Some bounds for the $A$-numerical radius of certain $2 \times 2$ operator matrices

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Abstract. For a given bounded positive (semidefinite) linear operator $A$ on a complex Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle)$, we consider the semi-Hilbertian space $(\mathcal{H}, \langle \cdot | \cdot \rangle_A)$ where $\langle x | y \rangle_A := \langle Ax | y \rangle$ for every $x, y \in \mathcal{H}$. The $A$-numerical radius of an $A$-bounded operator $T$ on $\mathcal{H}$ is given by

$$\omega_A(T) = \sup \{ |\langle Tx | x \rangle_A| ; x \in \mathcal{H}, \langle x | x \rangle_A = 1 \}.$$  

Our aim in this paper is to derive several $A$-numerical radius inequalities for $2 \times 2$ operator matrices whose entries are $A$-bounded operators, where $\mathcal{A} = \text{diag}(A, A)$.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot | \cdot \rangle$ and associated norm $\| \cdot \|$. Let $\mathcal{B}(\mathcal{H})$ stand for the $C^\ast$-algebra of all bounded linear operators on $\mathcal{H}$. The symbol $I$ denotes the identity operator on $\mathcal{H}$. Let $\mathcal{B}(\mathcal{H})^+$ be the cone of all positive (semi-definite) operators in $\mathcal{B}(\mathcal{H})$, i.e.,

$$\mathcal{B}(\mathcal{H})^+ = \{ A \in \mathcal{B}(\mathcal{H}) ; \langle Ax | x \rangle \geq 0, \forall x \in \mathcal{H} \}.$$ 

In all what follows, by an operator we mean a bounded linear operator. Moreover, for $T \in \mathcal{B}(\mathcal{H})$, we denote by $\mathcal{N}(T)$ and $\mathcal{R}(T)$ the kernel and the range of $T$, respectively. Furthermore, $T^*$ is the adjoint of $T$. For a given linear subspace $\mathcal{M}$ of $\mathcal{H}$, its closure in the norm topology of $\mathcal{H}$ will be denoted by $\overline{\mathcal{M}}$. In addition, let $P_S$ stand for the orthogonal projection onto a closed subspace $S$ of $\mathcal{H}$.

Let $A \in \mathcal{B}(\mathcal{H})^+$. Then, $A$ induces the following semi-inner product

$$\langle \cdot | \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}, (x, y) \mapsto \langle x | y \rangle_A := \langle Ax | y \rangle = \langle A^{1/2}x | A^{1/2}y \rangle.$$ 

Here $A^{1/2}$ stands for the square root of $A$. The seminorm induced by $\langle \cdot | \cdot \rangle_A$ is given by $\|x\|_A = \|A^{1/2}x\|$ for all $x \in \mathcal{H}$. One can verify that $\| \cdot \|_A$ is a norm if and only if $A$ is one-to-one, and that the seminormed space $(\mathcal{H}, \| \cdot \|_A)$ is complete if and only if $\overline{\mathcal{R}(A)} = \mathcal{R}(A)$. The semi-inner product $\langle \cdot | \cdot \rangle_A$ induces on the...
quotient $\mathcal{H}/\mathcal{N}(A)$ an inner product which is not complete unless $\mathcal{R}(A)$ is closed. However, a canonical construction due to de Branges and Rovnyak [10] shows that the completion of $\mathcal{H}/\mathcal{N}(A)$ is isometrically isomorphic to the Hilbert space $\mathcal{R}(A^{1/2})$ endowed with the following inner product

$$\langle A^{1/2}x, A^{1/2}y \rangle_{\mathcal{R}(A^{1/2})} := \langle P_{\mathcal{R}(A)}x \mid P_{\mathcal{R}(A)}y \rangle, \quad \forall x, y \in \mathcal{H}. \quad (1.1)$$

For the sequel, the Hilbert space $\big( \mathcal{R}(A^{1/2}), \langle \cdot, \cdot \rangle_{\mathcal{R}(A^{1/2})} \big)$ will be denoted by $\mathcal{R}(A^{1/2})$. It is worth noting that $\mathcal{R}(A)$ is dense in $\mathcal{R}(A^{1/2})$ (see [4]). For an account of results related to the Hilbert space $\mathcal{R}(A^{1/2})$, the reader is invited to consult [4] and the references therein. By using (1.1), it can be checked that

$$\langle Ax, Ay \rangle_{\mathcal{R}(A^{1/2})} = \langle x, y \rangle_A, \quad \forall x, y \in \mathcal{H}. \quad (1.2)$$

Let $T \in \mathcal{B}(\mathcal{H})$. An operator $S \in \mathcal{B}(\mathcal{H})$ is said to be an $A$-adjoint of $T$ if for all $x, y \in \mathcal{H}$, the identity $\langle Tx \mid y \rangle_A = \langle x \mid S y \rangle_A$ holds (see [2]). So, the existence of an $A$-adjoint of $T$ is equivalent to the existence of a solution of the equation $AX = T^*A$. Notice that this kind of equations can be investigated by using a well-known theorem due to Douglas [11] which briefly says that the operator equation $TX = S$ has a bounded linear solution $X$ if and only if $\mathcal{R}(S) \subseteq \mathcal{R}(T)$ if and only if there exists a positive number $\lambda$ such that $\|S^*x\| \leq \lambda \|T^*x\|$ for all $x \in \mathcal{H}$. Furthermore, among its many solutions it has only one, denoted $Q$, which satisfies $\mathcal{R}(Q) \subseteq \mathcal{R}(T^*)$. Such $Q$ is called the Douglas solution or the reduced solution of the equation $TX = S$. Clearly, the existence of an $A$-adjoint operator is not guaranteed. If we denote by $\mathcal{B}_A(\mathcal{H})$ the subspace of all operators admitting $A$-adjoints, then by Douglas theorem, we have

$$\mathcal{B}_A(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) : \mathcal{R}(T^*A) \subseteq \mathcal{R}(A) \} .$$

If $T \in \mathcal{B}_A(\mathcal{H})$, the reduced solution of the equation $AX = T^*A$ is a distinguished $A$-adjoint operator of $T$, which is denoted by $T^{\sharp_A}$. Note that, $T^{\sharp_A} = A^1T^*A$ in which $A^1$ is the Moore-Penrose inverse of $A$ (see [3]). Notice that if $T \in \mathcal{B}_A(\mathcal{H})$, then $T^{\sharp_A} \in \mathcal{B}_A(\mathcal{H})$, $(T^{\sharp_A})^{\sharp_A} = P_{\mathcal{R}(A)}TP_{\mathcal{R}(A)}$ and $((T^{\sharp_A})^{\sharp_A})^{\sharp_A} = T$. Moreover, If $S \in \mathcal{B}_A(\mathcal{H})$ then $TS \in \mathcal{B}_A(\mathcal{H})$ and $(TS)^{\sharp_A} = S^{\sharp_A}T^{\sharp_A}$. In addition for every $T \in \mathcal{B}_A(\mathcal{H})$ we have

$$\|T^{\sharp_A}T\|_A = \|TT^{\sharp_A}\|_A = \|T\|_A^2 = \|T^{\sharp_A}\|_A^2. \quad (1.3)$$

For results concerning $T^{\sharp_A}$, we refer the reader to [2, 3]. An operator $U \in \mathcal{B}_A(\mathcal{H})$ is called $A$-unitary if $\|Ux\|_A = \|U^{\sharp_A}x\|_A = \|x\|_A$ for all $x \in \mathcal{H}$. It should be mention that, an operator $U \in \mathcal{B}_A(\mathcal{H})$ is $A$-unitary if and only if $U^{\sharp_A}U = (U^{\sharp_A})^{\sharp_A}U^{\sharp_A} = P_{\mathcal{R}(A)}$ (see [2]).

