Integrable generalized Heisenberg ferromagnet equations in 1+1 dimensions: reductions and gauge equivalence

Zhanna Sagidullayeva$^{1,2,*}$, Kuralay Yesmakhanova$^{1,2,†}$, Nurzhan Serikbayev$^{1,2,‡}$, Gulgassyl Nugmanova$^{1,2,§}$, Koblandy Yerzhanov$^{1,2,¶}$, and Ratbay Myrzakulov$^{1,2,‖}$

$^1$Ratbay Myrzakulov Eurasian International Centre for Theoretical Physics, Nur-Sultan, 010009, Kazakhstan
$^2$Eurasian National University, Nur-Sultan, 010008, Kazakhstan

Abstract

The present work addresses the study and characterization of the integrability of some generalized Heisenberg ferromagnet equations (GHFE) in 1+1 dimensions. Lax representations for these GHFE are successfully obtained. The gauge equivalent counterparts of these integrable GHFE are presented.

Key words: Integrable equations, nonlinear Schrödinger equation, Heisenberg ferromagnet equation, Landau-Lifshitz equation, soliton equations, gauge equivalence, derivative nonlinear Schrödinger equation, Kaup-Newell equation, Chen-Lee-Liu equation, Gerdjikov-Ivanov equation, spin systems.

1 Introduction

The Heisenberg ferromagnet equation (HFE)

$$iS_t + \frac{1}{2} [S, S_{xx}] = 0$$

(1.1)

is one of the most fundamental integrable equations in soliton theory. It is well known that some integrable nonlinear evolution equations are related, often in an unexpected way, to each other by means of so-called gauge equivalences. For example, the gauge partner of the HFE is the following nonlinear Schrödinger equation (NLSE) $^{[1]-[2]}

$$iq_t + q_{xx} + 2\epsilon |q|^2 q = 0.$$  

(1.2)

Gauge equivalences between integrable nonlinear equations can be exploited to gain insight into either equation from its gauge partner, for instance by transforming solutions of one equation to solutions of the other. Formally the system can be cast into two gauge equivalent zero curvature conditions for the two sets of Lax operators. Returning to the HFE (1.1) and NLSE (1.2) we can note that they play important role in modern nonlinear physics and mathematics. They admit several integrable generalizations in 1+1 and 2+1 dimensions (see, e.g. $^{[3]-[7]}$ and references therein). In this paper, for
example among the integrable generalizations of HFE and NLSE, we are interested in the study of their derivative versions. In particular, for the NLSE (1.2) there exist the following three celebrated derivative nonlinear Schrödinger equations (DNLSE) of this kind, the DNLSE-I, DNLSE-II and the DNLSE-III (see, e.g. [11] and references therein):

i) the Kaup-Newell equation (KNE) or DNLSE-I [8]

\[ iq_t - q_{xx} - i|q|^2q_x = 0, \]  

(1.3)

ii) the Chen-Lee-Liu equation (CLLE) or DNLSE-II [9]

\[ iq_t - q_{xx} - i|q|^2q_x = 0, \]  

(1.4)

iii) the Gerdjikov-Ivanov equation (GIE) or DNLSE-III [10]

\[ iq_t - q_{xx} + iq^2\bar{q}_x - 0.5|q|^4q = 0. \]  

(1.5)

Through the \( U(1) \) - gauge transformation, these three DNLSE can be related to each other. In fact, if \( q(x, t) \) is a solution of the KNE (1.2), it is easy to find that the new field \( p(x, t) \) defined as (see, e.g. [11] and references therein)

\[ p(x, t) = q(x, t)e^{0.5\delta\zeta(x,t)}, \]  

(1.6)

with

\[ \zeta_x = |q|^2; \quad \zeta_t = i(q\bar{q}_x - \bar{q}q_x) + 1.5|q|^4, \quad \delta = \text{const} \]  

(1.7)

satisfies the CLLE (1.3) for \( \delta = 1 \), and the GIE (1.4) for \( \delta = 2 \).

In the modern mathematics and physics, gauge transformations constitute an useful tool to link integrable nonlinear evolution equations (INLEE) in soliton theory, since they provide some transformations (like Darboux, Bäcklund) between those INLEE as well as the relation of their associated linear problems (Lax representations) (see, e.g. [12]-[37] and references therein). In this paper, we exploit this gauge transformation property to construct the gauge equivalent counterpart of the Zhaidary equation and to find its integrable reductions and generalizations.

This paper is organized as follows. In section 2 we study the Zhaidary equation (ZE). In section 3, we construct the gauge equivalent partner of the ZE. The section 4, provides the particular reductions of the ZE and their gauge equivalents and gives the associated new Lax operators in the explicit form. Some integrable generalizations of the ZE is presented in section 5. Our conclusions are stated in section 6.

