Generalized integrability conditions and target space geometry

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Abstract

In some higher dimensional nonlinear field theories integrable subsectors with infinitely many conservation laws have been identified by imposing additional integrability conditions. Originally, the complex eikonal equation was chosen as integrability condition, but recently further generalizations have been proposed. Here we show how these new integrability conditions may be derived from the geometry of the target space and, more precisely, from the Noether currents related to a certain class of target space transformations.

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1 Introduction

Nonlinear field theories which allow for static, soliton type solutions are relevant in different branches of physics, ranging from elementary particle theory to condensed matter physics. Specifically for 3 + 1 dimensional space-time and a two-dimensional target space the presence of knot-like solitons can be expected provided that i) the condition of finite energy requires the field to approach a constant value at spatial infinity and ii) the target space has the topology of the two-sphere $S^2$. In general, it is notoriously difficult to obtain exact soliton solutions for such higher dimensional nonlinear field theories. On the other hand, in two dimensional space-time the concept of integrability is known to simplify the calculation of exact solutions significantly. A method to generalize the concept of integrability to higher dimensions was therefore developed in [1]. It was applied to the study of the Skyrme model [2], which has target space $S^3$ and consists, in addition to the usual sigma model type quadratic term, of an additional quartic term in the Lagrangian in order to circumvent the scaling instability (Derrick’s theorem) and allow for static soliton solutions. Further, the higher dimensional integrability was applied to several restrictions of the Skyrme model to target space $S^2$, like the Baby Skyrme model (in 2 + 1 dimensional space-time), the Faddeev–Niemi model [3, 4], the Aratyn–Ferreira–Zimerman (AFZ) model [5, 6] (which only contains the quartic term with the appropriate power to avoid Derrick’s theorem), or the Nicole model [7] (which only contains the quadratic term, again with the appropriate power to avoid Derrick’s theorem); see also [8], [9]. As all these models, except for the Skyrme model\footnote{Besides their particular applications, they contain the basic ideas of Skyrme of topological and scale stability in a simplified form, which facilitates their analysis.}, have a two-dimensional target space, their field content may be described by a complex field $u(r,t)$, which is the case which we want to study in this paper. Integrability in this context amounts to the construction of an infinite number of conserved currents. Indeed, if a current $K^\mu(u, \bar{u}, u_\nu, \bar{u}_\nu)$ can be found such that it obeys the following three conditions (we abbreviate $u_\mu \equiv \partial_\mu u$; further, the bar denotes complex conjugation)

\begin{align}
\text{Im}(\bar{u}_\mu K^\mu) &= 0 \quad (1) \\
u_\mu K^\mu &= 0 \quad (2) \\
\partial_\mu K^\mu &= 0 \quad (3)
\end{align}
then there exist the infinitely many conserved currents

\[ J^G_\mu = i(G_u K_\mu - G_{\bar{u}} \bar{K}_\mu) \]  

\[ \]  

(4)

where \( G \) is an arbitrary real function of \( u \) and \( \bar{u} \), and \( G_u \equiv \partial_u G \).

It turns out that for the AFZ model a current \( K_\mu \) can be determined which obeys all three conditions (1) - (3) without further constraints. As a consequence, the infinitely many currents (4) are conserved and generate infinitely many symmetries (the area-preserving diffeomorphisms on the target space two-sphere \( S^2 \)) of the AFZ model. In this model, therefore, integrability is realized and, further, infinitely many soliton solutions can be found analytically with the help of a separation of variables ansatz in toroidal coordinates, which is indicated by the base space symmetries of the model [6, 8].

On the other hand, for the Baby Skyrme, Nicole, and Faddeev–Niemi models no current \( K_\mu \) can be constructed which obeys all three conditions (1) - (3) without further constraining the allowed fields \( u \). It is, however, possible to find a current \( K_\mu \) which obeys conditions (1) - (3) provided that \( u \) obeys the additional constraint

\[ \partial_\mu u \partial^\mu u = 0, \]

(5)

the so-called complex eikonal equation. In the case of the Baby Skyrme model, this condition just corresponds to the Cauchy–Riemann equations which provide all the known instanton solutions of the two-dimensional sigma model and, at the same time, all soliton solutions of the Baby Skyrme model. More generally, the complex eikonal equation therefore defines integrable submodels for these three models where the infinitely many currents (4) are conserved. These infinitely many conserved quantities are, however, no longer related to symmetries of the submodels, because the eikonal equation is not of the Euler–Lagrange type [9].

