THE KATO SQUARE ROOT PROBLEM FOR DIVERGENCE FORM OPERATORS WITH POTENTIAL

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ABSTRACT. The Kato square root problem for divergence form elliptic operators with potential \( V : \mathbb{R}^n \to \mathbb{C} \) is the equivalence statement \( \| \mathcal{L}_V^1 \frac{1}{2} u \| \simeq \| \nabla u \| + \| V^\frac{1}{2} u \| \), where \( \mathcal{L}_V := -\text{div} (A \nabla) + V \) and the perturbation \( A \) is an \( L_\infty \) complex matrix-valued function satisfying an accretivity condition. This relation is proved for any potential satisfying \( \| V \| u \| \lesssim \| (|V| - \Delta) u \| \) for all \( u \in D(|V| - \Delta) \), with range contained in some positive sector. The class of potentials that will satisfy such a condition is known to contain the reverse Hölder class \( RH_q \) for any \( q \geq 2 \) and \( L_2^\frac{n}{2} (\mathbb{R}^n) \) in dimension \( n > 4 \). To prove the Kato estimate with potential, a non-homogeneous version of the framework developed by A. Axelsson, S. Keith and A. McIntosh in [6] is developed. In addition to applying this non-homogeneous framework to the scalar Kato problem with zero-order potential, it will also be applied to the Kato problem for systems of equations with zero-order potential and scalar equations with first-order potential.

1. Introduction

For Hilbert spaces \( \mathcal{H} \) and \( \mathcal{K} \), let \( \mathcal{L} (\mathcal{H}, \mathcal{K}) \) denote the space of bounded linear operators from \( \mathcal{H} \) to \( \mathcal{K} \) and set \( \mathcal{L} (\mathcal{H}) := \mathcal{L} (\mathcal{H}, \mathcal{H}) \). Fix \( n \in \mathbb{N}^* \) and let \( A \in L_\infty (\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n)) \). Consider the sesquilinear form \( I_A : H^1 (\mathbb{R}^n) \times H^1 (\mathbb{R}^n) \to \mathbb{C} \) defined by

\[
I_A [u, v] := \int_{\mathbb{R}^n} \langle A(x) \nabla u(x), \nabla v(x) \rangle \, dx
\]

for \( u, v \in H^1 (\mathbb{R}^n) \). Suppose that \( I_A \) satisfies the Gårding inequality

\[
\text{Re} (I_A [u, u]) \geq \kappa_A \| \nabla u \|^2
\]

for all \( u \in H^1 (\mathbb{R}^n) \), for some \( \kappa_A > 0 \). A well-known representation theorem from classical form theory (c.f. [17]) asserts the existence of an associated operator \( \mathcal{L}_A : D(\mathcal{L}_A) \subset L^2 (\mathbb{R}^n) \to L^2 (\mathbb{R}^n) \) for which

\[
I_A [u, v] = \langle \mathcal{L}_A u, v \rangle
\]

for all \( v \in H^1 (\mathbb{R}^n) \) and \( u \) in the domain of \( \mathcal{L}_A \),

\[
D (\mathcal{L}_A) = \{ u \in H^1 (\mathbb{R}^n) : \exists w \in L^2 (\mathbb{R}^n) \ s.t. \ I_A [u, v] = \langle w, v \rangle \ \forall v \in H^1 (\mathbb{R}^n) \}.
\]

The operator \( \mathcal{L}_A \) is denoted

\[
\mathcal{L}_A = -\text{div} A \nabla.
\]

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since the two sides of the above relation will naturally coincide whenever the right-hand side makes sense. The operator $L_A$ will be a densely defined maximal accretive operator. As such, it is possible to define a square root operator $\sqrt{L_A}$, with domain $D(L_A)$, that satisfies $\sqrt{L_A} \cdot \sqrt{L_A} = L_A$. A famous conjecture first posed by Kato asks whether the domain of this square root operator extends to all of $H^1(\mathbb{R}^n)$. This is the Kato square root problem. In essence, it amounts to proving that the estimate
\[
\left\| \sqrt{L_A} u \right\| \simeq \| u \|
\]
is true for any $u \in D(L_A)$. This long-standing problem withstood solution until \cite{3} where it was proved using local $T(\beta)$ methods. Then, in \cite{4} this solution was generalised to elliptic systems. We will be interested in an alternate method of proof that was built from similar principles and appeared a few years later.

Let $\Pi := \Gamma + \Gamma^*$ be a Dirac-type operator on a Hilbert space $\mathcal{H}$ and $\Pi_B := \Gamma + B_1 \Gamma^* B_2$ be a perturbation of $\Pi$ by bounded operators $B_1$ and $B_2$. Typically, $\Pi$ is considered to be a first-order system acting on $\mathcal{H} := L^2(\mathbb{R}^n; \mathbb{C}^N)$ for some $n, N \in \mathbb{N}^*$ and the perturbations $B_1$ and $B_2$ are multiplication by matrix-valued functions $B_1, B_2 \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^N))$. In their seminal paper \cite{6}, A. Axelsson, S. Keith and A. McIntosh developed a general framework for proving that the perturbed operator $\Pi_B$ possessed a bounded holomorphic functional calculus. This ultimately amounted to obtaining square function estimates of the form
\[
\int_0^\infty \| Q^B_t u \|^2 \frac{dt}{t} \simeq \| u \|^2,
\]
where $Q^B_t := t \Pi_B (I + t^2 \Pi_B^2)^{-1}$ and $u$ is contained in the range $R(\Pi_B)$. They proved that this estimate would follow entirely from a set of simple conditions imposed upon the operators $\Gamma, B_1$ and $B_2$, labelled (H1) - (H8). Then, by checking this list of simple conditions, the Axelsson-Keith-McIntosh framework, or AKM framework by way of abbreviation, could be used to conclude that the particular selection of operators
\[
\Gamma := \begin{pmatrix} 0 & 0 \\ \nabla & 0 \end{pmatrix}, \quad B_1 = I, \quad B_2 = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix},
\]
defined on $L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n; \mathbb{C}^n)$, would satisfy \cite{2} and therefore possess a bounded holomorphic functional calculus. The Kato square root estimate then followed almost trivially from this.

Let $V : \mathbb{R}^n \to \mathbb{C}$ be a measurable function that is finite almost everywhere on $\mathbb{R}^n$. $V$ can be viewed as a densely defined closed multiplication operator on $L^2(\mathbb{R}^n)$ with domain
\[
D(V) = \{ u \in L^2(\mathbb{R}^n) : V \cdot u \in L^2(\mathbb{R}^n) \}.
\]
The density of $D(V)$ follows from the measurability of $V$. Define the subspace
\[
H_V^1(\mathbb{R}^n) := H^1(\mathbb{R}^n) \cap D \left( V^{\frac{1}{2}} \right) := \left\{ u \in H^1(\mathbb{R}^n) : V^{\frac{1}{2}} \cdot u \in L^2(\mathbb{R}^n) \right\}.
\]
Here the complex square root $V^{\frac{1}{2}}$ is defined via the principal branch $\{ z \in \mathbb{C} : \Re(z) < 0 \}$. Let $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n))$ be as before with \cite{1} satisfied for some $\kappa_A > 0$. Consider the sesquilinear form $t_A^V : H_V^1(\mathbb{R}^n) \times H_V^1(\mathbb{R}^n) \to \mathbb{C}$ defined through
\[
t_A^V [u, v] := I_A [u, v] + \int_{\mathbb{R}^n} (V(x)u(x), v(x)) \, dx
\]
for \( u, v \in H^1_{V_1}(\mathbb{R}^n) \). Suppose that there exists some \( \kappa^V > 0 \) for which \( t^V_A \) satisfies the associated Gårding inequality

\[
\text{Re} \left( t^V_A [u, u] \right) \geq \kappa^V \left( \left\| V^{1/2} u \right\|^2 + \| \nabla u \|^2 \right),
\]

for all \( u \in H^1_{V_1}(\mathbb{R}^n) \).

**Remark 1.1.** If the range of \( V \) is contained in some sector

\[
S_{\mu^+} := \{ z \in \mathbb{C} \cup \{ \infty \} : |\arg(z)| \leq \mu \text{ or } z = 0, \infty \}
\]

for some \( \mu \in [0, \pi/2) \), then (5) will follow automatically from (1).

Once again, the accretivity of \( l^V_A \) implies the existence of a maximal accretive operator associated with this form denoted by

\[
L^V_A = -\text{div} A \nabla + V,
\]

defined on \( D(L^V_A) = D(L_A) \cap D(V) \).

Define \( \mathcal{W} \) to be the class of all finite almost everywhere measurable functions \( V : \mathbb{R}^n \rightarrow \mathbb{C} \) for which

\[
\sup_{u \in D((|V| - \Delta)^{-1})} \frac{|||V| u|| + \|(-\Delta) u\|}{||(|V| - \Delta) u||} < \infty.
\]

In this paper, our aim is to prove the potential dependent Kato estimate as presented in the following theorem.

**Theorem 1.1** (Kato with Potential). Let \( V \in \mathcal{W} \) and \( A \in L^\infty(\mathbb{R}^n; L(\mathbb{C}^n)) \). Suppose that the Gårding inequalities (1) and (5) are both satisfied with constants \( \kappa_A > 0 \) and \( \kappa^V_A > 0 \) respectively. There exists a constant \( C_V > 0 \) such that

\[
C_V^{-1} \cdot \left( \left\| V^{1/2} u \right\| + \| \nabla u \| \right) \leq \left\| \sqrt{L^V_A} u \right\| \leq C_V \left( \left\| V^{1/2} u \right\| + \| \nabla u \| \right)
\]

for all \( u \in D(L^V_A) \).

In direct analogy to the potential free case, the Kato problem with potential will be solved by constructing appropriate potential dependent Dirac-type operators and demonstrating that they retain a bounded holomorphic functional calculus under perturbation. In particular, this strategy will be applied to the Dirac-type operator

\[
\Pi_{|V|^{1/2}} := \Gamma_{|V|^{1/2}} + \Gamma^*_{|V|^{1/2}} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ |V|^{1/2} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & |V|^{1/2} & -\text{div} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

defined on \( L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n; \mathbb{C}^n) \), under the perturbation

\[
B_1 = I, \quad B_2 := \begin{pmatrix} I & 0 & 0 \\ 0 & e^{i\arg V} & 0 \\ 0 & 0 & A \end{pmatrix}
\]

It should be observed that the operator \( \Gamma_{|V|^{1/2}} \) will not necessarily satisfy the cancellation and coercivity conditions, (H7) and (H8), of \( \mathbb{C} \) due to the presence of the zero-order potential term. As such, the original framework developed by Axelsson, Keith and McIntosh cannot be directly applied. The key difficulty in proving our result is then to alter the original framework in order to allow for such operators. In
particular, a non-homogeneous version of the Axelsson-Keith-McIntosh framework will be developed to handle operators of the form

\[ \Gamma_J := \begin{pmatrix} 0 & 0 & 0 \\ J & 0 & 0 \\ D & 0 & 0 \end{pmatrix}, \]

where \( D \) and \( J \) are, respectively, homogeneous and non-homogeneous first-order differential operators. The technical challenge presented by the inclusion of the non-homogeneous operator \( J \) will be overcome by separating our square function norm into components and demonstrating that the non-homogeneous term will allow for the first two components to be bounded while the third component can be bound using an argument similar to the classical argument of [6].

Since the operator \( \Gamma_J \) is of a more general form than \( \Gamma_{|V|^{-\frac{1}{2}}} \), the non-homogeneous AKM framework that we develop will have applications not confined to zero-order scalar potentials. Indeed, the non-homogeneous framework will also be used to prove Kato estimates for systems of equations with zero-order potential and for scalar equations with first-order potentials. It takes no great leap of imagination to see that our framework could also be applied to a combination of these two situations. That is, it is possible to apply our framework to systems of equations with first-order potentials. This, however, will be left to the readers discretion.

The structure of this paper is as follows. Section 2 is quite classical in nature. It provides a brief survey of the natural functional calculus for bisectorial operators. Section 3 describes the non-homogeneous AKM framework and states the main results associated with it. Section 4 contains most of the technical machinery and is dedicated to a proof of our main result. Section 5 will apply the non-homogeneous AKM framework to the scalar Kato problem with potential, the Kato problem for systems with zero-order potential and the scalar Kato problem with first-order potential. It is here that a proof of Theorem 1.1 will be completed. Finally, in Section 6, we will provide a meta-discussion on the proof techniques used and compare our work with what has been previously accomplished on non-homogeneous Kato type estimates. Comparative strengths and weaknesses of our approach will be highlighted.

**Notation.** Throughout this article, the notation \( A \lesssim B \) and \( A \simeq B \) will be used to denote that there exists a constant \( C > 0 \) for which \( A \leq C \cdot B \) and \( C^{-1} \cdot B \leq A \leq C \cdot B \) respectively.

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While this paper was in its final stages of preparation, it was found that Andrew Morris and Andrew Turner from the University of Birmingham had obtained similar results. After meeting them and discussing their research, it appears that the two approaches differ in their assumptions and, more substantially, their proofs.

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2. Preliminaries

Let’s outline the construction of the natural functional calculus associated with a bisectorial operator. The treatment of functional calculi found here follows closely to [15] with significant detail omitted. Appropriate changes are made to account for the fact that we consider bisectorial operators instead of sectorial operators. Other thorough treatments of functional calculus for sectorial operators can be found in [16] or [1].

