Fractional matchings and component-factors of
(edge-chromatic critical) graphs

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Abstract

The paper studies component-factors of graphs which can be characterized in terms of their fractional matching number. These results are used to prove that every edge-chromatic critical graph has a $[1, 2]$-factor. Furthermore, fractional matchings of edge-chromatic critical graphs are studied and some questions are related to Vizing’s conjectures on the independence number and 2-factors of edge-chromatic critical graphs.

1 Introduction and Motivation

We consider finite simple graphs. For a graph $G$, $V(G)$ and $E(G)$ denote the set of vertices and the set of edges, respectively. For a vertex $v$ of $V(G)$, $E_G(v)$ denotes the set of edges which are incident to $v$. The degree of $v$, denoted by $d_G(v)$, is $|E_G(v)|$. The maximum degree of a vertex of $G$ is denoted by $\Delta(G)$ and the minimum degree of a vertex of $G$ is denoted by $\delta(G)$. If $\Delta(G) = \delta(G) = k$, then $G$ is $k$-regular. If $G$ is a 2-regular graph then it is also called a cycle, and if $G$ is a connected 2-regular graph, then we also call $G$ a circuit. For $v \in V(G)$, the set of neighbors of $v$ is denoted by $N_G(v)$. Clearly, $d_G(v) = |E_G(v)| = |N_G(v)|$, for graphs. For a set $X \subseteq V(G)$, the neighborhood of $X$ is defined as $N_G(X) = \bigcup_{x \in X} N_G(x)$. For $S \subseteq V(G)$, the set of edges with precisely one end in $S$ is denoted by $\partial_G(S)$. For $A, B \subseteq V(G)$, the set of edges with one end in $A$ and the other in $B$ is denoted by $E_G(A, B)$. Hence, $E_G(S, V(G) - S) = \partial_G(S)$. If there is no harm of confusion, then we will omit the indices.

A set $M$ ($M \subseteq E(G)$ or $M \subset V(G)$) is independent, if no two elements of $M$ are adjacent. An independent set of edges is also called a matching of $G$. The maximum cardinality of a matching of $G$ is the matching number of $G$, which is denoted by $\mu(G)$. A matching $M$ with

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\(|M| = \mu(G)\) is a maximum matching of \(G\). The number of vertices which are not incident to an edge of a maximum matching is the matching-deficiency of \(G\), and it is denoted by \(def(G)\). Clearly, \(def(G) = |V(G)| - 2\mu(G)\).

A fractional matching of \(G\) is a function \(f : E(G) \to [0,1]\) such that \(\sum_{e \in E_G(v)} f(e) \leq 1\) for all \(v \in V(G)\). If \(f(e) \in \{0,1\}\) for each edge, then \(f\) is the characteristic function of a matching of \(G\). The fractional matching number \(\mu_f(G)\) is \(\sup\{\sum_{e \in E(G)} f(e) : f\) is a fractional matching of \(G\}\}. Clearly, \(\mu_f(G) \leq \frac{1}{2}|V(G)|\) and if \(\sum_{e \in E(G)} f(e) = \frac{1}{2}|V(G)|\), then \(f\) is a fractional perfect matching. For a fractional matching \(f\) the set \(\{e : e \in E(G)\) and \(f(e) \neq 0\}\} is the support of \(f\) and it is denoted by \(supp(f)\).

**Theorem 1.1** \(\{14\}\) (Theorem 2.1.5)). For any graph \(G\), \(2\mu_f(G)\) is an integer. Moreover, there is a fractional matching \(f\) for which \(\sum_{e \in E(G)} f(e) = \mu_f(G)\) and \(f(e) \in \{0, \frac{1}{2}, 1\}\) for every \(e \in E(G)\).

Let \(G\) be a graph and \(g, f : V(G) \to \mathbb{Z}\) be two functions such that \(0 \leq g(v) \leq f(v)\) for all \(v \in V(G)\). A \((g, f)\)-factor is a spanning subgraph \(F\) of \(G\) that satisfies \(g(v) \leq df(v) \leq f(v)\) for all \(v \in V(G)\). If \(g(v) = a\) and \(f(v) = b\) for all \(v \in V(G)\), then \(F\) is a \([a, b]\)-factor, and if \(a = b = k\), then \(F\) is a \(k\)-factor of \(G\). Clearly, if \(F\) is a 1-factor, then \(E(F)\) is a perfect matching of \(G\). If \(F\) is a factor of a graph \(G\), then a path is \(F\)-alternating, if its edges are in \(F\) and \(E(G) - F\) alternately.

For a set \(S\) of connected graphs, a spanning subgraph \(F\) of \(G\) is called an \(S\)-factor if each component of \(F\) is isomorphic to an element of \(S\). If \(H \in S\), then a component of \(S\) which is isomorphic to \(H\) is called an \(H\)-component of \(F\). The number of components of \(G\) is denoted by \(c(G)\). A component is trivial if it consists of a single vertex and non-trivial otherwise. The set of trivial components of \(G\) is denoted by \(Iso(G)\) and \(iso(G)\) denotes \(|Iso(G)|\).

The complete bipartite graph with bipartition \((A, B)\) and \(|A| = r, |B| = s\) is denoted by \(K_{r,s}\). In case of \(r = 1\), \(K_{1,s}\) is called a star and the vertex of degree \(s\) is its center vertex. For \(K_{1,1}\), either of the two vertices can be regarded as its center vertex. A \(\{K_{1,1}, \ldots, K_{1,t}, C_m : m \geq 3\}\)-factor of \(G\) is called a star-cycle factor.

For a set \(S\) of vertices let \(G[S]\) and \(G - S\) be the subgraph of \(G\) induced by \(S\) and \(V(G) - S\), respectively. The following theorems characterize some component factors of graphs.

**Theorem 1.2** \(\{16\}\). A graph \(G\) has a \(\{K_{1,1}, C_m : m \geq 3\}\)-factor if and only if \(iso(G-S) \leq |S|\) for all \(S \subseteq V(G)\).

In terms of fractional perfect matchings, Theorem 1.2 is equivalent to the following formulation.

**Theorem 1.3** \(\{14\}\). A graph \(G\) has a fractional perfect matching if and only if \(iso(G-S) \leq |S|\) for all \(S \subseteq V(G)\).
The following theorems characterize graphs which satisfy relaxed conditions.

**Theorem 1.4** ([1]). A graph $G$ has a $\{K_{1,1}, K_{1,2}, C_m: m \geq 3\}$-factor if and only if $\text{iso}(G - S) \leq 2|S|$ for all $S \subseteq V(G)$.

**Theorem 1.5** ([2],[10]). Let $n \geq 2$ be an integer. A graph $G$ has a $\{K_{1,1}, \ldots, K_{1,n}\}$-factor if and only if $\text{iso}(G - S) \leq n|S|$ for all $S \subseteq V(G)$.

These results had been generalized by Berge and Las Vergnas [4] to star-cycle factors.

**Theorem 1.6** ([4]). Let $G$ be a graph and $f : V(G) \to \{1, 2, 3, \ldots\}$ be a function, and let $W = \{v : v \in V(G) \text{ and } f(v) = 1\}$. The graph $G$ has a star-cycle factor $F$ such that

(i) $d_F(v) \leq f(v)$ if $v$ is the center vertex of a star component of $F$, and

(ii) $V(C) \subseteq W$ for each circuit component $C$ of $F$ if and only if $\text{iso}(G - S) \leq \sum_{v \in S} f(v)$.

For each finite graph $G$ there is an integer $n$ such that $\text{iso}(G - S) \leq n|S|$ for all $S \subseteq V(G)$. Consequently, the following statement is proved.

**Corollary 1.7.** Every graph has a star-cycle factor.

In section 2 we characterize graphs with specific star-cycle factors in terms of their fractional matching number. In particular, we give an upper bound for the size of a star and for the number of star components which are different from $K_{1,1}$.

