NON-SINGULAR, VACUUM, STATIONARY SPACE-TIMES
WITH A NEGATIVE COSMOLOGICAL CONSTANT

PIOTR T. CHRUŚCIEL AND ERWANN DELAY

ABSTRACT. We construct infinite dimensional families of non-singular stationary space-times, solutions of the vacuum Einstein equations with a negative cosmological constant.

Contents

1. Introduction 1
2. Definitions, notations and conventions 3
3. Isomorphism theorems 4
  3.1. An isomorphism on two-tensors 4
  3.2. Two isomorphisms on one-forms 5
  3.3. An isomorphism on functions in dimension n 6
  3.4. An isomorphism on functions in dimension 3 6
4. The equations 8
  4.1. The linearised equation 9
  4.2. The modified equation 9
5. The construction 12
  5.1. The \( n \)-dimensional case 12
  5.2. The three-dimensional case 14
6. Uniqueness 15
7. Polyhomogeneity 15
Appendix A. “Dimensional reduction” of some operators 17
  A.1. Lichnerowicz Laplacian on two-tensors for a warped product metric 17
  A.2. The Laplacian on one-forms for a warped product metric 18
References 19

1. Introduction

A class of space-times of interest is that of vacuum metrics with a negative cosmological constant admitting a smooth conformal completion at infinity. It is natural to seek for stationary solutions with this property. In this paper we show that a large class of such solutions can be constructed by prescribing the conformal class of a stationary Lorentzian metric on the conformal boundary \( \partial \mathcal{M} \), provided that the boundary data are sufficiently close to, e.g., those of anti-de Sitter space-time.

Date: November 15, 2018.
We mention the recent papers [3, 4], where we have constructed infinite dimensional families of static, singularity free solutions of the vacuum Einstein equations with a negative cosmological constant. The main point of the current work is to remove the staticity restriction. This leads to new, infinite dimensional families of non-singular, stationary solutions of those equations.

We thus seek to construct Lorentzian metrics $g$ in any space-dimension $n \geq 2$, with Killing vector $X = \partial/\partial t$. In adapted coordinates those metrics can be written as

\begin{equation}
(1.1) \quad n+1 g = -V^2 (dt + \theta dx^i)^2 + g_{ij} dx^i dx^j,
\end{equation}

\begin{equation}
(1.2) \quad \partial_t V = \partial_t \theta = \partial_t g = 0.
\end{equation}

Our main result reads as follows (see below for the definition of non-degeneracy; the function $\rho$ in (1.3) is a coordinate near $\partial M$ that vanishes at $\partial M$):

**Theorem 1.1.** Let $n = \dim M \geq 2$, $k \in \mathbb{N} \setminus \{0\}$, $\alpha \in (0,1)$, and consider a static Lorentzian Einstein metric of the form (1.1)-(1.2) with strictly positive $V = \hat{V}$, $g = \hat{g}$, and $\theta = 0$, such that the associated Riemannian metric $\hat{g} = V^2 d\varphi^2 + \hat{g}$ on $S^1 \times M$ is $C^2$ compactifiable and non-degenerate, with smooth conformal infinity. For every smooth $\hat{\theta}$, sufficiently close to zero in $C^{k+2,\alpha}(\partial M, T_1)$, there exists a unique, modulo diffeomorphisms which are the identity at the boundary, nearby stationary vacuum metric of the form (1.1)-(1.2) such that, in local coordinates near the conformal boundary $\partial M$,

\begin{equation}
(1.3) \quad \hat{V} - \tilde{V} = O(\rho), \quad \theta_i = \tilde{\theta}_i + O(\rho), \quad g_{ij} - \tilde{g}_{ij} = O(1).
\end{equation}

Theorem 1.1 is more or less a rewording of Theorem 5.3 below, taking into account the discussion of uniqueness in Section 6. The $(n + 1)$-dimensional anti-de Sitter metric is non-degenerate in the sense above, so Theorem 1.1 provides in particular an infinite dimensional family of solutions near that metric.

The requirement of strict positivity of $\hat{V}$ excludes black hole solutions, it would be of interest to remove this condition.

The decay rates in (1.3) have to be compared with the leading order behavior $\rho^{-2}$ both for $\hat{V}^2$ and $\hat{g}_{ij}$. A precise version of (1.3) in terms of weighted function spaces (as defined below) reads

\begin{equation}
(1.4) \quad (V - \hat{V}) \in C^{k+2,\alpha}_1(S^1 \times M), \quad (g - \hat{g}) \in C^{k+2,\alpha}_2(S^1 \times M, S_2),
\end{equation}

\begin{equation}
(1.5) \quad \theta - \tilde{\theta} \in C^{k+2,\alpha}_2(S^1 \times M, T_1),
\end{equation}

and the norms of the differences above are small in those spaces.

Note that our hypothesis that the metric $\hat{g}$ is conformally $C^2$ implies that $\hat{g}$ is $C^{n-1,\alpha} \cap C^{3,\alpha}$-conformally compactifiable and polyhomogeneous [7]. We show in Section 7 that our solutions have complete polyhomogeneous expansions near the conformal boundary, see Theorem 7.1 for a precise statement. Since the Fefferman-Graham expansions are valid regardless of the signature of the boundary metric, the solutions are smooth in even space-time dimensions. In odd space-time dimensions the obstruction to smoothness
is the non-vanishing of the Fefferman-Graham obstruction tensor \([10, 11]\) of the (Lorentzian) metric obtained by restricting \(-(dt + \theta)^2 + V^{-2}g\) to the conformal boundary at infinity.

Theorem 1.1 is proved by an implicit-function argument. This requires the proof of isomorphism properties of an associated linearised operator. This operator turns out to be rather complicated, its mapping properties being far from evident. We overcome this by reinterpreting this operator as the Lichnerowicz operator \(\tilde{\Delta}_L + 2n\) in one-dimension higher. Our non-degeneracy condition above is then precisely the condition that \(\tilde{\Delta}_L + 2n\) has no \(L^2\)-kernel. While this is certainly a restrictive condition, large classes of Einstein metrics satisfying this condition are known \([1, 2, 4, 13]\).

2. Definitions, notations and conventions

Let \(\overline{N}\) be a smooth, compact \((n+1)\)-dimensional manifold with boundary \(\partial N\). Set \(N := \overline{N} \setminus \partial N\), \(N\) is thus a non-compact manifold without boundary. In our context the boundary \(\partial N\) will play the role of a boundary at infinity of \(N\). Let \(g\) be a Riemannian metric on \(N\), we say that \((N, g)\) is conformally compact if there exists on \(\overline{N}\) a smooth defining function \(\rho\) for \(\partial N\) (that is \(\rho \in C^\infty(\overline{N})\), \(\rho > 0\) on \(N\), \(\rho = 0\) on \(\partial N\) and \(d\rho\) nowhere vanishing on \(\partial N\)) such that \(\overline{g} := \rho^2g\) is a \(C^{2,\alpha}(\overline{N}) \cap C^\infty_0(N)\) Riemannian metric on \(\overline{N}\), we will denote by \(\tilde{g}\) the metric induced on \(\partial N\). Our definitions of function spaces follow \([13]\). Now if \(|d\rho|_{\overline{g}} = 1\) on \(\partial N\), it is well known (see \([14]\) for instance) that \(g\) has asymptotically sectional curvature \(-1\) near its boundary at infinity, in that case we say that \((N, g)\) is asymptotically hyperbolic. If we assume moreover than \((N, g)\) is Einstein, then asymptotic hyperbolicity enforces the normalisation

\[
(2.1) \quad \text{Ric}(g) = -ng ,
\]

where \(\text{Ric}(g)\) is the Ricci curvature of \(g\).

We recall that the Lichnerowicz Laplacian acting on a symmetric two-tensor field is defined as \([6, \S \text{1.143}]\)

\[
\Delta_L h_{ij} = -\nabla^k \nabla_k h_{ij} + R_{ik} h^{k}{}_{j} + R_{jk} h^{k}{}_{i} - 2R_{ijkj} h^{kl} .
\]

The operator \(\Delta_L + 2n\) arises naturally when linearising (2.1). We will say that \(g\) is non-degenerate if \(\Delta_L + 2n\) has no \(L^2\)-kernel.

While we seek to construct metrics of the form (1.1), for the purpose of the proofs we will often work with manifolds \(N\) of the form

\[
N = S^1 \times M,
\]
equipped with a warped product, asymptotically hyperbolic metric

\[
V^2 d\varphi^2 + g ,
\]

where \(V\) is a positive function on \(M\) and \(g\) is a Riemannian metric on \(M\). By an abuse of terminology, such metrics will be said static.

