SIGNATURES OF PARTITION FUNCTIONS AND THEIR COMPLEXITY REDUCTION THROUGH THE KP II EQUATION

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Abstract. A statistical amoeba arises from a real-valued partition function when the positivity condition for pre-exponential terms is relaxed, and families of signatures are taken into account. This notion lets us explore special types of constraints when we focus on those signatures that preserve particular properties. Specifically, we look at sums of determinantal type, and main attention is paid to a distinguished class of soliton solutions of the Kadomtsev-Petviashvili (KP) II equation. A characterization of the signatures preserving the determinantal form, as well as the signatures compatible with the KP II equation, is provided: both of them are reduced to choices of signs for columns and rows of a coefficient matrix, and they satisfy the whole KP hierarchy. Interpretations in term of information-theoretic properties, geometric characteristics, and the relation with tropical limits are discussed.

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1. Introduction

The concept of partition function encodes in a single object the statistical data compatible with the physical constraints of a system, e.g., conservation laws. Hence, it explicitly connects its probabilistic and physical characteristics. This is a fundamental principle in the investigation of composite systems [34, 21] and is now applied in many different branches of sciences [32]. The basic form of the partition function involves a sum of exponential terms

$$Z := \sum_{\alpha \in \mathbb{I}} g_{\alpha} \cdot e^{-\frac{\varepsilon_{\alpha}}{k_B T}}$$

(1.1)

where $k_B$ is Boltzmann’s constant, $T$ is the temperature, $\varepsilon_{\alpha}$ are the energy levels labelled by an indexing set $\mathbb{I}$ and $g_{\alpha}$ are the associated degenerations.

In practice, the implementation of this approach is limited by the intrinsic complexity of the model, since there are few cases where the partition function can be evaluated exactly, e.g., in a closed form [6]. Indeed, many phenomena in complex systems cannot be reduced to their individual components, thus (1.1) involves collections of objects and, consequently, the complexity of the calculation grows exponentially with the size of the system.

The occurrence of exact formulas for the partition function implies relations between the characterizing quantities of the system, e.g., correlation functions that are generated by the
partition function through derivation. This gives rise to a family of differential equations whose compatibility follows from the existence of the partition function. On the other hand, one can start from a family of differential equations and look for their compatibility. This naturally leads to the investigation on connections between the partition function formalism and the concept of integrability, especially integrable hierarchies. In such a context, the role of the partition function is similar to the notion of $\tau$-function, which provides one with a unifying framework for hierarchies of nonlinear partial differential equations and their remarkable behaviours, e.g., symmetries, infinite family of commuting flows, soliton solutions [28, 25].

Explicit links between the partition function formalism and integrable systems have already been identified and used to provide fundamental techniques in modern theoretical physics [6]. Remarkably, certain solutions of integrable PDEs can be interpreted as potentials, such as the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations in topological quantum field theories. A solution for the WDVV system defines a free energy for this theory, namely, a function which generates correlators by means of derivation [19].

Recent studies have been devoted to the combinatorics underlying these kinds of structures. One of most fruitful examples is the Kadomtsev-Petviashvili II (KP II) equation

$$\partial_x (-4 \cdot \partial_t u + \partial_{xxx} u + 6u \cdot \partial_x u) = -3 \partial_{yy} u$$

(1.2)

where $\partial_x u$ denotes the partial derivative of $u \equiv u(x,y,t)$ with respect to $x$. The KP II equation is one of most important $(2 + 1)$-integrable PDEs and is considered a universal model of two-dimensional integrable evolutionary equations. The combinatorial structures arise, for example, taking into account particular classes of solutions of the KP II equation, such as soliton solutions of determinantal (Wronskian) type [25]. These solutions are parametrized by points in the real Grassmannian space and their regularity and algebraic features have been extensively studied in the last years [8, 13, 16, 33, 1].

The aim of this work is to explore these requirements starting from a statistical perspective: we choose a class of combinatorial configurations related to sums of exponentials that generalise (1.1). Then, we investigate the information content encoded in constraints through the reduction of the class of allowed configurations. This purpose will be realised extending the method of statistical amoebas introduced in [5], i.e., considering a family of discrete “deformations” of the partition function and focusing on those that are compatible with some given constraints. A statistical amoeba is obtained relaxing the assumption of positive degenerations $g_{\alpha}$ in (1.1). Negative degenerations of energy levels can be related to an imaginary part of energies $\varepsilon_{\alpha}$ and, hence, used to describe metastable states [35, 41]. This approach has proved useful in statistical physics, where the locus of zeros of complex-valued partition functions is employed in the analysis of phase transitions [36, 37, 35, 41, 9, 14]. In our context, the partition function is real-valued, but indefinite signs for degenerations $g_{\alpha}$ open up the way to the study of stability ($Z > 0$), instability ($Z < 0$) and phase transitions ($Z = 0$).

In general, the requirement of compatibility of the choices of signs for $g_{\alpha}$ with a given constraint affects both the number and the form of allowed configurations. We will focus on determinantal relations, where the complexity of calculations of determinants is
polynomial via Gaussian reduction (while other inmanents, in general, have exponential complexity \[45\]), and integrability, which reduces the complexity through the occurrence of conserved quantities. We stress this point referring to the construction in [5], where all the combinations of signs for the \(N\) exponential terms in the partition function are allowed, as a free statistical amoeba. On the other hand, restrictions on the allowed combinations of signs give rise to a constrained statistical amoeba.

A strong relation between determinantal relations and integrability has been established by Sato (see, e.g., [44] and further developments in [40]), where an expansion in terms of Schur functions is found to be a \(\tau\)-function for the KP hierarchy if and only if its coefficients satisfy the Grassmann-Plücker relations [23, 29]. This has led to intensive research on equivalent reformulations of the KP hierarchy such as these involving the infinite Grassmannian [44, 40]. Whilst sharing some concepts with these issues, the present investigation is devoted to families of functions generated by a special class of solutions of the KP equation. It should be remarked that we concentrate only on the first equation (1.2) in the hierarchy for what concerns constraints reducing the complexity, and that we work in finite dimensionality. Moreover, the construction of statistical amoebas does not rely only on the global (determinantal) form of the initial function, but also on the individual terms in its expansion, because the generation of other functions goes through changes of signs for a given expression of the type (1.1). The consistency check with constraints and the construction of allowed configurations depend on relations between non-vanishing terms.

When we start from solitons in Wronskian form, the constraints come either from the determinantal relations or from the fulfilment of the KP II equation. In fact, we will see that both these assumptions reduce the allowed combinations of signs to the same family, i.e., the set of transformations that can be obtained by flipping the signs of some rows and/or columns of a coefficient matrix \(A\): this is showed in Theorem 11. The complexity reduction takes place because the information about the chosen signature for the terms in the exponential sum, which are labelled by subsets of \(\{1, \ldots, n\}\), is stored in the elements of \(\{0, 1, \ldots, n\}\): in general, this presentation is non-unique due to a certain equivalence relation (Proposition 15). The connection with other equations in the hierarchy follows from the special form of the transformations in this family (Theorem 12).

The main scope of this work is the formal derivation of the above-mentioned properties. However, it is important to highlight potential applications of this approach: for instance, choosing a coefficient matrix and constructing the corresponding soliton solution, one can encode a sequence of “bits” in the signs of the pre-exponential terms, and recover a special family of sequences checking that the KP II equation is satisfied. We will quantify the information content following from the check of the KP II equation through the Kullback-Leibler divergence between two probability distributions on the strings of signatures. These results also have geometric implications, and we will briefly discuss the links with oblique projections and their statistical relevance. Likewise, we will point out the connection with the tropical limit in statistical physics as introduced in [3] and developed in [2]. These issues could be of interest for a better understanding of the relation
between statistical physics, complex systems and learning methods [7, 31, 38]. We will outline some of these connections and postpone their detailed study to a separate article.

The paper is organized as follows: in Section 2 we fix the notation and summarize the basic notions that provide a starting point for our investigation. In Section 3 we introduce the link between soliton solutions of the KP hierarchy and statistical amoebas, generalizing the latter to higher dimensions. The occurrence of a particular structure to be preserved in the construction of statistical amoebas is the focus of the next sections: in Section 4 we prove that the compatibility of a choice of signs for the coefficients of exponential terms with the KP II equation implies that it is induced by rows and columns sign flips for a coefficient matrix. In Section 5 we consider the number of distinct configurations that can be obtained in this way. The relation with the strata of the corresponding free amoeba [5] is considered in Section 6. Two applications of the present framework are addressed in Section 7: the information content in the KP II constraint is studied via the Kullback-Leibler divergence (Subsection 7.1), while an intersection property is discussed in geometric terms (Subsection 7.2). Finally we draw conclusions and address some issues of potential interest for future investigations in Section 8.

2. Preliminaries

We briefly summarise the basic state of the art regarding Wronskian soliton solutions of the KP II equation and the statistical amoeba formalism which will be used in the rest of the paper. Before that, it is worth introducing some notations to enhance clearness.

2.1. Notation. In the following, we will denote by $P[n]$ the power set of $[n] := \{1, \ldots, n\}$, by $P_k[n]$ the collection of subsets of $[n]$ with $k$ elements and by $I \Delta J := (I \setminus J) \cup (J \setminus I)$ the symmetric difference of $I, J \in P[n]$. We will also use the notation $\{\pm 1\} := \{+1, -1\}$.

\[ I_\alpha := I \{\alpha\} \cup \{\beta\} \quad \alpha \in I, \beta \notin I \]

and, similarly, $I_\beta := I \cup \{\beta\}$, $I_\alpha := I \{\alpha\}$, $I_{\alpha_1 \alpha_2} := I \{\alpha_1, \alpha_2\}$, etc. The expression $I_\alpha$ implicitly assumes that $\alpha \in I$, and $\beta \notin I$, unless $\alpha = \beta$, in which case we have $I_\alpha := I$.

The symbol $\Delta_A(I)$ (respectively $\Delta_K(I)$) is the maximal minor of $A \in \mathbb{R}^{k \times n}$ (respectively, $K$) whose columns (respectively, rows) are indexed by $I \in P_k[n]$. We will occasionally use $\Delta(A; I)$ instead of $\Delta_A(I)$ for the sake of clearness. When a permutation $\pi$ of $[n]$ is involved, an additional sign comes from the parity of the number of inversions induced by $\pi$ on $H$. Particular attention will be paid to the set of pivot columns $V := \{\nu_1, \ldots, \nu_k\}$, which is the least element of $P_k[n]$ (with respect to the lexicographical order) associated with a non-vanishing minor $\Delta_A(V)$.

The $(n \times k)$-Vandermonde matrix relative to real parameters $\kappa_1, \ldots, \kappa_n$ is $K := (\kappa_\alpha^{-1})_{\alpha \in [n]}$. The determinant of a general (not necessarily maximal) minor of $K$ is

\[ \text{VdM}(\kappa; H) := \det \left( K_{H \times [k]} \right) = \prod_{\alpha < \beta} (\kappa_\beta - \kappa_\alpha), \quad H \subseteq [n]. \]

So $K$ has maximal rank in cases of pairwise distinct soliton parameters $\kappa_1, \ldots, \kappa_n$, as we will assume. We will look at exponential sums whose terms also depend on variables $x \in \mathbb{R}^d, d \geq 3$, where $x_1 = x$, $x_2 = y$, and $x_3 = t$. 
The real Grassmannian $\mathcal{G}R_{k,n}(\mathbb{R})$ is the space of $k$-dimensional linear subspaces of a $n$-dimensional vector space over $\mathbb{R}$. It can be presented as the quotient of the space of $k$-dimensional frames in $\mathbb{R}^n$ by the left action of $GL_k(\mathbb{R})$. If $\mathbb{R}^{k\times n}_{\text{max}}$ denotes the set of real $(k \times n)$-matrices of maximal rank $\min\{k,n\} = k$, one gets
\[ \mathcal{G}R_{k,n}(\mathbb{R}) \cong \mathbb{R}^{k\times n}_{\text{max}}/GL_k(\mathbb{R}) \] (2.3)
where $GL_k(\mathbb{R})$ acts as left multiplication. Greek indices $\alpha, \beta \in [n]$ will often represent columns, while Latin indices $i, j \in [k]$ will be used to label rows. Additional notation will be introduced in specific paragraphs.

2.2. KP II equation and Wronskian solutions. Determinantal solitons define a distinguished class of solutions that can be derived from Hirota’s direct method [28, 25] introducing derivatives $D_x$ acting on pairs of functions:
\[ D_x(f \cdot g) := (\partial_x f) \cdot g - f \cdot (\partial_x g) = (\partial_{x_1} - \partial_{x_2})f(x_1)g(x_2)|_{x_1=x_2}. \] (2.4)
So one can rewrite the KP II equation (1.2) in the following homogeneous bilinear form
\[ D_{\text{KP}}(\tau, \tau) := (D_x^4 - 4 \cdot D_x D_t + 3 \cdot D_y^2)\tau \cdot \tau = 0. \] (2.5)
By the same token, one can rewrite the other equations of the KP hierarchy in bilinear form [28, 40]. The tau-function $\tau(x)$ of the KP equation is related to $u$ via
\[ u(x) := 2 \cdot \frac{\partial^2}{\partial x_1^2} \ln \tau(x). \] (2.6)
It has been shown (see, e.g., [24]) that the Hirota derivative (2.4) can be characterized as a derivative operator with gauge invariance under the simultaneous action $f \mapsto e^{\varphi(x)} f$ and $g \mapsto e^{\varphi(x)} g$, i.e.,
\[ D_{\text{KP}}(e^{-\varphi(x)} \cdot f(x), e^{-\varphi(x)} \cdot g(x)) = e^{-2\varphi(x)} \cdot D_{\text{KP}}(f(x), g(x)) \] (2.7)
for any linear function $\varphi(x)$ of $x$. This is manifest in the antisymmetric form of $D_x$ in (2.4) and is also reflected in the expression of a special class of solutions [25]
\[ \text{Wr}(f_1, \ldots, f_n) = \det \left( \partial_{x_1}^{i-1} f_m \right)_{i,m \in [k]} \] (2.8)
where $f_\alpha, \alpha \in [n]$, are independent solutions for the following system of partial differential equations
\[ \frac{\partial}{\partial x_r} f = \frac{\partial \varphi}{\partial x_r} f, \quad r \in [d]. \] (2.9)
One can take a certain number, say $M$, of solutions of (2.9) in the form
\[ E_\alpha(x) = \exp \varphi_\alpha(x) := \exp \left( \sum_{r=1}^d \kappa_\alpha^r x_r \right) \] (2.10)
with real parameters $\kappa_\alpha$. A particular choice of solutions of (2.9) comes from linear combinations of these exponentials, with coefficients given by the entries of a matrix $A$:
\[ f_i(x) = \sum_{\alpha=1}^{n} A_{i\alpha} E_{\alpha}(x), \text{ that is } \vec{f} = \mathbf{A} \cdot \vec{E}. \] One has

\[
\begin{pmatrix}
E_1 & \partial_x E_1 & \cdots & \partial_x^{(k-1)} E_1 \\
\vdots & \vdots & \ddots & \vdots \\
E_n & \partial_x E_n & \cdots & \partial_x^{(k-1)} E_n
\end{pmatrix}
= \begin{pmatrix}
E_1 & \kappa_1 E_1 & \cdots & \kappa_1^{k-1} E_1 \\
\vdots & \vdots & \ddots & \vdots \\
E_n & \kappa_n E_n & \cdots & \kappa_n^{k-1} E_n
\end{pmatrix} = \mathbf{\Theta} \cdot \mathbf{K}
\] (2.11)

where \( \mathbf{\Theta} := \text{diag}(E_1, E_2, \ldots, E_n) \). Finally, the resulting soliton solution is equivalently expressed using the Cauchy-Binet expansion [29] as

\[
\tau(x) := \det(\mathbf{A} \cdot \mathbf{\Theta}(x) \cdot \mathbf{K})
= \sum_{\mathcal{I} \in \mathcal{P}_k[n]} \Delta_{\mathbf{A}}(\mathcal{I}) \cdot \Delta_{\mathbf{K}}(\mathcal{I}) \cdot e^{\sum_{\alpha \in \mathcal{I}} \psi_{\alpha}(x)}.
\] (2.12) (2.13)

It should be remarked that the left action of \( GL_k(\mathbb{R}) \) on \( \mathbb{R}^{k \times n} \), i.e., the multiplication of \( \mathbf{A} \) by a full-rank \( k \times k \) real matrix, induces the multiplication of \( \tau \) by a constant and, hence, leaves solutions (2.6) invariant. Such an action is equivalent to row operations on \( \mathbf{A} \), by a full-rank \( \mathbf{K} \).

The solution (2.6) is regular at \( \tau(x) > 0 \). If \( \tau(x) < 0 \), then \( \ln \tau(x) \) is multivalued, but its imaginary part does not depend on \( x \) and disappears after derivation in (2.6). Thus, possible singularities of soliton solutions are related to the points where \( \tau \) vanishes. If the order \( \kappa_1 < \cdots < \kappa_n \) for soliton parameters is fixed, then the locus of zeros of \( \tau \) is not empty if there exist maximal minors with opposite sign, i.e., if \( \mathbf{A} \) parametrises a point outside the totally non-negative part of the Grassmannian [33].

2.3. **Statistical amoebas.** The analysis of roots of the partition function is a fundamental technique in the study of stability, metastability and phase transitions in composite systems [36, 37, 41, 14]. Finite sums of exponentials of the type (1.1) are positive for real values of energies \( E_n \) and temperature \( T \) and positive degeneracies \( g_n \). So the zeros of the partition function define a singular locus in the complex domain and, in many cases, they approach the real line when the number of terms involved in (1.1) becomes large (thermodynamic limit).

If one restricts to the real domain, the partition function can vanish if not all the degeneracies \( g_n \) have the same sign. This corresponds to real partition function of indefinite signatures and relates to the concept of negative probabilities [46, 18, 20, 11]. This proposal has been developed in [5] for partition functions of the type

\[
\mathcal{Z} := \sum_{\alpha=1}^{N} g_{\alpha} \cdot e^{f_{\alpha}(x)}
\] (2.14)

where the functions \( f_{\alpha}(x) \) represent “micro-free energies” that depend on certain parameters \( x \in \mathbb{R}^d \) (e.g., temperature, external magnetic fields, etc.). When one fixes \( s \leq \frac{N}{2} \), the \( s \)-stratum of the statistical amoeba consists of the zero loci of the functions produced by any possible combination of \( s \) signs among the \( N \) terms in (2.14), i.e., \( g \in \{ \pm 1 \}^N \) with \( \# (g^{-1}(\{-1\})) = s \). In great generality, that is under the only assumption of polynomial functions \( f_{\alpha}(x) \), \( \alpha \in [N] \), a pattern can be found in the study of the singular locus: the
s-stratum is confined in a region $\mathbb{R}^d \setminus D_{s-}$ of the space of parameters $\mathbb{R}^d$, and the instability domains $D_{s-}$ obey the following inclusion property

$$D_{s-} \subseteq D_{\hat{s}-}, \quad 1 \leq s < \hat{s} < \frac{N}{2}. \quad (2.15)$$

The restriction $s < \frac{N}{2}$ also avoids the redundancy given by the simultaneous reversal of all the signs. This is equivalently expressed via the involution $g^{-1}(-1) \to [n] \setminus g^{-1}(-1)$, which preserves the singular locus and exchanges the role of equilibrium ($Z > 0$) and non-equilibrium ($Z < 0$) regions, since $Z(-g) = -Z(g)$. Thus, the strata associated with $s > \frac{N}{2}$ are said to generate the statistical antiamoeba.

When all the $(\begin{smallmatrix} N \\ s \end{smallmatrix})$ combinations of $s$ negative coefficients $g_\alpha$ are taken into account, the set $D_{s-}$ coincides with the locus of points where the maximal number of negative partition functions (2.14) is obtained. This maximum is the same for all the systems with polynomial $f_\alpha$ and equals $(N - 1)s - 1$.

