Consistency of the $AdS_7 \times S_4$ reduction and
the origin of self-duality in odd dimensions

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Abstract
We discuss the full nonlinear Kaluza-Klein (KK) reduction of the
original formulation of $d=11$ supergravity on $AdS_7 \times S_4$ to gauged maximal (N=4) supergravity in 7 dimensions. We derive the full nonlinear
embedding of the $d = 7$ fields in the $d = 11$ fields (“the ansatz”) and
check the consistency of the ansatz by deriving the $d=7$ supersymmetry
laws from the $d=11$ transformation laws in the various sectors. The
ansatz itself is nonpolynomial but the final $d = 7$ results are polynomial.
The correct $d = 7$ scalar potential is obtained. For most of our results
the explicit form of the matrix $U$ connecting the $d = 7$ gravitino to the
Killing spinor is not needed, but we derive the equation which $U$ has to
satisfy and present the general solution. Requiring that the expression
$\delta F = d\delta A$ in $d = 11$ can be written as $\delta d(\text{fields in } d = 7)$, we find the
ansatz for the 4-form $F$. It satisfies the Bianchi identities. The corre-
sponding ansatz for the 3-form $A$ modifies the geometrical proposal by
Freed et al. by including $d = 7$ scalar fields. A first order formulation
for $A$ in $d = 11$ is needed to obtain the $d=7$ supersymmetry laws and
the action for the nonabelian selfdual antisymmetric tensor field $S_{\alpha\beta\gamma,A}$.
Therefore selfduality in odd dimensions originates from a first order for-
amalism in higher dimensions.

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1 Introduction

The consistency of Kaluza Klein (KK) truncation at the nonlinear level of a massless gravitational higher-dimensional field theory to a massless gravitational lower-dimensional field theory has been a fundamental problem for a long time. In this article we present a complete solution for a particular model. Namely, we consider the original formulation of $d=11$ supergravity [1], without any string extensions, and write all components of all $d=11$ fields, denoted by $\Phi(y,x)$, as nonlinear expressions in terms of the massless $d=7$ fields $\phi(y)$ and spherical harmonics $Y(x)$ on $S_4$. All spherical harmonics will be expressed in terms of (products of) Killing spinors $\eta^I(x)$ with $I = 1, \ldots, 4$. Hence we present below explicit expressions for $\Phi(y,x) = \Phi(\phi(y), \eta(x))$. The consistency of this 'ansatz' amounts to proving that the $d=11$ supersymmetry variations of $\Phi$ produce the correct $d=7$ supersymmetry variations of $\phi$. A particular nonlinear combination of $\phi$'s and $\eta$'s enters as a $4 \times 4$ matrix $U^I_I(y,x)$. It appears in the ansatz for all fermionic $\Phi$'s and in the relation $\varepsilon(y,x) \sim \varepsilon_I(y)U^I_I(y,x)\eta^I(x)$ between the $d=11$ susy parameter $\varepsilon(y,x)$ and the $d=7$ parameter $\varepsilon(y)$. This matrix $U$ must satisfy a linear matrix equation, and we present explicit solutions. This is the first time a complete nonlinear KK reduction of an original supergravity theory to all massless modes in a lower dimension has been given. There exists in the literature also a modified version of $d=11$ supergravity with a local $SU(8)$ symmetry and for this theory de Wit and Nicolai [2, 3, 4, 5] have shown that the KK reduction is consistent. Because in their version of the theory the matrix $U$ only described gauge degrees of freedom, they did not try to determine it. (In an earlier study of the KK reduction of the original $d=11$ supergravity to $d=4$ they gave partial results on the matrix $U$ [2, 3], but then they abandoned this project. In fact, the matrix $U$ was first introduced in an implicit form in [6], while the contributions to $U$ quadratic in scalars were computed in [7]. Other KK reductions to a subset of massless fields have also in some cases been shown to be consistent [8, 9, 10, 11].

This article contains a detailed and self-contained derivation of our results, and is an expanded version of a previous letter [12]. The basic problem we must analyze is that in relations like $\delta \phi(y)Y(x) = \varepsilon(y,x)\Phi(y,x)$ one finds products of $Y(x)$'s on the r.h.s. Hence one must prove relations of the form $Y(x) = \Pi Y(x)$. (An example of such a relation is (4.30).) This requires a large number of properties of spherical harmonics. For the reader who is not an expert in these matters we have added a special section (section 3) in which we give a derivation of all identities we needed, starting from scratch. The basic object, in terms of which we express all spherical harmonics, is the Killing spinor. It is a square root of a Killing vector and is the spherical harmonic of the local supersymmetry parameters. The spherical harmonics of scalars, spinors and other fields are expressed into Killing spinors. We shall analyze all bosonic variations $\delta \Phi(\phi(y), \eta(x))$ up to terms with three $d=7$ fermion fields, and all fermionic variations up to terms with two $d=7$ fermion fields, but all results are to all orders in bosonic fields. Previous work in this area has also made the same restriction [3, 4, 5]. In principle one could also study the variations containing higher orders in fermion fields with our methods, but this is algebraically very complicated and given all small miracles we have encountered in our work, we expect that consistency holds there.
Initially, the aim of KK reductions of higher dimensional supergravities (sugras) was to find realistic field theories in $d=4$ dimensions, and the compactification on $\text{AdS}_4 \times S_7$ seemed promising ([13], see also [14]). However, it was later realized that it was impossible to obtain in this way chiral fermions and/or a vanishing cosmological constant [15]. Recent developments in the AdS-CFT correspondence have renewed interest in KK compactifications on anti-deSitter (AdS) spaces [16, 17, 18, 19, 20, 21, 22] (for a recent review see [23]).

While KK reduction (more precisely, truncation to the set of massless fields) on tori is always guaranteed to be consistent, this is not in general so. A simple example of an inconsistent truncation is Einstein gravity, which cannot be truncated to gravity plus gauge fields of the symmetry group $K$ of the compact manifold $M_K$. This was already noticed by Kaluza and others who studied the reduction from 5 to 4 dimensions. They realized that the field equation for the scalar contains source terms depending on the gauge fields; hence setting the scalar to zero was inconsistent. But even keeping the scalars in the theory is in general not enough to obtain consistency. In the Einstein equation, the terms involving the gauge fields will have as spherical harmonics $V^a_\mu V^\mu_b$, whereas the rest of the terms have spherical harmonic 1. However, if one restricts the gauge group $K$ to a subgroup $K'$ such that the subset of Killing vectors satisfies $V^a_\mu V^\mu_b = \delta^a_b$, then the KK reduction is consistent ($a, b \in K'$) [8] (In sugra, more fields are present - for instance in 11 dimensions we have the antisymmetric tensor $A_{\Lambda\Pi\Sigma}$ - and so the condition $V^a_\mu V^\mu_b = \delta^a_b$ gets modified in a complicated way). In the case of compactifications on a group manifold and on spheres $S^{4n+3}$, one can find such subsets of Killing vectors, whereas on $S^{2n}$ one can not (as shown in [8]). For the Maxwell-Einstein system, on the other hand, the truncation to massless fields is consistent as far as the gauge symmetries are concerned [24].

The study of KK truncations took a new direction with the advent of supergravity. The most intensively studied model was the $N=8$ sugra, which proved to be very challenging. In the course of its study, other criteria for consistent truncations were found in [9, 25]. In [9] it was realized that for a field theory invariant under a group $G$, a consistent truncation can be obtained by retaining only all fields which are singlets under a subgroup $G' \subset G$. In general, this gives an infinite set of fields (the untruncated set is of course always infinite). If $G$ is the isometry group of the KK compactification on a space $M_k$, and $G'$ acts transitively on $M_k$ (i.e. any point in $M_k$ can be reached from any other point by a $G'$ gauge transformation), the subset of fields is finite. This requirement implies in particular that $M_k$ is a coset manifold, $G/H = G'/H'$. In the case of the $\text{AdS}_4 \times S_7$ compactification, this mechanism gives a group theoretical explanation for the consistency of a particular truncation consisting of massless gauged $N=3$ supergravity coupled to some massive matter multiplets (this system is not contained in $N=8$ sugra). In [25] it was claimed that the truncation to 4d $N \geq 1$ sugra multiplets with second order formulation for the fields is always consistent, and it was speculated that the truncation to a sugra multiplet is always consistent.

The study of the consistency of the KK reduction of $d=11$ sugra at the nonlinear level was tackled in a series of papers by de Wit and Nicolai. In their pioneering work, they followed two approaches. One was to construct directly an ansatz for the 11d metric as opposed to the vielbein; this yielded an ansatz for the fermions only up to a
matrix $U$, depending on the 4d scalar fields. The dependence of this matrix on one of the scalars was even determined, but they did not succeed in finding the dependence on all other scalars. They then switched to another approach in which they reformulated 11d sugra as a theory with a local $SO(1,3) \times SU(8)$ tangent space symmetry group, and dimensionally reduced this theory. This latter approach was used to prove consistency of the reduction by reproducing the 4d susy transformations laws and equations of motion, but left the ansatz of the original 11 dimensional fields in terms of the 4d ones implicit. The ansatz for $F_{\Lambda\Pi\Sigma\Omega}$ was not explicitly given in the first approach. In the second approach no $F_{\Lambda\Pi\Sigma\Omega}$ appears, but the equations needed to construct $F_{\Lambda\Pi\Sigma\Omega}$ from the second approach were highly involved.

Lately, suggested by the need to embed $AdS_4$ and $AdS_7$ black holes as solutions of 11d sugra (and $AdS_5$ black holes as solutions of 10d IIB sugra), the authors of [10, 11] proposed nonlinear ansätze for the embedding in 11d sugra and 10d IIB sugra of a further truncation of the gauged sugras in 4, 7 and 5 dimensions to a set of U(1) gauge fields and scalars (in the $AdS_7$ case, two U(1) gauge fields and two scalars). The consistency of the truncations was proven by reproducing the lower dimensional equations of motion from the ones of d=11 and d=10 sugras (see next section). After that, a series of other papers looked at other truncations of 11d sugra on $AdS_7 \times S_4$ (a model with one scalar, one SU(2) gauge field and one antisymmetric tensor in [26], a model with 4 scalars in [27]). Consistent truncations of bosonic subsets of fields to other dimensions were also considered in [28, 29, 30, 31]. The authors of [32] also worked out an ansatz or an $S_3$ truncation of $N=1$ $d=10$ sugra which reproduced the bosonic action of $N=2$ $d=7$ gauged sugra; however, they did not explicitly checked consistency.

In a previous letter [12], we have given the full nonlinear ansätze for the KK reduction of 11d sugra on $AdS_7 \times S_4$. Here we present the details of the construction.

Crucial for the proof of consistency is a proper understanding of the origin of the concept of self-duality in odd dimensions – seven in this case. In sugra, this mechanism of selfduality in odd dimensions leads to a consistent coupling of a multiplet of massive antisymmetric tensor fields to nonabelian gauge fields. It was first discovered in a study of the KK reduction of d=11 sugra on $S_4$ [33], and found to give a formulation dual to Chern-Simons theory for the abelian case [34], but not for the nonabelian case [35]. It was found that the second order field equation for the field $A_{\alpha\beta\gamma,A}$, which one obtains from KK reduction of the field $A_{\Lambda\Pi\Sigma\Omega}$, was of the form $\Box A + m^2 A + m \epsilon \partial A = 0$. Adding in $d=7$ an auxiliary field $B_{\alpha\beta\gamma,A}$ with action $B^2$, a rotation from A,B to S,b yielded the final massless field $S_{\alpha\beta\gamma,A}$ and a massive mode $b_{\alpha\beta\gamma,A}$,

$$A = S + b, B = b - O(S + b)$$

where $O$ is an operator involving derivatives of the form $\epsilon D + m$, see (2.41). We have found that $\epsilon B$ is obtained by KK reduction of the auxiliary field $B_{\Lambda\Pi\Sigma\Omega}$ in $d=11$, which is obtained by using a first-order formulation for the 3-index photon in $d=11$.

This paper is organized as follows. In section 2.1 we discuss the issue of the consistency of KK truncations in general. We argue that it is sufficient to obtain the susy transformation laws of the dimensionally reduced theory from the susy transformation laws of the untruncated theory. In section 2.2 we present $d=11$ sugra with a first-order formulation for the 3-index photon $A_{\alpha\beta\gamma,A}$, and prove its local supersymmetry. The
linearized KK reduction is given in section 2.3. Although it was already given in [36], we summarize it here since it forms the starting point for the nonlinear extension. In section 3 we discuss Killing spinors and spherical harmonics. We begin in section 3.1 with a description of the spherical harmonics used for the massless multiplet and give various useful identities in section 3.2. Section 4 is the heart of this paper: it contains the proof of the consistency of KK reduction. In 4.1 we present the full ansatz for the d=11 vielbein and gravitino and we discuss the susy transformation laws of the vielbein. In 4.2 we derive an ansatz for the d=11 3-index antisymmetric tensor field $A_{\Lambda\Pi\Sigma}$ by a partial analysis of the susy laws for $A_{\Lambda\Pi\Sigma}$ and by a preliminary study of the susy transformation laws of the gravitino. It leads to the d=7 ‘self-duality in odd dimensions’. In 4.3 we give the reduction of the rest of the d=11 gravitino transformation laws and derive the transformation law of the 3-index d=7 tensor $S_{\alpha\beta\gamma,A}$. In section 5 we derive for completeness the seven dimensional bosonic equations of motion from the 11 dimensional bosonic equations of motion, with the gauge fields set to zero. We also derive the seven dimensional bosonic action and compare our ansatz with other ansätze for consistent truncation to bosonic subsets of fields. We finish in section 6 with conclusions and discussions. In Appendix 1 we give our conventions and some useful relations for Dirac matrices, including certain completeness relations. In Appendix 2 we give some details of the charge conjugation matrices in various dimensions and properties of modified Majorana spinors.

2 First-order d=11 supergravity and linearized KK reduction

2.1 KK consistency

Let us briefly discuss the issue of the consistency of the KK reduction. In general, if we truncate the fields in a Lagrangian by putting the ‘massive’ ones $\{\phi(n)\}$ to zero and keeping only the ‘massless’ ones $\{\phi(0)\}$, we have to check that the full equations of motion and transformation laws (for all the symmetries of the system) are consistent under this truncation.

More precisely, the equations of motion of the massive fields, $\delta S/\delta \phi(n)$ must not contain any term depending only on the massless fields (at the linear level this is true because $\phi(n)$ are eigenfunctions of the kinetic operator) because otherwise setting $\phi(n) = 0$ would be inconsistent. Similarly, in the transformation rules for the symmetries of the system, the massive fields should not transform into a term with only massless ones, again because then we cannot put the massive fields to zero.

In the case of the KK reduction of a D dimensional sugra action, $S^{(D)}(\{\Phi\})$, we usually know the Lagrangian in lower (d) dimensions $S^{(d)}(\{\phi(0)\})$ which we should obtain after truncation, but we need to find the nonlinear ansatz which specifies how the final massless fields $\phi(0)$ are embedded in the higher-dimensional fields $\Phi$. For a consistent truncation, there should be no term linear in massive fields in the untruncated action $\int d^d x \phi(n) f_n(\{\phi(0)\})$, because such a term would give an inconsistency of the equations of motion. Let’s assume that both the equations of motion and the susy laws are inconsistent, i.e. there exists a massive field which varies into a term involv-
ing only massless fields, \( \delta \phi(n) = g_n(\{\phi(0)\}) + \text{more} \). If we can reproduce the correct d-dimensional susy laws from the D dimensional susy laws, the variation of the term in the action which is linear in massive fields gives an extra piece depending only on massless fields

\[
\delta S^{(D)}(\{\phi(n)\}, \{\phi(0)\}) = \delta S^{(d)}(\{\phi(0)\}) + \int g_n(\{\phi(0)\}) f_n(\{\phi(0)\}) + O(\phi(n)) = 0 \quad (2.1)
\]

Since the d-dimensional theory is invariant, \( \delta S^{(d)}(\{\phi(0)\}) \) vanishes. Hence, for consistency \( g_n f_n = 0 \), i.e. the equations of motion and susy laws can't both contain purely massless terms for a given massive field. Since the commutator of the susy transformations gives the equations of motion, the two inconsistencies are presumably always equivalent. Therefore one criterion for a consistent truncation is to find a nonlinear ansatz which gives the susy transformation laws of \( S^{(d)} \) from the susy transformation laws of \( S^{(D)} \).

In [10, 11] the consistency of the KK reduction from the original \( d = 11 \) supergravity to \( AdS(7) \times S_4 \) to a small subset (two scalars and two \( U(1) \) gauge fields) of our fields was studied using the other criterion, namely consistency of the field equations. Only bosonic fields were considered. More recently, this work has been extended to other subsets of bosonic fields (a model with one scalar, one \( SU(2) \) gauge field and one antisymmetric \( S_{\alpha\beta\gamma} \), [26] and a model with only 4 scalars [27]). We do not restrict our attention to subsets, and consider both fermionic and bosonic fields. For completeness we shall also consider consistency of the bosonic field equations.

2.2 Supergravity and first order formulations

We start with the sugra Lagrangian in (10,1) dimensions with a first order formalism for the antisymmetric tensor field. (For our conventions and notation, see the appendix). This formulation was presented in our previous letter [12]. After that, a paper appeared which gave first order formulations of supergravities, in particular a formulation of 11d sugra which is first order in both the spin connection and the antisymmetric tensor field [37].

\[
\mathcal{L} = -\frac{1}{2k^2} ER(E, \Omega) - \frac{E}{2} \bar{\Psi}_\Lambda \Gamma^{\Lambda\Pi\Sigma} D_\Pi (\frac{\Omega + \bar{\Omega}}{2}) \Psi_\Sigma \\
+ \frac{E}{48} (\mathcal{F}_{\Lambda\Pi\Sigma\Omega} \mathcal{F}^{\Lambda\Pi\Sigma\Omega} - 48 \mathcal{F}_{\Lambda\Pi\Sigma\Omega} \partial_\Lambda A_{\Pi\Sigma\Omega}) \\
- k \frac{\sqrt{2}}{6} \epsilon^{\Lambda_1...\Lambda_4} \partial_{\Lambda_0} A_{\Lambda_1\Lambda_2\Lambda_3} \partial_{\Lambda_4} A_{\Lambda_5\Lambda_6\Lambda_7} A_{\Lambda_8\Lambda_9\Lambda_{10}} \\
- \frac{\sqrt{2}k}{8} E [\bar{\Psi}_\Pi \Gamma^{\Pi\Lambda_1...\Lambda_4} \Psi_\Sigma + 12 \bar{\Psi}_1 \Gamma^{A_2A_3} \Psi_{\Lambda_4} ] \frac{1}{24} \tilde{F}_{\Lambda_1...\Lambda_4} \quad (2.2)
\]

where \( \mathcal{F}_{\Lambda\Pi\Sigma\Omega} \) is an independent field with field equation \( \mathcal{F}_{\Lambda\Pi\Sigma\Omega} = F_{\Lambda\Pi\Sigma\Omega} \), and

\[
F_{\Lambda\Pi\Sigma\Omega} = \partial_\Lambda A_{\Pi\Sigma\Omega} + 23 \text{ terms} \quad (2.3)
\]

Furthermore \( E = \det E_\Lambda^M \) and

\[
R(E, \Omega) = R_{\Lambda\Pi} {}^{MN}(\Omega) E_\Pi^M E_N^\Lambda \quad (2.4)
\]
Here $\hat{F}$ is the supercovariant curl of $A\dagger$ and we use 1.5 order formalism for the spin connection, i.e., $\Omega$ is not independent, but rather it is the solution of its field equation (to which only the $\Omega$ term, but not the $\hat{\Omega}$ term, in the gravitino action contributes).

The supercovariant spin connection $\hat{\Omega}$ is obtained from $\Omega$ by adding terms bilinear in the fermions such that there are no $\partial\varepsilon$ terms in its susy transformations

$$\hat{\Omega}_{\Pi MN} = \Omega_{\Pi MN}(E) + \frac{k^2}{4}(\bar{\Psi}_\Pi \Gamma^M \Psi_N - \bar{\Psi}_\Pi \Gamma^M \Psi^N + \bar{\Psi}^M \Gamma_{\Pi} \psi^N)$$

The relation between $\Omega$ and $\hat{\Omega}$ is then given by

$$\text{R}{\text{L}}$$

It is useful to redefine

$$F_{\Lambda \Pi \Sigma \Omega} = \partial_{\Lambda} A_{\Pi \Sigma \Omega} + 23 \text{ terms} + \frac{B_{MNPQ} E^M_A \ldots E^Q}{\sqrt{E}}$$

Substituting this definition into the terms in the action involving $F$, we obtain

$$\text{R}{\text{L}}$$

The supersymmetry transformation rules which leave the action with the $B_{MNPQ}^2$ term invariant read

$$\text{R}{\text{L}}$$

where $R_{\Lambda}(\Psi)$ is the gravitino field equation,

$$\text{R}{\text{L}}$$

By replacing $24 F \partial A$ in (2.2) by $\mathcal{F} \hat{F}$, the terms $(\bar{\Psi}_\Gamma \Gamma \Psi) \hat{F}$ get absorbed. Then the $\mathcal{F}$ field equation reads $\mathcal{F} = \hat{F}$ and becomes supercovariant. We have not been able to absorb the remaining four-fermi terms by using our new first order formulation.
The expression $\hat{F}_{\Lambda\Pi\Sigma\Omega}$ denotes the usual supercovariantization of $\partial_\Lambda A_{\Pi\Sigma\Omega} + 23$ terms and $a$ and $b$ are free parameters. They will be fixed by the requirement of consistency of the KK truncation on $S_4$. The gravitational constant $k$ has dimensions $11/2$, hence $kA$ is dimensionless and $k\Psi$ has dimension $1/2$. Below we set $k$ equal to unity.

New in the transformation laws are the $B$ terms in $\delta \Psi_\Lambda$ and the expression for $\delta B$. Of course $\delta B$ is proportional to the gravitino field equation, because $B = 0$ is a field equation and field equations (usually) transform into field equations.

This theory admits a background solution which satisfies the equations of motion and which describes a geometry of the type $AdS_7 \times S_4$. The source is given by

$$F_{\mu\nu\rho\sigma} = \frac{3}{\sqrt{2}} m (\det e^m_\mu (x)) e_{\mu\nu\rho\sigma}$$

(2.17)

where $e^m_\mu$ is the vielbein on $S_4$ and $e^a_\alpha$ the vielbein on $AdS_7$. The parameter $m$ is the inverse radius of the sphere as will now be show. The field equations for the background geometry read then

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{1}{6} \left( F_{\mu...\nu} - \frac{1}{8} g_{\mu\nu} F^2 \right) = -\frac{9}{4} g_{\mu\nu} m^2$$

(2.18)

and

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{1}{48} g_{\alpha\beta} F^2 = \frac{9}{4} g_{\alpha\beta} m^2$$

(2.19)

The maximally symmetric solution is given by

$$R_{\mu\nu}^{(4)(\hat{e})} = m^2 \{ e^m_\mu (x) e^n_\nu (x) - e^m_\nu (x) e^n_\mu (x) \}$$

(2.20)

$$R_{\alpha\beta}^{(7)(\hat{e})} = -\frac{1}{4} m^2 (e^a_\alpha (y) e^b_\beta (y) - e^a_\beta (y) e^b_\alpha (y))$$

(2.21)

$$R = R^{(4)} + R^{(7)} = -\frac{3}{2} m^2$$

(2.22)

All other fields vanish in the background

$$\Psi^M_\Lambda = 0 ; F_{\alpha\Lambda\Pi\Sigma} = 0 ; B_{\Lambda\Pi\Sigma\Omega} = 0$$

(2.23)

First order formulations in sugra have been given before in $d=4$. Soon after the $N=1$ sugra was formulated with a second-order formulation for the spin connection [38], a first-order formulation for the spin connection (with an independent field $\omega^{mn}_\mu$ and an expression for $\delta \omega^{mn}_\mu$) was given [39]. For $d=11$ sugra, a first-order formulation for the spin connection was given in [40].

The reason we start with a first order formulation for the antisymmetric tensor field in $d=11$ is that at the linearized level one needs in 7 dimensions a 3-index auxiliary field to combine with the 3 index tensor to give the 7 dimensional supergravity field $S_{\alpha\beta\gamma,A}$ found in [41]. Since the 3-index auxiliary field can also be written as a 4-index auxiliary field by a duality transformation, this suggested to us that the auxiliary field in $d=11$ gives by reduction the auxiliary field in $d=7$. Our results confirm this suggestion.

Could we have chosen another first order formulation which would give us at the linearized level this auxiliary field? The other obvious choice is the first order formulation for the spin connection (Palatini formalism). The auxiliary field would then be
obtained as the totally antisymmetric part of the difference between the independent 
spin connection and the solution of its equation of motion, i.e.

\[ B_{\Lambda \Pi \Sigma} = (\Omega_{[\Lambda}^{\; MN} - \hat{\Omega}_{[\Lambda}^{\; MN}(E, \Psi)) E_{\Pi \Pi} E_{\Sigma]} \]  

(2.24)

When we dimensionally reduce, we obtain an auxiliary antisymmetric tensor field \( B_{\alpha \beta \gamma} \).
We will see later (in section 3) that this approach does not produce the 7 dimensional auxiliary field we need.

### 2.3 Linearized Kaluza-Klein reduction on \( S_4 \)

Any 11 dimensional field is written as a sum of the corresponding background field and the fluctuations. In the fluctuations we keep only 'massless' modes, i.e., fields which in 7 dimensions belong to the maximal (N=4) sugra multiplet. In full generality, the massless 7 dimensional fields may occur nonlinearly in the expansion of 11 dimensional fields.

The linearized terms in the expansion of the bosons have the generic form

\[ \Phi_{\alpha \mu}(y, x) = \sum I \phi_{\alpha, I}(y) Y^I_{\mu}(x) \]  

(2.25)

where \( \phi \) are 7 dimensional fields and \( Y \) are spherical harmonics. The 7 dimensional indices are attached to 7 dimensional fields, 4 dimensional indices to the spherical harmonics \( Y \) of the internal space. For fermions one finds a similar decomposition,

\[ \Psi^{\alpha}(y, x) = \sum I \Psi^I_{\alpha}(y) Y^{I, \alpha} \]  

(2.26)

where the spinor index \( \alpha = 1, 32 \) of an 11 dimensional spinor decomposes as \( \alpha = \tilde{a} \otimes \tilde{\tilde{a}} \) with \( \tilde{a} \) spinor indices on \( AdS_7 \) and \( \tilde{\tilde{a}} \) spinor indices on \( S_4 \). We shall suppress these spinor indices, and thus \( \tilde{a} \) and \( \tilde{\tilde{a}} \) will not be used below.

The linearized ansatz has been given in [36]. We reproduce it here for completeness. The vielbein \( E^A_{\alpha}(y, x) \) produces the metric \( g_{\alpha \beta} = E^A_{\alpha} E_{\beta A} = \hat{g}_{\alpha \beta} + k h_{\alpha \beta} \), where \( h_{\alpha \beta} \) are the fluctuations given by the following expressions in the various sectors

(i) In \( AdS_7 \) spacetime

\[ h_{\alpha \beta}(y, x) = h_{\alpha \beta}(y) - \frac{1}{5} \hat{g}^{\alpha \beta}(y)(h_{\mu \nu}(y, x) \hat{g}^{\mu \nu}(x)) \]  

(2.27)

where the redefinition of the graviton is needed in order to diagonalize its kinetic term. (The linearized kinetic term for the graviton is the Fierz-Pauli action for a massless spin 2 field and reads in any dimension \( \mathcal{L} = 1/2 h_{\mu \nu, \rho}^2 + h^2_{\mu \nu} - h^4 h_{\mu \nu} + 1/2 h_{\mu \nu}^2 \) where \( h_{\mu \nu} = \partial^\nu h_{\nu \mu} \) and \( h = h_{\mu \nu}^2 \).)

(ii) In the mixed sector,

\[ h_{\mu \alpha}(y, x) = B_{\alpha, IJ}(y)V^{IJ}_{\mu}(x); \; I, J = 1, 4 \]  

(2.28)

where \( V^{IJ}_{\mu}(x) = V^{JI}_{\mu}(x) \) are the ten Killing vectors on \( S_4 \).

(iii) In the \( S_4 \) sector

\[ h_{\mu \nu}(y, x) = S_{IJKL}(y) h^{IJKL}_{\mu \nu}(x) \]  

(2.29)
where the scalars $S_{IJKL}$ are in the 14 representation of USp(4) and $\eta^{IJKL}_{\mu}$ is the corresponding spherical harmonic. It has the symmetry of the Riemann tensor, but its symplectic trace and its totally antisymmetric part vanish (see below (3.6) and (3.7)).

The gravitino $\Psi_\Lambda$ splits into $\Psi_\mu$ and $\Psi_\alpha$, where the spin 1/2 fields are written as

$$\Psi_\mu(y,x) = \lambda_{IJKL}(y) \gamma^{1/2}_5 \eta_{IJKL}^{\mu}(x)$$ (2.30)

The $\lambda_{IJKL}$ with J,K,L=1,...,4 are the fermions in the 16 representation of USp(4), with $\eta_{IJKL}^{\mu}(x)$ the corresponding spherical harmonic. It has the symmetry of the Riemann tensor, but its symplectic trace and its totally antisymmetric part vanish (and so all its symplectic traces are zero). By definition, $\sqrt{\gamma_5} = \frac{i}{2}(1 + i\gamma_5)$, see appendix A1.

We took the scalars to be in the 14 representation and the spinors in the 16 representation because of the following reason. First, the total number of scalars has to be 14, and the total number of fermions 16, a fact which we know for instance from the case of toroidal compactification. Second, the complete mass spectrum on $S_4$ was found in [45], and the lowest mass modes are the singlet and the 14. The singlet has bigger mass, and is found to belong to another multiplet. The spinors have as lowest modes a 4 and the 16, but again the 4 has higher mass and belongs to the same multiplet as the scalar singlet.

The gravitini $\Psi_\alpha$ also have to be redefined in order to diagonalize their kinetic term

$$\Psi_\alpha(y,x) = \psi_{\alpha I}(y) \gamma^{1/2}_5 \eta^I(x) - \frac{1}{5} \tau_\alpha \gamma_5 \gamma^\mu \Psi_\mu(y,x)$$ (2.31)

where $\eta^I(x)$ is the Killing spinor. Clearly, $\gamma_5^{-1/2} = \frac{1}{2}(i\gamma_5 - 1)$.

