On the Distribution of Explosion Time of Stochastic Differential Equations

Jorge A. León, Liliana Peralta Hernández
Departamento de Control Automático
Cinvestav-IPN
Apartado postal 14-740
07000 México D.F., Mexico
and
José Villa-Morales
Departamento de Matemáticas y Física
Universidad Autónoma de Aguascalientes
Avenida Universidad 940, Ciudad Universitaria
20131 Aguascalientes, Ags., Mexico

Abstract

In this paper we use the Itô's formula and comparison theorems to study the blow-up in finite time of stochastic differential equations driven by a Brownian motion. In particular, we obtain an extension of Osgood criterion, which can be applied to some nonautonomous stochastic differential equations with additive Wiener integral noise. In most cases we are able to provide with a method to figure out the distribution of the explosion time of the involved equation.

Keywords: Iterated logarithm theorem for martingales, Itô’s formula, comparison theorems for integral and stochastic differential equations, Osgood criterion, partial differential equations of second order, time of explosion.

AMS MSC 2010: Primary 45R05, 60H10; Secondary 49K20.

1 Introduction

Consider the stochastic differential equation

\[ dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad t > 0, \]
\[ X_0 = x_0. \]  (1)

Here \( b, \sigma : \mathbb{R} \to \mathbb{R} \) are two locally Lipschitz functions, \( x_0 \in \mathbb{R} \) and \( \{W_t : t \geq 0\} \) is a Brownian motion defined on a complete probability space \( (\Omega, \mathcal{F}, P) \).
It is well-known that the solution $X$ of equation (1) may explode in finite time. That is, $|X_t|$ goes to infinite as $t$ approaches to a stopping time that could be finite with positive probability, which is called the explosion time of equation (1) (see McKean [12]). The Feller test is an important tool of the stochastic calculus to know if there is blow-up in finite time for (1) (see, for example, Karatzas and Shreve [10]). The reader can consult de Pablo et al. [5] (and references therein) for applications of blow-up.

In the case that $b$ is non-decreasing and positive, and $\sigma \equiv 1$, Feller test is equivalent to Osgood criterion [14], as it is proven in León and Villa [11]. It means, the solution of (1) explodes in finite time if and only if $\int_{x_0}^{\infty} (1/b(s))ds < \infty$. Also, when $\sigma \equiv 0$ and $b > 0$, Osgood [14] has stated that explosion time is finite if and only if $\int_{x_0}^{\infty} (1/b(s))ds < \infty$. In this case, the explosion time is equals to this integral.

Unfortunately, the distribution of the explosion time of equation (1) is not easy to calculate. One way to do it is using linear second-order ordinary differential equations. Indeed, Feller [7] has pointed out the Laplace transformation of this distribution is a bounded solution to some related ordinary differential equations (see Section 5.2 below for a generalization of this result). Also some numerical schemes have been analyzed in order to approximate the time of explosion (consult Dávila et al. [4]). In this paper, in Section 5.1, we also obtain the partial differential equation that has the distribution of the explosion time as a bounded solution.

Now consider the nonautonomous stochastic differential equation

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dW_t, \quad t > 0,$$

$$X_0 = x_0.$$

For this equation, Feller test and Osgood criterion are not useful anymore, but, in the case that $\sigma$ is independent of $x$, we are still able to associate the Laplace transformation of the distribution of the explosion time of (2) with a partial differential equation as Theorem [22] below establishes.

The main purpose of this paper is to deal with some extensions of Osgood criterion for some equations of the form (2). For instance, Lemma [7] provides a better understanding of Theorem 2.1 in [3], or if, in (2), $\sigma$ is independent of $x$, we obtain an extension of Osgood criterion by means of the law of iterated logarithm and comparison theorems. It is worth mentioning that versions of these important tools have been used to analyze global solutions of integral equations as it is done by Constantin [3], or to obtain an extension of Osgood criterion to integral equations with additive noise and with $0 < b(t, x) = b(x)$ non-decreasing (see León and Villa [11]).

The paper is organized as follows. Our comparison theorem for integral equations is introduced in Section 3. Some extensions of Osgood criterion are given in Sections 2, 3 and 4. Finally, the relation between partial differential equations and finite blow-up is considered in Section 5.
2 Osgood criterion for some stochastic differential equation with diffusion coefficient

Let $\sigma : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be a differentiable function and a continuous function, respectively. We consider the stochastic differential equation

$$X_t^\xi = \xi + \frac{1}{2} \int_0^t \sigma(X_s^\xi)\sigma'(X_s^\xi)h^2(s)ds + \int_0^t \sigma(X_s^\xi)h(s)dW_s, \quad t \geq 0,$$

where $\xi \in \mathbb{R}$. Here and in what follows, $W = \{W_t : t \geq 0\}$ is a Brownian motion.

Now we assume that there are $-\infty \leq x_1 < x_2 \leq \infty$ such that $\sigma \neq 0$ on $(x_1, x_2)$. Let $\xi \in (x_1, x_2)$ be fixed and define $\Psi_\xi : (x_1, x_2) \to \mathbb{R}$ as

$$\Psi_\xi(x) = \int_x^\xi \frac{dz}{\sigma(z)}.
$$

Set $l_\xi = \Psi_\xi(x_1) \wedge \Psi_\xi(x_2)$, $r_\xi = \Psi_\xi(x_1) \vee \Psi_\xi(x_2)$ and $Y_t = \int_0^t h(s)dW_s$, $t \geq 0$.

The following result is our first extension of Osgood criterion.

**Theorem 1** Let $\tau_\xi = \inf\{t \geq 0 : Y_t \not\in (l_\xi, r_\xi)\}$. Then, the process $X_t^\xi = \{\Psi_\xi^{-1}(Y_t) : 0 \leq t < \tau_\xi\}$ is a solution of equation (3).

