A Note on Fractional DP-Coloring of Graphs

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Abstract

DP-coloring (also called correspondence coloring) is a generalization of list coloring introduced by Dvořák and Postle in 2015. In 2019, Bernshteyn, Kostochka, and Zhu introduced a fractional version of DP-coloring. They showed that unlike the fractional list chromatic number, the fractional DP-chromatic number of a graph $G$, denoted $\chi^*_\text{DP}(G)$, can be arbitrarily larger than $\chi^*(G)$, the graph’s fractional chromatic number. We generalize a result of Alon, Tuza, and Voigt (1997) on the fractional list chromatic number of odd cycles, and, in the process, show that for each $k \in \mathbb{N}$, $\chi^*_\text{DP}(C_{2k+1}) = \chi^*(C_{2k+1})$. We also show that for any $n \geq 2$ and $m \in \mathbb{N}$, if $p^*$ is the solution in $(0,1)$ to $p = (1-p)^n$ then $\chi^*_\text{DP}(K_{n,m}) \leq 1/p^*$, and we prove a generalization of this result for multipartite graphs. Finally, we determine a lower bound on $\chi^*_\text{DP}(K_{2,m})$ for any $m \geq 3$.

Keywords. graph coloring, list coloring, fractional coloring, DP-coloring, correspondence coloring.

Mathematics Subject Classification. 05C15, 05C69

1 Introduction

In this paper all graphs are nonempty, finite, simple graphs unless otherwise noted. Generally speaking we follow West [14] for terminology and notation. The set of natural numbers is $\mathbb{N} = \{1, 2, 3, \ldots\}$. Given a set $A$, $\mathcal{P}(A)$ is the power set of $A$. For $m \in \mathbb{N}$, we write $[m]$ for the set $\{1, 2, \ldots, m\}$. If $G$ is a graph and $S, U \subseteq V(G)$, we use $G[S]$ for the subgraph of $G$ induced by $S$, and we use $E_G(S, U)$ for the subset of $E(G)$ with one endpoint in $S$ and the other endpoint in $U$. For $v \in V(G)$, we write $d_G(v)$ for the degree of vertex $v$ in the graph $G$, and we write $N_G(v)$ for the neighborhood of vertex $v$ in the graph $G$. Also, for $S \subseteq V(G)$, we let $N_G(S) = \bigcup_{v \in S} N_G(v)$. A graph $G$ is $d$-degenerate if every subgraph of $G$ has a vertex of degree at most $d$. We use $K_{n,m}$ to denote the complete bipartite graphs with partite sets of size $n$ and $m$. For a random variable $X$, we use $X \sim B(n, p)$ to indicate that $X$ is binomially distributed with $n$ trials each having probability of success $p$. For an event $E$, we use $\overline{E}$ to denote the complement of $E$.

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1.1 Fractional Coloring and Fractional Choosability

Before introducing fractional DP-coloring, we review some classical notions. Given a graph $G$, in the classical vertex coloring problem we wish to color the elements of $V(G)$ with colors from the set $[m]$ so that adjacent vertices receive different colors, a so-called proper $m$-coloring. We say $G$ is $m$-colorable when a proper $m$-coloring of $G$ exists. The chromatic number of $G$, denoted $\chi(G)$, is the smallest $k$ such that $G$ is $k$-colorable.

A set coloring of a graph $G$ is a function that assigns a set to each vertex of $G$ such that the sets assigned to adjacent vertices are disjoint. For $a, b \in \mathbb{N}$ with $a \geq b$, an $(a, b)$-coloring of graph $G$ is a set coloring $f$ of $G$ such that the codomain of $f$ is the set of $b$-element subsets of $[a]$. We say that $G$ is $(a, b)$-colorable when an $(a, b)$-coloring of $G$ exists. So, saying $G$ is $a$-colorable is equivalent to saying that it is $(a, 1)$-colorable. The fractional chromatic number, $\chi^*(G)$, of $G$ is defined by $\chi^*(G) = \inf\{a/b : G$ is $(a, b)$-colorable\}. Since any graph $G$ is $(\chi(G), 1)$-colorable, we have that $\chi^*(G) \leq \chi(G)$. This inequality may however be strict; for example, when $r \geq 2$, $\chi^*(C_{2r+1}) = 2 + 1/r < 3 = \chi(C_{2r+1})$ (see [11]). It is also well known that the infimum in the definition of $\chi^*(G)$ is actually a minimum [11].

List coloring is a variation on classical vertex coloring that was introduced independently by Vizing [13] and Erdős, Rubin, and Taylor [6] in the 1970’s. In list coloring, we associate with graph $G$ a list assignment $L$ that assigns to each vertex $v \in V(G)$ a list $L(v)$ of available colors. Graph $G$ is said to be $L$-colorable if there exists a proper coloring $f$ of $G$ such that $f(v) \in L(v)$ for each $v \in V(G)$ (we refer to $f$ as a proper $L$-coloring of $G$). A list assignment $L$ is called a $k$-assignment for $G$ if $|L(v)| = k$ for each $v \in V(G)$. We say $G$ is $k$-choosable if $G$ is $L$-colorable whenever $L$ is a $k$-assignment for $G$. The list chromatic number of $G$, denoted $\chi_L(G)$, is the smallest $k$ for which $G$ is $k$-choosable. Since a $k$-assignment can assign the same $k$ colors to every vertex of a graph, $\chi(G) \leq \chi_L(G)$.

