Higher order Hessian structures on manifolds

R DAVID KUMAR

Department of Collegiate Education, Government of Andhra Pradesh,
Hyderabad 500 001, India
E-mail: rdkumar1729@yahoo.com

MS received 9 February 2005; revised 9 June 2005

Abstract. In this paper we define nth order Hessian structures on manifolds and study them. In particular, when \( n = 3 \), we make a detailed study and establish a one-to-one correspondence between third-order Hessian structures and a certain class of connections on the second-order tangent bundle of a manifold. Further, we show that a connection on the tangent bundle of a manifold induces a connection on the second-order tangent bundle. Also we define second-order geodesics of special second-order connection which gives a geometric characterization of symmetric third-order Hessian structures.

Keywords. Hessian structure; connection; geodesic.

1. Introduction

Let \( M \) stand for a Hausdorff, paracompact, smooth \((C^\infty)\) manifold modelled on a finite dimensional Banach space \( E \). Let \( T_m M \) denote the tangent space to \( M \) at \( m \) [21]. We recall Lang’s characterization of \( T_m M \) to fix our notation [10]. For each chart \((U, \phi)\) at \( m \), consider triples of the form \((U, \phi, e)\) with \( e \in E \). If \((U, \phi)\) and \((V, \psi)\) are two charts at \( m \), define \((U, \phi, e)\) and \((V, \psi, \bar{e})\) to be equivalent if \( D(\psi \circ \phi^{-1})(\phi m) \cdot e = \bar{e} \). An equivalence class \( v \) is called a tangent vector at \( m \) and the set of all tangent vectors at \( m \) is denoted by \( T_m M \). If \( v \in T_m M \) and \((U, \phi, e) \in v \), we write \( v_\phi = e \) and \( T \phi(v) = (\phi m, e) \). If \( v, w \in T_m M \) and \( a, b \) are real numbers, we define \( av + bw = T \phi^{-1}(\phi m, a\phi e + b\phi w) \). This defines a linear structure on \( T_m M \) isomorphic to \( E \) and is independent of charts. Let \( TM = \bigcup_{m \in M} T_m M \) and let \( TU = \bigcup_{U \in \mathcal{U}} T_U M \). Given a chart \((U, \phi)\) at \( m, T \phi \): \( TU \rightarrow \phi(U) \times E \) is a bijection and \( \{TU, T \phi\}: (U, \phi) \) is a chart on \( M \) defines a vector bundle structure on \( TM \). Finally, if \( M = U' \) is an open subset of \( E \), then \( TU' \) can be identified with \( U' \times E \).

Let \( \mathcal{F}(M) \) denote the space of smooth real-valued functions on \( M \). More generally, for any open subset \( U \) of \( M \), \( \mathcal{F}(U) \) will stand for the space of smooth real-valued functions on \( U \). If \((U, \phi)\) is a chart on \( M \) and \( f \in \mathcal{F}(U) \) we write \( f_\phi = f \circ \phi^{-1} \). If \( m \in U \), we define \( df(m) \in T^*_m M \) by \( df(m)(v) = Df_\phi(\phi m) \cdot v_\phi \) where \( v_\phi \) is a local representation of \( v \) in the chart \((U, \phi)\). This definition is independent of charts. For each fixed \( v \) in \( T_m M \), the map \( f \mapsto df(m)(v) \) is a derivation in \( f \) and every derivation is of this form for a unique \( v \). Suppose \( X: m \mapsto X(m) \) is a cross section of \( TM \) over \( M \). Given a chart \((U, \phi)\), write \( X_\phi(\phi m) = [X(m)]_\phi \) for \( m \in U \). Thus \( T \phi(X(m)) = (\phi m, X_\phi(\phi m)) \) and \( X_\phi: \phi U \rightarrow E \). We call \( X \) a smooth vector field on \( M \) if \( X_\phi \) is smooth for all charts \((U, \phi)\). We denote the set of all vector fields on \( M \) by \( \mathcal{F}(M) \). If \( f \in \mathcal{F}(M) \) and \( X \in \mathcal{F}(M) \), we define \( X(f)(m) = df(m) \cdot X(m) \). Then \( X(f) \in \mathcal{F}(M) \) and the map \( X \) is a derivation on \( \mathcal{F}(M) \). Conversely,
Every derivation of $\mathcal{F}(M)$ is induced in this way by a unique $X \in \mathfrak{X}(M)$. Suppose $M = U'$ is open in $E$. Then vector fields $X$ and $Y$ on $U'$ can be identified with smooth maps from $U'$ to $E$. We define $DY \cdot X$ to be the smooth map given by $(DY \cdot X)(u) = DY(u) \cdot X(u)$.

We can see that a (second-order) Hessian structure on a manifold is equivalent to a symmetric connection on a manifold. In [3], this equivalence is established by first identifying Hessian structures with sprays and then sprays with symmetric connections. The difficulty of their proof is in the choice of defining a connection in terms of connection forms on the bundle of bases for the tangent spaces. When we take a connection $\nabla_X Y$ to be given by Koszul’s definition, we see that Hessian structures and symmetric connections can be directly related to each other.

Before proceeding further, we state certain results relating to higher order derivatives on manifolds. For $m \in M$, let $F^+_m$ denote the space of all those real-valued smooth functions whose domain is an open subset of $M$ containing the point $m$. $F_m$ stands for the elements of $F^+_m$ which vanish at $m$ and $F^m$ for the space of all sums of products of $n$ elements of $F_m$. $f \in F^m$ if and only if $f_0$ has vanishing partial derivatives of order up to and including $(n - 1)$ at $m$, for some and hence any chart $(U, \phi)$. If $f \in F^m$ then we say $f$ is $(n - 1)$-flat at $m$.

**Proposition 1.1.**

Let $f \in F^m$ and let $X_1, \ldots, X_n \in \mathfrak{X}(M)$. Then

(a) $[X_1 \cdot X_2 \ldots X_n \cdot f](m)$ is defined independently of the order of $X_i$'s.

(b) If $\psi_1, \ldots, \psi_n \in T_m M$ and $X_1, \ldots, X_n \in \mathfrak{X}(M)$ such that $X_i(m) = \psi_i$, then the $n$-th derivative of $f$ at $m$,

$$D^n f(m)(\psi_1, \ldots, \psi_n) \overset{\text{def}}{=} [X_1 \cdot X_2 \ldots X_n \cdot f](m)$$

is well-defined and is symmetric in the $\psi_i$'s.

(c) If $(U, \phi)$ is a chart at $m$, then

$$D^n f(m)(\psi_1, \ldots, \psi_n) = D^n f_\phi(\phi m)(\psi_1, \ldots, \psi_n, \phi m),$$

where $D^n f_\phi$ is the $n$-th order Frechet derivative of $f_\phi$ at $\phi m$.

If $f \in F^m$, then the $n$-th order derivative of $f$ is well-defined as a symmetric $n$-multilinear mapping on $T_{(m)M} M$. We define a $n$-th order Hessian structure $H^n$ (not necessarily symmetric), for every $n \in \mathbb{N}$ as a map $H^n : f \mapsto H^n f$ which is real linear in $f$ and associates with every $f \in \mathcal{F}(M)$, an $n$-th order covariant tensor $H^n f$ on $M$ such that if $f \in F^m$, $H^n f(m)$ is the $n$-th order derivative of $f$ at $m$. If $\nabla$ is a connection, then the definition of $\nabla^n f$ given by $\nabla f = df$, $\nabla^n f = \nabla(\nabla^{n-1} f)$ where the action of $\nabla$ denotes the covariant derivative acting on covariant tensors defines a Hessian structure $H^n$, for all $n$. Further we show that $\nabla^2 f$ and $\nabla^3 f$ are symmetric (for all $f$) if and only if $\nabla$ has torsion and curvature zero. In such a case $\nabla^n f$ is symmetric, for all $n$ and for all $f$. Thus we show how to define higher-order derivatives of functions on manifolds with a connection. It should be interesting to relate these higher derivatives to differential operator on the manifold.

Ambrose, Palais and Singer in [3] proved that, given any $H (= H^2)$ there exists a connection $\nabla$ such that $H^2 = \nabla^2$. We now raise the following question. Given $H^3$, does there exist a connection $\nabla$ such that $H^3 = \nabla^3$? This is not true in general. We prove that, just as a second-order Hessian structure arises from a connection on the tangent bundle, every
third-order Hessian structure on a manifold arises from a connection on the second-order tangent bundle. We characterize those $H^3$ which arise as $\nabla^3$. In this process, we show that a connection $\nabla$ on the tangent bundle of a manifold induces a connection $\tilde{\nabla}$ on the second-order tangent bundle. As far as we are aware, this observation is new. Also we introduce and discuss second-order geodesics which are related to third-order Hessian structures. We show that if $\nabla$ is a connection on the second-order tangent bundle induced from a connection $\tilde{\nabla}$ on the tangent bundle of a manifold, then every (first-order) geodesic of $\nabla$ is a second-order geodesic of $\tilde{\nabla}$.

