Time-Dependent Generalized Nash Equilibrium Problem

John Cotrina · Javier Zúñiga

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Abstract
We prove an existence result for the time-dependent generalized Nash equilibrium problem under generalized convexity without neither a quasi-variational inequality reformulation nor a quasi-equilibrium problem reformulation. Furthermore, an application to the time-dependent abstract economy is considered.

Keywords Generalized Nash equilibrium problem · Infinite-dimensional strategy spaces · Coerciveness · Generalized convexity · Abstract economy

Mathematics Subject Classification 91B55 · 91B50

1 Introduction
A time-dependent generalized Nash equilibrium problem (TD-GNEP for short) is a generalized Nash game, in which each player’s strategy and objective function depend on time. By working with continuous time, one needs to consider infinite-dimensional strategy spaces. These have been investigated in recent years (see, for instance, [1–3]). In [2,3] the concavity or convexity of all objective functions was considered in order to obtain an existence result. However, in [1] the authors gave an existence result under generalized convexity (more precisely, semistrict quasiconcavity), but for the jointly convex case. In the same vein, our existence result uses only quasiconcavity of all objective functions. Our proof uses a fixed point argument and requires a coerciveness condition. As a general application, we prove the existence of a time-dependent economic equilibrium of a time-dependent abstract economy as defined in [4]. More practical applications are considered in [5] and [6].
2 Preliminaries

Consider the Lebesgue space $L^2([0, T], \mathbb{R}^n)$ with the inner product

$$\langle \phi, \psi \rangle := \int_0^T \phi(t)\psi(t)dt.$$ 

To define what a TD-GNEP is, let us assume that we have $p$ players and, to each player $\nu \in \{1, 2, \ldots, p\}$, we associate a natural number $n_\nu$. Define $n := \sum_{\nu=1}^p n_\nu$. Each player has a strategy $x^\nu \in X_\nu(x^{-\nu}) \subset L^2([0, T], \mathbb{R}^{n_\nu})$, where by $x^{-\nu} \in L^2([0, T], \mathbb{R}^{n-n_\nu})$ we denote the vector formed by all players’ strategies except for those of player $\nu$. The set $X_\nu(x^{-\nu})$ is the strategy space of player $\nu$, given the strategy of the other players, as it is usually the case in non-cooperative games. We also write $x = (x^\nu, x^{-\nu}) \in \prod_{\nu=1}^p X_\nu(x^\nu) \subset L^2([0, T], \mathbb{R}^n)$, which is a shorthand (already used in many papers on the subject; see, e.g., [1,7]) to denote $x := (x^1, \ldots, x^{\nu-1}, x^\nu, x^{\nu+1}, \ldots, x^p)$ as a way to single out the strategy of player $\nu$ within the full strategy vector. Each $x^\nu(t) \in \mathbb{R}^{n_\nu}$ can be thought of as a strategy of player $\nu$ at time $t \in [0, T]$. Then, $x \in L^2([0, T], \mathbb{R}^n)$ is the full strategy vector and thus $x(t)$ is the vector of strategies of all players at a given time $t \in [0, T]$.

Given the strategy $x^{-\nu}$, player $\nu$ chooses a strategy $x^\nu$, that solves the following optimization problem

$$\max_{x^\nu} \theta_\nu(x^\nu, x^{-\nu}) \quad \text{subject to} \quad x^\nu \in X_\nu(x^{-\nu}), \quad (1)$$

where $\theta_\nu(x^\nu, x^{-\nu})$ denotes the payoff of player $\nu$, when the rival players have chosen the strategy $x^{-\nu}$. For every $x^{-\nu}$, the solution set of problem (1) is denoted by $S_\nu(x^{-\nu})$. For any $\nu \in \{1, 2, \ldots, p\}$, let $K_\nu$ be a nonempty subset of $L^2([0, T], \mathbb{R}^{n_\nu})$, $\theta_\nu : L^2([0, T], \mathbb{R}^n) \to \mathbb{R}$ be a function and $X_\nu : K_{-\nu} \rightrightarrows K_\nu$ be a set-valued map, where $K_{-\nu} := \prod_{i \neq \nu} K_i$. The TD-GNEP is the following problem:

Find $x_0$ such that $x_0^\nu \in S_\nu(x_0^{-\nu})$, for any $\nu \in \{1, 2, \ldots, p\}$.

