On the quasi-isometric rigidity of graphs of surface groups

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Abstract

Let Γ be a word hyperbolic group with a cyclic JSJ decomposition that has only rigid vertex groups, which are all fundamental groups of closed surface groups. We show that any group $H$ quasi-isometric to Γ is abstractly commensurable with Γ.

1 Introduction

The quasi-isometry class of a group is arguably its most fundamental geometric invariant and so it’s a central program in geometric group theory to understand which algebraic properties of groups are quasi-isometry invariants. We refer the reader to [Spa04, Kap14] for further background and a list of classical results. The strongest statements in this area are quasi-isometric rigidity statements, relating quasi-isometry to commensurability. Before stating the main result of the paper, we define the class of groups to which it applies.

Definition 1.1 (grsg-groups). A group Γ is called a graph of rigid surface groups, or a grsg-group, if it is word hyperbolic and it has a cyclic JSJ decomposition with only rigid type non-cyclic vertex groups, all of which are the fundamental groups of closed surfaces.

We also have the following constructive characterization of these groups, proved in Section 2.3.

Proposition 1.2. The class of grsg-groups consists of the word hyperbolic fundamental groups of graphs of spaces, whose vertex spaces are closed surfaces, edge spaces are circles, and immersions of edge spaces to vertex spaces map onto a set of filling curves in each vertex space.

Our main result is a strengthening of [MSW11, Theorem 1.2] in the word hyperbolic case.

Theorem A. Let Γ be a grsg-group and let $f : H \to \Gamma$ be a quasi-isometry. Then $H$ and Γ are abstractly commensurable.

In other words we show that every quasi-isometry class of grsg-groups consists of a single commensurability class. One aspect of novelty in this result is that quasi-isometric rigidity most often comes from being close to a Lie group, which is not the case here.
Theorem A is an important development in the context of the following thread of recent work. Haïssinsky in [Haï15] proved that hyperbolic groups with planar boundaries not containing Sierpinski carpets are virtually convex-cocompact Kleinian groups. Haïssinsky’s result is closely related to our own in that the lowest terms in a virtually cyclic Haken hierarchy of such groups will (virtually) be geometric amalgams of free groups (see [Laf07, Definition 2.2]), i.e. groups in which every non-elementary vertex group of the cyclic JSJ decomposition is quadratically hanging (QH) (these are also called flexible.)

Stark, in [Sta17], considers a special class of geometric amalgams of free groups, namely the groups $C_s$ whose JSJ decomposition consists of four QH subgroups (with orientable underlying surface) and one conjugacy class of edge groups, and shows that all such groups are quasi-isometric, but that $C_s$ partitions into infinitely many commensurability classes. Further study of the commensurability classes such groups is done by Dani, Stark, and Thomas in [DST16]. The phenomenon, first observed in [Why99, Theorem 1.6,1.7], is that flexible vertex groups in the JSJ decomposition give rise to many commensurability classes within a quasi-isometry class.

Our main result, involving groups without QH vertex groups, is therefore dual to these recent developments, and we believe it will be one of the pillars of the study of the commensurability classes of constructible hyperbolic groups, the smallest class of hyperbolic groups closed under amalgamations over finite and virtually cyclic subgroups.

1.1 Outline of the paper

In Section 2 we will gather some definitions and theorems, including basics of group theoretical JSJ theory. As a result of Papasoglu’s Theorem [Pap05, Theorem 7.1] we will show that a quasi-isometry between $H$ and $\Gamma$ gives rise to a quasi-isometry between trees of spaces associated to the cyclic JSJ decompositions of $H$ and $\Gamma$, coarsely mapping vertex spaces to vertex spaces and edge spaces to edge spaces. Such spaces are called churro-crêpe spaces; crêpes being, of course, copies of the hyperbolic Poincaré disc $\mathbb{H}^2$. We will also apply Wise theory [Wis11] to address issues related to torsion and subgroup separability.

In Section 3, drawing inspiration from [CM11], we consider the images of the edge spaces in the crêpes, these are line patterns. These line patterns give rise to wall spaces, which we cubulate. It is common knowledge that the cubulation of a crêpe produces a waffle. Our assumption that the line pattern comes from a filling set of curves enables us to show that the actions of vertex groups on waffles are properly discontinuous and cocompact; thus waffles are quasi isometric to crêpes. We further show that the full isometry group of a waffle is discrete, i.e. the full isometry group of a waffle also acts properly discontinuously. We also show that quasi-isometries of crêpes which coarsely preserve line patterns can be combinatorialized to isometries of the corresponding waffles; thus giving a pattern rigidity result analogous to Cashen and Macura’s [CM11].

In Section 4 we replace the crêpes with waffles in the tree of spaces, to obtain a churro-waffle space. By combining pattern rigidity and using a quasi-isometry invariants known as clutching ratios, we show that if $H$ is quasi-isometric to $\Gamma$, then the corresponding churro-waffle spaces are in fact isometric. Although we initiated this project before learning about the work of Cashen and Martin [CM17], many ideas in their paper are illustrated here.
For example our clutching ratios appear to be examples of their stretch factors and our waffles are their rigid models for vertex groups.

At this point, one would expect that having both $\Gamma$ and $H$ acting properly discontinuously and cocompactly by isometries on a common space would make them commensurable. It would certainly be the case if the churro-waffle space, denoted $\mathbb{P}^\#(\Gamma)$, had a discrete isometry group. Unfortunately $\text{Aut}(\mathbb{P}^\#(\Gamma))$ may be a non-discrete totally disconnected locally compact group (in particular an uncountable group.) We note that a similar problem to ours was recently considered by Stark and Woodhouse [SW17]. In their paper they consider hyperbolic geometric amalgams of surface groups whose JSJ decomposition has a single edge group. In their paper they construct a common model space for two groups and, by exhibiting the existence of a common rigid colouring, construct a discrete overgroup that contains both groups.

We do not follow such an approach. In Section 5 we define discrete groupings and, adapting the referee’s argument of the the Bass Conjugacy Theorem in [Bas93] to churro-waffle spaces, we prove that the existence of a flat discrete grouping (see Section 5.2) of $\mathbb{P}^\#(\Gamma)$ implies that all discrete cocompact subgroups of $\text{Aut}((\mathbb{P}^\#(\Gamma))$ are abstractly commensurable. This leads to an approach that is analogous to the Bass-Kulkarni proof, in [BK90], of Leightons’s Theorem [Lei82], but for churro-waffle spaces.

Having motivated discrete groupings, in Section 6 we construct them. Let $T$ be the tree underlying the tree of spaces structure of $\mathbb{P}^\#(\Gamma)$ and let $G$ be its full isometry group. We start with a collection of discrete group actions on the individual churros and waffles whose union constitutes $\mathbb{P}^\#(\Gamma)$. However the stabilizers of subspaces shared by churros and waffles (which are called strands) may not be isomorphic, thus producing an obstruction to forming a graph of groups. To resolve this discrepancy, we give a specific method, called augmentation, to enlarge the groups acting on churros and waffles, by adding finite abelian direct factors. We can then define the action of these new factors so that the subgroups stabilizing strands become isomorphic. We consider the case when $X = G\backslash T$ is a tree and when $X$ has cycles separately. The tree case is easier and this is a general phenomenon, see for example [Neu10, Theorem 2.4] which only works when the quotient is a tree. When the underlying graph $X = G\backslash T$ has cycles, the argument is more difficult (c.f [Neu10, Theorem 4.1]) and involves a point preimage counting argument for orbifold covers, and some finite group theory.

Finally in Section 7 we make some concluding remarks. We will address the issue of why we didn’t prove Theorem A for graphs of free groups and lay out other directions for further work.

2 Preliminaries

We start by defining the terms in the statement of the main result.

**Definition 2.1 (Quasi-isometric).** Let $X,Y$ be metric spaces. A map $f : X \to Y$ is called $(K,C)$-quasi-isometric embedding if for every $x,y \in X$

$$\frac{1}{K}d(x,y) - C \leq d(f(x),f(y)) \leq Kd(x,y) + C.$$ 

If the image of $f$ is $C$-quasi-dense, i.e. $Y$ is contained in the $C$-neighbourhood of $f(X)$, then $f$ is a quasi-isometry. Two groups $G,H$ are quasi-isometric if there is some quasi-isometry from a Cayley graph of $G$ to a Cayley graph of $H$. 

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The primary motivation for this definition is that, given two finite generating sets $S, S'$ of a group $\Gamma$, the resulting Cayley graphs will be quasi-isometric.

**Definition 2.2** (Abstractly commensurable). Two groups $H$ and $\Gamma$ are said to be *abstractly commensurable* or *virtually isomorphic* if there are finite normal subgroups $K_H \triangleleft H$, $K_\Gamma \triangleleft \Gamma$, and finite index subgroups $H^f_1 \triangleleft H/K_H$, $\Gamma^f_1 \triangleleft \Gamma/K_\Gamma$ such that

$$H^f_1 \approx \Gamma^f_1.$$

In our case $\Gamma, H$ will be shown to be virtually torsion-free, so that being virtually isomorphic amounts to having isomorphic finite index subgroups. Commensurability, or virtual isomorphism, is an equivalence relation and quasi-isometry classes get partitioned into finer commensurability classes.

**Definition 2.3** (Quasi-isometrically rigid). A group $\Gamma$ is *quasi-isometrically rigid* if its quasi-isometry class consists of a single commensurability class.

We assume that the reader has some familiarity with Bass-Serre theory and immediately give the definition of the JSJ decomposition.

**Definition 2.4** (Collapse and Refinement). A $G$-equivariant surjection of trees $c : S \to T$ is called a *collapse* if it is obtained by equivariantly collapsing subtrees $S^c \subset S$ to points. If such is the case we also say that $S$ is a *refinement* of $T$ and the preimage of a vertex $v$ is called its *blowup*.

**Definition 2.5.** A $G$-equivariant surjection of trees $c : S \to T$ is called a *folding* if it doesn’t collapse any edges to points.

There are many subtle variations of the group theoretical JSJ decomposition originally due to Rips and Sela [RS97] (see also [Bow98] for hyperbolic groups.) In [GL16] a construction for a canonical JSJ decomposition is given. For our purposes we shall use the following characterization of this canonical splitting, given in [DT13].

**Remark 2.6.** The following result also has an analogue that applies in the presence of torsion. For simplicity, however, we will restrict ourselves to the torsion-free case because the following formulation immediately gives a nice graph of spaces.

**Theorem 2.7** (The canonical cyclic JSJ decomposition for torsion-free hyperbolic groups. See [DT13, Proposition 3.6] and [GL16]). Let $\Gamma$ be a one-ended torsion free hyperbolic group. There is a canonical splitting as a graph of groups $J$ called the canonical cyclic JSJ decomposition with the following properties:

1. The underlying graph $J$ is bipartite with black and white vertex groups.
   
   (a) Black vertex groups are maximal cyclic.
   
   (b) White vertex groups are non-cyclic and fall into one of the following two categories:

   i. $\text{QH}$: The vertex group is isomorphic to the fundamental group of a compact surface $\Sigma$ and the images of attaching monomorphisms correspond coincide with the $\pi_1$-images of the connected components of $\partial \Sigma$.

   ii. $\text{Rigid}$: The vertex group does not admit any cyclic splitting relative to the images of the incident edge groups $\Gamma$. Furthermore the images of the edge groups of any two any distinct incident edges must have trivial intersection.
In particular we can direct every edge so that it points from a black vertex to a white vertex.

2. $T = T(\mathcal{J})$ is the corresponding Bass-Serre tree and any other $G$-tree $S$ with cyclic edge stabilizers can be obtained by blowing up $QH$ vertices, subdividing edges, folding, and then collapsing edges.

A grsg-group $\Gamma$ is therefore a hyperbolic group in which the black vertex groups of its cyclic JSJ decomposition are of type 1.(b).ii.

### 2.1 Graphs of groups and Bass-Serre theory: notation

The purpose of this section is to establish notation. We refer the reader to [SW79, Ser03] for details on graphs of groups, Bass-Serre theory, and graphs of spaces.

Let $G$ be a group and let $T$ be a simplicial $G$-tree. A subgroup or element of $G$ is called elliptic if it fixes a point of $T$, otherwise it is called hyperbolic. For every cell $\alpha$, either a vertex or an edge, we denote by $G_\alpha$ the (setwise) stabilizer of that cell. $G_\alpha$ is called an edge group or a vertex group if $\alpha$ is an edge or a vertex respectively. $G$ will always act on $T$ without inverting edges, i.e. the edge group $G_e$ is the pointwise stabilizer of $e$. This implies that we can $G$-equivariantly orient every edge of $T$, by choosing for each edge an initial and a terminal vertex. If $e$ is an edge with initial vertex $u$ and terminal vertex $v$ then we have monomorphisms

$$G_u \xrightarrow{\iota_{e,u}} G_e \xrightarrow{\tau_{e,v}} G_v. \quad (1)$$

Because the action of $G$ on $T$ is simplicial, the quotient $G\backslash T = J$ where $J$ is a graph. Every cell of $J$ corresponds to the $G$-equivalence class $[\alpha]$ of some cell of $T$, and all cells in the same orbit are $G$-conjugate. Furthermore every edge $[e]$ in the quotient $J$ inherits an orientation from the orientation of any representative $e \in [e]$ in $T$.

We can endow the directed graph $J$ with a graph of groups structure $\mathcal{J}$, where every vertex $[v]$ is equipped with a vertex group $J_{[v]} \approx G_v$ for some $v \in [v]$, every edge $[e]$ is equipped with an edge group $J_{[e]} \approx G_e$ for some $e \in [e]$. If the edge $[e]$ in $J$ has initial vertex $[v]$ and terminal vertex $[u]$ then we have attaching monomorphisms

$$J_{[v]} \xleftarrow{\iota_{e,v}} J_{[e]} \xrightarrow{\tau_{e,u}} J_{[v]}$$

which are copies of the monomorphisms (1), for an appropriate choice of lifts $u, e, v \in T$ where $e \in [e]$ and $u, v$ are adjacent to $e$.

Conversely, for any directed graph $J$ and such a graph of groups structure $\mathcal{J}$ we can associate a fundamental group of $\mathcal{J}$ as well as a dual Bass-Serre tree $T(\mathcal{J})$. Furthermore if $\mathcal{J}$ was obtained from the action of $G$ on a simplicial tree $T$, then there is a $G$-equivariant isomorphisms $T \to T(\mathcal{J})$. In this case we say $G$ splits as the graph of groups $\mathcal{J}$. Formally, the vertex and edge groups of $\mathcal{J}$ correspond to conjugacy classes of subgroups of $G$.

### 2.2 Graphs of spaces for grsg-groups

Let $\Gamma$ be a grsg-group and let $\mathcal{J}$ be its cyclic JSJ splitting. Then following [SW79, §3], we will construct a cell complex $X_\mathcal{J}$ with $\pi_1(X_\mathcal{J})$ called the graph of spaces associated to $\mathcal{J}$ as follows.
1. For each white vertex $v$ of $J$ take a closed surface $\Sigma_v$ such that $\pi_1(\Sigma_v) \approx J_v$. For each black vertex $v$ we take a a circle $S^1_v$. We call this the vertex space corresponding to $v$.

2. For each edge $e$ of $J$ take a copy of the annulus $\Sigma_e = S^1 \times [-1, 1]$. Note that $\pi_1(\Sigma_e) \approx J_e \approx \mathbb{Z}$. We call this the edge space.

3. Each edge $e$ of $J$ is directed, so there are initial and terminal vertices $v_0$ and $v_1$. We have corresponding attaching monomorphisms $\iota(u, e) : J_e \to J_{v_0}$ and $\tau(v, e) : J_e \to J_{v_1}$. Topologically, these monomorphisms can be realized as immersions

$$\iota^{top}(v_1, e) : S^1 \times \{-1\} \ni (v_1, e) \mapsto \Sigma_{v_1} \quad \tau^{top}(v_2, e) : S^1 \times \{1\} \ni (v_2, e) \mapsto \Sigma_{v_2}$$

where the image of $\iota^{top}(v_1, e)$ is a geodesic $\gamma_e$. For each white vertex $v$, let the surface line pattern $L_v$ be the set of $t^{top}(v, e)$ images in $\Sigma_v$ for all $e$ adjacent to $v$.

4. The graph of spaces $X_J$ is the quotient space obtained

$$X_J = \left( \left( \bigcup_{u \in \text{Vert}^w(J)} \Sigma_u \right) \cup \left( \bigcup_{v \in \text{Vert}^b(J)} S^1_v \right) \cup \left( \bigcup_{e \in \text{Edges}(J)} \Sigma_e \right) \right) / \sim$$

where $x \sim \tau^{top}(v, e)(x)$ or $x \sim \iota^{top}(u, e)(x)$, provided $x$ lies in the domain of the function.

We note that $\Gamma \approx \pi_1(X_J)$. The following result is standard, and we leave the proof to the reader, as an exercise in covering spaces and Bass-Serre theory. See also [SW79, Theorem 4.3].

**Proposition 2.8.** Let $\mathcal{J}$ be the JSJ splitting of $\Gamma$ and let $\widetilde{X}_J$ be the universal cover of the graph of spaces associated to $X_J$. Then

1. $\pi_1(X_J) = \Gamma$ making $\widetilde{X}_J$ quasi-isometric to $\Gamma$.

2. $\widetilde{X}_J$ is the union of the following lifts:

   a) Each white vertex space $\Sigma$ lifts to a disjoint union of copies of $\mathbb{H}^2$. In each copy there’s a set of geodesics $\mathcal{L}$, called a line pattern covering the surface line pattern $L$ in $\Sigma$. We refer to the vertex space with its line pattern as a crêpe $\otimes = (\mathbb{H}^2, \mathcal{L})$.

   b) Each black vertex space $S^1$ lifts in $\widetilde{X}_J$ to a disjoint union of copies of $\mathbb{R}$. We refer to these vertex spaces as churro cores, denoted by $\ast$.

   c) Each edge space lifts to a union of copies of $\mathbb{R} \times [-1, 1]$, called flaps, with each boundary component $\mathbb{R} \times \{1\}$ identified with some churro core $\otimes$ and $\mathbb{R} \times \{-1\}$ identified with some geodesic $\gamma \in \mathcal{L}$ in the line pattern of a crêpe $\otimes$, where $\otimes$ and $\ast$ cover vertex spaces connected by the edge space below. The union of each churro core with all of its attached flaps in $\widetilde{X}_J$ is called a churro, denoted by $\ast$.

3. The decomposition from item 2. makes $\widetilde{X}_J$ into a graph of spaces with underlying graph a bipartite tree $T_3$. In particular if we collapse every vertex space to a point and collapse every edge space $\mathbb{R} \times [-1, 1]$ onto its $[-1, 1]$-factor, then we get an identification map $\pi_T : \widetilde{X}_J \to T_3$.

4. Deck transformations of $\Gamma$ on $\widetilde{X}_J$ induce an action of $\Gamma$ on $T_3$, making $T_3 \Gamma$ - isomorphic to the Bass-Serre tree $T(\mathcal{J})$. 

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2.3 Constructing grsg-groups

grsg-groups are defined in terms of their JSJ decomposition and the latter is given by an existential result. We will now prove Proposition 1.2 which gives a straightforward method to construct these groups.

**Definition 2.9.** A set \( L \) of geodesic closed curves in the hyperbolic surface \( \Sigma \) is called **filling** if the complement

\[
\Sigma \setminus \left( \bigcup_{\gamma_i \in L} \gamma_i \right)
\]

is a disjoint union of simply connected hyperbolic polygons.

![Diagram](image)

Figure 1: Ensuring condition 1.(b).ii. of Theorem 2.7. Below, operations the underlying graphs: refinement, then folding. The universal cover of the wedge of cylinders on the right is a churro.

**Lemma 2.10** (see [DS99, Lemma 2.2]). If a closed surface group \( \pi_1(\Sigma) \) splits as a graph of groups \( \mathbb{X} \) with a cyclic edge groups, then there is an essential multicurve \( \Lambda \) such that every edge group of \( \mathbb{X} \) is conjugate to the \( \pi_1 \)-image of some component of \( \Lambda \).

**Proof of Proposition 1.2.** Let \( \Gamma \) be the fundamental group of a graph of spaces \( X \) given in the statement of the proposition. First note that by [Tou15, Corollary 1.5], \( \Gamma \) is one-ended.

We now show that the vertex groups satisfy the non-splitting property given in item 1.(b).ii. of Theorem 2.7. Suppose towards a contradiction that such a relative splitting of a vertex group existed. By Lemma 2.10, this implies the existence of an essential multicurve \( \Lambda \) which, after a homotopies, can be made disjoint from curves carrying the images of the incident edge groups. This is impossible, since \( \Lambda \) is essential and these curves are filling.

Now it may be that two distinct incident edges map onto the same curve in a vertex space of \( X \), violating the second clause of 2.(b). A “2-acylindrical move”, consisting of a refinement and a folding, can be used to remedy this. See Figure 1.

If an edge group maps to a non-maximal cyclic subgroup of a vertex group, a similar move may be performed. Since the graph of groups has no QH subgroups, [DT13, Theorem 3.11] immediately implies that the new graph of groups is the canonical JSJ decomposition. \( \Gamma \) is therefore a grsg-group. □
2.4 Consequences of grsg structure

The structure of grsg-groups makes it straightforward to apply [Pap05, Theorem 7.1] which connects the JSJ description and quasi-isometry.

**Theorem 2.11** ([Pap05, Theorem 7.1]). Let $G_1, G_2$ be one-ended finitely presented groups, let $J_1, J_2$ be their respective JSJ-decompositions and let $X_1, X_2$ be the Cayley graphs of $G_1, G_2$.

Suppose that there is a quasi-isometry $f : G_1 \rightarrow G_2$. Then there is a constant $C > 0$ such that if $A$ is a subgroup of $G_1$ conjugate to a vertex group, an orbifold hanging vertex group, or an edge group of the graph of groups $J_1$, then $f(A)$ contains in its $C$-neighbourhood (and is contained in the $C$-neighbourhood of) respectively a subgroup of $G_2$ conjugate to a vertex group, an orbifold hanging vertex group or an edge group of the graph of groups $J_2$.

In this next result we prove something stronger than quasi-isometric rigidity of the class of grsg-groups.

**Proposition 2.12** (The class of grsg-groups is QCERF and quasi-isometrically rigid). Let $H$ be a word hyperbolic group. The topology of the Gromov boundary $\partial H$ determines whether $H$ is virtually grsg. In which case, $H$ also happens to be QCERF.

**Proof.** By [Bow98, Theorem 5.28] (though the authors recommend [DT14, §2.3]) for a well thought-out exposition) the boundaries of the rigid vertex groups of the JSJ decomposition of $H$ are topologically distinguished subspaces of $\partial H$. Furthermore the topology of $\partial H$ determines whether the JSJ decomposition has any (virtually) QH subgroups.

Therefore, if $\partial H$ encodes the fact that the virtually cyclic JSJ decomposition of $H$ has only rigid-type non-elementary vertex groups (see Theorem 2.7 1.(b).ii.) and that the boundaries of all these vertex groups are circles, by [Tuk88, Gab92, CJ94] we may conclude that $H$ is a graph of one-ended virtually cocompact Fuchsian groups with virtually cyclic edge groups. Because virtually cyclic groups are quasiconvex in word-hyperbolic groups it follows from [Wis11, Theorem 14.5] that $H$ is virtually special. It therefore follows from [HW08] that $H$ is QCERF and therefore has a torsion-free finite index subgroup. In particular $H$ is virtually grsg.

