A $C^1$ generic condition for existence of symbolic extensions of volume preserving diffeomorphisms

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Abstract

We prove that a $C^1$-generic volume preserving diffeomorphism has a symbolic extension if and only if this diffeomorphism is partial hyperbolic. This result is obtained by means of good dichotomies. In particular, we prove Bonatti’s conjecture in the volume preserving scenario. More precisely, in the complement of Anosov diffeomorphisms we have densely robust heterodimensional cycles.

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1. Introduction and statement of the results

A system $(X, f)$ has a symbolic extension, if there exist a subshift $(Y, \sigma)$, which is a closed, shift invariant subset of a full shift over a finite alphabet, and a surjective continuous map $\pi : Y \to M$ such that $\pi \circ \sigma = f \circ \pi$. In this case, $(Y, \sigma)$ is called an extension of $(X, f)$ and $(X, f)$ a factor of $(Y, \sigma)$.

The complexity of a system $(X, f)$ could be seen by means of the topological entropy. Hence, if a system has a symbolic extension its complexity is bounded above by the complexity of a subshift. However, this system may contain additional information. The symbolic extension entropy of the system is the infimum of the topological entropy of all symbolic extensions of the system. And note that the topological entropy of a system is less than or equal to the symbolic extension entropy. The difference between these functions represents how entropy is hidden at finer and finer scales.

A symbolic extension of $(X, f)$ is a principal extension if the map $\pi$ is such that $h_\mu(\sigma) = h_{\pi \mu}(f)$ for every $\sigma$-invariant measure $\nu \in \mathcal{M}(\sigma|Y)$, where $h_\mu(\sigma)$ is the metric entropy of $\sigma$ with respect to $\nu$ and $\mathcal{M}(\sigma|Y)$ denotes the set of invariant measures of $\sigma|Y$. Note, the symbolic extension entropy is equal to the topological entropy if the system has a principal extension.
Let $M$ be a Riemannian, connected and compact manifold. A diffeomorphism $f: M \to M$ is asymptotically $h$-expansive if
\[
\lim_{\varepsilon \to 0} \sup_{x \in M} h(f|_{B_\varepsilon(x, \varepsilon)}) = 0,
\]
where $B_\varepsilon(x, \varepsilon) = \{ y \in M; d(f^j(x), f^j(y)) < \varepsilon \text{ for every } j \in \mathbb{N} \}$, and $h(\cdot)$ denotes the topological entropy. If there exists $\varepsilon_0$ such that $\sup_{x \in M} h(f|_{B_\varepsilon(x, \varepsilon)}) = 0$ for every $0 < \varepsilon < \varepsilon_0$, then $f$ is called $h$-expansive.

Boyle et al [9] showed that asymptotically $h$-expansive diffeomorphisms have principal extension. Hence, if a diffeomorphism has no symbolic extensions, in some sense, should there exist invariant subsets contained in balls with diameters arbitrary small having positive topological entropy bounded away from zero.

By a result of Buzzi [14], if every $C^\infty$ diffeomorphism over a compact manifold is asymptotically $h$-expansive, then it has a principal extension. A conjecture of Downarowicz and Newhouse [23] asserts that every $C^r$-diffeomorphism ($r > 1$) has a symbolic extension. This conjecture was solved by Downarowicz and Maass [22] in dimension one and for surface diffeomorphisms by Burguet [12]. Also, we would like to remark that recently Burguet and Fisher [13] extended this result to higher dimensions proving that every $C^2$ partially hyperbolic diffeomorphism with a two-dimensional centre bundle has a symbolic extension. See also [10].

Hence, it seems natural to try to relate the existence of symbolic extensions to the differential structure of a diffeomorphism. For instance, using shadowing we can easily find a symbolic extension for Anosov diffeomorphisms. Diaz et al [21] showed that every $C^1$ partially hyperbolic diffeomorphism is $h$-expansive if we define partial hyperbolic diffeomorphisms as in [19]. Hence it has a principal extension. See also [20]. A diffeomorphism $f$ exhibits a homoclinic tangency if there exists a hyperbolic periodic point $p$ of $f$ having a non-transversal homoclinic point. Thus, Liao et al [30] proved that if a diffeomorphism is not approximated by one exhibiting homoclinic tangency, then it is also $h$-expansive.

On the other hand, we can consider a problem about the existence of diffeomorphisms that has no symbolic extensions. We can note that such diffeomorphisms have a rich dynamics, since they are not asymptotically $h$-expansive. Moreover, if the conjecture of Downarowicz and Newhouse is right such diffeomorphisms can not be $C^r$ ($r > 1$).

In the symplectic scenario the author with Tahzibi [16] extended a result of Downarowicz and Newhouse [23], proving that $C^1$-generically either a symplectic diffeomorphism is Anosov or has no symbolic extensions. That is, in the symplectic setting we have a large set of diffeomorphisms having no symbolic extensions.

The aim of this paper is to obtain similar results in the conservative case.

We denote by $\text{Diff}^1_c(M)$ the set of $C^1$ volume preserving diffeomorphisms over $M$. Here, as in [19], a $f$-invariant subset $\Lambda$ is partial hyperbolic if there exists a continuous $Df$-invariant splitting $T_x M = E^c \oplus E^s_1 \oplus \cdots \oplus E^s_k \oplus E^a$ with non-trivial extremal bundles $E^c$ and $E^a$, such that each centre bundle $E^c_i$ is one-dimensional, and there exist constants $m \in \mathbb{N}, 0 < \lambda < 1$ such that for every $x \in M$:

- $\|Df^m v\| \leq 1/2$ for each unitary $v \in E^c$ (one says $E^c$ is (uniformly) contracted),
- $\|Df^{-m} v\| \leq 1/2$ for each unitary $v \in E^a$ (one says $E^a$ is (uniformly) expanded),
- $\|Df^m u\| \leq 1/2 \|Df^m(x)\|$ for each $x \in \Lambda$, each $i = 0, \ldots, k$ and each unitary vectors $u \in E^s_i \oplus \cdots \oplus E^s_k, v \in E^c_i \oplus \cdots \oplus E^c_k$ in $T_x M$.

If all centre bundles are trivial, then $\Lambda$ is called a hyperbolic set. We say that a volume preserving diffeomorphism $f: M \to M$ is partial hyperbolic if $M$ is a partial hyperbolic set. If $M$ is a hyperbolic set then we say that $f$ is an Anosov diffeomorphism.
The main result of this paper is the following:

**Theorem A.** There is a residual subset $\mathcal{R} \subset \text{Diff}^1_m(M)$ ($\dim M \geq 2$) such that if $f \in \mathcal{R}$ is a non-partial hyperbolic diffeomorphism then $f$ has no symbolic extension.

**Remark 1.1.** In dimension two, the previous theorem follows from Downarowicz and Newhouse’s result [23].

Now, theorem A and the result of Diaz et al [21] provide a generic intrinsic characterization of the existence of symbolic extensions in the volume preserving scenario.

**Theorem B.** There exists a residual subset $\mathcal{R} \subset \text{Diff}^1_m(M)$ ($\dim M \geq 2$) such that a diffeomorphism $f \in \mathcal{R}$ has a symbolic extension if and only if it is partial hyperbolic. In particular, if $f \in \mathcal{R}$ has a symbolic extension then it has a principal extension.

A direct consequence of this result is the following.

**Corollary C.** If a $C^1$ generic volume preserving diffeomorphism $f$ is conjugated to a $C^\infty$ diffeomorphism, then $f$ is partial hyperbolic.

In the papers [16, 23] the main tool to obtain their results is the existence of an ‘abundance’ of diffeomorphisms exhibiting homoclinic tangency in the complement of Anosov diffeomorphisms, since they are in the symplectic scenario. Hence, the relation between robustness of homoclinic tangency and non-existence of symbolic extensions is somewhat folkloric.

A diffeomorphism $f \in \text{Diff}^1(M)$ (respectively $\text{Diff}^1_m(M)$) exhibits a $C^1$ robust homoclinic tangency if there exist a hyperbolic basic set $\Lambda$ of $f$ and a small neighbourhood $\mathcal{U} \subset \text{Diff}^1(M)$ (respectively $\text{Diff}^1_m(M)$) of $f$ such that $W^s(\Lambda(g))$ has a non-transversal intersection with $W^u(\Lambda(g))$ for every $g \in \mathcal{U}$. Where $\Lambda(g)$ is the continuation of $\Lambda$ for $g$.

**Proposition D.** If $f \in \text{Diff}^1(M)$ (respectively $\text{Diff}^1_m(M)$) exhibits a $C^1$ robust homoclinic tangency, then there exists a residual subset $\mathcal{R}$ in some neighbourhood of $f$ in $\text{Diff}^1(M)$ (respectively $\text{Diff}^1_m(M)$), such that every $g \in \mathcal{R}$ has no symbolic extensions.

As a consequence of this result we will obtain theorem A. For that, we should investigate the relation between robustness of homoclinic tangency and partial hyperbolicity. More general, we should investigate the existence of good dichotomies.

