NON-ZERO SUM HEFFTER ARRAYS AND THEIR APPLICATIONS

SIMONE COSTA, STEFANO DELLA FIORE, AND ANITA PASOTTI

Abstract. In this paper we introduce a new class of partially filled arrays that, as Heffter arrays, are related to difference families, graph decompositions and biembeddings. A non-zero sum Heffter array NH(m, n; h, k) is an m × n p. f. array with entries in Z_{2nk+1} such that: each row contains h filled cells and each column contains k filled cells; for every x ∈ Z_{2nk+1} \ {0}, either x or −x appears in the array; the sum of the elements in every row and column is different from 0 (in Z_{2nk+1}). Here first we explain the connections with relative difference families and with path decompositions of the complete multipartite graph. Then we present a complete solution for the existence problem and a constructive complete solution for the square case and for the rectangular case with no empty cells when the additional, very restrictive, property of “globally simple” is required. Finally, we show how these arrays can be used to construct biembeddings of complete graphs.

1. Introduction

An m × n partially filled (p.f., for short) array on a set Ω is an m × n matrix whose elements belong to Ω and where some cells can be empty. In 2015 Archdeacon introduced a class of p.f. arrays which has been extensively studied: the Heffter arrays.

Definition 1.1. A Heffter array H(m, n; h, k) is an m × n p. f. array with entries in Z_{2nk+1} such that:
(a) each row contains h filled cells and each column contains k filled cells;
(b) for every x ∈ Z_{2nk+1} \ {0}, either x or −x appears in the array;
(c) the elements in every row and column sum to 0 (in Z_{2nk+1}).

Trivial necessary conditions for the existence of a H(m, n; h, k) are mh = nk, 3 ≤ h ≤ n and 3 ≤ k ≤ m. If m = n then h = k and a square H(n, n; k, k) will be simply denoted by H(n; k). The following result has been proved in [5, 14, 24].

Theorem 1.2. A H(n; k) exists for every n ≥ k ≥ 3.

In [21] this concept has been generalized as follows.

Definition 1.3. Let v = \frac{2nk}{\lambda} + t be a positive integer, where t divides \frac{2nk}{\lambda}, and let J be the subgroup of Z_v of order t. A λ-fold Heffter array A over Z_v relative to J, denoted by \lambda H_v(m, n; h, k), is an m × n p. f. array with elements in Z_v such that:
(a_1) each row contains h filled cells and each column contains k filled cells;
(b_1) the multiset \{±x \mid x ∈ A\} contains each element of Z_v \ J exactly \lambda times;

2010 Mathematics Subject Classification. 05B20; 05C10; 05C38.
Key words and phrases. Heffter array, orthogonal cyclic path decomposition, complete multipartite graph, embedding.
(c1) the elements in every row and column sum to 0 (in $\mathbb{Z}_v$).

It is easy to see that if $\lambda = t = 1$ we find again the arrays of Definition 1.1.

Classical Heffter arrays and the above generalization have been introduced also in view of their vast variety of applications and connections with other much studied problems and concepts. In particular there are some recent papers in which Heffter arrays are investigated to obtain new face 2-colorable embeddings (briefly biembeddings) see [3, 15, 20, 21, 23]. On the other hand several authors focused their attention on the existence problem, see [4, 5, 12, 14, 17, 24, 29, 30, 31].

Here we introduce a new class of p.f. arrays, which is related to the one of Heffter arrays.

**Definition 1.4.** Let $v = 2nk + t$ be a positive integer, where $t$ divides $2nk$, and let $J$ be the subgroup of $\mathbb{Z}_v$ of order $t$. A non-zero sum $\lambda$-fold Heffter array $A$ over $\mathbb{Z}_v$ relative to $J$, denoted by $\lambda NH_t(m,n; h,k)$, is an $m \times n$ p.f. array with elements in $\mathbb{Z}_v$ such that:

(a2) each row contains $h$ filled cells and each column contains $k$ filled cells;
(b2) the multiset $\{\pm x \mid x \in A\}$ contains each element of $\mathbb{Z}_v \setminus J$ exactly $\lambda$ times;
(c2) the sum of the elements in every row and column is different from 0 (in $\mathbb{Z}_v$).

Here we focus our attention on the case $\lambda = 1$, namely when the entries of the array are pairwise distinct. If the array is square then $h = k \geq 1$, while if $m \neq n$ then at least one of $h$ and $k$ has to be greater than 1. As done for Heffter arrays, a square non-zero sum Heffter array will be simply denoted by $NH(n;k)$.

This definition is motivated by Alspach’s partial sums conjecture that deals with sets whose sum is different from 0. In particular the truth of this conjecture would show that any $NH_t(m,n; h,k)$ provides two orthogonal path decompositions of $K_{\frac{2nk+t}{2}}$.

The relations between non-zero sum Heffter arrays, Alspach’s conjecture and orthogonal path decompositions will be explained in detail in Section 2. Then we will analyze the existence problem for $NH_t(m,n; h,k)$. In particular, in Section 3 we will show a construction that completely solves the square case when $t = 1$ (but, assuming the existence of a $H_t(n; k)$, it works also for $t > 1$) and, in Section 4 using a probabilistic approach, we will provide a complete solution to the general case. Moreover, in Section 5 we will construct $k$-diagonal square non-zero sum Heffter arrays and rectangular ones with no empty cells that satisfy the very restrictive property of being “globally simple”. Finally, in the last section, we will show how these arrays can be used to construct 2-colorable embeddings of complete graphs and we will prove some existence theorems about biembeddings.

2. Relations with orthogonal path decompositions

In this section we explain the relation between non-zero sum Heffter arrays, difference families and orthogonal decompositions. To do this we have to introduce some concepts and notation.

Given a finite subset $T$ of an abelian group $G$ and an ordering $\omega = (t_1, t_2, \ldots, t_k)$ of the elements in $T$, let $s_i = \sum_{j=1}^i t_j$, for any $i = 1, \ldots, k$, be the $i$-th partial sum of $T$. The ordering $\omega$ is said to be simple if $s_b \neq s_c$ for all $1 \leq b < c \leq k$ or, equivalently, if there is no proper subsequence of $\omega$ that sums to 0. If $\omega$ is a simple
Remark 2.2. We note that, if Conjecture 2.1 would be true, then any NH_{\lambda}\Gamma, respectively, and by R_{\text{edge}}\lambda q vertices, the complete multipartite graph with globally simple.

Moreover, if h,k ≥ 3, then every non-zero sum Heffter array is globally simple.

Conjecture 2.1 has been proved for subsets A of size k of \mathbb{Z}_v whenever

1. k ≤ 9 or k = 10 and v is a prime (see [3, 20]);
2. k \in \{v - 1, v - 2\} or k = v - 3 and v is a prime (see [3, 7, 20]);
3. v ≤ 25 (see [4]);
4. the prime power factors of v are greater than a suitable constant N and k ≤ 11 (see [22]).