**Remark 2** In this case, $\tau_\xi$ is called the explosion time of the solution to equation (3).

**Proof.** Applying Itô’s formula with $f(x) = \Psi_\xi^{-1}(x)$, $x \in (l_\xi, r_\xi)$ we have

$$f(Y_{t\wedge \tau_\xi}^\xi) - f(0) = \frac{1}{2} \int_0^{t\wedge \tau_\xi} f''(Y_s)h^2(s)ds + \int_0^{t\wedge \tau_\xi} f'(Y_s)h(s)dW_s,$$

where

$$\tau_\xi^k = \inf\{t > 0 : Y_t \not\in (l_\xi + k^{-1}, r_\xi - k^{-1})\}.$$

Letting $k \to \infty$ in (2) we get the result holds.

An immediate consequence of Theorem 1 is the following:

**Corollary 3** Let $\int_0^\infty h^2(s)ds = \infty$. Then the solution of equation (3) explodes in finite time if and only if either $l_\xi > -\infty$, or $r_\xi < \infty$. Moreover, if $l_\xi$ and $r_\xi$ are two real numbers, then

$$P(\tau_\xi \in dt) = \sum_{k=-\infty}^{\infty} (-1)^k \frac{r_\xi + k(r_\xi - l_\xi)}{\sqrt{2\pi(H(t))^{3/2}}} \exp \left( -\frac{(r_\xi + k(r_\xi - l_\xi))^2}{2H(t)} \right) dt,$$

with $H(t) = \int_0^t (h(s))^2ds$. 

3
Proof. It is well-known that there is a Brownian motion \( B = \{ B_t : t \geq 0 \} \) such that \( Y_t = B_{H(t)} \), \( t \geq 0 \), (see, for instance, Durrett [6]). Let \( \tilde{\tau}_\xi = \inf \{ t > 0 : B_t \notin (l_\xi, r_\xi) \} \). Then, it is easy to show that \( P(\tau_\xi \leq t) = P(\tilde{\tau}_\xi \leq H(t)) \). Consequently, the proof follows from Borodin and Salminen [1] (page 212).

Remark 4 Suppose that, for example, \( \sigma > 0 \), \( \Psi_\xi(x_1) = -\infty \) and \( \Psi_\xi(x_2) < \infty \). Then, as an immediate consequence of the proof of Corollary 3, we get that

\[
\tau_\xi = \inf \{ t : \int_0^t h(s) dW_s = \Psi_\xi(x_2) \}
\]

and

\[
P(\tau_\xi \leq t) = \Phi \left( \frac{\Psi_\xi(x_2)}{\sqrt{H(t)}} \right), \tag{4}
\]

where

\[
\Phi(x) = \frac{2}{\sqrt{2\pi}} \int_x^\infty e^{-z^2/2} dz.
\]

Observe that we get a similar result when \( \sigma \) is negative, or the involved interval has the form \((l_\xi, \infty)\).

Now we illustrate this remark with two examples.

Example 5 Let \( \sigma(x) = |x|^\alpha \), \( x \in \mathbb{R} \), \( \alpha > 1 \) and \( \xi \in \mathbb{R} \). Then

\[
\Psi_\xi(x) = \begin{cases} 
\frac{1}{1-\alpha}(|x|^{1-\alpha} - |\xi|^{1-\alpha}), & \xi > 0, \ x \geq 0, \\
\frac{1}{1-\alpha}(|\xi|^{1-\alpha} - |x|^{1-\alpha}), & \xi < 0, \ x \leq 0.
\end{cases}
\]

Hence,

\[
\Psi_\xi(-\infty) = \frac{|\xi|^{1-\alpha}}{1-\alpha} \quad \text{and} \quad \Psi_\xi(0) = \infty, \quad \text{for} \ \xi < 0,
\]

and

\[
\Psi_\xi(\infty) = \frac{|\xi|^{1-\alpha}}{\alpha - 1} \quad \text{and} \quad \Psi_\xi(0) = -\infty, \quad \text{for} \ \xi > 0.
\]

Therefore, there is explosion in finite time and

\[
P(\tau_\xi \leq t) = \Phi \left( \frac{|\xi|^{1-\alpha}}{(\alpha - 1)\sqrt{H(t)}} \right).
\]

Example 6 Let \( \sigma(x) = e^{\alpha x} \), \( x \in \mathbb{R} \), \( \alpha \neq 0 \) and \( \xi \in \mathbb{R} \). Then

\[
\Psi_\xi(x) = \frac{1}{\alpha}(e^{-\alpha \xi} - e^{-\alpha x}),
\]

\[
\Psi_\xi(-\infty) = \begin{cases} 
-\infty, & \alpha > 0, \\
\frac{1}{\alpha} e^{-\alpha \xi}, & \alpha < 0,
\end{cases} \quad \text{and} \quad \Psi_\xi(\infty) = \begin{cases} 
\frac{1}{\alpha} e^{-\alpha \xi}, & \alpha > 0, \\
\infty, & \alpha < 0.
\end{cases}
\]

Thus we deduce that there is explosion on the left for \( \alpha < 0 \), there is explosion on the right for \( \alpha > 0 \) and

\[
P(\tau_\xi \leq t) = \Phi \left( \frac{e^{-\alpha \xi}}{|\alpha|\sqrt{H(t)}} \right).
\]
3 An extension of Osgood criterion for integral equations

In this section we generalize recent results obtained in [2] and [11]. Now we study the following nonautonomous integral equation

\[ X_t^\xi = \xi + \int_0^t a(s)b(X_s^\xi)ds + g(t), \quad t \geq 0. \]  

(5)

The explosion time \( T_{x0}^\xi \) of this equation is defined as \( T_{x0}^\xi = \inf\{t \geq 0 : X_t^\xi \notin \mathbb{R}\} \).

