Moore-Penrose inverse of distance Laplacians of trees are $\mathbb{Z}$ matrices

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(In memory of Professor Michael Neumann)

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Abstract

We show that all off-diagonal entries in the Moore-Penrose inverse of the distance Laplacian matrix of a tree are non-positive.

Keywords: Trees, distance matrices, Laplacian matrices, Distance Laplacian matrices, Moore-Penrose inverse.

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1 Introduction

Let $T$ be a tree on $n$ vertices $\{1, \ldots, n\}$. Suppose to each edge $(p, q)$ of $T$, a positive number $w_{pq}$ is assigned. We say that $w_{pq}$ is the weight of $(p, q)$. The distance between any two vertices $i$ and $j$, denoted by $d_{ij}$, is the sum of all the weights in the path connecting $i$ and $j$. The distance matrix of $T$ is then the $n \times n$ matrix with $(i, j)^{th}$ entry equal to $d_{ij}$ if $i \neq j$ and $d_{ii} = 0$ for all $i = 1, \ldots, n$.

Define

$$\eta_i = \sum_{j=1}^{n} d_{ij} \quad \text{and} \quad \Delta := \text{Diag}(\eta_1, \ldots, \eta_n).$$

The distance Laplacian of $T$ is then the matrix $L_D := \Delta - D$. Distance Laplacian matrices are introduced and studied in [4]. Recall that the classical Laplacian matrix of $T$ is $L := \nabla - A$, where $A$ is the adjacency matrix of $T$ and $\nabla$ is the diagonal matrix with $\nabla_{ii}$ equal to degree of vertex $i$. We note some properties common to $L_D$ and $L$.

(i) $L_D$ and $L$ are positive semidefinite.

(ii) Row sums of $L_D$ and $L$ are equal to zero.

(iii) $\text{rank}(L) = \text{rank}(L_D) = n - 1$.

(iv) All off-diagonal entries of $L$ and $L_D$ are non-positive.

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In this paper, we deduce a property of distance Laplacian matrices of trees which the classical Laplacian matrices do not have. An \( n \times n \) real matrix \( A = [a_{ij}] \) is called a \( Z \) matrix if all the off diagonal entries of \( A \) are non-positive. The objective of this paper is to show that \( L_D^\dagger \) is always a \( Z \) matrix. In the following example, we see that \( L \) is the classical Laplacian of a path on four vertices, but \( L^\dagger \) is not a \( Z \) matrix.

\[
L = \begin{bmatrix}
1 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 1
\end{bmatrix}
\quad \text{and} \quad
L^\dagger = \begin{bmatrix}
\frac{7}{8} & \frac{1}{8} & \frac{3}{8} & \frac{5}{8} \\
\frac{3}{8} & \frac{1}{8} & \frac{3}{8} & \frac{5}{8} \\
\frac{3}{8} & \frac{1}{8} & \frac{3}{8} & \frac{5}{8} \\
\frac{3}{8} & \frac{1}{8} & \frac{3}{8} & \frac{5}{8}
\end{bmatrix}.
\]

Distance matrices are well studied and have many interesting properties and applications: see [1]. Numerical experiments reveal several new results on distance matrices. For example, a perturbation result says that if \( D \) is the distance matrix of a tree and \( L \) is the Laplacian of some connected graph (with same number of vertices), then all entries in \((D^{-1} - L)^{-1}\) are non-negative: See Theorem 4.6 in [2]. A far reaching generalisation of this result for matrix weighted trees is shown in Theorem 2.1 in [3]. Again this result was motivated by numerical experiments. Our result here is also motivated by numerical computations.

If \( A \) is a \( Z \) matrix, then we say it is an \( M \) matrix if all the eigenvalues of \( A \) have a non-negative real part. A question in [6] asks when the group inverse of a singular irreducible matrix \( M \) matrix is again an \( M \) matrix. Another question in resistive electrical networks [8] asks when is the Moore-Penrose inverse of a Laplacian of a connected graph is an \( M \) matrix. Kirkland and Neumann [7] characterized all weighted trees whose Laplacian have this property. The result in this paper says that distance Laplacians of trees are irreducible \( M \) matrices and their Moore-Penrose inverses are also \( M \) matrices. (We note that group inverse and Moore-Penrose inverse coincide for Laplacians and distance Laplacians.)

Our proof techniques in this paper works only for trees. To extend the result in a more general setting, for example, resistance Laplacian matrices of connected graphs, additional new techniques are certainly required.

## 2 Preliminaries

### 2.1 Notation

(i) If \( A = [a_{ij}] \) is an \( n \times n \) matrix with \((i, j)\)th entry equal to \( a_{ij} \), then the matrix \( A(i | j) \) will denote the submatrix of \( A \) obtained by deleting the \( i \)th row and the \( j \)th column of \( A \).

(ii) Let \( \Omega_1 := \{s_1, \ldots, s_k\} \) and \( \Omega_2 := \{t_1, \ldots, t_m\} \) be subsets of \( \{1, \ldots, n\} \). Then \( A[\Omega_1, \Omega_2] \) will be the \( k \times m \) matrix with \((i, j)\)th entry equal to \( a_{s_it_j} \). So, \( A = [a_{s_it_j}] \).
(iii) The vector of all ones in \( \mathbb{R}^n \) will be denoted by \( \mathbf{1} \). If \( m < n \), then \( \mathbf{1}_m \) will denote the vector of all ones in \( \mathbb{R}^m \). The notation \( J \) will stand for the symmetric matrix with all entries equal to 1.

(iv) The Moore-Penrose inverse of a matrix \( B \) is denoted by \( B^\dagger \) and its transpose by \( B' \).

(v) To denote the subgraph induced by a set of vertices \( W \), we use the notation \( [W] \). If \( a \) and \( b \) are any two vertices, then \( P_{ab} \) will be the path connecting \( a \) and \( b \). The set of all vertices of a subgraph \( H \) is denoted by \( V(H) \).

2.2 Basic results and techniques

We shall use the following results. Let \( T \) be a tree on \( n \) vertices labelled \( \{1, \ldots, n\} \) where \( n \geq 3 \). Let \( D = [d_{ij}] \) be the distance matrix of \( T \).

