THE FINITE-TIME RUIN PROBABILITY
FOR AN INHOMOGENEOUS RENEWAL RISK MODEL

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ABSTRACT. In the paper, we give an asymptotic formula for the finite-time ruin probability in a generalized renewal risk model. We consider the renewal risk model with independent strongly subexponential claim sizes and independent not necessarily identically distributed inter occurrence times having finite variances. We find out that the asymptotic formula for the finite-time ruin probability is insensitive to the homogeneity of inter-occurrence times.

1. Introduction. The renewal risk model has been extensively investigated in the literature since it was introduced by Sparre Andersen half a century ago (see [1]). In this risk model, the claim sizes $Z_1, Z_2, \ldots$ form a sequence of independent and identically distributed (i.i.d.) nonnegative random variables (r.v.s) with a common distribution function (d.f.) $F_Z(u) = P(Z_1 \leq u)$ and a finite mean $\beta = \mathbb{E}Z_1$, while the inter occurrence times $\theta_1, \theta_2, \ldots$ are i.i.d. nonnegative r.v.s with common finite positive mean $\mathbb{E}\theta_1 = 1/\lambda$. In addition, it is assumed that $\{Z_1, Z_2, \ldots\}$ and $\{\theta_1, \theta_2, \ldots\}$ are mutually independent. In this model, the number of accidents in the interval $[0, t]$ is given by a renewal counting process

$$\Theta(t) = \sup\{n \geq 1 : \theta_1 + \theta_2 + \ldots + \theta_n \leq t\}$$

which has a mean function $\lambda(t) = \mathbb{E}\Theta(t)$ with $\lambda(t) \sim \lambda t$ as $t \to \infty$. The surplus process of the insurance company is then expressed as

$$R(t) = x + ct - \sum_{i=1}^{\Theta(t)} Z_i, \ t \geq 0,$$

where $x \geq 0$ is the initial risk reserve, and $c > 0$ is the constant premium rate.

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The probability of ruin within time \( t \) is a bivariate function

\[
\psi(x, t) := \mathbb{P} \left( \inf_{0 \leq s \leq t} R(s) < 0 \right) = \mathbb{P} \left( \max_{1 \leq k \leq \Theta(t)} \sum_{i=1}^{k} (Z_i - c \theta_i) > x \right).
\]  

(3)

Under the assumptions that \( \mu = c \mathbb{E} \theta_1 - \mathbb{E} Z_1 = c/\hat{\lambda} - \beta > 0 \) and the equilibrium d.f.

\[
\frac{1}{\beta} \int_0^x F_Z(u) \, du
\]

is subexponential, Embrechts and Veraverbeke (see [7] and [18]) established a celebrated asymptotic relation for the ultimate ruin probability:

\[
\psi(x, \infty) \sim \frac{1}{\mu} \int_{x}^{\infty} F_Z(u) \, du, \quad x \to \infty.
\]  

(4)

We recall that a d.f. \( F \) supported on \([0, \infty)\) is subexponential (\( F \) belongs to the class \( S \)) if

\[
F^{*2}(x) \sim 2F(x), \quad x \to \infty,
\]

where \( F^{*2} \) denotes the convolution of \( F \) with itself.

In 2004, Tang showed that a formula similar to (4) holds for the finite-time ruin probability as well. More exactly, the following statement was proved in paper [17].

If d.f. \( F_Z \) has a consistent variation and \( \mathbb{E} \theta_1^p \) for some \( p > 1 + \mathbb{J}_F^{+} \) then

\[
\psi(x, t) \sim \frac{1}{\mu} \int_{x}^{x + \mu \lambda(t)} F_Z(u) \, du, \quad x \to \infty,
\]  

(5)

uniformly for all \( t \) such that \( t \in \Lambda = \{ t : \lambda(t) > 0 \} \).

Here and further

\[
\mathbb{J}_F^{+} = - \lim_{y \to \infty} \frac{1}{\log y} \liminf_{x \to \infty} \frac{F(xy)}{F(x)},
\]

denote the upper Matuszevska index of a d.f. \( F \).

Furthermore, we say that a d.f. \( F \) concentrated on \([0, \infty)\) (or on \( \mathbb{R} \)) has a consistent variation (\( F \) belongs to the class \( C \)) if

\[
\lim_{y \to 1} \limsup_{x \to \infty} \frac{F(xy)}{F(x)} = 1.
\]

If d.f. \( F \in C \) has a finite mean, then the equilibrium d.f. of \( F \) is subexponential (see, for instance, Proposition 1.4.4 in [6]). In addition, the upper Matuszevska index \( \mathbb{J}_F^{+} \) is finite for each d.f. \( F \in C \) (see, for instance, Section 2.1 in [4]).

In [10] and [12], it was proved that the asymptotic formula (5) holds uniformly for \( t \in [a(x), \infty) \) with an arbitrary unboundedly increasing function \( a(x) \) if d.f. \( F_Z \in S_* \).

A d.f. \( F \) belongs to class \( S_* \) (\( F \) is strongly subexponential according to Korshunov (see [11])) if

\[
\int_0^\infty F(u) \, du < \infty \quad \text{and} \quad \lim_{x \to \infty} \frac{F^{*2}(x)}{F(x)} = 2.
\]
uniformly in $v \in [1, \infty)$, where

$$F_v(x) = \begin{cases} \min \left\{ 1, \frac{x+v}{x} F(u) \, du \right\} & \text{if } x \geq 0, \\ 1 & \text{if } x < 0. \end{cases}$$

It follows from Lemma 4 of [11] that each d.f. $F \in C$ with finite mean value is strongly subexponential.

Wang et al. [19] generalized the above results. It was shown that the asymptotic formula (5) preserves its form in the case when the inter occurrence times $\theta_1, \theta_2, \ldots$ have certain dependence structure (without restriction that $E\theta_1^p < \infty$ for some $p > 1 + J^+_F$).

In this paper, we consider an inhomogeneous (in time) renewal risk model. We suppose that inter occurrence times $\theta_1, \theta_2, \ldots$ are independent but not necessarily identically distributed. We obtain that the asymptotic formula (5) preserves also its form for such inter occurrence times satisfying some additional requirements. In fact, we consider a renewal risk model defined by equations (1) and (2) under the following three main assumptions.