The antisymmetric tensor $F_{\Lambda\Pi\Sigma\Omega} = \hat{F}_{\Lambda\Pi\Sigma\Omega} + f_{\Lambda\Pi\Sigma\Omega}$ (where $f_{\Lambda\Pi\Sigma\Omega}$ is the fluctuation and $\hat{F}_{\Lambda\Pi\Sigma\Omega}$ the background solution) decomposes at the linearized level as follows

$$\frac{\sqrt{2}}{3} f_{\mu\nu\rho\sigma} = \sqrt{\text{det} \ g_{\mu\nu}} \epsilon_{\mu\nu\rho\sigma} h^\lambda_\lambda, \quad h^\lambda_\lambda \equiv h_{\mu\nu}(y,x) g^{\mu\nu}(x)$$ (2.32)

$$\frac{\sqrt{2}}{3} f_{\alpha\nu\rho\sigma} = \sqrt{\text{det} \ g_{\nu\rho}} \epsilon_{\nu\rho\sigma} \left[ \frac{1}{10} D^\tau D_\alpha h^\lambda_\lambda - h^\alpha_\alpha \right]$$ (2.33)

$$\frac{\sqrt{2}}{3} f_{\mu\nu\alpha\beta} = \frac{i}{3} \partial_{[\alpha} B^{IJ}_{\beta]} \eta^I \gamma_{\mu\nu} \gamma_5 \eta^J$$ (2.34)

$$\frac{\sqrt{2}}{3} f_{\nu\alpha\beta\gamma} = \sqrt{2} \frac{3}{4} A_{\alpha\beta\gamma,IJ} \partial_\nu \phi^I_J(x)$$ (2.35)

$$\frac{\sqrt{2}}{3} f_{\alpha\beta\gamma\delta} = \sqrt{2} \frac{3}{4} \partial_{[\alpha} A_{\beta\gamma\delta,IJ} \phi^I_J(x)$$ (2.36)

so that

$$A_{\mu\nu\rho}(y,x) = \frac{\sqrt{2}}{40} \sqrt{\text{det} \ g_{\mu\nu}} D^\sigma h^\lambda_\lambda$$ (2.37)

$$A_{\alpha\mu\nu}(y,x) = \frac{i}{12\sqrt{2}} B_{\alpha,IJ}(y) \eta^I(x) \gamma_{\mu\nu} \gamma_5 \eta^J(x)$$ (2.38)

$$A_{\alpha\beta\mu} = 0$$ (2.39)

$$A_{\alpha\beta\gamma}(y,x) = \frac{1}{6} A_{\alpha\beta\gamma,IJ} \phi^I_J(x)$$ (2.40)
where $A_{\alpha\beta\gamma,IJ}$ is a set of antisymmetric symplectic-traceless tensors in the 5 representation of USp(4) and $\phi^J_5 = \bar{\eta}^{\mu}\eta^J = -\bar{\eta}^J\gamma_5\eta^I$ the corresponding scalar spherical harmonic. The factor 1/10 in (2.33) was determined in [36] by solving the Bianchi identities, but we have here given the linearized KK formulas for the fields themselves. To show that (2.37) reproduces (2.32) one needs $\Box g \eta^{IJK\ell} = -10 g \eta^{IJK\ell}$, see (3.7). The rest of the result in (2.33) then follows if one uses $\tilde{D}_\mu \eta = \frac{i}{2} \gamma_\mu \eta$ and $V_\mu = \bar{\eta} \gamma_\mu \eta$, see (3.3).

When massive modes are put to zero, one would expect that $A_{\alpha\beta\gamma,IJ}$ becomes equal to the 7 dimensional supergravity field $S_{\alpha\beta\gamma,IJ}$. However, the field $A_{\alpha\beta\gamma,IJ}$ has an action with 2 derivatives, whereas the action of $S_{\alpha\beta\gamma,IJ}$ is linear in derivatives. That means that we need to introduce by hand an auxiliary field $B_{\alpha\beta\gamma,IJ}$, and rotate the $A$ and $B$ fields such that the sum of the action of $A$ with 2 derivatives and the action of $B$ with none gives two decoupled actions each linear in derivatives, one for $S_{\alpha\beta\gamma,IJ}$ and one for a massive field $b$. After setting the massive field $b$ to zero, the dependence of the auxiliary field $B_{\alpha\beta\gamma,IJ}$ on $S_{\alpha\beta\gamma,IJ}$ reads

$$B_{\alpha\beta\gamma,IJ} = \frac{1}{5}(S_{\alpha\beta\gamma,IJ} + \frac{1}{6}\epsilon_{\alpha\beta\gamma}^{\delta\epsilon\zeta} D_5 S_{\epsilon\zeta,IJ})$$

(2.41)

Note that if we want to understand $B$ as coming from a KK reduction, its spherical harmonic should be the same as for the field $A$. In general two fields in lower dimensions with the same index structure cannot have the same spherical harmonic, but in our case $B$ and $A$ have different mass dimensions, and therefore can have the same spherical harmonic.

## 3 Spherical harmonics and Killing spinors

### 3.1 Spherical harmonics for the massless multiplet

The coset representatives of a reductive coset manifold $L(x) = \exp(-x^a K_a)$ satisfy the equation $L^{-1} dL = e^a K_a + \omega^i H_i$. From $dL^{-1} = -(L^{-1} dL) L^{-1}$ with coset generators $K_a$ and subgroup generators $H_i$, one then obtains $(d + \omega^i H_i) L^{-1} = -(e^m K_m) L^{-1}$. Using the spinor representation of SO(5) with $H_m = 1/4 [\gamma_m, \gamma_m]$ and $K_m = -i c\gamma_m$, we define $\eta^I(x) \equiv L^{-1} \eta^I(0)$ where $\eta^I(0)$ is a constant spinor with $\eta^I(0)^\alpha = \Omega^{\alpha I}$. (The index $\alpha$ is a spinor index and runs from 1 to 4, while $\Omega$ is a constant antisymmetric $4 \times 4$ matrix. We suppress the spinor index $\alpha$ in most of the text.) In general, the difference between $\omega^i H_i$ and the spin connection term $1/2 \omega (spin)_a^{bc} J_{bc} = 1/4 c_{ab}^c c_{J}^b$, where $c_{ab}^c$ is defined by $[K_a, K_b] = c_{ab}^c H^c + c_{ab}^d K^d, c_{ab}^a = c_{db}^a + \delta^{ad}(c_{a'd'} b', \delta_{ab} + b \leftrightarrow d)$ [43]. Since $S_4 = SO(5)/SO(4)$ is a symmetric manifold, $c_{ab}^d = 0$ and thus $\omega^i H_i$ is the spin connection. For a treatment of spherical harmonics on $S_4$ and on general coset manifolds, where these issues are discussed, see [44, 45].

On $S_4$ we use $\mu = 1, 4$ instead of $a$ as curved coset indices, and $m = 1, 4$ as flat coset indices. Then the spinors $\eta^I(x)$ are Killing spinors as they satisfy

$$(\partial_\mu + \frac{1}{4} \omega_\mu^{mn}(x) \gamma_{mn}) \eta^I(x) = c e^m_m(x) (i \gamma_m) \eta^I(x),$$

(3.1)
On spheres there are two sets of Killing spinors \( \eta^\pm \), which can be derived from the integrability condition on the Killing spinor \( \bar{D}_\mu \bar{D}_\nu \eta = R_{\mu\nu} \gamma^{mn} \eta \), when we substitute the background curvature. In the following we choose to work with Killing spinors for which \( c = \frac{1}{2} m \); we shall also drop the factors of \( m \), understanding that we should replace everywhere \( \partial_\mu \) by \( \partial_\mu / m \) to get the correct dimensions.

The Killing spinors can be given an explicit form

\[
\eta^{\alpha I} = \{ \exp(\frac{-i}{2} x^\mu \delta^m \gamma^m) \}^{\alpha \beta} \Omega^{\beta I}
\]

(3.2)

where \( x^\mu \) are general coordinates in patches covering \( S_4 \). They satisfy

\[
\bar{D}_\mu \eta^I = \frac{i}{2} \gamma_\mu \eta^I; \quad \bar{\eta}^I \eta^I = \Omega^{I\bar{J}}; \quad \bar{D}_\mu \bar{\eta}^I = -\frac{i}{2} \bar{\eta}^I \gamma_\mu
\]

(3.3)

(In appendix A2 we discuss the definition \( \bar{\eta}^I = \eta^I \eta^4 \), where the charge conjugation matrix \( (C_4^I)_{\alpha \beta} \) on \( S_4 \) is equal to the symplectic metric \( \Omega_{\alpha \beta} \). It is shown there that \( \Omega^{I\alpha} = -(C^{-1})^{I\alpha} \). The first relation follows from our earlier discussion, while the second one is easy to check. A direct consequence of the second equation is a completeness type of relation satisfied by the Killing spinors

\[
\eta^I J \eta^J = -\delta^I_\beta \text{ with } \eta^I \equiv \eta^{I} \Omega_{I\bar{J}}
\]

(3.4)

One can directly verify the Killing equation by constructing the vielbein \( e^m \) and spin connection \( \omega^4 \) from \( L^{-1} dL \), expressing \( L \) in terms of a cosine and sine as in (3.11), and substituting into (3.1). All spherical harmonics can now be built from Killing spinors:

- Scalars 5, \( \phi^I_5 = -\phi_I^5 = \bar{\eta}^I \gamma_5 \eta^I \), \( \phi_I^5 \Omega_{IJ} = 0 \), or \( Y^A = \frac{1}{4} \phi^I_5 (\gamma^A)_{IJ} \) with \( I,J = 1,4 \) and \( \gamma^A \) defined in (A.11). (The matrices \( (\gamma^A)_{IJ} \) are antisymmetric and obtained from \( \{ \gamma_\mu \gamma_\nu, \gamma_\nu \}^I_J \) by lowering the index according to the rule (A.14)).

- Conformal Killing vectors 5, \( C^I_\mu = -C_{\mu}^I = \bar{\eta}^I \gamma_\mu \gamma_5 \eta^I \), \( C^I_\mu \Omega_{IJ} = 0 \), or \( C_{\mu}^I = \frac{i}{4} C_I (\gamma^A)_{IJ} = \bar{D}_\mu Y^A \), where \( \bar{D}_\mu Y^A = \partial_\mu Y^A \). They satisfy \( \bar{D}_\mu C^A_\nu = \frac{1}{4} \eta^{\mu \nu} (\bar{D}^A C_\rho) \). (In fact \( \partial_\mu C^A_\nu = \frac{2}{9} R^A_\nu \rho \partial_\rho C_\mu \).)

- Killing vectors 10, \( V^I_\mu = V^{JI} = \bar{\eta}^I \gamma_\mu \eta^J \) or \( V^{AB} = -V^{BA} = -\frac{4}{3} V^{I} (\gamma^{AB})_{IJ} \), satisfying \( \bar{D}_\mu V^A_\nu = 0 \) and \( V^I_\mu = i V^{AB} (\gamma^{AB})_{IJ} \). We show in (3.24) that \( V^{AB} = Y^{[A} \partial_\mu Y^{B]} \)

- Symmetric tensors \( \eta^I_{\mu \nu} K^L = \eta^I_{\mu \nu} K^L \), with Young tableau in the form of a \( 2 \times 2 \) box,

\[
\eta^I_{\mu \nu} K^L = \eta^I_{\mu \nu} K^L (-2) - \frac{1}{3} \eta^I_{\mu \nu} K^L (-10)
\]

(3.5)

It follows from this integrability condition that in general Killing spinors exist only on Einstein manifolds. On spheres there are two sets of Killing spinors \( \eta^\pm \) satisfying \( \bar{D}_\mu \eta^\pm = \pm c \gamma_\mu \eta^\pm \) which are related by \( \eta^+ = \gamma_5 \eta^- \) in even dimensions.
where $\Box IJKL(-2) = -2\eta_{\mu}^{IJKL}$ and $\Box IJKL(-10) = -10\eta_{\mu}^{IJKL}$. The first harmonic is traceless, while the second one is a pure trace. In terms of Killing spinors one has

$$
\eta_{\mu}^{IJKL}(-2) = C_{\mu}^{IJKL} - \frac{1}{4} g_{\mu\nu} C_{\lambda}^{IJKL} C_{\lambda}^{IJKL}
$$

(3.6)

$$
\eta_{\mu}^{IJKL}(-10) = g_{\mu\nu} (\phi_5^{IJ} \phi_5^{KL} + \frac{1}{4} C_{\lambda}^{IJKL})
$$

(3.7)

(Use $\Box \phi_5^{IJ} = -4\phi_5^{IJ}$ and $\Box C_{\mu}^{IJ} = -C_{\mu}^{IJ}$.) Both (3.6) and (3.7) are symplectic-traceless (use (3.4)) and without a totally antisymmetric part (use Fierz rearrangements).

- Vector-spinor $\eta_{\mu}^{JKL}$ with Young tableau in the form of a gun,

$$
\eta_{\mu}^{JKL} = \eta_{\mu}^{JKL}(-2) + \eta_{\mu}^{JKL}(-6)
$$

(3.8)

The first harmonic satisfies $\gamma^{\nu} \overset{\circ}{D}_{\nu} \eta_{\mu}^{JKL}(-2) = -2\eta_{\mu}^{JKL}(-2)$ and is gamma traceless, while the second one is a pure gamma trace and satisfies $\gamma^{\nu} \overset{\circ}{D}_{\nu} \eta_{\mu}^{JKL}(-6) = -6\eta_{\mu}^{JKL}(-6)$. Again these spherical harmonics can be expressed in terms of Killing spinors

$$
\eta_{\mu}^{JKL}(-2) = 3(\eta_{\mu}^{J} C_{\mu}^{KL} - \frac{1}{4} \gamma_{\mu}^{J} \gamma_{\nu}^{J} C_{\nu}^{KL})
$$

(3.9)

$$
\eta_{\mu}^{JKL}(-6) = \gamma_{\mu}^{J} (\eta_{\nu}^{J} \phi_5^{KL} - \frac{1}{4} \gamma_{\nu}^{J} C_{\nu}^{KL})
$$

(3.10)

The relative factors $1/3$ in (3.5) and 1 in (3.8) are not fixed by properties of the spherical harmonics themselves, but rather by consistency of the susy transformation rules and equations of motion at the linearized level [36].

### 3.2 Identities involving spherical harmonics

Substituting (3.2) into the definition of $Y^A$ in terms of the Killing spinors, one obtains the vectors $Y^A$ in terms of the coset parameters as:

$$
Y^A = -\frac{1}{4} \text{Tr} \left[ \gamma^A \gamma_5 \exp(-i \not{\xi}) \right]
$$

(3.11)

or, in components:

$$
Y^m = -\delta^m_{\mu} \frac{x^\mu}{x} \sin x
$$

(3.12)

$$
Y^5 = -\cos x
$$

(3.13)

where $x^2 = x^\mu x^\nu \delta_{\mu\nu}$. Clearly, $\sum (Y^A)^2 = 1$, and $x$ has a geometrical meaning: $-x$ is the azimuthal angle of a point on the unit sphere with coordinates $Y$.

On a sphere one can also define stereographic coordinates $\xi$ related to $Y^m$ by a conformal mapping. If the unit sphere lies on top of the stereographic plane one obtains

$$
\xi^\mu = \delta^m_{\mu} \frac{2Y^m}{1 - Y^5}
$$

(3.14)
In these $\xi$ coordinates the vielbein has a very simple form

$$e^m_\mu(\xi) = \delta^m_\mu \frac{4}{4 + \xi^2} \quad (3.15)$$

Also the spin connection and Killing spinors are simple rational functions of the $\xi$’s

$$\omega^{mn}_\mu = \frac{1}{1 + \xi^2} (\delta^m_\mu \xi^n + \delta^m_\mu \xi^n) \quad (3.16)$$

$$\eta^\pm(\xi) = \frac{4}{4 + \xi^2} (1 \pm \frac{i}{2} \gamma_m \xi^l \delta^m_\mu) \eta^\pm(0) \quad (3.17)$$

We have now 3 different coordinate systems on $S_4$: the coset coordinates $x$, the Euclidean coordinates $Y^A$ and the stereographic coordinates $\xi$. We shall mostly use the Euclidean coordinates $Y^A$, but note that the explicit expressions for the Killing spinors (which we do not use) are simplest in stereographic coordinates.

The scalars $Y^A$ parameterizing the sphere in a 5 dimensional space satisfy the identity

$$Y^A Y_A = 1 \quad (3.18)$$

It follows that

$$Y^A \overset{\circ}{D}_\mu Y_A = 0, \text{ or } Y^A C_{A\mu} = 0 \quad (3.19)$$

It is easy to check that

$$\overset{\circ}{D}_\mu \overset{\circ}{D}_\nu Y^A = -g_{\mu\nu} Y^A \quad (3.20)$$

Hence we obtain the completeness relation

$$\overset{\circ}{D}_\mu Y^A \overset{\circ}{D}_\nu Y_A = g_{\mu\nu}, \text{ or } C^A_{C A\mu} = g_{\mu\nu} \quad (3.21)$$

Another useful identity is

$$\overset{\circ}{D}_\mu Y^A \overset{\circ}{D}^\mu Y^B + Y^A Y^B = \delta^{AB} \quad (3.22)$$

It follows by using that $\partial_m Y^A = C^A_m$ and $Y^A \equiv C^A_5$ form an orthogonal $5 \times 5$ matrix according to (3.18), (3.19) and (3.21). We also give another proof because it illustrates the techniques we will use. Acting with $\overset{\circ}{D}_\lambda$ on both sides of (3.22); the r.h.s. is then zero trivially, and the l.h.s. is zero upon using (3.20). So the l.h.s. is constant, and at $x=0$, where $\eta^{\alpha} = \Omega^{\alpha I}$, we obtain

$$\overset{\circ}{D}_\mu Y^A|_{x=0} = \frac{1}{4} Tr(C^I \gamma_5 \gamma_\mu \gamma^A \tilde{\Omega}) = -\delta^A_\mu$$

$$Y^A|_{x=0} = \frac{1}{4} Tr(C^A \gamma_5 \gamma^A \tilde{\Omega}) = -\delta^A_5 \quad (3.23)$$

This proves (3.22).

In the same way one may prove that

$$V^{AB}_\mu = Y^A \overset{\circ}{D}_\mu Y^B \quad (3.24)$$
namely one may show that the r.h.s. satisfies the Killing vector equation, \( \hat{D}_\mu V^A_{\mu B} = 0 \), by using (3.20) and the antisymmetry in AB. The proportionality constant is fixed by taking \( x = 0 \) for a particular case, e.g. A=5, B=\( \nu \). Using the definition of \( V^A_{\mu B} \), the l.h.s. gives at \( x=0 = \frac{1}{2} g_{\mu \nu} \), and the r.h.s. gives with (3.23) also \( -\frac{1}{2} g_{\mu \nu} \).

Similarly, we may derive

\[
4\sqrt{g} \epsilon_{\mu \rho \sigma \sigma} \partial^\rho Y^{[A_1} \partial^\sigma Y^{A_2]} = +i \eta^I \gamma_{\mu \nu} \gamma^5 \eta^J (\gamma^{A_1 A_2})_{IJ} \tag{3.25}
\]

To prove this, we act with \( \hat{D}_\lambda \) on the last term and get \( +i \eta^I \gamma_{5 \mu \nu} \lambda \eta^J (\gamma^{A_1 A_2})_{IJ} \). We obtain the same result if we act with \( \hat{D}_\lambda \) on the first term. At \( x=0 \), using (3.23), the l.h.s. gives \( 4\sqrt{g} \epsilon_{\mu \nu A_1 A_2} \) and the r.h.s. gives \( Tr(C_{\gamma \rho \sigma} \gamma_5 \gamma^{A_1 A_2} \tilde{\Omega}) \). Evaluating the trace one finds that both results are equal. This proves (3.25).

\( \Rightarrow \) From (3.22) and (3.24) we find also

\[
V^A_{\mu B} V^B_{\mu A} = \frac{1}{2} \delta_{\mu \nu} V^I_{\mu} V^B_{\nu} = -4\delta^n \tag{3.26}
\]

\[
V^A_{\mu A} Y_A = \frac{1}{2} D_\mu Y^B \tag{3.27}
\]

\[
\frac{1}{2} [V^A_{\mu B} Y_B - A \leftrightarrow C] = \frac{1}{2} V^{AC} \tag{3.28}
\]

\[
V^A_{\mu B} V^\mu_{CD} = Y^{[A}_{C} \delta^B_D] \tag{3.29}
\]

\( \Rightarrow \) From SO(5) symmetry \( \dagger \) we deduce

\[
\int_{S_4} d^4 x \sqrt{g} Y^A Y^B = V_4 \frac{\delta^{AB}}{5}. \tag{3.30}
\]

where \( V_4 \) is the volume of \( S_4 \). In the same way we find

\[
\int_{S_4} d^4 x \sqrt{g} Y^A Y^B Y^C Y^D = \frac{V_4}{5} \left( \delta^{AB} \delta^{CD} + \delta^{AC} \delta^{BD} + \delta^{AD} \delta^{BC} \right) \tag{3.31}
\]

and integrals of any even number of \( Y^A \)'s will be proportional to symmetrized products of delta functions in a similar way. Integrals of an odd number of \( Y^A \)'s will give zero. See also the Appendix in [22]. A basic identity which we will use repeatedly is given by

\[
\epsilon_{A_1 \ldots A_5} = 5 \sqrt{g} \epsilon_{\mu \rho \sigma \sigma} \partial^\rho Y^{[A_1} \partial^\sigma Y^{A_1} \partial^\sigma Y^{A_5]} \tag{3.32}
\]

It also follows from the fact that \( C^A_m \) and \( C^A_5 \) form an orthogonal matrix. Another proof is obtained by acting with \( \hat{D}_\lambda \) on both sides. The l.h.s. gives zero, and because of (3.20), on the r.h.s. when \( \hat{D}_\lambda \) hits the \( \partial Y \)'s or \( Y^{A_5} \) gives zero by antisymmetry. Being constant, the r.h.s. should be proportional to the l.h.s., and the proportionality

\( \dagger \)The l.h.s. has to be an invariant tensor of SO(5) with 2 indices because the Haar measure \( d^4 x \sqrt{g} \) on \( S_4 \) is SO(5) invariant. This allows only \( \delta^{AB} \). The constant of proportionality is found by taking a trace with\( \delta^{AB} \).
constant is fixed by looking at \(x=0\). At \(x=0\), \(\eta^{\alpha I} = \Omega^{\alpha I}\), so from (3.23) we get on the r.h.s. \(5\sqrt{\bar{\gamma}}\epsilon_{\mu\nu\rho\sigma}\delta^{[\alpha_1}\delta^{\alpha_2}\delta^{A_3]} = \epsilon_{A_1...A_4}\delta_{A_5} + \epsilon_{A_2...A_5}\delta_{A_1} + ... + \epsilon_{A_3...A_5}\delta_{A_1} = \epsilon_{A_1...A_5}.

By repeatedly using (3.18),(3.19),(3.21), one derives from (3.32) the following further identities

\[
\epsilon_{A_1...A_5}dY^{A_1} = 4\sqrt{\bar{\gamma}}\epsilon_{\mu\nu\rho\sigma}\partial_\mu Y^{A_2}\partial_\nu Y^{A_3}\partial_\rho Y^{A_4}Y^{A_5} \tag{3.33}
\]

\[
\epsilon_{A_1...A_5}dY^{A_1} \wedge dY^{A_2} = 3\sqrt{\bar{\gamma}}\epsilon_{\mu\nu\rho\sigma}\partial_\mu Y^{A_3}\partial_\rho Y^{A_4}Y^{A_5} \tag{3.34}
\]

\[
\epsilon_{A_1...A_5}dY^{A_1} \wedge dY^{A_2} \wedge dY^{A_3} = 2\sqrt{\bar{\gamma}}\epsilon_{\mu\nu\rho\sigma}\partial_\mu Y^{A_4}Y^{A_5} \tag{3.35}
\]

\[
\epsilon_{A_1...A_5}dY^{A_1} \wedge ... \wedge dY^{A_4} = \sqrt{\bar{\gamma}}\epsilon_{\mu\nu\rho\sigma} \partial_\mu Y^{A_5} \tag{3.36}
\]

Contracting these identities with \(Y^{A_5}\), leads to a further chain of identities.

\[
\epsilon_{A_1...A_5}dY^{A_1} \wedge ... \wedge dY^{A_4}Y^{A_5} = \sqrt{\bar{\gamma}}\epsilon_{\mu\nu\rho\sigma} \partial_\mu Y^{A_5} \tag{3.37}
\]

\[
\epsilon_{A_1...A_5}dY^{A_1} \wedge dY^{A_2} \wedge Y^{A_5} = \sqrt{\bar{\gamma}}\epsilon_{\mu\nu\rho\sigma} \partial_\mu Y^{A_4} \tag{3.38}
\]

\[
\epsilon_{A_1...A_5}dY^{A_1} \wedge dY^{A_2}Y^{A_5} = \sqrt{\bar{\gamma}}\epsilon_{\mu\nu\rho\sigma} \partial_\mu Y^{A_3} \tag{3.39}
\]

\[
\epsilon_{A_1...A_5}dY^{A_1}Y^{A_5} = \sqrt{\bar{\gamma}}\epsilon_{\mu\nu\rho\sigma} \partial_\mu Y^{A_2}\partial_\rho Y^{A_3} \tag{3.40}
\]

\[
\epsilon_{A_1...A_5}Y^{A_5} = \sqrt{\bar{\gamma}}\epsilon_{\mu\nu\rho\sigma} \partial_\mu Y^{A_4} \tag{3.41}
\]

The last identity follows from (3.32). As a check on the normalization of the identities (3.33) to (3.41), we consider the point \(x=0\) and use (3.23). For example, the l.h.s. of (3.35) yields \(\epsilon_{A_1...A_5}\delta^{\alpha_1}\delta^{A_2}\delta^{A_3}\) which agrees with \(2\epsilon_{\mu\nu\rho\sigma}\delta^{[\alpha_1}\delta^{\alpha_2]}\) on the r.h.s.

By Fierz rearrangements, we get other identities. Fierzing \(\eta^{[K}(\bar{\eta}^{J]}\eta^{I})\), we get

\[
\gamma_5\eta^I\phi^K_{[I} - \gamma_\mu\gamma_5\eta^I C^K_{\mu J} = 4\eta^{[K}\Omega^{J]} - \eta^{I}\Omega^{JK} \tag{3.42}
\]

from which we retrieve (3.22) after multiplying with \(\bar{\eta}^L\). Fierzing \(\eta^{[K}(\bar{\eta}^{J]}\eta^{I})\), we obtain

\[
\gamma_\mu\eta^I V^K_{\mu J} = \frac{1}{2}\gamma_\mu\eta^I (\bar{\eta}^{J} \gamma_\mu\eta^K) = 4\eta^{[K}\Omega^{J]} \tag{3.43}
\]

In particular, (3.42) will be heavily used.

4 Derivation of the complete nonlinear Kaluza Klein reduction.

4.1 The dimensional reduction of the 11 dimensional vielbein transformation law

With the tools developed in the previous section, we turn to the main problem of this article, finding the complete nonlinear Kaluza-Klein ansatz.
For completeness, we give below the action and transformation laws of the dimensionally reduced theory, namely maximal gauged sugra in \( d = 7 \). The model has a local SO(5)\( _g \) gauge group for which A,B,... =1,5 are vector indices, while I,J,... =1,4 are spinor indices. The scalars \( \Pi_A^i \) parameterize the coset \( SL(5,\mathbf{R})/SO(5)_c \) but in the gauged model the \( SL(5,\mathbf{R}) \) rigid symmetry of the action is lost and replaced by the SO(5)\( _g \) gauge invariance. The subscripts \( g \) stand for gauge, and \( c \) for composite. The indices \( i,j,...=1,5 \) are SO(5)\( _c \) vector indices and \( I',J',...=1,4 \) are spinor indices. The model has the following fields: the vielbein \( e_a^\alpha \), the 4 gravitinos \( \psi_i^\alpha \), the SO(5)\( _g \) vector \( B_a^{AB} = -B_a^{BA} \), the scalars \( \Pi_A^i \), the antisymmetric tensor \( S_{\alpha\beta\gamma,A} \) and the spin 1/2 fields \( \lambda_i^\tau \) (vector-spinors under SO(5)\( _c \), satisfying \( \gamma^i\lambda_i^\tau = 0 \)). They have the correct mass-terms which ensure 'masslessness' in d=7 AdS space [46, 45]. The action reads

\[
e^{-1} \mathcal{L} = -\frac{1}{2} R + \frac{1}{4} m^2 (T^2 - 2T_{ij} T^{ij}) - \frac{1}{2} P_{\alpha ij} P^{\alpha ij} - \frac{1}{4} (\Pi^i A^i B^j f^{AB})^2
\]

\[
+ \frac{1}{2} (\Pi^i A^i S_{\alpha\beta\gamma,A})^2 + \frac{1}{48} me^{-1} e^{\alpha\beta\gamma\delta\epsilon\kappa} \mathcal{F}^{AB} S_{\alpha\beta\gamma,A} F_{\delta\epsilon\kappa,B} - \frac{1}{2} \bar{\psi}_a \gamma^\tau \gamma^\rho \psi^\tau \nabla_\rho \psi_\tau
\]

\[
- \frac{1}{2} \bar{\lambda}_a \gamma^\alpha \nabla_\alpha \lambda_b - \frac{1}{8} m (8T_{ij} T^{ij} - 8T^{ij} T_{ij}) \bar{\lambda}_a \lambda_b + \frac{1}{2} m T_{ij} \bar{\lambda}_a \lambda_b T^{ij} \psi_a + \frac{1}{2} \bar{\psi}_a \gamma^\tau \gamma^\rho \lambda^j \lambda_i
\]

\[
+ \frac{1}{8} m T \bar{\psi}_a \gamma^\tau \frac{1}{4} \bar{\psi}_a (\bar{\tau}^{\alpha\beta\gamma} - \frac{1}{2} g^{\alpha\beta} \bar{g}^{\gamma} \bar{\sigma}) \psi_a \Pi^i A^i B^j f^{AB}
\]

\[
+ \frac{1}{4} \bar{\psi}_a \gamma^{\beta\gamma} \gamma^{\alpha} \lambda_j \Pi^i A^i B^j f^{AB} \frac{1}{32} \bar{\lambda}_a \gamma^j \gamma^k \gamma^i \gamma^{\alpha} \lambda_j \Pi^i A^i B^j f^{AB} + \frac{1}{8} \bar{\psi}_a (\tau^{\alpha\beta\gamma} - \frac{1}{2} g^{\alpha\beta} \tau^{\gamma} \lambda^i \Pi^i A^i S_{\beta\gamma,A} + \frac{1}{16} \bar{\psi}_a (\tau^{\alpha\beta\gamma} - \frac{1}{2} g^{\alpha\beta} \tau^{\gamma} \lambda^i \Pi^i A^i S_{\beta\gamma,A})
\]

\[
- \frac{1}{8} \bar{\psi}_a \gamma^{\alpha} \gamma^{\beta} \gamma^{\gamma} \lambda_j \Pi^i A^i S_{\beta\gamma,A} \frac{1}{16} \bar{\psi}_a (\tau^{\alpha\beta\gamma} - \frac{1}{2} g^{\alpha\beta} \tau^{\gamma} \lambda^i \Pi^i A^i S_{\beta\gamma,A})
\]

\[
+ \frac{1}{8} \bar{\psi}_a \gamma^{\alpha} \gamma^{\beta} \gamma^{\gamma} \lambda_j \Pi^i A^i S_{\beta\gamma,A} \frac{1}{16} \bar{\psi}_a (\tau^{\alpha\beta\gamma} - \frac{1}{2} g^{\alpha\beta} \tau^{\gamma} \lambda^i \Pi^i A^i S_{\beta\gamma,A})
\]

The local supersymmetry transformation rules are given by

\[
\delta e^a_\alpha = \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\alpha
\]

\[
\Pi_A^j \Pi_B^j \delta B_a^{AB} = \frac{1}{4} \bar{\epsilon} \gamma^j \psi_\alpha + \frac{1}{8} \bar{\epsilon} \gamma^k \gamma^j \lambda_k
\]

\[
\delta S_{\alpha\beta\gamma,A} = -\frac{i \sqrt{3}}{8m} \Pi_A^i (2 \bar{\epsilon} \gamma^i \psi_\alpha + \bar{\epsilon} \gamma^i \gamma^j \lambda_j) \Pi_B^j \Pi_C^k F_{\beta\gamma}^{BC}
\]

\[
- \frac{i \sqrt{3}}{4m} \delta_{ij} \Pi_A^j D_\alpha (2 \bar{\epsilon} \gamma^i \psi_\alpha + \bar{\epsilon} \gamma^i \gamma^j \lambda_j)
\]

\[
+ \frac{i \sqrt{3}}{12} \delta_{AB} \Pi^{-1} \Pi^i (3 \bar{\epsilon} \gamma^i \psi_\alpha - \bar{\epsilon} \gamma^i \lambda_j)
\]

\[
\Pi^{-1} \alpha^i \delta \Pi_A^j = \frac{1}{4} \bar{\epsilon} \gamma^i \lambda^j + \bar{\epsilon} \gamma^i \lambda_i
\]

\[
\delta \psi_\alpha = \nabla_\alpha \epsilon - \frac{1}{20} m T \tau_\alpha \epsilon - \frac{1}{40} (\tau_\alpha^{\beta\gamma} - 8 \delta_\alpha^{\beta\gamma} \tau_\gamma) \gamma_{ij} \epsilon \Pi_A^i \Pi_B^j F^{AB}_{\beta\gamma}
\]

\[
+ \frac{im}{10 \sqrt{3}} (\tau_\alpha^{\beta\gamma} - \frac{9}{2} \delta_\alpha^{\beta\gamma} \tau_\gamma) \gamma^i \Pi^{-1} \alpha^i S_{\beta\gamma,A}
\]
\[ \delta \lambda_i = \frac{1}{16} \tau^{\alpha\beta}(\gamma_i \gamma_i - \frac{1}{2} \gamma \gamma_k \gamma_i) \epsilon \Pi^A_b \Pi^B_\beta F^{AB}_{\alpha\beta} + \frac{1}{20 \sqrt{3}} \tau^{\alpha\beta\gamma}(\gamma_i j - 4 \delta_i) \epsilon \Pi^{-1} A S_{\alpha\beta\gamma,A} + \frac{1}{2} m(T_{ij} - \frac{1}{5} T \delta_{ij}) \gamma^j \epsilon + \frac{1}{2} \tau^\gamma j^\gamma \epsilon P_{\alpha ij} \quad (4.7) \]

The contractions over indices \( I', J' \) are always as in \( \bar{\epsilon}_I \tau^\alpha \psi^{I'}_\alpha \) and \( \epsilon_I (\gamma^i) J' \psi^{J'}_\alpha \).

