus B_1\) such that \((\Theta_* g - \delta) \in W^{2,q}_{-\eta}(\mathbb{R}^n \setminus B_1)\).

Finally, the following statement concerning the conformal Laplacian will be important in our analysis. It is a variant of theorem 9.2 of [26].

Corollary A.1. Let \((M,g)\) an asymptotically flat manifold of order \(\tau > 0\) and assume \(R_g \geq 0\). If \(2 - n < \delta < 0\) then

\[
L_g = \Delta_g - c_n R_g : W^{2,q}_{\delta} \to L^{q}_{\delta-2}
\]

is an isomorphism.

Proof. Since \(L_g\) is a compact perturbation of the Laplacian, it is also Fredholm, hence thanks to its self-adjointness it suffices to proof injectivity to get the result. Let \(u \in W^{2,q}_{\delta}\) such that \(\Delta_g u = c_n R_g u\) since \(R_g u \in W^{2,q}_{\delta}\) for any \(\delta' \geq \delta - \tau - 2\), hence \(u \in W^{2,q}_{\delta'}\), since we can repeat the argument as much as necessary, we can assume that \(u \in W^{2,q}_{\delta'}\) with \(\delta' < n - 2\). Then we can multiply \(L_g(u) = 0\) by \(u\) and integrating by part, which gives

\[
\int_{M \setminus (\mathbb{R}^n \setminus B(0,R))} |\nabla u|^2 - c_n R_g u^2 \, dv = \int_{\partial B(0,R)} u \partial_u u \, d\sigma,
\]

Since \(u\) decrease enough, we can pass to the limit as \(R \to +\infty\), which gives that \(u \equiv 0\) since \(R_g \geq 0\) and \(u \to 0\) as \(|x| \to \infty\), which concludes the proof.

\(\square\)
B Appendix: Conventions on Q-curvature

Let us adopt the following a the general definition for the Q-curvature an arbitrary Riemannian manifold \((M^n, g)\) with \(n \geq 3\):

\[
Q_g = -\Delta_g \sigma_1(S_g) + 4\sigma_2(S_g) + \frac{n-4}{2} (\sigma_1(S_g))^2, \tag{B.1}
\]

where \(S_g \doteq \frac{1}{n-2} \left( \text{Ric}_g - \frac{1}{2(n-1)} R_g g \right)\) stands for the Schouten tensor and \(\sigma_k(S_g)\) stands for the \(k\)-th elementary symmetric function of the eigenvalues of \(S_g\). In this context, the Paneitz operator is defined by

\[
P_g u \doteq \Delta^2 u + \text{div}_g \left( \left( 4S_g - (n-2)\sigma_1(S_g)g \right) (\nabla u, \cdot) \right) + \frac{n-4}{2} Q_g u, \tag{B.2}
\]

for all \(u \in C^\infty(M)\). In this context, the following relations hold:

\[
\sigma_1(S_g) = \frac{R_g}{2(n-1)}, \tag{B.3}
\]

\[
\sigma_2(S_g) = \frac{1}{2} \left( \frac{n^2 - n}{4(n-1)^2(n-2)^2} R_g^2 - \frac{|\text{Ric}_g|^2}{(n-2)^2} \right),
\]

which implies that

\[
Q_g = -\frac{1}{2(n-1)} \Delta_g R_g - \frac{2}{(n-2)^2} |\text{Ric}_g|^2 + \frac{n^3 - 4n^2 + 16n - 16}{8(n-1)^2(n-2)^2} R_g^2,
\tag{B.4}
\]

\[
P_g u = \Delta^2 u + \text{div}_g \left( \left( \frac{4}{n-2} \text{Ric}_g - \frac{n^2 - 4n + 8}{2(n-1)(n-2)} R_g g \right) (\nabla u, \cdot) \right) + \frac{n-4}{2} Q_g u.
\]

In particular, for \(n \neq 4\), if \(\bar{g} = u^{\frac{4}{n}} g\), then

\[
Q_{\bar{g}} = \frac{2}{n-4} u^{\frac{n+4}{n-4}} P_g u. \tag{B.5}
\]