2 Zhaidary equation

The anisotropic Zhaidary equation (ZE) is given by

\[ (1 + 2\beta(c\beta + d))S_t - S \wedge S_{xt} - us_x + 4cwS_x - S \wedge JS = 0, \]  

(2.1)

\[ u_x + \frac{1}{2}(S^2)_t = 0, \]  

(2.2)

\[ w_x + \frac{1}{4(2\beta c + d)^2}(S^2)_t = 0, \]  

(2.3)
where \( \mathbf{S} = (S_1, S_2, S_3) \) is the unit spin vector, \( S^2 = 1, J = \text{diag}(J_1, J_2, J_3) \) \((J_1 \leq J_2 \leq J_3)\), \( u \) and \( v \) are scalar functions (potentials). The isotropic ZE has the form

\[
(1 + 2\beta(c\beta + d)) \mathbf{S}_t - \mathbf{S} \wedge \mathbf{S} = u \mathbf{S}_x - 2\beta(c\beta + d) \mathbf{S}_t + 4cw \mathbf{S}_x = 0,
\]

\[
u_x + \frac{1}{2} (S_x^2)_t = 0,
\]

\[
w_x + \frac{1}{4(2\beta c + d)^2} (S_x^2)_t = 0.
\]

The ZE (2.4)-(2.6) we sometime call the spin Zhaidary equation (sZE). The isotropic sZE is completely integrable that can be solved by the inverse scattering transformation method (IST). It possesses all the basic characters of integrable equations. The corresponding Lax representation (LR) has the form

\[
\Psi_x = U_1 \Psi,
\]

\[
\Psi_t = V_1 \Psi.
\]

Here

\[
U_1 = [ic(\lambda^2 - \beta^2) + id(\lambda - \beta)]S + \frac{c(\lambda - \beta)}{2c\beta + d} SS_x,
\]

\[
V_1 = \frac{1}{1 - 2c\lambda^2 - 2d\lambda} \{2c(\lambda^2 - \beta^2) + 2d(\lambda - \beta)]B + \lambda^2 F_2 + \lambda F_1 + F_0\},
\]

where

\[
F_2 = -4ic^2 wS, \quad F_1 = -4icdw S - \frac{4c^2}{2c\beta + d} wSS_x - \frac{ic}{2c\beta + d} S \{(SS)_t - [SS, B]\},
\]

\[
F_0 = -\beta F_1 - \beta^2 F_2, \quad B = 0.25([S, S_t] + 2iuS), \quad S = \mathbf{S} \cdot \sigma.
\]

If \( \beta = 0 \), the ZE takes the form

\[
S_t - \mathbf{S} \wedge \mathbf{S} = u \mathbf{S}_x + 4cw \mathbf{S}_x = 0,
\]

\[
u_x + \frac{1}{2} (S_x^2)_t = 0,
\]

\[
w_x + \frac{1}{4d^2} (S_x^2)_t = 0.
\]

or

\[
S_t - \mathbf{S} \wedge \mathbf{S}_t + (2cd^{-2} - 1)u \mathbf{S}_x = 0,
\]

\[
u_x + \frac{1}{2} (S_x^2)_t = 0.
\]

### 3 Gauge equivalent counterpart

Let us find the gauge equivalent counterpart of the ZE (2.4)-(2.6). For this aim, consider the gauge transformation

\[
\Phi = g \Psi, \quad g = \Phi|_{\lambda = \beta}, \quad S = g^{-1} \sigma_3 g.
\]

Then the new function \( \Phi \) satisfies the following set of the linear equations

\[
\Phi_x = U_2 \Phi,
\]

\[
\Phi_t = V_2 \Phi.
\]
where
\[ U_2 = i(c\lambda^2 + d\lambda)\sigma_3 + (2c\lambda + d)Q, \quad (3.4) \]
\[ V_2 = \frac{1}{1 - 2c\lambda^2 - 2d\lambda}(\lambda^2B_2 + \lambda B_1 + B_0). \quad (3.5) \]

Here
\[ B_2 = -4ic\sigma_3, \quad B_1 = -4icdv\sigma_3 - 2ic\sigma_3Q_t - 8c^2vQ, \quad B_0 = \frac{d}{2c}B_1 - \frac{d^2}{4c^2}B_2, \quad (3.6) \]
and
\[ Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad r = \epsilon \bar{q}, \quad \epsilon = \pm 1. \quad (3.7) \]

The compatibility condition
\[ U_2t - V_2x + [U_2, V_2] = 0 \quad (3.8) \]
gives the following equation
\[ iq_t - q_x t + 4ic(vq)_x - 2d^2vq = 0, \quad (3.9) \]
\[ i\bar{r}_t + r_x t + 4ic(\bar{v}r)_x + 2d^2vr = 0, \quad (3.10) \]
\[ v_x - (rq)_t = 0, \quad (3.11) \]
which is called the qZE that is the q-form of the Zhaidary equation.