Furthermore, the polynomial assumption is also suitable for the study of the tropical limit [3, 5], both in the linear and in the nonlinear cases (the latter is referred as a multi-scaling tropical limit [4]).

### 3. From soliton solutions to statistical amoebas

The explicit form of many soliton solutions of partial differential equations can be derived from a sum of exponentials [28]. In order to highlight the relation between statistical amoebas and $\tau$-functions, it is worth noting that the partition function (1.1) can be expressed as

$$Z = \det \left( g^T \cdot \Theta(\varepsilon) \cdot K_0 \right) \quad (3.1)$$

where $g := (g_1, g_2, \ldots, g_N)^T$, $\Theta(\varepsilon) := \text{diag} \left( e^{-\varepsilon_{\alpha_1} k_B T}, e^{-\varepsilon_{\alpha_2} k_B T}, \ldots, e^{-\varepsilon_{\alpha_N} k_B T} \right)$ and $K_0 := (1, \ldots, 1)^T$.

In particular, $g$ is a totally positive vector (all its entries are positive) and $K_0$ can be interpreted as a $n \times 1$ Vandermonde matrix. This formula for the partition function coincides with a Wronskian $\tau$-function (2.12).

More generally, we can express such a type of $\tau$-functions as a sum of exponentials through the Cauchy-Binet expansion of the determinant of a product: if one introduces

$$g_I := \Delta_A(I) \cdot \Delta_K(I), \quad \Lambda_I(x) := \Delta_A(I) \cdot \Delta_K(I) \cdot \exp \left( \sum_{\alpha \in I} \phi_\alpha(x) \right), \quad I \in \mathcal{P}_k[n] \quad (3.2)$$

then (2.13) is of the form (2.14),

$$\tau(x) = \det (A \cdot \Theta(x) \cdot K) = \sum_{I \in \mathcal{P}_k[n]} \Lambda_I(x). \quad (3.4)$$

In this way, we move from degenerations multiplicities $g$ to more general products of minors $g_I$. The dimension $k$, which coincides with the rank of $A$ and $K$ when $\tau$ does not identically vanish, indicates the number of line solitons at $x_2 \gg 0$, while $n - k$ is related to line solitons at $x_2 \ll 0$. In the statistical perspective, $\tau$ is the partition function for a statistical model whose configurations correspond to subsets $I \in \mathcal{P}_k[n]$ and have energies...
\[ \sum_{\alpha \in I} \varphi_\alpha. \] If \( \kappa_\alpha \in \mathbb{Z} \) and \( A \in \mathbb{Z}^{k \times n}_+ \) is a matrix with integer entries and maximal rank, then (3.2) is an integer, \( \Delta_A(I) \) and \( \Delta_K(I) \) can be seen as degenerations for independent events and \( \Delta_A(I) \cdot \Delta_K(I) \) is the joint degeneration. The \( \alpha \)-th column of \( A \) generalizes \( g_\alpha \) in (1.1), so it can be regarded as a degeneration vector relative to the \( \alpha \)-th energy level \( \varphi_\alpha(x), \alpha \in [n] \).

Regularity hypotheses for the case \( k = 1 \) can be extended to \( \tau \)-functions at \( k \geq 3 \) too. For example, the entries of \( g \) in (1.1) and (3.1) are assumed to be positive since they are a measure for degenerations associated with energy levels. At \( k > 1 \), this property generalizes to a real matrix \( A \in \mathbb{R}^{k \times n} \) where all the maximal minors are non-negative. If one fixes the ordering \( \kappa_1 < \cdots < \kappa_n \) for soliton parameters, then this request guarantees (indeed, it is equivalent to, see e.g. [12]) the positivity of the Cauchy-Binet expansion (2.13), hence the regularity of the solution of the original KP II equation.

In the next section, this kind of request will be relaxed: the matrix \( A \) is only assumed to obey the full-rank condition. However, some peculiarities of the total non-negative case will be discussed in Section 6.

**Definition 1.** A choice of signs, or signature, is a map

\[ \Sigma : \mathcal{G} \rightarrow \{\pm 1\} \quad (3.5) \]

where

\[ \mathcal{G} := \{I \in P_k[n] : \Delta_A(I) \neq 0\}. \quad (3.6) \]

\( \Sigma \) acts on (3.4) via

\[ g(I) \mapsto \Sigma(I) \cdot g(I), \]

which returns a new function

\[ \sum_{I \in P_k[n]} \Sigma(I) \cdot g_I \cdot e^{\sum_{\alpha \in I} \varphi_\alpha(x)}. \quad (3.7) \]

The set \( \mathcal{G} \) is a matroid on \([n]\): this means that the exchange relation holds, i.e.,

\[ \text{for all } A, B \in \mathcal{G}, \alpha \in A \setminus B, \text{ there exists } \beta \in B \setminus A \text{ such that } A_{\beta} \in \mathcal{G}. \quad (3.8) \]

In particular, we refer to a generic case as one where \( \mathcal{G} = P_k[n] \).

**4. Complexity reduction through the KP II constraint**

The determinantal partition function (3.1) at \( k = 1 \) reduces to the statistical amoebas studied in [5]. Cases at \( k > 1 \) have more complications, due to the occurrence of functional relations among the terms in the \( \tau \)-function (3.4). Indeed, the minors of a \((k \times n)\)-matrix have to satisfy the well-known Grassmann-Plücker relations (see, e.g., [23], §3.1). In particular, the three-terms Plücker relations

\[ \Delta_A(H_{\alpha \gamma}) \cdot \Delta_A(H_{\beta \delta}) = \Delta_A(H_{\alpha \beta}) \cdot \Delta_A(H_{\gamma \delta}) + \Delta_A(H_{\alpha \delta}) \cdot \Delta_A(H_{\beta, \gamma}) \quad (4.1) \]

hold for all \( 1 \leq \alpha < \beta < \gamma < \delta \leq n \) and \( \mathcal{H} \subset [n] \) with \( \# \mathcal{H} = k - 2 \) and \( \{\alpha, \beta, \gamma, \delta\} \cap \mathcal{H} = \emptyset \).

**Example 2.** For a generic matrix \( A \in \mathbb{R}^{k \times n} \), not all the signatures (3.5) for \( g \in \mathbb{R}^{(n)} \) in (3.4) correspond to another choice of matrix \( \tilde{A} \in \mathbb{R}^{k \times n} \) in (3.2). In fact, consider \( A \in \mathbb{R}^{k \times n} \) such that there exist two non-vanishing minors \( \Delta_A(H_{\alpha \gamma}) \) and \( \Delta_A(H_{\beta \delta}) \), e.g., taking \( A \) parametrizing a point in the totally positive part of the Grassmannian.
Suppose that the choice
\[ g_I \mapsto \begin{cases} -g_I, & I = \mathcal{H}_{\alpha\gamma} \\ g_I, & \text{otherwise} \end{cases}, \] (4.2)
which lies in the free statistical 1-amoeba relative to (3.4), corresponds to a certain matrix \( \tilde{A} \in \mathbb{R}^{k \times n} \). The relation (4.1) gives
\[
\Delta_A(\mathcal{H}_{\alpha\gamma}) \cdot \Delta_A(\mathcal{H}_{\beta\delta}) = \Delta_A(\mathcal{H}_{\alpha\beta}) \cdot \Delta_A(\mathcal{H}_{\gamma\delta}) + \Delta_A(\mathcal{H}_{\alpha\delta}) \cdot \Delta_A(\mathcal{H}_{\beta\gamma})
\]
that means \( \Delta_A(\mathcal{H}_{\alpha\gamma}) \cdot \Delta_A(\mathcal{H}_{\beta\delta}) = 0 \), a contradiction. It follows that, for points in the totally positive Grassmannian, no signature of the type (4.2) preserves the form (2.8).

In general, it is not a trivial task to check if a given map of the type (3.5) follows from maximal minors of a certain matrix. This issue is related to combinatorial structures behind Grassmann-Plücker relations, namely chirotopes \( \chi : [n]^k \rightarrow \{-1, 0, +1\} \) (see, e.g., [43, 10] for more details). In particular, chirotopes coming from \( A \in \mathbb{R}^{k \times n} \), in the sense that \( \chi(I) = \text{sign}(\Delta_A(I)) \) for all \( I \in [n]^k \), are said to be realizable. Both the check of the realizability of chirotopes and their enumeration have non-trivial complexity [26]. However, in the present framework, the data on the initial function, in particular the “lengths” of the exponential and pre-exponential terms, are given: starting from (3.2), these absolute values are known to be compatible with at least one determinantal (Wronskian) form, and can be used to explore other signatures.

We exclude situations where (3.4) identically vanishes, so \( A \) has maximal rank and there is no null row \( 0^T_n \). If there exists a null column, then the \( \tau \)-function does not depend on the corresponding soliton. This leads to the reduction of a \( n \)- to a \( (n-1) \)-soliton solution. So we can assume that there is no null column without loss of generality.

We now consider the conditions on choices of signs that map a \( \tau \)-function of the KP II equation (2.5) to another \( \tau \)-function, which will be called solitonic signatures. Let (3.5) be any choice of signs for coefficients \( g_I \) in (2.13), which is equivalent to a choice of a partition of \( G \) in two disjoint subsets \( \mathcal{G} = \mathcal{PS} \cup \mathcal{NS} \), \( \mathcal{PS} \cap \mathcal{NS} = \emptyset \), where
\[
\mathcal{PS} := \Sigma^{-1}(\{+1\}), \quad \mathcal{NS} := \Sigma^{-1}(\{-1\}). \tag{4.4}
\]
So one can write the resulting exponential sum as
\[
\sum_{I \in \mathcal{P}[n]} \Sigma(I) \cdot \Delta_A(I) \cdot \Delta_K(I) \cdot e^{\sum_{\alpha \in I} \varphi_\alpha(x)} = \tau(x) - 2 \cdot \tau_{\Sigma}(x) \tag{4.5}
\]
where
\[
\tau_{\Sigma}(x) := \sum_{I \in \mathcal{NS}} \Delta_A(I) \cdot \Delta_K(I) \cdot e^{\sum_{\alpha \in I} \varphi_\alpha(x)}. \tag{4.6}
\]
Moreover, we will say
\[
I \equiv J \Leftrightarrow \Sigma(I) = \Sigma(J), \quad I, J \in \mathcal{P}[n], \tag{4.7}
\]
which defines a relation on \( \mathcal{G} \).
We do not assume \textit{a priori} that a partition (4.4) returns a determinant (2.8): this means that the determinantal properties (i.e., Grassmann-Plücker relations) hold for the family \(\{\Delta_A(I) : I \in \mathcal{I}\}\), but not necessarily for \(\{\Sigma(I) \cdot \Delta_A(I) : I \in \mathcal{I}\}\).

The bilinearity of the Hirota derivative and the KP operator (2.5) gives

\[
\begin{align*}
D_{KP}(\tau - 2 \cdot \tau_\Sigma, \tau - 2 \cdot \tau_\Sigma) &= D_{KP}(\tau, \tau) + 4 \cdot D_{KP}(\tau_\Sigma, \tau_\Sigma) - 2 \cdot D_{KP}(\tau, \tau_\Sigma) - 2 \cdot D_{KP}(\tau_\Sigma, \tau) \\
&= 4 \cdot D_{KP}(\tau_\Sigma, \tau_\Sigma) - 2 \cdot D_{KP}(\tau, \tau_\Sigma) - 2 \cdot D_{KP}(\tau_\Sigma, \tau)
\end{align*}
\]

(4.8) since \(D_{KP}(\tau, \tau) = 0\). Therefore, one has

\[
0 = D_{KP}(\tau, \tau_\Sigma) - D_{KP}(\tau_\Sigma, \tau_\Sigma) = D_{KP}(\tau - \tau_\Sigma, \tau_\Sigma).
\]

(4.9)

We can say that \(\tau - \tau_\Sigma\) is “orthogonal” to \(\tau_\Sigma\) with respect to the \(D_{KP}\) bilinear operator. The bilinearity also implies that (4.9) is equivalent to

\[
0 = D_{KP}(\tau - \tau_\Sigma, \tau_\Sigma)
\]

\[
= \sum_{A \in \mathcal{PS}} \sum_{B \in \mathcal{NS}} g(A)g(B) \cdot D_{KP}\left[ \exp\left( \sum_{\alpha \in A} \sum_{u=1}^d \kappa^u_{\alpha} x_u \right) \right. \left. \exp\left( \sum_{\beta \in B} \sum_{u=1}^d \kappa^u_{\beta} x_u \right) \right]
\]

\[
= \sum_{A \in \mathcal{PS}} \sum_{B \in \mathcal{NS}} g(A)g(B) \cdot \left[ \exp\left( 2 \sum_{\beta \in B} \sum_{u=1}^d \kappa^u_{\beta} x_u \right) \right] D_{KP}\left[ 1, \exp\left( \sum_{u=1}^d \sum_{\alpha \in A} \kappa^u_{\alpha} x_u - \sum_{\beta \in B} \kappa^u_{\beta} x_u \right) \right].
\]

(4.10)

From

\[
\begin{align*}
D_{KP}\left[ 1, \exp\left( \sum_{u=1}^d \sum_{\alpha \in A} \kappa^u_{\alpha} x_u - \sum_{\beta \in B} \kappa^u_{\beta} x_u \right) \right]
&= \left( \partial_{x_1}^4 - 4 \cdot \partial_{x_1} \partial_{x_3} + 3 \cdot \partial_{x_3}^2 \right) \exp\left( \sum_{u=1}^d \sum_{\alpha \in A} \kappa^u_{\alpha} x_u - \sum_{\beta \in B} \kappa^u_{\beta} x_u \right) \\
&= C(A, B; \kappa) \cdot \exp\left( \sum_{u=1}^d \sum_{\alpha \in A} \kappa^u_{\alpha} x_u - \sum_{\beta \in B} \kappa^u_{\beta} x_u \right)
\end{align*}
\]

(4.11)

where

\[
C(A, B; \kappa) := \left[ \sum_{\alpha \in A} \kappa_{\alpha} - \sum_{\beta \in B} \kappa_{\beta} \right]^4 - 3 \cdot \left[ \sum_{\alpha \in A} \kappa_{\alpha}^2 - \sum_{\beta \in B} \kappa_{\beta}^2 \right]^2
\]

\[
- 4 \cdot \left[ \sum_{\alpha \in A} \kappa_{\alpha} - \sum_{\beta \in B} \kappa_{\beta} \right] \cdot \left[ \sum_{\alpha \in A} \kappa_{\alpha}^3 - \sum_{\beta \in B} \kappa_{\beta}^3 \right].
\]

(4.12)

The equation (4.10) is equivalent to

\[
\sum_{A \in \mathcal{PS}} \sum_{B \in \mathcal{NS}} g(A)g(B) \cdot C(A, B; \kappa) \cdot \exp\left( \sum_{u=1}^d \sum_{\alpha \in A} \kappa^u_{\alpha} x_u + \sum_{\beta \in B} \kappa^u_{\beta} x_u \right) = 0.
\]

(4.13)
The only occurrences of (4.14) that are satisfied for a \(A \cup B = \mathcal{C} \cup \mathcal{D}\) if and only if \(A \cap B = C \cap D\). So, if one assumes that there is no algebraic dependence that relates these sums when \(A \cup B \neq \mathcal{C} \cup \mathcal{D}\) or \(A \cap B \neq C \cap D\), then each exponential term in (4.14) is determined by the union \(A \cup B\) and the intersection \(A \cap B\), since

\[
\sum_{\alpha \in A} \kappa_{a} u + \sum_{\beta \in B} \kappa_{b} u = \sum_{\alpha \in A \cap B} \kappa_{\alpha} u + \sum_{\beta \in A \cup B} \kappa_{\beta} u,
\]

(4.15)

In particular, one has the following

**Lemma 3.** Assume that the minors \(\Delta_{A}(I)\) with \(I \in \mathcal{P}_{k}[n]\) are not vanishing, i.e., \(\mathcal{G} = \mathcal{P}_{k}[n]\). Then for each \(H \subseteq \mathcal{L} \subseteq [n]\) with \(#H + 4 = \#L = k + 2\), the number of pairs \((I_{+}, I_{-})\) associated with a term in (4.13) with union \(I_{+} \cup I_{-} = \mathcal{L}\) and intersection \(I_{+} \cap I_{-} = H\) is equal to (4.15) is 0 or 3. With the notation introduced in (2.1)-(4.7), this can be stated as

\[
I_{\gamma}^{\alpha} I_{\delta}^{\beta} \equiv I_{\gamma}^{\alpha} I_{\delta}^{\beta} \equiv I_{\delta}^{\alpha} I_{\gamma}^{\beta} \equiv I_{\gamma}^{\alpha} I_{\delta}^{\beta}.
\]

(4.16)

**Proof:** Suppose that there exist subsets \(I_{+} \in \text{PS}, I_{-} \in \text{NS}\) such that \(H := I_{+} \cap I_{-} =: \{\gamma_{1}, \ldots, \gamma_{k-2}\}, \gamma_{1} < \cdots < \gamma_{k-2}\), and \(L := I_{+} \cup I_{-}\), so the number of terms in (4.13) associated with \(H\) and \(L\) is not 0. In particular, \(#(I_{+} \Delta I_{-}) = 4\). Then, there exist \(\alpha_{+}, \beta_{+} \in I_{+}\) and \(\alpha_{-}, \beta_{-} \in I_{-}\) such that \(I_{+} = H \cup \{\alpha_{+}, \beta_{+}\}\). There are two additional pairs of subsets different from \(\{I_{+}, I_{-}\}\) with the same union and intersection. Thus, in order to have a vanishing coefficient for the associated term in (4.13), at least one of these two pairs has to belong to \(\text{PS} \times \text{NS}\). Let us suppose that only one of these two possibilities is in \(\text{PS} \times \text{NS}\), call it \((L_{+}, L_{-})\) with \(L_{\pm} = H \cup \{\alpha_{\pm}, \beta_{\pm}\}\). In such a case, the term \(C(I_{+}, I_{-}; \kappa)\) in (4.12) is

\[
C(I_{+}, I_{-}; \kappa) = 12 \cdot (k_{\alpha_{+}} - k_{\alpha_{-}})(k_{\beta_{+}} - k_{\alpha_{-}})(k_{\alpha_{+}} - k_{\beta_{-}})(k_{\beta_{+}} - k_{\beta_{-}})
\]

(4.17)

that is not vanishing since the soliton parameters are pairwise distinct by assumption. Using the parity \(P((\beta_{-}, \alpha_{-}, \beta_{+}, \alpha_{+})\)) of the permutation \((\beta_{-}, \alpha_{-}, \beta_{+}, \alpha_{+})\), we introduce

\[
\sigma(\alpha_{+}, \beta_{+}|\alpha_{-}, \beta_{-}) := \text{sign}[(\alpha_{+} - \alpha_{-})(\beta_{+} - \alpha_{-})(\alpha_{+} - \beta_{-})(\beta_{+} - \beta_{-})]
\]

(4.18)

which gives

\[
\Delta_{K}(I_{+}) \cdot \Delta_{K}(I_{-}) \cdot C(I_{+}, I_{-}; \kappa)
\]

\[
= 12 \cdot \prod_{\alpha_{+} < \alpha_{-}} (k_{\alpha_{+}} - k_{\alpha_{-}}) \cdot \prod_{\beta_{+} < \beta_{-}} (k_{\beta_{+}} - k_{\beta_{-}}) \cdot (k_{\beta_{+}} - k_{\alpha_{-}})(k_{\alpha_{+}} - k_{\beta_{-}})(k_{\beta_{+}} - k_{\beta_{-}})
\]

\[
= 12 \cdot \prod_{\gamma_{1} < \gamma_{2}} (k_{\gamma_{1}} - k_{\gamma_{2}}) \cdot \prod_{\delta_{1} < \delta_{2}} (k_{\delta_{1}} - k_{\delta_{2}}) \cdot \sigma(\alpha_{+}, \beta_{+}|\alpha_{-}, \beta_{-})
\]
\[ = 12 \cdot \text{VdM}(\kappa; I_+ \cap I_-) \cdot \text{VdM}(\kappa; I_+ \cup I_-) \cdot \sigma(\alpha_+, \beta_+ | \alpha_-, \beta_-). \quad (4.19) \]