Given an $a$-assignment $L$ for graph $G$ and $b \in \mathbb{N}$ such that $a \geq b$, we say that $f$ is an $(L, b)$-coloring of $G$ if $f$ is a set coloring of $G$ such that for each each $v \in V(G)$, $f(v) \subseteq L(v)$ with $|f(v)| = b$. We say that $G$ is $(L, b)$-colorable when an $(L, b)$-coloring of $G$ exists. Also, for $a, b \in \mathbb{N}$ with $a \geq b$, graph $G$ is $(a, b)$-choosable if $G$ is $(L, b)$-colorable whenever $L$ is an $a$-assignment for $G$. The fractional list chromatic number, $\chi^*_L(G)$, of $G$ is defined by $\chi^*_L(G) = \inf\{a/b : G$ is $(a, b)$-choosable\}. It is clear that if a graph is $(a, b)$-choosable, then it is $(a, b)$-colorable. So, $\chi^*(G) \leq \chi^*_L(G)$. In 1997, Alon, Tuza, and Voigt [1] famously proved that for any graph $G$, $\chi^*_L(G) = \chi^*(G)$. Moreover, they showed that for any graph $G$, there is an $M \in \mathbb{N}$ such that $G$ is $(M, M/\chi^*(G))$-choosable. So, the infimum in the definition of $\chi^*_L(G)$ is also actually a minimum.

In their 1979 paper Erdős et al. [6] asked: If $G$ is $(a, b)$-choosable and $c, d \in \mathbb{N}$ are such that $c/d > a/b$, must $G$ be $(c, d)$-choosable? A negative answer to this question is given in [6]. Erdős et al. also asked: If $G$ is $(a, b)$-choosable, does it follow that $G$ is $(at, bt)$-choosable for each $t \in \mathbb{N}$? Tuza and Voigt [12] showed that the answer to this question is yes when $a = 2$ and $b = 1$. However, in general, a negative answer to this question was recently given in [1]. It was shown that for each $a \geq 4$, a graph that is $(a, 1)$-choosable but not $(2a, 2)$-choosable can be constructed. We briefly consider the fractional DP-coloring analogues of both of these questions below.
1.2 Fractional DP-coloring

In 2015, Dvořák and Postle \cite{5} introduced DP-coloring (they called it correspondence coloring) in order to prove that every planar graph without cycles of lengths 4 to 8 is 3-choosable. Intuitively, DP-coloring is a generalization of list coloring where each vertex in the graph still gets a list of colors but identification of which colors are different can vary from edge to edge. We now give the formal definition. Suppose $G$ is a graph. A cover of $G$ is a pair $\mathcal{H} = (L, H)$ consisting of a graph $H$ and a function $L : V(G) \rightarrow \mathcal{P}(V(H))$ satisfying the following four requirements:

1. the set $\{L(u) : u \in V(G)\}$ is a partition of $V(H)$ with $|V(G)|$ parts;
2. for every $u \in V(G)$, the graph $L(u)$ is an independent set of vertices in $H$;
3. if $E_H(L(u), L(v))$ is nonempty, then $uv \in E(G)$;
4. if $uv \in E(G)$, then $E_H(L(u), L(v))$ is a matching (the matching may be empty).

Suppose $\mathcal{H} = (L, H)$ is a cover of $G$. We say $\mathcal{H}$ is $m$-fold if $|L(u)| = m$ for each $u \in V(G)$. An $\mathcal{H}$-coloring of $G$ is an independent set $I \subseteq V(H)$ such that $|I \cap L(u)| = 1$ for each $u \in V(G)$. The DP-chromatic number of a graph $G$, $\chi_{DP}(G)$, is the smallest $m \in \mathbb{N}$ such that $G$ admits an $\mathcal{H}$-coloring for every $m$-fold cover $\mathcal{H}$ of $G$.

Given an $m$-assignment $L$ for a graph $G$, it is easy to construct an $m$-fold cover $\mathcal{H}$ of $G$ such that $G$ has an $\mathcal{H}$-coloring if and only if $G$ has a proper $L$-coloring (see \cite{2}). It follows that $\chi_L(G) \leq \chi_{DP}(G)$. This inequality may be strict since it is easy to prove that $\chi_{DP}(C_n) = 3$ whenever $n \geq 3$, but the list chromatic number of any even cycle is 2 (see \cite{2} and \cite{6}).

We are now ready to introduce fractional DP-coloring. Suppose $\mathcal{H} = (L, H)$ is an $a$-fold cover of $G$ and $b \in \mathbb{N}$ such that $a \geq b$. Then, $G$ is $(\mathcal{H}, b)$-colorable if there is an independent set $S \subseteq V(H)$ such that $|S \cap L(v)| \geq b$ for each $v \in V(G)$. Equivalently, one could require $|S \cap L(v)| = b$ for each $v \in V(G)$; in this case we call $S$ an independent $b$-fold transversal. We refer to $S$ as an $(\mathcal{H}, b)$-coloring of $G$. For $a, b \in \mathbb{N}$ and $a \geq b$, we say $G$ is $(a, b)$-DP-colorable if for any $a$-fold cover $\mathcal{H}$ of $G$, $G$ is $(\mathcal{H}, b)$-colorable. The fractional DP-chromatic number, $\chi_{DP}^\ast(G)$, of $G$ is defined by

$$\chi_{DP}^\ast(G) = \inf\{a/b : G \text{ is } (a, b)\text{-DP-colorable}\}.$$ 

It is easy to prove that if $G$ is $(a, b)$-DP-colorable, then $G$ is $(a, b)$-choosable. Also, any graph $G$ must be $(\chi_{DP}(G), 1)$-DP-colorable. So, combining the facts we know, we have:

$$\chi^\ast(G) = \chi_L^\ast(G) \leq \chi_{DP}^\ast(G) \leq \chi_{DP}(G).$$

Theorem \ref{10} and Corollary \ref{7} below imply that both of the inequalities above can be strict. Furthermore, we know that $\chi_L^\ast(G) \leq \chi_L(G) \leq \chi_{DP}(G)$, and we will see below that it is possible for the list chromatic number of a graph to be either smaller ($K_{2,3}$ by Theorem \ref{10} below) or larger (odd cycles with at least five vertices by Theorem \ref{2} below) than the fractional DP-chromatic number of the graph.