Here \( T_{ij} = \Pi^{-1} A \Pi^{-1} B \beta_{AB} \) and \( \Omega_3[B] \) and \( \Omega_5[B] \) are the Chern-Simons forms for \( B^{AB}_\alpha \) (normalized to \( d\Omega_3[B] = (Tr F^2)^2 \) and \( d\Omega_5[B] = (Tr F^4) \)). The tensor \( P_{\alpha ij} \) and the connection \( Q_{\alpha ij} \) (appearing in the covariant derivatives as \( \nabla_\alpha = \partial_\alpha + Q_\alpha.. + P_\alpha.. \) in (4.4)) are the symmetric and antisymmetric parts of \( (\Pi^{-1})_i^\alpha_\beta B_\beta \partial_\alpha + g B_{\alpha A} B^A \) respectively. The curl of \( S_{\alpha\beta\gamma, A} \) has strength 4, so \( F_{\alpha\beta\gamma, A} = 4 \nabla_\alpha \left( S_{\beta\gamma, A} \right) \). Further, \( g = 2m \) (or rather \( g = 2mk \) but \( k = 1 \)) and (2.21) is the AdS\(_7\) solution of the field equations of (4.1) with \( T_{ij} = \delta_{ij} \). We put \( m = 1 \) most of the time, so the the field strength of the gauge field is defined as \( F_{\alpha\beta} = \partial_\alpha B_{\beta}^A + 2B_{\alpha A B}^C B^C_{\beta} - (\beta \leftrightarrow \alpha) \). However, note that the limit \( m \rightarrow 0 \) is singular due to the factors of \( m^{-1} \) in front of the Chern-Simons terms.

At the nonlinear level, we need an ansatz for the vielbein rather than the metric as we did in section 2.3. The ansatz for the vielbein \( E^M_\alpha \) is constructed as follows. In order to fix the off-diagonal part of the Lorentz group and remain with local \( SO(6,1) \times SO(4) \) invariance, we impose the gauge choice \( E^m_\mu = 0 \). The natural extension of the rescaling in (2.27), needed to obtain the usual Einstein action in \( d=7 \) without extra powers of \( \Delta \), is\( ^\dagger \):

\[ E^m_\alpha(y, x) = e^a_\alpha(y) \Delta^{-1/5}(y, x), \quad \Delta(y, x) \equiv \frac{\det E^m_\mu}{\det e^m_\mu} \quad (4.8) \]

This is a standard result in KK reduction of theories with gravity. The ansatz for \( E^m_\alpha(y, x) \) is also standard. It contains the gauge bosons

\[ E^m_\alpha(y, x) = B^m_\alpha(y, x) E^m_\mu(y, x) \quad (4.9) \]
\[ B^m_\alpha(y, x) = -2B^A_{\alpha B} V^{\mu AB}(x) \quad (4.10) \]

where \( B^A_{\alpha B}(y) \) is the 7 dimensional \( SO(5) \) gauge field and \( V^{\mu AB} \) the corresponding Killing vector. The factor of \(-2\) in the ansatz of \( B^m_\alpha \) comes from \( g = 2m \), and the minus sign is related to the sign convention in the definition of the covariant derivative \( D_\alpha = \partial_\alpha + g B_\alpha \). The only nontrivial step is finding an ansatz for \( E^m_\mu \).

Before we do that, let’s analyze the fermions. The gravitinos \( \Psi_M \equiv E^A_M \Psi_A \) split into \( \Psi_a \equiv E^A_M \Psi_A \) and \( \Psi_m \equiv E^m_M \Psi_A \) (We suppress the 32-dimensional spinor index.) In order to obtain diagonal kinetic terms for the fermions in \( d = 7 \), we proceed analogously to the vielbein and begin by introducing fields \( \Psi_a(y, x) \) and \( \Psi_m(y, x) \) as follows:

\[ \Psi_a = \Delta^{1/10}(\gamma_5)^{-p} \psi_a - A^{-1} \tau_a \gamma_5 \gamma^m \Delta^{1/10}(\gamma_5)^q \psi_m \quad (4.11) \]
\[ \Psi_m = \Delta^{1/10}(\gamma_5)^q \psi_m \quad (4.12) \]
\[ \varepsilon = \Delta^{-1/10}(\gamma_5)^{-p} \varepsilon, \quad \bar{\varepsilon} = \Delta^{-1/10}(\gamma_5)^{-p} \quad (4.13) \]

\( ^\dagger \)More generally, when dimensionally reducing a D dimensional supergravity theory to d dimensions, one has \( E^a(y, x) = e^a_\alpha(y) \Delta^{-1/(d-2)}(y, x) \)
where $A$, $p$ and $q$ will be fixed. The inverse relations, together with our ansatz are given by

$$
\psi_\alpha(y, x) = \Delta^{-1/10}(y, x)(\gamma_5)^p\epsilon^\alpha(y)(\Psi_a(y, x) + A\frac{1}{5}\tau_a\gamma_5\gamma^m\Psi_m(y, x))
$$

$$
\psi_m(y, x) = \Delta^{-1/10}(y, x)(\gamma_5)^{-q}\Psi_m(y, x)
$$

where $\psi_\alpha$ and $\psi_m$ becomes diagonal we find that $A=+1$ and $p = \pm\frac{1}{2}$, $q = \pm\frac{1}{2}$ (four combinations). However, by trying to match the 7 dimensional transformation laws, we will find that only $p = q$ works. The freedom in the signs of $p, q$ corresponds to a rescaling by $\gamma_5$. This freedom will be fixed in the following by the requirement of consistent truncation. We will find that only $p = q = -1/2$ works. (See for instance (4.137) below.) Incidentally we note that in the linearized ansatz used in [36] the other sign was chosen. Since in [36] only the leading fermionic transformation laws were studied, it did not matter at that point which rescaling by $\gamma_5$ was used.

We also note that the rotation in (4.11, 4.12) is the only one which diagonalizes the action for the seven dimensional gravitini and spin 1/2 fermions. If we try the more general rotation

$$
\Psi_a = A(\gamma_5)^p\psi_a + B\tau^a\gamma^m(\gamma_5)^q\psi_m
$$

$$
\Psi_m = C(\gamma_5)\psi_m + D\gamma_m(\gamma_5)^\epsilon\tau^a\psi_a
$$

by requiring a diagonal kinetic action for the gravitino and spin 1/2 fermions we recover (4.11) and (4.12).

In the field redefinitions, $\gamma_5^{-1/2}$ in $\psi_\alpha$ is needed to cancel the $\gamma_5$ coming from $\Gamma^{\alpha\beta\gamma} = \tau^{\alpha\beta\gamma} \otimes \gamma_5$ in the gravitino action; $\Delta^{-1/10}$ is needed to bring the gravitino action to the usual Rarita-Schwinger form with no extra powers of $\Delta$ (which would come from $\det E = \det e^{(7)}(\gamma_5)\Delta^{-1/7/5}\det e^{(4)}$); finally, the rotation of $\Psi_a, \Psi_m$ into $\psi_\alpha, \psi_m$ is needed to cancel the mixed terms $\Psi_a\Gamma^{\alpha\beta\gamma}D_\beta\Psi_m$. Since $\delta\Psi_A = D_A\epsilon +$ more, we need factors $\gamma_5^{1/2}$ and $\Delta^{-1/10}$ in $\epsilon$. For the linearized case we reobtain (2.31).

The $U$ matrix in the expansions of $\psi_\alpha, \psi_m$ and $\epsilon$ is a local (x and y dependent) SO(5) matrix in the spinor representation depending on the scalar fields in 7 dimensions $(\Pi_A^\alpha)$.

Various authors [2, 6, 7] found it necessary to add a U matrix for consistency of the truncation of the susy transformation laws, and we shall find the same need (see below (4.30)). One can either rotate the label index $I$ or the spinor index of the Killing spinors.

We choose the former, which is perhaps more natural, since the fermions have an SO(5) composite index, whereas the Killing spinors have naturally an SO(5) gauge index. (de Wit and Nicolai took the alternative rotation of spinor indices by U because they reformulated d=11 sugra in a form with local SU(8) invariance parameterized by an arbitrary $U(x, y)$, and in d=11 there are only spinorial indices are available. When one compactifies, the d=11 SU(8) invariance is fixed, only the 4d part remains unfixed.
There is a difference between the gauge $U = 1$ which gives the usual $d=11$ sugra (in the triangular gauge $E_a^\mu = 0$) and the gauge needed for $S_7$ compactification. Therefore, the gauge fixing produces also for us a given field-dependent matrix $U(y,x)$ which acts on the Killing spinors.

In $d=7$ one can always make $SO(5)$ gauge transformations on the fermions, and this would modify the $y$ dependence of $U(y,x)$ by a gauge factor. As we shall see in (4.22), consistency of our results for the transformation rules requires the matrix $U$ to satisfy the condition

$$U^T I \tilde{\Omega}^I J U' = \tilde{\Omega}^{I'} J' \rightarrow (\tilde{\Omega} \cdot U^T \cdot \Omega)^{I'} J' = -(U^{-1})^{I'} J'$$

(so that $U \tilde{\Omega} U^T = \tilde{\Omega}$; $U^T \tilde{\Omega} U = \tilde{\Omega}$) (4.19)

The need for this relation will occur in various other places as well. Since $\Omega$ is the charge conjugation matrix (see Appendix A.1) and $\tilde{\Omega} = -\Omega^{-1}$, this proves that $U$ is a $SO(5)$ matrix in the spinor representation. (Namely, an $U\text{Sp}(4)$ matrix as it is explained in the Appendix A.2).

Let us now return to $E_m^\mu$. We will determine the ansatz for $E_m^\mu$ by matching $\delta B_{[AB]}^\alpha$ with the expression in 7 dimensional gauged supergravity. The result is given in (4.29).

Since we are imposing the gauge $E_a^\mu = 0$, we need a compensating $SO(10,1)$ Lorentz transformation characterized by an antisymmetric matrix $\Omega_m^a$ in order to stay in this gauge

$$\delta E_a^\mu = 0 = \delta_{\text{SUSY}} E_a^\mu + E_m^\mu \Omega_m^a$$

$$\Omega_m^a = -\frac{1}{2} \bar{\xi} \Gamma^a \Psi_m = -\Omega_m^a$$

(4.20)

This Lorentz transformation acts on all vielbein components, for instance

$$\delta E^\mu_a = \delta_{\text{SUSY}} E^\mu_a + E^\mu_m \Omega_m^a, \text{etc.}$$

(4.21)

We can now require that we get the correct graviton transformation law in $d=7$. Using (2.12), we get from (4.8)

$$\delta' e^a_\alpha = \Delta^{1/5} [\delta(d = 11)E^a_\alpha + \frac{1}{5} E^a_\mu (\delta(d = 11)E^m_\mu E^a_\alpha + E^a_m \Omega_m^a)]$$

$$= \frac{1}{2} \Delta^{-1/10} \bar{\epsilon} \gamma_5 \tau^a \gamma_5 \Psi_\alpha^b \sigma^5 \epsilon^b_\delta + \frac{1}{5} \gamma^m \Psi_m e^a_\alpha$$

$$= \frac{1}{2} \bar{\epsilon} I' \tau_\alpha^a J_\mu U^{I'} J_\eta J' \eta' + ...$$

(4.22)

Since according to (3.3) $\tilde{\eta}' \eta'$ equals $\Omega^{I'} J$, we see the condition (4.19) appearing. Because this is not yet the final form of the vielbein law, we have introduced the notation $\delta(d = 11)$ for the $d=11$ susy transformations and $\delta'$ for the intermediate transformation.

In order to obtain agreement with (4.2), we add a field dependent $SO(6,1)$ rotation to the $d=11$ transformation laws. Adding a term $\frac{1}{5} \tau^{ab} \gamma^m \Psi_m e^b_\alpha$ inside the brackets, we obtain

$$\delta e^a_\alpha = \frac{1}{2} \bar{\epsilon} \tau^a \psi_\alpha = \frac{1}{2} \bar{\epsilon} I' \tau^a \psi_\alpha J \Omega^{I'} J + \frac{1}{2} \bar{\epsilon} I' \tau^a \psi_\alpha J (4.23)$$
This is the correct 7 dimensional transformation law of the $d=7$ vielbein.  

The transformation law for the gauge fields follows from (4.9). We need the compensating Lorentz transformation in (4.20) on both vielbeins in (4.9), but there is no SO(6,1) Lorentz transformation since $B_\alpha^i$ has no Lorentz index. One finds

$$\delta B_\alpha^i(y,x) = \delta(d=11)(E_\alpha^m E_\mu^m) + E_\alpha^m (E_\alpha^\mu \Omega^\alpha_m) + (E_\alpha^\mu \Omega_m^\alpha) E_\mu^m$$

$$= \frac{i}{2} \Delta^{-1/5} E_\mu^m \{-\bar{\psi}_\alpha \gamma^m \psi_\alpha - i \bar{\psi}_\alpha (\delta^{m n} + \frac{\gamma^m \gamma^n}{5}) \gamma_5 \psi_\alpha\}$$

(4.24)

where we used (A.9) in the last step. But in 7 dimensions, we have from (4.3)

$$\delta B_\alpha^{AB}(y) = (\Pi^{-1})_i^A (\Pi^{-1})_j^B \left[ \frac{1}{4} \bar{\psi}_\alpha \gamma^{ij} \psi_\alpha + \frac{1}{8} \bar{\psi}_\alpha \gamma^k \gamma^{ij} \lambda_k \right]$$

(4.25)

By multiplying (4.25) with $-2V_{AB}^m$ and equating the first term in (4.24) and (4.25), one obtains

$$-E_\mu^m \bar{\eta}^I \gamma^m \eta^J U^I U^J \bar{\epsilon}_I \psi_\alpha U^J = \frac{i}{2} \Delta^{-1/5} (\Pi^{-1})_i^A (\Pi^{-1})_j^B V_{AB}^\mu (\bar{\gamma}^{ij}) U^I U^J \bar{\psi}_\alpha U^J$$

(4.26)

Dropping a common factor $1/2 \bar{\epsilon}_I \psi_\alpha U^I$, we find the equation

$$i E_\mu^m (U V^m U^T)^{I^I J^J} = -\Delta^{1/5} (\Pi^{-1})_i^A (\Pi^{-1})_j^B V_{AB}^\mu (\bar{\gamma}^{ij}) U^I U^J$$

(4.27)

This equation can be solved for $E_\mu^m$ by multiplication with ($U V^m U^T$)$_{I^I J^J}$ and using (4.19) and (3.26)

$$E_\mu^m = \frac{i}{4} \Delta^{1/5} (\Pi^{-1})_i^A (\Pi^{-1})_j^B V_{AB}^\mu Tr(\bar{\gamma}^{ij} U V^m U^T \Omega)$$

(4.28)

(We used that $U \tilde{\Omega} U^T = \tilde{\Omega}$ implies that $U^T \Omega U = \Omega$ as follows from eliminating $U^T$).

To obtain an explicit expression for $E_\mu^m$, we move the factors $\Pi^{-1}$ and $\bar{\gamma}^{ij}$ in the r.h.s. of (4.27) to the left. Contracting with $V_{\nu AB}^I$ and $E_\mu^m$ and using (3.26), we find

$$E_\nu^m = \frac{i}{4} \Delta^{-1/5} Tr(\gamma_{ij} U V^m U^T \Omega) \Pi_i \Pi_j \nu_{\nu AB}$$

(4.29)

As a check on our results developed so far, we note that substituting (4.28) into (4.27) one should get an identity. We find

$$\frac{1}{4} (\Pi^{-1})_i^A (\Pi^{-1})_j^B V_{AB}^\mu [Tr(\gamma_{ij} U V^m U^T \Omega)] (U V^m U^T)^{I^I J^J} = (\Pi^{-1})_i^A (\Pi^{-1})_j^B V_{AB}^\mu (\bar{\gamma}^{ij}) U^I U^J$$

(4.30)

The matrices $U^{I^I J^J}$ are really needed for the consistency of (4.27). If one sets $U = 1$ in (4.30) and removes the factors $\Pi^{-1} y$ one is left with an x-dependent equation for the Killing vectors which is incorrect. When $U$ is y-dependent, one cannot factor off the $\Pi^{-1}$ and the relation becomes a condition on $U$. It will follow from the fundamental condition on $U$ which we obtain in (4.45).

---

The $d=7$ susy results are thus a combination of $d=11$ susy and $d=7$ Lorentz symmetry. Excluding 3-fermion terms, the $d=7$ Lorentz transformations do not show up anywhere else.
The 7 dimensional transformation law is equal to

\[ \bar{\eta}^I (\delta^{mn} + \frac{1}{3} \gamma^m \gamma^n) \gamma_5 n^{JKL} = 3 (\phi_5^{IJ} C_m^{KL} - C_m^{IJ} \phi_5^{KL}) + \frac{24}{5} V_m^{[I}[L} \Omega_{K]J} - \frac{6}{5} V_m^{IJ} \bar{\Omega}^{KL} \]  

(4.31)

where we have substituted the expression of \( n_m^{JKL} \) in (3.8) and used the Fierz relation (3.42). However, when projected on square (4.27). Using (4.19), we obtain

\[ \lambda \]  

will contribute because \( \lambda \) is symplectic traceless. Thus, the \( \psi_m \) term in the

d = 11 transformation law of the gauge field is:

\[ \delta B^\mu_A \frac{\psi_m}{\psi_m} = \frac{3}{2} E_m^\mu \Delta^{-1/5} \epsilon_I \tau_\alpha \lambda_{J'K'L'} U_{I'} U_{J'} U_{K'} U_{L'} \]

\[ (\phi_5^{IJ} (C_m)^{KL} - \phi_5^{KL} (C_m)^{IJ}) \]  

(4.32)

where we have used the ansatz in (4.15) for \( \psi_m \). Using \( Y^A = \frac{1}{3} \phi_5^{IJ} (\gamma^A)_{IJ} \) and \( C_m^B = \partial_m Y^B \), it follows from (3.24) that this expression is proportional to the Killing vector \( V_{AB}^\mu \), but there are still the extra \( U \) matrices. By starting with (3.42) contracted with \( \bar{\eta}^I \gamma_m \), we obtain the second line in (4.32) and a complicated term of the form \( (\bar{\eta} \gamma_{mn})^{KL} C_m^{IJ} \), which however vanishes since \( \lambda_{J'K'L'} \) is antisymmetric in \( K'L' \). We get

\[ \delta B^\mu_A \frac{\psi_m}{\psi_m} = -3 E_m^\mu \Delta^{-1/5} \epsilon_I \tau_\alpha \lambda_{J'K'L'} \bar{\Omega}_{I'} \bar{\Omega}_{J'} \epsilon_I \tau_\alpha \lambda_{J'K'L'} V_{AB}^\mu \]  

(4.33)

Using (4.27) we arrive at

\[ \delta B^\mu_A \frac{\psi_m}{\psi_m} = -3 i (\Pi^{-1})_i^A (\Pi^{-1})_j^B (\gamma^i)_{J'} (\gamma^j)_{K'} \bar{\Omega}_{I'} \bar{\Omega}_{J'} \epsilon_I \tau_\alpha \lambda_{J'K'L'} V_{AB}^\mu \]  

(4.34)

Since we have arrived at an expression proportional to \( V_{AB}^\mu \) we can compare with (4.25). The 7 dimensional transformation law is equal to

\[ -\frac{1}{4} (\Pi^{-1})_i^A (\Pi^{-1})_j^B \bar{\epsilon} \tau_\alpha \gamma^i \gamma^j \lambda_k V_{AB}^\mu \]  

(4.35)

and this should agree with (4.34). The two terms are equal if we assume the following normalization of the seven dimensional spin 1/2 fields:

\[ \lambda_{J'}^k = 3i (\gamma^k)_{J'} \lambda_{J'K'} \]  

(4.36)

At this point we have learned that if (4.27) is correct then the transformation of the gauge field \( B^A_{\alpha} \) follows from Kaluza-Klein reduction.

Next, we compute \( \Delta = det(E_m^\mu)/det(\bar{e}_m^\mu) \). To remove the dependence on \( U \), we square (4.27). Using (4.19), we obtain

\[ \Delta^{-2/5} g^{\mu\nu} = 2 V_{AB}^\mu V_{CD}^\nu (\Pi^{-1})_i^A (\Pi^{-1})_j^B (\Pi^{-1})_i^C (\Pi^{-1})_j^D \]  

(4.37)
The matrix $g^{\mu\nu}$ is the inverse of the metric $g_{\mu\nu} = G_{\mu\nu} = E^M_{\mu} E_{\nu M} = E^m_{\mu} E_{\nu m}$. It is given by $g^{\mu\nu} = E^\mu_{\nu} E^{\nu\mu}$ and differs from $G^{\mu\nu} = E^M_{\mu} E^{\nu M}$ by a term $E^m_{\mu} E^{\nu m}$. Using (3.36) and the fact that $\det T = 1$ we can evaluate directly the determinant of $g_{\mu\nu} = \Delta^{-\delta / 2} \partial_y A (T^{-1})^{AB} \partial_y B$:

$$
\det g_{\mu\nu} \equiv \det T_A^{A_i} \partial_y A (T^{-1})^{AB} \partial_y B \equiv \epsilon^{\mu_1 \ldots \mu_4 \mu_1' \ldots \mu_4} g_{\mu_1 \mu_1'} \ldots g_{\mu_4 \mu_4'}
\Delta^{16/5} \epsilon^{\mu_1 \ldots \mu_4} \partial_y A_1 \ldots \partial_y A_4 \epsilon^{\nu_1 \ldots \nu_4} \partial_y B_1 \ldots \partial_y B_4
\delta T_{A_1 B_1} \ldots T_{A_4 B_4}
= \delta T_{A_1 B_1} \ldots T_{A_4 B_4} = \delta T^{AB} Y_A Y_B
$$

conclude that

$$
\Delta^{-6/5} = (\Pi^{-1})_i^A (\Pi^{-1})_j^B \delta^{ij} Y_A Y_B
$$

(4.39)

The equation (4.39) is the starting point for obtaining the nonlinear metric ansatz for the compact dimensions.

Substituting the 7-d transformation law of the scalar fields in (4.39) we get:

$$
\delta (\Delta^{-6/5}) = \delta T^{AB} Y_A Y_B = 2 \delta (\Pi^{-1})_i^A (\Pi^{-1})_j^B \delta^{ij} Y_A Y_B
\frac{-1}{2} (\Pi^{-1})_i^A (\Pi^{-1})_j^B (\epsilon \gamma^i \lambda^j + \epsilon \gamma^j \lambda^i) Y_A Y_B
$$

(4.40)

where we recall the definition $T_{AB} = (\Pi^{-1})_i^A (\Pi^{-1})_j^B \delta^{ij}$ and used (4.5). On the other hand, from 11-d sugra we obtain

$$
\delta (\Delta^{-6/5}) = \frac{6}{5} \Delta^{-6/5} \delta E^\mu \cdot E^m = -\frac{3}{5} \Delta^{-6/5} \epsilon T^{mn} \Psi_m
$$

(4.41)

Again we find two expressions which should be equal, but whose spherical harmonics are not yet manifestly the same. We rewrite the spherical harmonic on the r.h.s. of (4.41) by using the Fierz relation (3.42)

$$
\bar{\eta}^I \gamma_m \gamma_5 \eta_m^{JKL} = -(5 \phi_5^{IJ} \phi_5^{KL} + \phi_5^{LJ} \phi_5^{KL} - 4 \phi_5^{[L} \phi_5^{K]} J)
$$

(4.42)

Again, only the terms with $\phi_5^{IJ} \phi_5^{KL}$ contribute because $\lambda_{J'K'}$ is symplectic-traceless and one finds ($\phi_5 \sim Y_A \gamma^A$)

$$
\delta (\Delta^{-6/5}) = -3 i \Delta^{-6/5} Y_A \bar{\gamma} C \bar{\epsilon} I \lambda_{J'K'L'} (U \gamma^A \bar{\Omega} U^T)^{J'K'L'} (U \gamma^C \bar{\Omega} U^T)^{K'L'}
$$

(4.43)

There are four $U$ matrices, three from $\psi_m$ and one from $\epsilon$. Substituting (4.43) we arrive at

$$
\delta (\Pi^{-1})_i^C \cdot (\Pi^{-1})_j^D \delta^{ij} Y_C Y_D
\frac{-1}{2} \Delta^{-6/5} Y_A \bar{\gamma} C \bar{\epsilon} I \lambda_{J'K'L'} (U \gamma^A \bar{\Omega} U^T)^{J'K'L'} (U \gamma^C \bar{\Omega} U^T)^{K'L'}
$$

(4.44)

In order that this result for $\delta \Pi^{-1}$ agrees with (4.40), the $U$-matrices must satisfy:

$$
Y_A (U \gamma^A \bar{\Omega} U^T)^{J'K'L'} = \pm \Delta^{3/5} (\Pi^{-1})_i^A (\gamma^i)^{J'K'L'} Y_A
$$

(4.45)

where we used the normalization relation (4.36) to remove the spinors. We will choose the minus sign, because we want that our matrix $U$ is equal to 1 on the background.
(and not to $-1$). Equation (4.45) will turn out to be the most important tool in our endeavor to find a consistent truncation of the 11 dimensional supergravity.

At this point we have come in 7 dimensions as far as de Wit and Nicolai in their first approach without a local $SU(8)$. (Actually, a relation corresponding to our (4.45) is not present in their work, but they did find the one corresponding to (4.30).) We have been able to find solutions for $U$ without having to extend the $d=11$ theory. First of all, we have been able to show that (4.30) follows from (4.45), so that (4.45) is the crucial equation. Secondly, we have found solutions to (4.45).

The proof that (4.45) implies (4.30) goes as follows. One begins with the l.h.s. of (4.45)

$$
\frac{1}{4}(\Pi^{-1})_i^A(\Pi^{-1})_j^B V_{\mu,AB} Tr(\gamma^{ij} U \gamma^{CD} U^{-1}) (U \gamma^{EF} U^{-1})_{i'}^{i'} V_{mCDV^mEF} (4.46)
$$

where one substitutes (3.29) to sum over the Killing vectors. Using $V_{\mu,AB} = Y^A[C_\mu^B]$ and writing $2\gamma^{ij} = \gamma^i \gamma^j - \gamma^j \gamma^i$, the r.h.s. of (4.45) appears twice, and using (4.45) once we obtain

$$
-\frac{1}{4} \Delta^{-3/5}(\Pi^{-1})_i^A V_{Y^A} Y^A (\Pi^{-1})_{i'}^{i'} (4.47)
$$

In the product of gamma matrices only the term $\delta^{AC} \gamma^D$ survives due to symmetry arguments. Splitting $(U \gamma^{ED} U^{-1})_{i'}^{i'}$ into $(U (\gamma^E \gamma^D - \delta^{ED}) U^{-1})_{i'}^{i'}$, eliminating the first $U^{-1}$ by (4.19) and applying (4.45), but now in the opposite direction, we get:

$$
\frac{1}{4}(\Pi^{-1})_i^A(\Pi^{-1})_j^E C_{\mu A} Y^A C_{\mu B} \delta^{ij} \tilde{\Omega}_{i'}^{i'} - \frac{1}{4}(\Pi^{-1})_i^A(\Pi^{-1})_j^E C_{\mu A} Y^A C_{\mu B} \delta^{ij} \tilde{\Omega}_{i'}^{i'} (4.48)
$$

What is left to do is the sum over $\gamma^D$ (use (A.12)) in order to get directly the r.h.s. of (4.30) (with $\gamma^{ij}$ written again as $\gamma^i \gamma^j - \delta^{ij}$).

