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Global notation

Joint work with Changchang Xi (CNU).

In the talk:
\(A\): Artin algebra (e.g. finite-dim. \(k\)-algebra over a field \(k\));
\(A\text{-Mod}\): the category of left \(A\)-modules;
\(A\text{-mod}\): the category of finitely generated (left) \(A\)-modules.
(NC) Nakayama Conjecture [Nakayama, 1958]: If $A$ has infinite dominant dimension $\Rightarrow A$: self-injective.

Dominant dimension of $A$:

$$\text{domdim}(A) := \sup\{n \mid I^j : \text{projective} \quad \forall \ 0 \leq j < n\}$$

where $0 \rightarrow_A A \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \rightarrow I^{n-1} \rightarrow I^n \rightarrow \cdots$ is a minimal injective coresolution.

$A$ is self-injective if projectives $=$ injectives.
How to describe infinite dominant dimension?

Morita-Tachikawa; Mueller(1968):

**Theorem**

Let $\Lambda$ be an algebra. Then $\text{domdim}(\Lambda) = \infty \iff \Lambda = \text{End}_A(M)$, where $M$ is

- **generator** (i.e. $A \in \text{add}(M)$);
- **cogenerator** (i.e. $D(A_A) \in \text{add}(M)$);
- **orthogonal** (i.e. $\text{Ext}_A^n(M, M) = 0$ for any $n \geq 1$).

$D$: the usual duality over $A$-mod (e.g. $D = \text{Hom}_k(-, k)$).
Tachikawa’s conjectures

(TC1) **Tachikawa’s First Conjecture** [Tachikawa, 1973]:
If $\text{Ext}^n_A(D(A), A) = 0 \quad \forall \; n \geq 1 \implies A$: self-injective.

In (TC1), the $A$-module $A \oplus D(A)$ is orthogonal.

(TC2) **Tachikawa’s Second Conjecture** [Tachikawa, 1973]:
If $A$: self-injective and $A_M$: finitely generated, orthogonal
$\implies M$: projective.

**Proposition**

(NC) holds for all algebras $\iff$ (TC1)+(TC2) hold for all algebras.
Tachikawa’s Second Conjecture (TC2)

TC2

If $A$: self-injective, $A_M$: finitely generated, orthogonal $\implies M$: projective.

In (TC2), we can assume $M$ is a generator (e.g. $A_M = A \oplus M_0$).

Lemma

The pair $(A, M)$ satisfies (TC2) $\iff$ $\text{End}_A(M)$ satisfies (NC).

(TC2) holds for $(A, M)$ where $A_M$ is arbitrary, but $A$ is

- symmetric alg./local self-injective alg. with radical$^3 = 0$ [Hoshino, 1984];
- group alg. of a finite group [Schulz, 1986];
- self-injective alg. of finite represent. type [Schulz, 1986].
Related conjectures

- Generalized Nakayama Conjecture
  [Auslander-Reiten, 1975]
- Auslander-Reiten Conjecture in commutative algebra
  [Avramov, Iyengar, Nasseh, Sather-Wagstaff, Takahashi, Yoshino, · · · · · · , 2017-2022.]
§ 2. Tachikawa’s Second Conjecture

Consider (TC2) for the pair \((A, M)\) where
- \(A\): arbitrary self-injective alg.
- \(M \in A\)-mod: generator.

**Aim of the talk**

Try to understand (TC2) by studying homological properties of orthogonal modules over self-injective algebras.

- Provide equivalent characterizations of (TC2).
- Introduce new homological conditions and Gorenstein-Morita algebras.
- Show that Gorenstein-Morita algebras satisfy the Nakayama Conjecture.
Gorenstein-projective modules

\( B \): (arbitrary) Artin algebra

**Definition**

A module \( Y \) over \( B \) is **Gorenstein-projective** if

\[ \exists \text{ exact complex of projective } B \text{-modules} \]

\[ P^\bullet : \cdots \to P^{-2} \to P^{-1} \to P^0 \xrightarrow{d^0} P^1 \to P^2 \to \cdots \]

such that \( \text{Im } (d^0) = Y \) and the complex \( \text{Hom}^\bullet_B(P^\bullet, B) \) is exact.