In \cite{3}, the following result is proven.

Theorem 1 (\cite{3}). Let $G$ be a connected graph. Then, $\chi_{DP}^\ast(G) \leq 2$ if and only if $G$ contains no odd cycles and at most one even cycle. Furthermore, if $G$ contains no odd cycles and
exactly one even cycle, then \( \chi_{DP}^*(G) = 2 \) even though 2 is not contained in the set \( \{a/b : G \text{ is } (a, b)\text{-DP-colorable}\} \).

So, unlike the fractional chromatic number and fractional list chromatic number, the infimum in the definition of the fractional DP-chromatic number is not always a minimum. In [3] it is also shown that if \( G \) is a graph of maximum average degree \( d \geq 4 \), then \( \chi_{DP}^*(G) \geq d/(2 \ln d) \). Since bipartite graphs have fractional chromatic number (and hence fractional list chromatic number) 2 and there exist bipartite graphs with arbitrarily high average degree, we see \( \chi_{DP}^*(G) \) and \( \chi^*(G) \) can be arbitrarily far apart and \( \chi_{DP}^*(G) \) cannot be bounded above by a function of \( \chi^*(G) \).

1.3 Outline of Results and Open Questions

We now present an outline of the results of this paper while also mentioning some open questions. In Section 2 we study the fractional DP-chromatic number of odd cycles. In 1997, Alon, Tuza, and Voigt [1] showed that \( C_{2r+1} \) is \((2r+1, r)\)-choosable. We generalize this result by showing \( \chi^*(C_{2r+1}) = \chi_{DP}^*(C_{2r+1}) = \chi_{DP}^*(C_{2r+1}) \).

**Theorem 2.** \( C_{2r+1} \) is \((2r+1, r)\)-DP-colorable. Consequently, \( \chi_{DP}^*(C_{2r+1}) = 2 + 1/r \).

Notice that by Theorem 2, we see it is possible for the list chromatic number of a graph to be larger than its fractional DP-chromatic number since \( \chi_{DP}^*(C_{2r+1}) < \chi^*(C_{2r+1}) = 3 \) when \( r \geq 2 \). Other classes of graphs with this strict inequality are shown by Corollary 7 and Theorem 10 below.

It is natural to ask analogues of the two questions posed about \((a, b)\)-choosability in [6].

**Question 3.** If \( G \) is \((a, b)\)-DP-colorable and \( c, d \in \mathbb{N} \) are such that \( c/d > a/b \), must \( G \) be \((c, d)\)-DP-colorable?

**Question 4.** If \( G \) is \((a, b)\)-DP-colorable, does it follow that \( G \) is \((at, bt)\)-DP-colorable for each \( t \in \mathbb{N} \)?

Question 4 is open. We suspect the answer is no because of Dvořák, Hu, and Sereni’s similar list coloring result [4]. The answer to Question 3 is no in a fairly strong sense. In particular, Corollary 1.12 in [3] implies that for any positive real number \( c \), there exist \( a, b, k \in \mathbb{N} \) such that \( k - a/b > c \), \( G \) is not \( k\)-colorable (and therefore not \((k, 1)\)-DP-colorable), and \( G \) is \((a, b)\)-DP-colorable[4]. By combining some known results, we quickly observe that the answer to Question 3 remains no even if we restrict our attention to only bipartite graphs.

**Proposition 5.** For each \( k \geq 149 \), there exists a \( k\)-degenerate bipartite graph \( G \), and \( a, b \in \mathbb{N} \) such that: \( k > a/b \), \( G \) is not \((k, 1)\)-DP-colorable, and \( G \) is \((a, b)\)-DP-colorable.

Proposition 5 is a consequence of two results. It follows from the randomized construction given in the proof of Theorem 1.10 in [3] that for \( d \geq 149 \), if \( G \) is a \( d\)-degenerate bipartite graph, then \( \chi_{DP}^*(G) \leq 5d/\ln(d) < d \). Note \( K_{d,t} \) is a \( d\)-degenerate bipartite graph, and it was shown in [10] that if \( t \geq 1 + (d^3/d!)(\ln(d) + 1) \), then \( \chi_{DP}(K_{d,t}) = d + 1 \).

4Thanks to an anonymous referee for this observation.
Theorem 9. If $G$ is a $d$-degenerate bipartite graph, then $\chi_{DP}^*(G) \leq (1 + o(1))d/\ln(d)$ as $d \to \infty$.

\footnote{An anonymous referee provided us an elegant proof of a special case of Corollary 7 which follows from Theorem 6.}
Clearly, $1 = 0$ since it is possible that none of the cycles in our decomposition are even. We also suppose that disjoint cycles: since adding additional edges to $H$ suppose that the vertices of $B$ odd cycles posed by Bernshteyn, Kostochka, and Zhu [3]. Question 11 could be extended to complete example of a graph whose fractional DP-chromatic number is irrational, answering a question achieved by Theorem 6 and Corollary 7 give the asymptotically tight result as the second partite \[ \frac{(d+2)^2(m+1)(1-d)^{-1}}{(d+2)(d^2)} < 1. \]

Then, $2 + d \leq \chi^*_D(G)$. For example, notice that when $m = 15$ and $d = 0.0959$, the inequality in the hypothesis is satisfied. So, by Theorems 6 and 10 we have that $2.0959 \leq \chi^*_D(K_{2,15}) \leq 2.619$. Since we suspect the probabilistic argument from Theorem 6 to be close to the truth for large values of $m$, we think that the lower bound provided by Theorem 10 can be improved by quite a bit for these $m$ values. We also have that $2.025 \leq \chi^*_D(K_{2,3}) \leq 2.619$ which provides another example of a graph whose fractional DP-chromatic number is larger than its list chromatic number since $2 = \chi_l(K_{2,3}) < \chi^*_D(K_{2,3})$.

