Are the singularities stable?

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Abstract

The spacetime singularities play a useful role in gravitational theories by distinguishing physical solutions from non-physical ones. The problem, we studying in this paper is: are these singularities stable? To answer this question, we have analyzed the general problem of stability of the family of the static spherically symmetric solutions of the standard Einstein-Maxwell model coupled to an extra free massless scalar field. We have obtained the equations for the axial and polar perturbations. The stability against axial perturbations has been proven.

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1 Introduction

Recently there has been considerable interest in so-called “dilaton fields”, i.e. neutral scalar fields whose background values determine the strength of the coupling constants in the effective four-dimensional theory. However, although the scalar field naturally arises in theory, its existence from the point of view of the general relativity is quite problematic. It has been shown that including a scalar field in the theory leads to a violation of the strong equivalence principle and modification of large-scale gravitational phenomena [1]. The presence of the scalar field affects the equations of motion of the other matter fields as well. Thus, for example, solutions which correspond to a pure electromagnetic field appear to be drastically modified by the scalar field. Such solutions were studied in [2]-[4], where it was shown that the scalar field generally destroys the horizons leading to the singularities in a scalar curvature on a finite radii. Special attention has been paid to the charged dilaton black hole solution [4]. Thus analysis of the perturbations around the extreme charged dilaton black hole solution performed in [4] demonstrates the analogy of the behavior of the black holes and elementary particles in the sense that there exists an energy gap in the excitation spectrum of the black hole.

From the other side, an interesting way to treat the problem of appearance of the spacetime singularities is to develop a theory of gravity including an extra spatial dimensions [5]. It turns
out that certain singularities can be resolved by simply passing into a higher-dimension theory of gravity for which spacetime is only effectively four-dimensional below some compactification scale. Moreover, while studying the decay of magnetic fields in Kaluza-Klein theory was argued in that for a physical four-dimensional magnetic field there are two ways it can decay: either by producing single naked singularities into which space “collapses,” or by producing pairs of monopole-anti-monopole pairs which accelerate off to infinity. Since many currently popular unified fields theories include an extra spatial dimensions, it is important to ask: could these singularities be stable in our four-dimensional world? Although, it was shown that static spherically symmetric solutions for the related case of Einstein-Klein-Gordon equations with a quadratic self-interaction term are unstable, it would be interesting to study this problem for the general case of the Einstein-Maxwell-scalar system.

In this paper we consider the problem of stability of the general class of static spherically-symmetric solutions of the standard Einstein-Maxwell model with an extra free scalar field \( \phi \) with four-dimentional action taken to be:

\[
S = -\frac{1}{16\pi} \int dx^4 \sqrt{-g} \left( R - 2g^{mn} \nabla_m \phi \nabla_n \phi + F^2 \right),
\]

where \( F_{ab} = \nabla_{[a} A_{b]} \) is the usual Maxwell field. The geometrical units \( c = \gamma = 1 \) are used through the paper as is the following metric convention \((+−−−)\). The fields equations corresponding to action (1) are easily calculated to be:

\[
R_{mn} = 2\nabla_m \phi \nabla_n \phi - 2F_{mk}F^k_n + \frac{1}{2}g_{mn}F^2,
\]

(2a)

\[
g^{ab} \nabla_a \nabla_b \phi = 0, \quad \nabla_a F^{ab} = 0
\]

(2b)

The general static spherically symmetric solution to system of equations (2) is well known and it might be given by the following relations

\[
ds^2 = u(r)dt^2 - v(r)dr^2 - w(r)d\Omega,
\]

(3a)

\[
v(r) = \frac{1}{u(r)} = q^2(r), \quad w(r) = (r^2 - \mu^2)q^2(r),
\]

(3b)

\[
\phi(r) = \frac{\phi_0}{2\mu} \ln \frac{r - \mu}{r + \mu}, \quad A_0'(r) = \frac{Q}{w(r)},
\]

(3c)

\[
g(r) = p_\pm \left( \frac{r - \mu}{r + \mu} \right)^k + p_\pm \left( \frac{r + \mu}{r - \mu} \right)^k,
\]

(3d)

\[
2p_\pm = 1 \pm \left( 1 + \frac{Q^2}{4\mu^2k^2} \right)^{1/2}, \quad \phi_0 = \mu\sqrt{1 - 4k^2},
\]

(3e)

where \( \mu, k, Q \) are the arbitrary constants, the prime denotes the derivative \( d/dr \) and the usual notation is accepted in (3a) for \( d\Omega = d\theta^2 + \sin^2 \theta d\varphi^2 \). The parameter \( \mu \) is related to physical mass \( \mu_0 > 0 \) and charge \( Q \) by

\[
\mu = \pm \frac{1}{2k} \sqrt{\mu_0^2 - Q^2},
\]

which saturates the bound \( |Q| \leq \mu_0 \). In the extreme limit \( |Q| = \mu_0 \), and the solution, independently on the scalar field, accepts the familiar form of the extreme Reisner-Nordström black hole solution:
ds^2 = \left(1 \mp \frac{\mu_0}{R}\right)^2 dt^2 - \left(1 \mp \frac{\mu_0}{R}\right)^{-2} dR^2 - R^2 d\Omega,

\phi(r) = 0, \quad R = r \pm \mu_0.