As an immediate consequence of Theorem 2.11 and Proposition 2.12.

**Lemma 2.13.** Let $\Gamma$ be a grsg-group and let $\phi : G \rightarrow \Gamma$ be a quasi-isometry. Then $G$ has a finite index subgroup $H$ such that:

1. $H$ is itself a grsg-group with cyclic JSJ decomposition $\mathbb{K}$, and
2. If $\mathcal{X}_\mathbb{K}$ and $\mathcal{X}_J$ are the complexes given in Proposition 2.8 for $H$ and $\Gamma$ respectively, then the quasi-isometry (again denoted by $\phi$)

$$\phi : \mathcal{X}_\mathbb{K} \rightarrow \mathcal{X}_J$$

maps every crêpe $\boxtimes$ of $\mathcal{X}_\mathbb{K}$ (Proposition 2.8 item 2.(a)) to a $C_{\boxtimes}$-neighbourhood of some crêpe of $\mathcal{X}_J$ and maps every churro core $\ast$ of $\mathcal{X}_\mathbb{K}$ (Proposition 2.8 item 2.(b)) to some $C_{\ast}$-neighbourhood of a churro core of $\mathcal{X}_J$.
3. This gives a graph isomorphism $T_\mathbb{K} \rightarrow T_J$ which preserves churro/crêpe-type of vertices.
Definition 2.14. A grsg-group $\Gamma$ is clean if for every churro stabilizer $\Gamma_\ast$ the following hold:

- If $\Gamma_\ast$ does not permute the churro flaps, and
- For every $\mathbb{H}^2$-space attached to $\ast$ the map $\Gamma_\ast \leq \text{Isom}(\mathbb{H}^2)$ does not have any orientation reversing images.

Proposition 2.15 (Cleaning). Any grsg-group $\Gamma$ has a clean finite index subgroup.

Proof. By 2.12 $\Gamma$ is QCERF, that is to say every quasiconvex subgroup is subgroup separable. For each churro $\ast$, $\Gamma_\ast$ admits a homomorphic image onto the finite group of permutations of the adjacent churro flaps, and for each crêpe attached to $\ast$ there is a homomorphism onto $\mathbb{Z}_2$ given as follows. Since $\Gamma_\ast$ acts hyperbolically it fixes an axis of translation, but it is possible that certain elements also reflect along that axis; thus giving non-trivial $\mathbb{Z}_2$ images.

We may already assume $\Gamma$ is torsion-free, which means that each $\Gamma_\ast$, being virtually cyclic, must in fact be of the form $\langle g \rangle$. There is some $N$ such that the image of $g^N$ is trivial in all these finite groups. Since $\langle g^N \rangle$ is quasiconvex we can find a finite index normal subgroup $\Gamma^{f.i.}$ that contains $\langle g^N \rangle$, but avoid $\{g, g^2, \ldots, g^{N-1}\}$. In particular no element in $\Gamma_\ast$ that stabilizes a $\Gamma$-translate of $\ast$ permutes any flaps of $\ast$ or reflects along any axes in adjacent crêpes.

If $\{\ast_i\}_{i=1}^n$ is a list of representatives of $\Gamma$-orbits of churros then the finite index subgroup

$$\bigcap_{i=1}^n \Gamma^{f.i.} \leq \Gamma$$

has the desired properties. \qed

3 Line patterns and cubulation

3.1 Crêpes to waffles: cubulating line patterns

3.1.1 Shadow graph of a line pattern

Each $\otimes \subset \tilde{X}_i$, as in item 2.(a) of Proposition 2.8, can be decomposed using its line pattern $L$, into a 2-complex CoSha($\otimes$) as follows: for every $\delta, \gamma \in L$, add a vertex $v = \delta \cap \gamma$, if the intersection is nonempty. Every segment on $\gamma \in L$ between two vertices is an edge and the connected components of the complement of this 1-skeleton are 2-cells called chambers. More succinctly: the set of geodesics $L$ divide $\otimes$ into a collection of chambers.

Definition 3.1. The shadow graph Sha($\otimes$) is the 1 skeleton dual of CoSha($\otimes$) obtained by adding a vertex at each chamber and connecting two vertices by an edge if the corresponding closed chambers intersect along a 1-cell in CoSha($\otimes$). Whenever that intersection is a segment of a geodesic $\gamma$ in $L$, label the corresponding edge in Sha($\otimes$) by $\gamma$.

Our choice of terminology foreshadows the fact that this graph will embed in, and can be seen as a lower dimension projection of, a CAT(0) cube complex.

Lemma 3.2. The shadow graph Sha($\otimes$) is locally finite and quasi-isometric to $\otimes$. 


Proof. Recall that each $\mathcal{L}$ covers a filling set of geodesics $L$ in some surface $\Sigma$. Since $L$ is filling then, by definition, the complement of the union of curves is a finite set of open polygons in $\Sigma$. By simple connectivity, each open polygon lifts in $\otimes$ to a disjoint union of homeomorphic copies; thus tiling $\otimes$ with finitely many $\pi_1(\Sigma)$-orbits of finite-sided open polygons, each with finitely many neighboring chambers. So dualizing, Sha ($\otimes$) is locally finite.

Sha ($\otimes$) can be geometrically realized by picking a point in the interior of each chamber of CoSha ($\otimes$), making it quasi-isometric to $\otimes$.

Sageev’s cube complex construction (originally for codimension-one subgroups) from [Sag97] has several varying generalizations, such as in [CN05] and [Nic04]. We review the construction from [CN05] and apply to obtain useful cubulations for each $\otimes$.

**Definition 3.3** (A space with walls (see [CN05, §1])). Consider a set $Y$ equipped with a (non-empty) collection $\mathcal{H}$ of non-empty subsets of $Y$ called **half-spaces**, closed under the involution

$$
\psi : \mathcal{H} \to \mathcal{H}
\quad h \mapsto Y \setminus h
$$

A **wall** is an unordered pair $\overrightarrow{h} = \overrightarrow{Y \setminus h} = \{h, Y \setminus h\}$ where $h \in \mathcal{H}$; denote the set of walls by $W = \mathcal{H}/\psi$. We say that points in $x, y \in Y$ are separated by a wall $\overrightarrow{h}$ if $x \in h$ and $y \in Y \setminus h$. $Y$ and $\mathcal{H}$ form a **space with walls** $(Y, \mathcal{H})$ if for any $p, q \in Y$ there are only finitely many walls that separate them (this is referred to as the **strong finite interval condition**). Define a pseudo-metric on the space with walls by letting $d(p, q)$ equal the number of walls separating $p, q \in Y$.

A group $G$ acting on the set $Y$ is said to **act properly on the space with walls** if $g \cdot h \in \mathcal{H}$ for all $g \in G$ and $h \in \mathcal{H}$, and the action is metrically proper with respect to the above pseudo-metric.

Two walls $\overrightarrow{h_1}, \overrightarrow{h_2}$ are said to **cross** if all four intersections

$$h_1 \cap h_2, h_1 \cap \psi(h_2), \psi(h_1) \cap h_2, \psi(h_1) \cap \psi(h_2)$$

are non-empty.

**Proposition 3.4** ([CN05, Theorem 3]). If a discrete group $G$ acts properly on a space with walls $(Y, \mathcal{H})$, then there is a $\text{CAT}(0)$ cube complex $X$ on which $G$ acts properly. Moreover, if there is a maximum number of pairwise crossing walls in $W$, then this number is the maximal dimension of cubes in $X$.

A section $\theta$ of the quotient map $\mapsto : \mathcal{H} \to W$ (i.e., $\mapsto (h) = \overrightarrow{h}$) satisfying

$$\theta(\overrightarrow{h_1}) \cap \theta(\overrightarrow{h_2}) \neq \emptyset$$

for all $\overrightarrow{h_1}, \overrightarrow{h_2} \in W$, is called a **consistent orientation** of the walls $W$ (in [CN05] these are called **admissible sections**). If there is a point $p \in Y$ such that $p \in \theta(\overrightarrow{h})$ for each $h \in \mathcal{H}$, say that $\theta$ is **realized** by $p$.

Consider the graph formed with a vertex for each consistent orientation of $W$, with vertices $\theta_1, \theta_2$ connected by an edge labelled by $\overrightarrow{h}$ if $\theta_1(\overrightarrow{h}) \neq \theta_2(\overrightarrow{h})$ and $\theta_1(\overrightarrow{e}) = \theta_2(\overrightarrow{e})$ for every $\overrightarrow{e} \neq \overrightarrow{h} \in W$. This graph has a connected component containing all realized consistent orientations ([CN05] refers to these as **special vertices**). This component forms the 1-skeleton of $X$. The $k$-cubes of $X$, for $k \geq 2$, are such that anytime there are $k$ edges (1-cubes)
Proposition 3.5. Let \( \mathfrak{X} = (\mathbb{H}^2, \mathcal{L}) \), where \( \mathcal{L} \) covers a surface line pattern in \( \Sigma \), and denote the set of vertices in \( \text{Sha}(\mathfrak{X}) \) by \( Y_L \). For each \( \gamma \in \mathcal{L} \), removing all edges in \( \text{Sha}(\mathfrak{X}) \) labelled by \( \gamma \) leaves two connected components, partitioning \( Y \) into two subsets. Let \( \mathcal{H}_L \) be the collection of all such subsets. This forms a space with walls \( (Y_L, \mathcal{H}_L) \) on which \( \pi_1(\Sigma) \) acts properly (with \( \mathcal{L} \) defining the set of walls). \( \pi_1(\Sigma) \) then acts properly on a CAT(0) cube complex \( \mathfrak{X} \) which contains \( \text{Sha}(\mathfrak{X}) \) in its 1-skeleton.

This space with walls is exactly the same as the set of chambers in \( \text{CoSha}(\mathfrak{X}) \) with walls given by the lines in \( \mathcal{L} \), so let \( \Pi_\mathcal{L} \) denote the hyperplane corresponding to the wall given by \( \gamma \). We call \( \mathfrak{X} \) the \\textit{waffle} constructed from \( \mathfrak{X} \).

Proof. Clearly the pairs of subsets of chambers separated by lines in \( \mathcal{L} \) are closed under complementation. Since \( \text{Sha}(\mathfrak{X}) \) is a connected, locally finite graph, with each edge labelled by exactly one wall, the strong finite interval condition is satisfied. The pseudo-metric for the space with walls is exactly that given by the path metric on \( \text{Sha}(\mathfrak{X}) \) with each edge having length 1. Since \( \text{Sha}(\mathfrak{X}) \) is locally finite, \( \pi_1(\Sigma) \) acts properly on \( \text{Sha}(\mathfrak{X}) \) with respect to this metric, and so Proposition 3.4 can be applied to produce \( \mathfrak{X} \). Each vertex of \( \text{Sha}(\mathfrak{X}) \) realizes a consistent orientation and each edge connect a pair of vertices realizing consistent orientations that differ on exactly on wall, so \( \text{Sha}(\mathfrak{X}) \) is contained in the 1-skeleton of \( \mathfrak{X} \).

\[ \square \]

3.2 Finiteness results for \( \mathfrak{X} \)

Although cube complexes are great we need a bit more, in particular we need to prove that the following always holds for a waffle \( \mathfrak{X} \) constructed from a crêpe.

- \( \mathfrak{X} \) is finite dimensional,
- the action of \( \pi_1(\Sigma) \) on \( \mathfrak{X} \) is cocompact, and
- the full isometry group \( \text{Isom}(\mathfrak{X}) \) is discrete; as opposed to being uncountable totally disconnected locally compact.

Although none of these results are surprising, they don’t seem to follow directly from the literature. Furthermore if we tessellate the Euclidean plane \( \mathbb{E}^2 \) with hexagons (see [Nic04, Example 4.12], [CN05, §3 Example 2]), then the canonical cubulation is the tessellation of \( \mathbb{E}^3 \) with 3-dimensional cubes, and therefore not \( \mathbb{Z}^2 \)-cocompact. If we weren’t requiring our surface \( \Sigma \) to be of negative Euler characteristic, then this example would arise in our construction. \( \delta \)-hyperbolic geometry therefore must play an important role.

The standard argument to prove cocompactness will also imply finite dimensionality. We say standard because we are essentially using the results of [Sag97, §3]. That being said the results of that section do not literally apply to our setting because we are dealing with multiple codimension 1 subgroups. Furthermore the proofs that are given refer to the technique of the proof in
Because we will use the visual metric on the boundary later on anyway and for the sake of self containment we will present a proof here.

We refer the reader to [Gro87, GdlH90, ABC+91] for everything about hyperbolic groups, δ-hyperbolic spaces and their boundaries. We identify $\mathbb{H}^2$ with the standard Poincaré open disc model. We note that $\mathbb{H}^2$ is a δ-hyperbolic metric space for some $\delta$ and that its Gromov boundary $\partial\mathbb{H}^2$ is the circle, which is also the topological closure of $\mathbb{H}^2$.

Given an observation point $o \in \mathbb{H}^2$ and points $x, y \in \mathbb{H}^2$ we have the Gromov inner product:

$$(x|y)_o = \frac{1}{2}((d(o, x) + d(o, y) - d(x, y)))$$

A key feature of δ-hyperbolicity is that, increasing δ if necessary, the inner product is to within an additive constant δ of the distance from $o$ to any geodesic $[x, y]$ connecting $x$ and $y$. If $a, b \in \partial\mathbb{H}^2$ and $o$ is an observation point in $\mathbb{H}^2$ then we can similarly define an inner product, which again is within δ of the distance from $o$ to $[a, b]$. See the left of Figure 2.

![Figure 2: On the left, the inner product on $\partial\mathbb{H}^2$. On the right, the ends and $o$-shadow of a geodesic.](image)

Given an $0 < \epsilon$ and $a, b \in \partial\mathbb{H}^2$ and $o \in \mathbb{H}^2$ we can define

$$q_\epsilon(a, b) = \exp(-\epsilon(a|b)_o)$$

and let

$$d_\epsilon(a, b) = \inf \sum_{i=1}^n q_\epsilon(a_{i-1}, a_i)$$

over all finite sequences $a_0 = a, a_1, \ldots, a_n = b$ of points in $\partial\mathbb{H}^2$.

**Proposition 3.6** ([GdlH90, §7.3, Proposition 10]). Let $\epsilon' = \exp(\epsilon\delta) - 1$. If $\epsilon' < \sqrt{2} - 1$ then $d_\epsilon$ is a metric on $\partial\mathbb{H}^2$ with

$$(1 - 2\epsilon')q_\epsilon(a, b) \leq d_\epsilon(a, b) \leq q_\epsilon(a, b)$$

for all $a, b \in \partial\mathbb{H}^2$. Furthermore this metric induces the standard topology (i.e. a circle) on $\partial\mathbb{H}^2$.

This metric is called a visual metric on $\partial\mathbb{H}^2$. 

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Lemma 3.7. For a bi-infinite geodesic $\gamma : (-\infty, \infty) \to \mathbb{H}^2$, with ends $\gamma^+, \gamma^- \in \partial \mathbb{H}^2$ and an observation point $o$ not on $\gamma$, then of the two connected components of $\mathbb{H}^2 \setminus \{\gamma^+, \gamma^-\}$, there is exactly one, called the o-shadow of $\gamma$, which contains a sequence of points $a_1, \ldots, a_{n-1}$ for which

$$q_\epsilon(\gamma^+, a_1) + \sum_{i=1}^{n} q_\epsilon(a_{i-1}, a_i) + q_\epsilon(a_{n-1}, \gamma^-) < q_\epsilon(\gamma^+, b_1) + \sum_{i=1}^{n} q_\epsilon(b_{i-1}, b_i) + q_\epsilon(b_{n-1}, \gamma^-)$$

for any sequence of points $b_1, \ldots, b_{n-1}$ in the other connected component of $\mathbb{H}^2 \setminus \{\gamma^+, \gamma^-\}$.

Proof. $d_\epsilon$ is a metric which induces the topology of $S^1$ on $\partial \mathbb{H}^2$, so if $o$ is not on $\gamma$ then $\gamma$ divides the circle into segments of unequal length, of which the shorter is the o-shadow.

See the right of Figure 2.

Lemma 3.8. If geodesics $\gamma, \eta$ in $\mathbb{H}^2$ cross then their o-shadows intersect for any observation point $o$ not on either $\gamma$ or $\eta$.

Certain experts will recognize that the following statement is related to the bounded packing property introduced in [HW09].

Lemma 3.9. There exists an $R$ such that any set $S$ of pairwise crossing geodesics in $\mathbb{H}^2$ all intersect a ball of radius $R$.

Proof. We will prove this result by using the visual metric and moving the observation point. Given an observation point $o$, a bi-infinite geodesic $\gamma$ and a visual metric $d_\epsilon$ call $d_\epsilon(\gamma^+, \gamma^-)$ the visual diameter of $\gamma$. The closer a geodesic is from the observation point, the bigger it’s visual diameter.

Fix now an $\epsilon$ so small and an $R$ so large that for any bi-infinite geodesic $\gamma$ at distance more than $R$ from $o$ the visual diameter $d_\epsilon(\gamma^+, \gamma^-) < 1/4$. This is possible since

$$d_\epsilon(\gamma^+, \gamma^-) \leq \exp(-\epsilon d_\epsilon(\gamma^+, \gamma^-)) = q_\epsilon(\gamma^+, \gamma^-) \leq \exp(-\epsilon(d(o, \gamma) - \delta)).$$

Now suppose that we have a set $\gamma_1, \ldots, \gamma_k$ of pairwise crossing geodesics that do not all intersect some ball of radius $R$. For an observation point $o$ set $\lambda_i^o = d_\epsilon(\gamma_i^+, \gamma_i^-)$.

Let $o \in \mathbb{H}^2$ realize

$$\inf_{o \in \mathbb{H}^2} \left( \max_i (d(o, \gamma_i)) \right).$$

Such a point exists since $\mathbb{H}^2$ is a proper metric space. Suppose that there is one $\gamma_i$ such that $\lambda_i^o < \lambda_j^o$ for all $i \neq j$. Then it follows that $\gamma_i$ is furthest from $o$. We can move $o$ slightly towards $\gamma_i$, increasing $\lambda_i^o$ but decreasing $\max_i (d(o, \gamma_i))$, contradicting the choice of $o$. It follows that there are at least 2 geodesics such that whose visual diameters are minimal.

Let $M$ denote the set of geodesics with minimal visual diameter. Suppose now that the union of the o-shadows of the geodesics in $M$ were properly contained in a half circle (i.e. a set of diameter 1) of $\partial \mathbb{H}^2$. Then again we could move $o$, see Figure 3 on the left, in a direction that decreases $\max_i (d(o, \gamma_i))$ and increases all the visual diameters of the geodesics in $M$, again contradicting the choice of $o$. 

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Figure 3: On the left moving $o$ to increase visual diameters. On the right, when $o$-shadows are small and too far apart to touch.

It therefore follows that the union of $o$-shadows of the geodesics in $M$ does not lie in a half circle. Since the $\lambda^o_i$ are minimal for the $\gamma_i$ in $M$, it follows that these are the geodesics furthest from $o$, and therefore by hypothesis are at distance more than $R$ from $o$. The choice of $R$ and $\epsilon$ ensure that for $\gamma_i$ in $M$, the visual diameters $\lambda^o_i < 1/4$. If all the $o$-shadows of the geodesics in $M$ had pairwise non-trivial intersection it would follow that the union of the shadows has diameter at most $3/4$ contradicting the fact that this union isn’t properly contained in a half circle. It follows that there are two disjoint $o$-shadows, but since the corresponding geodesics are supposed to cross this contradicts Lemma 3.8.

\[ \square \]

**Proposition 3.10.** Let $\mathbb{H}$ be constructed from $\mathbb{M} = (\mathbb{H}^2, \mathcal{L})$, where $\mathcal{L}$ covers a surface line pattern in $\Sigma$. Then the maximal dimension of cubes in $\mathbb{H}$ is finite and the action of $\pi_1(\Sigma)$ on $\mathbb{H}$ is cocompact. It follows that $\mathbb{H}$ is a hyperbolic metric space quasi-isometric to $\mathbb{H}^2$.

**Proof.** Let $R$ be as in Lemma 3.9, then any ball of radius $R$ intersects at most some number $N(R)$ of geodesics in $\mathcal{L}$ this simultaneously bounds the cardinality of any set of pairwise crossing geodesics in $\mathcal{L}$ as well as the number of $\pi_1(\Sigma)$ orbits of such sets. Seeing as these sets correspond exactly to the cubes in $\mathbb{H}$, cocompactness of the $\pi_1(\Sigma)$ action follows. The quasi-isometry result follows from the Milnor-Schwartz lemma. \[ \square \]

**Lemma 3.11** (two line condition). Let $G = \text{Isom}(\mathbb{H})$, let $x \in \mathbb{H}$ and let $\lambda_1, \lambda_2$ be hyperplanes. Any $g \in G$ that fixes $x$ and the endpoints of $\lambda_1, \lambda_2$ in $\partial \mathbb{H}$ is trivial. In particular the pointwise stabilizer of a square is trivial.

**Proof.** $G$ acts by permuting the hyperplanes $\Pi_\mathcal{L}$ of $\mathbb{H}$ and these hyperplanes are quasi-isometric to lines in $\mathcal{L}$. It follows that $g$ fixes four points in the Gromov boundary $\partial \mathbb{H}$, namely the endpoints of $\lambda_1, \lambda_2$. Take $x$ as an observation point.

It follows immediately from the definition of the visual metric $d_\epsilon$ (see [GdlH90, §7.3]) and the fact that $g$ induces an isometry of $\partial \mathbb{H}$ with respect to the visual metric $d_\epsilon$. On one hand $\partial \mathbb{H}$ is homeomorphic to the circle and $d_\epsilon$ induces this topology, on the other hand the only isometry of the circle that fixes 3 points is the identity. It follows that $g$ leaves every point on the boundary fixed, which means that it also fixes every hyperplane of $\mathbb{H}$. It is therefore a trivial isometry \[ \square \]
Corollary 3.12. The group $G = \text{Isom}(\mathbb{H})$ is discrete.

Note that discreteness is equivalent to saying that for some finite subset $S \subset \mathbb{H}$ the pointwise stabilizer of $S$ is finite.

Proof. Let $\square \subset \mathbb{H}$ be a cube that is at least of dimension 2. Let $K_{\square}$ be the pointwise stabilizer of $\square$. It follows that $K_{\square}$ fixes the hyperplanes traversing $\square$ pointwise, in particular there are at least two such hyperplanes. The result now follows immediately from Lemma 3.11.

\square

3.3 $\mathbb{H}$-isometry from quasimatching quasiline patterns

Definition 3.13. Let $\mathcal{L}$ be a collection of 2-ended quasiconvex subspaces $\gamma \subset X$ of a metric space $X$, with $\phi : X \to \mathbb{H}^2$ a quasi-isometry and each $\phi(\gamma)$ is a $(K,C)$-quasigeodesic for some uniform constants $K,C$. Let $\partial \mathcal{L}$ be the set of (unordered) pairs of ends $\partial \gamma$ of all $\gamma$ in $\mathcal{L}$. Each $\gamma \in \mathcal{L}$ has a tightening $\gamma'$, which is the geodesic in $\mathbb{H}^2$ with the same ends as $\phi(\gamma)$, and let $\mathcal{L}' = \{ \gamma' \}_{\gamma \in \mathcal{L}}$. Call $\mathcal{L}$ a quasiline pattern, and each $\gamma$ a quasiline if:

- $\partial \gamma_1 \neq \partial \gamma_2$ for all distinct $\gamma_1, \gamma_2 \in \mathcal{L}$ (equivalently $\partial \phi(\gamma_1) \neq \partial \phi(\gamma_2)$ since $\partial \phi : \partial X \to \partial \mathbb{H}^2$ is a homeomorphism)
- $\mathcal{L}'$ covers a filling set of closed geodesics in a closed surface.

