A bi-Hamiltonian approach to the Sine-Gordon and Liouville hierarchies

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Abstract: In this paper we study the sine-Gordon and the Liouville hierarchies in laboratory coordinates from a bi-Hamiltonian point of view. Besides the well-known local structure these hierarchies possess a second compatible non-local Poisson structure.

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1 Introduction

In [4] Dubrovin and Zhang started a classification program of integrable evolutionary PDEs based on the bi-Hamiltonian geometry.

As a first step in the realization of this program they studied bi-Hamiltonian hierarchies of PDEs of the form

\[ u_t^i = A^i_j(u)u_x^j + \sum_{k=1}^{\infty} \epsilon^k P_k(u, u_x, ...) \] (1.1)

where \( P_k(u, u_x, ...) \) are differential polynomials.

A central role in their approach is played by the bi-Hamiltonian structure of the dispersionless limit (\( \epsilon = 0 \)) of the hierarchy (1.1). Such structure consists of a pair of compatible local Poisson brackets of hydrodynamic type (see [4]).

Very important examples of integrable PDEs such as the KdV equation, the continuous limit of the Toda lattice and, after a suitable change of coordinates, the NLS equation belong to the class of hierarchies considered by Dubrovin and Zhang.

An open problem is to understand how to extend the classification scheme of [4] onto more wide class of integrable systems not admitting dispersionless limit. The first point to clarify is which kind of bi-Hamiltonian structure is involved in this case.
In this paper we consider two important examples of this class of systems: the sine-Gordon equation, one of the first PDE solved by inverse scattering transform (see [1] and [6]) and the Liouville equation:

\begin{align}
\omega_{tt} - \omega_{xx} &= -\sin \omega \\
\phi_{tt} - \phi_{xx} &= -e^{\phi}
\end{align}

(1.2)

(1.3)

(in laboratory coordinates).

Starting from the results of a paper of Boiti, Leon and Pempinelli ([2]), we show how to construct their bi-Hamiltonian structures.

In sine-Gordon case such structure consists of the following compatible pair of Poisson bivectors

\begin{align}
P_1 &= \begin{pmatrix} \partial & -2i \\ 2i & 0 \end{pmatrix} \\
P_2 &= \begin{pmatrix} \frac{\partial}{S_1^{(0)}} - S_1^{(0)} \partial^{-1} S_1^{(0)} + (\frac{u^2}{2} + s)\partial + \partial(\frac{u^2}{2} + s) - \frac{\partial^3}{4} & S_1^{(0)} \partial^{-1} S_2^{(0)} - 2i(u^2 + s) + \frac{i\partial^2}{2} \\
S_2^{(0)} \partial - S_1^{(0)} & S_2^{(0)} \partial^{-1} S_2^{(0)} - \partial \end{pmatrix}
\end{align}

(1.4)

(1.5)

with

\begin{align}
S_1^{(0)} &= i\left(\frac{i}{8} \sin \omega + u_x\right), \\
S_2^{(0)} &= -2u
\end{align}

while in the Liouville case we obtain the pair

\begin{align}
P_1' &= \begin{pmatrix} \partial & 2 \\ -2 & 0 \end{pmatrix} \\
P_2' &= \begin{pmatrix} \frac{1}{16} \left( \partial - S_1^{(0)} \partial^{-1} S_1^{(0)} + (8u^2 + e^\phi)\partial + \partial(8u^2 + e^\phi) - 4\partial^3 - S_1^{(0)} \partial^{-1} S_2^{(0)} + 32u^2 + 2e^\phi + 2 - 8\partial^2 \\
-32u^2 - 2e^\phi - 2 + 8\partial^2 & -S_2^{(0)} \partial^{-1} S_2^{(0)} + 16\partial \end{pmatrix}
\end{align}

(1.6)

(1.7)

where

\begin{align}
S_1^{(0)} &= \frac{\exp \phi}{2} + 4u_x, \\
S_2^{(0)} &= -8u.
\end{align}

Following the Lenard-Magri scheme ([12]) one can construct recursively the integrals of motion. In the sine-Gordon case these integrals usually appear in two distinct families corresponding to the asymptotic expansion of the reduced monodromy matrix \(T(\lambda)\) around \(\lambda = 0\) and \(\lambda = \infty\) ([7]). In the bi-Hamiltonian framework these two families are arranged in a single hierarchy, infinite in both directions.

Our observation is that, in both cases, the bivectors ([15]) and ([17]) are weakly nonlocal according to the definition of Maltsev and Novikov (see [11]).
This class of nonlocal Poisson bivectors appeared only recently in the mathematical literature as a generalization of certain nonlocal Poisson brackets of hydrodynamic type (see \[8\], \[13\] and \[14\]).

Nevertheless the list of \((1+1)\)-integrable systems admitting such kind of Poisson structures contains, besides the sine-Gordon and Liouville equations, many other well-known integrable systems as KdV (considering higher Poisson structures) and NLS (see \[11\]).

