0 Introduction

Recently, Seiberg and Witten [W] introduced new invariants of 4-manifolds, which are defined by counting solutions of a certain non-linear differential equation.

The new invariants are expected to be equivalent to Donaldson’s polynomial-invariants—at least for manifolds of simple type [KM 1]—and they have already found important applications, like e.g. in the proof of the Thom conjecture by Kronheimer and Mrowka [KM 2].

Nevertheless, the equations themselves remain somewhat mysterious, especially from a mathematical point of view.

The present paper contains our attempt to understand and to generalize the Seiberg-Witten equations by coupling them to connections in unitary vector bundles, and to relate their solutions to more familiar objects, namely stable pairs.

Fix a Spin$^c$-structure on a Riemannian 4-manifold $X$, and denote by $\Sigma^\pm$ the associated spinor bundles. The equations which we will study are:

\[
\begin{align*}
\mathcal{D}_{A,b}\Psi &= 0 \\
\Gamma(F^+_{A,b}) &= (\Psi \bar{\Psi})_0
\end{align*}
\]
This is a system of equations for a pair \((A, \Psi)\) consisting of a unitary connection in a unitary bundle \(E\) over \(X\), and a positive spinor \(\Psi \in A^0(\Sigma^+ \otimes E)\). The symbol \(b\) denotes a connection in the determinant line bundle of the spinor bundles \(\Sigma^\pm\) and \(D_{A,b} : \Sigma^+ \otimes E \to \Sigma^- \otimes E\) is the Dirac operator obtained by coupling the connection in \(\Sigma^+\) defined by \(b\) (and by the Levi-Civita connection in the tangent bundle) with the variable connection \(A\) in \(E\).

These equations specialize to the original Seiberg-Witten equations if \(E\) is a line bundle. We show that the coupled equations can be interpreted as a differential version of the generalized vortex equations [JT].

Vortex equations over Kähler manifolds have been investigated by Bradlow [B1], [B2] and by Garcia-Prada [G1], [G2]: Given a pair \((E, \varphi)\) consisting of a holomorphic vector bundle with a section, the vortex equations ask for a Hermitian metric \(h\) in \(E\) with prescribed mean curvature: more precisely, the equations—which depend on a real parameter \(\tau\)—are

\[i\Lambda F_h = \frac{1}{2}(\tau \text{id}_E - \varphi \otimes \varphi^*).\]

A solution exists if and only if the pair \((E, \varphi)\) satisfies a certain stability condition (\(\tau\)-stability), and the moduli space of vortices can be identified with the moduli space of \(\tau\)-stable pairs. A GIT construction of the latter space has been given by Thaddeus [T] and Bertram [B] if the base manifold is a projective curve, and by Huybrechts and Lehn [HL1], [HL2] in the case of a projective variety. Other constructions have been given by Bradlow and Daskalopoulos [BD1], [BD2] in the case of a Riemann surface, and by Garcia-Prada for compact Kähler manifolds [G2]. In this connection also [BD2] is relevant. In this note we prove the following result:

**Theorem 0.1** Let \((X, g)\) be a Kähler surface of total scalar curvature \(\sigma_g\), and let \(\Sigma\) be the canonical Spin^c-structure with associated Chern connection \(c\). Fix a unitary vector bundle \(E\) of rank \(r\) over \(X\), and define \(\mu_g(\Sigma^+ \otimes E) := \frac{\deg_g(E)}{r} + \sigma_g\).

Then for \(\mu_g < 0\), the space of solutions of the coupled Seiberg-Witten equation is isomorphic to the moduli space of stable pairs of topological type \(E\), with parameter \(\sigma_g\).

If the constant \(\mu_g(\Sigma^+ \otimes E)\) is positive, one simply replaces the bundle \(E\) with \(E^\vee \otimes K_X\), where \(K_X\) denotes the canonical line bundle of \(X\) (cf. Lemma 3.1).
Note that the above theorem gives a complex geometric interpretation of the moduli space of solutions of the coupled Seiberg-Witten equation associated to all Spin\textsuperscript{c}-structures on \( X \): The change of the Spin\textsuperscript{c}-structure is equivalent to tensoring \( E \) with a line bundle.

Notice also that in the special case \( r = 1 \) one recovers Witten’s result identifying the space of irreducible monopoles on a Kähler surface with the set of all divisors associated to line bundles of a fixed topological type; the stability condition which he mentions is the rank-1 version of the stable pair-condition.

Having established this correspondence, we describe some of the basic properties of the moduli spaces, and give a first application: We show that minimal surfaces of general type cannot be diffeomorphic to rational ones. This provides a short proof of one of the essential steps in Friedman and Qin’s proof of the Van de Ven conjecture [FQ]. More detailed investigations and applications will appear in a later paper.

We like to thank A. Van de Ven for very helpful questions and remarks.

1 Spin\textsuperscript{c}-structures and almost canonical classes

The complex spinor group is defined as \( \text{Spin}^c := \text{Spin} \times_{\mathbb{Z}_2} S^1 \), and there are two non-split exact sequences

\[
1 \longrightarrow S^1 \longrightarrow \text{Spin}^c \longrightarrow \text{SO} \longrightarrow 1
\]

In dimension 4, \( \text{Spin}^c(4) \) can be identified with the subgroup of \( U(2) \times U(2) \) consisting of pairs of unitary matrices with the same determinant, and one has two commutative diagrams:

\[
\begin{array}{cccccc}
1 & 1 \\
\downarrow & \downarrow \\
1 & \rightarrow & \mathbb{Z}_2 & \rightarrow & \text{Spin}(4) & \rightarrow & \text{SO}(4) & \rightarrow & 1 \\
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \rightarrow & S^1 & \rightarrow & \text{Spin}^c(4) & \rightarrow & \text{SO}(4) & \rightarrow & 1
\end{array}
\]

\[
\begin{array}{cccccc}
1 & \rightarrow & S^1 & \rightarrow & \text{Spin}^c(4) & \rightarrow & \text{SO}(4) & \rightarrow & 1
\end{array}
\]  \hspace{1cm} (1)

In dimension 4, \( \text{Spin}^c(4) \) can be identified with the subgroup of \( U(2) \times U(2) \) consisting of pairs of unitary matrices with the same determinant, and one has two commutative diagrams:

\[
\begin{array}{cccccc}
1 & 1 \\
\downarrow & \downarrow \\
1 & \rightarrow & \mathbb{Z}_2 & \rightarrow & \text{Spin}(4) & \rightarrow & \text{SO}(4) & \rightarrow & 1 \\
\downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \rightarrow & S^1 & \rightarrow & \text{Spin}^c(4) & \rightarrow & \text{SO}(4) & \rightarrow & 1
\end{array}
\]

\[
\begin{array}{cccccc}
1 & \rightarrow & S^1 & \rightarrow & \text{Spin}^c(4) & \rightarrow & \text{SO}(4) & \rightarrow & 1
\end{array}
\]  \hspace{1cm} (1)
where $\Delta : U(2) \rightarrow \text{Spin}^c(4) \subset U(2) \times U(2)$ acts by $a \mapsto \begin{pmatrix} \text{id} & 0 \\ 0 & \det a \end{pmatrix} a$, and

$$
\begin{array}{cccc}
1 & 1 & \mathbb{Z}_2 \\
\downarrow & & \\
S^1 & \text{Spin}^c(4) & \text{SO}(4) & 1 \\
\downarrow & & \downarrow & \downarrow \text{ad} \ (\lambda^+, \lambda^-) \\
S^1 \times S^1 & U(2) \times U(2) & \text{SO}(3) \times \text{SO}(3) & 1 \\
\downarrow & & \downarrow & \\
\mathbb{Z}_2 & S^1 & \mathbb{Z}_2 & 1 \\
\downarrow & & \downarrow & \\
1 & 1 & \\
\end{array}
$$

where $\lambda^\pm : \text{SO}(4) \rightarrow \text{SO}(3)$ are induced by the two projections of $\text{Spin}(4) = \text{SU}(2)^+ \times \text{SU}(2)^- [\text{HH}]$. $\lambda^\pm$ can be also be seen as the representations of $\text{SO}(4)$ in $\Lambda^\pm_2(\mathbb{R}^4) \simeq \mathbb{R}^3$ induced by the canonical representation in $\mathbb{R}^4$.