Other problems related to the previous one have been proposed in [2, 6, 18, 32].

Given an m × n p.f. array A, the rows and the columns of A are denoted by R_1, ..., R_m and by C_1, ..., C_n, respectively. We denote by \mathcal{E}(A) the list of the elements of the filled cells of A and by \mathcal{E}(R_i), \mathcal{E}(C_j), the elements of the i-th row, of the j-th column, respectively, of A. Also, by \omega_{R_i} and \omega_{C_j} we denote, respectively, an ordering of \mathcal{E}(R_i) and of \mathcal{E}(C_j). If for any i = 1, ..., m and for any j = 1, ..., n, the orderings \omega_{R_i} and \omega_{C_j} are simple, we define by \omega_r = \omega_{R_1} \circ ... \circ \omega_{R_m} the simple ordering for the rows and by \omega_c = \omega_{C_1} \circ ... \circ \omega_{C_n} the simple ordering for the columns. Also, by natural ordering of a row (column) of A one means the ordering from left to right (from top to bottom). A p.f. array A on an abelian group G is said to be

- simple if there exists a simple ordering for each row and each column of A;
- globally simple if the natural ordering of each row and each column of A is simple.

Remark 2.2. We note that, if Conjecture 2.1 would be true, then any NH_{(m, n; h, k)} would be simple. Moreover, if h,k ≤ 3, then every non-zero sum Heffter array is globally simple.

Given a graph \Gamma, we denote by V(\Gamma) and E(\Gamma) the vertex-set and the edge-set of \Gamma, respectively, and by \lambda \Gamma the multigraph obtained from \Gamma by repeating each edge \lambda times. Also by K_v, K_{q,r}, C_k, P_k we represent the complete graph on v vertices, the complete multipartite graph with q parts each of size r, the cycle of length k and the path of length k on k + 1 distinct vertices, respectively. Given a subgraph \Gamma of a graph K, a \Gamma-decomposition of K is a set of graphs (said blocks), all isomorphic to \Gamma, whose edge-sets partition the edge-set of K, see for instance [8]. Given an additive group G, a \Gamma-decomposition \mathcal{D} of a graph K is G-regular if, up to isomorphisms, V(K) = G and for any B ∈ \mathcal{D} also the graph B + g is a block of \mathcal{D} for any g ∈ G. Here we will work with cyclic P_k-decompositions, namely decompositions into paths regular under the cyclic group. It is well known that difference families are a very useful tool for constructing regular decompositions. We recall the definition, see [11, 9].

Definition 2.3. Let \Gamma be a graph with vertices in an additive group G. The multiset \Delta \Gamma = \{±(x - y) | (x, y) ∈ E(\Gamma)\} is called the list of differences from \Gamma.
More generally, given a set $W$ of graphs with vertices in $G$, by $\Delta W$ one means the union (counting multiplicities) of all multisets $\Delta \Gamma$, where $\Gamma \in W$.

**Definition 2.4.** Let $J$ be a subgroup of an additive group $G$ and let $\Gamma$ be a graph. A collection $\mathcal{F}$ of graphs isomorphic to $\Gamma$ and with vertices in $G$ is said to be a $(G, J, \Gamma, \lambda)$-difference family (briefly, DF) over $G$ relative to $J$ if each element of $G \setminus J$ appears exactly $\lambda$ times in the list of differences of $\mathcal{F}$ while no element of $J$ appears there.

If $\lambda = t = 1$ one speaks of a $(G, \Gamma, 1)$-DF. Moreover, if $t$ is a divisor of $v$, a $(\mathbb{Z}_v, \mathbb{Z}_v, \Gamma, \lambda)$-DF, where $\mathbb{Z}_v$ denotes the subgroup of $\mathbb{Z}$ of order $t$, is simply denoted by $(v, t, \Gamma, \lambda)$-DF. The connection between relative difference families and decompositions of a complete multipartite multigraph is given by the following result.

**Proposition 2.5.** [10] Proposition 2.6 If $\mathcal{F} = \{B_1, \ldots, B_\ell\}$ is a $(G, J, \Gamma, \lambda)$-DF, then $B = \{B_i + g \mid i = 1, \ldots, \ell; g \in G\}$ is a $G$-regular $\Gamma$-decomposition of $\lambda K_q \times r$, where $q = |G : J|$ and $r = |J|$. Hence, if $\mathcal{F}$ is a $(G, \Gamma, 1)$-DF there exists a $G$-regular $\Gamma$-decomposition of $K_{|G|}$.

It is well known, see for instance [19], that if there exists a simple $H_t(m, n; h, k)$, then there exist a $(2mh + t, t, C_h^h, 1)$-DF and a $(2nk + t, t, C_k^k, 1)$-DF. In a similar way one can prove the following.

**Proposition 2.6.** If there exists a simple $NH_t(m, n; h, k)$, then there exist a $(2mh + t, t, P_h, 1)$-DF and a $(2nk + t, t, P_k, 1)$-DF.

**Remark 2.7.** Let $\mathcal{F}_h$ and $\mathcal{F}_k$ be the relative DFs constructed in the previous proposition. Note that for any $P_h \in \mathcal{F}_h$ and any $P_k \in \mathcal{F}_k$, we have $|\Delta P_h \cap \Delta P_k| \in \{0, 2\}$.

We recall the following definition, see for instance [13].

**Definition 2.8.** Two graph decompositions $D$ and $D'$ of a simple graph $K$ are said orthogonal if and only if for any $B$ of $D$ and any $B'$ of $D'$, $B$ intersects $B'$ in at most one edge.

Heffter arrays are also used for getting orthogonal cycle decompositions, see for instance [11]. Analogously from non-zero sum Heffter arrays one can construct orthogonal path decompositions as shown by the following result.

**Proposition 2.9.** Let $A$ be a $NH_t(m, n; h, k)$ simple with respect to the orderings $\omega_r$ and $\omega_c$. Then:

1. there exists a cyclic $h$-path decomposition $D_{\omega_r}$ of $K_{2mh + t} \times 1$;
2. there exists a cyclic $k$-path decomposition $D_{\omega_c}$ of $K_{2nk + t} \times 1$;
3. the decompositions $D_{\omega_r}$ and $D_{\omega_c}$ are orthogonal.

**Proof.** (1) and (2) follow from Propositions 2.5 and 2.6. Then (3) follows from Remark 2.7.

We point out that the notion of simple array can be extended also to $\lambda NH_t(m, n; h, k)$ with $\lambda > 1$, similar to what was done in [21] for Heffter arrays. Then reasoning as above one can get two path decompositions of the $\lambda$-fold multipartite complete graph, but in this case the decompositions are not orthogonal.
Example 2.10. Consider the NH(3, 4; 4, 3), say $A$, below.

$$
\begin{array}{ccc}
1 & 10 & -11 \\
8 & 6 & -3 \\
-7 & 12 & 9 & 4
\end{array}
$$

Firstly we find simple orderings for the rows and the columns of $A$. Note that, for instance, the natural orderings from left to right of the first row and of the third row are not simple. While every ordering of the columns is simple since $k = 3$.