To the best of our knowledge the bi-Hamiltonian structure of the Liouville hierarchy was unknown, while the bi-Hamiltonian structure of the sine-Gordon hierarchy appeared the first time in the paper \[9\] (see remark 5 in section 3).

\section{Preliminary definitions}

Let \(L\) be the space of maps from the real line to some \(n\)-dimensional manifold \(M\) endowed with local coordinates \(u^1,\ldots,u^n\) and let \(A\) be the space of differential polynomials in \(u^{i,s}\), that is

\[ f \in A \iff f = \sum f_{i_1,s_1;\ldots;i_m,s_m} u^{(i_1,s_1)} \ldots u^{(i_m,s_m)}, \quad (2.8) \]

where

\[ u^{(i,1)} = u_x^i \]
\[ u^{(i,2)} = u_{xx}^i \]
\[ \vdots \]

The role of the functions on \(L\) is played by the local functionals

\[ I = \int_{-\infty}^{+\infty} f(u(x), u_x, u_{xx}, \ldots) dx \]

where \(f \in A\).

**Definition 1** A (non local) multivector \(X\) is a formal infinite sum of the type

\[ X = X^{i_1,s_1;\ldots;i_k,s_k} (x_1, \ldots, x_k; u(x_1), \ldots, u(x_k), u_x(x_1), \ldots) \frac{\partial}{\partial u^{i_1,s_1}(x_1)} \wedge \ldots \wedge \frac{\partial}{\partial u^{i_k,s_k}(x_k)} \]

where the coefficients satisfy the skew-symmetry condition with respect to simultaneous permutations

\[ i_p, s_p, x_p \leftrightarrow i_q, s_q, x_q \]

**Definition 2** A \(k\)-vector is called translation invariant if

\[ X^{i_1,s_1;\ldots;i_k,s_k} (x_1, \ldots, x_k; u(x_1), \ldots, u(x_k), u_x(x_1), \ldots) = \partial_x^{s_1} \ldots \partial_x^{s_k} X^{i_1,\ldots;i_k} (x_1, \ldots, x_k; u(x_1), \ldots, u(x_k), \ldots) \]

where \(X^{i_1,\ldots;i_k}(\ldots)\) means \(X^{i_1,0;\ldots;i_k,0}(\ldots)\) and for any \(t\)

\[ X^{i_1,\ldots;i_k} (x_1 + t, \ldots, x_k + t; u(x_1), \ldots, u(x_k), \ldots) = X^{i_1,\ldots;i_k} (x_1, \ldots, x_k; u(x_1), \ldots, u(x_k), \ldots) \]
It follows from this definition that a translation invariant multi-vector field is completely characterized by the “components”

\[ X^{i_1...i_k} := X(x_1, ..., x_k; u(x_1), ..., u(x_k), ...) \]

**Definition 3** A bivector \( P \) is called Poisson bivector if and only if

\[ [P, P]_{\text{schouten}} = 0 \] (2.9)

We recall that the Schouten bracket of two translation invariant bivectors is given by the formula (see [4]):

\[
[P, Q]_{ijk}^{xy} = \frac{1}{2} \sum_s \left( \frac{\partial P_{ij}^{xy}}{\partial u^s(x)} \frac{\partial^s Q_{xyz}^{kl}}{\partial x} + \frac{\partial Q_{ij}^{xy}}{\partial u^s(y)} \frac{\partial^s P_{xyz}^{kl}}{\partial y} + \frac{\partial P_{ij}^{xy}}{\partial u^s(x)} \frac{\partial^s Q_{xyz}^{kl}}{\partial y} + \frac{\partial Q_{ij}^{xy}}{\partial u^s(y)} \frac{\partial^s P_{xyz}^{kl}}{\partial x} + \frac{\partial P_{ij}^{xz}}{\partial u^s(z)} \frac{\partial^s Q_{xyz}^{kl}}{\partial z} + \frac{\partial Q_{ij}^{xz}}{\partial u^s(z)} \frac{\partial^s P_{xyz}^{kl}}{\partial x} \right) \] (2.10)

A Poisson bivector defines a Poisson bracket. Indeed we have the following

**Definition 4** The Poisson bracket of two local functionals \( I_1, I_2 \) is given by the formula

\[
\{I_1, I_2\} := \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta I_1 \frac{\partial P_{xy}^{ij}}{\partial u^i(x)} \delta I_2 \frac{\partial P_{xy}^{kl}}{\partial u^j(x)} dxdy \] (2.11)

**Definition 5** A pair of bivectors \((P_1, P_2)\) defines a bi-Hamiltonian structure if and only if

\[ [P_1, P_1]_{\text{Schouten}} = [P_1, P_2]_{\text{Schouten}} = [P_2, P_2]_{\text{Schouten}} = 0. \] (2.12)

**Definition 6** A Poisson bivector \( P \) of components \( P_{ij}^{xy} \) (and the corresponding Poisson bracket) is called local if it can be written as:

\[ P_{ij}^{xy} = \sum_k B_k^{ij}(u, u_x, ..., u_x) \delta^{(k)}(x - y) \] (2.13)