Let $X$ be a closed, oriented 4-manifold. Given any principal $\text{SO}(4)$-bundle $P$ over $X$, we denote by $P^\pm$ the induced principal $\text{SO}(3)$-bundles. If $\hat{P}$ is a $\text{Spin}^c(4)$-bundle, we let $\Sigma^\pm$ be the associated $U(2)$-vector bundles, and we set (via the vertical determinant-map in (1)) $\det(\hat{P}) = L$, so that $\det(\Sigma^\pm) = L$.

**Lemma 1.1** Let $P$ be a principal $\text{SO}(4)$-bundle over $X$ with characteristic classes $w_2(P) \in H^2(X, \mathbb{Z}_2)$, and $p_1(P), e(P) \in H^4(X, \mathbb{Z})$. Then
i) $P$ lifts to a principal $\text{Spin}^c(4)$-bundle $\hat{P}$ iff $w_2(P)$ lifts to an integral cohomology class.

ii) Given a class $L \in H^2(X, \mathbb{Z})$ with $w_2(P) \equiv L(\text{mod } 2)$, the $\text{Spin}^c(4)$-lifts $\hat{P}$ of $P$ with $\det \hat{P} = L$ are in 1-1 correspondence with the 2-torsion elements in $H^2(X, \mathbb{Z})$.

iii) Let $\hat{P}$ be a $\text{Spin}^c(4)$-principal bundle with $P \simeq \hat{P} / S^1$, and let $L = \det \hat{P}$. Then the Chern classes of $\Sigma^\pm$ are:

$$
c_1(\Sigma^\pm) = L \\
c_2(\Sigma^\pm) = \frac{1}{4}(L^2 - p_1(P) \mp 2e(P))
$$

**Proof:** [HH] and the diagrams above. ■
Consider now a Riemannian metric \( g \) on \( X \), and let \( P \) be the associated principal \( \text{SO}(4) \)-bundle. In this case the real vector bundles associated to \( P^\pm \) via the standard representations are the bundles \( \Lambda_\pm^2 \) of (anti-) self-dual 2-forms on \( X \).

The integral characteristic classes of \( P \) are given by \( p_1(P) = 3\sigma \) and \( e(P) = e \), where \( \sigma \) and \( e \) denote the signature and the Euler number of the oriented manifold \( X \). Furthermore, \( w_2(P) \) always lifts to an integral class, the lifts are precisely the characteristic elements in \( H^2(X, \mathbb{Z}) \), i.e. the classes \( L \) with \( x^2 \equiv x \cdot L \) for every \( x \in H^2(X, \mathbb{Z}) \) [HH].

Let \( T_X \) be the tangent bundle of \( X \), and denote by \( \Lambda_p \) the bundle of \( p \)-forms on \( X \). The choice of a Spin\(^c\)(4)-lift \( \hat{P} \) of \( P \) with associated \( U(2) \)-vector bundles \( \Sigma^\pm \) defines a vector bundle isomorphism \( \gamma : \Lambda^1 \otimes \mathbb{C} \to \text{Hom}_\mathbb{C}(\Sigma^+, \Sigma^-) \). There is also a \( \mathbb{C} \)-linear isomorphism \((\cdot)^\# : \text{Hom}_\mathbb{C}(\Sigma^+, \Sigma^-) \to \text{Hom}_\mathbb{C}(\Sigma^-, \Sigma^+)\) which satisfies the identity:

\[
\gamma(u)^\# \gamma(v) + \gamma(v)^\# \gamma(u) = 2g^\mathbb{C}(u, v)\text{id}_{\Sigma^+},
\]

and \( \gamma(u)^\# = \gamma(u)^* = g(u, u)\gamma(u)^{-1} \) for real non-vanishing cotangent vectors \( u \).

It is convenient to extend the homomorphisms \( \gamma(u) \) to endomorphisms of the direct sum \( \Sigma := \Sigma^+ \oplus \Sigma^- \). Putting \( \gamma(u)|_{\Sigma^-} := -\gamma(u)^\# \), we obtain a vector-bundle homomorphism \( \gamma : \Lambda^1 \otimes \mathbb{C} \to \text{End}_0(\Sigma) \), which maps the bundle \( \Lambda^1 \) of real 1-forms into the bundle of trace-free skew-Hermitian endomorphisms of \( \Sigma \). With this convention, we get:

\[
\gamma(u) \circ \gamma(v) + \gamma(v) \circ \gamma(u) = -2g^\mathbb{C}(u, v)\text{id}_\Sigma.
\]

Consider the induced homomorphism

\[
\Gamma : \Lambda^2 \otimes \mathbb{C} \to \text{End}_0(\Sigma)
\]

defined on decomposable elements by

\[
\Gamma(u \wedge v) := \frac{1}{2}[\gamma(u), \gamma(v)].
\]

The restriction \( \Gamma|_{\Lambda^2} \) identifies the bundle \( \Lambda^2 \) with the bundle \( \text{ad}_0(\hat{P}) \simeq \text{ad}(P) \) of skew-symmetric endomorphisms of the tangent bundle of \( X \).
\( \Lambda^2 \) splits as an orthogonal sum \( \Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_- \) and \( \Gamma \) maps the bundle \( \Lambda^2_+ \otimes \mathbb{C} \) (respectively \( \Lambda^2_- \)) isomorphically onto the bundle \( \text{End}_0(\Sigma^\pm) \subset \text{End}(\Sigma) \) of trace-free (trace free skew-Hermitian) endomorphisms of \( \Sigma^\pm \).

We give now an explicit description of the two spinor bundles \( \Sigma^\pm \) and of the map \( \Gamma \) in the case of a Spin\(^c\)(4)-structure coming from an almost Hermitian structure.

**Definition 1.2** A characteristic element \( K \in H^2(X, \mathbb{Z}) \) is an almost canonical class if \( K^2 = 3\sigma + 2e \).

Such classes exist on \( X \) if and only if \( X \) admits an almost complex structure. More precisely:

**Proposition 1.3** (Wu) \( K \in H^2(X, \mathbb{Z}) \) is an almost canonical class if and only if there exists an almost complex structure \( J \) on \( X \) which is compatible with the orientation, such that \( K = c_1(\Lambda^1_{J0}) \).

**Proof:** [HH]

Here we denote, as usual, by \( \Lambda^p_q \) the bundle of \((p, q)\)-forms defined by the almost complex structure \( J \).

Notice that any almost complex structure \( J \) on \( X \) can be deformed into a \( g \)-orthogonal one, and that \( J \) is \( g \)-orthogonal if \( g \) is \( J \)-Hermitian. The choice of a \( g \)-orthogonal almost complex structure \( J \) on \( X \) corresponds to a reduction of the \( \text{SO}(4) \)-bundle \( P \) of \( X \) to a \( U(2) \)-bundle via the inclusion \( U(2) \subset \text{SO}(4) \); since the inclusion factors through the embedding \( \Delta : U(2) \longrightarrow \text{Spin}^c(4) \) (see diagram (1)), this reduction defines a unique Spin\(^c\)(4)-bundle \( \hat{P}_J \) over \( X \). By construction we have \( \hat{P}_J/S^1 \simeq P \), and \( \det \hat{P}_J = -K \).

**Proposition 1.4** Let \( J \) be a \( g \)-orthogonal almost complex structure on \( X \), compatible with the orientation.

i) The spinor bundles \( \Sigma^\pm_J \) of \( \hat{P}_J \) are:

\[ \Sigma^+_J \simeq \Lambda^{00} \oplus \Lambda^{02}_J, \quad \Sigma^-_J \simeq \Lambda^{01}_J. \]

ii) The map \( \Gamma : \Lambda^2_+ \otimes \mathbb{C} \longrightarrow \text{End}_0(\Sigma^+_J) \) is given by

\[
\Lambda^{20}_J \oplus \Lambda^{02}_J \oplus \Lambda^{00}_J \omega_g \ni (\lambda^{20}, \lambda^{02}, \omega_g) \longmapsto \begin{bmatrix}
-i & -*(\lambda^{20} \wedge \cdot) \\
\lambda^{02} \wedge \cdot & i
\end{bmatrix} \in \text{End}_0(\Lambda^{00} \oplus \Lambda^{02}).
\]
**Proof:** i) \( c_1(\Sigma^+) = c_1(\Sigma^-) = -K \), \( c_2(\Sigma^+) = \frac{1}{4}[K^2 - 3\sigma - 2e] \), \( c_2(\Sigma^-) = \frac{1}{4}[K^2 - 3\sigma + 2e] = c_2(\Sigma^+) + e \), and \( U(2) \)-bundles on a 4-manifold are classified by their Chern classes.

ii) With respect to a suitable choice of the isomorphisms i), the Clifford map \( \gamma \) acts by
\[
\gamma(u)(\varphi + \alpha) = \sqrt{2} \left( \varphi u^{01} - i \Lambda_g u^{10} \wedge \alpha \right),
\]
\[
\gamma(u)^\#: (\varphi u^{10} \wedge \theta) - u^{01} \wedge \theta
\]
where \( \Lambda_g : \Lambda^p_{\Sigma} \rightarrow \Lambda^{p-1,q-1}_g \) is the adjoint of the map \( \cdot \wedge \omega_g \) [H1].