Set, for instance:

$$
\omega_1 = (1, 2, 10, -11), \quad \nu_1 = (1, 8, -7),
\omega_2 = (8, 6, -3, 5), \quad \nu_2 = (10, 6, 12),
\omega_3 = (4, -7, 12, 9), \quad \nu_3 = (-11, -3, 9),
\omega_4 = (2, 5, 4).
$$

Starting from these orderings we obtain the following paths:

$$
\begin{align*}
P_{\omega_1} &= [0, 1, 3, 13, 2], & P_{\nu_1} &= [0, 1, 9, 2], \\
P_{\omega_2} &= [0, 8, 14, 11, 16], & P_{\nu_2} &= [0, 10, 16, 3], \\
P_{\omega_3} &= [0, 4, 22, 9, 18], & P_{\nu_3} &= [0, 14, 11, 20], \\
P_{\omega_4} &= [0, 2, 7, 11]. &
\end{align*}
$$

Set $\mathcal{F}_\omega = \{P_{\omega_i} \mid i = 1, \ldots, 3\}$ and $\mathcal{F}_\nu = \{P_{\nu_i} \mid i = 1, \ldots, 4\}$; by the construction of the paths it immediately follows that $\Delta \mathcal{F}_\omega = \Delta \mathcal{F}_\nu = \mathbb{Z}_{25} \setminus \{0\}$. Hence $\mathcal{F}_\omega$ is a $(25, P_4, 1)$-DF and $\mathcal{F}_\nu$ is a $(25, P_3, 1)$-DF. Set $D_\omega = \{P_{\omega_i} + g \mid i = 1, \ldots, 3, g \in \mathbb{Z}_{25}\}$ and $D_\nu = \{P_{\nu_i} + g \mid i = 1, \ldots, 4, g \in \mathbb{Z}_{25}\}$. Then $D_\omega$ is a cyclic $P_4$-decomposition of $K_{25}$, $D_\nu$ is a cyclic $P_3$-decomposition of $K_{25}$. Moreover, $D_\omega$ and $D_\nu$ are orthogonal.

3. A constructive complete solution for the square case

In this section we present a complete solution for the existence of a NH($n; k$) for any $n \geq k \geq 1$.

Remark 3.1. The existence of a globally simple NH($n; 1$), say $A = (a_{i,j})$, for every $n \geq 1$ is trivial. In fact it is sufficient to set $a_{i,i} = i$ for every $i = 1, \ldots, n$, while all other cells are empty.

Proposition 3.2. For every $n \geq 2$, there exists a globally simple NH($n; 2$).

Proof. Let $A = (a_{i,j})$ be an $n \times n$ matrix with $n \geq 2$. Set $a_{i,i} = i$ and $a_{i,i+1} = n+i$ for every $i = 1, \ldots, n$, where the subscript are considered modulo $n$. It is immediate to see that $A$ is a NH($n; 2$).

We recall that given a square matrix $A$ of order $n$ a transversal of $A$ is a set of $n$ filled cells such that no two belong to the same row and no two belong to the same column.

Lemma 3.3. A square p.f. array with $k \geq 1$ filled cells in each row and in each column admits a transversal.

Proof. Let $A$ be an $n \times n$ array as in the statement. Associate to $A$ a bipartite graph, say $\Gamma$, on $2n$ vertices, in which the vertices of one part represent the $n$ rows and the vertices of the other part represent the $n$ columns of $A$ and there is an edge connecting two vertices if and only if the cell of the corresponding row and column
is not an empty cell. Note that a transversal of $A$ is nothing but a perfect matching of $\Gamma$ which, by construction, is $k$-regular and it is well-known that a regular and bipartite graph admits a perfect matching. □

**Theorem 3.4.** If there exists a $H_t(n; k)$ then there also exists a $NH_t(n; k)$.

**Proof.** Let $A$ be a $H_t(n; k)$. By Lemma 3.3 there exists a transversal $T$ of $A$ containing $n$ filled cells. Let $H$ be the matrix obtained from $A$ by changing the signs of the elements of $T$ and leaving all other cells unchanged. Note that to prove that $H$ is a $NH_t(n; k)$ we have only to check condition (c$2$) of Definition 1.4. Now we remark that if $t$ is odd, $\mathbb{Z}_{2nk+t}$ does not contain an involution, while if $t$ is even the involution of $\mathbb{Z}_{2nk+t}$ belongs to the subgroup of order $t$ of $\mathbb{Z}_{2nk+t}$. Hence, in any case, $A$ does not contain the involution. Also, we recall that $A$ is a $H_t(n; k)$ hence each row and each column of $A$ sums to zero. It follows that changing the sign of exactly one element in each row and in each column of $A$ all the sums are now different from zero. The result follows. □

**Corollary 3.5.** For every $n \geq k \geq 1$ there exists a $NH(n; k)$.

**Proof.** If $k = 1$ or $k = 2$ the result follows from Remark 3.1 and Proposition 3.2, respectively. For $k \geq 3$, the result follows from Theorems 1.2 and 3.4. □

In the next section we will present a generalization of the previous corollary, whose proof is not constructive.

**Example 3.6.** Here we have the $H(7; 5)$ constructed in [5]. Note that the cells with bold elements form a transversal of the array.

| -10 | 16 | -1 | -2 | -3 |
|-----|----|----|----|----|
| -4  | -6 | -7 | -8 | 22 |
| -30 | 29 | -9 | -8 | 18 |
| -11 | -12 | 28 | -31 | 26 |
| -14 | -15 | -13 | 17 | 25 |
| 27  | -34 | 20 | 19 | -32 |
| 24  | 23 | 21 | -35 | -33 |

Applying the proof of Theorem 3.4 we get the $NH(7; 5)$ below.

| 10  | 16 | -1 | -2 | -3 |
|-----|----|----|----|----|
| 4   | -6 | -7 | -5 | 22 |
| -30 | 29 | 9  | -8 | 18 |
| -11 | -12 | -28 | -31 | 26 |
| -14 | -15 | -13 | -17 | 25 |
| 27  | -34 | 20 | -19 | -32 |
| 24  | 23 | 21 | -35 | 33 |

### 4. A Complete Solution for the Rectangular Case

In this section we prove the existence of a $NH_t(m, n; h, k)$ whenever the trivially necessary conditions $m \geq k \geq 1$, $n \geq h \geq 1$, $nk = mh$ and $t|2nk$ are satisfied.

**Lemma 4.1.** Let $m \geq k \geq 1$ and $n \geq h \geq 1$ be such that $nk = mh$. Then there exists an $m \times n$ p.f. array $A$ that has exactly $h$ filled cells in each row and exactly $k$ filled cells in each column.

