EXPLICIT SOLUTIONS OF RICCI FLOW EQUATION FOR
LOCALLY HOMOGENEOUS $\mathbb{S}^1$-TRIPLES ON COMPACT
RIEMANN SURFACES

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Abstract. Let $(M, g)$ be a compact Riemann surfaces, and $\pi : P \to M$ be
a principal $\mathbb{S}^1$-bundle over $M$ endowed with a connection $A$. Fixing an inner
product on the Lie algebra of $\mathbb{S}^1$, the connection $A$ and metric $g$ define a Rie-
mannian metric $g_A$ on $P$. In this article, we show that the Ricci flow equation
of metric $g_A$ is equivalent to a system of differential equations. We will give
an explicit solution of normalized Ricci flow equation of metric $g_A$ in the case
where the base manifold is of constant curvature and the the initial connection
$A$ is Yang-Mills. Finally, we will describe some asymptotic behaviors of these
flows.

Keywords: Ricci flow, theory of connections, geometric structure

1. Locally homogeneous triples with structure group $K = \mathbb{S}^1$ over
Riemann surfaces

1.1. Metric connection on principal $K$-bundles. A Riemannian metric $g$ on
a differentiable manifold $M$ is defined to be locally homogeneous if, for every two
points $x, x' \in M$ there exist open neighborhoods $U \ni x, U' \ni x'$ and an isometry
$\varphi : U \to U'$ such that $\varphi(x) = x'$. Inspiring from this definition we consider the
following geometric objects:

Definition 1.1. Let $M$ be a smooth manifold and $K$ be a compact Lie group. A
locally homogeneous triple with structure group $K$ on $M$ is a triple $(g, P \xrightarrow{\pi} M, A)$,
where $\pi : P \to M$ is a principal $K$-bundle on $M$, $g$ is Riemannian metric on $M$, and
$A$ is connection on $P$ such that the following locally homogeneity condition is satisfied:
for every two points $x, x' \in M$ there exist an isometry $\varphi : U \to U'$ betweeen open neighborhoods $U \ni x, U' \ni x'$ with $\varphi(x) = x'$, and a $\varphi$-covering
bundle isomorphism $\Phi : P_U \to P_{U'}$ such that $\Phi^*(A_{U'}) = A_U$.

In these formulae, for an open set $U \subset M$, the subscript $U$ is used to denote
the restriction of the indicated objects to $U$. For connections on principal bundles
we adopt the conventions of [4], so in Definition 1.1 the symbol $A$ stands for a $K$-
invariant horizontal distribution of $P$. Fixing an ad-invariant inner product on the
Lie algebra $\mathfrak{k}$ of $K$, one can endow $P$ with a locally homogeneous Riemannian metric
g_A called the connection metric. Let $\langle \cdot, \cdot \rangle$ be an ad-invariant inner product on $\mathfrak{k}$
and define a Riemannian metric $g_A$ on $P$ characterized by the following condition
1. The canonical bundle isomorphism $V_P \simeq P \times \mathfrak{k}$ is an orthogonal bundle isomor-
phism with respect to the inner products defined by $g_A$ and $\langle \cdot, \cdot \rangle$.
2. The restriction of $\pi_*$ to the horizontal subbundle $A \subset TP$ gives an orthogonal
bundle isomorphism $A \to \pi^*(T_M)$. 

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3. The direct sum decomposition $T_P \simeq A \oplus V_P$ of the tangent bundle $T_P$ is $g_A$-orthogonal.

Remark 1. Let $(g, P \xrightarrow{\pi} M, A)$ be a locally homogeneous $K$-triple over $M$, and let $\langle \cdot, \cdot \rangle$ be an ad-invariant inner product on $\mathfrak{t}$. The connection metric $g_A$ on $P$ is locally homogeneous.

A very well-known theorem of Singer [3] assert that any locally homogeneous, complete, simply connected Riemannian manifold is homogeneous. Let $\pi : \tilde{M} \to M$ be the universal cover of $M$. The metric $\tilde{g} := \pi^* g$ defines a complete Riemannian metric on the simply connected manifold $\tilde{M}$. Since $(\tilde{M}, \tilde{g})$ is locally homogeneous, by theorem of Singer, it is also homogeneous. In particular, $\tilde{M}$ admits a connected transitive group of isometries denoted by $G$. Therefore, For a compact, locally homogeneous Riemannian manifold $(M, g)$, there exists a homogeneous model geometry given by $(\tilde{M}, G)$. Generalizing the method of Singer, one can prove a similar classification theorem for locally homogeneous triples [1], [2].

A model geometry is a simply connected manifold $X$ together with a transitive action of a Lie group $G$ on $X$ with compact stabilizers. In dimension three there exists (up to equivalence) eight maximal model geometries which admit compact quotients: $E^3_2$, $S^3$, $H^3$, $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $\text{SL}_2(\mathbb{R})$, Nil and Sol. We refer to [8], [7] for the explicit description of the pair $(X, G)$ corresponding to each symbol above. A geometric structure on a manifold $M$ is a diffeomorphism from $M$ to $X/\Gamma$ for some model geometry $(X, G)$, where $\Gamma \subset G$ is a discrete subgroup of $G$ acting freely on $X$. Excepting $H^3$ and Sol, for any model geometry of Thurston geometries there exist compact 3-manifolds with a geometric structure which are associated with locally homogeneous triples with structure group $S^3$. For instance any non-trivial $S^1$-bundle on a Riemann surface of genus $g$ has a geometric $(X, G)$-structure, where

1. $(X, G) = S^3$ when $g = 0$,
2. $(X, G) = \text{Nil}$ when $g = 1$,
3. $(X, G) = \text{SL}_2(\mathbb{R})$ when $g \geq 2$.

The trivial $S^1$-bundles over Riemann surfaces have geometric structures with model geometry $S^2 \times \mathbb{R}$, $E^3_2$, or $H^2 \times \mathbb{R}$. Moreover any $S^1$-bundle over a Riemann surface has a geometric metric which is associated with a locally homogeneous triple in the sense of Definition [1]. These examples shows the abundance of geometric principal $S^1$-bundles in the class of geometric three manifolds.

Suppose $M$ is a compact manifold, and $K$ is a compact Lie group, then any locally homogeneous triple $(g, P \xrightarrow{\pi} M, A)$ with structure group $K$ on $M$ can be identified with a $\Gamma$-quotient of the homogeneous triple $(\tilde{g} := \pi^* g, Q := \pi^* P \rightarrow \tilde{M}, B)$ on the universal cover $\tilde{M}$ [2]. Thanks to this result, instead of studying the Ricci flow of locally homogeneous triples over compact Riemann surfaces, it is enough to study the Ricci flow on compact Riemann surfaces with metric of constant curvature. We will study the (normalized) Ricci flow of locally homogeneous $S^1$-triples over compact surfaces $M$ with a metric of constant curvature. We will give an explicit solution of the (normalized) Ricci flow equations for the metric $g_A$ in the case where the initial connection $A$ is chosen to be Yang-Mills. Note that this is justified because it is proved that an $S^1$-triple $(g, P \xrightarrow{\pi} M, A)$ is locally homogeneous if and only if the connection $A$ is Yang-Mills [2]. Finally, we will study some asymptotic behaviors of these flows in several interesting cases as it is down in [3].
1.2. Levi-Civita connection associated with connection metric on circle bundles over Riemann surfaces. Let \( \pi : P \to M \) be a principal \( S^1 \)-bundle over a Riemann surface \( M \), \( g \) a Riemannian metric on \( M \) and \( A \) a connection on \( P \). We can endow \( P \) with the connection metric \( g_A \). We summarize these information by saying that \((g, P \xrightarrow{\pi} M, A)\) is a \( S^1 \)-triple. Identify the Lie algebra of \( S^1 \) with \( \mathbb{R} \) and denote the connection 1-form associated with \( A \) by \( \omega_A \in A^1(P, \mathbb{R}) \). Let \( \varphi = (\varphi_1, \varphi_2) \) be an orthonormal frame of Riemann surface \((M, g)\) over an open neighborhood \( U \subset M \) and let \((e_1 := \varphi_1, e_2 := \varphi_2)\) be the \( A \)-horizontal lifts of \( \varphi \) defined on \( P_U := \pi^{-1}(U) \). Therefore, we have

