Standard Dual Quaternion Functions and Standard Dual Quaternion Optimization

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Abstract

Dual quaternions now have wide applications in robotics, 3D motion modelling and control, and computer graphics. The magnitudes of dual quaternions and the 2-norms of dual quaternion vectors are dual numbers. A total order was defined for dual numbers. Thus, dual quaternion optimization problems, where objective and constraint functions have dual quaternion variables and dual number function values naturally arise. In this paper, we show that several common dual quaternion functions, such as the power function, the magnitude function, the 2-norm function and the $k$th largest eigenvalue function of dual quaternion Hermitian matrices, are standard dual quaternion functions, i.e., the standard parts of their function values depend upon only the standard parts of the dual quaternion variables. Furthermore, the sum, product, minimum, maximum and composite functions of two standard dual functions, the logarithm and the exponential of a standard unit dual quaternion functions, are still standard dual quaternion functions. To solve a standard dual quaternion optimization problem, we only need to solve two quaternion optimization problems. Thus, if the dual quaternion functions are standard, the related dual quaternion optimization problem is solvable.

Key words. Dual quaternion functions, dual quaternion optimization problems, standard dual quaternion functions, minimum logarithm, composite functions.

1 Introduction

Dual quaternions now have wide applications in robotics, 3D motion modelling and control, and computer graphics [1, 2, 3, 6, 7, 9, 11, 12, 13, 17].

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According to [14], the magnitudes of dual quaternions and the 2-norms of
dual quaternion vectors are dual numbers. A total order was also defined for
dual numbers in [14]. Thus, dual quaternion optimization problems, where ob-
jective and constraint functions have dual quaternion variables and dual num-
ber function values naturally arise. In particular, as in the quaternion case
[4, 5, 8, 10, 16], the least square approach for the completion problem results
optimization problems. However, how to solve a dual quaternion optimization
problem is still a problem. There is no calculus concepts for dual quaternion
functions. Even the limit concept is not established for dual quaternion func-
tions. These are difficulties.

Fortunately, some common dual quaternion functions are easy to handle. In
this paper, we show that several common dual quaternion functions, such as
the power function, the magnitude function, the 2-norm function and the $k$th
largest eigenvalue function of dual quaternion Hermitian matrices, are stan-
dard dual quaternion functions, i.e., the standard parts of their function values
depend upon only the standard parts of the dual quaternion variables. Fur-
thermore, the sum, product minimum maximum and composite functions of
two standard dual functions, the logarithm and the exponential of a standard
unit dual quaternion functions, are still standard dual quaternion functions. To
solve a standard dual quaternion optimization problem, we only need to solve
two quaternion optimization problems. Since quaternion optimization problems
have been studied intensively in these years [4, 5, 8, 10, 16], hence, if the dual
quaternion functions are standard, the related dual quaternion optimization
problem is solvable.

In the next section, we present some preliminary knowledge. The main
results of this paper are in Section 3.

Scalars, vectors and matrices are denoted by small letters, bold small let-
ters and capital letters, respectively. Dual numbers and dual quaternions are
distinguished by a hat symbol.
2 Preliminaries

2.1 Dual Numbers

The set of the real numbers, and the set of the dual numbers, are denoted as \( \mathbb{R} \) and \( \hat{\mathbb{R}} \), respectively. Following the literature such as [17], we use the hat symbol to denote dual numbers and dual quaternions. A dual number \( \hat{q} \) has the form \( \hat{q} = q + q_d \epsilon \), where \( q \) and \( q_d \) are real numbers, and \( \epsilon \) is the infinitesimal unit, satisfying \( \epsilon^2 = 0 \). The quaternion \( q \) is called the real part or the standard part of \( \hat{q} \), and the quaternion \( q_d \) is called the dual part or the infinitesimal part of \( \hat{q} \). The infinitesimal unit \( \epsilon \) is commutative in multiplication with real numbers, complex numbers and quaternion numbers. The dual numbers form a commutative algebra of dimension two over the reals. If \( q \neq 0 \), \( \hat{q} \) is said to be appreciable, otherwise, \( \hat{q} \) is said to be infinitesimal. Thus, a real number is a dual number with a zero dual part. Then the dual zero is still 0, and the dual identity is still 1.

