ON THE SAMPLING AND RECOVERY OF BANDLIMITED FUNCTIONS VIA SCATTERED TRANSLATES OF THE GAUSSIAN

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Abstract. Let $\lambda$ be a positive number, and let \( \{x_j : j \in \mathbb{Z}\} \subset \mathbb{R} \) be a fixed Riesz-basis sequence, namely, \((x_j)\) is strictly increasing, and the set of functions \( \mathbb{R} \ni t \mapsto e^{ix_j t} : j \in \mathbb{Z} \) is a Riesz basis \( \text{(i.e., unconditional basis)} \) for \( L^2[-\pi, \pi] \). Given a function \( f \in L^2(\mathbb{R}) \) whose Fourier transform is zero almost everywhere outside the interval \([-\pi, \pi]\), there is a unique square-summable sequence \((a_j : j \in \mathbb{Z})\), depending on \( \lambda \) and \( f \), such that the function
\[
I_\lambda(f)(x) := \sum_{j \in \mathbb{Z}} a_j e^{-\lambda(x-x_j)^2}, \quad x \in \mathbb{R},
\]
is continuous and square integrable on \((\mathbb{R}, \leq)\), and satisfies the interpolatory conditions \( I_\lambda(f)(x_i) = f(x_i), \quad j \in \mathbb{Z} \). It is shown that \( I_\lambda(f) \) converges to \( f \) in \( L^2(\mathbb{R}) \), and also uniformly on \( \mathbb{R} \), as \( \lambda \to 0^+ \). A multidimensional version of this result is also obtained. In addition, the fundamental functions for the univariate interpolation process are defined, and some of their basic properties, including their exponential decay for large argument, are established. It is further shown that the associated interpolation operators are bounded on \( \ell^p(\mathbb{Z}) \) for every \( p \in [1, \infty] \).

1. Introduction

This paper, one in the long tradition of those involving the interpolatory theory of functions, is concerned with interpolation of data via the translates of a Gaussian kernel. The motivation for this work is twofold. The first is the theory of Cardinal Interpolation, which deals with the interpolation of data prescribed at the integer lattice, by means of the integer shifts of a single function. This subject has a rather long history, and it enjoys interesting connections with other branches of pure and applied mathematics, e.g. Toeplitz matrices, Function Theory, Harmonic Analysis, Sampling Theory. When the underlying function (whose shifts form the basis for interpolation) is taken to be the so-called Cardinal B-Spline, one deals with Cardinal Spline Interpolation, a subject championed by Schoenberg, and taken up in earnest by a host of followers. More recently, it was discovered that there is a remarkable analogy between cardinal spline interpolation and cardinal interpolation by means of the (integer) shifts of a Gaussian, a survey of which topic may be found in [RS3]. The current article may also be viewed as a contribution in this vein; it too explores further connections between the interpolatory theory of splines and that of the Gauss kernel, but does so in the context of interpolation at point sets which are more general than the integer lattice. This brings us to

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the second, and principal, motivating influence for our work, namely the researches of Lyubarskii and Madych [LM]. This duo have considered spline interpolation at certain (infinite) sets of points which are generalizations of the integer lattice, and we were prompted by their work to ask if the analogy between splines and Gaussians, very much in evidence in the context of cardinal interpolation, persists in this ‘nonuniform’ setting also. Our paper seeks to show that this is indeed the case. The influence of [LM] on our work goes further. Besides providing us with the motivating question for our studies, it also offered us an array of basic tools which we have modified and adapted.

We shall supply more particulars – of a technical nature – concerning the present paper later in this introductory section, soon after we finish discussing some requisite general material.

A basic tool in our analysis is the Fourier Transform, so we assemble some basic and relevant facts about it here; our sources for this material are [Go] and [Ch]. If \( g \in L^1(\mathbb{R}) \), then the Fourier transform of \( g \), \( \hat{g} \), is defined as follows:

\[
\hat{g}(x) := \int_{-\infty}^{\infty} g(t) e^{-ixt} dt, \quad x \in \mathbb{R}.
\]

It is known that \( \hat{g} \) is uniformly continuous on \( \mathbb{R} \), and that \( \lim_{x \to \pm \infty} \hat{g}(x) = 0 \). In general \( \hat{g} \) need not be integrable, but if it is, and if \( g \in C(\mathbb{R}) \) (the space of functions which are continuous throughout the real line), then one obtains the following inversion formula:

\[
g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(x) e^{ixt} dx, \quad t \in \mathbb{R}.
\]

Suppose now that \( g \in L^2(\mathbb{R}) \). The Fourier transform of \( g \), denoted by \( \mathcal{F}[g] \), is the function in \( L^2(\mathbb{R}) \) for which \( \lim_{N \to \infty} \| h_N - \mathcal{F}[g] \|_{L^2(\mathbb{R})} = 0 \), where

\[
h_N(x) := \int_{-N}^{N} g(t) e^{ixt} dt, \quad x \in \mathbb{R}.
\]

The integral above is finite for every real number \( x \), because \( g \) is square integrable on \( \mathbb{R} \), hence locally integrable on \( \mathbb{R} \). As \( \mathcal{F}[g] \) is obtained \( \text{(a priori)} \) only as an element in \( L^2(\mathbb{R}) \), it is determined only almost everywhere. It is known that \( \mathcal{F} \) is a linear isomorphism on \( L^2(\mathbb{R}) \), and that the following hold:

\[
\mathcal{F}[g] \in L^2(\mathbb{R}), \quad g \in L^2(\mathbb{R}); \quad \mathcal{F}[g] = \hat{g}, \ g \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}).
\]

The inversion formula for the Fourier transform of square-integrable functions takes the following form: \( \lim_{N \to \infty} \| g - H_N \|_{L^2(\mathbb{R})} = 0 \), where

\[
H_N(t) := \frac{1}{2\pi} \int_{-N}^{N} \mathcal{F}[g](x) e^{ixt} dx, \quad t \in \mathbb{R}.
\]

If, in addition to being square integrable, \( \mathcal{F}[g] \) is also integrable on \( \mathbb{R} \), then the Dominated Convergence Theorem implies that

\[
\lim_{N \to \infty} H_N(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}[g](x) e^{ixt} dx =: H(t), \quad t \in \mathbb{R},
\]
so \( g \) must coincide with \( H \) almost everywhere. As \( H \) is continuous on \( \mathbb{R} \) (in fact, \( 2\pi H(t) = \mathcal{F}[g](-t) \)), we find that, if \( g \in L_2(\mathbb{R}) \cap C(\mathbb{R}) \) and \( \mathcal{F}[g] \in L_1(\mathbb{R}) \), then
\[
g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}[g](x)e^{ixt} \, dx, \quad t \in \mathbb{R}.
\]

The functions we seek to interpolate are the so-called \textit{bandlimited} or \textit{Paley–Wiener functions}. Specifically, we define
\[
PW_\pi := \{ g \in L_2(\mathbb{R}) : \mathcal{F}[g] = 0 \text{ almost everywhere outside } [-\pi, \pi] \}.
\]
The first equation in (2) leads to the finding that \( PW_\pi \) is a closed subspace of \( L_2(\mathbb{R}) \). Moreover, if \( g \in PW_\pi \), then \( \mathcal{F}[g] \in L_2(\mathbb{R}) \) and \( \mathcal{F}[g] = 0 \) almost everywhere outside \([-\pi, \pi]\), so \( \mathcal{F}[g] \in L_1(\mathbb{R}) \); hence the inversion formula discussed in the foregoing paragraph asserts that
\[
g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}[g](x)e^{ixt} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{F}[g](x)e^{ixt} \, dx,
\]
for almost every real number \( t \). Therefore, by altering the values of \( g \), if need be, on a Lebesgue-null set, we may assume that (6) holds for every real number \( t \), and \textit{this assumption will be in place throughout the article}. In particular we shall assume that every function in \( PW_\pi \) is continuous throughout \( \mathbb{R} \). Moreover, the Bunyakovskii–Cauchy–Schwarz Inequality and (2) combine to show that \( g \) is also a bounded function:
\[
|g(t)| \leq \frac{1}{\sqrt{2\pi}} \| \mathcal{F}[g] \|_{L_2[-\pi, \pi]} = \| g \|_{L_2(\mathbb{R})}, \quad t \in \mathbb{R}.
\]

Even though it is not relevant here, we mention, at least by way of explaining our terminology, that the celebrated Paley–Wiener Theorem proclaims that a function \( \pi \) is a Riesz-basis sequence if it satisfies the following conditions: \( x_j < x_{j+1} \) for every integer \( j \), and the sequence of functions \( (e_j(t) := e^{-ix_jt} : j \in \mathbb{Z}, \ t \in \mathbb{R}) \) is a \textit{Riesz basis} for \( L_2[-\pi, \pi] \). We recall that saying that a sequence \((\varphi_j : j \in \mathbb{Z})\) in a Hilbert space \( \mathcal{H} \) is a Riesz basis for \( \mathcal{H} \) means that every element \( h \in \mathcal{H} \) admits a unique representation of the form
\[
h = \sum_{j \in \mathbb{Z}} a_j \varphi_j, \quad \sum_{j \in \mathbb{Z}} |a_j|^2 < \infty,
\]
and that there exists a universal constant \( B \) such that
\[
B^{-1} \left( \sum_{j \in \mathbb{Z}} |c_j|^2 \right)^{1/2} \leq \sum_{j \in \mathbb{Z}} c_j \varphi_j \leq B \left( \sum_{j \in \mathbb{Z}} |c_j|^2 \right)^{1/2},
\]
for every square-summable sequence \((c_j : j \in \mathbb{Z})\). Classical examples of Riesz-basis sequences are given in \([LM]\), where it is also pointed out that if \((x_j : j \in \mathbb{Z})\) is an
Riesz-basis sequence, then there exist positive numbers $q$ and $Q$ such that
\begin{equation}
q \leq x_{j+1} - x_j \leq Q, \quad j \in \mathbb{Z}.
\end{equation}

The interpolation process we study here is one that arises from translating a fixed Gaussian. Specifically, let $\lambda > 0$ be fixed, and let $(x_j : j \in \mathbb{Z})$ be an Riesz-basis sequence (a relaxation of this condition will be considered in the final section of the paper). We show in the next section that given a function $f \in PW_\pi$, there exists a unique square-summable sequence $(a_j : j \in \mathbb{Z})$—depending on $\lambda$, $f$, and the sampling points $(x_j)$—such that the function
\begin{equation}
I_\lambda(f)(x) := \sum_{j \in \mathbb{Z}} a_j e^{-\lambda(x-x_j)^2}, \quad x \in \mathbb{R},
\end{equation}
is continuous and square-integrable on $\mathbb{R}$, and satisfies the interpolatory conditions
\begin{equation}
I_\lambda(f)(x_k) = f(x_k), \quad k \in \mathbb{Z}.
\end{equation}

The function $I_\lambda(f)$ is called the Gaussian Interpolant to $f$ at the data sites $(x_k : k \in \mathbb{Z})$. We also prove, again in the upcoming section, that the map $f \mapsto I_\lambda(f)$ is a bounded linear operator from $PW_\pi$ to $L^2(\mathbb{R})$. As expected, the norm of this operator $I_\lambda$—which we refer to as the Gaussian Interpolation Operator—is shown to be bounded by a constant depending on $\lambda$ and the choice of the Riesz-basis sequence. However, this is not sufficient for our subsequent analysis, in which we intend to vary the scaling parameter $\lambda$. So in Section 3 we demonstrate that, if the underlying RRB sequence is fixed, and if $\lambda \leq 1$ (the upper bound 1 being purely a matter of convenience), then the operator norm of $I_\lambda$ can be majorized by a number which is independent of $\lambda$. Armed with this finding, we proceed to Section 4, wherein we establish the following focal convergence result:

**Theorem 1.1.** Suppose that $(x_j : j \in \mathbb{Z})$ is a (fixed) Riesz-basis sequence, and let $I_\lambda$ be the associated Gaussian Interpolation Operator. Then for any $f \in PW_\pi$, we have
\begin{equation}
\lim_{\lambda \to 0^+} I_\lambda(f) \text{ in } L^2(\mathbb{R}) \text{ and uniformly on } \mathbb{R}.
\end{equation}

We note that, in the case when $x_j = j$, this theorem was proved in [BS].

Our proofs in Sections 2–5 rely heavily on the machinery and methods developed in [LM] for cardinal splines; indeed, as mentioned earlier, our primary task in this paper has been to adapt these to the study of the Gaussian. However, most of these arguments do not extend per se to the multidimensional situation, which occupies our attention in Section 5. The results presented in this section are far from complete, and should be viewed only as partial generalizations of their univariate counterparts. Nonetheless, it is not without interest to note that tackling even this simplified situation requires a combination of the results established in one dimension and some abstract functional-analytic techniques. The paper concludes with Section 6, in which we revisit univariate interpolation, but consider sampling points which satisfy a less restrictive condition than that of giving rise to a Riesz-basis sequence. We introduce here the fundamental functions for interpolation at such data sites, and prove that they decay exponentially for large argument. In addition to being of independent interest, as readers familiar with spline theory will readily attest, this result also paves the way towards a generalization of some of the main results of Section 2.

We have attempted to make this article as self contained as possible, and we request the indulgence of those readers who may find an abundance of detail between these pages.
2. Notations and basic facts

In this section we shall reintroduce the interpolation problem which concerns us, define the corresponding interpolant and interpolation operator, and establish some of their basic properties. We shall uncover these in a series of propositions, which begins with this simple observation.

