Fast Rate Generalization Error Bounds: Variations on a Theme

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Abstract—A recent line of works, initiated by [1] and [2], has shown that the generalization error of a learning algorithm can be upper bounded by information measures. In most of the relevant works, the convergence rate of the expected generalization error is in the form of $O(\sqrt{W/n})$ where $\lambda$ is an assumption-dependent coefficient and $I$ is some information-theoretic quantities such as the mutual information between the data sample and the learned hypothesis. However, such a learning rate is typically considered to be “slow”, compared to a “fast rate” of $O(1/n)$ in many learning scenarios. In this work, we first show that the square root does not necessarily imply a slow rate, and a fast rate result can still be obtained using this bound by evaluating $\lambda$ under an appropriate assumption. Furthermore, we identify the key conditions needed for the fast rate generalization error, which we call the $(\eta, c)$-central condition. Under this condition, we give information-theoretic bounds on the generalization error and excess risk, with a convergence rate of $O(1/n)$ for specific learning algorithms such as empirical risk minimization. Finally, analytical examples are given to show the effectiveness of the bounds.

I. INTRODUCTION

The generalization error of a learning algorithm lies in the core analysis of the statistical learning theory, and the estimation of which becomes remarkably crucial. Conventionally, many bounding techniques are proposed under different conditions and assumptions such as VC-dimension [3], algorithmic stability [4], PAC-Bayes [5] and robustness [6]. However, most bounds mentioned above are only concerned with the hypothesis or the algorithm solely. To fully characterize the intrinsic nature of a learning problem, it is shown in some recent works that the generalization error can be upper bounded using the information-theoretic quantities [1], [2] and the bound usually takes the following form:

$$
\mathbb{E}_{W, S_n}[\mathcal{E}(W, S_n)] \leq \sqrt{\frac{A(W; S_n)}{n}},
$$

(1)

where the expectation is taken w.r.t. the joint distribution of $W$ and $S_n$ induced by some algorithm $A$. Here, $\mathcal{E}(w, S_n)$ denotes the generalization error (properly defined in (4) in Section II) for a given hypothesis $w$ and data sample $S_n = (Z_i)_{i=1, \ldots, n}$, and $I(W; S_n)$ denotes the mutual information between the hypothesis and data sample, and $\lambda$ is some positive constant. In particular if the loss function is $\sigma$-sub-Gaussian\textsuperscript{1} under the distribution $P_W \otimes P_{S_n}$, $\lambda$ is equal to $2\sigma^2$. By introducing the mutual information, such a bound gives a data-algorithm dependent bound that can recover the previous results in terms of VC dimension [2], algorithmic stability [7], differential privacy [8] under mild conditions. Further, as pointed out by [9], the information-theoretic upper bound could be substantially tighter than the traditional bounds if we could exploit specific properties of the learning algorithm.

However, there are mainly two issues recognized from this bound. The first problem is that, with bounded mutual information, the convergence rate is usually $O(\sqrt{1/n})$, which is sub-optimal in some learning scenarios. The second issue is that the mutual information term can be arbitrarily large for some deterministic algorithms [10]. The latter can be addressed by introducing ghost samples [8] or using random subset methods [10]–[12]. Only a few works are dedicated to the former problem. In this work, we develop a general framework for the fast rate bounds using the mutual information following this line of works [13]–[15] and the contributions are listed as follows.

- We argue that the square root sign in (1) does not necessarily imply a slow rate and this bound can still achieve the fast rate. Specifically, under a proper assumption, the fast rate (e.g., $O(1/n)$) is attainable if $\lambda$ has the same order as the excess risk w.r.t. the sample size. In addition to removing the square root, we derive a novel form for the generalization error based on this variation.
- Inspired by the analysis under the sub-Gaussian case, we identify the key assumptions needed for a more general fast rate learning framework, which we call $(\eta, c)$-central condition. Compared with typical mutual information bounds, the convergence rate of the novel bound improves from $O(\sqrt{1/n})$ to $O(1/n)$ under some widely used algorithms such as empirical risk minimization (ERM) and regularized ERM. We could further extend our results to intermediate rates under the relaxed $(\eta, c)$-central conditions.
- The fast rate results are confirmed with a few simple examples both numerically and analytically, showing the effectiveness of the proposed bounds.

