On the critical exponents for a fractional diffusion-wave equation with a nonlinear memory term in a bounded domain

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Abstract

In this paper, we prove sharp blow-up and global existence results for a time fractional diffusion-wave equation with a nonlinear memory term in a bounded domain, where the fractional derivative in time is taken in the sense of Caputo type. Moreover, we also give a result for nonexistence of global solutions to a wave equation with a nonlinear memory term in a bounded domain. The proof of blow-up results is based on the eigenfunction method and the asymptotic properties of solutions for an ordinary fractional differential inequality.

Keywords: Fractional diffusion-wave equation; Blow-up; Global existence; Nonlinear memory

1 Introduction

This paper is mainly concerned with the blow-up and global existence of solutions for the following time fractional diffusion-wave equation with a nonlinear memory term:

$$\begin{cases}
0D_t^\alpha u - \Delta u = 0I_t^\gamma(|u|^p), & (t, x) \in (0, T) \times \Omega, \\
u(t, x) = 0, & (t, x) \in (0, T) \times \partial\Omega, \\
u(0, x) = u_0(x), & u_0(0, x) = u_1(x), & x \in \Omega,
\end{cases}$$

(1.1)

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $u_0, u_1 \in L^\infty(\Omega)$, $1 < \alpha < 2$, $\gamma > 0$, $p > 1$, and

$$0D_t^\alpha u = \frac{\partial^2}{\partial t^2}[0I_t^{2-\alpha}(u(t, x) - u_1(x) - u_0(x))].$$

We also give a result for nonexistence of global solutions to problem (1.1) with $\alpha = 2$.

In recent years, fractional differential equations have gained considerable popularity and importance, due to their demonstrated applications in seemingly widespread fields of science and engineering [15,21]. Hence, recently, there are a lot of papers on the existence and properties of solutions for fractional differential equations [1,4,12,16,19,25,29,31,36].

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Equation (1.1) interpolates the heat equation and the wave equation. Let us present a historical overview on some blow-up and global existence results for semilinear heat and wave equations. For the following Cauchy problem of semilinear heat equation

\[
\begin{cases}
    u_t - \Delta u = |u|^{p-1}u, & x \in \mathbb{R}^N, \ t > 0, \\
    u(0, x) = u_0(x), & x \in \mathbb{R}^N,
\end{cases}
\]

it is well known that all solutions of (1.2) with \(u_0 \geq 0, u_0 \not\equiv 0\), blow up in finite time if and only if \(p \leq 1 + \frac{2}{N}\), and if \(p > 1 + \frac{2}{N}\) and \(\|u_0\|_{L^q(R^N)}\) is small enough, where \(q_c = \frac{N(p-1)}{2}\), then the solution of (1.2) is global. The number \(1 + \frac{2}{N}\) is called the Fujita critical exponent of problem (1.2). We refer to [24] for the proof of these results. Recently, Zhang and Sun [35] considered problem (1.2) with \(0 < \alpha < 1\) instead of \(u_t\). They proved that if \(1 < p < 1 + \frac{2}{N}\), then any nontrivial positive solution blows up in finite time, while if \(p \geq 1 + \frac{2}{N}\) and the initial value is sufficiently small, the solution exists globally.

For the semilinear wave equation

\[
\begin{cases}
    u_{tt} - \Delta u = |u|^p, & x \in \mathbb{R}^N, \ t > 0, \\
    u(0, x) = u_0(x), \ u_t(0, x) = u_1(x), & x \in \mathbb{R}^N,
\end{cases}
\]

the critical exponent is \(p_c(N)\), which is the positive root of the quadratic equation \((N - 1)p^2 - (N + 1)p - 2 = 0\) for \(N > 1\), see [30] and the references therein. Recently, in [34], the authors determined the critical exponents of problem (1.3) with \(\partial D_\alpha^t u (0 < \alpha < 1)\) instead of \(u_{tt}\) when \(u_1 \equiv 0\) and \(u_1 \not\equiv 0\), respectively.

Let us now turn to the study of semilinear time fractional diffusion equations with a nonlinear memory term and semilinear wave equations with a nonlinear memory term. There have been many papers that considered the existence and nonexistence of global solutions for these problems [5–10, 18, 20, 32, 33].

In [5], Cazenave et al. studied the following semilinear heat equation with a nonlinear memory term

\[ u_t - \Delta u = 0\left(1 - \gamma \right)\left(|u|^{p-1}u\right) \]

on both \(\mathbb{R}^N\) and a bounded domain \(\Omega \subset \mathbb{R}^N\), and obtained the critical exponents of this problem. Recently, the authors of [20, 32, 33] generalized the results of [5] to the time fractional case \((0 < \alpha < 1)\), and gave the critical exponents of this problem when \(\alpha < \gamma\) and \(\alpha \geq \gamma\), respectively.

In [6], Chen and Palmieri established a generalized Strauss exponent \(p_0(N, \gamma)\) for problem (1.1) with \(\alpha = 2\) on \(\mathbb{R}^N\). They proved blow-up of energy solutions if \(1 < p \leq p_0(N, \gamma)\) for \(N \geq 2\), and \(p > 1\) for \(N = 1\). Fino and Jazar [9] proved that the nontrivial solution of problem (1.1) with \(\alpha = 2\) blows up in finite time if \(p(1 - \gamma) < 1\). In [18], the authors proved all solutions of (1.1) with \(\alpha = 2\) on \((0, \infty)\) blow up, provided that \(u_0, u_1\) have compact support and satisfy a certain positivity condition.

Motivated by the above results, in this paper, we investigate sharp blow-up and global existence of solutions for problem (1.1), and then extend the results in [5, 32, 33] to the time fractional case \((1 < \alpha < 2)\). Moreover, we also give a result for nonexistence of global solutions to a wave equation with a nonlinear memory term (i.e. (1.1) with \(\alpha = 2\)).
It should be mentioned that the study of the blow-up and global existence of solutions for (1.1) is not a simple generalization of those in the previous researches on semilinear diffusion equations with a nonlinear memory term. For one thing, our proof of blow-up results is based on the asymptotic properties of solutions for an ordinary fractional differential inequality. In the case $0 < \alpha < 1$, the proof of these properties depends on the nonnegativity of the Mittag-Leffler function $E_{\alpha,\alpha}(t)$ for $t \in \mathbb{R}$. But, $E_{\alpha,\alpha}(t)$ is not nonnegative on $(-\infty, 0]$ in the case $1 < \alpha < 2$, which gives us some technical difficulties in the treatments. To overcome these difficulties, we study the asymptotic behavior of $0^t \beta w$ for some $\beta > 0$ instead of $w$ (see Lemma 2.3, Corollary 2.5-2.6). This also allows us to obtain a result of the nonexistence of global solutions for problem (1.1) with $\alpha = 2$, which asserts that the case $p(1 - \gamma) = 1$ is in the blow-up category. For another, the estimates of the solution operators on $L^\infty(\Omega)$ are crucial to prove the global existence of solutions. In the case $0 < \alpha < 1$, one can easily obtain estimates of the solution operators on $L^\infty(\Omega)$, since the solution operators can be represented by a probability density function and the heat semigroup in $\Omega$ with the Dirichlet boundary condition. But, for the case $1 < \alpha < 2$, this representation is invalid. We prove the estimates of the solution operators on $L^\infty(\Omega)$ by the complex integral representations of the solution operators (see Lemma 3.3).

The remaining part of the paper is organized as follows. In Section 2, we present some results on the Mittag-Leffler function, the fractional derivatives and the fractional integrals, and show some results on an ordinary fractional differential inequality which will be used to prove the blow-up results. In Section 3, the local existence and uniqueness of the mild solution of problem (1.1) are given, and we prove sharp blow-up and global existence of solutions for problem (1.1).

For simplicity, in this paper, we use $C$ to denote a positive constant which may vary from line to line, but it is not essential to the analysis of the problem.

## 2 Preliminaries

In this section, we present some preliminaries that will be used in the next section.

First, we recall some properties of the Mittag-Leffler function with two parameters $\alpha, \beta \in \mathbb{C}$, $\alpha > 0$, $E_{\alpha}(z) = E_{\alpha,1}(z)$, $z \in \mathbb{C}$,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \alpha > 0, \quad E_{\alpha}(z) = E_{\alpha,1}(z), \quad z \in \mathbb{C},$$

which is an entire function. Let $\mu$ be a real number such that $\frac{\pi \alpha}{2} < \mu < \min\{\pi, \pi \alpha\}$. $E_{\alpha,\beta}(z)$ has different asymptotic behavior at infinity for $0 < \alpha < 2$ and $\alpha = 2$. If $0 < \alpha < 2$, then for $N \in \mathbb{N}$,

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{N} \frac{1}{\Gamma(\beta - \alpha k)} \frac{1}{z^k} + O\left(\frac{1}{z^{N+1}}\right)$$

(2.1)
Lemma 2.2 Assume that $1 < \alpha \leq 2$, $g \in AC^2([0,T])$, $g(T) = g'(T) = 0$, $f \in C^1([0,T])$, $0D_t^\alpha f$ exists almost everywhere for $t \in (0,T)$ and $0D_t^\alpha f \in L^1(0,T)$. Then

$$
\int_0^T 0D_t^\alpha f \cdot g dt = \int_0^T (f(t) - f'(0)t - f(0)) \, tD_T^{2-\alpha}g dt.
$$

(2.5)
then

\[ w(t) = 2 \] can be obtained by the classic formula of integration by parts.