A total order was introduced in [14] for dual numbers. Given two dual numbers \( \hat{p}, \hat{q} \in \hat{\mathbb{R}} \), \( \hat{p} = p + p_d \epsilon \), \( \hat{q} = q + q_d \epsilon \), where \( p, p_d, q \) and \( q_d \) are real numbers, we say that \( \hat{p} \leq \hat{q} \), if either \( p < q \), or \( p = q \) and \( p_d \leq q_d \). In particular, we say that \( \hat{p} \) is positive, nonnegative, nonpositive or negative, if \( \hat{p} > 0 \), \( \hat{p} \geq 0 \), \( \hat{p} \leq 0 \) or \( \hat{p} < 0 \), respectively.

2.2 Quaternions

The set of the quaternions is denoted by \( \mathbb{Q} \). A quaternion \( q \) has the form \( q = q_0 + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k} \), where \( q_0, q_1, q_2 \) and \( q_3 \) are real numbers, \( \mathbf{i}, \mathbf{j} \) and \( \mathbf{k} \) are three imaginary units of quaternions, satisfying

\[
\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, \quad \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.
\]

The real part of \( q \) is \( \text{Re}(q) = q_0 \). The imaginary part of \( q \) is \( \text{Im}(q) = q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k} \). The multiplication of quaternions satisfies the distribution law, but is noncommutative.
The conjugate of \( q = q_0 + q_1 i + q_2 j + q_3 k \) is \( q^* := q_0 - q_1 i - q_2 j - q_3 k \).

The magnitude of \( q \) is \( |q| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} \). It follows that the inverse of a nonzero quaternion \( q \) is given by \( q^{-1} = q^*/|q|^2 \). For any two quaternions \( p \) and \( q \), we have \( (pq)^* = q^*p^* \).

A quaternion is called imaginary if its real part is zero. If \( q \) is imaginary, then \( q^* = -q \). In the engineering literature [17], it is called a vector quaternion. Various 3D vectors, such as position vectors, displacement vectors, linear velocity vectors, and angular velocity vectors, can be represented as imaginary quaternions.

If \( |q| = 1 \), then \( q \) is called a unit quaternion, or a rotation quaternion. A spatial rotation around a fixed point of \( \theta \) radians about a unit axis \( (x_1, x_2, x_3) \) that denotes the Euler axis is given by the unit quaternion

\[
q = \cos(\theta/2) + x_1 \sin(\theta/2)i + x_2 \sin(\theta/2)j + x_3 \sin(\theta/2)k = e^\frac{\theta}{2}x.
\] (1)

where the unit axis \( x \) is an imaginary unit quaternion \( x = x_1 i + x_2 j + x_3 k \). We may also write

\[
\ln q = \frac{\theta}{2}x.
\] (2)

Note that a unit quaternion \( q \) is always invertible and \( q^{-1} = q^* \).

The collection of \( n \)-dimensional quaternion vectors is denoted by \( \mathbb{Q}^n \). For \( \mathbf{x} = (x_1, x_2, \ldots, x_n)^\top, \mathbf{y} = (y_1, y_2, \ldots, y_n)^\top \in \mathbb{Q}^n \), define \( \mathbf{x}^*\mathbf{y} = \sum_{i=1}^{n} x_i^*y_i \), where \( \mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*) \) is the conjugate transpose of \( \mathbf{x} \).

### 2.3 Dual Quaternions and Unit Dual Quaternions

The set of dual quaternions is denoted by \( \hat{\mathbb{Q}} \). A dual quaternion \( \hat{q} \in \hat{\mathbb{Q}} \) has the form

\[
\hat{q} = q + q_d \varepsilon,
\] (3)

where \( q, q_d \in \mathbb{Q} \) are the standard part and the dual part of \( \hat{q} \), respectively. If \( q \neq 0 \), then we say that \( \hat{q} \) is appreciable. If \( q \) and \( q_d \) are imaginary quaternions, then \( \hat{q} \) is called an imaginary dual quaternion.