(P1) (Theorem 3.4, [5]) If \( L \) is the Laplacian matrix of a weighted tree and \( L^\dagger = [\alpha_{ij}] \), then

\[
d_{ij} = \alpha_{ii} + \alpha_{jj} - 2\alpha_{ij}.
\]

Because \( \text{rank}(L) = n - 1 \) and \( L \) is positive semidefinite, it follows that for any \( 0 \neq x = (x_1, \ldots, x_n)' \in \mathbb{R}^n \) such that \( \sum_{i=1}^{n} x_i = 0 \),

\[
\sum_{i,j} x_i x_j d_{ij} = -2 \sum_{i,j} x_i x_j \alpha_{ij} < 0.
\]

(P2) Triangle inequality: If \( i, j, k \in \{1, \ldots, n\} \), then

\[
d_{ik} \leq d_{ij} + d_{jk}.
\]

(P3) Let \( \nu \) be a positive integer and the sets \( L_1, \ldots, L_N \) partition \( \{1, \ldots, \nu\} \). Let \( A = [a_{uv}] \) be a \( \nu \times \nu \) matrix such that \( A[L_i, L_j] = O \) for all \( i < j \). Then there exists a permutation matrix \( P \) such that

\[
P'AP = \begin{bmatrix}
A[L_1, L_1] & O & \cdots & O \\
A[L_2, L_1] & A[L_2, L_2] & \cdots & O \\
\vdots & \vdots & \ddots & \vdots \\
A[L_N, L_1] & A[L_N, L_2] & \cdots & A[L_N, L_N]
\end{bmatrix}.
\]

As a consequence,

(a) If \( a_{xy} = 0 \) for all \( x \in L_i, y \in L_j \) and \( i < j \), then \( A \) is similar to a block lower triangular matrix with \( i^{th} \) diagonal block equal to \( A[L_i, L_i] \).
(b) If \( a_{xy} = 0 \) for all \( x \in L_i, y \in L_j \) and \( i > j \), then \( A \) is similar to a block upper triangular matrix with \( i^{th} \) diagonal block equal to \( A[L_i, L_i] \).

(c) If \( a_{xy} = 0 \) for all \( x \in L_i, y \in L_j \) and \( i \neq j \), then \( A \) is similar to a block diagonal matrix with \( i^{th} \) diagonal block equal to \( A[L_i, L_i] \).

(P4) (Matrix determinant lemma) Let \( A \) be a \( m \times m \) matrix and \( x, y \) be \( m \times 1 \) vectors. Then

\[
\text{det}(A + xy') = \text{det}(A) + y' \text{adj}(A)x.
\]

3 Main result

Consider a tree \( T \) with vertices labelled \( \{1, \ldots, n\} \). If \( n = 2 \), then the result is easy to verify. We assume \( n > 2 \). Let \( D := [d_{ij}] \) denote the distance matrix of \( T \) and \( \eta_i \) be the \( i^{th} \) row sum of \( D \). Define \( \Delta := \text{Diag}(\eta_1, \ldots, \eta_n) \). Let

\[
S := \Delta - D.
\]

Each row sum of \( S \) is zero. So, all the cofactors of \( S \) are equal. Let this common cofactor be \( \gamma \). Henceforth, we fix the notation \( T, D, \Delta \) and \( S \). Our aim is to show that all the off-diagonal entries of \( S^\dagger = [s^\dagger_{ij}] \) are non-positive. We shall show that \( s^\dagger_{12} \leq 0 \). A similar argument can be repeated to any other off-diagonal entry of \( S^\dagger \). In the first step, we show that \( s^\dagger_{12} \leq 0 \) if and only if the determinant of a certain matrix constructed from \( D \) is non-negative.

3.1 Reformulation of the problem

We shall find a matrix \( R \) such that that \( s^\dagger_{12} \leq 0 \) if and only if \( \text{det}(R) \geq 0 \). The following lemma will be useful.

Lemma 1. Let \( S_* := S + J \). The following items hold.

(i) \( S \) is positive semidefinite and \( \text{rank}(S) = n - 1 \).

(ii) \( S_*^{-1} = S^\dagger + \frac{4}{n^2} \).

(iii) \( \text{det}(S_*) = n^2 \gamma \).

(iv) Let \( C = S(1|2) \). Then,

\[
s^\dagger_{12} = \frac{1'_{n-1}C^{-1}1_{n-1}}{n^2}.
\]

Proof. It follows from the definition of \( S \) that, \( S1 = 0 \). Let \( 0 \neq x \in 1^\perp \). By (P1), \( x'Dx > 0 \). So,

\[
x'Sx = x'\Delta x - x'Dx > 0.
\]
If $x \in \text{span}\{1\}$, then $Sx = 0$. Hence $S$ is positive semidefinite and $S$ has $n - 1$ positive eigenvalues. So, $\text{rank}(S) = n - 1$. This proves (i).

Since $S\mathbf{1} = 0$ and $\text{rank}(S) = n - 1$,

$$SS^\dagger = I - \frac{J}{n}.$$  

From a direct verification,

$$S^{-1}_* = S^\dagger + \frac{J}{n^2}.$$  

This proves (ii).

Writing $S_* = S + \mathbf{1}\mathbf{1}'$, by (P4),

$$\det(S_*) = \det(S) + 1'\text{adj}(S)\mathbf{1}.$$  

As $S\mathbf{1} = 0$, $\det(S) = 0$. Since all the cofactors of $S$ are equal to $\gamma$,

$$\text{adj}(S) = \gamma J.$$  

Therefore, $\det(S_*) = n^2\gamma$. Item (iii) is proved.

Each cofactor of $S$ is $\gamma$ and $C = S(1|2)$. So,

$$\det(C) = \det(S(1|2)) = -\gamma.$$  

In view of item (ii),

$$s^\dagger_{12} = (S^{-1}_*)_{12} - \frac{1}{n^2}$$  

$$= \frac{1}{\det(S_*)}(\text{adj}S_*)_{12} - \frac{1}{n^2}$$  

$$= -\frac{1}{n^2\gamma}(\det(S_*(1|2)) - \frac{1}{n^2}$$  

$$= -\frac{1}{n^2\gamma}(\det(S(1|2)) + J(1|2)) - \frac{1}{n^2}.$$  

By the matrix determinant lemma (P4),

$$s^\dagger_{12} = -\frac{1}{n^2\gamma}(\det(C) + \det(C)1'_{n-1}C^{-1}1_{n-1}) - \frac{1}{n^2}$$  

$$= -\frac{1}{n^2\gamma}(-\gamma - \gamma 1'_{n-1}C^{-1}1_{n-1}) - \frac{1}{n^2}$$  

$$= \frac{1'_{n-1}C^{-1}1_{n-1}}{n^2}.$$  

Item (iv) is proved and the proof is complete.
By numerical computations, we note that the sign of $s_{12}^\dagger$ depends on the sign of the determinant of a certain matrix defined from $D$. We introduce this matrix now.