For a pair $X,Y$ quasi-isometric to $\mathbb{H}^2$, quasiline patterns $\mathcal{L}_1, \mathcal{L}_2$ respectively, say that the quasi-isometry $\phi : X \to Y$ quasimatches $\mathcal{L}_1$ and $\mathcal{L}_2$ if $\partial \phi$ gives a bijection $\partial \mathcal{L}_1 \leftrightarrow \partial \mathcal{L}_2$. In this case let $\phi_q : \mathcal{L}_1 \to \mathcal{L}_2$ be the induced bijection of quasimatched quasiline patterns.

Remark 3.14. For $X$ quasi-isometric to $\mathbb{H}^2$ with quasiline pattern $\mathcal{L}$ we define the waffle $\boxplus$ of $(X,\mathcal{L})$ to just be the waffle constructed from $(\mathbb{H}^2,\mathcal{L}')$ as in Proposition 3.5, and so all of the results of Sections 3.2 still hold for such $\boxplus$. In particular, $\boxplus$ is quasi-isometric to $\mathbb{H}^2$ and the hyperplanes $\Pi_{\mathcal{L}}$ of $\boxplus$ are a quasiline pattern quasimatching $\mathcal{L}'$, and so also quasimatching $\mathcal{L}$.

For $\boxplus$ constructed from $(\mathbb{H}^2,\mathcal{L}')$, let $\mathbb{H}^0$ denote its 0-skeleton. Recall that each point $x \in \mathbb{H}^0$ corresponds to a consistent orientation $x : \mathcal{L}' \to \mathcal{H}_{\mathcal{L}'}$ of the lines (walls) in $\mathcal{L}'$, and that moreover, the two half-spaces of each wall $\gamma' \in \mathcal{L}'$ can be geometrically realized (see Lemma 3.2) as two sets of points in $\mathbb{H}^2$. Now removing $\gamma'$ from $\mathbb{H}^2$ leaves two connected components, each of which contains one of the (sets of points realizing the) half-spaces of $\gamma'$. The limit sets of the two connected components of $\mathbb{H}^2 \setminus \gamma'$ (without the shared boundary $\gamma$) are the two connected components of $\partial \mathbb{H}^2 \setminus \partial \gamma'$, which we call the half-arcs of $\gamma'$ (note that one of these half-arcs is the $o$-shadow of $\gamma$, as in Lemma 3.7). So a choice of half-space (for an orientation of the walls) is equivalent to a choice of half-arc. Let $x^\text{arc}(\gamma')$ denote the half-arc of $\gamma'$ corresponding to the half-space $x(\gamma')$.

In any given $n$-cube $\square$, the consistent orientations for distinct points in the 0-skeleton $\square^0$ only differ on the $n$-many walls corresponding to the hyperplanes through $\square$. In other words, the $2^n$ orientations of those walls all extend identically on the remaining walls, to give the consistent orientations defining all points in $\square^0$. 

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Proposition 3.15. Let $X, Y$ quasi-isometric to $\mathbb{H}^2$ with quasiline patterns $\mathcal{L}_1, \mathcal{L}_2$ respectively, and $\phi : X \to Y$ be a quasi-isometry, quasimatching $\mathcal{L}_1$ and $\mathcal{L}_2$. Let $\square_1$ and $\square_2$ be the waffles of $(X, \mathcal{L}_1)$ and $(Y, \mathcal{L}_2)$, respectively. There is an isometry $\bar{\gamma} : \square_1 \to \square_2$ with $\bar{\gamma}(\Pi_x) = \Pi_y$ for $\gamma \in \mathcal{L}_1$ and $\eta = \phi_\gamma(\gamma)$.

Proof. Let $\square_1$ be an $n$-cube in $\square_1$, not contained in any $(n+1)$-cube (call such an $n$-cube maximal), and let $\Pi_{\square_1} \subset \Pi_{\mathcal{L}_1}$ be the maximal collection of hyperplanes through $\square_1$. Recall that the set of lines in $\mathcal{L}_1$ corresponding to $\Pi_{\square_1}$ is a quasimatching $\mathcal{L}_2$. Inductively extending $\gamma$ by two $m$-cubes a $L$-cube in $\mathcal{L}_1$ is simply connected, $\mathcal{L}_1$ consists of $\gamma \in \mathcal{L}_1$ and $\mathcal{L}_2$ respectively, and $\Pi_{\mathcal{L}_2}$ is the maximal collection of hyperplanes through $\Pi_{\square_1}$.

The cardinalities of $\Pi_{\square_1}$ are exactly the number of walls on which $\phi_\gamma$, and $\Pi_{\mathcal{L}_2}$ are the respective sets of tightenings. Then there’s a quasi-isometry $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ quasimatching $\mathcal{L}_1 \to \mathcal{L}_2$, with $\phi_\gamma(\Pi_{\square_1}) = \Pi_{\mathcal{L}_2}$ if and only if $\phi_\gamma(\gamma) = \eta$. Let $\Pi_{\square_2} = \phi_\gamma(\Pi_{\square_1})$, which is itself the maximal collection of hyperplanes through a maximal $n$-cube $\square_2 \subset \square_2$ (since otherwise, additional hyperplanes would quasimatch additional hyperplanes through $\Pi_{\square_1}$). Let $\mathcal{L}_1 \subset \mathcal{L}_2$, be the set of quasimatches corresponding to $\Pi_{\square_2}$ for $i = 1, 2$, and $\mathcal{L}_1 \subset \mathcal{L}_2$, the respective sets of tightenings. Then there’s a quasi-isometry $\phi : \mathbb{H}^2 \to \mathbb{H}^2$ quasimatching $\mathcal{L}_1 \to \mathcal{L}_2$, with $\phi_\gamma(\gamma) = \eta$ if and only if $\phi_\gamma(\gamma) = \eta$. This defines a map $\bar{\gamma} : \square_1 \to \square_2$ in the following way. For $x \in \square_1$, let $y = \bar{\gamma}(x)$ be the orientation of the walls in $\mathcal{L}_1$, where $g^\sqrt{\gamma}(\eta) = \bar{\gamma}(\gamma)$ for each $\gamma \in \mathcal{L}_1$ and $\phi_\gamma(\gamma) = \eta \in \mathcal{L}_2$. So $\eta$ is a point in $\square_2$ (by extending $y$ on $\mathcal{L}_2$ with the orientations shared by all points in $\square_2$).

Now the cardinalities of $\Pi_{\square_1}$, $\Pi_{\square_2}$ are the same, so the dimensions of $\square_1, \square_2$, are equal. Since $\bar{\gamma}$ is injective on sets of half-arcs, $\bar{\gamma}$ is a bijection from the $0$-skeleton of $\square_1$ to the $0$-skeleton of $\square_2$.

Identify $\square_1$ and $\square_2$ with (copies of) $[-1,1]^n \subset \mathbb{R}^n$. $\bar{\gamma}$ is an $L^1$-isometry on the $0$-skeleton of $[-1,1]^n$, since the $L^1$ distance $\|x_1, x_2\|_1$ between any $0$-cubes $x_1, x_2$ is exactly the number of walls on which $x_1$ differ (i.e. those hyperplanes crossed by a geodesic in $\square_1$ between $x_1, x_2$), and by construction $\bar{\gamma}(x) = \gamma$ on exactly the same number of walls. Now each $m$-cube, $1 \leq m \leq n$, is spanned by two $m-1$ cubes a $L$-distance of 1 apart. Inductively extending $\bar{\gamma}^{m-1}$ to an $L^1$-isometry $\gamma^{\mathcal{L}_1}$, results in the $L^1$-isometry $\bar{\gamma}^{\mathcal{L}_1} = \gamma^{\mathcal{L}_1} : \square_1 \to \square_2$. $\bar{\gamma}$ is a homeomorphism from $\square_1$ to $\square_2$ as cell complexes, so it can be realized as an isometry (with respect to Euclidean metric) of $[-1,1]^n$.

Let $X_0 = \square_1$, and $X_{n+1} = X_n \cup \bigcup \square_k$ where $\square_k$ ranges over all cubes with $\square_k \cap \square_n > 1$ for some $\square_n \subset X_n$. Let $X_0 = \bigcup_{n=0}^\infty X_n$, call this the strongly connected component of $\square_1$ containing $\square_1$. Now if $\square_1 \neq X_0$, then $\square_1$ must be a union of multiple strongly connected components. However, since $\square_1$ is a connected cube complex, each strongly connected component must intersect another strongly connected component at a single vertex. Let $P$ be such a point of intersection. Since $\square_1$ is locally finite, any neighbourhood of $P$ intersects finitely many strongly connected components, each of which intersect any other at $P$ or not at all, so $P$ is a local cut point. However, since $\square_1$ is simply connected, $P$ is a global cut point, contradicting $\square_1$ being quasi-isometric to $\mathbb{H}^2$.

Thus $\bar{\gamma}_0 = \bar{\gamma}$ can be extended to $\bar{\gamma}_1$ on $X_1$, since if maximal cube $\square'$ and $\square_1$ in $\square_1$ share an $m$-cube $\square''$ for $1 \leq m < n$, then $\bar{\gamma}_1$ can be extended to $\square' \cup \square''$ (since maps defined above on $0$-skeletons agree and therefore extend to maximal common faces). Inductively, $\bar{\gamma}_n : X_n \to X_n$ can be extended to
\[ p_{n+1} \] for \( n = 0, 1, \ldots \), and since \( X_{\square} = \boxplus_1 \), this does in fact define a global map \( \hat{\varphi} : \boxplus_1 \to \boxplus_2 \), which is a local isometry (and so a covering map), and since both \( \boxplus_2 \) are simply connected, \( \hat{\varphi} \) is in fact an isometry.

**Definition 3.16.** Call \( \hat{\varphi} \) the combinatorialization of \( \phi \).

### 4 Churro-waffle space is quasi-isometry invariant

For \( \text{grsg} \)-group \( \Gamma \), the space \( \hat{X}_\Box \) has underlying tree structure \( T_\Box \), with churros attached to crêpes along lines of line patterns. From Proposition 3.15, for each crêpe \( \otimes = (\mathbb{H}^2, \mathcal{L}) \) there is waffle \( \boxplus \) with hyperplanes \( \Pi_{\mathcal{L}} \) quasimatching \( \mathcal{L} \). The vertex group \( \Gamma \otimes \) acts properly and cocompactly by isometries on \( \boxplus \), with each adjacent vertex group \( \Gamma \ast \) stabilizing a hyperplane \( \Pi_{\gamma} \), where \( \ast \subset \hat{X}_\Box \) is attached along \( \gamma \in \mathcal{L} \). Furthermore, if \( H \) and \( \Gamma \) are quasi-isometric \( \text{grsg} \)-groups, then waffles for corresponding vertices (as in Lemma 2.13) are isometric.

We would like to build a tree of spaces by attaching churros to waffles, following \( T_\Box \), in order to obtain a space on which \( \Gamma \) acts by isometry. However, this creates a problem: how are we supposed to attach churros to these hyperplanes?

It follows from Sageev’s work that these hyperplanes are themselves \( \text{CAT}(0) \)-cube complexes. We will therefore use \( \text{CAT}(0) \) geometry to show the existence of a canonical geodesic within each hyperplane along which we can attach a churro flap.

#### 4.1 Attaching churros to waffles: strands

**Proposition 4.1 (Strands of 2-ended \( \text{CAT}(0) \) spaces).** Let \( X \) be a 2-ended \( \text{CAT}(0) \) metric space and let \( \Gamma \) be a group that acts properly and cocompactly on \( X \) by isometries. There is a canonical geodesic \( \sigma : \mathbb{R} \to X \) called the \( \Gamma \)-strand of \( X \) with the property that \( \Gamma \cdot \sigma = \sigma \). Furthermore \( \sigma \) is an axis of translation for every hyperbolic isometry of \( X \).

**Definition 4.2 (Types of 2-ended groups).** Let \( \Gamma \) be 2-ended group, then either it maps onto \( \mathbb{Z} \), in which case we say it is of \( \mathbb{Z} \)-type or it maps onto \( D_8 = \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) in which case we say it is of \( \text{dihedral} \) type. If \( \Gamma \) is of dihedral type, we denote by \( \Gamma^+ \) the preimage of the maximal infinite cyclic subgroup of \( D_8 \), otherwise let \( \Gamma^+ = \Gamma \).

In the \( \mathbb{Z} \)-type case, Proposition 4.1 will follow almost immediately from the two results cited below. For the dihedral case a bit more work will be needed.

**Proposition 4.3 (see [BH11, Proposition II.2.7]).** Let \( X \) be a complete \( \text{CAT}(0) \) space. If \( Y \subset X \) is a bounded set or radius \( r_y \), then there exists a unique point \( c_Y \in X \), called the centre of \( Y \), such that \( Y \subset B(c_Y, r_y) \).

**Theorem 4.4 (see [BH11, Theorem II.6.8]).** Let \( X \) be a \( \text{CAT}(0) \) space. Let \( \gamma \) be a hyperbolic isometry of \( X \).

1. The axes of \( \gamma \) are parallel to each other an their union is \( \text{Min}(\gamma) \).
2. \( \text{Min}(\gamma) \) is isometric to a product \( Y \times \mathbb{R} \), and the restriction of \( \gamma \) to \( \text{Min}(\gamma) \) is of the form \( (y,t) \mapsto (y, t + |\gamma|) \), where \( y \in Y, t \in \mathbb{R} \).
Proof of Proposition 4.1. We start by considering $\Gamma$ surjection of $\Gamma$ onto $\mathbb{Z}$ be the set of points fixed pointwise by all elements of $\Gamma$. Let $g \in \Gamma$ be an element that maps to a generator of the maximal infinite cyclic subgroup. $K$ is finite and we have $\Gamma^+ = \langle g, K \rangle$. Conjugation by $g$ induces an automorphism of the finite group $K$ so if $n$ is the exponent of $\text{Aut}(K)$ then $g^n$ commutes with every element in $K$. Since $g^n$ commutes with $g$ as well we conclude that $g^n$ is central in $\Gamma^+$. It follows from Theorem 4.4 (3) that $\Gamma^+ = \langle g, K \rangle$ leaves $\text{Min}(g^n) = Y \times \mathbb{R}$ invariant.

Again, by Theorem 4.4 (3), every $\gamma \in \Gamma^+$ projects to an isometry $\alpha'$ of $Y$, which itself is a CAT(0) metric space. Since $X$ is 2-ended and $Y \times \mathbb{R}$ is a convex subset of $X$, $Y$ must be bounded. Let $\Gamma' = \langle \alpha' \mid \gamma \in \Gamma^+ \rangle$ and $Y_0 \subset Y$ be the set of points fixed pointwise by all elements of $\Gamma'$. Proposition 4.3 implies that $Y$ has a centre $c_\Gamma$ which must be fixed by $\Gamma'$ so $Y_0$ is nonempty.

Claim: The subset $Y_0 \times \mathbb{R} \subset X$ is independent of the choice of $g$. Let $g'$ be another lift of the image of $g$. Then $(g')^n = g^n k$ for some $k \in K$. $(g')^n$ maps $\text{Min}(g^n)$ to itself, but it may permute axes. Let $(g')^n$ act on $Y \times \mathbb{R}$ as $(\beta', \beta'')$ where $\beta''$ must be translation by $|g^n|$. By definition of $Y_0$, $Y_0 \times \mathbb{R} \subset \text{Min}((g')^n)$ since $\beta'$ acts trivially on $Y_0$. On the other hand if we repeat the construction with $(g')^n$ to get $Y_0' \times \mathbb{R} \subset \text{Min}((g')^n)$ we see that every element of $Y_0' \times \mathbb{R}$ must lie $\text{Min}(g^n)$ and in fact that $Y_0' \times \mathbb{R} \subset Y_0 \times \mathbb{R}$, so by symmetry both sets are equal. This completes the claim.

If $\Gamma = \Gamma^+$, i.e. $\Gamma$ is of $\mathbb{Z}$-type, then we take $\sigma = \{c_{Y_0}\} \times \mathbb{R} \subset Y_0 \times \mathbb{R}$, where $c_{Y_0}$ is the centre of $Y_0$ in the sense of Proposition 4.3, and we are done. We now turn to the dihedral case. Let $r \in \Gamma$ be such that $\Gamma = \langle \Gamma^+, r \rangle$, that is to say that $\pi(r) = \tilde{r}$ is a reflection in $D_x$. Let $x \in Y_0 \times \mathbb{R}$ and let $\rho_x = \{p\} \times \mathbb{R}$ be the $g^n$-axis containing $x$.

Claim: $r \cdot \rho_x \subset \text{Min}(g^n)$. We need to show that $g^n$ translates $r \cdot x$ minimally. Note that conjugation by $r$ gives an inverse modulo $k$ of $g^n$. Consider the following:

$$g^n r \cdot x = r (r^{-1} g^n r) \cdot x$$
$$= r g^{-n} k \cdot x$$
$$= r g^{-n} \cdot x$$

This means that $g^n$ sends the point $r \cdot x \in r \cdot \rho_x$ to $r g^{-n} \cdot x \in r \cdot \rho_x$. In particular $d(r \cdot x, g^n r \cdot x) = d(x, g^{-n} \cdot x)$, which means that $r \cdot x$ realizes the translation length of $g^n$. This proves the claim.

Now let $r'$ be another element such that $\pi(r) = \pi(r')$. Then $r' = k' r = rk$ for some $k, k' \in K$. On one hand $rk' \cdot \rho_x = r \cdot \rho_x = kr \cdot \rho_x$. Note that $k \in K$ can be chosen arbitrarily. Note also that

$$g r \cdot \rho_x = r g^{-1} k \cdot \rho_x = r \cdot \rho_x,$$

for some $k \in K$. It follows that $r \cdot \rho_x$, like $\rho_x$, is $\Gamma^+$-invariant and therefore in $Y_0 \times \mathbb{R}$; thus $Y_0 \times \mathbb{R} = \Gamma = \langle r, \Gamma^+ \rangle$-invariant. Consider the projection $Y_0 \times \mathbb{R} \to Y_0$. Since $\Gamma^+$ fixes fibres and $r$ maps fibres to fibres there is a canonical projection $\Gamma \to \text{Isom}(Y_0)$. Again there is a subset $Y_1 \subset Y_0$ that is fixed pointwise by the image of $\Gamma$. By Proposition 4.3 there is a unique centre $c_{Y_1} \in Y_1$ and the strand $\sigma = \{c_{Y_1}\} \times \mathbb{R} \subset X$ is canonical and $\Gamma$-invariant. □
We now classify strands by their ambient stabilizer groups.

Lemma 4.5. Let $\sigma \subset \boxplus$ be a strand corresponding to a hyperplane in a waffle \boxplus, and let $K \leq \text{Isom}(\boxplus)$ act cocompactly on \boxplus. Then the stabilizer $K_{\sigma}$ is one of the following types:

1. $\mathbb{Z}$-type,
2. dihedral type, or
3. isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$ or $(\mathbb{Z} \times \mathbb{Z}_2) \oplus \mathbb{Z}_2$.

In the latter case the $\mathbb{Z}_2$ factor fixes $\sigma$ pointwise.

Proof. If $K_{\sigma}$ acts freely on $\sigma$, then it is of $\mathbb{Z}$-type. Let $\sigma^\pm$ be the endpoints of $\sigma \in \partial \boxplus$ and suppose that some elements of $K_{\sigma}$ fix a point $x$ of $\sigma$. Then any such elements induce isometries of the circle $\partial \boxplus$ with respect to the visual metric based at $x$ and permute $\sigma^\pm$ or the connected components of the complement $\partial \boxplus \setminus \{\sigma^\pm\}$. Because elements of $K$ are fully determined by their action on $\partial \boxplus$, it follows that $(K_{\sigma})_x \leq \mathbb{Z} \oplus \mathbb{Z}_2$, so it either has cardinality 1, 2 or 4.

Now $K_{\sigma}$ acts discretely cocompactly on $\sigma$ by isometries, therefore the quotient $K_{\sigma}/\sigma$ is either a circle or a closed segment. If the quotient is a circle, then any element that fixes some point in $\sigma$, must fix $\sigma$ pointwise, so $K_{\sigma} \approx \mathbb{Z} \oplus \mathbb{Z}_2$. Otherwise there is some $K_x \leq K_{\sigma}$ that permutes the endpoints $\sigma^\pm$. If $|K_x| = 4$ then there must also be some element $k$ that fixes $\sigma^\pm$ and therefore $\sigma$ pointwise, but permutes the connected components of $\partial \boxplus \setminus \{\sigma^\pm\}$, so that $K_{\sigma} = (\mathbb{Z} \times \mathbb{Z}_2) \oplus \mathbb{Z}_2$. If $|K_x| = 2$ then $K_{\sigma} = \mathbb{Z} \times \mathbb{Z}_2$. 

Definition 4.6 (Strand types). Let $\sigma \subset \boxplus$ be a strand and let $K \leq \text{Isom}(\boxplus)$ act cocompactly on \boxplus. We say $\sigma$ is $K$-reflective if $K_{\sigma}$ is of the form given in Lemma 4.5 item 3. Otherwise it is called $K$-non-reflective. If $w$ say $\sigma$ is $K$-dihedral if $K_{\sigma}$ is dihedral type and otherwise $\sigma$ is $K$-\$\mathbb{Z}\$-type.

4.2 The churro-waffle space of grsg-group and its metric

Having found strands we can now define the churro-waffle spaces:

Definition 4.7 (Churro-waffle spaces). Let $\Gamma$ be a clean grsg-group, a churro-waffle space, denoted $\boxplus^*(\Gamma)$, for $\Gamma$ is the space obtained by $\Gamma$-equivariantly replacing every crêpe $\boxplus$ in $\widetilde{X}_j$ by the corresponding waffle $\boxplus$ and $\Gamma_{\boxplus}$-equivariantly gluing the sides of each churro $\ast$ to the corresponding strand $\sigma$ (given in Proposition 4.1) inside each adjacent waffle $\boxplus$.

4.2.1 Metrizing churros

Let $\ast$ be a churro that is attached to a collection $\{\boxplus_i \mid 1 \leq i \leq m_\ast\}$ of waffles. We want to endow $\ast$ with a specific path metric.