In section 3 we study edge-chromatic critical graphs. The edge-chromatic number $\chi'(G)$ of a graph $G$ is the minimum number $k$ of matchings which are needed to cover the edge set of $G$. In 1965, Vizing [18] proved that $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$ for a graph $G$. For $k \geq 2$, a graph $G$ is $k$-critical, if $\Delta(G) = k$, $\chi'(G) = k + 1$ and $\chi'(H) \leq k$ for each proper subgraph $H$ of $G$. We often say that $G$ is a critical graph, if there is a $k$, such that $G$ is a $k$-critical graph. The maximum cardinality of an independent set of vertices is the independence number of $G$ which is denoted by $\alpha(G)$. The following two conjectures are due to Vizing.

**Conjecture 1.8** ([19]). If $G$ is a critical graph, then $G$ has a 2-factor.

**Conjecture 1.9** ([17]). If $G$ is a critical graph, then $\alpha(G) \leq \frac{1}{2}|V(G)|$.

Clearly, if Conjecture 1.8 is true, then Conjecture 1.9 is also true. Conjectures on factors on critical graphs are surveyed in [3] where it was conjectured that every critical graph has a $[1,2]$ factor. We will prove this conjecture in section 3.

The article closes with section 4, where we study fractional matchings on critical graphs.
2 Fractional matching number and star-cycle factors

A graph $G$ is factor-critical if $G - v$ has a perfect matching for each $v \in V(G)$. Analogously, a matching is near perfect if it covers all vertices but one. Let $D(G)$ be the set of vertices of $G$ which are missed by at least one maximum matching of $G$, let $A(G) = N(D(G)) - D(G)$ and $C(G) = V(G) - (D(G) \cup A(G))$. We call the triple $(D(G), A(G), C(G))$ a Gallai-Edmonds decomposition of $G$. If there is no harm of confusion we shortly write $(D, A, C)$ instead of $(D(G), A(G), C(G))$. We will use the fundamental Gallai-Edmonds structure theorem.

**Theorem 2.1** ([7][8]). Let $G$ be a graph. If $(D, A, C)$ is a Gallai-Edmonds decomposition of $G$, then

1. every component of $G[D]$ is factor-critical,
2. $G[C]$ has a perfect matching,
3. every maximum matching consists of a near perfect matching on each component of $G[D]$, a perfect matching on $G[C]$, and a matching which matches every vertex of $A$ to one distinct component of $G[D]$, and
4. $\mu(G) = \frac{1}{2}(|V(G)| - c(G[D]) + |A|)$.

Next we formulate a sharpening of this result in the context of fractional matchings. Let $M$ be a maximum matching of a graph $G$ and $nc(M)$ be the number of non-trivial components of $G[D]$ that are not matched by an edge $e \in M \cap E(D, A)$, and $nc(G) = \max\{nc(M) : M \text{ is a maximum matching of } G\}$.

**Theorem 2.2** ([11]). Let $G$ be a graph and $n \geq 0$ be an integer. If $\mu_f(G) = \frac{1}{2}(|V(G)| - n)$, then $n = def(G) - nc(G)$.

Let $G$ be a graph with $\mu_f(G) = \frac{1}{2}(|V(G)| - n)$. Scheinerman [4] (Theorem 2.2.6) proved that $n = \max\{iso(G - S) - |S| : S \subseteq V(G)\}$. We call a set $S$ with $iso(G - S) = |S| + n$ a witness for $\mu_f(G)$. A crucial point in the proof of Theorem 2.2 is that every non-trivial component of $G[D]$ has a fractional perfect matching. The following theorem shows that they have even more structural properties.

**Theorem 2.3** ([3]). Let $G$ be a factor-critical graph with $|V(G)| > 1$. Then $G$ has a fractional perfect matching $f$ with $f(e) \in \{0, \frac{1}{2}, 1\}$ for every $e \in E(G)$ and the set $\{e : e \in E(G) \text{ and } f(e) = \frac{1}{2}\}$ forms exactly one odd circuit.

Furthermore, every maximum matching of $G$ is contained in the support of a fractional matching with values in $\{0, \frac{1}{2}, 1\}$. Let $M$ be a maximum matching with $nc(M) = nc(G)$. 

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A maximum fractional matching \( f \) with \( M \subseteq supp(f) \) is a canonical maximum fractional matching of \( G \) (with respect to \( M \)).

Theorem 2.2 shows that every graph has a canonical maximum fractional matching. A look into the proof details of Theorem 2.2 yields that it is also shown that \( A(G) \) contains a witness for \( \mu_f(G) \). We will state this fact in a more detailed manner in the following corollary.

**Corollary 2.4.** Let \( G \) be a graph, \( n \geq 0 \) be an integer, and \( \mu_f(G) = \frac{1}{2}(|V(G)| - n) \). If \( f \) is a canonical maximum fractional matching w.r.t. \( M \), then \( Iso(G[D]) \) contains two disjoint subsets \( D^+ \) and \( D^- \) with

1. \( D^- = \{ v : v \text{ is not matched by } M \} \) and \( |D^-| = n \),
2. \( D^+ = \{ w : \text{there is an } M \text{-alternating path from } w \text{ to some vertex of } D^- \} \),
3. \( M \) induces a perfect matching on \( D^+ \cup N(D^+ \cup D^-) \); in particular, \( |N(D^+ \cup D^-)| = |D^+| \), and
4. \( N(D^+ \cup D^-) \) is a witness for \( \mu_f(G) \).

If \( F \) is a star-cycle factor of \( G \), then \( t_i^F \) denotes the number of \( K_{1,i} \)-components of \( F \) and let \( l(G) = \min \{ \sum_{i=1}^{\infty} (i-1)t_i^F : F \text{ is a star-cycle factor of } G \} \). The next theorem gives a detailed insight into the structure of graphs with respect to their fractional matching number.

**Theorem 2.5.** Let \( G \) be a graph, \( n \geq 0 \) be an integer and \( \lambda \) be the minimum integer such that \( iso(G - S) \leq \lambda |S| \) for all \( S \subseteq V(G) \). If \( \mu_f(G) = \frac{1}{2}(|V(G)| - n) \), then \( \lambda \leq \left\lceil \frac{n}{\delta(G)} \right\rceil + 1 \) and \( G \) has a \( \{K_{1,1}, \ldots, K_{1,\lambda}, C_m : m \geq 3\} \)-factor \( F \), such that \( l(G) = \sum_{i=1}^{\lambda} (i-1)t_i^F = n \).

Furthermore, the \( K_{1,j} \)-components are induced subgraphs of \( G \), and for \( j \geq 2 \), their center vertices are in \( N(D^+ \cup D^-)(\subseteq A) \) and their leaves are in \( D^+ \cup D^- \).

**Proof.** Let \( f \) be a canonical maximum fractional matching w.r.t. \( M \). For \( n = 0 \) we have \( D^- = \emptyset \) and for \( n \geq 1 \) let \( D^- = \{d_1, \ldots, d_n\} \). Let \( V_0 = V(G) - D^- \), and for \( i \in \{1, \ldots, n\} \) let \( V_i = V_0 \cup \{d_1, \ldots, d_i\} \) and \( G_i = G[V_i] \). Clearly, \( G_i \) is a subgraph of \( G \) and \( f \) is a canonical maximum fractional matching of \( G_i \) w.r.t. \( M \).

We construct a sequence of subgraphs \( F_0, \ldots, F_n \) of \( G \), where the subgraph \( F_i \) is the desired \( \{K_{1,1}, \ldots, K_{1,t_i}, C_m : m \geq 3\} \)-factor on \( G_i \) \((t_i \leq \lambda)\) and \( G_n = G \).

If \( i = 0 \), then \( G[V_0] \) has a perfect fractional matching, \( iso(G_0 - S) \leq |S| \) for all \( S \subseteq V(G_0) \) and the statement follows with Theorem 2.2 that is, \( t_i^F = 0 \) for each \( i \geq 2 \) and therefore, \( l(G) = 0 \) and \( t_0 = 1 = \lambda \).