Such a point $x_0$ is called a time-dependent generalized Nash equilibrium, or a solution to the TD-GNEP.

For the sake of completeness, let us recall also that a real valued function $f : L^2([0, T], \mathbb{R}^n) \to \mathbb{R}$ is said to be:

- quasiconcave, if for any $x, y \in L^2([0, T], \mathbb{R}^n)$ and $\lambda \in [0, 1]$, we have 
  $$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\};$$
– *semistrictly quasiconcave*, if it is quasiconcave and for any \( x, y \in L^2([0, T], \mathbb{R}^n) \), such that \( f(x) \neq f(y) \), and \( \lambda \in ]0, 1[ \), we have

\[
f(\lambda x + (1 - \lambda)y) > \min\{f(x), f(y)\}.
\]

On the other hand, we now recall the continuity notions for a set-valued map \( F : L^2([0, T], \mathbb{R}^n) \rightarrow L^2([0, T], \mathbb{R}^n) \). The set-valued map \( F \) is said to be:

– *upper semicontinuous* at the point \( x \in L^2([0, T], \mathbb{R}^n) \), if for any open \( W \) such that \( F(x) \subset W \), there exists a neighborhood \( V \) of \( x \) such that, for all \( z \in V \), we have \( F(z) \subset W \);

– *lower semicontinuous* at the point \( x \in L^2([0, T], \mathbb{R}^n) \), if for any open \( W \) such that \( F(x) \cap W \neq \emptyset \), there exists a neighborhood \( V \) of \( x \) such that, for all \( z \in V \), we have \( F(z) \cap W \neq \emptyset \).

Our existence result will be obtained as a consequence of Kakutani’s Theorem, which is stated below and it can be found in [8].

**Theorem 2.1** (Kakutani) Let \( K \) be a nonempty, compact and convex subset of a locally convex space \( E \) and let \( T : K \rightrightarrows K \) be a set-valued map. If \( T \) is upper semicontinuous such that for all \( x \in K \), \( T(x) \) is nonempty, closed and convex, then \( T \) admits a fixed point.

### 3 Existence Result

As we have already mentioned earlier, we aim at proving the existence of a time-dependent Nash equilibrium, and this will be done thanks to a reformulation of the equilibrium problem into an associated fixed point problem.

With the previous notation of TD-GNEP, for any given optimal strategy \( x^{-\nu} \) of the rival players, let us define the set-valued map \( S : K \rightrightarrows K \) as

\[
S(x) = \prod_{\nu=1}^{p} S_{\nu}(x^{-\nu}),
\]

where \( K := \prod_{\nu=1}^{p} K_{\nu} \).

The following proposition connects the notions of equilibrium and fixed point. The proof is a direct consequence of the definitions.

**Proposition 3.1** Let \( \hat{x} \in K \), then \( \hat{x} \) is a time-dependent generalized Nash equilibrium, if and only if it is a fixed point of \( S \).

We now introduce an important definition. The TD-GNEP is said to satisfy the *coerciveness condition*, if the next two conditions hold.

1. For each \( u \in K \) there exists a convex, compact and nonempty subset \( K_u \) of \( L^2([0, T], \mathbb{R}^n) \) such that the following holds: for all \( x \in \left( \prod_{v} X_v(u^{-v}) \right) \setminus K_u \), there exists a \( y \in \left( \prod_{v} X_v(u^{-v}) \right) \cap K_u \) such that

\[
\theta_v(x) \leq \theta_v(y^v, x^{-v}),
\]

for all \( v \).
2. For all \( \nu \), there exists a convex, compact and nonempty subset \( R_\nu \) of \( L^2([0, T], \mathbb{R}^n_\nu) \) such that the following hold:

\[
\left( \prod_\nu X_\nu(u^{-\nu}) \right) \cap \prod_\nu R_\nu \neq \emptyset \text{ and } K_u \subset \prod_\nu R_\nu, \text{ for all } u \in K.
\]

Our main result generalizes [1, Corollary 4.1], where the objective functions are semistrict quasiconcave. Also, our result generalizes [7, Theorem 6] to the infinite dimensional case, and allows the constraint valued maps to possibly admit unbounded values.