**Definition 1.** For any $i, j \in \{3, \ldots, n\}$, define

$$R_{ij} := \begin{cases} -d_{21} + d_{i1} + d_{2j} - d_{ij} & i \neq j \\ -d_{21} + d_{i1} + d_{2i} + \sum_{k=1}^{n} d_{ik} & i = j. \end{cases}$$

Set $R := [R_{ij}]$.

The order of $R$ is $n - 2$. By triangle inequality,

$$-d_{21} + d_{i1} + d_{2i} \geq 0 \quad i = 3, \ldots, n.$$ 

Hence,

$$R_{ii} > 0 \quad i = 3, \ldots, n.$$ 

**Lemma 2.** $s_{12}^\dagger \leq 0$ if and only if $\det(R) \geq 0$.

**Proof.** Let $C = S(1|2)$ and $Q$ be the $(n - 1) \times (n - 1)$ matrix $\begin{bmatrix} 1 & 1_{n-2} \\ 0 & I_{n-2} \end{bmatrix}$. Then,

$$Q^{-1} = \begin{bmatrix} 1 & -1_{n-2} \\ 0 & I_{n-2} \end{bmatrix}.$$ 

By a direct computation,

$$Q'^{-1}CQ^{-1} = \begin{bmatrix} s_{21} & -s_{21} + s_{23} & \cdots & -s_{21} + s_{2n} \\ -s_{21} + s_{31} & s_{21} - s_{31} - s_{23} + s_{33} & \cdots & s_{21} - s_{31} - s_{2n} + s_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ -s_{21} + s_{n1} & s_{21} - s_{n1} - s_{23} + s_{n3} & \cdots & s_{21} - s_{n1} - s_{2n} + s_{nn} \end{bmatrix}. $$

The entries of $S$ are

$$s_{ij} = \begin{cases} -d_{ij} & i \neq j \\ \sum_{k=1}^{n} d_{ik} & i = j. \end{cases}$$

Hence,

$$Q'^{-1}CQ^{-1}(1|1) = R. \quad (1)$$

We note that

$$\det(C) = -\gamma < 0 \quad \text{and} \quad \det(Q) = 1.$$ 

So,

$$\det(Q'^{-1}CQ^{-1}) = -\gamma < 0.$$ 

By a simple computation,

$$(QC^{-1}Q')_{11} = 1'_{n-1}C^{-1}1_{n-1}. \quad (2)$$
Using (1),

\[
(QC^{-1}Q')_{11} = \frac{1}{\text{det}(Q'^{-1}CQ^{-1}) \text{det}(Q'^{-1}CQ^{-1}(1|1))} = -\frac{1}{\gamma} \text{det}(R).
\]  

(3)

By (2) and (3),

\[
1'_{n-1}C^{-1}1_{n-1} = -\frac{1}{\gamma} \text{det}(R).
\]

By item (iv) in Lemma 1, it now follows that \(s_{12}^t \leq 0\) if and only if \(\text{det}(R) \geq 0\).

\[\square\]

3.2 A property of \(R\)

We now proceed to show that \(\text{det}(R) \geq 0\). The following lemma will be useful in the sequel.

Lemma 3. Let \(\alpha \in V(P_{12})\). Suppose there exists a connected component \(\tilde{X}\) of \(T \setminus (\alpha)\) not containing 1 and 2. Let \(u \in V(\tilde{X})\) be the vertex adjacent to \(\alpha\). Consider a connected subgraph \(X\) of \(\tilde{X}\) containing \(u\). Put \(E := V(X)\). Then, \(R[E, E]\) is a positive semidefinite matrix.

Proof. If \(E = \{u\}\), then \(R[E, E] = [R_{uu}]\). As \(R_{uu} > 0\), the lemma is true in this case. Let \(|E| \geq 2\).

Claim 1: \(R[E, E]\) is symmetric.

Let \(r, s \in E\). Recall that

\[
R_{rs} = -d_{21} + d_{r1} + d_{2s} - d_{rs}.
\]  

(4)

By our assumption,

\[
d_{r1} = d_{r\alpha} + d_{\alpha 1} \quad \text{and} \quad d_{s2} = d_{s\alpha} + d_{\alpha 2}.
\]  

(5)

By (4) and (5),

\[
R_{rs} = -d_{21} + d_{r\alpha} + d_{\alpha 1} + d_{s\alpha} + d_{\alpha 2} - d_{rs}.
\]  

(6)

Again by our assumption,

\[
d_{r2} = d_{r\alpha} + d_{\alpha 2} \quad \text{and} \quad d_{s1} = d_{s\alpha} + d_{\alpha 1}.
\]  

(7)

By (6) and (7),

\[
R_{rs} = -d_{21} + d_{s1} + d_{r2} - d_{rs}.
\]

The right hand side of the above equation is \(R_{sr}\). The claim is proved.

We know that \(u \in E\) and is adjacent to \(\alpha\). Let \(\Omega\) be the set of all non-pendant vertices in \(T\). Since \(X\) is connected, and has at least two vertices, \(u\) is adjacent to a vertex in \(E\). Hence, \(u \in E \cap \Omega\). So, \(E \cap \Omega\) is a non-empty set.
Let $\delta \in E$ be such that 
\[ d_{\delta \alpha} = \max \{d_{x\alpha} : x \in E \cap \Omega\}. \]
Since $X$ is a tree, there exists a pendant vertex adjacent to $\delta$. Without loss of generality, let $E = \{x_1, \ldots, x_{t-1}, x_t\}$ where $x_1 = u$, $x_{t-1} = \delta$ and $x_t$ is pendant vertex adjacent to $x_{t-1}$.