The signs \( \sigma(\alpha_+, \beta_- | \alpha_-, \beta_+ \) and \( \sigma(\alpha_+, \alpha_- | \beta_+, \beta_-) \) can be found in the same way, taking into account that

\[ P((\beta_-, \alpha_-, \beta_+, \alpha_+)) = -P((\beta_-, \beta_+, \alpha_-, \alpha_+)) = -P((\beta_+, \alpha_-, \beta_-, \alpha_+)). \quad (4.20) \]

Furthermore, one can consider

\[ S(\alpha, H) := \max\{i \in [k] : \gamma_i < \alpha\}, \quad \alpha \in [n] \setminus H \quad (4.21) \]

where \( \gamma_0 := 0 \). In this way, it is easy to check that

\[ \Delta_A(H_{\alpha, \beta}) = \Delta_A(\alpha, \beta; H) \cdot (-1)^{1+S(\alpha, H)+S(\beta, H)} \cdot \text{sign } [\alpha - \beta]. \quad (4.22) \]

where \( \Delta_A(\alpha, \beta; H) \) is the product of \( \Delta_A(H_{\alpha, \beta}) \) and the parity of the permutation \((\alpha, \beta, \gamma_1, \ldots, \gamma_{k-2})\). Thus, one gets

\[ g(I_+) \cdot g(I_-) \cdot C_{I_+, I_-}(\kappa) = \quad (4.23) \]

The first term in braces in (4.23) is the same for all the pairs \((G_+, G_-)\) with \( G_+ \cap G_- = H \) and \( G_+ \cup G_- = L \). For the second term in braces, the antisymmetry (4.20) implies that

\[ g(L_+) \cdot g(L_-) \cdot C_{L_+, L_-}(\kappa) = \quad (4.24) \]

The three-terms Plücker relations (4.1), which are valid for minors \( \Delta_A(I) \), can be stated for any four pairwise distinct elements \( \delta_a \in [n] \setminus H, a \in [4], \) as

\[ \Delta_A(\delta_1, \delta_2 | H) \cdot \Delta_A(\delta_3, \delta_4 | H) = \Delta_A(\delta_1, \delta_3 | H) \cdot \Delta_A(\delta_2, \delta_4 | H) - \Delta_A(\delta_1, \delta_4 | H) \cdot \Delta_A(\delta_2, \delta_3 | H). \quad (4.25) \]

In particular, if one looks at the sum of (4.23) and (4.24) and applies (4.25) with \((\delta_1, \delta_2, \delta_3, \delta_4 \equiv (\alpha_+, \alpha_-, \beta_+, \beta_-), \) the result is

\[ g(I_+) \cdot g(I_-) \cdot C_{I_+, I_-}(\kappa) + g(L_+) \cdot g(L_-) \cdot C_{L_+, L_-}(\kappa) = \quad (4.26) \]
since we have assumed that all the minors are not vanishing. Hence, the coefficient in (4.13) associated with $H$ and $L$ is not vanishing and the associated $\tau$-function is not a solution of the KP equation (2.5).

On the contrary, if all the three terms are involved in (4.13), then their sum is

$$12 \cdot \text{VdM}(\kappa; I_+ \cap I_-) \cdot \text{VdM}(\kappa; I_+ \cup I_-)$$

$$- ( -1)^{S(\alpha_+,H)+S(\beta_+,H)+S(\alpha_-,H)+S(\beta_-,H)} \cdot \mathbb{P}(\beta_-,\alpha_-,\beta_+,\alpha_+)$$

$$(\Delta_\Lambda (\alpha_+,\beta_+ | H) \cdot \Delta_\Lambda (\alpha_-,\beta_- | H) - \Delta_\Lambda (\alpha_+,\beta_- | H) \cdot \Delta_\Lambda (\alpha_-,\beta_+ | H))$$

$$- \Delta_\Lambda (\alpha_+,\alpha_- | H) \cdot \Delta_\Lambda (\beta_+,\beta_- | H)) = 0 \quad (4.27)$$

since the second term in square brackets vanishes due to the three-terms Plücker relations (4.25). \hfill \Box

**Proposition 4.** Let $I \in \mathcal{P}_k[n]$, $\alpha_1,\alpha_2 \in I$ and $\delta_1,\delta_2 \in [n] \setminus I$ such that $I,T^{\alpha_1\alpha_2}_{\delta_1\delta_2} \in \mathcal{S}$.

Then at least one of the two products $\Delta_\Lambda (T^{\alpha_1}_{\delta_1}) \cdot \Delta_\Lambda (T^{\alpha_2}_{\delta_2})$, $\{T,U\} = \{1,2\}$, is not vanishing.

Further, if $\Delta_\Lambda (T^{\alpha_1}_{\delta_1}) \cdot \Delta_\Lambda (T^{\alpha_2}_{\delta_2}) \neq 0$, then $\Sigma(T^{\alpha_1}_{\delta_1}) \cdot \Sigma(T^{\alpha_2}_{\delta_2}) = \Sigma(I) \cdot \Sigma(T^{\alpha_1\alpha_2}_{\delta_1\delta_2})$.

**Proof:** First suppose that one of the product vanishes, e.g., $\Delta_\Lambda (T^{\alpha_1}_{\delta_1}) \cdot \Delta_\Lambda (T^{\alpha_2}_{\delta_2}) = 0$ without loss of generality. Then the three-terms Plücker relations (4.1) imply that $|\Delta_\Lambda (T^{\alpha_1}_{\delta_1}) \cdot \Delta_\Lambda (T^{\alpha_2}_{\delta_2})| = |\Delta_\Lambda (I) \cdot \Delta_\Lambda (T^{\alpha_1\alpha_2}_{\delta_1\delta_2})| \neq 0$. Further, $\Sigma(T^{\alpha_1}_{\delta_1}) \cdot \Sigma(T^{\alpha_2}_{\delta_2}) \neq \Sigma(I) \cdot \Sigma(T^{\alpha_1\alpha_2}_{\delta_1\delta_2})$ implies that there is exactly one non-vanishing term in (4.13) associated with a pair in $\mathbf{PS} \times \mathbf{NS}$ with intersection $T^{\alpha_1\alpha_2}_{\delta_1\delta_2}$ and union $\delta_1\delta_2$, hence (4.27) do not vanish. Thus $\Sigma(T^{\alpha_1}_{\delta_1}) \cdot \Sigma(T^{\alpha_2}_{\delta_2}) = \Sigma(I) \cdot \Sigma(T^{\alpha_1\alpha_2}_{\delta_1\delta_2})$. If instead both $\Delta_\Lambda (T^{\alpha_1}_{\delta_1}) \cdot \Delta_\Lambda (T^{\alpha_2}_{\delta_2})$ and $\Delta_\Lambda (T^{\alpha_1}_{\delta_1}) \cdot \Delta_\Lambda (T^{\alpha_2}_{\delta_2})$ are not vanishing, then (4.16) holds because of Lemma 3 (see in particular (4.26) and (4.27)). \hfill \Box

**Lemma 5.** Let $\mathcal{H}, \mathcal{K} \in \mathcal{S}$ with $r := \#(\mathcal{H} \setminus \mathcal{K})$. Then there exists a finite sequence $\mathcal{L}_0 := \mathcal{H}, \mathcal{L}_1, \ldots, \mathcal{L}_r := \mathcal{K}$ of elements of $\mathcal{S}$ such that $\#(\mathcal{L}_{u-1} \Delta \mathcal{L}_u) = 2$, $u \in [r]$.

**Proof:** Let $\mathcal{H} \setminus \mathcal{K} =: \{\gamma_1,\ldots,\gamma_r\}$ and $\mathcal{L}_0 := \mathcal{H}$, so the exchange property (3.8) implies that there exists $\Psi(\gamma_1) \in \mathcal{K} \setminus \mathcal{H}$ such that $\mathcal{L}_1 := \mathcal{H} \setminus \{\gamma_1\} \cup \{\Psi(\gamma_1)\} \in \mathcal{S}$. Note that $\#(\mathcal{L}_1 \Delta \mathcal{K}) = r - 1$. One can iterate the process, starting from $\mathcal{L}_u \in \mathcal{S}$ and finding $\Psi(\gamma_{u+1}) \in \mathcal{K} \setminus \mathcal{L}_u$ in order to define

$$\mathcal{L}_{u+1} := \mathcal{H} \setminus \{\gamma_1,\ldots,\gamma_{u+1}\} \cup \{\Psi(\gamma_1),\ldots,\Psi(\gamma_{u+1})\}, \quad u \in [r] \quad (4.28)$$

such that $\mathcal{L}_{u+1} \in \mathcal{S}$ too. At each step, the element $\Psi(\gamma_{u+1})$ is different from $\Psi(\gamma_t)$ for all $t \leq u$ because $\Psi(\gamma_t) \in \mathcal{L}_u$, while $\Psi(\gamma_{u+1}) \in \mathcal{K} \setminus \mathcal{L}_u$. So the map $\Psi : \mathcal{H} \setminus \mathcal{K} \rightarrow \mathcal{K} \setminus \mathcal{H}$ is injective and, hence, a bijection, since $\#\mathcal{H} = k = \#\mathcal{K}$ implies $\#(\mathcal{H} \setminus \mathcal{K}) = \#(\mathcal{K} \setminus \mathcal{H})$. All the subsets $\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_r$ are elements of $\mathcal{S}$ and $\#(\mathcal{L}_{u-1} \Delta \mathcal{L}_u) = 2$ for all $u \in [r]$.

Now we consider the following relations

$$\alpha \approx_A \beta \Leftrightarrow A \cong A^\beta_\alpha, \quad \alpha \in [n] \setminus I, \beta \in I. \quad (4.29)$$

If there is $\lambda \in A$ such that $I^\lambda, I^\beta_\alpha \in \mathcal{S}$ where $I = A^\lambda_\alpha$, then one can express $\alpha \approx I \beta$ as $I^\lambda \cong I^\beta_\alpha$. This does not depend on the choice of $\lambda \in [n] \setminus I$, as can be easily shown:
Remark 6. Let $I \in \mathcal{P}_k[n]$ and $\alpha, \beta \in I$. Then $I_\alpha^\gamma \cong I_\beta^\delta$ holds for a certain $\gamma \in [n]\setminus I$ if and only if $I_\delta^\gamma \cong I_\delta^\delta$ holds for all $\delta \in [n]\setminus I$ such that $I_\delta^\gamma, I_\delta^\delta \in \mathcal{G}$.

Proof: One implication is trivial. So assume that there exist $I \in \mathcal{P}_k[n]$, $\alpha, \beta \in I$ and $\gamma, \delta \in [n]\setminus I$ such that $I_\alpha^\gamma \cong I_\beta^\delta$ and $I_\delta^\gamma \cong -I_\delta^\delta$ (clearly $\gamma \neq \delta$). The case $0 \in \{I_\delta^\gamma, I_\delta^\delta\}$ is ruled out by the assumption $I_\delta^\gamma, I_\delta^\delta \in \mathcal{G}$. If $I_\alpha^\gamma \cong I_\beta^\delta$, then $I_\delta^\gamma \cong I_\delta^\delta$ by Proposition 4. This means that $I_\delta^\gamma \cong I_\delta^\gamma \cong I_\delta^\gamma \cong I_\delta^\delta \cong -I_\delta^\delta$, i.e., a contradiction.

Now suppose that $I_\alpha^\gamma \cong -I_\beta^\delta$, which means $I_\delta^\gamma \cong -I_\delta^\delta$ by Proposition 4. Thus $I_\delta^\gamma \cong -I_\delta^\delta \cong -I_\delta^\delta \cong -I_\delta^\delta$, i.e., a contradiction. □

In fact, the relations (4.29) are compatible for different choices of $A$ too.

Proposition 7. There are no subsets $A, B \in \mathcal{P}_k[n]$, $\alpha \in [n]\setminus (A \cup B)$, and $\beta \in A \cap B$ such that $A, A_\alpha^\beta, B, B_\alpha^\beta \in \mathcal{G}$ and $\Sigma(A) \cdot \Sigma((A)_\alpha^\beta) = -\Sigma(B) \cdot \Sigma((B)_\alpha^\beta)$.

Proof: Introduce $A_1 := A, B_1 := B, A_2 := (A)_\alpha^\beta, B_2 := (B)_\alpha^\beta$. Note that $A_1 \setminus B_1 = A_2 \setminus B_2, B_1 \setminus A_1 = B_2 \setminus A_2$ and $\alpha, \beta \notin A_1 \Delta B_1$.

We shall prove the statement by induction on the distance between $A_1$ and $B_1$, that is $r := \frac{1}{2} \cdot \#(A_1 \Delta B_1)$. First consider the case $r = 1$ and set $B_1 \setminus A_1 = \{\Psi(\gamma_1)\}$. In particular, the elements $\alpha, \beta, \gamma_1, \Psi(\gamma_1)$ are all distinct and the set $\mathcal{H} := A_1 \cap A_2 \cap B_1 \cap B_2 \in \mathcal{P}_{k-2}[n]$ satisfies $A_1 = H_{\beta \gamma_1}, A_2 = H_{\alpha \gamma_1}$, $B_1 = H_{\beta \Psi(\gamma_1)}$ and $B_2 = H_{a \Psi(\gamma_1)}$.

By hypothesis, $A_1, A_2, B_1, B_2 \in \mathcal{G}$, so Proposition 4 gives $\Sigma(A_1) \cdot \Sigma(B_2) = \Sigma(A_2) \cdot \Sigma(B_1)$. Multiplying both sides of this equation by $\Sigma(A_2) \cdot \Sigma(B_2)$ and applying $\Sigma(A_2)^2 = \Sigma(B_2)^2 = 1$ we find

$$\Sigma(A_1) \cdot \Sigma(A_2) = \Sigma(B_1) \cdot \Sigma(B_2). \quad (4.30)$$

Now assume that the statement holds whenever $\frac{1}{2} \cdot \#(A_1 \Delta B_1) < r$ and take $A_1, A_2, B_1, B_2 \in \mathcal{G}$ with $\frac{1}{2} \cdot \#(A_1 \Delta B_1) = r$. From Lemma 5 there exist a labelling $A_1 \setminus B_1 = \{\gamma_1, \ldots, r\}$ and two bijections $\Psi_T : A_T \setminus B_T \rightarrow B_T \setminus A_T, T \in \{1, 2\}$, such that

$$L_T^{(u)} := (A_T)^{\gamma_1, \ldots, r} \in \mathcal{G}, \ u \in [r], T \in \{1, 2\}. \quad (4.31)$$

Let us focus on the four sets $A_1, A_2, L_1^{(r-1)}, L_2^{(r-1)}$: $A_1, A_2 \in \mathcal{G}$ by hypothesis, and $L_1^{(r-1)}, L_2^{(r-1)} \in \mathcal{G}$ by construction. Further, one has $\beta \in (A_1 \cap L_1^{(r-1)}) \setminus (A_2 \cap L_2^{(r-1)})$ and $\alpha \in (A_2 \cap L_2^{(r-1)}) \setminus (A_1 \cap L_1^{(r-1)})$. Since the distance between $L_T^{(u)}$ and $A_T$ increases by 1 at each step, one gets $\frac{1}{2} \cdot \#(A_T \Delta L_T^{(r-1)}) = r - 1$. So the inductive hypothesis applies and

$$\Sigma(A_1) \cdot \Sigma(L_1^{(r-1)}) = \Sigma(A_2) \cdot \Sigma(L_2^{(r-1)}). \quad (4.32)$$

Likewise, $L_T^{(r)} = B_T, \beta \in (L_1^{(r-1)} \cap B_1) \setminus (L_2^{(r-1)} \cap B_2)$ and $\alpha \in (L_2^{(r-1)} \cap B_2) \setminus (L_1^{(r-1)} \cap B_1)$, so one can repeat the same argument as in (4.30) and get

$$\Sigma(L_1^{(r-1)}) \cdot \Sigma(B_1) = \Sigma(L_2^{(r-1)}) \cdot \Sigma(B_2). \quad (4.33)$$

Multiplying (4.32) and (4.33) side by side, and applying $\Sigma(L_1^{(r-1)})^2 = +1$, one finds

$$\Sigma(A_1) \cdot \Sigma(B_1) = \Sigma(A_2) \cdot \Sigma(B_2) \quad (4.34)$$
Definition 8. For each $\mathcal{I} := \{\beta_1, \ldots, \beta_k\} \in \mathfrak{G}$ take the $k \times (n-k)$ matrix $X(\mathcal{I})$ defined as $(X(\mathcal{I}))_{\alpha \beta} := \Sigma(\mathcal{I}) \cdot \Sigma(\mathcal{I}_{\alpha}^{\beta})$ if $\mathcal{I}_{\alpha}^{\beta} \in \mathfrak{G}$ and 0 otherwise ($\alpha \in [n] \setminus \mathcal{I}$). We refer to a path $\Phi_{\mathcal{I}}$ on $X(\mathcal{I})$ as a sequence of non-vanishing entries obtained moving alternately along rows and columns of $X(\mathcal{I})$, i.e., of the form

$$(i_1, \alpha_1) \rightarrow (i_2, \alpha_2) \rightarrow (i_3, \alpha_3) \rightarrow \ldots$$

with $i_T \neq i_{T+1}$, $\alpha_T \neq \alpha_{T+1}$ and $\mathcal{I}_{\alpha_T}^{i_T}, \mathcal{I}_{\alpha_{T+1}}^{i_{T+1}} \in \mathfrak{G}$ for all $T$. Two indices $\alpha, \beta \in [n]$ are said to be connected by a path $\Phi_{\mathcal{I}}$ if $\alpha$ is a component of the first element of $\Phi_{\mathcal{I}}$ but not of the second, and $\beta$ is a component of the last element of $\Phi_{\mathcal{I}}$ but not of the second-to-last.

For instance, if $\alpha \in \mathcal{I}$ and $\beta \in [n] \setminus \mathcal{I}$, then a path from $\alpha$ to $\beta$ starts with $(\alpha, \gamma)$ and ends with $(\beta, \delta)$, for some $\gamma \in [n] \setminus \mathcal{I}, \delta \in \mathcal{I}$; if instead both $\alpha$ and $\beta$ are elements of $\mathcal{I}$, a path starts with $(\alpha, \gamma)$ and ends with $(\beta, \delta)$, $\gamma, \delta \in [n] \setminus \mathcal{I}$. These paths induce a relation $\rightarrow_{\mathcal{I}}$ on $[n]$.

Definition 9. For all $\alpha, \beta \in [n]$, we say $\alpha \rightarrow_{\mathcal{I}} \beta$ if $\alpha = \beta$ or there is a path in $X(\mathcal{I})$ that connects $\alpha$ and $\beta$.

This is an equivalence relation: it is reflexive by definition; if $\alpha \rightarrow_{\mathcal{I}} \beta$, then the reverse path gives $\beta \rightarrow_{\mathcal{I}} \alpha$, so $\rightarrow_{\mathcal{I}}$ is symmetric; it is also transitive, as follows from the path $\alpha \rightarrow_{\mathcal{I}} \gamma$ obtained from the concatenation of $\alpha \rightarrow_{\mathcal{I}} \beta$ and $\beta \rightarrow_{\mathcal{I}} \gamma$ and the simplification of consecutive reverse subpaths. Moreover, the following result holds:

Proposition 10. The product of signs of edges $\chi$ along any closed path of $X(\mathcal{I}), \mathcal{I} \in \mathfrak{G}$, is $+1$.

