TWISTED KÄHLER-EINSTEIN METRICS ON FLAG VARIETIES

EDER M. CORREA AND LINO GRAMA

Abstract. In this paper, we describe invariant twisted Kähler-Einstein (tKE) metrics on flag varieties. We also explore some applications of the ideas involved in the proof of our main result to the existence of invariant twisted constant scalar curvature Kähler metrics. Also, we provide a precise description for the set of tKE metrics for several examples, including full flag varieties, the projectivization of the tangent bundle of $\mathbb{P}^n$, and families of flag varieties with Picard number 2.

1. Introduction

Let $X$ be a compact Kähler manifold and let $\xi$ be a Kähler class on $X$. Fixed a smooth $(1,1)$-form $\beta \in 2\pi\{c_1(X) - \xi\}$, the solutions of the equation

$$\text{Ric}(\omega) = \omega + \beta,$$

are called twisted Kähler-Einstein (tKE) metrics. As in the usual Fano case ($\xi = c_1(X)$ and $\beta = 0$), we have that Eq. (1.1) is not always solvable. In the particular case that $\xi = c_1(L)$ and $\beta > 0$, the existence of solutions of Eq. (1.1) was characterized in [BBJ21] using a twisted analogue of the $\delta$-invariant originally defined in [FO18]. As shown in [BJ18], [BBJ21], [BLZ19], the $\delta$-invariant is the right threshold to detect Ding-stability, an algebraic notion designed for the existence of tKE metrics. However, $\delta$-invariant is not easy to compute in general, see for instance [BJJ20]. Further, tKE metrics also appear, for instance, in the study of Kähler-Ricci flow through singularities, see [ST17], [ST12]. There is also a twisted version of the Kähler-Ricci flow which was studied in [Lin13], [LW15], [CS16]. In this paper, we restrict ourselves to the study of Eq. (1.1) in the setting of generalized flag varieties. A flag variety can be described as a quotient $X_P = G^\mathbb{C}/P$, where $G^\mathbb{C}$ is a semisimple complex algebraic group and $P$ is a parabolic subgroup (Borel-Remmert [BR62]). Regarding $G^\mathbb{C}$ as a complex analytic space, without loss of generality, we may assume that $G^\mathbb{C}$ is a connected simply connected complex Lie group. Fixed a compact real form $G \subset G^\mathbb{C}$, the main purpose in this work is to characterize the existence of $G$-invariant tKE metrics on flag varieties using essentially tools of Lie theory. As in the case of $G$-invariant Kähler-Einstein metrics, the main philosophy is to reduce the underlying non-linear PDE problem provided by Eq. (1.1) to an algebraic problem involving elements of the theory of semisimple Lie groups and Lie algebras. The advantage of this approach is that it allows us to describe $G$-invariant tKE metrics in a precise and explicit way. We also explore some applications of the ideas involved in the proof of our main result to the existence of certain invariant twisted constant scalar curvature Kähler metrics. Further, we provide a precise description for the greatest Ricci lower bound of every Kähler class on a flag variety. By means of this description, we establish some inequalities related to optimal volume upper bounds for Kähler metrics just using tools from Lie theory. Additionally, we describe the set of tKE metrics for several examples, including full flag varieties, the projectivization of the tangent bundle of $\mathbb{P}^n$, and families of flag varieties with Picard number 2.

1.1. Main results. Let $X_P = G^\mathbb{C}/P$ be a complex flag variety. Considering $\text{Lie}(G^\mathbb{C}) = g^\mathbb{C}$, if we choose a Cartan subalgebra $h \subset g^\mathbb{C}$, and a simple root system $\Sigma \subset h^*$, up to conjugation, we have that $P = P_\Theta$, for some $\Theta \subset \Sigma$, where $P_\Theta$ is a parabolic Lie subgroup determined by $\Theta$, see for instance [Akh95]. The main result which we prove in this work is the following

**Theorem A.** Let $L \in \text{Pic}(X_P)$ and let $\beta \in c_1(L)$ be a $G$-invariant $(1,1)$-form. Then there exists a (unique) $G$-invariant Kähler metric $\omega$ on $X_P$, satisfying

$$\text{Ric}(\omega) = \omega + \beta,$$

if, and only if,

$$\int_{\mathbb{P}^1_{\alpha}} \beta < 2\pi\langle \delta_P, h^\mathbb{C}_\alpha \rangle, \quad (1.3)$$

$\forall \alpha \in \Sigma\setminus\Theta$, where $\mathbb{P}^1_{\alpha} \subset X_P$, $\alpha \in \Sigma\setminus\Theta$, are generators of the cone of curves $\text{NE}(X_P)$. 

\textbf{Date:} October 18, 2022.
From the above result, one can prove that Eq. (1.1) can be always solved in the setting of flag varieties if \( \beta \) is \( G \)-invariant, i.e., we have the following.

**Corollary A.** Given a Kähler class \( \xi \) on \( X_P \) and a \( G \)-invariant (1, 1)-form \( \beta \in 2\pi (c_1(X_P) - \xi) \), then there exist a unique \( G \)-invariant Kähler metric \( \omega \in 2\pi \xi \), such that

\[
\text{Ric}(\omega) = \omega + \beta.
\] (1.4)

Given a semipositive (1, 1)-form \( \beta \) on a compact Kähler manifold \((X, \omega)\), denoting by \( S(\omega) \) the Chern scalar curvature of \( \omega \), if

\[
S(\omega) - \Lambda_\omega(\beta) = \text{const.},
\] (1.5)

we say that \( \omega \) is a \( \beta \)-twisted constant scalar curvature Kähler metric (\( \beta \)-twisted cscK metric). The equation above arises in the study and construction of constant scalar curvature Kähler metrics (cscK metrics), e.g. [Fin04], [Fin07], [ST07]. In [BB17, Theorem 4.5], it was shown that a \( \beta \)-twisted cscK metric is unique in each cohomology class. In the particular setting of flag varieties, by taking the trace with respect to \( \omega \) in Eq. (1.4), we obtain the following result.

**Corollary B.** Given a Kähler class \( \xi \) on \( X_P \) and a \( G \)-invariant (1, 1)-form \( \beta \in 2\pi (c_1(X_P) - \xi) \), then there exists a (unique) \( G \)-invariant Kähler metric \( \omega \in 2\pi \xi \) with constant \( \beta \)-twisted scalar curvature, such that

\[
S(\omega) - \Lambda_\omega(\beta) = \dim_G(X_P),
\] (1.6)

where \( S(\omega) \) denotes the Chern scalar curvature of \( \omega \).

Let \( K(X) \) be the Kähler cone of a compact Kähler manifold \( X \). For any Kähler class \( \xi \in K(X) \), one can define its greatest Ricci lower bound \( R(\xi) \) as being

\[
R(\xi) := \sup \{ r \in \mathbb{R} \mid \exists \text{Kähler form } \omega \in 2\pi \xi, \text{ s.t. } \text{Ric}(\omega) \geq r \omega \}. (1.7)
\]

This invariant was first studied by Tian in [Tia92] for the case \( \xi = c_1(X) \) and further studied, for instance, in [Rub08], [Rub09], [Szfrm[00]-1], [SW16]. As in the case of the \( \delta \)-invariant, few examples of explicit computations of the greatest Ricci lower bound are known so far, see for instance [Li11] for case when \( X \) is a toric Fano manifold. Using the previous results, we prove the following explicit expression for the greatest Ricci lower bound \( R(\xi) \) of any Kähler class \( \xi \) on a flag variety \( X_P \).

**Corollary C.** Let \( R : K(X_P) \to \mathbb{R} \), such that \( R(\xi) \) is the greatest Ricci lower bound of \( \xi \in K(X_P) \). Then, we have

\[
R(\xi) = \min \left\{ \frac{\langle \delta_P, h^\omega_{X_P} \rangle}{a_\alpha} \mid \alpha \in \Sigma \setminus \Theta \right\},
\] (1.8)

such that \( a_\alpha = \langle \xi, [\mathbb{P}^1_\alpha] \rangle \), \( \forall \alpha \in \Sigma \setminus \Theta \).