\[
\omega(e_i) = 0, \quad \pi_\ast(e_i) = \varphi_i \quad \text{for} \quad i = 1, 2.
\]

Let \( \xi \) be an element of the Lie algebra of \( S^1 \) and let \( \xi^\# : P \to T_P \) be the fundamental vector field defined on \( P \) trivializing the vertical bundle \( V_P \simeq P \times \xi \). The vertical section \( \xi^\# \in \mathfrak{X}^v(P) \) at a point \( y \in P \) is defined by

\[
\xi^\#_y := \left. \frac{d}{dt} \right|_{t=0} (y \exp(t\xi)).
\]

Note that we can define a smooth map \( f : P \to \mathbb{R} \) at a point \( y \in M \) by

\[
f(y) := \sqrt{g_A(\xi^\#_y, \xi^\#_y)} \quad \text{where} \quad y \in P_x
\]

For all \( y \in P_U \) set \( e_3(y) := f(y)^{-1} \xi^\#_y \) and note that \((e_1, e_2, e_3)\) is an orthonormal frame on \( P_U \) with respect to metric \( g_A \). Use \((\eta^1, \eta^2)\) to denote the dual frame associated with \((\varphi_1, \varphi_2)\) over \( U \subset M \). By a well-known theorem of differential geometry [5], there exists a unique skew-symmetric matrix \((\eta_i^j)\) of real-valued 1-forms \( \eta^j_i \in A^1(U, \mathbb{R}) \) such that

\[
d\eta^1 = - \eta^2_2 \wedge \eta^2, \quad d\eta^2 = - \eta^1_2 \wedge \eta^1.
\]

Putting \( \theta^1 = \pi^\ast \eta^1 \), \( \theta^2 = \pi^\ast \eta^2 \) and \( \theta^3 = f \omega \) we will obtain a co frame \((\theta^1, \theta^2, \theta^3)\) dual to \((e_1, e_2, e_3)\) on \( P_U \). Taking pullback of both sides of equations (1) we obtain

\[
\begin{align*}
d\theta^1 &= - \pi^\ast \eta^1_2 \wedge \theta^2, \\
d\theta^2 &= - \pi^\ast \eta^1_3 \wedge \theta^1.
\end{align*}
\]

The curvature 2-form of connection \( A \) is denoted by \( F_\omega = d\omega \in A^2(P, \mathbb{R}) \). Since \( S^1 \) is abelian \( F_\omega \) is \( S^1 \)-invariant and descend to a closed 2-form on \( M \). Using the dual frame \((\theta^1, \theta^2, \theta^3)\) on \( P_U \), the curvature 2-form \( F_\omega \) can be written as follow

\[
F_\omega = c \theta^1 \wedge \theta^2 \quad \text{where} \quad c \in C^\infty(P_U, \mathbb{R}).
\]

The map \( f \in C^\infty(P) \) is \( S^1 \)-invariant and it descends to a smooth map \( f : M \to \mathbb{R} \), so there exists the maps \( f_1, f_2 \in C^\infty(P_U) \) such that \( df = f_1 \theta^1 + f_2 \theta^2 \). Using this we deduce that

\[
d\theta^3 = d(f \omega) = df \omega + df \wedge \omega = f c \theta^1 \wedge \theta^2 + \frac{f_1}{f} \theta^1 \wedge \theta^3 + \frac{f_2}{f} \theta^2 \wedge \theta^3.
\]

The equations (2) and (4) lead to the following system of differential equations

\[
\begin{align*}
d\theta^1 &= - \pi^\ast \eta^1_2 \wedge \theta^2, \\
d\theta^2 &= - \pi^\ast \eta^1_3 \wedge \theta^1, \\
d\theta^3 &= f c \theta^1 \wedge \theta^2 + \frac{f_1}{f} \theta^1 \wedge \theta^3 + \frac{f_2}{f} \theta^2 \wedge \theta^3.
\end{align*}
\]
The equations (5) can be used to find the components of Levi-Civita connection for metric $g_A$. In fact, we search a unique skew-symmetric matrix $(\theta^i_j)$ of real-valued 1-forms $\theta^i_j \in A^1(P_U, \mathbb{R})$ satisfying the following differential equations

$$d\theta^i = -\sum_j \theta^i_j \wedge \theta^j.$$  
(6)

The solutions of these equations are the components of the Levi-Civita connection 1-form associated to the metric $g_A$. These solutions can be find in two steps:

**Step 1.** First of all, we try to find the real-valued 1-forms $\mu^i_j$ on $P_U$ which satisfy in (6). Unfortunately, there exists no reason to be sure that $(\mu^i_j)$ is skew-symmetric. Fortunately, we can modify this solution $(\mu^i_j)$ in an appropriate way such that the modified solution is skew-symmetric and still satisfies in (6).

**Step 2.** Put $\theta^i_j := \mu^i_j + \sum_k S^i_{jk} \theta^k$ and search a unique tensor $S$ such that:

1. $S^i_{jk} = S^i_{kj}$ for all $i, j, k$.
2. $(\theta^i_j)$ is skew-symmetric.

It is easy to verify that the components of skew-symmetric matrix $(\theta^i_j)$ defined by $\theta^i_j := \mu^i_j + \sum_k S^i_{jk} \theta^k$ satisfy in (6). Therefore, we just need to determined the unique tensor $S$ as above. These two steps give an algorithm to find the skew-symmetric matrix $(\theta^i_j)$ of real valued 1-forms which defines the Levi-Civita connection of metric $g_A$. First step is to find $\theta^i_j \in A^1(P_U, \mathbb{R})$ satisfying the equation (6). Thanks to the equations (5) we conclude that the following matrix of 1-forms has this property.

$$\mu^i_j := \begin{pmatrix} 0 & \pi^* \eta^1_2 & 0 \\ \pi^* \eta^1_2 & 0 & 0 \\ f c \theta^2 + \frac{f_1}{f} \theta^3 & \frac{f_1}{f} \theta^3 & 0 \end{pmatrix}.$$  
(7)