1.1 Second-order tangent bundle

We first introduce the concept of a second-order tangent vector at a point $m$ of the manifold $M$. We follow the notation of [3]. Let $F^+_m$ denote the space of all those real-valued smooth functions whose domain is an open subset of $M$ containing the point $m$. Let $F_m$ stand for the elements of $F^+_m$ which vanish at $m$ and $F^k_m$ for the space of all sums of products of $k$ elements of $F_m$. A tangent vector to $M$ at $m$ can be considered to be a linear functional on $F^+_m$ which vanishes on constant functions and on $F^2_m$ (i.e., a tangent vector at $m$ kills on 1-flat functions at $m$). We define a second-order tangent vector $t$ to $M$ as a linear form on $F^+_m$ which vanishes on constant functions and on $F^2_m$ (i.e., $t$ kills on 2-flat functions at $m$). The second-order tangent vectors form a linear subspace which we denote by $T^{(2)}_mM$. Observe that $T^{(2)}_mM \subset T^{(2)}_mM$. If $(U,x)$ is a chart at $m$, it is not difficult to show that the functionals $\frac{\partial}{\partial x^i}(m)$ and $\frac{\partial^2}{\partial x^i \partial x^j}(m)(i \leq j)$ constitute a basis for $T^{(2)}_mM$. Further, any $t \in T^{(2)}_mM$ can be written uniquely in the form $t = \Sigma t^i \frac{\partial}{\partial x^i}(m) + \Sigma t^{ij} \frac{\partial^2}{\partial x^i \partial x^j}(m)$ with the condition that $t^i = t^i$. The span of the vectors $\frac{\partial}{\partial x^i}(m)$ is $T_mM$. Let the vectors $\frac{\partial^2}{\partial x^i \partial x^j}(m)$ span the subspace $(T_mM)^\perp$. Clearly then $T^{(2)}_mM = T_mM \oplus (T_mM)^\perp$. Moreover $(T_mM)^\perp$ is not defined independently of the chart and there is no canonical way of choosing a complement of $T_mM$ of $T^{(2)}_mM$. This leads to the concept of a ‘dissection’ of second-order tangent bundle.

DEFINITION

A smooth second-order tangent vector field on $M$ is a map $\xi$ which associates to each point $m$ of $M$ a second-order tangent vector $\xi_m \in T^{(2)}_mM$ such that for every $f \in \mathcal{F}(M)$, the function $\xi f$ defined by $\xi f(m) = \xi_m(f) = \xi_m(f)$ is a smooth function on $M$.

Examples. $\frac{\partial}{\partial x^i}$ and $\frac{\partial^2}{\partial x^i \partial x^j}$ are smooth second-order vector fields on their domains.

Actually, we consider the problem of extending the theory of second-order structures to Banach manifolds. On Banach manifolds global chart-free methods are generally not available. One has to choose charts taking values in open sets of Banach spaces and use Frechet calculus methods. For this purpose, we have chosen coordinate-free methods, but use charts combined with Frechet calculus on Banach spaces. This method works both in finite and infinite dimensional cases and the proofs are exactly the same as far as the computational details are concerned. It is for establishing smoothness and taking into account such infinite dimensional phenomena as non-reflexivity etc, that one has to work harder in the case of Banach manifolds. So we use the language of Frechet calculus, but in a finite dimensional setting. For this reason our calculations become little complicated.
and not easily readable. Our definition of second-order tangent bundle seems to be new (it is a coordinate approach). Now we define the second-order tangent bundle of a manifold in the following way.

If $M = \mathbb{R}^2$, the second-order tangent vectors at a point are the linear functionals of the form $\Sigma a_i (\partial/\partial x_i) + \Sigma b_{ij}(\partial^2/\partial x_i \partial x_j)$. Calculating how these functionals transform under a change of coordinates and rewriting the transformation law in a coordinate-free language, we are led to the definition of second-order tangent vector at a point of a manifold.

Let $E \oplus E$ denote the symmetric tensor product of $E$ with itself. We shall identify symmetric bilinear maps from $E \times E$ to $E$ with linear maps from $(E \oplus E) \oplus E$ to $E$. If $A$ and $B$ are (linear) endomorphisms of $E$, then $A \Delta B$ denotes the endomorphism of $E \oplus E$ satisfying $(A \Delta B)(e_1 \Delta e_2) = A e_1 \Delta B e_2$. Let $E^{(2)}$ stand for $(E \oplus E) \oplus E$ and we denote a typical element of it by $x \oplus e$.

Let $m \in M$. Consider a chart $(U, \varphi)$ at $m$ and triples of the form $(U, \varphi, x \oplus e)$. Two such triples $(U, \varphi, x \oplus e)$ and $(V, \psi, y \oplus f)$ are said to be equivalent if

$$y = [DF(u) \Delta DF(u)] \cdot x$$

and

$$f = DF(u) \cdot e + D^2F(u) \cdot x,$$

where $F = \psi \circ \varphi^{-1}$ and $u = \varphi m$. It can be easily checked that we have really defined an equivalence relation and that equivalence preserves linearity. Equivalence classes are called second-order tangent vectors at $m$. The set of all such equivalence classes will be denoted by $T^2_m M$. A typical element of $T^2_m M$ will be denoted by $t$.

If $(U, \varphi, x \oplus e) \in t$, we write $t_\varphi = x \oplus e$ and $T^{(2)} \varphi(t) = (\varphi m, t_\varphi)$. As usual $T^{(2)} M = \bigcup_{m \in M} T^{(2)}_m M$, $T^{(2)} U = \bigcup_{m \in U} T^{(2)}_m M$ and $T^{(2)} \varphi : T^{(2)} U \rightarrow \varphi U \times E^{(2)}$ is a bijection. The collection $\{(T^{(2)} U, T^{(2)} \varphi) : (U, \varphi) \text{ is a chart on } M\}$ defines a vector bundle structure on $T^{(2)} M$. If $M = U'$ is an open subset of $E$ then $T^{(2)} U'$ can be identified with $U' \times E^{(2)}$.

For $m \in M$, let $(T^{(2)}_m M)^*$ denote the dual of $T^{(2)}_m M$ and let $(T^{(2)} M)^*$ denote the dual bundle.

Suppose $f$ is a smooth function defined in a neighborhood of $m$, $t \in T^{(2)}_m M$, $(U, \varphi)$ is a chart at $m$, $t_\varphi = x \oplus e$ and $\varphi m = u$. We define the second-order differential of $f$ as $d^{(2)} f(m)(t) = t(f)(m) = D^2 f \varphi(u) \cdot x + D f \varphi(u) \cdot e$. It is easy to check that $d^{(2)} f(m)(t)$ is well-defined and that $d^{(2)} f(m) \in (T^{(2)}_m M)^*$. Moreover if $f \in \mathcal{F}(U)$, $d^{(2)} f$ is a smooth cross-section of $(T^{(2)} U)^*$. Finally, note that $TM$ is a subbundle of $T^{(2)} M$.

**Note 1.2.** If $f$ is 2-flat at $m$ then $t(f)(m) = 0$. We can also see its dual approach. We then have a short exact sequence as given below.

$$0 \rightarrow F^2_m / F^3_m \xrightarrow{\ i \ } F_m / F^3_m \xrightarrow{\ j \ } F^2_m / F^3_m \rightarrow 0.$$
Higher order Hessian structures on manifolds

Let us now come to \( F_m^2/F_m^3 \). We show that this can be identified with \( S_m^2 \), the space of symmetric bilinear functionals on \( T_m M \times T_m M \). Let \( v, w \in T_m M \). Let \( X \) and \( Y \) be any two vector fields on \( M \) such that \( X(m) = v \) and \( Y(m) = w \). Let \( f \in F_m^2 \). Note that for any vector field \( Z \) on \( M \), \( Zf \in F_m \) and so \( Zf(m) = 0 \). Now we have \( X(Yf)(m) = Y(Xf)(m) = [X,Y]f(m) = 0 \), so that \( X(Yf)(m) = Y(Xf)(m) = v(Yf)(m) = w(Xf)(m) \). This shows that \( D^2 f(m)(v,w) = X(Yf)(m) \) is well-defined and belongs to \( S_m^2 \). Also it may be checked that if \((U, \phi)\) is any chart at \( m \), then \( D^2 f(m)(v,w) = D^2 f_\phi(m)(\phi v, \phi w) \), where \( D^2 f_\phi \) denotes the second-order Frechet derivative of \( f_\phi \) at \( \phi m \). \( D^2 f(m) \) is called the Hessian of \( f \) at \( m \) at the critical point \( m \). Clearly if \( f \in F_m^2 \) then \( Df(m) = 0 \) and \( D^2 f(m) = 0 \). It follows that \( F_m^3 \) is the kernel of the mapping \( f \mapsto D^2 f(m) \).

Let us get back to our short exact sequence which we write as

\[
F_m^2 
\xrightarrow{i} \quad S_m^2 
\xrightarrow{j} \quad J_m \xrightarrow{} 0.
\]

Here \( J_m \) is a vector space of dimension \( d \), \( J_m^2 \) is a vector space of dimension \( d + \left(\frac{d}{2}\right) \) and \( S_m^2 \) is a vector space of dimension \( \left(\frac{d}{2}\right) \). We can define in the usual way the three vector bundles \( J, J^2 \) and \( S^2 \) on \( M \) with fibers \( J_m, J_m^2 \) and \( S_m^2 \), respectively, to have a short exact sequence of vector bundles on \( M \):

\[
0 \xrightarrow{} S_m^2 \xrightarrow{i} J_m^2 \xrightarrow{j} J_m \xrightarrow{} 0.
\]

It is natural to ask when this exact sequence splits. This leads to the concept of a ‘dissection’ as a splitting of this exact sequence. Following this second-order Hessian structures and dissections are in one-to-one correspondence [3].

Also we can define the ‘tangent bundle of order 2’, denoted by \( T^2 M \) as a bundle of 2-jets \([11]\) defined in the following way, which is certainly a different approach to \( T^{(2)} M \) (the second-order tangent bundle as defined in §1.1).