**Assumption H1.** The claim sizes $\{Z_1, Z_2, \ldots\}$ are i.i.d. nonnegative r.v.s with common distribution function $F_Z$ and finite positive mean $\beta$.

**Assumptions H2.** The inter occurrence times $\{\theta_1, \theta_2, \ldots\}$ are independent non-negative r.v.s such that:

(H21) $\lim_{u \to \infty} \sup_{i \in \mathbb{N}} E\left( \theta_i \mathbb{1}_{\{\theta_i \geq u\}} \right) = 0$,

(H22) $\sum_{i=1}^{\infty} \frac{\text{Var}(\theta_i)}{i^2} < \infty$,

(H23) $\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} E\theta_i = \frac{1}{\lambda}$,

for some finite positive $\lambda$.

**Assumption H3.** The sequences $\{Z_1, Z_2, \ldots\}$ and $\{\theta_1, \theta_2, \ldots\}$ are mutually independent.

In the presented model analogously as in the classical Sparre Andersen model, the finite-time ruin probability $\psi(x,t)$ has expression (3), and we denote the mean function of the inhomogeneous renewal counting process $\Theta(t)$ by $\lambda(t)$, i.e. $\lambda(t) = E\Theta(t)$, where $t \geq 0$.

The model assumptions H1 and H3 are natural, while assumption H2 needs some additional comments. Hypothesis H21 requires that r.v.s $\{\theta_1, \theta_2, \ldots\}$ should be uniformly integrable. Such requirement is used sufficiently frequently in the study of non identically distributed r.v.s (see, for instance, [16] or Chapter II in [15]). We use assumption H21 together with H23 to obtain an asymptotic formula for the exponential moment tail of renewal process $\Theta(t)$ by $\lambda(t)$, i.e. $\lambda(t) = E\Theta(t)$, where $t \geq 0$.

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Example 1. Let \( \{\theta_1, \theta_2, \ldots\} \) be independent r.v.s, such that \( \theta_1, \theta_4, \theta_7, \ldots \) be distributed according to the Poisson law with parameter \( 1/\lambda_1 \), r.v.s \( \theta_2, \theta_5, \theta_8, \ldots \) be distributed according to the Poisson law with parameter \( 1/\lambda_2 \) and \( \theta_3, \theta_6, \theta_9, \ldots \) be distributed according to the Poisson law with parameter \( 1/\lambda_3 \). If \( \lambda_1 \neq \lambda_2 \neq \lambda_3 \) then the renewal counting process \( \Theta(t) \) is inhomogeneous but assumption \( H_2 \) holds with \( \lambda = 3\lambda_1\lambda_2\lambda_3/(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3) \).

Example 2. Let \( \{\theta_1, \theta_2, \ldots\} \) be independent r.v.s distributed in the following way:

\[
P(\theta_i = 0) = \frac{1}{2}, \quad P(\theta_i = 1) = \frac{1}{2} - \frac{1}{i+3}, \quad P(\theta_i = \sqrt{i+3}) = \frac{1}{i+3}, \quad i \in \mathbb{N}.
\]

The renewal process with such inter occurrence times is also inhomogeneous and assumption \( H_2 \) holds again with \( \lambda = 2 \) because:

\[
\sup_{i \in \mathbb{N}} E(\theta_i I\{\theta_i \geq u\}) \leq \frac{1}{u}, \quad u > 1,
\]

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} E \theta_i = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \left( \frac{1}{2} - \frac{1}{i+3} + \frac{1}{\sqrt{i+3}} \right) = \frac{1}{2},
\]

\[
\text{Var}(\theta_i) = \frac{5}{4} - \frac{1}{i+3} - \frac{1}{(i+3)^2} - \frac{i+1}{(i+3)\sqrt{(i+3)}} < \frac{5}{4}, \quad i \in \mathbb{N}.
\]

2. Main results. In this section, we present exact formulations of our assertions. Before these formulations we recall the definition of long tailed distribution.

A d.f. \( F \) supported on \([0, \infty)\) (or on \( \mathbb{R} \)) belongs to class \( L \) (is long tailed) if for each positive \( y \)

\[
\lim_{x \to \infty} F(x+y) F(x) = 1.
\]

The following two theorems and the corollary are the main results of the paper.

**Theorem 2.1.** Let assumptions \( H_1, H_2 \) and \( H_3 \) be satisfied, \( \mu > 0 \) and \( F_Z \in \mathcal{L} \). Then for each \( T \in \Lambda := \{t > 0 : \lambda(t) > 0\} \)

\[
\inf_{t \in [T, \infty)} \psi(x, t) \geq \frac{1}{\mu} \int_{x}^{x+\mu\lambda(t)} F_Z(u) \, du.
\]

**Theorem 2.2.** Let conditions \( H_1, H_{21}, H_{23}, H_3 \) be satisfied, \( \mu > 0 \) and \( F_Z \in \mathcal{S}_e \). Then

\[
\sup_{t \in [T, \infty)} \psi(x, t) \leq \frac{1}{\mu} \int_{x}^{x+\mu\lambda(t)} F_Z(u) \, du
\]

with an arbitrary \( T \in \Lambda \).

**Corollary 1.** If assumptions \( H_1, H_2 \) and \( H_3 \) hold, \( \mu := c/\lambda - \beta > 0 \) and d.f. \( F_Z \in \mathcal{S}_e \), then

\[
\psi(x, t) \sim \frac{1}{\mu} \int_{x}^{x+\mu\lambda(t)} F_Z(u) \, du
\]

uniformly for \( t \in [T, \infty) \), where \( T \in \Lambda \).
Obviously, Corollary 1 follows immediately from Theorems 2.1 and 2.2 because $\mathcal{S}_* \subset \mathcal{S} \subset \mathcal{L}$ due to Lemma 1 of [9], Lemma 2 of [5] and Lemma 1.3.5(a) of [6].