Suppose that \( F_k \) has been constructed in \( G_k \) for \( k \), with \( k \leq n - 1 \). We will construct \( F_{k+1} \) in \( G_{k+1} \).
There is a vertex \( a \in N_G(d_{k+1}) \) with \( a \not\in N\{d_1, \ldots, d_k\} \) or \( d_{F_k}(a) < \lambda \). Then \( F_k \cup \{d_{k+1}a\} \) is a \( \{K_1, \ldots, K_{t_{k+1}}, C_m : m \geq 3\} \)-factor of \( G_{k+1} \). The factor \( F_{k+1} \) is obtained from \( F_k \) by extending a \( K_{1,j+1} \)-component to a \( K_{1,j+1} \)-component. Hence, \( t_j^{F_k} = 1 = t_j^{F_{k+1}} \) and \( t_j^{F_{k+1}} + 1 = t_j^{F_{k+1}} \). Furthermore, \( t_{k+1} \leq t_k + 1 \). Therefore, \( \sum_{i=1}^\lambda (i-1) t_i^F = \sum_{i=1}^\lambda (i-1) t_i^{F_{k+1}} - (j-1) + j = k + 1. \)

**Case A:** There is a vertex \( a \in N_G(d_{k+1}) \) with \( a \not\in N\{d_1, \ldots, d_k\} \) or \( d_{F_k}(a) < \lambda \). Then in the above construction \( t \) is a \( \lambda \)-factor of \( G_{k+1} \). By Corollary 2.6, it remains to show that \( F \) is a star-cycle factor with \( t \) \( \lambda \)-component. Hence, \( (D) = 1 \) and \( t_k^{F_{k+1}} = 1 \) and \( t_{k+1} = \lambda \).

**Case B:** For all \( a \in N_G(d_{k+1}) \), \( d_{F_k}(a) = \lambda \). Let \( P \) be the set of all vertices of \( A(G) \) and \( D(G) \) for which there is an \( F_k \)-alternating path with initial vertex \( d_{k+1} \), \( T_D = P \cap D(G) \) and \( T_A = P \cap A(G) \). Note, that \( T_D \subseteq Iso(G[D]) \), since \( f \) is a canonical maximum fractional matching w.r.t. \( M \).

If \( d_{F_k}(a) = \lambda \) for all \( a \in T_A \), then, by the definition of \( T_A \) and \( T_D \), it follows that \( T_D \) is a set of isolated vertices in \( G - T_A \). But \( |T_D| = \lambda |T_A| + 1, \) a contradiction to the choice of \( \lambda \).

Hence, there is a \( a' \in T_A \) with \( d_{F_k}(a') < \lambda \). Let \( p = d_{k+1}, a', d^1, a^1, d^2, a^2 \) be a minimal \( F_k \)-alternating path \( (d^i \in D(G) \) and \( a^i \in A(G) \)) with end vertices \( d_{k+1} \) and \( a' \). Note that \( d_{F_k}(a') = \lambda, d_{F_k}(d') = 1, a'd' \in E(F_k) \) and \( d_{k+1}a^1, d'a^1+1, d'a' \not\in E(F_k) \). Let \( F_{k+1} \) be obtained from \( F_k \) by interchanging the edges of \( F_k \) and \( E(p) - E(F_k) \) in \( p \). Hence, \( F_{k+1} \) is a \( \{K_1, \ldots, K_{1;i}, C_m : m \geq 3\} \)-factor of \( G_{k+1} \). As in Case A it follows that \( \sum_{i=1}^\lambda (i-1) t_i^{F_{k+1}} = k + 1 \) and \( t_{k+1} \leq \lambda \).

Let \( F = F_{k+1} \). Then \( F \) is a \( \{K_1, \ldots, K_{1;i}, C_m : m \geq 3\} \)-factor of \( G \) and \( \sum_{i=1}^\lambda (i-1) t_i^{F} = n. \)

We cannot do better since \( f' : E(G) \to [0, 1] \) with \( f'(e) = \frac{1}{\lambda} \) if \( e \) is an edge of a \( K_{1,i} \) component of \( F, F'(e) = \frac{1}{2} \), if \( e \) is an edge of a circuit of \( F \), and \( f'(e) = 0 \) otherwise, is a fractional matching of \( G \) and \( \sum_{e \in E(G)} f'(e) = \frac{1}{2}(|V(G)| - n). \)

It remains to show that \( \lambda \leq \left\lceil \frac{n}{\sigma(G)} \right\rceil + 1. \) Without loss of generality we may assume that \( d_G(d_1) \leq \cdots \leq d_G(d_h) \). Let \( F \) be the constructed \( \{K_1, \ldots, K_{1;i}, C_m : m \geq 3\} \)-factor. Then in the above construction \( t \leq \lambda \) and \( t \) increases at most by \( 1 \) \( \delta(G) \) steps. Hence, \( t \leq \left\lceil \frac{n}{\sigma(G)} \right\rceil + 1. \) Therefore, \( iso(G - S) \leq \left\lceil \frac{n}{\sigma(G)} \right\rceil + 1 \) for all \( S \subseteq V(G) \). Since \( \lambda \) is minimum, the statement follows.

**Corollary 2.6.** For each graph \( G : l(G) = def(G) - nc(G) = \max\{iso(G - S) - |S| : S \subseteq V(G)\} = |V(G)| - 2\mu_f(G) \) and \( G \) has a \( \{K_1, \ldots, K_{1;i}, C_m : m \geq 3\} \)-factor with \( nc(G) \) circuits.

**Corollary 2.7.** Let \( G \) be a graph that has a \( \{K_1, \ldots, K_{1;i}, C_m : m \geq 3\} \)-factor. Then

\[
\alpha(G) \leq \frac{1}{2} \left( |V(G)| + (l(G) - nc(G)) \right).
\]

**Proof.** By Corollary 2.6, \( G \) has a star-cycle factor with \( nc(G) \) odd cycles and \( l(G) \) vertices extend \( K_{1,1} \)-components to a \( K_{1;i} \)-components, \( i > 1. \) Therefore, \( \alpha(G) \leq \frac{1}{2} \left( |V(G)| - l(G) \right) - \frac{1}{2} nc(G) + l(G). \)
Theorem 2.8. Let $G$ be a graph and $e' \in E(G)$. If there is a maximum fractional matching $f$ of $G$ with $f(e') \neq 0$, then there is a maximum fractional matching $f'$ with $f'(e) \in \{0, \frac{1}{2}, 1\}$ for all $e \in E(G)$ and $f'(e') \neq 0$, and the components of $\text{supp}(f')$ are $K_{1,1}$'s or odd circuits.

Proof. Let $f$ be a maximum fractional matching and $e' \in E(G)$ with $f(e') \neq 0$. By Theorem 1.1 we have that $\sum_{e \in E(G)} f(e) = \mu_f(G) = \frac{1}{2}(|V(G)| - n)$ for an integer $n \geq 0$. Let $f_0$ be a maximum fractional matching with $f_0(e') \neq 0$ and $|\{e : e \in E(G) \text{ and } f_0(e) = 0\}|$ maximal, and let $H = G[\text{supp}(f_0)]$. We will prove the statement by induction on $n$.

$n = 0$: In this case, $f$ and $f_0$ are fractional perfect matchings of $G$, and our proof of the statements closely follows the line of the proof of Theorem 1.1 given in [14].

If $H$ contains an edge $e_0 = vw$ with $d_H(v) = 1$, then $f_0(e_0) = 1$ and $e_0$ is the edge of a $K_{1,1}$-component of $H$. Hence, $f_0(e) = 0$ for all $e \in (E(v) \cup E(w)) \setminus \{e_0\}$. In particular, $e' \notin (E(v) \cup E(w)) \setminus \{e_0\}$.

Claim 1. $H$ does not contain an even circuit.

Suppose to the contrary that it contains an even circuit $C$. Let $E(C) = \{e_1, \ldots, e_{2k}\}$ and if $e' \in E(C)$, then let $e' = e_1$. Let $m = \min\{f_0(e_{2i}) : 1 \leq i \leq k\}$. Define $g : E(G) \to \{-1, 0, 1\}$, with $g(e) = 0$ if $e \in E(G) - E(C)$ and for $i, j \in \{1, \ldots, k\}$ let $g(e_{2i-1}) = 1$ and $g(e_{2j}) = -1$. Then $f_1 = f_0 + mg$ is a maximum fractional matching with $f_1(e') \neq 0$ and which assigns 0 to at least one more edge than $f_0$, a contradiction.

Claim 2. If $H$ contains an odd circuit $C_1$, then $C_1$ is a circuit component of $H$.