In the remaining of this paper we will need the following conditions:

**H1:** \( a : (0, \infty) \to (0, \infty) \) is a continuous function such that

\[ \lim_{t \to \infty} \int_t^{t+\eta} a(s)ds > 0, \quad \text{for some } \eta > 0. \]

**H2:** \( b : \mathbb{R} \to [0, \infty) \) is a continuous function such that there exist \( -\infty \leq l < \infty \) and \( -\infty < r < \infty \) satisfying that \( b > 0 \) and locally Lipschitz on \( (l, \infty) \), and \( b : [r, \infty) \to (0, \infty) \) is non-decreasing.

**H3:** \( g : [0, \infty) \to \mathbb{R} \) is a continuous function such that

\[ \limsup_{t \to \infty} \left( \inf_{0 \leq h \leq \tilde{\eta}} g(t + h) \right) = \infty, \quad \text{for some } \tilde{\eta} > 0. \]

Henceforth we utilize the convention

\[ A_t(x) = \int_t^x a(z)dz, \quad t \geq 0 \text{ and } x \in (t, \infty), \]

and

\[ B_t(x) = \int_\xi^x \frac{dz}{b(z)}, \quad x \in (l, \infty). \]

We begin with the following generalization of Osgood criterion.

**Lemma 7** Let **H1** and **H2** be satisfied and \( x_0 > l \). Consider the ordinary differential equation

\[ \begin{align*}
\frac{dy(t)}{dt} &= a(t)b(y(t))dt, \quad t > t_0, \\
y(t_0) &= x_0.
\end{align*} \]  

(6)

a) Assume that \( B_{x_0}(\infty) \geq A_{t_0}(\infty) \), then

\[ y(t) = B_{x_0}^{-1}(A_{t_0}(t)), \quad t \geq t_0. \]

b) If \( B_{x_0}(\infty) < A_{t_0}(\infty) \), then there is blow up in finite time and the time of explosion \( T_{y_{x_0}} \) is equal to \( A_{t_0}^{-1}(B_{x_0}(\infty)) \).
Remark 8. Observe that equation (6) (resp. equation (5)) has a unique solution for $x_0 > l$ (resp. for $\xi > l$) that may explode in finite time because of Hypotheses $H1$ and $H2$ (resp. $H1-H3$). This fact will be used in the proof of Theorem 10 below without mentioning.

Proof. From (6) we see that

$$
\int_{t_0}^{t} \frac{y'(s)}{b(y(s))} ds = \int_{t_0}^{t} a(s) ds.
$$

The change of variable $z = y(s)$ yields $B_{x_0}(y(t)) = A_{t_0}(t)$.

Now we deal with Statement a). If $B_{x_0}(\infty) \geq A_{t_0}(\infty)$, then $B_{x_0}(\infty) > A_{t_0}(t)$, for all $t > t_0$. Therefore $y(t) = B_{x_0}^{-1}(A_{t_0}(t))$, $t > t_0$ is well-defined.

Finally we consider Statement b). In this case we have $B_{x_0}^{-1}(A_{t_0}(t))$ is only defined for $t < A_{t_0}^{-1}(B_{x_0}(\infty)) < \infty$.

Also we are going to need the following elementary comparison result.

Lemma 9. Let $x_0 > r$ and $T > t_0$. Assume that $H1$ and $H2$ are satisfied, and that $u, v : [t_0, T] \rightarrow \mathbb{R}$ are two continuous functions.

a) Suppose that $u$ and $v$ are such that

$$
v(t) > x_0 + \int_{t_0}^{t} a(s)b(v(s))ds, \quad t \in [t_0, T],
$$

$$
u(t) = x_0 + \int_{t_0}^{t} a(s)b(u(s))ds, \quad t \in [t_0, T].
$$

Then $v(t) \geq u(t)$, for all $t \in [t_0, T]$.

b) If

$$r < v(t) < x_0 + \int_{t_0}^{t} a(s)b(v(s))ds, \quad t \in [t_0, T],
$$

$$u(t) = x_0 + \int_{t_0}^{t} a(s)b(u(s))ds, \quad t \in [t_0, T].
$$

Then $v(t) \leq u(t)$, for all $t \in [t_0, T]$.

Proof. We first deal with Statement a). Let $N = \{t \geq t_0 : b(v(s)) \leq b(u(s)), s \in [t_0, t]\}$. Since $t_0 \in N$, then the continuity of of $v$ and $u$, together with the fact that $b$ is non-decreasing on $(r, \infty)$, leads us to show that $\bar{T} = \sup N > t_0$. If $\bar{T} < T$ then

$$v(\bar{T}) - u(\bar{T}) > \int_{t_0}^{\bar{T}} a(s)[b(v(s)) - b(u(s))]ds \geq 0,$$

which is impossible due to the definition of $\bar{T}$.

Finally, we proceed similarly to prove that b) is also true and to finish the proof. ■

6
Theorem 10 Let $\xi \in \mathbb{R}$. Assume $H1$-$H3$. Then the explosion time $T^X_\xi$ of the solution $X^\xi$ of (5) is finite if and only if

$$\int_r^\infty \frac{ds}{b(s)} < \infty. \quad (7)$$

Proof. Suppose that $T^X_\xi < \infty$. Since $g$ is continuous, then

$$\int_0^t a(s)b(X^\xi_s)ds \begin{cases} < \infty, & t < T^X_\xi, \\ = \infty, & t = T^X_\xi. \end{cases}$$

Hence, there is $t_0 \in (0, T^X_\xi)$ such that

$$\xi + \int_0^{t_0} a(s)b(X^\xi_s)ds + \inf_{s \in [0, T^X_\xi]} g(s) > r,$$

and consequently $X_t > r$ for $t \in [t_0, T^X_\xi]$.