Given a Kähler class \( \xi \in K(X) \) of a Fano manifold \( X \), it was shown in [Zha22, Theorem 4.1] that

\[
R(\xi)^n \text{Vol}(\xi) \leq (n + 1)^n,
\] (1.9)

such that \( n = \dim_G(X) \). Also, if \( X \) is Fano and admits a Kähler-Einstein metric, it follows from [Fuj18] that

\[
(-K_X)^n \leq (n + 1)^n.
\] (1.10)

It is worth pointing out that the inequality above was first proved for the particular case of flag varieties in [Sno04]. From the explicit description provided in Corollary C for the greatest Ricci lower bound of Kähler classes of flag varieties, we prove the following result relating the inequalities above.

**Theorem B.** For every \( \xi \in K(X_P) \), the following inequalities hold

\[
R(\xi)^n \text{Vol}(\xi) \leq (-K_{X_P})^n \leq (n + 1)^n,
\] (1.11)

such that \( n = \dim_G(X_P) \).

It is worth mentioning that the proof which we present in this work for the result above is independent of the results provided in [Zha22, Theorem 4.1] and [Fuj18]. In fact, we prove Theorem B using essentially tools from Lie theory and the result provided in [Sno04].

### 1.2. Outline of the paper

In Section 2, we review some facts about the geometry of flag varieties. In Section 3, we prove our main results. In Section 4, we work out some examples in order to determine twisted Kähler-Einstein metrics. These examples include full flag varieties, projectivization of the tangent bundle of \( \mathbb{P}^{n+1} \), and families of flag varieties with Picard number 2.

### 2. Generalities on flag varieties

In this section, we review some basic facts about flag varieties. For more details on the subject presented in this section, we suggest [Akh95], [LB18], [Hum75], [BR62].
2.1. The Picard group of flag varieties. Let $G^\mathbb{C}$ be a connected, simply connected, and complex Lie group with simple Lie algebra $\mathfrak{g}^\mathbb{C}$. By fixing a Cartan subalgebra $\mathfrak{h}$ and a simple root system $\Sigma \subset \mathfrak{h}^*$, we have a decomposition of $\mathfrak{g}^\mathbb{C}$ given by

$$\mathfrak{g}^\mathbb{C} = n^- \oplus \mathfrak{h} \oplus n^+,$$

where $n^- = \sum_{\alpha \in \Pi^-} \mathfrak{g}_\alpha$ and $n^+ = \sum_{\alpha \in \Pi^+} \mathfrak{g}_\alpha$, here we denote by $\Pi = \Pi^+ \cup \Pi^-$ the root system associated to the simple root system $\Sigma = \{\alpha_1, \ldots, \alpha_m\} \subset \mathfrak{h}^*$. Let us denote by $\kappa$ the Cartan-Killing form of $\mathfrak{g}^\mathbb{C}$. From this, for every $\alpha \in \Pi^+$ we have $h_\alpha \in \mathfrak{h}$, such that $\alpha = \kappa(\cdot, h_\alpha)$, and we can choose $x_\alpha \in \mathfrak{g}_\alpha$ and $y_{-\alpha} \in \mathfrak{g}_{-\alpha}$, such that $[x_\alpha, y_{-\alpha}] = h_\alpha$. From these data, we can define a Borel subalgebra by setting $\mathfrak{b} = \mathfrak{h} \oplus n^-$. Now we consider the following result (see for instance [LB18], [Hum75]):

**Theorem 2.1.** Any two Borel subgroups are conjugate.

From the result above, given a Borel subgroup $B \subset G^\mathbb{C}$, up to conjugation, we can always suppose that $B = \exp(\mathfrak{b})$. In this setting, given a parabolic Lie subgroup $P \subset G^\mathbb{C}$, without loss of generality, we can suppose that

$$P = P_\Theta,$$

where $P_\Theta \subset G^\mathbb{C}$ is the parabolic subgroup which integrates the Lie subalgebra

$$\mathfrak{p}_\Theta = n^+ \oplus \mathfrak{h} \oplus n(\Theta)^{-},$$

with $n(\Theta)^{-} = \sum_{\alpha \in (\Theta)^{-}} \mathfrak{g}_\alpha$.

By definition, it is straightforward to show that $P_\Theta = N_{G^\mathbb{C}}(\mathfrak{p}_\Theta)$, where $N_{G^\mathbb{C}}(\mathfrak{p}_\Theta)$ is the normalizer in $G^\mathbb{C}$ of $\mathfrak{p}_\Theta \subset \mathfrak{g}^\mathbb{C}$. In what follows, it will be useful for us to consider the following basic chain of Lie subgroups

$$T^\mathbb{C} \subset B \subset P \subset G^\mathbb{C}.$$

For each element in the aforementioned chain of Lie subgroups we have the following characterization:

- $T^\mathbb{C} = \exp(\mathfrak{h})$; (complex torus)
- $B = N^+T^\mathbb{C}$, where $N^+ = \exp(n^+)$; (Borel subgroup)
- $P = P_\Theta = N_{G^\mathbb{C}}(\mathfrak{p}_\Theta)$, for some $\Theta \subset \Sigma \subset \mathfrak{h}^*$. (parabolic subgroup)

Now let us recall some basic facts about the representation theory of $\mathfrak{g}^\mathbb{C}$, more details can be found in [Hum72]. For every $\alpha \in \Sigma$, we set

$$h^\vee_\alpha = \frac{2}{\kappa(h_\alpha, h_\alpha)} h_\alpha.$$

The fundamental weights $\{\varpi_\alpha \mid \alpha \in \Sigma\} \subset \mathfrak{h}^*$ of $(\mathfrak{g}^\mathbb{C}, \mathfrak{h})$ are defined by requiring that $\varpi_\alpha(h_\beta^\vee) = \delta_{\alpha\beta}$, $\forall \alpha, \beta \in \Sigma$. We denote by

$$\Lambda^+ = \bigoplus_{\alpha \in \Sigma} \mathbb{Z}_{\geq 0} \varpi_\alpha,$$

the set of integral dominant weights of $\mathfrak{g}^\mathbb{C}$. Let $V$ be an arbitrary finite dimensional $\mathfrak{g}^\mathbb{C}$-module. By considering its weight space decomposition

$$V = \bigoplus_{\mu \in \Pi(V)} V_\mu,$$

such that $V_\mu = \{v \in V \mid h \cdot v = \mu(h)v, \forall h \in \mathfrak{h}\} \neq \{0\}$, $\forall \mu \in \Pi(V) \subset \mathfrak{h}^*$, from the Lie algebra representation theory we have the following facts:

1. A highest weight vector (of weight $\lambda$) in a $\mathfrak{g}^\mathbb{C}$-module $V$ is a non-zero vector $v_\lambda^+ \in V_\lambda$, such that $x \cdot v_\lambda^+ = 0$, ($\forall x \in n^+$).

   Such a $\lambda \in \Pi(V)$ satisfying the above condition is called highest weight of $V$;

2. $V$ irreducible $\implies \exists$ highest weight vector $v_\lambda^+ \in V$ (unique up to non-zero scalar multiples) for some $\lambda \in \Pi(V)$;

3. If $\lambda \in \Lambda^+$, then there exists a finite dimensional irreducible $\mathfrak{g}^\mathbb{C}$-module $V$ which has $\lambda$ as highest weight. In this case, we denote $V = V(\lambda)$;

4. For all $\lambda \in \Lambda^+$, we have $V(\lambda) = \mathfrak{U}(\mathfrak{g}^\mathbb{C}) \cdot v_\lambda^+$, where $\mathfrak{U}(\mathfrak{g}^\mathbb{C})$ is the universal enveloping algebra of $\mathfrak{g}^\mathbb{C}$;

5. The fundamental representations are defined by $V(\varpi_\alpha), \alpha \in \Sigma$;

6. Given $\lambda \in \Lambda^+$, such that $\lambda = \sum_\alpha n_\alpha \varpi_\alpha$, we have

   $$v_\lambda^+ = \bigotimes_{\alpha \in \Sigma} (v_{\varpi_\alpha}^+)^{\otimes n_\alpha} \quad \text{and} \quad V(\lambda) = \mathfrak{U}(\mathfrak{g}^\mathbb{C}) \cdot v_\lambda^+ \subset \bigotimes_{\alpha \in \Sigma} V(\varpi_\alpha)^{\otimes n_\alpha};$$