Recently, another class of nonlinear field theories with integrable submodels has been suggested by Wereszczyński [10], where a different, “generalized” first order constraint is imposed instead of the complex eikonal equation. The construction of the constraint consists essentially in choosing a vector-like quantity \( \bar{K}_\mu \), which is a function of \( u_\mu \) and \( \bar{u}_\mu \) (but not on higher than first derivatives), and in imposing the constraint

\[ u_\mu \bar{K}^\mu = 0. \]

(6)
For $\tilde{K}_\mu = u_\mu$ this leads to the eikonal equation, whereas for other choices a new integrability condition results. Further, some explicit Lagrangians were constructed in the same paper with the help of the quantities $\tilde{K}_\mu$, and explicit soliton solutions were provided for some particular members of this class of theories. These results have the special interest of being one of the rare cases where explicit solutions for the integrable submodels have been found.

In our paper we want to provide a unifying view on all these models with infinitely many conservation laws, both in the unconstrained case and in the cases of the eikonal and the generalized constraints. Our approach is based on a Noether current for target space transformations, essentially the current (4), and on the corresponding target space geometry. It turns out that the generalized integrability conditions and the corresponding explicit soliton solutions may be derived from a purely Lagrangian approach. Further, the generalized integrability conditions turn out to differ slightly from the eikonal equation in that the former depend on the specific Lagrangian chosen, whereas the eikonal equation only depends on the field contents (i.e., on the complex field $u$). In Section 2 we present our approach and clarify its geometric significance. In Section 3 we show that the soliton models of Wereszczynski are covered by our approach and easily rederive his results.

2 Conserved currents

As said, we consider field theories where the field content can be described by one complex field $u$ and its complex conjugate $\bar{u}$. Concretely, we allow for the class of Lagrangian densities

$$\mathcal{L}(u, \bar{u}, u_\mu, \bar{u}_\mu) = F(a, b, c)$$

where

$$a = u\bar{u}, \quad b = u_\mu \bar{u}^\mu, \quad c = (u_\mu \bar{u}^\mu)^2 - u_\mu^2 \bar{u}_\nu^2$$

and $F$ is at this moment an arbitrary real function of its arguments. That is to say, we allow for Lagrangian densities which depend on the fields and on

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2Observe that these $\tilde{K}_\mu$ are, in general, different from the current $K_\mu$ of Eqs. (1) - (3). In particular, they need no obey $\partial_\mu \tilde{K}_\mu = 0$ on-shell.
their first derivatives, are Lorentz invariant, real, and obey the phase symmetry $u \rightarrow e^{i\phi}u$ for a constant $\phi \in \mathbb{R}$. We could relax the last condition and allow for real Lagrangian densities which depend on $u$ and $\bar{u}$ independently, but this would just complicate the subsequent discussion without adding anything substantial. Further, all models we want to cover fit into the general framework provided by the class of Lagrangian densities (7), therefore we restrict our discussion to this class.

For the currents $K^\mu$ we choose

$$K^\mu = f(a)\bar{\Pi}^\mu$$

where $f$ is a real function of its argument, and $\Pi^\mu$ and $\bar{\Pi}^\mu$ are the conjugate four-momenta of $u$ and $\bar{u}$, i.e.,

$$\Pi_\mu \equiv \mathcal{L}_{u^\mu} = \bar{u}^\mu F_b + 2(u^\lambda \bar{u}_\lambda \bar{u}_\mu - \bar{u}^2 u_\mu)F_c. \quad (10)$$