These results lead to a natural question.

Question 11. Suppose $n \in \mathbb{N}$. If $p^*$ is the solution in $(0,1)$ to $p = (1-p)^n$, does $\chi^*_D(K_{n,m}) \rightarrow 1/p^*$ as $m \rightarrow \infty$?

The answer to Question 11 is clearly yes when $n = 1$. If the answer is yes for some $n \geq 2$, then Theorem 9 and Corollary 7 give the asymptotically tight result as the second partite set grows large. We could further ask whether, for some $n \geq 2$, there is an $m \in \mathbb{N}$ where we achieve $\chi^*_D(K_{n,m}) = 1/p^*$? If the answer to this follow-up question is yes it would give an example of a graph whose fractional DP-chromatic number is irrational, answering a question posed by Bernshteyn, Kostochka, and Zhu [3]. Question 11 could be extended to complete multipartite graphs where the sizes of all but one of the partite sets are held constant.

2 Odd Cycles

We now prove Theorem 2.

Proof. Suppose that the vertices of $G$ in cyclic order are: $v_1, v_2, \ldots, v_{2r+1}$. Suppose that $\mathcal{H} = (L, H)$ is an arbitrary $(2r + 1)$-fold cover of $G$. We must show that there is an $(\mathcal{H}, r)$-coloring of $G$. We may assume that $E_H(L(u), L(v))$ is a perfect matching whenever $uv \in E(G)$ since adding additional edges to $H$ only makes it harder to find an $(\mathcal{H}, r)$-coloring.

Clearly, $H$ is a 2-regular graph. This means that $H$ can be decomposed into vertex disjoint cycles: $B_1, B_2, \ldots, B_p$. The size of each of these cycles is a multiple of $2r + 1$. Let us suppose that $B_1, \ldots, B_l$ are even cycles and $B_{l+1}, \ldots, B_p$ are odd cycles (Note: we allow $l = 0$ since it is possible that none of the cycles in our decomposition are even. We also know the number of odd cycles in our decomposition must be odd since $|E(H)| = (2r + 1)^2$).

Clearly, $1 \leq p \leq 2r + 1$, and $|L(v_i) \cap V(B_j)| \geq 1$ for each $i \in [2r + 1]$ and $j \in [p]$.

Let $H'$ be the graph obtained from $H$ as follows: for each $j \in [p]$ we delete a vertex $d_j \in L(v_j) \cap V(B_j)$. We let $L'(v_j) = L(v_j) - \{d_j\}$ for each $j \in [p]$ and $L'(v_i) = L(v_i)$ for each
Let $t > p$. So, $H^*$ consists of $p$ vertex disjoint paths, and for each $j \in [p]$, let $P_j = B_j - \{d_j\}$. Note that if $1 < j < 2r + 1$, the endpoints of $P_j$ are in $L'(v_{j-1})$ and $L'(v_{j+1})$. Also for each $j \in [p]$, $|V(P_j)| = (2r + 1)k_j + 2r$ where $k_j$ is a nonnegative integer that is odd when $j \leq l$ and even when $j > l$. It is easy to see:

$$|V(H^*)| = (2r + 1)^2 - p = \sum_{j=1}^{p}((2r + 1)k_j + 2r) = 2rp + (2r + 1)\sum_{j=1}^{p}k_j.$$ 

Thus, $\sum_{j=1}^{p}k_j = 2r + 1 - p$. Now, we name the vertices of each path of $H^*$. Specifically, for $j \in [p]$ let the vertices of $P_j$ (in order) be: $a_1^j, a_2^j, \ldots, a_{(2r+1)k_j+2r}^j$ so that $a_1^j \in L'(v_{j+1})$ if $j < 2r + 1$ and $a_1^j \in L'(v_1)$ if $j = 2r + 1$. We call a vertex $a_m^j \in V(H^*)$ odd if $m$ is odd. Let $S$ consist of all the odd vertices in $H^*$. Clearly, $S$ is an independent set of $H$. We claim that $|S \cap L'(v_i)| \geq r$ for each $i \in [2r + 1]$.

In the case that $p = 1$, $P_1$ is a path of length $(2r + 1)^2 - 2$, and we have that for $i \in [2r + 1]$, $|S \cap V(P_1) \cap L'(v_i)| = r + 1$ when $i$ is even, and $|S \cap V(P_1) \cap L'(v_i)| = r$ when $i$ is odd. In the case that $p = 2r + 1$, each of $P_1, \ldots, P_{2r+1}$ is a path of length $2r - 1$, and we have that for $i \in [2r + 1]$, $|S \cap L'(v_i)| = r$.

So, we turn our attention to the case where $2 \leq p \leq 2r$. For each $j \in [l]$ notice that $P_j$ is a path with an odd number of vertices. So, when $j \in [l]$, $|S \cap V(P_j)| = (2r + 1)(k_j + 1)/2$. Moreover, since $G$ is an odd cycle, for each $j \in [l]$ and $i \in [2r + 1]$, we have that $|S \cap V(P_j) \cap L'(v_i)| = (k_j + 1)/2$.