Gorenstein-injective modules can be defined dually.

**Notation**: \( B\text{-GProj} \) (resp., \( B\text{-Gproj} \)): the cat. of (resp., finitely generated) Gorenstein-projective \( B \)-modules.
The stable module category $B$-$\text{Mod}$ of $B$:

Objects: all $B$-modules;
Morphisms: $\forall X, Y \in B$-$\text{Mod}$,

\[ \text{Hom}_B(X, Y) := \text{Hom}_B(X, Y)/\mathcal{P}(X, Y) \]

where $\mathcal{P}(X, Y)$ consists of homos. factorizing through projective $B$-modules.

- $B$-$\text{Mod}$ is triangulated for a self-injective algebra $B$.
- $B$-$\text{GProj}$ is always triangulated.
Compact objects in categories

\( \mathcal{C} \): additive category with set-indexed coproducts.

**Definition**

An object \( X \in \mathcal{C} \) is **compact** if \( \text{Hom}_\mathcal{C}(X, -) : \mathcal{C} \to \mathbb{Z}\text{-Mod} \) commutes with coproducts.

\( \mathcal{C}^c \): the subcat. of \( \mathcal{C} \) consisting of compact objects.

- \( \text{B-mod} = \text{B-Mod}^c \) for a self-injective algebra \( B \).
- \( \text{B-Gproj} \subseteq \text{B-GProj}^c \).
Notation

\(A\): **self-injective** algebra (i.e. Projectives = Injectives);
\(A\)-Proj: the cat. of projective \(A\)-modules;
\(M \in A\)-mod: generator;
add \((A M)\) (resp., Add \((A M)\)): direct summands of finite (resp.,
arbitrary) direct sums of copies of \(M\);
\(\perp > 0 M := \{X \in A\text{-Mod} | \Ext^n_A(X, M) = 0, \forall n > 0\}\);
\(M \perp > 0 := \{X \in A\text{-Mod} | \Ext^n_A(M, X) = 0, \forall n > 0\}\);
\[G := \perp > 0 M \cap M \perp > 0;\]
\[G^{\text{fin}} := G \cap A\text{-mod};\]
\(\lim\rightarrow G^{\text{fin}}\): filtered colimits in \(A\text{-Mod}\) of modules from \(G^{\text{fin}}\).

- \(\lim\rightarrow G^{\text{fin}} \subseteq G\).
- If \(M = A\), then \(G^{\text{fin}} = A\text{-mod}\) and \(G = A\text{-Mod} = \lim\rightarrow G^{\text{fin}}\).
Relative stable category

Definition

The $M$-stable category $A$-$\text{Mod}/[M]$ of $A$-$\text{Mod}$:

**Objects:** all $A$-modules;

**Morphisms:** $\forall X, Y \in A$-$\text{Mod},$

$$\text{Hom}_M(X, Y) := \text{Hom}_A(X, Y)/M(X, Y)$$

where $M(X, Y)$ consists of homos. factorizing through objects in $\text{Add}(M)$.

$A$-$\text{Mod}/[A] = A$-$\text{Mod}$ (stable module category of $A$)
In general, $\mathcal{D} := A\text{-Mod}/[M]$ is not a triangulated category, but a pretriangulated category.

$X \in A\text{-Mod};$

$\ell_X : X \to M^X$: minimal left $\text{Add}(M)$-approximation of $X;$

$r_X : M_X \to X$: minimal right $\text{Add}(M)$-approximation of $X.$

**Remark**

The $M$-cosyzygy and $M$-syzygy functors

$$\Omega_M : \mathcal{D} \to \mathcal{D}, \quad X \mapsto \text{Ker} \ (r_X),$$

$$\Omega^{-}_M : \mathcal{D} \to \mathcal{D}, \quad X \mapsto \text{Coker} \ (\ell_X)$$

are not necessarily equivalences.
Minimal left approximations

\[ \mathcal{C}: \text{additive category, } \mathcal{B}: \text{full subcat. of } \mathcal{C}. \]