Proposition 2.1. If $\alpha$ is a positive number, then

$$
\sum_{l \in \mathbb{Z}\{0\}} e^{-\alpha(2|l|-1)^2} \leq \frac{2e^{-\alpha}}{1 - e^{-\alpha}} =: \kappa(\alpha).
$$

Proof. 

$$
\sum_{l \in \mathbb{Z}\{0\}} e^{-\alpha(2|l|-1)^2} \leq 2\sum_{l=1}^{\infty} e^{-\alpha l} = \frac{2e^{-\alpha}}{1 - e^{-\alpha}}.
$$

\hfill \Box

In what follows we shall use the following notation: given a positive number $\lambda$, the Gaussian function with scaling parameter $\lambda$ is defined by

$$
g_\lambda(x) := e^{-\lambda x^2}, \quad x \in \mathbb{R}.
$$

We recall the well-known fact (see, for example, [Go, p. 43]) that

$$
(13) \quad \mathcal{F}[g_\lambda](u) = \tilde{g}_\lambda(u) = \sqrt{\frac{\pi}{\lambda}} e^{-u^2/(4\lambda)}, \quad u \in \mathbb{R}.
$$

We now record two results from the literature; both will be of use in this section and also in Section 5.

Proposition 2.2. cf. [NSW, Lemma 2.1]

Let $\lambda$ and $q$ be fixed positive numbers, and let $\| \cdot \|_2$ denote the Euclidean norm in $\mathbb{R}^d$. There exists a number $\nu$, depending only on $d$, $\lambda$, and $q$, such that the following holds: if $(x_j)$ is any sequence in $\mathbb{R}^d$ with $\|x_j - x_k\|_2 \geq q$ for $j \neq k$, and $x$ is any point in $\mathbb{R}^d$, then $\sum_j g_\lambda(\|x - x_j\|_2) \leq \nu$.

This next result is an important finding in the theory of radial-basis functions.

Theorem 2.3. cf. [NW, Theorem 2.3] 

Let $\lambda$ and $q$ be fixed positive numbers, and let $\| \cdot \|_2$ denote the Euclidean norm in $\mathbb{R}^d$. There exists a number $\theta$, depending only on $d$, $\lambda$, and $q$, such that the following holds: if $(x_j)$ is any sequence in $\mathbb{R}^d$ with $\|x_j - x_k\|_2 \geq q$ for $j \neq k$, then $\sum_{j,k} \xi_j \xi_k g_\lambda(\|x_j - x_k\|_2) \geq \theta \sum_j |\xi_j|^2$, for every sequence of complex numbers $(\xi_j)$.

Proposition 2.4. Suppose that $(x_j : j \in \mathbb{Z})$ is a sequence of real numbers satisfying the following condition: there exists a positive number $q$ such that $x_{j+1} - x_j \geq q$ for every integer $j$. Let $\lambda > 0$ be fixed, and let $(a_j : j \in \mathbb{Z})$ be a bounded sequence of complex numbers. Then the function $\mathbb{R} \ni x \mapsto \sum_{j \in \mathbb{Z}} a_j g_\lambda(x - x_j)$ is continuous and bounded throughout the real line.

Proof. Proposition 2.2 demonstrates that the series in question is uniformly convergent throughout $\mathbb{R}$, and that the sum is a bounded function of $x$. The apparent continuity of each summand and uniform convergence imply the continuity of the limit function. \hfill \Box
Remark 2.5. Suppose that $\lambda$ is a fixed positive number, and let $(x_j : j \in \mathbb{Z})$ be a sequence satisfying the conditions of Proposition 2.4. The latter conditions imply that $|x_j - x_k| \geq |j - k|q$ for every pair of integers $j$ and $k$, so the entries of the bi-infinite matrix $(g_{\lambda}(x_k - x_j))_{k,j \in \mathbb{Z}}$ decay exponentially away from its main diagonal. So the matrix is realizable as the sum of a uniformly convergent series of diagonal matrices. Hence it acts as a bounded operator on every $\ell_\infty(\mathbb{Z})$. Moreover, as the matrix is also symmetric, Theorem 2.3 ensures that it is bounded invertible on $\ell_2(\mathbb{Z})$. In particular, given a square-summable sequence $(d_k : k \in \mathbb{Z})$, there exists a unique square-summable sequence $(a(j, \lambda) : j \in \mathbb{Z})$ such that

$$\sum_{j \in \mathbb{Z}} a(j, \lambda)g_{\lambda}(x_k - x_j) = d_k, \quad k \in \mathbb{Z}.$$  

Suppose now that $(x_j : j \in \mathbb{Z})$ is a Riesz-basis sequence. Thus, given $h \in L_2[-\pi, \pi]$, there exists a square-summable sequence $(a_j : j \in \mathbb{Z})$ such that $h(t) = \sum_{j \in \mathbb{Z}} a_j e^{-ix_j t}$ for almost every $t \in [-\pi, \pi]$. We wish to extend this function to $\mathbb{R}$ as follows:

**Proposition 2.6.** Let $(x_j : j \in \mathbb{Z})$, $h$, and $(a_j : j \in \mathbb{Z})$ be as above. The function $H(u) := \sum_{j \in \mathbb{Z}} a_j e^{-ix_j u}$ is locally square integrable on $\mathbb{R}$; in particular it is well defined for almost every real number $u$.

**Proof.** Recall from the introductory section that there exists a constant $B$ such that

$$B^{-2} \sum_{j \in \mathbb{Z}} |c_j|^2 \leq \int_{-\pi}^{\pi} \left| \sum_{j \in \mathbb{Z}} c_j e^{-ix_j t} \right|^2 dt \leq B^2 \sum_{j \in \mathbb{Z}} |c_j|^2$$

for every square-summable sequence $(c_j : j \in \mathbb{Z})$. Let $u \in [(2l-1)\pi, (2l+1)\pi]$ for some integer $l$, and define, for every positive integer $N$, the (continuous) function $H_N(u) := \sum_{j=-N}^{N} a_j e^{-ix_j u}$. If $N > M$ are positive integers, then

$$\int_{(2l-1)\pi}^{(2l+1)\pi} |H_N - H_M|^2 = \int_{(2l-1)\pi}^{(2l+1)\pi} \left| \sum_{|j|=M+1}^{N} a_j e^{-ix_j u} \right|^2 du = \int_{-\pi}^{\pi} \sum_{|j|=M+1}^{N} a_j e^{-2\pi ilx_j} e^{-ix_j u} \right|^2 du \leq B^2 \sum_{|j|=M+1}^{N} |a_j|^2,$$

the final inequality coming from (14). Ergo, the square summability of the sequence $(a_j : j \in \mathbb{Z})$ shows that $(H_N : N \in \mathbb{N})$ is a Cauchy sequence, hence convergent, in $L_2[(2l-1)\pi, (2l+1)\pi]$. This proves the desired result. \hfill \Box

For future reference, we note that the argument leading up to (15) also provides the estimate

$$\int_{(2l-1)\pi}^{(2l+1)\pi} |H|^2 = \lim_{N \to \infty} \int_{(2l-1)\pi}^{(2l+1)\pi} |H_N|^2 \leq B^2 \sum_{j \in \mathbb{Z}} |a_j|^2, \quad l \in \mathbb{Z}.$$
whence the Bunyakovskii–Cauchy–Schwarz Inequality implies that
\[
\int_{(2l-1)\pi}^{(2l+1)\pi} |H| \leq \sqrt{2\pi} B \left( \sum_{j \in \mathbb{Z}} |a_j|^2 \right)^{1/2}, \quad l \in \mathbb{Z}.
\]

The next result is the first of the two main offerings of the current section.

**Theorem 2.7.** Suppose that \( \lambda \) is a fixed positive number, and let \((x_j : j \in \mathbb{Z})\) be a Riesz-basis sequence. Assume that \( \mathfrak{a} := (a_j : j \in \mathbb{Z}) \) is a square-summable sequence. The following hold:

(i) The function
\[
s(\mathfrak{a}, x) = s(x) := \sum_{j \in \mathbb{Z}} a_j e^{ix_j}, \quad x \in \mathbb{R},
\]
belongs to \( C(\mathbb{R}) \cap L_2(\mathbb{R}) \).

(ii) The function
\[
\hat{s}(\mathfrak{a}, u) = \hat{s}(u) := e^{-u^2/(4\lambda)} \sum_{j \in \mathbb{Z}} a_j e^{-i\lambda^2 u^2}
\]
is well defined for almost every real number \( u \), and \( \hat{s} \in L_2(\mathbb{R}) \cap L_1(\mathbb{R}) \).

(iii)
\[
\mathcal{F}[s] = \sqrt{\frac{\pi}{\lambda}} \hat{s}.
\]

(iv) The map \( \mathfrak{a} := (a_j : j \in \mathbb{Z}) \mapsto s(\mathfrak{a}, x) := \sum_{j \in \mathbb{Z}} a_j e^{ix_j}, \quad x \in \mathbb{R}, \) is a bounded linear transformation from \( \ell_2(\mathbb{Z}) \) into \( L_2(\mathbb{R}) \).

*Proof.* (i) The continuity of \( s \) on \( \mathbb{R} \) follows at once from the first inequality in (10) and Proposition 2.4. Define \( s_N(x) := \sum_{j=-N}^{N} a_j e^{ix_j}, \quad x \in \mathbb{R}, \) \( N \in \mathbb{N} \). As \( s_N \in L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \) for every positive integer \( N \), equation (2), (13), and a standard Fourier-transform calculation provide the following relations: if \( N > M \) are positive integers, then
\[
2\pi \| s_N - s_M \|_{L_2(\mathbb{R})}^2 = \| \hat{s}_N - \hat{s}_M \|_{L_2(\mathbb{R})}^2
\]
\[
= \frac{\pi}{\lambda} \int_{-\infty}^{\infty} e^{-u^2/(2\lambda)} \left( \sum_{|j|=M+1}^N a_j e^{-i\lambda^2 u^2} \right)^2 du
\]
\[
= \frac{\pi}{\lambda} \sum_{l \in \mathbb{Z}} \int_{(2l-1)\pi}^{(2l+1)\pi} e^{-u^2/(2\lambda)} \left( \sum_{|j|=M+1}^N a_j e^{-i\lambda^2 u^2} \right)^2 du.
\]

Let \( H_{M,N}(u) := \sum_{|j|=M+1}^N a_j e^{-i\lambda^2 u^2}, \quad u \in \mathbb{R}. \) Using the estimates \( e^{-u^2/(2\lambda)} \leq 1 \) for \( u \in [-\pi, \pi] \), and \( e^{-u^2/(2\lambda)} \leq e^{-(2l-1)^2\pi^2/(2\lambda)} \) for \( (2l-1)\pi \leq u \leq (2l+1)\pi, \) \( l \in \mathbb{Z} \setminus \{0\} \), we find from (18) that
\[
2\pi \| s_N - s_M \|_{L_2(\mathbb{R})}^2 \leq \frac{\pi}{\lambda} \| H_{M,N} \|_{L_2([0,\pi])}^2
\]
\[
+ \sum_{l \in \mathbb{Z} \setminus \{0\}} e^{-[(2l-1)^2\pi^2/(2\lambda)]} \| H_{M,N} \|_{L_2([-\pi,(2l+1)\pi])}^2.
\]
Now let \( B \) be the constant satisfying (14) and use (15) to estimate each of the integrals on the right side of (19). This yields the relation

\[
2\pi \| s_N - s_M \|_{L^2(\mathbb{R})}^2 \leq \left( \frac{\pi}{\lambda} \right) B^2 \left( \sum_{|j| = M+1}^N |a_j|^2 \right) \left[ 1 + \sum_{\ell \in \mathbb{Z} \setminus \{0\}} e^{-2(|\ell|-1)^2 \pi^2 / (2\lambda)} \right],
\]

whence Proposition 2.1 leads to the estimate

\[
2\pi \| s_N - s_M \|_{L^2(\mathbb{R})}^2 \leq \left( \frac{B^2 \pi}{\lambda} \right) \left( \sum_{|j| = M+1}^N |a_j|^2 \right) \left[ 1 + \kappa (\pi^2 / (2\lambda)) \right],
\]

As \((a_j : j \in \mathbb{Z})\) is square summable, we find that \((s_N : N \in \mathbb{N})\) is a Cauchy sequence in \(L^2(\mathbb{R})\), and hence that \(s \in L^2(\mathbb{R})\) as promised.

(ii) Let \( H(u) := \sum_{j \in \mathbb{Z}} a_j e^{-ix_j u} \). As observed in Proposition 2.6 \( H \) is defined almost everywhere on \( \mathbb{R} \), so the same is true of \( \tilde{s} \) as well. Now the argument in (i), combined with (16), shows that

\[
\| \tilde{s} \|_{L^2(\mathbb{R})}^2 \leq B^2 \left( \sum_{j \in \mathbb{Z}} |a_j|^2 \right) \left[ 1 + \kappa (\pi^2 / (2\lambda)) \right],
\]

whilst a slight, but obvious, variation on the theme, coupled with (17), demonstrates that

\[
\| \tilde{s} \|_{L^1(\mathbb{R})} \leq \sqrt{2\pi} B \left( \sum_{j \in \mathbb{Z}} |a_j|^2 \right)^{1/2} \left[ 1 + \kappa (\pi^2 / (2\lambda)) \right],
\]

and this completes the proof.

(iii) Let \((s_N : N \in \mathbb{N})\) be the sequence defined in the proof of (i). As each \( s_N \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \) and \( \lim_{N \to \infty} \| s_N - s \|_{L^2(\mathbb{R})} = 0 \), it suffices to show, thanks to (2), that \( \lim_{N \to \infty} \| \tilde{s}_N - \sqrt{(\pi / \lambda)} \tilde{s} \|_{L^2(\mathbb{R})} = 0 \). Calculations similar to the one carried out in (i) show that

\[
\| \tilde{s}_N - \sqrt{(\pi / \lambda)} \tilde{s} \|_{L^2(\mathbb{R})}^2 = \frac{\pi}{\lambda} \int_{-\infty}^{\infty} e^{-u^2 / (2\lambda)} \left| \sum_{|j| = N+1}^\infty a_j e^{-ix_j u} \right|^2 \, du
\]

\[
\leq B^2 \pi \left( \sum_{|j| = N+1}^\infty |a_j|^2 \right) \left[ 1 + \kappa (\pi^2 / (2\lambda)) \right],
\]

and the last term approaches zero as \( N \) tends to infinity, because the sequence \((a_j : j \in \mathbb{Z})\) is square summable.