4 Integrable reductions

The ZE admits several integrable reductions. Let us present these particular cases.

4.1 Kuralay equation

First, we assume that \( c = 0 \). Then from the ZE (2.4)-(2.6) we obtain the following Kuralay equation (KE):
\[ (1 + 2\beta d)S_t - S \wedge S_{xt} - uS_x = 0, \quad (4.1) \]
\[ u_x + \frac{1}{2}(S^2)_t = 0. \quad (4.2) \]

It is the spin form of the KE (sKE). If \( \beta = 0 \), this sKE takes the form
\[ S_t - S \wedge S_{xt} - uS_x = 0, \quad (4.3) \]
\[ u_x + \frac{1}{2}(S^2)_t = 0. \quad (4.4) \]

The KE is integrable by IST method. The corresponding LR looks like
\[ \Psi_x = U_3\Psi, \quad (4.5) \]
\[ \Psi_t = V_3\Psi. \quad (4.6) \]

Here
\[ U_3 = id(\lambda - \beta)S, \quad S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \quad S^2 = I, \quad S^\pm = S_1 \pm iS_2, \quad (4.7) \]
\[ V_3 = \frac{1}{1 - 2d\lambda}\{2d(\lambda - \beta)]B + \lambda^2F_2 + \lambda F_1 + F_0\}, \quad (4.8) \]
where
\[ F_2 = 0, \quad F_1 = 0, \quad F_0 = 0, \quad B = 0.25([S, S_t] + 2iuS) = 0.25Z. \tag{4.9} \]

Let us now present the gauge equivalent counterpart of the sKE (4.3)-(4.4). As usual, we consider the gauge transformation
\[ \Phi = g\Psi, \quad g = \Phi|_{\lambda=\beta}, \quad S = g^{-1}\sigma_3 g. \tag{4.10} \]
Then the new function \( \Phi \) satisfies the following set of the linear equations
\[ \Phi_x = U_4\Phi, \tag{4.11} \]
\[ \Phi_t = V_4\Phi, \tag{4.12} \]
where
\[ U_4 = [id\lambda\sigma_3 + dQ, \tag{4.13} \]
\[ V_4 = \frac{1}{1 - 2d\lambda}(\lambda^2B_2 + \lambda B_1 + B_0). \tag{4.14} \]
Here
\[ B_2 = 0, \quad B_1 = 0, \quad B_0 = \frac{d}{2}[-4idv\sigma_3 - 2i\sigma_3Q_t - 8cvQ], \tag{4.15} \]
and
\[ Q = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}. \tag{4.16} \]
The compatibility condition
\[ U_{4t} - V_{4x} + [U_4, V_4] = 0 \tag{4.17} \]
is equivalent to the following q-form of the KE (qKE):
\[ iq_t - q_{xt} - 2d^2vq = 0, \tag{4.18} \]
\[ ir_t + r_{xt} + 2d^2vr = 0, \tag{4.19} \]
\[ v_x - (rq)_t = 0. \tag{4.20} \]

### 4.2 Shynaray equation

Our next particular case is when \( d = 0 \). Then from the sZE 2.1)-(2.3) the following spin form of the Shynaray equation (sSE):
\[ (1 + 2c\beta^2)S_t - S \wedge S_{xt} - uS_x + 4cwS_x = 0, \tag{4.21} \]
\[ u_x + \frac{1}{2}(S^2/2)_t = 0, \tag{4.22} \]
\[ w_x + \frac{1}{16c^2\beta^2}(S^2/2)_t = 0. \tag{4.23} \]
As the particular case of the integrable equation, the sSE is also integrable by IST method. The corresponding LR is given by
\[ \Psi_x = U_5\Psi, \tag{4.24} \]
\[ \Psi_t = V_5\Psi. \tag{4.25} \]
Here
\[ U_5 = i c (\lambda^2 - \beta^2) S + \frac{\lambda - \beta}{2 \beta} S S_x, \]  
\[ V_5 = \frac{1}{1 - 2 c \lambda^2} \{ 2 c (\lambda^2 - \beta^2) B + \lambda^2 F_2 + \lambda F_1 + F_0 \}, \]  
where
\[ F_2 = -4 i c^2 w S, \quad F_1 = -\frac{4 c^2}{2 c \lambda} w S S_x - \frac{i c}{2 c \lambda} S \{ (S S_x)_t - [S S_x, B] \}, \]  
\[ F_0 = -\beta F_1 - \beta^2 F_2, \quad B = 0.25 ([S, S_t] + 2 i u S), \quad S = S \cdot \sigma. \]  