In some special cases this solution coincides with well known results which will support our future conclusions. This class of the solutions (3) describes the exterior region of the black holes and the naked singularities\(^3\). It is important to ask: could a distant observer study the objects, located under this spurious singularity at \(r = \mu\)? The answer appears to be no. Indeed, let us imagine that the observer will try to test this region using the perturbations of the fields involved. Can the infinite energy density (and corresponding singularity in the equation for the perturbations) be an opaque boundary for the perturbations, or there is a possibility that the perturbations might penetrate under the surface \(r = \mu\)? To answer this question, one might easily show that \(r = \mu\) is an effectively infinite point in the case of Schwatzchild \((k = \pm 1/2, Q = 0)\) and Reisner-Nordström \((k = \pm 1/2)\) solutions. In the vicinity of this surface there are solutions which propagate in both the “in” and “out” directions \((\sim e^{\pm \omega r^*})\)\(^4\). Then, for a distant observer, the time of the fall is logarithmically infinite because \(g_{00}/g_{11} \sim (r - \mu)^2\) (\(i.e.\) we have a horizon). This is true even though the distance to the horizon can be traversed in finite proper time. In the general case of the solution (3) with non-zero scalar field \((k \neq \pm 1/2 \text{ and } |Q| \neq \mu_0)\), there are no “in” or “out” going waves and one can see that because of the relation \(g_{00}/g_{11} \sim (r - m)^d\) with \(d < 2\) the time of the fall is finite (it has no horizon). From the analysis presented in this paper we will see that the perturbations will “stop” at the point \(r = \mu\) (the singular point of the equation for perturbations) and thus the observer will not be able to see the singularity. It is worth to note that, although the energy densities for both scalar and electromagnetic fields in solution (3) are infinite when \(r \rightarrow \mu\), one may show by straightforward calculation that the energy (and the mass) of the solution remain finite. This suggests that we may consider a small perturbations around the solution (3) and linearize the field equations (2). Moreover, it is reasonable to expect that the corresponding energy of perturbations will be small (and therefore also finite) as compared to the energy of the solution (3).

In this paper we will study the general problem of stability of the solution (3) which describes the “exterior” region of the black holes and the naked singularities. It is reasonable to note that one would not expect general solutions with naked singularities to be stable since the total mass can be negative. However, the analysis presented here will show that the solution (3) is stable at least against axial perturbations which, in the light of the results of\(^5\), makes this research specifically interesting for the general case of Einstein-Maxwell-scalar system, superstrings and Kaluza-Klein theories. The outline of this paper is as follows: In the next section we will introduce the definitions accepted throughout the paper and will obtain the system of equations for axial and polar perturbations. In section 3 we will study the problem of stability of the solution (3) against axial perturbations. In the following section 4 we will examine the problem of splitting of the obtained \(2 \times 2\) matrix equation into two independent equations. In the final section 5, we will summarize our results and suggest the perspectives of the research on the problem of stability of the static spherically symmetric solutions of the Einstein-Maxwell-scalar system. We will also discuss some possible experimental consequences of presence of the electromagnetic and scalar fields in the motion of the celestial bodies.

### 2 The general system of the equations for the fields perturbations

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\(^4\)In the special case \(k = \pm 1/2\) result (3) reduces to the Reisner-Nordström solution whose properties has been studied extensively\(^6\).
It is well-known that in the presence of the non-zero background matter field, the perturbations of matter and gravitational fields should be studied simultaneously. Otherwise, the equations of motion can appear to be inconsistent and, in any case, they can not be applied to the stability problem of the solution under consideration.

In this section we will obtain the general system of the equations for both axial and polar perturbations for the system of equations (2). In order to simplify the future calculations, let us introduce notation for the perturbations of scalar, electromagnetic and gravitational fields. Due to the symmetries of the background field, this can be done at a rather straightforward way [10]. Indeed, as far as the background field does not depend on time, we can write the perturbation of any component $f(t, \vec{x})$ for any field involved as:

$$\delta f(t, \vec{x}) = \exp(i\omega t)\delta f(\vec{x}).$$

Because of the spherical symmetry of the background field we, following [11], will define the spin-weighted spherical harmonics by the equations:

$$\left(-\partial_2 \pm s \cot \theta \mp \frac{i}{\sin \theta} \partial_3\right) Y_{lm}(\theta, \varphi) = \sqrt{(l \pm s + 1)(l \mp s)} Y_{l\pm 1 m}(\theta, \varphi),$$

where

$$0 Y_{lm}(\theta, \varphi) = Y_{lm}(\theta, \varphi).$$

Then we may expand all the spin-weighted perturbations through these spherical harmonics as follows:

(i). The "scalar" perturbations (i.e. the perturbations of the components without the angular indices) can be given by:

$$\delta \phi(\vec{x}) = \sum_{lm} z_{lm}(r) Y_{lm}(\theta, \varphi),$$

$$\delta A_0(\vec{x}) = \sum_{lm} k_{lm}(r) Y_{lm}(\theta, \varphi), \quad \delta A_1(\vec{x}) = \sum_{lm} n_{lm}(r) Y_{lm}(\theta, \varphi),$$

$$\delta g_{00}(\vec{x}) = \sum_{lm} a_{lm}(r) Y_{lm}(\theta, \varphi), \quad \delta g_{01}(\vec{x}) = \sum_{lm} b_{lm}(r) Y_{lm}(\theta, \varphi), \quad \delta g_{11}(\vec{x}) = \sum_{lm} c_{lm}(r) Y_{lm}(\theta, \varphi).$$