Now, let $L = \{l + 1, l + 3, \ldots, p - 2\}$ (Note: $L$ is empty if $l + 1 > p - 2$ and $|L| \leq r - 1$). For each $j \in L$ we consider $P_j$ and $P_{j+1}$ together. Note $P_j$ and $P_{j+1}$ are paths with an even number of vertices. So, when $j \in L$, $|S \cap (V(P_j) \cup V(P_{j+1}))| = (2r + 1)(k_j + k_{j+1})/2 + 2r$. Therefore, when $j \in L$ and $i \in [2r + 1] - \{j\}$, $|S \cap (V(P_j) \cup V(P_{j+1})) \cap L'(v_i)| = (k_j + k_{j+1})/2$, and for each $j \in L$, $|S \cap (V(P_j) \cup V(P_{j+1})) \cap L'(v_j)| = (k_j + k_{j+1})/2$ (since $a_1^j \in L'(v_{j+1})$ and $a_{(2r+1)k_{j+1}+2r}^j \in L'(v_j)$). Thus, for $i \in [2r + 1] - L$, we have:

$$\left| \bigcup_{j=1}^{p-1} (V(P_j) \cap S \cap L'(v_i)) \right| = \sum_{j=1}^{l} \frac{k_i + 1}{2} + \sum_{j \in L} \frac{k_j + k_{j+1} + 2}{2} = \frac{1}{2} \sum_{j=1}^{p-1} (k_i + 1) = \frac{p - 1}{2} + \frac{1}{2} \sum_{j=1}^{p-1} k_i = \frac{p - 1 + 2r + 1 - p - k_p}{2} = \frac{r - k_p}{2}.$$ 

Similarly, for $i \in L$, we have:

$$\left| \bigcup_{j=1}^{p-1} (V(P_j) \cap S \cap L'(v_i)) \right| = r - 1 - \frac{k_p}{2}.$$
It is easy to see that $|S \cap V(P_p) \cap L'(v_i)| \geq k_p/2$ for each $i \in [2r + 1]$. If $|\mathcal{L}| \geq 1$, notice that $a_{(2r + 1)k_p + 2r}^p \in L'(v_{p-1})$. So, when $|\mathcal{L}| \geq 1$, the last odd vertex in $V(P_p)$ is in $L'(v_{p-2})$. So, we know that for each $i \in \mathcal{L}$,

$$|S \cap V(P_p) \cap L'(v_i)| = \frac{k_p}{2} + 1.$$ 

It follows that $|S \cap L'(v_i)| \geq r$ for each $i \in [2r + 1]$ which implies that $S$ is an $(\mathcal{H}, r)$-coloring of $G$. \qed

3 Multpartite Graphs

We now work toward proving our bounds on the fractional DP-chromatic number of complete bipartite graphs. In order to prove Theorem 6 we will use two lemmas.

Lemma 12. Suppose $G$ is a graph where $\{U, W\}$ is a partition of $V(G)$, $d = \max_{w \in W} |N_G(w) \cap U|$, $p \in (0, 1)$, and $\epsilon > 0$. There is an $N \in \mathbb{N}$ such that for any $a \geq N$, if $\mathcal{H} = (L, H)$ is an $a$-fold cover of $G$, then there must exist $U' \subseteq \bigcup_{v \in U} L(v)$ and $W' \subseteq \bigcup_{w \in W} L(w)$ such that the following three conditions hold:

1. $|E_H(U', W')| = 0$,
2. $|U' \cap L(u)| \geq (p - \epsilon)a$ for all $u \in U$,
3. $|W' \cap L(w)| \geq ((1 - p)^d - \epsilon)a$ for all $w \in W$.

Proof. Let $n = |V(G)|$. We will now give a random procedure for constructing $C \subseteq \bigcup_{u \in U} L(u)$ and $D \subseteq \bigcup_{w \in W} L(w)$ which we will use to guarantee the existence of sets $U'$ and $W'$ as described in the statement. As in the proof of Theorem 2 we may assume that $E_H(L(u), L(v))$ is a perfect matching whenever $uv \in E(G)$.

For each $u \in U$ and each $x \in L(u)$ include $x$ in $C$ independently with probability $p$. For each $w \in W$ and each $y \in L(w)$ include $y$ in $D$ if $y$ is not adjacent to any of the vertices in $C$.

The probability that, for any $w \in W$, a vertex from $L(w)$ is included in $D$ is at least $(1 - p)^d$. For each $u \in U$, let $X_u$ be the random variable that is the number of vertices included in $C$ from $L(u)$. For each $w \in W$, let $Y_w$ be the random variable that is the number of vertices included in $D$ from $L(w)$. Let $E_{X_u}$ be the event that $X_u \geq (p - \epsilon)a$ and $E_{Y_w}$ be the event that $Y_w \geq ((1 - p)^d - \epsilon)a$. Since $X_u \sim B(a, p)$ and $Y_w \sim B(a, p_w)$ for some $p_w \geq (1 - p)^d$, we know that for any $u \in U$ and $w \in W$

$$\mathbb{P}(E_{X_u}) = \mathbb{P}(X_u < (p - \epsilon)a) \leq e^{-2ap^2},$$

$$\mathbb{P}(E_{Y_w}) = \mathbb{P}(Y_w < ((1 - p)^d - \epsilon)a) \leq \mathbb{P}(Y_w < (p_w - \epsilon)a) \leq e^{-2ap^2}.$$