2 The coupled Seiberg-Witten equations

Let \( P \) be the principal \( SO(4) \)-bundle associated with the tangent bundle of the oriented, closed Riemannian 4-manifold \((X, g)\), and fix a \( Spin^c(4) \) structure \( \hat{P} \) over \( P \) with \( L = \det(\hat{P}) \). The choice of a \( Spin^c(4) \)-connection in \( \hat{P} \) projecting onto the Levi-Civita connection in \( P \) is equivalent to the choice of a connection \( b \) in the unitary line bundle \( L \) [H1]. We denote by \( B(b) \) the \( Spin^c(4) \)-connection in \( \hat{P} \) corresponding to \( b \), and also the induced connection in the vector bundle \( \Sigma = \Sigma^+ \oplus \Sigma^- \). The curvature \( F_{B(b)} \) of the connection \( B(b) \) in \( \Sigma \) has the form
\[
F_{B(b)} = \frac{1}{2} F_{b^\text{id}_\Sigma} + F_g = \begin{bmatrix} \frac{1}{2} F_{b^\text{id}_\Sigma^+} + F_g^+ & 0 \\ 0 & \frac{1}{2} F_{b^\text{id}_\Sigma^-} + F_g^- \end{bmatrix},
\]
where \( F_g \) and \( F_g^\pm \) denote the Riemannian curvature operator, and its components with respect to the splitting \( \text{ad}(P) = \Lambda^2_+ \oplus \Lambda^2_- \).

Let now \( E \) be an arbitrary Hermitian bundle of rank \( r \) over \( X \), and \( A \) a connection in \( E \). We denote by \( A_b \) the induced connection in the tensor product \( \Sigma \otimes E \), and by \( \hat{\mathcal{D}}_{A,b} : A^0(\Sigma \otimes E) \rightarrow A^0(\Sigma \otimes E) \) the associated Dirac operator. \( \hat{\mathcal{D}}_{A,b} \) is defined as the composition:
\[
A^0(\Sigma \otimes E) \xrightarrow{\nabla_{A_b}} A^1(\Sigma \otimes E) \xrightarrow{m} A^0(\Sigma \otimes E)
\]
where \( m \) is the Clifford multiplication \( m(u, \sigma \otimes e) := (u)(\sigma) \otimes e \). \( \hat{\mathcal{D}}_{A,b} \) is an elliptic, self-adjoint operator and its square \( \hat{\mathcal{D}}_{A,b}^2 \) is related to the usual Laplacian \( \nabla_{A_b}^* \nabla_{A_b} \) by the Weitzenböck formula
\[
\hat{\mathcal{D}}_{A,b}^2 = \nabla_{A_b}^* \nabla_{A_b} + \Gamma(F_{A,b}).
\]
Here $\Gamma(F_{A,b}) \in A^0(\text{End}(\Sigma \otimes E))$ is the Hermitian endomorphism defined as the composition

$$A^0(\Sigma \otimes E) \xrightarrow{F_{A,b}} A^0(\Lambda^2 \otimes \Sigma \otimes E) \xrightarrow{\Delta} A^0(\text{End}_0(\Sigma) \otimes \Sigma \otimes E) \xrightarrow{\text{ev}} A^0(\Sigma \otimes E).$$

We set $F_{A,b} := F_A + \frac{1}{2} F_b \text{id}_E \in A^0(\Lambda^2 \otimes \text{End}(E))$.

**Proposition 2.1** Let $s$ be the scalar curvature of the Riemannian 4-manifold $(X, g)$. Fix a Spin$^c(4)$-structure on $X$ and choose connections $b$ and $A$ in $L$ and $E$ respectively. Then

$$\hat{D}_2^2 A,b = \nabla^* A b \nabla A b + \Gamma(F_{A,b}) + s \frac{4}{4} \text{id}_{\Sigma \otimes E}.$$

**Proof:** Since $\Gamma(F_g) = \frac{s}{4} \text{id}_\Sigma$ [H1], and $F_{A,b} = F_{B(b)} \otimes \text{id}_E + \text{id}_\Sigma \otimes F_A = \frac{1}{2} F_b \text{id}_\Sigma \otimes \text{id}_E + F_g \otimes \text{id}_E + \text{id}_\Sigma \otimes F_A = \text{id}_\Sigma \otimes (F_A + \frac{1}{2} F_b \text{id}_E) + F_g \text{id}_E$, we find $\hat{\Gamma}(F_{A,b}) = \Gamma(F_{A,b}) + \frac{s}{4} \text{id}_{\Sigma \otimes E}$. \hfill $\blacksquare$

**Remark 2.2** One has a Bochner-type result for spinors $\Psi$ on which $\Gamma(F_{A,b}) + \frac{s}{4} \text{id}_{\Sigma \otimes E}$ is positive: Such a spinor is harmonic if and only if it is parallel [H1].

Let $(\ , \ )$ be the pointwise inner product on $\Sigma \otimes E$, $| \ |$ the associated pointwise norm, and $\| \ |$ the corresponding $L^2$-norm. For a spinor $\Psi \in A^0(\Sigma^\pm \otimes E)$ we define $(\Psi \bar{\Psi})_0 \in A^0(\text{End}_0(\Sigma^\pm \otimes E))$ as the image of the Hermitian endomorphism $\Psi \otimes \bar{\Psi} \in A^0(\text{End}(\Sigma^\pm \otimes E))$ under the projection $\text{End}(\Sigma^\pm \otimes E) \longrightarrow \text{End}_0(\Sigma^\pm) \otimes \text{End}(E)$.

**Corollary 2.3** With the notations above, we have

$$(P_{A,b}^2 \Psi, \Psi) = (\nabla^*_A \nabla_A \Psi, \Psi) + (\Gamma(F_{A,b}^+), (\Psi + \bar{\Psi})_0) + (\Gamma(F_{A,b}^-), (\Psi - \bar{\Psi})_0) + \frac{s}{4} |\Psi|^2,$$

where $(F_{A,b}^-)$ $F_{A,b}^+$ is the (anti-)self-dual component of $F_{A,b}$.

**Proof:** Indeed, since $\Gamma(F_{A,b}^\pm)$ vanishes on $\Sigma^\mp$ and is trace free with respect to $\Sigma^\pm$, the inner product $(\Gamma(F_{A,b}), (\Psi \bar{\Psi}))$ in the Weitzenböck formula simplifies for a spinor $\Psi \in A^0(\Sigma^\pm \otimes E)$:

$$(\Gamma(F_{A,b}), (\Psi \bar{\Psi})) = (\Gamma(F_{A,b}^\mp), (\Psi \bar{\Psi})_0)$$
For a positive spinor $\Psi \in A^0(E \otimes \Sigma^+)$, the following important identity follows immediately:

$$\left(D^a_{A,b} \Psi, \Psi\right) + \frac{1}{2} |\Gamma(F^+_a) - (\Psi \bar{\Psi})_0|^2 = (\nabla^a_{A,b} \Psi, \Psi) + \frac{1}{2} |F^+_a|^2 + \frac{1}{2} |(\Psi \bar{\Psi})_0|^2 + \frac{s}{4} |\Psi|^2$$

(4)

If we integrate both sides of (4) over $X$, we get:

**Proposition 2.4** Let $(X, g)$ be an oriented, closed Riemannian 4-manifold with scalar curvature $s$, $E$ a Hermitian bundle over $X$. Choose a Spin$^c(4)$-structure on $X$ and a connection $b$ in the determinant line bundle $L = \det(\Sigma^+) = \det(\Sigma^-)$. Let $A$ be a connection in $E$. For any $\Psi \in A^0(\Sigma^+ \otimes E)$ we have:

$$\parallel D^a_{A,b} \Psi \parallel^2 + \frac{1}{2} \parallel \Gamma(F^+_a) - (\Psi \bar{\Psi})_0 \parallel^2 = 
= \parallel \nabla^a_{A,b} \Psi \parallel^2 + \frac{1}{2} \parallel F^+_a \parallel^2 + \frac{1}{2} \parallel (\Psi \bar{\Psi})_0 \parallel^2 + \frac{1}{4} \int_X s |\Psi|^2.$$

We introduce now our coupled variant of the Seiberg-Witten equations. The unknown is a pair $(A, \Psi)$ consisting of a connection in the Hermitian bundle $E$ and a section $\Psi \in A^0(\Sigma^+ \otimes E)$. The equations ask for the vanishing of the left-hand side in the above formula.

$$\left\{ \begin{array}{c} D^a_{A,b} \Psi = 0 \\ \Gamma(F^+_a) = (\Psi \bar{\Psi})_0 \end{array} \right. \quad (SW)$$

Proposition 2.4 and the inequality $|((\Psi \bar{\Psi})_0|^2 \geq \frac{1}{2} |\Psi|^4$ give immediately:

**Remark 2.5** If the scalar curvature $s$ is nonnegative on $X$, then the only solutions of the equations are the pairs $(A, 0)$, with $F^+_a = 0$.

If $L$ is the square of a line bundle $L^\frac{1}{2}$, and if we choose a connection $b^\frac{1}{2}$ in $L^\frac{1}{2}$ with square $b$, then $F^\pm_{A,b}$ is simply the curvature of the connection $A_{b^\pm}$ in $E \otimes L^\pm$. The solution of the coupled Seiberg-Witten equations on a manifold with $s \geq 0$ are in this case just $U(r)$-instantons on $E \otimes L^\pm$.

In the case of a Kähler surface $(X, g)$, the coupled Seiberg-Witten equation can be reformulated in terms of complex geometry. The point is that if
we consider the canonical Spin$^c$(4)-structure associated to the Kähler structure, the Dirac operator has a very simple form \([H1]\). The determinant of this Spin$^c$(4)-structure is the anti-canonical bundle \(K_X^\vee\) of the surface, which comes with a holomorphic structure and a natural metric inherited from the holomorphic tangent bundle.

Let \(c\) be the Chern connection in \(K_X^\vee\). With this choice, the induced connection \(B(c)\) in \(\Sigma = \Lambda^{00} \oplus \Lambda^{02} \oplus \Lambda^{01}\) coincides with the connection defined by the Levi-Civita connection. Recall that on a Kähler manifold, the almost complex structure is parallel with respect to the Levi-Civita connection, so that the splitting of the exterior algebra \(\bigoplus_p \Lambda^p \otimes \mathbb{C}\) becomes parallel, too.

**Proposition 2.6** Let \((X, g)\) be a Kähler surface with Chern connection \(c\) in \(K_X^\vee\). Choose a connection \(A\) in a Hermitian vector bundle \(E\) over \(X\) and a section \(\Psi = \varphi + \alpha \in A^0(E) + A^0(\Lambda^{02} \otimes E)\).

The pair \((A, \Psi)\) satisfies the Seiberg-Witten equations iff the following identities hold:

\[
\begin{align*}
F^{20}_{A,c} &= -\frac{1}{2} \varphi \otimes \bar{\alpha} \\
F^{02}_{A,c} &= \frac{1}{2} \alpha \otimes \varphi \\
i\Lambda_g F_{A,c} &= -\frac{1}{2} (\varphi \otimes \bar{\varphi} - *(\alpha \otimes \bar{\alpha})) \\
\bar{\partial}_A \varphi &= i \Lambda_g \partial_A \alpha
\end{align*}
\]

**Proof:** The Dirac operator is in this case \(D_{A,c} = \sqrt{2}(\bar{\partial}_A - i \Lambda_g \partial_A)\), and the endomorphism \((\Psi \bar{\Psi})_0\) has the form:

\[
\begin{pmatrix}
\frac{1}{2}(\varphi \otimes \bar{\varphi} - *(\alpha \otimes \bar{\alpha})) & *(\varphi \otimes \bar{\alpha} \wedge \cdot) \\
\alpha \otimes \bar{\varphi} & -\frac{1}{2}(\bar{\varphi} \otimes \bar{\varphi} - *(\alpha \otimes \bar{\alpha}))
\end{pmatrix}.
\]

Since \(\Gamma(F^+_{A,c}) = \Gamma(F^{20}_{A,c} + F^{02}_{A,c} + \frac{1}{2} \Lambda_g F_{A,c} : \omega_g)\) equals

\[
2 \begin{pmatrix}
-\frac{i}{2} \Lambda_g F_{A,c} & - * (F^{20}_{A,c} \wedge \cdot) \\
F^{20}_{A,c} \wedge \cdot & \frac{i}{2} \Lambda_g F_{A,c}
\end{pmatrix},
\]

the equivalence of the two systems of equations follows. \(\blacksquare\)
3 Monopoles on Kähler surfaces and the generalized vortex equation

Let $(X, g)$ be a Kähler surface with canonical Spin$^c(4)$-structure, and Chern connection $c$ in the anti-canonical bundle $K_X^\vee$.

We fix a unitary vector bundle $E$ of rank $r$ over $X$, and define $J(E) := \deg_g(\Sigma^+ \otimes E)$, i.e. $J(E) = 2r(\mu_g(E) - \frac{1}{2} \mu_g(K_X))$, where $\mu_g$ denotes the slope with respect to $\omega_g$.

Every spinor $\Psi \in \mathcal{A}^0(\Sigma^+ \otimes E)$ has the form $\Psi = \varphi + \alpha$ with $\varphi \in \mathcal{A}^0(E)$ and $\alpha \in \mathcal{A}^0(E^\vee)$.

We have seen that the coupled Seiberg-Witten equations are equivalent to the system:

$$\begin{cases}
F_{A,c}^{20} = -\frac{1}{2} \varphi \otimes \bar{\alpha} \\
F_{A,c}^{02} = \frac{1}{2} \alpha \otimes \varphi \\
i \Lambda_g F_{A,c} = -\frac{1}{2} (\varphi \otimes \varphi - *(\alpha \otimes \bar{\alpha})) \\
\bar{\partial}_A \varphi = i \Lambda_g \partial_A \alpha
\end{cases}$$

($SW^*$)

Lemma 3.1
A. Suppose $J(E) < 0$: A pair $(A, \varphi + \alpha)$ is a solution of the system ($SW^*$) if and only if

i) $F_{A,c}^{20} = F_{A,c}^{02} = 0$

ii) $\alpha = 0, \bar{\partial}_A \varphi = 0$

iii) $i \Lambda_g F_A + \frac{1}{2} \varphi \otimes \bar{\varphi} + \frac{1}{2} \text{Id}_E = 0$.

B. Suppose $J(E) > 0$, and put $a := \bar{\alpha} \in A^{20}(\bar{E}) = A^0(E^\vee \otimes K_X)$:

A pair $(A, \varphi + \bar{a})$ is a solution of the system ($SW^*$) if and only if

i) $F_{A,c}^{20} = F_{A,c}^{02} = 0$

ii) $\varphi = 0, \bar{\partial}_A a = 0$

iii) $i \Lambda_g F_A - \frac{1}{2} * (a \otimes \bar{a}) + \frac{1}{2} \text{Id}_E = 0$.