An operator $T$ is called $A$-bounded if there exists $\lambda > 0$ such that $\|Tx\|_A \leq \lambda \|x\|_A$, $\forall x \in \mathcal{H}$. An application of Douglas theorem shows that the subspace of all operators admitting $A^{1/2}$-adjoints, denoted by $\mathcal{B}_{A^{1/2}}(\mathcal{H})$, is equal the collection of all $A$-bounded operators, i.e.,

$$\mathcal{B}_{A^{1/2}}(\mathcal{H}) = \{ T \in \mathcal{B}(\mathcal{H}) : \exists \lambda > 0 ; \|Tx\|_A \leq \lambda \|x\|_A, \forall x \in \mathcal{H} \} .$$

Notice that $\mathcal{B}_A(\mathcal{H})$ and $\mathcal{B}_{A^{1/2}}(\mathcal{H})$ are two subalgebras of $\mathcal{B}(\mathcal{H})$ which are, in general, neither closed nor dense in $\mathcal{B}(\mathcal{H})$. Moreover, we have $\mathcal{B}_A(\mathcal{H}) \subseteq \mathcal{B}_{A^{1/2}}(\mathcal{H})$.
(see [2, 4]). Clearly, $\langle \cdot | \cdot \rangle_A$ induces a seminorm on $\mathcal{B}_{A^{1/2}}(\mathcal{H})$. Indeed, if $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, then it holds that
\[
\|T\|_A := \sup_{x \in \mathcal{K}(A) \setminus \{0\}} \frac{\|Tx\|_A}{\|x\|_A} = \sup \{ \|Tx\|_A ; \ x \in \mathcal{H}, \|x\|_A = 1 \} < \infty. \tag{1.4}
\]

Notice that it was proved in [16] that for $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ we have
\[
\|T\|_A = \sup \{ |\langle Tx | y \rangle_A| ; \ x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1 \}. \tag{1.5}
\]

Furthermore, the $A$-numerical radius of an operator $T \in \mathcal{B}(\mathcal{H})$ was firstly defined by Saddi in [19] by
\[
\omega_A(T) := \sup \{ |\langle Tx | x \rangle_A| ; \ x \in \mathcal{H}, \|x\|_A = 1 \}.
\]

It should be emphasized that it may happen that \(\|T\|_A\) and \(\omega_A(T)\) are equal to \(+\infty\) for some $T \in \mathcal{B}(\mathcal{H}) \setminus \mathcal{B}_{A^{1/2}}(\mathcal{H})$ (see [12]). However, these quantities are equivalent seminorms on $\mathcal{B}_{A^{1/2}}(\mathcal{H})$. More precisely, it was shown in [5] that for every $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$, we have
\[
\frac{1}{2} \|T\|_A \leq \omega_A(T) \leq \|T\|_A. \tag{1.6}
\]

Notice that if $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ and satisfies $AT^2 = 0$, then by [12, Corollary 2] we have
\[
\omega_A(T) = \frac{1}{2} \|T\|_A. \tag{1.7}
\]

Notice that the $A$-numerical radius of semi-Hilbertian space operators satisfies the weak $A$-unitary invariance property which asserts that
\[
\omega_A(U^*TU) = \omega_A(T), \tag{1.8}
\]

for every $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ and every $A$-unitary operator $U \in \mathcal{B}_A(\mathcal{H})$ (see [7, Lemma 3.8]).

For the sequel, for any arbitrary operator $T \in \mathcal{B}_A(\mathcal{H})$, we write
\[
\Re_A(T) := \frac{T + T^A}{2} \text{ and } \Im_A(T) := \frac{T - T^A}{2i}.
\]

It has recently been shown in [22, Theorem 2.5] that if $T \in \mathcal{B}_A(\mathcal{H})$, then
\[
\omega_A(T) = \sup_{\theta \in \mathbb{R}} \|\Re_A(e^{i\theta}T)\|_A = \sup_{\theta \in \mathbb{R}} \|\Im_A(e^{i\theta}T)\|_A. \tag{1.9}
\]

Let $T \in \mathcal{B}(\mathcal{H})$. Then, it was shown in [4, Proposition 3.6.] that $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ if and only if there exists a unique $\widetilde{T} \in \mathcal{B}(\mathcal{R}(A^{1/2}))$ such that $Z_AT = \widetilde{T}Z_A$. Here, $Z_A : \mathcal{H} \to \mathcal{R}(A^{1/2})$ is defined by $Z_Ax = Ax$. It has been shown in [12] that for every $T \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$ we have
\[
\|T\|_A = \|\widetilde{T}\|_{\mathcal{B}(\mathcal{R}(A^{1/2}))} \text{ and } \omega_A(T) = \omega(\widetilde{T}). \tag{1.10}
\]

Recently, the concept of the $A$-spectral radius of $A$-bounded operators has been introduced in [12] as follows:
\[
r_A(T) := \inf_{n \geq 1} \|T^n\|_A^{\frac{1}{n}} = \lim_{n \to \infty} \|T^n\|_A^{\frac{1}{n}}. \tag{1.11}
\]
We note here that the second equality in (1.11) is also proved in [12, Theorem 1]. Moreover, like the classical spectral radius of Hilbert space operators, it was shown in [12] that $r_A(\cdot)$ satisfies the commutativity property, which asserts that

$$r_A(TS) = r_A(ST),$$

(1.12)

for all $T, S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be $A$-selfadjoint if $AT$ is selfadjoint, that is, $AT = T^*A$. Moreover, it was shown in [12] that if $T$ is $A$-self-adjoint, then

$$\|T\|_A = \omega_A(T) = r_A(T).$$

(1.13)

In addition, an operator $T$ is called $A$-positive if $AT \geq 0$ and we write $T \geq_A 0$. Obviously, an $A$-positive operator is always an $A$-selfadjoint operator since $\mathcal{H}$ is a complex Hilbert space. If $T, S \in \mathcal{B}(\mathcal{H})$ and satisfies $T - S \geq_A 0$, then we will write $T \geq_A S$. For the sequel, if $A = I$ then $\|T\|_A$, $r(T)$ and $\omega(T)$ denote respectively the classical operator norm, the spectral radius and the numerical radius of an operator $T$. In recent years, several results covering some classes of operators on a complex Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle)$ were extended to $(\mathcal{H}, \langle \cdot | \cdot \rangle_A)$. Of course, the extension is not trivial since many difficulties arise. For instance, as it is mention above, it may happen that $\|T\|_A = \infty$ for some $T \in \mathcal{B}(\mathcal{H})$. Moreover, not any operator admits an adjoint operator for the semi-inner product $\langle \cdot | \cdot \rangle_A$. In addition, for $T \in \mathcal{B}_A(\mathcal{H})$ we have $(T^*_A)^{2A} = P_{\mathbb{R}(A)}TP_{\mathbb{R}(A)} T \neq T$. The reader is invited to see [5, 6, 7, 14, 18, 21, 22, 23] and the references therein.