According to Lemma 3.6 below $\lambda(t) \sim \lambda t$ if $t \to \infty$. Therefore, Corollary 1 implies more simple asymptotic formula for the finite-time ruin probability in the case when the horizon of time $t$ is restricted to a smaller region. Namely, under conditions of Corollary 1, we can obtain that

$$
\psi(x, t) \sim \frac{1}{\mu} \int_x^{x+\mu \lambda t} F_Z(u) \, du
$$

uniformly with respect to $t \in [a(x), \infty)$, where $a(x)$ is an unboundedly increasing function.

Possibly, the asymptotic formulas, presented in Theorems 2.1 and 2.2 (and so in Corollary 1), hold uniformly for all $t \in \Lambda$, not only for $t \in [T, \infty)$ with $T \in \Lambda$. At the moment, we do not know how we can extend the region of uniformity without additional requirements.

The rest of the paper is organized as follows. In Section 3 we collected all auxiliary results which we need to prove our Theorems 2.1 and 2.2. In Section 4 we obtain lower estimate of the finite-time ruin probability, while in the next Section 5 we prove the upper estimate for the same probability.

3. Auxiliary results. In this section, we present lemmas which we use in the proof of our main results.

**Lemma 3.1.** (see Lemma 1 in [11]) Let $\xi_1, \xi_2, \ldots$ be independent copies of r.v $\xi$ with d.f. $F_\xi$ and negative mean $\mathbb{E}\xi < 0$. If $F_\xi \in \mathcal{L}$, then

$$
\lim_{x \to \infty} \inf_{n \geq 1} \left\{ \mathbb{P} \left( \max_{1 \leq k \leq n} k \sum_{i=1}^{k} \xi_i > x \right) / \frac{1}{|\mathbb{E}\xi|} \int_x^{x+|\mathbb{E}\xi|} F_\xi(v) \, dv \right\} \geq 1.
$$

**Lemma 3.2.** (see Lemma 9 in [11]) Let $\xi_1, \xi_2, \ldots$ be independent copies of r.v $\xi$ with d.f. $F_\xi$ and negative mean $\mathbb{E}\xi < 0$. If $F_\xi \in \mathcal{S}_*$, then

$$
\limsup_{x \to \infty} \sup_{n \geq 1} \left\{ \mathbb{P} \left( \max_{1 \leq k \leq n} k \sum_{i=1}^{k} \xi_i > x \right) / \frac{1}{|\mathbb{E}\xi|} \int_x^{x+|\mathbb{E}\xi|} F_\xi(v) \, dv \right\} \leq 1.
$$

**Lemma 3.3.** (see Theorem 6.7 and Lemma 6.8 in [13]) If $\eta_1, \eta_2, \ldots$ are independent r.v.s such that $\sum_{i=1}^{\infty} \text{Var}(\eta_i) / i^2 < \infty$, then

$$
\frac{1}{n} \sum_{k=1}^{n} \eta_i - \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}\eta_i \to 0
$$

almost surely, or equivalently

$$
\lim_{n \to \infty} \mathbb{P} \left( \sup_{m \geq n} \left| \frac{1}{m} \sum_{k=1}^{m} \eta_i - \frac{1}{m} \sum_{k=1}^{m} \mathbb{E}\eta_i \right| > \epsilon \right) = 0
$$

for an arbitrary positive $\epsilon$.

**Lemma 3.4.** (see Lemma 1 in [2]) Let $\eta_1, \eta_2, \ldots$ be independent r.v.s such that:

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\eta_i = -d_1, \quad \lim_{u \to \infty} \sup_{i \in \mathbb{N}} \mathbb{E}(|\eta_i \mathbb{1}_{(\eta_i \leq u)}) = 0, \quad \eta_i \leq d_2, \ i \in \mathbb{N},
$$
for some positive constant constants constants \(d_1\) and \(d_2\). Then there exist positive constants \(d_3\) and \(d_4\), may depend on \(d_1\), \(d_2\), for which

\[
P\left( \sup_{k \geq 1} \sum_{i=1}^{k} \eta_i > x \right) \leq d_3 e^{-d_4 x}, \quad x > 0.\]

**Lemma 3.5.** (see Theorem 2.1 in [3]) Let \(\theta_1, \theta_2, \ldots\) be independent r.v.s satisfying Assumptions \(H_21\) and \(H_23\). Then, for every \(a > \lambda\) there exists \(b > 1\) such that

\[
\lim_{t \to \infty} \sum_{k > at} P(\Theta(t) \geq k) b^k = 0.
\]

**Lemma 3.6.** (see, for instance, Corollary 2.2 in [3]) If independent r.v.s \(\theta_1, \theta_2, \ldots\) satisfy Assumptions \(H_21\) and \(H_23\), then \(E\Theta^r(t) \sim N^t r\) for each \(r > 0\), where the renewal counting process \(\Theta(t)\) is defined by (1).

**Lemma 3.7.** (see, for instance, Corollary 2.3 in [4]) Under conditions of Lemma 3.6

\[
\Theta(t) \quad \text{p} \quad \frac{\Theta(t)}{E\Theta(t)} \quad t \to \infty \to 1.
\]

4. **Lower bound.** In this section, it is dealt with the proof of Theorem 2.1. Essentially, we keep in our proof the way of [19]. Let, as usual, \(\varepsilon, \delta \in (0, 1)\), \(L \in \mathbb{N}\) and \(\hat{Z}_i = Z_i - c(1 + \delta)/\lambda\), \(\hat{\theta}_i = (1 + \delta)/\lambda - \theta_i\) for \(i \in \mathbb{N}\). For such \(i\) we have \(\hat{Z}_i + c \hat{\theta}_i = Z_i - c \theta_i\). So, according to [3] we get that

\[
\psi(x, t) \geq P\left( \max_{1 \leq k \leq \Theta(t)} \sum_{i=1}^{k} \left( \hat{Z}_i + c \hat{\theta}_i \right) > x, \quad \min_{1 \leq k \leq \Theta(t)} \sum_{i=1}^{k} \hat{\theta}_i > -L \right)
\]

