Representing Attitudes Towards Ambiguity in Managerial Decisions

Sandro Sozzo* and Marta Gasparin†

Abstract

We provide here a general mathematical framework to model attitudes towards ambiguity which uses the formalism of quantum theory as a “purely mathematical formalism, detached from any physical interpretation”. We show that the quantum-theoretic framework enables modelling of the Ellsberg paradox, but it also successfully applies to more concrete human decision-making (DM) tests involving financial, managerial and medical decisions. In particular, we provide a faithful mathematical representation of various empirical studies which reveal that attitudes of managers towards uncertainty shift from ambiguity seeking to ambiguity aversion, and vice-versa, thus exhibiting hope effects and fear effects in management decisions. The present framework provides a new bold and promising direction towards the development of a unified theory of decisions in the presence of uncertainty.

Keywords: Quantum structures; Decision theory; Ellsberg paradox; Ambiguity attitudes; Hope and fear effects.

1 Introduction

Notwithstanding the remarkable success of expected utility theory (EUT), a number of fundamental problems are still unsolved before a single theory of human decision-making (DM) under uncertainty is unanimously accepted.

A first issue is exactly defining what one means by “uncertainty”. Following Knight [1], two kinds of uncertainty are possibly present: objective uncertainty, or risk, designates situations where probabilities are not known or knowable, i.e. can be estimated from past data or calculated by means of mathematical rules. By contrast, subjective uncertainty, or ambiguity, designates situations where probabilities are neither known, nor can they be derived, calculated or estimated in an objective way [2].

The predominant model of DM under risk was elaborated by von Neumann and Morgenstern [3], who identified the axioms allowing to uniquely represent human preferences over lotteries by maximization of the EU functional. On the one side, however, decision puzzles like the Allais paradox reveal that some of these axioms are violated in concrete choices [4]. And, on the other hand, this formulation does not account for ambiguity, which is prevalent in social science and likely influences social science decisions. The so-called Bayesian paradigm tries to fill this gap introducing the notion of subjective probability: even when the probabilities are not known, people may still form their own beliefs, or priors, which may vary from a person to another. People then update their beliefs according to the Bayes rule of standard probability theory, i.e. the one axiomatized by Kolmogorov (Kolmogorovian probability) [5].

*School of Business and Centre IQSCS, University Road LE1 7RH, Leicester (United Kingdom), Email address: ss831@le.ac.uk
†School of Business, University Road LE1 7RH, Leicester (United Kingdom), Email address: mg352@le.ac.uk
In 1950s, Savage extended von Neumann-Morgenstern EUT within the Bayesian paradigm, presenting a set of axioms which, once satisfied, “compel” decision-makers to behave as if they had a single Kolmogorovian probability with respect to which they maximize EU [6]. Savage’s formulation, also known as subjective EUT, provides the foundations of “rational behaviour”, that is, subjective EUT prescribes how people should behave in the presence of uncertainty, and it has been widely used in decision theory, economics and finance, because of its mathematical simplicity and predictive success.

Ellsberg showed in two simple experiments, the two-urn example and the three-colour example, that decision-makers do not always maximize EU but, rather, they prefer risky acts over ambiguous acts, a behaviour known as ambiguity aversion [7]. Ellsberg preferences particularly violate the famous sure thing principle, one of the building blocks of Savage’s axiomatization, and they have been empirically confirmed several times (see, e.g., the review in [9] [8]). These well-documented violations of EUT have led many scholars to elaborate alternative DM models, which even include representation of beliefs by more general structures than a single Kolmogorovian probability measure (see, e.g., the reviews in [8, 10, 9]).

Things are not cleared out if one considers more concrete DM tests, in which financial, managerial, medical, etc., decisions are taken in the presence of uncertainty. In these cases, indeed, different attitudes towards ambiguity, e.g., ambiguity seeking, arise in addition to the ambiguity aversion revealed by Ellsberg preferences. In order to realize the possibility of attitude reversal, consider the following example.

Suppose your doctor tells you that there is a defined probability that you have a serious disease. You then decide to consult other doctors. Some of them believe that the probability is much higher while others believe that the probability is less. Which option would you “prefer” – the former which is risky, or the latter which is ambiguous? Intuition suggests that the level of probability will play a crucial role in the final decision. Indeed, if the probability is low, it is reasonable to assume that a fear effect occurs and you prefer the risky option to the ambiguous one, thus showing an ambiguity aversion behaviour. On the other side, if the probability is high, it is reasonable to assume that a hope effect occurs and you instead prefer the ambiguous option, thus showing an ambiguity seeking behaviour [11].

A comparison between a risky and an ambiguous option, like the one presented in the two-urn example, was part of two experimental studies on managers, one on DM under environmental uncertainty performed by Viscusi and Chesson [11], and the other on investment decisions under performance uncertainty by Ho, Keller and Keltyka [12]. In both cases, a shift from hope to fear effects, and vice versa, was observed which is incompatible with a theory of rational preferences.

The present study fits an emergent research programme which applies with success the mathematical formalism of quantum theory to model complex cognitive phenomena, including categorization, decision, judgement, language and perception, where classical Bayesian, or Kolmogorovian, modelling techniques are problematical [13, 14, 15, 16, 17, 18]. This research has recently been extended to include the elaboration of quantum-based approaches to economics and finance [19, 20, 21, 22, 23, 24, 25]. In particular, we have recently worked out a theoretical framework to represent individuals’ preferences and choices under uncertainty (risk, ambiguity) that uses the formalism of quantum theory as a “pure mathematical formalism consisting in non-distributive probability measures and linear vector spaces over complex numbers”.

Indeed, this quantum mathematics in Hilbert spaces has some advantages in modelling the information uncertainty that is induced by a non-controllable context, like a cognitive one [26, 27, 28, 29, 30, 31]. A novel theoretical element of the quantum-theoretic framework is the introduction of the state of the conceptual entity under investigation (DM entity), which has a cognitive nature and can change under the interaction with the decision-maker. The notion of cognitive state, its representation and its connections with subjective probabilities through quantum probability, provide the conceptual and mathematical tools that enable capturing ambiguity and individual attitudes towards it. The quantum-theoretic framework has been successfully applied to model the Ellsberg paradox and recent variants, as recognised by some of the proponents (see, e.g., [32], p. 3836). This result can be seen as the first step towards the construction...
of a state-dependent extension of EUT (quantum EUT, or QEUT) where subjective probabilities are represented by quantum probabilities, rather than classical Kolmogorovian probabilities – structurally, quantum probability is more general than Kolmogorovian probability, as the latter rests on a distributive algebra, while the former does not.

In the present paper, we extend our previous findings, showing that the quantum-theoretic framework above enables successful modelling of the Ellsberg two-urn example, also providing a faithful mathematical representation of data collected on the two-urn DM test. This enables us to reproduce the ambiguity aversion pattern observed in the data. However, the quantum-theoretic framework can also incorporate different attitudes towards ambiguity, including ambiguity seeking behaviour, as well as shifts from one attitude to another. And, indeed, we show here that hope and fear effects in investment choices can be naturally incorporated into the quantum modelling of a DM scenario where a risky option is opposed to an ambiguous option, like in the DM tests in [29] and [12].

The results above provide a strong support to the systematic use of quantum mathematical structures in economics and to the development of a unitary quantum-based theory of DM under uncertainty.

For the sake of completeness, we summarize the content of this paper in the following.