In this paper, we consider the $2 \times 2$ operator diagonal matrix $A = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Clearly, $A \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})^+$. So, $A$ induces the following semi-inner product

$$\langle x, y \rangle_A = \langle Ax, y \rangle = \langle x_1 | y_1 \rangle_A + \langle x_2 | y_2 \rangle_A,$$

for all $x = (x_1, x_2) \in \mathcal{H} \oplus \mathcal{H}$ and $y = (y_1, y_2) \in \mathcal{H} \oplus \mathcal{H}$. Notice that if $T_{ij}$ are operators in $\mathcal{B}_A(\mathcal{H})$ for all $i, j \in \{1, 2\}$. Then, it was shown in [7, Lemma 3.1] that $(T_{ij})_{2 \times 2} \in \mathcal{B}_A(\mathcal{H} \oplus \mathcal{H})$ and

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}^{2A} = \begin{pmatrix} T_{11}^{2A} & T_{12}^{2A} \\ T_{21}^{2A} & T_{22}^{2A} \end{pmatrix}.$$  

(1.14)

Very recently, several inequalities for the $A$-numerical radius of $2 \times 2$ operator matrices have been established by P. Bhunia et al. (see [8]). This paper is devoted also to prove several new $A$-numerical radius inequalities of certain $2 \times 2$ operator matrices. Some of the obtained results cover and extend the following works [9, 17, 20].

2. Results

In this section, we present our results. Throughout this section $A$ is denoted to be the $2 \times 2$ operator diagonal matrix whose diagonal entry is the positive operator $A$. To prove our two next results, the following lemma concerning $A$-numerical radius inequalities is required. Notice that the first assertion is proved in [8] for operators in $\mathcal{B}_A(\mathcal{H})$.

**Lemma 2.1.** Let $P, Q, R, S \in \mathcal{B}_{A^{1/2}}(\mathcal{H})$. Then, the following assertions hold:
(a) $\omega_{A} \left[ \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} \right] = \max \{ \omega_{A}(P), \omega_{A}(S) \}$.

(b) $\omega_{A} \left[ \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} \right] \leq \omega_{A} \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right]$.

(c) $\omega_{A} \left[ \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \right] \leq \omega_{A} \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right]$.

Proof. (a) Follows by proceeding as in the proof of [8, Lemma 2.4].

(b) Clearly we have

$$\begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} = \frac{1}{2} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} + \frac{1}{2} \begin{pmatrix} P & -Q \\ -R & S \end{pmatrix}. \quad (2.1)$$

Let $U = \begin{pmatrix} -I & O \\ O & I \end{pmatrix}$. In view of (1.14) we have $U^\sharp A = \begin{pmatrix} -P_{R(A)} & O \\ O & P_{R(A)} \end{pmatrix}$. So, we verify that $\|Ux\|_{A} = \|U^\sharp A x\|_{A} = \|x\|_{A}$ for all $x = (x_1, x_2) \in H \oplus H$. Hence, $U$ is $A$-unitary operator. Thus, by (1.8) we have

$$\omega_{A} \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] = \omega_{A} \left[ U^\sharp A \left( \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right) U \right]$$

$$= \omega_{A} \left[ \begin{pmatrix} P_{R(A)} & O \\ O & P_{R(A)} \end{pmatrix} \begin{pmatrix} P & -Q \\ -R & S \end{pmatrix} \right]$$

$$= \omega_{A} \left[ \begin{pmatrix} P & -Q \\ -R & S \end{pmatrix} \right]$$

So, by taking into consideration (2.1) and the triangle inequality we prove the desired result.

(b) Let $U = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}$. By proceeding similarly as above, we prove that $U$ is $A$-unitary and

$$\omega_{A} \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] = \omega_{A} \left[ \begin{pmatrix} -P & Q \\ R & -S \end{pmatrix} \right].$$

Moreover, by using the fact that

$$\begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} P & Q \\ R & S \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -P & Q \\ R & -S \end{pmatrix},$$

and the subadditivity of the $A$-numerical radius $\omega_{A}(\cdot)$, we get the required result. \hfill \Box

Also, we need the following lemma.

Lemma 2.2. ([15]) Let $T, S \in B_{A^{1/2}}(H)$. Then,

$$\left\| \begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix} \right\|_{A} = \left\| \begin{pmatrix} T & 0 \\ 0 & S \end{pmatrix} \right\|_{A} = \max \{ \|T\|_{A}, \|S\|_{A} \}.$$ 

Now, we are in a position to prove our first result in this paper.
**Theorem 2.1.** Let $T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ be such that $P, Q, R, S \in B_{A^{1/2}}(\mathcal{H})$. Then,
\begin{equation}
\lambda_1 \leq \omega_{\mathcal{A}} \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] \leq \lambda_2,
\end{equation}
where
\begin{equation}
\lambda_1 = \max \left\{ \omega_{\mathcal{A}} \left[ \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \right], \max \{\omega_A(P), \omega_A(S)\} \right\}
\end{equation}
and
\begin{equation}
\lambda_2 = \frac{\|Q\|_A + \|R\|_A}{2} + \max \{\omega_A(P), \omega_A(S)\}.
\end{equation}

**Proof.** Clearly we have
\begin{equation}
\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} + \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix}.
\end{equation}
On the other, it is not difficult to see that $\mathcal{A} \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathcal{A} \begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. So, by (1.7) and Lemma 2.2 we have
\begin{equation}
\omega_{\mathcal{A}} \left[ \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} \right] = \frac{1}{2} \left\| \begin{pmatrix} 0 & Q \\ 0 & 0 \end{pmatrix} \right\|_A = \frac{1}{2} \|Q\|_A.
\end{equation}
Similarly, we have $\omega_{\mathcal{A}} \left[ \begin{pmatrix} 0 & 0 \\ R & 0 \end{pmatrix} \right] = \frac{1}{2} \|R\|_A$. So, by using the trivial observation (2.3) and the subadditivity of the $\mathcal{A}$-numerical radius $\omega_{\mathcal{A}}(\cdot)$ together with Lemma 2.1 (a), we get
\begin{equation}
\omega_{\mathcal{A}} \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] \leq \max \{\omega_A(P), \omega_A(S)\} + \frac{\|Q\|_A + \|R\|_A}{2}.
\end{equation}
On the other hand, by Lemmas 2.1 and 2.2 we have
\begin{equation}
\omega_{\mathcal{A}} \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] \geq \max \left\{ \omega_{\mathcal{A}} \left[ \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \right], \omega_{\mathcal{A}} \left[ \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix} \right] \right\} = \max \left\{ \omega_{\mathcal{A}} \left[ \begin{pmatrix} 0 & Q \\ R & 0 \end{pmatrix} \right], \max \{\omega_A(P), \omega_A(S)\} \right\}.
\end{equation}
By combining (2.4) together with (2.5), we reach the desired result. □

In order to prove our next result, we need the following lemma.