**Definition**

A morphism \( f : X \to B \) (or the object \( B \)) in \( \mathcal{C} \) is a **minimal left \( \mathcal{B} \)-approximation** of \( X \) if

- \( B \in \mathcal{B}, \)
- \( \text{Hom}_\mathcal{C}(f, Y) : \text{Hom}_\mathcal{C}(B, Y) \to \text{Hom}_\mathcal{C}(X, Y) \) is surjective \( \forall Y \in \mathcal{B}, \)
- \( g \in \text{End}_\mathcal{C}(B) \) is an isomorphism whenever \( f = fg. \)

Minimal right \( \mathcal{B} \)-approximations can be defined dually.
What happen if $M$ is orthogonal?

**Proposition**

Suppose $A M$: orthogonal (i.e. $M \in \perp > 0 M$). Then:

1. $\mathcal{G}$ (resp., $\mathcal{G}^\text{fin}$) is a Frobenius category. Its full subcategory of projective-injective objects equals $\text{Add}(M)$ (resp., $\text{add}(M)$).

2. Let $\Lambda := \text{End}_A(M)$. Then $\text{Hom}_A(M, -) : A\text{-Mod} \to \Lambda\text{-Mod}$ induces triangle equivalences

$$
\mathcal{G}/[M] \xrightarrow{\sim} \Lambda\text{-GProj} \quad \text{and} \quad \mathcal{G}^\text{fin}/[M] \xrightarrow{\sim} \Lambda\text{-Gproj}.
$$

$\mathcal{G}$ is called the $M$-Gorenstein subcategory of $A\text{-Mod}$. 
Nakayama-stable generators

Nakayama functor:

\[ \nu_A = A D(A) \otimes_A - : A\text{-Mod} \xrightarrow{\sim} A\text{-Mod}. \]

If \( A \): symmetric algebra (i.e. \( D(A) \simeq A A_A \)), then \( \nu_A \simeq \text{Id} \).

**Definition**

A generator \( _A M \) is **Nakayama-stable** if

\[ \text{add}(A M) = \text{add}(\nu_A(M)). \]

Auslander-Reiten formula:

\[ D \text{Hom}_A(M, -) \simeq \text{Hom}_A(-, \nu_A(M)[-1]). \]

where \([-1] := \Omega_A\) (autoequivalence of \( A\text{-Mod} \)).
Minimal left $\mathcal{G}$-approximations of modules

$A\ M$: orthogonal generator.
$\Omega^-_A(M) \to W$: minimal left $\mathcal{G}$-approximation of $\Omega^-_A(M)$;

$M$-$\text{resdim}(X) < \infty$ if $\exists$ exact sequence in $A$-$\text{mod}$

$$0 \to M_n \to \cdots \to M_1 \to M_0 \to X \to 0$$

with $M_i \in \text{add}(M)$ for $0 \leq i \leq n \in \mathbb{N}$;

$\mathcal{M} := \{ X \in A$-$\text{mod} \mid M$-$\text{resdim}(X) < \infty \}$.

Remk: $\mathcal{M} \subseteq \mathcal{G}^{\perp>0} \subseteq W^{\perp 1}$. 
Equivalent characterizations of (TC2)

**Assumptions:**

- $A$: self-injective Artin algebra;
- $M \in A\text{-mod}$: orthogonal, Nakayama-stable generator.

**Theorem**

The following statements are equivalent:

1. $AM$ is projective.
2. $G = \lim_{\leftarrow} G_{\text{fin}}$.
3. $W \in \lim_{\leftarrow} G_{\text{fin}}$.
4. $\text{Ext}_A^1(W, \bigoplus_{i \in \mathbb{N}} M_i) = 0$ for all $M_i \in \mathcal{M}$.
Modules of finite projective dimension

\[ \mathcal{P}^{<\infty}(B) := \{ Y \in B\text{-mod} \mid \text{proj.dim}(Y) < \infty \} \]

\[ \text{fin.dim}(B) := \sup\{ \text{proj.dim}(Y) \mid Y \in \mathcal{P}^{<\infty}(B) \} . \]

What is a “compact” version of the equality

\[ \mathcal{P}^{<\infty}(B) \cap B\text{-Gproj} = B\text{-proj} ? \]

Recall: \( B\text{-Gproj} \subseteq \overline{B\text{-Gproj}}^c. \)
Finitely generated to infinitely generated modules

**Definition**

A $B$-module $X$ is **compactly filtered** if it has a countable filtration in $B$-Mod

$$0 = X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X$$

such that $X = \bigcup_{n=0}^{\infty} X_n$ and $X_{n+1}/X_n \in \mathcal{P}^{<\infty}(B)$, $\forall n \in \mathbb{N}$.