7. For all $\lambda \in \Lambda^+$, we have the following equivalence of induced irreducible representations

   $$\varrho: G^\mathbb{C} \rightarrow \text{GL}(V(\lambda)) \iff \varrho_\lambda: \mathfrak{g}^\mathbb{C} \rightarrow \mathfrak{gl}(V(\lambda)),$$

   such that $\varrho(\exp(x)) = \exp(\varrho_\lambda(x))$, $\forall x \in \mathfrak{g}^\mathbb{C}$, notice that $G^\mathbb{C} = \langle \exp(\mathfrak{g}^\mathbb{C}) \rangle$. 
In what follows, for any representation \( \varphi: G^\mathbb{C} \to \text{GL}(V(\lambda)) \), for the sake of simplicity, we shall denote \( \varphi(g)v = gv \), for all \( g \in G^\mathbb{C} \), and all \( v \in V(\lambda) \). Let \( G \subset G^\mathbb{C} \) be a compact real form for \( G^\mathbb{C} \). Given a complex flag variety \( X_P = G^\mathbb{C}/P \), regarding \( X_P \) as a homogeneous \( G \)-space, that is, \( X_P = G/G \cap P \), the following theorem allows us to describe all \( G \)-invariant Kähler structures on \( X_P \) by means of elements of representation theory.

**Theorem 2.2** (Azad-Biswas, [AB03]). Let \( \omega \in \Omega^{1,1}(X_P)^G \) be a closed invariant real \((1,1)\)-form, then we have

\[
\pi^*\omega = \sqrt{-1}\partial\bar{\partial}\varphi,
\]

where \( \pi: G^\mathbb{C} \to X_P \), and \( \varphi: G^\mathbb{C} \to \mathbb{R} \) is given by

\[
\varphi(g) = \sum_{\alpha \in \Sigma \setminus \Theta} c_\alpha \log \left( \| g\varphi_{+}\alpha \|^2 \right), \quad (\forall g \in G^\mathbb{C})
\]

with \( c_\alpha \in \mathbb{R}, \forall \alpha \in \Sigma \setminus \Theta \). Conversely, every function \( \varphi \) as above defines a closed invariant real \((1,1)\)-form \( \omega_\varphi \in \Omega^{1,1}(X_P)^G \). Moreover, \( \omega_\varphi \) defines a \( G \)-invariant Kähler form on \( X_P \) if, and only if, \( c_\alpha > 0, \forall \alpha \in \Sigma \setminus \Theta \).

**Remark 2.3.** It is worth pointing out that the norm \( \| \cdot \| \) in the last theorem is a norm induced from some fixed \( G \)-invariant inner product \( \langle \cdot, \cdot \rangle_\alpha \) on \( V(\varpi_\alpha) \), for every \( \alpha \in \Sigma \setminus \Theta \).

**Remark 2.4.** An important consequence of Theorem 2.2 is that it allows us to describe the local Kähler potential for any homogeneous Kähler metric in a quite concrete way, for some examples of explicit computations we suggest [CG19], [Cor19].

By means of the above theorem we can describe the unique \( G \)-invariant representative in each integral class in \( H^2(X_P, \mathbb{Z}) \). In fact, consider the associated \( P \)-principal bundle \( P \to G^\mathbb{C} \to X_P \). By choosing a trivializing open covering \( X_P = \bigcup_{i \in I} U_i \), in terms of \( Cech \) cocycles we can write

\[
G^\mathbb{C} = \left\{ (U_i)_{i \in I}, \psi_{ij} : U_i \cap U_j \to P \right\}.
\]

Given a fundamental weight \( \varpi_\alpha \in \Lambda^+, \) we consider the induced character \( \chi_{\varpi_\alpha} \in \text{Hom}(T, \mathbb{C}^\times) \), such that \( (d\chi_{\varpi_\alpha})_\lambda = \varpi_\alpha \). From the homomorphism \( \chi_{\varpi_\alpha} : P \to \mathbb{C}^\times \) one can equip \( \mathbb{C} \) with a structure of \( P \)-space, such that \( Pz = \chi_{\varpi_\alpha}(p)^{-1}z, \forall p \in P \), and \( \forall z \in \mathbb{C} \). Denoting by \( \mathbb{C}_{-\varpi_\alpha} \) this \( P \)-space, we can form an associated holomorphic line bundle \( \mathcal{O}_\alpha(1) = G^\mathbb{C} \times_P \mathbb{C}_{-\varpi_\alpha} \), which can be described in terms of \( Cech \) cocycles by

\[
\mathcal{O}_\alpha(1) = \left\{ (U_i)_{i \in I}, \chi_{\varpi_\alpha}^{-1} \circ \psi_{ij} : U_i \cap U_j \to \mathbb{C}^\times \right\},
\]

that is, \( \mathcal{O}_\alpha(1) = \{ g_{ij} \} \in H^1(X_P, \mathcal{O}_\alpha^\times) \), such that \( g_{ij} = \chi_{\varpi_\alpha}^{-1} \circ \psi_{ij} \), for every \( i, j \in I \).

**Remark 2.5.** We observe that, if we have a parabolic Lie subgroup \( P \subset G^\mathbb{C} \), such that \( P = P_\Theta \), the decomposition

\[
P_\Theta = [P_\Theta, P_\Theta]T(\Sigma \setminus \Theta)^\mathbb{C}, \quad \text{such that} \quad T(\Sigma \setminus \Theta)^\mathbb{C} = \exp \left\{ \sum_{\alpha \in \Sigma \setminus \Theta} a_\alpha b_\alpha \mid a_\alpha \in \mathbb{C} \right\},
\]

e.g. [Akh95, Proposition 8], shows us that \( \text{Hom}(P, \mathbb{C}^\times) = \text{Hom}(T(\Sigma \setminus \Theta)^\mathbb{C}, \mathbb{C}^\times) \). Therefore, if we take \( \varpi_\alpha \in \Lambda^+, \) such that \( \alpha \in \Theta \), it follows that \( \mathcal{O}_\alpha(1) = X_P \times \mathbb{C} \), i.e., the associated holomorphic line bundle \( \mathcal{O}_\alpha(1) \) is trivial.

Given \( \mathcal{O}_\alpha(1) \in \text{Pic}(X_P) \), such that \( \alpha \in \Sigma \setminus \Theta \), as described above, if we consider an open covering \( X_P = \bigcup_{i \in I} U_i \) which trivializes both \( P \to G^\mathbb{C} \to X_P \) and \( \mathcal{O}_\alpha(1) \to X_P \), by taking a collection of local sections \( (s_i)_{i \in I} \), such that \( s_i : U_i \to G^\mathbb{C} \), we can define \( q_i : U_i \to \mathbb{R}^+ \), such that

\[
q_i = e^{-2\pi \varphi_{-\varpi_\alpha} s_i} = \frac{1}{\| \psi_{\varpi_\alpha} s_i \|^2},
\]

for every \( i \in I \). Since \( s_j = s_i \psi_{ij} \) on \( U_i \cap U_j \neq \emptyset \), and \( \psi_{-\varpi_\alpha} = \chi_{-\varpi_\alpha}(p)\psi_{\varpi_\alpha} \), for every \( p \in P \), such that \( \alpha \in \Sigma \setminus \Theta \), the collection of functions \( (q_i)_{i \in I} \) satisfy \( q_i = \chi_{-\varpi_\alpha}^{-1} \circ \psi_{ij}^2 q_i \) on \( U_i \cap U_j \neq \emptyset \). Hence, we obtain a collection of functions \( (q_i)_{i \in I} \) which satisfies on \( U_i \cap U_j \neq \emptyset \) the following relation

\[
q_i = |g_{ij}|^2 q_i,
\]

such that \( g_{ij} = \chi_{-\varpi_\alpha}^{-1} \circ \psi_{ij} \), where \( i, j \in I \). From this, we can define a Hermitian structure \( H \) on \( \mathcal{O}_\alpha(1) \) by taking on each trivialization \( f_i : L_{\chi_{\varpi_\alpha}} \to U_i \times \mathbb{C} \) a metric defined by

\[
H(f_i^{-1}(x, v), f_i^{-1}(x, w)) = q_i(x)\langle v, w \rangle,
\]
for \((x, v), (x, w) \in U_i \times \mathbb{C}\). The Hermitian metric above induces a Chern connection \(\nabla = d + \partial \log H\) with curvature \(F_\nabla\) satisfying (locally)

\[
\frac{-1}{2\pi} F_\nabla \bigg|_{U_i} = \frac{-1}{2\pi} \partial \partial \log \left( ||s_i v_{-\alpha}^\perp||^2 \right).
\] (2.6)