II. PROBLEM FORMULATION

We consider the following machine learning framework starting with a set of $n$ examples that $S_n = \{z_1, z_2, \ldots, z_n\}$, where each instance $z_i$ is i.i.d. drawn from some distribution $\mu$.

\textsuperscript{1}A random variable $X$ is $\sigma$-sub-Gaussian if $\log \mathbb{E} e^{\eta(X - \mathbb{E}[X])} \leq \frac{\sigma^2 \eta^2}{2}$, $\forall \eta \in \mathbb{R}$.
One may wish to learn a hypothesis \( w \) that exploits the properties of \( S_n \), with the aim of making predictions for new data correctly and efficiently. The hypothesis \( w \) is chosen from the set \( W \) with the possibly randomised algorithm \( A : Z^n \rightarrow W \) and we define the corresponding loss function \( \ell : W \times Z \rightarrow \mathbb{R} \). Particularly if we consider the supervised learning problem in the following context, we can write \( Z = X \times Y \) and \( z_i = (x_i, y_i) \) as a feature-label pair. Then the hypothesis \( w : X \rightarrow Y \) can be regarded as a predictor for the input sample. We will call \((\mu, \ell, W, A)\) a learning tuple. In a typical statistical learning problem, one may wish to minimize the expected loss function \( L_{\mu}(w) = E_{Z \sim \mu}[\ell(w, Z)] \). However, as the underlying distribution \( \mu \) is usually unknown in practice, one may wish to learn \( w \) by minimizing the empirical risk induced by the dataset \( S_n \), denoted as \( w_{\text{ERM}} \), such that

\[
w_{\text{ERM}} = \arg\min_{w \in W} \frac{1}{n} \sum_{i=1}^{n} \ell(w, z_i),
\]

which will be employed as a predictor for the new data. Here we define \( L(w, S_n) = \frac{1}{n} \sum_{i=1}^{n} \ell(w, z_i) \). To assess how this predictor performs on unseen samples, the generalization error can be upper bounded as

\[
E_{w} \left[ \ell(w, Z) \right] - E_{w} \left[ \ell(w^{*}, Z) \right] \leq \frac{1}{n} \sum_{i=1}^{n} \ell(w, z_i),
\]

Another important metric, the excess risk, is defined as

\[
R(w) := E_{Z \sim \mu}[\ell(w, Z)] - E_{Z \sim \mu}[\ell(w^{*}, Z)].
\]

The excess risk evaluates how well a hypothesis \( w \) performs with respect to \( w^{*} \) given the data distribution \( \mu \). We also define the corresponding empirical excess risk as

\[
\hat{R}(w, S_n) := \frac{1}{n} \sum_{i=1}^{n} r(w, z_i),
\]

where \( r(w, z) = \ell(w, z) - \ell(w^{*}, z) \). In the sequel, we are particularly interested in bounding the expected generalization error \( E_{W, S_n} \left[ \ell(W, S_n) \right] \) and the excess risk \( E_{W} [R(W)] \) for any \( W \) induced by the algorithm \( A \).

### III. Main Results

The recent advances show that under the sub-Gaussian assumption, the generalization error can be upper bounded using the information-theoretic quantities such as mutual information [2], [10], [11] or conditional mutual information [8], where the bound usually takes the following form.

**Theorem 1** ( [2], [10]). Suppose that \( \ell(W, Z) \) is \( \sigma \)-sub-Gaussian under the distribution \( P_{\mu} \otimes \mu \) where \( P_{\mu} \) is the marginal induced by the algorithm \( A \) and data distribution \( \mu \), then

\[
E_{W, S_n} \left[ \ell(W, S_n) \right] \leq \frac{1}{n} \sum_{i=1}^{n} \sqrt{2\sigma^2 I(W; Z_i)} \leq \sqrt{\frac{2\sigma^2 I(W; S_n)}{n}},
\]