Hence, using the definition of Caputo fractional derivative, Lemma 2.1 and the classic formula of integration by parts, one gets

\[
\left. \frac{d}{dt} \alpha I_t^{2-\alpha}(f(t) - f'(0)t - f(0)) \right|_0^T = 0,
\]

and

\[
\alpha I_t^{2-\alpha}[f(s) - f'(0)s - f(0)]g'(t) \bigg|_0^T = 0.
\]

Hence, using the definition of Caputo fractional derivatives, Lemma 2.1 and the classic formula of integration by parts, one gets

\[
\int_0^T 0D_\alpha^T f \cdot g dt = \int_0^T \frac{d^2}{dt^2} 0I_t^{2-\alpha}[f(s) - f'(0)s - f(0)]g(t) dt
\]

\[ = - \int_0^T \frac{d}{dt} 0I_t^{2-\alpha}[f(s) - f'(0)s - f(0)]g'(t) dt
\]

\[ = \int_0^T 0I_t^{2-\alpha}[f(s) - f'(0)s - f(0)]g''(t) dt
\]

\[ = \int_0^T \alpha I_t^{2-\alpha}[f(t) - f'(0)t - f(0)]_t^{2-\alpha}g''(t) dt
\]

\[ = \int_0^T \alpha I_t^{2-\alpha}[f(t) - f'(0)t - f(0)]D_\alpha^T g dt.
\]

This completes the proof of Lemma 2.2. \(\square\)

In order to prove blow-up results by the eigenfunction method due to [14], we prove the following results on an ordinary fractional differential equation, which are similar to ones in [5, 32].

Lemma 2.3 Let \( T > 0, \gamma > 0, 1 < \alpha < 2, p > 1, a,b > 0 \) and \( f \in L^1(0,T) \). If \( w \in C^1([0,T]) \), and \( w \neq 0 \) satisfies \( 0I_t^{2-\alpha}(w - w'(0)t - w(0)) \in AC^2([0,T]) \) and for almost every \( t \in [0,T] \),

\[
0D_\alpha^T w + aw = 0I_t^\gamma f, \quad f(t) \geq b|w(t)|^p, \quad (2.6)
\]

then \( w \) satisfies the following properties:

(i) There exist positive constants \( K_1 \) and \( K_2 \) independent of \( T \) such that

\[
Tw'(0) + w(0) \leq K_1 T^{\alpha+\gamma - \frac{\alpha}{p} \frac{1}{1-p}} + K_2 T^{-\frac{\alpha+\gamma}{p-1}}.
\]

(ii) If \( T = +\infty \), then \( \liminf_{t \to +\infty} |w(t)| = 0 \) and \( \liminf_{t \to +\infty} t^{\min\left\{ \frac{\alpha+\gamma}{p}, \frac{1}{p} \right\}} |w(t)| < +\infty. \)

(iii) If \( T = +\infty, \gamma \geq \frac{\alpha}{2} \) and \( \alpha + \gamma > 2 \), then \( \liminf_{t \to +\infty} t^{1-\gamma}w(t) > 0. \)

(iv) If \( T = +\infty, \gamma \geq \frac{\alpha}{2}, \alpha + \gamma \leq 2 \) and \( w'(0) = 0 \), then \( \liminf_{t \to +\infty} t^{1-\gamma}w(t) > 0. \)
(v) If \( T = +\infty, \gamma \geq \frac{\alpha}{2}, \alpha + \gamma \leq 2 \) and \( w'(0) > 0 \), then \( \liminf_{t \to +\infty} t^{\alpha-1} w(t) > 0 \).

(vi) If \( T = +\infty, 0 < \gamma < \frac{\alpha}{2} \), then for every \( \beta \) satisfying \( \frac{\alpha}{2} \leq \gamma + \beta < 1 \) and \( 0 < \beta < \alpha \), we have \( \liminf_{t \to +\infty} t^{1-\gamma-\beta} I^\beta_0 I^\gamma_t w > 0 \) if \( \alpha + \gamma > 2 \), or \( \alpha + \gamma \leq 2 \) and \( w'(0) = 0 \).

(vii) If \( T = +\infty, 0 < \gamma < \frac{\alpha}{2}, \alpha + \gamma \leq 2 \) and \( w'(0) > 0 \), then for every \( \beta \) satisfying \( \frac{\alpha}{2} \leq \gamma + \beta < 1 \) and \( 0 < \beta < \alpha \), we have \( \liminf_{t \to +\infty} t^{\alpha-\beta-1} I^\beta_0 I^\gamma_t w > 0 \).

(viii) If one of the following conditions holds:

(a) \( \alpha + \gamma > 2 \) and \( p(1-\gamma) \leq 1 \);
(b) \( \alpha + \gamma \leq 2 \), \( w'(0) = 0 \) and \( p(1-\gamma) \leq 1 \);
(c) \( \alpha + \gamma = 2 \), \( w'(0) > 0 \) and \( p(1-\gamma) \leq 1 \);
(d) \( \alpha + \gamma < 2 \), \( w'(0) > 0 \) and \( p < 1 + \frac{\gamma}{\alpha - 1} \),

then \( T < +\infty \).

**Proof.** The proofs of Property (i) and (ii) are similar to those of Lemma 5(i),(ii) and (iv) in [32]. For the convenience of the reader and the completeness of the paper, here we give sketchy proofs of Property (i) and (ii).

(i) We use the test function method to prove this conclusion. From (2.3), (2.5) and (2.6), we know that

\[
\int_0^T \left[ w(t)D^\gamma_T \varphi + aw \varphi \right] dt \geq b \int_0^T |w|^p(t) I^\gamma_T \varphi dt + w(0) \int_0^T t D^\gamma_T \varphi dt + w'(0) \int_0^T t \cdot t D^\gamma_T \varphi dt, \tag{2.7}
\]

for every \( \varphi \in AC^2([0,T]) \) with \( \varphi(T) = \varphi'(T) = 0, \varphi \geq 0 \).

Note that for \( l \geq \frac{p(\alpha+\gamma)}{\alpha+1} \),

\[
t^\gamma_T[1 \cdot t^{1-T} I^\gamma_T(t-T)^l] = \frac{\Gamma(l+1)}{\Gamma(l+1-\gamma)} T^{-l} I^\gamma_T(t-T)^l = T^{-l} t^{1-T} I^\gamma_T(t-T)^l, \tag{2.9}
\]

\[
\int_0^T w(t)D^\gamma_T(t) \varphi dt \leq \int_0^T w(t)D^\gamma_T(t) \varphi dt + \int_0^T |w|^p \psi_T dt \leq \int_0^T w(t)D^\gamma_T(t) \varphi dt + \int_0^T w(t)D^\gamma_T(t) \varphi dt \tag{2.10}
\]

(see, e.g., Property 2.16 in [15]). We take \( \varphi(t) = t D^\gamma_T \psi_T \) as the test function in (2.7), where \( \psi_T(t) = (1 - \frac{t}{T})^l \). It should be emphasized that choosing the test function of the type \( \psi_T(t) \) to prove the nonexistence of global solutions to fractional differential equations firstly appeared in [17]. By (2.7)-(2.10), we have

\[
w'(0) \int_0^T t \cdot t D^\gamma_T(t) \psi_T dt + w(0) \int_0^T t D^\gamma_T(t) \psi_T dt + \int_0^T |w|^p \psi_T dt \leq \int_0^T w(t)D^\gamma_T(t) \varphi dt + \int_0^T w(t)D^\gamma_T(t) \varphi dt \]
Therefore \( \lim \inf_{K} \) that there exist positive constants \((\text{see e.g. } [2, 15, 36])\) we know that there exist \(\tilde{\eta} = \eta\). Hence it follows from (2.11) that for \(t \geq \tilde{\eta}\),

\[
\frac{T \eta^p}{2(l + 1)} = \eta^p \int_T^\infty \psi_T(t) \, dt \leq \int_T^\infty |w|^p \psi_T \, dt \\
\leq K_1 T^{1 - \frac{p\gamma}{p-1}} + K_2 T^{1 - \frac{(\alpha+\gamma)p}{p-1}} + |w(0)| T^{1-(\alpha+\gamma)} + |w'(0)| T^{2-(\alpha+\gamma)}.
\]

In other words,

\[
\frac{1}{2(l + 1)} \eta^p \leq K_1 T^{1 - \frac{p\gamma}{p-1}} + K_2 T^{1 - \frac{(\alpha+\gamma)p}{p-1}} + |w(0)| T^{1-(\alpha+\gamma)} + |w'(0)| T^{1-(\alpha+\gamma)}.
\]

Letting \(T \to +\infty\), we have \(\eta = 0\) which contradicts \(\eta > 0\). Thus \(\lim \inf_{t \to +\infty} |w(t)| = 0\).

Next, we prove the second part in the statement. In fact, since \(\lim \inf_{t \to +\infty} |w(t)| = 0\), we know that there exist \(\tilde{\tau} \geq 0\) and a nondecreasing sequence \(\{t_n\}\) such that \(t_n \to +\infty\) and \(|w(t_n)| = \min_{\tilde{\tau} \leq t \leq t_n} |w(t)|\). Hence it follows from (2.11) that for \(t_n \geq \tilde{\tau}\),

\[
|w(t_n)|^p \int_{\tilde{\tau}}^{t_n} \psi_{t_n}(t) \, dt \leq \int_{\tilde{\tau}}^{t_n} |w(t)|^p \psi_{t_n}(t) \, dt \\
\leq K_1 t_n^{1 - \frac{p\gamma}{p-1}} + K_2 t_n^{1 - \frac{(\alpha+\gamma)p}{p-1}} + |w(0)| t_n^{1-(\alpha+\gamma)} + |w'(0)| t_n^{2-(\alpha+\gamma)}.
\]

This implies that

\[
|w(t_n)|^p \leq C[t_n^{1 - \frac{p\gamma}{p-1}} + t_n^{1 - \frac{(\alpha+\gamma)p}{p-1}} + |w(0)| t_n^{1-(\alpha+\gamma)} + |w'(0)| t_n^{1-(\alpha+\gamma)}] \\
\leq C[t_n^{1 - \frac{p\gamma}{p-1}} + t_n^{1-(\alpha+\gamma)}] \leq C t_n^{\min\{\alpha+\gamma-1, \frac{p\gamma}{p-1}\}}.
\]

Therefore \(\lim \inf_{n \to +\infty} t_n^{\min\{\frac{\alpha+\gamma-1}{p-1}, \frac{p\gamma}{p-1}\}} |w(t_n)| < +\infty\), which proves the desired conclusion.