The basic example of a non-degenerate, asymptotically hyperbolic, static Einstein space is the Riemannian counterpart of the AdS space-time. In that case \(M\) is the unit ball of \(\mathbb{R}^n\), with the hyperbolic metric

\[
g_0 = \rho^{-2} \delta ,
\]
$\delta$ is the Euclidean metric, $\rho(x) = \frac{1}{2}(1 - |x|^2)$, and
$$V_0 = \rho^{-1} - 1.$$ 

We denote by $\mathcal{T}^q_p$ the set of rank $p$ covariant and rank $q$ contravariant tensors. When $p = 2$ and $q = 0$, we denote by $\mathcal{S}_2$ the subset of symmetric tensors. We use the summation convention, indices are lowered and raised with $g_{ij}$ and its inverse $g^{ij}$.

3. ISOMORPHISM THEOREMS

Some of the isomorphism theorems we will use are consequence s of Lee’s theorems [13], it is therefore convenient to follow his notation for the weighted Hölder spaces $C^{k,\alpha}_\delta$. As described in the second paragraph before proposition B of [13], a tensor in this space corresponds to $\rho^\delta$ times a tensor in the usual $C^{k,\alpha}$ space as defined using the norm of the conformally compact metric. This implies that, in local coordinates near the conformal boundary, a function in $C^{k,\alpha}_\delta$ is $O(\rho^\delta)$, a one-form in $C^{k,\alpha}_\delta$ has components which are $O(\rho^{\delta-1})$, and a covariant two-tensor in $C^{k,\alpha}_\delta$ has components which are $O(\rho^{\delta-2})$.

We will use the isomorphism theorems of [13] in weighted $C^{k,\alpha}$ spaces, for $k \in \mathbb{N}$. Under the regularity conditions on the metric in our definition of asymptotically hyperbolic metric, those theorems apparently only apply to low values of $k$. However, under our hypotheses, one can use those theorems for $k = 2$, and use scaling estimates to obtain the conclusion for any value of $k$.

3.1. An isomorphism on two-tensors. We first recall a result of Lee (see Theorem C(c) and proposition D of [13], there is no $L^2$-kernel here by hypothesis):

**Theorem 3.1.** Let $\mathbb{S}^1 \times M$ be equipped with a non-degenerate asymptotically hyperbolic metric $\tilde{g}$. For $k \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\delta \in (0, n)$ the operator $\tilde{\Delta}_{L+2n}$ is an isomorphism from $C^{k+2,\alpha}_\delta(\mathbb{S}^1 \times M, \mathcal{S}_2)$ to $C^{k,\alpha}_\delta(\mathbb{S}^1 \times M, \mathcal{S}_2)$.

When the metric is static of the form $\tilde{g} = V^2 d\varphi^2 + g$ we deduce

**Corollary 3.2.** On $(M, g)$ we consider the operator

$$(W, h) \mapsto (l(W, h), L(W, h)),$$

where

$$l(W, h) = V[(\nabla^* \nabla + 2n + V^{-1} \nabla^* \nabla V + V^{-2} |dV|^2)W + V^{-1} \nabla_j V \nabla^j W - V^{-1} \nabla_j V \nabla^k V h_{kj} + \langle \text{Hess}_g V, h \rangle_g].$$
and

\[ L_{ij}(W, h) = \frac{1}{2} \Delta_L h_{ij} + nh_{ij} - \frac{1}{2} V^{-1} \nabla^k V \nabla_k h_{ij} + \frac{1}{2} V^{-2}(\nabla_i V \nabla^k V h_{kj} + \nabla_j V \nabla^k V h_{ki}) - \frac{1}{2} V^{-1}(\nabla_i \nabla^k V h_{kj} + \nabla_j \nabla^k V h_{ki}) + 2V^{-2} W(\text{Hess}_g V)_{ij} - 2V^{-3} \nabla_i V \nabla_j V W. \]

Then \((l, L)\) is an isomorphism from \(C^{k+2,\alpha}_{\delta-1}(M) \times C^{k+2,\alpha}_{\delta}(M, S_2)\) to \(C_k^{k,\alpha}_{\delta-2}(M) \times C_k^{k,\alpha}_{\delta}(M, S_2)\) when \(\delta \in (0, n)\).

**Proof.** First, it is easy to see that the Laplacian commutes with the Lie derivative operator in the Killing direction, so the operator \(\widetilde{\Delta}_L + 2n\) restricted to \(\phi\)-independent tensor field is again an isomorphism. Now, from Lemma A.2 below, if we define \(P\) to be the set of symmetric covariant two tensors of the form

\[ \tilde{h} = 2V W d\phi^2 + h_{ij} dx^i dx^j, \]

and if we let \(T\) denote the collection of tensors of the form

\[ \tilde{h} = 2\xi_i dx^i d\phi, \]

then the Lichnerowicz Laplacian preserves the decomposition \(P \oplus T\). In particular the operator \(\frac{1}{2}\widetilde{\Delta}_L + n\) restricted to \(P\) is an isomorphism, and this operator is \((l, L)\). \(\square\)

### 3.2. Two isomorphisms on one-forms

The proof of Corollary 3.2 also shows the following (note a shift in the rates of decay, as compared to the previous section, due to the fact that a tensor field \(\xi_i dx^i d\phi\) is in \(C^{m,\sigma}_\rho\) if and only if the one-form \(\xi_i dx^i\) is in \(C^{m,\sigma}_{\rho-1}\):)

**Corollary 3.3.** The operator on one-forms defined as

\[ L : \xi_i \mapsto -\nabla^k \nabla_k \xi_i + V^{-1} \nabla^k V \nabla_k \xi_i + 3V^{-2} \nabla_i V \nabla^k V \xi_k + R^l_i \xi_l - 3V^{-1} \nabla_i \nabla_j V \xi_j + 2n \xi_i, \]

is an isomorphism from \(C^{k+2,\alpha}_{\delta-1}(M, T_1)\) to \(C^{k,\alpha}_{\delta-1}(M, T_1)\) when \(\delta \in (0, n)\). If we let \(\xi = V^2 \theta\), we therefore obtain that the operator \(Q\) on one-forms defined as \(V^{-2} L(V^2 \theta_i)\)

\[ Q : \theta_i \mapsto -\nabla^k \nabla_k \theta_i - 3V^{-1} \nabla^k V \nabla_k \theta_i - 2V^{-1} \nabla^k V \nabla_k \theta_i + 3V^{-2} \nabla_i V \nabla^k V \theta_k + R^l_i \theta_l - 3V^{-1} \nabla_i \nabla_j V \theta_j + 2n \theta_i, \]

is an isomorphism from \(C^{k+2,\alpha}_{\delta+1}(M, T_1)\) to \(C^{k,\alpha}_{\delta+1}(M, T_1)\) when \(\delta \in (0, n)\).

We will appeal to yet another result of Lee (see [13] Theorem C(c), Proposition F and Corollary 7.4, there is again no \(L^2\)-kernel here because of the Ricci curvature condition):
Theorem 3.4. On $S^1 \times M$ equipped with an asymptotically hyperbolic metric $	ilde{g}$ with negative Ricci curvature, the operator $\tilde{\nabla}^* \tilde{\nabla} - \text{Ric}$ acting on one-forms is an isomorphism from $C^{k+2,\alpha}_{-\delta}(S^1 \times M, T_1)$ to $C^{k,\alpha}_{-\delta}(S^1 \times M, T_1)$ when $|\delta - \frac{n}{2}| < \sqrt{\frac{n^2}{4} + 1}$.

When the metric is static of the form $\tilde{g} = V^2 d\varphi^2 + g$ we deduce:

Corollary 3.5. Under the hypotheses of the preceding theorem, on $(M, g)$ consider the operator

$\Omega_i \mapsto B(\Omega)_i + R_{ij} \Omega^j - V^{-1} \nabla_i \nabla^j V \Omega_j =: B(\Omega)_i,$

where

$B(\Omega)_i := \nabla^k \nabla_k \Omega_i + V^{-1} \nabla^k V \nabla_k \Omega_i - V^{-2} \nabla_i \nabla^k V \nabla_k \Omega_i.$

Then $B$ is an isomorphism from $C^{k+2,\alpha}_{-\delta}(M, T_1)$ to $C^{k,\alpha}_{-\delta}(M, T_1)$ when $|\delta - \frac{n}{2}| < \sqrt{\frac{n^2}{4} + 1}$.