**Proof.** Let $m, n, h, k$ be as in the statement. Let $S$ be the subgroup of $\mathbb{Z}_m \times \mathbb{Z}_n$ generated by $(1, 1)$. It is not hard to see that the order of $S$ is equal to $\text{lcm}(m, n)$. Now consider the $m \times n$ empty array, say $A$, whose rows represent the elements...
of $\mathbb{Z}_m$ while the columns represent the elements of $\mathbb{Z}_n$. One can check that filling exactly the cells of $A$ corresponding to the elements of the subgroup $S$, we get a p.f. array with exactly $\frac{\text{lcm}(m,n)}{m}$ filled cells in each row and $\frac{\text{lcm}(m,n)}{n}$ filled cells in each column. Now note that $\text{lcm}(m,n)$ divides $nk = mh$ and set $r = \frac{nk}{\text{lcm}(m,n)}$. Consider $r$ distinct cosets of $S$ in $\mathbb{Z}_m \times \mathbb{Z}_n$. Filling exactly the cells of $A$ corresponding to the elements of these cosets we obtain the required array.

**Theorem 4.2.** For every $m \geq k \geq 1$, $n \geq h \geq 1$ and $t$ such that $nk = mh$ and $t|2nk$ there exists a NH$_3(m,n;h,k)$.

**Proof.** If $m = 1$, which implies $k = 1$ and $n = h$, take an arbitrary subset $S$ of $\mathbb{Z}_{2n+t} \setminus 2n+Z_{2n+t}$ of size $n$ which does not contain pairs of the form $\{x, -x\}$ for some $x \in \mathbb{Z}_{2n+t}$. Note that such a set $S$ exists since $\mathbb{Z}_{2n+t} \setminus 2n+Z_{2n+t}$ does not contain the involution (if it exists). Let $A$ be a $1 \times n$ array filled with the elements of $S$. If the row of $A$ does not sum to zero, $A$ is a NH$_3(1,n;n,1)$, otherwise, it is sufficient to change the sign of an element in $A$ and the new array is a NH$_3(1,n;n,1)$. Clearly if $n = 1$ one can reason in the same way.

So in the following we can assume $m, n \geq 2$. Let $S$ be a subset of $\mathbb{Z}_{2nk+t} \setminus 2nk+Z_{2nk+t}$ of size $nk$ and such that, for every $x \in \mathbb{Z}_{2nk+t} \setminus 2nk+Z_{2nk+t}$, exactly one of $x$ or $-x$ appears in $S$. As before, such a set $S$ exists since $\mathbb{Z}_{2nk+t} \setminus 2nk+Z_{2nk+t}$ does not contain the involution (if it exists). We also consider an $m \times n$ p.f. array $A$ having exactly $h$ filled cells in each row and exactly $k$ filled cells in each column whose existence follows by Lemma 4.1. Now, we prove that we can write the elements of $S$ in the filled cells of $A$ so that the sum over each row and column is non-zero. For this purpose, we fill $A$, uniformly at random, with the elements of $S$. Then we evaluate the expected value $E(X)$ of the random variable $X$ given by the number of rows that sum to zero.

Due to the linearity of $E(X)$ we have that:

$$E(X) = \sum_{i=1}^{m} P\left(\sum_{x \in R_i} x = 0\right).$$

For symmetry, $P(\sum_{x \in R_i} x = 0) = P(\sum_{x \in R_i} x = 0)$ for every $i = 1, \ldots, m$, and hence

$$E(X) = m \cdot P\left(\sum_{x \in R_1} x = 0\right).$$

Now we want to give an upper-bound for the term $P(\sum_{x \in R_i} x = 0)$. We note that, if $h \in \{1, 2\}$, this probability is zero since neither 0 nor pairs of the form $\{x, -x\}$ are contained in $S$. Therefore, we can suppose $h \geq 3$. Let us assume we have already chosen the first $h - 1$ elements, $x_1, \ldots, x_{h-1}$, of $R_1$. Then there exists at most one $\tilde{x} \in S \setminus \{x_1, \ldots, x_{h-1}\}$ that makes the sum zero. This means that

$$P\left(\sum_{x \in R_1} x = 0\right) \leq \frac{1}{mh - (h - 1)}$$

and hence

$$E(X) \leq \frac{m}{mh - (h - 1)}.$$

Note that, since $m \geq 2$ and $h \geq 3$, we obtain:

\begin{equation}
2m + (h - 1) \leq 2m + (m - 2) = mh.
\end{equation}
We also emphasize that in \((1.1)\) the equality holds only if \((m, h) = (2, 3)\) which implies \((n, k) = (3, 2)\) or \((n, k) = (6, 1)\) since \(mh = nk\) and \(n \geq h = 3\). This implies that \(\frac{m}{mn-(n-1)} < 1/2\), namely \(E(X) \leq 1/2\). Similarly, also the expected value \(E(Y)\) of the random variable \(Y\) given by the number of columns that sum to zero cannot exceed \(1/2\). By the previous remark on the equality in \((1.1)\), it immediately follows that if one of \(E(X)\) or \(E(Y)\) is equal to \(1/2\), then the other one is zero, which implies that \(E(X) + E(Y) < 1\). Therefore we can fill \(A\) with the elements of \(S\) so that the sum over each row and column is non-zero.

From Remark 2.2 and Theorem 4.2 we have the existence of a pair of orthogonal path decompositions whenever Conjecture 2.1 is satisfied. In particular, from the results of [3] and [26], it follows that:

**Proposition 4.3.** Let \(m \geq k \geq 1\), \(n \geq h \geq 1\) and \(t\) be such that \(nk = mh\) and \(t|2nk\). Then, assuming that either \(k, h \leq 9\) or \(k, h \leq 10\) and \(2nk + t\) is a prime, there exist a cyclic \(h\)-path decomposition \(D_{\omega_r}\) of \(K_{2nk+1 \times t}\) and a cyclic \(k\)-path decomposition \(D_{\omega_c}\) of \(K_{2nk+1 \times t}\). Also, \(D_{\omega_r}\) and \(D_{\omega_c}\) are a pair of orthogonal decompositions.

5. A CONSTRUCTIVE COMPLETE SOLUTION FOR GLOBALLY SIMPLE NH\((n; k)\) AND NH\((m, n; n, m)\)

In this section we construct a globally simple square non-zero sum Heffter array NH\((n; k)\) for every \(n \geq k \geq 1\) and a globally simple non-zero sum Heffter array NH\((m, n; n, m)\) for every \(m, n \geq 1\).

Given an \(n \times n\) p.f. array \(A\), for any \(1 \leq i \leq n\), the \(i\)-th diagonal of \(A\) is defined as follows:

\[
D_i = \{(i, 1), (i + 1, 2), \ldots, (i - 1, n)\}.
\]

All the arithmetic on the row and column indices is performed modulo \(n\), where the set of reduced residues is \(\{1, 2, \ldots, n\}\). The \(k + 1\) diagonals \(D_i, D_{i+1}, \ldots, D_{i+k}\) are said to be consecutive diagonals.