Before we state our main theorem, we need the following definition. Associated to a TD-GNEP that satisfies the coerciveness condition, for each \( \nu \), we consider the set-valued map \( Y_\nu : K_{-\nu} \cap \left( \prod_{i \neq \nu} R_i \right) \Rightarrow K_\nu \cap R_\nu \), defined as

\[
Y_\nu(x^{-\nu}) := X_\nu(x^{-\nu}) \cap R_\nu.
\]

These maps play an important role in our main result. We also need Berge’s maximum theorem, which can be found in [9].

**Theorem 3.1** (Berge’s maximum theorem) Let \( X, Y \) be two metric spaces, \( f : X \times Y \to \mathbb{R} \) be a function and \( F : X \Rightarrow Y \) a set-valued map. Assume that \( f \) is continuous, \( F \) is both upper and lower semicontinuous; and \( F \) is nonempty and compact valued. Then, the set-valued map \( \Phi : X \Rightarrow Y \), defined as

\[
\Phi(x) := \arg \max_{y \in F(x)} f(x, y),
\]

is upper semicontinuous with compact values.

**Theorem 3.2** With the previous notation, let us assume that:

1. \( K_\nu \) is a nonempty, closed and convex subset of \( L^2([0, T], \mathbb{R}^n_\nu) \), for all \( \nu \);
2. the TD-GNEP satisfies the coerciveness condition and the set-valued map \( Y_\nu \) is both upper and lower semicontinuous with nonempty, closed and convex values for any \( \nu \);
3. the function \( \theta_\nu \) is continuous for every player \( \nu \);
4. the function \( \theta_\nu(\cdot, x^{-\nu}) \) is quasiconcave for every player \( \nu \).

Then, there exists a time-dependent generalized Nash equilibrium.

**Proof** We define the set-valued map \( T_\nu : K_{-\nu} \cap \left( \prod_{i \neq \nu} R_i \right) \Rightarrow K_\nu \cap R_\nu \) by

\[
T_\nu(x^{-\nu}) := \arg \max_{x^\nu \in Y_\nu(x^{-\nu})} \theta_\nu(x^\nu, x^{-\nu}).
\]
Assumptions 1, 2 and 3, and Theorem 3.1 imply that \( T_v \) is upper semicontinuous with compact values, for all \( v \). Moreover, \( T_v \) has convex values due to Assumption 4. Therefore, the set-valued map \( T := \prod_v T_v \) is upper semicontinuous with closed and convex values. Since \( K \cap \prod_v R_v \) is compact, applying Theorem 2.1, \( T \) has at least one fixed point, i.e. there exists \( \hat{x} \in T(\hat{x}) \). We will show that \( \hat{x} \in S(\hat{x}) \). Indeed, suppose there is a \( v \) and a \( \bar{x}^v \in X_v(\bar{x}^{-v}) \) such that \[ \theta_v(\bar{x}^v, \bar{x}^{-v}) > \theta_v(\hat{x}^v, \hat{x}^{-v}). \] (2)

Since \( K_{\hat{x}} \subset \prod_v R_v \) due to the second part of the coerciveness condition, we deduce that \( x = (\bar{x}^v, \bar{x}^{-v}) \) does not belong to \( K_{\hat{x}} \). Indeed, if \( x \in K_{\hat{x}} \), then \( x \in K \cap \prod_v R_v \), which in turn implies \( \theta_v(x) \leq \theta_v(\hat{x}) \), and we get a contradiction with (2). Now, by the first part of the coerciveness condition, for this \( x \) there exists a \( y \in \prod_v X_v(\hat{x}^{-v}) \cap K_{\hat{x}} \subset Y(\hat{x}) \) such that \( \theta_v(\bar{x}^v, \bar{x}^{-v}) \leq \theta_v(y^v, \hat{x}^{-v}) \), which is a contradiction with (2). By Proposition 3.1 the result follows.

**Remark 3.1** Here we present a few remarks about the previous theorem.

1. It is important to note that we cannot apply [1, Corollary 4.1], because in our case the constraint set-valued maps are not defined by Rosen’s law. Also, it is not possible to apply [7, Theorem 6] due to the constraint set-valued maps \( X_v \) not necessarily being upper or lower semicontinuous.

