Efficient Algorithms for Outlier-Robust Regression

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March 12, 2018

Abstract

We give the first polynomial-time algorithm for performing linear or polynomial regression resilient to adversarial corruptions in both examples and labels.

Given a sufficiently large (polynomial-size) training set drawn i.i.d. from distribution \( D \) and subsequently corrupted on some fraction of points, our algorithm outputs a linear function whose squared error is close to the squared error of the best-fitting linear function with respect to \( D \), assuming that the marginal distribution of \( D \) over the input space is certifiably hypercontractive.

This natural property is satisfied by many well-studied distributions such as Gaussian, strongly log-concave distributions and, uniform distribution on the hypercube among others. We also give a simple statistical lower bound showing that some distributional assumption is necessary to succeed in this setting.

These results are the first of their kind and were not known to be even information-theoretically possible prior to our work.

Our approach is based on the sum-of-squares (SoS) method and is inspired by the recent applications of the method for parameter recovery problems in unsupervised learning. Our algorithm can be seen as a natural convex relaxation of the following conceptually simple non-convex optimization problem: find a linear function and a large subset of the input corrupted sample such that the least squares loss of the function over the subset is minimized over all possible large subsets.

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1 Introduction

An influential recent line of work has focused on developing \textit{robust} learning algorithms—algorithms that succeed on a data set that has been contaminated with adversarially corrupted outliers. It has led to important achievements such as efficient algorithms for robust clustering and estimation of moments \cite{LRV16, DKK16, CSV17, KS17, KS17a} in unsupervised learning and efficient learning of halfspaces \cite{KLS09, DKS17} with respect to malicious or “nasty noise” in classification. In this paper, we continue this line of work and give the first efficient algorithms for performing outlier-robust least-squares \textit{regression}. That is, given a training set drawn from distribution $D$ and arbitrarily corrupting an $\eta$ fraction of its points (by changing both labels and/or locations), our goal is to efficiently find a linear function (or polynomial in the case of polynomial regression) whose least squares loss is competitive with the best fitting linear function for $D$.

We give simple examples showing that unlike classical regression, achieving any non-trivial guarantee for robust regression is information-theoretically impossible without making assumptions on the distribution $D$. In this paper, we study the case where the marginal of $D$ on examples in the well-studied class of \textit{hypercontractive} distributions. Many natural distributions such as Gaussians, strongly log-concave distributions, and product distributions on the hypercube with bounded marginals fall into this category. To complement our algorithmic results, we also show that for the class of hypercontractive distributions, the bounds on the loss of the linear function output by our algorithm is optimal in its dependence on the fraction of corruptions $\eta$ up to multiplicative constants.

1.1 Outlier-Robust Regression

We formally define the problem next. In the following, we will use the following notations for brevity: For a distribution $D$ on $\mathbb{R}^d \times \mathbb{R}$ and for a vector $\ell \in \mathbb{R}^d$, let $\text{err}_D(\ell) = \mathbb{E}_{(x,y) \sim D}[\langle \ell, x \rangle - y]^2$ and let $\text{opt}(D) = \min_{\ell \in \mathbb{R}^d} \text{err}_D(\ell)$ be the least error achievable.

In the classical least-squares linear regression problem, we are given access to i.i.d. samples from a distribution $D$ over $\mathbb{R}^d \times \mathbb{R}$ and our goal is to find a linear function $\ell : \mathbb{R}^d \to \mathbb{R}$ whose squared-error—$\text{err}_D(\ell)$—is close to the best possible, $\text{opt}(D)$.

In outlier-robust regression, our goal is similar with the added twist that we only get access to a sample from the distribution $D$ where up to an $\eta$ fraction of the samples have been arbitrarily corrupted.

\textbf{Definition 1.1 ($\eta$-Corrupted Samples).} Let $D$ be a distribution on $\mathbb{R}^d \times \mathbb{R}$. We say that a set $U \subseteq \mathbb{R}^d \times \mathbb{R}$ is an $\eta$-corrupted training set drawn from $D$ if it is formed in the following fashion: generate a set $X$ of i.i.d samples from $D$ and arbitrarily modify any $\eta$ fraction to produce $U$.

Observe that the corruptions can be \textit{adaptive}, that is, they can depend on the original uncorrupted sample $X$ in an arbitrary way as long as $|U \cap X|/|X| \geq 1 - \eta$.\footnote{In unsupervised learning, this has been called the \textit{strong adversary} model of corruptions and is the strongest notion of robustness studied in the context.}

Our goal—which we term \textit{outlier-robust regression}—now is as follows: Given access to an $\eta$-corrupted training set $U$ drawn from $D$, find a linear function $\ell$ whose error $\text{err}_D(\ell)$ under the true distribution $D$ is small.
1.2 Statement of Results

Our main results give outlier-robust least-squares regression algorithms for hypercontractive distributions.

**Definition 1.2** (4-Hypercontractivity). A distribution \( D \) on \( \mathbb{R}^d \) is \((C, 4)\)-hypercontractive if for all \( \ell \in \mathbb{R}^d \), \( \mathbb{E}_{x \sim D} [\langle x, \ell \rangle^4] \leq C^2 \cdot \mathbb{E}_{x \sim D} [\langle x, \ell \rangle^2]^2 \).

In addition, we say that \( D \) is certifiably \((C, 4)\)-hypercontractive if there is a degree 4 sum-of-squares proof of the above inequality.

Observe that 4-hypercontractivity is invariant under arbitrary affine transformation, and in particular, doesn’t depend on the condition number of the covariance of the distribution.

We will elaborate on the notion of certifiability later on (once we have the appropriate preliminaries). For the time being, we note that many well-studied distributions including (potentially non-spherical) Gaussians, affine transformations of isotropic strongly log-concave distributions, the uniform distribution on the Boolean hypercube, and more generally, product distributions on bounded domains are known to satisfy this condition with \( C \) a fixed constant.

**Theorem 1.3.** [Informal] Let \( D \) be a distribution on \( \mathbb{R}^d \times [-M, M] \) and let \( D_X \) be its marginal distribution on \( \mathbb{R}^d \) which is certifiably \((C, 4)\)-hypercontractive. Let \( \ell^* = \arg \min_{\ell} \text{err}_D(\ell) \) have polynomial bit-complexity. Then for all \( \varepsilon > 0 \) and \( \eta < c/C^2 \) for a universal constant \( c > 0 \), there exists an algorithm \( \mathcal{A} \) with run-time \( \text{poly}(d, 1/\eta, 1/\varepsilon, M) \) that given a polynomial-size \( \eta \)-corrupted training set \( \mathcal{U} \), outputs a linear function \( \ell \) such that with probability at least \( 1 - \varepsilon \),

\[
\text{err}_D(\ell) \leq (1 + O(\sqrt{\eta})) \cdot \text{opt}(D) + O(\sqrt{\eta}) \mathbb{E}_{(x, y) \sim D} [(y - \langle \ell^*, x \rangle)^2] + \varepsilon.
\]

The above statement assumes that the marginal distribution is (certifiably) hypercontractive with respect to its fourth moments. Our results improve for higher-order certifiably hypercontractive distributions \( D_X \); see Theorem 5.1 for details. In the realizable case where \( (x, y) \sim D \) satisfies \( y = \langle \ell^*, x \rangle \) for some \( \ell^* \), the guarantee of Theorem 1.3 becomes \( \text{err}_D(\ell) \leq \varepsilon \); in particular, the error approaches zero at a polynomial rate.

We also get analogous results for outlier-robust polynomial regression. See Theorem A.3.

We also give a simple argument (see Section 6) to show that the above guarantee is optimal in its dependence on \( \eta \) up to the \( O(1) \) factors: even for distributions supported on \( \mathbb{R}^d \times [-1, 1] \), it is statistically impossible to achieve an error bound of \( (1 + o(\sqrt{\eta})) \text{opt} + o(\sqrt{\eta}) \) under the same assumptions.

Our result is a outlier-robust analog of agnostic regression problem - that is, the non-realizable setting. In addition, our guarantees makes no assumption about the condition number of the covariance of \( D_X \) and thus, in particular, holds for \( D_X \) with poorly conditioned covariances. Alternately, we give a similar guarantee for \( \ell_1 \) regression when the condition number of covariance of \( D_X \) is bounded without any need for hypercontractivity (see Theorem 5.9). We show that in the absence of distributional assumptions (such as hypercontractivity) it is statistically impossible to obtain any meaningful bounds on robust regression in Section 6.

**Application to Learning Boolean Functions under Nasty Noise.** Our work has immediate applications for learning Boolean functions in the nasty noise model, where the learner is presented with an \( \eta \)-corrupted training set that is derived from an uncorrupted training set of the form \((x, f(x))\)
with $x$ drawn from $\mathcal{D}$ on $\{0,1\}^n$ and $f$ is an unknown Boolean function. The goal is to output a hypothesis $h$ with $\Pr_x[h(x) \neq f(x)]$ as small as possible. The nasty noise model is considered the most challenging noise model for classification problems in computational learning theory.

Applying a result due to [KKMS08] (c.f. Theorem 5) for learning with respect to adversarial label noise only (standard agnostic learning) and a generalization of Theorem 1.3 to higher degree polynomials (see Theorem A.3) we obtain the following:

**Corollary 1.4.** Let $C$ be a class of Boolean functions on $n$ variables such that for every $c \in C$ there exists a (multivariate) polynomial $p$ of degree $d(\epsilon)$ with $\E_{x \sim \mathcal{D}}[(p(x) - c(x))^2] \leq \epsilon$. Assume that $d(\epsilon)$ is a constant for any $\epsilon = O(1)$ and that $\mathcal{D}$ is $(C, \eta)$ hypercontractive for polynomials of degree $d(\epsilon^2)$. Then $C$ can be learned in the nasty noise model in time $n^{O(d(\epsilon^2))}$ via an output hypothesis $h$ such that $\Pr_x[h(x) \neq c(x)] \leq O(\sqrt{\eta}) \E_{x \sim \mathcal{D}}[(p(x) - c(x))^4]^{1/2} + \epsilon$.

One of the main conclusions of work due to [KKMS08] is that the existence of low-degree polynomial approximators for a concept class $C$ implies learnability for $C$ in the agnostic setting. Corollary 1.4 shows that existence of low-degree polynomial approximators and hypercontractivity of $\mathcal{D}$ imply learnability in the harsher nasty noise model.

We note that Corollary 1.4 gives an incomparable set of results in comparison to recent work of [DKS17] for learning polynomial threshold functions in the nasty noise model.

**Concurrent Works.** Using a set of different techniques, Diakonikolas, Kamath, Kane, Li, Steinhardt and Stewart [DKK*18] and Prasad, Suggala, Balakrishnan and Ravikumar [PSBR18] also obtained robust algorithms for regression in the setting where data $(x, y)$ is generated via the process: $y = \langle w, x \rangle + e$ for an fixed unknown vector $w$ and zero mean noise $e$.

### 1.3 Our Approach

In this section, we give an outline of Theorem 1.3. At a high level, our approach resembles several recent works [MSS16, BM16, PS17, KS17c, HL17] starting with the pioneering work of [BKS15] that use the Sum-of-Squares method for designing efficient algorithms for learning problems. An important conceptual difference, however, is that previous works have focused on parameter recovery problems. For such problems, the paradigm involves showing that there’s a simple (in the “SoS proof system”) proof that a small sample uniquely identifies the underlying hidden parameters (referred to as “identifiability”) up to a small error.

In contrast, in our setting, samples do not uniquely determine a good hypothesis as there can be multiple hypotheses (linear functions) that all have low-error on the true distribution. Our approach thus involves establishing that there’s a “simple” proof that any low-error hypotheses that is inferred from the observed (corrupted) sample has low-error on the true distribution (we call this certifiability of a good hypothesis). To output a good solution in our approach (unlike in cases where there are uniqueness results), we have to crucially rely on the convexity (captured in the SoS proof system) of the empirical loss function.

