Threshold scattering of the $\eta$-meson off light nuclei

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Abstract

The scattering lengths of $\eta$-meson collisions with light nuclei $^2H$, $^3H$, $^3He$, and $^4He$ are calculated on the basis of few-body equations in coherent approximation. It is found that the $\eta$-nucleus scattering length depends strongly on the number of nucleons and the potential-range parameter. By taking into account the off-shell behavior of the $\eta N$ amplitude, the $\eta\alpha$ scattering length increases considerably.

In $\eta$-nucleus collisions at threshold energies, a large Final State Interaction (FSI) is expected due to the $N^*(1535) S_{11}$-resonance. This resonance, which is strongly coupled to the $\eta N$ and $\pi N$ channels, lies only $\sim 50$ MeV above the $\eta N$ threshold and has a very broad width of $\Gamma \approx 150$ MeV [1]. The $\eta$-nucleus dynamics, thus, is of interest from the point of view of both few-body and meson-nucleon physics.

In the energy region covering the $S_{11}$-resonance, the $\pi N$ and $\eta N$ scattering processes are to be considered as a coupled-channel problem [2–4]. The first of these channels has a long record of theoretical and experimental investigations, while the second one is understood only in a rudimentary way. The $\pi N$ and $\eta N$ channels are connected to each other primarily via the $N^*(1535)$ resonance. In fact, the $\eta NN$ coupling constant was shown to be negligible [5–7] as compared to the one of the $\eta NN^*$ vertex. This latter vertex constant, being hence crucial for determining the $\eta N$ interaction, is known only with a large uncertainty, so that the $\eta N$ scattering length inferred from it varies from $(0.27 + i 0.22)$fm [2] to $(0.55 + i 0.30)$fm [8]. The $\eta N$ near-threshold interaction, therefore, remains an interesting field of investigation. Of particular relevance in this context is the possibility of $\eta$-nucleus bound states [9,10]. Their existence would provide us with an excellent opportunity to answer the above-mentioned questions in a reliable manner.

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Various experimental groups have studied the near-threshold production of \( \eta \)-mesons in photo- and electro-nuclear processes \cite{11,12}, or in \( \pi \)-nucleus \cite{13}, \( N \)-nucleus \cite{14,15}, and nucleus-nucleus \cite{16} collisions. The expected strong \( \eta \)-nucleus FSI is evident in these measurements and must not be ignored in theoretical treatments.

A simple way of taking into account the low-energy FSI effects in \( \eta \)-production processes is described in Ref. \cite{17}. It consists in multiplying the reaction amplitude by an energy-dependent factor which contains the final-state scattering length as a parameter. This parameter, hence, is essential in analyzing the increasing number of such production data. To the best of our knowledge, only the scattering-length calculations by Wilkin \cite{8} for the \( \eta \) collision on \( ^3He, \; ^4He, \; \text{and} \; ^7Be \) exist in the literature. There, the lowest-order optical-potential method was used with a simple \( \eta N \) amplitude having no energy or off-shell dependence.

In the present work we calculate the \( \eta^2H, \; \eta^3H, \; \eta^3He, \; \text{and} \; \eta^4He \) scattering lengths on the basis of few-body equations with separable \( \eta N \) amplitudes of the type suggested in Refs. \cite{2,3}. The equations employed are obtained within the Finite-Rank Hamiltonian Approximation (FRHA), proposed in Refs. \cite{18,19} as an alternative to the multiple scattering or optical potential theory. Developed originally for the treatment of pion-nucleus scattering, this technique was later on also applied to other questions of few-body physics, such as the \( \Lambda \)-nucleus bound state problem \cite{20} or the \( NN \)-scattering in a six quark model \cite{19}.

The main idea of FRHA consists in separating the motion of the projectile and the internal motion of the nucleons inside the nucleus. Correspondingly the total Hamiltonian is split according to

\[
H = H_0 + V + H_A ,
\]

where \( H_0 \) denotes the kinetic energy operator of the \( \eta \)-meson relative to the center of mass of the nucleus, \( V = \sum_{i=1}^{A} V_i \) the sum of the individual \( \eta N \)-interactions, and \( H_A \) the Hamiltonian of the nucleus. The scattering amplitude for the transition from the initial state \( |\vec{k}, \psi_0> \) to the final state \( |\vec{k}', \psi_0> \), with \( |\psi_0> \) being the ground state of the nucleus,

\[
H_A |\psi_0> = \mathcal{E}_0 |\psi_0> ,
\]

is given by

\[
f(\vec{k}', \vec{k}; z) = -\frac{\mu}{2\pi} <\vec{k}', \psi_0|T(z)|\vec{k}, \psi_0> .
\]