Recently, Crovisier et al [19] proved that diffeomorphisms in $\text{Diff}^1(M)$ far from diffeomorphisms exhibiting homoclinic tangency, are approximated by partial hyperbolic diffeomorphisms. There they affirm that this result is also true in the conservative setting. Just for sake of completeness we will state it here and a sketch of the proof will appear inside the proof of lemma 3.5, see remark 3.6.

**Proposition 1.2.** Any diffeomorphism $f$ can be approximated in $\text{Diff}^1_m(M)$ by diffeomorphisms which exhibit a homoclinic tangency or by partially hyperbolic diffeomorphisms.

We define the index of a hyperbolic periodic point $p$ as the dimension of its stable manifold and we denote it by $\text{ind } p$. A diffeomorphism $f$ exhibits a heterodimensional cycle if there exist hyperbolic periodic points $p$ and $q$ with different indices such that $W^s(p) \cap W^u(q)$ and $W^u(p) \cap W^s(q)$ are non-empty intersections.

One open problem about dichotomies is Palis’s conjecture, which says that densely in $\text{Diff}^r(M)$ ($r \geq 1$) either a diffeomorphism is hyperbolic, or exhibits a homoclinic tangency, or exhibits a heterodimensional cycle. Palis’s conjecture was proved for $C^1$ surface...
diffeomorphisms by Pujals and Sambarino [37], and recently Crovisier and Pujals proved a remarkable result in this direction, by means of essential hyperbolicity, for this result see [18]. For symplectic and volume preserving diffeomorphisms, there are also complete proofs for Palis’s conjecture, see [2, 34, 17].

We would like to remark that in the volume preserving case what was proved, in fact, is that in the lack of hyperbolicity there are densely diffeomorphisms exhibiting heterodimensional cycles. Our next theorem is a generalization of this result.

A diffeomorphism \( f \in \text{Diff}^1(M) \) (respectively \( \text{Diff}^1_m(M) \)) exhibits a \( C^1 \) robust heterodimensional cycle if there exist a hyperbolic basic set \( \Lambda \) and a hyperbolic periodic point \( p \) of \( f \), with \( \text{ind} \, \Lambda \neq \text{ind} \, p \), such that \( \Lambda(g) \) and \( p(g) \) exhibit a heterodimensional cycle, i.e. \( W^s(\Lambda(g)) \cap W^u(p(g)) \) and \( W^u(\Lambda(g)) \cap W^s(p(g)) \) are non-empty intersections, for every diffeomorphism \( g \) in a neighbourhood of \( f \) in \( \text{Diff}^1(M) \) (respectively \( \text{Diff}^1_m(M) \)).

**Theorem E.** There is an open and dense subset \( A \subset \text{Diff}^1_m(M) \) (\( \dim M \geq 3 \)), such that if \( f \in A \) is a non-Anosov diffeomorphism, then \( f \) exhibits a \( C^1 \) robust heterodimensional cycle.

The previous theorem is in fact Bonatti’s conjecture restricted to the volume preserving scenario. See [7].

Theorem E is a consequence that in the conservative setting blender sets appear in abundance in the lack of hyperbolicity, proposition 2.1. A blender set is a powerful tool developed by Bonatti and Diaz [6]. In a heuristical way this set allows to find diffeomorphisms exhibiting quasi-transversal intersections between stable and unstable manifolds of hyperbolic sets of different indices \( C^1 \)-persistently. Using blenders Bonatti and Diaz built examples of diffeomorphisms exhibiting robust heterodimensional cycles and they also found a large class of examples of non-hyperbolic robustly transitive diffeomorphisms.

It is worth pointing out that Hertz et al in [27] have already proved the existence of blenders for volume preserving diffeomorphisms near conservative diffeomorphisms having a pair of hyperbolic periodic points with co-index one. Hence, since the author has proved with Arbieto [2] the existence of such pair of hyperbolic periodic points for diffeomorphisms near non-Anosov volume preserving diffeomorphisms, theorem E could also be seen as a consequence of these results.

Now, in a remarkable work, Bonatti and Diaz [7] also developed a way to obtain diffeomorphisms exhibiting robust homoclinic tangencies from blenders. In this paper, we also develop their techniques in the conservative case to prove the following result:

**Theorem F.** There is an open and dense subset \( D \subset \text{Diff}^1_m(M) \) (\( \dim M \geq 3 \)) such that if \( f \in D \) is a non-partial hyperbolic diffeomorphism then \( f \) exhibits a robust homoclinic tangency.

Note, theorem A is a direct consequence of the previous theorem and proposition D.

It’s worth pointing out that it’s not known if theorem F is true in dimension two. Moreover, the existence of diffeomorphisms exhibiting a robust homoclinic tangency in dimension two and \( C^1 \)-topology is not known. See [32], for examples of such diffeomorphisms in higher topology. Hence, related to this fact, we remark that the Smale conjecture is still open in dimension two and \( C^1 \)-topology. That is, hyperbolic diffeomorphisms are dense in \( \text{Diff}^1(M) \).

This paper is organized as follows: in the second section we recall some useful perturbation results, and we show how to build a special kind of blender in the volume preserving setting. This special kind of blender is a blender horseshoe introduced in [7]. In this section we will also prove theorem E. In section 3, we prove theorem F, and finally, in section 4, we prove proposition D and theorem A.
2. A blender horseshoe and some perturbation results

Let \( f \) be a \( C^1 \) diffeomorphism on \( M \). A hyperbolic transitive set \( \Gamma \) of \( f \) with \( \text{dim} \ W^u(\Gamma, f) = k \geq 2 \) is a \textit{cu-blender} if there exist a \( C^1 \)-neighbourhood \( \mathcal{U} \) of \( f \) and a \( C^1 \)-open set \( \mathcal{D} \) of embeddings of \((k-1)\)-dimensional discs \( D \) into \( M \) such that, for every diffeomorphism \( g \in \mathcal{U} \), every disc \( D \in \mathcal{D} \) intersects the local stable manifold \( W^s_{loc}(\Gamma(g)) \), where \( \Gamma(g) \) is the continuation of the hyperbolic set \( \Gamma \) for \( g \). \( \mathcal{D} \) is called the superposition region of the blender. Similarly, we can define a \( cs \)-blender with stable manifold replaced by unstable manifold.

The above definition was given in [7]. We would like to remark, that for a \( cs \)-blender, these \((k-1)\)-dimensional discs are usually \( uu \)-discs. See remark 2.13. We refer the reader to [6, 7] for more details about the geometry structure of this amazing set.

The main result of this section is the following.

**Proposition 2.1.** If \( f \in \text{Diff}^1_m(M) \) has two hyperbolic periodic points \( p_1 \) and \( p_2 \) of different indices, say \( i \) and \( i + j \), respectively, then for any neighbourhood \( \mathcal{U} \subset \text{Diff}^1_m(M) \) of \( f \) and any \( i \leq k \leq i + j - 1 \) there exists a diffeomorphism \( g \in \mathcal{U} \) having a \textit{cu-blender horseshoe} \( \Gamma \) with index \( k \).

**Remark 2.2.** A \( cu \)-blender horseshoe is a special kind of a \( cu \)-blender set, which will be defined in the proof of proposition 2.1.

**Remark 2.3.** A similar result still holds for \( cs \)-blenders. More precisely, if \( f \) is under the same hypotheses of proposition 2.1, then in any neighbourhood of \( f \) there exists a diffeomorphism \( g \) having a \( cs \)-blender horseshoe of index \( k \), for any \( i + 1 \leq k \leq i + j \).

**Remark 2.4.** As we have also remarked in the introduction, Hertz et al showed in [27] the existence of blenders in the conservative scenario. However, we observe that here the hypotheses are weaker and moreover we are interested in finding a special kind of blender. Also, we emphasize that our methods to prove proposition 2.1 are simpler. More precisely, we use a connecting lemma to find a strong homoclinic intersection while they use a heterodimensional cycle of co-index one like Bonatti and Diaz, see theorem 2.3 in [27].

We will recall now some useful perturbation results.

The first one is a pasting lemma of Arbieto and Matheus [3].

**Theorem 2.5 (Pasting lemma).** If \( f \) is a \( C^2 \) volume preserving diffeomorphism over \( M \), and \( x \in M \), then for every \( \varepsilon > 0 \) there exists a \( C^1 \) volume preserving diffeomorphism \( g \in \text{Diff}^1_M \) close to \( f \) such that for small neighbourhoods of \( x \), \( V \subset \mathcal{U} \), \( g|\mathcal{U}^c = f \) and \( g|V = Df(x) \) (in local coordinates).

**Remark 2.6.** If \( f \) is \( C^\infty \) then \( g \) could be taken \( C^\infty \), too.

A directly consequence of pasting lemma is a conservative version of Frank’s lemma, see [29].