**Lemma 2.3.** Let $T, S \in B(\mathcal{H})$ be two $\mathcal{A}$-positive operators. Then,
\begin{equation}
\omega_{\mathcal{A}} \left[ \begin{pmatrix} 0 & T \\ S & 0 \end{pmatrix} \right] = \frac{1}{2} \|T + S\|_A.
\end{equation}

**Proof.** Since $T$ and $S$ are $\mathcal{A}$-positive, then $T, S \in B_{A^{1/2}}(\mathcal{H})$. So, by [4, Proposition 3.6.] there exists two unique operators $\tilde{T}, \tilde{S} \in B(\mathcal{R}(A^{1/2}))$ such that $Z_A T = \tilde{T} Z_A$ and $Z_A S = \tilde{S} Z_A$. Moreover, since $T \geq_A 0$, then for all $x \in \mathcal{H}$ we have
\begin{equation}
\langle AT x, x \rangle \geq 0.
\end{equation}
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This implies, through (1.2), that

$$\langle Tx \mid x \rangle_A = \langle ATx, Ax \rangle_{R(A^{1/2})} = \langle \tilde{T}Ax, Ax \rangle_{R(A^{1/2})} \geq 0$$

for all $x \in \mathcal{H}$. Further, by using the density of $\mathcal{R}(A)$ in $R(A^{1/2})$, we obtain

$$\langle \tilde{T}A^{1/2}x, A^{1/2}x \rangle_{R(A^{1/2})} \geq 0, \quad \forall x \in \mathcal{H}.$$ 

So, $\tilde{T}$ is a positive operator on the Hilbert space $R(A^{1/2})$. Similarly, we prove that $\tilde{S} \geq 0$. Therefore, in view of [1, Corollary 3.] we have

$$\omega \left[ \begin{pmatrix} 0 & \tilde{T} \\ \tilde{S} & 0 \end{pmatrix} \right] = \frac{1}{2} \left\| \tilde{T} + \tilde{S} \right\|_{B(R(A^{1/2}))} = \frac{1}{2} \left\| \tilde{T} + S \right\|_{B(R(A^{1/2}))},$$

(2.7)

where the last equality follows since $\tilde{T} + S = \tilde{T} + \tilde{S}$. Moreover, by [7, Lemma 3.2], we have $\begin{pmatrix} 0 & \tilde{T} \\ \tilde{S} & 0 \end{pmatrix} \in B_{A^{1/2}}(\mathcal{H} \oplus \mathcal{H})$ and

$$\begin{pmatrix} 0 & \tilde{T} \\ \tilde{S} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tilde{T} \\ \tilde{S} & 0 \end{pmatrix}.$$ 

This proves the desired result by applying (2.7) together with (1.10). \qed

We are now in a position to state the following theorem.

**Theorem 2.2.** Let $\mathbb{T} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ be such that $P, Q, R, S \in B_A(\mathcal{H})$. Then,

$$\omega_A(\mathbb{T}) \leq \frac{1}{2} \left( \omega_A(P) + \omega_A(Q) \right) + \frac{1}{4} \left( \| I + PP^*A + QQ^*A \|_A + \| I + RR^*A + SS^*A \|_A \right).$$

**Proof.** We first prove that

$$\omega_A(\mathbb{S}) \leq \frac{1}{2} \omega_A(P) + \frac{1}{4} \| I + PP^*A + QQ^*A \|_A,$$

(2.8)

where $\mathbb{S} = \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix}$. Let $\theta \in \mathbb{R}$. It is not difficult to verify that $\mathbb{R}_A(e^{i\theta}\mathbb{S})$ is an $A$-self-adjoint operator. So, by (1.13) we have

$$r_A(\mathbb{R}_A(e^{i\theta}\mathbb{S})) = \| \mathbb{R}_A(e^{i\theta}\mathbb{S}) \|_A.$$
Now, by using (1.14), we see that
\[
\begin{align*}
    r_A \left[ R_A(e^{i\theta} S) \right] &= \frac{1}{2} r_A \left( e^{i\theta} S + e^{-i\theta} S^* \right) \\
    &= \frac{1}{2} r_A \left[ e^{i\theta} \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} + e^{-i\theta} \begin{pmatrix} P^* & 0 \\ Q^* & 0 \end{pmatrix} \right] \\
    &= \frac{1}{2} r_A \left[ \begin{pmatrix} e^{i\theta} P + e^{-i\theta} P^* & e^{i\theta} Q \\ e^{-i\theta} Q^* & 0 \end{pmatrix} \right] \\
    &= \frac{1}{2} r_A \left[ \begin{pmatrix} P^* & e^{i\theta} I \\ 0 & Q \\ Q^* & 0 \end{pmatrix} \begin{pmatrix} e^{-i\theta} I & 0 \\ 0 & P \end{pmatrix} \right] \\
    &= \frac{1}{2} r_A \left[ \begin{pmatrix} e^{i\theta} P^* & 0 \\ 0 & e^{i\theta} P \end{pmatrix} \begin{pmatrix} P^* & e^{i\theta} I \\ 0 & Q \\ Q^* & 0 \end{pmatrix} \right] \\
    &\leq \frac{1}{2} \omega_A \left[ \begin{pmatrix} e^{i\theta} P^* & 0 \\ 0 & e^{i\theta} P \end{pmatrix} \right] + \frac{1}{2} \omega_A \left[ \begin{pmatrix} 0 & e^{i\theta} I \\ P P^* + QQ^* & 0 \end{pmatrix} \right] \\
    &= \omega_A(P) + \frac{1}{2} \| I + PP^* + QQ^* \|_A, \text{ (by Lemmas 2.1 and 2.3)}. 
\end{align*}
\]

Hence,
\[
\| R_A(e^{i\theta} S) \|_A \leq \omega_A(P) + \frac{1}{2} \| I + PP^* + QQ^* \|_A. 
\]

So, by taking the supremum over all \( \theta \in \mathbb{R} \) and then applying (1.9) we obtain (2.8) as required. Let \( \mathbb{U} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). In view of (1.14) we have \( \mathbb{U}^* = \begin{pmatrix} 0 & P_{R(A)} \\ P_{R(A)} & 0 \end{pmatrix} \).