**Definition 5.1.** A square p.f. array \(A\) of size \(n \geq k \geq 1\) is said to be cyclically \(k\)-diagonal if the nonempty cells of \(A\) are exactly those of \(k\) consecutive diagonals.

**Theorem 5.2.** For any \(n \geq k \geq 1\) there exists a cyclically \(k\)-diagonal globally simple NH\((n; k)\).

**Proof.** Let \(n \geq k \geq 1\) and let \(A = (a_{i,j})\) be the \(n \times n\) array defined as follows. If \(j \geq i\), set \(j = i + r\), hence \(a_{i,j} = a_{i,i+r}\). If \(j < i\) instead of \(a_{i,j}\) we can consider \(a_{i,j+n}\) since, as usual, the subscripts are considered modulo \(n\). Then \(j + n = i + r\) for some \(r\) and hence, also in this case, \(a_{i;j}\) can be written in the form \(a_{i,i+r}\). Now, for any \(0 \leq r \leq k - 1\) we set \(a_{i,i+r} = \varepsilon[r + 1 + (i - 1)k]\) where \(\varepsilon\) is equal to 1 if \(r\) is even, while \(\varepsilon\) is equal to \(-1\) for \(r\) odd. All the other cells of \(A\) are left empty. One can check that the filled cells of \(A\) are exactly those of the diagonals \(D_1, D_2, \ldots, D_k\), so \(A\) is a cyclically \(k\)-diagonal array. Now we want to prove that \(A\) is a globally simple NH\((n; k)\). It is easy to see that conditions (a2) and (b2) of Definition 1.3 are satisfied.

We list the partial sums for each row starting from the element \(a_{i,i}\), for \(1 \leq i \leq n\). If \(k\) is even we have

\[
S(R_i) = ((i - 1)k + 1, -1((i - 1)k + 2, -2, \ldots, (i - 1)k + \frac{k}{2}, -\frac{k}{2}).
\]
while for $k$ odd it results
\[ S(R_i) = \left( (i-1)k + 1, -1, (i-1)k + 2, -2, \ldots, \frac{k-1}{2}, (i-1)k + \frac{k+1}{2} \right). \]

So it is immediate that the sum of the elements of each row is different from zero.

Now we list the partial sums for each column. We start from the first $k-1$ columns, and we have four cases according to the parity of $i$ and $k$. In each of these cases we list the sums starting from the element $a_{n-k+1+i,i}$.

Case 1. $k$ even, $i$ odd with $1 \leq i \leq k-1$.
\[ S(C_i) = \left( \left( -k(n - k + 1 + i), k - 1, (k - 1) - k(n - k + 1 + i), 2(k - 1), \ldots \right) \right). \]

Case 2. $k$ even, $i$ even with $1 \leq i \leq k-1$.
\[ S(C_i) = \left( \left( -k(n - k + 1 + i), k - 1, (k - 1) - k(n - k + 1 + i), 2(k - 1), \ldots \right) \right). \]

Case 3. $k$ odd, $i$ odd with $1 \leq i \leq k-1$.
\[ S(C_i) = \left( \left( k(n - k + 1 + i), (k - 1) + k(n - k + 1 + i), 2(k - 1), \ldots \right) \right). \]

Case 4. $k$ odd, $i$ even with $1 \leq i \leq k-1$.
\[ S(C_i) = \left( \left( k(n - k + 1 + i), (k - 1) + k(n - k + 1 + i), 2(k - 1), \ldots \right) \right). \]
Now we list the sums of the columns $C_j$ for $k \leq j \leq n$. Write $j$ as $k - 1 + i$ for $1 \leq i \leq n - k + 1$. We have two cases according to the parity of $k$.

If $k$ is even we have

$$S(C_{k+1}) = \left\{ -ik, k - 1 - (k - 1) - ik, 2(k - 1), -2(k - 1) - ik, 3(k - 1), \ldots, \right.$$  
$$\left. -\frac{k - 2}{2}(k - 1) - ik, \frac{k}{2}(k - 1) \right\},$$

while for $k$ odd it results

$$S(C_{k+1}) = \left\{ ik, -(k - 1), (k - 1) + ik, -2(k - 1), 2(k - 1) + ik, -3(k - 1), \ldots, \right.$$  
$$\left. -\frac{k - 1}{2}(k - 1), \frac{k - 1}{2}(k - 1) + ik \right\}.$$

Note that also the sum of the elements of each column is different from zero. Hence $A$ is a NH($n; k$). It can be easily verified that the partial sums for each row and each of the last $n - k + 1$ columns are pairwise distinct modulo $2nk + 1$. Instead proving that the partial sums for each of the first $k - 1$ columns are pairwise distinct is slightly tedious. For this reason, the partial sums distinctness was verified with Mathematica. So, we defer the reader to the following link for the source code.

http://stefano-dellafiore.me/additional_material_NH.zip

□

To help the reader to follow the proof of the previous theorem we present two examples with different parity of $k$.

**Example 5.3.** Applying the proof of Theorem 5.2 we get the NH(11; 8) and the NH(11; 9) below.

```plaintext
1 -2 3 -4 5 -6 7 -8
 9  -10 11  -12 13  -14 15  -16
17  -18 19  -20 21  -22 23  -24
25  -26 27  -28 29  -30 31  -32
-40  33  -34 35  -36 37  -38 39
47  -48  49  -50 51  -52 53
-54 55  -56  57  -58 59  -60
61  -62  63  -64  65  -66 67
75  -76  77  -78 79  -80  81
-82  83  -84 85  -86 87  -88
```

```plaintext
1 -2 3 -4 5 -6 7 -8
 9  -10 11  -12 13  -14 15  -17
19  -20 21  -22 23  -24 25  -26
36  -28 29  -30 31  -32 33  -34
-44 45  37  -38 39  -40 41  -42
52  -53 54  46  -47 48  -49 50  -51
-60  61  -62 63  55  -56 57  -58
68  -69 70  -71 72  64  -65 66  -67
76  -77 78  -80 81  73  -74 75
84  -85 86  -87 88  -89 90  -83
-92 93  -94 95  -96 97  -98 99
```

**Proposition 5.4.** For any $n \geq k \geq 1$, there exist a pair of orthogonal cyclic $P_k$-decompositions of $K_{2nk+1}$. 
is even we have $m$ of Theorem 5.2. So we already know, that the sum of the elements of each row is satisfied. Also, one can check that the partial sums of the rows, written starting from the element $a_{1,1}$, are exactly the same as the array constructed in the proof of Theorem 5.2. So we already know, that the sum of the elements of each row is not zero and that the partial sums are pairwise distinct.