In Sec. 2, we summarize the basic mathematics of subjective EUT (Sec. 2.1), together with the Ellsberg paradox in the two-urn example (Sec. 2.2) and analyse the empirical study of Ho, Keller and Keltyka [12] on attitudes reversal in the presence of ambiguity (Sec. 2.3). We then review in Sec. 3 the quantum-theoretic framework and its proposal of extension of subjective EUT and emphasize its novel elements with respect to more traditional approaches. Successively, we apply in Sec. 4 the quantum-theoretic framework, model the two-urn example (Sec. 4.1) and represent data collected in the two-urn experiment performed by ourselves and revealing an ambiguity aversion pattern (4.2). Finally, we elaborate in Sec. 5 a mathematical model in Hilbert space which accounts for the shifts between ambiguity aversion and ambiguity seeking behaviour in either direction. This allows us to represent hope and fear effects within a single theoretical framework. The mathematical framework works in both investment gain (Sec. 5.1) and loss (Sec. 5.2) scenarios of the empirical study in [12], whose data are also faithfully represented (Sec. 5.3).

2 Expected utility, paradoxes, ambiguity and its effects

We present in the following sections the essential definitions and results within subjective EUT, together with the Ellsberg paradox and some DM tests revealing individuals' behaviour in the presence of uncertainty. The reader who is interested to deepen these results can refer to [6, 9, 10].

The starting point which we will assume in the following is that human preferences are revealed by the decisions of individual agents (or, decision-makers).

2.1 Basic mathematical framework of subjective expected utility theory

The first axiomatization of decision-making under uncertainty was formulated by von Neumann and Morgenstern, who presented a set of axioms allowing to uniquely represent human decisions by means of the maximization of the EU functional with respect to a single Kolmogorovian probability measure.

A major limitation of von Neumann-Morgenstern’s formulation is that it only deals with the uncertainty that can be formalized by known probabilities (objective uncertainty, or risk). On the other hand, situations frequently occur where uncertainty cannot be formalized by known probabilities (subjective uncertainty, or ambiguity) [1]. As mentioned in Sec. 1 a Bayesian model would introduce the notion of subjective probability, thus minimizing the distinction between objective and subjective uncertainty. In a Bayesian model, if probabilities are not known, people anyway form their own beliefs (or priors), which may differ across individuals but are still formalized by Kolmogorovian probabilities [2]. As a matter of fact, Savage
presented an axiomatic formulation of subjective EUT which extends von Neumann-Morgenstern’s in agreement with a Bayesian model [6].

To present the mathematics of subjective EUT, we preliminarily introduce the following symbols.

Let $\mathcal{I}$ be the set of all physical states of nature and let $\mathcal{P}(\mathcal{I})$ be the power set of $\mathcal{I}$, that is, the set of all subsets of $\mathcal{I}$. Let $\mathcal{A} \subseteq \mathcal{P}(\mathcal{I})$ be a $\sigma$-algebra. An element $E \in \mathcal{A}$ denotes an event in the usual sense. Let $p : \mathcal{A} \subseteq \mathcal{P}(\mathcal{I}) \rightarrow [0, 1]$ be a Kolmogorovian probability measure over $\mathcal{A}$, that is, a normalized countably additive measure satisfying the axioms of Kolmogorov [7].

Then, let $\mathcal{X}$ be the set of all consequences, and let the function $f : \mathcal{I} \rightarrow \mathcal{X}$ denote an act. We denote the set of all acts by $\mathcal{F}$. Moreover, let $\succsim$ denote a weak preference relation, that is, a relation over the Cartesian product $\mathcal{F} \times \mathcal{F}$ which is complete and transitive. We adopt the usual interpretation of $\succ$ and $\sim$ as strong preference and indifference relations, respectively, namely, if a person strongly, or strictly, prefers act $f$ to act $g$, we write $f \succ g$; analogously, if a person is indifferent between $f$ and $g$, we write $f \sim g$.

Next, let $\mathbb{R}$ be the real line and $u : \mathcal{X} \rightarrow \mathbb{R}$ be a utility function, which we assume to be a strictly increasing and continuous function, as it is usual in the literature.

Let us now introduce some simplificative assumptions, which however do not affect our conclusions in this and the following sections. Firstly, we suppose that the set $\mathcal{I}$ is discrete and finite. Secondly, we suppose that an element $x \in \mathcal{I}$ denotes a monetary payoff, so that $\mathcal{X} \subseteq \mathbb{R}$.

Let $E_1, E_2, \ldots, E_n$ denote mutually exclusive and exhaustive elementary events, which thus form a partition of $\mathcal{I}$. If $x_i$ is the utility associated by the act $f$ to the event $E_i$, $i \in \{1, \ldots, n\}$, then $f$ can be equivalently represented by the $2n$-tuple $f = (E_1, x_1; \ldots; E_n, x_n)$, which is interpreted in the usual way as “we get the payoff $x_1$ if the event $E_1$ occurs, the payoff $x_2$ if the event $E_2$ occurs, \ldots, the payoff $x_n$ if the event $E_n$ occurs.

We finally define the expected utility functional associated with the act $f = (E_1, x_1; \ldots; E_n, x_n)$ with respect to the Kolmogorovian probability measure $p$ as $W(f) = \sum_{i=1}^{n} p(E_i) u(x_i)$.

Representation theorem. If the algebraic structure $(\mathcal{F}, \succsim)$ satisfies the axioms of ordinal event independence, comparative probability, non-degeneracy, small event continuity, dominance and the sure thing principle, then, for every $f, g \in \mathcal{F}$, a unique Kolmogorovian probability measure $p : \mathcal{A} \subseteq \mathcal{P}(\mathcal{I}) \rightarrow [0, 1]$ and a unique (up to positive affine transformations) utility function $u : \mathcal{X} \rightarrow \mathbb{R}$ exist such that $f$ is preferred to $g$. In symbols, $f \succsim g$ if and only if the expected utility of $f$ is greater than the expected utility of $g$, i.e. $W(f) \geq W(g)$. For every $i \in \{1, \ldots, n\}$, the utility value $u(x_i)$ depends on the individual’s attitudes towards risk, while $p(E_i)$ is interpreted as a subjective probability, expressing the individual’s belief that the event $E_i$ occurs [6].

Savage’s representation theorem is at the same time compelling at a normative level and testable at a descriptive level. Indeed, regardless of the former, if the axioms are intuitively reasonable and decision-makers agree with them, then they must all behave as if they maximized EU with respect to a single probability measure satisfying Kolmogorov’s axioms; furthermore, regarding the latter, the axioms suggest to perform concrete DM tests to confirm/disprove the general validity of subjective EUT, hence of the axioms themselves. This is why Savage’s EU formulation is typically accepted to prescribe “how rational agents should behave in the presence of uncertainty”. However, on the one side, the theory offers very little about where beliefs come from and how they should be calculated while, on the other side, DM tests, performed since the 1960s, have systematically found deviations from that rational behaviour in concrete situations. This will be the content of Secs. 2.2 and 2.3.