Further, it can be seen that \( \mathbb{U} \) is a unitary operator. So, by using (1.8) together with (2.8) we get
\[
\begin{align*}
    \omega_A(\mathbb{T}) &\leq \omega_A \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] + \omega_A \left[ \begin{pmatrix} 0 & 0 \\ R & S \end{pmatrix} \right] \\
    &= \omega_A \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] + \omega_A \left[ \mathbb{U}^* \begin{pmatrix} 0 & 0 \\ R & S \end{pmatrix} \mathbb{U} \right] \\
    &= \omega_A \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] + \omega_A \left[ \begin{pmatrix} 0 & 0 \\ P_{R(A)} & 0 \end{pmatrix} \begin{pmatrix} S & R \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ P_{R(A)} & 0 \end{pmatrix} \right] \\
    &= \omega_A \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] + \omega_A \left[ \begin{pmatrix} S & R \\ 0 & 0 \end{pmatrix} \right] \\
    &\leq \frac{1}{2} \left( \omega_A(P) + \omega_A(S) + \| PP^* + QQ^* \|_A^{1/2} + \| RR^* + SS^* \|_A^{1/2} \right).
\end{align*}
\]

This finishes the proof of the theorem. \(\square\)

The following lemma is useful in proving our next result.
Lemma 2.4. ([13]) Let $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ be such that $T_{ij} \in B_{A^{1/2}}(H)$ for all $i, j \in \{1, 2\}$. Then, $\mathbb{T} \in B_{A^{1/2}}(H \oplus H)$ and

$$r_{A}(\mathbb{T}) \leq r\left[\left(\|T_{11}\|_{A} \|T_{12}\|_{A} \|T_{21}\|_{A} \|T_{22}\|_{A}\right)\right].$$

Theorem 2.3. Let $T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ be such that $P, Q, R, S \in B_{A}(H)$. Then,

$$\omega_{A}(T) \leq \frac{1}{2} \left(\|P\|_{A} + \|S\|_{A} + \|PP^{*A} + QQ^{*A}\|_{A}^{1/2} + \|RR^{*A} + SS^{*A}\|_{A}^{1/2}\right). \tag{2.9}$$

Proof. We first prove that

$$\omega_{A}(S) \leq \frac{1}{2} \left(\|P\|_{A} + \|PP^{*A} + QQ^{*A}\|_{A}^{1/2}\right), \tag{2.10}$$

where $S = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$. Let $\theta \in \mathbb{R}$. By proceeding as in the proof of Theorem 2.2 we see that

$$\|\Re_{A}(e^{i\theta}S)\|_{A} = \Re_{A}(e^{i\theta}S) \leq \frac{1}{2} r_{A} \left[\begin{pmatrix} e^{-i\theta}P^{*A} + Q^{*A} & I \\ P^{*A} + QQ^{*A} & e^{i\theta}P \end{pmatrix}\right] \leq \frac{1}{2} r_{A} \left[\begin{pmatrix} \|P\|_{A} + \|PP^{*A} + QQ^{*A}\|_{A} & 1 \\ \|PP^{*A} + QQ^{*A}\|_{A} & \|P\|_{A} \end{pmatrix}\right] = \frac{1}{2} \left(\|P\|_{A} + \|PP^{*A} + QQ^{*A}\|_{A}^{1/2}\right).$$

Using an argument similar to that used in proof of Theorem 2.2, we get the desired result. \[\square\]

Before proving our next theorem we have to state the following lemma.

Lemma 2.5. ([6, Theorem 5.1]) Let $T \in B(H)$ be an $A$-selfadjoint operator. Then, for any positive integer $n$ we have

$$\|T^{n}\|_{A} = \|T\|_{A}^{n}.$$

Theorem 2.4. Let $T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ be such that $P, Q, R, S \in B_{A}(H)$. Then,

$$\omega_{A}(T) \leq \sqrt{\omega_{A}^{2}(P) + \frac{1}{2}\|Q\|_{A} \left(\omega_{A}(P) + \frac{1}{2}\|Q\|_{A}\right)} + \sqrt{\omega_{A}^{2}(S) + \frac{1}{2}\|R\|_{A} \left(\omega_{A}(S) + \frac{1}{2}\|R\|_{A}\right)}.$$

Proof. Let $S = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$. We first prove that

$$\omega_{A}(S) \leq \sqrt{\omega_{A}^{2}(P) + \frac{1}{2}\|Q\|_{A} \left(\omega_{A}(P) + \frac{1}{2}\|Q\|_{A}\right)} \tag{2.11}.$$
Let \( \theta \in \mathbb{R} \). A straightforward calculation shows that
\[
\Re_A(e^{i\theta} S) = \left( \Re_A(e^{i\theta} P) - \frac{1}{2} e^{i\theta} Q \right)
\]
\[
= \left( \Re_A(e^{i\theta} P) 0 \right) + \left( \frac{1}{2} e^{i\theta} Q \right).
\]
This implies that
\[
(\Re_A(e^{i\theta} S))^2 = \left( \Re_A(e^{i\theta} P) 0 \right) + \left( \frac{1}{4} Q^2 + \frac{1}{2} e^{i\theta} Q \right)
\]
\[
+ \left( 0 0 \right) + \left( \frac{1}{2} e^{i\theta} [\Re_A(e^{i\theta} P)] Q \right) + \left( \frac{1}{2} e^{-i\theta} Q^2 \right).
\]
Thus, by using (1.9) together with Lemma 2.2 we see that
\[
\| (\Re_A(e^{i\theta} S))^2 \|_A \leq \| \Re_A(e^{i\theta} P) \|_A^2 + \frac{1}{4} \| Q \|_A^2 + \frac{1}{2} \| \Re_A(e^{i\theta} P) \|_A \| Q \|_A
\]
\[
\leq \omega_A^2(P) + \frac{1}{4} \| Q \|_A^2 + \frac{1}{2} \omega_A(P) \| Q \|_A.
\]
Since \( \Re_A(e^{i\theta} S) \) is \( A \)-selfadjoint, then an application of Lemma 2.5 gives
\[
\| \Re_A(e^{i\theta} S) \|_A^2 \leq \omega_A^2(P) + \frac{1}{4} \| Q \|_A^2 + \frac{1}{2} \omega_A(P) \| Q \|_A.
\]
Taking the supremum over all \( \theta \in \mathbb{R} \) in the above inequality and then using (1.9) yields that
\[
\omega_A^2(S) \leq \omega_A^2(P) + \frac{1}{4} \| Q \|_A^2 + \frac{1}{2} \omega_A(P) \| Q \|_A.
\]
This proves (2.11). Using an argument similar to that used in proof of Theorem 2.2, we get the desired result.

Next we state the following useful lemmas related to \( A \)-selfadjoint operators.

**Lemma 2.6.** Let \( T, S \in \mathcal{B}(\mathcal{H}) \) be two \( A \)-selfadjoint operators. If \( T - S \geq_A 0 \), then
\[
\| T \|_A \geq \| S \|_A.
\]

**Proof.** Since \( T - S \geq_A 0 \), then \( \langle (T - S)x | x \rangle_A \geq 0 \) for all \( x \in \mathcal{H} \). This gives
\[
\langle Tx | x \rangle_A \geq \langle Sx | x \rangle_A, \quad \forall x \in \mathcal{H}.
\]
So, by taking the supremum over all \( x \in \mathcal{H} \) with \( \| x \|_A = 1 \) in the above inequality and then using (1.13) we obtain the desired result.