Proof: Let $\mathcal{I} := \{\gamma_1, \ldots, \gamma_k\}$ and denote $\gamma_{i_T}$ by $i_T$ in the rest of the proof for the sake of clearness. The same number of moves along rows and along columns is required to close a path, so it has an even length $2r$. Thus, the product of signs along the closed path is

$$\Phi_{\mathcal{I}}(i_1, \ldots, i_r | \alpha_1, \ldots, \alpha_r) := \prod_{T=1}^{r} \Sigma(\mathcal{I}_{\alpha_T}^{i_T}) \cdot \Sigma(\mathcal{I}_{\alpha_T}^{i_{T+1}})$$

(4.37)

where indices $T$ are taken modulo $r$, e.g., $i_{r+1} = i_1$. From this one can also see that the statement is invariant under the replacement of $X(\mathcal{I})$ by $X(\mathcal{J}) \in \mathfrak{G}$, as long as all the involved subsets $\mathcal{J}_{\alpha_T}^{i_T}$ and $\mathcal{J}_{\alpha_T}^{i_{T+1}}$ are in $\mathfrak{G}$: indeed, both sides of (4.37) can be multiplied by $(\Sigma(\mathcal{I}))^{2r} \cdot (\Sigma(\mathcal{J}))^{2r} = +1$, so the factors are expressed
as \( \Sigma(I^{T \cup r}_{\alpha T} \cup J(I^{T \cup r}_{\alpha T})) = \Sigma(J^{T \cup r}_{\alpha T}) \cdot \Sigma(I^{T \cup r}_{\alpha T} \cup J(I^{T \cup r}_{\alpha T})) \) by Proposition 7.

We prove the statement by induction on \( r \). The base cases are \( r = 1 \), which is trivial by symmetry of \( \chi(\alpha, \gamma_i) \) with respect to the interchange of its arguments, and \( r = 2 \). In the latter situation, there exist \( i \neq m \) and \( \alpha \neq \beta \) such that \( T^m_\alpha, T^m_\beta, T^m_\beta, T^m_\beta \in \mathfrak{G} \). By Proposition 4 the associated product of signs is +1. Now assume that there exist \( T \in [r], S \in [r-2] \), such that \( I^{T \cup r}_{\alpha T + S} \in \mathfrak{G} \). Then one can write

\[
\Phi_T(i_1, \ldots, i_r \mid \alpha_1, \ldots, \alpha_r) = \left( \prod_{W=0}^{r-2} \Sigma(I^{T \cup r}_{\alpha T + W} \cup J(I^{T \cup r}_{\alpha T + W})) \right) \cdot \Sigma(I^{T \cup r}_{\alpha T + S} \cup J(I^{T \cup r}_{\alpha T + S}))
\]

The lengths of these two closed path are \( 2S + 2 \) and \( 2(r-S) \) respectively, which lie in \( \{4, \ldots, 2r-2\} \); so the inductive hypothesis applies to both these paths and this gives the result.

On the other hand, if such \( T, S \) do not exist, then \( I^{T \cup r}_{\alpha T + S} \in \mathcal{P}_k[n] \setminus \mathfrak{G} \) for all \( T \in [r] \) and \( S \notin \{1, r\} \): by Proposition 4, the condition \( I^{T \cup r}_{\alpha T}, I^{T \cup r}_{\alpha T + S} \in \mathfrak{G} \), along with \( I^{T \cup r}_{\alpha T + S} \in \mathcal{P}_k[n] \setminus \mathfrak{G} \) at \( S \notin \{1, r\} \) and \( I^{T \cup r}_{\alpha T} \in \mathcal{P}_k[n] \setminus \mathfrak{G} \) at \( S \notin \{1, r\} \), implies that \( I^{T \cup r}_{\alpha T + S} \in \mathfrak{G} \) for all \( S \neq r \). Likewise, from \( I^{T \cup r}_{\alpha T}, I^{T \cup r}_{\alpha T + S} \in \mathfrak{G} \), \( I^{T \cup r}_{\alpha T + S} \in \mathcal{P}_k[n] \setminus \mathfrak{G} \) \((S \notin \{1, r\}) \) and \( I^{T \cup r+1}_{\alpha T + S} \in \mathcal{P}_k[n] \setminus \mathfrak{G} \) \((S \notin \{r-1, r\}) \), one gets \( I^{T \cup r+1}_{\alpha T + S+1} \in \mathfrak{G} \) whenever \( S \neq r \). Further, \( I^{T \cup r+1}_{\alpha T + S} \in \mathfrak{G} \) and \( I^{T \cup r+1}_{\alpha T + S} \in \mathcal{P}_k[n] \setminus \mathfrak{G} \) lead to \( I^{T \cup r+1}_{\alpha T + S} \in \mathfrak{G} \) for all \( T \notin \{1, 2\} \).

So we fix \( \mathcal{J} := J^{\alpha T}_{o_2} \in \mathfrak{G} \): by previous observations, one has \( J^{\alpha T}_{o_2}, J^{\alpha T}_{o_2} \in \mathfrak{G} \) for all \( T \notin \{1, 2\} \), \( J^{\alpha T}_{o_2} \in \mathfrak{G} \) and \( J^{\alpha T}_{o_2} = J^{\alpha T}_{o_2} \in \mathfrak{G} \). One can apply Proposition 4 to \( J^{\alpha T}_{o_2} \) and \( (J^{\alpha T}_{o_2})_{o_2} = J^{\alpha T}_{o_2} \), which are in \( \mathfrak{G} \), and write

\[
\Sigma(J^{\alpha T}_{o_2}) \cdot \Sigma(J^{\alpha T}_{o_2}) = \Sigma(J^{i_{i2}}_{o_1 o_2}) \cdot \Sigma(J^{i_{i2}}_{o_1 o_2}) = \Sigma(J^{i_{i2}}_{o_1 o_2}) \cdot \Sigma(J^{i_{i2}}_{o_1 o_2}) (4.39)
\]

Thus consider the path \( \Phi_{\mathcal{J}}(i_1, i_3, i_4, \ldots, i_r | \alpha_1, \alpha_3, \alpha_4, \ldots, \alpha_r) \) of length \( 2(r-1) \) on \( X(\mathcal{J}) \), which also satisfies \( \Sigma(J^{i_{i2}}_{o_1}) \cdot \Sigma(J^{i_{i2}}_{o_1}) = \Sigma(J^{i_{i3}}_{o_1 o_2}) \cdot \Sigma(J^{i_{i3}}_{o_1 o_2}) \) for all \( T \notin \{1, 2\} \) (all the subsets are in \( \mathfrak{G} \) and Proposition 7 holds) and \( \Sigma(J^{\alpha T}_{o_2}) \cdot \Sigma(J^{\alpha T}_{o_2}) \cdot \Sigma(J^{\alpha T}_{o_2}) \cdot \Sigma(J^{\alpha T}_{o_2}) \) by (4.39). Since the involved subsets are in \( \mathfrak{G} \), the inductive hypothesis applies to the path in \( X(\mathcal{J}) \) and that gives

\[
\Phi_T(i_1, \ldots, i_r | \alpha_1, \ldots, \alpha_r) = \Phi_{\mathcal{J}}(i_1, i_3, i_4, \ldots, i_r | \alpha_1, \alpha_3, \ldots, \alpha_r) = +1. (4.40)
\]
A special role is assumed by the pivot set \( \mathcal{V} \), since the hypothesis of no-null columns implies that each \( \alpha \in \{\nu_i \} \) is associated with an element \( \nu_i \in \mathcal{V} \) (possibly \( \alpha = \nu_i \)) such that \( \mathcal{V}_\alpha \in \mathfrak{S} \). So we can finally state the main result:

**Theorem 11.** A choice of signs \( \Sigma : \mathfrak{S} \rightarrow \{\pm 1\} \) returns a solution of the KP II equation (2.5) if and only if \( \Sigma \) is induced by a choice of signs for rows and columns of \( \mathbf{A} \) (up to the action of \( GL_k(\mathbb{R}) \)).

**Proof:** One implication is trivial, since a choice of signs for rows and columns of \( \mathbf{A} \) induces a (singular) soliton solution by construction. So consider any choice of signs \( \Sigma \) that returns a solution of the KP II equation and choose an arbitrary element in \( \mathfrak{S} \), e.g., \( \mathcal{V} \). For each \( \mathcal{I} \in \mathfrak{S} \) and \( \alpha \in \mathcal{I} \setminus \mathcal{V} \), given that \( \Delta_\mathbf{A}(\mathcal{I}) \cdot \Delta_\mathbf{A}(\mathcal{V}) \neq 0 \), the Plücker relations imply that there exists at least one non-vanishing term of the type \( \Delta_\mathbf{A}(\mathcal{I}_\nu) \cdot \Delta_\mathbf{A}(\mathcal{V}_\nu \cdot \nu) \). Thus one can always find \( \nu(\alpha) \in \mathcal{V} \setminus \mathcal{I} \) such that both \( \mathcal{I}_\nu \) and \( \mathcal{V}_\nu \cdot \nu \) are elements of \( \mathfrak{S} \). So let \( \mathcal{I} \setminus \mathcal{V} = \{\alpha_1, \ldots, \alpha_r\} \): as in Lemma 5, we start with \( \mathcal{L}_0 := \mathcal{I} \) and, from \( \mathcal{L}_{u-1} \in \mathfrak{S} \) and \( \alpha_u \in \mathcal{L}_{u-1} \setminus \mathcal{V} \), \( u \in [r] \), we find

\[
\nu(\alpha_u) \in \mathcal{V} \setminus \mathcal{L}_{u-1}, \quad \mathcal{L}_u := \mathcal{L}_{u-1} \setminus \{\alpha_u\} \cup \{\nu(\alpha_u)\}
\]

such that \( \mathcal{L}_u \), \( \mathcal{V}_\nu(\alpha_u) \), \( \mathcal{V} \subseteq \mathfrak{S} \). Also in this case, from \( \nu(\alpha_u) \in \mathcal{L}_{u-1} \) for all \( s < u \) and \( \nu(\alpha_u) \in \mathcal{V} \setminus \mathcal{L}_{u-1} \), it follows that all \( \nu(\alpha_u) \) are pairwise distinct. So we get

\[
\Sigma(\mathcal{I}) \cdot \Sigma(\mathcal{V}) = \prod_{u=1}^r \Sigma(\mathcal{L}_{u-1}) \cdot \Sigma(\mathcal{L}_u) = \prod_{u=1}^r \Sigma(\mathcal{L}_{u-1}) \cdot \Sigma((\mathcal{L}_{u-1})_{\nu(\alpha_u)}\alpha_u)
\]

\[
= \prod_{u=1}^r \Sigma(\mathcal{V}_{\nu(\alpha_u)}) \cdot \Sigma(\nu(\alpha_u))_{\nu(\alpha_u)} \quad \text{(from Proposition 7)}
\]

\[
= \prod_{u=1}^r \Sigma(\mathcal{V}_{\nu(\alpha_u)}) \cdot \Sigma(\mathcal{V}) = \prod_{u=1}^r \chi(\alpha_u, \nu(\alpha_u)).
\]

Now consider the equivalence \( \rightarrow \mathcal{V} \). Each class \( \mathcal{C}_p \) contains at least one element \( \nu_{i_p} \in \mathcal{V} \), since we have assumed that there are no null columns: fix a sign \( \chi(\nu_{i_p}) \in \{\pm 1\} \) for each of them. For any \( \alpha \in \mathcal{C}_p \), take a path \( \Phi(i_p \rightarrow \alpha) \) on \( \mathbf{X}(\mathcal{V}) \) connecting \( \nu_{i_p} \) and \( \alpha \), and set

\[
\chi(\alpha) := \chi(i_p) \cdot \left( \prod_{(m, \delta) \in \Phi(i_p \rightarrow \alpha)} \chi(\delta, \nu_m) \right).
\]

This definition is well-posed since it does not depend on the choice of the path by Proposition 10. If \( \mathcal{V}_\alpha \in \mathfrak{S} \), then \( \nu_i \in \mathcal{V} \) belongs to the same class of \( \alpha \), because the path with only one element \( \{\nu_i, \alpha\} \) connects them. The concatenation of \( \{\nu_i, \alpha\} \), the reverse of \( \Phi(i_p \rightarrow \alpha) \) and \( \Phi(i_p \rightarrow \nu_i) \) makes a closed path, whose product of signs is equal to \( +1 \) by Proposition 10. So

\[
\chi(\alpha, \nu_i) = \prod_{(m, \delta) \in \Phi(i_p \rightarrow \alpha)} \chi(\delta, \nu_m) \cdot \prod_{(l, \gamma) \in \Phi(i_p \rightarrow \nu_i)} \chi(\gamma, \nu_i)
\]
\[
\begin{align*}
\chi(i_P) \cdot \prod_{(m,\delta) \in \Phi(i_P \to \alpha)} \chi(\delta, \nu_m) & \cdot \prod_{(l,\gamma) \in \Phi(i_P \to \nu_i)} \chi(\gamma, \nu_l) \\
= \chi(\alpha) \cdot \chi(\nu_i). \quad (4.44)
\end{align*}
\]

Finally, since \( V \setminus I = \{ \nu(\alpha_1), \ldots, \nu(\alpha_r) \} \), we can express \( \Sigma(I) \) as
\[
\Sigma(I) = \Sigma(V) \cdot \prod_{u=1}^{r} \chi(\alpha_u, \nu(\alpha_u)) \quad \text{(from (4.42))}
\]
\[
= \Sigma(V) \cdot \prod_{u=1}^{r} \chi(\alpha_u) \cdot \chi(\nu(\alpha_u)) \quad \text{(from (4.44))}
\]
\[
= \Sigma(V) \cdot \left( \prod_{i=1}^{k} \chi(\nu_i) \right) \cdot \left( \prod_{i=1}^{k} \chi(\nu_i) \right) \cdot \left( \prod_{u=1}^{r} \chi(\nu(\alpha_u)) \right) \cdot \left( \prod_{u=1}^{r} \chi(\alpha_u) \right)
\]
\[
= R \cdot \left( \prod_{\nu_i \in I \cap V} \chi(\nu_i) \right) \cdot \left( \prod_{\alpha \in I} \chi(\alpha) \right). \quad (4.45)
\]

where
\[
R := \Sigma(V) \cdot \left( \prod_{i=1}^{k} \chi(\nu_i) \right). \quad (4.46)
\]

Hence \( \Sigma \) is induced by a choice of sign \( \chi(\alpha) \) for columns \( \alpha \in [n] \) and \( R \) for an arbitrary row of \( A \).

The previous results can be summarised in the following theorem, which relates the requirements coming from the determinantal form (3.1), the KP II equation and the whole KP hierarchy.

**Theorem 12.** For a generic choice of \( \kappa \), a signature (3.5) returns a solution \( \tau(x) - 2 \cdot \tau_{\Sigma}(x) \) of the KP II equation (2.5) if and only if it returns a solution \( \tau(x_1, x_2, x_3, \ldots) - 2 \cdot \tau_{\Sigma}(x_1, x_2, x_3, \ldots) \) for the whole KP hierarchy, if and only if \( \Sigma \) preserves the determinantal form (2.12).

**Proof:** A choice of signs (3.5) returns a solution of the KP II equation (2.5) if and only if it is induced by a choice of signs for rows and columns of \( A \) by Theorem 11. In this case, the associated function \( \tau(x_1, x_2, x_3, \ldots) - 2 \cdot \tau_{\Sigma}(x_1, x_2, x_3, \ldots) \) is a solution of the whole KP hierarchy.

If we focus on the determinantal structure of the \( \tau \)-function, all the choices of signs for rows and columns clearly preserve the form (2.12). On the other hand, if a signature \( \Sigma \) preserve this structure, then the KP II equation is satisfied (together with all the other members of the hierarchy). So, from Theorem 11, this solution can be expressed in terms of the initial function via a choice of signs for rows and columns of \( A \).

A more detailed analysis on the signatures preserving the determinantal constraints, which also includes the preservation of a specific subset of soliton parameters, is given in Appendix A.

A special situation is when \( \Sigma \) is defined over the whole set \( \mathcal{P}_k[n] \), i.e., \( \Theta = \mathcal{P}_k[n] \). In such a case, one can easily verify that Remark 6 implies that \( \approx_I \) is an equivalence. If some
vanishing minors occur, then the transitivity of the relation (4.29) is not guaranteed. So one could look for a transitive extension of all the relations ≈ at varying \( I \). An extension of \( \Sigma \), that is a map \( \tilde{\Sigma} : \mathcal{P}_k[n] \to \{ \pm 1 \} \) that satisfies \( \tilde{\Sigma}(I) = \Sigma(I) \) for all \( I \in \mathcal{G} \), can be obtained using (4.43) to label minors in \( \mathcal{P}_k[n] \setminus \mathcal{G} \) with a sign compatible with Proposition 4. So the following holds:

**Corollary 13.** For a solitonic choice of sign \( \Sigma \), there exists an extension \( \tilde{\Sigma} \) of \( \Sigma \) to the whole set \( \mathcal{P}_k[n] \) such that \( \tilde{\Sigma} \) is a solitonic choice of signs too.

It could be interesting to extend this approach to more general expressions for the \( \tau \)-function and to other hierarchies, in order to check the complexity reduction coming from the initial data and the specific requirements.

5. **Number of distinct configurations**

Having identified a family of signatures that preserve specific requirements, it is worth exploring some of its combinatorial aspects in order to clarify the effects of the initial data, e.g., the coefficient matrix \( A \), on the set of allowed configurations.

The overall sign given by row signature is obtained with the choice \( R \in \{ \pm 1 \} \) for an arbitrary row of \( A \) (see (4.46)). Sign flips for columns can be expressed as an action of \( \{ \pm 1 \}^n \) on \( R \times n \times n \) by right multiplication, i.e.,

\[
\sigma_\alpha := \begin{pmatrix} 1_{\alpha - 1} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1_{n-\alpha} \end{pmatrix}, \quad \alpha \in [n],
\]

(5.1)

\[
\sigma_S := \prod_{\alpha \in S} \sigma_\alpha, \quad S \in \mathcal{P}_k[n].
\]

(5.2)

Accordingly, (3.4) becomes

\[
\tau_S(x) := \det (\sigma_S(A) \cdot \Theta(x) \cdot K).
\]

(5.3)

If \( k = 1 \), then the minors of \( A \in \mathbb{R}^{1 \times n} \) are the entries of a row vector. Then, for every non-zero matrix \( B \in \mathbb{R}^{1 \times n} \) such that \( |A_\alpha| = |B_\alpha| \) for all \( \alpha \in [N] \), one can recover \( A \) from (5.2) choosing \( S = \{ \alpha \in [n] : A_{1,\alpha} \cdot B_{1,\alpha} < 0 \} \). If \( \mathcal{G} = \mathcal{P}_k[n] \), such a choice is unique. The use of (5.2) in the case \( k > 1 \) gives a suitable generalization.