(iv) The linearity of the map is evident, and that it takes \( \ell^2(\mathbb{Z}) \) into \( L^2(\mathbb{R}) \) is the content of part (i). Now (2), part (iii) above, and (20) combine to yield the relations

\[
2\pi \| s(\pi, \cdot) \|_{L^2(\mathbb{R})}^2 = \left\| \frac{\pi}{\lambda} \tilde{s}(\pi, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq \frac{B^2 \pi}{\lambda} \left[ 1 + \kappa (\pi^2 / (2\lambda)) \right] \| \pi \|_{\ell^2(\mathbb{Z})}^2.
\]
This next result points to a useful interplay between Riesz-basis sequences and bandlimited functions (see, for example, [Yo] pp. 29-32). It serves as a prelude to the second main theorem of this section.

**Proposition 2.8.** Suppose that \( (x_j : j \in \mathbb{Z}) \) is a Riesz-basis sequence, and that \( f \in \text{PW}_\pi \). Then the (sampled) sequence \( (f(x_j) : j \in \mathbb{Z}) \) is square summable. Moreover, there is an absolute constant \( C \) – depending only on \( (x_j) \), but not on \( f \) – such that \( \sum_{j \in \mathbb{Z}} |f(x_j)|^2 \leq C^2 \|f\|_{L_2(\mathbb{R})}^2 \).

**Proof.** Let

\[
\langle h_1, h_2 \rangle := \int_{-\pi}^{\pi} h_1(h_2) \, d\theta, \quad h_1, h_2 \in L_2[-\pi, \pi],
\]
denote the standard inner product in \( L_2[-\pi, \pi] \). Let \( e_j(t) := e^{-itj}, \; j \in \mathbb{Z}, \; t \in [-\pi, \pi], \) so that (3) implies the identities

\[
2\pi f(x_j) = \langle \mathcal{F}[f], e_j \rangle, \quad j \in \mathbb{Z}.
\]

Letting \( (\tilde{e}_j : j \in \mathbb{Z}) \) be the co-ordinate functionals of \( (e_j : j \in \mathbb{Z}) \) (which means that \( \tilde{e}_j = \sum_j \langle h, e_j \rangle e_j \) for any \( h \in L_2[\pi, \pi] \)), it follows that \( (\tilde{e}_j : j \in \mathbb{Z}) \) is also a Riesz basis whose co-ordinate functionals are \( (e_j : j \in \mathbb{Z}) \). Thus,

\[
g = \sum_{j \in \mathbb{Z}} (g, e_j)\tilde{e}_j, \quad g \in L_2[-\pi, \pi].
\]

So (3) provides a universal constant \( B \) such that

\[
\sum_{j \in \mathbb{Z}} |c_j|^2 \leq B^2 \left\| \sum_{j \in \mathbb{Z}} c_j\tilde{e}_j \right\|_{L_2[-\pi, \pi]}^2
\]

for every square summable sequence \( (c_j : j \in \mathbb{Z}) \). Hence (23) and (24) imply that

\[
4\pi^2 \sum_{j \in \mathbb{Z}} |f(x_j)|^2 \leq B^2 \left\| \mathcal{F}[f] \right\|_{L_2[-\pi, \pi]}^2 = 2\pi B^2 \|f\|_{L_2(\mathbb{R})}^2,
\]

the final equation stemming from (2), and the fact that \( \mathcal{F}[f] = 0 \) almost everywhere in \( \mathbb{R} \setminus [-\pi, \pi] \).

We now state the second of the two main results in this section. Most of our work has already been accomplished; what remains is to recast the findings in the context of our interpolation problem.

**Theorem 2.9.** Let \( \lambda \) be a fixed positive number, and let \( (x_j : j \in \mathbb{Z}) \) be a Riesz-basis sequence. The following hold:

(i) Given \( f \in \text{PW}_\pi \), there exists a unique square-summable sequence \( (a(j, \lambda) : j \in \mathbb{Z}) \) such that

\[
\sum_{j \in \mathbb{Z}} a(j, \lambda) g_\lambda(x_k - x_j) = f(x_k), \quad k \in \mathbb{Z}.
\]

(ii) Let \( f \) and \( (a(j, \lambda) : j \in \mathbb{Z}) \) be as in (i). The Gaussian interpolant to \( f \) at the points \( (x_j : j \in \mathbb{Z}) \), to wit,

\[
I_\lambda(f)(x) = \sum_{j \in \mathbb{Z}} a(j, \lambda) g_\lambda(x - x_j), \quad x \in \mathbb{R},
\]

belongs to \( C(\mathbb{R}) \cap L_2(\mathbb{R}) \).
(iii) Let $f$ and $I_\lambda(f)$ be as above. The Fourier transform of $I_\lambda(f)$ is given by

$$\mathcal{F}[I_\lambda(f)](u) = \left(\frac{\pi}{\lambda}\right)^{1/2} e^{-u^2/(4\lambda)} \sum_{j \in \mathbb{Z}} a(j, \lambda) e^{-ix_j u} =: \left(\frac{\pi}{\lambda}\right)^{1/2} e^{-u^2/(4\lambda)} \Psi_\lambda(u)$$

for almost every real number $u$. Moreover, $\mathcal{F}[I_\lambda(f)] \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$.

(iv) If $f$ and $I_\lambda(f)$ are as above, then

$$I_\lambda(f)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}[I_\lambda(f)](u) e^{ixu} \, du, \quad x \in \mathbb{R}.$$ 

In particular, $I_\lambda(f) \in C_0(\mathbb{R}) := \{g \in C(\mathbb{R}) : \lim_{|x| \to \infty} g(x) = 0\}$.

(v) The Gaussian interpolation operator $I_\lambda$ is a bounded linear operator from $PW_\pi$ to $L_2(\mathbb{R})$. That is, the map $PW_\pi \ni f \mapsto I_\lambda(f)$ is linear, and there exists a positive constant $D$, depending only on $\lambda$ and $(x_j : j \in \mathbb{Z})$, such that

$$\|I_\lambda(f)\|_{L_2(\mathbb{R})} \leq D \|f\|_{L_2(\mathbb{R})}$$

for every $f \in PW_\pi$.

Proof. Assertion (i) follows from the first inequality in (10), Remark 2.5, and Proposition 2.8. Assertion (ii) obtains from part (i) of Theorem 2.7 whilst assertion (iii) is a consequence of parts (ii) and (iii) of Theorem 2.7. Assertion (iv) follows from the fact that $I_\lambda(f)$ is continuous throughout $\mathbb{R}$, that $\mathcal{F}[I_\lambda(f)] \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$, and equation (6). Moreover, this representation and equation (1) show that $2\pi I_\lambda(f)(x) = \mathcal{F}[I_\lambda(f)](-x)$ for every real number $x$. So the Riemann–Lebesgue Lemma [Gr, Theorem 4A] ensures that $I_\lambda \in C_0(\mathbb{R})$. As to (v), let $T$ be the matrix $(g_\lambda(x_k - x_j))_{j,k \in \mathbb{Z}}$, let $f \in PW_\pi$, and let $\overrightarrow{d} := (f(x_j) : j \in \mathbb{Z})$. Then $a(j, \lambda)$ is the $j$-th component of the vector $T^{-1}\overrightarrow{d}$, and this demonstrates that $I_\lambda$ is linear. Now part (iv) of Theorem 2.7 asserts that

$$\|I_\lambda(f)\|_{L_2(\mathbb{R})} = O\left(\|T^{-1}\overrightarrow{d}\|_{\ell_2(\mathbb{Z})}\right) = O\left(\|\overrightarrow{d}\|_{\ell_2(\mathbb{Z})}\right),$$

where the Big-O constant depends only on $\lambda$ and the Riesz-basis sequence $(x_j : j \in \mathbb{Z})$. Furthermore, Proposition 2.8 reveals that

$$\|\overrightarrow{d}\|_{\ell_2(\mathbb{Z})} = O\left(\|f\|_{L_2(\mathbb{R})}\right),$$

with the Big-O constant here depending only on $(x_j : j \in \mathbb{Z})$. Combining (25) with (26) finishes the proof. □

3. Uniform boundedness of the interpolation operators

In the final result of the previous section, it was shown that, for a fixed scaling parameter $\lambda$, and a fixed Riesz-basis sequence $(x_j : j \in \mathbb{Z})$, the associated interpolation operator $I_\lambda$ is a continuous linear map from $PW_\pi$ into $L_2(\mathbb{R})$. As expected, the norm of this operator was shown to be bounded by a number which depends on both the scaling parameter and the choice of the Riesz-basis sequence. The goal in the current section is to demonstrate that, if the scaling parameter is bounded above by a fixed number (taken here to be 1 for convenience), then the norm of $I_\lambda$ can be bounded by a number which depends only on $(x_j : j \in \mathbb{Z})$. The proofs in this section (as well as in the next) are patterned after [LM].

Let $(x_j : j \in \mathbb{Z})$ be a Riesz-basis sequence, and let $B$ be the associated constant satisfying the inequalities in (13). Given $h \in L_2[-\pi, \pi]$, there is a square-summable
sequence \((a_j : j \in \mathbb{Z})\) such that \(h(t) = \sum_{j \in \mathbb{Z}} a_je^{-ixjt}\) for almost every \(t \in [-\pi, \pi]\). Let \(H\) denote the extension of \(h\) to almost all of \(\mathbb{R}\), as considered in Proposition 2.6. Given an integer \(l\), we define the following linear map \(A_l\) on \(L_2[-\pi, \pi]\):

\[
A_l(h)(t) := H(t + 2\pi l) = \sum_{j \in \mathbb{Z}} a_je^{-ixj(t + 2\pi l)}
\]

for almost every \(t \in [-\pi, \pi]\). We see from (16) and (14) that

\[
(A_l h)_{L_2[-\pi, \pi]} = \|H\|_{L_2([2l-1] \pi, (2l+1) \pi)} \leq B^2 \sum_{j \in \mathbb{Z}} |a_j|^2 \leq B^4 \|h\|_{L_2[-\pi, \pi]}^2.
\]

Thus every \(A_l\) is a bounded operator from \(L_2[-\pi, \pi]\) into itself; moreover, the associated operator norms of these operators are uniformly bounded:

\[
\|A_l\| \leq B^2.
\]

In what follows, we shall assume that \((x_j : j \in \mathbb{Z})\) is a (fixed) Riesz-basis sequence, and let \(e_j(t) := e^{-ixjt}\), \(t \in \mathbb{R}, j \in \mathbb{Z}\). We also denote by \(\langle \cdot , \cdot \rangle\) the standard inner product in \(L_2[-\pi, \pi]\), as defined via (22). Our first main task now is to exploit the presence of the Riesz basis \((e_j : j \in \mathbb{Z})\) in \(L_2[-\pi, \pi]\) to find an effective representation for the Fourier transform of the Gaussian interpolant to a given bandlimited function, on the interval \([-\pi, \pi]\). We begin with a pair of preliminary observations:

**Lemma 3.1.** Let \((x_j)\) and \((e_j)\) be as above, and let \(f\) be a given function in \(PW_\pi\). If \(\phi \in L_2[-\pi, \pi]\) satisfies the conditions

\[
2\pi f(x_k) = \int_{-\pi}^{\pi} \phi(t)e^{-ix_k t} \, dt, \quad k \in \mathbb{Z},
\]

then \(F[f]\) agrees with \(\phi\) in \(L_2[-\pi, \pi]\).

**Proof.** Equations (22) and (30) reveal that \(\langle F[f], e_k \rangle = \langle \phi, e_k \rangle\) for every integer \(k\), and the required result follows from (24).

**Lemma 3.2.** Let \((x_j)\) and \((e_j)\) be as above, and let \(B\) be the constant satisfying (14). Let \(h \in L_2[-\pi, \pi]\), and let \(\alpha > 0\). Define

\[
\phi_k = A_l^*(e^{-\alpha (l+2\pi l)^2} A_l(h)), \quad k \in \mathbb{Z}.
\]

Then

\[
\|\phi_0\|_{L_2[-\pi, \pi]} \leq \|h\|_{L_2[-\pi, \pi]} \quad \text{and} \quad \sum_{l \in \mathbb{Z}\setminus\{0\}} \|\phi_l\|_{L_2[-\pi, \pi]} \leq \|h\|_{L_2[-\pi, \pi]} B^4 \kappa^2 \alpha,
\]

where \(\kappa\) is the familiar function from Proposition 2.7. In particular, the series \(\sum_{l \in \mathbb{Z}} \phi_l\) converges in \(L_2[-\pi, \pi]\).

**Proof.** We note that

\[
\|\phi_0\|_{L_2[-\pi, \pi]} = \left\|e^{-\alpha (\cdot)^2} h\right\|_{L_2[-\pi, \pi]} \leq \|h\|_{L_2[-\pi, \pi]},
\]

\[
\sum_{l \in \mathbb{Z}\setminus\{0\}} \|\phi_l\|_{L_2[-\pi, \pi]} \leq B^4 \kappa^2 \alpha,
\]

\[
\|\phi_l\|_{L_2[-\pi, \pi]} \leq B^4 \kappa^2 \alpha,
\]

\[
\|A_l(h)\|_{L_2[-\pi, \pi]} \leq B^2 \|h\|_{L_2[-\pi, \pi]}.
\]

\[
\sum_{l \in \mathbb{Z}\setminus\{0\}} \|\phi_l\|_{L_2[-\pi, \pi]} \leq B^4 \kappa^2 \alpha,
\]

\[
\|\phi_l\|_{L_2[-\pi, \pi]} \leq B^4 \kappa^2 \alpha.
\]
Theorem 3.3. Let $\lambda > 0$ be fixed, and let $f \in PW_\pi$. Let $\psi_\lambda$ denote the restriction, to the interval $[-\pi, \pi]$, of the function $\Psi_\lambda$ given in part (iii) of Theorem 2.9. Then

\begin{equation}
\mathcal{F}[f] = \mathcal{F}[I_\lambda(f)] + \sqrt{\frac{\pi}{\lambda}} \sum_{l \in \mathbb{Z}\setminus\{0\}} A_l^* \left( e^{-(-2\pi l)^2/(4\lambda)} A_l(\psi_\lambda) \right)
onumber
\end{equation}

on $[-\pi, \pi]$.