By substituting (2.13) in (2.11) we get

\[
\{I_1, I_2\} := \int \delta I_1 \frac{\partial P_{ij}^{xy}}{\partial u^i(x)} \delta I_2 \frac{\partial P_{ij}^{xy}}{\partial u^j(x)} dx \] (2.14)

where

\[ P_{ij}^{xy} = \sum_k B_k^{ij}(u, u_x, ...) \partial^k \] (2.15)

\((\partial := \frac{d}{dx})\)
Definition 7 A Poisson bivector $P$ of components $P_{xy}^{ij}$ (and the corresponding Poisson bracket) is called weakly non-local (see [11]) if it can be written as:

$$P_{xy}^{ij} = \sum_k B_k^{ij}(u, u_x, ...) \delta^{(k)}(x - y) + \sum_{k,l} e_{kl} S_k^i(u, u_x, ...) \nu(x - y) S_l^j(u, u_y, ...) \quad (2.16)$$

where $\nu(x - y)$ is the step function ($= \frac{1}{2}$ if $x - y$ is positive and $= -\frac{1}{2}$ if $x - y$ is negative). The constants $e_{kl} = e_{lk}$ define a quadratic form in the space generated by the flows $S_k^i(u, u_x, ...)$. 

By substituting (2.16) in (2.11) we get

$$\{I_1, I_2\} := \int_{-\infty}^{+\infty} \frac{\delta I_1}{\delta u^i(x)} P_{ij} \frac{\delta I_2}{\delta u^j(x)} dx \quad (2.17)$$

where

$$P_{ij} = \sum_k B_k^{ij}(u, u_x, ...) \partial^k + \sum_{k,l} e_{kl} S_k^i(u, u_x, ...) \partial^{-1} S_l^j(u, u_x, ...) \quad (2.18)$$

and, by definition

$$\partial^{-1}(.) := \frac{1}{2} \left( \int_{-\infty}^{x} - \int_{x}^{+\infty} \right) dx(.) \quad (2.19)$$

Later on we will call local and weakly non local bivector also the corresponding differential and integro-differential operator.

Similarly it is possible to define the class of weakly nonlocal symplectic structures (see [11]).

Definition 8 Let $P$ be an invertible Poisson bivector. Then $P^{-1}$ is a weakly nonlocal symplectic structure if and only if it can be written in the following way:

$$(P^{-1})_{ij} = \sum_k B_{(k)ij}(u, u_x, ...) \partial^k + \sum_{k,l} e_{kl} Q_k^i(u, u_x, ...) \partial^{-1} Q_l^j(u, u_x, ...) \quad (2.20)$$

3 The sine-Gordon hierarchy

3.1 Useful results

- The starting point of the paper [2] is the hierarchy of equations represented by the equation of zero curvature:

$$U_t - V_x + [U, V] = 0 \quad (3.21)$$

where (see [2] for details)

$$U = -i \lambda \sigma_3 + u \sigma_1 + \left( i \lambda \right) \left( s \sigma_3 + iv \sigma_2 \right)$$

$$V = \sum_{j=0}^{n} V_j \lambda^{n-j} + \sum_{j=0}^{p} W_j \lambda^{p-j} \quad (3.23)$$
\( \sigma_j \) are the Pauli matrices and the functions \( u(x), v(x) \) and \( s(x) \) have the following behavior for large \(|x|\):

\[
\lim_{|x| \to \infty} u(x) = 0 \\
\lim_{|x| \to \infty} v(x) = 0 \\
\lim_{|x| \to \infty} s(x) = s_0
\]

The sine-Gordon hierarchy corresponds to the reduction of this hierarchy on the surface

\[
s^2 - v^2 = s_0^2 \tag{3.24}
\]

- The equations of this hierarchy can be written as

\[
\begin{pmatrix}
  u_t \\
  v_t
\end{pmatrix} = -i \sum_{j=0}^m \mu_j^{(+)} J \frac{\delta H_{2j+1}^{(+)}}{\delta q} + i \sum_{j=0}^k \mu_j^{(-)} J \frac{\delta H_{2j+1}^{(-)}}{\delta q} \tag{3.25}
\]

where

\[
\frac{\delta}{\delta q} = \left( \frac{\delta}{\delta u} + \frac{v}{s} \frac{\delta}{\delta s} \right) \tag{3.26}
\]

and

\[
J = \begin{pmatrix}
  \partial & 2s(x) \\
  -2s(x) & 0
\end{pmatrix} \tag{3.27}
\]

The first “Hamiltonians” are:

\[
H_1^{(+)} = i \int_{-\infty}^{+\infty} dx \left( s - s_0 + \frac{1}{2} u^2 \right) \tag{3.28}
\]

\[
H_1^{(-)} = -i \int_{-\infty}^{+\infty} dx \left( s - s_0 + \frac{1}{2} u^2 + \frac{v_x}{s} \left( \frac{v_x}{s} + 4u \right) \right) \tag{3.29}
\]