For $\mu \in [0, \pi)$ define the open and closed sectors
\[
S^o_{\mu+} := \{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \mu \} \quad \mu \in (0, \pi)
\]
\[
S_{\mu+} := \{ z \in \mathbb{C} \cup \{\infty\} : |\arg(z)| \leq \mu \text{ or } z = 0, \infty \} \quad \mu \in (0, \pi)
\]

Then, for $\mu \in \left[0, \frac{\pi}{2}\right)$, define the open and closed bisectors
\[
S^o_{\mu} := (S^o_{\mu+}) \cup (-S^o_{\mu+})
\]
and
\[
S_{\mu} := (S_{\mu+}) \cup (-S_{\mu+})
\]
respectively. Throughout this section we consider bisectorial operators defined on a Hilbert space $\mathcal{H}$.

**Definition 2.1 (Bisectorial Operator).** A linear operator $T : D(T) \subseteq \mathcal{H} \to \mathcal{H}$ is said to be $\omega$-bisectorial for $\omega \in \left[0, \frac{\pi}{2}\right)$ if the spectrum $\sigma(T)$ is contained in the bisector $S_\omega$ and if for any $\mu \in (\omega, \frac{\pi}{2})$, there exists $C_\mu > 0$ such that the resolvent bound
\[
|\zeta| \left\| (\zeta I - T)^{-1} \right\| \leq C_\mu
\]
holds for all $\zeta \in \mathbb{C} \setminus S_\mu$. $T$ is said to be bisectorial if it is $\omega$-bisectorial for some $\omega \in \left[0, \frac{\pi}{2}\right)$. Sectorial operators are defined identically except with the sector $S_{\mu+}$ performing the role of the bisector $S_\mu$. An important fact concerning bisectorial operators is the following decomposition result.

**Proposition 2.1 ([8 Thm. 3.8]).** Let $T : D(T) \subseteq \mathcal{H} \to \mathcal{H}$ be a bisectorial operator. Then $T$ is necessarily densely defined and the Hilbert space $\mathcal{H}$ admits the following decomposition
\[
\mathcal{H} = N(T) \oplus R(T).
\]

Let $T$ be an $\omega$-bisectorial operator for $\omega \in \left[0, \frac{\pi}{2}\right)$ and $\mu \in (\omega, \frac{\pi}{2})$. Let $\mathcal{M}(S^o_{\mu})$ denote the algebra of all meromorphic functions on the open bisector $S^o_{\mu}$ and define the following subalgebras,
\[
H(S^o_{\mu}) := \{ f \in \mathcal{M}(S^o_{\mu}) : f \text{ holomorphic on } S^o_{\mu} \},
\]
\[
H^\infty(S^o_{\mu}) := \{ f \in H(S^o_{\mu}) : \|f\|_\infty := \sup_{z \in S^o_{\mu}} |f(z)| < \infty \}.
\]
and
\[ H_0^\infty \left( S^\alpha_\mu \right) := \left\{ f \in H^\infty \left( S^\alpha_\mu \right) : \exists C, \alpha > 0 \text{ s.t. } |f(z)| \leq C \cdot \frac{|z|^\alpha}{1 + |z|^{2\alpha}} \forall z \in S^\alpha_\mu \right\}. \]

For any \( f \in H_0^\infty \left( S^\alpha_\mu \right) \), one can associate an operator \( f(T) \) as follows. For \( u \in \mathcal{H} \), define
\[ f(T)u := \frac{1}{2\pi i} \oint_{\gamma} f(z) (zI - T)^{-1} u \, dz, \]
where the curve
\[ \gamma := \{ \pm re^{i\nu} : 0 \leq r < \infty \} \]
for some \( \nu \in (\omega, \mu) \) is traversed anticlockwise.

**Theorem 2.1.** The map \( \Phi^T_0 : H_0^\infty \left( S^\alpha_\mu \right) \to \mathcal{L} (\mathcal{H}) \) defined through
\[ \Phi^T_0 (f) := f(T) \]

is a well-defined algebra homomorphism. Moreover, it is independent of the value of \( \nu \).

**Proof.** The resolvent bounds of our operator and the size estimates on \( f \) imply that the above integral will converge absolutely ensuring that \( f(T) \) is a well-defined bounded operator. An application of the Cauchy integral formula will give us the independence of the definition of \( f(T) \) from the value of \( \nu \). For a proof of the homomorphism property refer to [15, Lem. 2.3.1].

Since the functions in \( H_0^\infty \left( S^\alpha_\mu \right) \) approach zero at the origin we should expect that the null space of the newly formed operator will be larger than the null space of the original operator. This is indeed the case as stated in the below proposition.

**Proposition 2.2 ([15 Thm. 2.3.3]).** For a bisectorial operator \( T : D(T) \subseteq \mathcal{H} \to \mathcal{H} \), the null-space inclusion
\[ N(T) \subseteq N(f(T)) \]
holds for all \( f \in H_0^\infty \left( S^\alpha_\mu \right) \).

Define the subalgebra of functions
\[ \mathcal{E} \left( S^\alpha_\mu \right) := H_0^\infty \left( S^\alpha_\mu \right) \oplus \langle (z + i)^{-1} \rangle \oplus \langle (z - i)^{-1} \rangle \oplus \langle 1 \rangle. \]

\( \Phi^T_0 \) has an extension
\[ \Phi^T_p : \mathcal{E} \left( S^\alpha_\mu \right) \to \mathcal{L} (H) \]
defined through
\[ g(T) := \Phi^T_p (g) := f(T) + c \cdot (T + i)^{-1} + d \cdot (T - i)^{-1} + e \cdot I \]
for \( g = f + c \cdot (z + i)^{-1} + d \cdot (z - i)^{-1} + e \in \mathcal{E} \left( S^\alpha_\mu \right) \), where \( f \in H_0^\infty \left( S^\alpha_\mu \right) \) and \( c, d, e \in \mathbb{C} \).

**Theorem 2.2 ([15 Thm. 2.3.3]).** The map \( \Phi^T_p \) is an algebra homomorphism called the primary functional calculus associated with \( T \).

This map can be extended once more through the process of regularization. A function \( f \in \mathcal{M} \left( S^\alpha_\mu \right) \) is said to be regularizable with respect to the primary functional calculus \( \Phi^T_p : \mathcal{E} \left( S^\alpha_\mu \right) \to \mathcal{L} (H) \) if there exists \( e \in \mathcal{E} \left( S^\alpha_\mu \right) \) such that \( e(T) \) is injective and \( e \cdot f \in \mathcal{E} \left( S^\alpha_\mu \right) \). The notation \( \mathcal{E} \left( S^\alpha_\mu \right)_e \) will be used to denote the
Let $0$ algebra of regularizable functions. Let $\mathcal{C}(\mathcal{H})$ denote the set of closed operators from $\mathcal{H}$ to itself. Then define the extension

$$\Phi^T : \mathcal{E}(S_\mu^0) \to \mathcal{C}(\mathcal{H})$$

through

$$f(T) := \Phi^T(f) := \Phi^T_\mu(e)^{-1} \cdot \Phi^T_\mu(e \cdot f)$$

for $f \in \mathcal{E}(S_\mu^0)$ and $e \in \mathcal{E}(S_\mu^0)$ a regularizing function for $f$. This definition is independent of the chosen regularizer $e$ for $f$ and therefore $\Phi^T$ is well-defined. We have the following important theorem that establishes the desired properties of a functional calculus for this extension.

**Theorem 2.3 ([15 Thm. 1.3.2]).** Let $T$ be an $\omega$-bisectorial operator on a Hilbert space $\mathcal{H}$ for some $\omega \in [0, \frac{\pi}{2})$. Let $\mu \in (\omega, \frac{\pi}{2})$. The following assertions hold.

1. $\mathbf{1}(T) = I$ and $(z)(T) = T$, where $\mathbf{1} : S_\mu^0 \to \mathbb{C}$ is the constant function defined by $\mathbf{1}(z) := 1$ for $z \in S_\mu^0$.

2. Let $f, g \in \mathcal{E}(S_\mu^0)$. Then

$$f(T) + g(T) \subset (f + g)(T), \quad f(T)g(T) \subset (f \cdot g)(T)$$

and $D((f(T)g(T))) = D((f \cdot g)(T)) \cap D(g(T))$. One will have equality in these relations if $g(T) \in \mathcal{L}(\mathcal{H})$.

The following definition plays a vital role in the solution method to the Kato square root problem using the AKM framework.

**Definition 2.2.** Let $0 \leq \omega < \mu < \frac{\pi}{2}$. An $\omega$-bisectorial operator $T : D(T) \subset \mathcal{H} \to \mathcal{H}$ is said to have a bounded $H^\infty(S_\mu^0)$-functional calculus if there exists $c > 0$ such that

$$||f(T)|| \leq c \cdot ||f||_\infty$$

for all $f \in H^\infty(S_\mu^0)$. $T$ is said to have a bounded holomorphic functional calculus if it has a bounded $H^\infty(S_\mu^0)$-functional calculus for some $\mu$.

**Remark 2.1.** Note that a more intuitive definition for a bounded $H^\infty(S_\mu^0)$-functional calculus would be to require that [11] hold for all $f \in H^\infty(S_\mu^0)$. Unfortunately at this stage it is impossible to ascertain whether $H^\infty(S_\mu^0) \subset \mathcal{E}(S_\mu^0)$. When this inclusion does not hold, the operator $f(T)$ will not be well-defined for all $f \in H^\infty(S_\mu^0)$. If $T$ so happens to be injective, then each $f \in H^\infty(S_\mu^0)$ is in fact regularizable by $z(1 + z^2)$ and the estimate [11] makes sense for all $f \in H^\infty(S_\mu^0)$. Fortunately, in this situation the two definitions coincide. That is, [11] will be true for all $f \in H^\infty(S_\mu^0)$ if and only if it is true for all $f \in H^\infty(S_\mu^0)$ when $T$ is injective.

Let $q : S_\mu^0 \to \mathbb{C}$ be defined through

$$q(z) := \frac{z}{1 + z^2}, \quad z \in S_\mu^0.$$  

For $t > 0$, let $q_t$ denote the function $q_t(z) := q(tz)$ for $z \in S_\mu^0$. It is not too difficult to see that $q_t \in H^\infty(S_\mu^0)$ for any $t > 0$ (c.f. [15] pg. 29).
**Definition 2.3** *(Square Function Estimates).* A bisectorial operator $T$ on a Hilbert space $H$ is said to satisfy square function estimates if there exists a constant $C > 0$ such that

\begin{equation}
C^{-1} \cdot \|u\|^2 \leq \int_0^\infty \|q_t(T)u\|^2 \frac{dt}{t} \leq C \cdot \|u\|^2
\end{equation}

for all $u \in \overline{R(T)}$.

The above definition is the same as saying that the seminorm on $H$ defined through

\begin{equation}
\|u\|_{q,T} := \int_0^\infty \|q_t(T)u\|^2 \frac{dt}{t}
\end{equation}

is norm equivalent to $\|\cdot\|_H$ when restricted to the Hilbert subspace $R(T)$.

**Remark 2.2.** The function $q$ in the above definition of square function estimates is somewhat arbitrary. It can be replaced by any function $\psi \in H_0^\infty(\mu)$, not identically zero on either $S_\mu^+$ or $(-S_\mu^+)$, to obtain the equivalent quantity

\[
\int_0^\infty \|\psi_t(T)u\|^2 \frac{dt}{t},
\]

where $\psi_t(z) := \psi(tz)$ for $z \in S_\mu$ and $t > 0$. The fact that this square function norm is equivalent up to multiplicative constant to (13) on $\overline{R(T)}$ can be found in [15, Thm. 7.3.1].

**Proposition 2.3** *(Resolution of the Identity).* For any $u \in H$,

\begin{equation}
\frac{1}{2} (I - P_{N(T)}) u = \int_0^\infty (q_t(T))^2 u \frac{dt}{t},
\end{equation}

where $P_{N(T)}$ denotes the projection operator onto the subspace $N(T)$.

**Proof.** Equality follows from Proposition 2.2 for $u \in N(T)$. For $u \in \overline{R(T)}$ this is given by Theorem 5.2.6 of [15] in the sectorial case. The bisectorial case can be proved similarly.

**Corollary 2.1.** Suppose that $T$ is self-adjoint. Then for any $u \in H$,

\[
\int_0^\infty \|q_t(T)u\|^2 \frac{dt}{t} \leq \frac{1}{2} \|u\|^2.
\]

Equality will hold if $u \in \overline{R(T)}$.

**Proof.** As $T$ is self-adjoint, if follows from the definition of $q_t(T)$ that it must also be self-adjoint. On expanding the square function norm,

\[
\int_0^\infty \|q_t(T)u\|^2 \frac{dt}{t} = \int_0^\infty \langle q_t(T)u, q_t(T)u \rangle \frac{dt}{t} = \left\langle u, \int_0^\infty q_t(T)^2 u \frac{dt}{t} \right\rangle.
\]

The previous proposition then gives

\[
\int_0^\infty \|q_t(T)u\|^2 \frac{dt}{t} = \left\langle u, \frac{1}{2} (I - P_{N(T)}) u \right\rangle \leq \frac{1}{2} \|u\|^2.
\]
Equality will clearly hold in the above if \( u \in \mathcal{R}(T) \).

A fundamental result due to A. McIntosh is the equivalence of square function estimates with a bounded holomorphic functional calculus.

**Theorem 2.4 ([13] Thm. 7.3.1).** A bisectorial operator \( T \) will satisfy square function estimates if and only if it has a bounded holomorphic functional calculus.

Finally, the following Kato type estimate follows from the previous theorem using a well-known classical argument. This argument can be found, for example, in the proof of Corollary 3.4 in [11].