The current of Eq. (9) automatically obeys the reality condition (1) for real Lagrangian densities. For the other two conditions (2) and (3) we find

$$u_\mu K^\mu = fu^2_\mu F_b$$

and, with the help of the equations of motion

$$\partial^\mu \Pi_\mu = \mathcal{L}_u = \bar{u}F_a \quad (12)$$

and its complex conjugate,

$$\partial^\mu K_\mu = f \left( M' \bar{u}u^2_\mu F_b + u[M'(bF_b + 2cF_c) + F_a] \right) \quad (13)$$

where

$$M \equiv \ln f \quad (14)$$

and the prime denotes the derivative with respect to $a$. Before studying the conditions which make the r.h.s. of Eqs. (11) and (13) vanish, it is instructive to study the resulting expression for $\partial^\mu J^G_\mu$, because this will clarify the geometry behind our class of Lagrangian densities and the currents $J^G_\mu$. We find after a simple calculation

$$\partial^\mu J^G_\mu = i f \left( [(M'\bar{u}G_u + G_{uu})u^2_\mu - (M'\bar{u}G_{\bar{u}} + G_{\bar{u}\bar{u}})\bar{u}^2_\mu]F_b + (uG_u - \bar{u}G_{\bar{u}})[M'(bF_b + 2cF_c) + F_a] \right) \quad (15)$$
Now we want to ask under which circumstances this divergence may vanish. If no constraints are imposed neither on the Lagrangian (i.e., on $F$) nor on the allowed class of fields $u$, then we find the two equations for $G$,

$$uG_u - \bar{u}G_{\bar{u}} = 0,$$

and

$$M_u \bar{u}G_u + G_{uu} = 0 \Rightarrow \partial_u [f(u\bar{u})G_u] = 0$$

(17)

together with its complex conjugate. Eq. (16) implies that $G(u, \bar{u}) = \tilde{G}(u\bar{u})$, and then Eq. (17) leads to the general solution

$$G_u = k \frac{\bar{u}}{f}$$

(18)

where $k$ is a real constant. The corresponding current $J_\mu^G$ is the Noether current for the phase transformation

$$u \rightarrow e^{i\phi} u, \quad \bar{u} \rightarrow e^{-i\phi} \bar{u}$$

(19)

which is a symmetry of the Lagrangian by construction.

Next we restrict the possible Lagrangians by imposing (remember $M \equiv \ln f$)

$$M_a (b F_b + 2 c F_c) + F_a = 0.$$  

(20)

This equation can be solved easily by the method of characteristics and has the general solution

$$F(a, b, c) = \tilde{F} \left( \frac{b}{f}, \frac{c}{f^2} \right)$$

(21)

which has, in fact, a nice geometric interpretation. The point is that the resulting Lagrangian is a sigma model type of Lagrangian which can be expressed entirely in terms of the target space geometry. Indeed, trading the complex $u$ field for two real target space coordinates $\xi^\alpha, \ u \rightarrow (\xi^1, \xi^2)$, the expressions on which $\tilde{F}$ may depend can be expressed as follows. The first term is

$$\frac{b}{f} = \frac{u_\mu \bar{u}^\mu}{f} = g^{\alpha\beta}(\xi) \partial^\mu \xi^\alpha \partial_\mu \xi^\beta$$

(22)

where $\alpha = 1, 2$ etc, and the target space metric $g_{\alpha\beta}$ is diagonal for the coordinate choice $\xi^1 = \text{Re} u, \xi^2 = \text{Im} u$, i.e., $g_{\alpha\beta} = f^{-1} \delta_{\alpha\beta}$. For the second term we get

$$\frac{c}{f^2} = \tilde{\varepsilon}_{\alpha\beta} \varepsilon_{\gamma\delta} \partial^\mu \xi^\alpha \partial_\mu \xi^\gamma \partial^\nu \xi^\beta \partial_\nu \xi^\delta$$

(23)
where
\[ \tilde{\epsilon}_{\alpha\beta} = \det (g_{\gamma\delta}) \epsilon_{\alpha\beta}, \quad \det (g_{\gamma\delta}) = f^{-1} \] (24)
and \( \epsilon_{\alpha\beta} \) is the usual antisymmetric symbol in two dimensions. Observe that the two terms are different in that the first one, \( b/f \), depends on the target space metric, whereas the second one only depends on the determinant of the target space metric.