\[
= \sum_{n=1}^{\infty} P\left( \max_{1 \leq k \leq n} \sum_{i=1}^{k} \left( \hat{Z}_i + c \hat{\theta}_i \right) > x, \quad \min_{1 \leq k \leq n} \sum_{i=1}^{k} \hat{\theta}_i > -L, \Theta(t) = n \right)
\]

\[
\geq \sum_{n=1}^{\infty} P\left( \max_{1 \leq k \leq n} \sum_{i=1}^{k} \left( \hat{Z}_i - cL \right) > x, \quad \max_{1 \leq k \leq n} \sum_{i=1}^{k} \left( -\hat{\theta}_i \right) < L, \Theta(t) = n \right)
\]

\[
\geq \sum_{n=1}^{\infty} P\left( \max_{1 \leq k \leq n} \sum_{i=1}^{k} \hat{Z}_i > x + cL, \quad \sup_{k \geq 1} \sum_{i=1}^{k} \left( -\hat{\theta}_i \right) < L, \Theta(t) = n \right)
\]

\[
\geq \sum_{n \\geq 1} P\left( \max_{1 \leq k \leq n} \sum_{i=1}^{k} \hat{Z}_i > x + cL \right) P(\sup_{k \geq 1} \sum_{i=1}^{k} (-\hat{\theta}_i) < L, \Theta(t) = n) \tag{6}
\]

for all positive \(x\) and \(t\).

Since d.f. \(F_Z\) is long-tailed we obtain using Lemma 3.1 that

\[
P\left( \max_{1 \leq k \leq n} \sum_{i=1}^{k} \hat{Z}_i > x + cL \right) \geq (1 - \varepsilon) \frac{1}{|\beta|^n} \int_{x}^{x+|\beta|n} F_Z(u + cL + c(1 + \delta)/\lambda) \, du
\]

\[
\geq \frac{1 - \varepsilon}{|\beta|} \inf_{u \geq x} \frac{F_Z(u + cL + c(1 + \delta)/\lambda)}{F_Z(u)} \int_{x}^{x+\mu n} F_Z(u) \, du
\]
for \( n \geq 1 \) if \( x \) is sufficiently large \( (x \geq x_1 = x_1(\delta)) \), where
\[
\hat{\beta} = E\bar{Z}_1 = -\mu(1 + \delta + \delta\beta/\mu) < 0.
\]

Substituting the last estimate into (6) we get
\[
\lim_{x \to \infty} \inf_{t \geq T_1} \frac{\psi(x, t)}{\int_x F_Z(u) \, du} \geq \frac{(1 - \varepsilon)}{\mu(1 + \delta) + \delta\beta} \inf_{t \geq T_1} \mathbb{P}\left( \sup_{k \geq 1} \sum_{i=1}^k (-\hat{\theta}_i) < L, \Theta(t) \geq (1 - \varepsilon)\lambda t \right) \tag{7}
\]
for all for \( \varepsilon, \delta \in (0, 1), \ L \in \mathbb{N} \) and \( T_1 > 0 \).

It is obvious that
\[
\mathbb{P}\left( \sup_{k \geq 1} \sum_{i=1}^k (-\hat{\theta}_i) < L, \Theta(t) \geq (1 - \varepsilon)\lambda t \right) \geq \mathbb{P}\left( \sup_{k \geq 1} \sum_{i=1}^k (-\hat{\theta}_i) < L \right) + \mathbb{P}\left( \Theta(t) \geq (1 - \varepsilon)\lambda t \right) - 1. \tag{8}
\]

Conditions of Theorem 2.1 imply that
\[
\mathbb{P}\left( \sup_{k \geq 1} \sum_{i=1}^k (-\hat{\theta}_i) < L \right) \geq \mathbb{P}\left( \max_{1 \leq k \leq K} \sum_{i=1}^k (-\hat{\theta}_i) < L - 1 \right) \cap \left\{ \sum_{i=1}^k (-\hat{\theta}_i) < L - 1 \text{ for } k \geq K + 1 \right\}
\]
\[
\geq \mathbb{P}\left( \max_{1 \leq k \leq K} \sum_{i=1}^k (-\hat{\theta}_i) < L - 1 \right)
+ \mathbb{P}\left( \frac{1}{k} \sum_{i=1}^k (-\hat{\theta}_i) + \frac{1}{k} \sum_{i=1}^k \mathbb{E}\hat{\theta}_i < \frac{1 + \delta}{\lambda} - \frac{1}{k} \sum_{i=1}^k \mathbb{E}\theta_i \text{ for } k \geq K + 1 \right) - 1
\]
\[
\geq \mathbb{P}\left( \bigcap_{k=1}^K \left\{ \sum_{i=1}^k (-\hat{\theta}_i) < L - 1 \right\} \right)
+ \mathbb{P}\left( \frac{1}{k} \sum_{i=1}^k (-\hat{\theta}_i) + \frac{1}{k} \sum_{i=1}^k \mathbb{E}\hat{\theta}_i < \frac{1 + \delta}{\lambda} - \frac{1}{\lambda} - \frac{\delta}{2\lambda} \text{ for } k \geq K + 1 \right) - 1
\]
\[
\geq \mathbb{P}\left( \bigcap_{k=1}^K \left\{ \sum_{i=1}^k (-\hat{\theta}_i) < L - 1 \right\} \right)
+ \mathbb{P}\left( \sup_{k \geq K+1} \left| \frac{1}{k} \sum_{i=1}^k (-\hat{\theta}_i) + \frac{1}{k} \sum_{i=1}^k \mathbb{E}\hat{\theta}_i \right| < \frac{\delta}{2\lambda} \right) - 1.
\]

for each sufficiently large \( K = K(\delta) \) and \( L \geq 2 \)

So, due to Lemma 3.3
\[
\lim_{L \to \infty} \mathbb{P}\left( \sup_{k \geq 1} \sum_{i=1}^k (-\hat{\theta}_i) < L \right) = 1. \tag{9}
\]
In addition, Lemmas 3.6 and 3.7 show that
\[
\inf_{t \geq T_2} \mathbb{P}(\Theta(t) \geq (1 - \varepsilon) \lambda t) \geq 1 - \varepsilon
\]  
for some sufficiently large \(T_2 = T_2(\varepsilon, \delta)\).