2. If \( K_v \) is compact for all \( v \), and we put \( R_v = K_u = K_v \) for all \( v \) and \( u \), then \( Y_v = X_v \), and the TD-GNEP satisfies the coerciveness condition. Moreover, if the space is finite dimensional, we recover [7, Theorem 6].

3. If \( X_v \) has closed graph or it is upper semicontinuous, then \( Y_v \) is upper semicontinuous. Indeed, in the first case, this is a direct consequence of [10, Proposition 2.5.11]. In the second case, let \( U \) be an open set such that \( Y_v(x^{-v}) \subset U \). We consider \( U' := U \cup R_v^c \), where \( R_v^c \) is the complement of \( R_v \), and obtain \( X_v(x^{-v}) \subset U' \). By the upper semicontinuity of \( X_v \), there exists \( V_{x^{-v}} \), a neighborhood of \( x^{-v} \), such that \( X_v(z^{-v}) \subset U' \) for all \( z^{-v} \in V_{x^{-v}} \). Thus, \( Y_v(z^{-v}) \subset U' \cap R_v = U \cup R_v \subset U \), which in turn implies the upper semicontinuity of \( Y_v \).

4. The lower semicontinuity of \( X_v \) does not imply the lower semicontinuity of the set-valued map \( Y_v \). Consider for instance \( X_v : [0, 1] \rightarrow \mathbb{R}^2 \), defined by

\[
X_v(t) := [O, (1, t)],
\]

where \( O := (0, 0) \in \mathbb{R}^2 \), and consider the compact set \( K := [O, e_2] \) with \( e_2 := (0, 1) \). Clearly, \( X_v \) is lower semicontinuous at 0, but \( Y_v \) is not lower semicontinuous at 0. In this case, the lower semicontinuity of \( Y_v \) is guaranteed provided that \( X_v \) has an open graph (see [11, Lemma 4.2]), or if \( X_v \) is lower semicontinuous such that \( \text{int}(X_v(x^{-v})) \cap R_v \neq \emptyset \), for all \( x^{-v} \) (see [12, Corollary 1.3.10] or page 16 of [13]).

5. The key tool in the proof of the main theorem is the use of the coerciveness condition, which allows us to work with generalized Nash equilibrium problems.
having constraint set-valued maps that admit unbounded values. A similar notion to the coerciveness condition presented here was recently introduced in [14, Theorem 5].

4 Application: Time-dependent Abstract Economy

Arrow and Debreu in 1954 (see [15]) considered a general “economic system”, named abstract economy, along with a corresponding definition of equilibrium. After this pioneering work, several authors established the existence of an equilibrium, that included production and consumption; see, for instance, [5,16].

Recently, Donato et al. introduced in [4] the concept of time-dependent abstract economy, for which they gave an existence result through a variational reformulation. Motivated by this last work, we give an existence solution for a time-dependent abstract economy problem, using the TD-GNEP formulation. More precisely, we suppose that there are \( l \) distinct commodities (including all kinds of services). Each commodity can be bought or sold at a finite number of locations (in space and time). The commodities are produced in “production units” (companies), whose number is \( s \). For each production unit \( j \), there is a set \( A_j \) of possible production plans. An element \( a^j \in A_j \) is a vector in \( L^2([0, T], \mathbb{R}^l) \). We note that the sign of the \( h \)th component at time \( t \) of this last vector has a particular meaning. When the quantity \( a^j_h(t) \) is positive, it represents the commodity offered in the market by the production unit \( j \) at time \( t \), this is known as an output. When \( a^j_h(t) \) is negative, it represents the amount of this commodity, that will be used in the production process (like raw materials); this is known as an input. When \( a^j_h(t) \) equals zero, then the production unit \( j \) does not either produce the commodity \( h \) at time \( t \) or it is required in the production process (it is not either an input or and output). If we denote by \( p \in L^2([0, T], \mathbb{R}^l) \) the prices of the commodities, the production units will naturally aim at maximizing the total revenue.