**Corollary 2.2.** Suppose that the bisectorial operator \( T \) satisfies square function estimates. Then there exists a constant \( c > 0 \) such that

\[
(15) \quad c^{-1} \cdot \|Tu\| \leq \left\| \sqrt{T^2} u \right\| \leq c \cdot \|T\|
\]

for any \( u \in D(T) \).

### 3. Non-Homogeneous Axelsson-Keith-McIntosh

In this section we describe how the Axelsson-Keith-McIntosh framework can be altered to account for non-homogeneous operators of the form (9). Our main results for this framework will also be stated.

#### 3.1. AKM without Cancellation and Coercivity

The operators that we wish to consider, \( \Gamma_j \), will satisfy the first six conditions of [6]. However, they will not necessarily satisfy the cancellation condition (H7) and the coercivity condition (H8). It will therefore be fruitful to see what happens to the original AKM framework when the cancellation and coercivity conditions are removed.

Similar to the original result, we begin by assuming that we have operators that satisfy the hypotheses (H1) - (H3) from [6]. Recall these conditions for operators \( \Gamma, B_1 \) and \( B_2 \) on a Hilbert space \( \mathcal{H} \).

- **(H1)** \( \Gamma : D(\Gamma) \to \mathcal{H} \) is a closed, densely defined, nilpotent operator.
- **(H2)** \( B_1 \) and \( B_2 \) satisfy the accretivity conditions

\[
\Re \langle B_1 u, u \rangle \geq \kappa_1 \|u\|^2 \quad \text{and} \quad \Re \langle B_2 v, v \rangle \geq \kappa_2 \|v\|^2
\]

for all \( u \in R(\Gamma^\ast) \) and \( v \in R(\Gamma) \) for some \( \kappa_1, \kappa_2 > 0 \).
- **(H3)** The operators \( \Gamma \) and \( \Gamma^\ast \) satisfy

\[
\Gamma^\ast B_2 B_1 \Gamma^\ast = 0 \quad \text{and} \quad \Gamma B_1 B_2 \Gamma = 0.
\]

In [6] Section 4, the authors assume that they have operators that satisfy the hypotheses (H1) - (H3) and they derive several important operator theoretic consequences from only these hypotheses. As our operators \( \Gamma, B_1 \) and \( B_2 \) also satisfy (H1) - (H3), it follows that any result proved in [6] Section 4 must also be true for our operators and can be used with impunity. In the interest of making this article as self-contained as possible, we will now restate any such result that is to be used in this paper.
Proposition 3.1 (9). Define the perturbation dependent operators
\[ \Gamma_B^* := B_1 \Gamma^* B_2, \quad \Gamma_B := B_2^* \Gamma B_1^* \quad \text{and} \quad \Pi_B := \Gamma + \Gamma_B^*. \]
The Hilbert space \( \mathcal{H} \) has the following Hodge decomposition into closed subspaces:
\begin{equation}
\mathcal{H} = \mathcal{N}(\Pi_B) \oplus \mathcal{R}(\Gamma_B^*) \oplus \mathcal{R}(\Gamma).
\end{equation}
Moreover, we have \( \mathcal{N}(\Pi_B) = \mathcal{N}(\Gamma_B^*) \cap \mathcal{N}(\Gamma) \) and \( \mathcal{R}(\Pi_B) = \mathcal{R}(\Gamma_B^*) \oplus \mathcal{R}(\Gamma) \). When \( B_1 = B_2 = I \) these decompositions are orthogonal, and in general the decompositions are topological. Similarly, there is also a decomposition
\[ \mathcal{H} = \mathcal{N}(\Pi_B^*) \oplus \mathcal{R}(\Gamma_B) \oplus \mathcal{R}(\Gamma^*). \]

Proposition 3.2 (9). The perturbed Dirac-type operator \( \Pi_B \) is an \( \omega \)-bisectorial operator with
\[ \omega := \frac{1}{2} (\omega_1 + \omega_2) \]
where
\[ \omega_1 := \sup_{u \in \mathcal{R}(\Gamma^*) \setminus \{0\}} |\arg\langle B_1 u, u \rangle| < \frac{\pi}{2} \]
and
\[ \omega_2 := \sup_{u \in \mathcal{R}(\Gamma) \setminus \{0\}} |\arg\langle B_2 u, u \rangle| < \frac{\pi}{2}. \]
The bisectoriality of \( \Pi_B \) ensures that the following operators will be well-defined.

Definition 3.1. For \( t \in \mathbb{R} \setminus \{0\} \), define the perturbation dependent operators
\[ R_B^t := (I + t \Pi_B)^{-1}, \quad P_B^t := (I + t^2 (\Pi_B)^2)^{-1}, \]
\[ Q_B^t := t \Pi_B P_B^t \quad \text{and} \quad \Theta_B^t := t \Gamma_B^* P_B^t. \]
When there is no perturbation, i.e. when \( B_1 = B_2 = I \), the \( B \) will dropped from the superscript or subscript. For example, instead of \( \Theta_I^t \) or \( \Pi_I \) the notation \( \Theta_t \) and \( \Pi \) will be employed.

Remark 3.1. An easy consequence of Proposition 3.2 is that the operators \( R_B^t, P_B^t \) and \( Q_B^t \) are all uniformly \( L^2 \)-bounded in \( t \). Furthermore, on taking the Hodge decomposition Proposition 3.1 into account, it is clear that the operators \( \Theta_B^t \) will also be uniformly \( L^2 \)-bounded in \( t \).

The next result tells us how the operators \( \Pi_B \) and \( P_B^t \) interact with \( \Gamma \) and \( \Gamma_B^* \).

Lemma 3.1 (9). The following relations are true.
\[ \Pi_B \Gamma u = \Gamma_B^* \Pi_B u \quad \text{for all} \ u \in D(\Gamma_B^* \Pi_B), \]
\[ \Pi_B \Gamma_B^* u = \Gamma \Pi_B u \quad \text{for all} \ u \in D(\Gamma \Pi_B), \]
\[ \Gamma P_B^t u = P_B^t \Gamma u \quad \text{for all} \ u \in D(\Gamma), \quad \text{and} \]
\[ \Gamma_B^* P_B^t u = P_B^t \Gamma_B^* u \quad \text{for all} \ u \in D(\Gamma_B^*). \]

The subsequent lemma provides a square function estimate for the unperturbed Dirac-type operator \( \Pi \). When considering square function estimates for the perturbed operator, there will be several instances where the perturbed case can be reduced with the assistance of this unperturbed estimate. Its proof follows directly from the self-adjointness of the operator \( \Pi \) and Corollary 2.1.
Lemma 3.2 ([6]). The quadratic estimate

\[ \int_0^\infty \|Q_t u\|^2 \frac{dt}{t} \leq \frac{1}{2} \|u\|^2 \]

holds for all \( u \in \mathcal{H} \). Equality holds on \( R(\Pi) \).

The following result will play a crucial role in the reduction of the square function estimate (2).

Proposition 3.3 ([6]). Assume that the estimate

\[ \int_0^\infty \|\Theta B_t P_t u\|^2 \frac{dt}{t} \lesssim \|u\|^2 \]

holds for all \( u \in R(\Gamma) \), together with three similar estimates obtained on replacing \( \{\Gamma, B_1, B_2\} \) by \( \{\Gamma^*, B_2', B_1'\} \) and \( \{\Gamma, B_1^*, B_2^*\} \). Then \( \Pi_B \) satisfies the quadratic estimate

\[ \|u\|^2 \lesssim \int_0^\infty \|Q_B t u\|^2 \frac{dt}{t} \lesssim \|u\|^2 \]

for all \( u \in R(\Pi_B) \).

The following corollary is proved during the course of the proof of Proposition 4.8 of [6].

Corollary 3.1 (High Frequency Estimate). The estimate

\[ \int_0^\infty \|\Theta B_t (I - P_t) u\|^2 \frac{dt}{t} \lesssim \|u\|^2 \]

holds for all \( u \in R(\Gamma) \).

From this point onwards, it will also be assumed that our operators satisfy the additional hypotheses (H4) - (H6). These hypotheses are stated below for reference.

(H4) The Hilbert space is \( \mathcal{H} = L^2(\mathbb{R}^n; \mathbb{C}^N) \) for some \( n, N \in \mathbb{N}^* \).

(H5) The operators \( B_1 \) and \( B_2 \) represent multiplication by matrix-valued functions. That is,

\[ B_1(f)(x) = B_1(x) \cdot f(x) \quad \text{and} \quad B_2(f)(x) = B_2(x) \cdot f(x) \]

for all \( f \in \mathcal{H} \) and \( x \in \mathbb{R}^n \), where \( B_1, B_2 \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^N)) \).

(H6) For every bounded Lipschitz function \( \eta : \mathbb{R}^n \to \mathbb{C} \), we have that \( \eta D(\Gamma) \subset D(\Gamma) \) and \( \eta D(\Gamma^*) \subset D(\Gamma^*) \). Moreover, the commutators \( [\Gamma, \eta I] \) and \( [\Gamma^*, \eta I] \) are multiplication operators that satisfy the bound

\[ |[\Gamma, \eta I](x)|, \quad |[\Gamma^*, \eta I](x)| \leq c|\nabla \eta(x)| \]

for all \( x \in \mathbb{R}^n \) and some constant \( c > 0 \).

In contrast to the original result, our operators will not be assumed to satisfy the cancellation condition (H7) and the coercivity condition (H8). Without these two conditions, many of the results from Section 5 of [6] will fail. One notable exception to this is that the bounded operators associated with our perturbed Dirac-type operator \( \Pi_B \) will satisfy off-diagonal estimates.
Definition 3.2 (Off-Diagonal Bounds). Define $\langle x \rangle := 1 + |x|$ for $x \in \mathbb{C}$ and $\text{dist}(E, F) := \inf \{ |x - y| : x \in E, y \in F \}$ for $E, F \subset \mathbb{R}^n$.

Let $\{U_t\}_{t>0}$ be a family of operators on $\mathcal{H} = L^2(\mathbb{R}^n; \mathbb{C}^N)$. This collection is said to have off-diagonal bounds of order $M > 0$ if there exists $C_M > 0$ such that

$$\|U_t u\|_{L^2(E)} \leq C_M \langle \text{dist}(E, F)/t \rangle^{-M} \|u\|$$

whenever $E, F \subset \mathbb{R}^n$ are Borel sets and $u \in \mathcal{H}$ satisfies $\text{supp} u \subset F$.

Proposition 3.4 (\cite{[6]}). Let $U_t$ be given by either $R_t^B$, $R_{-t}^B$, $P^B_t$, $Q^B_t$ or $\Theta^B_t$ for every $t > 0$. The collection of operators $\{U_t\}_{t>0}$ has off-diagonal bounds of every order $M > 0$.

Introduce the following dyadic decomposition of $\mathbb{R}^n$. Let $\Delta = \cup_{j=-\infty}^{\infty} \Delta_{2^j}$, where $\Delta_j := \{ 2^j (k + (0,1)^n) : k \in \mathbb{Z}^n \}$ if $2^{j-1} < t \leq 2^j$. Define the averaging operator $A_t : \mathcal{H} \to \mathcal{H}$ through

$$A_t u(x) := \frac{1}{|Q(x,t)|} \int_{Q(x,t)} u(y) \, dy$$

for $x \in \mathbb{R}^n$, $t > 0$ and $u \in \mathcal{H}$, where $Q(x,t)$ is the unique dyadic cube in $\Delta_t$ that contains the point $x$.

For an operator family $\{U_t\}_{t>0}$ that satisfies off-diagonal bounds of every order, there exists an extension $U_t : L^\infty(\mathbb{R}^n; \mathbb{C}^N) \to L^2_{\text{loc}}(\mathbb{R}^n; \mathbb{C}^N)$ for each $t > 0$. This is constructed by defining

$$U_t u(x) := \lim_{r \to \infty} \sum_{R \in \Delta_t, \text{dist}(Q,R) < r} U_t (\mathbb{1}_R u)(x),$$

for $x \in Q \in \Delta_t$ and $u \in L^\infty(\mathbb{R}^n; \mathbb{C}^N)$. The convergence of the above limit is guaranteed by the off-diagonal bounds of $\{U_t\}_{t>0}$. Further detail on this construction can be found in ([6], [11], [18] or [12]). The above extension then allows us to introduce the principal part of the operator $U_t$.

Definition 3.3. Let $\{U_t\}_{t>0}$ be operators on $\mathcal{H}$ that satisfy off-diagonal bounds of every order. For $t > 0$, the principal part of $U_t$ is the operator $\zeta_t : \mathbb{R}^n \to \mathcal{L}(\mathbb{C}^N)$ defined through

$$[\zeta_t(x)](w) := (\Theta^B_t w)(x)$$

for each $x \in \mathbb{R}^n$ and $w \in \mathbb{C}^N$.

The following generalisation of Corollary 5.3 of \cite{[6]} will also be true with an identical proof.

Proposition 3.5. Let $\{U_t\}_{t>0}$ be operators on $\mathcal{H}$ that satisfy off-diagonal bounds of every order. Let $\zeta_t : \mathbb{R}^n \to \mathcal{L}(\mathbb{C}^N)$ denote the principal part of the operator $U_t$. Then there exists $c > 0$ such that

$$\int_Q |\zeta_t(y)|^2 \, dy \leq c$$

for all $Q \in \Delta_t$. Moreover, the operators $\zeta_t A_t$ are uniformly $L^2$-bounded in $t$.