1.2 The tangent bundle of order 2, \( T^2 M \)

Let \( M \) be a \( d \)-dimensional manifold (p. 368 of [11]). The tangent bundle of order 2, \( T^2 M \), of \( M \) is the \( 3d \)-dimensional manifold of 2-jets \( j^2 f \) at \( 0 \in \mathbb{R} \) of differentiable curves \( f : \mathbb{R} \to M \). \( (T^2 M \) is also called 2-velocity space.)

\( T^2 M \) has a natural bundle structure over \( M \), \( \pi^2 : sT^2 M \to M \) denotes the canonical projection. The tangent bundle \( TM \) is nothing but the manifold of 1-jets \( j^1 f \) at \( 0 \in \mathbb{R} \) of \( f : \mathbb{R} \to M \). If we denote \( \pi^{12} : T^2 M \to TM \) the canonical projection, then \( T^2 M \) has a bundle structure over \( TM \). Note that \( T^2 M \) is not a vector bundle. (The bundle of 2-jets defined above.) There is a result stating that ‘the linear connection \( \nabla \) on \( M \) determines a vector bundle structure on \( \pi^2 : T^2 M \to M \) and a vector bundle isomorphism \( T^2 M \to TM \oplus TM' \). Note that the space \( T^2 M \) of 2-velocities on \( M \) may be identified with a submanifold of \( T(TM) \), the tangent bundle to \( TM \) (see p. 372 of [11]).

2. Hessian structures

Let \( \Omega^r(M) \) denote the space of covariant tensors of \( r \)-th order on \( M \). Since we wish to associate \( r \)-th derivatives to smooth real-valued functions on manifolds, we have the following definition.
DEFINITION 2.1.
An n-th order Hessian structure $H^n$ on $M$ is a mapping $H^n: \mathcal{F}(M) \to \Omega^n(M)$, $f \mapsto H^n f$, such that (i) $H^n$ is real linear in $f$ and (ii) if $f \in F^n_m$ then $H^n f(m) = D^n f(m)$.

Note that the properties of $H^n$ imply that if $f \in \mathcal{F}(M)$ is constant on an open set $U$ in $M$, then $H^n f$ is zero on $U$. Consequently, $H^n$ is localizable so that it is meaningful to talk of $H^n f$ for $f \in \mathcal{F}(U)$. $H^n$ is said to be a symmetric Hessian structure if $H^n f$ is a symmetric covariant tensor of order $n$, for all $f$. In [3], a (second-order) Hessian structure was by definition symmetric. Here we drop the requirement of symmetry from the definition.

Suppose now $\nabla$ is a connection on $M$, i.e., a connection on the tangent bundle $TM$ of $M$. Recall [13] that if $\omega \in \Omega^n(M)$ then $\nabla \omega \in \Omega^{n+1}(M)$ is defined by $\nabla \omega(X_0, \ldots, X_n) = (\nabla X_0 \omega)(X_1, \ldots, X_n)$. Here $\nabla X_0 \omega$ is defined by
\[
(\nabla X_0 \omega)(X_1, \ldots, X_n) = \nabla X_0(\omega(X_1, \ldots, X_n)) - \sum_{i=1}^n \omega(X_1, \ldots, \nabla X_0 X_i, \ldots, X_n).
\]

$\nabla \omega$ is called the covariant derivative of $\omega$ with respect to the connection $\nabla$ on $M$. Also note the following:

1. Let $\omega$ be a symmetric covariant tensor of order $n$. Then $\nabla \omega$ is symmetric if and only if $(\nabla \omega)(X_0, X_1, \ldots, X_n) = (\nabla \omega)(X_1, X_0, \ldots, X_n)$, for all $X_i \in \mathfrak{X}(M)$.
2. Let $f \in \mathcal{F}(M)$. Define $\nabla f = df$, $\nabla^2 f = \nabla(df)$, and recursively $\nabla^{n+1} f = \nabla(\nabla^n f)$. For $X, Y \in \mathfrak{X}(M)$, $\nabla^2 f(X, Y) = X \cdot Y \cdot f - \nabla_X Y \cdot f$.

Theorem 2.2. Let $\nabla$ be a connection on $M$. Then $\nabla^n f$ is an n-th order Hessian structure on $M$, for all $n \geq 1$.

Proof. We first claim that if $f \in F^{n+k}_m$, then for fixed $X_1, \ldots, X_n$, $\nabla^n f(X_1, \ldots, X_n) = X_1 \cdot \ldots \cdot X_n \cdot f + g$, where $g \in F^{k+1}_m$. (This also implies that $\nabla^n f(X_1, \ldots, X_n) \in F^n_m$.) The proof is by induction on $n$. Clearly it is true for $n = 2$. Suppose it is true for $n$. Let $f \in F^{n+1}_m$. Then
\[
\nabla^{n+1} f(X_0, X_1, \ldots, X_n) = [\nabla X_0 (\nabla^n f)](X_1, \ldots, X_n)
\]
\[
= \nabla X_0 ([\nabla^n f(X_1, \ldots, X_n)] - \sum_{i=1}^n \nabla^{n+1} f(X_1, \ldots, \nabla X_0 X_i, \ldots, X_n)
\]
\[
= X_0 (X_1 \cdot X_2 \cdot \ldots \cdot X_n \cdot f + g) - \sum_{i=1}^n \nabla^n f(X_1, \ldots, \nabla X_0 X_i, \ldots, X_n)
\]
\[
= X_0 \cdot \ldots \cdot X_n \cdot f + X_0 \cdot g - \sum_{i=1}^n \nabla^n f(X_1, \ldots, \nabla X_0 X_i, \ldots, X_n).
\]

Here $g \in F^{k+2}_m$ and therefore $X_0 g \in F^{k+1}_m$. Also each term inside the summation on the right-hand side belongs to $F^{k+1}_m$. So the claim is proved.

Now we prove that $\nabla^n f$ is an n-th order Hessian structure on $M$. Now the linearity in $f$ is trivially true and so we have only to prove that if $f \in F^n_m$, then $\nabla^n f(X_1, \ldots, X_n)(m) = X_1 \cdot \ldots \cdot X_n \cdot f(m)$. But our claim implies that the difference $\nabla^n f(X_1, \ldots, X_n) - X_1 \cdot \ldots \cdot X_n \cdot f$, $f \in F^n_m$ if $f \in F^n_m$.

q.e.d.
In the case of $\mathbb{R}^n$, the $n$th derivative is symmetric. In case of a manifold, we ask for conditions under which the $n$th derivative is symmetric. We have the following results.

**Theorem 2.3.**

(a) $\nabla^2 f$ is symmetric if and only if $\nabla$ has torsion zero.

(b) $\nabla^2 f$ and $\nabla^3 f$ are both symmetric if and only if $\nabla$ has both torsion and curvature zero.

(c) If $\nabla$ has torsion and curvature zero then $\nabla^n f$ is symmetric, for all $n \geq 2$.

**Proof.** Recall the definitions of torsion $T$ and curvature $R$ from [4] defined by $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ and $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$. The proofs of (a) and (b) are trivial. (c) Assume that $\nabla$ has torsion and curvature zero. We need to prove that $\nabla^n f$ is symmetric, for all $n \geq 2$. The proof is by induction on $n$. By (a) and (b) the result is true for $n = 2, 3$. Suppose the result is true for $n$. We have to prove that the result is true for $(n+1)$. Consider

\[
\nabla^{n+1} f(X_0, X_1, \ldots, X_n) \\
= \nabla_{X_0} (\nabla^n f)(X_1, \ldots, X_n) \\
= \nabla_{X_0} (\nabla^n f(X_1, \ldots, X_n)) - \sum_{i=1}^{n} \nabla^n f(X_1, \ldots, \nabla_{X_0} X_i, \ldots, X_n) \\
= \nabla_{X_0} \left[ \nabla_{X_1} (\nabla^{n-1} f(X_2, \ldots, X_n)) \\
- \sum_{j=2}^{n} \nabla^{n-1} f(X_2, \ldots, \nabla_{X_1} X_j, \ldots, X_n) \right] \\
- \sum_{i=1}^{n} \nabla^n f(X_1, \ldots, \nabla_{X_0} X_i, \ldots, X_n) \\
= \nabla_{X_0} \left[ \nabla_{X_1} (\nabla^{n-1} f(X_2, \ldots, X_n)) \\
- \sum_{j=2}^{n} \nabla_{X_0} (\nabla^{n-1} f(X_2, \ldots, \nabla_{X_1} X_j, \ldots, X_n)) \right] \\
- \sum_{j=2}^{n} \nabla_{X_0} (\nabla^{n-1} f(X_2, \ldots, \nabla_{X_1} X_j, \ldots, X_n)) \\
- \left[ \nabla^n f(\nabla_{X_0} X_1, X_2, \ldots, X_n) \right] \\
+ \sum_{j=2}^{n} \nabla^n f(X_1, X_2, \ldots, \nabla_{X_0} X_i, \ldots, X_n) \\
= \nabla_{X_0} \nabla_{X_1} (\nabla^{n-1} f(X_2, \ldots, X_n)) \\
- \sum_{j=2}^{n} \nabla_{X_0} (\nabla^{n-1} f(X_2, \ldots, \nabla_{X_1} X_j, \ldots, X_n)) 
\]
\[ -\left[ \nabla_{X_0} X_1 (\nabla^{n-1} f(X_2, \ldots, X_n)) \right. \\
\left. - \sum_{j=2}^{n} \nabla^{n-1} f(X_2, \ldots, \nabla_{X_0} X_1, X_j, \ldots, X_n) \right] \\
- \sum_{i=2}^{n} \nabla^i f(X_1, X_2, \ldots, \nabla_{X_0} X_i, \ldots, X_n) \\
= [\nabla_{X_0} X_1 (\nabla^{n-1} f(X_2, \ldots, X_n)) - \nabla_{X_0} X_1 (\nabla^{n-1} f(X_2, \ldots, X_n))] \\
- \sum_{j=2}^{n} \nabla_{X_0} (\nabla^{n-1} f(X_2, \ldots, \nabla_{X_1} X_j, \ldots, X_n)) \\
+ \sum_{j=2}^{n} \nabla^{n-1} f(X_2, \ldots, \nabla_{X_0} X_1, X_j, \ldots, X_n) \\
- \sum_{i=2}^{n} \nabla^i f(X_1, X_2, \ldots, \nabla_{X_0} X_i, \ldots, X_n). \]