For Lagrangians which are of the form (21) the condition that the divergence (15) vanishes only leads to Eq. (17) for \( G \), that is, to
\[ G_u = \frac{\bar{H}(u)}{f}, \quad G_{\bar{u}} = \frac{H(u)}{f} \] (25)
where \( H(u) \) is an analytic function of \( u \) only. From what we found about the target space geometry, it will not come as a surprise that the solutions of Eq. (25) provide just the isometries of the corresponding target space metric \( g_{\alpha\beta} \). Indeed, from the reality of \( G \) and from the equality of the mixed second derivatives \( G_{u\bar{u}} \) one easily derives the equation
\[ H_u - \bar{H}_{\bar{u}} = M' (\bar{u}H - u\bar{H}). \] (26)
This equation always has the solution \( H = ku \), independently of \( M \), which just corresponds to the symmetry under the phase transformation (19). Further solutions depend on the explicit form of \( M \), that is, \( f \). E.g., for \( M' = 0 \) (i.e., for a Euclidean metric on target space), we find for \( H \) the general solution
\[ H = k_1 + ik_2 + k_3 u \quad \text{for} \quad k_i \in \mathbb{R} \] (27)
which generates the Euclidean group in \( \mathbb{R}^2 \) (the isometries of the flat, Euclidean metric in \( \mathbb{R}^2 \)). For \( f = (1 + u\bar{u})^2 \), which leads to the metric on \( S^2 \), we find for \( H \)
\[ H = k_1 \frac{i}{2}(1 - u^2) + k_2 \frac{1}{2}(1 + u^2) + k_3 u \quad \text{for} \quad k_i \in \mathbb{R} \] (28)
which generates the modular transformations (the isometries of the metric for the two-sphere \( S^2 \), when the latter is expressed in the coordinate \( u \) via stereographic projection). For more generic expressions for \( f \), Eq. (26) does not provide further solutions and, consequently, the isometries are exhausted by the phase symmetries (19).
Finally, we may further restrict the possible Lagrangians or the allowed field configurations to achieve that the current divergence (15) vanishes without requiring further restrictions on $G$. One way of achieving this is by assuming $F_b \equiv 0$ identically, i.e.,

$$F(a, b, c) \equiv \tilde{F}(\frac{c}{f^2})$$  

(this has already been pointed out in [8], using a slightly different approach). The AFZ model is precisely of this type. In this case the Lagrangian only depends on the determinant of the target space metric, and it easily follows that a general $G$, which is now no longer restricted (except for the condition of being real), is related to the area-preserving diffeomorphisms, i.e., $(iH\partial_u + \text{c.c.})$ generates area-preserving diffeomorphisms on functions of $u$ and $\bar{u}$, where $H \equiv fG\bar{u}$. For the case of the two-sphere as target space, area-preserving diffeomorphisms and their generators and Noether currents are discussed in detail e.g. in [8], [9], [11].

Alternatively, we may make Eq. (15) vanish by imposing restrictions on the allowed field configurations $u$. In this case the currents $J^G_{\mu}$ are still the Noether currents of area-preserving diffeomorphisms, but these transformations are no longer symmetry transformations of the pertinent Lagrangians in general. We may either require that $u$ obeys the complex eikonal equation (5), or we may require that $u$ obeys the (in general nonlinear) first order PDE which follows from the condition $F_b = 0$ in cases when this condition does not hold identically (i.e., for Lagrangians which do depend on the term $b = u^a \bar{u}_\mu$). The first case provides the integrability condition for the integrable submodels of the Faddeev–Niemi, Nicole and Baby Skyrme model, as was discussed, e.g., in [1], [9]. The second case provides the generalized integrability conditions which were introduced by Wereszczynski in [10], as we shall discuss in the next section. Observe that the first condition, the complex eikonal equation, is model independent, whereas the second condition $F_b = 0$ depends on the model, i.e., on the Lagrangian.
3 Soliton models of Wereszczyński