The conjugate of \( \hat{q} \) is

\[
\hat{q}^* = q^* + q_d^* \varepsilon.
\] (4)
Thus, if $\hat{q} = \hat{q}^*$, then $\hat{q}$ is a dual number. If $\hat{q}$ is imaginary, then $\hat{q}^* = -\hat{q}$.

The magnitude of $\hat{q}$ was defined in [14] as

$$|\hat{q}| := \begin{cases} |q| + \frac{(qq^* + q_dq^*)}{2|q|} \epsilon, & \text{if } q \neq 0, \\ |q_d| \epsilon, & \text{otherwise,} \end{cases} \quad (5)$$

which is a dual number.

For two dual quaternions $\hat{p} = p + p_d \epsilon$ and $\hat{q} = q + q_d \epsilon$, their addition and multiplications are defined as

$$\hat{p} + \hat{q} = (p + q) + (p_d + q_d) \epsilon$$

and

$$\hat{p} \hat{q} = pq + (pq_d + p_dq) \epsilon.$$  

See [11, 12]. Under these arithmetic rules, a dual number is commutative with a dual quaternion or a dual quaternion vector.

A dual quaternion $\hat{q}$ is called invertible if there exists a quaternion $\hat{p}$ such that $\hat{p} \hat{q} = \hat{q} \hat{p} = 1$. A dual quaternion $\hat{q}$ is invertible if and only if $\hat{q}$ is appreciable. In this case, we have

$$\hat{q}^{-1} = q^{-1} - q^{-1} q_d q^{-1} \epsilon.$$  

If $|\hat{q}| = 1$, then $\hat{q}$ is called a unit dual quaternion. A unit dual quaternion $\hat{q}$ is always invertible and we have $\hat{q}^{-1} = \hat{q}^*$. The 3D motion of a rigid body can be represented by a unit dual quaternion. We have

$$\hat{q} \hat{q}^* = (q + q_d \epsilon)(q^* + q_d^* \epsilon) = qq^* + (qq_d^* + q_d q^*) \epsilon = \hat{q}^* \hat{q}.$$  

Thus, $\hat{q}$ is a unit dual quaternion if and only if $q$ is a unit quaternion, and

$$qq_d^* + q_d q^* = q^* q_d + q_d^* q = 0. \quad (6)$$

Suppose that there is a rotation $q \in \mathbb{Q}$ succeeded by a translation $p^b \in \mathbb{Q}$, where $p^b$ is an imaginary quaternion. Here, following [17], we use superscripts $b$ and $s$ to represent the relation of the rigid body motion with respect to the body frame attached to the rigid body and the spatial frame which is relative
to a fixed coordinate frame. Then the whole transformation can be represented using unit dual quaternion \( \hat{q} = q + q_d \epsilon \), where \( q_d = \frac{1}{2} q p^b \). Note that we have
\[
qq^* + q_d q^* = \frac{1}{2} [q(p^b)^*q^* + q p^b q^*] = \frac{1}{2} q [(p^b)^* + p^b] q^* = 0.
\]
Thus, a transformation of a rigid body can be represented by a unit dual quaternion
\[
\hat{q} = q + \frac{\epsilon}{2} q p^b,
\]
where \( q \) is a unit quaternion to represent the rotation, and \( p^b \) is the imaginary quaternion to represent the translation or the position. Here, unit quaternion \( q \) serves as a rotation, taking coordinates \( r_o \) of a point in the original frame to coordinates \( r^n \) in the new frame by
\[
r^n = q^* r^o q,
\]
where \( r^o \) and \( r^n \) are two imaginary quaternions, their superscripts \( o \) and \( n \) represent “original” and “new” respectively. On the other hand, every attitude of a rigid body which is free to rotate relative to a fixed frame can be identified by a unique unit quaternion \( q \). Thus, in (7), \( q \) is the attitude of the rigid body, while \( \hat{q} \) represents the transformation.

Combining (7) with (2), we have
\[
\ln \hat{q} = \frac{1}{2} (\theta x + \epsilon p^b).
\]
A unit dual quaternion \( \hat{q} \) serves as both a specification of the configuration of a rigid body and a transformation taking the coordinates of a point from one frame to another via rotation and translation. In (7), if \( \hat{q} \) is the configuration of the rigid body, then \( q \) and \( p^b \) are the attitude of and position of the rigid body respectively.