The sine-Gordon equation corresponds to the choice \( m = k = 0, \mu_0^{(+)} = \mu_0^{(-)} = 1, \mu_j^{(+)} = \mu_j^{(-)} = 0 \) for \( j \) different from 0. In fact by straightforward calculation one gets:

\[
\begin{pmatrix}
  u_t \\
  v_t
\end{pmatrix} = \begin{pmatrix}
  u_x + 4v \\
  -v_x - 4us
\end{pmatrix} \tag{3.30}
\]

On the surface \( s_0 = 1 \) \( v \) and \( s \) can be written as functions of \( \omega \):

\[
v = is_0 \sin \omega \tag{3.31}
\]

\[
s = s_0 \cos \omega \tag{3.32}
\]
By inserting (3.32) and (3.32) in the system (3.30) one obtains sine-Gordon equation
\[ \omega_{tt} - \omega_{xx} = -16s_0 \sin \omega \] (3.33)

- The recursion operators for this hierarchy are given by the formulae
\[
N = \begin{pmatrix}
2(\nu + u_x)\partial^{-1}u + \nu^2 + s - \frac{\partial^2}{4} & (2(\nu + u_x)\partial^{-1}\frac{\nu}{s} + \frac{\nu}{s} - \frac{\partial}{2}) \\
-2su\partial^{-1}u + \frac{1}{2}s\partial & -2su\partial^{-1}\frac{\nu}{s} + s
\end{pmatrix}
\] (3.34)
\[
N^{-1} = \frac{1}{s^2_0} \begin{pmatrix}
-v\partial^{-1}w + s & -2v\partial^{-1}\left(\frac{w}{s} + \frac{1}{2}s\partial\frac{1}{s}\right) + \frac{1}{2}s\partial\frac{1}{s} \\
\frac{1}{2}s(-\partial + \omega\partial^{-1}w) & s\omega\partial^{-1}\left(\frac{w}{s} + \frac{1}{2}s\partial\frac{1}{s}\right) - \frac{1}{2}s\partial^2\frac{1}{s} + s
\end{pmatrix}
\] (3.35)
where \( w = \frac{\nu}{s} + 2u \).

- The hierarchy is local, that is all the equations are differential.

3.2 From the recursion operator to the bi-Hamiltonian structure

In this section we show how to construct the sine-Gordon bi-Hamiltonian structure starting from the pair \((J, NJ)\), where \( NJ = \)
\[
\begin{pmatrix}
-(2(\nu + u_x)\partial^{-1}(2(\nu + u_x)) + (\frac{1}{2}u^2 + s)\partial + \partial(\frac{1}{2}u^2 + s) - \frac{\partial^3}{4} & 2(\nu + u_x)\partial^{-1}us + 2s(u^2 + s) - \frac{\partial^2}{2}s \\
2us\partial^{-1}(2(\nu + u_x) - 2s(u^2 + s) + \frac{\partial^2}{2} & -4su\partial^{-1}us + s\partial s
\end{pmatrix}
\] (3.36)

We begin with the following

Remark 1 The system
\[
\begin{pmatrix}
\nu_t \\
v_t
\end{pmatrix} = \begin{pmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{pmatrix} \begin{pmatrix}
\frac{\delta H}{\delta u} \\
\frac{\delta H}{\delta v} + \frac{\nu}{s}\frac{\delta H}{\delta s}
\end{pmatrix}
\] (3.37)
is equivalent, on the cylinder (3.24), to the Hamiltonian system
\[
\begin{pmatrix}
\nu_t \\
v_t \\
\nu_t
\end{pmatrix} = \begin{pmatrix}
P_{11} & P_{12} & P_{12}\frac{\nu}{s} \\
P_{21} & P_{22} & P_{22}\frac{\nu}{s} \\
\frac{\nu}{s}P_{21} & \frac{\nu}{s}P_{22} & \frac{\nu}{s}P_{22}\frac{\nu}{s}
\end{pmatrix} \begin{pmatrix}
\frac{\delta H}{\delta u} \\
\frac{\delta H}{\delta v} \\
\frac{\delta H}{\delta s}
\end{pmatrix}
\] (3.38)

In fact the third equation
\[
s_t = \frac{\nu}{s}P_{21}\frac{\delta H}{\delta u} + \frac{\nu}{s}P_{22}\frac{\delta H}{\delta v} + \frac{\nu}{s}P_{22}\frac{\nu}{s}\frac{\delta H}{\delta s} = \frac{\nu}{s}v_t
\] (3.39)
is compatible with the constraint (3.24).
Then we have to substitute the $2 \times 2$ matrices $J$ and $NJ$ with the corresponding $3 \times 3$ matrices
\[
\tilde{J} = \begin{pmatrix}
J_{11} & J_{12} & J_{12} \frac{v}{s} \\
J_{21} & J_{22} & J_{22} \frac{u}{s} \\
\frac{w}{s} J_{21} & \frac{w}{s} J_{22} & \frac{w}{s} J_{22} \frac{u}{s} \\
\end{pmatrix}
\]
\[\text{(3.40)}\]
and $\tilde{NJ} =$
\[
\begin{pmatrix}
(NJ)_{11} & (NJ)_{12} & (NJ)_{12} \frac{v}{s} \\
(NJ)_{21} & (NJ)_{22} & (NJ)_{22} \frac{u}{s} \\
\frac{w}{s} (NJ)_{21} & \frac{w}{s} (NJ)_{22} & \frac{w}{s} (NJ)_{22} \frac{u}{s} \\
\end{pmatrix}
\]
\[\text{(3.41)}\]