Now set

$$M = \sup\{|g(t)| : 0 \leq t \leq T^X_\xi\} + \xi + \int_0^{t_0} a(s)b(X^\xi_s)ds.$$ 

This yields

$$X^\xi_t < M + 1 + \int_0^t a(s)b(X^\xi_s)ds, \quad t \in [t_0, T^X_\xi].$$

On the other hand, we consider the integral equation

$$u(t) = (M + 1) + \int_0^t a(s)b(u(s))ds, \quad t \geq t_0.$$ 

Because $M > r$, Lemmas 7 and 9 give $T^{u}_{M+1} = A_{0}^{-1}(B_{M+1}(\infty)) \leq T^X_\xi < \infty$. Whence

$$\int_{M+1}^{\infty} \frac{ds}{b(s)} < \infty.$$ 

The continuity and positivity of $b$ in $[r, \infty)$ implies (7).

Reciprocally, suppose that $X^\xi$ does not explode in finite time. From Hypotheses $H1$ and $H3$, we can find a sequence $\{t_n : n \in \mathbb{N}\}$ such that $t_n \uparrow \infty$ and

$$r + 1 < \xi + \inf_{0 \leq h \leq \eta} g(t_n + h) \uparrow \infty, \quad \text{as } n \to \infty.$$ 

Observe that

$$X^\xi_{t+\eta_n} > \xi + \inf_{0 \leq h \leq \eta} g(t_n + h) - 1 + \int_0^t a(s+t_n)b(X^\xi_{s+t_n})ds, \quad t \in [0, \tilde{\eta}].$$

Now consider the integral equation

$$u(t) = \xi + \inf_{0 \leq h \leq \eta} g(t_n + h) - 1 + \int_0^t a(s+t_n)b(u(s))ds, \quad t \in [0, \tilde{\eta}].$$
Therefore Lemmas 7 and 9 yield
\[
\int_{\xi + \inf_{s \leq t} g(s) + 1}^{\infty} \frac{ds}{b(s)} > \int_{t_n}^{t_n + \tilde{\eta}} a(s) ds.
\]
Whence \( H1 \) implies \( \int_{r}^{\infty} \frac{ds}{b(s)} = \infty \).

We finish this section with the following result for bounded noise.

**Proposition 11** Assume that Hypotheses \( H1 \) and \( H2 \) are true. Also assume that \( g \) in equation (5) is a bounded function and that \( \xi + \inf_{s \geq 0} g(s) > r \). Then, we have the following statements:

a) \( \int_{r}^{\infty} (1/b(s)) ds = \infty \) implies that the solution of equation (5) does not explode in finite time.

b) \( \int_{r}^{\infty} (1/b(s)) ds < \infty \) yields that the solution of equation (5) blows up in finite time and

\[
T_{\xi}^{X} \in (A_{0}^{-1}(B_{\xi + \sup_{s \geq 0} g(s)}(\infty)), A_{0}^{-1}(B_{\xi + \inf_{s \geq 0} g(s)}(\infty))).
\]

**Proof.** Let \( \varepsilon > 0 \) be such that \( \xi + \inf_{s \geq 0} R(s) > r + \varepsilon \). Set

\[
Z_{i}^{\xi} = \xi + \sup_{s \geq 0} g(s) + \varepsilon + \int_{0}^{t} a(s)b(Z_{i}^{\xi}) ds
\]

and

\[
Y_{i}^{\xi} = \xi + \inf_{s \geq 0} g(s) - \varepsilon + \int_{0}^{t} a(s)b(Y_{i}^{\xi}) ds.
\]

By Lemma 8 we have,

\[
Y_{i}^{\xi} < X_{i}^{\xi} < Z_{i}^{\xi}, \quad t < T_{\xi + \sup_{s \geq 0} g(s) + \varepsilon}^{Z}.
\]

Letting \( \varepsilon \downarrow 0 \) the proof is an immediate consequence of Lemma 7 and Hypothesis \( H1 \) and \( H2 \).

**4 Stochastic differential equation with additive Wiener integral noise**

In this section we study equation (5) when the noise \( g \) is a Wiener integral. More precisely, here we study the stochastic differential equation

\[
X_{i}^{\xi} = \xi + \int_{0}^{t} a(s)b(X_{i}^{\xi}) ds + I_{t},
\]

where \( I_{t} = \int_{0}^{t} f(s)dW_{s} \) and \( f : [0, \infty) \to \mathbb{R} \) is a square-integrable function on \( [0, M] \), for any \( M > 0 \).

In the remaining of this section we utilize the following assumption:
**H4:** \( \int_0^\infty f^2(s)ds = \infty \) and

\[
\sum_{n=M}^{\infty} \frac{1}{\Upsilon^p(n)} \left( \int_n^{n+2} f^2(s)ds \right)^{p/2} < \infty,
\]

(9)

for some \( M, p > 0 \), where

\[ \Upsilon(t) = \sqrt{2 \left( \int_0^t f^2(s)ds \right) \log \log \left( e^e \vee \int_0^t f^2(s)ds \right)} \].

**Remark 12** Observe that (9) holds if, for example,

\[ t \mapsto \left( \int_0^{t+2} f^2(s)ds \right) \left( \int_0^t f^2(s)ds \right)^{-1} - 1. \]

is a decreasing function in \( L^p([M, \infty)) \) for some \( M, p > 0 \).

On the other hand, as a consequence of iterated logarithm theorem for locally square integrable martingales, we can now state the following:

**Lemma 13** Under the fact that \( \int_0^\infty f^2(s)ds = \infty \), we have

\[ \limsup_{t \to \infty} \frac{I_t}{\Upsilon(t)} = 1 \quad \text{with probability one.} \quad (10) \]

**Proof.** The result is Theorem 1.1 in Qing Gao [9].

The following theorem is the main result of this section.