Denote the collection of \( n \)-dimensional dual quaternion vectors by \( \hat{Q}^n \).

For \( \hat{x} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)^\top \), \( \hat{y} = (\hat{y}_1, \hat{y}_2, \cdots, \hat{y}_n)^\top \in \hat{Q}^n \), define \( \hat{x}^* \hat{y} = \sum_{i=1}^{n} \hat{x}_i^* \hat{y}_i \), where \( \hat{x}^* = (\hat{x}_1^*, \hat{x}_2^*, \cdots, \hat{x}_n^*) \) is the conjugate transpose of \( \hat{x} \). We say \( \hat{x} \) is appreciable if at least one of its component is appreciable. We also say that \( \hat{x} \) and \( \hat{y} \) are orthogonal to each other if \( \hat{x}^* \hat{y} = 0 \). By [14], for any \( \hat{x} \in \hat{Q}^n \), \( \hat{x}^* \hat{x} \)
is a nonnegative dual number, and if \( \hat{x} \) is appreciable, \( \hat{x}^* \hat{x} \) is a positive dual number.

For \( \hat{x} = (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)^T \), by [14], if not all of \( \hat{x}_i \) are infinitesimal, its 2-norm is defined as

\[
\|\hat{x}\|_2 = \sqrt{\sum_{i=1}^{n} |\hat{x}_i|^2}.
\] (10)

If all \( \hat{x}_i \) are infinitesimal, we have \( \hat{x}_i = (x_i)_{d}\epsilon \) for \( i = 1, 2, \ldots, n \). Then we have

\[
\|\hat{x}\|_2 = \sqrt{\sum_{i=1}^{n} |(x_i)_{d}|^2\epsilon}.
\] (11)

### 2.4 Eigenvalues of Dual Quaternion Hermitian Matrices

The collection of \( m \times n \) dual quaternion matrices is denoted by \( \hat{Q}^{m\times n} \). A dual quaternion matrix \( \hat{A} \in \hat{Q}^{m\times n} \) can be written as

\[
\hat{A} = A + A_{d}\epsilon,
\]

where \( A, A_{d} \in Q^{m\times n} \) are the standard part and the infinitesimal part of \( \hat{A} \), respectively.

The conjugate transpose of \( \hat{A} \) is denoted as

\[
\hat{A}^* = A^* + A_{d}^*\epsilon.
\]

A square dual quaternion matrix \( \hat{A} \in \hat{Q}^{n\times n} \) is called Hermitian if \( \hat{A}^* = \hat{A} \); unitary if \( \hat{A}^* \hat{A} = I_n \), where \( I_n \) is the \( n \times n \) identity matrix.

Suppose that \( \hat{A} \in \hat{Q}^{n\times n} \) is a Hermitian matrix. For any \( \hat{x} \in \hat{Q}^n \), we have \((\hat{x}^* \hat{A} \hat{x})^* = \hat{x}^* \hat{A} \hat{x} \). Thus, \( \hat{x}^* \hat{A} \hat{x} \) is a dual number. A dual quaternion Hermitian matrix \( \hat{A} \in \hat{Q}^{n\times n} \) is called positive semidefinite if for any \( \hat{x} \in \hat{Q}^n \), \( \hat{x}^* \hat{A} \hat{x} \geq 0 \); \( \hat{A} \) is called positive definite if for any \( \hat{x} \in \hat{Q}^n \) with \( \hat{x} \) being appreciable, we have \( \hat{x}^* \hat{A} \hat{x} > 0 \) and is appreciable. Then, \( \hat{A} \) is a dual quaternion Hermitian matrix if and only if \( A \) and \( A_{d} \) are two quaternion Hermitian matrices.