**Part One: Certifying that a linear function has low loss.** Let $X$ be an uncorrupted sample from the underlying distribution $\mathcal{D}$ and suppose we are given an $\eta$-corruption $U = \{(u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n)\}$ of $X$. Let $\hat{\mathcal{D}}^2$ be the uniform distribution on $X$. Our goal is to

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*We use superscript * to denote empirical quantities and superscript ’ to denote quantities on corrupted samples.*
come up with a linear function $\ell$ that has low error on $\hat{D}$ given access only to $U$. By standard generalization bounds, this will also imply that $\ell$ has low error on $D$ with high probability.

It is important to observe that even without computational constraints, that is, information theoretically, it is unclear why this should at all be possible. To see why, let’s consider the following natural strategy: brute-force search over all subsets $T$ of $U$ of size $(1 - \eta)|U|$ and perform least-squares regression to obtain linear function $\ell_T$ with empirical loss $\varepsilon_T$. Then, output $\ell_T$ with minimal empirical loss $\varepsilon_T$ over all subsets $T$.

Since some subset $T^*$ of size $(1 - \eta)|U|$ will be a proper subset of the uncorrupted sample, the empirical loss of $\ell_{T^*}$ will clearly be small. However, a priori, there’s nothing to rule out the existence of another subset $R$ of size $(1 - \eta)|U|$ such that the optimal regression hypothesis $\ell_R$ on $R$ has loss smaller than that of $\ell_{T^*}$ while $\ell_R$ has a large error on the $\hat{D}$.

This leads to the following interesting question on certifying a good hypothesis: given a linear function $\ell$ that has small empirical loss with respect to some subset $T$ of $(1 - \eta)$ fraction of the corrupted training set $U$, can we certify that its true loss with respect to $X$ is small?

We can phrase this as a more abstract “robust certification” question: given two distributions $D_1$ (=uniform distribution on $X$ above) and $D_2$ (=uniform distribution on $T$ above) on $\mathbb{R}^d \times \mathbb{R}$ that are $\eta$ close in total variation distance, and a linear function $\ell$ that has small error on $D_2$, when can we certify a good upper bound on the error of $\ell$ on $D_1$?

Without making any assumptions on $D_1$, it is not hard to construct examples where we can give no meaningful bound on the error of a good hypothesis $\ell$ on $D_1$ (see Section 6). More excitingly, we show an elementary proof of a “robust certifiability lemma” that proves a statement as above whenever $D_1$ has hypercontractive one dimensional marginals. The loss with respect to $D_1$ increases as a function of the statistical distance and the degree of hypercontractivity.

Applying our certification lemma, it thus suffices to find a subset $T$ of $U$ of size $\geq (1 - \eta)|U|$ and a linear function $\ell$ such that the least squares error of $\ell$ over $T$ is small.

**Part Two: Inefficient Algorithm via Polynomial Optimization.** Coming back to the question of efficient algorithms, the above approach can appear hopeless in general since simultaneously finding $\ell$ and a subset $T$ of size $(1 - \eta)|U|$ that minimizes the error of $\ell$ w.r.t. uniform distribution on $T$ is a non-convex quadratic optimization problem. At a high-level, we will be able to get around this intractability by observing that the proof of our robust certifiability lemma is “simple” in a precise technical sense. This simplicity allows us to convert such a certifiability proof into an efficient algorithm in a principled manner. To describe this connection, we will first translate the naive idea for an algorithm above into a polynomial optimization problem.

For concreteness in this high-level description, we suppose that for $(x, y) \sim D$, the distribution on $x$ is $(C, 4)$-hypercontractive for a fixed constant $C$ and $\mathbb{E}[y^4] = O(1)$. Further, it can also be shown that, with high probability, $\hat{D}$ is also $(O(1), 4)$-hypercontractive as long as the size of the original uncorrupted sample $X$ is large enough.

Following the certification lemma, our goal is to use $U$ to find a distribution $D'$ and a linear function $\ell$ such that 1) the loss of $\ell$ with respect to $D'$ is small and 2) $D'$ is close to $\hat{D}$. It is easy to phrase this as a polynomial optimization problem.

To do so we will look for $X' = \{(x'_1, y'_1), \ldots, (x'_n, y'_n)\}$ and weights $w_1, w_2, \ldots, w_n \in \{0, 1\}$ with $\sum_i w_i \geq (1 - \eta)n$ and $(x'_i, y'_i) = (u_i, v_i)$ if $w_i = 1$. Let $D'$ be the uniform distribution on $D$. Clearly, the condition on weights $w$ ensures that the statistical distance between $\hat{D}, D'$ is at most $\eta$. Ideally, we intend $w_i$'s to be the indicators of whether or not the $i$'th sample is corrupted. We now try
to find \( \ell \) that minimizes the least squares error on \( \mathcal{D}' \). This can be captured by the following optimization program: \( \min_{w, \ell, X'} (1/n) \sum_i (y'_i - \langle \ell, x'_i \rangle)^2 \) where \((w, \ell, X')\) satisfy the polynomial system of constraints:

\[
\mathcal{P} = \left\{ \begin{array}{l}
\sum_{i=1}^n w_i = (1 - \eta) \cdot n \\
w_i^2 = w_i & \forall i \in [n]. \\
w_i \cdot (u_i - x'_i) = 0 & \forall i \in [n]. \\
w_i \cdot (v_i - y'_i) = 0 & \forall i \in [n].
\end{array} \right. \tag{1.1}
\]

In this notation, our robust certifiability lemma implies that for any \((w, \ell, X')\) satisfying \(\mathcal{P}\),

\[
\text{err}_{\mathcal{D}}(\ell) \leq (1 + O(\sqrt{\eta})) \cdot \text{err}_{\mathcal{D}'}(\ell) + O(\sqrt{n}). \tag{1.2}
\]

It is easy to show that the minimum of the optimization program \(\text{opt}(\mathcal{D}) \leq \text{opt}(\mathcal{D})\) (up to standard generalization error) by setting \(X' = X\) and \(w_i = 1\) if and only if \(i\)'th sample is uncorrupted. By the above arguments, solutions to the above program satisfy the bound stated in Theorem 1.3. Unfortunately, this is a quadratic optimization problem and is NP-hard in general.

We are now ready to describe the key idea that allows us to essentially turn this hopelessly inefficient algorithm into an efficient one. This exploits a close relationship between the simplicity of the proof of robust certifiability and the success of a canonical semi-definite relaxation of (1.1).

**Part Three: From Simple Proofs to Efficient Algorithms.** Suppose that instead of finding a single solution to the program in (1.1), we attempt to find a distribution \(\mu\) supported on \((w, \ell, X')\) that satisfies \(\mathcal{P}\) and minimizes \(\mathbb{E}_\mu[1/n \sum_i (y'_i - \langle \ell, x'_i \rangle)^2]\). Let \(\text{opt}_\mu\) be the minimum value. Then, as Equation 1.2 holds for all \((w, \ell, X')\) satisfying \(\mathcal{P}\), it also follows that

\[
\mathbb{E}_{(w, \ell, X') \sim \mu} [\text{err}_{\mathcal{D}}(\ell)] \leq (1 + O(\sqrt{n})) \cdot \text{opt}_\mu + O(\sqrt{n}). \tag{1.3}
\]

A priori, we appear to have made our job harder. While computing a distribution on solutions is no easier than computing a single solution, even describing a distribution on solutions appears to require exponential resources in general. However, by utilizing the convexity of the square loss, we can show that having access to just the first moments of \(\mu\) is enough to recover a good solution.

Formally, by the convexity of the square loss, the above inequality yields:

\[
\text{err}_{\mathcal{D}} \left( \mathbb{E}_\mu[\ell] \right) \leq \mathbb{E}_{(w, \ell, X') \sim \mu} [\text{err}_{\mathcal{D}}(\ell)] \leq (1 + O(\sqrt{n})) \cdot \text{opt}_\mu + O(\sqrt{n}). \tag{1.4}
\]

All of the above still doesn’t help us in solving program 1.1 as even finding first moments of distributions supported on solutions to a polynomial optimization program is NP-Hard.

The key algorithmic insight is to observe that we can replace distributions \(\mu\) by an efficiently computable (via the SoS algorithm) proxy called as pseudo-distributions without changing any of the conclusions above.

In what way is a pseudo-distribution a proxy for an actual distribution \(\mu\) satisfying \(\mathcal{P}\)? It turns out that if a polynomial inequality (such as the one in (1.2)) can be derived from \(\mathcal{P}\) via a low-degree sum-of-squares proof, then (1.3) remains valid even if we replace \(\mu\) in (1.3) by a pseudo-distribution \(\bar{\mu}\) of large enough degree. Roughly speaking, the SoS degree of a proof measures the “simplicity” of the proof (in the “SoS proof system”). In other words, facts with simple proofs holds not just for distributions but also for pseudo-distributions.
Thus, the important remaining steps are to show that 1) the inequality (1.2) (which is essentially the conclusion of our robust certifiability lemma) and 2) the convexity argument in (1.4) has a low-degree SoS proof. We establish both these claims by relying on standard tools such as the SoS versions of the Cauchy-Schwarz and Hölder’s inequalities.

We give a brief primer to the SoS method in Section 4 that includes rigorous definitions of concepts appearing in this high-level overview.

1.4 Related Work

The literature on grappling with outliers in the context of regression is vast, and we do not attempt a survey here. Many heuristics have been developed modifying the ordinary least squares objective with the intent of minimizing the effect of outliers (see [RL87]). Another active line research is concerned with parameter recovery, where each label y in the training set is assumed to be from a generative model of the form \(\theta^T x + e\) for some (usually independent) noise parameter \(e\) and unknown weight vector \(\theta \in \mathbb{R}^d\). For example, the recovery properties of LASSO and related algorithms in this context have been intensely studied (see e.g., [XCM10], [LW11]). For more challenging noise models, recent work due to Du, Balakrishnan, and Singh [DBS17] studies sparse recovery in the Gaussian generative setting in Huber’s \(\epsilon\)-contamination model, which is similar but formally weaker than the noise model we consider here.

It is common for “robust regression” to refer to a scenario where only the labels are allowed to be corrupted adversarially (for example, see [BJKK17] and the references therein), or where the noise obeys some special structure (e.g., [HS10]) (although there are some contexts where both the covariates (the \(x\)’s) and labels may be subject to a small adversarial corruption [CCM13]).

What distinguishes our setting is 1) we do not assume the labels come from a generative model; each \((x, y)\) element of the training set is drawn iid from \(\mathcal{D}\) and 2) we make no assumptions on the structure or type of noise that can affect a training set (other than that at most an \(\eta\) fraction of points may be affected). In contrast to the parameter recovery setting, our goal is similar to that of agnostic learning: we will output a linear function whose squared error with respect to \(\mathcal{D}\) is close to optimal.

From a technical standpoint, as discussed before our work follows the recent paradigm of converting certifiability proofs to algorithms. Previous applications in machine learning have focused on various parameter-recovery problems in unsupervised learnings. Our work is most closely related to the recent works on robust unsupervised learning (moment estimation and clustering) [KS17c, HL17, KS17b].

2 Preliminaries and Notation

2.1 Notation

We will use the following notations and conventions throughout: For a distribution \(\mathcal{D}\) on \(\mathbb{R}^d \times \mathbb{R}\) and function \(f : \mathbb{R}^d \to \mathbb{R}\), we define \(\text{err}_\mathcal{D}(f) = \mathbb{E}_{(x, y) \sim \mathcal{D}}[(f(x) - y)^2]\). For a vector \(\ell \in \mathbb{R}^d\), we abuse notation and write \(\text{err}_\mathcal{D}(\ell)\) for \(\mathbb{E}_{(x, y) \sim \mathcal{D}}[(\langle \ell, x \rangle - y)^2]\). For a real-valued random variable \(X\), and integer \(k \geq 0\), we let \(\|X\|_k = \mathbb{E}[X^k]^{1/k}\).

\(^3\)Even the term “robust” is very overloaded and can now refer to a variety of different concepts.
2.2 Distribution Families

Our algorithmic results for a wide class of distributions that include Gaussian distributions and others such as log-concave and other product distributions. We next define the properties we need for the marginal distribution on examples to satisfy.

