Algebras with one operation including Poisson and other Lie-admissible algebras

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Abstract. We investigate algebras with one operation. We study when these algebras form a monoidal category and analyze Koszulness and cyclicity of the corresponding operads. We also introduce a new kind of symmetry for operads, the dihedrality, responsible for the existence of dihedral cohomology.

The main trick, which we call the polarization, will be to represent an algebra with one operation without any specific symmetry as an algebra with one commutative and one anticommutative operations. We will try to convince the reader that this change of perspective might sometimes lead to new insights and results.

This point of view was used in [15] to introduce a one-parametric family of operads whose specialization at 0 is the operad for Poisson algebras, while at a generic point it equals the operad for associative algebras. We study this family and explain how it can be used to interpret the deformation quantization (⋆-product) in a neat and elegant way.

Table of content: 1 Some examples to warm up – page 3
2 Monoidal structures – page 9
3 Koszulness, cyclicity and dihedrality – page 13

Introduction

If not stated otherwise, all algebraic objects in this paper will be defined over a fixed field \( \mathbb{K} \) of characteristic 0. We assume basic knowledge of operads as it can be gained for example from [20], though the first two sections can be read without this knowledge. We are going to study classes of algebras with one operation \( \cdot : V \otimes V \to V \) and axioms given as linear combinations of terms of the form \( v_{\sigma(1)} \cdot (v_{\sigma(2)} \cdot v_{\sigma(3)}) \) and/or \( (v_{\sigma(1)} \cdot v_{\sigma(2)}) \cdot v_{\sigma(3)} \), where \( \sigma \in \Sigma_3 \) is a permutation. All classical examples of algebras, such as associative, commutative, Lie and, quite surprisingly, Poisson algebras, are of this type.

In a fancier, operadic, language this means that we are going to consider algebras over operads \( \mathcal{P} \) of the form \( \mathcal{P} = \Gamma(E)/(R) \), where \( \Gamma(E) \) denotes the free operad generated by a \( \Sigma_2 \)-module \( E \) placed in arity 2 and \( (R) \) is the ideal generated by a \( \Sigma_3 \)-invariant subspace \( R \) of \( \Gamma(E)(3) \). Operads of this form are called quadratic [20, Definition 3.31]. We moreover

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assume that the $\Sigma_2$-module $E$ is *generated by one element*. This gives a precise meaning to what we mean by an “algebra with one operation.”

It follows from an elementary representation theory that, as a $\Sigma_2$-module,

$$ \text{either (1) } E \cong \mathbb{I}_2, \quad \text{or (2) } E \cong \text{sgn}_2, \quad \text{or (3) } E \cong \mathbb{K}[\Sigma_2], $$

where $\mathbb{I}_2$ is the one-dimensional trivial representation, $\text{sgn}_2$ is the one-dimensional signum representation and $\mathbb{K}[\Sigma_2]$ is the two-dimensional regular representation. In terms of the product these cases can be characterized by saying that $\cdot : V \otimes V \to V$ is

1. commutative, that is $x \cdot y = y \cdot x$ for all $x, y \in V$,
2. anticommutative, that is $x \cdot y = -y \cdot x$ for all $x, y \in V$,
3. without any symmetry, which means that there is no relation between $x \cdot y$ and $y \cdot x$.

Any multiplication $\cdot : V \otimes V \to V$ of type (3) can decomposed into the sum of a commutative multiplication $\bullet$ and an anti-commutative one $[-,-]$ via the *polarization* given by

$$ (1) \quad x \bullet y := \frac{1}{\sqrt{2}}(x \cdot y + y \cdot x) \quad \text{and} \quad [x,y] := \frac{1}{\sqrt{2}}(x \cdot y - y \cdot x), \quad \text{for } x, y \in V. $$

The inverse process of *depolarization* assembles a type (1) multiplication $\bullet$ with a type (2) multiplication $[-,-]$ into

$$ (2) \quad x \cdot y := \frac{1}{\sqrt{2}}(x \bullet y + [x,y]), \quad \text{for } x, y \in V. $$

The coefficient $\frac{1}{\sqrt{2}}$ was chosen so that the polarization followed by the depolarization (and vice versa) is the identity. On the operadic level, the above procedure reflects the decomposition

$$ \mathbb{K}[\Sigma_2] \cong \mathbb{I}_2 \oplus \text{sgn}_2 $$

of the regular representation into the trivial and signum representations.

The polarization enables one to view structures with a type (3) multiplication (such as associative algebras in Example 1) as structures with one commutative and one anticommutative operation, while the depolarization interprets structures with one commutative and one anticommutative operation (such as Poisson algebras in Example 2) as structures with one type (3) operation. We will try to convince the reader that this change of perspective might sometimes lead to new insights and results. The polarization-depolarization trick was independently employed by Livernet and Loday in their unpublished preprint [15].

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1. Some examples to warm up

In this section we give a couple of examples to illustrate the (de)polarization trick. We will usually omit the • denoting a commutative multiplication and write simply $xy$ instead of $x \cdot y$.

**Example 1.** *Associative algebras* are traditionally understood as structures with one operation of type (3). Let us see what happens with the associativity

$$ (x \cdot y) \cdot z = x \cdot (y \cdot z), \text{ for } x, y, z \in V, $$

if we polarize the multiplication $\cdot : V \otimes V \to V$. We claim that decomposing $x \cdot y = \frac{1}{\sqrt{2}} (xy + [x, y])$, the associativity (3) becomes equivalent to the following two axioms:

$$ [x, yz] = [x, y]z + y[x, z], $$

$$ [y, [x, z]] = (xy)z - x(zy). $$

To verify this, observe that (3) implies

$$ u_1(x, y, z) := A(x, y, z) - A(y, x, z) + A(z, y, x) + A(x, z, y) + A(y, z, x) - A(z, x, y) = 0, $$

$$ u_2(x, y, z) := A(x, y, z) + A(y, x, z) - A(z, y, x) + A(x, z, y) - A(y, z, x) - A(z, x, y) = 0, $$

with

$$ A(x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z) $$

the associator. Now, axiom (6) (resp. (5)) can be obtained from $u_1(x, y, z) = 0$ (resp. from $u_2(x, y, z) = 0$) by the depolarization, that is replacing $A(x, y, z)$ by

$$ \frac{1}{2} \left\{ (xy)z + [xy, z] + [x, y]z + [[x, y], z] - x(yz) - x[y, z] - x[yz] - [x, [y, z]] \right\}. $$

To prove that (6) together with (5) imply (3), observe that

$$ A(x, y, z) = \frac{1}{4} \left\{ (u_1 + u_2)(x, z, y) - (u_1 - u_2)(z, x, y) \right\}. $$

Let us remark that the summation of (5) over cyclic permutations gives the Jacobi identity

$$ [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0. $$

**Example 2.** *Poisson algebras* are usually defined as structures with two operations, a commutative associative one and an anti-commutative one satisfying the Jacobi identity. These operations are tied up by a distributive law

$$ [x, yz] = [x, y]z + y[x, z] $$

which we already saw in (4). The depolarization reinterprets Poisson algebras as structures with one type (3) operation $\cdot : V \otimes V \to V$ and one axiom:

$$ x \cdot (y \cdot z) = (x \cdot y) \cdot z - \frac{1}{3} \left\{ (x \cdot z) \cdot y + (y \cdot z) \cdot x - (y \cdot x) \cdot z - (z \cdot x) \cdot y \right\}. $$
The Jacobi identity gives of Poisson algebras in [12].