**Remark 1.** Throughout this paper, we focus on the case when \( \sum_{i=1}^{n} I(W; Z_i) \leq I(W; S_n) \leq \epsilon \) for some \( \epsilon > 0 \) following the virtue of mutual information stability [2]. Specifically, we focus on the case when \( I(W; Z_i) \sim O(1/n) \) in the sequel so the bound in (7) gives a convergence rate of \( O(\sqrt{1/n}) \) for a constant \( \sigma^2 \). This assumption holds for many learning settings such as ERM in regression problem [7], the Gibbs algorithm with mild assumptions [7], [16] and any algorithms under the general VC hypothesis classes (up to \( \log n \)) [2], [15].

From the above result, it is usually recognized that the square root sign prevents us from the fast rate, even in the following simple Gaussian mean estimation problem considered in [10].

**Example 1.** Let \( \ell(w, z_i) = (w - z_i)^2 \), each sample is drawn from some Gaussian distribution, \( Z_i \sim \mathcal{N}(\mu, \sigma^2) \). We consider the ERM algorithm that gives,

\[
W_{\text{ERM}} = \frac{1}{n} \sum_{i=1}^{n} Z_i \sim \mathcal{N}(\mu, \frac{\sigma^2 N}{n}).
\]

The true generalization error can be calculated to be

\[
E_{W, S_n} \left[ \ell(W_{\text{ERM}}, S_n) \right] = \frac{2\sigma^2 N}{n},
\]

To evaluate the upper bound in Theorem 1 for this example, we notice that for any \( i, \ell(W, Z_i) \sim \frac{n+1}{n} \sigma^2_{X \chi^2} \) where \( \chi^2 \) denotes the chi-squared distribution with 1 degree of freedom. Hence, the cumulant generating function can be calculated as,

\[
\log E_{P_{\mu} \otimes \mu} \left[ e^{\ell(w, Z_i) - \mathbb{E}[\ell(W, Z_i)]} \right] = -\sigma^2_{W} \eta - \frac{1}{2} \log (1 - 2\sigma^2_{W} \eta),
\]

where \( \eta \leq \frac{1}{2\sigma^2_{W}} \) and \( \sigma^2_{W} = \frac{n+1}{n} \frac{\sigma^2}{n} \) to simplify the notation. In this case, it can be proved that,

\[
-\sigma^2_{W} \eta - \frac{1}{2} \log (1 - 2\sigma^2_{W} \eta) \leq \sigma^4_{W} \eta^2.
\]

Thus the loss is \( \sqrt{2\sigma^2_{W} \text{sub-Gaussian under } P_{W} \otimes \mu} \). We can also calculate the mutual information as

\[
I(W; Z_i) = \frac{1}{2} \log \frac{n}{n-1}.
\]

Then the bound becomes

\[
E_{W, S_n} \left[ \ell(W, S_n) \right] \leq \frac{\sigma^2 N}{n} \sum_{i=1}^{n} \sqrt{\frac{(n+1)^2}{n^2} \log \frac{n}{n-1}},
\]

which will be of the order \( O\left(\frac{1}{\sqrt{n}}\right) \) as \( n \) goes to infinity.
the learning algorithm must be “good” enough compared to the optimal hypothesis \( w^* \). Here we encode the notion of goodness in terms of the cumulant generating function by controlling the gap between \( \ell(w, Z) \) and \( \ell(w^*, Z) \). To facilitate such an idea, we make the sub-Gaussian assumption w.r.t. the excess risk and bound the generalization error as follows.

**Theorem 2.** Suppose that \( r(W, Z) \) is \( \sigma \)-sub-Gaussian under distribution \( P_W \otimes \mu \), then

\[
E_{W,S_n} [\mathcal{E}(W, S_n)] \leq \frac{1}{n} \sum_{i=1}^{n} \sqrt{2\sigma^2 I(W; Z_i)}. \tag{9}
\]

Furthermore, the excess risk can be bounded by,

\[
E_W [\mathcal{R}(W)] \leq E_{W,S_n} [\hat{\mathcal{R}}(W, S_n)] + \frac{1}{n} \sum_{i=1}^{n} \sqrt{2\sigma^2 I(W; Z_i)}. \tag{10}
\]

All proofs and calculation details in this paper can be found in [17]. Now we evaluate the bound in Theorem 2 for the Gaussian example. Notice that (9) is identical to (7), and the only difference between Theorem 1 and Theorem 2 is the assumption.