Proof. Let $\phi$ denote the function on the right side of (34). In view of Lemma 3.1 it suffices to show that

\begin{equation}
2\pi f(x_k) = \langle \phi, e_k \rangle, \quad k \in \mathbb{Z}.
onumber
\end{equation}

Now Theorem 2.9 implies the relations

\begin{equation}
2\pi f(x_k) = 2\pi I_\lambda(f)(x_k) = \sqrt{\frac{\pi}{\lambda}} \int_{-\infty}^{\infty} e^{-u^2/(4\lambda)} \Psi_\lambda(u) e^{ix_k u} du, \quad k \in \mathbb{Z},
\end{equation}

whilst

\begin{align}
\int_{-\infty}^{\infty} e^{-u^2/(4\lambda)} \Psi_\lambda(u) e^{ix_k u} du &= \sum_{l \in \mathbb{Z}} \int_{(2l-1)\pi}^{(2l+1)\pi} e^{-u^2/(4\lambda)} \Psi_\lambda(u) e^{ix_k u} du \\
&= \sum_{l \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{-(t+2\pi l)^2/(4\lambda)} \Psi_\lambda(t) e^{ix_k (t+2\pi l)} dt \\
&= \sum_{l \in \mathbb{Z}} \int_{-\pi}^{\pi} e^{-(t+2\pi l)^2/(4\lambda)} A_l(\psi_\lambda(t)) A_l(e_k(t)) dt \\
&= \sum_{l \in \mathbb{Z}} \langle A_l^* e^{-(t+2\pi l)^2/(4\lambda)} A_l(\psi_\lambda(t)), e_k(t) \rangle \\
&= \sum_{l \in \mathbb{Z}} A_l^* e^{-(t+2\pi l)^2/(4\lambda)} A_l(\psi_\lambda), e_k(t) \rangle,
\end{align}

the final step being justified by Lemma 3.2. Noting that

\begin{align}
\sum_{l \in \mathbb{Z}} A_l^* e^{-(t+2\pi l)^2/(4\lambda)} A_l(\psi_\lambda) &= e^{-(t)^2/(4\lambda)} \psi_\lambda + \sum_{l \in \mathbb{Z}\setminus\{0\}} A_l^* e^{-(t+2\pi l)^2/(4\lambda)} A_l(\psi_\lambda),
\end{align}

we find that (37), (36), and part (iii) of Theorem 2.9 yield (34), and with it the proof.

Combining (34) and (31) leads directly to the following:
Corollary 3.4. Suppose that $\kappa$, $\lambda$, $f$, $\psi$, and $B$ are as before. Then

$$\|\mathcal{F}[I_\lambda(f)]\|_{L_2[-\pi,\pi]} \leq \|\mathcal{F}[f]\|_{L_2[-\pi,\pi]} + \sqrt{\frac{\pi}{\lambda}} B^4 \kappa(\pi^2/(4\lambda)) \|\psi\|_{L_2[-\pi,\pi]}.$$

The preceding corollary shows that, if $f$ is bandlimited, then the energy of the Fourier transform of (its Gaussian interpolant) $I_\lambda(f)$ on the interval $[-\pi, \pi]$ is controlled by that of the Fourier transform of $f$ on that interval, plus another term which involves the energy of $\psi$ on the interval. Our next task is to show that this second term can also be bounded effectively via the energy of $\mathcal{F}[f]$ on $[-\pi, \pi]$. Before proceeding with this, however, we pause to consider formally, an operator which has already made its debut, albeit indirectly, in Theorem 3.3. This operator will also play a role later in this section, and a larger one in the next.

Given a positive number $\alpha$, we define the operator $T_{[\alpha]}$ on $L_2[-\pi, \pi]$ as follows:

$$T_{[\alpha]}(h) := e^{\pi^2 \alpha} \sum_{l \in \mathbb{Z}\setminus\{0\}} A_l \left( e^{-\alpha(\pi^2 t^2)} A_l(h) \right), \quad h \in L_2[-\pi, \pi].$$

That this operator is well defined is guaranteed by Lemma 3.2 whilst its linearity is plain. The following properties of $T_{[\alpha]}$ are easy to verify.

Proposition 3.5. The operator $T_{[\alpha]}$ is self adjoint, positive, and its norm is no larger than $e^{-\pi^2 \alpha} B^4 \kappa(\pi^2 \alpha)$, where $\kappa$ is the function defined through Proposition 2.1, and $B$ is the familiar constant associated to the given Riesz-basis sequence $(x_j)$.

We now return to the task of carrying forward the estimate in Corollary 3.3. The first order of business is to attend to $\|\psi\|_{L_2[-\pi,\pi]}$:

Proposition 3.6. The following holds:

$$\|\psi\|_{L_2[-\pi,\pi]} \leq \sqrt{\frac{\pi}{\lambda}} e^{\pi^2/(4\lambda)} \|\mathcal{F}[f]\|_{L_2[-\pi,\pi]}.$$

Proof. Equation (33) asserts that

$$\mathcal{F}[f] = \mathcal{F}[I_\lambda(f)] + e^{-\pi^2/(4\lambda)} \sqrt{\frac{\pi}{\lambda}} T_{[\pi/(4\lambda)]} (\psi).$$

As

$$\langle \mathcal{F}[I_\lambda(f)], \psi \rangle = \sqrt{\frac{\pi}{\lambda}} \int_{-\pi}^{\pi} e^{-u^2/(4\lambda)} |\psi(u)|^2 \, du \geq 0,$$

and $T_{[\pi/(4\lambda)]}$ is a positive operator, we find from (33) that $\langle \mathcal{F}[f], \psi \rangle$ is nonnegative, and also that $\langle \mathcal{F}[f], \psi \rangle \geq \langle \mathcal{F}[I_\lambda(f)], \psi \rangle$. Hence the Bunyakovskii–Cauchy–Schwarz inequality and (10) lead to the relations

$$\|\mathcal{F}[f]\|_{L_2[-\pi,\pi]} \|\psi\|_{L_2[-\pi,\pi]} \geq \sqrt{\frac{\pi}{\lambda}} \int_{-\pi}^{\pi} e^{-u^2/(4\lambda)} |\psi(u)|^2 \, du \geq \sqrt{\frac{\pi}{\lambda}} e^{-\pi^2/(4\lambda)} \|\psi\|_{L_2[-\pi,\pi]}^2,$$

and the required result follows directly. \qed

The upcoming corollary is obtained via a combination of Corollary 3.4. Proposition 3.6. Equation (2) and the fact that $e^{\pi^2/4\lambda} \kappa(\pi^2/(4\lambda)) = (1 - e^{-\pi^2/(4\lambda)})^{-1} \leq 2$, whenever $\lambda \leq 1$. 

Corollary 3.7. Suppose that $\kappa$, $\lambda$, $f$, $\psi$, and $B$ are as before. Then

$$\|\mathcal{F}[I_\lambda(f)]\|_{L_2[-\pi,\pi]} \leq \|\mathcal{F}[f]\|_{L_2[-\pi,\pi]} + \sqrt{\frac{\pi}{\lambda}} B^4 \kappa(\pi^2/(4\lambda)) \|\psi\|_{L_2[-\pi,\pi]}.$$
Corollary 3.7. Assume that $0 < \lambda \leq 1$, and let $(x_j)$ and $f$ be as above. The following holds:

$$
\|F[I_\lambda(f)]\|_{L_2[-\pi,\pi]} \leq B^2 \sqrt{\frac{2\pi}{\lambda}} \|\Psi\|_{L_2[-\pi,\pi]} \sqrt{\kappa(\pi^2/(2\lambda))} \leq \sqrt{8\pi} B^2 \|f\|_{L_2(\mathbb{R})}.
$$

The preceding result accomplishes the first half of what we set about to do in this section. The second part will be dealt with next; our deliberations will be quite brief, for the proof is now familiar terrain.

Proposition 3.8. Let $0 < \lambda \leq 1$. The following holds:

$$
\|F[I_\lambda(f)]\|_{L_2(\mathbb{R}\setminus[-\pi,\pi])} \leq B^2 \sqrt{\frac{2\pi}{\lambda}} \|\Psi\|_{L_2[-\pi,\pi]} \sqrt{\kappa(\pi^2/(2\lambda))} \leq \sqrt{8\pi} B^2 \|f\|_{L_2(\mathbb{R})}.
$$

Proof. We begin by noting that

$$
\|F[I_\lambda(f)]\|_{L_2(\mathbb{R}\setminus[-\pi,\pi])}^2 \leq B^2 \sqrt{\frac{2\pi}{\lambda}} \|\Psi\|_{L_2[-\pi,\pi]} \sqrt{\kappa(\pi^2/(2\lambda))} \leq \sqrt{8\pi} B^2 \|f\|_{L_2(\mathbb{R})}.
$$

The last term in (41) may be bounded as follows:

$$
\frac{\pi}{\lambda} \sum_{l \in \mathbb{Z}\setminus\{0\}} \int_{-\pi}^{\pi} e^{-(l+2\pi)t^2/(2\lambda)} \|A_l(\psi_\lambda)(t)\|^2 dt
$$

$$
\leq \frac{\pi}{\lambda} \sum_{l \in \mathbb{Z}\setminus\{0\}} e^{-(2(\psi^2/4\lambda) - 1)} \|A_l(\psi_\lambda)\|^2_{L_2[-\pi,\pi]}
$$

$$
\leq \frac{B^4\pi}{\lambda} \|\psi\|_{L_2[-\pi,\pi]}^2 \kappa(\pi^2/(2\lambda)),
$$

the last inequality being consequent upon (29). This proves the first of the two stated inequalities.

The second inequality follows from the first, by way of Proposition 3.6 (2) and the fact that $(1 - e^{-\pi^2/4\lambda})^{-1} \leq 2$, whenever $\lambda \leq 1$. 

We close this section by summarizing the findings of Corollary 3.7 and Proposition 3.8.

Theorem 3.9. Suppose that $(x_j : j \in \mathbb{Z})$ is a fixed Riesz-basis sequence. Then 

$$
\{I_\lambda : 0 < \lambda \leq 1\}
$$

is a uniformly-bounded family of linear operators from $PW_2$ to $L_2(\mathbb{R})$.

4. Convergence of $I_\lambda$

This section is devoted to the proof of the convergence result stated in the introduction (Theorem 1.1). We begin by laying some requisite groundwork. Let $\alpha$ be a fixed positive number. Recall the linear operator $T_{[\alpha]}$ from (35):

$$
T_{[\alpha]}(h) := e^{\pi^2 \alpha} \sum_{l \in \mathbb{Z}\setminus\{0\}} A_l^* \left( e^{-\alpha((\cdot+2\pi)^2)} A_l(h) \right), \quad h \in L_2[-\pi,\pi].
$$

We now define the following (multiplier) operator on $L_2[-\pi,\pi]$:

$$
M_{[\alpha]}(h) := e^{-\alpha((\pi^2/(\cdot)^2))} h, \quad h \in L_2[-\pi,\pi].
$$

The following properties of $M_{[\alpha]}$ are easy to verify.
Then there is an following:

Proposition 4.1. The operator $\mathcal{M}_{[\alpha]}$ is a bounded linear operator on $L_2[\pi, \pi]$, whose norm does not exceed 1. Moreover, it is self adjoint, strictly positive, and invertible.

In what follows, we let $(x_j : j \in \mathbb{Z})$ be a fixed Riesz-basis sequence, and let $r : L_2(\mathbb{R}) \to L_2[-\pi, \pi]$ denote the map which sends a function in $L_2(\mathbb{R})$ to its restriction to the interval $[-\pi, \pi]$; note that $r$ is a bounded linear map with unit norm.

Let $f \in PW_\pi$. Recall (from part (iii) of Theorem 2.9) that, if $I_\lambda(f)$ denotes the Gaussian interpolant to $f$ at the points $(x_j)$, then

\begin{equation}
\psi_\lambda(t) = \sqrt{(\lambda/\pi)} e^{t^2/(4\lambda)} \mathcal{F}[I_\lambda(f)](t)
\end{equation}

for almost every $t$ in $[-\pi, \pi]$ (remembering that $\psi_\lambda$ is the restriction of $\Psi_\lambda$ to the interval $[-\pi, \pi]$). With all this in mind, we find from the definitions of $\mathcal{T}_{[\alpha]}$ and $\mathcal{M}_{[\alpha]}$ that equation (14) may be cast in the following form:

\begin{equation}
(\mathcal{I} + \mathcal{T}_{[1/(4\lambda)]} \mathcal{M}_{[1/(4\lambda)]}) r(\mathcal{F}[I_\lambda(f)]) = r(\mathcal{F}[f]),
\end{equation}

where $\mathcal{I}$ denotes the identity on $L_2[-\pi, \pi]$.