Definition 4.8 (Horostrip). Consider the upper half plane model of $\mathbb{H}^2$, the hyperbolic 2-space. For $0 < y_1 < y_2$ the horostrip $\mathcal{P}(y_1, y_2)$ is the closed region bounded by the lines

$$y = y_1 \text{ and } y = y_2.$$ 

The lines in $\mathcal{P}(y_1, y_2)$ parallel to the $y$-axis are called vertical leaves. The quantity $|\ln(y_2/y_1)|$ is called the width of $\mathcal{P}(y_1, y_2)$. The boundary component contained in $\{(x, y) \mid y = y_1\}$ is called the long side.
Proposition 4.9. Consider a quadrilateral $Q$ contained in the horostrip $P(y_1, y_2)$ whose boundary is given by the union

$$\alpha \cup \ell_1 \cup \beta \cup \ell_2$$

where $\alpha, \beta \subset \partial P(y_1, y_2)$ and $\ell_1, \ell_2$ are vertical leaves. Then, with respect to the metric induced by $\mathbb{H}^2$, $\ell_1, \ell_2$ are geodesics with length

$$\left| \ln \left( \frac{\text{length}(\alpha)}{\text{length}(\beta)} \right) \right| = \text{width}(P(y_1, y_2)).$$

Proof. If $\gamma(t) = (x(t), y(t))$ is a parametric curve $\gamma : [a, b] \to \{(x, y) \in \mathbb{R}^2 | y > 0\}$ and $\gamma = \gamma([a, b])$ is a path in $\mathbb{H}^2$, then its length is given by

$$\text{length}(\gamma) = \int_a^b ds_{\mathbb{H}^2} = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt.$$

Without loss of generality $\alpha \subset \{(x, y) | y = y_1\}$ and $y_1 < y_2$. The result now follows from the paramaterizations:

$$\alpha(t) = (t, y_1); t \in [a, b]$$
$$\ell_1(t) = (b, t); t \in [y_1, y_2]$$
$$\beta(t) = (t, y_2); t \in [a, b]$$
$$\ell_2(t) = (a, t); t \in [y_1, y_2]$$

and some calculus. \qed

We also have the following:

Lemma 4.10. Two horostrips of equal width are isometric.

Because $\Gamma$ is clean, $\Gamma_\ast = \langle h_\ast \rangle$, and $h_\ast(\mathbb{H}_i) = \mathbb{H}_i$, this enables the following definition.

Definition 4.11 (Translation lengths). Let $\ast$ be attached to the waffles in $\{\mathbb{H}_i | 1 \leq i \leq m_\ast\}$ with strands $\mathbb{H}_i \cap \ast = \sigma_i$. We have $\Gamma_\ast = \langle h_\ast \rangle$ which maps each $\sigma_i$ to itself and we define the strand translation lengths:

$$\tau_{\sigma_i} = d_{\mathbb{H}_i}(x, h_\ast x), \text{ for any } x \in \sigma_i$$

where $d_{\mathbb{H}_i}$ is the standard unit edge length CAT(0) metric on $\mathbb{H}_i$. We further define the core translation length to be the average

$$\tau_{\ast} = \frac{1}{m_\ast} \sum_{i=1}^{m_\ast} \tau_{\sigma_i}.$$

Definition 4.12 (The churro metric). Let $\ast$ be as described above, then the flap $F_i$ connecting the core $\ast$ to the waffle $\mathbb{H}_i$ can be metrized as:

1. A horostrip of width

$$\left| \ln \left( \frac{\tau_{\ast}}{\tau_{\sigma_i}} \right) \right|$$

with the long side identified with $\ast$ if $\tau_{\ast} > \tau_{\sigma_i}$, or the long side identified with $\sigma_i \subset \mathbb{H}_i$ if $\tau_{\ast} < \tau_{\sigma_i}$; provided $\tau_{\ast} \neq \tau_{\sigma_i}$.
2. The Euclidean unit strip \( F_\mathbb{E} = \{(x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1 \} \) equipped with the Euclidean \( \mathbb{E}^2 \)-metric, if \( \tau_\mathbb{E} = \tau_{\sigma_i} \).

We then identify the sides of all the \( \ast \) sides of the flaps so that the lengths of any arc \( \alpha \) contained in \( \ast \) coincide in \( (\mathbb{H}^2 \text{ or } \mathbb{E}^2) \)-metric on each attached flap.

**Lemma 4.13.** Let \( \{\Box_i \mid 1 \leq i \leq m_\Box\} \) be the waffles attached to a churro \( \ast \), then the metric on \( \ast \) given in Definition 4.12 is determined by the collection of clutching ratios of \( \ast \):

\[
r_{ij} = \frac{\tau_{\sigma_i}}{\tau_{\sigma_j}}; 1 \leq i, j \leq m_\ast.
\]

**Proof.** Once \( r_{ij} \) is given, all the \( \tau_{\sigma_i} \) can be determined as \( \mu \alpha \) up to a common unknown scaling factor \( \mu \), and therefore the average \( \overline{\tau}_\ast \) also be determined as \( \mu \overline{\alpha} \) up to the same scaling factor \( \mu \). The metric on \( \ast \) is given by the ratios \( \frac{\tau_\ast}{\tau_{\sigma_i}} = \frac{\alpha}{\alpha_i} \), which do not depend on \( \mu \). \qed

### 4.2.2 The path metric on a churro-waffle space

Before continuing we note that churros, with the metric of Definition 4.12, may not have isometrically embedded boundaries in the following sense: the path metric restricted to the boundary (i.e. given by paths lying in the boundary) may not agree with the induced subspace metric; the reason being that horospheres are not convex.

**Definition 4.14** (Global path metric). Let \( \boxplus^\ast(\Gamma) \) be the churro-waffle space for a clean group \( \Gamma \) where each waffle \( \Box \) is equipped with the standard piecewise Euclidean path metric \( ds_\Box \) and each churro \( \ast \) is equipped with the path metric \( ds_\ast \) given in Definition 4.12. If a churro \( \ast \) is adjacent to a waffle \( \Box \) with \( \ast \cap \Box = \sigma \) we identify the component \( \gamma \subset \partial \ast \) with \( \sigma \subset \Box \) so that for any arc \( \gamma \subset \sigma \) we have

\[
\int_\gamma ds_\Box = \int_\gamma ds_\ast.
\]

This induces the **global path metric** \( d \) on \( \boxplus^\ast(\Gamma) \).

**Proposition 4.15.** \( \Gamma \) can be made to act by isometries on \( \boxplus^\ast(\Gamma) \) with respect to the global path metric \( d \).

**Proof.** The only fact we actually need to show is that for a churro \( \ast \), the stabilizer \( \Gamma_\ast = \langle h_\ast \rangle \) can act by isometries. Indeed, we already know that for each waffle \( \Box \) the path metric is \( \Gamma_\Box \)-equivariant, if this is also the case for every churro then, by construction and definition, \( \Gamma \) will carry geodesics to geodesics.

Let \( \{\Box_i \mid 1 \leq i \leq m_\Box\} \) be the set of waffles attached to the churro \( \ast \) and let \( x \in \ast \) be some point in the churro core. Let \( x_i \in \Box_i \) be the point in connected to \( x \) via a vertical leaf in the flap \( F_i \) connecting \( \ast \) to \( \sigma_i \subset \Box_i \).

Let \( y_i \) be the point in \( \ast \) connected to \( h_\ast \cdot x_i \) by a vertical leaf in the flap \( F_i \). We note that with respect to the path metric defined in Definition 4.14 the length of the arc \( \alpha_i \subset \sigma_i \) connecting \( x_i \) to \( h_\ast \cdot x_i \) is exactly \( \tau_{\sigma_i} \) and by Definition 4.12 and Proposition 4.9, with respect to the path metric, in each flap we have that the length of the arc \( \beta \) between \( x \) and \( y_i \) is \( \tau_\ast \). It follows that restricted to each flap \( F_i \) the action of \( h_\ast \) can be realized as the restriction of a parabolic isometry of the upper half plane model of \( \mathbb{H}^2 \) of the type:

\[
(x, y) \mapsto (x + h, y); \text{ for some } h \in \mathbb{R}.
\]
which agree on
\[ \mathfrak{m} \mathfrak{c}_1 \mathfrak{F}_i. \]

It follows that the length of any parameterized curve can be in \( \mathbb{H}^\#(\Gamma) \) can be computed using the local path metrics in the churro and waffle pieces and that such lengths are preserved under actions by \( \Gamma \).

### 4.2.3 The cell-complex structure on a churro-waffle space

**Definition 4.16** (Strand subdivision). Let \( \Box \) be a waffle, then a **strand subdivision** is a combinatorial subdivision of the cube complex \( \Box \) so that each strand lies inside the 1-skeleton of the subdivided waffle. The **waffle subdivision** of a strand is the simplicial structure on a strand induced by the strand subdivision of the ambient waffle.

**Definition 4.17** (Flap subdivision, core and strand vertices). Let a waffle \( \Box \) be attached to a churro \( \mathfrak{m} \) and if \( \hat{\sigma} \) is a strand subdivision of \( \Box \) and \( \hat{\sigma} \sigma \) is the corresponding waffle subdivision of the strand \( \sigma = \Box \cap \mathfrak{m} \), then we subdivide the flap \( F \) connecting \( \sigma \) and the churro core \( \mathfrak{m} \) so that each vertical leaf \( \ell \) containing a vertex of \( \hat{\sigma} \) becomes a 1-cell called a **subdivision edge**. Extremities of these 1-cells in \( \mathfrak{m} \) are called **core vertices** if they lie in the churro core, and **strand vertices** if they lie in a strand.

**Definition 4.18** (Generalized Dehn twists). Consider the map \( \alpha : Z \to \mathfrak{m} \) identifying a component of the boundary \( Z \subset \partial F \subset \mathfrak{m} \) of a flap of churro \( \mathfrak{m} \) to the churro core \( \mathfrak{m} \). A **generalized Dehn twist** is the result of reattaching \( F \) to \( \mathfrak{m} \) via a \( \Gamma \)-equivariant post composition with a translation:

\[
(t \mapsto t + \epsilon) \circ \alpha; \epsilon \in \mathbb{R}.
\]

An **equivariant generalized Dehn twist** of a churro-waffle space \( \mathbb{H}^*(\Xi) \) is the result of \( \text{Aut}(\mathbb{H}^*(\Xi)) \)-equivariantly redefining the attaching maps of flaps to churro cores via generalized Dehn twists. If \( D \) is a finite set of equivariant generalized Dehn twists, then denote the space obtained by successively applying these as

\[ \mathbb{H}^*(\Xi)^D. \]

**Lemma 4.19.** There is a \( \Gamma \)-equivariant quasi-isometry between \( \mathbb{H}^*(\Gamma) \) and the result of finitely many equivariant generalized Dehn twist \( D \), furthermore \( \Gamma \) still acts on the resulting space \( \mathbb{H}^*(\Gamma)^D \) by automorphisms. Also, given an equivariant strand subdivision of each waffle in \( \mathbb{H}^*(\Gamma) \), equivariant generalized Dehn twists can only produce a finite number of distinct (up to isomorphism) flap subdivisions that minimize the number of \( \Gamma \)-orbits of core vertices.

**Proof.** \( \Gamma \) acts freely on \( \mathbb{H}^*(\Gamma) \) and is in bijective correspondence with homotopy classes of paths from some basepoint \( b \in \mathbb{H}^*(\Gamma) \) not lying in a churro core and some other point \( \gamma \cdot b \in \mathbb{H}^*(\Gamma) \) where \( \gamma \in \Gamma \). Let \( \rho \) be a path connecting \( b \) and \( \gamma \cdot b \). We can subdivide \( \rho \) into

\[
\rho = \rho_1 \ast \cdots \ast \rho_k
\]

where each \( \rho_i \) is maximal with respect to containment with the property that the interior of \( \rho_i \) is disjoint from the set of churro cores. Let \( \delta \) be a generalized
equivariant Dehn twist. Each subpath of $\rho$ given in (3) has an image in the resulting space, the push-forward of $\rho$ is the concatenation
\[
\delta^*(\rho) = \rho_1 \ast \mu_1 \ast \cdots \ast \mu_{k-1} \ast \rho_k,
\]
where the $\mu_i$ are shortest paths contained in churro cores connecting the endpoints of the $\rho_i$ to $\rho_{i+1}$. It is a consequence of the definition of fundamental groups and covering spaces that in this way $\delta^*$ gives an action of $\Gamma$ by deck transformations, and by construction the path metric remains $\Gamma$-equivariant.

Now note that the number of $\Gamma$ orbits of core vertices is minimized whenever the endpoints of different subdivision edges in different flaps coincide in $\bar{\mathcal{X}}$. Since $\Gamma$ acts cocompactly on $\mathbb{A}^\mathbb{R}(\Gamma)$, there are only finitely many possible cellular decompositions of churros that can arise from generalized equivariant Dehn twists, up to automorphism.

**Lemma 4.20** (Assembly Lemma). Let $H, \Gamma$ be clean grsg-groups and suppose there is a quasi-isometry $\phi : \mathbb{A}^\mathbb{R}(H) \to \mathbb{A}^\mathbb{R}(\Gamma)$ that satisfies the following:

1. $\phi$ coarsely maps churros to churros and waffles to waffles, which induces an graph isomorphism of the underlying tree structures of $\mathbb{A}^\mathbb{R}(H)$ and $\mathbb{A}^\mathbb{R}(\Gamma)$.
2. The restriction of $\phi$ to a waffle $\mathbb{A} \subset \mathbb{A}^\mathbb{R}(H)$ induces an isometry $\circ\mathbb{A}$ of $\mathbb{A}$ with respect to the intrinsic metric.
3. $\phi$ preserves the clutching ratios of a churro, in that
\[
\frac{\tau_{\sigma_1}}{\tau_{\sigma_2}} = \frac{\tau_{\sigma_1'}}{\tau_{\sigma_2'}}.
\]

for any pair $\mathbb{A}_1, \mathbb{A}_2 \subset \mathbb{A}^\mathbb{R}(H)$ of waffles attached to a churro $\ast$ along strands $\sigma_1, \sigma_2$ respectively, where $\sigma_1', \sigma_2'$ are the respective strands along which $\mathbb{A}_1(\mathbb{A}_1), \mathbb{A}_2(\mathbb{A}_2)$ are attached to their shared adjacent churro (which is a bounded distance from $\phi(\ast)$).

Then there are families of equivariant generalized Dehn twists $D, E$ such that $\mathbb{A}^\mathbb{R}(H)^D$ and $\mathbb{A}^\mathbb{R}(\Gamma)^E$ are isomorphic. Thus both $H$ and $\Gamma$ can be made to act freely by automorphisms on the common churro-waffle space $\mathbb{A}^\mathbb{R}(H)^D \cong \mathbb{A}^\mathbb{R}(\Gamma)^E$.

**Proof.** If we metrize $\mathbb{A}^\mathbb{R}(H)$ and $\mathbb{A}^\mathbb{R}(\Gamma)$ as in Definition 4.14 then by Lemma 4.13 and equation (4), the local metrics defined on the churros make $\phi$-correspondent churros isometric. Note that the strands in the waffles only depend on the isometry classes of the waffles, so we can pick strand subdivisions for every waffle that is $\text{Isom} (\mathbb{A})$-equivariant, and pick such a subdivision for every isometry class of waffles occurring in $\mathbb{A}^\mathbb{R}(H)$ and $\mathbb{A}^\mathbb{R}(\Gamma)$. It now follows by Lemma 4.19, that after performing finitely many equivariant generalized Dehn twists, and redefining the flap subdivision, we can make $\mathbb{A}^\mathbb{R}(H)$ and $\mathbb{A}^\mathbb{R}(\Gamma)$ isometric as metric spaces and isomorphic as cell complexes.

**Remark 4.21.** Any quasi-isometry $\phi : H \to \Gamma$ between clean grsg-groups induces a quasi-isometry between $\mathbb{A}^\mathbb{R}(H)$ and $\mathbb{A}^\mathbb{R}(\Gamma)$ satisfying 1. and 2. of Lemma 4.20. Since $\mathbb{A}^\mathbb{R}(H)$ and $\mathbb{A}^\mathbb{R}(\Gamma)$ are quasi-isometric to $\mathbb{X}_K$ and $\mathbb{X}_J$, respectively, 1. follows from Proposition 2.8. Proposition 3.15 gives 2.

We will show that 3. of Lemma 4.20 is also satisfied in Section 4.4.
4.3 Quasi-isometry is close to combinatorialization

Throughout this section let \( \mathbb{H} \) be the waffle constructed from a crêpe \( \mathbb{H}^2, \mathcal{L} \), as in Section 3.3.

**Definition 4.22.** Let \( \alpha, \beta \) be geodesics then \( \alpha \cap_D \beta \), the \( D \)-coarse intersection of \( \alpha \) and \( \beta \), is the set of points that are simultaneously at distance \( D \) from \( \alpha \) and \( \beta \), i.e.

\[
\alpha \cap_D \beta = \{ x \mid d(x, \alpha), d(x, \beta) \leq D \} = N_D(\alpha) \cap N_D(\beta).
\]

**Lemma 4.23.** If \( \alpha, \beta \) are bi-infinite geodesics with distinct endpoints in \( \partial X \), then \( \alpha \cap_D \beta \) has finite diameter \( M(\alpha, \beta, D) \).

**Proof.** If \( \alpha \cap_D \beta \) has infinite diameter, then \( \alpha, \beta \) must fellow travel to the same point in \( \partial X \), contradicting that they have distinct endpoints. \( \square \)

**Lemma 4.24.** Let \( \sigma_1, \sigma_2 \) be strands in \( \mathbb{H} \), then for any \( D > 0 \) there is an \( N(D) \) such that

\[
diam(\sigma_1 \cap_D \sigma_2) \leq N(D),
\]

where the diameter of the empty set is 0.

**Proof.** Denote \( G = Isom(\mathbb{H}) \). Let \( A \subseteq \mathbb{H} \) be a compact set such that \( GA = \mathbb{H} \) (i.e. \( A \) is a coarse fundamental domain) and let \( \hat{A} = N_{2D}(A) \) be the \( 2D \) neighbourhood of \( A \).

**Claim:** Any pair \( (\alpha, \beta) \) of strands of \( \mathbb{H} \) such that \( \alpha \cap_D \beta \neq \emptyset \) is in the same \( G \)-orbit as a pair of geodesics \( (g\alpha, g\beta) \), which both intersect \( \hat{A} \). Indeed, let \( (\alpha, \beta) \) be such a pair, then there is some \( x \in \mathbb{H} \) such that \( d(x, \alpha), d(x, \beta) \leq D \).

Let \( g \in G \) be such that \( gx \in \mathbb{H}_1 \). Then \( g\alpha \) and \( g\beta \) must intersect \( \hat{A} \) in nonempty sets. The claim follows.

We may therefore take

\[
N(D) = \max_{(\alpha, \beta)} M(\alpha, \beta, D)
\]

where \( (\alpha, \beta) \) ranges over the pairs of strands that intersect \( \hat{A} \), and \( M(\alpha, \beta, D) \) is given in Lemma 4.23. This must be finite since there are only finitely many strands that intersect \( \hat{A} \) . \( \square \)

**Proposition 4.25** (Pattern preserving is close to isometry). Let \( \phi : \mathbb{H}_1 \to \mathbb{H}_2 \) be a \((K, C)\)-quasi-isometry which quasimatches the set of strands in \( \mathbb{H}_1 \) to the set of strands in \( \mathbb{H}_2 \), and let \( \hat{\phi} \) be the combinatorialization of \( \phi \). Then there is some \( D = D(K, C, \mathbb{H}_1) \) such that

\[
\sup_{x \in \mathbb{H}_1} d(\phi(x), \hat{\phi}(x)) \leq D.
\]

**Proof.** Every \( x \in \mathbb{H}_1 \) is contained in some cube \( \square \). Let \( D_1 \) be the largest cube diameter in \( \mathbb{H}_1 \). Then there are strands \( \sigma_1, \sigma_2 \) corresponding to crossing hyperplanes going through \( \square \) and we have \( x \in \sigma_1 \cap_{D_1} \sigma_2 \). By Lemma 4.24, the diameter of \( \sigma_1 \cap_{D_1} \sigma_2 \) is bounded by some \( D_2 \).

\( \phi \) sends \( \sigma_1, \sigma_2 \) to \((K, C)\)-quasigeodesics which fellow-travel to strands \( \sigma_1', \sigma_2' \in \mathbb{H}_2 \). By the canonicity of strands, \( \hat{\phi} \) sends the strands \( \sigma_1, \sigma_2 \) to \( \sigma_1', \sigma_2' \) respectively. It follows that \( \hat{\phi}(x) \in \sigma_1' \cap_{D_2} \sigma_2' \), which is also of diameter bounded by \( D_2 \). Now by [BH11, Theorem III.H.1.7] the quasigeodesics \( h(\sigma_1), h(\sigma_2) \) are contained in \( D_3 \) neighbourhoods of \( \sigma_1', \sigma_2' \). We have for \( i = 1, 2 \),

\[
d(\phi(x), \phi(\sigma_i')) \leq KD_1 + C = D_4.
\]

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It follows that for \( i = 1, 2 \),

\[
d(\phi(x), \sigma'_i) \leq D_4 + D_5 = D_5 \Rightarrow \phi(x) \in \sigma'_1 \cap D_5, \sigma'_2.
\]

Since

\[
\phi(x) \in \sigma'_1 \cap D_5, \sigma'_2 \subset \sigma'_1 \cap D_5, \sigma'_2 \ni \phi(x)
\]

and by Lemma 4.24 the diameter of \( \sigma'_1 \cap D_5, \sigma'_2 \) is at most some number \( N(D_5) \).

This gives the desired bound. \( \square \)

### 4.4 Quasi-isometry preserves clutching ratios

The following is well known, but we couldn’t find a precise reference.

**Proposition 4.26.** Let \( X \) be a \( \delta \)-thin hyperbolic space, and let \( Y \subset X \) be a \( R \)-quasiconvex subset. Then any two closest point projections \( \pi_Y, \pi'_Y \) that send an element \( x \) a closest element in \( Y \) differ by at most \( 2(\delta + R) \), i.e. for all \( x \in X \)

\[
d(\pi_Y(x), \pi'_Y(x)) \leq 2(\delta + R).
\]

**Proof.** Consider Figure 4, obtained by mapping the geodesic triangle to a tripod (see [ABC*91, Definition 1.5]). Because \( \pi_Y(x), \pi'_Y(x) \) are both closest point projections we have that the concatenated paths \( \alpha_1 * \alpha_2 \) and \( \beta_1 * \beta_2 \) have the same length, and \( \alpha_1, \beta_1 \) have the same length. Let \( \gamma_1 * \gamma_2 \) be a geodesic joining \( \pi_Y(x), \pi'_Y(x) \). By construction \( |\alpha_2| = |\gamma_1| = |\gamma_2| = |\beta_2| \). Because \( Y \) is \( R \)-quasiconvex, there is a point \( z \in Y \) a distance \( R \) from the point in the middle of the geodesic \( \gamma_1 * \gamma_2 \), and we have \( d(x, z) \leq |\alpha_1| + \delta + R \). Since \( \pi_Y \) and \( \pi'_Y \) are nearest point projections we have

\[
|\alpha_1| + \delta + R \geq |\alpha_1| + |\alpha_2| \Rightarrow \delta + R \geq |\alpha_2| = |\gamma_1|.
\]

It follows that the \( d(\pi_Y(x), \pi'_Y(x)) = |\gamma_1 * \gamma_2| \) is at most \( 2(\delta + R) \). \( \square \)

We now show that quasi-isometries of clean grsg-groups satisfy 3. of Lemma 4.20.

**Proposition 4.27.** Let \( H \) and \( \Gamma \) be clean grsg-groups, and \( \phi : \bigoplus^*(H) \to \bigoplus^*(\Gamma) \) be a quasi-isometry. Then

\[
\frac{\tau_{\sigma_i}}{\tau_{\sigma_j}} = \frac{\tau_{\sigma_i'}}{\tau_{\sigma_j'}}.
\]
for any $\varpi_i, \varpi_j \subset \varpi^*(H)$ attached to a churro $\ast$ along strands $\sigma_i, \sigma_j$ respectively, where $\sigma_i', \sigma_j'$ are the respective strands along which $\varpi_i(\varpi_i), \varpi_j(\varpi_j)$ are attached to their shared adjacent churro $\ast'$.