The idea is that the sine-Gordon bi-Hamiltonian structure is the restriction of the pair $(\tilde{J}, \tilde{NJ})$ on the surface $3.24$. By replacing the coordinates $(u, v, s)$ by the coordinates $(u, \omega, s_0)$ where
\[
v = is_0 \sin \omega \tag{3.42}
\]
\[
s = s_0 \cos \omega \tag{3.43}
\]
and taking into account that, after a change of coordinates $u^i = u^j(u^k)$ a bivector $P^{ij}$ transforms as
\[
P^{ij'} = \frac{\partial u^{i'}}{\partial u^k} P^{kl} \frac{\partial u^{j'}}{\partial u^l} \tag{3.44}
\]
we obtain
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & -i \frac{s}{s_0} & i \frac{v}{s_0} \\
0 & -\frac{w}{s_0} & \frac{s}{s_0} \\
\end{pmatrix}
\begin{pmatrix}
J_{11} & J_{12} & J_{12} \frac{v}{s} \\
J_{21} & J_{22} & J_{22} \frac{u}{s} \\
\frac{w}{s} J_{21} & \frac{w}{s} J_{22} & \frac{w}{s} J_{22} \frac{u}{s} \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & -i \frac{s}{s_0} & -\frac{w}{s_0} \\
0 & i \frac{w}{s_0} & \frac{s}{s_0} \\
\end{pmatrix}
= \begin{pmatrix}
\partial & -2i & 0 \\
2i & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & -i \frac{s}{s_0} & i \frac{v}{s_0} \\
0 & -\frac{w}{s_0} & \frac{s}{s_0} \\
\end{pmatrix}
\begin{pmatrix}
(NJ)_{11} & (NJ)_{12} & (NJ)_{12} \frac{v}{s} \\
(NJ)_{21} & (NJ)_{22} & (NJ)_{22} \frac{u}{s} \\
\frac{w}{s} (NJ)_{21} & \frac{w}{s} (NJ)_{22} & \frac{w}{s} (NJ)_{22} \frac{u}{s} \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & -i \frac{s}{s_0} & -\frac{v}{s_0} \\
0 & i \frac{v}{s_0} & \frac{s}{s_0} \\
\end{pmatrix}
= \begin{pmatrix}
\partial & -2i & 0 \\
-2i & 0 & -2i \frac{u_2}{2} \\
-2i u \frac{u_2}{2} + 2i(u^2 + s) - \frac{i \omega}{2} & 4\omega \frac{u_2}{2} & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\]

From these formulae it follows immediately that the restriction on the surface $3.24$ is given by the expressions $1.41$ and $1.5$.

Then we have the following theorem.
Theorem 1 The pair \((\tilde{J}, \tilde{N}J)\) and its restriction on the cylinder \((3.22)\) are bi-Hamiltonian structures.

We omit the proof of this theorem consisting of a very long but straightforward calculation based on the formula \((2.10)\).

The most difficult part of this kind of computation is to check whether certain expressions containing products of distributions vanish. The trick is to use some identities between distributions like

\[
f(y)\delta^{(s)}(x-y) = \sum_{q=0}^{s} \binom{s}{q} f^{(q)}(x)\delta^{(s-q)}(x-y)\]

\[
\delta(x-y)\delta(x-z) = \delta(y-x)\delta(y-z) = \delta(z-x)\delta(z-y)\]

\[
\nu(y-z)\delta(z-x) = \partial_{y}^{-1}(\delta(z-x)\delta(y-z)) = \partial_{y}^{-1}(\delta(x-y)\delta(x-z)) = -\nu(x-y)\delta(x-z)\]

\[
\nu(x-y)\delta(y-z) = \partial_{x}^{-1}(\delta(y-x)\delta(y-z)) = \partial_{x}^{-1}(\delta(z-x)\delta(z-y)) = -\nu(z-x)\delta(z-y)\]

\[
\nu(z-x)\delta(x-y) = \partial_{z}^{-1}(\delta(x-y)\delta(x-z)) = \partial_{z}^{-1}(\delta(y-x)\delta(y-z)) = -\delta(y-x)\nu(y-z)\]

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In other words we have the Lenard-Magri chain:

\[ P_1 \rightarrow I_1 \rightarrow P_1 \]
\[ I_2 \rightarrow P_2 \rightarrow I_2 \rightarrow P_2 \]
\[ P_2 \delta I_{-2} = P_1 \delta I_{-1} \]
\[ P_2 \delta I_{-1} = P_1 \delta I_0 \]
\[ P_2 \delta I_0 = P_1 \delta I_1 \]
\[ P_2 \delta I_1 = P_1 \delta I_2 \]
\[ P_2 \delta I_2 = P_1 \delta I_3 \]
\[ \ldots \]