Suppose that $C_1$ contains a vertex $v$ with $d_H(v) > 2$. Let $P$ be a path which starts in $v$ with an edge which is not an edge of $C_1$. This path cannot return to $C_1$, since then $H$ would contain an even circuit. It can also not have an end vertex $x$ of degree 1, since then $f_0(e) = 1$ for the edge which is incident to $x$ in $H$. Hence, it ends at a vertex $w$ with $N(w) \subseteq V(P)$. Thus, $H$ contains a graph $B$ which consists of two odd circuits $C_1$ and $C_2$ which are connected by a path (possibly of length 0). Let $g : E(H) \to \{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$ be a function with $g(e) = 0$ if $e \notin E(B)$ and $\pm 1$ alternately on the path which connects the two odd circuits of $B$ and $\pm \frac{1}{2}$ alternately around the circuits such that $\sum_{e \in E(v)} g(e) = 0$ for each $v \in V(B)$. If $e' \in E(B)$, then choose $g$ such that $g(e') > 0$. Let $m$ be the smallest number such that there is an edge $e \in E(B)$ with $f_1(e) = (f_0 + mg)(e) = 0$. Then $f_1$ is fractional perfect matching of $G$ which assigns the value 0 to more edges that $f_0$. Furthermore, the value 0 can only achieved on an edge $e$ with $g(e) < 0$. Hence, $f_1(e') \neq 0$ and we obtain a contradiction to the definition of $f_0$. Thus, the claim is proved.

Hence, the components of $H$ are odd circuits or $K_{1,1}$’s. The function $f' : E(G) \to \{0, \frac{1}{2}, 1\}$ with $f'(e) = \frac{1}{2}$, if $e$ is an edge of a circuit component of $H$, $f'(e) = 1$, if $e$ is an edge of a
Proof. Let $\mu_f(G) = \frac{1}{2}(|V(G)| - n)$ for an integer $n \geq 0$. By Theorem 2.8 there is a fractional maximum matching $f'$ with $f'(e) \in \{0, \frac{1}{2}, 1\}$ for all $e \in E(G)$ and $f'(e') \neq 0$. Hence, $e'$ is an edge of a circuit or a $K_{1,1}$-component of $G[\text{supp}(f')]$. Furthermore, there are precisely $n$ vertices $v_1, \ldots, v_n$ with $\sum_{e \in E(v_i)} f'(e) = 0$. Let $x \in N(v_i)$. Then $\sum_{e \in E(x)} f'(e) = 1$. If $x$ is a vertex of a circuit component $C$ of $G[\text{supp}(f')]$, then, since $C$ is of odd order, we easily deduce a contradiction to the maximality of $f'$. Hence, $x \in N(v_i)$ is a vertex of a $K_{1,1}$-component of $G[\text{supp}(f')]$. Furthermore, at most one endvertex of a $K_{1,1}$-component can be in $\bigcup_{i=1}^n N(v_i)$, since for otherwise we again can deduce a contradiction to the maximality of $f'$. Extending $G[\text{supp}(f')]$ by connecting each $v_i$ to one of its neighbors yields the desired fractional maximum matching of $G$ with $f'(e') \neq 0$.
\{K_{1,1}, \ldots, K_{1,t}, C_m : m \geq 3\}\)-factor of \(G\).

The other direction of the statement is trivial. \(\square\)

If \(iso(G - S) \leq \lambda|S|\), with \(\lambda\) minimal, then the cycle-star factor \(F\) in Corollary 2.9 is not necessarily a \(\{K_{1,1}, \ldots K_{1,t}, C_m : m \geq 3\}\)-factor with \(t \leq \lambda\).

Let \(\min(G, K_{1,2}) = \min\{t_F : F\) is a \(\{K_{1,1}, K_{1,2}, C_m : m \geq 3\}\)-factor of \(G\}\). The following corollary will be used in section III.

**Corollary 2.10.** Let \(G\) be a graph, that has a \(\{K_{1,1}, K_{1,2}, C_m : m \geq 3\}\)-factor and let \(n\) be a natural number. Then, \(\min(G, K_{1,2}) = n\) if and only if \(\mu_f(G) = \frac{1}{2}(|V(G)| - n)\).

**Proof.** The result follows directly from Theorem 2.5 and Corollary 2.6. \(\square\)

Theorem 1.2 is the special case \(m = n\) of the following corollary.

**Corollary 2.11.** Let \(G\) be a graph and let \(n, m\) be integers with \(0 < n \leq m \leq 2n\). If \(iso(G - S) \leq \frac{m-n}{m+n}|S|\) for all subsets \(S \subseteq V(G)\), then

\(i\) \(\min(G, K_{1,2}) \leq \frac{m-n}{m+n}|V(G)|\),

\(ii\) \(\alpha(G) \leq \frac{m-n}{m+n}|V(G)|\).

**Proof.** \(i\) Since \(1 \leq \frac{m-n}{m+n} \leq 2\) it follows with Theorem 1.4 that \(G\) has a \(\{K_{1,1}, K_{1,2}, C_m : m \geq 3\}\)-factor. Furthermore, for all \(S \subseteq V(G)\):

\[
iso(G - S) \leq \frac{m}{m+n}|S| = \frac{2m}{2n}|S|
\]

\(\iff\)

\[
\frac{2n}{m+n} iso(G - S) \leq \frac{2m}{m+n}|S|
\]

\(\iff iso(G - S) = \frac{m-n}{m+n} iso(G - S) \leq |S| + \frac{m-n}{m+n}|S|
\]

\(\iff iso(G - S) \leq |S| + \frac{m-n}{m+n} (iso(G - S) + |S|).
\]

Since \(iso(G - S) + |S| \leq |V(G)|\) for all \(S \subseteq V(G)\) it follows that

\[
iso(G - S) \leq |S| + \frac{m-n}{m+n}|V(G)|.
\]

Now, the result follows with Corollaries 2.6 and 2.10.

\(ii\) By \(i\), \(G\) has as a \(\{K_{1,1}, K_{1,2}, C_m : m \geq 3\}\)-factor \(F\) with \(\min(G, K_{1,2}) \leq \frac{m-n}{m+n}|V(G)|\).

Then, for all \(S \subseteq V(G)\) we have

\[
iso(G - S) \leq iso(F - S) \leq \frac{m-n}{m+n}|V(G)| + \frac{1}{2} \left(|V(G)| - 3 \frac{m-n}{m+n}|V(G)|\right)
\]

\[
\leq \frac{m-n}{2(m+n)}|V(G)| + \frac{1}{2}|V(G)| = \frac{m}{m+n}|V(G)|.
\]

\(\square\)
In the following we will apply Lovász’ \((g, f)\)-factor Theorem. This is the only theorem, where multigraphs are allowed. Here a multigraph is a graph that may have loops and multiple edges.

**Theorem 2.12** ([12]). Let \( G \) be a multigraph and let \( g, f : V(G) \to \mathbb{Z} \) be functions such that \( g(v) \leq f(v) \) for all \( v \in V(G) \). Then \( G \) has a \((g, f)\)-factor if and only if for all disjoint subsets \( S \) and \( T \) of \( V(G) \),

\[
\gamma(S, T) = \sum_{v \in S} f(v) + \sum_{v \in T} (d_G(v) - g(v)) - E_G(S, T) - q^*(S, T)
\]

where \( q^*(S, T) \) denotes the number of components \( C \) of \( G - (S \cup T) \) such that \( g(v) = f(v) \) for all \( v \in V(C) \) and

\[
\sum_{v \in V(C)} f(v) + e_G(C, T) \equiv 1 \mod 2.
\]

Notice that \( q^*(S, T) = 0 \) for all disjoint subsets \( S \) and \( T \) of \( V(G) \), if \( g(v) < f(v) \) for all \( v \in V(G) \).

The following theorem extends a result of Berge and Las Vergnas (Theorem 7 in [1]) from \([1, 2]\)-factors to \( \{K_{1, 1}, K_{1, 2}, C_m : m \geq 3\}\)-factors of a graph.