(ii). The "vector" perturbations (i.e. the perturbations of the components with one angular index only) can be expanded [11] with respect to the spin-weighted spherical harmonics with the spin $\pm 1$ as:

$$\delta g_{02}(\vec{x}) \pm \frac{i}{\sin \theta} \delta g_{03}(\vec{x}) =$$

$$= \sum_{lm} -l(l + 1)[d_{lm}(r) \mp ie_{lm}(r)] Y_{lm}(\theta, \varphi),$$

$$\delta g_{12}(\vec{x}) \pm \frac{i}{\sin \theta} \delta g_{13}(\vec{x}) =$$

$$= \sum_{lm} -l(l + 1)[f_{lm}(r) \mp ig_{lm}(r)] Y_{lm}(\theta, \varphi),$$

$$\delta g_{22}(\vec{x}) \pm \frac{i}{\sin \theta} \delta g_{23}(\vec{x}) =$$

$$= \sum_{lm} -l(l + 1)[h_{lm}(r) \mp ig_{lm}(r)] Y_{lm}(\theta, \varphi).$$
\[ \delta A_2(\vec{x}) \pm \frac{i}{\sin \theta} \delta A_3(\vec{x}) = \]
\[ = \sum_{lm} -l(l+1)(\alpha_{lm}(r) \mp is_{lm}(r))\pm_1 Y_{lm}(\theta, \varphi). \quad (6c) \]

(iii). And, finally, the "tensor" perturbations can be expanded [11] with respect to the spin-weighted spherical harmonics with the spin \( \pm 2, 0 \) as:
\[ \delta g_{22}(\vec{x}) + \frac{1}{\sin^2 \theta} \delta g_{33}(\vec{x}) = \]
\[ = \sum_{lm} [h_{lm}(r) - l(l+1)\epsilon_{lm}(r)]\pm_2 Y_{lm}(\theta, \varphi), \quad (7a) \]
\[ \delta g_{22}(\vec{x}) - \frac{1}{\sin^2 \theta} \delta g_{33}(\vec{x}) \pm \frac{2i}{\sin \theta} \delta g_{23}(\vec{x}) = \]
\[ = \sum_{lm} \sqrt{(l-1)l(l+1)(l+2)}[\epsilon_{lm}(r) \mp 2ij_{lm}(r)]\pm_2 Y_{lm}(\theta, \varphi). \quad (7b) \]

In order to reduce the effective number of the variables we will perform the gauge transformation:
\[ x_a \rightarrow x_a + \xi_a, \quad (8a) \]
where the components of the four-vector \( \xi_a(t, \vec{x}) \) are given by the relations:
\[ \xi_0(t, \vec{x}) = \sum_{lm} \alpha_{lm}(r)Y_{lm}(\theta, \varphi), \]
\[ \xi_1(t, \vec{x}) = \sum_{lm} \beta_{lm}(r)Y_{lm}(\theta, \varphi), \]
\[ \xi_2(t, \vec{x}) \pm \frac{i}{\sin \theta} \xi_3(t, \vec{x}) = \]
\[ = \sum_{lm} -l(l+1)[\gamma_{lm}(r) \mp i\delta_{lm}(r)]\pm_1 Y_{lm}(\theta, \varphi). \quad (8b) \]

We will impose the same conditions on the coefficients as in [10]:
\[ \gamma_{lm} = \frac{1}{2} i\omega \gamma_{lm}, \quad \delta_{lm} = j_{lm}, \]
\[ \alpha_{lm} = d_{lm} - i\omega \gamma_{lm}, \quad \beta_{lm} = f_{lm} - \gamma'_{lm} + \gamma_{lm} \frac{w'}{w}. \quad (9) \]

Furthermore, we will introduce an additional set of convenient notations (with tildas) given by the following relations:
\[ a_{lm} = \tilde{a}_{lm} + 2i\omega \alpha_{lm} - \beta_{lm} \frac{w'}{v}, \]
\[ b_{lm} = \tilde{b}_{lm} + i\omega \beta_{lm} + \alpha'_{lm} - \alpha_{lm} \frac{w'}{u}, \]
\[ c_{lm} = \tilde{c}_{lm} + 2\beta'_{lm} - \beta_{lm} \frac{w'}{v}, \]
\[ b_{lm} = \tilde{b}_{lm} + 2i\omega \alpha_{lm} - \beta_{lm} \frac{u'}{v}, \]

\[ e_{lm} = \tilde{e}_{lm} + i\omega \delta_{lm}, \]

\[ g_{lm} = \tilde{g}_{lm} + \delta'_{lm} - \delta_{lm} \frac{w'}{w}, \]

\[ h_{lm} = \tilde{h}_{lm} + \beta_{lm} \frac{w'}{v}, \]

\[ z_{lm} = \tilde{z}_{lm} - \frac{\phi'}{2v} \left( 2f_{lm} - \epsilon'_{lm} + \epsilon_{lm} \frac{w'}{w} \right). \]