Now we list the partial sums of the columns starting from $a_{1,j}$, for $1 \leq j \leq n$. If $m$ is even we have

$$S(C_j) = \pm \left( j, -n, n+j, -2n, 2n+j, -3n, 3n+j, \ldots, \frac{m-2}{2}n+j, -\frac{m}{2}n \right),$$

while for $m$ odd it results

$$S(C_j) = \pm \left( j, -n, n+j, -2n, 2n+j, -3n, 3n+j, \ldots, -\frac{m-1}{2}n+j, \frac{m-1}{2}n+j \right),$$

where the sign is $+$ if $j$ is odd, $-$ if $j$ is even. Note that also the sum of the elements of each column is different from zero. Hence $A$ is a NH$(m,n; n,m)$. Finally, by a direct check, it is not hard to see that the partial sums of each column are pairwise distinct in $\mathbb{Z}_{2mn+1}$, hence $A$ is a globally simple NH$(m,n; n,m)$.

**Example 5.6.** Applying the proof of Theorem 5.5 we get the NH$(8, 5; 5, 8)$ and the NH$(9, 4; 4, 9)$ below.

|     | 1   | -2  | 3   | -4  | 5   |
|-----|-----|-----|-----|-----|-----|
| 1   | 1   | -2  | 3   | -4  | 5   |
| -1   | -11 | -12 | -13 | -14 | -15 |
| -16  | -17 | -18 | -19 | -20 | -20 |
| 21   | 22  | 23  | 24  | 25  | 25  |
| -26  | -27 | -28 | -29 | -30 | -30 |
| -31  | -32 | -33 | -34 | -35 | -35 |
| -36  | -37 | -38 | -39 | -40 | -40 |

|     | 1   | -2  | 3   | -4  | 5   |
|-----|-----|-----|-----|-----|-----|
| 1   | 1   | -2  | 3   | -4  | 5   |
| -1   | -11 | -12 | -13 | -14 | -15 |
| -16  | -17 | -18 | -19 | -20 | -20 |
| 21   | 22  | 23  | 24  | 25  | 25  |
| -26  | -27 | -28 | -29 | -30 | -30 |
| -31  | -32 | -33 | -34 | -35 | -35 |
| -36  | -37 | -38 | -39 | -40 | -40 |

**Proposition 5.7.** For any $m, n \geq 1$, there exist a cyclic $P_m$-decomposition $D_{\omega_r}$ of $K_{2mn+1}$ and a cyclic $P_n$-decomposition $D_{\omega_c}$ of $K_{2mn+1}$. Also, $D_{\omega_r}$ and $D_{\omega_c}$ are a pair of orthogonal decompositions.

Proof. The result follows by Proposition 2.9 and Theorem 5.5.

6. Relation with embeddings

As already remarked in the Introduction, Archdeacon introduced Heffter arrays also because of their applications and, in particular, since they are useful for finding biembeddings of cycle decompositions. In this section, generalizing some of Archdeacon’s results we show how starting from a non-zero sum Heffter array it is possible to obtain suitable biembeddings.

We recall the following definition, see [27].

**Definition 6.1.** An embedding of a graph $\Gamma$ in a surface $\Sigma$ is a continuous injective mapping $\psi : \Gamma \to \Sigma$, where $\Gamma$ is viewed with the usual topology as 1-dimensional simplicial complex.
The connected components of $\Sigma \setminus \psi(\Gamma)$ are called $\psi$-faces. If each $\psi$-face is homeomorphic to an open disc, then the embedding $\psi$ is said to be cellular.

**Definition 6.2.** A biembedding of two circuit decompositions $D$ and $D'$ of a simple graph $\Gamma$ is a face 2-colorable embedding of $\Gamma$ in which one color class is comprised of the circuits in $D$ and the other class contains the circuits in $D'$.

Following the notation given in [3], for every edge $e$ of a graph $\Gamma$, let $e^+$ and $e^-$ denote its two possible directions and let $\tau$ be the involution swapping $e^+$ and $e^-$ for every $e$. Let $D(\Gamma)$ be the set of all directed edges that correspond to edges of $\Gamma$, and, for any $v \in V(\Gamma)$, call $D_v$ the set of edges directed out of $v$. A local rotation $\rho_v$ is a cyclic permutation of $D_v$. If we select a local rotation for each vertex of $\Gamma$, then all together they form a rotation of $D(\Gamma)$. We recall the following result, see [3, 25, 28].

**Theorem 6.3.** A rotation $\rho$ on $\Gamma$ is equivalent to a cellular embedding of $\Gamma$ in an orientable surface. The face boundaries of the embedding corresponding to $\rho$ are the orbits of $\rho \circ \tau$.

Now our goal is to provide a construction of a biembedding starting from a non-zero sum Heffter array. In order to characterize the faces of this embedding, we introduce some notations.

Given a cyclic path decomposition $D$ of $K_v$ we want to define an associated circuit decomposition $C(D)$. As usual, we identify the vertex-set of $K_v$ with $\mathbb{Z}_v$. Let us consider $P = (x_0, x_1, \ldots, x_k) \in D$, and let $\lambda_P$ be the minimum positive integer such that $\lambda_P(x_k - x_0) = 0$ in $\mathbb{Z}_v$. Then we set $C_P := \lambda_P^{-1} \bigcup_{i=0}^{\lambda_P-1} P + i(x_k - x_0)$.

Note that, because of the definition, $C_P$ is a circuit and we say that the path $P$ generates $C_P$. Now we consider the set of all circuits of type $C_P$ without repeating them. More formally, we define $C(D) := \{C_P : P \in D\}$.

Since $C(D)$ is a set, chosen from a set $T$ of generating paths for those circuits, we have that $C(D) = \{C_P : P \in T\}$. Moreover, since $\bigcup_{P \in D} P = \bigcup_{P \in T} C_P$, we have that $C(D)$ is a decomposition of $K_v$ into circuits.

Then, adapting the proof of Theorem 3.4 of [20], we show that the Archdeacon embedding (see [3]) is well-defined also in this case. For this purpose we recall the following definition. Given a non-zero sum Heffter array $A = \text{NH}(m, n; h, k)$, the orderings $\omega_r$ and $\omega_c$ are said to be compatible if $\omega_c \circ \omega_r$ is a cycle of length $|E(A)|$.

**Theorem 6.4.** Let $A$ be a non-zero sum Heffter array $\text{NH}(m, n; h, k)$ that is simple with respect to the compatible orderings $\omega_r$ and $\omega_c$. Then there exists a cellular biembedding of the circuit decompositions $C(D_{\omega_r})$ and $C(D_{\omega_c})$ of $K_{2nk+1}$ into an orientable surface.