(iii) In terms of \(\partial D_t^\alpha w + a w = a I_\alpha^t f(t)\), the solution \(w\) is explicitly expressed as follows (see e.g. [2],[15],[36])

\[
w(t) = E_\alpha(-\delta t^\alpha)w(0) + t E_{\alpha,2}(-\delta t^\alpha)w'(0) + \int_0^t (t - s)^{\alpha-1} E_{\alpha,\alpha}(-\delta(t - s)^\alpha) a I_\alpha^s f \, ds.
\]
Then it follows from Fubini’s theorem and (2.14) that
\[ w(t) = E_\alpha(-at^\alpha)w(0) + tE_{\alpha,2}(-at^\alpha)w'(0) + \frac{1}{\Gamma(\gamma)} \int_0^t \int_0^{t-\tau} h^{\alpha-1}(t-\tau-h)^{\gamma-1}E_{\alpha,\alpha}(-ah^\alpha)dhf(\tau)d\tau \]
\[ = E_\alpha(-at^\alpha)w(0) + tE_{\alpha,2}(-at^\alpha)w'(0) + \int_0^t (t-s)^{\gamma+\alpha-1}E_{\alpha,\alpha+\gamma}(-a(t-s)^\alpha)f(s)ds. \]
(2.12)

Furthermore, observing \( E_{\alpha,\rho}(z) > 0 \) for all \( z \in \mathbb{R} \) when \( 1 < \alpha < 2 \) and \( \rho \geq \frac{3\alpha}{2} \) (see Theorem 2 in [23]), we derive from (2.6) and (2.12) that
\[ w(t) \geq E_\alpha(-at^\alpha)w(0) + tE_{\alpha,2}(-at^\alpha)w'(0) + b \int_0^t (t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-a(t-s)^\alpha)|w(s)|^pds. \]
(2.13)

Since \( w \neq 0 \), we suppose \( w(t_0) \neq 0 \) for some \( t_0 \in [0, T) \). Then there exists \( \delta > 0 \) such that \( w(t) \neq 0 \) for \( t \in [t_0, t_0 + \delta] \). In view of (2.14), we have
\[ \lim_{t \to +\infty} t^\alpha E_{\alpha,\alpha+\gamma}(-at^\alpha) = \frac{1}{a\Gamma(\gamma)}. \]
This implies
\[ \lim_{t \to +\infty} \int_{t_0}^{t_0+\delta} (t-\tau)^\alpha E_{\alpha,\alpha+\gamma}(-a(t-\tau)^\alpha)d\tau = \frac{\delta}{a\Gamma(\gamma)}. \]
Hence, for \( t \) large enough, we deduce from (2.13) that
\[ w(t) \geq E_\alpha(-at^\alpha)w(0) + tE_{\alpha,2}(-at^\alpha)w'(0) \]
\[ + b \min_{t \in [t_0, t_0 + \delta]} |w(t)|^p \int_{t_0}^{t_0+\delta} (t-s)^{\alpha+\gamma-1}E_{\alpha,\alpha+\gamma}(-a(t-s)^\alpha)ds \]
\[ \geq E_\alpha(-at^\alpha)w(0) + tE_{\alpha,2}(-at^\alpha)w'(0) + \frac{b\delta t^{\gamma-1}}{2a\Gamma(\gamma)} \min_{t \in [t_0, t_0 + \delta]} |w(t)|^p. \]
(2.14)

Note that (2.1) yields that
\[ w(0)E_\alpha(-at^\alpha) = \frac{w(0)}{a\Gamma(1-\alpha)} \frac{1}{t^{\alpha}} + O(\frac{1}{t^{2\alpha}}), \quad w'(0)E_{\alpha,2}(-at^\alpha) = \frac{w'(0)}{a\Gamma(2-\alpha)} \frac{1}{t^{\alpha-1}} + O(\frac{1}{t^{2\alpha-1}}), \]
(2.15)
as \( t \to +\infty \). Moreover \( \alpha + \gamma > 2 \) implies \( 1 - \gamma < \alpha - 1 \). Hence it follows from (2.14) and (2.15) that \( \liminf_{t \to +\infty} t^{1-\gamma}w(t) > 0 \).

(iv) Since \( w'(0) = 0 \), it follows from (2.14) that
\[ w(t) \geq E_\alpha(-at^\alpha)w(0) + \frac{b\delta t^{\gamma-1}}{2a\Gamma(\gamma)} \min_{t \in [t_0, t_0 + \delta]} |w(t)|^p. \]
Thus we can obtain the desired conclusion by (2.15).

(v) In this case, we have \( 1 - \gamma \geq \alpha - 1 \). Thus, using (2.14), (2.15) and the fact that \( w'(0) > 0 \), we get \( \liminf_{t \to +\infty} t^{\alpha-1}w(t) > 0 \).
(vi) Since $0 < \gamma < \frac{\alpha}{2}$ and $1 < \alpha < 2$, we can choose $\beta$ satisfying $\frac{\alpha}{2} \leq \gamma + \beta < 1$ and $0 < \beta < \alpha$. From (2.12) and (2.4), we know that

$0 I^\beta_t w = 0 I^\beta_t E_\alpha(-at^\alpha)w(0) + 0 I^\beta_t [t E_\alpha,2(-at^\alpha)w'(0)] + 0 I^\beta_t \int_0^t (s) \gamma + \alpha - 1 E_{\alpha,\alpha+\gamma}(-a(t-s)^\alpha)f(s)ds$

$= t^\beta E_{\alpha,1+\beta}(-at^\alpha)w(0) + t^{1+\beta} E_{\alpha,2+\beta}(-at^\alpha)w'(0)$

$+ \int_0^t (t-s)\gamma + \alpha - 1 E_{\alpha,\alpha+\gamma+\beta}(-a(t-s)^\alpha)f(s)ds,$

(2.16)

where we have used the fact that

$0 I^\beta_t \int_0^t (t-s)\gamma + \alpha - 1 E_{\alpha,\alpha+\gamma}(-a(t-s)^\alpha)f(s)ds$

$= \frac{1}{\Gamma(\beta)} \int_0^t \int_0^t (s)\gamma + \alpha - 1 E_{\alpha,\alpha+\gamma}(-a(s-t)^\alpha)f(s)dsd\tau$

$= \frac{1}{\Gamma(\beta)} \int_0^t \int_0^t (s)\gamma + \alpha - 1 E_{\alpha,\alpha+\gamma}(-a(s-t)^\alpha)f(s)d\tau ds$

$= \frac{1}{\Gamma(\beta)} \int_0^t \int_0^t (t-s)\gamma + \alpha - 1 E_{\alpha,\alpha+\gamma+\beta}(-a(t-s)^\alpha)f(s)ds$

$= \int_0^t (t-s)\gamma + \alpha - 1 E_{\alpha,\alpha+\gamma+\beta}(-a(t-s)^\alpha)f(s)ds.$

(2.17)

Furthermore, since $\gamma + \beta \geq \frac{\alpha}{2}$, we know $E_{\alpha,\alpha+\gamma+\beta}(-z) > 0$ for every $z \in \mathbb{R}$ (see Theorem 2 in [23]). Thus, using (2.16), and by an argument similar to the proof of Property(iii), we have that for $t$ large enough,

$0 I^\beta_t w \geq t^\beta E_{\alpha,1+\beta}(-at^\alpha)w(0) + t^{1+\beta} E_{\alpha,2+\beta}(-at^\alpha)w'(0)$

$+ \int_0^t (t-s)\gamma + \alpha - 1 E_{\alpha,\alpha+\gamma+\beta}(-a(t-s)^\alpha)|w(\tau)|^p d\tau$

$\geq t^\beta E_{\alpha,1+\beta}(-at^\alpha)w(0) + t^{1+\beta} E_{\alpha,2+\beta}(-at^\alpha)w'(0) + \frac{\delta t^{\gamma + \beta - 1}}{2a\Gamma(\gamma + \beta)} \min_{t \in [0,t_0+\delta]} |w(t)|^p.$

(2.18)

Observe that $\beta < \alpha$ and

$w(0)t^\beta E_{\alpha,1+\beta}(-at^\alpha) = \frac{w(0)}{a\Gamma(1+\beta-\alpha)} \frac{1}{t^{\alpha-\beta}} + O(\frac{1}{t^{2\alpha-\beta}}),$

(2.19)

$w'(0)t^{1+\beta} E_{\alpha,2+\beta}(-at^\alpha) = \frac{w'(0)}{a\Gamma(2+\beta-\alpha)} \frac{1}{t^{\alpha-\beta-1}} + O(\frac{1}{t^{2\alpha-\beta-1}}),$

(2.20)

as $t \to +\infty$. Then it follows from (2.18), (2.19) and (2.20) that $\liminf_{t \to +\infty} t^{1-\gamma-\beta} 0 I^\beta_t w > 0$ if $\alpha + \gamma > 2$, or $\alpha + \gamma \leq 2$ and $w'(0) = 0$.

(vii) The assumption that $\alpha + \gamma \leq 2$ yields $\alpha - \beta - 1 \leq 1 - \gamma - \beta$. Thus the desired conclusion can be obtained by (2.18), (2.19) and (2.20).
(viii) Suppose that $T = +\infty$. The proof is divided into four cases.

**Case 1.** Assume that condition (a) is satisfied. We deduce from (2.11) that

$$
\int_0^{T} |w|^p \psi_T dt \leq K_1 T^{1 - \frac{\alpha}{p-1}} + K_2 T^{1 - \frac{\alpha+\gamma}{p-1}} + |w(0)|T^{1 - \alpha - \gamma} + |w'(0)|T^{2 - \alpha - \gamma}.
$$

Thus, if $p(1 - \gamma) < 1$, then taking $T \to +\infty$, we get $\int_0^{+\infty} |w|^p dt = 0$ by (2.21). This implies $w \equiv 0$, which contradicts the assumption that $w \not\equiv 0$.