(10)

The notations introduced above significantly simplify the future analysis of the perturbations of the equations of motion. Thus, by expanding the equations of motion (2) over the field variations and then separating the terms with different angular dependence (i.e. terms, proportional to \( Y_{lm}(\theta, \varphi) \), \( \pm 1 Y_{lm}(\theta, \varphi) \), ...), one can easily find the correspondent equations for the perturbations. In particular, from expressions for the components \( R_{22}, R_{23} \) and \( R_{33} \) given by the equations (2a) we will obtain the following relations:

\[ \tilde{g}'_{lm} = \tilde{g}_{lm} \left( \frac{v'}{v} - \frac{u'}{u} \right) + i\omega \frac{v}{u} \tilde{e}_{lm}. \]

(11a)

Another equation might be obtained from the expressions for the components \( R_{12} \) and \( R_{13} \) (2a), namely:

\[ \tilde{e}'_{lm} = \tilde{e}_{lm} \frac{w'}{w} + \tilde{g}_{lm} \frac{u}{i\omega w} \left( l(l+1) - 2 - \omega^2 \frac{w}{u} \right) + 2\tilde{s}_{lm} A'_0. \]

(11b)

And finally from the second equation in (2b) one may find the last equation:

\[ \tilde{s}''_{lm} = \tilde{s}'_{lm} \left( \frac{v'}{2v} - \frac{u'}{2u} \right) + \]

\[ + \tilde{s}_{lm} \frac{w}{w} \left( l(l+1) - \omega^2 \frac{w}{u} \right) + \left( \tilde{e}'_{lm} - \tilde{e}_{lm} \frac{w'}{w} - i\omega \tilde{g}_{lm} \right) \frac{A'_0}{u}. \]

(11c)

Thus, we have obtained three independent components of the perturbations. These components are interacting only with each other \([10]\) and hence they have no influence on the other components. This is the trivial consequence of the fact that these components are axial, i.e. when the spatial coordinates are inverted, their transformation rules appears to be \(-(-)^l\) rather then \((-)^l\).

Analogously, the general system for the polar perturbations takes the form:

\[ \tilde{a}'_{lm} = \tilde{a}_{lm} \left( \frac{u'}{2u} + \frac{w'}{2w} \right) + \tilde{b}_{lm} \frac{u}{i\omega w} \left( \frac{l(l+1)}{2} - \omega^2 \frac{w}{u} \right) - \]

\[ - \tilde{c}_{lm} \frac{u'}{2v} + \tilde{h}_{lm} \frac{u}{w} \left( \frac{u'}{2u} - \frac{w'}{2w} \right) + \tilde{k}_{lm} 2A'_0 - \tilde{z}_{lm} u\phi', \]

\[ \tilde{b}'_{lm} = \tilde{b}_{lm} \left( \frac{v'}{2v} - \frac{u'}{2u} \right) + i\omega \left( \tilde{c}_{lm} + \tilde{h}_{lm} \frac{v}{w} \right) + \tilde{n}_{lm} 2A'_0, \]
\[ \tilde{c}_{lm} = \tilde{a}_{lm} \frac{v}{u}, \]

\[ \tilde{h}'_{lm} = \tilde{h}_{lm} \left( \frac{u' + w'}{2w} \right) + \tilde{b}_{lm} \frac{l(l+1)}{2i\omega} + \tilde{c}_{lm} \frac{w'}{2u} + \tilde{z}_{lm} w\phi', \]

\[ \tilde{k}_{lm} = \tilde{a}_{lm} \frac{A'_0}{2u} - \tilde{c}_{lm} \frac{A'_0}{2w} + \tilde{h}_{lm} \frac{A'_0}{w} + \tilde{n}_{lm} \frac{u}{i\omega w} \left( l(l+1) - \omega^2 \frac{w}{u} \right), \]

\[ \tilde{n}'_{lm} = \tilde{n}_{lm} \left( \frac{v'}{2v} - \frac{u'}{2u} \right) + \tilde{k}_{lm} \frac{i\omega v}{u}, \]

\[ \tilde{z}'_{lm} = -\tilde{z}_{lm} \left( \frac{w'}{w} + \frac{u'}{2u} \right) - \frac{\tilde{a}_{lm}}{u\phi'} \left[ \frac{(l(l+1)}{2} - 1 \right] \frac{v}{w} + \frac{A'_0}{2u} + \frac{3u'w'}{4uw} \right] + \frac{\tilde{h}_{lm}}{w\phi'} \left[ \frac{(l(l+1)}{2} - 1 \right] \frac{v}{w} + \frac{u'}{2u} \left( \frac{w'}{2w} - \frac{u'}{2u} \right) + \frac{A'_0}{u} - \frac{\omega^2 v}{u} \right] + \frac{\tilde{b}_{lm}}{uw\phi'} \frac{i}{2\omega} \left( \frac{l(l+1)}{2} u' - \omega^2 w' \right) + \tilde{k}_{lm} \frac{A'_0}{w u\phi'} + \tilde{n}_{lm} \frac{l(l+1)A'_0}{i\omega w u\phi'}. \] (12)

This is the system of independent equations for polar perturbations. All the other equations appear to be a consequence of them.