Finally, we look at the solutions of equation (4.45) for the matrix $U$. The $4 \times 4$ matrix $U$ can be expanded into the complete basis $1, \gamma^A, \gamma^A \gamma^B$ as follows

$$
U = N + N_A \gamma_A + N_{AB} \gamma_{AB} (4.49)
$$

Then, the condition that it is an $SO(5)$ matrix in the spinor representation, equation (4.19), says that

$$
U^{-1} = -\tilde{\Omega} U^T \Omega = N + N_A \gamma_A - N_{AB} \gamma_{AB} (4.50)
$$

Unitarity of $U$ implies that $N, N_A$ and $N_{AB}$ are real. The fact that $UU^{-1} = U^{-1}U = 1$ gives then the conditions

$$
N^2 + N_A^2 + 2N_{AB}^2 = 1 (4.51)
$$

$$
N_A N_{AB} = 0 (4.52)
$$

$$
2NN_A = \epsilon_{ABCDEF} N_{BC} N_{DE} (4.53)
$$

On the other hand (4.45), which we can rewrite as

$$
UY^i = \tilde{\varphi} U, \quad \tilde{\varphi} = (4.54)
$$

$$
v_i = \Pi^{-1} A Y^A \Delta^{3/5} \quad ; \quad v^2 = Y^2 = 1 (4.55)
$$

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implies the equations

\[ N_A(Y_A - v_A) = 0 \]  \hspace{1cm} (4.56)
\[ N(Y_A - v_A) = -2N_{AB}(Y_B + v_B) \]  \hspace{1cm} (4.57)
\[ N_A(Y + v_B) = \frac{1}{2}\varepsilon_{ABCDE}(Y_C - v_C)N_{DE} \]  \hspace{1cm} (4.58)

The first observation is that if \( U \) is an SO(5) solution of \( UY' = \neq \) then also \( UY' \) is a solution. But we want solutions which are equal to 1 on the background, so we will disregard the solutions which are equal to \( Y' \) on the background. From (4.57) we get that

\[ N = \frac{2N_{AB}v_AY_B}{1 - v \cdot Y} \]  \hspace{1cm} (4.59)

whereas by multiplying (4.58) with \( Y_B + v_B \) we get

\[ N_A = \frac{\epsilon_{ABCDE}v_BY_CN_{DE}N_A}{1 + v \cdot Y} + \frac{N'}{2(1 + v \cdot Y)} \]  \hspace{1cm} (4.60)

where at this point \( N' \) is arbitrary. This satisfies (4.56) automatically. If we take \( N \neq 0 \), then from (4.52) we obtain after contraction with \( Y_B \)

\[ N' = -\frac{2}{N_{A'B'}v_AY_B}\epsilon_{ABCD}v_BY_CY_A \]  \hspace{1cm} (4.61)
\[ N_A = \frac{\epsilon_{BCD}v_BY_CN_{DE}N_{AF}Y_F}{1 + v \cdot Y} \]  \hspace{1cm} (4.62)

(Notice that if \( N = 0 \) one divides in (4.61) by zero, so the case \( N = 0 \) should be treated separately). The solutions which are equal to 1 on the background (i.e. have \( N \neq 0 \)) are parameterized by an antisymmetric matrix \( N_{AB} \). The SO(5) conditions become now constraints on \( N_{AB} \)

\[ \epsilon_{ABCD}v_BY_CY_A'N_{DE}N_{AF} \left( \delta_{FG} - \frac{v_A}{N_{A'B'y_A'y_F}} \right) = 0 \]  \hspace{1cm} (4.63)
\[ \epsilon_{ABCD}v_BY_CY_A'N_{DE} \left( \frac{1 - (v \cdot Y)^2}{2(N_{A'B'y_A'y_F})}N_B - 2v_BY_C \right) = -2(Y_A + v_A)\epsilon_{BCD}v_BY_CY_A'N_{DE}N_{FG}Y_G \]  \hspace{1cm} (4.64)
\[ N^2 + N_A^2 + 2N_A^2 = 1 \]  \hspace{1cm} (4.65)

where the last condition fixes the normalization. The \( UY' = \neq \) conditions yield further constraints on \( N_{AB} \)

\[ \frac{N_{BC}v_BY_C}{1 - v \cdot Y}(Y_A - v_A) + N_{AB}(Y_B + v_B) = 0 \]  \hspace{1cm} (4.66)
\[ (Y + v)_{(F}v_{A]BCD}v_BY_CY_A'N_{DE} \left( \frac{1}{1 + v \cdot Y} \right) = \frac{1}{2}\epsilon_{AFCDE}(Y_C - v_C)N_{DE} \]  \hspace{1cm} (4.67)

The most general solution of these remaining five equations can be obtained by noting that in 5d space we have, besides \( Y_A \) and \( v_A \), another 3 independent vectors
\[ Z_1, Z_2, Z_3. \text{ We can then make out of them an orthonormal set, also orthogonal to } Y \text{ and } v. \text{ Then the most general solution for } N_{AB} \text{ will be written as} \]
\[ N_{AB} = \sum_{m \neq n=1}^{5} a_{mn} X_m^A X_n^B \]  
\[ (4.68) \]
where \( \{X_i^A\} = \{Y^A, v^A, Z_1^A, Z_2^A, Z_3^A\} \). We define
\[ S_A = \frac{Y_A + v_A}{\sqrt{2(1 + v \cdot Y)}} \]  
\[ D_A = \frac{Y_A - v_A}{\sqrt{2(1 - v \cdot Y)}} \]  
\[ (4.69) \]
\[ (4.70) \]
satisfying \( S^2 = D^2 = 1, S \cdot D = 0. \) Then (4.66) and (4.67) yield (after multiplying (4.67) with \( \epsilon_{AFC'D'E'} \))
\[ (S \cdot N \cdot D) D_A + N_{AB} S_B = 0 \]  
\[ (4.71) \]
\[ D_{[C'} N_{D'E']} + 4 S_D N_{[E'SC'D']} D_{[E']} = 0 \]  
\[ (4.72) \]
Substituting (4.71) into (4.72) implies
\[ D_{[C'} N_{D'E']} = 0, \text{ of which the most general solution is} \]
\[ N_{AB} = D_{[A} X_{B]} \]  
\[ (4.73) \]
with \( X_A \) an arbitrary vector. Then (4.63) and (4.64) are satisfied trivially, because \( \epsilon_{ABCDE} Y^B D^C X^D = 0 \), whereas (4.65) fixes the normalization. We notice that \( D_A, S_A, Z_1A, Z_2A, Z_3A \) make an orthonormal set, and any component of \( X_A \) along \( D_A \) will drop out of \( N_{AB} \), so that we have
\[ X_A = a S_A + b_i Z_i^A \]  
\[ (4.74) \]
The normalization condition (4.65) gives
\[ \sum b_i^2 + a^2 v \cdot Y = 1 \]  
\[ (4.75) \]
Thus the most general solution contains 3 parameters. We will now look at particular cases. We note that the covariant vectors at our disposal are of the form
\[ X_i^{(z)} = \frac{[(\Pi^{-1})^z]_i^A Y_A}{Y \cdot T^z \cdot Y} \]  
\[ (4.76) \]
where \( z \in \mathbb{Z} \) and if \( \Pi_A^i \neq \delta_A^i \), then the \( X^{(z)} \)'s will generate the whole sphere (they will not lie in a lower dimensional hyperplane), so we can build \( Z_1, Z_2, Z_3 \) out of 3 suitably chosen \( X_i^{(z)} \)'s \( (z \neq 0, 1) \).

The simplest possibility is to build \( U \) only out of \( Y^A \) and \( v^A \). Then \( N_{AB} = av_{[A} Y_{B]}, \) so that \( N = a(1 + v \cdot Y) \) and \( N_A = 0. \) The normalization condition (4.51) fixes then \( a = \pm (2(1 + v \cdot Y))^{-1/2}. \) The unique covariant solution built out of only \( Y_A \) and \( v_A \) then reads
\[ U^{(1)} = \frac{1 + v \cdot Y + v_A Y_B \gamma_{AB}}{\sqrt{2(1 + v \cdot Y)}} = \frac{1 + \gamma Y}{\sqrt{2(1 + v \cdot Y)}} \]  
\[ (4.77) \]
It clearly satisfies $UY_f = \not \! U$ and is and $USp(4)$ matrix.

We note that the general solution (4.73) can be rewritten as

$$U(X) = \pm \frac{X \cdot (Y + v) + X^A(Y^B - v^B)\gamma_{AB}}{\sqrt{2}\sqrt{1 - v \cdot Y + 2X \cdot Y \cdot X \cdot v}} = \frac{XY \ + \not \! Y}{\sqrt{2}\sqrt{1 - v \cdot Y + 2X \cdot Y \cdot X \cdot v}}$$

(4.78)

where $X$ now is still arbitrary, but has unit norm. We can easily check that if $\vec{X}$ is in the $(Y,v)$ plane we reproduce $U(1)$. However, now we can take $X$ to be a noncovariant vector, which we can take to be $X = (0,0,0,1)$. Then

$$U_{\text{noncov}} = \frac{\gamma_5 Y \ + \not \! \gamma_5}{\sqrt{2}\sqrt{1 - v \cdot Y + 2Y_5v_5}} = \frac{Y_5 + v_5 - i\gamma_m(Y_m - v_m)}{\sqrt{(Y_5 + v_5)^2 + (Y_m - v_m)^2}}, \ m = 1, 4$$

(4.79)

which can be written as an exponent

$$U = \exp[-i\gamma_m(Y_m - v_m)u]; \quad u = [(Y_m - v_m)^2]^{-1/2} \arctan - \frac{\sqrt{(Y_m - v_m)^2}}{Y_5 + v_5}$$

(4.80)

We can also generate the most general noncovariant solution made out of only $Y^A$ and $v^A$, by choosing

$$X = \frac{Y + \beta v + \alpha(\vec{0},1)}{\sqrt{1 + \beta^2 + \alpha^2 + 2\alpha\beta v \cdot Y + 2\alpha Y_5 + 2\alpha\beta v_5}}$$

(4.81)

to obtain

$$U_{\text{noncov}}^{(X)} = \frac{1 + \not \! Y + \alpha'(\gamma_5 Y \ + \not \! \gamma_5)}{\sqrt{2}\sqrt{1 + v \cdot Y + 2\alpha(Y_5 + v_5) + \alpha^2(1 - v \cdot Y + 2Y_5v_5)}}$$

(4.82)

where $\alpha' = \alpha/(\beta + 1)$ is an arbitrary parameter. We note that taking $\alpha'$ to infinity, we get the maximally noncovariant solution $U_{\text{noncov}}$.

Maybe it’s interesting to analyze what happens at the linearized level. Since $(\Pi^{-1})^i_A = (e^{-\delta \pi^A})^i_A$, the vectors $X_i^{(z)} = [(\Pi^{-1})^z]_i^A Y_A / Y \cdot T^z \cdot Y$ become $X_i^{(z)} \simeq Y^A - z\delta v_A$, where $\delta v_A = y_A Y \cdot \delta \pi \cdot Y - \delta \pi_{AB} Y_B$. And so the only vectors at our disposal are $Y^A$ and $\delta v_A$, so the most general solution for $U$ which is equal to 1 on the background is obtained from $U_{\text{noncov}}^{(X)}$

$$U \simeq 1 + \frac{1}{2(1 + \alpha Y_5)} (\delta v_A Y_B \gamma_{AB} + \alpha'\delta v_A \gamma_{A5})$$

$$= 1 + \frac{1}{2} \delta v_A Y_B \gamma_{AB} + \frac{\alpha_1}{2} Y^m C_m^\alpha A \delta v_B \gamma_{AB}$$

(4.83)

with $\alpha_1$ parameterizing the deviation from covariance. Actually the most general linearized solution is found by adding also a term $\beta Y^m C_m^\alpha A \delta v_B \gamma_{AB} Y^B$; because of the linearity of the equations we can add the arbitrary noncovariant piece coming from the $UY_f$ solution.

We end this section with a discussion of the metric. It is independent of $U$ and has simple geometrical properties as we shall see. From (4.28) and (4.45) one can in principle obtain an expression for the vielbein. Squaring this result would then lead
to an expression for the metric $g_{\mu\nu}$. It is easier to obtain the result for $g_{\mu\nu}$ by directly verifying that it is the inverse of $g^{\mu\nu}$ in (4.37).

$$g_{\mu\nu} = \Delta^{4/5} C_{\mu}^A C_{\nu}^B T_{AB}^{-1}$$

$$g^{\mu\nu} = \Delta^{2/5} \left( C_{A}^{\mu} C_{B}^{\nu} T^{AB} Y_C Y_D T^{CD} - C_{A}^{\mu} Y_B T^{AB} C_{B}^{\nu} Y_D T^{CD} \right)$$

Since we are in the gauge where $E_{\mu}^{a} = 0$, one easily checks that $G_{\mu\nu} g^{\nu\rho} = g_{\mu\nu} g^{\nu\rho} = \delta_{\mu}^{\rho}$. Recalling that $C^{A}_{\mu} = D_{\mu} Y^{A} = \partial_{\mu} Y^{A}$, we see that the metric $g_{\mu\nu} = G_{\mu\nu}$ describes an ellipsoid with a conformal factor $\Delta^{4/5}$, whose axes at a specific point $y$ in the $d=7$ space time are determined by the eigenvalues of $T_{AB}^{-1}$.

To summarize, the metric of the internal space reads

$$ds_{4}^{2} = G_{\mu\nu} dx^{\mu} dx^{\nu} = \Delta^{4/5} T_{AB}^{-1} \partial_{\mu} Y^{A} \partial_{\nu} Y^{B} dx^{\mu} dx^{\nu}$$

$$G_{\alpha\mu} dx^\alpha dx^\mu = 2\Delta^{4/5} T_{AB}^{-1} Y^{C} B^{BC} dY^{A}$$

while the metric of the 7 dimensional space-time is

$$ds_{7}^{2} = G_{\alpha\beta} dy^{\alpha} dy^{\beta}$$

$$= \Delta^{-2/5} g_{\alpha\beta} dy^{\alpha} dy^{\beta} + 4\Delta^{4/5} Y_{A} T_{BD}^{-1} B^{BC} Y_{A} B^{CD} Y_{C}$$

where $G_{\alpha\beta}$ denotes the 11 d metric. Then we can write the 11 dimensional metric in a concise form

$$ds_{11}^{2} = G_{\alpha\beta} dx^{\alpha} dx^{\beta}$$

Note that, since after compactification the gauge coupling is $g = 2m$, and we set $m = 1$, the gauge covariant derivative is $dY + 2B \cdot Y \equiv DY$. Thus the metric is manifestly gauge invariant

$$ds_{11}^{2} = \Delta^{-2/5} g_{\alpha\beta} dy^{\alpha} dy^{\beta} + 4\Delta^{4/5} (DY)^{A} T_{AB}^{-1} (DY)^{B}$$

For later use for the gravitino transformation law, we rewrite the vielbein and its inverse in a simpler form, involving only the conformal Killing vectors

$$E_{\mu}^{m} = \frac{1}{4} \Delta^{2/5} \Pi_{A}^{i} C^{A}_{\mu} C^{mB} Tr(\Pi^{-1} \gamma^{i} U \gamma_{B})$$

$$E^{\mu n} = \frac{1}{4} (\Pi^{-1})_{A}^{i} C^{A}_{\mu} C^{nB} \Delta^{-2/5} Tr(\Pi^{-1} \gamma^{i} U \gamma_{B})$$

For the derivation of this result one may use (4.111).

### 4.2 The ansatz for the 11 dimensional antisymmetric tensor and auxiliary field and self-duality in odd dimensions

At this point we have obtained the complete nonlinear ansatz for the metric and the gravitino and we checked the transformation rules of the $d=7$ bosonic fields which are.
embedded in the d=11 vielbein, viz. the graviton, the gauge fields and the scalars. We now turn to the antisymmetric tensor $A_{\Lambda\Pi\Sigma}$.

We begin by presenting the final form for our ansatz for the KK reduction on $S_4$ of the field strength $F_{\Lambda\Pi\Sigma\Omega}$; a discussion of how we arrived at this result will be given afterwards.

\[
\frac{\sqrt{2}}{3m} F_{\mu_\nu_\rho_\sigma} = \epsilon_{\mu_\nu_\rho_\sigma} \sqrt{\det g} \left[ 1 + \frac{1}{3} \left( \frac{T}{Y_A Y_B T^{AB}} - 5 \right) - \frac{2}{3} \left( \frac{Y_A (T^2)^{AB} Y_B}{(Y_A T^{AB} Y_B)^2} - 1 \right) \right] \quad (4.92)
\]

\[
\frac{\sqrt{2}}{3m} F_{\mu_\nu_\alpha_\beta} = \partial_{\mu} \left( \epsilon_{ABCDE} B^A_{\alpha} C^B_{\beta} C^C_{\gamma} C^D_{\delta} T^{E Y_F} \frac{Y^{T Y_G}}{Y^{T Y_H}} \right) + \frac{\sqrt{g} \epsilon_{\mu_\nu_\rho_\sigma} \epsilon_{\alpha_\beta_\gamma_\delta}}{3} \left( \frac{\partial_\beta T^{AB} Y_B}{Y_A T^{AB} Y_B} - \frac{T^{AB} Y_B}{Y_A T^{AB} Y_B} \right)^2 (Y_C \partial_\alpha T^{CD} Y_D) \quad (4.93)
\]

\[
\frac{\sqrt{2}}{3m} F_{\mu_\alpha_\beta_\gamma} = \partial_{\mu} A_{\alpha_\beta_\gamma} + \frac{4}{3} \partial_{\mu} \left( \epsilon_{ABCDE} B^A_{\alpha} B^B_{\beta} C^C_{\gamma} C^D_{\delta} Y_F T^{E Y_H} \frac{Y^{T Y_G}}{Y^{T Y_H}} \right) - \frac{2}{3} \partial_{\alpha} \left( \epsilon_{ABCDE} B^A_{\beta} B^B_{\gamma} C^C_{\delta} Y_F C^D_{\eta} T^{E Y_H} \frac{Y^{T Y_G}}{Y^{T Y_H}} \right) + \partial_{\mu} \left( \epsilon_{ABCDE} \partial_{\alpha} B^A_{\beta} + \frac{4}{3} B^A_{\beta} B^B_{\beta} B^C_{\delta} B^D_{\gamma} Y_F T^{E Y_H} \frac{Y^{T Y_G}}{Y^{T Y_H}} \right) \quad (4.94)
\]

\[
\frac{\sqrt{2}}{3m} F_{\alpha_\beta_\gamma_\delta} = 4 \partial_{\alpha} A_{\beta_\gamma_\delta} + 4 \partial_{\alpha} \epsilon_{ABCDE} \left( \frac{4}{3} B^A_{\beta} B^B_{\gamma} B^C_{\delta} B^D_{\epsilon} Y_F T^{E Y_H} \frac{Y^{T Y_G}}{Y^{T Y_H}} \right) + (\partial_{\beta} B^A_{\gamma} + \frac{4}{3} B^A_{\gamma} B^B_{\gamma} B^C_{\delta} B^D_{\gamma} Y_F T^{E Y_H} \frac{Y^{T Y_G}}{Y^{T Y_H}}) \quad (4.95)
\]

The independent fluctuation, $A_{\alpha_\beta_\gamma}$, mixes with the auxiliary field $B_{\alpha_\beta_\gamma_\delta}$ where $k$ is an arbitrary scale factor which will be fixed below (see below (5.31). All the other components of $B_{\Lambda\Pi\Sigma\Omega}$ are set to zero because in d=11 they vanish on-shell, so that, in order that they also vanish on d=7 on-shell, they would have to be proportional to d=7 field equations. Field equations of the form $\partial^2 \phi$ are ruled out because they would lead to $(\partial^2 \phi)^2$ terms in the action. In principle, the other components of $B_{\Lambda\Pi\Sigma}$ could still depend on the field equation for $S_{\alpha_\beta_\gamma}$ because that one is linear in
derivatives. However, this produces incorrect answers for the susy transformation rules of the fermions, as we shall show.

Both $A_{\alpha \beta \gamma}$ and $\tilde{B}_{\alpha \beta \gamma}$ will be written as $A_{\alpha \beta \gamma} \sim S_{\alpha \beta \gamma, A}(y)Y^A(x)$ and $B_{\alpha \beta \gamma} \sim T^{AB}S_{\alpha \beta \gamma, A}Y^B$. Since they contain the same spherical harmonic $Y^A$ they will mix. This mixing of $A_{\alpha \beta \gamma}$ with $\tilde{B}_{\alpha \beta \gamma}$ is needed to convert the second order field equation obeyed by $A_{\alpha \beta \gamma}$ into the first order one obeyed by the antisymmetric tensor field of 7 dimensional gauged sugra, $S_{\alpha \beta \gamma, A}$.

The ansatz for $F_{\Lambda \Omega \Sigma \Pi}$ is obtained by requiring consistency of susy laws. It is the same as the geometrical ansatz of [47] if the scalars are set to zero. One would expect that this should be the case, since the ansatz proposed by the authors of [47] was constructed such that the Chern-Simons terms in the 7-dimensional action are obtained after integrating the Chern-Simons term of the 11-dimensional action on $S_4$. The two terms in $F_{\mu \nu \rho \sigma}$ and $F_{\mu \nu \rho a}$ which only depend on scalar fields are components of a separately closed form. We note that any closed form $f_4$ which appears only in the $F_{\mu \nu \rho \sigma}$, $F_{\mu \nu \rho a}$ sector, must be of the form $f_{\mu \nu \rho a} = \epsilon_{\mu \nu \rho a} f$, $f_{\mu \nu \rho a} = \epsilon_{\mu \nu \rho a} D^a D_{\alpha} f$ and will not contribute in the 11 dimensional Chern-Simons action $\epsilon_{FFA}$. This is so because, if we work for simplicity in 12 dimensions (where the Chern-Simons form becomes $\epsilon_{FFA}$), then $f_4$ contributes only to a term of the type

$$e^{\mu \nu \rho \sigma} \epsilon_{\alpha_1 \ldots \alpha_8} (f_{\mu \nu \rho a} F_{\alpha_1 \ldots \alpha_4} F_{\alpha_5 \ldots \alpha_8} - 32 f_{\mu \nu \rho a} F_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} F_{\alpha_5 \ldots \alpha_8})$$

which vanishes if we partially integrate the $D_{\alpha_1}$ in $f_{\mu \nu \rho a}$, use the Bianchi identity, and then partially integrate back the resulting $D_{\alpha}$. The precise expression of the ansatz in the $F_{\mu \nu \rho \sigma}$ sector is highly constrained. It must reproduce the linearized term in (2.32), and it must yield the correct scalar potential in d=7 after integrating over the compact space. In order to perform this integral to which both the Einstein action and the kinetic action of the 3-index photon contribute we start with the d=4 scalar curvature associated with the conformal metric $\tilde{g}_{\mu \nu} = \Delta^{-4/5} g_{\mu \nu}$, namely

$$\tilde{R}^{(4)} = \tilde{g}^{\mu \nu} R_{\lambda \mu \nu} \lambda = -2 Y_A Y_B (T^3)^{AB} + 2 T Y_A Y_B (T^3)^{AB} + Y_A Y_B T^{AB} (Tr(T^2) - T^2)$$

and its relation with the d=4 scalar curvature

$$R^{(4)} = \Delta^{-4/5} \tilde{R}^{(4)} - 6 \tilde{g}^{\mu \nu} D_{\mu} D_{\nu} \ln(\Delta^{-2/5}) - 6 \tilde{g}^{\mu \nu} D_{\mu} \ln(\Delta^{-2/5}) D_{\nu} \ln(\Delta^{-2/5})$$

Using (4.39) to convert $\Delta$ into $T^{AB}$, and adding the other contributions from the Einstein action, the integral over the compact space of the Einstein action, when setting the gauge fields to zero, and disregarding terms with d=7 space time derivatives yields

$$-\frac{1}{2} \int d^3 x \sqrt{\left| \det \tilde{g} \right| (x)} \left( \Delta^{-6/5} \tilde{R}^{(4)} + 2 \frac{Y_A Y_B (T^3)^{AB} Y_C Y_D T^{CD} - (Y_A Y_B (T^2)^{AB})^2}{(Y_A Y_B T^{AB})^2} \right)$$

**We used a symbolic manipulation program to obtain (4.99).**
We further use that

\[
\int d^4x \sqrt{\det \tilde{g}}(x) \frac{Y_A Y_B}{Y_C Y_D T_{CD}} = \frac{1}{5} T_{AB}^{-1} \text{Vol}(S_4) \quad \tag{4.102}
\]

\[
\int d^4x \sqrt{\det \tilde{g}}(x) \frac{Y_A Y_B Y_C Y_D}{(Y_E Y_F T^{EF})^2} = \frac{\text{Vol}(S_4)}{5 \cdot 7} (T_{AB}^{-1} T_{CD}^{-1} + T_{AC}^{-1} T_{BD}^{-1} + T_{AD}^{-1} T_{BC}^{-1}) \quad \tag{4.103}
\]

(This becomes clear after diagonalizing \(T^{CD}\).) At this point, the integrated Einstein action contribution is of the desired form, namely a linear combination of \(T^2\) and \(T_r(T^2)\): \((-31/70)T_r(T^2) + (23/70)T^2\).

On the other hand, the integrated kinetic action of the 3 index photon has the form

\[F_{\mu
u\rho}^2 \sim (Y_E Y_F T^{EF})^2 (1 + S)^2,\]

where \(\frac{2}{\sqrt{3}} \sqrt{\det \tilde{g}}(x) \epsilon_{\mu\nu\rho\sigma}(1 + S) = F_{\mu\nu\rho\sigma}\). (Use that \((\det g^{\mu\nu})(\det \tilde{g}_{\mu\nu}) = \Delta^{-2}\) and substitute (4.39).) The function \(S\) must be homogeneous of degree zero in \(T\), because in \(d=7\) the potential is proportional to \(T^2 - 2T_{ij} T^{ij} = T^2 - 2T_{AB} T^{AB}\) and the leading term \((Y_E Y_F T^{EF})^2\) has already two \(T\)'s. Furthermore, \(S\) depends only on the scalar fluctuations in \((\Pi^{-1})^A_i\). In fact, the most general expression \(S\) can have is

\[S = a\left(\frac{YT^2 Y}{(YTY)^2} - 1\right) + b\left(\frac{T}{YTY} - \frac{5}{4}\right) \quad \tag{4.104}\]

because higher powers of \(T\) appearing in the ratios would yield terms proportional to \(T_r(T^{-1})\) in the \(7\)-dimensional action. To be more explicit, let’s consider a term of the form \(YT^3 Y/(YTY)^3\). Then, when integrating \(S^2\), such a term also generates a integral of the form \(\int d^4x(YT^3 Y)^2/(YTY)^4\) which clearly will produce unwanted \(T_r(T^{-1})\)'s (apply (4.103)). One requires then that the linearized limit of (4.92) yields the results of [36], and that when adding the Einstein action contribution we recover upon integration the \(d=7\) scalar potential. The last term in (4.92) has no linearized contributions, while the coefficient of the second term gets fixed to 1/3 by requiring agreement with [36]. Hence, \(b = 1/3\) while \(a\) is still a free parameter. However, \(a\) gets also fixed after evaluating the integral over \(S_4\) of the square of \((1 + S)\) using (4.102), (4.103) and (4.104) and requiring that after adding the Einstein action contribution one obtains \((T^2 - 2T_r(T^2))/4\). Thus, \(a\) has to satisfy a quadratic equation, namely \(a(a + 2/3) = 0\). It will turn out that the consistency of the gravitino transformation law excludes the first solution \((a = 0)\), and requires the second \((a = -2/3)\).

The result for \(F_{\mu
u\rho\sigma}\) is rather surprising. Initially, we made the ansatz

\[F_{\mu
u\rho\sigma} = \sqrt{\tilde{g}} \epsilon_{\mu
u\rho\sigma} \tilde{D}^{\alpha} \partial_\alpha (1 + S) + \text{gauge field dependent terms} \quad \tag{4.105}\]

where \(1 + S\) contains the ansatz for \(F_{\mu
u\rho\sigma}\) in 4.92). This ansatz satisfies the Bianchi identity

\[4 \tilde{D}_{[\mu} F_{\nu\rho\sigma]} - \partial_\alpha F_{\mu
u\rho\sigma} = 0 \quad \tag{4.106}\]

However, our original expression for \(S\) contained no term with \((Y \cdot T \cdot Y)^{-2}\) (it corresponded to the solution \(a = 0)\), and for this choice of \(S\), the ansatz for \(F_{\mu
u\rho\sigma}\) was found to be truly nonlocal already in a perturbative expansion. With our present ansatz \((a = -2/3)\), the apparently nonlocal expression is, in fact, local and given by
(4.93). It seems very unlikely that the first solution, containing the $d = 11$ nonlocal term $\tilde{D}_\sigma \frac{1}{2} D_\alpha (1 + S)$ would lead to the correct $d = 7$ action, since its presence requires that we expand $\frac{1}{2} D_\alpha (1 + S)$ into an infinite series of spherical harmonics, and though the factor $\frac{1}{2}$ by itself may not be fatal, being integrated over $S_4$, the fact that infinitely many spherical harmonics would enter could lead to inconsistencies in the KK reduction.

The gauge field dependence of the $F_{\Lambda \Sigma \Pi \Omega}$ is dictated by the 11-dimensional susy laws. We begin with the $F_{\mu \nu \rho \alpha}$ sector, and for simplicity, after substituting the 11-dimensional susy variation in terms of our fermionic ansätze, we keep only the gravitino terms. Then, we get

$$\delta F_{\mu \nu \rho \alpha} |\psi = -\frac{\sqrt{2}}{8} \left( \partial_{[\mu} \bar{\epsilon} \Gamma_{\nu]\rho} \Psi_\alpha - \partial_\alpha \bar{\epsilon} \Gamma_{[\nu\rho} \Psi_\mu - 2 \partial_{[\mu} \Gamma_{\alpha \rho] \Psi_\nu} \right) |\psi \hspace{1cm} (4.107)$$

$$= \frac{3\sqrt{2}}{4} \partial_{[\mu} \left( \Delta^{-1/5} \tilde{\epsilon}_{\nu\rho} \Psi_\alpha \delta^j \gamma^i \gamma^j \gamma^5 \eta^j U_{[I}^I U_{J]}^J E_{[\rho]}^\nu E_{\mu]}^\nu \right) \hspace{1cm} (4.108)$$

$$= \frac{3\sqrt{2}}{2} \partial_{[\mu} \left( \delta B_{AB}^{\alpha \beta} C_{\nu C}^{\alpha \rho} \frac{T^E Y_F}{Y T Y} \right) \epsilon_{ABCD} \hspace{1cm} (4.109)$$

$$= \delta \left[ \frac{3\sqrt{2}}{2} \partial_{[\mu} \left( B_{\alpha \beta}^{AB} C_{\nu C}^{\alpha \rho} \frac{T^E Y_F}{Y T Y} \right) \epsilon_{ABCD} \right] \hspace{1cm} (4.110)$$

where to go from (4.108) to (4.109) we used (3.4) and the identity

$$E_{[\mu}^m \Delta^{-2/5} U_{I}^{I'} U_{J}^{J'} C_{\mu]}^{IJ} = -i \Pi_{AB}(\gamma_{I})^I_{J'} C_{\mu}^A \hspace{1cm} (4.111)$$

(Substitute into (4.29) that $V^m \sim Y^C Y^D C_0^m Y_0 D$, replace $2\gamma^C D\gamma^D - \gamma^C \gamma^D$ and use (4.45) to eliminate $\gamma^C$. One obtains then the commutator $[\gamma_{ij}, \gamma_k] \sim \delta_{jk} \gamma_i - \delta_{ik} \gamma_j$. The $\Pi^{-1}$ from (4.45) cancels then one of the $\Pi$’s in $E_{\mu}^m$). The total susy variation $\delta$ could be pulled out in (4.110) because we are keeping only gravitino dependent terms, and scalars vary into spin 1/2 fields.

Hence (4.110) allows us to read off the gauge field dependence of $F_{\mu \nu \rho \alpha}$ (see eq. (4.93)). Another nontrivial check of this ansatz is that the susy variation of the scalars in the l.h.s. of (4.93) reproduces the $B_{\alpha \beta}^{AB}$-dependent terms which we get from the 11-dimensional susy variation of $F_{\mu \nu \rho \alpha}$.