**Lemma 14.** Let \( A \in \mathbb{R}^{k \times n} \) and \( C_1, \ldots, C_P \) be the equivalence classes associated with the relation \( \to_V \) in Definition 9. For all \( \mathcal{H}_1, \mathcal{H}_2 \subseteq [n] \), the equalities

\[
\text{sign} [\Delta (\sigma_{\mathcal{H}_1}(A); I)] = \text{sign} [\Delta (\sigma_{\mathcal{H}_2}(A); I)], \quad I \in \mathcal{G}
\]

(5.4)

imply that there exists \( p \subseteq [P] \) such that

\[
\mathcal{H}_1 \Delta \mathcal{H}_2 = \bigcup_{q \in p} C_q.
\]

(5.5)

**Proof:** First note that the condition (5.4) holds for \( A \) if and only if it holds for \( \sigma_{\mathcal{H}_2}(A) \): indeed, for all \( S \subseteq [n] \), \( A \) and \( \sigma_S(A) \) have the same set of non-vanishing maximal minors and, in particular, the action of \( \sigma_{\mathcal{H}_2} \) does not affect the relation \( \to_V \) relative
to the pivot set \( \mathcal{V} \). In the substitution \( A \mapsto \sigma_{H_2}(A) \), the additional signs coming from \( \sigma_\alpha, \alpha \in H_2 \), appear on both sides of (5.4), then

\[
\sigma_{H_1}(A) \mapsto \sigma_{H_1}(\sigma_{H_2}(A)) = (\sigma_{H_1 \cap H_2})^2(\sigma_{H_1 \Delta H_2}(A)) = \sigma_{H_1 \Delta H_2}(A),
\]

\[
\sigma_{H_2}(A) \mapsto (\sigma_{H_2})^2(A) = A
\]

(5.6)
since \( \sigma_{H_1 \cap H_2} = \sigma_{H_2} = 1_n \). So (5.4) holds for the pair \( (H_1, H_2) \) if and only if it holds for \( (H_1 \Delta H_2, \emptyset) \).

Take any class \( C_q, q \in [P] \), and two elements of \( \alpha, \beta \in C_q \). Then, there exists a path \( \Phi_{\mathcal{V}}(\alpha \rightarrow \beta) \) in \( X(\mathcal{V}) \), which consists of a chain of pairs \( (\nu_T, \gamma_T), \nu_T, \gamma_T \in C_q \), associated with subsets \( \mathcal{V}_{\nu_T} \subseteq \mathcal{G} \). If the condition (5.4) holds for such subsets, one gets

\[
\nu_T \in H_1 \Delta H_2 \Leftrightarrow \gamma_T \in H_1 \Delta H_2
\]

(5.7)
for all \( q \in [P] \) and \( \alpha, \beta \in C_q \), which is equivalent to (5.5).

Thus the redundancy in the representation of signatures by subsets of \([n]\) is due to the elements of

\[
\left\{ \bigcup_{q \in \mathcal{P}} C_q, \mathcal{P} \subseteq [P] \right\}.
\]

(5.8)
Note that \( \bigcup_{q \in \mathcal{P}} C_q \) are pairwise distinct for different choices of \( \mathcal{P} \) since \( (C_1, \ldots, C_P) \) is a partition. This still holds for the elements \( H_2 \) in

\[
\left\{ H_1 \Delta \left( \bigcup_{q \in \mathcal{P}} C_q \right), \mathcal{P} \subseteq [P] \right\}.
\]

(5.9)
satisfying (5.5), since the symmetric difference is invertible and, hence, the mapping \( \bigcup_{q \in \mathcal{P}} C_q \mapsto H_1 \Delta \left( \bigcup_{q \in \mathcal{P}} C_q \right) \) is bijective.

**Proposition 15.** The number of distinct signatures obtained from sign choices (5.2) for columns and (4.46) for a row is \( 2^{n+1-P} \), where \( P \) is the number of classes of \( \rightarrow_{\mathcal{V}} \) associated with \( A \).

**Proof:** For each subset \( I \in \mathcal{G} \) consider a map constructed as in (4.41) that associates \( \nu(\alpha) \in I \setminus \mathcal{V} \) to a unique \( \alpha \in I \setminus \mathcal{V} \) so that \( \mathcal{V}_{\nu(\alpha)} = \mathcal{G} \). This also implies \( \alpha \rightarrow_{\mathcal{V}} \nu(\alpha) \).

Thus, the non-vanishing minors of \( A \) correspond to subsets which intersect all the classes \( C_1, \ldots, C_P \) and, in particular,

\[
\#(I \cap C_q) = \#(\mathcal{V} \cap C_q) =: k_q, \quad q \in [P]
\]

(5.10)
does not depend on \( I \in \mathcal{G} \). Then, the involutions \( \Delta_q \) defined by

\[
H \mapsto \Delta_q(H) := H \Delta C_q, \quad H \subseteq [n], q \in [P]
\]

(5.11)
Hence, if subsets associated with a single signature, and the number of allowed signatures is equalities non-vanishing minors of $A$ and denote the new column by the index $H$ components model describes the original one using only column operations, up to a common new signature for columns only, i.e., $\sigma_0 := (1)$. Using the partition $P$, one can uniquely express $H \subseteq \{n\}$ in terms of its components $H_q := H \cap C_q$ and get, for each $p := \{q_1, \ldots, q_T\} \subseteq [P]$, the following equalities

$$H \Delta \left( \bigcup_{q \in p} C_q \right) = \left( \bigcup_{s \in [P] \setminus p} H_q \right) \cup \left( \bigcup_{q \in p} C_q \setminus H_q \right) = \Delta_{q_1} \circ \Delta_{q_2} \circ \cdots \circ \Delta_{q_T}(H).$$

Hence, if $H_2 = H_1 \Delta \left( \bigcup_{q \in p} C_q \right)$ (as in (5.5)), then from (5.12) and (5.13) one gets

$$\sign[\Delta(\sigma_{H_2}(A); I)] = \prod_{q \in p} (-1)^{k_q} \cdot \sign[\Delta(\sigma_{H_1}(A); I)].$$

Now choose any ancillary soliton parameter $\kappa_0 \notin \{\kappa_1, \ldots, \kappa_n\}$, introduce the matrix

$$\tilde{A} := (1) \oplus A = \begin{pmatrix} 1 & 0_T \\ 0_k & A \end{pmatrix},$$

and denote the new column by the index 0. There exists a bijection between the non-vanishing minors of $A$ and those of $\tilde{A}$, that is $I \mapsto \{0\} \cup I$. Any choice of signs $\sigma_H$ for the columns of $A$, $H \subseteq \{n\}$, can be extended to a choice for $\tilde{A}$ as $\tilde{\sigma}_H := (1) \oplus \sigma_H$. Furthermore, the choice $R = -1$ in (4.46) is restated as a new signature for columns only, i.e., $(-1) \oplus 1_n =: \tilde{\sigma}(0)$. Hence this equivalent model describes the original one using only column operations, up to a common multiplicative factor for the terms $A_I(x)$. The pivot set for $\tilde{A}$ is $\mathcal{V} := \{0\} \cup \mathcal{V}$, and this extends the relation $\rightarrow_Y$ to the equivalence $\rightarrow_{\tilde{Y}}$ whose classes are $C_1, \ldots, C_P$ and $\{0\}$, since $\mathcal{V}_0^\alpha \in \mathcal{P}_k[n] \setminus \emptyset$ for all $\alpha \in [n]$.

So there are $2^{n+1}$ choices of signs for the columns of $\tilde{A}$ and $P + 1$ distinct equivalence classes for $\rightarrow_{\tilde{Y}}$. The cardinality of one of them, i.e., $\#\{0\} = 1$, is odd: hence, from (5.12), the substitution $p \mapsto \{0\} \cup p$ induces the mapping

$$\sign[\Delta(\sigma_{H_2}(\tilde{A}); I)] \mapsto \sign[\Delta(\sigma_{H_2\Delta\{0\}}(\tilde{A}); I)] = -\sign[\Delta(\sigma_{H_2}(\tilde{A}); I)]$$

for any subset $p \subseteq [P]$. In conclusion, Lemma 14 states that, for each $H_1 \subseteq \{0\} \cup \{n\}$, the possible sets $H_2$ satisfying (5.4) lie in (5.9); each element of this family satisfies (5.14) independently on $I \in \emptyset$; finally, from (5.16) it follows that each signature $p \subseteq [P]$ corresponds to the opposite one $\{0\} \cup p$, so they occur in equal numbers. This means that exactly half of the terms in (5.9) have the same signature of $H_1$, while the other half have opposite signature. Hence there are $\frac{1}{2}2^{P+1} = 2^P$ subsets associated with a single signature, and the number of allowed signatures is $\frac{2^{n+1}}{2^P} = 2^{n+1-P}$.
Remark 16. The previous discussion also implies that the sets (5.9) are equipollent and pairwise disjoint, since each of them contains all the possible combinations of subsets of $[n]$ that induce a given signature $\Sigma$ or the opposite $-\Sigma$.

The freedom in the choice of signs generalizes free statistical amoebas (including an additional row sign flip), which fall within the case $P = 1$. Indeed, let $\mu_s$ denote the number of distinct signatures induced by the sign flip of exactly $s$ columns of $A$, $s \in [n]$, without limitations on $R \in \{\pm 1\}$. Each combination of signs for row and columns can be labelled by an element in $P_s[n] \times \{\pm 1\}$, hence $\mu_s \leq 2 \cdot \binom{n}{s}$. At $s = \frac{n}{2}$, $H \in P_{n/2}[n]$ produces the same signature of its complement $[n] \setminus H \in P_{n/2}[n]$ for an appropriate choice of $R$, so $\mu_{n/2} \leq \binom{n}{n/2}$. These bounds, along with the action of (5.14) and the result in Proposition 15, give

$$2^n = \sum_{s=0}^{n/2} \mu_s = \sum_{s=0}^{n/2} \frac{\mu_s + \mu_{n-s}}{2} \leq \sum_{s=0}^{n} \binom{n}{s} = 2^n.$$  
Thus all the bounds are in fact equalities, i.e., $\mu_s = 2 \cdot \binom{n}{s}$ at $s \neq \frac{n}{2}$ and $\mu_{n/2} = \binom{n}{n/2}$.

6. Levels of constrained amoebas

The previous discussion leads to an extension of the concept of statistical amoeba to higher dimensional cases. Following the construction for free statistical amoebas in [5], one can focus on the family of functions (or the associated locus of zeros) obtained from $\tau$ through all the combinations of $s$ sign flips for columns at $s \in [n]$ fixed.

We can associate with each $S \subseteq [n]$ the vector $v(S) \in \mathbb{F}_2^n$ defined as $v(\alpha) = 1$ if $\alpha \in S$ and $v(\alpha) = 0$ otherwise. The intersection of $S_1, S_2 \in P[n]$ is given by the componentwise product $v(S_1 \cap S_2)_\alpha = v(S_1)_\alpha \cdot v(S_2)_\alpha$, hence the parity of $\#(S_1 \cap S_2)$ is equal to the dot product

$$\#(S_1 \cap S_2) \equiv \sum_{\alpha=1}^{n} v(S_1)_\alpha \cdot v(S_2)_\alpha =: v(S_1) \cdot v(S_2) \mod 2. \quad (6.1)$$

Motivated by this, we adopt the following notation

$$S \parallel T \iff \#(S \cap T) \equiv 1 \mod 2, \quad (6.2)$$
$$S \perp T \iff \#(S \cap T) \equiv 0 \mod 2 \quad (6.3)$$

and $H \perp \mathcal{L} := \{L \in \mathcal{L} : H \perp L\}$ with $H \in P[n]$ and $\mathcal{L} \subseteq P[n]$.

Unlike the free statistical amoeba at $k = 1$, only some strata are visible under choices of signs allowed by determinantal/integrability requirements. For instance, a single change of sign preserves neither the determinantal structure (see Example 2) nor the solution of the KP II equation in generic situations. Hence the 1-stratum for the free statistical amoeba is not part of the constrained amoeba.

Let us focus on the case $\Phi = P_k[n]$. For all the choices of $\sigma_S$, $s \in [n]$ and $S \in P_s[n]$, the number $\Omega(n, k; s)$ of $-$ signs generated by $S$ is equal to the cardinality of $S \parallel P_k[n]$. This holds for all possible choices of $S \in P_s[n]$ by permutation symmetry. The singular locus corresponding to $S$ is a subset of all the singular loci in the free $\Omega(n, k; s)$-statistical amoeba.
Proposition 17. If $\mathcal{G} = \mathcal{P}_k[n]$, then a choice (5.2) with $\# \mathcal{S} = s$ induces a signature with

$$\Omega(n, k; s) := \left\{ \begin{array}{ll} \omega(n, k; s), & m \geq k + s, \\ \frac{1}{2} \left( \frac{(-1)^{n-k-s}}{2} \right) + (-1)^{n-k-s} \omega(n, n - k; n - s), & m < k + s \end{array} \right. \quad (6.4)$$

where

$$\omega(n, k; s) := \frac{1}{2} \binom{n}{k} - \frac{1}{2} \binom{n - s}{k} \ {}_2 F_1(-s, -k; n - s - k + 1; -1). \quad (6.5)$$

Proof: Let $\mathcal{I} \in \mathcal{P}_s[n]$. The number of subsets $\mathcal{A} \in \mathcal{P}_k[n]$ satisfying $\mathcal{A} \parallel \mathcal{I}$ is

$$\frac{1}{2} \min(k, s) \sum_{a=0}^{s} \binom{s}{2a+1} \binom{n-s}{k-2a+1} = \sum_{a=0}^{s} \frac{(-1)^{a+1} + 1}{2} \binom{s}{a} \binom{n-s}{k-a}$$

$$= \frac{1}{2} \cdot \sum_{a=0}^{k} (-1)^{a+1} \binom{s}{a} \binom{n-s}{k-a} + \frac{1}{2} \sum_{a=0}^{s} \binom{s}{a} \binom{n-s}{k-a} \quad (6.6)$$

where $\binom{t}{w} = 0$ at $t < w$ or $w < 0$. We observe that $\mathcal{A} \parallel \mathcal{S}$ is equivalent to

$$\# (\{[n] \setminus \mathcal{S}) \cap ([n] \setminus \mathcal{A})) = \# [n] \setminus (\mathcal{S} \cup \mathcal{A}) \cong n - s - k + 1 \mod 2 \quad (6.7)$$

where the principle of inclusion-exclusion has been used in the second line. So, at even $n - s - k$ (respectively, odd $n - s - k$), the enumeration of sets $\mathcal{A} \in \mathcal{P}_k[n]$ with $\mathcal{A} \parallel \mathcal{S}$ is equivalent to the enumeration of subsets $[n] \setminus \mathcal{A} \in \mathcal{P}_{n-s}[n]$ with $[n] \setminus \mathcal{A} \parallel [n] \setminus \mathcal{S}$ (respectively, $[n] \setminus \mathcal{A} \perp [n] \setminus \mathcal{S}$). Furthermore, exactly one holds between $n - s - k \geq 0$ or $n - (n-s) - (n-k) = s + k - n > 0$.

So we can look at the situations where $n - s - k \geq 0$ and derive the required quantity in other cases by the previous observation. First note that

$$\sum_{a=0}^{s} \binom{s}{a} \binom{n-s}{k-a} = \binom{n}{k}. \quad (6.8)$$

counts the number of elements of $\mathcal{P}_k[n]$. For the first summation, one has

$$\sum_{a=0}^{k} (-1)^{a+1} \binom{s}{a} \binom{n-s}{k-a}$$

$$= - \binom{n-s}{k} \cdot \sum_{a=0}^{k} \binom{k}{a} \frac{(-1)^{a} \cdot s!(n-s-k)!}{(s-a)!(n-s-k+a)!}$$

$$= - \binom{n-s}{k} \ {}_2 F_1(-s, -k; n - s - k + 1; -1) \quad (6.9)$$

where $\ {}_2 F_1$ is the Gaussian hypergeometric function, and all the equivalences are well-posed due to the condition $n - s - k \geq 0$. Adding the two contributions, we get $\Omega(n, k; s) = \omega(n, k; s)$. From this we also find that, at $n - s - k < 0$, the number of $(n-k)$-subsets $\mathcal{B} \subseteq [n]$ with $\mathcal{B} \parallel ([n] \setminus \mathcal{S})$ is $\omega(n, n-k; n-s)$, while those with $\mathcal{B} \perp ([n] \setminus \mathcal{S})$ are $\binom{n}{k} - \omega(n, n-k; n-s)$. By (6.7) and subsequent observations,
these two quantities respectively enumerate the number of \(s\)-subsets \(\mathcal{A} \subseteq [n]\) with \(\mathcal{A} \parallel \mathcal{S}\) at even \(n - k - s\) and at odd \(n - k - s\). These results can be expressed as in (6.4), which concludes the proof. \(\square\)

It is worth remarking that there is a duality between the dimension \(k\) and the level \(s\): the number of pairs \((\mathcal{H}_1; \mathcal{H}_2) \in \mathcal{P}_k[n] \times \mathcal{P}_s[n]\) such that \(\mathcal{H}_1 \parallel \mathcal{H}_2\) can be enumerated in two ways, i.e., fixing one of the two components \(\mathcal{H}_i, i \in \{1, 2\}\) and considering all the subsets \(\mathcal{I}\) with \(\mathcal{H}_i \parallel \mathcal{I}\). This double counting implies the following identity

\[
\binom{n}{s} \cdot \Omega(n, k; s) = \binom{n}{k} \cdot \Omega(n, s; k). \tag{6.10}
\]

From this, one also finds

\[
\Omega(n, k; s) < \frac{1}{2} \binom{n}{k} \Leftrightarrow \frac{\binom{n}{s}}{\binom{n}{k}} \Omega(n, k; s) < \frac{1}{2} \binom{n}{s} \Leftrightarrow \Omega(n, s; k) < \frac{1}{2} \binom{n}{s}. \tag{6.11}
\]

In this sense, the distinction between (free) amoebas and antiamoebas, as defined in Section 2.3, is compatible with such a duality. Furthermore, when \(g \tau > 0\) for all \(\mathcal{I} \in \mathcal{P}_k[n]\), the quantity \(\Omega(n, s; k)\) dual to \(\Omega(n, k; s)\) related to the behaviour of the constrained amoeba at large values of \(x\), namely, to its tropical limit [5]: outside the locus where \(\max_{\mathcal{H} \in \mathcal{P}_s[n]} \Lambda_{\mathcal{H}}(x) =: \Lambda_D(x)\) is attained more than once, the sign of \(\tau_\mathcal{S}(x)\) at \(||x|| \to \infty\) coincides with the induced sign for this dominant term, that is \((-1)^{\#(\mathcal{S} \cap \mathcal{D})}\). So \(\Omega(n, s; k)\) represents the number of subsets \(\mathcal{S} \in \mathcal{D} \parallel \mathcal{P}_s[n]\) where \(\tau_\mathcal{S}(x) < 0\).

7. Applications

7.1. Application to information transfer via message coding. The previous results suggest using an initial function (2.13) to encode information. Specifically, we can think at each signature as a message encoded in a \(G\)-bits string, where \(G := \#\mathcal{G}\). Here we assume that the order of the bits in the string corresponds to a given (e.g., lexicographical) order for the elements of \(\mathcal{P}_k[n]\). Analogously, we can represent (2.13) as a string with \(\binom{n}{s}\) entries in \(\{0, 1, \bot\}\), where the entries label the minors of \(\mathcal{A}\) and the symbol \(\bot\) is associated with vanishing minors.

If one knows that the original function \(\tau(x)\) solves the KP II equation and receive a new function \(\tau(x) - 2 \cdot \tau_\Sigma(x)\), then the fulfillment of the KP II equation implies that a particular choice of signs has been sent. We stress that all the signs have to be checked to confirm that \(\Sigma\) is induced by row/column operations: indeed, in the generic case \(G = \binom{n}{k}\), a single switch of sign converts a constrained signature to a non-constrained one, as shown in Example 2. However, in the present approach, one can check if \(\Sigma\) is induced by a choice (5.2) indirectly, i.e., without having to find such a configuration \(\sigma_\mathcal{S}\) and, hence, avoiding the effort to get this additional knowledge.