Compactly filtered modules $\Rightarrow$ countably generated.

**Definition**

A $B$-module $X$ is **compactly Gorenstein-projective** if it is compact in $B$-$\text{GProj}$.

Finitely generated Gorenstein-projective modules $\Rightarrow$ compactly Gorenstein-projective.
Two homological conditions

Notation:
$B$-CF: compactly filtered $B$-modules.
$B$-Proj$\omega$: countably generated projective $B$-modules.
$B$-GProj$^c$: compactly Gorenstein-projective $B$-modules.
$B$-GProj$\omega$: countably generated Gorenstein-projective $B$-modules.

(HC1): $\text{Ext}_B^{>0}(B\text{-GProj}_\omega, \bigoplus_{i \in \mathbb{N}} M_i) = 0$ for all $M_i \in \mathcal{P}^{<\infty}(B)$.

(HC2): $B$-CF $\cap$ $B$-GProj$^c$ = $B$-Proj$\omega$.

Remk: $\mathcal{P}^{<\infty}(B) \subseteq B\text{-GProj}^{\perp>0}$. 
Lemma

(1) \( \text{fin.dim} (B) < \infty \implies (HC1) \implies (HC2) \).

(2) \( B\text{-GProj} \perp > 0 \subseteq B\text{-Mod}: \text{closed under direct sums} \implies (HC1) \).

(3) If \( B \) is virtually Gorenstein, then \( (HC1) \) holds.

(4) (HC2) is preserved under

- derived equivalences,
- stable equivalences of Morita type,
- certain singular equivalences of Morita type with level.

Open question:
Does \( (HC1) \) (resp., \( HC2 \)) hold for all Artin algebras?

- Finitistic Dimension Conjecture [Rosenberg, Zelinsky; Bass, 1960]: \( \text{fin.dim} (B) < \infty, \forall B \).
Definition (Beligiannis, 2005)

\( B \) is virtually Gorenstein if \( B \text{-GProj}^{\perp} > 0 = \perp^{0}B\text{-GInj} \).

\( B\text{-GInj} \): the cat. of Gorenstein-injective \( B \)-modules.

Theorem (Beligiannis, 2005)

Virtually Gorenstein algebras satisfy the Gorenstein Symmetric Conjecture.

\((\text{GSC})\): If \( \text{inj.dim} (B_B) < \infty \), then \( \text{inj.dim} (B_B) < \infty \).

[A. Beligiannis, Cohen-Macaulay modules, (co)torsion pairs and virtually Gorenstein algebras, \textit{J. Algebra.} 288 (2005), 137-211.]
Gorenstein-Morita algebras

**Definition**

An algebra $B$ is called a **Gorenstein-Morita algebra** if

- $B = \text{End}_A(M)$ where
  - $A$: self-injective algebra;
  - $M$: Nakayama-stable generator for $A$-mod.
- $B$ satisfies the condition (HC2):

$$B \text{-CF} \cap B \text{-GProj}^c = B \text{-Proj}_\omega.$$ 

**Corollary**

Let $B$ be a Gorenstein-Morita algebra.

If $\text{domdim}(B) = \infty$, then $B$ is self-injective.

Thus $B$ satisfies the Nakayama Conjecture.
§ 3. Recollements of (relative) stable categories

$A$: (arbitrary) self-injective algebra, $M \in A\text{-mod}$: generator.

