Normal $p$-complements and irreducible character codegrees

Jiakuan Lu, Yu Li, Boru Zhang
School of Mathematics and Statistics, Guangxi Normal University,
Guilin 541006, Guangxi, P. R. China

Abstract

Let $G$ be a finite group and $p \in \pi(G)$, and let $\text{Irr}(G)$ be the set of all irreducible complex characters of $G$. Let $\chi \in \text{Irr}(G)$, we write $\text{cod}(\chi) = |G : \ker \chi|/\chi(1)$, and called it the codegree of the irreducible character $\chi$. Let $N \triangleleft G$, write $\text{Irr}(G|N) = \{ \chi \in \text{Irr}(G) \mid N \nsubseteq \ker \chi \}$, and $\text{cd}(G|N) = \{ \text{cod}(\chi) \mid \chi \in \text{Irr}(G|N) \}$. In this paper, we prove that if $N \triangleleft G$ and every member of $\text{cd}(G|N')$ is not divisible by some fixed prime $p \in \pi(G)$, then $N$ has a normal $p$-complement and $N$ is solvable.

Keywords: Finite groups; Codegree of a character; Normal $p$-complement.
MSC(2000): 20C20, 20C15

1 Introduction

Throughout this paper, $G$ always denotes a finite group and, as usual, let $\text{Irr}(G)$ be the set of all irreducible complex characters of $G$ and $\text{cd}(G) = \{ \chi(1) \mid \chi \in \text{Irr}(G) \}$. The structure of $G$ is heavily determined by $\text{cd}(G)$, and there are a lot of classical theorems on this subject. For example, Thompson proved that if there is some prime $p$ that divides every member of $\text{Irr}(G)$ exceeding 1, then $G$ has a normal $p$-complement (See [10] or [4, Corollary 12.2]).

Given that $N \leq G$, we write $\text{Irr}(G|N) = \{ \chi \in \text{Irr}(G) \mid N \nsubseteq \ker \chi \}$ and $\text{cd}(G|N) = \{ \chi(1) \mid \chi \in \text{Irr}(G|N) \}$.

Berkovich [1] proved a analog of the theorem of Thompson. He showed that $N \leq G$ has a normal $p$-complement if all nonlinear members of $\text{Irr}(G|N)$ have degree divisible by some fixed prime $p$.

Isaacs and Knutson [5] proved a generalized version of Berkovich’s theorem. They proved that $N \leq G$ is solvable and has a normal $p$-complement if every member of $\text{cd}(G|N')$ is divisible by $p$.

Let $\chi \in \text{Irr}(G)$. In [8], the authors defined $\text{cod}(\chi) = |G : \ker \chi|/\chi(1)$, and called it the codegree of the irreducible character $\chi$. Many facts about codegree of the irreducible characters in finite groups have been obtained. For
example, see [8] [6] [8] [11]. Recently, Chen and Yang [2] proved that if \( G \) is a \( p \)-solvable group and \( \text{cod}(\chi) \) is a \( p' \)-number for every monolithic, monomial \( \chi \in \text{Irr}(G) \), then \( G \) has a normal \( p \)-complement.

Let \( H \) be a maximal subgroup of \( G \) and let \( \chi \) be an irreducible constituent of \((1_H)^G\). Following [2], we call \( \chi \) a \( \mathcal{P} \)-character of \( G \) with respect to \( H \), and denote by \( \text{Irr}_\mathcal{P}(G) \) the set of \( \mathcal{P} \)-characters of \( G \). In [9], Qian and Yang showed many interesting facts about \( \mathcal{P} \)-characters in a finite solvable group. Lu, Wu and Meng [11] proved that if \( G \) is a \( p \)-solvable group and \( \text{cod}(\chi) \) is a \( p' \)-number for every \( \chi \in \text{Irr}_\mathcal{P}(G) \), then \( G \) is \( p \)-nilpotent.

In this paper, we may take one more step. We write

\[
\text{cod}(G|N) = \{ \text{cod}(\chi) \mid \chi \in \text{Irr}(G|N) \}.
\]

Our main result is the analog of the theorem of Isaacs and Knutson.

**Theorem 1.1** Let \( N \trianglelefteq G \) and suppose that every member of \( \text{cod}(G|N) \) is not divisible by some fixed prime \( p \in \pi(G) \). Then \( N \) has a normal \( p \)-complement and \( N \) is solvable.

## 2 Proof

The following result is useful in the proof of the solvability in Theorem 1.1.

**Lemma 2.1** ([4] Lemma 2.2) Suppose that \( A \) acts on \( G \) via automorphisms and that \( (|A|, |G|) = 1 \). If \( C_G(A) = 1 \), then \( G \) is solvable.

**Proof of the theorem** We first prove that \( N \) has a normal \( p \)-complement. Write \( M = O^p(N) \), let \( P \in \text{Syl}_p(M) \). Assume that \( P > 1 \) and we work to obtain a contradiction.