**Proof.** Since the orderings $\omega_r$ and $\omega_c$ are compatible, we have that $\omega_c \circ \omega_r$ is a cycle of length $|E(A)|$. Let us consider the permutation $\rho_0$ on $\pm E(A) = \mathbb{Z}_{2nk+1} \setminus \{0\}$
correspond to the orbits of \( \rho \) and we name by \( \bar{\rho}_0(a) \) \( \omega_{c_i} \) \( a \in \mathcal{E}(A) \), \( \omega\circ\omega_{r_i}(a) \) and hence \( \bar{\rho}_0 \) acts cyclically on \( \mathcal{E}(A) \) and we define the map \( \rho \) on the set of its oriented edges so that:

\[
\rho((x, x + a)) = (x, x + \bar{\rho}_0(a)).
\]

Since \( \bar{\rho}_0 \) acts cyclically on \( \pm\mathcal{E}(A) = \mathbb{Z}_{2n+1} \setminus \{0\} \), the map \( \rho \) is a rotation of \( K_{2n+1} \). Hence, by Theorem 6.3, there exists a cellular embedding \( \sigma \) of \( K_{2n+1} \) in an orientable surface (i.e. the Archdeacon embedding) so that the face boundaries correspond to the orbits of \( \rho \circ \tau \) where \( \tau((x, x + a)) = (x + a, x) \).

Let us consider the oriented edge \( (x, x + a) \) with \( a \in \mathcal{E}(A) \), and let \( C \) be the column containing \( a \). Moreover, let us denote by \( \lambda_r \) the minimum positive integer such that \( \sum_{i=0}^{\lambda_r|\mathcal{E}(C)|-1} \omega_{r}^i(a) = 0 \) in \( \mathbb{Z}_{2n+1} \). Since \( a \in \mathcal{E}(A) \), \( -a \notin \mathcal{E}(A) \) and we have that:

\[
\rho \circ \tau((x, x + a)) = \rho((x + a, (x + a) - a)) = (x + a, x + a + \omega_r(a)).
\]

Thus \( (x, x + a) \) belongs to the boundary of the face \( F_1 \) delimited by the oriented edges:

\[
(x + a), (x + a + \omega_r(a), x + a + \omega_r(a) + \omega_{c_i}(a)), \ldots
\]

\[
(\ldots, x + \sum_{i=0}^{\lambda_r|\mathcal{E}(C)|-2} \omega_{c_i}(a), x).
\]

We note that the circuit associated to the face \( F_1 \) is:

\[
(x + a, x + a + \omega_r(a), \ldots, x + \sum_{i=0}^{\lambda_r|\mathcal{E}(C)|-2} \omega_{c_i}(a)).
\]

Let us now consider the oriented edge \( (x, x + a) \) with \( a \notin \mathcal{E}(A) \). Hence \( -a \in \mathcal{E}(A) \), and we name by \( R \) the row containing the element \( -a \). Moreover, let us denote by \( \lambda_r \) the minimum positive integer such that \( \sum_{i=0}^{\lambda_r|\mathcal{E}(R)|-1} \omega_{r}^i(-a) = 0 \) in \( \mathbb{Z}_{2n+1} \). Since \( -a \in \mathcal{E}(A) \) we have that:

\[
\rho \circ \tau((x, x + a)) = \rho((x + a, (x + a) - a)) = (x + a, x + a + \omega_r(-a)).
\]

Thus \( (x, x + a) \) belongs to the boundary of the face \( F_2 \) delimited by the oriented edges:

\[
(x + a), (x - (-a), x - (-a) - \omega_r(-a)), \ldots
\]

\[
(x - (-a) - \omega_r(-a), x - (-a) - \omega_r(-a) - \omega_{r}^2(-a), \ldots, (x - \sum_{i=0}^{\lambda_r|\mathcal{E}(R)|-2} \omega_{r}^i(-a), x).
\]

Since \( A \) is a non-zero sum Heffter array and \( \omega_r \) acts cyclically on \( \mathcal{E}(R) \), for any \( j \in [1, \lambda_r|\mathcal{E}(R)|] \) we have that:

\[
-\sum_{i=0}^{j-1} \omega_{r}^i(-a) = \sum_{i=j}^{\lambda_r|\mathcal{E}(R)|-1} \omega_{r}^i(-a) = \sum_{i=1}^{\lambda_r|\mathcal{E}(R)|-j} \omega_{r}^{\lambda_r|\mathcal{E}(R)|-i}(-a) = \sum_{i=1}^{\lambda_r|\mathcal{E}(R)|-j} \omega_{r}^{-i}(-a).
\]
It follows that the circuit associated to the face $F_2$ can be written also as:

$$
\left( x, x + \sum_{i=1}^{\lambda_r |\mathcal{E}(R)|-1} \omega_r^{-i}(-a), x + \sum_{i=1}^{\lambda_r |\mathcal{E}(R)|-2} \omega_r^{-i}(-a), \ldots, x + \omega_r^{-1}(-a) \right).
$$

Therefore any nonoriented edge $\{x, x + a\}$ belongs to the boundaries of exactly two faces: one of type $F_1$ and one of type $F_2$. Hence the embedding is 2-colorable.

Moreover, it is easy to see that those face boundaries are the circuits of the decompositions $\mathcal{C}(D_{\omega_r^{-1}})$ and $\mathcal{C}(D_{\omega_c})$. □

**Remark 6.5.** Note that the proof of Theorem 6.4 can be adapted to the case of a $\lambda \text{NH}_t(m, n; h, k)$ that is simple with respect to the compatible orderings $\omega_r$ and $\omega_c$. Here we obtain a cellular biembedding of the circuit decompositions $\mathcal{C}(D_{\omega_r^{-1}})$ and $\mathcal{C}(D_{\omega_c})$ of $\lambda K_{2nk+1}$ into an orientable surface. However, since the notations would get much more complicated and since our existence results of Theorems 5.2 and 5.5 cover the case $t = \lambda = 1$ we have written the proof only in that case.

We note that the existence results on compatible orderings of [16] do not depend on the sum of the elements in each row and each column of the array. Hence, from Proposition 4.8, Theorem 4.12 and Theorem 4.15 of [16], it follows that, given a cyclically $k$-diagonal non-zero sum $\text{NH}(n; k)$, there exists a pair of compatible orderings in the following cases:

1. if $nk$ is odd and $\gcd(n, k-1) = 1$;
2. if $nk$ is odd and $3 < k < 200$;
3. if $nk$ is odd and $n \geq (k-2)(k-1)$.

Furthermore, due to Theorem 5.2 for all such values of $n$ and $k$, there exists a biembedding of $K_{2nk+1}$ into an orientable surface whose face lengths are multiples of $k$. It follows that:

**Theorem 6.6.** Let $n \geq k$ be odd integers such that one of the following holds:

1. $\gcd(n, k-1) = 1$;
2. $3 < k < 200$;
3. $n \geq (k-2)(k-1)$.

Then there exists a biembedding of $K_{2nk+1}$ into an orientable surface $\Sigma$ whose face lengths are multiples of $k$.

Moreover, from Theorem 3.3 of [16], it follows that given a $\text{NH}(m, n; n, m)$, there exists a pair of compatible orderings whenever $m$ and $n$ are not both odd. As a consequence of Theorem 5.5 we have that:

**Theorem 6.7.** Let $m$ and $n$ be integers and assume that $m$ and $n$ are not both odd. Then there exists a biembedding of $K_{2mn+1}$ into an orientable surface $\Sigma$ whose face lengths are multiples of $m$ or multiples of $n$.