The \( T \)-operator in this amplitude satisfies the Lippmann-Schwinger equation

\[
T(z) = V + V \frac{1}{z - H_0 - H_A} T(z) .
\]

Introducing an auxiliary operator \( T^0 \) by
\[ T^0(z) = V + V \frac{1}{z - H_0} T^0(z), \quad (5) \]
we obtain instead of Eq. (4)

\[ T(z) = T^0(z) + T^0(z) \frac{1}{z - H_0} H_A \frac{1}{z - H_0 - H_A} T(z). \quad (6) \]

Our main approximation consists in restricting the spectral decomposition of \( H_A \) to the ground state \( |\psi_0> \),

\[ H_A \approx \mathcal{E}_0 |\psi_0> <\psi_0|. \quad (7) \]

Physically this so-called coherent approximation [21], which is widely used in multiple scattering and optical potential approaches, means that during the multiple scattering of the \( \eta \)-meson the nucleus remains unexcited. Equation (6) then reads

\[ T(z) = T^0(z) + \mathcal{E}_0 T^0(z) |\psi_0> \frac{1}{(z - H_0)(z - H_0 - \mathcal{E}_0)} <\psi_0|T(z), \quad (8) \]

and, after sandwiching it between the ground state and the relative-momentum states, we obtain for the matrix elements

\[ T(\vec{k}', \vec{k}; z) = <\vec{k}', \psi_0|T^0(z)|\vec{k}, \psi_0> \]

\[ + \mathcal{E}_0 \int \frac{d\vec{k}''}{(2\pi)^3} \frac{V(\vec{k}', \vec{k''}; \vec{r})}{(z - \vec{k''}^2/2\mu)(z - \mathcal{E}_0 - \vec{k''}^2/2\mu)} T(\vec{k}'', \vec{k}; z). \quad (9) \]

Our problem thus is split into two steps. The first consists in evaluating the auxiliary amplitude \( <\vec{k}', \psi_0|T^0(z)|\vec{k}, \psi_0> \) which determines the inhomogeneity and the kernel of Eq. (9); in the second step this equation is to be solved.

It is easily seen that the operator \( T^0 \) describes the scattering of the \( \eta \)-meson off the nucleons which are fixed in their position within the nucleus. This is due to the fact that Eq. (5) does not contain any operator which acts on the internal nuclear Jacobi coordinates \( \{\vec{r}\} \). All operators in Eq. (5), hence, are diagonal in these variables, so that its momentum representation reads

\[ T^0(\vec{k}', \vec{k}; \vec{r}; z) = V(\vec{k}', \vec{k}; \vec{r}) + \int \frac{d\vec{k}''}{(2\pi)^3} \frac{V(\vec{k}', \vec{k''}; \vec{r})}{z - \vec{k''}^2/2\mu} T^0(\vec{k}'', \vec{k}; \vec{r}; z). \quad (10) \]

It depends, in other words, only parametrically on the coordinates \( \{\vec{r}\} \). After solving this integral equation and determining the ground state wave function \( \psi_0(\vec{r}) \) by any appropriate bound-state method, the input \( <\vec{k}', \psi_0|T^0(z)|\vec{k}, \psi_0> \) to Eq. (8) is obtained via

\[ <\vec{k}', \psi_0|T^0(z)|\vec{k}, \psi_0> = \int d\vec{r}|\psi_0(\vec{r})|^2 T^0(\vec{k}', \vec{k}; \vec{r}; z). \quad (11) \]

Since we are interested finally only in the on-shell amplitudes, it suffices to consider the half-on-shell restriction of Eq. (9). That is, we put in all the above relations the parameter \( z \) onto the initial energy.
\[ z = \frac{k^2}{2\mu} - |E_0| + i0. \]  

(12)

Therefore, the \( T^0 \)-matrix enters Eq. (3) off the energy shell, differing thus from the conventional fixed-scatter amplitude for which \( z = \frac{k^2}{2\mu} \). In scattering-length calculations we have \( k = 0 \) and hence \( z = -|E_0| + i0 \), so that Eqs. (10) and (13) become nonsingular and easy to handle. From the practical point of view it is convenient to rewrite Eq. (10) by using the Faddeev-type decomposition

\[ T^0(k', k; \vec{r}; z) = \sum_{i=1}^{A} T^0_i(k', k; \vec{r}; z), \]

with

\[ T^0_i(k', k; \vec{r}; z) = t_i(k', k; \vec{r}; z) + \int \frac{d\vec{k}'}{(2\pi)^3} t_i(k', \vec{k}'; \vec{r}; z) \frac{z - \frac{k'^2}{2\mu}}{\sum_{j \neq i} T^0_j(\vec{k}', k; \vec{r}; z)}. \]  

(13)

Here, \( t_i \) is the t-matrix for the scattering of the \( \eta \)-meson off the \( i \)-th nucleon.