Let us find the gauge equivalent counterpart of the sSE (4.21)-(4.23). Consider the gauge transformation
\[ \Phi = g \Psi, \quad g = \Phi|_{\lambda=\beta}, \quad S = g^{-1} \sigma_3 g. \]  
Then the function $\Phi$ satisfies the following set of the linear equations
\[ \Phi_x = U_6 \Phi, \]  
\[ \Phi_t = V_6 \Phi, \]  
where
\[ U_6 = i c \lambda^2 \sigma_3 + 2 c \lambda Q, \]  
\[ V_6 = \frac{1}{1 - 2 c \lambda^2} (\lambda^2 B_2 + \lambda B_1 + B_0). \]  

Here
\[ B_2 = -4 i c \sigma_3, \quad B_1 = -2 i c \sigma_3 Q_t - 8 c^2 v Q, \quad B_0 = 0. \]  

Hence, the compatibility condition of the equations (4.31)-(4.32)
\[ U_6 t - U_6 x + [U_6, V_6] = 0 \]  
gives the following nonlinear evolution equation
\[ i q_t - q_{x x} + 4 i c (v q)_x = 0, \]  
\[ i r_t + r_{x x} + 4 i c (v r)_x = 0, \]  
\[ v_x - (r q)_t = 0. \]  
It is the q-form of the SE or shortly the sSE.

5 Integrable generalizations
In the previous section we have presented some integrable reductions of the ZE. It is interesting to note that the ZE also admits several integrable extensions/generalizations. Now we are going to present some of these integrable generalizations of the ZE.
5.1 Nurshuak equation

One of such integrable generalizations of the ZE (2.4)-(2.6) is the following Nurshuak equation (NE)

\[
iS_t + 2\epsilon_1Z_x + i\epsilon_2(S_{xt} + [S_x, Z])_x + (wS)_x + \frac{1}{\omega}[S, W] = 0,
\]

(5.1)

\[
u_x - \frac{i}{4}tr(S \times [S_x, S_t]) = 0,
\]

(5.2)

\[
w_x - \frac{i}{4}\epsilon_2[tr(S_x^2)]_t = 0,
\]

(5.3)

\[
iW_x + \omega[S, W] = 0,
\]

(5.4)

where

\[
Z = \frac{1}{4}([S, S_t] + 2iuS).
\]

(5.5)

As integrable equation, the NE admits the LR of the form

\[
\Psi_x = U_7\Psi,
\]

(5.6)

\[
\Psi_t = V_7\Psi.
\]

(5.7)

Here

\[
U_7 = -i\lambda S,
\]

(5.8)

\[
V_7 = \frac{1}{1-2\epsilon_1\lambda - 4\epsilon_2\lambda^2}\{(2\epsilon_1\lambda + 4\epsilon_2\lambda^2)Z + \lambda V_1 + \frac{i}{\lambda + \omega}W - \frac{i}{\omega}W\},
\]

(5.9)

where

\[
V_1 = wS + i\epsilon_2(S_{xt} + [S_x, Z]),
\]

(5.10)

\[
W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}, \quad S = \begin{pmatrix} S_3 & S^- \\ S^+ & -S_3 \end{pmatrix}, \quad S^\pm = S_1 \pm is_2.
\]

(5.11)

The compatibility condition \(\Psi_{xt} = \Psi_{tx}\) gives the NE (5.1)-(5.4). The gauge partner of the NE (5.1)-(5.4) has the form

\[
iq_t + \epsilon_1q_{xt} + i\epsilon_2(q_{xt} + [q_x, Z])_x + (wq)_x + (wq)_t - 2i\eta = 0,
\]

(5.12)

\[
ir_t - \epsilon_1r_{xt} + i\epsilon_2(r_{xt} + [r_x, Z])_x + (wr)_x + (wr)_t - 2ik = 0,
\]

(5.13)

\[
v_x + 2\epsilon_1(rq)_t - 2i\epsilon_2(rq_t - rq_{xt}) = 0,
\]

(5.14)

\[
w_x - 2i\epsilon_2(rq)_t = 0,
\]

(5.15)

\[
p_x - 2i\omega p - 2\eta q = 0,
\]

(5.16)

\[
k_x + 2i\omega k - 2\eta r = 0,
\]

(5.17)

\[
\eta_x + rp + kq = 0,
\]

(5.18)

where \(r = \delta_1\tilde{q}, \quad k = \delta_2\tilde{p}, \quad \delta_j = \pm 1\). As the gauge equivalent of the integrable NE, this set of equations (5.12) - (5.18) is also integrable. Its LR reads as

\[
\Psi_x = U_8\Psi,
\]

(5.19)

\[
\Psi_t = V_8\Psi.
\]

(5.20)