By the union bound, we know the probability that $|C \cap L(u)| \geq (p - \epsilon)a$ for each $u \in U$ and that $|D \cap L(w)| \geq ((1 - p)^d - \epsilon)a$ for each $w \in W$ is

$$\mathbb{P}\left(\bigcap_{u \in U} E_{X_u} \bigcap \bigcap_{w \in W} E_{Y_w}\right) = 1 - \mathbb{P}\left(\bigcup_{u \in U} \overline{E_{X_u}} \bigcup \bigcup_{w \in W} \overline{E_{Y_w}}\right) \geq 1 - ne^{-2ap^2}.$$
We now show that this probability is positive for large enough \( a \). Let \( N \) be any integer larger than \( \ln(n)/(2e^2) \). For any \( a \geq N \) we see

\[
1 - ne^{-2a^2} > 1 - ne^{-2(\ln(n)/(2e^2))^2} = 0.
\]

Therefore, there must be sets \( U' \) and \( W' \) as described in the statement. \( \square \)

**Lemma 13.** Suppose \( G \) is a graph where \( \{U,W\} \) is a partition of \( V(G) \) and \( U \) is an independent set of vertices. Let \( d = \max_{w \in W} |N_G(w) \cap U| \). Fix \( p' \in (0,1] \) and suppose \( G[W] \) has the property that for any \( \epsilon' \in (0,p') \) there is an \( N_{\epsilon'} \in \mathbb{N} \) such that \( G[W] \) is \((a',[(p' - \epsilon')a'])\)-DP-colorable for all \( a' \geq N_{\epsilon'} \).

Suppose \( p^* \) is the unique element in \((0,1)\) satisfying \( p = p'(1 - p)^d \). Then \( G \) has the property that for any \( \epsilon \in (0,p^*) \) there is an \( N \in \mathbb{N} \) such that \( G \) is \((a,[(p - \epsilon)a])\)-DP-colorable for all \( a \geq N \). Consequently, \( \chi_{DP}(G) \leq 1/p^* \).

**Proof.** Consider an arbitrary \( a \)-fold cover \( H = (L,H) \) of \( G, \epsilon' \in (0,p') \), and \( \epsilon^* \in (0,p^*) \).

By Lemma 12 there exists some \( N_{\epsilon'} \) such that if \( a \geq N_{\epsilon'} \) then we must be able to get a set \( U' \subseteq \bigcup_{u \in U} L(u) \) that contains a \([(p^* - \epsilon')a]\)-fold transversal of \( H_U \), and a set \( W' \subseteq \bigcup_{w \in W} L(w) \) that contains a \([(1 - p^*)^d - \epsilon^*)a]\)-fold transversal of \( H_W \). Note that \([(1 - p^*)^d - \epsilon^*)a]\) = \((p^*/p' - \epsilon^*)a\). Moreover, \(|E_H(U',W')| = 0\).

By the lemma statement, there is an \( N_{\epsilon'} \in \mathbb{N} \) such that \( G[W] \) is \((a',[(p' - \epsilon')a'])\)-DP-colorable for all \( a' \geq N_{\epsilon'} \).

For each \( w \in W \) let \( L'(w) = W' \cap L(w) \). If \( a \geq N_{\epsilon'} \) then we know that \((L',H[W'])\) contains a \([(p^*/p' - \epsilon^*)a]\)-fold cover of \( G[W] \). So \((L',H[W'])\) must have an independent \([(p' - \epsilon')p'/p' - \epsilon^*)a]\)-fold transversal if \( a \) also satisfies \([(p^*/p' - \epsilon^*)a]\) \( \geq N_{\epsilon'} \). Let \( N'_{\epsilon'} \) \( \equiv \frac{(N_{\epsilon'} + 1)}{(p^*/p' - \epsilon^*)} \). Note that \([(p^*/p' - \epsilon^*)a]\) \( \geq N'_{\epsilon'} \) is satisfied if \( a \geq N'_{\epsilon'} \).

Notice

\[
[(p' - \epsilon')(p^*/p' - \epsilon^*)a]\) \( \geq (p' - \epsilon')(p^*/p' - \epsilon^*)a - 2 \geq (p^*/p' - \epsilon^*)a - 2.
\]

Given \( \epsilon \in (0,p^*) \) we can fix an \( \epsilon' \in (0,p') \) and an \( \epsilon^* \in (0,\epsilon) \) such that

\[
\epsilon > \epsilon^* + \epsilon'p^*/p' \geq \epsilon.
\]

Therefore, there must be some \( N^* \in \mathbb{N} \) such that for all \( a \geq N^* \),

\[
[(p' - \epsilon')(p^*/p' - \epsilon^*)a]\) \( \geq (p^*/p' - \epsilon^*)a - 2 \geq (p^*/p' - \epsilon^*)a - 2 \geq (p^*/p' - \epsilon^*)a.
\]

If \( a \geq \max\{N_{\epsilon'},N'_{\epsilon'},N^*\} \) we know that \( W' \) must contain an independent \([(p^*/p' - \epsilon^*)a]\)-fold transversal of \( H_W \). Call this \([(p^*/p' - \epsilon^*)a]\)-fold transversal \( T_W \). And since we chose \( \epsilon^* \) to be less than \( \epsilon \), we also know that \( U' \) must contain a \([(p^*/p' - \epsilon^*)a]\)-fold transversal of \( H_U \). Call this \([(p^*/p' - \epsilon^*)a]\)-fold transversal \( T_U \). Since \( U \) is an independent set of vertices, we know the vertices in \( T_U \) form an independent set of vertices of \( H_U \). We know \(|E_H(U',W')| = 0\). Therefore, \( T_U \cup T_W \) is an independent \([(p^*/p' - \epsilon^*)a]\)-fold transversal of \( H \) and \( G \) is \((a,[p^*/p' - \epsilon^*)a]\)-DP-colorable.