Choose \( S \in \text{Syl}_p(G) \) such that \( P \leq S \). Then \( P = S \cap M \leq S \), and \( S \) permutes \( \text{Lin}(P) \), where \( \text{Lin}(P) = \{ \lambda \in \text{Irr}(P) \mid \lambda(1) = 1 \} \). Since \( |S| \) and \( |\text{Lin}(N)| \) are \( p \)-powers, we may choose a nonprincipal linear character \( \lambda \) of \( P \) such that \( \lambda \) is stabilized by \( S \).

Now \( S \) stabilizes \( \lambda^M \), and thus \( S \) permutes the irreducible constituents of \( \lambda^M \). Since \( \lambda^M(1) = |M : P| \) is not divisible by \( p \), \( \lambda^M \) must have some \( S \)-invariant irreducible constituent \( \alpha \in \text{Irr}(M) \) with degree not divisible by \( p \). Clearly, \( \alpha \) is stabilized by \( MS \), and \( |MS : S| \) is \( p \)-power and so is relatively prime to \( \alpha(1) \). Furthermore, the determinantal order \( o(\alpha) \) is not divisible by \( p \) since \( M = O^p(M) \). and thus \( o(\alpha) \) is also relatively prime to \( |MS : S| \). It follows from [4] Corollary 6.28 that \( \alpha \) extends to some irreducible character \( \beta \in \text{Irr}(MS) \).

Next, \( \beta^G(1) = \beta(1)|G : MS| = \alpha(1)|G : MS| \) is not divisible by \( p \), and hence it has some constituent \( \chi \in \text{Irr}(G) \) with degree not divisible by \( p \). Assume that \( \text{cod}(\chi) = |G : \ker \chi|/\chi(1) \) is not divisible by \( p \). Then we deduce that \( S \leq \ker \chi \), in particular, \( P \leq \ker \chi \). Thus, the irreducible constituents of \( \chi_P \) are trivial. By Frobenius Reciprocity, \( \beta \) is an irreducible constituent of \( \chi_{MS} \) and so \( \alpha \) is an
irreducible constituent of $\chi_M$. Similarly, $\lambda$ is an irreducible constituent of $\alpha_P$. Thus, $\lambda$ is an irreducible constituent of $\chi_P$ and is trivial, a contradiction.

Assume that $\text{cod}(\chi)$ is divisible by $p$. By hypothesis, $\chi$ is not a member of $\text{Irr}(G|N')$ and thus $N' \leq \ker \chi$, and so the irreducible constituents of $\chi_N$ are linear. The irreducible constituents of $\chi_M$ are therefore linear, and in particular, $\alpha$ is linear. Thus $\alpha$ is an extension of $\lambda$ to $M$, and it follows that $o(\lambda)$ divides $o(\alpha)$, which is not divisible by $p$, and therefore $o(\lambda)$ is not divisible by $p$. This is a contradiction since $\lambda$ is a nontrivial linear character of a $p$-group.

Now, we prove that $N$ is solvable. By the above arguments, we know that $N$ has a normal $p$-complement $K$, and we write $M = N' \cap K$. Then $p$ does not divide $|M|$ and every member of $\text{cod}(G|M)$ is not divisible by $p$. Let $S \in \text{Syl}_p(G)$ and note that if $\theta \in \text{Irr}(M)$ is $S$-invariant, then $\theta$ is extendible to some character $\varphi \in \text{Irr}(MS)$. Since $p$ does not divide $\varphi^G(1) = |G : MS|\varphi(1)$, we see that $\varphi^G$ has an irreducible constituent $\chi$ with degree not divisible by $p$.

Assume that $\text{cod}(\chi) = |G : \ker \chi|/\chi(1)$ is not divisible by $p$. Then $S \leq \ker \chi$ and thus $S \leq \ker \varphi$. We deduce that $\varphi(ms) = \varphi(m)$ for all $m \in M$ and $s \in S$. So every $\theta \in \text{Irr}(M)$ is $S$-invariant. It follows from Glauberman’s theorem that $C_M(S) = M$. Thus $MS = M \times S$ and $\varphi = \theta \times 1_S$. Since $S \neq 1$, we may choose again $\varphi = \theta \times \xi \in \text{Irr}(MS)$ for some nonprincipal character $\xi \in \text{Irr}(S)$. So $S \nsubseteq \ker \varphi$, which is a contradiction.

Thus $\text{cod}(\chi)$ is divisible by $p$. It follows that $\chi \notin \text{cod}(G|N')$ and hence $M \leq \ker \chi$ and $\theta = 1_M$. It follows again from Glauberman’s theorem that $C_M(S) = 1$. By Lemma 2.1, $M$ is solvable, and thus $N$ is solvable, as claimed.

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