Proof. The argument is identical to the proof of Corollary 3.2 using Lemma A.3 and the fact that, in the notation of Lemma A.3,

$\tilde{R}_{ic} \tilde{\Omega}^c = R_{ij} \Omega^j - V^{-1} \nabla_i \nabla^j V \Omega_j.$

3.3. An isomorphism on functions in dimension $n$. If we assume that $V^2 d\varphi^2 + g$ is a static asymptotically hyperbolic metric on $S^1 \times M$, then it is easy to check that at infinity $V^{-2} |dV|^2 = 1$ and $V^{-1} \nabla_i \nabla V = n$. In dimension $n$, we will need an isomorphism property for the following operator acting on functions:

$\sigma \mapsto T \sigma := V^{-3} \nabla_i (V^3 \nabla_i \sigma) = \nabla_i \nabla_i \sigma + 3V^{-1} \nabla^i V \nabla_i \sigma.$

From [5, Theorem 7.2.1 (ii) and Remark (i), p. 77] we obtain:

Theorem 3.6. Let $(V, g)$ be close in $C^{k+2,\alpha}_{-\delta}(M) \times C^{k+2,\alpha}_{-\delta}(M, S_2)$ to an asymptotically hyperbolic static metric. Then $T$ is an isomorphism from $C^{k+2,\alpha}_{-\delta}(M)$ to $C^{k,\alpha}_{-\delta}(M)$ when $0 < \delta < n + 2$.

Remark 3.7. Theorem 3.6 will be used with $\sigma = O(\rho^2)$, note that $\delta = 2$ verifies the inequality above since $n \geq 2$.

3.4. An isomorphism on functions in dimension 3. In dimension $n = 3$, we will also be interested in the following operator acting on functions:

$\omega \mapsto Z \omega := V^3 \nabla_i (V^{-3} \nabla_i \omega) = \nabla^i \nabla_i \omega - 3V^{-1} \nabla^i V \nabla_i \omega.$

The indicial exponents for this equation are $\mu_- = -1$ and $\mu_+ = 0$ (see [5, Remark (i), p. 77]). As $\mu_+ \neq 0$ we cannot invoke [5, Theorem 7.2.1] to conclude. Instead we appeal to the results of Lee [13]. For this we need to have a formally self-adjoint operator, so we set $\omega = V^{\frac{3}{2}} f$, thus

$Z \omega = V^{\frac{3}{2}} \left[ \nabla^i \nabla_i f - \left( \frac{15}{4} V^{-2} |dV|^2 - \frac{3}{2} V^{-1} \nabla^i V \nabla_i V \right) f \right] =: V^{\frac{3}{2}} Z f.$
At infinity $V^{-2}|dV|^2 = 1$ and $V^{-1}\nabla^i \nabla_i V = 3$, leading to the following indicial exponents
\[ \delta = \frac{1}{2}, \frac{3}{2}. \]

We want to show that $Z$ satisfies condition (1.4) of [13],
\[ \|u\|_{L^2} \leq C \|Z u\|_{L^2}, \tag{3.2} \]
for smooth $u$ compactly supported in a sufficiently small open set $\mathcal{U} \subset M$ such that $\overline{\mathcal{U}}$ is a neighborhood of $\partial M$. We will need the following, well known result; we give the proof for completeness:

**Lemma 3.8.** On an asymptotically hyperbolic manifold $(M, g)$ with boundary defining function $\rho$ we have, for all compactly supported $C^2$ functions,
\[ \int u \nabla^* \nabla u \geq \left( \frac{n-1}{2} \right)^2 \int (1 + O(\rho)) u^2. \]

**Proof.** Let $f$ be a smooth function to be chosen later, then
\[ \int |f^{-1} d(f u)|^2 = \int |du|^2 + f^{-2} |df|^2 u^2 + 2 f^{-1} u \langle df, du \rangle \geq 0 \]
An integration by parts shows that
\[ \int 2 f^{-1} u \langle df, du \rangle = \int u^2 f^{-2} |df|^2 + u^2 f^{-1} \nabla^* \nabla f. \]
This leads to
\[ \int u \nabla^* \nabla u = \int |du|^2 \geq \int (- f^{-1} \nabla^* \nabla f - 2 f^{-2} |df|^2) u^2. \]
When $f = \rho^{-n-1}$ the last term equals $(n-1)^2 \|u(1 + O(\rho))\|^2_{L^2}/4$, which concludes the proof. \qed

Lemma 3.8 combined with the fact that $V^{-2}|dV|^2 = 1 + O(\rho)$ and that $V^{-1}\nabla^* \nabla V = -3 + O(\rho)$ shows that
\[ \|u\|_{L^2} \|Z u\|_{L^2} \geq - \int u Z u \geq \left( \frac{(3-1)^2}{4} + \frac{15}{4} - \frac{9}{2} \right) \int (1 + O(\rho)) u^2, \]
which shows that $Z$ satisfies the condition (3.2) with
\[ C = \left( \frac{(3-1)^2}{4} + \frac{15}{4} - \frac{9}{2} \right)^{-1/2} = 2. \]
We recall that the critical weight to be in $L^2$ is $O(\rho^1)$ so the function $f = V^{-3/2} = O(\rho^{3/2})$, corresponding to $\omega = 1$, is in the $L^2$-kernel of $Z$. We prove now that this kernel equals
\[ \ker Z = V^{-3/2} \mathbb{R}. \]
Assume $f$ is in the $L^2$-kernel of $Z$, by elliptic regularity $f$ is smooth on $M$. Let $\varphi_k \in W^{1,\infty}$ be any function on $M$ such that $\varphi_k = 1$ on the geodesic ball $B_p(k)$ of radius $k$ centred at $p$, with $\varphi_k = 0$ on $M \setminus B_p(k+1)$, and $|\nabla \varphi_k| \leq C$ independently of $k$. Such functions can be constructed by composing the
geodesic distance from $p$ with a test function on $\mathbb{R}$. Integrating by parts one has

\[ 0 = - \int V^3 \varphi_k^2 f Z f = - \int \varphi_k^2 f \nabla^i (V^{-3} \nabla_i f) \]
\[ = \int \varphi_k^2 V^{-3} |\nabla f|^2 + 2V^{-3} f \varphi_k \nabla_i \varphi_k \nabla_i f \]

Using Hölder’s inequality, the second integral can be estimated from below by

\[ -2 \left( \int \varphi_k^2 V^{-3} |\nabla f|^2 \right)^{1/2} \left( \int f^2 V^{-3} |\nabla \varphi_k|^2 \right)^{1/2} , \]

leading to

\[ \int \varphi_k^2 V^{-3} |\nabla f|^2 \leq 4 \int f^2 V^{-3} |\nabla \varphi_k|^2 . \]

By Lebesgue’s dominated convergence theorem, the right-hand side converges to zero as $k$ tends to infinity because $f \in L^2$, while $V^{-1}$ is uniformly bounded, and $\nabla \varphi_k$ is supported in $B_p(k + 1) \setminus B_p(k)$. So $f$ is a constant.

Using [13], Theorem C(c), we thus obtain

**Theorem 3.9.** Let $(V, g)$ be close in $C^{k+2,\alpha}_{-1}(M) \times C^{k+2,\alpha}_0(M, S^2)$ to an asymptotically hyperbolic static metric. Then $Z$ is an isomorphism from $C^{k,\alpha}_\delta(M)/V^{-3/2}\mathbb{R}$ to

\[ \left\{ f \in C^{k,\alpha}_\delta(M) : \int_M V^{-3/2} f = 0 \right\} . \]

when $1/2 < \delta < \frac{3}{2}$. Equivalently, $Z$ is an isomorphism from $C^{k+2,\alpha}_\delta(M)/\mathbb{R}$ to

\[ (3.3) \left\{ f \in C^{k,\alpha}_\delta(M) : \int_M V^{-3} f = 0 \right\} . \]

when $-1 < \delta < 0$.

**4. The equations**

Rescaling the metric to achieve a convenient normalisation of the cosmological constant, the vacuum Einstein equations for a metric satisfying (1.1)-(1.2) read (see, e.g., [9])

\[
\begin{cases}
V(\nabla^* \nabla V + nV) = \frac{1}{4} |\lambda|^2 , \\
\text{Ric}(g) + ng - V^{-1} \text{Hess}_g V = \frac{1}{2V} \lambda \circ \lambda , \\
\text{div}(V \lambda) = 0 ,
\end{cases}
\]

where

\[ \lambda_{ij} = -V^2 (\partial_i \theta_j - \partial_j \theta_i) , \quad (\lambda \circ \lambda)_{ij} = \lambda_{ij} \lambda_{kl} . \]

In dimension $n = 3$ an alternative set of equations can be obtained by introducing the *twist potential* $\omega$. Writing $d\omega = \omega_i dx^i$ one sets

\[ \omega_i = \frac{V}{2} \varepsilon_{ijk} \lambda^j_k \quad \iff \quad \lambda^j_k = \frac{1}{V} \varepsilon_{jk\ell} \omega^\ell . \]
This leads to (compare [12])
\[
\begin{align*}
V(\nabla^* \nabla V + 3V) &= \frac{1}{2\pi^2} |d\omega|^2, \\
\text{Ric}(g) + 3g - V^{-1} \text{Hess}_g V &= \frac{1}{2\pi^2} (d\omega \otimes d\omega - |d\omega|^2 g), \quad (4.2) \\
\nabla^*(V^{-3} \nabla \omega) &= 0.
\end{align*}
\]