**Definition 2.1 (Certifiable hypercontractivity).** For a function \( C : [k] \to \mathbb{R}_+ \), we say a distribution \( D \) on \( \mathbb{R}^d \) is \( k \)-certifiably \( C \)-hypercontractive if for every \( r \leq k/2 \), there’s a degree \( k \) sum of squares proof of the following inequality in variable \( v \):

\[
\mathbb{E}_D (x, v)^{2r} \leq \left( C(r) \mathbb{E}_D (x, v)^2 \right)^r.
\]

Many natural distribution families satisfy certifiable hypercontractivity with reasonably growing functions \( C \). For instance, Gaussian distributions, uniform distribution on Boolean hypercube satisfy the definitions with \( C(r) = cr \) for a fixed constant \( c \). More generally, all distributions that are affine transformations of isotropic distributions satisfying the Poincaré inequality [KS17a], are also certifiably hypercontractive. In particular, this includes all strongly log-concave distributions. Certifiable hypercontractivity also satisfies natural closure properties under simple operations such as affine transformations, taking bounded weight mixtures and taking products. We refer the reader to [KS17c] for a more detailed overview where certifiable hypercontractivity is referred to as certifiable subgaussianity.

3 Robust Certifiability

The conceptual core of our results is the following **robust certifiability** result: for *nice* distributions (e.g., as defined in Definition 2.1), a regression hypothesis inferred from a large enough \( \epsilon \)-corrupted sample has low-error over the uncorrupted distribution.

3.1 Robust Certifiability for Arbitrary Distributions

We begin by giving a robust certifiability claim for arbitrary distributions for L1 regression.

The error that we incur depends on the L2 squared loss of the best fitting regression hypothesis, and in particular, we do not obtain *consistency* in the statistical sense: i.e, the error incurred by the regression hypothesis does not vanish even in the “realizable” case when, in the true uncorrupted distribution, there’s a linear function that correctly computes all the labels. In Section 6, we show that if we make no further assumption on the distribution, then this is indeed inherent and that achieving consistency under adversarial corruptions is provably impossible without making further assumptions. In the following subsection, we show that assuming that the moments of the underlying uncorrupted distribution are “bounded” (i.e., linear functions of the distribution are hypercontractive), one can guarantee consistency even under the presence of adversarial outliers.

While the certifiability statements are independently interpretable, for the purpose of robust regression, it might be helpful to keep in mind that \( D \) corresponds to uniform distribution on large enough sample from the unknown uncorrupted distribution while \( D' \) corresponds to the uniform distribution on the sample that serves as the “certificate”.

**Lemma 3.1 (Robust Certifiability for L1 Regression).** Let \( D, D' \) be two distributions on \( \mathbb{R}^d \times \mathbb{R} \) with marginals \( D, D' \) on \( \mathbb{R}^d \), respectively. Suppose \( \|D - D'\|_{TV} \leq \eta \) and further, that the ratio of the largest
The main result of this section is the following lemma.

1. \[ \mathbf{E}_D|\langle \ell, x \rangle - y| \leq \mathbf{E}_D|\langle \ell, x \rangle - y| + 2\kappa^{1/2}\eta^{1/2} \sqrt{\mathbf{E}_D y^2 + 2\kappa^{1/2}\eta^{1/2} \cdot \mathbf{E}_D(y - \langle \ell^*, x \rangle)^2}. \]

**Proof.** Let \( G \) be a coupling between \( D, D' \). That is, \( G \) is a joint distribution on \((x, y), (x', y')\) such that the marginal on \((x', y')\) is \( D' \) and the marginal on \((x, y)\) is \( D \) satisfying \( \mathbf{P}_G 1 \{ (x, y) = (x', y') \} = 1 - \eta \). Let \( \text{err}_D(\ell) = \mathbf{E}_D|y - \langle \ell, x \rangle| \). We have:

\[
\mathbf{E}_D|y - \langle \ell, x' \rangle| = \mathbf{E}_G 1 \{ (x, y) = (x', y') \} |y - \langle \ell, x \rangle| + \mathbf{E}_G 1 \{ (x, y) \neq (x', y') \} \cdot |y - \langle \ell, x \rangle| \\
\leq \text{err}_D(\ell) + \frac{\mathbf{E}_G 1 \{ (x, y) \neq (x', y') \} \cdot \mathbf{E}_D(y - \langle \ell, x \rangle)^2}{\sqrt{\mathbf{E}_D(y - \langle \ell, x \rangle)^2}} \\
= \text{err}_D(\ell) + \sqrt{\mathbf{E}_D(y - \langle \ell, x \rangle)^2}. 
\]

Now, we must have: \( \mathbf{E}_D(y - \langle \ell, x \rangle)^2 \leq 2 \mathbf{E}_D y^2 + 2 \mathbf{E}_D(\ell, x)^2 \).

For any \( \ell^* \), \( \mathbf{E}_D(\ell^*, x)^2 \leq 2 \mathbf{E}_D y^2 + 2 \mathbf{E}_D(y - \langle \ell^*, x \rangle)^2 \).

Since the all eigenvalues of \( \mathbf{E}_D xx^T \) are within \( \kappa \) of each other and \( \|\ell^*\|_2 \geq \|\ell\| \), \( \mathbf{E}_D(\ell, x)^2 \leq \kappa \cdot \mathbf{E}_D(\ell^*, x)^2 \). Plugging in the above estimate gives the lemma.

\[ \square \]

### 3.2 Robust Certifiability for Hypercontractive Distributions

The main result of this section is the following lemma.

**Lemma 3.2** (Robust Certifiability for L2 Regression). Let \( D, D' \) be distributions on \( \mathbb{R}^d \times \mathbb{R} \) such that \( \|D - D'\|_{TV} \leq \varepsilon \) and the marginal \( D_X \) of \( D \) on \( x \) is \( k \)-certifiably \( C \)-hypercontractive for some \( C : [k] \to \mathbb{R}_+ \) and for some even integer \( k \geq 4 \).

Then, for any \( \ell, \ell^* \in \mathbb{R}^d \) and any \( \eta \) such that \( 2C(k/2)\eta^{1-2/k} < 0.9 \), we have:

\[
\text{err}_D(\ell) \leq (1 + O(C(k/2))\eta^{1-2/k}) \cdot \text{err}_{D'}(\ell) + O(C(k/2))\eta^{1-2/k} \cdot \left( \mathbf{E}_D(y - \langle \ell^*, x \rangle)^k \right)^{2/k}. 
\]

**Proof.** Fix a vector \( \ell \in \mathbb{R}^d \); for brevity, we write \( \text{err}_D \) for \( \text{err}_D(\ell) \) and \( \text{err}_{D'} \) for \( \text{err}_{D'}(\ell) \) in the following.

Let \( G \) be a coupling between \( D, D' \). That is, \( G \) is a joint distribution on \((x, y), (x', y')\) such that the marginal on \((x', y')\) is \( D' \) and the marginal on \((x, y)\) is \( D \) satisfying \( \mathbf{P}_G 1 \{ (x, y) = (x', y') \} = 1 - \eta \).

Let \( ((x, y), (x', y')) \sim G \). Writing \( 1 = 1 \{ (x, y) = (x', y') \} + 1 \{ (x, y) \neq (x', y') \} \), we obtain:

\[
\mathbf{E}[(y - \langle \ell, x \rangle)^2] = \mathbf{E}[1 \{ (x, y) = (x', y') \} (y - \langle \ell, x \rangle)^2] + \mathbf{E}[1 \{ (x, y) \neq (x', y') \} \cdot (y - \langle \ell, x \rangle)^2] \\
= \mathbf{E}[1 \{ (x, y) = (x', y') \} (y' - \langle \ell, x' \rangle)^2] + \mathbf{E}[1 \{ (x, y) \neq (x', y') \} \cdot (y - \langle \ell, x \rangle)^2] \\
\leq \text{err}_D + \left( \mathbf{E}[1 \{ (x, y) \neq (x', y') \}^{k-2]} \right)^{1-2/k} \left( \mathbf{E}(y - \langle \ell, x \rangle)^k \right)^{2/k} \\
\leq \text{err}_{D'} + \eta^{1-2/k} \cdot \left( \mathbf{E}(y - \langle \ell, x \rangle)^k \right)^{2/k}. 
\]

(3.1)
Here, the inequality uses the Hölder’s inequality for the second term and the fact that
\[ \mathbb{E}_{G} 1 \{ (x, y) = (x', y') \} (y - \langle \ell, x \rangle)^2 \leq \mathbb{E}_{D'} (y - \langle \ell, x \rangle)^2 = \text{err}_{D'}(\ell) \] for the first term.

We next bound \( \| y - \langle \ell, x \rangle \|_k \). By Minkowski’s inequality,
\[
\| y - \langle \ell, x \rangle \|_k \leq \| y - \langle \ell^*, x \rangle \|_k + \| \langle \ell - \ell^*, x \rangle \|_k.
\]

Now, by using hypercontractivity of \( D \), we get
\[
\| \langle \ell - \ell^*, x \rangle \|_k \leq \sqrt{C(k/2)} \cdot \| \langle \ell - \ell^* \rangle \|_2.
\] (3.2)

Further,
\[
\| \langle \ell - \ell^*, x \rangle \|_2 \leq \| y - \langle \ell^*, x \rangle \|_2 + \| y - \langle \ell, x \rangle \|_2 \leq \| y - \langle \ell^*, x \rangle \|_k + \| y - \langle \ell, x \rangle \|_2.
\]

Combining the above three inequalities, we get
\[
\| y - \langle \ell, x \rangle \|_k \leq (1 + \sqrt{C(k/2)}) \| y - \langle \ell^*, x \rangle \|_k + \sqrt{C(k/2)} \| y - \langle \ell, x \rangle \|_2.
\]

Therefore, as \( (a + b)^2 \leq 2a^2 + 2b^2 \) and \( 2(1 + \sqrt{C(k/2)})^2 \leq 8C(k/2) \),
\[
\| y - \langle \ell, x \rangle \|_k^2 \leq 8C(k/2) \| y - \langle \ell^*, x \rangle \|_k^2 + 2C(k/2) \text{err}_{D'}.
\]

Substituting the above into Equation 5.4, we get
\[
\text{err}_{D} \leq \text{err}_{D'} + 8\eta^{1-2/k} C(k/2) \cdot \| y - \langle \ell^*, x \rangle \|_k^2 + 2\eta^{1-2/k} C(k/2) \text{err}_{D'}.
\]

Rearranging the inequality and observing that \( 1/(1 - 2\eta^{1-2/k} C(k/2)) \leq 1 + O(C(k/2))\eta^{1-2/k} \) gives us
\[
\text{err}_{D} \leq (1 + O(C(k/2))\eta^{1-2/k})\text{err}_{D'} + O(C(k/2))\eta^{1-2/k} \cdot \| y - \langle \ell^*, x \rangle \|_k^2,
\]
proving the claim.

\[ \square \]

The argument for the above lemma also extends straightforwardly to polynomial regression (see Appendix A):

4 Sum of Squares proofs and Sum of Squares Optimization

In this section, we define pseudo-distributions and sum-of-squares proofs. See the lecture notes [BS16] for more details and the appendix in [MSS16] for proofs of the propositions appearing here.

Let \( x = (x_1, x_2, \ldots, x_n) \) be a tuple of \( n \) indeterminates and let \( \mathbb{R}[x] \) be the set of polynomials with real coefficients and indeterminates \( x_1, \ldots, x_n \). We say that a polynomial \( p \in \mathbb{R}[x] \) is a sum-of-squares (sos) if there are polynomials \( q_1, \ldots, q_r \) such that \( p = q_1^2 + \cdots + q_r^2 \).
4.1 Pseudo-distributions

Pseudo-distributions are generalizations of probability distributions. We can represent a discrete (i.e., finitely supported) probability distribution over \( \mathbb{R}^n \) by its probability mass function \( D : \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( D \geq 0 \) and \( \sum_{x \in \text{supp}(D)} D(x) = 1 \). Similarly, we can describe a pseudo-distribution by its mass function. Here, we relax the constraint \( D \geq 0 \) and only require that \( D \) passes certain low-degree non-negativity tests.