Then associativity of the commutative part becomes

\[ v_1(x, y, z) := (x \cdot y) \cdot z + (y \cdot x) \cdot z - (z \cdot y) \cdot x - (y \cdot z) \cdot x - x \cdot (y \cdot z) - x \cdot (z \cdot y) + z \cdot (y \cdot x) + z \cdot (x \cdot y) = 0. \]

The Jacobi identity gives

\[ v_2(x, y, z) := (x \cdot y) \cdot z - (y \cdot x) \cdot z - (z \cdot y) \cdot x - (x \cdot z) \cdot y + (y \cdot z) \cdot x + (z \cdot x) \cdot y - x \cdot (y \cdot z) + y \cdot (x \cdot z) + z \cdot (y \cdot x) + x \cdot (z \cdot y) - y \cdot (z \cdot x) - z \cdot (x \cdot y) = 0. \]

Finally, the distributive law gives

\[ v_3(x, y, z) := (x \cdot y) \cdot z - (y \cdot x) \cdot z + (z \cdot y) \cdot x + (x \cdot z) \cdot y + (y \cdot z) \cdot x - (z \cdot x) \cdot y - x \cdot (y \cdot z) + y \cdot (x \cdot z) - z \cdot (y \cdot x) - y \cdot (z \cdot x) + z \cdot (x \cdot y) = 0. \]

Now it is straightforward to verify that the vector

\[ v(x, y, z) := (x \cdot y) \cdot z - x \cdot (y \cdot z) - \frac{1}{3} \{(x \cdot z) \cdot y + (y \cdot z) \cdot y - (y \cdot x) \cdot z - (z \cdot x) \cdot y\}. \]

determined by (8) can be expressed as

\[ v(x, y, z) = \frac{1}{6} \{2v_1(x, y, z) + v_2(x, y, z) + v_3(x, y, z) + 2v_3(z, x, y)\}. \]

This shows that the depolarized multiplication indeed fulfills (8). To prove that (8) implies the usual axioms of Poisson algebras, one might use a similar straightforward method as in Example 1. Operadically this means the following. The operad $P$ for Poisson algebras can be presented as

\[ Poiss = \Gamma(\mathbb{K} \langle \Sigma_2 \rangle) / (R), \]

where $R$ is the 6-dimensional $\Sigma_3$-invariant subspace of $\Gamma(\mathbb{K} \langle \Sigma_2 \rangle)(3)$ generated by $v_1$ (the associativity), $v_2$ (the Jacobi) and $v_3$ (the distributive law). Equation (8) implies that $v \in R$ but we must prove that $v$ in fact generates $R$. We leave this as an exercise to the reader.

The depolarization form of the axioms was already used to study deformations and rigidity of Poisson algebras in [12].
Example 3. In this example, the ground field will be the complex numbers \( \mathbb{C} \). In their unpublished note \[15\], Livernet and Loday considered a one-parametric family of algebras with the axioms

\[
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \\
[x, yz] = [x, y]z + y[x, z], \\
(xy)z - x(yz) = q[y, [x, z]], \\
\]

depending on a complex parameter \( q \). Observe that, for \( q \neq 0 \), the first axiom (the Jacobi identity) is implied by the third one. Let us call algebras satisfying the above axioms \( LL_q \)-algebras (from Livernet-Loday).

For \( q = 0 \), (11) becomes the associativity and we recognize the usual definition of Poisson algebras. If \( q = 1 \), we get associative algebras, in the polarized form of Example 1. Furthermore, one may also consider the limit for \( q \to \infty \):

\[
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \\
[x, yz] = [x, y]z + y[x, z], \\
[y, [x, z]] = 0. \\
\]

In this case, the first identity trivially follows from the last one. These \( LL_\infty \)-algebras are algebras with a two-step-nilpotent anticommutative bracket and a commutative multiplication, related by a distributive law (the second equation).

The depolarization allows one to interpret \( LL_q \)-algebras as algebras with one type (3) operation \( \cdot : V \otimes V \to V \). The corresponding calculation was made in \[15\]. One must distinguish two cases. For \( q \neq -3 \) we get the axiom

\[
x \cdot (y \cdot z) = (x \cdot y) \cdot z + \frac{q - 1}{q + 3} \left\{ (x \cdot z) \cdot y + (y \cdot z) \cdot x - (y \cdot x) \cdot z - (z \cdot x) \cdot y \right\},
\]

while for \( q = -3 \) we get a structure with three axioms

\[
(x \cdot z) \cdot y + (y \cdot z) \cdot x - (y \cdot x) \cdot z - (z \cdot x) \cdot y = 0, \\
A(x, y, z) + A(z, y, x) = 0, \\
A(x, y, z) + A(y, z, x) + A(z, x, y) = 0,
\]

where \( A \) denotes, as usual, the associator (6). It can be easily verified that the formula

\[
x \ast y := \frac{1 + \sqrt{q}}{2} x \cdot y + \frac{1 - \sqrt{q}}{2} y \cdot x
\]

converts \( LL_q \)-algebras, for \( q \neq 0, \infty \), into associative algebras. Operadically this means that the operad \( LL_q \) for \( LL_q \)-algebras is, for \( q \notin \{0, \infty\} \), isomorphic to \( LL_1 = Ass \). This fact was observed also in \[15\].
Example 4. In this example we explain how Livernet-Loday algebras can be used to interpret deformation quantization of Poisson algebras. The ground ring here will be the ring \( \mathbb{K}[[t]] \) of formal power series in \( t \). Let us recall [2] that a \(*\)-product on a \( \mathbb{K}\)-vector space \( A \) is a \( \mathbb{K}[[t]]\)-linear associative unital multiplication \( * : A[[t]] \otimes A[[t]] \to A[[t]] \) which is commutative mod \( t \). Expanding, for \( u, v \in A \),

\[
u * v = u *_0 v + t \ u *_1 v + t^2 \ u *_2 v + \cdots, \quad \text{with} \quad u *_i v \in A \quad \text{for} \quad i \geq 0,
\]

one easily verifies that the operations \( \cdot_0 \) and \( [-,-]_0 \) defined by

\[
u \cdot_0 v := u *_0 v \quad \text{and} \quad [u,v]_0 := u *_1 v - v *_1 u, \quad u, v \in A,
\]

are such that \( P := (A, \cdot_0, [-,-]_0) \) is a Poisson algebra. The object \( (A[[t]], *) \) is sometimes also called the deformation quantization of the Poisson algebra \( P \). In applications, \( P \) is the \( \mathbb{R}\)-algebra \( C^\infty(M) \) of smooth functions on a Poisson manifold \( M \) that represents the phase space of a classical physical system. One moreover assumes that all products \( *_i, i \geq 0, \) in (13) are bilinear differential operators, see again [2] for details. The relevance of \( LLq\)-algebras for quantization is explained in the following theorem.