Here

\[
U_8 = -i\lambda \sigma_3 + A_0,
\]

(5.21)

\[
V_8 = \frac{1}{1-(2\epsilon_1\lambda + 4\epsilon_2\lambda^2)}\{\lambda B_1 + B_0 + \frac{i}{\lambda + \omega}B_{-1}\},
\]

(5.22)
where

\[ B_1 = w\sigma_3 + 2i\epsilon_2\sigma_3A_0t, \quad A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \tag{5.23} \]

\[ B_0 = -i\frac{v}{2}\sigma_3 + \begin{pmatrix} 0 & i\epsilon_1q_t - \epsilon_2q_{xt} + iwq \\ i\epsilon_1r_t + \epsilon_2r_{xt} - iwr & 0 \end{pmatrix}, \tag{5.24} \]

\[ B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \tag{5.25} \]

### 5.2 Aizhan equation

The spin form of the Aizhan equation is given by

\[ iS_t + i\epsilon_2(S_{xt} + [S_x, Z])_x + (wS)_x + \frac{1}{\omega}[S, W] = 0, \tag{5.26} \]

\[ u_x - \frac{i}{4}tr(S \times [S_x, S_t]) = 0, \tag{5.27} \]

\[ w_x - \frac{i}{4}\epsilon_2[tr(S^2)]_t = 0, \tag{5.28} \]

\[ iW_x + \omega[S, W] = 0. \tag{5.29} \]

The Aizhan equation is one of integrable generalizations of the ZE. The Lax representation of the Aizhan equation has the form

\[ \Psi_x = U_9\Psi, \tag{5.30} \]

\[ \Psi_t = V_9\Psi, \tag{5.31} \]

where

\[ U_9 = -i\lambda S, \tag{5.32} \]

\[ V_9 = \frac{1}{1 - 4\epsilon_2\lambda^2}\{4\epsilon_2\lambda^2Z + \lambda V_1 + \frac{i}{\lambda + \omega}W - \frac{i}{\omega}W\}. \tag{5.33} \]

with

\[ V_1 = wS + i\epsilon_2(S_{xt} + [S_x, Z]), \tag{5.34} \]

\[ W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}. \tag{5.35} \]

The gauge equivalent equation for the Aizhan equation reads as

\[ iq_t + i\epsilon_2q_{xt} - vq + (wq)_x - 2ip = 0, \tag{5.36} \]

\[ ir_t + i\epsilon_2r_{xt} + vr + (wr)_x - 2ik = 0, \tag{5.37} \]

\[ v_x - 2i\epsilon_2(r_xq - rq_x) = 0, \tag{5.38} \]

\[ w_x - 2i\epsilon_2(qr)_t = 0, \tag{5.39} \]

\[ p_x - 2i\omega p - 2\eta q = 0, \tag{5.40} \]

\[ k_x + 2i\omega k - 2\eta r = 0, \tag{5.41} \]

\[ \eta_x + rp + kq = 0. \tag{5.42} \]

For this equation, the corresponding Lax representation is given by

\[ \Phi_x = U_{10}\Phi, \tag{5.43} \]

\[ \Phi_t = V_{10}\Phi, \tag{5.44} \]
where

$$U_{10} = -i\lambda \sigma_3 + A_0,$$

$$V_{10} = \frac{1}{1 - 4\epsilon_2\lambda^2} \{\lambda B_1 + B_0 + \frac{i}{\lambda + \omega} B_{-1}\}. \tag{5.45}$$

Here

$$B_1 = w\sigma_3 + 2i\epsilon_2\sigma_3 A_0 t,$$ \tag{5.47}

$$A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix},$$ \tag{5.48}

$$B_0 = -\frac{i}{2}v\sigma_3 + \begin{pmatrix} 0 & -\epsilon_2 q_{xt} + iwq \\ \epsilon_2 r_{xt} - iwr & 0 \end{pmatrix},$$ \tag{5.49}

$$B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}. \tag{5.50}$$

### 5.3 Zhanbota equation

In this section, we consider the Zhanbota equation. It has the form

$$S_t - S \land S_{xt} - wS_x - \frac{1}{\omega} S \land W = 0,$$ \tag{5.51}

$$u_x + S \times (S_x \land S_t) = 0,$$ \tag{5.52}

$$W_x - \omega S \land W = 0. \tag{5.53}$$

The Zhanbota equation is also one of integrable generalization of the ZE. The matrix form of the Zhanbota equation reads as

$$iS_t + \frac{1}{2} [S, S_{xt}] + iuS_x + \frac{1}{\omega} [S, W] = 0,$$ \tag{5.54}