If $p(1 - \gamma) = 1$, then using (2.21) and taking $T \to +\infty$, we obtain that $\int_0^{+\infty} |w|^p dt < +\infty$. Additionally, if $\gamma \geq \frac{\alpha}{2}$, Property (iii) implies that there exist constants $C > 0$ and $L > 0$ such that $w(t) \geq C t^{\gamma - \beta - 1}$ for $t \geq L$, which yields that

$$
C \int_L^{+\infty} t^{-1} dt = C \int_L^{+\infty} t^{-p(1 - \gamma)} \leq \int_0^{+\infty} |w|^p dt.
$$

This contradicts $\int_0^{+\infty} |w|^p dt < +\infty$. On the other hand, if $0 < \gamma < \frac{\alpha}{2}$, Property (vi) implies that there exist constants $C > 0$ and $\tilde{L} > 0$ such that $0 \int_0^\beta w(t) \geq C t^{\gamma - \beta - 1}$ for $t \geq \tilde{L}$, where $\beta$ satisfies $0 < \beta \leq \gamma + \beta < 1$ and $\beta < \alpha$. By the weighted estimate of the operator $0 I_t^\beta$, the inequality

$$
\left( \int_0^{+\infty} t^{-\beta p} \frac{0 I_t^\beta w}{|w|^p} dt \right)^\frac{1}{p} \leq \frac{\Gamma(1 - 1/p)}{\Gamma(\beta + 1 - 1/p)} \left( \int_0^{+\infty} |w(t)|^p dt \right)^\frac{1}{p}
$$

is valid (see Lemma 2.3 (a) in [15] or Exercise 28 in Chapter 6 of [11]). As a result,

$$
C \int_L^{+\infty} t^{-1} dt = C \int_L^{+\infty} t^{-(\beta p - p(1 - \gamma)) + p \beta} dt \leq \left( \frac{\Gamma(1 - 1/p)}{\Gamma(\beta + 1 - 1/p)} \right)^p \int_0^{+\infty} |w(t)|^p dt,
$$

which again contradicts $\int_0^{+\infty} |w|^p dt < +\infty$.

**Case 2.** Assume that condition (b) is satisfied, that is $\alpha + \gamma \geq 2$, $w'(0) = 0$ and $p(1 - \gamma) \leq 1$. Then it follows from (2.11) that

$$
\int_0^{T} |w|^p \psi_T dt \leq K_1 T^{1 - \frac{\alpha}{p-1}} + K_2 T^{1 - \frac{\alpha+\gamma}{p-1}} + |w(0)|T^{1 - \alpha - \gamma}.
$$

If $p(1 - \gamma) < 1$, then we have $w \equiv 0$ by taking $T \to +\infty$. This contradicts the assumption that $w \not\equiv 0$. If $p(1 - \gamma) = 1$, then using Property(iv)(vi) and repeating the arguments in case 1, we can also obtain a contradiction.

**Case 3.** Assume that condition (d) is satisfied. Then from (2.11) we have

$$
|w'(0)|T^{2 - \alpha - \gamma} \leq K_1 T^{1 - \frac{\gamma}{p-1}} + K_2 T^{1 - \frac{\alpha+\gamma}{p-1}} + |w(0)|T^{1 - \alpha - \gamma}.
$$

In other words,

$$
w'(0) \leq K_1 T^{-1 - \frac{\gamma}{p-1} + \alpha} + K_2 T^{-1 - \frac{\alpha+\gamma}{p-1}} + |w(0)|T^{-1}.
$$
Observe that $\alpha - 1 < \frac{\gamma}{p-1}$. Thus taking $T \to +\infty$, we deduce from (2.22) that $w'(0) = 0$, which contradicts the assumption that $w'(0) > 0$.

**Case 4.** Assume that condition (c) holds. For the case $p(1 - \gamma) < 1$, the assumption $\alpha + \gamma = 2$ yields $\alpha - 1 < \frac{\gamma}{p-1}$. Thus $T < +\infty$ by case 3. For the case $p(1 - \gamma) = 1$, we can also obtain a contradiction by repeating the arguments in case 1 and using Property(v), (vii). Hence $T < +\infty$.

**Remark 2.4** In [3], for problem (2.6) with $\alpha = 1$, some similar results were proved by comparison technique and shifting the time. The authors of [2] overcame the technical difficulty, given by the memory effect of the equation, and extended the results to the time fractional case ($p > 1$). Their proof depends on the nonnegativity of the Mittag-Leffler function $E_{\alpha,0}(t)$ for $t < 0$. But, in the case $1 < \alpha < 2$, $E_{\alpha,0}(t)$ is not nonnegative on $(-\infty, 0)$. Hence, the method using in [2] could not be applied to study problem (2.6).

Here, we consider the asymptotic behavior of $I_0^\beta w$ for some $\beta > 0$ instead of $w$ in some cases. Accordingly, we can prove Property(viii), which is crucial to prove our blow-up results.

**Corollary 2.5** Let $T > 0$, $\gamma > 0$, $1 < \alpha < 2$, $p > 1$, $a, b > 0$, and $\beta \in [\frac{2}{3}, \alpha)$. If $w \in C^2([0, T])$, and $w \neq 0$ satisfies that $I_0^{\alpha-\alpha}(w - w'(0)t - w(0)) \in AC^2([0, T])$ and for almost every $t \in [0, T],
\begin{align*}
0D_t^\alpha w + aw \geq b \cdot I_t^\beta |w|^p,
\end{align*}
then the following properties hold.

(i) There exist positive constants $K_1$ and $K_2$ independent of $T$ such that
\begin{align*}
Tw'(0) + w(0) & \leq K_1 T^{\alpha+\gamma-\frac{p\gamma}{p-1}} + K_2 T^{-\frac{\alpha+\gamma}{p-1}},
\end{align*}

(ii) If $T = +\infty$, then $\liminf_{t \to +\infty} |w(t)| = 0$ and $\liminf_{t \to +\infty} t^{\min\{\frac{\alpha+\gamma}{p-1}, \frac{\gamma}{p-1}\}} |w(t)| < +\infty$.

(iii) If $T = +\infty$, $w'(0) = 0$, then $\liminf_{t \to +\infty} t^{1-\gamma-\beta} I_t^\beta w > 0$.

(iv) If $T = +\infty$, $\alpha + \gamma \leq 2$ and $w'(0) > 0$, then $\liminf_{t \to +\infty} t^{\alpha-\beta-1} I_t^\beta w > 0$.

(v) If $T = +\infty$, $\alpha + \gamma > 2$, then $\liminf_{t \to +\infty} t^{1-\gamma-\beta} I_t^\beta w > 0$.

(vi) If one of the following conditions is satisfied:

(a) $\alpha + \gamma > 2$ and $p(1 - \gamma) \leq 1$;
(b) $\alpha + \gamma \leq 2$, $w'(0) = 0$ and $p(1 - \gamma) \leq 1$;
(c) $\alpha + \gamma = 2$, $w'(0) > 0$ and $p(1 - \gamma) \leq 1$;
(d) $\alpha + \gamma < 2$, $w'(0) > 0$ and $p < 1 + \frac{\gamma}{\alpha-1}$,
then $T < +\infty$. 
Proof. According to the proof of Lemma 2.3(i)(ii), we know that Property (i) and (ii) also hold in this case.

Denote \( \partial_0D_t^\beta w + aw = g(t) \). Then

\[
w(t) = E_\alpha(-at^\alpha)w(0) + tE_{\alpha,2}(-at^\alpha)w'(0) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-a(t-s)^\alpha)g(s)ds.
\]

Take \( \beta \in [\frac{\alpha}{2}, \alpha) \). It follows from (2.24), (2.4) and (2.17) that

\[
0I_t^\beta w = t^\beta E_{\alpha,1+\beta}(-at^\alpha)w(0) + t^{1+\beta}E_{\alpha,2+\beta}(-at^\alpha)w'(0) + \int_0^t (t-s)^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}(-a(t-s)^\alpha)g(s)ds.
\]

Noting that \( 2 + \beta > \alpha + \beta > \frac{3\alpha}{2} \), we have \( E_{\alpha,2+\beta}(-z) > 0 \) and \( E_{\alpha,\alpha+\beta}(-z) > 0 \) for every \( z \in \mathbb{R} \) (see Theorem 2 in [23]). Hence we conclude from (2.24), (2.25) and (2.12) that

\[
0I_t^\beta w \geq t^\beta E_{\alpha,1+\beta}(-at^\alpha)w(0) + t^{1+\beta}E_{\alpha,2+\beta}(-at^\alpha)w'(0)
\]

\[
+ b \int_0^t (t-s)^{\alpha+\beta-1}E_{\alpha,\alpha+\beta}(-a(t-s)^\alpha)I_s^\beta |w|^pds.
\]

The rest of proof is similar to that for Lemma 2.3(vi)-(viii), so we omit it. \( \square \)

When \( \alpha = 2 \), we have the following results.

**Corollary 2.6** Let \( T > 0, \gamma > 0, \ p > 1, \ a, b > 0, \) and \( \beta \geq 2 \). If \( w \not\equiv 0 \) satisfies that for almost every \( t \in [0, T] \),

\[
w'' + aw \geq b \cdot 0I_t^\beta |w|^p,
\]

then the following properties hold.

(i) There exist positive constants \( K_1 \) and \( K_2 \) independent of \( T \) such that

\[
Tw'(0) + w(0) \leq K_1T^{2+\gamma - \frac{p\gamma}{p-1}} + K_2T^{-\frac{2+\gamma}{p-1}}.
\]

(ii) If \( T = +\infty \), then \( \lim \inf_{t \to +\infty} |w(t)| = 0 \) and \( \lim \inf_{t \to +\infty} t^{\min\left\{ \frac{1+\gamma}{p}, \frac{\gamma}{p-1} \right\}} |w(t)| < +\infty \).

(iii) If \( T = +\infty \), then \( \lim \inf_{t \to +\infty} t^{1-\gamma - \frac{\beta}{p}} 0I_t^\beta w > 0 \).

(iv) If \( p(1-\gamma) \leq 1 \), then \( T < +\infty \).