Concretely, a level-\( \ell \) pseudo-distribution is a finitely-supported function \( D : \mathbb{R}^n \rightarrow \mathbb{R} \) such that \( \sum_x D(x) = 1 \) and \( \sum_x D(x)f(x)^2 \geq 0 \) for every polynomial \( f \) of degree at most \( \ell/2 \). (Here, the summations are over the support of \( D \).) A straightforward polynomial-interpolation argument shows that every level-\( \infty \)-pseudo distribution satisfies \( D \geq 0 \) and is thus an actual probability distribution. We define the pseudo-expectation of a function \( f \) on \( \mathbb{R}^d \) with respect to a pseudo-distribution \( D \), denoted \( \mathbb{E}_{D(x)} f(x) \), as

\[
\mathbb{E}_{D(x)} f(x) = \sum_x D(x) f(x). \tag{4.1}
\]

The degree-\( \ell \) moment tensor of a pseudo-distribution \( D \) is the tensor \( \mathbf{E}_{D(x)}(1, x_1, x_2, \ldots, x_n)^{\otimes \ell} \). In particular, the moment tensor has an entry corresponding to the pseudo-expectation of all monomials of degree at most \( \ell \) in \( x \). The set of all degree-\( \ell \) moment tensors of probability distribution is a convex set. Similarly, the set of all degree-\( \ell \) moment tensors of degree \( d \) pseudo-distributions is also convex. Key to the algorithmic utility of pseudo-distributions is the fact that while there can be no efficient separation oracle for the convex set of all degree-\( \ell \) moment tensors of an actual probability distribution, there’s a separation oracle running in time \( n^{O(\ell)} \) for the convex set of the degree-\( \ell \) moment tensors of all level-\( \ell \) pseudodistributions.

**Fact 4.1** ([Sho87, Par00, Nes00, Las01]). For any \( n, \ell \in \mathbb{N} \), the following set has a \( n^{O(\ell)} \)-time weak separation oracle (as defined in [GLS81]):

\[
\{ \mathbb{E}_{D(x)}(1, x_1, x_2, \ldots, x_n)^{\otimes d} \mid \text{degree-}d \text{ pseudo-distribution } D \text{ over } \mathbb{R}^n \}.
\tag{4.2}
\]

This fact, together with the equivalence of weak separation and optimization [GLS81] allows us to efficiently optimize over pseudo-distributions (approximately)—this algorithm is referred to as the sum-of-squares algorithm.

The level-\( \ell \) sum-of-squares algorithm optimizes over the space of all level-\( \ell \) pseudo-distributions that satisfy a given set of polynomial constraints—we formally define this next.

**Definition 4.2** (Constrained pseudo-distributions). Let \( D \) be a level-\( \ell \) pseudo-distribution over \( \mathbb{R}^n \). Let \( \mathcal{A} = \{ f_1 \geq 0, f_2 \geq 0, \ldots, f_m \geq 0 \} \) be a system of \( m \) polynomial inequality constraints. We say that \( D \) satisfies the system of constraints \( \mathcal{A} \) at degree \( r \), denoted \( D \models_r \mathcal{A} \), if for every \( S \subseteq [m] \) and every sum-of-squares polynomial \( h \) with \( \deg h + \sum_{i \in S} \max \{ \deg f_i, r \} \),

\[
\mathbb{E}_{D(x)} h \cdot \prod_{i \in S} f_i \geq 0.
\]

We write \( D \models \mathcal{A} \) (without specifying the degree) if \( D \models_0 \mathcal{A} \) holds. Furthermore, we say that \( D \models \mathcal{A} \) holds approximately if the above inequalities are satisfied up to an error of \( 2^{-n^{\ell}} \cdot ||h|| \cdot \prod_{i \in S} ||f_i|| \), where \( ||\cdot|| \) denotes the Euclidean norm of the coefficients of a polynomial in the monomial basis.\(^4\)

\(^4\)The choice of norm is not important here because the factor \( 2^{-n^\ell} \) swamps the effects of choosing another norm.
We remark that if $D$ is an actual (discrete) probability distribution, then we have $D \models \mathcal{A}$ if and only if $D$ is supported on solutions to the constraints $\mathcal{A}$.

We say that a system $\mathcal{A}$ of polynomial constraints is \textit{explicitly bounded} if it contains a constraint of the form $\{\|x\|^2 \leq M\}$. The following fact is a consequence of Fact 4.1 and [GLS81].

**Fact 4.3** (Efficient Optimization over Pseudo-distributions). There exists an $(n + m)^{O(1)}$-time algorithm that, given any explicitly bounded and satisfiable system$\ddag$ $\mathcal{A}$ of $m$ polynomial constraints in $n$ variables, outputs a level-$\ell$ pseudo-distribution that satisfies $\mathcal{A}$ approximately.

A property of pseudo-distributions that we will use frequently is the following:

**Fact 4.4** (Hölder’s inequality). Let $f, g$ be SoS polynomials. Let $p, q$ be positive integers so that $1/p + 1/q = 1$. Then, for any pseudo-distribution $\tilde{\mu}$ of degree $r \geq pq \cdot \deg\left(f \cdot g\right)$, we have:

$$\left(\tilde{\mathbb{E}}_{\tilde{\mu}}[f \cdot g]\right)^{pq} \leq \tilde{\mathbb{E}}[f^n]^q \cdot \tilde{\mathbb{E}}[g^q]^p.$$ 

In particular, for all even integers $k \geq 2$, and polynomial $f$ with $\deg(f) \leq r$,

$$\left(\tilde{\mathbb{E}}_{\tilde{\mu}}[f]\right)^k \leq \tilde{\mathbb{E}}_{\tilde{\mu}}[f^k].$$

### 4.2 Sum-of-squares proofs

Let $f_1, f_2, \ldots, f_r$ and $g$ be multivariate polynomials in $x$. A sum-of-squares proof that the constraints $\{f_1 \geq 0, \ldots, f_m \geq 0\}$ imply the constraint $\{g \geq 0\}$ consists of polynomials $(p_S)_{S \subseteq [m]}$ such that

$$g = \sum_{S \subseteq [m]} p_S \cdot \prod_{i \in S} f_i. \tag{4.3}$$

We say that this proof has degree $\ell$ if for every set $S \subseteq [m]$, the polynomial $p_S \prod_{i \in S} f_i$ has degree at most $\ell$. If there is a degree $\ell$ SoS proof that $\{f_i \geq 0 | i \leq r\}$ implies $\{g \geq 0\}$, we write:

$$\{f_i \geq 0 | i \leq r\} \models_\ell \{g \geq 0\}. \tag{4.4}$$

Sum-of-squares proofs satisfy the following inference rules. For all polynomials $f, g : \mathbb{R}^n \to \mathbb{R}$ and for all functions $F : \mathbb{R}^n \to \mathbb{R}^m, G : \mathbb{R}^n \to \mathbb{R}^k, H : \mathbb{R}^p \to \mathbb{R}^q$ such that each of the coordinates of the outputs are polynomials of the inputs, we have:

| Rule | Inference
|---|---
| $\frac{\mathcal{A} \models_\ell \{f \geq 0, g \geq 0\}}{\mathcal{A} \models_\ell \{f \geq 0\}, \mathcal{A} \models_{\ell + r} \{g \geq 0\}}$ | (addition and multiplication)
| $\frac{\mathcal{A} \models_\ell B, \mathcal{B} \models_{\ell + r} C}{\mathcal{A} \models_{\ell, \ell+r} C}$ | (transitivity)
| $\frac{\mathcal{A} \models \{F \geq 0\}}{\mathcal{A} \models_{\ell \cdot \deg(H)} \{G(H) \geq 0\}}$ | (substitution)

Low-degree sum-of-squares proofs are sound and complete if we take low-level pseudo-distributions as models.

Concretely, sum-of-squares proofs allow us to deduce properties of pseudo-distributions that satisfy some constraints.

$\ddag$Here, we assume that the bitcomplexity of the constraints in $\mathcal{A}$ is $(n + m)^{O(1)}$. 

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Fact 4.5 (Soundness). If \( D \Vdash A \) for a level-\( \ell \) pseudo-distribution \( D \) and there exists a sum-of-squares proof \( A \Vdash B \), then \( D \Vdash_{r',r} B \).

If the pseudo-distribution \( D \) satisfies \( A \) only approximately, soundness continues to hold if we require an upper bound on the bit-complexity of the sum-of-squares \( A \Vdash B \) (number of bits required to write down the proof).

In our applications, the bit complexity of all sum of squares proofs will be \( n^{O(1)} \) (assuming that all numbers in the input have bit complexity \( n^{O(1)} \)). This bound suffices in order to argue about pseudo-distributions that satisfy polynomial constraints approximately.

The following fact shows that every property of low-level pseudo-distributions can be derived by low-degree sum-of-squares proofs.

Fact 4.6 (Completeness). Suppose \( d \geq r' \geq r \) and \( A \) is a collection of polynomial constraints with degree at most \( r \), and \( A \vdash \sum_{i=1}^{n} x_i^2 \leq B \) for some finite \( B \).

Let \( \{ g \geq 0 \} \) be a polynomial constraint. If every degree-\( d \) pseudo-distribution that satisfies \( D \Vdash A \) also satisfies \( D \Vdash_r \{ g \geq 0 \} \), then for every \( \varepsilon > 0 \), there is a sum-of-squares proof \( A \Vdash \{ g \geq -\varepsilon \} \).

We will use the following standard sum-of-squares inequalities:

Fact 4.7 (SoS Hölder’s Inequality). Let \( f_1, f_2, \ldots, f_n \) and \( g_1, g_2, \ldots, g_n \) be SoS polynomials over \( \mathbb{R}^d \). Let \( p, q \) be integers such that \( 1/p + 1/q = 1 \). Then,

\[
\left( \frac{1}{n} \sum_{i=1}^{n} f_i g_i \right)^{pq} \leq \left( \frac{1}{n} \sum_{i=1}^{n} f_i^p \right)^{q} \left( \frac{1}{n} \sum_{i=1}^{n} g_i^q \right)^{p} \]

Fact 4.8. For any \( a_1, a_2, \ldots, a_n \),

\[
\left\{ \sum_{i} a_i^k \leq n^k \left( \sum_{i} a_i \right)^2 \right\}
\]

5 Algorithm

In this section, we present and analyze our robust regression algorithms. We begin by setting some notation that we will use throughout this section:

1. \( D \) denotes the uncorrupted distribution on \( \mathbb{R}^d \times \mathbb{R} \). In general, calligraphic letters will denote distributions on example-label pairs. \( D = D_x \) will denote the marginal distribution on \( x \).

2. We will write \( X = ((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)) \) to denote the uncorrupted input sample of size \( n \) drawn according to \( D \). For some bound \( B \) on the bit-complexity of linear functions, we will write \( \text{opt}(D) \) for the optimum least squares error of any linear function of bit complexity \( B \) on \( D \). Recall that the bit complexity of a linear function is the number of bits required to write down all of its coefficients.

3. We will write \( \hat{D} \) for the uniform distribution on the sample \( X \). \( \hat{D} = \hat{D}_x \) will denote the marginal distribution on \( x \). Note that our algorithm does not get direct access to \( D \) or \( \hat{D} \). We will write \( \text{opt}(\hat{D}) \) for the optimum least squares error of any linear function of bit complexity \( B \) on \( \hat{D} \).
4. We will write $U = ((u_1, v_1), (u_2, v_2), \ldots, (u_n, v_n))$ to denote an $\eta$-corruption of $X$, i.e., $U$ is obtained by changing $\eta$ fraction of the example-label pairs. Our algorithm gets access to $U$.

5. For $\ell \in \mathbb{R}^d$, and $M > 0$, let $\ell_M : \mathbb{R}^d \to \mathbb{R}$ denote the truncated linear function defined as follows:

$$
\ell_M(x) = \begin{cases}
\langle \ell, x \rangle & \text{if } |\langle \ell, x \rangle| \leq M \\
\text{sign}(\langle \ell, x \rangle) \cdot M & \text{otherwise}.
\end{cases}
$$

### 5.1 Robust Least Squares Regression

In this section, we present our Robust Least Squares Regression algorithm. The main goal of this section is to establish the following result.