$$u_x - \frac{i}{4} tr(S[S_x, S_t]) = 0,$$ \tag{5.55}

$$iW_x + \omega [S, W] = 0. \tag{5.56}$$

where, $S = S_i\sigma_i$, $W = W_i\sigma_i$, ($i = 1, 2, 3$) and $\omega$ is a constant parameter. The $W = (W_1, W_2, W_3)$ is the vector potential. The Zhanbota equation possesses the following Lax representation:

$$\Psi_x = U_{11}\Psi,$$ \tag{5.57}

$$\Psi_t = V_{11}\Phi.$$ \tag{5.58}

Here, the matrix operators $U_{11}$ and $V_{11}$ have the forms

$$U_{11} = -i\lambda S,$$ \tag{5.59}

$$V_{11} = \frac{1}{1 - 2\lambda} \{\lambda V_1 + \frac{i}{\lambda + \omega} W - \frac{i}{\omega} W\},$$ \tag{5.60}

where

$$V_1 = 2Z = \frac{1}{2} ([S, S_t] + 2iuS),$$ \tag{5.61}

$$W = \begin{pmatrix} W_3 & W^- \\ W^+ & -W_3 \end{pmatrix}. \tag{5.62}$$
Let us find the gauge equivalent counterpart of the Zhanbota equation \( (5.51) - (5.53) \). It is not difficult to verify that the gauge equivalent counterpart of the Zhanbota equation is given by
\[
q_t + \frac{\kappa}{2t}q_{xt} + ivq - 2p = 0, \\
r_t - \frac{\kappa}{2t}r_{xt} - ivr - 2k = 0, \\
v_x + \frac{\kappa}{2}(rq)_t = 0, \\
p_x - 2i\omega p - 2\eta q = 0, \\
k_x + 2i\omega k - 2\eta r = 0, \\
\eta_x + rp + kq = 0,
\]
where \( q, r, p, k \) are some complex functions; \( v, \eta \) are real potential functions and \( \kappa \) is a constant parameter. It is the \( q \)-form of the Zhanbota equation. The Lax representation for the equations \( (5.63) - (5.68) \) reads as
\[
\Phi_x = U_{12}\Phi, \\
\Phi_t = V_{12}\Phi,
\]
where
\[
U_{12} = -i\lambda\sigma_3 + A_0, \\
V_{12} = \frac{1}{1 - \kappa\lambda}\{B_0 + \frac{i}{\lambda + \omega}B_{-1}\}.
\]
Here
\[
A_0 = \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \\
B_0 = \frac{i}{2}\upsilon\sigma_3 - \frac{\kappa}{2i}\begin{pmatrix} 0 & q_y \\ r_y & 0 \end{pmatrix}, \\
B_{-1} = \begin{pmatrix} \eta & -p \\ -k & -\eta \end{pmatrix}.
\]
Next, we consider the reduction \( r = \delta_1\bar{q}, \quad k = \delta_2\bar{p} \) with \( \kappa = 2, \quad \delta_j = \pm 1 \), where the bar means the complex conjugate. Then, the system \( (5.63) - (5.68) \) takes the following more compact form
\[
iq_t + q_{xt} - vq - 2ip = 0, \\
v_x + 2\delta_1(|q|^2)_t = 0, \\
p_x - 2i\omega p - 2\eta q = 0, \\
\eta_x + (\delta_1\bar{q}p + \delta_2\bar{p}q) = 0.
\]
Note that it is nothing but the \( q \)-form of the Zhanbota equation.

### 5.4 Akbota equation

In this subsection, we consider the following Akbota equation
\[
S_t - S \wedge (\alpha S_{xx} + \beta S_{xt}) - uS_x = 0, \\
u_x + S \times (S_x \wedge S_t) = 0.
\]
This Akbota equation is one of the integrable generalizations of the ZE. It has the following Lax representation

\[ \Psi_x - U_{13} \Psi = 0, \quad (5.82) \]
\[ \Psi_t - V_{13} \Psi = 0, \quad (5.83) \]

where

\[ U_{13} = \frac{i}{2} \lambda S, \quad V_{13} = \frac{1}{1 - \lambda \beta} \{ \alpha (\frac{1}{2} i \lambda^2 S + \frac{1}{4} [S, S_x]) + \beta \lambda Z \}. \quad (5.84) \]

Note that the gauge equivalent counterpart of the Akbota equation is the following nonlinear evolution equation

\[ iq_t + \alpha q_{xx} + \beta q_{xt} + vq = 0, \quad (5.85) \]
\[ v_x - 2 [\alpha (|q|^2)_x + \beta (|q|^2)_t] = 0. \quad (5.86) \]