**Lemma 2.7 (Frank’s lemma).** Let \( f \in \text{Diff}^1_m(M) \) and \( \mathcal{U} \) be a \( C^1 \) neighbourhood of \( f \) in \( \text{Diff}^1_m(M) \). Then, there exists a smaller neighbourhood \( \mathcal{U}_0 \subset \mathcal{U} \) of \( f \) and \( \delta > 0 \) such that if \( g \in \mathcal{U}_0(f) \), \( S = \{x_1, \ldots, x_m\} \subset M \) be any finite piece of orbit and \( \{L_i : T_{x_i}M \to T_{x_{i+1}}M\}_{i=1}^m \) conservative linear maps satisfying \( \|L_i - Dg(x_i)\| \leq \delta \) for every \( i = 1, \ldots, m \), then for any small fixed neighbourhood \( V \) of \( S \) there exist \( h \in \mathcal{U}(f) \) in the same class of differentiability of \( g \), such that \( h = g \) in \( V^c \), moreover \( h(x_i) = g(x_i) \) and \( Dh(x_i) = L_i \).

The next result is a connecting lemma of Hayashi [26]. A conservative version was proved by Wen and Xia [42].
Theorem 2.8 (C⁴-connecting lemma). Let \( f \in \text{Diff}^1_m(M) \) and \( p_1, p_2 \) hyperbolic periodic points of \( f \), such that there exist sequences \( y_n \in M \) and positive integers \( k_n \) such that:

- \( y_n \rightarrow y \in W^s_{loc}(p_1, f) \), \( y \neq p_1 \); and
- \( f^{k_n}(y_n) \rightarrow x \in W^s_{loc}(p_2, f) \), \( x \neq p_2 \).

Then, there exists a \( C^4 \) volume preserving diffeomorphism \( g \) \( C^4 \)-close to \( f \) such that \( W^s(p_1, g) \) and \( W^u(p_2, g) \) have a non-empty intersection close to \( y \).

The following technical result will be needed in the proof of proposition 2.1.

Lemma 2.9. If \( f \in \text{Diff}^1_m(M) \) has two hyperbolic periodic points \( p_1 \) and \( p_2 \) of different indices, say \( i \) and \( i + j \), respectively, then for any neighbourhood \( U \subset \text{Diff}^1_m(M) \) of \( f \), \( i \leq k \leq i + j - 1 \) and \( \varepsilon > 0 \) there exists a diffeomorphism \( g \in U \) with a hyperbolic periodic point \( p \), such that \( p \) has index \( k \), \( Dg^{\tau(p,g)} \) has only real eigenvalues with multiplicity one, say \( \lambda_1 < \ldots < \lambda_d \), and moreover \( |\lambda_{k+1} - 1| < \varepsilon \). Where \( \tau(p,g) \) denotes the period of \( p \).

This lemma follows by the same method as in proposition 3.2 in [2]. However, provided this method will be useful later we will give a sketch of the proof. It is worth pointing out, as commented in [2], that this result is similar to results in [1].

Before we prove the above lemma, let us recall some definitions.

Recall, two hyperbolic periodic points \( p \) and \( q \), having the same index are homoclinically related if there exist a transversal intersection between \( W^s(p, f) \) and \( W^u(q, f) \), and \( W^u(p, f) \) and \( W^s(q, f) \). We denote by \( H(p, f) \) the closure of the hyperbolic periodic points homoclinically related to \( p \), which is called the homoclinic class of \( p \). We say that a homoclinic class is non-trivial if contains more than one saddle. Similarly, we can define the homoclinic relation between hyperbolic periodic points and hyperbolic sets.

A continuous \( Df \)-invariant splitting \( T_{\Lambda}M = E_1 \oplus \cdots \oplus E_k \) for a \( f \)-invariant subset \( \Lambda \) is dominated if the third condition in the partial hyperbolic definition is satisfied.

For abbreviation, sometimes we use expressions such as ‘after a perturbation’, or ‘there exists a diffeomorphism \( C^4 \)-close’, which mean that these perturbations could be done as small as we desire.

Proof of lemma 2.9. By a result of Xia [40], a generic volume preserving diffeomorphism has all homoclinic classes non-trivial. Thus, after a perturbation, we can assume \( H(p_1, f) \) and \( H(p_2, f) \) are non-trivial. Now, by results of Bonatti et al [8], and Frank’s lemma we can perturb \( f \) to \( f_1 \) in order to obtain \( \tilde{p}_1 \) and \( \tilde{p}_2 \) hyperbolic periodic points homoclinically related to \( p_1(f_1) \) and \( p_2(f_1) \), respectively, such that \( Df_{1}^{\tau(\tilde{p}_1,f_1)}(\tilde{p}_1) \) and \( Df_{1}^{\tau(\tilde{p}_2,f_1)}(\tilde{p}_2) \) have only real eigenvalues with multiplicity one.

To simplify the notation we replace \( \tilde{p}_1 \) and \( \tilde{p}_2 \) by \( p_1 \) and \( p_2 \), respectively. And moreover, we will continue to write \( p_1 \) and \( p_2 \) for their continuations.

In the sequence we perturb \( f_1 \) in order to find a diffeomorphism exhibiting a heterodimensional cycle between \( p_1 \) and \( p_2 \). For that, we use a result of Bonatti and Crovisier [5]:

Proposition 2.10 (Bonatti and Crovisier). There exists a residual subset \( \mathcal{R} \) of \( \text{Diff}^1_m(M) \) such that if \( g \in \mathcal{R} \) then there exists a hyperbolic periodic point \( p \) of \( g \) such that \( M = H(p, g) \). In particular, \( g \) is transitive.

After a perturbation, we can assume \( f_1 \in \mathcal{R} \), i.e., \( f_1 \) is transitive. Then, using a connecting lemma we can perturb \( f_1 \) to \( f_2 \) such that there is a transversal intersection between \( W^u(p_1, f_2) \) and \( W^s(p_2, f_2) \) (see proof of theorem D in [42] for details of this kind of perturbation). Since this intersection is robust, we can repeat the above process and perturb \( f_2 \) to \( f_3 \) such
that $W'(p_1, f_3)$ and $W''(p_2, f_3)$ have also a non-empty intersection, which implies that $f_3$

exhibits a heterodimensional cycle between $p_1$ and $p_2$. Moreover, $f_3$ can be taken such that

$Df^3_{|E_{1,p_1}}(p_1)$ and $Df^3_{|E_{2,p_2}}(p_2)$ have only real eigenvalues with multiplicity one.

Let $x \in W'(p_1, f_3) \cap W''(p_2, f_3)$ and $y \in W'(p_1, f_3) \cap W''(p_2, f_3)$ be two heteroclinic

points of the cycle. Recall $\text{ind } p_1 = i$ and $\text{ind } p_2 = i + j$. Without loss of generality we can

assume $y$ is a transversal heteroclinic point, and $x$ is a quasi-transversal heteroclinic point, i.e.

$T_x W'(p_1, f_3) \cap T_y W''(p_2, f_3) = \{0\}$. By the regularization result of Ávila [4] (which says we

can suppose $f_3$ to be $C^\infty$) and the pasting lemma, we can linearize the diffeomorphism in a

small neighbourhood $U_{p_1}$ and $U_{p_2}$ of $p_1$ and $p_2$, respectively. More precisely, we can assume

$f_3$ is equal to $Df_3(p_1)$ and $Df_3(p_2)$ (in local coordinates) in the neighbourhoods $U_{p_1}$ and $U_{p_2}$,

respectively.

For simplicity of notation, in the remainder of this proof we assume that $p_1$ and $p_2$ are fixed

points, and we will look at $U_{p_1}$ and $U_{p_2}$ in local coordinates. Since $Df_3(p_1)$ and $Df_3(p_2)$ have

only real eigenvalues with multiplicity one, we can find a decomposition of $\mathbb{R}^d$ by eigenspaces

of $Df_3(p_1)$ (respectively $Df_3(p_2)$), which we denote by $E_{1,p_1} \oplus \cdots \oplus E_{d,p_1}$ (respectively

$E_{p_2} \oplus \cdots \oplus E_{d,p_2}$). We set $\lambda_k$ (respectively $\alpha_i$), $k = 1, \ldots, d$, the eigenvalue of

$Df_3(p_1)$ of $E_{1,p_1}$ (respectively $Df_3(p_2)$ of $E_{p_2}$). We can also suppose the eigenvalues are in an increasing order.

In order to be more precise, we will make the following assumptions, we consider $E_{1,p_1}(\cdot)$

the extension of the direction $E_{1,p_1}$ in the neighbourhood $U_{p_1}$, the same for $E_{1,p_2}(\cdot)$. We remark

these decompositions are all dominated splittings, indeed.

Since $\text{ind } p_1 = i$, it follows that the stable and unstable directions of $p_1$ are $E_{i,p_1} = E_{1,p_1}(p_1) \oplus \cdots \oplus E_{i,p_1}(p_1)$ and $E_{p_1}(p_1) = E_{1,p_1}(p_1) \oplus \cdots \oplus E_{d,p_1}(p_1)$, respectively. Similarly, the stable and unstable directions of $p_2$ are $E_{p_2} = E_{1,p_2}(p_2) \oplus \cdots \oplus E_{i+p_2}(p_2)$ and $E_{p_2}(p_2) = E_{1+p_2}(p_2) \oplus \cdots \oplus E_{d,p_2}(p_2)$, respectively, since $\text{ind } p_2 = i + j$.