As an input information we need these off-shell amplitudes and the ground-state wave functions of the nuclei involved. Due to the dominance of the \( S_{11} \)-resonance near the threshold energy, we can restrict ourselves to the S-wave \( \eta N \)-interaction. In Ref. [9] it was demonstrated that the role played by the higher partial waves is indeed negligible. The \( \eta NN \) coupling constant is much smaller than the \( \eta NN^* \) one [4,4]. This means that the \( \eta N \) collision goes predominantly via a virtual formation of \( N^*(1535) \), that is, via the \( \eta N \to N^* \to \eta N \) reaction. Hence, the corresponding amplitude must contain two \( \eta N \leftrightarrow N^* \) vertex functions and an \( N^* \)-propagator in between. We employ the Yamaguchi-type form

\[ t_{\eta N}(k', k; z) = \frac{\lambda}{(k'^2 + \alpha^2)(z - E_0 + i\Gamma/2)(k^2 + \alpha^2)}, \]  

(14)

commonly used in few-body physics, with a simple Breit-Wigner propagator. Two of the four parameters of the t-matrix [14] are immediately fixed, namely the resonance energy \( E_0 = 1535 \text{ MeV} - (M_N + M_\eta) \) and the width \( \Gamma = 150 \text{ MeV} \) [1]. The parameter \( \lambda \) is chosen to provide the correct zero-energy on-shell limit, i.e., to reproduce the known \( \eta N \) scattering length \( a_{\eta N} \),

\[ t_{\eta N}(0, 0, 0) = -\frac{2\pi}{\mu_{\eta N}} a_{\eta N}. \]  

(15)

Finally, in order to fix the parameter \( \alpha \), we make use of the results of Refs. [2,3], where the same \( \eta N \to N^* \) vertex function \((k'^2 + \alpha^2)^{-1}\) was employed with \( \alpha \) being determined via a two-channel fit to the \( \pi N \to \pi N \) and \( \pi N \to \eta N \) experimental data.

Due to experimental uncertainties and differences between the models of the physical processes, there are three different values available for the scattering length in the literature: \( a_{\eta N} = (0.27 + i 0.22) \text{ fm} \) [2], \( a_{\eta N} = (0.28 + i 0.19) \text{ fm} \) [3], and \( a_{\eta N} = (0.55 + i 0.30) \text{ fm} \) [8]. For the range parameter \( \alpha \) one also finds three different values: \( \alpha = 2.357 \text{ fm}^{-1} \) [2],
\( \alpha = 3.316 \text{ fm}^{-1} \) [3], and \( \alpha = 7.617 \text{ fm}^{-1} \) [4]. Since there is no criterium for singling out one of them, we use all 9 combinations of \( a_{\eta N} \) and \( \alpha \) in our calculation. For the bound states we employed simple Gaussian-type functions, which were constructed to be symmetric with respect to nucleon permutations, and to reproduce the experimental mean square radii: \( \sqrt{<r_d^2>} = 1.956 \text{ fm} \) [22], \( \sqrt{<r_{3H}^2>} = 1.755 \text{ fm} \) [23], \( \sqrt{<r_{3He}^2>} = 1.959 \text{ fm} \) [23], and \( \sqrt{<r_{4He}^2>} = 1.671 \text{ fm} \) [24]. For masses and binding energies of the nuclei we used the experimental values [25].

The method described above involves only the coherent approximation (7), according to which all excitations of the target nucleus are neglected. This approximation could be shown [26], as expected, to be justified at low collision energies and in the case of a wide energy gap between the ground and first excited nuclear state. In our zero-energy calculation the method should, therefore, work particularly well. Among the target nuclei \( ^2\text{H} \), \( ^3\text{H} \), \( ^3\text{He} \), and \( ^4\text{He} \) the most accurate results are expected to be those for the \( ^4\text{He} \) nucleus, since its first excited state at 20.21 MeV [24] lies comparatively high.