Proof: (cf. [W]) The splitting $\Sigma^+ \otimes E = \Lambda^{00} \otimes E \oplus \Lambda^{02} \otimes E$ is parallel with respect to $\nabla_{A,c}$, so that, by Proposition 2.4

$$\| P_{A,c} \Psi \|^2 + \frac{1}{2} \| \Gamma(F_{A,c}^+) - (\Psi \bar{\Psi})_0 \|^2 =$$

$$= \| \nabla_{A,c} \varphi \|^2 + \| \nabla_{A,c} \alpha \|^2 + \frac{1}{2} \| F_{A,c}^+ \|^2 + \frac{1}{2} \| (\Psi \bar{\Psi})_0 \|^2 + \frac{1}{4} \int_X s(|\varphi|^2 + |\alpha|^2).$$

11
The right-hand side is invariant under the transformation \((A, \varphi, \alpha) \mapsto (A, \varphi, -\alpha)\), hence any solution \((A, \varphi + \alpha)\) must have \(F^0_A = F^0_A = 0\) and \(\varphi \otimes \bar{\alpha} = \alpha \otimes \bar{\varphi} = 0\); the latter implies obviously \(\alpha = 0\) or \(\varphi = 0\). Integrating the trace of the equation \(i \Lambda g F_{A,c} = -\frac{1}{2} (\varphi \otimes \bar{\varphi} - *(\alpha \otimes \bar{\alpha}))\), we find:

\[
J(E) = c_1(\Sigma^+ \otimes E) \cup [\omega_g] = (2c_1(E) - rc_1(K_X)) \cup [\omega_g] = 2 \int_X i \frac{1}{2\pi} \text{Tr}(F_{A,c}) \wedge \omega_g = \frac{1}{4\pi} \int_X \text{Tr}(i\Lambda F_{A,c}) \omega^2_g = \frac{1}{8\pi} \int_X \text{Tr}(-\varphi \otimes \bar{\varphi} + *(\alpha \otimes \bar{\alpha})) \omega^2_g
\]

This equation shows that we must have \(\alpha = 0\), if \(J(E) < 0\), and \(\varphi = 0\), if \(J(E) > 0\). Notice that, replacing \(E\) by \(E^\vee \otimes K_X\), the second case reduces to the first one.

The assertion follows now immediately from the identity \(i\Lambda g F_c = s\).

Notice that the last equation

\[i\Lambda g F_A + \frac{1}{2} \varphi \otimes \bar{\varphi} + \frac{1}{2} \text{id}_E = 0\]

has the form of a generalized vortex equation as studied by Bradlow [B1], [B2] and by Garcia-Prada [G2]; it is precisely the vortex equation with constant \(\tau = -s\), if \((X, g)\) has constant scalar curvature.

Let \(s_m\) be the mean scalar curvature defined by \(\int_X s \omega^2_g = s_m \int_X \omega^2 = 2s_m \text{Vol}_g(X)\).

We are going to prove that the system

\[
\begin{cases}
\bar{\partial}_A^2 = 0 \\
\bar{\partial}_A \varphi = 0 \\
i\Lambda g F_A + \frac{1}{2} \varphi \otimes \bar{\varphi} + \frac{1}{2} \text{id}_E = 0
\end{cases}
\]

for the pair \((A, \varphi)\) consisting of a unitary connection in \(E\), and a section in \(E\), is always equivalent to the vortex system with parameter \(\tau = -s_m\), i.e. to the system obtained by replacing the third equation with

\[i\Lambda g F_A + \frac{1}{2} \varphi \otimes \bar{\varphi} + \frac{1}{2} s_m \text{id}_E = 0.\]

"Equivalent" means here that the corresponding moduli spaces of solutions are naturally isomorphic.
Let generally $t$ be a smooth real function on $X$ with mean value $t_m$, and consider the following system of equations:

\[
\begin{align*}
\bar{\partial}_A^2 & = 0 \\
\bar{\partial}_A \varphi & = 0 \\
i \Lambda_g F_A + \frac{1}{2} \varphi \otimes \bar{\varphi} - \frac{i}{2} t \text{id}_E & = 0
\end{align*}
\]  

$(V_t)$ is defined on the space $\mathcal{A}(E) \times A^0(E)$, where $\mathcal{A}(E)$ is the space of unitary connections in $E$. The product $\mathcal{A}(E) \times A^0(E)$ (completed with respect to sufficiently large Sobolev indices) carries a natural $L^2$ Kähler metric $\tilde{g}$ and a natural right action of the gauge group $U(E)$: $(A, \varphi)^f := (A^f, f^{-1} \varphi)$, where $d_{A^f} := f^{-1} \circ d_A \circ f$.

For every real function $t$ let

\[ m_t : \mathcal{A}(E) \times A^0(E) \rightarrow A^0(\text{ad}(E)) \]

be the map given by $m_t := \Lambda_g F_A - \frac{1}{2} \varphi \otimes \bar{\varphi} + \frac{i}{2} t \text{id}_E$.

**Proposition 3.2** $m_t$ is a moment map for the action of $U(E)$ on $\mathcal{A}(E) \times A^0(E)$.

**Proof:** Let $a^\#$ be the vector field on $\mathcal{A}(E) \times A^0(E)$ associated with the infinitesimal transformation $a \in A^0(\text{ad}(E)) = \text{Lie}(U(E))$, and define the real function $m_t^a : \mathcal{A}(E) \times A^0(E) \rightarrow \mathbb{R}$ by:

\[ m_t^a(x) = \langle m_t(x), a \rangle_{L^2}. \]

We have to show that $m_t$ satisfies the identities:

\[ \iota_{a^\#} \omega_{\tilde{g}} = \text{d} m_t^a, \quad m_t^a \circ f = m_t^{\text{ad}_f(a)} \text{ for all } a \in A^0(\text{ad}(E)), \quad f \in U(E). \]

It is well known that, in general, a moment map for a group action in a symplectic manifold is well defined up to a constant central element in the Lie algebra of the group. In our case, the center of the Lie algebra $A^0(\text{ad}(E))$ of the gauge group is just $iA^0 \text{id}_E$, hence it suffices to show that $m_0$ is a moment map. This has already been noticed by Garcia-Prada [G1], [G2].

Note also that in our case every moment map has the form $m_t$ for some function $t$, which shows that from the point of view of symplectic geometry, the natural equations are the generalized vortex equations $(V_t)$. 

13
In order to show that Bradlow’s main result [B2] also holds for the generalized system \((V_t)\), we have to recall some definitions.

Let \(E\) be a holomorphic vector bundle of topological type \(E\), and let \(\phi \in H^0(\mathcal{E})\) be a holomorphic section. The pair \((\mathcal{E}, \phi)\) is \(\lambda\)-stable with respect to a constant \(\lambda \in \mathbb{R}\) iff the following conditions hold:

1. \(\mu_g(\mathcal{E}) < \lambda\) and \(\mu_g(\mathcal{F}) < \lambda\) for all reflexive subsheaves \(\mathcal{F} \subset \mathcal{E}\) with \(0 < \text{rk}(\mathcal{F}) < r\).
2. \(\mu_g(\mathcal{E}/\mathcal{F}) > \lambda\) for all reflexive subsheaves \(\mathcal{F} \subset \mathcal{E}\) with \(0 < \text{rk}(\mathcal{F}) < r\) and \(\phi \in H^0(\mathcal{F})\).

**Theorem 3.3** Let \((X, g)\) be a closed Kähler manifold, \(t \in A^0\) a real function, and \((\mathcal{E}, \phi)\) a holomorphic pair over \(X\). Set \(\lambda := \frac{1}{4\pi t} \text{Vol}_g(X)\). \(\mathcal{E}\) admits a Hermitian metric \(h\) such that the associated Chern connection \(A_h\) satisfies the vortex equation

\[
iA_g F_A + \frac{1}{2} \phi \otimes \bar{\phi} - \frac{1}{2} \text{tr}E = 0
\]

iff one of the following conditions holds:

1. \((\mathcal{E}, \phi)\) is \(\lambda\)-stable
2. \(\mathcal{E}\) admits a splitting \(\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''\) with \(\phi \in H^0(\mathcal{E}')\) such that \((\mathcal{E}', \phi)\) is \(\lambda\)-stable, and \(\mathcal{E}''\) admits a weak Hermitian-Einstein metric with factor \(\frac{t}{2}\). In particular \(\mathcal{E}''\) is polystable of slope \(\lambda\).

**Proof:** In the case of a constant function \(t = \tau\), the theorem was proved by Bradlow [B2], and his arguments work in the general context, too: The fact that the existence of a solution of the vortex equation implies \((i)\) or \((ii)\) follows by replacing the constant \(\tau\) in [B2] everywhere with the function \(t\).