Finally, the ensuing partial result will also be valid. Its proof follows in an identical manner to the first part of the proof of Proposition 5.5 of \cite{[6]}.
Proposition 3.6. Let \( \{ U_t \}_{t > 0} \) be operators on \( \mathcal{H} \) that satisfy off-diagonal bounds of every order. Let \( \zeta_t : \mathbb{R}^n \to \mathcal{L}(\mathbb{C}^N) \) denote the principal part of \( U_t \). Then there exists \( c > 0 \) such that
\[
\| (U_t - \zeta_t A_t) v \| \leq c \cdot \| t \nabla v \|.
\]
for any \( v \in H^1(\mathbb{R}^n; \mathbb{C}^N) \subset \mathcal{H} \) and \( t > 0 \).

3.2. Additional Structure. At this point, further structure will be imposed upon our operators in order to generalise the non-homogeneous operator \( \Gamma_{|V|^{1/2}} \) defined in (7). This additional structure will later be exploited in order to obtain square function estimates.

Let \( \mathbb{C}^N = V_1 \oplus V_2 \oplus V_3 \) where \( V_1, V_2 \) and \( V_3 \) are finite-dimensional complex Hilbert spaces. Let \( \mathbb{P}_i : \mathbb{C}^N \to \mathbb{C}^N \) be the projection operator onto the space \( V_i \) for \( i = 1, 2 \) and 3. Our Hilbert space will have the following orthogonal decomposition
\[
\mathcal{H} := L^2(\mathbb{R}^n; \mathbb{C}^N) = L^2(\mathbb{R}^n; V_1) \oplus L^2(\mathbb{R}^n; V_2) \oplus L^2(\mathbb{R}^n; V_3).
\]
The notation \( \mathbb{P}_i \) will also be used to denote the natural projection operator from \( \mathcal{H} \) onto \( L^2(\mathbb{R}^n; V_i) \). For a vector \( v \in \mathcal{H} \), \( v_i \in L^2(\mathbb{R}^n; V_i) \) will denote the \( i \)th component for \( i = 1, 2 \) or 3.

Let \( \Gamma_J \) be an operator on \( \mathcal{H} \) of the form
\[
\Gamma_J := \begin{pmatrix} 0 & 0 & 0 \\ J & 0 & 0 \\ D & 0 & 0 \end{pmatrix},
\]
where \( J \) and \( D \) are closed densely defined operators
\[
J : L^2(\mathbb{R}^n; V_1) \to L^2(\mathbb{R}^n; V_2), \quad D : L^2(\mathbb{R}^n; V_1) \to L^2(\mathbb{R}^n; V_3).
\]
Define the operators
\[
\Gamma_0 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ D & 0 & 0 \end{pmatrix}, \quad M_J := \begin{pmatrix} 0 & 0 & 0 \\ J & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
\[
\Pi_0 := \Gamma_0 + \Gamma_0^*, \quad S_J := M_J + M_J^* \quad \text{and} \quad \Pi_J := \Gamma_J + \Gamma_J^*.
\]

Let \( B_1, B_2 \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^N)) \) be matrix-valued multiplication operators. The following key assumption will be imposed on our operators throughout the entirety of this article.

**Key Assumption.** The family of operators \( \{ \Gamma_0, B_1, B_2 \} \) satisfies the conditions (H1) - (H8) of [6] while \( \{ \Gamma_J, B_1, B_2 \} \) satisfies only (H1) - (H6).

For reference, the cancellation condition (H7) and the coercivity condition (H8) are shown below for the operator \( \Gamma_0 \).

(H7) For any \( u \in D(\Gamma_0) \) and \( v \in D(\Gamma_0^*) \), both compactly supported,
\[
\int_{\mathbb{R}^n} \Gamma_0 u = 0 \quad \text{and} \quad \int_{\mathbb{R}^n} \Gamma_0^* v = 0.
\]
(H8) There exists $c > 0$ such that
\[
\|\nabla u\| \leq c \cdot \|\Pi_0 u\|
\]
for all $u \in R(\Pi_0) \cap D(\Pi_0)$.

**Example 3.1.** Typical examples of operators that satisfy the previous key assumption are when both $D$ and $J$ are first-order partial differential operators. If the perturbations $B_1$ and $B_2$ satisfy suitable accretivity conditions then the families of operators $\{\Gamma_0, B_1, B_2\}$ and $\{\Gamma_j, B_1, B_2\}$ will both satisfy (H1) - (H6). If, in addition, $D$ is homogeneous and there exists $c > 0$ for which
\[
\|\nabla u\| \leq c \cdot \|D u\|
\]
for all $u \in R(D^*) \cap D(D)$ and
\[
\|\nabla u\| \leq c \cdot \|D^* u\|
\]
for all $u \in R(D) \cap D(D^*)$ then $\{\Gamma_0, B_1, B_2\}$ will also satisfy (H7) and (H8). A particular example of such a situation is given by the operator $\Gamma_{V_1}^2$ together with perturbations $B_1$ and $B_2$ as defined in (7) and (8) with (1) and (5) satisfied.

**Remark 3.2.** Since the operator $\Gamma_0$, together with the perturbations $B_1$ and $B_2$, satisfy all eight conditions (H1) - (H8) of [6], it follows that any result from that paper must be valid for these operators.

**Definition 3.4.** For $t \in \mathbb{R} \setminus \{0\}$, define the perturbation dependent operators
\[
\Gamma_{J,B}^t := B_1 \Gamma_{J}^* B_2, \quad \Pi_{J,B}^t := \Gamma_{J} + \Gamma_{J,B}^* B_1.
\]
\[
R_t^{J,B} := (I + it \Pi_{J,B})^{-1}, \quad P_t^{J,B} := \left(I + t^2 \left(\Pi_{J,B}\right)^2\right)^{-1},
\]
\[
Q_t^{J,B} := t \Pi_{J,B} P_t^{J,B}, \quad \Theta_t^{J,B} := t \Pi_{J,B} P_t^{J,B}.
\]
When there is no perturbation, i.e. when $B_1 = B_2 = I$, the $B$ will dropped from the superscript or subscript. For example, instead of $\Theta_t^{J,I}$ the notation $\Theta_t^J$ will be employed.

We now introduce a coercivity condition to serve as a replacement for (H8) for the operators $\{\Gamma_{J,B}^t, B_1, B_2\}$. This condition will not be automatically imposed upon our operators but, rather, will be taken as a hypothesis for our main results.

(H8J) $B_2$ is of the form
\[
B_2 = \begin{pmatrix}
I & 0 & 0 \\
0 & A_{22} & A_{23} \\
0 & A_{32} & A_{33}
\end{pmatrix},
\]
where $A_{ij} \in L^\infty(\mathbb{R}^n; \mathcal{L}(V_j, V_j))$ for $i, j = 2$ or 3. The inclusion
\[
D(\Gamma_{J} J + D^* D) \subset D(J^* A_{22} J + D^* A_{32} J)
\]
is satisfied. Furthermore, there exists a constant $C > 0$ such that for all $u \in D(J^* J + D^* D)$,
\[
\|(J^* A_{22} J + D^* A_{32} J) u\| + \|D^* D u\| \leq C \cdot \|(J^* J + D^* D) u\|.
\]
Remark 3.3. The situation of most interest to us is when $A_{32} = 0$ and

$$\| J^* A_{22} J u \| = \| J^* J u \|$$

for all $u \in D(J^* A_{22} J) = D(J^* J)$. In this case, the Riesz transform condition of (H8J) becomes the perturbation free condition

$$\| J^* J u \| \leq C \cdot \| (J^* J + D^* D) u \|$$

for all $u \in D(J^* J + D^* D)$ and the domain inclusion $D(J^* J + D^* D) \subset D(J^* A_{22} J + D^* A_{32} J)$ becomes trivially satisfied. Furthermore, when this occurs, (H8J) will be equivalent to the condition

$$\| S_J u \| \leq C \cdot \| \Pi_J u \|$$

or equivalently

$$\| \Pi_0 u \| \leq C \cdot \| \Pi_J u \|$$

for all $u \in D(\Pi_J)$.

The main result of the non-homogeneous AKM framework can now be stated.

Theorem 3.1. Suppose that the condition (H8J) is satisfied. Then the estimate

(23) \[
\int_0^\infty \left\| \Theta^{I,B}_t P^I_t u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2
\]

holds for all $u \in R(\Gamma_J)$.

The proof of this theorem will be reserved for Section 4. For now, let’s prove an estimate that serves as a dual to the above estimate.

Proposition 3.7. For $t > 0$, define the operator

$$\hat{P}^{I,B}_t := \left( I + t^2 (\Gamma^*_J + B_2 \Gamma_J B_1)^2 \right)^{-1}.$$ 

Suppose that $B_1 = I$. Then

(24) \[
\int_0^\infty \left\| \hat{P}^{I,B}_t B_2 \Gamma_J B_1 P^I_t u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2
\]

for all $u \in R(\Gamma_J^*)$.

Proof. As $\{\Gamma^*_J, B_2, B_1\}$ satisfy (H1) - (H6), it follows from Remark 3.1 that the operators $\hat{P}^{I,B}_t$ are well-defined and uniformly $L^2$-bounded. On applying this to the left-hand side of (24), we have

\[
\begin{align*}
\int_0^\infty \left\| \hat{P}^{I,B}_t B_2 \Gamma_J B_1 P^I_t u \right\|^2 \frac{dt}{t} & \lesssim \int_0^\infty \left\| t B_2 \Gamma_J P^I_t u \right\|^2 \frac{dt}{t} \\
& \lesssim \int_0^\infty \left\| t \Gamma_J P^I_t u \right\|^2 \frac{dt}{t} \\
& \leq \int_0^\infty \left\| t \Pi_J P^I_t u \right\|^2 \frac{dt}{t} \\
& = \int_0^\infty \left\| Q^I_t u \right\|^2 \frac{dt}{t} \\
& \lesssim \|u\|^2,
\end{align*}
\]
where the inequality $\|\Gamma_J v\| \leq \|\Pi_J v\|$ for $v \in D(\Pi_J)$ follows immediately from the three-by-three matrix form of the operators and Lemma 3.2 was used in the last line.

From our main result, Theorem 3.1 and Proposition 3.7 the upper and lower square function estimates for $Q_{tJ,B}^J$ can be proved using the results of [4].

**Theorem 3.2.** Suppose that the condition (H8J) is valid and $B_1 = I$. Then

$$\|u\|^2 \lesssim \int_0^\infty \|Q_t^{J,B} u\|^2 \frac{dt}{t} \lesssim \|u\|^2$$

for all $u \in \overline{R(\Pi_J)}$.

**Proof.** Proposition 3.3 states that in order to prove the square function estimate (25), it is sufficient for the estimate (23) to be valid for the permutations of operators $\{\Gamma_J, B_1, B_2\}$, $\{\Gamma_J, *B_1, *B_2\}$, $\{*J, B_1, *B_2\}$ and $\{*J, *B_1, *B_2\}$. The permutations $\{\Gamma_J, B_1, B_2\}$ and $\{\Gamma_J, B_1, B_2\}$ both come under the umbrella of Theorem 3.1 and the permutations $\{\Gamma_J, B_2, B_1\}$ and $\{\Gamma_J, *B_1, *B_2\}$ are handled by Proposition 3.7.

From the upper and lower estimate of the previous theorem, Theorem 2.4 then implies that $\Pi_{J,B}$ has a bounded holomorphic functional calculus. The following Corollary is then readily deduced from Corollary 2.2.

**Corollary 3.2.** Suppose that $B_1 = I$ and that (H8J) is satisfied. The operator

$$L_{B}^{J} := J^*A_{22}J + D^*A_{32}J + J^*A_{23}D + D^*A_{33}D$$

is a sectorial operator with a bounded holomorphic functional calculus. Moreover

$$\left\| \sqrt{L_B^J} u \right\| \simeq \|Ju\| + \|Du\|$$

for all $u \in D(\Pi_{J,B})$.

**Proof.** The bounded holomorphic functional calculus of $L_B^J$ follows from the bounded holomorphic functional calculus of $\Pi_{J,B}$ and that $\Pi_{J,B}^2$ is of the form

$$\Pi_{J,B}^2 = \begin{pmatrix} L_B^J & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}.$$ 

The estimate (26) follows from Corollary 2.2 applied to the operator $\Pi_{J,B}$ and an element $(u, 0, 0) \in H$ with $u \in D(L_B^J)$.

### 4. Square Function Estimates

In this section, a proof of our main result, Theorem 3.1, will be provided. As stated in the introduction, the technical challenge presented by the inclusion of the non-homogeneous operator $J$ will be overcome by separating our square function norm into components. In this manner,

$$\int_0^\infty \left\| \Theta_{tJ,B} P_i^J u \right\|^2 \frac{dt}{t} \lesssim \sum_{i=1,2,3} \int_0^\infty \left\| \Theta_{tJ,B} P_i^J u \right\|^2 \frac{dt}{t},$$

where the notation $P_i$, introduced in Section 3, denotes the projection operator onto the subspace $L^2(\mathbb{R}^n; V_i) \subset H$ for $i = 1, 2$ and 3. Notice that for $P_1^J P_i^J u = 0$
for any $u \in R(\Gamma_J)$ and thus the boundedness of the first component is trivial. The boundedness of the second component relies on the non-homogeneous term $J$ and is given in the following lemma.