4.1. The linearised equation. We first consider the operator from the set of functions times symmetric two tensor fields to itself, defined as
\[
\begin{pmatrix} V \\ g \end{pmatrix} \mapsto \begin{pmatrix} V(\nabla^* \nabla V + nV) \\ \text{Ric}(g) + ng - V^{-1} \text{Hess}_g V \end{pmatrix}.
\]

The two components of its linearisation at \((V, g)\) are
\[
p(W, h) = V \left[ (\nabla^* \nabla + 2n + V^{-1} \nabla^* \nabla)V \right] W + (\text{Hess}_g V, h)_g - (\text{div grav}, dV)_g,
\]
\[
P_{ij}(W, h) = \frac{1}{2} \Delta_L h_{ij} + nh_{ij} + \frac{1}{2} V^{-1} \nabla^k V (\nabla_i h_{kj} + \nabla_j h_{ki} - \nabla_k h_{ij}) - (\text{div}^* \text{div grav} h)_{ij} + V^{-2} W (\text{Hess}_g V)_{ij} - V^{-1} (\text{Hess}_g W)_{ij}.
\]

We let \(\text{Tr}\) denote the trace and we set
\[
\text{grav} = h - \frac{1}{2} \text{Tr}_g h g, \quad (\text{div} h)_i = -\nabla^k h_{ik}, \quad (\text{div}^* w)_{ij} = \frac{1}{2} (\nabla_i w_j + \nabla_j w_i),
\]
(note the geometers’ convention to include a minus in the definition of divergence). It turns out to be convenient to introduce the one-form
\[
w_j = V^{-1} \nabla^k V h_{kj} + \nabla^k h_{kj} - \frac{1}{2} \nabla_j (\text{Tr} h) - V^{-1} \nabla_j W - V^{-2} \nabla_j \nabla W,
\]
which allows us to rewrite \(P(W, h)\) as
\[
P(W, h) = L(W, h) + \text{div}^* w,
\]
where \(L\) is as in Corollary 3.2. Similarly, \(p(W, h)\) can be rewritten as
\[
p(W, h) = l(W, h) + V (w, dV)_g.
\]

4.2. The modified equation. We want to use the implicit function theorem to construct our solutions. As is well known, the linearisation of the Ricci tensor does not lead to well behaved equations, and one adds “gauge fixing terms” to take care of this problem. Our choice of those terms arises from harmonic coordinates for the vacuum Einstein equations in one dimension higher.

In dimension 3, we start by solving the following system of equations
\[
\begin{align*}
q(V, g) &:= V(\nabla^* \nabla V + 3V + \langle \Omega, dV \rangle) - \frac{1}{2\pi^2} |d\omega|^2 = 0, \\
Q(V, g) &:= \text{Ric}(g) + 3g - V^{-1} \text{Hess}_g V + \text{div}^* \Omega \\
&= \frac{1}{2\pi^2} (d\omega \otimes d\omega - |d\omega|^2 g) = 0, \\
\nabla^*(V^{-3} \nabla \omega) &= 0,
\end{align*}
\]
with
\[
-\Omega_j = -\langle \Omega(V, g, U, b) \rangle_j \\
:= \hat{g}_{j\mu} \hat{g}^{i\beta} \left( \hat{\Gamma}_{\alpha\beta}^{\mu} - \Gamma_{\alpha\beta}^{\mu} \right) \\
= g_{jk} g^{\ell m} (\hat{\Gamma}_{\ell m}^{k} - \Gamma_{\ell m}^{k}) + V^{-2} g_{jk} (U \nabla^j U - V \nabla^j V) \\
= g^{\ell m} (\nabla_m g_{j\ell} - \frac{1}{2} \nabla^j g_{j\ell}) + V^{-2} g_{jk} (U \nabla^j U - V \nabla^j V)
\]
\]
\[
(4.4)
\]
where \( \nabla \)-derivatives are relative to a fixed metric \( b \) with Christoffel symbols \( \Gamma^\alpha_{\beta\gamma} \), \( U \) is a fixed positive function, latin indices run from 0 to \( n \), and \( \tilde{g} := V^2(dx^0)^2 + g \) with Christoffel symbols \( \tilde{\Gamma}^\alpha_{\beta\gamma} \), while the \( \tilde{\Gamma}^\alpha_{\beta\gamma} \)'s are the Christoffel symbols of the metric \( U^2(dx^0)^2 + b \), compare (A.1) below. The co-vector field \( \Omega \) has been chosen to contain terms which cancel the “non-elliptic terms” in the Ricci tensor, together with some further terms which will ensure bijectivity of the operators involved. The second line of the equation above makes clear the relation of \( \Omega \) to the \( n+1 \)-dimensional metric \( \tilde{g} \) and its \((U, b)\)-equivalent.

In dimension \( n \), as a first step we will solve the system

\[
\begin{align*}
q(V, g) &:= V(\nabla^* \nabla V + nV + \langle \Omega, dV \rangle) - \frac{1}{4}|\lambda|_g^2 = 0 , \\
Q(V, g) &:= \text{Ric}(g) + ng - V^{-1} \text{Hess}_g V + \text{div}^* \Omega - \frac{1}{2V^2} \lambda \circ \lambda = 0 , \\
\text{div}(V \lambda) &:= V^{-3} \nabla^i (V^3 \theta_i) ,
\end{align*}
\]

where \( \Omega \) is as in dimension 3, while the “Lorenz-gauge fixing function” \( \sigma \) equals

\[
\sigma = V^{-3} \nabla^i (V^3 \theta_i) .
\]

A calculation shows

\[
\text{div}(V \lambda) + V^3 d\sigma = V^3 [-Q + 2(V^{-1} \nabla^* \nabla V + n)](\theta) ,
\]

where \( Q \) is as in Corollary 3.3, which makes clear the elliptic character of the third equation in (4.5).

The derivative of \( \Omega \) with respect to \((V, g)\) at \((U, b)\) is

\[
D_{(V, g)} \Omega(U, b)(W, h) = -w ,
\]

where \( w \) is the one-form defined in Section 4.1 with \((V, g)\) replaced with \((U, b)\). Thus, the linearisation of \((q, Q)\) at \((U, b)\) is

\[
D(q, Q)(U, b) = (l, L) ,
\]

where \((l, L)\) is the operator defined in Section 4.1 with \((V, g)\) replaced with \((U, b)\). We will show that, under reasonable conditions, solutions of (4.3) (resp. (4.5)) are solutions of (4.2) (resp. (4.1)). If \((\omega, V, g)\) solves (4.3) (resp. if \((\theta, V, g)\) solves (4.5)), we set

\[
\Phi := \text{div}^* \Omega , \\
a := \begin{cases} 
\frac{1}{2V^2} |d\omega|^2 & \text{in the context of (4.3)}, \\
\frac{1}{4V^2} |\lambda|_g^2 & \text{in the context of (4.5)}, 
\end{cases}
\]

\[
A := \begin{cases} 
\frac{1}{2V^2} (d\omega \omega - |d\omega|^2 g) & \text{in the context of (4.3)}, \\
\frac{1}{2V^2} \lambda \circ \lambda & \text{in the context of (4.5)}. 
\end{cases}
\]

With this notation, the first two equations in both (4.3) and (4.5) take the form

\[
\begin{align*}
\nabla^* \nabla V + nV + \langle \Omega, dV \rangle & = Va , \\
\text{Ric}(g) + ng - V^{-1} \text{Hess}_g V + \Phi & = A .
\end{align*}
\]

If we take the trace of the second equation in (4.6) we obtain

\[
0 = R(g) + n^2 + V^{-1} \nabla^* \nabla V + \text{Tr} \Phi - \text{Tr} A \\
= R(g) + n^2 - n - V^{-1} \langle \Omega(V, g), dV \rangle + \text{Tr} \Phi + a - \text{Tr} A .
\]
Then
\[ E(g) := \text{grav}_g \text{Ric}(g) \]
\[ = -ng + V^{-1} \text{Hess} V - \Phi - \frac{1}{2}(-n(n - 1) + V^{-1}\langle \Omega(V, g), dV \rangle - \text{Tr} \Phi)g + \text{grav}_g A + \frac{a}{2}g. \]