Suppose that \( \hat{A} \in \hat{Q}^{n\times n} \). If there are \( \hat{\lambda} \in \hat{Q}, \hat{x} \in \hat{Q}^n \), where \( \hat{x} \) is appreciable, such that

\[
\hat{A} \hat{x} = \hat{x} \hat{\lambda},
\] (12)
then \( \hat{\lambda} \) is called a right eigenvalue of \( \hat{A} \), with \( \hat{x} \) as an associated right eigenvector. If there are \( \hat{\lambda} \in \hat{\mathbb{Q}} \), \( \hat{x} \in \hat{\mathbb{Q}}^n \), where \( \hat{x} \) is appreciable, such that
\[
\hat{A}\hat{x} = \hat{\lambda}\hat{x},
\]
then \( \hat{\lambda} \) is called a left eigenvalue of \( \hat{A} \), with \( \hat{x} \) as an associated left eigenvector. If \( \hat{\lambda} \) is a dual number and a right eigenvalue of \( \hat{A} \), then it is also a left eigenvalue of \( \hat{A} \), as a dual number is commutative with a dual quaternion vector. In this case, it is simply called an eigenvalue of \( \hat{A} \), with \( \hat{x} \) as an associated eigenvector.

The following theorems were proved in [15].

**Theorem 2.1.** Suppose that \( \hat{\lambda} = \lambda + \lambda_d \varepsilon \in \hat{\mathbb{Q}} \) is a right eigenvalue of \( \hat{A} \in \hat{\mathbb{Q}}^{n \times n} \), with associated right eigenvector \( \hat{x} = x + x_d \varepsilon \in \mathbb{Q}^n \). Then
\[
\hat{\lambda} = \frac{\hat{x}^* \hat{A}\hat{x}}{\hat{x}^* \hat{x}},
\]
\( \lambda \) is a right eigenvalue of the quaternion matrix \( A \) with a right eigenvector \( x \), i.e., \( x \neq 0 \) and
\[
A x = x \lambda.
\]
We also have
\[
\lambda = \frac{x^* A x}{x^* x}.
\]

**Theorem 2.2.** A right eigenvalue \( \hat{\lambda} \) of a Hermitian matrix \( \hat{A} = A + A_d \varepsilon \in \hat{\mathbb{Q}}^{n \times n} \) must be a dual number, hence an eigenvalue of \( \hat{A} \), and its standard part \( \lambda \) is a right eigenvalue of the quaternion Hermitian matrix \( A \). Furthermore, assume that \( \hat{\lambda} = \lambda + \lambda_d \varepsilon \), \( \hat{x} = x + x_d \varepsilon \in \mathbb{Q}^n \) is an eigenvector of \( \hat{A} \), associate with the eigenvalue \( \hat{\lambda} \), where \( x, x_d \in \mathbb{Q}^n \). Then we have
\[
\lambda_d = \frac{x^* A_d x}{x^* x}.
\]

A dual quaternion Hermitian matrix has at most \( n \) dual number eigenvalues and no other right eigenvalues.

An eigenvalue of a positive semidefinite Hermitian matrix \( \hat{A} \in \hat{\mathbb{Q}}^{n \times n} \) must be a nonnegative dual number. In that case, \( A \) must be positive semidefinite. An eigenvalue of a positive definite Hermitian matrix \( \hat{A} \in \hat{\mathbb{Q}}^{n \times n} \) must be an appreciable positive dual number. In that case, \( A \) must be positive definite.
Theorem 2.3. Suppose that $\hat{A} = A + A_d \epsilon \in \hat{Q}^{n \times n}$ is a Hermitian matrix. Then there are unitary matrix $\hat{U} \in \hat{Q}^{n \times n}$ and a diagonal matrix $\hat{\Sigma} \in \hat{R}^{n \times n}$ such that

$$\hat{\Sigma} = \hat{U}^* \hat{A} \hat{U},$$

(17)

with the diagonal entries of $\hat{\Sigma}$ being $n$ eigenvalues of $\hat{A}$,

$$\hat{A}\hat{u}_{i,j} = \hat{u}_{i,j}(\lambda_i + \lambda_{i,j} \epsilon),$$

(18)

for $j = 1, \cdots, k_i$ and $i = 1, \cdots, r$, $\hat{U} = (\hat{u}_{1,1}, \cdots, \hat{u}_{1,k_1}, \cdots, \hat{u}_{r,k_r})$, $\lambda_1 > \lambda_2 > \cdots > \lambda_r$ are real numbers, $\lambda_i$ is a $k_i$-multiple right eigenvalue of $A$, $\lambda_{i,1} \geq \lambda_{i,2} \geq \cdots \geq \lambda_{i,k_i}$ are also real numbers, $\sum_{i=1}^{r} k_i = n$. Counting possible multiplicities $\lambda_{i,j}$, the form $\hat{\Sigma}$ is unique.