**Remark 6.8.** Let us assume that we are under the hypothesis of Theorem 6.6 and that $2nk + 1$ is a prime. Then each face is a circuit of length $k(2nk + 1)$ and hence we have $2n$ faces. It follows that the genus of the surface $\Sigma$ is:

$$
g = \frac{2 - (2nk + 1) + (2nk + 1)nk - 2n}{2}.
$$
Similarly, if we are under the hypothesis of Theorem 6.7 and $2mn + 1$ is a prime, then we have $m + n$ faces. It follows that the genus of the surface $\Sigma$ is:

$$g = \frac{2 - (2mn + 1) + (2mn + 1)mn - (m + n)}{2}.$$ 

**Acknowledgements**

The first and the third authors were partially supported by INdAM–GNSAGA.

**References**

[1] R.J.R. Abel and M. Buratti, *Difference families*, in: *Handbook of Combinatorial Designs*. Edited by C. J. Colbourn and J. H. Dinitz. Second edition. Discrete Mathematics and its Applications. Chapman & Hall/CRC, Boca Raton, 2007.

[2] B. Alspach and G. Liversidge, *On strongly sequenceable abelian groups*, Art Discrete Appl. Math. 3 (2020) #P1.02.

[3] D.S. Archdeacon, *Heffter arrays and biembedding graphs on surfaces*, Electron. J. Combin. 22 (2015) #P1.74.

[4] D.S. Archdeacon, T. Boothby and J.H. Dinitz, *Tight Heffter arrays exist for all possible values*, J. Combin. Des. 25 (2017), 5–35.

[5] D.S. Archdeacon, J.H. Dinitz, D.M. Donovan and E.S. Yazıcı, *Square integer Heffter arrays with empty cells*, Des. Codes Cryptogr. 77 (2015), 409–426.

[6] D.S. Archdeacon, J.H. Dinitz, A. Mattern and D.R. Stinson, *On partial sums in cyclic groups*, J. Combin. Math. Combin. Comput. 98 (2016), 327–342.

[7] J.P. Bode and H. Harborth, *Directed paths of diagonals within polytopes*, Discrete Math. 299 (2005), 3–10.

[8] D. Bryant and S. El-Zanati, *Graph decompositions*, in: *Handbook of Combinatorial Designs*. Edited by C. J. Colbourn and J. H. Dinitz. Second edition. Discrete Mathematics and its Applications. Chapman & Hall/CRC, Boca Raton, 2007.

[9] M. Buratti, *Recursive constructions for difference matrices and relative difference families*, J. Combin. Des. 6 (1998), 165–182.

[10] M. Buratti and A. Pasotti, *Graph decompositions with the use of difference matrices*, Bull. Inst. Combin. Appl. 47 (2006), 23–32.

[11] A.C. Burgess, N.J. Cavenagh, D.A. Pike, *Mutually orthogonal cycle systems*, preprint.

[12] K. Burrage, N.J. Cavenagh, D. Donovan and E.S. Yazici, *Globally simple Heffter arrays $H(n;k)$ when $k \equiv 0, 3 \pmod{4}$*, Discrete Math. 343 (2020), 111787.

[13] Y. Caro and R. Yuster, *Orthogonal decomposition and packing of complete graphs*, J. Combin. Theory Ser. A 88 (1999), 93–111.

[14] N.J. Cavenagh, J. Dinitz, D. Donovan and E.S. Yazıcı, *The existence of square non-integer Heffter arrays*, Ars Math. Contemp. 17 (2019), 369–395.

[15] N.J. Cavenagh, D. Donovan and E.S. Yazıcı, *Biembeddings of cycle systems using integer Heffter arrays*, J. Combin. Des. 28 (2020), 900–922.

[16] S. Costa, M. Dalai and A. Pasotti, *A tour problem on a toroidal board*, Austral. J. Combin. 76 (2020), 183–207.

[17] S. Costa, F. Morini, A. Pasotti and M.A. Pellegrini, *A generalization of Heffter arrays*, J. Combin. Des. 28 (2020), 171–206.

[18] S. Costa, F. Morini, A. Pasotti and M.A. Pellegrini, *A problem on partial sums in abelian groups*, Discrete Math. 341 (2018), 705–712.

[19] S. Costa, F. Morini, A. Pasotti and M.A. Pellegrini, *Globally simple Heffter arrays and orthogonal cyclic cycle decompositions*, Austral. J. Combin. 72 (2018), 549–593.

[20] S. Costa, A. Pasotti and M.A. Pellegrini, *Relative Heffter arrays and biembeddings*, Ars Math. Contemp. 18 (2020), 241–271.

[21] S. Costa and A. Pasotti, *On $\lambda$-fold relative Heffter arrays and biembedding multigraphs on surfaces*, Europ. J. Combin. 97 (2021), 103370.

[22] S. Costa and M.A. Pellegrini, *Some new results about a conjecture by Brian Alspach*, Archiv der Mathematik 115 (2020), 479–488.

[23] J.H. Dinitz and A.R.W. Mattern, *Biembedding Steiner triple systems and $n$-cycle systems on orientable surfaces*, Austral. J. Combin. 67 (2017), 327–344.
[24] J.H. Dinitz and I.M. Wanless, The existence of square integer Heffter arrays, Ars Math. Contemp. 13 (2017), 81–93.
[25] J.L. Gross and T.W. Tucker, Topological Graph Theory, John Wiley, New York, 1987.
[26] J. Hicks, M.A. Ollis and J.R. Schmitt, Distinct partial sums in cyclic groups: polynomial method and constructive approaches, J. Combin. Des. 27 (2019), 369–385.
[27] B. Mohar, Combinatorial local planarity and the width of graph embeddings, Canad. J. Math. 44 (1992), 1272–1288.
[28] B. Mohar and C. Thomassen, Graphs on surfaces, Johns Hopkins University Press, Baltimore, 2001.
[29] F. Morini and M.A. Pellegrini, On the existence of integer relative Heffter arrays, Discrete Math. 343 (2020), 112088.
[30] F. Morini and M.A. Pellegrini, Magic rectangles, signed magic arrays and integer $\lambda$-fold relative Heffter arrays, Austral. J. Combin. 80 (2021), 249–280.
[31] F. Morini and M.A. Pellegrini, Rectangular Heffter arrays: a reduction theorem, preprint available at https://arxiv.org/abs/2107.08857v2.
[32] M.A. Ollis, Sequences in dihedral groups with distinct partial products, Austral. J. Combin. 78 (2020), 35–60.

DICATAM - Sez. Matematica, Università degli Studi di Brescia, Via Branze 43, I-25123 Brescia, Italy
Email address: simone.costa@unibs.it

DII, Università degli Studi di Brescia, Via Branze 38, I-25123 Brescia, Italy
Email address: s.dellafiore001@unibs.it

DICATAM - Sez. Matematica, Università degli Studi di Brescia, Via Branze 43, I-25123 Brescia, Italy
Email address: anita.pasotti@unibs.it