As usual, we will use the vanishing of the divergence of \( E \) to obtain an equation for \( \Omega \). For a solution to the modified equation, the divergence of \( E(g) \) equals
\[ \text{div} E(g)_j = V^{-2} \nabla^i V \nabla_i \nabla_j V - V^{-1} \nabla^i \nabla_j V + \nabla^i \Phi_{ij} - \frac{1}{2} \nabla_j \text{Tr} \Phi \\
+ \frac{1}{2} \nabla_j (V^{-1}\langle \Omega(V, g), dV \rangle) + \left( \text{div} \left( \text{grav}_g A + \frac{a}{2}g \right) \right)_j \\
= V^{-1} \nabla^i V (R_{ij} + ng_{ij} + \Phi_{ij} - A_{ij}) - V^{-1} \nabla^i V R_{ij} \\
- V^{-1} \nabla_j (nV + \langle \Omega(V, g), dV \rangle + V a) + \nabla^i \Phi_{ij} - \frac{1}{2} \nabla_j \text{Tr} \Phi \\
+ \frac{1}{2} \nabla_j (V^{-1}\langle \Omega(V, g), dV \rangle) + \left( \text{div} \left( \text{grav}_g A + \frac{a}{2}g \right) \right)_j \\
= \nabla^i \Phi_{ij} + V^{-1} \nabla^i V \Phi_{ij} - \frac{1}{2} \nabla_j \text{Tr} \Phi - \frac{1}{2} V^{-1} \nabla_j \langle \Omega(V, g), dV \rangle \\
- \frac{1}{2} V^{-2} \nabla_j V \langle \Omega(V, g), dV \rangle + V^{-1} \nabla j (V a) - V^{-1} \nabla^i V A_{ij} + \left( \text{div} \left( \text{grav}_g A + \frac{a}{2}g \right) \right)_j \\
= \nabla^i \Phi_{ij} + V^{-1} \nabla^i V \Phi_{ij} - \frac{1}{2} \nabla_j \text{Tr} \Phi - \frac{1}{2} V^{-1} \nabla_j (V \langle \Omega(V, g), dV \rangle) + \beta_j \\
= \frac{1}{2} [\nabla^k \nabla_k \Omega_j + V^{-1} \nabla^i V \nabla_i \Omega_j - V^{-2} \nabla_j V \nabla^i V \Omega_i + R_{ij} \Omega^i - V^{-1} \nabla_j \nabla^i V \Omega_i] + \beta_j \\
= \frac{1}{2} [B(\Omega)_j + R_{ij} \Omega^i - V^{-1} \nabla_j \nabla^i V \Omega_i] + \beta_j \\
= \frac{1}{2} B(\Omega)_j + \beta_j. \]

We now claim that \( \beta_j \) vanishes when \( \sigma \) does. For (4.3) this is a straightforward computation. For (4.5) we have
\[ -\beta_j = \frac{1}{2} V \lambda^k \nabla_k \sigma - \frac{1}{2} V^{-3} \nabla_j V |\lambda|^2 + \frac{1}{8} V^{-2} \nabla_j |\lambda|^2 \\
+ \frac{1}{2} V^{-2} \lambda^i \lambda^k \lambda_{ij} + V^{-3} \nabla^i V \lambda^k \lambda_{jk}. \]

From the definition of \( \lambda_{ij} \) one has
\[ \nabla_{[i} (V^{-2} \lambda_{kj]) = 0. \]

This gives
\[ \lambda_{ik} (2 \nabla_i \lambda_{kj} + \nabla_j \lambda_{ik} - 6 V^{-1} (\nabla_{[i} V) \lambda_{kj]) = 0, \]
which can also be rewritten as
\[ \lambda^i \nabla_i \lambda^j = -\frac{1}{4} \nabla_j |\lambda|^2 + 2V^{-1} \nabla_i V \lambda^i \lambda^k \lambda^j + V^{-1} \nabla_j V |\lambda|^2, \]
and our claim follows.

We will see during the construction to follow that solutions of the third
equation in (4.5) which decay sufficiently fast satisfy \( \sigma = 0 \). The Bianchi
identity \( \text{div} E(g) = 0 \) shows then that \( \Omega \) is in the kernel of \( \mathcal{B} \). It follows from
Corollary 3.5 that the only solution of this equation which decays sufficiently
fast is zero.

5. The construction

5.1. The \( n \)-dimensional case. We consider \( \tilde{V}^2 d\varphi^2 + \hat{g} \), an asymptoti-
cally hyperbolic Einstein static metric on \( S^1 \times M \). We prescribe
\( \hat{\theta} \in C^{k+2,\alpha}(\partial M, T_1) \), and we seek a solution
\[ \theta = \theta(\hat{\theta}, V, g) \in C_{1}^{k+2,\alpha}(M, T_1) \]
of the problem
\begin{equation}
\begin{cases}
\text{div}(V \lambda) + V^3 d\sigma = -Q + 2V^{-1} \nabla^* \nabla V + n, \\
\theta - \hat{\theta} \in C_{2}^{k+2,\alpha}(M, T_1);
\end{cases}
\end{equation}
recall that
\begin{equation}
\lambda_{ij} = -V^2 (\partial_i \theta_j - \partial_j \theta_i) \text{ and } \sigma = V^{-3} \nabla^i (V^3 \theta_i).
\end{equation}
Such solutions can be obtained by solving the following equation for \( \theta - \hat{\theta} \):
\[ [-Q + 2(V^{-1} \nabla^* \nabla V + n)](\theta - \hat{\theta}) = -[-Q + 2(V^{-1} \nabla^* \nabla V + n)]\hat{\theta}. \]
When \( V = \tilde{V} \) in (5.1), then the term \( V^{-1} \nabla^* \nabla V + n \) vanishes and the
operator is an isomorphism by Corollary 3.3 with \( \delta = 1 \). Thus, the
operator appearing there is an isomorphism for all nearby \( V \)'s. In fact,
for any Riemannian metric \( g \) on \( M \), close to \( \hat{g} \) in \( C_{0}^{k+2,\alpha}(M, S_2) \), with
\( g - \hat{g} \in C_{1}^{k+2,\alpha}(M, S_2) \), and for any function \( V \) on \( M \), close to \( \tilde{V} \) in
\( C_{-1}^{k+2,\alpha}(M) \), with \( V - V_0 \in C_{0}^{k+2,\alpha}(M) \) a unique solution exists. Moreover
the map \( (\hat{\theta}, V, g) \mapsto \theta - \hat{\theta} \) is smooth.

Let us denote by \( \tilde{\theta} \) the solution of (5.1) with \( (V, g) = (\tilde{V}, \hat{g}) \).

\textbf{Remark 5.1.} \( \hat{\theta} \) is polyhomogenous when \( \tilde{V} \) and \( \hat{g} \) are by the results in [5].
Applying the second line of (5.1) twice we obtain
\[ \theta - \hat{\theta} = \theta - \hat{\theta} + \hat{\theta} - \hat{\theta} \in C_{2}^{k+2,\alpha}(M, T_1). \]
Furthermore, one has directly that \( \sigma - \hat{\sigma} \in C_{2}^{k+1,\alpha}(M) \); \( \hat{\sigma} \) is in fact also in
\( C_{2}^{k+1,\alpha}(M) \) by expanding near the boundary.

Suppose that \( \theta \) solves \( \text{div}(V \lambda) + V^3 d\sigma = 0 \), then clearly
\[ \text{div}[\text{div}(V \lambda) + V^3 d\sigma] = 0. \]
Since the double divergence of any anti-symmetric tensor vanishes identically it holds that $\text{div div}(V\lambda) = 0$, so that under (5.1) $\sigma$ is in $C^{k+1,\alpha}_{2}(M)$ and verifies

$$\nabla^i(V^3\nabla_i\sigma) = 0.$$  

It follows from Theorem 3.6 that $\sigma = 0$ when $n \geq 2$.

Let us define a map $F$, from one-forms on $\partial_{\infty}M$ times functions on $M$ times symmetric two-tensor fields on $M$ to functions on $M$ times symmetric two-tensor fields, which to $(\hat{\theta}, V, g)$ associates

$$\left( V(\nabla^*\nabla V + nV + \Omega(V, g, \hat{V}, \hat{g}, d\hat{V})) - \frac{1}{4}\lambda|\lambda|^2 - \frac{1}{2V^2} \lambda \circ \lambda \right).$$

**Proposition 5.2.** Let $\tilde{V}^2d\varphi^2 + \hat{g}$ be an asymptotically hyperbolic static Einstein metric on $S^1 \times M$, $k \in \mathbb{N}$, $\alpha \in (0, 1)$. The map $F$ defined as

$$C^{k+2,\alpha}(\partial M, T_1) \times C^{k+2,\alpha}_{1}(M) \times C^{k+2,\alpha}_{2}(M, S_2) \rightarrow C^{k,\alpha}_{0}(M) \times C^{k,\alpha}_{2}(M, S_2)$$

$$(\hat{\theta}, W, h) \mapsto F(\hat{\theta}, \hat{V} + W, \hat{g} + h)$$

is smooth in a neighborhood of zero.