An immediate consequence of the last remark is the following

**Theorem 2** The sine-Gordon equation belongs to the hierarchy \( \{ H_k = s_0 I_k + I_{k+1} \}_{k \in \mathbb{Z}} \).

where

\[ H_{1\text{SG}} = \int \left( u^2 + \frac{i}{2}u \omega_x - \frac{\omega^2}{8} + 2s_0 \cos \omega \right) dx \quad (3.46) \]

\[ H_{2\text{SG}} = \int \left( \frac{1}{80} \left( \frac{1}{2}u^2 + \frac{i}{2}u \omega_x - \frac{\omega^2}{8} + s_0 \cos \omega \right) + \frac{1}{s_0} \left( \frac{s_0^2}{4} \cos(2\omega) - \frac{3s_0}{8} \cos \omega \omega_x^2 - \frac{\omega_x^2}{32} + \frac{\omega^4}{128} ight) + \frac{s_0}{2} u^2 \cos \omega + ius_0 \cos \omega \omega_x - \frac{i}{8}u \omega_{xxx} + \frac{1}{16} u_x^2 - \frac{i}{16} u \omega_x^3 - \frac{3}{16} u^2 \omega_x^2 + \frac{i}{4} u^3 \omega_x + \frac{1}{8} u^4 \right) dx \quad (3.47) \]

The momentum integrals are given by the following expressions:
\[ P_{1SG} = \frac{1}{8} \int (4i\omega_x - \omega_x^2) \, dx \]  
\[ P_{2SG} = \int \frac{1}{s_0} \left( \frac{1}{2} u^2 + \frac{1}{2} iu\omega_x - \omega_x^2 + s_0 \cos \omega \right) dx \]  
\[ + \frac{s_0}{2} u^2 \cos \omega + i u s_0 \cos \omega \omega_x - \frac{i}{8} u \omega_{xxx} + \frac{1}{8} u_x^2 - \frac{i}{16} u^3 \omega_x + \frac{3}{16} u^2 \omega_x^2 + \frac{i}{4} u^3 \omega_x + \frac{1}{8} u^4 \]  
\[ \int (3.49) \]

Moreover the Poisson bivectors \( P_1 \) and \( P_2 \) are invertible. More precisely:

\[ P_1^{-1} = \begin{pmatrix} 0 & -i \\ \frac{i}{2} & -\frac{2}{3} \end{pmatrix} \]  
\[ (3.50) \]

and \( P_2^{-1} = \frac{1}{s_0} \times \)

\[ \begin{pmatrix} T_1 \partial^{-1} T_1 + \frac{i}{4} \partial & T_1 \partial^{-1} T_2 - i \left( \frac{s_0}{4} \cos \omega - \frac{\partial^2}{8} - \frac{T_2^2}{2} \right) \\ T_2 \partial^{-1} T_1 + i \left( \frac{s_0}{4} \cos \omega - \frac{\partial^2}{8} - \frac{T_2^2}{2} \right) & T_2 \partial^{-1} T_2 + \left( \frac{T_2^2}{4} - \frac{s_0}{2} \cos \omega \right) \partial + \frac{\left( \frac{T_2^2}{4} + \frac{s_0}{2} \cos \omega \right)}{2} \end{pmatrix} \]  
\[ (3.51) \]

with

\[ T_1 = \frac{i}{2} (i\omega_x + 2u) \]
\[ T_2 = - \left( i s_0 \sin \omega - \frac{i}{4} \omega_{xx} - \frac{1}{2} u_x \right) \]

**Proof** By adding the Lenard-Magri chain for the Hamiltonians \( \{ I_k \}_{k \in \mathbb{Z}} \) multiplied by \( s_0 \) to the same
chain shifted by one we obtain

\[
\begin{align*}
P_1 \rightarrow H_{-2} \quad & P_2 \delta H_{-2} = P_1 \delta H_{-1} \\
P_2 \rightarrow H_{-1} \quad & P_2 \delta H_{-1} = P_1 \delta H_0 \\
P_2 \rightarrow H_0 \quad & P_2 \delta H_0 = P_1 \delta H_1 \\
P_2 \rightarrow H_1 \quad & P_2 \delta H_1 = P_1 \delta H_2 \\
P_2 \rightarrow H_2 \quad & \ldots
\end{align*}
\]

In order to find the recursion operators of this hierarchy we need the following

**Remark 3** Let \( N \) and \( M = N^{-1} \) be the \( 2 \times 2 \) recursion operator introduced above. If \( M \) and \( N \) satisfy the equations

\[
\begin{align*}
\begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} J_{11}^2 & J_{12}^2 \\ J_{21}^2 & J_{22}^2 \end{pmatrix} &= \begin{pmatrix} J_{11}^1 & J_{12}^1 \\ J_{21}^1 & J_{22}^1 \end{pmatrix} \\
(3.52)
\end{align*}
\]

and

\[
\begin{align*}
\begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix} \begin{pmatrix} J_{11}^1 & J_{12}^1 \\ J_{21}^1 & J_{22}^1 \end{pmatrix} &= \begin{pmatrix} J_{11}^2 & J_{12}^2 \\ J_{21}^2 & J_{22}^2 \end{pmatrix} \\
(3.53)
\end{align*}
\]