2.2 The Ellsberg paradox

In 1961, Daniel Ellsberg presented various thought experiments in which he suggested that decision-makers would prefer acts with known (or objective) probabilities over acts with unknown (or subjective) probabil-
we presented a sample of 200 participants with a questionnaire in which they had to choose between the pairs of acts “$f_1$ versus $f_2$” and “$f_3$ versus $f_4$” in Table 1. Respondents were provided with a paper similar to the one in Figure 1. In the two-urn test, 26 participants chose acts $f_1$ and $f_3$, 10 chose $f_1$ and $f_4$, 6 chose $f_2$ and $f_3$, and 158 chose $f_2$ and $f_4$. Hence, overall 164 respondents over 200 preferred act $f_2$ over act $f_1$, for a preference rate of $164/200 = 0.82$ (the for the sake of simplicity, we assumed that each choice concerned two alternatives, hence indifference between acts was not a possible option.)
difference is significant, $p = 1.49E-24$). Moreover, 168 respondents over 200 preferred act $f_4$ over act $f_3$, for a preference rate of $168/200=0.84$ (the difference is again significant, $p = 1.25E-28$). Finally, 16 respondents over 200 preferred either $f_1$ and $f_4$ or $f_2$ and $f_3$, for an inversion rate of $184/200=0.92$. This pattern accords with Ellsberg preferences and straightly points towards ambiguity aversion, thus confirming existing results in empirical literature.

The Ellsberg paradox and other Ellsberg-type puzzles put at stake both the descriptive and normative foundations of subjective EUT, which led various scholars to propose alternatives to subjective EUT, in which more general, also non-Kolmogorovian, mathematical structures are used to represent subjective uncertainty. Major non-EU models include Choquet EU, cumulative prospect theory, maxmin EU, alpha maxmin EU, smooth preferences, variational preferences, etc. (see, e.g., the reviews in [9, 10]).

However, Mark Machina elaborated in 2009 two variants of Ellsberg examples, the 50/51 example and the reflection example, which challenge major non-EU models in a similar way as Ellsberg examples challenge subjective EUT [33, 34]. Machina preferences have been confirmed in two tests against the predictions of both subjective EUT and its non-EU generalizations [29] and [35]. The implication of Ellsberg and Machina paradoxes is that a unified theoretic approach to represent human preferences and choices under uncertainty is still an unachieved goal [35].

2.3 Experimental studies on ambiguity attitudes

We have seen in Sec. 2.2 that, in each pair of acts of the two-urn example, people are asked to choose between a risky option, that is, an option with known probability of getting a given consequence, and an ambiguous option, that is, an option with unknown probability, but belonging to a given range, of getting the same consequence. This is exactly the experimental setting that is designed to test individual attitudes

Figure 1. A sample of the questionnaire on the two-urn example. It corresponds to the choice between acts $f_1$ and $f_2$ in Table 1.
towards ambiguity in more concrete DM situations, involving medical, managerial and financial decisions.

Tests in scenarios different from urns have revealed different attitudes towards ambiguity, namely, ambiguity neutral and ambiguity seeking, in addition to the ambiguity aversion identified in the two-urn example (see, e.g., [8, 9] for a review of these studies). In these cases, attitudes towards ambiguity depend on:

(i) likelihood of uncertain events;
(ii) domain of the consequences;
(iii) source that generates ambiguity [36].

We review some of these studies in the present section. To this end, we preliminarily introduce two notions which express the success or failure of a financial operation, as follows. A gain is realized when the result of an operation made by the decision-maker is above expectations. A loss is realized when the result of an operation made by the decision-maker is below expectations.

For example, Viscusi and Chesson presented an experimental study on insurance decisions in the presence of environmental uncertainty [11]. The study consisted in asking participants to compare a risky option with a given probability \( \bar{p} \) to realize a gain (respectively, a loss), with an ambiguous option with a probability to realize a gain (respectively, a loss) ranging between \( \bar{p} - \Delta \) and \( \bar{p} + \Delta \). The authors found that (i) plays a fundamental role in determining ambiguity attitudes. Indeed, if the probability of a gain is high, then a fear effect occurs in which people tend to be ambiguity averse. But, as the probability of a gain decreases, people tend to be less ambiguity averse, reaching a crossover point in which they become ambiguity seeking, which indicates a shift from a fear to a hope effect. Viceversa, if the probability of a loss is high, then a hope effect occurs in which people tend to be ambiguity seeking. But, as the probability of a loss decreases, people tend to be less ambiguity seeking, reaching a crossover point in which they become ambiguity averse, which indicates a shift from a hope to a fear effect.

This empirical pattern was confirmed at high probabilities by a test performed by Ho, Keller and Keltyka [12] on 40 MBA students. Managers are typically provided with a target, or a benchmark, to measure their performance. Thus, a management decision corresponds to a gain (loss) if it leads to a higher (lower) outcome than the benchmark. The authors considered two experiments in a “within subjects design”, as follows.

ROI experiment. In this experiment, the return on investment (ROI) had a benchmark of 16%. In the loss scenario, participants had to compare a risky option with a probability of a loss equal to 66%, with an ambiguous option with a probability of a loss ranging between 40% and 80%. In the gain scenario, participants had instead to compare a risky option with a probability of a gain equal to 63%, with an ambiguous option with a probability of a gain ranging between 42% and 84%.

In the ROI experiment, the authors found that, in the loss scenario, 59% of the participants preferred the ambiguous option, thus revealing an ambiguity seeking attitude, hence the presence of a hope effect (41% preferred instead the risky option). On the contrary, in the gain scenario, 66% of the participants preferred the risky option, thus revealing an ambiguity averse attitude, hence the presence of a fear effect. These findings agree with those in [11].

IRR experiment. In this experiment, the internal rate of return (IRR) had a benchmark of 15%. In the loss scenario, participants had to compare a risky option with a probability of a loss equal to 65%, with an ambiguous option with a probability of a loss ranging between 45% and 85%. In the gain scenario, participants had instead to compare a risky option with a probability of a gain equal to 68%, with an ambiguous option with a probability of a gain ranging between 47% and 89%.

In the IRR experiment, the authors found that, in the loss scenario, 65% of the participants preferred the ambiguous option, thus revealing an ambiguity seeking attitude, hence the presence of a hope effect (35% preferred instead the risky option). On the contrary, in the gain scenario, 62% of the participants preferred the risky option, thus revealing an ambiguity averse attitude, hence the presence of a fear effect.
These findings are consistent with those in the ROI experiment and agree with the patterns in [11].

Additional experiments were performed in [12] in which other sources of uncertainty, like outcome ambiguity, were investigated. However, we will not deal with these additional experiments here, for the sake of brevity.

3 A quantum-theoretic framework for human decision-making

We present in this section the essentials of the general quantum-theoretic framework we have recently elaborated to model human decisions in the presence of uncertainty, which proposes a unitary solution to the puzzles in Secs. 2.2 and 2.3 [26, 27, 28, 29, 30, 31].

Before proceeding further, we remind here that this approach only needs the essential mathematics of quantum theory, which mainly consists in the use of a finite dimensional linear vector spaces over complex numbers, called Hilbert space, and in the introduction of a more general probability than Kolmogorovian probability, called quantum probability. These mathematical notions are indeed more suited to represent cognitive states and context-induced changes of cognitive states which must be added to physical states of nature in order to incorporate the uncertainty surrounding the decision process.