Second step is to modify $\mu^i_j$ by adding a tensor field $S^i_{jk}$ such that $S^i_{jk} = S^i_{kj}$ for all $i, j, k$. Therefore, we have to solve the following system of equations

$$\begin{align*}
\theta^i_j &= \mu^i_j + \sum_k S^i_{jk} \theta^k \\
\theta^i_j &= - \theta^i_j \\
S^i_{jk} &= S^i_{kj}
\end{align*}$$  
(8)

If $i = j$, then by the skew-symmetry property of $\theta^i_i$ we have

$$0 = \theta^i_i = \mu^i_i + \sum_k S^i_{ik} \theta^k = \sum_k S^i_{ik} \theta^k.$$  
(9)

So, for all $i$ and $k$ we have $S^i_{ik} = S^i_{ki} = 0$. By the first equation of (7) we have

$$\theta^2_2 = \mu^2_2 + \sum_k S^2_{2k} \theta^k = \pi^* \eta^2_2 + \sum_k S^2_{2k} \theta^k,$$  
(10)

$$\theta^2_1 = \mu^2_1 + \sum_k S^2_{1k} \theta^k = \pi^* \eta^2_1 + \sum_k S^2_{1k} \theta^k.$$  
(11)

Using (9), (10) and $\theta^2_2 = - \theta^2_1$ we get $S^2_{2k} = - S^2_{1k}$. In the same way, the first equation of (7) implies

$$\theta^3_3 = \mu^3_3 + \sum_k S^3_{3k} \theta^k = \sum_k S^3_{3k} \theta^k,$$  
(12)

$$\theta^3_1 = \mu^3_1 + \sum_k S^3_{1k} \theta^k = f c \theta^2 + \frac{f_1}{f} \theta^3 + \sum_k S^3_{1k} \theta^k.$$  
(13)
Using (11), (12) and \( \theta_2^1 = -\theta_1^1 \) we obtain
\[
S^1_{11} = 0, \quad S^1_{22} + S^1_{32} = -fc, \quad S^1_{33} = -\frac{f^3}{T}.
\]
In the same way, we have the following equations
\[
\theta_2^2 = \mu_2^2 + \sum_k S^2_{3k} \theta^k = \sum_k S^2_{3k} \theta^k,
\]
\[
\theta_2^3 = \mu_3^3 + \sum_k S^3_{2k} \theta^k = \frac{f^2}{f} \theta^3 + \sum_k S^3_{2k} \theta^k,
\]
Using (13), (14) and \( \theta_3^3 = -\theta_2^3 \) we obtain
\[
S^3_{22} = 0, \quad S^3_{23} = -\frac{f^2}{f}, \quad S^3_{31} = S^3_{32} = 0.
\]
Using the above equations, we deduce that \( S^1_{12} = S^1_{32} = -\frac{4}{f}fc \) and \( S^3_{13} = \frac{4}{f}fc \). So, we have proved the next proposition.

**Proposition 1.2.** Let \( \pi : P \to M \) be a principal \( S^1 \)-bundle over a Riemann surface \((M, g)\) endowed with a connection \( A \). Using the same notation as above, the Levi-Civita connection \((\theta_i^j)\) of connection metric \( g_A \) is given by

1. \( \theta_2^1 = \pi^*(\eta_1^2) - \frac{1}{2} fc \theta^3 \).
2. \( \theta_3^3 = -\frac{1}{2} fc \theta^2 - \frac{1}{3} \theta^3 \).
3. \( \theta_3^3 = \frac{1}{2} fc \theta^3 - \frac{f^2}{f^3} \theta^3 \).

1.3. The curvature tensors of connection metric on \( S^1 \)-bundles over Riemann surfaces. Let \( \pi : P \to M \) be a principal \( S^1 \)-bundle over a Riemann surface \((M, g)\) endowed with connection metric \( g_A \). Let \((\theta_i^j)\) be the skew-symmetric matrix of 1-forms which defines the Levi-Civita connection of \( g_A \) as in Proposition 1.2. Using the co-frame \((\theta^1, \theta^2, \theta^3)\) defined on \( P_U \), the curvature 2-form \( F_A \) can be written as \( F_A = c \theta^1 \wedge \theta^2 \), where \( c \) is a smooth function on \( P_U \). We shall use the following decomposition in next calculations.

\[
dc = c \theta^1 \wedge \theta^2, \quad df = f_1 \theta^1 + f_2 \theta^2, \quad df = f_1 \theta^1 + f_2 \theta^2, \quad df = f_1 \theta^1 + f_2 \theta^2, \quad df = f_1 \theta^1 + f_2 \theta^2 \quad \text{for} \quad i = 1, 2.
\]

If \( \Omega^\theta = (\Omega^\theta_i) \) denotes the curvature 2-form of Levi-Civita connection \( \theta = (\theta^i_j) \), then we will use the structure equation \( \Omega^\theta_i = \theta^i_j + \sum_k \theta^k \wedge \theta^k \) to determine the components of \( \Omega^\theta \) \(\text{[4]}\). In order to calculate \( \Omega^\theta_1 = d\theta^1_2 + \theta^3_2 \wedge \theta^3_2 \), we need the following
\[
d\theta^1_2 = \pi^* d\eta^1_2 - \frac{1}{2} d(fc \theta^3).
\]
we also know that
\[
d\theta^1 = d(f \omega) = f d\omega + df \wedge \omega = fc \theta^1 \wedge \theta^2 + \frac{f_1}{f} \theta^1 \wedge \theta^3 + \frac{f_2}{f} \theta^2 \wedge \theta^3.
\]
Using this equation, it follows that
\[
d(fc \theta^3) = f(d \theta^1 + \theta^3) + c(df \wedge \theta^3) + fc d\theta^3
\]
\[
= \frac{1}{2} \theta^1 \wedge \theta^3 + f \beta \theta^2 \wedge \theta^3 + cf_1 \theta^1 \wedge \theta^3
\]
\[
+ cf_2 \theta^2 \wedge \theta^3 + f^2 c^2 \theta^3 \wedge \theta^2 + cf_1 \theta^1 \wedge \theta^3 + cf_2 \theta^2 \wedge \theta^3.
\]
If $K_g$ denotes the Gauss curvature of surface $(M, g)$, then $d\eta_2^2 = K(\eta^1 \wedge \eta^2)$ and hence $d\theta_2^2 = \pi^* K_g(\theta^1 \wedge \theta^2)$. Putting $\hat{K} = \pi^* K_g$ to save on notation and using the equations (16), (17) we get

\begin{equation}
(18) \quad d\theta_2^2 = (\hat{K} - \frac{1}{2} f^2 c^2) \theta^1 \wedge \theta^2 - \left(\frac{1}{2} f \alpha + c f_1\right) \theta^1 \wedge \theta^3 - \left(\frac{1}{2} f \beta + c f_2\right) \theta^2 \wedge \theta^3.
\end{equation}

We have also

\begin{equation}
(19) \quad \theta_3^1 \wedge \theta_3^2 = \left(-\frac{1}{2} f c \theta^2 - \frac{f_1}{f} \theta^3\right) \wedge \left(-\frac{1}{2} f c \theta^1 + \frac{f_2}{f} \theta^3\right)
= -\frac{1}{4} f^2 c^2 \theta^1 \wedge \theta^2 - \frac{1}{2} c f_2 \theta^2 \wedge \theta^3 - \frac{1}{2} c f_1 \theta^1 \wedge \theta^3.
\end{equation}