**Claim 2:**
\[ R_{ix_{t-1}} = R_{ix_t} \text{ for all } i \in \{x_1, x_2, \ldots, x_{t-2}\}. \] (8)

By the definition of $R$,
\[ R_{ix_{t-1}} - R_{ix_t} = -d_{21} + d_{i1} + d_{ix_{t-1}} - d_{ix_t} - (-d_{21} + d_{i1} + d_{2x_t} - d_{ix_t}) \]
\[ = (d_{2x_{t-1}} - d_{2x_t}) - (d_{ix_{t-1}} - d_{ix_t}). \] (9)

Since $x_t$ is a pendant vertex and is adjacent to $x_{t-1}$, we have 
\[ d_{2x_{t-1}} - d_{2x_t} = -d_{x_t x_{t-1}}, \]
and since $i \in \{x_1, \ldots, x_{t-2}\}$, we have 
\[ d_{ix_{t-1}} - d_{ix_t} = -d_{x_t x_{t-1}}. \]

By (9), we now get 
\[ R_{ix_{t-1}} = R_{ix_t}. \]
The claim is true.

**Claim 3:** $R_{ix_{t-1}} = 2d_{\delta \alpha} = 2d_{x_{t-1} \alpha}$. 

By definition,
\[ R_{ix_{t-1}} = -d_{21} + d_{x_{t-1}1} + d_{2x_t} - d_{x_{t-1}x_t}, \] (10)

Vertices 2 and $x_t$ belong to different components of $T \setminus (\alpha)$. Also, $x_t$ and $x_{t-1}$ are adjacent and $x_t$ is pendant in $X$. Hence, 
\[ d_{2x_t} = d_{2\alpha} + d_{\alpha x_t} = d_{2\alpha} + d_{\alpha x_{t-1}} + d_{x_{t-1}x_t}, \] (11)

By (10) and (11),
\[ R_{ix_{t-1}} = -d_{21} + d_{x_{t-1}1} + d_{2\alpha} + d_{\alpha x_{t-1}}. \]

As $\alpha \in V(P_{12})$, $d_{21} = d_{2\alpha} + d_{\alpha 1}$. Hence 
\[ R_{ix_{t-1}} = -d_{\alpha 1} + d_{x_{t-1}1} + d_{\alpha x_{t-1}}. \] (12)

As $1 \not\in V(\bar{X})$, $d_{x_{t-1}} = d_{x_{t-1}1} + d_{\alpha 1}$. Hence by (12),
\[ R_{ix_{t-1}x_{t}} = 2d_{\alpha x_{t-1}}. \] (13)

This proves the claim.
Claim 4: Diagonal entries of $R[E, E]$ are greater than or equal to $2d_{\delta\alpha}$.

Let $r \in E$. By definition,

$$R_{rr} = -d_{21} + d_{r1} + d_{2r} + \sum_{k=1}^{n} d_{rk}. \quad (14)$$

Since $r \in E$, $1 \notin E$ and $2 \notin E$,

$$d_{r1} = d_{r\alpha} + d_{1\alpha} \text{ and } d_{r2} = d_{r\alpha} + d_{2\alpha}. \quad (15)$$

By (14) and (15),

$$R_{rr} = -d_{21} + d_{r\alpha} + d_{1\alpha} + d_{r\alpha} + d_{2\alpha} + \sum_{k=1}^{n} d_{rk}. \quad (16)$$

As $d_{21} = d_{2\alpha} + d_{\alpha1}$, (16) simplifies to

$$R_{rr} = 2d_{r\alpha} + \sum_{k=1}^{n} d_{rk}. \quad (17)$$

Case 1: Suppose $r \notin \{\delta, x_t\}$.

Then

$$R_{rr} = 2d_{r\alpha} + \sum_{k=1}^{n} d_{rk} \geq 2d_{r\alpha} + d_{r\delta} + d_{rx_t}. \quad (18)$$

By triangle inequality,

$$d_{r\alpha} + d_{r\delta} \geq d_{\delta\alpha} \text{ and } d_{r\alpha} + d_{rx_t} \geq d_{x_t\alpha}.$$ 

In view of (18),

$$R_{rr} \geq d_{x_t\alpha} + d_{\delta\alpha}.$$ 

As $x_t$ is adjacent only to $\delta$,

$$R_{rr} \geq d_{x_t\alpha} + d_{\delta\alpha} = d_{\delta\alpha} + d_{\delta x_t} + d_{\delta\alpha} \geq 2d_{\delta\alpha}.$$ 

Case 2: If $r = \delta$, then it is immediate from (17).

Case 3: Suppose $r = x_t$.

By (17),

$$R_{x_tx_t} \geq 2d_{x_t\alpha}.$$ 

Since $x_t$ is pendant and adjacent to $\delta$,

$$d_{x_t\alpha} = d_{\delta\alpha} + d_{x_t\delta} = d_{\delta\alpha} + d_{\delta x_t} \geq d_{\delta\alpha}.$$ 

By the two previous inequalities,

$$R_{x_tx_t} \geq 2d_{\delta\alpha}.$$
The claim is proved.
Define a $t \times t$ matrix

$$P := \begin{bmatrix}
2d_{\delta} & R_{x_1x_2} & \cdots & R_{x_1x_t} \\
R_{x_2x_1} & 2d_{\delta} & \cdots & R_{x_2x_t} \\
\vdots & \vdots & \ddots & \vdots \\
R_{x_{t-1}x_1} & R_{x_{t-1}x_2} & \cdots & 2d_{\delta}
\end{bmatrix}.$$ 

Because $R[E, E]$ is a symmetric matrix, $P$ is symmetric.