We also assume the existence of “consumption units”, typically families or individuals, whose number is \( r \). Associated to each consumption unit \( i \), we have a vector \( b^i \in L^2([0, T], \mathbb{R}^l) \), where \( b^i_h(t) \) represents the quantity of the \( h \)th commodity consumed by the \( i \)th individual at time \( t \). When \( b^i_h(t) \) is positive, it represents the amount of commodity \( h \) being consumed in the market by the consumption unit \( i \) at time \( t \). When it is negative, it represents a labor service being offered by the consumption unit \( i \) at time \( t \). When this quantity is zero, the consumption unit \( i \) does not either consume the commodity \( h \) at time \( t \) or it offers it as a labor service. In general, \( b^i \) must belong to a certain set \( B_i \subset L^2([0, T], \mathbb{R}^l) \), which is convex, closed and bounded from below, i.e., there is a vector \( \beta^i \) such that \( \beta^i_h(t) \leq b^i_h(t) \) a.e. in \([0, T]\), for all \( h \). The set \( B_i \) includes all consumption vectors among which the individual could choose one, if there were no budgetary constraints (the latter constraints will be explicitly formulated below). We also assume that the \( i \)th consumption unit is endowed with a vector \( \xi^i \in L^2([0, T], \mathbb{R}^l_+) \) of initial holding of commodities, and has a contractual claim to the share \( \alpha_{ij} \) of the profit of the \( j \)th production unit such that \( \alpha_{ij} \geq 0 \) and \( \sum_{i=1}^{r} \alpha_{ij} = 1 \), for all \( j \). Under these conditions it is then clear that, given a vector of
prices \( p, \) which belongs to a convex and closed set \( C \subset L^2([0, T], \mathbb{R}^l), \) the choice of the \( i \)th unit is further restricted to those vectors \( b^i \in B_i \) such that
\[
\langle p, b^i \rangle \leq \langle p, \xi^i \rangle + \sum_{j=1}^s \alpha_{ij} \langle p, a^j \rangle.
\]

As it is standard in economic theory, the consumption units aim is to maximize a utility function \( u_i(b^i) \).

**Definition 4.1** A time-dependent economic equilibrium is a vector of the form
\[
(h^1, \ldots, h^s, b^1, \ldots, b^r, p)
\]
such that
\[
\langle p, h^j \rangle = \max_{a^j \in A_j} \langle p, a^j \rangle, \quad \text{for all } j,
\]
\[
u_i(b^i) = \max_{b^i \in D_i(h^i, p)} u_i(b^i), \quad \text{for all } i,
\]
\[
\sum_{i=1}^r \langle \hat{p}, b^i - \xi^i \rangle - \sum_{j=1}^s \langle \hat{p}, h^j \rangle = \max_{p \in P} \sum_{i=1}^r \langle p, b^i - \xi^i \rangle - \sum_{j=1}^s \langle p, h^j \rangle,
\]
where
\[
D_i(a, p) = \left\{ b^i \in B_i : \langle p, b^i \rangle \leq \langle p, \xi^i \rangle + \max \left\{ 0, \sum_{j=1}^s \alpha_{ij} \langle p, a^j \rangle \right\} \right\}
\]
and
\[
P = \left\{ p \in C : p \geq 0, \frac{1}{T} \int_T^l p^h(t)dt = 1 \right\}.
\]

This definition of time-dependent economic equilibrium is motivated by [7].

The following lemma says that the consumption units’ demand must be always satisfied in average by the production units. Moreover, it is only possible for the price of a commodity to be zero a.e. in \([0, T]\), if the supply exceeds the demand a.e. in \([0, T]\).

In other words, Definition 4.1 implies the time-dependent allocation-price equilibrium introduced in [4].

**Lemma 4.1** Let \( A_j \) be a set such that \( 0 \in A_j, \) for all \( j. \) If \((\hat{a}, \hat{b}, \hat{p})\) is a time-dependent economic equilibrium, then
\[
\int_0^T \left[ \sum_{i=1}^r \left( \hat{b}^i_h(t) - \xi^i_h(t) \right) - \sum_{j=1}^s \hat{a}^j_h(t) \right] dt \leq 0, \quad \text{for all } h.
\]
Moreover, if all utility functions $u_i$ have no maximum in $B_i$ (non-satiation), i.e., for all $b^i \in B_i$ there exists a $\hat{b}^i \in B_i$ such that $u_i(\hat{b}^i) > u_i(b^i)$; and they all are semistrictly quasiconcave, then
\[
\sum_{i=1}^{r} \langle \hat{p}, \hat{b}^i - \xi^i \rangle - \sum_{j=1}^{s} \langle \hat{p}, a^j \rangle = 0. \tag{7}
\]