Interchanging \( X_0 \) and \( X_1 \) in the above formula and using
\[ T(X, Y) \cdot f = \nabla_Y (X \cdot f) - \nabla_X (Y \cdot f) - [X, Y] \cdot f, \]
we consider the difference
\[ \nabla^{n+1} f(X_0, X_1, \ldots, X_n) - \nabla^{n+1} f(X_1, X_0, \ldots, X_n) \]
\[ = T(X_1, X_0) \cdot \nabla^{n-1} f(X_2, \ldots, X_n) \\
+ \left[ \sum_{j=2}^{n} \nabla_{X_1} (\nabla^{n-1} f(X_2, \ldots, \nabla_{X_0} X_j, \ldots, X_n)) \right] \\
\left. - \sum_{j=2}^{n} \nabla_{X_0} (\nabla^{n-1} f(X_2, \ldots, \nabla_{X_1} X_j, \ldots, X_n)) \right] \\
+ \left[ \sum_{j=2}^{n} \nabla^{n-1} f(X_2, \ldots, \nabla_{X_0} X_1 - \nabla_{X_1} X_0, X_j, \ldots, X_n) \right] \\
+ \left[ \sum_{i=2}^{n} \nabla^i f(X_0, X_2, \ldots, \nabla_{X_1} X_i, \ldots, X_n) \right] \\
- \sum_{i=2}^{n} \nabla^i f(X_1, \ldots, \nabla_{X_0} X_i, \ldots, X_n). \quad (1) \]

Now we consider
\[ \sum_{i=2}^{n} \nabla^i f(X_1, \ldots, \nabla_{X_0} X_i, \ldots, X_n) \]
Higher order Hessian structures on manifolds

\begin{align*}
= \sum_{i=2}^{n} [\nabla_{X_{i}} (\nabla^{n-1} f)](X_{2}, \ldots, \nabla_{X_{0}} X_{i}, \ldots, X_{n}) \\
= \sum_{i=2}^{n} \left[ \nabla_{X_{i}} (\nabla^{n-1} f(X_{2}, \ldots, \nabla_{X_{0}} X_{i}, \ldots, X_{n})) \right. \\
- \left. \sum_{k=2}^{n} \nabla^{n-1} f(X_{2}, \ldots, \nabla_{X_{0}} X_{i}, \ldots, \nabla_{X_{l}} \nabla_{X_{0}} X_{k}, \ldots, X_{n}) \right]. 
\end{align*}

(2)

Similarly we have,

\begin{align*}
\sum_{i=2}^{n} \nabla^{n} f(X_{0}, X_{2}, \ldots, \nabla_{X_{i}} X_{l}, \ldots, X_{n}) \\
= \sum_{i=2}^{n} \left[ \nabla_{X_{i}} (\nabla^{n-1} f(X_{2}, \ldots, \nabla_{X_{i}} X_{l}, \ldots, X_{n})) \right. \\
- \left. \sum_{k=2}^{n} \nabla^{n-1} f(X_{2}, \ldots, \nabla_{X_{i}} X_{l}, \ldots, \nabla_{X_{0}} \nabla_{X_{i}} X_{k}, \ldots, X_{n}) \right]. 
\end{align*}

(3)

Substitute (2) and (3) in (1) to get

\begin{align*}
\nabla^{n+1} f(X_{0}, X_{1}, \ldots, X_{n}) - \nabla^{n+1} f(X_{1}, X_{0}, X_{2}, \ldots, X_{n}) \\
= T(X_{1}, X_{0}) \cdot \nabla^{n-1} f(X_{2}, \ldots, X_{n}) \\
+ \sum_{j=2}^{n} \nabla^{n-1} f(X_{2}, \ldots, \nabla_{X_{0}} X_{j}, \ldots, \nabla_{X_{1}} X_{j}, \ldots, X_{n}) \\
+ \sum_{i=2}^{n} \sum_{k=2}^{n} \nabla^{n-1} f(X_{2}, \ldots, \nabla_{X_{0}} X_{i}, \ldots, \nabla_{X_{i}} \nabla_{X_{0}} X_{k}, \ldots, X_{n}) \\
- \sum_{i=2}^{n} \sum_{k=2}^{n} \nabla^{n-1} f(X_{2}, \ldots, \nabla_{X_{i}} X_{i}, \ldots, \nabla_{X_{0}} \nabla_{X_{i}} X_{k}, \ldots, X_{n}) \\
= T(X_{1}, X_{0}) \cdot \nabla^{n-1} f(X_{2}, \ldots, X_{n}) \\
+ \sum_{j=2}^{n} \nabla^{n-1} f(X_{2}, \ldots, \nabla_{T(X_{0}, X_{1})} X_{j}, \ldots, X_{n}) \\
- \sum_{i=2}^{n} \nabla^{n-1} f(X_{2}, \ldots, R(X_{0}, X_{1}) X_{i}, \ldots, X_{n})
\end{align*}

This is enough to prove the theorem. \textit{q.e.d.}

Ambrose, Palais and Singer in \cite{3} proved that every symmetric second-order Hessian structure $H^{2}$ is of the form $\nabla^{2}$, for a symmetric connection $\nabla$. We see below that the condition of symmetry is unnecessary. Before this, we prove the following Lemma 2.4.
Lemma 2.4. Fix \( v, w \in T_m M \). Let \( Bf(m)(v, w) = Df^2(m)(v, w) - Hf(m)(v, w) \). Then

(i) the map \( B: f \to Bf(m)(v, w) \) is real linear in \( f \),
(ii) if \( df(m) = 0 \) then \( Bf(m) = 0 \),
(iii) \( B \) is a derivation in \( f \), i.e., \( B(fg) = fB(g) + gB(f) \) for all \( f, g \in \mathcal{F}(M) \).

Proof. Lemma 2.4 (i) and (ii) follow from the relevant definitions. Lemma 2.4 (iii) follows from Lemma 2.4 (ii) by noting that if we set \( h = (f - fm)(g - gm) \), then \( dh(m) = 0 \) so that \( Bh(m) = 0 \). q.e.d.

Theorem 2.5. If \( H^2 \) is a second-order Hessian structure on \( M \), then there exists a unique connection \( \nabla \) on \( M \) such that \( H^2 f = \nabla^2 f \) for all \( f \). Thus there is a one-to-one correspondence between second-order Hessian structures on \( M \) and connections on \( M \).

Proof. Suppose \( H^2 \) is a second-order Hessian structure on \( M \). Fix \( X, Y \in \mathcal{X}(M) \) and \( f \in \mathcal{F}(M) \). Define \( Bf(X, Y) = X \cdot Y \cdot f - H^2 f(X, Y) \). Then \( B: f \to Bf(X, Y) \) is a derivation. Hence it defines a vector field on \( M \) which we denote it by \( \nabla_X Y \) such that \( Bf(X, Y) = \nabla_X Y \cdot f \). Then we have \( \nabla_X Y \cdot f = X \cdot Y \cdot f - H^2 f(X, Y) \). It can be easily checked that \( \nabla \) satisfies all the properties of a connection and that \( \nabla \) is symmetric if \( H^2 \) is symmetric. q.e.d.

It is now natural to ask the following question. Does every third-order Hessian structure \( H^3 \) arise as \( \nabla^3 \), for a connection \( \nabla \)? We show below that this need not be the case. However, every \( H^3 \) comes from a connection on the second-order tangent bundle \( T^{(2)} M \) of \( M \). We characterize the connections on \( T^{(2)} M \) which arise in this way. We also characterize the smaller class of connections which are associated to \( H^3 \) of the form \( \nabla^3 \). Incidentally, we prove that every connection \( \nabla \) on \( T^{(2)} M \) induces a connection \( \nabla \) on \( T^{(2)} M \). We do all this in the language of charts and Frechet calculus methods.

3. Second-order connections

We define vector bundles now, to recall notation [2]. Let \( E \) and \( F \) be finite dimensional Banach spaces with \( U \) open in \( E \). We call the product \( U \times F \) a local vector bundle (l.v.b.). We call \( U \) the base space, which can be identified with \( U \times \{0\} \), which is called the zero section. For \( u \in U \), \( \{u\} \times F \) is called the fiber over \( u \), which can be endowed with the Banach space structure of \( F \). The map \( \pi: U \times F \to U \) given by \( \pi(u, x) = u \) is called the projection of \( U \times F \). Note that \( U \times F \) is an open subset of \( E \times F \) and so is a local manifold. Let \( U \times F \) and \( U' \times F' \) be two l.v.b.'s. A map \( \varphi: U \times F \to U' \times F' \) is called a l.v.b. mapping if \( \varphi \) is smooth and it has the form \( \varphi(u, x) = (\varphi_1(u), \varphi_2(u) \cdot x) \) where \( \varphi_1: U \to U' \) and \( \varphi_2: U \to L(F, F') \) are smooth. A l.v.b. map that is a diffeomorphism is called a l.v.b. isomorphism. For example, (i) any linear map \( A \in L(E, F) \) defines a vector bundle map \( \varphi_A: E \times E \to E \times F \) by \( \varphi_A(u, e) = (u, A \cdot e) \) and (ii) if \( U \) is open in \( E, V \) is open in \( F \) and \( f: U \to V \) is smooth, then the map \( Tf: U \times E \to V \times F \) given by \( Tf(u, e) = (f(u), Df(u) \cdot e) \) is a l.v.b. map.