**Claim 5:** $P$ is positive semidefinite.

We will prove this by using induction on $|E|$. Suppose $|E|$ has only two vertices. Write $E = \{x_1, x_2\}$, where $u = x_1$. By a simple verification,

$$P = \begin{bmatrix}
2d_{\alpha} & 2d_{\alpha} & 2d_{\alpha} \\
2d_{\alpha} & 2d_{\alpha} & 2d_{\alpha}
\end{bmatrix}.$$ 

So, the claim is true in this case. Suppose the result is true if $|E| < t$. Define a $t \times t$ matrix by

$$Q_1 := \begin{bmatrix}
I_{t-2} & 0 & 0 \\
0 & 1 & -1 \\
0 & 0 & 1
\end{bmatrix}.$$ 

We show that $Q_1^tPQ_1$ is positive semidefinite. Claim 2 and Claim 3 imply that the last two columns of $P$ are equal. Hence, by direct computation,

$$Q_1^tPQ_1 = \begin{bmatrix}
P(x_t|x_t) & 0 \\
0' & 0
\end{bmatrix}.$$ 

(19)

Define

$$X' := X \setminus \{x_t\}.$$ 

Because $x_t$ is pendant, $X'$ is a connected subgraph of $X$ and $u \in V(X')$. Set

$$E' := V(X') = \{x_1, \ldots, x_{t-1}\}, \text{ where } x_1 = u.$$ 

Define

$$d_{\mu_1} := \max\{d_{\alpha x} : x \in \Omega \cap E'\}.$$ 

By induction hypothesis,

$$P_1 := \begin{bmatrix}
2d_{\mu} & R_{x_1x_2} & \cdots & R_{x_1x_{t-1}} \\
R_{x_2x_1} & 2d_{\mu} & \cdots & R_{x_2x_{t-1}} \\
\vdots & \vdots & \ddots & \vdots \\
R_{x_{t-1}x_1} & R_{x_{t-1}x_2} & \cdots & 2d_{\mu}
\end{bmatrix}.$$ 

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is positive semidefinite. Put

\[
P_2 := \begin{bmatrix}
2d_{\delta \alpha} - 2d_{\mu \alpha} & 0 & \ldots & 0 \\
0 & 2d_{\delta \alpha} - 2d_{\mu \alpha} & \ldots & 0 \\
\ldots & \ldots & \ddots & \ldots \\
0 & 0 & \ldots & 2d_{\delta \alpha} - 2d_{\mu \alpha}
\end{bmatrix}.
\]

Then,

\[P(x_t|x_t) = P_1 + P_2.\]

Since \(d_{\delta \alpha} - d_{\mu \alpha} \geq 0\), \(P(x_t|x_t)\) is positive semidefinite and so is \(P\). This proves the claim.

Define

\[\Lambda := \text{Diag}(R_{x_1x_1} - 2d_{\delta \alpha}, \ldots, R_{x_tx_t} - 2d_{\delta \alpha}).\]

Then,

\[R[E,E] = P + \Lambda.\]

By Claim 4, \(\Lambda\) is positive semidefinite and by the previous claim, \(P\) is positive semidefinite. So, \(R[E,E]\) is positive semidefinite. The proof is complete.

\[\square\]

### 3.3 Partitioning \(\{3, \ldots, n\}\)

Let the degree of vertex 1 be \(m\). We denote the vertex sets of \(m\) components of \(T \setminus (1)\) by

\[V_1', V_2', \ldots, V_m'.\]

Assume \(2 \in V_1'\). Define

\[V_1 := V_1' \setminus \{2\}.\]

**Illustration**

Consider Figure 1: \(T_{16} \setminus (1)\) has three components. Vertex set of these components

![Figure 1: Tree \(T_{16}\) on 16 vertices](image-url)
are

\[ V'_1 = \{3, 13, 11, 12, 4, 14, 2, 5, 6, 15, 16\}, \quad V_2 = \{10\}, \quad V_3 = \{7, 9, 8\}. \]

Now,

\[ V_1 = V'_1 \setminus \{2\} = \{3, 13, 11, 12, 4, 14, 5, 6, 15, 16\}. \]

### 3.4 A canonical form of \( R \)

We now show that \( R \) is similar to a block lower triangular matrix.

**Lemma 4.** \( R \) is similar to a block lower triangular matrix with diagonal blocks equal to \( R[V_1, V_1], R[V_2, V_2], \ldots, R[V_m, V_m] \).

**Proof.** We know that 

\[ V_1 \cup \cdots \cup V_m = \{3, \ldots, n\} \text{ and } V_i \cap V_j = \emptyset. \]

Since

\[ R = [R_{\alpha\beta}] \quad 3 \leq \alpha, \beta \leq n, \]

by item (a) in (P3), it suffices to show that if \( i < j \), \( x \in V_i \) and \( y \in V_j \), then

\[ R_{xy} = 0. \]

By definition,

\[ R_{xy} = -d_{21} + d_{x1} + d_{2y} - d_{xy}. \quad (20) \]

Since \( x \) and \( y \) belong to different components of \( T \setminus (1) \),

\[ d_{xy} = d_{x1} + d_{y1}. \quad (21) \]

Using (21) in (20),

\[ R_{xy} = -d_{21} + d_{x1} + d_{2y} - d_{x1} - d_{y1} \]

\[ = -d_{21} + d_{2y} - d_{y1}. \quad (22) \]

We recall that \( 2 \in V'_1 \) and \( y \in V_j \). Since \( i < j \), we see that \( 1 < j \). Hence, 2 and \( y \) belong to different components of \( T \setminus (1) \). Thus, \( d_{2y} = d_{21} + d_{y1} \). By (22), \( R_{xy} = 0 \).

The proof is complete.

The following is immediate.

**Corollary 1.**

\[ \det(R) = \prod_{i=1}^{m} \det(R[V_i, V_i]). \]

**Lemma 5.**

\[ \det(R[V_j, V_j]) \geq 0 \quad j = 2, \ldots, m. \]

**Proof.** Let \( j \in \{2, \ldots, m\} \). Put \( \tilde{X} = X = [V_j], \quad E = V_j \) and \( \alpha = 1 \) in Lemma 3. The result now follows. \( \square \)
### 3.5 A canonical form of $R[V_1, V_1]$

We partition $V_1$. Define

$$V_A := \{y \in V_1 : 2 \notin V(P_1y)\}.$$  

$$V_B := \{y \in V_1 : 2 \in V(P_1y)\}.$$  

Then,

$$V_1 = V_A \cup V_B \quad \text{and} \quad V_A \cap V_B = \emptyset.$$  

For example, in Figure 1,

$$V_A = \{3, 4, 14, 11, 13, 12\} \quad \text{and} \quad V_B = \{5, 6, 15, 16\}.$$  

**Lemma 6.** Let $x \in V_B$ and $y \in V_A$. Then $2 \in V(P_{xy})$.

**Proof.** Suppose

$$2 \notin V(P_{xy}). \tag{23}$$  

Since $y \in V_A$,

$$2 \notin V(P_1y). \tag{24}$$  

Equations (23) and (24) imply $2 \notin V(P_{1x})$. This contradicts $x \in V_B$. The proof is complete. \hfill \Box

**Lemma 7.** $R[V_1, V_1]$ similar to a block upper triangular matrix with diagonal blocks equal to $R[V_A, V_A]$ and $R[V_B, V_B]$.