**Theorem 5.1.** Let $\mathcal{D}$ be a distribution on $\mathbb{R}^d \times [-M, M]$ for some positive real $M$ such the marginal on $\mathbb{R}^d$ is $(C, k)$-certifiably hypercontractive distribution. Let $\text{opt}_B(\mathcal{D}) = \min_{\ell} \mathbb{E}_\mathcal{D}[(y - \langle \ell, x \rangle)^2]$ where the minimum is over all $\ell \in \mathbb{R}^d$ of bit complexity $B$. Let $\ell^*$ be any such minimizer.

Fix any even $k \geq 4$ and any $\epsilon > 0$. Let $X$ be an i.i.d. sample from $\mathcal{D}$ of size $n \geq n_0 = \text{poly}(d^k, B, M, 1/\epsilon)$. Then, with probability at least $1 - \epsilon$ over the draw of the sample $X$, given any $\eta$-corruption $U$ of $X$ and $\eta$ as input, there is a polynomial time algorithm (Algorithm 5.2) that outputs a $\ell \in \mathbb{R}^d$ such that for $C = C(k/2)$,

$$
\text{err}_D(\ell_M) < (1 + O(C)\eta^{1-2/k}) \text{opt}_B(\mathcal{D}) + O(C)\eta^{1-2/k} \left( \mathbb{E}_{\mathcal{D}}(y - \langle \ell^*, x \rangle)^k \right)^{2/k} + \epsilon.
$$

By an entirely analogous argument, we also get a similar guarantee for outlier-robust polynomial regression. We defer the details to Section A.

We need the boundedness assumption on the labels $y$ (that they lie in $[-M, M]$) and the bounded bit-complexity assumption on the linear functions ($B$) mainly to obtain generalization bounds for linear regression as are often used even for regression without corruptions. Further note that specializing the above to the case $k = 4$ gives Theorem 1.3.

Following the outline described in the introduction, we first define a set of polynomial inequalities which will be useful in our algorithm: Let $\eta > 0$ be a parameter and consider the following system of polynomial inequalities in variables $w \in \mathbb{R}^n$, $\ell \in \mathbb{R}^d$, $x'_1, \ldots, x'_n \in \mathbb{R}^d$:

\begin{equation}
\mathcal{P}_{U, \eta} = \left\{ \begin{array}{l}
\sum_{i=1}^n w_i = (1 - \eta) \cdot n \\
\sum_{i=1}^n w_i^2 = w_i \\
w_i \cdot (u_i - x'_i) = 0 \\
w_i \cdot (v_i - y'_i) = 0
\end{array} \right\} \quad \forall i \in [n].
\end{equation}

Observe that this system is feasible: use $w_i = 1$ if $(x_i, y_i) = (u_i, v_i)$ and 0 otherwise (i.e., $w_i = 1$ if and only if the $i$'th example was corrupted) and taking $(x'_i, y'_i) = (x_i, y_i)$ for all $i \in [n]$.

We are now ready to describe our algorithm for robust L2 regression.

**Algorithm 5.2 (Algorithm for Robust L2 Linear Regression via sum-of-squares).**

**Given:**
- $\eta$: A bound on the fraction of adversarial corruptions.
- $U$: An $\eta$-corruption of a labeled sample $X$ of size $n$ sampled from a $(C, k)$-certifiably hypercontractive distribution $\mathcal{D}$.
5.2 Analysis of the Algorithm

We now analyze the algorithm and prove Theorem 5.1. The analysis can be broken into two modular steps: (1) Bounding the optimization error (roughly translates to bounding the empirical error) and (2) Bounding the generalization error. Concretely, we break down the analysis into the following two steps. Let \( \hat{\text{opt}}_k = ((1/n) \sum_{i=1}^{n} (y_i - \langle \ell^*, x_i \rangle)^k)^{2/k} \) and \( \text{opt}_k(D) = \mathbb{E}_{(x,y) \sim D}[(y - \langle \ell^*, x \rangle)^k]^{2/k} \).

**Lemma 5.3** (Bounding the optimization error). Under the assumptions of Theorem 5.1 (and following the above notations), with probability at least \( 1 - \varepsilon \),

\[
\text{err}_D(\hat{\ell}) \leq (1 + C(k/2)\eta^{-2/k}) \cdot \hat{\text{opt}}_{\text{SOS}} + O(C(k/2)) \cdot \eta^{-1/2} \cdot \hat{\text{opt}}_k.
\]

**Lemma 5.4** (Bounding the generalization error). Under the assumptions of Theorem 5.1, with probability at least \( 1 - \varepsilon \), the following hold:

1. \( \hat{\text{opt}}_{\text{SOS}} \leq \text{opt}(D) + \varepsilon. \)
2. \( \text{err}_D(\hat{\ell}_M) \leq \text{err}_D(\hat{\ell}) + \varepsilon. \)

Ideally, we would liked to have \( \hat{\text{opt}}_k \leq \text{opt}_k(D) + \varepsilon \). Given such an inequality, Theorem 5.1 would follow immediately from the above two lemmas. A small technical issue is that we cannot prove such an inequality as we don’t have good control on the moments of \( (y - \langle \ell^*, x \rangle)^k \). However, we can exploit the robust setting to get around this issue by essentially truncating large values - since the distribution with truncated values will be close in statistical distance to the actual distribution. We remark that the proof of Lemma 5.4 follows standard generalization arguments for the most part.

We defer the proofs of the above lemmas to the appendix and proceed to finish analyzing our algorithm. With Lemma 5.3, 5.4 in hand, we are now ready to prove our main theorem. We just need the following lemma to get around bounding \( \hat{\text{opt}}_k \).

**Lemma 5.5.** For every distribution \( D \) on \( \mathbb{R}^d \times \mathbb{R} \) such that \( v = E_D(y - \langle \ell^*, x \rangle)^k < \infty \), there exists a distribution \( F \) such that \( \|D - F\|_{TV} < \eta \) and \( (y - \langle \ell^*, x \rangle)^k \) is bounded absolutely bounded in the support of \( F \) by \( v/\eta \).

**Proof.** Set \( F = D \mid (y - \langle \ell^*, x \rangle)^k \leq v/\eta \). Then, by definition \( F \) satisfies the property that \( (y - \langle \ell^*, x \rangle)^k \) is bounded by \( v/\eta \) in the support of \( F \). Further, by Markov’s inequality, the probability of the event we conditioned on is at least \( 1 - \eta \). This completes the proof. \( \square \)

Observe that an \( \eta \) corrupted sample from \( D \) can be thought of as an \( 2\eta \) corrupted sample from \( F \). Since \( (y - \langle \ell^*, x \rangle)^k \) is bounded in \( F \), it allows us to use Hoeffding bound for concentration to show that the empirical expectation of \( (y - \langle \ell^*, x \rangle)^k \) converges to its expectation under \( D \).
We now prove Lemma 5.3.2, viewing the above inequality as a polynomial inequality in variables \((y - \langle \ell', x \rangle)^k\) that satisfies the inequalities in \(-\)at the beginning of this sketch, a pseudo-density \(\tilde{q}\) polynomial inequality is \(\tilde{q}\) on \(X\). Let \(\tilde{q}\) be the uniform distribution on \(X\). We defer the proof of the lemma to Section 5.2.2.

### Proof Outline

The rough idea is to exploit the following abstract property of pseudo-distributions: If a collection of polynomial inequalities \(\mathcal{P} = \{p_i(z) \geq 0, i \in [r]\}\) SOS-imply another polynomial inequality \(q(z) \geq 0\), then for any pseudo-distribution \(\tilde{\mu}\) of appropriately high degree (depending on the degree of the SOS proof) that satisfies the inequalities in \(\mathcal{P}\) also satisfies \(q\), that is, \(\mathbb{E}_{\tilde{\mu}}[q] \geq 0\). Further, the SoS algorithm allows us to compute pseudo-distributions satisfying a set of polynomial inequalities efficiently.

Now, let \((w, \ell, X')\) satisfy the inequalities \(\mathcal{P}_{U, \eta}\). Then, by Lemma A, applied to \(\tilde{D}\) and the uniform distribution on \(X'\), we get

\[
\text{err}_{\tilde{D}}(\ell) \leq (1 + c C \eta^{-1/2}) \left( \frac{1}{n} \sum_{i=1}^{n} (y'_i - \langle \ell, x'_i \rangle)^2 \right) + c C \eta^{-1/2} \cdot \text{opt}_k,
\]

for some universal constant \(c > 0\).

To view the above inequality as a polynomial inequality in variables \(w, \ell, X'\), we rephrase it as follows. For brevity, let \(\text{err}(w, \ell, X') = (1/n) \sum_{i=1}^{n} (y'_i - \langle \ell, x'_i \rangle)^2\). Then,

\[
(\text{err}_{\tilde{D}}(\ell) - \text{err}(w, \ell, X'))^{k/2} \leq \eta^{k/2 - 1} \cdot 2^{O(k)} C^k \text{err}(w, \ell, X')^{k/2} + \eta^{k/2 - 1} \cdot 2^{O(k)} C^k \cdot \text{opt}_k^{k/2}.
\]

We show that the above version of the robust certifiability lemma has a SOS proof; that is, viewing the above inequality as a polynomial inequality in variables \(w, \ell, X'\), this inequality has a SOS proof starting from the polynomial inequalities \(\mathcal{P}_{U, \eta}\). Thus, by the property of pseud-densities at the beginning of this sketch, a pseudo-density \(\tilde{\mu}\) as in our algorithm satisfies an analogue of the above inequality which after some elementary simplifications gives us a bound of the form

\[
\mathbb{E}_{\tilde{\mu}}[\text{err}_{\tilde{D}}(\ell)] \leq (1 + c C \eta^{-1/2}) \cdot \text{opt}_{\text{SOS}} + c C \eta^{-1/2} \text{opt}_k.
\]

As it stands, the above inequality is not very useful for us as it does not tell us which \(\ell\) to choose. However, for any degree at most \(k/2\) polynomial \(p\), we also have that \((\mathbb{E}_{\tilde{\mu}}[p(w, \ell)])^2 \leq \mathbb{E}[p(w, \ell)^2]\) (see Fact 4.4). Applying this to each \((y_i - \langle \ell, x_i \rangle)\), we get that

\[
\text{err}_{\tilde{D}}(\mathbb{E}_{\tilde{\mu}}[\ell]) \leq \mathbb{E}_{\tilde{\mu}}[\text{err}_{\tilde{D}}(\ell)] \leq (1 + c C \eta^{-1/2}) \cdot \text{opt}_{\text{SOS}} + c C \eta^{-1/2} \text{opt}_k,
\]

proving the claim.

We next formalize the above approach starting with a SOS proof of Lemma A. We defer the proof of the lemma to Section 5.2.2.
Lemma 5.6 (SoS Proof of Robust Certifiability of Regression Hypothesis). Let $X$ be a collection of $n$ labeled examples in $\mathbb{R}^d \times \mathbb{R}$ such that $\tilde{D}$, the uniform distribution on $x_1, x_2, \ldots, x_n$ is $k$-certifiably hypercontractive and all the labels $y_1, y_2, \ldots, y_n$ are bounded in $[-M, M]$. Let $U$ be an $\eta$-corruption of $X$.

Let $(w, \ell, X')$ satisfy the set of system of polynomial equations $\mathcal{P}_{U, \eta}$. Let $\text{err}_D(\ell)$ be the quadratic polynomial $\mathbb{E}_{(x,y)\sim D}(y-\langle \ell, x \rangle)^2$ in vector valued variable $\ell$. Let $\text{err}(w, \ell, X')$ be the polynomial $\frac{1}{n} \sum_{i \in n} (y_i' - \langle \ell, x_i \rangle)^2$ in vector valued variables $w, \ell, x_1', \ldots, x_n'$.

Then, for any $\ell^* \in \mathbb{R}^d$ of bit complexity at most $B < \text{poly}(n, d^k)$, $C = C(k/2)$ and any $\eta$ such that $100C\eta^{1-2/k} < 0.9$,

$$\mathcal{A}_{U, \eta} [\ell, \text{err}_D(\ell) - \text{err}(w, \ell, X')]^{k/2} \leq \eta^{k/2-1} \cdot 2^{\Theta(k)} C^k \text{err}(w, \ell, X')^{k/2}$$

$$+ \eta^{k/2-1} \cdot 2^{\Theta(k)} C^k \left( \sum_{i=1}^n (y_i - \langle \ell^*, x_i \rangle) \right)^k \quad \text{(5.2)}$$

Moreover, the bit complexity of the proof is polynomial in $n$ and $d^k$.