**Example 2** (Continuing from Example 1). Consider the settings in Example 1. First we note that the expected risk minimizer \( w^* \) is calculated as \( \mu \). Then we have,

\[
r(w, z_i) = (w - z_i)^2 - (\mu - z_i)^2.
\]

The expected excess risk can be calculated as,

\[
E_W[\mathcal{R}(W)] = \frac{\sigma^2}{n}.
\]

Then we can bound the cumulant generating function as,

\[
\log E_{P_W \otimes \mu} \left[ e^{\eta (r(W,Z) - E[r(W,Z)])} \right] \leq \frac{2\eta^2 \sigma^2}{n},
\]

for any \( \eta \in \mathbb{R} \). Hence \( r(W, Z) \) is \( \sqrt{\frac{4\eta^2}{n}} \)-sub-Gaussian under the distribution \( P_W \otimes \mu \). Then the bounds becomes,

\[
E_{W,S_n} [\mathcal{E}(W, S_n)] \leq \frac{\sigma^2}{n} \sum_{i=1}^{n} \left[ \frac{1}{n} \log \frac{n}{n-1} \right],
\]

which is \( O(1/n) \), yielding a fast rate characterization.

Unlike typical information-theoretic results where the bounds are based on the assumption that the loss function is \( \sigma \)-sub-Gaussian, we assume that the excess risk is \( \sigma \)-sub-Gaussian. Even though the bound in (7) has exactly the same form as in (9), the key difference is that under our assumption, \( \sigma \) can depend on the sample size and will converge to 0 as the sample size increases, while this is not the case under the previous assumption as we see in Example 1. Moreover, the excess risk can be straightforwardly upper bounded as in (10).

To make above “fast rate” result more explicit, we provide an alternative bound based on the same sub-Gaussian assumption. The key feature of the following bound is that it does not contain the square root.

**Theorem 3** (Fast Rate with Sub-Gaussian Condition). Assume that \( r(W, Z) \) is \( \sigma \)-sub-Gaussian under the distribution \( P_W \otimes \mu \). Then it holds that

\[
E_{W,S_n} [\mathcal{E}(W, S_n)] \leq \frac{1 - a_n}{a_n} E_{W,S_n} [\hat{\mathcal{R}}(W, S_n)] + \frac{1}{n\eta a_n} \sum_{i=1}^{n} I(W; Z_i). \tag{11}
\]

for any \( 0 < \eta < \frac{2E_{P_W E_{\mu}}[r(W, Z)_i]}{\sigma^2} \) and \( a_n = 1 - \frac{2E_{P_W E_{\mu}}[r(W, Z)_i]}{\sigma^2} \). Furthermore, the expected excess risk is bounded by,

\[
E_W[\mathcal{R}(W)] \leq \frac{1}{a_n} E_{W,S_n} [\hat{\mathcal{R}}(W, S_n)] + \frac{1}{n\eta a_n} \sum_{i=1}^{n} I(W; Z_i).
\]

**Remark 2.** The bound in Theorem 3 appears to provide a "fast rate" result if \( I(W; Z) \) scales as \( O(1/n) \), assumed throughout the paper. However, notice that both \( \eta \) and \( \alpha_n \) depend on the expected excess risk \( E_{P_W \otimes \mu}[r(W, Z)] \) and \( \sigma^2 \), which potentially depend on \( n \) as well. Hence a more careful examination is needed. Specifically, it can be seen that if the ratio of the two quantities remains a constant independent of \( n \), the fast rate result will then hold.