In order to quantify the amount of information that can be acquired through this single check, we consider the well-known Kullback-Leibler divergence [15]. After the reception of the message, but before any additional check on the signs, one can recover the data related to dependence relations among the columns of \(\mathcal{A}\), namely \(\mathcal{G}\), \(\mathcal{G}\) and, for any fixed \(\mathcal{V} \in \mathcal{G}\), the relation \(\rightarrow\mathcal{V}\) and the dimensions \(k_1, \ldots, k_p\). At this point, the prior information
is based on strings of bits indexed by $\mathcal{S} \subseteq P_k[n]$. With no additional constraint, we can assume a prior distribution $u$ where all the $2^G$ $G$-bits strings have the same statistical weight $2^{-G}$. In the generic case $G = \binom{n}{k}$ there are $\exp\left(\binom{n}{k} \cdot \ln 2\right)$ such strings, but only $2^n$ of them satisfy the KP II equation by Theorem 11 and Proposition 15. This restricted family of strings associated with solitonic signatures is the support for the posterior distribution $u_{KP}$. So the Kullback-Leibler divergence is given by

$$D_{KL}^*(u_{KP} || u) := \sum_{S \subseteq [n]} 2^{-n} \cdot \ln \left( \frac{\exp(-n \cdot \ln 2)}{\exp(-\binom{n}{k} \cdot \ln 2)} \right)$$

$$= \ln 2 \cdot \left\lfloor \frac{n}{k} \right\rfloor - n. \quad (7.1)$$

When $\#\mathcal{S} < \binom{n}{k}$, the number of distinct general strings is $2^G$, while the number of those satisfying the KP II equation is $2^{n+1-P}$ by Proposition 15. So one gets

$$D_{KL}^*(u_{KP} || u) = \ln 2 \cdot (G - n + P - 1) \quad (7.2)$$

which is a real non-negative quantity, hence we also find

$$G + P \geq n + 2. \quad (7.3)$$

In the previous cases, the assumptions of equiprobability for strings in $\{\pm 1\}^\mathcal{S}$ and for choices of signs induced by (5.2) are consistent as a result of Proposition 15, since the equivalence classes associated to the different signatures are equipollent (also see Remark 16). One can get the quantities (7.1)-(7.2) through the choices $(R, \sigma_S)$, $R \in \{\pm 1\}$ and $S \subseteq [n]$, which reduces to the same multiple-counting of both general and solitonic signatures. This type of equivalence does not necessarily hold when further restrictions affect the prior. In particular, we consider a situation where the number $s = \#\mathcal{S}$ for allowed $\sigma_S$ is fixed and known, as in the discussion on statistical amoebas in [5].

In order to explore some features of this setting, we first estimate the information content associated with the check of the KP II equation in the generic situation $G = \binom{n}{k}$. Let us look at the effects of the knowledge of a fixed value for $s$ at $s \neq \frac{n}{2}$ to avoid the occurrence of solitonic signatures related by the involutions (5.11) and, hence, to single out the contribution of the evidence on $s$. This enables us to consider a preliminary check before looking at the KP II equation: if neither the received string $\Sigma$ nor its opposite $-\Sigma$ satisfy $\#\text{NS} = \Omega(n, k; s)$, then $\Sigma$ is not obtained through (5.2) with $R \in \{\pm 1\}$ and, hence, does not satisfy the KP II equation. Furthermore, at $\Omega(n, k; s) \neq \frac{1}{2} \binom{n}{k}$, also the row sign $R$ is fixed by this preliminary check, and this restricts the support of the uniform distributions for the prior $u$ and the posterior $u_{KP}$ to $\mathcal{P}_{\Omega(n,k;s)}[P_k[n]]$ and $\mathcal{P}_s[n]$, respectively. The uncertainty on $R$ still remains at $\Omega(n, k; s) = \frac{1}{2} \binom{n}{k}$. So the Kullback-Leibler divergence is equal to

$$D_{KL}^*(u_{KP} || u) = \ln \left[ \left( \frac{n}{s} \right)^{-1} \cdot \left( \frac{\binom{n}{k}}{\Omega(n, k; s)} \right) \right] - \ln 2 \cdot \delta_{2\Omega(n,k;s),\binom{n}{k}}. \quad (7.4)$$

The last term, which involves the cases $\Omega(n, k; s) = \frac{1}{2} \binom{n}{k}$, does not appear if one also knows that only column operations can be performed.
We present a graphical representation of the Kullback-Leibler divergence at $G = \binom{n}{k}$: the behaviour of (7.4) at different values of $s$ is shown in Figure 7.1, and can be compared with (7.1) in Figure 7.2a. The analytic continuations of both these formulas have been used.

The gain or loss of information can be quantified by the difference $D_{\text{KL}}^{s}(u_{KP}||u) - D_{\text{KL}}^{\ast}(u_{KP}||u)$ at $n = 10$ (see Figure 7.2b). It depends on the values $n, k, s$, as shown in the following example.

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Example 18. Let us consider the generic case $\mathcal{G} = \mathcal{P}_k[n]$. At $(n,k,s) := (7,3,1)$ we find that $G = 35$, $\Omega(7,3,1) = 15$, $D_{KL}(35-7) \cdot \log 2 \approx 19.4081$ and $D_{KL}(1) = \ln \left( \frac{1}{7} \cdot \binom{35}{15} \right) \approx 19.9554$, so $D_{KL}^* - D_{KL}(1) < 0$.

On the other hand, moving to the case $(n,k,s) = (9,4,4)$, the quantities $G = 126$, $\Omega(9,4,4) = 60$, $D_{KL}(126-9) \cdot \log 2 \approx 81.0982$ and $D_{KL}(4) = \ln \left( \frac{1}{9} \cdot \binom{126}{60} \right) \approx 79.7126$, so $D_{KL}^* - D_{KL}(4) > 0$.

In contrast to the generic case, situations where $\mathcal{G} \neq \mathcal{P}_k[n]$ may exhibit different combinatorial features, and the equivalence between solitonic signatures (3.5) and the induced representations (5.2) is weakened. In particular, the constraint on the number of negative signs $\# \text{NS}$ in $\Sigma$ does not correspond to a fixed number of signs $s$ for $\mathcal{S}$ in (5.2), and vice versa. Furthermore, if one assumes a uniform prior distributions on $\mathcal{P}_s[n]$ in the construction of solitonic signatures by the sender, then it is natural to assume that the solitonic signatures are not equiprobable. In fact, moving from signatures in $\{\pm 1\}^6$ to elements of $\mathcal{P}_s[n]$, the assumption that choices for $\mathcal{S} \in \mathcal{P}_s[n]$ occur with the same probability $\binom{n}{s}^{-1}$ endows the associated $G$-bits solitonic strings with multiplicity weights. Each weight expresses the redundancy of the representation (5.2) counting the number of subsets $\mathcal{S} \in \mathcal{P}_s[n]$ that induce the same signature. At fixed $s$, these weights do not coincide for all the signatures in general. For instance, if there is no subset $p \subseteq [P]$ such that

$$\sum_{q \in p} k_q = 2 \sum_{q \in p} s_q, \quad (7.5)$$

then each involution (5.11) makes the index $s$ of the signature change. On the other hand, if (7.5) is satisfied, then both $\mathcal{S}$ and $\mathcal{S} \Delta \bigcup_{q \in p} \mathcal{H}_q$ have $s$ negative signs and induce the same signature. We clarify these issues through the following examples.

Example 19. Consider the matrix

$$\mathbf{P}_1 := \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}. \quad (7.6)$$

The non-vanishing minors for $\mathbf{P}_1$ can be indexed by elements in $\{1,2,3\} \times \{4,5\} \times \{6,7\}$, so $G = 12$, while it is easily seen that $P = 3$. Not all the choices in $\mathcal{P}_2[7] \times \{\pm 1\}$ correspond to distinct signatures, due to coincidences from (5.11) and $R \in \{\pm 1\}$. In fact, there are 22 distinct solitonic signatures associated with as many classes in a partition of $\mathcal{P}_2[n]$, where each class includes all the subsets $\mathcal{S}$ that returns the same signature. Assuming a uniform distribution on $\mathcal{P}_2[7] \times \{\pm 1\}$, we get the following induced distribution for solitonic signatures

$$\left\{ \frac{1}{42} \cdot \frac{1}{42} \cdot \frac{1}{21} \cdot \frac{1}{21} \cdot \frac{1}{21} \cdot \frac{1}{21} \cdot \frac{1}{21} \cdot \frac{1}{21} \cdot \frac{1}{21} \cdot \frac{1}{21} \cdot \frac{1}{21} \cdot \frac{1}{21} \cdot \frac{1}{21} \cdot \frac{1}{21} \cdot \frac{2}{21} \cdot \frac{2}{21} \cdot \frac{2}{21} \cdot \frac{2}{21} \right\}. \quad (7.7)$$

The Kullback-Leibler divergence from the uniform prior over $2^6$ to (7.7) is $D_{KL}(\mathbf{P}_1; s = 2) \approx 5.30625$. Note that the relative entropy, starting from $\mathbf{P}_1$ but without the information on $s = 2$, is $7 \cdot \ln 2 \approx 4.85203 < D_{KL}(\mathbf{P}_1; s = 2)$ by (7.2). On the other hand, using (7.1) and
we find that $D_{\text{KL,unconstrained}} \approx 19.4081$ and $D_{\text{KL,s=2}} \approx 18.8568$ in the generic case. Thus, while the unconstrained case is favourable in the generic case, the unconstrained one may be preferred when $\mathcal{S} \neq \mathcal{P}_k[n]$.

**Example 20.** Consider the matrix

$$P_2 := \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \end{pmatrix}.$$  

(7.8)

It is easily seen that $P_1$ is a rectangular block matrix whose three blocks are full-rank Vandermonde matrices, call them $k_1, k_2, k_3$. Thus one can enumerate the non-vanishing minors of $P_1$ as in the previous example and find $G = \binom{4}{2} \cdot \binom{3}{2}^2 = 54$.

First we note that the constraints on $\#S$ in (5.2) and on the negative signs in $\Sigma$ do not match: at $s = 2$, the choice $S = \{1, 2\}$ returns a signature $\Sigma$ with 36 negative signs, while $S = \{1, 5\}$ gives 27 negative signs. The operation (4.46) preserves the parity $\#\mathcal{NS}$, since $G$ is even, so the choices $\{1, 2\}$ and $\{1, 5\}$ still have different parity when the choice for $R$ is included. Vice versa, a fixed number of negative signs for solitonic signatures does not correspond to the same constraint for $\#S$: both $S_1 = \{1, 5\}$ and $S_2 = \{1, 5, 9\}$ generate signatures with 27 negative signs, despite having different cardinalities.

Let us focus on the case $s = 2$. Before undertaking any check on the signature, we choose a uniform prior on the $2^G = 2^{54}$ possible ones. In contrast to Example 19, distinct choices of $R$ produce distinct signatures: indeed, two subsets $S_1, S_2 \in \mathcal{P}_s[n]$ inducing opposite signatures $\Sigma_1 = -\Sigma_2$ has to be related by the action of some involutions (5.11) respecting (7.5). But any operation of this type returns the same signature, since the ranks of the matrices $k_1, k_2, k_3$ are all even. However, also in this case the uniform distribution on $\mathcal{P}_2[n]$ induces a non-uniform distribution on the solitonic signatures: for instance, the signature associated with $\{8, 9\}$ does not coincides with other ones, while $\{1, 2\}$ and $\{3, 4\}$ lie in the same equivalence class, and this results in different weights. Computing the relative entropy as in the previous example, one gets $D_{\text{KL}}(P_2; s = 2) \approx 33.0226$, while the unconstrained case gives $46 \cdot \ln(2)$.

The features of non-generic cases and the dependence on the parameters $(n, k, s)$ can be used to adapt the amount of information content in the two cases of unconstrained and fixed cardinality for $S$ in (5.2). A specific analysis in this regard will be carried on in a separate work.

7.2. **Intersection property and its geometric interpretation.** The introduction of the families

$$\mathcal{M}(x) := \{S \in \mathcal{P}[n] : \tau_S(x) < 0\}, \quad \mathcal{M}_s := \mathcal{M} \cap \mathcal{P}_s[n]$$

plays a significant role in the identification of the stratified structure of statistical amoebas at $k = 1$. This comes from a simple combinatorial property (see Proposition 5 in [5]), which can be extended to the case $k > 1$ as follows.
Proposition 21. If $\Delta_A(I) \cdot \Delta_K(I) \geq 0$ for all $I \in \mathcal{P}_k[n]$, then there are no 2k pairwise disjoint sets in $\mathcal{O}(x)$.

Proof: Let us suppose that such 2k sets exist, i.e., $\{I_1, \ldots, I_{2k}\} \subseteq \mathcal{O}(x)$ such that $I_a \cap I_b = \emptyset$ for all $a \neq b$. By definition, $\tau_{I_a}(x) < 0$ is equivalent to the inequalities

$$\sum_{H \parallel I_a} \Lambda_H(x) > \sum_{K \perp I_a} \Lambda_K(x) \quad a \in [2k].$$

(7.10)

Let us divide these 2k subsets in two classes $\mathcal{S}_1 := \{I_a : 1 \leq a \leq k\}$ and $\mathcal{S}_2 := \{I_a : k+1 \leq a \leq 2k\}$. Adding the inequalities (7.10) for $\mathcal{S}_1$ term by term, one gets

$$\sum_{w=0}^{\lfloor k/2 \rfloor} (k-2w) \cdot \sum_{H \parallel \mathcal{S}_1} \Lambda_H(x) > \sum_{w=0}^{\lfloor k/2 \rfloor} (k-2w) \cdot \sum_{K \perp \mathcal{S}_1} \Lambda_K(x).$$

(7.11)

Similarly, for $\mathcal{S}_2$ one has

$$\sum_{w=0}^{\lfloor k/2 \rfloor} (k-2w) \cdot \sum_{H \parallel \mathcal{S}_2} \Lambda_H(x) > \sum_{w=0}^{\lfloor k/2 \rfloor} (k-2w) \cdot \sum_{K \perp \mathcal{S}_2} \Lambda_K(x).$$

(7.12)

If $\Lambda_H(x) \neq 0$ appears in the left hand side of (7.11) (respectively (7.12)), then it has non-empty intersection with more than $\lfloor k/2 \rfloor$ elements in $\mathcal{S}_1$ (respectively, $\mathcal{S}_2$).

In the same way, if $H \parallel \mathcal{S}_2 = \{I_b, b \in [k-w]\}$ with $w < \lfloor k/2 \rfloor$ and distinct $I_b$, then $w_1 := \#(H \perp \mathcal{S}_1) \leq w < \lfloor k/2 \rfloor$. So $\#(H \parallel \mathcal{S}_1) = k-w_1$ and the coefficient of $\Lambda_H(x)$ in the right hand side of (7.11) is $k-2w_1 \geq k-2w$. Since $\Lambda_H(x) \geq 0$ by hypothesis, this means that the right hand side of (7.11) is an upper bound for the left hand side of (7.12), which implies

$$\sum_{w=0}^{\lfloor k/2 \rfloor} (k-2w) \cdot \sum_{H \parallel \mathcal{S}_1} \Lambda_H(x) > \sum_{w=0}^{\lfloor k/2 \rfloor} (k-2w) \cdot \sum_{H \parallel \mathcal{S}_2} \Lambda_H(x).$$

(7.13)

Arguing in the same way for (7.12), one gets

$$\sum_{w=0}^{\lfloor k/2 \rfloor} (k-2w) \cdot \sum_{H \parallel \mathcal{S}_2} \Lambda_H(x) > \sum_{w=0}^{\lfloor k/2 \rfloor} (k-2w) \cdot \sum_{H \parallel \mathcal{S}_1} \Lambda_H(x)$$

(7.14)

which is incompatible with (7.13), i.e., a contradiction. \hfill \Box

The previous bound also holds in the restriction from $\mathcal{O}(x)$ to each individual family $\mathcal{O}_s(x)$, and this provides an extension of a geometric property that can be stressed in the case $s = 1$ to higher levels $s > 1$. At this purpose, we introduce the matrices

$$\zeta_\alpha := \vec{e}_\alpha \cdot \vec{e}^T_\alpha = (\delta_{\beta \gamma} \cdot \delta_{\alpha \beta})_{\beta, \gamma \in [n]}, \quad \alpha \in [n]$$

(7.15)

where $\{\vec{e}_\alpha : \alpha \in [n]\}$ is the standard basis for $\mathbb{R}^n$, and

$$L := (A \cdot \Theta(x) \cdot K)^{-1} \cdot A \cdot \Theta(x).$$

(7.16)

Note that $(A \cdot \Theta(x) \cdot K)^{-1}$ exists at det$(A \cdot \Theta(x) \cdot K) \neq 0$ (i.e., outside the singular locus), and $L$ is a left-inverse of $K$. In particular, $(K \cdot L)^2 = K \cdot (L \cdot K) \cdot L = K \cdot 1_K \cdot L = K \cdot L$, so $K \cdot L$ is idempotent. Then, the role played by $\alpha \in [n]$ in the behaviour of (3.4) can be
assessed by the sign of

\[
\frac{\tau(\alpha)(x)}{\tau(x)} = \det((A \cdot \Theta(x) \cdot K)^{-1}) \cdot \det(A \cdot \sigma_\alpha \cdot \Theta(x) \cdot K)
\]

\[
= \det \left( (A \cdot \Theta(x) \cdot K)^{-1} \cdot A \cdot \Theta(x) \cdot (\mathbb{1}_n - 2 \cdot \zeta_\alpha) \cdot K \right)
\]

\[
= \det(\mathbb{1}_k - 2L \cdot \zeta_\alpha \cdot K)
\]  \quad (7.17)

Taking into account that \(\zeta_\alpha^2 = \zeta_\alpha\), one can apply Sylvester’s determinant identity [22] twice and get

\[
\det(\mathbb{1}_k - 2L \cdot \zeta_\alpha \cdot K) = \det(\mathbb{1}_n - 2 \cdot \zeta_\alpha \cdot K \cdot L)
\]

\[
= \det(\mathbb{1}_n - 2 \cdot \zeta_\alpha^2 \cdot K \cdot L)
\]

\[
= \det(\mathbb{1}_n - 2 \cdot \zeta_\alpha \cdot K \cdot L \cdot \zeta_\alpha).
\]  \quad (7.18)

Note that \(\zeta_\alpha \cdot K \cdot L \cdot \zeta_\alpha = \langle K_\alpha | L^\alpha \rangle \cdot \tilde{e}_\alpha \cdot \tilde{e}_\alpha^T\), where \(K_\alpha\) (respectively \(L^\alpha\)) is the vector corresponding to the \(\alpha\)th row of \(K\) (respectively column of \(L\)) and \(\langle \ | \rangle\) is the usual Euclidean scalar product. From the matrix determinant lemma [22], one finds

\[
\det(\mathbb{1}_n - 2 \cdot \zeta_\alpha \cdot K \cdot L \cdot \zeta_\alpha) = 1 - 2 \cdot \langle K_\alpha | L^\alpha \rangle \cdot (\tilde{e}_\alpha \cdot \tilde{e}_\alpha^T) = 1 - 2 \langle K_\alpha | L^\alpha \rangle.
\]  \quad (7.19)

So \(\frac{\tau(\alpha)(x)}{\tau(x)} < 0\) if and only if \(\tilde{e}_\alpha^T \cdot (KL\tilde{e}_\alpha) = \langle K_\alpha | L^\alpha \rangle > \frac{1}{2}\). Thus, the result in Proposition 21 is equivalent to the property that there exist at most \(2k - 1\) diagonal entries of \(K \cdot L\) such that \(\langle K_\alpha | L^\alpha \rangle > \frac{1}{2}\), independently on \(n\).

8. Discussion and future perspectives

In this work we investigated the effects of particular requirements connected to a type of complexity reduction, namely determinantal and integrability constraints, on real-valued partition functions. Such a reduction can be observed through the statistical amoeba associated with an initial sum of exponentials (2.13) fulfilling the given constraints. In such a framework, the family of allowed choices of signs for pre-exponential terms (3.2) coincides with the signs induced by row/column sign choices for the coefficient matrix \(A\). In particular, the consistency with the KP II equation returns \(\tau\)-functions for the whole KP hierarchy. This led to the exploration of the number of distinct signatures for a general \(A\), levels of constrained statistical amoebas, and their applications in the information-theoretic and geometric settings.