Now, if $U$ is a small enough neighbourhood of a hyperbolic periodic point $p$ of $f$ we can define the local stable and unstable manifolds of $p$ in $U$ as $W'_U(p, f) = \{x : \text{such that } f^n(x) \in U \text{ for every } n \geq 0\}$ and $W''_U(p, f) = \{x : \text{such that } f^n(x) \in U \text{ for every } n \leq 0\}$, respectively.

Hence, by the choice of $f_3$, $W'_{loc}(p_1, f_3) = E_{p_1} \cap U_{p_1}$, $W''_{loc}(p_1, f_3) = E_{p_1} \cap U_{p_1}$,

$W'_{loc}(p_2, f_3) = E_{p_2} \cap U_{p_2}$ and $W''_{loc}(p_2, f_3) = E_{p_2} \cap U_{p_2}$.

Claim. There is a diffeomorphism $f_3 \in C^1$-close to $f_3$ such that the $f_3$-invariant subset

$\Lambda = O(x) \cup O(y) \cup \{p_1, p_2\}$ still is $f_3$-invariant and moreover has a dominated splitting

by one-dimensional sub-bundles.

We define $E(y) := T_y (W''(p_1, f_3) \cap W'(p_2, f_3))$. Since $y$ belongs to unstable manifold

of $p_1$ and $f_3|U_{p_1} = Df_3(p_1)$, if $n$ is large enough, it follows that $Df^{3n}(y)(E(y))$ is in

$E_{1+p_1}(f^n_{3}(y)) \oplus \cdots \oplus E_{d,p_1}(f^n_{3}(y))$. Moreover, by transversality we can assume that

$Df^{3n}(y)(E(y)) \cap E_{1+p_1}(f^{3n}(y)) \oplus \cdots \oplus E_{d,p_1}(f^{3n}(y)) = \{0\}$. Provided we have a dominated splitting in $U_{p_1}$, $Df^{3n}(y)(E(y))$ converges to $E_{1+p_1}(p_1) \oplus \cdots \oplus E_{1+p_1}(p_1)$ when $n \to \infty$. Then, choosing $n$ large enough and using Frank’s lemma, after a perturbation we can assume $f_3$ such that $Df^{3n}(y)(E(y)) = E_{1+p_1}(f^{3n}(y)) \oplus \cdots \oplus E_{1+p_1}(f^{3n}(y))$. Note, the perturbation necessary here is local, and moreover keeps unchanged the orbit of $y$.

We now apply this argument again, considering the future orbit of $y$, to obtain a perturbation of $f_3$ such that we have also $Df^{3n}_3(y)(E(y)) = E_{1+p_1}(f^{3n}_3(y)) \oplus \cdots \oplus E_{1+p_1}(f^{3n}_3(y))$. This perturbation of $f_3$ which we continue denoting by the same letter has a $Df_3$-invariant sub-bundle on $O(y) \cup \{p_1, p_2\}$ which we will denote by $E$, for convenience.
As before, we have that $Df_3^m(f_3^{-n}(y))(E_{i+j+1,p_1}(f_3^{-n}(y)) \oplus \cdots \oplus E_{d,p_1}(f_3^{-n}(y)))$ converges to $E_{i+j+1,p_1}(p_2) \oplus \cdots \oplus E_{d,p_1}(p_2)$ if $m \to \infty$, since we have a dominated splitting in $U_{p_2}$. Then by the same argument again we can perturb $f_3$ such that $Df_3^m(f_3^{-n}(y))(E_{i+j+1,p_1}(f_3^{-n}(y)) \oplus \cdots \oplus E_{d,p_1}(f_3^{-n}(y))) = E_{i+j+1,p_1}(f_3^{-m-n}(y)) \oplus \cdots \oplus E_{d,p_1}(f_3^{-m-n}(y))$, and the sub-bundle $E$ is still $Df$-invariant. Replacing $m$ and $n$ by large positive integers if necessary, and applying once more the argument, $f_3$ could be assumed such that $Df_3^m(f_3^{-n}(y))(E_{1,p_1}(f_3^{-m-n}(y)) \oplus \cdots \oplus E_{i,p_1}(f_3^{-m-n}(y)) = E_{1,p_1}(f_3^{-n}(y)) \oplus \cdots \oplus E_{i,p_1}(f_3^{-n}(y)).$

Therefore, $f_3$ is such that there exists a $Df_3$-invariant splitting over $O(y) \cup \{p_1, p_2\}$.

Moreover, if we repeat this process finitely many times inside each invariant sub-bundle, $f_3$ could be assumed such that

$$Df_3^m(f_3^{-n}(y))(E_{k,p_1}(f_3^{-n}(y))) = E_{k,p_1}(f_3^j(y)), \quad k = 0, \ldots, d; \quad \text{for } n \text{ large enough.}$$

Finally, applying the above arguments, with $y$ replaced by $x$, $f_3$ can also be chosen such that

$$Df_3^m(f_3^{-n}(x))(E_{k,p_1}(f_3^{-n}(x))) = E_{k,p_1}(f_3^j(x)), \quad k = 0, \ldots, d; \quad \text{for } n \text{ large enough,}$$

which finishes the proof of the claim, since this $Df$-invariant splitting is natural dominated.

We fix now an arbitrary $i \leq k \leq i+j-1$, and we consider the diffeomorphism $f_3$ given by the previous claim. Using the heteroclinic points $x$ and $y$, we can perform a perturbation of $f_3$ to obtain a periodic orbit in a small neighbourhood of $\Lambda$, with an arbitrary large period. In fact, this could be done such that this periodic orbit has as many points as we want in the neighbourhoods $U_{p_1}$ and $U_{p_2}$, being fixed the number of points outside these neighbourhoods. See figure 1. Hence, by continuity of the dominated splitting over $\Lambda$, and since $\|Df_3|E_{k+1}(p_1)\| > 1$ and $\|Df_3|E_{k+1}(p_2)\| < 1$, it follows there exists a diffeomorphism $f_3$ $C^1$-close to $f_3$ having a hyperbolic periodic point $p$ in a neighbourhood of $\Lambda(f_3)$ with index $k$ and such that $Df_3^{\tau(p,f_3)}(p)$ has only real eigenvalues with multiplicity one. Moreover, this could be done so that $\|Df_3^{\tau(p,f_3)}|E_{k+1}(p)\|$ is as close to one as we desire. For details we refer the reader to [2].

□
Proof of proposition 2.1. We fix an arbitrary \( i \leq k \leq i + j - 1 \). By lemma 2.9, after a perturbation, we can assume there exists a hyperbolic periodic point \( p \) of \( f \) such that \( p \) has index \( k \), \( Df^{\tau(p,f)}(p) \) has only real eigenvalues with multiplicity one, say \( \lambda_1 < \cdots < \lambda_d \), and moreover \( \lambda_{k+1} \) is so close to one as we want.

If \( E_{\lambda_i} \) is the corresponding eigenspace to \( \lambda_i \), then we have on \( p \) a natural partially hyperbolic splitting \( T_pM = E^s \oplus E^c_u \oplus E^u \), where \( E^s = \bigcup_{1 \leq i \leq k} E_{\lambda_i} \) is the stable direction of dimension \( k \), and the unstable direction is divided in two subspaces, \( E^c_u = E_{\lambda_{k+1}} \) (the centre-unstable direction), and \( E^u = \bigcup_{i > k+1} E_{\lambda_i} \) (the strong unstable direction). By Hirsch et al [28], the strong directions are integrable, which means here the existence of a local invariant submanifold \( W^u_{loc}(p,f) \), the local strong unstable manifold of \( p \), which varies \( C^1 \)-continuously with respect to the diffeomorphism and so that \( T_pW^u_{loc}(p,f) = E^u \). We define the strong unstable manifold by \( W^u(p,f) = \bigcup_{n \in \mathbb{N}} f^n(W^u_{loc}(p,f)) \).

As we did in the proof of lemma 2.9 in order to find a heterodimensional cycle, we can perform finitely many \( C^1 \) perturbations of \( f \), using proposition 2.10, a connecting lemma, the regularization result of Ávila and the pasting lemma, to find a \( C^1 \) diffeomorphism \( f_1 \) such that:

1. \( W^s(p(f_1), f_1) \cap W^u(p(f_1), f_1) \neq \emptyset \), and
2. \( f_1^{\tau(p(f_1),f_1)} = Df_1^{\tau(p(f_1),f_1)}(p(f_1)) \) (in local coordinates).

Now, after a local perturbation, if necessary, by Frank’s lemma we can assume this strong homoclinic intersection is quasi-transversal. More precisely, for a point \( x \), let \( \dim T_xW^s(p(f_1), f_1) + T_xW^u(p(f_1), f_1) = d - 1 \) since \( \dim E^c_u(p(f_1)) = 1 \).

By abuse of notation, we write just \( p \) instead of \( p(f_1) \). Also, since \( \| Df^{\tau(p,f)}(p)E^c_u \| \) is as close to one as we desire, after another perturbation, we can suppose \( |\tilde{\lambda}_c| = \| Df^{\tau(p,f)}(p)E^c_u \| = 1 \).

From now on, we look at \( U \) in local coordinates. Then, in \( U \) the local stable and strong unstable manifolds of \( p \) coincide with their directions, i.e., \( W^s_{loc}(p,f) = E^s(p,f) \cap U \) and \( W^u_{loc}(p,f) = E^u(p,f) \cap U \).