**Example 3.** Since the expected excess risk can be calculated as \( E_W[\mathcal{R}(W)] = \frac{\sigma^2}{n} \), and \( r(W, Z) \) is \( \sqrt{\frac{4\eta^2}{n}} \)-sub-Gaussian, then we require that \( 0 < \eta < \frac{1}{2\sigma^2} \), which is independent of the sample size. For simplicity, we can consider the case \( \eta = \frac{1}{4\sigma^2} \) as an example, then \( a_n \) is calculated to be \( \frac{1}{2} \). Then we have the generalization error bound.

\[
\frac{1 - a_n}{a_n} E_{W,S_n} [\hat{\mathcal{R}}(W_{ERM}, S_n)] + \frac{1}{n\eta a_n} \sum_{i=1}^{n} I(W; Z_i) \leq \frac{3\sigma^2}{n},
\]

where the empirical excess risk \( E_{W,S_n} [\hat{\mathcal{R}}(W_{ERM}, S_n)] \) is calculated as \( \frac{\sigma^2}{n} \) and the bound has the rate of \( O(1/n) \).

**A. Fast Rate Bound**

As discussed above, although the bound in Theorem 3 takes the form of a "fast rate", it is still not very satisfying because it contains quantities \( (\eta, \alpha_n) \) that could scale with \( n \), making it hard to determine the actual convergence rate, the same as in the original bound in Theorem 2. To this end, we propose a different "fast rate" bound to alleviate this drawback. In particular, this bound does not contain extra quantities that depend on \( n \). The key to this bound is the so-called the expected \((\eta, c)\)-central condition (or we simply say \((\eta, c)\)-central condition for short), inspired by the works [13]–[15], [18], which is the key condition leading to the fast rate.

**Definition 1** (Expected \((\eta, c)\)-Central Condition). Let \( \eta > 0 \) and \( 0 < c \leq 1 \) be two constants. We say that \((\mu, \ell, W, A)\)
satisfies the expected \((\eta, c)\)-central condition if the following inequality holds for the optimal hypothesis \(w^*\):
\[
\log \mathbb{E}_{P_W \otimes \mu} \left[ e^{-\eta(\ell(W,Z) - \ell(w^*,Z))} \right] \leq c \mathbb{E}_{P_W \otimes \mu} [\ell(W,Z) - \ell(w^*,Z)].
\] (12)

Compared to the conventional \(\eta\)-central condition [13, Def. 3.1] by setting \(c = 0\) in (12) as
\[
\log \mathbb{E}_{P_W \otimes \mu} \left[ e^{-\eta(\ell(w,Z) - \ell(w^*,Z))} \right] \leq 0,
\] (13)

the RHS of (12) is negative and has a tighter control than (13) of the tail behaviour for some \(\eta'\). We firstly show that the Bernstein condition [19]–[22] implies the (\(\eta, c\))-central condition for certain \(\eta\) and \(c\) in the following corollary.

Corollary 1. Let \(\beta \in [0,1]\) and \(B \geq 1\). For a learning tuple \((\mu, \ell, W, A)\), we say that the Bernstein condition holds if the following inequality holds for the optimal hypothesis \(w^*\):
\[
\mathbb{E}_{P_W \otimes \mu} \left[ \left( \ell(W,Z) - \ell(w^*,Z) \right)^2 \right] \leq B \left( \mathbb{E}_{P_W \otimes \mu} [\ell(W,Z) - \ell(w^*,Z)] \right)^\beta.
\]

Then, if \(\beta = 1\) and \(\mathbb{E}_{P_W \otimes \mu} \left[ \left( \ell(W,Z) - \ell(w^*,Z) \right)^2 \right] \leq B \left( \mathbb{E}_{P_W \otimes \mu} [\ell(W,Z) - \ell(w^*,Z)] \right)\) holds if the learning tuple also satisfies
\[
\min \left\{ \frac{1}{2}, \frac{1}{2B(c-w)}, \frac{1}{2} \right\}\)-central condition.

The Bernstein condition is usually recognized as a characterization of “easiness” of the learning problem under various \(\beta\) where \(\beta = 1\) corresponds to the “easiest” learning case. For bounded loss functions, the Bernstein condition will automatically hold with \(\beta = 0\). The standard Bernstein condition requires that the inequality holds for any \(w \in \mathcal{W}\), which is usually difficult to satisfy even in some trivial examples as we will see in Example 1. Different from the standard setting, we only require that the learned (randomised) hypothesis \(W\) satisfy the inequality in expectation. This is a weaker but more natural condition in the sense that we do not expect any \(w \in \mathcal{W}\) will work but hope that the algorithm outputs the hypothesis that performs well in average.