**Lemma 4.1.** For any $u \in R(\Gamma_J)$,

$$
\int_0^\infty \left\| \Theta_t^{J,B} P_2 P_t^J u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2.
$$

**Proof.** First it will be proved that for $u \in R(\Gamma_J)$ we have $P_2 P_t^J u \in D(J_{B,J}^*)$.

Note that this is equivalent to

$$(P_t u)_2 \in D(J^* A_{22} + D^* A_{32}).$$

Since $u \in R(\Gamma_J)$, $u = \Gamma_J v$ for some $v \in D(\Gamma_J)$. As $P_t^J u = P_t^J \Gamma_J v = \Gamma_J P_t^J v$ by Lemma 3.1 and $P_t^J u \in D(\Pi_J)$, it follows that $(P_t^J v)_1 \in D(J^* J + D^* D)$, which by (H8J) is contained in $D(J^* A_{22} J + D^* A_{32} J)$. Therefore $J (P_t^J v)_1 = (P_t^J u)_2 \in D(J^* A_{22} + D^* A_{32}).$

Since $P_2 P_t^J u \in D(J_{B,J}^*)$, it follows from Lemma 3.1 that

$$
\Theta_t^{J,B} P_2 P_t^J u = P_t^{J,B} \Gamma_J \Theta P_2 P_t^J u.
$$

The estimate in (H8J) gives

$$
\|\Gamma_{J,B}^* P_2 v\| \lesssim \|\Gamma_J v\|
$$

for any $v \in R(\Gamma_J)$. Since $P_t^J$ and $\Gamma_J$ commute by Lemma 3.1, it follows that

$$
\|\Gamma_{J,B}^* P_2 P_t^J u\| \lesssim \|\Gamma_J P_t^J u\|
$$

for $u \in R(\Gamma_J)$. On applying the uniform $L^2$-boundedness of the $P_t^{J,B}$ operators,

$$
\int_0^\infty \left\| \Theta_t^{J,B} P_2 P_t^J u \right\|^2 \frac{dt}{t} = \int_0^\infty \left\| P_t^{J,B} \Gamma_J \Theta P_2 P_t^J u \right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\| P_t^{J,B} \Gamma_J \Theta P_2 P_t^J u \right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\| \Gamma_{J,B}^* P_2 P_t^J u \right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\| \Gamma_J P_t^J u \right\|^2 \frac{dt}{t}.
$$

On successively applying Proposition 3.1 and Lemma 3.2 we obtain

$$
\int_0^\infty \left\| \Theta_t^{J,B} P_2 P_t^J u \right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \left\| \Gamma_J \Theta P_t^J u \right\|^2 \frac{dt}{t} = \int_0^\infty \left\| \Theta_t^J u \right\|^2 \frac{dt}{t} = \frac{1}{2} \|u\|^2.
$$

It remains to bound the third component of our square function estimate,

(27) $$
\int_0^\infty \left\| \Theta_t^{J,B} P_3 P_t^J u \right\|^2 \frac{dt}{t} \lesssim \|u\|^2.
$$

This will be handled in a similar manner to the classical proof in [6] but the effect of the projection $P_3$ must be accounted for. Introduce the notation $\Theta_t^{J,B}$ to denote...
the operators $\tilde{\Theta}^J_{t,B} := \Theta^J_{t,B} P_3$. Let $\gamma_t^J B$ and $\tilde{\gamma}^J_t B$ denote the principal parts of
the operators $\Theta^J_{t,B}$ and $\tilde{\Theta}^J_{t,B}$ respectively. That is, they are the multiplication
operators defined through

$$
\gamma_t^J B(x) w := \Theta^J_{t,B}(w)(x) \quad \text{and} \quad \tilde{\gamma}_t^J B(x)(w) := \left( \Theta^J_{t,B} P_3 \right)(w)(x),
$$

for $w \in C^N$ and $x \in \mathbb{R}^n$. Evidently we must have $\tilde{\gamma}^J_t B(x) w = \gamma_t^J B(x) P_3 w$.

Our square function norm can be reduced to this principal part by applying the
splitting

$$
\int_0^\infty \| \tilde{\Theta}^J_{t,B} P^J_t u \|^2 \frac{dt}{t} \leq \int_0^\infty \left\| \left( \tilde{\Theta}^J_{t,B} - \tilde{\gamma}^J_t B A_t \right) P^J_t u \right\|^2 \frac{dt}{t} + \int_0^\infty \left\| \tilde{\gamma}^J_t B A_t P^J_t u \right\|^2 \frac{dt}{t}.
$$

Since the operator $\Theta^J_{t,B}$ satisfies the conditions of Proposition 3.6, it follows that

$$
\int_0^\infty \left\| \left( \tilde{\Theta}^J_{t,B} - \tilde{\gamma}^J_t B A_t \right) P^J_t u \right\|^2 \frac{dt}{t} = \int_0^\infty \left\| \left( \Theta^J_{t,B} - \gamma_t^J B A_t \right) P_3 P^J_t u \right\|^2 \frac{dt}{t}
\leq \int_0^\infty \left\| t \nabla P_3 P^J_t u \right\|^2 \frac{dt}{t}.
$$

As $u \in R(\Gamma_J)$ we must have $u = \Gamma_J v$ for some $v \in D(\Gamma_J)$. It then follows from
Lemma 3.1 that

$$
P_3 P^J_t u = P_3 \Gamma_J P^J_t v = \Gamma_0 P^J_t v \in R(\Gamma_0).
$$

This allows us apply (H8) for the operator $\Gamma_0$ to obtain

$$
\| t \nabla P_3 P^J_t u \| \leq \| \Pi_0 P^J_t u \|.
$$

It is not too difficult to see, simply by expanding out the operators, that (H8J) implies $\| \Pi_0 \Gamma_J \tilde{v} \| \leq \| \Pi_J \Gamma_J \tilde{v} \|$ for any $\tilde{v} \in D(\Gamma_J)$. Therefore

$$
\int_0^\infty \| t \Pi_0 P^J_t u \|^2 \frac{dt}{t} = \int_0^\infty \| t \Pi_0 \Gamma_J P^J_t v \|^2 \frac{dt}{t}
\leq \int_0^\infty \| t \Pi_J \Gamma_J P^J_t v \|^2 \frac{dt}{t}
= \int_0^\infty \| t \Pi_J P^J_t u \|^2 \frac{dt}{t}
= \frac{1}{2} \| u \|^2.
$$

Our theorem has therefore been reduced to a proof of the following square function estimate

$$
\int_0^\infty \| \tilde{\gamma}^J_t B A_t P^J_t u \|^2 \frac{dt}{t} \leq \| u \|^2.
$$

On splitting from above using the triangle inequality,

$$
\int_0^\infty \| \tilde{\gamma}^J_t B A_t P^J_t u \|^2 \frac{dt}{t} \leq \int_0^\infty \| \tilde{\gamma}^J_t B A_t (P^J_t - I) u \|^2 \frac{dt}{t} + \int_0^\infty \| \tilde{\gamma}^J_t B A_t u \|^2 \frac{dt}{t}.
$$

To proceed any further, the following result is required.

**Proposition 4.1.** For any $u \in H$,

$$
\int_0^\infty \| P_3 A_t (P^J_t - I) u \|^2 \frac{dt}{t} \leq \| u \|^2.
$$
Next, suppose that \( t > s \) for any \( t > 0 \). So suppose that \( u \in R(\Pi_J) \). On applying the resolution of the identity, equation (17) of [3],

\[
\int_0^\infty \| P^J_t - I \| u \|^2 \frac{dt}{t} = \int_0^\infty \left( \int_0^\infty \| P^J_t - I \| (Q^J_s)^2 \frac{ds}{s} \right)^2 \frac{dt}{t}.
\]

The Cauchy-Schwarz inequality leads to

\[
\int_0^\infty \| P^J_t - I \| u \|^2 \frac{dt}{t} \lesssim \left( \int_0^\infty \left( \int_0^\infty \| P^J_t - I \| (Q^J_s)^2 \frac{ds}{s} \right)^2 \frac{dt}{t} \right)^{1/2}.
\]

Let’s estimate the term \( \| P^J_t - I \| Q^J_s \|. \) First assume that \( t \leq s \). On noting that \( (I - P^J_t) Q^J_s = \frac{1}{t} Q^J_s (I - P^J_s) \) we obtain

\[
\| P^J_t - I \| Q^J_s \| \leq \| (P^J_t - I) Q^J_s \| \lesssim \frac{t}{s} \| Q^J_s (I - P^J_s) \| \lesssim \frac{t}{s}.
\]

Next, suppose that \( t > s \). Then the equality \( P^J_t Q^J_s = \frac{s}{t} Q^J_s P^J_s \) gives

\[
\| P^J_t - I \| Q^J_s \| \lesssim \| P^J_t Q^J_s \| + \| P^J_s Q^J_s \| \lesssim \frac{s}{t} + \| P^J_s Q^J_s \|.
\]

For the second term we have

\[
\| A_t \frac{\partial_t}{\partial t} u \|^2 = \sum_{Q \in \Delta_t} |Q| \left( \int_Q \frac{\partial_t}{\partial t} \frac{\partial_t}{\partial t} u \right)^2 \lesssim \sum_{Q \in \Delta_t} \frac{|Q|^2}{t} \left( \int_Q \frac{\partial_t}{\partial t} \frac{\partial_t}{\partial t} u \right)^2 \lesssim \sum_{Q \in \Delta_t} \frac{s}{t} \left( \int_Q \frac{\partial_t}{\partial t} u \right)^2 \lesssim \frac{s}{t} \left\| u \right\|^2.
\]
where the inequality \( \| \Gamma_0 v \| \leq \| \Pi_J v \| \) for \( v \in D(\Pi_J) \), used to obtain the third line of the above equation, follows trivially from the matrix form of the operators \( \Gamma_0 \) and \( \Pi_J \). Putting everything together gives

\[
\| P_3 A_t (P_t^J - I) Q_s^J \| \lesssim \min \left\{ \frac{t}{s}, \frac{s}{t} \right\}^{\frac{1}{2}}.
\]

This bound can then be applied to (31) to give (30).

Recall from Proposition 3.5 that the uniform estimate \( \| \tilde{\gamma}_{J,B} t A_t \| \lesssim 1 \) is true for all \( t > 0 \). Furthermore, notice that \( A_t^2 = A_t \) and \( P_3 A_t = A_t P_3 \) for all \( t > 0 \). These facts combine together with the above proposition to produce

\[
\int_0^\infty \left\| \tilde{\gamma}_{J,B} t A_t u \right\|^2 \frac{dt}{t} = \int_0^\infty \left\| \tilde{\gamma}_{J,B} t A_t P_3 A_t (P_t^J - I) u \right\|^2 \frac{dt}{t} \lesssim \int_0^\infty \| P_3 A_t (P_t^J - I) u \|^2 \frac{dt}{t} \lesssim \| u \|^2.
\]

For the second term in (29), apply Carleson’s theorem ([20] Theorem 2, page 59) to obtain

\[
\int_0^\infty \left\| \tilde{\gamma}_{J,B} t A_t u \right\|^2 \frac{dt}{t} \lesssim \| \mu \|_c \cdot \| u \|^2,
\]

where \( \mu \) is the measure on \( \mathbb{R}^{n+1} \) defined through

\[
d\mu(x,t) := \left| \tilde{\gamma}_{J,B} t (x) \right|^2 dx \frac{dt}{t}
\]

for \( x \in \mathbb{R}^n \) and \( t > 0 \), and \( \| \mu \|_c \) is the Carleson norm of \( \mu \),

\[
\| \mu \|_c := \sup_{Q \in \Delta} \frac{\mu(R_Q)}{|Q|}, \quad R_Q := Q \times [0,l(Q)).
\]

The proof of our theorem has thus been reduced to showing that the measure \( \mu \) is a Carleson measure.

4.1. Carleson Measure Estimates. The aim of this section is to prove the following Carleson measure estimate,

\[
\sup_{Q \in \Delta} \frac{1}{|Q|} \int_{l(Q)} \left( \int_Q \left| \tilde{\gamma}_{J,B} t (x) \right|^2 dx \right) \frac{dt}{t} < \infty.
\]

Let \( \mathcal{L}_3 \) denote the subspace

\[
\mathcal{L}_3 := \{ \nu \in \mathcal{L}(\mathbb{C}^N) \setminus \{0\} : \nu P_3 = \nu \}.
\]

By construction, we have \( \tilde{\gamma}_{J,B} t (x) \in \mathcal{L}_3 \) for any \( t > 0 \) and \( x \in \mathbb{R}^n \) since

\[
\tilde{\gamma}_{J,B} t (x) P_3 w = \left( \Theta_{J,B} t P_3 \right) (P_3 w) (x) = \left( \Theta_{J,B} t P_3 \right) (w) (x) = \gamma_{J,B} t (x) (w).
\]
Let $\sigma > 0$ be a constant to be determined at a later time. Let $\mathcal{V}$ be a finite set consisting of $\nu \in L_3$ with $|\nu| = 1$ such that $\cup_{\nu \in \mathcal{V}} K_{\nu} = L_3 \setminus \{0\}$, where

$$
K_{\nu} := \{ \nu' \in L_3 \setminus \{0\} : \frac{|\nu'|}{|\nu|} - \nu \leq \sigma \}.
$$

Then, in order to prove our Carleson measure estimate (35), it is sufficient to fix $\nu \in \mathcal{V}$ and prove that

$$
(37) \quad \sup_{Q \in \Delta} \frac{1}{|Q|} \int \int_{(x,t) \in R_Q} \left| \tilde{\gamma}^{J,B}_t(x) \right|^2 \frac{dx \, dt}{t} < \infty.
$$

Recall the John-Nirenberg lemma for Carleson measures as applied in [6] and [3].