The Lax representation of this set of equations is given by

\[ \Phi_x = U_{14} \Phi, \quad (5.87) \]
\[ \Phi_t = V_{14} \Phi, \quad (5.88) \]

where

\[ U_{14} = \frac{i \lambda}{2} \sigma_3 + Q, \quad Q = \begin{pmatrix} 0 & \bar{q} \\ \bar{q} & 0 \end{pmatrix}, \quad V_{14} = \frac{1}{1 - \lambda \beta} \left\{ \frac{i \lambda^2}{2} \alpha \sigma_3 + \alpha \lambda Q + V_0 \right\} \quad (5.89) \]

with

\[ V_0 = \begin{pmatrix} \alpha i |q|^2 + i \beta \partial_x^{-1} |q|^2 & -i \beta \bar{q}_t - i \alpha \bar{q}_x \\ i \beta q_y + \alpha i q_x & -\alpha i |q|^2 + i \beta \partial_x^{-1} |q|^2 \end{pmatrix}. \quad (5.90) \]

6 Conclusion

In this paper, we have shown that the Zhaidary equation is integrable by the IST method. Its gauge equivalent equation is constructed. Some integrable particular cases (reductions) is studied. Also several integrable generalizations of the ZE are presented. For these reductions and generalizations their gauge partners and their Lax representations are found. Various issues are worthy of further exploration. In particular, we hope to investigate the discrete, nonlocal and fractional integrable versions of the ZE, its reductions and generalizations as well as solutions and geometry of the above presented integrable equations in the future. For example, the ZE and related integrable equations admit a geometric interpretation that directly relates the curvature and torsion of a vector field to any their solution. We will demonstrate that these geometrical relations and interpretations can be extended to the extended discrete, nonlocal and fractional integrable versions of the ZE.

7 Acknowledgments

This work was supported by the Ministry of Education and Science of Kazakhstan, Grant AP08857372.

References

[1] M. Lakshmanan. Phys. Lett. A, 64, 53-54 (1977)
[2] V. E. Zakharov, L. A. Takhtajan, Theor. Math. Phys., 38, 17-23 (1979)

[3] Y. Ishimori. Multi-vortex solutions of a two-dimensional nonlinear wave equation, Prog. Theor. Phys., 72, 33-37 (1984)

[4] Chen Chi, Zhou Zi-Xiang. Darboux Transformation and Exact Solutions of the Myrzakulov-I Equation, Chin. Phys. Lett., 26, N8, 080504 (2009)

[5] Chen Hai, Zhou Zi-Xiang. Darboux Transformation with a Double Spectral Parameter for the Myrzakulov-I Equation, Chin. Phys. Lett., 31, N12, 120504 (2014)

[6] Hai Chen, Zi-Xiang Zhou. Global explicit solutions with n double spectral parameters for the Myrzakulov-I equation, Modern Physics Letters B, 30, N29, 1650358 (2016)

[7] Hai-Rong Wang, Rui Guo. Soliton, breather and rogue wave solutions for the Myrzakulov-Lakshmanan-IV equation, Optik, 242, 166353 (2021)

[8] D. Kaup and A. C. Newell. An exact solution for a derivative nonlinear Schrödinger equation, J. Math. Phys. 19, 798-801 (1978)

[9] H. H. Chen, Y. C. Lee and C. S. Liu. Integrability of Nonlinear Hamiltonian Systems by Inverse Scattering Method, Phys. Scr., 20, 490-492 (1979)

[10] V. Gerdjikov and I. Ivanov. A quadratic pencil of general type and nonlinear evolution equations. II. Hierarchies of Hamiltonian structures, Bulg. J. Phys. 10, 130-143 (1983)

[11] Paz Albares. Integrability and rational soliton solutions for gauge invariant derivative nonlinear Schrödinger equations, [arXiv:2102.12183]

[12] Li-Yuan Ma, Shou-Feng Shen, Zuo-Nong Zhu. Integrable nonlocal complex mKdV equation: soliton solution and gauge equivalence, [arXiv:1612.06723]

[13] Li-Yuan Ma, Zuo-Nong Zhu. Nonlocal nonlinear Schrödinger equation and its discrete version: soliton solutions and gauge equivalence, [arXiv:1503.06909]

[14] Julia Cen, Francisco Correa, Andreas Fring. Nonlocal gauge equivalence: Hirota versus extended continuous Heisenberg and Landau-Lifschitz equation, [arXiv:1910.07272]

[15] V. S. Gerdjikov, G. G. Grahovski, N. A. Kostov. On N-wave Type Systems and Their Gauge Equivalent, [arxiv:0111027]

[16] A. Kundu. Landau-Lifshitz and higher-order nonlinear systems gauge generated from nonlinear Schrödinger-type equations, J. Math. Phys., 25, 3433-3438 (1984)

[17] M. Wadati and K. Sogo. Gauge Transformations in Soliton Theory, J. Phys. Soc. Jpn., 52, 394-398 (1983)

[18] R. Myrzakulov, A. Danlybaeva and G. Nugmanova. Theor. and Math. Phys., V.118, 13, P. 441-451 (1999).