The second condition is the central condition with the witness condition [13], [14], which also implies the \((\eta, c)\)-central condition. We say \((\mu, \ell, W, A)\) satisfies the \(\eta\)-central condition [13], [14] if for the optimal hypothesis \(w^*\), the following inequality holds,
\[
\mathbb{E}_{P_W \otimes \mu} \left[ e^{-\eta(\ell(W,Z) - \ell(w^*,Z))} \right] \leq 1.
\]

We also say the learning tuple \((\mu, \ell, W, A)\) satisfies the \((u, c)\)-witness condition [14] if for constants \(u > 0\) and \(c \in (0,1]\), the following inequality holds,
\[
\mathbb{E}_{P_W \otimes \mu} [\ell(W,Z) - \ell(w^*,Z)] \cdot 1_{\{\ell(W,Z) - \ell(w^*,Z) \leq u\}} \geq c \mathbb{E}_{P_W \otimes \mu} [\ell(W,Z) - \ell(w^*,Z)],
\]
where \(1_{\{\cdot\}}\) denotes the indicator function. Then we have the following corollary.

Corollary 2. If the learning tuple satisfies both \(\eta\)-central condition and \((u, c)\)-witness condition, then the learning tuple also satisfies the \((\eta', c/c)\)-central condition for any \(0 < \eta' < \eta\).

The standard \(\eta\)-central condition is a key condition for proving the fast rate [13], [14], [18]. Some examples are exponential concave loss functions (including log-loss) with \(\eta = 1\) (see [18], [23] for examples) and bounded loss functions with Massart noise condition with different \(\eta\) [13]. Again, different from the standard central condition, we only require that it holds in expectation w.r.t. the distribution induced by the algorithm \(A\). The witness condition [14, Def. 12] is imposed to rule out situations in which learnability simply cannot hold. The intuitive interpretation of this condition is that we exclude bad hypothesis \(w\) with negligible probability (but still can contribute to the expected loss), which we will never witness empirically. With the definitions in place, we derive the fast rate bounds under the \((\eta, c)\)-central condition as follows.

Theorem 4 (Fast Rate with \((\eta, c)\)-central condition). Assume the learning tuple \((\mu, \ell, W, A)\) satisfies the \((\eta, c)\)-central condition for some constants \(\eta > 0\) and \(0 < c \leq 1\). Then, for all \(\eta' \in (0, \eta]\), it holds that,
\[
\mathbb{E}_W \mathbb{E}_{S_n} \mathbb{E}(W,S_n) \leq \frac{1 - c}{c} \mathbb{E}_W \mathbb{E}_{S_n} [\mathcal{R}(W,S_n)] + \frac{1}{c\eta' n} \sum_{i=1}^{n} I(W; Z_i).
\]

Furthermore, the excess risk is bounded by,
\[
\mathbb{E}_W [\mathcal{R}(W)] \leq \frac{1}{c} \mathbb{E}_W \mathbb{E}_{S_n} [\mathcal{R}(W,S_n)] + \frac{1}{c\eta' n} \sum_{i=1}^{n} I(W; Z_i).
\]

Such a bound has similar form with [15, Eq. (3)] which consists of the empirical excess risk and mutual information terms and the first term is negative for some algorithms such as ERM. Notice that different from the bound in Theorem 3, the bound in Theorem 4 contains constants \(c\) and \(\eta'\) that do not depend on the sample size \(n\). By absorbing the necessary dependence on \(n\) in the definition of the \((\eta, c)\)-central condition, now it is instructive to compare the different assumptions used in the above bounds. A summary of the key technical conditions is presented in Table I for easier comparisons, while some comments are provided in Section III-C. In this case, the convergence rate will depend on the mutual information \(I(W; Z_i)\), which can achieve fast rate of \(O(1/n)\) for appropriate learning problems and algorithms [8], [15], [16]. In the following we analytically examine our bounds in Gaussian mean estimation, and we also empirically verify our bounds with a logistic regression problem in [17].