Let us now specify the Lagrangian to

$$L = \left( \frac{\lambda_1 b^3}{f^3} + \frac{\lambda_2 bc}{f^3} \right)^{\frac{1}{2}} = f^{-\frac{3}{2}} \left( \lambda_1 b^3 + \lambda_2 bc \right)^{\frac{1}{2}}$$  \hspace{1cm} (30)$$

where as above \( f = f(a) \) and \( \lambda_1, \lambda_2 \) are two real constants. This Lagrangian is of the type (21). Further, the noninteger power in the Lagrangian is chosen precisely such as to render the energies of field configurations scale invariant, avoiding thereby Derricks theorem and allowing for static, solitonic solutions. In addition, it is equal to the Lagrangian studied by Wereszczyński (see Eq. (30) of Ref. [10]) when the identifications

$$f = G^{-\frac{3}{4}}, \quad \lambda_1 = \alpha + \beta + 1, \quad \lambda_2 = -\beta - 1$$  \hspace{1cm} (31)$$

are made. The condition \( F_b = 0 \) leads to the condition

$$3\lambda_1 b^2 + \lambda_2 c = 0$$  \hspace{1cm} (32)$$

or, more explicitly,

$$3\lambda_1 (u_\mu \bar{u}_\mu)^2 + \lambda_2 ((u_\mu \bar{u}_\mu)^2 - u_\mu^2 \bar{u}_\mu^2) = 0$$  \hspace{1cm} (33)$$

which coincides with the integrability condition Eq. (35) of Ref. [10].

Further, once the integrability condition (33) is imposed, the equation of motion is equivalent to the condition

$$\partial_\mu K^\mu = 0,$$  \hspace{1cm} (34)$$

where \( K^\mu \) is defined as before, \( K^\mu = f \bar{\Pi}^\mu \) with

$$\bar{\Pi}_\mu \equiv \mathbf{L}_{\bar{u}_\mu} = \frac{1}{2} f^{-\frac{3}{2}} \left( \lambda_1 b^3 + \lambda_2 bc \right)^{-\frac{3}{2}} \left[ (3\lambda_1 + 2\lambda_2) b^2 u_\mu + \lambda_2 c u_\mu - 2\lambda_2 b u_\mu^2 \bar{u}_\mu \right].$$  \hspace{1cm} (35)$$

The equation of motion (e.o.m.) (34) coincides with Eq. (36) of Ref. [10].

Having unravelled the geometric nature of these further generalizations of integrability, let us finally derive in this framework the explicit soliton solutions of Wereszczyński of the integrable submodels, that is, simultaneous solutions of the generalized constraint (33) and of the e.o.m. (34). Notice that
in this systematic approach one also derives the corresponding Lagrangians (i.e., specific choices for the target space metric function $f$ such that the field configurations solving the constraint solve at the same time the e.o.m.). The starting point for the construction of the solutions is the observation that both the integrability condition Eq. (33) and the e.o.m. (divergence condition) Eq. (34) have, in the static case, the conformal transformations on the base space $\mathbb{R}^3$ as symmetries. The symmetry under scale transformations is obvious for both equations, whereas the symmetry under the remaining conformal transformations can be checked without difficulty (the general method of calculating the symmetries of PDEs of the above types is explained, e.g., in [8], [9], [12]). As a consequence, both equations are compatible with a separation of variables ansatz in toroidal coordinates. That is to say, if we introduce toroidal coordinates ($\eta, \xi, \varphi$) via

$$
\begin{align*}
  x &= q^{-1} \sinh \eta \cos \varphi, \\
  y &= q^{-1} \sinh \eta \sin \varphi, \\
  z &= q^{-1} \sin \xi; \\
  q &= \cosh \eta - \cos \xi.
\end{align*}
$$

(36)

then the ansatz

$$
  u = \rho(\eta) e^{im\varphi+in\xi}, \quad m, n \in \mathbb{Z}
$$

(37)

is compatible with both equations and leads, in both cases, to an ODE for $\rho(\eta)$. A detailed explanation for this ansatz and its relation to the conformal symmetry of the static equations is provided in [8]. Further, fields $u$ within this ansatz can be interpreted as Hopf maps $S^3 \rightarrow S^2$, and soliton solutions within this ansatz are therefore topological in nature (“Hopf solitons”). For details on the pertinent geometry and topology we refer, e.g., to [13], [14], [15].