Therefore, by considering the \(G\)-invariant \((1,1)\)-form \(\Omega_\alpha \in \Omega^{1,1}(X_P)^G\), which satisfies \(\pi^*\Omega_\alpha = \frac{-1}{2\pi} \partial \partial \varphi_{-\alpha}\), where \(\pi: G^\mathbb{C} \to G^\mathbb{C}/P = X_P\), and \(\varphi_{-\alpha}(g) = \frac{1}{2\pi} \log ||g v_{-\alpha}^\perp||^2\), \(\forall g \in G^\mathbb{C}\), we have

\[
\Omega_\alpha|_{U_i} = (\pi \circ s_i)^* \Omega_\alpha = \frac{-1}{2\pi} F_\nabla \bigg|_{U_i},
\] (2.7)

i.e., \(c_1(\mathcal{O}_\alpha(1)) = [\Omega_\alpha]\), \(\forall \alpha \in \Sigma \setminus \Theta\). By considering \(\text{Pic}(X_P) = H^1(X_P, \mathcal{O}_{X_P})\), from the ideas described above we have the following result.

**Proposition 2.6.** Let \(X_P\) be a complex flag variety associated to some parabolic Lie subgroup \(P = P_\Theta\). Then, we have

\[\text{Pic}(X_P) = H^1(X_P, \mathbb{Z}) = H^2(X_P, \mathbb{Z}) = \bigoplus_{\alpha \in \Sigma \setminus \Theta} \mathbb{Z}[\Omega_\alpha].\] (2.8)

**Proof.** Let us sketch the proof. The last equality on the right-hand side of Eq. (2.8) follows from the following facts:

(i) \(\pi_2(X_P) \cong \pi_1(T(\Sigma \setminus \Theta)^\mathbb{C}) = \mathbb{Z}^{|\Sigma \setminus \Theta|}\), where \(T(\Sigma \setminus \Theta)^\mathbb{C}\) is given as in Remark 2.5;

(ii) Since \(X_P\) is simply connected, it follows that \(H_2(X_P, \mathbb{Z}) \cong \pi_2(X_P)\) (Hurewicz’s theorem);

(iii) By taking \(P_\alpha^1 \hookrightarrow X_P\), such that

\[P_\alpha^1 = \exp(g_{-\alpha})x_0 \subset X_P,\] (2.9)

for all \(\alpha \in \Sigma \setminus \Theta\), where \(x_0 = eP \in X_P\), it follows that

\[\langle c_1(\mathcal{O}_\alpha(1)), [P_\beta^1] \rangle = \int_{P_\beta^1} c_1(\mathcal{O}_\alpha(1)) = \delta_{\alpha\beta},\] (2.10)

for every \(\alpha, \beta \in \Sigma \setminus \Theta\). Hence, we obtain

\[\pi_2(X_P) = \bigoplus_{\alpha \in \Sigma \setminus \Theta} \mathbb{Z}[P_\alpha^1],\] and \(H^2(X_P, \mathbb{Z}) = \bigoplus_{\alpha \in \Sigma \setminus \Theta} \mathbb{Z}c_1(\mathcal{O}_\alpha(1)).\)

Moreover, from above we also have \(H^{1,1}(X_P, \mathbb{Z}) = H^2(X_P, \mathbb{Z})\). In order to conclude the proof, from the Lefschetz theorem on \((1,1)\)-classes [Huy05], and from the fact that \(\text{rk}(\text{Pic}^0(X_P)) = 0\), we obtain the first equality in Eq. (2.8). \(\square\)

**Remark 2.7** (Harmonic 2-forms on \(X_P\)). Given any \(G\)-invariant Riemannian metric \(g\) on \(X_P\), let us denote by \(\mathscr{H}^2(X_P, g)\) the space of real harmonic 2-forms on \(X_P\) with respect to \(g\), and by \(\mathscr{J}^1_1(X_P)\) the space of closed invariant \((1, 1)\)-forms. Combining the result of Proposition 2.6 with [Tak78, Lemma 3.1], we obtain

\[\mathscr{J}^1_1(X_P) = \mathscr{H}^2(X_P, g).\] (2.11)

Therefore, the closed \(G\)-invariant \((1, 1)\)-forms described in Theorem 2.2 are harmonic with respect to any \(G\)-invariant Riemannian metric on \(X_P\).

**Remark 2.8** (Kähler cone of \(X_P\)). It follows from Eq. (2.8) and Theorem 2.2 that the Kähler cone of a complex flag variety \(X_P\) is given explicitly by

\[\mathcal{K}(X_P) = \bigoplus_{\alpha \in \Sigma \setminus \Theta} \mathbb{R}^+ [\Omega_\alpha].\] (2.12)

**Remark 2.9** (Cone of curves of \(X_P\)). It is worth observing that the cone of curves \(\text{NE}(X_P)\) of a flag variety \(X_P\) is generated by the rational curves \([P_\alpha^1] \in \pi_2(X_P), \alpha \in \Sigma \setminus \Theta\), see for instance [Tim11, §18.3] and references therein.

### 2.2. The first Chern class of flag varieties.

In this subsection, we will review some basic facts related to the Ricci form of \(G\)-invariant Kähler metrics on flag varieties.

Let \(X_P\) be a complex flag variety associated to some parabolic Lie subgroup \(P = P_\Theta \subset G^\mathbb{C}\). By considering the identification \(T^{1,0}_x X_P \cong \mathfrak{m} \subset \mathfrak{g}^\mathbb{C}\), such that

\[\mathfrak{m} = \sum_{\alpha \in H^1(\mathfrak{g}); (\mathfrak{g})^+} \mathfrak{g}_{-\alpha},\]

where \(x_0 = eP \in X_P\), we can realize \(T^{1,0}_x X_P\) as being a holomorphic vector bundle associated to the \(P\)-principal bundle \(P \to G^\mathbb{C} \to X_P\) given by
The twisted product on the right-hand side above is obtained from the isotropy representation \( \text{Ad}: P \to \text{GL}(m) \). From this, a straightforward computation shows us that

\[
K^{-1}_{X_P} = \det \left( T^{1,0}X_P \right) = \det \left( G^C \times_P m \right) = L_{\chi_{s_P}},
\]

where \( \det(\text{Ad}(g)) = \chi_{s_P}^{-1}(g), \forall g \in P \), so \( \det \circ \text{Ad} = \chi_{s_P}^{-1} \). Hence, from the previous results we have

\[
\chi_{s_P} = \prod_{\alpha \in \Sigma \setminus \Theta} \chi^{(\delta_P, h^\alpha_\omega)}_{\Sigma_\alpha} \implies \det \left( T^{1,0}X_P \right) = \otimes_{\alpha \in \Sigma \setminus \Theta} \theta_\alpha(\ell_\alpha),
\]

such that \( \ell_\alpha = \langle \delta_P, h^\alpha_\omega \rangle, \forall \alpha \in \Sigma \setminus \Theta \). In the above computation, notice that

\[
\delta_P = \sum_{\alpha \in \Pi^+ \setminus \Theta^+} \alpha.
\]

If we consider the invariant Kähler metric \( \rho_0 \in \Omega^{-1}(X_P)^G \), locally describe by

\[
\rho_0|_U = \sum_{\alpha \in \Sigma \setminus \Theta} \langle \delta_P, h^\alpha_\omega \rangle \sqrt{-1} \partial \bar{\partial} \log\left( ||s_U v^\alpha_\omega||^2 \right),
\]

for some local section \( s_U: U \subset X_P \to G^C \), it is straightforward to see that

\[
c_1(X_P) = \frac{[\rho_0]}{2\pi}.
\]

By the uniqueness of \( G \)-invariant representative of \( c_1(X_P) \), it follows that

\[
\text{Ric}(\rho_0) = \rho_0,
\]

i.e., \( \rho_0 \in \Omega^{-1}(X_P)^G \) defines a \( G \)-invariant Kähler–Einstein metric on \( X_P \) (cf. [Mat72]).