The derived estimates (7) – (10) imply that
\[
\liminf_{x \to \infty} \inf_{t \geq T_2} \left( \psi(x, t) \int_x^{x + \mu(1 - \varepsilon)t} F_Z(u) \, du \right) \geq \frac{(1 - \varepsilon)^2}{\mu(1 + \delta) + \delta \beta}
\]
for all for \(\varepsilon, \delta \in (0, 1)\) and sufficiently large \(T_2\).

According to Lemma 3.6 \(\mathbb{E} \Theta(t) \sim \lambda t\). Therefore
\[
\liminf_{x \to \infty} \inf_{t \geq T_3} \left( \psi(x, t) \int_x^{x + \mu(1 - \varepsilon)t} F_Z(u) \, du \right) \geq 1 - 3\varepsilon
\]
if \(x > 0\), \(\varepsilon \in (0, 1/3)\) and \(t \geq T_3\) (\(T_3 \geq T_2\)).

The last estimate substituting into (11) we obtain
\[
\liminf_{x \to \infty} \inf_{t \geq T_3} \left( \psi(x, t) \int_x^{x + \mu(1 - \varepsilon)t} F_Z(u) \, du \right) \geq \frac{(1 - 3\varepsilon)}{1 - \varepsilon}
\]
for all for \(\varepsilon \in (0, 1/3)\), \(\delta \in (0, 1)\) and sufficiently large \(T_3\).

Now let \(T\) be such that \(\lambda(T) > 0\). If \(x > 0\) and \(t \in [T, T_3]\), then due to expression (3) we have
\[
\psi(x, t) \geq \mathbb{P} \left( \sum_{i=1}^{\Theta(t)} (Z_i - c \theta_i) > x \right)
\]
\[
\geq \sum_{n=1}^{\infty} \mathbb{P} \left( \sum_{i=1}^{n} Z_i - c \sum_{i=1}^{n} \theta_i > x, \sum_{i=1}^{n} \theta_i < t, \sum_{i=1}^{n+1} \theta_i > t \right)
\]
\[
\geq \sum_{n=1}^{\infty} \mathbb{P} \left( \sum_{i=1}^{n} Z_i > x + c t, \Theta(t) = n \right)
\]
\[
\geq \sum_{n=1}^{\infty} \mathbb{P} \left( \max_{1 \leq m \leq n} \sum_{i=1}^{m} (Z_i - c/\lambda) > x + c T_3 \right) \mathbb{P}(\Theta(t) = n).
\]

Suppose that \(\varphi(x) \geq 1\) is some unboundedly increasing function under condition
\[
\frac{F_x(x + \mu \varphi(x))}{F_x(x)} \sim 1, \quad x \to \infty.
\]
The existence of such function follows from condition \( F_Z \in \mathcal{L} \). According to Lemma 3.1 we have

\[
\psi(x, t) \geq \frac{1 - \varepsilon}{\mu} \sum_{n=1}^{\infty} \mathbb{P}(\Theta(t) = n) \int_{x + c T_3 + \mu n}^{x + c T_3 + \mu n} F_Z(u + c/\lambda) \, du
\]

\[
\geq (1 - \varepsilon) \sum_{n=1}^{\infty} n \mathbb{P}(\Theta(t) = n) F_Z(x + c T_3 + c/\lambda + \mu \varphi(x))
\]

\[
\geq (1 - \varepsilon)^2 F_Z(x) \mathbb{E} \Theta(t) I_{\{\Theta(t) > \varphi(x)\}}
\]

if \( t \in [T, T_3] \) and \( x \geq x_2 = x_3(\varepsilon, T, T_3) \geq x_2 \).

The Hölder inequality implies that for sufficiently large \( x \) (\( x \geq x_3 = x_3(\varepsilon, T, T_3) \geq x_2 \))

\[
\mathbb{E} \Theta(t) I_{\{\Theta(t) > \varphi(x)\}} \leq \left( \mathbb{E} \Theta^2(t) \right)^{1/2} \sqrt{\mathbb{P}(\Theta(t) > \varphi(x))}
\]

\[
\leq \left( \mathbb{E} \Theta^2(T_3) \right)^{1/2} \sqrt{\mathbb{P}(\Theta(T_3) > \varphi(x))} \frac{\lambda(t)}{\lambda(T)}
\]

\[
\leq \varepsilon \lambda(t).
\]

The last estimate and (14) imply that

\[
\psi(x, t) \geq (1 - \varepsilon)^3 F_Z(x) \lambda(t) \geq \frac{(1 - \varepsilon)^3}{\mu} \int_{x}^{x + \mu \lambda(t)} F_Z(u) \, du
\]

for all \( \varepsilon \in (0, 1) \), \( x \geq x_3 \) and \( t \in [T, T_3] \). Consequently,

\[
\liminf_{x \to \infty} \inf_{t \in [T, T_3]} \left( \psi(x, t) \sqrt[3]{\frac{1}{\mu}} \int_{x}^{x + \mu \lambda(t)} F_Z(u) \, du \right) \geq (1 - \varepsilon)^3
\]

The desired lower bound of Theorem 2.1 follows now from (12) and (16) immediately because of arbitrariness of \( \varepsilon \in (0, 1/3) \) and \( \delta \in (0, 1) \).

5. Upper bound. In this section, we obtain the assertion of Theorem 2.2. The proof of the assertion consists of two parts. In the first part of the proof we use the way from [12]. In the second part of proof we mainly use the consideration from [19].