**Lemma 2.7.** ([13]) Let \( T \in \mathcal{B}_A(\mathcal{H}) \) be an \( A \)-selfadjoint operator. Then, \( T^{2n} \geq_A 0 \) for any positive integer \( n \).

We are now in a position to prove the following theorem.

**Theorem 2.5.** Let \( T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \) be such that \( P, Q, R, S \in \mathcal{B}_A(\mathcal{H}) \). Then,
\[
\omega_A(T) \leq \sqrt{2\omega_A^2(P) + \frac{1}{2} (\| P^2 A Q \|_A + \| Q \|_A^2) + \sqrt{2\omega_A^2(S) + \frac{1}{2} (\| S^2 A R \|_A + \| R \|_A^2)}}.
\]
Some bounds for the $\rho$-numerical radius of certain $2 \times 2$ operator matrices

**Proof.** We first prove that

$$
\omega_\rho \left[ \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] \leq \sqrt{2\omega_\rho^2(P) + \frac{1}{2}(\|P^{\sharp A}Q\|_A + \|Q\|_A^2)}.
$$

(2.12)

Let $\theta \in \mathbb{R}$. By using (1.14), it can be verified that

$$
\Re_\rho \left[ e^{i\theta} \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} \Re_\rho(e^{i\theta}P) \frac{1}{2}e^{i\theta}Q \\ 0 \end{pmatrix}
$$

and

$$
\Im_\rho \left[ e^{i\theta} \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] = -i \begin{pmatrix} -\frac{1}{2}e^{-i\theta}Q^{\sharp A} \frac{1}{2}e^{i\theta}Q \\ 0 \end{pmatrix}.
$$

Moreover, by Lemma 2.7, $\Im_\rho^2 \left[ e^{i\theta} \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] \geq_\rho \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. So, we have

$$
\Re_\rho^2 \left[ e^{i\theta} \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] + \Im_\rho^2 \left[ e^{i\theta} \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] - \Re_\rho^2 \left[ e^{i\theta} \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] \geq_\rho \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
$$

Hence, it follows from Lemma 2.6 that

$$
\| \Re_\rho^2 \left[ e^{i\theta} \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] \|_A \leq \| \Re_\rho^2 \left[ e^{i\theta} \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] + \Im_\rho^2 \left[ e^{i\theta} \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] \|_A.
$$

On the other hand, a short calculation reveals that

$$
\Re_\rho^2 \left[ e^{i\theta} \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] + \Im_\rho^2 \left[ e^{i\theta} \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right]
$$

$$
= \left( \Re_\rho^2(e^{i\theta}P) + \Im_\rho^2(e^{i\theta}P) \frac{1}{2}P^{\sharp A}Q + \frac{1}{2}Q^{\sharp A}P \frac{1}{2} \right) + \left( \frac{0}{2} \frac{0}{2} \right).
$$

Hence, by using Lemma 2.2 and (1.9) we see that

$$
\| \Re_\rho^2 \left[ e^{i\theta} \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] \|_A^2
$$

$$
\leq \| \Re_\rho^2(e^{i\theta}P) + \Im_\rho^2(e^{i\theta}P) \|_A + \frac{1}{2} \max\{\|P^{\sharp A}Q\|_A, \|Q^{\sharp A}P\|_A\} + \frac{1}{2} \|Q\|_A^2
$$

$$
\leq 2\omega_\rho^2(P) + \frac{1}{2} \left( \max\{\|P^{\sharp A}Q\|_A, \|Q^{\sharp A}P\|_A\} + \|Q\|_A^2 \right).
$$

(2.13)
On the other hand, one observes that $P_{\mathcal{R}(A)}A = AP_{\mathcal{R}(A)} = A$. Moreover, by (1.5), we see that

$$\|P^{2_A}Q\|_A = \|Q^{2_A}P_{\mathcal{R}(A)}PP_{\mathcal{R}(A)}\|_A$$

$$= \sup \left\{ |\langle AP_{\mathcal{R}(A)}x, (Q^{2_A}P_{\mathcal{R}(A)}P)^{2_A}y \rangle| ; x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1 \right\}$$

$$= \sup \left\{ |\langle Q^{2_A}P_{\mathcal{R}(A)}Px, y \rangle| ; x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1 \right\}$$

$$= \sup \left\{ |\langle AP_{\mathcal{R}(A)}x, Qy \rangle| ; x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1 \right\}$$

$$= \sup \left\{ |\langle Q^{2_A}Px, y \rangle| ; x, y \in \mathcal{H}, \|x\|_A = \|y\|_A = 1 \right\}$$

$$= \|Q^{2_A}P\|_A.$$ 

So, by taking into account (2.13), it follows that

$$\|\mathcal{R}_\theta\left[ e^{i\theta} \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] \|_A^2 \leq 2\omega^2_A(P) + \frac{1}{2} (\|P^{2_A}Q\|_A + \|Q\|_A^2).$$

By taking the supremum over all $\theta \in \mathbb{R}$ in the above inequality we obtain (2.12) as required. Finally, by using an argument similar to that used in proof of Theorem 2.2, we get the desired inequality. 

Our next result reads as follows.

**Theorem 2.6.** Let $T = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ be such that $P, Q, R, S \in B_A(\mathcal{H})$. Then,

$$\omega_A(T) \leq \min \{\mu, \nu\},$$

where

$$\mu = \sqrt{\min\{\|P + Q\|_A^2, \|P - Q\|_A^2\} + 2\omega_A(PQ^{2_A})} \quad + \sqrt{\min\{\|R + S\|_A^2, \|R - S\|_A^2\} + 2\omega_A(SR^{2_A})},$$

and

$$\nu = \sqrt{\min\{\|P + R\|_A^2, \|P - R\|_A^2\} + 2\omega_A(P^{2_A}R)} \quad + \sqrt{\min\{\|Q + S\|_A^2, \|Q - S\|_A^2\} + 2\omega_A(S^{2_A}Q)}.$$
Proof. By using (1.6) together with (1.3) and Lemma 2.2 we see that

\[
\omega_A \left[ \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] \leq \left\| \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right\|_A^A^{\frac{1}{2}}
\]

\[
= \left\| \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right\|_A^{\frac{1}{2}}
\]

\[
= \left\| \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \begin{pmatrix} P^A & 0 \\ 0 & Q^A \end{pmatrix} \right\|_A^{\frac{1}{2}}
\]

\[
= \left\| \begin{pmatrix} PP^A & QQ^A \\ 0 & 0 \end{pmatrix} \right\|_A^{\frac{1}{2}}
\]

\[
= \| PP^A + QQ^A \|_A^{\frac{1}{2}}.
\]  \hspace{1cm} (2.14)