12
then the $3 \times 3$ matrices

\[
\tilde{M} = \begin{pmatrix}
M_{11} & M_{12} & 0 \\
M_{21} & M_{22} & 0 \\
\frac{v}{s}M_{21} & 0 & \frac{v}{s}M_{22}\frac{s}{v}
\end{pmatrix}
\]  

(3.54)

and

\[
\tilde{N} = \begin{pmatrix}
N_{11} & N_{12} & 0 \\
N_{21} & N_{22} & 0 \\
\frac{v}{s}N_{21} & 0 & \frac{v}{s}N_{22}\frac{s}{v}
\end{pmatrix}
\]  

(3.55)

satisfy the equations:

\[
\tilde{M}\tilde{J}^2 = \tilde{J}^1
\]  

(3.56)

and

\[
\tilde{N}\tilde{J}^1 = \tilde{J}^2
\]  

(3.57)

where

\[
\tilde{J}^1 = \begin{pmatrix}
J_{11}^1 & J_{12}^1 & \frac{1}{s}J_{12}\frac{v}{s} \\
J_{21}^1 & J_{22}^1 & \frac{1}{s}J_{22}\frac{v}{s} \\
\frac{v}{s}J_{21}^1 & \frac{v}{s}J_{22}^1 & \frac{v}{s}J_{22}\frac{v}{s}
\end{pmatrix}
\]  

(3.58)

and

\[
\tilde{J}^2 = \begin{pmatrix}
J_{11}^2 & J_{12}^2 & \frac{1}{s}J_{12}\frac{v}{s} \\
J_{21}^2 & J_{22}^2 & \frac{1}{s}J_{22}\frac{v}{s} \\
\frac{v}{s}J_{21}^2 & \frac{v}{s}J_{22}^2 & \frac{v}{s}J_{22}\frac{v}{s}
\end{pmatrix}
\]  

(3.59)

(in our case $\tilde{J}^1 = (3.40)$ and $\tilde{J}^2 = (3.41)$)

Since $\tilde{J}^1$ and $\tilde{J}^2$ are not invertible in the space $(u, v, s)$ we can try to express $\tilde{M}$ and $\tilde{N}$ in terms of them only after the restriction on the surface (3.24).

Taking into account that after a change of coordinates $u'' = u''(u^k)$ $\tilde{M}$ transforms according to the rule

\[
\tilde{M}' = \frac{\partial u''}{\partial u'^k} \tilde{M} \frac{\partial u'^l}{\partial u''}
\]  

(3.60)

it is easy to see that in the coordinates $(u, \omega, s_0)$ $\tilde{M}$ becomes:

\[
\frac{1}{s_0^2} \begin{pmatrix}
s - v\partial^{-1}w & \frac{i}{2}s\partial - 2iv\partial^{-1}(v + \frac{1}{4}w\partial) & * \\
-\frac{i}{2}(-\partial + w\partial^{-1}w) & s - \frac{1}{4}\partial^2 + w\partial^{-1}(v + \frac{1}{4}w\partial) & * \\
0 & 0 & *
\end{pmatrix}
\]
The equations (3.56) in these coordinates becomes
\[
\frac{1}{s_0^2} \begin{pmatrix}
  s - v \partial^{-1} w & \frac{i}{2} s \partial - 2iv \partial^{-1} (v + \frac{1}{4} w \partial) \\
  -\frac{i}{2} (-\partial + w \partial^{-1} w) & s - \frac{1}{4} \partial^2 + w \partial^{-1} (v + \frac{1}{4} w \partial)
\end{pmatrix}
\begin{pmatrix}
P_{11}^1 & P_{12}^1 \\
P_{21}^2 & P_{22}^2
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
P_{11}^1 & P_{12}^1 \\
P_{21}^2 & P_{22}^2
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

From these equations it follows immediately that
\[
M' = \frac{1}{s_0} \begin{pmatrix}
  s - v \partial^{-1} w & \frac{i}{2} s \partial - 2iv \partial^{-1} (v + \frac{1}{4} w \partial) \\
  -\frac{i}{2} (-\partial + w \partial^{-1} w) & s - \frac{1}{4} \partial^2 + w \partial^{-1} (v + \frac{1}{4} w \partial)
\end{pmatrix}
\]
satisfies the equation
\[
M' P_2 = P_1
\]
and therefore it is a recursion operator for the sine-Gordon hierarchy.

It is easy to check that the bivector (3.50) is the inverse of \( P_1 \) and that the bivector (3.51) = \( P_1^{-1} M' \) is the inverse of \( P_2 \).