**Remark 2.10.** From the uniqueness of the \( G \)-invariant representative for \( c_1(X_P) \), given any \( G \)-invariant Kähler metric \( \omega_\varphi \), we have that \( \text{Ric}(\omega_\varphi) = \rho_0 \). Therefore, the scalar curvature \( S(\omega_\varphi) \) of \( \omega_\varphi \) is given by

\[
S(\omega_\varphi) = \Lambda_{\omega_\varphi}(\text{Ric}(\omega_\varphi)) = \text{tr}_{\omega_\varphi}(\rho_0).
\]

Since \( \rho_0 \) is harmonic with respect to any \( G \)-invariant Kähler metric, we have that \( S(\omega_\varphi) \) is constant.

Given any two \( G \)-invariant Kähler metrics \( \omega_1 \) and \( \omega_2 \) on \( X_P \), we have \( \text{Ric}(\omega_1) = \text{Ric}(\omega_2) = \rho_0 \). Thus, it follows that the smooth function \( \frac{\det(\omega_1)}{\det(\omega_2)} \) is constant. Moreover, we have

\[
\text{Vol}(X_P, \omega_1) = \frac{\det(\omega_1)}{\det(\omega_2)} \text{Vol}(X_P, \omega_2).
\]

In particular, denoting \( V_0 = \text{deg}(X_P, -K_{X_P}) = (-K_{X_P})^n \), we can show from above the following result.

**Theorem 2.11** (Azad-Biswas, [AB03]). The volume of \( X_P \) with respect to an arbitrary \( G \)-invariant Kähler metric \( \omega = \sum_{\alpha \in \Sigma \setminus \Theta} c_\alpha \Omega_\alpha \), such that \( c_\alpha > 0 \), \( \forall \alpha \in \Sigma \setminus \Theta \), is given by

\[
\text{Vol}(X_P, \omega) = \frac{V_0}{n!} \prod_{\gamma \in \Pi^+ \setminus \Theta^+} \left[ \sum_{\alpha \in \Sigma \setminus \Theta} c_\alpha \langle \omega_\alpha, h^\gamma_\omega \rangle \right]^{\frac{n}{2}}.
\]

**Remark 2.12.** It is worth pointing out that the expression given in Eq. (2.20) is slightly different from [AB03]. The reason for this is that we consider the volume of \( X_P \) with respect to an arbitrary Kähler metric \( \omega \) as being \( \frac{1}{n!} \int_{X_P} \omega^n \), instead of \( \int_{X_P} \omega^n \). Given a Kähler class \( \xi \in \mathcal{K}(X_P) \), we define the volume of \( \xi \) as being

\[
\text{Vol}(\xi) := n! \text{Vol}(X_P, \omega),
\]

for some \( \omega \in \xi \). Thus, according to our convention, the formula presented in [AB03] for the volume of \( X_P \) with respect to an arbitrary \( G \)-invariant Kähler metric \( \omega \) corresponds to \( \text{Vol}(\omega) \).

### 3. Proof of Main Results

**Theorem 3.1.** Let \( L \in \text{Pic}(X_P) \) and let \( \beta \in c_1(L) \) be a \( G \)-invariant (1,1)-form. Then there exists a (unique) \( G \)-invariant Kähler metric \( \omega \) on \( X_P \), satisfying

\[
\text{Ric}(\omega) = \omega + \beta,
\]

if, and only if,

\[
\int_{X_P} \beta < 2\pi \langle \delta_P, h^\alpha_\omega \rangle,
\]

\( \forall \alpha \in \Sigma \setminus \Theta \), where \( \mathbb{P}^1_\alpha \subset X_P, \alpha \in \Sigma \setminus \Theta \), are generators of the cone of curves \( \text{NE}(X_P) \).
Proof. Given $L \in \text{Pic}(X_P)$, it follows that

$$L = \bigotimes_{\alpha \in \Sigma \setminus \Theta} \mathcal{O}_\alpha(\ell_\alpha),$$

(3.3)
such that $\ell_\alpha \in \mathbb{Z}$, $\forall \alpha \in \Sigma \setminus \Theta$. If $\omega$ is a $G$-invariant Kähler metric on $X_P$ satisfying Eq. (3.1), it follows that

$$2\pi c_1(X_P) = [\omega] + c_1(L).$$

(3.4)
Hence, we have

$$[\omega] = \sum_{\alpha \in \Sigma \setminus \Theta} (2\pi \langle \delta_p, h_\alpha^\vee \rangle - \ell_\alpha) c_1(\mathcal{O}_\alpha(1)).$$

(3.5)
Since $\omega$ is a positive real $(1,1)$-form, it follows that $2\pi \langle \delta_p, h_\alpha^\vee \rangle - \ell_\alpha > 0$, $\forall \alpha \in \Sigma \setminus \Theta$. Thus, we conclude that

$$\int_{P_\alpha^1} \beta = \ell_\alpha < 2\pi \langle \delta_p, h_\alpha^\vee \rangle,$$

(3.6)
for all $\alpha \in \Sigma \setminus \Theta$. On the other hand, given a $G$-invariant $(1,1)$-form $\beta \in c_1(L)$, such that $\int_{P_\alpha^1} \beta < 2\pi \langle \delta_p, h_\alpha^\vee \rangle$, $\forall \alpha \in \Sigma \setminus \Theta$, we set

$$\omega := \sum_{\alpha \in \Sigma \setminus \Theta} \left(2\pi \langle \delta_p, h_\alpha^\vee \rangle - \int_{P_\alpha^1} \beta\right) \Theta.$$

(3.7)
By the above definition, we have that $\omega$ defines a $G$-invariant Kähler metric on $X_P$. Moreover, it is straightforward to verify that $\text{Ric}(\omega) = \omega + \beta$. The uniqueness of $\omega$ follows from the fact that it is $G$-invariant. \hfill \square

In the setting of the theorem above, if we replace $c_1(L)$ by $2\pi \left(c_1(X_P) - \xi\right)$, for some Kähler class $\xi$ on $X_P$, observing that every $\beta \in 2\pi \left(c_1(X_P) - \xi\right)$ satisfies

$$\int_{P_\alpha^1} \beta = 2\pi \langle c_1(X_P) - \xi, P_\alpha^1 \rangle < 2\pi \int_{P_\alpha^1} c_1(X_P) = 2\pi \langle \delta_p, h_\alpha^\vee \rangle,$$

(3.8)
for all $\alpha \in \Sigma \setminus \Theta$, from Theorem 3.1 we obtain the following corollary.

**Corollary 3.2.** Given a Kähler class $\xi$ on $X_P$ and a $G$-invariant $(1,1)$-form $\beta \in 2\pi \left(c_1(X_P) - \xi\right)$, then there exist a unique $G$-invariant Kähler metric $\omega \in 2\pi \xi$, such that

$$\text{Ric}(\omega) = \omega + \beta.$$  

(3.9)
By taking the trace with respect to $\omega$ in Eq. (3.9), we obtain the following result.

**Corollary 3.3.** Given a Kähler class $\xi$ on $X_P$ and a $G$-invariant $(1,1)$-form $\beta \in 2\pi \left(c_1(X_P) - \xi\right)$, then there exists a (unique) $G$-invariant Kähler metric $\omega \in 2\pi \xi$ with constant $\beta$-twisted scalar curvature, such that

$$S(\omega) - \Lambda_\omega(\beta) = \dim_G(X_P),$$

(3.10)
where $S(\omega)$ denotes the Chern scalar curvature of $\omega$.

From Corollary 3.2 we can show the following.

**Corollary 3.4.** Let $R : \mathcal{K}(X_P) \to \mathbb{R}$, such that $R(\xi)$ is the greatest Ricci lower bound of $\xi \in \mathcal{K}(X_P)$. Then, we have

$$R(\xi) = \min \left\{ \frac{\langle \delta_p, h_\alpha^\vee \rangle}{a_\alpha} \right\}$$

(3.11)
such that $a_\alpha = \langle \xi, [P_\alpha^1] \rangle$, $\forall \alpha \in \Sigma \setminus \Theta$.