Now we proceed in two steps, analogously to the calculation in [10]. Firstly, we insert this ansatz into the integrability condition Eq. (33) and find one solution for each value of $m$ and $n$. Then we assume that this solution is also a solution of the e.o.m (34), insert it into the e.o.m., and determine the target space metric function $f$ accordingly (at this point our presentation differs slightly from the one chosen in [10], where the solution for $f$ was given at the beginning). This determination of $f$ is always possible, because the solution $\rho(\eta)$ of the constraint (33) allows to express $\eta$ in terms of $\rho$, i.e., to express the e.o.m. as an ODE in the independent variable $\rho$ and in the dependent variable $f$ (remember that $f = f(a)$ and $a = \rho^2$; we use the same letter $f$ also for $f(\rho)$, which should not cause any confusion).
We need the gradient in toroidal coordinates

\[
\nabla = (\nabla \eta) \partial_\eta + (\nabla \xi) \partial_\xi + (\nabla \varphi) \partial_\varphi = q(\dot{\eta} \partial_\eta + \dot{\xi} \partial_\xi + \frac{1}{\sinh \eta} \dot{\varphi} \partial_\varphi)
\]

where \((\hat{e}_\eta, \hat{e}_\xi, \hat{e}_\varphi)\) form an orthonormal frame in \(\mathbb{R}^3\). Further we need the relations

\[
\nabla \cdot \hat{e}_\eta = -\sinh \eta + \frac{1 - \cosh \eta \cos \xi}{\sinh \eta}, \quad \nabla \cdot \hat{e}_\xi = -2 \sin \xi, \quad \nabla \cdot \hat{e}_\varphi = 0.
\]

Inserting now the ansatz (37) into the constraint equation (33) we find after a brief calculation the equation

\[
\left(\frac{L_\eta}{\Gamma}\right)^4 - 2(\lambda - 1) \left(\frac{L_\eta}{\Gamma}\right)^2 + 1 = 0
\]

where

\[
L \equiv \ln \rho, \quad \Gamma \equiv \left(n^2 + \frac{m^2}{\sinh^2 \eta}\right)^{\frac{1}{2}}
\]

and

\[
\lambda \equiv -\frac{2\lambda_2}{3\lambda_1}.
\]

Eq. (40) is an algebraic second order equation for the quantity \((L_\eta/\Gamma)^2\) with the solution

\[
\left(\frac{L_\eta}{\Gamma}\right)^2 = A^2(\lambda) \equiv \lambda - 1 + \sqrt{\lambda^2 - 2\lambda}
\]

and the condition that \(A^2\) must be real and positive leads to the restriction

\[
\lambda \geq 2.
\]

[Remark: this restriction of \(\lambda\) also determines the signs of \(\lambda_1\) and \(\lambda_2\), which must have opposite signs according to (42). The point is that the expression within the square root in the Lagrangian (30) must be positive which, together with the constraint (44), implies \(\lambda_1 < 0\) and \(\lambda_2 > 0\).]

Taking now the square root of Eq. (43) and choosing either of the two signs \(\pm A\) on the r.h.s. we may integrate the expression for \(L_\eta\) with the result
\( \rho^{(\pm)} \), where \( \rho^{(-)} = 1/\rho^{(+)} \) and

\[
\rho^{(+)} = k \sinh^{A[m]} \eta \left( |n| \cosh \eta + \sqrt{m^2 + n^2 \sinh^2 \eta} \right)^{A[n]} 
\]  

(45)

where \( k \) is a constant. This result coincides precisely with Eq. (24) of Ref. [10]. It is interesting to observe that, for \( A = 1 \), these field configurations also solve the static complex eikonal equation, see [16].

Now we insert this solution into the e.o.m. (34) and determine \( f \) such that Eq. (34) holds. We restrict to the solutions \( \rho^{(+)} \) and find, after some calculation, for the spatial part \( \vec{K} \) of the current \( K^\mu \)

\[
\vec{K} = \frac{3}{2} A f^{-\frac{1}{2}} q^2 u \rho \Gamma \left( \hat{e}_\eta \Gamma A \mathcal{B} + i \left( n \hat{e}_\xi + \hat{e}_\varphi \frac{m}{\sinh \eta} \right) C \right) 
\]  

(46)

where frequent use has been made of the relation \( L_\eta = A \Gamma \). Further,

\[
\mathcal{A} = \left( 6 \lambda A^2 (A^2 + 1) - (A^2 + 1)^3 \right)^{-\frac{1}{2}} \\
\mathcal{B} = -A^4 + 2(2\lambda - 1)A^2 + 2\lambda - 1 \\
\mathcal{C} = (2\lambda - 1)A^4 + 2(2\lambda - 1)A^2 - 1 
\]  