These results give rise to questions on further links between the combinatorics of complex structures and integrability, some of which have already been pointed out. In particular, it is worth exploring in more detail the redundancy in the description of signatures (3.5) through subsets (5.2) and the concept of instability domains. These issues are also related to the investigation of the tropical limit of constrained statistical amoebas, as briefly mentioned in Section 6. In fact, these matrix models provide a natural framework for the realization of different tropical concepts, in particular for the nested tropical expansion and the tropical symmetry introduced in [2]. The nested expansion relies on the extension of the ordering of individual phases \(\varphi_\alpha(x)\) at points where \(|x| | \rightarrow \infty\) to an ordering of collective phases \(\sum_{\alpha \in I} \varphi_\alpha, I \in \mathcal{P}_k[n]\), while the tropical copies of elements of \([n]\) can be
represented as copies of columns of $A$. Some additional remarks in this regard are given in Appendix B. A careful analysis of these subjects could be useful in the development of concrete models for the thermodynamic and statistical systems mentioned in [2], hence it fits within the increasing number of applications of tropical techniques in the description of physical systems (see, e.g., [39, 30, 42, 17]).

Besides the theoretical interest, the previous points prompt a search for new applications of soliton-like structures to the propagation of information. Furthermore, the concept of dimensionality reduction can be studied in more depth in the context of subspaces classification. Indeed, the bounds discussed in Section 7.2 may be implemented for the purpose of statistical regression [27], in particular when generalized/weighted least square methods are employed (see, e.g., [38]). These structures involve bounds for diagonal elements of a projection matrix (also called leverages for orthogonal projections) and might be applied in signal processing and machine learning [7, 31]. More generally, the presentation of statistical amoebas as families of partitions of the type (4.4) could be combined with cross-validation techniques. The links between these physical and information-theoretic concepts deserve further investigations for a better understanding, and they will be explored in more detail in a separate paper.

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APPENDIX A. PRESERVATION OF DETERMINANTAL CONSTRAINTS

Remark 22. There exist sets $v \subseteq [n]$, $R(v) \subseteq [k]$, and $a \in \mathbb{R}^{(k-\#R(v)) \times (n-\#v)}$ that preserve the Cauchy-Binet expansion (3.4) up to a factor independent on $I \in \mathcal{P}_k[n]$, namely

$$\tau(x) = C \cdot \exp \left( \sum_{\nu \in v} \varphi_{\nu}(x) \right) \cdot \det \left( a \cdot \Theta(x)[[n] \setminus v][n] \setminus v \right) \cdot K[[n] \setminus v; [k] \setminus R(v)]$$

with $C$ constant.

Proof: Given $A \in \mathbb{R}^{k \times n}$, let $A_0$ be the reduced row-echelon form of $A$ with $D \cdot A_0 = A$, $D \in GL_k(\mathbb{R})$. Introduce the set

$$v := \bigcap_{I \in \Theta} \{ \alpha \in [n] : (\alpha \notin I \Rightarrow I \in \mathcal{P}_k[n] \setminus \Theta) \}.$$  

(A.2)

In particular, from $V \in \Theta$ one finds $v \subseteq V$, so we can consider $R(v) := \{ i \in [k] : v_i \in v \}$. The dependence of $v$ and $R(v)$ on $A$ will be implicit when no ambiguity arises.

Let $\pi$ be the permutation of $[n]$ such that both the restrictions $\pi^{-1}|_v$ and $\pi^{-1}|_{[n] \setminus v}$ are increasingly monotone, and $\pi^{-1}(\alpha) < \pi^{-1}(\beta)$ for each $\alpha, \beta \in [n] \setminus v$. Similarly, take the permutation $\varpi$ of $[k]$ such that $\varpi^{-1}|_{R(v)}$ and $\varpi^{-1}|_{[k] \setminus R(v)}$ are both order-preserving and, for any $i \in R(v), j \in [k] \setminus R(v)$, $\varpi^{-1}(i) < \varpi^{-1}(j)$. The action of $\pi$ on $A$, $\Theta(x)$ and $K$ via the matrix representation $\pi := (\delta_{\pi(\alpha), \beta})_{\alpha, \beta \in [n]}$ is

$$A \mapsto A \cdot \pi^{-1}, \quad \Theta(x) \mapsto \pi \cdot \Theta(x) \cdot \pi^{-1}, \quad K \mapsto \pi \cdot K,$$

(A.3)

which preserves the product $A \cdot \Theta(x) \cdot K$. The matrix $\pi \cdot \Theta(x) \cdot \pi^{-1}$ is still diagonal, since $\pi$ lies in the normalizer of diagonal matrices, and its entries are the same of $\Theta(x)$. The parity of the number of inversions induced by $\pi$ on $I \in \mathcal{P}_k[n]$ is the same for $\Delta(A \cdot \pi^{-1}; I)$ and $\Delta(\pi \cdot K; I)$. So the action of $\pi$ preserves the terms in the Cauchy-Binet expansion (3.4) up to their permutation denoted by $A_I(x) \mapsto A_{\pi(I)}(x)$. Hence, we can express $g_I$ using the order given by $\pi$ for columns, and an additional relabelling of the rows via the left action of the matrix $\varpi$ representing $\varpi$:

$$g_I = \det(D) \cdot \det(\varpi) \cdot \det \left( A_0[[k] \setminus R(v); I \setminus v] \right) \cdot \text{VdM}(\kappa; v) \cdot \text{VdM}(\kappa; I \setminus v) \cdot \prod_{\alpha, \beta \in [n] \setminus v} (\kappa_\beta - \kappa_\alpha).$$

(A.4)

Introduce

$$P_\beta := \prod_{\alpha \in v} (\kappa_\beta - \kappa_\alpha), \quad \beta \in [n] \setminus v$$

(A.5)

which are non-vanishing under the hypothesis of distinct parameters $\kappa$. So the matrix

$$a := \text{diag} \left( P_\nu^{-1} : \nu \in V \setminus v \right) \cdot A_0[[k] \setminus R(v); [n] \setminus v] \cdot \text{diag} \left( P_\alpha : \alpha \in [n] \setminus v \right)$$

(A.6)

is well-defined. One can reformulate (A.4) as

$$g_I = C \cdot \Delta_a(I \setminus v) \cdot \text{VdM}(\kappa; I \setminus v)$$

(A.7)
where
\[ C := \det(D) \cdot \det(\varpi) \cdot \left( \prod_{\nu \in V \setminus v} P_\nu \right) \cdot \text{VdM}(\kappa; \nu). \quad (A.8) \]
does not depend on \( \mathcal{I} \). Since the soliton parameters \( \kappa \) are pairwise distinct, all the terms in (A.8) are non-vanishing by definition, thus \( C \neq 0 \). All the non-vanishing terms in (3.4) have a common factor \( C \cdot \exp \left( \sum_{\nu \in V} \varphi_\nu (x) \right) \), and the Cauchy-Binet expansion, along with (A.7) and the correspondence
\[ \mathcal{I} \mapsto \mathcal{I} \setminus v, \quad \mathcal{I} \in \mathcal{G}, \quad (A.9) \]
allows to formulate the \( \tau(x) \) as in (A.1).

The erasure of rows and columns associated with indices \( \nu \in v \) in (A.7) preserves the reduced row-echelon form: this is inherited by \( a \) when the normalization \( \text{diag}(P^{-1}_\nu : \nu \in V \setminus v) \) in (A.6) is taken into account. The equality (A.1) also implies that \( a \) has maximal rank and no null columns. Note that \( \#(\mathcal{I} \setminus v) = k - \#v \) for all the terms \( \mathcal{I} \in \mathcal{G} \). In particular, \( V \setminus v \) is still the minimum element of \( P_k([n] \setminus v) \) associated with a non-vanishing minor of \( a \) with respect to the lexicographic order on \([n] \setminus v\) induced by \([n]\). Accordingly, a signature \( \Sigma \) for \( \tau \) induces a signature for the reduced model (A.1), which will still be denoted by \( \Sigma \).

\[ \Sigma(\mathcal{I} \setminus v) := \text{sign}(C) \cdot \Sigma(\mathcal{I}). \quad (A.10) \]

As remarked in the Introduction, we focus on Cauchy-Binet expansions that generate statistical amoebas relative to the exponential functions in (3.4), rather than determinants. Indeed, each function \( f(x) \) can be trivially expressed as the determinant of a diagonal matrix \( \text{diag}(1, \ldots, 1, f(x)) \), and no constraints arise. Also the number of degrees of freedom, which is related to the dimensions of the matrices involved in the expansion, has to be bounded in order to get non-trivial constraints. In fact, any sum of functions \( \sum_{W=1}^W f_t(x), W \in \mathbb{N} \), can be expressed via (2.13) as
\[ \det \left[ (f_1(x) \ldots f_W(x)) \oplus \mathbb{1}_\ell \right] \cdot (1 \ldots 1)^T \oplus \mathbb{1}_\ell \right) \].

This leads us to define determinantal choices of signs, i.e. signatures preserving the determinantal expansion (2.12), as follows.

**Definition 23.** We call a signature \( \Sigma \) a **determinantal choice of signs**, acting on a determinant (2.12), if there exist a set \( \{ \bar{\kappa}_\alpha : \alpha \in [n] \setminus v \} \) and a matrix \( \bar{a} \in \mathbb{R}^{\#(k-v) \times \#(n-v)} \) such that \( \Sigma \) is induced by these data through (2.13), up to a common scale factor \( \lambda(x) \).

Specifically, this means that
\[ \Delta_{\bar{a}}(\mathcal{I} \setminus v) \cdot \Delta_{\bar{K}}(\mathcal{I} \setminus v) \cdot \exp \left( \sum_{\alpha \in \mathcal{I} \setminus v} \bar{\varphi}_\alpha (x) \right) = \lambda(x) \cdot \Sigma(\mathcal{I}) \cdot \Delta_A(\mathcal{I}) \cdot \Delta_K(\mathcal{I}) \cdot \exp \left( \sum_{\alpha \in \mathcal{I}} \varphi_\alpha (x) \right), \quad \mathcal{I} \in \mathcal{G} \quad (A.12) \]

where \( \bar{\varphi}_\alpha (x) := \sum_{u=1}^d \bar{\kappa}_\alpha^u x_u \) and \( \lambda(x) \neq 0 \).

This definition only relies on the data provided by the terms in the expansion (2.12) indexed by \( \mathcal{G} \), as it is shown in the next lemma.
Lemma 24. Assuming that the components of $\kappa$ are pairwise distinct, the data $n - \#v$, $k - \#v$ and $\{\kappa_\alpha : \alpha \in [n] \setminus v\}$ are uniquely determined. Furthermore, a determinantal choice of signs preserves the absolute values of the maximal minors $\Delta_\alpha(I)$ of the matrix $a$ defined in (A.6), up to a multiplicative factor independent on $I$.

Proof: Let us start from $A$ and the associated sets $\Theta$ and $v$, and take any $\alpha \in [n] \setminus v$. From the lack of null columns, there exist $I, J \in \Theta$ with $\alpha \in I \setminus J$. The exchange property (3.8) implies that there exists $\beta \in J \setminus I$ with $I, I_\beta^\alpha \in \Theta$. We are looking at transformations that preserve each exponential $\exp \left( \sum_{u=1}^{d} \kappa_{\alpha u} x_u \right)$ in (3.4) with $I \in \Theta$, then the ratio of the exponentials corresponding to $I$ and $I_\beta^\alpha$

$$\exp \left( \sum_{u=1}^{d} \left( \kappa_{\alpha u} - \kappa_{\beta u} \right) \cdot x_u \right) = \exp \left( \sum_{\gamma \in I} \left( \sum_{u=1}^{d} \kappa_{\gamma u} x_u \right) \right)^{-1} \cdot \exp \left( \sum_{\delta \in I_\beta^\alpha} \left( \sum_{u=1}^{d} \kappa_{\delta u} x_u \right) \right)$$

(A.13)

is preserved too. In particular, the coefficients $\kappa_{\alpha} - \kappa_{\beta}$ of $x_1$ and $\kappa_{\alpha}^2 - \kappa_{\beta}^2$ of $x_2$ are left unchanged. The assumption $\kappa_{\alpha} \neq \kappa_{\beta}$ for all $\alpha \neq \beta$ implies that both the quantities $\kappa_{\alpha} - \kappa_{\beta}$ and $\kappa_{\alpha}^2 - \kappa_{\beta}^2 = \kappa_{\alpha} + \kappa_{\beta}$ are well-defined and fixed. This means that we can recover the values of data $\kappa_{\alpha}$ (and $\kappa_{\beta}$) for all $\alpha \in [n] \setminus v$ from the form of the exponential terms.

Now look at another full-rank matrix $A_1 \in \mathbb{R}^{k_1 \times n_1}$ without null columns, which generates the data $\Theta_1$ as in (3.6) and $v_1$ as in (A.2), and a vector $\kappa_1 \in \mathbb{R}^{n_1}$ such that (A.12) also holds after the substitution $A \mapsto A_1$ and $\kappa \mapsto \kappa_1$. Same as above, for each $\alpha \in [n_1] \setminus v_1$ we can find $I \in \Theta_1$ and $\beta \in [n_1] \setminus I$ such that $\alpha \in I$ and $I_\beta^\alpha \in \Theta_1$ and, from the previous observations, we recover the same set $\{\kappa_{\alpha} : \alpha \in [n] \setminus v\} = \{\tilde{\kappa}_{\alpha} : \alpha \in [n_1] \setminus v_1\}$ of distinguishable soliton parameters. This establishes a correspondence $w : \Theta \rightarrow \Theta_1$ defined by $I \setminus v = w(I) \setminus v_1$ for all $I \in \Theta$.

For each $\alpha \in [n] \setminus v$, we can choose $I(\alpha) \in \Theta$ with $\alpha \in I(\alpha)$, which exists by the lack of null columns in $A$, then $\alpha \in w(I(\alpha))$ too. From (A.7) and (A.12), this means that $\Delta_{A_1}(w(I(\alpha))) \cdot \Delta_{K_1}(w(I(\alpha))) \neq 0$, which implies that the parameters $(\kappa_1)_\nu$, $\nu \in v_1$, and $(\kappa_1)_\alpha$ are pairwise distinct. Since this holds for all $\alpha \in [n] \setminus v$, all the components of $\tilde{\kappa}$ are pairwise distinct too. So one has $\Delta_{A}(I) = 0$ if and only if $\Delta_{A_1}(w(I)) = 0$. These data induce the same form (A.1) with matrices $a$ and $a_1$ and multiplicative constants $C, C_1 \neq 0$. From the previous discussion $\text{VdM}(\kappa; I \setminus v) = \text{VdM}(\kappa_1; w(I) \setminus v_1)$ for all $I \in \Theta$, so the equalities (A.12) imply $|\Delta_{a}(I \setminus v)| = C\text{VdM}(\Delta_{a_1}(I \setminus v))$. $\square$

Theorem 25. Assuming that the parameters $\kappa_1, \ldots, \kappa_n$ are pairwise distinct, a choice of signs is determinantal if and only if it is induced by a choice of sign for the rows and the columns of $A$ (up to the action of $GL_k(\mathbb{R})$).

Proof: We fix the gauge given by the $GL_k(\mathbb{R})$-action setting $\frac{C_1 \cdot \lambda(x)}{C \cdot \lambda(y)} = 1$. By Remark 22 and Lemma 24, a signature is determinantal only if the absolute values of maximal minors of $a$ are preserved. If $v \neq \emptyset$, then the overall sign($C$) in (A.10) can be
expressed as a choice of sign for a row in \( R(\nu) \), and the study is reduced to \([k] \setminus R(\nu) \) and \([n] \setminus \nu \) through the map (A.9). Therefore, to simplify the notation, we can focus on the case \( \mathbf{a} \in \mathbb{R}^{k \times n}_+ \) with pivot set \( \mathcal{V} \) without loss of generality.

The equality

\[
|a_{i\beta}| = \left| \frac{\Delta_a (\nu_\beta)}{\Delta_a (\mathcal{V})} \right| = \left| \frac{\Delta_a (\nu_\beta)}{\Delta_a (\mathcal{V})} \right| = |\bar{a}_{i\beta}|, \quad i \in [k], \beta \in [n] \setminus \mathcal{V}
\]  

(A.14)

means that the absolute values of the entries of \( \mathbf{a} \) are fixed too. So, the transformation (3.5) is induced by a choice of signs

\[
\sigma : \{(i, \alpha) \in [k] \times [n] : a_{i\alpha} \neq 0\} \rightarrow \{\pm 1\}
\]  

(A.15)

for the non-vanishing entries of \( \mathbf{a} \), which will be denoted by \( \sigma(\mathbf{a}) \) as well.

We now construct a sequence of operations to label the rows and columns of \( \mathbf{a} \) with signs \( \eta : [k] \rightarrow \{\pm 1\} \) and \( \chi : [n] \rightarrow \{\pm 1\} \) respectively. Let \( \mathcal{G} := ([k], E) \) be a graph whose vertices label the rows of \( \mathbf{a} \). The pair \((i, j)\) is an edge if and only if there exists \( \gamma \in [n] \) such that \( a_{i\gamma} \neq 0 \neq a_{j\gamma} \), i.e.

\[
(i, j) \in E \iff \exists \gamma \in [n] : \Delta_a (\nu_\gamma) \neq 0 \neq \Delta_a (\chi_\gamma).
\]  

(A.16)

So fix an arbitrary row \( h_1 \in [k] \) of \( \mathbf{a} \), e.g. \( h_1 = 1 \). Without loss of generality, we can set \( \eta(h_1) := +1 \). Define \( c_1 := \{\alpha \in [n] : a_{h_1\alpha} \neq 0\} \) and \( \chi(\alpha) := \sigma(h_1, \alpha) \) for all \( \alpha \in c_1 \). Note that, for each \( i \in [k] \), all the products \( \sigma(h_1, \alpha) \cdot \sigma(i, \alpha) \), \( \alpha \in c_1 \) and \( a_{i\alpha} \neq 0 \), coincide: indeed, this is trivially true at \( i = h_1 \). If \( i \neq h_1 \) and \( (i, \alpha) \), \( (i, \beta) \) are such that \( \alpha, \beta \in c_1 \), then \( 0 \notin \{a_{h_1\alpha}, a_{h_1\beta}\} \) by definition. Hence, at \( a_{i\alpha} \neq 0 \neq a_{i\beta} \), the constraint \( |\Delta_{\sigma(a)} (\mathcal{V} \setminus \{\nu_{h_1}, \nu_i \cup \{\alpha, \beta\}\})| = |\Delta_a (\mathcal{V} \setminus \{\nu_{h_1}, \nu_i \cup \{\alpha, \beta\}\})| \) is equivalent to

\[
|\sigma(h_1, \alpha) \cdot \sigma(i, \beta) \cdot a_{h_1\alpha} \cdot a_{i\beta} - \sigma(i, \alpha) \cdot \sigma(h_1, \beta) \cdot a_{i\alpha} \cdot a_{h_1\beta}| = |a_{h_1\alpha} \cdot a_{i\beta} - a_{i\alpha} \cdot a_{h_1\beta}|.
\]  

(A.17)

From \( a_{h_1\alpha} \cdot a_{i\beta} - a_{i\alpha} \cdot a_{h_1\beta} \neq 0 \), one gets \( |a_{h_1\alpha} \cdot a_{i\beta} - a_{i\alpha} \cdot a_{h_1\beta}| \neq |a_{h_1\alpha} \cdot a_{i\beta} + a_{i\alpha} \cdot a_{h_1\beta}| \), thus

\[
\begin{align*}
\sigma(h_1, \alpha) \cdot \sigma(i, \beta) &= \sigma(i, \alpha) \cdot \sigma(h_1, \beta) \\
&\iff \sigma(h_1, \alpha) \cdot \sigma(h_1, \beta) \cdot \sigma(h_1, \alpha) \cdot \sigma(i, \beta) \\
&\iff \sigma(h_1, \alpha) \cdot \sigma(h_1, \beta) \cdot \sigma(i, \alpha) \cdot \sigma(h_1, \beta) \\
&\iff \sigma(h_1, \alpha) \cdot \sigma(i, \beta) \\
&= \sigma(h_1, \alpha) \cdot \sigma(i, \alpha).
\end{align*}
\]  