**Lemma 4.2** (The John-Nirenberg Lemma for Carleson Measures). Let $\rho$ be a measure on $\mathbb{R}^{n+1}_+$ and $\beta > 0$. Suppose that for every $Q \in \Delta$ there exists a collection $\{Q_k\}_{k \in \mathcal{V}} \subset \Delta$ of disjoint subcubes of $Q$ such that $E_Q := Q \cup \cup_k Q_k$ satisfies $|E_Q| > \beta |Q|$ and such that

$$
(38) \quad \sup_{Q \in \Delta} \frac{\rho(E^*_Q)}{|Q|} \leq C
$$

for some $C > 0$, where $E^*_Q := R_Q \setminus \cup_k R_{Q_k}$. Then

$$
(39) \quad \sup_{Q \in \Delta} \frac{\rho(R_Q)}{|Q|} \leq \frac{C}{\beta}.
$$

**Proof.** Fix $Q \in \Delta$ and let $\{Q_{k_1}\}_{k_1}$ be a collection of subcubes as in the hypotheses of the lemma. Apply the bound (38) to the decomposition

$$
\rho(R_Q) = \rho(E^*_Q) + \sum_{k_1} \rho(R_{Q_{k_1}})
$$

to obtain

$$
\rho(R_Q) \leq C|Q| + \sum_{k_1} \rho(R_{Q_{k_1}}).
$$

For each $k_1$, let $\{Q_{k_1,k_2}\}_{k_2}$ be a collection of subcubes of $Q_{k_1}$ that satisfy the hypotheses of the lemma. Decompose $\rho(Q_{k_1})$ and once again apply (38) to obtain

$$
\rho(R_Q) \leq C|Q| + \sum_{k_1} \left( \rho(E^*_{Q_{k_1}}) + \sum_{k_2} \rho(Q_{k_1,k_2}) \right)
$$

$$
\leq C|Q| + \sum_{k_1} C|Q_{k_1}| + \sum_{k_1,k_2} \rho(Q_{k_1,k_2})
$$

$$
\leq C|Q| + C|Q|(1 - \beta) + \sum_{k_1,k_2} \rho(Q_{k_1,k_2}).
$$

Iterating this process and summing the resulting geometric series gives (39). □

With this tool at our disposal, the proof of our theorem can be reduced to the following proposition.

**Proposition 4.2.** There exists $\beta > 0$ and $\sigma > 0$ that will satisfy the following conditions. For every $\nu \in \mathcal{V}$ and $Q \in \Delta$, there is a collection $\{Q_k\}_{k \in \mathcal{V}} \subset \Delta$ of disjoint
subcubes of $Q$ such that $E_{Q,\nu} = Q \setminus \cup_k Q_k$ satisfies $|E_{Q,\nu}| > \beta |Q|$ and such that

$$\sup_{Q \in \Delta} \frac{1}{|Q|} \int \int_{(x,t) \in \mathcal{E}_{Q,\nu}} |\hat{\gamma}^j_{i,B}(x)|^2 \frac{dx dt}{t} < \infty,$$

where $E_{Q,\nu} := R_Q \setminus \cup_k R_{Q_k}$.

For now, fix $\nu \in \mathcal{V}$ and $Q \in \Delta$. Let $w^\nu$, $\hat{w}^\nu \in \mathcal{C}^N$ with $|\hat{w}^\nu| = |w^\nu| = 1$ and $\nu^* (\hat{w}^\nu) = w^\nu$. To simplify notation, when superfluous, this dependence will be kept implicit by defining $w := w^\nu$ and $\hat{w} := \hat{w}^\nu$. Notice that since $\nu$ satisfies $\nu = \nu \mathcal{P}_3$, $w$ must satisfy $\mathcal{P}_3 w = w$.

For $\epsilon > 0$ the function $f_{Q,\epsilon}^w$ can be defined in an identical manner to $[6]$. Specifically, let $\eta_Q : \mathbb{R}^N \rightarrow [0, 1]$ be a smooth cutoff function equal to 1 on $2Q$, with support in $4Q$ and with $\| \nabla \eta_Q \| \leq \frac{1}{2}$ where $l := l(Q)$. Then define $w_Q := \eta_Q \cdot w$ and

$$f_{Q,\epsilon}^w := w_Q - \epsilon i \Gamma_j (I + \epsilon i \Pi_{J,B})^{-1} w_Q = (I + \epsilon i \Gamma_j) (I + \epsilon i \Pi_{J,B})^{-1} w_Q.$$

**Lemma 4.3.** There exists a constant $C > 0$ that satisfies $\|f_{Q,\epsilon}^w\| \leq C|Q|^\frac{1}{2}$ and

$$\left| \int_Q \mathcal{P}_3 f_{Q,\epsilon}^w - w \right| \leq C \epsilon^2,$$

for any $\epsilon > 0$. Moreover, $C$ will not depend on $Q$, $\sigma$, $\nu$, $w$ or $\epsilon$.

**Proof.** The first claim follows from

$$\|f_{Q,\epsilon}^w\| \lesssim \|w_Q\| + \left\| \epsilon i \Gamma_j (I + \epsilon i \Pi_{J,B})^{-1} w_Q \right\| \lesssim |Q|^\frac{1}{2} + \left\| \epsilon i \Pi_{J,B} (I + \epsilon i \Pi_{J,B})^{-1} w_Q \right\| \lesssim |Q|^\frac{1}{2}.$$

On recalling that $w$ is zero in the first two components,

$$\left| \int_Q \mathcal{P}_3 f_{Q,\epsilon}^w - w \right|^2 = \left| \int_Q \mathcal{P}_3 \epsilon i \Gamma_j (I + \epsilon i \Pi_{J,B})^{-1} w_Q \right|^2 = \left| \int_Q \epsilon i \Gamma_0 (I + \epsilon i \Pi_{J,B})^{-1} w_Q \right|^2.$$

At this point, apply Lemma 5.6 of [6] to the operator $\Upsilon = \Gamma_0$ to obtain

$$\left| \int_Q \epsilon i \Gamma_0 (I + \epsilon i \Pi_{J,B})^{-1} w_Q \right|^2 \lesssim \left( \frac{\epsilon t}{l} \right)^2 \left( \int_Q \left| (I + \epsilon i \Pi_{J,B})^{-1} w_Q \right|^2 \right)^\frac{1}{2} \cdot \left( \int_Q \left| \Gamma_0 (I + \epsilon i \Pi_{J,B})^{-1} w_Q \right|^2 \right)^\frac{1}{2} \lesssim \epsilon \left( \int_Q \left| \epsilon i \Gamma_0 (I + \epsilon i \Pi_{J,B})^{-1} w_Q \right|^2 \right)^\frac{1}{2} \lesssim \epsilon \left( \int_Q \left| \epsilon i \Gamma_j (I + \epsilon i \Pi_{J,B})^{-1} w_Q \right|^2 \right)^\frac{1}{2} \lesssim \epsilon,$$

where the inequality $\|\Gamma_0 v\| \leq \|\Gamma_j v\|$ for $v \in D(\Gamma_j)$ follows trivially from the matrix form of $\Gamma_0$ and $\Gamma_j$. \qed
Lemma 4.4. There exists a constant $D > 0$ such that
\[ \int_R \int_Q |\Theta^{I,B}_{\Omega(x)}|^2 \frac{dx\,dt}{t} \leq D\frac{|Q|}{\epsilon^2}. \]

Moreover, $D$ will not depend on $Q$, $\sigma$, $\nu$, $w$ or $\epsilon$.

**Proof.** First observe that
\[
\Theta^{I,B}_{\Omega(x)} = P^{I,B}_{t} \tau \Gamma^{*}_{j,B}(I + \epsilon \Im \Gamma^{*}_{j,B})^{-1} w_Q,
\]
Therefore
\[
\int_R \int_Q |\Theta^{I,B}_{\Omega(x)}|^2 \frac{dx\,dt}{t} = \int_{0}^{t} \left( \frac{t}{\epsilon}\right)^2 \int_Q |P^{I,B}_{t} \tau \Gamma^{*}_{j,B}(I + \epsilon \Im \Gamma^{*}_{j,B})^{-1} w_Q|^2 \frac{dt}{t} \leq \frac{|Q|}{(\epsilon l)^2} \int_{0}^{t} t \,dt \approx \frac{|Q|}{\epsilon^2}.
\]

From this point forward, with $C$ as in Lemma 4.3, set $\epsilon := \frac{1}{4C^2}$ and introduce the notation $f^w_Q := f^w_{Q,\epsilon}$. With this choice of $\epsilon$ it must be true that
\[ \left| \int_Q \mathbb{P}_3 f^w_Q - w \right| \leq \frac{1}{2}. \]

That is,
\[
1 - 2\text{Re} \left( \int_Q \mathbb{P}_3 f^w_Q, w \right) = |w|^2 - 2\text{Re} \left( \int_Q \mathbb{P}_3 f^w_Q, w \right) \leq \left| \int_Q \mathbb{P}_3 f^w_Q - w \right|^2 \leq \frac{1}{4}.
\]

On rearranging we find that
\[
\text{Re} \left( \int_Q \mathbb{P}_3 f^w_Q, w \right) \geq \frac{1}{4}.
\]

In this context, Lemma 5.11 of [6] will take on the below form.

Lemma 4.5. There exists $\beta, c_1, c_2 > 0$ and a collection $\{Q_k\}$ of dyadic cubes of $Q$ such that $|E_{Q,\nu}| > \beta |Q|$ and such that
\[ \text{Re} \left( w, \int_Q \mathbb{P}_3 f^w_Q \right) \geq c_1 \quad \text{and} \quad \int_{Q'} |\mathbb{P}_3 f^w_{Q'}| \leq c_2 \]
for all dyadic subcubes $Q' \in \Delta$ of $Q$ which satisfy $R_{Q'} \cap E_{Q,\nu}^* = \emptyset$. Moreover, $\beta$, $c_1$ and $c_2$ are independent of $Q$, $\sigma$, $\nu$ and $w$. 
The proof of this statement follows in an identical manner to the argument in [6]. If we set $\sigma = \frac{c_1}{c_2}$, then the following pointwise estimate can be deduced.

**Lemma 4.6.** If $(x, t) \in E_{Q, \nu}^*$ and $\tilde{\gamma}_t^{J, B}(x) \in K_{\nu}$ then

$$\left| \tilde{\gamma}_t^{J, B}(x) \left( A_t f_Q^w(x) \right) \right| \geq \frac{1}{2} c_1 \left| \tilde{\gamma}_t^{J, B}(x) \right|.$$  

**Proof.** First observe that

$$\left| \nu \left( A_t f_Q^w(x) \right) \right| \geq \text{Re} \left\langle \hat{w}, \nu \left( A_t f_Q^w(x) \right) \right\rangle = \text{Re} \left\langle w, A_t f_Q^w(x) \right\rangle = \text{Re} \left\langle w, A_t \mathcal{P}_3 f_Q^w(x) \right\rangle \geq c_1.$$

Then

$$\left| \frac{\tilde{\gamma}_t^{J, B}(x)}{\gamma_t^{J, B}(x)} \right| \left( A_t f_Q^w(x) \right) = \left| \frac{\tilde{\gamma}_t^{J, B}(x)}{\gamma_t^{J, B}(x)} \right| \left( A_t \mathcal{P}_3 f_Q^w(x) \right) \geq \nu \left( A_t f_Q^w(x) \right) - \left| \frac{\tilde{\gamma}_t^{J, B}(x)}{\gamma_t^{J, B}(x)} \right| - \nu \left| A_t \mathcal{P}_3 f_Q^w(x) \right| \geq c_1 - \sigma c_2 = \frac{1}{2} c_1.\qed$$

**Proof of Proposition 4.2.** From the pointwise bound of the previous lemma,

$$\int \int_{(x, t) \in E_{Q, \nu}^*} \left| \tilde{\gamma}_t^{J, B}(x) \right| dx dt \lesssim \int \int_{R_Q} \left| \tilde{\gamma}_t^{J, B}(x) A_t f_Q^w(x) \right|^2 dx dt \lesssim \int \int_{R_Q} \left| \Theta_t^{J, B} f_Q^w(x) - \tilde{\gamma}_t^{J, B}(x) A_t f_Q^w(x) \right|^2 dx dt \lesssim \int \int_{R_Q} \left| \Theta_t^{J, B} f_Q^w(x) \right|^2 dx dt.$$

At this stage we can begin to unravel our square function norm,

$$\int \int_{R_Q} \left| \tilde{\gamma}_t^{J, B}(x) A_t f_Q^w(x) \right|^2 dx dt \lesssim \int \int_{R_Q} \left| \Theta_t^{J, B} f_Q^w(x) - \tilde{\gamma}_t^{J, B}(x) A_t f_Q^w(x) \right|^2 dx dt + \int \int_{R_Q} \left| \Theta_t^{J, B} f_Q^w(x) \right|^2 dx dt.$$

Lemma 4.4 states that the final term in the above estimate will be bounded from above by a multiple of $|Q|$. This reduces the task of proving the proposition to bounding the first term of the above splitting. Recall that $f_Q^w$ can be expressed in the form

$$f_Q^w := w_Q - u_Q^w,$$

where $u_Q^w \in R(\Gamma_J)$ is given by

$$u_Q^w := \epsilon I \Gamma_J \left( I + \epsilon \mathcal{P}_J \right)^{-1} w_Q.$$
An application of the triangle inequality then leads to
\[
\int \int_{R_Q} \left| \Theta_t^{J,B} f_Q^w(x) - \tilde{\gamma}_t^{J,B}(x) A_t w_Q(x) \right|^2 \frac{dx \, dt}{t} \lesssim \int \int_{R_Q} \left| \Theta_t^{J,B} w_Q(x) - \tilde{\gamma}_t^{J,B}(x) A_t w_Q(x) \right|^2 \frac{dx \, dt}{t} + \int \int_{R_Q} \left| \Theta_t^{J,B} w_Q^w(x) - \tilde{\gamma}_t^{J,B}(x) A_t w_Q^w(x) \right|^2 \frac{dx \, dt}{t}. 
\]
(46)