The ansatz in the next sector, namely, $F_{\mu \nu \alpha \beta}$ will be fixed again by requiring the consistency of susy laws. The complete answer is given in (4.94). The procedure to determine the ansatz remains the same: vary $F_{\mu \nu \alpha \beta}$ under 11-dimensional susy laws, look for simplicity only at gravitino dependent terms, and rewrite (after substituting the various fermionic and vielbein ansätze) the 11-dimensional susy variation in terms of a total 7-dimensional susy variation of 7-dimensional bosonic (gauge and scalar) fields. So, varying the r.h.s. one gets

$$\delta F_{\mu \nu \alpha \beta} |\psi = -\frac{\sqrt{2}}{8} (2!)^2 \left( 2 \partial_{[\mu} \bar{\epsilon} \Gamma_{\nu] (\alpha} \Psi_{\beta]} - \partial_{[\mu} \bar{\epsilon} \Gamma_{\alpha \beta}] \Psi_\nu \right) \hspace{1cm} (4.93)$$
\[ -\partial_{[\alpha} \varepsilon \Gamma_{\nu\mu} \Psi_{\beta}] + 2 \partial_{[\beta} \varepsilon \Gamma_{\alpha][\nu} \Psi_{\mu]} \bigg|_\psi \]

\[ = -\sqrt{2} \partial_\mu \left( -i \varepsilon_I \gamma_{\alpha} \gamma_{\beta} J^I \Delta^{-2/5} \eta^I \gamma_{5\gamma} \eta^J U_{I} U_{J} E_\nu \right.
-2 \varepsilon_I \gamma_{\beta} J^I \Delta^{-1/5} \eta^I \gamma_{\alpha} \gamma_{5\gamma} \eta^J U_{I} U_{J} E_\nu E_\rho B_{\alpha}^{AB} V_{AB} \rho 
+ \frac{\sqrt{2}}{2} \partial_{[\alpha} \left( \varepsilon_I \gamma_{\beta} J^I \Delta^{-1/5} \eta^I \gamma_{\alpha} \gamma_{5\gamma} \eta^J E_\nu E_\rho \right) \bigg) \]

\[ = \frac{\sqrt{2}}{2} \partial_{[\alpha} \left( \varepsilon_I \gamma_{\beta} J^I \Delta^{-1/5} \eta^I \gamma_{\alpha} \gamma_{5\gamma} \eta^J E_\nu E_\rho \right) \bigg) \]

\[(4.112) \]

\[ -\partial_{[\alpha} \varepsilon \Gamma_{\nu\mu} \Psi_{\beta}] + 2 \partial_{[\beta} \varepsilon \Gamma_{\alpha][\nu} \Psi_{\mu]} \bigg|_\psi \]

\[ = \sqrt{2} \partial_\mu \left( \varepsilon_I \gamma_{\beta} J^I \Delta^{-1/5} \eta^I \gamma_{\alpha} \gamma_{5\gamma} \eta^J E_\nu E_\rho \right) \bigg) \]

\[ = \frac{\sqrt{2}}{2} \partial_{[\alpha} \left( \varepsilon_I \gamma_{\beta} J^I \Delta^{-1/5} \eta^I \gamma_{\alpha} \gamma_{5\gamma} \eta^J E_\nu E_\rho \right) \bigg) \]

\[ = \frac{\sqrt{2}}{2} \partial_{[\alpha} \left( \varepsilon_I \gamma_{\beta} J^I \Delta^{-1/5} \eta^I \gamma_{\alpha} \gamma_{5\gamma} \eta^J E_\nu E_\rho \right) \bigg) \]

\[ = \frac{\sqrt{2}}{2} \partial_{[\alpha} \left( \varepsilon_I \gamma_{\beta} J^I \Delta^{-1/5} \eta^I \gamma_{\alpha} \gamma_{5\gamma} \eta^J E_\nu E_\rho \right) \bigg) \]

\[ = \frac{\sqrt{2}}{2} \partial_{[\alpha} \left( \varepsilon_I \gamma_{\beta} J^I \Delta^{-1/5} \eta^I \gamma_{\alpha} \gamma_{5\gamma} \eta^J E_\nu E_\rho \right) \bigg) \]

\[ = \frac{\sqrt{2}}{2} \partial_{[\alpha} \left( \varepsilon_I \gamma_{\beta} J^I \Delta^{-1/5} \eta^I \gamma_{\alpha} \gamma_{5\gamma} \eta^J E_\nu E_\rho \right) \bigg) \]

To go from (4.112) to (4.113) we again made use of (3.4) and (4.111). Furthermore, in (4.113) the first term vanishes trivially, and the last one can be written as a total 7-dimensional susy variation. With the use of the Schouten identity

\[ (\delta(B_{[\beta}^{AB}) B_{\alpha]}^{CF} C_\nu^D T^{EY}_{YG} (\cdot Y,T,Y) \cdot Y) \sum_{[\gamma] \varepsilon_{ABCDE} = 0 \] (4.114) \]

which yields

\[ \delta(B_{[\beta}^{AB}) B_{\alpha]}^{CF} C_\nu^D T^{EY}_{YG} (\cdot Y,T,Y) \cdot Y) \sum_{[\gamma] \varepsilon_{ABCDE} = 0 \] (4.115) \]

we get that

\[ \delta(B_{[\beta}^{AB}) B_{\alpha]}^{CF} C_\nu^D T^{EY}_{YG} (\cdot Y,T,Y) \cdot Y) \sum_{[\gamma] \varepsilon_{ABCDE} = 0 \] (4.116) \]

After acting with a \( \partial_\mu \) on (4.116), the remaining terms can be cast into a total susy variation. Then we substitute back into (4.113) and we recognize that the 11-dimensional susy variation of \( F_{\alpha \beta \mu \nu} \) is the same as the 7-dimensional susy variation of the ansatz given in (4.94).

To derive the ansatz of the next sector \( F_{\mu \alpha \beta \gamma} \) is a bit more laborious. We follow the procedure outlined previously, and vary under susy the r.h.s. of (4.95)

\[ \delta F_{\mu \alpha \beta \gamma} \psi = -\frac{\sqrt{2}}{8} 3! \left( \partial_\mu \varepsilon \Gamma_{[\alpha \beta} \Psi_{\gamma]} - 2 \partial_{[\alpha} \varepsilon \Gamma_{\beta \gamma} \Psi_{\mu]} \right) \Psi_{\mu] \bigg|_\psi \]

\[ = -\frac{3 \sqrt{2}}{4} \partial_\mu \left[ \varepsilon \Delta^{-1/10} \sqrt{75} \left( \tau_{\alpha \beta} \Delta^{-2/5} + 2 \gamma_m \tau_{\beta \gamma} E_m^{\mu} \Delta^{-1/5} + \gamma_{mn} E_m^{\mu} E_n^{\mu} \right) \Delta^{-1/10} \sqrt{75} \psi_{\gamma] \bigg|_\psi \]

\[ + \frac{3 \sqrt{2}}{2} \partial_{[\alpha} \left[ \varepsilon \Delta^{-1/10} \sqrt{75} \left( 2 \gamma_m \tau_{\beta \gamma} E_m^{\mu} \Delta^{-1/5} + \gamma_{mn} E_m^{\mu} E_n^{\mu} \right) \Delta^{-1/10} \sqrt{75} \psi_{\gamma] \bigg|_\psi \]

(4.117)
Then we substitute our ansätze in the l.h.s. of (4.117) and with the same tricks as we used before we arrive at

\begin{equation}
\delta F_{\mu \alpha \beta \gamma} \mid \psi = \frac{3\sqrt{2}}{4} (\Pi^{-1})^{i} A^{A \mu} \epsilon \tau_{[\alpha \beta} \gamma^i \psi_{\gamma]} + \frac{3\sqrt{2}}{2} D_{[\alpha} \left( \epsilon \tau_{\beta} \gamma^i \psi_{\gamma]} \Pi A^{i} \right) C^{A} + 6\sqrt{2} \partial_{\mu} \left[ \epsilon_{\alpha \beta \gamma \lambda \mu} \delta (B^{[\lambda}_{\gamma}] (B_{\alpha} \cdot Y)^{C} (B_{\beta}] \cdot Y)^{D} \frac{T^{EF} Y_{F}}{Y \cdot Y} \right] + 6\sqrt{2} \partial_{\mu} \left[ \epsilon_{\alpha \beta \gamma \lambda \mu} \delta (B^{[\lambda}_{\gamma}] (B_{\beta}] \cdot Y)^{C} C^{D} \frac{T^{EF} Y_{F}}{Y \cdot Y} \right] (4.118)
\end{equation}

To get the correct gauge-field dependence in the \( F_{\mu \alpha \beta \gamma} \) sector we take into account the susy variation of the independent fluctuation \( A_{\mu \beta \gamma} \) whose ansatz in terms of 7-dimensional fields is (see below) \(-\frac{2i}{\sqrt{3}} S_{\alpha \beta \gamma} A(y) Y^{A}(x)\). Then the 7-dimensional susy variation of the gauge field dependent part of \( F_{\mu \alpha \beta \gamma} \) will be the difference between (4.118) and the 7 dimensional susy variation of \( \partial_{\mu} A_{\alpha \beta \gamma} \). Note that the first two terms in (4.118) are already parts of the susy variation of \(-i \sqrt{3} S_{\alpha \beta \gamma}\). We are ready now to extract the susy variation of the gauge field dependent piece of \( F_{\mu \alpha \beta \gamma} \):

\begin{equation}
(\langle \delta (F_{\mu \alpha \beta \gamma}) \mid B \text{ terms only} \rangle \mid \psi = 6\sqrt{2} \left[ \partial_{\mu} \left( \epsilon_{\alpha \beta \gamma \lambda \mu} \delta (B^{[\lambda}_{\gamma}] (B_{\alpha} \cdot Y)^{C} (B_{\beta}] \cdot Y)^{D} \frac{T^{EF} Y_{F}}{Y \cdot Y} \right) + \partial_{[\alpha} \left( \epsilon_{\alpha \beta \gamma \lambda \mu} \delta (B^{[\lambda}_{\gamma}] (B_{\beta}] \cdot Y)^{C} C^{D} \frac{T^{EF} Y_{F}}{Y \cdot Y} \right) + \frac{3\sqrt{2}}{2} \epsilon_{\alpha \beta \gamma \lambda \mu} \delta (B^{[\lambda}_{\gamma}] F^{C D}_{[\beta \gamma]} C^{E}_{\mu} \right) (4.119)
\end{equation}

We are going to rewrite the first two terms as total 7-dimensional susy variations. Then we shall “peel off” a \( \delta \) from both sides of (4.119) and read off the ansatz for \( (F_{\alpha \beta \mu}) \mid B \text{ terms only} \).

We begin by working out the first term. First, we shall pull outside a total \( \delta \) susy variation, then we use a Schouten identity to release a susy gauge field variation from its contraction with the spherical harmonic.

\begin{equation}
\partial_{\mu} \left( \epsilon_{\alpha \beta \gamma \lambda \mu} \delta (B^{[\lambda}_{\gamma}] (B_{\alpha} \cdot Y)^{C} (B_{\beta}] \cdot Y)^{D} \frac{T^{EF} Y_{F}}{Y \cdot Y} \right) = \delta \partial_{\mu} \left( \epsilon_{\alpha \beta \gamma \lambda \mu} B^{A B}_{[\lambda} (B_{\alpha} \cdot Y)^{C} (B_{\beta}] \cdot Y)^{D} \frac{T^{EF} Y_{F}}{Y \cdot Y} \right) - 2\partial_{\mu} \left( \epsilon_{\alpha \beta \gamma \lambda \mu} B^{A B}_{[\gamma} \delta (B_{\alpha} \cdot Y)^{C} (B_{\beta}] \cdot Y)^{D} \frac{T^{EF} Y_{F}}{Y \cdot Y} \right) \]
\begin{equation}
= \frac{1}{3} \delta \partial_{\mu} \left( \epsilon_{\alpha \beta \gamma \lambda \mu} B^{A B}_{[\gamma} (B_{\alpha} \cdot Y)^{C} (B_{\beta}] \cdot Y)^{D} \frac{T^{EF} Y_{F}}{Y \cdot Y} \right) + \frac{1}{3} \partial_{\mu} \left( \epsilon_{\alpha \beta \gamma \lambda \mu} B^{A B}_{[\gamma} \delta (B_{\alpha}^{CD}) (B_{\beta}] \cdot Y)^{E} \right) (4.120)
\end{equation}
Schoutenizing the second term of (4.119) we get:

\[
\partial_\alpha \left( \epsilon_{ABCDEF} \delta(B^{AB}_\gamma (B^C_\beta \cdot Y)^D C^E_\mu \frac{T^{EF} Y_F}{Y \cdot T \cdot Y} \right) \\
= \frac{1}{2} \epsilon_{ABCDEF} \left[ \delta \partial_\alpha \left( B^{AB}_\gamma (B^C_\beta \cdot Y)^D C^E_\mu \frac{T^{EF} Y_F}{Y \cdot T \cdot Y} \right) \right] + \frac{1}{2} \partial_{[\alpha} \left( B^{AB}_\gamma \delta(B^C_\beta)^E_\mu \right) \right]
\] (4.121)

The last step is to substitute (4.120) and (4.121) into (4.119)

\[(\delta(F_{\mu \alpha \beta \gamma})|_B \text{ terms only}) |_{\psi} = \epsilon_{ABCDEF} \left\{ 3\sqrt{2} \delta \left( \frac{2}{3} B^{AB}_\gamma (B^C_\beta \cdot Y)^D C^E_\mu \frac{T^{EF} Y_F}{Y \cdot T \cdot Y} \right) + \partial_{[\alpha} \left( B^{AB}_\gamma (B^C_\beta \cdot Y)^D C^E_\mu \frac{T^{EF} Y_F}{Y \cdot T \cdot Y} \right) + 3\sqrt{2} \delta \partial_\alpha \left( B^{AB}_\gamma \delta(B^C_\beta)^E_\mu \right) + 2\sqrt{2} \partial_\mu \left( B^{AB}_\gamma \delta(B^C_\beta \cdot Y)^E + 3\sqrt{2} \delta(B^{AB}_\gamma) \left( \partial_\beta B^C_\gamma + 2(B_\beta \cdot B_\gamma)^C_\mu \right) \right) \right\} \]

\[
= \epsilon_{ABCDEF} \left[ 2\sqrt{2} \partial_\mu \left( B^{A\beta}_\gamma (B_\alpha \cdot Y)^C B^B_\gamma \delta(B^E_\mu \cdot Y) \frac{T^{EF} Y_F}{Y \cdot T \cdot Y} \right) \\
- 3\sqrt{2} \partial_{[\alpha} \left( B^{A\beta}_\gamma (B_\alpha \cdot Y)^C B^B_\gamma \frac{T^{EF} Y_F}{Y \cdot T \cdot Y} \right) + \frac{3\sqrt{2}}{2} \partial_\mu \left( \partial_{[\alpha} B^{A\beta}_\gamma + \frac{4}{3} (B_{[\alpha \cdot B_\beta])^A}_\gamma \right) \right]
\] (4.122)

To achieve the last expression on the l.h.s. of (4.122) we need to perform one more trick, namely to use the Schouten identity to rewrite

\[
\epsilon_{ABCDEF} B^{A\beta}_\gamma \delta(B^C_\beta \cdot Y)^E C^F_\mu = \epsilon_{ABCDEF} \left[ \delta \left( B^{A\beta}_\gamma (B_\alpha \cdot B_\beta)^C_\mu \right) + 3\delta(B^C_\gamma) \right. \\
-3\delta(B^E_\gamma) \left. \right]
\] (4.123)

We conclude that, indeed, from the requirement that the susy laws are consistent we obtain the ansatz for $F_{\mu \alpha \beta \gamma}$ given in (4.95).

At this moment, we can (again for simplicity of the argument) solve $F_{\alpha \beta \gamma \delta}$ from the Bianchi identities. The Bianchi identities

\[
\partial_{[\alpha} F_{\beta \gamma \delta \epsilon] = 0
\] (4.124)

are trivially satisfied in the $F$ sectors which we derived so far, because the gauge field dependent terms $F$ can be easily cast into an exact form, while the purely scalar sectors are components of a separately closed form. Thus we use the last Bianchi identity

\[
\partial_\mu F_{\alpha \beta \gamma \delta} = 4\partial_{[\alpha} F_{\mu \beta \gamma \delta]
\] (4.125)

which yields

\[
0 = \partial_\mu \left[ F_{\alpha \beta \gamma \delta} - 4 \epsilon_{ABCDEF} \partial_{[\alpha} \left( 2\sqrt{2} B^{A\beta}_\gamma \delta(B^C_\beta \cdot Y)^D C^E_\mu \frac{T^{EF} Y_F}{Y \cdot T \cdot Y} + \frac{3\sqrt{2}}{2} \left( \partial_\beta B^{A\beta}_\gamma + \frac{4}{3} (B_\beta \cdot B_\gamma)^A_\mu \right) \right) \right]
\] (4.126)
to solve for \( F_{\alpha\beta\gamma\delta} \). The result, which is given in (4.96) is compatible with the susy laws, and satisfies trivially \( \partial_{\mu} F_{\alpha\beta\gamma\delta} = 0 \).

A brief inspection of the ansatz given in (4.92-4.96) suggests that we can also give the ansatz for the 11-dimensional 3-index field, \( A_{\Lambda\Pi\Sigma} \):

\[
\begin{align*}
A_{\mu\nu\rho} &= -\frac{1}{6\sqrt{2}} \epsilon_{ABCDEF} C^A_{\mu} C^B_{\nu} C^D_{\rho} Y^E \frac{(T \cdot Y)^E}{Y \cdot T \cdot Y} - \frac{1}{2\sqrt{2}} \epsilon_{\mu\nu\rho\sigma} \hat{g} \\
A_{\nu\rho} &= \frac{1}{6\sqrt{2}} \epsilon_{ABCDEF} B^A_{\alpha} C^B_{\nu} C^D_{\rho} \frac{(T \cdot Y)^E}{Y \cdot T \cdot Y} \\
A_{\alpha\beta\gamma} &= \frac{i\sqrt{6}}{6} S_{\alpha\beta\gamma, A} Y^A \\
A_{\alpha\beta} &= \frac{1}{3\sqrt{2}} \epsilon_{ABCDEF} (B_{[\alpha} \cdot Y)^A B_{\beta]} C^A_{\nu} \frac{(T \cdot Y)^E}{Y \cdot T \cdot Y} \\
A_{\alpha\beta} &= \frac{1}{2\sqrt{2}} \left( \partial_{[\alpha} B^A_{\beta]} + \frac{4}{3} (B_{[\alpha} \cdot B_{\beta])^A B_{\gamma]} Y^E \right) (4.127)
\end{align*}
\]

Also, to ease the comparison between the ansatz necessary to achieve a consistent truncation (given in (4.92-4.96)) and the geometrical ansatz of [47] we present the expression of \( F_{\Lambda\Pi\Sigma\Omega} \) in form language. It is gauge invariant.

\[
\frac{\sqrt{2}}{3} F^{(4)} = \epsilon_{ABCDEF} \left( \frac{1}{3} D Y^A D Y^B D Y^C D Y^D \frac{(T \cdot Y)^E}{Y \cdot T \cdot Y} \\
+ \frac{4}{3} D Y^A D Y^B D Y^C D \frac{(T \cdot Y)^D}{Y \cdot T \cdot Y} Y^E \\
+ 2 F^{(2)}_{AB} D Y^C D Y^D \frac{(T \cdot Y)^E}{Y \cdot T \cdot Y} + F^{(2)}_{AB} F^{(2)}_{CD} Y^E \right) + d(A) (4.128)
\]

where \( F^{(2)}_{AB} = 2(DB^{AB} + 2(B \cdot B)^{AB}) \) and \( D Y^A = dY^A + 2(B \cdot Y)^A \).

To find the ansatz for \( A \) and \( \tilde{B} \), we require that we obtain all the \( S_{\alpha\beta\gamma} \) terms in the \( d=7 \) gravitino susy transformation law from KK reduction. Consider the linearized gravitino transformation law. This should already display the self-duality mechanism, and also will fix the free parameters \( a \) and \( b \) in the \( d=11 \) susy transformation law of the auxiliary field \( B \). The \( F_{\Lambda\Pi\Sigma\Omega} \) term in \( \delta \psi_A \) reads:

\[
\begin{align*}
\delta \psi_{\zeta} \mid_{\text{lin., } F \text{ term, } B=0} &= \frac{\sqrt{2}}{288} (\Gamma_{\alpha\beta\gamma\delta} \zeta - 88 \tilde{\delta}_{\alpha}^{[\alpha} \Gamma_{\beta\gamma\delta]} \zeta) F_{\alpha\beta\gamma\delta} \\
+ \frac{\sqrt{2}}{288} (4\Gamma^{\mu\alpha\beta\gamma} \zeta + 3 \cdot 8\delta^{\alpha}_{\zeta} \Gamma^{\mu\beta\gamma} \zeta) F_{\mu\alpha\beta\gamma} (4.129)
\end{align*}
\]

\[
\begin{align*}
\delta \psi_{m} \mid_{\text{lin., } F \text{ term, } B=0} &= \frac{\sqrt{2}}{288} \Gamma_{\alpha\beta\gamma\delta} m F_{\alpha\beta\gamma\delta} \\
+ \frac{\sqrt{2}}{288} (4\Gamma^{\mu\alpha\beta\gamma} m - 8e^{m}_{\alpha} \Gamma^{\alpha\beta\gamma} \zeta) F_{\mu\alpha\beta\gamma} (4.130)
\end{align*}
\]

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The result for $\delta \psi_\epsilon$ can be worked out further by decomposing the 11 dimensional gamma matrices and using the commutation relations of $\sqrt{\gamma}$

$$\delta \psi_\epsilon|_{\text{lin.}, F\, \text{term}, B=0} = \frac{\sqrt{2}}{288} \frac{\gamma_5}{5} \left\{ 3(3\tau^{\alpha\beta\gamma\delta} \epsilon - 8\delta^{[\alpha \tau^{\beta\gamma\delta}]}) F_{\alpha\beta\gamma\delta} - 24i(\tau^{\alpha\beta\gamma\delta} \epsilon + \frac{9}{2} \delta^{[\alpha \tau^{\beta\gamma\delta}]) \gamma^\mu F_{\mu\alpha\beta\gamma} \right\} \epsilon$$  

(4.131)

We introduce a normalization constant $c$ for $S_{\alpha\beta\gamma,A}$ by embedding the 7d field $S$ into the 11d curvature $F$ as follows:

$$F_{\mu\alpha\beta\gamma}|_{B=0} = c S_{\alpha\beta\gamma,A} \partial_\mu Y^A$$  

(4.132)

and a corresponding relation for the auxiliary field,

$$\tilde{B}_{\alpha\beta\gamma} = B_{\alpha\beta\gamma,A} Y^A$$  

(4.133)

Substituting the linearized ansatz for $B_{\alpha\beta\gamma,A}$ in (2.41) into the transformation law of $\psi_\epsilon$, we find

$$\delta \psi_\epsilon|_{\text{lin.}, B\, \text{term}} = \frac{1}{24}(b \Gamma^{\alpha\beta\gamma\delta} B_{\alpha\beta\gamma\delta} - a \Gamma^{\alpha\beta\gamma} B_{\epsilon\alpha\beta\gamma}) \epsilon$$

$$+ \frac{1}{2} \tau_5 \gamma_5 \frac{b}{24} \Gamma^{\alpha\beta\gamma\delta} m B_{\alpha\beta\gamma\delta} \epsilon$$

$$= \frac{k}{24} \epsilon_{\alpha\beta\gamma\delta} e^{\kappa} B_{\epsilon\kappa,A} Y^A \left( \frac{9}{5} b \tau_\epsilon^{\alpha\beta\gamma} - (a - \frac{16}{5} b) \delta^{[\alpha \tau^{\beta\gamma\delta}]) \gamma_5 \epsilon \right)$$

$$= \frac{\gamma_5}{5} \left\{ 9 \frac{kb}{5} \frac{4!}{24} \frac{1}{5} \delta^{[\alpha \tau^{\beta\gamma\delta}]} S_{\alpha\beta\gamma,A} Y^A \right\}$$

$$+ \frac{9}{5} \frac{kb}{24} 6 \tau_\epsilon^{\alpha\beta\gamma\delta} 4 D_\epsilon S_{\beta\gamma\delta,A} Y^A$$

$$- \frac{m}{24} (a - \frac{16}{5} b) 3 \tau_\epsilon^{\alpha\beta\gamma} S_{\beta\gamma\delta,A} Y^A$$

$$- \frac{k}{24} (a - \frac{16}{5} b) 6 \tau_\epsilon^{\alpha\beta\gamma} 4 D_\epsilon S_{\beta\gamma\delta,A} Y^A \right\} \epsilon$$  

(4.134)

In order to reproduce the 7 dimensional sugra result (and have a consistent truncation of 11 dimensional sugra to the massless 7 dimensional fields), we must reproduce the gauged sugra result for $\delta \psi_I$, which does not have any $DS$ terms, so we must require $kb = -\frac{5\sqrt{2}}{12}$, $ka = -\frac{5\sqrt{2}}{9}$. Then we obtain

$$\delta \psi_\epsilon|_{\text{lin.}, S\, \text{terms}} = c \left[ \frac{\sqrt{2}}{60} (\tau_\epsilon^{\alpha\beta\gamma} - \frac{9}{2} \delta^{[\alpha \tau^{\beta\gamma}]) \gamma_5 (m + i \gamma^\mu D_{\mu}) Y^A \epsilon \right]$$  

(4.135)

Using (3.42), we get by contracting with $(\gamma^A)_{JK}$ and using $1/4 \phi_{JK} (\gamma^A)_{JK} = Y^A$, according to section (3.1),

$$\frac{1}{m} (m + i \gamma^\mu D_{\mu}) Y^A \gamma_5 \epsilon I = -\epsilon_I (\gamma^A)_{K\eta^K}$$  

Hence

$$\delta \psi_\epsilon|_{\text{lin.}, S\, \text{terms}} = c \left[ \frac{\sqrt{2}}{60} \left( \tau_\epsilon^{\alpha\beta\gamma} - \frac{9}{2} \delta^{[\alpha \tau^{\beta\gamma}]) \epsilon \gamma_5 \right) (\gamma^A)_{K\eta^K} \right] S_{\alpha\beta\gamma,A}$$  

(4.137)
Incidentally, we note that if we would use the p=q=+1/2 solution in (4.16) instead of p = q = −1/2 we would get \((i\gamma^\mu D_\mu + m)\) in (4.135) (because the commutation relation of \(\gamma^\pm \) past \(\gamma_\mu\) gives a ± sign). That means that at this moment the sign is fixed by requiring a consistent truncation.

In order to match this result with the \(S\) term in \(\delta \psi_e\) in (4.6), we fix \(c = -i\sqrt{6}\). We can then check this value of \(c\) by putting (4.132) in the 11d action for \(F_{\Lambda\Pi\Sigma\Omega}\). We obtain the correct normalization for the 7d action of \(S_{\alpha\beta\gamma\Lambda} \).

At this moment, one can see why a first order formulation for gravity instead of the antisymmetric tensor field does not work. Consider the spin connection \(\Omega\) as an independent field, as done in [40]; then the 11 dimensional gravitino transformation law is the same, with \(D_\Lambda(\Omega)\) replaced by \(D_\Lambda(\Omega)\) where

\[
\Omega_\Lambda^{MN} = \tilde{\Omega}_\Lambda^{MN}(E, \Psi) + B_{\Lambda\Pi\Sigma\Omega}E_{\Pi}E_{\Sigma} + \Delta \Omega_\Lambda^{MN} \tag{4.138}
\]

and \(\Delta \Omega_\Lambda^{MN}E_{\Pi}E_{\Sigma} = 0\). Thus the \(B_{\alpha\beta\gamma}\) term we get in \(\delta_{\text{susy},11-\text{dim}} \Psi_\alpha\) is at the linearized level is \(\frac{1}{2}B_{\alpha\beta\gamma}T^{\beta\gamma} \varepsilon\). But this has no \(\gamma_\gamma\) with respect to (4.134), and since the susy law is fixed by the definition of \(B\), we can’t introduce by hand the missing \(\gamma_5\). Therefore it will be impossible to cancel the \(F\) terms in (4.131), as we did in (4.135). In conclusion, we need a 4-index tensor to mix with the independent fluctuation \(S_{\alpha\beta\gamma}\).

Now, let us analyze the nonlinear level. The nonlinear KK reduction gives for the terms involving \(S\)

\[
\delta \psi_e|_{\text{S term}} = \Delta^{-1/10} \tau_5^{1/2} \epsilon^e_a [E_a^{\Lambda} \delta \Psi_\Lambda|_{\text{S term}} + \frac{1}{5} \tau_5 \gamma^m E^\Lambda_m \delta \Psi_\Lambda|_{\text{S term}}]
\]

and \(\Delta \Omega_\Lambda^{MN}E_{\Pi}E_{\Sigma} = 0\). Thus the \(B_{\alpha\beta\gamma}\) term we get in \(\delta_{\text{susy},11-\text{dim}} \Psi_\alpha\) is at the linearized level is \(\frac{1}{2}B_{\alpha\beta\gamma}T^{\beta\gamma} \varepsilon\). But this has no \(\gamma_\gamma\) with respect to (4.134), and since the susy law is fixed by the definition of \(B\), we can’t introduce by hand the missing \(\gamma_5\). Therefore it will be impossible to cancel the \(F\) terms in (4.131), as we did in (4.135). In conclusion, we need a 4-index tensor to mix with the independent fluctuation \(S_{\alpha\beta\gamma}\).