**Proof.** For $i = 1, 2$ let $\varpi_i = \varpi_i(\varpi_i)$ be the combinatorializations of $\varphi$ restricted to $\varpi_i$, $\varpi_i' = \varpi_i(\varpi_i)$. Notice that $\varpi_i$ sends $\sigma_i$ to $\sigma_i'$, for $i = 1, 2$, as a consequence of Proposition 4.25. Let $a = \tau_{\sigma_i}, b = \tau_{\sigma_j}, a' = \tau_{\sigma_i'}, b' = \tau_{\sigma_j'}$. Define the involution $\omega : \sigma_1 \cup \sigma_2 \rightarrow \sigma_1 \cup \sigma_2$, where the geodesic between a point $P \in \sigma_1 \cup \sigma_2$ and $\omega(P)$ is a vertical leaf in $\ast$ (as in Definition 4.8). Similarly define $\omega' : \sigma_1' \cup \sigma_2' \rightarrow \sigma_1' \cup \sigma_2'$.

Fix a point $x$ on $\sigma_1$, and let $x' = \varpi_1(x)$ on $\sigma_1'$, $y = \omega(x)$ on $\sigma_2$ and $y' = \omega'(x')$ on $\sigma_2'$. Since $H$ is clean, $H_* = \langle h_* \rangle$, and let $h = h^*_r$ for some $r$ which is sufficiently large so that for any coarse inverse $\phi^{-1} : \varpi^*(\Gamma) \rightarrow \varpi^*(H)$, the points $\phi^{-1}(\phi(h^nx))$ are distinct for all integers $n$ (as are the points $\phi^{-1}(\phi(h^ny))$). For any coarse inverse $\phi^{-1} : \varpi^*(\Gamma) \rightarrow \varpi^*(H)$, pattern matching projections $\pi_i$ from neighbourhoods of $\varpi_i$ to $\varpi_i$ can be arranged (by perturbing a bounded amount), where $\pi_i\phi^{-1} : \varpi_i' \rightarrow \varpi_i$ and $\pi_1\phi^{-1}$, $\pi_2\phi^{-1}$ fix all $h^nx, h^ny$ respectively.

Let $g = g_{\varphi'} \ast \varpi_2'$ generate $\langle g \rangle = \Gamma_{\varphi'}$. There are pattern matching (bounded) projections $\pi_i' : \varphi(\varpi_i) \rightarrow \varpi_i$ with $\pi_i' \varphi(\sigma_i) = \{g^n x' \}$ and $\pi_2' \varphi(\sigma_2) = \{g^n y' \}$. Let $\pi_{\varphi'} : \mathcal{N}(\varpi_i') \rightarrow \mathcal{N}(\varpi_i')$, where $\mathcal{N}(\varpi_i')$ is an appropriate neighbourhood of $\varpi_i'$ containing $\phi(\varpi_i)$, be a bounded map defined so that it is the identity on $\mathcal{N}(\varpi_i') \setminus \sigma_{\varphi'}$ and sends $\sigma_{\varphi'}$ to $\varphi(\sigma_1)$ with $\pi_{\varphi'}^{-1}(x') = \phi(x)$.

For each positive integer $n$, let $\psi_1^{(n)} = \pi_{\varphi'}\phi h^n \pi_{\varphi'}^{-1} : \varpi_1 \rightarrow \mathcal{N}(\varpi_1')$. The restriction of $\psi_1^{(n)}$ to $\varpi_1'$ is a quasi-isometry of $\varpi_1'$ which quasimatches $\sigma_1$ to itself, and now let $h_1^{(n)}$ be the combinatorialization of $\psi_1^{(n)} |_{\varpi_1'}$. Since each $h_1^{(n)}$ stabilizes $\sigma_{\varphi'}$ (preserving orientation and fixing no point), there are positive integers $m = m_n$ for which $d_{\varpi_1'}(g^m x', h_1^{(n)}x') \leq \frac{a'}{2}$. Notice that since $d_{\varpi_1'}(\phi(x), \phi(h^nx)) \rightarrow \infty$, $d(x', g^m x') \rightarrow \infty$, and so $m_n \rightarrow \infty$ as $n$ grows.

Now $d_{\varpi_1'}(\psi_1^{(n)}(x'), \phi(h^nx))$ is uniformly bounded for all $n$, since $\psi_1^{(n)}(x') = \psi_1^{(n)}(\phi(x))$ and $\psi_1^{(n)}(\phi(x)) = \pi_{\varphi'}\phi(h^nx)$, and so, by Proposition 4.25, the distance between $\phi(h^nx)$ and $h_1^{(n)}x'$ must also be uniformly bounded for all $n$.

Let $Q_n = g^m x'$. By construction, $d_{\varpi_1'}(y', \phi(Q_n)) = m_n b'$. If $\frac{a'}{b'} \neq \frac{a}{b}$, without loss of generality suppose $\frac{a}{b} > \frac{a'}{b'}$ and let $P_n$ be the point on $\sigma_{\varphi'}$ between $y'$ and $\omega'(Q_n)$ where $d_{\varpi_1'}(y', P_n) = m_n b'a'_2$. Now $\phi(h^ny)$ is bounded distance to $\phi(h^n x)$, and so there’s a uniform bound, independent of $n$, from $\phi(h^ny)$ to $\omega'(Q_n)$. Furthermore, $\varpi_1^{-1}(Q_n)$ is a bounded
distance from \( h^n x \) and \( \omega \left( \hat{\phi}_1^{-1}(Q_n) \right) \) is a bounded distance from \( \hat{\phi}_2^{-1}(P_n) \) (since \( y \) is bounded distance from \( \hat{\phi}_2^{-1}(y') \)). But this implies that \( h^n y \) is a bounded distance from \( \hat{\phi}_2^{-1}(P_n) \), and so \( \phi(h^n y) \) is a bounded distance from both \( P_n \) and \( \omega(Q_n) \), but \( d_{\hat{\phi}_2^{-1}}(P_n, \omega'(Q_n)) = m_n \left( \frac{a}{b} - \frac{a'}{b'} \right) \). However \( m_n \to \infty \) as \( n \to \infty \), so \( \frac{a}{b} = \frac{a'}{b'} \).

So by Lemma 4.20 and Proposition 2.15, we have the following.

**Corollary 4.28.** If \( \phi : H \to \Gamma \) is a quasi-isometry of grsg.-groups, then there are families of equivariant Dehn twists \( D, E \) such that \( \Delta \) and \( \Gamma \) both virtually act by automorphism on the metric cell complex \( \boxplus^*(\Delta)^D \cong \boxplus^*(\Gamma)^E \), where \( \bar{\Delta}, \bar{\Gamma} \) are clean finite index subgroups of \( \Delta, \Gamma \) respectively.

## 5 Commensurability from discrete groupings

For this section denote by \( \boxplus^*(\Gamma) \) the churro-waffle space associated to \( \Gamma \). Let \( G = \text{Isom}(\boxplus^*(\Gamma)) \) denote the full isometry group, which by construction coincides with the automorphism group of \( \boxplus^*(\Gamma) \) as a cell complex. \( \boxplus^*(\Gamma) \) has the underlying structure of a tree of spaces, dual to the JSJ splitting of \( \Gamma \), denote this tree by \( T \), recall that this exactly the tree \( T_3 \) constructed in Proposition 2.8 item 3. \( T \) is bipartite with vertices either corresponding to waffles (i.e. cubings of crepes) or to churros. The edges of \( T \) correspond to the intersections of churros and waffles along strands. It follows that vertex groups are waffle and churro stabilizers and edge groups are the stabilizers of churro flaps. If \( v \) is a vertex of \( T \) then the vertex group \( G_v \), in general, is a non-discrete totally disconnected locally compact group. On the other hand the automorphism groups of churros and waffles are discrete. We therefore denote by \( G_v \) the image of \( G_v \) in the discrete isometry group obtained by forgetting the action outside of those subspaces.

**Definition 5.1 (see [BK90]).** Let \( X \to Y \) be a topological quotient map. A *grouping* is the action of a group \( H \circlearrowright X \) such that \( Y = H \backslash X \). A grouping is *discrete* if \( H \) acts as a discrete group of isometries.

**Definition 5.2.** A graph of actions \( Y \circlearrowright \) consists of a graph of groups \( Y \), with underlying graph \( Y \), along with the following additional data.

1. For each vertex \( v \in \text{Verts}(Y) \), there is a metric cell complex \( p_v \) equipped with an action \( Y_v \circlearrowright p_v \).

2. For each edge \( e \in \text{Edges}(Y) \) connecting vertices \( u, v \in \text{Verts}(Y) \) a pair of subspaces \( S^e_u \subset p_v, S^e_v \subset p_y \) whose stabilizers are the images of \( Y_e \) in the respective vertex groups an identification isometry \( S^e_u \sim S^e_v \).

The tree of spaces obtained by attaching copies of the pieces along identification isometries, following the Bass-Serre covering tree is called the *covering complex* associated to \( Y \circlearrowright \).

**Definition 5.3.** A *piece* \( p \subset \boxplus^*(\Gamma) \) is a subset consisting either of an entire churro or an entire waffle.
The quotient $G \setminus \mathbb{R}^\Gamma$ decomposes as a graph of spaces with underlying graph $G \setminus T = X$. The vertex spaces either correspond to quotients of waffles or churros and then we take the edge spaces to be the images of strands. This naturally gives rise to a graph of actions structure $X^\Gamma$ with underlying group $G$. We will first give specific constructions of discrete groupings of churros which, later in Section 6, will be used to construct a discrete grouping of $\mathbb{R}^\Gamma(\Gamma) \to G \setminus \mathbb{R}^\Gamma(\Gamma)$.

5.1 The basic grouping of a churro

Recall that a churro $\ast$ consists of a core $\ast$ to which flaps $F_i$ are attached. The side of a flap $s_F$ that is not attached to a churro core is called a flap side. This is depicted in Figure 5. We note that churros are equipped with a cocompact cell complex structure, making their automorphism group discrete.

We will now construct a discrete grouping of $\ast \to G_\ast \setminus \ast$. We note that although the automorphism group $\text{Aut}(\ast)$ is discrete the stabilizer $G_\ast$ is in general not discrete, due to the fact that the natural representation $G_\ast \to \text{Aut}(\ast)$ may have a very large kernel.

**Definition 5.4.** Denote by $G^0_\ast \leq G_\ast$ the subgroup of elements that act trivially on the churro core $\ast$. An element $g \in G^0_\ast$ is called purely flapping.

**Definition 5.5 (Flap families).** We say that two flaps $F_1, F_2 \subset \ast$ in a churro are in the same flap family if there is some purely flapping $g \in G^0_\ast$ such that $g \cdot F_1 = F_2$. We denote by $\mathcal{F}_F$ the set of flaps in the flap family of $F$.

**Definition 5.6 (Churro class).** We say two flaps $F_1, F_2 \subset \ast$ are in the same churro class, $\text{Class}_\ast(F_1)$, if there is some $g \in G_\ast$ such that $g \cdot F_1 = F_2$. We say that two flap families $\mathcal{F}_1, \mathcal{F}_2$ are in the same churro class $\text{Class}_\ast(\mathcal{F}_1)$ if there is some $g \in G_\ast$ such that $g \cdot \mathcal{F}_1 = \mathcal{F}_2$.

In this way $G_\ast$ acts by permutations on the collection of flap families. The following is obvious.

**Lemma 5.7.** If $g \in G_\ast$ acts non-trivially on the collection of flap families then it acts non-trivially on $\ast$.

We will now construct a finitely generated group $B_\ast$ equipped with an action $B_\ast \acts \ast$ that gives a discrete grouping of $\ast \to G_\ast \setminus \ast$. Informally, it will be a direct product of a group, either $\mathbb{Z}$ or $\mathbb{Z} \rtimes \mathbb{Z}_2$, acting on the churro core $\ast$ and some finite groups acting transitively on flap families. For reasons that will only become fully apparent in Section 6.3 it will be advantageous to use a specific class of finite groups.

![Figure 5: The anatomy of a churro](image-url)
Definition 5.8 (Low power abelian groups). Let \( n \) be a positive integer and let \( n = p_1^{e_1} \cdots p_k^{e_k} \) be its prime decomposition. The low power abelian group of order \( n \) is the group
\[
H(n) = \bigoplus_{i=1}^{k} \left( \bigoplus_{j=1}^{e_i} C_{p_i} \right),
\]
where \( C_{p_i} \) is the cyclic group of order \( p_i \).

Lemma 5.9. Low power abelian groups satisfy the following properties:

- \( H(n) \) acts freely and transitively on any set of order \( n \).
- The class of low power abelian groups is closed under direct products and passing to subgroups.
- Within the class of low power abelian groups, isomorphism is determined by cardinality.

Proof. The first assertion is a consequence of Cayley’s Theorem, the second assertion follows from the definition, the third assertion is an immediate consequence of the classification of finitely generated abelian groups and the prime decomposition theorem.

5.1.1 The construction of the basic grouping of a churro

Denote by \( G_{\#} = Z_{\#} \leq \text{Aut}(\#) \) the image of \( G_{\#} \) in a discrete group of isometries of the real line. \( Z_{\#} \) is either isomorphic to \( \mathbb{Z} \) in which case it is said to be of \( \mathbb{Z} \)-type, otherwise \( Z_{\#} \) is infinite dihedral and said to be of \( (\mathbb{Z} \times \mathbb{Z}_2) \)-type. Let \( \{F_1, \ldots, F_k\} \) be a minimal set of representatives of \( G_{\#} \) orbits of the set of flaps attached to \( \# \).

For each \( F_i \) we will construct a \( Z_{\#} \)-orbit. By construction there is a well-defined action of \( Z_{\#} \) on \( \text{Class}_{\#}(\mathcal{F}_{F_i}) \), the churro class of a flap family. Denote by \( \rho_{F_i} : Z_{\#} \to \text{Perm}(\text{Class}_{\#}(\mathcal{F}_{F_i})) \) the corresponding permutation representation, and denote the images \( \rho_{F_i}(g) = \bar{g} \). We can now express \( G_{\#} \cdot F_i \) as a disjoint union of flap families
\[
G_{\#} \cdot F_i = \bigsqcup_{\bar{g} \in \rho_{F_i}(Z_{\#})} \bar{g} \cdot \mathcal{F}_{F_i}
\]
and for each \( \bar{g} \in \rho_{F_i}(Z_{\#}) \) we chose chose an element \( F_i^{\bar{g}} \in \bar{g} \cdot \mathcal{F}_{F_i} \) and define the action
\[
g \cdot F_i = F_i^{\bar{g}},
\]
which is well defined since \( \rho_{F_i} \) is a homomorphism. Now if \( g \cdot F_i = F_i \), then \( g \) extends naturally from an isometry of \( \# \) to an isometry of the entire flap \( F_i \), which is either a translation if \( g \) has infinite order, or a reflection if \( g \) has order 2. In this way we have constructed an action by automorphisms
\[
Z_{\#} \cup \left( \# \cup \left( \bigcup_{g \in Z_{\#}} g \cdot F_i \right) \right).
\]

Now suppose the \( F_i \)’s flap family has \( |\mathcal{F}_{F_i}| = n_i \) elements then let \( H(n_i) \) denote the low power abelian group of cardinality \( n_i \). For each flap family
\[
g \cdot \mathcal{F}_{F_i} = \mathcal{F}_{F_i}^{\bar{g}}; \bar{g} \in Z_{\#}
\]
pick a transitive action $H(n_i) \triangleleft \mathcal{F}_{F_i}^g$, this naturally extends to an action

$$H(n_i) \triangleleft \left( \star \cup \left( \bigcup_{F \in \mathcal{F}_{F_i}^g} F \right) \right)$$

(6)

that fixes $\star$ pointwise. We can now combine these action to get an action, defined as a map for each element in $Z_{\star} \oplus H(n_i)$,

$$Z_{\star} \oplus H(n_i) \triangleleft \left( \star \cup \left( \bigcup_{g \in G_{\star}} g \cdot F_i \right) \right)$$

$$(z, h)|_{F_i}: F_i \rightarrow h(z(F_i)) = h(F_i^z),$$

where the elements $z \in Z_{\star}$ and $h \in H(n_i)$ are viewed as maps given by the actions (5) and (6). Now for each representative $F_j \in \{F_1, \ldots, F_k\}$ we take a copy $H(\mathcal{F}_{F_k}) \approx H(n_k)$ and repeat this construction to get an action on the churro

$$Z_{\star} \oplus \left( \bigoplus_{i=1}^{k} H(\mathcal{F}_{F_k}) \right) \circ \star,$$

(7)

where a direct summand $H(\mathcal{F}_{F_k})$ permutes the flaps the flap families $g \cdot \mathcal{F}_{F_k}, g \in Z_{\star}$ and fixes the flaps in the other flap families pointwise.

**Definition 5.10** (Basic grouping of a churro). The group

$$B_{\star} = Z_{\star} \oplus \left( \bigoplus_{i=1}^{k} H(n_k) \right)$$

given in (7) is called the **basic churro group for $G_{\star} \cup \star$** and the action (7) is called the **basic discrete grouping of $\star \rightarrow G_{\star} \backslash \star$**. The direct summand $Z_{\star}$ is called the **core factor** and the direct summand $\left( \bigoplus_{i=1}^{k} H(n_k) \right)$ is called the **flap factor**.

**Lemma 5.11.** $B_{\star} \cup \star$ is a discrete grouping of $\star \rightarrow G_{\star} \backslash \star$.

*Proof.* By construction we have ensured that $\star$ has the same orbits via the actions of $B_{\star}$ and $G_{\star}$. \qed

Finally, we record some terminology that will be useful later.

**Definition 5.12** (Flap group, core stabilizer). Let $F \subset \star$ be a flap. The direct summand $H(\mathcal{F}_F) \leq B_{\star}$, which we denote $H(F)$ is called the **flap group for $F$**, and the finite index subgroup $Z_F \leq Z_{\star} \leq B_{\star}$ that maps $F$ to itself is called the **core stabilizer of $F$**.

We also record the following observation which we will use later:

**Lemma 5.13** (Flap stabilizer). Let $\{F_1, \ldots, F_k\}$ be the set of representatives of the $B_{\star}$-orbits of the set of flaps attached to $\star$ and let $F \subset B_{\star} \cdot F_0(F)$ then the $B_{\star}$-stabilizer of $F$ is

$$(B_{\star})_F = Z_F \oplus \left( \bigoplus_{j \neq 0(F)} H(\mathcal{F}_{F_j}) \right),$$

(8)

where $Z_F$ is the core stabilizer of $F$ and the other direct summand is the product of all the flap groups fixing $F$. 

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5.2 Flat groupings

We note that given discrete grouping $\Delta$ of $G \to G\setminus \Gamma^*$, the action of $\Delta$ on $T$, the tree underlying $\Gamma^*$, splits $\Delta$ into a graph of groups, vertex groups being either churro or waffle stabilizers. Since the action of $\Delta$ is discrete and churros and waffles have discrete isometry groups, it follows that each vertex group $\Delta_v$ is a discrete group which acts discretely on each vertex space.

In the previous section given a churro $\ast$ in $\Gamma^*$ we described a basic grouping $B\ast$ which is a discrete subgroup of $\text{Isom}(\ast)$.

**Definition 5.14** (Basic grouping of a waffle). The basic grouping of a $\boxminus$ is the image

$$\overline{G\boxminus} = B\boxminus \leq \text{Aut}(\boxminus)$$

of $G\boxminus$ in $\text{Aut}(\boxminus)$ equipped with natural action.

**Lemma 5.15.** $B\boxminus \circ \boxminus$ is a discrete grouping of $\boxminus \to G\boxminus \setminus \boxminus$.

**Proof.** By Corollary 3.12 $\text{Aut}(\boxminus)$ is discrete; so $B\boxminus$ must be discrete as well. The fact that it is a grouping is follows from the definition. □

As we shall see later, there is no reason to expect that the vertex groups $\Delta_v$ of a discrete grouping are simply isomorphic to $B\boxminus$ or $B\ast$. That being said we will construct discrete groupings that are close enough. We shall first give some additional technical criteria for discrete groupings, then show that these added criteria enable us to obtain a type of conjugacy result. Finally we will actually go and prove the existence of these discrete groupings.

**Definition 5.16.** A basic grouping $\Delta$ is said to be **directly augmented** if for each churro or waffle piece $p$ of $\Gamma^*$ we have that the stabilizer is given by

$$\Delta_p \approx B_p \oplus A_p$$

where the $B_p$ factor acts naturally on $p$ and the factor $A_p$ called the augmentation is a finite abelian group which acts trivially on $p$. The subgroup corresponding to

$$B_p \oplus \{1\}$$

is called the **basic core**.

**Definition 5.17.** Let $\Delta$ be a discrete directly augmented grouping of $\Gamma^*$ and let $\delta$ be an isometry that is elliptic on the action of $T$, i.e. it stabilizes a churro or a waffle piece $p$. We say $\delta \in \Delta \setminus \{1\}$ is **clean in $p$** if the following hold.

- **If $p$ is a churro $\ast$ then we require:**
  1. that $\delta$ lies inside the basic core $B\ast \oplus \{1\} \leq \Delta\ast$,
  2. that $\delta$ doesn’t interchange the ends of $\ast$, and
  3. that $\delta$ doesn’t permute any of the flaps of $\ast$.

- **If $p$ is a waffle $\boxminus$ then we require:**
  1. that $\delta$ lies inside the basic core $B\boxminus \oplus \{1\} \leq \Delta\boxminus$,
  2. that the extension of $\delta$ to a homeomorphism of the Gromov boundary $\partial \boxminus \approx S^1$ is orientation preserving, and
  3. that $\delta$ doesn’t fix a point in $\boxminus$. 

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Finally we have our last criterion for groupings. Note that if $\Delta$ splits as a graph of groups, or, equivalently, acts without inversions on a tree, then edges give rise to isomorphisms between subgroups of vertex groups of adjacent vertices, via the identification of the edge stabilizer within the stabilizers of the adjacent vertices.

**Definition 5.18.** We say that a discrete directly augmented grouping $\Delta$ of $\hat{\oplus}^*(\Gamma) \to G \backslash \hat{\oplus}^*(\Gamma)$ is flat if, via the action $\Delta \cup T$, the isomorphisms induced by identifications of edge stabilizers in the adjacent vertex stabilizers send isometries that are clean in one piece to isometries that are clean in the adjacent piece.

### 5.3 Commensurability via flat discrete groupings

$\hat{\oplus}^*(\Gamma)$ is obtained by gluing flap sides contained in churros to strands contained in waffles. By Lemma 4.5 the stabilizers of a strand $\sigma$ is either infinite cyclic, infinite dihedral, or the direct product of $\mathbb{Z}_2$ and such groups, whereas (8) in Lemma 5.13 indicates that the stabilizer of a flap side $s_F$ can be considerably different. In particular it may not be possible to assemble the actions given by basic groupings of the pieces into a graph of actions.

Consider however the following special case: every strand $\sigma \subset \hat{\oplus}$ is $B_{\hat{\oplus}}$-non-reflective (see Definition 4.6) and the flap factor of each basic churro group (see Definition 5.10) is trivial. In this case for every strand $\sigma$ and flap side $s_F$ to which it is attached we have an isomorphism

$$(B_{\hat{\oplus}})_\sigma \approx (B_{\hat{\oplus}})_F$$

with which the identification map $\sigma \to s_F$ is equivariant. By the decomposition of $\hat{\oplus}^*(\Gamma)$ as a tree of spaces this gives rise precisely to graph of actions $X^\cup$ (see Definition 5.2) with a finitely generated (and therefore discrete) fundamental group acting on the covering complex $\hat{\oplus}^*(\Gamma)$ giving, by Lemmas 5.11 and 5.15, a discrete grouping of $\hat{\oplus}^*(\Gamma) \to G \backslash \hat{\oplus}^*(\Gamma)$. In fact in this case the full automorphism group $G$ itself is discrete, therefore finitely generated and $\Gamma$ and $H$ are obviously commensurable.