**Theorem 14** Assume that \( H1, H2 \) and \( H4 \) are true. Then the stochastic differential equation (8) blows up in finite time with probability 1 if and only if

\[ \int_r^\infty \frac{ds}{b(s)} < \infty. \]

**Proof.** We first observe that, by Theorem [10], we only need to show that the paths of \( I \) satisfy Hypothesis \( H3 \) almost surely.

Burkholder-Davis-Gundy inequality (see, for instance, Theorem 3.5.1 in [6]) yields

\[ E \left[ \left( \sup_{s,t \in [n,n+2]} |I_t - I_s| \right)^p \right] \leq c_p \left( \int_n^{n+2} f^2(s)ds \right)^{p/2}, \]

where \( c_p \) is a constant depending only on \( p \). Then, by [9],

\[ E \left[ \sum_{n=M}^{\infty} \left( \sup_{s,t \in [n,n+2]} \frac{|I_t - I_s|}{\Upsilon(n)} \right)^p \right] \leq c_p \sum_{n=M}^{\infty} \frac{1}{\Upsilon^p(n)} \left( \int_n^{n+2} f^2(s)ds \right)^{p/2} < \infty. \]
Therefore, it is enough to prove that $I(\omega)$ satisfies $H_3$ for $\omega \in \Omega$ for which there exists $n_0 \in \mathbb{N}$ such that
\[
\sup_{s,t \in [n, n+2]} \frac{|I_t(\omega) - I_s(\omega)|}{\Upsilon(n)} \leq \frac{1}{4}, \quad \text{for } n \geq n_0
\]
and (10) is satisfied. Hence, we can find a sequence \(\{t_n: n \in \mathbb{N}\}\) such that
\[
t_n > n_0 \quad \text{and} \quad \frac{I_{t_n}(\omega)}{\Upsilon(t_n)} \geq \frac{1}{2} \quad \text{for all } n \in \mathbb{N}.
\]
Finally, using the properties established in this proof, we are able to write, for $n \geq n_0$,
\[
\inf_{s \in [t_n, t_n+1]} I_s(\omega) = I_{t_n}(\omega) + \inf_{s \in [t_n, t_n+1]} (I_s(\omega) - I_{t_n}(\omega))
\]
\[
\geq I_{t_n}(\omega) + \inf_{s \in [t_n, t_n+1]} \left(-|I_s(\omega) - I_{t_n}(\omega)|\right)
\]
\[
\geq I_{t_n}(\omega) - \left(\sup_{s,t \in [t_n, t_n+2]} \frac{|I_s(\omega) - I_t(\omega)|}{\Upsilon([t_n])}\right) \Upsilon([t_n])
\]
\[
\geq \frac{1}{2} \Upsilon(t_n) - \frac{1}{4} \Upsilon([t_n]) \geq \frac{1}{4} \Upsilon(t_n) \to \infty,
\]
as $n \to \infty$, where $[t]$ is the integer part of $t$ and, in the last inequality, we have used that $\Upsilon$ is a non-decreasing function.

Now, in order to state a consequence of Theorem 14, we consider the equation
\[
Y_t = \xi + \int_0^t \tilde{b}(s, Y_s) ds + I_s, \quad t \geq 0. \tag{11}
\]
Here, for each $T > 0$, the function $\tilde{b} : [0, \infty) \times \mathbb{R} \to [0, \infty)$ is locally Lipschitz (uniformly on $s \in [0, T]$), $\tilde{b}'(\cdot, x)$ is continuous, for $x \in \mathbb{R}$, and $I$ satisfy Hypothesis $H_4$ with $f$ continuous. Remember that, in this case, equation (11) has a unique solution that may explode in finite time.

**Corollary 15** Let $a$ and $b$ satisfy Conditions $H_1$ and $H_2$, respectively. Assume that $\xi \in \mathbb{R}$, $b$ is locally Lipschitz, \(\int_{-\infty}^{\infty} (1/b(x)) dx < \infty \) (resp. $\int_{-\infty}^{\infty} (1/b(x)) dx = \infty$) and \(a(s)b(x) \leq \tilde{b}(s, x)\) (resp. $\tilde{b}(s, x) \leq a(s)b(x)$, $(s, x) \in [0, \infty) \times \mathbb{R}$). Then, the solution to equation (11) explodes (resp. does not explode) in finite time.

**Proof.** We only consider the case that $\int_{-\infty}^{\infty} (1/b(x)) dx = \infty$ and $\tilde{b}(s, x) \leq a(s)b(x)$, since the proof is similar for the other one.

Let $X_\xi$ and $Y$ be the solutions of equations (8) and (11), respectively. Then, from Milian [13] (Theorem 2), we get
\[
Y_t \leq X_\xi^\xi, \quad t \geq 0.
\]
Thus, by Theorem 14, the solution $Y$ of equation (11) cannot explode in finite time because it cannot go to $-\infty$ in finite time since $\tilde{b}$ is $\mathbb{R}_+$-valued and $I$ has continuous paths and, consequently, bounded paths on compact intervals of $[0, \infty)$. Therefore the proof is complete.

**Example 16** Take

$$a(x) = x^\alpha, \quad x \in (0, \infty),$$

$$b(x) = 8x^2 - 36x + 48, \quad x \in \mathbb{R},$$

$$f(x) = x^\beta, \quad x \in (0, \infty), \quad \beta > -\frac{1}{2}.$$  

Hence

$$\lim_{t \to \infty} \int_t^{t+1} x^\alpha \, dx = \begin{cases} +\infty, & \alpha > 0, \\ 1, & \alpha = 0, \\ 0, & \alpha < 0, \end{cases}$$

and

$$\frac{(t+2)^{2\beta+1}}{t^{2\beta+1}} - 1 = \left(1 + \frac{2}{t}\right)^{2\beta+1} - 1 \leq C_1 \frac{1}{t}.$$  

The last function belongs to $L^p([1, \infty))$, for any $p > 1$. Thus $f$ satisfied (9) due to Remark 12.