### 3 The stability against the axial perturbations

In this section we will concentrate on the stability of the solution (3) against the axial perturbations. In order to approach this problem, we must rewrite the system (11) as an eigenproblem with respect to \( \omega^2 \). There exists only one way to combine the first two equations of the system (11) into a single equation of the Shrödinger-type where \( \omega^2 \) playing the role of energy. To show this, let us define the following combination:

\[ C_{lm} = \lambda_1 \tilde{c}_{lm} + \lambda_2 \tilde{g}_{lm}. \]

It is straightforward to check that the equation for \( C''_{lm} \) acquires the form of the Shrödinger equation with that substitution only when \( \lambda_1 = 0 \).

To present the system of equations for \( \tilde{g}_{lm} \) and \( \tilde{s}_{lm} \) in the hermitian form, it convenient to introduce new functions \( \psi_1 \) and \( \psi_2 \) as follows:

\[ \tilde{g}_{lm} = \psi_1 w^{1/2} \left( \frac{v}{u} \right)^{3/4} 8i\omega, \tag{13a} \]

\[ \tilde{s}_{lm} = 4\psi_2 \left( \frac{v}{u} \right)^{1/4} \sqrt{l(l+1) - 2}. \tag{13b} \]

Then the equations for the column \( \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \) takes the form:

\[ \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right)'' + \omega^2 \rho(r) \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) + \left( \begin{array}{cc} d(r) + a(r) & b(r) \\ b(r) & d(r) \end{array} \right) \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) = 0, \tag{14a} \]
where the functions $a(r), b(r), d(r)$ and weight $\rho(r)$ are defined from the relations (3) and given as follows:

$$a(r) = \frac{3\mu^2(1 + 4k^2)}{(r^2 - \mu^2)^2} + \frac{6r}{r^2 - \mu^2} \frac{q(r)'}{q(r)}, \quad (14b)$$

$$b(r) = -\frac{2Q}{q(r)} \sqrt{l(l+1)-\frac{2}{(r^2 - \mu^2)^{3/2}}}, \quad (14c)$$

$$d(r) = -\frac{\mu^2(3 + 16k^2)}{(r^2 - \mu^2)^2} - \frac{6Q^2}{q(r)^2(r^2 - \mu^2)^2} - \frac{l(l+1)}{(r^2 - \mu^2)} - \frac{8r}{r^2 - \mu^2} \frac{q'(r)}{q(r)}, \quad (14d)$$

$$\rho(r) = q(r)^4. \quad (14e)$$

In order to prove the stability of the solution (3) with respect to axial perturbations, following Wald [12] it is necessary to show that: (i) the spectrum of the differential operator (14) is positive, (ii) the differential operator (14) is not only hermitian, but also self-adjoint.

Concerning the eigenvalues of the operator (14), by straightforward computation one can prove that for any $\mu < r < \infty$ both eigenvalues of the potential matrix are positive and then that the eigenvalues of the total operator (14) are also positive.

Now let us analyze the self-adjointness of (14). The boundary conditions at the spatial infinity ($r \to +\infty$) are fixed by the means of the standard procedure:

$$\int dr q(r)^4 \psi^+ \psi < \infty \quad (15)$$

and need no further consideration. However, the condition (15) with $r \to \mu$ permits both possible asymptotics for the function $\psi(r)$:

$$\psi_1(r) = \text{const} \cdot (r - \mu)^{1/2 \pm (s-1)}, \quad (16a)$$

$$\psi_2(r) = \text{const} \cdot (r - \mu)^{1/2 \pm (s-1)/2}, \quad (16b)$$

where $s = |2k|$. By using the condition (15) one might immediately conclude that because of the relation:

$$q(r \to \mu) = \text{const} \cdot (r - \mu)^{-2s},$$

the positive sign in (16a) is forbidden. It means that in order to make the differential operator in (14) self-adjoint, we must impose some reasonable boundary condition that will suppress one of possible asymptotics of $\psi_2$ (16b) for $r \to \mu$. The appropriate restriction appears to be quite natural: to impose the condition of finiteness of the energy of electromagnetic perturbations. For the positive sign in the condition (16b), the energy density is proportional to $(r - \mu)^{-1}$, and the corresponding total energy becomes infinite.

This result completes the proof of the stability of the solution (3) against the axial perturbations with the finite value of the initial energy.
4 The separation of the equations for $\psi_1(r)$ and $\psi_2(r)$

Now the question arises whether the system (14) can be split into two independent equations for some linear combinations of $\psi_1$, $\psi'_1$, $\psi_2$ and $\psi'_2$. Let us obtain the general condition on the coefficients $a$, $b$, $d$ and $\rho$ in (14) which will permit one to say whether $2 \times 2$ system can be split or not.