On noticing that for every \( x \in Q \) and \( 0 < t < l(Q) \),
\[
\Theta_t^{J,B} w_Q(x) - \tilde{\gamma}_t^{J,B}(x) A_t w_Q(x) = \Theta_t^{J,B} w_Q(x) - \Theta_t^{J,B} (A_t w_Q(x))(x) = \Theta_t^{J,B} ((\eta_Q - 1) w)(x),
\]

it is clear that the first term in (46) can be handled in an identical manner as in the proof of Proposition 5.9 from [6]. Specifically, since \( (\text{supp}(\eta_Q - 1)) \cap 2Q = \emptyset \), the off-diagonal estimates of the operator \( \Theta_t^{J,B} \) lead to
\[
\int_Q \left| \Theta_t^{J,B} ((\eta_Q - 1) w)(x) \right|^2 \, dx \lesssim \frac{t \, |Q|}{t},
\]
which implies that
\[
\int \int_{R_Q} \left| \Theta_t^{J,B} w_Q(x) - \tilde{\gamma}_t^{J,B}(x) A_t w_Q(x) \right|^2 \frac{dx \, dt}{t} \lesssim |Q|.
\]
As for the second term in (46),
\[
\int \int_{R_Q} \left| \Theta_t^{J,B} w_Q^w(x) - \tilde{\gamma}_t^{J,B}(x) A_t w_Q^w(x) \right|^2 \frac{dx \, dt}{t} \lesssim \int \int_{R_Q} \left| \Theta_t^{J,B} (I - P_t^J) w_Q^w(x) \right|^2 \frac{dx \, dt}{t} + \int \int_{R_Q} \left| \Theta_t^{J,B} P_t^J w_Q^w(x) - \tilde{\gamma}_t^{J,B}(x) A_t w_Q^w(x) \right|^2 \frac{dx \, dt}{t}. 
\]
(47)

Since \( w_Q^w \in R(\Gamma J) \), Corollary 3.1 gives
\[
\int \int_{R_Q} \left| \Theta_t^{J,B} (I - P_t^J) w_Q^w \right|^2 \frac{dx \, dt}{t} \lesssim \|w_Q^w\|^2 \lesssim |Q|.
\]

For the remaining term in (47),
\[
\int \int_{R_Q} \left| \Theta_t^{J,B} P_t^J w_Q^w(x) - \tilde{\gamma}_t^{J,B}(x) A_t w_Q^w(x) \right|^2 \frac{dx \, dt}{t} \lesssim \int \int_{R_Q} \left| \Theta_t^{J,B} (I - P_3) P_t^J w_Q^w \right|^2 \frac{dx \, dt}{t} + \int \int_{R_Q} \left| \Theta_t^{J,B} P_t^J w_Q^w(x) - \tilde{\gamma}_t^{J,B}(x) A_t w_Q^w(x) \right|^2 \frac{dx \, dt}{t}. 
\]
(48)

Since the first and second components of our main square function estimate have already been proved to be bounded we have
\[
\int \int_{R_Q} \left| \Theta_t^{J,B} (I - P_3) P_t^J w_Q^w \right|^2 \frac{dx \, dt}{t} \lesssim \|w_Q^w\|^2 \lesssim |Q|.
\]
For the second term in (48),
\[
\int \int_{R_Q} \left| \tilde{\Theta}_t^{J,B} P_t^{J} u_Q^w(x) - \gamma_t^{J,B} (x) A_t u_Q^w(x) \right|^2 \frac{dx dt}{t} \lesssim \int \int_{R_Q} \left| \tilde{\Theta}_t^{J,B} P_t^{J} u_Q^w(x) - \gamma_t^{J,B} (x) A_t P_t^{J} u_Q^w(x) \right|^2 \frac{dx dt}{t} \\
+ \int \int_{R_Q} \left| \gamma_t^{J,B} (x) P_t^{J} (A_t - A_t) u_Q^w(x) \right|^2 \frac{dx dt}{t}.
\]

To bound the first term on the right-hand side of the above estimate notice that
\[
\tilde{\Theta}_t^{J,B} P_t^{J} u_Q^w(x) - \gamma_t^{J,B} (x) A_t P_t^{J} u_Q^w(x) = \Theta_t^{J,B} P_3 P_t^{J} u_Q^w(x) - \gamma_t^{J,B} (x) A_t P_3 P_t^{J} u_Q^w(x) \\
= \left( \Theta_t^{J,B} - \gamma_t^{J,B} (x) A_t \right) v(x),
\]
where \( v := P_3 P_t^{J} u_Q^w \). Proposition 3.6 then gives
\[
\left\| \left( \Theta_t^{J,B} - \gamma_t^{J,B} A_t \right) v \right\| \lesssim \| t \nabla v \| = \| t \nabla P_3 P_t^{J} u_Q^w \|.
\]
Since \( u_Q^w \in R(\Gamma_J) \) it follows that \( u_Q^w = \Gamma_J \tilde{v} \) for some \( \tilde{v} \in D(\Gamma_J) \). Lemma 3.1 then implies that
\[
P_3 P_t^{J} u_Q^w = P_3 \Gamma_J P_t^{J} \tilde{v} = \Gamma_0 P_t^{J} \tilde{v} \in R(\Gamma_0).
\]
This allows us to apply (H8) for the operator \( \Gamma_0 \) to obtain
\[
\left\| \left( \Theta_t^{J,B} - \gamma_t^{J,B} A_t \right) P_3 P_t^{J} u_Q^w \right\| \lesssim \| t \Pi_0 P_t^{J} u_Q^w \|.
\]
An application of (H8J) then leads to
\[
\left\| \left( \Theta_t^{J,B} - \gamma_t^{J,B} A_t \right) P_3 P_t^{J} u_Q^w \right\| \lesssim \| t \Pi_J P_t^{J} u_Q^w \|.
\]
Therefore
\[
\int \int_{R_Q} \left| \tilde{\Theta}_t^{J,B} P_t^{J} u_Q^w(x) - \gamma_t^{J,B} (x) A_t P_t^{J} u_Q^w(x) \right|^2 \frac{dx dt}{t} \lesssim \int_0^{\infty} \| Q_t^{J} u_Q^w \|^2 \frac{dt}{t} \\
= \frac{1}{2} \| u_Q^w \|^2 \\
\lesssim |Q|.
\]
Finally, the boundedness of the term
\[
\int \int_{R_Q} \left| \gamma_t^{J,B} (x) P_3 A_t \left( P_t^{J} - I \right) u_Q^w(x) \right|^2 \frac{dx dt}{t} = \int \int_{R_Q} \left| \gamma_t^{J,B} A_t P_3 A_t \left( P_t^{J} - I \right) u_Q^w(x) \right|^2 \frac{dx dt}{t}
\]
follows immediately from Proposition 4.1 and the uniform \( L^2 \)-boundedness of the operators \( \gamma_t^{J,B} A_t \).

5. Applications

Our non-homogeneous framework will now be applied to three different contexts. We begin with the case that serves as the primary motivation for this article, the scalar Kato square root problem with zero-order potential.
5.1. **Scalar Kato with Zero-Order Potential.** Theorem [1.1] the promised result of the introductory section, will now be proved. Brand the definition of the operators $\Gamma_{|V|^{\frac{1}{2}}}$, $B_1$ and $B_2$ to be as follows. Define our Hilbert space to be

$$\mathcal{H} := L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n; \mathbb{C}^n),$$

for some $n \in \mathbb{N}^*$. Let $V : \mathbb{R}^n \rightarrow \mathbb{C}$ be a complex-valued measurable function that is finite almost everywhere on $\mathbb{R}^n$. Set $J = |V|^{\frac{1}{2}}$ and $D = \nabla$. Our operator $\Gamma_J$ is then given by

$$\Gamma_J = \Gamma_{|V|^{\frac{1}{2}}} = \begin{pmatrix} 0 & 0 & 0 \\ |V|^{\frac{1}{2}} & 0 & 0 \\ \nabla & 0 & 0 \end{pmatrix},$$

defined on the dense domain $H^1_V(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n; \mathbb{C})$, where $H^1_V(\mathbb{R}^n)$ is as defined in the introductory section. The density of $H^1_V(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$ follows from the measurability of $|V|^{\frac{1}{2}}$. The adjoint of this operator is given by

$$\Gamma^*_{|V|^{\frac{1}{2}}} = \begin{pmatrix} 0 & |V|^{\frac{1}{2}} & -\text{div} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$ 

Let $A \in L^\infty(\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n))$ be a matrix-valued multiplication operator and suppose that the Gårding inequalities (1) and (5) are satisfied with constants $\kappa_A > 0$ and $\kappa^V_A > 0$ respectively. Define our perturbations $B_1$ and $B_2$ through

$$B_1 = I \quad \text{and} \quad B_2 := \begin{pmatrix} I \\ 0 \\ e^{i \text{arg} V} \\ 0 \\ 0 \\ A \end{pmatrix}.$$ 

Our perturbed Dirac-type operator then becomes

$$\Pi_{|V|^{\frac{1}{2}}, B} := \Gamma_{|V|^{\frac{1}{2}}} + \Gamma^*_{|V|^{\frac{1}{2}}} \begin{pmatrix} I & 0 & 0 \\ 0 & e^{i \text{arg} V} & 0 \\ 0 & 0 & A \end{pmatrix} = \begin{pmatrix} 0 & |V|^{\frac{1}{2}} & -\text{div}A \\ |V|^{\frac{1}{2}} & 0 & 0 \\ \nabla & 0 & 0 \end{pmatrix}.$$ 

It is straightforward to check that

$$\left( V - \text{div}A \nabla \right) \cdot \left( |V|^{\frac{1}{2}} e^{i \text{arg} V} \right) = V - \text{div}A \nabla = \mathcal{L}_V^A,$$

with correct domains. Thus the square of our perturbed Dirac-type operator is

$$\Pi^2_{|V|^{\frac{1}{2}}, B} = \begin{pmatrix} V - \text{div}A \nabla & 0 & 0 \\ 0 & V & -|V|^{\frac{1}{2}} \text{div}A \\ 0 & \nabla |V|^{\frac{1}{2}} e^{i \text{arg} V} & -\nabla \text{div}A \end{pmatrix}.$$ 

It is clear from the form of our operator $\Gamma_0$ and the fact that $A$ satisfies (1) that the operators $\{\Gamma_0, B_1, B_2\}$ satisfy (H1) - (H8). Similarly, since $A$ and $V$ satisfy (5), it follows that $\{\Gamma_J, B_1, B_2\}$ will satisfy the properties (H1) - (H6).

The proof of Theorem [1.1] can now be concluded. Let $V \in \mathcal{W}$. Then (H8J) will clearly be satisfied. Theorem [1.1] then follows immediately from Corollary [3.2]
5.1.1. *Scalar Potentials that Satisfy the Kato Estimate.* At this stage the unperturbed condition \( V \in W \) is still in quite an abstract form. It will therefore be instructive to unpack this condition and compare \( W \) with other commonly used classes of potentials. Recall the definition of the reverse Hölder class of potentials.

**Definition 5.1.** A non-negative and locally integrable function \( V : \mathbb{R}^n \to \mathbb{R} \) is said to satisfy the reverse Hölder inequality with index \( 1 < q < \infty \) if there exists \( C > 0 \) such that

\[
\left( \frac{1}{|B|} \int_B V^q \, dx \right)^{\frac{1}{q}} \leq C \left( \frac{1}{|B|} \int_B V \, dx \right)
\]

holds for every ball \( B \subset \mathbb{R}^n \). Let \( RH_q \) denote the class of all potentials that satisfy the reverse Hölder inequality of index \( q \).

**Remark 5.1.** It is obvious that any potential bounded both from above and below must be contained in \( RH_q \) for any \( 1 < q < \infty \). It is also well-known that for any polynomial \( P \), \( |P| \) will be contained in \( RH_q \) for any \( 1 < q < \infty \) (this is given as an exercise in [20] on pg. 219 for example).

The reverse Hölder classes \( RH_q \) have played a very influential role in the development of the harmonic analysis of Schrödinger operators. These potentials form a natural class for the construction of numerous harmonic analytic objects associated with the Schrödinger operator. Indeed, to name a few important results, this development led to the construction of both a Hardy space ([9]) and a Muckenhoupt weight class ([7]) associated with \( V - \Delta \). The most important result for our purposes is the boundedness of Riesz transforms associated with the Schrödinger operator for reverse Hölder potentials. The following result was first proved by Z. Shen in the seminal paper [19] for dimension \( n \geq 3 \) and \( q \geq \frac{n}{2} \). This result was later improved and extended to arbitrary dimension by P. Auscher and B. Ali in [2].

**Theorem 5.1 ([19], [2]).** For any \( V \in RH_q \) with \( q \geq 2 \) there exists a \( c_V > 0 \) for which

\[
\|Vu\|_2 \leq c_V \cdot \|(V - \Delta) u\|_2
\]

for all \( u \in D(V - \Delta) \). That is,

\[
RH_q \subseteq W.
\]

Notice that the above inclusion implies, in particular, that the Kato estimate holds for the absolute value of any polynomial potential. The ensuing proposition demonstrates that the inclusion of the reverse Hölder potentials in \( W \) is strict, at least in dimension \( n > 4 \).