In the same way as it happened at the linear level, the \(\partial_\alpha S_{\beta\gamma}\) terms in \(B_{\alpha\beta\gamma}\) will cancel the \(F_{\alpha\beta\gamma}\) terms, whereas the \(S_{\alpha\beta}\) terms in \(B_{\alpha\beta\gamma}\) and \(F_{\mu\alpha\beta\gamma}\) add, and moreover the term \(B_\alpha^{AB} S^\beta_{\gamma\delta B}\) in \(B_{\alpha\beta\gamma}\) cancel the \(B_\delta^\mu F_{\mu\alpha\beta\gamma}\) term, provided we choose the ansatz (neglecting “massive” fields which are put to zero)

\[
\frac{B_{\alpha\beta\gamma}}{\sqrt{E}} = \frac{6k}{5} (F_{\alpha\beta\gamma} + 4D_\delta^\mu F_{\mu\alpha\beta\gamma}) - \frac{k}{5} \varepsilon_{\alpha\beta\gamma} \epsilon_\kappa \tilde{B}_{\epsilon\kappa}
\]

\[
\frac{B_\alpha^{AB} S^\beta_{\gamma\delta B}}{\sqrt{E}} = \frac{24kc}{5} \nabla_{[\alpha} S_{\beta\gamma\delta]} Y^A - \frac{k}{5} \varepsilon_{\alpha\beta\gamma} \epsilon_\kappa \tilde{B}_{\epsilon\kappa} \tag{14.40}
\]

where \(\tilde{B}\) contains only \(S\) terms (no \(\partial_\alpha S_{\beta\gamma\delta}\) or \(B_\alpha^{AB} S^\beta_{\gamma\delta B}\) and we have used (3.22) and (3.24). In the following, the parameters \(a, b, c\) take the values which we determined previously. Then we obtain

\[
\delta \psi_e|_{\text{S term}} = -\frac{\sqrt{2}}{60} (\tau_5 \alpha^{\epsilon\beta\gamma} - \frac{9}{2} \delta^{[\alpha_{\epsilon\beta\gamma}]}_\gamma_5) \Delta^{3/5}
\]
\[(\Delta^{-1/5}E^\mu_m)(-i\gamma^m)F_{\mu\alpha\beta\gamma} + \tilde{B}_{\alpha\beta\gamma}\] 

We further work out the \( F_{\mu\alpha\beta\gamma} \) term in \( \delta\psi_\epsilon \) by substituting our ansätze for various fields, where in \( F_{\mu\alpha\beta\gamma} \) we consider only the term with \( S 

\begin{align*}
F_{\mu\alpha\beta\gamma}|_{B=0} &= cS_{\alpha\beta\gamma,A}\partial_\mu Y^A 
\end{align*}

(4.142)

So, by substituting for \( F_{\mu\alpha\beta\gamma} \), using the ansätze for \( E^\mu_m \) in (4.28) and introducing the identity as \( \bar{\eta}_N^\alpha \eta_\beta^N = \delta_\beta^\alpha \), we get

\begin{align*}
\Delta^{-1/5}E^\mu_m\gamma_5(-i\gamma^m)\Delta^{3/5}F_{\mu\alpha\beta\gamma}\epsilon 
= -\frac{1}{4}\Delta^{3/5}cS_{\alpha\beta\gamma,C}C^C_mC_mTr(\gamma^iUV^mU^T\Omega) 
(\Pi^{-1})^A_i(\Pi^{-1})^B_jB^mAB^mN\eta_NU^{i'}I\epsilon^{i'} 
\end{align*}

(4.143)

Then, using (3.42), we have

\begin{align*}
V^PQC^n_{N|} &= 4\phi_5^P[I\Omega^N|Q - (\Omega^PQ\phi_5^NI + \Omega^{NI}\phi_5^P) 
\end{align*}

(4.144)

and substituting this relation for the two products \( V^PC_n \) in (4.143), and doing a bit of algebra, we get

\begin{align*}
\Delta^{3/5}(\Pi^{-1})^A_i(\Pi^{-1})^B_jB^mAC^n_{\eta_NU^{i'}I\epsilon^{i'}} 
\frac{1}{2} \left[ \phi_5^P(U^T\gamma^iU)_{PN} - \phi_5^P(U^T\gamma^iU)_{PI} \right] \eta_NU^{i'}I\epsilon^{i'} 
\end{align*}

(4.145)

But now we can use the condition we got on \( U \) from matching the scalar transformation law (4.45), and get

\begin{align*}
cS_{\alpha\beta\gamma,A}(\Pi^{-1})^A_i(\Pi^{-1})^B_jB^mAC^n_{\eta_NU^{i'}I\epsilon^{i'}} - \Delta^{3/5}T^{AB}cS_{\alpha\beta\gamma,B}Y^A\gamma_5\eta_NU^{i'}I\epsilon^{i'} 
\end{align*}

(4.146)

We now see that the first term is exactly what we want to have in 7 dimensional supergravity (the \( S_{\alpha\beta\gamma,A} \) term of \( \delta\psi_\epsilon \) in [41]), whereas the second term gets canceled by the \( \tilde{B}_{\alpha\beta\gamma} \) term, if we choose the ansatz for \( \tilde{B}_{\alpha\beta\gamma} \) to be:

\begin{align*}
\tilde{B}_{\alpha\beta\gamma} &= cT^{AB}S_{\alpha\beta\gamma,B}Y^A 
\end{align*}

(4.147)

However, let us pause and comment on the ansatz for \( B_{MNPQ} \). We know that \( B_{MNPQ} = 0 \) is a solution of the 11d equations of motion. So we can say that the ansatz for the various components of \( B_{MNPQ} \) has to be such that the various components of \( B_{MNPQ} = 0 \) correspond to various 7d equations of motion. Then the first remark is that we have to add to (4.140) the necessary bilinears in fermions and gauge field strength to complete the 7-dimensional equation of motion for \( S_{\alpha\beta\gamma,A} \) (remember that we always dropped in the susy transformation rules 3- and 4-fermi terms). Using that \( k = 5(6\sqrt{2}) \) (as it will be determined below (5.31)), the ansatz of the auxiliary field \( B_{MNPQ} \) is

\begin{align*}
\frac{B_{\alpha\beta\gamma\delta}}{\sqrt{E}} &= -24\sqrt[3]{3}i\nabla_{[\alpha}S_{\gamma\beta\delta]}A^Y^A + \sqrt[3]{3}i\epsilon_{\alpha\beta\gamma}\epsilon^\kappa T^{AB}S_{\epsilon\kappa,B}Y^A 
+ 9\epsilon_{ABCD}F_{[\alpha^BC}^{DE}F_{\gamma^\delta]}^Y^A + 2 - fermi\ terms 
\end{align*}

(4.148)
The necessity of adding the gauge field terms in the ansatz for the auxiliary field $B_{MNPQ}$ can be seen if we try to derive the 7-dimensional action from the 11-dimensional one. The ansatz for $F_{\alpha\beta\gamma\delta}$ is already bilinear in the gauge field strength, and when integrating out $(F_{\alpha\beta\gamma\delta})^2$ on $S_4$ we get as a result terms with $(F_{AB\alpha\beta})^4$ which are not present in the $d = 7$ action. However, they are precisely canceled by the gauge field strengths which enter in the ansatz of $B_{MNPQ}$, after integrating out $(B_{MNPQ})^2$. The second observation is that no other bosonic equation of motion can appear in $B_{MNPQ}$, since all other bosonic equations have 2 derivatives, and so we would get terms with 4 derivatives in the 7d action. After finding the correct ansatz for $B_{\alpha\beta\gamma\delta}$, we can easily find the nonlinear terms in $\delta\lambda_i^J$:

$$\delta\psi_m|_{S\ term} = \Delta^{-1/10} \gamma_5^{1/2} E_m^A \delta\Psi_A$$

$$\quad = \frac{\sqrt{2}}{288} \Delta^{-1/10} \gamma_5^{1/2} \left[ \Gamma^{\alpha\beta\gamma\delta} \{ m F_{\alpha\beta\gamma\delta} + 4(\Gamma^{\mu\alpha\beta\gamma} m - 2E_{\mu}^m \Gamma^{\alpha\beta\gamma}) F_{\mu\alpha\beta\gamma} - \frac{5}{6k} \Gamma^{\alpha\beta\gamma\delta} m B_{\alpha\beta\gamma\delta} \right] \epsilon$$

where we have substituted the value of $b$ found before. If we now use the ansatz for $B_{\alpha\beta\gamma\delta}$ and $B_{\alpha\beta\gamma}$, the $F_{\mu\alpha\beta\gamma}$ and $B_{\alpha\beta\gamma}$ terms cancel again, as in $\delta\psi_\epsilon$, and we are left with

$$\delta\psi_m|_{S\ term} = \frac{i\sqrt{2}}{288} 4\gamma F_{\alpha\beta\gamma} \Delta^{3/5}$$

$$\quad \left( (\Delta^{-1/5} E_m^\alpha)(\gamma_m n - 2\delta_m^n)(-iF_{\mu\alpha\beta\gamma}) - \gamma_m \tilde{B}_{\alpha\beta\gamma} \right) \epsilon$$

We next substitute for $F_{\mu\alpha\beta\gamma}$ and $\tilde{B}_{\alpha\beta\gamma}$. Then we write $E_m^\alpha$ in terms of scalars via (4.91). Using (3.21) for the contraction of $C_{\mu}$'s which we get, and the identity

$$(\Pi^{-1})_i^C Y_{C,TR}(U^{-1}\gamma^iU\gamma^D)C_{\mu}^D = 0$$

which we easily get by using (4.45), we can rewrite (4.150) as follows

$$\delta\psi_m^{(11d)} = \frac{1}{48\sqrt{3}} \gamma^{\alpha\beta\gamma} S_{\alpha\beta\gamma, A} \left[ -i(\Pi^{-1})_i^B C_{\mu}^D T_{(U^{-1}\gamma^{i}U\gamma^{D})} (\gamma_m^n - 2\delta_m^n) - 4\Delta^{3/5} T^{AB} Y_{B,7m} \right] \eta^{I'J'K'L}$$

Next we want to put in a similar form the expression which we get for $\delta\psi_m$ from the 7 dimensional result for $\delta\lambda_i^J$. From the ansatz in (4.15) and the normalization in (4.36), and substituting the 7 dimensional transformation law of $\delta\lambda_i^J$, we get

$$\delta\psi_m^{(7d)} = -\frac{1}{240\sqrt{3}} \gamma^{\alpha\beta\gamma} S_{\alpha\beta\gamma, A} (\Pi^{-1})_j^A (\gamma^{ij} + 4\delta^{ij})_{J'}$$

$$\quad \left( (\gamma_i)_{K'L'} \epsilon_{I'J'} U_{K'}^j U_{L'}^l \eta_{m}^{JKL} \right)$$

Then we write the spherical harmonic $\eta^{IKL}_m$ as

$$\eta^{IKL}_m = [i(\gamma m - 2\delta m p) C_{m}^{pA} + \gamma m Y^{A}] \eta^{I} (\gamma A)^{KL}$$

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Using the U matrix relation (4.45), we find the identity

\[ Tr(U\gamma_i U^{-1}\gamma_i)Y_A = 4\Delta^{3/5}(\Pi^{-1})_i^A Y_A \]  \hspace{1cm} (4.155)

which will be heavily used in the following. Using this relation and the expression derived for \( \eta^I_{mK} \), we find that

\[ \delta\psi_m^{(7d)} = \frac{1}{48\sqrt{3}} \epsilon^{\alpha\beta\gamma} S_{\alpha\beta\gamma,B} [i Tr(U\gamma_i U^{-1}\gamma_i)(\Pi^{-1})_i^B (\gamma_{mp} - 2\delta_{mp})C^A + 4\Delta^{3/5}T^{AB} Y_A \gamma_m] \epsilon 
- \frac{1}{60\sqrt{3}} \epsilon^{\alpha\beta\gamma} S_{\alpha\beta\gamma,B} (\Pi^{-1})_j^B [i (\gamma_{mp} - 2\delta_{mp})C^A 
+ \gamma_m Y^A \eta^I (\gamma_j U\gamma_i U^{-1}) j' j ' \epsilon \epsilon] \]  \hspace{1cm} (4.156)

Now we work on the last term (the last two lines). For that, we need a few identities: By multiplying the Fierzing relation (3.42) with \( Y_A (\gamma^A) \), we deduce that

\[ \gamma_5 \eta^I = -Y_A (\gamma^A) \gamma_5 \eta^I \]  \hspace{1cm} (4.157)

If we multiply the same relation (3.42) with \( \gamma_5 (\gamma^A) \), we get

\[ -i\gamma_5 \eta^I C_p^A + Y^A \eta^I = Y_B (\gamma^A \gamma^B) \gamma_5 \eta^I \]  \hspace{1cm} (4.158)

from which we also derive, by multiplying with \( C_A^m \), that

\[ \gamma_m \eta^I = iC_m^A Y^B (\gamma_A \gamma_B) \gamma_5 \eta^I \]  \hspace{1cm} (4.159)

From (4.158) and (4.159) we also deduce

\[ \gamma_5 \gamma^p C_p^A \eta^I (U\gamma_A) j = 4C_m^A \eta^J (U\gamma_A) j \]  \hspace{1cm} (4.160)

Using (4.159) and (4.160) it follows that the last term in (4.156) is zero, so that we are left with the correct 11 dimensional result.

### 4.3 The gravitino and \( S_{\alpha\beta\gamma,A} \) transformation laws

Let us recapitulate what we have done so far. We have checked that using our ansätze for the 11 dimensional fields we reproduce the 7 dimensional transformation laws for the graviton \( e^a_\alpha \), the gauge field \( B_{\mu}^{AB} \), the scalars \( \Pi^I \), and the terms involving \( S_{\alpha\beta\gamma} \) in the transformation laws of the gravitini \( \psi^I_\alpha \) and fermions \( \lambda^I \). We have also checked the gauge field dependence of \( F_{\Lambda\Pi\Sigma\Omega} \) by matching terms in \( \delta^{(11d)}_{\text{susy}} F_{\Lambda\Pi\Sigma\Omega} \) with the variation of all the fields in the ansatz for \( F_{\Lambda\Pi\Sigma\Omega} \). The analysis of the independent fluctuation \( S_{\alpha\beta\gamma,A} \) was implicitly done when we studied the \( F_{\alpha\beta\gamma} \) sector. However, for completeness of our work we give a separate check on \( S_{\alpha\beta\gamma} \) susy transformation law.

We begin by considering the case when \( B_{\mu}^{AB} = 0 \) in the \( S_{\alpha\beta\gamma,A} \) transformation law and we find \( \delta S_{\alpha\beta\gamma,A} \) from

\[ \frac{i}{\sqrt{6}} \delta F_{\mu\alpha\beta\gamma} |_{B=0} = \delta S_{\alpha\beta\gamma,A} |_{B=0} \partial_\mu Y^A \]  \hspace{1cm} (4.161)
because the $B^A_{\alpha\beta}$ terms in $F_{\mu\alpha\beta\gamma}$ appear at least quadratically, (so $\delta B^A_{\alpha\beta}$ will always be multiplied by $B^C_D$, which we put to zero). Using the transformation law for $A_{\Lambda\Pi\Sigma}$ in (2.14) and the fact that

\[ \bar{\partial}(4.45), \]

we get

\[ \delta S_{\alpha\beta\gamma,A}|_{B=0} = \partial_{\mu}Y^A = -i\sqrt{3}\partial_{\mu}\varepsilon\Gamma_{\alpha\beta}\Psi_{\gamma} \]

\[ = -i\sqrt{3}{4} \left\{ \partial_{\mu}[\bar{\varepsilon}\Delta^{-3/5}(\tau_{\alpha\beta\gamma}\gamma\psi_{\gamma}) + \frac{i}{5}\bar{\tau}_{\alpha\beta\gamma}\gamma^m\psi_m] \right\} \]

\[ -\partial_{\alpha}[E^{n}_{\mu}\Delta^{-2/5}\bar{\varepsilon}(-2i\gamma_m\gamma_{\beta}\psi_{\gamma}) + \frac{1}{5}\bar{\tau}_{\beta\gamma}(-2\gamma_{nm} + 3\delta_{nm})\psi^m)] \]  \hspace{1cm} (4.163)

We split this contribution into 3 terms, corresponding to the terms in the 7 dimensional transformation law, one with $\partial_{\alpha}$ acting on $\psi_{\gamma}l'$ and $\lambda_{i}l'$, one with $\partial_{\alpha}$ acting on $\Pi_{A}i$, and one with no $\partial_{\alpha}$. Indeed, in 7 dimensions we have

\[ \delta S_{\alpha\beta\gamma,A}|_{B=0} \]

\[ = \frac{i\sqrt{3}}{12}\delta_{AB}(\Pi^{-1})^i B(3\bar{\varepsilon}\tau_{[\alpha\beta]}^i\psi_{\gamma}) - \bar{\varepsilon}\tau_{\alpha\beta\gamma}\lambda^i \]

\[ -i\sqrt{3}{4}\delta_{ij}\Pi_{A}j \nabla_{[\alpha} (2\bar{\varepsilon}\tau_{\beta\gamma}^l\psi_{\gamma}) + \bar{\varepsilon}\tau_{\beta\gamma}\lambda^i) \]  \hspace{1cm} (4.164)

where $\nabla_{\alpha}(...)_{i}$ acts as $\partial_{\alpha}(...)_{i} + Q_{\alpha}\bar{\psi}_{\gamma_{i}}(\Pi^{-1})_{i}^A (\partial_{\alpha}\Pi_{A}^J)(...)j$. (If $B^A_{\alpha\beta}$ is not zero, $Q_{\alpha ij}$ will contain also a B term).

The term with no $\partial_{\alpha}$ in (4.163) is given by

\[ -\frac{i\sqrt{3}}{4}\Delta^{-3/5}[\Delta^{3/5}(\partial_{\mu}\Delta^{-3/5}) + \partial_{\mu}][\bar{\varepsilon}\tau_{[\alpha\beta\gamma]}\psi_{\gamma}] + \frac{i}{5}\bar{\varepsilon}\tau_{\alpha\beta\gamma}\gamma^m\psi^m] \]  \hspace{1cm} (4.165)

To evaluate the contribution with the gravitino $\psi_{\gamma}$, we make use of the equations (4.39) and (4.45). From $\bar{\varepsilon}\tau_{\alpha\beta\gamma}\psi_{\gamma}$ we obtain a factor $U\bar{\gamma}\eta U^T \sim U\phi_5 U^T$. This is the combination which appears on the l.h.s. of (4.45) (because $Y_{A}^\gamma \sim \phi_{5}$). Substituting (4.45), the $\partial_{\mu}$ derivatives of $\Delta^{3/5}$ cancel and we are left with

\[ -\frac{i\sqrt{3}}{4}\bar{\varepsilon}i^l_{\alpha\beta\gamma}^l\psi_{\gamma}i^l_{\alpha\beta\gamma}^j \Pi^{-1}_{i} A_{l} \partial_{\mu}Y_{A} \]  \hspace{1cm} (4.166)

Similarly, the contribution of the spin 1/2 fermions in (4.165) reads

\[ \frac{i\sqrt{3}}{4\cdot3}(\Pi^{-1})^i_{A} \bar{\Omega}i^l_{\alpha\beta\gamma}^l \varepsilon_{i^l_{\alpha\beta\gamma}^j} \partial_{\mu}Y_{A} \]  \hspace{1cm} (4.167)

We compare the sum of these two contributions with the 7-dimensional result

\[ \frac{i\sqrt{3}}{12}(\Pi^{-1})^i_{A} \varepsilon(3\bar{\tau}_{[\alpha\beta\gamma]}^i\psi_{\gamma} - \tau_{\alpha\beta\gamma}\lambda^i)\partial_{\mu}Y_{A} \]  \hspace{1cm} (4.168)

and we can conclude that we find agreement.
The term with $\partial_\alpha$ acting on the scalars is given by

$$
\frac{i\sqrt{3}}{4} \left( \partial_{\alpha}(E^\mu_\mu \Delta^{-2/5}) \left( -2i\varepsilon_\gamma \gamma_5 \tau_\beta \psi_\gamma + \frac{1}{5} \varepsilon_{\tau_\beta |}(-2\gamma_{nm} + 3\delta_{nm})\psi^m \right) 
+ \Delta^{-2/5}E^m_\mu \left( -2iC^IJ_n \varepsilon_I \tau_\beta \psi_\gamma \partial_{\alpha}(U^{I'}_I U^{J'}_J) + 
\frac{1}{5} \varepsilon_I \tau_\beta \lambda_{J'K'L'} \eta^I(-2\gamma_{nm} + 3\delta_{nm})\partial_{\alpha}(U^{I'}_I U^{J'}_J U^{K'}_K U^{L'}_L) \eta^{J'K'L'}_m \right) \right) (4.169)
$$

The spherical harmonic of the spin 1/2 fields, contracted with $\lambda_{J'K'L'}$, yields

$$
\frac{1}{5} \eta^I(-2\gamma_{nm} + 3\delta_{nm})\eta^{J'K'L'} \lambda_{J'K'L'} U^{J'}_J U^{K'}_K U^{L'}_L 
= 3\Omega^{IJ}C^{KL}_n \lambda_{J'K'L'} U^{J'}_J U^{K'}_K U^{L'}_L \eta^{J'K'L'}_m (4.170)
$$

Together with (4.170) we find for (4.169)

$$
-\frac{i\sqrt{3}}{4} (\partial_{\alpha} \Pi^I)(\Pi^{-I})B(\partial_{\alpha} \Pi^B)\varepsilon(2\tau_\gamma \gamma_5 \psi_\gamma + \varepsilon_I \tau_\beta \psi_\gamma \lambda_{J'K'}) \partial_{\mu} Y^A (4.171)
$$

This agrees with the result in 7 dimensional gauged supergravity

$$
-\frac{i\sqrt{3}}{4} \Pi^I(\Pi^{-1})B(\partial_{\alpha} \Pi^B)\varepsilon(2\tau_\gamma \gamma_5 \psi_\gamma + \varepsilon_I \tau_\beta \psi_\gamma \lambda_{J'K'}) \partial_{\mu} Y^A (4.172)
$$

Finally, we consider the terms with $\partial_\alpha$ acting on $\psi_{\alpha I'}$ and $\lambda_{I'}$

$$
\frac{i\sqrt{3}}{4} \Delta^{-3/5} \Delta^{1/5} E^m_\mu \left( -\partial_\alpha \varepsilon_2 \gamma_5 \tau_\beta \psi_\gamma \lambda_{J'K'}(U^{I'}_I U^{J'}_J + \frac{1}{5} \tau_\beta \lambda_{J'K'}(U^{I'}_I U^{J'}_J U^{K'}_K U^{L'}_L \eta^{J'K'L'}_m \right) \right) (4.173)
$$

In the last term we use (4.170) and get

$$
-\frac{i\sqrt{3}}{4} \Delta^{-2/5} E^m_\mu (\partial_\alpha \varepsilon_2 \gamma_5 \tau_\beta \psi_\gamma) U^{I'}_I U^{J'}_J C^{IJ}_n 
+ \frac{i\sqrt{3}}{4} \Delta^{-2/5} E^m_\mu \Omega^{IJ}C^{KL}_n U^{I'}_I U^{J'}_J U^{K'}_K U^{L'}_L \eta^{J'K'L'}_m \partial_{\alpha} \varepsilon_I \tau_\beta \lambda_{J'K'} (4.174)
$$

Using (4.111) and $U^T \tilde{\Omega} U = \tilde{\Omega}$ we obtain

$$
-\frac{i\sqrt{3}}{2} \Pi^I \partial_\alpha \varepsilon_2 \gamma_5 \tau_\beta \psi_\gamma \lambda_{J'K'} \partial_{\mu} Y^A - \frac{i\sqrt{3}}{4} \Pi^I \partial_\alpha \varepsilon_2 \gamma_5 \tau_\beta \lambda_{J'K'} \partial_{\mu} Y^A (4.175)
$$

This agrees with the d=7 result

$$
-\frac{i\sqrt{3}}{4} \Pi^I \partial_\alpha (2\varepsilon_2 \gamma_5 \gamma_5 \gamma_5 \psi_\gamma \lambda_{J'K'} \partial_{\mu} Y^A \delta_{ij} (4.176)
$$

This was the last term to be checked in the transformation law of the 3 index tensor field $S_{\alpha \beta \gamma A}$ with $B^A_{AB}$ set to zero, which therefore is in complete agreement with the 11 dimensional transformation laws.
Now, we let the gauge fields to take non-zero values. Since the dependence on the gravitini and gauge fields was already checked (see the analysis performed for the susy law in the $F_{\alpha\beta\gamma\mu}$ sector), we will only check the $F_{\alpha\beta}$ 7-dimensional susy law. Hence we keep from the 11-dimensional susy variation of $S_{\alpha\beta\gamma}$ only the terms with potential contribution to the spin 1/2 and $F_{\alpha\beta}$ dependent terms in the $d=7$ susy law:

$$
-i\sqrt{6}\delta S_{\alpha\beta\gamma,A} Y^A |_{F_{\alpha\beta} \text{ terms}} = \frac{6}{\sqrt{2}} \epsilon_{ABCD} F_{\alpha\beta} \delta(B_{\alpha\beta}^{EF}) |_{\lambda \text{ terms}} Y_F C_D^\mu (T \cdot Y)^E \cdot \cdot \cdot Y
$$

After we substitute the ansätze of various 11-dimensional fields, we need to evaluate the spherical harmonic of

$$\bar{\eta}^I \left( \frac{1}{5} \gamma_{mn} \gamma^m \psi_m + 2 \gamma_{[m} \psi_{n]} \right)$$

which is

$$3(\bar{\eta}^I \gamma_{mn} \eta^J \phi_5^{KL} + 2 \bar{\eta}^I \gamma_{[m} \eta^J C_{n]}^{KL}).$$

The contribution of the last two terms in (4.177) is

$$-\frac{9i\sqrt{2}}{2} \Delta^{-1/5} \epsilon_{\mu \nu \chi \delta} \gamma_{\mu \nu \chi \delta} (\lambda i_{[m} \eta_{j]}^{KL}) C_{n]}^{KL}$$

$$= \frac{3\sqrt{2}}{5} \bar{\eta}^I \gamma_{mn} \eta^J \phi_5^{KL} + 2 \bar{\eta}^I \gamma_{[m} \eta^J C_{n]}^{KL})$$

Use now the identity

$$\gamma^{[ij \chi k]} = \frac{1}{3} (\gamma^{ij \chi k} \lambda_l - \gamma^{ij \chi k} \lambda_l )$$

(4.179)

to rewrite (4.178)

$$-\sqrt{2} F_{[\alpha\beta}^{DE} (\epsilon_{\tau \gamma} | \gamma^{ij \chi k} \lambda_l + \epsilon_{\tau \gamma} | \gamma^{ij \chi k} \lambda_l ) \Pi_A^{i j} (\Pi^{(\mu)}_{B} \Pi^{(\tau)}_{C}) C_{\alpha}^{A} \gamma_{\gamma}^{\chi} Y_D$$

$$= \frac{6}{\sqrt{2}} \epsilon_{ABCD} (F_{[\alpha\beta} Y) A \delta(B_{\gamma}^{D}) C_{\mu} D (T \cdot Y)^E \cdot \cdot \cdot Y$$

(4.180)

where to reach the final result in (4.180) we replaced $\gamma^{ij \chi k}$ by $-1/2 \epsilon_{ijkmn} \gamma_{mn}$ and we also used the property of the scalar fields $\Pi_{A}^{m}$ of having determinant one.

The first term in (4.177) can be written as the sum of two terms upon using the Schouten identity to release the variation $\delta(B_{\gamma}^{CF})$ from its contraction with $Y_F$.

One of these terms cancels precisely the contribution of (4.180), and the other, $(F_{[\alpha\beta} Y) A \delta(B_{\gamma}^{BC}) C_{\mu} Y F \epsilon_{ABCD} Y$ will give the 7-dimensional susy variation $\delta(d = 7)$ $S_{\alpha\beta\gamma,A} \lambda_F \text{ terms} Y^A$. Since the susy variation of a gauge invariant object must be gauge
invariant, we will not show explicitly the cancelation of terms produced by the 11-dimensional susy variation which are proportional with bare $D^A_{\alpha}$ gauge fields. The same argument implies that the partial derivative $\partial_{\alpha}$ which we obtained in (4.176) is in fact, a gauge covariant $D_{\alpha}$ derivative. (That this is the case, we already know from the analysis of the sector $F_{\alpha\beta\gamma\mu}$ where we derived the gravitino dependent terms in (4.176), and we obtained in fact a gauge covariant derivative).

Let us now turn back to $\delta \psi_{\alpha}$. In 7 dimensions, we have:

$$\delta \psi_{\alpha}''|_{S=B=0} = D_{\alpha} \epsilon'' + \frac{1}{4} (\Pi^{-1})_{i}^{A} (\partial_{\alpha} \Pi^{i}) (\gamma_{j})_{I'} J' \epsilon_{I'} - \frac{1}{20} (\Pi^{-1})_{i}^{A} (\Pi^{-1})_{i}^{A} \tau_{\alpha} \epsilon''$$

But in 11 dimensions we have got

$$\delta \Psi_{\alpha}|_{B=S=0} = \partial_{\alpha} \epsilon + \frac{1}{4} \Omega_{\alpha}^{MN}(E) \Gamma_{MN}|_{B=0} \epsilon + F... terms + 3-fermi terms$$

By direct substitutions one obtains

$$\Omega_{\alpha}^{MN} \Gamma_{MN}|_{B=0} = \omega \alpha^{ab} \tau_{ab} + \gamma_{mn} E^m_{\mu \phi} \partial_{\alpha} E^m_{\mu}$$
$$\Omega_{\mu}^{MN}(E) \Gamma_{MN}|_{B=0} = \gamma_{mn} E^m_{\mu \nu \rho} (2 \partial_{[\mu} E_{\nu]} - E_{\mu \nu} E_{\rho \sigma} \partial_{\rho} E_{\sigma})$$
$$\Omega_{\mu}^{MN}(E) \Gamma_{MN}|_{B=0} = \gamma_{mn} E^m_{\mu \nu \rho} (2 \partial_{[\mu} E_{\nu]} - E_{\mu \nu} E_{\rho \sigma} \partial_{\rho} E_{\sigma}) + \tau^\alpha \gamma_{5} (\Delta_{1/5} \partial_{\alpha} E^m_{\mu} + (\Delta_{1/5} E^m_{\mu} \partial_{\alpha} E_{\rho \sigma}))$$

For the terms involving $F_{\mu \nu \rho \sigma}$ and $F_{\alpha \mu \nu \rho}$, we find

$$E_{\alpha}^{A} \delta \Psi_{\alpha}|_{F_{\mu \nu \rho \sigma} term, B=S=0} = \frac{3}{288} \Delta^{-1/10} \gamma_{5}^{-1/2} \epsilon_{\alpha}$$
$$+ \frac{2}{3} \left( \frac{Y_{A}(T_{2})^{AB} Y_{B}}{(Y_{A}Y_{B})^{2}} - 1 \right) - \frac{1}{4 \Delta^{1/5}} (E_{\sigma m} - 2 E_{\sigma \gamma_{5} \gamma_{m}}) \tau^{\alpha} C^{\sigma}$$

$$E_{\alpha}^{A} \delta \Psi_{\alpha}|_{F_{\mu \nu \rho \sigma} term, B=S=0} = \frac{3}{288} \Delta^{-1/10} \gamma_{5}^{-1/2} \epsilon_{\alpha}$$
$$+ \frac{2}{3} \left( \frac{Y_{A}(T_{2})^{AB} Y_{B}}{(Y_{A}Y_{B})^{2}} - 1 \right) - \frac{1}{4 \Delta^{1/5}} (E_{\sigma m} - 2 E_{\sigma \gamma_{5} \gamma_{m}}) \tau^{\alpha} C^{\sigma}$$

We also have that

$$\delta \psi_{\alpha}|_{S=B=0} = \Delta^{-1/10} \gamma_{5}^{-1/2} \epsilon_{\alpha}$$

Putting everything together, we find, besides the usual term $\partial_{\alpha} \epsilon'' + \omega \alpha^{ab} \tau_{ab} \epsilon''$, terms with $\tau_{\alpha}^{\beta}$, terms with $\tau_{\alpha}$, and terms without any $\tau_{\alpha}$'s. After some algebra, the terms
with $\tau^\beta_\alpha$ cancel, as they should, whereas the terms without any $\tau_\alpha$’s simplify to
\[
\frac{1}{4}\gamma_{mn}E^m_{\mu}(\partial_\alpha E^\mu_{\mu})\epsilon + \epsilon_1'(\partial_\alpha U^I_{\sigma})\eta^I + \frac{3i}{4}\gamma_m(\Delta^{1/5}E^m_{\sigma})\epsilon\Delta^{-6/5}C_A
\]
\[
\left(\frac{\partial_\alpha T_{AB}Y_B}{Y_A T_{AB}Y_B} - \frac{T_{AB}Y_B}{(Y_A T_{AB}Y_B)^2}(Y_C\partial_\alpha T_{CD}Y_D)\right)
\]  
(4.187)
to be compared to
\[
-\frac{1}{4}(\Pi^{-1})_i^jA^i(\partial_\alpha Y_B)\gamma^{j}\gamma_{5} - Y_D(\gamma_B)^I_{J}\gamma_{5}
\]  
(4.188)
We will first of all work on the first term in (4.187). Using the Fierzing relation (3.42) twice for each $\gamma$ matrix acting on $\eta^I$, we prove that
\[
C^{mB}C^{nD}\gamma_{mn}\eta^I = \left[(\gamma_{DB})^I_J + Y_B(\gamma_B)^I_J\gamma_{5} - Y_D(\gamma_B)^I_J\gamma_{5}\right]\eta^I
\]  
(4.189)
Using (4.90) and (4.91) to rewrite the vielbeins, the previous relation to get rid of $\gamma_{mn}$ and (A.12) and (4.45) to get rid of the resulting traces we obtain, after some algebra,
\[
\frac{1}{4}E^m_{\nu}\partial_\alpha E^\nu_{\mu}\gamma_{mn}\epsilon = -\partial_\alpha U^I_{\nu}\eta^I\epsilon_\nu - \frac{1}{4}\Pi\partial_\alpha(\Pi^{-1})^A_{j}(\gamma_{ij})^I_{J}\nu U^I_{\nu}\eta^J\epsilon_\nu
\]
\[
-\frac{1}{2}Y_B(\Delta^{3/5}(\Pi^{-1})_i^jA^i(\gamma_{i})^{I}_{J}\nu U^I_{J}\gamma_{5}\eta^J\epsilon_\nu
\]
\[
+\frac{1}{2}\partial_\alpha(\Pi^{-1})_{j}^{A}(\Pi^{-1})_{j}^{B}Y_B
\]
\[
\Delta^{3/5}(\gamma^{j})^I_J\nu U^I_{J}\gamma_{5}\eta^J\epsilon_\nu
\]  
(4.190)
The first term cancels against the second term in (4.187), and the second term gives the correct 7 dimensional result in (4.188). That means that the last two terms, together with the last term in (4.187) should give zero.