**Proof.** Let $x \in V_B$ and $y \in V_A$. In view of item (b) in (P3), it suffices to show that $R_{xy} = 0$.

By the previous Lemma

$$d_{2y} + d_{2x} = d_{xy}. \tag{25}$$  

By Definition 1,

$$R_{xy} = -d_{21} + d_{x1} + d_{2y} - d_{xy}. \tag{26}$$  

As $x \in V_B$,

$$d_{1x} - d_{21} = d_{2x}. \tag{27}$$  

By (26) and (27),

$$R_{xy} = d_{2x} + d_{2y} - d_{xy}.$$  

Equation (25) now gives

$$R_{xy} = 0.$$  

The proof is complete. \hfill \Box

The following is now immediate.

**Corollary 2.**

$$\det(R[V_1, V_1]) = \det(R[V_A, V_A])\det(R[V_B, V_B]).$$
3.6 Partitioning $V_A$

We partition $V_A$. Let $P_{12}$ be the path with vertices $\{1, u_1, \ldots, u_q, 2\}$ and edges 

$$(1, u_1), (u_1, u_2), \ldots, (u_q, 2).$$

Define 

$$U_i := \{y \in V_A : d_{u_i y} \leq d_{u_j y} \text{ for all } i \neq j\} \quad i = 1, \ldots, q.$$ 

(We can think of $U_i$ as the collection of vertices in $V_A$ which are nearer to $u_i$ than $u_j$.) Clearly $u_i \in U_i$. To illustrate, consider Figure 1. Here, 

$$u_1 = 3, \quad U_1 = \{3, 11, 12, 13\}, \quad u_2 = 4, \quad U_2 = \{4, 14\}.$$ 

Lemma 8. The following items hold.

(i) If $y \in U_i$, then $u_i \in V(P_{1y}) \cap V(P_{y1}).$

(ii) If $y \in U_i$, then $u_i \in V(P_{yu_j})$ for $j \neq i$.

(iii) $U_1, \ldots, U_q$ partition $V_A$.

(iv) Let $y \in U_i$ and $z \in U_j$. If $i \neq j$, then 

$$P_{yz} = P_{yu_i} \cup P_{u_iu_j} \cup P_{u_jz}.$$ 

(v) Each $[U_i]$ is a tree.

Proof. We prove (i) now. Assume that $u_i \notin V(P_{1y})$. Because $u_1$ is the only vertex in $V_1$ adjacent to 1, $i \neq 1$. So, $u_i \notin V(P_{u_1y})$. Then, $P_{u_1y} \cup P_{u_1u_i}$ contains $P_{yu_i}$. Now, $u_{i-1} \in V(P_{yu_i})$. This implies 

$$d_{u_{i-1}y} < d_{u_i y}.$$ 

But this cannot happen as $y \in U_i$. So, 

$$u_i \in V(P_{1y}).$$

Using a similar argument, 

$$u_i \in V(P_{2y}).$$

This proves (i).

Let $j > i$. By (i) and from the definition of $u_i$ and $u_j$, 

$$u_i \in V(P_{2y}) \text{ and } u_j \in V(P_{2u_i}).$$

Thus, 

$$P_{2y} = P_{2u_j} \cup P_{u_ju_i} \cup P_{u_iy}.$$
The above equation implies
\[ u_i \in V(P_{yu_j}) \text{ for all } j > i. \]

A similar argument leads to
\[ u_i \in V(P_{yu_j}) \text{ for all } j < i. \]

The proof of (ii) is complete.

Let \( y \in U_i \cap U_j \), where \( j \neq i \). By (ii), it now follows that
\[ u_i \in V(P_{u_jy}) \text{ and } u_j \in V(P_{u_iy}). \]

As these two cannot happen simultaneously, \( y \notin U_i \cap U_j \). Thus,
\[ U_i \cap U_j = \emptyset. \]

By definition of \( U_1, \ldots, U_q \),
\[ U_1 \cup \cdots \cup U_q \subseteq V_A. \]

Let \( x \in V_A \) and \( k \in \{1, \ldots, q\} \) be such that
\[ d_{xuk} := \min(d_{xu_1}, \ldots, d_{xu_q}). \]

Then, \( x \in U_k \). Hence
\[ V_A \subseteq U_1 \cup \cdots \cup U_q. \]

So,
\[ V_A = U_1 \cup \cdots \cup U_q. \]

The proof of (iii) is complete.

The proof of (iv) follows from (ii).

We now show that \([U_i]\) is a tree. Let \( y \in U_i \). Since \( y, u_i \in V'_{1} \) and \([V'_{1}]\) is a tree, we have
\[ V(P_{yu_i}) \subseteq V'_1. \tag{28} \]

To show that \([U_i]\) is a tree, it now suffices to show that \( V(P_{yu_i}) \subseteq U_i \). Let \( x \in V(P_{yu_i}) \). Assuming \( x \notin U_i \), we shall get a contradiction. By (28), we now have only three cases:

(a) \( x \in U_j \) for some \( j \neq i \)  
(b) \( x = 2 \)  
(c) \( x \in V_B \).

Assume (a). In view of item (iv) above, we get \( u_i \in V(P_{xy}) \). But then \( x \notin V(P_{yu_i}) \). This is a contradiction. So, (a) is not true.

If (b) is true, then \( 2 \in V(P_{yu_i}) \). However (i) implies \( u_i \in V(P_{y2}) \). This is a contradiction.

If we assume (c), then \( x \in V(P_{yu_i}) \). By Lemma 6, \( 2 \in V(P_{yx}) \) and therefore \( 2 \in V(P_{yu_i}) \) implying case (b) is true which is a contradiction.

Hence, \( V(P_{yu_i}) \subseteq U_i \). So, \([U_i]\) is a tree. This completes the proof. \( \square \)
3.7 A canonical form of $R[V_A, V_A]$

We now show that $R[V_A, V_A]$ is similar to a block upper triangular matrix.