In the case of \(n = 4\) we can apply the above definitions to get

\[
Q_g = -\frac{1}{6} \Delta_g R_g - \frac{1}{2} |\text{Ric}_g|^2 + \frac{1}{6} R_g^2,
\tag{B.6}
\]

\[
P_g u = \Delta^2 u + \text{div}_g \left( \left( \frac{2}{3} \text{Ric}_g - \frac{2}{3} R_g g \right) (\nabla u, \cdot) \right),
\]

and in this case, if \(\bar{g} = e^{2u} g\), we have that \([28]\)

\[
Q_{\bar{g}} = e^{-4u} \left( P_g u + Q_g \right). \tag{B.7}
\]

Now, let us notice that it is also quite standard to redefine the Q-curvature in dimension 4 via (see \([13, 27]\))

\[
Q^{(4)}_g = \frac{1}{2} Q^G = -\frac{1}{12} \Delta_g R_g - \frac{1}{4} |\text{Ric}_g|^2 + \frac{1}{12} R_g^2 \quad \tag{B.8}
\]

and in this case, it follows that

\[
2Q^{(4)}_g = e^{-4u} \left( P_g u + Q^{(4)}_g \right). \tag{B.9}
\]

We will not adopt this redefinitions and keep a unified notation via \((B.1)\) along this paper.
C Appendix: Conformal normal coordinates

In order to deliver a presentation as self-contained as possible, this section is meant to summarize some of the results concerning conformal normal coordinates presented in [26], which are used in the core of this paper. The basic construction is given on a smooth Riemannian manifold \((M^n, g)\) where we intend to expand \(g\) around a fixed point \(p \in M\). In particular, we are interested in finding an element \(\tilde{g}\) within the conformal class \([g]\) for which, in \(\tilde{g}\)-normal coordinates \(\{x^i\}\), the following expansion holds around \(p\):

\[
\det(\tilde{g}) = 1 + O(r^N) \tag{C.1}
\]

for any chosen \(N \geq 2\), where \(r = |x|\) (see Theorem 5.1 in [26]). This type of expansion is achieved by first noticing that, for any Riemannian metric \(g\), in \(g\)-normal coordinates \(\{x^i\}\) around \(p \in M\), the following holds

\[
g_{ij}(x) = \delta_{ij} + \frac{1}{3}R_{iklj}x^k x^l + \frac{1}{6}R_{iklj,a}x^k x^l x^a + \left(\frac{1}{20}R_{iklj,ab} + \frac{2}{45}R_{iklc}R_{jab} - \frac{1}{18}R_{ij}R_{kl}\right)x^k x^l x^a x^b + O(r^5),
\]

where all the coefficients at evaluated at \(p\). From this expression it is possible to compute \(\det(g)\) around \(p\) for any such metric, so as to get

\[
det(g)(x) = 1 - \frac{1}{3}R_{ij}x^i x^j - \frac{1}{6}R_{ij,k}x^i x^j x^k \tag{C.2}
\]

where again all the coefficients are evaluated at \(p\). Assuming an expansion of the form \(\det(g) = 1 + O(r^N)\) with \(N \geq 2\) (the case \(N = 2\) is valid for any metric in its own normal coordinates) and appealing to Theorem 5.2 in [26], the authors can find a homogeneous polynomial \(f \in \mathcal{P}_N\) so that the expansion of \(\det(\tilde{g}) = 1 + O(r^{N+1})\) for \(\tilde{g} = e^{2f}g\), which establishes an inductive proof (see the proof of Theorem 5.1). But notice that this implies that the symmetrization of the coefficients in [C.2] of order up to \(N\) must vanish for \(\tilde{g}\).

Thus, in the case we do this construction for \(N \geq 4\), we see that this implies

\[
\tilde{R}_{ij}(p) = 0, \quad \tilde{R}_{ij,k} + \tilde{R}_{ki,j} + \tilde{R}_{jk,i}(p) = 0. \tag{C.3}
\]

Putting together the second condition above with the contracted Bianchi identities, we also see that

\[
\tilde{R}_{k}(p) = 0. \tag{C.4}
\]

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