We also need the following lemma (that follows from appropriate matrix concentration results) from [KS17c] stating that the uniform distribution on a sufficiently large set of i.i.d samples from a hypercontractive distribution also satisfy hypercontractivity. This allows us to argue that the uncorrupted empirical distribution $\tilde{D}$ is also hypercontractive when $D$ is.

**Lemma 5.7** (Lemma 5.5 of [KS17c]). Let $D$ be a $(C, k)$-certifiably hypercontractive distribution on $\mathbb{R}^d$. Let $X$ be an i.i.d. sample from $D$ of size $n \geq \Omega((d^k \log (d/\delta))^{k/2})$. Then, with probability at least $1 - \delta$ over the draw of $X$, the uniform distribution $D$ over $X$ is $(2C, k)$-certifiably hypercontractive.

We can now prove Lemma 5.3.

**Proof of Lemma 5.3.** If $n \geq \tilde{\Theta}(d^k \log (d/\epsilon)^{k/2})$, then by Lemma 5.7, with probability at least $1 - \epsilon$, the uniform distribution $\tilde{D}$ on $X$ is $(2C, k)$-hypercontractive.

Since $\tilde{\mu}$ is a pseudo-distribution of level $k$, combining Fact 4.5 and Lemma 5.6, we must have for $C = C(k/2)$,

$$\tilde{\mathbb{E}}_{\tilde{\mu}} (\text{err}_D(\ell) - \text{err}(w, \ell, X'))^{k/2} \leq O(C^{k/2} \eta^{k/2-1}) \tilde{\mathbb{E}}_{\tilde{\mu}} \text{err}(w, \ell, X')^{k/2} + O(C^{k/2} \eta^{k/2-1}) \left( \mathbb{E}_{\tilde{D}} (y - \langle \ell', x \rangle) \right)^k \quad \text{(5.3)}$$

Taking $2/k$th powers of both sides of the above equation and recalling the definition of $\tilde{\text{opt}}_{\text{SOS}}, \text{opt}_k$, we get

$$\langle \tilde{\mathbb{E}}_{\tilde{\mu}} \text{err}_D(\ell) - \text{err}_{D}(\ell) \rangle^{2/k} \leq O(C) \eta^{1-2/k} \cdot \tilde{\text{opt}}_{\text{SOS}} + O(C) \eta^{1-2/k} \text{opt}_k^{2/k} \quad \text{.}$$

Now, by Fact 4.4, $\langle \tilde{\mathbb{E}}_{\tilde{\mu}} [\text{err}_D(\ell) - \text{err}(w, \ell, X')] \rangle^{k/2} \leq \mathbb{E}_{\tilde{\mu}} [\text{err}_D(\ell) - \text{err}(w, \ell, X')]^{k/2}$ and thus,

$$\tilde{\mathbb{E}}_{\tilde{\mu}} \text{err}_D(\ell) \leq (1 + O(C) \eta^{1-2/k}) \cdot \tilde{\text{opt}}_{\text{SOS}} + O(C) \eta^{1-2/k} \text{opt}_k$$

Finally, by another application of Fact 4.4, we have that for every $i$, $(y_i - \langle x_i, \tilde{\mathbb{E}}_{\tilde{\mu}} [\ell] \rangle)^2 \leq \tilde{\mathbb{E}}_{\tilde{\mu}} [y_i (x_i, \ell)^2]$; in particular, $\text{err}_D(\tilde{\mathbb{E}}_{\tilde{\mu}} [\ell]) \leq \mathbb{E}_{\tilde{\mu}} \text{err}_D(\ell)$. Thus, we have

$$\text{err}_D(\tilde{\mathbb{E}}_{\tilde{\mu}} [\ell]) \leq (1 + O(C) \eta^{1-2/k}) \cdot \tilde{\text{opt}}_{\text{SOS}} + O(C) \eta^{1-2/k} \text{opt}_k$$

proving the lemma. \qed
5.2.2 Proof of Lemma 5.6

Here we prove Lemma 5.6. The proof is similar in spirit to that of Lemma A but we need to adapt the various steps to a form suitable for SOS proof system.

Proof of Lemma 5.6. For brevity, we write $\text{err}_D$ for $\text{err}_D(\ell)$.

Let $w' \in \{0, 1\}^n$ be given by $w'_i = w_i$ iff $i$th sample is uncorrupted in $U$ and 0 otherwise. Then, observe that $\sum_i w'_i = s$ for $s \geq (1 - 2\eta)n$.

Then,

$$\left| \frac{2}{w'} \left\{ \frac{1}{n} \sum_i (1 - w'_i) \leq 2\eta \right\} \right|.$$

Let $\text{err}_{w'}(\ell) = \frac{1}{n} \sum_{i=1}^n w'_i (v_i - \langle \ell, u_i \rangle)^2$. We have:

$$\left| \frac{4}{w, \ell} \text{err}_D(\ell) = \frac{1}{n} \sum_{i=1}^n w'_i (y_i - \langle \ell, x_i \rangle)^2 + \sum_{i=1}^n (1 - w'_i) \cdot (y_i - \langle \ell, x_i \rangle)^2 \right|.$$

On the other hand, we also have:

$$\left| \frac{4}{w, \ell} \frac{1}{n} \sum_{i=1}^n w'_i (y_i - \langle \ell, x_i \rangle)^2 \leq \sum_{i=1}^n (y'_i - \langle \ell, x'_i \rangle)^2 = \text{err}_{D'}(\ell). \right|$$

Combining the above and using the sum-of-squares vesion of the Hölder’s inequality, we have:

$$\left| \frac{k}{w, \ell} (\text{err}_D(\ell) - \text{err}_{D'}(\ell))^{k'} \right| = \left( \frac{1}{n} \sum_{i=1}^n (1 - w'_i) \cdot (y_i - \langle \ell, x_i \rangle)^2 \right)^{k'} \leq \left( \frac{1}{n} \sum_{i=1}^n (1 - w'_i) \right)^{k'-1} \left( \frac{1}{n} \sum_{i=1}^n (y_i - \langle \ell, x_i \rangle)^k \right) \leq 2^{k'-1} \eta^{k'-1} \left( \frac{1}{n} \sum_{i=1}^n (y_i - \langle \ell, x_i \rangle)^k \right). \tag{5.4}$$

Next, using the sum-of-squares inequality $(a + b)^k \leq 2^k a^k + 2^k b^k$, we have:

$$\left| \frac{k}{\ell} \left\{ \frac{1}{n} \sum_{i=1}^n (y_i - \langle \ell, x_i \rangle)^k \right\} \leq 2^k \left( \frac{1}{n} \sum_{i=1}^n (y_i - \langle \ell', x_i \rangle)^k \right) + 2^k \left( \frac{1}{n} \sum_{i=1}^n (\ell - \ell', x_i)^k \right) \tag{5.5} \right.$$

By certifiable hypercontractivity of $D_x = D$, we have:

$$\left| \frac{k}{\ell} \left\{ \frac{1}{n} \sum_{i=1}^n (\ell - \ell', x_i)^k \right\} \leq C(k)k' \left( \frac{1}{n} \sum_{i=1}^n (\ell - \ell', x_i)^2 \right)^{k'} \right.$$  

Again, by using the sum-of-squares inequality $(a + b)^k \leq 2^k a^k + 2^k b^k$, we have:

$$\left| \frac{k}{\ell} \left\{ \frac{1}{n} \sum_{i=1}^n (\ell - \ell', x_i)^k \right\} \leq 2^{k'} \left( \frac{1}{n} \sum_{i=1}^n (y_i - \langle \ell, x_i \rangle)^2 \right)^{k'} + 2^{k'} \left( \frac{1}{n} \sum_{i=1}^n (y_i - \langle \ell, x_i \rangle)^2 \right)^{k'} \right.$$
Finally, using the sum-of-squares version of Hölder’s inequality again, we have:

\[
\left\| \frac{1}{n} \sum_{i=1}^{n} (y_i - \langle \ell^*, x_i \rangle)^k \right\|_2^k \leq \frac{1}{n} \sum_{i=1}^{n} (y_i - \langle \ell^*, x_i \rangle)^k \right\|
\]

Combining the above with (5.5), we have:

\[
\left\| \frac{1}{n} \sum_{i=1}^{n} (y_i - \langle \ell, x_i \rangle)^k \right\|_2^k \leq O(C(k/2))^k \left( \frac{1}{n} \sum_{i=1}^{n} (y_i - \langle \ell^*, x_i \rangle)^k \right) + O(C(k/2))^k \left( \frac{1}{n} \sum_{i=1}^{n} (y_i - \langle \ell^*, x_i \rangle)^2 \right)^k \right\|
\]

Thus, together with (5.4), we have:

\[
\left\| \frac{1}{n} (\text{err}_D(\ell) - \text{err}_D(\ell^*)) \right\|_2^k \leq \eta^{k-1} \cdot O(C(k/2))^k (\text{err}_D(\ell))^k' + \eta^{k-1} \cdot O(C(k/2))^k \left( \frac{1}{n} \sum_{i=1}^{n} (y_i - \langle \ell^*, x_i \rangle)^k \right) \right\|
\]

Using the sum of squares inequality \( \delta^k a^k \leq (2\delta)^k (a - b)^k + (2\delta)^k b^k \) for any \( a, b \) and even \( k \), and applying it with \( a = \text{err}_D(\ell), b = \text{err}_D(\ell^*) \) and \( \delta = \eta^{k-1} \cdot O(C(k/2))^k \) and rearranging, we have:

\[
\left\| \frac{1}{n} (1 - \delta) (\text{err}_D(\ell) - \text{err}_D(\ell^*)) \right\|_2^k \leq \eta^{k-1} \cdot O(C(k/2))^k (\text{err}_D(\ell))^k' + \eta^{k-1} \cdot O(C(k/2))^k \left( \frac{1}{n} \sum_{i=1}^{n} (y_i - \langle \ell^*, x_i \rangle)^k \right) \right\|
\]

For \( \delta < 0.9 \), this implies:

\[
\left\| \frac{1}{n} (\text{err}_D(\ell) - \text{err}_D(\ell^*)) \right\|_2^k \leq \eta^{k-1} \cdot O(C(k/2))^k (\text{err}_D(\ell))^k' + \eta^{k-1} \cdot O(C(k/2))^k \left( \frac{1}{n} \sum_{i=1}^{n} (y_i - \langle \ell^*, x_i \rangle)^k \right) \right\|
\]

This completes the proof.

\[\square\]

5.2.3 Bounding the Generalization Error

In this section we prove Lemma 5.4. The lemma follows from standard concentration inequalities combined with standard generalization bounds for linear regression.

Proof of Lemma 5.4(1). Let \( \ell^* \) be a linear function of bit complexity at most \( B \) that achieves the optimum least squares regression error on \( D \). We will first show that \( \tilde{\text{err}}_{\text{SOS}} \leq \text{err}_D(\ell^*) \) by exhibiting a feasible pseudo-density. To see this, consider the point-mass density, \( \tilde{\mu} \), supported on the following point: \((w, \ell^*, X')\) where \( w_i = 1 \) if \( (x_i, y_i) = (u_i, v_i) \) and 0 otherwise (i.e., \( w_i = 1 \) if and
only if the \( i \)'th example was uncorrupted) and \((x'_i, y'_i) = (x_i, y_i)\) for all \( i \in [n] \). Clearly, \( \tilde{u} \) is a feasible solution to the optimization program \( \text{5.2} \) and \( \tilde{u} \in [\mathcal{Y}] \). Now, the \( \text{5.2} \) follows by hypercontractivity. The claim now follows. \( \square \)

**Theorem 5.9.** Let \( \mathcal{D} \) be a distribution on \( \mathbb{R}^d \times [M, M] \) for some positive real \( M \) and let \((x_1, y_1), \ldots, (x_n, y_n)\) be independent samples from \( \mathcal{D} \) for \( n > n_0 = O(M^2B \log (1/\delta)/\epsilon^2) \).