The weight $\rho$ can be eliminated from the equation (14) using the substitution $r \rightarrow \tilde{r}(r)$. Thus it appears to be sufficient to study the case $\rho = 1$ only. With this restriction one will get the following equation:

$$\psi'' + \omega^2 \psi + \left(\frac{d + a}{b} \frac{b}{d}\right) \psi = 0. \tag{17}$$

Let us suppose that there exists set of coefficients $\eta_1$, $\eta_2$, $\eta_3$ and $\eta_4$ that the linear combination

$$\zeta = \eta_1 \psi_1 + \eta_2 \psi_2 + \eta_3 \psi'_1 + \eta_4 \psi'_2 \tag{18}$$

satisfies the following equation

$$\zeta'' + \omega^2 \zeta + \Omega \zeta = 0 \tag{19}$$

Note that the coefficients in the substitution (18) should not depend on $\omega$, otherwise the problem of the construction of the coefficients $\eta_1$, $\eta_2$, $\eta_3$ and $\eta_4$ becomes trivial. Moreover, the result obtained in this case appears to be practically useless. Indeed, due to the "shadowing" produced by the functions $\eta_1$, $\eta_2$, $\eta_3$ and $\eta_4$, the behavior of function $\zeta$ after the substitution (18) will not be directly connected with the behavior of the initial function $\psi$.

By comparing the equations (17)-(19), and separating the terms proportional to $\psi_1$, $\psi_2$, $\psi'_1$, $\psi'_2$ and $\omega^2$, one can easily find that

$$\eta_3 = \text{const}, \quad \eta_4 = \text{const}. \tag{20}$$

It should be noted that the presence of the arbitrary constants $\eta_3$ and $\eta_4$ corresponds to the orthogonal rotation with the constant coefficients in the $(\psi_1, \psi_2)$ plane performed before the definition of the function $\zeta$ given by (18). Consequently, keeping in mind the possibility of the preliminary constant orthogonal rotation, we can choose $\eta_3 = 1$, $\eta_4 = 0$. Then we can find explicit expressions for $\eta_1$ and $\eta_2$:

$$\eta_1 = -\frac{1}{2} \int b(r) \, dr + \frac{1}{2} \int a(r) \, dr - \frac{\int (b(r) \int a(r) \, dr) \, dr}{2 \int b(r) \, dr}, \tag{21a}$$

$$\eta_2 = \frac{1}{2} \int b(r) \, dr. \tag{21b}$$

And, finally, the last equation of the system might be presented as:

$$a(r) + 2d(r) + \text{const}_1 = \frac{b'}{b} - \frac{b}{2 \int b(r) \, dr} + \left(\int b(r) \, dr\right)^2 + \frac{\text{const}_2}{(\int b(r) \, dr)^2} + \frac{\int [a(r) \int b(r) \, dr] \, dr}{2(\int b(r) \, dr)^2}. \tag{22}$$

This equation is the consistency condition. It means that if it is fulfilled, then the coefficients $\eta_1$, $\eta_2$, $\eta_3$ and $\eta_4$ given by (20)-(21) satisfy the equations (17)-(19) simultaneously. By restoring the weight $\rho$ (14e), we might obtain from (22) the general form of the consistency condition as follows:
\[ a(r) + 2d(r) + \text{const}_1 \cdot q^4(r) = \]
\[ \frac{\sigma''}{\sigma} - \frac{1}{2} \left( \frac{\sigma'}{\sigma} \right)^2 + \rho^2 \sigma^2 + \frac{\text{const}_2}{\sigma^2} + \frac{\int a(r)\sigma(r)dr}{2\sigma^2}, \]

where the function \( \sigma(r) \) is defined by
\[ \sigma(r) = -\frac{1}{2\sqrt{\rho(r)}} \int \frac{b(r)dr}{\sqrt{\rho(r)}} = -\frac{1}{2\rho(r)^{1/2}} \int \frac{b(r)dr}{q^2(r)}. \]

Let us clarify the nature of the equation (23a). It is well-known that any equation of type (14) has a "dual" equation [13]. The simplest way to obtain the dual equation is to rewrite the system (14) as the first-order equations:
\[ \begin{pmatrix} \psi' \\ \chi' \end{pmatrix} = \begin{pmatrix} \hat{M} - (q'/q)\hat{E} & i\omega q^2\hat{E} \\ i\omega q^2\hat{E} & -\hat{M} - (q'/q)\hat{E} \end{pmatrix} \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \]

where \( \hat{M} \) is \( 2 \times 2 \) matrix, \( \hat{E} \) is identity \( 2 \times 2 \) matrix. Note that the matrix \( \hat{M} \) always exists because it can be directly constructed as
\[ \hat{M} = \hat{A}' \hat{A}^{-1}, \]

with the matrix \( \hat{A} \) given by
\[ \hat{A} = \begin{pmatrix} \psi_1^{(1)} & \psi_1^{(2)} \\ \psi_2^{(1)} & \psi_2^{(2)} \end{pmatrix}. \]

where \( \psi_1^{(1)} \) and \( \psi_1^{(2)} \) are two linearly independent solutions of the equation (14). The straightforward calculation permits us to verify that the function \( \psi \) from the equation (24) satisfies the system (14), and
\[ \begin{pmatrix} d + a \\ b \\ d + a \\ b \end{pmatrix} = \left( \hat{M} - (q'/q)\hat{E} \right)^2 + \left( \hat{M} - (q'/q)\hat{E} \right)'. \]

Similarly, the function \( \chi \) is governed by the "dual" equation
\[ \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}'' + \omega^2 q^4 \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} + \begin{pmatrix} \tilde{d} + \tilde{a} \\ \tilde{b} \end{pmatrix} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0, \]

where
\[ \begin{pmatrix} \tilde{d} + \tilde{a} \\ \tilde{b} \end{pmatrix} = \left( \hat{M} + (q'/q)\hat{E} \right)^2 - \left( \hat{M} + (q'/q)\hat{E} \right)'. \]

One can verify that equation (23a) is equivalent to the condition \( \tilde{b} = 0 \). It means that equations (17) can be separated only if the "dual" system has a diagonal form.