The object of the decision, which the decision-maker interacts with, defines a conceptual DM entity $\Omega_{DM}$ which is supposed to be always in a defined state $p_v$. This state expresses the “different potential ways $\Omega_{DM}$ can be influenced by the decision-maker”, hence it has a cognitive nature and must be distinguished from a physical state of nature (see Sec. 2.1) [14, 18]. More, $p_v$ captures, both mathematically and conceptually, aspects of ambiguity and ambiguity attitudes, as we will see in the following. The notions of “conceptual entity” and “cognitive state” may sound a bit vague and abstract to the reader who is not trained in the foundations of quantum physics. However, the meaning of these notions will be clarified to the specific examples we will consider in Secs. 4 and 5.

Let $\Sigma_{DM}$ be the set of all states of $\Omega_{DM}$ and let $\mathcal{E}$ be the set of all events which may occur. For every $p_v \in \Sigma_{DM}$, let $\mu(E,p_v)$ be the (subjective) probability that $E$ occurs when $\Omega_{DM}$ is in the state $p_v$.

Then, let $E_1, E_2, \ldots, E_n \in \mathcal{E}$ denote mutually exclusive and exhaustive elementary events, let $\mathcal{X} \subseteq \mathbb{R}$ be the set of all consequences (assumed to indicate monetary payoffs), and let, for every $i \in \{1, \ldots, n\}$, the act $f \mapsto$ map the elementary event $E_i \in \mathcal{E}$ into the payoff $x_i \in \mathbb{R}$, so that $f = (E_1, x_1; \ldots; E_n, x_n)$. Finally, let $u : \mathcal{X} \rightarrow \mathbb{R}$ be a continuous strictly increasing utility function expressing individual attitudes towards risk.

We use the canonical notation and representation of quantum theory (see, e.g., [14, 18]) and associate $\Omega_{DM}$ with a Hilbert space $\mathcal{H}$, that is, a finite-dimensional vector space endowed with a scalar product $\langle \cdot | \cdot \rangle$, over the field $\mathbb{C}$ of complex numbers. The number $n$ of mutually exclusive and exhaustive elementary events entails that $\mathcal{H}$ can be chosen to be isomorphic to the Hilbert space $\mathbb{C}^n$ of all n-tuples of complex numbers. Let $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$ be the canonical orthonormal (ON) basis of $\mathbb{C}^n$, where $|e_1\rangle = (1, 0, \ldots 0)$, $|e_2\rangle = (0, 1, \ldots)$, $|e_n\rangle = (0, 0, \ldots 1)$.

A state $p_v \in \Sigma_{DM}$ of $\Omega_{DM}$ is represented by a vector $|v\rangle \in \mathbb{C}^n$ such that $\sqrt{\langle v | v \rangle} = 1$, or unit vector.

An event $E \in \mathcal{E}$ is represented by an orthogonal projection operator $\hat{P}$ over $\mathbb{C}^n$, that is, an operator $\hat{P} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that, for every $v, w \in \mathbb{C}$ and $|v\rangle, |w\rangle \in \mathbb{C}^n$, $\hat{P}(v|v\rangle + w|w\rangle) = v\hat{P}|v\rangle + w\hat{P}|w\rangle$ (linearity), $\hat{P}^2|v\rangle = \hat{P}|v\rangle$ (idempotency) and $\langle v|\hat{P}|w\rangle = \langle w|\hat{P}|v\rangle^*$ (self-adjointness), where $^*$ denotes complex conjugate.
conjunction. The set $\mathcal{L}(\mathbb{C}^n)$ of all orthogonal projection operators over $\mathbb{C}^n$ forms a non-distributive lattice structure, unlike a Boolean algebra which is distributive.

It follows from the above quantum representation of events that, for every $i \in \{1, \ldots, n\}$, the elementary event $E_i$ is represented by the 1-dimensional orthogonal projection operator $\hat{P}_i = |e_i\rangle\langle e_i|$

For every state $p_v \in \Sigma_{DM}$ of $\Omega_{DM}$, represented by the unit vector $|v\rangle = \sum_{i=1}^{n} (\alpha_i |v\rangle |e_i\rangle \in \mathbb{C}^n$, the function

$$\mu_v : \hat{P} \in \mathcal{L}(\mathbb{C}^n) \mapsto \mu_v(\hat{P}) \in [0, 1]$$

induced by the Born rule, is a quantum probability measure over $\mathcal{L}(\mathbb{C}^n)$. In particular, we identify $\mu_v(\hat{P})$ with the (subjective) probability $\mu(E, p_v)$ that the event $E$, represented by the orthogonal projection operator $\hat{P}$, occurs when $\Omega_{DM}$ is in the state $p_v$. Thus, in particular, for every $i \in \{1, \ldots, n\}$,

$$\mu(E_i, p_v) = \langle v|\hat{P}_i|v\rangle = |\langle e_i|v\rangle|^2$$

Let us now represent acts by using the quantum mathematical formalism. The act $f = (E_1, x_1; \ldots; E_n, x_n)$ is represented by the self-adjoint operator

$$\hat{F} = \sum_{i=1}^{n} u(x_i)\hat{P}_i = \sum_{i=1}^{n} u(x_i)|e_i\rangle\langle e_i|$$

Then, we introduce, for every $p_v \in \Sigma_{DM}$, the functional “EU in the state $p_v$”, $W_v : \mathcal{F} \rightarrow \mathbb{R}$, as follows. For every $f \in \mathcal{F}$,

$$W_v(f) = \langle v|\hat{F}|v\rangle = \langle v|\left(\sum_{i=1}^{n} u(x_i)\hat{P}_i\right)|v\rangle = \sum_{i=1}^{n} u(x_i)\langle e_i|v\rangle |\langle e_i|v\rangle|^2 = \sum_{i=1}^{n} \mu(E_i, p_v)u(x_i)$$

where we have used (2) and (3). Equation (4) generalizes the usual EU formula of Sec. 2.2.

We observe that the EU generally depends on the state $p_v$ of the DM entity $\Omega_{DM}$. When $W_v(f)$ does (not) explicitly depend on the state $p_v$, then the act $f$ is (un)ambiguous, in the sense specified in Sec. 2.2. This agrees with the insight above that the state $p_v$ mathematically and conceptually incorporates the presence of ambiguity.

Let us now come to the dynamical part of the DM process. The state of the DM entity can change under the effect of a context, which has again a cognitive nature. An example of such a context-dependence is the presence of ambiguity. In symbols, for every $f, g \in \mathcal{F}$, states $p_v, p_w \in \Sigma_{DM}$ may exist such that $W_v(f) > W_v(g)$, whereas $W_w(f) < W_w(g)$, depending on decision-makers’ attitudes towards ambiguity. This suggests introducing a state-dependent preference relation $\succ_v$ on the set of acts $\mathcal{F}$, as follows. For every $f, g \in \mathcal{F}$ and $p_v \in \Sigma_{DM}$, $f \succ_v g$ if and only if $W_v(f) \geq W_v(g)$.
We have recently proved that a context-induced state change explains the inversion of preferences observed in the Ellsberg and Machina paradox situations, which can be successfully modelled in the quantum-theoretic framework [27]. In addition, we have provided a quantum representation of various DM tests, including the three-color example [26, 29, 31], the 50/51 example [26, 29] and the reflection example [28, 29].

The results above are important, in our opinion, because the quantum-theoretic framework provides a unitary solution to several paradoxes and pitfalls of rational decision theory, opening at the same time the way towards a quantum-based state-dependent generalization of subjective EUT, or QEUT [28]. Furthermore, the quantum-theoretic framework allows us to draw the following conciliatory result.