Let \( \varepsilon, \delta \in (0, 1) \), \( T \in \Lambda \) and \( \bar{Z}_i = Z_i - c(1 - \delta)/\lambda, \bar{\theta}_i = (1 - \delta)/\lambda - \theta_i \) for each \( i \in \mathbb{N} \). According to (3) we have that

\[
\psi(x, t) \leq \mathbb{P}\left( \max_{1 \leq k \leq (1 + \varepsilon)\lambda(t)} \sum_{i=1}^{k} \bar{Z}_i + c \sup_{k \geq 1} \sum_{i=1}^{k} \bar{\theta}_i > x \right)
\]

\[
+ \mathbb{P}\left( \max_{1 \leq k \leq \Theta(t)} \sum_{i=1}^{k} (Z_i - c \theta_i) > x, \Theta(t) > (1 + \varepsilon)\lambda(t) \right)
\]

\[
:= \psi_1(x, t) + \psi_2(x, t)
\]

if \( x > 0 \) and \( t \geq T \). Denoting

\[
\zeta_t = \max_{1 \leq k \leq (1 + \varepsilon)\lambda(t)} \sum_{i=1}^{k} \bar{Z}_i, \quad \chi = \sup_{k \geq 1} \sum_{i=1}^{k} \bar{\theta}_i, \quad \chi^+ = \chi I_{\{\chi > 0\}},
\]
we obtain
\[ \psi_1(x, t) = \mathbb{P}(\zeta_t + \chi > x) \]
\[ \leq \int_{[0, x-y]} \mathbb{P}(\zeta_t > x-u) d\mathbb{P}(\chi^+ \leq u) + \mathbb{P}(\chi^+ > x-y) \]
\[ := \psi_{11}(x, y, t) + \psi_{12}(x, y, t), \tag{18} \]
where 0 < y \leq x/2.

If 0 < \delta < 1 - \lambda\beta/c = \mu/(\mu + \beta)$, then $\tilde{\beta} := \mathbb{E}Z_1 = -\mu + \delta(\mu + \beta) < 0$. In addition, we have that d.f. $\mathbb{P}(Z_1 \leq u) = F_Z(u + c(1 - \delta)/\lambda)$ belongs to the class $S_\star$ due to Lemma 3 from [11]. So, applying Lemma 3.2, we get that

\[ \psi_{11}(x, y, t) \leq \frac{1 + \varepsilon}{|\beta|} \int_{[0, x-y]} \left( \int_x^{x-u+|\beta|(1+\varepsilon)\lambda(t)} F_Z(w+c(1-\delta)/\lambda) dw \right) dF_{\chi^+}(u), \]
where $x \geq 2y$, $y$ is sufficiently large ($y \geq x_1 = x_1(\delta, \varepsilon)$) and $F_{\chi^+}$ denotes the d.f. of r.v. $\chi^+$.

By the Fubini-Tonelli theorem
\[ \psi_{11}(x, y, t) \leq \frac{1 + \varepsilon}{|\beta|} \int_{[0, x-y]} \left( \int_x^{x+\mu(1+\varepsilon)\lambda(t)} F_Z(w-u) dw \right) dF_{\chi^+}(u) \]
\[ = \frac{1 + \varepsilon}{|\beta|} \int_x^{x+\mu(1+\varepsilon)\lambda(t)} F_Z * F_{\chi^+}(w) dw. \tag{19} \]

Conditions of Theorem 2.2 imply that:
\[ \frac{1}{n} \sum_{i=1}^n \mathbb{E} \tilde{\theta}_i \rightarrow \frac{-\delta}{\lambda}; \]
\[ \tilde{\theta}_i \leq \frac{1 - \delta}{\lambda} \text{ for each } i \in \mathbb{N}; \]
\[ \lim_{u \to \infty} \sup_{i \in \mathbb{N}} \mathbb{E} \left( \tilde{\theta}_i \mathbb{I}_{\{\tilde{\theta}_i \leq -u\}} \right) \leq 2 \lim_{u \to \infty} \sup_{i \in \mathbb{N}} \mathbb{E} \left( \tilde{\theta}_i \mathbb{I}_{\{\tilde{\theta}_i \geq u\}} \right) = 0. \]

So, due to Lemma 3.4
\[ F_{\chi^+}(w) = \mathbb{P}(\chi > w) \leq c_1 e^{-c_2 w}, \tag{20} \]
for some positive constants $c_1 = c_1(\delta)$ and $c_2 = c_2(\delta)$. Applying, for instance, Corollary 2 from [14] we obtain
\[ F_Z * F_{\chi^+}(w) \sim F_Z(w). \]

because $F_Z \in S_\star \subset \mathcal{L}$.

Therefore, estimate (19) implies that
\[ \psi_{11}(x, y, t) \leq \frac{(1 + \varepsilon)^2}{|\beta|} \int_x^{x+\mu(1+\varepsilon)\lambda(t)} F_Z(w) dw, \tag{21} \]
where $\varepsilon \in (0, 1)$, $\delta \in (0, \mu/(\mu + \beta))$, $t \geq T$ and $x \geq 2y$ and $y$ is sufficiently large, i.e. $y \geq x_2(\delta, \varepsilon) \geq x_1$.
If \( t \geq T \), then
\[
\int_x^{x+\mu(1+\varepsilon)\lambda(t)} F_Z(w) \, dw = \int_x^{x+\mu\lambda(t)} F_Z(w) \, dw \left( 1 + \frac{\int_x^{x+\mu(1+\varepsilon)\lambda(t)} F_Z(w) \, dw}{\int_x^{x+\mu\lambda(t)} F_Z(w) \, dw} \right) \leq (1 + \varepsilon) \int_x^{x+\mu\lambda(t)} F_Z(w) \, dw.
\]

The last inequality and estimate (21) imply that
\[
\limsup_{x \to \infty} \sup_{t \in [T, \infty)} \psi_{11}(x, y, t) \leq \frac{(1 + \varepsilon)^3}{\mu - \delta(\mu + \beta)}
\]
if \( \varepsilon \in (0, 1) \), \( \delta \in (0, \mu/(\mu + \beta)) \) and \( y \geq x_2 \).

To estimate the term \( \psi_{12}(x, y, t) \) from (18) we use (20) again. If \( y \geq x_2 \), then we have
\[
\limsup_{x \to \infty} \sup_{t \in [T, \infty)} \psi_{12}(x, y, t) \leq \limsup_{x \to \infty} \frac{\mathbb{P}(\chi^+ > x/2)}{\mu\lambda(T) F_Z(x + \mu\lambda(T))} \leq \frac{c_1}{\mu\lambda(T)} \limsup_{x \to \infty} \frac{e^{-c_2x/2}}{F_Z(x + \mu\lambda(T))} = 0
\]
because of \( F_Z \in S_\ast \subset S \subset L \) and Lemma 1.3.5 (b) from [6].