The straightforward verification of the consistency of the equation (23a) for the values of \( a, b,d \) and \( \rho \) given by (14b) - (14c) leads to rather complicated calculations. From the other side, by analyzing the asymptotic behavior of condition (23a) in the limit \( r \to \mu \), we might conclude that (even taking into account the preliminary constant orthorgonal rotation in \( (\psi_1, \psi_2) \) plane) equation (23a) can’t be fulfilled in the limit \( r \to \mu \). It leads us to the conclusion that the system (14) is "essentially" two-dimensional and can not be split into two independent equations.
5 Discussion.

We have analyzed the problem of stability of the exact solution of the standard Einstein-Maxwell gravity coupled to an extra free massless scalar field. It was shown that, although the solution (3) contains naked singularities, it is stable at least against axial perturbations. The problem of the stability of this solution against polar perturbations is much harder to analyze. One unexpected complication of these studies is that the differential operator for corresponding $3 \times 3$ eigenproblem appears to be non-Hermitian. However, this research is currently in progress and the obtained results will be reported in a subsequent publication.

Anticipating the possible questions we would like to note that the correspondence of our analysis to the existing results on the perturbations of the Reisner-Nordström solution is not quite straightforward. The reason for this comes from the conclusion that no smooth limit of our analysis exists for $\phi \to 0$ and, although we can reduce our relations to the case $k = \pm 1/2$, the final results will be degenerate at the point $r = \mu$. To show this, one may examine the case $g_{00}/g_{11} \sim (r - \mu)^d$ with $d = 2$ and see that the frequency term in the equation (14a) is influencing the asymptotic behaviour of the function $\psi(r)$ at the vicinity of the surface $r = \mu$. Moreover, the matrix structure of the eigenproblem becomes: $\hat{E} \cdot a(r) + \hat{\sigma}_3 \cdot b(r) \cdot \text{const}_1 + \hat{\sigma}_1 \cdot b(r) \cdot \text{const}_2$, where $\hat{\sigma}_1$ and $\hat{\sigma}_3$ are corresponding Dirac matrices. As a result, the equations for perturbations in the case of the Reisner-Nordström solution appear to be split into the following three groups (reconstructing the already known result [9]): (i) the scalar perturbations, which don’t interact with other perturbations; (ii) the two independent modes of the axial perturbations, where both the gravity and the Maxwell field are mixed together, and (iii) the two independent modes of the polar perturbations, which also contain a mixture of gravitational and electromagnetic fields.

Concluding this part, we would like to note that the stability of the general solution which contains the naked singularities is well fit to the scenario proposed in the multi-dimentional extensions of the general relativity. Thus, under the certain circumstances, the Kaluza-Klein vacuum may decay by endlessly producing naked singularities. This process from the five-dimensional point of view corresponds to Witten’s “bubbles of nothing” which must eventually collide [5], and so in four-dimensions the singularities will coalesce. However, it should be emphasized that we have explored just the four-dimensional solutions and the further analysis of this problem should include the non-trivial coupling of the scalar field in the higher dimensions.

The proven stability of the exterior static solution (3), makes it interesting to study whether the scalar and electromagnetic fields might be detected through the space gravitational experiments. Thus, following the standard procedure of the PPN formalism [14], one will find that only the parameter $\beta$ deviates from its general relativistic value, namely

$$\beta = 1 + \frac{\gamma Q^2}{c^2 2\mu_0^2}$$

where $\mu_0$ is the Newtonian mass of the source and we have restored the dimensional constants $\gamma$ and $c$. This result coincide with one for the Reissner-Nordström solution and as long as the metric (3) in post-Newtonian limit doesn’t contains parameter $k$, the scalar field (defined by the action (1)), can not be detected from the data processing of the modern relativistic celestial mechanical experiments.

The second term in the expression (28) represents the ratio of the electrostatic energy contribution in the gravitational field produced by the same charged massive body. The presence of this term might lead to an observable discrepancy in the motion of the celestial bodies. For example, it will contribute to the Nordtvedt effect, which was extensively studied in the Moon’s motion [14]. In the

\footnote{For the general case with non-zero scalar field ($k \neq \pm 1/2$ and $|O| \neq \mu_0$), as we saw, $d < 2$ and this influence is absent.}
recent analysis of data obtained in Lunar Laser Ranging which was carried out to detect the Nordtvedt effect, a very tight [13] limitation on the parameterized post-Newtonian parameter $\beta$ was obtained:

$$\beta = 0.9999 \pm 0.0006$$  \hspace{1cm} (29)

This result suggests that within the accuracy one part in ten thousand, the contribution of the ratios of electrostatic to self-graviational energies presented by (29) for both Moon and Earth is negligible small.