**Theorem 2.13.** Let \( G \) be a graph that has a \( \{K_{1, 1}, K_{1, 2}, C_m : m \geq 3\}\)-factor. For \( e \in E(G) \), say \( e = uv \), there exists no \( \{K_{1, 1}, K_{1, 2}, C_m : m \geq 3\}\)-factor which contains \( e \) if and only if there exists a subset \( S \) of \( V(G) \) that satisfies

(i) \( u, v \in S \)

(ii) \( 2|S| - 2 \leq iso(G - S) \leq 2|S| \).

Furthermore, the inequalities of (ii) are tight.

**Proof.** The condition \( iso(G - S) \leq 2|S| \) in (ii) is satisfied, since \( G \) has a \( \{K_{1, 1}, K_{1, 2}, C_m : m \geq 3\}\)-factor. Therefore, it remains to prove that \( 2|S| - 2 \leq iso(G - S) \). We first consider the graph \( G' \) which is obtained from \( G \) by contracting \( e \), that is \( V(G') = (V(G) \setminus \{u, v\}) \cup \{w\} \) and \( E(G') \) is obtained from \( E(G[V(G) \setminus \{u, v\}]) \cup \{xw : xu \in E(G) \text{ or } xv \in E(G)\} \). Notice, that \( G' \) is not necessarily a simple graph. Let \( S \) be a subset of \( V(G) \) and \( S' \) a subset of \( V(G') \). Then we call the sets \( S \) and \( S' \) corresponding sets, if \( u, v \in S \) if and only if \( w \in S' \).

**Claim 1.** \( G \) has a \( \{K_{1, 1}, K_{1, 2}, C_m : m \geq 3\}\)-factor \( F \) with \( e \in F \) if and only if \( G' \) has a \((g', f')\)-factor with \( g'(x) = 1 \), \( f'(x) = 2 \) for all \( x \in V(G') \setminus \{w\} \), \( g'(w) = 0 \) and \( f'(w) = 1 \).
If $G$ has a $\{K_{1,1}, K_{1,2}, C_m : m \geq 3\}$-factor $F$ with $e \in F$ and $e$ is contained in a $C_m$-component, then decompose this component into $K_{1,1}$ and $K_{1,2}$-components. So $e$ is either contained in a $K_{1,1}$-component or in a $K_{1,2}$-component. Contract $e$, and the remaining edges of $F$ in $G'$ obviously form a $(g', f')$-factor of $G'$.

If $G'$ has a $(g', f')$-factor $F'$, then $g'(w) \in \{0, 1\}$. The set $F$, with $F = F' \cup \{u, v\}$, is a $[1, 2]$-factor of $G$ and in any case, $e$ is an end edge of a path. If we decompose all paths of length at least three into paths of length one or two, then we get a $\{K_{1,1}, K_{1,2}, C_m : m \geq 3\}$-factor $F$ of $G$ with $e \in F$, and the claim is proved.

$\implies$: Let $S$ be a set of $V(G)$ with $u, v \in S$ and $2|S| - 2 \leq \text{iso}(G - S)$. Let $S'$ be the corresponding set of $S$. Since $u, v \in S$, we have $w \in S'$. Further $|S| = |S'| + 1$, $\text{iso}(G - S) = \text{iso}(G' - S')$ and $2|S'| \leq \text{iso}(G' - S')$.

Let $T' := \text{iso}(G' - S')$. Then it follows

$$\sum_{x \in S'} f'(x) + \sum_{x \in T'} (d_{G' - S'}(x) - g(x)) = 2|S'| - 1 - |T'| \leq -1.$$ 

By Theorem 2.12 $G'$ has no $(g', f')$-factor and by Claim 1 $G$ has no $\{K_{1,1}, K_{1,2}, C_m : m \geq 3\}$-factor that contains $e$.

$\implies$: Let $e$ be an edge of $E(G)$, say $e = uv$, that is not contained in all $\{K_{1,1}, K_{1,2}, C_m : m \geq 3\}$-factors of $G$.

Since $G$ has a $\{K_{1,1}, K_{1,2}, C_m : m \geq 3\}$-factor, $G$ also has a $(g, f)$-factor with $g(x) = 1$ and $f(x) = 2$ for all $x \in V(G)$ and by Theorem 2.12 for all disjoint subsets $X$ and $Y$ of $V(G)$ we have

$$\gamma(X, Y) = \sum_{x \in X} f(x) + \sum_{y \in Y} (d_{G - X}(y) - g(y)) \geq 0. \quad (1)$$

Since $e$ is not contained in all $\{K_{1,1}, K_{1,2}, C_m : m \geq 3\}$-factors of $G$, by Claim 1 and Theorem 2.12 there exits two disjoint subsets $X'$ and $Y'$ of $V(G')$ with $\gamma(X', Y') < 0$ (with respect to $g'$ and $f'$). Let $S'$ and $T'$ be two subsets of $V(G')$ satisfying $\gamma(S', T') < 0$.

Case 1: $w \notin S' \cup T'$. We have

$$\gamma(S', T') = \sum_{x \in S'} f'(x) + \sum_{x \in T'} (d_{G' - S'}(x) - g'(x)) = \sum_{x \in S} f(x) + \sum_{x \in T} (d_{G - S}(x) - g(x)) = \gamma(S, T).$$

This is a contradiction, since by inequality (1) it follows that $\gamma(S', T') \geq 0$. 

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Case 2: $w \in T'$. We have
\[
\gamma(S', T') = \sum_{x \in S'} f'(x) + \sum_{x \in T'} (d_{G' - S'}(x) - g'(x))
\]
\[
= \sum_{x \in S'} f'(x) + \sum_{x \in T' \setminus w} (d_{G' - S'}(x) - g'(x)) + d_{G' - S'}(w) - g'(w)
\]
\[
= \sum_{x \in S} f(x) + \sum_{x \in T \setminus \{u, v\}} (d_{G - S}(x) - g(x)) + d_{G' - S'}(w) - 0 \geq 0
\]

again a contradiction.

Case 3: $w \in S'$. We have
\[
\gamma(S', T') = \sum_{x \in S'} f'(x) + \sum_{x \in T'} (d_{G' - S'}(x) - g'(x))
\]
\[
= 2|S'| - 1 - |T'| + \sum_{x \in T'} d_{G' - S'}(x) < 0
\]

and, since $\gamma(S', T')$ is a natural number, it follows, that
\[
\sum_{x \in T'} d_{G' - S'}(x) \leq |T'| - 2|S'|.
\] (2)

Since $\sum_{x \in T'} d_{G' - S'}(x) \geq 0$, we have $|T'| \geq 2|S'|$.

Suppose $iso(G' - S') < 2|S'|$. It follows, that $\sum_{x \in T'} d_{G' - S'}(x) \geq |T'| - 2|S'| + 1$, a contradiction by the right side of inequality (2). Therefore, $iso(G' - S') \geq 2|S'|$.

We have $|S| = |S'| + 1$ and $iso(G - S) = iso(G' - S')$. Therefore, there exits a subset $S$ of $V(G)$ with $u, v \in S$ and $2|S| - 2 \leq iso(G - S)$, if there exits no $\{K_{1,1}, K_{1,2}, C_m: m \geq 3\}$-factor that contains $e$.

We give some examples to show that the inequalities of (ii) are tight.

- For the given graph there exits no $\{K_{1,1}, K_{1,2}, C_m: m \geq 3\}$-factor that contains the edge $e = uv$ and for $S = \{u, v\}$ we have $iso(G - S) = 2|S|

- For the given graph there exits no $\{K_{1,1}, K_{1,2}, C_m: m \geq 3\}$-factor that contains the edge $e = uv$ and for $S = \{u, v\}$ we have $iso(G - S) = 2|S| - 1

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For the given graph there exits no \( \{K_{1,1}, K_{1,2}, C_m: m \geq 3\} \)-factor that contains the edge \( e = uv \) and for \( S = \{u, v, v_1, v_2\} \) we have \(|S| = 4\), \( iso(G - S) = 6 \). Thus, \( iso(G - S) = 2|S| - 2 \)

\[
\begin{array}{c}
\bullet \text{Corollary 2.14. Let } G \text{ be a graph that has a } \{K_{1,1}, K_{1,2}, C_m: m \geq 3\} \text{-factor and } e \in E(G). \text{ If } e \text{ is not contained in any } \{K_{1,1}, K_{1,2}, C_m: m \geq 3\} \text{-factor, then } f(e) = 0 \text{ for every maximum fractional } f \text{ matching of } G. \\
\end{array}
\]

\[\square\]

**3 Component factors of edge-chromatic critical graphs**

Woodall [21] proved that \( \alpha(G) \leq \frac{2}{5}|V(G)| \) for a critical graph \( G \). Using his proof approach we generalize some of his results to deduce that every critical graph has a \([1,2]\)-factor. Clearly, every \([1,2]\)-factor can be decomposed into a \( \{K_{1,1}, K_{1,2}, C_m: m \geq 3\} \)-factor. We will use this fact to prove an upper bound for \( \min(G, K_{1,2}) \) for critical graphs.