Relations (18), (21) and (23) hold for all \( y \geq x_2 \). So, these relations imply that
\[
\limsup_{x \to \infty} \sup_{t \in [T, \infty)} \psi_1(x, t) \leq \frac{(1 + \varepsilon)^3}{\mu - \delta(\mu + \beta)}
\]
for all \( \varepsilon \in (0, 1) \), \( \delta \in (0, \mu/(\mu + \beta)) \) and \( t \in \Lambda \).

It remains to get a similar inequality for \( \psi_2(x, t) \). Lemma 3.6 implies that
\[
\psi_2(x, t) \leq \sum_{n > (1+\varepsilon)\lambda(t)} \mathbb{P}\left( \max_{1 \leq k \leq n} \sum_{i=1}^k Z_i > x, \Theta(t) = n \right) \leq \sum_{n > (1+\varepsilon/2)\lambda} \mathbb{P}(\Theta(t) = n),
\]
where \( x > 0 \) and \( t \) is sufficiently large \( t \geq T_4 = T_4(\varepsilon) \geq T \). According to the Kesten estimate for d.f. \( F_Z \in S_\ast \subset S \) (see, for instance, Lemma 1.3.5 (c) in [6]) we have that
\[
\sup_{x \geq 0} \frac{F_Z^n(x)}{F_Z(x)} \leq c_3(1 + \Delta)^n,
\]
where \( \Delta > 0 \) and \( c_3 = c_3(\Delta) \) is a suitable positive constant.
For each $x > 0$ and $T_5 \geq T_4$ relations (25), (26) imply that

$$
\sup_{t \in (T_5, \infty)} \frac{\psi_2(x, t)}{\int_x \frac{F_Z(w)}{w} \, dw} \leq \frac{1}{\mu \lambda(T_5)} \sup_{t \in (T_5, \infty)} \sum_{n > (1 + \varepsilon/2) \lambda} \sup_{x \geq 0} \frac{F_Z(x)}{F_Z(x + \mu \lambda(T_5))} \sum_{n > (1 + \varepsilon/2) \lambda} (1 + \Delta)^n \mathbb{P}(\Theta(t) = n).
$$

If $b = 1 + \Delta$ is chosen for $a = (1 + \varepsilon/2) \lambda$ according to Lemma 3.5, then the last inequality implies that

$$
\limsup_{x \to \infty} \sup_{t \in (T_5, \infty)} \frac{\psi_2(x, t)}{\int_x \frac{F_Z(w)}{w} \, dw} \leq \varepsilon
$$

where $T_5 = T_5(\varepsilon) \in \Lambda$ is sufficiently large.

The last inequality together with equality (17) and estimate (24) implies that

$$
\limsup_{x \to \infty} \sup_{t \in (T_5, \infty)} \frac{\psi(x, t)}{\int_x \frac{F_Z(w)}{w} \, dw} \leq \frac{(1 + \varepsilon)^3}{\mu - \delta(\mu + \beta)} + \varepsilon
$$

(27)

It remains to estimate $\psi(x, t)$ in the case when $t \in [T, T_5]$. Suppose that function $1 \leq \varphi(x) \leq \sqrt{x}$, $x \geq 1$, satisfies property (13). If $x \geq 1$ and $t \geq T$, then due to (3) we have

$$
\psi(x, t) \leq \mathbb{P}\left( \max_{1 \leq k \leq \Theta(t)} \sum_{i=1}^{k} Z_i > x, \Theta(t) \leq \varphi(x) \right)
$$

$$
+ \mathbb{P}\left( \max_{1 \leq k \leq \Theta(t)} \sum_{i=1}^{k} Z_i + \sum_{k \geq 1} \bar{\theta}_i > x, \Theta(t) > \varphi(x) \right)
$$

$$
:= \tilde{\psi}_1(x, t) + \tilde{\psi}_2(x, t)
$$

(28)

Applying Lemma 3.2 we obtain

$$
\tilde{\psi}_1(x, t) = \mathbb{P}\left( \sum_{i=1}^{\Theta(t)} Z_i > x, \Theta(t) \leq \varphi(x) \right)
$$

$$
\leq \sum_{n \leq \varphi(x)} \mathbb{P}\left( \sum_{i=1}^{n} Z_i > x, \Theta(t) = n \right)
$$

$$
\leq \sum_{n \leq \varphi(x)} \mathbb{P}\left( \max_{1 \leq k \leq n} \sum_{i=1}^{k} \left( Z_i - \frac{c}{\lambda} \right) > x - \frac{c \varphi(x)}{\lambda} \right) \mathbb{P}(\Theta(t) = n)
$$

$$
\leq (1 + \varepsilon) \frac{\mu}{\lambda(t)} \sum_{n=1}^{\infty} \mathbb{P}(\Theta(t) = n) \int_{x - \varepsilon \varphi(x)/\lambda}^{x - \varphi(x)/\lambda} \frac{F_Z(u)}{u} \, du
$$

$$
\leq (1 + \varepsilon) \lambda(t) F_Z\left( x - \frac{c \varphi(x)}{\lambda} \right)
$$
because of condition (13).

On the other hand, if $x$ is sufficiently large, then Lemma 3.2 implies

$$\limsup_{x \to \infty} \sup_{t \in [T, T_3]} \frac{\tilde{\psi}_1(x,t)}{\int x \ F_Z(w) \ dw} \leq 1 + \varepsilon$$  \hfill (29)$$

because of condition (13).