The difficult part consists in showing that every \(\lambda\)-stable pair \((\mathcal{E}, \phi)\) admits a metric \(h\) such that \((A_h, \phi)\) satisfies the vortex equation \((V_t)\). To this end let \(\text{Met}(\mathcal{E})\) be the space of Hermitian metrics in \(\mathcal{E}\), and fix a background metric \(k \in \text{Met}(\mathcal{E})\). Bradlow constructs a functional \(M_{\phi, \tau}(\cdot, \cdot) : \text{Met}(\mathcal{E}) \times \text{Met}(\mathcal{E}) \rightarrow \mathbb{R}\), which is convex in the second argument, such that any critical point of \(M_{\phi, \tau}(k, \cdot)\) is a solution of the vortex equation; the point is then to find an absolute minimum of \(M_{\phi, \tau}(k, \cdot)\). The existence of an absolute minimum follows from the following basic \(C^0\) estimate:

**Lemma 3.4** (Bradlow) Let \(\text{Met}^2(E, B) := \{h = ke^a | a \in L^2_p(\text{End}(E)), a^*k = a, \| \mu_t(A_h, \phi) \|_{L^p} \leq B\}\). If \((\mathcal{E}, \phi)\) is \(\frac{1}{4\pi t} \text{Vol}_g(X)\)-stable, then there exist positive
constants $C_1, C_2$ such that
\[
\sup |a| \leq C_1 M_{\varphi, \tau}(k, ke^a) + C_2,
\]
for all $k$-Hermitian endomorphisms $a \in L^2_{\nu}(\text{End}(E))$. Moreover, any absolute minimum of $M_{\varphi, t}(k, \cdot)$ on $\text{Met}^0_{L^2}(E, B)$ is a critical point of $M_{\varphi, t}(k, \cdot)$, and gives a solution of the vortex equation $V_\tau$.

Let now $t$ be a real function on $X$, and choose a solution $v$ of the Laplace equation $i\Lambda g \bar{\partial} \partial v = \frac{1}{2}(t - t_m)$. If we make the substitution $h = h' e^v$, then $h$ solves the vortex equation $(V_t)$ iff $h'$ is a solution of
\[
i\Lambda g F h' + \frac{1}{2} e^v \varphi \otimes \bar{\varphi} h' - \frac{1}{2} t_m \text{id}_E = 0.
\]
Define $\mu_{t_m, v}(h') := i\Lambda g F h' + \frac{1}{2} e^v \varphi \otimes \bar{\varphi} h' - \frac{1}{2} t_m \text{id}_E = 0$, and
\[
M_{\varphi, t_m, v}(k, h) := M_D(k, h) + \| e^{\frac{\varphi}{2}} \varphi \|_h^2 - \| e^{\frac{\varphi}{2}} \varphi \|_k^2 - t_m \int_X \text{Tr} \log(k^{-1} h)),
\]
where $M_D$ is the Donaldson functional [D]. Then it is not difficult to show that all arguments of Bradlow remain correct after replacing $\mu_{t_m}$ and $M_{\varphi, t_m}$ with $\mu_{t_m, v}$ and $M_{\varphi, t_m, v}$ respectively. Indeed, let $l$ be a positive bound from below for the map $e^v$. Then
\[
M_{\varphi, t_m}(k, ke^{a + \log t}) \leq M_D(k, ke^a) + M_D(k e^a, lke^a) + \| l \varphi \|_h^2 - t_m \int_X \text{Tr} \log(kek^{-1} h) \\
\leq M_{\varphi, t_m, v}(k, ke^a) + \| e^{\frac{\varphi}{2}} \varphi \|_h^2 + 2 \log \text{deg}_g(E) - rt_m \log \text{Vol}_g(X) \\
\leq M_{\varphi, t_m, v}(k, ke^a) + C'(k, \varphi, v, l).
\]
Similarly, we get constants $n > 0$, $C''$ and an inequality
\[
M_{\varphi, t_m, v}(k, ke^{a + \log n}) \leq M_{\varphi, t_m}(k, ke^a) + C'',$
which shows that the basic $C^0$ estimate in the Lemma above holds for $M_{\varphi, t_m, v}$ iff it holds for Bradlow’s functional $M_{\varphi, t_m}$.

Remark 3.5 In the special case of a rank-1 bundle $E$, a much more elementary proof based on [B1] is possible.
4 Moduli spaces of monopoles, vortices, and stable pairs

Let \((X, g)\) be a closed Kähler manifold of arbitrary dimension, and fix a unitary vector bundle \(E\) of rank \(r\) over \(X\). We denote by \(\mathcal{A}(E)\) the affine space of semiconnection of type \((0,1)\) in \(E\). The complex gauge group \(GL(E)\) acts on \(\mathcal{A}(E) \times A^0(E)\) from the right by \((\tilde{\partial}_A, \varphi)^g := (g^{-1} \circ \tilde{\partial}_A \circ g, g^{-1} \varphi)\); this action becomes complex analytic after suitable Sobolev completions. We denote by \(\mathcal{S}(E)\) the set of pairs \((\tilde{\partial}_A, \varphi)\) with trivial isotropy group. Notice that \(\varphi \neq 0\) when \((\tilde{\partial}_A, \varphi) \in \mathcal{S}(E)\), and that \(\mathcal{S}(E)\) is an open subset of \(\mathcal{A}(E) \times A^0(E)\), by elliptic semi-continuity [K].

The action of \(GL(E)\) on \(\mathcal{S}(E)\) is free, by definition, and we denote the Hilbert manifold \(\mathcal{S}(E)/GL(E)\) by \(\mathcal{B}^s(E)\). The map \(p : \mathcal{A}(E) \times A^0(E) \rightarrow A^{02}(\text{End}(E)) \oplus A^{01}(E)\) defined by \(p(\tilde{\partial}_A, \varphi) = (F^{02}_A, \tilde{\partial}_A \varphi)\) is equivariant with respect to the natural actions of \(GL(E)\), hence it gives rise to a section \(\hat{p}\) in the associated vector bundle \(\mathcal{S}(E) \times_{GL(E)} (A^{02}(\text{End}(E)) \oplus A^{01}(E))\) over \(\mathcal{B}^s(E)\). We define the moduli space of simple pairs of type \(E\) to be the zero-locus \(Z(\hat{p})\) of this section. \(Z(\hat{p})\) can be identified with the set of isomorphism classes consisting of a holomorphic bundle \(E\) of differentiable type \(E\), and a holomorphic section \(\varphi \neq 0\), such that the kernel of the evaluation map \(ev(\varphi) : H^0(\text{End}(E)) \rightarrow H^0(\mathcal{E})\) is trivial.

In a similar way we define the moduli space \(\mathcal{V}_t^0\) of gauge-equivalence classes of irreducible solutions of the generalized vortex equation \(V_t\):

Let \(B^+\) denote as usual the subbundle \(((\Lambda^{02} + \Lambda^{20}) \cap \Lambda^2) \oplus \Lambda^0 \omega\) of the bundle \(\Lambda^2\) of real 2-forms on \(X\). We denote by \(\mathcal{D}\) the open subset of the product \(\mathcal{D} := \mathcal{A}(E) \times A^0(E) \simeq \mathcal{A}(E) \times A^0(E)\) consisting of pairs with trivial isotropy group with respect to the action of the gauge group \(U(E)\). The quotient \(\mathcal{B}^s(E) := \mathcal{D}^s(E)/U(E)\) comes with the structure of a real-analytic manifold.

Let \(v : \mathcal{D}(E) \rightarrow A^0(B^+ \otimes \text{ad}(E)) \oplus A^{01}(E)\) be the map given by:

\[
v(A, \varphi) = (F^{20} + F^{02}, m_t(A, \varphi) \omega_B \text{id}_E, \tilde{\partial}_A \varphi).\]

Again \(v\) is \(U(E)\)-equivariant, and the moduli space \(\mathcal{V}_t^0\) of \(t\)-vortices is defined to be the zero-locus \(Z(v)\) of the induced section \(\hat{v}\) of \(\mathcal{D}^s(E) \times_{U(E)} A^0(B^+ \otimes \text{ad}(E)) \oplus A^{01}(E)\) over \(\mathcal{B}^s(E)\).
Notice now that by Proposition 3.2, the second component \( v^2 \) of \( v \) is a moment map for the \( U(E) \) action. It is easy to see that (at least in a neighbourhood of \( Z(v) \cap D^* \)) it has the general property of a moment map in the finite dimensional Kähler geometry: Its zero locus \( Z(v^2) \) is smooth and intersects every \( GL(E) \) orbit along a \( U(E) \) orbit, and the intersection is transversal. This means that the natural map \( A \to \bar{\partial}_A \) defines in a neighbourhood of \( Z(\hat{v}^2) \cap D^* \) an open embedding \( i : Z(\hat{v}^2) \to \bar{B}^s \) of smooth Hilbert manifolds.