**Lemma 3.1 (Vizing’s Adjacency Lemma [18]).** Let \( G \) be a critical graph. If \( e = xy \in E(G) \), then at least \( \Delta(G) - d_G(y) + 1 \) vertices in \( N(x) \setminus \{y\} \) have degree \( \Delta(G) \).

Let \( G \) be a critical graph. We denote by \( \sigma(v,w) \) the number of vertices in \( N(w) \setminus \{v\} \) that have degree at least \( 2\Delta(G) - d_G(v) - d_G(w) + 2 \), for an edge \( vw \) of \( G \). We have
\( \sigma(v, w) \leq \Delta(G) \), since in a critical graph \( G \), \( d_G(v) + d_G(w) \geq \Delta(G) + 2 \). Further, we have
\[
\sigma(v, w) \geq \Delta(G) - d_G(v) + 1,
\]
(3)
since by Lemma 3.1 \( w \) has at least \( \Delta(G) - d_G(v) + 1 \) neighbors different from \( v \) with degree \( \Delta(G) \).

**Lemma 3.2** ( [20] ). Let \( G \) be a critical graph and \( v \in V(G) \) and let
\[
p_{\min} := \min_{w \in N(v)} \sigma(v, w) - \Delta(G) + d_G(v) - 1 \quad \text{and} \quad p := \min \left\{ p_{\min}, \left\lfloor \frac{1}{2} d_G(v) \right\rfloor - 1 \right\}.
\]
(4)
Then \( v \) has at least \( d_G(v) - p - 1 \) neighbors \( w \) for which \( \sigma(v, w) \geq \Delta(G) - p - 1 \).

**Theorem 3.3.** Let \( G \) be a critical graph and let \( S \) be an arbitrary subset of \( V(G) \). Then
\[
iso(G - S) < \left( \frac{3}{2} - \frac{1}{\Delta(G)} \right)|S|.
\]
Proof. Let \( G \) be a critical graph, \( S \) be an arbitrary subset of \( V(G) \) and \( T = Iso(G - S) \). Further let \( T^- = \{ t \in T : 2 \leq d_G(t) < \frac{1}{2} \Delta(G) \} \), \( T^+ = \{ t \in T : \frac{1}{2} \Delta(G) \leq d_G(t) < \Delta(G) \} \), and \( T^{++} = \{ t \in T : d_G(t) = \Delta(G) \} \). In a critical graph there are no vertices of degree less than 2, so \( T = T^- \cup T^+ \cup T^{++} \).

We define two functions \( f_i : T \to \mathbb{R} \) with \( f_i(t) = g_i(d_G(t)) \) for all vertices \( t \in T \) and \( i \in \{1, 2\} \), where \( g_i : \mathbb{N} \to \mathbb{R} \) and
\[
g_1(k) := \frac{2(\Delta(G) - k)}{k} \quad \text{and} \quad g_2(k) := \frac{\Delta(G) - 2}{k - 1}.
\]
The functions \( g_1 \) and \( g_2 \) are both decreasing functions of \( k \).

**Claim 1.** For all \( t \in T^+ \), \( f_1(t) \leq f_2(t) \).

Let \( t \) be a vertex of \( T^+ \) and \( k := d_G(t) \). Then
\[
f_2(t) - f_1(t) = g_2(k) - g_1(k) = \frac{((\Delta(G) - 2)k - 2(\Delta(G) - k)(k - 1))}{k(k - 1)} = \frac{(2k - \Delta(G))(k - 2)}{k(k - 1)}
\]
If \( (2k - \Delta(G))(k - 2) = 2k^2 + (-4 - \Delta(G))k + 2\Delta(G) \geq 0 \) and therefore, if \( k \geq \frac{1}{2} \Delta(G) \), then the fraction is nonnegative. Thus the claim is proved.

We now define three charge functions \( M_i, i \in \{1, 2, 3\} \) on \( V(G) \) as follows: \( M_i : V(G) \to \mathbb{N} \) with
\[
M_0(t) = 0, \quad M_1(t) = 2d_G(t), \quad M_2(t) = 2\Delta(G) \quad \text{if} \ t \in T,
\]
\[
M_0(s) = 3\Delta(G) - 2, \quad M_1(s) = \Delta(G) - 2, \quad M_2(s) = 0 \quad \text{if} \ s \in S,
\]
\[
M_0(v) = 0, \quad M_1(v) = 0, \quad M_2(v) = 0 \quad \text{if} \ v \in V(G) - (S \cup T).
\]

We prove that the functions \( M_1 \) and \( M_2 \) satisfy
This implies and therefore,

\[ (ii) \sum M_1(v) < (3\Delta(G) - 2)|S|, \]

\[ (ii) \sum v \in V(G) M_2(v) \leq \sum v \in V(G) M_1(v). \]

This implies

\[ 2\Delta(G)|T| = \sum v \in V(G) M_2(v) \leq \sum v \in V(G) M_1(v) < (3\Delta(G) - 2)|S| \]

and therefore,

\[ iso(G - S) = |T| < \left(\frac{3}{2} - \frac{1}{\Delta(G)}\right)|S|. \]

(i) Starting with the distribution \(M_0\), let each vertex in \(T\) receive charge 2 from each of its neighbors in \(S\). Let the resulting charge distribution be called \(M_0^*\). We have \(M_0^*(t) = 2d_G(t)\) for all \(t \in T\) and for all \(s \in S\), \(M_0^*(s) = 3\Delta(G) - 2 - 2|N(s) \cap T| \geq \Delta(G) - 2\). So \(M_0^*(v) \geq M_1(v)\) for all \(v \in V(G)\), with strict inequality if \(s\) is a vertex of \(S\) with fewer than \(\Delta(G)\) neighbors in \(T\). There exists such a vertex \(s\), since either \(s\) has a neighbor in \(V(G) \setminus (S \cup T)\) or \(S \cup T = V(G)\) and \(S\) is not an independent set. Thus,

\[ \sum v \in V(G) M_1(v) < \sum v \in V(G) M_0^*(v) = \sum v \in V(G) M_0(v) = (3\Delta(G) - 2)|S|. \]

(ii) Starting with the distribution \(M_1\), we will redistribute charge according to the following discharging rule:

- Step 1: Each vertex \(s \in S\) gives charge \(f_1(t)\) to each vertex \(t \in N(s) \cap T^+\).
- Step 2: Each vertex \(s \in S\) distributes its remaining charge equally among all vertices (if any) in \(N(s) \cap T^-\).

The resulting charge distribution we denote by \(M_1^*\).

First we show that \(M_1^*(s) \geq 0 = M_2(s)\) for all \(s \in S\). We compare the above discharging rule, the actual discharging rule, with the equitable discharging rule in which each vertex \(s \in S\) distributes its charge of \(M_1(s) = \Delta(G) - 2\) equally among all its neighbors (if any) in \(T^- \cup T^+\). We show, that, for a vertex \(s \in S\), every vertex of \(N(s) \cap T^+\) receives no more charge from \(s\) in Step 1 of the actual discharging rule than it would receive under the equitable discharging rule, so that the remaining charge referred in Step 2 is nonnegative.