**Proof.** By the proof of Lemma 2.3 we know that Property (i) and (ii) also hold in the case \( \alpha = 2 \).
(iii) Denote \( w'' + aw = g(t) \). Then
\[
    w(t) = E_2(-at^2)w(0) + tE_{2,2}(-at^2)w'(0) + \int_0^t (t-s)E_{2,2}(-a(t-s)^2)g(s)ds.
\]
Since \( 2 + \beta > 1 + \beta \geq 3 \), we know \( E_{2,1+\beta}(-z) \geq 0 \), \( E_{2,2+\beta}(-z) > 0 \) and \( E_{2,2+\beta}(-z) > 0 \) for every \( z \in \mathbb{R} \) (see Theorem 2 in \([23]\)). Then it follows from (2.25) and (2.26) that
\[
    0\int^t_0 w \geq t^\beta E_{2,1+\beta}(-at^2)w(0) + t^{1+\beta} E_{2,2+\beta}(-at^2)w'(0) + b \int_0^t (t-s)^{1+\beta} E_{2,2+\beta}(-a(t-s)^2)0I^\gamma_s |w|^pds
\]
\[
= t^\beta E_{2,1+\beta}(-at^2)w(0) + t^{1+\beta} E_{2,2+\beta}(-at^2)w'(0) + b \int_0^t (t-s)^{1+\beta+\gamma} E_{2,2+\beta+\gamma}(-a(t-s)^2)|w(s)|^pds.
\]
Since \( w \neq 0 \), there exist \( t_0 \in [0,T) \) and \( \delta > 0 \) such that \( |w(t)| \neq 0 \) for \( t \in [t_0,t_0+\delta] \).
Observe that \( \beta - 1 < \beta + \gamma - 1 \), and (2.2) yields that
\[
t^\beta E_{2,2+\beta+\gamma}(-at^2) = a^{-\frac{1+\beta+\gamma}{2}} t^{1-\beta-\gamma} \cos(t - \frac{(1+\beta+\gamma)\pi}{2}) + \frac{1}{a\Gamma(\beta+\gamma)} + O\left(\frac{1}{t^2}\right),
\]
as \( t \to +\infty \). Then by an argument similar to the proof of estimate (2.14), we have
\[
0\int^t_0 \beta w \geq t^\beta E_{2,1+\beta}(-at^2)w(0) + t^{1+\beta} E_{2,2+\beta}(-at^2)w'(0) + \frac{b\delta t^{\gamma+\delta-1}}{2a\Gamma(\beta+\gamma)} \min_{t \in [t_0,t_0+\delta]} |w(t)|^p.
\]
Furthermore, invoking \( \beta - 1 < \beta + \gamma - 1 \) and noting that (2.2) implies that
\[
t^\beta E_{2,1+\beta}(-at^2) = a^{-\frac{\beta}{2}} \cos(t - \frac{\beta \pi}{2}) + \frac{1}{a\Gamma(\beta-\frac{1}{2})} t^{\beta-2} + O\left(\frac{1}{t^{1-\beta}}\right),
\]
\[
t^{1+\beta} E_{2,2+\beta}(-at^2) = a^{-\frac{1+\beta}{2}} \cos(t - \frac{(1+\beta)\pi}{2}) + \frac{1}{a\Gamma(\beta)} t^{\beta-1} + O\left(\frac{1}{t^{1-\beta}}\right),
\]
as \( t \to +\infty \), we can obtain \( 0\int_t^\delta w \geq C t^{\beta+\gamma-1} \) for \( t \) large enough. This completes the proof.
(iv) The proof is similar to that of Lemma (2.3 (viii)), so we omit it. \( \square \)

3 Finite time blow-up and Global existence

In this section, we give the local existence result of problem (1.1), and prove sharp results on the blow-up and global existence of solution for problem (1.1).

Let \( \lambda_1 > 0 \) be the first eigenvalue of \(-\Delta\) in \( H^1_0(\Omega)\). We denote by \( \varphi_1 \) the corresponding positive eigenfunction with \( \int_\Omega \varphi_1(x)dx = 1 \). Denote \(-A = \Delta\). Let \( 1 < q \leq +\infty \), and consider the operator \( A \) defined on
\[
D(A) = \begin{cases} 
\{ u \in \bigcap_{r \geq 1} W^{2,r}_0(\Omega) \mid u, Au \in L^\infty(\Omega), u = 0 \text{ on } \partial\Omega \}, & \text{if } q = \infty, \\
\{ u \in W^{1,q}_0(\Omega) \mid Au \in L^q(\Omega) \}, & \text{if } 1 < q < +\infty.
\end{cases}
\]

First, we give the definitions of the solution operators, which are similar to those in \([34]\).
Definition 3.1 Let $\alpha \in (1,2)$, $1 < q \leq +\infty$. For every $u_0 \in L^q(\Omega)$, we define the operators $P_\alpha(t)$ and $S_\alpha(t)$ by

$$P_\alpha(t)u_0 = \frac{1}{2\pi i} \int_\Gamma E_\alpha(\lambda t^\alpha)(\lambda I + A)^{-1}u_0 d\lambda, \quad t > 0, \quad \text{and} \quad P_\alpha(0)u_0 = u_0,$$  \hspace{1cm} (3.1)

$$S_\alpha(t)u_0 = \frac{1}{2\pi i} \int_\Gamma E_{\alpha,\alpha}(\lambda t^\alpha)(\lambda I + A)^{-1}u_0 d\lambda, \quad t > 0, \quad \text{and} \quad S_\alpha(0)u_0 = \frac{u_0}{\Gamma(\alpha)},$$  \hspace{1cm} (3.2)

where $\Gamma \in \{\gamma(\varepsilon, \varphi) \subseteq \rho(-A) \mid \varepsilon > 0, \varphi \text{ satisfies } 0 < \varphi < \pi, \frac{\pi}{2} < \Im(\lambda_1 + \varepsilon e^{i\varphi}) < \pi\}$. Here $\gamma(\varepsilon, \varphi) = \{re^{i\arg(-\lambda_1 + \varepsilon e^{i\varphi})} \mid |r| \geq |\lambda_1 + \varepsilon e^{i\varphi}|\} \cup \{-\lambda_1 + \varepsilon e^{i\theta} \mid -\theta < \theta < \varphi\}$. 

Remark 3.2 According to [24] and Cauchy’s integral theorem, $P_\alpha(t)$ and $S_\alpha(t)$ are independent of $\varphi$ and $\varepsilon$, and then are well defined.

By making use of the complex integral representations of the solution operators, we can derive some estimates of the operators $P_\alpha(t)$ and $S_\alpha(t)$.

Lemma 3.3 The operators $P_\alpha(t)$ and $S_\alpha(t)$ have the following properties.

(i) For $u_0 \in L^\infty(\Omega)$, we have $P_\alpha(t)u_0 \in C((0, +\infty), L^\infty(\Omega)) \cap C([0, +\infty), L^q(\Omega))$ for every $q \in (1, +\infty)$, $\lim_{t \to 0^+} P_\alpha(t)u_0 = u_0$ in the weak-star topology of $L^\infty(\Omega)$. Moreover, if $u_0 \in L^s(\Omega)$ ($1 < s \leq +\infty$), then there exists a constant $C > 0$ such that for $t \geq 0$,

$$\|P_\alpha(t)u_0\|_{L^s(\Omega)} \leq \frac{C}{1 + t^{\alpha - 1}} \|u_0\|_{L^s(\Omega)}, \quad \|0t^{\alpha - 1} P_\alpha(t)u_0\|_{L^s(\Omega)} \leq \frac{C}{1 + t^{\alpha - 1}} \|u_0\|_{L^s(\Omega)}.$$  \hspace{1cm} (3.3)

(ii) For $u_0 \in L^\infty(\Omega)$, $0 < \gamma \leq 1$ and $t > 0$, we have $S_\alpha(t)u_0 = t^{\alpha - 1} \frac{\alpha}{\alpha + 1} P_\alpha(t)u_0$,

$$0t^{\alpha - 1} S_\alpha(t)u_0 = 0t^{\alpha + \gamma - 1} P_\alpha(t)u_0 = \frac{t^{\alpha + \gamma - 1}}{2\pi i} \int_\Gamma E_{\alpha,\alpha}(\lambda t^\alpha)(\lambda I + A)^{-1}u_0 d\lambda,$$

$t^{\alpha - 1} S_\alpha(t)u_0 \in C((0, +\infty), L^\infty(\Omega))$ and $S_\alpha(t)u_0 \in C([0, +\infty), L^q(\Omega))$ for every $q \in (1, +\infty)$. Moreover, if $u_0 \in L^s(\Omega)$ ($1 < s \leq +\infty$), then there exists a constant $C > 0$ such that for $t \geq 0$,

$$\|S_\alpha(t)u_0\|_{L^s(\Omega)} \leq \frac{C}{1 + t^{2\alpha}} \|u_0\|_{L^s(\Omega)}, \quad \|0t^{\alpha - 1} S_\alpha(t)u_0\|_{L^s(\Omega)} \leq \frac{C}{1 + t^{1 - \gamma}} \|u_0\|_{L^s(\Omega)}.$$  \hspace{1cm} (3.4)

(iii) If $u_0 \in L^\infty(\Omega)$, then $P_\alpha(t)u_0 \in C^1((0, +\infty), L^\infty(\Omega))$. Moreover, $\int_\Omega (P_\alpha(t)u_0) v dx \in C^1((0, +\infty))$ for every $v \in D(A)$.

(iv) Let $T > 0$, $q \geq 1$ and $w = \int_0^T (t - s)^{\alpha - 1} S_\alpha(t - s) f(s) ds$. If $f \in L^q((0, T), L^r(\Omega))$ for some $r \in (1, +\infty)$ then $w \in C([0, T], L^r(\Omega))$. Furthermore, if $q(\alpha - 1) > 1$, then $w \in C^{1,\alpha - 1,\frac{r}{2}}(0, T], L^r(\Omega))$. 