Let \( x \in W^s(p,f_1) \cap W^u(p,f_1) \) be a quasi-transversal strong homoclinic point of \( p \). Hence, there exist positive integers \( n \) and \( m \) such that \( f_1^n(x) = (x_0^n, 0, 0) \) and \( f_1^m(x) = (0, 0, x_0^m) \in U \). Here, we are considering the natural extension to \( U \) of the partial hyperbolic splitting \( T_pM = E^s \oplus E^c_u \oplus E^u \). Also, without loss of generality we can suppose this decomposition orthogonal.

By the same method as in the claim in the proof of lemma 2.9, we can find a diffeomorphism \( f_2 C^1 \)-close to \( f_1 \), such that shrinking \( U \) if necessary \( f_2 \) satisfies the following conditions.

1. \( f_2^{\tau(p,f_2)} = Df_2^{\tau(p,f_2)}(p) = Df_1^{\tau(p,f_1)}(p) \) in \( U \), keeping invariant the directions \( E^j \cap U \), \( j = s, cu, uu \);
2. \( x \) is still a strong homoclinic point of \( p \), and moreover

\[
Df_2^{m,n}(f_2^{-m}(x))(E^j(f_2^{-m}(x))) = E^j(f_2^m(x)), \quad j = s, cu, uu.
\]

Provided \( f_2 \) is obtained through finitely many perturbations of \( f_1 \) using Frank’s lemma, \( f_2 \) is in the same class of differentiability of \( f_1 \), which implies \( f_2 \) is \( C^\infty \). Hence, we can use the pasting lemma in order to linearize \( f_2 \) in a segment of the orbit of \( x \). More precisely, we can choose neighbourhoods \( U_m, U_n \subset U \) of \( f_2^{-m}(x) \) and \( f_2^n(x) \), respectively, and perturb \( f_2 \) to \( f_3 \) such that \( f_3^{-m}(E^j(y) \cap U_m) = E^j(f_3^m(y)) \cap U_n \), for every \( y \in U_m \) and \( j = s, cu, uu \).

Using the fact that \( f_2^{\tau(p,f_2)} \) is linear in \( U \) and \( \tilde{\lambda}_c = 1 \), replacing \( m \) and \( n \) with larger ones, and after once more perturbation, we can suppose \( f_3 \) satisfying
(3) $f_{3}^{mn}: U_m \rightarrow U_n$ is an affine map. More precisely,

$$f_{3}^{mn}(x', x^c, x^u) = (x'_0 + A_\varepsilon(x'), \lambda_c x^c, A_u(x^u - x'_0)),$$

where $A_\varepsilon$ is a linear contraction, $A_u$ a linear expansion and $1 < \lambda_c < 1 + \varepsilon$, for some small $\varepsilon > 0$.

(4) $E^s \oplus E^{uu}$ is invariant for both maps $f_{3}^{(p,f_t)}|U$ and $f_{3}^{mn}|U_m$.

Hence, if $D \subset (E^s \oplus E^{uu}) \cap U$ is a small enough rectangle containing $p$ and $f_{3}^{(p,f_t)}(x)$ in its interior, then $f_{3}^{(p,f_t)+mn}(D) \cap D$ has two non-empty disjoint connected components for some $l$ large enough, one of them containing $p$ and another one $f_{3}^{l}(x)$, which we denote by $A$ and $B$, respectively.

For simplicity of notation, we set $\tilde{F} = f_{3}^{l(p,f_t)+mn}|D$, $\tilde{A} = \tilde{F}^{-1}(A)$ and $\tilde{B} = \tilde{F}^{-1}(B)$.

Note, $\tilde{F}$ is a linear map on $\tilde{A} \cup \tilde{B}$, and the stable and strong unstable directions are $\tilde{F}$-invariant. Moreover, taking $l$ larger if necessary $\tilde{F}|E^s$ and $\tilde{F}^{-1}|E^{uu}$ are contractions, for every point in $\tilde{A} \cup \tilde{B}$, and $\tilde{A} \cup \tilde{B}$, respectively. Hence, the maximal invariant set in $D$ for $\tilde{F}$,

$$\Sigma = \bigcap_{n \in \mathbb{Z}} \tilde{F}^n(D)$$

is a hyperbolic set conjugated to the full shift of two symbols. We denote by $q \in \tilde{B}$ the other fixed point of $\tilde{F}$. Note, $E^s \oplus E^{uu}$ is the hyperbolic splitting over $\Sigma$.

Fixing any arbitrary small $\delta > 0$, we set $R = D \times [-\delta, \delta] \subset U$, and replace $\tilde{A}$ and $\tilde{B}$ by $\tilde{A} \times [-\delta, \delta]$ and $\tilde{B} \times [-\delta, \delta]$, respectively. Taking $\delta$ smaller, $F := f_{3}^{l(p,f_t)+mn}|A \cup B$ is then well defined. Moreover, taking the centre coordinate as the last one, we have

$$F(x', x^c, x^u) = (\tilde{F}(x', x^u), \lambda_c x^c).$$

Since $\lambda_c > 1$, it follows that $\Lambda_0 = \Sigma \times 0$ is the maximal $F$-invariant set in $R$. Also, provided $E^s$, $E^{uu}$ and $E^{uu}$ are $F$-invariant, we have a natural partial hyperbolic splitting on $\Lambda_0$. In particular, $\Lambda_0$ is a hyperbolic set with index $k$ since $\|F|E^{uu}\| > 1$.

After a coordinate change, we can suppose $R = [-1, 1]^{l} \times [-1, 1] \times [-1, 1]^{u}$ in local coordinates, and $p = (0, 0, 0)$ in this chart.

For every $t > 0$ small enough, using the pasting lemma we can find a perturbation $h_t$ of the identity map such that $h_t(x', x^c, x^u) = (x', x^c - t, x^u)$ for every point in $\Lambda_0$, and $h_t = Id$ outside a small neighbourhood of $U_0$. We define $f_t = h_t \circ f_3$, which is $C^1$-close to $f_3$.

Shrinking $U_0$ if necessary, the above perturbation $f_t$ in terms of $F_t$ is the following

(1) $F_t = F$, \hspace{1cm} if $x \in \tilde{A}$

(2) $F_t = F + (0, -t, 0)$, \hspace{1cm} if $x \in \tilde{B}$.

Provided $t$ is small, the maximal $F_t$-invariant set $\Lambda_t$ in $R$ is the continuation of the hyperbolic set $\Lambda_0$ of $F$, hence $\Lambda_t$ is also hyperbolic. Moreover, note $E^s \oplus E^{uu} \oplus E^{uu}$ is still the hyperbolic splitting on $\Lambda_t$, and $p$ is still a hyperbolic fixed point of $F_t$. We denote by $q_t$ the continuation of the hyperbolic fixed point $q$ of $F$.

This set $\Lambda_t$ is defined as a cu-blender horseshoe.

In the sequence, we will describe some properties of $\Lambda_t$ which characterize, in fact, a cu-blender horseshoe.

For $\alpha \in (0, 1)$ we denote by $C_{\alpha}^{s}$ and $C_{\alpha}^{uu}$ the following cone-fields in $R$:

$$C_{\alpha}^{s}(x) = \{ v = (v', v^c, v^u) \in E^s \oplus E^{uu} \mid \|v' + v^u\| \leq \alpha \|v^u\| \},$$

$$C_{\alpha}^{uu}(x) = \{ v = (v', v^c, v^u) \in E^{uu} \oplus E^{uu} \mid \|v' + v^u\| \leq \alpha \|v^u\| \}.$$
Proof of theorem E. Let $f$ be a non-Anosov volume preserving diffeomorphism. By theorem 1.1 in [2], there exists a diffeomorphism $f_1 \in \text{Diff}^+_C(M)$ $C^1$-close to $f$ having a non-hyperbolic periodic point $p$. After a bifurcation of $p$ we can assume that $f_1$ has two hyperbolic periodic points of different indices, say $p_1$ and $p_2$, with $\text{ind } p_1 = i$ and $\text{ind } p_2 = i+j$, $i, j > 0$. 

We now prove theorem E.
By proposition 2.1 we can find a volume preserving diffeomorphism \( f_2 \in C^1 \)-close to \( f_1 \) such that \( f_2 \) has a blender horseshoe \( \Lambda \) with index \( i + j - 1 \). We replace now \( p_1 \) by one of the two reference saddles of \( \Lambda \).

As in the proof of proposition 2.1, we can perturb \( f_2 \) to \( f_3 \) to obtain a heterodimensional cycle between \( p_1 \) and \( p_2 \). Let \( z \) denote a point of non-transversal intersection between \( W^s(p_1, f_3) \) and \( W^u(p_2, f_3) \), which we can assume to be a quasi-transversal intersection. Provided the partial hyperbolic structure in the superposition region \( C \) of the blender, replacing \( z \) by a positive iterated, the connected disc in \( W^u(p_2, f_3) \cap C \) containing \( z \) is in fact a \( uu \)-disc which is in between of the two reference saddles of \( \Lambda \), as defined in the proof of proposition 2.1. Note, this could be done such that \( W^u(p_1, f_3) \cap W^s(p_2, f_3) \) has a transversal intersection.