**Theorem 5.** A \(*\)-product on a \( \mathbb{K}\)-vector space \( A \) is the same as an \( LLq^2\)-algebra structure on the \( \mathbb{K}[[t]]\)-module \( V := A[[t]] \).

**Proof.** Given a \(*\)-product, define \( \cdot : V \otimes V \to V \) and \( [-,-] : V \otimes V \to V \) as the polarization (1) of \( * : V \otimes V \to V \). Commutativity of \( * \) mod \( t \) means that \( [-,-] = 0 \) mod \( t \), therefore there exists a bilinear antisymmetric map \( \{ -,- \} : V \otimes V \to V \) such that \( [-,-] = t \{ -,- \} \). It is immediate to check that \( (V, \cdot, \{,-,-\}) \) is an \( LLq^2\)-algebra. On the other hand, given an \( LLq^2\)-algebra \( (V, \cdot, \{-,-\}) \), then

\[
u * v := \frac{1}{\sqrt{2}}(u \cdot v + t\{u,v\}), \quad \text{for} \quad u, v \in V,
\]

clearly defines a \(*\)-product on \( A \).

**Example 6.** Recall [14] that a type (3) product \( \cdot : V \otimes V \to V \) is called Lie-admissible if the commutator of this product is a Lie bracket or, equivalently, that the antisymmetric part \( [-,-] \) of its polarization fulfills the Jacobi identity (7). This observation suggests that the polarization might be particularly suited for various types of Lie-admissible algebras.

Some important classes of Lie-admissible algebras were studied in [13]. Before we recall the definitions, we note that a type (3) product \( \cdot : V \otimes V \to V \) is Lie-admissible if and only if its associator (6) satisfies

\[
\sum_{\sigma \in \Sigma_3} (-1)^{\xi(\sigma)} A(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = 0,
\]

(15)
where $\epsilon(\sigma)$ denotes the signature of the permutation $\sigma$. Now $G$-associative algebras, where $G$ is a (not necessary normal) subgroup of $\Sigma_3$, are algebras with a type (3) multiplication whose commutator satisfies a condition which is, for $G \neq \Sigma_3$, stronger than (15), namely

$$\sum_{\sigma \in G} (-1)^{\epsilon(\sigma)} A(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = 0.$$ 

Therefore we have six different types of $G$-associative algebras corresponding to the following six subgroups of $\Sigma_3$:

\[ G_1 := \{1\}, \ G_2 := \{1, \tau_{12}\}, \ G_3 := \{1, \tau_{23}\}, \ G_4 := \{1, \tau_{13}\}, \ G_5 := A_3 \text{ and } G_6 := \Sigma_3, \]

where $\tau_{ij}$ denotes the transposition $i \leftrightarrow j$ and $A_3$ is the alternating subgroup of $\Sigma_3$. $G_1$-associative algebras are clearly associative algebras whose polarization we discussed in Example 1. $G_2$-associative algebras are the same as Vinberg algebras, also called left-symmetric algebras, see [23].

In the polarized form, $G_2$-associative algebras are structures with a commutative multiplication and a Lie bracket related by the axiom:

$$2[x, y]z + [[x, y], z] - x(yz) + y(xz) - x[y, z] + y[x, z] - [x, yz] + [y, xz] = 0.$$ 

As suggested by Loday, (16) can be written as the sum

$$\{(xz)y - x(zy) - [z, [x, y]]\} + \{(x, y)z + y[x, z] - [x, yz]\} + \{[y, xz] - x[y, z] - [y, x]z\} = 0$$

of three terms which vanish separately if the multiplication is associative, see [4] and [5]. $G_3$-associative algebras are also classical objects, known as right-symmetric algebras [21] or pre-Lie algebras [7].

The operad Pre-Lie associated to pre-Lie algebras is isomorphic to the operad Vinb for Vinberg algebras via the operadic homomorphism determined by

$$xy \mapsto xy, \ [x, y] \mapsto -[x, y].$$

This homomorphism converts (16) into

$$2[x, y]z - [[x, y], z] + x(yz) - y(xz) - x[y, z] + y[x, z] - [x, yz] + [y, xz] = 0.$$ 

After polarizing, we identify $G_4$-associative algebras as structures satisfying

$$(xy)z - x(yz) = [[x, z], y],$$

which clearly implies the Jacobi (17). $G_5$-associative algebras have a commutative multiplication and a Lie bracket tied together by

$$[xy, z] + [yz, x] + [zx, y] = 0.$$
Figure 1: The composition $f \circ_3^i g \in \text{Lin}(V^\otimes 10, V^\otimes 7)$ of functions $f \in \text{Lin}(V^\otimes 6, V^\otimes 3)$ and $g \in \text{Lin}(V^\otimes 5, V^\otimes 5)$.

The polarization of $G_6$-associative algebras, which are sometimes confusingly called just Lie-admissible algebras, reveals that the category of these objects is a dull one, consisting of structures with a commutative multiplication and a Lie bracket, with no relation between these two operations.

Let us close this example by observing that axiom (10) of $LL_q$-algebras implies axiom (17) of $G_5$-associative algebras, therefore $LL_q$-algebras form, for each $q$, a subcategory of the category of $G_5$-associative algebras. An equally simple observation is that the polarized product of $G_4$-associative algebras satisfies the first and the third identities of $LL_{-1}$-algebras but not the distributive law.

**Example 7.** Lie-admissible structures mentioned in Example 6 are rather important. As it was shown in the seminal paper [7], there exists a natural pre-Lie structure on the Hochschild cochain complex of every associative algebra induced by an even more elementary structure christened later, in [8], a brace algebra. This pre-Lie structure is responsible for the existence of the intrinsic bracket on the Hochschild cohomology, see again [7].

We offer the following generalization of this structure. For a vector space $V$, denote by

$$X := \bigoplus_{m,n \geq 1} \text{Lin}(V^\otimes m, V^\otimes n)$$

the space of all multilinear maps. For $f \in \text{Lin}(V^\otimes b, V^\otimes a)$ and $g \in \text{Lin}(V^\otimes d, V^\otimes c)$ define $f \circ_j^i g \in \text{Lin}(V^\otimes a+c-1, V^\otimes b+d-1)$ to be the map obtained by composing the $j$-th output of $g$ into the $i$-th input of $f$ and arranging the remaining outputs and inputs as indicated in Figure 4. Define finally

$$f \circ g := \sum_{1 \leq i \leq b, 1 \leq j \leq c} (-1)^{i(b+1)+j(c+1)} f \circ_j^i g.$$

We leave to the reader to verify that $(X, \circ)$ is a $G_6$-associative algebra.