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**Proof.** (i) To prove the first estimate in (3.3), we take \( \varepsilon_0 > 0 \) small enough such that \( \pi - \arcsin \frac{\pi}{2} > \frac{\pi}{2} \), and choose \( \Gamma = \gamma(\varepsilon_0, \varphi) \in \rho(-A) \) with \( \arg(-\lambda_1 + \varepsilon_0 e^{i\varphi}) \in (\frac{\pi}{2}, \pi) \). Denote \( \varphi_0 = \arg(-\lambda_1 + \varepsilon_0 e^{i\varphi}) \), \( r_0 = |\lambda_1 + \varepsilon_0 e^{i\varphi}| \) and \( \Gamma = \gamma(\varepsilon_0, \varphi) = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \), where \( \Gamma_1 = \{ re^{i\varphi_0} \mid r \geq r_0 \}, \Gamma_2 = \{ -\lambda_1 + \varepsilon_0 e^{i\theta} \mid -\varphi \leq \theta \leq \varphi \} \) and \( \Gamma_3 = \{ re^{-i\varphi_0} \mid r \geq r_0 \} \). Note that \( \frac{\pi}{2} < \pi - \arcsin \frac{\pi}{2} \leq |\arg z| \leq \pi \) for \( z \in \Gamma \). Then, for \( u_0 \in L^s(\Omega)(1 < s \leq +\infty) \) and \( t > 0 \), we deduce from (2.14) that there exists a constant \( C > 0 \) such that

\[
\left\| \frac{1}{2\pi i} \int_{\Gamma_k} E_\alpha(t^\alpha \lambda)(\lambda I + A)^{-1} u_0 d\lambda \right\|_{L^s(\Omega)} \leq C \int_{\Gamma_k} \frac{1}{1 + t^\alpha |\lambda|} \frac{1}{|\lambda + \lambda_1|} \| u_0 \|_{L^s(\Omega)} d\lambda \leq C \int_{r_0}^{+\infty} \frac{dr}{(1 + t^\alpha r)^{1/2}} \| u_0 \|_{L^s(\Omega)} \leq C \int_{r_0}^{+\infty} \frac{1}{1 + t^\alpha r} \frac{1}{r} \| u_0 \|_{L^s(\Omega)} \leq \frac{C}{t^\alpha} \| u_0 \|_{L^s(\Omega)}, \quad k = 1, 3, \quad (3.4)
\]

and

\[
\left\| \frac{1}{2\pi i} \int_{\Gamma_2} E_\alpha(t^\alpha \lambda)(\lambda I + A)^{-1} u_0 d\lambda \right\|_{L^s(\Omega)} \leq C \int_{\Gamma_2} \frac{1}{1 + t^\alpha |\lambda|} \frac{1}{|\lambda + \lambda_1|} \| u_0 \|_{L^s(\Omega)} d\lambda \leq C \int_{-\varphi}^{\varphi} \frac{1}{1 + t^\alpha (\lambda_1 - \varepsilon_0)} \| u_0 \|_{L^s(\Omega)} d\theta \leq \frac{C}{t^\alpha} \| u_0 \|_{L^s(\Omega)}. \quad (3.5)
\]

On the other hand, by Cauchy’s integral theorem, we can take \( \Gamma = \tilde{\Gamma}_1 \cup \tilde{\Gamma}_2 \cup \tilde{\Gamma}_3 \), where \( \tilde{\Gamma}_1 = \{ re^{i\tilde{\varphi}} \mid r \geq \frac{1}{t} \}, \tilde{\Gamma}_2 = \{ \frac{1}{t^\alpha} e^{i\theta} \mid -\tilde{\varphi} \leq \omega \leq \tilde{\varphi} \} \) and \( \tilde{\Gamma}_3 = \{ re^{-i\tilde{\varphi}} \mid r \geq \frac{1}{t} \}, \tilde{\varphi} \in (\frac{\pi}{2}, \pi) \). Then there exists a constant \( C > 0 \) such that

\[
\left\| \frac{1}{2\pi i} \int_{\tilde{\Gamma}_k} E_\alpha(t^\alpha \lambda)(\lambda I + A)^{-1} u_0 d\lambda \right\|_{L^s(\Omega)} \leq C \int_{r_0}^{+\infty} \frac{\| u_0 \|_{L^s(\Omega)}}{r(1 + t^\alpha r)} dr \leq C \| u_0 \|_{L^s(\Omega)}, \quad k = 1, 3, \quad (3.6)
\]

and

\[
\left\| \frac{1}{2\pi i} \int_{\tilde{\Gamma}_2} E_\alpha(t^\alpha \lambda)(\lambda I + A)^{-1} u_0 d\lambda \right\|_{L^s(\Omega)} \leq C \int_{-\tilde{\varphi}}^{\tilde{\varphi}} \frac{1}{|\lambda|} \| u_0 \|_{L^s(\Omega)} d\lambda \leq C \| u_0 \|_{L^s(\Omega)}. \quad (3.7)
\]

Combining (3.4)–(3.7), we get that \( \| P_\alpha(t) u_0 \|_{L^s(\Omega)} \leq \frac{C}{1 + t^\alpha} \| u_0 \|_{L^s(\Omega)} \) for some \( C > 0 \).

Next, we prove the second estimate in (3.3). By (2.14) and \( 4 > \frac{3\alpha}{2} \), there exists a constant \( C > 0 \) such that \( \alpha, \lambda(-t^\alpha) \geq \frac{C}{1 + t^\alpha} \). Thus

\[
\| P_\alpha(t) u_0 \|_{L^s(\Omega)} \leq \| T_1 P_\alpha(t) u_0 \|_{L^s(\Omega)} \leq C_0 I_1 E_\alpha(-t^\alpha) \| u_0 \|_{L^s(\Omega)} = C t E_\alpha,5(-t^\alpha) \| u_0 \|_{L^s(\Omega)} \leq \frac{C}{1 + t^\alpha} \| u_0 \|_{L^s(\Omega)}.
\]
Using the dominated convergence theorem, we can find that $P_\alpha(t)u_0 \in C((0, +\infty), L^\infty(\Omega))$. Moreover, a density argument and (3.3) show that $\lim_{t \to 0^+} P_\alpha(t)u_0 = u_0$ in $L^q(\Omega)$. Thus $u \in C([0, +\infty), L^q(\Omega))$.

Finally, we prove $\lim_{t \to 0^+} P_\alpha(t)u_0 = u_0$ in the weak-star topology of $L^\infty(\Omega)$. In fact, by the definition of the operator $P_\alpha(t)$ and taking $\varepsilon > \lambda_1$, we have

$$P_\alpha(t)u_0 - u_0 = \frac{1}{2\pi i} \int_\Gamma E_\alpha(t^\alpha \lambda)(\lambda I + A)^{-1}u_0d\lambda - u_0 = -\frac{1}{2\pi i} \int_\Gamma E_\alpha(t^\alpha \lambda)\frac{1}{\lambda}(\lambda I + A)^{-1}u_0d\lambda.$$

This and the dominated convergence theorem imply that for every $v \in C_0^\infty(\Omega)$,

$$\int_\Omega [P_\alpha(t)u_0 - u_0]vdx = -\int_\Omega \frac{1}{2\pi i} \int_\Gamma E_\alpha(t^\alpha \lambda)\frac{1}{\lambda}(\lambda I + A)^{-1}u_0d\lambda Avdx \to 0,$$

as $t \to 0^+$. Thus we can complete the proof by (3.3) and the fact that $C_0^\infty(\Omega)$ is dense in $L^1(\Omega)$.

(ii) The proof is similar to that of (i), so we omit it.

(iii) Using the dominated convergence theorem, we find

$$P'_\alpha(t)u_0 = \frac{t^{\alpha-1}}{2\pi i} \int_\Gamma \lambda E_{\alpha,\alpha}(t^\alpha \lambda)(\lambda I + A)^{-1}u_0d\lambda = -\frac{t^{\alpha-1}}{2\pi i} \int_\Gamma E_{\alpha,\alpha}(t^\alpha \lambda)A(\lambda I + A)^{-1}u_0d\lambda.$$

Then $P_\alpha(t)u_0 \in C^1((0, +\infty), L^\infty(\Omega))$ by the dominated convergence theorem. Furthermore, for $v \in D(A)$, we deduce from (3.8) that

$$\int_\Omega [P'_\alpha(t)u_0]vdx = -\frac{t^{\alpha-1}}{2\pi i} \int_\Omega \int_\Gamma E_{\alpha,\alpha}(t^\alpha \lambda)(\lambda I + A)^{-1}u_0d\lambda Avdx \to 0,$$

as $t \to 0^+$. Note that $\int_\Omega [P_\alpha(t)u_0]vdx \in C([0, +\infty))$ by (i). Consequently, $\int_\Omega (P_\alpha(t)u_0)vdx \in C^1([0, +\infty))$.

(iv) The proof is similar to that of Lemma 3.5 in [34], so we omit it. □

According to the results in [34], we can give the definition of the mild solution of (1.1).

**Definition 3.4** Let $T > 0$ and $u_0, u_1 \in L^\infty(\Omega)$. A function $u \in C((0, T], L^\infty(\Omega))$ is said to be a mild solution of problem (1.1) if $u$ satisfies $\lim_{t \to 0^+} \|u(t) - P_\alpha(t)u_0\|_{L^\infty(\Omega)} = 0$ and

$$u = P_\alpha(t)u_0 + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)\lambda I_{\alpha}(|u|^p)ds.$$

For problem (1.1), we have the following local existence result.