The issue arises when $G$ doesn’t act discretely on $\hat{\oplus}^*(\Gamma)$. A similar situation was studied by Bass in the case of trees, which generally have non-discrete automorphism groups. Our idea is to follow the general approach for the commensurability result [BK90, Corollary 4.8]. More specifically we will draw inspiration from the referee’s proof of [BK90, Corollary 4.8], included in that paper, to prove a similar result in the context of churro-waffle spaces. Our goal is to show that even if $G$ is not discrete, a discrete flat grouping $\Delta$ of $\hat{\oplus}^*(\Gamma) \to G \backslash \hat{\oplus}^*(\Gamma)$ will be sufficient for our purposes. Specifically we will prove the following:

**Corollary 5.19.** If $\hat{\oplus}^*(\Gamma) \to G \backslash \hat{\oplus}^*(\Gamma)$ admits a discrete flat grouping $\Delta \cup \hat{\oplus}^*(\Gamma)$ then $\Gamma$ is quasi-isometrically rigid.

**Definition 5.20** (Compact Fundamental Domain). Let $H$ act freely and cocompactly on $\hat{\oplus}^*(\Gamma)$. A compact fundamental domain $D_{H \hat{\oplus}^*}(\Gamma)$ is a compact closed simply connected subcomplex such that

$$\hat{\oplus}^*(\Gamma) = H \cdot D_{H \hat{\oplus}^*}(\Gamma).$$
Definition 5.21 ($\mathcal{D}_{H \mathbb{G}^*} (\Gamma)$-generating sets and intersection pairings). A $\mathcal{D}_{H \mathbb{G}^*} (\Gamma)$-generating set of $H$ is a generating set $S = \{h_1, \ldots, h_n\}$ such that

$$\mathcal{D}_{H \mathbb{G}^*} (\Gamma) \cap h_i \cdot \mathcal{D}_{H \mathbb{G}^*} (\Gamma) \neq \emptyset$$

and the equivalence relation obtained by closing

$$x \sim y \Leftrightarrow \exists h_i (h_i \cdot x = y)$$

under reflectivity and symmetry gives

$$\mathcal{D}_{H \mathbb{G}^*} (\Gamma)/\sim_S = H \mathbb{G}^* (\Gamma).$$

We further call the induced mapping

$$\rho_i : h_i^{-1} (\mathcal{D}_{H \mathbb{G}^*} (\Gamma) \cap h_i \cdot \mathcal{D}_{H \mathbb{G}^*} (\Gamma)) \rightarrow \mathcal{D}_{H \mathbb{G}^*} (\Gamma) \cap h_i \cdot \mathcal{D}_{H \mathbb{G}^*} (\Gamma)$$

the intersection pairing of $\mathcal{D}_{H \mathbb{G}^*} (\Gamma)$ induced by $h_i$.

Proposition 5.22 (Weak Conjugacy Theorem). Let $H$ act cleanly on $\mathbb{G}^* (\Gamma)$ and let $\Delta \circledcirc \mathbb{G}^* (\Gamma)$ be a flat discrete grouping, then there is a subgroup $H\Delta \leq \Delta$ that is isomorphic to $H$. Furthermore $[\Delta : H\Delta] < \infty$.

Proof. We will prove this by an incremental construction of a compact fundamental domain $\mathcal{D}_{H \mathbb{G}^*} (\Gamma)$ and a $\mathcal{D}_{H \mathbb{G}^*} (\Gamma)$-generating set $S = \{h_1, \ldots, h_n\} \subset H$. As we do this we will also find a companion set $Y = \{\delta_1, \ldots, \delta_n\} \subset \Delta$ with the property that the partial identification of $\mathcal{D}_{H \mathbb{G}^*} (\Gamma)$ induced by $h_i \in S$ coincides with the partial identification induced by $\delta_i$. Furthermore we will show that $\langle Y \rangle \leq \Delta$ acts freely on $\mathbb{G}^* (\Gamma)$. From this it will immediately follow that

$$H \mathbb{G}^* (\Gamma) = \langle Y \rangle \mathbb{G}^* (\Gamma)$$

and $\pi_1 (H \mathbb{G}^* (\Gamma)) \approx \pi_1 (\langle Y \rangle \mathbb{G}^* (\Gamma)) \approx \langle Y \rangle$, giving the desired isomorphic subgroup. To see finite index, it is enough to notice that $H\Delta = \langle Y \rangle$ acts cocompactly on $\mathbb{G}^* (\Gamma)$, which means that $[\Delta : H\Delta] < \infty$ since $\Delta \circledcirc \mathbb{G}^* (\Gamma)$ is discrete and cocompact. We now proceed with the proof.

$\mathbb{G}^* (\Gamma)$, being built out of churros and waffles, has the structure of a bipartite tree $T$. Vertices of $T$ are either of waffle type or of churro type. Edges correspond to strands: the intersection of waffle and churro pieces. If $v \in \text{Verts}(T)$ is a vertex, denote by $p_v$ the corresponding waffle or churro. We denote by $\overline{\Pi}_{p_v} \leq \text{Isom} (p_v)$ the image of the stabilizer $H_{p_v} = H_v$ in $\text{Isom} (p_v)$.

Let $w_1$ be a waffle-type vertex of $T$ and denote by $\overline{\Pi}_1$ the corresponding piece. Consider the compact fundamental domain $\mathcal{D}^1 = \mathcal{D}_{H \overline{\Pi}_1 \mathbb{G}}$ and let $S^1 = \{h_{11}, \ldots, h_{n1}\}$ be a $\mathcal{D}^1$-generating set. We note that $\mathcal{D}^1$ intersects a finite collection $\Sigma_1$ of strands, and this collection naturally surjects onto the collection of $H_{\overline{\Pi}_1}$-orbits of edges of $T$ attached to $w_1$. We note that $S^1$ has the property that if $h_j^1 \cdot \sigma \cap \sigma \neq \emptyset$ for some $\sigma \in \Sigma_1$, then $h_j^1$ acts on the churro * containing $\sigma$ as a clean translation.

Since $\Delta$ is a flat discrete grouping of $G \rightarrow \mathbb{G}^* (\Gamma)$, $\overline{\Delta_{\overline{\Pi}_1}} \geq \overline{\Pi_{\overline{\Pi}_1}}$, we can find element $\delta_{11}, \ldots, \delta_{n1} \subset \Delta$ that are clean in $\overline{\Pi}_1$ (Definition 5.17) such that the images $\overline{\delta_j^1} = h_j^1$ in $\text{Isom} (\overline{\Pi}_1)$. If $h_j^1$ maps a strand $\sigma \in \Sigma_1$ to itself then the corresponding $\delta_j^1$, because it is chosen to be clean in $\overline{\Pi}_1$, and because $\Delta$ is a flat grouping (Definition 5.18), also gives an isometry that is clean in * the
churro attached to the strand \( \sigma \). Since \( H \) acts cleanly, we must have \( \delta_j^1 = \hat{h}_j^1 \) in \( \text{Isom}(\mathbb{H}) \). Denote \( Y^1 = \{ \delta_1^1, \ldots, \delta_{n_1}^1 \} \).

Now we pick \( \Sigma_1' \subset \Sigma_1 \) to be a subset of strands so that the attached churros form a complete collection of representatives of the \( H \)-orbits (not \( H_{\mathbb{H}} \)) of churros attached to \( \mathbb{H}_1 \). We enlarge \( D^1 \) to \( (D^1)' \) by attaching compact fundamental domain of the churros attached to \( \Sigma_1' \) under the action of the \( h_j^1 \)'s that map strands to themselves. At this point we observe the following:

- The intersection pairings on \( (D^1)' \) induced by the elements of \( S^1 \) coincide with the intersection pairings induced by the corresponding elements of \( Y^1 \).
- The group \( \langle Y^1 \rangle \) acts freely because the elements were chosen to be clean in \( \mathbb{H}_1 \).

Let \( D_{n}^1 \) be the subtree of \( T \) corresponding to the pieces containing the churros and waffles of \( (D^1)' \). Set \( S_1 = S^1, Y_1 = Y^1 \) and \( D_1 = (D^1)' \). We have completed the base case.

Suppose now that we have a connected subcomplex \( D_n \) which is a union of subcomplexes of waffles and churros. Let \( \mathbb{H}_1, \ldots, \mathbb{H}_n \) be the waffles that intersect \( D_n \) in more than an a strand, and suppose that for each \( i \), \( D_n \cap \mathbb{H}_i = D_i^1 \) is a compact fundamental domain \( D_{H_{\mathbb{H}_i} \cap \mathbb{H}} \). Moreover the \( \mathbb{H}_i \) all lie in distinct \( H \)-orbits. Suppose furthermore that there is a set \( S_n = S^1 \cup \ldots \cup S^n \) where each \( S^i \) is a \( D^i \)-generating set of \( H_{\mathbb{H}_i} \) and a corresponding \( Y^n = Y_1 \cup \ldots \cup Y_n \subset \Delta \) of matched elements such that the set of intersection pairings on \( D_n \) induced by the \( h_j^1 \) \( S_n \) coincided with the intersection pairings of the corresponding \( \delta_j \) and that \( \langle Y^n \rangle \) acts freely on \( \mathbb{H} \{ \Gamma \} \).

**Case 1: we can still add waffles.** Let \( D_{n}^T \subset T \) be the subtree corresponding to the union of churros and waffles intersecting \( D_n \) in more than a strand. Note that by construction the finite subtree \( D_{n}^T \) may not be closed: some edges are half open as they are missing a vertex. Suppose that \( D_{n}^T \) still doesn’t contain a full set of representatives of \( H \)-orbits of waffle-type vertices. Then there is some churro such that \( \ast \cap D_n \) intersects a waffle piece \( \mathbb{H}_{n+1} \) in a strand \( \sigma \), furthermore \( \mathbb{H}_{n+1} \) is not in the same \( H \)-orbit as any other waffle in the collection \( \mathbb{H}_1, \ldots, \mathbb{H}_n \).

There are some elements \( Y^* \subset Y^n \subset \Delta \) that act on \( \ast \) as clean isometries, and therefore by flatness they also act cleanly on \( \mathbb{H}_{n+1} \). By hypothesis, there are corresponding elements \( S^* \subset S^n \subset H \) that act cleanly on \( \ast \) and also cleanly on \( \mathbb{H}_{n+1} \). Since the actions are clean for every \( h \in S^* \) and corresponding \( \delta \in Y^* \) we have an equality \( h_{\mathbb{H}_{n+1}} = \delta_{\mathbb{H}_{n+1}} \) in \( \text{Isom}(\mathbb{H}_{n+1}) \). Let \( \sigma' = \mathbb{H}_{n+1} \cap D_n \) be the subset of the strand \( \sigma \). We note that \( \sigma' \) is a compact fundamental domain for the action of \( \langle Y^* \rangle \) or \( \langle S^* \rangle \) on \( \sigma \subset \ast \). Let \( D_n^{n+1} \supset \sigma' \) be a compact fundamental domain \( D_{H_{\mathbb{H}_{n+1}} \cap \mathbb{H}_{n+1}} \) containing \( \sigma' \). We can extend \( S^* \subset S^{n+1} \) to a \( D^{n+1} \)-generating set of \( H_{\mathbb{H}_{n+1}} \), we may further assume that no element of \( S^{n+1} \setminus S^* \) maps \( \sigma \) to itself, as such an element could be removed from \( S^{n+1} \) and we would still have a \( D_{H_{\mathbb{H}_{n+1}} \cap \mathbb{H}_{n+1}} = D^{n+1} \)-generating set.

Denote by \( h_{1}^{n+1}, \ldots, h_{n_{n+1}}^{n+1} \) the new elements of \( S^{n+1} \setminus S^* \). We note that every new element added moves \( \sigma \) off itself so that for every \( j \) we have

\[
h_j^{n+1}.(D_n \cup D^{n+1}) \cap (D_n \cup D^{n+1}) \subset D^{n+1} \subset \mathbb{H}_{n+1}.
\]

In other words, the new intersection pairings give rise to identifications in \( D_n^{n+1} \subset \mathbb{H}_{n+1} \). Set \( S_{n+1} = S_n \cup S^{n+1} \).

Let \( \Sigma_{n+1} \) be the set of strands intersecting \( D^{n+1} \). As before we chose corresponding elements \( \{ \delta_{1}^{n+1}, \ldots, \delta_{n_{n+1}}^{n+1} \} = Y^{n+1} \setminus Y^* \) that act cleanly on \( \mathbb{H}_{n+1} \)}
and therefore, by flatness, also act cleanly on churros when fixing strands of $\Sigma_{n+1}$. Set $Y_{n+1} = Y_n \cup Y^{n+1}$. Let $\Sigma'_{n+1}$ be the set of strands attached to churros that are not in the $H$-orbit of any of the other churros intersecting $D_n \cup D^{n+1}$, and attach compact fundamental domains of churros to the strand pieces $\sigma \cap D^{n+1}, \sigma \in \Sigma'_{n+1}$. This gives $\langle D^{n+1} \rangle'$. Set $D_{n+1} = D_n \cup \langle D^{n+1} \rangle'$. Let $D_{n+1}'$ be the subtree of $T$ containing the vertices corresponding to waffles and churros that intersect $D_{n+1}$ in more than a strand.

Let $Y_{n+1} = Y_n \cup Y^{n+1}$. We first note that that $\langle Y^{n+1} \rangle$ acts freely. Indeed consider the tree $T_{n+1} = \langle Y_{n+1} \rangle \cdot D^T_{n+1}$, and suppose that some $\delta \in \langle Y_{n+1} \rangle$ fixes a point in $\boxplus^\#(\Gamma)$. Then $\delta$ fixes a vertex of $T_{n+1}$. Note that how $Y_{n+1}$ was constructed, every piece is stabilized by clean isometries, which implies that $\delta = 1$. Thus, $\langle Y_{n+1} \rangle$ acts freely. It is also clear from the construction that the intersection pairings of $D_{n+1}$ coming from $Y^{n+1}$ are the same as those coming from $S^{n+1}$. We have therefore managed to pass to a larger $D_{n+1}$ This completes case 1 of the induction.

Case 2: $D_m$ has a representative for every waffle. At this point $D_m$ is in fact a compact fundamental domain $D_{H\boxplus^\#(\Gamma)}$ as it contains a compact fundamental domain for every $H$-orbit of a waffle and churro. Furthermore we have subsets $(S_m) \subset H$ and $(Y_m) \subset \Delta$, the latter acting freely and inducing the same intersection pairings as $(S_m)$. We note that if the quotient $G \backslash T = X$ is a tree then we are done. If this is not the case, then $(S_m)$ is generated by $T$-elliptic elements, and there are missing stable letters (or so-called Bass-Serre generators) to generate $H$. Let $\Sigma$ be the collection of strands that intersect $D_m$ that have no churros attached to them. We call the subset $\Sigma_{\text{int}} \subset \Sigma$ the set of internal strands, and define it to consist of those strands that are in the $H$-orbit of some other strand crossing $D_m$ to which a churro is attached. The complement $\Sigma_{\text{ext}} = \Sigma \setminus \Sigma_{\text{int}}$ of external strands have the following property: for each strand $\sigma \subset \boxplus_j$ in $\Sigma_{\text{ext}}$ there is a strand $\sigma'$ that lies in a churro $\star$ that intersects $D_m$ in more than a strand and there is some $h \in H$ such that $h \cdot \sigma' = \sigma$. Furthermore if $\star$ is the waffle containing $\sigma'$, then $\star \cap D_m \subset \sigma'$, since by hypothesis $D_m$ only intersects each $H$-orbit of a waffle exactly once in more than an strand. By choosing $h_\star \in H_\star$ and $h_i \in H_{\boxplus_j}$ appropriately we can take $h_\sigma = h_\star h h_i$ so that

$$\emptyset \neq h_\sigma \cdot (\sigma \cap D_m) \cap D_m \subset \sigma,$$

giving another intersection pairing of $D_m$. For every such $h_\sigma, \sigma \in \Sigma_{\text{ext}}$ we pick a corresponding element $\delta_\sigma \in \Delta$, with the property that $\delta_\sigma$ conjugates clean elements of $\Delta$ to clean elements, which is fine since in each vertex group the set of clean isometries is closed under conjugation.

From this it follows that $Y = Y_m \cup \{ \delta_\sigma \mid \sigma \in \Sigma_{\text{ext}} \}$ generates a group that acts freely on $\boxplus^\#(\Gamma)$. Let $S = S_m \cup \{ h_\sigma \mid \sigma \in \Sigma_{\text{ext}} \}$. By analysis of the intersection pairings it is clear that

$$H \setminus \boxplus^\#(\Gamma) = \langle S \rangle \setminus \boxplus^\#(\Gamma) = \langle Y \rangle \setminus \boxplus^\#(\Gamma).$$

Since $\langle Y \rangle$ acts freely we conclude that $H \approx \langle Y \rangle = \langle H \rangle^\Delta$. \hfill \Box

Proof of Corollary 5.19. Suppose that $\Gamma$ is quasi isometric to $H$. By Proposition 2.15, $H$ and $\Gamma$ have finite index clean subgroups. Without loss of generality we may assume that $H$ and $\Gamma$ are themselves clean. By Proposition 5.22 there are subgroups $H^\Delta, \Gamma^\Delta \leq \Delta$
isomorphic to $H, \Gamma$ (respectively.) Furthermore since $[\Delta : H^\Delta], [\Delta : \Gamma^\Delta] < \infty$ we have a bound:

$$[H^\Delta : \Gamma^\Delta \cap H^\Delta], [\Gamma^\Delta : \Gamma^\Delta \cap H^\Delta] < \infty.$$  

Thus $\Gamma$ and $H$ are abstractly commensurable.

\[
\square
\]

6 Constructing a flat discrete grouping of $\boxplus^* (\Gamma)$ for the win

We now fix some notation for the remainder of the paper. From $\boxplus^* (\Gamma)$ we may construct a tree $T$ whose vertices are pieces, either churros or waffles. We draw an edge $e \in \text{Edges}(T)$ for every pair of adjacent pieces. Note that $T$ is bipartite, its vertices being either churros or waffles. We adopt the following:

**Convention 6.1 (Edges point to churros).** If $e$ is an edge between a churro $\ast$ and a waffle $\square$ then we orient $e$ so that it points to the churro, i.e. $i(e) = \square$, and $t(e) = \ast$.

Further to $e$ we associate an identification map:

\[
\square \ni \sigma_e \sim s_e \subset F_e \subset \ast
\]

where in $\boxplus^*(\Gamma)$ the strand $\sigma_e$ and flap side $s_e$ are identified:

$$\sigma_e = s_e = \square \cap \ast.$$

Seeing as strands are always denoted $\sigma_e \subset \square$ and flap sides are always denoted $s_e \subset F_e \subset \ast$, we can denote the stabilizers

$$B_{\sigma_e} = (B_{\square})_{\sigma_e}, B_{s_e} = (B_{\square})_{F_e} = (B_{\square})_{s_e}.$$

We also note that there is a natural $G$-equivariant quotient map $\boxplus^*(\Gamma) \rightarrow T$ obtained by collapsing waffles to points and churros to “asterisks”.

**Definition 6.2 (Reflective edges and flap fixators).** Let $e \in \text{Edges}(T)$. If the strand $\sigma_e$ is reflective (see Definition 4.6) then we say $e$ is reflective.

Consider the quotient space $G \backslash \boxplus^*(\Gamma)$, because of the tree of spaces structure, it has a graph of spaces structure with underlying graph $G \backslash T = X$. The vertex spaces being either orbifold quotients of waffles or churros. In this way the vertices of $X$ correspond to $G$-orbits of churros and waffles. In particular $\text{Verts}(X)$ is partitioned in to the set of churro-type and waffle-type vertices and is bipartite. The edges $\text{Edges}(X)$ correspond to $G$-orbits of identified strand and flap sides, and always point to churro type vertices. Because the edges adjacent to a churro type vertex are in natural correspondence with the flap orbits of a corresponding lift, this gives convenient notation to express stabilizers:

**Definition 6.3 (Groups associated to edges in the quotient).** Let $e \in \text{Edges}(X)$ and let $\tilde{e}$ be a lift of $e$ in $T$. Let $F_{\tilde{e}}$ be the flap corresponding to $\tilde{e}$ and let $\mathcal{F}_{F_{\tilde{e}}}$ be its flap family. Then the low power abelian group, $H(\mathcal{F}_{\tilde{e}})$, corresponding to $e$ is an isomorphic copy of $H(\mathcal{F}_e)$ (see Definition 5.12.) We define $H(e)$ to be an isomorphic copy of $H(\tilde{e})$ and we define $Z_e$ to be a copy of core stabilizer (see Lemma 5.13) $Z_{F_{\tilde{e}}}$.  

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Lemma 6.4 (Basic edge groups). Let $e \in \text{Edges}(X)$ and let $\bar{e}$ be a lift of $e$ in $T$. Then we define the basic group

$$B_{s\bar{e}} \approx Z_e \oplus \left( \bigoplus_{f \in \text{star}(t(e)) \setminus \{e\}} H(f) \right) \leq B_t(e)$$

where

$$\text{star}(t(e)) \setminus \{e\} = \{f \in \text{Edges}(X) \mid t(f) = t(e) \text{ but } f \neq e\}.$$

For strands we have the basic group

$$B_{\sigma e} \approx Z_e \oplus \langle \tau_e \rangle \leq B_t(e)$$

where $\langle \tau_e \rangle \cong \mathbb{Z}_2$ if $e$ is reflective and $\langle \tau_e \rangle = \{1\}$ otherwise.

At this point we have a graph $X$ such that at each $v \in \text{Verts}(X)$ we have a group action on some space, furthermore for each edge $e \in \text{Edges}(X)$ we have an identification map between orbits of strands and flap sides. This is already enough information to reconstruct $\mathbb{F}(\Gamma)$ as a tree of spaces, the only obstruction to obtaining a graph of actions $\mathbb{F}(\Gamma)$ is the discrepancy between the flap side stabilizers (10) and the strand stabilizers (11).

To remedy this problem we will augment the basic discrete groupings of churros and waffles so that the stabilizers coincide.

Definition 6.5 (Augmentation). Consider a faithful group action $\Xi \otimes p$. A direct augmentation $\hat{\Xi} \otimes p$ consists of an extension

$$1 \to A \to \Xi \oplus A \xrightarrow{\pi} \Xi \to 1$$

where $A$ is a finite group and the action $(\Xi \oplus A) \otimes p$ is given by

$$\xi \cdot x = \pi(\xi) \cdot x,$$

i.e. by ignoring the $A$-factor. The subgroup $A$ is called the augmentation subgroup.