Using that $\Delta^{-6/5} = Y \cdot T \cdot Y$ and the identity
\[
\Delta^{6/5}\partial_\alpha(Y \cdot T \cdot Y)T_{AB}Y_D - \partial_\alpha T_{AB}Y_D = \Delta^{6/5}\partial_\alpha(Y \cdot T \cdot Y)T_{BD}Y_D(\delta^{AB} - Y_A Y_B)
\]
\[
-\partial_\alpha T_{BD}Y_D(\delta^{AB} - Y_A Y_B)
\]  
(4.191)
and rewrite the last two terms in (4.190) as
\[
\Delta^{-3/5}(\delta^{AB} - Y_A Y_B)\frac{1}{4Y \cdot T \cdot Y}\left[\partial_\alpha(Y \cdot T \cdot Y)T_{BD}Y_D
\right.
\]
\[
- \partial_\alpha T_{BD}Y_D\]  
\left[\Pi A^i(\gamma_{i})^{I}_{J}\nu U^I_{J}\gamma_{5}\eta^J\epsilon_\nu
\right]
\]  
(4.192)
Now taking the last term in (4.187), substituting for $E^m_{\sigma}$ from (4.90) , using the summation relation (A.12) and the Fierzing relation
\[
C^{mB}C^{nD}\gamma_{mn}\eta^I = -i[(\gamma_{B})^I_J\gamma_{5}\eta^J + Y_B\eta^I]
\]  
(4.193)
which we can prove by using (3.42), we get the same result as in (4.192), but with the opposite sign, as we should. So all the extra contributions cancel and we are left with the 7 dimensional result (4.188). Note that at this moment if we chose the $a=0$
solution for (4.104), it will not work. So, as promised, the ansatz for the scalar field
dependence of $F_{\mu
u\rho\sigma}$ is fixed by the consistency of the gravitino transformation law.

The hardest term in $\delta \psi_\alpha$ is the $\tau_\alpha$ term, which becomes (after some algebra)

$$
\tau_\alpha (\Delta^{-1/5} E^{\mu\nu\rho\sigma} \gamma_m) \left[ i \frac{5}{50} (\Delta^{-1}\partial_\mu \Delta) \epsilon + \frac{1}{10} \gamma_\mu \epsilon - i \frac{1}{5} \epsilon U_i \partial U^i \right] 
- \frac{3}{20} \Delta^{-6/5} \tau_\alpha \left[ 1 + \frac{1}{3} \left( \frac{T}{Y\Lambda} F_{\mu\nu\rho\sigma} - 5 \right) - \frac{2}{3} \left( \frac{Y A (T^2)_{AB} Y_B}{Y A (T^2)_{AB} Y_B^2} - 1 \right) \right] \epsilon
- \frac{i}{20} \tau_\alpha \left( \Delta^{-1/5} E^{\mu\nu} E^{\rho\sigma} \left( \epsilon \partial_\mu \Delta \right) \left( \gamma_{mnq} - 2 \delta_{m\nu} \gamma_n \right) \right) \epsilon
$$

(4.194)

to be compared with

$$
\frac{1}{20} (\Pi^{-1})_A^A \epsilon \tau_\alpha \epsilon
$$

(4.195)

This last problem turns out to be surprisingly complicated. We have only partial
results; they seem to involve the explicit expression of the matrix $U$ and we prefer to
devote a separate paper to this issue. The same holds for the scalar dependent terms
in the susy law of the 7 dimensional spin 1/2 fields.

### 5 Bosonic equations of motion and action

In this section we will do some further checks on our ansatze by looking at the bosonic
action and equations of motion, and reproducing some of the corresponding equations
of motion and terms in the action in seven dimensions.

First, we give the 11 dimensional bosonic equations of motion following from the
action in (2.2):

$$
R_{\Lambda\Pi} = -\frac{1}{6} (F_{\Lambda\Lambda_1\Lambda_2\Lambda_3} F_{\Pi\Lambda_1\Lambda_2\Lambda_3} - \frac{1}{12} G_{\Lambda\Pi} F^2) \tag{5.1}
$$

$$
\partial_\Lambda (EF_{\Lambda\Lambda_1\Lambda_2\Lambda_3} \rho) = \frac{k}{\sqrt{2(24)^2}} \epsilon_{\Lambda_1...\Lambda_{11}} F_{\Lambda_4...\Lambda_7} F_{\Lambda_8...\Lambda_{11}} \tag{5.2}
$$

$$
B_{\Lambda\Pi\Sigma\Omega} = 0 \tag{5.3}
$$

In seven dimensions we have the bosonic action

$$
e^{-1} \mathcal{L}'_7 = -\frac{1}{2} R + \frac{1}{4} m^2 (T^2 - 2T_{ij} T^{ij}) - \frac{1}{2} P_{\alpha ij} P^{\alpha ij} - \frac{1}{4} (\Pi_A^i \Pi_B^j F_{\alpha ij}^{AB})^2 + \frac{1}{2} (\Pi^{-1} A S_{\alpha\gamma A})^2 + \frac{1}{48} m e^{-1} \epsilon_{\alpha\beta\gamma\delta\eta\zeta} \delta^{AB} S_{\alpha\beta\gamma, A} F_{\delta\eta\zeta, B} + \frac{m}{8} e^{-1} \Omega_5 [B]

- \frac{m^{-1}}{16} e^{-1} \Omega_3 [B] + \frac{ie^{-1}}{16\sqrt{3}} \epsilon_{\alpha\beta\gamma\delta\eta\zeta} F_{ABCDE} [\delta^{AG} S_{\alpha\beta\gamma, G} F_{\delta\eta\zeta, E} [B] \tag{5.4}

For the equations of motion, we will put the gauge field $B_{\alpha}^{AB}$ to zero, so in the action
we need to keep only terms at most linear in $B_{\alpha}^{AB}$ which we can rewrite as follows.

$$
e^{-1} \mathcal{L}'_7 = -\frac{1}{2} R + \frac{1}{4} m^2 (T^2 - 2T_{ij} T^{ij}) - \frac{1}{2} P_{\alpha ij} P^{\alpha ij} + \frac{1}{2} (\Pi^{-1} A S_{\alpha\gamma A})^2 + \frac{1}{48} m e^{-1} \epsilon_{\alpha\beta\gamma\delta\eta\zeta} \delta^{AB} S_{\alpha\beta\gamma, A} F_{\delta\eta\zeta, B} \tag{5.4}
$$
= -\frac{1}{2} R + \frac{1}{4} m^2 (T^2 - 2T_{AB} T^{AB}) + \frac{1}{8} Tr(\partial_\alpha T^{-1} \partial^\alpha T)
- \frac{1}{2} B^\alpha_{AB} (T \partial_\alpha T^{-1})_{AB} + \frac{1}{2} T^{AB} S_{\alpha \beta \gamma, A} S^{\alpha \beta \gamma}_{B}
+ \frac{1}{48} me^{-1} \epsilon^{\alpha \beta \gamma \delta \kappa \xi} \delta^{AB} S_{\alpha \beta \gamma, A} F_{\delta \kappa \xi, B}
\quad (5.5)

We notice that the action is now expressed in terms of $T_{AB}$ instead of $\Pi_\alpha^i$, so we obtain the following bosonic equations of motion for $T_{AB}$, $B^\alpha_{AB}$ and $S_{\alpha \beta \gamma, A}$ and the metric $g_{\alpha \beta}$, respectively

\begin{align*}
\frac{1}{2} m^2 (T \delta_{AB} - 2T_{AB}) + \frac{1}{4} (T^{-1} \partial_\alpha (T \partial_\alpha T^{-1}))_{AB} + \frac{1}{2} S_{\alpha \beta \gamma, A} S^{\alpha \beta \gamma}_{B}
- \frac{1}{5} T_{AB} \frac{1}{2} m^2 (T^2 - 2T \partial_\alpha T^2) + \frac{1}{4} \partial_\alpha T (\partial_\alpha T^{-1}) + \frac{1}{2} S_{\alpha \beta \gamma, A} S^{\alpha \beta \gamma}_{B} T^{AB} = 0 \tag{5.6}

\end{align*}

\begin{align*}
-\frac{1}{12} me^{-1} \epsilon^{\alpha \beta \gamma \delta \kappa \xi} S_{\beta \gamma \delta, A} S_{\delta \kappa \xi, B} - \frac{1}{2} (T \partial_\alpha T^{-1})_{[AB]} = 0 \tag{5.7}

T^{AB} S_{\alpha \beta \gamma, B} + \frac{1}{24} me^{-1} \epsilon^{\alpha \beta \gamma \delta \kappa \xi} F_{\delta \kappa \xi, A} = 0 \tag{5.8}

R^{(7)}_{\alpha \beta \gamma \delta \kappa \xi} - \frac{1}{10} g_{\alpha \beta} [m^2 (T^2 - 2T_{AB}) - 4T^{AB} S_{\alpha \gamma \delta, A} S_{\gamma \delta \beta, B}]
- \frac{1}{4} T \partial_\alpha T^{-1} \partial_\beta T - 3T^{AB} S_{\alpha \gamma \delta, A} S_{\gamma \delta \beta, B} = 0 \tag{5.9}

\end{align*}

We note here that in (5.6) we have used Lagrange multipliers for the condition $\det T_{AB} = 1$. From (5.6) we obtain also

\begin{align*}
\partial_\alpha (\partial_\alpha T^{-1})_{[AB]} = 0 \tag{5.10}
\end{align*}

and from (5.10), together with (5.7) we also get that

\begin{align*}
\epsilon^{\alpha \beta \gamma \delta \kappa \xi} \delta^{AB} S_{\alpha \beta \gamma, [A} \partial_{\delta} S_{\delta \kappa \xi, B]} = 0 \tag{5.11}
\end{align*}

Now we want to see that we reproduce these equations from the 11 dimensional equations of motion. First, we notice that (5.3) was already discussed. The only nontrivial component is the $\{\alpha \beta \gamma \delta\}$, which is just the equation of motion of the antisymmetric tensor, $S_{\alpha \beta \gamma, A}$. Let’s look now at (5.2), setting the gauge field to zero.

Substituting the ansatze for $F_{\Lambda \Pi \Sigma \Omega}$ and $g_{\Lambda \Pi}$ the $\{\nu \rho \sigma\}$ component of (5.2) becomes

\begin{align*}
Y_{(A} \partial_\mu Y_{B)} T_{CB} \left[ 2(T \delta_{AC} - 2T_{AC}) + (T^{-1} \partial^\alpha (\partial_\alpha T^{-1}))_{AC} + 2S_{\alpha \beta \gamma, A} S^{\alpha \beta \gamma, C} \right]
- Y_{(A} \partial_\mu Y_{B)} S_{\alpha \beta \gamma, A} (S_{\alpha \beta \gamma, C} T_{CB} + \frac{1}{6} m e^{-1} \epsilon^{\alpha \beta \gamma \delta \kappa \xi} \partial_\delta S_{\delta \kappa \xi, B})
- \frac{1}{3} Y_{(A} \partial_\mu Y_{B)} m e^{-1} \epsilon^{\alpha \beta \gamma \delta \kappa \xi} S_{\alpha \beta \gamma, A} \partial_\delta S_{\delta \kappa \xi, B} = 0 \tag{5.12}
\end{align*}

We notice that the first line is the scalar equation of motion (5.6), the second line is the antisymmetric tensor equation of motion in (5.8) and the third is (5.11) which is a combination of the scalar and gauge field equations of motion.

Similarly, by substituting the ansatze for $F_{\Lambda \Pi \Sigma \Omega}$ and $g_{\Lambda \Pi}$, the $\{\nu \rho \sigma\}$ component of (5.2) becomes

\begin{align*}
\partial_\mu Y^{A} \partial_\alpha Y^{B} [(T^{-1} \partial_\alpha T)_{BA} + \frac{1}{6} \epsilon^{\alpha_1 \ldots \alpha_6} s_{\alpha_1 \alpha_2 \alpha_3, A} s_{\alpha_4 \alpha_5 \alpha_6, B}] = 0 \tag{5.13}
\end{align*}
which is just the equation of motion for the gauge field, (5.7). The \(\{\beta\gamma\delta\}\) component of (5.2) is

\[
Y^A[4\delta^T\partial_\alpha S_{\alpha\beta\gamma},A + \frac{1}{6}\epsilon^{\alpha\beta\gamma\alpha_1...\alpha_4}(\partial_\alpha T^{AB})S_{\alpha_2\alpha_3\alpha_4,B} - S_{\alpha\beta\gamma},A T^{2AB}Y_B]
\]

\[
\frac{Y^A}{Y \cdot T \cdot Y}[\frac{4}{\frac{Y^A}{Y \cdot T \cdot Y}}[4\delta^T\partial_\alpha S_{\alpha\beta\gamma},A Y \cdot \partial_\delta \cdot Y + \frac{1}{6}\epsilon^{\alpha\beta\gamma\alpha_1...\alpha_4}S_{\alpha_2\alpha_3\alpha_4,B} T^{AB}(Y \cdot \partial_\alpha T \cdot Y)]
\]

\[
\left(T - 2 \frac{Y \cdot T^2 \cdot Y}{Y \cdot T \cdot Y}\right) Y^A \left[\frac{1}{6} \epsilon^{\alpha\beta\gamma\alpha_1...\alpha_4} \partial_\alpha S_{\alpha_2\alpha_3\alpha_4,A} + S_{\alpha\beta\gamma},B T^{AB}\right] = 0 \tag{5.14}
\]

where all three lines are now zero due to the antisymmetric tensor equation of motion, (5.8). The \(\{\alpha\beta\mu\}\) component of (5.2) gives

\[
C^A_\mu \left\{ \left(T_{AC} - \frac{T^{AD}Y_BT^{CE}Y_E}{Y \cdot T \cdot Y}\right)(\partial_\gamma S_{\alpha\beta\gamma},B \delta^{BC} + S_{\alpha\beta\gamma},B(\partial_\gamma T_{BD})T^{-1}_{DC})
\right.
\]

\[
- (S_{\alpha\beta\gamma,C}T^{BC} + 1 \epsilon^{\alpha\beta\gamma\alpha_1...\alpha_5})
\]

\[
\frac{1}{Y \cdot T \cdot Y} \left(\partial_\gamma T^{AD}Y_B - T^{AD}Y_B \frac{Y \cdot \partial_\gamma T \cdot Y}{Y \cdot T \cdot Y}\right) \right\} \tag{5.15}
\]

In the first line, the second bracket is zero upon using the equation of motion for the 7d antisymmetric tensor, (5.8), to convert both \(S_{\alpha\beta\gamma},B\)’s into \(\partial_\alpha S_{\alpha_2\alpha_3\alpha_4,C}T^{-1}_{BC}\), whereas the second line gives directly the antisymmetric tensor equation.

The Einstein’s equations (5.1) are considerably more involved. We have computed all the terms for zero gauge field, but the task of reconstructing the seven dimensional field equations is quite laborious, and we have not completed it, but we can see that we get nontrivial combinations of these seven dimensional field equations. The fact that already from the antisymmetric field equation (5.2) we get all the seven dimensional field equations (except Einstein’s equation) is already quite nontrivial. Moreover, the eleven dimensional Einstein’s equation for \(\Lambda \Pi = \alpha\beta\) contains the seven dimensional Einstein’s equation, (5.9), together with many more terms. So we can say that the seven dimensional bosonic equations of motion solve the eleven dimensional equations of motion in all the sectors we have checked. We leave the complete check of the Einstein’s equations to the diligent reader.

Let’s now turn to the bosonic action in (5.4). Part of the action was already calculated. We have also reproduced the scalar field potential, and this is how we fixed the dependence of \(F_{AIP\Omega i}\) on the scalars \(\Pi_A^i\). The condition that the Einstein action in 11 dimensions gives us the Einstein action in 7 dimensions gave us the ansatz for the 11 dimensional vielbein component \(E_3^A\). Incidentally, we note also that we found the kinetic terms for the gravitini \(\psi_\alpha^T\) and spin 1/2 fermions \(\lambda_\nu^A\) (without the Q-connection piece). By requiring that we reproduce them we have fixed the ansatz for the components of the 11 dimensional gravitino. So we need to reproduce the scalar field and gauge field kinetic terms, \(-\frac{1}{2}P_{\alpha ij}P^{\alpha ij} - \frac{1}{4}(\Pi_A^i\Pi_B^j F_{\alphaij}^{AB})^2\), and also the \(S_{\alpha\beta\gamma,A}\) terms. The \(S_{\alpha\beta\gamma,A}\) terms give us the mechanism for ‘self-duality in odd dimensions’ at the level of the action (we have deduced this mechanism from the susy laws).

Let us first look at the \(P_{\alpha ij}P^{\alpha ij}\) term. It is equal to

\[
\frac{1}{8}Tr(\partial_\alpha T^{-1}\partial^\alpha T) - \frac{1}{2}B_{\alpha}^{AB}(T^{-1}\partial_\alpha T)_{AB} - \frac{1}{4}(T_{AC}T^{-1}_{BD}B_{\alpha}^{AB}B_{\alpha}^{CD} - (B_{\alpha}^{AB})^2) \tag{5.16}
\]
We will look only at the first two terms, the last two being inferred as being the gauge invariant completion (already the term linear in $B$ should be dictated by gauge invariance, so this will be a nontrivial check on the algebra). The contributions to this term come from the 11 dimensional Einstein action and the Maxwell term, $-1/48F^2$. The contributions to $P_{\alpha i}P^{\alpha i}$ from $-1/48F^2$ come from the components

\[ -\frac{1}{48} \times 4E \left[ F_{\alpha\mu\rho}F_{\alpha'\mu'\rho'}G^{\alpha\alpha'}G^{\mu\mu'}G^{\rho\rho'} + 2F_{\alpha\mu\nu}\sigma G^{\mu\mu'}G^{\nu\nu'}G^{\sigma\sigma'} + \frac{1}{4}F_{\mu\nu\rho\sigma}F^{\rho\nu\sigma\sigma'}G^{\mu\mu'}G^{\nu\nu'}G^{\rho\rho'} \right] \] (5.17)

We note that the metric $G^{\mu\nu}$ gives also a gauge field dependence:

\[ G^{\mu\nu} = g^{\mu\nu} + \Delta^{2/5}B^{\mu\rho}B^{\nu}_\rho = \Delta^{2/5} \left( C^\mu_C B^T A B Y C Y D T^{C D} - C^\mu_A Y B T^{A B} C^\nu_C Y D T^{C D} + B^{\mu\rho}B^{\nu}_\rho \right) \] (5.18)

but since it already has two gauge fields it can contribute only in the last term in (5.17). The first term contributes (after substituting the ansatze for the various fields)

\[ \frac{V_4}{4} \left[ \frac{1}{7} T r (\partial_\alpha T^{-1} \partial^\alpha T) + \frac{1}{7 \cdot 5} (T r (T^{-1} \partial_\alpha T))^2 + \frac{12}{5} B^{A B} (T^{-1} \partial_\alpha T)_{A B} \right] \] (5.19)

whereas the second term contributes

\[ + \frac{4}{5 \cdot 7} B^{A B} (T^{-1} \partial_\alpha T)_{A B} \] (5.20)

Finally, the third term in (5.17) contributes only to the $B^2$ term in $P^2_{\alpha i}$, so we will neglect it. We 'see' that the two contributions add up to the combination required by gauge invariance,

\[ \frac{V_4}{4 \cdot 7} \left[ T r (\partial_\alpha T^{-1} \partial^\alpha T) - 8B^{A B} (T^{-1} \partial_\alpha T)_{A B} \right] + \frac{1}{7 \cdot 5 \cdot 4} (T r (T^{-1} \partial_\alpha T))^2 \] (5.21)

where the last term is gauge invariant by itself (we can freely replace $\partial_\alpha$ with $\nabla_\alpha$).

One would think that we need to add also the term $F_{\alpha\mu\rho}F_{\alpha'\mu'\rho'}G^{\alpha\alpha'}G^{\mu\mu'}G^{\rho\rho'}$ which has also two gauge fields and scalars, however it has both two gauge fields and two derivatives on scalars, so it should cancel with similar terms coming from the 11 dimensional Einstein action. For the same reason, the gauge field contribution of $G^{\mu\nu}$ to the first term in (5.17) has not been taken into account.

The Einstein action $\int -\frac{1}{2} R$, gives for the term with two derivatives on scalars and no gauge fields

\[ -\frac{1}{5 \cdot 7 \cdot 4} T r (T^{-1} \partial_\alpha T)^2 + \frac{5}{7 \cdot 8} T r (\partial_\alpha T \partial^\alpha T^{-1}) \] (5.22)

But we know that the metric is gauge invariant, and so we expect that $\int R$ is a gauge invariant object too, that means that we can complete the previous terms in a gauge invariant way. Then the sum of the $\int F^2$ and $\int R$ contributions reproduces (5.16).
Next we try to reproduce the kinetic term for the gauge field, \(-\frac{1}{4}(\Pi_A^i\Pi_B^j F^{AB})^2\). It has also contributions from the Einstein action and from the antisymmetric tensor kinetic term, more precisely the component

\[-\frac{1}{48}E F_{\mu\nu\alpha\beta} F_{\mu'\nu'\alpha'\beta'} G^{\mu\mu'} G^{\nu\nu'} G^{\alpha\alpha'} G^{\beta\beta'} \]  
(5.23)

which gives

\[-\frac{3V_4}{20} T AC T BD F_{\alpha\beta} F^{\alpha\beta} \]  
(5.24)

We note that if we didn’t have any scalar field dependence in \(F_{\mu\nu\alpha\beta}\), we would get a term of the type

\[\left[4T AC T BD F_{\alpha\beta} F^{\alpha\beta} + F_{\alpha\beta} F^{\alpha\beta} (T_{AB} - Tr T^{-1} T_{AB})\right] \]  
(5.25)

From the Einstein action, if we look only at \((\partial B)^2\) terms, we get

\[-\frac{3V_4}{20} T AC T BD \partial_{\alpha} B_{\beta} B^{\alpha\beta} \]  
(5.26)

In the same manner as before, we assume that we can complete this term in a gauge invariant manner to an \(F^2\) term. When we add the two contributions, we get the correct term,

\[-\frac{V_4}{4} T AC T BD F_{\alpha\beta} F^{\alpha\beta} \]  
(5.27)

Finally, let’s come to the \(S\) terms in the bosonic action. They come from the \(F^2,\epsilon FFA\) and \(B^2\) terms in the 11 dimensional action. We will treat first the terms with no \(F_{\alpha\beta}\), and afterwards the term with \(F_{\alpha\beta}\), namely \(\epsilon^{\alpha\beta\gamma\delta\epsilon\zeta} \epsilon^{ABC} \epsilon^{DEF} S_{\alpha\beta\gamma\delta\epsilon\zeta} \epsilon^{ABC} F_{BDE}^C\).

The \(F^2\) terms contributing to the part with no \(F_{\alpha\beta}\) are

\[-\frac{1}{48}E \left[4F_{\mu\alpha\beta\gamma} F_{\mu'\alpha'\beta'\gamma'} (G^{\mu\mu'} G^{\alpha\alpha'} G^{\beta\beta'} G^{\gamma\gamma'} - 3G^{\mu\alpha'} G^{\beta\beta'} G^{\gamma\gamma'} G^{\cdot\cdot\cdot}) + F_{\alpha\beta\delta\gamma} F_{\alpha'\beta'\gamma'} G^{\alpha'\alpha'} G^{\beta'\beta'} G^{\gamma'\gamma'} G^{\cdot\cdot\cdot} + F_{\mu\alpha\beta\gamma} F_{\delta'\delta'\gamma'} G^{\alpha'\alpha'} G^{\beta'\beta'} G^{\gamma'\gamma'} G^{\cdot\cdot\cdot}\right] \]  
(5.28)

When we substitute the ansatz for the \(S_{\alpha\beta\gamma\delta}^A\) dependence of \(F\) and the ansatz for the metric \((G^{\mu\nu}\) from (5.17)), we get

\[\frac{2}{5}[S_{\alpha\beta\gamma\delta}^B S^{\alpha\beta\gamma\delta} B T_{AB} + T_{AB}^{-1}(\nabla_{[\alpha} S_{\beta\gamma\delta]\gamma\delta}] A \nabla_{[\alpha} S^{\beta\gamma\delta\delta}] B) \]  
(5.29)

The \(B^2\) terms give

\[\frac{1}{5}(\frac{6k}{3})^2 [S_{\alpha\beta\gamma\delta}^B S^{\alpha\beta\gamma\delta} B T_{AB} - 4T_{AB}^{-1}(\nabla_{[\alpha} S_{\beta\gamma\delta]\gamma\delta}] A \nabla_{[\alpha} S^{\beta\gamma\delta\delta}] B + \frac{1}{3} \epsilon^{\alpha\beta\gamma\delta\epsilon\zeta} S_{\epsilon\zeta\delta}^A \nabla_{[\alpha} S^{\beta\gamma\delta\delta}] A) \]  
(5.30)

Finally, the \(\epsilon FFA\) term contributes

\[\frac{1}{20} \epsilon^{\alpha\beta\gamma\delta\epsilon\zeta} S_{\epsilon\zeta\delta}^A \nabla_{[\alpha} S^{\beta\gamma\delta\delta}] A \]  
(5.31)
The condition of cancelation of the terms with 2 derivatives gives \( k = 5/(6\sqrt{2}) \), and we are left with

\[
\frac{1}{2} S_{\alpha\beta\gamma,A} S^\alpha_{\beta\gamma} T_{AB} + \frac{1}{12} \epsilon^{\alpha\beta\gamma\delta\eta\zeta} S_{\eta\zeta,A} \nabla_{\alpha} S_{\beta\gamma\delta,A} \tag{5.32}
\]

exactly as we have in the seven dimensional action. So we have still to recover only the \( \epsilon SFF \) term. It comes from two contributions,

\[
- \frac{3\sqrt{2}}{6(24)^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu\rho\sigma} \epsilon^{\alpha\beta\gamma\delta\eta\zeta} A_{\alpha\beta\gamma} F_{\delta\eta\zeta} + \frac{B^2}{48} \tag{5.33}
\]

and they sum up to

\[
\frac{i}{16\sqrt{3}} \epsilon^{\alpha\beta\gamma\delta\eta\zeta} S_{\alpha\beta\gamma,A} \epsilon_{ABCDE} F_{\delta\epsilon}^B F_{\eta\zeta}^D \tag{5.34}
\]

as it should. This finishes the analysis of the bosonic action.

In the last part of this section we will make some comments on other ansätze for bosonic fields which appeared in the literature. Most notably, we will try to see the relation with the truncation considered in [26] to the bosonic sector of N=1 gauged supergravity with gauge group SU(2). A series of papers [10, 11, 27] also looked at other bosonic truncations of 11d sugra to 7 dimensions with abelian gauge groups. Consistent truncations to other dimensions were considered in [29, 28, 30, 31].