Since gravity attracts positive and negative charges equally, then the matter accreted on a massive astrophysical object will be nearly neutral. For the case of the celestial bodies with the gravitational energy dominating over the electromagnetic one, the parameter $\beta$ might be presented as:

$$\beta = 1 + \frac{\gamma_0}{c^2} \frac{Q^2}{4n^2M_\odot^2}. \hspace{1cm} (30)$$

where the mass of the star $\mu_0$ was expressed in terms of the solar masses $M_\odot$: $\mu_0 = n M_\odot$. The constraints imposed on new weak forces from the behavior of the astrophysical objects for the maximum possible electric charge $Q_{\text{max}}$ carried by celestial bodies the following estimation: $Q_{\text{max}} \leq 10^{36} e$ [16]. This gives the following estimation for the electrostatic energy contribution in the parametrized post-Newtonian parameter $\beta$:

$$\Delta \beta = \beta - 1 \leq \frac{2.17}{n^2} \times 10^{-7}. \hspace{1cm} (31)$$

Unfortunately, even with $n = 1$ this result gives practically unmeasurable value for the contribution of the electrostatic energy of the charged astrophysical body to the generated gravitational field. According to this result, the detection of the electrostatic field contribution in the relativistic celestial mechanics experiments performed in the weak gravitational field is presently impossible.

Thus, we have shown that the influence of both the electromagnetic and the scalar fields (given by the action (1)) on the motion of the asprophysical bodies is practically unmeasureable in modern gravitational experiments. However, a wide class of multi-dimensional theories of gravity with an arbitrary number of massless scalar fields coupled to the usual tensor gravitational field has been recently considered in [17]. This investigation was performed in order to analyze the cosmological consequences of an inclusion of the multi-scalar field terms in the theory. As a result, the authors of this paper illustrated that although these theories might have coinciding post-Newtonian limits with general relativity, they predict non-Einsteinian behavior of the stellar objects in a strong gravitational field. In particular, it was noted that this discrepancy will lead to observable effects, for example, for the binary pulsars. This result makes it specifically interesting to study the multi-scalar field extensions of the general Einstein-Maxwell-scalar model together with the condition to meet the experimental constraints based on the tests of general relativity performed to date.

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References

[1] T. Damour and A. M. Polyakov, *GRG*, Vol.26 (1994) 1171;

T. Damour, G. W. Gibbons and G. Gundlach, *Phys. Rev. Lett.* 64(2) (1990) 123
[2] G. W. Gibbons, Nucl. Phys. B207 (1982) 337;
G. W. Gibbons and K. Maeda, Nucl. Phys. B298 (1988) 741;
D. Garfinke, G. T. Horowitz and A. Strominger, Phys. Rev. D43 (1991) 314;
J. H. Horne and G. T. Horowitz, Phys. Rev. D46 (1992) 134;
S. J. Poletti and D. L. Wilshire, Phys. Rev. D50, 7260 (1994).

[3] P. Silaev, Theor. Math. Fiz. (in Russian) 91 (1989) 418

[4] C. F. E. Holzhey and F. Wilczek, Nucl. Phys. B380 (1992) 447

[5] E. Witten, Nucl. Phys. B443 (1995) 85

[6] G. W. Gibbons, G. T. Horowitz and P. K. Townsend, Class. Quantum Grav. 12 (1995) 297

[7] J. H. Horne and G. T. Horowitz, Phys. Rev. D46 (1992) 134;
S. J. Poletti and D. L. Wilshire, Phys. Rev. D50, 7260 (1994).

[8] Ph. Jetzer and D. Scialom, Phys. Lett. 169A (1992) 12

[9] S. Chandrasekhar, The mathematical theory of black holes (Claredon Press, Oxford, 1963)

[10] T. Regge, J. A. Wheeler, Phys. Rev. 108 (1957) 1063;
F. J. Zerilli, Phys. Rev. Lett. 24 (1970) 737
F. J. Zerilli, Phys. Rev. D2 (1970) 2141;
L. A. Edelstein, C. V. Vishveshwara, Phys. Rev. D1 (1970) 3514

[11] E. T. Newman, R. Penrose, JMP 7 (1966) 863;
J. N. Goldberg et al., JMP 8, (1967) 2155

[12] R. M. Wald, JMP 20 (1979) 1056;
C. V. Vishveshwara, Phys. Rev. D1 (1970) 2870

[13] S. Chandrasekhar, Proc. Roy. Soc. A343 (1975) 289

[14] K. Nordtvedt, Phys. Rev. D7 (1973) 2347;
C. M. Will, Theory and Experiment in Gravitational Physics (Cambridge Univ. Press, Cambridge 1993)

[15] J. O. Dickey et al., Science, 265 (1994) 482

[16] D. E. Krause, H. T. Kloor and E. Fischbach, Phys. Rev. D49 (1994) 6892

[17] T. Damour, and G. Esposito-Far'ese, Class. Quantum Grav. 9 (1992) 2093;
T. Damour and J. H. Taylor, Phys. Rev. D45 (1992) 1840;
T. Damour, K. Nordtvedt, Phys. Rev. Lett. 70 (1993) 2217; Phys. Rev. D48 (1993) 3436;
A. L. Berkin and R.W. Hellings, Phys. Rev. D49 (1994) 6442