Let $Y_{\text{Hoch}} \subset X$ be the subspace $Y_{\text{Hoch}} := \bigoplus_{m \geq 1} \text{Lin}(V^\otimes m, V)$ and let dually $Y_{\text{coHoch}} := \bigoplus_{n \geq 1} \text{Lin}(V, V^\otimes n)$. Clearly both $Y_{\text{Hoch}}$ and $Y_{\text{coHoch}}$ are $\circ$-closed. It turns out that $(Y_{\text{Hoch}}, \circ)$ is a $G_3$-associative algebra (= pre-Lie algebra) and $(Y_{\text{coHoch}}, \circ)$ a $G_2$-associative algebra (= Vinberg algebra). We recognize $(Y_{\text{Hoch}}, \circ)$ as the underlying space of the Hochschild cochain
complex $C^*_\text{Hoch}(A; A)$ of an associative algebra $A = (V, \cdot)$ with the classical pre-Lie structure $[7]$. The space $(Y_{\text{coHoch}}, \circ)$ has a similar interpretation in terms of the Cartier cohomology of coassociative coalgebras $[8]$.

To interpret $X$ in a similar way, we need to recall that an infinitesimal bialgebra $[1]$ (also called a mock bialgebra in $[5]$) is a triple $(V, \mu, \delta)$, where $\mu$ is an associative multiplication, $\delta$ is a coassociative comultiplication and

$$\delta(\mu(u, v)) = \delta_1(u) \otimes \mu(\delta_2(u), v) + \mu(u, \delta_1(v)) \otimes \delta_2(v)$$

for each $u, v \in V$, with the standard Sweedler’s notation for the comultiplication used. It turns out that $X$ is the underlying space of the cochain complex defining the cohomology of an infinitesimal bialgebra and $[f, g] := f \circ g - g \circ f$ is the intrinsic bracket, see $[19]$, on this cochain complex. The reader is encouraged to verify that the axioms of infinitesimal bialgebras can be written as the ‘master equation’

$$[\mu + \delta, \mu + \delta] = 0,$$

with $\mu : V \otimes V \to V$ and $\delta : V \to V \otimes V$ interpreted as elements of $X$.

2. Monoidal structures

Consider the category $\mathcal{P}\text{-alg}$ of algebras over a fixed operad $\mathcal{P}$. Following $[9]$ we say that $\mathcal{P}$ is a Hopf operad, if the category $\mathcal{P}\text{-alg}$ admits a strict monoidal structure $\odot : \mathcal{P}\text{-alg} \times \mathcal{P}\text{-alg} \to \mathcal{P}\text{-alg}$ such that the forgetful functor $\Box : \mathcal{P}\text{-alg} \to \text{Vect}_K$ to the category of $K$-vector spaces with the standard tensor product, is a strict monoidal morphism, see $[17, VII.1]$ for the terminology. This condition can be expressed solely in terms of $\mathcal{P}$ as in the following definition.

**Definition 8.** An operad $\mathcal{P}$ is a Hopf operad if there exists an operadic map $\Delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$ (the diagonal) which is coassociative in the sense that

$$\Delta \otimes 1_P \Delta = (1_P \otimes \Delta) \Delta,$$

where $1_P : \mathcal{P} \to \mathcal{P}$ denotes the identity. We also assume the existence of a counit $e : \mathcal{P} \to \text{Com}$, where $\text{Com}$ is the operad for commutative associative algebras, satisfying

$$e \otimes 1_P \Delta = (1_P \otimes e) \Delta = 1_P.$$

The last equation uses the canonical identification $\mathcal{P} \cong \text{Com} \otimes \mathcal{P} \cong \mathcal{P} \otimes \text{Com}$. Our terminology slightly differs from the original one of $[9]$ which did not assume the counit. Our assumption about the existence of the counit rules out trivial diagonals.

The diagonal $\Delta : \mathcal{P} \to \mathcal{P} \otimes \mathcal{P}$ induces a product $\odot : \mathcal{P}\text{-alg} \times \mathcal{P}\text{-alg} \to \mathcal{P}\text{-alg}$ in a way described for example in $[20$, page 197]. Equation (18) is equivalent to the coassociativity of this product. To interpret (19), observe that, since $\text{Com}$ is isomorphic to the endomorphism
operad $\mathcal{E}nd_K$ of the ground field, the counit $e$ equips $K$ with a $\mathcal{P}$-algebra structure. Equation (19) then says that $K$ with this structure is the unit object for the monoidal structure induced by $\Delta$.

In the rest of this section we want to discuss the existence Hopf structures on quadratic operads $\mathcal{P} = \Gamma(E)/(R)$ with one operation. Let us look more closely at the map $e : \mathcal{P} \to \mathcal{C}om$ first. Since $\mathcal{C}om = \Gamma(\mathbb{I}_2)/(R_{ass})$, with $\mathbb{I}_2$ the trivial representation of $\Sigma_2$ and $(R_{ass})$ the ideal generated by the associativity, the counit $e$ is determined by a $\Sigma_2$-equivariant map

\[ e(2) : E \to \mathbb{I}_2. \]  

If $E = \mathbb{I}_2$ (case (1) of the nomenclature of the introduction), such a map is the multiplication by a scalar $\alpha$. If $E = sgn_2$ (case (2)), the only equivariant $e(2)$ is the zero map. Finally, if $E = K[\Sigma_2]$ (case (3)), $e(2)$ must be the projection $K[\Sigma_2] \to \mathbb{I}_2$ multiplied by some $\alpha \in K$.

Equation (19) implies the non-triviality of $e(2)$. This excludes case (2) and implies that $\alpha \neq 0$ in cases (1) and (3). In these two cases we may moreover assume the normalization $\alpha = 1$, the general case can be brought to this form by rescaling $e \mapsto \alpha^{-1}e$, $\Delta \mapsto \alpha \Delta$.

Let us introduce the following useful pictorial language. Denote by $\bigcirc \in \mathcal{P}(2)$ the operadic generator for a type (3) operation (a multiplication with no symmetry). Similarly, we denote the generator for a commutative operation by $\bigotimes$ and for an anti-commutative one by $\bigcirc$. The right action of the generator $\tau \in \Sigma_2$ on $\mathcal{P}(2)$ is, in this language, described by

\[ \tau \bigotimes_1 \bigotimes_2 = \bigotimes_2 \bigotimes_1 \quad \text{and} \quad \tau \bigcirc_1 \bigcirc_2 = \bigcirc_2 \bigcirc_1. \]

The polarization (11) is then given by

\[ \bigotimes_1 \bigotimes_2 = \frac{1}{\sqrt{2}} \left( \bigotimes_1 \bigotimes_2 + \bigotimes_2 \bigotimes_1 \right) \quad \text{and} \quad \bigcirc_1 \bigcirc_2 := \frac{1}{\sqrt{2}} \left( \bigcirc_1 \bigcirc_2 - \bigcirc_2 \bigcirc_1 \right), \]

and the depolarization (2) by

\[ \bigotimes_1 \bigotimes_2 := \frac{1}{\sqrt{2}} \left( \bigotimes_1 \bigotimes_2 + \bigotimes_2 \bigotimes_1 \right). \]

In the rest of this section we investigate the existence of diagonals for quadratic operads with one operation. Since the diagonal is, by assumption, an operadic homomorphism, it is uniquely determined by its value on a chosen generator of $\mathcal{P}(2)$. Let us see what can be concluded from this simple observation. As before, we distinguish three cases.