Lemma 9. $R[V_A, V_A]$ is similar to a block upper triangular matrix with $i$th diagonal block equal to $R[U_i, U_i]$.

Proof. Let $i > j$. Pick any two elements $r \in U_i$ and $s \in U_j$. By item (c) in (P3), it suffices to show that

$$R_{rs} = 0.$$  \hfill (29)

We recall that

$$R_{rs} = -d_{21} + d_{r1} + d_{2s} - d_{rs}.$$  \hfill (29)

By item (i) and (iv) of Lemma 8

$$d_{r1} = d_{ru1} + d_{u1}, \quad d_{2s} = d_{2u_j} + d_{uj}s \quad \text{and} \quad d_{rs} = d_{ru1} + d_{ujui} + d_{su_j}.$$  \hfill (30)

Thus (29) and (30) give

$$R_{rs} = -d_{21} + d_{ru1} + d_{u1} + d_{2u_j} + d_{uj}s - (d_{ru1} + d_{ujui} + d_{su_j})$$

$$= -d_{21} + d_{u1} + d_{2u_j} - d_{ujui}.$$  

Since $i > j$ and $P_{12}$ has edges $(u_k, u_{k+1}),$

$$-d_{21} + d_{u1} = -d_{u2} \quad \text{and} \quad d_{2u_j} - d_{ujui} = d_{2u_i}.$$  

Thus, $R_{rs} = 0$. This completes the proof. \hfill \Box

The following is immediate now.

Corollary 3.

$$\det(R[V_A, V_A]) = \prod_{i=1}^{q} \det(R[U_i, U_i]).$$

3.8 Computation of $\det(R[U_i, U_i])$

Fix $i \in \{1, \ldots, q\}$. We further partition $U_i$ into disjoint sets. Let $u_i$ have $p_i$ adjacent vertices in $[U_i]$. Then, $[U_i] \setminus (u_i)$ have $p_i$ components:

$$G_{i1}, \ldots, G_{ip_i}.$$  

Define $Q_{ik} := V(G_{ik})$. For example in Figure 1 for $i = 1$, we have

$$u_1 = 3, \quad U_1 = \{12, 11, 3, 13\}.$$  

There are two components in $[U_1] \setminus (3)$. The vertices of these components are

$$Q_{11} = \{12, 11\} \quad \text{and} \quad Q_{12} = \{13\}.$$  

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Lemma 10. The following items hold.

(i) $\det(R[U_i, U_i]) = R_{u_i u_i} \left( \prod_{k=1}^{p_i} \det(R[Q_{ik}, Q_{ik}]) \right)$.

(ii) $G_{i1}, \ldots, G_{ip_i}$ are connected components of $T \setminus (u_i)$.

(iii) $\det(R[U_i, U_i]) \geq 0$.

Proof. Let $a \in Q_{ir}, b \in Q_{is}$ and $r \neq s$. By definition,

$$R_{ab} = -d_{a1} + d_{21} + d_{2b} - d_{ab}.$$

Since $u_i \in V(P_{12})$,

$$R_{ab} = -d_{a1} + d_{21} + d_{2b} - d_{ab}. \quad (31)$$

As $a \in U_i$, it follows from item (i) of Lemma 8 that

$$d_{au_i} = d_{a1} - d_{1a_i}. \quad (32)$$

Using (32) in (31),

$$R_{ab} = -d_{2a_i} + d_{a1} + d_{2b} - d_{ab}. \quad (33)$$

As $b \in U_i$, it follows from item (i) of Lemma 8 that

$$d_{bu_i} = d_{2b} - d_{2u_i}. \quad (34)$$

Using (34) in (33),

$$R_{ab} = d_{bu_i} + d_{a1} - d_{ab}. \quad (35)$$

Finally, since $a$ and $b$ belong to different components of $[U_i] \setminus (u_i)$,

$$d_{au_i} + d_{bu_i} = d_{ab}. \quad (36)$$

Using (36) in (35),

$$R_{ab} = 0.$$

We now show that $R_{ui_x} = 0$ for any $x \in Q_{is}$. By definition,

$$R_{ui_x} = -d_{u1} + d_{u1} + d_{x2} - d_{ux}. \quad (38)$$

Since $u_i$ lies on $P_{12}$,

$$R_{ui_x} = -d_{u2} + d_{2x} - d_{ux}. \quad (37)$$

As $x \in Q_{is} \subset U_i$, by part (i) of Lemma 8,

$$d_{ux} + d_{2u_i} = d_{2x}. \quad (38)$$

By (37) and (38),

$$R_{ui_x} = 0.$$
Similarly,
\[ R_{xu_i} = 0. \]

By item (c) in (P3), we now conclude that \( R[U_i, U_i] \) is similar to a block diagonal matrix with diagonal blocks
\[ R_{u_iu_i}, \quad R[Q_{ik}, Q_{ik}] \quad k = 1, \ldots, p_i. \]

Hence
\[ \det(R[U_i, U_i]) = R_{u_iu_i} \left( \prod_{k=1}^{p_i} \det(R[Q_{ik}, Q_{ik}]) \right). \]

This completes the proof of (i).

By definition \( G_1, \ldots, G_{p_i} \) are connected components of \( [U_i] \setminus (u_i) \). So, each \( G_{ik} \) is connected. Suppose \( G_{ik} \) is not a connected component of \( T \setminus (u_i) \). Then, there exists \( v \in V(T) \setminus \{u_i\} \) but not in \( Q_{ik} \) such that \( v \) is adjacent to a vertex \( g \in Q_{ik} \).

Suppose \( v \in Q_{ij} \) for some \( j \neq k \). But \( Q_{ik} \) and \( Q_{ij} \) are components of \( [U_i] \setminus (u_i) \) and hence \( u_i \in V(P_{gv}) \). This is not possible.

Suppose \( v \in U_j \) where \( j \neq i \). Then, item (iv) in Lemma 8 implies \( u_i \in V(P_{gv}) \). This is not possible.

Suppose \( v \in V_B \). Then, Lemma 9 gives \( 2 \in V(P_{vg}) \). Again, this is not possible.

Let \( v \in V_2 \cup \cdots \cup V_{m} \). Since \( g \in V_1, 1 \in P_{gv} \). This is a contradiction. Thus, \( G_{ik} \) is a connected component of \( T \setminus (u_i) \). The proof of (ii) is complete.