Let \(s \in S\) and let \(\delta\) be the minimum degree of a neighbor of \(s\). By Lemma 3.1 the vertex \(s\) has at least \(\Delta(G) - \delta + 1\) neighbors of degree \(\Delta(G)\), and hence, at most \(\delta - 1\) neighbors in \(T^- \cup T^+\). Thus, under the equitable discharging rule, each vertex \(t \in N(s) \cap (T^- \cup T^+)\) receives from \(s\) at least

\[ \frac{\Delta(G) - 2}{\delta - 1} \geq \frac{\Delta(G) - 2}{d_G(t) - 1} = f_2(t). \]
By Claim 1, every vertex of $N(s) \cap T^+$ receives no more charge from $s$ in Step 1 of the actual discharging rule than it would receive under the equitable discharging rule. Thus, $M_1^*(s) \geq 0 = M_2(s)$ for all $s \in S$.

It remains to show that $M_1^*(t) \geq 2\Delta(G) = M_2(t)$ for all $t \in T$. For all $t \in T^+$, $M_1(t) = 2\Delta(G) = M_2(t)$. Further for all $t \in T^+$, $M_1^*(t) = 2d_G(t) = d_G(t)\cdot f_1(t) = 2\Delta(G) = M_2(t)$. It remains to consider vertices in $T^-$.

We fix a vertex $t \in T^-$ and denote by $k$ the degree of $t$, so $k = d_G(t)$. Further we define a function $h$ with $h : \mathbb{N} \times \mathbb{N}_0 \rightarrow \mathbb{R}$ and

$$h(k, l) = \frac{1}{k - l - 1} (\Delta(G) - 2 - l g_1(\Delta(G) - k + 2))$$

$$= \frac{1}{k - l - 1} (\Delta(G) - 2 - l \frac{2(k - 2)}{\Delta(G) - k + 2}) .$$

**Claim 2.** If $l$ is a nonnegative integer and a vertex $s \in S$ is a neighbor of $t$ such that the number of vertices in $N(s) \setminus \{t\}$ with degree at least $2\Delta(G) - k - d_G(s) + 2$ is at least $\Delta(G) - k + l + 1$, so $\sigma(s, t) \geq \Delta(G) - k + l + 1$, then $s$ gives $t$ at least charge $h(k, l)$ in Step 2.

By definition of $\sigma(s, t)$, vertex $s$ has $\sigma(s, t)$ neighbors with degree at least $2\Delta(G) - k - d_G(s) + 2$. Since $d_G(s) \leq \Delta(G)$ and $t \in T^-$ and therefore, $k < \frac{1}{2}\Delta(G)$,

$$2\Delta(G) - k - d_G(s) + 2 \geq \Delta(G) - k + 2 \geq \frac{1}{2}\Delta(G).$$

By Lemma 3.1 vertex $s$ has at least $\Delta(G) - k + 1$ neighbors with degree $\Delta(G)$. Let $L^{++}$ be a set of $(\Delta(G) - k + 1)$ neighbors of $s$ with degree $\Delta(G)$, and let $L^+$ be a set, disjoint from $L^{++}$, of $l$ neighbors of $s$ with degree at least $\Delta(G) - k + 2$, which exists since $\sigma(s, t) \geq \Delta(G) - k + l + 1$ by hypothesis. So $L^{++} \subseteq T^{++} \cup S \cup (V(G) \setminus (S \cup T))$ and $L^+ \subseteq T^{++} \cup T^+ \cup S \cup (V(G) \setminus (S \cup T))$.

Applying the actual discharging rule, vertex $s$ gives nothing to any vertex in $L^{++}$ and in Step 1 $s$ gives each vertex in $L^+$ at most charge $g_1(\Delta(G) - k + 2)$, since $g_1$ is a decreasing function and the degree of any vertex in $L^+$ is at least $\Delta(G) - k + 2$. So the remaining charge of $s$ is at least $\Delta(G) - 2 - lg_1(\Delta(G) - k + 2)$ and there are $d_G(s) - (\Delta(G) - k + l + 1) \leq k - l - 1$ remaining neighbors of $s$. Therefore, any vertex in $T^-$ gets as least as much of it as any other neighbor of $s$ and therefore, at least $h(k, l)$. Thus, the claim is proved.

We prove, that vertex $t$ gets at least $2(\Delta(G) - k)$ charge in Step 2. This implies that $M_1^*(t) \geq M_1(t) + 2(\Delta(G) - k) = 2\Delta(G) = M_2(t)$.

We define $p$ as in (3) of Lemma 3.2. It follows, that $t$ has at least $k - p - 1$ neighbors $s \in S$ with $\sigma(s, t) \geq \Delta(G) - p - 1$. Let $N^+(t)$ be a set of such $k - (p + 1)$ neighbors.
and let $N^{-}(t) = N(t) \setminus N^{+}(t)$. The set $N^{-}(t)$ contains $p + 1$ neighbors $s$ of $t$, for each with $\sigma(s, t) \geq \Delta(G) - k + p + 1$, by the definition of $p$. Applying Claim 2 to the vertices $N^{-}(t)$ with $l = p$ for the vertices in $N^{-}(t)$ and $l = k - p - 2$ for the vertices in $N^{+}(t)$, we see that $t$ receives charge of at least $M^{+}(k, p)$ in Step 2, where

$$M^{+}(k, p) := (p + 1)h(k, p) + (k - (p + 1))h(k, k - p - 2).$$

It remains to show that the minimal value of $M^{+}$ is at least $2(\Delta(G) - k)$. Let $r = p + 1$, so that $1 \leq r \leq \frac{1}{2}k$, since $0 \leq p \leq \frac{1}{2}k - 1$ by (3) and (4). Setting

$$b := \frac{2(k - 2)}{\Delta(G) - k + 2} \quad \text{and} \quad a := \Delta(G) - 2 + b$$

we can write

$$M^{+}(k, p) = \frac{r(a - br)}{k - r} + \frac{(k - r)(a - b(k - r))}{r}.$$

The derivative of this with respect to $r$ is

$$\frac{ak - bk^2 + b(k - r)^2}{(k - r)^2} - \frac{ak - bk^2 + br^2}{r^2} = \frac{ak - bk^2}{(k - r)^2} - \frac{ak - bk^2}{r^2}.$$

This is zero if and only if $r = \frac{1}{2}k$ (unless $ak - bk^2 = 0$, if $M^{+}(k, p)$ is independent of $p$); thus, $M^{+}(k, p)$, regarded as a function of $p$, has only one stationary point (for positive $p$), when $p + 1 = \frac{1}{2}k$. Substituting this value of $p$ gives

$$M^{+}\left(k, \frac{1}{2}k - 1\right) = 2\left(\Delta(G) - 2 - \frac{(k - 2)^2}{\Delta(G) - k + 2}\right) \geq 2(\Delta(G) - k),$$

where the inequality holds, because $k < \frac{1}{2}\Delta(G)$ and so

$$\frac{(k - 2)^2}{\Delta(G) - k + 2} \leq k - 2.$$

To complete the proof, we must consider also the other extreme value of $p$, $p = 0$, and show that $M^{+}(k, 0) \geq 2(\Delta(G) - k)$, so we have to show that

$$\frac{\Delta(G) - 2}{k - 1} + (k - 1)\left(\Delta(G) - 2 - \frac{2(k - 2)^2}{\Delta(G) - k + 2}\right) \geq 2(\Delta(G) - k). \quad (5)$$

This evidently holds with equality if $k = 2$; so we may assume that $k \geq 3$. Since $k < \frac{1}{2}\Delta(G)$, we can write $\Delta(G) = 2k + q$, where $q \geq 1$. Ignoring the first term of (5), and dividing through by $k - 1$ and rearranging, it suffices to show that

$$2k + q - 2 - \frac{2(k + q)}{k - 1} - \frac{2(k - 2)^2}{k + q + 2} \geq 0 \quad (6)$$
Since the left side of (11) is clearly an increasing function of \( q \), it suffices to verify inequality (11) for \( s = 1 \), when the left side becomes

\[
2k - 1 - \frac{2(k+1)}{k-1} - \frac{2k^2 - 8k + 8}{k+3} = 2k - 1 - 2 - \frac{4}{k-1} - 2k + 14 - \frac{50}{k+3} \\
= 11 - \frac{4}{k-1} - \frac{50}{k+3},
\]

which is positive since \( k \geq 3 \).