Therefore, (18) and (19) imply

\begin{equation}
\Omega_2^1 = d\theta_2^1 + \theta_3^1 \wedge \theta_3^2 = (\hat{K} - 3 f^2 c^2) \theta^1 \wedge \theta^2 - \left(\frac{1}{2} f \alpha + \frac{3}{2} c f_1\right) \theta^1 \wedge \theta^3 - \left(\frac{1}{2} f \beta + \frac{3}{2} c f_2\right) \theta^2 \wedge \theta^3.
\end{equation}

To determine $\Omega_3^1$, we need to calculate $d\theta_3^1 = -\frac{1}{2} d(f c \theta^2) - d(f_1 \omega)$. But

\begin{equation}
(20) \quad d(f c \theta^2) = f dc \wedge \theta^2 + c df \wedge \theta^2 + f c d\theta^2 = f \alpha \theta^1 \wedge \theta^2 + c f_1 \theta^1 \wedge \theta^2 + f c (-\pi^* \eta_2^1 \wedge \theta^1) + 3 f^2 c^2 \theta^1 \wedge \theta^2.
\end{equation}

and

\begin{equation}
(21) \quad (f_1 \omega) = f_1 d\omega + df_1 \wedge \omega = c f_1 \theta^1 \wedge \theta^2 + (f_{11} \theta^1 + f_{12} \theta^2) \wedge \frac{1}{f} \theta^3
= c f_1 \theta^1 \wedge \theta^2 + \frac{f_{11}}{f} \theta^1 \wedge \theta^3 + \frac{f_{12}}{f} \theta^2 \wedge \theta^3.
\end{equation}

Therefore, (20) and (21) give

\begin{equation}
(22) \quad d\theta_3^1 = -\left(\frac{1}{2} f \alpha + \frac{3}{2} c f_1\right) \theta^1 \wedge \theta^2 - \frac{f_{11}}{f} \theta^1 \wedge \theta^3 - \frac{f_{12}}{f} \theta^2 \wedge \theta^3 - \frac{1}{2} f c (\pi^* \eta_2^1) \wedge \theta^1.
\end{equation}

We have also

\begin{equation}
(23) \quad \theta_3^1 \wedge \theta_3^2 = (\pi^* \eta_2^1 - \frac{1}{2} f c \theta^3) \wedge \left(\frac{1}{2} f c \theta^1 - \frac{f_2}{f} \theta^3\right)
= \frac{1}{2} f c (\pi^* \eta_2^1) \wedge \theta^1 - \frac{f_2}{f} (\pi^* \eta_2^1) \wedge \theta^3 + \frac{1}{4} f^2 c^2 \theta^1 \wedge \theta^3.
\end{equation}

Using the decompositions $\pi^* \eta_2^1 = \eta_{2,1} \theta^1 + \eta_{2,2} \theta^2$, and the equations (22) and (23) we obtain

\begin{equation}
\Omega_3^1 = -\left(\frac{1}{2} f \alpha + \frac{3}{2} c f_1\right) \theta^1 \wedge \theta^2 + (\frac{1}{4} f^2 c^2 - \frac{f_{11}}{f}) \eta_{2,1} \wedge \theta^1 \wedge \theta^3 - \frac{f_{12}}{f} \eta_{2,2} \wedge \theta^2 \wedge \theta^3.
\end{equation}

In order to determine $\Omega_2^3$, we need first to calculate

\begin{equation}
(24) \quad d\theta_3^2 = \frac{1}{2} d(f c \theta^1) - d(f_2 \omega)
\end{equation}

But, we have

\begin{equation}
(25) \quad d(f c \theta^1) = f c d\theta^1 + f dc \wedge \theta^1 + c df \wedge \theta^1
= -f c (\pi^* \eta_2^1) \wedge \theta^2 - f \beta \theta^1 \wedge \theta^2 - c f_2 \theta^1 \wedge \theta^2.
\end{equation}

and

\begin{equation}
(26) \quad d(f_2 \omega) = f_2 d\omega + df_2 \wedge \omega = c f_2 \theta^1 \wedge \theta^2 + (f_{21} \theta^1 + f_{22} \theta^2) \wedge \frac{1}{f} \theta^3
= c f_2 \theta^1 \wedge \theta^2 + \frac{f_{21}}{f} \theta^1 \wedge \theta^3 + \frac{f_{22}}{f} \theta^2 \wedge \theta^3.
\end{equation}
Secondly, we need the following calculation

\[
\theta_1^2 \wedge \theta_3^3 = \left( - \pi^* \eta_2^1 + \frac{1}{2} fc \theta^3 \right) \wedge \left( - \frac{1}{2} fc \theta^2 - \frac{f_1}{f} \theta^3 \right) = \frac{1}{2} fc (\pi^* \eta_2^1) \wedge \theta^2 + \frac{f_1}{f} (\pi^* \eta_2^1) \wedge \theta^3 + \frac{1}{4} f^2 c^2 \theta^2 \wedge \theta^3.
\]

Finally, the equations (24), (25), (26) and (27) imply that

\[
\Omega_2^2 = d \theta_3^3 + \theta_1^2 \wedge \theta_3^3 = - \left( \frac{1}{2} f \beta + \frac{3}{2} cf_2 \right) \theta^1 \wedge \theta^2 + \frac{f_1}{f} \eta_{2,1} - \frac{f_2}{f} \theta^1 \wedge \theta^3 + \left( \frac{1}{4} f^2 c^2 \right) \theta^2 \wedge \theta^3.
\]

Using these notations, we have proved the next proposition.

**Proposition 1.3.** Let \( \pi : P \to M \) be a principal \( S^1 \)-bundle over a Riemann surface \((M, g)\) endowed with a connection \( A \). Let \( \theta = (\theta_1^1) \) be the connection 1-form of the Levi-Civita connection of \( g_A \). Using the same notation as above, the components of curvature 2-form \( \Omega^\theta = (\Omega^\theta_1) \) corresponding to \( \theta \) is given by

1. \( \Omega_1^2 \) = \( \tilde{K} - \frac{1}{2} f^2 c^2 \theta^2 \wedge \theta^2 + \left( \frac{1}{2} f \alpha - \frac{3}{2} cf_1 \right) \theta^1 \wedge \theta^3 + \left( \frac{1}{2} f \beta - \frac{3}{2} cf_2 \right) \theta^2 \wedge \theta^3 \)

2. \( \Omega_3^2 \) = \( - \frac{1}{2} f \alpha - \frac{3}{2} cf_1 \theta^1 \wedge \theta^2 + \left( \frac{1}{2} f^2 c^2 - \frac{1}{4} f \eta_{2,1} \right) \theta^1 \wedge \theta^3 - \left( \frac{1}{4} f^2 c_2 + \frac{1}{4} f \eta_{2,2} \right) \theta^2 \wedge \theta^3 \)

3. \( \Omega_3^2 \) = \( - \frac{1}{2} f \beta - \frac{3}{2} cf_2 \theta^1 \wedge \theta^2 + \left( \frac{1}{4} f \eta_{2,1} - \frac{1}{4} f \eta_{2,2} \right) \theta^1 \wedge \theta^3 + \left( \frac{1}{4} f^2 c_2 + \frac{1}{4} f \eta_{2,2} \right) \theta^2 \wedge \theta^3 \)