On the other hand, it is clear that $\int_1^\infty \frac{4s}{8s^2 - 36s + 48} \, ds < \infty$, $\xi > 0$. Then

$$X_t^\xi = \xi + \int_0^t s^\alpha (8(X_s^\xi)^2 - 36(X_s^\xi) + 48) \, ds + \int_0^t s^\beta \, dW_s,$$

explodes in finite time when $\alpha \geq 0$. Notice that $b$ is not necessarily increasing as in [11] or [2]. Moreover, we can improve Theorem 14 in some particular cases, see [15].

**Example 17** The function $Y_t \equiv 1$ is solution to

$$Y_t = 1 + \int_0^t (Y_s)^2 \, ds - t, \quad t \geq 0.$$ 

Although $\int_1^\infty (1/s^2) \, ds < \infty$, $Y$ does not blow-up in finite time because $g(t) = -t$, $t \geq 0$, does not satisfies Hypothesis $H3$.

Also notice that $f(t) = \exp(\exp(t))$, $t \geq 0$, does not satisfies (9). We intuitively understand that in this case the noise is to strong and we have also blow up in finite time, for any initial condition. We have a contrary effect as in Example 17.

**Proposition 18** Let $f$ and $I$ be defined in equation (8). Suppose $H1$, $H2$ and $\int_0^\infty f^2(s) \, ds < \infty$ are satisfied. Then $I$ is bounded with probability one and, under the assumption $\xi + \inf_{s \geq 0} I_s > r$, the stochastic differential equation (8) blows up in finite time if and only if $B_r(\infty) < \infty$.

**Remark** Observe that $\xi + \inf_{s \geq 0} I_s$ depends on $\omega$.

**Proof.** The result follows from [6] (Lemma 3.4.7 and Theorem 3.4.9), and Proposition 11.
An approach to obtain the distribution of the explosion time of a stochastic differential equation

Now we study some stochastic differential equations of the form

\[ X^\xi_t = \xi + \int_0^t b(s, X^\xi_s)ds + \int_0^t \sigma(s, X^\xi_s)dW_s, \quad t \geq 0. \] (12)

Namely, we propose a method to figure out the distribution of the explosion time \( \tau^\xi \) of \( X^\xi \). Intuitively, \( \tau^\xi \) is a stopping time such that (12) has a solution up to this stopping time and \( \lim \sup_{t \uparrow \tau^\xi} |X_t| = \infty \).

5.1 Autonomous case

This section is devoted to deal with the stochastic differential equation

\[ X^\xi_t = \xi + \int_0^t b(X^\xi_s)ds + \int_0^t \sigma(X^\xi_s)dW_s, \quad t \geq 0, \]

with \( b, \sigma \in C^1(\mathbb{R}) \). In this case, McKean [12] has shown that \( X^\xi_{(\tau^\xi)^-} \in \{-\infty, \infty\} \) on \( [\tau^\xi < \infty] \). So, henceforth, we can utilize the convention

\[ \tau^\xi_+ = \inf\{t > 0 : X^\xi_t = \infty\} \quad \text{and} \quad \tau^\xi_- = \inf\{t > 0 : X^\xi_t = -\infty\}. \]

Theorem 19 Consider a bounded function \( u : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R} \) that satisfies the following boundary value problem:

\[ \frac{\partial u}{\partial t}(t, x) = \frac{1}{2}\sigma^2(x)\frac{\partial^2 u}{\partial x^2}(t, x) + b(x)\frac{\partial u}{\partial x}(t, x), \quad t > 0 \text{ and } x \in \mathbb{R}, \] (13)

\[ u(0, x) = 0, \quad \text{for all } x \in \mathbb{R}. \] (14)

a) Assume that \( u(t, \infty) = u(t, -\infty) = 1 \). Then \( P(\tau^\xi \leq t) = u(t, \xi) \).

b) \( u(t, \infty) = 1 \) and \( u(t, -\infty) = 0 \) implies that \( P(\tau^\xi_+ \leq t) = u(t, \xi) \).

c) If \( u(t, \infty) = 0 \) and \( u(t, -\infty) = 1 \), we have \( P(\tau^\xi_- \leq t) = u(t, \xi) \).

Remarks

1) Maximum principle provides with conditions on \( b \) and \( \sigma \) that guarantee the solution of equation (13) is bounded (see Friedman [8]).

2) It is quite interesting to observe that (13) is related to transition density of process \( X^\xi \), or related to the fundamental solution of the associated Cauchy problem (see [8]). On the other hand, (14) and the conditions in Statement a)-c) are intuitively clear. In fact, (14) establishes that if we begin at a real point (\( \xi \in \mathbb{R} \)), then we need some time to get blow-up. And other conditions mean that if we begin at cementery state (\( \pm \infty \)), then the time to blow-up is less than any time.
3) Observe that \( P(\tau^\xi \leq t) = P(\tau^\xi_+ \leq t) + P(\tau^\xi_- \leq t) \) and that, for example in Statement a), we have \( P(\tau^\xi < \infty) = u(\infty, \xi) \).

4) If \( X^\xi \) does not explodes in finite time, then equation (13)-(14) has not a bounded solution satisfying conditions established in either Statement a), b), or c).