Theorem 2.4. Suppose that $\hat{A} \in \hat{Q}^{n \times n}$ is Hermitian. Then $\hat{A}$ has exactly $n$ eigenvalues, which are all dual numbers. There are also $n$ eigenvectors, associated with these $n$ eigenvalues. The Hermitian matrix $\hat{A}$ is positive semidefinite or definite if and only if all of these eigenvalues are nonnegative, or positive and appreciable, respectively.

3 Standard Dual Quaternion Functions and Standard Dual Quaternion Optimization

We call a function $\hat{f} : \hat{Q}^n \to \hat{R}$ a dual quaternion function. Let

$$\hat{f}(\hat{x}) = f(\hat{x}) + \epsilon f_d(\hat{x})$$

and

$$\hat{x} = x + \epsilon x_d,$$

where $f(\hat{x}), f_d(\hat{x}) \in Q^n$, $x, x_d \in Q^n$ are the standard parts and dual parts of $\hat{f}(\hat{x})$ and $\hat{x}$, respectively. If $f(\hat{x}) \equiv f(x)$ for all $\hat{x} \in \hat{Q}^n$, then we say that $\hat{f}$ is a standard dual quaternion function.

Consider the following standard dual quaternion optimization problem:

$$\min \{ \hat{f}(\hat{x}) : \hat{g}_i(\hat{x}) \leq 0, \ i = 1, \cdots, m, \ \hat{h}_j(\hat{x}) = 0, \ j = 1, \cdots, p \},$$

(19)
where \( \hat{f}, \hat{g}_i \) for \( i = 1, \cdots, m \) and \( \hat{h}_j \) for \( j = 1, \cdots, p \) are standard dual quaternion functions. Denote the standard parts of \( \hat{f}, \hat{g}_i, \hat{h}_j \) by \( f, g_i, h_j \) and \( f_d, (g_i)_d, (h_j)_d \) respectively. We may first solve a quaternion optimization problem

\[
\min \{ f(x) : g_i(x) \leq 0, \ i = 1, \cdots, m, \ h_j(x) = 0, \ j = 1, \cdots, p \}, \quad (20)
\]

to find an optimal or approximate optimal solution \( x^* \in \mathbb{Q}^n \). Denote

\[
I = \{ i : 1 \leq i \leq m, g_i(x^*) = 0 \}.
\]

Then we may solve another quaternion optimization problem

\[
\min \{ f_d(x^* + \epsilon x_d) : (g_i)_d(x^* + \epsilon x_d) \leq 0, \ i \in I, \ h_j(x^* + \epsilon x_d) = 0, \ j = 1, \cdots, p \}, \quad (21)
\]

to find an optimal or approximate optimal solution \( (x^*)_d \in \mathbb{Q}^n \). In this way \( \hat{x}^* := x^* + \epsilon x_d \) is an optimal or approximate optimal solution of (19).

Thus, we may solve a standard dual quaternion optimization problem by solving two quaternion optimization problems.

Next, we show that many common dual quaternion functions are standard dual quaternion optimization functions.

A simple example is the power function. The power function \( \hat{f} = \hat{x}^m \) for a positive integer \( m \). Let \( \hat{x} = x + \epsilon x_d \). Then \( \hat{f} = (\hat{x})^m = x^m + m \epsilon x^{m-1} x_d \), i.e., \( f = x^m = \hat{f}(x) \). By (5), the magnitude function is also a standard quaternion function. By (5), (10) and (11), the 2-norm function is also a standard quaternion function. A fourth example is \( \hat{f} = \hat{\lambda}_k(\hat{A}) \), where \( \hat{A} \) is the an \( n \times n \) dual quaternion Hermitian matrix, \( \hat{\lambda}_k(\hat{A}) \) is the \( k \)th largest eigenvalue of \( A \), \( 1 \leq k \leq n \). By Theorem 2.3, this is also a standard dual quaternion function.