1.4. The Ricci curvature of metric connection on principal \( S^1 \)-bundles over Riemann surfaces. Let \((g, P \to M, A)\) be a \( S^1 \)-triple over \( M \) and, let \( g_A \) be the connection metric on \( P \). Let \((e_1, e_2, e_3)\) be the \( g_A \)-orthonormal frame on \( P \) chosen as above. For an arbitrary point \( y \in P \) and two vectors \( v, w \in T_y P \) the Ricci curvature tensor is given by \( \text{Ric}(v, w) = \sum_{m=1}^3 g_A \left( R^\theta(v, e_m) e_m, w \right) \), where \( R^\theta \in A^2(P, \text{End}(T_P)) \) is the Riemann curvature tensor associated to Levi-Civita connection \( \nabla^\theta \) on tangent bundle \( T_P \). The components of Ricci curvature tensor in \( g_A \)-orthonormal frame \((e_1, e_2, e_3)\) is given by

\[
R_{ik} = \text{Ric}(e_i, e_k) = \sum_m g_A \left( R^\theta(e_i, e_m) e_m, e_k \right).
\]

The Riemann curvature tensor \( R^\theta \) is related to curvature 2-form \( \Omega^\theta = (\Omega^\theta_1) \) by

\[
R^\theta(e_i, e_m) e_m = \sum_i \Omega^\theta_i (e_i, e_m) e_i.
\]

Since \((e_1, e_2, e_3)\) is \( g_A \)-orthonormal, we have

\[
R_{ik} = \sum_m g_A \left( \Omega^\theta_i (e_i, e_m) e_i, e_k \right) = \sum m \Omega^\theta_i (e_i, e_m).
\]

Using Proposition 1.2 and Proposition 1.3 and (28), the components of Ricci tensor of \( g_A \) can be computed. If in (28) we substitute \( l = 1 \), then \( R_{ik} = \Omega^\theta_2 (e_1, e_2) + \Omega^\theta_3 (e_1, e_3) \), calculating this for \( k = 1, 2, 3 \) and using Proposition 1.3 we obtain

\[
R_{11} = \tilde{K} - \frac{1}{2} f^2 c^2 - \frac{f_1}{f} \eta_{2,1}^1, \quad R_{12} = \frac{f_1}{f} \eta_{2,1}^1 - \frac{f_2}{f} \theta^3, \quad R_{13} = \frac{1}{2} f \beta + \frac{3}{2} cf_2.
\]

If we substitute \( l = 2 \) and \( l = 3 \) in (28), then in a similar way we obtain

\[
R_{21} = - \frac{f_1}{f} \eta_{2,2}^1, \quad R_{22} = \tilde{K} - \frac{1}{2} f^2 c^2 + \frac{f_1}{f} \eta_{2,2}^1, \quad R_{23} = - \frac{1}{2} f \alpha - \frac{3}{2} cf_1.
\]
and
\[ R_{31} = \frac{1}{2} f \beta + \frac{3}{2} c f_2, \quad R_{32} = -\frac{1}{2} f \alpha - \frac{3}{2} c f_1, \quad R_{33} = \frac{1}{2} f^2 c^2 - \frac{f_{11}}{f} f_2 \eta_{2,1} + \frac{f_1}{f} f_1 \eta_{2,2} - \frac{f_2}{f}. \]

With the same notation as previous, we have \( \Delta f = -f_{11} - f_{22} - f_2 \eta_{2,1} + f_1 \eta_{2,2} \), where \( \Delta f = -d \star df \) is the laplacian of function \( f : P \to \mathbb{R} \). Using \( df = f_1 \theta^1 + f_2 \theta^2 \) we obtain
\[ \Delta f = -d \star (f_1 \theta^1 + f_2 \theta^2) = -f_{11} - f_{22} - f_2 \eta_{2,1} + f_1 \eta_{2,2}. \]

So, we can write \( R_{33} = \frac{1}{2} f^2 c^2 + \Delta f \). The Ricci tensor of metric \( g_A \) with respect the frame \( (e_1, e_2, e_3) \) has the following representation
\[ \text{Ric}(g_A) = \begin{pmatrix} R & S' & \varphi \\ S^t & \varphi \end{pmatrix}. \]

where
\[ R = \begin{pmatrix} \bar{K} - \frac{1}{2} f^2 c^2 & 0 \\ 0 & \bar{K} - \frac{1}{2} f^2 c^2 \end{pmatrix}, \quad S = \begin{pmatrix} \frac{1}{2} f \beta + \frac{3}{2} c f_2 \\ \frac{1}{2}(f \alpha - \frac{3}{2} c f_1) \end{pmatrix}, \quad \varphi = \frac{1}{2} f^2 c^2 + \frac{\Delta f}{f}. \]

**Remark 2.** The equation \( d^2 f = 0 \) implies \( f_{21} - f_1 \eta_{2,1} = f_{12} + f_2 \eta_{2,2} \) which is equivalent to \( R_{12} = R_{21} \).

**Remark 3.** Let \( \nabla \) denote the Linear connection on tangent bundle \( T_P \) induced by the Levi-Civita connection \( \theta \). Since, \( df = f_1 \theta^1 + f_2 \theta^2 \) one has the next formula
\[ \nabla df = \nabla(f_1 \theta^1 + f_2 \theta^2) = df_1 \otimes \theta^1 + f_1 \nabla \theta^1 + df_2 \otimes \theta^2 + f_2 \nabla \theta^2 = f_{11} \theta^1 \otimes \theta^1 + f_{22} \theta^2 \otimes \theta^2 + f_1 (-\pi^* \eta_2 \otimes \theta^2) + f_2 (\pi^* \eta_2 \otimes \theta^1). \]

Thus, the matrix representation of \( \nabla df \) with respect to \( (\theta^1, \theta^2, \theta^3) \) is given by
\[ \nabla df = \begin{pmatrix} f_{11} + f_2 \eta_{2,1} & f_{21} - f_1 \eta_{2,1} \\ f_{12} + f_2 \eta_{2,2} & f_{22} - f_1 \eta_{2,2} \end{pmatrix}. \]

Using these remarks and the fact that \( |F_\omega|_g^2 = c^2 \) we can rewrite \( R \) in the following way:
\[ R = \left( \bar{K} - \frac{1}{2} f^2 |F_\omega|_g^2 \right) \pi^* g - \frac{1}{f} df \nabla df = \pi^* \text{Ric}(g) - \frac{1}{2} f^2 |F_\omega|_g^2 \pi^* g - \frac{1}{f} df \nabla df. \]

We can also rewrite \( S \) in the invariant way as
\[ S = \frac{1}{2} (f \beta + 3 c f_2) \theta^1 \otimes \theta^1 - \frac{1}{2} (f \alpha + 3 c f_1) \theta^2 \otimes \theta^3 = -\frac{1}{2} f^2 (d^* F_\omega) \otimes \omega - \frac{3}{2f} (d^* F_\omega) \otimes (d^* df). \]

Since these formula are written in an invariant way they are independent of chosen frame \( (e_1, e_2, e_3) \) and the next proposition is proved.