- Given $(A, M)$, we construct two pairs of triangle endofunctors of the stable module category of $A$:

  $$\begin{align*}
  (\Diamond) \quad (\Phi, \Psi) \text{ and } (\Phi', \Psi') : \, A\text{-Mod} & \longrightarrow A\text{-Mod}.
  \end{align*}$$

- If $AM$ is orthogonal or $\Omega$-periodic, then these functors can be embedded into a recollement of $A\text{-Mod}$.
  ($M$ is called $\Omega$-periodic if $\Omega^n_A(M) \simeq M$ in $A\text{-Mod}$ for some $n \geq 1$.)

- If $AM$ is orthogonal and Nakayama-stable, then this recollement can be restricted to a recollement of the $M$-stable category of $A\text{-Mod}$.
Recollements of triangulated categories

Beilinson, Bernstein and Deligne [1982]:
A recollement \((\mathcal{Y}, \mathcal{D}, \mathcal{X})\):

\[
\begin{array}{c}
\mathcal{Y} \\
\downarrow \quad i_! = i_* \\
\mathcal{D} \\
\downarrow \quad j！ = j^* \\
\mathcal{X}
\end{array}
\]

- 6 triangle functors;
- 4 adjoint pairs: \((i^*, i_*)\), \((i_, i!)\), \((j!, j^!)\) and \((j^*, j_*)\);
- 3 fully faithful functors (pointing to \(\mathcal{D}\), e.g. \(i_*\));
- 3 zeros of composition (along the same level, e.g. \(i^* j! = 0\));
- 2 triangles: \(\forall X \in \mathcal{D}, \exists\) triangles in \(\mathcal{D}\):
  \[j! j^!(X) \xrightarrow{\text{counit}} X \xrightarrow{\text{unit}} i_* i^*(X) \longrightarrow j! j^!(X)[1],\]
  \[i! i^!(X) \xrightarrow{\text{counit}} X \xrightarrow{\text{unit}} j_* j^*(X) \longrightarrow i! i^!(X)[1].\]
Thick subcategories of module categories

**Definition**

A full subcat. $\mathcal{U} \subseteq A$-Mod is **thick** if

- it is closed under direct summands in $A$-Mod;
- $\forall 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ exact seq. in $A$-Mod with two terms in $\mathcal{U}$, the third term also belongs to $\mathcal{U}$.

**Notation:**

$\mathcal{I}$: the smallest **thick** subcat. of $A$-Mod containing $M$ and being closed under **direct sums**.

- $\mathcal{I}$: localizing subcat. of $A$-Mod containing $M$.
- If $M = A$, then $\mathcal{I} = A$-Proj.
A recollement of $A$-$\text{Mod}$ with explicit functors

**Theorem**

*If $A^{}M$ is orthogonal or $\Omega$-periodic, then $\exists$ a recollement:*

$$
\begin{array}{c}
\overset{\sim}{\Psi} \\
\text{inc}
\end{array}
\xleftarrow{\text{inc}} 
\overset{\sim}{\Psi}'
\quad \quad 
\overset{\sim}{\Phi} \\
\text{inc}
\xrightarrow{\Phi''}
\mathcal{S}
\end{array}
$$

where $M^\perp := \{ X \in A$-$\text{Mod} \mid \text{Hom}_A(M, X[n]) = 0 \ \forall \ n \in \mathbb{Z} \}$, 

$\Phi = \text{inc} \circ \tilde{\Phi}$, \ $\Psi = \text{inc} \circ \tilde{\Psi}$, \ $\Psi' = \text{inc} \circ \tilde{\Psi}'$ \ and \ $\Phi'' = \Phi' \circ \text{inc}$.