Example 4. We can examine whether the Gaussian mean estimation satisfies the \((\eta, c)\)-central condition. It can be checked that for all \(n\),
\[
\log \mathbb{E}_{P_W \otimes \mu} \left[ e^{-\eta r(W,Z)} \right] \leq \frac{2\beta^2 \sigma_N^4 - \eta \sigma_N^2 r}{n} \leq - \frac{c \eta \sigma_N^2}{n}.
\]
From the above inequality, this learning problem satisfy the
\((\eta, c)\)-central condition for any \(0 < \eta < \frac{1}{2\sigma_N^2}\) and any \(c \geq 1 - 2n\eta\sigma_N^2\), which is independent of the sample size and thus
does not affect the convergence rate. Similarly, take \(\eta = \frac{1}{2\sigma_N^2}\)
and \(c = \frac{1}{2}\), the bound becomes
\[
\frac{1 - c}{c} E_{W,S_n} [\hat{R}(W, S_n)] + \frac{1}{cn} \sum_{i=1}^{n} I(W; Z_i) = \frac{3\sigma_N^2}{n},
\]
which coincides with the bound in Example 3 and we can
arrive at the fast rate since \(I(W; Z_i) \sim O(1/n)\).

Moreover, the learning bound in Theorem 4 can be applied to
the regularized ERM algorithm as:
\[ w_{\text{ERM}} = \arg\min_{w \in \mathcal{W}} L(w, S_N) + \frac{\lambda}{n} g(w), \]
where \(g : \mathcal{W} \to \mathbb{R}\) denotes the regularizer function and \(\lambda\)
is some coefficient. We define \(\hat{R}_{\text{reg}}(w, S_n) = \hat{R}(w, S_n) + \frac{\lambda}{n}(g(w) - g(w^*))\), then we have the following lemma.

**Lemma 1.** We assume conditions in Theorem 4 hold and also
assume \(|g(w_1) - g(w_2)| \leq B\) for any \(w_1\) and \(w_2\) in \(\mathcal{W}\) with
some \(B > 0\). Then for \(W_{\text{ERM}}\):
\[
E_W[\hat{R}(W_{\text{ERM}})] \leq \frac{1}{c} E_{W,S_n} [\hat{R}_{\text{reg}} (W_{\text{ERM}}, S_n)] + \frac{\lambda B}{cn} + \frac{1}{cn} \sum_{i=1}^{n} I(W_{\text{ERM}}; Z_i).
\]

As \(\hat{R}_{\text{reg}}(w, S_n)\) will be negative for \(w_{\text{ERM}}\), the regularized
ERM algorithm can lead to the fast rate if \(I(W_{\text{ERM}}; Z_i) \sim O(1/n)\), which coincides with results in [24].

**B. Intermediate Rate Bound**

From Theorem 4 we can achieve the fast rate if the
mutual information between the hypothesis and data example
is converging with \(O(1/n)\). To further relax the \((\eta, c)\)-central
condition, we can also derive the intermediate rate with the
order of \(O(n^{-\alpha})\) for \(\alpha \in \left[\frac{1}{2}, 1\right]\). Similar to the \(v\)-central
condition, which is a weaker condition of the \(\eta\)-central condition,[13], [14], we propose the \((v, c)\)-central condition first and
derive the intermediate rate results in Theorem 5.

**Definition 2 ((v, c)-Central Condition).** We say that
\((\mu, \ell, W, A)\) satisfies the \((\eta, c)\)-central condition up to some \(c > 0\) if the following inequality holds for the optimal hypothesis \(w^*:\)
\[
\log E_{P_{W} \otimes \mu} \left[ e^{-\eta(\ell(W,Z) - \ell(w^*, Z))} \right] \leq -c\eta E_{P_{W} \otimes \mu} [\ell(W, Z) - \ell(w^*, Z)] + \eta c. \tag{14}
\]
Let \(v : [0, \infty) \to [0, \infty)\) is a bounded and non-decreasing
function satisfying \(v(\epsilon) > 0\) for all \(\epsilon > 0\). We say that
\((\mu, \ell, W, A)\) satisfies the \((v, c)\)-central condition if for all \(c \geq 0\) such that (14) is satisfied with \(\eta = v(\epsilon)\).