**Proposition 1.4.** Let \( (g, P \to M, A) \) be a \( G^1 \)-triple on \( M \). If \( \omega \) and \( F_\omega \) denote, respectively, the connection 1-form and the curvature 2-form associated with \( A \), then the matrix representation of Ricci curvature tensor of \( g_A \) with respect to orthonormal frame \( (\theta^1, \theta^2, \theta^3) \) is given by
\[ \text{Ric}(g_A) = \begin{pmatrix} R & S' & \varphi \\ S^t & \varphi \end{pmatrix}. \]
where, \( R = (\pi^*K_g - \frac{1}{2}f^2|F_\omega|^2)\pi^*g - \frac{1}{f}\nabla df, \ S = -\frac{1}{2}f^2(d^*F_\omega)\otimes\omega - \frac{3}{2f}(\ast F_\omega)(\ast df)\otimes\omega \) and \( \varphi = \frac{1}{2}f^2|F_\omega|^2 + \frac{\Delta f}{f}. \)

**Corollary 1.5.** Suppose that \( f : M \to \mathbb{R} \) is a constant function on \( M \) and \( A \) is a Yang-Mills connection. Then

1. If \( R_g = 2K_g \) denotes the scalar curvature of \( g \), then the scalar curvature of \( g_A \) is given by
   \[ R_{g_A} = \pi^*R_g - \frac{1}{2}f^2|F_\omega|^2. \]
2. The horizontal and vertical subspaces are orthogonal relative to the Ricci tensor of metric \( g_A \) and we have
   \[ \text{Ric}(g_A) = (\pi^*K_g - \frac{1}{2}f^2|F_\omega|^2)\pi^*g + \frac{1}{2}f^4|F_\omega|^2\omega \otimes \omega. \]

2. The Ricci flow equation of metric connection on principal \( S^1 \)-bundles over compact Riemann surfaces

2.1. Ricci flow equation on principal \( S^1 \)-bundles over compact Riemann surfaces. Let \( (M,g) \) be a connected, compact, oriented Riemann surface and, let \( (g,P \to M, A) \) be a \( S^1 \)-triple over \( M \). The connection metric on \( P \) is defined by \( g_A = \pi^*g + f^2\omega \otimes \omega \), where \( \omega \in \Lambda^1(P, \mathbb{R}) \) and \( f \in C^\infty(P) \) is the smooth \( S^1 \)-invariant map defined as in Section 1.2. The variation of connection metric \( g_A(t) \) with time is as follow

\[ \dot{g}_A(t) = \pi^*g_t + f_t^2\dot{\omega}_t \otimes \omega_t + f_t^2\omega_t \otimes \dot{\omega}_t + 2f_t\dot{f}_t\omega_t \otimes \omega_t. \]

The Ricci flow equation of metric \( g_A \) is the following evolution equation

\[ \dot{g}_A(t) = -2\text{Ric}(g_A(t)). \]

Using Proposition 1.4, the Ricci flow equation of \( g_A \) is equivalent to the following system of differential equations:

\[ \begin{aligned}
\dot{g}_t &= -2\text{Ric}(g_t) + f_t^2|F_\omega|^2 g_t + \frac{\Delta f}{f} \nabla df_t.
\omega_t &= d^*F_{\omega_t} + 3f_t^{-3}(\ast F_\omega)(\ast df_t).
\dot{f}_t &= -\frac{1}{2}f_t^3|F_\omega|^2 - \Delta f_t.
\end{aligned} \] (29)

2.2. Ricci flow equation of locally homogeneous \( S^1 \)-triples on compact Riemann surfaces with constant curvature. Let \( (M,g_0) \) be a connected, oriented, compact Riemann surface with constant Gauss curvature \( K_0 \), and let \( \pi : P \to M \) be a principal \( S^1 \)-bundle endowed with a Yang-Mills connection \( A_0 \). Suppose that \( \omega_0 \in \Lambda^1(P, \mathbb{R}) \) (resp. \( F_{\omega_0} = df_0 \in \Lambda^2(P, \mathbb{R}) \)), denotes the connection 1-form (resp. the curvature 2-form) associated with connection \( A_0 \). Let \( g_t = \lambda_t g_0 \) be a one-parameter family of metric on \( M \). If \( f_t \) denotes the size of the fiber at time \( t \geq 0 \) and \( \omega_t = \omega_0 \), then we search a solution of equations (29) with the initial conditions \( \lambda(0) = f(0) = 1 \). Moreover, we suppose that \( f_t : M \to \mathbb{R} \) is constant with respect to \( M \) and hence \( df_t = \Delta f_t = 0 \). Since \( A_0 \) is Yang-Mills and \( \omega_t = \omega_0 \) we have

\[ 0 = \omega_t = d^*F_{\omega_1} = d^*F_{\omega_0} = 0. \]
This means that the second equation of Ricci flow equations (29) is automatically verified and they are equivalent to

\begin{align}
\dot{g}_t &= -2\text{Ric}(g_t) + f_t^2 |F_{\omega_0}|_{g_t}^2 g_t, \\
\dot{f}_t &= -\frac{1}{2} f_t^4 |F_{\omega_0}|_{g_t}^2 .
\end{align}

(30)

To save on notation denote the norm square of curvature 2-form \( F_{\omega_0} \in A^2(M, \mathbb{R}) \) by \( |F_{\omega_0}|_{g_0}^2 = : F_0 \) and note that

\begin{equation}
|F_{\omega_0}|_{g_t}^2 = g_t(F_{\omega_0}, F_{\omega_0}) = \lambda_t^{-2} g_0(F_{\omega_0}, F_{\omega_0}) = \lambda_t^{-2} |F_{\omega_0}|_{g_0}^2 = \lambda_t^{-2} F_0 .
\end{equation}

(31)

The Ricci tensor is scale invariant, i.e. \( \text{Ric}(g_t) = \text{Ric}(\lambda(t)g_0) = \text{Ric}(g_0) = K_0 g_0 \).

Hence, the equations (30) take the form

\begin{align}
\dot{\lambda}_t &= -2K_0 + F_0 f_t^2 \lambda_t^{-1} , \\
\dot{f}_t &= -\frac{1}{2} f_t^4 \lambda_t^{-2} .
\end{align}

(32)

Lemma 2.1. Let \( \pi : P \to M \) be a \( S^3 \)-bundle on a compact Riemann surface \( (M, g) \) and \( A \) be a Yang-Mills connection on \( P \). The curvature 2-form \( F_A \in A^2(M, \mathbb{R}) \) is a constant multiple of the volume form of \( (M, g) \). More precisely, there exists a constant \( c \in \mathbb{R} \) such that

\[ F_A = c \text{Vol}_g . \]

Proof: Since the curvature \( F_A \) is a 2-form on \( M \), there exists a smooth function \( f_A \in C^\infty(M) \) such that \( F_A = f_A \text{Vol}_g \). Since \( A \) is Yang-Mills \( d^* F_A = 0 \), we know also that \( dF_A = 0 \). Therefore

\[ 0 = (dd^* + d^* d) F_A = (dd^* + d^* d) f_A \text{Vol}_g = (\Delta f_A) \text{Vol}_g . \]

This means \( \Delta f_A = 0 \) on \( M \) and as \( M \) is compact, there exists a constant \( c \in \mathbb{R} \) such that \( f_A \equiv c \). This constant can be determined using \( c \int_M \text{Vol}_g = 2\pi c_1(P) \).