Case (1). In this case, the operad $\mathcal{P}$ is generated by one commutative bilinear operation $\bigotimes \in \mathcal{P}(2)$. The diagonal must necessarily satisfy

\[ \Delta(\bigotimes_1 \bigotimes_2) = A \left( \bigotimes_1 \bigotimes_2 \otimes \bigotimes_1 \bigotimes_2 \right), \quad \text{for some } A \in K. \]

The coassociativity (18) is fulfilled automatically while (19) implies $A = 1$.  

Case (2). Analyzing the counit, we already observed that operads with one anti-symmetric operation do not admit a (counital) diagonal. An easy argument shows that non-trivial diagonals for type (2) operads do not exist even if we do not demand the existence of a counit. Indeed, in case (2) we have 
\[ P(2) \cong \text{sgn}_2 \] while 
\[ P(2) \otimes P(2) \cong \text{sgn}_2 \otimes \text{sgn}_2 \cong \mathbb{I}_2, \]
therefore \( \Delta(2) : P(2) \to P(2) \otimes P(2) \) is trivial, as is any \( \Sigma_2 \) equivariant map \( \text{sgn}_2 \to \mathbb{I}_2 \). Let us formulate this observation as:

**Theorem 9.** There is no non-trivial diagonal on a quadratic operad generated by an anti-symmetric product. In particular, the operad \( \mathcal{L} \text{ie} \) for Lie algebras is not an Hopf operad.

Case (3). As an operadic homomorphism, the diagonal (if exists) is uniquely determined by an element \( \Delta(\underline{1,2}) \in P(2) \otimes P(2) \). The following proposition characterizes which choices of \( \Delta(\underline{1,2}) \) may lead to a coassociative counital diagonal.

**Proposition 10.** Let \( \mathcal{P} \) be a quadratic Hopf operad generated by a type (3) product \( \underline{1,2} \). Then there exists \( B \in \mathbb{K} \) such that the polarized form of the diagonal \( \Delta \) is given by

\[
\Delta(\underline{1,2}) = \underline{1,2} \otimes \underline{1,2} - B \left\{ \left( \underline{1,2} - \underline{2,1} \right) \otimes \left( \underline{1,2} - \underline{2,1} \right) \right\}.
\]

The polarized version of this equation reads

\[
\begin{align*}
\Delta(\underline{1,2}) &= \frac{1}{\sqrt{2}} \left\{ \underline{1,2} \otimes \underline{1,2} + (1 - 4B) \left( \underline{1,2} \otimes \underline{1,2} \right) \right\}, \\
\Delta(\underline{1,2}) &= \frac{1}{\sqrt{2}} \left\{ \underline{1,2} \otimes \underline{1,2} + \underline{1,2} \otimes \underline{1,2} \right\}.
\end{align*}
\]

**Proof.** A simple bookkeeping. The most general choice for \( \Delta(2) : P(2) \to P(2) \otimes P(2) \) is

\[
\Delta(\underline{1,2}) = A(\underline{1,2} \otimes \underline{1,2}) + B(\underline{1,2} \otimes \underline{2,1}) + C(\underline{1,2} \otimes \underline{1,2}) + D(\underline{2,1} \otimes \underline{1,2})
\]

with some \( A, B, C, D \in \mathbb{K} \). A straightforward calculation shows that the coassociativity \( (\Delta \otimes \mathbb{I}_P) \Delta = (\mathbb{I}_P \otimes \Delta) \Delta \) for \( \Delta \) defined by (23) has the following four families of solutions:

(i) \( D = C = 0, \ A = B \),  
(ii) \( B = C = -D, \ A \) arbitrary,  
(iii) \( D = B = 0, \ A = C \),  
(iv) \( B = C = D = A \).

The counit condition (19) leads to the system:

\[
A + C = 1, \ B + D = 0, \ A + B = 1, \ \text{and} \ C + D = 0.
\]

We easily conclude that the only solution is a type (ii) one with \( B = C = -D \) and \( A = 1 - B \). This gives (21) whose polarization is (22).

Proposition 10 offers a mighty tool to investigate the existence of Hopf structures for type (3) operads. It says that such an operad \( \mathcal{P} \) is a Hopf operad if and only if there exists
$B \in \mathbb{K}$ such that the diagonal defined by (21) (resp. (22) in the polarized form) extends to an operad map, i.e. preserves the relations $R$ in the quadratic presentation $\Gamma(E)/(R)$ of $\mathcal{P}$.

Regarding the existence of diagonals in general, an operad might admit no Hopf structure at all (examples of this situation are provided by Theorem 9), it might admit exactly one Hopf structure (see Example 12 for operads with this property), or it might admit several different monoidal structures, as illustrated in Example 13.

Let us formulate another simple proposition whose proof we leave as an exercise. We say that $\mathcal{P}$ is a set-operad, if there exists an operad $\mathcal{S}$ in the monoidal category of sets such that, for any $n \geq 1$, $\mathcal{P}(n)$ is the $\mathbb{K}$-linear span of $\mathcal{S}(n)$, and that the operad structure of $\mathcal{P}$ is naturally induced from the operad structure of $\mathcal{S}$.

**Proposition 11.** Every set-operad $\mathcal{P}$ admits an Hopf structure given by the formula $\Delta(p) := p \otimes p$, for any $p \in \mathcal{P}$.