**Proposition 5.1.** For \( n > 4 \),

\[
L^{\frac{2n}{n+2}}(\mathbb{R}^n) \subset W.
\]

**Proof.** Fix \( V \in L^{\frac{2n}{n+2}}(\mathbb{R}^n) \). Hölder’s inequality gives us

\[
\left\| \frac{V}{(V - \Delta)^{-1}} u \right\|_2 \leq \|V\|_{\frac{2n}{n+2}} \cdot \left\| (V - \Delta)^{-1} u \right\|_{\frac{2n}{n+2}}.
\]

It is well-known that the Riesz potential \( (V - \Delta)^{-1} \) is bounded from \( L^2(\mathbb{R}^n) \) to \( L^{\frac{2n}{n+2}}(\mathbb{R}^n) \) (see for example Theorem 4 of [7]). There must then exist some \( C_V > 0 \) for which

\[
\left\| \frac{V}{(V - \Delta)^{-1}} u \right\|_2 \leq \|V\|_{\frac{2n}{n+2}} \cdot C_V \cdot \|u\|_2
\]
5.2. Systems with Zero-Order Potential. Fix \( m \in \mathbb{N}^* \) and \( A \in L^\infty (\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n \otimes \mathbb{C}^m)) \). Let \( V : \mathbb{R}^n \to \mathcal{L}(\mathbb{C}^m) \) be a measurable matrix-valued function with coefficients that are finite almost everywhere. \( V \) can be viewed as a densely defined closed multiplication operator on \( L^2 (\mathbb{R}^n; \mathbb{C}^m) \) with domain
\[
D(V) = \{ u \in L^2 (\mathbb{R}^n; \mathbb{C}^m) : V \cdot u \in L^2 (\mathbb{R}^n; \mathbb{C}^m) \}.
\]
It will be assumed that there exists some \( U \in L^\infty (\mathbb{R}^n; \mathcal{L}(\mathbb{C}^n \otimes \mathbb{C}^m)) \) for which \( V \) has the decomposition
\[
V(x) = |V(x)|^{\frac{1}{2}} \cdot U(x) \cdot |V(x)|^{\frac{1}{2}}
\]
for each \( x \in \mathbb{R}^n \), where \(|V(x)| := \sqrt{V(x)^* V(x)}\). Similar to the scalar case, one can define forms \( I_A \) and \( I_A^V \) defined respectively through
\[
I_A [u, v] := \int_{\mathbb{R}^n} \langle A(x) \nabla u(x), \nabla v(x) \rangle \, dx
\]
for \( u, v \in H^1 (\mathbb{R}^n) \) and
\[
I_A^V [u', v'] := I_A [u', v'] + \int_{\mathbb{R}^n} \langle V(x) u'(x), v'(x) \rangle \, dx
\]
for \( u' \) and \( v' \) contained in \( H^1_V (\mathbb{R}^n; \mathbb{C}^m) := H^1 (\mathbb{R}^n; \mathbb{C}^m) \cap D(|V|^{\frac{1}{2}}) \).
Assume that the forms \( I_A \) and \( I_A^V \) satisfy the Gårding inequalities [4] and [5] with constants \( \kappa_A > 0 \) and \( \kappa_A^V > 0 \) respectively. Then \( I_A \) and \( I_A^V \) will both have a unique associated maximal accretive operator, \( \mathcal{L}_A \) and \( \mathcal{L}_A^V \). In the below theorem, our non-homogeneous framework will be applied to determine the domain of \( \sqrt{\mathcal{L}_A^V} \) for a wide class of potentials.

**Theorem 5.2.** Suppose that there exists \( c_V > 0 \) such that
\[
(49) \quad \| \Delta u \| + \| V \cdot u \| \leq c_V \cdot \| (|V| - \Delta) u \|
\]
for all \( u \in D(|V| - \Delta) \). Then there must exist some \( C_V > 0 \) such that
\[
C_V^{-1} \left( \| |V|^{\frac{1}{2}} u \| + \| \nabla u \| \right) \leq \| \sqrt{\mathcal{L}_A^V} u \| \leq C_V \cdot \left( \| |V|^{\frac{1}{2}} u \| + \| \nabla u \| \right)
\]
for all \( u \in D(\mathcal{L}_A^V) \).

**Proof.** Set
\[
D := \nabla : H^1 (\mathbb{R}^n; \mathbb{C}^m) \subset L^2 (\mathbb{R}^n; \mathbb{C}^m) \to L^2 (\mathbb{R}^n; \mathbb{C}^m \otimes \mathbb{C}^m)
\]
and
\[
J := |V|^{\frac{1}{2}} : D(|V|^{\frac{1}{2}}) \subset L^2 (\mathbb{R}^n; \mathbb{C}^m) \to L^2 (\mathbb{R}^n; \mathbb{C}^m),
\]
both defined as operators on \( L^2 (\mathbb{R}^n; \mathbb{C}^m) \). Define the perturbation matrices
\[
B_1 := I \quad \text{and} \quad B_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & A \end{pmatrix}.
\]
It is not too difficult to see that the operators \( \{ \Gamma, B_1, B_2 \} \) will satisfy conditions (H1) - (H8) and \( \{ \Gamma, B_1, B_2 \} \) will satisfy (H1) - (H6). Indeed, the only non-trivial...
condition for both sets of operators is (H2) and this follows from the respective Gårding inequalities \([1]\) and \([5]\). It is also clear from (49) that (H8J) will be satisfied. The Kato estimate then follows from Corollary 3.2.

5.3. **First Order Potentials.** Let \(b: \mathbb{R}^n \to \mathbb{C}^m\) be measurable and finite almost everywhere and \(A \in L^\infty(\mathbb{R}^n; \mathbb{C}^n)\). We will prove two different Kato estimates for first order potentials.

5.3.1. **First Kato Estimate.** Define the Hilbert space \(\mathcal{H}\) to be

\[
\mathcal{H} := L^2(\mathbb{R}^n; \mathbb{C}) \oplus L^2(\mathbb{R}^n; \mathbb{C}^{3n}) \oplus L^2(\mathbb{R}^n; \mathbb{C}^n).
\]

Set

\[
J := \begin{pmatrix} b \\ b \\ \nabla \end{pmatrix} : L^2(\mathbb{R}^n; \mathbb{C}) \to L^2(\mathbb{R}^n; \mathbb{C}^{3n}) \quad \text{and} \quad D := \nabla : L^2(\mathbb{R}^n; \mathbb{C}) \to L^2(\mathbb{R}^n; \mathbb{C}^n).
\]

Let \(B_1 = I\) as usual and

\[
B_2 := \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & A & 0 \\ 0 & 0 & 0 & A \end{pmatrix}.
\]

The operator \(L_J^b\) as in Corollary 3.2 is

\[
L_J^b := L_J^b = b^* A b - \text{div} A b + b^* A \nabla - \text{div} A \nabla = (\nabla + b)^* A (\nabla + b).
\]

Suppose that \(A\) satisfies the ellipticity condition

\[
\Re \langle Au, u \rangle \geq \kappa \|u\|^2
\]

for all \(u \in L^2(\mathbb{R}^n; \mathbb{C}^n)\), for some \(\kappa > 0\). Then (H2) will be satisfied for both sets of operators \(\{\Gamma_0, B_1, B_2\}\) and \(\{\Gamma_J, B_1, B_2\}\). Therefore \(\{\Gamma_0, B_1, B_2\}\) will satisfy (H1) - (H8) and \(\{\Gamma_J, B_1, B_2\}\) will satisfy (H1) - (H6). The below theorem then follows as an immediate application of Corollary 3.2.

**Theorem 5.3.** Suppose that there exists some \(c_b > 0\) for which

\[
\|b^* b u\| + \|\Delta u\| \leq c_b \left((\nabla + b)^* (\nabla + b) u\right)
\]

for all \(u \in D ((\nabla + b)^* (\nabla + b))\). Then there exists some constant \(C_B > 0\) for which

\[
C_b^{-1} \cdot (\|b^* u\| + \|\nabla u\|) \leq \sqrt{L_A^b u} \leq C_B \cdot (\|b^* u\| + \|\nabla u\|)
\]

for all \(u \in D (L_A^b)\).

Indeed, the unperturbed Riesz transform bound in the above theorem implies (H8J) holds in this context.
5.3.2. Second Kato Estimate. For a result of a slightly different flavour, one could alternatively set the Hilbert space to be

\[ \mathcal{H} := L^2(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n; \mathbb{C}^n) \oplus L^2(\mathbb{R}^n; \mathbb{C}^n). \]

Then set

\[ J := \nabla + b : L^2(\mathbb{R}^n; \mathbb{C}) \to L^2(\mathbb{R}^n; \mathbb{C}^n) \quad \text{and} \quad D := \nabla : L^2(\mathbb{R}^n; \mathbb{C}) \to L^2(\mathbb{R}^n; \mathbb{C}^n). \]

Also let

\[ B_1 = I \]

and

\[ B_2 = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & A \end{pmatrix}. \]

The operator \( L^J_{B_2} \) as in Corollary 3.2 is then given by

\[ \tilde{L}^b_A := L^J_{B_1} = (b + \nabla)^* (b + \nabla) - \div A \nabla. \]

Suppose that \( A \) satisfies the standard Gårding inequality

\[ \Re \int_{\mathbb{R}^n} (A(x) \nabla u(x), \nabla u(x)) \, dx \geq \kappa \cdot \| \nabla u \|^2 \]

for all \( u \in H^1(\mathbb{R}^n; \mathbb{C}) \), for some \( \kappa > 0 \). Then \( \{ \Gamma_0, B_1, B_2 \} \) and \( \{ \Gamma_J, B_1, B_2 \} \) will both satisfy (H2). This in turn implies that \( \{ \Gamma_0, B_1, B_2 \} \) satisfies (H1) - (H8) and \( \{ \Gamma_J, B_1, B_2 \} \) satisfies (H1) - (H6). The non-homogeneous framework, in the form of Corollary 3.2 applied to these operators then produces the following theorem.

**Theorem 5.4.** Suppose that there exists some \( c_b > 0 \) such that

\[ \| \Delta u \| \leq c_b \left\| \left[ (\nabla + b)^* (\nabla + b) - \Delta \right] u \right\| \]

for all \( u \in D \left( (\nabla + b)^* (\nabla + b) - \Delta \right) \). Then there exists some constant \( C_b > 0 \) for which

\[ C_b^{-1} \cdot \left( \| (\nabla + b) u \| + \| \nabla u \| \right) \leq \sqrt{\tilde{L}^b_A u} \leq C_b \cdot \left( \| (\nabla + b) u \| + \| \nabla u \| \right) \]

for all \( u \in D \left( \tilde{L}^b_A \right) \).

To see that the above theorem is true, simply note that (50) implies (H8J) in this context.

6. Final Remarks

It is important to note that this is not the first time that Kato type estimates have been studied for non-homogeneous operators. We will now take some time to outline how our article differs in techniques and results from each of these previous forays.

Recently, in \([13]\) and \([14]\), F. Gesztesy, S. Hofmann and R. Nichols studied the domains of square root operators using techniques distinct from those developed in \([6]\). The article \([13]\) considers potentials in the class \( L^p + L^\infty \) but is not directly relevant since it considers bounded domains. On the other hand, \([14]\) does not impose a boundedness assumption on the domain and considers the potential class \( L^{\frac{n}{2}} + L^\infty \). There is already an immediate comparison with our potential class since it was shown in Proposition 5.1 that \( L^{\frac{n}{2}} \subset \mathcal{W} \) in dimension \( n > 4 \). It is not immediately clear whether \( L^\infty \) is contained within our class.
Axelsson, Keith and McIntosh themselves considered non-homogeneous operators on Lipschitz domains with mixed boundary conditions in [5]. The potentials that they considered were, however, bounded both from above and below and thus contained in \( RH_2 \subset W \). In [10] and [11], M. Egert, R. Haller-Dintelmann and P. Tolksdorf generalised this to certain non-smooth domains.

The articles [5], [11] and [10] are all built upon the original AKM framework, similar to this one. A key step in the original proof of the AKM framework is the proof of the estimate

\[
\int_0^\infty \| (A_t - P_t) u \|^2 \frac{dt}{t} \lesssim \| u \|^2.
\]

This estimate allows for the \( A_t \) and \( P_t \) operators to be freely interchanged at several stages in the proof granting use of some of the more enviable properties of the \( A_t \) operator. This equivalence will no longer hold in the potential dependent setting and presents a significant obstruction. The articles [5], [11] and [10] circumvent this problem by imposing boundedness of the potential from above and below. The boundedness of the potential from below allows one to reduce the main square function estimate to the local square function estimate

\[
\int_0^1 \left\| \Theta^{B,|V|^12} P_t^{|V|^12} u \right\|^2 \frac{dt}{t} \lesssim \| u \|^2
\]

for all \( u \in R \left( \Gamma_{|V|^12} \right) \). Then one only requires a local version of (51) to hold, namely

\[
\int_0^1 \left\| \left( A_t - P_t^{|V|^12} \right) u \right\|^2 \frac{dt}{t} \lesssim \| u \|^2
\]

for all \( u \in R \left( \Gamma_{|V|^12} \right) \). Such an estimate will be true for any potential bounded from above.

This is a crude explanation as to why the techniques developed in [5] cannot be directly applied for a general potential that is not bounded both from above and below. There are similar obstructions, for example in the selection of test functions in the Carleson measure proof. However, these also disappear when the potential is bounded both from above and below.

In this paper, our method has been to instead exploit the algebraic structure of the operators \( \Gamma_{|V|^12} \), \( B_1 \) and \( B_2 \). This exploitation has allowed us to conclude that the estimate (51) will at least hold on the third component. Similar obstructions in the proof of the main square function estimate also vanish when considered component-wise. As a consequence of this three-by-three mindset we have been able to obtain square function estimates for potentials that aren’t necessarily bounded from above or below and, moreover, are not contained in \( L^p (\mathbb{R}^n) \) for any \( 1 \leq p \leq \infty \).

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