**Theorem 5.** Assume the learning tuple \((\mu, \ell, W, A)\) satisfies the \((v, c)\)-central condition up to \(c\) for some function \(v\) as
defined in Def. 2 and \(0 < c < 1\). Then it holds that for any
\(c \geq 0\) and any \(0 < \eta' \leq v(\epsilon)\),
\[
E_{W,S_n} [E(W,S_n)] \leq \frac{1 - c}{c} E_{W,S_n} [\hat{R}(W, S_n)] + \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{\eta' c} I(W; Z_i) + \frac{\epsilon}{c} \right).
\]

In particular, if \(v(\epsilon) \approx e^{1-\beta}\) for some \(\beta \in [0, 1]\), then the
generalization error is bounded by,
\[
E_{W,S_n} [E(W,S_n)] \leq \frac{1 - c}{c} E_{W,S_n} [\hat{R}(W, S_n)] + \frac{2}{nc} \sum_{i=1}^{n} I(W; Z_i)^{1/2}.
\]
Thus, the expected generalization is found to have an order of
\(I(W; Z_i)^{1/2}\), which corresponds to the results under Bernstein’s condition [15], [21], [22].

**TABLE I: Technical Conditions Comparisons**

| Condition             | Key Inequality |
|-----------------------|----------------|
| \((\eta, c)\)-Central | \(\log \mathbb{E} \left[ e^{-\eta r(W,Z)} \right] \leq -c\eta \mathbb{E}[r(W,Z)]\) |
| Bernstein with \(\beta = 1\) | \(\log \mathbb{E} \left[ e^{-\eta r(W,Z)} \right] \leq -\frac{1}{2}\eta \mathbb{E}[r(W,Z)]\) |
| Central + Witness     | \(\log \mathbb{E} \left[ e^{-\eta r(W,Z)} \right] \leq -\frac{1}{2}\eta \mathbb{E}[r(W,Z)]\) |
| Central Only          | \(\log \mathbb{E} \left[ e^{-\eta r(W,Z)} \right] \leq 0\) |
| Sub-Gaussian          | \(\log \mathbb{E} \left[ e^{-\eta r(W,Z)} \right] \leq -\eta \mathbb{E}[r(W,Z)] + \frac{a^2 \sigma^2}{2}\) |

**C. Connection to other works**

Fast rate conditions are widely investigated under different
learning frameworks and conditions [13]–[15], [18], [22]–[24]. We propose the \((\eta, c)\)-central condition, a stronger condition
than \(\eta\)-central condition, that can lead to the fast rate for
the generalization error in expectation, which also coincides
with many existing works such as [14] and [15] for certain
choices of \(c\) and \(\eta\). In particular, with bounded loss, \(\beta = 1\) in
the Bernstein condition is equivalent to the central condition
with the witness condition for fast rate, from which \((\eta, c)\)-
central condition follows. As an example of unbounded loss
functions, the log-loss will satisfy the central and witness
conditions under well-specified model [14], [25], which also
consequently implies the \((\eta, c)\)-central condition. As suggested
by Theorem 3, the sub-Gaussian condition can also satisfy the
\((\eta, c)\)-central condition if it satisfies that \(\eta \sigma^2 = a \mathbb{E}[r(W,Z)]\)
for some constant \(a \in (0, 2)\).

As the most relevant work, our bound is similar to that found
in [15] which applies conditional mutual information [8], but
their results are derived under the PAC-Bayes framework and
rely on the exchangeable data-dependent priors. Our result
applies to general algorithms with the mutual information
and our assumptions on the hypothesis are weaker since we only
require the proposed conditions hold in expectation w.r.t. \(P_W\),
instead of for all \(w \in \mathcal{W}\). Our results also have the benefit of
allowing the convergence factors to be further improved
by using different metrics and data-processing techniques, see
[11], [26], [27] for examples.
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