Therefore, by the properties of blenders, remark 2.13, and the continuity of the unstable manifold of \( p_2 \), every volume preserving diffeomorphism \( g \) in a small neighbourhood of \( f_3 \) has a heterodimensional cycle between \( p_2(g) \) and \( \Lambda(g) \).

\[ \square \]

3. Robustness of homoclinic tangency

In this section we prove theorem F. For that, it will be necessary to introduce folded submanifolds, introduced by Bonatti and Diaz [7].

**Definition 3.1.** Let \( f \) be a diffeomorphism on \( M \) having a blender-horseshoe set \( \Lambda \) of index \( u + 1 \) with reference cube \( C \), reference saddles \( p \) and \( q \), and \( N \subset M \) be a submanifold of dimension \( u + 1 \). We say that \( N \) is folded with respect to \( \Lambda \) if the interior of \( N \) contains a submanifold \( S \subset C \cap N \) of dimension \( u + 1 \), satisfying the following properties:

- There are \( 0 < \alpha' < \alpha \) and a family \( (S_t)_{t \in [0,1]} \) of discs tangent to the cone field \( C^uu \), depending continuously on \( t \), such that \( S = \bigcup_{t \in [0,1]} S_t \). Here, \( \alpha \) comes from the definition of a blender horseshoe, in particular \( S_t \) is a \( uu \)-disc;
- \( S_0 \cap W^s_{loc}(A) \) and \( S_1 \cap W^u_{loc}(A) \) are non-empty transverse intersection points between \( N \) and \( W^s_{loc}(A) \), where \( A \in \{p, q\} \);
- for every \( t \in (0, 1) \), the \( uu \)-disc \( S_t \) is in between of \( W^s_{loc}(p) \) and \( W^u_{loc}(q) \).

To emphasize the reference saddle \( \Lambda \) of the blender we have considered, we say a submanifold \( N \) is folded with respect to \( (\Lambda, A) \).

**Theorem 3.2 (Theorem 2, p 18, [7]).** Let \( f \) be a \( C^r \) (\( r \geq 1 \)) diffeomorphism over \( M \), and \( N \subset M \) be a folded submanifold with respect to a blender-horseshoe \( \Lambda \) of \( f \). Then \( N \) and \( W^s_{loc}(\Lambda) \) have a non-empty \( C^r \)-robust non-transversal intersection.

To prove theorem F the following results will also be needed.

Firstly, Wen [41] has proved in a dichotomy between diffeomorphisms having a dominated splitting and diffeomorphisms exhibiting a homoclinic tangency in the space of \( C^1 \) diffeomorphisms. Using the pasting lemma, Liang et al [29] proved this dichotomy also in the volume preserving scenario, theorem 1.3 in [29]. This, together with the fact that generically in the conservative setting the whole manifold is a homoclinic class, proposition 2.10, and the two homoclinic classes are either disjoint or equal, see [15], we have directly the following result.

**Proposition 3.3.** There exists a residual subset \( \mathcal{R} \subset \text{Diff}_m^1(M) \), such that if \( f \in \mathcal{R} \), then we have the following dichotomy:

1. Either the homoclinic class of \( p \) is the whole manifold, \( H(p, f) = M \), and has a dominated splitting \( TM = E \oplus F \), with \( \dim E = \text{ind}(p) \), for every hyperbolic periodic point \( p \) of \( f \).
(2) or there exists a positive integer \(1 \leq k \leq \dim M\) such that for any hyperbolic periodic point \(p\) of index \(k\) there exists a diffeomorphism \(g\) \(C^1\)-close to \(f\) exhibiting a homoclinic tangency for \(p(g)\).

It is worth pointing out that a stronger result than the previous proposition should be true. More precisely, Gourmelon’s result [25] should be true in the conservative setting. He proved that if the homoclinic class of a hyperbolic periodic point \(p\) has no dominated splitting of same index as \(p\), then after a \(C^1\) perturbation we can create a homoclinic tangency to \(p\).

The following result is a conservative version of theorem 1 in [1]. It may be proved using the same arguments as in lemma 2.9, see [2] for details.

**Proposition 3.4.** There is a residual subset \(\mathcal{R} \in \text{Diff}_m^1(M)\) of diffeomorphisms \(f\) such that, for every \(f \in \mathcal{R}\) containing hyperbolic periodic points of indices \(i\) and \(j\) contains hyperbolic periodic points of index \(k\) for all \(1 \leq k \leq j\).

**Proof of theorem F.** Let \(f\) be a volume preserving diffeomorphism which is not approximated by a partial hyperbolic diffeomorphism in \(\text{Diff}_m^1(M)\). In particular, \(f\) is a non-Anosov diffeomorphism, and then after a perturbation if necessary as in the proof of theorem E, we can assume \(f\) has hyperbolic periodic points of different indices.

We set \(i\) the smallest positive integer such that for any neighbourhood \(\mathcal{V}\) of \(f\) every hyperbolic periodic point \(p\) of any diffeomorphism \(g \in \mathcal{V}\) has \(\text{ind } p \geq i\). Thus, given a small neighbourhood \(\mathcal{V}\) of \(f\), we can suppose \(f\) has a hyperbolic periodic point \(p_1\) with index \(i\), after a perturbation if necessary. Now, we set \(j\) the largest positive integer such that for any neighbourhood \(\mathcal{V}_1 \subset \mathcal{V}\) of \(f\) every hyperbolic periodic point \(p\) of any diffeomorphism \(g \in \mathcal{V}\) has \(\text{ind } p \leq i + j\). Hence, if we consider \(\mathcal{V}_1\) small enough such that the hyperbolic periodic point \(p_1\) persists, then after a perturbation we can also suppose \(f\) has a hyperbolic periodic point \(p_{i+j}\) with index \(i + j\). Moreover, by construction every hyperbolic periodic point \(p\) of any diffeomorphism \(g \in \mathcal{V}_1\) has \(\text{ind } p \leq i + j\).

Now, by proposition 3.4 we can assume \(f\) such that there are hyperbolic periodic points of index \(k\), for every \(1 \leq k \leq i + j\). Hence, there are \(q_0, q_1, \ldots, q_j\) hyperbolic periodic points of \(f\) of indices \(i, i + 1, \ldots, i + j\), respectively. Also, by proposition 2.1 we can find a diffeomorphism \(f_1\) \(C^1\)-close to \(f\), which is not approximated by partial hyperbolic diffeomorphisms, such that there exist blender-horseshoe subsets \(\Lambda_k\) with \(\text{ind } \Lambda_k = i + k\), for every \(k = 0, \ldots, j - 1\). By remark 2.3, we can also assume there is a \(cs\)-blender horseshoe \(\Lambda_j\) with \(\text{ind } \Lambda_j = i + j\).

By proposition 2.10 and a result of Carballo et al [15], we can also suppose that \(H(q_0(f_1), f_1) = \cdots = H(q_j(f_1), f_1) = M\), i.e. hyperbolic periodic points of every index are dense in the whole manifold \(M\). We would like to note that although the result in [15] is in dissipative setting, provided it is a consequence of the connecting lemma, it is still true in the volume preserving scenario.

**Lemma 3.5.** There exists \(p \in \{q_0(f_1), \ldots, q_j(f_1)\}\) and a diffeomorphism \(f_2\) \(C^1\)-close to \(f_1\) such that \(f_2\) exhibits a homoclinic tangency for \(p(f_2)\).

**Proof.** Suppose, contrary to our claim, that every diffeomorphism in a small neighbourhood of \(f_1\) exhibits no homoclinic tangency for any hyperbolic periodic points \(q_0(f_1), \ldots, q_j(f_1)\). Then, \(f_1\) is in fact not approximated by diffeomorphisms exhibiting homoclinic tangency.

Hence, since \(f_1\) could be taken in the residual subset given by proposition 3.3 we should have dominated splittings \(TM = E_k \oplus F_k\) for \(f_1\), with \(\dim E_k = i + k\), for every \(k = 0, \ldots, j\).
Which implies we have a dominated splitting $TM = E_i \oplus E^1 \oplus \ldots E^{j-1} \oplus E_{i+j}$, where $\dim E^k = 1$ for every $1 \leq k \leq j - 1$.

Since for diffeomorphisms in $\mathcal{V}_1$ any hyperbolic periodic point $p$ has $\text{ind } p \leq i + j$, it follows by the same method as in lemma 2.1 in [36], that there exists $K > 0$, $m \in \mathbb{N}$ and $0 < \lambda < 1$ such that every hyperbolic periodic point $p$ of a diffeomorphism $g \in \mathcal{U}$ with index $i + j$ and sufficiently large period one has

$$
\prod_{l=0}^{\infty} \prod_{r=0}^{m-1} \| Dg^{-l}\vert E_{i+j}(g^{-lm-r}(p)) \| \leq K \lambda^k,
$$

where $k = \bigg\lfloor \frac{\tau(p, g)}{m} \bigg\rfloor$. \hfill (3.1)

To obtain this, Potrie [36] uses a result for uniformly contracting sequences introduced by Mañe, lemma II.5 in [31].