Explicit constructions of the functors $(\Phi, \Psi)$ and $(\Phi', \Psi') : A$-$\text{Mod} \rightarrow A$-$\text{Mod}$. 
Orthogonal generators over self-injective algebras

$A$: self-injective Artin algebra;  
$M$: orthogonal, Nakayama-stable generator for $A$-mod.  
In this case,

$$\mathcal{G} = \{ X \in A\text{-Mod} \mid \text{Hom}_A(M[n], X) = 0, \forall n \neq 0, 1 \}.$$ 

Now, let

$\Lambda := \text{End}_A(M)$: the endomorphism algebra of $M$ in $A$-Mod;  
$\Gamma := \text{End}_A(M)$: the endomorphism algebra of $M$ in $A$-Mod;  
$\mathcal{E} := \{ X \in \mathcal{G} \mid \text{Hom}_A(M, X), \text{Hom}_A(M[1], X) \in \Gamma\text{-mod} \}$;  
$\pi : \mathcal{G} = \mathcal{G}/[A] \rightarrow \mathcal{G}/[M]$: the quotient functor.
Recollements of relative stable categories

Theorem

\[ \exists \text{ a recollement of triangulated categories:} \]

\[
\begin{align*}
M \perp & \xrightarrow{\pi \circ \text{inc}} \mathcal{G}/[M] \xleftarrow{\tilde{\Psi}} (\mathcal{G} \cap \mathcal{S})/[M] \\
& \xleftarrow{\tilde{\Psi}'} \xrightarrow{\text{inc}} \xleftarrow{\tilde{\Phi}} \xrightarrow{\Phi''} (\mathcal{G} \cap \mathcal{S})/[M]
\end{align*}
\]

which restricts to a recollement

\[
\begin{align*}
M \perp & \xrightarrow{\text{inc}} \mathcal{E}/[M] \xleftarrow{\tilde{\Phi}} (\mathcal{E} \cap \mathcal{S})/[M].
\end{align*}
\]

Remk: \( A M \) is projective \( \Leftrightarrow (\mathcal{G} \cap \mathcal{S})/[M] = 0 \Leftrightarrow (\mathcal{E} \cap \mathcal{S})/[M] = 0. \)
Compact objects and generating sets

**Proposition**

1. Each nonzero object of $(\mathcal{E} \cap \mathcal{I})/[M]$ is compact in $\mathcal{G}/[M]$ and **infinitely generated** as an $A$-module.

2. Let $\mathcal{S}$ be the set of isomorphism classes of simple objects of the heart of a torsion pair in $A\text{-Mod}$ determined by $M$. Then

   $$(\mathcal{G} \cap \mathcal{I})/[M] = \langle \text{Add}(\mathcal{S}) \rangle_{2n}^{\{0,1\}},$$

   $$((\mathcal{G} \cap \mathcal{I})/[M])^c = (\mathcal{E} \cap \mathcal{I})/[M] = \langle \mathcal{S} \rangle_{2n}^{\{0,1\}}.$$

   where $n$ is the Loewy length of $\Gamma$.

3. $\dim ((\mathcal{E} \cap \mathcal{I})/[M]) \leq \min\{2n - 1, 2m + 1\} < \infty$, where $m$ is the global dimension of $\Gamma$. 
Heart of a torsion pair in $A$-Mod defined by $M$

Let

$\mathcal{Y} := \{ Y \in A\text{-Mod} \mid \text{Hom}_A(M[n], Y) = 0 \ \forall \ n \geq 0 \}$,

$\mathcal{X} := \{ X \in A\text{-Mod} \mid \text{Hom}_A(X, Y) = 0 \ \forall \ Y \in \mathcal{Y} \}$.

$\implies (\mathcal{X}, \mathcal{Y})$: torsion pair in $A$-Mod.

[Beilinson, Bernstein and Deligne(1982)]:
The category $\mathcal{H} := \mathcal{X} \cap \mathcal{Y}[1]$, called the heart of $(\mathcal{X}, \mathcal{Y})$, is an abelian category.

Lemma

$\exists$ equivalence of abelian categories:

$\mathcal{H} \xrightarrow{\sim} \Gamma\text{-Mod}$.

[M. Hoshino, Y. Kato and J-I. Miyachi, On t-structures and torsion theories induced by compact objects, *J. Pure Appl. Algebra* 167 (2002) 15-35.]
Construction of \((\Phi, \Psi) : A\text{-Mod} \to A\text{-Mod}\)

Let \(A_M = A \oplus M_0\), \(\Lambda := \text{End}_A(M)\),
\(e^2 = e \in \Lambda\) corresponding to the direct summand \(A\) of \(M\),
\(S_e : \Lambda\text{-Mod} \to A\text{-Mod}, Y \mapsto eY\) the **Schur functor** \((A = e\Lambda e)\).