Suppose that $g \in L_2[-\pi, \pi]$, and let

$$
\tilde{g}(t) = \begin{cases} 
  g(t) & \text{if } t \in [-\pi, \pi]; \\
  0 & \text{if } t \in \mathbb{R} \setminus [-\pi, \pi]. 
\end{cases}
$$

Then there is an $f \in PW_\pi$ such that $\mathcal{F}[f] = \tilde{g}$; in fact, we may take $f$ to be the following:

$$
f(x) := \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{2\pi i xt} dt, \quad x \in \mathbb{R}.
$$

Let $I_\lambda(f)$ denote the interpolant to $f$ at the points $(x_j)$, and define $L_\lambda(g) := r(\mathcal{F}[I_\lambda(f)])$; in other words, $L_\lambda = r \circ \mathcal{F} \circ I_\lambda \circ \mathcal{F}^{-1}$. As the Fourier transform $\mathcal{F}$ is a linear isomorphism on $L_2(\mathbb{R})$, and the maps $I_\lambda$ and $r$ are linear and continuous, the map $g \mapsto L_\lambda(g)$ is a continuous linear operator on $L_2[-\pi, \pi]$; moreover, equation (15) affirms that

\begin{equation}
(\mathcal{I} + \mathcal{T}_{[1/(4\lambda)]} \mathcal{M}_{[1/(4\lambda)]}) L_\lambda(g) = g.
\end{equation}

This being true for every $g \in L_2[-\pi, \pi]$, we deduce that the map $\mathcal{I} + \mathcal{T}_{[1/(4\lambda)]} \mathcal{M}_{[1/(4\lambda)]}$ is surjective on $L_2[-\pi, \pi]$, and that $L_\lambda$ is a right inverse of $\mathcal{I} + \mathcal{T}_{[1/(4\lambda)]} \mathcal{M}_{[1/(4\lambda)]}$. We now show that $\mathcal{I} + \mathcal{T}_{[1/(4\lambda)]} \mathcal{M}_{[1/(4\lambda)]}$ is, in fact, invertible on $L_2[-\pi, \pi]$.

Proposition 4.2. The map $\mathcal{I} + \mathcal{T}_{[1/(4\lambda)]} \mathcal{M}_{[1/(4\lambda)]}$ is injective, hence invertible, on $L_2[-\pi, \pi]$. Moreover, there is a constant $\Delta$, depending only on the sequence $(x_j)$, such that

$$
\left\| (\mathcal{I} + \mathcal{T}_{[1/(4\lambda)]} \mathcal{M}_{[1/(4\lambda)]})^{-1} \right\| \leq \Delta, \quad 0 < \lambda \leq 1.
$$

Proof. Suppose that $(\mathcal{I} + \mathcal{T}_{[1/(4\lambda)]} \mathcal{M}_{[1/(4\lambda)]}) g = 0$ for some $g \in L_2[-\pi, \pi]$. Then

$$
0 = \langle (\mathcal{I} + \mathcal{T}_{[1/(4\lambda)]} \mathcal{M}_{[1/(4\lambda)]}) g, \mathcal{M}_{[1/(4\lambda)]} g \rangle
$$

$$
= \langle g, \mathcal{M}_{[1/(4\lambda)]} g \rangle + \langle \mathcal{T}_{[1/(4\lambda)]} \mathcal{M}_{[1/(4\lambda)]} g, \mathcal{M}_{[1/(4\lambda)]} g \rangle
$$

$$
\geq \langle g, \mathcal{M}_{[1/(4\lambda)]} g \rangle \geq 0,
$$

where we have used the positivity of the operators $\mathcal{T}_{[1/(4\lambda)]}$ and $\mathcal{M}_{[1/(4\lambda)]}$ to obtain the two inequalities above. It follows that $\mathcal{M}_{[1/(4\lambda)]} g = 0$, and, as $\mathcal{M}_{[1/(4\lambda)]}$ is strictly
positive, \textit{g} must be zero. Hence \( I + T_{[1/(4\lambda)]}M_{[1/(4\lambda)]} \) is injective, and therefore invertible. So (46) may now be stated as follows:

\[
(I + T_{[1/(4\lambda)]}M_{[1/(4\lambda)]})^{-1} = L_{\lambda}.
\]

Consequently, the uniform boundedness of \( \| (I + T_{[1/(4\lambda)]}M_{[1/(4\lambda)]})^{-1} \| \) for \( \lambda \in (0, 1) \) obtains from recalling the equation \( L_{\lambda} = r \circ F \circ I_{\lambda} \circ F^{-1} \), along with Theorem 3.9.

We are now ready for the first of the two focal results of this section.

**Theorem 4.3.** If \( f \in PW_\pi \), then

\[
\lim_{\lambda \to 0^+} \| f - I_{\lambda}(f) \|_{L_2(\mathbb{R})} = 0.
\]

**Proof.** In view of the first identity in (2), it is sufficient to show that

\[
\lim_{\lambda \to 0^+} \| F[f] - F[I_{\lambda}(f)] \|_{L_2(\mathbb{R})} = 0.
\]

Assume that \( 0 < \lambda \leq 1 \), and let \( \lambda' := 1/(4\lambda) \). As \( F[f] \) is zero almost everywhere outside \([-\pi, \pi] \), we see that

\[
\| F[f] - F[I_{\lambda}(f)] \|_{L_2(\mathbb{R})}^2 = \| F[f] - F[I_{\lambda}(f)] \|_{L_2([-\pi, \pi]')}^2 + \| F[I_{\lambda}(f)] \|_{L_2(\mathbb{R} \setminus [-\pi, \pi])}^2.
\]

On the interval \([-\pi, \pi] \), we have, via (45), that

\[
F[f] - F[I_{\lambda}(f)] = \left[ I - \left( I + T_{[\lambda']} M_{[\lambda']} \right)^{-1} \right] F[f] = \left( I + T_{[\lambda']} M_{[\lambda']} \right)^{-1} T_{[\lambda']} M_{[\lambda']} (F[f]),
\]

where the second step is a matter of direct verification. Consequently

\[
\| F[f] - F[I_{\lambda}(f)] \|_{L_2([-\pi, \pi])} \leq \left\| \left( I + T_{[\lambda']} M_{[\lambda']} \right)^{-1} \right\| \left\| T_{[\lambda']} \right\| \left\| M_{[\lambda']} (F[f]) \right\|_{L_2([-\pi, \pi])}.
\]

Now Proposition 1.2 provides a positive constant \( \Delta \) which bounds the first term on the right of (49) for every \( \lambda \in (0, 1) \). As \( \kappa(\pi^2/(4\lambda)) = O(e^{-\pi^2/(4\lambda)}) \) for \( 0 < \lambda \leq 1 \), Proposition 3.9 implies that

\[
\left\| T_{[\lambda']} \right\| = O \left( e^{\pi^2/(4\lambda)} \kappa(\pi^2/(4\lambda)) \right) = O(1), \quad 0 < \lambda \leq 1,
\]

for some Big-O constant which is independent of \( \lambda \). Using this pair of estimates in (49) provides

\[
\| F[f] - F[I_{\lambda}(f)] \|_{L_2([-\pi, \pi])} \leq O \left( \| M_{[\lambda']} (F[f]) \|_{L_2([-\pi, \pi])} \right), \quad 0 < \lambda \leq 1.
\]
Turning to the second term on the right of (48), we see from Proposition 3.8, (44), (51) that

\[
\| \mathcal{F}[I_\lambda(f)] \|_{L^2([-\pi, \pi])}^2 = O \left( \frac{\| \psi \|_{L^2} \| \kappa(\pi^2/(2\lambda)) \|}{\lambda} \right)
\]

\[
= O \left( \| e^{(\cdot)^2/(4\lambda)} \mathcal{F}[I_\lambda(f)] \|_{L^2} \| \kappa(\pi^2/(2\lambda)) \| \right)
\]

\[
= O \left( e^{\pi^2/(2\lambda)} \| \mathcal{M}_{\lambda'} \mathcal{F}[I_\lambda(f)] \|_{L^2} \| \kappa(\pi^2/(2\lambda)) \| \right)
\]

\[
= O \left( \| \mathcal{M}_{\lambda'} \mathcal{F}[I_\lambda(f)] \|_{L^2} \right), \quad 0 < \lambda \leq 1,
\]

the final step resulting from the (oft cited) estimate \( \kappa(\pi^2/(2\lambda)) = O(e^{-\pi^2/(2\lambda)}) \), \( 0 < \lambda \leq 1 \). Now

\[
\| \mathcal{M}_{\lambda'} \mathcal{F}[I_\lambda(f)] \|_{L^2}^2 = O \left( \| \mathcal{M}_{\lambda'} \mathcal{F}[f] \|_{L^2}^2 \right),
\]

because \( \| \mathcal{M}_{\lambda'} \| \leq 1 \) (Proposition 4.1), and (50) holds. Combining (52) and (50) with (48), we find that

\[
\| \mathcal{F}[f] - \mathcal{F}[I_\lambda(f)] \|_{L^2}^2 = O \left( \| \mathcal{M}_{\lambda'} \mathcal{F}[f] \|_{L^2}^2 \right) = o(1), \quad \lambda \to 0^+,
\]

the last assertion being a consequence of the Dominated Convergence Theorem. This establishes (47), and the proof is complete. \( \square \)

The final theorem of the section deals with uniform convergence.

**Theorem 4.4.** If \( f \in PW_\pi \), then \( \lim_{\lambda \to 0^+} I_\lambda(f)(x) = f(x), \ x \in \mathbb{R} \), and the convergence is uniform on \( \mathbb{R} \). In particular, the operators \( I_\lambda, 0 < \lambda \leq 1 \), are uniformly bounded as operators from \( PW_\pi \) to \( C_0(\mathbb{R}) \), via the Uniform Boundedness Principle.

**Proof.** Assume that \( 0 < \lambda \leq 1 \), and let \( x \in \mathbb{R} \). From (6), part (iv) of Theorem 2.3, and the fact that \( \mathcal{F}[f] = 0 \) almost everywhere on \( \mathbb{R} \setminus [-\pi, \pi] \), we see that

\[
f(x) - I_\lambda(f)(x) = \frac{1}{2\pi} \int_{-\pi}^\pi (\mathcal{F}[f](u) - \mathcal{F}[I_\lambda(f)](u)) e^{ixu} \, du
\]

\[
- \frac{1}{2\pi} \int_{\mathbb{R} \setminus [-\pi, \pi]} \mathcal{F}[I_\lambda(f)](u) e^{ixu} \, du.
\]

The modulus of the first term on the right side of (54) is no larger than

\[
\| \mathcal{F}[f] - \mathcal{F}[I_\lambda(f)] \|_{L^2}^2 = O \left( \| \mathcal{F}[f] - \mathcal{F}[I_\lambda(f)] \|_{L^2}^2 \right) = o(1), \ \lambda \to 0^+,
\]

and the final step resulting from the (oft cited) estimate \( \kappa(\pi^2/(2\lambda)) = O(e^{-\pi^2/(2\lambda)}) \), \( 0 < \lambda \leq 1 \). Now

\[
\| \mathcal{M}_{\lambda'} \mathcal{F}[I_\lambda(f)] \|_{L^2}^2 = O \left( \| \mathcal{M}_{\lambda'} \mathcal{F}[f] \|_{L^2}^2 \right),
\]

because \( \| \mathcal{M}_{\lambda'} \| \leq 1 \) (Proposition 4.1), and (50) holds. Combining (52) and (50) with (48), we find that

\[
\| \mathcal{F}[f] - \mathcal{F}[I_\lambda(f)] \|_{L^2}^2 = O \left( \| \mathcal{M}_{\lambda'} \mathcal{F}[f] \|_{L^2}^2 \right) = o(1), \quad \lambda \to 0^+,
\]

the last assertion being a consequence of the Dominated Convergence Theorem. This establishes (47), and the proof is complete. \( \square \)
via (2) and Theorem 4.3, The second term on the right side of (54) is estimated in a familiar way:

$$\left| \int_{\mathbb{R}\setminus[-\pi,\pi]} \mathcal{F}\left[I_\lambda(f)\right](u)e^{iuw} \, du \right|$$

$$\leq \int_{\mathbb{R}\setminus[-\pi,\pi]} |\mathcal{F}\left[I_\lambda(f)\right](u)| \, du$$

$$= \frac{\sqrt{\pi}}{\lambda} \sum_{l \in \mathbb{Z}\setminus\{0\}} \int_{(2l-1)\pi}^{(2l+1)\pi} e^{-u^2/(4\lambda)} |\Psi_\lambda(u)| \, du$$

$$\leq \frac{\sqrt{\pi}}{\lambda} \sum_{l \in \mathbb{Z}\setminus\{0\}} e^{-(2l\pi-1)^2}\pi^2/(4\lambda) \|A_1(\psi_\lambda)\|_{L_1[-\pi,\pi]}$$

$$= O\left( \frac{1}{\lambda} \sum_{l \in \mathbb{Z}\setminus\{0\}} e^{-(2l\pi-1)^2}\pi^2/(4\lambda) \|A_1(\psi_\lambda)\|_{L_2[-\pi,\pi]} \right)$$

$$= O\left( \|\psi_\lambda\|_{L_2[-\pi,\pi]} \sqrt{\pi(\pi^2/(4\lambda))} \right).$$

Borrowing the argument which led up to (51), (52), and the final conclusion of (53), we deduce that $I_\lambda(f)(x)$ converges to $f(x)$ uniformly in $\mathbb{R}$. \qed

5. A Multidimensional Extension

We consider now the multidimensional Gaussian interpolation operator. Let $d \in \mathbb{N}$, and let $(x_j : j \in \mathbb{N}) \subset \mathbb{R}^d$. We say that $(x_j : j \in \mathbb{N}) = (x_{(j,1)}, x_{(j,2)}, \ldots, x_{(j,d)} : j \in \mathbb{N}) \subset \mathbb{R}^d$ is a \emph{d-dimensional Riesz-basis sequence} if the sequence $(e^{(j)} : j \in \mathbb{N})$, with

$$e_j : [-\pi, \pi]^d \to C, \quad e_j(t_1, t_2, \ldots, t_d) := e^{-i(x_j, t)} = e^{-i\sum_{i=1}^d x_{(j,i)} t_i},$$

is a Riesz basis of $L_2[-\pi, \pi]^d$.

In general there is no \emph{natural indexing} of the elements of $\{x_j\}$ by $\mathbb{Z}$ or $\mathbb{Z}^d$ if $d \geq 2$; so we index generic d-dimensional Riesz-basis sequences by $\mathbb{N}$. Later we shall concentrate on \emph{grids} in $\mathbb{R}^d$, i.e., on Riesz bases in $\mathbb{R}^d$ which are products of one-dimensional Riesz-basis sequences. In that case the natural index set is $\mathbb{Z}^d$. We note that, as in the 1-dimensional case, a Riesz basis-sequence in $\mathbb{R}^d$ also has to be separated [LM].