**Theorem 3.5** Let $1 < \alpha < 2$, $\gamma > 0$ and $p > 1$. For given $u_0, u_1 \in L^\infty(\Omega)$, there exists $T = T(u_0, u_1) > 0$ such that problem (1.1) admits a unique mild solution $u$ in $C((0, T], L^\infty(\Omega)) \cap L^\infty((0, T], L^\infty(\Omega)) \cap C([0, T], L^q(\Omega))$ for every $q \in (1, \infty)$. The solution $u$ can be extended to a maximal interval $[0, T^*)$, and either $T^* = +\infty$ or $T^* < +\infty$ and $\lim_{t \to T^*-} \|u(t)\|_{L^\infty(\Omega)} = +\infty.$
Proof. For given $T > 0$, let $E_T = L^\infty((0,T),L^\infty(\Omega))$. Let $B_K$ denote the closed ball in $E_T$ with center 0 and radius $K$. For given $u_0, u_1 \in L^\infty(\Omega)$, we define the operator $G$ on $E_T$ as

$$G(u)(t) = P_\alpha(t)u_0 + 0I_t^\alpha P_\alpha(t)u_1 + \int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)_0I_s^\gamma(|u|^p)ds. \quad (3.9)$$

Choose $M \geq \|u_0\|_{L^\infty(\Omega)} + T\|u_1\|_{L^\infty(\Omega)}$. Then it follows from Lemma 3.7 that for $u \in B_K$, $t \in (0,T)$ and some constant $C > 0$,

$$\|G(u)(t)\|_{L^\infty(\Omega)} \leq C[\|u_0\|_{L^\infty(\Omega)} + T\|u_1\|_{L^\infty(\Omega)}] + C \int_0^t (t-\tau)^{\alpha-1}\tau^\gamma\|u(\tau)\|^p_{L^\infty(\Omega)}d\tau \leq C(M + T^{\alpha+\gamma}K^p), \quad (3.10)$$

$$\|G(u) - G(v)\|_{L^\infty(\Omega)} \leq CTK^{p-1}T^{\alpha+\gamma}\|u(t) - v(t)\|_{L^\infty((0,T),L^\infty(\Omega))}. \quad (3.11)$$

From (3.10) and (3.11), we can choose $T$ and $M$ such that $G$ is a strict contractive mapping on $B_K$. Thus $G$ possesses a unique fixed point $u \in B_K$. Note that $\int_0^t (t-s)^{\alpha-1}S_\alpha(t-s)_0I_s^\gamma(|u|^p)ds \in C([0,T],L^\infty(\Omega))$ by the dominated convergence theorem. Furthermore, using Lemma 3.3 we know that $u \in C([0,T],L^\infty(\Omega)) \cap C((0,T),L^q(\Omega))$ for every $q \in (1, +\infty)$, and $\lim_{t \to 0^+} \|u(t) - P_\alpha(t)u_0\|_{L^\infty(\Omega)} = 0$. The uniqueness of the mild solution follows from Gronwall’s inequality.

Set

$$T^* = \sup\{T \mid u \in L^\infty((0,T),L^\infty(\Omega)) \cap C([0,T],L^q(\Omega)) \text{ is a mild solution of (1.1)}\}.$$

By an analogous argument to that of Theorem 4.5 in [34], we can prove that if $T^* < +\infty$ and $\|u\|_{L^\infty((0,T^*),L^\infty(\Omega))} < +\infty$, then $\lim_{t \to T^* -} u(t)$ exists in $L^\infty(\Omega)$. This implies that $u$ can be extended after $T^*$, which contradicts the definition of $T^*$. Thus we obtain the desired conclusion.

Recalling the formula of integration by parts, we can give the following definition of weak solution of (1.1). Moreover, we also obtain the relationship between weak solutions and mild solutions of (1.1).

**Definition 3.6** Let $1 < \alpha \leq 2$, $u_0, u_1 \in L^1(\Omega)$ and $T > 0$. We say that $u \in L^p((0,T),L^p(\Omega))$ is a weak solution of (1.1) if

$$\int_0^T \int_\Omega [0I_t^\alpha(|u|^p)\varphi + (u_0 + tu_1)(tD_t^\alpha\varphi)]dtdx = \int_0^T \int_\Omega u(-\Delta\varphi)dtdx + \int_0^T \int_\Omega u(tD_t^\alpha\varphi)dtdx$$

for every $\varphi \in C^{2,2}([0,T] \times \Omega)$ with $\varphi = 0$ on $\partial\Omega$ and $\varphi(T,x) = \varphi_1(T,x) = 0$ for $x \in \overline{\Omega}$. Moreover, we call $u$ a global weak solution of (1.1) if $T > 0$ can be arbitrarily chosen.

**Lemma 3.7** Let $T > 0$, $u_0, u_1 \in L^\infty(\Omega)$. If $u \in C((0,T],L^\infty(\Omega))$ is a mild solution of (1.1) obtained by Theorem 3.6 then $u$ is also a weak solution of (1.1).

**Proof.** By Theorem 3.5 we know $u \in C([0,T],L^q(\Omega))$ for every $q \in (1, +\infty)$. Thus an argument similar to the proof of Lemma 5.2 in [34] shows that $u$ is also a weak solution of (1.1).

Then we give blow-up results of problem (1.1) with $1 < \alpha < 2$. 

\[\square\]
Due to the arbitrariness of \( \psi \) \( \text{From Jensen's inequality, we have} \)

\[ t \]

Denote \( w \) in the sense of distributions. In addition, the fact that \( u \), \( q \) \[ \text{in the sense of distributions. In addition, the fact that} \]

\( u \), \( q \) \[ \text{in the sense of distributions. In addition, the fact that} \]

then all nonzero mild solutions of \( (1.1) \) by Lemma 3.7. For \( u \), \( C \) \[ \text{in the sense of distributions. In addition, the fact that} \]

Proof. By the regularity theory of elliptic equations [13], we know the eigenfunction \( \varphi_1 \in C^2(\Omega) \) and \( \varphi_1(x) = 0 \) on \( \Omega \). Suppose that \( u \) is a global mild solution of \( (1.1) \). Then \( u \in C([0, +\infty), L^q(\Omega)) \) for every \( q \in (1, +\infty) \), and \( u \) is also a global weak solution of \( (1.1) \) by Lemma 3.7. For \( T > 0 \), we take \( \varphi(t, x) = \psi_T(t)\varphi_1(x) \) as a test function, where \( \psi_T \in C^2([0, T]) \) satisfies \( \psi_T(T) = \psi_T'(T) = 0 \), and then

\[ 
\int_0^T \int_\Omega [0 I_1^\gamma(|u|^p)\varphi_1 \psi_T + (u_0 + t u_1)\varphi_1(t D_T^\alpha \psi_T)] dt dx = \int_0^T \int_\Omega [\lambda_1 u \varphi_1 \psi_T + u \varphi_1(t D_T^\alpha \psi_T)] dt dx. 
\]

(3.12)

Denote \( w(t) = \int_\Omega u \varphi_1 dt \). From Lemma 3.3, we deduce \( w \in C^1([0, T]) \). Then (3.12) and \( \text{Lemma 2.3} \) yield that

\[ 
\int_0^T \int_\Omega [0 I_1^\gamma(|u|^p)\varphi_1 \psi_T'' + \lambda_1 w \psi_T'] dt dx \]

\[ 
= \int_0^T \int_\Omega [w - w(0) - tw'(0)] \lambda D_T^\alpha \psi_T dt dx + \lambda_1 \int_0^T \int_\Omega f \psi_T dt dx 
\]

\[ 
= \int_0^T \int_\Omega [0 I_1^\gamma(|u|^p)\varphi_1 dx \psi_T dt dx]. 
\]

(3.13)

Due to the arbitrariness of \( \psi_T \), we obtain

\[ 
\frac{d^2}{dt^2} \int_\Omega [0 I_1^\gamma(|u|^p)\varphi_1 dx \psi_T + \lambda_1 w(t) = \int_\Omega 0 I_1^\gamma(|u|^p)\varphi_1 dx 
\]

(3.14)

in the sense of distributions. In addition, the fact that \( u \in C([0, +\infty), L^q(\Omega)) \) for every \( q \in (1, +\infty) \) implies \( \int_\Omega 0 I_1^\gamma(|u|^p)\varphi_1 dx \in C([0, T]) \), and Lemma 3.3 yields \( 0 I_1^\gamma(|w - w(0) - tw'(0)|) \in C([0, T]) \). Thus it follows from the regularity theory that the equality (3.14) holds for \( t \in [0, T] \) in the classical sense. In other words,

\[ 
0 D_t^\alpha w + \lambda_1 w(t) = \int_\Omega 0 I_1^\gamma(|u|^p)\varphi_1 dx, \quad t \in [0, T]. 
\]

From Jensen’s inequality, we have

\[ 
0 D_t^\alpha w + \lambda_1 w(t) = \int_\Omega 0 I_1^\gamma(|u|^p)\varphi_1 dx \geq 0 I_1^\gamma \left( \int_\Omega |u| \varphi_1 dx \right)^p \geq 0 I_1^\gamma |w|^p, \quad t \in [0, T]. 
\]

(3.15)

Then (3.15) and Corollary 2.5 vii yield a contradiction. This completes the proof. \( \square \)

Next, we give a blow-up result of the semilinear wave equation (i.e., (1.1) with \( \alpha = 2 \)).
Theorem 3.9 Let $p > 1$, $\gamma > 0$, $\alpha = 2$ and $u_0, u_1 \in L^1(\Omega)$. Assume $u \in C([0, T], L^p(\Omega))$ is a weak solution of (1.1). If $p(1 - \gamma) \leq 1$, then $T < +\infty$.

Proof. In this case, we know that for every $\psi_T \in C^2([0, T])$ with $\psi_T(T) = \psi_T'(T) = 0$, 

$$
\int_\Omega \int_0^T [0 \int_0^T (|u|^p) \varphi_1 \psi_T + (u_0 + tu_1) \varphi_1 \psi_T'] dt dx = \int_\Omega \int_0^T [\lambda u \varphi_1 \psi_T + u \varphi_1 \psi_T'] dt dx.
$$

Denote $w(t) = \int_\Omega \psi_1 dx$. Since $u \in C([0, T], L^p(\Omega))$, we have $w \in C([0, T])$. Then 

$$w'' + \lambda_1 w(t) = \int_0^T (|u|^p) \varphi_1 \psi_1 \psi dx$$

in the sense of distributions. Furthermore, since $w \in C([0, T])$ and $\int_\Omega 0 \int_0^T (|u|^p) \varphi_1 dx \in C([0, T])$, we deduce from the regularity theory that $w \in C^2([0, T])$. Then in terms of Jensen’s inequality, we have 

$$w'' + \lambda_1 w(t) \geq 0 \int_0^T |w|^p, \quad t \in [0, T],$$

which implies $T < +\infty$ by Corollary 2.6 (iv).