**Example 12.** Let $LL_q$ denote the operad for $LL_q$ algebras. Then the one-parametric family $\{LL_q\}_{q \neq \infty}$ is a family of Hopf operads, with the diagonal given by

\[
\Delta(\begin{array}{c} \Box \\ 1 \end{array}, \begin{array}{c} \Box \\ 2 \end{array}) = \begin{array}{c} \Box \\ 1 \end{array} \otimes \begin{array}{c} \Box \\ 2 \end{array} - \frac{1-q}{4} \left\{ \left( \begin{array}{c} \Box \\ 1 \end{array} - \begin{array}{c} \Box \\ 2 \end{array} \right) \otimes \left( \begin{array}{c} \Box \\ 1 \end{array} - \begin{array}{c} \Box \\ 2 \end{array} \right) \right\}.
\]

The polarized version of this equation reads

\[
\begin{cases}
\Delta(\begin{array}{c} \Box \\ 1 \end{array}) = \frac{1}{\sqrt{2}} \left\{ \begin{array}{c} \Box \\ 1 \end{array} \otimes \begin{array}{c} \Box \\ 1 \end{array} + q \left( \begin{array}{c} \Box \\ 1 \end{array} \otimes \begin{array}{c} \Box \\ 1 \end{array} \right) \right\}, \\
\Delta(\begin{array}{c} \Box \\ 1 \end{array}) = \frac{1}{\sqrt{2}} \left\{ \begin{array}{c} \Box \\ 1 \end{array} \otimes \begin{array}{c} \Box \\ 1 \end{array} + \begin{array}{c} \Box \\ 1 \end{array} \otimes \begin{array}{c} \Box \\ 1 \end{array} \right\}.
\end{cases}
\]

The above normalized diagonal is moreover unique for each $q \neq \infty$. Observe that the limit for $q \to \infty$ of formulas (24) (resp. (25)) does not make sense and, indeed, it can be easily shown that the operad $LL_{\infty}$ is not Hopf.

**Example 13.** We give a funny example of a category of algebras which admits several non-equivalent monoidal structures. Let us consider a type (3) product $x, y \mapsto x \cdot y$, with the axiom

\[
(x \cdot y) \cdot z = z \cdot (y \cdot x).
\]

Then

\[
\Delta(\begin{array}{c} \Box \\ 1 \end{array}) := \begin{array}{c} \Box \\ 1 \end{array} \otimes \begin{array}{c} \Box \\ 2 \end{array} - B \left[ \left( \begin{array}{c} \Box \\ 1 \end{array} - \begin{array}{c} \Box \\ 2 \end{array} \right) \otimes \left( \begin{array}{c} \Box \\ 1 \end{array} - \begin{array}{c} \Box \\ 2 \end{array} \right) \right].
\]

defines an Hopf structure for any $B \in \mathbb{K}$.

Above we saw an algebra admitting a one-parametric family of non-equivalent monoidal structures. It would be interesting to see a structure that admits a discrete family of non-equivalent Hopf structures.

**Example 14.** It can be shown that the only $G$-admissible algebras that admit a monoidal structure are associative algebras. In particular, neither Vinberg nor pre-Lie algebras form a monoidal category.
3. Koszulness, cyclicity and dihedrality

In this section we study cyclicity \[10\] of operads mentioned in the previous sections. We then introduce the notion of dihedrality of operads and investigate this property. To complete the picture, we also list results concerning Koszulness \[11\] of some operads with one operation.

Let us recall first what is a cyclic operad. Let \(\Sigma_n^+\) be the group of automorphisms of the set \(\{0, \ldots, n\}\). This group is, of course, isomorphic to the symmetric group \(\Sigma_{n+1}\), but the isomorphism is canonical only up to an identification \(\{0, \ldots, n\} \cong \{1, \ldots, n+1\}\). We interpret \(\Sigma_n\) as the subgroup of \(\Sigma_n^+\) consisting of permutations \(\sigma \in \Sigma_n^+\) with \(\sigma(0) = 0\). If \(\gamma_n^+ \in \Sigma_n^+\) denotes the cycle \((0, \ldots, n)\), that is, the permutation with \(\gamma_n^+(0) = 1, \gamma_n^+(1) = 2, \ldots, \gamma_n(n) = 0\), then \(\gamma_n^+\) and \(\Sigma_n\) generate \(\Sigma_n^+\).

By definition, each operad \(\mathcal{P}\) has a natural right action of \(\Sigma_n\) on each piece \(\mathcal{P}(n), n \geq 1\). The operad \(\mathcal{P}\) is cyclic if this action extends, for any \(n \geq 1\), to a \(\Sigma_n^+\)-action in a way compatible with structure operations. See \[20\] Definition II.5.2 or the original paper \[10\] for a precise definition.

We already recalled in the introduction that an (ordinary) operad \(\mathcal{P}\) is quadratic if it can be presented as \(\mathcal{P} = \Gamma(E)/(R)\), where \(E = \mathcal{P}(2)\) and \(R \subset \Gamma(E)(3)\). The action of \(\Sigma_2\) on \(E\) extends to an action of \(\Sigma_2^+\), via the sign representation \(\text{sgn} : \Sigma_2^+ \to \{\pm 1\} \cong \Sigma_2\). It can be easily verified that this action induces a cyclic operad structure on the free operad \(\Gamma(E)\). In particular, \(\Gamma(E)(3)\) is a right \(\Sigma_3^+\)-module. An operad \(\mathcal{P}\) as above is called cyclic quadratic if the space of relations \(R\) is invariant under the action of \(\Sigma_3^+\). Since \(R\) is, by definition, \(\Sigma_3\)-invariant, \(\mathcal{P}\) is cyclic quadratic if and only if \(R\) is preserved by the action of the generator \(\gamma_3^+\).

Remark 15. There are operads that are both quadratic and cyclic but not cyclic quadratic. The simplest example of this exotic phenomenon is provided by the free operad \(\Gamma(V_{2,2})\) generated by the 2-dimensional irreducible representation \(V_{2,2}\) of \(\Sigma_3 \cong \Sigma_2^+\) placed in arity 2. In general, an operad \(\mathcal{P}\) is cyclic quadratic if and only if it is both quadratic and cyclic and if the \(\Sigma_2^+\)-action on \(\mathcal{P}(2)\) is induced from the operadic \(\Sigma_2\)-action on \(\mathcal{P}(2)\) via the homomorphism \(\text{sgn} : \Sigma_2^+ \to \{\pm 1\} \cong \Sigma_2\).

Let us turn our attention to the cyclicity of operads for algebras with one operation. Since, as proved in \[10\] Proposition 3.6, each quadratic operad with one operation of type (1) or (2) is cyclic quadratic, we shall focus on operads with a type (3) multiplication. The right action of the generator \(\gamma_3^+ \in \Sigma_3^+\) on \(\Gamma(\mathbb{K}[\Sigma_2])/(3)\) is described in the following table:

\[
\begin{align*}
((x \cdot y) \cdot z)\gamma_3^+ &= x \cdot (y \cdot z), & (x \cdot (y \cdot z))\gamma_3^+ &= (x \cdot y) \cdot z, \\
((y \cdot z) \cdot x)\gamma_3^+ &= (y \cdot x) \cdot z, & (y \cdot (z \cdot x))\gamma_3^+ &= y \cdot (x \cdot z), \\
((z \cdot x) \cdot y)\gamma_3^+ &= y \cdot (z \cdot x), & (z \cdot (x \cdot y))\gamma_3^+ &= (y \cdot z) \cdot x, \\
((y \cdot x) \cdot z)\gamma_3^+ &= x \cdot (z \cdot y), & (y \cdot (x \cdot z))\gamma_3^+ &= (x \cdot z) \cdot y, \\
((z \cdot y) \cdot x)\gamma_3^+ &= z \cdot (y \cdot x), & (z \cdot (y \cdot x))\gamma_3^+ &= (z \cdot y) \cdot x, \\
((x \cdot z) \cdot y)\gamma_3^+ &= (z \cdot x) \cdot y, & (x \cdot (z \cdot y))\gamma_3^+ &= z \cdot (x \cdot y).
\end{align*}
\]
| Operad           | Type of algebras | Koszul | Cyclic | Dihedral | Hopf |
|------------------|------------------|--------|--------|----------|------|
| $Ass = L\mathcal{L}_1 = G_1$-ass | associative      | yes    | yes    | yes      | yes  |
| $P_{oiss} = L\mathcal{L}_0$     | Poisson          | yes    | yes    | yes      | yes  |
| $L\mathcal{L}_q$, $q \neq 0, \infty$ | $LL_q$-algebras  | yes    | yes    | yes      | yes  |
| $L\mathcal{L}_\infty$          | $LL_\infty$-algebras | yes    | yes    | yes      | no   |
| $Vinb = G_2$-ass               | Vinberg          | yes    | no     | no       | no   |
| $pre-Lie = G_3$-ass            | pre-Lie          | yes    | no     | no       | no   |
| $G_4$-ass                   | $G_4$-associative | no     | yes    | yes      | no   |
| $G_5$-ass                   | $G_5$-associative | no     | no     | yes      | no   |
| $G_6$-ass                   | Lie-admissible   | yes    | yes    | yes      | no   |

Figure 2: Koszulness, quadratic cyclicity, dihedrality and Hopfness of operads with one type (3) operation.

Using this table, it is easy to investigate the cyclicity of operads with one operation, see Example 19 where the corresponding analysis was done for $G_5$-associative algebras. The results are summarized in Figure 2.

In Definition 16 below we single out a property of quadratic operads responsible for the existence of the dihedral cohomology [6, 16] of associated algebras. As far as we know, this property has never been considered before. Let $\mathcal{P} = \Gamma(E)/(R)$ be a quadratic operad. Let $\lambda \in \Sigma_2$ be the generator and define a left $\Sigma_2$-action on $E$ using the operadic right $\Sigma_2$-action by $\lambda e := e\lambda$, for $e \in E$. It follows from the universal property of free operads that this action extends to a left $\Sigma_2$-action on $\Gamma(E)$.

**Definition 16.** We say that a quadratic operad $\mathcal{P} = \Gamma(E)/(R)$ is dihedral if the left $\Sigma_2$-action on $\Gamma(E)$ induces a left $\Sigma_2$-action on $\mathcal{P}$. A quadratic operad is cyclic dihedral, if it is both cyclic and dihedral and if these two structures are compatible, by which we mean that

$$(\lambda u)\sigma = \lambda(u\sigma),$$

for each $u \in \mathcal{P}(n)$, $\lambda \in \Sigma_2$, $\sigma \in \Sigma_n^+$ and $n \geq 1$. In other words, the cyclic and dihedral actions make each piece $\mathcal{P}(n)$ of a cyclic dihedral operad a left $\Sigma_2$-right $\Sigma_n^+$-bimodule.

**Remark 17.** We emphasize that dihedrality is a property defined only for quadratic operads. We do not know how to extend this definition for a general operad. Observe that the left $\Sigma_2$-action on $\Gamma(E)$ induces an action on $\mathcal{P}$ as required in Definition 16 if and only if the space of relations $R \subset \Gamma(E)(3)$ is $\Sigma_2$-stable.

The operad $\Gamma(V_{2,2})$ considered in Remark 15 is quadratic, cyclic and dihedral, but not cyclic dihedral, because the left $\Sigma_2$-action on $V_{2,2}$ is clearly not compatible with the right $\Sigma_2^+$-action. On the other hand, each cyclic quadratic operad which is dihedral is cyclic dihedral.
We leave as an exercise to prove that all quadratic operads generated by one operation of type (1) or (2) are dihedral. Therefore again the only interesting case to investigate is a type (3) operation. The dihedrality is then easily understood if we write the axioms in the polarized form as follows. Let $E = \mathbb{K}[\Sigma_2]$ and decompose

\begin{equation}
\Gamma(E)(3) = \Gamma_+(E)(3) \oplus \Gamma_-(E)(3),
\end{equation}

where $\Gamma_+(E)(3)$ is the $\Sigma_3$-subspace of $\Gamma(E)(3)$ generated by compositions $x(yz)$ and $[x, [y, z]]$, and $\Gamma_-(E)(3)$ is the $\Sigma_3$-subspace of $\Gamma(E)(3)$ generated by compositions $x[y, z]$ and $[x, yz]$.

In the pictorial language of Section 2, $\Gamma_+(E)(3)$ is the $\Sigma_3$-invariant subspace generated by compositions of the following two types

- \hspace{1cm}

while $\Gamma_-(E)(3)$ is the $\Sigma_3$-invariant subspace generated by

- \hspace{1cm}

Decomposition (26) is obviously $\Sigma_3^+$-invariant. It is almost evident that $\lambda$ acts trivially on $\Gamma_+(E)(3)$ while on $\Gamma_-(E)(3)$ it acts as the multiplication by $-1$. We therefore get for free the following:

**Proposition 18.** A quadratic operad $\mathcal{P} = \Gamma(E)/(R)$ generated by a type (3) multiplication is dihedral if and only if the space of relations $R$ decomposes as

$$R = R_+ \oplus R_-,$$

with $R_+ \subset \Gamma_+(E)(3)$ and $R_- \subset \Gamma_-(E)(3)$.

**Example 19.** In this example we show that the operad $G_5$-ass for $G_5$-associative algebras is dihedral but not cyclic. Recall from Example 6 that the polarized form of the axioms for these algebras consists of the Jacobi identity

$$[[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

and equation (17)

$$[xy, z] + [yz, x] + [zx, y] = 0.$$ 

Since the left-hand side of the Jacobi identity belongs to $\Gamma_+(E)(3)$ and the right-hand side of (17) to $\Gamma_-(E)(3)$, the space of relations obviously decomposes as required by Proposition 18. Therefore $G_5$-ass is dihedral.

Let us inspect the cyclicity. By definition, the unpolarized form of the axiom for $G_5$-associative algebras reads

\begin{equation}
A(x, y, z) + A(y, z, x) + A(z, x, y) = 0,
\end{equation}

where $A(x, y, z)$ is an associative product.

\[\text{Example 20.}\]
where \( A \) denotes, as usual, the associator \( \gamma_3^- \). The action of \( \gamma_3^- \) converts this equation to
\[
-A(x, y, z) + A(y, x, z) - A(y, z, x) = 0.
\]
The sum of the above equations gives
\[
A(y, x, z) + A(z, x, y) = 0.
\]
It is then a simple linear algebra to prove that this equation does not belong to the \( \Sigma_3 \)-closure of \( \mathcal{G}_5 \). Therefore \( \mathcal{G}_5 \)-ass is not cyclic.