The \emph{d-dimensional Gaussian function with scaling parameter $\lambda > 0$} is defined by

$$g^{(d)}_\lambda(x_1, x_2, \ldots, x_d) = e^{-\lambda\|x\|^2} = e^{-\lambda\sum_{i=1}^d x_i^2}, \quad x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d.$$

The \emph{Fourier transform} on $L_1(\mathbb{R}^d)$ and that in $L_2(\mathbb{R}^d)$ are defined as in the 1-dimensional case, and we denote it by $\hat{g}$, if $g \in L_1(\mathbb{R}^d)$, and by $\mathcal{F}^{(d)}[g]$, if $g \in L_2(\mathbb{R}^d)$.

The Paley-Wiener functions on $\mathbb{R}^d$ are given by

$$PW^{(d)}_{\pi} := \{g \in L_2(\mathbb{R}^d) : \mathcal{F}^{(d)}[g] = 0 \text{ almost everywhere outside } [-\pi, \pi]^d\}.$$
Let $d \in \mathbb{N}$, let $\lambda$ be a fixed positive number, and let $(x_j : j \in \mathbb{N})$ be a Riesz-basis sequence in $\mathbb{R}^d$. For any $f \in \text{PW}_2^{(d)}$ there exists a unique square-summable sequence $(a(j, \lambda) : j \in \mathbb{N})$ such that

$$
\sum_{j \in \mathbb{N}} a(j, \lambda)g_\lambda^{(d)}(x_k - x_j) = f(x_k), \quad k \in \mathbb{N}.
$$

The Gaussian Interpolation Operator $I_\lambda^{(d)} : \text{PW}_2^{(d)} \rightarrow L_2[-\pi, \pi]^d$, defined by

$$
I_\lambda^{(d)}(f)(\cdot) = \sum_{j \in \mathbb{N}} a(j, \lambda)g_\lambda^{(d)}(\cdot - x_j),
$$

where $(a(j, \lambda) : j \in \mathbb{N})$ satisfies (57), is a well-defined, bounded linear operator from $\text{PW}_2^{(d)}$ to $L_2[-\pi, \pi]^d$. Moreover, $I_\lambda^{(d)}(f) \in C_0(\mathbb{R}^d)$.

We now generalize Theorems 4.3 and 4.4 to the multidimensional case assuming that the underlying Riesz-basis sequence is a grid. For simplicity we restrict ourselves to the case when $d = 2$, but note that our arguments carry over readily to all other values of $d$. It ought to be noted, however, that even this simplistic situation, namely that our data sites form a grid, cannot be handled via a straightforward multivariate extension of the crucial ingredients from Section 3. In particular, the proofs of Corollary 3.7 and Proposition 3.8 do not extend to higher dimensions. So we pursue a different tack below.

We recall some basic tools from tensor products. Let $X$ and $Y$ be two Banach spaces and $X^*$ and $Y^*$ their dual spaces. The algebraic tensor product of $X$ and $Y$ is denoted by $X \otimes Y$, and consists of the vector space of all linear combinations of elementary products $x \otimes y$ with $x \in X$ and $y \in Y$. All our Banach spaces are spaces of functions, and we can therefore identify elements of $X \otimes Y$ with functions on a product of sets.

A norm $\alpha(\cdot)$ on $X \otimes Y$ is called a reasonable cross norm if $\alpha(x \otimes y) \leq \|x\| \cdot \|y\|$, and if for $\phi \in X^*$ and $\psi \in Y^*$, the map $\phi \otimes \psi : X \otimes Y \rightarrow C$, $\sum x_i \otimes y_i \mapsto \sum \phi(x_i)\psi(y_i)$, is bounded on $(X \otimes Y, \alpha)$, and $\alpha(\phi \otimes \psi) = \sup_{u \in X \otimes Y, \alpha(u) \leq 1} |\phi \otimes \psi(u)| \leq \|\phi\| \cdot \|\psi\|$. If $\alpha(\cdot)$ is a norm on $X \otimes Y$, we denote the completion of $X \otimes Y$ with respect to $\alpha(\cdot)$ by $X \otimes_\alpha Y$.

**Proposition 5.2.** cf. [Ry] Section 3.1, Proposition 6.1

Let $X$ and $Y$ be Banach spaces. For $u \in X \otimes Y$ define

$$
\varepsilon(u) := \sup \{\phi \otimes \psi(u) : \phi \in X^*, \psi \in Y^*, \|\phi\|, \|\psi\| \leq 1\}
$$

(i) $\varepsilon(\cdot)$ is a reasonable cross norm, and we call $\varepsilon$ the injective tensor norm on $X \otimes Y$.

(ii) If $\alpha(\cdot)$ is any reasonable cross norm on $X \times Y$, then

$$
\varepsilon(u) \leq \alpha(u), \quad u \in X \otimes Y.
$$

(iii) For Banach spaces $V$ and $W$, and bounded operators $S : X \rightarrow U$ and $T : Y \rightarrow W$, the map $S \otimes T : X \otimes Y \rightarrow V \otimes W$, defined by $S \otimes T(\sum_{i=1}^n x_i \otimes y_i) = \sum_{i=1}^n S(x_i) \otimes T(y_i)$, extends (uniquely, by density) to a bounded operator $S \otimes_\varepsilon T : X \otimes_\varepsilon Y \rightarrow V \otimes_\varepsilon W$, and $\|S \otimes_\varepsilon T\| = \|S\| \cdot \|T\|$.  

On tensor products of Hilbert spaces we can define a unique tensor norm for which the completions are again Hilbert spaces.
Proposition 5.3. cf. [KR] p. 125 ff

Let $H$ and $K$ be Hilbert spaces.
(i) There is a unique inner product $\langle \cdot, \cdot \rangle_{H\otimes K}$ on $H \otimes K$ for which

$$\langle h_1 \otimes k_1, h_2 \otimes k_2 \rangle_{H\otimes K} = \langle h_1, h_2 \rangle_H \langle k_1, k_2 \rangle_K, \quad h_1, h_2 \in H \quad \text{and} \quad k_1, k_2 \in K,$$

where $\langle \cdot, \cdot \rangle_H$ and $\langle \cdot, \cdot \rangle_K$ denote the inner products in $H$ and $K$, respectively. We denote the completion of $H \otimes K$ with respect to the corresponding Hilbert norm $\| \cdot \|_h$ (i.e., $\| u \|_h = \langle u, u \rangle_{H\otimes K}^{1/2}$, for $u \in H \otimes K$) by $H \otimes_h K$.

(ii) $\| \cdot \|_h$ is a reasonable cross norm on $H \otimes K$.

(iii) If $V$ and $W$ are two Hilbert spaces, and $S : H \to V$ and $T : K : W$ are two bounded operators then the map $S \otimes T : H \otimes K \to V \otimes W$, defined via $S \otimes T(\sum_{i=1}^n x_i \otimes y_i) = \sum_{i=1}^n S(x_i) \otimes T(y_i)$, extends to a bounded operator $S \otimes_h T : X \otimes_h Y \to V \otimes_h W$, and $\| S \otimes_h T \| = \| S \| \cdot \| T \|$.

Let $(e_i : i \in \mathbb{N}) \subset H$ and $(f_i : i \in \mathbb{N}) \subset K$. Then $(e_i : i \in \mathbb{N})$ and $(f_i : i \in \mathbb{N})$ are Riesz bases, or orthogonal bases for $H$ and $K$, respectively, if and only if $(e_i \otimes f_j : i, j \in \mathbb{N})$ is a Riesz basis, or an orthogonal basis, of $H \otimes_h K$.

If $(\Omega, \Sigma, \mu)$ and $(\Omega', \Sigma', \mu')$ are two measure spaces, and $H = L_2(\mu)$, and $K = L_2(\mu')$, then $H \otimes_h K$ is, via the identification of $f \otimes g$ with the function $\Omega \times \Omega' \ni (\omega, \omega') \mapsto g(\omega)f(\omega')$, isometrically isomorphic to $L_2(\mu \times \mu')$.

Using Proposition 5.3 and the identification of $L_2(\mathbb{R}) \otimes L_2(\mathbb{R})$ with $L_2(\mathbb{R}^2)$, we deduce that $F^{(2)}[\cdot] = F[\cdot] \otimes_h F[\cdot]$. Consequently,

$$PW_{\pi}^{(2)} = (F^{(2)})^{-1}(L_2[-\pi, \pi]^2) = (F \otimes_h F)^{-1}(L_2[-\pi, \pi] \otimes_h L_2[-\pi, \pi]) = PW_\pi \otimes_h PW_\pi.$$

For the remainder of this section we fix two Riesz-basis sequences $(x_j : j \in \mathbb{Z})$ and $(y_j : j \in \mathbb{Z})$, and we put $z_{(l,k)} = (x_l, y_k)$ for $l, k \in \mathbb{Z}$. By Proposition 5.3 $(z_{(l,k)} : l, k \in \mathbb{Z})$ is a Riesz-basis sequence on $\mathbb{R}^2$, and we denote the 1-dimensional Gaussian interpolation operator corresponding to $\lambda > 0$ and $(x_j : j \in \mathbb{Z})$ by $I_\lambda$, and the one associated to $\lambda > 0$ and $(y_j : j \in \mathbb{Z})$ by $I'_{\lambda}$. The 2-dimensional Gaussian interpolation operator corresponding to $(z_{(l,k)} : l, k \in \mathbb{Z})$ is denoted by $I^{(2)}_{\lambda}$.

Proposition 5.4. For any positive number $\lambda$,

$$I^{(2)}_{\lambda} = I_\lambda \otimes_h I'_{\lambda}.$$

Proof. From Theorem 2.9, Theorem 5.1, and Proposition 5.3 we find that the operators $I_\lambda \otimes_h I'_{\lambda}$ and $I^{(2)}_\lambda$ are bounded linear operators from $PW_{\pi}^{(2)}$ into $L_2[-\pi, \pi]^2$. So we only need to verify that

$$I^{(2)}_\lambda(f \otimes h) = I_\lambda(f) \otimes_h I'_{\lambda}(h), \quad f, h \in PW_\pi.$$

Recall from Theorem 2.9 that there exists a unique pair of square-summable sequences $(a(j, \lambda) : j \in \mathbb{Z})$ and $(b(j, \lambda) : j \in \mathbb{Z})$ such that

$$\sum_{j \in \mathbb{Z}} a(j, \lambda)g(\lambda(x_k - x_j)) = f(x_k) \quad \text{and} \quad \sum_{j \in \mathbb{Z}} b(j, \lambda)g(\lambda(y_k - y_j)) = h(y_k), \quad k \in \mathbb{Z}.$$
This yields for $k, l \in \mathbb{Z}$
\[
f(x_k)h(y_l) = \sum_{j \in \mathbb{Z}} a(j, \lambda)g_\lambda(x_k - x_j) \sum_{j \in \mathbb{Z}} b(j, \lambda)g_\lambda(y_l - y_j)
= \sum_{j, m \in \mathbb{Z}} a(j, \lambda)b(m, \lambda)g_\lambda(x_k - x_j)g_\lambda(y_l - y_m)
= \sum_{j, m \in \mathbb{Z}} a(j, \lambda)b(m, \lambda)g_\lambda^{(2)}(z_{(k,l)} - z_{(j,m)}).
\]

Therefore the uniqueness in Theorem 5.1 implies that, for every $(x, y) \in \mathbb{R}^2$,
\[
I^{(2)}(f \otimes h)(x, y) = \sum_{j, m \in \mathbb{Z}} a(j, \lambda)b(m, \lambda)g_\lambda^{(2)}((x, y) - (j, m))
= \sum_{j, m \in \mathbb{Z}} a(j, \lambda)b(m, \lambda)g_\lambda(x - x_j)g_\lambda(y - y_m) = I_x(f)(x)I_y(h)(y),
\]
and this finishes the proof. □

Proposition 5.4 allows us to transfer Theorems 4.3 and 4.4 to the multidimensional case.

**Theorem 5.5.** Suppose that $F \in \mathcal{P}W^{(2)}$. Then \( \lim_{\lambda \to 0^+} \|I_\lambda^{(2)}(F) - F\|_{L_2(\mathbb{R}^2)} = 0 \), and \( \lim_{\lambda \to 0^+} I_\lambda^{(2)}(F)(z) = F(z) \) uniformly for $z \in \mathbb{R}^2$.

**Proof.** Let $F \in \mathcal{P}W^{(2)}$. The first assertion follows from the aforementioned fact that $\mathcal{P}W \otimes \mathcal{P}W$ is dense in $\mathcal{P}W^{(2)}$, Proposition 5.3, and Theorem 4.3.

In order to show the second statement we first note (cf. [Ry, page 50]) that the injective tensor product $C_0(\mathbb{R}) \otimes C_0(\mathbb{R})$ is, via the natural map, isometrically isomorphic to $C_0(\mathbb{R}^2)$. By Theorem 4.4 and Proposition 5.2 (iii) the operators
\[
I_x \otimes I_y : \mathcal{P}W \otimes \mathcal{P}W \to C_0(\mathbb{R}^2)
\]
are uniformly bounded. So Propositions 5.2 (ii) and 5.3 (ii) imply that $I_x \otimes hI_y^{(2)} = I_\lambda^{(2)}$ are also uniformly bounded operators from $\mathcal{P}W^{(2)}$ to $C_0(\mathbb{R}^2)$. As $\mathcal{P}W \otimes \mathcal{P}W$ is dense in $\mathcal{P}W^{(2)}$, we argue as in the proof of the first statement, and conclude from Theorem 4.3 that $I_\lambda^{(2)}(F)$ converges uniformly to $F$, for $F \in \mathcal{P}W^{(2)}$. □

6. FURTHER RESULTS ON UNIVARIATE GAUSSIAN INTERPOLATION

In this final section we return to univariate interpolation, in order to discuss extensions of some results obtained in Section 2. We begin with a general result concerning bi-infinite matrices which appears to be folkloric. We have seen two articles which cite a well-known treatise for it, but our search of the latter came up emptyhanded. A proof of the said result is indicated in [JAI], but for sake of completeness and record, we include a fairly self-contained and expanded rendition of this argument here.