**Proof.** Given $\xi \in \mathcal{K}(X_P)$, we have $\xi = \sum_{\alpha \in \Sigma \setminus \Theta} a_\alpha [\Omega_\alpha]$, with $a_\alpha > 0$, $\forall \alpha \in \Sigma \setminus \Theta$. Let us denote

$$R_0 = \min \left\{ \frac{\langle \delta_p, h_\alpha^\vee \rangle}{a_\alpha} \right\},$$

For any $\epsilon > 0$, we can always choose $s > 0$, such that $R_0 - \epsilon < s \leq R_0$. By taking a $G$-invariant $(1,1)$-form $\beta \in 2\pi \left(c_1(X_P) - s\xi\right)$, it follows from Corollary 3.2 that there exist a unique $G$-invariant tKE metric $\omega \in 2\pi s\xi$. From this, we have

$$\text{Ric}(\omega) = \omega + \beta = s\left(\frac{\omega}{s}\right) + \beta \Rightarrow \text{Ric}(\omega) - s\left(\frac{\omega}{s}\right) = \beta,$$

Since $s \leq R_0 \Rightarrow a_\alpha s \leq \langle \delta_p, h_\alpha^\vee \rangle$, $\forall \alpha \in \Sigma \setminus \Theta$, we have

$$c_1(X_P) - s\xi = \sum_{\alpha \in \Sigma \setminus \Theta} (\langle \delta_p, h_\alpha^\vee \rangle - s a_\alpha)[\Omega_\alpha] \geq 0,$$
thus \( \text{Ric}(\omega) - s(\omega) = \beta \geq 0 \), with \( \frac{s}{\beta} \in 2\pi \xi \). From this, we obtain
\[
s \in \{ r \in \mathbb{R} \mid \exists \text{ Kähler form } \omega \in 2\pi \xi, \text{ s.t. } \text{Ric}(\omega) \geq r\omega \}.
\]
Therefore, by definition of \( R(\xi) \), we conclude that \( R_0 = R(\xi) \).

From the description provided above for the greatest Ricci lower bound, we can prove the following result using essentially tools from Lie theory.

**Theorem 3.5.** For every \( \xi \in \mathcal{K}(X_P) \), the following inequalities hold
\[
R(\xi)^n \text{Vol}(\xi) \leq (-K_{X_P})^n \leq (n + 1)^n,
\]
such that \( n = \dim_\mathbb{C}(X_P) \).

**Proof.** Given \( \xi \in \mathcal{K}(X_P) \), it follows that \( \xi = \sum_{\alpha \in \Sigma \setminus \Theta} a_\alpha [\Omega_\alpha] \), with \( a_\alpha > 0 \), \( \forall \alpha \in \Sigma \setminus \Theta \). By considering \( \omega = \sum_{\alpha \in \Sigma \setminus \Theta} a_\alpha \Omega_\alpha \), we obtain from Theorem 2.20 the following
\[
\text{Vol}(\xi) = n! \text{Vol}(X_P, \omega) = V_0 \frac{\prod_{\gamma \in \Pi^+ \setminus \{\Theta\}^+} \left[ \sum_{\alpha \in \Sigma \setminus \Theta} a_\alpha \langle \omega_\alpha, h_\gamma^\prime \rangle \right]}{\prod_{\gamma \in \Pi^+ \setminus \{\Theta\}^+} \left[ \sum_{\alpha \in \Sigma \setminus \Theta} \langle \delta_\alpha, h_\gamma^\prime \rangle \langle \omega_\alpha, h_\gamma^\prime \rangle \right]}.
\]
Since \( R(\xi) \leq \frac{\langle \delta_\alpha, h_\gamma^\prime \rangle}{a_\alpha} \), \( \forall \alpha \in \Sigma \setminus \Theta \), it follows that \( a_\alpha R(\xi) \leq \langle \delta_\alpha, h_\gamma^\prime \rangle \), \( \forall \alpha \in \Sigma \setminus \Theta \). Therefore, we obtain
\[
R(\xi)^n \text{Vol}(\xi) = V_0 \frac{\prod_{\gamma \in \Pi^+ \setminus \{\Theta\}^+} \left[ \sum_{\alpha \in \Sigma \setminus \Theta} a_\alpha R(\xi) \langle \omega_\alpha, h_\gamma^\prime \rangle \right]}{\prod_{\gamma \in \Pi^+ \setminus \{\Theta\}^+} \left[ \sum_{\alpha \in \Sigma \setminus \Theta} \langle \delta_\alpha, h_\gamma^\prime \rangle \langle \omega_\alpha, h_\gamma^\prime \rangle \right]} \leq V_0 = \text{deg}(X_P, -K_{X_P}),
\]
notice that \( n = \dim_\mathbb{C}(X_P) = |\Pi^+ \setminus \{\Theta\}^+| \). Following [BH59, Theorem 24.10] and [Tm11, Example 18.13], we have
\[
\text{deg}(X_P, -K_{X_P}) = (-K_{X_P})^n = \int_{X_P} c_1(X_P)^n = n! \prod_{\gamma \in \Pi^+ \setminus \{\Theta\}^+} \langle \delta_\alpha, h_\gamma^\prime \rangle \langle \omega_\alpha, h_\gamma^\prime \rangle.
\]
Therefore, from the result provided in [Sno04, Theorem 1] we obtain the desired inequalities. \( \square \)

### 4. Examples

In order to compute the numbers \( \langle \delta_\alpha, h_\gamma^\prime \rangle \), \( \alpha \in \Sigma \setminus \Theta \), which appear in the formulas of our results, it is worth recalling that \( \delta_\alpha = c_1 \omega_\alpha + \ldots + c_n \omega_\alpha \), where \( c_i > 0 \) and \( \alpha_i \in \Sigma \setminus \Theta \) (cf. Borel-Hirzebruch [BH58]). The numbers \( c_i \) are called Koszul numbers, these numbers characterize the Kähler-Einstein metrics on flag varieties (see [Kim09], [AC10], [ACS13]). In what follows, we present some examples which illustrate how the existence of \( G \)-invariant tKE metrics can be characterized using Koszul numbers.

#### 4.1. Full flag variety \( X_B = G^\mathbb{C}/B \).

In the setting of Theorem 3.1, in the particular case that \( P = B \) (i.e. \( \Theta = \emptyset \)), it follows that \( \delta_B = 2g^+, \) where
\[
g^+ = \frac{1}{2} \sum_{\alpha \in \Pi^+} \alpha = \sum_{\alpha \in \Sigma} \omega_\alpha.
\]
Therefore, considering a \( G \)-invariant \((1,1)\)-form \( \beta = c_1(L) \), for some \( L \in \text{Pic}(X_B) \), from Theorem 3.1 we have that the necessary and sufficient condition over \( \beta \) for the existence of a \( G \)-invariant Kähler metric \( \omega \) on \( X_B \), satisfying \( \text{Ric}(\omega) = \omega + \beta \), is given by
\[
\int_{X_B} \beta < 2\pi \langle \delta_B, h_\alpha^\prime \rangle = 2\pi \langle 2g^+, h_\alpha^\prime \rangle = 4\pi \sum_{\gamma \in \Sigma} \omega_\gamma \langle h_\alpha^\prime, h_\gamma^\prime \rangle = 4\pi, \quad \forall \alpha \in \Sigma.
\]