Example. Suppose \( (M, g_0) \) is a Riemann surface of constant positive curvature \( K_0 \geq 0 \) and \( A_0 \) is a non flat Yang-Mills connection on \( S^3 \)-bundle \( P \to M \) such that \( F_0 = K_0 \). Note that there exists always such a connection \( A_0 \), in fact using Lemma 2.1 we can scale an arbitrary non flat Yang-Mills connection in an appropriately way to obtain a Yang-Mills connection \( A_0 \) for which \( F_0 = K_0 \). The initial metric \( g_{A_0} = \pi^* g + \omega_0 \otimes \omega_0 \) is an Einstein metric of positive scalar curvature. Putting \( \varphi_t := f_t^2 \), the Ricci flow equation is equivalent to

\begin{align}
\dot{\lambda}_t &= -2K_0 + K_0 \varphi_t \lambda_t^{-1} , \\
\dot{\varphi}_t &= -K_0 \varphi_t^2 \lambda_t^{-2} .
\end{align}

(33)

Supposing that \( \varphi_t \lambda_t^{-1} = 1 \) and integrating (33) we obtain \( \varphi_t = \lambda_t = 1 - K_0 t \).

Hence, the Ricci flow solution exists for all finite time and

\[ g_A(t) = (1 - K_0 t) g_{A_0} \quad \text{for all} \quad 0 \leq t < 1/K_0 . \]

Therefore, the metric shrinks to a point as \( t \) approaches to \( K_0 \), while the curvatures becomes infinite. For instance, let \( \pi : S^3 \to S^2 \) be the famous \( S^3 \)-bundle of Hopf fibration endowed with a Yang-Mills connection \( A_0 \). Suppose \( g_{S^2} \) is the standard metric on 2-sphere induced by the embedding \( S^2 \subset \mathbb{R}^3 \) and put \( g_0 := \frac{1}{4} g_{S^2} \). Using Lemma 2.1 the curvature \( F_{A_0} \) of Yang-Mills connection \( A_0 \) satisfies \( F_{A_0} = 2 \text{Vol}_{g_0} \).

The metric \( g_{A_0} := \pi^* g_0 + \omega_0 \otimes \omega_0 \) induced by the connection \( A_0 \) and metric \( g_0 \) coincides with the standard metric \( g_{S^2} \) on 3-sphere \( S^3 \). Since \( F_0 = K_0 = 4 \), the
Ricci-flow solution is given by $g_A(t) = (1 - 4t)g_{S^3}$. The sphere $S^3$ is collapsing to a point and the curvatures becomes infinite as $t \to 1/4$.

2.3. Explicit solutions of normalized Ricci flow equation for locally homogeneous $S^3$-triples on compact Riemann surfaces. Let $(M, g)$ be a connected, oriented, compact Riemann surface endowed with a metric of constant curvature. An $S^3$-triple $(g, P \to_s M, A)$ is locally homogeneous if and only if the connection $A$ is Yang-Mills [2]. Suppose that $(g, P \to_s M, A)$ is a locally homogeneous $S^3$-triple over $M$, then one can endow the total space $P$ with a locally homogeneous metric $g_A$. The total space $P$ is a compact manifold of dimension three and the normalized Ricci-flow equation of metric $g_A$ on $P$ is the next evolution equation [3]

$$
(34) \quad \frac{\partial}{\partial t} g_A(t) = -2Ric \ g_A(t) + \frac{2}{3} r(t) g_A(t).
$$

where $r(t) := \int_P R(t) d\mu_t/\int_P d\mu_t$ is the average of scalar curvature $R(t)$ of $g_A(t)$ and $\mu_t$ denotes its volume form. The factor $r(t)$ is used to normalize the equation so that the volume is constant. Let $\omega_0 \in A^1(P, \mathbb{R})$ (resp. $F_\omega_0 = d\omega_0 \in A^2(P, \mathbb{R})$), be the connection 1-form (resp. the curvature 2-form) associated with connection $A_0$. Consider the one-parameter family of metrics $g_t = \lambda_t g_0$ on $M$ and, let $f_t$ be the size of the fiber at time $t$. Let $\omega_t = \omega_0$ be the initial Yang-Mills connection, for all $t \geq 0$. We search a solution of the normalized Ricci-flow equation (34) with the initial conditions $\lambda(0) = f(0) = 1$. Since $A$ is Yang-Mills, Corollary 1.5 implies that $R(t) = 2K_{g_0} - \frac{1}{2} f_t^2 |F_{\omega_t}|_{g_t}^2 = 2K_0 \lambda_t^{-1} - \frac{1}{2} f_t^2 \lambda_t^{-2}$. Therefore, the scaler curvature $R(t)$ is constant on $P$ and we have $r(t) = R(t)$. The normalized Ricci flow equation (34) take the form

$$
\begin{align*}
\dot{g}_t &= -2Ric(g_t) + f^2 |F_{\omega_t}|_{g_t}^2 g_t + \frac{2}{3} (2K_{g_t} - \frac{1}{2} f_t^2 |F_{\omega_t}|_{g_t}^2) g_t, \\
2 \dot{f}_t &= 2(\frac{1}{2} f_t^2 |F_{\omega_t}|_{g_t}^2) + \frac{2}{3} (2K_{g_t} - \frac{1}{2} f_t^2 |F_{\omega_t}|_{g_t}^2).
\end{align*}
$$

Or, equivalently

$$
\begin{align*}
\dot{g}_t &= -\frac{2}{3} Ric(g_t) + \frac{2}{3} f_t^2 |F_{\omega_t}|_{g_t}^2 g_t, \\
\dot{f}_t &= \frac{2}{3} K_{g_t} f_t - \frac{2}{3} f_t^2 |F_{\omega_t}|_{g_t}^2.
\end{align*}
$$

Since $g_t = \lambda_t g_0$ we deduce $K_{g_t} = \lambda_t^{-1} K_0$, $\text{Ric}(g_t) = \text{Ric}(g_0) = K_0 g_0$. By (32) we have $|F_{\omega_t}|_{g_0}^2 = \lambda_t^{-2} F_0$. Make use of these formulae in (35) leads to the following system of ordinary differential equations

$$
\begin{align*}
\dot{\lambda}_t &= -\frac{2}{3} K_0 + \frac{2}{3} F_0 f_t^2 \lambda_t^{-1}, \\
\dot{f}_t &= \frac{2}{3} K_0 f_t \lambda_t^{-1} - \frac{2}{3} F_0 f_t^3 \lambda_t^{-2}.
\end{align*}
$$

In the remaining part of the paper, we give an explicit solutions of these differential equations and, we describe some asymptotical behaviors of their solutions. A simple observation is that

$$
\dot{\lambda}_t f_t + \lambda_t \dot{f}_t = -\frac{2}{3} K_0 f_t + \frac{2}{3} F_0 f_t^3 \lambda_t^{-1} + \frac{2}{3} K_0 f_t - \frac{2}{3} F_0 f_t^3 \lambda_t^{-1} = 0.
$$