**Theorem 6.1.** Suppose that $(A(j, k))_{j,k \in \mathbb{Z}}$ is a bi-infinite matrix which, as an operator on $l^2(\mathbb{Z})$, is self adjoint, positive and invertible.

Assume further that there exist positive constants $\kappa$ and $\gamma$ such that $|A(j, k)| \leq \kappa e^{-\gamma|j-k|}$ for every pair of integers $j$ and $k$. Then there exist constants $\tilde{\kappa}$ and $\tilde{\gamma}$ such that $|A^{-1}(s, t)| \leq \tilde{\kappa} e^{-\tilde{\gamma}|s-t|}$ for every $s, t \in \mathbb{Z}$. 

For the proof of Theorem 6.1 we shall need the following pair of lemmata.

**Lemma 6.2.** Let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert space, and let \(A : H \to H\) be a bounded linear, self-adjoint, positive and invertible operator. Let \(R := I - \frac{A}{\|A\|}\), where \(I\) denotes the identity. Then \(R = R^*\), \(\langle x, Rx \rangle \geq 0\) for every \(x \in H\), and \(\|R\| < 1\).

**Remark 6.3.** It is a deep fact (cf. [KR, Proposition 2.4.6]) that if \(A : H \to H\) is a bounded linear operator such that \(\langle x, Ax \rangle \in \mathbb{R}\) for all \(x \in H\), then \(A\) is self-adjoint. Thus a bounded linear operator \(A : H \to H\) is self-adjoint, positive, and invertible if and only if

\[
\inf_{x \in H, \|x\| = 1} \langle x, Ax \rangle > 0.
\]

**Proof of Lemma 6.2.** The symmetry of \(R\) is evident. If \(\|x\| = 1\), then

\[
\langle x, Rx \rangle = \|x\|^2 - \left\langle x, \frac{A}{\|A\|} x \right\rangle.
\]

By the assumptions on \(A\) and the BCS inequality we see that the term on the right of the preceding equation is between 0 and 1. Therefore \(\|R\| = \sup \{\langle x, Rx \rangle : \|x\| = 1\} \leq 1\). If \(\|R\| = 1\), then there is a sequence \((x_n : n \in \mathbb{N})\) such that \(\|x_n\| = 1\) for every \(n\), and

\[
1 = \lim_{n \to \infty} \langle x_n, Rx_n \rangle = \lim_{n \to \infty} \left(1 - \left\langle x_n, \frac{A}{\|A\|} x_n \right\rangle\right),
\]

which contradicts the invertibility of \(A\).

**Lemma 6.4.** Suppose that \((R(s, t))_{s,t \in \mathbb{Z}}\) is a bi-infinite matrix satisfying the following condition: there exist positive constants \(C\) and \(\gamma\) such that \(|R(s, t)| \leq Ce^{-\gamma|s-t|}\) for every pair of integers \(s\) and \(t\). Given \(0 < \gamma' < \gamma\), there is a constant \(C(\gamma, \gamma')\), depending on \(\gamma\) and \(\gamma'\), such that \(|R^n(s, t)| \leq C^n(\gamma, \gamma') (n+1) e^{-\gamma'|s-t|}\) for every \(s, t \in \mathbb{Z}\).

**Proof.** Suppose firstly that \(s \neq t \in \mathbb{Z}\), and assume without loss that \(s < t\). Note that

\[
\sum_{u=-\infty}^{\infty} e^{-\gamma|s-u|} e^{-\gamma'|t-u|} = \sum_{u=s}^{t} e^{-\gamma(u-s)} e^{-\gamma'(t-u)} + \sum_{u=-\infty}^{s-1} e^{-\gamma(s-u)} e^{-\gamma'(t-u)} + \sum_{u=t+1}^{\infty} e^{-\gamma(u-s)} e^{-\gamma'(u-t)}
\]

\[
=: \Sigma_1 + \Sigma_2 + \Sigma_3.
\]

Now

\[
\Sigma_1 = e^{-\gamma'(t-s)} e^{\gamma' s} \sum_{u=s}^{t} e^{-u(\gamma' - \gamma)} = e^{-\gamma'(t-s)} e^{\gamma' s} \sum_{v=0}^{t-s} e^{-v(\gamma' - \gamma)} = e^{-\gamma'(t-s)} \sum_{v=0}^{t-s} e^{-v(\gamma' - \gamma)} \leq e^{-\gamma'(t-s)} \frac{e^{-\gamma'(t-s)}}{1 - e^{-\gamma'(s-t)}}.
\]

(58)

(59)
Moreover,

\[ \Sigma_2 = \sum_{v=1}^{\infty} e^{-\gamma v} e^{-\gamma(t-s+v)} = e^{-\gamma(t-s)} \sum_{v=1}^{\infty} e^{-v(\gamma+\gamma')} \leq \frac{e^{-\gamma'(t-s)}}{1 - e^{-(\gamma+\gamma')}}, \]

whereas

\[ \Sigma_3 = \sum_{v=1}^{\infty} e^{-\gamma v} e^{-\gamma(v+t-s)} = e^{-\gamma(t-s)} \sum_{v=1}^{\infty} e^{-v(\gamma+\gamma')} \leq \frac{e^{-\gamma'(t-s)}}{1 - e^{-(\gamma+\gamma')}}, \]

If \( s = t \), then

\[ \sum_{u=-\infty}^{\infty} e^{-\gamma|s-u|} e^{-\gamma'|t-u|} = \sum_{u=-\infty}^{\infty} e^{-(\gamma+\gamma')|s-u|} \leq \frac{2}{1 - e^{-(\gamma+\gamma')}}, \]

From [63] we conclude that

\[ |R^2(s,t)| \leq C^2 \left[ \frac{1}{1 - e^{-(\gamma-\gamma')}} + \frac{2}{1 - e^{-(\gamma+\gamma')}} \right] =: C^2 C(\gamma, \gamma'). \]

The general result follows from this via induction.

**Proof of Theorem 5.5.** Let \( R = I - \frac{A}{\|A\|} \) be the matrix given in that lemma. As

\[ R(j, k) = \frac{A(j, k)}{\|A\|} \text{ if } j \neq k, \quad \text{and} \quad R(k, k) = \frac{\|A\| - A(k, k)}{\|A\|}, \]

there is some constant \( C \) such that \( |R(s, t)| \leq C e^{-\gamma|s-t|} \) for every pair of integers \( s \) and \( t \). As \( A = \|A\|(I - R) \), and \( r := \|R\| < 1 \) (Lemma 6.2), the standard Neumann series expansion yields the relations

\[ A^{-1} = \|A\|^{-1} \sum_{n=0}^{\infty} R^n \]
\[ = \|A\|^{-1} \sum_{n=0}^{N-1} R^n + \|A\|^{-1} R^N \sum_{n=0}^{\infty} R^n = \|A\|^{-1} \sum_{n=0}^{N-1} R^n + R^N A^{-1}, \]

for any positive integer \( N \). As \( R^0(s, t) = I(s, t) = 0 \) if \( s \neq t, s, t \in \mathbb{Z} \), we see from [63] that

\[ A^{-1}(s, t) = \|A\|^{-1} \sum_{n=0}^{N-1} R^n(s, t) + [R^N A^{-1}](s, t), \quad s \neq t. \]

Choose and fix a positive number \( \gamma' < \gamma \), and recall from Lemma [6.4] that there is a constant \( C(\gamma, \gamma') \) such that \( |R^n(s, t)| \leq C^n C(\gamma, \gamma')^{n-1} e^{-\gamma|s-t|} \) for every positive integer \( n \), and every pair of integers \( s \) and \( t \). So we may assume that there is some constant \( D := D(\gamma, \gamma') > 1 \) such that \( |R^n(s, t)| \leq D^n e^{-\gamma|s-t|} \) for every positive integer \( n \), and every pair of integers \( s \) and \( t \). Using this bound in [64] provides the following estimate for every \( s \neq t \):

\[ |A^{-1}(s, t)| \leq \|A\|^{-1} e^{-\gamma|s-t|} \sum_{n=1}^{N-1} D^n + \|A^{-1}\| r^N \]
\[ \leq \|A\|^{-1} e^{-\gamma|s-t|} \frac{D^N}{D - 1} + \|A^{-1}\| r^N. \]
Let $m$ be a positive integer such that $e^{-\gamma}D^{1/m} < 1$, and let $s, t \in \mathbb{Z}$ with $|s-t| \geq m$.
Writing $|s-t| = Nm + k$, $0 \leq k \leq m - 1$, we find that

\begin{equation}
\tag{66}
 e^{-\gamma|s-t|}D^N = \left[ e^{-\gamma}D^{1/m} \right]^{s-t} \leq [e^{-\gamma}D^{1/m}]^{s-t}.
\end{equation}

Further,

\begin{equation}
\tag{67}
 r^N = \left[ \frac{1}{r^{m+1/N}} \right]^{s-t} \leq \frac{1}{r^{1/2m}} [s-t],
\end{equation}

and combining (66) and (67) with (65) leads to the following bounds for every $|s-t| \geq m$ and an appropriately chosen $\gamma > 0$:

\begin{equation}
\tag{68}
 |A^{-1}(s,t)| \leq \frac{\|A^{-1}\|}{D-1} \left[ e^{-\gamma}D^{1/m} \right]^{s-t} + \|A^{-1}\| \frac{1}{r^{1/2m}}[s-t] = O(e^{-\gamma|s-t|}).
\end{equation}

On the other hand, if $|s-t| < m$, we obtain

\begin{equation}
\tag{69}
 |A^{-1}(s,t)| \leq \|A^{-1}\| \leq (\|A^{-1}\| e^{\alpha \gamma}) e^{-\gamma|s-t|},
\end{equation}

and combining (68) with (69) finishes the proof. \qed

A direct consequence of the previous theorem is the following:

**Corollary 6.5.** Let $\lambda > 0$, and let $(x_j : j \in \mathbb{Z})$ be a sequence of real numbers satisfying the following condition: there exists a positive number $q$ such that $x_{j+1} - x_j \geq q$ for every $j \in \mathbb{Z}$. Let $A := A_{\lambda}$ be a bi-infinite matrix whose entries are given by $A(j,k) := e^{-\lambda(x_j-x_k)^2}$, $j, k \in \mathbb{Z}$. Then there exist positive constants $\beta_1$ and $\gamma_1$, depending on $\lambda$ and $q$, such that $|A^{-1}(s,t)| \leq \beta_1 e^{-\gamma_1 |s-t|}$, $s, t \in \mathbb{Z}$.

**Proof.** The hypothesis on the $x_j$’s implies that $|x_j - x_k| \geq |j-k|q$, for $j, k \in \mathbb{Z}$. \qed

**Remark 6.6.** The foregoing result implies, in particular, that $A^{-1}$ is a bounded operator on every $l_p(\mathbb{Z})$, $1 \leq p \leq \infty$.

We turn now to the second half of this section, in which we introduce the fundamental functions for Gaussian interpolation (at scattered data sites), and set out some of their basic properties.

**Theorem 6.7.** Let $\lambda > 0$ be fixed, and let $(x_j : j \in \mathbb{Z})$ be a sequence of real numbers satisfying the following condition: there exists $q > 0$ such that $x_{j+1} - x_j \geq q$ for every integer $j$. Let $A = A_{\lambda}$ be the bi-infinite matrix whose entries are given by $A(j,k) = e^{-\lambda(x_j-x_k)^2}$, $j, k \in \mathbb{Z}$. Given $l \in \mathbb{Z}$, let the $l$-th fundamental function be defined as follows:

\[ L_{l,\lambda}(x) := L_l(x) := \sum_{k \in \mathbb{Z}} A^{-1}(k,l)e^{-\lambda(x-x_k)^2}, \quad x \in \mathbb{R}. \]

The following hold:

(i) The function $L_l$ is continuous throughout $\mathbb{R}$.

(ii) Each $L_l$ obeys the fundamental interpolatory conditions $L_l(x_m) = \delta_{lm}$, $m \in \mathbb{Z}$.

(iii) If, in addition, there exists a positive number $Q$ such that $x_{j+1} - x_j \leq Q$ for every $Q$, then there exist positive constants $\beta_2$ and $\rho$, depending on $\lambda$, $q$, and $Q$ such that $|L_l(x)| \leq \beta_2 e^{-\rho |x-x_l|}$ for every $x \in \mathbb{R}$ and every $l \in \mathbb{Z}$.

(iv) Assume that $(x_j : j \in \mathbb{Z})$ satisfies the condition stipulated in (iii). Let $(b_l : l \in \mathbb{Z})$ be a sequence satisfying the following condition: there exists a positive number $K$ and a positive integer $P$ such that $|b_l| \leq K ||l||^P$ for every integer $l$. Then the function $\mathbb{R} \ni x \mapsto \sum_{l \in \mathbb{Z}} b_lL_l(x)$ is continuous on $\mathbb{R}$. 