Fix \( k \in \{1, \ldots, p_i\} \). We use Lemma 6. Set \( X = X = G_{ik}, E = Q_{ik} \) and \( \alpha = u_i \).

By (ii), \( X \) is a connected component of \( T \setminus (u_i) \). Hence \( \det(R[Q_{ik}, Q_{ik}]) \) \geq 0. In view of item (i), \( \det(R[U_i, U_i]) \) \geq 0. The proof is complete.

From Corollary 3 and Lemma 10, we get the following.

Lemma 11.
\[ \det(R[V_A, V_A]) \geq 0. \]

3.9 Computation of \( \det(R[V_B, V_B]) \)

Let \( [V_B] \) have \( s \) components and let the vertex sets of these components be
\[ W_1, \ldots, W_s. \]

For example in Figure 11 \( [V_B] \) has two components
\[ W_1 = \{5, 6\}, \quad W_2 = \{15, 16\}. \]

Lemma 12. Let \( z_i \in W_i \) and \( z_j \in W_j \) where \( i \neq j \). Then \( 2 \in V(P_{z_iz_j}) \).
Proof. Since \( z_i \) and \( z_j \) belong to different components of \( [V_B] \), there must exist a vertex \( x \) such that
\[
x \in V'_1, \quad x \notin V_B, \quad \text{and} \quad x \in V(P_{z_i z_j}).
\]
If \( x = 2 \), then we are done. Now, assume \( x \neq 2 \). Then, \( x \in V_A \). Since \( z_i \in V_B \), Lemma 6 implies that \( 2 \in V(P_{z_i x}) \). This implies \( 2 \in V(P_{z_i z_j}) \). The proof is complete. \( \square \)

Lemma 13. \([W_1], \ldots, [W_s]\) are connected components of \( T \setminus (2) \).

Proof. Each \([W_j]\) is connected. Suppose \([W_j]\) is not a component in \( T \setminus (2) \). Then there exists a vertex \( g \in W_j \) adjacent to \( v \in V(T \setminus (2)) \setminus W_j \).

Let \( v \in W_k \), where \( k \neq j \). Then, by Lemma 12, \( 2 \in V(P_{vg}) \). This is not possible.

Suppose \( v \notin V_1' \). Then, \( v \in V_2 \cup \cdots \cup V_m \); hence \( 1 \in V(P_{vg}) \). This is a contradiction.

Thus, \([W_j]\) is a component in \( T \setminus (2) \). This completes the proof. \( \square \)

Lemma 14. The following items hold.

(i) \( \det(R[V_B, V_B]) = \prod_{\nu=1}^{s} \det(R[W_\nu, W_\nu]) \).

(ii) \( \det(R[W_i, W_i]) \geq 0 \quad i = 1, \ldots, s. \)

(iii) \( \det(R[V_B, V_B]) \geq 0. \)

Proof. The sets \( W_1, \ldots, W_s \) partition \( V_B \). Let \( a \in W_i \) and \( b \in W_j \). We claim that if \( i \neq j \), then \( R_{ab} = 0 \). By definition
\[
R_{ab} = -d_{21} + d_{a1} + d_{2b} - d_{ab}.
\] (39)

By Lemma 12, \( 2 \in V(P_{ab}) \). Hence
\[
d_{ab} = d_{a2} + d_{2b}.
\] (40)

By (39) and (40),
\[
R_{ab} = -d_{21} + d_{a1} + d_{2b} - d_{a2} - d_{2b} = -d_{21} + d_{a1} - d_{a2}.
\] (41)

Since \( a \in V_B \), \( 2 \in V(P_{a1}) \),
\[
d_{a1} = d_{a2} + d_{21}.
\] (42)

By (41) and (42),
\[
R_{ab} = 0.
\]

By (P3), \( R[V_B, V_B] \) is similar to a block diagonal matrix with diagonal blocks
\[
R[W_1, W_1], \ldots, R[W_s, W_s].
\]
Therefore,
\[
\det(R[V_B, V_B]) = \prod_{i=1}^{s} \det(R[W_i, W_i]).
\]

This completes the proof of (i).

The proof of (ii) follows by substituting \( \tilde{X} = X = [W_i], E = W_i \) and \( \alpha = 2 \) in Lemma 3.

Item (iii) is immediate from (i) and (ii). \( \square \)

Corollary 2, Lemma 11 and Lemma 14 give the following.

**Lemma 15.**
\[
\det(R[V_1, V_1]) \geq 0.
\]

### 3.10 Proof of main result

**Theorem 1.** The Moore-Penrose inverse of the distance Laplacian matrix of a tree is a \( \mathbb{Z} \) matrix.

**Proof.** The proof follows from Lemmas 2, 5 and 15. \( \square \)

### 4 An example

We can ask if the result true for any Euclidean distance matrix. Here is a counter example. Let
\[
D = \begin{bmatrix}
0 & 1 & 4 & 9 & 16 \\
1 & 0 & 1 & 4 & 9 \\
4 & 1 & 0 & 1 & 4 \\
9 & 4 & 1 & 0 & 1 \\
16 & 9 & 4 & 1 & 0
\end{bmatrix}.
\]

Using standard techniques, it can be verified that \( D \) is an Euclidean distance matrix. Let
\[
\Delta := \text{Diag}(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) \quad \text{and} \quad S := \Delta - D.
\]

Then,
\[
S^\dagger = \begin{bmatrix}
\frac{2}{81} & \frac{-1}{81} & \frac{-1}{90} & \frac{-1}{405} & \frac{1}{810} \\
\frac{-1}{81} & \frac{-1}{81} & \frac{1}{405} & \frac{-1}{405} & \frac{-1}{405} \\
\frac{90}{81} & \frac{45}{45} & \frac{-1}{16} & \frac{-1}{16} & \frac{90}{90} \\
\frac{-1}{405} & \frac{-1}{405} & \frac{-1}{16} & \frac{81}{81} & \frac{-1}{81} \\
\frac{1}{810} & \frac{1}{810} & \frac{-1}{81} & \frac{-1}{81} & \frac{81}{81}
\end{bmatrix}.
\]

We see that \( s^\dagger_{15} > 0 \).

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