Remark 3.10 By Corollary 2.3 (i) and Corollary 2.6(i), we know that for given $T > 0$, if

$$T \int_\Omega u_1(x) \varphi_1(x) dx + \int_\Omega u_0(x) \varphi_1(x) dx > K_1 T^{\alpha + \gamma - \frac{\alpha - 1}{p}} + K_2 T^{- \frac{\alpha - 1}{p}} + K,$$

then the corresponding solution of (1.1) does not exist globally in time and $T^* < T$.

Finally, we have the following results of global existence of solutions for sufficiently small initial values.

Theorem 3.11 Let $p > 1$, $1 < \alpha < 2$ and $u_0, u_1 \in L^\infty(\Omega)$.

(i) If $\alpha + \gamma \geq 2$, $p(1 - \gamma) > 1$ and $\|u_0\|_{L^\infty(\Omega)} + \|u_1\|_{L^\infty(\Omega)}$ is sufficiently small, then the mild solution $u$ of (1.1) is global.

(ii) If $\alpha + \gamma < 2$, $p(1 - \gamma) > 1$, $u_1 \equiv 0$ and $\|u_0\|_{L^\infty(\Omega)}$ is sufficiently small, then the mild solution of (1.1) exists globally.

(iii) If $\alpha + \gamma < 2$, $p \geq 1 + \frac{\gamma}{\alpha - 1}$ and $\|u_0\|_{L^\infty(\Omega)} + \|u_1\|_{L^\infty(\Omega)}$ is sufficiently small, then the mild solution of (1.1) exists globally.

Proof. (i) We prove the global existence of solutions for problem (1.1) by the contraction mapping principle.

Let $X = \{u \in L^\infty((0, \infty), L^\infty(\Omega)) \mid \|u\| < \infty\}$, where $\|u\| = \sup_{t>0} (1 + t)^{-\frac{\gamma}{\alpha - 1}} \|u(t)\|_{L^\infty(\Omega)}$. Then $X$ is a Banach space. For given $u \in X$, we define

$$\Psi(u)(t) = P_\alpha(t) u_0 + \alpha \int_0^t P_\alpha(t-s) u_1 + \int_0^t (t-s)^{\alpha - 1} S_\alpha(t-s) \int_0^t \varphi_1 (|u|^p) ds, \quad t \geq 0.$$
Let \( B_M \) denote the closed ball in \( X \) with center 0 and radius \( M \), where \( M > 0 \) is to be chosen sufficiently small.

To prove our result, it suffices to show that \( \Psi \) is a contractive mapping on \( \mathcal{E} \) when 
\[
\|u_0\|_{L^\infty(\Omega)}, \|u_1\|_{L^\infty(\Omega)} \text{ and } M \text{ are chosen sufficiently small.}
\]

The assumptions that \( \alpha + \gamma \geq 2 \) and \( p > \frac{1}{\gamma} \) imply \( p > \frac{1}{\gamma} \geq 1 + \frac{\alpha}{2\alpha - 1}, \frac{\gamma}{\gamma - 1} < \frac{\gamma}{\gamma - 1} + 1 \leq \alpha \) and \( \frac{p}{2 - \gamma} < 1 \). Hence, by Lemma 3.33 there exists a constant \( C > 0 \) such that for any \( u \in B_M \) and \( t \geq 0 \),
\[
(1 + t)^{\frac{\gamma}{2 - \gamma}}\|P_\alpha(t)u_0\|_{L^\infty(\Omega)} \leq C(1 + t)^{\frac{\gamma}{2 - \gamma} - \alpha}\|u_0\|_{L^\infty(\Omega)} \leq C\|u_0\|_{L^\infty(\Omega)},
\]
(3.16)

and
\[
(1 + t)^{\frac{\gamma}{2 - \gamma}}\|P_\alpha(t)u_1\|_{L^\infty(\Omega)} \leq C(1 + t)^{\frac{\gamma}{2 - \gamma} - 1 - \alpha}\|u_1\|_{L^\infty(\Omega)} \leq C\|u_1\|_{L^\infty(\Omega)},
\]
(3.17)

Next, we estimate (3.18). In terms of (2.1), we know that there exist positive constants \( C_1, C_2 \) and \( L \) such that for \( t > L \)
\[
\frac{C_1}{1 + t^{2\alpha}} \leq -E_{\alpha, \alpha}(-t^{\alpha}) \leq \frac{C_2}{1 + t^{2\alpha}}.
\]

This implies that for \( t > 2L \)
\[
\int_0^t \frac{(t - s)^{\alpha - 1}}{1 + (t - s)^{2\alpha}} s^{-\frac{\gamma}{\gamma - 1}} ds
\]
\[
\leq C \int_0^{t-L} -(t - s)^{\alpha - 1} E_{\alpha, \alpha}(-(t - s)^{\alpha}) s^{-\frac{\gamma}{\gamma - 1}} ds + \int_t^{t-L} \frac{(t - s)^{\alpha - 1}}{1 + (t - s)^{2\alpha}} s^{-\frac{\gamma}{\gamma - 1}} ds
\]
\[
= C \int_0^t -(t - s)^{\alpha - 1} E_{\alpha, \alpha}(-(t - s)^{\alpha}) s^{-\frac{\gamma}{\gamma - 1}} ds + C \int_{t-L}^t (t - s)^{\alpha - 1} E_{\alpha, \alpha}(-(t - s)^{\alpha}) s^{-\frac{\gamma}{\gamma - 1}} ds
\]
\[
+ \int_{t-L}^t \frac{(t - s)^{\alpha - 1}}{1 + (t - s)^{2\alpha}} s^{-\frac{\gamma}{\gamma - 1}} ds.
\]
(3.19)

Furthermore, since \( \frac{\gamma}{\gamma - 1} < 1 \) and \( \alpha > 1 \), we know that \( (t - s)^{\alpha - 1} E_{\alpha, \alpha}(-(t - s)^{\alpha}) s^{-\frac{\gamma}{\gamma - 1}} \in L^1((0, t)) \) for given \( t > 0 \). Thus it follows from the dominated convergence theorem that
\[
\int_0^t (t - s)^{\alpha - 1} E_{\alpha, \alpha}(-(t - s)^{\alpha}) s^{-\frac{\gamma}{\gamma - 1}} ds = \sum_{k=0}^{\infty} \int_0^t (-1)^k (t - s)^{\alpha k + \alpha - 1} s^{-\frac{\gamma}{\gamma - 1}} ds
\]
\[
= \sum_{k=0}^{\infty} \frac{(-1)^k (t - s)^{\alpha k + \alpha - 1} s^{-\frac{\gamma}{\gamma - 1}} ds}{\Gamma(\alpha k + \alpha)}
\]
On the critical exponents for a fractional diffusion-wave equation

\[
= \Gamma\left(1 - \frac{\gamma}{p-1}\right) t^{\alpha - \frac{\gamma}{p-1}} E_{\alpha, \alpha + 1 - \frac{\gamma}{p-1}}(-t^{\alpha}).
\]

Consequently, we conclude from (3.19) that for \( t > 2L \)

\[
\int_0^t \frac{(t-s)^{\alpha-1}}{1+(t-s)^{2\alpha}} s^{-\frac{\gamma}{p}} \, ds \leq -C t^{\alpha - \frac{\gamma}{p-1}} E_{\alpha, \alpha + 1 - \frac{\gamma}{p-1}}(-t^{\alpha}) + C t^{-\frac{\gamma}{p-1}} \leq C t^{-\frac{\gamma}{p-1}}.
\]

Hence

\[
(1 + t)^{\frac{\gamma}{p-1}} \| \Psi(u) - P_{\alpha}(t) u_0 - a I^1_{\alpha} P_{\alpha}(t) u_1 \|_{L^\infty(\Omega)} \leq C M^p.
\] (3.20)

On the other hand, an argument similar to the above proof shows that there exists a constant \( C > 0 \) such that for any \( u, v \in B_M \) and \( t \geq 0 \),

\[
(1 + t)^{\frac{\gamma}{p-1}} \| \Psi(u) - \Psi(v) \|_{L^\infty(\Omega)} \leq C M^p - 1 \int_0^t \frac{(t-s)^{\alpha-1}}{1+(t-s)^{2\alpha}} s^{-\frac{\gamma}{p}} \, ds \| u - v \|
\]

\[
\leq C M^p - 1 \int_0^t \frac{(t-s)^{\alpha-1}}{1+(t-s)^{2\alpha}} s^{-\frac{\gamma}{p}} \, ds \| u - v \|
\]

\[
\leq C M^p - 1 \| u - v \|. \quad (3.21)
\]

Combining (3.16), (3.17), (3.20) and (3.21), we know that \( \Psi \) is a strict contractive map on \( B_M \) if \( \| u_0 \|_{L^\infty(\Omega)}, \| u_1 \|_{L^\infty(\Omega)} \) and \( M \) are chosen small enough. Then the contraction mapping principle implies that \( \Psi \) has a unique fixed point \( u \in B_M \). In addition, from Lemma 3.3 we know that \( u \in C((0, +\infty), L^\infty(\Omega)) \) and \( \lim_{t \to 0^+} \| u(t) - P_{\alpha}(t) u_0 \|_{L^\infty(\Omega)} = 0 \).

Thus problem (1.1) admits a global mild solution.

(ii) Since \( p > 1 \) implies that \( \frac{\gamma}{p-1} < \frac{\alpha}{p-1} < 1 < \alpha \), we know that the estimates (3.16), (3.18), (3.20) and (3.21) also hold in this case. Thus we can obtain the desired conclusion.

(iii) The assumption that \( \alpha + \gamma < 2 \) and \( p \geq 1 + \frac{\gamma}{\alpha - 1} \) imply that \( p \geq 1 + \frac{\gamma}{\alpha - 1} > \frac{1}{1-\gamma} \).

\[
\frac{\gamma}{p-1} < \frac{\gamma}{p-1} + 1 \leq \alpha \text{ and } \frac{\gamma}{p-1} < 1. \text{ Then, repeating the arguments in the proof of case (i), we can complete the proof.} \]

\[\square\]

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