Finally from
\[
P_1 \delta P_{1SG} = P_2 \delta P_{2SG} \tag{3.62}
\]
\[
P_1 \delta H_{1SG} = P_2 \delta H_{2SG} \tag{3.63}
\]
it follows immediately
\[
\delta P_{2SG} = P_2^{-1} P_1 \delta P_{1SG} \tag{3.64}
\]
\[
\delta H_{2SG} = P_2^{-1} P_1 \delta H_{1SG} \tag{3.65}
\]

By straightforward calculation we get
\[
\frac{\delta H_{2SG}}{\delta u} = -\frac{1}{4} u_{xx} + i s_0 \cos \omega \omega_x - \frac{3}{8} \omega_x^2 - \frac{3}{8} i u_2 \omega_x + \frac{1}{2} u^3 + \frac{1}{2} i s_0 \omega_x + u s_0 \cos \omega + u s_0 - \frac{1}{8} i \omega_{xxx} +
\]
\[
-\frac{1}{16} \omega_x^3
\]
\[
\frac{\delta H_{2SG}}{\delta \omega} = -i s_0 \cos \omega u_x - \frac{1}{2} s_0^2 \sin 2\omega + \frac{3}{4} i u \omega_x \omega_{xx} + \frac{3}{4} u u_x \omega_x + \frac{3}{8} u^2 \omega_x + s_0 \sin \omega +
\]
\[
+ \frac{3}{4} s_0 \cos \omega \omega_{xx} + \frac{1}{4} s_0 \omega_x - \frac{1}{2} i s_0 u_x + \frac{1}{8} i u_{xxx} - \frac{3}{8} s_0 \sin \omega_x^2 + \frac{3}{16} i u_x \omega_x^2 - \frac{3}{32} \omega_x^4 + \frac{3}{4} i u^2 u_x +
\]
\[
- \frac{1}{16} \omega_{xxx} - \frac{1}{2} u^2 s_0 \cos \omega
\]
that is the differential of the Hamiltonian (3.47). Similarly we get (3.49).

**Corollary 2** The operator (3.51) defines a weakly nonlocal symplectic structure.
Remark 4 In order to obtain the usual symplectic structure of the sine-Gordon hierarchy

\[
\Theta_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]  

(3.66)

it is sufficient to make the change of variables \( q = \omega, p = 4iu - q_x \): in the new coordinates the Poisson bivector (1.4) becomes the usual one while the second one coincide with the Poisson bivector found by Fuchssteiner and Oevel (9).

Remark 5 The geometrical structures (recursion operator, bi-Hamiltonian structure) of the sine-Gordon equation are well known in light-cone coordinates (\( \xi = \frac{t+x}{2}, \eta = \frac{t-x}{2} \)) (see for example [3]).

Starting from these structures it is not obvious how to get their counterpart in laboratory coordinates.

A possible way was suggested by Fuchssteiner and Oevel in (9) and it is based on the properties of a particular class of time-dependent vector fields called master symmetries. In the case of integrable PDEs admitting a hamiltonian formulation with respect to some Poisson bivector \( P \) a master symmetry \( \tau \) has the property that the Lie derivative \( \text{Lie}_\tau P \) defines a new Poisson bivector compatible with \( P \).

The main idea of the paper [9] is that the master symmetry of the sine-Gordon equation in light-cone coordinates can be modified in such a way that the new vector field coincides with it on the submanifold of the solutions and gives rise to an evolution symmetry with respect to the laboratory coordinates.

Such evolution symmetry is the master symmetry of the sine-Gordon hierarchy in laboratory coordinates. Therefore the Lie derivative of the symplectic structure (3.66) along it defines a second compatible Poisson bivector \( \Theta_2 \) and the operator \( \Phi = \Theta_2 \Theta_1^{-1} \) is an hereditary symmetry, in other words for any vector fields \( A \) and \( B \) it satisfies the equation:

\[
\Phi^2 [A, B] + [\Phi A, \Phi B] = \Phi ([\Phi A, B] + [A, \Phi B])
\]  

(3.67)

The proof of the hereditaryness of the operator \( \Phi \) is based on some computer algebra methods introduced in the paper [10].

4 The Liouville hierarchy

4.1 Useful results of [2]

• The Liouville hierarchy corresponds to the reduction of the hierarchy of equations (3.21) on the plane in the space \( (u, v, s) \) given by the equation \( s = v \).

• The difference with respect to the sine-Gordon case is that the recursion operator (3.34) is no longer invertible and as a consequence only the local Hamiltonians \( \{H_j^+\}_{j \in \mathbb{N}} \) and \( H_1^- \) survive.
4.2 The bi-Hamiltonian structure of the Liouville hierarchy

We have seen that the bi-Hamiltonian structure (1.4, 1.5) of the sine-Gordon hierarchy is simply the restriction of the pair \((\tilde{J}, \tilde{NJ})\) on the surface \(s^2 - v^2 = s_0^2\). Similarly it is possible to obtain the bi-Hamiltonian structure of the Liouville hierarchy as restriction of the pair \((\tilde{J}, \frac{1}{16} \tilde{J} + \tilde{NJ})\) on the plane \(s = v = \frac{\phi}{16}\). The only difference in the Liouville case is that, in order to have a second local Hamiltonian, we have to choose \(\frac{1}{16} \tilde{J} + \tilde{NJ}\) instead of \(\tilde{NJ}\).

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