**Proof.** Using Itô’s formula on \( 0 \leq s < t \) and that \( u \) is solution to (13) we obtain

\[
u(t, \xi) = E \left[ u(t - (s \wedge \tau^m_\xi), X^\xi_{s \wedge \tau^m_\xi}) \right],
\]

where \( \tau^m_\xi = \inf\{t > 0 : |X^\xi_t| > m\} \). Since \( u \) is bounded, then the above stochastic integral is a martingale. Therefore

\[
u(t, \xi) = E \left[ u(t - (s \wedge \tau^m_\xi), X^\xi_{s \wedge \tau^m_\xi}) \right].
\]

Taking \( s \uparrow t \), then continuity of \( X^\xi \) and the boundedness of \( u \), together with the dominated convergence theorem, allow us to write

\[
u(t, \xi) = E \left[ u(t - \tau_\xi, X^\xi_{\tau_\xi}), \tau_\xi \leq t \right] + E \left[ u(t - \tau_\xi, X^\xi_{\tau_\xi}), \tau_\xi > t \right].
\]

Now we consider Statement a),

\[
u(t, \xi) = E \left[ u(t - \tau_\xi, X^\xi_{\tau_\xi}), \tau_\xi \leq t \right] = P(\tau_\xi \leq t).
\]

Statement b) is proven as follows. From equality (13) we get

\[
u(t, \xi) = E \left[ u(t - \tau_\xi, X^\xi_{\tau_\xi}), \tau_\xi \leq t, \tau_\xi^+ < \tau_\xi^- \right] + E \left[ u(t - \tau_\xi, X^\xi_{\tau_\xi}), \tau_\xi \leq t, \tau_\xi^- < \tau_\xi^+ \right] = P(\tau_\xi^- \leq t).
\]

Finally, Statement c) is proven similarly. So the proof is complete. \( \blacksquare \)

**Examples 20** \( a) \) In Example 20 with \( f \equiv 1 \), we have

\[
\bar{u}(t, \xi) = \begin{cases} 
\Phi \left( \frac{|\xi|^{\frac{1-\alpha}{(\alpha-1)^2}}} {((\alpha-1)^2)\xi} \right), & \xi > 0, \\
0, & \xi \leq 0,
\end{cases}
\]
and

\[ u(t, \xi) = \begin{cases} 
0, & \xi > 0, \\
\Phi \left( \frac{|\xi|^{1-\alpha}}{(\alpha-1)\sqrt{t}} \right), & \xi \leq 0,
\end{cases} \]

satisfy Statements b) and c) of Theorem 19, respectively. In particular, if \( \xi > 0 \), then \( \bar{u}(\infty, \xi) = 1 \), therefore we have a positive blow-up. This phenomenon is explained by Milan [13] (Theorem 1) due to the involved solution being a nonnegative process when \( \xi > 0 \).

b) For \( \beta > 0 \), the partial differential equation

\[
\frac{\partial u}{\partial t}(t, x) = \frac{a^2}{2} e^{2\beta x} \frac{\partial^2 u}{\partial x^2}(t, x) + \beta a^2 e^{2\beta x} \frac{\partial u}{\partial x}(t, x),
\]

\[ u(0, x) = 0, \quad \forall x \in \mathbb{R}, \]

has solution,

\[ u(t, x) = \exp \left( -\frac{e^{-2\beta x}}{2(a\beta)^2 t} \right). \]

Since \( u(t, \infty) = 1 \) and \( u(t, -\infty) = 0 \), then the distribution of explosion time to the stochastic differential equation

\[ X_t^x = x + \int_0^t \beta a^2 e^{2\beta X_s^x} ds + \int_0^t ae^{\beta X_s^x} dW_s, \]

is given by

\[ P(\tau_\xi \leq t) = \exp \left( -\frac{e^{-2\beta x}}{2(a\beta)^2 t} \right) \]

because of \( P(\tau_\xi^+ < \infty) = 1 \).

**Remark** It is not difficult to see that Examples 5 and 6 are solution of the corresponding partial differential equations (PDEs), then we conjecture that the distribution of the explosion time is the solution of such a PDE. If this is true, then we have the following criterion of explosion: There is explosion in finite time if and only if the corresponding PDE has a bounded solution. Moreover, this criterion could be applied in more dimensions and for non autonomous processes (see [8]).

### 5.2 Laplace transform of the distribution of the explosion time

Finally, in this subsection we indicate how we could calculate the Laplace transformation of the distribution of the explosion time \( \tau_\xi \) of the solution to equation (11). It means, we assume that the equation

\[ X_t^\xi = \xi + \int_0^t b(s, X_s^\xi) ds + I_t, \quad t \geq 0, \tag{16} \]
has a unique solution that may blow-up in finite time, where $b$ takes values in $\mathbb{R}^+$ and $I_t = \int_0^t f(s) dW_s$. Note that if $\omega \in \Omega$ is such that $\tau_\xi(\omega) < \infty$, then $X_\xi^t > \xi + \inf_{0 \leq s \leq \tau_\xi(\omega)} I_t(\omega), \ t \leq \tau_\xi(\omega)$, and consequently $\int_{\tau_\xi(\omega)}^{\tau_\xi(\omega)^+} b(s, X_s^\xi) ds = \infty$. Thus, $X_{\tau_\xi^+} = \infty$, on $[\tau_\xi < \infty]$, and $\tau_\xi = \tau_\xi^+$.

We begin with an auxiliary result.

**Lemma 21** let $\lambda > 0$ and $\tau_\xi$ the explosion time of the solution of equation (16). Then

$$E\left(e^{-\lambda \tau_\xi}\right) = \lambda \int_0^\infty P(\tau_\xi \leq u) e^{-\lambda u} du.$$  

**Proof.** Let us denote the distribution of $\tau_\xi$ by $F_{\tau_\xi}$. Fubini theorem leads to justify

$$E\left(e^{-\lambda \tau_\xi}\right) = \int_{(0, \infty)} e^{-\lambda s} F_{\tau_s}(ds) = \lambda \int_0^\infty \left( \int_s^\infty e^{-\lambda u} du \right) F_{\tau_s}(ds) = \lambda \int_0^\infty F_{\tau_s}(ds) e^{-\lambda s} = \lambda \int_0^\infty F_{\tau_s}(u) e^{-\lambda u} du.$$  

Consequently, the proof is complete. □

Now we can state the main result of this subsection.