The results of our calculations are given in Table I. As can be seen, the \( \eta \)-nucleus scattering length depends strongly on the number of nucleons \( A \), and is sensitive to the range parameter \( \alpha \), i.e. to the off-shell continuation of the \( \eta N \)-amplitude. Prior to this work, Wilkin [8] performed calculations of the \( \eta \text{A} \) scattering lengths. In these investigations the \( \eta N \) amplitude was simply assumed to be constant and equal to the value of its zero-energy limit, \( a_{\eta N} = (0.55 + i 0.30) \text{ fm} \). It is easily seen that such an assumption is equivalent to using a zero-range, i.e. a \( \delta \)-type interaction. The values of the \( \eta \text{He}^3 \) and \( \eta \text{He}^4 \) scattering lengths obtained by Wilkin are \( a(\eta \text{He}^3) = (-2.31 + i 2.57) \text{ fm} \) and \( a(\eta \text{He}^4) = (-2.00 + i 0.97) \text{ fm} \). As our results show, the non-zero-range character of the \( \eta N \) interaction, i.e., its off-shell properties change these values significantly.

More recently [27], Wilkin used the multiple-scattering expansion in order to incorporate the nucleon-nucleon correlations. In this approach the \( \eta \alpha \) and \( \eta N \) scattering lengths are connected via the simple relation

\[
a(\eta, \text{He}^4) = (0.181/a_{\eta N} - 0.281 \text{ fm}^{-1})^{-1}.
\]

With the three values of \( a_{\eta N} \) listed in Table I, this formula gives \( a(\eta \text{He}^4) = (0.99 + i 2.67) \text{ fm} \), \( (1.39 + i 2.58) \text{ fm} \), or \( (-1.38 + i 6.95) \text{ fm} \), respectively. As can be seen, the agreement between this simple formula and our microscopic calculation is very poor, especially for large values of the range-parameter \( \alpha \).

If we admit the maximal size of the \( \eta N \)-interaction region found in the literature, which corresponds to \( \alpha = 2.357 \text{ fm}^{-1} \), and use the rather reasonable estimate \( a_{\eta N} = (0.55 + i 0.30) \text{ fm} \) by Wilkins, then the corresponding large value of \( a(\eta \text{He}^4) = (-4.41 + i 2.86) \text{ fm} \) indicates that the condition \( A \geq 12 \) obtained in Ref. [4] for the existence of \( \eta \)-nucleus bound states must be reduced to lower values of \( A \).

Since the \( \eta N \) interaction is isospin-independent, the difference between the \( \eta \text{H}^3 \) and \( \eta \text{He}^3 \) is caused by differences of the sizes and binding energies of these nuclei. Therefore
the use of more accurate nuclear wave functions is of utmost importance. Such realistic wave functions can be constructed, for instance, by means of the Integrodifferential Equation Approach (IDEA) \[28,29\], and are in progress.

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### Table I. The $\eta$-nucleus scattering length results (in fm) obtained by using the 9 combinations of the range parameter $\alpha$ and the $a_{\eta N}$ scattering length.

|       | $\alpha=2.357$ (fm$^{-1}$) | $\alpha=3.316$ (fm$^{-1}$) | $\alpha=7.617$ (fm$^{-1}$) | $a_{\eta N}$ (fm) |
|-------|-----------------|-----------------|-----------------|-----------------|
| $^2$H | 0.66+i0.82      | 0.65+i0.85      | 0.62+i0.89      |                 |
| $^3$H | 0.91+i1.80      | 0.81+i1.88      | 0.63+i1.93      | 0.27+i0.22      |
| $^3$He| 0.96+i1.72      | 0.89+i1.78      | 0.76+i1.84      |                 |
| $^4$He| 0.90+i3.32      | 0.48+i3.45      | -0.04+i3.40     |                 |
| $^2$H | 0.75+i0.73      | 0.74+i0.76      | 0.72+i0.81      |                 |
| $^3$H | 1.19+i1.70      | 1.11+i1.81      | 0.93+i1.92      | 0.28+i0.19      |
| $^3$He| 1.21+i1.59      | 1.16+i1.68      | 1.04+i1.78      |                 |
| $^4$He| 1.67+i3.43      | 1.23+i3.78      | 0.53+i3.94      |                 |
| $^2$H | 1.53+i2.00      | 1.38+i2.15      | 1.14+i2.22      |                 |
| $^3$H | -0.69+i5.13     | -1.21+i4.50     | -1.30+i3.79     | 0.55+i0.30      |
| $^3$He| 0.08+i5.22      | -0.52+i4.83     | -0.79+i4.21     |                 |
| $^4$He| -4.41+i2.85     | -3.73+i2.18     | -3.12+i1.85     |                 |