#### 4.2. Projetivization of \( TP^{n+1} \).

Let us consider the flag variety \( X_P = \text{SU}(n + 2)/\text{SU}(n + 1) \times \text{SU}(1) \). This space can be viewed as the projetivization of the tangent bundle of \( P^{n+1} \), so let us denote \( X_P = P(T^n P^{n+1}) \), see [Hir05]. Let \( g^G \) be the Lie algebra \( \mathfrak{sl}(n + 2, \mathbb{C}) \) with Cartan subalgebra given by the diagonal traceless matrices. Recall that the roots of the Lie algebra \( g^G = \mathfrak{sl}(n + 2, \mathbb{C}) \) is given by the functionals \( \alpha_{ij} = \lambda_i - \lambda_j \), where \( \lambda_i(\text{Diag}(a_1, \ldots, a_{n+2})) = a_i, i = 1, \ldots, n + 2 \). The set of simple roots \( \Sigma \) is given \( \alpha_i = \lambda_i, i = 1, \ldots, n + 1 \). The space \( P(T^n P^{n+1}) \) is characterized by \( P = P_\Theta \), such that \( \Sigma \setminus \Theta = \{ \alpha_{n+1}, \alpha_{n+2} \} \). In particular, we have rank \( H^2(P(T^n P^{n+1}), \mathbb{Z}) = 2 \). A straightforward computation gives in this case
\[
\langle \delta_P, h_\alpha^\prime \rangle = \langle \delta_P, h_\alpha^\prime \rangle = n + 1, \quad \langle \delta_P, h_\alpha^\prime \rangle = 2.
\]
Therefore, considering a $SU(n+2)$-invariant $(1,1)$-form $\beta \in c_1(L)$, for some $L \in \text{Pic}(\mathbb{P}(T\mathbb{P}^{n+1}))$, then there exists a $SU(n+2)$-invariant Kähler metric $\omega$ on $\mathbb{P}(T\mathbb{P}^{n+1})$, such that $\text{Ric}(\omega) = \omega + \beta$ if, and only if,

$$
\int_{\mathbb{P}^{n+1}_{\alpha_{n,n+1}}} \beta < 2\pi \langle \delta_p, h^\vee_{\alpha_{n,n+1}} \rangle = 2\pi(n+1) \quad \text{and} \quad \int_{\mathbb{P}^{n+1}_{\alpha_{n+1,n+2}}} \beta < 2\pi \langle \delta_p, h^\vee_{\alpha_{n+1,n+2}} \rangle = 4\pi. \quad (4.4)
$$

4.3. Flag varieties with Picard number two and few isotropy summands. According to Proposition 2.6, flag varieties with Picard number 2 are parameterized by $\Theta \subset \Sigma$, such that $\Sigma \setminus \Theta = \{\alpha, \gamma\}$. In what follows, we will characterize tKE metrics for some families of such spaces parameterized in terms of the number of components of the isotropy representation as follows (see [Kim90], [AC10], [ACS13]): let $\mu = n_1 \alpha_1 + \cdots + n_m \alpha_m$ be the maximal root of II and recall the height of a simple root $\alpha_i$ with respect to $\mu$ is the positive number $ht(\alpha_i) = n_i$. In this setting, we have the following classification:

1. **Type I**: flags with three isotropy summands: $\Sigma \setminus \Theta = \{\alpha, \gamma : ht(\alpha) = ht(\gamma) = 1\}$
2. **Type II**: flags with four isotropy summands: $\Sigma \setminus \Theta = \{\alpha, \gamma : ht(\alpha) = ht(\gamma) = 1, ht(\gamma) = 2\}$
3. **Type III**: flags with five isotropy summands: $\Sigma \setminus \Theta = \{\alpha, \gamma : ht(\alpha) = 1, ht(\gamma) = 2 \text{ or } ht(\alpha) = 2, ht(\gamma) = 2\}$

The next proposition summarizes the classification of flag varieties of type I, II, III.

**Proposition 4.1** ([AC10],[ACS13],[Kim90]). The flag varieties of type I, II and III (up to equivalence) and the corresponding Koszul numbers are listed in the Table 1. The black dots on the Dynkin diagram represent the simple roots on $\Sigma \setminus \Theta$ and the labels on the Dynkin diagram denotes the height of the simple root with respect to the maximal root $\mu$.

| Type | $X_P = G/G \cap P$ | $\Sigma \setminus \Theta = \{\alpha_1, \alpha_2\}$ | $\delta_p = \ell_1 \omega_{\alpha_1} + \ell_2 \omega_{\alpha_2}$ |
|------|-------------------|-------------------|-------------------|
| I    | $SO(2\ell)/U(1) \times U(\ell - 1)$ \((\ell \geq 4)\) | $\bullet$ 2 2 2 2 2 2 1 | $\ell \omega_{\alpha_1} + \ell \omega_{\alpha_2}$ |
| I    | $SO(2\ell)/U(1) \times U(\ell - 1)$ \((\ell \geq 4)\) | $\bullet$ 2 2 2 2 2 1 | $\ell \omega_{\alpha_1} + 2(\ell - 2) \omega_{\alpha_2}$ |
| I    | $SO(2\ell)/U(1) \times U(\ell - 1)$ \((\ell \geq 4)\) | $\bullet$ 2 2 2 2 2 1 | $\ell \omega_{\alpha_1} + 2(\ell - 2) \omega_{\alpha_2}$ |
| I    | $SU(\ell + m)/SU(U(\ell) \times U(m) \times U(n))$ \((\ell, m, n \geq 1)\) | $\bullet$ 2 2 2 2 2 1 1 1 1 | $(\ell + m) \omega_{\alpha_1} + (m + n) \omega_{\alpha_2}$ |
| I    | $E_6/U(1) \times U(1) \times \text{Spin}(8)$ | $\bullet$ 2 2 2 2 2 2 1 | $4 \omega_{\alpha_1} + 4 \omega_{\alpha_2}$ |
| II   | $SO(2\ell + 1)/SO(2\ell - 3) \times U(1) \times U(1)$ | $\bullet$ 2 2 2 2 2 2 1 | $2 \omega_{\alpha_1} + (2\ell - 3) \omega_{\alpha_2}$ |
| II   | $Sp(\ell)/U(p) \times U(\ell - p)$ \((1 \leq p \leq \ell - 1)\) | $\bullet$ 2 2 2 2 2 2 2 2 2 2 2 2 2 2 | $\ell \omega_{\alpha_1} + (\ell - p + 1) \omega_{\alpha_2}$ |
| II   | $SO(2\ell)/SO(2\ell - 2) \times U(1) \times U(1)$ | $\bullet$ 2 2 2 2 2 2 2 2 2 2 2 2 2 2 | $2\ell \omega_{\alpha_1} + 2(\ell - 2) \omega_{\alpha_2}$ |
| II   | $SO(2\ell)/U(p) \times U(\ell - p)$ \((2 \leq p \leq \ell - 2)\) | $\bullet$ 2 2 2 2 2 2 2 2 2 2 2 2 2 2 | $\ell \omega_{\alpha_1} + 2(\ell - p - 1) \omega_{\alpha_2}$ |
| II   | $E_6/SU(5) \times U(1) \times U(1)$ | $\bullet$ 2 2 2 2 2 2 2 2 2 2 2 2 2 2 | $2 \omega_{\alpha_1} + 8 \omega_{\alpha_2}$ |
| II   | $E_7/SO(10) \times U(1) \times U(1)$ | $\bullet$ 2 2 2 2 2 2 2 2 2 2 2 2 2 2 | $2 \omega_{\alpha_1} + 12 \omega_{\alpha_2}$ |
| III  | $SO(2\ell + 1)/U(1) \times U(p) \times SO(2(\ell - p - 1) + 1)$ \((\ell \geq 5, 3 \geq p \geq \ell - 3)\) | $\bullet$ 2 2 2 2 2 2 2 2 2 2 2 2 2 2 | $(p + 1) \omega_{\alpha_1} + (2\ell - p - 2) \omega_{\alpha_2}$ |

Applying Theorem 3.1 joint with Proposition 4.1, one can characterize and classify tKE metrics on these families of flag varieties as follows.
Corollary 4.2. Let us consider a flag variety $X_P$ of type I, II, III, and $\delta_P = \ell_1 \omega_{\alpha_1} + \ell_2 \omega_{\alpha_2}$ as listed on Table 1. Let $L \in \text{Pic}(X_P)$ and let $\beta \in c_1(L)$ be a $G$-invariant (1,1)-form, then there exist a (unique) $G$-invariant Kähler metric $\omega$ on $X_P$, such that $\text{Ric}(\omega) = \omega + \beta$ if, and only if,
\[
 n_1 := \int_{\mathbb{P}_1^{\ell_1}} \beta < 2\pi \ell_1, \quad \text{and} \quad n_2 := \int_{\mathbb{P}_2^{\ell_2}} \beta < 2\pi \ell_2. \tag{4.5}
\]
In this case, the Kähler metric is given by $\omega = (2\pi \ell_1 - n_1)\Omega_{\alpha_1} + (2\pi \ell_2 - n_2)\Omega_{\alpha_2}$, where the forms $\Omega_{\alpha_i}, i = 1, 2$, are the generators of $H^2(X_P, \mathbb{Z})$ given in Eq. (2.8).