Proof. (i) As $A^{-1}$ is a bounded operator on $l_\infty(\mathbb{Z})$, the sequence $(A^{-1}(k, l) : k \in \mathbb{Z})$ is bounded. Hence the continuity of $L_l$ follows from Proposition 2.3.
(ii) Given $m \in \mathbb{Z}$, we have

$$L_l(x_m) = \sum_{k \in \mathbb{Z}} A^{-1}(k, l) e^{-\lambda(x_m-x_k)^2} = \sum_{k \in \mathbb{Z}} A^{-1}(k, l) A(m, k) = I(m, l) = \delta_{l,m}.$$ 

(iii) The assumption $x_{j+1} - x_j \leq Q$ for every integer $j$ implies that $|x_k - x_l| \leq |k - l|Q$ for every pair of integers $k$ and $l$. Therefore Corollary 6.5 leads to the bound $|A^{-1}(k, l)| \leq \beta_1 e^{-\gamma_2|x_k-x_l|}$, $k, l \in \mathbb{Z}$, where $\gamma_2 := \gamma_1/Q$. Consequently,

$$|L_l(x)| \leq \beta_1 \sum_{k \in \mathbb{Z}} e^{-\gamma_2|x_k-x_l|} e^{-\lambda(x-x_k)^2}$$

where $\rho := \min\{\lambda, \gamma_2\}$. Therefore

$$e^{\rho|x-x_l|}|L_l(x)| \leq \beta_1 \sum_{k \in \mathbb{Z}} e^{\rho|x-x_l|} e^{-\rho(x-x_k)^2}$$

$$\leq \beta_1 \sum_{k \in \mathbb{Z}} e^{-\rho^2} e^{-\rho(x-x_k)^2}, \quad x \in \mathbb{R}.$$ 

Fix $x \in \mathbb{R}$, and let $s$ be the integer such that $x_s \leq x < x_{s+1}$. From (71) we obtain

$$e^{\rho|x-x_l|}|L_l(x)| \leq \beta_1 \sum_{k=s}^{s+1} e^{\rho|x-x_k|} e^{-\rho(x-x_k)^2}$$

$$+ \beta_1 \sum_{k \in \mathbb{Z} \setminus \{s, s+1\}} e^{\rho|x-x_k|} e^{-\rho(x-x_k)^2}$$

$$\leq 2\beta_1 e^{\rho Q} + \beta_1 \sum_{k \in \mathbb{Z} \setminus \{s, s+1\}} e^{\rho|x-x_k|} e^{-\rho(x-x_k)^2}.$$ 

As $mq \leq |x - x_{s-m}| \leq (m + 1)Q$ for every positive integer $m$, and $(m - 1)q \leq |x - x_{s-m+1}| \leq mQ$ for every positive integer $m \geq 2$, the final sum on the right of (72) is no larger than

$$\beta_1 \sum_{m=1}^{\infty} e^{\rho(m+1)q} e^{-\rho^2 q^2} + \beta_1 \sum_{m=2}^{\infty} e^{\rho mq} e^{-\rho(m-1)^2 q^2} =: \beta_2.$$ 

Combining this with (72) and (71) finishes the proof.
(iv) Each summand is continuous by assertion (i). Let $x \in \mathbb{R}$, and let $x_s \leq x < x_{s+1}$. Assertion (iii) implies that

$$\left| \sum_{l \in \mathbb{Z}} b_l L_l(x) \right| \leq K \beta_2 \left[ \sum_{l=s}^{s+1} |l|^P + \sum_{l \in \mathbb{Z} \setminus \{s, s+1\}} |l|^P e^{-\rho|x-x_l|} \right]$$

$$\leq K \beta_2 \left[ \sum_{l=s}^{s+1} |l|^P + \sum_{m=2}^{\infty} |l|^P e^{-\rho q m} + \sum_{m=1}^{\infty} |l|^P e^{-\rho(m-1)q} \right].$$ 

It follows that the series $\sum_{l \in \mathbb{Z}} b_l L_l(x)$ is locally uniformly convergent, whence the stated result follows. \hfill \Box
The counterpart of Part (iii) of the foregoing theorem for Gaussian cardinal interpolation was obtained in [RS1], and its analogue for spline interpolation was proved in [deB].

Earlier in this paper we have discussed Gaussian interpolation operators associated to functions, specifically to bandlimited functions. Here our perspective changes slightly, as we begin to think of these interpolation operators acting on sequence spaces.

**Theorem 6.8.** Let \( \lambda > 0 \) be fixed. Suppose that \( (x_j : j \in \mathbb{Z}) \) is a real sequence satisfying the following condition: there exist positive numbers \( q \) and \( Q \) such that \( q \leq x_{j+1} - x_j \leq Q \) for every integer \( j \). Let \( A = A_\lambda \) be the bi-infinite matrix whose entries are given by \( A(j,k) = e^{-\lambda(x_j-x_k)^2} \), \( j, k \in \mathbb{Z} \). Given \( p \in [1, \infty] \), and \( \overline{y} := (y_l : l \in \mathbb{Z}) \in \ell_p(\mathbb{Z}) \), define

\[
I_\lambda(\overline{y}, x) := \sum_{k \in \mathbb{Z}} (A^{-1}\overline{y})_k e^{-\lambda(x-x_k)^2}, \quad x \in \mathbb{R},
\]

where \( (A^{-1}\overline{y})_k \) denotes the \( k \)-th component of the sequence \( A^{-1}\overline{y} \). The following hold:

(i) The function \( \mathbb{R} \ni x \mapsto I_\lambda(\overline{y}, x) \) is continuous on \( \mathbb{R} \).

(ii) If \( x \) is any real number, then

\[
I_\lambda(\overline{y}, x) = \sum_{l \in \mathbb{Z}} y_l L_l(x),
\]

where \( (L_l : l \in \mathbb{Z}) \) is the sequence of fundamental functions introduced in the preceding theorem.

(iii) There is a constant \( \beta_3 \), depending on \( \lambda, q, Q, \) and \( p \), such that

\[
\|I_\lambda(\overline{y}, \cdot)\|_{L_p(\mathbb{R})} \leq \beta_3 \|\overline{y}\|_{\ell_p(\mathbb{Z})},
\]

for every \( \overline{y} \in \ell_p(\mathbb{Z}) \).

**Proof.** (i) As \( A^{-1} \) is a bounded operator on \( \ell_p(\mathbb{Z}) \), the sequence \( A^{-1}\overline{y} \) is bounded. Hence the continuity of \( I_\lambda(\overline{y}, \cdot) \) follows from Proposition 2.4.

(ii) Let \( x \in \mathbb{R} \). Then

\[
I_\lambda(\overline{y}, x) = \sum_{k \in \mathbb{Z}} (A^{-1}\overline{y})_k e^{-\lambda(x-x_k)^2} = \sum_{k \in \mathbb{Z}} \left[ \sum_{l \in \mathbb{Z}} y_l A^{-1}(k,l) \right] e^{-\lambda(x-x_k)^2}.
\]

The required result obtains by interchanging the order of summation, which is justified by the following series of estimates, the first of which is consequent upon Corollary 6.5 and the last of which follows from Proposition 2.4.

\[
\sum_{k \in \mathbb{Z}} \left[ \sum_{l \in \mathbb{Z}} |y_l A^{-1}(k,l)| \right] e^{-\lambda(x-x_k)^2} = O\left( \sum_{k \in \mathbb{Z}} \left[ \sum_{l \in \mathbb{Z}} e^{-\gamma_1 |k-l|} \right] e^{-\lambda(x-x_k)^2} \right)
\]

\[
= O\left( \sum_{k \in \mathbb{Z}} e^{-\lambda(x-x_k)^2} \right) = O(1).
\]
(iii) We begin with \( p = \infty \). If \( x_s \leq x < x_{s+1} \), assertion (ii), the triangle inequality, and a now familiar argument lead to the following:

\[
|I_x(y, x)| = \left| \sum_{l \in \mathbb{Z}} y_l L_l(x) \right|
\]

\[
\leq \beta_2 \left[ |y_s| + |y_{s+1}| + \sum_{m=1}^{\infty} |y_{s-m}| e^{-\rho m q} + \sum_{m=2}^{\infty} |y_{s+m}| e^{-\rho (m-1) q} \right]
\]

\[
\leq \frac{2 \beta_2}{1 - e^{-\rho q}} \|y\|_{\ell_{\infty}(\mathbb{Z})} =: \beta_3 \|y\|_{\ell_{\infty}(\mathbb{Z})}.
\]

It is immediate that \( \|I_x(y, \cdot)\|_{L_\infty(\mathbb{R})} \leq \beta_3 \|y\|_{\ell_{\infty}(\mathbb{Z})} \).

Suppose now that \( 1 \leq p < \infty \). Let \( J_s := [x_s, x_{s+1}) \), \( s \in \mathbb{Z} \), and recall from (73) that, if \( x \in J_s \), then

\[
|I_x(y, x)| \leq \beta_2 \left[ |y_s| + |y_{s+1}| + \sum_{m=1}^{\infty} |y_{s-m}| e^{-\rho m q} + \sum_{m=2}^{\infty} |y_{s+m}| e^{-\rho (m-1) q} \right].
\]

Therefore, as \( |x_{s+1} - x_s| \leq Q \), we have

\[
\int_{J_s} |I_x(y, x)|^p \, dx \leq Q \beta_2^p \left[ |y_s| + |y_{s+1}| + \sum_{m=1}^{\infty} |y_{s-m}| e^{-\rho m q} + \sum_{m=2}^{\infty} |y_{s+m}| e^{-\rho (m-1) q} \right]^p.
\]

Let \( \varpi = (u_k : k \in \mathbb{Z}) \) and \( \upsilon = (v_k : k \in \mathbb{Z}) \) be a pair of sequences defined by \( u_k = |y_k|, k \in \mathbb{Z} \), and

\[
v_k = \begin{cases} 
1 & \text{if } k \in \{-1, 0\}; \\
e^{-\rho k q} & \text{if } k \geq 1; \\
e^{\rho (k+1) q} & \text{if } k \leq -2.
\end{cases}
\]

Then (74) may be recast as follows:

\[
\int_{J_s} |I_x(y, x)|^p \, dx \leq Q \beta_2^p \left| \sum_{m \in \mathbb{Z}} u_{s-m} v_m \right|^p,
\]

so that

\[
\|I_x(y, \cdot)\|_{L_p(\mathbb{R})} = \left[ \sum_{s \in \mathbb{Z}} \int_{J_s} |I_x(y, x)|^p \, dx \right]^{1/p} \leq Q^{1/p} \beta_2 \left[ \sum_{s \in \mathbb{Z}} \left| \sum_{m \in \mathbb{Z}} u_{s-m} v_m \right|^p \right]^{1/p}.
\]

Now the Generalized Minkowski Inequality (cf. [HLP, p. 123]) implies that

\[
\left[ \sum_{s \in \mathbb{Z}} \left| \sum_{m \in \mathbb{Z}} u_{s-m} v_m \right|^p \right]^{1/p} \leq \sum_{m \in \mathbb{Z}} |v_m| \left[ \sum_{s \in \mathbb{Z}} \left| u_{s-m} \right|^p \right]^{1/p} \leq \frac{2}{1 - e^{-\rho q}} \|y\|_{\ell_{p}(\mathbb{Z})},
\]

and a combination of (76) and (75) completes the proof. \( \square \)
We conclude the paper with a few remarks. Suppose that $f$ is a bandlimited function, and let $d_k = f(x_k)$, $k \in \mathbb{Z}$. We have seen in Proposition 2.8 that this sequence $d = (d_k : k \in \mathbb{Z})$ is square summable. Furthermore, as observed in the course of the proof of Theorem 2.9(v), the Gaussian interpolant $I_d(f)\cdot$ studied earlier coincides with the function $I_d(d,\cdot)$ introduced in this section. Thus the final part of the previous theorem presents a twofold generalization of the estimate (25): to values of $p \in [1, \infty]$ other than 2, whilst requiring only that the underlying set of sampling points satisfies condition (10). In particular we do not assume in Theorem 6.8(iii) that $(x_j : j \in \mathbb{Z})$ is a Riesz-basis sequence. However, it is not without interest to note that the main convergence theorems obtained in Section 4 do not hold for data sites which merely satisfy the quasi-uniformity condition (10). For example, let $X := \mathbb{Z} \setminus \{0\}$, and let $f$ be the bandlimited function given by $f(x) := \sin(\pi x)/(\pi x)$, $x \in \mathbb{R}$. As $f(x_j) = 0$ for every $x_j \in X$, $I_d(f)$ is identically zero, so it is manifest that $I_d(f)$ does not converge to $f$.

Counterparts of the result obtained in part (iii) of the preceding theorem, for the case when $x_j = j$, may be found in [RS2] and [RS1]. However, those estimates are much more precise in nature.

Suppose that $(x_j : j \in \mathbb{Z})$ is a strictly increasing sequence of real numbers satisfying the following condition:

\begin{equation}
|x_j - j| \leq D < 1/4, \quad j \in \mathbb{Z}.
\end{equation}

Then $(x_j)$ is a Riesz-basis sequence [K]. Let $(L_{l,\lambda} : l \in \mathbb{Z})$ be the associated sequence of fundamental functions defined in Theorem 6.7. Define

$$G(x) := (x - x_0) \prod_{j=1}^{\infty} \left(1 - \frac{x}{x_j}\right) \left(1 - \frac{x}{x_{j+1}}\right), \quad x \in \mathbb{R},$$

and let

$$G_l(x) := \frac{G(x)}{(x - x_l)G'(x_l)}, \quad x \in \mathbb{R}, \quad l \in \mathbb{Z}.$$ 

It is shown in [K] each $G_l$ is a bandlimited function satisfying the interpolatory conditions $G_l(x_m) = \delta_{m,l}$, $m \in \mathbb{Z}$. Thus we find that $I_{d_l}(G_l) = L_{l,\lambda}$, and conclude from Theorems 4.3 and 4.4 that $\lim_{\lambda \to 0^+} L_{l,\lambda} = G_l$ in $L_2(\mathbb{R})$ and uniformly on $\mathbb{R}$. As a particular example, we learn from [H1] that, if $x_0 = 0$ and $x_j = x_{j-1} = j + c^2/j$, $j \geq 1$, $0 < c < 1/2$, then $(x_j)$ fulfills (17), and that the corresponding function $G$ is given in closed form:

$$G(x) = x[\cos(\pi x^2 - 4c^2)^{1/2} - \cos(\pi x)]/2\sinh(\pi c), \quad x \in \mathbb{R}.$$ 

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