The SU(2) truncation has the metric, one scalar \( X \), an SU(2) gauge field \( A^i_{(1)} \) and a 3-form field \( A^{i}_{(3)} \). The reduction ansatz in [26], written in form language, is

\[
ds^2_{11} = \hat{\Delta}^{1/3} ds^2_7 + 2g^{-2}X^{-3} \hat{\Delta}^{1/3} d\xi^2 + \frac{1}{2} g^{-2} \hat{\Delta}^{-2/3} X^{-1} \cos^2 \xi \sum_i (\sigma^i - g A^i_{(1)})^2 \tag{5.35}
\]

\[
\hat{F}_{(4)} = - \frac{1}{2\sqrt{2}} g^{-3} (X^{-8} \sin^2 \xi - 2X^2 \cos^2 \xi + 3X^{-3} \cos^2 \xi - 4X^{-3}) \hat{\Delta}^{-2/3} X^{-1} \sin^2 \xi d\xi \wedge \epsilon_{(3)}
\]

\[
- \frac{5}{2\sqrt{2}} g^{-3} \hat{\Delta}^{-2} X^{-4} \sin^2 \xi \cos^2 \xi dX \wedge \epsilon_{(3)} + \sin^2 \xi F_{(4)}
\]

\[
+ \sqrt{2} g^{-1} \sin^2 \xi X^4 \ast F_{(4)} \wedge d\xi - \frac{1}{\sqrt{2}} g^{-2} \cos^2 \xi F_{(2)}^i \wedge d\xi \wedge h^i
\]

\[
- \frac{1}{4\sqrt{2}} g^{-2} X^{-4} \hat{\Delta}^{-1} \sin^2 \xi \cos^2 \xi F_{(2)}^i \wedge h^j \wedge h^k \epsilon_{ijk} \tag{5.36}
\]

where \( h^i = \sigma^i - g A^i_{(1)}, \epsilon_{(3)} = h^1 \wedge h^2 \wedge h^3, \sigma^i \) are the three left-invariant forms on \( S_3 \), and

\[
\hat{\Delta} = X^{-4} \sin^2 \xi + X \cos^2 \xi \tag{5.37}
\]

and where the selfduality of \( F_{(4)} \) is imposed by hand

\[
X^4 \ast F_{(4)} = - \frac{1}{\sqrt{2}} g A_{(3)} + \frac{1}{2} \omega_{(3)}, \quad \omega_{(3)} = A^i_{(1)} \wedge F_{(2)}^i - \frac{1}{6} g \epsilon_{ijk} A^j_{(1)} \wedge A^k_{(1)} \wedge A^i_{(1)} \tag{5.38}
\]

The corresponding seven dimensional action is

\[
\mathcal{L} = R \ast 1 - \frac{1}{2} * d\phi \wedge d\phi - g^2 \left( \frac{1}{4} e^{-\frac{2}{\sqrt{10}}} - 2 \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{10}} \right) * 1
\]

\[
- \frac{1}{2} e^{-\frac{2}{\sqrt{10}}} \ast F_{(4)} \wedge F_{(4)} - \frac{1}{2} e^{\frac{2}{\sqrt{10}}} * F_{(2)}^i \wedge F_{(2)}^i
\]

\[
+ \frac{1}{2} F_{(2)}^i \wedge F_{(2)}^i \wedge A_{(3)} - \frac{1}{2\sqrt{2}} g F_{(4)} \wedge A_{(3)} \tag{5.39}
\]
where \( X = e^{-\frac{\phi}{\sqrt{10}}} \). On the other hand our bosonic action is

\[
e^{-1} \mathcal{L}'_{7d} = \frac{1}{2} R + \frac{1}{4} m^2 (T^2 - 2 T_{AB} T^{AB}) + \frac{1}{8} Tr(\partial_\alpha T^{-1} \partial^\alpha T) - \frac{1}{2} B_{\alpha}^{AB} (T^{-1} \partial_\alpha T)_{AB} + \frac{1}{2} T^{AB} S_{\alpha \beta \gamma, A} \epsilon^{\alpha \beta \gamma}
\]

\[+ \frac{1}{48} m e^{-1} \epsilon^{\alpha \beta \gamma \delta \epsilon \eta \zeta} \delta^{AB} S_{\alpha \beta \gamma, A} F_{\delta \epsilon \eta \zeta, B} - \frac{1}{4} (T_{AC} T_{BD} B_{\alpha}^{AB} B_{\alpha}^{CD} - (B_{\alpha}^{AB})^2) + \frac{m^{-1}}{8} e^{-1} \Omega_5 [B] - \frac{m^{-1}}{16} e^{-1} \Omega_3 [B]
\]

\[+ \frac{i}{16 \sqrt{3}} \epsilon^{\alpha \beta \gamma \delta \epsilon \eta \zeta} S_{\alpha \beta \gamma, A} \epsilon_{\delta \epsilon \eta \zeta, B} F_{B C}^{D E} F_{D E}^{F G} F_{G H}^{I J}
\]

(5.40)

and the metric and field strength in form language are given in (4.88) and (4.128), respectively. A comparison of the seven dimensional part of the metric tells us that \( \hat{\Delta} = \Delta^{-6/5} \) and suggests the ansatz for embedding \( X \) into \( T_{AB} = diag\{X, X, X, X, X^{-4}\} \), and breaking the SO(5) invariance by writing the 4-sphere in terms of 3-spheres as \( Y^A = \{\cos \xi \hat{Y}, \sin \xi \hat{\psi}\} \), with \( \hat{Y}^2 = 1 \). Indeed, then we reproduce the form of \( \hat{\Delta} \), and if we also choose \( B^{\hat{\mu}} = 0 \), we get

\[ds_{11}^2 = \hat{\Delta}^{1/3} ds_{7}^2 + \hat{\Delta}^{1/3} X^3 d\xi^2 + \hat{\Delta}^{-2/3} X^{-1} \cos^2 \xi (d\hat{Y}^2 + 2 B^{\hat{\mu}} \hat{Y}^{\hat{\nu}})^2
\]

(5.41)

and we also reproduce the ansatz for the 11 dimensional antisymmetric tensor at zero gauge field. The gauge field dependence looks somewhat different. We are thankful to C.N. Pope, H.Lü and A. Sadrzadeh for pointing to us that the ansatz in (5.35) is in fact contained in our general ansatz and is the same as the one given in (5.41). We know that \( SO(4) \simeq SU(2) \times SU(2) \). One can restrict then the set of six gauge fields \( B^{\hat{\mu}} \) to one of the two sets of \( SU(2) \) gauge fields by imposing a (anti)self-duality condition

\[B^{\hat{\mu}} = -\frac{1}{2} \epsilon^{\hat{\mu} \hat{\nu} \hat{\rho}} B_{\hat{\rho}}
\]

(5.42)

As a consequence, with this constraint imposed on our initial set of gauge fields, we are left with only three independent gauge fields, namely \( B^{i} = -1/2 A^{i}_{(1)} \), with \( i = 1, 2, 3 \). Also, (5.42) implies that

\[B^{i} \bar{B}^{\hat{i}} B^{\hat{i}} = \frac{1}{4} \delta^{i \hat{i}} B^2 = \frac{1}{4} \delta^{i \hat{i}} B^{\hat{\mu}} B^{\hat{\mu}}
\]

(5.43)

Thus, when squaring \( B^{\hat{\mu}} Y^{\hat{\nu}} \) in (5.41) we get

\[4 B^{\hat{\mu}} Y^{\hat{\nu}} B^{\hat{\nu}} Y^{\hat{\nu}} = \delta^{i \hat{i}} Y^{i} Y^{i} B^2 = \sum_{i=1}^{3} (A^{i}_{(1)})^2
\]

(5.44)

Using the connection between Euclidean coordinates on \( S_3 \) and Euler angles

\[Y_{1} = \cos \frac{\varphi + \psi}{2} \cos \frac{\theta}{2}, \quad Y_{2} = \sin \frac{\varphi + \psi}{2} \cos \frac{\theta}{2}, \quad Y_{3} = \cos \frac{\psi - \varphi}{2} \sin \frac{\theta}{2}, \quad Y_{4} = \sin \frac{\psi - \varphi}{2} \sin \frac{\theta}{2}
\]

(5.45)
we can now easily check that in (5.41) another term in the 4-dimensional line element, namely \( \sum_{\hat{\mu} = 1}^{4} (dY_{\hat{\mu}})^{2} \) corresponds to the term \( 1/4(\bar{d} \partial^{2} + d\bar{c}^{2} + d\bar{\psi}^{2} + 2d\bar{\psi}d\bar{\psi}\cos \theta) = 1/4 \sum_{i=1}^{3} d\sigma_{i}^{2} \) in (5.35). What remains to be shown is that the cross terms, linear in the gauge field also coincide in both relations. This is indeed the case, as we can deduce by using the set of relations (5.45)

\[
4B^{\hat{\mu}\hat{\nu}} dY^{\hat{\mu}} dY^{\hat{\nu}} = 4B^{1\hat{4}}(dY^{1}Y^{\hat{4}} - dY^{\hat{4}}Y^{1} + dY^{3}Y^{2} - dY^{2}Y^{3}) + B^{2\hat{3}}(\ldots) + B^{3\hat{1}}(\ldots)
\]

In fact, we can understand the previous result by organizing the four \( Y_{\hat{\mu}} \) in a \( 2 \times 2 \) SU(2) matrix called \( \mathcal{G} \). Then we obtain the SU(2) invariant one-forms \( \sigma_{i} \) by considering the product \( \mathcal{G}^{-1}d\mathcal{G} \).

Our ansatz for the field strength of the 11-dimensional 3-index antisymmetric tensor up to one gauge field and no \( S_{(3)} \) reduces in the truncated case to

\[
\frac{1}{\sqrt{2}}g^{-3}(X^{-3}sin^{2}\xi - 2X^{2}cos^{2}\xi + 3X^{-3}cos^{2}\xi - 4X^{-3})\tilde{\Delta}^{-2}
\]

\[
\epsilon_{A_{1}...A_{5}}(dY^{A_{1}}...dY^{A_{4}}M^{A_{4}}B^{A_{5}}) - \frac{5}{\sqrt{2}}\tilde{\Delta}^{-2}X^{-4}sin\xi cos\xi dX\epsilon_{A_{1}...A_{5}}dY^{A_{1}}dY^{A_{2}}dY^{A_{3}}dM^{A_{4}}B^{A_{5}}
\]

where \( M^{A_{4}} \equiv (cos\xi, sin\xi, Y_{\hat{\mu}}) \) and we have used here a certain rewriting of \( F_{\mu\nu\rho\alpha} \), namely:

\[
\frac{\sqrt{2}}{3}F_{\mu\nu\rho\alpha} = \frac{1}{3}\epsilon_{ABCDE}C^{A}_{\mu}C^{B}_{\nu}C^{C}_{\rho} \left[ Y^{D}\partial_{\alpha} \left( \frac{T^{EF}Y_{F}}{Y \cdot T \cdot Y} \right) - B^{DE}_{\alpha} \right] + \frac{2}{3}\epsilon_{\mu\nu\rho}B^{AB}_{\alpha} \partial^{C} \left( \frac{Y^{AD}B^{CD}Y_{C}}{Y \cdot T \cdot Y} \right)
\]

Working along the same lines as before, it can be shown that this ansatz coincides with the one proposed by [26] (for a complete discussion see [50]). The field equations obtained from our action (5.40) also coincide with the field equations and the constraints of (5.39). Hence, as observed by C.N.Pope, H.Lü and A. Sadrzadeh, our general KK ansatz [12] for the maximal sugra in \( d=7 \) contains as a special case the N=2 model in [26]. In conclusion, there is one consistent embedding which contains all (known) subcases in \( d = 7 \) [50].

## 6 Discussion and conclusions

In this paper we discuss the consistency of the KK reduction of the original formulation of 11d sugra [1] on \( AdS_{7} \times S_{4} \) [36] using the nonlinear ansatz presented in a previous letter [12] for the embedding of \( d=7 \) fields in \( d=11 \). Our ansatz for the metric factorizes into a rescaled 7 dimensional metric and a gauge invariant two form which depends on
the composite tensor $T^{AB} = \Pi^{-1}_i A^{\alpha} \Pi^{-1}_k B^{\alpha}$, where $\Pi^{\alpha}_A$ are the coset elements for the scalar fields.

$$ds^2_{11} = \Delta^{-2/5} \left[ g_{\alpha\beta} dy^\alpha dy^\beta + (DY)^A Y^{-1}_A T^{AB}(DY)^B \right]$$

$$DY^A = dY^A + 2B^{AB}Y_B$$

The scale factor satisfies $\Delta^{-6/5} = Y \cdot T \cdot Y$. We note that the full internal metric $g_{\mu\nu} = \Delta^{4/5} \partial_\mu Y \cdot T^{-1} Y$, $\partial_\nu Y^B$ has the geometrical meaning of a metric on an ellipsoid multiplied by a conformal factor, namely $g_{\mu\nu} = \Delta^{4/5} \partial_\mu Z^A \partial_\nu Z_A$ where $Z^A$ is constrained to lie on an ellipsoid. Therefore the overall effect of all scalar fluctuations on the internal metric is to deform the background sphere into a conformally rescaled ellipsoid.

The 4-form field strength is given by

$$\frac{\sqrt{2}}{3} F_{(4)} = \epsilon_{A_1 \ldots A_5} \left[ -\frac{1}{3} (DY)^{A_1} \ldots (DY)^{A_5} (T \cdot Y)^{A_5} \right.$$

$$\left. + \frac{4}{3} (DY)^{A_1} (DY)^{A_2} (DY)^{A_3} D \left( (T \cdot Y)^{A_4} \right) (T \cdot Y)^{A_5} \right.$$

$$+ 2F_{(2)}^{A_1 A_2} (DY)^{A_3} (DY)^{A_4} \frac{(T \cdot Y)^{A_5}}{Y \cdot T \cdot Y} + F_{(2)}^{A_1 A_2} F_{(2)}^{A_3 A_4} Y^{A_5} \right] + d(S_{(3)}BY^B)$$

where $S_{(3)}B_{\alpha\beta\gamma} = -\frac{8i}{\sqrt{3}} S_{\alpha\beta\gamma,B}$ is real. It is again gauge invariant but differs from the geometric proposal of Freed, Harvey, Minasian and Moore [47] by the dependence on T. Setting T=1 we recover their result but our expression follows from consistency of the KK program and still satisfies $DF_{(4)} = 0$. To prove $DF_{(4)} = 0$, one may use the Schouten identity $\epsilon_{[A_1 \ldots A_5} Y_{B]} = 0$. This fixes all relative coefficients in $F_{(4)}$. Because in this process we can at most convert one factor $T \cdot Y/(Y \cdot T \cdot Y)$ into a $Y$, there do not seem possible deformations with two factors $T \cdot Y/(Y \cdot T \cdot Y)$.

A confirmation of our ansatz for $F_{(4)}$ is obtained by evaluating the term $\epsilon FFA$ in the 11 dimensional action. We begin with $\epsilon FFF$ in $d=12$. The terms without bare B’s contain $F_{(2)}$, Y, $\partial_\mu Y$ and T’s. Using (3.41) in reverse order and the orthogonality relations in (4.103), integration over $S_4$ produces $2TrF^4 - (TrF^2)^2$, which is indeed the exterior derivative in the $d=7$ Chern Simons term. The B terms should not affect this result because both $F_{(4)}$ and the final result are gauge invariant.

The 4-index antisymmetric auxiliary field $B$ has only a 7-dimensional part.

$$\frac{B_{\alpha\beta\gamma\delta}}{\sqrt{E}} = \frac{i}{2\sqrt{3}} \epsilon_{\alpha\beta\gamma\delta\epsilon\zeta} \delta S^{(7)}_{\epsilon\zeta A} Y^A$$

On $d=7$ shell, $B_{\alpha\beta\gamma\delta}$ should vanish and since it should contain at most one derivative to exclude higher derivative terms in the $d=7$ action, it can only be proportional to the field equation of $S_{\alpha\beta\gamma,A}$. Since it should mix with $A_{\alpha\beta\gamma}$ to produce selfduality in odd dimensions, it should have the same spherical harmonic as $A_{\alpha\beta\gamma}$, and this explains the factor $Y^A$ in (6.4) and rules out an alternative $(T \cdot Y)^A/(Y \cdot T \cdot Y)$.

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Our ansatz for the fermions (4.14-4.16) is the standard one except for the following aspects:

(i) There are U matrices connecting the SO(5) subscripts of the d=7 fermions to the SO(5) subscripts of the Killing spinors.

(ii) Factors of $\sqrt{\gamma^5}$. They are fixed by requiring that $\psi_\alpha(y,x)$ varies only into the Killing spinor ($\delta(d=11)\psi_\alpha(y,x) = \delta(d=7)\psi_\alpha(y)U^I(y,x)\eta^I(x)$).

The consistency of the truncation is proven by obtaining the correct d=7 susy transformation laws from d=11 ones. We checked all transformation rules except: (i) the variation with more than one fermion field and (ii) some terms in the variation of the fermions which depend only on scalars. No authors have ever determined the higher order fermionic terms, but we would like to come back to them in the future. Presumably, they fix the remaining freedom in $U$. As to the scalar terms in $\delta\psi_\alpha$ and $\delta\lambda$, many consistency checks are satisfied, but this is a complicated problem which deserves a separate study. Also here we expect that the remaining freedom in $U$ will get fixed.

We have followed de Wit and Nicolai [2, 3] by introducing a matrix $U(y,x)$ which connects the d=7 fermions to the Killing spinors in the ansatz for the gravitino. This matrix must satisfy equation (4.45), $UY = \nu U$ with $v_i = \Pi^{-1}_i A Y A \Delta^{3/5}$, in order that the susy transformation laws $\delta B_{AB}$ for the gauge fields and $\delta\Pi^i_A$ for the scalars come out correctly. We have found general solutions to this equation, but for most of our results, the explicit form of $U$ is not needed.

The ansatz for the antisymmetric tensor field strength $F_{\Lambda\Pi\Sigma\Omega}$ was determined in 3 steps: first, the dependence on the gauge fields was uniquely fixed by requiring that the 11 dimensional susy variation of $F_{\Lambda\Pi\Sigma\Omega}$ matches the 7 dimensional susy variation of our ansatz for $F_{\Lambda\Pi\Sigma\Omega}$. Next, the ansatz for the embedding of the independent d = 7 fluctuation $S_{\alpha\beta\gamma,A}$ was derived together with the ansatz for the d = 11 auxiliary field $B_{MNPQ}$. Finally, the dependence of $F_{\Lambda\Pi\Sigma\Omega}$ on the d = 7 scalar fields was fixed by requiring that one obtains the correct scalar field potential in d=7 and the correct dependence of the d=7 sugra transformation rules on the tensor $T_{ij} = (\Pi^{-1}_i A (\Pi^{-1})_j B \delta_{AB}$ where $\Pi_A^i$ is the group vielbein for the scalars.

For completeness of our results, we have derived the 7 dimensional bosonic action and bosonic equations of motion at zero gauge field from the corresponding objects in 11 dimensions. Since in the bosonic sector all criteria for consistency are satisfied (the consistency of the transformation rules, equations of motion and even the action agrees) we infer that our results also prove the consistency of the bosonic truncation. We have shown the relation to other ans"atze found in the literature for consistent truncation to subsets of bosonic fields.

We have also explained the origin of self-duality in odd dimensions. One has to use a first order action for $F_{\Lambda\Pi\Sigma\Omega}$ in d=11 to obtain a self-dual action for $S_{\alpha\beta\gamma,A}$ in d=7 which is linear in derivatives. The d = 7 fluctuation $S_{\alpha\beta\gamma,A}$ not only appears in the d = 11 curvatures $F_{\mu\alpha\beta\gamma}$ and $F_{\alpha\beta\gamma\delta}$, but also in the d = 11 auxiliary field $B$ in (6.4), namely in the form $B \sim S + \epsilon DS$, and substituting the ansatz for $B$ and $F(S)$ into $d = 11$ action, we recovered the selfdual action in d = 7. We believe that in all cases, self-dual actions can be obtained from KK reduction of a first order formalism for the
corresponding field.

It may be useful to point out why in our opinion the original $d = 11$ sugra theory is to be preferred for purposes of compactification over the $SU(8)$ version of de Wit and Nicolai.

(i) Their theory has a local $SO(3,1) \times SU(8)$ group in tangent space. The $SO(3,1)$ becomes the Lorentz group in $d = (3,1)$ dimensions, and thus it becomes immediately clear that for compactifications to other dimensions we cannot use this theory. (For compactification to $d = (6,1)$ we would need an 11d theory with a $SO(6,1) \times SO(5)$ tangent group. For compactifications to other dimensions one would need other versions of 11d sugra than the $SU(8)$ version, namely versions with $SO(2,1) \times SO(16), SO(4,1) \times USp(8)$ and $SO(5,1) \times SO(5) \times SO(5)$ tangent group. These were found in [52].) On the other hand, the standard version of $d=11$ sugra can be used for compactifications to any dimension.

(ii) If we gauge fix in the $SU(8)$ theory the group $SU(8)$ to $SO(7)$ and compare to the standard $d = 11$ sugra with $SO(10,1)$ gauged fixed to $SO(3,1) \times SO(7)$ then in the $SU(8)$ theory the equations of motion correspond to both equations of motion and Bianchi identities in the usual $d = 11$ sugra. This means that the two gauged fixed theories are only equivalent on-shell. Of course, also in Cremmer and Julia’s theory for $N=8$ sugra in $d=4$, one needs to go temporarily on-shell to dualize certain fields. Note that in our paper we never need to dualize and always stay off-shell.

(iii) In fact, in the work of de Wit and Nicolai there is no action for their $SU(8)$ theory, only field equations. To quote ref. [4], page 389: “... the $d=11$ lagrangean depends explicitly on the antisymmetric tensor field $A_{MNP}$ for which no expression exists in terms of the $SU(8)$ covariant expressions used in this paper.” In the field equations only the field strength appears, but in the action bare $A$’s appear.

(iv) To lift solutions of $d = 4$ gauged sugra to solutions of the usual $d = 11$ sugra such as in [10], one would need the analog of our results for $d=4$. The ansatz of de Wit and Nicolai could be used to lift such solutions to their $SU(8)$ theory. However, all recent work on the consistency of truncations of the KK reductions to $d = (3,1)$ has used the standard $d=11$ theory [51]. In particular, all work on M theory and membranes is based on the standard sugra theory.

Acknowledgements. We would like to thank I. Park for collaboration at the early stages of this work, B. de Wit and H. Nicolai for useful discussions on their work and C.N. Pope, H. Lü and A. Sadrzadeh for pointing out that our bosonic ansätze reduce to theirs [26] when we further truncate $\mathcal{N} = 4$ $d = 7$ to $\mathcal{N} = 2$ $d = 7$ gauged sugra.

Appendix A

A.1 Conventions and gamma matrix algebra

We denote in 7 dimensional Minkowski space the coordinates by $y$, flat vector indices by $a,b,c,...$ and curved vector indices by $\alpha, \beta, \gamma,...$. Similarly, we denote in 4 dimensional Euclidean space the coordinates by $x$, flat indices by $m,n,p$, and curved vector indices by $\mu, \nu,...$. Eleven dimensional vector indices are denoted by $M,N,P$ for flat indices and $\Lambda, \Pi, \Sigma...$ for curved indices. The $SO(5) \simeq USp(4)$ gauge group is denoted by $SO(5)_g$ and $A,B,... = 1,5$ are $SO(5)_g$ vector indices and $I,J,... = 1,4$ are $USp(4)$ indices (or
$SO(5)_g$ spinor indices). The scalars form a coset $SL(5,R)/SO(5)$ and to avoid confusion we denote the composite subgroup by $SO(5)_c$, with vector indices $i,j,\ldots$ and spinor indices $I',J',\ldots=1,4$.

The d=11 metric $g_{MN}$ has signature mostly plus, $(-\ldots+)$. The Clifford algebra in d=11 reads

$$\Gamma M\Gamma N + \Gamma N\Gamma M = 2\eta_{MN}; \quad M,N = 0,\ldots,10$$  \hspace{1cm} (A.1)

We introduce Dirac matrices in d=7 and d=4.

$$\Gamma a\Gamma \tau_{ab} + \Gamma \tau_{ab}\Gamma a = 2\eta_{ab}; \quad a,b = 0,\ldots,6$$  \hspace{1cm} (A.2)

$$\gamma_m\gamma_n + \gamma_n\gamma_m = 2\delta_{mn}; \quad m,n = 1,\ldots,4$$  \hspace{1cm} (A.3)

They are used to construct the d=11 Dirac matrices

$$\Gamma a = \tau_a \otimes \gamma_5 \quad \text{for} \quad a = 0,6 \quad \text{and} \quad \Gamma_{6+m} = 1 \otimes \gamma_m \quad \text{for} \quad m = 1,4$$  \hspace{1cm} (A.4)

where $\gamma_5 = \gamma_1 \ldots \gamma_4$ hence $\gamma_5^2 = 1$. We choose $\tau_0 = \tau_1 \ldots \tau_6$, hence $\Gamma_0 = \Gamma_1 \ldots \Gamma_{10}$. We normalize the $\epsilon$ symbols as $\epsilon^{0,1,2,3} = \epsilon^{0,7} = 1$, so that in d=7

$$\Gamma^{[a_1 \ldots \tau] \alpha} = \Gamma^{a_1 \ldots \alpha_7} \rightarrow \Gamma^{a_1 \ldots \alpha_k} = \epsilon^{a_1 \ldots \alpha_7} \Gamma_{a_1+1 \ldots \alpha_7} (\epsilon^{k/2+1})^l = \epsilon^{k/2+1} (7 - k)!$$  \hspace{1cm} (A.5)

where all antisymmetrizations are with strength one. A similar duality relation between $\Gamma$ matrices holds in d=11:

$$\Gamma^{\Lambda_1 \ldots \Lambda_k} = \frac{(-1)^{[k/2+1]}}{(11 - k)!} \epsilon^{\Lambda_1 \ldots \Lambda_{11}} \Gamma^{\Lambda_{12} \ldots \Lambda_{11}}$$  \hspace{1cm} (A.6)

Also, for general d,

$$\epsilon^{a_1 \ldots \alpha_k \alpha_{k+1} \ldots \alpha_d} \epsilon_{\beta_1 \ldots \beta_k \alpha_{k+1} \ldots \alpha_d} = -k! (d - k)! \delta^{[a_1 \ldots \alpha_k]}_{[\beta_1 \ldots \beta_k]}$$  \hspace{1cm} (A.7)

A definition of $\sqrt{\gamma_5}$ is obtained from $\epsilon^{\alpha \gamma_5} = \cos \alpha + i \sin \alpha \gamma_5$. Choosing $\alpha = \pi/2$ we get $\gamma_5 = -ie^{i\pi/2} \gamma_5$, and thus

$$\gamma_5^{12} = -\frac{1}{2}(1 + i\gamma_5), \quad \gamma_5^{-12} = -\frac{1}{2}(1 - i\gamma_5)$$  \hspace{1cm} (A.8)

Then we also have

$$C\gamma_5^{12}C^{-1} = \Gamma_5^{12}T$$

$$\gamma_5^{12} = -i\gamma_5^{-12}$$  \hspace{1cm} (A.9)

where $\gamma_5^{12} = \gamma_5^{12}$ and $C\gamma_\mu C^{-1} = -\gamma_\mu^T$, see next appendix. Some useful formulas for gamma matrices in d dimensions are

$$\Gamma a\Gamma b \ldots b_n \Gamma^a = \Gamma a\Gamma b \ldots b_n$$

$$\Gamma a\Gamma b \ldots b_n = \Gamma a\Gamma b \ldots b_n + n\delta_{[a}^{[b_1} \Gamma^{b_2} \ldots b_n]$$

$$\Gamma a_{a_1} a_{a_2} \Gamma b \ldots b_n = \Gamma a_{a_1} a_{a_2} \Gamma b \ldots b_n + 2n\delta_{[a_1}^{[b_1} \Gamma_{a_2} b \ldots b_n]$$

$$\Gamma a_{a_1} \ldots a_m \Gamma b_1 \ldots b_n = \Gamma a_{a_1} \ldots a_m \Gamma b_1 \ldots b_n + 2n(1 - n)\delta_{[a_1}^{[b_1} \delta_{a_2} b_1 \ldots b_n]$$

$$\Gamma a_{a_1} \ldots a_m \Gamma b_1 \ldots b_n = \Gamma a_{a_1} \ldots a_m \Gamma b_1 \ldots b_n + 2n(1 - n - 2)\delta_{a_3} b_3 \ldots b_n \Gamma a_1(1 - n - 2)\delta_{a_3} b_3 \ldots b_n \Gamma a_1$$  \hspace{1cm} (A.10)
As representation for the SO(5) Clifford algebra we take
\[ \gamma^A = \{i\gamma^\mu, \gamma^5\} \] (A.11)

In 5 dimensions,
\[ (C\gamma^A)_{IJ}(C\gamma^A)_{KL} = +2(\Omega_{IK}\Omega_{JL} - \Omega_{JK}\Omega_{IL}) - \Omega_{IJ}\Omega_{KL} \] (A.12)
\[ (C\gamma^{AB})_{IJ}(C\gamma^{AB})_{KL} = 4(\Omega_{IK}\Omega_{JL} + \Omega_{IL}\Omega_{JK}) \] (A.13)

because these tensors are USp(4) invariant tensors with definite symmetry properties. Hence they should be constructed from the symplectic metric \( C_{IJ} \). We choose a gamma matrix representation such that
\[ C_{IJ} = \Omega_{IJ}. \] (This matrix \( C_{IJ} \) is thus the charge conjugation matrix \( C^{(5)} = C^{(5)}_+ \) in 5 dimensions, see below. It equals \( C^{(4)}_- \)).

We lower USp(4) indices with \( \Omega_{IJ} \) as follows
\[ \lambda^I = \lambda^J \Omega_{JI} \] (A.14)
and raise them with \( \tilde{\Omega}^{IJ} \) as
\[ \lambda^I = \tilde{\Omega}^{IJ} \lambda_J \] (A.15)
where \( \tilde{\Omega}^{IJ} \) is defined by \( \tilde{\Omega}^{IJ}\Omega_{JK} = \delta^I_K \). Of course, \( \tilde{\Omega} \) is \( -\Omega^{-1} \) and can be obtained from \( \Omega_{IJ} \) by raising indices with \( \tilde{\Omega}^{IJ} \). The result is \( \tilde{\Omega}^{IJ} = \Omega^{IJ} \). We have found it convenient to use a different symbol for \( \Omega_{IJ} \), namely \( \tilde{\Omega}^{IJ} \).

A.2 Charge conjugation matrices and modified Majorana spinors

In even dimensions there are two charge conjugation matrices, \( C^{(\pm)} \) and \( C^{(-)} \), satisfying \( C^{(\pm)}\gamma^\mu C^{(\pm)-1} = \pm\gamma^\mu T \). In odd dimensions there is only one charge conjugation matrix, either \( C^{(\pm)} \) or \( C^{(-)} \). They are either symmetric or antisymmetric, and all their properties are independent of the representation chosen. For a general discussion of Majorana and modified Majorana spinors in Minkowski or Euclidean space and charge conjugation matrices, see [48].

In 11 Minkowski dimensions, there is only one \( C \) matrix. It satisfies
\[ C^{(11)}\Gamma_M C^{(11)-1} = -\Gamma^T_M \] (A.16)
hence \( C^{(11)} = C^{(11)}_- \). Furthermore, \( C^{(11),T} = -C^{(11)} \).

In 7 Minkowski dimensions, \( C^{(7)} = C^{(7)}_- \), but is symmetric
\[ C^{(7)}_{\tau a} C^{(7)-1} = -\tau_a^T, \quad C^{(7),T} = C^{(7)} \] (A.17)

In \( d=4 \) both a \( C^{(4)}_+ \) and a \( C^{(4)}_- \) exist. Both are antisymmetric, but \( C^{(11)} = C^{(7)} \otimes C^{(4)} \), where \( C^{(4)} = C^{(4)}_- \).

Then \( C\gamma_\mu \) is symmetric and \( C\gamma_5 \) antisymmetric (since \( C^T = -C \)). It also follows that for the 5-dimensional gamma matrices, \( C\gamma^A \) in (A.12) are antisymmetric and the matrices \( C\gamma^{[AB]} \) in (A.13) are symmetric. This means that \( (C\gamma^A)_{[IJ]} \) and \( (C\gamma^{[AB]})_{(IJ)} \)
give the isomorphisms between the 5, respectively the 10 representations of SO(5) and Usp(4) \((SO(5) \simeq Usp(4))\).

The Majorana condition in 11 dimensions is

\[
\Psi^T C^{(11)} = i \Psi^\dagger \Gamma^0
\]

(A.18)

In 7 Minkowski dimensions we can only define a modified Majorana condition,

\[
(\lambda_I)^T C^{(7)} \tilde{\Omega}^{IJ} = i (\lambda_J)^\dagger \tau^0 \equiv \bar{\lambda}^J
\]

(A.19)

In the text we always define \(\bar{\lambda}_I \equiv \lambda^T C(\gamma)\). Using (A.19), the spacetime spinors satisfy

\[
\bar{\lambda}_I = (\lambda_J)^\dagger i \tau^0 \Omega_{JI}
\]

(A.20)

To determine which Majorana condition we need in \(d=4\) Euclidean space, we decompose the anticommuting 11 dimensional Majorana spinors \(\Psi\) into 4 anticommuting 7-dimensional modified Majorana spinors times corresponding commuting 4-dimensional hspinors \(\eta^I\). The spinors \(\eta\) must satisfy the following modified Majorana condition

\[
(\eta^K)^T C^{(4)}_{(-)} \Omega_{KJ} = - (\eta^J)^\dagger \gamma_5
\]

(A.21)

Since in 4 Euclidian dimensions there are 2 choices for \(C^{(4)}\), we can use \(C^{(4)}_{(+)} = C^{(4)}_{(-)} \gamma_5\), satisfying \(C^{(4)}_{(+)} \gamma_m C^{(4)-1}_{(+)} = i T^m\), in terms of which (A.21) takes the same form as (A.19),

\[
(\eta^K)^T C^{(4)}_{(+)} \Omega_{KJ} = (\eta^K)^\dagger
\]

(A.22)

Note that both (A.22) as well as the same condition with \(C^{(4)}_{(-)}\) instead of \(C^{(4)}_{(+)}\) satisfy the consistency condition obtained by taking its complex conjugate and applying the relation twice [48].

Also, in the text we always define \(\bar{\eta}^I \equiv (\eta^I)^T C^{(4)}_{(-)}\). Using (A.21), the Killing spinors satisfy

\[
\bar{\eta}^I = - (\eta^J)^\dagger \gamma_5 \bar{\Omega}^{IJ}
\]

(A.23)

In the fermionic ansätze we decompose the \(d=11\) spinors into \(\lambda_I y U^{I'}_{IJ} \eta^I(x)\). Using the Majorana conditions in the (10,1), (6,1) and (4,0) dimensional spaces one obtains the condition

\[
(U^{I'}_{IJ})^* = \Omega^{I'}_{IJ'} U^{J'}_{J'} \tilde{\Omega}^{IJ}
\]

(A.24)

Combined with the relation \(U \tilde{\Omega} U^T = \tilde{\Omega}\) derived in the text, we deduce that \(U\) is unitary. Hence the matrices \(U\) are matrices of the group USp(4), namely the intersection of U(4) and Sp(4,C).

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