Then, with probability at least \( 1 - \delta \) over the draw of \( X \), for every \( \ell \in \mathcal{H} \),

\[
\mathbb{E}_{(x,y) \sim \mathcal{D}} (y - h(x))^2 \leq (1/n) \sum_{i=1}^n (y_i - h(x_i))^2 + \epsilon.
\]

**5.3 Robust L1 Regression**

In this section, we present our robust L1 regression algorithm. Our main goal is the following theorem.

**Theorem 5.9.** Let \( \mathcal{D} \) be an arbitrary distribution on \( \mathbb{R}^d \times \mathcal{Y} \) for \( \mathcal{Y} \subseteq [-M, M] \) for a positive real \( M \). Let \( \kappa \) be the ratio of the maximum to the minimum eigenvalue of the covariance matrix of \( \mathcal{D} \), the marginal of \( \mathcal{D} \) on \( x \). Let \( \text{opt}(\mathcal{D}) \) be the minimum of \( \mathbb{E}_{\mathcal{D}}|y - (\ell, x)| \) over all \( \ell \) that have bit complexity bounded above by \( B \). Let \( \ell^* \) be any such minimizer and \( \eta > 0 \) be an upper bound on the fraction of corruptions.

For any \( \epsilon > 0 \), let \( X \) be an i.i.d. sample from \( \mathcal{D} \) of size \( n \geq n_0 \) for some \( n_0 = O(1/\epsilon^3) \cdot (M^2 \|\ell\|^4_2 + d \log (d)/\eta) \).

Then, with probability at least \( 1 - \epsilon \) over the draw of the sample \( X \), given any \( \eta \)-corruption \( U \) of \( X \) and \( \eta \) as input, there’s a polynomial time algorithm (Algorithm \( \text{5.11} \)) that outputs a function \( f : \mathbb{R}^d \times \mathbb{R} \) such that:

\[
\mathbb{E}_{(x,y) \sim \mathcal{D}} |y - f(x)| < \text{opt}(\mathcal{D}) + O(\sqrt{\kappa \eta})(\sqrt{\mathbb{E}_{\mathcal{D}}y^2} + \sqrt{\mathbb{E}_{\mathcal{D}}(y - (\ell^*, x))^2}) + \epsilon.
\]

**Remark 5.10.** The lower bound example in Lemma \( \text{6.2} \) also shows that the above bound is tight in the dependence on \( \eta \) and \( \kappa \).
As in the previous section, our algorithm will find pseudo-distributions satisfying a set of polynomial inequalities that encode the hypotheses of the robust certifiability lemma and the “error” polynomial.

Let $\mathcal{A}_{U,\eta,Q}$ be the following system of polynomial equations:

$$
\begin{align*}
\forall i & \in [n], \quad \sum_{i=1}^{n} w_i = (1 - \eta) \cdot n, \\
\forall i & \in [n], \quad w_i^2 = w_i, \\
\forall i & \in [n], \quad w_i \cdot (u_i - x_i') = 0, \\
\forall i & \in [n], \quad w_i \cdot (v_i - y_i') = 0, \\
\|\ell\|_2^2 & \leq Q^2, \\
\forall i & \in [n], \quad \tau_i' \geq (y_i' - \langle \ell, x_i' \rangle), \\
\forall i & \in [n], \quad \tau_i' \geq -(y_i' - \langle \ell, x_i' \rangle).
\end{align*}
$$

(5.10)

This system of equations takes as parameters the input sample $U$ and a bound on the fraction of outliers $\eta$.

We can now describe our algorithm for robust L1 regression.

---

**Algorithm 5.11 (Algorithm for Robust L1 Linear Regression via Sum-of-Squares).**

**Given:** An $\eta$-corruption $U$ of a labeled sample $X$ of size $n$ from an arbitrary distribution $D$. The Euclidean norm of the best fitting L1 regression hypothesis for $D$, $Q$.

**Operation:**

1. find a level-4 pseudo-distribution $\tilde{\mu}$ that satisfies $\mathcal{A}_{U,\eta,Q}$ and minimizes $\left(\frac{1}{n} \sum_{i=1}^{n} \tau_i\right)^2$.
2. Return $\hat{\ell} = \mathbb{E}_{\tilde{\mu}} \ell$.

---

**Analysis of Algorithm.** The plan of this subsection and the proofs are essentially analogous to the ones presented in the previous subsection. We will split the analysis into bounding the optimization and generalization errors as before. Let $\text{opt}_{\text{SoS}}$ be the L1 error of $\hat{\ell}$ output by Algorithm 5.11 and let $\ell^*$ be the optimal hypothesis for $D$.

**Lemma 5.12 (Bounding the Optimization Error).** Under the assumptions of Theorem 5.9 (and following the above notations),

$$
\text{err}_D(\hat{\ell}) \leq \text{opt}_{\text{SoS}} + 2\kappa^{1/2} \eta^{1/2} \sqrt{\sum_{i=1}^{n} y_i^2} + 2\kappa^{1/2} \eta^{1/2} \text{err}_D(\ell^*).
$$

**Lemma 5.13 (Bounding the Generalization Error).** Under the assumptions of Theorem 5.9, with probability at least $1 - \varepsilon$,

1. $\text{opt}_{\text{SoS}} \leq \text{opt}(D) + \varepsilon$.
2. $\text{err}_D(\hat{\ell}(M)) \leq \text{err}_D(\hat{\ell}) + \varepsilon$.
3. $\frac{1}{n} \sum_{i=1}^{n} y_i^2 \leq \mathbb{E}_D y^2$. 

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The proofs of the above two lemmas are entirely analogous to the ones presented in the previous section. The main technical ingredient as before is a SoS version of the robust certifiability result. Since this is the only technical novelty in this subsection, we present the statement and proof of this result below and omit the other proofs.

**Lemma 5.14 (SoS Proof of Robust Certifiability for L1 Regression).** Let $X$ be a collection of $n$ labeled examples in $\mathbb{R}^d$ such that $D$, the uniform distribution on $x_1, x_2, \ldots, x_n$ has 2nd moment matrix with all eigenvalues within a factor $\kappa$ of each other. Let $U$ be an $\eta$-corruption of $X$.

Let $w, \ell, X', \tau'$ satisfy the set of system of polynomial equations $A_{U, \eta, Q}$. Let $\tau_i$ satisfy $\tau_i^2 = (y_i - \langle \ell, x_i \rangle)^2$ and $\tau_i \geq 0$ for every $i$. Then, for any $\ell' \in \mathbb{R}^d$ such that $\|\ell\|_2 \leq Q$,

$$A_{U, \eta, Q} \left[ \frac{w, \ell', X'}{4} \right] = \frac{1}{n} \sum_{i=1}^{n} \tau_i \leq \frac{1}{n} \sum_{i=1}^{n} \tau_i' + 2\kappa^{1/2}\eta^{1/2} \sqrt{\frac{1}{n} \sum_{i=1}^{n} y_i^2} + 2\kappa^{1/2}\eta^{1/2} \cdot \sqrt{\frac{1}{n} \sum_{i=1}^{n} (y_i - \langle \ell', x_i \rangle)^2}.$$ 

**Proof of Lemma 5.14.** For every $i \in [n]$, define $w_i' = w_i$ iff $(x_i, y_i)$ is uncorrupted in $U$. Then, observe that $\sum_i w_i' = s$ for $s \geq (1-2\epsilon)n$ and that $\frac{1}{n} \sum_i (w_i' - w_i) = 0$. Then,

$$\left| \frac{1}{n} \sum_i (1 - w_i')^2 \right| \leq 2\epsilon.$$ 

Thus, we have:

$$\frac{1}{n} \sum_i \tau_i = \frac{1}{n} \sum_i w_i' \tau_i + \frac{1}{n} \sum_i (1 - w_i') \cdot \tau_i$$

Further, it’s easy to verify by direct expansion that:

$$\{ w_i(x_i - x_i') = 0, w_i(y_i - y_i') = 0 \mid \forall i \} \left| \frac{4}{w_i} \right| \{ w_i' \tau_i' = 0 \mid \forall i \}$$

As a result, we have:

$$\frac{4}{w_i, \ell', \tau'} \left| \frac{1}{n} \sum_i \tau_i = \frac{1}{n} \sum_i w_i' \tau_i' + \frac{1}{n} \sum_i (1 - w_i') \cdot \tau_i \right.$$ 

For brevity, let’s write $\text{err}_D(\ell) = \sum_{i=1}^{n} \sum_{i=1}^{n} \tau_i$ and $\text{err}_{D'}(\ell) = \sum_{i=1}^{n} \sum_{i=1}^{n} \tau_i'$. Then, we have: Using the sum-of-squares version of the Cauchy-Shwarz inequality, we have:

$$\left| \frac{4}{w_i, \ell', \tau'} (\text{err}_D(\ell) - \text{err}_{D'}(\ell))^2 \right| = \left( \frac{1}{n} \sum_{i=1}^{n} (1 - w_i') \cdot \tau_i \right)^2 \leq \left( \frac{1}{n} \sum_{i=1}^{n} (1 - w_i') \right)^2 \frac{1}{n} \sum_{i=1}^{n} (y_i - \langle \ell, x_i \rangle)^2 \leq \epsilon \frac{1}{n} \sum_{i=1}^{n} (y_i - \langle \ell, x_i \rangle)^2.$$ 

(5.11)

Next, we have:

$$\left| \frac{4}{w_i} \right| \left\{ \frac{1}{n} \sum_{i=1}^{n} (y_i - \langle \ell, x_i \rangle)^2 \leq 2 \frac{1}{n} \sum_{i=1}^{n} (y_i - \langle \ell', x_i \rangle)^2 + 2 \frac{1}{n} \sum_{i=1}^{n} (\ell - \ell', x_i)^2 \right\}.$$
Further, we also have:

\[
\left\{ \frac{1}{n} \sum_{i=1}^{n} (\ell - \ell^*, x_i)^2 \right\} \leq \frac{2}{n} \sum_{i=1}^{n} (\ell, x_i)^2 + \frac{2}{n} \sum_{i=1}^{n} (\ell^*, x_i)^2 \]

Using that for any PSD matrix \( A \), we have the SoS inequality \( \|x\|^2_2 A_{\text{min}} \leq x^T A x \leq \|x\|^2_2 A_{\text{max}} \) where \( A_{\text{max}} \) and \( A_{\text{min}} \) are the largest and smallest singular values of \( A \), respectively, we have:

\[
\left\{ \frac{1}{n} \sum_{i=1}^{n} (\ell, x_i)^2 \right\} \leq \kappa \frac{1}{n} \sum_{i=1}^{n} (\ell^*, x_i)^2 \]

Finally, we also have:

\[
\left\{ \frac{1}{n} \sum_{i=1}^{n} (\ell^*, x_i)^2 \right\} \leq \frac{1}{n} \sum_{i=1}^{n} (y_i - (\ell^*, x_i))^2 + \frac{2}{n} \sum_{i=1}^{n} y_i^2 \]

Combining the above inequalities with (5.11) yields the lemma. \( \Box \)

6 Statistical Limits of Outlier-Robust Regression

Here we exhibit statistical lower bounds for what can be achieved for outlier-robust regression. In particular, these simple examples illustrate strong separations between regression and regression in the presence of contamination and also demonstrate that several of the assumptions we make are necessary.

Necessity of Distributional Assumptions. A classical result in analysis of regression is consistency of the least-squares estimator when the labels are bounded. Concretely, let \( D \) be a distribution on \( \mathbb{R}^d \times [-1, 1] \). Let \((x_1, y_1), \ldots, (x_n, y_n)\) be i.i.d samples from \( D \). Let \( \hat{\ell} = \arg \min_{\ell} \sum_{i=1}^{n} (y_i - (\ell, x_i))^2 \), be the least-squares estimator. Then, (say, via Theorem 11.3 in [GKKW02]) \( \text{err}_D(\hat{\ell}) \leq O(d/n) + 8 \cdot \arg \min_{\ell} \text{err}_D(\ell) \).