In this setting, let \( \pi: E \to M \) be a vector bundle, where \( M \) is a manifold modelled on a finite dimensional Banach space \( E \) and each fibre of \( E \) is modelled on a finite dimensional Banach space \( F \). For the local structure of vector bundles, we follow the notation given below. Let \( m \in M \). If \( (U, \varphi) \) is a local chart at \( m \), then the vector bundle chart is a triple \( (\Phi, \varphi, U) \) where the following diagram commutes:
Thus, if \( \tau: U \to E \) is a local section of \( \pi \), then the principal part \( \tau_\phi: \varphi U \to F \) with respect to the vector bundle chart \((\Phi, \varphi, U)\) is given by \((\Phi \circ \tau \circ \varphi^{-1})(u) = (u, \tau_\phi(u))\); \((u \in \varphi U)\). Similarly, for a vector field \( X \) of \( M \), the local representative \( X_\phi: \varphi U \to E \) with respect to the chart \((U, \varphi)\) is given by \((T \varphi \circ X \circ \varphi^{-1})(u) = (u, X_\varphi(u))\); \((u \in \varphi U)\). For any two vector bundle charts \((\Phi, \varphi, U)\) and \((\Psi, \psi, V)\) at \( e \in E \), we have the l.v.b. mapping given by \((\Psi \circ \Phi^{-1})(u, x) = (\phi_1(u), \phi_2(u) \cdot x)\) for all \((u \in \varphi U, x \in F)\) where \( \phi_1 = \psi \circ \varphi^{-1} \) and \( \phi_2 : \varphi U \to L(F, F) \) given by \( \phi_2(u) \cdot x = Pr_2 \circ \Psi \circ \Phi^{-1} \circ (u, x) \in F \) is smooth. Let \( \exists \in \mathcal{E}(M) \) denote the set of all smooth vector fields on \( M \). Following (4) and (5), we have the definition given below.

**DEFINITION 3.1.**

A connection \( \nabla \) on a vector bundle \( \pi: E \to M \) is a mapping \( \nabla: \exists \in \mathcal{E}(M) \times \exists \in \mathcal{E}(M) \to \exists \in \mathcal{E}(M), (X, \tau) \mapsto \nabla_X \tau \) for all \( X \in \exists \in \mathcal{E}(M) \), \( \tau \in \exists \in \mathcal{E}(M) \) such that it satisfies the following properties:

(i) \( \nabla \) is real bilinear in \( X \) and \( \tau \),

(ii) \( \nabla_{fX} \tau = f \nabla_X \tau \) and \( \nabla_X (f \tau) = X(f) \tau + f \nabla_X \tau \), for \( f \in \mathcal{F}(M) \) and \( \tau \in \exists \in \mathcal{E}(M) \).

Note that in a local vector bundle chart \((\Phi, \varphi, U)\) of \( E \), a connection has the form \((\nabla_X \tau)_\varphi = D\tau_\varphi \cdot X_\varphi - \Gamma_\varphi(\varphi m)(X_\varphi, \tau_\varphi) \) for all \( m \in U \) where \( \Gamma_\varphi: \varphi U \to L^2(E \times F, F) \) is the Christoffel symbol of \( \nabla \).

Let us consider the vector bundle \( E = T^{(2)}M \) over \( M \) (the second-order tangent bundle of \( M \)). Locally, if \( M = U'' \) is open in \( E \), then \( T^{(2)}U'' = U'' \times E \oplus (E \Lambda E) \). We define a connection \( \nabla \) on \( T^{(2)}M \) as above and we now find the transformation property of \( \Gamma \). Let \((U, \varphi)\) and \((V, \psi)\) be two charts at \( m \) and suppose \((\Phi, \varphi, U)\) and \((\Psi, \psi, V)\) are vector bundle charts at \( e \in E \) and let \( F = \psi \circ \varphi^{-1}, \varphi m = u, \psi m = v \). Then \( F(u) = v \). If \( \tau: U \to T^{(2)}M \) is a local section of \( T^{(2)}M \), then the principal part is \( \tau_\phi: \varphi U \to F \) with respect to the vector bundle chart \((\Phi, \varphi, U)\), where \( F = E \oplus (E \Lambda E) \). Then we have the following rules. We write \( \tau_\phi = V_\phi \oplus S_\phi \), where \( V_\phi(u) \in E \) and \( S_\phi(u) \in E \Lambda E \). \( \tau_\phi(v) = \varphi_2(u) \cdot \tau_\phi(u) \) and \( X_\phi(v) = DF(u) \cdot X_\phi(u) \).

Then by the definition of second-order tangent vectors, we have

\[
V_\psi(v) = DF(u) \cdot V_\phi(u) + D^2F(u) \cdot S_\phi(u)
\]  

and

\[
S_\psi(v) = [DF(u) \Delta DF(u)] \cdot S_\phi(u).
\]

From the definition of \( \nabla \), we have in a local chart \((U, \varphi)\) of \( M \),

\[
(\nabla_X \tau)_\varphi = D\tau_\varphi \cdot X_\varphi - \Gamma_\varphi(X_\varphi, \tau_\varphi).
\]
We write $\tilde{\Gamma}_\varphi(X_\varphi, \tau_\varphi) = \tilde{\Gamma}_\varphi^1(X_\varphi, \tau_\varphi) \oplus \tilde{\Gamma}_\varphi^2(X_\varphi, \tau_\varphi)$, where $\tilde{\Gamma}_\varphi^1(X_\varphi, \tau_\varphi)(u) \in E$ and $\tilde{\Gamma}_\varphi^2(X_\varphi, \tau_\varphi)(u) \in E \Delta E$. Therefore

$$(\tilde{\nabla}_X \tau)_\varphi = [DV_\varphi \cdot X_\varphi - \tilde{\Gamma}_\varphi^1(X_\varphi, \tau_\varphi)] \oplus [DS_\varphi \cdot X_\varphi - \tilde{\Gamma}_\varphi^2(X_\varphi, \tau_\varphi)] \in F.$$  

But

$$(\tilde{\nabla}_X \tau)_\psi = \varphi_2(\tilde{\nabla}_X \tau)_\varphi$$

$$= DF \cdot [DV_\varphi \cdot X_\varphi - \tilde{\Gamma}_\varphi^1(X_\varphi, \tau_\varphi)]$$

$$+ D^2F \cdot [DS_\varphi \cdot X_\varphi - \tilde{\Gamma}_\varphi^2(X_\varphi, \tau_\varphi)]$$

$$\oplus (DF \Delta DF)[DS_\varphi \cdot X_\varphi - \tilde{\Gamma}_\varphi^2(X_\varphi, \tau_\varphi)].$$

On the other hand, $(\tilde{\nabla}_X \tau)_\psi = D \tau_\psi \cdot X_\varphi - \tilde{\Gamma}_\psi(X_\varphi, \tau_\varphi)$. Now

$$[DV_\psi \cdot X_\varphi](v) = DV_\psi(v) \cdot DF(u) \cdot X_\varphi(u)$$

$$= D(V_\psi \circ F)(u) \cdot X_\varphi(u)$$

$$= D[DF \cdot V_\varphi + D^2F \cdot S_\varphi](u) \cdot X_\varphi(u) \quad \text{(by eqs (4) and (5))}$$

$$= D^2F(u)(X_\varphi(u)\Delta V_\varphi(u)) + DF(u) \cdot DV_\varphi(u) \cdot X_\varphi(u)$$

$$+ D^3F(u)(X_\varphi(u)\Delta S_\varphi(u)) + D^2F(u) \cdot DS_\varphi(u) \cdot X_\varphi(u).$$

Again,

$$[DS_\psi \cdot X_\varphi](v) = DS_\psi(v) \cdot DF(u) \cdot X_\varphi(u)$$

$$= D[S_\psi \circ F](u) \cdot X_\varphi(u)$$

$$= D[(DF \Delta DF) \cdot S_\varphi](u) \cdot X_\varphi(u)$$

$$= D(DF \Delta DF)(u)(X_\varphi(u)\Delta S_\varphi(u))$$

$$+ (DF \Delta DF)(u) \cdot DS_\varphi(u) \cdot X_\varphi(u).$$

Therefore

$$[DV_\psi \cdot X_\varphi](v) \oplus [DS_\psi \cdot X_\varphi](v)$$

$$= [D^2F(u)(X_\varphi(u)\Delta V_\varphi(u)) + DF(u) \cdot DV_\varphi(u) \cdot X_\varphi(u)$$

$$+ D^3F(u)(X_\varphi(u)\Delta S_\varphi(u)) + D^2F(u) \cdot DS_\varphi(u) \cdot X_\varphi(u)]$$

$$\oplus \{D(DF \Delta DF)(u)(X_\varphi(u)\Delta S_\varphi(u))$$

$$+ (DF \Delta DF)(u) \cdot DS_\varphi(u) \cdot X_\varphi(u)\}$$

$$= \{[DF(u) \cdot DV_\varphi(u) \cdot X_\varphi(u) + D^2F(u) \cdot DS_\varphi(u) \cdot X_\varphi(u)]$$

$$\oplus [(DF \Delta DF)(u) \cdot DS_\varphi(u) \cdot X_\varphi(u))]\} + \{D^3F(u)(X_\varphi(u)\Delta S_\varphi(u))$$

$$+ D^2F(u)(X_\varphi(u)\Delta V_\varphi(u))] \oplus [D(DF \Delta DF)(u)(X_\varphi(u)\Delta S_\varphi(u))]\}$$
\[ \Delta \] \[ \phi \] \[ u \]

\[ \in \]

\[ \Delta \]

\[ \phi \]

\[ \psi \]

\[ \Delta F(\phi)(X_\phi(\Delta S_\phi(u))) + D^2F(\phi)(X_\phi(\Delta V_\phi(u))) \]

\[ \oplus [D(DF\Delta DF)(u)(X_\phi(\Delta S_\phi(u)))] \].