Now, by a known Mae’s argument also introduced in [31], we can find a positive integer $n$ such that $\| Df^n\vert E_{i+j} \|$ contracts for every $x \in M$. This argument consists in use Mañe’s ergodic closing lemma to obtain a hyperbolic periodic point of index $i + j$ that doesn’t satisfy equation (3.1), if $\| Df^n\vert E_{i+j} \|$ does not contract for every $n > 0$. All of these arguments, including the results for uniformly contracting sequences are done for volume preserving diffeomorphisms in [2].

Similarly, we can prove that $\| Df^n\vert E_i \|$ contracts for a large enough positive integer $n$. Then $f_i$ is partial hyperbolic, which is a contradiction and then finishes the proof of the lemma.

\hfill $\Box$

**Remark 3.6.** We point out that the above arguments give in particular a proof of proposition 1.2.

Hence, let $f_2$ and $p$ given by lemma 3.5. After a perturbation, we can suppose $p$ is one of the two reference saddles of the blender horseshoe $\Lambda = \Lambda_k(f_2)$, if $\text{ind } p = i + k$.

By proposition 3.2, the proof is completed by showing that:

**Lemma 3.7.** There is a diffeomorphism $g$ arbitrarily $C^1$-close to $f_2$ such that $W^u(p(g))$ is a folded submanifold with respect to the continuation $\Lambda(g)$ of the blender horseshoe $\Lambda$ for $g$.

This lemma is a volume preserving version of lemma 4.9 in [7].

**Proof.** Let $B$ denote a point of homoclinic tangency for $p(f_2)$, which we can suppose to be in $\mathbb{C}$, i.e. $B$ is in the reference cube of the blender horseshoe $\Lambda$. As in the proof of lemma 2.9, after a perturbation, we can assume $T_B W^u(p, f_2) \cap T_B W^u(p, f_2) = E^{uu}(B)$ is the one-dimensional centre-unstable subspace. Hence, let $B \subset T_B W^u(p, f_2)$ be such that $T_B W^u(p, f_2) = B \oplus E^{uu}$.

Provided we have a partial hyperbolic splitting in $\mathbb{C}$, $Df^n(B)(\mathbb{V})$ converges to $E^{uu}(p)$, if $n$ goes to infinity. Hence, if $U \subset W^u(p, f_2)$ is a small enough disc containing $B$ and $n$ is a large enough positive integer, then $S = f^n(U)$ is foliated by $uu$-discs. More precisely, $S = \cup_{t \in [0,1]} S_t$ and $S_t$ is a $uu$-disc.

We could have assumed that $B$ is a point of quadratic homoclinic tangency. Hence, to finish we need to analyse two cases.

In the first one, see figure 2, we can unfold the homoclinic tangency to obtain $t_1$ and $t_2$ such that $S_{t_1} \cap W^u_{loc}(p, f_2)$ and $S_{t_1} \cap W^u_{loc}(p, f_2)$ are non empty, and $S_{t_1} \cap W^u_{loc}(p, f_2)$ is a $uu$-disc in between of $p$ and $q$. Therefore, $\tilde{S} = \cup_{t \in [t_1, t_2]} S_t$ is a folded manifold inside the unstable manifold as we wanted.

In the second case, replacing $S$ by a positive iterated, should exist $t_1$ and $t_2$ such that $S_{t_1} \cap W^u_{loc}(q, f_2)$ and $S_{t_1} \cap W^u_{loc}(q, f_2)$ are non-empty intersections. Finally, unfolding the tangency as before, we also obtain a folded manifold. See figure 3. \hfill $\Box$
4. Non-existence of symbolic extensions

In this section we prove proposition D, and at the end we prove theorem A.

Proof of proposition D. We give the proof only for volume preserving diffeomorphisms. The general case is completely similar.

Let $f$ be a $C^1$ volume preserving diffeomorphism and $U \subset \text{Diff}^1_{m}(M)$ be a neighbourhood of $f$ as in the assumptions.

By Robinson [38], Kupka–Smale’s result is still true in the volume preserving setting. Hence, there exists an open and dense subset $U_n \subset U$ of diffeomorphisms $g$ such that every periodic point of $g$ with period smaller than or equal to $n$ is hyperbolic (or elliptic if $M$ is a surface). Since the proposition for area preserving diffeomorphisms is contained in the Downarowicz–Newhouse’s result [23], to avoid elliptic periodic points we suppose $\dim M \geq 3$.

We set $R_1 = \cap U_n$ which is a residual subset in $U$. Let $R_{1,m,n}$ be the open set of diffeomorphisms $g$ in $U_n$ where $m$ is the smallest one such that $\text{Per}_m(g) \neq \emptyset$. Hence,

$$A_n = \bigcup_{j=1}^{n} R_{1,j,n}.$$
Let $\Lambda$ be the hyperbolic basic set which exhibits robust homoclinic tangency for $f$. For every $g \in U$, we denote by $H(\Lambda(g))$ the set of hyperbolic periodic points of $g$ homoclinically related with $\Lambda(g)$, and we set

$$H_n(\Lambda(g)) = H(\Lambda(g)) \cap \text{Per}_n(g).$$

If $p$ is a hyperbolic periodic point of $g$, $|\mu(p, g)| < 1 < |\lambda(p, g)|$ denote the two eigenvalues of $Dg^{\tau(p,g)}$ the nearest one, i.e. if $\nu$ is an eigenvalue of $Dg^{\tau(p,g)}$ then either $|\nu| \leq |\mu(p, g)|$ or $|\nu| \geq |\lambda(p, g)|$. We define

$$\chi(p, g) = \frac{1}{\tau(p, g)} \log \min\{|\lambda(p, g)|, |\mu(p, g)|^{-1}\}.$$

for every hyperbolic periodic point $p$ of $g$.

Finally, for any positive integer $n$, we say that a diffeomorphism $g \in U$ satisfies property $S_n$ if for every $p \in H_n(\Lambda(g))$ there exists a hyperbolic basic set of zero dimension $\Lambda(p, n)$ such that

1. There exists an ergodic measure $\mu \in \mathcal{M}(f|\Lambda(p, n))$ such that
   $$h_\mu(g) > \chi(p, g) - \frac{1}{n}.$$

2. For every ergodic measure $\mu \in \mathcal{M}(f|\Lambda(p, n))$, we have
   $$\rho(\mu, \mu_p) < \frac{1}{n},$$
   where $\mu_p$ is the dirac measure on the orbit of $p$. 

Figure 3. Folded manifold: second case.
(3) For every periodic point $q \in \Lambda(p, n)$, we have
\[ \chi(q, g) > \chi(p, g) - \frac{1}{n}. \]

For positive integers $m \leq n$, let $D_{m,n} \subset \mathcal{R}_{1,m,n}$ be the subset of diffeomorphisms satisfying property $S_n$.

**Claim.** $D_{m,n}$ is open and dense in $\mathcal{R}_{1,m,n}$.

This claim is a conservative version of lemma 3.3 in [16], which is an extension for symplectic diffeomorphisms of lemma 5.1 in [23]. By the claim the proof is similar in spirit to theorem 1.3 in [23]. See also [16]. However, using the following result of Burguet the proof could be simplified.

**Proposition 4.1 (Burguet, corollary 1 in [11]).** Let $(X, f)$ a dynamical system admitting a symbolic extension. Then the entropy function $h : \mathcal{M}(f) \to \mathbb{R}$ is a difference of nonnegative upper semicontinuous functions. In particular the entropy function $h$ restrict to any compact set of measures has a large set of continuity points.

We set
\[ R = \bigcap_{n \geq 0} \bigcup_{m=0}^{n} D_{m,n}, \]
which is a residual subset in $\mathcal{U}$ by the claim.

Now, let $f \in R$ and define $\chi(f) = \sup\{\chi(p, f), p \in H(\Lambda(f))\}$. We denote by $E$ the closure of
\[ E_1 = \{\mu_p : p \in H(\Lambda(f)) \text{ and } \chi(p, f) > \chi(f)/2\}. \]

Since $f \in R$, for any periodic point $p$ such that $\mu_p \in E_1$, it follows there exist ergodic measures $\nu_n \to \mu_p$ such that $h_{\nu_n}(f) > \chi(f)/2$. Moreover, since $\nu_n \in \mathcal{M}(f|\Lambda(p, n))$, by Sigmund [39] $\nu_n$ is approximated by hyperbolic periodic measures also supported in the hyperbolic set $\Lambda(p, n)$, which by item 3 of property $S_n$ should be in $E_1$. Hence, $\nu_n \in E$ for every $n$, and thus
\[ \limsup_{\nu_n \to \mu_p, \nu_n \in E} h_{\nu_n}(f) > \frac{\chi(f)}{2}. \]

Therefore, $E_1$ has no continuity point for the entropy function $h$, since $p$ is arbitrary and $E_1$ has densely periodic measures. Which implies by proposition 4.1 that $f$ has no symbolic extensions.