**Proof.** The function $\hat{V} \in C^{k+2,\alpha}_{-1}(M)$ is strictly positive, so the same is true for $\hat{V} + W$ if $W$ is sufficiently small in $C^{k+2,\alpha}_{1}(M) \subset C^{k+2,\alpha}_{-1}(M)$. Similarly, the symmetric two-tensor field $\hat{g} + h \in C^{k+2,\alpha}_{0}(M, S_2)$ is positive definite when $h$ is small in $C^{k+2,\alpha}_{2}(M, S_2) \subset C^{k+2,\alpha}_{0}(M, S_2)$. The map $(\hat{\theta}, V, g) \mapsto \theta$ is smooth. Now, for $\theta \in C^{k+2,\alpha}_{1}(M)$, by (5.2) and by Remark 5.1 we have

$$\lambda_{ij} = O(\rho^{-2}),$$

which further implies

$$\frac{1}{2V^2} \lambda \circ \lambda \in C^{k,\alpha}_{2}(M, S_2).$$

The fact that the remaining terms in $F(\hat{\theta}, \hat{V} + v, \hat{g} + h)$ are in the space claimed, and that the map is smooth is standard (see [13] for instance).

We can conclude now as follows:

**Theorem 5.3.** Let $n \geq 2$, and let $\tilde{V}^2d\varphi^2 + \hat{g}$ be a polyhomogenous non-degenerate asymptotically hyperbolic static Einstein metric on $S^1 \times M$, $k \in \mathbb{N}\setminus\{0\}$, $\alpha \in (0, 1)$. For all $\hat{\theta}$ close to zero in $C^{k+2,\alpha}(\partial M, T_1)$, there exists a unique solution

$$(\theta, V, g) = (\hat{\theta} + \vartheta, \hat{V} + W, \hat{g} + h)$$

to (4.1) with $\hat{\theta} - \hat{\theta} \in C^{k+2,\alpha}_{2}(M)$ and

$$(\vartheta, W, h) \in C^{k+2,\alpha}_{3}(M) \times C^{k+2,\alpha}_{1}(M) \times C^{k+2,\alpha}_{2}(M, S_2),$$

close to zero, satisfying the gauge conditions $\Omega = \sigma = 0$. Moreover, the maps $\hat{\theta} \mapsto \hat{\theta} - \hat{\theta}$ and $\hat{\theta} \mapsto (\vartheta, W, h)$ are smooth maps of Banach spaces near zero.
5.2. The three-dimensional case. In three dimensions an alternative construction can be given, as follows. We consider again an asymptotically hyperbolic Einstein static metric $\hat{V}^2 d\varphi^2 + \hat{g}$ on $S^1 \times M$. We use Theorem 3.9, with $g$ — a Riemannian metric on $M$ close to $\hat{g}$ in $C^{k+2,\alpha}_0(M, S_2)$, and $V$ — a function on $M$ close to $\hat{V}$ in $C^{k+2,\alpha}_1(M)$. For our purposes there is no preferred value of parameter $\delta$ there. It is convenient to set $\delta = s - 1$, and arbitrarily choose some $s \in (0, 1)$. For any $\hat{\omega} \in C^{k+1,\alpha}(\partial M)$ satisfying

$$
\int_{\partial M} \hat{\omega} = 0,
$$

there exists a unique, modulo constant, solution

$$
\omega = \omega(\hat{\omega}, V, g) \in C^{k+2,\alpha}_{-1}(M)
$$

to

$$
\begin{cases}
\nabla^*(V^{-3} \nabla \omega) = 0, \\
\omega - \hat{\omega} \rho^{-1} \in C^{k+2,\alpha}_{s-1}(M).
\end{cases}
$$

(This can be seen by writing $Z\delta \omega = -Z(\hat{\omega} \rho^{-1})$, and checking that the source term in the equation for $\delta \omega$ satisfies the integrability condition (3.3) when (5.3) holds.) Moreover, the map $(\hat{\omega}, V, g) \mapsto \omega - \hat{\omega} \rho^{-1}$ is smooth in the $C^{k+2,\alpha}_{s-1}(M)$ topology.

We define a new map $F$, defined on the set of functions on $\partial M$ times functions on $M$ times symmetric two-tensor fields, mapping to functions on $M$ times symmetric two-tensor fields, which to $(\hat{\omega}, V, g)$ associates

$$
V(\nabla^* \nabla V + 3V + (\Omega(V, g, \hat{V}, \hat{g}), dV)) - \frac{1}{2V^4} |d\omega|^2 - \frac{1}{2V^4} (d\omega d\omega - |d\omega|^2 g).
$$

PROPOSITION 5.4. Let $\hat{V}^2 d\varphi^2 + \hat{g}$ be an asymptotically hyperbolic static Einstein metric on $S^1 \times M$, $k \in \mathbb{N}$, $\alpha \in (0, 1)$. The map $F$ defined as

$$
C^{k+2,\alpha}(\partial M) \times C^{k+2,\alpha}_1(M) \times C^{k+2,\alpha}_2(M, S_2) \to C^{k,\alpha}_0(M) \times C^{k,\alpha}_2(M, S_2)
$$

is smooth in a neighborhood of zero.

Proof. The proof is essentially the same as that of Proposition 5.2. We simply note that for all $s \in (0, 1)$ we have, by direct estimations,

$$
V^{-4} (d\omega d\omega - |d\omega|^2 g) \in C^{k,\alpha}_2(M, S_2).
$$

\[\square\]
We are ready to formulate now:

**Theorem 5.5.** Let \( \dim M = 3 \) and let \( \hat{V}^2 d\varphi^2 + \hat{g} \) be a non-degenerate asymptotically hyperbolic static Einstein metric on \( S^1 \times M \), \( k \in \mathbb{N} \), \( \alpha \in (0, 1) \), \( s \in (0, 1) \). For all \( \hat{\omega} \) close to zero in \( C^{k+2,\alpha} (\partial M) \) and satisfying (5.3) there exists a unique solution \( (\omega, V, g) = (\hat{\omega} \rho^{-1} + w, \hat{V} + W, \hat{g} + h) \) to (4.2) with \( (w, W, h) \in C^{k+2,\alpha}(M_1) \times C^{k+2,\alpha}(M, S^2) \), close to zero, satisfying the gauge condition \( \Omega = 0 \). Moreover, the map \( \hat{\omega} \mapsto (w, W, h) \) is a smooth map of Banach spaces near zero.

**Proof.** The proof is identical to that of Theorem 5.3, making use of Proposition 5.4. \( \square \)

6. **Uniqueness**

So far we have shown that solutions are unique in the gauge \( \Omega = 0 \), together with the condition \( \sigma = 0 \) in the context of (4.1). We claim that any metrics satisfying the hypotheses of Theorem 1.1 can be brought to this gauge.

First, consider \( \sigma \); note that the one-form \( \theta \) of (1.1) is defined modulo the differential of a function \( f \) defined on \( M \); indeed, the replacement \( t \to t + f \) leads to \( \theta \to \theta + df \). We can then use Theorem 3.6 to find a unique \( f \) such that the function \( \sigma \) associated with \( \theta + df \) vanishes.

The vanishing of \( \Omega \) requires a smallness hypothesis, as well as some work. Suppose that we are given a couple \((V, g)\) near to \((\hat{V}, \hat{g})\). The second line of (4.4) shows that, in the notation of [7], the condition \( \Omega = 0 \) is exactly the condition \( \Delta_{\hat{g}} \hat{g} \text{Id} = 0 \), where \( \hat{g}' = \hat{V}^2 d\varphi^2 + \hat{g} \). The proof that \( \Omega \) can be made to vanish is established by inspection of the arguments of Section 4 of [7]. We simply note that the implicit function theorem, as invoked there, can be applied globally on \( M \) (rather than in a collar neighborhood of the boundary, as in [7]) if we assume that \((\hat{V}/V, V^{-2}\hat{g})\) is close to \((1, \hat{V}^{-2}\hat{g})\) in \( C^2(\overline{M}) \). Indeed, the linearised operator, denoted by \( L \) in [7], is again an isomorphism by the results of [13], as follows from the fact that \( \hat{g} \) is Einstein, with negative scalar curvature. (Actually, the Einstein equations are irrelevant for the question of \( \Omega = 0 \) gauge, as long as the Ricci tensor of \( \hat{g} \) is negative definite.)

Uniqueness of solutions up-to-diffeomorphism (which is the identity on the boundary) is a direct consequence of the above.

7. **Polyhomogeneity**

Let \( U_0 \subset \mathbb{R}^n \) be an open set, and let \( U = U_0 \times (0, \varepsilon) \) with coordinates \((x, y)\). For any \( \delta \in \mathbb{R} \), we denote by \( \mathcal{C}^\delta \) the space of functions \( f \in C^\infty(U) \) that satisfy, on any subset \( K \times (0, \varepsilon_0) \) with \( K \subset U_0 \) compact and \( 0 < \varepsilon_0 < \varepsilon \), estimates of the following form for all integers \( r \geq 0 \) and all multi-indices \( \alpha \):

\[
| (y \partial_y)^r \partial_x^\alpha f(x, y) | \leq C_{r,\alpha} y^\delta.
\]
(We use the multi-index notations $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\partial_x^\alpha = (\partial_{x_1})^{\alpha_1} \cdots (\partial_{x_n})^{\alpha_n}$.)