(47)

are some functions of the parameter \( \lambda \). For the divergence \( \nabla \cdot \vec{K} \) we find, after some more calculation (remember \( M = \ln f \)),

\[
\nabla \cdot \vec{K} = \frac{3}{2} A q^3 f^{-\frac{1}{2}} u \rho A \mathcal{B} \left( -\frac{1}{2} \rho M_\rho A \Gamma^3 + 2 A \Gamma^3 + \frac{\cosh \eta}{\sinh \eta} \left( \Gamma^2 - \frac{2m^2}{\sinh^2 \eta} \right) \right) \\
-\frac{3}{2} A q^3 f^{-\frac{1}{2}} u \rho \Gamma^3 C. 
\]  

(48)

Before solving the conservation equation \( \nabla \cdot \vec{K} = 0 \), we restrict the integers \( m \) and \( n \) to the case \( m = n \). This we do because we need the function inverse to \( \rho^{(+)}(\eta) \). This would be extremely complicated for \( m \neq n \), whereas it leads to the simple expression

\[
\sinh \eta = \rho^{-1}_{m|A} 
\]  

(49)
for \( m = n \). Further we have for \( m = n \) that

\[
\Gamma = |m| \frac{\cosh \eta}{\sinh \eta}
\]

(50)

and find, with the help of the easily verified identity \( C = A^2 B \), that the condition \( \nabla \cdot \vec{K} = 0 \) leads to the equation

\[
\frac{1}{2} \rho M_\rho = 1 + \frac{1}{|m|A} \frac{\sinh^2 \eta - 1}{\cosh^2 \eta}
\]

(51)

or, using Eq. (49),

\[
M_\rho = \frac{2}{\rho} \left( 1 - \frac{1}{|m|A} \right) + \frac{4}{|m|A \rho} \frac{\rho^{|m|A}}{1 + \rho^{|m|A}}
\]

(52)

with the solution

\[
f = \rho^{2-|m|A} \left( 1 + \rho^{|m|A} \right)^2
\]

(53)

which coincides exactly with the solution Eq. (40) of Ref. [10].

To finish, let us briefly comment on the construction of more complicated scale-invariant models of the same type, which is discussed in Section 4 of Ref. [10]. There the possibility of constructing further models was pointed out, and it was observed that the integrability conditions for these models have the same solutions (45) within the ansatz (37) and, therefore, in this sense do not give rise to new field configurations. Here we just want to comment that these models certainly are covered by our approach and, further, that it can be easily understood from our methods why the corresponding integrability conditions lead to the same solutions (45). In fact, the condition of scale invariance dictates that the corresponding Lagrangian densities have to be of the form

\[
\mathcal{L} = f^{-\frac{2}{k}} \left( \alpha_k b^k + \alpha_{k-2} b^{k-2} + \ldots + \alpha_1 b c \right)^{\frac{2}{k}} \ldots \quad k \text{ odd}
\]

(54)

or

\[
\mathcal{L} = f^{-\frac{2}{k}} \left( \alpha_k b^k + \alpha_{k-2} b^{k-2} + \ldots + \alpha_0 c \right)^{\frac{2}{k}} \ldots \quad k \text{ even}
\]

(55)
and the case discussed more explicitly in Section 4 of Ref. [10] correlates to \( k = 4 \). Further, the integrability condition \( \mathcal{L}_b = 0 \) leads in all cases to the equation

\[
\tilde{\lambda}_1 b^2 + \tilde{\lambda}_2 c = 0 \tag{56}
\]

(where the \( \tilde{\lambda}_i \) are some functions of the parameters \( \alpha_j \) in the Lagrangian), as may be checked easily. This integrability condition is the same as Eq. (33), and, therefore, has the same solutions (45). The only change is that the dependence of the parameter \( A \) on the original parameters in the Lagrangian is, of course, different for different Lagrangians.

In conclusion, we see that the specific generalizations of the complex eikonal equation, proposed in Ref. [10] as integrability conditions in the sense of [1], can be in fact understood in general geometric terms, which allows to obtain directly the explicit results of [10], and to understand also its limitations.

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