For \(\mathcal{X} \subseteq A\text{-Mod}\) and \(\mathcal{Y} \subseteq \Lambda\text{-Mod}\), define

\[
\mathcal{K}_{\text{ac}}(\mathcal{X}) := \{X^\bullet \in \mathcal{K}(\mathcal{X}) \mid X^\bullet \text{ is exact}\},
\]

\[
\mathcal{K}_{e\text{-ac}}(\mathcal{Y}) := \{Y^\bullet \in \mathcal{K}(\mathcal{Y}) \mid S_e(Y^\bullet) \text{ is exact}\}.
\]

If \(S_e : \mathcal{Y} \xrightarrow{\simeq} \mathcal{X}\) as additive categories, then

\[
S_e : \mathcal{K}_{e\text{-ac}}(\mathcal{Y}) \xrightarrow{\simeq} \mathcal{K}_{\text{ac}}(\mathcal{X}).
\]
Construction of \((\Phi, \Psi) : \text{A-Mod} \rightarrow \text{A-Mod}\)

\[
\begin{array}{ccc}
\text{A-Mod} & \xrightarrow{S} & \mathcal{H}_{\text{ac}}(\text{A-Proj}) & \xrightarrow{\text{Hom}_{A}(M,-)} & \mathcal{H}_{\text{e-ac}}(\text{Add}(\Lambda e)) & \xleftarrow{\text{inc}} & \mathcal{H}_{\text{e-ac}}(\Lambda-\text{Proj}) \\
\Phi & (\text{Id}) & \downarrow & \downarrow & \downarrow & \downarrow & \Phi \\
\text{A-Mod} & \xleftarrow{Z^0} & \mathcal{H}_{\text{ac}}(\text{A-Proj}) & \xleftarrow{\ell_M} & \mathcal{H}_{\text{ac}}(\text{Add}(M)) & \xleftarrow{\text{inc}} & \mathcal{H}_{\text{e-ac}}(\Lambda-\text{Proj}) \\
\Psi & & \downarrow\text{inclusion} & & & & \\
\end{array}
\]

\[
\begin{array}{cc}
\mathcal{H}_{\text{ac}}(\Lambda-\text{Proj}) & \xrightarrow{I_{\Lambda}} \mathcal{H}(\Lambda-\text{Proj}) \\
\text{I:inclusion} & \text{Q:localization} \\
\end{array}
\]

\[
\begin{array}{cc}
\mathcal{H}_{\text{e-ac}}(\Lambda-\text{Proj}) & \xrightarrow{Q_{\Lambda}} \mathcal{D}(\Lambda) \\
\end{array}
\]

\(\mathcal{H}\): homotopy category; 
\(\mathcal{D}\): derived category; 
\(Q_{\Lambda}\): taking homotopically projective resolutions; 
\(\ell_M\): taking total complexes of Cartan-Eilenberg injective coresolutions of complexes.

[H.X.Chen, Applications of hyperhomology to adjoint functors, *Comm. Algebra* **50** (1) (2022) 19-32.]
Related papers

[1] H.X.Chen and C.C.Xi, Homological theory of orthogonal modules, 1-40, arXiv:2208.14712.

[2] H.X.Chen, Ming Fang and C.C.Xi, Mirror-reflective algebras and Tachikawa’s second conjecture, 1-27, arXiv:2211.08037.

Theorem (Chen-Fang-Xi)

The following are equivalent for a field $k$.

(1) Tachikawa’s Second Conjecture holds for all symmetric algebras over $k$.

(2) Each indecomposable symmetric algebra over $k$ has no stratifying ideal apart from itself and 0.

(3) The supremum of stratified ratios of all indecomposable symmetric algebras over $k$ is less than 1.

$I := AeA$ with $e^2 = e$ is a stratifying ideal of $A$ if $Ae \otimes_{eAe} eA \simeq AeA$ and $\text{Tor}^{eAe}_n(Ae, eA) = 0$, $\forall n > 0$. 

Thank you very much!