For example if we augment the basic grouping of a waffle we find the stabilizer of a strand $\sigma \subset \mathbb{F}$ becomes

$$(B_{\mathbb{F}} \oplus A)_{\sigma} \approx (B_{\mathbb{F}})_{\sigma} \oplus A.$$
Definition 6.7 (Augmentation contributions). For each $e \in \text{Edges}(X)$, the reflector contribution $\langle \tau_e \rangle$ is defined follows:

$$
\langle \tau_e \rangle \approx \begin{cases} 
\mathbb{Z}_2 & \text{if } e \text{ is reflective,} \\
\{1\} & \text{otherwise.}
\end{cases}
$$

(13)

For each churro type vertex $c \in \text{Verts}(X)$ and any other vertex $c \neq v \in \text{Verts}(X)$ the churro contribution $H(c)$ is the corresponding flap factor, given in Definition.

We remark that $H(c) \approx \bigoplus_{e \in \text{star}(c)} H(e)$, where $H(e)$ is given in Definition 6.3. Denote by $\text{Verts}^{\text{churro}}(X)$ the collection of churro type vertices. Then we have the following:

Definition 6.8 (Augmentation, simply connected case). Let $v \in \text{Verts}(X)$. Then the augmentation $A_v$ is the low power abelian group

$$
A_v = \left( \bigoplus_{\{c \in \text{Verts}^{\text{churro}}(X) | v \neq c\}} \frac{H(c)}{H(e)} \right) \bigoplus \left( \bigoplus_{\{f \in \text{Edges}(X) | \bar{c} \to v\}} \langle \tau_f \rangle \right). 
$$

(14)

The left direct factor is called the flap contribution and the right direct factor is called the reflector bank.

Lemma 6.9. The augmentation $A_v$ is well-defined. Furthermore the flap contribution can be rewritten as

$$
\bigoplus_{\{c \in \text{Verts}^{\text{churro}}(X) | v \neq c\}} H(c)/H(e_{c,v}) = \bigoplus_{\{e \in \text{Edges}(X) | \bar{c} \to v\}} H(t(e))/H(e)
$$

(15)

where $e_{c,v}$ is the first edge on the unique path in $X$ from $c$ to $v$.

Proof. Because low power groups are abelian and determined by cardinality, the “fraction of groups” given in (14) will be well defined (up to isomorphism) if the order of the “denominator” divides the order of the “numerator.”

We start with the second statement. By Convention 6.1 and the fact that $X$ is a tree to each edge $e \in \text{Edges}(E)$ such that $\bar{e} \to v$ there is a unique corresponding $c \in \text{Verts}^{\text{churro}}(X) \setminus \{v\}$, specifically the vertex $t(e)$. In fact, $e = e_{c,v}$. It follows on the one hand that in the flap contribution in (14) we can match each term in the “numerator” to a term in the denominator. On the other hand, by definition, $H(e) = H(e_{c,v})$ is a direct summand of $H(c)$. The second statement now follows. It is now clear that (14) is well defined.

Proposition 6.10. Let $X$ be the graph underlying the graph of spaces decomposition of $G \setminus \mathbb{H}^* (\Gamma)$ and suppose that $X$ is a tree. Each vertex $v \in \text{Verts}(X)$ corresponds to a $G$ orbit of some piece, either a waffle or a churro, $p_v \subset \mathbb{H}^* (\Gamma)$ and for each edge $e \in \text{Edges}(X)$ connecting vertices $i(e)$ and $t(e)$ there is an identification isometry between a strand and a flap side. Given the augmentations in Definition 6.8, the corresponding stabilizers of identified strands and flap sides are isomorphic. If we equip every vertex $v \in \text{Verts}(X)$ with the action given in Definition 6.5, then for any isomorphism

$$
(B_{i(e)} \oplus A_{i(e)}))_{\sigma_v} \simeq (B_{t(e)} \oplus A_{t(e)})_{s_e}
$$

we can construct a graph of actions $X^\sigma$ realizing a direct flat discrete grouping of $\mathbb{H}^* (\Gamma) \to G \setminus \mathbb{H}^* (\Gamma)$. 

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Proof. Let \( e \in \text{Edges}(X) \) and let \( w = i(e), c = t(e) \). Let \( \sigma_e \subset \mathbb{H}_w \) and \( s_e \subset \ast_c \) be the strand and flap side identified via the isometry (9). All that we need to show is that given the augmentation

\[
B_w \oplus A_w \sqcup \mathbb{H}_w \text{ and } B_c \oplus A_c \sqcup \ast_c
\]
given in Definition 6.5, that we have an isomorphism

\[
(B_w \oplus A_w)_{\sigma_e} \cong (B_c \oplus A_c)_{s_e}.
\]

Finally must show that this isomorphism can be arranged to be flat, i.e. clean isometries are identified with clean isometries.

Let us first compare \( A_w \) and \( A_c \). By definition and since the edges are oriented towards churros the flap augmentations \( A_w \) and \( A_c \) only differ by \( \ast \) as given in (10) and the reflector bank (see (14) in Definition 6.8) in \( A_c \) has an extra \( \mathbb{Z}_2 \) factor if and only if \( e \) is reflective. It follows that there is some group \( A \) such that

\[
A_w = A \oplus \left( \sum_{f \in \text{star}(t(e)) \setminus \{e\}} H(f) \right) \text{ and } A_c = A \oplus \langle \tau_e \rangle
\]

with \( \tau_e \) either 1 or of order 2 exactly if \( e \) is reflective. By the definitions of the basic group augmentations and by Lemma 6.4 we get

\[
(B_w \oplus A_w)_{\sigma_e} \cong Z_e \oplus \langle \tau_e \rangle \oplus A \oplus \left( \sum_{f \in \text{star}(t(e)) \setminus \{e\}} H(f) \right) \text{ and } (B_c \oplus A_c)_{s_e} \cong Z_e \oplus \left( \sum_{f \in \text{star}(t(e)) \setminus \{e\}} H(f) \right) \oplus A \oplus \langle \tau_e \rangle
\]

which are isomorphic. Next we show that we can chose these to be flat.

Indeed note that any clean isometry \( \mathbb{H}_w \) fixing \( \sigma_e \) lies inside the subgroup \( Z_e \oplus \{1\} \leq (B_w \oplus A_w)_{\sigma_e} \) and any clean isometry \( \ast \), fixing \( s_e \) also lies inside the subgroup isomorphic to \( Z_e \oplus \{1\} \leq (B_c \oplus A_c)_{s_e} \). Furthermore the only possible non clean element in this subgroup is an elliptic isometry that interchanges the ends of \( \sigma_e \) or \( s_e \), which is impossible. It therefore follows that any isomorphism \( (B_w \oplus A_w)_{\sigma_e} \rightarrow (B_c \oplus A_c)_{s_e} \) extending this identification of subgroups will enable us to assemble a graph of actions \( X^\Gamma \) realizing a direct flat discrete grouping.

\[ \square \]

### 6.2 Orbifold maps

When the quotient \( G \backslash T = X \) has no cycles, the augmentations are obtained by propagation through the tree \( X \), which terminates. If \( X \) has cycles, however, it may very well be that the augmentations must get bigger and bigger. In the next section we will prove that if some group \( \Gamma \) acts freely, and cocompactly on \( \mathbb{H}^k(\Gamma) \) then it will be possible to balance augmentations around loops. The argument will rely on counting points preimages in orbifold covers.

A first difficulty is that unlike classical orbifolds, which are fairly homogeneous (i.e. all points in a manifold have homeomorphic neighbourhoods), cube
complexes are more “lumpy”: points can have very different looking neighbourhoods. The main result of this section is that in spite of this non-homogeneity, in our specific context, we will still be able to define a global sheet number for orbifold covers which enables us to count preimages.

**Definition 6.11** (Orbifold maps, covers). Let $Z$ be a locally finite metrized cell complex and let $K \triangleleft Z$ by automorphisms. The quotient

$$Z \rightarrow K\backslash Z$$

is called an orbifold map. The weight of a point $x \in K\backslash Z$ is 1 divided by the cardinality of the stabilizer of a preimage $\tilde{x} \in Z$ of $x$. If all points have weight 1 then $Z \rightarrow K\backslash Z$ is a covering map.

It is obvious that a covering map in this context coincides with the standard definition. In general for a covering map, if $Z$ is simply connected, we can recover $K \triangleleft Z$ via the action on the universal cover $K \approx \pi_1(K\backslash Z) \triangleleft K\backslash Z = Z$. For a general orbifold map we cannot recover $K$ only from the quotient $K\backslash Z$. There is a category of orbifold maps originating from $Z$ which is and there is a Galois-type duality between the subgroup lattice of $\text{Aut}(Z)$ and given by the well-defined quotient map

$$H\backslash Z \rightarrow K\backslash Z$$

for any pair $H \leq K$. The following is immediate from the metrization hypothesis and the locally finite cell complex structure.

**Lemma 6.12** (Standard neighbourhoods). Let $Z$ be a locally finite metrized cell complex. Then for every point $x \in K\backslash Z$ there is a neighbourhood $U$ called a standard neighbourhood that lifts in $Z$ to a disjoint union of metric balls in $Z$ centred at some preimage of $x$.

### 6.2.1 Orbifold quotients of waffles

For this section fix a waffle $\boxplus$ and a group $B$ acting faithfully and cocompactly on $\boxplus$. The definitions we will give in this section are chosen to be suitable to this context.

**Definition 6.13.** A point $x \in \boxplus$ is called singular if $B_x$ is non-trivial. Otherwise it is called regular. The $B$-weight of $x$ is the ratio

$$\text{wt}_B (x) = \frac{1}{|B_x|}.$$ 

For a point $\tilde{x} \in B\backslash \boxplus$, we define $\text{wt}_B (\tilde{x})$ to the weight $\text{wt}_B (x)$ of any of its preimages in $\boxplus$.

The following terminology differs when discussing cube complexes or general cell complexes. In this context we mean it in the general cell complex sense.

**Definition 6.14** (Without inversions). Let $Z$ be a locally finite polyhedral complex. We say an action $G \triangleleft Z$ is without inversions if for every cell $\sigma$, the stabilizer $G_\sigma$ fixes $\sigma$ pointwise.
Lemma 6.15. Let $c$ be a cube. If we pass to the first barycentric subdivision (this is now a simplicial complex), then the action of $\text{Isom}(c)$ is without inversions. Furthermore, if $\tau$ is a maximal simplex of a cube $c$, then the setwise stabilizer of $\tau$ coincides with the pointwise stabilizer of $c$.

Proof. We define the barycentric subdivision inductively. Let $c$ be an $n$-cube, and suppose that for each of its faces we have performed a barycentric subdivision so that $\partial c$ is a union of closed simplices

$$\partial c = \bigcup_{i=1}^{K} \sigma_i.$$ 

As an induction hypothesis we assume that for every face $f$ of $c$ if some isometry of $f$ fixes $\sigma_i$, then the isometry fixes the simplex pointwise. This is true for a point.

Let $c_0$ be the centre of $c$, we realize $c$ as a simplicial complex by making it the union of cones

$$c = \bigcup_{i=1}^{k} \text{Cone}_{c_0}(\sigma_i).$$ 

Any isometry of $c$ fixes $c_0$. It follows that if any isometry maps a simplex $\tau$ to itself, it must map the base $\sigma_i \subset \partial \tau$ to itself, and therefore maps the face $f$ containing the base $\sigma_i$ to itself. The restriction of the isometry of $c$ to $f$ is therefore an isometry of $f$. By the inductive hypothesis, the restriction of the isometry fixes $\sigma_i$ pointwise so, by the cone structure, it fixes $\tau$ pointwise.

The second assertion of the proof follows from the fact that every isometry of a cube $c$ can be realized by embedding $c$ into $\mathbb{R}^n$ and acting on it with permutation matrix. Since maximal simplices have full dimensions, their pointwise stabilizers must be trivial. \qed

From this we immediately get

Corollary 6.16. After passing to barycentric subdivisions, we may assume that actions on waffles are without inversions.

Lemma 6.17. Maximal cells of $\mathbb{W}$ have trivial (setwise) stabilizers.

Proof. Because $\mathbb{W}$ is one ended, there cannot be any cut vertices and every edge must be contained in a higher dimensional cube. Let $\sigma \subset \mathbb{W}$ be maximal cell. If $g \cdot \sigma = \sigma$ then it fixes $\sigma$ pointwise. It therefore follows that if $\hat{\sigma} \subset \mathbb{W}$ is the cube that contains $\sigma$ then $g$ also fixes $\hat{\sigma}$ pointwise. In particular $g$ fixes some square in $\mathbb{W}$, so by Lemma 3.11 $g$ is trivial. \qed

Corollary 6.18 (Dense open regular points). The set of $B$–regular points of $\mathbb{W}$ form a dense open set.

Proof. Let $U$ be the union of the maximal cubes of $\mathbb{W}$, this is a set of regular points of $\mathbb{W}$ whose closure is $\mathbb{W}$. \qed

The following is immediate from the faithfulness of $B \odot \mathbb{W}$ and our definition of regular points.

Lemma 6.19 (The regularity of regular points). Let $x \in \mathbb{W}$ be singular and let $x \in U$ be a metric open ball centred at $x$. For any regular points $r \in U$ the cardinalities of the orbit $|B_x \cdot r|$ is exactly $|B_x| = \frac{1}{\text{wt}_B(x)}$.
Proposition 6.20 (Sheet numbers and counting preimages). Let $\boxplus \to H\backslash \boxplus$ be a covering map and let $B \supseteq H$. Consider the commutative diagram of quotient maps:

\[
\begin{array}{ccc}
\boxplus & \to & \\ \\
\downarrow & & \downarrow \\
H\backslash \boxplus & \to & B\backslash \boxplus.
\end{array}
\]

There is a well-defined number

\[
\text{sh}_p(\boxplus) = k
\]

called the sheet number such that every regular point has exactly $\text{sh}_p(\boxplus) = k$ $p$-preimages. More generally any (possibly singular) point $\bar{x}$ with $B$-weight $\text{wt}_B(\bar{x}) = w$ has $\text{sh}_p(\boxplus) \cdot \text{wt}_B(\bar{x}) = wk$ $p$-preimages.

Proof. Consider the weight function $\text{wt}_B(-) : B\backslash \boxplus \to \mathbb{Q}$. We pass to the first barycentric subdivision of $\boxplus$ to ensure that $B \supseteq \boxplus$ without inversion. This change of structure does not change the set of regular points. Since we are acting without inversions, the function $\text{wt}_B(-)$ is constant on open cells, and therefore gives a well defined map from the set of open cells of $B\backslash \boxplus$. We also have the following monotonicity condition: if one open cell is in the boundary of another open cell, say $\sigma \subset \partial \tau$ then, by containment of stabilizers we have, $\text{wt}_B(\sigma) \leq \text{wt}_B(\tau)$.

Let $1 = \epsilon_1 > \epsilon_2 > \cdots > \epsilon_t$ be the finite collection of realized weights and consider the sets.

\[
C_i = \{ \bar{x} \in B\backslash \boxplus \mid \text{wt}_B(\bar{x}) \leq \epsilon_i \}
\]

Since weight is constant on open cells, and by the monotonicity result $C_{i+1}$ is a (combinatorially and topologically) closed subcomplex of $C_i$. $R = C_1 \backslash C_2$ is therefore the open set of regular points, which by Corollary 6.18 is dense in $B\backslash \boxplus$. Let $R^1, \ldots, R^k$ be the connected components of $R$. Consider the mapping $p|_{p^{-1}(R)} : p^{-1}(R) \to R$. Since it consists only of regular points, it is a standard covering map so the $p$-degree function $\text{deg}_p$ that counts the number of $p$-preimages must be constant on each connected component $R^\xi$.

Suppose towards a contradiction that the $p$-degree function was not constant on $R$. Then $R$ has many connected components, two of which, say $R^1$ and $R^2$ have different degrees and, since $B\backslash \boxplus$ is connected and $R$ is dense, $R^1$ and $R^2$ can be chosen to have intersecting closures. Let $\hat{z}$ be a singular point with a regular neighbourhood $U$, i.e. a metric ball small enough so that it lifts to a disjoint union of metric balls, containing $\hat{x}_1 \in R^1$ and $\hat{x}_2 \in R^2$. Let $\hat{U}_j$ be a connected component $p^{-1}(U)$. Since $\boxplus \to H\backslash \boxplus$ is a covering map, for some lift $z \in \boxplus$ of $\hat{z}$ there is an induced action of $B_z$ on $\hat{U}_j$ so that the quotient $\hat{U}_j \to U$ is given by $B_z \backslash (\hat{U}_j)$. By Lemma 6.19 it follows that $\hat{x}_1$ and $\hat{x}_2$ both have $|B_z|$ preimages in $\hat{U}_j$. It follows that $\bar{x}_1$ and $\bar{x}_2$ have the same number of preimages in $H\backslash \boxplus$. This contradicts the assumption that $R^1$ and $R^2$ had different $p$-degrees.

It therefore follows that the $p$-degree function is constant on the entire set $R$ of regular points. This gives the desired sheet number. Finally, Lemma 6.19 gives the cardinality of the singular point preimages. \qed
We end this section by a detailed examination of the weights of points in strands.

**Definition 6.21 (Density).** Let \( Y \subset \mathbb{E} \) be a subset. We say that \( Y \) has \( B \)-density \( \rho \), denoted \( \text{Density}_B(Y) \), if

\[
\text{wt}_B(z) = \rho, \text{ for all } z \in D \subset Y,
\]

where \( D \subset Y \) is a dense open subset.

We chose the term density since we are ignoring the weights of certain points on a set of measure 0. To illustrate density, consider the following example. Let \( \mathbb{Z} \times \mathbb{Z}_2 \) act faithfully by isometries on \( \mathbb{R} \). This action induces a discrete set \( D \subset \mathbb{R} \) of “reflection points”. The \( (\mathbb{Z} \times \mathbb{Z}_2) \)-weights of the reflection points is \( \frac{1}{2} \), whereas the weights of the other points is 1. This therefore gives a density of 1. Suppose now that instead we have \( (\mathbb{Z} \times \mathbb{Z}_2) \times \mathbb{Z}_2 \) acting on \( \mathbb{R} \times [-1, 1] \), with the \( (\mathbb{Z} \times \mathbb{Z}_2) \) factor acting as above on the \( \mathbb{R} \)-factor and the \( \mathbb{Z}_2 \) factor acting as multiplication by \( \pm 1 \) on \( [-1, 1] \). In this case the density of the core \( \mathbb{R} \times \{0\} \subset \mathbb{R} \times [-1, 1] \) is \( \frac{1}{2} \). This is illustrated in Figure 8.

Recall in Section 4.1, we constructed for each hyperplane in \( \mathbb{E} \) a strand \( \sigma \). We now want to compute the densities of strands.

**Lemma 6.22.** Let \( c \subset \mathbb{E}^\ast(\Gamma) \) be an \( n \)-cube. Then the image \( (B_{\mathbb{E}})_c \leq \text{Isom}(c) \) is naturally isomorphic to a group of isometries of the circle \( \mathbb{E} \) that permutes the \( 2n \) points in \( \mathbb{E} \) corresponding to the endpoints (or limit sets) of the hyperplanes crossing \( c \). In particular it embeds into the dihedral group \( D_{2,2,n} \) of order \( n \).

**Proof.** Any isometry of \( c \) fixes the centre \( O_c = (0, \ldots, 0) \in [-1, 1]^n \) of \( c \); so, using \( O_c \) as an observation point, we find that \( (B_{\mathbb{E}})_c \) acts by isometries on \( \mathbb{E} \) with respect to the \( O_c \)-visual metric. Since the action of \( \text{Isom}(\mathbb{E}) \) is fully determined by the action on the endpoints of hyperplanes in \( \mathbb{E} \) and, in particular, the cyclic ordering of these points must be preserved the result follows. \( \square \)

**Corollary 6.23.** Let \( c \subset \mathbb{E}^\ast(\Gamma) \) be an \( n \)-cube. The subgroup of \( (B_{\mathbb{E}})_c \leq \text{Isom}(c) \) that maps a hyperplane \( \mathcal{H} \) to itself sits inside \( \mathbb{Z}_2 \mathbb{E} \mathbb{Z}_2 \) and the subgroup that fixes the endpoints \( \Lambda(\mathcal{H}) \subset \mathbb{E} \) sits inside \( \mathbb{Z}_2 \).

At this point it would be tempting to conclude that almost all points on a strand have weight 1 or 1/2. Unfortunately this result doesn’t follow immediately since hyperplanes are high dimensional objects and the group we describe in Corollary 6.23 actually has an \( \lfloor n/2 \rfloor \) dimensional fixed set. In particular we must exclude the possibility that a strand has a non-trivial subsegment \( \sigma' = \sigma \cap c \) sitting inside some cube \( c \) that consists of points of weight less than \( \frac{1}{4} \). This desirable condition will hold if \( \sigma' \) does not lie in the intersection of two or more hyperplanes in a cube.

**Proposition 6.24 (Reflectivity characterizes density).** Let \( \sigma \subset \mathbb{E} \) be a strand. If \( \sigma \) is \( B \)-reflective, then \( \text{Density}_B(\sigma) = \frac{1}{2} \). Otherwise, \( \text{Density}_B(\sigma) = 1 \).

**Proof.** By Corollary 6.23, it is enough to show that in any cube \( c \subset \mathbb{E} \) the strand fragment \( \sigma \cap c \) doesn’t sit in the intersection of two distinct hyperplanes. Indeed if \( \sigma \cap c \) has density less than \( 1/2 \), then it must be in the intersection of many hyperplanes of \( c \) and the stabilizer of \( \sigma \cap c \) must permute these hyperplanes and fix their intersection.
Suppose towards a contradiction that this was the case. Then for some cube \( c' \), there is a hyperplane \( \mathcal{H}' \) such that \( \sigma \cap c' \subset \mathcal{H}(\sigma) \cap \mathcal{H}' \cap c' \). There must be some cube \( c_u \) in which, ultimately, \( \sigma \) leaves \( \mathcal{H}' \) and some penultimate cube \( c_p \), also traversed by \( \sigma \), and adjacent to \( c_u \), in which \( \sigma \cap c_p \subset \mathcal{H}(\sigma) \cap \mathcal{H}' \cap c_p \). The boundary of a cube \([-1,1]^n\) corresponds to points in which some of the coordinates is \( \pm 1 \). If the cubes \( c_p \) and \( c_u \) are attached along a \( k \)-dimensional face then we may pick coordinates so that \( c_p = [-1,1]^p \), \( c_u = [-1,1]^u \) so that \( c_p \cap c_u \) is given by:

\[
\{(x_1, \ldots, x_k, 1, \ldots, 1) \in C_p \mid -1 \leq x_1, \ldots, x_k \leq 1\}
\]

and

\[
\{(x_1, \ldots, x_k, -1, \ldots, -1) \in C_u \mid -1 \leq x_1, \ldots, x_k \leq 1\}.
\]

In particular we may set the \( x_1 \)-coordinate to be dual to \( \mathcal{H}(\sigma) \) in both \( c_u \) and \( c_p \), i.e. \( \mathcal{H}(\sigma) \) is the set of all points where the \( x_1 \)-coordinate is 0. We may also set the \( x_2 \)-coordinate to be dual to \( \mathcal{H}' \) in both \( c_u \) and \( c_p \).