**Theorem 22** Consider $\lambda > 0$, the explosion time $\tau_\xi$ of the solution of (16) and a bounded function $u : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ that is a solution of the partial differential equation

$$-\frac{\partial u}{\partial t}(t, x) = \frac{1}{2} f^2(t) \frac{\partial^2 u}{\partial x^2}(t, x) + b(t, x) \frac{\partial u}{\partial x}(t, x) - \lambda u(t, x), \ t > 0 \text{ and } x \in \mathbb{R},$$

$$u(t, \infty) = 1.$$  

Then, $\lambda \int_0^\infty P(\tau_\xi \leq u) e^{-\lambda u} du = u(0, \xi).$  

**Proof.** As in the proof of Theorem 19 Itô’s formula gives

$$u(0, \xi) = E\left( u(t \wedge \tau_m, X_t^{\xi, \tau_m}) \exp(-\lambda(t \wedge \tau_m)) \right).$$  

We can now easily complete the proof of this result by combining Lemma 21, the arguments used in the last part of the proof of Theorem 19 and the fact that $X_{\tau_m}^\xi \to \infty$ as $m \to \infty$ on $[\tau_\xi < \infty]$. Indeed we first take $m \uparrow \infty$, and then $t \uparrow \infty$. □

**Remark** In some cases we have the converse of Theorem 22. For example, consider the stochastic differential equation

$$X_t^\xi = \xi + \int_0^t (g(X_s^\xi) + a - b) ds + c W_t, \ t \geq 0,$$
where \( a, b, c \in \mathbb{R} \), \( a > b \) and \( g : \mathbb{R} \to [0, \infty) \). Then the associated ordinary differential equation is

\[
\frac{c^2}{2} w''(x) + (g(x) + a - b)w'(x) - \lambda w(x) = 0, \quad t > 0, \quad (17)
\]
\[
w(\infty) = 1.
\]

Therefore, if \( X^\xi \) explodes in finite then (17) has a bounded solution, in fact it is the Laplace transform of the explosion time (see [7]). Then the solution of

\[
-\frac{\partial u}{\partial t}(t, x) = \frac{c^2}{2} \frac{\partial^2 u}{\partial x^2}(t, x) + (g(x - bt) + a) \frac{\partial u}{\partial x}(t, x) - \lambda u(t, x), \quad t > 0, \quad x \in \mathbb{R},
\]
\[
u(t, \infty) = 1.
\]

is given by \( u(t, x) = w(x - bt) \).

Acknowledgements: Professor Villa-Morales was partially supported by grant 118294 of CONACyT and grant PIM13-3N of UAA. The authors Thanks Universidad Autónoma de Aguascalientes y Cinvestav-IPN for their hospitality and economical support. Other authors were also partially supported by a CONACyT grant.

References

[1] A. N. Borodin, P. Salminen (2002). Handbook of Brownian Motion-Facts and Formulae, Second Edition, Birkhäuser.

[2] M.J. Ceballos-Lira, J.E. Macías-Díaz, J. Villa (2011). A generalization of Osgood’s test and a comparison criterion for integral equations with noise, Electronic Journal of Differential Equations 2011, no. 05, 1–8.

[3] A. Constantin (1995). Global existence of solutions for perturbed differential equations, Ann. Mat. Pura Appl. CLXVIII, no. IV, 237-299.

[4] J. Dávila, J.F. Bonder, J.D. Rossi, P. Groisman, M. Sued (2005). Numerical analysis of stochastic differential equations with explosions, Stoch. Anal. Appl. 23, no.4, 809-825.

[5] A. de Pablo, R. Ferreira, F. Quirós, J.L. Vázquez (2005). Blow-up. El problema matemático de explosión para ecuaciones y sistemas de reacción-difusión. Bol. Soc. Esp. Mat. Apl. 32, 75-111.

[6] R. Durrett (1996). Stochastic Calculus: A Practical Introduction, CRC Press.

[7] W. Feller (1954). Diffusion processes in one dimension. Trans. Amer. Math. Soc. 77, 1-31.
[8] A. Friedman (1964). Partial Differential Equations of Parabolic Type, Dover.

[9] Fu Qing GAO (2009). Laws of the Iterated Logarithm for Locally Square Integrable Martingales, Acta Mathematica Sinica, English Series 25, no. 2, 209–222.

[10] I. Karatzas, S.E. Shreve (1991). Brownian Motion and Stochastic Calculus. Second Edition, Springer-Verlag.

[11] J.A. León, J. Villa (2011). An Osgood criterion for integral equations with applications to stochastic differential equations with an additive noise, Statistics & Probability Letters 81, no. 4, 470–477.

[12] H.P. McKean (1969). Stochastic Integrals. Academic Press.

[13] A. Milian (1995). Stochastic Viability and comparison theorem, Colloquium Mathematicum 68, no. 2, 297–316.

[14] W.F. Osgood (1898). Beweis der Existenz einer Lösung der Differentialgleichung $dy/dx = f(x, y)$ ohne Hinzunahme der Cauchy-Lipschitz'schen Bedingung Monatsh. Math. Phys. (Vienna) 9, no. 1, 331-345.

[15] J. Villa (2011). Un ejemplo de explosión en ecuaciones diferenciales estocásticas con ruido aditivo, Aportaciones Matemáticas SMM, Comunicaciones 44, 187–194.