Moreover, it is not difficult to verify that

\[
PP^A + QQ^A = (P \pm Q)(P \pm Q)^A \mp (PQ^A + QP^A).
\]

So, since \( PP^A + QQ^A \geq A \), it follows from (1.13) that

\[
\| PP^A + QQ^A \|_A = \omega_A(PP^A + QQ^A)
\]

\[
= \omega_A((P \pm Q)(P \pm Q)^A \mp (PQ^A + QP^A))
\]

\[
\leq \omega_A((P \pm Q)(P \pm Q)^A) + \omega_A(PQ^A) + \omega_A(QP^A)
\]

\[
= \| P \pm Q \|_A^2 + \omega_A(PQ^A) + \omega_A(QP^A),
\]

where the last equality follows by using (1.13) together with (1.3) since the operator \((P \pm Q)(P \pm Q)^A\) is \(A\)-positive. Further, one observes that

\[
\omega_A(PQ^A) = \omega_A \left( (Q^A)^A P^A \right)
\]

\[
= \omega_A(P_{R(A)} \overline{Q} P_{R(A)} P^A) = \omega_A(P_{R(A)} Q P^A).
\]

This yields that \( \omega_A(PQ^A) = \omega_A(QP^A) \). Thus, we get

\[
\| PP^A + QQ^A \|_A \leq \| P \pm Q \|_A^2 + 2 \omega_A(PQ^A).
\]

This implies that

\[
\| PP^A + QQ^A \|_A \leq \min \left( \| P + Q \|_A^2, \| P - Q \|_A^2 \right) + 2 \omega_A(PQ^A).
\]

So, by taking into account (2.14), we get

\[
\omega_A \left[ \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] \leq \sqrt{\min \left( \| P + Q \|_A^2, \| P - Q \|_A^2 \right) + 2 \omega_A(PQ^A)}.
\]  \hspace{1cm} (2.15)
By considering the $A$-unitary operator $U = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, we see that
\[
\omega_A \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] \leq \omega_A \left[ \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] + \omega_A \left[ \begin{pmatrix} 0 & 0 \\ R & S \end{pmatrix} \right]
\]
\[
= \omega_A \left[ \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] + \omega_A \left[ V \left( \begin{pmatrix} S & R \\ 0 & 0 \end{pmatrix} \right) U \right]
\]
\[
= \omega_A \left[ \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] + \omega_A \left[ \begin{pmatrix} S & R \\ 0 & 0 \end{pmatrix} \right] \quad \text{(by (1.8))}
\]
\[
\leq \min \left( \|P + Q\|_A^2, \|P - Q\|_A^2 \right) + 2\omega_A(PQ^*A)
\]
\[
+ \min \left( \|R + S\|_A^2, \|R - S\|_A^2 \right) + 2\omega_A(SR^*A)
\]

By observing that $\omega_A \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] = \omega_A \left[ \begin{pmatrix} P^*A & R^*A \\ Q^*A & S^*A \end{pmatrix} \right]$ and using similar arguments as above we get
\[
\omega_A \left[ \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \right] = \omega_A \left[ \begin{pmatrix} P^*A & R^*A \\ Q^*A & S^*A \end{pmatrix} \right]
\]
\[
\leq \min \left( \|P^*A + R^*A\|_A^2, \|P^*A - R^*A\|_A^2 \right) + 2\omega_A(P^*A(R^*A)^*A)
\]
\[
+ \min \left( \|Q^*A + S^*A\|_A^2, \|Q^*A - S^*A\|_A^2 \right) + 2\omega_A(S^*A(Q^*A)^*A)
\]
\[
= \min \left( \|P + R\|_A^2, \|P - R\|_A^2 \right) + 2\omega_A(R^*AP)
\]
\[
+ \min \left( \|Q + S\|_A^2, \|Q - S\|_A^2 \right) + 2\omega_A(Q^*AS).
\]

Hence, the proof is complete since $\omega_A(R^*AP) = \omega_A(P^*A R)$ and $\omega_A(Q^*AS) = \omega_A(S^*A Q)$. \qed

In order to prove a lower bound for $\omega_A \left[ \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right]$, we need the following lemmas.

**Lemma 2.8.** Let $T, S \in B_A(\mathcal{H})$. Then
\[
\max \left\{ \|T + S\|_A^2, \|T - S\|_A^2 \right\} - \|TT^*A + SS^*A\|_A \leq 2\omega_A \left( TS^*A \right). \quad (2.16)
\]

**Proof.** Let $x \in \mathcal{H}$ be such that $\|x\|_A = 1$. We obviously have
\[
\|Tx + SX\|_A^2 = \|Tx\|_A^2 + 2R \langle (Tx) | Sx \rangle_A + \|Sx\|_A^2
\]
\[
\leq \langle (T^*A T + S^*A S) x | x \rangle_A + 2 \langle (S^*A T) x | x \rangle_A
\]
\[
\leq \omega_A \left( T^*A T + S^*A S \right) + 2\omega_A \left( S^*A T \right)
\]
\[
= \|T^*A T + S^*A S\|_A + 2\omega_A \left( S^*A T \right),
\]
where the last equality follows since $T^{z_A}T + S^{z_A}S \geq_A 0$. So, by taking the supremum over all $x \in \mathcal{H}$ with $\|x\|_A = 1$ in the above inequality we get

$$\|T + S\|^2_A \leq \omega_A \left(T^{z_A}T + S^{z_A}S\right) + 2\omega_A \left(S^{z_A}T\right).$$

Similarly, we prove that

$$\|T - S\|^2_A \leq \|T^{z_A}T + S^{z_A}S\|_A + 2\omega_A \left(S^{z_A}T\right).$$

Hence, we get the desired result. \hfill \Box

**Lemma 2.9.** Let $T, S \in B(\mathcal{H})$. Then, the following assertions hold

1. If $T \geq_A 0$ and $S \geq_A 0$, then

   $$\|T - S\|_A \leq \max\{\|T\|_A, \|S\|_A\}.$$  \hspace{1cm} (2.17)

2. If $T, S \in B_A(\mathcal{H})$, then

   $$2\|T S^{z_A}\|_A \leq \|TT^{z_A} + SS^{z_A}\|_A.$$  \hspace{1cm} (2.18)

**Proof.** (1) Let $Q = T - S$. It is not difficult to see that

$$\|T\|_A I \geq_A T \geq_A Q \quad \text{and} \quad \|S\|_A I \geq_A S \geq_A -Q.$$ 

This implies, by Lemma 2.6, that $\|Q\|_A \leq \|T\|_A$ and $\|Q\|_A \leq \|S\|_A$. This proves the desired property.