**Theorem 20.** Cyclic quadratic operads generated by one operation are dihedral.

**Proof.** The claim is obvious when \( \mathcal{P} \) is generated by one operation of type (1) or (2). Suppose \( \mathcal{P} \) is a quadratic operad of the form
\[
\mathcal{P} = \Gamma(E)/(R), \quad \text{where } E =: \mathbb{K}[\Sigma_2].
\]
It was calculated in \[10\] that, as \( \Sigma_3^+ \)-modules,
\[
\Gamma_+(E)(3) = \mathbb{1}_3 \oplus V_{2,2} \oplus \text{sgn} \oplus V_{2,2} \quad \text{and} \quad \Gamma_-(E)(3) = V_{3,1} \oplus V_{2,1,1},
\]
where the irreducible representations \( \mathbb{1}, \text{sgn}, V_{2,2}, V_{3,1} \) and \( V_{2,1,1} \) are given by the following character table:

| \( I \) | \( (01) \) | \( (012) \) | \( (0123) \) | \( (01)(23) \) |
|--------|----------|----------|----------|----------|
| \( \mathbb{1} \) | 1 1 1 1 1 |
| \( \text{sgn} \) | 1 -1 1 -1 1 |
| \( V_{2,2} \) | 2 0 -1 0 2 |
| \( V_{3,1} \) | 3 1 0 -1 -1 |
| \( V_{2,1,1} \) | 3 -1 0 1 -1 |

Observe that there are no common factors in \( \Gamma_+(E)(3) \) and \( \Gamma_-(E)(3) \), therefore it follows from an elementary representation theory that each \( \Sigma_3^+ \)-invariant subspace \( R \) of \( \Gamma(E)(3) \) decomposes as \( R = R_+ \oplus R_- \) with \( R_+ \subset \Gamma_+(E)(3) \) and \( R_- \subset \Gamma_-(E)(3) \). This means that \( \mathcal{P} \) is cyclic, by Proposition 18.

Theorem 20 was a consequence of the fact that for operads generated by one operation, the \( \Sigma_3^+ \)-spaces \( \Gamma_+(E)(3) \) and \( \Gamma_-(E)(3) \) do not contain a common irreducible factor. The following example shows that this is not longer true for general quadratic operads.

**Example 21.** Consider the quadratic operad \( \mathcal{P} = \Gamma(E)/(R) \), where \( E =: \mathbb{K}[\Sigma_2] \oplus \mathbb{1}_2 \) and where \( (R) \) is the operadic ideal generated by the relations
\[
\begin{align*}
\mathcal{r}_1 &:= x \cdot (yz) + y \cdot (zx) + z \cdot (xy) = 0 \quad \text{and} \\
\mathcal{r}_2 &:= (xy) \cdot z + (z \cdot x)y + x(z \cdot y) = 0.
\end{align*}
\]
In the above display, \( \cdot \) denotes the multiplication corresponding to a generator of \( \mathbb{K}[\Sigma_2] \) and we, as usual, omit the symbol for the commutative multiplication corresponding to a generator of \( \mathbb{I}_2 \). Then \( \mathcal{P} \) is cyclic but not dihedral.

Let us explain how this example was constructed. It can be calculated that in decomposition (26) of the 27-dimensional space \( \Gamma(E)(3) \),

\[
\begin{align*}
\Gamma_+(E)(3) &= 3\mathbb{I}_2 \oplus \text{sgn} \oplus 4V_{2,2} \oplus V_{3,1} \text{ and } \\
\Gamma_-(E)(3) &= 2V_{3,1} \oplus 2V_{2,1,1}.
\end{align*}
\]

There is a common irreducible factor \( V_{3,1} \) which occurs both in \( \Gamma_+(E)(3) \) and in \( \Gamma_-(E)(3) \). Therefore, to construct an operad which is cyclic but not dihedral, it is enough to choose a generator \( e_+ \) of \( V_{3,1} \) in \( \Gamma_+(E) \) and a generator \( e_- \) of \( V_{3,1} \) in \( \Gamma_-(E) \) and define \( R \) to be the \( \Sigma_3^+ \)-subspace of \( \Gamma(E)(3) \) generated by \( e_+ + e_- \). Operad \( \mathcal{P} \) above corresponds to one of these choices.

In Figure 2 we also recalled the following more or less well-known results about Koszulness of operads considered in this paper. The operad \( \text{Ass} \) is Koszul by [11] and the operad \( \text{Poiss} \) by [18, Corollary 4.6]. The operad \( \text{Vinb} \) is Koszul, because it is isomorphic to the operad \( \text{pre-Lie} \) which is known to be Koszul [4]. The operads \( \mathfrak{g}_4\text{-ass} \) and \( \mathfrak{g}_5\text{-ass} \) are not Koszul, as proved in [22].

The Koszulness of the operad \( \mathfrak{g}_6\text{-ass} \) can be proved as follows. In Example 6 we observed that \( \mathfrak{g}_6\)-associative algebras consist of a commutative multiplication and a Lie bracket, with no relation between these two operations. Therefore \( \mathfrak{g}_6\text{-ass} \) is the free product

\[
\mathfrak{g}_6\text{-ass} \cong \text{Lie} * \Gamma(\mathbb{I}_2)
\]

of the operad \( \text{Lie} \) for Lie algebras and the free operad \( \Gamma(\mathbb{I}_2) \) generated by one commutative operation. The Koszulity of the operad \( \mathfrak{g}_6\text{-ass} \) now follows from the obvious fact that the free product of two quadratic Koszul operads is again quadratic Koszul.

Let us turn our attention to the operad \( \mathcal{L}\mathcal{L}_q \) governing \( LL_q\)-algebras. For \( q = 0 \), the Koszulness of \( \mathcal{L}\mathcal{L}_q \) follows from the isomorphism \( \mathcal{L}\mathcal{L}_0 \cong \mathcal{P}_{\text{ois}} \). For \( q \notin \{0, \infty\} \) we argue as follows. The Koszulness of an operad \( \mathcal{P} \) is characterized by the acyclicity, in positive dimensions, of the cobar dual of \( \mathcal{P} \) [11]. This means that Koszulness is not affected by a field extension. We may therefore assume that the ground field \( \mathbb{K} \) is algebraically closed. In this case the operad \( \mathcal{L}\mathcal{L}_q \) is isomorphic, via the isomorphism [12], to the operad \( \text{Ass} \) which is [18].

It remains to analyze the case \( q = \infty \). It immediately follows from the definition of \( LL_\infty\)-algebras that the corresponding operad \( \mathcal{L}\mathcal{L}_\infty \) is constructed from Koszul quadratic operads \( \Gamma(\mathbb{I}_2) \) and \( \Gamma(\text{sgn}_2)/(\Gamma(\text{sgn}_2)(3)) \) via a distributive law, it is therefore Koszul by [18, Theorem 4.5].
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