Acknowledgment: E. M. Correa is supported by FAEP/Unicamp grant 2528/22. L. Grama is partially supported by São Paulo Research Foundation FAPESP grants 2018/13481-0, 2021/04003-0, 2021/04065-6 and CNPq grant no. 305036/2019-0.

References

[AB03] Hassan Azad and Indranil Biswas. Quasi-potentials and Kähler-Einstein metrics on flag manifolds. II. J. Algebra, 269(2):480–491, 2003.

[AC10] Andreas Arvanitoyeorgos and Ioannis Chrysikos. Invariant Einstein metrics on flag manifolds with four isotropy summands. Ann. Global Anal. Geom., 37(2):185–219, 2010.

[ACS13] Andreas Arvanitoyeorgos, Ioannis Chrysikos, and Yusuke Sakane. Homogeneous Einstein metrics on generalized flag manifolds. Internat. J. Math., 24(10):1350077, 2013.

[AB03] A. Borel and F. Hirzebruch. Characteristic classes and homogeneous spaces. II.

[BB17] Robert J. Berman and Bo Berndtsson. Convexity of the $K$-energy on the space of Kähler metrics and uniqueness of extremal metrics. J. Amer. Math. Soc., 30(4):1165–1196, 2017.

[BB21] Robert J. Berman, Sławomir Dinovici, and Mattias Jonsson. A variational approach to the Yau-Tian-Donaldson conjecture. J. Amer. Math. Soc., 34(3):605–652, 2021.

[BH58] A. Borel and F. Hirzebruch. Characteristic classes and homogeneous spaces. I. Amer. J. Math., 80:458–538, 1958.

[BH59] A. Borel and F. Hirzebruch. Characteristic classes and homogeneous spaces. II. Amer. J. Math., 81:315–382, 1959.

[BJ20] Harold Blum and Mattias Jonsson. A non-archimedean approach to k-stability, 2018.

[BJ18] Sébastien Boucksom and Mattias Jonsson. A non-archimedean approach to k-stability, 2018.

[BLZ19] Harold Blum, Yuchen Liu, and Chuyu Zhou. Optimal destabilization of k-unstable fano varieties via stability thresholds, 2019.

[BR62] A. Borel and R. Remmert. Über kompakte homogene Kählersche Mannigfaltigkeiten. Math. Ann., 145:429–439, 1961/62.

[CG19] Eder M. Correa and Lino Grama. Calabi-Yau metrics on canonical bundles of complex flag manifolds. J. Algebra, 527:109–135, 2019.

[Cor19] Eder M. Correa. Homogeneous contact manifolds and resolutions of Calabi-Yau cones. Comm. Math. Phys., 367(3):1095–1151, 2019.

[CS16] Tristan C. Collins and Gábor Székelyhidi. The twisted Kähler-Ricci flow. J. Reine Angew. Math., 716:179–205, 2016.

[Fin04] Joel Fine. Constant scalar curvature Kähler metrics on fibred complex surfaces. J. Differential Geom., 68(3):397–432, 2004.

[Fin07] Joel Fine. Fibrations with constant scalar curvature Kähler metrics and the CM-line bundle. Math. Res. Lett., 14(2):239–247, 2007.

[FO18] Kento Fujita and Yuji Odaka. On the $K$-stability of Fano varieties and anticanonical divisors. Tohoku Math. J. (2), 70(4):511–521, 2018.

[Fuj18] Kento Fujita. Optimal bounds for the volumes of Kähler-Einstein Fano manifolds. Amer. J. Math., 140(2):391–414, 2018.

[Hir05] Friedrich Hirzebruch. The projective tangent bundles of a complex three-fold. Pure Appl. Math. Q., 1(3, Special Issue: In memory of Armand Borel, Part 2):441–448, 2005.

[Hum72] James E. Humphreys. Introduction to Lie algebras and representation theory. Graduate Texts in Mathematics, Vol. 9. Springer-Verlag, New York-Berlin, 1972.

[Hum75] James E. Humphreys. Linear algebraic groups. Graduate Texts in Mathematics, No. 21. Springer-Verlag, New York-Heidelberg, 1975.

[Huy05] Daniel Huybrechts. Complex geometry. Universitext. Springer-Verlag, Berlin, 2005. An introduction.

[Kim90] Masahiro Kimura. Homogeneous Einstein metrics on certain Kähler $C$-spaces. In Recent topics in differential and analytic geometry, volume 18 of Adv. Stud. Pure Math., pages 303–320. Academic Press, Boston, MA, 1990.

[LB18] V. Lakshmibai and Justin Brown. Flag varieties, volume 53 of Texts and Readings in Mathematics. Hindustan Book Agency, Delhi, 2018. An interplay of geometry, combinatorics, and representation theory, Second edition of [ MR2474907].

[Li11] Chi Li. Greatest lower bounds on Ricci curvature for toric Fano manifolds. Adv. Math., 226(6):4921–4932, 2011.

[Liu13] Chao Li. The generalized Kähler Ricci flow. J. Math. Anal. Appl., 408(2):751–761, 2013.

[LW15] Jiawei Liu and Yue Wang. Convergence of the generalized Kähler-Ricci flow. Commun. Contemp. Math., 17(2):1550007, 2015.

[Mats72] Yozo Matsushima. Remarks on Kähler-Einstein manifolds. Nagoya Math. J., 46:161–173, 1972.

[Rub08] Yanir A. Rubinstein. Some discretizations of geometric evolution equations on the Ricci iteration on the space of Kähler metrics. Adv. Math., 218(5):1526–1565, 2008.

[Rub09] Yanir A. Rubinstein. On the construction of Nadel multiplier ideal sheaves and the limiting behavior of the Ricci flow. Trans. Amer. Math. Soc., 361(11):5839–5850, 2009.
[Sno04] Dennis Snow. Bounds for the anticanonical bundle of a homogeneous projective rational manifold. *Doc. Math.*, 9:251–263, 2004.

[ST07] Jian Song and Gang Tian. The Kähler-Ricci flow on surfaces of positive Kodaira dimension. *Invent. Math.*, 170(3):609–653, 2007.

[ST12] Jian Song and Gang Tian. Canonical measures and Kähler-Ricci flow. *J. Amer. Math. Soc.*, 25(2):303–353, 2012.

[ST17] Jian Song and Gang Tian. The Kähler-Ricci flow through singularities. *Invent. Math.*, 207(2):519–595, 2017.

[SW16] Jian Song and Xiaowei Wang. The greatest Ricci lower bound, conical Einstein metrics and Chern number inequality. *Geom. Topol.*, 20(1):49–102, 2016.

[Szfrm[o]–1] Gábor Székelyhidi. Greatest lower bounds on the Ricci curvature of Fano manifolds. *Compos. Math.*, 147(1):319–331, 2011.

[Tak78] Masaru Takeuchi. Homogeneous Kähler submanifolds in complex projective spaces. *Japan. J. Math. (N.S.*), 4(1):171–219, 1978.

[Tia92] Gang Tian. On stability of the tangent bundles of Fano varieties. *Internat. J. Math.*, 3(3):401–413, 1992.

[Tim11] Dmitry A. Timashev. *Homogeneous spaces and equivariant embeddings*, volume 138 of *Encyclopaedia of Mathematical Sciences*. Springer, Heidelberg, 2011. Invariant Theory and Algebraic Transformation Groups, 8.

[Zha22] Kewei Zhang. On the optimal volume upper bound for Kähler manifolds with positive Ricci curvature (with an appendix by Yuchen Liu). *Int. Math. Res. Not. IMRN*, (8):6135–6156, 2022.

University of Campinas (UNICAMP), Institute of Mathematics, Statistics and Scientific Computing (IMECC), Campinas, Brazil.

Email address: ederc@unicamp.br, lgrama@unicamp.br