**Proof of claim.** As in the proof of the technical proposition in [16], the procedure is to find Newhouse’s snakes (see remark 4.2 for a definition) after a perturbation of a diffeomorphism exhibiting a homoclinic tangency, to obtain from them nice hyperbolic sets satisfying the conditions in the property $S_n$. However, to find a diffeomorphism satisfying property $S_n$ it is necessary to have an argument to obtain Newhouse’s snakes related to any arbitrary hyperbolic periodic point. To do this we will use robustness of homoclinic tangency.

We denote by $\Lambda$ the hyperbolic set of $f$ exhibiting robust homoclinic tangency.

Let $g \in A_n$ be an arbitrary diffeomorphism. By definition of $A_n$, there exists a small neighbourhood $V$ of $g$ where the cardinality of periodic points of period smaller than or equal to $n$ is constant.

Let $p \in H_n(\Lambda(g))$. Since $\Lambda(g)$ has a robust homoclinic tangency and $p$ is homoclinic related with $\Lambda(g)$, after a perturbation we can assume that $g$ exhibits a homoclinic tangency
for the hyperbolic periodic point $p$. By the regularization result of Ávila [4] we can suppose that $g$ is $C^\infty$ and then using the pasting lemma we can assume that $g^{\tau(p,g)} = Dg^{\tau(p,g)}$ in some neighbourhood of $p$ (in local coordinate), say $V$. That is, $g^{\tau(p,g)}(V \cap g^{-\tau(p,g)}(V))$ is linear. Hence, $W_{loc}^s(p,g)$ and $W_{loc}^u(p,g)$ coincide with stable and unstable directions restrict to $V$.

For simplicity we suppose $p$ is a hyperbolic fixed point of $g$.

Let $q$ be the point of homoclinic tangency between $W_{loc}^s(p,g)$ and $W^u(p,g)$, such that $q \in V$ and $g^{-1}(q) \notin V$. Hence, there is a small neighbourhood $U \subset V$ of $q$ such that $g^{-1}(U) \cap V = \emptyset$. We denote by $D$ the connected component of $W^u(p,g) \cap U$ that contains $q$. For convenience we suppose $dim(TqW_{loc}^s(p,g) \cap TqW^u(p,g)) = 1$, which can be done after a perturbation.

We look to $U$ in a local coordinate, being $q$ the zero of this chart, and we consider the following splitting of space $\mathbb{R}^d = T_qD \oplus T_qD^\perp$. Since $D \cap U$ is an open disc inside $W^u(p,g)$, it follows that $D$ is a graph of a map $r : T_qD \rightarrow T_qD^\perp$, which is as regular as $g$. That is, $D = \{(x,r(x)) \mid r \in C^\infty\}$ map. Moreover, such map $r$ is such that $Dr(0) = 0$. Defining $\phi(x, y) = (x, y - r(x))$, if $U$ is small enough, this map is a $C^\infty$ volume preserving diffeomorphism from $U$ to $\phi(U)$, and is $C^1$ close to identity in a small enough neighbourhood of $q$. Therefore, by the pasting lemma we can find a $C^\infty$ volume preserving diffeomorphism $h$ on $M$, $C^1$-close to identity such that $h = \phi$ in some small neighbourhood of $q$, and $h = Id$ outside $U$.

We define $g_1 = h \circ g$ which is a $C^1$ perturbation of $g$. Note, $g_1$ is such that $T_qD \cap U \subset W^u(p,g_1)$.

Let $I$ be this interval of homoclinic tangency. By a coordinate change in $U$, we can suppose that $W_{loc}^u(p,g_1) \cap U = E^s(p,g_1) \cap U \subset \mathbb{R}^\prime \times \{0\}^u$, and $I \subset \{(x_1, 0, ..., 0), -3\delta \leq x_1 \leq 3\delta\}$, for some $\delta > 0$ small enough. We are now considering the usual coordinates in $\mathbb{R}^d$.

Let $N$ be any large positive integer. By the same method as in the construction of $h$, for any $\delta > 0$ small enough we can find a volume preserving diffeomorphism $\Theta : M \rightarrow M$, $\delta - C^1$ near $Id$, such that $\Theta = Id$ in the complement of $B(q, 2\delta)$ and

$$\Theta(x, y) = \left( x_1, \ldots, x_s, y_1 + A \cos \frac{\pi x_1 N}{2\delta}, y_2, \ldots, y_u \right), \quad \text{for } (x, y) \in B(0, \delta) \subset U,$$

for $A = \frac{2\delta K}{\pi N}$, where $K$ is a constant depending only on the local coordinate on $U$.

We define $g_2 = \Theta \circ g_1$ which is $\delta - C^1$ close to $g_1$ and moreover $g_2 = g_1$ in the complement of $g_1^{-1}(U)$. Note, $g_2$ exhibits $N$ transversal homoclinic points for $p$ inside $U$. More precisely, these points belong to $g_2(g_1^{-1}(I))$.

**Remark 4.2.** This kind of perturbation is the so called Newhouse’s snake.

We remark that $g_2$ depends on $N$, but by abuse of notation, we use the same letter $g_2$ for every $N$.

From now on we will look to $V$ in local coordinate. Moreover, we assume $E^s_p = \mathbb{R}^\prime \times \{0\}^u$ and $E^u_p = \{0\}^\prime \times \mathbb{R}^u$, where $E^s_p$ and $E^u_p$ are the stable and unstable directions of $p$ with dimensions $s$ and $u$, respectively. Observe $g_2$ is linear in $V$ since $g_2$ is equal to $g_1$ in $V$.

For any positive large integer $t$, we can define a small rectangle $D_t = D^s \times D^u_t$, where $D^s = W_{loc}^s(p,g_2) \cap U$ and $D^u_t$ is a small disc in $\{0, \ldots, 0, y_1, \ldots, y_u\}$, $y_i \in \mathbb{R}^s$ and $|y_i| < A/4$, such that $g_2^t(D_t)$ is a disc $A/4 - C^1$ close to the connected component of $W^u(p,g_2) \cap U$ containing the $N$ transversal homoclinic points of $g_2$ in $g_2(g^{-1}(I))$. We fix $t$ as being the
A generic condition for existence of symbolic extension

It is clearly that \( t \) depends on \( N \), and goes to infinity if \( N \) does.

Since \( N \) is large, it follows that \( A \) is as small which implies \( D_t \) is such that \( g_2(D_t) \cap D_t \) has \( N \) disjoint connected components. Moreover, taking \( N \) larger if necessary, the maximal invariant set in \( D_t \) for \( g_2 \)

\[
\tilde{\Lambda}(p, N) = \bigcap_{j \in \mathbb{Z}} g_2^j(D_t)
\]
is a hyperbolic set.

Let

\[
\Lambda(p, N) = \bigcup_{0 \leq j \leq t} g_2(\tilde{\Lambda}((p, N)))
\]

be the hyperbolic periodic set of \( g_2 \) induced by \( \tilde{\Lambda}(p, N) \).

Now, if \( n \) is an arbitrary large positive integer, we can proceed in the same way as in the proof of technical proposition in [16] to find a large positive integer \( N \), such that \( \Lambda(p, N)(\tilde{g}) \) satisfies all items of property \( S_n \) for every diffeomorphism \( \tilde{g} \) \( C^1 \)-close to \( g_2 \).

Finally, since the cardinality of \( \text{Per}_n \) is finite and constant for diffeomorphisms in \( V \), we can repeat the same process finitely many times to obtain an open set \( C^1 \)-close to \( g \) of diffeomorphisms satisfying property \( S_n \).

We finish the paper with the proof of theorem A.

**Proof of theorem A.** Since \( \text{Diff}^1_m(M) \) is a separable space, it follows there exists an enumerable dense subset \( \{ f_1, \ldots, f_n, \ldots \} \) of diffeomorphisms in \( \text{Diff}^1_m(M) \).

If \( f_i \) is not partial hyperbolic, then by theorem A we can suppose \( f_i \) exhibits a robust homoclinic tangency, after a perturbation. Then, by proposition D there exists a neighbourhood \( \mathcal{U}_i \) of \( f_i \) and a residual subset \( \mathcal{R}_i \subset \mathcal{U}_i \) such that every diffeomorphism \( g \in \mathcal{R}_i \) has no symbolic extensions. Hence, \( \mathcal{R}_i \) contains an enumerable intersection of open and dense subset in \( \mathcal{U}_i \), say \( \mathcal{R}_i = \cap \mathcal{B}_n \). We define \( \mathcal{B}_n = \mathcal{B}_n \cup (\text{cl}(\mathcal{U}_i))^c \), which is in fact an open and dense subset of \( \text{Diff}^1_m(M) \).

If \( f_i \) is partial hyperbolic we define \( \mathcal{B}_n \) as being the set of partial hyperbolic diffeomorphisms and the ones that are not approximated by partial hyperbolic diffeomorphisms, which is also an open and dense subset of \( \text{Diff}^1_m(M) \).

Hence, we define

\[
\mathcal{R} = \bigcap_{i, n \in \mathbb{N}} \mathcal{B}_n
\]

which is a residual subset in \( \text{Diff}^1_m(M) \). Finally, note by construction that if \( f \in \mathcal{R} \) is not partial hyperbolic, then \( f \) has no symbolic extensions. Which proves the theorem.

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