Furthermore, many operations preserve standard dual quaternion functions.

**Theorem 3.1.** Suppose that \( \hat{f}, \hat{g} : \mathbb{Q}^n \to \mathbb{R} \) are two standard dual quaternion functions. Then their sum, product, minimum and maximum functions are still standard dual quaternion functions.
Proof. Let \( \hat{h}(\hat{x}) = \hat{f}(\hat{x}) + \hat{g}(\hat{x}) \). Then \( \hat{h} = h + \epsilon h_d \), where

\[
h(x) = f(x) + g(x) = f(x) + g(x) = h(x),
\]

i.e., \( \hat{h} \) is also a standard dual quaternion function. This proves the first conclusion.

Let \( \hat{h}(\hat{x}) = \hat{f}(\hat{x}) \hat{g}(\hat{x}) \). Then \( \hat{h} = h + \epsilon h_d \), where

\[
h(x) = f(x)g(x) = f(x)g(x) = h(x),
\]

i.e., \( \hat{h} \) is also a standard dual quaternion function. This proves the second conclusion.

Let \( \hat{h}(\hat{x}) = \min\{\hat{f}(\hat{x}), \hat{g}(\hat{x})\} \). Then \( \hat{h} = h + \epsilon h_d \), where

\[
h(x) = \min\{f(x), g(x)\} = \min\{f(x), g(x)\} = h(x),
\]

i.e., \( \hat{h} \) is also a standard dual quaternion function. This proves the third conclusion.

The fourth conclusion can be proved similarly. □

**Corollary 3.2.** Suppose that \( \hat{f}_1, \ldots , \hat{f}_m : \hat{Q}^n \to \hat{\mathbb{R}} \) are \( m \) standard dual quaternion functions, where \( m \) is a positive integer. Then their sum, product, minimum and maximum functions are still standard dual quaternion functions.

**Corollary 3.3.** Suppose that \( \hat{f} : \hat{Q}^n \to \hat{\mathbb{R}} \) is a standard dual quaternion function, and \( m \) is a positive integer. Then \((\hat{f})^m\) is still a standard dual quaternion function.

**Theorem 3.4.** Suppose that \( \hat{f} : \hat{Q} \to \hat{\mathbb{R}} \) is a standard unit dual quaternion function. Then its logarithm and exponential functions are still standard dual quaternion functions.

**Proof.** We have

\[
\hat{f}(\hat{x}) = f(\hat{x}) + \epsilon f_d(\hat{x}) = f(x) + \epsilon f_d(\hat{x}),
\]
as \( \hat{f} \) is a standard dual quaternion function. Since \( f \) is a unit dual quaternion function, we have \( |f(x)| = 1 \). Thus, we may write
\[
\hat{f}(\hat{x}) = f(x) + \frac{\epsilon}{2} f(x)p^b.
\]
Write
\[
f(x) = \cos(\theta/2) + y_1 \sin(\theta/2)i + y_2 \sin(\theta/2)j + y_3 \sin(\theta/2)k.
\]
Then \( \theta/2 \) and \( y = y_1i + y_2j + y_3k \) are functions of \( x \). By (9), we have
\[
\ln \hat{f}(\hat{x}) = \frac{1}{2}(\theta y + \epsilon p^b).
\]
Thus, \( \ln \hat{f} \) is still a standard dual quaternion function. Similarly, we may show that \( e^{\hat{f}} \) is a standard dual quaternion function too.

Theorem 3.5. Suppose that \( \hat{f} : \hat{Q} \to \hat{R} \) and \( \hat{g} : \hat{Q}^n \to \hat{R} \) are two standard unit dual quaternion functions. Then their composite function \( \hat{h} = \hat{f} \circ \hat{g} : \hat{Q}^n \to \hat{R} \) is also a standard dual quaternion function.

Proof. For \( \hat{x} \in \hat{Q}^n \),
\[
h(\hat{x}) = f(\hat{g}(\hat{x})) = f(g(\hat{x})) = f(g(x)) = h(x),
\]
where the second and the third equalities hold because \( \hat{f} \) and \( \hat{g} \) are standard unit dual quaternion functions respectively. Thus, \( \hat{h} \) is also a standard dual quaternion function.

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