**Proof** This result follows from some arguments that appear in [4, Theorem 3] and [15, Section 1.4.2]. Since $0 \in A_j$ and (3) is satisfied, $\langle \hat{p}, \hat{a}^j \rangle \geq 0$ and thus
\[
\sum_{j=1}^{s} \langle \hat{p}, \alpha_{ij} \hat{a}^j \rangle \geq 0.
\]
On the other hand, as $\hat{b} \in D(\hat{a}, \hat{p})$, we deduce that
\[
\sum_{i=1}^{r} \langle \hat{p}, \hat{b}^i - \xi^i \rangle - \sum_{j=1}^{s} \langle \hat{p}, \hat{a}^j \rangle \leq 0. \tag{8}
\]
Now, consider $p$ defined as
\[
p^h = \begin{cases} 0, & h \neq h_0, \\ 1, & h = h_0; \end{cases}
\]
clearly $p \in P$. Therefore, by (5) we have
\[
\left\langle p, \sum_{i=1}^{r} (\hat{b}^i - \xi^i) - \sum_{j=1}^{s} \hat{a}^j \right\rangle = \int_{0}^{T} \left[ \sum_{i=1}^{r} (\hat{b}^i_{h_0}(t) - \xi^i_{h_0}(t)) - \sum_{j=1}^{s} \hat{a}^j_{h_0}(t) \right] dt \leq 0.
\]
This finishes the proof of (6). For the next part, if for all $i$ we have the following equality in the definition of $D_i(\hat{a}, \hat{p})$
\[
\langle \hat{p}, \hat{b}^i \rangle = \langle \hat{p}, \xi^i \rangle + \max \left\{ 0, \sum_{j=1}^{s} \alpha_{ij} \langle \hat{p}, \hat{a}^j \rangle \right\},
\]
then
\[
\sum_{i=1}^{r} \langle \hat{p}, \hat{b}^i - \xi^i \rangle = \sum_{i=1}^{r} \max \left\{ 0, \sum_{j=1}^{s} \alpha_{ij} \langle \hat{p}, \hat{a}^j \rangle \right\} \geq \sum_{j=1}^{s} \langle \hat{p}, \hat{a}^j \rangle.
\]
Using the last inequality and (8), we deduce (7)
\[
\sum_{i=1}^{r} \langle \hat{p}, \hat{b}^i - \xi^i \rangle - \sum_{j=1}^{s} \langle \hat{p}, \hat{a}^j \rangle = 0.
\]
On the other hand, consider the existence of a $i_0$ such that the strict inequality in the definition of $D_{i_0}(\hat{a}, \hat{p})$ is satisfied

$$\langle \hat{p}, \hat{b}^{i_0} \rangle < \langle \hat{p}, \xi^{i_0} \rangle + \max \left\{ 0, \sum_{j=1}^{s} \alpha_{i_0 j} \langle \hat{p}, \hat{a}^j \rangle \right\}.$$  

Since the utility function $u_{i_0}$ has no maximum in $B_{i_0}$, and it is semistrictly quasiconcave, then for $\hat{b}^{i_0}$ there exists $b^{i_0}$ such that $u_{i_0}(b^{i_0}) > u_{i_0}(\hat{b}^{i_0})$ and $u_{i_0}(\lambda b^{i_0} + (1 - \lambda)\hat{b}^{i_0}) > u_{i_0}(\hat{b}^{i_0})$, for all $\lambda \in ]0, 1[$. Suppose the strict inequality holds in (8), i.e.

$$\sum_{i=1}^{r} \langle \hat{p}, \hat{b}^i - \xi^i \rangle - \sum_{j=1}^{s} \langle \hat{p}, a^j \rangle < 0.$$  

If this is the case, we could choose a $\lambda$ small enough for which the inequality is still satisfied and $\lambda b^{i_0} + (1 - \lambda)\hat{b}^{i_0} \in D_{i_0}(\hat{a}, \hat{p})$, which is a contradiction that again implies (7).