In particular, in the realizable-case, i.e., when \((x, y) \sim D \) satisfies \( y = (\ell^*, x) \), the error of the least-squares estimator approaches zero as \( n \to \infty \) irrespective of the marginal distribution \( D_X \).

Given the above bound, it is natural to ask if we could get a similar marginal-distribution independent bound in the presence of outliers: Does there exist an estimator which achieves error \( h(\eta) \) with \( \eta \)-fraction of corruptions for some function \( h : \mathbb{R} \to \mathbb{R} \) with \( h \to 0 \) as \( \eta \to 0 \)? It turns out that this is statistically impossible in the presence of sample contamination.

**Lemma 6.1.** There is a universal constant \( c > 0 \) such that the following holds. For all \( \eta > 0 \), there is no algorithm that given \( \eta \)-corrupted samples\(^6\) \((x, y)\) from distributions \( D \) on \( \mathbb{R}^d \times [-1, 1] \) finds a hypothesis vector \( \ell \in \mathbb{R}^d \) such that \( \mathbb{E}[\text{err}_D(\ell)] < c \).

**Proof.** Suppose there is an algorithm as above. Let \( \delta = \eta/(2 - \eta) \) and let \( \kappa \geq 2 \) be sufficiently large to be chosen later. Let \( D \) be the distribution of the random variable on \( \mathbb{R}^2 \times \mathbb{R} \) samples as follows: 1) Sample \( \alpha \) uniformly at random from \([-1, 1] \); 2) With probability \( 1 - \eta \) output \((\alpha, \alpha), \alpha \); 3) With probability \( \eta \) output \((\kappa \cdot \alpha, \alpha), \alpha \). Note that for \((x, y) \sim D \), \( y = (\ell, x) \) for \( \ell = (0, 1) \).

\(^6\)The lemma also holds in the weaker Huber’s contamination model even though we do not study this model in this work.
Similarly, let $\mathcal{D}'$ be the distribution of the random variable on $\mathbb{R}^2 \times \mathbb{R}$ samples as follows: 1) Sample $\alpha$ uniformly at random from $[-1, 1]$; 2) With probability $1 - \eta$ output $((a,\alpha),a)$; 3) With probability $\eta$ output $((a,\kappa \cdot \alpha),\alpha)$. Note that for $(x',y') \sim \mathcal{D}, y' = \langle \ell', x' \rangle$ for $\ell = (1, 0)$.

It follows from a few elementary calculations that for any $w \in \mathbb{R}^2$, $\text{err}_{\mathcal{D}}(w) + \text{err}_{\mathcal{D}'}(w) \geq \Omega(1) \cdot \sqrt{\frac{\kappa}{1 - \eta}}$.

It follows that for some universal constant $c > 0$, and $\kappa = 1/\sqrt{\eta}$, $\min(\text{err}_{\mathcal{D}}(w), \text{err}_{\mathcal{D}'}(w)) \geq c$.

Finally, let $\mathcal{D}''$ be the distribution of the random variable sampled as follows: 1) Sample $\alpha$ uniformly at random from $[-1, 1]$; 2) With probability $1 - \delta$ output $((a,\alpha),\alpha)$; 3) With probability $\delta/2$ output $((\kappa \cdot \alpha,\alpha),\alpha)$; 4) With probability $\delta/2$ output $((a,\kappa \cdot \alpha),\alpha)$.

Note that $\mathcal{D}''$ can be obtained by a $(\eta/2)$-corruption of $\mathcal{D}$ as well as $\mathcal{D}'$. On the other hand, for any $w \in \mathbb{R}^2$, one of $\text{err}_{\mathcal{D}}(w), \text{err}_{\mathcal{D}'}(w)$ is at least $c$ where $c$ is the constant from the previous lemma. Thus no algorithm can output a good hypothesis for both $\mathcal{D}$ or $\mathcal{D}'$. The claim now follows. □

**Optimality of Our Algorithm for Hypercontractive Distributions.** We also show that the additive $O(\sqrt{\eta})$ dependence in our bounds is necessary for 4-hypercontractive distributions.

**Lemma 6.2.** There is a universal constants $c, C > 0$ such that the following holds. For all $\eta > 0$, there is no algorithm that given $\eta$-corrupted samples from distributions $\mathcal{D}$ on $\mathbb{R}^d \times \mathbb{R}$ with (1) $\mathcal{D}_X$ (C,4)-certifiably hypercontractive, and (2) $E[y^2] \leq C$, finds a hypothesis vector $w \in \mathbb{R}^d$ such that $E[\text{err}_{\mathcal{D}}(w)] = (1 + o(\sqrt{\eta})) \cdot (\min_{c \in \mathbb{R}^d} \text{err}_{\mathcal{D}}(c^*)) + o(\sqrt{\eta})$.

The above should be compared with Theorem 1.3 which for $c D$ as in the lemma efficiently finds a $\ell$ with $E[\text{err}_{\mathcal{D}}(\ell)] = (1 + O(\sqrt{\eta})) \cdot (\min_{c \in \mathbb{R}^d} \text{err}_{\mathcal{D}}(c^*)) + O(\sqrt{\eta})$. Thus Theorem 1.3 is optimal for the class of 4-hypercontractive distributions up to multiplicative constants.

**Proof Sketch.** Fix some $\delta = \Theta(\eta)$ to be chosen later. For brevity, let $U$ be the uniform distribution on $[-1, 1]$ and let $Z$ be the distribution of the random variable sampled as follows: 1) With probability $1 - \delta$, sample $\alpha$ uniformly at random from $[-1, 1]$ and output $\eta \cdot \alpha$; 2) With probability $\delta/2$ output $1/\delta^{1/4}$; 3) With probability $\delta/2$ output $-1/\delta^{1/4}$.

It follows from standard calculations that $Z$ is $(C_1, 4)$-certifiably hypercontractive for a fixed constant $C_1$. Further, if we let $X = (U,Z)$ for $U,Z$ generated independently, then $X$ is also $(C_1, 4)$-certifiably hypercontractive; we skip the details from this sketch.

Now let $\ell = (1, 1)$ and $\ell' = (1, -1)$. Let $\mathcal{D}$ be the distribution of $(X, \langle \ell, X \rangle + U)$ for $U$ independent from $X$; similarly, let $\mathcal{D}'$ be the distribution of $(X, \langle \ell', X \rangle + U)$.

We first claim that $||\mathcal{D} - \mathcal{D}'||_{TV} = O(\delta)$. To see this, let us consider a coupling between $\mathcal{D}, \mathcal{D}'$ obtained by choosing the same $X$ for both. Then, with probability $1 - \delta$ over $X, |\langle \ell, X \rangle - \langle \ell', X \rangle| \leq 2\delta$; further, whenever this happens, the statistical distance between $\langle \ell, X \rangle + U$ and $\langle \ell', X \rangle + U$ is at most $O(\delta)$. Therefore, $||\mathcal{D} - \mathcal{D}'||_{TV} = O(\delta)$.

Finally, note that

$$E[(\langle \ell, X \rangle - \langle \ell', X \rangle)^2] = 4 E[Z^2] \geq 4\delta(1/\delta^{1/4})^2 = 4\sqrt{\delta}.$$

Therefore, for any $w$,

$$E[(\langle w, X \rangle - \langle \ell, X \rangle)^2] + E[(\langle w, X \rangle - \langle \ell', X \rangle)^2] \geq 4\sqrt{\delta}.$$

In particular, for any $w$, max $(E[(\langle w, X \rangle - \langle \ell, X \rangle)^2], E[(\langle w, X \rangle - \langle \ell', X \rangle)^2]) \geq 2\sqrt{\delta}$. Now, note that for any $w$, $\text{err}_{\mathcal{D}}(w) = E[U^2] + E[(\langle w, X \rangle - \langle \ell, X \rangle)^2]$ and a similar equality holds for $\text{err}_{\mathcal{D}'}(w)$.
Combining the above we get that for any $w$, \( \max(\text{err}_D(w), \text{err}_{D'}(w)) \geq \mathbb{E}[U^2] + 2\sqrt{\delta} \). Note that \( \min_{\ell} \text{err}_D(\ell) = \mathbb{E}[U^2] \) and the same holds for $D'$ as well. Finally, as \( \|D - D'\|_{TV} = O(\delta) \leq \eta \), by setting $\delta = \Theta(\eta)$ appropriately. Therefore, $D, D'$ could be $\eta$-corruptions of each other and no $w$ could get error better than $\mathbb{E}[U^2] + 2\sqrt{\delta} = \mathbb{E}[U^2] + \Omega(\sqrt{\eta})$ on both of them. The claim now follows. \(\square\)

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A Outlier-Robust Polynomial Regression

Our arguments also extend straightforwardly to get similar guarantees for polynomial regression. We elaborate on these next.

The following extends the definition of hypercontractivity to polynomials.

**Definition A.1** (Certifiable polynomial hypercontractivity). For a function $C: [k] \to \mathbb{R}_+$, we say a distribution $D$ on $\mathbb{R}^d$ is $k$-certifiably $(C, t)$-hypercontractive if for every $r$ such that $2rt \leq k$, there is a degree $k$ sum of squares proof of the following inequality in variable $P$ where $p$ stands for $\langle P, x^{\otimes t} \rangle$.

$$
\mathbb{E}_D p(x)^{2r} \leq \left( (C(r) \mathbb{E}_D p(x))^2 \right)^r.
$$

Many natural distributions satisfy certifiably hypercontractivity [KOTZ14] for polynomials such as gaussian distributions and the product distributions on the hypercube $\{0, 1\}^n$ with all coordinate marginals in $(0, 1)$. Our results will apply to all such distributions.

Next, we state an extension of our robust certification lemma for polynomial regression. The proof is essentially the same as that of .

**Lemma A.2** (Robust Generalization for polynomial regression). Fix $k, t \in \mathbb{N}$ and let $\mathcal{D}, \mathcal{D}'$ be distributions on $\mathbb{R}^d \times \mathbb{R}$ such that $\| \mathcal{D} - \mathcal{D}' \|_{TV} \leq \epsilon$ and the marginal $\mathcal{D}_x$ of $\mathcal{D}$ on $x$ is $k$-certifiably $(C, t)$-hypercontractive for some $C: [k] \to \mathbb{R}_+$ and for some even integer $k \geq 4$.

Then, for any degree at most $t$ polynomials $p, p^* : \mathbb{R}^d \to \mathbb{R}$, and any $\epsilon$ such that $2C(k/2)\epsilon^{1-2/k} < 0.9$, we have:

$$
\text{err}_{\mathcal{D}}(p) \leq (1 + O(C(k/2))\epsilon^{1-2/k}) \cdot \text{err}_{\mathcal{D}'}(p) + O(C(k/2))\epsilon^{1-2/k} \cdot \left( \mathbb{E}_{\mathcal{D}} (y - p^*(x))^2 \right)^{2/k}.
$$

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Theorem A.3. Let $\mathcal{D}$ be a distribution on $\mathbb{R}^d \times [-M, M]$ for some positive real $M$ such the marginal on $\mathbb{R}^d$ is $(C, k)$-certifiably hypercontractive distribution for degree $t$ polynomials. Let $\text{opt}_B(\mathcal{D}) = \min_p \mathbb{E}_{\mathcal{D}}[(y - p(x))^2]$ where the minimum is over all polynomials $p$ of degree $t$ and bit complexity $B$. Let $p^*$ be any such minimizer.

Fix any even $k \geq 4$ and any $\varepsilon > 0$. Let $X$ be an i.i.d. sample from $\mathcal{D}$ of size $n \geq n_0 = \text{poly}(d^k, B, M, 1/\varepsilon)$. Then, with probability at least $1 - \varepsilon$ over the draw of the sample $X$, given any $\eta$-corruption $U$ of $X$ and $\eta$ as input, there is a polynomial time algorithm (Algorithm 5.2) that outputs a $\ell \in \mathbb{R}^d$ such that for $C = C(k/2),

$$\text{err}\mathcal{D}(p, M) < (1 + O(C)\eta^{1-2/k}) \text{opt}_B(\mathcal{D}) + O(C)\eta^{1-2/k} \left( \mathbb{E}_{\mathcal{D}}(y - p^*(x))^k \right)^{2/k} + \varepsilon.$$