Therefore, \( \sum_{v \in V(G)} M_2(v) \leq \sum_{v \in V(G)} M_1^*(v) = \sum_{v \in V(G)} M_1(v) \) and the proof is complete.

\[\square\]

**Theorem 3.4.** Let \( G \) be a critical graph. Then \( G \) has a \( \{K_{1,1}, K_{1,2}, C_m : m \geq 3\} \)-factor with \( \min(G, K_{1,2}) \leq \frac{1}{3}|V(G)| \) and \( \alpha(G) \leq \frac{3}{5}|V(G)| \) for all \( \Delta(G) \geq 2 \).

*Proof.* Let \( G \) be a critical graph and let \( S \) be an arbitrary subset of \( V(G) \). By Theorem 3.3, \( iso(G - S) < \left( \frac{3}{2} - \frac{1}{\Delta(G)} \right)|S| < \frac{3}{2}|S| \), and the statement follows with Corollary 2.11. \(\square\)

For a proof of the existence of a \([1, 2]\)-factor in a critical graph \( G \), it is, by Theorem 1.4, sufficient to prove that \( iso(G - S) \leq 2|S| \) for all \( S \subseteq V(G) \). This can also be achieved by modifying the proof of Theorem 2.2 in [5], where it is proved that \( \alpha(G) \leq \frac{3}{5}|V(G)| \).

**Corollary 3.5.** Every critical graph has a \([1, 2]\)-factor.

**Theorem 3.6.** Let \( G \) be a critical graph. For every edge \( e \) there is a \( \{K_{1,1}, K_{1,2}, C_m : m \geq 3\} \)-factor \( F \) with \( e \in E(F) \).

*Proof.* Let \( G \) be a critical graph and let \( e = uv \). Suppose to the contrary that there is no \( \{K_{1,1}, K_{1,2}, C_m : m \geq 3\} \)-factor that contains \( e \). By Theorems 3.3 and 2.13 there exists a subset \( S \) of \( V(G) \) with \( u, v \in S \) and \( 2|S| - 2 \leq iso(G - S) < \left( \frac{3}{2} - \frac{1}{\Delta(G)} \right)|S| < \frac{3}{2}|S| \). Since \( u, v \in S \), \( |S| \geq 2 \).

If \( \Delta(G) = 3 \), then \( 2|S| - 2 \leq \left( \frac{3}{2} - \frac{1}{3} \right)|S| = \frac{7}{6}|S| \Leftrightarrow \frac{7}{6}|S| \leq 2 \Leftrightarrow |S| \leq \frac{12}{7} \).

Since \( |S| \) and \( iso(G - S) \) are integers, \( |S| = 2 \) and \( iso(G - S) = 2 \). Let \( v_1, v_2 \) be the isolated vertices of \( G - S \). Since \( G \) is critical and \( |S| = 2 \), \( d(v_i) = 2 \) and \( N_G(v_i) = S, i \in \{1, 2\} \). This is a contradiction, since in a critical graph vertices of degree two have no common neighbor.

If \( \Delta(G) \geq 4 \), then \( 2|S| - 2 < \frac{3}{2}|S| \Leftrightarrow \frac{1}{2}|S| < 2 \Leftrightarrow |S| < 4 \).

Since \( |S| \) and \( iso(G - S) \) are integers, there are the following two possibilities. If \( |S| = 2 \), then \( iso(G - S) = 2 \). Again a contradiction. If \( |S| = 3 \), then \( iso(G - S) = 4 \). Since in a critical graph, there are no vertices of degree less than 2, the number of edges in \( E_G(S, Iso(G - S)) \geq 8 \). Since the degree of a vertex in \( iso(G - S) \) is at most 3, with Lemma 3.1 a vertex of \( S \) has a least \( \Delta(G) - 2 \) vertices of degree \( \Delta(G) (\Delta(G) \geq 4) \). Therefore, \( E_G(S, Iso(G - S)) \leq 6 \). A contradiction. \(\square\)
4 Fractional matchings on edge-chromatic critical graphs

The study of fractional matchings of critical graphs gives insight into the structure of critical graphs. Our studies of component factors of critical graphs uses the concept of fractional matchings. We propose the following conjecture.

Conjecture 4.1. If \( G \) is a critical graph, then \( G \) has a fractional perfect matching.

Conjecture 4.1 is in between Conjectures 1.8 and 1.9. We have: Conjecture 1.8 implies Conjecture 4.1 which implies Conjecture 1.9. Clearly, Conjecture 4.1 is true for 2-critical graphs.

Let \( G \) be a graph with \( \Delta(G) = k \). The \( k \)-deficiency \( s(G) \) of \( G \) is \( k|V(G)| - 2|E(G)| \). The function \( f(e) = \frac{1}{k} \) for each \( e \in E(G) \) is a fractional matching on \( G \). Hence, with Corollaries 2.10 and 3.5 we obtain the following corollary.

Corollary 4.2. If \( G \) is a critical graph, then \( \mu_f(G) \geq \frac{1}{k}|V(G)| - \lfloor \frac{s(G)}{k} \rfloor \), and therefore, \( \min(G, K_{1,2}) \leq \lfloor \frac{s(G)}{k} \rfloor \), and \( \alpha(G) \leq \frac{1}{k}|V(G)| + \lfloor \frac{s(G)}{k} \rfloor \).

Let \( k \geq 2 \) be an integer and \( G \) be a graph with \( \Delta(G) = k \). Let \( v \in V(G) \) with \( d_G(v) = d \) and let \( N_G(v) = \{v_1, v_2, \ldots, v_d\} \). Let \( u_1, \ldots, u_k \) be vertices of degree \( k - 1 \) in a complete bipartite graph \( K_{k,k-1} \). Graph \( G' \) is a Meredith extension [13] of \( G \) (applied on \( v \)), if it is obtained from \( G - v \) and \( K_{k,k-1} \) by adding edges \( v_i u_i \) for each \( i \in \{1, \ldots, d\} \). In [9] it is proved, that \( G \) is critical if and only if \( G' \) is critical. Similar to the proofs of the corresponding statements for Conjectures 1.8 and 1.9 [3, 15] we can apply Meredith extension to prove the following statement.

Theorem 4.3. The following two statements are equivalent for each \( k \geq 3 \):

1. Every \( k \)-critical graph \( G \) has a fractional perfect matching.

2. Every \( k \)-critical graph \( G \) with \( \delta(G) = k - 1 \) has a fractional perfect matching.

Proof. Let \( G \) be a \( k \)-critical graph. Apply Meredith extension to all vertices \( v \) of \( G \) with \( d_G(v) < k - 1 \). The resulting graph \( H \) has \( \delta(H) = k - 1 \) and it has a fractional perfect matching \( f \). By Theorem 1.1 we can assume that \( f(e) \in \{0, \frac{1}{2}, 1\} \) for all \( e \in E(H) \). If \( u \) is a vertex to which Meredith extension was applied on, then \( |\text{supp}(f) \cap \partial_G(V(K_{k,k-1}))| \in \{1, 2\} \). In both cases it is easy to see that the contraction of the \( K_{k,k-1} \) yields a critical graph which has a fractional perfect matching. So eventually \( G \) has one.

Let \( G \) be a graph with Gallai-Edmonds decomposition \( (D, A, C) \). Lui and Liu [11] proved that \( \mu_f(G) = \mu(G) \) if and only if \( D \) is an independent set. In particular, \( \mu_f(G) = \mu(G) \) if \( G \) has a 1-factor. Furthermore, if \( G \) has a 1- or a 2-factor, then \( G \) has a fractional perfect
matching. In [9] it is shown that for all \( k \geq 3 \) there are \( k \)-critical graphs of even order which have no 1-factor, and that there are \( k \)-critical graphs \( G \) of odd order and \( G - v \) does not have a 1-factor, where \( d_G(v) = \delta(G) \). We close with the conjecture which is unsolved even for critical graphs which have a near perfect matching. However, it is true if Conjecture 4.1 is true.

**Conjecture 4.4.** Let \( k \geq 3 \) and \( G \) be a \( k \)-critical graph. If \( G \) does not have a 1-factor, then \( \mu_f(G) > \mu(G) \).

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