According to subjective EUT, decision makers should maximize EU with respect to a single Kolmogorovian probability measure. The quantum-theoretic framework shows that decision makers actually maximize EU with respect to a non-Kolmogorovian, namely quantum, probability measure.

Regarding the two-urn example in Sec. 2.2 the quantum-theoretic framework can be successfully applied too, as we have proved in [30]. However, due to its paradigmatic nature to represent ambiguity attitudes, we want to dedicate a separate section to the modelling.

4 A quantum model for ambiguity aversion effects

In this section, we particularize the quantum-theoretic framework in Sec. 3 to the two-urn example, proving that it enables faithful representation of the empirical data in Sec. 2.2 and that it enables modelling of hope and fear effects in management decision tests.

4.1 Quantum representation of the two-urn example

The two-urn example defines two conceptual entities, DM entity $\Omega_{DM}^{I}$ which is the urn with 100 red or black balls in unknown proportion, and DM entity $\Omega_{DM}^{II}$ which is the urn with 50 red balls and 50 black balls. In the absence of any interaction with a decision-maker, both entities are initially in the same state $p_{v_{0}}$, which has a cognitive nature, as we have seen in Sec. 3. This initial state does not depend on any cognitive context and it uniquely determined by symmetry considerations on physical urns.

Let $E_{R}$ and $E_{B}$ denote the exhaustive and mutually exclusive elementary events “a red ball is drawn” and “a black ball is drawn”, respectively. Both $\Omega_{DM}^{I}$ and $\Omega_{DM}^{II}$ are thus associated with a 2-dimensional complex Hilbert space, which we choose to be $\mathbb{C}^{2}$. Let $|e_{1}\rangle = (1, 0)$ and $|e_{2}\rangle = (0, 1)$ be the unit vectors of the canonical ON basis of $\mathbb{C}^{2}$. The elementary events $E_{R}$ and $E_{B}$ are represented by the projection operators $\hat{P}_{R} = |e_{1}\rangle \langle e_{1}|$ and $\hat{P}_{B} = |e_{2}\rangle \langle e_{2}| = 1 - \hat{P}_{R}$, projecting onto the 1-dimensional subspace generated by $|e_{1}\rangle$ and $|e_{2}\rangle$, respectively.

Laplacian indifference considerations on physical urns suggest that the initial state $p_{v_{0}}$ of both $\Omega_{DM}^{I}$ and $\Omega_{DM}^{II}$ is represented by the unit vector

$$|v_{0}\rangle = \frac{1}{\sqrt{2}}|e_{1}\rangle + \frac{1}{\sqrt{2}}|e_{2}\rangle = \frac{1}{\sqrt{2}}(1, 1)$$

in the ON basis $\{|e_{1}\rangle, |e_{2}\rangle\}$. A generic state $p_{v}$ of both $\Omega_{DM}^{I}$ and $\Omega_{DM}^{II}$ is instead represented by the unit vector

$$|v\rangle = \rho_{R}e^{i\theta_{R}}|e_{1}\rangle + \rho_{B}e^{i\theta_{B}}|e_{2}\rangle = (\rho_{R}e^{i\theta_{R}}, \rho_{B}e^{i\theta_{B}})$$

where $\rho_{R}, \rho_{B} \geq 0$, $\rho_{R}^{2} + \rho_{B}^{2} = 1$, and $\theta_{R}, \theta_{B} \in \mathbb{R}$.

For every $i \in \{R, B\}$, the (subjective) probability $\mu_{v}(E_{i})$ of drawing a ball of color $i$ in the state $p_{v}$ of either $\Omega_{DM}^{I}$ or $\Omega_{DM}^{II}$ is then

$$\mu(E_{i}, p_{v}) = \langle v|\hat{P}_{i}|v\rangle = |\langle e_{i}|v\rangle|^{2} = \rho_{i}^{2}$$

where $\rho_{R}$ and $\rho_{B}$.
Let us now consider the quantum representation of acts. For given utility values $u(0)$ and $u(100)$, the acts $f_1$, $f_2$, $f_3$ and $f_4$ in Table 1, Sec. 2.2 are respectively represented by the self-adjoint operators

$$
\hat{F}_1 = u(100)\hat{P}_R + u(0)\hat{P}_B \tag{8}
$$

$$
\hat{F}_2 = u(100)\hat{P}_R + u(0)\hat{P}_B \tag{9}
$$

$$
\hat{F}_3 = u(0)\hat{P}_R + u(100)\hat{P}_B \tag{10}
$$

$$
\hat{F}_4 = u(0)\hat{P}_R + u(100)\hat{P}_B \tag{11}
$$

The EU of $f_1$, $f_2$, $f_3$ and $f_4$ in a state $p_0$ of both entities $\Omega^I_{DM}$ and $\Omega^{II}_{DM}$ is given by

$$
W_v(f_1) = \langle v|\hat{F}_1|v\rangle = \rho_R^2 u(100) + \rho_B^2 u(0) = \rho_R^2 u(100) + (1 - \rho_R^2)u(0) \tag{12}
$$

$$
W_v(f_2) = \langle v|\hat{F}_2|v\rangle = \rho_R^2 u(100) + \rho_B^2 u(0) = \rho_R^2 u(100) + (1 - \rho_R^2)u(0) \tag{13}
$$

$$
W_v(f_3) = \langle v|\hat{F}_3|v\rangle = \rho_R^2 u(0) + \rho_B^2 u(100) = \rho_R^2 u(100) + (1 - \rho_R^2)u(100) \tag{14}
$$

$$
W_v(f_4) = \langle v|\hat{F}_4|v\rangle = \rho_R^2 u(0) + \rho_B^2 u(100) = \rho_R^2 u(100) + (1 - \rho_R^2)u(100) \tag{15}
$$

respectively, where we have used (6) and (8)–(11).

Coming to the dynamical part of the decision, when a decision-maker is asked to compare acts $f_1$ and $f_2$, the comparison itself, before a decision is taken, defines a cognitive context, which may change the state of entities $\Omega^I_{DM}$ and $\Omega^{II}_{DM}$. Analogously, when a decision-maker is asked to compare acts $f_3$ and $f_4$, the comparison itself, before a decision is taken, defines a new cognitive context, which may change the state of $\Omega^I_{DM}$ and $\Omega^{II}_{DM}$. However, a comparison between $f_1$ and $f_2$ (and also a comparison between $f_3$ and $f_4$) will have different effects on $\Omega^I_{DM}$ and $\Omega^{II}_{DM}$. Indeed, since act $f_1$ is ambiguous whereas $f_2$ is unambiguous, a comparison between $f_1$ and $f_2$ will determine a change of $\Omega^I_{DM}$ from the state $p_{v_0}$ to a generally different state $p_{v_{12}}$, whereas the same comparison will leave $\Omega^{II}_{DM}$ in the initial state $p_{v_0}$. Analogously, since $f_3$ is ambiguous whereas $f_4$ is unambiguous, a comparison between $f_3$ and $f_4$ will determine a change of $\Omega^I_{DM}$ from the state $p_{v_0}$ to a generally different state $p_{v_{34}}$, whereas the same comparison will leave $\Omega^{II}_{DM}$ in the initial state $p_{v_0}$.