(2) Let $T = \begin{pmatrix} T & S \\ 0 & 0 \end{pmatrix}$. In view of (1.14) we see that

$$TT^{z_A} = \begin{pmatrix} TT^{z_A} + SS^{z_A} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad T^{z_A}T = \begin{pmatrix} T^{z_A}T & T^{z_A}S \\ S^{z_A}T & S^{z_A}S \end{pmatrix}.$$ 

Let $U = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}$. By using (1.14), one gets $U^{z_A} = \begin{pmatrix} R_{\mathbb{A}(T)} & O \\ O & -R_{\mathbb{A}(T)} \end{pmatrix}$. So, we verify that $\|UX\|_A = \|U^{z_A}x\|_A = \|x\|_A$ for all $x = (x_1, x_2) \in \mathcal{H} \oplus \mathcal{H}$. Hence, $U$ is $A$-unitary operator. Moreover, clearly we have $(U^{z_A})^{z_A} = U^{z_A}$. In addition, a short calculation shows that

$$(T^{z_A}T)^{z_A} = U^{z_A}(T^{z_A}T)^{z_A}U^{z_A} = \begin{pmatrix} 0 & 2(T^{z_A}S)^{z_A} \\ 2(S^{z_A}T)^{z_A} & 0 \end{pmatrix}.$$ 

So, by applying Lemma 2.2 and then using (2.17) we get

$$2\|T^{z_A}S\|_A = \|\left( (T^{z_A}T)^{z_A} - U^{z_A}(T^{z_A}T)^{z_A}U^{z_A} \right) \|_A \leq \max \{ \| (T^{z_A}T)^{z_A} \|_A, \| U^{z_A}(T^{z_A}T)^{z_A}U^{z_A} \|_A \} \leq \max \{ \| T^{z_A}T \|_A, \| U(T^{z_A}T)U \|_A \} \leq \| T^{z_A}T \|_A \text{ (since } \|U\|_A = 1) \leq \| TT^{z_A} \|_A \text{ (by Lemma 2.2).}$$

Hence, we prove the desired result. \hfill \Box
Lemma 2.10. Let $T, S \in B_A(\mathcal{H})$. Then,

$$\max \left\{ \|T + S\|_A^2, \|T - S\|_A^2 \right\} \geq \frac{\|T + S\|_A^2 - \|T - S\|_A^2}{2} + \max \left\{ \|T^2 + S^2\|_A, \|TT^\sharp A + S^2\|_A \right\}.$$  

Proof. Notice that for any two real numbers $x$ and $y$ we have

$$\max\{x, y\} = \frac{1}{2} (x + y + |x - y|). \quad (2.19)$$

Now, by using (1.3) together with (2.19) we see that

$$\max \left\{ \|T + S\|_A^2, \|T - S\|_A^2 \right\}$$

$$= \frac{1}{2} (\|T + S\|_A^2 + \|T - S\|_A^2 + \|T + S\|_A^2 - \|T - S\|_A^2)$$

$$\geq \frac{1}{2} (\|T^\sharp A + S^\sharp A\|_A + \|T^\sharp A - S^\sharp A\|_A + \|T + S\|_A^2 - \|T - S\|_A^2)$$

$$\geq \frac{1}{2} \left( \|T^\sharp A + S^\sharp A\|_A + \parallel T + S\parallel_2^2 - \|T - S\|_A^2 \right)$$

$$= \|T^\sharp A + S^\sharp A\|_A + \frac{\|T + S\parallel_2^2 - \|T - S\parallel_2^2}{2}. \quad (2.20)$$

By replacing $T$ and $S$ by $T^\sharp A$ and $S^\sharp A$, respectively, in (2.20) and then using the fact that $\|X\|_A = \|X^\sharp A\|_A$ for every $X \in B_A(\mathcal{H})$ we get

$$\max \left\{ \|T + S\|_A^2, \|T - S\|_A^2 \right\} \geq \|TT^\sharp A + S^2\|_A + \frac{\|T + S\parallel_2^2 - \|T - S\parallel_2^2}{2}.$$

On the other hand, by (2.19) one has

$$\max \left\{ \|T + S\|_A^2, \|T - S\|_A^2 \right\}$$

$$= \frac{1}{2} (\|T + S\|_A^2 + \|T - S\|_A^2 + \|T + S\|_A^2 - \|T - S\|_A^2)$$

$$\geq \frac{1}{2} (\|T + S\|_A^2 - \|T - S\|_A^2 + \|T + S\|_A^2 - \|T - S\|_A^2)$$

$$\geq \frac{1}{2} \left( \|T + S\parallel_2^2 + \|T - S\parallel_2^2 \right)$$

$$= \|T^2 + S^2\|_A + \frac{\|T + S\parallel_2^2 - \|T - S\parallel_2^2}{2}.$$

So, the proof of the lemma is complete. \hfill \Box

Now we are ready to prove the following theorem.

Theorem 2.7. Let $P, Q \in B_A(\mathcal{H})$. Then,

$$\omega_A \left[ \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] \leq \frac{1}{2} \sqrt{\max \left\{ \|P + Q\|_A^2, \|P - Q\|_A^2 \right\} - 2\omega_A (PQ^\sharp A)}.$$  

(2.21)
Proof. We first prove that
\[
\max \left( \| P + Q \|_A^2, \| P - Q \|_A^2 \right) - 2\omega_A(PQ^\sharp_A) \geq 0. \tag{2.22}
\]
By applying (2.18) together with the second inequality in (1.6), one observes
\[
2\omega_A(PQ^\sharp_A) \leq \| P^\sharp_A P + Q^\sharp_A Q \|_A.
\]
This implies, by applying Lemma 2.10, that
\[
\max \left( \| P + Q \|_A^2, \| P - Q \|_A^2 \right) \geq \| PP^\sharp_A + QQ^\sharp_A \|_A \geq 2\omega_A(PQ^\sharp_A) + \frac{\| P + Q \|_A^2 - \| P - Q \|_A^2}{2}.
\]
Hence, (2.22) holds. Now, by using the first inequality in (1.6) we get
\[
\omega_A \left[ \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] \geq \frac{1}{4} \left\| \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right\|_A^2 = \frac{1}{4} \left\| \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \left( \begin{pmatrix} P^\sharp_A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Q^\sharp_A & 0 \\ 0 & 0 \end{pmatrix} \right) \right\|_A \geq \frac{1}{4} \left\| PP^\sharp_A + QQ^\sharp_A \right\|_A \geq \frac{1}{4} \max \left\{ \| T + S \|_A^2, \| T - S \|_A^2 \right\} - 2\omega_A(TS^\sharp_A),
\]
where the last inequality follows from Lemma 2.8. This finishes the proof of the theorem. \(\square\)

The following corollary is an immediate consequence of Theorem 2.7 and (2.15).

**Corollary 2.1.** Let \( P, Q \in \mathcal{B}(\mathcal{H}) \) be such that \( APQ^\sharp_A = 0 \). Then,
\[
\frac{1}{2} \max \left( \| P + Q \|_A, \| P - Q \|_A \right) \leq \omega_A \left[ \begin{pmatrix} P & Q \\ 0 & 0 \end{pmatrix} \right] \leq \min \left( \| P + Q \|_A, \| P - Q \|_A \right).
\]
In particular, if \( Q = 0 \) we get
\[
\frac{1}{2} \| P \|_A \leq \omega_A(P) \leq \| P \|_A.
\]

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