The initial conditions $f(0) = \lambda(0) = 1$ imply that $f(t) \lambda(t) = 1$, for all $t \geq 0$. So, to solve the equations (36) it is enough to solve the system below

$$
\dot{\lambda}_t = -\frac{2}{3} K_0 + \frac{2}{3} F_0 \lambda_t^{-3}, \quad \dot{f}_t = \lambda_t^{-1}.
$$
Thanks to Corollary 1.5 the scalar curvature of $g_{A_0}(t)$ can be computed by

\[
R_t = 2Kg_t - \frac{1}{2}f_t^2|\omega_t|^2_{g_t} = 2K_0\lambda_t^{-1} - \frac{1}{2}F_0\lambda_t^{-4}.
\]

To study the equations (37), we will consider two situations. Firstly, we suppose that $K_0 \neq 0$ and we set $a_0 := \sqrt[3]{\frac{F_0}{K_0}}$. The equations (37) is equivalent to

\[
\lambda_t = -\frac{2}{3}K_0 + \frac{2}{3}F_0\lambda_t^{-3} = -\frac{2}{3}K_0(1 - \frac{F_0}{K_0}\lambda_t^{-3}).
\]

Solving this equation leads to the following explicit solution

\[
a_0 \ln \frac{\sqrt[3]{|\lambda_t - a_0|}}{\sqrt[3]{\lambda_t^2 + a_0\lambda_t + a_0^2}} - a_0 \ln \frac{\pi}{3} \tan^{-1}\left(\frac{2\lambda_t + a_0}{\sqrt{3a_0}}\right) + \lambda_t = -\frac{2}{3}K_0t + c_0.
\]

Here, $c_0 \in \mathbb{R}$ is a real constant which can be determined by the initial condition $\lambda(0) = 1$. The rest of this work is dedicated to describe the asymptotical behavior of (normalized) Ricci flow in some interesting cases:

**Spherical geometry:** if $K_0 > 0$ and $A_0$ is non flat, then $\lambda_t \to a_0$ and $f_t \to a_0^{-1}$ as $t$ goes to infinity and the Ricci flow converge to some Einstein metric. The scalar curvature of metric $g_{A_0}(t)$ is computed due to equations (38) and $R_t \to \frac{2}{9}K_0a_0^{-1}$ as $t$ goes to infinity. In this case, the circle bundle $\pi : P \to M$ has a geometric structure with model geometry $\text{SL}_2(\mathbb{R})$.

**Geometry of the universal cover of $\text{SL}(2, \mathbb{R})$:** if $K_0 < 0$ and $A_0$ is non flat, then from the first equation of (30) we have $\lambda_t = -\frac{2}{3}K_0 + \frac{2}{3}F_0f_t^2\lambda_t^{-1}$ which implies that $\lambda_t \geq \frac{2}{3}F_0f_0^2\lambda_t^{-1}$. Use $f_0\lambda_0 = 1$ and integrate this to obtain $\lambda_t \geq \frac{2}{3}F_0t + C_0$. Therefore, $\lambda_t \to \infty$ and $f_t \to 0$ as $t$ goes to infinity. This means that the metric on the base expand without bound while the fiber of the bundle shrinks to zero. Using the equation (38), the scalar curvature falls off to zero as $t$ goes to infinity. In this case, the circle bundle $\pi : P \to M$ has a geometric structure with model geometry $\text{SL}_2(\mathbb{R})$.

**Euclidean geometry:** If $K_0 = 0$ and $A_0$ is flat ($F_{A_0} = 0$), then for all $t \geq 0$ we have $\lambda_t = f_t = 1$. Hence, the Ricci flow solution is trivial, $g_A(t) = g_A(0)$ for all $t \geq 0$. In this case, the circle bundle $\pi : P \to M$ has a geometric structure with model geometry $\mathbb{E}^3$.

Now, suppose that $K_0 \neq 0$ and $A_0$ is flat, then the Ricci flow equations are equivalent to

\[
\lambda_t = -\frac{2}{3}K_0, \quad f_t = \frac{2}{3}K_0f_t\lambda_t^{-1}
\]

Integrating this, it follows that $\lambda_t = 1 - \frac{2}{3}K_0t$ and $f_t = (1 - \frac{2}{3}K_0t)^{-1}$. Therefore, the solution of Ricci flow equation is given by

\[
g_A(t) = (1 - \frac{2}{3}K_0t)\pi^*g_0 + (1 - \frac{2}{3}K_0t)^{-2}\omega_0 \otimes \omega_0.
\]

The scalar curvature of this metric is given by $R_t = 6K_0(3 - 2K_0t)^{-1}$. Now, we can consider two cases:

**Geometry of $S^2 \times \mathbb{R}$:** if $K_0 > 0$ (and $A_0$ is flat), then at time $t = \frac{3}{2K_0}$ the metric vanishes and a curvature singularity appear. Also, we have $f_t \to \infty$, which means that the size of the fiber expands. In this case, the circle bundle $P$ has a
geometric structure with model geometry $S^2 \times \mathbb{R}$. This is the only class of locally homogeneous geometries whose Ricci flows do achieve curvature singularities [3].

**Geometry of $H^2 \times \mathbb{R}$:** If $K_0 < 0$ (and $A_0$ is flat), then the Ricci flow solution exists for any time $t \geq 0$. If $t$ goes to infinity, then $\lambda_t \to \infty$ and $f_t \to 0$, which means that $g_t$ is nonsingular and the fiber collapses. The scalar curvature falls off to zero when $t$ goes to infinity. In this case, the circle bundle $\pi : P \to M$ has a geometric structure with model geometry $H^2 \times \mathbb{R}$.

**Geometry of the Heisenberg group:** If $K_0 = 0$ and $A_0$ is non-flat ($F_{A_0} \neq 0$), then the Ricci flow equations take the form

$$\dot{\lambda}_t = \frac{2}{3} F_0 \lambda_t^{-3}, \quad f_t = \lambda_t^{-1}.$$ 

The solutions are given by $\lambda(t) = (1 + \frac{8}{3} F_0 t)^{1/4}$ and $f(t) = (1 + \frac{8}{3} F_0 t)^{-1/4}$. Hence, the Ricci flow solution is given by

$$g_A(t) = (1 + \frac{8}{3} F_0 t)^{1/4} \pi^* g_0 + (1 + \frac{8}{3} F_0 t)^{-1/2} \omega_0 \otimes \omega_0.$$ 

If $t$ goes to infinity, then $\lambda_t \to \infty$ and $f_t \to 0$, which means that the metric $g_t$ on the base manifold $M$ expands and the fiber collapses. Using remark [1,3] the scalar curvature of $g_A(t)$ is given by

$$R_t = -\frac{1}{2} f_t^2 |F_{\omega}|^2_{g_t} = \frac{1}{2} F_0 f_t^2 \lambda_t^{-2} = -\frac{1}{2} f_t^4 = -\frac{1}{2} F_0 (1 + \frac{8}{3} F_0 t)^{-1}.$$ 

Hence, the scalar curvature falls off to zero as $t$ goes to infinity. In this case, the circle bundle $\pi : P \to M$ over surface $M$ has a geometric structure with model geometry $\text{Nil}$.

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