A smooth function $f : U \to \mathbb{R}$ is said to be polyhomogeneous (cf. [5, 15]) if there exists a sequence of real numbers $s_i \to +\infty$, a sequence of nonnegative integers $\{q_i\}$, and functions $f_{ij} \in C^\infty(U_0)$ such that

\begin{equation}
    f(x, y) = \sum_{i=1}^\infty \sum_{j=0}^{q_i} y^{s_i} (\log y)^j f_{ij}(x)
\end{equation}

in the sense that for any $\delta > 0$, there exists a positive integer $N$ such that

\begin{equation}
    f(x, y) - \sum_{i=1}^N \sum_{j=0}^{q_i} y^{s_i} (\log y)^j f_{ij}(x) \in \mathcal{C}^\delta.
\end{equation}

A function or tensor field on $M$ is said to be polyhomogeneous if its coordinate representation in local coordinates near the conformal boundary is polyhomogeneous. (We refer the reader to [8] for a discussion of equivalence of alternative definitions of polyhomogeneity.)

In this section, we apply the theory of [5] to conclude that solutions to (4.5) are polyhomogeneous. The key step in the proof is a regularity result for the linearised operator $(l, L, \mathcal{L})$. Following [5], we say that an interval $(\delta_-, \delta_+) \subset \mathbb{R}$ is a weak regularity interval for a second-order linear operator $P$ on the spaces $C^{0,\lambda}_\delta(M_R; S_2)$ if whenever $u$ is a locally $C^2$ section of $S_2$ such that $u \in C^{0,0}_\delta(M_R; S_2)$ and $Pu \in C^{0,\lambda}_\delta(M_R; S_2)$ with $\lambda \in (0, 1)$ and $\delta_- < \delta_0 < \delta < \delta_+$, it follows that $u \in C^{2,\lambda}_\delta(M_R; S_2)$. We use the notation of [7].

**Theorem 7.1.** Solutions given by Theorems 5.3 and 5.5 are polyhomogeneous. Similarly, solutions of (4.1) and (4.2) with smooth boundary data such that $\theta$ and $\rho^2(V^2d\varphi^2 + g)$ are in $C^2(\overline{M})$ are polyhomogeneous.

**Proof.** We start by noting that metrics such that $\theta$ and $\rho^2(V^2d\varphi^2 + g)$ are in $C^2(\overline{M})$ can be brought, near the boundary, to a gauge in which the equations are elliptic by setting $\sigma$ to zero as in Section 6, and then applying the results of Section 4 of [7] to the metric $V^2d\varphi^2 + g$. On the other hand, solutions given by Theorems 5.3 and 5.5 are directly in the closely related gauge $\Omega = 0$; those two gauges do not coincide, but the proof works in both gauges. Alternatively one could use the analysis in Section 4 of [7] to transform a $C^2$–compactifiable $V^2d\varphi^2 + g$ to the gauge $\Omega = 0$. A polyhomogeneous approximate solution $\tilde{V}^{-2}(dt + \tilde{\theta})^2 + \tilde{g}$ can then be constructed using a Fefferman-Graham expansion up-to-not-including the critical exponent.

For any $\phi = (0, \phi^2, \phi^1, \phi)$, function, two-tensor, one-form on $M$, define

$$ F[\phi] := (\rho\tilde{V}^{-1}q, \rho^2Q, Q)(\tilde{V} + \rho^{-1}0\phi, \tilde{g} + \rho^{-2}2\phi, \tilde{\theta} + 1\phi) $$

with $(q, Q, Q)$ as in (4.5), while for the solutions arising from Theorems 5.3 and 5.5 the one-form $\tilde{\theta}$ can be taken as the solution of the third equation in (4.5) with $V = \tilde{V}$ and $g = \tilde{g}$. ($F$ should not be confused with the map $F$ of the previous section.) Then $\phi$ satisfies $F[\phi] = 0$. One can apply [5, Theorem 5.1.1] to $F$, and thereby conclude that $\phi$ is polyhomogeneous. The argument proceeds as in [7, Section 5] and will not be repeated here. We
simply mention that the property, that the interval \( (0, n) \) is a weak regularity interval for the operator \( L^*[\phi_0] \) on the spaces \( AC^0_{k+\lambda}(M_R) \), is an immediate consequence of Corollaries 3.2 and 3.3.

\[ \square \]

**APPENDIX A. “Dimensional reduction” of some operators**

**A.1. Lichnerowicz Laplacian on two-tensor for a warped product metric.** We shall use the following coordinate systems on \( S^1 \times M \):

\[ (x^a) = (\varphi, x^j) = (x^0, x^i) = (x^0, ..., x^n). \]

**Lemma A.1.** Let \((M, g)\) be a Riemannian manifold, let \( V, W \) be two functions on \( M \), let \( h \) be a symmetric covariant two-tensor on \( M \) and let \( \xi \) be a one-form on \( M \). On \( S^1 \times M \) we consider the Riemannian metric \( \bar{g} = V^2 d\varphi^2 + g \) and the symmetric covariant two-tensor

\[ \bar{h} = 2V W d\varphi^2 + 2\xi_i d\varphi^i dx^j + h_{ij} dx^i dx^j \]

satisfying \( \mathcal{L}_\xi \bar{h} = 0 \), where \( \mathcal{L} \) denotes a Lie derivative. Then, in local coordinates, the Laplacian of \( \bar{h} \) has the following components:

\[
\begin{align*}
\bar{\nabla}^c \bar{\nabla}_c \bar{h}_{00} &= 2[V \nabla^k \nabla_k W - \nabla^k \nabla_k V W - \nabla^k V \nabla_k W - V^{-1} dV]^2 W + \nabla^k V \nabla^l V h_{kl}], \\
\bar{\nabla}^c \bar{\nabla}_c \bar{h}_{i0} &= \nabla^k V \nabla_k \xi_i - V^{-1} \nabla^k \nabla_k V \xi_i - V^{-1} \nabla^k V \nabla_k \xi_i - 3V^{-2} \nabla_i V \nabla^k V \xi_k, \\
\bar{\nabla}^c \bar{\nabla}_c \bar{h}_{ij} &= \nabla^k V \nabla_k h_{ij} + V^{-1} \nabla^k V \nabla_k h_{ij} - V^{-2} (\nabla_i V \nabla^k V h_{kj} + \nabla_j V \nabla^k V h_{ki}) + 4V^{-3} \nabla_i V \nabla_j V W.
\end{align*}
\]

**Proof.** The Christoffel symbols of the metric \( \bar{g} = V^2 d\varphi^2 + g \) are

\[ (A.1) \]

\[ \bar{\Gamma}^0_{0i} = \bar{\Gamma}^0_{ij} = \bar{\Gamma}^{0}_{0j} = 0, \quad \bar{\Gamma}^k_{ij} = \Gamma^k_{ij}, \quad \bar{\Gamma}^0_{i0} = V^{-1} \nabla_i V, \quad \bar{\Gamma}^k_{00} = -V \nabla^k V. \]

The covariant derivatives of \( \bar{h} \), in local coordinates, read

\[
\begin{align*}
\bar{\nabla}_0 \bar{h}_{00} &= 2V \nabla^k V \xi_k, \\
\bar{\nabla}_0 \bar{h}_{ij} &= -V^{-1} (\nabla_i V \xi_j + \nabla_j V \xi_i), \\
\bar{\nabla}_k \bar{h}_{ij} &= \nabla_k \xi_i - V^{-1} \nabla_k V \xi_i, \\
\bar{\nabla}_0 \bar{h}_{i0} &= V \nabla^k V h_{ki} - 2 \nabla_i V W, \\
\bar{\nabla}_k \bar{h}_{00} &= 2V \nabla_k W - 2 \nabla_k V W, \\
\bar{\nabla}_k \bar{h}_{ij} &= \nabla_k h_{ij}.
\end{align*}
\]

The result is obtained by substitution. \( \square \)

We recall that the Lichnerowicz Laplacian is

\[ (A.2) \]

\[ \bar{\Delta}_L \bar{h}_{ab} = -\bar{\nabla}^c \bar{\nabla}_c h_{ab} + R_{ac} \bar{h}^c_b + R_{bc} \bar{h}^c_a - 2 \bar{R}_{abcd} \bar{h}^{cd}. \]