Finally, we can prove the existence of a time-dependent economic equilibrium using Theorem 3.2.

**Theorem 4.1** Let the following hold:

(i) The set $A_j$ is convex and compact, for all $j$.
(ii) The set $C$ is compact.
(iii) For all $i$

(a) $D_i$ is upper and lower semicontinuous;
(b) $u_i$ is quasiconcave and continuous;
(c) there exists a convex, compact and nonempty set $M_i$ such that for all $(a, p) \in A \times P$ and all $b_i \in D_i(a, p) \setminus M_i$, there is a $b_i^t \in D_i(a, p) \cap M_i$ with $u_i(b_i^t) \geq u_i(b_i)$;
(d) $\text{int}(D_i(a, p)) \cap M_i \neq \emptyset$, for all $(a, p) \in A \times P$.

Then, there exists a time-dependent economic equilibrium.

**Proof** We use the following notation:

- $x := (a, b, p) = (a^1, \ldots, a^s, b^1, \ldots, b^r, p)$ and

$$x^v := \begin{cases}  
\begin{array}{ll}
\begin{cases}  \alpha^{v-s}, & v \in \{s + 1, \ldots, s + t\}, \\
p, & v = s + t + 1.
\end{cases}
\end{array}
\end{cases}$$
– The objective functions $\theta_v : L^2([0, T], \mathbb{R}^n) \to \mathbb{R}$ are defined as:

$$
\theta_v(x) := \begin{cases}
\langle \langle p, a^v \rangle \rangle, & v \in \{1, \ldots, s\}, \\
\eta_{v-s}(b^{v-s}, u^v), & v \in \{s + 1, \ldots, s + t\}, \\
\left\langle p, \sum_{i=1}^t (b^i - \xi^i) - \sum_{j=1}^s a^j \right\rangle, & v = s + t + 1.
\end{cases}
$$

– For each $v$ we set:

$$
K_v := \begin{cases}
A_v, & v \in \{1, \ldots, s\}, \\
B_{v-s} \cap M_{v-s}, & v \in \{s + 1, \ldots, s + t\}, \\
P, & v = s + t + 1.
\end{cases}
$$

– The strategy set of player $v$ is defined as:

$$
X_v(x^{-v}) := \begin{cases}
A_v, & v \in \{1, \ldots, s\}, \\
D_{v-s}(a, p) \cap M_{v-s}, & v \in \{s + 1, \ldots, s + t\}, \\
P, & v = s + t + 1.
\end{cases}
$$

Clearly, each objective function is quasiconcave and continuous. Also, as the set-valued maps $D_i$ are upper and lower semicontinuous and $(iii) - (d)$ holds, it is clear that the set-valued map $X_v : K^{-v} \Rightarrow K_v$ is upper and lower semicontinuous too. Moreover, since the sets $C_i$, $A_j$’s and $M_i$’s are compact, we have that $K_v$ is compact for all $v$. Also, for each $x^{-v}$ the strategy set $X_v(x^{-v})$ is convex and closed. Thus, by Theorem 3.2 the TD-GNEP has a solution which is a time-dependent economic equilibrium, due to Assumption $(iii) - (c)$. □

**Remark 4.1** Theorem 4.1 does not require neither hypothesis 2(a) and 2(b) of [4, Theorem 4] nor the semistrict quasiconcavity of all objective functions. Also, they work with the weak topology. Instead, we require the coerciveness condition for every utility function and we only work with the strong topology. An interpretation of the coerciveness condition is the following: for each $i$, there exists $M_i$ such that for all production vectors $a_i$, all price vectors $p$ and all consumption units $b^{i}$ in $D_i(a, p) \setminus M_i$, there exists another consumption unit $\tilde{b}^i$ in $D_i(a, p) \cap M_i$ which it is at least as preferable as the previous one. In other words, all utility functions have a nice set where their maximum can be found.

**5 Conclusions**

In this article, we prove an existence result (Theorem 3.2) for time-dependent generalized Nash equilibrium problems under only quasiconcavity assumptions of all objective functions. An application to the time-dependent abstract economy is presented (Theorem 4.1).

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