For each \( \epsilon > 0 \) suppose \( M_\epsilon \) satisfies: \( G \) is \((a,[(p^*/p' - \epsilon^*)a]\)-DP-colorable for any \( a \geq M_\epsilon \), consider the sequences \( \epsilon_k = 1/k \) and \( a_k = \max\{k,M_\epsilon k\} \). For sufficiently large \( k \), guaranteeing \((p^*/p' - 1/k)k - 1 > 0\), we know

\[
\chi_{DP}(G) \leq \frac{a_k}{[(p^*/p' - \epsilon_k)a_k]} \leq \frac{a_k}{(p^*/p' - \epsilon_k)a_k - 1} = \frac{1}{p^*/p' - 1/a_k - 1/a_k} \leq \frac{1}{p^*/p' - 1/k - 1/k} = \frac{1}{p^*/p' - 1/2/k}.
\]
By taking the limit as \( k \to \infty \), it follows that \( \chi_{DP}^*(G) \leq 1/p^* \).

We are now ready to prove Theorem 6.

**Theorem 6.** Suppose \( G \) is an \( m \)-partite graph with partite sets \( A_1, A_2, \ldots, A_m \). Let

\[
d_j = \max \left\{ |N_G(v) \cap A_j| : v \in \bigcup_{k=j+1}^m A_k \right\},
\]

\( p^*_m = 1 \) and \( p^*_j \) be the unique solution in \((0, 1)\) to \( p = p^*_j(1-p)^{d_j} \) for all \( j \in [m-1] \). Then \( \chi_{DP}^*(G) \leq \frac{1}{p^*_1} \).

**Proof.** We will prove the stronger statement: \( G \) has the property that for any \( \epsilon \in (0, p^*_1) \) there is an \( N \in \mathbb{N} \) such that \( G \) is \((a, \lfloor (p^*_1 - \epsilon) a \rfloor)\)-DP-colorable for all \( a \geq N \), and consequently \( \chi_{DP}^*(G) \leq 1/p^*_1 \). This follows from induction on the number of partite sets \( m \) using Lemma 13. For the base case, when \( m = 2 \), \( G \) is the bipartite graph with partite sets \( A_1 \) and \( A_2 \). Let \( U = A_1 \) and \( W = A_2 \). Since \( G[W] \) is an independent set, it has the property that every \( a' \)-fold cover of \( G[W] \) has an independent \( a' \)-fold transversal (satisfying the requirement for Lemma 13 with \( p' = 1 \)). Let \( d = d_1 = \max_{w \in W}|N_G(w) \cap U| \) and \( p^*_1 \) be the unique element in \((0, 1)\) satisfying \( p = 1(1-p)^d \). Applying Lemma 13 completes the base case.

Next, we assume that our result holds for any given \( k \)-partite graph for some fixed \( k \geq 2 \).

Consider \( G \), a \((k+1)\)-partite graph with partite sets \( A_1, A_2, \ldots, A_{k+1} \). Let \( U = A_1 \) and \( W = \bigcup_{i=2}^{k+1} A_i \). Notice by the induction hypothesis that \( G[W] \) is a \( k\)-partite graph with partite sets \( A_2, \ldots, A_{k+1} \) that satisfies the hypothesis of Lemma 13 with \( p' = p^*_2 \). Applying Lemma 13 with \( d = d_1 \) and \( p^* = p^*_1 \) shows that the statement holds for \( G \).

Therefore, our result holds for any \( m \)-partite graph with \( m \geq 2 \) by induction.

**Corollary 7** follows immediately from Theorem 6.

### 4 Lower Bound for Complete Bipartite Graphs

From this point forward, when considering a copy of the complete bipartite graph \( K_{n,m} \), we will always assume that the partite sets are \( A = \{v_1, \ldots, v_n\} \) and \( B = \{u_1, \ldots, u_m\} \). We will now use a probabilistic argument to prove Theorem 10.

**Proof.** Throughout this proof suppose \( m \in \mathbb{N} \) is fixed and \( m \geq 3 \). Since \( G \) contains more than one even cycle we know that \( \chi_{DP}^*(G) \geq 2 \) by Theorem 10. Our goal for this proof is to show that \( \chi_{DP}^*(G) \geq 2 + d \). So, suppose that \( a \) and \( t \) are arbitrary natural numbers such that \( 2 < a/t \leq 2 + d \). Also, let \( r = a/t \) and \( \delta = r - 2 \) so that \( \delta \in (0, d) \). To prove the result, it is sufficient to show that \( G \) is not \((a, t)\)-DP-colorable.

We form an \( a \)-fold cover \((L, H)\) of \( G \) by the following (partially random) process. We begin by letting \( L(v_i) = \{(v_i, l) : l \in [a]\} \) and \( L(u_j) = \{(u_j, l) : l \in [a]\} \) for each \( i \in [2] \) and \( j \in [m] \). Let the graph \( H \) have vertex set

\[
\left( \bigcup_{i=1}^2 L(v_i) \right) \cup \left( \bigcup_{j=1}^m L(u_j) \right).
\]
Let $L(v)$ be an independent set of vertices in $H$ for each $v \in V(G)$. Finally, for each $i \in [2]$ and $j \in [m]$, uniformly at random choose a perfect matching between $L(v_i)$ and $L(u_j)$ from the $a!$ possible perfect matchings. It is easy to see that $\mathcal{H} = (L, H)$ is an $a$-fold cover of $G$.