Consider now the segments \( \sigma_u = \sigma \cap c_u \) and \( \sigma_p = \sigma \cap c_p \). Then the concatenation \( \sigma_p\star\sigma_u \subset \sigma \) is connected and therefore geodesic. \( \sigma_p \) is the straight line from point \( a = (0,0,c_3,\ldots,a_{n_u}) \) to \( b = (0,0,b_3,\ldots,b_n) \) in \( c_p \). Now by our coordinate convention \( \sigma_u \) is a straight line from a point \( c = (0,0,c_3,\ldots,c_{n_u}) \), which is identified with \( b \) in \( \mathbb{Z}^3 \), to a point \( d = (0,d_3,\ldots,d_{n_u}) \) in which \( d_2 \neq 0 \) since \( \sigma \) must exit \( \mathcal{H}' \) in \( c_u \). The Euclidean length formulas give

\[
|\sigma_p| = \sqrt{0 + 0 + L}
\]

where \( L = (b_3 - a_3)^2 + \cdots + (b_{n_p} - a_{n_p})^2 \) and

\[
|\sigma_u| = \sqrt{0 + d_2^2 + M}
\]

where \( M = (d_3 - c_3)^2 + \cdots + (d_{n_u} - c_{n_u})^2 \). Consider now the \( \epsilon \)-nudge (depicted in Figure 6) obtained by perturbing the \( x_2 \)-coordinate in the point \( c_p \equiv b = c \in c_u \):

\[
b = (0,0,b_3,\ldots,b_{n_p}) \mapsto (0,\epsilon,b_3,\ldots,b_{n_p})
\]

and

\[
c = (0,0,c_3,\ldots,c_{n_u}) \mapsto (0,\epsilon,c_3,\ldots,c_{n_u}).
\]

On the one hand the \( \epsilon \)-nudge is still in \( c_u \cup c_p \) and this gives a curve \( \eta(\epsilon) \) with the same endpoints as \( \sigma_p \cup \sigma_u \). Consider now the formula for the length of \( \eta(\epsilon) \):

\[
\ell(\epsilon) = \sqrt{\epsilon^2 + L} + \sqrt{(d_2 - \epsilon)^2 + M}
\]

\[
\Rightarrow \ell'(\epsilon) = \frac{\epsilon}{\sqrt{\epsilon^2 + L}} + \frac{\epsilon - d_2}{\sqrt{(d_2 - \epsilon)^2 + M}}
\]

\[
\Rightarrow \ell'(0) = \frac{-d_2}{\sqrt{d_2^2 + M}} \neq 0
\]

Therefore \( \epsilon = 0 \) is not a critical point of \( \ell(\epsilon) \), in particular it is not a local minimum. It follows that for some \( \epsilon \) close enough to 0 we can make \( \eta(\epsilon) \) shorter than \( \sigma_p \star \sigma_u \), while preserving endpoints, contradicting that it is geodesic. \( \square \)
Figure 6: An $\epsilon$ nudge seen in a projection. $\sigma_p \ast \sigma_u$ is shown as a solid line, the $\epsilon$ nudge is shown as a dotted line. The common $x_1$ coordinate of both cubes is projected to the page, and the boxes are spanned by their common $x_2$ coordinates and their third respective coordinate.

Figure 7: Dihedral group acting on the churro core, flap families are singletons. The quotient is shown below.

### 6.2.2 Orbifold quotients of churros

Let $\ast$ be a churro and let $B_{\ast}$ act on $\ast$ cocompactly. We will first present two basic examples of two flap churros. Consider first Figure 7. $B_{\ast}$ is the infinite dihedral group, which is generated by two elements of order two which reflect $\bar{\ast}$ and interchange the flaps, preserving the markings shown in Figure 7. Note that although there is one orbit of flaps, there are no purely flapping (recall Definition 5.4) automorphisms, so each flap family (recall Definitions 5.5) consists of a single flap, and the flap group $H(F)$, whose cardinality coincides with the size of the flap family $|F_F|$, (recall Definition 5.12) of each flap $F \subset \ast$ is trivial. The quotient $B_{\ast} \backslash \ast$ is shown below. The core consists of a segment with weight $\frac{1}{2}$ endpoints, and a single flap, whose side consists of a circle. Because the churro core can only be acted on by $\mathbb{Z}$ or $\mathbb{Z} \rtimes \mathbb{Z}_2$, the following makes sense.

**Definition 6.25** (Core sheet number). Let $H \leq B_{\ast}$ act freely on $\ast$ so that $\ast \to H \backslash \ast$ is a covering map. Consider the commutative diagram of quotient maps:

![Diagram]

The core sheet number

$$k = \text{sh}_{p}(\bar{\ast})$$
is the number of $p$-preimages of almost every $\bar{x}$ in the image of the churro core $\star$ in $B_{\star}\backslash\star$.

We note that in the example given in Figure 7, for a subgroup $H \leq B_{\star}$ acting on $\star$, for the quotient map

$$p : H\backslash\star \rightarrow B_{\star}\backslash\star,$$

almost every point of $B_{\star}\backslash\star$ has $k$ $p$-preimages, where $k$ is the core sheet number. We note that in this example, there is a difference between the core group and the stabilizer of a side, which sits inside as an index 2 subgroup.

Let us now consider another group acting on the same churro. This time let

$$B_{\star} \cong \mathbb{Z} \times \mathbb{Z}_2 \cong \langle t \rangle \times \langle s \rangle$$

and consider Figure 8. In this case there is the purely flapping isometry $s$ that

![Figure 8: $\mathbb{Z} \times \mathbb{Z}_2$ acting on a churro. The infinite order element $t$ translates the churro core $\star$ and fixes the flaps. The element $s$ of order 2 switches the flaps, but fixes $\star$ pointwise.](image)

switches the flaps. There is therefore a unique flap family consisting of those two flaps. If $F$ is either flap we have $H(F) = H(F_F)$ and $|F_F| = 2$. The quotient $B_{\star}\backslash\star$ consists the image of $\star$ with a single flap attached. Looking again at Figure 8, we see that for a subgroup $H \leq B_{\star}$ acting on $\star$, for the quotient map

$$p : H\backslash\star \rightarrow B_{\star}\backslash\star,$$

the flap interiors and sides have $k \cdot |F_F| = 2$, where $k$ is the core sheet number.

We now have the following:

**Definition 6.26** (Weight of a flap). Let $B_{\star} \supset \star$ and let $F \subset \star$ be a flap. The weight $\text{wt}_{B_{\star}}(F)$ of $F$ is the cardinality $|F_F|$ of its flap family, or the cardinality of the corresponding flap group $H(F)$.

We note that unlike for waffles, weights in churros are always at least 1. The following is now an immediate consequence of counting orbits.

**Proposition 6.27** (Churro preimages). Let $H \leq B_{\star}$ act freely on $\star$ so that $\star \rightarrow H\backslash\star$ is a covering map. Consider the commutative diagram of quotient maps:

![Diagram](image)
Then for every flap quotient $\bar{F}$ in $B_{\#}\setminus \star$, the number of $p$-preimages of almost every point $\bar{x}$ in the quotient of a flap side is $\bar{s}_F$ is

$$|p^{-1}(\bar{x})| = \text{sh}_p(\star) \cdot \text{wt}_{B_{\#}}(F),$$

where $\text{sh}_p(\star)$ is the core sheet number, and $F$ is a lift in $\star$ of $\bar{F}$.

### 6.3 When $G\setminus T = X$ contains cycles

In Section 6.1 when $G\setminus T$ was a tree, we were able to augment all the basic vertex group actions in such a way as to obtain a discrete grouping, or a tree of action with discrete vertex groups. Suppose now that $G\setminus T = X$ is not a tree. To start we pick a spanning tree $Y \subset X$ and augment the vertex groups as in Section 6.1. Recall (Convention 6.1) that each edge $e \in \text{Edges}(X)$ is oriented and points towards a churro type vertex. Recall also that $e$ corresponds to an identification of a orbit of sides $G \cdot s_{\#e}$ to an orbit of strands $G \cdot \sigma_e$.

What remains to be shown is that for the edges in $X \setminus Y$, after performing the augmentations from $Y$, the stabilizers of strands and corresponding flap sides are isomorphic. Our argument will rely on counting preimages of points in flap sides and in strands and the fact that the isomorphism types of low power abelian groups are determined by their cardinalities.

We will first set some notation for this last part of the paper. Every vertex of $X$, either corresponds to an orbit of a waffle piece or a churro piece in $\boxtimes^*(\Gamma)$. If $w = \text{Verts}(X)$ is of waffle type then we define $B_w = B_{\#w}$ where $\boxtimes_w \subset \boxtimes^*(\Gamma)$ is a waffle piece whose $G$-orbit corresponds to $w$. Similarly, we define $B_c = B_{\#c}$ for a churro type vertex $c \in \text{Verts}(X)$. In both cases these basic vertex groups also come equipped with an action on the corresponding churro or waffle pieces.

Recall that $X$ is bipartite and by Convention 6.1 that edges are always oriented to point to churro type vertices. We say that $e \in \text{Verts}(X)$ is reflective if all (or, equivalently, one) of its lifts are reflective (recall Definition 6.2). We can now assign the following pair of numbers to an edge:

**Definition 6.28 (Edge weights and coweights).** Let $e \in \text{Edges}(X)$ correspond to the orbit of a pairing

$$\boxtimes_e \supset \sigma_e \sim s_e \subset F_e \subset \star_e$$

as given in (9). Then the weight and coweight of $e$ are:

- $\text{wt}(e) = \text{Density}_{B_{\#e}}(\sigma_e)$ (see Definition 6.21), and

- $\text{coWt}(e) = \text{wt}_{B_{\#e}}(F_e)$ (see Definition 6.26).

We remark that $\text{wt}(e) = \frac{1}{2}$ if and only if $e$ is reflective, otherwise $\text{wt}(e) = 1$.

**Proposition 6.29.** Let $\Gamma \leq G$ act cleanly on $\boxtimes^*(\Gamma)$ (recall Definition 2.14) and let

$$p : \Gamma \setminus \boxtimes^*(\Gamma) \to G\setminus \boxtimes^*(\Gamma)$$

be the induced quotient map. If $e$ is an edge of $X$, the graph underlying the graph of spaces decomposition of $G\setminus \boxtimes^*(\Gamma)$, connecting a strand quotient $\bar{\sigma} \subset B_{\boxtimes} \boxtimes$ with a flap side quotient $\bar{s}_F \subset \bar{F} \subset B_{\#}\setminus \star$ then,

$$\text{sh}_p(\boxtimes) \cdot \text{wt}(e) = \text{sh}_p(\star) \cdot \text{coWt}(e)$$

(16)
Proof. By Proposition 6.20 and Definition 6.21 of density, on a dense open subset of the quotient \( \bar{\sigma} \) of a strand \( \sigma \), every point has

\[ \text{sh}_p(\bar{\mathbb{H}}) \cdot \text{wt}(e) = \text{sh}_p(\mathbb{H}) \cdot \text{Density}_{B_{\mathbb{H}}}(\sigma) \]

\( p \)-preimages. Similarly Proposition 6.27 implies that a dense open set of points in the corresponding flap side has \( \text{sh}_p(\mathbb{H}) \cdot \text{coWt}(e) \) preimages. Since these sets coincide (16) holds.

Corollary 6.30 (Clean subgroup implies balance). For every edge \( e \in \text{Edges}(X) \) we assign number

\[ \nu(e) = \frac{\text{coWt}(e)}{\text{wt}(e)}. \]

If \( G \) admits a clean cocompact subgroup \( \Gamma \leq G \), then for any cycle \( c = e_1^{-1}, e_2, \ldots, e_k \) starting and ending at a churro-type vertex \( c_1 \in \text{Verts}(X) \) we must have the equivalent equalities:

\[ \frac{\nu(e_2)}{\nu(e_1)} \cdots \frac{\nu(e_k)}{\nu(e_{k-1})} = 1 \quad (17) \]

\[ \nu(e_1)\nu(e_3)\cdots\nu(e_{k-1}) = \nu(e_2)\nu(e_4)\cdots\nu(e_k). \quad (18) \]

\[ \prod_{\text{odd } j} \text{coWt}(e_j) \prod_{\text{even } i} \text{wt}(e_i) = \prod_{\text{even } i} \text{coWt}(e_i) \prod_{\text{odd } j} \text{wt}(e_j) \quad (19) \]

Proof. Let \( v_1, v_2, \ldots, v_k, v_{k+1} = c_1 \) be the vertices visited in order along the cycle \( c \). Because \( X \) is bipartite \( c_i \) is churro type for all odd indices, and waffle type for all even indices. Recall that all edges point to churro type vertices. Let \( p \) be the orbifold cover corresponding to \( \Gamma \leq G \) and let \( d_i = \text{sh}_p(\mathbb{H})_i \) or \( \text{sh}_p(\mathbb{H})_i \) be appropriate sheet number. Then by Proposition 6.29, for every even index \( j \) we have

\[ d_j = \nu(e_{j-1})d_{j-1} \]

\[ d_j = \nu(e_{j+1})d_{j+1} \]

\[ \Rightarrow d_{j-1} = \frac{\nu(e_{j+1})}{\nu(e_{j-1})}d_{j+1} \]

Since \( d_1 = d_{k+1} \), we obtain (17). The other equalities are obviously equivalent.

Remark 6.31. After having worked out this argument and looking back at [BK90], we noticed a striking resemblance between the Corollary above, which is a consequence of the existence of a subgroup acting freely and cocompactly, and [BK90, Proposition 1.2] characterizing the unimodularity of actions on trees.

Finally we can prove our main result.

Proposition 6.32. Suppose there is some \( \Gamma \leq G \) which is a clean cocompact subgroup. Then \( \mathbb{H}^*(\Gamma) \to G \setminus \mathbb{H}^*(\Gamma) \) admits a direct flat discrete grouping.

Proof. We will first construct a direct discrete grouping and then observe at the end that it can be used to make a flat grouping.

Let \( X = G \setminus T \). If \( X \) is a tree then the result holds from Proposition 6.10. We may therefore assume that \( X \) contains cycles. Let \( Y \subset X \) be a spanning tree, for every \( v \in \text{Verts}(X) = \text{Verts}(Y) \) we construct the augmentation \( A_v \) given in Definition 6.8 using the edges \( e \in \text{Edges}(Y) \). We require that for \( c \in ...
The groups $H(c)$ are the full flap factors of the corresponding churro-type vertices, i.e. some of them will contain direct factors corresponding to edges that are not in $\text{Edges}(Y)$.

We need to verify that with the augmented actions, the stabilizers of churro sides and strands identified by edges in $\text{Edges}(X\setminus Y)$ are isomorphic. Let $e \in \text{Edges}(X\setminus Y)$ connect a churro-type vertex $c$ and a waffle-type vertex $w$. Let $p : e_1, e_2, \ldots, e_{k-1}$ be the unique path in $Y$ connecting $w$ and $c$. The addition of the edge $e = e_k$ makes a minimal cycle. We have the following picture

$$c = c_1 \xrightarrow{e_1} w_2 \xrightarrow{e_2} c_3 \xrightarrow{e_{k-1}} w = w_k$$

with $c_1 = c$ and $w_k = w$. The edge $e = e_k$ is drawn as dotted as it does not lie in the spanning tree $Y$ and therefore does not use contribute to the augmentations. Let $A_w, A_c$ be the augmentations of $B_w, B_c$. Our goal is to show that the subgroups corresponding to strand or flap side stabilizers are isomorphic.

Recall that by Definition 6.8 and equation (15) in Lemma 6.9 that the augmentations depend only on which edges in $Y$ point towards and point away from a given vertex. This means that the discrepancy between the augmentations of $B_c$ and $B_w$ arises exactly from the edges along the path $e_1 \cdots e_{k-1}$ shown in (20). Specifically there is some common group $A$ such that $A_w = A \oplus A'_w$ and $A_c = A \oplus A'_c$ and the groups $A'_w, A'_c$ can be computed from the vertex groups shown along the path $e_1 \cdots e_{k-1}$ in (20).

Recall that in the sequence $c_1, w_2, c_3, \ldots, w_k$ odd indices correspond to churro type and even indices correspond to waffle type. For each churro-type vertex $c_j$, denote by $N_j$ the cardinality of the flap factor (Definition 5.10) of $B_{c_j}$, or equivalently of the churro contribution $H(c_j)$ (Definition 6.7). By Definition 6.8, augmentations $A_{\{f\}}$ get an extra $\mathbb{Z}_2$ factor, for every reflective edge $f$ such that $L_f = v$ and, by Lemma 6.9, every churro type vertex $c_l$ contributes a $H(\frac{N_l}{\text{coWt}(c_l,v)})$ factor, where $e_{c_l,v}$ is the edge adjacent to $c_l$ contained in the unique path from $c_l$ to $v$ (recall the relationship between coweights and cardinalities of flap families, Definition 6.26.) In particular $e_{c_l,v}$ points away from $v$. Now all the odd-indexed edges point to $c$ and the even-indexed edges point to $e$. It follows that in (20),

$$A'_w = \left( \bigoplus_{\text{odd } j} H\left(\frac{N_j}{\text{coWt}(e_j)}\right) \right) \oplus \left( \bigoplus_{\text{even } i} \text{wt}(e_i)^{-1} \right)$$

and

$$A'_c = \left( \bigoplus_{\text{even } i} H\left(\frac{N_{i+1}}{\text{coWt}(e_i)}\right) \right) \oplus \left( \bigoplus_{\text{odd } j} \text{wt}(e_j)^{-1} \right).$$

In particular

$$|A'_w| = \frac{N_1 \cdots N_{k-1}}{\prod_{\text{odd } j} \text{coWt}(e_j) \prod_{\text{even } i} \text{wt}(e_i)}$$

where indices range from 1 to $k - 1$ and

$$|A'_c| = \frac{N_2 \cdots N_{k-1}}{\prod_{\text{even } i} \text{coWt}(e_i) \prod_{\text{odd } j} \text{wt}(e_j)}$$

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where indices range from 1 to \(k - 1\). In particular (19) and the fact that we are missing the \(e = e_k\)-factors implies that

\[
\frac{|A'_w|}{|A'_c|} = \frac{N_1 \cdots N_{k-1}}{N_2 \cdots N_{k-1}} \cdot \prod_{\text{even } i} \text{coWt} (e_i) \prod_{\text{odd } j} \text{wt} (e_j) = \frac{N_1 \text{wt} (e_k)}{\text{coWt} (e_k)}.
\]

This gives:

\[
|A'_w| \cdot \text{wt} (e_k)^{-1} = |A'_c| \cdot \frac{N_1}{\text{coWt} (e_k)}.
\]

It therefore follows, since low-power abelian groups are determined by cardinality, that

\[
A'_w \approx A'_c \oplus H \left( \frac{N_1}{\text{coWt} (e_k)} \right) \quad \text{if } e_k \text{ is non-reflective}
\]

\[
A'_w \oplus Z_2 \approx A'_c \oplus H \left( \frac{N_1}{\text{coWt} (e_k)} \right) \quad \text{if } e_k \text{ is reflective.}
\]

In particular note that if \(e_k\) is reflective then the augmentation \(A'_c\) is larger in cardinality with respect to \(A'_w\). It follows that there is a group \(A''_c \leq A'_c \approx A''_c \oplus \langle \tau_{e_k} \rangle\) such that:

\[
(B_w \oplus A_w) \approx B_w \oplus A \oplus A''_c \oplus H \left( \frac{N_1}{\text{coWt} (e_k)} \right)
\]

\[
(B_c \oplus A_c) \approx B_c \oplus A \oplus A''_c \oplus \langle \tau_{e_k} \rangle.
\]

Where \(\tau_{e_k} = Z_2\) if \(e_k\) is reflective, and trivial otherwise. Consider now all edges in \(X\). Since \(c_1 = c\) we have \(N_1 = \prod_{c \in \text{star}(e)} |H(e)|\) and \(\text{coWt} (e_k) = |H(e_k)|\) so that

\[
H \left( \frac{N_1}{\text{coWt} (e_k)} \right) \approx \bigoplus_{f \in \text{star}(t(e)) \setminus \{e\}} H(f).
\]

Analogously to the proof of Proposition 6.10, we use the description given by Lemma 6.4 to write out the stabilizers with respect to the augmented action given Definition 6.5 to get:

\[
(B_w \oplus A_w)_{\sigma_e} \approx Z_e \oplus \langle \tau_e \rangle \oplus \left[ A \oplus A''_c \oplus \left( \bigoplus_{f \in \text{star}(t(e)) \setminus \{e\}} H(f) \right) \right]
\]

\[
(B_c \oplus A_c)_{\sigma_e} \approx Z_e \oplus \left( \bigoplus_{f \in \text{star}(t(e)) \setminus \{e\}} H(f) \right) \oplus \left[ A \oplus A''_c \oplus \langle \tau_e \rangle \right].
\]

The factors in square brackets correspond to the augmentations. These two groups are clearly isomorphic, which means that we can construct a graph of actions \(X^G\) that realizes a direct discrete grouping of \(\boxplus^* (\Gamma) \to G \setminus \boxplus^* (\Gamma)\).

Furthermore both of these groups contain a subgroup isomorphic to \(Z_e \oplus \{1\}\), which contains all clean isometries fixings these subspaces of the respective pieces. As observed in the proof of Proposition 6.10 any isomorphism of these groups identifying these subgroups will give rise to a flat discrete grouping.

\[\text{proof of Theorem A. This follows from Proposition 6.32 and Corollary 5.19.}\]
7 Conclusion

As mentioned in the introduction, we believe this paper plays a foundational role in the classification of constructible hyperbolic groups, the smallest class of hyperbolic groups closed under amalgamations over finite and virtually cyclic subgroups. After free groups, the most basic constructible groups are graphs of free groups with cyclic edge groups (which incidentally includes the class of closed surface groups). One may therefore wonder why we didn’t prove Theorem A for graphs of rigid free groups with cyclic edge groups.

The overall approach for this case should be the same. The first step is to apply pattern rigidity [CM11]. In their paper however, instead of obtaining a waffle with a discrete isometry group, one obtains a cube complex \( X \) that is quasi-isometric to a tree. Applying [HT17, Theorem C] to this tree, we can replace \( X \) with a tree equipped with a line pattern \( \mathcal{L} \). Things break down due to the fact that \( \text{Aut}(X, \mathcal{L}) \), unlike the automorphism group of a waffle, may be an uncountable totally disconnected locally compact group, as shown by the example in [CM11, §6.3]. Indeed, a key feature of waffles that gets used throughout Section 6 is Corollary 3.12, which immediately gives waffles discrete groupings. So even though \( \text{Aut}(\mathbb{F}^p(\Gamma)) \) is not discrete in general, this “local discreteness” enables us to make progress.

To be able to construct a satisfactory grouping for a graph of free groups we would first need a variation of Leighton’s theorem, specifically for the case of symmetry-restricted graphs (see [Neu10, Theorem 4.1]). While this paper was being finalized Woodhouse in [Woo18] proved this necessary generalization of Leighton’s Theorem, applicable to trees equipped with line patterns. Although Woodhouse’s result doesn’t immediately solve the problem, it should play a key role in the proof of the following:

Conjecture 7.1. If \( \Gamma \) is the fundamental group of a graph of groups with a JSJ decomposition in which all non-cyclic vertex groups are rigid free groups, then \( \Gamma \) is quasi-isometrically rigid (in the sense of Definition 2.3).

All that being said, Theorem A remains fundamental because, although surfaces are fundamental groups of graphs of free groups, our main result would not follow from a positive answer to Conjecture 7.1.

A further direction of interest is investigation across constructible groups. Our result relates quasi-isometric rigidity to rigidity in JSJ decomposition. Two natural questions are:

1. Is there a quasi-isometric rigidity combination theorem for rigid cyclic amalgams?

2. Is the presence of QH subgroups the only possible cause of multiple commensurability classes within a given quasi-isometry class of constructible hyperbolic groups.

Ultimately we hope that this line of investigation will lead to a complete understanding of the commensurability classes within the quasi-isometry classes of constructible hyperbolic groups.

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