(A.18)

So, for any \((h_1, i) \in E\), the ith row can be labelled with a sign \( \eta(i) := \sigma(i, \alpha) \cdot \chi(\alpha) \) for some \( \alpha \in c_1 \) assuming that \( c_1 \neq \emptyset \). Then let \( r_1 := \{i \in [k] \setminus \{h_1\} : (i, h_1) \in E\} \) and \( r_2 := \min r_1 \). By the previous observation, one has a definite sign \( \eta(h_2) \). Let \( c_2 := \{\alpha \in [n] \setminus c_1 : a_{h_2\alpha} \neq 0\} \) and assign \( \chi(\alpha) := \sigma(h_2, \alpha) \cdot \eta(h_2) \) for any \( \alpha \in c_2 \). As before, for each fixed \( i \in [k] \), all the signs \( \sigma(h_2, \alpha) \cdot \sigma(i, \alpha), \alpha \in c_2 \) and \( a_{i\alpha} \neq 0 \), coincide. So the sign \( \eta_2(i) := \sigma(i, \alpha) \cdot \chi(\alpha) \), for any \( \alpha \in c_2 \), is well-defined. It may exist \( g \in [k] \setminus \{h_1, h_2\} \) such that \( a_{g\alpha} \neq 0 \) and \( a_{g\beta} \neq 0 \) for some \( \alpha \in c_1, \beta \in c_2 \).
We can check that \( \varrho_2(g) = \varrho(g) \): in fact, the signs restricted to rows \( h_1, h_2, g \) and columns \( \alpha, \beta \) can be depicted as

\[
\begin{array}{ccc}
\text{column } \alpha & \text{column } \beta & \text{column } \gamma \\
\hline
\text{row } h_1 & g(h_1) \cdot \chi(\alpha) & 0 \\
\text{row } h_2 & W & g(h_2) \cdot \chi(\beta) \\
\text{row } g & g(g) \cdot \chi(\alpha) & \varrho_2(g) \cdot \chi(\beta) \\
\end{array}
\]

(A.19)

If \( W \neq 0 \), then it equals \( g(h_2) \cdot \chi(\alpha) \) by definition. Thus, the constraint \( |\Delta_a(I)| = |\Delta_{\sigma(a)}(I)| \) at \( I = V \setminus \{ \nu_{h_2}, \nu_g \} \cup \{ \alpha, \beta \} \) implies that

\[
\varrho(h_2) \cdot \chi(\alpha) \cdot \varrho_2(g) \cdot \chi(\beta) = \varrho(h_2) \cdot \chi(\beta) \cdot \varrho(g) \cdot \chi(\alpha)
\]

(A.20)

that is \( \varrho_2(g) = \varrho(g) \). Now assume that \( W = 0 \). Since \( (h_2, h_1) \in E \), there exists \( \gamma \in [n] \) such that \( a_{h_1\gamma} \neq 0 \neq a_{h_2\gamma} \), and \( \gamma \) is clearly different from \( \alpha \) and \( \beta \) since \( a_{h_1\beta} = 0 = a_{h_2\alpha} \), then consider the extended scheme

\[
\begin{array}{ccc}
\text{column } \alpha & \text{column } \beta & \text{column } \gamma \\
\hline
\text{row } h_1 & g(h_1) \cdot \chi(\alpha) & 0 & g(h_1) \cdot \chi(\gamma) \\
\text{row } h_2 & 0 & g(h_2) \cdot \chi(\beta) & g(h_2) \cdot \chi(\gamma) \\
\text{row } g & g(g) \cdot \chi(\alpha) & \varrho_2(g) \cdot \chi(\beta) & X \\
\end{array}
\]

(A.21)

If \( X \neq 0 \) (hence \( |\varrho_\gamma| \neq 0 \) ), then \( X = \varrho(g) \cdot \chi(\gamma) \) and, applying \( |\Delta_a(I)| = |\Delta_{\sigma(a)}(I)| \) at \( I = V \setminus \{ \nu_{h_2}, \nu_g \} \cup \{ \beta, \gamma \} \), one gets \( \varrho_2(g) = \varrho(g) \). Also note that this can be found without evaluating \( X \) from the requirement \( |\Delta_a(I)| = |\Delta_{\sigma(a)}(I)| \) at both \( I = V \setminus \{ \nu_{h_1}, \nu_g \} \cup \{ \alpha, \gamma \} \) and \( I = V \setminus \{ \nu_{h_2}, \nu_g \} \cup \{ \beta, \gamma \} \) simultaneously, i.e.

\[
+1 = \varrho(h_1) \cdot \chi(\alpha) \cdot X \cdot \varrho(h_1) \cdot \chi(\gamma) \cdot \varrho(g) \cdot \chi(\alpha) = \varrho(h_2) \cdot \chi(\beta) \cdot X \cdot \varrho(h_2) \cdot \chi(\gamma) \cdot \varrho_2(g) \cdot \chi(\beta)
\]

(A.22)

that gives

\[
\varrho(h_1) \cdot \chi(\alpha) \cdot \varrho(h_1) \cdot \chi(\gamma) \cdot \varrho(g) \cdot \chi(\alpha) = \varrho(h_2) \cdot \chi(\beta) \cdot \varrho(h_2) \cdot \chi(\gamma) \cdot \varrho_2(g) \cdot \chi(\beta).
\]

(A.23)

If instead \( X = 0 \), considering the constraint \( |\Delta_a(I)| = |\Delta_{\sigma(a)}(I)| \) in the case \( I = V \setminus \{ \nu_{h_1}, \nu_{h_2}, \nu_g \} \cup \{ \alpha, \beta, \gamma \} \), we get

\[
\begin{align*}
| - a_{h_1\gamma} \cdot a_{h_2\beta} \cdot a_{g\alpha} - a_{h_2\gamma} \cdot a_{g\beta} \cdot a_{h_1\alpha} | \\
= & \quad | \varrho(h_1) \cdot \chi(\gamma) \cdot \varrho(h_2) \cdot \chi(\beta) \cdot \varrho(g) \cdot \chi(\alpha) - a_{h_1\gamma} \cdot a_{h_2\beta} \cdot a_{g\alpha} | \\
+ & \quad | \varrho(h_2) \cdot \chi(\gamma) \cdot \varrho_2(g) \cdot \chi(\beta) \cdot \varrho(h_1) \cdot \chi(\alpha) - a_{h_2\gamma} \cdot a_{g\beta} \cdot a_{h_1\alpha} | \\
\end{align*}
\]

(A.24)

and, from \( 0 \neq a_{h_1\gamma} \cdot a_{h_2\beta} \cdot a_{g\alpha} \cdot a_{h_2\gamma} \cdot a_{g\beta} \cdot a_{h_1\alpha} \), we obtain

\[
\varrho(h_1) \cdot \chi(\gamma) \cdot \varrho(h_2) \cdot \chi(\beta) \cdot \varrho(g) \cdot \chi(\alpha) = \varrho(h_2) \cdot \chi(\gamma) \cdot \varrho_2(g) \cdot \chi(\beta) \cdot \varrho(h_1) \cdot \chi(\alpha).
\]

(A.25)

Since all the factors in (A.23) and (A.25) belong to \( \{ \pm 1 \} \) and, in particular, they are non-vanishing and idempotent, these expressions simplify as

\[
\varrho_2(g) = \varrho(g).
\]

(A.26)
and the sign $g_2(g) \cdot \chi(\gamma) = \rho(g) \cdot \chi(\gamma)$ can be consistently fixed. This construction can be extended through the iterations

$$
\tau_{r-1} := \{ i \in [k] \setminus \{h_1, \ldots, h_{r-1}\} : (i, h_{r-1}) \in E \},
$$

$$
h_r := \min \tau_{r-1},
$$

$$
\varsigma_r := \left\{ \alpha \in [n] \setminus \bigcup_{t=1}^{r-1} \varsigma_t : a_{h_r \alpha} \neq 0 \right\}
$$

(A.27)

and the signs $g_{r-1}(h_r)$ and $\chi(\alpha) := \sigma(h_r, \alpha) \cdot g_{r-1}(h_r)$ for any $\alpha \in \varsigma_r$. If the signs $g_s$, as long as $\chi$, are uniquely defined at steps $s < r$, then one can check the compatibility of $g_r$ with $\rho$ too, formally $g_r(g) = \rho(g)$ for all rows $g$ connected to both $h_r$ and $h_s$, $s < r$. At this purpose, it is worth introducing paths on $\alpha$, that are defined as finite sequences of non-vanishing elements of $\alpha$ connected by alternate moves along rows and columns, i.e. chains of the type (4.36). The process that links the signs $g_r(g)$, derived from $h_r$, and $\rho(g)$, obtained in a previous step $h_s$, can be represented by a path $\Phi$ starting with $(g, \alpha_1) \rightarrow (h_{i_2}, \alpha_1) \rightarrow \ldots$, passing through some signed rows $\Phi_g := \{g, h_{i_2}, \ldots, h_{i_L}\}$ and columns $\Phi_\chi := \{\alpha_1, \ldots, \alpha_L\}$, and ending with $\cdots \rightarrow (h_{i_L}, \alpha_L) \rightarrow (g, \alpha_L)$, where $\alpha_1 = \alpha_s \in \varsigma_s$, $\alpha_L = \alpha_r \in \varsigma_r$ and $a_{g_\alpha s} \neq 0 \neq a_{g_\alpha r}$.

Note that the column of $\alpha$ associated with any $\alpha_T \in \Phi_\chi$ has at least two non-vanishing entries $(h_{\alpha_T}, \alpha_T)$ and $(h_{\alpha_T+1}, \alpha_T)$, while there is only one non-vanishing entry in pivot columns by definition, hence $V \cap \Phi_\chi = \emptyset$. Suppose that there is a row $\tilde{h} \in \Phi_g$ with $0 \notin \{a_{h_{\alpha_T-1}}, a_{h_{\alpha_T}}, a_{h_{\alpha_T+1}}\}$, $X \notin \{T-1, T\}$, and first assume that $T < X$: since the column $\alpha_X$ is reached at $(h_{i_X}, \alpha_X)$, we can substitute the chain $(\tilde{h}, \alpha_{T-1}) \rightarrow (\tilde{h}, \alpha_T) \rightarrow \cdots \rightarrow (h_{i_{X-1}}, \alpha_X) \rightarrow (h_{i_X}, \alpha_X)$ with $(\tilde{h}, \alpha_{T-1}) \rightarrow (\tilde{h}, \alpha_X) \rightarrow (h_{i_{X+1}}, \alpha_X)$ in the path. At $X < T-1$, likewise, we can change $(h_{i_X}, \alpha_X) \rightarrow (h_{i_{X+1}}, \alpha_X) \rightarrow \cdots \rightarrow (h_{i_{T-1}}, \alpha_X) \rightarrow (\tilde{h}, \alpha_T)$ with $(h_{i_X}, \alpha_X) \rightarrow (\tilde{h}, \alpha_T)$. In both cases, the result is a path connecting $\alpha_s$ to $\alpha_r$ with shorter length, which can still be used to compare $g_r(g)$ and $\rho(g)$. Similar substitutions can be carried out for columns $\tilde{c} \in \Phi_\chi$ that appear as components of more than two elements of the path. So we focus on paths of minimal length: the previous construction shows that the submatrix extracted from $\alpha$ selecting the rows in $\Phi_g$ and columns in $\Phi_\chi$ associated with a minimal path $\Phi$ from $(g, \alpha_s)$ to $(g, \alpha_r)$ has exactly two non-vanishing elements per row and column. Hence, the condition $|\Delta_\alpha(V \setminus \{\nu_i : i \in \Phi_g \cup \Phi_\chi\})| = |\Delta_\sigma(a)(V \setminus \{\nu_i : i \in \Phi_g \cup \Phi_\chi\})|$ can be expressed as in (A.25), that is

$$
\rho(g) \cdot \chi(\alpha_1) \cdot \left( \prod_{T=2}^L \rho(h_{i_{T-1}}) \cdot \chi(\alpha_T) \right) = \left( \prod_{T=1}^{L-1} \rho(h_{i_{T+1}}) \cdot \chi(\alpha_T) \right) \cdot g_2(g) \cdot \chi(\alpha_L)
$$

(A.28)

which gives $g_r(g) = \rho(g)$.

These steps can be repeated while $\tau_r \neq \emptyset \neq \varsigma_r$: in this process we start from a node $h_1$ of the graph $G$ and follow a path of adjacent vertices. If $\tau_r = \emptyset$, then we can follow this path backwards until we reach $\tau_u, u \in [r-1]$, such that there exists $\tilde{h}_u \in \tau_u$ with $\tilde{h}_u \neq h_w, w \in [r-1]$. Then, these operations can be repeated along
another path of adjacent nodes starting from $\tilde{h}_u$. At each stage an additional sign is selected compatibly with the previous ones. Such a process explores each node in the connected component of $G$ containing $h_1$ exactly once. Repeating these steps for all the connected components of $G$ we give a sign to all the rows of $a$. Since each column $\alpha$ contains at least one non-vanishing element $a_{h_{0,\alpha}} \neq 0$ and all the rows are visited, the index $\alpha$ belongs to $c_r$ at certain step $r$. Hence, all the columns are labelled by a sign as well. By construction, one has $\sigma(i, \alpha) = g(i) \cdot \chi(\alpha)$ for all $(i, \alpha)$ such that $a_{i\alpha} \neq 0$. Clearly, this assignment produces a determinantal choice of signs.

Remark 26. The proof of Theorem 25 is constructive: it generates one of the possible sign configurations that induce a given choice $\Sigma$. This configuration is not unique, e.g. switching the sign of two distinct rows of $A$ or $a$ returns the same $\Sigma$. The uniqueness of the previous construction follows from the choice of a (arbitrary) $\sigma$ in (A.15) for the entries of the reduced row echelon form $a$ and signs of distinguished nodes (such as $h_1$) in the connected components of $G$. Furthermore, the algorithm attributes a sign to vanishing entries of $a$ too. The labelling $+0$ or $-0$ can be thought as the sign of the associated entry in a perturbed matrix $b$ such that $\text{sign}(\Delta_a(I)) = \text{sign}(\Delta_b(I))$ for all $I \in P_k[n]$ with $\Delta_a(I) \neq 0$. These points are relevant when $a$ has many vanishing entries.

Appendix B. Remarks on the tropical limit

With regard to the issues dealt with in the present work, tropical methods have proved useful in the study of algebraic amoebas [39, 42] and the analysis of KdV and KP II soliton solutions and their singularities [16, 17, 33]. Here we briefly discuss how the models we have described give a concrete realization to the concepts discussed in [2], especially the role of order and enumeration in the tropical limit in statistical physics. Relevant quantities, like the free energy $-k_BT \cdot \ln Z$ associated with (1.1), fit naturally in this process, and their tropical limit is suitable for the description of phenomena like exponential degenerations of energy levels and limiting temperatures, see [3].

Determinantal partition functions (2.12) give a concrete realization of the nested tropical limit discussed in [2]. For the sake of concreteness, let us consider the case when $g_{IL} \geq 0$ for all $I \in P_k[n]$. Outside the locus of points $x \in \mathbb{R}^d$ where $\varphi_\alpha(x) = \varphi_\beta(x)$ for some $\alpha \neq \beta$, one has $\varphi_{\pi(1)}(x) > \cdots > \varphi_{\pi(n)}(x)$ for a permutation $\pi \in S_n$. Here, for large values of $x$, there exists exactly one dominant term $\Lambda_D(x) > \Lambda_H(x)$, $H \in P_k[n] \setminus \{D\}$, as follows from the polynomial form of $\varphi_\alpha(x)$. In particular, $D$ is the least set in $P_k[n]$ associated with a non-vanishing $g_D$ with respect to the lexicographical order induced by $(\pi(1), \ldots, \pi(n))$. Hence one has $\Lambda_H(x) \ll \Lambda_D(x)$ for all $H \neq D$ and $\tau_\pi(x) < 0$ if and only $D \parallel I$. Moreover, if $g_{IL} > 0$ for all $I \in P_k[n]$, then the number of such subsets $I$ with odd intersection with $D$ is $\Omega(n, s; k)$, that is the dual quantity of $\Omega(n, k; s)$ as in (6.10) and (6.11). The nested form (see [2], §6) comes from the identification of dominant term $D$: if one sets $D(r) := \{H \in P_k[n] : \max\{i \in [k] \cup \{0\} : \{\pi(1), \ldots, \pi(i)\} \subseteq H \cap D\} = r\}$, then $\tau$ can be expressed as

$\tau(x)$
the Grassmannian. Further, the effect of the tropical copy of a single soliton function gets a multiplicative factor.

The soliton solution is preserved. Thus, there is an intrinsic tropical global symmetry in the singular locus defined by

\[ \sum_{\alpha \in I} \varphi_{\alpha} \]

If the same number of copies, say \( k \), are created for all \( \alpha \in [n] \), then one can introduce a “copy” the \( \alpha \)th soliton as a copy of the associated row/column, formally

\[
\begin{align*}
A & \mapsto T_\alpha(A) := A \cdot T_\alpha, \\
K & \mapsto T_\alpha(K) := T_\alpha^T \cdot K, \\
\Theta(x) & \mapsto T_\alpha(\Theta(x)) := \pi_{\text{cyc},\alpha}^{-1} \cdot \left( \Theta(x) \oplus (e^{\varphi_\alpha(x)}) \right) \cdot \pi_{\text{cyc},\alpha}
\end{align*}
\]

(\text{B.2})

where \( \pi_{\text{cyc},\alpha} \) is the matrix associated with the cyclic permutation \((\alpha + 1 \alpha + 2 \ldots n + 1)\) and

\[
(T_\alpha)^{\gamma \beta} := \sum_{\omega=1}^{\alpha} \delta_{\gamma,\beta} \cdot \delta_{\gamma,\omega} + \sum_{\omega=\alpha}^{n} \delta_{\gamma,\beta-1} \cdot \delta_{\gamma,\omega}, \quad \alpha, \gamma \in [n], \beta \in [n + 1].
\]

(\text{B.3})

If the same number of copies, say \( \ell \), are created for all \( \alpha \in [n] \), then the resulting \( \tau \)-function gets a multiplicative factor \((\ell + 1)^k\) which disappears in the derivation (2.6), so the soliton solution is preserved. Thus, there is an intrinsic tropical global symmetry in this particular class of solutions, which is also compatible with the \( GL_k(\mathbb{R}) \)-action defining the Grassmannian. Further, the effect of the tropical copy of a single soliton \( \kappa_{\alpha_0} \) on the singular locus defined by \( \det (A \cdot \Theta(x) \cdot K) = 0 \) is the same as its erasure, since

\[
\begin{align*}
\det & \left( T^{(\ell)}_{\alpha_0}(A) \cdot T^{(\ell)}_{\alpha_0}(\Theta(x)) \cdot T^{(\ell)}_{\alpha_0}(K) \right) \\
& = \ell \cdot \sum_{\alpha_0 \in I} \Delta_I(A) \Delta_I(K) \exp \left( \sum_{\beta \in I} \varphi_\beta(x) \right) \\
& = -\ell \cdot \sum_{\alpha_0 \in I} \Delta_I(A) \Delta_I(K) \exp \left( \sum_{\beta \in I} \varphi_\beta(x) \right).
\end{align*}
\]

(\text{B.4})
This process is independent on the choice of $\ell > 1$, since the factor $\ell$ in (B.4) disappears in the derivation (2.6). In this sense, the invariance under tropical copies is local on the singular locus.