Again

\[ \Gamma_{\psi}(X_\psi, \tau_\psi) \]

\[ = D\tau_\psi \cdot X_\psi - (\nabla_X \tau)_\psi \]

\[ = [DV_\psi \cdot X_\psi \oplus DS_\psi \cdot X_\psi] - (\nabla_X \tau)_\psi \]

\[ = \phi_2[\phi_2 [DV_\psi \cdot X_\psi \oplus DS_\psi \cdot X_\psi]] + [D^3F(X_\phi \Delta S_\phi) + D^2F(X_\phi \Delta V_\phi) \]

\[ \oplus D(DF\Delta DF)(X_\phi \Delta S_\phi)] - \phi_2 \cdot [D\tau_\psi \cdot X_\phi - \Gamma_{\phi}(\tau_\phi)(X_\phi, \tau_\phi)] \]

\[ = \phi_2 \cdot [\Gamma_{\phi}(X_\phi, \tau_\phi)] + [D^3F(X_\phi \Delta S_\phi) + D^2F(X_\phi \Delta V_\phi) \]

\[ \oplus D(DF\Delta DF)(X_\phi \Delta S_\phi)] \]

\[ = [DF \cdot \Gamma_{\phi}^1(X_\phi, \tau_\phi) + D^2F \cdot \Gamma_{\phi}^2(X_\phi, \tau_\phi) \oplus (DF\Delta DF) \cdot \Gamma_{\phi}^2(X_\phi, \tau_\phi)] \]

\[ + [D^3F(X_\phi \Delta S_\phi) + D^2F(X_\phi \Delta V_\phi) \oplus D(DF\Delta DF)(X_\phi \Delta S_\phi)] \]

\[ = [DF \cdot \Gamma_{\phi}^1(X_\phi, \tau_\phi) + D^2F \cdot \Gamma_{\phi}^2(X_\phi, \tau_\phi) + D^3F(X_\phi \Delta S_\phi) \]

\[ + D^2F(X_\phi \Delta V_\phi)] \oplus [(DF\Delta DF) \cdot \Gamma_{\phi}^2(X_\phi, \tau_\phi) \]

\[ + D(DF\Delta DF)(X_\phi \Delta S_\phi)] \].

This is the transformation formula for \( \Gamma \). In what follows we simplify the presentation of this formula. As usual we write \( \tau_\phi = V_\phi \oplus S_\phi \), where \( V_\phi(u) \in E \) and \( S_\phi(u) \in EAE \), \( u \in \phi U \). We define

\[ A_\phi(X_\phi, V_\phi) = Pr_1 \circ \Gamma_{\phi}(X_\phi, V_\phi \oplus 0), \]

\[ R_\phi(X_\phi, S_\phi) = Pr_1 \circ \Gamma_{\phi}(X_\phi, 0 \oplus S_\phi), \]

\[ B_\phi(X_\phi, V_\phi) = Pr_2 \circ \Gamma_{\phi}(X_\phi, V_\phi \oplus 0), \]

\[ C_\phi(X_\phi, S_\phi) = Pr_2 \circ \Gamma_{\phi}(X_\phi, 0 \oplus S_\phi), \]

where \( A_\phi(X_\phi, \cdot) : E \rightarrow E, R_\phi(X_\phi, \cdot) : EAE \rightarrow E, B_\phi(X_\phi, \cdot) : E \rightarrow EAE \) and \( C_\phi(X_\phi, \cdot) : EAE \rightarrow EAE \). Now we can represent \( \Gamma \) in the matrix form as

\[ \Gamma_{\phi}(X_\phi, \tau_\phi) = \begin{pmatrix} A_\phi(X_\phi, V_\phi) & R_\phi(X_\phi, S_\phi) \\ B_\phi(X_\phi, V_\phi) & C_\phi(X_\phi, S_\phi) \end{pmatrix} \]

i.e., we can write \( \Gamma_{\phi}^1(X_\phi, \tau_\phi) = A_\phi(X_\phi, V_\phi) + R_\phi(X_\phi, S_\phi) \) and \( \Gamma_{\phi}^2(X_\phi, \tau_\phi) = B_\phi(X_\phi, V_\phi) + C_\phi(X_\phi, S_\phi) \). Therefore
[A_\psi(X_\psi, V_\psi) + R_\psi(X_\psi, S_\psi)] \oplus [B_\psi(X_\psi, V_\psi) + C_\psi(X_\psi, S_\psi)] \\
= [DF \cdot (A_\varphi(X_\varphi, V_\varphi) + R_\varphi(X_\varphi, S_\varphi)) \\
+ D^2 F \cdot (B_\varphi(X_\varphi, V_\varphi) + C_\varphi(X_\varphi, S_\varphi)) \\
+ D^3 F (X_\varphi \Delta S_\varphi) + D^2 F (X_\varphi \Delta V_\varphi)] \\
\oplus [(DF \Delta DF) \cdot (B_\varphi(X_\varphi, V_\varphi) + C_\varphi(X_\varphi, S_\varphi)) \\
+ D(DF \Delta DF)(X_\varphi \Delta S_\varphi)].

Hence

A_\psi(X_\psi, V_\psi) + R_\psi(X_\psi, S_\psi) = DF \cdot [A_\varphi(X_\varphi, V_\varphi) + R_\varphi(X_\varphi, S_\varphi)] \\
+ D^2 F \cdot [B_\varphi(X_\varphi, V_\varphi) + C_\varphi(X_\varphi, S_\varphi)] \\
+ D^3 F (X_\varphi \Delta S_\varphi) + D^2 F (X_\varphi \Delta V_\varphi)

and

B_\psi(X_\psi, V_\psi) + C_\psi(X_\psi, S_\psi) = (DF \Delta DF) \cdot (B_\varphi(X_\varphi, V_\varphi) \\
+ C_\varphi(X_\varphi, S_\varphi)) + D(DF \Delta DF)(X_\varphi \Delta S_\varphi).

We now have the following formulas for $A_\varphi$, $B_\varphi$, $R_\varphi$ and $C_\varphi$.

Case i. If $S_\varphi = 0$, then $S_\psi = 0$ and $V_\psi = DF \cdot V_\varphi$. Therefore from above, we have

\begin{equation}
A_\psi(X_\psi, V_\psi) = DF \cdot A_\varphi(X_\varphi, V_\varphi) + D^2 F \cdot B_\varphi(X_\varphi, V_\varphi) + D^2 F (X_\varphi \Delta V_\varphi) \tag{6}
\end{equation}

and

\begin{equation}
B_\psi(X_\psi, V_\psi) = (DF \Delta DF) \cdot B_\varphi(X_\varphi, V_\varphi). \tag{7}
\end{equation}

Case ii. If $V_\varphi = 0$ then $V_\psi = D^2 F \cdot S_\varphi$. Then from above, we have

\begin{equation}
R_\psi(X_\psi, S_\psi) = DF \cdot R_\varphi(X_\varphi, S_\varphi) \\
+ D^2 F \cdot C_\varphi(X_\varphi, S_\varphi) + D^3 F (X_\varphi \Delta S_\varphi) \\
- A_\psi(X_\psi, D^2 F \cdot S_\varphi)
\end{equation}

and

\begin{equation}
C_\psi(X_\psi, S_\psi) = (DF \Delta DF) \cdot C_\varphi(X_\varphi, S_\varphi) \\
+ D(DF \Delta DF)(X_\varphi \Delta S_\varphi) - B_\psi(X_\psi, D^2 F \cdot S_\varphi).
\end{equation}

Note 3.2. If we choose $A_\varphi(X_\varphi, V_\varphi) = 0$ and $B_\varphi(X_\varphi, V_\varphi) = -X_\varphi \Delta V_\varphi$, where $\tau_\varphi = V_\varphi \oplus S_\varphi$, $V_\varphi(u) \in E$, $S_\varphi(u) \in E \Delta E \ (u \in U)$, then $A_\psi(X_\psi, V_\psi) = 0$ and $B_\psi(X_\psi, V_\psi) = -X_\psi \Delta V_\psi$.