Thus, the EUs in (13) and (15) in the state $p_{v_0}$ of $\Omega^{II}_{DM}$ become

$$
W_{v_0}(f_2) = W_{v_0}(f_4) = \frac{1}{2}(u(100) + u(0)) \tag{16}
$$

which do not depend on the cognitive state of $\Omega^{II}_{DM}$, in agreement with the fact that $f_2$ and $f_4$ are unambiguous acts, while the EUs in (12) and (14) do depend on the final state of $\Omega^I_{DM}$, again in agreement with the fact that $f_1$ and $f_3$ are ambiguous acts.

Let us then prove that two ambiguity averse final states $p_{v_{12}}$ and $p_{v_{34}}$ of $\Omega^I_{DM}$ can be found such that the corresponding EUs satisfy the Ellsberg preferences in Sec. 2.2 that is, $W_{v_{12}}(f_1) < W_{v_{12}}(f_2)$ and $W_{v_{34}}(f_3) < W_{v_{34}}(f_4)$. Indeed, consider the states $p_{v_{12}}$ and $p_{v_{34}}$ respectively represented, in the canonical ON basis of $\mathbb{C}^2$, by the unit vectors

$$
|v_{12}\rangle = (\sqrt{\alpha}, \sqrt{1-\alpha}) \tag{17}
$$

$$
|v_{34}\rangle = (\sqrt{1-\alpha}, -\sqrt{\alpha}) \tag{18}
$$

where $0 \leq \alpha < \frac{1}{2}$. One preliminarily observes that the vectors $|v_{12}\rangle$ and $|v_{34}\rangle$ are orthogonal, that is, $\langle v_{12}|v_{34}\rangle = 0$. Moreover, using (12), (18), we get

$$
W_{v_{12}}(f_1) = \alpha u(100) + (1 - \alpha)u(0) < \frac{1}{2}(u(100) + u(0)) = W_{v_0}(f_2) \tag{19}
$$

$$
W_{v_{34}}(f_3) = (1 - \alpha)u(0) + \alpha u(100) < \frac{1}{2}(u(100) + u(0)) = W_{v_0}(f_4) \tag{20}
$$

Hence, the ambiguity averse states $p_{v_{12}}$ and $p_{v_{34}}$ satisfy Ellsberg preferences in the two-urn example within the quantum-theoretic framework. We have thus elaborated a working model for the two-urn example.
4.2 Data representation in the two-urn example

To represent the data in Sec. 2.2, let us consider the decision between acts $f_1$ and $f_2$ and represent it by the pair of orthogonal projection operators $\{M, \mathds{1} - M\}$, also called a spectral measure. The orthogonal projection operator $M$ projects onto the 1-dimensional subspace generated by the unit vector $|m\rangle = (\rho_m e^{i\theta_m}, \tau_m e^{i\phi_m})$, with $\rho_m, \tau_m \geq 0, \rho_m^2 + \tau_m^2 = 1, \theta_m, \phi_m \in \mathbb{R}$. Thus, the operator $M$ can be written as

$$M = |m\rangle\langle m| = \left(\begin{array}{cc} \rho_m^2 & \rho_m \tau_m e^{i(\theta_m - \phi_m)} \\ \rho_m \tau_m e^{-i(\theta_m - \phi_m)} & \tau_m^2 \end{array}\right)$$

(21)

Analogously, suppose that the decision between acts $f_3$ and $f_4$ is represented by the spectral family $\{N, \mathds{1} - N\}$, where the orthogonal projection operator $N$ projects onto the 1-dimensional subspace generated by the unit vector $|n\rangle = (\rho_n e^{i\theta_n}, \tau_n e^{i\phi_n})$, with $\rho_n, \tau_n \geq 0, \rho_n^2 + \tau_n^2 = 1, \theta_n, \phi_n \in \mathbb{R}$. Thus, the operator $M$ can be written as

$$N = |n\rangle\langle n| = \left(\begin{array}{cc} \rho_n^2 & \rho_n \tau_n e^{i(\theta_n - \phi_n)} \\ \rho_n \tau_n e^{-i(\theta_n - \phi_n)} & \tau_n^2 \end{array}\right)$$

(22)

To find a quantum mathematical representation, we have to determine the unit vectors $|v_{12}\rangle$, $|v_{34}\rangle$, $|m\rangle$ and $|n\rangle$ which satisfy the following conditions

\begin{align*}
|v_{12}|M|v_{12}\rangle &= |\langle m|v_{12}\rangle|^2 = 0.82 \quad (23) \\
|v_0|M|v_0\rangle &= |\langle m|v_0\rangle|^2 = 0.50 \quad (24) \\
|m|m\rangle &= 1 \quad (25) \\
|v_{34}|N|v_{34}\rangle &= |\langle n|v_{34}\rangle|^2 = 0.84 \quad (26) \\
|v_0|N|v_0\rangle &= |\langle n|v_0\rangle|^2 = 0.50 \quad (27) \\
|n|n\rangle &= 1 \quad (28)
\end{align*}

The system of 6 equations must be satisfied by the parameters $0 \leq \alpha < \frac{1}{2}, \rho_m, \tau_m, \rho_n, \tau_n \geq 0, \theta_m, \phi_m, \theta_n, \phi_n \in \mathbb{R}$. Equations (23) and (26) are determined by empirical data, (24) and (27) are determined by normalization conditions, while (25) and (28) are determined by the fact that decision-makers who are not sensitive to ambiguity should overall be indifferent between $f_1$ and $f_2$, as well as between $f_3$ and $f_4$. Hence, on average, half respondents are expected to prefer $f_1$ ($f_3$) and the other half $f_2$ ($f_4$). To simplify the analysis, let us set $\theta_m = 90^\circ, \theta_n = 270^\circ, \phi_m = \phi_n = 0$. Hence, we are left with a system of 6 equations in 5 unknown variables whose solution is

\[\begin{aligned}
\alpha &= 0.14815 \\
\rho_m &= 0.21274 \\
\tau_m &= 0.97711 \\
\rho_n &= 0.99155 \\
\tau_n &= 0.12975
\end{aligned}\]

(29)

Hence, the ambiguity averse states $p_{v_{12}}$ and $p_{v_{34}}$ of DM entity $\Omega_{DM}$ are respectively represented by the unit vectors

\[\begin{aligned}
|v_{12}\rangle &= (0.38490, 0.92296) \\
|v_{34}\rangle &= (0.92296, -0.38490)
\end{aligned}\]

(30)

(31)

which determine, through the quantum probability formula (1), the subjective probability distributions underlying the DM test in Sec. 2.2.
Table 2. Representation of events, payoffs and acts in the gain scenario.

| Acts   | Option I                     | Option II                    |
|--------|------------------------------|------------------------------|
|        | \(E_1\): gain \(p_1 \in [\bar{p} - \Delta, \bar{p} + \Delta]\) | \(E_2\): not gain \(p_2 = 1 - p_1\) |
|        | \(E_1\): gain \(p_1 = \bar{p}\) | \(E_2\): not gain \(p_2 = 1 - \bar{p}\) |
| \(f_1\) | \(G\)                        | 0                            |
| \(f_2\) |                              | \(G\)                        |

The orthogonal projection operators in (21) and (22) reproducing the preference rates in the same test are instead given by

\[ M = \begin{pmatrix} 0.04526 & 0.20787i \\ -0.20787i & 0.95474 \end{pmatrix} \]  
(32)

\[ N = \begin{pmatrix} 0.98316 & -0.12865i \\ 0.12865i & 0.01684 \end{pmatrix} \]  
(33)

The ambiguity averse states reproduce Ellsberg preferences and represent the ambiguity aversion pattern identified in the DM test on the two-urn example, which completes the construction of a quantum mathematical representation for the data. As we can see, genuine quantum structures, namely, context-dependence, superposition and intrinsically non-deterministic state change are responsible of ambiguity aversion, while quantum probabilities crucially represent subjective probabilities.