Regard now \( V_g \) as the subspace of \( Z(\hat{v}^2) \subset B^*(E) \) defined by the equation \( \hat{v}^1, \hat{v}^3 = 0 \). On the other hand, the pullback of the equation \( \hat{p} = 0 \), cutting out the moduli space \( Z(\hat{p}) \) of simple holomorphic pairs, via the open embedding \( i \) is precisely the equation \( \hat{v}^1, \hat{v}^3 = 0 \), cutting out \( V_g \). We get therefore an open embedding \( i_0 : V_g \to Z(\hat{p}) \) of real analytic spaces induced by \( i \), and by Theorem 3.3 the image of \( i_0 \) consists of the set of \( \lambda \)-stable pairs, with \( \lambda := \frac{1}{4\pi} t_m \text{Vol}_g(X) \).

Finally we denote by \( \mathcal{M}_X^*(E, \lambda) \subset Z(\hat{p}) \) the open subspace of \( \lambda \)-stable pairs, with the induced complex space-structure. Putting everything together, we have:

**Theorem 4.1** Let \((X, g)\) be a closed Kähler manifold, \( t \in A^0 \) a real function, and \( \lambda := \frac{1}{4\pi} t_m \text{Vol}_g(X) \). Fix a unitary vector bundle \( E \) of rank \( r \) over \( X \). There are natural real-analytic isomorphisms of moduli spaces

\[
V^p_t(E) \simeq V^p_{tm}(E) \simeq \mathcal{M}_X^*(E, \lambda).
\]

Let us come back now to the monopole equation \((\text{SW}^*)\) on a Kähler surface. In this case the function \( t \) is the negative of the scalar curvature \( s \), so that the corresponding constant \( \lambda \) becomes:

\[
\lambda = \frac{-s_m}{4\pi} \text{Vol}_g(X) = -\frac{1}{8\pi} \int_X s\omega^2 = -\frac{1}{8\pi} \int_X (i\Lambda F_c)\omega^2 = -\frac{1}{4\pi} \int_X iF_c \wedge \omega = \frac{1}{2} \mu_g(K).
\]

This yields our main result:

**Theorem 4.2** Let \((X, g)\) be a Kähler surface with canonical \( \text{Spin}^c(4) \)-structure, and Chern connection \( c \) in \( K_X \). Fix a unitary vector bundle \( E \) of rank \( r \) over \( X \), and suppose \( J(E) = \deg_g(\Sigma^+ \otimes E) < 0 \). The moduli space of solutions of the coupled Seiberg-Witten equations is isomorphic to the moduli space \( \mathcal{M}_X^*(E, \frac{1}{2} \mu_g(K)) \) of \( \frac{1}{2} \mu_g(K) \)-stable pairs of topological type \( E \).
At this point it is natural to study the properties of the moduli spaces $\mathcal{M}_X^g(E, \lambda)$. We do not want to go into details here, and we content ourselves by describing some of the basic steps.

The infinitesimal structure of the moduli space around a point $[(A, \varphi)]$ is given by a deformation complex $(C^*_q, d^*_A, \partial_A, \varphi)$ which is the cone over the evaluation map $ev^*, ev^q(\varphi) : A^0q(End(E)) \rightarrow A^0q(E)$. More precisely $C^*_q = A^0q(End(E)) \oplus A^0q-1(E)$ and the differential $d^*_A, \varphi$ is given by the matrix

$$d^*_A, \varphi = \begin{bmatrix} -\bar{D}_A & 0 \\ ev(\varphi) & \bar{\partial}_A \end{bmatrix},$$

where $\bar{\partial}_A$ and $\bar{D}_A$ are the operators of the Dolbeault complexes $(A^0, \bar{\partial}_A)$ and $(A^0End(E), \bar{D}_A)$ respectively.

Associated to the morphism $ev^*(\varphi)$ is an exact sequence

$$\ldots \rightarrow H^q(End(\mathcal{E}_A)) \xrightarrow{ev^q(\varphi)} H^q(\mathcal{E}_A) \rightarrow H^{q+1}_\partial A, \varphi \rightarrow H^{q+1}_0(End(\mathcal{E}_A)) \rightarrow \ldots$$

with finite dimensional vector spaces

$$H^q_\partial A, \varphi = \ker(ev^q(\varphi)) \oplus \coker(ev^{q-1}(\varphi)).$$

$H^0_\partial A, \varphi$ vanishes for a simple pair $(\bar{\partial}_A, \varphi)$, and $H^1_\partial A, \varphi$ is the Zariski tangent space of $\mathcal{M}_X^g(E, \lambda)$ at $[\bar{\partial}_A, \varphi]$.

A Kuranishi type argument yields local models of the moduli space, which can be locally described as the zero loci of holomorphic map germs

$$K[\bar{\partial}_A, \varphi] : H^1_\partial A, \varphi \rightarrow H^2_\partial A, \varphi$$

at the origin.

One finds that $H^2_\partial A, \varphi = 0$ is a sufficient smoothness criterion in the point $[\bar{\partial}_A, \varphi]$ of the moduli space, and that the expected dimension is $\chi(E) - \chi(End(E))$. The necessary arguments are very similar to the ones in [BD1], [BD2].

The moduli spaces $\mathcal{M}_X^g(E, \lambda)$ will be quasi-projective varieties if the underlying manifold $(X, g)$ is Hodge, i.e. if $X$ admits a projective embedding such that a multiple of the Kähler class is a polarisation [G1].

A GIT construction for projective varieties of any dimension has been given in [HL2]. The spaces $\mathcal{M}_X^g(E, \lambda)$ vary with the parameter $\lambda$, and flip-phenomena occur just like in the case of curves [T].
5 Applications

The equations considered by Seiberg and Witten are associated to a Spin$^c$(4)-structure, and correspond to the case when (in our notations) the unitary bundle $E$ is the trivial line bundle. Alternatively, we can fix a Spin$^c$(4)-structure $s_0$ on $X$, and regard the Seiberg-Witten equations corresponding to the other Spin$^c$(4)-structures as coupled Seiberg-Witten equations associated to $s_0$ and to a unitary line bundle $E$. The Spin$^c$(4)-structure we fix will always be the canonical structure defined by a Kähler metric. In the most interesting case of rank-1 bundles $E$ over Kähler surfaces the central result is:

**Proposition 5.1** Let $(X, g)$ be a Kähler surface with canonical class $K$, and let $L$ be a complex line bundle over $X$ with $L \equiv K \pmod{2}$. Denote by $W^\text{g}_X(L)$ the moduli space of solutions of the Seiberg-Witten equation for all Spin$^c$(4)-structures with determinant $L$. Then

i) If $\mu(L) < 0$, $W^\text{g}_X(L)$ is isomorphic to the space of all linear systems $|D|$, where $D$ is a divisor with $c_1(O_X(2D - K)) = L$.

ii) If $\mu(L) > 0$, $W^\text{g}_X(L)$ is isomorphic to the space of all linear systems $|D|$, where $D$ is a divisor with $c_1(O_X(2D - K)) = -L$.

**Proof:** Use Theorem 4.2 and Bradlow’s description of the moduli spaces of stable pairs in the case of line bundles [B1].

We have already noticed (Remark 2.5) that in the case of a Riemannian 4-manifold with nonnegative scalar curvature $s_g$, the Seiberg-Witten equations have only reducible solutions. In the Kähler case, the same result can be obtained under the weaker assumption $\sigma_g \geq 0$ on the total scalar curvature.

**Corollary 5.2** Let $(X, g)$ be a Kähler surface with nonnegative total scalar curvature $\sigma_g$. Then all solutions of the Seiberg-Witten equations in rank 1 are reducible. If moreover the surface has $K^2 > 0$, then for every almost canonical class $L$, the corresponding Seiberg-Witten equations are incompatible.