The curvature tensor of the warped product metric \( \bar{g} = V^2 d\varphi^2 + g \) has the following components [16, Prop. 42, Chap. 7] (note, however, that our curvature tensor is the negative of the one in [16]):

\[
\begin{align*}
\bar{R}^l_{ijk} &= R^l_{ijk}, \quad \bar{R}^l_{0j0} = -V \nabla^j \nabla^l V, \quad \bar{R}^0_{ijk} = 0, \\
\bar{R}_{ik} &= R_{ik} - V^{-1} \nabla_k \nabla_i V, \quad \bar{R}_{0k} = 0, \quad \bar{R}_{00} = -V \nabla^i \nabla_i V.
\end{align*}
\]
The zero order terms in \((A.2)\) are thus
\[
\begin{align*}
\tilde{R}_{0c}\tilde{h}^c_0 &= 2\nabla^*\nabla V W, \\
2\tilde{R}_{0cd}\tilde{h}^{cd} &= -2V \nabla^i \nabla^j V h_{ij}, \\
\tilde{R}_{ic}\tilde{h}^c_0 + \tilde{R}_{0c}\tilde{h}^c_i &= R^l_i \xi_l - V^{-1} \nabla_i \nabla^l V \xi_l + V^{-1} \nabla^* \nabla V \xi_i, \\
2\tilde{R}_{icd}\tilde{h}^{cd} &= 2V^{-1} \nabla_i \nabla_j V \xi_j, \\
\tilde{R}_{icj}\tilde{h}^j_i + \tilde{R}_{jic}\tilde{h}^i_c &= R_{ik} h^c_k + R_{jk} h^i_k - V^{-1} \nabla_i \nabla_k V h^c_k - V^{-1} \nabla_j \nabla_k V h^i_k, \\
2\tilde{R}_{icijd}\tilde{h}^{ijd} &= 2R_{ikjhl} - 4V^{-2} \nabla_i \nabla_j V W.
\end{align*}
\]

Lemma A.1 implies now:

**Lemma A.2.** Under the hypotheses of Lemma A.1, the Lichnerowicz Laplacian of \(\tilde{h}\) is
\[
\begin{align*}
\tilde{\Delta}_{L}\tilde{h}_{00} &= 2 \left[ -V \nabla^k \nabla_k W - \nabla^k \nabla_k V W + \nabla^k V \nabla_k W + V^{-1} |dV|^2 W \\
&\quad - \nabla^k V \nabla^l \nabla h_{kl} + V \nabla^i \nabla^j V h_{ij} \right], \\
\tilde{\Delta}_{L}\tilde{h}_{ij} &= -\nabla^k \nabla_k \xi_i + V^{-1} \nabla^k V \nabla_k \xi_i + 3V^{-2} \nabla_i V \nabla^k V \xi_k \\
&\quad + R^l_i \xi_l - 3V^{-1} \nabla_i \nabla_j V \xi_j, \\
\tilde{\Delta}_{L}\tilde{\xi}_{ij} &= \Delta_L h_{ij} - V^{-1} \nabla^k V \nabla_k h_{ij} + V^{-2} (\nabla_i V \nabla^k V h_{kj} + \nabla_j V \nabla^k V h_{ki}) \\
&\quad - 4V^{-3} \nabla_i V \nabla_j V W - V^{-1} (\nabla_i \nabla^k V h_{kj} + \nabla_j \nabla^k V h_{ki}) \\
&\quad + 4V^{-2} \nabla_i \nabla_j V W.
\end{align*}
\]

\(\Box\)

**A.2. The Laplacian on one-forms for a warped product metric.**

**Lemma A.3.** Let \((M, g)\) be a Riemannian manifold, let \(V, f\) be two functions on \(M\) and let \(\Omega\) be a one-form on \(M\). Let us consider, on \(S^1 \times M\), the Riemannian metric \(\tilde{g} = V^2 d\varphi^2 + g\) and the one-form
\[
\tilde{\Omega} = f d\varphi + \Omega_i d\tilde{x}^i.
\]

Then in local coordinates, the Laplacian of \(\tilde{\Omega}\) equals
\[
\begin{align*}
\tilde{\nabla}^i \tilde{\nabla}_i \tilde{\Omega}_0 &= \nabla^k \nabla_k f - V^{-1} f \nabla^k \nabla_k V - V^{-1} \nabla^k V \nabla_k f, \\
\tilde{\nabla}^c \tilde{\nabla}_c \tilde{\Omega}_i &= \nabla^k \nabla_k \Omega_i + V^{-1} \nabla^k V \nabla_k \Omega_i - V^{-2} \nabla_i V \nabla^k V \Omega_k =: B(\Omega)_i.
\end{align*}
\]

**Proof.** We have
\[
\begin{align*}
\tilde{\nabla}_0 \tilde{\Omega}_0 &= V \nabla^k V \Omega_k, \\
\tilde{\nabla}_0 \tilde{\Omega}_i &= -V^{-1} \nabla_i V f, \\
\tilde{\nabla}_i \tilde{\Omega}_0 &= \partial_i f - V^{-1} \nabla_i V f, \\
\tilde{\nabla}_i \tilde{\Omega}_i &= \nabla_j \tilde{\Omega}_i,
\end{align*}
\]
and the result easily follows. \(\Box\)

**Acknowledgements** The research of PTC was supported in part by a Polish Research Committee grant 2 P03B 073 24. We are grateful to the Isaac Newton Institute, Cambridge, for hospitality and financial support during part of work on this paper.
References

1. M.T. Anderson, *Einstein metrics with prescribed conformal infinity on 4-manifolds*, (2001), math.DG/0105243.

2. ______, *Boundary regularity, uniqueness and non-uniqueness for AH Einstein metrics on 4-manifolds*, Adv. in Math. 179 (2003), 205–249, math.DG/0104171.

3. M.T. Anderson, P.T. Chruściel, and E. Delay, *Non-trivial, static, geodesically complete vacuum space-times with a negative cosmological constant*, JHEP 10 (2002), 063, 22 pp., gr-qc/0211006.

4. ______, *Non-trivial, static, geodesically complete space-times with a negative cosmological constant. II. n ≥ 5, AdS/CFT correspondence: Einstein metrics and their conformal boundaries*, IRMA Lect. Math. Theor. Phys., vol. 8, Eur. Math. Soc., Zürich, 2005, pp. 165–204.

5. L. Andersson and P.T. Chruściel, *On asymptotic behavior of solutions of the constraint equations in general relativity with “hyperboloidal boundary conditions”*, Dissert. Math. 355 (1996), 1–100.

6. A.L. Besse, *Einstein manifolds*, Ergebnisse d. Math. 3. Folge, vol. 10, Springer, Berlin, 1987.

7. P.T. Chruściel, E. Delay, J.M. Lee, and D.N. Skinner, *Boundary regularity of conformally compact Einstein metrics*, Jour. Diff. Geom. 69 (2005), 111–136, math.DG/0401386.

8. P.T. Chruściel and S. Łeśki, *Polynomial solutions of nonlinear wave equations without corner conditions*, Jour. Hyp. PDE 3 (2006), 81–141, math.ap/0506423.

9. R. Coquereaux and A. Jadczak, *Riemannian geometry, fiber bundles, Kaluza-Klein theories and all that*, World Sci. Lect. Notes Phys. 16 (1988), 1–345.

10. C. Fefferman and C.R. Graham, *Conformal invariants*, Élie Cartan et les mathématiques d’aujourd’hui, The mathematical heritage of Élie Cartan, Sémin. Lyon 1984, Astérisque, No.Hors Sér. 1985, 95-116.

11. C.R. Graham and K. Hirachi, *The ambient obstruction tensor and Q-curvature*, AdS/CFT correspondence: Einstein metrics and their conformal boundaries, IRMA Lect. Math. Theor. Phys., vol. 8, Eur. Math. Soc., Zürich, 2005, pp. 59–71.

12. D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact solutions of Einstein’s field equations*, Cambridge University Press, Cambridge, 1980.

13. J.M. Lee, *Fredholm operators and Einstein metrics on conformally compact manifolds*, Memoirs AMS, in press, math.DG/0105046.

14. R. Mazzeo, *The Hodge cohomology of a conformally compact metric*, Jour. Diff. Geom. 28 (1988), 309–339.

15. ______, *Elliptic theory of differential edge operators I*, Commun. Partial Diff. Equations 16 (1991), 1615–1664.

16. B. O’Neill, *Semi–Riemannian geometry*, Academic Press, New York, 1983.

Piotr T. Chruściel, LMPT, Fédération Denis Poisson, Faculté des Sciences, Parc de Grandmont, F37200 Tours, France.

E-mail address: piotr.chrusciel@lmpt.univ-tours.fr
URL: http://www.phys.univ-tours.fr/~piotr

Erwann Delay, Laboratoire d’analyse non linéaire et géométrie, Faculté des Sciences, 33 rue Louis Pasteur, 84000 Avignon, France

E-mail address: Erwann.Delay@univ-avignon.fr
URL: http://www.math.univ-avignon.fr/Delay