We want to show that with positive probability there is no $(\mathcal{H}, t)$-DP-coloring of $G$. For $i = 1, 2$, let $A_i$ be the set of $t$-element subsets of $L(v_i)$. We say $(A_1, A_2) \in A_1 \times A_2$ is good for $u_j$ if $|L(u_j) \cap (N_H(A_1 \cup A_2))| \geq t$, meaning there is a $t$-element subset of $L(u_j)$ that is independent of $A_1 \cup A_2$. We know we can find a $(\mathcal{H}, t)$-coloring of $G$ if $(A_1, A_2) \in A_1 \times A_2$ is good for each vertex in $\{u_j : j \in [m]\}$. Let $E_j$ be the event that $(A_1, A_2)$ is good for $u_j$. In order for $E_j$ to occur we need at least $3t - a$ of the vertices in $N_H(A_1) \cap L(u_j)$ to also be in $N_H(A_2) \cap L(u_j)$. So,

$$P[E_j] = \binom{a}{t}^{-1} \sum_{i=3t-a}^{t} \binom{t}{i} \binom{a-t}{t-i} = \binom{a}{t}^{-1} \sum_{i=0}^{a-2t} \binom{t}{i} \binom{a-t}{t-i}$$

$$= \binom{a}{t}^{-1} \binom{a-2t}{t} \sum_{i=0}^{a-2t} \binom{a-t}{i}.$$

Since $r \leq 2.125 < 2.5$, it easily follows that $a-2t < r/2$ and $a-2t < (a-t)/2$. Using a well known bound on the partial sum of binomial coefficients (see [2] Thm 3.1), we obtain:

$$P[E_j] = \binom{a}{t}^{-1} \sum_{i=0}^{a-2t} \binom{t}{i} \binom{a-t}{i} \leq \binom{a}{t}^{-1} \left( \sum_{i=0}^{a-2t} \binom{t}{i} \right) \left( \sum_{i=0}^{a-2t} \binom{a-t}{i} \right)$$

$$\leq \binom{a}{t}^{-1} \left( \frac{t}{a-2t} \right)^{a-2t} \left( \frac{t}{3t-a} \right)^{3t-a} \left( \frac{a-t}{a-2t} \right)^{a-2t} \left( \frac{a-t}{t} \right)^t$$

$$= \binom{a}{t}^{-1} \left( \frac{r-1}{(r-2)^2} \right)^{t(r-2)} \left( \frac{1}{3-r} \right)^{t(3-r)} (r-1)^t$$

$$= \binom{a}{t}^{-1} \left( \frac{\delta+1}{\delta^2} \right)^{t\delta} \left( \frac{1}{1-\delta} \right)^{t(1-\delta)} (\delta+1)^t.$$

The probability $(A_1, A_2)$ is good for each vertex in $\{u_j : j \in [m]\}$ is then $(P[E_1])^m$. Since $|A_1 \times A_2| = \binom{a}{2}$, we can guarantee the existence of an $a$-fold cover, $\mathcal{H}^*$, for $G$ such that there is no $(\mathcal{H}^*, t)$-coloring of $G$ if

$$\binom{a}{t}^2 (P[E_1])^m < 1.$$

Using a well known bound on binomial coefficients, we compute

$$\binom{a}{t}^2 (P[E_1])^m \leq \binom{a}{t}^{2-m} \left( \frac{\delta+1}{\delta^2} \right)^{mt\delta} \left( \frac{1}{1-\delta} \right)^{mt(1-\delta)} (\delta+1)^{mt}$$

$$\leq \left( \frac{t}{a} \right)^{t(m-2)} \left( \frac{\delta+1}{\delta^2} \right)^{mt\delta} \left( \frac{1}{1-\delta} \right)^{mt(1-\delta)} (\delta+1)^{mt}$$
Thus, to prove the desired it suffices to show that:

\[
\left( \frac{\delta + 2}{\delta + 2}(\delta + 1)^{\delta + 1}(1 - \delta)^{\delta - 1} \right) < 1.
\]

Consider the function \( f : (0, 1) \to (0, \infty) \) given by \( f(x) = \frac{(x + 2)^{2/m(x + 1)^{x + 1}(1 - x)^{x - 1}}}{(x + 2)(x^2)} \). It is easy to verify that \( f \) is increasing on \((0, 0.5)\). So, since \( 0 < \delta \leq d < 0.5 \),

\[
\left( \frac{\delta + 2}{\delta + 2}(\delta + 1)^{\delta + 1}(1 - \delta)^{\delta - 1} \right) = f(\delta) \leq f(d) = \frac{(d + 2)^{2/m(d + 1)^{d + 1}(1 - d)^{d - 1}}}{(d + 2)(d^2)} < 1
\]
as desired. \( \square \)

Notice that in our argument above the upper bound: \( \left( \binom{a}{t} \right)^{-1} \left( \sum_{i=0}^{a-2t} \binom{t}{i} \right) \left( \sum_{i=0}^{a-2t} \binom{a-t}{i} \right) \) used for \( \left( \binom{a}{t} \right)^{-1} \sum_{i=0}^{a-2t} \binom{t}{i} \left( \binom{a-t}{i} \right) \) is a fairly weak upper bound. So, our result may be able to be improved significantly with a better upper bound on \( \left( \binom{a}{t} \right)^{-1} \sum_{i=0}^{a-2t} \binom{t}{i} \left( \binom{a-t}{i} \right) \). For a concrete application of Theorem 10, notice that when \( m = 15 \) and \( d = 0.0959 \), the inequality in the hypothesis is satisfied. So, by Corollary 7 and Theorem 10, we have that \( 2.0959 \leq \chi^*_{DP}(K_{2, 15}) \leq 2.619 \).

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