**Proof:** The first assertion follows directly from the theorem, since the condition $\sigma_g \geq 0$ is equivalent to $K \cup [\omega_g] \leq 0$. For the second assertion, note that if $L$ is an almost canonical class, then $L^2 = K^2 > 0$, hence (regarded as line bundle) it cannot admit anti-selfdual connections.
Remark 5.3 The Seiberg-Witten invariants associated to almost canonical classes are well-defined for oriented, closed 4-manifolds $X$ satisfying $3\sigma + 2e > 0$.

Proof: Recall that if $L$ is an almost canonical class, then the expected dimension of the moduli space of solutions of the perturbed Seiberg-Witten equations $[W, KM]$ corresponding to a Spin$^c$-structure of determinant $L$ is 0. Seiberg and Witten associate to every such class $L$ the number $n_L$ of points (counted with the correct signs $[W]$) of such a moduli space chosen to be smooth and of the expected dimension. In the case $b_+ \geq 2$, using the same cobordism argument as in Donaldson theory, it follows that these numbers are well-defined, i.e. independent of the metric, provided the moduli space has the expected dimension $[KM]$. The point is that the space of $L$-good metrics $[KM]$ (i.e. metrics with the property that the space of harmonic anti-selfdual forms does not contain the harmonic representative of $c_1^g(L)$) is in this case path-connected. On the other hand, under the assumption $3\sigma + 2e > 0$, it follows that $L^2 > 0$ for any almost canonical class $L$, hence all metrics are $L$-good.

Proposition 5.4 Let $(X, H_0)$ be a polarised surface with $K$ nef and big, and choose a Kähler metric $g$ with Kähler class $[\omega_g] = H_0 + nK =: H$ for some $n \geq KH_0$. Then $W^0_X(L)$ is empty for all almost canonical classes, except for $L = \pm K$, when it consists of a simple point.

Proof: Let $L$ be an almost canonical class with $LH < 0$. Suppose $D$ is an effective divisor with $c_1(O_X(2D - K)) = L$, so that $D(D - K) = 0$. Then $D^2 = DK \geq 0$ since $K$ is nef. If $D^2$ were strictly positive, the Hodge index theorem would give $(D - K)^2 \leq 0$, i.e. $K^2 \leq D^2$. But from $LH < 0$ we get $0 > (2D - K)(H_0 + nK) = (2D - K)H_0 + n(2D^2 - K^2) \geq (2D - K)H_0 + n$, which leads to the contradiction $n < (K - 2D)H_0 \leq KH_0$. Therefore $D^2 = DK = 0$, so that, again by the Hodge index theorem, $D$ must be numerically zero. Since $D$ is effective, it must be empty, and $L = -K$.

Replacing $L$ by $-L$ if $L$ is an almost canonical class with $LH > 0$, we find $L = K$ in this case. The corresponding Seiberg-Witten moduli spaces are simple points in both cases, since $H^3_{\partial, \varphi} H^1(O(D)|_D) = 0$. 

20
Corollary 5.5 There exists no orientation-preserving diffeomorphism between a rational surface and a minimal surface of general type.

Proof: Indeed, any rational surface $X$ admits a Hodge metric with positive total scalar curvature [H2]. If $X$ was orientation-preservingly diffeomorphic to a minimal surface of general type, then $K^2 > 0$, hence the Seiberg-Witten invariants are well defined (Remark 5.3), and vanish by Corollary 5.2. Proposition 5.4 shows, however, that the Seiberg-Witten invariants of a minimal surface of general type are non-trivial for two almost canonical classes.

Witten has already proved [W] that for a minimal surface of general type with $p_g > 0$ ($b_+ \geq 2$), the only almost canonical classes which give non-trivial invariants are $K$ and $-K$. Their proof uses the moduli space of solutions of the perturbation of the Seiberg-Witten equation with a holomorphic form. Proposition 5.4 shows that a stronger result can be obtained with the non-perturbed equations by choosing the Hodge metric $H = H_0 + nK$, $n \gg 0$.

For the proof of Corollary 5.5, we need in fact only the mod. 2 version of the Seiberg-Witten invariants [KM2].
Bibliography

[AHS] Atiyah M., Hitchin N. J., Singer I. M.: *Selfduality in four-dimensional Riemannian geometry*, Proc. R. Lond. A. 362, 425-461 (1978)

[BPV] Barth, W., Peters, C., Van de Ven, A.: *Compact complex surfaces*, Springer Verlag (1984)

[B] Bertram, A.: *Stable pairs and stable parabolic pairs*, J. Alg. Geometry 3, 703-724 (1994)

[B1] Bradlow, S. B.: *Vortices in holomorphic line bundles over closed Kähler manifolds*, Comm. Math. Phys. 135, 1-17 (1990)

[B2] Bradlow, S. B.: *Special metrics and stability for holomorphic bundles with global sections*, J. Diff. Geom. 33, 169-214 (1991)

[BD1] Bradlow, S. B.; Daskalopoulos, G.: *Moduli of stable pairs for holomorphic bundles over Riemann surfaces I*, Intern. J. Math. 2, 477-513 (1991)

[BD2] Bradlow, S. B.; Daskalopoulos, G.: *Moduli of stable pairs for holomorphic bundles over Riemann surfaces II*, Intern. J. Math. 4, 903-925 (1993)

[D] Donaldson, S.: *Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc. 3, 1-26 (1985)

[DK] Donaldson, S.; Kronheimer, P.B.: *The Geometry of four-manifolds*, Oxford Science Publications (1990)

[FM] Friedman, R.; Morgan, J.W.: *Smooth 4-manifolds and Complex Surfaces*, Springer Verlag 3. Folge, Band 27 (1994)

[FQ] Friedman, R.; Qin, Z.: *On complex surfaces diffeomorphic to rational surfaces*, Preprint (1994)

[G1] Garcia-Prada, O.: *Dimensional reduction of stable bundles, vortices and stable pairs*, Int. J. of Math. Vol. 5, No 1, 1-52 (1994)

[G2] Garcia-Prada, O.: *A direct existence proof for the vortex equation over a compact Riemann surface*, Bull. London Math. Soc., 26, 88-96 (1994)

[HH] Hirzebruch, F., Hopf H.: *Felder von Flächenelementen in 4-dimensionalen 4-Mannigfaltigkeiten*, Math. Ann. 136 (1958)

[H1] Hitchin, N.: *Harmonic spinors*, Adv. in Math. 14, 1-55 (1974)

[H2] Hitchin, N.: *On the curvature of rational surfaces*, Proc. of Symp. in Pure Math., Stanford, Vol. 27 (1975)

[HL1] Huybrechts, D.; Lehn, M.: *Stable pairs on curves and surfaces*, J. Alg. Geometry 4, 67-104 (1995)
[HL2] Huybrechts, D.; Lehn, M.: *Framed modules and their moduli*. Int. J. Math. 6, 297-324 (1995)

[JT] Jaffe, A., Taubes, C.: *Vortices and monopoles*, Boston, Birkhäuser (1980)

[K] Kobayashi, S.: *Differential geometry of complex vector bundles*, Princeton University Press (1987)

[KM1] Kronheimer, P.; Mrowka, T.: *Recurrence relations and asymptotics for four-manifold invariants*, Bull. Amer. Math. Soc. 30, 215 (1994)

[KM2] Kronheimer, P.; Mrowka, T.: *The genus of embedded surfaces in the projective plane*, Preprint (1994)

[OSS] Okonek, Ch.; Schneider, M.; Spindler, H: *Vector bundles on complex projective spaces*, Progress in Math. 3, Birkhäuser, Boston (1980)

[Q] Qin, Z.: *Equivalence classes of polarizations and moduli spaces of stable locally free rank-2 sheaves*, J. Diff. Geom. 37, No 2 397-416 (1994)

[S] Simpson, C. T.: *Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization*, J. Amer. Math. Soc. 1 867-918 (1989)

[T] Thaddeus, M.: *Stable pairs, linear systems and the Verlinde formula*, Invent. math. 117, 181-205 (1994)

[UY] Uhlenbeck, K. K.; Yau, S. T.: *On the existence of Hermitian Yang-Mills connections in stable vector bundles*, Comm. Pure App. Math. 3, 257-293 (1986)

[W] Witten, E.: *Monopoles and four-manifolds*, Mathematical Research Letters 1, 769-796 (1994)

Authors addresses:

Mathematisches Institut, Universität Zürich,
Winterthurerstrasse 190, CH-8057 Zürich
e-mail: okonek@math.unizh.ch
teleman@math.unizh.ch