Using equations (6) and (7), we can easily see that the choice of $A_\varphi$ and $B_\varphi$ are independent of charts and hence they define a subclass of second-order connections on $T^{(2)}M$. We have the following definition.
Higher order Hessian structures on manifolds

DEFINITION 3.3.
A connection $\nabla$ on the vector bundle $T^{(2)}M$ is called a special second-order connection on $M$ if the Christoffel symbols of $\nabla$ in the given chart $(U, \varphi)$ are given by the matrix of symbols

$$\Gamma^\varphi_{\varphi}(X_{\varphi}, \tau_{\varphi}) = \begin{pmatrix}
0 & R_{\varphi}(X_{\varphi}, S_{\varphi}) \\
-X_{\varphi}\Delta V_{\varphi} & C_{\varphi}(X_{\varphi}, S_{\varphi})
\end{pmatrix},$$

where $\tau_{\varphi} = V_{\varphi} \oplus S_{\varphi}, V_{\varphi}(u) \in E, S_{\varphi}(u) \in \mathbb{E}\Delta E$ $(u \in \varphi U)$. We call $R_{\varphi}$ and $C_{\varphi}$ as the Christoffel symbols of $\nabla$ in the chart $(U, \varphi)$.

Note 3.4. Suppose $\nabla$ is a special second-order connection on $M$. Then in any other chart $(V, \psi)$ at $m \in M$, the transformation laws for the Christoffel symbols are given by

$$R_{\psi}(X_{\psi}, S_{\psi}) = DF \cdot R_{\varphi}(X_{\varphi}, S_{\varphi}) + D^2F \cdot C_{\varphi}(X_{\varphi}, S_{\varphi}) + D^3F(X_{\varphi}\Delta S_{\varphi}) \tag{8}$$

and

$$C_{\psi}(X_{\psi}, S_{\psi}) = (DF \Delta DF) \cdot C_{\varphi}(X_{\varphi}\Delta S_{\varphi}) + D(DF \Delta DF)(X_{\varphi}\Delta S_{\varphi}) + X_{\varphi}\Delta D^2F \cdot S_{\varphi}. \tag{9}$$

If $S_{\varphi} = Y_{\varphi}\Delta Z_{\varphi}$, we write $R_{\varphi}(X_{\varphi}, Y_{\varphi}, Z_{\varphi}) = R_{\varphi}(X_{\varphi}, Y_{\varphi}\Delta Z_{\varphi})$ and similarly for $C_{\varphi}$.

4. Third-order Hessian structures

Let $K$ be a third-order Hessian structure on $M$ (the case $n = 3$ of Definition 2.1). In what follows we state that the following lemma and the proof is easy and hence left to the reader.

Lemma 4.1. Let $F: U \to V$ be a smooth map, where $U, V$ are open $E$ and let $\Delta F = DF \Delta DF: U \to L(E\Delta E, E\Delta E)$. Then $D(\Delta F): U \to L(E, L(E\Delta E, E\Delta E))$ is given by

$$D(\Delta F)(u)(x, y\Delta z) = [DF(u) \cdot y\Delta D^2F(u)(x\Delta z)]$$

$$+ [D^2F(u)(x\Delta y)\Delta DF(u) : z].$$

PROPOSITION 4.2.

There is a one-to-one correspondence between third-order Hessian structures on $M$ and special second-order connections on $T^{(2)}M$ such that if $K$ and $\nabla$ corresponds to each other, then for every chart $(U, \varphi)$ on $M, X, Y, Z \in \mathbb{E}(M)$ and $f \in \mathcal{F}(U)$ we have $K f(X, Y, Z) = D^3f_{\varphi}(X_{\varphi}, Y_{\varphi}, Z_{\varphi}) + D^2f_{\varphi} \cdot C_{\varphi}(X_{\varphi}, Y_{\varphi}, Z_{\varphi}) + Df_{\varphi} \cdot R_{\varphi}(X_{\varphi}, Y_{\varphi}, Z_{\varphi})$, where $R_{\varphi}$ and $C_{\varphi}$ are the Christoffel symbols of $\nabla$ in the chart $(U, \varphi)$.

Proof. Suppose $K$ is a third-order Hessian structure on $M$. Fix a chart $(U, \varphi)$ on $M$, fix $m \in U$ and $X, Y, Z \in \mathbb{E}(U)$. Let $\varphi m = u$. For $f \in \mathcal{F}(U)$, define $\Lambda f(X, Y, Z)(m) = K f(X, Y, Z)(m) - D^3f_{\varphi}(X_{\varphi}, Y_{\varphi}, Z_{\varphi})$ $(u)$. Then (i) $\Lambda$ is real linear in $f$ and (ii) $\Lambda f(X, Y, Z)(m) = 0$ if $Df(m) = 0$ and $D^2f(m) = 0$. It follows that $\Lambda: f \to \Lambda f(X, Y, Z)(m)$ is a second-order tangent vector at $m$. Hence there exists a second-order tangent vector in
\[ E \oplus \Delta E \] which we denote by \( R_\phi(X_\phi, Y_\phi, Z_\phi)(u) \oplus C_\phi(X_\phi, Y_\phi, Z_\phi)(u) \) such that \( \Lambda(f) = D^2 f_\phi \cdot C_\phi(X_\phi, Y_\phi, Z_\phi)(u) + Df_\phi \cdot R_\phi(X_\phi, Y_\phi, Z_\phi)(u) \). Therefore, if \( X, Y, Z \in \infty(U) \),
\[
[R_\phi(X_\phi, Y_\phi, Z_\phi) \oplus C_\phi(X_\phi, Y_\phi, Z_\phi)](f_\phi) = [K f(X, Y, Z) \circ \phi^{-1} - D^3 f_\phi(X_\phi, Y_\phi, Z_\phi)].
\]
We may conclude from this that \( R_\phi \oplus C_\phi \) is a smooth trilinear map on \( \phi U \). Therefore
\[
K f(X, Y, Z)(m) = D^3 f_\phi(X_\phi, Y_\phi, Z_\phi)(u) + D^2 f_\phi \cdot C_\phi(X_\phi, Y_\phi, Z_\phi)(u) + Df_\phi \cdot R_\phi(X_\phi, Y_\phi, Z_\phi)(u).
\]
We can easily verify the transformation laws for
\[
\text{Conversely, if } R_\phi \text{ and } C_\phi \text{ and hence we obtain}
\]
\[
K f(X, Y, Z)(m) = DF \Delta DF \cdot C_\phi(X_\phi, Y_\phi, Z_\phi)(u) + [X_\phi(\Delta F) \Delta F(Y_\phi, Z_\phi)(u) + \text{cyclic terms}]
\]
and
\[
R_\phi(X_\phi, Y_\phi, Z_\phi) = DF(u) \cdot R_\phi(X_\phi, Y_\phi, Z_\phi)(u) + D^2 F(u) \cdot C_\phi(X_\phi, Y_\phi, Z_\phi)(u) + D^3 F(X_\phi, Y_\phi, Z_\phi)(u).
\]
These are precisely the relations satisfied by the Christoffel symbols (C and R) of a special second-order connection \( \nabla \), so that \( K \) determines a special second-order connection \( \nabla \) on \( M \).

Conversely, if \( \nabla \) is a second-order connection on \( M \) and \( C, R \) are the associated Christoffel symbols of \( \nabla \) in a chart, then defining \( K f(X, Y, Z)(m) = D^3 f_\phi(X_\phi, Y_\phi, Z_\phi)(u) + D^2 f_\phi \cdot C_\phi(X_\phi, Y_\phi, Z_\phi)(u) + Df_\phi \cdot R_\phi(X_\phi, Y_\phi, Z_\phi)(u) \), it is not difficult to verify, by reversing the order of steps in the above proof, that \( K \) defines a third-order Hessian structure on \( M \). Thus we have a one-to-one correspondence between special second-order connections and third-order Hessian structures.

q.e.d.

In the next section we discuss the concept of a geodesic of special second-order connection and we prove a theorem below which gives a very satisfying geometric characterization of symmetric third-order Hessian structures.

5. Second-order geodesics

Let \( c \) be a smooth curve in \( M \) such that \( c(0) = m \). For \( f \in \mathcal{F}(M) \), define \( \bar{c}(f) = (f \circ c)''(0) \). That is in a local chart \( (U, \phi) \) at \( m \in U \), we have \( \bar{c}(f) = D^2 f_\phi(\phi m)(c'_{\phi}(0), c'_{\phi}(0)) + Df_\phi(\phi m) \cdot c''_{\phi}(0) \). If \( f \in F^3_m \) then \( \bar{c}(f) = 0 \). Hence \( \bar{c} \in T^{(2)}_m M \). Let \( (U, \phi) \) be chart at \( m \).

Define \( [\bar{c}]_\phi \equiv c''_{\phi} + c'_{\phi} \Delta c'_{\phi} \). This is well-defined and is a second-order tangent vector at \( m \).

Note that, if \( \nabla \) is a special second-order connection on \( T^{(2)} M \) then \( \nabla_{v_{\tau}} \tau \) depends only on the behavior of \( \tau \) near \( \pi(v) \) and also that \( \nabla_{v_{\tau}} \tau \) can be calculated once \( \tau \) is known along any smooth curve \( c \) in \( M \) with initial tangent vector \( v \). Therefore \( \nabla_{c} \bar{c} \) is well-defined.

Suppose \( \nabla \) is a special second-order connection with Christoffel symbols \( R \) and \( C \). Let \( (U, \phi) \) be a chart at \( m \). Then we obtain

\[
(\nabla_{\bar{c}} \bar{c})_\phi = [c''_{\phi} - R_\phi(c'_{\phi}, c'_{\phi}, c'_{\phi})] + [C_\phi(c'_{\phi}, c'_{\phi}, c'_{\phi}) = 3c'_{\phi} \Delta c'_{\phi}].
\]

DEFINITION 5.1.