[19] R. Myrzakulov, G. Mamyrbekova, G. Nugmanova, M. Lakshmanan. Symmetry, 7(3), 1352-1375 (2015).

[20] Z. S. Yersultanova, M. Zhassymbayeva, K. Yesmakhanova, G. Nugmanova, R. Myrzakulov. Darboux transformation and exact solutions of the integrable Heisenberg ferromagnetic equation with self-consistent potentials, Int. Jour. Geom. Meth. Mod. Phys., 13, N1, 1550134 (2016).

[21] R. Myrzakulov, G. N. Nugmanova, R. N. Syzdykova. Gauge equivalence between (2+1)-dimensional continuous Heisenberg ferromagnetic models and nonlinear Schrödinger-type equations, J. Phys. A: Math. Gen., 31, N47, 9535-9545 (1998)
[22] R. Myrzakulov, S. Vijayalakshmi, R. Syzdykova, M. Lakshmanan, *On the simplest (2+1) dimensional integrable spin systems and their equivalent nonlinear Schrödinger equations*, J. Math. Phys., 39, 2122-2139 (1998).

[23] V.G. Makhankov, R. Myrzakulov. *σ-Model Representation of the Yajima-Oikawa Equation System*, Preprint P5-84-719, JINR, Dubna, 1984

[24] V. G. Makhankov, R. Myrzakulov, O. K. Pashaev. *Gauge Equivalence, Supersymmetry and Classical Solutions of the ospu(1,1/1) Heisenberg Model and the Nonlinear Schrödinger Equation*, Letters in Mathematical Physics, 16, 83-92 (1988).

[25] R. Myrzakulov, S. Vijayalakshmi, G.N. Nugmanova, M. Lakshmanan. *A (2+1)-dimensional integrable spin model: Geometrical and gauge equivalent counterpart, solitons and localized coherent structures*, Phys. Lett. A, 233, 391 (1997). [arXiv:solv-int/9704005]

[26] Nevin Ertug Gürbüz, R. Myrzakulov, Z. Myrzakulova. *Three anholonomy densities for three formulations with anholonomic coordinates with hybrid frame in R^3_1*, Optik, 169161 (2022)

[27] R. Myrzakulov, O. K. Pashaev, Kh. T. Kholmurodov. *Particle-Like Excitations in Many Component Magnon-Phonon Systems*, Physica Cripita, 33, N4, 378 (1986)

[28] R. Myrzakulov, M. Lakshmanan, S. Vijayalakshmi, A. Danlybaeva, J. Math. Phys., 39, 3765-3771 (1998).

[29] R. Myrzakulov, G. Nugmanova, R. Syzdykova, Journal of Physics A: Mathematical & Theoretical, 31 147, 9535-9545 (1998).

[30] R. Myrzakulov, M. Daniel, R. Amuda, Physica A., 234, 13-4, 715-724 (1997).

[31] S.C. Anco, R. Myrzakulov, Journal of Geometry and Physics, v.60, 1576-1603 (2010).

[32] R. Myrzakulov, G. K. Mamyrbekova, G. N. Nugmanova, K. Yesmakhanova, M. Lakshmanan. Physics Letters A, 378, 30-31, 2118-2123 (2014).

[33] Z. Myrzakulova, G. Nugmanova, K. Yesmakhanova, N. Serikbayev, R. Myrzakulov. *Integrable generalized Heisenberg ferromagnet equations with self-consistent potentials and related Yajima-Oikawa type equations*, DOI: 10.13140/RG.2.2.19991.85923 [https://www.researchgate.net/publication/359768563 Integrable generalized Heisenberg ferromagnet equations with self-consistent potentials and related Yajima-Oikawa type equations...]

[34] L. Martina, Kur. Myrzakul, R. Myrzakulov, G. Soliani, Journal of Mathematical Physics, V.42, 13, P.1397-1417 (2001).

[35] A. Myrzakul, G. Nugmanova, N. Serikbayev, R. Myrzakulov. *Surfaces and Curves Induced by Nonlinear Schrödinger-Type Equations and Their Spin Systems*, Symmetry, 13, N10, 1827 (2021)

[36] A. Myrzakul and R. Myrzakulov. *Integrable geometric flows of interacting curves/surfaces, multilayer spin systems and the vector nonlinear Schrödinger equation*, International Journal of Geometric Methods in Modern Physics, 14, N10, 1750136 (2017)

[37] A. Myrzakul and R. Myrzakulov. *Integrable motion of two interacting curves, spin systems and the Manakov system*, International Journal of Geometric Methods in Modern Physics, 14, N7, 1750115 (2017)