Using (20), the fact that $F_Z \in \mathcal{L}$ and Lemma 1.3.5 (b) from [6] we have

$$\limsup_{x \to \infty} \sup_{t \in [T, T_3]} \frac{\tilde{\psi}_{22}(x,t)}{\int x \ F_Z(w) \ dw} \leq \limsup_{x \to \infty} \frac{c_1 e^{-c_2(x - \varphi(x))}}{\mu \lambda(T) F_Z(x + \mu \lambda(T_3))} = 0.$$  \hfill (31)$$

If $x$ is sufficiently large, then Lemma 3.2 implies

$$\hat{\psi}_{21}(x,t) = \sum_{n > \varphi(x)} \int_{[0, x - \varphi(x)]} \mathbb{P} \left( \max_{1 \leq k \leq n} \sum_{i=1}^{k} \tilde{Z}_i + \chi^+ > x, \chi^+ \leq x - \varphi(x), \Theta(t) = n \right) \ d\mathbb{P} \left( \chi^+ \leq y, \Theta(t) = n \right)$$

$$\leq \frac{1 + \varepsilon}{|\beta|} \sum_{n > \varphi(x)} \int_{[0, x - \varphi(x)]} \left( \int_{x-y}^{x-y+|\beta|n} F_Z(w) \ dw \right) \ d\mathbb{P} \left( \chi^+ \leq y, \Theta(t) = n \right)$$

$$\leq (1 + \varepsilon) \sum_{n > \varphi(x)} n \int_{[0, x - \varphi(x)]} F_Z(x-y) \ d\mathbb{P} \left( \chi^+ \leq y, \Theta(t) = n \right)$$

$$= (1 + \varepsilon) \sum_{n > \varphi(x)} n \mathbb{P} \left( Z + \chi^+ > x, \chi^+ \leq x - \varphi(x), \Theta(t) = n \right)$$

$$\leq (1 + \varepsilon) \sum_{n > \varphi(x)} n \mathbb{P} \left( Z + \chi^+ > x, \chi^+ \leq x - \varphi(x), Z \leq x - \varphi(x), \Theta(t) = n \right)$$

$$+ (1 + \varepsilon) \sum_{n > \varphi(x)} n \mathbb{P} \left( Z > x - \varphi(x), \Theta(t) = n \right)$$

$$:= (1 + \varepsilon)(\hat{\psi}_{211}(x,t) + \hat{\psi}_{212}(x,t)).$$  \hfill (32)$$

Using the Hölder inequality we get

$$\hat{\psi}_{212}(x,t) = \frac{F_Z(x - \varphi(x)) \mathbb{E} \Theta(t) \mathbbm{1}_{\Theta(t) > \varphi(x)}}{\int x \ F_Z(w) \ dw} \leq \frac{F_Z(x - \varphi(x)) \left( \mathbb{E} \Theta^2(t) \right)^{1/2} \left( \mathbb{P}(\Theta(t) > \varphi(x)) \right)^{1/2}}{\int x \ F_Z(w) \ dw}.$$
Therefore,

\[
\limsup_{x \to \infty} \sup_{t \in [T, T_5]} \frac{\hat{\psi}_{212}(x, t)}{\int_x F_Z(w) \, dw} \leq \left( \frac{\mathbb{E} \Theta^2(T_5)}{\mu \lambda(T)} \right)^{1/2} \limsup_{x \to \infty} \frac{F_Z(x - \varphi(x))}{F_Z(x + \mu \lambda(T_5))} \left( \mathbb{P}(\Theta(T_5) > \varphi(x)) \right)^{1/2}
\]

\[
= 0 \quad (33)
\]

given property [13].

Finally, if \( t \in [T, T_5] \) and \( x \) is sufficiently large, then

\[
\frac{\hat{\psi}_{211}(x, t)}{\int_x F_Z(w) \, dw} \leq \sum_{n > \varphi(x)} n \mathbb{P}(Z + \chi^+ > x, \varphi(x) < Z \leq x - \varphi(x), \Theta(t) = n) F_Z(x - \varphi(x))
\]

\[
= \int_{\varphi(x)}^{x - \varphi(x)} \sum_{n > \varphi(x)} n \mathbb{P}(\chi^+ > x - y, \Theta(t) = n) \, dF_Z(y)
\]

\[
= \int_{\varphi(x)}^{x - \varphi(x)} \mathbb{E} \left( \Theta(t) I_{\{\chi^+ > x - y\} \{\Theta(t) > \varphi(x)\}} \right) \, dF_Z(y)
\]

\[
\leq \left( \mathbb{E} \Theta^2(t) \right)^{1/2} \int_{\varphi(x)}^{x - \varphi(x)} \left( \mathbb{P}(\chi^+ > x - y) \right)^{1/2} \, dF_Z(y)
\]

\[
\leq \left( c_1 \mathbb{E} \Theta^2(T_5) \right)^{1/2} \int_{\varphi(x)}^{x - \varphi(x)} e^{-c_2(x - y)/2} \, dF_Z(y)
\]

\[
\leq \varepsilon \int_{\varphi(x)}^{x - \varphi(x)} F_Z(x - y) \, dF_Z(y)
\]

because of the Hölder inequality, estimate (20) and Lemma 1.3.5 (b) from [6]. Therefore, using property [13] and Theorem 3.7 from [8] we get

\[
\limsup_{x \to \infty} \sup_{t \in [T, T_5]} \frac{\hat{\psi}_{211}(x, t)}{\int_x F_Z(w) \, dw} \leq \frac{\varepsilon}{\mu \lambda(T)} \limsup_{x \to \infty} \frac{F_Z(x)}{F_Z(x + \mu \lambda(T_5))} \int_{\varphi(x)}^{x - \varphi(x)} F_Z(x - y) \, dF_Z(y)
\]

\[
= 0.
\]
The last inequality together with relations (28) – (33) imply that

$$\limsup_{x \to \infty} \sup_{t \in [T, T_5]} \frac{\psi(x, t)}{x + \mu \lambda(t)} \lesssim \frac{1 + \varepsilon}{\mu}.$$  

Consequently, due to estimate (27), we have that

$$\limsup_{x \to \infty} \sup_{t \in [T, \infty)} \frac{\psi(x, t)}{x + \mu \lambda(t)} \lesssim \frac{1 + \varepsilon}{\mu} + \int_x F_Z(w) \, dw,$$

where \( \varepsilon \in (0, 1) \), \( \delta \in (0, \mu/(\mu + \beta)) \) and \( T \in \Lambda \). We obtain the assertion of Theorem 2.2 by letting \( \varepsilon \) and \( \delta \) to zero in the last inequality.

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