\( c \) is called a second-order geodesic of \( \nabla \) if \( \nabla_{t} \bar{c} = 0 \) for all \( t \). Thus \( c \) is a second-order geodesic if and only if it satisfies the following equations:
Proof. Let \( \overline{\nabla} \) be a special second-order symmetric connection on \( T^{(2)}M \) and let \( K \) be the unique symmetric third-order Hessian structure on \( M \) associated to \( \overline{\nabla} \). Then \( Kf(v, v, v) = (f \circ c)^"(0) \) for all \( m \in M, v \in T_mM \) if there is a second-order geodesic \( c \) such that \( c(0) = m, c'(0) = v \).

(The proof is easy by using the definition and hence left to the reader.)

Given a connection \( \nabla \) on \( TM \), there are standard theorems on how \( \overline{\nabla} \) induces a connection on tensor bundles associated to \( M \) by covariant differentiation. But we are not aware of any observation in the literature that \( \overline{\nabla} \) induces a connection on \( T^{(2)}M \). We can now show that such a thing is possible.

\( \text{DEFINITION 5.2.} \)

Let \( \nabla \) be a connection on \( TM \). Then \( Kf = \nabla^2 f \) defines a third-order Hessian structure on \( M \) and associated to \( K \), there is a special second-order connection \( \overline{\nabla} \) on \( T^{(2)}M \). We call \( \overline{\nabla} \) the connection induced by \( \nabla \).

\( \text{PROPOSITION 5.5.} \)

Suppose \( \nabla \) has the Christoffel symbol \( \Gamma_\varphi \) in a chart \( (U, \varphi) \). Then the induced connection \( \overline{\nabla} \) on \( T^{(2)}M \) has Christoffel symbols given by

\[
C_\varphi(X_\varphi, Y_\varphi, Z_\varphi) = X_\varphi \Delta \Gamma_\varphi(Y_\varphi, Z_\varphi) + Y_\varphi \Delta \Gamma_\varphi(X_\varphi, Z_\varphi) + Z_\varphi \Delta \Gamma_\varphi(X_\varphi, Y_\varphi)
\]

and

\[
R_\varphi(X_\varphi, Y_\varphi, Z_\varphi) = D[\Gamma_\varphi(Y_\varphi, Z_\varphi)] \cdot X_\varphi - \Gamma_\varphi((\nabla_X Y)_\varphi, Z_\varphi) - \Gamma_\varphi(Y_\varphi, (\nabla_X Z)_\varphi).
\]

Proof. We know that in a local chart \( (U, \varphi) \) at \( m \in U, \nabla \) has the form \( (\nabla_X Y)_\varphi = D\varphi \cdot X_\varphi - \Gamma_\varphi(X_\varphi, Y_\varphi) \), where \( \Gamma_\varphi \) is the Christoffel symbol of \( \nabla \) which has transformation property \( \overline{\nabla} \). Then in a local chart \( (U, \varphi) \) at \( m, \nabla^3 f \) has the expression (and by using Leibnitz rule for higher derivatives, \( \overline{\nabla} \))

\[
\nabla^3 f(X, Y, Z) = [\nabla_X (\nabla^2 f)](Y, Z)
\]

\[
= [\nabla_X (\nabla^2 f(Y, Z))] - [\nabla^2 f(Y, \nabla_X Z)]
\]

\[
= [X \cdot (Y \cdot Z \cdot f - \nabla_Y Z \cdot f)] - [\nabla_X Y \cdot Z \cdot f - \nabla_Y Z \cdot f]
\]

\[
= X \cdot \nabla_Y Z \cdot f - \nabla_Y \nabla_X Z \cdot f
\]

\[
= D^3 f_\varphi(X_\varphi, Y_\varphi, Z_\varphi) + [D^2 f_\varphi(X_\varphi, \Gamma_\varphi(Y_\varphi, Z_\varphi))
\]

\[
+ D^2 f_\varphi(Y_\varphi, \Gamma_\varphi(X_\varphi, Z_\varphi)) + D^2 f_\varphi(Z_\varphi, \Gamma_\varphi(X_\varphi, Y_\varphi))]
\]

\[
+ [D f_\varphi \cdot D[\Gamma_\varphi(Y_\varphi, Z_\varphi)] \cdot X_\varphi - D f_\varphi \cdot \Gamma_\varphi((\nabla_X Y)_\varphi, Z_\varphi)
\]

\[
- D f_\varphi \cdot \Gamma_\varphi(Y_\varphi, (\nabla_X Z)_\varphi)]
\]
That is,
\[
\nabla^3 f(X,Y,Z) = D^3 f_\varphi(X_\varphi,Y_\varphi,Z_\varphi) + D^2 f_\varphi \cdot [X_\varphi \Delta \Gamma_\varphi(Y_\varphi,Z_\varphi) + Y_\varphi \Delta \Gamma_\varphi(X_\varphi,Z_\varphi) + Z_\varphi \Delta \Gamma_\varphi(X_\varphi,Y_\varphi)] + D f_\varphi \cdot [\Gamma_\varphi(Y_\varphi,Z_\varphi)] \cdot (X_\varphi \Delta \Gamma_\varphi)(Y_\varphi,Z_\varphi) - \Gamma_\varphi((\nabla_X Y)_\varphi,Z_\varphi).
\]

Hence we have \( C_\varphi(X_\varphi,Y_\varphi,Z_\varphi) \) and \( R_\varphi(X_\varphi,Y_\varphi,Z_\varphi) \) as mentioned in Proposition 5.5.

q.e.d.

Note 5.6. If \( \nabla \) has torsion and curvature zero, then \( \nabla^3 f \) is symmetric and hence the corresponding \( \tilde{\nabla} \) is symmetric.

6. Conclusion

Not every \( K \) arises like \( \nabla^3 \), for some \( \nabla \), whereas in the second-order case, every \( H^2 \) arises like \( \nabla^2 \) for a unique \( \nabla \).

PROPOSITION 6.1.

If \( \tilde{\nabla} \) is a special second-order connection on \( T^{(2)} M \) induced from a (first order) connection \( \nabla \) on \( M \), then every geodesic of \( \nabla \) is also a geodesic of \( \tilde{\nabla} \).

Proof. Let \( \nabla \) be a connection on \( M \). Recall that \( c \) is a geodesic of \( \nabla \) iff \( c''_\varphi = \Gamma_\varphi(c'_\varphi,c'_\varphi) \), for all charts \( \varphi \). We have to prove that \( c \) is a second-order geodesic of \( \tilde{\nabla} \). Using the equations for \( C_\varphi \) and \( R_\varphi \) in Proposition 5.5, we obtain \( C_\varphi(c'_\varphi,c'_\varphi,c'_\varphi) = 3c'_\varphi \Delta \Gamma_\varphi(c'_\varphi,c'_\varphi) = 3c'_\varphi \Delta c''_\varphi \) and \( R_\varphi(c'_\varphi,c'_\varphi,c'_\varphi) = c''_\varphi \).

q.e.d.

Acknowledgment

I am deeply grateful to my guru Prof. K Viswanath, Department of Mathematics, University of Hyderabad, for suggesting this problem as part of my Ph.D. thesis and my gratitude to him for his continuous support and guidance.

References

[1] Abraham R and Marsden J E, Foundations of Mechanics, 2nd edition (Massachusetts: The Benjamin/Cummings Publishing Company Inc.) (1978)
[2] Abraham R, Marsden J E and Ratiu T, Manifolds, tensor analysis and applications, first edition (Massachusetts: Addison-Wesley Publishing Company Inc.) (1983); and second edition (Springer-Verlag) (1987)
[3] Ambrose W, Palais R S and Singer I M, Sprays (Anais da Academia Brasileira de Ciencias) (1960) vol. 32
[4] Brickell F and Clark R S, Differentiable manifolds – an Introduction (London: Van Nostrand Reinhold Company) (1970)
[5] David Kumar R, Second and higher order structures on manifolds, Ph.D. thesis (Hyderabad, India: University of Hyderabad) (1995) (unpublished)
[6] David Kumar R and Viswanath K, Local expression of Hessian structures and dissections on manifolds, J. Ramanujan. Math. Soc. 2(2) (1987) 173–184
Higher order Hessian structures on manifolds

[7] Fomin V E, Differential geometry of Banach manifolds, Russian version (Publishing House of Kazan University) (1983)
[8] Flaschel P and Klingenberg W, Riemannsche Hilbert-mannigfalting keiten. Periodische Geodatische, LNM 282 (Berlin-Heidelberg, New York: Springer-Verlag) (1972)
[9] Juhani Fiskaali, Sauli Luukkonen and Eljas Maatta, On the differential geometry of bounded projections on Banach spaces, Preprint (July, 1987)
[10] Lang S, Differential manifolds (New York: Springer-Verlag) (1972)
[11] Libermann P and Charles Michael Marle, Sympletic geometry and analytical mechanics (Dordrecht, Holland: D. Reidel Publishing Company) (1987)
[12] Nelson E, Topics in dynamics I: Flows, Preliminary Informal Notes of University Courses and Seminars in Mathematics. Mathematical Notes (Princeton: Princeton University Press and the University of Tokyo Press) (1969)
[13] Nelson Edward, Tensor analysis (Princeton, New Jersey: Princeton University Press and the University of Tokyo Press) (1967)
[14] Nijenhuis A, Jacobi-type identities for bilinear differential concomitants of certain tensor fields I, Indag. Math. 17 (1955) 390–403
[15] Nickerson H K, Spencer D C and Steenrod N E, Advanced Calculus (London: D. Van Nostrand Company Inc.) (1959)
[16] Poor A Walter, Differential geometric structures (New York: Mc Graw-Hill Book Company) (1981)
[17] Spivak Michael, A comprehensive introduction to differential geometry, second edition (Houston, Texas: Publish or Perish Inc.) (1970) vol. II