5 A general model of hope and fear effects

We apply in this section the quantum-theoretic framework expounded in Secs. 3 and 4 to model hope effects and fear effects in management decisions involving comparison of a risky with an ambiguous option. To this aim, we need to convert the DM tests in Sec. 2.3 into a version of the Ellsberg two-urn example.

We split our analysis into two parts, a gain (Sec. 5.1) and a loss scenario (Sec. 5.2).

5.1 Gain scenario

We consider the experimental design in Sec. 2.3 and denote by \(\bar{p}\) the probability that the value of the financial parameter \(\lambda\) in a given investment is above a benchmark \(\lambda_{\text{benchmark}}\).

We introduce an ambiguous option I with a probability of realizing a gain \(G\) which ranges between \(\bar{p} - \Delta\) and \(\bar{p} + \Delta\), and a risky option with a probability \(\bar{p}\) of realizing the gain \(G\). This is equivalent to the scenario presented in Table 2. A choice has to be made between acts \(f_1\) and \(f_2\).

The empirical pattern found in [11] and confirmed in [12] for high probabilities is the following:
(i) most people will prefer \(f_2\) to \(f_1\) if \(\bar{p}\) is high, thus indicating ambiguity aversion and a fear effect;
(ii) most people will prefer \(f_1\) to \(f_2\) if \(\bar{p}\) is low, thus indicating ambiguity seeking and a hope effect.

To reproduce (i) and (ii) in a quantum-theoretic model, we refer to the mathematical formalism in Sec. 4.1. More precisely, we introduce a DM entity \(\Omega_{DM}^I\), corresponding to the ambiguous option, whose initial state \(p_{v_0}\) in the absence of any context is represented by the unit vector

\[ |v_0\rangle = \sqrt{\bar{p}}|e_1\rangle + \sqrt{1 - \bar{p}}|e_2\rangle = (\sqrt{\bar{p}}, \sqrt{1 - \bar{p}}) \]  
(34)

in the canonical ON basis \(\{|e_1\}, |e_2\}\) of \(\mathbb{C}^2\). This initial state will however change under interaction with a context, e.g., the decision-maker pondering between \(f_1\) and \(f_2\), into a final state \(p_v\) represented by the
Hence, the cognitive state $p$ where $0 \leq p \leq 1$, $\theta_1, \theta_2 \in \mathbb{R}$.

Then, we introduce a DM entity $\Omega_{DM}^I$, corresponding to the risky option, whose initial state $p_{v_0}$ in the absence of any context is again represented by the unit vector

$$|v_0\rangle = \sqrt{p}|e_1\rangle + \sqrt{1-p}|e_2\rangle = (\sqrt{p}, \sqrt{1-p})$$

This time, however, $p_{v_0}$ is not supposed to change under the interaction with a context, e.g., decision-maker pondering between $f_1$ and $f_2$.

Acts $f_1$ and $f_2$ are instead represented by the self-adjoint operators

$$\hat{F}_1 = u(G)|e_1\rangle\langle e_1| + u(0)|e_2\rangle\langle e_2|$$

$$\hat{F}_2 = u(G)|e_1\rangle\langle e_1| + u(0)|e_2\rangle\langle e_2|$$

respectively, where $u(\cdot)$ is the corresponding utility function, such that $u(G) > u(0)$.

We now distinguish between two cases.

*Case with high probability $\bar{p}$.** Let us construct a final ambiguity averse state $p_{v_GH}$ of DM entity $\Omega_{DM}^I$ which reproduces a fear effect, hence such that $W_{v_GH}(f_1) < W_{v_0}(f_2)$.

Firstly, the EU of act $f_2$ in the state $p_{v_0}$ of $\Omega_{DM}^I$ is, using (4), (36) and (38),

$$W_{v_0}(f_2) = \bar{p}u(G) + (1-\bar{p})u(0)$$

We choose the final state of $\Omega_{DM}^I$ to be the state represented by the unit vector

$$|v_{GH}\rangle = \sqrt{\bar{p}-\alpha}|e_1\rangle + \sqrt{1-\bar{p}+\alpha}|e_2\rangle = (\sqrt{\bar{p}-\alpha}, \sqrt{1-\bar{p}+\alpha})$$

where $0 \leq \alpha \leq \Delta$. This vector represents an ambiguity averse state. Indeed, the EU of act $f_1$ in the state $p_{v_GH}$ of $\Omega_{DM}^I$ is, using (4), (37) and (40),

$$W_{v_GH}(f_1) = (\bar{p} - \alpha)u(G) + (1-\bar{p}+\alpha)u(0) = \bar{p}u(G) + (1-\bar{p})u(0) - \alpha(u(G) - u(0)) < W_{v_0}(f_2)$$

Hence, the cognitive state $p_{v_GH}$ will generate a fear effect in the gain scenario with high probability $\bar{p}$.

*Case with low probability $\bar{p}$.** Let us construct a final ambiguity seeking state $p_{v_GL}$ of DM entity $\Omega_{DM}^I$ which reproduces a hope effect, hence such that $W_{v_GL}(f_1) > W_{v_0}(f_2)$.

We choose the final state of $\Omega_{DM}^I$ to be the state represented by the unit vector

$$|v_{GL}\rangle = \sqrt{\bar{p}+\alpha}|e_1\rangle + \sqrt{1-\bar{p}-\alpha}|e_2\rangle = (\sqrt{\bar{p}+\alpha}, \sqrt{1-\bar{p}-\alpha})$$

where $0 \leq \alpha \leq \Delta$. This vector represents an ambiguity seeking state. Indeed, the EU of act $f_1$ in the state $p_{v_GL}$ of $\Omega_{DM}^I$ is, using (4), (37) and (42),

$$W_{v_GL}(f_1) = (\bar{p} + \alpha)u(G) + (1-\bar{p}-\alpha)u(0) = \bar{p}u(G) + (1-\bar{p})u(0) + \alpha(u(G) - u(0)) > W_{v_0}(f_2)$$

Hence, the cognitive state $p_{v_GL}$ will generate a hope effect in the gain scenario with low probability $\bar{p}$. 

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Table 3. Representation of events, payoffs and acts in the loss scenario.

5.2 Loss scenario

We again consider the experimental design in Sec. 2.3 and proceed as in Sec. 5.1, with obvious changes.

We denote by \( \bar{p} \) the probability that the value of the financial parameter \( \lambda \) in a given investment is below a benchmark \( \lambda_{\text{benchmark}} \).

We introduce an ambiguous option I with a probability of realizing a loss \( L \) which ranges between \( \bar{p} - \Delta \) and \( \bar{p} + \Delta \), and a risky option with a probability \( \bar{p} \) of realizing the loss \( L \). This is equivalent to the scenario presented in Table 3. A choice has been made between acts \( f_1 \) and \( f_2 \).

The empirical pattern found in [11] and confirmed in [12] for high probabilities is the following:
(i) most people will prefer \( f_1 \) to \( f_2 \) if \( \bar{p} \) is high, thus indicating ambiguity seeking and a hope effect;
(ii) most people will prefer \( f_2 \) to \( f_1 \) if \( \bar{p} \) is low, thus indicating ambiguity aversion and a fear effect.

We again distinguish between two cases and use the symbols and procedures in Sec. 5.1. In particular, (36)–(35) still hold and (37) and (38 hold with \( L \) in place of \( G \) and \( u(L) < u(0) \). It is then easy to prove that the ambiguity seeking state of DM entity \( \Omega_D^I \) generating a hope effect in the loss scenario with high probability \( \bar{p} \) is the state \( p_{v_{LH}} \) represented by the unit vector

\[
|v_{LH}\rangle = \sqrt{\bar{p} - \alpha}|e_1\rangle + \sqrt{1 - \bar{p} + \alpha}|e_2\rangle = (\sqrt{\bar{p} - \alpha}, \sqrt{1 - \bar{p} + \alpha})
\]

where \( 0 \leq \alpha \leq \Delta \), while the ambiguity averse state of DM entity \( \Omega_D^I \) generating a fear effect in the loss scenario with low probability \( \bar{p} \) is the state \( p_{v_{LL}} \) represented by the unit vector

\[
|v_{LL}\rangle = \sqrt{\bar{p} + \alpha}|e_1\rangle + \sqrt{1 - \bar{p} - \alpha}|e_2\rangle = (\sqrt{\bar{p} + \alpha}, \sqrt{1 - \bar{p} - \alpha})
\]

where again \( 0 \leq \alpha \leq \Delta \).

We have thus provided a quantum-theoretic model of both hope and fear effects underlying individual attitudes towards ambiguity in management decisions, like those performed in [11] and [12].

5.3 Data representation

We elaborate here a quantum mathematical representation of the DM test in [12], along the lines Secs. 4.1, 5.1 and 5.2. To this end, we preliminarily note that only high probabilities of gains and losses were considered those studies.

We start by the IRR experiment in Sec. 2.3 which has the IRR as a parameter of financial performance. Here, we have \( IRR_{\text{benchmark}} = 15\% \). The authors set a probability \( \bar{p} = 0.68 \) and a range \( \Delta = 0.21 \) in the gain scenario, and a probability \( \bar{p} = 0.65 \) and a range \( \Delta = 0.20 \) in the loss scenario. The rate of preference of \( f_2 \) over \( f_1 \) was 0.62 in the gain scenario and 0.35 in the loss scenario.

Following the procedure in Sec. 4.2 and the symbols in Secs. 5.1 and 5.2 in a quantum-theoretic representation we need to determine two final states \( p_{v_{GH}} \) and \( p_{v_{LH}} \), respectively represented in the canonical ON basis of \( C^2 \) by the unit vectors

\[
|v_{GH}\rangle = (\sqrt{0.68 - \alpha}, \sqrt{0.32 + \alpha})
\]

\[
|v_{LH}\rangle = (\sqrt{0.65 - \alpha}, \sqrt{0.35 + \alpha})
\]
0 ≤ α ≤ 0.20, and two 1-dimensional orthogonal projection operators \( M = |m⟩⟨m| \) and \( N = |n⟩⟨n| \) such that
\[
\langle v_{GH}|M|v_{GH}⟩ = 0.62 \quad (48)
\]
\[
\langle v_{LH}|N|v_{LH}⟩ = 0.35 \quad (49)
\]
One can show that a solution is obtained with \( α = 0.05 \), thus
\[
|v_{GH}⟩ = (0.79373, 0.60828) \quad (50)
\]
\[
|v_{LH}⟩ = (0.77460, 0.63246) \quad (51)
\]
and
\[
M = \begin{pmatrix}
0.96154 & 0.19231i \\
-0.19231i & 0.03846
\end{pmatrix} \quad (52)
\]
\[
N = \begin{pmatrix}
0.93733 & -0.24236 \\
0.24236 & 0.06267
\end{pmatrix} \quad (53)
\]
We finally consider the ROI experiment in Sec. 2.3, where \( ROI_{benchmark} = 16\% \). The authors set a probability \( \bar{p} = 0.63 \) and a range \( Δ = 0.21 \) in the gain scenario, and a probability \( \bar{p} = 0.66 \) and a range \( Δ = 0.14 \) in the loss scenario. The rate of preference of \( f_2 \) over \( f_1 \) was 0.66 in the gain scenario and 0.41 in the loss scenario.

Proceeding as above, we need to determine two final states \( p_{v_{GH}} \) and \( p_{v_{LH}} \), respectively represented by the unit vectors
\[
|v_{GH}⟩ = (\sqrt{0.63 - \alpha}, \sqrt{0.37 + \alpha}) \quad (54)
\]
\[
|v_{LH}⟩ = (\sqrt{0.66 - \alpha}, \sqrt{0.34 + \alpha}) \quad (55)
\]
0 ≤ α ≤ 0.14, and two 1-dimensional orthogonal projection operators \( M = |m⟩⟨m| \) and \( N = |n⟩⟨n| \) such that
\[
\langle v_{GH}|M|v_{GH}⟩ = 0.66 \quad (56)
\]
\[
\langle v_{LH}|N|v_{LH}⟩ = 0.41 \quad (57)
\]
One can show that a solution is obtained again with \( α = 0.05 \), thus
\[
|v_{GH}⟩ = (0.76158, 0.64807) \quad (58)
\]
\[
|v_{LH}⟩ = (0.78102, 0.62450) \quad (59)
\]
and
\[
M = \begin{pmatrix}
0.99312 & 0.08215 \\
-0.08215 & 0.00679
\end{pmatrix} \quad (60)
\]
\[
N = \begin{pmatrix}
0.09091 & 0.28748i \\
-0.28748i & 0.90909
\end{pmatrix} \quad (61)
\]
This completes the construction of a quantum representation of the experimental study presented in in [12]. As we can see, the influence of psychological factors, like hopes and fears, on decision inherent investments, can be explained within a quantum-theoretic framework, while aversion to ambiguity and ambiguity seeking behaviour are natural manifestations of genuine quantum structures, like context-dependence, superposition
and intrinsically non-deterministic state change, in human reasoning. The presence of quantum structures explains the departure of rationality, in either direction (aversion, seeking) and also the switch between a direction and another.

We conclude this paper with a remark that is important, in our opinion. One may argue that the quantum models in Secs. 3, 4 and 5 are “ad hoc”, in the sense that they require introduction of new parameters which, according to some behavioural scientists, would describe empirical data, without necessarily explaining them. We believe that this is not the case here. The quantum-theoretic framework in Sec. 3 follows from the investigation of quantum theory in Hilbert space as a unitary and general theoretical paradigm for human decisions under uncertainty. As such, the quantum-theoretic framework does not only deal with the two-urn example or a more general choice between a risky and an ambiguous option. It rather is meant to apply to any decision, in any DM test with any outcome. In addition, the quantum models in Secs. 4 and 5 follow from straight and necessary application of the quantum-theoretic framework to specific cases. Hence, these models have to satisfy the mathematical prescriptions and epistemological constraints of quantum theory, in addition to satisfy empirical data. Being “theory-based”, rather “data-based”, models, they are not “ad hoc”. Of course, new DM tests have to be performed to check whether the quantum-theoretic framework and ensuing models work in general, or whether we instead need more complex Kolmogorovian or more general non-Hilbertian structures in a unified theory of human decisions.

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